Affine diffeomorphism groups are undistorted

Robert Tang

ABSTRACT
The affine diffeomorphism group $\text{Aff}(S, q)$ of a half-translation surface $(S, q)$ comprise the self-diffeomorphisms with constant differential away from the singularities. This group coincides with the stabiliser of the associated Teichmüller disc under the action of the mapping class group on Teichmüller space. We prove that any finitely generated subgroup of $\text{Aff}(S, q)$ is undistorted in the mapping class group. We also show that the systole map restricted to the associated electrified Nielsen core in the Teichmüller disc is a quasi-isometric embedding into the curve graph.

1. Introduction
For a finite type surface $S$, the mapping class group $\text{MCG}(S)$ is the (finitely generated) group of orientation-preserving self-homeomorphisms of $S$ up to isotopy. There has been considerable interest in understanding the large-scale geometry of mapping class groups. In particular, determining the distortion properties of naturally occurring subgroups plays a central role in this regard; see [5, Problem 3.7]. Given a finitely generated group $G$, a finitely generated subgroup $H \leq G$ is said to be undistorted if the inclusion $H \hookrightarrow G$ is a quasi-isometric embedding with respect to any (hence every) word metrics on $H$ and $G$; otherwise we say $H$ is distorted. On the one hand, subsurface mapping class groups [11, 15] and convex cocompact subgroups [8] are undistorted, while on the other, handlebody groups [12] and Torelli groups [2] have been shown to be distorted.

Our focus is on the distortion of subgroups that stabilise Teichmüller discs. Recall that the Teichmüller space $\mathcal{T}(S)$ parameterises the marked hyperbolic structures on $S$ up to isotopy. The mapping class group acts by isometries on $\mathcal{T}(S)$ with respect to the Teichmüller metric. A Teichmüller disc is a geodesically embedded copy of the hyperbolic plane in $\mathcal{T}(S)$ that arises from the $\text{SL}(2, \mathbb{R})$–orbit of a quadratic differential; its stabiliser under the action of $\text{MCG}(S)$ on $\mathcal{T}(S)$ can be naturally identified with the affine diffeomorphism group of any quadratic differential generating the given Teichmüller disc. These form an important class of groups, and are closely related to the study of billiard dynamics and translation surfaces [10, 16, 28].

**Theorem 1.1.** Any finitely generated subgroup of $\text{MCG}(S)$ stabilising a Teichmüller disc in $\mathcal{T}(S)$ is undistorted. In particular, affine diffeomorphism groups of Veech surfaces are undistorted.

**Remark 1.2.** In the cases where $\text{MCG}(S) \cong \text{SL}(2, \mathbb{Z})$, this is a consequence of the fact that finitely generated subgroups of a virtually free group are quasiconvex, and hence undistorted.

**Remark 1.3.** The subgroups under consideration in Theorem 1.1 are virtually free. However, there exist distorted free subgroups of $\text{MCG}(S)$; see for example, the point-pushing subgroups in the case where $S$ is a punctured surface [2].
Much of this paper is devoted to establishing bounded geometry results for Nielsen cores in Teichmüller discs. We assume throughout that $\Gamma \leq \text{MCG}(S)$ is a finitely generated subgroup stabilising a Teichmüller disc $H(\Gamma)$. We also assume that $\Gamma$ is not virtually cyclic. Viewing $H(\Gamma)$ as a copy of the hyperbolic plane, $\Gamma$ acts as a finitely generated Fuchsian group (upon passing to a finite-index quotient) $[28]$; the Nielsen core $N(\Gamma) \subseteq H(\Gamma)$ is the convex hull of its limit set in $\partial H(\Gamma)$. The finite generation assumption implies that the quotient $N(\Gamma)/\Gamma$ is a finite-area hyperbolic orbifold (possibly with geodesic boundary) and hence has finitely many cusps. Choose a sufficiently small horocyclic neighbourhood of each cusp so that their pre-images in $N(\Gamma)$ give a collection of pairwise disjoint horodiscs. We construct two variants of the Nielsen core using this family of horodiscs. The electrified Nielsen core $N_{el}(\Gamma)$ is obtained from $N(\Gamma)$ by forcing each horodisc to have uniformly bounded diameter (see Section 4 for the definition of an electrified space). The truncated Nielsen core $N^{tr}(\Gamma)$ is the complement of the interiors of all horodiscs in $N(\Gamma)$ equipped with the path metric. The action of $\Gamma$ on $N^{tr}(\Gamma)$ is geometric (properly discontinuous and cocompact), and so any orbit map $\Gamma \to N^{tr}(\Gamma)$ is a quasi-isometry by the Švarc–Milnor Lemma.

We prove that the electrified and truncated Nielsen cores, respectively, quasi-isometrically embed into the curve graph $C(S)$ and marking graph $M(S)$ (see Section 2.1 for background on combinatorial complexes). The systole map $\sigma: T(S) \to C(S)$ sends a hyperbolic surface to its set of shortest curves. A celebrated theorem of Masur and Minsky is that the systole map is a quasi-isometry, where $T(S)$ is equipped with the electrified Teichmüller metric and $C(S)$ with the combinatorial metric $[14]$. Leininger asks whether this still holds if the systole map is restricted to an electrified Teichmüller disc arising from a Veech surface (in which case the Nielsen core is the full Teichmüller disc). We give a positive answer in a more general setting.

**Theorem 1.4.** The restricted systole map $\sigma: N_{el}(\Gamma) \to C(S)$ is a $\Gamma$–equivariant quasi-isometric embedding.

As a consequence, the natural inclusion $N(\Gamma) \hookrightarrow T(S)$ is a quasi-isometric embedding when both spaces are equipped with their respective electrified metrics (see Corollary 4.5).

Next, we consider the analogous statement for the truncated Nielsen core and the marking graph. Masur and Minsky show that the mapping class group acts geometrically on $M(S)$ [15]. They also define an $\text{MCG}(S)$–equivariant short marking map $\mu: T(S) \to M(S)$; see Section 2.3 for details.

**Theorem 1.5.** The restricted short marking map $\mu: N^{tr}(\Gamma) \to M(S)$ is a $\Gamma$–equivariant quasi-isometric embedding.

There is an analogous consequence: the inclusion of $N^{tr}(\Gamma)$ into the thick part of Teichmüller space is a quasi-isometric embedding (see Corollary 4.2).

**Proof of Theorem 1.1.** If $\Gamma$ is virtually cyclic, then it is undistorted in $\text{MCG}(S)$ [6], so we may assume otherwise. Choose a basepoint $x_0 \in N^{tr}(\Gamma)$ and let $\mu_0$ be a short marking at $x_0$. Using the above theorem and the fact that the action of $\Gamma$ on $N^{tr}(\Gamma)$ is geometric, we deduce that the orbit map $\Gamma \to M(S)$ given by $g \mapsto g \cdot \mu_0 = \mu(g \cdot x_0)$ is a quasi-isometric embedding. Since $\text{MCG}(S)$ acts geometrically on $M(S)$, it follows that the inclusion $\Gamma \hookrightarrow \text{MCG}(S)$ is a quasi-isometric embedding. □

The proofs of Theorems 1.4 and 1.5 rely on the following technical result involving subsurface projections; see Section 2.1 for the definition of the map $\pi_Y: C(S) \to C(Y)$ where $Y \subseteq S$ is an essential subsurface. Any parabolic subgroup of $\Gamma$ has an invariant multicurve on $S$; call an annulus on $S$ parabolic for $\Gamma$ if its core curve is a component of such an invariant multicurve.
Proposition 1.6. There exists a constant $D = D(S, \Gamma)$ such that for any essential subsurface $Y \subseteq S$, the image $\pi_Y(\mu(N(\Gamma)))$ has infinite diameter in $C(Y)$ if $Y$ is a parabolic annulus for $\Gamma$; and diameter at most $D$ in $C(Y)$ otherwise.

This statement originally appears as Lemma 5.16 in a paper of Durham, Hagen, and Sisto [4], and is a key ingredient in proving that $\Gamma$ is hierarchically hyperbolic with respect to its parabolic subgroups. However, after discussions with the authors, it became apparent that there is a mistake in their proof, and so we shall give an alternative proof in Section 3.6. (In their paper, $\Gamma$ is called a Veech subgroup but we shall not use this term in order to avoid confusion with Veech groups which are subgroups of PSL(2, $\mathbb{R}$).)

2. Background

We begin by recalling some standard notions from coarse geometry. Given $a, b \geq 0$ and $K > 0$, write $a \prec_K b$ to mean $a \leq K \cdot b + K$, and $a \asymp_K b$ when $a \prec_K b$ and $b \prec_K a$. When the constant $K$ can be chosen to depend only on the topology of a surface $S$, we shall also write $a \prec b$ and $a \asymp b$ for simplicity.

A map between metric spaces $f: X \to Y$ is called coarsely Lipschitz if there exists some $K > 0$ such that $d_Y(f(x), f(y)) \prec_K d_X(x, y)$ for all $x, y \in X$. Furthermore, if $d_Y(f(x), f(y)) \asymp_K d_X(x, y)$ for all $x, y \in X$, then we call $f$ a quasi-isometric embedding. A quasi-isometric embedding with coarsely dense image is called a quasi-isometry. A quasigeodesic is a quasi-isometric embedding of an interval. These notions also make sense if $f$ is multi-valued, so long as we assume that the image of each $x \in X$ is non-empty and has uniformly bounded diameter. When $X$ is a combinatorial complex, we shall adopt the convention
\[ d_X(X, Y) := \text{diam}_X(X \cup Y) \]
for sets $X, Y \subseteq X$; this ensures that the triangle inequality holds.

If $G$ is a finitely generated group, then a finitely generated subgroup $H \leq G$ is undistorted if the inclusion map $H \to G$ is a quasi-isometric embedding with respect to (any of) their word metrics.

Given $c \geq 0$, the cutoff function is defined by $[t]_c = t$ if $t \geq c$, and $[t]_c = 0$ otherwise.

2.1. Combinatorial complexes

Throughout this paper, we shall assume that $S$ is a connected, orientable surface (without boundary) of finite genus and with a finite set $Z$ of punctures. Furthermore, we assume that its complexity
\[ \xi(S) := 3 \cdot \text{genus}(S) + |Z| - 3 \]
is at least 2. The mapping class group $\text{MCG}(S)$ is the group of orientation-preserving self-homeomorphisms of $S$ up to isotopy. Mapping class groups have been fruitfully studied via their actions on various graphs associated to $S$. Each of these graphs is endowed with the standard combinatorial metric, where each edge is isometrically identified with an interval of unit length. For further reference, see [14, 15].

A simple closed curve on $S$ is an (isotopy class of an) embedded loop on $S$ that is not homotopic to a point or into a puncture. For brevity, we shall use the term curve to mean simple
closed curve unless otherwise specified. An arc is (a proper isotopy class of) an interval on $S$ that has embedded interior, with endpoints contained in the set of punctures, and cannot be homotoped into a puncture. The arc-and-curve graph $\mathcal{AC}(S)$ has as vertices the arcs and curves on $S$, with edges connecting pairs of vertices whenever the corresponding arcs or curves have disjoint representatives. The curve graph $\mathcal{C}(S)$ is the induced subgraph of $\mathcal{AC}(S)$ whose vertices are the curves. Both of these graphs are connected, locally infinite, have infinite diameter, and are Gromov hyperbolic. Furthermore, the inclusion map $\mathcal{C}(S) \to \mathcal{AC}(S)$ is a quasi-isometry.

There is a modified definition of the arc-and-curve graph in the case of the (closed) annulus $\hat{A}$: vertices of $\mathcal{AC}(\hat{A})$ are embedded arcs connecting the two boundary components of $\hat{A}$ considered up to isotopy fixing their endpoints; while edges are defined as usual. The resulting graph $\mathcal{AC}(\hat{A})$ is quasi-isometric to $\mathbb{Z}$.

Next, we consider the marking graph $\mathcal{M}(S)$ which was first introduced by Masur and Minsky [15]. We shall not recall the full definition; instead, we state some facts that suffice for our purposes. A marking $\mu$ on $S$ consists of a pants decomposition $\text{base}(\mu)$ of $S$, called the set of base curves, and for each $\beta \in \text{base}(\mu)$, another curve, called a transversal, that intersects $\beta$ exactly once or twice and is disjoint from all other base curves. Each marking on $S$ has diameter at most 3 as a subset of $\mathcal{C}(S)$. The set of markings on $S$ form the vertices of $\mathcal{M}(S)$, with edges defined using a rule which guarantees the following.

