Locally discrete expanding groups of analytic diffeomorphisms of the circle

Bertrand Deroin

Abstract

We show that a finitely generated subgroup of $\text{Diff}^{\infty}(S^1)$ that is expanding and locally discrete in the analytic category is analytically conjugated to a uniform lattice in $\tilde{\text{PGL}}_2(\mathbb{R})$ acting on the $k$th covering of $\mathbb{RP}^1$ for a certain integer $k > 0$.

1. Introduction

In the study of the dynamics of finitely generated groups acting by analytic diffeomorphisms on the circle (or more generally in analytic unidimensional dynamics), the dichotomy discreteness versus nondiscreteness is very useful and important. Many interesting dynamical properties can be easily established in the nonlocally discrete regime, for instance, concerning ergodicity or minimality of the action. However, in the locally discrete regime things are not completely understood yet, even if a conjectural classification is expected (see, for example, the survey [3]).

The goal of this note is to provide such a classification under the additional assumption that the action is expansive, as announced in [3]. Expansive means that for every point of the circle there exists an element of the group whose derivative at that point is greater than 1 in modulus. Our main result (Corollary 9.2) shows that up to analytic conjugacy, only cocompact lattices of the finite cyclic coverings of $\text{PGL}_2(\mathbb{R})$ acting on the corresponding finite cyclic covering of the real projective line $\mathbb{RP}^1$ are at the same time expansive and locally discrete in the analytic category. The precise definitions of expansiveness and local discreteness in the analytic category are exposed in Sections 3 and 5, respectively.

This result is part of a more general one concerning the dynamics of pseudo-groups of holomorphic maps on Riemann surfaces having both local discreteness and hyperbolicity properties. However, its proof in the particular case of the circle group action is considerably simpler (essentially because of the use of a combination of the convergence group theorem by Gabai [4] and Casson-Jungreis [2], and of the differentiable rigidity theory of Fuchsian groups by Ghys [5]), and deserves a special interest for the theory of circle group actions.

Organisation of the article. Section 2 is devoted to review some aspects of the theory of Gromov hyperbolic groups that will be needed in our argument. In Sections 3 and 4 (respectively, Section 5), we present the definition of expansiveness (respectively, local discreteness). Section 6 is devoted to the main technical tool of our method, namely, the convergence property of the lines of expansion. Sections 7, 8 and 9 are devoted to the proof of our main result: Corollary 9.2.

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2. Preliminaries of geometric group theory

We review some notions of geometric group theory that will be useful for our argument. We refer the reader to [7] or [10] for a more detailed exposition.

Let $G$ be a finitely generated group. Given a finite symmetric generating subset $S \subset G$, we associate the norm $\|g\|$ of an element $g \in G$ as being the minimum number of elements of $S$ that is needed to write $g$.

The Cayley graph of the pair $(G, S)$ is the nonoriented graph whose vertices are the elements of $G$ and the edges are the pairs $(g, sg)$ with $g \in G$ and $s \in S$. The group $G$ acts naturally on its Cayley graph by right multiplications. The combinatorial distance $d$ associated to this graph — namely, the one defined as the minimum number of edges one has to cross to go from a vertex to another one — is given by the formula $d(g_1, g_2) := \|g_2g_1^{-1}\|$. Another finite symmetric generating set gives rise to another graph whose set of vertices is $G$, for which the identity map is bi-Lipschitz with respect to the associated distances.

Given a number $\delta \geq 0$, a graph is called $\delta$-hyperbolic if its triangles are thin, in the sense that there exists a constant $\delta > 0$ such that for any triple of vertices $g_1, g_2, g_3$, and any collection of geodesics $[g_1, g_2], [g_2, g_3]$ and $[g_3, g_1]$ between these points, we have that

$$[g_1, g_3] \subset [g_1, g_2]^{\delta} \cup [g_2, g_3]^{\delta},$$

where $A^{\delta}$ is the set of points at a distance from a point of $A$ less than $\delta$. A triangle in a graph satisfying an inequality such as (1) is called $\delta$-thin. A graph is called Gromov hyperbolic if there exists a number $\delta$ such that it is $\delta$-hyperbolic; this notion is invariant by quasi-isometry. A finitely generated group $G$ is called Gromov hyperbolic if its Cayley graphs (with respect to symmetric finite generating sets) are Gromov hyperbolic.

A geodesic ray parameterized by an interval $I \subset \mathbb{Z}$ is a sequence $\{g_n\}_{n \in I}$ of elements of $G$ such that $d(g_k, g_l) = |k - l|$ for every $k, l \in I$. The set of equivalence classes of geodesic rays parameterized by $\mathbb{N}$ up to bounded Hausdorff distance, is the geometric boundary of $G$, and is denoted by $\partial G$. It is equipped with the quotient of the topology of simple convergence. In the case where $G$ is Gromov hyperbolic, this topological space is a compact metric space. Moreover, in this latter case, the group $G$ acts naturally on its boundary by homeomorphisms, and the action is minimal unless $G$ is virtually cyclic.

Given constants $\alpha \geq 1$ and $\beta \geq 0$, an $(\alpha, \beta)$-quasi-geodesic on $G$ is a sequence $\{g_n\}_{n \in \mathbb{N}}$ of elements of $G$ such that for every $m, n \in \mathbb{N}$, we have

$$\alpha^{-1}|n - m| - \beta \leq d(g_m, g_n) \leq \alpha |n - m| + \beta. \tag{2}$$

We recall that there exists a constant $\eta = \eta(\alpha, \beta, \delta)$ such that two $(\alpha, \beta)$-quasi-geodesics having the same extremities are bounded apart in Hausdorff distance by less than $\eta$. As a consequence of this, up to enlarging the constant $\delta$, any $(\alpha, \beta)$-quasi-geodesic triangle is $\delta$-thin.

The Gromov product is defined by

$$(h_1, h_2)_g := \frac{d(g, h_1) + d(g, h_2) - d(h_1, h_2)}{2}, \tag{3}$$

for every $g, h_1, h_2 \in G$. Its geometrical significance in a Gromov hyperbolic graph is that, up to some constant depending only on the constants of hyperbolicity of the graph, the Gromov product $(h_1, h_2)_g$ is the distance from $g$ to a (any) geodesic between $h_1$ and $h_2$. Given two points $p, q \in \partial G$, this Gromov product can be extended to points of the boundary of $G$ by the following formula

$$(p, q)_g := \sup_{(h_m)_m, (k_n)_n} \limsup_{m, n \to \infty} (h_m, k_n)_g, \tag{4}$$

where the first supremum is taken over all geodesics $\{h_m\}_{m \in \mathbb{N}}$ and $\{k_n\}_{n \in \mathbb{N}}$ that tend to $p$ and $q$, respectively. In fact, if $(h_m)_m$ and $(k_n)_n$ are geodesics tending to $p$ and $q$, respectively,
then if \( m, n \) are sufficiently large, the quantity \((h_m, k_n)_g\) differs from \((p, q)_g\) by an additive term which is bounded by a constant that depends only on the hyperbolicity constants of \( G \). Then two points \( p, q \in \partial G \) are close to each other if and only if the Gromov product \((p, q)_e\) is large.

