TOPOLOGICAL RADICALS OF SEMICROSSED PRODUCTS

G. ANDREOLAS, M. ANOUSSIS, AND C. MAGIATIS

ABSTRACT. We characterize the hypocompact radical of a semicrossed product in terms of properties of the dynamical system. We show that an element $A$ of a semicrossed product is in the hypocompact radical if and only if the Fourier coefficients of $A$ vanish on the closure of the recurrent points and the 0-Fourier coefficient vanishes also on the largest perfect subset of $X$.

1. Introduction and Preliminaries

Let $B$ be a Banach algebra. An element $a$ of $B$ is said to be compact if the map $M_{a,a} : B \to B, x \mapsto axa$ is compact. Following Shulman and Turovskii \[17, 3.2\] we will call a Banach algebra $B$ hypocompact if any nonzero quotient $B/J$ by a closed ideal $J$ contains a nonzero compact element. We will say that an ideal $J$ of a Banach algebra $B$ is hypocompact if it is hypocompact as an algebra. Shulman and Turovskii have proved that any Banach algebra $B$ has a largest hypocompact ideal \[17, Corollary 3.10\]. This ideal is closed and is called the hypocompact radical of $B$. We will denote it by $B_{hc}$.

The hypocompact radical of Banach algebras was studied within the framework of the theory of topological radicals \[17, 18\]. This theory originated with Dixon \[6\] and was further developed by Shulman and Turovskii in a series of papers \[13, 14, 15, 17, 18\] and by Kissin, Shulman and Turovskii \[16\]. The theory of topological radicals has applications to various problems of Operator Theory and Banach algebras.

It follows from \[4, Lemma 8.2\], that the hypocompact radical contains the ideal generated by the compact elements. If $X$ is a Banach space, we shall denote by $B(X)$ the Banach algebra of all bounded linear operators on $X$ and by $K(X)$ the Banach subalgebra of all compact operators on $X$. Vala has shown in \[19\] that an element $a \in B(X)$ is a compact element if and only if $a \in K(X)$. It follows that if $H$ is a separable Hilbert space, the hypocompact radical of $B(H)$ is $K(H)$. Indeed, the ideal $K(H)$ is the only proper ideal of $B(H)$ while the Calkin algebra $B(H)/K(H)$ does not have any non-zero compact element \[8, section 5\].

Shulman and Turovskii observe in \[17, p. 298\] that there exist Banach spaces $X$, such that the hypocompact radical $B(X)_{hc}$ of $B(X)$ contains all the weakly compact operators and contains strictly the ideal of compact operators $K(X)$.

Argyros and Haydon constructed in \[3\] a Banach space $X$ such that every operator in $B(X)$ is a scalar multiple of the identity plus a compact operator. It follows that $B(X)/K(X)$ is finite-dimensional and hence the hypocompact radical of $B(X)$ coincides with $B(X)$.

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A nest $\mathcal{N}$ on a Hilbert space $\mathcal{H}$ is a totally ordered family of closed subspaces of $\mathcal{H}$ containing $\{0\}$ and $\mathcal{H}$, which is closed under intersection and closed span. If $\mathcal{N}$ is a nest on a Hilbert space $\mathcal{H}$, the nest algebra associated to $\mathcal{N}$ is the (non selfadjoint) algebra of all operators $T \in \mathcal{B}(\mathcal{H})$ which leave each member of $\mathcal{N}$ invariant. The hypocompact radical of a nest algebra was characterized in [1].

We recall the construction of the semicrossed product we will consider in this work. Let $X$ be a locally compact metrizable space and $\phi : X \to X$ a homeomorphism. The pair $(X, \phi)$ is called a dynamical system. An action of $\mathbb{Z}_+$ on $C_0(X)$ by isometric $*$-automorphisms $\alpha_n$, $n \in \mathbb{Z}_+$ is obtained by defining $\alpha_n(f) = f \circ \phi^n$. We write the elements of the Banach space $\ell^1(\mathbb{Z}_+, C_0(X))$ as formal series $A = \sum_{n \in \mathbb{Z}_+} U^n f_n$ with the norm given by $\|A\|_1 = \sum_{n \in \mathbb{Z}_+} \|f_n\|_{C_0(X)}$. The multiplication on $\ell^1(\mathbb{Z}_+, C_0(X))$ is defined by setting

$$U^n f U^m g = U^{n+m}(a^n(f)g)$$

and extending by linearity and continuity. With this multiplication, $\ell^1(\mathbb{Z}_+, C_0(X))$ is a Banach algebra.

