**SAILING OVER THREE PROBLEMS OF KOSZMIDER**

FÉLIX CABELLO SÁNCHEZ, JESÚS M.F. CASTILLO, WITOLD MARCISZEWSKI, GRZEGORZ PLEBANEK, AND ALBERTO SALGUERO-ALARCÓN

**Abstract.** We discuss three problems of Koszmider on the structure of the spaces of continuous functions on the Stone compact $K_A$ generated by an almost disjoint family $\mathcal{A}$ of infinite subsets of $\omega$ — we present a solution to two problems and develop a previous result of Marciszewski and Pol answering the third one. We will show, in particular, that assuming Martin’s axiom the space $C(K_A)$ is uniquely determined up to isomorphism by the cardinality of $\mathcal{A}$ whenever $|\mathcal{A}| < c$, while there are $2^c$ nonisomorphic spaces $C(K_A)$ with $|\mathcal{A}| = c$. We also investigate Koszmider’s problems in the context of the class of separable Rosenthal compacta and indicate the meaning of our results in the language of twisted sums of $c_0$ and some $C(K)$ spaces.

1. **Introduction**

Koszmider poses in [23] five problems about the structure of the spaces of continuous functions on the Stone compact $K_A$ generated by an almost disjoint family $\mathcal{A}$ of infinite subsets of $\omega$. Problem 2, that Koszmider himself solves, is the existence of an almost disjoint family $\mathcal{A}$ such that under either the Continuum Hypothesis CH or Martin’s Axiom MA,

$$C(K_A) \simeq c_0 \oplus C(K_A)$$

is the only possible decomposition of $C(K_A)$ in two infinite dimensional subspaces. Problem 1 asks whether a similar separable space exists. Argyros and Raikoftsalis [1] call a Banach space $\mathcal{X}$ quasi-prime if there exists an infinite dimensional subspace $\mathcal{Y}$ such that $\mathcal{X} \simeq \mathcal{Y} \oplus \mathcal{X}$ is the only possible nontrivial decomposition of $\mathcal{X}$. If, moreover $\mathcal{Y}$ is not isomorphic to $\mathcal{X}$ then $\mathcal{X}$ is called strictly quasi-prime. Argyros and Raikoftsalis [1] show the existence, for each $\ell_p$, $p \geq 1$, (resp. $c_0$) of a separable strictly quasi-prime space $\mathcal{X}_p \simeq \ell_p \oplus \mathcal{X}_p$ (resp. $\mathcal{X}_0 \simeq c_0 \oplus \mathcal{X}_0$).

We are concerned in this paper with the other three:

- **Problem 3** Assuming MA, is it true that if $|\mathcal{A}| = |\mathcal{B}| < c$ then $C(K_A) \simeq C(K_B)$?
- **Problem 4** Assuming MA and $|\mathcal{A}| < c$ is $C(K_A) \simeq C(K_A) \oplus C(K_A)$?
- **Problem 5** Are there two almost disjoint families $\mathcal{A}, \mathcal{B}$ of the same cardinality such that $C(K_A)$ is not isomorphic to $C(K_B)$?

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Koszmider’s questions are mentioned by Hrušák [19, 9.2] in his survey on applications of almost disjoint families. Let us point out that Problem 5 was actually solved by Marciszewski and Pol [28] who gave an example of a pair of almost disjoint families \( A, B \) of cardinality \( c \) such that \( C(K_A) \not\cong C(K_B) \) — this is a direct consequence of [28, Theorem 3.4]. It was, moreover, briefly outlined in [28, 7.4] how one can prove, using some ideas from [25], that there are \( 2^c \) isomorphism types of Banach spaces of the form \( C(K_A) \).

We will solve affirmatively Problems 3 and 4 and present a detailed, self-contained solution to Problem 5. Summing up, we will obtain

**Theorem 1.1.**

(a) Under \( \text{MA}(\aleph) \), if \( A,B \) are almost disjoint families such that \( |A| = |B| = \aleph \), then \( C(K_A) \cong C(K_B) \).

(b) Under \( \text{MA}(\aleph) \), if \( A \) is an almost disjoint family with \( |A| = \aleph \), then \( C(K_A) \cong C(K_A) \oplus C(K_A) \).

(c) There are \( 2^c \) nonisomorphic spaces \( C(K_A) \) for almost disjoint families \( A \) of size \( c \).

Here \( \text{MA}(\aleph) \) denotes Martin’s axiom for ccc partial orders and \( \aleph \) many dense sets (recall that if we assume \( \text{MA}(\aleph) \) then, automatically, \( \aleph < c \)).

Our proof of Theorem 1.1(c) follows a relatively simple counting argument suggested in [28]. Such a reasoning does not provide ‘concrete’ examples of pairs of nonisomorphic Banach spaces. We show, however, that building on descriptive properties of separable Rosenthal compacta, one can name an uncountable sequence of almost disjoint families \( \{A_\xi : \xi < \omega_1\} \) such that every \( K_{A_\xi} \) is a Rosenthal compactum and the Banach spaces \( C(K_{A_\xi}) \) are pairwise nonisomorphic.

We also analyze the implications of these problems regarding the study of twisted sums of \( c_0 \) and \( C(K) \) spaces. It follows from our results that there are \( 2^c \) pairwise nonisomorphic twisted sums of \( c_0 \) and \( C(K) \) where \( K \) is either the one point compactification of the discrete space \( c \) or \( K \) is the classical Rosenthal compact called the double arrow space. The latter partially extends [3, Corollary 4.11] stating that \( c_0 \) admits a nontrivial twisted sum with \( C(K) \) for every nonmetrizable separable Rosenthal compact \( K \).

2. Preliminaries on twisted sums of Banach spaces

We will write \( A \cong B \) to mean that the Banach spaces \( A \) and \( B \) are isomorphic. An exact sequence \( z \) of Banach spaces is a diagram

\[
0 \longrightarrow Y \xrightarrow{j} Z \xrightarrow{\rho} X \longrightarrow 0
\]

formed by Banach spaces and linear continuous operators in which the kernel of each arrow coincides with the image of the preceding one. The middle space \( Z \) is usually called a twisted sum of \( Y \) and \( X \). By the open mapping theorem, \( Y \) must be isomorphic to a subspace of \( Z \) and \( X \) to the quotient \( Z/Y \). Two exact sequences \( z \) and \( s \) are said to be equivalent, denoted \( z \equiv s \),
if there is an operator $T : Z \to S$ making commutative the diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & Y & \xrightarrow{j} & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow T & & \downarrow & & \\
0 & \longrightarrow & Y & \xrightarrow{j_s} & S & \xrightarrow{\rho_s} & X & \longrightarrow & 0 \\
\end{array}
$$

The sequence $z$ is said to be trivial, or to split, if the injection $j$ admits a left inverse; i.e., there is a linear continuous projection $P : Z \to Y$ along $j$. Equivalently, if $z \equiv 0$ where 0 denotes the exact sequence $0 \to Y \to Y \oplus X \to X \to 0$. Given an exact sequence $z$ and an operator $\gamma : X' \to X$ the pull-back exact sequence $z\gamma$ is the lower sequence in the diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & Y & \xrightarrow{j} & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \xrightarrow{j_s} & PB & \xrightarrow{\gamma} & X' & \longrightarrow & 0 \\
\end{array}
$$

where $PB = \{(z, x') \in Z \oplus \infty X' : \rho z = \gamma x'\}$ endowed with the subspace norm. Dually, given an operator $\alpha : Y \to Y'$ the push-out exact sequence $\alpha z$ is the lower sequence in the diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & Y & \xrightarrow{j} & Z & \xrightarrow{\rho} & X & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \alpha & & \downarrow \gamma & & \\
0 & \longrightarrow & Y' & \xrightarrow{j_s} & PO & \xrightarrow{\gamma} & X & \longrightarrow & 0 \\
\end{array}
$$

where $PO = (Y' \oplus_1 Z) / \Delta$, where $\Delta = \{(\alpha y, -jy) : y \in Y\}$ endowed with the quotient norm. We need to keep in mind the following two facts:

**Proposition 2.1.** Let $z$ and $s$ be two exact sequences.

1. If $s \rho z \equiv 0$ then there is an operator $\tau$ such that $\tau z \equiv s$.
2. If $s \rho z \equiv 0 \equiv z \rho s$ then $Y \oplus S \simeq Y \oplus Z$.

