The 1-Point Cluster Distribution Function and its Moments

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Abstract

We derive the 1-point probability density function of the smoothed Abell-ACO cluster density field and we compare it with that of artificial cluster samples, generated as high peaks of a Gaussian field in such a way that they reproduce the low-order (2- and 3-point) correlation functions and the observed cluster selection functions. We find that both real and simulated pdfs are well approximated by a log-normal distribution even when the Gaussian smoothing radius is as large as 40 $h^{-1}$ Mpc. Furthermore the low-order moments of the pdf are found to obey a relation $\gamma \propto \sigma^4$, with $\gamma$ being the skewness. Since clusters have not had enough time to significantly depart from their original birth-place positions, these results are consistent with them being high-peaks of an underlying initial Gaussian density field.

A by-product of our analysis is that when we rescale the pdf cluster moments to those of the QDOT-IRAS galaxies, using linear biasing with $b_{cI} \sim 4.5$ and for the common smoothing radius of 20 $h^{-1}$ Mpc, we find them to be significantly smaller than those directly estimated from the QDOT data by Saunders et al. (1991).
1 Introduction

The study of the large-scale galaxy distribution is of fundamental importance in modern cosmology. A complete knowledge of the present cosmic density fluctuation field $\delta_m \equiv \frac{\rho - \langle \rho \rangle}{\langle \rho \rangle}$, in the linear regime, would yield informations on the shape of the spectrum of primordial perturbations which gave rise, in the gravitational instability picture, to the present cosmic structures and their distribution. Within the framework of the most common galaxy formation theories, the standard two assumptions are (i) that the initial $\delta_m$ field is described by a Gaussian distribution and (ii) that some sort of biasing mechanism is at work. In the most popular biasing models, galaxies form at the high peaks of the underlying matter field (Kaiser 1984; Bardeen et al. 1986). This model was introduced to reconcile, among other things, the observed low values of $\Omega$ with the inflationary model (Olive 1990).

Many statistical approaches have been used to study the distribution of extragalactic structures. The most common one is the two–point correlation function, $\xi(r)$, (cf. Peebles 1980) which has been found to have the form:

$$\xi(r) = (r/r_o)^{-\gamma},$$  \hspace{1cm} (1)

with $r_o \approx 5 \, h^{-1} \, \text{Mpc}$ and $\gamma \approx 1.8$ from $10 \, h^{-1} \, \text{Kpc}$ up to $10 \, h^{-1} \, \text{Mpc}$ (Davis & Peebles 1983). Cluster of galaxies exhibit similar kind of correlations up to $\sim 50 \, h^{-1} \, \text{Mpc}$ with the same slope and $r_o \approx 20 - 25 \, h^{-1} \, \text{Mpc}$ (cf. Bahcall & Soneira 1983; Klypin & Kopylov 1983). Another statistical measure is the 3–point correlation function, for which a non-zero signal implies non–Gaussian statistics. The analysis of both 2-D and 3-D samples has shown that galaxies do have a non-zero connected 3-point correlation function with a hierarchical coefficient, $Q \simeq 1$ (Groth & Peebles 1977; Sharp, Bonometto & Lucchin 1984). Similar results hold also for clusters of galaxies (Tóth et al. 1989, Jing & Zhang 1989, Jing & Valdarnini 1991 [hereafter JV]; Borgani, Jing & Plionis 1992).

These results are of great interest, since non-Gaussianity might arise either from gravitational clustering, or from biasing mechanisms, or finally from intrinsic non-Gaussian statistics of the initial matter field. A complete knowledge of the N–point correlation function would allow one, using the linear biasing assumption, to recover the statistical properties of the $\delta_m$ field. For $N > 3$, however, this approach becomes impractical due to the limited extension of the available extragalactic samples.

An alternative method, which is becoming increasingly popular, is the study of the probability density function (pdf). Theoretically the knowledge of the N–point correlation function is equivalent to that of the pdf and its moments, but for real data sets the latter are easier to measure. Because of the discrete nature of the available samples of extragalactic objects, and in order to obtain a continuous density field one is forced to smooth the discrete distribution with some function (usually a top-hat or a Gaussian one is used). For a normally distributed primordial density field and as long as the variance $\sigma^2 \equiv \langle \delta^2 \rangle$ is small, under the action of gravity, the deviation of the initial pdf from the Gaussian shape is well approximated by the Edgeworth expansion (Colombi 1994).

In the non–linear regime N-body simulations have shown the the pdf becomes non-Gaussian as
the clustering evolves (Bouchet, Schaffer & Davis 1991; Bouchet & Hernquist 1992; Lahav et al. 1993; Juszkiewicz et al. 1993; Kofman et al. 1994). For \( \sigma \ll 1 \) the shape of the pdf is well fitted by a lognormal distribution. The pdf of different galaxy samples has been estimated by Bouchet et al. (1993), Gatza˜naga and Yokoyama (1993). Kofman et al. (1994) have compared the pdf derived form CDM N-body simulations with that of the IRAS sample and the one recovered using the POTENT procedure with \( \Omega = 1 \). Their main conclusion is that, if galaxies trace the mass, the observed pdf can be fitted using Gaussian initial conditions.