- $\mathcal{M}(S)$ is connected, locally finite, and admits a geometric action by $\text{MCG}(S)$.
- If $\mu, \mu' \in \mathcal{M}(S)$ are adjacent, then $\text{diam}_{\mathcal{C}(S)}(\mu \cup \mu') \leq 4$.

By the Švarc–Milnor Lemma, any orbit map $\text{MCG}(S) \to \mathcal{M}(S)$ is a quasi-isometry.

Let $Y \subseteq S$ be a non-pants essential subsurface. For any $\alpha \in \mathcal{AC}(S)$ intersecting $Y$ essentially, the subsurface projection $\pi_Y(\alpha) \subseteq \mathcal{AC}(Y)$ is defined as follows. Equip $S$ with a complete hyperbolic metric (the choice of metric does not matter for this construction). Let $\bar{S}^Y$ be the cover corresponding to $\pi_1(Y)$ and $\tilde{\alpha}$ be the pre-image of the geodesic representative of $\alpha$. The Gromov compactification $\bar{S}^Y$ admits a natural identification with $Y$; we then set $\pi_Y(\alpha)$ to be all essential arcs and curves appearing in the closure of $\tilde{\alpha}$ in $\bar{S}^Y$. We extend subsurface projections to subsets of $\mathcal{AC}(S)$ by taking the union of the images of the individual arcs and curves. Subsurface projections can also be defined for a (measured) foliation $F$ on $S$: set $\pi_Y(F)$ to be the set of all essential arcs or curves on $\bar{S}^Y$ that descend to a leaf of $F$. (It may be the case that $Y$ is the subsurface filled by some non-compact leaf of $F$, in which case $\pi_Y(F)$ is empty.)

**Lemma 2.1 [15].** For any essential subsurface $Y \subseteq S$, the map $\pi_Y : \mathcal{M}(S) \to \mathcal{AC}(Y)$ is uniformly coarsely Lipschitz.

The following is the celebrated distance formula for the mapping class group. We shall use $d_Y$ as shorthand notation for $d_{\mathcal{AC}(Y)}$. In the case where $Y$ is an annulus with core curve $\alpha$, we also write $d_{\alpha}$ in place of $d_Y$.

**Theorem 2.2 [15].** There exists a constant $c_1 = c_1(S)$ such that for all $c \geq c_1$, there exists $A_1 > 0$ such that

$$d_{\mathcal{M}(S)}(\mu, \mu') \asymp_{A_1} \sum_Y \left[ d_Y(\mu, \mu') \right]_c$$

for all markings $\mu, \mu' \in \mathcal{M}(S)$, where the sum is taken over all essential subsurfaces $Y \subseteq S$.

We shall also recall a distance bound for the arc-and-curve graph in terms of geometric intersection numbers. For closed surfaces, Hempel showed that

$$d_S(\alpha, \beta) \leq 2 \log i(\alpha, \beta) + 2 \quad (2.3)$$
whenever \(\alpha, \beta\) are curves satisfying \(\iota(\alpha, \beta) \neq 0\) [13]; Schleimer extended this result to all non-annular surfaces [23]. By a standard argument, the bound also holds (at the cost of increasing the additive constant) if \(\alpha\) or \(\beta\) are arcs. The distance between two arcs \(\alpha\) and \(\beta\) in \(\mathcal{AC}(\mathbb{R})\) agrees with \(\iota(\alpha, \beta)\) up to a uniform additive error.

2.2. Teichmüller space, quadratic differentials, and Teichmüller discs

We now recall some background on Teichmüller theory and half-translation surfaces; refer to [7, 9, 25] for further details. The Teichmüller space \(T(S)\) of \(S\) is the space of marked complete hyperbolic metrics on \(S\) (up to isotopy). By the Uniformisation theorem, this is equivalent to the space of marked conformal structures on \(S\). The mapping class group acts on \(T(S)\) by change of marking. Moreover, \(T(S)\) is homeomorphic to \(\mathbb{R}^{2\xi(S)}\).

The cotangent bundle to \(T(S)\) is naturally identified with the space \(QD(S)\) of quadratic differentials (up to isotopy). By a quadratic differential on \(S\), we mean a Riemann surface \(x \in T(S)\) equipped with an integrable meromorphic quadratic differential that has at most simple poles at (and only at) the punctures. We shall use \(q \in QD(S)\) to denote a quadratic differential, with the underlying conformal structure implicit in the notation. Let \(QD(x)\) be the space of quadratic differentials with underlying conformal structure \(x \in T(S)\). A quadratic differential \(q \in QD(x)\) gives rise to natural co-ordinates: these are defined in a neighbourhood of a point \(z_0\) by

\[
z \mapsto \int_{z_0}^{z} \sqrt{q(w)} dw
\]

in terms of the complex co-ordinates from \(x\). These natural co-ordinates give an atlas away from the zeroes of \(q\) where the transition maps are of the form \(z \mapsto \pm z + c\) for some \(c \in \mathbb{C}\).

Pulling back the Euclidean metric on \(\mathbb{C}\) via this atlas endows \(S\) with a locally Euclidean metric away from the zeroes of \(q\), together with a preferred choice of vertical slope. The metric completion yields a singular Euclidean metric, known as a half-translation structure, where each zero of order \(p\) becomes a Euclidean cone point with cone angle \((p + 2)\pi\); in particular, poles have cone angle \(\pi\). We shall use \((S, q)\), or simply \(q\), to denote \(S\) equipped with this half-translation structure (with the choice of vertical slope). The integrability assumption ensures that the Euclidean area of \((S, q)\) is finite; we shall use \(QD^1(S)\) to denote the space of unit-area half-translation structures on \(S\).

Next, we consider geodesic representatives of a curve \(\alpha\) on a half-translation surface \((S, q)\). In order to deal with punctured surfaces, we allow \(\alpha\) to be homotoped so that it passes through punctures at the final moment of the homotopy, but not at any prior time. There are two possibilities: either \(\alpha\) has a unique geodesic representative and is formed by concatenating a sequence of saddle connections (straight-line segments connecting singularities with no interior singularities), or there is a unique maximal (open) Euclidean cylinder foliated by the geodesic representatives of \(\alpha\). In the latter case, we refer to \(\alpha\) as a cylinder curve on \((S, q)\). The Euclidean (or flat) length \(l_q(\alpha)\) is the length of any geodesic representative of \(\alpha\) on \((S, q)\) with respect to the Euclidean metric. The horizontal and vertical lengths of \(\alpha\) on \((S, q)\), denoted by \(l^H_q(\alpha)\) and \(l^V_q(\alpha)\), are obtained by, respectively, integrating \(|\Im(\sqrt{q})|\) and \(|\Re(\sqrt{q})|\) along a geodesic representative of \(\alpha\).

There is a natural \(\text{SL}(2, \mathbb{R})\)–action on \(QD^1(S)\) defined by \(\mathbb{R}\)–linear transformations of the natural co-ordinates. Two natural restrictions of this action yield the Teichmüller geodesic flow and unipotent flow, respectively, defined by taking orbits under

\[
t \mapsto g_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \quad \text{and} \quad s \mapsto u_s := \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}.
\]
in particular, orbits under $g_i$ descend to (unit-speed) geodesics in $T(S)$ called Teichmüller geodesics. By Teichmüller’s theorem, there exists a unique Teichmüller geodesic connecting any given pair of distinct points in $T(S)$.

The $SL(2, \mathbb{R})$–orbit of quadratic differential $q \in \text{QD}^1(S)$ descends to an isometrically embedded copy of the hyperbolic plane (of curvature $-4$) in $T(S)$, called a Teichmüller disc, which we shall denote by $H(q)$. The setwise stabiliser of $H(q) \subset T(S)$ under the action of $\text{MCG}(S)$ is naturally identified with the affine diffeomorphism group $\text{Aff}(q)$; an affine (self-)diffeomorphism of $(S, q)$ acts bijectively on the set of singularities, and restricts to a diffeomorphism with constant differential away from the singular set. The differential homomorphism $D: \text{Aff}(q) \to PSL(2, \mathbb{R})$ determines a short exact sequence
\[ 1 \to \text{Aut}(q) \to \text{Aff}(q) \to PSL(q) \to 1, \]
where the image $PSL(q)$ is a Fuchsian group called the Veech group [28]. The kernel is the pointwise stabiliser of $H(q)$, and coincides with the finite group of slope-preserving isometries of $(S, q)$. In the case where $PSL(q)$ is a lattice, then $(S, q)$ is called a Veech surface.

Affine diffeomorphisms are classified as follows. Let $T_\alpha \in \text{MCG}(S)$ denote a Dehn twist about a curve $\alpha$.

**Proposition 2.4** [27, 28]. Let $\phi \in \text{Aff}(q)$ be non-trivial. Then:

1. $D\phi$ is hyperbolic if and only if $\phi$ is pseudo-Anosov;
2. $D\phi$ is parabolic if and only if some power of $\phi$ coincides with $T_{\alpha_1}^{k_1} \circ T_{\alpha_2}^{k_2} \circ \ldots \circ T_{\alpha_j}^{k_j}$ for some set of pairwise disjoint curves $\alpha_1, \ldots, \alpha_j$ and (non-zero) integers $k_1, \ldots, k_j$; and
3. $D\phi$ is elliptic if and only if $\phi$ is periodic (finite order).

In the case where $D\phi$ is parabolic, there is an associated cylinder decomposition of $(S, q)$. The foliation on $(S, q)$ by geodesics parallel to the unique eigenslope of $D\phi$ is invariant under $\phi$; indeed, all separatrices with this slope are saddle connections. Cutting $(S, q)$ along these saddle connections decomposes it into a finite union of (maximal) Euclidean cylinders. The diffeomorphism $\phi$ acts by (possibly) permuting the cylinders, and by some power of a Dehn twist about the core curve of each cylinder.

2.3. Systoles and short markings

Let us now review some natural maps from Teichmüller space to the curve complex and the marking complex. For the rest of this paper, we shall fix a sufficiently small constant $\epsilon_0 > 0$. This value is chosen so that it satisfies the statements in this subsection, and to also ensure that Theorem 3.6 holds.

The extremal length of $\alpha$ on $x \in T(S)$ is
\[ \text{Ext}_x(\alpha) := \sup_{\rho} \frac{l_\rho(\alpha)^2}{\text{area}(\rho)}, \]
where $\rho$ runs over all metrics in the conformal class $x$. We shall also write $\text{Ext}_q(\alpha)$ to mean $\text{Ext}_x(\alpha)$ whenever $q \in \text{QD}(x)$. Given $\epsilon > 0$, we call $x \in T(S)$ $\epsilon$-thick if $\text{Ext}_x(\alpha) \geq \epsilon$ for all curves $\alpha \in C(S)$, and write $T_{>\epsilon}(S)$ for the $\epsilon$-thick part of Teichmüller space. When $\epsilon$ is sufficiently small, $T_{>\epsilon}(S)$ is connected. Moreover, by Mumford’s Compactness Criterion [1, 19], $\text{MCG}(S)$ acts geometrically on $T_{>\epsilon}(S)$.

The systole map $\sigma: T(S) \to C(S)$ is defined by assigning $x \in T(S)$ its set of minimal length curves $\sigma(x) \subset C(S)$. The systole sets for $x \in T(S)$ defined using extremal length, hyperbolic length, or Euclidean length for any $q \in \text{QD}(x)$ agree in $C(S)$ up to uniformly bounded Hausdorff distance (see [26, Lemma 4.7]), and so, for our purposes, we may use whichever definition is most convenient.
The short marking map $\mu : \mathcal{T}(S) \to \mathcal{M}(S)$ is defined as follows. The base curves of $\mu(x)$ are chosen to minimise hyperbolic length on $x \in \mathcal{T}(S)$ according to the greedy algorithm. A transversal for each base curve $\beta$ is chosen to minimise $d_\beta(a, a)$, where $a$ is a geodesic arc perpendicular to $\beta$ with respect to the hyperbolic metric. The choice of short marking may not be unique, however, all the possible choices for $\mu(x)$ form a uniformly bounded diameter set in $\mathcal{M}(S)$. We shall also write $\mu_x$ to stand for $\mu(x)$.

