YOUNG’S SEMINORMAL FORM AND SIMPLE MODULES FOR S_n IN CHARACTERISTIC p.

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Abstract. We realize the integral Specht modules for the symmetric group S_n as induced modules from the subalgebra of the group algebra generated by the Jucys-Murphy elements. We deduce from this that the simple modules for \( \mathbb{F}_p S_n \) are generated by reductions modulo \( p \) of the corresponding Jucys-Murphy idempotents.

1. Introduction.

This article is a continuation of the investigation pursued in [RH1-2] that seeks to demonstrate the importance of Young’s seminormal basis for the modular, that is characteristic \( p \), representation theory of the symmetric group \( S_n \). A main obstacle is here that Young’s seminormal basis is defined over the field \( \mathbb{Q} \) and indeed there seems to be a general consensus that this obstacle makes Young’s seminormal basis a characteristic zero phenomenon, essentially. Still we believe that Young’s seminormal basis is a fundamental object for the modular representation theory as well, and we think that the results of our works provide strong evidence in favor of this claim.

Let \( \text{Par}_n \) be the set of partitions of \( n \) and let \( S(\lambda) \) be the integral Specht module for \( S_n \) associated with \( \lambda \in \text{Par}_n \). Then, as has been known for a long time, the set of \( S_\mathbb{Q}(\lambda) := S(\lambda) \otimes_{\mathbb{Z}} \mathbb{Q} \) classifies the irreducible \( \mathbb{Q} S_n \)-modules when \( \lambda \in \text{Par}_n \), whereas the reduced Specht modules \( S(\lambda) \otimes_{\mathbb{Z}} \mathbb{F}_p \) are reducible in general. In fact the irreducible modules \( D(\lambda) \) for \( \mathbb{F}_p S_n \) are classified by the set of \( p \)-regular partitions \( \text{Par}_n^{\text{reg}} \) and are obtained as \( D(\lambda) = S(\lambda) / \text{rad}(\cdot, \cdot) \) where \( \langle \cdot, \cdot \rangle \) is a certain symmetric bilinear and \( S_n \)-invariant form on \( S(\lambda) \).

The decomposition numbers \([ S(\lambda) : D(\mu) ] \) for \( \mathbb{F}_p S_n \) have been the topic of much research activity in recent years, but still remain unknown in general and even the dimensions of \( D(\mu) \) are not known in general. But using the theory of Young’s seminormal form we obtain in this work, as our main Theorem 5, a construction of \( D(\mu) \) that may be a good starting point for obtaining combinatorial expressions for \( \dim D(\mu) \).

The basic principles behind this construction are parallels of standard methods in the modular representation theory of algebraic groups. Indeed, let \( S(\lambda)^{\circ} \) denote the contragredient dual of \( S(\lambda) \). Then \( \langle \cdot, \cdot \rangle \) corresponds to a homomorphism \( c_\lambda : S(\lambda) \to S(\lambda)^{\circ} \). Moreover, for \( \lambda \in \text{Par}_n^{\text{reg}} \) we have that \( D(\lambda) = \text{im} \overline{c_\lambda} \) where \( \overline{c_\lambda} \) is the reduced homomorphism modulo \( p \). Passing to the representation theory of an algebraic group \( G \) over an algebraically closed field of characteristic \( p \), the Weyl module \( \Delta(\lambda) \), the dual Weyl module \( \nabla(\lambda) \) and the simple module \( L(\lambda) \) correspond to \( S(\lambda), S(\lambda)^{\circ} \) and \( D(\lambda) \) and the bilinear form \( \langle \cdot, \cdot \rangle \) on \( S(\lambda) \) corresponds to a form on \( \Delta(\lambda) \) that we denote

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the same way. It induces a $G$-linear homomorphism $c_\lambda : \Delta(\lambda) \to \nabla(\lambda)$ and the simple module satisfies $L(\lambda) = \text{im} \, c_\lambda$. But in the $G$-module setting, $\nabla(\lambda)$ can also be constructed as the module of global sections of a line bundle on the associated flag manifold, and using this, one obtains a new construction of $c_\lambda$ without using the bilinear form. The properties of this new construction of $c_\lambda$ then provide a useful method for obtaining information on $L(\lambda)$, see eg. [A, Jan].

Returning to the symmetric group, we then look for a different construction of $c_\lambda$. For this we prove in our Theorem 3 that $S(\lambda)$ is induced from a certain subalgebra, denoted $G\mathbb{Z}_n$, of the group algebra, corresponding to the fact that $\nabla(\lambda)$ is induced from a Borel subgroup of $G$. Given this, our new construction of $c_\lambda$ is obtained from a Frobenius reciprocity argument.

At the basis of our work are the famous Jucys-Murphy elements $L_k$, $k = 1, 2, \ldots, n$ that were introduced independently by Jucys and Murphy in [Ju1-3] and [Mu81]. They give rise to idempotents $E_t$ of $\mathbb{Q}S_n$, the Jucys-Murphy idempotents, indexed by $\lambda$-tableaux $t$, that are closely related to Young’s seminormal basis of the Specht module $S_{\mathbb{Q}}(\lambda)$. Moreover they commute with each other and therefore generate a commutative subalgebra of the group algebra. This is the algebra $G\mathbb{Z}_n$ that was mentioned above, the Gelfand-Zetlin algebra. In the case of the ground field $\mathbb{Q}$ it was considered by Okounkov and Vershik in [OV] as a kind of Cartan subalgebra of a semisimple Lie algebra, but for us it is important to work with an integral version of $G\mathbb{Z}_n$, where the analysis of [OV] fails.

We have now formulated the main ingredients of our result. The surprisingly simple final result is that $D(\lambda)$ is generated by $a_\lambda E_\lambda$ where $E_\lambda = E_{t,\lambda}$ and $a_\lambda$ is the least common multiple of the denominators of $E_t$. It should be noted that, even though it appears to be a very natural idea to investigate the $\mathbb{Z}_n$ or $\mathbb{F}_p S_n$-submodule of $S(\lambda)$ generated by $a_\lambda E_\lambda$, the only reference in the literature along these lines is [RH2], as far as we know.

In an important recent paper [BK], J. Brundan and A. Kleshchev showed that $\mathbb{F}_p S_n$ is a $\mathbb{Z}$-graded algebra in a nontrivial way by establishing an isomorphism between $\mathbb{F}_p S_n$ and the cyclotomic KLR-algebra, i.e. cyclotomic Khovanov-Lauda-Rouquier algebra, of type $A$. Their results work in greater generality than $\mathbb{F}_p S_n$ but we shall only consider this case. J. Hu and A. Mathas refined in [HuMa] this graded structure on $\mathbb{F}_p S_n$ to a graded cellular algebra structure by constructing an explicit graded cellular basis. A second goal of our paper is to show that key features of their constructions can be carried out entirely within the theory of Young’s seminormal form, as developed by Murphy. We hope that this approach to their results, together with our main Theorem 5, may provide a combinatorial expression for $\text{dim} \, D(\lambda)$.

The generators of the cyclotomic KLR-algebra are

\[ \{ e(i) \mid i \in (\mathbb{F}_p)^n \} \cup \{ y_1, \ldots, y_n \} \cup \{ \psi_1, \ldots, \psi_{n-1} \} \]

and [BK] prove their Theorem by constructing elements in $\mathbb{F}_p S_n$, denoted the same way, that verify the cyclotomic KLR-algebra relations. The $y_i$ are essentially Jucys-Murphy operators and $e(i)$ are certain idempotents, not necessarily nonzero. In fact they can be identified with the idempotents constructed in [Mu83] by summing Jucys-Murphy idempotents $E_t$ over tableaux.
classes. The elements $\psi_i$ are the most difficult to handle and [BF] take as starting point for this certain explicitly given intertwining elements $\phi_i$. These intertwiners, together with the $e(i)$ and $y_i$, already satisfy relations that are close to the cyclotomic KLR-relations but still need to be adjusted to get the complete match.

We here give a natural construction of the intertwining elements $\phi_i$ within Murphy’s theory for the seminormal basis. Indeed, we see them as natural analogues of certain elements $\Psi_i$ of the Hecke algebra that appear in [Mu92], although only in the semisimple case. We show that Murphy’s ideas, in a suitable sense, can be carried out over $\mathbb{F}_p$ as well. From this we obtain a cellular basis for $\mathbb{F}_pS_n$ using a modification of the constructions done in [HuMa].

Let us sketch the layout of the paper. In section 2 we fix the basic notation of the paper. It is mostly standard, except possibly for the notion of tableau class which was introduced in [Mu83]. We also review the construction from [Mu83] of the tableau class idempotents. Section 3 contains the construction of the intertwiners $\Psi_{L,i}$. This requires a control of the denominators of the Jucys-Murphy idempotents that are involved in the tableau class idempotents. In section 4 we construct the cellular basis. In section 5 we first introduce the Gelfand-Zetlin algebra $GZ_n$ and then set up the induction functor. We then prove that the Specht module is induced up from a “rank one” module of $GZ_n$. An important ingredient for this is a uniqueness statement, due to James [J], of the integral Specht module. Finally in section 6 we deduce our main results.

Note that the notation used throughout the paper may vary slightly from the one used in the introduction.

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2. Basic notation and idempotents in positive characteristic.