3. Expanding property and derivative cocycle

**Definition 3.1.** A subgroup \( G \subset \text{Diff}^\omega(S^1) \) is expanding if for every point \( x \in S^1 \), there exists an element \( g \in G \) such that \( \log |Dg(x)| > 0 \).

**Remark 3.2.** By compactness of \( S^1 \), if \( G \subset \text{Diff}^\omega(S^1) \) is a subgroup which is expanding, there exists a finite subset \( S \subset G \) and a constant \( \varepsilon > 0 \) such that for every point \( x \in S^1 \) there exists \( s \in S \) such that \( \log |Ds(x)| \geq \varepsilon \). If \( G \) is finitely generated, we can assume furthermore that \( S \) is a symmetric generating set.

**Definition 3.3.** If \( G \) is a subgroup of \( \text{Diff}^\omega(S^1) \), for every \( x \in S^1 \), we define the derivative cocycle \( D_x \) by the formula

\[
D_x(g_1, g_2) := \log |Dg_2(x)| - \log |Dg_1(x)| = \log |D(g_2 \circ g_1^{-1})(g_1(x))|.
\]

**Remark 3.4.** The derivative cocyle satisfies the obvious equivariance

\[
D_{g(x)}(g_1 \circ g^{-1}, g_2 \circ g^{-1}) = D_x(g_1, g_2),
\]

for every \( x \in S^1 \), and \( g, g_1, g_2 \in G \).

Moreover, if \( G \) is generated by a finite symmetric subset \( S \), the derivative cocycle satisfies

\[
D_x(g_1, g_2) < \tau d(g_1, g_2),
\]

where

\[
\tau := \max_{x \in S^1, s \in S} \log |Ds(x)|.
\]

**Definition 3.5.** Let \( G \subset \text{Diff}^\omega(S^1) \) be a finitely generated subgroup, \( S \subset G \) a finite symmetric generating subset, \( c > 0 \), and \( x \in S^1 \). A \( c \)-line of expansion relative to the point \( x \) is a sequence \( \{E^n_x\}_{n \geq 0} \) of elements of \( G \) such that for every \( n > 0 \) one can write \( E^n_x = s^n_x E^{n-1}_x \) for some \( s^n_x \in S \), and for every \( 0 \leq m \leq n \) we have

\[
D_x(E^m_x, E^n_x) \geq c(n - m).
\]

**Remark 3.6.** Observe that if \( S \) and \( \varepsilon \) are given by Remark 3.2, for any \( 0 < c \leq \varepsilon \), for any point \( x \in S^1 \), and for every \( g \in G \), there exists a \( c \)-line of expansion relative to the point \( x \) starting at \( g \). In the sequel, we will choose either \( c = \varepsilon \) or \( c = \varepsilon/2 \).

**Lemma 3.7.** Given a finitely generated group \( G \subset \text{Diff}^\omega(S^1) \), a finite symmetric generating subset \( S \subset G \), and a constant \( c > 0 \), there exists \( \alpha \geq 1 \) such that any \( c \)-line of expansion is a \((\alpha, 0)\)-quasi-geodesic.

**Proof.** We clearly have \( d(E^m_x, E^n_x) \leq |m - n| \) for every \( m, n \) so the right hand side of (2) is satisfied with \( \alpha = 1 \) and \( \beta = 0 \). Moreover, for any \( m, n \in \mathbb{N} \) with \( m \leq n \), (7), (8) together with (9) show

\[
c(n - m) \leq D_x(E^m_x, E^n_x) \leq \tau d(E^m_x, E^n_x),
\]

so that the left-hand side of (2) is satisfied with \( \alpha = c/\tau \) and \( \beta = 0 \). \( \square \)
4. Bounded distortion along lines of expansion

Definition 4.1. The distortion of a differentiable analytic (respectively, holomorphic) map \( g : U \to V \) between open subsets of \( S^1 \) (respectively, \( \mathbb{C} \)) in restriction to a subset \( E \subset U \) is the quantity

\[
\kappa(g, E) := \max_{x, y \in E} \log \frac{|Dg(y)|}{|Dg(x)|}.
\]

Lemma 4.2. Let \( G \) be a subgroup of \( \text{Diff}^\omega(S^1) \) generated by a finite symmetric set \( S \), and let \( c > 0 \) be some constant. There exists \( r > 0 \) such that the following holds. For any \( c \)-line of expansion \( (E_n^n)_{n \geq 0} \) relative to some \( x \in S^1 \), and for every \( n \in \mathbb{N} \), the element \( (E_n^n)^{-1} \in G \) has a holomorphic univalent extension \( (E_n^n)^{-1} \) to the ball \( B(x_n, r) \) of center \( x_n = E_n^x(x) \).

Proof. Choose a real number \( r > 0 \) small enough, so that the following condition holds: every \( s \in S \) extends as a univalent holomorphic map \( \tilde{s} \) defined on the \( r \)-neighborhood \( A_r \) of \( S^1 \) in \( \mathbb{C}/\mathbb{Z} \), and moreover for every \( y \in S^1 \)

\[
\kappa(\tilde{s}, B(y, r)) \leq c. \tag{11}
\]

Let \( 0 \leq m \leq n \) be an integer. Since \( (E_n^n)_{n \geq 0} \) is a \( c \)-line of expansion at \( x \), we have that \( |D(s_m^n)^{-1}(x_m)| \leq e^{-c} \). So (11) shows that for every \( y \in B(x_m, r) \)

\[
|D(s_m^n)^{-1}(y)| \leq e^c \cdot |D(s_m^n)^{-1}(x_m)| \leq 1.
\]

Hence, the ball \( B(x_m, r) \) is sent by \( (s_m^n)^{-1} \) inside the ball \( B(x_{m-1}, r) \).

Since \( (E_n^n)^{-1} := (s_1^n)^{-1} \ldots (s_m^n)^{-1} \), it has a holomorphic extension to \( B(x_n, r) \), which maps this latter to \( B(x, r) \).

To end this paragraph, recall the following distortion estimate, due to Koebe [9].

