NOTES ON FOLD MAPS OBTAINED BY SURGERY OPERATIONS AND ALGEBRAIC INFORMATION OF THEIR REEB SPACES

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Abstract. The theory of Morse functions and their higher dimensional versions or fold maps on manifolds and its application to geometric theory of manifolds is one of important branches of geometry and mathematics. Studies related to this were started in 1950s by differential topologists such as Thom and Whitney and have been studied actively.

One of fundamental, important, interesting and difficult problems on such studies is construction of explicit fold maps and studying algebraic and differential topological information of source manifolds. One of explicit studies on construction of fold maps is construction by introducing and using surgery operations by Kobayashi since 1990s and motivated by this, the author introduced generalized versions and have succeeded in constructing explicit fold maps and knowing homological information of source manifolds.

In this paper, we study fold maps obtained by surgery operations to fundamental fold maps, especially algebraic topological information of Reeb spaces, defined as the spaces of connected components of inverse images, often inheriting fundamental and important algebraic invariants such as (co)homology groups and fundamental and important in studying manifolds. The author has already studied about homology groups of Reeb spaces and source manifolds in the introduced studies before. In this paper, we study about cohomology rings as more precise information.

1. INTRODUCTION AND FUNDAMENTAL NOTATION AND TERMINOLOGIES.

Fold maps are smooth maps regarded as higher dimensional versions of Morse functions and fundamental and important tools in studying manifolds by investigating singular points and their values of generic smooth maps: the study is regarded as a general version of differential topological theory of Morse theory.

Definition 1. Let $m$ and $n$ be integers satisfying the relation $m \geq n \geq 1$. A smooth map between $m$-dimensional smooth manifold without boundary and

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2010 Mathematics Subject Classification. Primary 57R45. Secondary 57N15.

Key words and phrases. Singularities of differentiable maps; generic maps. Differential topology. Reeb spaces.

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$n$-dimensional smooth manifold without boundary is said to be a fold map if at each singular point $p$, the form is

$$(x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_{n-1}, \sum_{k=n}^{m-i} x_k^2 - \sum_{k=m-i+1}^{m} x_k^2)$$

for an integer $0 \leq i(p) \leq \frac{m-n+1}{2}$. A singular point such that at the point a smooth map is of such a form is a fold point of the map.

**Proposition 1.** For a fold map on a closed manifold, the following hold.

1. For any singular point $p$, the $i(p)$ as in Definition 1 is unique ($i(p)$ is called the index of $p$).
2. The set consisting of all singular points (the singular set) of a fixed index of the map is a closed smooth submanifold of dimension $n-1$ of the source manifold.
3. The restriction map to the singular set is a smooth immersion of codimension 1.

For fundamental theory of fold maps and more general generic maps, see [3] for example.

In studies of fold maps and application to differential topology of manifolds, constructing explicit fold maps is fundamental, important and difficult, where there are several fundamental examples as presented in Example 1.

**Example 1.**

1. (Discussed in [15] etc.) FIGURE 1 represents a Morse function with just 2 singular points, characterizing a homotopy sphere topologically, and the canonical projection of a unit sphere.
2. (Discussed in [15] etc.) FIGURE 2 represents images of fold maps into the plane such that the restrictions to the singular sets are embedding and that the images of the restriction maps are the boundaries of the images.
3. (Discussed in [18] and later in [4], [5] and [7] etc.) FIGURE 3 represents a fold map into the plane or $\mathbb{R}^n$ ($n \geq 3$) such that the restriction to the singular set is embedding, that the singular set is a disjoint union of two spheres and that the inverse image of each regular value is $S^{m-n}$ or a disjoint union of two copies of an $(m-n)$-dimensional homotopy sphere $\Sigma$ represented as a manifold obtained by gluing two copies of $D^{m-n+1}$ (we call such a homotopy sphere an almost-sphere) on the boundaries by a diffeomorphism as shown. Note also that such maps can characterize manifolds represented as total spaces of smooth bundles over $S^n$ with fibers diffeomorphic to $\Sigma$ with a condition on the structure of the map on the inverse image of the target space with the interior of an $n$-dimensional standard closed disc in the innermost connected component of the regular value set removed. See the three papers by the author cited before.
The first two examples are special generic: a special generic map is a fold map such that the index of each singular point is 0. Studies of special generic maps and their source manifolds were started by Furuya and Porto, Saeki and Sakuma have obtained various results and Nishioka and Wrazidlo recently obtained interesting results. Interesting points on special generic maps will be presented in Remark 1. See also [2], [16], [17] and see also [13] and [20]).

The author has succeeded in constructing explicit fold maps, in [4], [5], [7], [8] and [9] for example. In [10] and [11], Kobayashi has also succeeded in such works independently, for example.
Figure 2. Images of fold maps on $S^3 \times S^{m-1}$ ($m \geq 2$) and a manifold represented as a connected sum of two copies of the manifold into the plane such that the restriction maps to the singular sets are embedding and that the images of the restriction maps are the boundaries of the images.

Figure 3. An explicit fold map into $\mathbb{R}^n$ ($n \geq 2$): the manifolds represent inverse images of regular values.
In this paper, we further study about construction in [9]. In the paper, the author has constructed fold maps by surgery operations called bubbling operations. Such operations were introduced by the author based on [10], [11] etc. and regarded as general versions of surgery operations introduced and used in these studies. The author has also studied about Reeb spaces, defined as all connected components of inverse images of maps, inheriting fundamental important such as homology groups of source manifolds and fundamental and important tools in studying the manifolds. More precisely, the author has also investigated changes of homology groups of Reeb spaces by the operations. In this paper, we study about more precise algebraic invariants of Reeb spaces and source manifolds: we study about cohomology rings.

The organization of the paper is as the following and before the explanation, we define several terminologies used throughout this paper. The singular set of a smooth map is the set of all singular points, the singular value set of the map is the image of the singular set and each point in the set is a singular value, the regular value set is the complement of the singular value set in the target manifolds and each point is a regular value.

In section 2, we review Reeb spaces and bubbling operations; in [9], we introduced such operations and we revise definitions a little in the present paper. A bubbling operation is based on a bubbling surgery introduced in [11] ([10] is also closely related to the study). It is an operation of removing a closed disc in the regular value set and its inverse image and attaching a new smooth map whose singular set is a sphere of dimension smaller than the target space by 1 and consists of fold points and whose restriction to the singular set is an embedding. Note that the operation does not change the other part of the map. In a word, a bubbling operation is defined by replacing a closed disc by a regular neighborhood of a bouquet of closed, connected and orientable submanifolds and regarded as a higher dimensional version. After introducing the definition of the operation, we introduce explicit cases and for example, we see that several simple examples including ones presented in FIGURES 1–3 are obtained by finite iterations of such operations. We also review results on the changes or differences of homology groups. In addition, we also introduce results stating that for suitable fold maps such that inverse images of regular values are disjoint unions of spheres, we can know algebraic invariants of source manifolds from the Reeb spaces, first shown in [18] and later shown in [5] and [6].