**Proof.** The first fact can be seen in [5] and is an easy consequence of the homology sequence; the second is the so-called diagonal principle [11].

The space of exact sequences between two given spaces $Y$ and $X$ modulo equivalence will be denoted $\text{Ext}(X,Y)$ (in reverse order). We will write $\text{Ext}(X,Y) = 0$ to mean that all elements $z \in \text{Ext}(X,Y)$ are $z \equiv 0$.

In this paper we are mainly concerned with twisted sums of $c_0$ and $C(K)$, where $K$ is a compact space. These objects have a long tradition; for instance, if $A$ is the family of branches of the dyadic tree then the space $C(K_A)$ was introduced in [21] and is usually called a Johnson-Lindenstrauss space. The classical Sobczyk theorem, $c_0$ is complemented in any separable Banach superspace, implies that $\text{Ext}(C(K), c_0) = 0$ whenever $K$ is metrizable. The question of whether $\text{Ext}(C(K), c_0) \neq 0$ for every non-metrizable $K$ was posed in [8, 9] and has been intensively studied [10, 11, 27, 3, 14, 16]. While for some classes of compacta $K$ one can demonstrate that
Ext($C(K), c_0 \neq 0$ more or less effectively, the problem cannot be decided within the usual set theory. Indeed, one has

**Theorem 2.2.**

(a) Assuming CH, Ext($C(K), c_0 \neq 0$ for every non-metrizable compact space $K$ (see [3, Theorem 5.8]).

(b) Assuming MA($\aleph_1$), Ext($C(K_\mathcal{A}), c_0 = 0$ whenever $|\mathcal{A}| = \aleph_1$ ([27, Corollary 5.3]).

(c) Assuming MA($\aleph_1$), Ext($C(K), c_0 = 0$ whenever $K$ is a separable scattered space of weight $\aleph_1$ and finite height (this extension of (b) is due to Correa and Tausk [16, Corollary 4.2]).

3. Preliminaries on compacta

The most natural method of constructing twisted sums of $c_0$ and $C(K)$ is to consider a countable discrete extension $L = K \cup \omega$, that is, a compact space containing $K$ as a subspace and containing a countable set of isolated points $\omega$ (we tacitly assume that $K \cap \omega = \emptyset$). Then the subspace $Z = \{ g \in C(L) : g|K \equiv 0 \}$ is a natural copy of $c_0$, whereas $C(L)/Z$ may be identified with $C(K)$. This was fully discussed in [27] and [3]. Recall the following observation (see [27, Theorem 2.8(a)] or [3, Theorem 4.13]).

**Theorem 3.1.** Suppose that $L = K \cup \omega$ is a countable discrete extension of a compact space such that $\omega$ is dense in $L$ (in other words, $L$ is a compactification of $\omega$ whose remainder is homeomorphic to $K$). If $K$ does not carry a strictly positive measure, then $C(L)$ is a nontrivial twisted sum of $c_0$ and $C(K)$.

Our solution to Problem 5 is motivated by examining the variety of twisted sums of $c_0$ and $C(K)$, where $K$ is either the Stone space constructed from an almost disjoint family of subsets of $\omega$, or $K = S$, where $S$ denotes the classical double arrow space (see below).

3.1. The Stone compact of an almost disjoint family. Recall that a topological space is scattered if every of its subsets contains an isolated point (see [32] for the basic properties and further references). We consider here some scattered compact spaces $K$; $K'$ is the first derivative (the set of nonisolated points in $K$). Higher Cantor-Bendixon derivatives $K^{(\alpha)}$ are defined inductively; the height of $K$ is the first ordinal number $\alpha$ for which $K^{(\alpha)} = \emptyset$.

Recall that a family $\mathcal{A}$ of infinite subsets of $\omega$ is almost disjoint if $A \cap B$ is finite for any distinct $A, B \in \mathcal{A}$. To every almost disjoint family $\mathcal{A}$ one can associate the Stone space of the algebra of subsets of $\omega$ generated by $\mathcal{A}$ and all finite sets. This is a scattered compact space of height 3. Indeed,

$$K_\mathcal{A} = \omega \cup \{ A : A \in \mathcal{A} \} \cup \{ \infty \},$$

where points in $\omega$ are isolated, basic open neighborhoods of a given point $A$ are of the form $\{ A \} \cup (A \setminus F)$ with $F \subseteq \omega$ finite, and $K_\mathcal{A}$ is the one point compactification of the locally compact space $\omega \cup \{ A : A \in \mathcal{A} \}$, where $\infty$ is the point at infinity.

We write $A(\aleph_1)$ for the (Aleksandrov) one-point compactification of a discrete space of cardinality $\aleph_1$. The space $K_\mathcal{A}$ might be called the Aleksandrov-Urysohn compactum associated to an
almost disjoint family \( \mathcal{A} \); this terminology was used in \([28, 27] \) and \([3] \); see also \([18, \text{section 2}] \) (most often, spaces of the form \( K_{\mathcal{A}} \) are called Mrówka spaces). Hrušák \([19] \) offers a survey on various applications of that construction to topology and functional analysis.

Note that the Banach space \( c_0(\aleph) \) may be seen as a hyperplane in \( C(\aleph) \) and that \( c_0(\aleph) \) is isomorphic to \( C(\aleph) \). Moreover, if \( |A| = \aleph \) then \( K_A \) is a countable discrete extension of the space \( A(\aleph) \). Hence, the following is a direct consequence of Theorem 3.1.

**Theorem 3.2.** Given any almost disjoint family \( \mathcal{A} \) of cardinality \( \aleph > \omega \), the space \( C(K_{\mathcal{A}}) \) is a nontrivial twisted sum of \( c_0 \) and \( c_0(\aleph) \).

### 3.2. The double arrow space.
Actually, we mention here a single compactum, the double arrow space, and later we shall consider the class of its countable discrete extensions. We denote this classical space by \( S \); recall that
\[
S = \left( (0, 1] \times \{0\} \right) \cup \left( [0, 1) \times \{1\} \right),
\]
is equipped with the order topology given by the lexicographical order
\[
(s, i) < (t, j) \text{ if either } s < t, \text{ or } s = t \text{ and } i < j.
\]

The space \( S \) is a nonmetrizable separable compactum having a countable local base at every point. It will be convenient to see \( S \) as the Stone space of some algebra of subsets of a countable set. Let \( Q \) be a countable dense subset of \((0, 1)\), and for each \( x \in (0, 1) \) put \( P_x = \{ q \in Q : q \leq x \} \). Let \( \mathfrak{A} \) be the algebra of subsets of \( Q \) generated by the chain \( \{ P_x : x \in (0, 1) \} \). Then \( \text{ult}(\mathfrak{A}) \) is homeomorphic to \( S \).

