Maximum Principle of Forward-Backward Stochastic Differential System of Mean-Field Type with Observation Noise *

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Abstract

This paper is concerned with the partial information optimal control problem of mean-field type under partial observation, where the system is given by a controlled mean-field forward-backward stochastic differential equation with correlated noises between the system and the observation, moreover the observation coefficients may depend not only on the control process and but also on its probability distribution. Under standard assumptions on the coefficients, necessary and sufficient conditions for optimality of the control problem in the form of Pontryagin’s maximum principles are established in a unified way.

Keywords Mean-Field, FBSDE, Partial Observation, Girsanovs Theorem, Maximum Principle

Introduction

In recent years, the systems with interacting behavior have attracted increasing attention in the stochastic control theory. The so-called mean-field models are designed to study such systems. The history of the mean-field models can be dated back to the early works of [10, 11]. Since then, the mean-field models have been found useful to describe the aggregate behavior of a large number of mutually interacting particles in diverse areas of physical sciences, such as statistical mechanics, quantum mechanics and quantum chemistry. Recent interest is to study the stochastic maximum principle under the mean-field models. Previous works include [12, 13, 14, 15, 16, 17, 18], and references therein.

It is well known that forward-backward stochastic differential equations (FBSDEs in short) consists of a forward stochastic differential equation (SDE in short) of Itô type and a backward stochastic differential equation (BSDE in short) of Pardoux-Peng (for details see [6, 3]). FBSDEs are not only encountered in stochastic optimal control problems when applying the stochastic maximum principle but also used in mathematical finance (see Antonelli [1], Duffie and Epstein [2], El Karoui, Peng and Quenez [3] for example). It now becomes more clear that certain important problems in mathematical economics and mathematical finance, especially in the optimization problem, can be formulated to be FBSDEs. There are two important approaches to the general stochastic optimal control problem. One is the Bellman dynamic programming principle, which results in the Hamilton-Jacobi-Bellman equation. The other is the maximum principle. Now the maximum principle of forward-backward stochastic systems driven by Brownian motion have been studied extensively in the literature. We refer to [7, 8, 9] and references therein.

The main contribution of this paper is that one sufficient (a verification theorem) and one necessary optimality conditions for the existence of optimal controls of FBSDE of mean field type are established in a unified way. The main idea is to get directly variation formula in terms of the Hamiltonian and the associated adjoint system which is governed by a linear mean-field forward-backward stochastic differential equation of mean-field and neither the corresponding Taylor type expansions of the state process and the cost functional nor the variational systems will be used.

The paper is organized as follows. In section 2, the partial information optimal problem of mean-field systems for FBSDE is formulated and various assumptions used throughout the paper is presented. In Section 3 we establish sufficient and necessary stochastic maximum principles in a unified way.

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2 Formulation of Problem

Let $\mathcal{T} := [0, T]$ denote a fixed time interval of finite length, i.e., $T < \infty$. We equip $(\Omega, \mathcal{F}, \mathbb{P})$ a complete probability space with a right-continuous, $\mathbb{P}$- complete filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \in \mathcal{T}}$, to be specified below. By $\mathcal{P}$ we denote the predictable $\sigma$ field on $\Omega \times [0, T]$ and by $\mathcal{B}(\Lambda)$ the Borel $\sigma$-algebra of any topological space $\Lambda$. Furthermore, we assume that $\mathcal{F}_t = \mathcal{F}$. Denote by $E[\cdot]$ be the expectation taken with respect to $\mathbb{P}$. Let $\{W(t), t \in \mathcal{T}\}$ and $\{Y(t), t \in \mathcal{T}\}$ be two one-dimensional standard Brownian motions on $(\Omega, \mathcal{F}, \mathbb{P})$. Denote by $\{\mathcal{F}^W_t\}_{t \in \mathcal{T}}$ and $\{\mathcal{F}^Y_t\}_{t \in \mathcal{T}}$ be the natural filtration generated by $\{W(t), t \in \mathcal{T}\}$ and $\{Y(t), t \in \mathcal{T}\}$, respectively. Assume that $\mathbb{F}$ is the $\mathbb{P}$- augmentation of the natural filtration generated by $\{\mathcal{F}^W_t\}_{t \in \mathcal{T}}$ and $\{\mathcal{F}^Y_t\}_{t \in \mathcal{T}}$.

Let $E$ be a Euclidean space. Denote the inner product in $E$ by $(\cdot, \cdot)$, the norm in $E$ by $|\cdot|$, the transpose of the matrix or vector $A$ by $A^T$. For a function $\psi : \mathbb{R}^n \to \mathbb{R}$, denote by $\psi_x$ its gradient. If $\psi : \mathbb{R}^n \to \mathbb{R}^k$ (with $k \geq 2$), then $\psi_x = \left(\frac{\partial \psi}{\partial x}\right)$ is the corresponding $(k \times n)$- Jacobian matrix. By $\mathcal{P}$ we denote the predictable $\sigma$ field on $\Omega \times [0, T]$ and by $\mathcal{B}(\Lambda)$ the Borel $\sigma$-algebra of any topological space $\Lambda$. In the follows, $K$ represents a generic constant, which can be different from line to line.

Next we introduce some spaces of random variable and stochastic processes. For any $\alpha, \beta \in [1, \infty)$, let

- $M^{\beta}_{\mathcal{P}}(0,T;E)$: the space of all $E$-valued and $\mathcal{F}_t$-adapted processes $f = \{f(t, \omega), (t, \omega) \in \mathcal{T} \times \Omega\}$ satisfying $\|f\|_{M^{\beta}_{\mathcal{P}}(0,T;E)} \triangleq \left(\mathbb{E}\left[\int_0^T |f(t)|^\beta dt\right]\right)^{\frac{1}{\beta}} < \infty$;
- $S^{\beta}_{\mathcal{P}}(0,T;E)$ : the space of all $E$-valued and $\mathcal{F}_t$-adapted càdlàg processes $f = \{f(t, \omega), (t, \omega) \in \mathcal{T} \times \Omega\}$ satisfying $\|f\|_{S^{\beta}_{\mathcal{P}}(0,T;E)} \triangleq \left(\mathbb{E}\left[\sup_{t \in \mathcal{T}} |f(t)|^\beta\right]\right)^{\frac{1}{\beta}} < +\infty$,
- $L^\beta(\Omega, \mathcal{F}, P; E) :$ the space of all $E$-valued random variables $\xi$ on $(\Omega, \mathcal{F}, P)$ satisfying $\|\xi\|_{L^\beta(\Omega, \mathcal{F}, P; E)} \triangleq \sqrt{\mathbb{E}|\xi|^\beta} < \infty$,
- $M^{\alpha,\beta}_{\mathcal{P}}(0,T;L^\alpha(0,T;E))$ : the space of all $L^\alpha(0,T;E)$-valued and $\mathcal{F}_t$-adapted processes $f = \{f(t, \omega), (t, \omega) \in [0, T] \times \Omega\}$ satisfying $\|f\|_{M^{\alpha,\beta}_{\mathcal{P}}(0,T;L^\alpha(0,T;E))} \triangleq \left\{\mathbb{E}\left[\left(\int_0^T |f(t)|^\alpha dt\right)^{\frac{\beta}{\alpha}}\right]\right\}^{\frac{\alpha}{\beta}} < \infty$.