**Proposition 2.5** [14, 15]. The maps $\sigma : \mathcal{T}(S) \to \mathcal{C}(S)$ and $\mu : \mathcal{T}(S) \to \mathcal{M}(S)$ are both coarsely Lipschitz.

Since $\mu$ is (coarsely) $\text{MCG}(S)$–equivariant, it follows that the restriction $\mu : \mathcal{T}_\geq \epsilon(S) \to \mathcal{M}(S)$ is a quasi-isometry.

Subsurface projections can be defined on Teichmüller space by composing with the short marking map. Note that $\pi_Y \circ \mu : \mathcal{T}(S) \to \mathcal{AC}(Y)$ is also coarsely Lipschitz. As shorthand notation, write

$$\pi_Y(q) = \pi_Y(x) := \pi_Y(\mu_x)$$

and

$$d_Y(x, y) := \text{diam}_{\mathcal{AC}(Y)}(\pi_Y(x) \cup \pi_Y(y))$$

for $x, y \in \mathcal{T}(S)$ and $q \in \text{QD}(x)$.

**Theorem 2.6** [21]. Fix a sufficiently small $\epsilon > 0$. Then there exists a constant $c_2 = c_2(S, \epsilon)$ such that for any $c \geq c_2$, there exists $\Delta_2 > 0$ such that

$$d_{\mathcal{T}(S)}(x, y) \geq \Delta_2 \sum_{Y \not\in \mathcal{A}} [d_Y(x, y)]_c + \sum_{\alpha} \log [d_\alpha(x, y)]_c$$

whenever $x, y \in \mathcal{T}(S)$ are $\epsilon$–thick. Here, the sums are, respectively, taken over all essential non-annular subsurfaces $Y \subseteq S$ and all essential simple closed curves $\alpha$.

Let us now discuss the behaviour of the systole map along a Teichmüller geodesic. As shorthand notation, we shall write $t$ as subscript when referring to lengths on $q_t := g_t \cdot q$ (or the underlying conformal structure $x_t$).

**Theorem 2.7** [22]. Let $G : \mathbb{R} \to \mathcal{T}(S)$ be a Teichmüller geodesic and $Y \subseteq S$ be a subsurface. Then $\pi_Y \circ G : \mathbb{R} \to \mathcal{AC}(Y)$ is a uniform reparameterised quasigeodesic. Furthermore, if $Y \neq S$, then there exists a (possibly empty) interval $I_Y \subset \mathbb{R}$ such that:

- each component of $\mathbb{R} \setminus I_Y$ has uniformly bounded image under $\pi_Y \circ G : \mathbb{R} \to \mathcal{AC}(Y)$; and
- if $\alpha \subseteq \partial Y$ is a boundary component, then $\text{Ext}_t(\alpha) \leq \epsilon_0$ for all $t \in I_Y$.

Let $|I|$ denote the length of an interval $I \subseteq \mathbb{R}$.

**Corollary 2.8.** Let $\nu^+$ and $\nu^-$ be the horizontal and vertical foliations associated to $q \in \text{QD}(S)$. For every $L > 0$, there exists a constant $D = D(S, L) > 0$ such that the following holds. Suppose $Y$ is a proper subsurface that has a boundary component $\alpha$ satisfying $l_t(\alpha) \geq L$ for all $t \in \mathbb{R}$. Then $d_Y(\nu^+, \nu^-) \lesssim |I_Y| < D$.

**Proof.** The condition $l_t(\alpha) \geq L$ for all $t \in \mathbb{R}$ implies that $\alpha$ cannot be completely horizontal nor completely vertical. The flat length of $\alpha$ satisfies

$$l_\alpha(e^{\frac{t}{2}}l^H_\alpha + e^{\frac{t}{2}}l^V_\alpha) \approx L_0 \cosh(t - t_0) \geq L \cosh(t - t_0),$$
where \( t = t_0 \) is a time at which \( l_t(\alpha) \) attains a global minimum \( L_0 \). Using the definition of extremal length, we have

\[
\text{Ext}_t(\alpha) \geq L^2 \cosh^2(t - t_0).
\]

By the previous theorem, \( \text{Ext}_t(\alpha) \leq \epsilon_0 \) for all \( t \in I_Y \), and so \( |I_Y| \) is bounded from above by some function of \( \frac{\epsilon_0}{L^2} \). Combining this with the above theorem and the fact that \( \pi_Y : \mathcal{T}(S) \to \mathcal{AC}(Y) \) is uniformly coarsely Lipschitz, we may bound \( d_Y(\nu^+, \nu^-) \prec |I_Y| \) from above by some function of \( \frac{\epsilon_0}{L^2} \).

\[\square\]

The no-backtracking property follows from hyperbolicity of the curve complex and the fact that Teichmüller geodesics descend to (reparameterised) quasigeodesics in \( \mathcal{C}(S) \).

**Lemma 2.9** [14]. There exists a constant \( C = C(S) \) such that the following holds. Let \( \mathcal{G} : \mathbb{R} \to \mathcal{T}(S) \) be a Teichmüller geodesic. Then

\[
d_S(\mathcal{G}(s), \mathcal{G}(u)) \geq d_S(\mathcal{G}(s), \mathcal{G}(t)) + d_S(\mathcal{G}(t), \mathcal{G}(u)) - C
\]

for all \( s \leq t \leq u \).

In the case of thick Teichmüller geodesic segments, we have stronger control over their images in \( \mathcal{C}(S) \) using Theorem 2.6 and Rafi’s characterisation of short curves along Teichmüller geodesics.

**Proposition 2.10** [20, 21]. Let \( \epsilon > 0 \) be sufficiently small, and suppose \( \mathcal{G} : I \to \mathcal{T}(S) \) is an \( \epsilon \)-thick Teichmüller geodesic segment. Then \( \sigma \circ \mathcal{G} : I \to \mathcal{C}(S) \) is a \( K \)-quasigeodesic, where \( K = K(S, \epsilon) \).

We finish off the section by briefly recalling the notion of the geodesic representative \( Y_q \) of a subsurface \( Y \subseteq S \) on a half-translation structure \((S, q)\) due to Rafi [20] (see also [18]). First suppose \( Y \) is an annulus. If the core curve of \( Y \) is a cylinder curve, then set \( Y_q \) to be the unique geodesic representative of its core curve. Now suppose \( Y \) is non-annular. The idea is to ‘pull tight’ each boundary component of \( \partial Y \) to their geodesic representatives to obtain \( Y_q \). If a component of \( \partial Y \) is a cylinder curve, then we require \( Y_q \) to be disjoint from the interior of the associated maximal cylinder. If some component \( \alpha \) of \( \partial Y \) is homotopic to a puncture of \( S \), then the geodesic representative of \( \alpha \) to degenerates to the completion point on \((S, q)\) associated to the puncture. Following Minsky–Taylor, we say that \( Y \) is \( q \)-compatible if there is a homotopy between \( Y \) and \( Y_q \) that restricts to an isotopy between their interiors. If this holds, then cutting \((S, q)\) along all saddle connections appearing on \( \partial Y_q \) yields \( Y_q \) as a complementary component.

**Proposition 2.11** [18]. Let \( \nu^+, \nu^- \) be the horizontal and vertical foliations associated to \( q \in \mathcal{QD}^1(S) \). If a subsurface \( Y \) is not \( q \)-compatible, then \( d_Y(\nu^+, \nu^-) \leq 3 \) (and the subsurface projections \( \pi_Y(\nu^\pm) \) are non-empty).

### 3. Bounded geometry and Nielsen cores

For the rest of this paper, we shall fix a finitely generated subgroup \( \hat{\Gamma} \leq \text{MCG}(S) \) that stabilises a Teichmüller disc \( \mathcal{H}(\Gamma) \subset \mathcal{T}(S) \). Furthermore, we assume that \( \hat{\Gamma} \) is not virtually cyclic, as such groups are known to be undistorted in \( \text{MCG}(S) \) [6]. This also guarantees that the Teichmüller disc is unique; see the remark below.
The group $\hat{\Gamma}$ can also be viewed as a subgroup of $\text{Aff}(q)$ for any $q \in \text{QD}(S)$ generating $\mathbf{H}(\Gamma)$. Restricting the differential homomorphism to $\hat{\Gamma}$ gives rise to a short exact sequence

$$1 \to \hat{\Gamma} \cap \text{Aut}(q) \to \hat{\Gamma} \to \Gamma \to 1,$$

where the image $\Gamma \leq \text{PSL}(q)$ is a finitely generated non-elementary Fuchsian group. Since the kernel is finite, the quotient map $\hat{\Gamma} \to \Gamma$ is a quasi-isometry with respect to their word metrics. Therefore, for our purposes, we may equally work with the actions of $\hat{\Gamma}$ or $\Gamma$ on the associated Teichmüller disc $\mathbf{H}(\Gamma)$ regarded, respectively, as either a subset of $\mathcal{T}(S)$ or as a copy of $\mathbb{H}^2$.

All quadratic differentials $q$ shall henceforth be assumed to belong to the $\text{SL}(2, \mathbb{R})$–orbit in $\text{QD}^1(S)$ descending to $\mathbf{H}(\Gamma)$ unless otherwise specified.

**Remark 3.1.** Any non-elementary Fuchsian group contains a hyperbolic element, and so $\hat{\Gamma}$ contains a pseudo-Anosov element. Any pseudo-Anosov element stabilises a unique Teichmüller geodesic, and thus determines a unique Teichmüller disc. Therefore, the Teichmüller disc $\mathbf{H}(\Gamma)$ is uniquely determined by the subgroup $\Gamma$.

**Definition 3.2.** The Nielsen core $\mathcal{N}(\Gamma) \subseteq \mathbf{H}(\Gamma)$ of $\Gamma$ is the convex hull of the limit set $\Lambda(\Gamma) \subseteq \partial \mathbf{H}(\Gamma)$.

The inclusion $\mathcal{N}(\Gamma) \hookrightarrow \mathbf{H}(\Gamma)$ is an isometric embedding. Since the action of $\text{MCG}(S)$ on $\mathcal{T}(S)$ is properly discontinuous, the same holds for the action of $\Gamma$ on $\mathcal{N}(\Gamma)$. The quotient $\mathcal{N}(\Gamma)/\Gamma$ is a finite area hyperbolic orbifold, possibly with geodesic boundary. Since $\Gamma$ is finitely generated, $\mathcal{N}(\Gamma)/\Gamma$ has empty boundary precisely when $\Lambda(\Gamma) = \partial \mathbf{H}(\Gamma)$, in which case $\Gamma$ is a lattice.

The goal of this section is to establish positive constants which control the geometry of quadratic differentials appearing over $\mathcal{N}(\Gamma)$.

### 3.1. Parabolic subgroups

The (finite) set of cusps of $\mathcal{N}(\Gamma)/\Gamma$ is in one-to-one correspondence with the conjugacy classes of maximal parabolic subgroups of $\Gamma$. Let $\mathcal{P}(\Gamma)$ be the set of all maximal parabolic subgroups of $\Gamma$. For each $H \in \mathcal{P}(\Gamma)$, let $\text{PCyl}(H)$ be the set of core curves of the cylinders associated to $H$. Define $\text{PCyl}(\Gamma) := \bigcup_{H \in \mathcal{P}(\Gamma)} \text{PCyl}(H)$ to be the set of parabolic cylinder curves associated to $\Gamma$. Furthermore, if a saddle connection is parallel to some parabolic slope, then we shall call it a parabolic saddle connection.

**Remark 3.3.** If $\Gamma$ has at least one (maximal) parabolic subgroup $H$, then it must have infinitely many; these can be obtained, for example, by conjugating $H$ by powers of a hyperbolic element of $\Gamma$. It may be the case that $\mathcal{P}(\Gamma)$ is empty, occurring precisely when $\mathcal{N}(\Gamma)/\Gamma$ has no cusps; in this situation, $\mathcal{N}(\Gamma)/\Gamma$ has at least one geodesic boundary component as $\Gamma$ cannot act cocompactly on $\mathbf{H}(\Gamma)$.