Lemma 4.3 (Koebe). There is a constant \( \kappa > 0 \) such that the following holds. Let \( z_1, z_2 \) be points of \( \mathbb{C} \), \( r > 0 \) be a positive real number, and \( g \) be a univalent nonconstant holomorphic map from \( B(z_1, r) \) to \( \mathbb{C} \) sending \( z_1 \) to \( z_2 \). Then for every \( 0 \leq r' \leq r/2 \),

\[
\kappa(g, B(z_1, r')) \leq \kappa.
\]

Moreover, \( g(B(z_1, r')) \) contains the ball \( B(z_2, r' e^{-\kappa}|Dg(z_1)|) \).

5. Local discreteness

For the next definition, we think of the circle as being the one-dimensional submanifold \( S^1 = \mathbb{R}/\mathbb{Z} \) contained in the Riemann surface \( \mathbb{C}/\mathbb{Z} \).

Definition 5.1. A subgroup \( G \subset \text{Diff}^\omega(S^1) \) is nonlocally discrete in the analytic category at a point \( x \in S^1 \), if there exists a neighborhood \( V \subset \mathbb{C}/\mathbb{Z} \) of \( x \) and a sequence \( \{g_n\}_{n \geq 1} \) of elements of \( G \setminus \{\text{id}\} \) that extend as univalent holomorphic maps from \( V \) to \( \mathbb{C}/\mathbb{Z} \) and whose extensions to \( V \) tend uniformly to the identity on \( V \) when \( n \) tends to infinity. Otherwise, \( G \) is locally discrete in the analytic category at the point \( x \).

We have the following simple characterization of the local discreteness of \( G \) everywhere in the analytic category. For every \( x \in \mathbb{C}/\mathbb{Z} \), and every nonnegative real number \( r \), we denote by \( B(x, r) \) the ball of radius \( r \) centered at \( x \) in \( \mathbb{C}/\mathbb{Z} \), for the euclidean distance.
Lemma 5.2. Let $G \subset \Diff^\omega(S^1)$ be a finitely generated subgroup, which is locally discrete in the analytic category at every point. Let $S$ be a finite symmetric generating set for $G$. Then, given numbers $r > 0$ and $a < b$, there is an integer $\gamma \in \mathbb{N}^*$ such that the following holds. Let $y \in S^1$ and $f \in G$. Suppose that $f$ can be extended as a univalent holomorphic map $\tilde{f}$ defined on the ball $B(y, r)$ that satisfies $a \leq \log |D\tilde{f}| \leq b$ on $B(y, r)$. Then $f$ is a composition of at most $\gamma$ elements of $S$, namely, $\|f\| \leq \gamma$.

Proof. By contradiction, suppose that this is not true. Then there is a sequence $(f_k)_k$ of elements of $G$ and a sequence $(y_k)_k$ of points of $S^1$ such that for every $k$, $f_k$ has a holomorphic univalent extension $\tilde{f}_k$ defined on $B(y_k, r)$ whose derivative on this latter satisfies $a \leq \log |D\tilde{f}_k| \leq b$, and whose word-length tends to infinity. Taking a subsequence if necessary, we can suppose that $(y_k)_k$ converges to a point $y \in S^1$ when $k$ tends to infinity. Then, the maps $\tilde{f}_k$ are defined on the ball $B(y, r/2)$ when $k$ is large enough. Since logarithm of derivatives are bounded on $B(y, r/2)$, Montel’s theorem shows that, taking a subsequence if necessary, the maps $\tilde{f}_k$ converge uniformly to a holomorphic univalent map $\tilde{f} : B(y, r/4) \to \mathbb{C}/\mathbb{Z}$. In particular, the maps $\tilde{f}_k \circ \tilde{f}_{k+1}^{-1}$ are well defined on $B(y, r/8)$ if $k$ is large enough, and converge to the identity in the uniform topology when $k$ tends to infinity. The discreteness assumption implies that for $k$ large enough, $f_{k+1} = f_k$; this contradicts the fact that the word-length of $f_k$ tends to infinity.

The following result shows that for a finitely generated expanding group of analytic diffeomorphisms of the circle, being locally discrete in the analytic category somewhere is the same notion as being locally discrete in the analytic category everywhere.

Lemma 5.3. If a finitely generated subgroup $G \subset \Diff^\omega(S^1)$ is expanding, then it is either nonlocally discrete in the analytic category everywhere, or locally discrete in the analytic category everywhere.

Proof. The set of points where the group is nonlocally discrete in the analytic category is obviously an open set. Its complement, the set where the group is locally discrete in the analytic category, is a closed invariant subset that we denote $\Lambda$.

Suppose by contradiction that both sets ($\Lambda$ and its complement) are nonempty.

Claim 1. The group $G$ is not virtually abelian.

Proof. Otherwise, it would not be abelian. Indeed, by contradiction, suppose that $G$ is an abelian finitely generated subgroup of $\Diff^\omega(S^1)$ which is expansive.

The rotation number function $\text{rot} : G \to \mathbb{R}/\mathbb{Z}$ (see, for example, the survey [6, p. 349] for its definition and its basic properties) is a morphism since $G$ is abelian. If this function takes an irrational value, the group $G$ preserves a unique nonatomic probability measure of full support on the circle, which is incompatible with the expanding assumption (see the combination of Lemma 4.2 and Koebe’s Lemma) and leads to a contradiction.

Hence, the morphism $\text{rot}$ takes only rational values, so considering a subgroup of finite index, if necessary (this operation does not alter the expanding property), we can assume that each element of $G$ has vanishing rotation number, or equivalently each of them has a nontrivial set of fixed points. By the analytic nature of the action, any nontrivial element $f$ of $G$ has a finite set of fixed points, denoted by $\text{Fix}(f)$.

We claim that for a pair $f, g \in G$ of nontrivial elements, we have $\text{Fix}(f) = \text{Fix}(g)$. Indeed, since $g$ commutes with $f$, the set $\text{Fix}(f)$ is invariant by $g$. The map $g$ acts as a cyclic permutation on $\text{Fix}(f)$, and has vanishing rotation number. Hence, $g$ is the identity on $\text{Fix}(f)$, or in other
words, \( \text{Fix}(f) \subset \text{Fix}(g) \). Since the opposite inclusion is also valid by exchanging the role of \( f \) and \( g \), we infer \( \text{Fix}(f) = \text{Fix}(g) \).

