Thermodynamics, photon sphere and thermodynamic geometry of Eloy Ayon Beato - Garcia Spacetime.

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Abstract

We study the thermodynamics of Ayon Beato-Garcia black hole and the relation between the photon orbits and the thermodynamic phase transition of the black hole. Then we study the interaction between the microstructures of the black hole using Ruppeiner geometry. The radius of photon orbit and minimum impact parameter behave non-monotonously below the critical point, which mimics the behaviour of Hawking temperature and pressure in extended thermodynamics. Their change during the large black hole and small black hole phase transition serve as the order parameter, having a critical exponent $1/2$. The results show that gravity and thermodynamics of Ayon Beato-Garcia black hole are closely related. Following this we study the thermodynamic geometry which gives insight of the microstructure interaction of the black hole. We find that the large black hole phase is analogous to bosonic gas with a dominant attractive interaction, and the small black hole phase behaves like an anyonic gas with both attractive and repulsive interactions.

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I. INTRODUCTION

In recent years it is a well-established notion that a black hole system is not merely a gravitating system but also has thermal properties. In parallel to ordinary thermodynamic systems we can establish laws of black hole mechanics, by identifying temperature and entropy from surface gravity and area, respectively \[1, 2\]. Black holes exhibit a wide variety of phase transitions which are interesting from several aspects. Particularly, the identification of dynamical cosmological constant with the pressure term \[3, 4\], leading to extended thermodynamics, which eventually lead to the development of the area of black hole chemistry \[5–7\].

For a gravitational system, its characteristic features can be determined by observing the behaviour of test particles around it. It is interesting to note that photon approaching the black hole can orbit around it depending on its physical conditions. Many such photons orbiting the black hole form a photon sphere, which has several theoretical and observational importance, like, in black hole shadow, gravitational lensing, quasinormal mode (QNM) etc. It was discovered by Wei et al. \[8, 9\], that the photon orbit parameters are encoded with the phase transition details of a black hole in AdS background. The key parameters in connecting the gravitational and thermodynamic aspects of the black hole are the radius of the photon orbit and minimum impact parameter. Their behaviour directly correlates to the phase structure of the black hole. These parameters undergo a sudden change during the phase transition and this change can be seen as the order parameter that characterises the phase transition. The connection between null geodesics and thermodynamics has been widely studied for Kerr-AdS \[9\], Born-Infeld with AdS background \[10\], regular AdS black holes \[11\], massive gravity \[12\], Born-Infeld-dilaton black hole \[13\], five-dimensional Gauss-Bonnet black holes \[14\] etc. Related studies in other contexts have also been carried out in subsequent works \[15–17\]. In this article a similar investigation is carried out for ABG-AdS black hole along with the microstructure study.

To probe the underlying microstructure of the black hole phase transition, a novel Ruppeiner geometric method was introduced by constructing a line element in the parameter space of fluctuating coordinates, temperature and volume \[18, 19\]. In thermodynamic geometry, the curvature scalar is an indicator of nature and strength of microscopic interaction determined from its sign and magnitude, respectively \[20–24\]. Even though the phase
transition resembles that of a van der Waals system, the black hole microstructure has a distinct microstructure. For example charged AdS black hole shows distinct repulsive interaction along with attractive interaction\textsuperscript{[18, 19]}, whereas, 5D Gauss-Bonnet AdS has only attractive interaction \textsuperscript{[25]}. In contrast to conventional approach, where the microstructure determines the phase transition, in this approach interaction among the constituents are sought from phase structure studies \textsuperscript{[26]}. There exists numerous studies concerning black hole microstructure from this perspective \textsuperscript{[27–41]}. 

