On a class of retarded integrodifferential equations

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Abstract

The following class of retarded integro-differential equations in a Banach space

\[ \dot{x}(t) = Ax(t) + \int_0^t b(t - \tau) Lx_{\tau} d\tau + Kx_t; \quad t \geq 0, \]

are taken into consideration in this study. The delay term \( Lx_{\tau} \) of this equation is inserted into the integral as a convolution product with a scalar kernel. We prove the well-posedness of the problem under investigation using the Miyadera-Voigt perturbation and the theory of semigroups. We also explore the spectral analysis of an associated abstract Cauchy problem.

Keywords: Integro-differential equations, \( C_0 \)-semigroup, delay equations, Miyadera-Voigt perturbation.

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I. Introduction.

It is commonly acknowledged that equations in Banach spaces with delay present special challenges in terms of well-posedness. Delayed differential equations are a type of dynamical system that is significant for modelling hereditary phenomena in physics, biology, chemistry, economics, ecology, and other fields. Numerous articles have addressed the analysis of these equations using a semigroup method; for instance, J. Hale and G. Webb [9, 19] were among the pioneers in this area. We also cite [14, 4, 21, 15, 11, 1, 2, 8, 3, 10] for references from more recent works.

In this paper, we focus on the following retarded integro-differential equation in a Banach space \( X \)

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + \int_0^t b(t - \tau) Lx_{\tau} d\tau + Kx_t; \\
x(0) &= x, \quad x_0 = \varphi.
\end{align*}
\]

(1)

Here, \( A : D(A) \subset X \rightarrow X \) generates a \( C_0 \)-semigroup \((T(t))_{t \geq 0}\) on \( X \), \( b(\cdot) \) are scalar kernels in \( W^{1,p}(\mathbb{R}_+, \mathbb{C}) \), where \( \mathbb{R}_+ \) denotes the half line \([0, +\infty)\) and \( 1 < p < +\infty \). The delay operator \( L \in \mathcal{L}(W^{1,p}([-1, 0], X); X) \), the initial condition \((x, \varphi) \in X \times L^p([-1, 0], X)\) and for each \( t \geq 0 \), the history function \( x_t : [-1, 0] \rightarrow X \) of \( x(\cdot) \) is defined by \( x_t(\theta) = x(t + \theta) \) for \( \theta \in [-1, 0] \).

To the best of our knowledge, a semigroup technique has not yet been used to study this class of retarded equations where the delay term is in the convolution product. The difficulty
of this equation lies in the delay term \( Lx_t \) which is one of the term of this convolution product \( \int_0^t b (t - \tau) Lx_\tau d\tau \). Knowing in advance that \( x_t \) satisfies the following equation

\[
\begin{aligned}
\frac{d}{dt}v(t, \theta) &= \frac{d}{d\theta}v(t, \theta), \\
v(t, 0) &= x(t), \\
v(0, \theta) &= \varphi(\theta).
\end{aligned}
\tag{2}
\]

To overcome this difficulty, we were inspired by the paper [6], by introducing the function \( g(t, s) \in L^p(\mathbb{R}_+, X) \) defined by

\[
g(t) := g(t, s) = \int_0^t b(t + s - \tau) Lx_\tau d\tau, \quad t, s \geq 0.
\tag{3}
\]

This paper is organized as follows: In section 2, we rewrite the equation (1) in associated product spaces to an abstract Cauchy problem as follows \( \dot{Y}(t) = AY(t) \), and it is well known that this system is well-posed if and only if \( \mathcal{A} \) generates a \( C_0 \)-semigroup. In our situation, the idea is to write \( \mathcal{A} \) as a perturbation of a generator of a \( C_0 \)-semigroup by an unbounded operator, then we apply Miyadera-Voigt perturbation to obtain a sufficient condition for \( \mathcal{A} \) to generate a \( C_0 \)-semigroup. In section 3, we study the spectral properties of \( \mathcal{A} \), by calculating the resolvent \( R(\lambda, \mathcal{A}) \) and the resolvent set \( \rho(\mathcal{A}) \) of the operator \( \mathcal{A} \). In section 4, we give two applications of the retarded integro-differential Volterra equations to illustrate the main results of this paper.

II. WELL-POSEDNESS.

In this section, we denote by \( X_1 := (D(A), \| \cdot \|_1) \) the Banach space equipped with the graph norm \( \| x \|_1 = \| x \| + \| Ax \| \), and by \( \rho(A) \) the resolvent set of \( A \), \( \sigma(A) = \mathbb{C} \setminus \rho(A) \) the spectrum of \( A \), \( R(\lambda, A) := (\lambda I - A)^{-1} \) for \( \lambda \in \rho(A) \) the resolvent operator of \( A \), \( \mathcal{C}_\omega = \{ \lambda \in \mathbb{C} \mid \Re(\lambda) > \omega \} \) with \( \omega \in \mathbb{R} \), \( \Re(\lambda) \) the real part of \( \lambda \) and let \( Y \) be another Banach space,

**Definition II.1.** [20] An operator \( B \in \mathcal{L}(X_1, Y) \) is called an \( q \)-admissible observation operator for \( (\mathbf{T}(t))_{t \geq 0} \), with \( q \geq 1 \) if

\[
\int_0^T \| B\mathbf{T}(t)x \|^q dt \leq \gamma^q(\tau)\| x \|^q
\tag{4}
\]

for all \( x \in D(A) \) and for some constants \( \tau > 0 \) and \( \gamma(\tau) > 0 \). In the special case when \( Y = X \), \( q = 1 \) and the constant \( \gamma := \gamma(\tau) < 1 \).

\[
\int_0^T \| B\mathbf{T}(t)x \| dt \leq \gamma \| x \|,
\tag{5}
\]

the operator \( B \) is called Miyadera-Voigt perturbation for \( (\mathbf{T}(t))_{t \geq 0} \).

