Abstract

This paper deals with the estimation of the hidden factor in Dynamic Generalized Factor Analysis via a generalization of Kalman filtering. Asymptotic consistency is discussed and it is shown that the Kalman one-step predictor is not the right tool while the pure filter yields a consistent estimate.

Key words: Generalized Factor Analysis, Factor estimation, Kalman filtering, Kalman predictor, Dynamic Factor models.

1 Introduction

Factor Analysis has a long history; it has apparently first been introduced by psychologists and successively it has been studied and applied in various branches of Statistics and Econometrics: Spearman (1904); Burt (1909); Ledermann (1937, 1939); Bekker & de Leeuw (1987); Lawley & Maxwell (1971). Several dynamic versions of these models have also been introduced in the econometric literature already a long time ago, see e.g. Geweke (1977); Picci & Pinzoni (1986); Peña & Box (1987); Peña & Poncelpa (2006) and references therein.

With a few exceptions, see e.g. Ciccone et al. (2018, 2020), little attention has been paid to Factor Analysis models in the system and control engineering community until recently, when we have been witnessing a revival of interest due to certain generalizations which bypass the problem of intrinsic unidentifiability of these models by relaxing a crucial independence condition on the noise variables. These models, called Generalized Factor Analysis (GFA) models, although initially motivated by econometric applications, seem to have a potential to be quite useful also in engineering applications due to their ability of modeling time series data of large cross-sectional dimension. The static generalization was first rigorously discussed by Chamberlain (1983); Chamberlain & Rothschild (1983). Later, Forni, Lippi and collaborators introduced Dynamic Generalized Factor Analysis (DFGA) models and their estimation in a series of widely quoted papers: Forni et al. (2000a); Forni & Lippi (2001); Forni et al. (2004, 2005). Although Generalized Factor Analysis models have been primarily of interest to econometricians, this modeling paradigm has stimulated interest in the System Identification community, see Anderson & Deistler (2008); Anderson et al. (2018); Deistler & Zimmer (2007); Deistler et al. (2010); Bottegal & Picci (2013). Indeed, modern engineering applications are characterized by interconnected systems with hundreds or thousands of variables which are mainly “driven” by a small number of hidden factors. In plain words, generalized factor models can describe a large amount of sensors measuring quantities which provide information on some (few) variables of interest. An example of urban pollution monitoring is illustrated at the end of the paper where the hidden factor vector describes the average concentration of few pollutants over time in a certain city while the observed variables model the measurements taken from the sensors spread in the city. Other example from the engineering literature may be found in Bottegal & Picci (2013, p. 760-761).
As this paper is mainly addressed to the System and Control community, we shall not discuss Econometric applications which are abundantly referred to in the quoted literature. Our target is the growing interest of Control Engineering researchers in large or complex networked systems acting in the presence of uncertainty and disturbances. Modeling and identification of these systems often brings to models with such a high dimension to involve more parameters than available data. Factor Analysis is believed to provide an answer to this issue as it concentrates the explication of high dimensional data or observables in a (hopefully) small-dimensional vector of latent factors. State space analysis of DGFA models seems to offer the ideal approach to this end. Indeed, it provides natural tools to estimate the hidden factor variable of a DGFA model which is the important problem addressed in this paper. Classical state space estimation techniques originated with the work of R.E. Kalman seem to be the perfect device to tackle the problem. Some other works in recent literature address factor estimation by state-space techniques, see e.g. Kapetanios & Marcellino (2009), Marcellino (2017), Doz et al. (2011), Doz et al. (2011) and Giannone et al. (2008). We refer to Stock & Watson (2011, 2015) for a survey mentioning some of the themes treated in these papers, especially the important problem of dynamic factor estimation which will be the main concern of this paper. For simplicity, we make the same stationarity assumptions as in these papers, although the main arguments of our paper can be shown to hold also if we replace stationarity by some form of weak time dependence. The main difference in the setup of this paper with respect to Doz et al. (2011) is that we discuss the Kalman filter and the one-step ahead predictor for DGFA models instead of the fixed-interval smoother. The essential ideas behind the motivation and structure of Generalized Factor Analysis are shortly reviewed in the next two sections.

1.1 Contributions of the paper

We consider a class of state-space DGFA models and analyze the capability of Kalman estimators (filter and predictor) to provide a perfect asymptotic estimate namely estimates for which the covariance of the estimation error tends to zero when the number of observed outputs diverges. The main contributions of this paper are twofold:

(A) We show that the one-step ahead Kalman predictor does not provide a perfect asymptotic estimate of the hidden factors. As a consequence, we show that a generalized dynamic Factor Analysis model cannot be a predictor-based innovation model.

(B) We prove that the pure filter, on the contrary, provides a perfect asymptotic estimate of the latent state variable. Moreover, under reasonable assumptions, the estimation error converges weakly to the idiosyncratic noise generating the data so that generalized dynamic Factor Analysis models are weakly equivalent to a pure-filter type innovation models.

Besides this two main contributions we derive some ancillary results on static Factor Analysis models that are interesting, per se. Notice that, by combining our two main results, we come to a significant warning concerning the application of standard subspace identification methods to the identification of G DFA models. In fact, subspace methods are normally based on the one-step ahead Kalman predictor model which we show cannot produce a generative model exhibiting G DFA features. On the other hand, considering the pure filter estimate in place of the predictor can overcome this problem.

Notations: In this paper boldface symbols will normally denote random variables or random arrays, either finite or infinite. All random variables will be real, zero-mean and with finite variance. The symbol $H(v)$ denotes the standard Hilbert space of random variables linearly generated by the scalar components \{v_1 , \ldots , v_n , \ldots \} of a (possibly infinite) family of random variables which we denote by v. For $\xi, \eta \in H(v)$, the inner product is the mathematical expectation $\langle \xi, \eta \rangle := \mathbb{E} [\xi \eta]$ which induces the (variance) norm of random variables by setting $\|\xi\|^2 = \mathbb{E} [\xi^2]$. Convergence of random sequences will always be understood with respect to this norm. Finally, \text{diag}(a_1 , \ldots , a_n ) denotes the diagonal matrix whose elements in the main diagonal are $a_1 , \ldots , a_n$.

This paper is concerned with model-based estimation, in particular, with the problem of the model-state estimation by a Kalman one-step ahead predictor, or by a Kalman filter. We highlight this point because in the econometric literature the term “estimation” normally refers to a parameter estimation procedure for estimating the model and consequently the involved time-variables on the sole basis of the observed data.