In the following, under partial observations, we formulate a class of optimal control problems of mean-field type. Consider the following controlled FBSDE of mean-field type:

\begin{equation}
\begin{aligned}
&dx(t) = b(t, x(t), u(t), \mathbb{E}[x(t)], \mathbb{E}[u(t)])dt + \sigma_1(t, x(t), u(t), \mathbb{E}[x(t)], \mathbb{E}[u(t)])dW(t) + \sigma_2(t, x(t), u(t), \mathbb{E}[x(t)], \mathbb{E}[u(t)])dW^u(t), \\
&dy(t) = f(t, x(t), y(t), z_1(t), z_2(t), u(t), \mathbb{E}[x(t)], \mathbb{E}[y(t)], \mathbb{E}[z_1(t)], \mathbb{E}[z_2(t)], \mathbb{E}[u(t)])dt + z_1(t)dW(t) + z_2(t)dW^u(t), \\
&x(0) = x_0, \\
y(T) = \phi(x(T), \mathbb{E}[x(T)]),
\end{aligned}
\end{equation}

where $u(\cdot)$ is our admissible control process taking values in $U$ being a nonempty convex subset of $\mathbb{R}^k$; $(x(\cdot), y(\cdot), z_1(\cdot), z_2(\cdot))$, the solution of $\mathbb{P}$ is the state process with initial state $x_0$ and $W^u$, taking values in $\mathbb{R}$, is a stochastic process depending on the control process $u(\cdot)$. In the above, $b : \mathcal{T} \times \Omega \times (\mathbb{R}^n \times U) \to \mathbb{R}^n$, $\sigma_1 : \mathcal{T} \times \Omega \times (\mathbb{R}^n \times U)^2 \to \mathbb{R}^n$, $\sigma_2 : \mathcal{T} \times \Omega \times (\mathbb{R}^n \times U)^2 \to \mathbb{R}^n$, $f : \mathcal{T} \times \Omega \times (\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m U)^2 \to \mathbb{R}^n$, $\phi : \Omega \times (\mathbb{R}^n)^2 \to \mathbb{R}^m$ are given random mapping.

Suppose that the state process $(x(\cdot), y(\cdot), z_1(\cdot), z_2(\cdot))$ cannot be directly. Instead, we can observe a related process $Y(\cdot)$ is given by the following SDE of mean-field type

\begin{equation}
\begin{aligned}
&dY(t) = h(t, x(t), u(t), \mathbb{E}[x(t)], \mathbb{E}[u(t)])dt + dW^u(t), \\
&Y(0) = 0,
\end{aligned}
\end{equation}

where $h : \mathcal{T} \times \Omega \times (\mathbb{R}^n \times U)^2 \to \mathbb{R}$ is a given random mapping. In the above equations, $u(\cdot)$ is our admissible control process defined as follows.

**Definition 2.1.** An control process $u(\cdot)$ is called admissible control process if it is $\mathcal{F}^{Y}_t$-adapted and valued $U$ such that
\[
\mathbb{E}\left[\int_0^T |u(t)|^2 dt\right] < \infty.
\]
Denote by $\mathcal{A}$ the set of all admissible controls.
Lemma 2.1. For any admissible control following basic estimate holds: 

\[ (x, u, x', u') \rightarrow \psi(t, \omega, x, u, x', u') \]

is continuous differentiable with respect to \((x, u, x', u')\) with continuous and uniformly bounded derivatives, where \(\psi = b, \sigma_1, \sigma_2\) and \(h\). (ii) The mapping \(f\) is \(\mathcal{P} \otimes (\mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(U))^2\)-measurable such that \(f(0, 0, 0, 0, 0) \in M_{\mathbb{P}}^{2}(0, T; L^2(0, T; E))\). For almost all \((t, \omega) \in \mathcal{T} \times \Omega\), the mapping 

\[ (x, y, z_1, z_2, u, x', y', z_1', z_2', u') \rightarrow f(t, \omega, x, y, z_1, z_2, u, x', y', z_1', z_2', u') \]

is continuous differentiable with respect to \((x, y, z_1, z_2, u, x', y', z_1', z_2', u')\) with appropriate growths. More precisely, there is a constant \(C > 0\) such that for all \((x, y, z_1, z_2, u, x', y', z_1', z_2', u') \in (\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times U)^2\) and a.e. \((t, \omega) \in \mathcal{T} \times \Omega\),

\[
\begin{align*}
(1 + |x| + |y|) & \cdot |z_1| + |z_2| + |u| + |x'| + |y'| + |z_1'| + |z_2'| + |u'|^{-1} |f(t, x, y, z_1, z_2, u, x', y', z_1', z_2', u')| \\
& + |f_x(t, x, y, z_1, z_2, u, x', y', z_1', z_2', u')| + |f_y(t, x, y, z_1, z_2, u, x', y', z_1', z_2', u')| + |f_{z_1}(t, x, y, z_1, z_2, u, x', y', z_1', z_2', u')| \\
& + |f_{z_2}(t, x, y, z_1, z_2, u, x', y', z_1', z_2', u')| + |f_{x'}(t, x, y, z_1, z_2, u, x', y', z_1', z_2', u')| \\
& + |f_{y'}(t, x, y, z_1, z_2, u, x', y', z_1', z_2', u')| + |f_{z_1'}(t, x, y, z_1, z_2, u, x', y', z_1', z_2', u')| + |f_{z_2'}(t, x, y, z_1, z_2, u, x', y', z_1', z_2', u')| \\
& + |f_{x'}(t, x, y, z_1, z_2, u, x', y', z_1', z_2', u')| \leq C.
\end{align*}
\]

(iii) The mapping \(\phi\) is \(\mathcal{F}_T \otimes (\mathcal{B}(\mathbb{R}^n))^2\)-measurable. For almost all \((t, \omega) \in [0, T] \times \Omega\), the mapping 

\[ x \rightarrow \phi(\omega, x, x') \]

is continuous differentiable with respect to \((x, x')\) with appropriate growths, respectively. More precisely, there exists a constant \(C > 0\) such that for all \((x, x') \in (\mathbb{R}^n)^2\) and a.e. \(\omega \in \Omega\),

\[ (1 + |x| + |x'|)^{-1} |\phi(x, x')| + |\phi_x(x, x')| + |\phi_{x'}(x, x')| \leq C. \tag{4} \]

Inserting (2) into (1) get that 

\[
\begin{align*}
\frac{dx(t)}{(b - \sigma h)(t, x(t), u(t), E[x(t)], E[u(t)])dt} + \sigma_1(t, x(t), u(t), E[x(t)], E[u(t)])dW(t) + \sigma_2(t, x(t), u(t), E[x(t)], E[u(t)])dY(t), \\
\frac{dy(t)}{(f(t, x(t), y(t), z_1(t), z_2(t), u(t), E[x(t)], E[y(t)], E[z_1(t)], E[z_2(t)], E[u(t)]) - z_2(t)h(t, x(t), u(t), E[x(t)], E[u(t)]))dt} \\
+ z_1(t)dw(t) + z_2(t)dy(t), \\
x(0) &= x, \\
y(T) &= \Phi(x(T), E[x(T)]).
\end{align*}
\]

For any admissible control \(u(\cdot) \in \mathcal{A}\), by Assumption (2), it is easy to have the following important result on (5).