**Remark II.2.** By Hölder’s inequality it’s easy to see that an \( q \)-admissible observation operator for \( (\mathbf{T}(t))_{t \geq 0} \) with \( q > 1 \) from \( \mathcal{L}(X_1, X) \) is also a Miyadera-Voigt perturbation.

We will use in the sequel the following perturbation theorem (see. [7] Corollary 3.16, page 199).

**Theorem II.3** ([12][13]). Let \( A \) be the generator of \( C_0 \)-semigroup \( (\mathbf{T}(t))_{t \geq 0} \) on a Banach space \( X \) and \( B \) is a Miyadera-Voigt perturbation for \( (\mathbf{T}(t))_{t \geq 0} \) satisfy (5), then the sum \( A + B \) with domain \( D(A + B) = D(A) \), generates a strongly continuous semigroup \( (\mathcal{F}(t))_{t \geq 0} \) on \( X \). Moreover \( (\mathcal{F}(t))_{t \geq 0} \) satisfies

\[
\mathcal{F}(t)x = \mathbf{T}(t)x + \int_0^t \mathbf{T}(t - s)B\mathcal{F}(s)xd\tau
\]
and
\[ \int_0^\tau \|B^\tau(t)x\|dt \leq \frac{\gamma}{1-\gamma} \|x\| \]
for \( x \in D(A) \) and \( t \geq 0 \).

Our current goal is to investigate the well-posedness of (1) with the semigroup approach as given in [1]. To do this we will rewrite (1) as an abstract Cauchy problem
\[ \dot{W}(t) = AW(t) \]
in an appropriate Banach space \( X \). To obtain this assertion, we use the function \( g \) defined in (3), as a consequence, one has
\[ \frac{\partial}{\partial t} g(t,s) = b(s)Lx_t + \frac{\partial}{\partial s}g(t,s). \]
Taking into consideration the equation (2), we can rewrite (1) as follows
\[
\begin{cases}
\dot{x}(t) = Ax(t) + Kx_t + g(t)(0) \\
\dot{x}_t = D_0x_t \\
\dot{g}(t) = D_sg(t) + b(\cdot)Lx_t \\
x(0) = x, x_0 = \varphi \text{ and } g(0) = 0.
\end{cases}
\]
with
\[ D_\theta = \frac{\partial}{\partial \theta}; \quad D(D_\theta) = \{ \varphi \in W^{1,p}([-1,0],X) | \varphi(0) = 0 \} \]
and
\[ D_s = \frac{\partial}{\partial s}; \quad D(D_s) = W^{1,p}(\mathbb{R}_+,X). \]
Having chosen the product space
\[ \mathcal{X} = X \times L^p([-1,0],X) \times L^p(\mathbb{R}_+,X), \]
which is a Banach space with the norm
\[ \| (x,\psi,h) \| = \|x\| + \|\psi\|_{L^p([-1,0],X)} + \|h\|_{L^p(\mathbb{R}_+,X)}, \quad (x,\psi,h) \in \mathcal{X}, \]
and set \( Y(t) = \begin{pmatrix} x(t) \\ x_t \\ g(t) \end{pmatrix} \), equation (1) is rewritten in \( \mathcal{X} \) by the following Cauchy problem
\[
\begin{cases}
\dot{Y}(t) = \mathcal{A}Y(t) \\
Y(0) = \begin{pmatrix} x(0) \\ \varphi \\ 0 \end{pmatrix} \quad ; \quad t \geq 0,
\end{cases}
\]
with
\[ \mathcal{A} = \begin{pmatrix} A & K & \delta_0 \\ 0 & D_\theta & 0 \\ 0 & b(\cdot)L & D_s \end{pmatrix} \]
(7)
and

$$D(A) = \left\{ \left( \begin{array}{c} \chi \\ \psi \\ h \end{array} \right) \in D(A) \times W^{1,p}([-1,0], X) \times W^{1,p}(\mathbb{R}_+, X) ; \psi(0) = \chi \right\}. \quad (8)$$

Here $\delta_0$ denotes the Dirac distribution, i.e., $\delta_0(f) = f(0)$ for each $f \in W^{1,p}(\mathbb{R}_+, X)$.

When $A$ generates a $C_0$-semigroup $(\mathcal{T}(t))_{t \geq 0}$ on $\mathcal{X}$, the mild solution of the abstract Cauchy problem (6) is written by $Y(t) = \mathcal{T}(t)Y(0)$, and it is a classical solution when $Y(0) \in D(A)$. For more background on Semigroup theory we refer the reader to [13, 5, 7, 16, 17].

Now, $A$ can be expressed as a perturbation of the $C_0$-semigroup generator $\mathcal{A}_0$ by the operator $B$, as follows

$$A = \mathcal{A}_0 + B \quad (9)$$

with

$$\mathcal{A}_0 = \left( \begin{array}{cc} A & 0 \\ 0 & -D_x \end{array} \right), \quad B = \left( \begin{array}{cc} 0 & K \\ 0 & \delta_0 \end{array} \right).$$

and $D(A) = D(A_0) = D(B) = D(A) \times W^{1,p}(\mathbb{R}_+, X)$.

