Unconditional security of continuous-variable quantum key distribution

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The unconditional security of continuous-variable quantum key distribution is established for all schemes based on the estimation of the channel loss and excess noise. It is proved that, in the limit of large keys, Gaussian attacks are asymptotically optimal among the most general (coherent) attacks, where the transmission is tapped using arbitrary ancillas and stored in a quantum memory as a whole. Then, it is shown that the previously derived bounds on the achievable secret key rates against collective attacks remain asymptotically valid for arbitrary coherent attacks.

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Quantum key distribution (QKD) is probably to date the most successful application of quantum information sciences. This technique [1], based on the transmission of quantum signals between two authorized parties (Alice and Bob), enables them to generate a random bit string, called secret key, which signals between two authorized parties (Alice and Bob), enables them to generate a random bit string, called secret key, which most successful application of quantum information sciences.

where the exact definition of the conditional entropy $H$ depends on the type of attacks considered: individual, collective or coherent. The second term of the r.h.s. of Eq. (1) is simply measured while running the protocol since it is accessible to the legitimate parties, so only the first term needs to be estimated, or, more precisely, bounded below. For individual attacks, $H(a|c)$ denotes Shannon conditional entropy, and can be calculated explicitly for a Gaussian attack. For collective attacks, it must be replaced by $S(a|E)$, which denotes von Neumann conditional entropy and can again be computed exactly for a Gaussian attack. Physically, this means that Eve accesses the Holevo information about Alice’s variable $a$ by making an appropriate measurement on her quantum system $E$ (instead of accessing the Shannon information between $a$ and her measurement $c$). For individual and collective attacks, Gaussian attacks have been proved to be optimal [2 3 4] as they minimize $H(a|c)$ and $S(a|E)$ for a given channel transmission and excess noise, which largely simplifies the security analysis.

To address unconditional security, one must consider the most general class of attacks, namely coherent attacks, and the first term of the r.h.s. of Eq. (1) must be replaced by the quantum smooth min-entropy $S_{\text{min}}^a(E)$, as introduced in [5]. The min-entropy (or Rényi entropy of parameter $\infty$) is particularly relevant for the security study of cryptographic protocols as it quantifies the guessing probability, i.e., the probability that Eve correctly guesses the value of the classical variable $a$. Replacing Shannon or von Neumann entropies by min-entropies encapsulates the idea that the entire transmission, made of $n$ symbols, is tapped as a whole. The min-entropy can be viewed as a one-shot quantity, while Shannon or von Neumann entropies are computed on a single-symbol basis and get a meaning only by assuming that there are many identical transmissions. Not surprisingly, these two types of entropies become asymptotically equal in the latter case, the quantum smooth min-entropy of a power state $\rho^\otimes n$ tending to $(n$ times) the von Neumann entropy of $\rho$ in the limit $n \to \infty$.

QKD protocols can be classified in discrete-variable protocols, based on photon counting (e.g., BB84 [6]), and continuous-variable protocols, based on homodyne detection [8]. We will be concerned with this latter class of protocols in the following, in particular those based on the Gaussian modulation of Gaussian (coherent or squeezed) states. For discrete-variable protocols, the unconditional security can be proved by using a quantum de Finetti theorem stating that symmetric states are “close to” product states [9]. An $n$-partite state is said to be symmetric if it is invariant under any permutation of its subsystems. If the protocol is symmetric, one concludes that the smooth min-entropy of the symmetric state shared by Alice and Bob is asymptotically equal to $(n$ times) the von Neumann entropy, as used for calculating the secret key rates against collective attacks. This proves that coherent attacks are not more powerful than collective attacks. Unfortunately, this approach cannot be applied as such to the security of continuous-variable protocols against coherent attacks as it would require extending a de Finetti theorem to infinite dimensional Hilbert spaces. Such an extension, however, has just been shown to hold provided that experimentally verifiable conditions are fulfilled [17].