To see this, note first that every \( \mathcal{F} \in \text{ult}(\mathfrak{A}) \) is uniquely determined by the set
\[
I(\mathcal{F}) = \{ x \in (0, 1) : P_x \in \mathcal{F} \},
\]
which is a subinterval of \((0, 1)\) of the form \([y, 1)\) or \((y, 1)\) for some \( y \in [0, 1] \). Put \( \mathcal{F} = \mathcal{F}^+_y \) if \( I(\mathcal{F}) = [y, 1) \) and \( \mathcal{F} = \mathcal{F}^-_y \) if \( I(\mathcal{F}) = (y, 1) \), and define \( h : S \to \text{ult}(\mathfrak{A}) \) by
\[
h(y, i) = \begin{cases} 
\mathcal{F}^-_y & \text{if } y \in [0, 1), i = 1; \\
\mathcal{F}^+_y & \text{if } y \in (0, 1], i = 0.
\end{cases}
\]
Then \( h \) is an homeomorphism of \( S \) and \( \text{ult}(\mathfrak{A}) \).

### 4. Solution to Problems 3 and 4

We start by recalling the following well-known observation:

**Remark 4.1.** If \( K \) is an infinite scattered compact space then \( K \) contains a nontrivial convergent sequence, and so \( C(K) \simeq c_0 \oplus C(K) \).

**Proof of Theorem 1.1 (a) and (b).** Pick two almost disjoint families \( \mathcal{A}, \mathcal{B} \) of subsets of \( \omega \) of cardinality \( \aleph < c \). Under \( \text{MA}(\aleph) \), we have
\[
\text{Ext}(C(K_{\mathcal{A}}), c_0) = 0 = \text{Ext}(C(K_{\mathcal{B}}), c_0),
\]

thanks to Theorem 2.2 (b). Hence, if we call \( a \) (resp. \( b \)) the two exact sequences in the diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & c_0 & \longrightarrow & C(K_A) & \longrightarrow & c_0(\aleph) & \longrightarrow & 0 \\
\downarrow & & & & \downarrow & & & & \\
0 & \longrightarrow & c_0 & \longrightarrow & C(K_B) & \longrightarrow & c_0(\aleph) & \longrightarrow & 0
\end{array}
\]

then \( a\rho_B \equiv 0 \equiv b\rho_A \) and thus by Proposition 2.1(2) and Remark 4.1

\[ C(K_A) \cong c_0 \oplus C(K_A) \cong c_0 \oplus C(K_B) \cong C(K_B). \]

Part (b) is consequence of the well known algebraic identity

\[ \text{Ext}(C(K_A) \oplus C(K_A), c_0) = \text{Ext}(C(K_A), c_0) \times \text{Ext}(C(K_A), c_0), \]

and the following

**Claim.** *If \( X \) is a twisted sum of \( c_0 \) and \( c_0(\aleph) \) so that \( \text{Ext}(X, c_0) = 0 \) then \( X \cong C(K_A) \).*

**Proof of the Claim.** Assume the existence of an exact sequence

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & c_0 & \longrightarrow & X & \longrightarrow & c_0(\aleph) & \longrightarrow & 0
\end{array}
\]

The same argument as before yields \( X \oplus c_0 \cong C(K_A) \oplus c_0 \cong C(K_A) \). All that is left to see is that the space \( X \) has a complemented copy of \( c_0 \). Indeed, it follows from [12] that every twisted sum space \( X \) as above has Pełczyński’s property (V). The quotient operator \( \rho \) is therefore an isomorphism on some copy of \( c_0 \). Since \( \rho(c_0) \) must be necessarily complemented in \( c_0(\aleph) \), \( c_0 \) will also be complemented in \( X \). Hence \( X \cong c_0 \oplus X \) and the proof concludes. \( \square \)

We do not know whether every twisted sum space

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & c_0 & \longrightarrow & X & \longrightarrow & c_0(\aleph) & \longrightarrow & 0
\end{array}
\]

must be isomorphic to a \( C(K) \)-space. However,

**Proposition 4.2.** *Let \( \mathcal{A} \) be an almost disjoint family of subsets of \( \omega \) such that \( |\mathcal{A}| = \aleph < c \). Under \( \text{MA}(\aleph) \), every twisted sum space \( X \)

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & c_0 & \longrightarrow & X & \longrightarrow & c_0(\aleph) & \longrightarrow & 0
\end{array}
\]

is a quotient of \( C(K_A) \).*

**Proof.** Since \( \text{Ext}(C(K_A), c_0) = 0 \) it follows from Proposition 2.1(1) that there is a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & c_0 & \longrightarrow & C(K_A) & \longrightarrow & c_0(\aleph) & \longrightarrow & 0 \\
\downarrow & & & & \downarrow & & & & \\
0 & \longrightarrow & c_0 & \longrightarrow & X & \longrightarrow & c_0(\aleph) & \longrightarrow & 0
\end{array}
\]

By the definition of the push-out space one has an exact sequence

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & c_0 & \longrightarrow & c_0 \oplus C(K_A) & \longrightarrow & X & \longrightarrow & 0
\end{array}
\]
Now, the obvious isomorphism $C(K_{A}) \simeq c_{0} \oplus C(K_{A})$ proves the assertion.

The following result contains a generalization of Theorem 1.1(a).

**Theorem 4.3.** Assume $\text{MA}(\mathbb{N})$ and let $K_{i}$, $i = 0, 1$ be separable scattered compacta of finite height and weight $\aleph_{0}$.

1. If $C(K'_{0}) \simeq C(K'_{1})$ then $C(K_{0}) \simeq C(K_{1})$.
2. If $C(K_{0}) \simeq C(K_{1})$ then $X_{0} \simeq X_{1}$ whenever $X_{i}$, $i = 0, 1$, is a twisted sum $0 \to c_{0} \to X_{i} \to C(K_{i}) \to 0$.
3. Moreover, all iterated twisted sums of $X_{K_{i}}$, $i = 0, 1$ and $c_{0}$ are also isomorphic.

**Proof.** By Remark 4.1, $C(K_{i}) \simeq c_{0} \oplus C(K_{i})$. Moreover, one has exact sequences

$$0 \longrightarrow c_{0} \longrightarrow C(K_{i}) \overset{R}{\longrightarrow} C(K'_{i}) \longrightarrow 0$$

where $R$ is the natural restriction map, which yields ker $R \simeq c_{0}$. Consider the two exact sequences

$$0 \longrightarrow c_{0} \longrightarrow C(K_{0}) \overset{\alpha R}{\longrightarrow} C(K'_{1}) \longrightarrow 0$$

$$0 \longrightarrow c_{0} \longrightarrow C(K_{1}) \overset{R}{\longrightarrow} C(K'_{1}) \longrightarrow 0$$

where $\alpha : C(K'_{0}) \to C(K'_{1})$ is an isomorphism. Let $z_{1}$ denote the lower sequence and $z_{0}$ the upper sequence. Since $K_{i}$ is a separable scattered compact of finite height, Theorem 2.2(c) implies $\text{Ext}(C(K_{i}), c_{0}) = 0$, and thus $z_{0}R \equiv 0 \equiv z_{1}\alpha R$. Therefore,

$$C(K_{0}) \simeq c_{0} \oplus C(K_{0}) \simeq c_{0} \oplus C(K_{1}) \simeq C(K_{1}),$$

and this proves (1). To prove (2), consider the two exact sequences $0 \to c_{0} \to X_{i} \overset{Q_{i}}{\to} C(K_{i}) \to 0$ and let $\alpha : C(K_{0}) \to C(K_{1})$ be an isomorphism. Recall that $\text{Ext}(X_{i}, c_{0}) = 0$ by a 3-space argument (see [5]) and thus a similar reasoning as above can be used with the two sequences

$$0 \longrightarrow c_{0} \longrightarrow X_{0} \overset{\alpha Q_{0}}{\longrightarrow} C(K_{1}) \longrightarrow 0$$

$$0 \longrightarrow c_{0} \longrightarrow X_{1} \overset{Q_{1}}{\longrightarrow} C(K_{1}) \longrightarrow 0$$