Furthermore, the time evolution of the moments of the pdf has been studied both analytically and with N-body simulations (Goroff et al. 1986; Coles & Frenk 1991; Juszkiewicz et al. 1993; Juszkiewicz, Bouchet & Colombi 1993; Kofman et al. 1994; Colombi 1994). For a given filter and smoothing radius, the normalized skewness of the matter distribution, \( S_3 \equiv \langle \delta^3 \rangle / \langle \delta^2 \rangle^2 \), will depend on the shape of the power spectrum \( P(k) \). Juszkiewicz, Bouchet & Colombi (1993), using second order theory, found \( S_3 \simeq 34/7 - (3 + n) \) for a top-hat filter and scale-free spectra. These results agree with those obtained from N-body simulations (Kofman et al. 1994). The proportionality \( \langle \delta^3 \rangle \propto \sigma^4 \) holds also at a fixed time when one changes the smoothing radius (Coles & Frenk 1991; Kofman et al. 1994).

In this paper we estimate the pdf for the distribution of Abell+ACO clusters of galaxies (Abell 1958; Abell, Corwin & Olowin 1989). Since clusters of galaxies are the brightest objects in the sky they can probe the very large scales, reliably up to \( \approx 300 h^{-1} \) Mpc. On these scales the density fluctuations are well within the linear regime and the deviations of the pdf from a Gaussian distribution is determined either by the fact that they are biased tracers of the underlying field or due to non-Gaussian initial conditions. The main aim of this paper is to establish whether the present cluster pdf can be derived entirely under the assumption that clusters form at the high peaks of a Gaussian background field. The basic procedure is to compare the observed cluster pdf with that measured from the distribution of simulated clusters, generated as peaks of a initially Gaussian background field and having the same boundaries and selection effects as the real data.

## 2 Data Samples

### 2.1 Real cluster samples

We apply our analysis to the combined Abell-ACO \( R \geq 0 \) cluster sample, as defined in Plionis & Valdarnini (1991) [hereafter PV91] and analysed in Plionis, Valdarnini & Jing (1992) [hereafter PVJ]. The northern sample, with \( \text{dec} \geq -17^{\circ} \) (Abell), is defined by those clusters that have measured \( z \lesssim 0.1 \), while the southern sample, (ACO) with \( \text{dec} \leq -17^{\circ} \), is defined by those clusters with \( m_{10} \leq 16.4 \) (note that with this definition and due to the availability of many new cluster redshifts only 7 ACO clusters have \( m_{10} \) estimated redshifts from the \( m_{10} - z \) relation derived in PV91). Both samples are limited in Galactic latitude by \( |b| \gtrsim 30^{\circ} \). The redshifts we have used are mainly from Struble & Rood (1987) and Postman, Huchra & Geller (1992) with additions from taken Rhee &
Katgert (1988) and Batuski et al. (1991), the original ACO paper, Vettolani et al. (1989); Cappi et al. (1991) and Muriel et al. (1991) as well as by our own cross-correlation analysis of the Fairall & Jones (1991) galaxy redshift catalogue with the cluster catalogues. The total number of clusters in our samples is 357 Abell and 157 ACO ones.

We take into account the effect of Galactic absorption, which is assumed to follow the usual cosecant law:

$$P(|b|) = \text{dex} \left[ \alpha (1 - \csc |b|) \right]$$  \hspace{1cm} (2)

with \( \alpha \approx 0.3 \) for the Abell sample (Bahcall & Soneira 1983; Postman et al. 1989) and \( \alpha \approx 0.2 \) for the ACO sample (Batuski et al. 1989), by weighting each cluster by \( 1/P(|b|) \), a correction which explicitly assumes that the unobserved (obscured) clusters are correlated with the observed ones, a reasonable assumption under the well known correlation properties of clusters.

We have derived the cluster-redshift selection function, \( P(z) \), by fitting the cluster density, as a function of \( z \) (cf. Postman et al. 1989; PVJ), by:

$$P(z) = \begin{cases} 
1 & \text{for } z \leq z_c \\
\alpha \exp(-z/z_o) & \text{for } z > z_c 
\end{cases}$$  \hspace{1cm} (3)

where \( z_c \) is the maximum redshift at which the sample exhibits a roughly constant density (for more details see PVJ). We find \( z_c \approx 0.079 \) and 0.066 for the Abell and ACO samples, which correspond to \( R \approx 230 \) and 190 \( h^{-1} \) Mpc respectively. Cluster comoving distances are estimated using the standard relation (Mattig 1958):

$$R = \frac{c}{H_o q_o^2 (1 + z)} \left( q_o z + (1 - q_o)(1 - \sqrt{2q_o z + 1}) \right)$$  \hspace{1cm} (4)

with \( H_o = 100 \) km sec\(^{-1}\) Mpc\(^{-1}\) and \( q_o = 0.2 \).