Lemma 2.1. Suppose that Assumption (2) hold. Then associated with admissible control \(u(\cdot) \in \mathcal{A}\), (3) have a unique strong solution \((x(\cdot), y(\cdot), z_1(\cdot), z_2(\cdot)) \in S_{\mathbb{P}}^2(0, T; \mathbb{R}^n) \times S_{\mathbb{P}}^2(0, T; \mathbb{R}^m) \times M_{\mathbb{P}}^{2}(0, T; L^2(0, T; \mathbb{R}^m)) \times M_{\mathbb{P}}^{2}(0, T; L^2(0, T; \mathbb{R}^m))\). Moreover, the following basic estimate holds:

\[
\begin{align*}
\mathbb{E}\left[\sup_{t \in \mathcal{T}} |x(t)|^2\right] + \mathbb{E}\left[\sup_{t \in \mathcal{T}} |y(t)|^2\right] & + \mathbb{E}\left[\left(\int_0^T |z_1(t)|^2 dt\right)^2\right] + \mathbb{E}\left[\left(\int_0^T |z_2(t)|^2 dt\right)^2\right] \\
& \leq K \left\{1 + |x|^2 + \mathbb{E}\left[\left(\int_0^T |u(t)|^2 dt\right)^2\right]\right\}. \tag{6}
\end{align*}
\]

Moreover, if \((\bar{x}(\cdot), \bar{y}(\cdot), \bar{z}_1(\cdot), \bar{z}_2(\cdot))\) is the unique strong solution associated with another admissible control \(\bar{u}(\cdot) \in \mathcal{A}\), we have

\[
\begin{align*}
\mathbb{E}\left[\sup_{t \in \mathcal{T}} |x(t) - \bar{x}(t)|^2\right] + \mathbb{E}\left[\sup_{t \in \mathcal{T}} |y(t) - \bar{y}(t)|^2\right] & + \mathbb{E}\left[\left(\int_0^T |z_1(t) - \bar{z}_1(t)|^2 dt\right)^2\right] + \mathbb{E}\left[\left(\int_0^T |z_2(t) - \bar{z}_2(t)|^2 dt\right)^2\right] \\
& \leq K \mathbb{E}\left[\int_0^T |u(t) - \bar{u}(t)|^2 dt\right]. \tag{7}
\end{align*}
\]
Proof. The proof can be proved similar to Proposition 2.1 in [5] and Lemma 2 in [1].

For any given admissible control \( u(\cdot) \in A \) and the corresponding strong solution \((x^u(\cdot), y^u(\cdot), z_1^u(\cdot), z_2^u(\cdot))\) of the equation (5), define a stochastic process by

\[
\rho^u(t) = \exp \left\{ \int_0^t h(s, x^u(s), u(s), E[x^u(s)], E[u(s)])dY(s) - \frac{1}{2}h^2(s, x^u(s), u(s), E[x^u(s)], E[u(s)])ds \right\},
\]

which is the solution of the SDE

\[
\begin{align*}
d\rho^u(t) &= \rho^u(t)h(s, x^u(s), u(s), E[x^u(s)], E[u(s)])dY(s) \\
\rho^u(0) &= 1.
\end{align*}
\]

The following basic result is on the stochastic process \( \rho^u(\cdot) \).

Lemma 2.2. Suppose that Assumption 2.1 holds. Then for any \( u(\cdot) \in A \) and any \( \alpha \geq 2 \), it follows that

\[
E \left[ \sup_{t \in T} |\rho^u(t)|^\alpha \right] \leq K.
\]

Further, if \( \bar{\rho}(\cdot) \) is the solution of (9) associated with another admissible control \( \bar{u}(\cdot) \in A \), we get that

\[
E \left[ \sup_{t \in T} |\rho^u(t) - \bar{\rho}(t)|^2 \right] \leq K\left\{ E \left[ \int_0^T |u(t) - \bar{u}(t)|^2 dt \right] \right\}^{\frac{1}{2}}.
\]

Proof. The proof can be proved similarly to the proof of Proposition 2.1 in [5].

Under Assumption 2.1, \( \rho^u(\cdot) \) is an \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \leq T}, \mathbb{P})\)-martingale. we thus can introduce a new probability measure \( \mathbb{P}^u \) on \((\Omega, \mathcal{F})\) by

\[
d\mathbb{P}^u = \rho^u(1)d\mathbb{P}.
\]

Then using Girsanov’s theorem and (2), \((W(\cdot), W^u(\cdot))\) is an \(\mathbb{R}^2\)-valued standard Brownian motion on the new probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P}^u)\). So \((\mathbb{P}^u, x^u(\cdot), y^u(\cdot), z_1^u(\cdot), z_2^u(\cdot), \rho^u(\cdot), W^u(\cdot))\) is a weak solution on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \leq T})\) of (1) and (2).

Give the cost functional by

\[
J(u(\cdot)) = E^u \left[ \int_0^T l(t, x(t), y(t), z_1(t), z_2(t), u(t), E[x(t)], E[y(t)], E[z_1(t)], E[z_2(t)], E[u(t)])dt + \Phi(x(T), E[x(T)]) + \gamma(y(0)) \right],
\]

where \(E^u\) stands for the mathematical expectation on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P}^u)\) and the following assumption on \((t, \omega) \in [0, T] \times \Omega\), \(l, \Phi, \gamma : \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}\) will be needed:

Assumption 2.2. \( \Phi \) is \(\mathcal{F}_T \otimes (\mathcal{B}(\mathbb{R}^n))^2 \)-measurable, and \( \gamma \) is \(\mathcal{F}_0 \otimes (\mathcal{B}(\mathbb{R}^n))^2 \)-measurable, \(l \) is \(\mathcal{F} \otimes (\mathcal{B}(\mathbb{R}^n)^2 \otimes \mathcal{B}(\mathbb{R}^m)^2 \otimes \mathcal{B}(\mathbb{R}^m)^2) \)-measurable. For almost all \((t, \omega) \in [0, T] \times \Omega\), the mappings \(l, \Phi, \gamma\) are continuous differentiable with respect to \((x, y, z_1, z_2, u, x', y', z_1', z_2', u')\) with appropriate growths, respectively. More precisely, for all \((x, y, z_1, z_2, u, x', y', z_1', z_2', u') \in (\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m)\) and e.e. \((t, \omega) \in [0, T] \times \Omega\), it follows that

\[
\begin{align*}
\left(1 + |x| + |y| + |z_1| + |z_2| + |u| + |x'| + |y'| + |z_1'| + |z_2'| + |u'|\right)^{-1} &|l_x(t, x, y, z_1, z_2, u, x', y', z_1', z_2', u')| \\
&
+ |l_y(t, x, y, z_1, z_2, u, x', y', z_1', z_2', u')| \\
&
+ |l_z(t, x, y, z_1, z_2, u, x', y', z_1', z_2', u')| \\
&
+ |l_u(t, x, y, z_1, z_2, u, x', y', z_1', z_2', u')| \\
&
+ |l_{x'}(t, x, y, z_1, z_2, u, x', y', z_1', z_2', u')| \\
&
+ |l_{y'}(t, x, y, z_1, z_2, u, x', y', z_1', z_2', u')| \\
&
+ |l_{z'}(t, x, y, z_1, z_2, u, x', y', z_1', z_2', u')| \\
&
+ |l_{u'}(t, x, y, z_1, z_2, u, x', y', z_1', z_2', u')| \\
&
+ (1 + |x|^2 + |y|^2 + |z_1|^2 + |z_2|^2 + |u|^2 + |x'|^2 + |y'|^2 + |z_1'|^2 + |z_2'|^2 + |u'|^2)^{-1} |l(t, x, y, z_1, z_2, u, x', y', z_1', z_2', u')| \leq C; \\
(1 + |x|^2)^{-1} \Phi(x, x') + (1 + |x'|)^{-1} \Phi_x(x, x') \leq C; \\
(1 + |y|^2)^{-1} \gamma(y) + (1 + |y'|)^{-1} \gamma_y(y) \leq C.
\end{align*}
\]

Under Assumption 2.1 and 2.2, by the estimates (9) and (10), it is easy to check that the cost functional is well-defined.