In section 3, based on the results of the previous section, as main works of the present paper, we investigate changes of cohomology rings of Reeb spaces; more precisely, as explicitly depicted in FIGURE 5 later, we change the topology of the Reeb space one after another and investigate the resulting topology. We can know structures of cohomology rings of source manifolds by virtue of the result just before in the cases where inverse images of regular
values are disjoint unions of spheres and where additional suitable differential
topological conditions on the maps are assumed.

Throughout this paper, we assume that $M$ is a smooth, closed manifold of
dimension $m$, that $N$ is a smooth manifold of dimension $n$ without boundary,
that $f : M \to N$ is a smooth map and that $m > n \geq 1$. We denote the
singular set of $f$ by $S(f)$.

Moreover, in the proceeding sections, manifolds are of class $C^\infty$ and maps
between manifolds are smooth and class $C^\infty$ and fold maps unless otherwise
stated. In addition, the structure groups of bundles such that the fibers are
(smooth) manifolds are assumed to be (subgroups of) diffeomorphism groups.

The author is a member of the project and supported by the project Grant-
in-Aid for Scientific Research (S) (17H06128 Principal Investigator: Osamu
Saeki)” Innovative research of geometric topology and singularities of different-
tiable mappings” (https://kaken.nii.ac.jp/en/grant/KAKENHI-PROJECT-
17H06128/).

2. Reeb spaces, bubbling operations and fold maps such that
inverse images of regular values are disjoint unions of
spheres.

2.1. Definitions and fundamental properties of Reeb spaces and bub-
bling operations.

Definition 2. Let $X$ and $Y$ be topological spaces. For $p_1, p_2 \in X$ and for a
continuous map $c : X \to Y$, we define as $p_1 \sim_c p_2$ if and only if $p_1$ and $p_2$ are
in a same connected component of $c^{-1}(p)$ for some $p \in Y$. Thus $\sim_c$ is an
equivalence relation. We denote the quotient space $X/\sim_c$ by $W_c$ and call $W_c$
the Reeb space of $c$.

We denote the induced quotient map from $X$ into $W_c$ by $q_c$. We can define
$\bar{c} : W_c \to Y$ uniquely so that the relation $c = \bar{c} \circ q_c$ holds.

For a (stable) fold map $c$, the Reeb space $W_c$ is regarded as a polyhedron.
For example, for a Morse function and more generally, a smooth function on
a closed manifold, the Reeb space is a graph and for a special generic map,
the Reeb space is regarded as a smooth manifold immersed into the target
manifold. We present this; see also section 2 of [15]. A linear bundle whose
fiber is a standard closed (unit) disc or its boundary is a smooth bundle whose
structure group acts on the fiber linearly.

Proposition 2. There exists a special generic map $f : M \to \mathbb{R}^n$ if and only if
$M$ is obtained by gluing the following two manifolds by a bundle isomorphism
between the $S^{m-n}$-bundles over the boundary $\partial P$ of a compact manifold $P$,
which appears in the explanation.

1. A smooth $S^{m-n}$-bundle over a compact smooth manifold $P$ satisfying
$\partial P \neq \emptyset$ we can immerse into $\mathbb{R}^n$. 
(2) A linear $D^{m-n+1}$-bundle over $\partial P$.

Note that $P$ is regarded as the Reeb space of a special generic map on the manifold.

For Reeb spaces, see also [14] for example.

Next, we review bubbling operations, introduced in [9].

Definition 3 ([9]). For a fold map $f : M \rightarrow N$, let $P$ be a connected component of the regular value set $\mathbb{R}^n - f(S(f))$. Let $S$ be a connected and orientable closed submanifold of $P$ and $N(S)$, $N(S)_i$ and $N(S)_o$ be small closed regular neighborhoods of $S$ in $P$ such that $N(S)_i \subset N(S) \subset N(S)_o$ holds. Furthermore, we can naturally regard $N(S)_o$ as a linear bundle whose fiber is an $(m-n+1)$-dimensional disc of radius 1 and $N(S)_i$ and $N(S)$ are subbundles of the bundle $N(S)$ whose fibers are $(m-n+1)$-dimensional discs of radii $\frac{1}{2}$ and $\frac{3}{2}$, respectively. Let $f^{-1}(N(S)_o)$ have a connected component $Q$ such that $f|_Q$ makes $Q$ a bundle over $N(S)_o$.

Let us assume that there exist an $m$-dimensional closed manifold $M'$ and a fold map $f' : M' \rightarrow \mathbb{R}^n$ satisfying the following.

1. $M - \text{Int} Q$ is a compact submanifold (with non-empty boundary) of $M'$ of dimension $m$.
2. $f|_{M - \text{Int} Q} = f'|_{M - \text{Int} Q}$ holds.
3. $f'(S(f'))$ is the disjoint union of $f(S(f))$ and $\partial N(S)$.
4. $(M' - (M - Q)) \cap f^{-1}(N(S)_i)$ is empty or $f'|_{(M' - (M - Q)) \cap f^{-1}(N(S)_i)}$ makes $(M' - (M - Q)) \cap f^{-1}(N(S)_i)$ a bundle over $N(S)$.

These assumptions enable us to consider the procedure of constructing $f'$ from $f$ and we call it a normal bubbling operation to $f$ and, $f^{-1}(S) \cap q_f(Q)$, which is homeomorphic to $S$, the generating manifold of the normal bubbling operation.

Furthermore, let us suppose additional conditions.

1. $f'|_{(M' - (M - Q)) \cap f^{-1}(N(S)_i)}$ makes $(M' - (M - Q)) \cap f^{-1}(N(S)_i)$ the disjoint union of two bundles over $N(S)$, then the procedure is called a normal $M$-bubbling operation to $f$.
2. $f'|_{(M' - (M - Q)) \cap f^{-1}(N(S)_i)}$ makes $(M' - (M - Q)) \cap f^{-1}(N(S)_i)$ the disjoint union of two bundles over $N(S)$ and the fiber of one of the bundles is an almost-sphere, then the procedure is called a normal $S$-bubbling operation to $f$.

In the definition above, let $S$ be the bouquet of finite connected and orientable closed submanifolds whose dimensions are smaller than $n$ of $P$ and $N(S)$, $N(S)_i$ and $N(S)_o$ be small regular neighborhoods of $S$ in $P$ such that $N(S)_i \subset N(S) \subset N(S)_o$ holds and that these three are isotopic as regular neighborhoods. By a similar way, we define a similar operation and call the operation a bubbling operation to $f$. We call $Q_0 := f^{-1}(S) \cap q_f(Q)$, which is homeomorphic to $S$, the generating polyhedron of the bubbling operation.
Last, in the case where a fold map is not given and only a smooth manifold \( N \) is given, we can define a bubbling operation naturally. We thus obtain a special generic map \( f : M \to N \) such that \( f|_{S(f)} \) is an embedding. We call this a \textit{default} bubbling operation.

In the following example, as in [9], we present explicit and important facts on bubbling operations.

