Weighted prime geodesic theorems

Anton Deitmar

Received: 6 June 2022 / Revised: 26 January 2023 / Accepted: 3 March 2023 / Published online: 28 June 2023
© The Author(s) 2023

Abstract
Prime geodesic theorems for weighted infinite graphs and weighted building quotients are given. The growth rates are expressed in terms of the spectral data of suitable translation operators inspired by a paper of Bass.

Keywords  Ihara zeta function · Weighted graphs · Bruhat–Tits buildings

Mathematics Subject Classification  11N80 · 11F72 · 11N05 · 20E08 · 20F65 · 51E24 · 53C22

1 Introduction

The first prime geodesic theorem was given by Huber in [19]. It states that for a compact hyperbolic surface $X$ the number $N(T)$ of prime closed geodesics of length $\leq T$ satisfies

$$N(T) \sim \frac{e^{2T}}{2T}, \quad \text{as } T \to \infty.$$

It has been sharpened by giving estimates on the error term and it has been extended to other manifolds [8, 18, 20–22, 26]. Applications to class numbers are in [24], and, extending this result, in [7, 11]. It was extended to more general dynamical systems, culminating in Margulis’s celebrated result on Anosov flows, stating that the number $N(T)$ of closed orbits of length $\leq T$ satisfies $N(T) \sim \frac{e^{hT}}{hT}$, where $h$ is the entropy of the system [23].

An extension to the graph case was formulated in [9, 17]. In [10], zeta functions of graphs with weights were introduced. Here weights are representing resistance to a flow along the edges or an individual distribution of the flow at the nodes. The paper
[14] gives prime geodesic theorems for compact quotients of buildings, generalizing the graph case. This generalization is natural, as in number theoretical situations, graphs and building quotients both turn up in the $p$-adic setting. In the present paper the latter two ideas are combined in stating prime geodesic theorems for weighted infinite graphs and weighted building quotients. In the latter case, a full expansion of the numbers of closed geodesics of a given length is presented.

The first section treats an extension of the Perron–Frobenius Theorem to the infinite-dimensional case, which is suitable for our purposes. The next two sections deal with the graph case and the last three with building quotients.

2 $E$-Positive operators

Definition 2.1 Let $H$ be a Hilbert space and $E$ an orthonormal basis. Let $H^+_E$ denote the cone of all $\sum_{e \in E} c_e e \in H$ with $c_e \geq 0$ for every $e \in E$. A bounded linear operator $T$ on $H$ is called $E$-positive if $T(\sum_{e} c_e e) \in H^+_E$. This is equivalent to

$$\langle Te, f \rangle \geq 0$$

for all $e, f \in E$. Note that the adjoint $T^*$ is $E$-positive if $T$ is. Further, if $S$ and $T$ are $E$-positive, then so are $S + T$ and $ST$.

Definition 2.2 We say that a bounded operator $T : H \to H$ is $E$-reducible if there exists a proper subset $\emptyset \neq F \subseteq E$ such that $T(F) \subset \ell^2(F)$, where by $\ell^2(F)$ we mean the closure of the span of $F$. This is in accordance with the isomorphism $H \cong \ell^2(E)$. In this case, the closed subspace $\ell^2(F)$ is $T$-stable.

If $T$ is not $E$-reducible, we say that $T$ is $E$-irreducible.

Definition 2.3 For a compact operator $T$ and an eigenvalue $\lambda$ the geometric multiplicity is the dimension of the eigenspace $\ker(T - \lambda)$. In this situation, the sequence $\ker(T - \lambda)^n, n \in \mathbb{N}$, is eventually stationary. The algebraic multiplicity is the limit

$$\lim_{n \to \infty} \dim \ker(T - \lambda)^n \in \mathbb{N}.$$ 

Theorem 2.4 Let $T$ be an $E$-positive compact operator on $H$ with positive spectral radius $r = r(T) > 0$. Then the spectral radius $r$ is an eigenvalue of $T$. Assume that $T$ is $E$-irreducible and trace class. Then the algebraic multiplicity of $r$ is 1. The eigenvalues $\lambda$ with $|\lambda| = r$ distribute evenly over the unit circle times $r$. More precisely, if $r = \lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ are all eigenvalues of $T$ with $|\lambda_j| = r$, then we can order them in a way that $\lambda_j = re^{2\pi ij/n}$. Finally, every $\lambda_j$ has algebraic multiplicity 1.

Proof If $\dim H < \infty$, this is the Theorem of Frobenius, see [15]. So we assume $H$ to be infinite-dimensional. We first show that if $T$ is $E$-irreducible, then so is the adjoint.
operator $T^*$. For this assume $T$ is $E$-irreducible and let $F \subset E$ be such that $T^*$ maps $\ell^2(F)$ to itself. Then for every $f \in F$ and every $e \in E \setminus F$ we have

$$0 = \langle T^*f, e \rangle = \langle f, Te \rangle.$$

This implies that $T$ maps $\ell^2(E \setminus F)$ to itself, hence $F$ is either empty or equals $E$, which means that $T^*$ is $E$-irreducible.

The fact that $r = r(A)$ is an eigenvalue is known as the Krein–Rutman Theorem, see for instance [2, Theorem 7.10]. This theorem also says that there exists an eigenvector $v_0$ for the eigenvalue $\lambda_0 = r$ which is $E$-positive in the sense that $\langle v_0, e \rangle \geq 0$ for every $e \in E$. Let $F$ be the set of all $f \in E$ such that $\langle v_0, f \rangle = 0$. For $f \in F$ we write

$$T^*f = \sum_{e \in E} c_e e.$$

We have

$$0 = r \langle v_0, f \rangle = \langle Tv_0, f \rangle = \langle v_0, T^*f \rangle = \sum_{e \in E} c_e \langle v_0, e \rangle.$$

Since $T^*$ is positive, $c_e \geq 0$ and so we have $e \in F$ if $c_e \neq 0$. This means that $\ell^2(F)$ is $T^*$-stable, hence, since $v_0 \neq 0$, we get $F = \emptyset$. So we conclude that $\langle v_0, e \rangle > 0$ for every $e \in E$. This means that $v_0$ is a totally $E$-positive vector.

