Linear level repulsions near exceptional points of non-Hermitian systems

C. Wang\textsuperscript{1,\textcopyright} and X. R. Wang\textsuperscript{2,3,\textcopyright}  \\
\textsuperscript{1}Center for Joint Quantum Studies and Department of Physics, \textsuperscript{2}Physics Department, The Hong Kong University of Science and Technology (HKUST), Clear Water Bay, Kowloon, Hong Kong \textsuperscript{3}HKUST Shenzhen Research Institute, Shenzhen 518057, China  \\
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The nearest-neighbor level-spacing distributions are a fundamental quantity of disordered systems and are classified into different universality classes. They are the Wigner-Dyson and the Poisson functions for extended and localized states in Hermitian systems, respectively. The distributions follow the Ginibre functions for the non-Hermitian systems whose eigenvalues are complex and away from exceptional points (EPs). However, the level-spacing distributions of disordered non-Hermitian systems near EPs are still unknown, and a corresponding random matrix theory is absent. Here, we show a new class of universal level-spacing distributions in the vicinity of EPs of non-Hermitian Hamiltonians. Two distribution functions, $P_{\text{symm}}(s)$ for the symmetry-preserved phase and $P_{\text{symm}}(s)$ for the symmetry-broken phase, are needed to describe the nearest-neighbor level-spacing distributions near EPs. Surprisingly, both $P_{\text{symm}}(s)$ and $P_{\text{symm}}(s)$ are proportional to $s$ for small $s$, or linear level repulsions, in contrast to cubic level repulsions of the Ginibre ensembles. For disordered non-Hermitian tight-binding Hamiltonians, $P_{\text{symm}}(s)$ and $P_{\text{symm}}(s)$ can be well described by a surmise $P_{\text{symm}}(s) = \tilde{c}_1 s \exp[-\tilde{c}_2 s^2]$ in the thermodynamic limit (infinite systems) with a constant $\tilde{a}$ that depends on the localization nature of states at EPs rather than the dimensionality of non-Hermitian systems and the order of EPs.

Symmetries are powerful concepts for classifying disordered quantum systems described by random Hermitian matrices. The nearest-neighbor level-spacing distribution of a disordered metal follows one of three well-known Wigner-Dyson distributions, called symmetry classes, according to time-reversal and spin-rotational symmetries \cite{1,3}. Later, Altland and Zirnbauer proved that the Wigner-Dyson classes, which are invariant by adding a constant potential, do not exhaust all possibilities \cite{4}. The Wigner-Dyson classes can be further subdivided into seven new groups according to chiral and particle-hole symmetries: three chiral ensembles with chiral symmetry and four Bogoliubov-de Gennes ensembles with particle-hole symmetry. In total, there are ten symmetry classes for Hermitian random matrices.

Each symmetry class has its specific energy-spectral statistics and features, which are independent of the details of Hamiltonians \cite{5}. Energy-spectral statistics have been studied in many fields of physics, including nuclear physics \cite{6}, condensed-matter physics \cite{7}, information theory \cite{8}, and many fundamental phenomena in quantum physics \cite{9}. One example is the Anderson localization transitions. The distribution $P(s)$ of level spacing $s$ of two nearest-neighbor extended states is well described by the Wigner-Dyson functions of different symmetry classes \cite{10}. In contrast, $P(s)$ for localized states follows the Poisson distribution. Another example is that energy-spectral statistics can distinguish integrable quantum systems from chaotic ones: the Poisson distribution for quantum integrable systems \cite{11} and the Wigner-Dyson distributions for quantum chaotic systems \cite{12}.

Non-Hermiticity has a unique position in physics, especially in disordered \cite{13-15} and topological systems \cite{16,18}. Level spacing $s$ between two complex eigenenergies is defined as the Euclidean distance in the complex-energy plane such that $P(s)$ is properly defined. A pioneering work by Grobe, Haake, and Sommers shows that $P(s)$ is the Poisson distribution in the complex-energy plane for an integrable system and are the so-called Ginibre distributions of corresponding symmetry classes \cite{19} for a fully chaotic system \cite{20}. The three Gaussian Ginibre (orthogonal, unitary, and symplectic) ensembles display a universally cubic level repulsion \cite{21}, $\lim_{s \to 0} P(s) \sim s^3$, while non-Ginibre distributions also appear in some symmetry classes with transpose symmetry \cite{22,23}.