Here, $A$ denotes the operator $\left( \begin{array}{cc} A & 0 \\ 0 & D_{\theta} \end{array} \right)$ with $D(A) = \left\{ \left( \begin{array}{c} x \\ \varphi \end{array} \right) \in D(A) \times W^{1,p}([-1,0], X) ; \varphi(0) = x \right\}$.

From [1], $A$ generates a $C_0$-semigroup $(\mathcal{T}(t))_{t \geq 0}$ on $X = X \times L^p([-1,0], X)$, with

$$\mathcal{T}(t) = \left( \begin{array}{cc} \mathcal{T}(t) \\ \mathcal{T}_t \end{array} \right) \omega(0(t)),$$

where $(S^0(t))_{t \geq 0}$ is the nilpotent left shift semigroup on $L^p([-1,0], X)$ and $\mathcal{T}_t : X \to L^p([-1,0], X)$ is defined by

$$(\mathcal{T}_t x)(\tau) = \begin{cases} \mathcal{T}(t + \tau)x, & -t < \tau \leq 0, \\ 0, & -1 \leq \tau \leq -t. \end{cases}$$

The diagonal of $\mathcal{A}_0$ implies that $\mathcal{A}_0$ generates the $C_0$-semigroup $(\mathcal{T}_0(t))_{t \geq 0}$ defined by

$$\mathcal{T}_0(t) = \left( \begin{array}{cc} \mathcal{T}(t) & 1 \\ 0 & \mathcal{T}_t \end{array} \right),$$

here $(S(t)f)(r) = f(t+r)$ denotes the translation semigroup on $L^p(\mathbb{R}_+, X)$.

**Assumption** : A bounded operator $\Xi$ from $W^{1,p}([-1,0], X)$ to $X$ satisfies the assumption $(M_q)$ for $q \geq 1$, if there exist, $\alpha > 0$ and $\gamma_\Xi(\alpha) \geq 0$ such that

$$(M_q) \quad \int_0^\alpha \| \Xi (T_\tau x + S^0(\tau) \varphi) \|^q d\tau \leq \gamma_\Xi^q(\alpha) \left\| \left( \begin{array}{c} x \\ \varphi \end{array} \right) \right\|^q$$

for all $\left( \begin{array}{c} x \\ \varphi \end{array} \right) \in D(A)$.

In the case when $q = 1$, we assume also that $\lim_{\alpha \to 0} \gamma_\Xi(\alpha) = 0$. 

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In the following we will see two examples of operators verifying assumption \((M_\theta)\), and we refer to \([1]\) for many other interesting examples.

**Example II.4.** 1. Let \(\Xi : W^{1,2}([1,0], X) \to X\) be the operator defined by \(\Xi = \delta_{-1}\). For \(f \in D(A)\) and \(\varphi \in W^{1,2}([-1,0], X)\) satisfying \(\varphi(0) = f\), it is easy to show that there is \(0 < \alpha < 1\) such that

\[
\int_0^\alpha \|\Xi \left( T_\tau f + S^0(\tau) \varphi \right) \|^2 \, d\tau = \int_0^\alpha \|\varphi(\tau - 1)\|^2 \, d\tau,
\]

\[
< 1, 2 \alpha \varphi \Xi \left( T_\tau f + S^0(\tau) \varphi \right) \| d \tau \leq \alpha^\frac{\gamma}{2} M [\varphi([-1,0])] \left\| \begin{pmatrix} x \\ \varphi \end{pmatrix} \right\|
\]

This implies that assumption \((M_\alpha)\) is satisfied by \(\Xi\).

2. Let \(\eta : [-1,0] \to X\) be of bounded variation and \(\Xi : C([-1,0], X) \to X\) be the bounded linear operator given by the Riemann-Stieltjes integral \(\Xi(\varphi) = \int_{-1}^{\eta} d\eta(\theta) \varphi(\theta)\) for all \(\varphi \in C([-1,0], X)\). Since \(W^{1,2}([-1,0], X)\) is continuously embedded in \(C([-1,0], X)\), \(\Xi\) defines a bounded operator from \(W^{1,2}([-1,0], X)\) to \(X\), and

\[
\int_0^\alpha \|\Xi \left( T_\tau x + S^0(\tau) \varphi \right) \| d \tau \leq \alpha^\frac{\gamma}{2} M [\varphi([-1,0])] \left\| \begin{pmatrix} x \\ \varphi \end{pmatrix} \right\|
\]

for all \(0 < \alpha < 1\), where \(M = \sup_{\tau \in [0,1]} \|\tau\|\) and \(\varphi\) is the positive Borel measure on \([-1,0]\) defined by the total variation of \(\eta\). It is clear that \(\lim_{\alpha \to 0} \gamma(\alpha) = 0\) with \(\gamma(\alpha) = \alpha^\frac{\gamma}{2} M [\varphi([-1,0])]\).

Then assumption \((M_1)\) is satisfied.

The following Lemmas are necessary to establish the section’s main result,

**Lemma II.5.** If \(L\) and \(K\) satisfy assumption \((M_1)\), then \(B\) is a Miyadera-Voigt perturbation for \((T_0(t))_{t \geq 0}\).

