ON HYPERBOLIC MIXED PROBLEMS WITH DYNAMIC AND WENTZELL BOUNDARY CONDITIONS

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Dedicated to Gisele Ruiz Goldstein in occasion of her sixtieth birthday

Abstract. We study mixed hyperbolic systems with dynamic and Wentzell boundary conditions. The boundary condition contains a tangential operator which is strongly elliptic on the boundary. We prove results of generation of strongly continuous groups and well-posedness.

1. Introduction. The aim of this paper is the study of a problem in the form

\[
\begin{align*}
D^2_t u(t,x) &= A(x,D_x)u(t,x) + f(t,x), \quad (t,x) \in (0,T) \times \Omega, \\
D^2_t \gamma u(t,x') &= \nabla_{\tau} \cdot (B(x')\nabla_{\tau} u)(t,x') + F(x',D_x)u(t,x') + h(t,x'), \\
(t,x') &\in (0,T) \times \partial \Omega, \\
u(0,x) &= u_0(x), \quad x \in \Omega, \\
D_t u(0,x) &= u_1(x), \quad x \in \Omega.
\end{align*}
\]

(1)

Roughly speaking (precise assumptions will be given in the following), \( A(x,D_x) \) is a strongly elliptic differential operator in divergence form in the bounded domain \( \Omega \) with smooth boundary \( \partial \Omega \); \( \gamma f \) is the trace of \( f \) on \( \partial \Omega \); \( \nabla_{\tau} \) is the tangential gradient in \( \partial \Omega \), \( \nabla_{\tau} \cdot \) is the divergence operator in \( \partial \Omega \), \( B(x') \) is a positive definite symmetric operator in the tangent space \( T_{x'}(\partial \Omega) \), with \( x' \in \partial \Omega \), \( F(x',D_x) \) is a linear differential operator of order not exceeding one (not necessarily tangential) and coefficients defined in \( \partial \Omega \).

(1) is strictly connected with the problem

\[
\begin{align*}
D^2_t u(t,x) &= A(x,D_x)u(t,x) + f(t,x), \quad (t,x) \in (0,T) \times \Omega, \\
A(x',D_x)u(t,x') &= \nabla_{\tau} \cdot (B(x')\nabla_{\tau} u)(t,x') + F(x',D_x)u(t,x') + h(t,x'), \\
(t,x') &\in (0,T) \times \partial \Omega, \\
u(0,x) &= u_0(x), \quad x \in \Omega, \\
D_t u(0,x) &= u_1(x), \quad x \in \Omega.
\end{align*}
\]

(2)

formally obtained replacing in (1) \( D^2_t \gamma u(t,x') \) in the second equation with the trace of the second term in the first equation. In case (2) one usually speaks of Wentzell boundary conditions.

A physical interpretation of (2) is given in [10], Chapter 6.

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In our knowledge, problems (1) and (2) have been always considered in the particular case that
\[ F(x', D_x) = -\beta(x') \frac{\partial}{\partial \nu_A} - c(x'), \]
where we indicate with \( \frac{\partial}{\partial \nu_A} \) the conormal derivative associated with \( A(x, D_x) \). See, for example, [1], [9], [12], often connected with problems of control.

The most general results are contained in [2], where \( F(x', D_x) \) is in the form (3) with \( \beta(x') > 0 \), which is allowed (to some extent) to be unbounded and with infimum equal to 0. The authors do not even assume that the coefficients of \( A(x, D_x) \) and \( B(x') \) are continuous; they need to be just measurable and bounded. They work in the basic space \( L^2(\Omega, d\mu) := L^2(\Omega) \times L^2(\partial\Omega, d\beta). \) with \( F(x', D_x) \) as in (3). They show that a certain operator connected with (1) and (2) is self-adjoint and upper bounded. This allows to formulate theorems of well-posedness in a certain generalized sense. They consider also the case when \( D \) is replaced by \( D_t^2 + aD_t \) (this is the telegraph equation).

Roughly speaking, in this paper we want to show that, at least in case of “regular coefficients” for \( A(x, D_x) \) and \( B(x') \), (1) and (2) are well posed whenever the operator \( F(x', D_x) \) has bounded and measurable coefficients in \( \partial\Omega \).

This is the plan of this paper: Section 2 is dedicated to the proof of Theorem 2.1. We begin by considering a particular case, with \( F(x', D_x) = -\frac{\partial}{\partial \nu_A} - \gamma \). In this situation the result is essentially known (see for this also [3]), but we have decided to give a complete proof in order to make the paper more or less self-contained. The general statement is obtained by combining an estimate of the conormal derivative of the solution to a hyperbolic Cauchy-Dirichlet system (see Theorem 2.5) with a perturbation theorem of Miyadera type (Theorem 2.6). The estimate is inspired by a nice result due to I. Lasiecka, J. L. Lions, R. Triggiani (see [8]).

The final Section 3 contains developments and applications of Theorem 2.1 to a generalization of (1), and to (2).

To conclude this preliminary section, we describe some notations we are going to use.