### 3.2. Cylinder widths

Given a parabolic subgroup $H \in \mathcal{P}(\Gamma)$, let $W_H(q) > 0$ be the minimum width of all (maximal) cylinders on $(S,q)$ whose core curve belongs to $\text{PCyl}(H)$. Since $(S,q)$ has unit area, we deduce that each curve in $\text{PCyl}(H)$ has flat length at most $\frac{1}{W_H(q)}$ on $(S,q)$. Note that $W_H(q) \to \infty$ as $q$ tends towards the fixed point of $H$ on $\partial \mathbf{H}(\Gamma)$. It follows that the function

$$q \mapsto \sup_{H \in \mathcal{P}(\Gamma)} W_H(q)$$

defined on $\mathcal{N}(\Gamma)$ descends to a continuous proper function on $\mathcal{N}(\Gamma)/\Gamma$, and thus attains a positive minimum value $W_\Gamma > 0$ at some $q_0 \in \mathcal{N}(\Gamma)$. Now, there are only finitely many curves
on $(S,q_0)$ whose flat length is bounded above by any given positive constant. Therefore, there are finitely many $H \in \mathcal{P}(\Gamma)$ for which $W_H(q_0)$ is bounded from below by any given positive constant, and so the supremum is attained by some parabolic subgroup.

**Lemma 3.4.** Every non-parabolic saddle connection on $q \in N(\Gamma)$ has length at least $W_\Gamma$. In particular, if $\alpha \in \mathcal{C}(S)$ is not parallel to a parabolic slope, then $l_q(\alpha) \geq W_\Gamma$ throughout $N(\Gamma)$.

*Proof.* For each $q \in N(\Gamma)$, there exists a parabolic subgroup $H \in \mathcal{P}(\Gamma)$ for which every cylinder in $\text{PCyl}(H)$ has width at least $W_\Gamma$ on $(S,q)$. Then any non-parabolic saddle connection on $(S,q)$ must intersect some cylinder in $\text{PCyl}(H)$ transversely, and thus has length at least $W_\Gamma$. If $\alpha$ is not parallel to a parabolic slope, then its geodesic representative on $(S,q)$ must use at least one saddle connection not parallel to the slope of $H$. The result follows. \qed

**Corollary 3.5.** If $\alpha \in \mathcal{C}(S)$ is not parallel to a parabolic slope, then $\text{Ext}_q(\alpha) \geq W_\Gamma^2$ throughout $N(\Gamma)$.

### 3.3. Expanding annuli

Let us turn our attention to curves that are parallel to some parabolic slope for $\Gamma$. These curves can have arbitrarily short flat length on $N(\Gamma)$. Our goal is to show that when such a curve is not itself a parabolic cylinder curve, then its extremal length is bounded below by some constant depending only on $\Gamma$.

We shall briefly recall the notions of flat and expanding annuli from Minsky [17]. Given a curve $\alpha \in \mathcal{C}(S)$, define its (possibly degenerate) flat annulus $F_\alpha(q)$ as follows: If $\alpha$ is a cylinder curve on $(S,q)$, then let $F_\alpha(q)$ be the associated maximal flat cylinder; otherwise let $F_\alpha(q)$ be the (unique) geodesic representative of $\alpha$ on $(S,q)$. Note that $F_\alpha(q)$ contains all geodesic representatives of $\alpha$.

Next, equip the annular cover $S^\alpha$ with the pullback metric from $(S,q)$. The flat annulus $F_\alpha(q)$ lifts to a unique flat annulus $\tilde{F}_\alpha(q)$ on $S^\alpha$. Cutting $S^\alpha$ along the two (possibly coincident) boundary curves of $\tilde{F}_\alpha(q)$ yields exactly two components $S^\alpha_+ \text{ and } S^\alpha_-$ that are not flat cylinders. Given $r > 0$, let $\tilde{E}_\alpha^\pm(r) \subset S^\alpha_\mp$ be the intersection of $S^\alpha_\pm$ with the closed $r$–neighbourhood of $\tilde{F}_\alpha(q)$ in $S^\alpha$. For $r > 0$, $\tilde{E}_\alpha^\pm(r)$ is topologically a closed annulus.

Now, consider the projection of $\tilde{E}_\alpha^\pm(r)$ to $(S,q)$. If the projection map is injective on the interior of $E_\alpha^\pm(r)$, then we call the image $E_\alpha^\pm(r)$ a (regular) expanding annulus of $\alpha$ on $(S,q)$ of radius $r$. Furthermore, if $E_\alpha^\pm(r)$ has no singularities in its interior, then it is called primitive; let $r^\pm_\alpha(\alpha) \geq 0$ be the largest value of $r$ for which this holds. Call the boundary curve of $E_\alpha^\pm(r)$ that coincides with a boundary curve of $F_\alpha(q)$ the inner boundary, and the other boundary curve the outer boundary (these will coincide when $r = 0$).

The following theorem gives an estimate for the extremal length of short curves in terms of the geometry of their maximal flat and primitive expanding annuli. Recall that the *modulus* $\text{Mod}(C)$ of a flat cylinder $C$ is its width divided by the length of its core curve.

**Theorem 3.6 [3, 17].** If a curve $\alpha$ satisfies $\text{Ext}_q(\alpha) < \epsilon_0$ then

$$\frac{1}{\text{Ext}_q(\alpha)} \leq \max \left\{ \text{Mod}(F_\alpha(q)), \log \left( \frac{r^\pm_\alpha(\alpha)}{l_q(\alpha)} \right) \right\}.$$ 

Let us now focus our attention on primitive expanding annuli associated to curves parallel to some parabolic slope for $\Gamma$. Given a parabolic subgroup $H \in \mathcal{P}(\Gamma)$, let $\gamma_H, \gamma'_H$, respectively, be a shortest and longest saddle connections on $(S,q)$ parallel to the slope corresponding to $H$, respectively.
for some \( q \in \mathbf{H}(\Gamma) \). (These saddle connections will, respectively, remain shortest and longest throughout \( \mathbf{H}(\Gamma) \).) Define
\[
\rho_H := \frac{l_q(\gamma_H')}{l_q(\gamma_H)} > 0,
\]
for any \( q \in \mathbf{H}(\Gamma) \). Note that this ratio is constant under \( \text{SL}(2, \mathbb{R}) \)-deformations. Since there are finitely many conjugacy classes of maximal parabolic subgroups, it follows that the supremum
\[
\rho^\Gamma := \sup_{H \in P(\Gamma)} \rho_H < \infty.
\]
is finite and attained.

**Lemma 3.7.** Let \( \alpha \) be a curve that is parallel to some parabolic slope for \( \Gamma \). Then
\[
\frac{r^\pm_q(\alpha)}{l_q(\alpha)} \leq \rho^\Gamma
\]
for all \( q \in N(\Gamma) \).

**Proof.** By applying a rotation, we may assume that \( \alpha \) is horizontal on \((S, q)\). Since the horizontal slope is parabolic, \((S, q)\) admits a horizontal cylinder decomposition. In particular, every horizontal separatrix is a saddle connection.

Let \( E \) be a maximal primitive expanding annulus for \( \alpha \) on \((S, q)\). If \( r^\pm_q(\alpha) = 0 \), then we are done, so we may assume that \( E \) has non-empty interior. The inner boundary of \( E \) is a geodesic representative of \( \alpha \) passing through at least one singularity. Since \( \alpha \) is horizontal, the interior angle (inside \( E \)) at each singularity on the inner boundary is an integer multiple of \( \pi \). We claim that at least one such singularity has interior angle at least \( 2\pi \). If not, then, for sufficiently small \( 0 < r < r^\pm_q(\alpha) \), the annulus \( E^\pm_q(r) \subset E \) is isometric to a Euclidean cylinder; this contradicts the construction of flat and expanding annuli. Therefore, there exists a horizontal saddle connection \( \beta \) starting on the inner boundary of \( E \) with an initial segment lying in the interior of \( E \). Since \( E \) is primitive, we deduce that \( r^\pm_q(\alpha) \leq l_q(\beta) \), for otherwise \( E \) will contain an interior singularity. On the other hand, \( l_q(\alpha) \) is at least the length of the shortest horizontal saddle connection. The desired result follows using the definition of \( \rho^\Gamma \). \( \square \)

Any curve parallel to a parabolic slope that is not itself a parabolic cylinder curve has a degenerate flat annulus. Thus, combining the preceding results with Corollary 3.5 yields the following.

**Proposition 3.8.** There exists a constant \( 0 < \epsilon^\Gamma \leq \epsilon_0 \) such that for every curve \( \alpha \in \mathcal{C}(S) \), either
\begin{itemize}
  \item \( \alpha \) is a parabolic cylinder curve for \( \Gamma \) and \( \inf_{q \in N(\Gamma)} \text{Ext}_q(\alpha) = 0 \); or
  \item \( \alpha \) is not a parabolic cylinder curve for \( \Gamma \) and \( \text{Ext}_q(\alpha) \geq \epsilon^\Gamma \) for all \( q \in N(\Gamma) \).
\end{itemize}

In the following subsections, we may need to take \( \epsilon^\Gamma \) sufficiently small to ensure that desired properties hold.

### 3.4. Horodiscs

We now choose a preferred family of horodiscs in \( \mathbf{H}(\Gamma) \) associated to the family of parabolic subgroups. For each \( H \in P(\Gamma) \), let \( \gamma_H \) be a shortest saddle connection with slope corresponding to \( H \), then define a pair of nested horodiscs
\[
U(H) = \{ q \in \mathbf{H}(\Gamma) : l_q(\gamma_H) \leq \sqrt{\epsilon^\Gamma} \} \quad \text{and} \quad U'(H) = \{ q \in \mathbf{H}(\Gamma) : l_q(\gamma_H) \leq \sqrt{\epsilon_0} \}. 
\]
Each $U(H)$ descends to a neighbourhood of a cusp on $N(\Gamma)/\Gamma$. Since there are finitely many cusps, we may choose $\epsilon_{\Gamma}$ sufficiently small to ensure that
\[ d_{H(\Gamma)}(U(H), U(K)) \geq 1 \]
for all distinct $H, K \in \mathcal{P}(\Gamma)$. This also ensures that the cusp neighbourhood arising from each $U(H)$ is topologically an annulus. Next, define the truncated Nielsen core of $\Gamma$ to be
\[ N^{tr}(\Gamma) := N(\Gamma) \setminus \bigcup_{H \in \mathcal{P}(\Gamma)} \text{int}(U(H)). \]
By Proposition 3.8, the $\epsilon_{\Gamma}$–thin part of $N(\Gamma)$ is contained in $\bigcup_{H \in \mathcal{P}(\Gamma)} U(H)$, and so $N^{tr}(\Gamma)$ is $\epsilon_{\Gamma}$–thick.

**Lemma 3.9.** There exists a constant $R_{\Gamma} > 0$ such that for all distinct $H, K \in \mathcal{P}(\Gamma)$, we have $\text{diam}(U'(H) \cap U'(K)) \leq R_{\Gamma}$.

**Proof.** Let $t = \frac{1}{2} \log(\epsilon_0/\epsilon_{\Gamma}) \geq 0$. Suppose $q \in U'(H)$. By applying a rotation, we may assume $\gamma_H$ is vertical on $(S, q)$. Then
\[ l_{g, q}(\gamma_H) = e^{-l_q(\gamma_H)} \leq \sqrt{\epsilon_{\Gamma}/\epsilon_0} \sqrt{\epsilon_0} = \sqrt{\epsilon_{\Gamma}}. \]
Therefore, $U'(H)$ is contained in the $t$–neighbourhood of $U(H)$ for all $H \in \mathcal{P}(\Gamma)$. The diameter bound then follows using elementary hyperbolic geometry and the fact that the distance between distinct horodiscs $U(H)$ and $U(K)$ is at least 1. \hfill $\Box$

### 3.5. Virtual triangle areas

Smillie and Weiss characterise Veech surfaces as the half-translation surfaces whose virtual triangle spectrum is discrete [24]. Motivated by their work, we consider the parabolic virtual triangle spectrum of $\Gamma$ obtained by restricting to the parabolic saddle connections, and prove that it is always discrete. This is not necessary for our main theorem, however, we include it as it may be of independent interest.