**Proof.** Let \(\begin{pmatrix} x \\ \varphi \end{pmatrix} \in D(A)\), \(L\) and \(K\) satisfy assumption \((M_1)\) then there is \(\alpha > 0\) such that

\[
\int_0^\alpha \|B T_0(\tau) \begin{pmatrix} x \\ \varphi \end{pmatrix}\| d \tau = \int_0^\alpha \left\| \begin{pmatrix} K (T_\tau x + S^0(\tau) \varphi + h(\tau)) \\ b(\cdot) L (T_\tau x + S^0(\tau) \varphi) \end{pmatrix} \right\| d \tau
\]

\[
\leq \alpha^{1-\frac{\gamma}{2}} \|h\|_{L^p(\mathbb{R}^+, X)} + \int_0^\alpha \left\| K (T_\tau x + S^0(\tau) \varphi) \right\| d \tau
\]

\[
\leq \alpha^{1-\frac{\gamma}{2}} \|h\|_{L^p(\mathbb{R}^+, X)} + \gamma_K(\alpha) \left\| \begin{pmatrix} x \\ \varphi \end{pmatrix} \right\| + \gamma_L(\alpha) \|b\|_{L^p(\mathbb{R}^+)} \left\| \begin{pmatrix} x \\ \varphi \end{pmatrix} \right\|
\]

With \(\gamma(\alpha) = \max \left( \gamma_K(\alpha); \gamma_L(\alpha) \|b\|_{L^p(\mathbb{R}^+)}; \alpha^{1-\frac{\gamma}{2}} \right)\), choose now \(\alpha\) small enough such that \(0 \leq \gamma(\alpha) < 1\), then \(B\) is a Miyadera-Voigt perturbation for \((T_0(t))_{t \geq 0}\). □
Lemma II.6. If $L$ and $K$ satisfy assumption $(M_p)$ then $B$ is a $p$-admissible observation operator for $(T_0(t))_{t \geq 0}$.

Proof. Let \((\chi \psi \ h) \in D(A)\), $L$ and $K$ satisfy the Assumption $(M_p)$ then there is $\alpha > 0$ such that

\[
\int_0^a \left\| B T_0(\tau) \begin{pmatrix} \chi \\ \psi \\ h \end{pmatrix} \right\|_p^p d\tau = \int_0^a \left\| \begin{pmatrix} K (T_{\tau} \chi + S^0(\tau) \psi) + h(\tau) \\ b(\cdot)L (T_{\tau} \chi + S^0(\tau) \psi) \end{pmatrix} \right\|_p^p d\tau \\
\leq 4^p \left( \|h\|_{L^p(R^+,X)} + \int_0^a \left\| K (T_{\tau} \chi + S^0(\tau) \psi) \right\|_p^p d\tau \\
+ \|b\|_{L^p(R^+)} \int_0^a \left\| L (T_{\tau} \chi + S^0(\tau) \psi) \right\|_p^p d\tau \right) \\
\leq 4^p \left( \|h\|_{L^p(R^+,X)} + \gamma_K(\alpha) \left\| \begin{pmatrix} \chi \\ \psi \end{pmatrix} \right\|_p^p + \gamma_L(\alpha) \|b\|_{L^p(R^+)} \left\| \begin{pmatrix} \chi \\ \psi \end{pmatrix} \right\|_p^p \right) \\
\leq \gamma^p(\alpha) \left\| \begin{pmatrix} \chi \\ \psi \end{pmatrix} \right\|_p^p
\]

With $\gamma(\alpha) = 4 \times \max \left(1; \gamma_K(\alpha); \gamma_L(\alpha) \|b\|_{L^p(R^+)}\right)$. Then $B$ is a $p$-admissible observation operator for $(T_0(t))_{t \geq 0}$.

The main result of this section is the following Theorem.

Theorem II.7. If $L$ and $K$ satisfy conditions of one of Lemma II.5 and Lemma II.6 then the equation (1) is well-posed and the solution is writing by

\[
x(t) = T(t)x + \int_0^t T(t-s) \int_0^s b(s-\tau)Lx_\tau d\tau ds + \int_0^t T(t-s)Kx_s ds, \tag{12}
\]

for \((x \rho \ h) \in D(A) \times W^{1,p}([-1,0], X)\).

Proof. Let \((x \rho \ h) \in D(A) \times W^{1,p}([-1,0], X)\), we now that $A = A_0 + B$ and $A_0$ generates the following semigroup

\[
T_0(t) = \begin{pmatrix} T(t) & 0 & 0 \\ T_t & S^0(t) & 0 \\ 0 & 0 & S(t) \end{pmatrix},
\]

by Lemma II.5 or Lemma II.6 Remark II.2 and applying Theorem II.3 we conclude that $A$ generates a strongly continuous semigroup $(T(t))_{t \geq 0}$ satisfying

\[
T(t) \begin{pmatrix} x \\ \rho \\ 0 \end{pmatrix} = T_0(t) \begin{pmatrix} x \\ \rho \\ 0 \end{pmatrix} + \int_0^t T_0(t-s)BT(s) \begin{pmatrix} x \\ \rho \\ 0 \end{pmatrix} ds
\]

and by equation (6) we have

\[
\begin{pmatrix} x(t) \\ x_t \\ g(t) \end{pmatrix} = T_0(t) \begin{pmatrix} x \\ \rho \\ 0 \end{pmatrix} + \int_0^t T_0(t-s)B \begin{pmatrix} x(s) \\ \rho(s) \\ g(s) \end{pmatrix} ds
\]
which implies
\[
\begin{pmatrix}
  x(t) \\
  x_1 \\
  g(t)
\end{pmatrix} = \begin{pmatrix}
  T(t)x \\
  T(x+S^0(t)\phi) + \int_0^1 T(t-s)(Kx_s + \delta_0g(s))ds \\
  S(t-s)b(\cdot)Lx_s
\end{pmatrix}.
\]

The solution is the first component given by
\[
x(t) = T(t)x + \int_0^1 T(t-s)(Kx_s + \delta_0g(s))ds
\]
which ends the proof.

\[\square\]