The motivation for the present study is due to the interest in singularity free solutions to Einstein’s equations. Typically a black hole possesses a physical singularity at the center which is usually considered as a limitation of classical gravity as there is no complete quantum explanation for it. Black holes without these central physical singularity are known as regular black holes, which was initially proposed by Bardeen \textsuperscript{[42, 43]}, where the central singularity is replaced with a de-Sitter core. Many more regular solutions followed this \textsuperscript{[44–51]}, out of which the solution due to Eloy Ayon Beato and Garcia is quite interesting as it is an exact solution to Einstein’s field equation coupled to a non-linear electromagnetic field \textsuperscript{[48]}. This non-linear field goes to Maxwell’s field in the weak field limit and the solution admits weak energy condition. Another interesting feature of this solution is it does not possess a Cauchy surface, and hence it does not contradict Penrose singularity theorem. Nevertheless, there is no detailed study on the thermodynamics of this solution in the literature, so we begin with it.

The article is organised as follows. In the next section, we discuss the thermodynamics and phase structure of the black hole. In section III the photon orbit and its relation to phase transition is presented. Then the microstructure of the black hole is probed using the novel Ruppeiner geometry (section IV). At the end we discuss the results in section V.

II. PHASE STRUCTURE OF THE AYON-BEATO GARCIA ADS BLACK HOLE

The Ayon-Beato Garcia (ABG) black hole solution in the 4-dimensional AdS background is given by \textsuperscript{[48, 52]},

\[
ds^2 = - f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \tag{1}
\]
where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$ is the 2-sphere, and the metric function,

$$f(r) = 1 - \frac{2mr^2}{(q^2 + r^2)^{3/2}} + \frac{q^2r^2}{(q^2 + r^2)^2} - \frac{\Lambda r^2}{3}. \tag{2}$$

With the electric field given by,

$$E = qr^2 \left( \frac{r^2 - 5q^2}{(r^2 + q^2)^4} + \frac{15m}{2(r^2 + q^2)^{7/2}} \right). \tag{3}$$

The spacetime is regular, with no singularity at the center of the black hole when $r = 0$. This can be verified by observing that the Kretschmann scalar does not have a singularity at $r = 0$,

$$k = R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \frac{8 \left( 36M^2 + 12Mq(\Lambda q^2 - 3) + q^2(\Lambda q^2 - 3)^2 \right)}{3q^6}. \tag{4}$$

In this section we study the extended thermodynamics by considering dynamic cosmological constant as thermodynamic pressure, $P = -\Lambda/8\pi$. The thermodynamic quantities of the system, namely temperature, volume and entropy, are obtained as,

$$T = \frac{q^2r^4(16\pi Pr^2 - 1) + q^4r^2(8\pi Pr^2 - 3)}{4\pi r^3}, \tag{5}$$

$$V = \frac{4}{3}\pi \left( \frac{q^2 + r^2}{3} \right), \tag{6}$$

and

$$S = 2\pi \left[ \left( \frac{r}{2} - \frac{1}{r} \right) \sqrt{r^2 + 1} + \frac{3}{2} \log \left( \sqrt{r^2 + 1} + r \right) \right]. \tag{7}$$

These quantities satisfy the first law of black hole mechanics,

$$dM = TdS + \Psi dQ + VdP, \tag{8}$$

and its integral form, the Smarr relation,

$$M = 2(TS - VP) + \Psi Q. \tag{9}$$

The expression for Hawking temperature is obtained from the surface gravity $\kappa$ as $T = \kappa/2\pi$. The entropy $S$ does not follow from the Bekenstein’s area law, instead it follows from the first law. This is a characteristic feature of some regular black holes [53]. The black hole solution we consider is a spherically symmetric one, where entropy and volume are interdependent, and hence the heat capacity vanishes,

$$C_V = T \left( \frac{\partial S}{\partial T} \right)_V = 0. \tag{10}$$
The equation of state is obtained by rearranging the expression for Hawking temperature (Eq. 5),

$$P = \frac{3q^4 r^2 (4\pi rT + 1) + q^2 r^4 (12\pi rT + 1) + q^6 (4\pi rT + 2) + r^6 (4\pi rT - 1)}{8\pi r^4 (q^2 + r^2)^2}. \quad (11)$$