If \( \Omega \) is a domain with smooth boundary and \( x' \in \partial\Omega \), we shall indicate with \( \nu(x') \) the unit normal vector to \( \partial\Omega \) in \( x' \), pointing outside \( \Omega \), with \( \frac{\partial}{\partial \nu} \) the corresponding normal derivative. \( T_{x'}(\partial\Omega) \) will be the tangent space to \( \partial\Omega \) in \( x' \) and \( T(\partial\Omega) \) the tangent bundle. If \( A \) is the differential operator
\[ \sum_{i=1}^{n} \sum_{j=1}^{n} D_{x_i}(a_{ij}(x)D_{x_j}), + \sum_{j=1}^{n} a_j(x)D_{x_j} + \alpha_0(x), \]
and \( x' \in \partial\Omega \), we set
\[ D_{\nu_A} u(x') = \frac{\partial u}{\partial \nu_A}(x') = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x')D_{x_j} u(x') \nu_i(x'). \]

\( C \) will indicate a positive constant we are not interested to precise. In a sequence of formulas we shall write \( C_1, C_2, \ldots \). If the constants depend on \( T \), we shall write \( C(T), C_1(T), \ldots \).

If \( X \) and \( Y \) are normed spaces, we shall indicate with \( \mathcal{L}(X, Y) \) the space of linear bounded operators from \( X \) to \( Y \). If \( X = Y \), we shall write \( \mathcal{L}(X) \). If \( V \) is a Hilbert space, we shall indicate with \( V^* \) the space of antilinear bounded functionals in \( V \).
2. The main theorem. As we said, in this section we shall study a simplified version of (1). We begin by stating our assumptions.

(A1) \( \Omega \) is an open bounded subset of \( \mathbb{R}^n \) lying on one side of its boundary \( \partial \Omega \), which is a submanifold of \( \mathbb{R}^n \) of dimension \( n - 1 \) and class \( C^2 \).

(A2) \( A_0(x, D_x) = \sum_{i=1}^{n} \sum_{j=1}^{n} D_x[a_{ij}(x)D_x] \), with \( a_{ij} \in C^1(\overline{\Omega}) \) \( (1 \leq i, j \leq n) \), real valued, \( a_{ij} = a_{ji} \),

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \nu |\xi|^2,
\]

for any \( x \in \overline{\Omega} \), \( \xi \in \mathbb{R}^n \), for some \( \nu \) positive.

(A3) \( \forall x' \in \partial \Omega \) \( B(x') \) is is symmetric and positive definite element of \( \mathcal{L}(T_{x'}(\partial \Omega)) \).

(A4) \( B(x') \) depends smoothly on \( x' \), in the sense that, if \( u \) is a \( C^1 \) section with values in \( T(\partial \Omega) \), \( B(\cdot)u(\cdot) \) is a \( C^1 \) section.

(A5) \( F(x', D_x)u(x') = \sum_{j=1}^{n} f_j(x')D_xu(x') + f_0(x')u(x') \), with \( f_j \in \mathcal{L}_\infty(\partial \Omega) \) \( (0 \leq j \leq n) \).

We set

\[
H = L^2(\Omega) \times L^2(\partial \Omega),
\]

Of course, \( H \) is a Hilbert space with the usual scalar product

\[
((f_0, h_0), (f_1, h_1))_H = \int_{\Omega} f_0(x)f_1(x)dx + \int_{\partial \Omega} h_0(x)h_1(x)d\sigma,
\]

where \( \sigma \) is the standard Riemannian measure in \( \partial \Omega \). We set also

\[
V = \{(\phi, \psi) \in H^1(\Omega) \times H^1(\partial \Omega) : \gamma \phi = \psi \}.
\]

We equip \( V \) with the scalar product

\[
((\phi_0, \psi_0), (\phi_1, \psi_1))_V := \int_{\Omega} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x)D_x\phi_0(x)D_x(\bar{\phi_1}(x))dx
\]

\[
+ \int_{\partial \Omega} [B(x')\nabla_x \psi_0(x') \cdot \nabla_x \bar{\psi_1}(x') + \psi_0(x)\bar{\psi_1}(x')]d\sigma.
\]

We introduce the following operator \( A_2 \) in \( H \times H \):

\[
\begin{aligned}
D(A_2) = W &:= \{(\phi, \psi) \in H^2(\Omega) \times H^2(\partial \Omega) : \psi = \gamma \phi \}, \\
A_2(\phi, \psi) &= (A_0(\cdot, D_x)\phi, \nabla_x \cdot (B(\cdot)\nabla_x \psi) + F(\cdot, D_x)\phi).
\end{aligned}
\]

The main result if this section is the following

**Theorem 2.1.** Suppose that (A1)-(A5) are fulfilled. We introduce the following operator \( M \):

\[
\begin{aligned}
D(M) &= W \times V, \\
M(\phi, \psi) &= (f, h).
\end{aligned}
\]

Then \( M \) is the infinitesimal generator of a strongly continuous group in \( V \times H \).