Let \( I \) be a connected component of the complement of the common set of fixed points of elements of \( G \). The interval \( I \) is invariant by \( G \) and the action of \( G \) on \( I \) is free. Hence, \( G \) preserves a Radon measure of full support on \( I \), by a theorem of H"older (see, for example, [6, p. 376]). Given \( x \in I \), Koeb’s Lemma shows that the sequence \( x_n = E_x^n(x) \) given by Lemma 4.2 sends an interval of size exponentially small in term of \( n \) around \( x \) to an interval of bounded size around \( x_n \). This contradicts the existence of an invariant Radon measure of full support on \( I \), and ends the proof of the claim.

\[ \square \]

**Claim 2.** The finite orbits are contained in the nonlocally discrete part, or equivalently \( \hat{\Lambda} \) does not contain any finite orbit.

**Proof.** Indeed, by the expansiveness assumption any point \( p \) in a finite orbit is a hyperbolic fixed point of some element of \( G \) which takes the form \( f : z \mapsto \lambda z \) with \( |\lambda| < 1 \) in some linearizing coordinates with \( z(p) = 0 \). In particular, since \( G \) is not virtually abelian, the stabilizer of \( p \) contains an element tangent to the identity at \( p \) of the form \( g : z \mapsto z + a z^k + \cdots \) with \( a \neq 0 \) and \( k \geq 2 \). Now, we have for a positive integer \( n \)

\[ f^{-n} \circ g \circ f^n = z + a\lambda^{(k-1)n} z^k + \cdots \] (12)

which shows that there exists a neighborhood of \( z = 0 \) in \( \mathbb{C} \) where \( f^{-n} \circ g \circ f^n \) converges to the identity when \( n \) tends to \( +\infty \). This proves our claim. \( \square \)

The set \( \hat{\Lambda} \) contains a minimal nonempty compact invariant subset \( \Lambda \) which is strict and not reduced to a finite orbit by the claim. Such a set is called an exceptional minimal set. A result of Hector [8] shows that the stabilizers of the components of the complement of an exceptional minimal set are virtually cyclic. Hence, every point in the complement of \( \Lambda \) is locally discrete in the analytic category (in fact in the compact open topology). In particular, the group is locally discrete everywhere, and that contradicts the fact that the complement of \( \hat{\Lambda} \) is nonempty. The result follows. \( \square \)

In view of this result, we will call an expanding group **locally discrete in the analytic category** if it is locally discrete in the analytic category at some point, and hence at every point.

6. **Convergence of lines of expansion in the Cayley graph**

**Proposition 6.1.** Let \( G \subset \text{Diff}^\omega(S^1) \) be a finitely generated subgroup which is expanding and locally discrete in the analytic category. Let \( S \) be a finite symmetric subset of generators of \( G \),

\[ \tau := \sup_{x \in S^1, s \in S} \log |Ds(x)|, \]

and let \( c > 0 \) be some constant. Then, there are constants \( \gamma_1, \gamma_2, \gamma_3 > 0 \) such that the following holds. Let \( x \in S^1 \), and \( (E^x_m)_m, (F^x_n)_n \) be \( c \)-lines of expansion relative to the point \( x \). Then for every nonnegative integers \( n, m \geq \gamma_1 d(E^x_0, F^x_0) + \gamma_2 \), such that

\[ D_x(E^x_m, F^x_n) \leq c, \] (13)

we have \( d(E^x_m, F^x_n) \leq \gamma_3 \).
Proof. Moving $x$ to $E_0^x(x)$ if necessary, we can assume $E_0^x = e$ (see Remark 3.4). We will then write $g := F_0^x$, $y = g(x)$ and $E_n^x := F_n^x \circ g^{-1}$. The sequence $\{E_n^x\}_{n \geq 0}$ is then a c-line of expansion relative to the point $y$, which explains the notation. Observe then that $\|g\| = d(E_0^x, F_0^x)$. We denote $x_m := E_m^x(x)$ and $y_n := F_n^x(x) = E_n^x(y)$.

The map $(E_m^x)^{-1}$ is defined on $B(x_m, r)$, where $r$ is the constant given by Lemma 4.2, and Koebe’s Lemma shows
\[
\kappa((E_m^x)^{-1}, B(x_m, r/2)) \leq \kappa,
\]
and
\[
(E_m^x)^{-1}(B(x_m, r/2)) \subseteq B\left(x, \frac{e^\kappa r}{2|DE_m^x(x)|}\right).
\]

Since the map $g$ is a composition of $\|g\|$ elements of $S$, it has a univalent holomorphic extension $\tilde{g}$ defined on the ball $B(x, re^{-\frac{(\|g\|-1)\sigma}{2}})$, that takes values in the ball $B(y, r)$. Koebe’s Lemma shows
\[
\kappa\left(\tilde{g}, B\left(x, re^{-\frac{(\|g\|-1)\sigma}{2}}\right)\right) \leq \kappa.
\]
In order that the composition $\tilde{g} \circ (E_m^x)^{-1}$ be defined on $B(x_m, r/2)$ with distortion bounded by $2\kappa$, a sufficient condition is then
\[
\frac{e^\kappa r}{2|DE_m^x(x)|} \leq re^{-\frac{(\|g\|-1)\sigma}{2}},
\]
or equivalently
\[
\log |DE_m^x(x)| \geq (\|g\| - 1)\sigma + \kappa.
\]
In this case, we have for every $r' \leq r/2$
\[
\tilde{g} \circ (E_m^x)^{-1}(B(x_m, r')) \subseteq B(y, e^{2\kappa r'}|D(g \circ (E_m^x)^{-1})(x_m)|),
\]
because $(E_m^x)_m$ is a c-line of expansion. Observe that condition (15) is fulfilled if
\[
m \geq \gamma_1\|g\| + \gamma_2, \text{ where } \gamma_1 = \sigma/c, \ \gamma_2 = \frac{\kappa - \sigma}{c}.
\]
Similarly, the map $(E_n^y)^{-1}$ is defined on the ball of radius $r$, and we have
\[
B\left(y, \frac{e^{-\kappa r}}{2|DE_n^y(y)|}\right) \subseteq (E_n^y)^{-1}(B(y_n, r/2))
\]
and
\[
\kappa((E_n^y)^{-1}, B(y_n, r/2)) \leq \kappa.
\]
Then the extension $E_n^y$ is well defined and univalent on the ball $B(y, \frac{e^{-\kappa r}}{2|DE_n^y(y)|})$ and its distortion is bounded by
\[
\kappa\left(E_n^y, B\left(y, \frac{e^{-\kappa r}}{2|DE_n^y(y)|}\right)\right) \leq \kappa.
\]
In particular, for every $r' \leq r/2$, we have, under the condition
\[
e^{2\kappa r'}|D(g \circ (E_m^x)^{-1})(x_m)| \leq \frac{e^{-\kappa r}}{2|DE_n^y(y)|},
\]
(17)
or equivalently

\[ 3\kappa + D_x(E^x_m, E^y_n \circ g) \leq \log(r/2r'), \quad (18) \]

that the composition \( \tilde{E}_y \circ g \circ (E^x_m)^{-1} \) is defined on the ball \( B(x_m, r') \) and its distortion is bounded by \( 3\kappa \). Condition \( (18) \) is satisfied if \( \log(r/2r') \geq 3\kappa + \varepsilon \).

