Efficiency nonminimally supported design for three parameters weighted exponential model

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Abstract. The weighted exponential distribution function has a specific curve shape, the curve from zero reaches to the maximum point then down, and at a certain time it is relatively constant close to zero. This function can be used to describe the growth curve model. Minimally supported design is a design with the number of supported design equal to the number of parameters in the model. Locally D-optimal design for the weighted exponential model is minimally supported design with uniform weight. The standardized variance of D-optimal design is less than or equal to the number of parameters and maximized the standardized variance at the supported designs. We propose an alternative design by adding one supported design. Nonminimally supported design is obtained from supported design of D-optimal design plus one supported design in three ways, by adding one of them, by adding the average of them or by adding one supported design around them. We compare nonminimally supported designs in terms of efficiency, standardized variance, and propose a design that is efficient and practically convenient for practitioners.

1. Introduction

The D-optimal design usually uses minimally supported design with uniform weight [1,2,3]. Supported designs of D-optimal design are obtained by maximizing the determinant of the information matrix. The aim of D-optimality is minimizing the variance of parameter estimates so that the parameters in the model are significant. The D-optimal design also depends on the model which we used. In nonlinear models the determination of D-optimal design becomes complicated. This is because the information matrix in the nonlinear model contains an unknown parameter so it needs information about the value of the parameter.

The weighted exponential distribution function has a specific curve shape, the curve from zero reaches the maximum point then down, and at a certain time, it is relatively constant close to zero. This function can be used to describe the growth curve model. Gupta and Kundu [4] introduced the weighted exponential (WE) distribution as an alternative to Gamma or Weibull distribution. The distribution function of weighted exponentials is:

\[ f(x) = \frac{1+\beta}{\beta} \theta e^{-\alpha x} (1 - e^{-\alpha \beta x}), \quad x > 0, \alpha, \beta > 0 \]  

(1)

The curve of the model (1) has a maximum point at \( x = \frac{\ln(1+\beta)}{\alpha \beta} \). The model in equation (1) is a nonlinear model.
D-optimal design for the exponential model has been studied. There are many models that have been studied and each of them is used in different cases. In the homoscedastic case [5] developed model exponential and Poisson. Lall et al.[6] used a logistic model that was applied to agriculture. Detteet al.[7] use exponential, log-linear and EMAX models, the first model was applied in agriculture and called the Mitscherlichs growth model, the second and third models were applied for legume growth analysis, and the fourth model was applied in the field of pharmacokinetics. Atkinson [8] used an exponential model that is applied in the health field, which is determining the concentration of drugs in the blood. Widiharihet al.[9,10, 11] developed an exponential model with a mean weighted into a weighted exponential model and a generalized exponential model.

Based on the equation (1), we construct a design for the three-parameter weighted exponential that fulfills D-optimality as follows:

\[ y = \theta_3 e^{-\theta_1 x} (1 - e^{-\theta_2 x}) + \varepsilon, \quad x \geq 0, \theta_1, \theta_2, \theta_3 > 0 \]  

(2)

with homoscedastic errors assumption.

In this paper, we propose the nonminimally supported design of the model (2) by adding one supported design to minimally supported design that obtained from D-optimal design. We assume that all of the supported designs have uniform weight. The determinant of the information matrix of the model (2) for minimally and nonminimally supported design is calculated then determines the efficiency of nonminimally supported design. The highest efficiency for nonminimally supported design is designed with adding one supported design which is taken one of the supported designs of D-optimal design.