We now assume that $T$ is $E$-irreducible, $E$-positive and trace class. Then the Fredholm determinant $\det(1 - uT)$ is an entire function. For $u \in \mathbb{C}$ such that $\frac{1}{u} \notin \sigma(T)$ we set

$$B(u) = \det(1 - uT)(1 - uT)^{-1}.$$

For each $\lambda \in \mathbb{C}$ let $H(\lambda)$ be the largest $T$-stable subspace on which $T$ has spectrum $\{\lambda\}$. If $\lambda \neq 0$, then $H(\lambda)$ is finite-dimensional and

$$H(\lambda) = \bigcup_{n=1}^{\infty} \ker(T - \lambda)^n.$$

Let $H^r = \bigoplus_{\lambda \neq r} H(\lambda)$. Then $H(r)$ and $H^r$ are closed, $T$-stable subspaces and

$$H = H(r) \oplus H^r.$$

Then $\det(1 - uT|_{H(r)})(1 - uT|_{H(r)})^{-1} = (1 - uT|_{H^r})^\#$, where the $\#$ indicates the adjugate matrix. This implies that $B(u)$ extends continuously to $u = 1/r$. Applying $B(1/r)$ to a vector in $H_{<r}$ we see that $B(1/r) \neq 0$. For $0 < u < 1/r$ and $e, f \in E$ we have

$$\langle B(u)e, f \rangle = \det(1 - uT)(1 - uT)^{-1} e, f \rangle = \det(1 - uT) \sum_{n=0}^{\infty} u^n \langle T^n e, f \rangle \geq 0.$$
Letting \( u \) tend to \( 1/r \) we conclude that there is a sign \( \sigma \in \{ \pm 1 \} \) such that \( \sigma B(1/r) \) is \( E \)-positive. Since \( v_0 \) is totally positive, we have

\[
0 \neq B(1/r) v_0 = \lim_{u \nearrow 1/r} \det(1 - uT)(1 - uT)^{-1} v_0 = \lim_{u \nearrow 1/r} \det(1 - uT) \frac{1}{1 - ur} v_0.
\]

This implies that \( 1/r \) is a simple zero of \( \det(1 - uT) \), which is to say that the algebraic multiplicity is 1. Therefore the condition (G) of [25, Definition 4.7] is satisfied, and hence the remaining points of the theorem follow from [25, Theorem 5.2].

We finally consider the situation without the condition of irreducibility.

**Proposition 2.5** Let \( T \) be a compact operator with positive spectral radius \( r(T) > 0 \). Then for a given orthonormal basis \( E \) there exists a disjoint decomposition \( E = E_1 \cup \cdots \cup E_n \) with the following properties:

(a) The space \( \ell^2(E_1 \cup \cdots \cup E_k) \) is \( T \)-stable for each \( k \).

(b) For each \( k \) let \( P_k \) denote the orthogonal projection onto \( \ell^2(E_k) \) then the operator

\[
T_k = P_k T : \ell^2(E_k) \to \ell^2(E_k)
\]

either has a spectral radius \( r(T_j) < r(T) \) or is \( E \)-irreducible.

If \( T \) is \( E \)-positive, then \( T_k \) is \( E_k \)-positive for each \( k \). If \( T \) is trace class, then each \( T_k \) is as well and for the Fredholm determinant one has

\[
\det(1 - uT) = \prod_{k=1}^n \det(1 - uT_k), \quad u \in \mathbb{C}.
\]

**Proof** If \( T \) is \( E \)-irreducible, we are done. Otherwise there is a \( \emptyset \neq F \subsetneq E \) such that \( \ell^2(F) \) is \( T \)-stable. We consider \( T_F = T|_{\ell^2(F)} \) and \( T_{E \setminus F} = PT : \ell^2(E \setminus F) \to \ell^2(E \setminus F) \), where \( P \) is the orthogonal projection to \( \ell^2(E \setminus F) \). We have \( \det(1 - uT) = \det(1 - uT_F) \det(1 - uT_{E \setminus F}) \). If either of the operators \( T_F \) or \( T_{E \setminus F} \) has spectral radius less than \( r(T) \) or is irreducible, we leave this factor in peace and continue with the other, which we then decompose further. This process will stop, as there are only finitely many spectral values \( \lambda \) with \( |\lambda| = r(T) \) and these are eigenvalues of finite multiplicity. By the fact that the Fredholm determinant distributes it follows that the spectral values distribute and by the finiteness, this process will terminate, yielding the proposition. □

**3 The Ihara zeta function**

**Definition 3.1** Let \( X \) denote an oriented graph. This means that \( X \) consists of the following data: a set \( N(X) \) of nodes and a set \( E(X) \subset X \times X \) of oriented edges. We call \( x \) the source of the oriented edge \( e = (x, y) \) and \( y \) the target and we write this as \( x = s(e), \quad y = t(e) \). The nodes \( x, y \) are called the endpoints of the edge \( (x, y) \).
Definition 3.2 The valency, $\text{val}(x)$ of a node $x$ is the number of edges having $x$ for one of their endpoints. Throughout, we will assume that $X$ has bounded valency, i.e., that there exists $M > 0$ such that

$$\text{val}(x) \leq M$$

holds for all nodes $x$.

We introduce the notion of a weight. Usually, a weight $w(e) > 0$ is put on an edge representing a length or a resistance or the reciprocal of that. In order to be more flexible, it is more convenient for us to consider a transition weight, which may be interpreted as the likelihood of a particle or a current of choosing a certain edge.