### III. Spectral analysis.

In this section, we consider (1) is well-posed and we calculate the resolvent set \(\rho(\mathcal{A})\) and resolvent operator \(R(\lambda,\mathcal{A})\) of the generator \(\mathcal{A}\) and we derive an interesting formula similar that have been obtained in [2 Proposition 3.19, page 56]. We will need throughout this section the Laplace transform of the scalar kernel \(b(\cdot)\), for this we must assume that, There is \(\beta \in \mathbb{R}\) such that \(b(\cdot)e^{-\beta} \in L^1(\mathbb{R}^+)\) and we note this Laplace transform by \(\hat{b}(\cdot)\) defined by
\[
\hat{b}(\lambda) = \int_0^\infty e^{-\lambda t} b(t)dt, \lambda \in \mathbb{C}_\beta.
\]
The main result of this section is the following Theorem.

**Theorem III.1.** Let \((\mathcal{A}, D(\mathcal{A}))\) be the operator defined by (7) and (8). For \(\lambda \in \mathbb{C}_0 \cap \mathbb{C}_\beta \cap \rho(\mathcal{A})\) we have
\[
\lambda \in \rho(\mathcal{A}) \iff 1 \in \rho \left( e_\lambda R(\lambda,\mathcal{A}) \left( K + \hat{b}(\lambda)L \right) \right) \iff \lambda \in \rho \left( A + (K + \hat{b}(\lambda)L)e_\lambda \right)
\]
where \((e_\lambda x)(\theta) = e^{\lambda \theta} x\) for \(x \in X\) and \(\theta \in [-1,0]\). Moreover, for \(\lambda \in \rho(\mathcal{A})\) the resolvent
\[
R(\lambda, \mathcal{A}) =
\]

\[
\begin{pmatrix}
R_\lambda & R_\lambda \Sigma_\lambda R(\lambda, D_\theta) & \left[ R_\lambda (K_\lambda + L_\lambda) R(\lambda, A) + R(\lambda, A) \right] R(\lambda, D_\theta) \\
e_\lambda R_\lambda & e_\lambda R_\lambda \Sigma_\lambda & e_\lambda R_\lambda \delta_0 R(\lambda, D_\theta) \\
R(\lambda, D_\lambda) L_\lambda R_\lambda R(\lambda, D_\lambda) & R(\lambda, D_\theta) & \left[ R(\lambda, D_\lambda) \right] \left[ \left[ R(\lambda, D_\lambda) \right] \left[ R(\lambda, D_\lambda) \right] \left[ R(\lambda, D_\lambda) \right] \right] \\
+ L_\lambda R_\lambda \Sigma_\lambda R(\lambda, D_\theta) & R(\lambda, D_\theta) & \left[ R(\lambda, D_\lambda) \right] \left[ R(\lambda, D_\lambda) \right] \left[ R(\lambda, D_\lambda) \right] \\
+ b(\cdot) LR_\lambda \delta_0 R(\lambda, D_\theta) & \left[ R(\lambda, D_\lambda) \right] \left[ R(\lambda, D_\lambda) \right] \left[ R(\lambda, D_\lambda) \right] \\
\end{pmatrix}
\]
with
\[
\begin{align*}
\Sigma_\lambda &= K + \hat{b}(\lambda)L, \\
R_\lambda &= R(\lambda, A + \Sigma_\lambda e_\lambda), \\
L_\lambda &= \hat{b}(\lambda) L e_\lambda, \\
K_\lambda &= K e_\lambda.
\end{align*}
\]
Proof. Let $\lambda \in \mathbb{C}_0 \cap \rho(A)$, we have

$$R(\lambda, A_0) = \int_0^{+\infty} e^{-\lambda t} T_0(t) \, dt = \begin{pmatrix} R(\lambda, A) & 0 & 0 \\ e_\lambda R(\lambda, A) & R(\lambda, D_0) & 0 \\ 0 & 0 & R(\lambda, D_0) \end{pmatrix}.$$ 

Then $\lambda \in \rho(A)$ if and only if the bounded operator $I - R(\lambda, A_0) B$ is invertible, and we have

$$R(\lambda, A) = (I - R(\lambda, A_0) B)^{-1} R(\lambda, A_0).$$

(13)

It is interesting to know the conditions for which the bounded operator $I - R(\lambda, A_0) B$ is invertible, then we calculate

$$I - R(\lambda, A_0) B = \begin{pmatrix} I & -R(\lambda, A) K & -R(\lambda, A) \delta_0 \\ 0 & (I - e_\lambda R(\lambda, A) K) & -e_\lambda R(\lambda, A) \delta_0 \\ 0 & -R(\lambda, D_0) b(\cdot) L & I \end{pmatrix}$$

and suppose that $I - R(\lambda, A_0) B$ is invertible, we have

$$(I - R(\lambda, A_0) B)^{-1} \begin{pmatrix} f \\ g \\ h \end{pmatrix} \iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} \iff (I - R(\lambda, A_0) B) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} f \\ g \\ h \end{pmatrix}$$

$$\iff \begin{cases} f = x - R(\lambda, A) Ky - R(\lambda, A) \delta_0 z \\ g = (I - e_\lambda R(\lambda, A) K) y - e_\lambda R(\lambda, A) \delta_0 z \\ h = -R(\lambda, D_0) b(\cdot) Ly + z \end{cases}$$