Now we pose an optimal control problem of forward-backward stochastic differential systems with partial information in its weak formulation, i.e., with changing the reference probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P}^u)\), as follows.
Problem 2.1. Seek \( \bar{u}(\cdot) \in \mathcal{A} \) such that

\[
J(\bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{A}} J(u(\cdot)),
\]

subject to (1), (2) and (13).

Using Bayes’ formula, it is easy to check that we can rewrite the cost functional (13) as

\[
J(u(\cdot)) = \mathbb{E} \left[ \int_{0}^{T} \rho^u(t) l(t, x(t), y(t), z_1(t), z_2(t), u(t), \mathbb{E}[x(t)], \mathbb{E}[y(t)], \mathbb{E}[z_1(t)], \mathbb{E}[z_2(t)], \mathbb{E}[u(t)]) dt + \rho^u(T) \Phi(x(T), \mathbb{E}[x(T)]) + \gamma(y(0)) \right].
\]

Therefore, Problem 2.1 can be translated into the following equivalent optimal control problem in its strong formulation, i.e., without changing the reference probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})\). Here we will regard \( \rho^u(\cdot) \) as an additional state process besides the state process \((x^u(\cdot), y^u(\cdot), z^u_1(\cdot), z^u_2(\cdot))\).

Problem 2.2. Seek \( \bar{u}(\cdot) \in \mathcal{A} \) such that

\[
J(\bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{A}} J(u(\cdot)),
\]

subject to (14) and the following state equation

\[
\begin{aligned}
dx(t) &= (b - \sigma_2 h)(t, x(t), u(t), \mathbb{E}[x(t)], \mathbb{E}[u(t)]) dt + \sigma_1(t, x(t), u(t), \mathbb{E}[x(t)], \mathbb{E}[u(t)]) dW(t) \\
y(t) &= f(t, x(t), y(t), z_1(t), z_2(t), u(t), \mathbb{E}[x(t)], \mathbb{E}[y(t)], \mathbb{E}[z_1(t)], \mathbb{E}[z_2(t)], \mathbb{E}[u(t)]) - z_2(t) h(t, x(t), u(t), \mathbb{E}[x(t)], \mathbb{E}[u(t)]) dt \\
d\rho(t) &= \rho(t) h(s, x(s), u(s), \mathbb{E}[x^u(s)], \mathbb{E}[u(s)]) dY(s), \\
\rho^u(0) &= 1, \\
x(0) &= x, \\
y(T) &= \Phi(x(T), \mathbb{E}[x(T)]).
\end{aligned}
\]

Here we call \( \bar{u}(\cdot) \in \mathcal{A} \) satisfying above an optimal control process of Problem 2.2 and the corresponding state process \((\bar{x}(\cdot), \bar{y}(\cdot), \bar{z}_1(\cdot), \bar{z}_2(\cdot), \bar{\rho}(\cdot))\) the optimal state process. Correspondingly \((\bar{u}(\cdot); \bar{x}(\cdot), \bar{y}(\cdot), \bar{z}_1(\cdot), \bar{z}_2(\cdot), \bar{\rho}(\cdot))\) is said to be an optimal pair of Problem 2.2.

3 A Variation Formulation for the Cost Functional

In this section, we will establish a variation formulation for the cost functional by using the Hamiltonian and adjoint process.

We first define the Hamiltonian by \( \mathcal{H} : \Omega \times \mathcal{T} \times (\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times U)^2 \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R} \) as follows:

\[
\begin{aligned}
H(t, x, y, z_1, z_2, u, x', y', z_1', z_2', u', k, p, q_1, q_2, R_2) \\
= l(t, x, y, z_1, z_2, u, x', y', z_1', z_2', u') + \langle b(t, x, u, x', u'), p \rangle + \langle \sigma_1(t, x, u, x', u'), q_1 \rangle + \langle \sigma_2(t, x, u, x', u'), q_2 \rangle \\
+ \langle f(t, x, y, z_1, z_2, u, x', y', z_1', z_2', u'), k \rangle + \langle R_2, h(t, x, u, x', u') \rangle.
\end{aligned}
\]

For any given admissible control pair \((\bar{u}(\cdot); \bar{x}(\cdot), \bar{y}(\cdot), \bar{z}_1(\cdot), \bar{z}_2(\cdot))\), we define the corresponding adjoint process as the
solution as the solution of the following FBSDE:

\[
\begin{align*}
\dd x(t) &= -l(t, \tilde{\Theta}(t), x(t), x(\tilde{u}(t)), E[\tilde{u}(t)])dt + \tilde{R}_1(t) dW(t) + \tilde{R}_2(t) dW^u(t), \\
\dd \tilde{p}(t) &= -\left[ H_x(t, \tilde{\Theta}(t), \tilde{u}(t), E[\tilde{u}(t)], \tilde{\lambda}(t), \tilde{R}_2(t)) + \frac{1}{\rho(t)} E^u[H' x(t, \tilde{\Theta}(t), \tilde{u}(t), E[\tilde{u}(t)], \tilde{\lambda}(t), \tilde{R}_2(t))] \right] dt \\
&\quad + \tilde{q}_1(t) dW(t) + \tilde{q}_2(t) dW^u(t), \\
\dd \tilde{k}(t) &= -\left[ H_y(t, \tilde{\Theta}(t), \tilde{u}(t), E[\tilde{u}(t)], \tilde{\lambda}(t), \tilde{R}_2(t)) + \frac{1}{\rho(t)} E^u[H' y(t, \tilde{\Theta}(t), \tilde{u}(t), E[\tilde{u}(t)], \tilde{\lambda}(t), \tilde{R}_2(t))] \right] dt \\
&\quad - \left[ H_z(t, \tilde{\Theta}(t), \tilde{u}(t), E[\tilde{u}(t)], \tilde{\lambda}(t), \tilde{R}_2(t)) + \frac{1}{\rho(t)} E^u[H' z(t, \tilde{\Theta}(t), \tilde{u}(t), E[\tilde{u}(t)], \tilde{\lambda}(t), \tilde{R}_2(t))] \right] dW(t) \\
&\quad - H_z(\tilde{\Theta}(t), \tilde{u}(t), E[\tilde{u}(t)], \tilde{\lambda}(t), \tilde{R}_2(t)) + \frac{1}{\rho(t)} E^u[H' z(t, \tilde{\Theta}(t), \tilde{u}(t), E[\tilde{u}(t)], \tilde{\lambda}(t), \tilde{R}_2(t))] dW(t),
\end{align*}
\]

where

\[
\tilde{\Theta}(t) := (\tilde{x}(t), \tilde{y}(t), \tilde{z}_1(t), \tilde{z}_2(t)),
\]

\[
E[\tilde{\Theta}(t)] := (E[\tilde{x}(t)], E[\tilde{y}(t)], E[\tilde{z}_1(t)], E[\tilde{z}_2(t)]),
\]

\[
\tilde{\lambda}(t) := (\tilde{k}(t), \tilde{p}(t), \tilde{q}_1(t), \tilde{q}_2(t)),
\]

\[
\tilde{\Gamma}(t) := (\tilde{r}(t), \tilde{R}_1(t), \tilde{R}_2(t)).
\]