Definition 3.3 A transition weight on $X$, henceforth simply called a weight, is a map

$$w : E(X) \times E(X) \to [0, \infty)$$

such that

$$w(e, f) \neq 0 \Rightarrow t(e) = s(f)$$

and

$$\sum_{e, f \in E(X)} w(e, f) < \infty.$$

Example 3.4 A natural example is given in the case of $X$ being a quotient $X = \Gamma \backslash Y$, where $Y$ is a tree of bounded valency and $\Gamma$ is a tree lattice [5]. We let $\Gamma_f$ denote the stabiliser of $f$ in the group $\Gamma$. In this case the weight

$$w(e, f) = \frac{1}{|\Gamma_f|}$$

is a natural choice which fits the approach of Bass [4] to the Ihara zeta function in case of ramified quotients, see also [12].

Definition 3.5 An oriented path of length $n$ in $X$ is an $n$-tuple $p = (e_1, e_2, \ldots, e_n)$ of oriented edges such that $t(e_j) = s(e_{j+1})$ holds for all $j = 1, \ldots, n - 1$. We write the length as $\ell(p) = n$. The path $p$ is called a closed path if $t(e_n) = s(e_1)$. For a closed path $p = (e_1, \ldots, e_n)$ we define its weight as

$$w(p) = w(e_1, e_2)w(e_2, e_3) \ldots w(e_{n-1}, e_n)w(e_n, e_1).$$

Definition 3.6 (Shifting the starting point) On the set $CP(X)$ of all closed paths we install an equivalence relation $\sim$ generated by

$$(e_1, e_2, \ldots, e_n) \sim (e_2, e_3, \ldots, e_n, e_1).$$
An equivalence class \( c = [p] \) of closed paths is called a \textit{cycle}. The length and weight functions factor through the quotient of this equivalence, so \( \ell(c) \) and \( w(c) \) are well-defined for a cycle \( c \).

**Definition 3.7** For a cycle \( c \) and a natural number \( k \) we define \( c^k \) to be the cycle one gets by iterating the cycle \( c \) for \( k \) times. One has

\[
\ell(c^k) = k \ell(c) \quad \text{and} \quad w(c^k) = w(c)^k.
\]

A cycle \( c \) is called \textit{primitive} if \( c \) is not a power \( c_1^k \) of some shorter cycle \( c_1 \). For every cycle \( c \) there is a uniquely determined primitive \( c_0 \) and a uniquely determined number \( \mu(c) \in \mathbb{N} \) such that

\[
c = c_0^{\mu(c)}.
\]

The cycle \( c_0 \) is called the \textit{underlying primitive} and \( \mu(c) \) is called the \textit{multiplicity} of the cycle \( c \).

**Definition 3.8** The \textit{Ihara zeta function} of the weighted oriented graph \((X, w)\) is defined by the product

\[
Z(u) = \prod_c \left( 1 - w(c)u^{\ell(c)} \right)^{-1},
\]

where the product runs over all primitive cycles in \( X \).

**Definition 3.9** Let \( H = \ell^2(E) \) be the Hilbert space of all \( \ell^2 \)-functions on \( E(X) \) the elements of which we write as formal series \( \sum_{e \in E(X)} c_e e \) with \( c_e \in \mathbb{C} \) satisfying \( \sum_{e \in E(X)} |c_e|^2 < \infty \).

**Definition 3.10** Inspired by [4], we consider the \textit{Bass operator} \( T : H \to H \) given by

\[
T(e) = \sum_{f \in E(X)} w(e, f) f.
\]

If for two edges we have \( t(e) = s(f) \), we write \( e \to f \), so we have \( T(e) = \sum_{e \to f} w(e, f) f \).

The following theorem is a straightforward generalization of [10, Theorem 1.6].

**Theorem 3.11** The operator \( T \) is of trace class. The product \( Z(u) \) converges for \(|u|\) sufficiently small. The function \( Z(u) \) extends to a meromorphic function, more precisely, \( Z(u)^{-1} \) is entire and satisfies

\[
Z(u)^{-1} = \det(1 - uT),
\]

where \( \det \) is the Fredholm-determinant.
Proof The proof is essentially the same as the proof of [10, Theorem 1.6]. We repeat it here for the convenience of the reader. We show that the operator $T$ is of trace class, and for every $n \in \mathbb{N}$ we have

$$\text{tr } T^n = \sum_{l(c)=n} l(c_0) w(c),$$

where the sum runs over all cycles $c$ of length $n$ and $c_0$ is the underlying primitive cycle to $c$.

For this we consider the natural orthonormal basis of $\ell^2(E)$ given by $E$. Using this orthonormal basis, one easily sees that $T^n$ has the claimed trace, once we know that $T$ is of trace class. For this we estimate

$$\sum_{e \in E} \|Te\| = \sum_{e} (\langle Te, Te \rangle)^{\frac{1}{2}} = \sum_{e} \left( \sum_{f \in E} \langle Te, f \rangle \langle f, Te \rangle \right)^{\frac{1}{2}} \leq \sum_{e} \sum_{f} |\langle Te, f \rangle|^2 = \sum_{e, f} w(e, f)^2 < \infty.$$

This implies that $T$ is of trace class. For small values of $u$ we have

$$\text{det}(1 - uT) = \exp \left( - \sum_{n=1}^{\infty} \frac{u^n}{n} \text{tr } T^n \right) = \exp \left( - \sum_{n=1}^{\infty} \frac{u^n}{n} \sum_{l(c)=n} l(c_0) w(c) \right) = \exp \left( - \sum_{c_0} \sum_{m=1}^{\infty} \frac{u^{ml(c_0)}}{m} w(c_0)^m \right) = \prod_{c_0} \left( 1 - w(c_0)u^{l(c_0)} \right) = Z(u)^{-1}.$$

