Dual jet geometrical objects of momenta in the
time-dependent Hamilton geometry

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Abstract

The aim of this paper is to obtain on the dual 1-jet space $J^1_\ast (\mathbb{R}, M)$ the
main geometrical objects used in the dual jet geometry of time-dependent
Hamiltonians. We talk about distinguished (d-) tensors, time-dependent
semisprays, nonlinear connections and their mathematical connections.

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semisprays of momenta, nonlinear connections, adapted bases.

1 Introduction

According to Olver’s opinion [6], we recall that the 1-jet spaces and their duals
are the fundamental ambient mathematical spaces used in the study of classical
and quantum field theories in their Lagrangian and Hamiltonian approaches
(see also [3]). For this reason, the studies of Miron [4] and Atanasiu ([1], [2]) led
to the development of the Hamilton geometry of cotangent bundles exposed by
Miron, Hrimiuc, Shimada and Sabău in the monograph [5]. We emphasize that,
via the Legendre duality of the Hamilton spaces with the Lagrange spaces, the
preceding authors have shown in [5] that the theory of Hamilton spaces has the
same symmetry as the Lagrange geometry, giving thus a geometrical framework
for the Hamiltonian theory of Analytical Mechanics.

According to this physical and geometrical context, suggested by the cotan-
gent bundle framework of the Miron et al., this paper is devoted to exposing a
particular case of the time-dependent covariant Hamilton geometry studied in
[3] on dual 1-jet spaces (in the sense of d-tensors, time-dependent semisprays of
momenta and nonlinear connections), which is a natural dual jet extension of
the Hamilton geometry on the cotangent bundle from [5].

2 The dual 1-jet space

In our geometrical study we start with a smooth real manifold $M^n$ of dimension
$n$, whose local coordinates are $(x^i)_{i=1,n}$. Let us also consider the dual 1-jet
vector bundle (i.e., the time-dependent phase space of momenta)

\[ J^1(R, M) \equiv R \times T^* M \to R \times M, \]

whose local coordinates are denoted by \((t, x^i, p^1_i)\), where the coordinates \(p^1_i\) have the physical meaning of momenta.

The coordinate transformations \((t, x^i, p^1_i) \leftrightarrow (\tilde{t}, \tilde{x}^i, \tilde{p}_{1}^{j})\) induced from \(R \times M\) on the dual 1-jet space \(J^1(R, M)\) are given by

\[
\begin{cases}
\tilde{t} = \tilde{t}(t) \\
\tilde{x}^i = \tilde{x}^i(x^j) \\
\tilde{p}_{1}^{j} = \partial x^j \frac{d\tilde{t}}{dt} \partial \tilde{p}_{1}^{j},
\end{cases}
\]

(1)

where \(d\tilde{t}/dt \neq 0\) and \(\det(\partial \tilde{x}^i/\partial x^j) \neq 0\). It follows that, in our dual jet geometrical approach, we use a "relativistic" time \(t\).

By comparison, in the cotangent Hamiltonian approach from [5], the authors use the trivial bundle \(R \times T^* M \to T^* M\), whose coordinates are \((t, x^i, p_i)\). In this context, the changes of coordinates are given by

\[
\begin{cases}
\tilde{t} = t \\
\tilde{x}^i = \tilde{x}^i(x^j) \\
\tilde{p}_i = \partial x^j \partial \tilde{x}^i p_j,
\end{cases}
\]

emphasizing the absolute character of the time \(t\). In such a context, a time dependent Hamiltonian is a real valued function \(H\) on \(R \times T^* M\), which is also called rheonomic, or non-autonomous Hamiltonian. A geometrization of these Hamiltonians was realized by Miron, Atanasiu and their co-workers in the works [1], [2], [4] and [5].