It is well known that, charged AdS black holes show critical behaviour similar to the van der Waals system in conventional thermodynamic system. The critical parameters associated with the black hole system are obtained by using the condition $\partial P/\partial V = 0$ and $\partial^2 P/\partial V^2 = 0$, as,

$$T_c = \frac{0.02208}{q}, \quad P_c = \frac{0.0008926}{q^2}, \quad V_c = 417.157 q^2. \quad (12)$$

With the help of these we define the reduced parameters,

$$\tilde{T} = \frac{T}{T_c}, \quad \tilde{P} = \frac{P}{P_c}, \quad \tilde{V} = \frac{V}{V_c}. \quad (13)$$

In the reduced parameter space, the equation of state takes the form,

$$\tilde{P} = \frac{0.00978 \left(217354\tilde{T}\sqrt{85.941\tilde{V}^{2/3} - 4\tilde{V}^2 - 6785.63\tilde{V}^{2/3} + 291582\tilde{V}^{4/3} - 1.5661 \times 10^6 \tilde{V}^2 + 16\pi^2}\right)}{\left(85.941\tilde{V}^{2/3} - 4\right)^{2\tilde{V}^{4/3}}} \tilde{V}^{4/3}. \quad (14)$$

The reduced equation of state is independent of the electric charge parameter $q$. In 1(a) we plot the reduced pressure $\tilde{P}$ against reduced volume $\tilde{V}$, which shows an oscillatory behaviour for a range of $\tilde{P}$ and $\tilde{V}$, this oscillatory behaviour decreases as the temperature increases, and completely vanishes for temperatures greater than the critical temperature $T_c$. The same behaviour can be observed from the Gibbs free energy $\tilde{G}$ ($G = M - TS$).
plotted against reduced temperature $\tilde{T}$. It shows a swallow tail behaviour, which corresponds to the oscillatory section of $\tilde{P}-\tilde{V}$ diagram. This section is unphysical, and the system follows the curve without entering the swallow tail section. The point of intersection gives the coexistence point, which signifies that, at that particular point, two phases are in coexistence. As the pressure increases, the swallow tail behaviour vanishes at the critical values. The plot of these coexistence points gives the coexistence curve, which terminates at the critical point. We fit this curve and find the fitting equation of the coexistence curve to be,

$$\tilde{P} = -51.1418\tilde{T}^{10} + 287.367\tilde{T}^9 - 701.533\tilde{T}^8 + 977.194\tilde{T}^7 - 857.004\tilde{T}^6 + 492.65\tilde{T}^5$$
$$- 187.017\tilde{T}^4 + 46.3054\tilde{T}^3 - 6.39543\tilde{T}^2 + 0.595748\tilde{T} - 0.0211245. \quad (15)$$

We can observe a small black hole (SBH)- large black hole (LBH) phase transition in the black hole system, with the horizon radius as the order parameter. This behaviour is similar to that of van der Waals system of liquid-vapour phase transition.

Next, we obtain the spinodal curve, using the condition,

$$(\partial_{\tilde{V}} \tilde{P})_{\tilde{T}} = 0. \quad (16)$$

The explicit form of the spinodal curve is,

$$\tilde{T}_{sp} = A \frac{1417.65 \sqrt{\tilde{V}^{3/2}}}{\tilde{V}^{3/2} - \frac{182752 \sqrt{\tilde{V}}}{\tilde{T}^{5/2}}}, \quad (17)$$

where,

$$A = \frac{177.032}{\tilde{V}^{5/3}\tilde{T}^3} - \frac{44.258}{\tilde{V}^{5/3}\tilde{T}^2} + \frac{2.05992}{\tilde{V}^{7/3}\tilde{T}^2} - \frac{1.75579 \times 10^6 \sqrt{\tilde{V}}}{\tilde{V}\tilde{T}^3} - \frac{7607.16}{\tilde{V}\tilde{T}^3} + \frac{326883}{3\sqrt{\tilde{V}}\tilde{T}^3} + \frac{4.54747 \times 10^{-13}}{\tilde{V}\tilde{T}^2} + \frac{10215.1}{3\sqrt{\tilde{V}}\tilde{T}^2}. \quad (18)$$