The Abell and ACO cluster densities, out to their limit of completeness \( (z_c) \), is \( \sim 1.4 \times 10^{-5} \) \( h^3 \) Mpc\(^{-3}\) and \( \sim 2.1 \times 10^{-5} \) \( h^3 \) Mpc\(^{-3}\), corresponding to a characteristic length scale of \( \langle \rho \rangle^{-1} \approx 41 \) \( h^{-1} \) Mpc and 36 \( h^{-1} \). The higher space-density of ACO clusters is partly due to the huge Shapley concentration (Shapley 1930; Scaramella et al. 1989; Raychaudhury 1989), but a significant part is also due to systematic density differences between the Abell and ACO cluster samples, as a function also of \( z \), which have been noted in a number of studies (cf. PV91 and references therein) and which could be attributed to the high sensitivity of the IIIa-J emulsion plates.

Since we want to use a cluster sample covering the whole sky we need to take into account the density variations of the Abell and ACO cluster samples. We will do this by renormalizing the ACO density to the Abell one using a radial matching function, \( W(R) \), as in PV91. Since in this paper we are mostly interested in the cluster density fluctuations \( \delta \rho / \rho \), it is equivalent to normalize the density of the Abell to the ACO or vice-versa. In order to test the robustness of the results in the choice of the \( W(R) \) function, we will use a number of such functions, among which the \( W(R) = 1 \) (no matching) and the \( W(R) = \text{constant} \) (no radial dependance) cases.

Note that in PVJ we had estimated the 2-point correlation function for our samples of clusters and we found that the slope of the derived 2-p function has a value \( \sim 1.8 \pm 0.2 \) for both Abell
and ACO samples. However, their amplitudes are slightly different, with Abell clusters having $r_0 \approx 18 \pm 4$ $h^{-1}$ Mpc (bootstrap errors used) out to $\lesssim 50$ $h^{-1}$ Mpc while the ACO clusters have $r_0 \approx 22 \pm 10$ $h^{-1}$ Mpc but only out to $\sim 30$ $h^{-1}$ Mpc. This could be, however, due to the fact that the ACO sample is defined in a relatively small solid angle and therefore the large wavelengths may be undersampled. Based on our previous analysis (Jing et al. 1992), we believe that the cluster correlations are reliable and mostly unaffected by cluster contamination effects (cf. Sutherland 1988).

In PVJ we also estimated the 3-point correlation function using the moment method (cf. Sharp, Bonometto & Lucchin 1984; Jing & Valdarnini 1991; Borgani, Jing & Plionis 1991). We found a good fit to the spatial 3-point function provided by the hierarchical expression:

$$\zeta(r_1, r_2, r_{12}) = Q\left[\xi(r_1)\xi(r_2) + \xi(r_1)\xi(r_{12}) + \xi(r_2)\xi(r_{12})\right],$$

(5)

with $Q_{\text{Abell}} \approx 0.6$ and $Q_{\text{ACO}} \approx 0.7$, with relative bootstrap error of $\sim \pm 0.2$.

Finally, in order to minimize the uncertainties due to the approximate character of the redshift selection function, $P(z)$, and of the radial function, $W(R)$, especially at large distances, we will restrict our analysis to $R_{\text{max}} = 240$ $h^{-1}$ Mpc.

### 2.2 Simulated cluster samples

The procedure used to generate our artificial cluster catalogues has been presented in PVJ. Here we only remind the reader the main steps of our procedure.

We generate our simulated cluster catalogues using a method similar to that of Postman et al. (1989). We identify the position $\vec{x}$ of points corresponding to clusters of galaxies according to the following prescription: $N_p$ points are randomly placed in a cube of size $640$ $h^{-1}$ Mpc with $N_g = 64^3$ grid points. We keep points only within the sphere of radius $r_{\text{max}} = 320$ $h^{-1}$ Mpc centered at the cube centre. The points are then displaced from their original positions $\vec{x}_r$ using appropriately the Zel’dovich approximation. The power spectrum of the density fluctuations, $P(k)$, has a Gaussian distribution with random phases. $P(k)$ is chosen so that the perturbed particle positions give a correlation function in the desired range. In accordance with Postman et al. (1989), we use the following power spectrum:

$$P(k) = A k^n \exp(-|\vec{k}|^2/\Lambda^2) \Theta(|\vec{k}|)$$

(6)

where $\Lambda^{-1} = 0.1$ $h^{-1}$ Mpc, $\Theta(|\vec{k}|) = 0$ for $|\vec{k}| > 2\pi/80$ $h^{-1}$ Mpc, $A$ is a normalization constant and $n$ is the spectral index.