We are concerned with the representation theory of the symmetric group $S_n$ in positive characteristic. Let us first set up the basic notation. Let $p$ be a prime. We use the ground rings $R$, $\mathbb{Q}$ and $\mathbb{F}_p$, where $R$ is the localization of $\mathbb{Z}$ at the prime ideal $(p)$ and where $\mathbb{F}_p$ is the finite field of $p$ elements. Then $R$ is a local ring with maximal ideal $pR$ and $R/pR = \mathbb{F}_p$. Let $n$ be a positive integer and let $S_n$ be the symmetric group on $n$ letters. An $n$-composition is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of positive integers with sum $n$. An $n$-partition is an $n$-composition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that $\lambda_i \geq \lambda_{i+1}$ for all $i$. The set of $n$-partitions is denoted $\text{Par}_n$. For $\lambda \in \text{Par}_n$, the associated Young diagram, also denoted $\lambda$, is the graphical representation of $\lambda$ through $n$ empty boxes in the plane. The first $\lambda_1$ boxes are placed in the first row, the next $\lambda_2$ boxes are placed in the second row, left aligned with respect to the first row, etc. This is the English notation for Young diagrams. The boxes are denoted the nodes of $\lambda$ and are indexed using matrix convention. Thus the node of $\lambda$ indexed by $[2, 3]$ is the one situated in the second row and
the third column of $\lambda$. The $p$-residue diagram of a partition $\lambda$ is obtained by writing $j - i \mod p$ in the $[i, j]$'th node of $\lambda$. The $[i, j]$'th node is called a $k$-node of $\lambda$ if $k = j - i \mod p$.

Let $t$ be a $\lambda$-tableau, i.e. a filling of the nodes of $\lambda$ using the numbers of $\{1, 2, \ldots, n\}$, each once. We write $t[i, j] = k$ if the $[i, j]$'th node of $t$ is filled in with $k$ and $r_1(k) = j - i$ if $t[i, j] = k$. Then $r_1(k)$ is also referred to as the content of $t$ at the node containing $k$. For $k \in \{1, 2, \ldots, n\}$ we define $t(k) := [i, j]$ where $t[i, j] = k$. A tableau $t$ is said to be row standard if $t[i, j] \leq t[i, j + 1]$ for all $i, j$ such that the terms are defined. The set of row standard tableaux of all $n$-partitions is denoted $RStd(n)$. A tableau $t$ is said to be standard if $t[i, j] \leq t[i, j + 1]$ and $t[i, j] \leq t[i + 1, j]$ for all $i, j$ such that the terms are defined. For $\lambda$ an $n$-partition, we define $Shape(t) := \lambda$ if $t$ is a $\lambda$-tableau. The set of standard tableaux of $n$-partitions is denoted $Std(n)$ and the set of standard tableaux of shape $\lambda$ is denoted $Std(\lambda)$.

If $t$ and $s$ are tableaux we write $t \sim_p s$ if $r_1(k) = r_s(k) \mod p$ whenever $t[i, j] = s[i_1, j_1] = k$. This defines an equivalence relation on the set of all tableaux which can be restricted to an equivalence relation on the set of standard tableaux. When we refer to a tableau class we always mean a class with respect to the last relation, consisting of standard tableaux. If $t$ is a standard tableau we denote its class by $[t]$. In general we refer to tableau classes using capital letters, like $T$ or $S$. We denote by $C_n$ the set of tableau classes of all $n$-partitions.

For $t$ a tableau, the multiset of residues $\{r_1(k) \mid k = 1, \ldots, n\}$ depends only on $Shape(t)$ and this induces an equivalence relation on $Par_n$ that we also denote $\sim_p$. The blocks of $FS_n$ are given by this equivalence relation according to the Nakayama conjecture, see for example Wildon’s notes [W]. Clearly, for two tableaux $s, t$ we have that $t \sim_p s$ implies $Shape(t) \sim_p Shape(s)$.

We use the convention that $S_n$ acts on the right on $\{1, \ldots, n\}$ and hence on tableaux. In other words, we multiply cycles in $S_n$ from the left to the right.

For $t$ a $\lambda$-tableau, we define the associated element $d(t) \in S_n$ by

$$t^\lambda d(t) = t$$

where $t^\lambda$ denotes the highest $\lambda$-tableau, having the numbers $\{1, 2, \ldots, n\}$ filled in along rows. Highest refers to the dominance order $\leq$ on tableaux. It is derived from the dominance order $\leq$ on compositions given by

$$\lambda \leq \mu \text{ if } \sum_{i=1}^m \lambda_i \leq \sum_{i=1}^m \mu_i \text{ for } m = 1, 2, \ldots, \min(k, l)$$

for $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_l)$ by viewing tableaux as series of compositions. Similarly the dominance order can be extended to pairs of tableaux in the following way

$$(s, t) \leq (s_1, t_1) \text{ if } s \leq s_1 \text{ and } t \leq t_1.$$  

In [HuMa] this order on pairs of tableaux is called the strong dominance order and is written $\blacktriangleleft$. The dominance orders are all partial.

Let $t$ be a $\lambda$-tableau with node $(i, j)$. The $(i, j)$-hook consists of the nodes to the right and below the $(i, j)$ node, its cardinality is the hook-length
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The product of all hook-lengths only depends on $\lambda$ and is denoted $h_\lambda$. The hook-quotient is $\gamma_{t,n} = \frac{\prod h_{i,j}}{h_{i,j}-1}$ with the product taken over all nodes in the row of $\lambda$ that contains $n$, omitting hooks of length one. For general $i$, we define $\gamma_{t,i}$ similarly, by first deleting from $t$ the nodes containing $i+1$, $i+2$, $\ldots$, $n$. We set $\gamma_t = \prod_{i=2}^n \gamma_{t,i}$.

In general, when we use $\lambda$ as a subscript it refers to the tableau $t^\lambda$. In this situation we have $\gamma_\lambda = \gamma_{t^\lambda} = \prod_{i} \lambda_i!$.

For $k = 1, 2, \ldots, n$ the Jucys-Murphy elements $L_k \in \mathbb{Z}S_n$ are defined by

$$L_k := (1, k) + (2, k) + \ldots + (k-1, k)$$

with the convention that $L_1 := 0$. They commute with each other and satisfy the following commutation relations with the simple transpositions

$$
\begin{align*}
(k - 1, k)L_k &= L_{k-1}(k - 1, k) + 1 \\
(k - 1, k)L_{k-1} &= L_k(k - 1, k) - 1 \\
(k - 1, k)L_l &= L_l(k - 1, k) & \text{if } l \neq k - 1, l \neq k.
\end{align*}
$$

These elements are a key ingredient for understanding the representation theory of $S_n$. Their generalizations appear in many contexts of representation theory, for example as the degenerate affine Hecke algebra, where the $L_k$ are commuting generators that satisfy the above relations with the simple transpositions. In the original works of Jucys and Murphy, [Ju1], [Ju2], [Ju3] and [Mu81], the $L_k$’s were used to construct orthogonal idempotents $E_t \in \mathbb{Q}S_n$, indexed by tableaux $t$, and to derive Young’s seminormal form from them. We denote these idempotents the Jucys-Murphy idempotents.

Their construction is as follows

$$E_t := \prod_{\{c|-n < c < n\}} \prod_{\{i|\tau(i) \neq c\}} \frac{L_i - c}{r_{\tau(i)} - c}.$$ 

For $t$ standard we have $E_t \neq 0$, whereas for $t$ nonstandard either $E_t = 0$, or $E_t = E_s$ for some standard tableau $s$ related to $t$, see [Mu83] page 260. For example, if $t$ is obtained from the standard tableau $s$ by interchanging $k - 1$ and $k$ that occur in the same row or column of $s$ then $E_t = 0$. Running over all standard tableaux, the $E_t$ form a set of primitive and complete idempotents, that is their sum is 1. Moreover, they are eigenvectors for the action of the Jucys-Murphy operators in $\mathbb{Q}S_n$, since

$$L_k - r_{\tau(k)}E_t = 0 \text{ or equivalently } L_k = \sum_{t \in \text{Std}(n)} r_{\tau(k)}E_t$$

which is the key formula for deriving Young’s seminormal basis from them. In this situation (1) gives Young’s seminormal form for the action of $\sigma_i$ on the seminormal basis.

Unfortunately, the $E_t$ contain many denominators and hence it is not possible to reduce them modulo $p$. In order to overcome this obstacle, Murphy introduced in [Mu83] certain elements $E_T$ for each tableau class $T$. They are defined as follows

$$E_T := \sum_{t \in T} E_t.$$
He showed that the $E_T$’s, with $T$ varying over all classes, give a set of complete orthogonal idempotents in $RS_n$. The most difficult part of this is to show that $E_T \in RS_n$ since they are clearly orthogonal, idempotent and complete (note at this point that in [Mu83] it is not stated clearly that the sum in (3), going over the elements of the tableau class $T$, should only involve standard tableaux). We now present his proof that $E_T \in RS_n$, in our notation. Several of its ingredients will be important for us. See also [MaSo] for a presentation of this and related results from an abstract point of view.

A key point is to consider $F_t$ for $t$ any tableau, given by

$$F_t := \prod_{\{ c \mid -n < c < n \} \setminus \{ i \mid r_t(i) \equiv c \mod p \} } L_i - c \prod_{i \mid r_t(i) \equiv c \mod p} \frac{L_i - c}{r_t(i) - c}. \quad (4)$$

It is clear that $F_t \in RS_n$ and that all $F_t$’s and $E_T$’s commute. The denominator of $F_t$ depends only on the underlying partition $\text{Shape}(t) = \lambda$ of $t$ and is denoted $w^\lambda$. Although $w^\lambda$ is not constant on the classes, we have that $w^\lambda = w^\mu$ modulo $p$ if $\lambda \sim_p \mu$. Especially, if $s \sim_p t$ we get that $w^\text{Shape}(s) = w^\text{Shape}(t)$ modulo $p$. The numerator of $F_t$ only depends on the class $[t]$ of $t$ and so we have

$$F_s = F_t \quad \text{if } s \sim_p t \text{ and } \text{Shape}(s) = \text{Shape}(t)$$

Suppose that $t \in T$. Using (2) we get that

$$F_tE_s = \begin{cases} w^{\text{Shape}(s)} / w^{\text{Shape}(t)} E_s & \text{if } s \sim_p t \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

and so we deduce

$$F_t = \frac{1}{w^\lambda} \sum_{s \in T} w^{\text{Shape}(s)} E_s. \quad (6)$$

Hence $E_T F_t = F_t$ where we set $T = [t]$. Using this we get for any positive integer $m$ that

$$(E_T - F_t)^m = \sum_{i=0}^{m} \binom{m}{i} (-1)^{m-i} E_T^i F_t^{m-i} = E_T - 1 + 1 + \sum_{i=0}^{m-1} \binom{m}{i} (-1)^{m-i} E_T^i F_t^{m-i} = E_T - 1 + \sum_{i=0}^{m} \binom{m}{i} (-1)^{m-i} F_t^{m-i} = E_T - 1 + (1 - F_t)^m.$$

Combining this with equation (6) we arrive at the formula

$$E_T = 1 - (1 - F_t)^m + \sum_{s \in T} \left( 1 - \frac{w^{\text{Shape}(s)}}{w^\lambda} \right)^m E_s. \quad (7)$$

Using it, the proof that $E_T \in RS_n$ follows by taking $m$ big enough for

$$(1 - \frac{w^{\text{Shape}(s)}}{w^\lambda})^m E_s \in RS_n$$

to hold for all $s \in T$. 
3. Commutation rules.