Example 2. (1) The case where the generating manifold is point is a \textit{bubbling surgery}, introduced in [11], based on ideas of [10]. [12] is closely related to such operations: as surgery operations, \textit{R-operations} are defined as operations deforming stable maps from closed manifolds whose dimensions are larger than 2 into the plane and preserving topologies and smooth structures of the source manifolds. Note that for example, maps presented in \textsc{FIGURE}s 1–3 are obtained by finite iterations of bubbling surgeries starting from default bubbling operations whose generating manifolds are points.

(2) By suitable default normal bubbling operations whose generating manifolds are points, we can obtain maps in \textsc{FIGURE} 1. Moreover, for example, by a suitable bubbling default bubbling operation, we can obtain every manifold admitting a special generic map into the plane as the source manifold (see [15]).

Let \( m > n > 1 \). Note that on a manifold represented as a connected sum of the product \( S^{k_1} \times S^{k_2} \) satisfying the relations \( k_1 + k_2 = m \), \( 0 < k_1 \leq n - 1 \) and \( k_2 > 0 \), by default bubbling operations, we can construct a special generic map into \( \mathbb{R}^n \) by taking generating polyhedron as a bouquet of standard spheres in a suitable family \( \{ S^{k_i} \} \).

\textsc{FIGURE} 4 represents several simple default bubbling operations. It also accounts for general bubbling operations in case the inverse images are not empty.

(3) Let us define a function \( f_{m,n,S} \) satisfying the following.

(a) \( f_{m,n,S} \) is a Morse function on a compact manifold one of connected components of whose boundary is the fiber \( F \) of the bundle \( S' \) over \( S \) and with just one singular point.

(b) The inverse image of the maximal value is the component, diffeomorphic to \( F \), and the inverse image of the minimal value is the disjoint union of connected components of the boundary except the previous one if the disjoint union is not empty and is the singular point of \( f_{m,n,S} \) if the disjoint union is empty (or the index of the singular point is 0).

Then, by a bubbling operation to \( f \) such that the generating manifold is \( S' \), we can obtain a new fold map \( f' : M' \to \mathbb{R}^n \) satisfying the
Figure 4. Default bubbling operations and the images of resulting special generic maps.

following conditions where we abuse notation in Definition 3. We call this operation a \textit{trivial} normal bubbling operation.

(a) $f|_{M - \operatorname{Int} Q} = f'|_{M - \operatorname{Int} Q}$.
(b) $f'|_{f' - 1(N(S)_i) \cap (M - (M - Q))}$ gives a trivial bundle over $N(S)_i$.
(c) There exists a connected component of $f^{-1}(N(S)_o \sm \operatorname{Int} N(S)_i)$ such that the restriction map of $f'$ to the component is regarded as the product of the Morse function $f_{m,n,S}$ and $\operatorname{id}_{\partial N(S)}$.

Moreover, let the normal bundle or tubular neighborhood of $S$ be trivial. $N_o(S)$ is represented by $S \times D^{n-\dim S}$ and $S$ is regarded as $S \times \{0\} \subset S \times D^{n-\dim S}$. For example, let $S$ be the standard sphere embedded as an unknot in the interior of an open ball in the interior of $P$. Furthermore, let the restriction of $f'$ to $f'^{-1}(N(S)_o)$ is regarded as the product of a surjective map $f'|_{f' - 1(D^{n-\dim S})} : f'^{-1}(D^{n-\dim S}) \rightarrow$
$D^{n-\dim S}$ where $D^{n-\dim S}$ is a fiber of the trivial bundle $N(S)\circ \text{id}_S$. Then we call the previous operation a strongly trivial normal bubbling operation. Bubbling surgeries, presented in Example 2 (1), are strongly trivial normal bubbling operations, for example. Last, we can extend trivial normal bubbling operations to the cases of bubbling operations.

(4) In the previous example, if $F_1$ and $F_2$ are closed and connected manifolds so that the manifold $F$ is represented as a connected sum of $F_1$ and $F_2$, then, we can consider $f_{m,n,S}$ so that the boundary of its source manifold consists of three connected components and the boundary with the connected component $F$ removed is the disjoint union of $F_1$ and $F_2$. In this case, the bubbling operation is an $M$-bubbling operation. We can take $F_1$ as any almost-sphere of dimension $m-n$ and $F_2$ suitably and in this case, the operation is an $S$-bubbling operation. FIGURE 5 later accounts for an explicit $M$-bubbling operation and the change of the topology of the Reeb space.

(5) If we perform an $S$-bubbling operation to a fold map such that the inverse images of regular values are always disjoint unions of almost-spheres (standard spheres), then we obtain a fold map satisfying the same condition. A simple fold map is a fold map such that the map $q_f|_{S(f)} : S(f) \subset M \to W_f$ is injective. Such maps were systematically studied in [19] for example. Any special generic map and fold map such that the map $f|_{S(f)}$ is an embedding are simple fold maps. If we perform an $M$-bubbling operation to a simple fold map (such that the restriction map to the singular set is embedding), then the resulting fold map is also such a map.

The following is a key lemma in section 3 and it follows immediately from the definition of an $M$-bubbling operation.

**Lemma 1.** Let $f$ be a fold map. If an $M$-bubbling operation is performed to $f$ and a new map $f'$ is obtained, then $W_f$ is a proper subset of $W_{f'}$ such that for the map $\bar{f}' : W_{f'} \to N$, the restriction to $W_f$ is $\bar{f} : W_f \to N$.

FIGURE 5 represents an example of Lemma 1 in the case where $n = 3$ holds.

The following proposition is a fundamental and key tool in the present paper.

**Proposition 3 ([9]).** Let $f : M \to N$ be a fold map. Let $f' : M' \to N$ be a fold map obtained by an $M$-bubbling operation to $f$. Let $S$ be the generating polyhedron of the $M$-bubbling operation. Let $k$ be a positive integer and $S$ be represented as the bouquet of submanifolds $S_j$ where $j$ is an integer satisfying $1 \leq j \leq k$. Then, for any integer $0 \leq i < n$, we have
Lemma 1 in a case of $n = 3$.

$$H_i(W_f'; R) \cong H_i(W_f; R) \oplus \bigoplus_{j=1}^{k} (H_{i-(n-\dim S_j)}(S_j; R))$$

and we also have $H_n(W_f'; R) \cong H_n(W_f; R) \oplus R$.

A more precise proof with explanations on Mayer-Vietoris sequences, homology groups of product bundles etc. is presented in [9] and we present a shorter proof here. We also note that in discussions later, for example ones in section 3 including the proofs of results of the present paper, such precise explanations are omitted.