This a fortiori also proves the convergence of the product. \hfill \square

4 The prime geodesic theorem for a weighted graph

Theorem 4.1 Let $(X, w)$ be a weighted oriented graph. For $m \in \mathbb{N}$ let

$$N_m = \sum_{c: l(c)=m} w(c) l(c_0).$$

Then there are $r > 0$ and natural numbers $n_1, \ldots, n_s$ such that, as $m \to \infty$,

$$N_m = r^m \sum_{k=1}^{s} n_k 1_{n_k \mathbb{N}}(m) + O((r - \varepsilon)^m)$$

for some $\varepsilon > 0$. \hfill \square
Proof Let $Z(u)$ be the Ihara zeta function of $X$. By the definition of $Z$ we get

$$u \frac{Z'}{Z}(u) = \sum_{m=1}^{\infty} N_m u^m.$$ 

The formula $Z(u) = \det (1 - uT)^{-1}$ on the other hand yields

$$N_m = \text{tr}(T^n) = r^m \sum_{k=1}^{s} \sum_{j=0}^{n_k-1} e^{\frac{2\pi i jm}{n_k}} + \sum_{|\lambda| < r} m(\lambda) \lambda^m,$$

where $r = r(T) > 0$ is the spectral radius of $T$ and the last sum runs over all eigenvalues $\lambda$ of $T$ with $|\lambda| < r$. The number $m(\lambda)$ is the algebraic multiplicity of $\lambda$. Finally, the numbers $s$ is the number of components as in Proposition 2.5 with $r(T_j) = r(T)$ and $n_j$ is the number of eigenvalues $\lambda$ of that component, satisfying $|\lambda| = r(T)$. Since the sum $\sum_{j=0}^{n_k-1} e^{\frac{2\pi i jm}{n_k}}$ is zero unless $m$ is a multiple of $n_k$, the theorem follows. \qed

Let

$$\vartheta(n) = \sum_{c_0 : l(c_0) \leq n} w(c_0) l(c_0),$$

$$\psi(n) = \sum_{c : l(c) \leq n} w(c) l(c),$$

$$\pi(n) = \sum_{c_0 : l(c_0) \leq n} w(c_0).$$

Proposition 4.2 Assume that $r = r(T) > 1$. Let $n_1, n_2, \ldots, n_s$ be the numbers of Theorem 4.1, let $K$ be their least common multiple and let $C = \sum_{k=1}^{s} n_k \frac{r^{n_k}}{r^{n_k} - 1}$. Then, as $n \to \infty$ one has

$$\vartheta(nK) \sim \psi(nK) \sim r^{nK} C$$

and

$$\pi(nK) \sim \frac{r^{nK}}{nK} C.$$ 

Proof Let

$$l_1 = \liminf_n \frac{\vartheta(nK)}{r^{nK}},$$

$$L_1 = \limsup_n \frac{\vartheta(nK)}{r^{nK}},$$

$$l_2 = \liminf_n \frac{\psi(nK)}{r^{nK}},$$

$$L_1 = \limsup_n \frac{\psi(nK)}{r^{nK}},$$

$$l_3 = \liminf_n \frac{nK \pi(nK)}{r^{nK}},$$

$$L_1 = \limsup_n \frac{nK \pi(nK)}{r^{nK}}.$$
We compute

\[ \vartheta(n) \leq \psi(n) = \sum_{c_0 : l(c_0) \leq n} \sum_{i = 1}^{\left\lfloor \frac{n}{l(c_0)} \right\rfloor} w(c_0)^i l(c_0) \leq n \sum_{c_0 : l(c_0) \leq n} w(c_0) = n\pi(n) \]

and this implies \( l_1 \leq l_2 \leq l_3 \) as well as \( L_1 \leq L_2 \leq L_3 \). Next we let \( 0 < \alpha < 1 \) and we get

\[ \vartheta(n) \geq \sum_{\alpha n < l(c_0) \leq n} w(c_0)^i l(c_0) \geq \alpha n \sum_{\alpha n < l(c_0) \leq n} w(c_0) = \alpha n(\pi(n) - \pi(\alpha n)). \]

So that

\[ \frac{\vartheta(n)}{r^n} \geq \alpha \frac{n\pi(n)}{r^n} - \frac{\alpha n\pi(\alpha n)}{r^{\alpha n}} r^{(\alpha-1)n}. \]

Since \( r^{(\alpha-1)n} \) tends to zero, this implies and this finally implies \( \alpha l_3 \leq l_1 \) and \( \alpha L_3 \leq L_1 \).

As \( \alpha \) was arbitrary it follows \( l_3 \leq l_1 \) and \( L_3 \leq L_1 \), so \( l_1 = l_2 = l_3 \) and \( L_1 = L_2 = L_3 \). The claim follows if we finally show that \( l_2 = L_2 = C \). We have

\[
\psi(nK) = \sum_{m=1}^{nK} N_m = \sum_{m=1}^{nK} r^m \sum_{k=1}^{s} n_k 1_{n_k \mid n}(m) + O((r - \varepsilon)^m)
\]

\[
= \sum_{m=1}^{nK} r^m \sum_{k \mid n_k \mid m} n_k + O((r - \varepsilon)^m)
\]

\[
= \sum_{k=1}^{s} n_k \sum_{m \leq nK \atop n_k \mid m} r^m + O((r - \varepsilon)^m)
\]

\[
= \sum_{k=1}^{s} n_k \sum_{1 \leq j \leq nK \atop n_k \mid m} r^{jn_k} + O((r - \varepsilon)^{jn_k})
\]

\[
= \sum_{k=1}^{s} n_k r^{nK} \frac{r^{nK} - 1}{r^{n_k} - 1} + O((r - \varepsilon)^{nK})
\]

and so

\[
\frac{\psi(nK)}{r^{nK}} = \sum_{k=1}^{s} n_k r^{n_k} \frac{1 - r^{-nK}}{r^{n_k} - 1} + O((r - \varepsilon)^{nK}) = \sum_{k=1}^{s} n_k \frac{r^{n_k}}{r^{n_k} - 1} + o(1).
\]
Proposition 4.3  (a) For the spectral radius $r = r(T)$ one has

$$r = \sup \left\{ u > 0 : \sum_c w(c)u^{l(c)} < \infty \right\}.$$ 

(b) If there exist two different primitive cycles $c_0, d_0$ with a common oriented edge $e$ and $w(c_0) = w(d_0) \geq 1$, then $r > 1$.