Now, doing a transformation of coordinates (1) on \(J^1(R, M)\), we obtain the following results:

**Proposition 1** The elements of the local natural basis of vector fields

\[
\left\{ \frac{\partial}{\partial t}, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial p^1_i} \right\} \subset \mathcal{X}(J^1(R, M))
\]

transform by the rules

\[
\frac{\partial}{\partial t} = \frac{d\tilde{t}}{dt} \frac{\partial}{\partial \tilde{t}} + \frac{\partial \tilde{p}_{1}^{j}}{\partial \tilde{t}} \frac{\partial}{\partial \tilde{p}_{1}^{j}},
\]

\[
\frac{\partial}{\partial x^i} = \frac{\partial \tilde{x}^i}{\partial x^j} \frac{\partial}{\partial \tilde{x}^j} + \frac{\partial \tilde{p}_{1}^{j}}{\partial \tilde{x}^j} \frac{\partial}{\partial \tilde{p}_{1}^{j}},
\]

\[
\frac{\partial}{\partial p^1_i} = \frac{\partial \tilde{x}^i}{\partial \tilde{x}^j} \frac{d\tilde{t}}{dt} \frac{\partial}{\partial \tilde{p}_{1}^{j}},
\]

(2)
Proposition 2 The elements of the local natural basis of covector fields
\[ \{ dt, dx^i, dp^1_i \} \subset \mathcal{X}^* (J^1 R, M) \]
transform by the rules
\begin{align*}
    dt &= \frac{dt}{dt} \, \tilde{t}, \\
    dx^i &= \frac{\partial x^i}{\partial \tilde{x}^j} \, d\tilde{x}^j, \\
    dp^1_i &= \frac{\partial p^1_i}{\partial \tilde{t}} \, d\tilde{t} + \frac{\partial p^1_i}{\partial \tilde{x}^j} \, d\tilde{x}^j + \frac{\partial \tilde{x}^j}{\partial x^i} \, dt \, d\tilde{t} \, \tilde{p}^1_j.
\end{align*}

3 Time-dependent semisprays of momenta

As in the book [5], a central role in our dual jet geometrical study is played by \textit{d-tensors}.

Definition 3 A geometrical object
\[ T = \left( T_{Ij(1)(1)}^{1(i)(1)} \right) \] on the dual jet space \( J^1 R, M \), whose local components change according to the rules
\[ T_{Ij(1)(1)}^{1(i)(1)} = T_{Ij(1)(1)}^{1(p)(1)} \frac{dt}{dt} \frac{dx^i}{dt} \left( \frac{\partial x^k}{\partial \tilde{x}^r} \, dt \right) \left( \frac{\partial \tilde{x}^q}{\partial \tilde{x}^j} \, dt \right) \cdots \]
with respect to a transformation of coordinates \( (I) \) on \( J^1 R, M \), is called a \textit{d-tensor} or a \textit{distinguished tensor field} on \( J^1 R, M \).

Remark 4 The placing between parentheses of certain indices of the local components \( T_{Ij(1)(1)}^{1(i)(1)} \) is necessary for clearer future contractions.

Example 5 If \( H : J^1 R, M \rightarrow R \) is a Hamiltonian function depending on the momenta \( p^1_i \), then the local components
\[ G_{(1)(1)}^{i(j)} = \frac{1}{2} \frac{\partial^2 H}{\partial p^1_i \partial p^1_j} \]
represent a d-tensor field \( G = \left( G_{(1)(1)}^{i(j)} \right) \) which is called the \textit{vertical fundamental metrical d-tensor} produced by \( H \).

Example 6 The distinguished tensor \( C = \left( C_{(1)}^{i(i)} \right) \), where \( C_{(1)}^{i(i)} = p^1_i \), is called the \textit{Liouville-Hamilton d-tensor field of momenta} on the dual jet space \( J^1 R, M \).