**Proof.** For each $S_j$, we can take a small closed tubular neighborhood, regarded as the total space of a linear $D^{n-\dim S_j}$-bundle over $S_j$. By the definition of an M-bubbling operation, we can see that a small regular neighborhood of $S_j$ is represented as a boundary connected sum of the tubular neighborhoods. We may consider that $W_f'$ is obtained by attaching a manifold represented as a connected sum of total spaces of linear $S^{n-\dim S_j}$-bundles over $S_j$ ($1 \leq j \leq k$) by considering $D^{n-\dim S_j}$ in the beginning of this proof as a hemisphere of $S^{n-\dim S_j}$ and identifying the subspace obtained by restricting the space to fibers $D^{n-\dim S_j}$ with the original regular neighborhood. Lemma 1 and FIGURE 5 may help us to understand the topology of the resulting space $W_f' \supset W_f$. For the manifold represented as a connected sum of total spaces of linear $S^{n-\dim S_j}$-bundles over $S_j$ ($1 \leq j \leq k$), the bundles are regarded as products in considering the homology groups since they admit sections,
corresponding to the submanifolds $S_j$ and regarded as sections corresponding to the origin in the fiber $D^{n-\dim S_j} \subset S^{n-\dim S_j}$. From this observation on the topologies of $W_f$ and $W_{f'}$, we have the result. □

The following has been shown in [9]. We can show this by applying Proposition 3 one after another. We can show the result by taking a suitable family of generating polyhedra, which are bouquets of finite numbers of standard spheres.

**Proposition 4 ([9]).** Let $R$ be a PID. For any integer $0 \leq j \leq n$, we define $G_j$ as a free and finitely generated module over $R$ so that $G_0$ is trivial and that $G_n$ is not zero. Then, by a finite iteration of $M$-bubbling operations to a map $f$, we obtain a fold map $f'$ such that $H_j(W_f; R)$ is isomorphic to $H_j(W_{f'}; R) \oplus G_j$.

We restrict $M$-bubbling operations in Proposition 3 and Proposition 4 to normal ones and thus we have the following. We omit the proofs. In fact, for example, Proposition 5 is a specific case of Proposition 3.

**Proposition 5 ([9]).** For a fold map $f : M \to N$, let $f' : M' \to N$ be a fold map obtained by a normal $M$-bubbling operation to $f$ and let $S$ be the generating manifold of the normal $M$-bubbling operation and of dimension $k < n$. Then for any PID $R$ and integer $i$, we have $H_i(W_f; R) \cong H_i(W_f'; R) \oplus (H_{i-(n-k)}(S; R))$.

**Proposition 6 ([9]).** For any integer $0 \leq j \leq n$, we define $G_j$ as a free finitely generated module over a PID $R$ so that $G_0$ is a trivial module and $G_n$ is not a trivial module. Let the sum $\sum_{j=1}^{n-1} \text{rank } G_j$ of the ranks of $G_j$ is not larger than the rank of $G_n$. Then, by a finite iteration of normal $M$-bubbling ($S$-bubbling) operations starting from $f$, we obtain a fold map $f'$ and $H_j(W_f; R)$ is isomorphic to $H_j(W_{f'}; R) \oplus G_j$.

### 2.2. Fold maps such that inverse images of regular values are disjoint unions of spheres

In the end of this section, we review a proposition for fold maps such that inverse images of regular values are disjoint unions of spheres. Such maps are important and often appear: special generic maps satisfy the property and a map in Example 1 (3) does. The following is a proposition for simple fold maps, appearing in Example 2 (5), satisfying the property and this is a key proposition to know algebraic invariants of source manifolds from Reeb spaces under appropriate constraints. Several statements such as one on an isomorphism of subrings of cohomology rings were not shown in the cited articles, however, we can show this in a manner similar to the used manners: the key is fundamental theory of handle decompositions.

**Proposition 7 ([18] ([5])).** Let $m$ and $n$ be integers satisfying $m > n \geq 1$. Let $M$ be a closed and connected orientable manifold of dimension $m$ and $N$ be an $n$-dimensional manifold without boundary.
Then, for a simple fold map \( f : M \to N \) such that inverse images of regular values are always disjoint unions of almost-spheres and that indices of singular points are always 0 or 1, two induced homomorphisms \( q_\ast : H_\ast(M) \to H_\ast(W_f; R) \), \( q_f^\ast : H^\ast(W_f; R) \to H^\ast(M; R) \) are isomorphisms for \( 0 \leq j \leq m - n - 1 \) and for any ring \( R \). Furthermore, let \( J \) be a set of integers not smaller than 0 and not larger than \( m - n - 1 \) and if \( \bigoplus_{j \in J} H^\ast(W_f; R) \) is a subring of the cohomology ring \( H^\ast(W_f; R) \), then \( q_f^\ast \) induces an isomorphism between \( \bigoplus_{j \in J} H^\ast(W_f; R) \) and \( \bigoplus_{j \in J} H^\ast(M; R) \).

Furthermore, if \( R \) is PID and the relation \( m = 2n \) holds, then the rank of \( M \) is twice the rank of \( W_f \). In addition, if \( H_{n-1}(W_f; R) \), which is isomorphic to \( H_{n-1}(M; R) \), is free, then they are also free.

3. On cohomology rings of Reeb spaces and source manifolds.

We investigate not only homology groups, but also cohomology rings of the resulting Reeb spaces and present these results as main results.

3.1. A connected sum of two smooth maps whose codimensions are negative. First we introduce a connected sum of two smooth maps whose codimensions are negative. This is also a fundamental operation in constructing maps. See [15] and see also [12], in which such operations were used to construct new maps from given pairs of special generic maps into fixed Euclidean spaces and generic maps on closed manifolds of dimensions larger than 2 into the plane.

Let \( \pi_{m+1,n} : \mathbb{R}^{m+1} \to \mathbb{R}^n \) be the canonical projection. Set \( \mathbb{R}^n^+ := \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \geq 0 \} \). The restriction of the map \( \pi_{m+1,n} \) to the unit sphere \( S^m \) is the canonical projection as presented in Example 1 (1) and we denote its restriction to the inverse image of \( \mathbb{R}^n^+ \) by \( \pi_{S^m,n} \); its source manifold is diffeomorphic to \( D^m \) (see Figure 6).

Let \( m > n \geq 1 \) be integers, \( M_i (i = 1, 2) \) be a closed and connected manifold of dimension \( m \) and \( f_i : M_1 \to \mathbb{R}^n \) and \( f_2 : M_2 \to \mathbb{R}^n \) be smooth maps. Let \( P_i (i = 1, 2) \) be the closure of a region obtained by a hyperplane in \( \mathbb{R}^n \) such that for the map \( f_i|_{f_i^{-1}(P_i)} : f_i^{-1}(P_i) \to P_i \), there exist diffeomorphisms \( \Phi \) and \( \phi \) satisfying

\[
\phi \circ f_i|_{f_i^{-1}(P_i)} = \pi_{S^m,n} \circ \Phi
\]

(two maps are \( C^\infty \) equivalent; see [3] for example).