The Banach algebra $\ell^1(\mathbb{Z}_+, C_0(X))$ can be faithfully represented as a (concrete) operator algebra on a Hilbert space. This is achieved by assuming a faithful action of $C_0(X)$ on a Hilbert space $\mathcal{H}_0$. Then, we can define a faithful contractive representation $\pi$ of $\ell_1(\mathbb{Z}_+, C_0(X))$ on the Hilbert space $\mathcal{H} = \mathcal{H}_0 \otimes \ell^2(\mathbb{Z}_+)$ by defining $\pi(U^n f)$ as

$$\pi(U^n f)(\xi \otimes e_k) = \alpha^n(f)(\xi \otimes e_{k+n}).$$

The semicrossed product $C_0(X) \times_{\phi} \mathbb{Z}_+$ is the closure of the image of $\ell^1(\mathbb{Z}_+, C_0(X))$ in $\mathcal{B}(\mathcal{H})$ in the representation just defined, where $\mathcal{B}(\mathcal{H})$ is the algebra of bounded linear operators on $\mathcal{H}$. Note that the semicrossed product is in fact independent of the faithful action of $C_0(X)$ on $\mathcal{H}_0$ (up to isometric isomorphism) [7]. We will denote the semicrossed product $C_0(X) \times_{\phi} \mathbb{Z}_+$ by $A$ and an element $\pi(U^n f)$ of $A$ by $U^n f$ to simplify the notation. We refer to [12, 7, 5], for more information about the semicrossed product.

For $A = \sum_{n \in \mathbb{Z}_+} U^n f_n \in \ell^1(\mathbb{Z}_+, C_0(X))$, we call $f_n = E_n(A)$ the $n$th Fourier coefficient of $A$. The maps $E_n : \ell^1(\mathbb{Z}_+, C_0(X)) \to C_0(X)$ are contractive in the (operator) norm of $A$, and therefore they extend to contractions $E_n : A \to C_0(X)$. An element $A$ of the semicrossed product $A$ is 0 if and only if $E_n(A) = 0$ for all $n \in \mathbb{Z}_+$ and thus $A$ is completely determined by its Fourier coefficients. We will denote $A$ by the formal series $A = \sum_{n \in \mathbb{Z}_+} U^n f_n$, where $f_n = E_n(A)$. Note however that the series $\sum_{n \in \mathbb{Z}_+} U^n f_n$ does not in general converge to $A$ [12, II.9, IV.2 Remark].

In this paper we characterize the hypocompact radical of a semicrossed product in terms of properties of the dynamical system. We show that an element $A$ of a semicrossed product is in the hypocompact radical if and only if the Fourier coefficients of $A$ vanish on the closure of the recurrent points and the 0-Fourier coefficient vanishes also on the largest perfect subset of $X$.

## 2. The Hypocompact Radical

To obtain the characterization of the hypocompact radical of a semicrossed product we recall the following properties of the hypocompact radical of a Banach algebra proved by Shulman and Turovskii in [17].
Theorem 2.1. Let \( \mathcal{B} \) be a Banach algebra and \( \mathcal{I} \) a closed ideal of \( \mathcal{B} \).

1. If \( \mathcal{B} \) is hypocompact, then \( \mathcal{I} \) and \( \mathcal{B}/\mathcal{I} \) are hypocompact \([17]\) Corollary 3.9).
2. If \( \mathcal{I} \) and \( \mathcal{B}/\mathcal{I} \) are hypocompact, then \( \mathcal{B} \) is hypocompact \([17]\) Corollary 3.9).
3. Let \( p : \mathcal{B} \to \mathcal{B}/\mathcal{I} \) be the quotient map. Then \( p(\mathcal{B}_{nc}) \subseteq (\mathcal{B}/\mathcal{I})_{hc} \)[17] Corollary 3.13].