We plot the spinodal curve in 2(a), the blue dashed lines are the spinodal curves, which are on the either sides of the coexistence curve, and terminate with it at the critical point. The area between the coexistence curve and spinodal curves signify the metastable phases. The area below the coexistence curve and above the spinodal curve corresponds to the supercooled LBH, and the area above the coexistence curve and between the spinodal curve, to the superheated SBH. Beyond the critical point, the phase transition is second order.
where SBH and LBH are indistinguishable, referred as the supercritical black hole phase. The coexistence curve and spinodal curves in $T-V$ plane is plotted in 2(b).

As the black hole undergoes phase transition, its radius changes abruptly which is depicted in 3(a), a characteristic feature of first-order phase transition. The change in radius as a function of temperature is shown in fig. 3(b), which acts an order parameter. As the horizon radius characterises the phase transition properties of the system, one can think of a similar correlation between thermodynamics and the quantities dependent on horizon radius, such as null geodesics around the black hole which we study in the next section.
III. PHOTON SPHERE AND PHASE TRANSITION

A. Geodesic equations

Now we study the relationship between the phase transition and the photon sphere of the black hole. We begin by considering the equatorial ($\theta = \pi/2$) orbit of a photon around the black hole. The motion of the photon is described by the following Lagrangian,

$$2\mathcal{L} = -f(r)\dot{t}^2 + \frac{\dot{r}^2}{f(r)} + r^2\dot{\phi}^2.$$  (19)

where $\dot{x}^\mu = dx^\mu/d\lambda$, $\lambda$ is the affine parameter. The symmetries of spacetime leads to Killing fields $\partial_t$ and $\partial_\phi$, which are associated with the conserved quantities of the particle motion, namely energy and orbital angular momentum of the photon. $p_a = g_{ab}\dot{x}^b$ gives the generalised momenta of the system. They are,

$$p_t = -f(r)\dot{t} \equiv E$$  (20)  
$$p_\phi = r^2\dot{\phi} \equiv L$$  (21)  
$$p_r = \dot{r}/f(r).$$  (22)

The equations of motion for $r$ and $t$ can be written using the above generalised momenta as,

$$\dot{t} = \frac{E}{f(r)}$$  (23)  
$$\dot{\phi} = \frac{L}{r^2\sin^2\theta}.$$  (24)

We note that the Hamiltonian of the system vanishes,

$$2\mathcal{H} = -E\dot{t} + L\dot{\phi} + \dot{r}^2/f(r) = 0.$$  (25)

Using the expressions for $r$ and $\phi$, the radial $r$ motion can be expressed as,

$$\dot{r}^2 + V_{eff} = 0,$$  (26)

where the effective potential is given by,

$$V_{eff} = \frac{L^2}{r^2}f(r) - E^2.$$  (27)

The trajectory of the photon around the black hole is determined by the effective potential. Orbital motion of a photon is only possible when $V_{eff} < 0$, as $\dot{r}^2 > 0$. For a particular value
of angular momentum, photon has a circular orbit around the black hole, forming a photon sphere. For values less/greater than this critical value of angular momentum, the photon gets absorbed/scattered by the black hole. The photon orbit is characterised by,

\[ V_{\text{eff}} = 0, \quad V'_{\text{eff}} = 0, \quad V''_{\text{eff}} < 0, \tag{28} \]

where \( V' = \partial V/\partial r \) and \( V'' = \partial^2 V/\partial r^2 \). In this orbit, the radial velocity of the photon is zero. The equation \( V'_{\text{eff}} = 0 \) can be expanded as,

\[ 2f(r_{ps}) - r_{ps}\partial_r f(r_{ps}) = 0. \tag{29} \]