We can glue the maps \( f_i|_{f_i^{-1}(\mathbb{R}^n - \text{Int}P_i)} : f_i^{-1}(\mathbb{R}^n - \text{Int}P_i) \to \mathbb{R}^n - \text{Int}P_i \) \((i = 1, 2)\) to obtain a new map regarded as a smooth map into \( \mathbb{R}^n \) so that the source manifold is represented as a connected sum of the original source manifolds. The resulting map is said to be a connected sum of \( f_1 \) and \( f_2 \). See also Figure 7.
Example 3. Most of maps presented in Example 2 (2) (and FIGURE 2) are obtained by finite iterations of connected sums of maps obtained by strongly trivial default normal bubbling operations.

3.2. **Results.** Related to the explanation just before, we can show a result on cohomology rings of Reeb spaces.
We have the following,

**Proposition 8.** Let $k, l$ and $n$ be positive integers satisfying $n \geq 2$. Let $W$ be a compact $n$-dimensional manifold represented as a boundary connected sum of $k$ manifolds each of which is represented as the product of a standard sphere and a standard closed disc. Let $j$ be an integer satisfying $1 \leq j \leq k$ and $k_j$ be an integer satisfying $1 \leq k_j \leq n - 2$. We represent each of the $k$ manifolds by $S^{k_j} \times D^{n-k_j}$. Let $P$ be a compact polyhedron represented as a bouquet of $l$ standard spheres the dimension of each of which is not smaller than 1 and not larger than $n-1$. Let $j$ be an integer satisfying $1 \leq j \leq l$ and $l_j$ be an integer satisfying $1 \leq l_j \leq n - 1$. We denote each of $l$ spheres by $S_j := S^{l_j}$.

We can realize $P$ as a polyhedron in $W$ (we denote this by $P$ and each corresponding sphere by $S_j$). So that the following hold.

1. $P$ is in the interior of a collar neighborhood of $\partial W$ in $W$ and each of $l$ spheres $S_j$ is regarded as a closed submanifold of $W$.
2. For each $1 \leq j \leq l$, $W_j$ be the set of all integers $1 \leq j' \leq k$ satisfying $l_j = k_{j'}$. We can set a $l_j$-cycle $\nu_{j'}$ with coefficient ring $\mathbb{Z}$ represented...
Figure 8. A case where for the dimension, \( n = 3 \) holds:
the three arrows in the right-hand side point the circle in the polyhedron \( P \) and the other arrows point the 2-dimensional spheres, respectively.

by the canonical sphere \( S^{k_j'} \times \{0\} \subset S^{k_j'} \times D^{n-k_j'} \subset W \) for each \( j' \in W_j \) with an integer \( n_{j,j'} = 0, -1, 1 \) and the cycle represented by \( S_j \) is represented as the sum of \( n_{j,j'} \nu_{j'} \) for \( j' \in W_j \). Moreover, in the case where the relation \( 2l_j \leq n \) holds, we can take an arbitrary integer as \( n_{j,j'} \) for each \( j' \in W_j \).

We omit the strict proof. Instead, in FIGURE 8, we explicitly present a case where \( n = 3 \) holds. In this case, \( k = 2 \) holds (the manifold \( W \) is represented as a boundary connected sum of \( S^2 \times D^1 \) and \( S^1 \times D^2 \)) and \( l = 3 \) holds: the polyhedron \( P \) is a bouquet of a circle and two copies of the 2-dimensional standard sphere: note that the cycle represented by the circle is \( k \) times the cycle represented by \( S^1 \times \{0\} \subset W \) where \( k \) is an arbitrary integer and that the cycles represented by the 2-dimensional spheres are zero and the cycle or the minus the cycle represented by \( S^2 \times \{0\} \subset W \), respectively. It can be generalized immediately.

Note also that the last part is based on Whitney’s theory on embeddings and arguments and results on realizations of generic maps on surfaces into surfaces as embeddings into 4-dimensional spaces in [1] and [21] (in the case \( n = 4 \) and \( l_j = 2 \)).

For a finite set \( X \), we denote the cardinality of \( X \) by \( \sharp X \). Based on Proposition 8, we present a proposition.

**Proposition 9.** Let \( m > n \geq 3 \) be integers. Let \( f \) be a fold map on an \( m \)-dimensional closed and connected \( M \) into \( \mathbb{R}^n \) obtained by a finite iteration of bubbling operations starting from a fold map such that the restriction map to the set of all singular points of index 0 is an embedding, that the image and the Reeb space are a compact \( n \)-dimensional manifold represented as a boundary connected sum of \( k \) manifolds each of which is represented as the product of a standard sphere and a standard closed disc as the manifold of Proposition
8. and that the image of the restriction map to the set of singular points of indices 0 before is the boundary. Let \( j \) be an integer satisfying \( 1 \leq j \leq k \) and \( k_j \) be an integer satisfying \( 1 \leq k_j \leq n - 1 \) and we represent each of the \( k \) manifolds by \( S^{k_j} \times D^{n-k_j} \) as Proposition 8.

If a fold map \( f' \) is obtained by an \( M \)-bubbling operation whose generating polyhedron is \( P \) in the explanation of the result of Proposition 8 to \( f \), then we have the following where we abuse notation in Proposition 8 together with identifications of Reeb spaces as subpolyhedra of Reeb spaces obtained as the results of \( M \)-bubbling operations to the original maps as Lemma 1 and where \( R \) is a PID having the identity element \( 0 \neq 1 \in R \).

1. The isomorphisms \( H_k(W_f; R) \cong H_k(W_f'; R) \oplus R^{1 \leq j \leq l_{i_j} = n-k} \) and \( H^k(W_f; R) \cong H^k(W_f'; R) \oplus R^{1 \leq j \leq l_{i_j} = n-k} \) hold for \( 2 \leq k \leq n - 1 \). Moreover, \( H_1(W_f; R) \cong H_1(W_f'; R) \) and \( H^1(W_f; R) \) hold and \( H_n(W_f; R) \cong H_n(W_f'; R), H^n(W_f; R) \cong H^n(W_f'; R) \) hold.

2. Let \( \nu_j^* \in H^{k_j}(W_f; R) \) be a cocycle such that \( \nu_j^*(\nu_j) = 1 \) and that for any chain not representing a cycle represented as \( \alpha \nu_j \) where \( \alpha \) is an integer, the value is 0. Then for any pair \( (\nu_j^*, \nu_{j_2}^*) \in H^{k_1}(W_f; R), \nu_{j_2}^* \in H^{k_2}(W_f; R) \), the product in \( H^*(W_f; R) \) vanishes.

3. For any element of \( R^{1 \leq j \leq l_{i_j} = n-k_1} \) in \( H^{k_1}(W_f; R) \cong H^{k_1}(W_f'; R) \oplus R^{1 \leq j \leq l_{i_j} = n-k_1} \) and for any element \( R^{1 \leq j \leq l_{i_j} = n-k_2} \) in \( H^{k_2}(W_f; R) \cong H^{k_2}(W_f'; R) \oplus R^{1 \leq j \leq l_{i_j} = n-k_2} \) in (1), the product in \( H^*(W_f; R) \) vanishes.