We begin the proof of Theorem 2.1 by recalling the well known procedure of identifying the element \((f, h)\) of \( H \) with the element \( J(f, h) \) of \( V^* \) defined as

\[
(J(f, h), (\phi, \psi))_H = \int_{\Omega} f(x)\phi(x)dx + \int_{\partial \Omega} h(x')\psi(x')d\sigma,
\]

where \( \phi, \psi \in V \).
Remark 1. By Lemma 2.3, in case $A$ is the infinitesimal generator of an analytic semigroup in $X$, we deduce
\begin{align*}
|\langle J(f, h), (\phi, \psi) \rangle| & \leq \|(f, h)\|_H \|((\phi, \psi))_H \leq C_0 \|(f, h)\|_H \|((\phi, \psi))_V|,
\end{align*}
for any $(\phi, \psi)$ in $V$. We deduce that $\|(f, h)\|_V ≤ C_0 \|(f, h)\|_H$. So the identification of $(f, h)$ with $J(f, h)$ carries to $\|(f, h)\|_{V'} ≤ C_0 \|(f, h)\|_H$ and $H \hookrightarrow V^*$. We introduce the operator $A_0$, defined as follows:
\begin{align}
D(A_0) = & \{ (u, v) \in V : \exists (f, h) \in H : ((u, v), (\psi, \psi))_V = ((f, h), (\phi, \psi))_H \}, \\
A_0(u, v) = & (f, h).
\end{align}
(4)
The following result is well known (for a proof, see [11], Chapter 2.2).

Lemma 2.2. If $A_0$ is the linear operator defined in (4), $D(A_0)$ is dense in $H$, $A_0$ is self-adjoint and positive and $D(A_0^{1/2}) = V$.

Concerning $D(A_0)$, we have:

Lemma 2.3. Suppose that (A1)-(A4) hold. Then
\begin{align*}
D(A_0) &= W \\
\forall (u, v) \in W & \quad A_0(u, v) = (-A_0(\cdot, D_x)u, -\nabla \cdot (B(\cdot)\nabla v) + \frac{\partial u}{\partial v_A} + v).
\end{align*}

Proof. We consider the operator $A_1 : W \rightarrow H$, $A_1(u, v) = (-A_0(\cdot, D_x)u, -\nabla \cdot (B(\cdot)\nabla v) + \frac{\partial u}{\partial v_A} + v)$.

It is immediately seen, employing Green’s formula, that
\begin{align*}
A_1 \subseteq A_0.
\end{align*}
(5)
On the other hand, as $A_0$ is self-adjoint and positive, $-A_0$ is the infinitesimal generator of an analytic semigroup in $H$. By Theorem 4.1 in [6], the same happens for $A_1$. So $\rho(A_0) \cap \rho(A_1) \neq \emptyset$. This, together with (5), implies the conclusion. \hfill \Box

Remark 1. By Lemma 2.3, in case $F(\cdot, D_x) = -\frac{\partial}{\partial v_A} - \gamma$, $A_2 = -A_0$.

Now we are able to employ the following result (see [7], Theorem 7.2):

Theorem 2.4. Let $B$ be the infinitesimal generator of a strongly continuous group in the Banach space $X$. Assume that $0 \in \rho(B)$. Define
\begin{align*}
\begin{dcases}
D(M_0) = D(B^2) \times D(B), \\
M_0(u, v) = (v, B^2 u).
\end{dcases}
\end{align*}
Then $M_0$ is the infinitesimal generator of a strongly continuous group in the Banach space $D(B) \times X$.

Corollary 1. Suppose that (A1)-(A4) hold. We introduce the following operator $M_0$:
\begin{align}
\begin{dcases}
D(M_0) = W \times V, \\
M_0((\phi, \psi), (f, h)) = ((f, h), -A_0(\phi, \psi)).
\end{dcases}
\end{align}
(6)
Then $M_0$ is the infinitesimal generator of a strongly continuous group in $V \times H$.

Proof. We set $B := iA_0^{1/2}$. Then $B$ is skew-adjoint and $D(B) = V$. By Stone’s theorem, $B$ is the infinitesimal generator of a strongly continuous group of isometries in $H$. We have $B^2 = -A_0$. So the conclusion follows from Theorem 2.4. \hfill \Box
We shall indicate with \((e^{tM_0})_{t \in \mathbb{R}}\) the group generated by \(M_0\) in \(V \times H\).

**Remark 2.** If \((\phi, \psi) \in V\) and \((\alpha, \beta) \in H\), we shall often write \((\phi, \psi, \alpha, \beta)\) instead of \(((\phi, \psi), (\alpha, \beta))\).

**Remark 3.** If \((\phi, \psi, \alpha, \beta)\) belongs to \(V \times H\) and its components are real valued, then the components of \(e^{tM_0}(\phi, \psi, \alpha, \beta)\) are real valued. In case \((\phi, \psi, \alpha, \beta) \in W \times V\), this can be easily deduced from the uniqueness of the solution of the problem

\[
\begin{aligned}
D^2_t u(t, x) &= A_0(x, D_x)u(t, x), \quad (t, x) \in \mathbb{R} \times \Omega, \\
D^2_t \gamma u(t, x') &= \nabla_x \cdot (B(x') \nabla \gamma u)(t, x') - \frac{\partial u}{\partial \nu}(t, x') - \gamma u(t, x'), \\
(t, x') &\in \mathbb{R} \times \partial \Omega, \\
u(0, x) &= \phi(0, x), \quad x \in \Omega, \\
D_t \nu(0, x) &= \alpha(x), \quad x \in \Omega,
\end{aligned}
\]

which follows from Corollary 1. The general case follows by continuity.