2 Review of static Factor Analysis models

A classical (static) Factor Analysis model is a representation of the form

$$y = Fx + e,$$
of $N$ observable random variables $\mathbf{y} = [\mathbf{y}(1) \ldots \mathbf{y}(N)]^\top$, as linear combinations of $q$ random common factors $\mathbf{x} = [\mathbf{x}_1 \ldots \mathbf{x}_q]^\top$, plus uncorrelated “noise” or “error” terms $\mathbf{e} = [\mathbf{e}(1) \ldots \mathbf{e}(N)]^\top$. The columns $\{f_1, f_2, \ldots, f_q\}$ of matrix $F$, called the factor loadings, can be chosen to be linearly independent. Moreover, the common factors can be normalized in such a way that $\mathbb{E}[\mathbf{xx}^\top] = \mathbf{I}$. An essential part of the model specification is that the $N$ components of the error $\mathbf{e}$ should be (zero-mean and) mutually uncorrelated random variables, i.e.

$$
\mathbb{E}[\mathbf{xe}^\top] = 0, \quad \mathbb{E}[\mathbf{ee}^\top] = \text{diag}\{\sigma_1^2, \ldots, \sigma_N^2\}.
$$

The aim of these models is to provide an “explanation” of the mutual correlations of the observable variables $\mathbf{y}(i)$ in terms of a small number $q$ of common factors, in the sense that, setting: $\hat{\mathbf{y}}(k) := \sum f_i(k)\mathbf{x}_i$, where $f_i(k)$ is the $k$-th component of $f_i$, one has exactly $\mathbb{E}[\mathbf{y}(i)\mathbf{y}(j)] = \mathbb{E}[\hat{\mathbf{y}}(i)\hat{\mathbf{y}}(j)]$, for all $i \neq j$. Note that a Factor Analysis representation then induces a decomposition of the covariance matrix $\Sigma$ of $\mathbf{y}$ as

$$
\Sigma = FF^\top + \text{diag}\{\sigma_1^2, \ldots, \sigma_N^2\}
$$

which can be seen as a special kind of low rank plus sparse decomposition of a covariance matrix, see Chandrasekaran et al. (2012), a diagonal covariance matrix being as sparse as one could possibly ask for.

Although providing a quite natural and useful data compression scheme, factor models in many circumstances suffer from a serious non-uniqueness problem coming from the fact that, even for a fixed dimension $q$ there are in general many (possibly infinitely many) statistically non-equivalent Factor Analysis models describing the same family of observables $\mathbf{y}(1), \ldots, \mathbf{y}(N)$. In addition, determining the minimal integer $q$ for which a decomposition as in (1) holds for a given symmetric positive definite matrix $\Sigma$ has been an open problem since the beginning of the last century. Moreover, there are in general many minimal Factor Analysis models (say with $F$’s of the same rank $q$ and normalized factors) representing a fixed $N$-tuple of random variables $\mathbf{y}$. This inherent nonuniqueness of Factor Analysis models has been called “factor indeterminacy” and corresponds to unidentifiability in the systems and control language. The overlooking of this unidentifiability issue and the acritical usage of Factor Analysis models have been vehemently criticized by Kalman in a series of papers, see e.g. Kalman (1983)[4].

One may try to get uniqueness by giving up or mitigating the requirement of uncorrelatedness of the components of $\mathbf{e}$. Obviously this tends to make the problem ill-defined as the basic goal of uniquely splitting the external signal into a noiseless component plus “additive noise” is made vacuous, unless some extra assumptions are made on the model and on the very notion of “noise”. Quite surprisingly, for models describing an infinite number of observables a meaningful weakening of the property of uncorrelation of the components of $\mathbf{e}$ can be introduced, so as to guarantee the uniqueness of the decomposition.

### 3 Generalized Factor Analysis models

The covariance matrix of an infinite-dimensional zero mean vector $\mathbf{y} = [\mathbf{y}(k)]_{k=1,2,\ldots}$ is formally written as $\Sigma := \mathbb{E}\mathbf{yy}^\top$. We let $\Sigma_N$ indicate the top-left $N \times N$ block of $\Sigma$, equal to the covariance matrix of the subvector made of the first $N$ components of $\mathbf{y}$ denoted $\mathbf{y}_N$. The inequality $\Sigma > 0$ means that all submatrices $\Sigma_N$ of $\Sigma$ are positive definite, which we shall always assume in the following.

Let $\ell^2(\Sigma)$ denote the Hilbert space of infinite sequences $a := \{a(k), k \in \mathbb{N}\}$ such that $\|a\|^2_\Sigma := a^\top \Sigma a < \infty$. When $\Sigma = \mathbf{I}$, we simply use the symbol $\ell^2$ and denote the corresponding norm with the symbol $\| \cdot \|$.

A side question discussed in the appendix is the relation between $\ell^2$ and $\ell^2(\Sigma)$: indeed, for $N$ finite it is obvious that $\ell^2 = \ell^2(\Sigma)$ since $\lambda_{\max}\mathbf{I}_N > \Sigma > \lambda_{\min}\mathbf{I}_N$, $\lambda_{\max}$ and $\lambda_{\min}$ being the maximum and minimum eigenvalue of $\Sigma$, respectively.

**Definition 1 (Forni & Lippi (2001))** Let $a := \{a_n, n \in \mathbb{N}\}$ be a sequence of elements of the space $\ell^2 \cap \ell^2(\Sigma)$. We say that $a = \{a_n, n \in \mathbb{N}\}$ is an averaging sequence (AS) if

$$
\lim_{n \to \infty} \|a_n\| = 0.
$$
Example 1 The sequence of elements in $\ell^2$

$$a_n = \frac{1}{n} \left[ 1 \cdots 1 0 \ldots \right]^	op$$

is an averaging sequence.

An AS can be seen just as a sequence of linear functionals in $\ell^2 \cap \ell^2(\Sigma)$ converging strongly to zero. The definition is instrumental to the concept of idiosyncratic sequence of random variables which will be introduced next.

Definition 2 (Forni & Lippi (2001)) We say that the random sequence $y$ is idiosyncratic if for any averaging sequence $\mathbf{a} = \{a_n \in \ell^2 \cap \ell^2(\Sigma)\}$. The limit is understood in mean square.

Remark 1 In econometrics, “idiosyncratic error” is used to describe unobserved factors that impact the dependent variable. The mathematical definition of idiosyncratic disturbance was introduced in the static GFA case by Chamberlain (1983), and Chamberlain & Rothschild (1983). Since then, models with infinite cross-sectional size and idiosyncratic errors have been analyzed by many authors. Beside Forni & Lippi (2001), that we take as our reference for the definition of idiosyncratic sequence, we mention Forni & Reichlin (1996a), Stock & Watson (1998), Forni & Reichlin (1996b), Forni et al. (2000b), Forni et al. (2004), and Forni et al. (2005). We also mention that in Forni & Lippi (2001), in place of averaging sequences of elements in $\ell^2$, a more general concept of dynamic averaging sequences is introduced. This concept seems to be useful only when the observable $y$ is described in the spectral domain by spectral densities in place of covariances.