Example 7 If \( h_{11}(t) \) is a semi-Riemannian metric on \( R \), then the geometrical object
\[ L = \left( L_{(1)(1)}^{i(j)} \right) \], where \( L_{(1)(1)}^{i(j)} = h_{11} \tilde{p}^1_i \tilde{p}^1_j \), is called the \textit{momentum Liouville-Hamilton d-tensor} associated with the metric \( h_{11}(t) \).
Example 8 Using the preceding metric \( h_{11}(t) \), the distinguished tensor \( J = (J^{(i)}_{(1)ij}) \), where \( J^{(i)}_{(1)ij} = h_{11}\delta_j \), is called the \textbf{d-tensor of h-normalization} on the dual 1-jet space \( J^{1*}(\mathbb{R}, M) \).

It is obvious that any d-tensor on \( J^{1*}(\mathbb{R}, M) \) is a tensor field on \( J^{1*}(\mathbb{R}, M) \). Conversely, the opposite is not true. As examples, we construct two tensors on \( J^{1*}(\mathbb{R}, M) \), which are not d-tensors on \( J^{1*}(\mathbb{R}, M) \).

**Definition 9** A global tensor \( G \) on \( J^{1*}(\mathbb{R}, M) \), locally expressed by

\[
G = p_i^1 dx^i \otimes \frac{\partial}{\partial t} - 2G^{(1)}_{(j)i} dx^i \otimes \frac{\partial}{\partial p_j^1},
\]

is called a \textbf{temporal semispray} on the dual 1-jet space \( J^{1*}(\mathbb{R}, M) \).

Taking into account that the temporal semispray \( G \) is a global tensor on \( J^{1*}(\mathbb{R}, M) \), by a direct calculation, we obtain

**Proposition 10** (i) Under a transformation of coordinates (4) the local components \( G^{(1)}_{(j)i} \) of the global tensor \( G \) change according to the rules

\[
2G^{(1)}_{(k)j} = 2G^{(1)}_{(j)i} \frac{dt}{dx^i} \frac{\partial x^j}{\partial x^k} - \frac{\partial x^j}{\partial x^k} \frac{\partial p^1_k}{\partial t} p^1_i.
\]  

(ii) Conversely, to give a temporal semispray on \( J^{1*}(\mathbb{R}, M) \) is equivalent to give a set of local functions \( G = (G^{(1)}_{(j)i}) \) which transform by the rules (4).

Example 11 If \( H^1_{11}(t) = (h^{11}/2)(dh_{11}/dt) \) is the Christoffel symbol of a semi-Riemannian metric \( h_{11}(t) \) of the temporal manifold \( \mathbb{R} \), then the local components

\[
0 G^{(1)}_{(j)i} = \frac{1}{2} H^1_{11} p^1_j p^1_k
\]

represent a temporal semispray \( G \) on the dual 1-jet space \( J^{1*}(\mathbb{R}, M) \), which is called the \textbf{canonical temporal semispray associated with the metric} \( h_{11}(t) \).

A second example of tensor on the dual 1-jet space \( J^{1*}(\mathbb{R}, M) \), which is not a distinguished tensor, is given by

**Definition 12** A global tensor \( G \) on \( J^{1*}(\mathbb{R}, M) \), locally expressed by

\[
G = \delta_i^j dx^j \otimes \frac{\partial}{\partial x^i} - 2G^{(1)}_{(j)i} dx^i \otimes \frac{\partial}{\partial p^1_j},
\]

is called a \textbf{spatial semispray} on the dual 1-jet space \( J^{1*}(\mathbb{R}, M) \).
Like in the case of a temporal semispray, we can prove without difficulties the following statement:

**Proposition 13** To give a spatial semispray on $J^1\ast(\mathbb{R}, M)$ is equivalent to give a set of local functions $G = \left( G^{(1)}_{2(j)i} \right)$ which transform by the rules

$$2 \tilde{G}^{(1)}_{2(j)k} = 2 G^{(1)}_{2(j)i} \frac{dt}{dt} \frac{\partial x^i}{\partial \tilde{x}^j} \frac{\partial \tilde{p}_1}{\partial \tilde{x}^i} - \frac{\partial x^i}{\partial \tilde{x}^j} \frac{\partial \tilde{p}_1}{\partial \tilde{x}^i}. \tag{6}$$

**Example 14** If $\gamma^{ij}_k(x)$ are the Christoffel symbols of a semi-Riemannian metric $\varphi_{ij}(x)$ of the spatial manifold $M$, then the local components

$$0 G^{(1)}_{2(j)k} = -\frac{1}{2} \gamma^{ij}_k p_i \tag{7}$$

define a spatial semispray $G^0_2$ on the dual 1-jet space $J^1\ast(\mathbb{R}, M)$, which is called the **canonical spatial semispray** associated with the metric $\varphi_{ij}(x)$.