In this Letter, it is shown that there exists another way to address the security of continuous-variable QKD, which circumvents the need for a de Finetti theorem in infinite dimension. The idea is to exploit the extremality of Gaussian states with respect to the (non smooth) min-entropy. The extremality of Gaussian states has led to powerful results in the past, as it was
used, e.g., to prove the optimality of Gaussian attacks among collective attacks [3]. The point of using Gaussian states here is that even if symmetric Gaussian states are not known to be exponentially close to product states (although some preliminary results in this direction have been obtained [11]), their min-entropy is equal to the min-entropy of a well-defined product state. Then, using the link between the smooth min-entropy and the von Neumann entropy, one shows that the secret key rate against coherent attacks can be asymptotically bounded below by a secret key rate against Gaussian collective attacks. This establishes the proof of the unconditional security of continuous-variable QKD against the most general attacks.

Note that unconditional security of QKD protocols can also be proved by showing their equivalence to some entanglement purification protocols. This strategy can be used both for discrete-variable [12] and continuous-variable [13] protocols but, unfortunately, does not allow to derive a secret key rate.

**Sketch of the proof.** The central object of the proof is thus the smooth min-entropy $S^\epsilon_{\min}(\rho|E)$, which we will also often denote as $S^\epsilon_{\min}(\rho)$ in the following, with $\rho$ being the $2n$-mode state shared by Alice and Bob. As in all security proofs of QKD, Eve is given a purification of the state $\rho$ on Alice’s side and can then extend to the smooth min-entropy of a power state with the von Neumann entropy of the state, or, more precisely, $S^\epsilon_{\min}(\rho^{(n-1)}) \geq (n-1)|S(\rho|E) - \delta|$, where, for any $\epsilon > 0$, the parameter $\delta > 0$ tends to zero at the limit $n \to \infty$. This concludes the proof that coherent attacks are not more powerful than collective attacks, exactly as in the finite-dimensional case. Let us now proceed with the details of our proof.

**Non-smooth min-entropy.** Our starting point is the non-smooth quantum (conditional) min-entropy $S_{\min}(\rho|E)$ as introduced in [15]. Remember first that the min-entropy (or Rényi entropy of parameter $\infty$) of a state $\rho$ is defined as

$$S_{\infty}(\rho) := - \log \lambda_{\max}(\rho),$$

where $\lambda_{\max}(\rho)$ stands for the maximum eigenvalue of $\rho$. We may then evaluate $S_{\infty}$ for the Hermitian operator

$$\rho_{AE} := \sigma_E^{\epsilon/2} \rho_{AE} \sigma_E^{-\epsilon/2},$$

with $\rho_{AE}$ being the density operator on $\mathcal{H}_A \otimes \mathcal{H}_E$ characterizing Alice and Eve’s joint state, and $\sigma_E$ being a nonnegative trace-one operator on $\mathcal{H}_E$ whose support contains the support of $\rho_{AE} = tr_A(\rho_{AE})$. By taking the supremum of the resulting quantity over all nonnegative trace-one operators $\sigma_E$, one obtains the (conditional) min-entropy of $A$ given $E$, namely

$$S_{\min}(A|E) := \sup_{\sigma_E} \left( \frac{\rho_{AE}}{\sigma_E} \right).$$

In the present context, we consider Alice’s system $A$ to be classical, which boils down to substituting $A$ with the classical variable $a$ in the above expressions. In the entanglement-based equivalent picture of QKD, this means that Alice’s measurement is treated as a unitary process and her measurement device as a physical system, not $a$. Thus, $a$ corresponds either to a real variable for homodyne-detection protocols or a pair of real variables for heterodyne-detection protocols.