Within Ginibre’s framework \cite{19}, the eigenstates of Hamiltonians are non-orthogonal, and their eigenvalues are generally complex. Nevertheless, a large class of non-Hermitian Hamiltonians possesses exceptional points (EPs) and exceptional lines that separate domains of real eigenenergies from that of complex ones if either parity-time symmetry (PT-symmetry) \cite{24} or pseudo-Hermiticity \cite{25} is presented. $P(s)$ near EPs, where right eigenstates are mutually orthogonal and their duals are the corresponding left eigenstates, may lead to different energy-spectral statistics than those of Gaussian Ginibre ensembles. However, no careful study of level statistics near EPs is available, and a rigorous extension of random matrix theory (RMT) for EPs is needed.

Our goal is to investigate $P(s)$ near EPs of non-Hermitian systems. We find that the Ginibre distributions are no longer applicable there. The nearest-neighbor level-spacing distributions of small random matrices with EPs, denoted as $P_{\text{ep}}(s)$, are different in the symmetry-preserved and symmetry-broken phases where eigenvalues are real and complex, respectively. Secondly and importantly, level repulsions are linear near EPs, instead of cubic in the Ginibre distributions, irrespective of symmetries of non-Hermitian matrices. Thirdly, in the thermodynamic limit, $P_{\text{ep}}(s)$ in both the symmetry-preserved and symmetry-broken phases agree with a surmise of $P_{\text{ep}}(s) = \tilde{c}_1 s \exp[-\tilde{c}_2 s^2]$ with $\tilde{c}_{1,2}$ being normalized constants and $\tilde{a} = 2$ and $3$ if the state at the EP is extended and localized, respectively. Our surmise $P_{\text{ep}}(s)$ is applicable to a large family of disordered non-Hermitian systems with different orders of
Symmetry classes with EPs.—We first need to find symmetry classes with EPs. There are eight classes of non-Hermitian Hamiltonians according to four possible symmetry operators $O$ satisfying $[H,O]_{c=±1} = HO - cOH = 0$, where $O$ are K, Q, P, or C symmetry transformations in the Bernard-LeClair classification. The four allowed transformations are beyond unitary and unitary operators required by Hermitian Hamiltonian [26]. Out of the eight non-Hermitian classes, only three of them support real spectra where $O$ is antilinear, see an analysis in Supplementary Information [27].

The first two classes are non-Hermitian Hamiltonians with K symmetry, defined by $[H,Θ_k]_{c=1} = 0$, where $Θ_k = U_kK$ consists of complex-conjugate operation $K$ and unitary operator $U_k$. Eigenvalues $ε$ of such a $H$ are either real $R$ or appear in pairs $(ε,ε')$, and the critical points separating real and complex eigenvalues are EPs. $Θ_k^2 = ±I$ distinguish two K-symmetric class with EPs. Here, $I$ is the unit matrix. The eigenstates of a K-symmetric system with $Θ_k^2 = -I$ must be double degenerated, see a proof in Supplementary Information [27].

The third class is Q-symmetric (also known as pseudo-Hermitian) Hamiltonians satisfying $[H,Θ_q]_{c=1} = 0$, where $Θ_q = U_qη$ is the product of a unitary operator $U_q$ and Hermitian-conjugate operator $η$. $σ^1/σ^2 = I$ and $η = η^\dagger$ [25]. One should not confuse the Hermitian-conjugate operator $η$ with the complex-conjugate operation $K$. There is only one Q-symmetric class since $Θ_q^2 = I$, and Hermitian Hamiltonians belong to the trivial Q-symmetric class for $U_q = I$. The remaining five classes featured by P and C symmetries do not imply real spectra and are not considered in this work.