With this choice of \( r' > 0 \), we can then apply Lemma 5.2 with \( \gamma_3 = \gamma \) to get the conclusion.

\[ \Box \]

7. Gromov hyperbolicity of \( G \)

PROPOSITION 7.1. Let \( G \subset \text{Diff}^\omega(S^1) \) be a finitely generated subgroup which is expanding and locally discrete in the analytic category. Then, \( G \) is Gromov hyperbolic.

Proof. We will use the following result \([10, \text{Theorem 2.11}]\) of Nekrashevych.

Fix \( x \in S^1 \), and denote by \( \Gamma^x \) the directed graph whose vertices are the elements of \( G \), and whose directed edges are the couples \( (g_0, g_1) \in G^2 \) such that \( D_x(g_0, g_1) \geq c/2 \) and \( d(g_0, g_1) \leq 2 \),

\[ \text{(19)} \]

where \( \varepsilon > 0 \) is the constant given by Remark 3.2. The set \( G \) is equipped with the combinatorial metric \( d_{\Gamma^x} \) induced by the underlying undirected graph induced by \( \Gamma^x \): \( d_{\Gamma^x}(g_1, g_2) \) is the minimum number of undirected edges of \( \Gamma^x \) necessary to go from \( g_1 \) to \( g_2 \). Hence, the combination of Proposition 6.1 and of \([10, \text{Theorem 1.2.9}]\) with the following constants \( \eta = c/4 \), \( \Delta = \varepsilon \), show that \( \Gamma^x \) equipped with its distance \( d_{\Gamma^x} \) is Gromov hyperbolic.

Claim. The inclusion \((G, d_{\Gamma^x}) \to (G, d)\) is a quasi-isometry

Proof. From the definition of \( \Gamma^x \), we immediately have

\[ d \leq 2d_{\Gamma^x}. \quad \text{(20)} \]

Let \( \{g_1, g_2\} \) be an edge of \( G \), that is, \( g_2 \in Sg_1 \). We can assume that \( D_x(g_1, g_2) \geq 0 \) up to exchanging \( g_1 \) and \( g_2 \). If \( D_x(g_1, g_2) \geq c/2 \), then the directed arrow \( g_1 \to g_2 \) belongs to \( \Gamma^x \). If not, let \( s \in S \) be an element such that \( D_x(g_1, sg_1) \geq \varepsilon \) given by Remark 3.2. Then \( \Gamma^x \) contains the directed edges: \( g_1 \to sg_1 \) (by definition of \( s \)) and \( g_2 \to sg_1 \). So, the \( d_{\Gamma^x} \)-distance between \( g_1 \) and \( g_2 \) in the graph is bounded by 2. In particular, we get

\[ d_{\Gamma^x} \leq 2d. \quad \text{(21)} \]

The claim follows from equations \( (20) \) and \( (21) \). \( \Box \)

The proposition follows from the claim and the fact that Gromov hyperbolicity is a quasi-isometric invariant. \( \Box \)

8. The group \( G \) is virtually a Fuchsian group

A consequence of Proposition 7.1 is that we have a well-defined map

\[ \Omega : S^1 \to \partial G, \quad \text{(22)} \]

which associates to a point \( x \in S^1 \) the equivalence class in \( \partial G \) of a \( \varepsilon \)-line of expansion at the point \( x \) (here \( \varepsilon \) is the constant appearing in Remark 3.2). Indeed, Proposition 6.1 shows that two such lines of expansion are at a bounded Hausdorff distance from each other.
 Proposition 8.1. The map $\Omega: S^1 \to \partial G$ is a finite covering.

Proof. The proof is organized as a sequence of claims.

Claim. The map $\Omega: S^1 \to \partial G$ is equivariant and continuous.

Proof. The equivariance is immediate. Let us prove the continuity at a point $x \in S^1$. Suppose $\{E^x_m\}_m$ is a $c$-line of expansion at $x$, where $m > 0$ is the constant given by Remark 3.2. Let $m_0$ be a large integer. For $y \in S^1$ in a sufficiently small neighborhood of $x$, denoting $y_m := E^x_m(y)$, and recalling that $E^x_m = s^x_m \circ E^x_{m-1}$ (see Definition 3.5), we have

$$\log |D s^x_m(y)| \geq c/2$$

for every $0 \leq m \leq m_0$.

Hence, one can define a $c/2$-line of expansion $\{E^y_m\}_{m \geq 0}$ relative to $y$ by

$$E^y_m := E^x_m$$

and by taking for $\{E^y_m\}_{m \geq m_0}$ a $c$-line of expansion relative to $y$. The two $c/2$-lines of expansion $\{E^x_m\}_m$ and $\{E^y_m\}_m$ relative to $x$ and $y$, respectively, coincide for $m \leq m_0$ and converge, respectively, to $\omega_x$ and $\omega_y$. Hence, being $(\alpha,0)$-quasi-geodesics for a constant $\alpha$ depending only on $\tau$ and $S$ (see Lemma 3.7) their limit points $\omega_x$ and $\omega_y$ are close to each other in $\partial G$. This proves continuity of $\Omega$. \hfill \Box

Claim 1. There are constants $c, d > 0$ such that, for every $x \in S^1$, any geodesic ray $\{g_n\}_{n \geq 0}$ in $G$ starting at $g_0 = e$ and tending to $\omega \in \partial G$ satisfies

$$D_x(e, g_n) \geq cn - d \quad \text{for} \quad n \leq (\Omega(x), \omega)_c$$

and

$$D_x(e, g_n) \leq -cn + \tau \cdot (\Omega(x), \omega)_c + d \quad \text{for} \quad n \geq (\Omega(x), \omega)_c.$$

Proof. Let $\{E^x_k\}_{k \geq 0}$ and $\{F^x_m\}_{m \geq 0}$ be $c$-lines of expansion relative to $x$ beginning at $E^x_0 = e$ and $F^x_0 = g_n$, respectively. By Proposition 6.1, there exist integers $L$ and $M$ such that

$$d(E^x_k, F^x_M) \leq \gamma_3.$$  

Let $\alpha$ be given by Lemma 3.7, and let $\beta = \gamma_3$. The $(\alpha, \beta)$-quasi-geodesic triangle formed by the segments

$$\{g_s\}_{0 \leq s \leq n}, \quad \{E^x_k\}_{0 \leq k \leq L} \quad \text{and} \quad \{F^x_m\}_{0 \leq m \leq M},$$

is $\delta$-thin for a certain constant $\delta$ depending only on $(\alpha, \beta)$ and the hyperbolicity constants of $(G, d)$. So, for some integer $N$, the segments $\{g_s\}_{0 \leq s \leq (\Omega(x), \omega)_c}$ and $\{g_s\}_{n \leq s \geq (\Omega(x), \omega)_c}$ are $\delta$-close to $c$-segments of expansion relative to $x$. The conclusion follows. \hfill \Box

Claim 2. There is a number $M \in \mathbb{N}^*$ such that the level subsets $\Omega^{-1}(\omega)$, for $\omega \in \partial G$, have cardinality less than $M$.