Example 2 A zero-mean sequence whose variance is a bounded operator in $\ell^2$ is idiosyncratic. In fact let the operator norm $\|\Sigma\|$ be bounded by $\alpha > 0$. Then $\Sigma \leq \alpha I$, where $I$ is the identity operator, so that,

$$\mathbb{E}[(a_n \top y)^2] = a_n \top \Sigma a_n \leq \alpha \|a_n\|^2 \to 0$$

for any sequence $\{a_n\}$ tending to zero in norm.

The characterization actually goes both ways:

Proposition 1 A zero-mean sequence of random variables is idiosyncratic if and only if its variance matrix is a bounded operator in $\ell^2$.

It follows, as remarked in Forni & Lippi (2001), that a sequence $y$ is idiosyncratic if and only if there exists $M$ such that $\forall N > 0$, $\|\Sigma_N\| < M$, where $\Sigma_N$ is the covariance of the vector $y_N$ obtained by selecting the first $N$ components of $y$ (see Bottegal & Picci (2013) for more details). In particular, uncorrelated (white) sequences of zero-mean random variables having a uniformly bounded variance are idiosyncratic.

Consider a zero-mean finite variance stochastic process $y := \{y(k), k \in \mathbb{Z}_+\}$ represented as a random column vector with an infinite number of components.

Definition 3 A Generalized Factor Analysis Model (GFA) of the process $y$ is a representation by a finite linear combination of unobserved random variables plus idiosyncratic noise, of the form

$$y(k) = \sum_{i=1}^q f_i(k)x_i + \tilde{y}(k), \quad k = 1, 2, \ldots$$

where the random variables $x_i, i = 1, \ldots, q,$ are called the common factors and the deterministic infinite-dimensional real vectors $f_i$, called the factor loadings, are strongly linearly independent (the definition being reviewed in the appendix). The components $\tilde{y}(k)$'s of the idiosyncratic noise are zero mean random variables orthogonal to $x$.

We order the linear combinations $\tilde{y}(k) := \sum f_i(k)x_i$, with $k = 1, 2, \ldots$, into an infinite random vector $\tilde{y}$ and likewise for the noise terms $\tilde{y}(k)$ so that (2) can for short be written $y = \tilde{y} + \tilde{y}$ where the hidden or latent component $\tilde{y}$ is a linear function of $x$ e.g. $\tilde{y} = Fx$. Since the columns of $F$ are linearly independent, $H(\tilde{y}) = H(x)$ and the common factors are just obtained by picking an orthonormal basis in $H(\tilde{y})$. 


The key characteristics which qualifies the idiosyncratic noise process \( \hat{y} \) is that by a deterministic linear averaging operation on the components of the observation vector \( y \), one can (asymptotically) eliminate the additive noise and reveal the common factors. The question is how to construct a suitable AS from the model specifications to achieve this goal. This will be one of the the main themes of the next sections.

### 3.1 Application and interpretation of GFA models

We now suggest an interpretation of GFA models in the framework of applications to ensembles of a large number of agents distributed in space and interacting in a random fashion. We consider two distinct mechanisms:

(A) **Short distance interaction.** The idiosyncratic covariances \( \{\hat{\sigma}(k, j)\} \) describe the mutual influence of neighbouring units having coordinates \( \hat{y}(k), \hat{y}(j) \). Let \( \hat{\Sigma} \) denote the covariance matrix of \( \hat{y} \). Since \( \hat{\Sigma} \) is a bounded operator in \( \ell^2 \), it is a known fact (Akhiezer & Glazman, 1961, Section 26) that \( \hat{\sigma}(k, j) \rightarrow 0 \) as \( |k - j| \rightarrow \infty \)

so in a sense the idiosyncratic component \( \hat{y} \) models only short range interactions among the agents, which are decaying with distance. Agents which are far away from each other are not affected by mutual influence.

(B) **Factor loadings and long range influence.** Since \( \mathbb{E}[y(k)y(j)] = \sum_{i=1}^{n} f_i(k)f_i(j) \) and the elements of the column vectors \( f_i \in \mathbb{R}^\infty \) do not decay with distance, the products \( f_i(k)f_i(j) \) do not vanish when \( |k - j| \rightarrow \infty \).

Hence the factor loadings describe “long range” correlation between the factor components and the \( \hat{y} \) component of \( y \) can be interpreted as variables produced by long range interaction among agents.

In dynamic GFA models the components of the infinite vector process \( \hat{y}(t) \) move in time like a rigidly connected set of points in space. For this reason \( \{\hat{y}(t)\} \) is called the flocking component of the process \( \{y(t)\} \) (Bottegal & Picci, 2015).

### 4 Dynamic State-Space DGFA models

Let us introduce the time variable \( t \in \mathbb{Z} \) and denote now by \( y := \{y(t), t \in \mathbb{Z}\} \) a zero-mean, stationary, vector process of infinite cross-sectional dimension so that at each time \( t \) the random vector \( y(t) \) has countably infinite random components. We consider the following finite-dimensional\(^1\) dynamic model where \( y(t) \) depends on a vector of \( n \) common factors, \( x \), evolving according to a linear dynamics of the form \(^2\)

\[
\begin{align*}
\begin{cases}
x(t+1) = Ax(t) + v(t) \\
y(t) = Cx(t) + w(t).
\end{cases}
\end{align*}
\]

(3)

The study of GDFA’s (3) will be undertaken by considering sequences of truncated models of increasing cross-sectional dimension \( N \), each describing the subvector \( y_N \) made of the first \( N \) components of the original output vector \( y \). More precisely, we consider a class of truncated models of the form

\[
\begin{align*}
\begin{cases}
x(t+1) = Ax(t) + v(t) \\
y_N(t) = C_Nx(t) + w_N(t),
\end{cases}
\end{align*}
\]

(4)

where the state dimension \( n \) of the model (3) is fixed (and therefore does not grow with the output dimension \( N \)). Each output matrix \( C_N \in \mathbb{R}^{N \times n} \) is the top submatrix of \( C \) of dimension \( N \times n \) so that \( C_N \) has the nested structure

\[
C_{N+k} = \begin{bmatrix} C_N^\top & C_{N+1}^\top \\ \end{bmatrix}^\top ,
\]

\(^1\) We warn the reader that in the dynamic setting the notation \( y(t) \) denotes an infinite random vector and must not be confused with the notation of the static setting where \( y(k) \) denotes a scalar random variable.

\(^2\) Of course, more general infinite-dimensional DGFA models are possible that are not included in our systems’ class.

\(^3\) For consistency with standard system-engineering notations (see e.g. Lindquist & Picci, 2015), we have denoted the factor loading (also called output or observation) matrix by \( C \) instead of \( F \) as in the Factor Analysis literature.
and the noise vectors \( w_N(t) \) have a similar nested structure. Notice that, by these assumptions each \( y_N(t) \) is a \( N \)-vector stationary process.