Small random matrices.—Let us first follow Wigner’s wisdom to analytically derive $P_{EP}(s)$ for small random matrices [11]. We concentrate on Gaussian ensembles whose probability functions are $P(H)dH ∝ \exp[-Tr[HH^\dagger]/σ^2]dH$ with $σ$ being a real positive number. Consider non-Hermitian Hamiltonians with K symmetry of $Θ_k^2 = I$ and for a specific choice of $Θ_k = σ^1K$, a 2 × 2 random matrix with the designed symmetry can be constructed as

$$H_{maj}^{K+} = aI + bσ_1 + cσ_2 + idσ_3,$$  (1)

where $σ_{1,2,3}$ are Pauli matrices and $a, b, c, d$ are independent real random numbers with Gaussian distributions of zero means and variance $σ^2$. Eigenvalues of $H_{maj}^{K+}$ are $ε_{1,2} = a ± \sqrt{b^2 + c^2 - d^2}$, which are real if $b^2 + c^2 ≥ d^2$ and appear in pair, $(ε, ε')$, if $b^2 + c^2 < d^2$. The domain with real eigenvalues is termed as the symmetry-preserved phase, and the others known as the symmetry-broken phase [28]. The two phases are separated by an EP at $b^2 + c^2 - d^2 = 0$.

Clearly, $ε_{1,2}$ are closest at the EP whose level-spacing distributions are $P_{EP}(s)$. Since the term inside the square root of $ε_{1,2}$ changes signs at the EP, $P_{EP}(s)$ should be determined by separately integrating over $a, b, c, d$ in the symmetry-preserved and symmetry-broken phases because of different constraints. Let us consider the symmetry-preserved phase first and redefine $b = t\sin[φ], c = t\cos[φ], t = [0, ∞), φ ∈ [0, 2π]$. In the symmetry-preserved phase, $ε_1^2 = a ± \sqrt{t^2 - d^2}$ and $s^2 > d^2$. Conservation of probability requires

$$P(a, b, c, d)da db dc dφ = P(ε_1^+, t, φ)f(ε_1^+, t, φ) df(ε_1^+, t, φ) df(ε_1^+, t, φ),$$  (2)

where $f = (ε_1^+ - ε_2^-)/2\sqrt{4t^2 - (ε_1^+ - ε_2^-)^2}$ is the Jacobian. Then, we have

$$P(ε_1^+, t) = \int_0^{2π}dφ \int_0^{t}dtP(ε_1^+, t, φ)J = \frac{1}{Z}(ε_1^+ - ε_2^-)e^{-[ε_1^++(ε_1^--ε_2^-)^2]/σ^2}$$  (3)

with $Z$ being the normalized constant to be determined. Then, we set $u = ε_1^+ + ε_2^-$ and $s = ε_1^+ - ε_2^-$ and obtain $P_{ep}^{K+}(s)$ by integrating over $u$ and applying the normalization conditions

$$\int_{-∞}^{∞} P_{EP}^{K+}(s)ds = 1.$$  (4)

Through the same approach, we find $P_{EP}(s)$ for the symmetry-broken phase is [27]

$$P_{SP}^{K-}(s) = c_1sErfc[\sqrt{2c_2}s]exp[c_2s^2]$$  (5)

with $Erfc[x] = (2/\sqrt{π})\int_x^{∞} e^{-t^2}dt$ being the complementary error function $lim_{x→0} Erfc[x] = 1$ and $c_1 ≃ 2.54, c_2 ≃ 0.526$.

Equations (4) and (5) accord perfectly with numerical results obtained by directly diagonalizing Eq. (1), see Figs. 1(a) and (d), as well as those for different choices of $Θ_k$, see evidence in Supplementary Information [27]. From Eqs. (4) and (5), we find $P_{EP}(s)$ of the two phases exhibit linear level repulsions: $lim_{s→0} P_{SP/SP}^{K+}(s) ∼ s$. To the best of our knowledge, linear level repulsions of non-Hermitian random matrices