We assign ‘clusters’ to the peaks of the background field according to the following prescription: each particle inside the cube is assigned a value $\nu$ such that $\delta_{\text{g}} > \nu \sigma$, where $\sigma$ is the $r_{\text{rms}}$ density fluctuation within the cube and $\text{g}$ is the nearest grid point to the particle. Here $\delta_{\text{g}}$ is the density field smoothed with a Gaussian filter function having a smoothing radius of $10$ $h^{-1}$ Mpc. The parameters $A$ and $n$ and $\nu$ must be chosen such that the peak-peak correlation function in the range $10 - 60$ $h^{-1}$ Mpc, corresponds to the observed $\xi_{cc}$. We find that for $A \approx 1.9 \times 10^6$, $n \approx -1.5$.
and \( \nu \sim 1.3 \) we obtain a distribution of simulated clusters with \( \langle r_c \rangle_{\text{sim}} = 21 \ h^{-1} \) Mpc. Furthermore these clusters have by construction non-zero higher order correlations and we have found in PVJ, using the direct triplets-counting method, \( Q_{\text{sim}} \simeq 0.6 \pm 0.16 \), where the uncertainty is the ensemble one.

Finally we note that our simulated cluster catalogues have the same geometrical boundaries, the same redshift and Galactic obscuration selection functions, the same low-order correlation properties as well as the same mean space density as the real clusters.

### 3 Method

In order to obtain a continuous cluster density field we smooth the cluster distribution in a \( 24^3 \) cube \( (480^3 \ h^{-3} \) Mpc\(^3 \) using a Gaussian kernel:

\[
W(x_i - x_g) = \left( \frac{2\pi R_{sm}^2}{2} \right)^{-3/2} \exp \left( -\frac{|x_i - x_g|^2}{2R_{sm}^2} \right) \tag{7}
\]

The smoothed cluster density, at the grid-cell positions \( x_g \), is then:

\[
\rho(x_g) = \frac{\sum_i \rho(x_i)W(x_i - x_g)}{\int W(|x - x_g|)d^3x} \tag{8}
\]

where the sum is over the distribution of clusters at positions \( x_i \). In order to study the cluster probability density function, \( P(\rho) \), at different scales, we use four smoothing radii; \( R_{sm} = 20, 30, 40 & 50 \ h^{-1} \) Mpc with \( |x_i - x_g| \leq 3R_{sm} \). In this case the integral at the denominator of eq.(8) is 
\( \simeq 0.97 \). There are, however, two problems which we have to resolve:

- Due to the geometrical boundaries of the observation volume and the zone of avoidance, each Gaussian sphere is incomplete. Using a Monte-Carlo method, we estimate this incompleteness, by defining a grid completeness factor:

  \[
f(x_g) = \frac{\int n(x)p(x)W(|x - x_g|)d^3x}{\int n(x)W(|x - x_g|)d^3x} \tag{9}
\]

  where \( n(x) \) is the density of a random distribution of points (thus \( n(x) = \text{constant} \)) and \( p(x) \) represents the selection functions and geometric boundaries. Note that by definition \( 0 \leq f(x_g) \leq 1 \). The smoothed cluster density is then corrected, at each grid point \( x_g \), by \( 1/f(x_g) \). However, to reduce the uncertainty due to the approximate nature of this correction we use in our analysis only those grid-cells with \( f(x_g) \geq 0.8 \). Due to the small number of such cells for large smoothing radii, we use also \( f(x_g) \geq 0.7 \) but only for the \( R_{sm} \geq 40 \) cases.

- Discreteness effects can be introduced for small \( R_{sm} \), if the number of cluster counts, in the Gaussian sphere, is small or if the smoothing fails to create a continuous density field. The latter occurs only for \( R_{sm} = 20 \ h^{-1} \) Mpc where some cells have \( \rho_{sm} = 0 \). This biases the \( P(\rho) \) at \( \rho/\langle \rho \rangle \ll 1 \). This brings up the issue of which is the optimal smoothing radius to derive.
the $P(\rho)$ of a discrete distribution of points. From considerations presented in the following section we find that the optimal radius is of the order the mean interparticle separation, ie. $\sim 30 - 40 \, h^{-1} \, \text{Mpc}$ for the Abell+ACO clusters.

By applying the selection functions and the geometric boundaries of the real cluster distribution to that of the simulated clusters we have tested the robustness of our correction procedure by comparing bin-by-bin the original simulation cluster density contrast, $\delta_{\text{ori}}$ (whole cube), with the recovered density contrast, $\delta_{\text{rec}}$. We find an excellent agreement (Figure 1) and we have also verified that the pdf’s, before and after the application of the selection functions and the correction procedure, are identical.