Proof  (a) We have

$$u \frac{Z'}{Z}(u) = \sum_c w(c)I(c_0)u^{l(c)}.$$ 

Hence it follows that $\frac{1}{r}$ equals the supremum of all $u > 0$ for which $\sum_c w(c)I(c_0)u^{l(c)} < \infty$. Let $l = \sup \left\{ u > 0 : \sum_c w(c)u^{l(c)} < \infty \right\}$. We show that $l = \frac{1}{r}$. So suppose that $\sum_c w(c)u^{l(c)} < \infty$. Then, as power series may be differentiated element-wise and on the other hand they converge locally uniformly and may be integrated, it follows that

$$\sup \left\{ u > 0 : \sum_c w(c)u^{l(c)} < \infty \right\} = \sup \left\{ u > 0 : \sum_c w(c)I(c_0)u^{l(c)} < \infty \right\}.$$ 

As $1 \leq l(c_0) \leq l(c)$, the claim follows.

(b) The cycles $c_0d_0$ and $d_0c_0$ are primitive, distinct, and of the same length. Replacing $c_0, d_0$ with these, we assume that $l(c_0) = l(d_0) = l$. Next we fix representing closed paths of $c_0$ and $d_0$ which start with the edge $e$. For $n \in \mathbb{N}$ we have

$$\langle T^{nl}e, e \rangle \geq \sum_{p : l(p) = nl} 1,$$ 

where the sum runs over all closed paths starting with $e$ and being of the form $c_0^{m_1}d_0^{m_1} \cdots c_0^{m_s}d_0^{m_s}$ for some $m_jn_j \in \mathbb{N}_0$. This implies that

$$\langle T^{nl}e, e \rangle \geq \langle S^{3n}f, f \rangle,$$ 

where $S$ is the Bass operator of the oriented graph with constant weight $w = 1$.

One sees that $S^{3n}f = 2^n f$. Therefore $\langle T^{nl}e, e \rangle \geq 2^n$ and so the operator norm satisfies $\|T^{nl}\| \geq 2^n$. The spectral radius therefore satisfies

\(\square\) Springer
$$r = \lim_n \|T^{nl}\|^{\frac{1}{nl}} \geq 2^{\frac{1}{l}} > 1.$$

\[\square\]

## 5 Affine buildings

For background on this section, the reader may consult [13]. Let $X$ be a locally finite affine building. By this we understand a polysimplicial complex which is the union of a given family of affine Coxeter complexes, called apartments, such that any two chambers (= cells of maximal dimension, which is fixed) are contained in a common apartment and for any two apartments $a, b$ containing chambers $C, D$ there is a unique isomorphism $a \rightarrow b$ fixing $C$ and $D$ point-wise. A chamber is called \textit{thin} if at every wall it has a unique neighbor chamber, it is called \textit{thick}, if at each wall it neighbors at least two other chambers. The building is called thin or thick if all its chambers are. For the ease of presentation, we will always assume that the building $X$ is simplicial instead of polysimplicial.

Note that our definition includes buildings which are not Bruhat–Tits. In higher dimensions, buildings tend to be of Bruhat–Tits type [6]. For buildings of dimension at most two the situation is drastically different. Indeed, Ballmann and Brin proved that every 2-dimensional simplicial complex in which the links of vertices are isomorphic to the flag complex of a finite projective plane has the structure of a building [3].

When speaking of “points” in $X$, we identify the complex $X$ with its geometric realization. Note that the latter carries a topology as a CW-complex. In this topology, a set is compact if and only if it is closed and contained in a finite union of chambers. Note that an affine building is always contractible, see [16, Section 14.4].

**Definition 5.1** Generally, there are different families of apartments which make $X$ a building, but there is a unique maximal family [1, Theorem 4.54]. In this paper, we will always choose the maximal family. Let $\text{Aut}(X)$ be the automorphism group of the building $X$, that is, the set of all automorphisms $g : X \rightarrow X$ of the complex $X$ which map apartments to apartments. In the geometric realisation these are cellular maps which are affine on each cell.

**Definition 5.2** A choice of types is a labelling that attaches to each vertex $v$ a label $\text{lab}(v)$, or type in $\{0, 1, \ldots, d\}$ such that for each chamber $C$ the set $V(C)$ of vertices of $C$ is mapped bijectively to $\{0, 1, \ldots, d\}$.

Restricting the labelling gives a bijection between the set of all choices of types and the set of all bijections $V(C_0) \rightarrow \{0, 1, \ldots, d\}$, where $C_0$ is any given chamber. Therefore the number of different choices of types is $(d + 1)!$. We fix a choice of types such that each vertex of type zero is a special vertex. This means that the set of reflection hyperplanes containing it, meets every parallelity class of reflection hyperplanes of the ambient apartment, see [13, Definition 1.2.3].