**Definition 15** A pair $G = \left( G_1, G_2 \right)$, consisting of a temporal semispray $G_1$ and a spatial semispray $G_2$, is called a **time-dependent semispray of momenta** on the dual 1-jet space $J^1\ast(\mathbb{R}, M)$.

### 4 Nonlinear connections and adapted bases

In what follows, we study the important geometrical concept of nonlinear connection on the dual 1-jet space $J^1\ast(\mathbb{R}, M)$, which is intimately related by the concept of time-dependent semispray.

**Definition 16** A pair of local functions $N = \left( N^{(1)}_{1(k)j}, N^{(1)}_{2(k)i} \right)$ on $J^1\ast(\mathbb{R}, M)$, which transform by the rules

$$\tilde{N}^{(1)}_{1(j)1} = N^{(1)}_{1(k)j} \frac{\partial x^k}{\partial \tilde{x}^j} \frac{dt}{\tilde{t}} - \frac{\partial \tilde{p}_1}{\partial \tilde{x}^i} \tag{8}$$

$$\tilde{N}^{(1)}_{2(j)v} = N^{(1)}_{2(k)v} \frac{d\tilde{t}}{dt} \frac{\partial x^k}{\partial \tilde{x}^j} \frac{\partial \tilde{p}_1}{\partial \tilde{x}^i} - \frac{\partial x^i}{\partial \tilde{x}^j} \frac{\partial \tilde{p}_1}{\partial \tilde{x}^i} \tag{9}$$

is called a nonlinear connection on the dual 1-jet bundle $J^1\ast(\mathbb{R}, M)$. The geometrical entity $N_1 = \left( N^{(1)}_{1(j)1} \right)$ (respectively $N_2 = \left( N^{(1)}_{2(j)v} \right)$ ) is called a **temporal** (respectively spatial) nonlinear connection on $J^1\ast(\mathbb{R}, M)$.

Now, let us expose the connection between the time-dependent semisprays of momenta and nonlinear connections on the dual 1-jet space $J^1\ast(\mathbb{R}, M)$. For this, let us consider that $\varphi_{ij}(x)$ is a semi-Riemannian metric on the spatial manifold $M$. Thus, using the transformation rules (4), (6) and (8) of the geometrical objects taken in study, we can easily prove the following statements:
Proposition 17 (i) The connection between the temporal semisprays $G = \left( G^{(1)}_{1(i)j} \right)$ and the temporal components of nonlinear connections $N_{\text{temporal}} = \left( N^{(1)}_{1(r)1} \right)$ is given by the relations

$$N^{(1)}_{1(r)1} = \phi^{jk} \frac{\partial G^{(1)}_{1(i)j}}{\partial p_i^1}, \quad G^{(1)}_{1(i)j} = \frac{1}{2} N^{(1)}_{1(i)1} p_j^1.$$  

(ii) The connection between spatial semisprays $G = \left( G^{(1)}_{2(j)i} \right)$ and the spatial components of nonlinear connections $N_{\text{spatial}} = \left( N^{(1)}_{2(i)j} \right)$ is given via the relations

$$N^{(1)}_{2(j)i} = 2 G^{(1)}_{2(j)i}, \quad G^{(1)}_{2(j)i} = \frac{1}{2} N^{(1)}_{2(j)1} p_i^1.$$  

Remark 18 It is obvious that on the 1-jet space $J^1(R, M)$ a time-dependent semispray of momenta $G$ naturally induces a nonlinear connection $N_G$ and vice-versa, a nonlinear connection $N$ induces a time-dependent semispray $G_N$. The nonlinear connection $N_G$ is called the canonical nonlinear connection associated with the time-dependent semispray of momenta $G$ and vice-versa.