**Remark 4.** Suppose that the assumptions \((A1)-(A4)\) are satisfied. Let \(u_0 \in H^2(\Omega), \gamma u_0 \in H^2(\partial \Omega), u_1 \in H^1(\Omega), \gamma u_1 \in H^1(\partial \Omega)\), so that \((u_0, \gamma u_0, u_1, \gamma u_1) \in W \times V\). Let

\[
(\phi(t), \psi(t), \alpha(t), \beta(t)) = e^{tM_0}(u_0, \gamma u_0, u_1, \gamma u_1). \tag{7}
\]

Then \(\phi \in C^{2-j}(\mathbb{R}; H^1(\Omega)), \psi \in C^{2-j}(\mathbb{R}; H^1(\partial \Omega)), \psi = \gamma \phi, \alpha = D_t \phi, \beta = D_t \psi = \gamma \alpha\). Moreover,

\[
D_t \phi(t, x) = A_0(x, D_x)\phi(t, x), \quad (t, x) \in \mathbb{R} \times \Omega.
\]

So \(\phi\) is also the solution of the mixed Cauchy-Dirichlet problem

\[
\begin{aligned}
D^2_t \phi(t, x) &= A_0(x, D_x)\phi(t, x), \quad (t, x) \in (a, b) \times \Omega, \\
\phi(t, x') &= \psi(t, x'), \quad (t, x') \in (a, b) \times \partial \Omega, \\
\phi(0, x) &= u_0(x), \quad x \in \Omega, \\
D_t \phi(0, x) &= u_1(x), \quad x \in \Omega. \tag{8}
\end{aligned}
\]

Now we want to replace \(-\frac{\partial u}{\partial \nu} - \gamma\) with an essentially arbitrary differential operator of order not exceeding one. To this aim, the key fact is the following result, following the idea of [8], Theorem 2.1. For a slightly different situation, see also [5].

**Theorem 2.5.** Suppose that the conditions \((A1)-(A4)\) are fulfilled. Let \(a, b \in \mathbb{R}\), with \(a < b\). Then there exists \(C(a, b)\) positive such that, \(\forall (u_0, \gamma u_0, u_1, \gamma u_1)\) belonging to \(W \times V\),

\[
\|\frac{\partial \phi}{\partial t}\|_{L^2((a,b) \times \partial \Omega)} \leq C(a, b)(\|u_0, \gamma u_0, u_1, \gamma u_1\|_{W \times H}),
\]

with \(\phi(t)\) first component of \(e^{tM_0}(u_0, \gamma u_0, u_1, \gamma u_1)\) (see Remark 4).

**Proof.** We continue to employ the notation (7). Concerning the proof, it suffices to consider the case that the components of \((u_0, \gamma u_0, u_1, \gamma u_1)\) are real valued, so that, by Remark 3, \(\phi\) is real valued.

We set

\[
N := \|u_0\|_{H^1(\Omega)}^2 + \|\gamma u_0\|_{H^1(\partial \Omega)}^2 + \|u_1\|_{L^2(\Omega)}^2 + \|\gamma u_1\|_{L^2(\partial \Omega)}^2.
\]

By simplicity of notation, we shall write, given a certain expression \(E\), that \(E = O(N)\) if there exists \(C\) positive, possibly depending on \((a, b)\), but not \((u_0, \gamma u_0, u_1, \gamma u_1)\), such that

\[
|E| \leq C\|u_0, \gamma u_0, u_1, \gamma u_1\|_{W \times H}^2.
\]
For example,
\[
\max_{t \in [a,b]} \{ \phi(t) \| H^1(\Omega) + \| \psi(t) \| H^1(\partial \Omega) + \| \alpha(t) \| L^2(\Omega) + \| \beta(t) \| L^2(\partial \Omega) \} = O(N).
\]

We introduce a function \( h \in C^1(\overline{\Omega}; \mathbb{R}^n) \) such that, for each \( j \in \{1, \ldots, n\} \), if \( x' \in \partial \Omega \),
\[
h_j(x') = \sum_{i=1}^n a_{i,j}(x') \nu_i(x').
\]

If \( b \in H^1(\Omega) \), we have, by Green’s formula,
\[
\int_{\Omega} h(x) \cdot \nabla_x b(x) dx = \int_{\partial \Omega} A(x') b(x') d\sigma(x') - \int_{\Omega} \text{div}_x(h)(x)b(x)dx,
\]
with
\[
A(x') = \sum_{i=1}^n \sum_{j=1}^n a_{i,j}(x') \nu_i(x') \nu_j(x'), \quad x' \in \partial \Omega.
\]