(17) \]

Here the following short hand notation have been used:

\[
H_a(t, \tilde{\Theta}(t), \tilde{u}(t), E[\tilde{u}(t)], \tilde{\lambda}(t), \tilde{R}_2(t))
\]

\[
= H_a(t, \tilde{\Theta}(t), \tilde{u}(t), E[\tilde{u}(t)], \tilde{\lambda}(t), \tilde{R}_2(t)) - \sigma^2(t, \tilde{x}(t), \tilde{u}(t), E[\tilde{x}(t)], E[\tilde{u}(t)]) \tilde{p}(t) - \tilde{z}_2^2(t) \tilde{k}(t) \tag{19}
\]

where \(a = x, y, z_1, z_2, u, x', y', z_1', z_2', u'\). Here the FBSDE (17) is said to be the adjoint equation whose solution consists of a 7-tuple process \((\tilde{p}(\cdot), \tilde{q}_1(\cdot), \tilde{q}_2(\cdot), \tilde{k}(\cdot), \tilde{r}(\cdot), \tilde{R}_1(\cdot), \tilde{R}_2(\cdot))\). In view of Assumptions \(2.1\) and \(2.2\) from Lemma 2 in \(3\) and Proposition 2.1 in \(3\), the adjoint equation \(17\) has a unique strong solution \((\tilde{p}(\cdot), \tilde{q}_1(\cdot), \tilde{q}_2(\cdot), \tilde{k}(\cdot), \tilde{r}(\cdot), \tilde{R}_1(\cdot), \tilde{R}_2(\cdot)) \in S^2_{\mathbb{F}}(0, T; \mathbb{R}^n) \times M^2_{\mathbb{F}}(0, T; L^2(0, T; \mathbb{R}^n)) \times M^2_{\mathbb{F}}(0, T; L^2(0, T; \mathbb{R}^n)) \times S^2_{\mathbb{F}}(0, T; \mathbb{R}^m) \times M^2_{\mathbb{F}}(0, T; L^2(0, T; \mathbb{R}^n)) \times M^2_{\mathbb{F}}(0, T; L^2(0, T; \mathbb{R}^n))\)

also said to be the adjoint process corresponding to the admissible pair \((\tilde{u}(\cdot); \tilde{x}(\cdot), \tilde{y}(\cdot), \tilde{z}_1(\cdot), \tilde{z}_2(\cdot), \tilde{\rho}(\cdot)); \tilde{\Theta}(\cdot)\).

For any two admissible pairs \((u(\cdot); \Theta^u(\cdot), \rho^u(\cdot)) = (u(\cdot); x^u(\cdot), y^u(\cdot), z_1^u(\cdot), z_2^u(\cdot), \rho^u(\cdot))\) and \((u(\cdot); \Theta(\cdot), \rho(\cdot)) = (u(\cdot); \tilde{x}(\cdot), \tilde{y}(\cdot), \tilde{z}_1(\cdot), \tilde{z}_2(\cdot), \tilde{\rho}(\cdot))\), we give a presentation for the difference \(J(u(\cdot)) - J(\tilde{u}(\cdot))\) in terms of the adjoint process \((\tilde{\lambda}(\cdot), \tilde{\Gamma}(\cdot)) = (\tilde{k}(\cdot), \tilde{p}(\cdot), \tilde{q}_1(\cdot), \tilde{q}_2(\cdot), \tilde{r}(\cdot), \tilde{R}_1(\cdot), \tilde{R}_2(\cdot))\) and the Hamiltonian \(H\) and as well as other relevant expressions.

In the following, to simplify the notation, we denote by:

\[
\begin{align*}
\gamma^u(0) &= \gamma(y^u(0)), \\
\tilde{\gamma}(0) &= \gamma(\tilde{y}(0)), \\
\phi^u(T) &= \phi(x^u(T), E[x^u(T)]), \\
\tilde{\phi}(T) &= \phi(\tilde{x}(T), E[\tilde{x}(T)]), \\
\Phi^u(T) &= \Phi(x^u(T), E[x^u(T)]), \\
\tilde{\Phi}(T) &= \Phi(\tilde{x}(T), E[\tilde{x}(T)]), \\
\alpha^u(t) &= \alpha(t, x^u(t), u(t), E[x^u(t)], E[u(t)]), \\
\tilde{\alpha}(t) &= \alpha(t, \tilde{x}(t), \tilde{u}(t), E[\tilde{x}(t)], E[\tilde{u}(t)]), \\
\beta^u(t) &= \beta(t, \Theta^u(t), u(t), E[\Theta^u(t)], E[u(t)]), \\
\tilde{\beta}(t) &= \beta(t, \tilde{\Theta}(t), \tilde{u}(t), E[\tilde{\Theta}(t)], E[\tilde{u}(t)]), \\
\tilde{\mathcal{H}}(t) &= \mathcal{H}(t, \tilde{\Theta}(t), \tilde{u}(t), E[\tilde{\Theta}(t)], E[\tilde{u}(t)]), \quad \tilde{\lambda}(t), \tilde{R}_2(t)).
\end{align*}
\]

(20)
Lemma 3.1. Suppose that Assumptions $\mathcal{A}$ and $\mathcal{B}$ holds. Using the abbreviation $\mathcal{M}$ and $\mathcal{N}$, it follows that

$$J(u(\cdot)) - J(\bar{u}(\cdot)) = \mathbb{E}^\mathcal{N}\left[ \int_0^T \mathcal{H}(t, \Theta^u(t), u(t), \mathbb{E}[\Theta^u(t)], \mathbb{E}[u(t)], \bar{\lambda}(t), \bar{R}_2(t)) - \bar{\mathcal{H}}(t) - \langle \mathcal{H}_x(t), x^u(t) - \bar{x}(t) \rangle \, dt \right]$$

(21)

$$+ \mathbb{E}^\mathcal{N}\left[ \mathcal{H}_y(t) + \frac{1}{\rho(t)} \mathbb{E}^\mathcal{N}\left[ \mathcal{H}_y(t), y^u(t) - \bar{y}(t) \right] \right]$$

$$+ \mathbb{E}^\mathcal{N}\left[ \nu_x(t) \left( \mathcal{H}_{z_1}(t), z_1^u(t) - \bar{z}_1(t) \right) \right]$$

$$+ \mathbb{E}^\mathcal{N}\left[ \nu_2(t) \left( \mathcal{H}_{z_2}(t), z_2^u(t) - \bar{z}_2(t) \right) \right]$$

(22)

Proof. In view of the definition of the cost function $J(u(\cdot))$, we get that

$$J(u(\cdot)) - J(\bar{u}(\cdot)) = \mathbb{E}^\mathcal{M}\left[ \int_0^T l^u(t)dt + \mathcal{F}^u(T) + \gamma^u(0) \right] - \mathbb{E}^\mathcal{N}\left[ \int_0^T \bar{l}(t)dt + \bar{\mathcal{F}}(T) + \bar{\gamma}(0) \right]$$

$$= \mathbb{E}\left[ \int_0^T (\rho^u(t)\bar{l}^u(t) - \bar{\rho}(t)\bar{l}(t))dt \right] + \mathbb{E}[\rho^u(T)\mathcal{F}^u(T) - \bar{\rho}(T)\bar{\mathcal{F}}(T)] + \mathbb{E}[\gamma^u(0) - \bar{\gamma}(0)]$$