\[\text{FIG. 1. The nearest-neighbor level-spacing distributions of small random matrices in the symmetry-preserved (labelled as } P_{ep}(s) \text{) and symmetry-broken (labelled as } P_{sp}(s) \text{) phases for the K-symmetric classes of } Θ_k^2 = I \text{ (a,d) and } Θ_k^2 = -I \text{ (b,e) and the Q-symmetric class (c,f). Empty and filled circles are numerical data obtained by diagonalizing Eqs. (1), (6), (8) for } σ = 1 \text{ and } 10^6 \text{ random ensembles, and solid lines are Eqs. (5), (7), (9). For comparisons, } P(s) \text{ of the Ginibre unitary distributions of } 2 \times 2 \text{ matrices are also plotted (blue lines) [20].} \]
have never been reported before, and the well-known Ginibre distributions predict a cubic level repulsions, \(\lim_{s \to 0} P(s) \sim s^3\) [19, 21].

Cubic level repulsions are universal in the Ginibre distributions [21]. Naturally, the universality of the linear level repulsions should be tested. Recall that there are two additional classes supporting EPs. The first ones are K-symmetric systems of \(\mathbf{Q}_k = -I\), where a two-fold degeneracy is required to obtain EPs [27]. Hence, the minimal model is a \(4 \times 4\) matrix that can be constructed as

\[
H_{\text{small}}^{K^-} = aI + ib\Gamma^1 + c\Gamma^2 + id\Gamma^3 + ie\Gamma^4 + if\Gamma^5,
\]

where \(a, b, c, d, e, f\) are independent real random numbers with the same Gaussian distributions. The five anticommuted Gamma matrices are \(\Gamma^1, \Gamma^2, \Gamma^3 = (I \otimes \tau_3, I \otimes \tau_1, \sigma_1 \otimes \tau_2, \sigma_2 \otimes \tau_2, \sigma_2 \otimes \tau_2)\) with \(\tau_{1,2,3}\) being Pauli matrices. One can see that \(H_{\text{small}}^{K^-}\) preserves K symmetry since \([H_{\text{small}}^{K^-}, \Theta_k] = 0\) with \(\Theta_k = (i\sigma_2 \otimes \tau_1)\mathcal{K}\) and \(\mathcal{K}^2 = -I\). Eigenvalues of \(H_{\text{small}}^{K^-}\) are doubly degenerated: \(\epsilon_k^\pm = a \pm \sqrt{c^2 - b^2 - d^2 - e^2 - f^2}\).

The two degenerated eigenvalues \(\epsilon_k^\pm\) coalesce at an EP where \(c^2 = b^2 + d^2 + e^2 + f^2\). Analytically, we find \(P_{\text{ep}}(s)\) in the symmetry-preserved and symmetry-broken phases are

\[
P_{\text{SP}}^{K^-}(s) = c_3 \left( \frac{3s^2 - c_4s^4}{c_4} + \sqrt{3} s \text{Erfc}[\sqrt{3}c_4s](1 - 4c_4^2s^2)e^{c_4s^2} \right) / (\sqrt{2\pi}c_4^3),
\]

\[
P_{\text{SB}}^{K^-}(s) = c_5 s(1 + 4c_6s^2)e^{-c_6s^2},
\]

with \(c_3 = 1.35, c_4 = 0.600, c_5 \approx 0.616, c_6 \approx 1.54\), as well as linear level repulsions \(\lim_{s \to 0} P_{\text{SP(SB)}}^{K^-}(s) \sim s\). As shown in Figs. 1(b) and (e) and Supplemental Information [27]. Eq. 7 above agrees perfectly with numerical results and is valid for a different \(\Theta_k\).

The third symmetry class with EPs is the Q-symmetric class where \([H, \Theta_q] = 0\). For simplicity, we choose a specific symmetry operator \(\Theta_q = \sigma_3\eta\) such that the corresponding random matrix reads

\[
H_{\text{small}}^{Q} = aI + ic\sigma_1 + id\sigma_2 + b\sigma_3,
\]

where \(a, b, c, d\) are the same as those in \(H_{\text{small}}^{K^-}\). The eigenvalues are \(\epsilon_k^\pm = a \pm \sqrt{b^2 - c^2 - d^2}\). \(H_{\text{small}}^{Q}\) undergoes a transition from the symmetry-protected phase to the symmetry-broken phase at an EP \(b^2 - c^2 - d^2 = 0\), where \(P_{\text{ep}}(s)\) in the two phases are derived analytically [27]