4 Results & Discussion

In figure 2 we present the simulation-cluster mean pdf’s (over the 50 simulations), for all four smoothing radii, after the application of the selection functions and the subsequent corrections, discussed in the previous section. The errorbars are the 1σ scatter around the ensemble mean. We plot the Gaussian fit (dashed line) to the cluster pdf, given by:

$$P(\varrho) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[ -\frac{(\varrho - 1)^2}{2\sigma^2} \right]$$

where $\sigma$ is the standard deviation of $\varrho(\equiv \rho/\langle \rho \rangle)$, and we also plot the log-normal fit, which has been found to give an extremely good fit to the CDM density and the IRAS galaxy pdf in the weakly-linear regime (Kofman et al. 1994). The log-normal distribution is given by:

$$P(\varrho) = \frac{1}{\sqrt{2\pi\sigma_L^2}} \exp \left[ -\frac{(\ln \varrho - \mu_L)^2}{2\sigma_L^2} \right] \frac{1}{\varrho}$$

where $\mu_L$ and $\sigma_L$ is the mean and standard deviation of $\ln \varrho$.

From figure 2 it is evident that the log-normal distribution fits well the simulation cluster pdf for all four $R_{sm}$, while the Gaussian one is rejected and although it becomes increasingly a better fit to the cluster $P(\varrho)$, as $R_{sm}$ increases, it always provides a worse fit than the corresponding log-normal distribution, even for $R_{sm} = 50 \, h^{-1} \, \text{Mpc}$. Note, however, that for $R_{sm} = 20 \, h^{-1} \, \text{Mpc}$ and $\varrho \lesssim 1$ the log-normal distribution does not fit well the pdf. The reason is that at this smoothing scale and range of $\varrho$ there are significant shot-noise contributions (see discussion in previous section).

The log-normal distribution has been argued (Coles & Jones 1991) to describe the distribution of density perturbations resulting from Gaussian initial conditions in the weakly non-linear regime. It has been shown to fit well the Lick Euler-Poincaré characteristic (a measure related to the genus) and the 2-dimensional Lick counts in cells at a resolution which corresponds, at the characteristic depth of the Lick catalogue, to scales of $\sim 4 - 10 \, h^{-1} \, \text{Mpc}$ (Coles & Plionis 1991). However, Bernardeau & Kofman (1994) have shown that the log-normal distribution is not a universal form of the cosmic density pdf but a very convenient fit only in some portion of the $(\sigma, n)$-plane (ie, $\sigma \ll 1$
and $n \approx -1$). In this context it seems that it is not a coincidence that the simulation cluster PDF is fitted quite well by a log-normal distribution but it could be a consequence of the fact that we used a power spectrum with exponent near to $-1$ ($n_{\text{sim}} \approx -1.5$). Alternatively, it could be a generic feature of high-peak biasing.

In figure 3 we present the one-point PDF of the Abell+ACO smoothed cluster distribution, for three $R_{\text{sm}}$ values[7] and for two different values of $f(x_g)$, the latter to check the effect of the limited number of cells used to derive the PDF. The uncertainty here is given by Poisson sampling errors and thus they should be considered only a lower limit to the intrinsic uncertainty. A more reasonable indication of the scatter is the ensemble errors seen in figure 2. Again we also plot the best Gaussian (dashed line) and log-normal (solid line) fits. We verify that also the real cluster smoothed PDF is fitted quite well by a log-normal distribution, especially for $R_{\text{sm}} \geq 30 \ h^{-1} \ Mpc$ where discreteness effects are minimized.

We now proceed to estimate the moments of the simulated and real cluster distributions, which are defined by:

$$\langle \delta^n \rangle = \int_{-1}^{\infty} \delta^n f(\delta) d\delta$$

where $\delta = \varrho - 1$. Note that if $f(\delta)$ is a Gaussian then it is defined in an infinite interval and thus it always has $f(\leq -1) \neq 0$. However by definition $\delta > -1$ and therefore $f(\delta)$ can be a Gaussian only in the limit $\sigma \to 0$. The gravitational evolution of the $\delta$ field acts in a way to increase the variance, $\sigma^2 \equiv \langle \delta^2 \rangle$, and thus in order for $\delta > -1$ to hold, $f(\delta)$ becomes skewed. Note that the skewness, $\gamma \equiv \langle \delta^3 \rangle$ of a Gaussian distribution (defined in an infinite interval) is zero. In the case of the $\delta$ field, however, we have that:

$$\gamma = \int_{-1}^{\infty} \frac{\delta^3}{\sigma \sqrt{2\pi}} \exp \left[ -\frac{\delta^2}{2\sigma^2} \right] d\delta = \frac{\sigma(1 + 2\sigma^2)}{\sqrt{2\pi} \exp (1/2\sigma^2)}$$

and thus $\gamma \neq 0$ (unless $\sigma \to 0$). We have already introduced the variance and the skewness of $f(\delta)$. In a similar fashion we can define higher order moments; for example the $4^{th}$ moment is the kurtosis, $K \equiv \langle \delta^4 \rangle$ (cf. Lahav et al. 1993; Kofman et al. 1994).