We assume that

(A) The \( n \)-dimensional latent factor \( x \equiv \{ x(t); t \in \mathbb{Z} \} \) follows a stationary Markov evolution described by the first of equation in (3), where \( A \in \mathbb{R}^{n \times n} \) is an asymptotically stable matrix (all its eigenvalues have modulus strictly less than one) and \( v(t) \) a white noise process of dimension \( n \) whose covariance is denoted by \( Q \). Hence the steady-state variance of \( x(t) \) is the unique solution of the Stein (discrete-time Lyapunov) equation \( \Sigma = A \Sigma A^\top + Q \).

(B) The infinite-dimensional white noise vector \( \{ w(t) \} \) is not assumed to have uncorrelated components as in standard Factor Analysis models and (without loss of generality) is assumed to be uncorrelated with \( \{ v(t) \} \) at all times.

The following assumptions regarding the asymptotic behaviour for \( N \to \infty \) of the sequence of models (4) will be made:

(C) \( C \) is an \( \infty \times n \) matrix with strongly linearly independent columns. This means that

\[
\lim_{N \to \infty} \lambda_{\min}[C_N^\top C_N] = +\infty,
\]

where \( \lambda_{\min}[\cdot] \) denotes the smallest eigenvalue, see the appendix for a discussion and for more details on this.

(D) The noise \( w(0) \) is an idiosyncratic sequence (with respect to the cross sectional dimension).

These conditions can easily be shown to be equivalent to conditions (7) and (8) in Stock & Watson (2011). Since \( w_N(t) \) is a stationary white noise process (in \( t \)) for all \( N \), it is easy to check that Assumption D implies that for any given \( t \), \( w(t) \) is also idiosyncratic.

To avoid technicalities, we also assume that

(E) The model is wide-sense stationary and minimal so that the pair \((A, Q)\), where \( Q \) is the covariance of the white noise \( v \), is reachable.

(F) The covariance \( R_N \) of the output noise \( w_N(t) \) is positive definite, i.e. \( R_N > 0, \forall N \).

As observed by Deistler et al. (2010), minimality of the model (3) does not guarantee that the state is a minimal static factor of the observed process.

5 The Static case: latent variable estimation

Are these models realistic and well posed and if so, how can we use them to compute estimates of the latent vector \( x \) based on observations of \( y \) (and knowledge of the parameters)? In order to get a hint on how to answer this basic question we shall look at the following simple but illuminating example.

Example 3 (Estimation by averaging) Let \( \mathbb{I} \) be an infinite column vector of 1’s and let \( x \) be a scalar random variable uncorrelated with \( w \), an infinite-dimensional vector whose components form a zero-mean weakly stationary sequence having finite variance. Consider the static GFA model

\[
y = \mathbb{I}x + w.
\]

It is easy to show that \( \lim_{n \to \infty} a_n^\top w = 0 \) for any averaging sequence so that \( w \) is idiosyncratic. In particular, for the AS of Example 1, we have \( L^2 - \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n y(k) = x \) and we can recover the latent factor by averaging. \( \square \)

In this example we can asymptotically estimate without error the factor by averaging. This method exploits the two basic properties of GFA models, in particular idiosyncrasy of the noise, by passing to the limit in \( N \). But how general is this procedure? Below we shall consider a general static model and estimate \( x \) by the standard linear Bayes rule. The idea is to show that Bayes rule provides indeed a way to construct averaging sequences.

Consider the sequence of static finite-dimensional linear models

\[
y_N = C_N x + w_N \quad N = 1, 2, 3, \ldots
\]
where \( \mathbf{x} \) is a fixed random vector (not depending on \( N \)) and \( \mathbf{w}_N \) are zero-mean random vectors uncorrelated with \( \mathbf{x} \). Without loss of generality the variance matrix of \( \mathbf{x} \) is normalized to the identity. The question is how to estimate (or reconstruct) the hidden variable \( \mathbf{x} \) in the representation (6) based on an infinite string of observations.

Let \( \hat{\Sigma}_N := \mathbb{E}[\mathbf{w}_N \mathbf{w}_N^\top] \) so that,

\[
\Sigma_N = C_N C_N^\top + \hat{\Sigma}_N
\]

**Proposition 2** Consider a class of truncated models of the form (6), where the sequences \( C_N \) and \( \hat{\Sigma}_N \) are given, and assume that (5) holds and that \( \mathbf{w}_N \) converges to an idiosyncratic process. Then, for \( N \to \infty \), the optimal Bayes estimate \( \hat{\mathbf{x}}_N := \hat{\mathbb{E}}[\mathbf{x} \mid \mathbf{y}_N] \) converges in mean square to the actual vector \( \mathbf{x} \) which can therefore be reconstructed with arbitrary precision provided that \( N \) is sufficiently large.

**Proof.** The classical orthogonal projection formula yields

\[
\hat{\mathbf{x}}_N := \hat{\mathbb{E}}[\mathbf{x} \mid \mathbf{y}_N] = (I + C_N^\top \hat{\Sigma}_N^{-1} C_N)^{-1} C_N^\top \hat{\Sigma}_N^{-1} \mathbf{y}_N
\]

The covariance of the estimation error \( \mathbf{e}_N := \mathbf{x} - \hat{\mathbf{x}}_N \) is given by

\[
\mathbb{E}[\mathbf{e}_N \mathbf{e}_N^\top] = (I + C_N^\top \hat{\Sigma}_N^{-1} C_N)^{-1}
\]

because the estimation error \( \mathbf{e}_N \) and the estimate \( \hat{\mathbf{x}}_N \) are orthogonal. Since \( \Sigma_N \leq \alpha I_N \), we have \( \hat{\Sigma}_N^{-1} \geq \alpha^{-1} I_N \), so that

\[
\mathbb{E}[\mathbf{e}_N \mathbf{e}_N^\top] \leq (I + \alpha^{-1} C_N^\top C_N)^{-1} \leq \alpha (C_N^\top C_N)^{-1}
\]

and since the columns of \( C_N \) are strongly linearly independent, Equation (5) holds so that

\[
\lim_{N \to \infty} \mathbb{E}[\mathbf{e}_N \mathbf{e}_N^\top] = 0.
\]

In other words, the estimation error of the hidden variables converges to zero in mean square. \( \Box \)

Notice that for simplicity, we have assumed that the sequence \( \hat{\Sigma}_N \) is known but, with an argument similar to that in [Doz et al. 2011] we may show that the same result holds also when this is not the case.

Since the finite-data innovation (i.e., the estimation error \( \mathbf{e}_N := \mathbf{y}_N - C_N \hat{\mathbf{x}}_N \)) is orthogonal to \( \hat{\mathbf{x}}_N \), one also has the orthogonal (innovation) representation

\[
\mathbf{y}_N = C_N \hat{\mathbf{x}}_N + \mathbf{e}_N
\]

and, since under the stated assumptions on the model (6), \( \hat{\mathbf{x}}_N \to \mathbf{x} \) as \( N \to \infty \), we have the following result.