Proof. Indeed, let $x^k \in \Omega^{-1}(\omega)$, $k = 1, \ldots, M$, be distinct points, and let $\{g_n\}_n$ be a geodesic ray from $e$ to $\omega \in \partial G$. For every $k = 1, \ldots, M$, the sequence $\{g_n\}_n$ is $O(\delta)$-close to a $c$-line of expansion $\{E^x_{m_k}\}_{m \geq 0}$ relative to $x^k$ and starting at $E^x_{m_k} = e$.

Let $n$ be a large integer. For every $k = 1, \ldots, M$, there exists $m_k$ such that $d(g_n, E^x_{m_k}) = O(\delta)$. Let $r > 0$ be the constant given by Lemma 4.2. Denoting $x^k_n = E^x_{m_k}(x^k)$ for $n \geq 0$ and $k = 1, \ldots, M$, Lemma 4.2 shows that the map $(E^x_{m_k})^{-1}$ extends as a univalent map defined on $B(x^k_n, r/2)$ whose image lies in the ball $B(x^k_n, r')$. In particular, there exists $r' > 0$
developed by Ghys in 

It is presumably well known that Corollary 8.2, together with the differentiable rigidity theory

□

a Cantor set.

is cocompact since otherwise the group $G$ to the circle, hence the result follows from the convergence group theorem $2, 4$

$\square$

$(G$ and locally discrete in the analytic category. Then $G$ is topologically conjugate to a finite group on the circle. Using the same notations as in Claim 1, the Hausdorff distance between $\Omega^{-1}(\omega^k)$ and the set $\{\log |Dg_n| \geq 0\}$ is less than $\varepsilon$ for every $n \geq n(\varepsilon)$, for every $k \in \mathbb{N} \cup \{\infty\}$. Applying this to $k \geq k(n(\varepsilon))$, we get that the Hausdorff distance between $\Omega^{-1}(\omega^k)$ and $\Omega^{-1}(\omega^\infty)$ is less than $2\varepsilon$, which proves the claim.

□

Claim 3. The map $\omega \in \partial G \mapsto \Omega^{-1}(\omega) \in K$ is continuous.

Proof. Let $\{\omega^k\}_k$ be a sequence of points of $\partial G$ tending to $\omega^\infty \in \partial G$. For every $k \in \mathbb{N}$, let $\{g^k_n\}_{n \geq 0}$ be a geodesic ray tending to $\omega^k$. Up to extracting if necessary, one can assume that for each $n$, the sequence $\{g^k_n\}_{n \geq 0}$ is stationary, namely for $k \geq k(n)$, $g^k_n = g^\infty_n$. The sequence $\{g^\infty_n\}_n$ is a geodesic ray tending to $\omega^\infty$. Using the same notations as in Claim 1, the Hausdorff distance between $\Omega^{-1}(\omega^k)$ and the set $\{\log |Dg^k_n| \geq 0\}$ is less than $\varepsilon$ for every $n \geq n(\varepsilon)$, for every $k \geq k(\varepsilon)$, and the claim follows.

□

Claim 4. The function $k : \omega \in \partial G \mapsto |\Omega^{-1}(\omega)| \in \mathbb{N}$ is constant.

Proof. Note that $G$ cannot be virtually cyclic since otherwise its action on the circle could not be expanding. By [7, Chapitre 8], it acts minimally on its boundary. Claim 3 shows that $k$ is lower semi-continuous, and it is $G$-invariant. In particular, the subset $\{k = \min k\} \subset \partial G$ is closed, nonempty, and $G$-invariant. The action of $G$ on $\partial G$ being minimal, it must be the whole $\partial G$, hence the conclusion holds.

□

Claims 2, 3 and 4 show that $\Omega$ is a covering.

□

Corollary 8.2. The action of $G$ on $S^1$ is topologically conjugated to the action of a (cocompact) lattice of $\text{PGL}_2^k(\mathbb{R})$ on the kth covering $\mathbb{R}P^1_k$ for a certain integer $k > 0$.

Proof. By Proposition 8.1, the action of $G$ on $S^1$ is topologically conjugate to a finite covering of its action on its boundary. This proves that the boundary of $G$ is homeomorphic to the circle, hence the result follows from the convergence group theorem [2, 4]. The lattice is cocompact since otherwise the group $G$ would be virtually free, and its boundary would be a Cantor set.

□

9. Differentiable rigidity

It is presumably well known that Corollary 8.2, together with the differentiable rigidity theory developed by Ghys in [5], implies our main result, that is, Corollary 9.2. However, this implication is not directly stated in this form in the literature, so we provide a detailed proof below.

Proposition 9.1. Let $G \subset \text{Diff}^+(S^1)$ be a finitely generated subgroup which is expanding and locally discrete in the analytic category. Then $G$ preserves an analytic $\mathbb{R}P^1$-structure.

Proof. We first observe that we can assume that the group $G$ preserves the orientation on $S^1$. Indeed, suppose that we know that the subgroup $G^+$ of elements of $G$ that preserves the
orientation on \( S^1 \) preserves an analytic projective structure \( \sigma \) on \( S^1 \). If \( G^+ = G \) we are done. If not, let \( g \) be an element of \( G \) that reverses the orientation. Since \( G \) is generated by \( G^+ \) and \( g \), it suffices to prove that \( g \) preserves the projective structure \( \sigma \) as well. For this, let us consider the quadratic differential defined by \( S(g) := \{g, z\}dz^2 \), where \( \{g, z\} \) is the Schwarzian derivative \( \{g, z\} = \frac{D^3 g}{Dz^3} - \frac{3}{2} \frac{(D^2 g)^2}{(Dz)^2} \) computed in projective coordinates of \( \sigma \). The cocycle relations satisfied by the Schwarzian derivative shows that \( S(g) \) is a well-defined quadratic differential on the circle, namely it is independent of the chosen projective coordinates. We need to prove that \( S(g) = 0 \) identically. For every \( f \in G^+ \), there exists \( h \in G^+ \) such that \( g \circ f = h \circ g \). The cocycle relation satisfied by the Schwarzian derivative, together with the fact that both \( f \) and \( h \) preserve \( \sigma \), implies \( S(g) = f^* S(g) \), hence proving that \( S(g) \) is invariant by the whole group \( G^+ \). In particular, if \( S(g) \) does not vanish identically, the volume \( \sqrt{|S(g)|} \) is \( G^+ \)-invariant, which contradicts the fact that \( G^+ \) is expanding on the support of \( S(g) \).