An important problem in determining the moments of $f(\delta)$ is the contribution of discreteness effects which can dominate in low-density discrete distributions. If the point-like distribution represents a Poisson sampling of an underlying continuous density field then the shot-noise contributions can be easily corrected (cf. Peebles 1980). However, the galaxy-cluster distribution can be hardly considered a poisson sampling since it rather represents a strongly biased sampling of an underlying galaxy density field. Therefore the poisson shot-noise corrections should not be expected to provide a reasonable correction of the discreteness effects. In fact, Borgani et al. (1994) found for their 2-d cluster analysis that indeed the poisson shot-noise correction resulted in extremely noisy moments and they preferred not to correct for shot-noise effects, which resulted in stable, well-behaved moments.

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1We have not used the $R_{\text{sm}} = 50 \ h^{-1} \ Mpc$ because there are only a few cells with $f \geq 0.7$. In the simulation case we overcame this problem due to the large number of available realizations.
Furthermore, the smoothing process itself suppresses the shot-noise effects considerably (cf. Gaztañaga & Yokoyama 1993). From eq.(8) it is easy to show that the power spectrum of the smoothed cluster density is related to that of the cluster $P_{cl}(k)$ by
\[ P_s(k) = e^{-k^2 R_{sm}^2} \left( \frac{1}{N} + P_{cl}(k) \right) \simeq e^{-k^2 R_{sm}^2} \left[ 1 + \frac{4\pi}{\langle l \rangle^2} \int_0^{\pi/k} \xi(s)s^2 ds \right], \tag{14} \]
where $\langle l \rangle$ is the average cluster separation. Then the best representation of $P_{cl}(k)$ is for $R_{sm} \simeq \langle l \rangle$, the original field being oversmoothed for $R_{sm} \gg \langle l \rangle$ and undersampled for $R_{sm} \ll \langle l \rangle$.

In Table 1 we present the second, third and fourth moment of the cluster pdf as a function of smoothing scale, for the simulations as well as the real data. It is apparent that the cluster moments are significantly smaller than those of the simulation clusters. Although the latter are constructed in such a way to reproduce the correct slope and roughly the correct amplitude of the 2-point spatial correlation function of the Abell and ACO ($R \geq 0$) clusters (see PVJ), there are differences in the amplitude. For example the correlation length of the Abell and ACO clusters is $r_\circ \simeq 18$ and $22 \ h^{-1} \ Mpc$, respectively while of the simulated clusters $\langle r_\circ \rangle_{sim} \simeq 21 \ h^{-1} \ Mpc$ and since the Abell sample is the dominant one we expect that the overall cluster variance will be smaller than that of the simulated clusters. In the same direction works also the fact that the simulated clusters have a non-zero $\xi(r)$ at separations larger than those of the real clusters and therefore since the smoothing process mixes a lot of different scales, the non-zero $\xi(r)$ at large $r$ will tend to increase the variance.

As we discussed in the Introduction many authors found, on the basis of a variety of clustering scenarios from Gaussian initial conditions, that the skewness is simply related to the variance of the mass distribution by:
\[ \gamma = B \left( \sigma^2 \right)^A \tag{15} \]
This result holds for $\sigma \lesssim 1$ (cf. Fig.5 of Kofman et al. 1994) while $\sigma$ changes either due to the clustering evolution or due to different smoothing radii. We therefore plot in figure 4(a) the skewness, $\gamma$, versus the variance, $\sigma^2$ for the real cluster pdf and in panel (b) for the simulated cluster pdf. For the simulated cluster case we plot the values for each of the 50 simulations and for each simulation at 4 different smoothing scales (as discussed in the previous section). Similarly for the real clusters we plot the values corresponding to different Abell-ACO normalization functions, $W(R)$, and for 3 smoothing scales. It is evident that the plotted points in both, simulation and real cluster, cases lie on the same line. To check whether eq.(15) is fulfilled we have used a weighted least-square fit to determine the slope and the intercept of the line:
\[ \log \gamma = A \log \sigma^2 + \log B \tag{16} \]
and we have found for the cluster case:
\[ A = 2.09 \pm 0.05 \quad B = 1.94 \pm 0.15, \]
and for the real cluster case:

$$A = 2.04 \pm 0.06 \quad B = 1.8 \pm 0.2,$$

which imply that indeed eq.(13) is fulfilled. This result is also in agreement with the hierarchical relation ($\gamma = 3Q\sigma^4$) with $Q \approx 0.6$, consistent with previous cluster correlation function analysis (cf. JV, PVJ). The parameter $B$ is also called \textit{normalized skewness}, $S_3$. In general, the normalized moments are given by:

$$S_n = \frac{\langle \delta^n \rangle_c}{\langle \delta^2 \rangle_c^{(n-1)}}$$

(17)

where $\langle \delta^n \rangle_c$ are the cumulants. The normalized skewness, $S_3$, plotted in Figure 5, for both the simulation and real cluster \textit{pdf} has roughly the same value, and is constant as a function of $R_{sm}$. Such a behaviour is consistent with density distributions evolving from Gaussian initial conditions (Coles & Frenk 1991; Kofman et al. 1994). Furthermore, eq.(13) is valid also in the high-peak biasing model in CDM universes (Coles & Frenk 1991). This is worth stressing since what we measure here is not the skewness induced on the density field by gravity but that arising from a threshold peak-selection.