Let \( X \) be a locally compact metrizable space. We shall use the characterization of the hypocompact radical of \( C_0(X) \) which may be obtained using \([18]\) Corollary 8.19 & Theorem 8.22. We provide a proof for completeness.

A point \( x \in X \) is called \textit{accumulation point} of \( X \), if \( x \in X \setminus \{x\} \). The set of the accumulation points of \( X \) is denoted \( X_a \). If \( x \in X \setminus X_a \), then the point \( x \) is called an \textit{isolated point}. A subset \( Y \) of a topological space is said to be \textit{dense in itself}, if it contains no isolated points. If \( Y \) is closed and dense in itself, it is said to be a \textit{perfect set}. The set \( Y \) is said to be a \textit{scattered set}, if it does not contain dense in themselves subsets.

It is well known that every space is the disjoint union of a perfect and a scattered one, and this decomposition is unique \([9]\) Theorem 3, p 79]. If \( X \) is a locally compact metrizable space, we write \( X = X_p \cup X_s \) where \( X_p \) is the perfect set and \( X_s \) is the scattered set.

Theorem 2.2. If \( X \) is a locally compact metrizable space, then 
\[
C_0(X)_{hc} = \{ f \in C_0(X) : f(X_p) = \{0\} \}.
\]

Proof. Let \( \mathcal{I} \) be the ideal \( \{ f \in C_0(X) : f(X_p) = \{0\} \} \) of \( C_0(X) \). The ideal \( \mathcal{I} \) is isomorphic to \( C_0(X_a) \). We show that every non-zero quotient of \( \mathcal{I} \) by a closed ideal has a non-zero compact element. Let \( \mathcal{J} \) be a closed ideal of \( \mathcal{I} \) and \( S \) a closed subset of \( X_a \) such that \( \mathcal{J} = \{ f \in C_0(X_a) : f(S) = \{0\} \} \). The quotient algebra \( \mathcal{I}/\mathcal{J} \) is isomorphic to \( C_0(S) \). Hence it suffices to prove that the algebra \( C_0(S) \) has a non-zero compact element. Since the set \( S \) is contained in \( X_a \) it is scattered, and it contains an isolated point \( y \). Let \( \chi_{\{y\}} \) be the characteristic function of \( \{y\} \). Then, the operator \( M_{\chi_{\{y\}}, \chi_{\{y\}}} : C_0(S) \to C_0(S) \) is a rank-one operator and hence, \( \chi_{\{y\}} \) is a compact element of the algebra \( C_0(S) \). It follows that \( \mathcal{I} \subseteq C_0(X)_{hc} \).

We show now that \( \mathcal{I} = C_0(X)_{hc} \). Assuming that \( \mathcal{I} \neq C_0(X)_{hc} \) we will prove that the quotient algebra \( C_0(X)_{hc}/\mathcal{I} \) contains no non-zero compact elements. This implies that \( \mathcal{I} = C_0(X)_{hc} \) by Theorem 2.1. Let \( f \in C_0(X)_{hc} \setminus \mathcal{I} \). There exists \( x_p \in X_p \), such that \( f(x_p) \neq 0 \) and an open neighborhood \( U_p \) of \( x_p \), such that 
\[
|f(x)| > \frac{|f(x_p)|}{2}, \quad \forall x \in U_p.
\]

Consider a sequence of points \( \{x_i\}_{i \in \mathbb{N}} \subseteq U_p \cap X_p \) and a sequence of open subsets \( \{V_i\}_{i \in \mathbb{N}} \) of \( X \), such that \( x_i \in V_i \subseteq U_p \) and \( V_i \cap V_j = \emptyset \) for \( i \neq j \).