So, from now on we will assume that \( G \) preserves orientation on \( S^1 \). Denote by \( \phi : \partial G \to \mathbb{R}P^1 \) a homeomorphism that conjugates the \( G \)-action on its boundary to the action of a Fuchsian group \( \Gamma \subset \text{PSL}_2(\mathbb{R}) \) on \( \mathbb{R}P^1 \) (see Corollary 8.2 for the existence of \( \phi \)). We can assume that \( \phi \) preserves orientation, so that the lattice \( \Gamma \) is indeed contained in \( \text{PSL}_2(\mathbb{R}) \). We denote by \( \rho : G \to \Gamma \) the representation that satisfies

\[
\phi(g\omega) = \rho(g)\phi(\omega) \quad \text{for every} \quad g \in G, \ \omega \in \partial G.
\]

Let \( U \subset S^1 \times \mathbb{R}P^1 \) be the complement of the graph of \( \Phi = \phi \circ \Omega \), namely, the set of points \((x, z) \in S^1 \times \mathbb{R}P^1 \) such that \( z \neq \Phi(x) \). Consider the analytic action of \( G \) on \( U \times \mathbb{R} \) defined by:

\[
g(x, z, t) = (g(x), \rho(g)(z), t + \log |Dg(x)|).
\]  

**Claim.** The action (23) is free, proper discontinuous and cocompact. Hence, the quotient of \( U \) by \( G \) is a closed analytic 3-manifold \( M \).

**Freeness.** Assume that there exists a point \((x, z, t)\) which is fixed by an element \( g \in G \). The theory of Gromov hyperbolic groups (see, for example, [7, Chapitre 8, §3]) tells us that \( g \) is either of finite order, or a hyperbolic element, in which case the sequence \( \{g^n\}_{n \in \mathbb{N}} \) is a quasi-geodesic. Assume by contradiction that \( g \) is hyperbolic. It has a fixed point \( x \) on \( S^1 \) having a derivative equal to 1 at \( x \) — recall that we assume the orientation is preserved by \( G \) — so all the numbers \( D_x(e, g^n) \) are zero. This contradicts the Claim 1 of the proof of Proposition 8.1 since the sequence \( \{g^n\}_{n \geq 0} \) is a quasi-geodesic. This contradiction shows that \( g \) has finite order in \( G \). Its image \( \rho(g) \) is an element of \( \text{PSL}_2(\mathbb{R}) \) of finite order that must be the identity since it fixes the point \( z \in \mathbb{R}P^1 \). Hence, \( g \) lies in the kernel of \( \rho \) which is a cyclic subgroup of \( G \) acting freely on \( S^1 \). Since \( g \) fixes the point \( x \), it is therefore the identity map. Hence, the action is free as claimed.

**Proper discontinuity.** Any compact set in \( U \) is contained in some compact set of the sort

\[
K := \{(x, z, t) \in U \times \mathbb{R} \mid (\Omega(x), \phi^{-1}(z))_c \leq C \text{ and } |t| \leq T\},
\]

for some constants \( C, T \). We must prove that there is only a finite number of elements \( g \) of \( G \) such that \( gK \cap K \neq \emptyset \). Let \( g \in G \), and suppose that for some \((x, z, t) \in K \) we have \( g(x, z, t) \in K \). Since both \((\Omega(x), z)_c\) and \((g\Omega(x), g\phi^{-1}(z))_c\) are bounded by \( C \), \( g \) must lie at a distance from a geodesic between \( \phi^{-1}(z) \) and \( \Omega(x) \) bounded by some constant depending only on \( C \) and the constants of hyperbolicity of \((G, d)\). But a geodesic from \( \phi^{-1}(z) \) to \( \Omega(x) \) lies at a finite distance from a \( c \)-line of expansion between \( \phi^{-1}(z) \) and \( \Omega(x) \) relative to \( x \), the Hausdorff distance being bounded by some constant depending only on \( c \) and \( G, S \). Hence, we have \(|\log |Dg(x)||| \geq c\|g\| + cst \) for some positive constants depending only on \( c \) and \( G \). However, \(|\log |Dg(x)||| \leq 2T\) since both \((x, z, t)\) and \( g(x, z, t) \) lie in \( K \). This proves that the norm of
g is bounded, hence there is only a finite number of \( g \in G \) such that \( gK \cap K \neq \emptyset \), and the properness of the action of \( G \) on \( U \times \mathbb{R} \) follows.

**Cocompactness.** The cocompactness of the action of \( G \) on \( U \times \mathbb{R} \) follows from cohomological reasons, but it is instructive to prove it by hands. We will prove that any point \((x,z,t) \in U \times \mathbb{R}\) can be sent by an element of \( G \) to a point \((x'',z'',t'')\) belonging to some compact set defined by equation (24), for some constants \(C,T\) that depend only on \( G \).

Let us first find \( g \in G \) such that \( \rho(g)(\Phi(x)) \) and \( \rho(g)(z) \) are separated by a constant that depends only on \( G \). Let \( \{g_n\}_{n \geq 0} \) be a geodesic ray on \( G \) tending to \( \Omega(x) \). A consequence of Lemma 4.2 is that, if \( n \) is large enough, so that \( |Dg_n(x)| \) is larger than the inverse of the distance between \( x \) and \( \Phi^{-1}(z) \), the distance between \( g_n(x) \) and the set \( \Phi^{-1}(\rho(g_n)z) \) is larger than some constant depending only on \( G \). In particular, the points \( \Phi(g_n(x)) \) and \( \rho(g_n)(z) \) are separated by a constant that depends only on \( G \). Similarly, suppose that a geodesic ray \( \{g_n\}_{n \geq 0} \) tends to \( \phi^{-1}(z) \) when \( n \) tends to infinity. Then, the sequence \( \{g_n\}_{n \geq 0} \) lies at a finite distance from a line of expansion at \( z \) (for the \( p \)-action of \( G \) on \( \mathbb{R}P^1 \) given by \( \rho \), which is locally discrete and expansive), and the same argument applies: namely, for \( n \) large enough, the distance between \( \rho(g_n)(\Omega(z)) \) and \( \rho(g_n)(z) \) is bounded from below by some constant depending only on \( G \). So, we are done.