Associated to any saddle connection on $(S, q)$ is a holonomy vector in $\mathbb{C}$ that has the same slope and length; this is well defined up to scaling by $\pm 1$. Let $\text{hol}_{\Gamma}(q)$ be the set of holonomy vectors associated to parabolic saddle connections on $(S, q)$ (with respect to $\Gamma$). Define the parabolic virtual triangle spectrum of $\Gamma$ to be
\[ PVT(\Gamma) := \{|u \wedge v| : u, v \in \text{hol}_{\Gamma}(q)\} \subset \mathbb{R}, \]
for some (hence all) $q \in \mathcal{H}(\Gamma)$. Let $v_H(q)$ be the holonomy vector of a shortest saddle connection parallel to the slope corresponding to $H \in \mathcal{P}(\Gamma)$.

**Lemma 3.10.** Let $H, K \in \mathcal{P}(\Gamma)$. Then
\[ d_{H(\Gamma)}(U(H), U(K)) = \log \left( \frac{|v_H \wedge v_K|}{\epsilon_{\Gamma}} \right). \]

**Proof.** Let $\mathcal{G}(t) = q_t$ be the infinite Teichmüller geodesic whose horizontal and vertical slopes correspond to the slopes of $H$ and $K$, respectively, and where $q_0 \in \partial U(H)$. Note that the unique geodesic segment connecting $U(H)$ and $U(K)$ is a subinterval of $\mathcal{G}$. Using the definition of the horodiscs, we have $v_H(q_0) = \sqrt{\epsilon_{\Gamma}}$ and $v_K(q_0) = e^t v_K(q_0) = e^t \sqrt{\epsilon_{\Gamma}}$ when $t = d_{H(\Gamma)}(U(H), U(K))$. The result follows. \hfill $\Box$

**Lemma 3.11.** The set $PVT(\Gamma)$ is discrete in $\mathbb{R}$. 

Proof. It suffices to show that $PVT(\Gamma) \cap [0, b]$ contains finitely many values for all $b \geq 0$. By the above lemma, this is equivalent to proving that there are finitely many geodesic segments on $N^{tr}(\Gamma)/\Gamma$ orthogonal to cusp boundaries of length less than any given positive constant. This follows from a standard argument; for example, by doubling $N^tr(\Gamma)/\Gamma$ along its boundary to obtain a closed compact surface, and using the fact that there are finitely many closed curves whose geodesic length is bounded above by any given constant. □

3.6. Bounded projection image

The main technical result of this paper is the following dichotomy for subsurface projections.

Call an annulus on $S$ parabolic for $\Gamma$ if its core curve is a parabolic cylinder curve for $\Gamma$.

Proposition 3.12. There exists a constant $D_\Gamma > 0$ such that given any subsurface $Y \subseteq S$, the set $\pi_Y(\mu(N(\Gamma)))$ has:

- infinite diameter in $\mathcal{AC}(Y)$ if $Y$ is a parabolic annulus for $\Gamma$; and
- diameter at most $D_\Gamma$ in $\mathcal{AC}(Y)$ otherwise.

Proof. For brevity, write $M(\Gamma) := \mu(N(\Gamma))$.

First, we consider the case where $Y$ is a parabolic annulus. Let $H$ be the parabolic subgroup containing $Y$ in its associated cylinder decomposition. Let $\phi \in H$ be a non-trivial element that preserves $Y$. Note that $\phi|_Y$ is a power of a Dehn twist about the core curve of $Y$. Furthermore, $\pi_Y(\phi^n \cdot \mu) = (\phi|_Y)^n \cdot \pi_Y(\mu)$ for any marking $\mu \in M(\Gamma)$. Since orbits in $\mathcal{AC}(Y)$ under $\langle \phi|_Y \rangle$ have infinite diameter, it follows that $\pi_Y(M(\Gamma)) \supseteq \pi_Y((\phi) \cdot \mu)$ is unbounded.

We may henceforth assume $Y \neq S$ is not a parabolic annulus. By Proposition 2.11, we may also assume that $Y$ is $q$–compatible.

Let $G(\Gamma)$ be the set of bi-infinite geodesics on $H(\Gamma)$ with both endpoints in the limit set $\Lambda(\Gamma)$. Note that $N(\Gamma)$ is the convex hull of $\bigcup_{G \in G(\Gamma)} G$ in $H(\Gamma)$. For concreteness, we take the images of $U(H)$ on $N(\Gamma)/\Gamma$ to be a set of preferred cusp neighbourhoods, where $H$ runs over a set of representatives for each conjugacy class in $P(\Gamma)$.

Lemma 3.13. There exists some $r_\Gamma > 0$ such that $N(\Gamma)$ is contained in the $r_\Gamma$–neighbourhood of $\bigcup_{G \in G(\Gamma)} G$ in $H(\Gamma)$.

Proof. It suffices to prove the result in the case where $H(\Gamma)/\Gamma$ is a hyperbolic surface; the general case can be dealt with by taking a finite orbifold cover. In this situation, every geodesic $G \in G(\Gamma)$ descends to a complete geodesic on $N(\Gamma)/\Gamma$; moreover, every complete geodesic on $N(\Gamma)/\Gamma$ arises this way. Note that a complete geodesic on $N(\Gamma)/\Gamma$ cannot be contained in any cusp neighbourhood, and so must have non-empty intersection with $N^{tr}(\Gamma)/\Gamma$. It follows that $N^{tr}(\Gamma)/\Gamma$ is contained in the $r$–neighbourhood of any complete geodesic on $N(\Gamma)/\Gamma$ for $r \geq \text{diam}(N^{tr}(\Gamma)/\Gamma)$.

We now deal with the cusp neighbourhoods of $N(\Gamma)/\Gamma$. Since $\Gamma$ is non-elementary, it either has no (maximal) parabolic subgroups, or infinitely many. Therefore, for every $H \in P(\Gamma)$ there exists some geodesic $G \in G(\Gamma)$ with one end contained in $U(H)$. The associated cusp neighbourhood in $N(\Gamma)/\Gamma$ is then contained in the $r$–neighbourhood of the image of $G$, as long as $r > 0$ is greater than the length of the cusp boundary.

The desired result holds for any value of $r_\Gamma > 0$ greater than both the diameter of $N^{tr}(\Gamma)/\Gamma$, and its longest cusp boundary length. □

Using Proposition 2.5, the following is immediate.
Corollary 3.14. There exists some $r'_1 > 0$ such that for all $Y$, the set $\pi_Y(M(\Gamma))$ is contained in the $r'_1$–neighbourhood of $\bigcup_{G \in \mathcal{G}(\Gamma)} \pi_Y(G)$ in $\mathcal{AC}(Y)$.

Our strategy is to now show that $\pi_Y(\Lambda(\Gamma))$ has diameter in $\mathcal{AC}(Y)$ bounded above by a constant independent of the choice of $Y$. Then for any $G \in \mathcal{G}(\Gamma)$ with endpoints $\nu^+$ and $\nu^-$, the uniform reparameterised quasigeodesic $\pi_Y \circ G$ (coarsely) connects $\pi_Y(\nu^+)$ and $\pi_Y(\nu^-)$ in $\mathcal{AC}(Y)$ and hence has uniformly bounded diameter (see Theorem 2.7). Observe that for any pair $G, G' \in \mathcal{G}(\Gamma)$, there exists some $G'' \in \mathcal{G}(\Gamma)$ sharing at least one endpoint with each of $G$ and $G'$; this implies that $\bigcup_{G \in \mathcal{G}(\Gamma)} \pi_Y(G)$ has diameter bounded from above by a constant depending only on $\Gamma$. Appealing to the above corollary completes the proof of Proposition 3.12.

Proposition 3.15. There exists some $D > 0$ such that for any subsurface $Y \neq S$, not a parabolic annulus, the set $\pi_Y(\Lambda(\Gamma))$ has diameter at most $D$ in $\mathcal{AC}(Y)$.

Proof. Let $Y \neq S$ be a subsurface that is not a parabolic annulus. The proof proceeds in three cases depending on the geodesic representatives of $\partial Y$.

Case 1: $\partial Y$ has a boundary component $\gamma$ that is not parallel to some parabolic slope. Let $\nu^+, \nu^- \in \Lambda(\Gamma)$ be a pair of distinct foliations, and $G \in \mathcal{G}(\Gamma)$ be the Teichmüller geodesic in N(Γ) connecting them. By Lemma 3.4, we have $l_t(\gamma) \geq W_\Gamma$ for all $t$. Applying Corollary 2.8, we deduce that $d_Y(\nu^+, \nu^-) \preceq |\partial Y| \preceq D_1$ for some $D_1 = D_1(S, W_\Gamma)$.

Case 2: Each boundary curve of $Y$ has a parabolic slope, but they are not all parallel. Let $\gamma_1, \gamma_2 \subset \partial Y$ be boundary curves parallel to distinct parabolic slopes, and let $H_1, H_2 \in \mathcal{P}(\Gamma)$, respectively, be the corresponding parabolic subgroups. By Theorem 2.7 and the definition of $U''(H_i)$, we have $G(I_Y) \subseteq U''(H_1) \cap U''(H_2)$ for any $G \in \mathcal{G}(\Gamma)$. Then $|I_Y| \leq R_\Gamma$, by Lemma 3.9, and so the result follows using Corollary 2.8.

Case 3: All boundary curves of $Y$ are parallel and have parabolic slope. For the remaining case, note that $Y$ cannot be an annulus. We shall prove a stronger statement in order to bound $d_Y(\nu^+, \nu^-)$. Given $q \in \text{QD}(S)$, let $\mathcal{PMF}(q)$ be the set of projectivised measured foliations arising as the horizontal foliation of $e^{i\theta}q$ for some $\theta \in \mathbb{R}P^1$. Recall that $Y_q$ is the geodesic representative of $Y$ on $(S, q)$.

Lemma 3.16. Let $q \in \text{QD}(S)$ and suppose $Y \neq S$ is a non-annular $q$–compatible subsurface with horizontal boundary. Let $\gamma \in \mathcal{C}(Y)$ be a horizontal curve. Then

$$\text{diam}_Y \mathcal{PMF}(q) \prec \log \left( \frac{l_q(\gamma)}{l_q(\partial Y)} \right).$$

Note that this bound is silent if there exist no essential horizontal curves on $Y$.

Proof. The strategy is to bound $d_Y(\gamma, \nu)$ from above for all $\nu \in \mathcal{PMF}(q)$. If $\nu$ is horizontal, then $d_Y(\gamma, \nu) \leq 1$, so we may assume otherwise. By applying an appropriate $\text{SL}(2, \mathbb{R})$–deformation, we may arrange so that $\nu$ is vertical while preserving the horizontal slope. Cut $Y_q$ along all vertical separatrices that start either at a boundary singularity with internal angle at least $2\pi$ or an interior singularity, and end either at a singularity or on the boundary of $Y_q$. This decomposes $Y_q$ into a union of (at least one) Euclidean rectangles (with horizontal and vertical sides), and a (possibly empty) set of subsurfaces with vertical boundary. The number of separatrices that were cut along is bounded above in terms of $|\chi(Y)| \leq |\chi(S)|$, and so the number of rectangles is also bounded above in terms of $|\chi(Y)|$. Note that $\partial Y_q$ is the union of
the horizontal sides of these rectangles, and so the widths of these rectangles sum to $2l_q(\partial Y)$. Therefore, there exists a rectangle $R$ of width($R$) $> l_q(\partial Y)$. Let $\eta$ be the vertical arc in $R$ that connects the midpoints of its two horizontal sides. Then

$$i(\eta, \gamma) \leq \frac{l_q(\gamma)}{\text{width}(R)} < \frac{l_q(\gamma)}{l_q(\partial Y)},$$

and so by Hempel’s bound (2.3) we have

$$d_Y(\eta, \gamma) \propto \log \left( \frac{l_q(\gamma)}{l_q(\partial Y)} \right).$$

Since $\nu$ is vertical, it has no transverse intersection with $\eta$, and so $d_Y(\eta, \nu) \leq 1$. □

It remains to show that $Y$ has an essential curve that is not too long compared to the length of $\partial Y$. By assumption, $Y$ is $q$–compatible and thus has embedded interior. Since the horizontal slope is parabolic, cutting $(S, q)$ along all horizontal saddle connections decomposes it into a union of horizontal cylinders. In particular, $Y_q$ can be formed by taking a non-empty subset of these cylinders, then gluing them along some horizontal saddle connections; let $\gamma$ be a shortest core curve of a horizontal cylinder contained in $Y_q$. Note that $\gamma$ cannot be peripheral on $Y$, for if any boundary component of $Y$ is a cylinder curve, then the interior of the associated maximal cylinder must be disjoint from $Y_q$.