**Proposition 3** Under the assumptions of Proposition 2 and assuming also that \( \hat{\Sigma}_N \) is uniformly coercive\(^4\) and that \( C_N \) is uniformly bounded,\(^5\) any model (6) tends, for \( N \to \infty \), to become an innovation model. In particular, each entry of the finite-data innovation \( \mathbf{e}_N \) tends, in mean-square sense, to the corresponding entry of \( \mathbf{w}_N \) and the limit innovation process tends to be idiosyncratic.

**Proof.** Let \( \delta_N := \mathbf{e}_N - \mathbf{w}_N = C_N \mathbf{e}_N \). We have

\[
\mathbb{E}[\delta_N \delta_N^\top] = C_N (I + C_N^\top \hat{\Sigma}_N^{-1} C_N)^{-1} C_N^\top \leq \alpha C_N (C_N^\top C_N)^{-1} C_N^\top
\]

\(^4\) This means that there exists \( c > 0 \) independent of \( N \) such that \( \hat{\Sigma}_N \geq c I_N \). This assumption is only made to avoid technical issues and might be weakened.

\(^5\) This means that \( \exists \beta \) independent of \( N \) such that \( \text{max}_{i,j} |[C_N]_{ij}| \leq \beta \), where \( [C_N]_{ij} \) is the entry in row \( i \) and column \( j \) of \( C_N \).
where the last inequality follows from the same argument that led to \((8)\). Let \(i \in \{1, 2, \ldots, N\}\) and consider the \(i\)-th component \([\delta_N]_i\) of \(\delta_N\). Its covariance \(\mathbb{E} \left[ (\delta_N[i])^2 \right]\) satisfies

\[
\mathbb{E} \left[ (\delta_N[i])^2 \right] \leq \frac{\alpha}{\sigma_{\min}(C_N^\top C_N)} [C_N]_i [C_N]^\top,[N][[N]
\]

where \([C_N]_i\) is the \(i\)-th row of \(C_N\). Since by assumption \(C_N\) is uniformly bounded, \([C_N]_i[C_N]^\top,[i]\) is also uniformly bounded so that there exists \(\gamma\) independent of \(N\) such that

\[
\max_i \mathbb{E} \left[ (\delta_N[i])^2 \right] \leq \frac{\alpha\gamma}{\sigma_{\min}(C_N^\top C_N)} \rightarrow 0
\]

because \(\sigma_{\min}[C_N^\top C_N]\) diverges in view of the strongly linear independence assumption. To see that \(\delta_N\) tends to be idiosyncratic, notice that \(\epsilon_N = \delta_N + \omega_N\) so that

\[
\mathbb{E} \left[ \epsilon_N \epsilon_N^\top \right] \leq 2\Sigma_N + 2\mathbb{E} [\delta_N \delta_N^\top] \\
\leq 2\alpha(I_N + C_N(C_N^\top C_N)^{-1} C_N^\top) \\
\leq 4\alpha I_N
\]

which shows that the covariance \(\mathbb{E} \left[ \epsilon_N \epsilon_N^\top \right]\) converges monotonically to a bounded operator in \(\ell^2\).

**Remarks**

1. The mean-square entry-wise convergence of \(\epsilon_N\) to \(\omega_N\) is a weak form of convergence that may be interpreted as convergence in a mixed mean-square and infinity-type norm. It is natural to ask whether a stronger convergence holds in which the entire vector \(\epsilon_N\) converges to \(\omega_N\) or, equivalently, whether \(\delta_N\) does converge to zero in mean square. This is indeed not the case. In fact, for \(N\) sufficiently large,

\[
I + C_N^\top C_N^{-1} C_N \leq I + c^{-1} C_N^\top C_N \leq 2c^{-1} C_N^\top C_N,
\]

where the last inequality depends, in view of \((5)\), on the fact that the smallest singular value of \(C_N^\top C_N\) diverges. Then,

\[
\alpha C_N(C_N^\top C_N)^{-1} C_N^\top \geq \mathbb{E} [\delta_N \delta_N^\top] \geq \frac{\alpha}{2} C_N(C_N^\top C_N)^{-1} C_N^\top.
\]

Finally notice that \(C_N(C_N^\top C_N)^{-1} C_N^\top\) is the matrix orthogonally projecting on the image of \(C_N\), so that it has rank equal to \(n\) and \(n\) eigenvalues equal to 1. In conclusion, for any (large enough) \(N\) the covariance matrix of \(\delta_N\) is a matrix of rank \(n\) whose \(n\) largest eigenvalues are bounded above by \(\alpha\) and are greater than \(c/2\). Therefore, \(\epsilon_N\) does not tend to \(\omega_N\) in mean square.

2. Notice that the assumption of Proposition 2 that \(C_N\) is uniformly bounded can be weakened but cannot be completely eliminated. In fact, it is easy to see that the result still holds, for example, for \(C_N = [1 \ 2 \ 3 \ \ldots \ N]^\top\). Indeed, in the former case, max \(\mathbb{E} \left[ (\delta_N[i])^2 \right] \leq \frac{6N^2\alpha}{N(N+1)(2N+1)} \rightarrow 0\), but in the latter case max \(\mathbb{E} \left[ (\delta_N[i])^2 \right] \geq \frac{3c/4}{2(N^2-1)} \rightarrow 3c/8 > 0\).

3. In general, the error process \(\epsilon_N\) does not need to have uncorrelated components and hence this procedure can in principle be applied to GFA models and extended to the dynamic case. The key question however is to check if \(\epsilon_N\) can converge to an idiosyncratic sequence.

In the static case the answer to this question is affirmative as shown before. Interestingly and somehow surprisingly, this is not guaranteed to happen in the dynamic case.

### 6 The Dynamic case: the Kalman Predictor

Asymptotic factor (actually factor space) estimation by linear Bayesian averaging can be generalized to Dynamic GFA models. This is also implicitly hinted at in (Stock & Watson, 2011, p. 9). For each finite \(N\) one can estimate the latent variable \(x(t)\) in the GDA model \((4)\) by Kalman filtering, see e.g. Kapetanios & Marcellini (2009). The usual understanding of Kalman filtering leads to compute the one-step ahead estimate \(\hat{x}_N(t|t-1)\) of the hidden

\[
\hat{x}_N(t|t-1) = x(t) + \frac{1}{(1+c)}
\]

\[
\text{At the price of some technical complication in the derivation we could refine the lower bound to } c/(1 + \epsilon) \text{ for any positive } \epsilon.
\]
variable $x(t)$ based on previous outputs up to time $t - 1$. The estimator can obviously be implemented for finite truncations of the model, of increasing dimension $N$ and, assuming steady state, leads to the following sequence of innovation models:

$$\begin{cases} 
\dot{x}_N(t+1|t) = Ax_N(t|t-1) + K_N e_N(t) \\
y_N(t) = C_N x_N(t|t-1) + e_N(t),
\end{cases} \quad (9)$$