Now write \((x',z',t') = g(x,z,t)\), with \( \Omega(x') \) and \( z' \) separated by a constant depending only on \( G \). If \( t' \leq 0 \), then the \( t''\)-variable of the point \((x'',z'',t'') := E_n'(x',z',t')\) grows linearly with \( n \), while keeping the distance between \( \Phi(x'') \) and \( z'' \) separated by a constant depending only on \( G \). Hence, for some \( n \) the point \((x'',z'',t'')\) belongs to the compact set defined in equation (24) for some constants \(C,T\) depending only on \( G \). If \( t'' > 0 \), let \( \{g_n\}_{n \geq 0} \) be a geodesic ray beginning at \( g_0 = e \) and tending to \( \phi^{-1}(z') \) when \( n \) tends to \( \infty \). Because \( \Phi(x') \) and \( z' \) are separated by some constant depending only on \( G \), the Gromov product \( (\Omega(x'),\phi^{-1}(z'))_{e} \) is bounded by some constant depending only on \( G \) as well, and the Claim 1 of the proof of the Proposition 8.1 shows that \( \log |Dg_{n}(x')| \) decreases linearly when \( n \) tends to \( \infty \). Letting \((x'',z'',t'') := g_n(x',z',t')\), and reasoning as before, we infer that the Gromov product \( (\Omega(x''),\phi^{-1}(z''))_{e} \) remains bounded by some constant depending only on \( G \), and for some \( n \), the \( t''\)-coordinates enters in some fixed interval of the form \([-T,T]\) for some constant \( T \) depending only on \( G \) as well. The conclusion follows.

**Claim 1.** The flow on \( M \) induced by the nonsingular vector fields \( V := \frac{\partial}{\partial t} \) is Anosov, and its weak stable foliation \( \mathcal{F}^+ \) is given by the equation \( dx = 0 \).

**Proof.** Let \( \Delta \subset U \times \mathbb{R} \) be a fundamental domain for the \( G \)-action given by (23). Let \((x,z,t) \in \Delta \) be some point, and \( s \) be a negative real number. There exists an element \( g_s \) of \( G \) such that \( g_s(x,z,t) \in \Delta \). This element \( g_s \) can be chosen as the \( n \)th term of an expansive line at the point \( x \) beginning at \( e \), with \( n \) growing linearly with \( -s \) and constants depending only on \( G \). Hence, \( \log |Dg_s(x)| \) (respectively, \( \log |D\rho(g_s)(z)| \)) increases linearly to \( +\infty \) with \( -s \) (respectively, decreases linearly to \( -\infty \) with \( -s \)). As a consequence: when \( s \) tends to \( +\infty \), the flow \( \exp(sV) \) contracts exponentially a metric on the bundle \( T\mathcal{F}^-/\mathbb{R}V \), where \( \mathcal{F}^- \) is the foliation defined on the covering \( U \) by \( z = cst \), whereas it expands exponentially a metric on the foliation \( \mathcal{F}^{++} \) defined on the covering \( U \) by \( (x,t) = cst \). The conclusion follows.

The weak unstable foliation \( \mathcal{F}^+ \) defined on the covering \( U \) by \( x = cst \) is analytic, hence a theorem of Ghys [5, Théorème 4.1] shows that \( \mathcal{F}^+ \) has an analytic transverse projective structure. This latter lifts to a transverse projective structure on the foliation of \( U \) defined by the submersion \( x : U \to \mathbb{S}^1 \), which is invariant by \( G \), hence gives an analytic projective structure on \( \mathbb{S}^1 \) which is invariant by \( G \). The proof of Proposition 9.1 is complete.
Corollary 9.2. A finitely generated subgroup of $\text{Diff}^\omega(S^1)$ which is expanding and locally discrete in the analytic category is analytically conjugated to a uniform lattice in $\widetilde{\text{PGL}}_2^k(R)$ acting on the $k$th covering of $RP^1$ for a certain integer $k > 0$.

Proof. By Proposition 9.1, the group $G$ preserves an analytic $RP^1$-structure on $S^1$. Let $D : R \to RP^1$ be a developing map for this structure: this is an analytic local diffeomorphism which is equivariant with respect to an element $A \in \text{PGL}_2(R)$, namely we have

$$D(\bar{x} + 1) = AD(\bar{x}) \text{ for every } \bar{x} \in R.$$  

(25)

The group $G$ lifts to a subgroup $\tilde{G} \subset \text{Diff}^\omega(R)$ that commutes with the translation $\bar{x} \mapsto \bar{x} + 1$. Since the $RP^1$-structure is invariant by $G$, there exists a representation $\tilde{\rho} : \tilde{G} \to \text{PGL}_2(R)$ which is such that

$$D \circ \tilde{g} = \tilde{\rho}(\tilde{g}) \circ D.$$  

(26)

We then have $A = \tilde{\rho}'(\bar{x} \mapsto \bar{x} + 1)$. Hence, $A$ commutes with $\tilde{\rho}'(\tilde{G})$. We will prove that $A = Id$. Assume by contradiction that $A$ has no fixed point on $RP^1$. In this case, $\tilde{G}$ is contained in a conjugate of $PO_2(R)$, and in particular preserves a measure on $RP^1$ with an analytic density. Its preimage by $D$ is invariant by $\tilde{G}$, in particular by the translation $\bar{x} \mapsto \bar{x} + 1$, hence produces on $S^1$ a measure invariant by $G$ that has an analytic density as well. This contradicts the expanding property for $G$. Assume by contradiction that $A$ has some fixed point but is not the identity. Then $D^{-1}(\text{Fix}(A))$ is a discrete $\tilde{G}$-invariant subset of $R$, and projects in $S^1$ to a $G$-invariant finite orbit. This is a contradiction since the image of this latter by the map $\Omega$ would be a finite $G$-orbit in $DG$. The only remaining possibility is that $A = Id$ as claimed.

In particular, the representation $\tilde{\rho}'$ induces a representation $\rho' : G \to \text{PGL}_2(R)$, and (25) shows that $D$ induces a finite covering from $S^1$ to $RP^1$ which is $\rho'$-equivariant. Since $G$ is a finitely generated, locally discrete and expanding subgroup of $\text{Diff}^\omega(S^1)$, the same is true for the image of $\rho'$, hence this latter is a uniform lattice in $\text{PGL}_2(R)$. The result follows. \qed

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