By Urysohn’s lemma there exists a sequence of functions \( \{h_i\}_{i \in \mathbb{N}} \) such that \( h_i(x_i) = 1 \) and \( h_i(X \setminus V_i) = \{0\} \). Let \( q : C_0(X)_{hc} \to C_0(X)_{hc}/\mathcal{I} \) be the quotient map. We estimate for \( i \neq j \):
\[
|M_{q(f), q(h_i)}(q(h_j)) - M_{q(f), q(h_j)}(q(h_j))| = \inf_{g \in \mathcal{I}} \|f^2 h_i - f^2 h_j + g\| \geq \inf_{g \in \mathcal{I}} |(f^2 h_i - f^2 h_j + g)(x_i)| = |f^2(x_i)| > \frac{|f(x_p)|^2}{4}.
\]
Hence, the sequence \( \{ M_{\delta(f), \delta(f)}(q(h_i)) \}_{i \in \mathbb{N}} \) has no convergent subsequence, which implies that the element \( q(f) \) is non compact. □

Recall that a set \( Y \subseteq X \) is called wandering if the sets \( \phi^{-1}(Y), \phi^{-2}(Y), \ldots \) are pairwise disjoint. Since \( \phi \) is a homeomorphism, this condition is equivalent to the condition that \( \phi^m(Y) \cap \phi^n(Y) = \emptyset \), for all \( m, n \in \mathbb{Z}_+ \), \( m \neq n \). A point \( x \in X \) is called wandering if it possesses an open wandering neighborhood. Otherwise it is called non wandering. We will denote by \( X_w \) the set of wandering points of \( X \). It is clear that \( X_w \) is the the union of all open wandering subsets of \( X \).

Let \( X_1 \) be the set of non wandering points of \( X \) and set \( \phi_1 = \phi|_{X_1} \) the restriction of \( \phi \) to \( X_1 \). We thus obtain a dynamical system \( (X_1, \phi_1) \). Define by transfinite recursion a family \( (X_\gamma, \phi_\gamma) \) of dynamical systems. If \( (X_\gamma, \phi_\gamma) \) is defined, then set \( X_{\gamma+1} \) the set of non wandering points of the dynamical system \( (X_\gamma, \phi_\gamma) \) and \( \phi_{\gamma+1} = \phi|_{X_{\gamma+1}} \). If \( \gamma \) is a limit ordinal and the systems \( (X_\beta, \phi_\beta) \) have been defined for all \( \beta < \gamma \), set \( X_\gamma = \cap_{\beta < \gamma} X_\beta \) and \( \phi_\gamma = \phi|_{X_\gamma} \) the restriction of \( \phi \) to \( X_\gamma \). This process must stop at some ordinal \( \gamma_0 \), since the cardinality of the family cannot exceed the cardinality of the power set of \( X \). The following is [4 Lemma 13].

**Proposition 2.3.** The set \( X_{\gamma_0} \) is the closure of the set of recurrent points \( X_r \) of the system \( (X, \phi) \).

If \( \gamma \) is an ordinal \( \gamma \leq \gamma_0 \), we will denote by \( I_\gamma \) the ideal
\[
\{ A \in \mathcal{A} : E_0(A) = 0, E_n(A)(X_\gamma) = \{0\}, \forall n \in \mathbb{Z}_+, n \geq 1 \}.
\]

The proof of the following lemma is straightforward, and is omitted.

**Lemma 2.4.** If \( \gamma \) is a limit ordinal, then \( I_\gamma = \cup_{\beta < \gamma} I_\beta \).

It is known that the ideal generated by the compact elements of \( \mathcal{A} \) is contained in the hypocompact radical [4]. We will need the following characterization of this ideal which is proved in [2].

**Theorem 2.5.** The ideal generated by the compact elements of \( \mathcal{A} \) is the set
\[
\{ A \in \mathcal{A} | E_n(A)(X \setminus X_w) = \{0\}, \forall n \in \mathbb{Z}_+ \text{ and } E_0(A)(X_a) = \{0\} \}.
\]

The following is the main result of the paper.