(23)

$$= \mathbb{E}^\mathcal{M}\left[ \int_0^T [\bar{l}(t)]dt \right] + \mathbb{E}^\mathcal{N}[\mathcal{F}^u(T) - \bar{\mathcal{F}}(T)] + \mathbb{E}\left[ \int_0^T (\rho^u(t) - \bar{\rho}(t)){\bar{l}}(t)dt \right]$$

$$+ \mathbb{E}[\rho^u(T) - \bar{\rho}(T)]\mathcal{F}^u(T) + \mathbb{E}[\gamma^u(0) - \bar{\gamma}(0)].$$

On the other hand, by (17), it follows that $(\bar{\rho}(\cdot), \bar{q}_1(\cdot), \bar{q}_2(\cdot), \bar{k}(\cdot))$ satisfies the following FBSDE

$$d\bar{\rho}(t) = - \mathbb{E}[\mathcal{H}_x(t), x^u(t) - \bar{x}(t)]dt + \bar{q}_1(t) dW(t) + \bar{q}_2(t) dW^u(t),$$

$$d\bar{k}(t) = - \mathbb{E}[\mathcal{H}_{z_1}(t), z_1^u(t) - \bar{z}_1(t)]dt - \mathbb{E}[\mathcal{H}_{z_2}(t), z_2^u(t) - \bar{z}_2(t)]dt + W^u(t),$$

(24)

$$\bar{\rho}(T) = \bar{\mathcal{F}}(T) - \mathbb{E}[\bar{\phi}_x(T)\bar{k}(T)] + \mathbb{E}[\bar{\phi}_z^u(T)\bar{k}(T)],$$

and $(\bar{r}(\cdot), \bar{R}_1(\cdot), \bar{R}_2(\cdot))$ solves the following BSDE
\[
\begin{align*}
\begin{cases}
d\tilde{r}(t) = -[\tilde{r}(t) + \tilde{R}_2(t)\tilde{h}(t)]dt + \tilde{R}_1(t)\,dW(t) + \tilde{R}_2(t)\,dY(t), \\
\tilde{r}(T) = \Phi(T).
\end{cases}
\end{align*}
\] (25)

Moreover, by (2), it is easy to check that the \((x^n(\cdot), y^n(\cdot), z^n_1(\cdot), z^n_2(\cdot))\) satisfies the following FBSDE:

\[
\begin{align*}
\begin{cases}
dx(t) = [b^n(t) + \sigma^n_2(t)(\bar{h}(t) - h^n(t))]\,dt + \sigma^n_1(t)\,dW(t) + \sigma^n_2(t)\,dW_\bar{u}(t) \\
dy(t) = [f^n(t) + z_2(t)(\bar{h}(t) - h^n(t))]\,dt + z_1(t)\,dW(t) + z_2(t)\,dW_\bar{u}(t) \\
x(0) = x, \\
y(T) = \phi^n(T) - \tilde{\phi}(T).
\end{cases}
\end{align*}
\] (26)

Therefore \((x^n(t) - \bar{x}(t), y^n(t) - \bar{y}(t), z^n_1(t) - \bar{z}_1(t), z^n_2(t) - \bar{z}_2(t))\) solves the following FBSDE:

\[
\begin{align*}
\begin{cases}
dx(t) - \bar{x}(t) = [b^n(t) - \bar{b}(t) + \sigma^n_2(t)(\bar{h}(t) - h^n(t))]\,dt + [\sigma^n_1(t) - \bar{\sigma}_1(t)]\,dW(t) + [\sigma^n_2(t) - \bar{\sigma}_2(t)]\,dW_\bar{u}(t) \\
dy(t) - \bar{y}(t) = [f^n(t) - \bar{f}(t) + z_2(t)(\bar{h}(t) - h^n(t))]\,dt + [z_1(t) - \bar{z}_1(t)]\,dW(t) + [z_2(t) - \bar{z}_2(t)]\,dW_\bar{u}(t) \\
x(0) - \bar{x}(0) = 0, \\
y(T) - \bar{y}(T) = \phi^n(T) - \tilde{\phi}(T).
\end{cases}
\end{align*}
\] (27)

By the definition of \(\mathcal{H}\), it follows that

\[
E^\tilde{P} \left[ \int_0^T (l^n(t) - \tilde{l}(t))\,dt \right] = E^\tilde{P} \left[ \int_0^T \left( \mathcal{H}(t, \Theta^n(t), u(t), E[\Theta^n(t)], E[u(t)], \tilde{h}(t), \tilde{R}_2(t)) - \tilde{H}(t) \right) \,dt \right]
\]

\[
- E^\tilde{P} \left[ \int_0^T \left( (\tilde{p}(t), b^n(t) - \bar{b}(t)) + \langle \tilde{q}_1(t), \sigma^n_1(t) - \bar{\sigma}_1(t) \rangle + \langle \tilde{q}_2(t), \sigma^n_2(t) - \bar{\sigma}_2(t) \rangle \\
+ \langle \tilde{k}(t), f^n(t) - \bar{f}(t) \rangle + \langle \tilde{R}_2(t) - \bar{\sigma}_2(t), \tilde{p}(t) - \bar{z}_2(t) \rangle \hat{k}(t), h^n(t) - \tilde{h}(t) \rangle \,dt \right) \right]
\] (28)

By using Itô formula to \((\tilde{p}(t), x^n(t) - \bar{x}(t), \tilde{k}(t), y^n(t) - \bar{y}(t))\) and taking expectation with respect to \(P^\tilde{u}\), we obtain that

\[
E^\tilde{P} \left[ \Phi_x(T) + \frac{1}{\hat{\rho}(T)} E^\tilde{P} \left[ \Phi_x(T) - [\tilde{\sigma}^u_x(T)\tilde{k}(T) + \frac{1}{\hat{\rho}(T)} E^\tilde{P} [\tilde{\sigma}^u_x(T)\tilde{k}(T)]], x^n(T) - \bar{x}(T) \right] \right] = E^\tilde{P} \left[ \langle \tilde{k}(t), \phi^n(T) - \tilde{\phi}(T) \rangle \right]
\]

\[
- E^\tilde{P} \left[ \int_0^T \langle \tilde{h}_x(t) + \frac{1}{\hat{\rho}(T)} E^\tilde{P} [\tilde{h}_x(t)], x^n(T) - \bar{x}(t) \rangle \,dt \right]
\]

\[
- E^\tilde{P} \left[ \int_0^T \langle \tilde{h}_y(t) + \frac{1}{\hat{\rho}(T)} E^\tilde{P} [\tilde{h}_y(t)], y^n(T) - \bar{y}(t) \rangle \,dt \right]
\]

\[
- E^\tilde{P} \left[ \int_0^T \langle \tilde{h}_{z_1}(t) + \frac{1}{\hat{\rho}(T)} E^\tilde{P} [\tilde{h}_{z_1}(t)], z^n_1(t) - \bar{z}_1(t) \rangle \,dt \right]
\]

\[
- E^\tilde{P} \left[ \int_0^T \langle \tilde{h}_{z_2}(t) + \frac{1}{\hat{\rho}(T)} E^\tilde{P} [\tilde{h}_{z_2}(t)], z^n_2(t) - \bar{z}_2(t) \rangle \,dt \right]
\]

\[
- E \left[ \langle y^n(0) - \bar{y}(0), \gamma_y(0) \rangle \right]
\] (29)