\[
P_{\text{SP}}^{Q}(s) = c_7 s \text{Erfc}[\sqrt{3}c_8s]e^{c_8s^2}/(\sqrt{2\pi}c_8^3),
\]

\[
P_{\text{SB}}^{Q}(s) = \left(\pi/2\right)se^{-2s^2/4},
\]

with \(c_7 \approx 2.42, c_8 \approx 0.271\). Again, we have a linear level repulsion, \(\lim_{s \to 0} P_{\text{SP(SB)}}^{Q}(s) \sim s\), and Eq. 9 describes numerical data excellently as shown in Figs. 1(c) and (f).

Results of small random matrices are simple and meaningful. Although \(P_{\text{ep}}(s)\) bifurcate into the symmetry-protected and symmetry-broken phases and are quantitatively different for different symmetry classes, the level repulsions are always linear. It is widely believed that RMT-statistics lead to cubic level repulsions in non-Hermitian systems, and one would expect that RMT gives cubic level repulsions for all non-Hermitian random Hamiltonians [20]. However, Eqs. 4, 5, 7, and 9 indicate that cubic level repulsions are not true at least near EPs.

Large random matrices. While Wigner-Dyson distributions for small matrices (known as Wigner surmise) are good approximations for random \(N \times N\) matrices with \(N \gg 1\) [29], Ginibre distributions show significant \(N\)- dependences [5]. Hence, it is important to investigate \(P_{\text{ep}}(s)\) and whether linear level repulsions holds for large matrices near EPs. To calculate \(P_{\text{ep}}(s)\), one needs to accurately know EPs. This is easy for small random matrices because analytical expression of eigenvalues are available, but is highly non-trivial for large random matrices in general [29]. Thus, we consider three special Hamiltonians with K symmetry and with known EPs.

The first one is a tight-binding model in two-dimensional (2D) square lattices of size \(L \times L\) whose Hamiltonian in the momentum space and in the absence of disorders is

\[
h_{2D}(k) = v_0 l + a \sin k_1\sigma_2 - a \sin k_2\sigma_1 + ik_3\]

with \(v_0, \alpha, \kappa\) being real positive numbers. The effective \(k \cdot p\) Hamiltonian of Eq. (10) near \(k = 0\) reads \(v_0 l + a(p \times \sigma) \cdot \hat{z} + ik_3\). The second term describes a Rashba-like spin-orbit coupling with strength \(\alpha\) [31], the third term is an imaginary Zeeman term \(ik_3\) distinguishing lifetimes of two orbitals [32]. Possible physical realizations of Eq. (10) include a large family of ferromagnetic semiconductors such as MnGaAs and other III-V host materials [33].

Equation (10) preserves K symmetry with \(\Theta_k = \sigma_1\mathcal{K}\). The disorders are introduced through an on-site random potential \(V_{2D} = \sum_{i} c_i^2 v_i \sigma_2 c_i\), where \(c_i\) (\(c_i^\dagger\)) is particle creation (annihilation) operator at site \(i\) and \(v_i\) is an uncorrelated Gaussian distribution of zero mean and variance \(\sigma^2\) [27]. \(P_{\text{ep}}(s)\) is obtained by numerically solving \(H_{2D} + V_{2D}\), where \(H_{2D}\) is Hamiltonian Eq. (10) in real space whose expression is given in Supplementary Information [27]. Disorders break lattice-translational symmetry but preserve K symmetry. For \(\alpha > \kappa/\sqrt{2}, N = 2L^2\) eigenvalues of \(H_{2D}\) distribute in a cross region in the complex-energy plane with the EP at \(\epsilon = v_0 + 0i\), see Fig. 2(a). \(P_{\text{ep}}(s)\) curves are obtained from two nearest-neighbor eigenvalues to the EP for the many random configurations, where the conventional unfolding procedures are used [34].