where the innovation

$$e_N(t) := y_N(t) - C_N \dot{x}_N(t|t-1)$$

has covariance

$$\Lambda_N = C_N P_N C_N^\top + R_N \quad (10)$$

with $P_N$ being the stabilizing solution of the Algebraic Riccati Equation (ARE)

$$P_N = A[P_N - P_N C_N^\top (C_N P_N C_N^\top + R_N)^{-1} C_N P_N] A^\top + Q \quad (11)$$

from which one can compute the Kalman gain,

$$K_N := A P_N C_N^\top \Lambda_N^{-1}.$$

The question is how the estimates behave for $N \to \infty$. In particular the question is if for $N \to \infty$ the (stationary) innovation representation (9) of $y_N(t)$ is a legitimate DGFA representation. Given the standing assumption on $C$, this will be true if and only if the limit innovation process $e_N(t)$ is idiosyncratic. Now the (steady-state) innovation covariance is given by (10) and since the original model is assumed DGFA, by Assumption (D) the output noise covariance $R_N := \mathbb{E}[w_N(t)w_N(t)^\top]$ tends for $N \to \infty$ to a bounded covariance operator $R$. The term $C_N P_N C_N^\top$ can be interpreted as a perturbation of $R_N$ and it looks like such a perturbation tends to be unbounded. In fact, under the minimality (and hence by the reachability) assumption, for each finite $N$, $P_N > 0$, and since $C_N$ has strongly linearly independent columns one may be led to conjecture that $C_N P_N C_N^\top$ converges to an unbounded operator of finite rank $n$. This fact casts doubts on the model (9) converging to a legitimate DGFA model for $N \to \infty$. The argument can be made rigorous as stated in the following theorem.

**Theorem 1** Consider a class of truncated models of the form (4) with Assumptions (A) to (F). For $N \to \infty$ the innovation process $e_N$ in the steady-state innovation representation (9) does not tend to an idiosyncratic process. Hence for $N \to \infty$ the innovation model (9) does not tend to a legitimate DGFA model.

**Proof.** By minimality of the model (4), $(A, Q)$ is reachable and hence the stabilizing solution $P_N$ of the ARE (11) is positive definite for each fixed $N$. Then we can rewrite the ARE (11) as

$$P_N = A \left( P_N^{-1} + C_N^\top R_N^{-1} C_N \right)^{-1} A^\top + Q \quad (12)$$

and hence, for each fixed $N$, we have $P_N \geq Q$. Since in the original GDFA model $w$ is idiosyncratic, the noise covariances $R_N$ are uniformly bounded, i.e. there exists $\alpha$ (independent of $N$) such that $R_N \leq \alpha I$. Therefore,

$$P_N^{-1} + C_N^\top R_N^{-1} C_N \geq \frac{1}{\alpha} C_N^\top C_N \geq \frac{\lambda_{\min}[C_N^\top C_N]}{\alpha} I,$$

and, as a consequence,

$$\left( P_N^{-1} + C_N^\top R_N^{-1} C_N \right)^{-1} \leq \frac{\alpha I}{\lambda_{\min}[C_N^\top C_N]} \xrightarrow{N \to \infty} 0. \quad (13)$$

This inequality together with (12) implies that $P_N$ converges monotonically to $Q$. Hence, the perturbation term $C_N P_N C_N^\top$ of $R_N$ in (10) must have at least one eigenvalue tending to infinity (actually as many as the rank of $Q$) and must therefore tend to an unbounded operator so that $e_N(t)$ is not an idiosyncratic process. In conclusion, the innovation model (9) does not satisfy the conditions of a GDFA model. \( \Box \)

Since, as shown in the previous proof, $P_N$ converges to $Q$, which is not the zero matrix, we have the following corollary.
Corollary 1 Under the assumptions of Theorem 1, the steady state prediction error of the state does not converge to zero (in mean square) as $N$ diverges. In particular, the one step ahead predictor of the common component vector $\chi_N(t) := C_N x(t)$ does not converge neither to $C x(t)$ nor to the measured signal $y(t)$, as $N \to \infty$.

**Proof.** The one step ahead predictor of $\chi_N(t)$, $\hat{\chi}(t | t - 1) := C_N x_N(t | t - 1)$ is just the one step ahead predictor of $y_N(t)$ and one can write

$$y_N(t) = \hat{\chi}_N(t | t - 1) + e_N(t)$$

where $e_N(t)$ is the output prediction error i.e. the innovation. Since, as we have shown, the covariance matrix of the prediction error $e_N$ becomes unbounded as $N \to \infty$, the predictor $\hat{\chi}_N(t | t - 1)$ cannot be consistent in mean square. In fact, as we have already seen, the term $C_N P_N C_N^\top$, namely the steady state covariance matrix of the prediction error $C_N x(t) - C_N x_N(t | t - 1)$ must have at least one eigenvalue tending to infinity and there must then be at least one direction along which the error covariance diverges in mean square. This is a fortiori true for the covariance matrix $C_N P_N C_N^\top + R_N$ of the difference $y_N(t) - \hat{\chi}_N(t | t - 1)$.

Notice that while $e_N(t)$ is not idiosyncratic, the “model noise” process $\bar{v}_N(t) := K_N e_N(t)$ does converge in mean-square to a finite covariance noise $\bar{v}(t)$. In fact its covariance is $\bar{Q}_N := K_N A_N R_N^\top$ and by rearranging (11) one immediately sees that

$$\bar{Q}_N = A P_N A^\top - P_N + Q$$

which clearly converges to $A Q A^\top$ because $P_N \to Q$. Therefore, the (steady-state) covariance of $\hat{x}_N(t | t - 1)$ converges to the unique solution of the Stein equation

$$\Sigma = A \Sigma A^\top + A Q A^\top.$$

**Remark 2** That the innovation model (9) cannot in the limit be interpreted as a valid GDFA model has important consequences. Among them it suggests that the application of standard subspace identification methods to the identification of GDFA models (see e.g. Marcellino (2017)), while effective in deriving a good generative model for the observed data, may lack the capability of extracting the very GDFA feature of the model itself. In fact, these identification of GDFA models (see e.g. Marcellino (2017)), while effective in deriving a good generative model for the observed data, may lack the capability of extracting the very GDFA feature of the model itself. Indeed, the results of the following section seem to suggest that considering the pure filter estimator in place of the predictor, may be advantageous.

**Remark 3** We may attempt a frequency-domain analysis of our result. Obviously the matrix transfer function of each model of the type (9) is the unique outer (i.e. stable and minimum-phase) spectral factor of the spectral density of the truncated process $y_N(t)$. For $N \to \infty$ these spectral factors (which are $N \times N$ rational outer matrices of full rank $N$) have zeros inside the open unit circle. Hence, the limit spectral factor will have all of its zeros (at most) in the closed unit circle. Therefore the limit spectral factor could also be called outer, or (weakly) minimum phase. However, since the innovation process corresponding to this factor is not idiosyncratic, the limit model cannot be a DGFA model. This could be stated by saying that there cannot exist prediction-error innovation models in the class of GDFA descriptions.