The solution of the above equation on substituting the metric function yields the expression for radius of photon orbit \( r_{ps} \), which is a function of black hole parameters \((M, q, P)\). Another key parameter of the photon sphere, the minimum impact parameter, is obtained by solving the equation \( (V_{\text{eff}} = 0) \), which reads,

\[ u_{ps} = \frac{L_c}{E} = \frac{r}{\sqrt{f(r)}} \bigg|_{r_{ps}}. \tag{30} \]

In a reduced parameter space, observing the photon sphere radius and minimum impact parameter with respect to Hawking temperature and pressure, we find the correlation between photon sphere and black hole phase transition. The isobars in \( \tilde{T} - r_{ps} \) and \( \tilde{T} - u_{ps} \) shows similar behaviour as that of van der Waals system (Fig. 5). Below the critical pressure, they show an oscillating behaviour, and above the critical pressure this behaviour
disappears. Similarly, an oscillatory behaviour below critical temperature is shown by the isotherms in $\tilde{P} - \tilde{r}_{ps}$ and $\tilde{P} - \tilde{u}_{ps}$ planes (Fig. 6). However, the increasing and decreasing trends of the functions are flipped. These behaviours are characteristic features of the van der Waals like phase transition of the black hole. This clearly shows the connection between photon orbit and phase transition of the black hole.

B. Critical behaviour of the photon sphere

The AGB black hole in AdS spacetime exhibits a van der Waals like first order phase transition below the critical point. At the critical point this behaviour ceases, and a second-order phase transition is observed. In the previous subsection, we have seen that the photon sphere parameters aptly demonstrate these features, therefore, it is reasonable to see change
in those parameters during the phase transition. We construct equal area law to obtain the behaviour of radius of photon orbit and minimum impact parameter. This construction is possible by the fact that the isobars in $\tilde{T} - \tilde{r}_{ps}$ and $\tilde{T} - \tilde{u}_{ps}$ planes are similar to the isobars in the conventional $\tilde{T} - \tilde{S}$ isobars of the black hole. The results are shown in fig. 7. Both the photon orbit parameters $r_{ps}$ and $u_{ps}$ have similar behaviours. The parameters increase with increase in temperature in coexistence SBH phase and decrease with increase in temperature in coexistence LBH phase. At the critical point $\tilde{T} = 1$, the values of LBH and SBH phases match. In fig. 8 we show the change in photon orbit parameters with the phase transition temperature. For first-order phase transition, the change is finite, whereas at the second order transition point the differences vanish. The behaviour of both $\Delta r_{ps}$ and $\Delta u_{ps}$ can serve as an order parameter to characterise the black hole phase transition. We examine the critical behaviour associated with $\Delta r_{ps}$ and $\Delta u_{ps}$ near the critical point (shown in the insets of fig. 8). We numerically obtain the critical exponent $\delta$ as follows,

$$\Delta \tilde{r}_{ps}, \quad \Delta \tilde{u}_{ps} \sim a \times (1 - \tilde{T})^\delta. \quad (31)$$

Taking logarithm on both side, we have,

$$\ln \Delta \tilde{r}_{ps}, \quad \ln \Delta \tilde{u}_{ps} \sim \delta \ln(1 - \tilde{T}) + \ln a. \quad (32)$$

Which means that $\ln \Delta \tilde{r}_{ps}$ and $\ln \Delta \tilde{u}_{ps}$ linearly vary with $\ln(1 - \tilde{T})$. A simple numerical fit of the curve by varying $\tilde{T}$ from 0.99 to 0.9999 yields the results, which are shown in fig 9. The critical exponent of $\Delta \tilde{r}_{ps}$ and $\Delta \tilde{u}_{ps}$, near the point $\tilde{T} = 1$ are obtained to be,

$$\Delta \tilde{r}_{ps} = 1.28653 \times (1 - \tilde{T})^{0.487368} \quad (33)$$

and

$$\Delta \tilde{u}_{ps} = 0.430121 \times (1 - \tilde{T})^{0.513824}. \quad (34)$$