For states in the symmetry-protected phase far from the EP, say \(\epsilon \in (\epsilon_0 - \Delta \epsilon, \epsilon_0 + \Delta \epsilon)\) with \(\epsilon_0 = 3.5\) and \(\Delta \epsilon \sim 10^{-2}\), \(P(s)\) in Fig. 2(b) is well described by the Wigner surmise of Gaussian unitary ensemble [Here, \(\lim_{s \to 0} P(s) \sim s^3\)] [40]. This is because non-Hermitian systems in the symmetry-protected phase behave like a Hermitian system without the time-reversal symmetry due to \(V_{2D}\). Near the EP, say \(\epsilon_0 = 3.99\), \(P(s) \sim P_{\text{SP}}(s)\) that deviates from the Wigner-Dyson distribution and shows a linear level repulsion in the limit of \(s \to 0\).
see Fig. 2(b). This also happens for $P_{\text{SB}}(s)$ in the symmetry-broken phase. Interestingly, for a small system size $L = 20$, $P_{\text{SB}}(s)$ is different from $P_{\text{SB}}(s)$, but they merge for a large system size of $L = 200$, see Figs. 2(c) and (d), respectively.

Our surmise of the nearest-neighbor level-spacing distributions near the EPs is

$$\tilde{P}_{\text{ep}}(s) = \tilde{c}_{1,2} s \exp[-\tilde{c}_{2,3} s^2].$$

(11)

Here, $\tilde{a} > 0$, and $\tilde{c}_{1,2}$ are normalized constants. The surmise has the linear level repulsion for small $s$ and an exponential decay $\propto \exp[-\tilde{c}_{2,3} s^2]$ for large $s$. For $L = 200$, $\tilde{P}_{\text{ep}}(s)$ fits well to the numerically-calculated $P_{\text{SP}}(s)$ and $P_{\text{SB}}(s)$ of Hamiltonian Eq. (10) with $\tilde{a} = 2.99 \pm 0.02$ and $\tilde{\alpha} = 3.02 \pm 0.03$, respectively, see black lines in Figs. 2(b) and (d).

$P_{\text{SP}}(s)$ and $P_{\text{SB}}(s)$ for various system sizes $L$ are numerically obtained. The goodness-of-fit $Q$ of our data to Eq. (11) is $Q > 10^{-3}$ for $L > 10$ such that Eq. (11) is a satisfactory description of $P_{\text{ep}}(s)$ for $L > 10$. Figure 3(a) depicts the exponent $\tilde{a}$ as a function of $L$. Similar to small random matrices $[L = 1]$, $P_{\text{SP}}(s)$ and $P_{\text{SB}}(s)$ for relatively small sizes [say $10 < L < 40$] are different as $\tilde{a}$ in the symmetry-preserved phase is not equal to that in the symmetry-broken phase. With the increase of $L$, $P_{\text{SP}}(s)$ and $P_{\text{SB}}(s)$ merge and approach $\tilde{P}_{\text{ep}}(s)$ with $\tilde{a} = 3$.

To test whether the exponent $\tilde{a}_{L,\infty}$ is universal, we consider a one-dimensional (1D) tight-binding model of length $L$ with K symmetry whose Hamiltonian is

$$h_{1D}(k) = v_0 I + \alpha \sin k_1 \sigma_2 + i k \sigma_3.$$  

(12)

Equation (12) satisfies $[h_{1D}, \Theta_k] = 0$ with $\Theta_k = \sigma_1 k^0$. The Hamiltonian in real-space is $H_{1D}$ given in the Supplementary Information. Random on-site potentials $V_{1D} = \sum c_i^0 v_i \sigma_2 c_i$ with $v_i$ following the Gaussian distribution of the zero mean and variance $\sigma^2$ are used for studying the level statistics, see