**Theorem 2.6.** The hypocompact radical \( \mathcal{A}_{hc} \) of \( \mathcal{A} \) is equal to
\[
I = \{ A \in \mathcal{A} : E_0(A)(X_p) = 0, E_n(A)(X_{\gamma_0}) = \{0\}, \forall n \in \mathbb{Z}_+ \}.
\]

**Proof.** 1st step

We shall prove that \( I \) is contained in \( \mathcal{A}_{hc} \). We first prove that \( I_{\gamma_0} \) is contained in \( \mathcal{A}_{hc} \). Assume the contrary.

It follows from Theorem 2.5 that \( I_1 \) is contained in the ideal generated by the compact elements. The hypocompact radical contains the ideal generated by the compact elements [4], and hence \( I_1 \) is contained in \( \mathcal{A}_{hc} \).

Let \( \beta \) be the least ordinal \( \beta \leq \gamma_0 \) such that \( I_\beta \) is not contained in \( \mathcal{A}_{hc} \). We show that \( \beta \) is a successor. If not, since \( I_\gamma \subseteq \mathcal{A}_{hc} \) for all \( \gamma < \beta \), we obtain from Lemma 2.4 that \( I_\beta = \cup_{\gamma < \beta} I_\gamma \subseteq \mathcal{A}_{hc} \), which is absurd. Hence, \( \beta \) is a successor.

We are going to prove that \( I_\beta \) is a hypocompact algebra. Consider the algebra \( I_\beta/\mathcal{I}_{\beta-1} \). It suffices to show that \( I_\beta/\mathcal{I}_{\beta-1} \) is hypocompact, since the class of hypocompact algebras is closed under extensions and the ideal \( I_{\beta-1} \) is hypocompact (Theorem 2.1).
We show that the algebra $\mathcal{I}_\beta/\mathcal{I}_{\beta-1}$ is generated by the compact elements it contains and hence is a hypocompact algebra by [4].

Let $A \in \mathcal{I}_\beta$. It follows from the condition defining $\mathcal{I}_\beta$, that $U^n E_n(A) \in \mathcal{I}_\beta$, for all $n \in \mathbb{Z}_+, n \geq 1$. Hence, it suffices to show that the image of $U^n E_n(A)$ under the natural map $\pi : \mathcal{I}_\beta \to \mathcal{I}_\beta/\mathcal{I}_{\beta-1}$ is contained in the ideal generated by the compact elements of $\mathcal{I}_\beta/\mathcal{I}_{\beta-1}$. It suffices to see this for an element of $\mathcal{I}_\beta$ of the form $U^n f$ with $f$ compactly supported. It follows from [7] Lemma 14], that $f$ can be written as a finite sum $f = \sum f_i$ where each $f_i$ has compact support contained in an open set $V_i$ such that $V_i \cap X_{\beta-1}$ is wandering for the system $(X_{\beta-1}, \phi_{\beta-1})$ and $U^n f_i \in \mathcal{I}_\beta$, for all $i$.

Hence, it suffices to prove that $\pi(U^n f)$ is a compact element, where $f$ has compact support contained in an open set $V$, such that $V \cap X_{\beta-1}$ is wandering for the system $(X_{\beta-1}, \phi_{\beta-1})$.

We calculate:

$$U^n f(\sum U^m g_m)U^n f = \sum U^{2n+m} f \circ \phi^{m+n} g_m \circ \phi^n f,$$

for $\sum U^m g_m \in \mathcal{I}_{\beta}$.

Since $n \geq 1$, we have $n + m \geq 1$, for all $m \in \mathbb{Z}_+$, and consequently $f \circ \phi^{m+n} f = 0$ on $X_{\beta-1}$, for all $m \in \mathbb{Z}^+$ since $V \cap X_{\beta-1}$ is wandering. Hence, $U^n f(U^m g_m)U^n f = U^{2n+m} f \circ \phi^{m+n} g_m \circ \phi^n f \in \mathcal{I}_{\beta-1}$.