The spaces $X_{i}$ obviously contain complemented copies of $c_{0}$ since $C(K_{i})$ does so and the pull-back sequence

$$0 \longrightarrow c_{0} \longrightarrow X_{i} \overset{Q_{i}}{\longrightarrow} C(K_{i}) \longrightarrow 0$$

$$0 \longrightarrow c_{0} \longrightarrow c_{0} \oplus c_{0} \longrightarrow c_{0} \longrightarrow 0$$

splits by Sobczyk’s theorem. Therefore $X_{i} \simeq c_{0} \oplus X_{i}$. 

Example 4.4. The separability assumption in Proposition 4.3 is essential, and for that reason we cannot extend the result to higher derived sets in an obvious way. Indeed, take $\aleph < \mathfrak{c}$ and let $K_0$ be the subset of those elements $x$ of the Cantor cube $2^\aleph$ for which the support $\{\xi < \aleph : x(\xi) \neq 0\}$ has at most 2 elements. Take any almost disjoint family $\mathcal{A}$ of size $\aleph$ and $K_\mathcal{A}$ as the second compactum. Then $K_\mathcal{A}' = \mathcal{A}(\aleph) = K_\mathcal{A}'$; however, $C(K_0) \not\cong C(K_\mathcal{A})$ since $C(K_0)$ is weakly compactly generated (in other words, $K_0$ is Eberlein compact) while the latter space is not ($K_\mathcal{A}$ is not Eberlein compact since it is separable, but not metrizable).

Since each scattered compact space $K$ of height 2 is a finite sum of one-point compactifications of discrete spaces, the corresponding function space $C(K)$ is isomorphic to $c_0(|K|)$. Hence, from Theorem 4.3 we obtain the following corollaries:

**Corollary 4.5.** Assuming $\mathsf{MA}(\aleph)$, if $K_0$ and $K_1$ are separable scattered compact spaces of height 3 and weight $\aleph$, then $C(K_0)$ and $C(K_1)$ are isomorphic.

**Corollary 4.6.** Assuming $\mathsf{MA}(\aleph)$, if $K$ is a separable scattered compact space of height 3 and weight $\aleph$, then $c_0(C(K))$ (the $c_0$-direct sum of $C(K)$) is isomorphic to $C(K)$. In particular, $C(K)$ is isomorphic to its square.

**Proof.** Note that $c_0(C(K))$ is isomorphic to $C(\mathcal{A}(\omega) \times K)$ and such compact space has finite height, so we infer from [16, Corollary 4.2] that $\text{Ext}(c_0(C(K)), c_0) = 0$. On the other hand, we have the following diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & c_0 & \longrightarrow & c_0(C(K)) & \longrightarrow & c_0(c_0(|K|)) & \longrightarrow & 0 \\
0 & \longrightarrow & c_0 & \longrightarrow & C(K) & \longrightarrow & c_0(|K|) & \longrightarrow & 0 \\
\end{array}
\]

Hence, the diagonal principles (see Proposition 2.1) yield $c_0 \oplus C(K) \sim c_0 \oplus c_0(C(K))$. Since $C(K)$ contains $c_0$ complemented, and so does $c_0(C(K))$, we are done. In particular, $C(K)$ is isomorphic to its square, because $c_0(C(K))$ has this property. \qed

5. Counting non-isomorphic $C(K)$-spaces

For any compact space $K$ we identify, as usual, the dual space $C(K)^*$ with the space $M(K)$, of all signed regular Borel measures on $K$ having finite variation. We start by the following general result on the isomorphism types of $C(K)$ spaces.

**Theorem 5.1.** Let $\mathcal{K}$ be a family of compact spaces such that

(i) $K$ is separable and $|M(K)| = \mathfrak{c}$ for every $K \in \mathcal{K}$;

(ii) For every pair of distinct $K, L \in \mathcal{K}$ one has $C(K) \cong C(L)$ and $K, L$ are not homeomorphic.

Then $\mathcal{K}$ is of cardinality at most $\mathfrak{c}$.

**Proof.** Since every $K \in \mathcal{K}$ is separable, there is a continuous surjection $\pi_K : \beta \omega \to K$. Consequently, $C(K)$ can be isometrically embedded into $C(\beta \omega)$ via the mapping $\pi_K^*(g) = g \circ \pi_K$, $g \in C(K)$.
Take $K, L \in \mathcal{K}$, $K \neq L$. Note that if the condition

$$\pi_K(p_1) = \pi_K(p_2) \iff \pi_L(p_1) = \pi_L(p_2),$$

was satisfied for any $p_1, p_2 \in \beta\omega$ then the formula $h(\pi_K(p)) = \pi_L(p)$ would properly define a homeomorphism $h : K \rightarrow L$. Hence, for instance, there are $p_1, p_2 \in \beta\omega$ such that $\pi_K(p_1) = \pi_K(p_2)$ while $\pi_L(p_1) \neq \pi_L(p_2)$. This immediately implies that $\pi_K^*\{C(K)\} \neq \pi_L^*\{C(L)\}$ — consider a function $g \in C(L)$ that distinguishes $\pi_L(p_1)$ and $\pi_L(p_2)$.

Fix now $K_0 \in \mathcal{K}$; for every $K \in \mathcal{K}$ there is an isomorphism $T_K : C(K_0) \rightarrow C(K)$ and therefore $S_K : C(K_0) \rightarrow C(\beta\omega)$, where $S_K = \pi_K^* \circ T_K$ is a bounded linear operator. It follows that

$S_K \neq S_{K'}$ whenever $K \neq K'$. To conclude the proof, it is therefore sufficient to show that the space $\mathcal{L}(C(K_0), C(\beta\omega))$ of all bounded operators, is of size at most $c$.

Note that any operator $R : C(K_0) \rightarrow C(\beta\omega)$ is uniquely determined by the sequence

$$\langle R^* (\delta_n) : n \in \omega \rangle,$$

of measures from $M(K_0)$, where $R^* : M(\beta\omega) \rightarrow M(K_0)$ is the adjoint operator. By our assumption, $|M(K_0)| = c$ so $|M(K_0)|^\omega = c$, and we are done. \hfill \Box

Let us remark that, in the setting of the above theorem, if we know only that $|M(K_0)| = c$ for some $K_0 \in \mathcal{K}$, then automatically $|M(K)| = c$ for every $K \in \mathcal{K}$ (since $C(K) \approx C(K_0)$). However, separability of the domain is not preserved by isomorphisms between the spaces of continuous functions. For instance, $\ell_\infty \approx L_\infty[0,1]$ by Pełczyński theorem, so $C(\beta\omega) \approx C(K)$, where $K$ is the Stone space of the measure algebra (recall that such $K$ is not separable).

