Representations of Algebraic Groups and Principal Bundles on Algebraic Varieties

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Abstract

In this talk we discuss the relations between representations of algebraic groups and principal bundles on algebraic varieties, especially in characteristic $p$. We quickly review the notions of stable and semistable vector bundles and principal $G$-bundles, where $G$ is any semisimple group. We define the notion of a low height representation in characteristic $p$ and outline a proof of the theorem that a bundle induced from a semistable bundle by a low height representation is again semistable. We include applications of this result to the following questions in characteristic $p$:

1) Existence of the moduli spaces of semistable $G$-bundles on curves.

2) Rationality of the canonical parabolic for nonsemistable principal bundles on curves.

3) Luna’s etale slice theorem.

We outline an application of a recent result of Hashimoto to study the singularities of the moduli spaces in (1) above, as well as when these spaces specialize correctly from characteristic 0 to characteristic $p$. We also discuss the results of Laszlo-Beauville-Sorger and Kumar-Narasimhan on the Picard group of these spaces. This is combined with the work of Hara and Srinivas-Mehta to show that these moduli spaces are $F$-split for $p$ very large. We conclude by listing some open problems, in particular the problem of refining the bounds on the primes involved.

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1. Some Definitions

We begin with some basic definitions:

Let $V$ be a vector bundle on a smooth projective curve $X$ of genus $g$ over an algebraically closed field (in any characteristic).

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Definition 1.1: $V$ is stable (respectively semi-stable) if for all subbundles $W$ of $V$, we have

\[
\mu(W) \coloneqq \deg \frac{W}{\text{rk } W} < (\leq) \mu(V) \coloneqq \deg \frac{V}{\text{rk } V}.
\]

For integers $r$ and $d$ with $r > 0$, one constructs the moduli spaces $U^s(r, d)(U(r, d))$ of stable (semistable) vector bundles of rank $r$ and degree $d$, using Geometric Invariant Theory (G.I.T.).

If the ground field is $\mathbb{C}$, the complex numbers, one has the basic (genus $X \geq 2$):

**Theorem 1.2:** Let $V$ have degree 0. Then $V$ is stable $\iff$ $V \simeq V_\sigma$, for some irreducible representation $\sigma : \pi_1(X) \to U(n)$.

This is due to Narasimhan-Seshadri. Note that $H \to X$ is a principal $\pi_1(X)$ fibration, where $H$ is the upper-half plane. Any $\sigma : \pi_1(X) \to GL(n, \mathbb{C})$ gives a vector bundle of rank $n$ on $X, V_\sigma = H \times^{\pi_1(X)} \mathbb{C}^n$.

**Remark 1.3:** It follows from Theorem 1.2 that if $V$ is a semistable bundle on a curve $X$ over $\mathbb{C}$, then $\otimes^n(V), S^n(V)$, in fact any bundle induced from $V$ is again semistable. By Lefschetz, this holds for any algebraically closed field of characteristic 0.

**Remark 1.4:** In general, a subbundle $W$ of a vector bundle $V$ is a reduction of the structure group of the principal bundle of $V$ to a maximal parabolic of $GL(n), n = \text{rank } V$. This is in turn equivalent to a section $\sigma$ of the associated fibre space:

\[E \times^{GL(n)} GL(n)/P.\]

Now let $X$ be a smooth curve and $E \overset{\pi}{\to} X$ a principal $G$-bundle on $X$, where $G$ is a semisimple (or even a reductive) group in any characteristic.

**Definition 1.5:** $E$ is stable (semistable) $\iff \forall$ maximal parabolics $P$ of $G, \forall$ sections $\sigma$ of $E(G/P)$, we have degree $\sigma^\# T_\pi > 0(\geq 0)$, where $T_\pi$ is the relative tangent bundle of $E(G/P) \overset{\pi}{\to} X$.

Over $\mathbb{C}$, we have the following [18]:

**Theorem 1.6:** $E \to X$ is stable $\iff E \simeq E_\sigma$ for some irreducible representation $\sigma : \pi_1(X) \to K$, the maximal compact of $G$.

The analogue of Remark 1.3 is valid in this general situation.

**Remark 1.7:** One can analogously define stable and semistable vector bundles and principal bundles on normal projective varieties of dimension $> 1$. Again, in characteristic 0, bundles induced from semistable bundles continue to be semistable.

**Remark 1.8:** In characteristic $p$, bundles induced from semistable bundles need not be semistable, in general[7]. In this lecture we shall examine some conditions when this does hold, and also discuss some applications to the moduli spaces of principal $G$-bundles on curves.