We have also that in \((a, b) \times \partial \Omega\), for \( j = 1, \ldots, n \), employing the notation (7),
\[
D_{x_j} \phi(t, x') = \nu_j(x') \frac{\partial \phi}{\partial \nu}(t, x') + T_j \psi(t, x'),
\]
with \( T_j \) differential operator of order one in \( \partial \Omega \), with coefficients in \( C^1(\partial \Omega) \). Then, by Remark 4,
\[
\int_{(a,b) \times \Omega} D_t^2 \phi(t, x) h(x) \cdot \nabla_x \phi(t, x) dt dx = \int_{(a,b) \times \Omega} A_0(x, D_x) \phi(t, x) h(x) \cdot \nabla_x \phi(t, x) dt dx.
\]

Now,
\[
\int_{(a,b) \times \Omega} D_t^2 \phi(t, x) h(x) \cdot \nabla_x \phi(t, x) dt dx = \int_{\Omega} \alpha(b, x) h(x) \cdot \nabla_x \phi(b, x) dx - \int_{\Omega} \alpha(a, x) h(x) \cdot \nabla_x \phi(a, x) dx - \int_{(a,b) \times \Omega} \alpha(t, x) h(x) \cdot \nabla_x \alpha(t, x) dt dx.
\]

We have
\[
| \int_{\Omega} \alpha'(b, x) h(x) \cdot \nabla_x \phi(b, x) dx - \int_{\Omega} \alpha(a, x) h(x) \cdot \nabla_x \phi(a, x) dx | \leq C(a, b)(\| \alpha(b, \cdot) \| L^2(\Omega))^{\| \phi(b, \cdot) \| H^1(\Omega)} + \| \alpha(a, \cdot) \| L^2(\Omega))\| \phi \| C([a,b]; H^1(\Omega))
\]
\[
= O(N).
\]

Moreover, by (9),
\[
| \int_{(a,b) \times \Omega} \alpha(t, x) h(x) \cdot \nabla_x \alpha(t, x) dt dx | = \frac{1}{2} | \int_{(a,b) \times \Omega} \alpha(t, x) h(x) \cdot \nabla_x \alpha(t, x) dt dx |
\]
\[
= \frac{1}{2} | \int_a^b \int_{\Omega} A(x') \beta(t, x')^2 d\sigma(t) dt - \int_a^b \int_{\Omega} \text{div}_x h(x) \alpha(t, x')^2 dx dt | \leq C(|| \beta ||_{C([a,b]; L^2(\Omega))} + || \alpha ||_{C([a,b]; L^2(\Omega))})
\]
\[
= O(N).
\]

So, by (11),
\[
\int_{(a,b) \times \Omega} A_0(x, D_x) \phi(t, x) h(x) \cdot \nabla_x \phi(t, x) dt dx = O(N).
\]
We have

\[
I_1 = \int_{(a,b) \times \Omega} A_0(x, D_x) \phi(t, x) h(x) \cdot \nabla_x \phi(t, x) \, dt \, dx
\]

Moreover,

\[
I_2 = -\int_{(a,b) \times \Omega} \phi(t, x) D_{x_k} \phi(t, x) h_k(x) \, dt \, dx
\]

so that

\[
\sum_{k=1}^{n} \int_{(a,b) \times \Omega} a_{ij}(x) D_{x_j} \phi(t, x) h_k(x) \, dt \, dx = \sum_{j=1}^{n} \int_{(a,b) \times \Omega} a_{ij}(x) D_{x_j} \phi(t, x) h_k(x) \, dt \, dx
\]

\[
\sum_{k=1}^{n} \int_{(a,b) \times \Omega} a_{ij}(x) D_{x_j} \phi(t, x) h_k(x) \, dt \, dx = \sum_{j=1}^{n} \int_{(a,b) \times \Omega} a_{ij}(x) D_{x_j} \phi(t, x) h_k(x) \, dt \, dx
\]

Moreover,

\[
I_2 = -\int_{(a,b) \times \Omega} \int_{\Omega, \nu} \phi(t, x) D_{x_k} \phi(t, x) h_k(x) \, dt \, dx
\]

We deduce that

\[
I_1 - \frac{I_2}{2} = O(N). \tag{12}
\]

We have

\[
I_3 = \int_{(a,b) \times \Omega} \int_{\Omega, \nu} \phi(t, x) D_{x_k} \phi(t, x) h_k(x) \, dt \, dx
\]

so that

\[
I_3 = \int_{(a,b) \times \Omega} \int_{\Omega, \nu} \phi(t, x) D_{x_k} \phi(t, x) h_k(x) \, dt \, dx
\]

with \( S_1 \) differential operator of order one in \( \partial \Omega \), while

\[
I_3 = \int_{(a,b) \times \Omega} \int_{\Omega, \nu} \phi(t, x) D_{x_k} \phi(t, x) h_k(x) \, dt \, dx
\]

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x) \nu_i(x') T_j \psi(t, x')
\]

so that

\[
I_3 = \int_{(a,b) \times \Omega} \int_{\Omega, \nu} \phi(t, x) D_{x_k} \phi(t, x) h_k(x) \, dt \, dx
\]