**Corollary 5.2.** There are $2^c$ pairwise nonisomorphic twisted sums of $c_0$ and $c_0(c)$.

**Proof.** In what follows, $\mathcal{A, B}$ (with possible indices) denote almost disjoint families of subsets of $\omega$ of cardinality $c$. For every almost disjoint family $\mathcal{A}$, $K_\mathcal{A}$ is a separable compact space and, since $K_\mathcal{A}$ is scattered, $M(K_\mathcal{A}) = c$.

Note first that any homeomorphism $h : K_\mathcal{A} \rightarrow K_\mathcal{B}$ is determined by a permutation of $\omega$. Therefore, we can define a sequence $\langle A_\xi : \xi < 2^c \rangle$ such that the spaces $K_{A_\xi}$ are pairwise not homeomorphic. By a direct application of Theorem 5.1 we conclude that there is $I \subseteq 2^c$ of cardinality $2^c$ such that $C(K_{A_\xi})$ is not isomorphic to $C(K_{A_\eta})$ whenever $\xi, \eta \in I$ and $\xi \neq \eta$. Now, the assertion follows from Theorem 3.2 \hfill \Box

The argument that proves Theorem 5.1 and Corollary 5.2 can be generalized to some higher cardinals, see the appendix.

Recall that, given a scattered compact space $K$, we have the equality $w(K) = |K|$ (see [32]). Since, for infinite compacta $K$, weights of $K$ and $C(K)$ are equal, we obtain the following simple observation.

**Proposition 5.3.** Let $\mathcal{A}$ and $\mathcal{B}$ be infinite almost disjoint families of subsets of $\omega$. If $C(K_\mathcal{A})$ and $C(K_\mathcal{B})$ are isomorphic then $|\mathcal{A}| = |\mathcal{B}|$. 

Using the construction from the proof of Theorem 8.7 from [27], we shall now define a large family of compactifications of $\omega$ with remainders homeomorphic to $S$. We follow here the notation introduced in subsection 5.2.

For every $x \in (0, 1)$, fix a nondecreasing sequence $(q^n_x)_{n \in \omega}$ in $Q$ such that $\lim_n q^n_x = x$, and put $S_x = \{q^n_x : n \in \omega\}$. Take any function $\theta : (0, 1) \to 2$ and define

$$
R^\theta_x = \begin{cases} 
P_x & \text{if } \theta(x) = 0, \\ 
P_x \setminus S_x & \text{if } \theta(x) = 1. 
\end{cases}
$$

Then $R^\theta_x \subseteq R^\theta_y$ whenever $x < y$ since, in such a case, $P_x \cap S_y$ is finite.

Let $\mathfrak{B}^\theta$ be a subalgebra of $\mathcal{P}(Q)$ generated by $\{R^\theta_x : x \in (0, 1)\} \cup \text{fin}$, where fin denotes the family of all finite subsets of $Q$: set $L^\theta = \text{ult}(\mathfrak{B}^\theta)$.

Note that every space $L^\theta$ may be seen as a compactification of the discrete set $Q$ with the remainder homeomorphic to $S$. Indeed, we may think that $Q \subseteq L^\theta$ by identifying $q \in Q$ with $F \in \text{ult}(\mathfrak{B}^\theta)$ containing $\{q\}$. If $F \in \text{ult}(\mathfrak{B}^\theta)$ contains no finite subset of $Q$ then it is uniquely determined by the set $\{x \in (0, 1) : R_x \in F\}$. Now the point is that the family $\{R_x : x \in (0, 1)\}$ forms a chain with respect to almost inclusion. Hence the Boolean algebra $\mathfrak{B}/\text{fin}$ is isomorphic to $\mathfrak{A}$ which means that $L^\theta \setminus Q$ is homeomorphic to $\text{ult}(\mathfrak{A}) \simeq S$ (see also [27, Theorem 8.7]). It follows that every space $C(L^\theta)$ is a twisted sum of $c_0$ and $C(S)$.

**Corollary 5.4.** There are $2^\omega$ pairwise nonisomorphic twisted sums of $c_0$ and $C(S)$.

**Proof.** We can follow the idea of the proof of Corollary 5.2. We have a family $\{L^\theta : \theta \in 2^{(0,1)}\}$ of separable compact spaces. For any $\theta$, $|M(L^\theta)| = \mathfrak{c}$ because $L^\theta$ may be identified with $S \cup \omega$ and $M(S)$ has cardinality $\mathfrak{c}$. The latter follows from the fact that a compact separable linearly ordered space carries at most $\mathfrak{c}$ regular measures, see e.g. Mercourakis [30].

Again, for a fixed space $L^\theta$, the set $\{\eta \in 2^{(0,1)} : L^\theta \simeq L^\eta\}$ is of size at most $\mathfrak{c}$ (since any homeomorphism of such spaces fixes $Q$ and, as before, we can single out $2^\omega$ many pairwise nonisomorphic spaces of the form $C(L^\theta)$).

6. Separable Rosenthal compacta

The purpose of this section is to collect several subtle results concerning applications of descriptive set theory to separable Rosenthal compacta that are needed in the next section.

A compact space $K$ is **Rosenthal compact** if embeds into $B_1(\omega^\omega)$, the space of Baire-one functions on the Polish space $\omega^\omega$ (homeomorphic to the irrationals), equipped with the topology of pointwise convergence; recall that a Baire-one function is a pointwise limit of a sequence of continuous functions. We refer to [26] for basic properties of Rosenthal compacta and further references.

Recall that in a Polish space $T$, the Borel $\sigma$-algebra $\text{Bor}(T)$ can be written as

$$
\text{Bor}(T) = \bigcup_{1 \leq \alpha < \omega_1} \Sigma^0_\alpha(T) = \bigcup_{1 \leq \alpha < \omega_1} \Pi^0_\alpha(T),
$$
where $\Sigma^0_1(T)$ and $\Pi^0_1$ are the families of open and closed sets, respectively. Additive classes $\Sigma^0_n$ are defined inductively as all countable unions of elements from $\bigcup_{\beta<\alpha} \Pi^0_\beta$ and so on, see Kechris [22, 11.B] for details. Also recall that a subset of a Polish space is analytic if it is a continuous image of $\omega^\omega$. In a Polish space, every Borel set is analytic.

If $K$ is any separable compact space, $D \subseteq K$ is any countable dense subset and $X$ is a topological space, then we denote by $C_{D}(K, X)$ the space of all continuous functions from $K$ into $X$, equipped with the topology of pointwise convergence on $D$. We write $C_{D}(K)$ for $C_{D}(K, \mathbb{R})$. In other words, $C_{D}(K)$ is a subset of $\mathbb{R}^{D}$ which is the image of the restriction map $C(K) \ni g \to g|_{D} \in \mathbb{R}^{D}$. We extend this notation for countable subsets $M \subseteq M(K)$ separating the functions in $C(K)$. For such $M$, $C_{M}(K)$ stands for the space $C(K)$ endowed with the weak topology generated by $M$; here, we can identify $f \in C_{M}(K)$ with $(\mu(f))_{\mu \in M} \in \mathbb{R}^{M}$.

We briefly mention some properties of Rosenthal compacta. Godefroy showed in [20] that if $K$ is Rosenthal compact, then $M_{1}(K)$, the dual unit ball, is Rosenthal compact in its weak* topology. On the other hand, Bourgain, Fremlin and Talagrand [4] proved that every Rosenthal compactum $K$ is Fréchet-Urysohn, i.e., for any $A \subseteq K$ and $x \in A$, there is a sequence $(x_{n})_{n \in \omega}$ of points from $A$ which converges to $x$. Those properties, in particular, imply that if $D \subseteq K$ is a countable dense set then every $\mu \in M(K)$ is a Baire-one function on $C_{D}(K)$; see the proof of Theorem 3.1 in [25].