Our first aim is to generalize certain results valid for $E_t$ to $E_T$. We are especially looking for a generalization for $E_T$ of the elements denoted $\Psi_t$ in [Mu92]. For this we need to work out the commutation relations between $E_T$ and the simple transpositions $\sigma_k = (k-1,k)$.

Assume that $t$ and $t\sigma_k$ are standard tableaux and write $T := [t]$. We first consider the case where $[t\sigma_k] = [t]$, that is $r_t(k-1) = r_t(k)$ mod $p$. We prove the following Lemma.

**Lemma 1.** In the above situation $[t\sigma_k] = [t] = T$ we have

$$\sigma_k E_T = E_T \sigma_k.$$

**Proof.** We consider the commutator $[\sigma_k, E_T]$. By the previous section it belongs to $RS_n$. We show that it actually belongs to $p^N RS_n$ for any positive (big) integer $N$, from which the result follows. Fix therefore such an $N$. We use formula (7) and first consider the individual terms of that sum. We choose $m$ big enough for $(1 - \frac{w^{\text{Shape}(t_1)}}{w^\mu})^m E_{t_1} \in p^N RS_n$ to hold for all $t_1 \in T$. From this we get that

$$\left[ \sigma_k, \sum_{t_1 \in T} \left( 1 - \frac{w^{\text{Shape}(t_1)}}{w^\mu} \right)^m E_{t_1} \right] \in p^N RS_n$$

and so by (7) it is enough to prove that $\sigma_k$ commutes with $F_i$.

Now by the commutation rules (1), we have that $\sigma_k$ commutes with all terms of $F_i$ of the form $L_i - c$ where $i \neq k - 1, k$. The remaining terms may be grouped together in pairs of the form

$$(L_{k-1} - c)(L_k - c)$$

since by assumption $r_t(k-1) = r_t(k)$ mod $p$. But these expressions are symmetric in $L_{k-1}$ and $L_k$ and therefore commute with $\sigma_k$ by the commutation rules (1). The Lemma is proved. \qed

We next consider the case where $s, t$ are both standard and $s = t\sigma_k \notin [t]$, that is $r_t(k-1) \neq r_t(k)$ mod $p$. We set $S := [s]$ and $T := [t]$. In order to work out the commutation rule between $\sigma$ and $E_T$ in this case, we first consider $E := E_S + E_T$. We need the following auxiliary Lemma.

**Lemma 2.** $E$ belongs to $RS_n$ and commutes with $\sigma_k$.

**Proof.** Clearly $E$ belongs to $RS_n$ since $E_S$ and $E_T$ do. For each $s \in S$ we have that either $s\sigma_k \in T$ or $s\sigma_k$ is nonstandard, and conversely for each $t \in T$ we have that either $t\sigma_k \in S$ or $t\sigma_k$ is nonstandard. Accordingly, the sum $E = \sum_{t \in T} E_t + \sum_{s \in S} E_s$ may be split into terms $E_s + E_{s\sigma_k}$ with $s \in S$ and $s\sigma_k \in T$, terms $E_s$ with $s\sigma_k$ nonstandard and terms $E_t$ with $t\sigma_k$ nonstandard.

The first kind of terms are symmetric in $L_{k-1}$ and $L_k$ and therefore commutes with $\sigma_k$. For the second kind of terms, $k - 1$ and $k$ occur in the same row or column of $s$, next to each other. Thus, using Theorem 6.4 of [Mu92] together with the formula for $x_t$ of loc. cit. appearing eight lines above that Theorem 6.4, we get that $E_s \sigma_k = \pm E_s$, where the sign is positive iff $k - 1$ and $k$ are in the same row of $s$. Applying the anti-automorphism $* of
For each tableau class $T$ we choose an arbitrary $t \in T$ and define

$$r_T(i) := r_t(i).$$

Thus $r_T(i) \in \mathbb{Z}$, hence also $r_T(i) \in R$, but it is only well defined modulo $p$. With this notation we can formulate our next Lemma.

**Lemma 3.** Assume that $S$ and $T$ are chosen as above. Then there is a positive integer $m_1$ such that the following formulas hold for $m \geq m_1$

$$E_T = \left( \frac{L_k - r_S(k)}{r_T(k) - r_S(k)} \right)^m E, \quad E_S = \left( \frac{L_k - r_S(k)}{r_T(k) - r_S(k)} \right)^m E.$$

**Proof.** For tableaux $t$ and $s$ we define $E_{t,s} = E_t + E_s$. Then obviously $E_{t,s}$ is idempotent and by (2), we have that $L_k = \sum_{u \in \text{Std}(n)} r_u(k) E_u$. Hence for standard tableaux $s$ and $t$ such that $s = \sigma_k t$ we get that

$$E_t = \left( \frac{L_k - r_s(k)}{r_t(k) - r_s(k)} \right) E_{t,s}. \quad (8)$$

Similarly we have that

$$E_s = \left( \frac{L_{k-1} - r_s(k)}{r_t(k) - r_s(k)} \right) E_{t,s}. \quad (9)$$

Note that (8) and (9) hold even if only $s$ or $t$ is standard, as long as $s = \sigma_k t$. The formulas are used in Murphy’s papers but unfortunately they do not generalize to $E_T$ or $E_S$ since we do not have $L_k = \sum_{U \in \mathbb{S}_n} r_U(k) E_U$ even though $r_u(k) = r_{u_1}(k)$ mod $p$ for $u \sim_p u_1$. The problem is that the individual $E_u$ lie in $\mathbb{Q}S_n$ rather than $RS_n$.

In order to solve this problem we proceed as follows. Consider first an $a \in \mathbb{Z}R$. From the binomial expansion we get the following formula in $R[x]$, valid for any positive integer $m$

$$(x + a)^m = x^m + p^m R[x]. \quad (10)$$

We deduce from it the formula

$$\left( \frac{x + a}{c + d} \right)^m = \left( \frac{x}{c} \right)^m + p^m R[x] \quad (11)$$

for any $c \in R^x$ and $a, d \in \mathbb{Z}R$.

Set $T' := T \cup S\sigma_k$. Then $T' \setminus T$ consists of nonstandard tableaux and so $E_T = \sum_{t \in T'} E_t$. Using (8), for $m$ is large enough we get

$$E_T = \sum_{t \in T'} E_t = \sum_{t \in T', s = t \sigma_k} \left( \frac{L_k - r_s(k)}{r_t(k) - r_s(k)} \right) E_{t,s} = \sum_{t \in T', s = t \sigma_k} \left( \frac{L_k - r_s(k)}{r_t(k) - r_s(k)} \right)^m E_{t,s} = \sum_{t \in T', s = t \sigma_k} \left( \frac{L_k - r_s(k)}{r_T(k) - r_S(k)} \right)^m E_{t,s}$$

The last equality follows from (11), since for any $N$ we may choose $m$ big enough to make the difference of the two sides belong to $p^N RS_n$. From the last expression we then get that $E_T$ is equal to

$$\left( \frac{L_k - r_S(k)}{r_T(k) - r_S(k)} \right)^m (E_S + E_T) = \left( \frac{L_k - r_S(k)}{r_T(k) - r_S(k)} \right)^m E$$

$RS_n$ that fixes the simple transpositions we get $\sigma_k E_s = \pm E_s$, with the same sign as before, and so indeed $E_s$ and $\sigma_k$ commute. Finally, the third kind of terms involving $E_s$ are treated in the same way.
as claimed. The other equality is proved the same way. By choosing $m$ even bigger we obtain an $m_1$ that works for both equations. \hfill \Box

At this stage Murphy constructs in [Mu92], using the formulas (8) and (9), elements $\Psi_t$ and $\Phi_t$ of $\mathbb{Q}S_n$ satisfying

$$E_\lambda \Phi_t = \Psi_t E_t.$$  

(12)

The construction is as follows. Let $t$ be any $\lambda$-tableau and let $k$ be an integer between 1 and $n$. The radial length between the nodes $t[k]$ and $t[k-1]$ is defined as $h_{t,k} = h_k := r_t(k-1) - r_t(k)$. Let $d(t) = \sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_N}$ be a reduced expression of $d(t)$. We associate with it a sequence of tableaux $t_1 = t^\lambda, t_2, \ldots, t_{N+1} = t$ by setting recursively $t_{k+1} := t_k s_{i_k}$. Then $\Phi_t$ and $\Psi_t$ are given by the formulas

$$\Phi_t := \left( \sigma_{i_1} - \frac{1}{h_{i_1,i_1}} \right) \left( \sigma_{i_2} - \frac{1}{h_{i_2,i_2}} \right) \ldots \left( \sigma_{i_N} - \frac{1}{h_{i_N,i_N}} \right)$$

$$\Psi_t := \left( \sigma_{i_1} + \frac{1}{h_{i_1,i_1}} \right) \left( \sigma_{i_2} + \frac{1}{h_{i_2,i_2}} \right) \ldots \left( \sigma_{i_N} + \frac{1}{h_{i_N,i_N}} \right)$$  

(13)

As noted in [Mu92], $\Phi_t$ and $\Psi_t$ actually do depend on the chosen decomposition of $d(t)$, and not just on $d(t)$, and so the notation is slightly misleading. On the other hand, the key property (12) holds independently of the choice of reduced expression of $d(t)$, and so we just take anyone.