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x) \nu_i(x') T_j \psi(t, x')
\]

so that

\[
I_3 = \int_{(a,b) \times \Omega} \int_{\Omega, \nu} \phi(t, x) D_{x_k} \phi(t, x) h_k(x) \, dt \, dx
\]

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(x) \nu_i(x') T_j \psi(t, x')
\]
with $S_2$ differential operator of order one in $\partial \Omega$. From (12) we deduce
\[
\frac{1}{2} \int_{(a,b) \times \partial \Omega} A(x')^2 \frac{\partial \phi}{\partial \nu}(t,x')dtd\sigma + \int_{(a,b) \times \partial \Omega} \frac{\partial \phi}{\partial \nu}(t,x')(S_1 - \frac{1}{2} S_2)\psi(t,x')dtd\sigma = O(N),
\]
and, as $A(x')^2$ is lower bounded by a positive constant, for some $C_0$ positive independent of $u_0,u_1$,
\[
\int_{(a,b) \times \partial \Omega} \frac{\partial u}{\partial \nu}(t,x')^2dtd\sigma \leq C_0[N + N^{1/2} \int_{(s,T) \times \partial \Omega} \frac{\partial u}{\partial \nu}(t,x')^2dtd\sigma]^{1/2},
\]
implying (supposing $C_0 \geq 4$)
\[
\int_{(a,b) \times \partial \Omega} \frac{\partial u}{\partial \nu}(t,x')^2dtd\sigma \leq \left( \frac{C_0 + \sqrt{C_0^2 - 4C_0}}{4} \right)
\]
\end{proof}

**Corollary 2.** Suppose (A1)-(A4) hold. Let $T \in \mathbb{R}^+$, $u_0 \in H^2(\Omega)$ with $\gamma u_0 \in H^2(\partial \Omega)$, $u_1 \in H^1(\Omega)$ with $\gamma u_1 \in H^1(\partial \Omega)$, so that $(u_0, \gamma u_0, u_1, \gamma u_1) \in W \times V$. Let
\[
(\phi(t), \psi(t), \alpha(t), \beta(t)) = e^{tM_0}(u_0, \gamma u_0, u_1, \gamma u_1)
\]
Moreover, if $x' \in \partial \Omega$, let $G(x',D_x)u(x') = \sum_{j=1}^n g_j(x')D_j u(x') + g_0(x'u(x')$, with $g_j \in L^\infty(\partial \Omega)$. Then there exists $C(T)$ positive, independent of $u_0$ and $u_1$, such that
\[
\|G(\cdot, D_x)\phi\|_{L^2([-T,T] \times \partial \Omega)} \leq C(T)(\|u_0, \gamma u_0, u_1, \gamma u_1\|_{V \times H}).
\]

**Proof.** If we set
\[
k(x') := g(x') \cdot \nu(x'),
\]
then $t(x') := g(x') - k(x')\nu(x')$ is tangential to $\partial \Omega$ in $x'$. So
\[
G(x', D_x)\phi(t,x') = k(x') \frac{\partial \phi}{\partial \nu}(t,x') + t(x') \cdot \nabla_x \psi(t,x') + g_0(x')\psi(t,x').
\]
and
\[
\|G(\cdot, D_x)\phi(\cdot, \cdot)\|_{L^2(\partial \Omega)} \leq C_0(\|\frac{\partial \phi}{\partial \nu}(\cdot, \cdot)\|_{L^2(\partial \Omega)} + \|\psi(\cdot, \cdot)\|_{H^1(\partial \Omega)}).
\]

Now we recall the following perturbation result of Miyadera type (see [4], Corollary 3.16):

**Theorem 2.6.** Let $A$ be the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ on a Banach space $X$ and let $B \in \mathcal{L}(D(A), X)$ satisfy, for some $t_0 > 0$, $q \in [0,1)$,
\[
\int_{-t_0}^{t_0} \|BT(t)x\|dt \leq q\|x\|, \quad \forall x \in D(A).
\]

Then $A + B$, with domain $D(A)$, is the infinitesimal generator of a strongly continuous semigroup in $X$.

**Corollary 3.** Let $A$ be the infinitesimal generator of a strongly continuous group $(T(t))_{t \in \mathbb{R}}$ on a Banach space $X$ and let $B \in \mathcal{L}(D(A), X)$ satisfy, for some $t_0 > 0$, $q \in [0,1)$,
\[
\int_{-t_0}^{t_0} \|BT(t)x\|dt \leq q\|x\|, \quad \forall x \in D(A).
Then $A + B$, with domain $D(A)$, is the infinitesimal generator of a strongly continuous group in $X$.

**Proof.** As $((T(t))_{t \in \mathbb{R}}$ is a group, if we set $T_-(t) := T(-t)$, with $t \geq 0$, $(T_-(t))_{t \geq 0}$ is a strongly continuous semigroup with infinitesimal generator $-A$. By Theorem 2.6, $-A - B$, with domain $D(A)$, is the infinitesimal generator of a strongly continuous semigroup. As both $\pm (A + B)$ are infinitesimal generators of strongly continuous semigroups, $A + B$ is the infinitesimal generator of a strongly continuous group. □

Now we are able to prove Theorem 2.1.