The following result can be used as a characterization of separable Rosenthal compacta; part (a) of Theorem 6.1 is due to Godefroy [20], while part (b) is Corollary 2.4 in Dobrowolski and Marciszewski [17].

**Theorem 6.1.** (a) A separable compact space $K$ is Rosenthal compact if and only if $C_{D}(K)$ is analytic for every countable dense set $D \subseteq K$.

(b) If $K$ is a compact space and $D \subseteq K$ is a countable dense set such that $C_{D}(K)$ is analytic then either $K$ is Rosenthal compact or $K$ contains a copy of $\beta\omega$.

Following Godefroy’s characterization of separable Rosenthal compacta, Marciszewski [25] introduced an index measuring the complexity of such spaces. This kind of Rosenthal’s index will be denoted here by $\text{ri}(\cdot)$; note that in [25] the working notation $\eta(\cdot)$ was introduced.

**Definition 6.2.** We define the index $\text{ri}$ on the class of separable Rosenthal compacta as follows. Set $\text{ri}(K) = \omega_{1}$ if $C_{D}(K)$ is Borel in $\mathbb{R}^{D}$ for no countable dense set $D \subseteq K$. Otherwise, set $\text{ri}(K) = \alpha$, where $\alpha$ is the least ordinal number $< \omega_{1}$ such that

$$C_{D}(K) \in \Sigma_{1+\alpha}^{0}(\mathbb{R}^{D}) \cup \Pi_{1+\alpha}^{0}(\mathbb{R}^{D}),$$

for some countable dense $D \subseteq K$.

The reader should be warned that the difference between Definition 6.2 and that from [25] is connected with the fact that older tradition was to count Borel classes starting from 0 rather than 1 (for instance, $F_{\alpha\delta}$ is $\Pi_{\alpha}^{0}$). Recall that $\text{ri}(K) \geq 2$ whenever $K$ is infinite, see [25, Theorem 2.1]. The double arrow space $\mathbb{S}$ is a classical nonmetrizable Rosenthal compactum with $\text{ri}(\mathbb{S}) = 2$. 

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The main feature of the index $\ri$ is that it is almost preserved by isomorphisms of Banach spaces. The following result is a particular case of Corollary 3.2 in Marciszewski [25]. We outline the main idea of the proof from [25] because the argument is a much shorter in our setting.

**Theorem 6.3.** If $K$ and $K'$ are separable Rosenthal compacta and $C(K) \simeq C(K')$ then
\[
\ri(K) \leq 1 + \ri(K') \quad \text{(and, by symmetry, } \ri(K') \leq 1 + \ri(K)) \text{.}
\]

In particular, $\ri(K) = \ri(K')$ whenever $\ri(K) \geq \omega$.

**Proof.** Let $T : C(K) \to C(K')$ be an isomorphism such that $c \cdot \|g\| \leq \|Tg\| \leq \|g\|$ for every $g \in C(K)$, where $c > 0$. Fix countable dense sets $D \subseteq K$ and $D' \subseteq K'$ realizing the values of $\ri(K)$ and $\ri(K')$, respectively, and write $\Delta_D = \{\delta_d : d \in D\}$, $\Delta_{D'} = \{\delta_d : d \in D'\}$.

**Claim 1.** There are countable sets $M, M'$, where $\Delta_D \subseteq M \subseteq M_1(K)$, $\Delta_{D'} \subseteq M' \subseteq M_1(K')$ such that
\[
C_M(K) \ni g \to Tg \in C_{M'}(K'),
\]
is a homeomorphism.

**Proof of the claim:** Note that $T^*$ sends $M_1(K')$ into $M_1(K)$. Likewise, if we consider $S : C(K') \to C(K)$ given by $S = c \cdot T^{-1}$ then $S^*$ sends $M_1(K)$ into $M_1(K')$. To define $M$ and $M'$ put $M(0) = \Delta_D$ and $M'(0) = \Delta_{D'}$ and define inductively
\[
M_{n+1} = T^*[M'(n)], \quad M'(n + 1) = S^*[M(n)].
\]
Then the sets $M = \bigcup_n M(n)$ and $M' = \bigcup_n M'(n)$ are as required. For instance, suppose that a sequence $(g_k)_k$ converges in $C_M(K)$ and consider any $\nu \in M'$. Then $\nu \in M'(n)$ for some $n$ and therefore $\mu = T^*\nu \in M_{n+1} \subseteq M$. Hence $\nu(Tg_k) = T^*\nu(g_k) \to T^*\nu(g) = \nu(Tg)$. \hfill \Box

Now we can examine the mapping $\varphi : C_D(K) \to C_{D'}(K')$ closing the following diagram
\[
\begin{array}{ccc}
C_M(K) & \overset{T}{\longrightarrow} & C_{M'}(K') \\
\downarrow{id} & & \downarrow{id} \\
C_D(K) & \overset{\varphi}{\longrightarrow} & C_{D'}(K')
\end{array}
\]

**Claim 2.** $\varphi$ and $\varphi^{-1}$ are mapping of the first Baire class.

**Proof of the claim:** Indeed, while the mapping $id : C_M(K) \to C_D(K)$ is continuous (as $\Delta_D \subseteq M$), its inverse is of the first Baire class since every $\mu \in M_1(K)$ is a weak* limit of a sequence from the absolute convex hull of $\Delta_D$. This shows that $\varphi$ is of the first Baire class; the argument for $\varphi^{-1}$ is symmetric. \hfill \Box

The final step is to use Kuratowski’s theorem [24, par. 35 VII] which assures that there are $F_{\sigma \delta}$ sets $A, B$ containing $C_D(K)$ and $C_{D'}(K')$ respectively, and an extension $\tilde{\varphi}$ of $\varphi$ to a Baire-one isomorphism $A \to B$. This, together with $\ri(K), \ri(K') \geq 2$ gives $\ri(K) \leq 1 + \ri(K')$ and $\ri(K') \leq 1 + \ri(K)$. \hfill \Box
7. Twisted sums and Rosenthal compacta

Denote by $2^{<\omega}$ the full dyadic tree and, as usual, $2^\omega$ is the Cantor set. For any $x \in 2^\omega$, set $B(x) = \{ x|n : n \in \omega \} \subset 2^{<\omega}$. It is clear that for any $Z \subseteq 2^\omega$, $A_Z = \{ B(x) : x \in Z \}$ is an almost disjoint family of infinite subsets of $2^{<\omega}$.

The mapping $x \mapsto B(x)$ is a homeomorphic embedding of $2^\omega$ into $2^{2^{<\omega}}$ (we identify $\mathcal{P}(2^{<\omega})$ with $2^{2^{<\omega}}$). Therefore, for any $Z \subseteq 2^\omega$, $A_Z$ is an almost disjoint family (in $2^{2^{<\omega}}$) which is homeomorphic to $Z$. Moreover, for Borel $Z$, we have the following (Marciszewski [25, 4.2]).

**Theorem 7.1.** If $Z \in \Sigma^0_{1+\alpha}(2^\omega)$, where $\alpha \geq 2$, then $K_{A_Z}$ is Rosenthal compact and $\alpha \leq r_i(K_{A_Z}) \leq 1 + \alpha + 1$.

Using Theorem 7.1 and Theorem 6.3 we arrive at the following.