Our aim is to construct similar elements for $E_T$ and $E_S$. For this we need the following commutation rules between $\sigma_k$ and the powers $L_k^m$ and $(L_k - a)^m$.

**Lemma 4.** For $m \in \mathbb{N}$ and $a \in R$ the following formulas hold:

a) $\sigma_k L_k^m = L_{k-1}^m \sigma_k + \sum_{i=0}^{m-1} L_{k-1}^i L_k^{m-i-1}$

b) $\sigma_k (L_k - a)^m = (L_{k-1} - a)^m \sigma_k + \sum_{i=0}^{m-1} (L_{k-1} - a)^i (L_k - a)^{m-i-1}$.

*Proof.* Formula a) is proved using a straightforward induction on the commutation rules given in (1). Formula b) is proved the same way, since $L_k - a$ satisfies the same commutation rules with $\sigma_k$ as $L_k$ does. \hfill \Box

We generalize the concept of radial length to tableaux classes by setting

$$h_{T,k} = h_k := r_T(k-1) - r_T(k) \in \mathbb{Z} \subseteq R$$

for $k$ any integer between 1 and $n$. It depends on the choices of $r_T(k)$ and is therefore only unique modulo $p$.

We are now going to construct certain elements $\Psi_L, t$, verifying a generalization of (12) for the $E_T$'s. Set first $h_L(k) = h_L := L_{k-1} - L_k$. Modelled on $\Psi_t$, we shall construct $\Psi_L, t$ as products of expressions of the form

$$\sigma_k - \frac{1}{h_L}.$$  

On the other hand, for such expressions to make sense in general, one would need to consider an appropriate completion of the group ring, and define $\frac{1}{h_L}$ inside it as a power series. We here take a simpler approach, always considering $L_k$ and $L_{k-1}$ as elements of $\text{End}_{\mathbb{F}_p}(V)$ for $V$ a $\mathbb{F}_p$-vector space such that $L_{k-1} - L_k + \alpha \in \text{End}_{\mathbb{F}_p}(V)$ is nilpotent for some $\alpha \in \mathbb{F}_p^\times$. Under that assumption, $\frac{1}{h_L}$ can be defined as the corresponding geometric series, which is finite. The next Lemma should be seen in this light.
Lemma 5. Suppose that $s, t$ are standard tableaux with $s = t \sigma_k$ and that $T := [t]$ and $S := [s]$ are different tableaux classes. Let $h := h_T$. Then $L_{k-1} - L_k - h$ acts nilpotently in $E_T(\mathbb{F}_p S_n)$. Especially, $L_{k-1} - L_k$ is invertible as an element of $\text{End}_{\mathbb{F}_p}(E_T(\mathbb{F}_p S_n))$.

Proof. Notice that since $E_T \in RS_n$, we have that the product $E_T(\mathbb{F}_p S_n)$ is well defined. Consider first $L_{k-1} - L_k - h$ as an element of $RS_n$. Using formula (2) we have that

$$(L_{k-1} - L_k - h)^N = \sum_u (r_u(k-1) - r_u(k) - r_T(k-1) + r_T(k))^N E_u.$$ 

Multiplied by $E_T$ it gives the formula

$$(L_{k-1} - L_k - h)^N E_T = \sum_{u \in T} (r_u(k-1) - r_u(k) - r_T(k-1) + r_T(k))^N E_u.$$ 

Each coefficient of $E_u$ is here a multiple of $p$. Hence we may take $N$ large enough for $(L_{k-1} - L_k - h)^N E_T$ to belong to $p^m RS_n$. We reduce modulo $p$ and get the statement of the Lemma. 

We can now prove the following Lemma.

Lemma 6. Let $s, t, S, T, h, m$ be as in the previous Lemma and let $h_L := L_{k-1} - L_k$. View $1/h_L$ as an element of $\text{Hom}_{\mathbb{F}_p}(E_T(\mathbb{F}_p S_n), \mathbb{F}_p S_n)$ via the previous Lemma. Then for $N \in \mathbb{N}$ and $a \in \mathbb{F}_p$ the following formulas hold in $\text{Hom}_{\mathbb{F}_p}(E_T(\mathbb{F}_p S_n), \mathbb{F}_p S_n)$

a) $$(\sigma_k - \frac{1}{h_L}) L_k^N = L_{k-1}^N (\sigma_k - \frac{1}{h_L})$$
b) $$(\sigma_k - \frac{1}{h_L}) (L_k - a)^N = (L_{k-1} - a)^N (\sigma_k - \frac{1}{h_L}).$$

Proof. Using the previous Lemma and the fact that $E_T$ commutes with $L_k$ and $(L_k - a)^r$ we first notice that the expressions are well defined transformations of $E_T(\mathbb{F}_p S_n)$. Let us now show a). Since $L_k$ and $L_{k-1}$ commute it is equivalent to

$$h_L (\sigma_k L_k^N - L_{k-1}^N \sigma_k) = L_k^N - L_{k-1}^N$$

and hence, using Lemma 4, to the valid expression

$$(L_k - L_{k-1}) \sum_{i=0}^{N-1} L_{k-1}^i L_k^{N-i-1} = L_k^N - L_{k-1}^N.$$ 

Formula b) is proved the same way. 

We now obtain the following result.

Lemma 7. Let the notation be as above. Then we have

$$(\sigma_k - \frac{1}{h_L(k)}) E_T = E_S (\sigma_k - \frac{1}{h_L(k)}$$

in $\text{Hom}_{\mathbb{F}_p}(E_T(\mathbb{F}_p S_n), \mathbb{F}_p S_n)$.

Proof. The proof is obtained by combining Lemma 2, 3 and 6.

The Lemma is a generalization of Lemma 6.2 from [Mu92], where $L_k, L_{k-1}$ and hence $h_L$ act semisimply. Note that the second minus sign is there a plus sign, corresponding to the fact that the eigenvalues of $h_L$ on $E_s$ and $E_t$ are equal but with opposite signs.
Set $T^\lambda := [t^\lambda]$. For $d(t) = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_N}$ in reduced form we define

$$
\Psi_{L,d(t)} := \left( \sigma_{i_1} - \frac{1}{h_L(i_1)} \right) \left( \sigma_{i_2} - \frac{1}{h_L(i_2)} \right) \cdots \left( \sigma_{i_N} - \frac{1}{h_L(i_N)} \right)
$$

where $\frac{1}{h_L(i_j)}$ is set to 1 when $[t_j] = [t_{j-1}]$. Combining Lemma 1 and 7 we get the following Theorem.

**Theorem 1.** $E_T \Psi_{L,d(t)} = \Psi_{L,d(t)} E_T$.

We view $\sigma_k - \frac{1}{h_L(k)}$ as an analogue of the Khovanov-Lauda generator $\psi_k$, or more precisely of the element denoted $\phi_i$ in [BK]. These intertwining elements are the starting point of their work. In our approach the $\phi$-elements have a representation theoretical interpretation coming from the theory of the seminormal basis whereas they appear somewhat pulled out of the sleeve in [BK].

4. A cellular basis.

In this section we use the results from the previous sections to construct a cellular basis for $FP\, S_n$. Our construction is inspired by the one given by J. Hu and A. Mathas in [HuMa].

Let us first introduce some notation. For $\lambda$ a partition of $n$ we let $S_\lambda$ denote the row stabilizer of $t^\lambda$. Let $x_\lambda$ and $y_\lambda$ be the elements of $RS_n$ given by

$$
x_\lambda = \sum_{\sigma \in S_\lambda} \sigma \quad \text{and} \quad y_\lambda = \sum_{\sigma \in S_\lambda} (-1)^{|\sigma|} \sigma
$$

where $|\sigma|$ is the sign of $\sigma$. For a pair $(s, t)$ of $\lambda$-tableaux we define

$$
x_{st} = d(s)^{-1} x_\lambda d(t) \quad \text{and} \quad y_{st} = d(s)^{-1} y_\lambda d(t).
$$

If $s$ is a $\lambda$-tableau we get that $x_{ss}$ is the sum of the elements of the row-stabilizer of $s$. A similar comment applies to $y_{ss}$.

The set $\{x_{st}\}$ with $(s, t)$ running over pairs of standard $\lambda$-tableaux and $\lambda$ over partitions of $n$ gives Murphy’s standard basis for $RS_n$. Similarly $\{y_{st}\}$ gives the dual standard basis. They are cellular bases in the sense of Graham and Lehrer, [GL], with respect to the dominance order. This implies that $RS_n^{\geq \lambda}$ and $RS_n^{> \lambda}$, defined by

$$
RS_n^{\geq \lambda} := \text{span}_R \{x_{st} \mid (s, t) \text{ pair of } \mu\text{-tableaux with } \mu \unlhd \lambda\}
$$

and

$$
RS_n^{> \lambda} := \text{span}_R \{x_{st} \mid (s, t) \text{ pair of } \mu\text{-tableaux with } \mu \nleq \lambda\}
$$

are ideals of $RS_n$. The associated left cell module is

$$
C(\lambda) := R\{x_{s\lambda} \mid s \text{ is a } \lambda\text{-tableau}\} \mod RS_n^{> \lambda}.
$$

In modern terminology it is often referred to as the Specht module, although it rather corresponds to the dual Specht module defined via Young symmetrizers.