Thus, $\pi(U^n f)$ is a compact element of $\mathcal{I}_{\beta}/\mathcal{I}_{\beta-1}$, and $\mathcal{I}_{\beta}$ is a hypocompact ideal which is a contradiction. We conclude that $\mathcal{I}_{\gamma}$ is contained in $\mathcal{A}_{hc}$. Now, $\mathcal{I}/\mathcal{I}_{\gamma}$ is isomorphic to $\{f \in C_0(X) : f(X_p \cup X_{\gamma}) = \{0\}\}$ which is a hypocompact algebra by Theorem [2.2]. It follows from Theorem [2.1] that $\mathcal{I}$ is a hypocompact ideal, and hence it is contained in $\mathcal{A}_{hc}$.

2nd step

We show now that $\mathcal{A}_{hc} = \mathcal{I}$. We will suppose that $\mathcal{I} \subsetneq \mathcal{A}_{hc}$ and we will prove that the quotient algebra $\mathcal{A}_{hc}/\mathcal{I}$ contains no non-zero compact elements. This implies that $\mathcal{A}_{hc} = \mathcal{I}$ by Theorem [2.1].

Let $A \in \mathcal{A}_{hc} \setminus \mathcal{I}$ and set $E_m(A) = f_m$, for all $m \in \mathbb{Z}_+$. Since the map $E_0$ is a continuous homomorphism from $\mathcal{A}$ onto $C_0(X)$, it follows from Theorem [2.1] that $E_0(\mathcal{A}_{hc}) \subseteq C_0(X)_{hc}$ and hence by Theorem [2.2] we have $E_0(A)(X_p) = \{0\}$.

Since $A \notin \mathcal{I}$, it follows from Proposition [2.3] that there exists $m \in \mathbb{Z}_+$ such that $f_m(X_p) \neq \{0\}$. We set

$$m_0 = \min\{m \in \mathbb{Z}_+ : f_m(X_p) \neq \{0\}\},$$

and we consider $x_0 \in X_p$ such that $f_{m_0}(x_0) \neq 0$. There exists an open neighborhood $U_0$ of $x_0$ such that

$$|f_{m_0}(x)| > \frac{|f_{m_0}(x_0)|}{2}, \quad \forall x \in U_0. \quad (1)$$

Since $x_0$ is a recurrent point, there exist an open neighborhood $V_0$ of $x_0$ such that $\overline{V_0} \subseteq U_0$ and a strictly increasing sequence $\{n_i\}_{i=1}^\infty \subseteq \mathbb{N}$ such that

$$\phi^{n_i}(x_0) \in V_0, \quad \forall i \in \mathbb{N}. \quad (2)$$

Choosing, if necessary, a subsequence, we may assume that $n_1 > m_0$ and $n_{i+1} > 3n_i$. By Urysohn’s lemma there is $u_0 \in C_0(X)$ such that $u_0(x) = 1$, for all $x \in \overline{V_0}$ and $u_0(X \setminus U_0) = \{0\}$. We thus have

$$u_0(x_0) = u_0 \circ \phi^{n_i}(x_0) = 1, \quad \forall i \in \mathbb{N}. \quad (3)$$
By [10] Proposition 2.1, we have that \( U^{m_0} f_{m_0} \in \mathcal{A}_{hc} \), (see also [7] p. 133). Hence, if we consider the sequence \( \{B_i\}_{i=1}^\infty \), where

\[
B_i = (U^{n_i+1-n_i-m_0} u_0 \circ \phi^{-m_0})(U^{m_0} f_{m_0})(U^{n_i-m_0} u_0 \circ \phi^{-m_0}) = U^{n_i+1-m_0} u_0 \circ \phi^{n_i-m_0} f_{m_0} \circ \phi^{n_i-m_0} u_0 \circ \phi^{-m_0},
\]

it follows that \( \{B_i\}_{i=1}^\infty \subseteq \mathcal{A}_{hc} \).

Let \( \pi : \mathcal{A}_{hc} \to \mathcal{A}_{hc}/\mathcal{I} \) be the quotient map. To prove that the element \( \pi(A) \) is not a compact element of \( \mathcal{A}_{hc}/\mathcal{I} \), we will prove that the sequence \( \{M_{\pi(A), \pi(A)}(\pi(B_i))\}_{i \in \mathbb{N}} \) has no Cauchy subsequence.