which implies that

\[
E^\tilde{P} \left[ \int_0^T \left( (\tilde{p}(t), b^n(t) - \bar{b}(t)) + \langle \tilde{q}_1(t), \sigma^n_1(t) - \bar{\sigma}_1(t) \rangle + \langle \tilde{q}_2(t), \sigma^n_2(t) - \bar{\sigma}_2(t) \rangle + \langle \tilde{k}(t), f^n(t) - \bar{f}(t) \rangle \right) \,dt \right]
\]
Applying Itô formula to \((\rho^n(t) - \bar{\rho}(t))\tilde{r}(t)\), yields that

\[
\mathbb{E}[(\rho^n(T) - \bar{\rho}(T))\tilde{\Phi}(T)] = -\mathbb{E} \left[ \int_0^T (\rho^n(t) - \bar{\rho}(t))\tilde{l}(t) + \bar{R}_2(t)\bar{h}(t) dt \right] + \mathbb{E} \left[ \int_0^T \bar{R}_2(t)(\rho^n(t)h^n(t) - \bar{\rho}(t)\bar{h}(t)) dt \right],
\]

which implies that

\[
\mathbb{E}[(\rho^n(T) - \bar{\rho}(T))\tilde{\Phi}(T)] + \mathbb{E} \left[ \int_0^T (\rho^n(t) - \bar{\rho}(t))\tilde{l}(t) dt \right] = \mathbb{E} \left[ \int_0^T \bar{R}_2(t)\rho^n(t)(h^n(t) - \bar{h}(t)) dt \right].
\]

Putting (30) into (28), it follows that

\[
\mathbb{E}^u \left[ \int_0^T (l^n(t) - \bar{l}(t)) dt \right] = \mathbb{E}^u \left[ \int_0^T \left( \mathcal{H}(t, \Theta^n(t), \mathbb{E}[\Theta^n(t)], u(t), \mathbb{E}[u(t)], \tilde{\Lambda}(t), \bar{R}_2(t)) - \tilde{\mathcal{H}}(t) 
\right.
\]

\[
- \left( \tilde{\mathcal{H}}_x(t) + \frac{1}{\bar{\rho}(t)}\mathbb{E}^u[\mathcal{H}_x(t)], x^n(t) - \bar{x}(t) \right)
- \left( \tilde{\mathcal{H}}_y(t) + \frac{1}{\bar{\rho}(t)}\mathbb{E}^u[\mathcal{H}_y(t)], y^n(t) - \bar{y}(t) \right)
- \left( \tilde{\mathcal{H}}_{z_1}(t) + \frac{1}{\bar{\rho}(t)}\mathbb{E}^u[\mathcal{H}_{z_1}(t)], z^n_1(t) - \bar{z}_1(t) \right)
- \left( \tilde{\mathcal{H}}_{z_2}(t) + \frac{1}{\bar{\rho}(t)}\mathbb{E}^u[\mathcal{H}_{z_2}(t)], z^n_2(t) - \bar{z}_2(t) \right)
- \left( (\sigma^n_2(t) - \bar{\sigma}_2(t))(h^n(t) - \bar{h}(t)), \bar{\rho}(t) \right)
- \left( (z^n_2(t) - \bar{z}_2(t))(h^n(t) - \bar{h}(t)), \bar{k}(t) \right) \right) dt
\]

\[
\left. \mathbb{E}^u \left[ (\tilde{\Phi}_x(T) + \frac{1}{\bar{\rho}(T)}\mathbb{E}^u[\tilde{\Phi}_x(T)], x^n(T) - \bar{x}(T) \right) \right]
\]

\[
\mathbb{E}^u \left[ \left( \tilde{\Phi}_x(T) + \frac{1}{\bar{\rho}(T)}\mathbb{E}^u[\tilde{\Phi}_x(T)], x^n(T) - \bar{x}(T) \right) \right]
\]

\[
\mathbb{E}^u \left[ \left( \tilde{\Phi}_x(T) + \frac{1}{\bar{\rho}(T)}\mathbb{E}^u[\tilde{\Phi}_x(T)], x^n(T) - \bar{x}(T) \right) \right]
\]

Then by putting (33) and (34) into (23), we obtain (21). The proof is complete. □

Since the control domain \(U\) is convex, for any given admissible controls \(u(\cdot) \in A\), the following perturbed control process \(u^\varepsilon(\cdot)\):

\[
u^\varepsilon(\cdot) := \bar{u}(\cdot) + \varepsilon(u(\cdot) - \bar{u}(\cdot)), \quad 0 \leq \varepsilon \leq 1,
\]
Lemma 3.2. Suppose Assumptions 2.1 and 2.2 hold. Then

\[ E \left[ \sup_{t \in T} |u^* - \bar{u}(t)|^2 \right] + E \left[ \sup_{t \in T} |y^* - \bar{y}(t)|^2 \right] = O(\epsilon^4). \]

and

\[ E \left[ \sup_{t \in T} |\rho^* - \bar{\rho}(t)|^2 \right] = O(\epsilon^2). \]

Proof. The proof can be obtained directly by Lemmas 2.1 and 2.2. \( \square \)

Now we are in the position to apply Lemma 3.1 and Lemma 3.2 to derive the variational formula for the cost functional \( J(u(\cdot)) \) in terms of the Hamiltonian \( \mathcal{H} \).

Theorem 3.1. Suppose that Assumptions 2.1 and 2.2 holds. Then for any admissible control \( u(\cdot) \in \mathcal{A} \), a variation formula for the cost functional \( J(u(\cdot)) \) at \( \bar{u}(\cdot) \) is given by

\[
\frac{d}{d\epsilon} J(\bar{u}(\cdot) + \epsilon(u(\cdot) - \bar{u}(\cdot)))|_{\epsilon=0} = \lim_{\epsilon \to 0^+} \frac{J(\bar{u}(\cdot) + \epsilon(u(\cdot) - \bar{u}(\cdot))) - J(\bar{u}(\cdot))}{\epsilon} = E \left[ \int_0^T \langle \bar{\rho}(t) \mathcal{H}_u(t) + E^\mathbb{R}[\mathcal{H}_u(t)], u(t) - \bar{u}(t) \rangle \, dt \right].
\]

(35)

Proof. To simplify our notations, denote by

\[
\beta^\epsilon := E^u \left[ \int_0^T \left\{ \mathcal{H}(t, \Theta^u(t), E[\Theta^u(t)], u(t), E[u(t)], \Lambda(t), \bar{\Lambda}(t), \bar{R}(t)) - \bar{\mathcal{H}}(t) \\
- \langle \bar{\mathcal{H}}_x(t) + \frac{1}{\bar{\rho}(t)} E^\mathbb{R}[\mathcal{H}_x(t)], x^u(t) - \bar{x}(t) \rangle \rangle_{t \in T} \right\}
\]

(36)
In view of Lemma 3.1 we get
\[ J(u^*(\cdot)) - J(\tilde{u}(\cdot)) = \beta^\epsilon + \epsilon E \left[ \int_0^T \langle \tilde{p}(t) \tilde{H}_u(t) + E^u[\tilde{H}_u(t)], u(t) - \tilde{u}(t) \rangle dt \right]. \]  
(38)

By Assumptions 2.1 and 2.2 combining the Taylor Expansions, Lemma 3.2, and the dominated convergence theorem, we obtain that
\[ \beta^\epsilon = o(\epsilon). \]  
(39)

Putting (39) into (38) gives
\[ \lim_{\epsilon \to 0^+} \frac{J(u^*(\cdot)) - J(\tilde{u}(\cdot))}{\epsilon} = E \left[ \int_0^T \langle \tilde{p}(t) \tilde{H}_u(t) + E^u[\tilde{H}_u(t)], u(t) - \tilde{u}(t) \rangle dt \right]. \]

This completes the proof. \(\Box\)

4 Main Results

This section is devoted to establishing the necessary condition and sufficient maximum principles for Problem 2.1 or 2.2. We first prove the necessary condition of optimality for the existence of an optimal control.