The next step is to show that the min-entropy $S_{\min}(\rho|E)$ of a conditional on $E$ is a well-defined function of the $2n$-mode state $\rho$ shared by Alice and Bob; this property will then extend to the smooth min-entropy $S^\epsilon_{\min}(\rho|E)$. Without loss of generality, Eve is assumed to hold a purification of $\rho$. Then, since all purifications are unitarily equivalent on Eve’s
side and since the min-entropy is invariant under unitaries, the quantities $S_{\min}(a|E)$ and $S_{\min}(a^2|E)$ only depend on $\rho$. Now, to make both these quantities well-defined, it is necessary to assume that $a$ is discretized, albeit with an arbitrarily small step. This is consistent with the actual experiments since any practical measurement has a finite precision. We note that the exact value of the discretization step does not matter, as $a$ is discretized similarly in the second term of the r.h.s. of Eq. (1), namely $H(a|b)$, and the terms linked to this discretization cancel each other when one computes the key rate. Note also that the (practical) discretization of $a$ is not contradictory with the fact that we consider an infinite-dimensional Hilbert space.

**Extremality of Gaussian states.** Let us show that the min-entropy is extremal for Gaussian states, that is, $H_{\min}(\rho)$ is bounded below by the same quantity computed with the Gaussian state $\rho^G$. This is done using the following Lemma [14]:

**Lemma 1.** Let $f : B(H_{\otimes N}) \to \mathbb{R}$ be a lower semi-continuous functional (w.r.t. trace norm) which is strongly superadditive and invariant under local unitaries $f(U_{\otimes N}^{\dagger} \rho U_{\otimes N}^{\dagger}) = f(\rho)$. Then, for every density operator $\rho$, we have

$$f(\rho) \geq f(\rho^G)$$

where $\rho^G$ is the Gaussian state with the same first- and second-order moments as $\rho$.

We apply Lemma 1 to the functional $f : \rho \mapsto S_{\min}(\rho)$, with $N = 2$ and $\rho$ being the 2-mode state shared by Alice and Bob. The three necessary conditions are verified following the same procedure as in [3], where the extremality of Gaussian states for the quantity $S(a|E)$ was proved.

(i) **Lower semi-continuity.** It is known that for two quantum states $\rho_{AB}$ and $\sigma_{AB}$, there exist respective purifications $\rho_{ABE}$ and $\sigma_{ABE}$ of these states such that $\|\rho_{AB} - \sigma_{AB}\| \leq \epsilon$ implies $\|\rho_{ABE} - \sigma_{ABE}\| \leq 2\sqrt{\epsilon}$. Thus, since $f$ does not depend on the choice of purification of $\rho$, it is sufficient to prove the continuity of $f(\rho_{ABE})$ to infer that of $f(\rho_{AB})$. Furthermore, since partial trace can only decrease the trace norm and since the operator norm $\|\cdot\|_{\infty}$ is a lower bound of $\|\cdot\|_1$, we have $\|\rho_{AB} - \sigma_{AB}\|_{\infty} \leq \|\rho_{ABE} - \sigma_{ABE}\|_1$. Hence, one only needs to establish the semi-continuity of $f(\rho_{ABE})$ with respect to the operator norm.

Note first that, for a given $\sigma_{E}$, the function $\rho_{AB} \mapsto \rho_{A}\rho_{B}/\sigma_{E}$ is clearly continuous, where $\sigma_{E}$ is the Hermitian operator defined in Eq. (6). Then, the function $\rho_{AB} \mapsto \lambda_{\max}(\rho_{AB}/\sigma_{E}) = \|\rho_{AB}/\sigma_{E}\|_{\infty}$ is continuous with respect to the operator norm, as well as the function $\rho_{AB} \mapsto S_{\infty}(\rho_{AB}/\sigma_{E})$ defined in Eq. (5). Finally, regarding $\sigma_{E}$ as a parameter, the min-entropy $S_{\min}(a|E)$ is a lower semi-continuous function of $\rho_{AB}$, being the upper envelope of a family of continuous functions, see Eq. (7).