FIG. 2. (a) Eigenvalues of the real-space Hamiltonian $H_{1D}$ of Eq. (10) with disorders in the complex-energy plane for $L = 20$. (b) $P(s)$ of $H_{1D}$ of $L = 200$ in two energy windows $[\epsilon_0 - 3 \Delta \epsilon, \epsilon_0 + 3 \Delta \epsilon]$ with $\epsilon_0 = 3.5$ (triangles) and 3.99 (circles) and $\Delta \epsilon \sim 10^{-3}$. The red line in (b) is the Wigner surmise for Gaussian unitary ensemble. The black lines in (b) and (d) are $\tilde{P}_{\text{ep}}(s) = \tilde{c}_1 s \exp[-\tilde{c}_2 s^2]$ with $\tilde{a} = 3$. (c), (d) $P_{\text{SP}}(s)$ and $P_{\text{SB}}(s)$ of $H_{1D}$ for (c) $L = 20$ and (d) $L = 200$. Other model parameters are $v_0 = 4, \alpha = 0.2, \kappa = 0.1, \sigma = 0.1$. Each point in (b)-(d) is averaged over more than $10^4$ ensembles.

Supplemental Information [27]. The model has an EP at $\epsilon = v_0 + i \theta$. From fitting $P_{\text{ep}}(s)$ to Eq. (11), $\tilde{a} = 2$, instead of $\tilde{a}_{L,\infty} = 3$ in 2D, is obtained for the symmetry-preserved and symmetry-broken phases as shown in Fig. 3(b). Interestingly, $\tilde{a} = 2$ equals to the Brody distribution in 2D for independently uniformly distributed random energy levels in the complex-energy plane [36, 37].

The reason for two $\tilde{a}_{L,\infty}$ in Fig. 3 is as follows: For Hermitian systems, $P(s)$ at an Anderson transition point universally decays as a stretched-exponential, $\propto \exp[-\tilde{c}_2 s^\tilde{\alpha}]$, for large $s$, and becomes a Gaussian ($\tilde{a} = 2$) or a Poisson ($\tilde{a} = 1$) that is the Brody distribution in 1D for the extended and localized states, respectively [50]. Based on this fact, we conjecture $P_{\text{ep}}(s)$ for localized EPs follows the Brody distribution in 2D since levels of localized states are uncorrelated. However, $P_{\text{ep}}(s)$ for levels near the extended EPs, which are correlated, has a faster decay rate at the tail, i.e., a larger exponent $\tilde{a}_{L,\infty} = 3$. We have partially confirm this argument by proving the following issues in Supplementary Information [27]: (i) EPs of Eq. (10) undergo an Anderson localization transition at $\sigma_c = 0.63 \pm 0.05 > \sigma = 0.1$ used in Fig. 3(a). (ii) EPs of Eq. (12) are localized by infinitesimal disorders.

FIG. 3. (a) $\tilde{a}$ as a function of $L$ for $H_{1D}$ in the symmetry-preserved (the blue circles) and symmetry-broken (the red squares) phases. (b) $\tilde{a}$ as a function of $\ln L$ for $H_{1D}$. Here, $v_0 = 4, \alpha = 0.2, \kappa = 0.1, \sigma = 0.1$. The black dashed lines in (a) and (b) locate $\tilde{a} = 3$ and 2, respectively.

FIG. 4. (a) Eigenvalues of $H_{1D}$ of Hamiltonian Eq. (13) with disorders in the complex-energy plane for $\alpha = 0.2, \kappa = 0.1, \sigma = 0.1, L = 8$. (b) $P_{\text{SP}}(s)$ (the orange circles) and $P_{\text{SB}}(s)$ (the purple squares) of levels near the EP in (a). The solid lines are fitted by Eq. (11) with $\tilde{a} = 2.96 \pm 0.08$ and $2.6 \pm 0.2$ for $P_{\text{SP}}(s)$ and $P_{\text{SB}}(s)$, respectively. Insert: Zoom-in of the peaks of $P_{\text{SP}}(s)$ and $P_{\text{SB}}(s)$. (c) Same as (b) but for $L = 12$. The black solid line is Eq. (11) of $\tilde{a} = 3$. (d) $\tilde{a}$ as a function of $L$ for the symmetry-preserved and symmetry-broken phases. The black dashed line is $\tilde{a} = 3$. Each data is average over more than $10^4$ ensembles.
Higher-order EPs.—EPs in Eqs. (10) and (12) are second-order. It is important to check whether level-spacing distributions near a higher-order EP exhibit also the linear level repulsion. For this purpose, we consider the following three-dimensional (3D) model of size $L \times L \times L$ whose clean Hamiltonian in the momentum space is

$$h_{3D}(k) = v_0 I + \alpha \sum_{\mu=1,2,3} \sin k_\mu \Gamma^\mu + i \kappa \Gamma^3. \quad (13)$$