Let \( k, l \in \mathbb{N} \) with \( k > l \). If \( r < (n_{k+1} - m_0) \), the \( r \)th Fourier coefficient of \( B_k \) is 0, and this also holds for \( M_{A,A}(B_k) \). It follows that

\[
E_{n_{i+1}+m_0}(M_{A,A}(B_k)) = 0,
\]

since \( n_{i+1} + m_0 < 3n_{i+1} - m_0 < n_{k+1} - m_0 \).

Therefore, it follows that

\[
||M_{\pi(A), \pi(A)}(\pi(B_k) - B_l))|| = \inf_{N \in \mathcal{I}} ||M_{A,A}(B_k - B_l) + N||
\]

\[
\geq \inf_{N \in \mathcal{I}} ||E_{n_{i+1}+m_0}(M_{A,A}(B_k) - B_l) + N||
\]

\[
\geq \inf_{N \in \mathcal{I}} ||E_{n_{i+1}+m_0}(M_{A,A}(B_l) + N)(x_0)|| = ||E_{n_{i+1}+m_0}(M_{A,A}(B_l))(x_0)||
\]

since \( x_0 \in X_r \) and thus, for all \( N \in \mathcal{I} \), we have \( E_{n_{i+1}+m_0}(N)(x_0) = 0 \).

We calculate \( |E_{n_{i+1}+m_0}(M_{A,A}(B_l))(x_0)| \).

We have

\[
|E_{n_{i+1}+m_0}(M_{A,A}(B_l))(x_0)| = \sum_{n=0}^{2m_0} |f_{2m_0-n} \circ \phi^{n_{i+1}+n-m_0} u_0 \circ \phi^{n_{i}+n-m_0} f_{m_0} \circ \phi^{n_{i}+n-m_0} u_0 \circ \phi^{-m_0} f_n(x_0)|.
\]

For \( n < m_0 \) we have \( f_n(x_0) = 0 \). Also, for \( n > m_0 \) and \( n \leq 2m_0 \) we have \( f_{2m_0-n} \circ \phi^{n_{i+1}+n-m_0}(x_0) = 0 \), since \( 2m_0 - n < m_0 \) and \( \phi^{n_{i+1}+n-m_0}(x_0) \in X_r \).

Finally,

\[
|E_{n_{i+1}+m_0}(M_{A,A}(B_l))(x_0)| = \frac{|f_{m_0} \circ \phi^{n_{i+1}} u_0 \circ \phi^{n_{i}} f_{m_0} \circ \phi^{n_{i}} u_0 f_{m_0}(x_0)|}{8}.
\]

It follows that the sequence \( \{M_{\pi(A), \pi(A)}(\pi(B_i))\}_{i \in \mathbb{N}} \) contains no Cauchy subsequence, and hence \( \pi(A) \) is not a compact element of \( \mathcal{A}_{hc}/\mathcal{I} \).

\[ \square \]

3. THE SCATTERED RADICAL

The following are taken from [18] 8.2. A Banach algebra is called scattered if the spectrum of every element \( a \in A \) is finite or countable. A Banach algebra \( A \) has a largest scattered ideal denoted by \( \mathcal{R}_s(A) \). This ideal is closed and is called the scattered radical of \( A \) [18] Theorem 8.10].

Since all \( C^* \)-algebras are semisimple and their quotients are again \( C^* \)-algebras, it follows from [18] Theorem 8.22 that \( \mathcal{C}_0(X)_{hc} = \mathcal{C}_0(X)_s \).
Donsig, Katavolos and Manousos proved in [7] a characterization of the Jacobson radical for more general semicrossed products. The next theorem follows from their result [7, Theorem 18].

**Theorem 3.1.** The Jacobson radical of $A$ coincides with the set of operators

$$\{ A \in A \mid E_n(A)(X_r) = \{0\}, \forall n \in \mathbb{Z}_+ \text{ and } E_0(A) = 0 \}.$$  

It follows from Theorem 2.6 and the above characterization, that the Jacobson radical of $A$ is contained in $A_{hc}$. Hence, from [18, Theorem 8.15] we obtain the following.