Working over the ground fields $F_p$ and $Q$, we get ideals $FP\, S_n^{\geq \lambda}, FP\, S_n^{> \lambda}$ and $QS_n^{\geq \lambda}, QS_n^{> \lambda}$ of $FP\, S_n$ and $QS_n$, using constructions similar to the ones for $RS_n^{\geq \lambda}, RS_n^{> \lambda}$. Similarly, we get cell modules $C(\lambda)$ and $C_Q(\lambda)$ for $FP\, S_n$ and $QS_n$. We use the same notation $x_{s\lambda}$ for the classes of $x_{s\lambda}$ in $C(\lambda)$, $C(\lambda)$ or $C_Q(\lambda)$. They form bases for $C(\lambda)$ over $R$, for $C(\lambda)$ over $F_p$ and for
$C_Q(\lambda)$ over $\mathbb{Q}$, when $s \in \text{Std}(\lambda)$. Hence we have the base change properties

$C(\lambda) := C(\lambda) \otimes_R \mathbb{F}_p$ and $C_Q(\lambda) := C(\lambda) \otimes_R \mathbb{Q}$.

We need to recall another basis for $RS_n$ that was also introduced by Murphy, see [Mu92]. For $\lambda \in \text{Par}_n$, we define

$$\xi_\lambda = \prod_{i=1}^n (L_i + \rho_\lambda(i))$$  \hspace{1cm} (15)$$

where $\rho_\lambda(i) = k$ for $t^\lambda(i) = [k,l]$, that is $\rho_\lambda(i)$ is the row number of the $i$-node of $t^\lambda$. For any pair $(s,t)$ of $\lambda$-tableau we set

$$\xi_{st} := d(s)^{-1} \xi_\lambda d(t).$$

Then $\{\xi_{st}\}$ is a basis for $RS_n$ when $(s,t)$ runs over the same parameter set as above. This follows from Theorem 4.5 of [Mu92], saying that $\xi_\lambda = x_\lambda + \sum_{t \in R\text{Std}(n), t \succeq t^\lambda} c_{uv} x_{uv}$ (16)

and Theorem 4.18 of [Mu95], saying that for any two tableaux $u, v \in R\text{Std}(n)$ of the same shape, the element $x_{uv}$ can be expanded in terms of standard basis elements $x_{u'v'}$ where $(u',v') \succeq (u,v)$. We also get from this and (16), using that $RS_n^{\lambda}$ is an ideal in $RS_n$, that the images of $\{\xi_{st}\}$ in $C(\lambda)$ coincide with $\{x_{st}\}$, for $s$ standard $\lambda$-tableaux.

Motivated by the construction done by Hu and Mathas in [HuMa] we now introduce for each pair of standard tableaux $(s,t)$ of the same shape $\lambda$ the following elements of $\mathbb{F}_p S_n$

$$\psi_{st} := \Psi_{L,d(s)}^* \xi_\lambda E_{T^\lambda} \Psi_{L,d(t)}$$  \hspace{1cm} (17)$$

where $\Psi_{L,d(s)}, \Psi_{L,d(t)}$ are as in (14) and where once again $*$ is the antiautomorphism that fixes the transpositions. Since $\frac{1}{h_{L,k}(k)}$ is a polynomial expression of Jucys-Murphy elements, $\Psi_{L,d(s)}^*$ is obtained from $\Psi_{L,d(s)}$ by reversing the factors. Note that in $\psi_{st}$ the two middle factors $\xi_\lambda$ and $E_{T^\lambda}$ commute.

We aim at proving that the set of $\psi_{st}$ is a cellular basis for $\mathbb{F}_p S_n$ when $(s,t)$ runs over pairs of standard tableaux of the same shape. We begin with the following preparatory Lemma.

**Lemma 8.** For every partition $\lambda$ of $n$ we have triangular expansions

$$\xi_\lambda = x_\lambda + \sum_{(u,v) \succ (s,t)} c_{uv} x_{uv}, \quad \xi_\lambda E_{T^\lambda} = x_\lambda + \sum_{(u,v) \succ (s,t)} d_{uv} x_{uv}$$

where $c_{uv} \in R, d_{uv} \in \mathbb{F}_p$.

**Proof.** The first expansion follows from (16) and Theorem 4.18 of [Mu95].

In order to obtain the second expansion, we first recall Corollary 2.15 of [HuMa1] saying that for any pair of standard tableaux $(s,t)$ of the same shape we have

$$x_{st} L_k = r_t(k)x_{st} + \sum_{(u,v) \succ (s,t)} c_{uv} x_{uv}$$  \hspace{1cm} (18)$$
where \((u, v)\) runs over pairs of standard tableaux of the same shape and \(c_{uv} \in R\). Especially, we get
\[
 x_\lambda L_k = r_\lambda(k)x_\lambda + \sum_{(u, v) > (\lambda, \lambda)} c_{uv}x_{uv}.
\]
From this we get via the definition of \(E_t\) that
\[
 x_\lambda E_t = \delta_{t\lambda} x_\lambda + \sum_{(u, v) > (\lambda, \lambda)} d_{uv}x_{uv}
\]
for certain \(d_{uv} \in \mathbb{Q}\) where \(\delta_{t\lambda}\) is the Kronecker delta. Using the definition of \(E_T\) we deduce from this that
\[
 x_\lambda E_T = x_\lambda + \sum_{(u, v) > (\lambda, \lambda)} d_{uv}x_{uv}
\]
where \((u, v)\) still is a pair of standard tableaux of the same shape but where we may now assume that \(d_{uv} \in R\). We then finish the proof of the Lemma by reducing modulo \(p\). □

Remark. For \(\mu := Shape(\sigma)\) and \(\nu := Shape(s)\) we define \((\sigma, \tau) \succeq (s, t)\) if either \((\sigma, \tau) \succeq (s, t)\) and \(\mu = \nu\), or if \(\mu \triangleright \nu\). Then the triangularity property (18) appears already in [AMR] if \(\succeq\) is replaced by \(\succeq\).

The next result gives the promised cellularity property for \(\{\psi_{st}\}\).

**Theorem 2.** For pairs of standard tableaux \((s, t)\) of the same shape we have
\[
 \psi_{st} = x_{st} + \sum_{(\sigma, \tau) > (s, t)} a_{\sigma \tau}x_{\sigma \tau}, \quad a_{\sigma \tau} \in \mathbb{F}_p,
\]
where \(\succeq\) is the partial order introduced in the above remark. Moreover, with respect to the dominance order, the set
\[
 \{\psi_{st} \mid (s, t) \text{ pair of standard tableaux of the same shape}\}
\]
defines a cellular basis for \(\mathbb{F}_p S_n\) with cell modules \(C(\lambda)\).

**Proof.** We have
\[
 \psi_{st} := \Psi^*_L_{d(s)}E_{T_\lambda}\xi_\lambda \Psi^*_{L,d(t)}.
\]
For \(d(s) = \sigma_{i_1}\sigma_{i_2} \ldots \sigma_{i_N}\) in reduced form, the rightmost term of \(\Psi^*_{L,d(s)}\) is \(\sigma_{i_1} - \frac{1}{h_{L(i_1)}}\). Let us consider its action on \(E_{T_\lambda}\xi_\lambda\). Assume first that \(\frac{1}{h_{L(i_1)}} \neq 1\). Now \(\frac{1}{h_{L(i_1)}} = \frac{L_{i_1-1} - L_{i_1}}{L_{i_1-1} - L_{i_1}}\) is a linear combination of terms of the form
\[
 (L_{i_1-1} - L_{i_1} - r_\lambda(i_1 - 1) + r_\lambda(i_1))^t
\]
and hence by (18) it acts upper triangularly. Thus \(\sigma_{i_1} - \frac{1}{h_{L(i_1)}}\) maps \(E_{T_\lambda}\xi_\lambda\) to \(x_{\sigma_{i_1}T_\lambda\Lambda}\) plus higher terms with respect to \(\succeq\). If \(\frac{1}{h_{L(i_1)}} = 1\) the same conclusion holds trivially. We repeat this argument for the other terms of \(\Psi^*_{L,d(s)}\) and then for the terms of \(\Psi^*_{L,d(t)}\), and obtain the triangularity statement of the Theorem.

From this we deduce that
\[
 \mathbb{F}_p S_n^{\prec \lambda} = \text{span}_{\mathbb{F}_p} \{\psi_{st} \mid (s, t) \text{ pair of } \mu\text{-tableaux with } \mu \triangleright \lambda\}.
\]
An argument similar to the one given in Theorem 5.8 of [HuMa] now gives the cellularity of \(\{\psi_{st}\}\) with *-involution satisfying \(\psi^*_{st} = \psi_{ts}\). □
From the general theory of cellular algebras there is an associated bilinear invariant form on $C(\lambda)$, that we denote $\langle \cdot, \cdot \rangle_\lambda$. It is given by

$$\psi_{\lambda s} \psi_{t\lambda} = \langle \psi_{s\lambda}, \psi_{t\lambda} \rangle_\lambda \psi_\lambda \mod \mathbb{F}_p S_n^\lambda$$

Its radical $\text{rad}_\lambda$ is a submodule of $C(\lambda)$ and $C(\lambda)/\text{rad}_\lambda$ is either simple or zero. An important point of the theory of cellular algebras is that this gives rise to a classification of the simple modules for $\mathbb{F}_p S_n$. Indeed, every simple module for $\mathbb{F}_p S_n$ is of the form $C(\lambda)/\text{rad}_\lambda$ for a unique $\lambda$.

Our next Lemma shows that $\langle \cdot, \cdot \rangle_\lambda$ is in block form with respect to our basis. Note that bases that block diagonalise the bilinear form have also been found in [MaSo] and in [BKW].