### 2. Low height representations
Here we introduce the basic notion of a low height representation in characteristic $p$. Let $f : G \to SL(n) = SL(V)$ be a representation of $G$ in char $p$, $G$ being reductive. Fix a Borel $B$ and a Torus $T$ in $G$. Let $L(\lambda_i), 1 \leq i \leq m$, be the simple $G$-modules occurring in the Jordan-Holder filtration of $V$. Write each $\lambda_i$ as $\sum q_{ij} \alpha_j$, where $\{\alpha_j\}$ is the system of simple roots corresponding to $B$ and $q_{ij} \in Q \forall i,j$. Define $ht\lambda_i = \sum q_{ij}$. Then one has the basic [9,20]:

**Definition 2.1:** $f$ is a low-height representation of $G$, or $V$ is a low-height module over $G$, if $2ht(\lambda_i) < p \forall i$.

**Remark 2.2:** If $2ht(\lambda_i) < p \forall i$, then it easily follows that $V$ is a completely reducible $G$-module. In fact for any subgroup $\Gamma$ of $G$, $V$ is completely reducible over $\Gamma \Leftrightarrow \Gamma$ itself is completely reducible in $G$. By definition, an abstract subgroup $\Gamma$ of $G$ is completely reducible in $G \Leftrightarrow$ for any parabolic $P$ of $G$, if $\Gamma$ is contained in $P$ then $\Gamma$ is contained in a Levi component $L$ of $P$. These results were proved by Serre [20] using the notion of a saturated subgroup of $G$.

In general, denote $\sup (2ht \lambda_i)$ by $ht_GV$. If $V$ is the standard $SL(n)$ module, then $ht_{SL(n)}^\lambda(V) = i(n-i), 1 \leq i \leq n-1$. More generally, $ht_G(V_1 \otimes V_2) = ht_GV_1 + ht_GV_2$. The following theorem is the key link between low-height representations and semistability of induced bundles [9]:

**Theorem 2.3:** Let $E \to X$ be a semistable $G$-bundle, where $G$ is semisimple and the base $X$ is a normal projective variety. Let $f : G \to SL(n)$ be a low-height representation. Then the induced bundle $E(SL(n))$ is again semistable.

The proof is an interplay between the results of Bogomolov, Kempf, Rousseau and Kirwan in G.I.T. on one hand and the results of Serre mentioned earlier on the other. The group scheme $E(G)$ over $X$ acts on $E(SL(n)/P)$ and assume that $\sigma$ is a section of the latter. Consider the generic point $K$ of $X$ and its algebraic closure $\overline{K}$. Then $E(G)_{\overline{K}}$ acts on $E(SL(n)/P)_{\overline{K}}$, and $\sigma$ is a $K$-rational point of the latter. There are 2 possibilities:

1) $\sigma$ is G.I.T semistable. In this case, one can easily prove that $\deg \sigma^\# T_\sigma \geq 0$.

2) $\sigma$ is G.I.T. unstable, i.e., not semistable. Let $P(\sigma)$ be the Kempf-Rousseau parabolic for $\sigma$, which is defined over $\overline{K}$. For $\deg \sigma^\# T_\sigma$ to be $\geq 0$ it is sufficient that $P(\sigma)$ is defined over $K$. Note that since $V$ is a low-height representation of $G$, one has $p \geq h$. One then has ([20]).

**Proposition 2.4:** If $p \geq h$, there is a unique $G$-invariant isomorphism $\log : G^u \to g_{nilp}'$, where $G^u$ is the unipotent variety of $G$ and $g_{nilp}'$ is the nilpotent variety of $g_{nilp} = \text{Lie } G$.

Proposition 2.4 is used in

**Proposition 2.5:** Let $H$ be any semisimple group and $W$ a low-height representation of $H$. Let $W_1 \subset W$ and assume that $\exists X \in \text{Lie } H$, $X$ nilpotent such that $X \in \text{Lie } (\text{Stab } (W_1))$. Then in fact one has $X \in \text{Lie } [\text{Stab } (W_1)_{\text{red}}]$.

Along with some facts from G.I.T, Proposition 2.5 enables us to prove that $P(\sigma)$ is in fact defined over $K$, thus finishing the sketch of the proof of Theorem 2.3. See also Ramanathan-Ramanan [19]. One application of low-height representations...
is in the proof of a conjecture of Behrend on the rationality of the canonical parabolic or the instability parabolic. If \( V \) is a nonsemistable bundle on a variety \( X \), then one can show that there exists a flag \( V \),

\[
0 = V_0 \subset V_1 \subset V_2 \cdots \subset V_n = V
\]
of subbundles of \( V \) with the properties:

1. Each \( V_i/V_{i-1} \) is semistable and \( \mu (V_i/V_{i-1}) > \mu (V_{i+1}/V_i) \), \( 1 \leq i \leq n - 1 \).
2. The flag \( V \) as in (1) is unique and infinitesimally unique, i.e., \( V \) is defined over any field of definition of \( X \) and \( V \). Such a flag corresponds to a reduction to a parabolic \( P \) of \( GL(n) \) and properties (1) and (2) may be expressed as follows: the elementary vector bundles on \( X \) associated to \( P \) all have positive degree and \( H^0(X, E(g)/E(p)) = 0 \), where \( g = \text{Lie GL}(n) \) and \( p = \text{Lie } P \).