(ii) **Strong super-additivity.** This property results from the recombination-chain-rule property of the min-entropy [15]:

$$S_{\min}(AB|C) \geq S_{\min}(A|BC) + S_{\min}(B|C). \quad (8)$$

One has:

$$S_{\min}(a_1, a_2|E) \geq S_{\min}(a_1|a_2E) + S_{\min}(a_2|E)$$

$$\geq S_{\min}(a_1|A_2B_2E) + S_{\min}(a_2|A_1B_1E)$$

$$\geq S_{\min}(a_1|E_1) + S_{\min}(a_2|E_2)$$

where the second and third inequalities follow respectively from the subadditivity of the min-entropy [13] and the fact that $A_2B_2E$ (resp. $A_1B_1E$) purifies $A_1B_1$ (resp. $A_2B_2$). The additivity of $f(\rho)$, also needed for Lemma 1 to hold, is a consequence of the additivity of the min-entropy [5].

(iii) **Invariance under local unitaries.** This property is proved following the same steps as in [3]. The crucial remark is that one can restrict the proof of Lemma 1 to some Gaussianization operations that do not mix the $x$ and $p$ quadratures. Then, using the fact that the min-entropy is invariant under local unitaries, we infer the same property for $f(\rho)$.

**Smooth min-entropy.** The function $f(\rho)$ is actually interesting in is $S_{\min}^\epsilon(\rho)$, i.e., the smooth version of the min-entropy $S_{\min}(\rho)$, see Eq. (2). Unfortunately, Lemma 1 cannot directly be applied to $S_{\min}^\epsilon(\rho)$ because this functional is not known to be additive. Nevertheless, we can use Lemma 1 for $S_{\min}(\rho)$ and introduce the smoothness by hand in order to prove that Gaussian states are approximately extremal for $S_{\min}^\epsilon(\rho)$ provided that $\epsilon$ is small enough. More precisely, if we can prove that the function $\epsilon \mapsto S_{\min}^\epsilon(\rho)$ is continuous in $\epsilon$ for a given $\rho$, then one has $S_{\min}^\epsilon(\rho) \geq S_{\min}^\epsilon(\rho^G) - \epsilon'$, with $\epsilon' > 0$ continuously tending to zero as $\epsilon \to 0$.

We address this issue by taking into account a physical constraint on the mean energy. We first write

$$S_{\min}(\rho) \geq \sup_{\sigma_{E} \in K} S_{\infty}(\frac{\rho_{AE}}{\sigma_{E}}). \quad (9)$$

which is analogous to Eq. (7) except that we have restricted the supremum to the compact set $K$ of states $\sigma_{E}$ of bounded energy; hence, we have a lower bound. Second, we note that the function $\rho_{AE} \mapsto S_{\infty}(\rho_{AE}/\sigma_{E})$ is continuous in its two inputs. Then, taking the supremum of a continuous function $S_{\infty}(\rho_{AE}/\sigma_{E})$ over the compact set $K$ as in Eq. (9) yields a continuous function, $\rho \mapsto S_{\min}(\rho)$. Thus, in view of Eq. (2), we conclude that the function $\epsilon \mapsto S_{\min}^\epsilon(\rho)$ is continuous, as long as we put an energy constraint in the supremum over $\sigma_{E}$. As is obvious from Eq. (9), this energy constraint can only decrease $S_{\min}^\epsilon(\rho^G)$, so it goes on the safe side.

**Symmetric Gaussian states.** Up to here, we have proved that the min-entropy can be safely bounded by assuming that the state $\rho$ shared by Alice and Bob is Gaussian. Now, let us exploit the fact that $\rho$ (and $\rho^G$) can be considered symmetric (this property can be enforced with a random permutation of Alice and Bob’s strings after the quantum transmission, although this permutation is not needed for symmetric protocols such as those of interest here). The permutation-invariance of $\rho$ (or $\rho^G$) implies that its covariance matrix $\Gamma$ has a symmetric form, Eq. (4). If a de Finetti theorem held for infinite dimension, one would conclude that such a symmetric state is “close
to a product state. Here, however, we just exploit the Gaussianity of the state. We apply the symplectic transformation $S = \mathbb{1}_2 \otimes X$, where $X$ is any orthogonal $n \times n$ matrix for which $X_{i,1} = 1/\sqrt{n}$, in order to diagonalize $\Gamma$, namely \( \Gamma' = S \Gamma S^T = \begin{pmatrix} \gamma + (n-1)\sigma & 0 & \cdots & 0 \\ 0 & \gamma - \sigma & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \gamma - \sigma \end{pmatrix} \) \number{10}.