2. Material and methods

The nonlinear model is denoted by:

\[ y = \eta(x, \theta) + \varepsilon \]  

(3)

with independent \( \varepsilon \sim N(0, \sigma^2) \)

\[ E(Y|x) = \eta(x, \theta) \]  

(4)

The aim of this paper is to construct the design \( \xi \) containing the supported designs and their proportions. Designs \( \xi \) of supported designs \( (x_i, i = 1, 2, 3, \ldots, p)\) and their proportions \( (w_i, i = 1, 2, 3, \ldots, p)\) is denoted by:

\[ \xi = (x_1 \ x_2 \ \cdots \ x_p) \]  

(5)

where: \( w_i = \frac{r_i}{n} \) : number of observations of the supported design \( x_i \), \( n = \sum_{i=1}^{p} r_i \), \( \sum_{i=1}^{p} w_i = 1 \).

The information matrix of design \( \xi \) for the model (4) is:

\[ M(\xi, \theta) = \sum_{i=1}^{p} w_i h(x_i, \theta) h^T(x_i, \theta) \]  

(6)

where \( h(x, \theta) = \frac{\partial \eta(x, \theta)}{\partial \theta} = (h_1(x, \theta), h_2(x, \theta), \ldots, h_k(x, \theta))^T \) is the vector of partial derivatives of the conditional expectation \( E(Y|x) \) with respect to the parameters \( \theta \). \( M(\xi, \theta) \) is \( k \times k \) (\( k \): number of parameters) symmetric matrix. The information matrix in the nonlinear model contains unknown parameters so that it needs prior information about those parameter values. A D-optimal design maximizes \( |M(\xi, \theta)| \), which is the determinant of the information matrix. The standardized variance \( d(\xi, x) \) is:

\[ d(\xi, x) = h^T(x, \theta) M^{-1}(\xi, \theta) h(x, \theta) \]  

(7)
Determine of design $\xi$ which satisfy D-optimality based on The Equivalence Theorem that can be written as follows:

$$\xi^* \text{is D-optimal design } \Leftrightarrow d(\xi^*, x) \leq k \quad (8)$$

Algorithm to construct nonminimally supported design and their efficiency:

1. Determine the minimally supported design $(x_1, x_1, x_3)$ for the model (2) [9], by maximizing:

$$|M(\xi, \theta)| \propto e^{-2\theta_3 \Sigma_{i=1}^3 x_i[A + B]} \quad (9)$$

where:

$$A = \sum_{i=1}^2 (x_i x_j)^2 (1 - e^{-\theta_2 x_i})^2 (e^{-\theta_2 x_i} - e^{-\theta_2 x_j})^2, \quad i \neq j, j = 2, 3$$

$$B = 2 \sum_{i=1}^2 x_i x_j x_k^2 (1 - e^{-\theta_2 x_i}) (1 - e^{-\theta_2 x_j}) (e^{-\theta_2 x_i} - e^{-\theta_2 x_k}) (e^{-\theta_2 x_j} - e^{-\theta_2 x_k}), \quad i \neq j, j = 2, 3, k = 1, 2, 3$$

2. Based on step (1) we can find $x_1, x_1, x_3$ and the value of $|M(\xi, \theta)|$ (determination of minimally supported design).

3. Adding one supported design $(x_4)$ in three ways design as follows:

   I. $x_4$ is taken one of $x_1, x_2$ or $x_3$

   II. $x_4$ is taken of average $x_1, x_2$ and $x_3$

   III. $x_4$ is taken randomly around $x_1, x_2, x_3$ and they're average

4. Determine the determinant of the information matrix for all alternatif nonminimally supported design.

$$M(\xi, \theta) = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{bmatrix}$$

where:

$$m_{11} = \frac{1}{4} \sum_{i=1}^4 \theta_3 x_i^2 e^{-2\theta_1 x_i} (1 - e^{-\theta_2 x_i})^2$$

$$m_{22} = \frac{1}{4} \sum_{i=1}^4 \theta_3 x_i^2 e^{-2(\theta_1 + \theta_2) x_i}$$

$$m_{33} = \frac{1}{4} \sum_{i=1}^4 \theta_3 x_i^2 e^{-2\theta_1 x_i} (1 - e^{-\theta_2 x_i})^2$$