**Lemma 9.** The basis $\{\psi_{s\lambda} | s \text{ standard } \lambda\text{-tableau}\}$ of $C(\lambda)$ is in block form with respect to $\langle \cdot, \cdot \rangle_\lambda$ with blocks given by the tableau classes.

**Proof.** Suppose that $s, t$ are standard $\lambda$-tableaux and that the tableau classes $S := [s]$ and $T := [t]$ are different. Then we have that

$$\psi_{\lambda s} \psi_{t\lambda} = \xi_\lambda E_T^{\lambda} \Psi_{L,d(s)} \Psi_{L,d(t)}^* E_T^{\lambda} \xi_\lambda.$$

Using Theorem 1 and its $*$-version, and noting that $E_U^{\ast} = E_U$ for all $U$ since $E_U$ is a sum of products of Jucys-Murphy operators, we get that this is equal to

$$\xi_\lambda \Psi_{L,d(s)} E_S E_T^{\lambda} \Psi_{L,d(t)}^* \xi_\lambda.$$

But $E_S E_T = 0$ and the Lemma follows. $\Box$

### 5. Specht modules and Jucys-Murphy operators.

In this section we give a new realization of the Specht modules, using Jucys-Murphy operators.

An essential ingredient of our construction is the use of what we denote the Gelfand-Zetlin subalgebra of $RS_n$ as a kind of Cartan subalgebra of a semisimple Lie algebra. This is in accordance with ideas promoted by Okounkov and Vershik in the article “A new approach to the representation theory of the symmetric group”, [OV]. Their approach also applies to a wider class of algebras than the group algebra of the symmetric group, but relies heavily on the algebras being semisimple. Moreover, the Specht modules themselves have in their approach apparently so far only been treated from the “old” point of view. In this section we realize the Specht module as induced modules from the Gelfand-Zetlin subalgebra, at least over $\mathbb{R}$ and $\mathbb{Q}$ and thus partially remedy these deficiencies. It would be interesting to investigate to what extent these results hold in positive characteristic.

Define $GZ_n \subseteq RS_n$ to be the Gelfand-Zetlin algebra, the $R$-subalgebra of $RS_n$ generated by the Jucys-Murphy operators:

$$GZ_n := \langle L_i | i = 1, \ldots, n \rangle.$$

This definition is not quite equivalent to the one used by for example Okounkov and Vershik in [OV]. They first of all work over a field of characteristic zero and even in that case, our definition of the Gelfand-Zetlin algebra is actually a Theorem in [OV] that characterizes the subalgebra.
GZ\_n is a commutative subalgebra of RS\_n and it contains the center \(Z(RS\_n)\) of RS\_n – indeed by Theorem 1.9 of [Mu83] we know that \(Z(RS\_n)\) consists of the symmetric polynomials in the \(L_k\).

We aim at defining an induction functor from GZ\_n-modules to RS\_n-modules. For this we first need to state a few categorical generalities on \(R\)-modules.

For an \(R\)-module \(M\) we define \(M^* := \text{Hom}_R(M, R)\). If \(M\) is also a left RS\_n-module, \(M^*\) becomes a right RS\_n-module and vice-versa. Let \(R\)-modfg denote the category of finitely generated \(R\)-modules and let \(RS\_n\)-modfg denote the subcategory whose objects are also left RS\_n-modules.

Since \(R\) is Euclidean, we have for \(M \in R\)-modfg that

\[ M = \text{Fr}(M) \oplus \text{Tor}(M) \]

where \(F(M)\) is the free part of \(M\) and \(\text{Tor}(M)\) the torsion part of \(M\). If \(f : M \to N\) is a morphism in \(R\)-modfg then clearly \(f(\text{Tor}(M)) \subseteq \text{Tor}(N)\) and from this we deduce that \(M \mapsto \text{Tor}(M)\) is a left exact functor on \(R\)-modfg. On the other hand \(M \mapsto \text{Fr}(M)\) is an exact functor. Indeed, we may define it as \(\text{Fr}(M) := M^{**}\) which shows that it is a covariant functor in the first place. But for \(M \in R\)-modfg the canonical map \(M \to M^{**}\) induces an isomorphism \(M/\text{Tor}(M) \to \text{Fr}(M)\). This gives a natural transformation from the functor \(M \mapsto M/\text{Tor}(M)\) to \(\text{Fr}\) and hence, since \(M \mapsto M/\text{Tor}(M)\) is right exact, we get that \(\text{Fr}\) is right exact as well, whereas left exactness follows directly from the definitions.

From this we get that \(\text{Fr}\) induces an exact functor on \(RS\_n\)-modfg. Indeed, if \(M\) is a left \(RS\_n\)-module then also \(\text{Fr}(M) = M^{**}\) is a left \(RS\_n\)-module and exactness follows from exactness at \(R\)-modfg level.

We let \(GZ\_n\)-modfg denote the subcategory of \(R\)-modfg whose objects are also \(GZ\_n\)-modules. Finally, we define \(R\)-modfr as the category of finitely generated free \(R\)-modules and \(RS\_n\)-modfr as the subcategory whose objects are also left \(RS\_n\)-modules.

After these preparations we are in position to define the induction functor. For \(M \in GZ\_n\)-modfg we define

\[ \text{Ind}(M) := \text{Fr}(RS\_n \otimes_{GZ\_n} M) \]

Then \(\text{Fr}(RS\_n \otimes_{GZ\_n} M) \in RS\_n\)-modfr. Furthermore, by the above considerations we have that \(M \mapsto \text{Ind}(M)\) is a right exact functor from \(GZ\_n\)-modfg to \(RS\_n\)-modfr.

An important property of \(\text{Ind}\) is the following Frobenius reciprocity rule

\[ \text{Hom}_{GZ\_n}(M, N) \cong \text{Hom}_{RS\_n}(\text{Ind}(M), N) \]

for \(M \in GZ\_n\)-modfg and \(N \in GZ\_n\)-modfr. It follows from

\[ \text{Hom}_{R}(M, N) \cong \text{Hom}_{R}(F(M), N) \]

for \(M \in R\)-modfg, \(N \in R\)-modfr and the usual Frobenius reciprocity for induction.
For us the most important case of the above construction is the following. Let $\lambda$ be a partition of $n$ and let $I_{\lambda}$ be the ideal of $GZ_n$ generated by $L_i - r_{\lambda}(i)$ for $i = 1, \ldots, n$. Set

$$1_{\lambda} := GZ_n / I_{\lambda}.$$ 

Then we may consider $1_{\lambda}$ as a left $GZ_n$-module. As we point out in the final remarks of this section, it is free of rank one over $R$ with generator 1. The action of $L_i$ on 1 is multiplication by $r_{\lambda}(i)$. We next define

$$\text{Ind}(\lambda) := \text{Ind}(1_{\lambda}).$$

We aim at studying $\text{Ind}(\lambda)$ at some depth, our main result being a proof of the isomorphism $\text{Ind}(\lambda) \cong C(\lambda)$. The following Lemma is a first step towards this.

Let $t_{\lambda}$ be the lowest $\lambda$-tableau having $1, 2, \ldots, n$ filled in along columns and define $s_{\lambda} := d(t_{\lambda}) \in S_n$. Set

$$z_{\lambda} := x_{\lambda} s_{\lambda} y_{\lambda'} = x_{\lambda} s_{\lambda}^{-1} y_{\lambda'} \in RS_n. \quad (19)$$

Then $x_{\lambda} RS_n$ is isomorphic to the right permutation module studied in [J] where the isomorphism maps the tabloid $\{t^\lambda\}$ to $x_{\lambda}$. The alternate column sum of $t^\lambda$ is $\kappa_{t_{\lambda}} = s_{\lambda}^{-1} y_{\lambda'} s_{\lambda'} = s_{\lambda} y_{\lambda'} s_{\lambda}^{-1}$ and so the isomorphism maps the polytableau $e_{t_{\lambda}}$ of [J] to $x_{\lambda} s_{\lambda'}^{-1} y_{\lambda'} s_{\lambda}$. In other words $x_{\lambda} s_{\lambda'} y_{\lambda'} RS_n$ identifies with the Specht module $S^\lambda$ of [J]. For $s$ a $\lambda$-tableau we set

$$z_{\lambda s} := x_{\lambda} s_{\lambda} y_{\lambda} d(s').$$

Then $\{z_{\lambda s} \mid s \in \text{Std}(\lambda)\}$ is a basis for $z_{\lambda} RS_n$, see eg. Lemma 5.1 and Theorem 5.6 of [DJ].

**Lemma 10.** With the above notation we have

$$\{x \in RS_n \mid L_i x = r_{\lambda}(i) x \text{ for all } i\} = z_{\lambda} RS_n.$$

**Proof.** Let us denote by $\lambda LS_n$ the left hand side of the Lemma. We first prove that $\lambda LS_n \supseteq z_{\lambda} RS_n$. Now $\lambda LS_n$ certainly is a right submodule of $RS_n$ and it is known, see for example [Mu92] page 498, that

$$x_{ab} s_{\lambda} y_{\lambda'} = 0 \quad \text{unless } \mu \subseteq \lambda \quad (20)$$

where $\mu = \text{Shape}(s) = \text{Shape}(b)$. Combining this with the fact that $L_i$ acts upper triangularly on the $\{x_{st}\}$-basis, as is seen by applying $*$ to (18), we find that $z_{\lambda}$ belongs to $\lambda LS_n$, from which the inclusion $\supseteq$ indeed follows.