**Proof of Theorem 2.1.** We set $X = V \times H$, $A = M_0$ and we introduce the following operator $B$:

$$
\begin{cases}
B : W \times V \rightarrow V \times H, \\
B(u_0, \gamma u_0, u_1, \gamma u_1) = (0, 0, 0, F(\cdot, D_x)u_0 + \frac{\partial u_0}{\partial \nu_A} + \gamma u_0).
\end{cases}
$$

Setting $G(\cdot, D_x) = F(\cdot, D_x) + \frac{\partial}{\partial \nu_A} + \gamma$, we have, taking $t_0 \in (0, 1]$, with the position (13),

$$
\int_{t_0}^{t_0} \|B e^{tM_0}(u_0, \gamma u_0, u_1, \gamma u_1)\|_{V \times H} dt
\leq (2t_0)^{1/2} \int_{t_0}^{t_0} \|G(\cdot, D_x) \phi(t)\|_2^2 dt)^{1/2} \leq C(1)(2t_0)^{1/2} \|(u_0, \gamma u_0, u_1, \gamma u_1)\|_{V \times H}.
$$

So the assumptions of Corollary 3 are satisfied and the conclusion follows from the fact that $M = M_0 + B$.

3. **Developments of Theorem 2.1.** We shall employ the following well known fact, concerning strongly continuous semigroups:

**Proposition 1.** Let $A$ be the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ in the Banach space $X$. Let $x \in D(A)$ and $f \in W^{1,1}(0, T; X) + C([0, T]; X) \cap L^1(0, T; D(A))$. Then the Cauchy problem

$$
\begin{cases}
u'(t) = Au(t) + f(t), & t \in [0, T], \\
u(0) = x
\end{cases}
$$

has a unique solution $u$ in $C^1([0, T]; X) \cap C([0, T]; D(A))$ given by the variation of parameter formula

$$
u(t) = T(t)x + \int_0^t T(t-s)f(s)ds.
$$

We consider the following problem:

$$
\begin{cases}
D_t^2 u(t, x) + a(x)D_t u(t, x) = A(x, D_x)u(t, x) + f(t, x), & (t, x) \in (0, T) \times \Omega, \\
D_t^2 \gamma u(t, x') + b(x')D_t \gamma u(t, x') = \nabla_x \cdot (B(x')\nabla \gamma u)(t, x') + F(x', D_x)u(t, x') + h(t, x'), & (t, x') \in (0, T) \times \partial \Omega, \\
u(0, x) = u_0(x), & x \in \Omega, \\
D_t u(0, x) = u_1(x), & x \in \Omega.
\end{cases}
$$

We introduce the following assumptions:
Suppose, moreover, that:

Proposition 3.

(B1) (A1), (A3), (A4), (A5) hold;
\[ A(x, D_x) = A_0(x, D_x) + \sum_{j=1}^{n} a_j(x) D_{x_j} + a_0(x), \]
with \( A_0 \) as in (A2), \( a_j \in L^\infty(\Omega) \) \((0 \leq j \leq n)\);

(B2) \( a \in L^\infty(\Omega) \), \( b \in L^\infty(\partial\Omega) \).

Then we have:

Proposition 2. Suppose that (B1)-(B2) hold. We introduce the following operator \( M_1 : V \times H \to V \times H \):
\[
\begin{aligned}
M_1(v_0, v_1, w_0, w_1) &= (0, 0, \sum_{j=1}^{n} a_j(\cdot) D_{x_j} v_0 + a_0(\cdot) v_0 - a(\cdot) w_0, -b(\cdot) w_1).
\end{aligned}
\]

Then

(I) \( M + M_1 \), with domain \( W \times V \), is the infinitesimal generator of a strongly continuous group in \( V \times H \);

(II) consider the problem (15), with \( T \in \mathbb{R}^+ \). Suppose, moreover, that:

(a) \( u_0 \in H^2(\Omega) \), \( \gamma u_0 \in H^2(\partial\Omega) \), \( v_1 \in H^1(\Omega) \), \( \gamma u_1 \in H^1(\partial\Omega) \);

(b) \( f(t, x) = f_1(t, x) + f_2(t, x) \), with \( f_1 \in C([0, T]; L^2(\Omega)) \cap L^1(0, T; H^1(\Omega)) \), \( \gamma f_1 \in C([0, T]; L^1(\partial\Omega)) \cap L^1(0, T; H^1(\partial\Omega)) \), \( f_2 \in W^{1,1}(0, T; L^2(\Omega)) \);

(c) \( h(t, x') = \gamma f_1(\cdot, x') + h_1(t, x') \), with \( h_1 \in W^{1,1}(0, T; L^2(\partial\Omega)) \).

Then (15) has a unique solution \( u \) belonging to \( \cap_{j=0}^{2} C^{2-j}([0, T]; H^j(\Omega)) \), with \( \gamma u \) belonging to \( \cap_{j=0}^{2} C^{2-j}([0, T]; H^j(\partial\Omega)) \).