We wish to bound $\frac{l_g(\gamma)}{l_q(\partial Y)}$ from above. Let $l_0$ and $l_1$, respectively, be the lengths of the shortest and the longest horizontal saddle connection on $(S, q)$. Note that $l_q(\partial Y) \geq l_0$. Consider the maximal cylinder $C$ with core curve $\gamma$. Each boundary component of $C$ runs over any given saddle connection on $(S, q)$ at most twice. Since the number of horizontal saddle connections on $(S, q)$ is bounded above in terms of $|\chi(S)|$, we deduce that $l_q(\gamma) \propto l_1$. Therefore

$$\frac{l_q(\gamma)}{l_q(\partial Y)} \propto \frac{l_1}{l_0} \leq \rho \Gamma.$$

Applying the above lemma completes the proof of Proposition 3.15, and hence Proposition 3.12. □

3.7. Cusp winding

In this section, we estimate the annular projection distance $d_\alpha(q, q')$ for points $q, q' \in N^{1\Gamma}(\Gamma)$ and $\alpha \in PCyl(\Gamma)$ in terms of the amount of winding about the associated cusp.

Let us recall Rafi’s estimate for annular projection distance in terms of the relative twisting of a pair of quadratic differentials about a curve $\alpha \in C(S)$. Given $q \in QD(S)$, let $\eta_\alpha(q)$ be a complete Euclidean geodesic on $(S, q)$ orthogonal to the geodesic representative of $\alpha$. (We may also assume that $\eta_\alpha(q)$ does not hit any singularities.) Let $\tilde{\eta}_\alpha(q)$ be a lift of $\eta_\alpha(q)$ on the annular cover $S^\alpha$ that intersects the unique closed lift of $\alpha$ essentially. Then the relative twisting $\text{tw}_\alpha(q, q')$ is defined to be the geometric intersection number between $\tilde{\eta}_\alpha(q)$ and $\tilde{\eta}_\alpha(q')$: this is well defined up to a uniform additive error.

**Proposition 3.17** [22]. For all $q, q' \in QD(S)$ and $\alpha \in C(S)$, we have $\text{tw}_\alpha(q, q') \simeq d_\alpha(q, q')$.

We now focus on the case where $\alpha$ is a parabolic cylinder curve for $\Gamma$, and where the quadratic differentials are restricted to $H(\Gamma)$. Choose $H \in P(\Gamma)$ so that $\alpha \in PCyl(H)$, and suppose $G$ is a Teichmüller geodesic on $H(\Gamma)$ orthogonal to $\partial U(H)$. By applying a suitable rotation, we may assume that all cylinders in $PCyl(H)$ are horizontal on $q_t \in G$ for all $t \in \mathbb{R}$. Therefore, any geodesic on $(S, q_t)$ orthogonal to the geodesic representative of $\alpha$ is vertical, and hence is a leaf of the vertical foliation on $(S, q_t)$. In particular, $\eta_\alpha(q_t)$ can be chosen to be the same topological leaf for all $t \in \mathbb{R}$. It follows that $\text{tw}_\alpha(q, q') = 0$ for all $q, q' \in G$. 

Next, we define the cusp winding with respect to a parabolic subgroup \( H \in \text{PCyl}(\Gamma) \) as follows: Given \( q, q' \in \text{H}(\Gamma) \), let \( d_H(q, q') \) be the distance between their respective nearest point projections to the horocycle \( \partial U(H) \), measured along \( \partial U(H) \). Note that if \( \mathcal{G} \) is a geodesic in \( \text{H}(\Gamma) \) orthogonal to \( \partial U(H) \), then all points along \( \mathcal{G} \) project to a common point on \( \partial U(H) \); thus \( d_H \) gives a notion of distance between pairs of such geodesics.

We shall show that the relative twisting and cusp winding agree up to uniform additive and multiplicative error depending only on \( \Gamma \). Let us introduce some more constants. Given a parabolic subgroup \( H \in \mathcal{P}(\Gamma) \), let

\[
m_H := \min_{\alpha \in \mathcal{P}(\Gamma)} \{ \text{Mod}(C_q(\alpha)) \} \quad \text{and} \quad m_H' := \max_{\alpha \in \mathcal{P}(\Gamma)} \{ \text{Mod}(C_q(\alpha)) \},
\]

where \( q \in \partial U(H) \). Define

\[
m_{\Gamma} := \inf_{H \in \mathcal{P}(\Gamma)} m_H \quad \text{and} \quad m_{\Gamma}' := \sup_{H \in \mathcal{P}(\Gamma)} m_H'.
\]

Since there are finitely many parabolic subgroups up to conjugation, it follows that

\[
0 < m_{\Gamma} \leq m_{\Gamma}' < \infty.
\]

**Lemma 3.18.** There exists a constant \( K = K(\Gamma) \) such that the following holds. Let \( H \in \mathcal{P}(\Gamma) \) be a parabolic subgroup and suppose \( \alpha \in \text{PCyl}(H) \). Then for all \( q, q' \in \text{H}(\Gamma) \), we have

\[
d_H(q, q') \approx K d_{\alpha}(q, q').
\]

**Proof.** If \( \mathcal{G} \) is a Teichmüller geodesic on \( \text{H}(\Gamma) \) orthogonal to \( \partial U(H) \) then \( \eta_{\alpha}(q) \) can be chosen to be the same topological leaf for all \( q \in \mathcal{G} \). Therefore, it suffices to prove the desired result for pairs of points on \( \partial U(H) \). Fix some \( q_0 \in \partial U(H) \) and assume, by applying a rotation, that \( \alpha \) is horizontal on \((S, q_0)\). Then every point on \( \partial U(H) \) has the form \( q_s = u_s \cdot q_0 \) for some \( s \in \mathbb{R} \), where \( u_s \) is the unipotent flow on \( \text{QD}(S) \).

We first estimate the relative twisting under the unipotent flow. For brevity, write \( \eta_s \) for \( \eta_{\alpha}(q_s) \). Observe that \( \eta_0 \) is vertical on \((S, q_0)\), and has slope \( \frac{1}{2} \) on \((S, q_s)\). Let \( \mathcal{C} \) be the cylinder with core curve \( \alpha \) on \((S, q_s)\), and let \( m \) be its modulus. Equip the annular cover \( S^\alpha \) with the metric obtained by pulling back the half-translation structure from \((S, q_s)\), and let \( \mathcal{C} \) be the unique Euclidean cylinder on \( S^\alpha \) projecting to \( \mathcal{C} \). The number of intersections between \( \mathcal{C} \) and \( \eta_s \) that occur on \( \mathcal{C} \) is equal to \( |m|s| \) up to a uniform additive error; whereas the number of intersections occurring outside of \( \mathcal{C} \) is at most two (this follows, for example, using the Gauss–Bonnet Theorem). Since \( m_{\Gamma} \leq m \leq m_{\Gamma}' \), it follows that

\[
m_{\Gamma}|s| \prec tw_{\alpha}(q_0, q_s) \prec m_{\Gamma}'|s|.
\]

Next, we estimate the cusp winding. Identify \( \text{H}(q) \) with the upper half-plane model of \( \mathbb{H}^2 \) so that \( \partial U(H) \) coincides with the horizontal line \( \Im(z) = 1 \). Under this identification, the unipotent flow acts as \( u_s(z) = z + s \) for all \( z \in \mathbb{H}^2 \). In particular, we have \( q_s = q_0 + s \) (viewed as points on the complex plane). Since \( q_s \) is the closest point projection of itself to \( \partial U(H) \), we deduce that

\[
d_H(q_0, q_s) \approx |s|.
\]

Combining the above estimates with Proposition 3.17 completes the proof. \( \square \)

**4. Quasi-isometric embeddings**

We are now ready to prove the main results.

**Theorem 4.1.** The short marking map \( \mu : \text{N}^{\Gamma}(\Gamma) \rightarrow \mathcal{M}(S) \) is a \( \Gamma \)-equivariant quasi-isometric embedding. Furthermore, there exists a constant \( c_\Gamma > 0 \) such that for all \( c \geq c_\Gamma \),
there exists some $K = K(S, \Gamma, c)$ such that

$$d^r_{N(\Gamma)}(x, y) \asymp K \: d_S(\mu_x, \mu_y) + \sum_{\alpha \in PCyl(\Gamma)} [d_\alpha(\mu_x, \mu_y)]_c$$

for all $x, y \in N^{tr}(\Gamma)$.

Since $\mu : \mathcal{T}_{\beta, \epsilon}(S) \to \mathcal{M}(S)$ is a MCG(S)-equivariant quasi-isometry, the following is immediate.

**Corollary 4.2.** For $\epsilon > 0$ sufficiently small, the inclusion $N^{tr}(\Gamma) \hookrightarrow \mathcal{T}_{\beta, \epsilon}(S)$ is a quasi-isometric embedding.

The next result answers a question of Leininger, who originally posed it in the case of electrified Teichmüller discs arising from lattice Veech groups. Our result holds more generally for all finitely generated Veech groups.

Let us recall the construction of an electrified space. Given a length space $\mathcal{X}$ and a collection $\mathcal{U}$ of non-empty subsets of $\mathcal{X}$, the electrification $\mathcal{X}^{el}$ of $\mathcal{X}$ along $\mathcal{U}$ is defined as follows. For each $U \in \mathcal{U}$, introduce a new point $*_{U}$, called an electrification point. Then for each $x \in U$, add an interval of length $\frac{1}{2}$ connecting $x$ to $*_{U}$. The metric on $\mathcal{X}^{el}$ is declared to be the induced path metric. This procedure forces each subset $U \in \mathcal{U}$ to have diameter at most 1 in $\mathcal{X}^{el}$. Note that the inclusion $\mathcal{X} \hookrightarrow \mathcal{X}^{el}$ is 1-Lipschitz.

Define the **electrified Nielsen core** $N^{el}(\Gamma)$ to be $N(\Gamma)$ electrified along the collection of horodiscs $\{U(H) : H \in \mathcal{P}(\Gamma)\}$. We shall use $*_{H}$ to denote the electrification point $*_{U(H)}$. The systole map can be extended to $N^{el}(\Gamma)$ by declaring $\sigma(x) := PCyl(H)$ for all $x$ lying in the open $\frac{1}{2}$-neighbourhood of $*_{H}$ in $N^{el}(\Gamma)$.

**Theorem 4.3.** The systole map $\sigma : N^{el}(\Gamma) \to \mathcal{C}(S)$ is a quasi-isometric embedding.

This result has a natural counterpart for the embedding of the electrified core into electrified Teichmüller space. For a sufficiently small $\epsilon > 0$, define $\mathcal{T}^{el}(S)$ to be the electrification of $\mathcal{T}(S)$ along the collection of thin regions $V(\alpha) := \{x \in \mathcal{T}(S) : \text{Ext}_x(\alpha) \leq \epsilon\}$, where $\alpha$ runs over $\mathcal{C}(S)$. Write $*_{\alpha}$ for the associated electrification point. The natural inclusion $\iota : N(\Gamma) \to \mathcal{T}(S)$ can be extended to an embedding between their respective electrifications as follows. By choosing $c_{\Gamma}$ and $\epsilon$ appropriately, we have $U(H) \subseteq V(\alpha) \cap N(\Gamma)$ whenever $H \in \mathcal{P}(\Gamma)$ and $\alpha \in PCyl(H)$. For each $H \in PCyl(\Gamma)$, choose some $\alpha(H) \in PCyl(H)$ then set $\iota(\epsilon) = *_{\alpha(H)}$. We then define $\iota$ on each interval connecting $*_{H}$ to some $x \in U(H)$ in $N^{el}(\Gamma)$ by mapping it to the unique interval connecting $x$ to $*_{*_{H}}$ in $\mathcal{T}^{el}(S)$. The map $\iota : N^{el}(\Gamma) \to \mathcal{T}^{el}(S)$ clearly defines the choice of cylinder curve for each parabolic subgroup, however, it is well defined up to bounded error.