**Theorem 4.1 (Necessary Stochastic Maximum principle).** Suppose that Assumptions 2.1 and 2.2 holds and \((\tilde{u}(\cdot); \tilde{x}(\cdot), \tilde{y}(\cdot), \tilde{z}_1(\cdot), \tilde{z}_2(\cdot), \tilde{\rho}(\cdot))\) is an optimal pair of Problem 2.2. Then
\[ \langle E[\tilde{p}(t)\tilde{H}_u(t)]|\mathcal{F}_t^Y] + E^u[\tilde{H}_u(t)], u - \tilde{u}(t) \rangle \geq 0, \quad \forall u \in U, \text{ a.e. a.s.} \]
(40)

**Proof.** Because all admissible controls are \{\mathcal{F}_t^Y\}_{t \in \mathcal{T}}-adapted processes, using the property of conditional expectation, Theorem 3.1 and the optimality of \(\tilde{u}(\cdot)\), we obtain that
\[
E \left[ \int_0^T \langle E[\tilde{p}(t)\tilde{H}_u(t)]|\mathcal{F}_t^Y] + E^u[\tilde{H}_u(t)], u(t) - \tilde{u}(t) \rangle dt \right]
\]
\[
= E \left[ \int_0^T \langle E[\tilde{p}(t)\tilde{H}_u(t)]|\mathcal{F}_t^Y], u(t) - \tilde{u}(t) \rangle dt \right]
\]
\[
= E \left[ \int_0^T \langle \tilde{p}(t)\tilde{H}_u(t) + E^u[\tilde{H}_u(t)], u(t) - \tilde{u}(t) \rangle dt \right]
\]
\[
= \lim_{\epsilon \to 0^+} \frac{J(\tilde{u}(\cdot)) - \epsilon(u(\cdot) - \tilde{u}(\cdot)) - J(\tilde{u}(\cdot))}{\epsilon} \geq 0,
\]
which proves (40) is satisfied. The proof is complete. \(\Box\)

Next we prove the sufficient condition of optimality for the existence of an optimal control of Problem 2.2 in the case when the observation process does not contains the control process and the state process. Assume that
\[ h(t,x,u) = h(t) \]
is an \(\mathcal{F}_t^Y\) - adapted bounded process. Introduce a new probability measure \(Q\) on \((\Omega, \mathcal{F})\) by
\[ dQ = \rho(1)dP, \]
(41)
where
\[
\rho \begin{cases} 
\rho(t) = \rho(t)h(s)dY(s) \\
\rho(0) = 1.
\end{cases}
\]
(42)

**Theorem 4.2. [Sufficient Maximum Principle]** Suppose that Assumptions 2.1 and 2.2 hold and \((\tilde{u}(\cdot); \tilde{\Theta}(\cdot)) = (\tilde{u}(\cdot); \tilde{x}(\cdot), \tilde{y}(\cdot), \tilde{z}_1(\cdot), \tilde{z}_2(\cdot))\) is an admissible pair with \(\phi(x) = \phi x\), where \(\phi\) is \(\mathcal{F}_T\)-measurable bounded random variable. Assume that
(i) $\Phi$ and $\gamma$ is convex in $x$ and $y$, respectively,

(ii) the Hamiltonian $\mathcal{H}$ is convex in $(x, y, z_1, z_2, u)$,

(iii) the optimal pair of Problem $2.2$. is completed.

Proof. Let $(u(\cdot); x^u(\cdot), y^u(\cdot), z_1^u(\cdot), z_2^u(\cdot), \rho^u(\cdot))$ be an arbitrary admissible pair. From Lemma 3.1, the difference $J(u(\cdot)) - J(\bar{u}(\cdot))$ can be represented as follows

$$
J(u(\cdot)) - J(\bar{u}(\cdot)) = \mathbb{E}^Q \left[ \int_0^T \left\{ \mathcal{H}(t, \Theta^u(t), E[\Theta^u(t)], u(t), E[u(t)], \bar{\Lambda}(t), \bar{R}_2(t)) - \bar{\mathcal{H}}(t) \\
- \langle \bar{\mathcal{H}}_x(t), x^u(t) - \bar{x}(t) \rangle - \langle \bar{\mathcal{H}}_y(t), y^u(t) - \bar{y}(t) \rangle \\
- \langle \bar{\mathcal{H}}_{\tilde{z}_1}(t), \tilde{z}_1^u(t) - \bar{\tilde{z}}_1(t) \rangle - \langle \bar{\mathcal{H}}_{\tilde{z}_2}(t), \tilde{z}_2^u(t) - \bar{\tilde{z}}_2(t) \rangle \right\} dt \right].
$$

In view of the convexity of $\mathcal{H}$, $\Phi$ and $\gamma$ (i.e. Conditions (i) and (ii)), we obtain

$$
\mathcal{H}(t, \Theta^u(t), E[\Theta^u(t)], u(t), E[u(t)], \bar{\Lambda}(t), \bar{R}_2(t)) - \bar{\mathcal{H}}(t) \\
\geq \langle \bar{\mathcal{H}}_x(t), x^u(t) - \bar{x}(t) \rangle + \langle \bar{\mathcal{H}}_y(t), E[x^u(t)] - E[\bar{x}(t)] \rangle \\
+ \langle \bar{\mathcal{H}}_{\tilde{z}_1}(t), \tilde{z}_1^u(t) - \bar{\tilde{z}}_1(t) \rangle + \langle \bar{\mathcal{H}}_{\tilde{z}_2}(t), E[z_2^u(t)] - E[\bar{z}_2(t)] \rangle \\
+ \langle \bar{\mathcal{H}}_u(t), u(t) - \bar{u}(t) \rangle + \langle \bar{\mathcal{H}}_{\bar{u}}(t), E[u(t)] - E[\bar{u}(t)] \rangle
$$

and

$$
\gamma^u(0) - \bar{\gamma}(0) \geq \langle y^u(0) - \bar{y}(0), \gamma_y(0) \rangle.
$$

Furthermore, by the convex optimization principle (see Proposition 2.21 of [?]) and the optimality condition (iii), we obtain that

$$
\langle u(t) - \bar{u}(t), E[\mathcal{H}_u(t)|\mathcal{F}_t^Y] \rangle + \langle E[u(t)] - E[\bar{u}(t)], E^Q[\mathcal{H}_{\bar{u}}(t)|\mathcal{F}_t^Y] \rangle \geq 0.
$$

which imply that

$$
E^Q \left[ (u(t) - \bar{u}(t), \mathcal{H}_u(t))] + E^Q[ (E[u(t)] - E[\bar{u}(t)], \mathcal{H}_{\bar{u}}(t))] \right] \geq 0.
$$

Inserting (43), (44), (45) into (43), we get

$$
J(u(\cdot)) - J(\bar{u}(\cdot)) \geq 0.
$$

Since $u(\cdot)$ is arbitrary, we get that $\bar{u}(\cdot)$ is an optimal control process and thus $(\bar{u}(\cdot), \bar{x}(\cdot), \bar{y}(\cdot), \bar{z}_1(\cdot), \bar{z}_2(\cdot))$ is an optimal pair. The proof is completed. $$\square$$
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