Proof. (I) follows from Theorem 2.1 and the fact that \( M_1 \) belongs to \( \mathcal{L}(V \times H) \).

(II) We set \( \phi := u, \psi := \gamma u, \alpha := D_t u, \beta := \gamma \alpha = D_t \psi \). Then (15) can be written in the equivalent form
\[
(\phi(t), \psi(t), \alpha(t), \beta(t)) = (M + M_1)(\phi(t), \psi(t), \alpha(t), \beta(t)) + (0, 0, f(t), h(t)),
\]
for \( t \in [0, T] \),
\[
(\phi(0), \psi(0), \alpha(0), \beta(0)) = (u_0, \gamma u_0, v_1, \gamma u_1).
\]
Then \( (u_0, \gamma u_0, \alpha(0), \beta(0)) \) belongs to \( W \times V \), while
\[
(0, 0, f(t), h(t)) = (0, 0, f_1(t), \gamma f_1(t)) + (0, 0, f_2(t), h_1(t)),
\]
with the first summand in
\[
C([0, T]; V \times H) \cap L^1(0, T; W \times V) = C([0, T]; V \times H) \cap L^1(0, T; D(M + M_1)),
\]
the second summand in \( W^{1,1}(0, T; V \times H) \). By Proposition 1, (16) has a unique solution in \( C^1([0, T]; V \times H) \cap C([0, T]; W \times V)) \). \( \square \)

We conclude with an application to (2).

Proposition 3. Consider the problem (2), with the assumption (B1) and \( T \in \mathbb{R}^+ \). Suppose, moreover, that:

(a) \( u_0 \in H^2(\Omega) \), \( \gamma u_0 \in H^2(\partial\Omega) \), \( v_1 \in H^1(\Omega) \), \( \gamma u_1 \in H^1(\partial\Omega) \);

(b) \( f \in C([0, T]; L^2(\Omega)) \cap L^1(0, T; H^1(\Omega)) \), \( \gamma f \in C([0, T]; L^2(\partial\Omega)) \cap L^1(0, T; H^1(\partial\Omega)) \);

(c) \( h \in W^{1,1}(0, T; L^2(\partial\Omega)) \).
Then (2) has a unique solution \( u \) belonging to \( \cap_{j=0}^{2} C^{2-j}([0,T]; H^j(\Omega)) \), with \( \gamma u \) belonging to \( \cap_{j=0}^{2} C^{2-j}([0,T]; H^j(\partial \Omega)) \). Here \( A(x', D_x)u(t,x') \) is intended as \( D_t^2 \gamma u - \gamma f \).

Proof. The problem is equivalent to (15) with \( a \equiv 0, b \equiv 0 \) and \( h \) replaced by \( \gamma f + h \). So the conclusion follows from Proposition 2. \( \square \)

REFERENCES

[1] M. Cavalcanti, A. Khemmoudj and M. Medjden, Uniform stabilisation of the damped Cauchy-Ventcel problem with variable coefficients and dynamic boundary conditions, J. Math. Anal. Appl., 328 (2007), 900–930.
[2] R. Clendenen, G. R. Goldstein and J. A. Goldstein, Degenerate flux for dynamic boundary conditions in parabolic and hyperbolic equations, Discr. Cont. Dynam. Syst. Ser. S, 9 (2016), 651–660.
[3] G. M. Coclite, A. Favini, G. R. Goldstein, J. A. Goldstein and S. Romanelli, Continuous dependence in hyperbolic problems with Wentzell boundary conditions, Commun. Pure Applied Anal., 13 (2004), 419–433.
[4] K.-J. Engel and R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Graduate Texts in Mathematics, 194, Springer, 2000.
[5] C. Giorgi and D. Guidetti, Reconstruction of kernel depending also on a space variable, Math. Methods Appl. Sci., 41 (2018), 4560–4588.
[6] G. R. Goldstein, J. A. Goldstein, D. Guidetti and S. Romanelli, Maximal regularity, analytic semigroups, and dynamic and general Wentzell boundary conditions with a diffusion term on the boundary, Annali di Matematica Pura ed Applicata, 2019.
[7] J. A. Goldstein, Semigroups of Linear Operators & Applications, Dover Publications, Inc. (Second Edition), 2017.
[8] I. Lasiecka, J. L. Lions and R. Triggiani, Non homogeneous boundary value problems for second order hyperbolic operators, J Math Pures et Appl., 65 (1986), 149–192.
[9] S. Nicaise and K. Laoubi, Polynomial stabilization of the wave equation with Ventcel’s boundary conditions, Math. Nachr., 283 (2010), 1428–1438.
[10] G. Ruiz Goldstein, Derivation and physical interpretation of general boundary conditions, Adv. Diff. Eq., 11 (2006), 457–480.
[11] H. Tanabe, Equations of Evolution, Monographs and Studies in Mathematics, 6. Pitman (Advanced Publishing Program), Boston, Mass.-London, 1979.
[12] E. Vitillaro, On the wave equation with hyperbolic dynamical boundary conditions, interior and boundary damping and supercritical sources, Journ. Diff. Eq., 265 (2018), 4873–4941.

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