**Definition 5.3** Pick a chamber $C$ and an apartment $a$ containing $C$. Let $v_0, v_1, \ldots, v_d$ be the vertices of $C$ with $\text{lab}(v_j) = j$. Pick $v_0$ as origin to give the affine space $a$ the structure of a vector space. Let $\text{Cone}(C, a)$ denote the open cone in $a$ spanned by the
interior $\hat{C}$ of the chamber $C$, i.e.

$$\text{Cone}(C, a) = (0, \infty) \cdot \hat{C}.$$ 

Further let

$$\text{Cone}(C) = \bigcup_{a \supset C} \text{Cone}(C, a).$$

For two chambers $C, D$ we finally write

$$C \sim D$$

if and only if

$$C \neq D \text{ and } \text{Cone}(D) \subset \text{Cone}(C).$$

**Definition 5.4** Let $v_1, \ldots, v_d$ be the other vertices of $C$ and let

$$\Lambda = \bigoplus_{j=1}^{d} \mathbb{Z} e_j \subset \mathfrak{a},$$

where $e_j = r_j v_j$ and $r_j > 0$ for each $j = 1, \ldots, d$ is the largest rational number such that $\Lambda_0 \subset \Lambda$, where $\Lambda_0$ is the lattice of vertices of type zero.

Let

$$\mathbb{N}^d(\Lambda_0)$$

denote the set of all $k = (k_1, \ldots, k_d) \in \mathbb{N}^d$ such that $\sum_{j=1}^{d} k_j e_j$ lies in $\Lambda_0$. Analogously define

$$\mathbb{N}_0^d(\Lambda_0).$$

For given $k \in \mathbb{N}_0^d$, the element $\sum_{j=1}^{d} k_j e_j$ is contained in a unique chamber $C(k) \subset \text{Cone}(C)$ such that $\text{Cone}(C(k)) \subset \text{Cone}(C)$ as in the picture.
Note that this construction depends on the apartment $a$ which has been fixed so far. For a given chamber $D$ in the building, we say that $D$ is in relative position $k$ to $C$, if there exists an apartment $a$ such that $D = C(k)$ in that apartment. We write this as

$$C \leadsto_k D.$$ 

We further write

$$C \leadsto D,$$

if there exists some $k$ with $C \leadsto_k D$.

### 6 Discrete groups

The automorphism group $G = \text{Aut}(X)$ of the building carries a natural topology, the compact-open topology. It is a totally disconnected locally-compact group. A subgroup $\Gamma \subset G$ is discrete if and only if for each chamber $C$ the stabilizer group

$$\Gamma_C = \{\gamma \in \Gamma : \gamma C = C\}$$

is finite. A discrete group $\Gamma$ is a lattice in $G$, i.e., there exists a finite $G$-invariant Radon measure on $G/\Gamma$ if and only if

$$\sum_{C \in \mathcal{C}} \frac{1}{|\Gamma_C|} < \infty,$$

where the sum runs over the set $\mathcal{C}$ of all labelled chambers and $\Gamma_C$ is the stabiliser in $\Gamma$ of $C$.

We fix a discrete group $\Gamma \subset \text{Aut}(X)$. The subgroup $\Gamma^{\text{lab}}$ of all $\gamma \in \Gamma$ which preserve a given labelling, is normal and of finite index in $\Gamma$. Replacing $\Gamma$ by $\Gamma^{\text{lab}}$ we will henceforth assume that $\Gamma$ preserves labellings.
Definition 6.1 Fix a discrete, label preserving subgroup $\Gamma \subset G$. A $\Gamma$-weight is a map

$$w : \mathcal{C} \times \mathcal{C} \to [0, \infty)$$

such that

- $w(\gamma C, D) = w(C, \gamma D) = w(C, D)$ for all $C, D \in \mathcal{C}$ and all $\gamma \in \Gamma$,
- $w(C, D) \neq 0 \Rightarrow C \leadsto \gamma D$ for some $\gamma \in \Gamma$,
- $\sum_{C, D \in \mathcal{C} \setminus \mathcal{C}} w(C, D) < \infty$,
- $w(C, D)w(D, E) = w(C, E)$ holds for all $C, D, E \in \mathcal{C}$ which satisfy $C \leadsto D \leadsto E$.

Definition 6.2 Let $w$ be a $\Gamma$-weight. For $k \in \mathbb{N}_0^d(\Lambda_0)$ we define an operator

$$T_k : \ell^\infty(\mathcal{C}) \to \ell^\infty(\mathcal{C})$$

by

$$T_k(C) = \sum_{C'} w(C, C')C',$$

where the sum runs over all chambers $C'$ in relative position $k$ to $C$. Note that for given $k \in \mathbb{N}_0^d$ in each apartment $a$ containing $C$ there is at most one $C'$ in position $k$, but the same $C'$ can lie in infinitely many apartments containing $C$. As we assume the building to be locally finite, the sum defining $T_k$ is actually finite.

Lemma 6.3 For $k, l \in \mathbb{N}_0^d(\Lambda_0)$ we have

$$T_kT_l = T_{k+l}.$$

In particular, the operators $T_k$ and $T_l$ commute.

Proof Fix a chamber $D$ in the double sum $T_k(T_l(C))$. By definition, there exists a uniquely determined chamber $C_l$ in relative position $l$ to $C$ such that $D$ occurs in the sum $T_k(C_l)$. We want to show that $D$ is in relative position $k+l$ to $C$. For this, consider an apartment $a$, containing both. All we need to show is, that $a$ also contains $C_k$. Let $b(C)$ denote the barycentric center of $C$ and $b(D)$ the one of $D$. Let $S$ be the intersection of all apartments which contain $C$ and $D$. Then $S$ is a compact convex subset of $a$, the boundary of which consists of finitely many hypersurfaces along which the building ramifies. Then $S$ contains

$$S_0 = (b(C) + \text{Cone}(C, a)) \cap (b(D) - \text{Cone}(D, a)),$$

since any hypersurface $H$, which meets the interior of $S_0$, has a translate passing through the special point $v_0$ and this translate also passes through the interior of $\text{Cone}(C, a)$, but that cannot happen by the minimality of the cone. We claim that the chamber $C_l$ also intersects $S_0$. If not, there must be a hypersurface parting $C_l$ from $S_0$
and then, either a translate of it in a would meet the interior of Cone \((C, a)\), or a translate in an apartment containing \(C_l\) would pass through the interior of the corresponding cone. In either case we get a contradiction. Therefore, \(C_l\) has a common point with \(S_0\), hence \(C_l\) lies in the apartment \(a\).