In order to show the other inclusion $\subseteq$ we first work in $\mathbb{Q} S_n$, and define

$$\lambda LS_{\mathbb{Q}, n} := \{x \in \mathbb{Q} S_n \mid L_i x = r_{\lambda}(i) x \text{ for all } i\}.$$ 

Setting $t = t_{\lambda}$ we recall from [Mu92] page 511 that

$$E_{\lambda} = h_{\lambda}^{-1} z_{\lambda} \Psi_i^* \quad (21)$$

and from this we deduce that $\lambda LS_{\mathbb{Q}, n} = z_{\lambda} \mathbb{Q} S_n$. We then get that

$$\lambda LS_n = z_{\lambda} \mathbb{Q} S_n \cap RS_n = z_{\lambda} \mathbb{Q} S_n \cap x_{\lambda} RS_n.$$

Here the last equality follows from the facts that $\{x_{st}\}$ is an $R$-basis of $RS_n$ and that $z_{\lambda} \mathbb{Q} S_n \subseteq x_{\lambda} \mathbb{Q} S_n$. Finally, since $z_{\lambda} \mathbb{Q} S_n = S_{\mathbb{Q}}(\lambda)$ is the Specht module defined over $\mathbb{Q}$, we get from Corollary 8.9 of [J], which is based on the Garnir relations, that

$$z_{\lambda} \mathbb{Q} S_n \cap x_{\lambda} RS_n \subseteq z_{\lambda} RS_n.$$
We can now prove the promised result on $\text{Ind}(\lambda)$.

Recall that $RS_n$ is equipped with a symmetric nondegenerate bilinear form $\langle \cdot, \cdot \rangle$, given by

$$\langle a, b \rangle := \text{coeff}_1(ab)$$

where $\text{coeff}_1(x)$ is the coefficient of 1 when $x \in RS_n$ is expanded in the canonical basis of $RS_n$. It is associative in the following sense

$$\langle ab, c \rangle = \langle a, bc \rangle \quad \text{for all } a, b, c \in RS_n.$$

The form induces an $RS_n$-bimodule isomorphism $RS_n \cong RS_n^* = \text{Hom}_R(RS_n, R)$ where the $RS_n$-bimodule structure on $RS_n^*$ is given as follows

$$af(x)b := f(bxa) \quad \text{for all } a, b, x \in RS_n \text{ and } f \in RS_n^*.$$

We can now prove the promised result on $\text{Ind}(\lambda)$.

**Theorem 3.** For $\lambda$ any partition of $n$ there is an isomorphism of $RS_n$-modules

$$\text{Ind}(\lambda) \cong C(\lambda).$$

**Proof.** Define $\text{LGZ}_\lambda := \sum_i RS_n(L_i - r_{\lambda}(i))$. Then $\text{LGZ}_\lambda$ is a left ideal of $RS_n$ and by the definitions we have that

$$\text{Ind}(\lambda) = \text{Fr}(RS_n/\text{LGZ}_\lambda) = (RS_n/\text{LGZ}_\lambda)^{**}. \quad (22)$$

But $\langle \cdot, \cdot \rangle$ is nondegenerate, and therefore it induces an isomorphism of the right $RS_n$-modules

$$(RS_n/\text{LGZ}_\lambda)^* \cong (\text{LGZ}_\lambda)^\perp$$

where $(\text{LGZ}_\lambda)^\perp := \{ x \in RS_n \mid \langle x, \text{LGZ}_\lambda \rangle = 0 \}$. On the other hand, using the symmetry, associativity and nondegeneracy of $\langle \cdot, \cdot \rangle$ we find that $x \in (\text{LGZ}_\lambda)^\perp$ iff $(L_i - r_{\lambda}(i))x = 0$ for all $i$. We then deduce from the previous Lemma that

$$(\text{LGZ}_\lambda)^\perp = z_\lambda RS_n.$$

Thus, we are reduced to showing that $z_\lambda RS_n^* \cong C(\lambda)$. This is a little variation of a well-known fact, that normally is presented using either two left or two right modules. In our setting, with one left and one right module, the pairing $z_\lambda RS_n \times C(\lambda) \mapsto R$ is given by the rule $(z_\lambda u, x_\lambda) \mapsto \text{coeff}(z_\lambda u x_\lambda)$ where for any $u \in RS_n$ we define $\text{coeff}(u)$ as the coefficient of $x_\lambda$ when $u$ is expanded in the $x_{st}$-basis.

We now deduce the following universal property of $C(\lambda)$. We consider it analogous to the universal property for the Weyl module of an algebraic group, which is a consequence of the Kempf’s vanishing Theorem of the cohomology of the line bundle on the flag manifold given by a dominant weight, see eg. [A2], [RH3].

**Theorem 4.** Let $M \in RS_n$-modfr. Let

$$\chi M := \{ m \in M \mid L_i m = r_{\lambda}(i)m \text{ for all } i \}.$$

Then $\text{Hom}_{RS_n}(C(\lambda), M) = \chi M$.

**Proof.** Any $m \in \chi M$ induces a map in $\text{Hom}_{\text{GZ}}(1_\lambda, M)$ and then by Frobenius reciprocity a map in $\text{Hom}_{RS_n}(\text{Ind}(\lambda), M) = \text{Hom}_{RS_n}(C(\lambda), M)$. On the other hand, any element of $f \in \text{Hom}_{\text{GZ}}(1_\lambda, M)$ gives rise to an element of $\chi M$, namely the image $f(1)$.

\[ \square \]
Remark. Let $G_{Z_Q}$ be the original Gelfand-Zetlin algebra introduced in [OV], that is the $Q$-subalgebra of $Q S_n$ generated by the Jucys-Murphy elements $L_i, i = 1, \ldots, n$. We let $I_{Q, \lambda}$ denote the ideal of $G_{Z_Q}$ generated by the $L_i - r_\lambda(i)$ for $i = 1, \ldots, n$ and set $1_{Q, \lambda} := G_{Z_Q} / I_{Q, \lambda}$ and can then define $\text{Ind}_Q(\lambda)$ as the $Q S_n$-module induced up from $1_{Q, \lambda}$. Now the same series of arguments as the one used above, even with some simplifications in Lemma 8, leads to the isomorphism

$$\text{Ind}_Q(\lambda) \cong C_Q(\lambda).$$

But in this case the result could actually also have been obtained as follows. From (2) we have that

$$G_{Z_Q} = \langle E_t \mid t \in \text{Std}(n) \rangle.$$

(23)

Here the $\{E_t\}$ even form a $Q$-basis for $G_{Z_Q}$ since they are orthogonal idempotents. On the other hand, since $E_\lambda \xi_\lambda = \gamma_\lambda E_\lambda$, as is proved on page 508 of [Mu92], we get that the basis for $Q S_n$ constructed in the previous section, in this case takes the form

$$\{\Psi_t^* E_\lambda \Phi_t \mid s, t \in \text{Std}(\lambda), \lambda \in \text{Par}_n\}.$$

Let $e v_\lambda : G_{Z_Q} \to 1_{Q, \lambda}$ be the quotient map. Then one checks, using the fact that the contents $r_t(i)$ determine $t$ uniquely, that

$$e v_\lambda(E_t) = \begin{cases} 1 & \text{if } t = t^\lambda \\ 0 & \text{otherwise.} \end{cases}$$

(24)

It now follows from $E_\lambda \Phi_t = \Psi_t E_t$ that $\text{Ind}_Q(\lambda)$ has basis

$$\{\Psi_t^* E_\lambda \mid s \in \text{Std}(\lambda), \lambda \in \text{Par}_n\}$$

and the claim of the Remark follows from this.

Remark. From (24) we also get that $1_{Q, \lambda}$ is of dimension one over $Q$ with basis $\{E_\lambda\}$, and from this we conclude that $1_\lambda$ is free of rank one over $R$. Indeed, using $1_\lambda = G_{Z_n} / I_\lambda$ one checks that $1_\lambda$ is cyclic over $R$, generated by $1$. On the other hand, using exactness of the tensor functor $M \to M \otimes_R Q$ we get that $1_{Q, \lambda} = 1_\lambda \otimes_R Q$. Hence $\text{dim}_Q(1_\lambda \otimes_R Q) = 1$ and so $1_\lambda$ is torsion free, and thus free of rank one over $R$.

Remark. In general $RS_n$ is not free over $G_{Z_n}$. Indeed, if $RS_n$ were free over $G_{Z_n}$ then $Q S_n$ would be free over $G_{Z_Q}$. Since $\{E_t \mid t \in \text{Std}(n)\}$ is a basis of $G_{Z_Q}$ we can determine the dimension of $G_{Z_Q}$. For instance, for $n = 3$ we find $\dim G_{Z_Q} = 4$ which does not divide $\dim Q S_n = 6$.

6. Simples.

Let $G$ be an algebraic group over an algebraically closed field $k$ of characteristic $p$. Let $B$ be a Borel subgroup of $G$ with maximal torus $T \subset B$ and let $X(T)$ (resp. $X(T)^+$) be the set of weights (resp. dominant weights) with respect to $B$ and $T$. For $\lambda \in X(T)^+$ there is an associated Weyl module $\Delta(\lambda)$ with unique simple quotient $L(\lambda)$. It is the reduction modulo $p$ of a $Z$-form of a module for the corresponding complex group. The finite dimensional simple modules for $G$ are classified by $L(\lambda)$ where $\lambda \in X(T)^+$. We write $\nabla(\lambda) := \Delta(\lambda)^*$ where $*$ is the contravariant duality functor on finite dimensional $G$-modules. We may realize $\nabla(\lambda)$ as the $G$-module $H^0(\lambda)$ of global sections of the line bundle on $G/B$ associated with $\lambda$. 
Let $\langle \cdot , \cdot \rangle_\lambda$ be a nonzero contravariant form on $\Delta(\lambda)$. It induces a $G$-linear map $c_\lambda : \Delta(\lambda) \to \nabla(\lambda)$. As a matter of fact, since $\langle \cdot , \cdot \rangle_\lambda$ is unique up to multiplication by a nonzero scalar, we have that $c_\lambda$ generates $\text{Hom}_G(\Delta(\lambda), \nabla(\lambda))$ and that $im c_\lambda$ is isomorphic to $L(\lambda)$. In this sense, $\Delta(\lambda)$ and $\nabla(\lambda)$ give rise to a realization of $L(\lambda)$.