The systole map $\sigma : \mathcal{T}(S) \to \mathcal{C}(S)$ can be extended to $\mathcal{T}^{el}(S)$ by declaring $\sigma(x) := \alpha$ for all $x$ in the open $\frac{1}{2}$-neighbourhood of $*_{\alpha}$. Thus, $\sigma \circ \iota$ and $\sigma$ coarsely agree as maps from $N^{el}(\Gamma)$ to $\mathcal{C}(S)$.

**Theorem 4.4 [14].** The map $\sigma : \mathcal{T}^{el}(S) \to \mathcal{C}(S)$ is a quasi-isometry.

**Corollary 4.5.** The map $\iota : N^{el}(\Gamma) \to \mathcal{T}^{el}(S)$ is a quasi-isometric embedding.

The specific choice of horodiscs is not crucial for the above theorems, so long as they are chosen in a $\Gamma$–equivariant manner.
4.1. Relevant subsurfaces

We require a technical result in order to control the number of terms appearing in sums for our arguments in the following subsections.

**Theorem 4.6** [15]. There exists a constant \( c_0 = c_0(S) > 0 \) for which the following holds. Given any threshold \( c \geq c_0 \), there exists \( N > 0 \) such that for any subsurface \( Y \subseteq S \) and \( \alpha, \beta \in \mathcal{C}(S) \), the poset

\[
\mathcal{R}_c^Y(\alpha, \beta) := \{ Z \subseteq Y : d_Z(\alpha, \beta) \geq c \}
\]

has at most \( N \cdot d_Y(\alpha, \beta) + N \) maximal elements (with respect to subsurface inclusion).

This result is a consequence of the Large Links Theorem and the Existence of Hierarchies from Masur–Minsky [15]. Their original statement only asserts the existence of a fixed threshold \( c_0 \) for which the above holds. We shall sketch a proof of the general statement, allowing for a variable threshold \( c \), assuming the original statement.

**Proof.** Fix a constant \( c \geq c_0 \). By assumption, there exists a constant \( N_0 = N_0(S) \geq 1 \) such that for every essential subsurface \( Y \subseteq S \), the set \( \mathcal{R}_{c_0}^Y(\alpha, \beta) \) has at most \( N_0 \cdot d_Y(\alpha, \beta) + N_0 \) maximal elements. Fix a subsurface \( Y \) and let \( Z \) be a maximal element of \( \mathcal{R}_{c_0}^Y(\alpha, \beta) \). Then there exists a maximal chain \( Y = Y_0 \supseteq Y_1 \supseteq \ldots \supseteq Y_k \supseteq Z \) in \( \mathcal{R}_{c_0}^Y(\alpha, \beta) \) where \( k \leq \xi(Y) \) and \( Y_i \notin \mathcal{R}_c^Y(\alpha, \beta) \) for each \( 0 < i \leq k \). We shall bound the number of possible chains of this form. Since we consider only maximal chains, each \( Y_{i+1} \) is a maximal element of \( \mathcal{R}_{c_0}^{Y_i}(\alpha, \beta) \). Also note that

\[
c_0 \leq d_{Y_i}(\alpha, \beta) < c
\]

for each \( 0 < i \leq k \). Applying the assumption, there are at most \( N_0 \cdot d_Y(\alpha, \beta) + N_0 \) maximal elements in \( \mathcal{R}_{c_0}^Y(\alpha, \beta) \); while \( \mathcal{R}_c^Y(\alpha, \beta) \) has at most \( N_0 \cdot c + N_0 \) maximal elements for each \( i \geq 1 \). Therefore, by induction, there are at most

\[
(N_0 \cdot d_Y(\alpha, \beta) + N_0)(N_0 \cdot c + N_0)^{\xi(Y)}
\]

possible chains of the desired form. Setting \( N := N_0(N_0 \cdot c + N_0)^{\xi(S)} \) completes the proof. \( \square \)

4.2. Proof of Theorem 4.1

First, observe that the inclusion \( N^{tr}(\Gamma) \rightrightarrows T(S) \) is 1–Lipschitz. Since the map \( \mu : T(S) \rightrightarrows \mathcal{M}(S) \) is coarsely Lipschitz, it follows that

\[
d_{\mathcal{M}(S)}(\mu_x, \mu_y) < d_{N^{tr}(\Gamma)}^{tr}(x, y) \tag{4.7}
\]

for all \( x, y \in N^{tr}(\Gamma) \). Thus, it remains to prove the reverse coarse inequality.

We shall establish an upper bound for distances in the truncated core in terms of Teichmüller distance and cusp winding.

**Lemma 4.8.** For all \( c \geq 1 \), there exists a constant \( A_3 > 1 \) such that

\[
d_{N^{tr}(\Gamma)}^{tr}(x, y) \leq A_3 \cdot d_{H(\Gamma)}(x, y) + \sum_{H \in \mathcal{H}(\Gamma)} [d_H(x, y)]_c
\]

for all \( x, y \in N^{tr}(\Gamma) \). Moreover, the sum contains finitely many terms.
Proof. Let \( \pi_\Gamma : H(\Gamma) \to N^r(\Gamma) \) be the nearest point projection map. Let \( \mathcal{G} \) be the Teichmüller geodesic connecting \( x \) to \( y \) in \( H(\Gamma) \), and \( \mathcal{G}' \) be its image under \( \pi_\Gamma \). The map \( \pi_\Gamma \) replaces each (maximal) subsegment of \( \mathcal{G} \) contained in some horodisc \( U(H) \) with a detour running along the associated horocycle \( \partial U(H) \) (with the same endpoints as the given subsegment). Our strategy is to bound the length of each detour in terms of the length of the original subsegment. This will give an upper bound on the length of \( \mathcal{G}' \), and hence \( d^{r(\Gamma)}_{N(\Gamma)}(x, y) \).

For notational convenience, when dealing with any particular parabolic subgroup \( H \in P(\Gamma) \), we shall choose an identification of \( H(\Gamma) \) with the upper half-plane \( \mathbb{H}^2 \) so that \( \partial U(H) \) is the vertical projection to \( \partial U(H) \). For notational convenience, when dealing with any particular parabolic subgroup \( H \in P(\Gamma) \), we shall choose an identification of \( H(\Gamma) \) with the upper half-plane \( \mathbb{H}^2 \) so that \( \partial U(H) \) is the vertical projection to \( \partial U(H) \). Furthermore, we have \( d_H(x, y) = \frac{1}{2} |\Re(x) - \Re(y)| \), viewing \( x \) and \( y \) as points on the complex plane (the factor of \( \frac{1}{2} \) comes from the fact that we are working with the hyperbolic plane of curvature \(-4\)).

Consider those \( H \in P(\Gamma) \) where \( d_H(x, y) < \epsilon \). By elementary circle geometry, \( \mathcal{G} \) lies below the horizontal line \( \Im(z) = \sqrt{1 + \epsilon^2} \). The projection \( \pi_\Gamma \) restricted to the region lying between this line and \( \partial U(H) \) is \( A_3 \)-Lipschitz, where \( A_3 = A_3(\epsilon) \). Therefore, the length of the (possibly empty) segment \( \mathcal{G} \cap U(H) \) increases by at most a multiplicative factor of \( A_3 \) under \( \pi_\Gamma \).

Now, consider those \( H \in P(\Gamma) \) where \( d_H(x, y) \geq \epsilon \). Since \( \epsilon \geq 1 \), we have \( |\Re(x) - \Re(y)| \geq 2 \) and so the geodesic \( \mathcal{G} \) intersects \( U(H) \) non-trivially. This can only occur for at most finitely many \( H \in P(\Gamma) \) as the pairwise distance between distinct horodiscs is at least 1. Observe that \( \pi_Y(\mathcal{G} \cap U(H)) \) lies inside the horizontal line segment connecting \( \Re(x) + i \) to \( \Re(y) + i \) in \( \mathbb{H}^2 \). Therefore, the length of \( \mathcal{G} \cap U(H) \) increases by an additive factor of at most \( d_H(x, y) \) under \( \pi_Y \).

The above two cases account for all possible detours. The desired result follows using the fact that \( \mathcal{G} \) has length \( d_H(\Gamma)(x, y) \).

Next, we choose a sufficiently large threshold \( c_\Gamma > 2 \) to ensure that the following all hold.

- \( c_\Gamma \geq c_0 \) from Theorem 4.6.
- \( c_\Gamma \geq c_1 \) from Theorem 2.2.
- \( c_\Gamma \geq c_2 \) from Theorem 2.6, where we fix \( \epsilon_\Gamma \) as the choice of \( \epsilon \), and
- \( c_\Gamma \geq D_\Gamma \) from Proposition 3.12; this implies that the only proper subsurfaces \( Y \) with \( d_Y(x, y) \geq c_\Gamma \) are parabolic annuli.

Using Lemma 3.18, we may choose a constant \( c' = c'(S, \Gamma, c_\Gamma) \geq 2 \) so that for any \( H \in P(\Gamma) \) and \( \alpha \in PCyl(H) \), we have \( d_\alpha(x, y) \geq c_\Gamma \) whenever \( d_H(x, y) \geq c' \). By Lemma 4.8, we have

\[
d^r_{N(\Gamma)}(x, y) \leq A_3 \cdot d^r_T(S)(x, y) + \sum_{H \in P(\Gamma)} [d_H(x, y)]_{c'} \tag{4.9}
\]

where \( A_3 = A_3(\Gamma, c') \). By Lemma 3.18, there exists some \( K > 0 \) such that

\[
d_H(x, y) \leq \sum_{\alpha \in PCyl(H)} (K \cdot d_\alpha(x, y) + K) \tag{4.10}
\]

for all \( H \in P(\Gamma) \). We shall use (4.10) to replace the sum over parabolic subgroups in (4.9) with a sum over parabolic annuli, however, we need to control the number of terms that appear due to the additive factor of \( K \). As we are considering only those \( H \in P(\Gamma) \) where \( d_H(x, y) \geq c' \), the corresponding \( \alpha \)-terms must satisfy \( d_\alpha(x, y) \geq c \). Thus, all such (annuli with core curve) \( \alpha \) are elements of \( R^S_{c_2}(\mu_x, \mu_y) \); moreover, they are maximal by Proposition 3.12. Therefore, by Theorem 4.6, there are at most \( N \cdot d_S(x, y) + N \) relevant parabolic annuli appearing in the sum, where \( N = N(S, c_\Gamma) \). Consequently, there exists some \( B = B(S, \Gamma) > 0 \) such that

\[
d^r_{N(\Gamma)}(x, y) \leq_B d^r_T(S)(x, y) + [d_S(x, y)]_{c_\Gamma} + \sum_{\alpha \in PCyl(\Gamma)} [d_\alpha(x, y)]_{c_\Gamma}. \tag{4.11}
\]
Since $N^r(\Gamma)$ is $\epsilon_\Gamma$–thick, we may apply Theorem 2.6 and Proposition 3.12 to obtain

$$d_{\mathcal{T}(S)}(x, y) \asymp_{A_2} \sum_{Y \in \mathcal{A}} [d_Y(x, y)]_{c_T} + \sum_{\alpha \in \mathcal{C}(S)} [\log(d_{\alpha}(x, y))]_{c_T}$$

(4.12)

$$= [d_S(x, y)]_{c_T} + \sum_{\alpha \in \mathrm{PCyl}(\Gamma)} [\log(d_{\alpha}(x, y))]_{c_T}$$

(4.13)

for some $A_2 = A_2(S, \epsilon_\Gamma)$. Now, observe that

$$[\log t]_c + [t]_c \leq 2[t]_c$$

(4.14)

for all $t \geq 2$. Combining (4.11)–(4.14), we deduce that

$$d_{N(\Gamma)}^{el}(x, y) \asymp_{B'} [d_S(x, y)]_{c_T} + \sum_{\alpha \in \mathrm{PCyl}(\Gamma)} [d_{\alpha}(x, y)]_{c_T}$$

(4.15)

$$= \sum_{Y \subseteq S} [d_Y(x, y)]_{c_T}$$

(4.16)

for some $B' = B'(S, \Gamma)$. Finally, combining the above with Theorem 2.2 and (4.7), we may conclude that

$$d_{N(\Gamma)}^{el}(x, y) \asymp_{A} d_{\mathcal{M}(S)}(\mu_x, \mu_y)$$

(4.17)

for some $A = A(S, \Gamma)$. The desired distance formula, where we allow for any threshold $c \geq c_\Gamma$, follows immediately using Theorem 2.2.