On the other hand, for any chamber \(D\) in relative position \(k + l\) to \(C\) there exist uniquely determined chambers \(C_k\) and \(C_l\) in relative positions \(k\) and \(l\) such that each apartment containing \(C\) and \(D\) also contains \(C_k\) and \(C_l\). This and the transitivity of \(w\) proves the claim. \(\square\)

**Example 6.4** The integral cone \(\mathbb{N}_0(\Lambda_0)\) has unique generators \(k^{(1)}, \ldots, k^{(d)}\). Let \(T_{(j)} = T_{k^{(j)}}\). Fix some \(u \in \mathbb{C}^d\) and set

\[
w(C, D) = u_1^{n_1} \ldots u_d^{n_d}\]

if \(\gamma D = T_{(1)}^{n_1} \cdots T_{(d)}^{n_d} C\) holds for some \(\gamma \in \Gamma\). This defines a \(\Gamma\)-weight, which is a direct generalisation of the graph case in Definition 3.3.

**Definition 6.5** A *quasicharacter* on \(\mathbb{N}^d(\Lambda_0)\) is a map \(\chi: \mathbb{N}^d(\Lambda_0) \rightarrow \mathbb{C}^\times\) with \(\chi(k + l) = \chi(k)\chi(l)\). For a given quasicharacter \(\chi\) let

\[
\ell^2(\mathbb{C})_{\chi}
\]

be the generalized eigenspace, i.e., the set of all \(v \in \ell^2(\mathbb{C})\) such that for every \(k \in \mathbb{N}^d(\Lambda_0)\) one has

\[
(T_k - \chi(k))^m v = 0
\]

for some \(m \in \mathbb{N}\). For every non-zero \(\chi\) the space \(\ell^2(\mathbb{C})_{\chi}\) is finite-dimensional. Let \(m(\chi) \in \mathbb{N}_0\) denote its dimension. We then get

\[
\text{tr}(T_k) = \sum_{\chi \in \Omega} m(\chi) \chi(k),
\]

where the sum runs over the set \(\Omega\) of all quasi-characters \(\chi\).

**Proposition 6.6** For \(k \in \mathbb{N}^d(\Lambda_0)\) let \(a_k \in \mathbb{C}\) be bounded, i.e., there exists \(M > 0\) such that \(|a_k| \leq M\) holds for all \(k\). Then the operator

\[
T = \sum_{0 \neq k \in \mathbb{N}^d(\Lambda_0)} a_k T_k
\]

is well defined and maps \(\ell^2(\Gamma \backslash \mathbb{C}) = \ell^2(\mathbb{C})^\Gamma\) to itself. On this Hilbert space, \(T\) is a trace class operator.
Proof  We compute
\[ \sum_{C \mod \Gamma} \|TC\| \leq \sum_{C,D \mod \Gamma} M \omega(C,D) < \infty. \]
This implies that \( T \) is trace class. \( \square \)

Definition 6.7  Let \( \mathcal{T} \) denote the unital subring of \( \text{End}(\ell^\infty(\mathcal{C})) \) generated by the translation operators \( T_k \) with \( k \in \mathbb{N}^d(\Lambda_0) \). This is a commutative integral domain. Let \( \mathcal{K} \) denote its quotient field.

For indeterminates \( u_1, \ldots, u_d \) we define the formal power series
\[ T(u) = \sum_{k \in \mathbb{N}^d(\Lambda_0)} u^k T_k \in \mathcal{T}[u_1, \ldots, u_d], \]
where \( u^k = u_1^{k_1} \cdots u_d^{k_d} \). Note that the summation only runs over the set \( \mathbb{N}^d(\Lambda_0) \) of all \( k \in \mathbb{N}^d \) such that \( \sum_{j=1}^d k_j e_j \in \Lambda_0 \).

Theorem 6.8  \( T(u) \) is a rational function in \( u \). More precisely, there exists a finite set \( E \subset \Lambda_0^+ \) and \( k(e) \in \mathbb{N}_0^d \) for every \( e \in E \) as well as \( k(1), \ldots, k(d) \in \mathbb{N}_0^d \setminus \{0\} \) such that
\[ T(u) = \frac{\sum_{e \in E} u^{k(e)} T_{k(e)}}{(1 - T_{k(1)} u^{k(1)}) \cdots (1 - T_{k(d)} u^{k(d)})}. \]
Proof  The case of trivial weight is [13, Theorem 3.1.4]. The proof given there extends without problems to the case of general weight. \( \square \)

Definition 6.9  We say that \( u \in \mathbb{C}^d \) is singular if there exists \( 1 \leq j \leq d \) such that \( u^{-k(j)} \) is an eigenvalue of \( T_k(j) \). Otherwise, \( u \) is regular. The singular set is a countable union of complex submanifolds of codimension 1 in \( \mathbb{C}^d \), so the regular set is connected, open and dense.

Proposition 6.10  The family \( u \mapsto T(u) \) is a meromorphic family of trace class operators on \( \mathbb{C}^d \). It is holomorphic on the regular set. The map
\[ Z(u) = \text{tr} T(u) \]
is meromorphic on \( \mathbb{C}^d \) and holomorphic on the regular set. We have
\[ Z(u) = \sum_{\chi \in \mathcal{Q}} m(\chi) \frac{\sum_{e \in E} u^{k(e)} \chi(k(e))}{(1 - u^{k(1)} \chi(1)) \cdots (1 - u^{k(d)} \chi(d))}. \]
Proof  This is clear from the theorem and the fact that all \( T_k \) are trace class. \( \square \)
7 The prime geodesic theorem for a weighted building quotient

Let $\mathbb{D}^\times = \{z \in \mathbb{C} : 0 < |z| < 1\}$.