$h_{3D}(k)$ has $K$ symmetry since $[h_{3D}(k), \Theta_k]_{\sigma=1} = 0$ with $\Theta_k = i \sigma_2 \otimes \tau_3 K$. For $\kappa = 0$, $h_{3D}(k)$ is Hermitian and display a quadruple degeneracy, whereby two doubly degenerate bands touch the other two at high-symmetry points in the first Brillouin zone. For finite $\kappa$, the degeneracy points split into forth-order EPs at $\sigma^2(\sin^2 k_1 + \sin^2 k_2 + \sin^2 k_3) = \kappa^2$, see Supplementary Information \[27\]. The forth-order EPs form a closed exceptional sphere of radius $\kappa/\alpha$ in the Brillouin zone.

We study the real-space Hamiltonian $H_{3D}$ of Eq. (13) with an additional random on-site potential $V_{3D} = \sum_i c_i^\dagger V_i c_i^\dagger$ where $V_i$ is the Gaussian distribution of zero mean \[27\]. The disordered potential does not break $K$ symmetry, and the EP is at $\epsilon = v_0 + i \alpha$ in the complex-energy plane, see Fig. [3]a.

Akin to those of the second-order EPs, $P_{3D}(s)$ and $P_{3D}(s)$ of the forth-order EPs of $L = 8$ can be fitted by Eq. (11) with $\tilde{\alpha} = 2.96 \pm 0.08$ and $2.6 \pm 0.2$, respectively, see Fig. [4]b. It is about $\tilde{\alpha} = 3$ for a larger size $L = 12$ as shown in Fig. [4]c. Furthermore, as shown in Fig. [4]d, $\tilde{\alpha}$ of $P_{3D}(s)$ and $P_{3D}(s)$ merge and approach to 3 in the thermodynamic limit, similar to the cases of $h_{3D}$ shown in Fig. [3]a.

Generally speaking, states in 3D models are much more extended than those in 2D models. We have proven that the EPs of $H_{3D}$ of the same disorder strength $\sigma = 0.1$ are extended. It is reasonable to assert that the EPs of $H_{3D}$ are extended as well. Hence, Fig. [3]a and Fig. [4]d strongly indicate that the order of EPs and the dimensionality of non-Hermitian systems do not change $P_{3D}(s)$ where $\lim_{s \to 0} P_{3D}(s) \sim s$ and $\tilde{\alpha} = 3$, as long as the EPs are extended.

Discussions.—With the rapid advance in Hamiltonian engineering in optical \[33\], mechanical \[39\], electric \[40\] systems, to name a few, where EPs are realized by suitably controlling gain and loss, the reported linear level repulsion can be tested experimentally. Here, we suggest cavity-magnon-polaritons as feasible platforms for observing linear level repulsions at EPs, whose effective Hamiltonians are non-Hermitian due to the inevitable loss. The $PT$-symmetric systems with EPs have already been realized experimentally \[41\]–[44], and quasi-particles due to strong couplings between magnons and cavity photons were detected. Our prediction should be easily detectable in this well-developed system, see Supplementary Information \[27\].

Conclusion.—In summary, the nearest-neighbor level-spacing distributions near EPs display linear level repulsions for small random matrices. We generalize this finding by investigating 1D, 2D, and 3D disordered tight-binding Hamiltonians with either the second-order or the forth-order EPs and find that the profile Eq. (11) of $P_{3D}(s)$ describes our numerical data for large enough sizes well. One interesting open question is whether there exist other classes of EPs with non-linear level repulsions. Non-Hermitian systems have, in total, thirty-eight symmetry classes if multiple symmetries are considered, in which twenty-eight classes support EPs \[26\]. We speculate that all of them exhibit linear level repulsions, but a comprehensive study of all symmetry classes is needed before making a definite statement about the question.

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