$$m_{12} = \frac{1}{4} \sum_{i=1}^4 -\theta_3 x_i^2 e^{-2(\theta_1 + \theta_2) x_i} (1 - e^{-\theta_2 x_i})$$

$$m_{13} = \frac{1}{4} \sum_{i=1}^4 -\theta_3 x_i e^{-2\theta_1 x_i} (1 - e^{-\theta_2 x_i})^2$$

$$m_{23} = \frac{1}{4} \sum_{i=1}^4 \theta_3 x_i e^{-2(\theta_1 + \theta_2) x_i} (1 - e^{-\theta_2 x_i})$$

5. Determine the efficiency of nonminimally supported design. The formula of efficiency as follows:

$$Eff = \frac{|M(\xi, \theta)|\text{of nonminimally supported design}}{|M(\xi, \theta)|\text{of minimally supported design}}$$

3. Results and discussions

Consider model (2):

$$y = \theta_3 e^{-\theta_1 x} (1 - e^{-\theta_2 x}) + \varepsilon, \quad x \geq 0, \theta_1, \theta_2, \theta_3 > 0$$

Model (2) has a unimodal curve, the maximum point occurs at point $x = \frac{\ln(1 + \theta_2)}{\theta_1}$. The curve of the
model (2) for $\theta_3 = 73.1146$ ; $\theta_1 = 0.07119$ at several of $\theta_2$ and for $\theta_3 = 73.1146$ ; $\theta_2 = 0.1561$ at several of $\theta_1$ are presented in Figure 1 and Figure 2.

**Figure 1.** The curve of the model (2) at several of $\theta_2$ for $\theta_3 = 73.1146$ ; $\theta_1 = 0.07119$

**Figure 2.** The curve of the model (2) at several of $\theta_1$ for $\theta_3 = 73.1146$ ; $\theta_2 = 0.1561$

Based on Figure 1, if $\theta_3$ and $\theta_1$ fixed with varying $\theta_2$ value and close each other, then each curve has a maximum relatively the same. Based on Figure 2, if $\theta_3$ and $\theta_2$ fixed and several of $\theta_1$ each curve has a different maximum.

In this paper, we propose nonminimally supported design when we have prior information those parameter value $\theta_1 = 0.05, \theta_2 = 0.025, \theta_3 = 1$ and $\theta_1 = 0.04, \theta_2 = 0.065, \theta_3 = 1$, the design region is $[0, 65]$.

1. Minimally supported design based on (9) as follows:

   **Table 1.** Minimally supported design model (2) for $\theta_1 = 0.05, \theta_2 = 0.025, \theta_3 = 1$ and $\theta_1 = 0.04, \theta_2 = 0.065, \theta_3 = 1$, the design region is $[0, 65]$

   | $\theta_1$ | $\theta_2$ | $\theta_3$ | $x_1$   | $x_2$   | $x_3$   | $|M(\xi, \theta)|$ |
   |------------|------------|------------|--------|--------|--------|----------------|
   | 0.05       | 0.025      | 1          | 7.5053 | 26.6819| 63.4042| 0.000054558481 |
   | 0.04       | 0.065      | 1          | 6.5366 | 23.9444| 60.3841| 0.029602510611 |

2. Adding one supported design ($x_4$) from Table 1 in three ways design, the result is presented in Table 2.

   I. $x_4$ is taken one of $x_1, x_2$ or $x_3$
   II. $x_4$ is taken of average $x_1$ , $x_2$ and $x_3$
   III. $x_4$ is taken randomly around $x_1$ , $x_2$ , $x_3$ and they're average.
Table 2. Nonminimally supported design model (2) for $\theta_1 = 0.05, \theta_2 = 0.025, \theta_3 = 1$ and $\theta_1 = 0.04, \theta_2 = 0.065, \theta_3 = 1$, the design region is [0, 65]