**Theorem 7.1** (Prime geodesic theorem) For $k \in \mathbb{N}^d(\Lambda_0)$ let

$$N(k) = \sum_{C \mod \Gamma} w(C, \gamma C).$$

There are $z_1, \ldots, z_N \in \mathbb{C}^d$ such that

$$N(k) \sim \sum_{j=1}^N z_j^k,$$

as $k_j \to \infty$ independently. Moreover, there exists a sequence $z_j \in (\mathbb{C}^\times)^d$ with $\lim_j z_j = 0$ such that for every $k \in \mathbb{N}^d(\Lambda_0)$ we have, with absolute convergence of the sum,

$$N(k) = \sum_{j=1}^\infty z_j^k.$$

**Proof** We show the last assertion first. By definition we get $N(k) = \text{tr}(T_k)$ and so

$$N(k) = \sum_{\chi \in \Omega} m(\chi) \chi(k).$$

As $\chi \in \Omega$ is a quasi-character, there exists $z_\chi \in (\mathbb{C}^\times)^d$ with

$$\chi(k) = z_\chi^k.$$

Now let $(z_j)$ be the sequence that runs through all $z_\chi$ with $m(\chi) \neq 0$, where each $z_\chi$ is repeated with multiplicity $m(\chi)$. The claim follows. The first claim follows from the fact that $z_j \to 0$ in $\mathbb{C}^d$. $\square$

Writing $z_j = (z_{j,1}, \ldots, z_{j,d}) \in (\mathbb{C}^\times)^d$ we write the statement of the theorem as

$$N(k_1, k_2, \ldots, k_d) = \sum_{j=1}^\infty z_{j,1}^{k_1} \cdots z_{j,d}^{k_d}.$$

**Funding** Open Access funding enabled and organized by Projekt DEAL.
Open Access  This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

References

1. Abramenko, P., Brown, K.S.: Buildings. Graduate Texts in Mathematics, vol. 248. Springer, New York (2008)
2. Abramovich, Y.A., Aliprantis, C.D.: An Invitation to Operator Theory. Graduate Studies in Mathematics, vol. 50. American Mathematical Society, Providence (2002)
3. Ballmann, W., Brin, M.: Orbihedra of nonpositive curvature. Inst. Hautes Études Sci. Publ. Math. 82, 169–209 (1995)
4. Bass, H.: The Ihara–Selberg zeta function of a tree lattice. Internat. J. Math. 3(6), 717–797 (1992)
5. Bass, H., Lubotzky, A.: Tree Lattices. Progress in Mathematics, vol. 176. With Appendices by H. Bass, L. Carbone, A. Lubotzky, G. Rosenberg and J. Tits. Birkhäuser, Boston (2001)
6. Bruhat, F., Tits, J.: Groupes réductifs sur un corps local. Inst. Hautes Études Sci. Publ. Math. 41, 5–251 (1972)
7. Deitmar, A.: Class numbers of orders in cubic fields. J. Number Theory 95(2), 150–166 (2002)
8. Deitmar, A.: A prime geodesic theorem for higher rank spaces. Geom. Funct. Anal. 14(6), 1238–1266 (2004)
9. Deitmar, A.: Ihara zeta functions and class numbers. Adv. Stud. Contemp. Math. 24(4), 439–450 (2014)
10. Deitmar, A.: Ihara zeta functions of infinite weighted graphs. SIAM J. Discrete Math. 29(4), 2100–2116 (2015)
11. Deitmar, A., Hoffmann, W.: Asymptotics of class numbers. Invent. Math. 160(3), 647–675 (2005)
12. Deitmar, A., Kang, M.-H.: Tree-lattice zeta functions and class numbers. Michigan Math. J. 67(3), 617–645 (2018)
13. Deitmar, A., Kang, M.-H., McCallum, R.: Building lattices and zeta functions. Adv. Geom. 20(2), 249–272 (2020)
14. Deitmar, A., McCallum, R.: A prime geodesic theorem for higher rank buildings. Kodai Math. J. 41(2), 440–455 (2018)
15. Gantmacher, F.R.: The Theory of Matrices, vol. 1,2. Chelsea Publishing, New York (1959)
16. Garrett, P.: Buildings and Classical Groups. Chapman & Hall, London (1997)
17. Hashimoto, K.: Artin type L-functions and the density theorem for prime cycles on finite graphs. Internat. J. Math. 3(6), 809–826 (1992)
18. Hejhal, D.A.: The Selberg Trace Formula for PSL(2, R). Vol. I. Lecture Notes in Mathematics, vol. 548. Springer, Berlin (1976)
19. Huber, H.: Zur analytischen theorie hyperbolischen Raumformen und Bewegungsgruppen. Math. Ann. 138, 1–26 (1959)
20. Iwaniec, H.: Prime geodesic theorem. J. Reine Angew. Math. 349, 136–159 (1984)
21. Katsuda, A., Sunada, T.: Homology and closed geodesics in a compact Riemann surface. Amer. J. Math. 101(1), 145–155 (1988)
22. Koyama, S.: Prime geodesic theorem for arithmetic compact surfaces. Int. Math. Res. Notices 1998(8), 383–388 (1998)
23. Margulis, G.A.: On Some Aspects of the Theory of Anosov Systems. Springer Monographs in Mathematics, Springer, Berlin (2004)
24. Sarnak, P.: Class numbers of indefinite binary quadratic forms. J. Number Theory 15(2), 229–247 (1982)
25. Schaefer, H.H.: Banach Lattices and Positive Operators. Die Grundlehren der Mathematischen Wissenschaften, vol. 215. Springer, New York (1974)
26. Soundararajan, K., Young, M.P.: The prime geodesic theorem. J. Reine Angew. Math. 676, 105–120 (2013)

**Publisher's Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.