**Corollary 7.2.** There is a family $\{ K_\xi : \xi < \omega_1 \}$ of separable Rosenthal compacta such that $C(K_\xi) \not\cong C(K_\eta)$ whenever $\xi \not= \eta$ and every $C(K_\xi)$ is a (nontrivial) twisted sum of $c_0$ and $c_0(\mathbb{C})$.

**Proposition 7.3.** Let

$$0 \rightarrow c_0 \rightarrow C(L) \rightarrow C(K) \rightarrow 0$$

be a twisted sum, where $K$ is an infinite separable Rosenthal compact space, and $L$ is a compact space. If this twisted sum is trivial, then $L$ is a separable Rosenthal compact space, and $r_i(L) \leq 1 + r_i(K)$.

**Proof.** Triviality of our twisted sum gives us the following string of isomorphisms

$$C(L) \simeq c_0 \oplus C(K) \simeq C(\omega + 1) \oplus C(K) \simeq C((\omega + 1) \oplus K).$$

Godefroy [20] proved that the class of separable Rosenthal compacta is preserved by isomorphisms of function spaces; since $(\omega + 1) \oplus K$ belongs to this class, so does $L$. By Theorem 6.3 it is enough to verify that $r_i((\omega + 1) \oplus K) \leq r_i(K)$. Let $D$ be a countable dense subset of $K$ realizing the value of $r_i(K)$. By [25, Theorem 2.1] $C_D(K)$ is not a $G_{\delta\sigma}$-subset of $\mathbb{R}^D$, and it is well known that $C_\omega(\omega + 1)$ is an $F_{\sigma\delta}$-subset of $\mathbb{R}^\omega$. Hence, for the countable dense subset $E = \omega \cup D$ of $(\omega + 1) \oplus K$, the space $C_E((\omega + 1) \oplus K)$ can be identified with the product $C_\omega(\omega + 1) \times C_D(K)$ which is a Borel subset of $\mathbb{R}^\omega \times \mathbb{R}^D$ of the class $r_i(K)$. \hfill $\square$

We turn to examining twisted sums of $c_0$ and $C(S)$; below we follow the notation introduced in [5.2] and section 5.

**Lemma 7.4.** For any $Z \subseteq (0, 1)$, the space $C_Q(L^\infty_x)$ contains a $G_\delta$-subset $X$ homeomorphic to $(0, 1) \setminus (Z \cup Q)$.

**Proof.** Denote $\chi_Z$ by $\theta$. We will look for $X$ inside the closed subset $C_Q(L^\theta, 2) = C_Q(L^\theta) \cap 2^Q$ of $C_Q(L^\theta)$, which may be seen as traces of clopen subsets of $L^\theta$ on $Q$, i.e., the algebra $\mathcal{B}^\theta$. Define

$$P_1 = \{ (x, 1) \cap Q : x \in [0, 1] \setminus Q \} \subseteq 2^Q$$

$$P_2 = \{ (x, 1) \cap Q : x \in Q \} \subseteq 2^Q$$

$$P_3 = \{ (x, 1) \cap Q : x \in Q \} \subseteq 2^Q$$
where we identify $\mathcal{P}(Q)$ with $2^Q$. Observe that the union $P = P_1 \cup P_2 \cup P_3$ is closed in $2^Q$.
Indeed, we have $s \in 2^Q \setminus P$ if and only if there exists $p < q$ in $Q$ with $s(p) = 1$ and $s(q) = 0$. Since $P_2 \cup P_3$ is countable, $P_1$ is $G_δ$-subset of $2^Q$. A routine verification shows that the mapping $x \mapsto (x, 1) \cap Q$ is a homeomorphism of $[0, 1] \setminus Q$ onto $P_1$. Analyzing the description of the algebra $\mathfrak{B}^θ$ defining $L^θ$ one can verify that $[(x, 1) \cap Q] \in P_1 \cap C_Q(L^θ)$ if and only if $x \in [0, 1] \setminus (Z \cup Q)$. Hence, the set $X = P_1 \cap C_Q(L^θ)$ has the required properties.

**Corollary 7.5.** Let $Z \subseteq (0, 1)$ be such that $Z \notin \Sigma_θ^0((0, 1)) \cup \Pi_θ^0((0, 1))$. Then $C(L^{xz})$ is a nontrivial twisted sum of $C_0$ and $C(\mathbb{S})$.

**Proof.** Suppose that $C(L^{xz})$ represents a trivial twisted sum of $c_0$ and $C(\mathbb{S})$. Since $\mathfrak{r}(\mathbb{S}) = 2$, from Proposition 7.3 we obtain that $L^{xz}$ is Rosenthal compact with $\mathfrak{r}(L^{xz}) \leq 3$. Theorem 2.2 from [25] says that, for a separable Rosenthal compact space $K$, and any two countable dense subsets $D$, $E$ of $K$, the Borel classes of $C_D(K)$ and $C_E(K)$ can differ by at most 1. Therefore $C_Q(L^{xz}) \in \Sigma_θ^0([\mathbb{R}^Q] \cup \Pi_θ^0([\mathbb{R}^Q])$. Hence, any $G_δ$-subset of $C_Q(L^{xz})$ is also Borel of the same class, a contradiction with Lemma 7.4 and our assumption on $Z$.

To prove the next Lemma 7.8 we need to employ a more effective description of the sets $S_x$ used in section 5. We now consider $Q = \mathbb{Q} \cap (0, 1)$; for $x \in (0, 1)$, let $0.\bar{i}_0^1i_1^2\ldots$ be the binary expansion of $x$ using infinitely many 1’s, i.e., $i_k^r \in \{0, 1\}, k \in \omega$ and $x = \sum_{k=0}^{\infty} (i_k^r/2^{k+1})$. We define

$$S_x = \left\{ \sum_{k=0}^{n} i_k^r/2^{k+1} : n \in \omega \right\} \setminus \{0\}.$$

Using the fact that, for $x \in (0, 1) \setminus Q$, the expansion $0.\bar{i}_0^1i_1^2\ldots$ also has infinitely many 0’s, one can easily verify that the mapping $x \mapsto S_x$ is continuous on $(0, 1) \setminus Q$ (actually, it is a homeomorphic embedding of $(0, 1) \setminus Q$ into $2^Q$).

We also need the following two auxiliary results. The first one is well known, and follows easily from the fact that Boolean operations on $\mathcal{P}(\omega)$ are continuous.

**Proposition 7.6.** Each subalgebra of $\mathcal{P}(\omega)$ with an analytic set of generators is analytic.

The second result is probably also well known, it can be derived from the results of Godefroy [20]. For the sake of completeness we include an (alternative) proof of it.

**Proposition 7.7.** Let $K$ be a separable zero-dimensional compact space and $D$ a countable dense subspace of $K$. If $C_D(K, 2) = C_D(K) \cap 2^D$ is analytic, so is $C_D(K)$.

**Proof.** Since $\mathbb{R}$ is homeomorphic to $(0, 1)$ and $C_D(K, (0, 1)) = C_D(K,[0,1]) \cap (0,1)^D$ is a $G_δ$-subset of $C_D(K,[0,1])$, it is enough to prove that $C_D(K,[0,1])$ is analytic. The countable product $C_D(K,2^\omega)$ can be identified with the space $C_D(K,2^\omega)$. Hence, $C_D(K,2^\omega)$ is also analytic, and it is enough to show that $C_D(K,[0,1])$ is a continuous image of $C_D(K,2^\omega)$. By [31] Lemma 1] there exists a continuous map $\phi : 2^\omega \rightarrow [0,1]$ such that

$$\forall f \in C(K,[0,1]) \exists g \in C(K,2^\omega) f = \phi \circ g$$
Lemma 7.8. If $\theta : (0, 1) \rightarrow 2$ is Borel then $L^\theta$ is Rosenthal compact.