Example 19 The canonical nonlinear connection $\hat{N} = \left( \begin{array}{c} 0 N^{(1)}_{1(i)1} \\ 0 N^{(1)}_{2(i)j} \end{array} \right)$ produced by the canonical time-dependent semispray of momenta $G = \left( \begin{array}{c} 0 \\ 0 G_1 \\ 0 G_2 \end{array} \right)$ has the local components

$$\hat{N}^{(1)}_{1(i)1} = H_{11} p_i^1, \quad \hat{N}^{(1)}_{2(i)j} = -\gamma_{ik} p_k^1.$$

This nonlinear connection is called the canonical nonlinear connection on $J^1(R, M)$, associated with the semi-Riemannian metrics $h_{11}(t)$ and $\varphi_{ij}(x)$.

Taking into account the complicated transformation rules (2) and (3), we need a horizontal distribution on the dual 1-jet space $J^1(R, M)$, in order to construct some adapted bases of vector and covector fields, whose transformation rules are simpler (tensorial ones, for instance).

In this direction, let $u^* = (t, x^i, p_i^1) \in J^1(R, M)$ be an arbitrary point and let us consider the differential map

$$\pi^* \subset u^*: T_u^* J^1(R, M) \to T_{(t,x)} (R \times M)$$

of the canonical projection

$$\pi^*: J^1(R, M) \to R \times M, \quad \pi^* (u^*) = (t, x),$$
together with its vector subspace $W_{u^*} = \text{Ker} \pi^{*,u^*} \subset T_{u^*} J^{1*}(\mathbb{R}, M)$. Because the differential map $\pi^{*,u^*}$ is a surjection, we find that we have $\dim_{\mathbb{R}} W_{u^*} = n$ and, moreover, a basis in $W_{u^*}$ is determined by \( \left\{ \frac{\partial}{\partial p_1} \right\}_{u^*} \).

So, the map $W: u^* \in J^{1*}(\mathbb{R}, M) \mapsto W_{u^*} \subset T_{u^*} J^{1*}(\mathbb{R}, M)$ is a differential distribution, which is called the \textit{vertical distribution} on the dual 1-jet space $J^{1*}(\mathbb{R}, M)$.

**Definition 20** A differential distribution

\[ H: u^* \in J^{1*}(\mathbb{R}, M) \mapsto H_{u^*} \subset T_{u^*} J^{1*}(\mathbb{R}, M), \]

which is supplementary to the vertical distribution $W$, that is we have

\[ T_{u^*} J^{1*}(\mathbb{R}, M) = H_{u^*} \oplus W_{u^*}, \quad \forall \ u^* \in J^{1*}(\mathbb{R}, M), \]

is called a \textbf{horizontal distribution} on the dual 1-jet space $J^{1*}(\mathbb{R}, M)$.

The above definition implies that $\dim_{\mathbb{R}} H_{u^*} = n + 1$, $\forall \ u^* \in J^{1*}(\mathbb{R}, M)$. Moreover, the Lie algebra of the vector fields $X J^{1*}(\mathbb{R}, M)$ can be decomposed in the direct sum $X J^{1*}(\mathbb{R}, M) = S(H) \oplus S(W)$, where $S(H)$ (respectively $S(W)$) is the set of differentiable sections on $H$ (respectively $W$).