7 The Dynamic case: the pure filter estimator

Since the output noise covariance of the one step-ahead innovation model is in a sense “too big” as it has an unbounded component, one may wonder if to obtain a viable GDFA model, one could choose the (pure) filter estimate $E [x(t) | y_N]$ instead of the one-step ahead state predictor. It is in fact well-known that this estimate leads to a model with smaller state error variance.

To verify this conjecture, consider the steady state estimate $\hat{x}_N(t) := E [x(t) | y_N]$ given the infinite past, which satisfies the recursion

$$\hat{x}_N(t + 1) = E [x(t + 1) | y_N] + E [x(t + 1) | e_N(t + 1)]$$

which for a model with uncorrelated state and output noises, yields

$$\hat{x}_N(t + 1) = A \hat{x}_N(t) + L_N e_N(t + 1)$$

7 Up to uninteresting multiplication on the right side by an orthogonal matrix.
where $L_N := P_N C_N^T \Lambda_N^{-1}$. This yields the filtered innovation model

$$
\begin{cases}
\dot{x}_N(t+1) = Ax_N(t) + L_N e_N(t+1) \\
y_N(t) = C_N x_N(t) + e_N(t)
\end{cases}
$$

(14)

where (note the hatted symbol) $\hat{e}_N := y_N(t) - C_N \tilde{x}_N(t)$ is the filter innovation which is a white noise process. In fact $e_N$ and $\hat{e}_N$ are related by the formula

$$
e_N(t) = [I - C_N L_N]^{-1} \hat{e}_N(t)
$$

(15)

which follows from (14) as

$$
\hat{e}_N(t) := y_N(t) - C_N (\tilde{x}_N(t \mid t-1) + L_N e_N(t)) \\
= e_N(t) - C_N L_N e_N(t).
$$

This agrees with the fact that the noise term in the second equation of (14) is uncorrelated with $y'_N$ and hence with $\tilde{x}_N(t)$. The variance of $\hat{e}_N(t)$ has the representation

$$
\hat{\Lambda}_N := E[\hat{e}_N(t) \hat{e}_N(t)^T] \\
= [I - C_N P_N C_N^T \Lambda_N^{-1}] \Lambda_N [I - C_N P_N C_N^T \Lambda_N^{-1}]^T \\
= R_N \Lambda_N^{-1} R_N.
$$

**Theorem 2** Consider a class of truncated models of the form (4) with Assumptions (A) to (F). Then, the state and output noises in the associated model (14) are uncorrelated and the output noise variance $\Lambda_N$ tends to a bounded operator as $N \to \infty$. Therefore (14) converges to a legitimate DGFA representation of the process $y$.

**Proof.** That $v_N(t) := e_N(t+1)$ and $w_N(t) := \hat{e}_N(t)$ are uncorrelated follows readily from the equation (15) because $\hat{e}_N(t)$ is white. Moreover, all the eigenvalues of $\Lambda_N = C_N P_N C_N^T + R_N$ are positive and bounded below. Hence, $\hat{\Lambda}_N$ remains bounded for $N \to \infty$. \[ \square \]

A remarkable property of the pure filter realization which follows already from the calculation in (13) of the previous paragraph, is recast in the following statement.

**Corollary 2** Under the assumptions of Theorem 2, consider the (steady-state) covariance matrix of the state filtering error

$$
\Pi_N := E[\tilde{x}(t) - \hat{x}_N(t)] [\tilde{x}(t) - \hat{x}_N(t)]^T \\
= [P_N^{-1} + C_N^T R_N^{-1} C_N]^{-1}.
$$

Then, as $N \to \infty$, the filter error covariance $\Pi_N$ converges to the zero matrix.

This result implies that the limit for $N \to \infty$ of the (steady-state) filtered state estimate must converge to the true state $x(t)$, in other words we may say that the filtered estimate is a consistent estimator. This should not be surprising since it agrees with the previous general observation that in any bona-fide GDFA model (4) the hidden variable $x(t)$ can asymptotically be recovered exactly as a linear functional of the infinite cross sectional history of the process.

**Remark 4** Observe that even if the filtered state estimation error converges to zero (in mean square), in general we cannot recover the original idiosyncratic noise. In fact, if the output noise covariance is uniformly coercive (that is there is a $c > 0$ independent of $N$ such that $R_N \geq c I_N$) then the steady-state covariance of $\hat{\delta}_N(t) := \hat{e}_N(t) - w_N(t)$ does not converge to zero as it is a rank $n$ matrix whose $n$ non-zero eigenvalues are bounded from below by a positive constant. However, by using also in this case the arguments developed for the static case, we can show that if $C_N$ is uniformly bounded then for each fixed $i$ the $i-th$ component of $\hat{\delta}_N(t)$ converges to zero in mean square.
8 Examples

We consider the problem of estimating the average concentration over time of two pollutants, namely benzene (C6H6) and carbon monoxide (CO), in a certain city by means of a large number of sensors spread all over the area. This situation can be described by a DGFA model where the hidden factor vector describes these concentrations while the observed variables models the measurements taken from the sensors.

More in details, let $x_1(t)$ and $x_2(t)$ describe the average concentration in the city of C6H6 (in $10 \mu g/m^3$) and of CO (in $mg/m^3$), respectively, with sampling time equal to 1 hour, and let $x(t) = [x_1(t), x_2(t)]^T$ be the vector of latent factors. A good representation of the dynamics of the factors is given by the first equation of (4) where the state matrix $A$ and the covariance $Q$ of the white Gaussian noise $\{v(t)\}$ are

$$A = \begin{bmatrix} 0.9692 & -0.0442 \\ 0.2582 & 0.7707 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.1682 & 0.2806 \\ 0.2806 & 0.7531 \end{bmatrix}.$$  

This model has been identified from the time series collected in the period 11 March 2004–3 April 2005 by the regional environmental protection agency (ARPA) within an Italian city (for more details see De Vito et al. (2008)). Suppose that $N$ sensors are available and that each sensor measures either the concentration of C6H6 or CO; the placement of the sensors is shown in Figure 8. For simplicity, we assume that $N$ is a multiple of 4. Since the concentration of the pollutants varies considerably within the city, the sensor output at time $t$ is a measure of the average concentration corrupted by a local random fluctuation related to its position. In addition, each sensor is affected by an accidental measurement error which is independent from the other sensors. Then, we can describe the observation process by the second equation of (4) where $C_N = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & \ldots \\ 0 & 1 & 0 & 1 & 0 & 1 & \ldots \end{bmatrix}^T$ and the idiosyncratic process $\{w_N(t)\}$ is modeled as follows. We define the idiosyncratic noise vector as $w_N(t) = [w_{1,1}(t), \ldots, w_{1,4}(t), w_{2,1}(t), \ldots, w_{N,4}(t)]^T$ and we assume that the component $w_{l,k}(t)$ is given by a term proportional to the average of the noise affecting the sensors located in the preceding square plus an uncorrelated white Gaussian noise. Mathematically, we have for $k = 1, \ldots, 4$ and $l = 2, 3, \ldots, N/4$

$$w_{1,k}(t) = n_{1,k}(t),$$

$$w_{l,k}(t) = 0.5 \sum_{h=0}^{4} w_{l-1,h} + n_{l,k}(t).$$

Here, $\{n_{l,k}(t)\}$ is a normalized white Gaussian noise uncorrelated at all times with $\{v(t)\}$ and with $\{n_{\bar{k},\bar{l}}(t)\}$ for $(k,l) \neq (\bar{k},\bar{l})$. 