Therefore, the above results, together with the fact that both the simulated and real cluster \textit{pdf} are fitted by a log-normal distribution, imply:

- the Abell+ACO clusters are compatible with being high-peaks of a Gaussian background and
- the spectral index of density fluctuation spectrum, from which the cluster distribution has emerged, is compatible with $\approx -1.5$.

We emphasize that they are \textit{only} consistent because a thorough analysis of the expected behaviour of peaks of a non-Gaussian background in 3-D has not yet been investigated thoroughly and therefore we cannot exclude the possibility that such peaks may have a similar behaviour. However, there are indications from the study of the moments of simulated \textit{galaxy} distributions emerging from non-Gaussian initial conditions (Weinberg & Cole 1992; Coles et al. 1993), that they scale in a manner similar to that of distributions emerging from Gaussian initial conditions. In fact, Coles et al. (1993) find that in their initially positive skewed models, $S_3$ is also a constant but has a higher value than that of the Gaussian models. Initially negative skewed models are difficult to distinguish from Gaussian models on the basis of their normalized moments since they produce similar values of $S_3$ (see also Table 2 of Weinberg & Cole 1992).

A similar analysis for clusters of galaxies has been performed, but only in 2-dimensions (Borgani et al. 1994) and it was found that not only the skewness and the variance, in both Gaussian and non-Gaussian models, scale in a similar manner but also $S_3 \approx 1.8$ for all the models. Since however the projection from 3 to 2 dimensions Gaussianizes the distribution, as expected from the central limit theorem, these results may not be representative of the 3-D case.

It would be interesting to compare the cluster \textit{pdf} moments with those of galaxies for similar smoothing scales. In fact, we have one smoothing scale ($R_{sm} = 20 \ h^{-1} \ \text{Mpc}$) in common with the
analysis by Saunders et al. (1991) of the QDOT IRAS redshift sample, and it is this smoothing scale for which they found large values of the variance and the skewness and claimed that the CDM model is ruled out at a 97% confidence limit. Assuming that the linear biasing model is correct, on the relevant scales, and using a biasing factor between clusters and IRAS galaxies of $b_{c,I} \approx 4.5$ (Peacock & Dodds 1994) we predict

$$\sigma_{I,\text{pred.}}^2 = b_{c,I}^{-2}\sigma_c^2 = 0.0245 \pm 0.005$$

$$\gamma_{I,\text{pred.}} = b_{c,I}^{-3}\gamma_c = 0.0049 \pm 0.0018$$

while the corresponding measured values of the QDOT sample and of the CDM QDOT look-alikes (from Saunders et al. 1991) are $(\sigma^2, \gamma) = (0.0669, 0.025)$ and $(0.0192, -0.001)$ respectively. It is evident that the predicted IRAS moments from the cluster moments and from linear biasing are significantly smaller than the Saunders et al. (1991) direct estimation but still larger, although marginally so, than the CDM ones. Alternatively we can ask whether there is a unique biasing value, $b_{c,I}$, for which the rescaled cluster moments, to the IRAS level, are equal to the QDOT-IRAS ones. In fact there is; using $b_{c,I} \approx 2.7$ we succeeded in the above task. However, such a low value of $b_{c,I}$ does not seem to be supported by many different studies (cf. Jing & Valdarnini 1993; Peacock & Dodds 1994).

These results may be viewed as suggestive that the linear biasing model cannot be applied straightforwardly to the moments of the pdf. However, Juszkiewicz et al. (1993) have found that eq.(15) still holds for a biased population, if the linear fluctuations are Gaussian and the biased density is a local function of the matter density (see also Coles & Frenk 1991).

Jing & Valdarnini (1993) have found that the quadrupole anisotropy, observed in the COBE experiment, can be reproduced by the measured Abell cluster and IRAS galaxies power spectra, if their bias factors are $b_{c,I} \approx 3 - 4$ and $b_I \approx 1.5$, respectively (see also Peacock and Dodds 1994). These results, together with the value $S_3 \approx 1.8$ which we obtain in this work for Abell+ACO clusters and using the linear biasing framework, we estimate that for the matter distribution $S_{3m} \approx 8 - 12$. These values look unrealistically large since Goroff et al. (1986) obtain $S_{3m} \approx 3$ for a CDM spectra and a Gaussian smoothing with $R_{sm} \geq 20$ Mpc.