4. For any cocycle \( \nu_j^* \in H^{k_j}(W_f; R) \) such that \( \nu_j^*(\nu_j) = 1 \) as before, for any \( k \neq k_j \) and for any element represented as \( (0, p) \in H^{n-k}(W_f; R) \oplus R^{1 \leq j \leq l_{i_j} = k} \cong H^{n-k}(W_f; R) \) where \( p \) is a sequence of \( \geq 1 \leq j \leq l_j = k \) integers such that just one number is 1 and that the others are 0, the product of the two elements in \( H^*(W_f; R) \) vanishes. Moreover, if \( k = k_j \) here and \( p \) is the element corresponding to \( j' \) satisfying \( l_{j'} = k_j \), then the product is represented \( \nu_{j', j} \) times a generator of \( H^n(W_f; R) \).

**Proof.** The first statement follows immediately from (the proofs of) Propositions 3 and 5. To know strictly, see [9]. The key ingredient in the proofs is that the Reeb space obtained by a bubbling operation to the original map can be regarded as a polyhedron obtained by attaching manifolds represented as connected sums of products of spheres along the polyhedron in a natural way (see FIGURE 9) in knowing homology groups and cohomology rings: in
Figure 9. A bouquet of two circles and a manifold represented as a connected sum of two copies of $S^1 \times S^{n-1}$ attached naturally along this.

Knowing more precise topological information, we may not simply argue in this way.

The second statement follows from Lemma 1 and the topology of the Reeb spaces.

In proving the third and the fourth statements, the following observation is a key: $R^k \{1 \leq j \leq l, l_j = n-k\}$ in $H^k(W_f; R) \cong H^k(W_f; R) \oplus R^k \{1 \leq j \leq l, l_j = n-k\}$ is represented by linear combinations of cycles represented by spheres normal to $S_j (1 \leq j \leq l)$; see Figures 10 and 11. This leads us to complete the proof of the last two statements.

□

For example, we can construct a map satisfying the assumption by a suitable default bubbling operation or a finite iteration of connected sums of maps obtained by suitable strongly trivial default normal bubbling operations.

We have the following.

**Theorem 1.** Let $R$ be a PID having the identity element $0 \neq 1 \in R$. Let $m > n \geq 3$ be integers. Let $f$ be a fold map on an $m$-dimensional closed and connected $M$ into $\mathbb{R}^n$. Let $\{G_j\}_{j=1}^n$ be a family of free finitely generated
Figure 10. A bouquet of two circles $S_1$ and $S_2$ and corresponding elements in $R \oplus R$ ($R^\oplus \{1 \leq j \leq l_j = n-k\}$ in $H^k(W_f'; R) \cong H^k(W_f; R) \oplus R^\oplus \{1 \leq j \leq l_j = n-k\}$).

$R$-module satisfying $G_1 = \{0\}$ and $G_n \neq \{0\}$. Let the sequence of integers 
\{rank $G_j\}_{j=1}^{n-1}$ and let \{rank $G_j\}_{j=1}^{n-1}$ be a family of sequences of non-negative integers whose lengths are all $n-1$ such that the sum of all the sequences and 
\{rank $G_j\}_{j=1}^{n-1}$ coincide. We denote the $j_2$-th element of $A_{j_1}$ by $A_{j_1,j_2}$. Let 
\{s_j\}_{j=1}^{n-1} be another family of non-negative integers satisfying $s_{n-1} = 0$. For any integer $1 \leq j_2 \leq n-1$ and any integer $1 \leq j_3 \leq A_{j_1,j_2}$, for each integer 
$1 \leq j' \leq s_{n-j_2}$, let $A_{j_1,j_3,j',j''}$ be an integer if the relation $2(n-j_2) \leq n$ holds

STEP 1 Obtain a map satisfying the assumption of the original map in Proposition 9.
STEP 2 Consider a connected sum of $f$ and the map obtained in the previous step.
STEP 3 To the map obtained in the previous step, perform a finite iteration of $M$-bubbling operations.
Moreover, we can obtain the map $f'$ satisfying the following.

1. The isomorphisms

$$H_k(W_{f'}; R) \cong H_k(W_f; R) \oplus R^{s_k} \oplus G_k$$

and

$$H^k(W_{f'}; R) \cong H^k(W_f; R) \oplus R^{s_k} \oplus G_k$$

hold and the summand $R^{s_k} \oplus G_k$ of the cohomology group is regarded as $\text{Hom}_R(R^{s_k} \oplus G_k, R)$ where $R^{s_k} \oplus G_k$ here is regarded as the summand of the homology group for $1 \leq k \leq n-1$. Moreover, $H_n(W_{f'}; R) \cong H_n(W_f; R) \oplus G_n$ and $H^n(W_{f'}; R) \cong H^n(W_f; R) \oplus G_n$ hold.

2. For arbitrary positive integers $k_1 \leq n-1$ and $k_2 \leq n-1$, we restrict the cohomology group $H^{k_1}(W_{f'}; R) \cong H^{k_1}(W_f; R) \oplus R^{s_{k_1}} \oplus G_{k_1}$ to the summand $H^{k_1}(W_f; R)$ and the cohomology group $H^{k_2}(W_{f'}; R) \cong H^{k_2}(W_f; R) \oplus R^{s_{k_2}} \oplus G_{k_2}$ to the summand $R^{s_{k_2}} \oplus G_{k_2}$. Take one
element from each group and consider the product in $H^*(W_f; R)$. Then it vanishes.

(3) We restrict each cohomology group $H^k(W_f; R) \cong H^k(W_f; R) \oplus R^{s_k} \oplus G_k$ to the summand $R^{s_k}$ and consider the resulting subgroup of $H^*(W_f; R)$. Then for any pair of elements whose degrees are positive, the product vanishes.

(4) For any element of $G_{k_1}$ in

$$H^{k_1}(W_f; R) \cong H^{k_1}(W_f; R) \oplus R^{s_{k_1}} \oplus G_{k_1}$$

and for any element of $G_{k_2}$ in

$$H^{k_2}(W_f; R) \cong H^{k_2}(W_f; R) \oplus R^{s_{k_2}} \oplus G_{k_2}$$

in (1), the product in $H^*(W_f; R)$ vanishes.

(5) Take an element of the summand $R^{A_{j,k_1}} \subset \bigoplus_{j=1}^{\text{rank}} R^{G_{j,k_1}} \cong G_{k_1}$ (1 \leq k_1 \leq n - 1) such that the only one component (corresponding to the $k_2$-th element) is 1 and the other components are 0: we regard $G_{k_1}$ as a summand of

$$H^{k_1}(W_f; R) \cong H^{k_1}(W_f; R) \oplus R^{s_{k_1}} \oplus G_{k_1}$$

canonically. If for any non-zero element in $H^k(W_f; R)$ satisfying $k \neq n - k_1$, consider the product with the previous given element, then it vanishes. For any element in $H^{n-k_1}(W_f; R)$ of the form

$$(0, p, 0) \in H^{n-k_1}(W_f; R) \oplus R^{s_{n-k_1}} \oplus G_{n-k_1}$$

where just one component (the $k_3$-th component) of $p$ is 1 and the others are 0, consider the product with the previous given element, then it is $A_{j,k_1,k_2,k_3}$ times a generator of the $j$-th summand of $G_{n} \subset H^n(W_f; R)$.