Remark that $\delta_0 R(\lambda, D_0) b(\cdot) L = \hat{b}(\lambda) L$ for $\lambda \in \mathbb{C}_\beta$ and from the second and third line of the above system we have

$$(I - e_\lambda R(\lambda, A) \left(K + \hat{b}(\lambda) L\right)) y = g + e_\lambda R(\lambda, A) \delta_0 h,$$

then $I - R(\lambda, A_0) B$ is invertible if and only if $1 \in \rho \left(e_\lambda R(\lambda, A) \left(K + \hat{b}(\lambda) L\right)\right)$ and by a simple computation we have

$$R(\lambda, A) \left(I - \left(K + \hat{b}(\lambda) L\right) e_\lambda R(\lambda, A)\right)^{-1} = R \left(\lambda, A + \left(K + \hat{b}(\lambda) L\right) e_\lambda \right).$$

Then

$$\lambda \in \rho(A) \iff 1 \in \rho \left(e_\lambda R(\lambda, A) \left(K + \hat{b}(\lambda) L\right)\right) \iff \lambda \in \rho \left(A + \left(K + \hat{b}(\lambda) L\right) e_\lambda \right).$$

For simplicity of notation, we use $Q(\lambda) = \left(I - e_\lambda R(\lambda, A) \left(K + \hat{b}(\lambda) L\right)\right)^{-1}$ and we have

$$\begin{cases} x = f + R(\lambda, A) Ky + R(\lambda, A) \delta_0 z \\ y = Q(\lambda) g + Q(\lambda) e_\lambda R(\lambda, A) \delta_0 h \\ z = R(\lambda, D_0) b(\cdot) Ly + h, \end{cases}$$
thus

\[
\begin{cases}
  x = f + R(\lambda, A) \left( K + \hat{b}(\lambda) L \right) Q(\lambda) g + R(\lambda, A) \delta_0 h \\
  + R(\lambda, A) \left( K + \hat{b}(\lambda) L \right) Q(\lambda) e_1 R(\lambda, A) \delta_0 h \\
  y = Q(\lambda) g + Q(\lambda) e_1 R(\lambda, A) \delta_0 h \\
  z = R(\lambda, D_s) b(\cdot) LQ(\lambda) g + (R(\lambda, D_s) b(\cdot) LQ(\lambda) e_1 R(\lambda, A) \delta_0 + I) h.
\end{cases}
\]

Thus led to

\[(I - R(\lambda, A_0) B)^{-1} = \]

\[
\begin{pmatrix}
  I & R(\lambda, A) \left( K + \hat{b}(\lambda) L \right) Q(\lambda) e_1 R(\lambda, A) \\
  0 & R(\lambda, D_s) b(\cdot) LQ(\lambda)
\end{pmatrix}
\]

and from (13) one has

\[R(\lambda, A) = \begin{pmatrix}
  R_{11} & R_{12} & R_{13} \\
  R_{21} & R_{22} & R_{23} \\
  R_{31} & R_{32} & R_{33}
\end{pmatrix}\]

with

\[
\begin{cases}
  R_{11} = R(\lambda, A) + R(\lambda, A) \left( K + \hat{b}(\lambda) L \right) Q(\lambda) e_1 R(\lambda, A) \\
  R_{12} = R(\lambda, A) \left( K + \hat{b}(\lambda) L \right) Q(\lambda) R(\lambda, D_0) \\
  R_{13} = [R(\lambda, A) \left( K + \hat{b}(\lambda) L \right) Q(\lambda) e_1 + I] R(\lambda, A) \delta_0 R(\lambda, D_s) \\
  R_{21} = Q(\lambda) e_1 R(\lambda, A) \\
  R_{22} = Q(\lambda) R(\lambda, D_0) \\
  R_{23} = Q(\lambda) e_1 R(\lambda, A) \delta_0 R(\lambda, D_s) \\
  R_{31} = R(\lambda, D_s) b(\cdot) LQ(\lambda) e_1 R(\lambda, A) R(\lambda, D_s) \\
  R_{32} = R(\lambda, D_s) b(\cdot) LQ(\lambda) R(\lambda, D_0) \\
  R_{33} = (R(\lambda, D_s) b(\cdot) LQ(\lambda) e_1 R(\lambda, A) \delta_0 + I) R(\lambda, D_s).
\end{cases}
\]
A simple computation gives the desired resolvent operator components

\[ R_{11} = R(\lambda, A) + R(\lambda, A) \left( K + \hat{b}(\lambda) L \right) \left( I - e_\lambda R(\lambda, A) \left( K + \hat{b}(\lambda) L \right) \right)^{-1} e_\lambda R(\lambda, A) \]

\[ = R(\lambda, A) \left( I - \left( K + \hat{b}(\lambda) L \right) e_\lambda R(\lambda, A) \right)^{-1} \]

\[ = R \left( \lambda, A + \left( K + \hat{b}(\lambda) L \right) e_\lambda \right) \left( K + \hat{b}(\lambda) L \right) R(\lambda, D_\theta) \]

\[ R_{12} = R(\lambda, A) \left( K + \hat{b}(\lambda) L \right) \left( I - e_\lambda R(\lambda, A) \left( K + \hat{b}(\lambda) L \right) \right)^{-1} R(\lambda, D_\theta) \]

\[ = R(\lambda, A) \left( I - \left( K + \hat{b}(\lambda) L \right) e_\lambda R(\lambda, A) \right)^{-1} \left( K + \hat{b}(\lambda) L \right) R(\lambda, D_\theta) \]

\[ = R \left( \lambda, A + \left( K + \hat{b}(\lambda) L \right) e_\lambda \right) \left( K + \hat{b}(\lambda) L \right) R(\lambda, D_\theta) \}