This behaviour, i.e. $\Delta \tilde{r}_{ps} \sim (1 - \tilde{T})^{1/2}$ and $\Delta \tilde{u}_{ps} \sim (1 - \tilde{T})^{1/2}$, show that $\Delta \tilde{r}_{ps}$ and $\Delta \tilde{u}_{ps}$ are the order parameters of black hole phase transition with a critical exponent $1/2$. The reflection of critical behaviour in $\Delta \tilde{r}_{ps}$ and $\Delta \tilde{u}_{ps}$ further asserts the connection between that photon orbits and thermodynamic phase transitions.
Figure 7: Variation of photon sphere radius and minimum impact parameter.

Figure 8: Change in photon sphere radius and minimum impact parameter.

Figure 9: Fitting curves for radius and impact parameter against temperature of the black hole.
IV. RUPPEINER GEOMETRY AND INTERACTING MICROSTRUCTURES

For a spherically symmetric black hole a novel Ruppeiner geometry method was proposed by Wei et al [18], which gives the microstructure of the black hole. In this construction the parameter space is made of the fluctuation coordinates temperature $T$ and volume $V$. The corresponding line element is given by,

$$dl^2 = \frac{C_V}{T^2}dT^2 - \frac{(\partial_V P)_T}{T}dV^2.$$  \hspace{1cm} (35)

However the Ruppeiner curvature scalar constructed from this line element is divergent, which arises due to the vanishing heat capacity $C_V$. To overcome this, a normalised curvature scalar is considered as follows,

$$R_N = C_V R.$$  \hspace{1cm} (36)

We have obtained the normalised Ruppeiner scalar $R_N$ for ABG black hole, which is a lengthy expression. The behaviour of $R_N$ with respect to $\tilde{V}$ for different temperatures $\tilde{T}$ is shown in Fig. 10. From this figure it is clear that $R_N$ has an extremal point at $\tilde{V} = 1$. Below the critical temperature $R_N$ shows two divergences, which merge at the critical temperature $\tilde{T} = 1$. Above which there are no divergences. In all the cases there exists repulsive interaction at small values of $\tilde{V}$ (shown in insets). However, we need to consider the thermodynamic stability of the black hole.

We obtain the sign changing curve of $R_N$, by setting $R_N = 0$. The expression we obtained satisfies,

$$T_0 = \frac{T_{rsp}}{2},$$  \hspace{1cm} (37)

which is a universal relation. This is the temperature at which $R_N$ changes sign, which is half the spinodal curve temperature.
In fact, along the spinodal curve, the normalised Ruppeiner scalar diverges. In our case, we note that the region between the spinodal curve and the sign changing curve, $R_N$ has a positive sign indicating repulsive interaction among the black hole microstructures. In contrast to van der Waals fluid and charged AdS black holes, the region below the sign changing curve is attractive. However these regions are part of the unstable states of the system which lie under the spinodal curve and is of no significance. The physically relevant regions of parameter space are regions above the spinodal curve, in which two regions are of interest namely region $I$ and $II$ as shown in Fig. 11(a). As in case of RN-AdS black hole, the region $II$ to the left of coexistence curve shows a repulsive interaction in the small black hole phase. The shaded region $I$ between the sign changing and coexistence curves is metastable SBH phase with repulsive interaction. Elsewhere, the microstructure is similar to that of van der Waals fluid.