4.3. Proof of Theorem 4.3

The proof proceeds in a similar fashion to the previous subsection.

**Lemma 4.18.** The extended systole map $\sigma: N^{el}(\Gamma) \to \mathcal{C}(S)$ is coarsely Lipschitz.

**Proof.** Since $N^{el}(\Gamma)$ is a path space, it suffices to prove that any set $V \subset N^{el}(\Gamma)$ of diameter at most $\frac{1}{2}$ has uniformly bounded diameter under $\sigma$. If $V$ is contained in $\mathcal{N}(\Gamma)$, then this is immediate from the fact that the usual systole map $\sigma: \mathcal{T}(S) \to \mathcal{C}(S)$ is coarsely Lipschitz.

Now suppose otherwise. Then $V$ non-trivially intersects the $\frac{1}{2}$–neighbourhood of some electrification point $*H$, and is thus contained in the $1$–neighbourhood of $*H$. Note that distinct electrification points have disjoint $1$–neighbourhoods as the pairwise distance between horodiscs is at least $1$ in $N^{el}(\Gamma)$. By taking $\epsilon_\Gamma$ small if necessary, the set $\sigma(x)$ is contained in the simplex $\mathrm{PCyl}(H) \subset \mathcal{C}(S)$ for all $x \in U(H)$. By definition, the same is true for any $x$ in the (open) $\frac{1}{2}$–neighbourhood of $*H$. Any other $x \in N^{el}(\Gamma)$ in the $1$–neighbourhood of $*H$ not accounted for in the previous two cases must lie in the $\frac{1}{2}$–neighbourhood of $U(H)$ in $\mathcal{N}(\Gamma)$. The desired result follows using the coarse Lipschitz property of the usual systole map. \hfill \Box

It remains to bound $d_{N(\Gamma)}^{el}(x, y)$ from above by some linear function of $d_S(x, y)$ for all pairs of points $x, y \in N^{el}(\Gamma)$. Since $N(\Gamma)$ is $1$–dense in $N^{el}(\Gamma)$, it suffices to prove this for $x, y \in N(\Gamma)$.

Choose a constant $c' = c'(S, \Gamma, c_T) \geq 2$, as in the previous section, so that whenever $d_H(x, y) \geq c'$ for some $H \in \mathcal{P}(\Gamma)$, we have $d_\alpha(x, y) \geq c_T$ for all $\alpha \in \mathrm{PCyl}(H)$. Let $\mathcal{G}$ be the Teichmüller geodesic in $\mathcal{N}(\Gamma)$ connecting $x$ to $y$. Consider the (finite) set of $H \in \mathcal{P}(\Gamma)$ for which $d_H(x, y) \geq c'$. We shall order this set $H_1, \ldots, H_n$ according to the order in which the horodiscs $U(H_i)$ appear along $\mathcal{G}$. Arguing as in the previous section, we may deduce that

$$n \prec d_S(x, y),$$

(4.19)
where \( \mathbb{N} = \mathbb{N}(S,c_\Gamma) \). Let \( x_i \) and \( y_i \) be the endpoints of the subinterval \( G \cap \overline{U(H_i)} \), with \( x_i \) chosen to be closer to \( x \) along \( G \), and set \( y_0 = x \) and \( x_{n+1} = y \). Applying the no-backtracking property (Lemma 2.9) to \( G \) at each of the \( x_i \) and \( y_i \) for \( 1 \leq i \leq n \), we deduce that

\[
\sum_{i=0}^{n} d_S(y_i, x_{i+1}) \leq \sum_{i=0}^{n} d_S(y_i, x_{i+1}) + \sum_{i=1}^{n} d_S(x_i, y_i) \quad (4.20)
\]

\[
\leq d_S(x, y) + 2Cn \quad (4.21)
\]

\[
\lesssim_A d_S(x, y), \quad (4.22)
\]

where \( C = C(S) \) and \( A = A(C, \mathbb{N}) \).

Let \( G_i \subseteq G \) be the subinterval connecting \( y_i \) to \( x_{i+1} \), for \( 0 \leq i \leq n \).

**Lemma 4.23.** There exists a constant \( \epsilon' = \epsilon'(\Gamma, \epsilon_\Gamma, c') > 0 \) such that each segment \( G_i \) is \( \epsilon' \)-thick.

**Proof.** Suppose \( H \in \mathcal{P}(\Gamma) \) satisfies \( d_H(x, y) < c' \). Using elementary hyperbolic geometry, there exists some \( r = r(c') \) such that \( G \cap \overline{U(H)} \) lies in the \( r \)-neighbourhood of \( \partial U(H) \). Therefore, each \( G_i \) is contained in the \( r \)-neighbourhood of \( N^{tr}(\Gamma) \) in \( N(\Gamma) \); by cocompactness, all such subintervals are \( \epsilon' \)-thick for some \( \epsilon' = \epsilon'(\Gamma, \epsilon_\Gamma, c') > 0 \). \( \square \)

Applying Proposition 2.10, we deduce that \( \sigma \circ G_i \) is a parameterised quasigeodesic, and so

\[
|G_i| \leq C'd_S(y_i, x_{i+1}) + C' \quad (4.24)
\]

for some \( C' = C'(S, c') \).

Next, we construct a modified path \( G' \) in \( N^c(\Gamma) \) by replacing each subsegment \( G \cap \overline{U(H)} \) of \( G \) with the path of length 1 from \( x_i \) to \( y_i \) passing through \( *_{H_i} \). Since the horodiscs are pairwise disjoint, this procedure can be done simultaneously for all such horodiscs. By construction, the length of \( G' \) is

\[
|G'| = \sum_{i=0}^{n} |G_i| + n \quad (4.25)
\]

Finally, combining the inequalities above, we deduce that

\[
d^{c}_{N(\Gamma)}(x, y) \leq \sum_{i=0}^{n} |G_i| + n \quad (4.26)
\]

\[
\leq C' \left( \sum_{i=0}^{n} d_S(y_i, x_{i+1}) \right) + C'(n+1) + n \quad (4.27)
\]

\[
\lesssim A'd_S(x, y) \quad (4.28)
\]

for some \( A' = A'(A, C', \mathbb{N}) \).

**Acknowledgements.** The author thanks Chris Leininger for interesting discussions and for asking whether Theorem 1.4 is true. The author would also like to thank Matthew Durham, Mark Hagen, and Alessandro Sisto for engaging conversations regarding their work, and to Richard Webb for helpful comments. We also thank the anonymous referee for providing useful feedback.
References

1. L. Bers, ‘A remark on Mumford’s compactness theorem’, Israel J. Math 12 (1972) 400–407.
2. N. Broaddus, B. Farb and A. Putman, ‘Irreducible Sp-representations and subgroup distortion in the mapping class group’, Comment. Math. Helv. 86 (2011) 537–556.
3. Y.-E. Choi, K. Rafi and C. Series, ‘Lines of minima and Teichmüller geodesics’, Geom. Funct. Anal. 18 (2008) 698–754.
4. M. Durham, M. Hagen and A. Sisto, ‘Boundaries and automorphisms of hierarchically hyperbolic spaces’, Geom. Topol. 21 (2017) 3659–3758.
5. B. Farb, ‘Some problems on mapping class groups and moduli space’, Problems on mapping class groups and related topics, Proceedings of Symposia in Pure Mathematics 74 (ed. B. Farb; American Mathematical Society, Providence, RI, 2006) 11–55. MR2264130
6. B. Farb, A. Lubotzky and Y. Minsky, ‘Rank-1 phenomena for mapping class groups’, Duke Math. J. 106 (2001) 581–597.
7. B. Farb, ‘Some problems on mapping class groups and moduli space’, Problems on mapping class groups and related topics, Proceedings of Symposia in Pure Mathematics 74 (ed. B. Farb; American Mathematical Society, Providence, RI, 2006) 11–55. MR2264130
8. B. Farb, A. Lubotzky and Y. Minsky, ‘Rank-1 phenomena for mapping class groups’, Duke Math. J. 106 (2001) 581–597.
9. B. Farb and D. Margalit, A primer on mapping class groups, Princeton Mathematical Series, vol. 49, (Princeton University Press, Princeton, NJ, 2012). MR2850125 (2012h:57032)
10. B. Farb and L. Mosher, ‘Convex cocompact subgroups of mapping class groups’, Geom. Topol. 6 (2002) 91–152.
11. E. Gutkin, P. Hubert and T. A. Schmidt, ‘Affine diffeomorphisms of translation surfaces: periodic points, Fuchsian groups, and arithmeticity’, Ann. Sci. Éc. Norm. Supér. 36 (2003) 847–866.
12. U. Hamenstädt, ‘Geometry of the mapping class groups II: boundary amenability’, Invent. Math. 175 (2009) 545–609.
13. U. Hamenstädt and S. Hensel, ‘The geometry of the handlebody groups I: distortion’, J. Topol. Anal. 04 (2012) 71–97.
14. H. A. Masur and Y. N. Minsky, ‘Geometry of the complex of curves. I. Hyperbolicity’, Invent. Math. 138 (1999) 103–149. 1714338 (2000i:57027)
15. H. A. Masur and Y. N. Minsky, ‘Geometry of the complex of curves. II. Hierarchical structure’, Geom. Funct. Anal. 10 (2000) 902–974. MR1791145 (2001k:57020)
16. H. Masur and S. Tabachnikov, ‘Rational billiards and flat structures’, Handbook of dynamical systems, vol. 1A (eds B. Hasselblatt and A. Katok; North-Holland, Amsterdam, 2002) 1015–1089. MR1928530 (2003j:37002)
17. Y. N. Minsky, ‘Harmonic maps, length, and energy in Teichmüller space’, J. Differential Geom. 35 (1992) 151–217.
18. Y. N. Minsky and S. J. Taylor, ‘Fibered faces, veering triangulations, and the arc complex’, Geom. Funct. Anal. 27 (2017) 1450–1496.
19. D. Mumford, ‘A remark on Mahler’s compactness theorem’, Proc. Amer. Math. Soc. 28 (1971) 289–294.
20. K. Rafi, ‘A characterization of short curves of a Teichmüller geodesic’, Geom. Topol. 9 (2005) 179–202.
21. K. Rafi, ‘A combinatorial model for the Teichmüller metric’, Geom. Funct. Anal. 17 (2007) 936–959. MR2346280
22. K. Rafi, ‘Hyperbolicity in Teichmüller space’, Geom. Topol. 18 (2014) 3025–3053.
23. S. Schleimer, ‘Notes on the complex of curves’, http://homepages.warwick.ac.uk/~masgar/Maths/notes.pdf.
24. J. Smillie and B. Weiss, ‘Characterizations of lattice surfaces’, Invent. Math. 180 (2010) 535–557. MR2600249 (2012c:37072)
25. K. Strebel, Quadratic differentials, Ergebnisse der Mathematik und ihrer Grenzgebiete; Folge 3, Bd. 5, (Springer, Berlin, 1984).
26. R. Tang and R. C. H. Web, ‘Shadows of Teichmüller discs in the curve graph’, Int. Math. Res. Not. 2018 (2018) 3301–3341.
27. W. P. Thurston, ‘On the geometry and dynamics of diffeomorphisms of surfaces’, Bull. Amer. Math. Soc. (N.S.) 19 (1988) 417–431.
28. W. A. Veech, ‘Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards’, Invent. Math. 97 (1989) 553–583. MR1005006 (91h:58083a)
Robert Tang
Department of Pure Mathematics
Xi’an Jiaotong–Liverpool University
111 Ren’ai Road, Suzhou Industrial Park
Suzhou, Jiangsu Province 215123
China

robert.tang@xjtlu.edu.cn