Furthermore, a map constructed in STEP 1 can be special generic and can be replaced by an arbitrary map obtained by a finite iteration of bubbling operations which are not M-bubbling operations and whose generating polyhedra are of dimensions smaller than $n - 1$.

Proof. For STEP 1 and STEP 2, we consider a connected sum of the map $f$ and a map obtained by a default trivial bubbling operation whose generating polyhedron is a bouquet consisting of $s_j$ $j$-dimensional standard spheres (1 \leq j \leq n - 2), or equivalently as explained in Example 3 by a finite iteration of default strongly trivial normal bubbling operations and connected sums of them: we can consider more general map for the latter map satisfying the assumption of Proposition 9, which proves the last part of the statement.

The proof of the main statement is completed by applying Proposition 9 one after another. At each step, we perform an M-bubbling operation whose generating polyhedron is a bouquet consisting of $A_{j,k_1} (n - k_1)$-dimensional
standard spheres for $2 \leq k_1 \leq n - 1$. Moreover, for the $k_2$-th $(n - k_1)$-dimensional standard sphere, the corresponding homology class is a linear combination of the cycles represented by spheres as in Proposition 8 and each coefficient is $A_{j,k_1,k_2}$.

□

Remark 1. In the situation of Proposition 9 and Theorem 1, let $R$ be $\mathbb{Z}$ or $\mathbb{Q}$.

According to this work, cohomology rings of Reeb spaces seem to be various. In fact, as an easy observation, by a bubbling operation of Theorem 1 to the map explained just before, we can obtain cohomology rings of Reeb spaces such that there exists no pair of cocycles satisfying the following.

1. The cocycles are not 0-cycles.
2. The product of these two cocycles does not vanish.

and ones such that there exist such pairs.

By changing a non-zero coefficient number of the cycle represented by a sphere whose dimension is not so large in the generating polyhedron into another non-zero number. By this step, we cannot change the resulting cohomology ring whose coefficient ring is $\mathbb{Q}$. On the other hand, the resulting cohomology ring whose coefficient ring is $\mathbb{Z}$ changes (in general).

In short, we can easily obtain families of Reeb spaces such that cohomology rings are isomorphic in the cases where $R = \mathbb{Q}$ and that cohomology rings are mutually not isomorphic in the cases where $R = \mathbb{Z}$.

Proposition 6 implies that from cohomology rings of Reeb spaces, we can know cohomology rings of source manifolds completely in several cases.

These facts explicitly show that difference of topology (cohomology rings) of Reeb spaces are closely related to difference of topology (cohomology rings) of source manifolds.

According to [15], [16], [17] and [20], it is explicitly found that special generic maps often restrict topology and differentiable structures of source manifolds strictly. For example, in considerable cases exotic homotopy spheres do not admit special generic maps into Euclidean spaces whose dimensions are larger than 2, where homotopy spheres except exotic 4-dimensional spheres, being undiscovered, admit special generic maps into $\mathbb{R}$ and (in the cases where the dimensions are larger than 1) $\mathbb{R}^2$. Later, on 7-dimensional homotopy spheres, stable fold maps into $\mathbb{R}^4$ such that the singular value sets are concentric spheres (or round fold maps, introduced in [6]) and that inverse images of regular values are disjoint unions of spheres, satisfying the assumption of Proposition 6, were constructed by the author in [4] and [5] and the author has also explicitly found that the numbers of components of singular value sets are closely related to differentiable structures of the source manifolds. As a new work, in the present paper, we first succeeded in similar works related to cohomology rings of manifolds.

Theorem 2. In the situation of Theorem 1, if $m$ is sufficiently large, then in STEP 1 we do not need to use special generic maps obtained by default
trivial bubbling operations whose generating polyhedra are bouquets consisting of standard spheres or we do not need to assume "trivial".

Proof. By an S-bubbling operation, as presented in FIGURE 5 for example, locally we obtain a connected component of an inverse image containing just one singular point of a small arc containing a value of the quotient map to the Reeb space of a singular point which is branched in the interior and for a small neighborhood of which and points being not values of singular points, inverse images are standard spheres, as presented in FIGURE 12.

We explain precisely about the figure. The three objects lined vertically mean an arc intersecting once with the set of singular values considered here, the inverse image of the arc in the Reeb space and the inverse image of the arc in the source manifold. At the top of the third figure, arrows, two circles and a bouquet of two circles are equipped and indicate structure groups of the bundles over a submanifold consisting of all new singular values with fibers being the subspace of a canonically embedded torus in the one-dimensional higher Euclidean space represented as the space between the two dotted vertical lines. We need to note that circles and a bouquet of circles represent spheres and a bouquet of spheres appearing as cross sections obtained by observing the underlying corresponding lines. Last, note also that we assume the source manifold as a sufficiently high dimensional manifold and that this makes the structure group of the bundle locally regarded as rotations of these spheres and bouquets as indicated. This helps us to do S-bubbling operations which are not trivial. □

Remark 2. By applying Theorem 2, we can obtain family of source manifolds whose cohomology rings are isomorphic and whose characteristic classes such as Stiefel Whitney classes, Pontrjagin classes etc. are distinct, making corresponding manifolds in the family not homeomorphic or homotopy equivalent.

We present a sketch about this. We can construct a submanifold represented as an inverse image of a collar neighborhood of the boundary of the Reeb space flexibly from the viewpoint of characteristic classes (see also Proposition 2); note that they are homotopy equivalent to the bouquet appearing as the generating polyhedron. In the step of S-bubbling operations, we can do operations to the maps without bearing new non-trivial characteristic classes.

Last, we construct further examples.

Theorem 3. Let $R$ be a PID having the identity element $0 \neq 1 \in R$. Let $u$ be an element representing $0$ or $1$ in $R$.

Let $m > n \geq 2$ be integers.

Let $f$ be a fold map on an $m$-dimensional closed and connected $M$ into $\mathbb{R}^n$.

Let $\{G_j\}_{j=1}^n$ be a family of free finitely generated $R$-module satisfying $G_n = R$. 
Figure 12. The connected component of an inverse image containing just one singular point around a value of the quotient map to the Reeb space of a singular point which is branched and for a small neighborhood of which and points being not values of singular points, inverse images are standard spheres.

Let \{s_j\}_{j=1}^{n-1} be a family of sufficiently large non-negative integers. For any integer \(1 \leq j \leq n - 1\) and any integer \(1 \leq j_1 \leq \text{rank}G_j\), for each integer \(1 \leq j_2 \leq s_{n-j}\), let \(G_{j,j_1,j_2}\) be an integer if the relation \(2(n-j) \leq n\) holds and \(0, 1\) or \(-1\) if the relation \(2(n-j) > n\) holds. Let \{t_j\}_{j=1}^{n-1} be another family of sufficiently large non-negative integers. Then by the following steps, we can obtain a fold map \(f'\).