| Supported design | $\theta_1$ | $\theta_2$ | $\theta_3$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $| M(\xi, \theta) |$ | Efficiency |
|------------------|------------|------------|------------|-------|-------|-------|-------|----------------|------------|
|                  | 0.05       | 0.025      | 1          | 7.5053| 26.6819| 63.4042| 7.5053| 0.00000460      | 0.843102   |
|                  |            |            |            | 26.6819| 0.0000460| 0.843102|
|                  |            |            |            | 63.4042| 0.0000460| 0.843105|
|                  |            |            |            | 32.5305| 0.0000448| 0.820127|
|                  |            |            |            | 31.5305| 0.0000451| 0.826299|
|                  |            |            |            | 33.5305| 0.0000444| 0.813671|
|                  |            |            |            | 5.0503 | 0.0000444| 0.812339|
|                  |            |            |            | 6.0503 | 0.0000457| 0.836300|
|                  |            |            |            | 8.0503 | 0.0000458| 0.838059|
|                  |            |            |            | 24.6819| 0.0000458| 0.839455|
|                  |            |            |            | 25.6819| 0.0000460| 0.842214|
|                  |            |            |            | 27.6819| 0.0000460| 0.842254|
|                  |            |            |            | 28.6819| 0.0000459| 0.839808|
|                  |            |            |            | 61.4042| 0.0000460| 0.841742|
|                  |            |            |            | 62.4042| 0.0000460| 0.842759|
|                  |            |            |            | 64.4042| 0.0000460| 0.842757|
|                  |            |            |            | 65.0000| 0.0000460| 0.842219|

|                  | 0.04       | 0.065      | 1          | 6.5366| 23.9444| 60.3841| 6.5366| 0.0249771       | 0.84375    |
|                  |            |            |            | 23.9444| 0.0249771| 0.84375|
|                  |            |            |            | 60.3841| 0.0249771| 0.84375|
|                  |            |            |            | 30.2884| 0.0240555| 0.81262|
|                  |            |            |            | 29.2884| 0.0242739| 0.81999|
|                  |            |            |            | 31.2884| 0.0233857| 0.80519|
|                  |            |            |            | 5.5366 | 0.0247062| 0.83460|
|                  |            |            |            | 4.5366 | 0.0237388| 0.80192|
|                  |            |            |            | 7.5366 | 0.0247840| 0.83723|
|                  |            |            |            | 8.5366 | 0.0243439| 0.82236|
|                  |            |            |            | 22.9444| 0.0249440| 0.84263|
|                  |            |            |            | 21.9444| 0.0248428| 0.83921|
|                  |            |            |            | 25.9444| 0.0248575| 0.83971|
|                  |            |            |            | 26.9444| 0.0247210| 0.83510|
|                  |            |            |            | 59.3841| 0.0249669| 0.84341|
|                  |            |            |            | 58.3841| 0.0249366| 0.84238|
|                  |            |            |            | 61.3841| 0.0249669| 0.84341|
|                  |            |            |            | 62.3841| 0.0249364| 0.84237|

Based on Table (1) and (2), shows that minimally supported design is the highest value for the determinant of the information matrix. The highest efficiency for nonminimally supported design is designed I, $x_4$ is taken one of $x_1, x_2$ or $x_3$. The standardized variance ($d(\xi, x)$) for minimally supported design is less than or equal three, while the standardized variance ($d(\xi, x)$) for nonminimally supported design is less than or equal four. Therefore, Design I is the practical design we recommend over other alternatives for nonminimally supported design due to its high efficiency.
4. Conclusion

The results of this research indicate that the more supported designs used there are no guarantee of better results based on the determinant of the value of the information matrix. Minimally supported design with uniform weight has the highest determinant of the information matrix. The highest efficiency for nonminimally supported design is designed I, $x_4$ is taken one of $x_1, x_2$ or $x_3$. The design I is the practical design we recommend over other alternatives for nonminimally supported design due to its high efficiency.

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