\[ R_{13} = \left[ R(\lambda, A) \left( K + \hat{b}(\lambda) L \right) \left( I - e_\lambda R(\lambda, A) \left( K + \hat{b}(\lambda) L \right) \right)^{-1} e_\lambda + I \right] R(\lambda, A) \delta_0 R(\lambda, D_\theta) \]

\[ = \left[ R \left( \lambda, A + \left( K + \hat{b}(\lambda) L \right) e_\lambda \right) \left( K + \hat{b}(\lambda) L \right) e_\lambda + I \right] R(\lambda, A) \delta_0 R(\lambda, D_\theta) \}

\[ R_{21} = \left( I - e_\lambda R(\lambda, A) \left( K + \hat{b}(\lambda) L \right) \right)^{-1} e_\lambda R(\lambda, A) \]

\[ = e_\lambda R(\lambda, A) \left( K + \hat{b}(\lambda) L \right) \]

\[ = e_\lambda R \left( \lambda, A + \left( K + \hat{b}(\lambda) L \right) e_\lambda \right) \]

\[ R_{22} = \left( I - e_\lambda R(\lambda, A) \left( K + \hat{b}(\lambda) L \right) \right)^{-1} R(\lambda, D_\theta) \]

\[ = \left[ I + e_\lambda R(\lambda, A) \left( I - \left( K + \hat{b}(\lambda) L \right) e_\lambda R(\lambda, A) \right)^{-1} \left( K + \hat{b}(\lambda) L \right) \right] R(\lambda, D_\theta) \]

\[ = \left[ I + e_\lambda R(\lambda, A) \left( I - \left( K + \hat{b}(\lambda) L \right) e_\lambda R(\lambda, A) \right)^{-1} \left( K + \hat{b}(\lambda) L \right) \right] R(\lambda, D_\theta) \]

\[ = \left[ I + e_\lambda R \left( \lambda, A + \left( K + \hat{b}(\lambda) L \right) e_\lambda \right) \left( K + \hat{b}(\lambda) L \right) \right] R(\lambda, D_\theta) \]

\[ R_{23} = R_{21} \delta_0 R(\lambda, D_\theta) \]

\[ = e_\lambda R \left( \lambda, A + \left( K + \hat{b}(\lambda) L \right) e_\lambda \right) \delta_0 R(\lambda, D_\theta) \]

\[ R_{31} = R(\lambda, D_\theta) b(\cdot) LR_{23} \]

\[ = R(\lambda, D_\theta) b(\cdot) L e_\lambda R \left( \lambda, A + \left( K + \hat{b}(\lambda) L \right) e_\lambda \right) R(\lambda, D_\theta) \]

\[ R_{32} = R(\lambda, D_\theta) b(\cdot) LR_{22} \]

\[ = R(\lambda, D_\theta) b(\cdot) L \left[ I + e_\lambda R \left( \lambda, A + \left( K + \hat{b}(\lambda) L \right) e_\lambda \right) \left( K + \hat{b}(\lambda) L \right) \right] R(\lambda, D_\theta) \]

and

\[ R_{33} = R(\lambda, D_\theta) b(\cdot) LR_{23} \delta_0 + I \]

\[ = R(\lambda, D_\theta) b(\cdot) L e_\lambda R \left( \lambda, A + \left( K + \hat{b}(\lambda) L \right) e_\lambda \right) \delta_0 + I \]
which complete the proof.

**Remark III.2.** If \( b = 0 \) we have

\[
R(\lambda, A) = \begin{pmatrix}
R_{11} & R_{12} & R_{13} \\
R_{21} & R_{22} & R_{23} \\
R_{31} & R_{32} & R_{33}
\end{pmatrix}
\]

with

\[
\begin{align*}
R_{11} &= R(\lambda, A + K_\lambda) \\
R_{12} &= R(\lambda, A + K_\lambda) KR(\lambda, D_\theta) \\
R_{13} &= [R(\lambda, A + K_\lambda) K_\lambda + I] R(\lambda, A) \delta_0 R(\lambda, D_s) \\
R_{21} &= \epsilon \lambda R(\lambda, A + K_\lambda) \\
R_{22} &= [\epsilon \lambda R(\lambda, A + K_\lambda) K + I] R(\lambda, D_\theta) \\
R_{23} &= \epsilon \lambda R(\lambda, A + K_\lambda) K_\lambda \delta_0 R(\lambda, D_s) \\
R_{31} &= R_{32} = 0 \\
R_{33} &= R(\lambda, D_\theta)
\end{align*}
\]

As \( b(\cdot) = 0 \), the function \( g \) defined in (3) is equal to 0, then if we replace the third space \( L^p(\mathbb{R}_+, X) \) by \( \{0\} \), we obtain the same result as in [2, Proposition 3.19, page 56]

\[
R(\lambda, A) \begin{pmatrix} x \\
\varphi \\
0
\end{pmatrix} = \begin{pmatrix}
-\frac{R(\lambda, A + K_\lambda)}{\epsilon \lambda R(\lambda, A + K_\lambda)} & -\frac{R(\lambda, A + K_\lambda) KR(\lambda, D_\theta)}{\epsilon \lambda R(\lambda, A + K_\lambda) K + I} & R(\lambda, A + K_\lambda) \delta_0 R(\lambda, D_s) \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \begin{pmatrix} x \\
\varphi \\
0
\end{pmatrix} \times \{0\}.
\]

### IV. Application.

In order to apply the results obtained in Sections 2 and 3, we consider the two following examples.