Proof. Let $\theta = \chi_Z$, obviously $Z$ is a Borel subset of $(0, 1)$. We will consider the Borel sets $Z_0 = Z \setminus Q$ and $Z_1 = (0, 1) \setminus (Z \cup Q)$.

First, we will show that the set $\{R^\theta_x : x \in (0, 1)\} \subset \mathcal{P}(Q)$ of generators of the algebra $\mathfrak{B}^\theta$ is analytic. We can neglect the countable set $\{R^\theta_x : x \in Q\}$, so it is enough to verify that both sets $G_i = \{R^\theta_x : x \in Z_i\}$, $i = 0, 1$, are analytic. We have $G_0 = \{P_x \setminus S_x : x \in Z_0\}$ and $G_1 = \{P_x : x \in Z_1\}$. Since the map $A \mapsto Q \setminus A$ is a homeomorphism of $\mathcal{P}(Q)$ and $Q \setminus P_x = (x, 1) \cap Q$, the argument from the proof of Lemma 7.4 shows that $G_1$ is homeomorphic to $Z_1$, hence analytic. Observe that, by the same argument and the continuity of the mapping $x \mapsto S_x$, the map $\varphi : Z_0 \rightarrow G_0$ defined by $\varphi(x) = P_x \setminus S_x$ is continuous and onto, therefore $G_0$ is also analytic.

Lemma 7.3 implies that the algebra $\mathfrak{B}^\theta$, which can be identified with $C_Q(L^\theta, 2)$, is analytic. In turn, from Lemma 7.7 we infer that $C_Q(L^\theta)$ is also analytic. Now, Theorem 6.1(b) gives us the desired conclusion.

Corollary 7.9. If the set $Z \subseteq (0, 1)$ is not coanalytic then

(i) $L^{X_z}$ is not Rosenthal compact;
(ii) $C(L^{X_z})$ is a nontrivial twisted sum of $c_0$ and $C(S)$;
(iii) $C(L^{X_z})$ is not isomorphic to $C(L^\theta)$ whenever the function $\theta$ is Borel.

Proof. Part (i) follows directly from Lemma 7.4 (ii) is a consequence of (i) and Proposition 7.3. The last statement follows from Lemma 7.8.

It is likely that twisted sums of $c_0$ and $C(S)$ that are of the form $C(K)$ with $K$ Rosenthal compact can be examined more closely using the Rosenthal index. However, the following problem is open to us.

Problem 7.10. Let $\theta : (0, 1) \rightarrow 2$ be Borel. Is $\text{ri}(L^\theta) < \omega_1$? Can we find an effective estimate of $\text{ri}(L^\theta)$ using the class of $\theta$?

Appendix A. Counting non-isomorphic $C(K)$-spaces one more time

If $I$ is an arbitrary set and $\mathcal{A}$ is a family of subsets of $I$ then $\mathcal{A}$ is an almost disjoint family if, again, $A \cap B$ is finite for any distinct $A, B \in \mathcal{A}$. It is well-known that on a set of cardinality $\aleph$ there is an almost disjoint family of cardinality $\aleph^{\omega}$. In fact, $\aleph^{\omega}$ is the largest possible size of such a family: if $\mathcal{A} \subseteq \mathcal{P}(\aleph)$ is almost disjoint them we can assume that every $A \in \mathcal{A}$ is infinite for any $A \in \mathcal{A}$ pick a countable infinite set $\varphi(A) \subseteq A$. Then $\varphi$ is one-to-one, so $|\mathcal{A}| \leq \aleph^{\omega}$. For that reason the arguments used in Theorem 5.1 and Corollary 5.2 can be generalized to obtain information about the cardinality of Ext($c_0(\aleph^{\omega}), c_0(\aleph)$) only when $\aleph < \aleph^{\omega}$ — we outline it here.
Given an almost disjoint $\mathcal{A}$ of subsets of $I$, we can form a version of Aleksandrov-Urysohn space $K_{\mathcal{A}}$, where

$$K_{\mathcal{A}} = I \cup \{A : A \in \mathcal{A}\} \cup \{\infty\}.$$ 

As before, points in $I$ are isolated, basic open neighborhoods of a given point $A$ are of the form $\{A\} \cup (A \setminus F)$ with $F \subseteq I$ finite, and the point $\infty$ compactifies the locally compact space $I \cup \{A : A \in \mathcal{A}\}$. Then $K_{\mathcal{A}}$ is a scattered compact space of height 3 and density $|I|$. We actually consider families $\mathcal{A}$ consisting of countable infinite sets, as in Dow and Vaughan [18], since in our context the only feature of $\mathcal{A}$ that matters is the cardinal number $|\mathcal{A}|$.

**Theorem A.1.** Assume that $\aleph < \aleph_\omega$ and let $\mathcal{A}$ be an almost disjoint family of subsets of $\aleph$ of cardinality $\aleph_\omega$. Then $C(K_{\mathcal{A}})$ is a non-trivial twisted sum of $c_0(\aleph)$ and $c_0(\aleph_\omega)$.

**Proof.** We shall see that there is no extension operator $E : c_0(\aleph_\omega) \to C(K_{\mathcal{A}})$. Assume the existence of such $E$; then, for every $A \in \mathcal{A}$, there must be $\xi_A \in \aleph$ so that $|E\chi_A(\xi_A)| > \frac{1}{2}$. Since $|\mathcal{A}| = \aleph_\omega > \aleph$, we infer the existence of a point $\xi \in \aleph$ such that $\xi = \xi_{A_n}$ for a sequence of distinct $A_n \in \mathcal{A}$. Then, for every natural number $m$ we obtain

$$\left| \sum_{j=1}^{m} E\chi_{A_j}(\xi) \right| \geq \frac{m}{2} \quad \text{and} \quad \left\| \sum_{j=1}^{m} \chi_{p_j} \right\| = 1,$$

which contradicts continuity of $E$. \qed

The next results can be proved by similar arguments to those used in [5.1] and [5.2].

**Theorem A.2.** Fix an infinite cardinal $\aleph$, and let $\mathcal{K}$ be a family of compact spaces such that

(i) every $K$ has density $\aleph$ and $|M(K)| \leq 2^{\aleph}$;

(ii) For every pair of distinct $K, L \in \mathcal{K}$ one has $C(K) \simeq C(L)$ and $K, L$ are not homeomorphic.

Then $\mathcal{K}$ is of cardinality at most $2^\aleph$.

**Corollary A.3.** If $2^{\aleph_\omega} > 2^\aleph$ then there are $2^{\aleph_\omega}$ pairwise non-isomorphic twisted sums of $c_0(\aleph)$ and $c_0(\aleph_\omega)$.

The above corollary can be applied to any (infinite) $\aleph < \mathfrak{c}$ under Martin’s axiom since then $2^\aleph = \mathfrak{c}$.

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Instituto de Matemáticas Imuex, Universidad de Extremadura, Avenida de Elvas, 06071-Badajoz, Spain
E-mail address: fcabello@unex.es, castillo@unex.es, salgueroalarcon@unex.es

Institute of Mathematics, University of Warsaw, Banacha 2
02–097 Warszawa, Poland
E-mail address: wmarcisz@mimuw.edu.pl

Mathematical Institute, University of Wrocław, pl.Grunwaldzki, 2/4, 50-384 Wrocław, Poland,
E-mail address: grzes@math.uni.wroc.pl