In this section we try to carry over this realization of the simple $G$-modules to the case of the symmetric group. As we shall see, the results of the previous section provide a suitable solution to this problem.

Let $M$ be a left $RS_n$-module. The contragredient dual $M^\ast$ of $M$ is defined to be $M^* := \text{Hom}_R(M, R)$ with $RS_n$-action given by $(\sigma f)(x) := f(\sigma^{-1}x)$ for $\sigma \in S_n, x \in M$ and $f \in M^*$. It is a left $RS_n$-module as well.

Using Theorem 5.3 of [Mu95], with a small modification since we are working with left modules, we have that the contragredient dual of $C(\lambda)$ is

$$C(\lambda)^\ast = RS_n y_\lambda s_\lambda^{-1} x_\lambda = RS_n y_\lambda s_\lambda x_\lambda.$$  

(25)

This isomorphism is also valid in the specialized situation

$$C(\lambda)^\ast = \mathbb{F}_p S_n y_\lambda s_\lambda x_\lambda. \quad (26)$$

Let $\langle \cdot , \cdot \rangle_\lambda$ be the bilinear form on $C(\lambda)$ associated with Murphy’s standard basis, following [Mu95] or the general cellular algebra theory, see [GL]. It is given by

$$(x_{s\lambda}, x_{t\lambda})_{\lambda} = \text{coeff}_\lambda(x_{s\lambda}, x_{t\lambda})$$

where once again $\text{coeff}_\lambda(u)$ is the coefficient of $x_\lambda$ when $u$ is expanded in the $x_{s\lambda}$-basis. It induces an $RS_n$-homomorphism $c_\lambda : C(\lambda) \to C(\lambda)^\ast$, or setting $z'_{\lambda} := y_\lambda s_\lambda x_\lambda$ and using (26) and Theorem 3

$$c_\lambda : \text{Ind}(\lambda) = \text{Fr}(RS_n \otimes_{GZ_n} 1_\lambda) \to RS_n z'_{\lambda}.$$ 

In general $c_\lambda$ is injective since $\langle \cdot , \cdot \rangle_\lambda$ is nondegenerate over $R$, but not surjective. We can now state and prove our main result.

**Theorem 5.** a) There is $a_\lambda \in \mathbb{Q}$ such that $a_\lambda E_\lambda \in RS_n$ and such that $c_\lambda$ corresponds to $1 \mapsto a_\lambda E_\lambda$ under Frobenius reciprocity.

b) The simple $\mathbb{F}_p S_n$-module $D(\lambda)$ associated with $\lambda$ is given by $D(\lambda) = \mathbb{F}_p S_n a_\lambda E_\lambda$.

**Proof.** By Theorem 3 and 4, and the fact that $C(\lambda)^\ast$ is free over $R$, we have

$$\text{Hom}_{RS_n}(\text{Ind}(\lambda), C(\lambda)^\ast) = \text{Hom}_{GZ_n}(1_\lambda, C(\lambda)^\ast)$$

hence $c_\lambda$ is given by $1 \mapsto m_\lambda$ where $m_\lambda \in (RS_n z'_{\lambda})$. From Lemma 10 we then have

$$m_\lambda \in RS_n z'_{\lambda} \cap z_\lambda RS_n = RS_n s_\lambda z'_{\lambda} \cap z_\lambda s_\lambda^{-1} RS_n.$$ 

But the Young preidempotent $e := z_\lambda s_\lambda^{-1}$ satisfies $e^2 = \gamma_\lambda \gamma_{\lambda'} e$ since it can be rewritten as $e = x_{\lambda} \kappa_{\lambda}$ with $\kappa_{\lambda}$ as above. Hence we get

$$m_\lambda = \frac{1}{\gamma_\lambda \gamma_{\lambda'}} z_\lambda s_\lambda^{-1} m s_\lambda z'_{\lambda} = \frac{1}{\gamma_\lambda \gamma_{\lambda'}} x_{\lambda} s_\lambda y_{\lambda'} s_\lambda^{-1} m s_\lambda y_{\lambda'} s_\lambda^{-1} x_{\lambda}$$

for some $m \in RS_n$. On the other hand, it is known that the $R$-module $x_{\lambda} RS_n y_{\lambda'}$ is free of rank one, generated by $x_{\lambda} s_\lambda y_{\lambda'}$, see for example [Mu92] page 498, and so we may rewrite $m_\lambda$ as follows

$$m_\lambda = a_\lambda x_{\lambda} s_\lambda y_{\lambda'} s_\lambda^{-1} x_{\lambda}$$
for some $a_\lambda \in \mathbb{Q}$. We now recall the expression for $z_\lambda$ given on page 511 of loc. cit. which in our notation becomes

$$x_\lambda s_\lambda y_\lambda s_\lambda^{-1} = b_\lambda E_\lambda s_\lambda E_t$$

where $b_\lambda \in \mathbb{Q}$ and $t$ is the lowest $\lambda$-tableau. Applying $\ast$ to it we get

$$s_\lambda y_\lambda s_\lambda^{-1} x_\lambda = b_\lambda E_t s_\lambda^{-1} E_\lambda.$$

Combining these expressions and using that $y_\lambda$ is a preidempotent, we find the following formula for $m_\lambda$, up to a scalar in $\mathbb{Q}$

$$m_\lambda = E_\lambda s_\lambda E_t s_\lambda^{-1} E_\lambda.$$

We then finally use the version of Young’s seminormal form that is developed on page 152 of [RH1] and obtain

$$m_\lambda = a_\lambda E_\lambda$$

where $a_\lambda$ is a (new) scalar in $\mathbb{Q}$. This finishes the proof of $a$.

We next show $b)$. From the definitions we have that

$$RS_n/\text{LGZ}_\lambda = RS_n \otimes_{\text{GZ}_n} 1_\lambda = \text{Ind}(\lambda) \oplus \text{Tor}(RS_n \otimes_{\text{GZ}_n} 1_\lambda)$$

and so by $a$) we have that $c_\lambda : \text{Ind}(\lambda) \to C(\lambda)^\otimes$ is given by $w \in RS_n \mapsto a_\lambda w E_\lambda$ since $C(\lambda)^\otimes$ is torsion-free. Reducing $c_\lambda$ modulo $p$ we get the homomorphism $\overline{c_\lambda}$

$$\overline{C(\lambda)} = \text{Ind}(\lambda) \otimes_R \mathbb{F}_p \xrightarrow{\overline{c_\lambda}} C(\lambda)^\otimes \otimes_R \mathbb{F}_p = \overline{C(\lambda)^\otimes}$$

given by $w \otimes 1 \mapsto a_\lambda w E_\lambda \otimes 1$ for $w \in RS_n$. We deduce from this that the image of $\overline{c_\lambda}$ is the submodule of $\overline{C(\lambda)^\otimes}$ generated by $\overline{a_\lambda E_\lambda} = a_\lambda E_\lambda \otimes_R \mathbb{F}_p$. But from the general principles explained above, this is equal to $D(\lambda)$. The Theorem is proved.

\textbf{Remark.} So far we do not have an exact formula for $a_\lambda$. On the other hand, since $c_\lambda$ is unique up to multiplication by an element of $R$, and since $\overline{c_\lambda}$ is nonzero iff $\lambda$ is $p$-restricted, we may simply choose for $a_\lambda$ the least common multiple of the denominators of the coefficients of $E_\lambda$ when expanded in the canonical basis of $RS_n$. The case where $\lambda$ is not $p$-restricted is not relevant for us, of course.

\textbf{Remark.} The Theorem gives rise to an algorithm for calculating $\dim D(\lambda)$ that goes as follows. Let $D(\lambda)$ be the $\dim S(\lambda) \times n!$ matrix over $\mathbb{F}_p$ that has $\overline{a_\lambda E_\lambda}$ in the first row and $\overline{d(t)^{-1} a_\lambda E_\lambda}$ for $t \in \text{Std}(\lambda) \setminus \{t^\lambda\}$ in the other rows. Then $\dim D(\lambda) = \text{rank} D(\lambda)$. Note that $E_\lambda$ can be calculated using formula (21). We have implemented this algorithm in the GAP system. We have checked $n < 8$ for all relevant primes and found complete match with the known dimensions for $D(\lambda)$, as given by Mathas’s Specht-package.

On the other hand, although the first row of the matrix $D(\lambda)$ easily gives the rest of it, the algorithm cannot expected to perform better than the usual algorithm for calculating $\dim D(\lambda)$, via the $\dim S(\lambda) \times \dim S(\lambda)$ matrix associated with the bilinear form on $S(\lambda)$ – since, after all, $n!$ in general is much bigger than $\dim S(\lambda)$.

\textbf{Remark.} As was pointed out to us by A. Mathas, a generator for $D(\lambda)$ is given on page 41 in [J]. In our terminology it is $x_\lambda s_\lambda y_\lambda s_\lambda^{-1} x_\lambda$ and hence, by the arguments of the Theorem, it coincides with our generator. Our final
expression of it is somewhat shorter, but still does not permit calculations much beyond the ones already indicated.

**Remark.** It is known from [Mu92] that \( \text{coeff}_1(E_\lambda) = \frac{1}{h_\lambda} \) where \( h_\lambda \) is the hook-product as above. In fact since \( RS_n \) is a symmetric algebra we get from Proposition 9.17 of [CR] that this fact also holds for \( E_t \) when \( t \in \text{Std}(\lambda) \). Based on GAP calculations we conjecture that the coefficient of any \( w \in S_n \) in \( E_\lambda \) is either zero or on the form \( \frac{1}{k_w h_\lambda} \) for some nonzero integer \( k_w \).

According to our GAP-calculations, a similar statement does not hold for the general \( E_t \).

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