**Example IV.1.** Consider the following diffusion equation. Let \( t \geq 0 \) and \( x \in \mathbb{R} \)

\[
\begin{align*}
\frac{\partial z(x,t)}{\partial t} &= \Delta z(x,t) + \int_0^t e^{-(t-s)} \int_{-\mu(t)}^0 d\mu(s) z(x, s + \theta) d\theta ds + z(x, t - 1); \\
z(x, 0) &= z^0(x) \\
z(x, \theta) &= \varphi(x, \theta); \quad \theta \in [-1, 0].
\end{align*}
\]

In order to write the system (14) as the abstract form of system (1), we take

- the state space \( X = L^2(\mathbb{R}) \),
- the operator \( A = \Delta \), with \( D(A) = W^{2,2}(\mathbb{R}) \),
- the function \( \mathbb{R}_+ \ni t \mapsto z(t) = z(\cdot, t) \in L^2(\mathbb{R}) \), and the history function \( z_1 : [-1, 0] \to L^2(\mathbb{R}) \); \( z_1(s) = z(t + s) \),
- the state delay operator \( L : W^{1,2}([-1, 0], L^2(\mathbb{R})) \to L^2(\mathbb{R}) \), defined by \( L \varphi = \int_{-1}^0 d\mu(\theta) \varphi(\theta) \), with \( \mu : [-1, 0] \to X \) be of bounded variation,
- the state delay operator \( K : W^{1,2}([-1, 0], L^2(\mathbb{R})) \to L^2(\mathbb{R}) \), defined by \( K = \delta_{-1} \).

It is well known that \( A \) generates an analytic \( C_0 \)-semigroup on \( X \), denoted by \( \mathbb{T}(t) \) called the diffusion semigroup (see [7] Chapter II, page 69). Moreover, \( \mathbb{T}(t) \) is expressed as follows

\[
(\mathbb{T}(t)f)(s) := \begin{cases}
(4\pi t)^{-1/2} \int_{\mathbb{R}} e^{-(s-\theta)^2/4t} f(\theta) d\theta & t > 0 \\
f(s) & t = 0
\end{cases} \quad \text{where } s \in \mathbb{R}.
\]
From (11) we have already mentioned that condition \( (M_1) \) is satisfied by \( L \). It is well-known see [13] Theorem 6.13, page 74 that \((-\Delta)^\gamma \) satisfied the following estimation \( \| f^\gamma (-\Delta)^\gamma \mathbf{T}(t) \| \leq M \), with \( M \) and \( \gamma \) are positive constants. By a simple computation with \( \gamma \in (0, \frac{1}{2}) \) we have:

\[
\int_0^\alpha \| (-\Delta)^\gamma \mathbf{T}(t)x \| ^2 dt \leq M^2 \frac{1}{1-2\gamma} \alpha^{1-2\gamma} \| x \| ^2 ,
\]

which implies that \( F \) is a 2-admissible observation operator for \( (\mathbf{T}(t)) \). Then the well-posedness of the system (14) is reached by Theorem II.7.

**Example IV.2.** Consider the following retarded heat equation with the Dirichlet boundary condition on the state space \( X = L^2[0, \pi] \), for \( x \in [0, \pi] \) and \( t \geq 0 \)

\[
\begin{cases}
\frac{\partial}{\partial t} z(x, t) = \Delta z(x, t) + \int_0^t e^{-(t-s)}z(x, s-1)ds + z(x, t-1); \\
z(0, t) = z(\pi, t) = 0; \\
z(x, 0) = z_0(x); \\
z(x, s) = \varphi(x, s); & s \in [-1, 0].
\end{cases}
\]

Here we take

- the operator \( A = \Delta \), with \( D(A) = \{ f \in H^2[0, \pi] : f(0) = f(\pi) = 0 \} \),
- the state delay operators \( L, K : H^1([-1, 0], X) \rightarrow X \), defined by \( L = K = \delta_{-1} \),
- the scalar kernel is defined by \( b(t) = e^{-t} \).

The well-posedness of this system is due to the fact that \( A \) generates a \( C_0 \)-semigroup and from (10), we have already mentioned that condition \( (M_2) \) is satisfied by \( L \), these conditions verify Theorem II.7. Moreover, \( A \) generates a compact \( C_0 \)-semigroup \( \mathbf{T}(t) \) on \( X \) (see, e.g., [20, Example 1.1], [17, Example 2.6.8] and [13, Example 2.1.1]), the compactness of \( \mathbf{T}(t) \) guarantees that \( \sigma(A) = \sigma_p(A) \). In the following orthonormal basis \( \phi_n(x) = \sqrt{\frac{2}{\pi}} \cos(nx) \) for \( n \geq 1 \), we have \( \sigma_p(A) = \{-n^2 : n = 1, 2, \ldots \} \). Then from Theorem III.7 one has for \( \lambda \notin \mathbb{C}_0 \cap \rho(A) \):

\[
\lambda \in \sigma(A) \iff \lambda \in \sigma \left( A + \delta_{-1}e_\lambda + \hat{b}(\lambda) \delta_{-1}e_\lambda \right) \\
\iff \left( \lambda - \delta_{-1}e_\lambda - \hat{b}(\lambda) \delta_{-1}e_\lambda \right) \in \sigma(A)
\]

then

\[
\sigma(A) = \left\{ \lambda \notin \mathbb{C}_0 \mid \lambda - \frac{2+\lambda}{1+\lambda} e^{-\lambda} = -n^2 \right\} \\
= \left\{ \lambda \notin \mathbb{C}_0 \mid \left( \lambda + n^2 \right)(\lambda + 1) = (2+\lambda)e^{-\lambda} \right\}.
\]

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