Although, in the range of interest the cluster power spectrum has an effective slope steeper than that of the CDM, the values of $S_{3m}$ cannot exceed an upper limit of $\approx 5$ obtained from perturbation theory (cf. Fig.1 of Juszkiewicz, Bouchet & Colombi 1993). Taken together all these results point toward a more cautious approach for the linear bias assumption. On the other hand, however, the optical cluster dipole (PV91; Scaramella et al. 1991) is found to be well aligned with that of the CBR, which means that clusters do effectively trace the underlying gravitational field.

We therefore suggest that the linear bias assumption is a good approximation for spatial regions where $\delta \gtrsim 0$ but breaks down in underdense regions where $\delta \lesssim 0$. The skewness is a measure of the asymmetry of the distribution with respect the mean and therefore one cannot simply relate the moments of the biased field to that of matter field by a constant factor.
5 Conclusions

We have derived the Abell+ACO cluster probability density function at different smoothing scales and we found it to be consistent with a log-normal distribution at all scales considered. Similar results were obtained from the analysis of synthetic clusters generated as high-peaks of a Gaussian background density field having the same low-order correlation properties and selection functions as the real clusters. Furthermore we determined the moments of the pdf and we found them to scale according to: \( \gamma = S_3 \sigma^4 \). Such a scaling is expected within the framework of Gaussian initial conditions and gravitational instability. However, since the clusters probe the linear field, the scaling is attributed to the fact that they are biased tracers of the underlying density field. Furthermore, this scaling is in agreement with the hierarchical clustering model with \( S_3 \equiv 3Q \approx 0.6 \), consistent with previous correlation function analysis (cf. VJ, PVJ). These results, together with the arguments of Kofman et al. (1994), indicate that the observed cluster distribution is consistent with it being generated from a initial Gaussian fluctuation spectrum.

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Table 1: The low-order moments of the Abell+ACO (subfix \( A \)) and simulated cluster (subfix \( sim \)) pdfs, as a function of smoothing scale \( R_{sm} \). The uncertainty in the real cluster moments is given from the scatter induced by using different \( W(R) \) functions (see text), while in the simulation case the uncertainties are due to the scatter around the ensemble mean values. Note that we also give the values for the fourth moment, \( K \), although in the text we do not discuss it due to the large uncertainty associated with its estimation. Note, however, that for the \( R_{sm} = 20 \ h^{-1} \) Mpc case, where we have a relatively more reliable estimation, \( S_4 (\equiv \frac{K - 3\sigma^4}{\sigma^4}) \approx 5.1 \) and 5.3 for the real and simulated cluster data, respectively.

| \( R_{sm} \) | \( \sigma^2_A \)   | \( \gamma_A \)   | \( K_A \)       | \( \sigma^2_{sim} \) | \( \gamma_{sim} \) | \( K_{sim} \)   |
|--------------|-----------------|-----------------|----------------|----------------------|-----------------|----------------|
| 20           | 0.496±0.057     | 0.4456±0.154    | 1.363±0.7      | 0.85±0.19            | 1.44±0.68       | 5.4±3.1        |
| 30           | 0.136±0.023     | 0.0327±0.0168   | 0.365±0.566    | 0.36±0.11            | 0.29±0.2        | 0.67±0.5       |
| 40           | 0.048±0.013     | 0.0040±0.005    | --             | 0.2±0.078            | 0.095±0.087     | 0.173±0.154    |
**Figure Captions**

**Figure 1.**
Comparison, cell-by-cell, of the original simulated cluster fluctuation field, $\delta_{ori}$, with the one recovered after applying the real cluster selection functions and geometrical boundaries and using the method described in section 3 (for $f(x_g) \geq 0.8$). A random subsample of 10%, from each of the 50 simulations, has been plotted.

**Figure 2.**
The simulated cluster pdf at the 4 smoothing scales, indicated in each panel (filled symbols). The errorbars are the scatter around the ensemble mean (over 50 simulations). The solid line is the log-normal fit to the data while the dashed line is the Gaussian fit.

**Figure 3.**
The real cluster pdf at the 3 smoothing scales indicated in each panel. Filled symbols correspond to $f(x_g) \geq 0.8$ while open symbols to $f(x_g) \geq 0.9$, for the $R_{sm} = 20$ and $30$ $h^{-1}$ Mpc cases, respectively. For $R_{sm} = 40$ $h^{-1}$ Mpc they correspond to 0.7 and 0.6, respectively. The errorbars are Poisson errors and the line fits are as in figure 2.

**Figure 4.**
Variance versus skewness plot. The open circles, solid squares, open stars and open squares correspond to $R_{sm} = 20, 30, 40$ and $50$ $h^{-1}$ Mpc respectively. The solid line is the best fitted line given by eq.(16). (a) Real clusters (b) Simulated clusters.

**Figure 5.**
The normalized skewness, $S_3$, versus smoothing scale for both real and simulated clusters.