**STEP 1**

Obtain a map by a default bubbling operation.
STEP 2
Consider a connected sum of \( f \) and the map obtained in the previous step.

STEP 3
To the map obtained in the previous step, perform a finite iteration of \( M \)-bubbling operations.

Moreover, we can obtain the map \( f' \) satisfying the following.

1. The isomorphisms

\[
H_k(W_f; R) \cong H_k(W_f; R) \oplus R^{s_k} \oplus R^{t_k} \oplus G_k
\]

and

\[
H^k(W_f; R) \cong H^k(W_f; R) \oplus R^{s_k} \oplus R^{t_k} \oplus G_k
\]

hold and the summand \( R^{s_k} \oplus R^{t_k} \oplus G_k \) of the cohomology group is regarded as \( \text{Hom}_R(R^{s_k} \oplus R^{t_k} \oplus G_k, R) \) where \( R^{s_k} \oplus R^{t_k} \oplus G_k \) here is regarded as the summand of the homology group for \( 1 \leq k \leq n-1 \). Moreover, \( H_n(W_f; R) \cong H_n(W_f; R) \oplus R \) and \( H^n(W_f; R) \cong H^n(W_f; R) \oplus R \) hold.

2. For arbitrary positive integers \( k_1 \leq n-1 \) and \( k_2 \leq n-1 \), we restrict the cohomology group \( H^{k_1}(W_f; R) \cong H^{k_1}(W_f; R) \oplus R^{s_{k_1}} \oplus R^{t_{k_1}} \oplus G_{k_1} \) to the summand \( H^{k_1}(W_f; R) \) and the cohomology group \( H^{k_2}(W_f; R) \cong H^{k_2}(W_f; R) \oplus R^{s_{k_2}} \oplus R^{t_{k_2}} \oplus G_{k_2} \) to the summand \( R^{s_{k_2}} \oplus R^{t_{k_2}} \oplus G_{k_2} \). Take one element from each group and consider the product in \( H^*(W_f; R) \). Then it vanishes.

3. We restrict each cohomology group \( H^k(W_f; R) \cong H^k(W_f; R) \oplus R^{s_k} \oplus R^{t_k} \oplus G_k \) to the summands \( R^{s_k} \) and \( R^{t_k} \) and consider the resulting subgroups of \( H^*(W_f; R) \). Then for any pair of elements whose degrees are positive, one of which is in the former group and the remaining element of which is in the latter group, the product vanishes.

4. For any element of \( G_{k_1} \) in \( H^{k_1}(W_f; R) \cong H^{k_1}(W_f; R) \oplus R^{s_{k_1}} \oplus R^{t_{k_1}} \oplus G_{k_1} \) and for any element of \( G_{k_2} \) in \( H^{k_2}(W_f; R) \cong H^{k_2}(W_f; R) \oplus R^{s_{k_2}} \oplus R^{t_{k_2}} \oplus G_{k_2} \) of degree \( 1 \leq k_1 \leq n-1 \) such that the only one component corresponding to the \( k_2 \)-th element is 1 and the other components are 0: we regard \( G_{k_1} \) as a summand of \( H^{k_1}(W_f; R) \cong H^{k_1}(W_f; R) \oplus R^{s_{k_1}} \oplus R^{t_{k_1}} \oplus G_{k_1} \) canonically. If for any non-zero element in \( H^k(W_f; R) \) satisfying \( k \neq n-k_1 \), consider the product with the previous given element, then it vanishes. For any element in \( H^{n-k_1}(W_f; R) \) of the form \( (0, p, 0, 0) \in H^{n-k_1}(W_f; R) \oplus R^{s_{n-k_1}} \oplus R^{t_{n-k_1}} \oplus G_{n-k_1} \) where just one component
Consider a finite number of sets of pairs of integers satisfying the following.

(a) For each pair of integers of each set, the first component is positive.

(b) Each set is a finite set.

(c) For each set, consider the sum of all of first components. Then the sum is smaller than \( n \).

(d) For each pair of integers of each set, let the first component be \( j \), then the third component is not smaller than \( 1 \) and not larger than \( s_j \).

(e) For each pair \( (p, q) \) of integers of each set and for some \( r, G_{p,r,q} \) is not 0.

(f) For any two distinct sets, there is no common pair.

Then for each set \( \{(p_j, q_j)\}_{j \in I} \) where \( I \) is a finite set consisting of \( |I| \) elements, we can consider a corresponding natural product of \( |I| \) elements where each factor is \((0, p_j, 0, 0) \in H^{p_j}(W_f; R) \oplus R^{s_j} \oplus R^{t_j} \oplus G_{p_j} \) and where \( p_j, 0 \) is a unit vector consisting of 0 or 1 with 1 as the \( q_j \)-th component and the resulting element is represented as 
\[(0, p_0, 0, 0) \in H^{\sum_{j \in I} p_j}(W_f; R) \oplus R^{\sum_{j \in I} p_j} \oplus R^{\sum_{j \in I} s_j} \oplus G_{\sum_{j \in I} p_j}, \]
where \( p_0 \) is a unit vector consisting of 0 or 1.

Consider a finite number of finite sequences of positive integers; for each sequence \( \{p_j\}_{j \in I} \) such that \( I \) is a finite set, that the sum is not smaller than \( n - 1 \) and that the length is \( |I| \), we can consider a corresponding natural product of \( |I| \) elements where each factor is 
\[(0, 0, p_j, 0) \in H^{p_j}(W_f; R) \oplus R^{s_j} \oplus R^{t_j} \oplus G_{p_j}, \]
and where \( p_j, 0 \) is a unit vector consisting of 0 or 1 and the resulting element is represented as 
\[(0, 0, p_0, 0) \in H^{\sum_{j \in I} p_j}(W_f; R) \oplus R^{\sum_{j \in I} p_j} \oplus R^{\sum_{j \in I} s_j} \oplus G_{\sum_{j \in I} p_j}, \]
where \( p_0 \) is a unit vector consisting of 0 or 1.

Proof. Corresponding to \((p_j, q_j)\) in each set \( I \) of the assumption (6), we correspond \( \prod_{j \in I} S^{p_j} \) and construct a bouquet of a family of manifolds including these manifolds such that the \( k \)-th homology group is free and of rank \( s_k \). Based on the assumption (7), similarly to the previous case, we construct a bouquet such that the \( k \)-th homology group is free and of rank \( t_k \). We consider the bouquet of these two bouquets. Then we perform a trivial default bubbling operation whose generating polyhedron is PL homeomorphic to the last bouquet.
Then we perform a trivial default bubbling operation whose generating polyhedron is PL homeomorphic to the bouquet.

Thus, by constructing a connected sum of the original map and this obtained map and only performing a trivial bubbling operation in a way similar to the proof of Proposition 9, we obtain the desired map.

□

We can make comments as Remark 1 and Remark 2 on Theorem 3.

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