Supposing that $H$ is a fixed horizontal distribution on $J^{1*}(\mathbb{R}, M)$, we have the isomorphism

\[ \pi^{*,u^*}|_{H_{u^*}} : H_{u^*} \rightarrow T_{\pi^*(u^*)}(\mathbb{R} \times M), \]

which allows us to prove the following result:

**Theorem 21** (i) There exist unique linear independent horizontal vector fields $\frac{\delta}{\delta t}, \frac{\delta}{\delta x^i} \in S(H)$, having the properties

\[ \pi^* \left( \frac{\delta}{\delta t} \right) = \frac{\partial}{\partial t}, \quad \pi^* \left( \frac{\delta}{\delta x^i} \right) = \frac{\partial}{\partial x^i}. \]  

(ii) The horizontal vector fields $\frac{\delta}{\delta t}$ and $\frac{\delta}{\delta x^i}$ can be uniquely written in the form

\[ \frac{\delta}{\delta t} = \frac{\partial}{\partial t} - N^{(1)}_{1(j)1} \frac{\partial}{\partial p^j_1}, \quad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^{(1)}_{2(j)i} \frac{\partial}{\partial p^j_2}. \]  

(iii) The local coefficients $N^{(1)}_{1(j)1}$ and $N^{(1)}_{2(j)i}$ obey the rules \( \Box \) of a nonlinear connection $N$ on $J^{1*}(\mathbb{R}, M)$.

(iv) On the 1-jet space $J^{1*}(\mathbb{R}, M)$ to give a horizontal distribution $H$ is equivalent to give a nonlinear connection $N = \left( N^{(1)}_{1(j)1}, N^{(1)}_{2(j)i} \right)$.
Proof. Let $\frac{\delta}{\delta t}, \frac{\delta}{\delta x_i} \in \mathcal{X}(J^1(\mathbb{R}, M))$ be vector fields on $J^1(\mathbb{R}, M)$, locally expressed by

$$\frac{\delta}{\delta t} = A^1_1 \frac{\partial}{\partial t} + A^1_j \frac{\partial}{\partial x^j} + A^{(1)}_{(j)1} \frac{\partial}{\partial p^1_j},$$

$$\frac{\delta}{\delta x_i} = X^1_i \frac{\partial}{\partial t} + X^j_i \frac{\partial}{\partial x^j} + X^{(1)}_{(j)i} \frac{\partial}{\partial p^1_j},$$

which verify the relations (10). Then, taking into account the local expression of the map $\pi^*\pi$, we get

$$A^1_1 = 1, \quad A^1_j = 0, \quad A^{(1)}_{(j)1} = -N^{(1)}_{1(j)1},$$

$$X^1_i = 0, \quad X^j_i = \delta^j_i, \quad X^{(1)}_{(j)i} = -N^{(1)}_{2(j)i}.$$ 

These equalities prove the form (11) of the vector fields from Theorem, together with their linear independence. The uniqueness of the coefficients $N^{(1)}_{1(j)1}$ and $N^{(1)}_{2(j)i}$ is obvious.

Because the vector fields $\frac{\delta}{\delta t}$ and $\frac{\delta}{\delta x^i}$ are globally defined, we deduce that a change of coordinates (11) on $J^1(\mathbb{R}, M)$ produces a transformation of the local coefficients $N^{(1)}_{1(j)1}$ and $N^{(1)}_{2(j)i}$ by the rules (8).

Finally, starting with a set of functions $N = \left(N^{(1)}_{1(j)1}, N^{(1)}_{2(j)i}\right)$, which satisfy the rules (8), we can construct the horizontal distribution $\mathcal{H}$, taking

$$H_u^* = \text{Span}\left\{ \frac{\delta}{\delta t} \bigg|_U, \frac{\delta}{\delta x^i} \bigg|_U \right\}.$$ 

The decomposition $T_u^*J^1(\mathbb{R}, M) = H_u^* \oplus W_u^*$ is obvious now. □

**Definition 22** The set of the linear independent vector fields

$$\left\{ \frac{\delta}{\delta t}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial p^1_j} \right\} \subset \mathcal{X}(J^1(\mathbb{R}, M))$$

is called the *adapted basis of vector fields produced by the nonlinear connection* $N = \left(N_1, N_2\right)$.