Since $S_{\min}^e(\rho^G)$ is invariant under $S$, we may consider instead of $\rho^G$ a Gaussian state which has the diagonal covariance matrix $\Gamma'$ and is thus a genuine product state. Then, the subadditivity of the smooth min-entropy implies that $S_{\min}^e(\rho^G) \geq S_{\min}^e(\rho_{\text{ind}}) \otimes (n-1)$ where $\rho_{\text{ind}}$ is a bimodal Gaussian state with covariance matrix $\gamma - \sigma$.

Gaussian product states. We are left now with the problem of estimating the smooth min-entropy of a Gaussian product state. This is done using the following Lemma, stating that the smooth min-entropy of a product state asymptotically converges towards its von Neumann entropy \[^{[5]}\]:

**Lemma 2.** Let $\rho_{aE} \in \mathcal{H}_A \otimes \mathcal{H}_E$ be a density operator which is classical on $\mathcal{H}_A$. Then, for any $\epsilon > 0$,

$$
\frac{1}{n} S_{\min}^e(\rho_{aE}^\otimes n) \geq S(a|E) - \delta, \number{11}
$$

where $\delta := (2S_{\max}(\rho_a) + 3) \sqrt{\frac{1 + \log(1/\epsilon)}{n}}$.

Note that the variable $a$ must be discretized in order to use Lemma 2; otherwise, the entropies are ill-defined, as already emphasized. Interestingly, Lemma 2 can be rewritten with $S_{\min}^e(\rho_{aE}^\otimes n)$ and $S(a|E)$ being expressed both as a supremum over $\sigma_E$ and, in addition, inequality \number{11} holds for any $\sigma_E \in \mathcal{K}$ with a bounded energy does not affect the convergence towards the von Neumann entropy.

Finally, applying Lemma 2 to the state $\rho_{\text{ind}} \otimes (n-1)$ and combining it with the previous inequalities, we get an explicit lower bound on the smooth min-entropy of the initial 2-mode state $\rho$ shared by Alice and Bob, namely

$$
S_{\min}^e(\rho) \geq (n-1)[S(a|E) - \delta] - \epsilon \number{12}
$$

where $S(a|E)$ is the conditional von Neumann entropy measuring Eve’s uncertainty when Alice and Bob’s state is a Gaussian state whose covariance matrix is $\gamma - \sigma$. Therefore, in the asymptotic limit $n \to \infty$, the smooth min-entropy is simply bounded by an expression similar to that used when addressing the security against Gaussian collective attacks. This concludes the unconditional security proof.

Note that the secret key rate against collective attacks is obtained by considering the Gaussian state with the observed covariance matrix $\gamma$, whereas $S(a|E)$ must be evaluated here based on the covariance matrix $\gamma - \sigma$. Nevertheless it can be easily computed since both $\sigma$ and $\gamma$ are experimentally accessible. In addition, for a QKD protocol with coherent states and homodyne detection, one can check that asymptotically, $K(\gamma - \sigma) \geq K(\gamma)$. This is proved using the fact that the submatrices in \number{10} should be valid covariance matrices for any $n$. Therefore there is no need to measure $\sigma$ to establish unconditional security, but monitoring $\sigma$ may lead to a better key rate.

**Conclusion.** We have proved that the most general attacks against continuous-variable quantum key distribution protocols cannot beat Gaussian collective attacks, up to some finite-size corrections which vanish in the limit of a large key size.

Our proof holds for all protocols based on probing the quantum channel via the second-order moments of Alice and Bob’s continuous data. For readability, we only considered direct reconciliation, but the generalization of the proof to reverse reconciliation \[^{[16]}\] is also immediate (the result of Bob’s measurement $b$ should be interchanged with $a$).

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\[^{[17]}\] This was shown in work done in parallel to ours \[^{[10]}\].