**Theorem 3.2.**

$$A_{hc} = A_s.$$  

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**References**

1. G. Andréolas and M. Anoussis, *Topological radicals of nest algebras*, Studia Math. 237 (2017), no. 2, 177–184.
2. G. Andréolas, M. Anoussis and C. Magiatis *Compact multiplication operators on semicrossed products*, preprint arXiv:2110.07684 2021.
3. S. Argyros and R. G. Haydon, *A hereditarily indecomposable $\mathcal{L}_\infty$-space that solves the scalar–plus–compact–problem*, Acta Math. 206 (2011), no. 1, 1–54.
4. M. Brešar and Yu. V. Turovskii, *Compactness conditions for elementary operators*, Studia Math. 178 (2007), no. 1, 1–18.
5. K. R. Davidson, A. H. Fuller and E. T. A. Kakariadis, *Semicrossed products of operator algebras: a survey* New York J. Math. 24A (2018), 56–86.
6. P. G. Dixon, *Topologically irreducible representations and radicals in Banach algebras*, Proc. London Math. Soc. (3) 74 (1997), no. 1, 174–200.
7. A. Donsig, A. Katavolos and A. Manoussos, *The Jacobson radical for analytic crossed products*, J. Funct. Anal. 187 (2001), no. 1, 129–145.
8. C. K. Fong and A. R. Sourour, *On the operator identity $\sum A_kXB_k = 0$*, Can. J. Math. 31 (1979), 845–857.
9. K. Kuratowski, *Topology. Vol. I, New edition, revised and augmented*. Translated from the French by J. Jaworowski, Academic Press, New York-London; Państwowe Wydawnictwo Naukowe, Warsaw, 1966.
10. P. S. Muhly, *Radicals, crossed products, and flows*, Ann. Polon. Math. 43 (1983), 35–42.
11. J. Peters, *Semicrossed products of $C^*$-algebras*, J. Funct. Anal. 59 (1984), no. 3, 498-534.
12. J. Peters, *The ideal structure of certain nonselfadjoint operator algebras*, Trans. Amer. Math. Soc. 305 (1988), no. 1, 333–352.
13. Turovskii, Yu. V., Shulman, V. S. *Radicals in Banach algebras, and some problems in the theory of radical Banach algebras*. (Russian) Funktsional. Anal. i Prilozhen. 35 (2001), no. 4, 88–91; translation in Funct. Anal. Appl. 35 (2001), no. 4, 312–314
14. V. S. Shulman and Yu. V. Turovskii, *Topological radicals. I. Basic properties, tensor products and joint quasinilpotence*, Topological algebras, their applications, and related topics, 293–333, Banach Center Publ., 67, Polish Acad. Sci., Warsaw, 2005.
15. V. S. Shulman and Yu. V. Turovskii, *Topological radicals. II. Applications to spectral theory of multiplication operators. Elementary operators and their applications*, 45–114, Oper. Theory Adv. Appl., 212, Birkhäuser/Springer Basel AG, Basel, 2011.
16. E. Kissin, V. S. Shulman and Yu. V. Turovskii, *Topological radicals and Frattini theory of Banach Lie algebras*, Integral Equations Operator Theory 74 (2012), no. 1, 51–121.

17. Turovskii, Yu. V., Shulman, V. S. *Topological radicals and the joint spectral radius*. (Russian) Funktsional. Anal. i Prilozhen. 46 (2012), no. 4, 61–82; translation in Funct. Anal. Appl. 46 (2012), no. 4, 287–304.

18. V. S. Shulman and Yu. V. Turovskii, *Topological radicals, V. From algebra to spectral theory*, Algebraic methods in functional analysis, 171–280, Oper. Theory Adv. Appl., 233, Birkhäuser/Springer, Basel, (2014).

19. K. Vala, *On compact sets of compact operators*, Ann. Acad. Sci. Fenn. Ser. A I No. 351 (1964).

20. K. Vala, *Sur les éléments compacts d’une algèbre normée*, Ann. Acad. Sci. Fenn. Ser. A I No. 407 (1967).