![Fig. 1. Placement of the N sensors all over the city. The sensors, represented by red circles, are located on the vertices of concentric squares. In particular, the (l, k) sensor is situated on the k-th vertex of the l-th square.](image)
We estimate the latent variable \( x(t) \) by the Kalman one-step ahead predictor (9) and the pure filter (14) for an increasing number \( N \) of sensors. The results are summarized in Figure 2 and 3 and they confirm that the pure filter yields to a consistent estimate, while the state prediction error does not converge to zero as \( N \) diverges. To better illustrate Remark 4, we also compute the “Euclidean” and the “infinity” norms of \( \delta_N(t) := \hat{o}_N(t) - w_N(t) \), i.e.

\[
\|\delta_N(t)\| := \sqrt{\mathbb{E}[\delta_N(t)^\top \delta_N(t)]} = \sqrt{\text{tr}[C_N \Pi_N C_N^\top]}
\]

\[
\|\delta_N(t)\|_\infty := \max_i \sqrt{\mathbb{E}[\delta_N(t)_i]^2} = \max_i \sqrt{[C_N \Pi_N C_N^\top]_{ii}}
\]

with \( \delta_N(t)_i \) being the \( i \)-th component of \( \delta_N(t) \) and \( [C_N \Pi_N C_N^\top]_{ii} \) the \( i \)-th diagonal element of \( C_N \Pi_N C_N^\top \). The results are shown in Figure 4 and they reveal that, according to what observed in Remark 4, the overall vector \( \delta_N(t) \) does not converge to zero in mean square even if its \( i \)-th component does converge to zero for each \( i \).

9 Conclusion

We have shown that the (pure) Kalman filter leads to a legitimate GDFA model while the standard Kalman predictor does not. As a consequence only the former model allows a consistent estimation of the factor process as the cross-sectional dimension tends to infinity.

Appendix

A About the space \( \ell^2(\Sigma) \)

We analyse the relation between \( \ell^2 \) and \( \ell^2(\Sigma) \). In general this relation depends on \( \Sigma \). Let us first consider the case where the components of \( y \) are zero-mean uncorrelated random variables with unbounded variance, say \( \Sigma \succeq I \) (where \( I \) is the identity operator); for example let \( y \) have uncorrelated components with \( \mathbb{E}y(k)^2 = k^2 \). Then \( \|a\|_2^2 \geq \|a\|^2 \) and hence \( \ell^2 \supseteq \ell^2(\Sigma) \). On the other hand, let \( a := \{a(k) = 1/k, k \in \mathbb{N}\} \); clearly, \( a \in \ell^2 \) but \( a \notin \ell^2(\Sigma) \). Hence, in this case, \( \ell^2 \nsubseteq \ell^2(\Sigma) \).

When the \( y(k) \)'s are zero-mean random variables with a bounded variance matrix, say \( \Sigma \preceq I \) we have \( \|a\|_2^2 \leq \|a\|^2 \) and hence \( \ell^2 \subseteq \ell^2(\Sigma) \). Consider in particular the case when \( \mathbb{E}y(k)^2 = \frac{1}{n^2} \). Then, let \( a := \{a(k) = 1, \forall k \in \mathbb{N}\} \); clearly, \( a \notin \ell^2 \) but \( a \in \ell^2(\Sigma) \). Hence, in this case, \( \ell^2 \nsubseteq \ell^2(\Sigma) \).

B The notion of strong linear independence

In this section we collect some relevant facts discussed in Bottegal & Picci (2015) where more details and the proofs of the collected results are available. Let \( C \in \mathbb{R}^{\infty \times n} \) and \( c^i, i = 1, \ldots, n \) be the columns of \( C \). Let \( C_N \) be the (finite) submatrix of \( C \) obtained by extracting the first \( N \) rows of \( C \). Let \( c_N^i \) be the \( i \)-th column of \( C_N \) and define

\[
c_N^i := c_N^i - \Pi[c_N^i \bigg | \tilde{C}_N^i]
\]

where \( \Pi \) is the orthogonal projection onto the Euclidean space \( \tilde{C}_N^i = \text{span} \{c_N^j, j \neq i\} \) of dimension (at most) \( n-1 \).

Definition 4 The column vectors \( c^i, i = 1, \ldots, n \) in \( \mathbb{R}^{\infty} \) are strongly linearly independent if

\[
\lim_{N \to \infty} \|c_N^i\|_2 = +\infty \quad i = 1, \ldots, n.
\]

If \( n = 1 \) the condition (B.1) is equivalent to \( \|c\|_2 = \infty \). In a sense, this condition says that the tails of two strongly linearly independent vectors in \( \mathbb{R}^{\infty} \) cannot get “too close” asymptotically.
Fig. 2. Euclidean norm of the steady-state prediction error $x(t) - \hat{x}_N(t|t-1)$ and filtering error $x(t) - \hat{x}_N(t)$ for increasing dimension $N$. 
Fig. 3. Euclidean norm of the steady-state prediction error covariance matrix $P_N$ and of the steady-state filter error covariance matrix $Π_N$ for increasing dimension $N$.

Fig. 4. Euclidean and infinity norm of the random variable $\hat{δ}_N(t) = \hat{e}_N(t) - w_N(t)$ for increasing dimension $N$.

**Theorem 3** Let $y$ be a purely deterministic sequence of rank $n$, i.e. let

$$y(k) = \sum_{i=1}^{n} c^i(k) x_i, \quad k \in \mathbb{Z}_+;$$

with uncorrelated latent variables $x_i$. Then $y$ is not idiosyncratic if and only if, the vectors $c^i, i = 1, \ldots, n$ are strongly linearly independent.

**Corollary 3** $C \in \mathbb{R}^{\infty \times n}$ has strongly linearly independent columns if and only if $
lim_{N \to \infty} \lambda_{\min}[C_N^T C_N] = \infty$.

**Corollary 4** The covariance $Σ$ has a GFA decomposition with $n$ latent factors if and only if it can be decomposed as the sum of a matrix $\hat{Σ}$ which defines a bounded operator in $ℓ^2$ and a rank $n$ perturbation $\hat{Σ} = CC^T$ where $C \in \mathbb{R}^{\infty \times n}$ has strongly linearly independent columns.

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