With respect to the coordinate transformations (11), the elements of the adapted basis (12) have their transformation laws as tensorial ones (in contrast
with the transformations rules (2):

\[
\frac{\delta}{\delta t} = \frac{dt}{d\tilde{t}} \frac{\delta}{\delta \tilde{t}}, \\
\frac{\delta}{\delta x^i} = \frac{\partial x^i}{\partial \tilde{x}^j} \frac{\delta}{\delta \tilde{x}^j}, \\
\frac{\partial}{\partial p^i_1} = \frac{dt}{d\tilde{t}} \frac{\partial}{\partial \tilde{x}^j} \frac{\partial}{\partial \tilde{p}^j_1}.
\]

The dual basis (of covector fields) of the adapted basis (12) is given by

\[
\{dt, dx^i, \delta p^1_i\} \subset \mathcal{X}^* (J^{1*}(\mathbb{R}, M))
\]

(13)

where

\[
\delta p^1_i = dp^1_i + N^{(1)}_{1(i)} dt + N^{(1)}_{2(i),j} dx^j.
\]

**Definition 23** The dual basis of covector fields (13) is called the adapted cobasis of covector fields of the nonlinear connection \(N = \left( N_1, N_2 \right)\).

Moreover, with respect to transformation laws (1), we obtain the following tensorial transformation rules:

\[
dt = \frac{dt}{d\tilde{t}}, \\
dx^i = \frac{\partial x^i}{\partial \tilde{x}^j} d\tilde{x}^j, \\
\delta p^1_i = \frac{dt}{d\tilde{t}} \frac{\partial \tilde{x}^j}{\partial x^i} \delta \tilde{p}^j_1.
\]

As a consequence of the preceding assertions, we find the following simple result:

**Proposition 24** (i) The Lie algebra of vector fields on \(J^{1*}(\mathbb{R}, M)\) decomposes in the direct sum \(\mathcal{X} (J^{1*}(\mathbb{R}, M)) = \mathcal{X} (\mathcal{H}_\mathbb{R}) \oplus \mathcal{X} (\mathcal{H}_M) \oplus \mathcal{X} (\mathcal{W})\), where

\[
\mathcal{X} (\mathcal{H}_\mathbb{R}) = \text{Span} \left\{ \frac{\delta}{\delta t} \right\}, \quad \mathcal{X} (\mathcal{H}_M) = \text{Span} \left\{ \frac{\delta}{\delta x^i} \right\}, \quad \mathcal{X} (\mathcal{W}) = \text{Span} \left\{ \frac{\partial}{\partial p^i_1} \right\}.
\]

(ii) The Lie algebra of covector fields on \(J^{1*}(\mathbb{R}, M)\) decomposes in the direct sum \(\mathcal{X}^* (J^{1*}(\mathbb{R}, M)) = \mathcal{X}^* (\mathcal{H}_\mathbb{R}) \oplus \mathcal{X}^* (\mathcal{H}_M) \oplus \mathcal{X}^* (\mathcal{W})\), where

\[
\mathcal{X}^* (\mathcal{H}_\mathbb{R}) = \text{Span} \{dt\}, \quad \mathcal{X}^* (\mathcal{H}_M) = \text{Span} \{dx^i\}, \quad \mathcal{X}^* (\mathcal{W}) = \text{Span} \{\delta p^1_i\}.
\]

**Definition 25** The distributions \(\mathcal{H}_\mathbb{R}\) and \(\mathcal{H}_M\) are called the \(\mathbb{R}\)-horizontal distribution and \(M\)-horizontal distribution on \(J^{1*}(\mathbb{R}, M)\).
5 Discussion

The results of this paper represent the basics for a subsequent geometrization (in the sense of nonlinear connection, canonical d-linear connection, d-torsions and d-curvatures) on dual jet spaces of the time-dependent Hamiltonians regarded as real-valued functions on the 1-jet space $J^1(R, M)$. This Hamilton geometrization is similar with that developed on cotangent bundles ([5], [4], [1] and [2]), but is characterized by a "relativistic" time in the study. In contrast the time-dependent Hamilton geometrization on cotangent bundles is characterized by an absolute time.

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