Next we consider the behaviour of $R_N$ along the coexistence curve. The analytical expres-
Figure 11: 11(a): The sign changing curve of $R_N$, coexistence curve and spinodal curve. 11(b): The normalised curvature scalar $R_N$ along the coexistence saturated SBH and LBH phases. The blue (solid) line and red (dashed) line corresponds to the small black hole and large black hole, respectively. The region where $R_N$ is positive is depicted in the inset.

The solution for the coexistence curve is not feasible due to the complexity of spacetime, and hence numerical solution is obtained. The result is shown in fig. 11(b). As expected, the curvature scalar diverges near the critical point for both the SBH and LBH phases. Towards the lower end of temperature there exists a repulsive interaction for SBH phase, as shown by the curve crossing $R_N = 0$ line. However, at higher temperature SBH phase has dominant attractive interaction. This shows that the SBH phase behaves as an anyon gas. For the LBH phase the nature of interaction is always attractive, similar to boson gas. From these observations we can say that, at lower temperatures, during the phase transition both the microstructure and type of interaction undergoes a change. Whereas at higher temperatures, the nature of interaction remains unchanged even though the microstructure changes. Compared to this, in case of van der Waals fluid, the system always posses a dominant attractive interaction, which remains unchanged during phase transition.

Finally, we examine the near critical point behaviour of the curvature scalar, which is carried out numerically. The numerical fit is obtained by assuming the following functional dependence,

$$R_N \sim (1 - Tr)^p.$$  \hspace{1cm} (38)

Which can be written as,

$$\ln |R_N| = -p \ln(1 - Tr) + q.$$ \hspace{1cm} (39)
Performing the numerical fit for the SBH and LBH branches separately we get,

\[
SBH: \quad \ln R_N = -1.79837 \ln(1 - T_r) - 0.823466, \quad (40)
\]

\[
LBH: \quad \ln R_N = -2.19002 \ln(1 - T_r) - 3.25666, \quad (41)
\]

These are shown in Fig. 12 along with the numerical fitting data. From the numerical study we can conclude that \( p \approx 2 \), which we set as \( p = 2 \), taking into account the numerical errors. Combining Eq. (40) and Eq. (41) we have,

\[
R_N(1 - T_r)^2 = -e^{-(0.823466+3.25666)/2} = -0.130021 \approx -1/8. \quad (42)
\]

This is in agreement with the universal result for vdW fluid and other AdS black holes [18, 19, 25], that \( R_N \) has a universal exponent 2 and the relation \( R_N(1 - T_r)^2 = -1/8 \), near the critical point.

V. DISCUSSIONS

In this article, we have studied the thermodynamics, correlation between photon orbit and phase transition, following this the microstructure of the ABG black hole using Ruppeiner geometry. The black hole shows van der Waals like critical behaviour, which is evident from the swallow tail behaviour of the free energy. The black hole exhibits a first-order phase transition between the small black hole and large black hole phases, which terminates at the critical point where the phase transition is second order. The coexistence curve for the system is obtained numerically from the behaviour of Gibbs free energy. Studying the
photon sphere around the black hole establishes a correlation between the thermodynamics and gravity of the black hole. The phase transition behaviour is mirrored in the photon orbit parameters, such as radius and minimum impact parameter. These parameters undergo a sudden change during the phase transition, and the differences $\Delta r_{ps}$ and $\Delta u_{ps}$ serve as order parameters with a critical exponent of 1/2. Due to the complexity of the non-linearly coupled electromagnetic field, the analysis is carried out numerically.

At the end, we study the underlying microstructure using novel Ruppeiner geometric method, which shows a deviation from typical van der like system. The curvature scalar behaviour is analysed along the coexistence curve which shows a dominant repulsive interaction at certain interval of parameter space of temperature and volume. Our study shows that the large black hole phase behaves as a bosonic system, with only dominant attractive interaction, and the small black hole phase resembles an anyonic system, with both dominant repulsive and attractive interactions. The behaviour of curvature scalar near the critical point is analysed numerically, and it is found to satisfy the universal relation $R_N(1 - T_r)^2 = -1/8$.

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