One may ask whether there is a such a canonical reduction for a non semistable principal \( G \) bundle \( E \rightarrow X \). Such a reduction was first asserted first by Ramanathan [18], and then by Atiyah-Bott[1] both over \( \mathbb{C} \) and both without proofs. It was Behrend [5], who first proved the existence and uniqueness of the canonical reduction to the instability parabolic in all characteristics. Further, Behrend conjectured that \( H^0(X, E(g)/E(p)) = 0 \).

In characteristic zero, one can check that all three definitions of the instability parabolic coincide and that Behrend’s conjecture is valid. In characteristic \( p \), one uses low-height representations to show the equality of the three definitions and prove Behrend’s conjecture [14].

**Theorem 2.6:** Let \( E \rightarrow X \) be a nonsemistable principal \( G \)-bundle in char \( p \). Assume that \( p > 2 \dim G \). Then all the 3 definitions coincide and further we have \( H^0(X, E(g)/E(p)) = 0 \), where \( p = \text{Lie } P \) and \( P \) is the instability parabolic.

Theorem 2.6 is useful, among other things, for classifying principal \( G \)-bundles on \( \mathbb{P}^1 \) and \( \mathbb{P}^2 \) in characteristic \( p \).

If \( V \) is a finite-dimensional representation of a semisimple group \( G \) (in any characteristic), then the G.I.T. quotient \( V//G \) parametrizes the closed orbits in \( V \). Now, let the characteristic be zero and let \( v_0 \in V \) have a closed orbit. Then Luna’s étale slice theorem says that \( \exists \) a locally closed non-singular subvariety \( S \) of \( V \) such that \( v_0 \in S \) and \( S//G_{v_0} \) is isomorphic to \( V//G \), locally at \( v_0 \), in the étale topology. Here \( G_{v_0} \) is the stabilizer of \( v_0 \). The proof uses the fact that \( G_{v_0} \) is a reductive subgroup of \( G \) (not necessarily connected!), hence \( V \) is a completely reducible \( G \) module. In characteristic \( p \), one has to assume that \( V \) is a low-hi representation of \( G \). Then the conclusion of Luna’s étale slice theorem is still valid: to be more precise, let \( V \) be a low-hi representation of \( G \) and let \( v_0 \in V \) have a closed orbit. Put \( H = \text{Stab } (v_0) \). The essential point, as in characteristic 0, is to prove the complete reducibility of \( V \) over \( H \). Using the low-hi assumption, one shows that every \( X \in \text{Lie } H \) with \( X \) nilpotent can be integrated to a homomorphism \( G_{\alpha} \rightarrow H \) with tangent vector \( X \). Now, under the hypothesis of low-hi, one shows that \( H_{\text{red}} \) is a saturated subgroup of \( G \) and \( (H_{\text{red}} : H^0_{\text{red}}) \) is prime to \( p \). This shows that \( V \)
is a completely reducible $H_{\text{red}}$ module. Further, one shows that $H_{\text{red}}$ is a normal subgroup of $H$ with $H/H_{\text{red}}$ a finite group of multiplicative type, i.e. a finite subgroup of a torus. Now the complete reducibility of $V$ over $H$ follows easily [11].

Just as in characteristic zero, one deduces the existence of a smooth $H$-invariant subvariety $S$ of $V$ such that $v_0 \in S$ and $S//H$ is locally isomorphic to $V//G$ at $v_0$. This result is used in the construction of the moduli space $M_G$ to be described in the next section.

3. Construction of the moduli spaces

The moduli spaces of semistable $G$-bundles on curves were first constructed by Ramanathan over $\mathbb{C}$ [16,17], then by Faltings and Balaji-Seshadri in characteristic 0 [3,6]. There are 3 main points in Ramanathan’s construction:

1. If $E \to X$ is semistable, then the adjoint bundle $E(g)$ is semistable.
2. If $E \to X$ is polystable, then $E(g)$ is also polystable.
3. A semisimple Lie Algebra in char 0 is rigid.

The construction of $M_G$ in char $p$ was carried out in [2,15]. We describe the method of [15] first: points (1) and (2) are handled by Theorem 2.3 and the following [11]:

**Theorem 3.1:** Let $E \to X$ be a polystable $G$-bundle over a curve in char $p$. Let $\sigma: G \to SL(n) = SL(V)$ be a representation such that all the exterior powers $\wedge^i V, 1 \leq i \leq n - 1$, are low-height representations. Then the induced bundle $E(V)$ is also polystable.

The proof uses Luna’s étale slice theorem in char $p$ and Theorem 2.3.

Now one takes a total family $T$ of semistable $G$ bundle on $X$ and takes the good quotient of $T$ to obtain $M_G$ in char $p$. Theorem 3.1 is used to identify the closed points of $M_G$ as the isomorphism classes of polystable $G$-bundles, just as in char 0. The semistable reduction theorem is proved by lifting to characteristic 0 and then applying Ramanathan’s proof (in which (3) above plays a crucial role).

This construction follows Ramanathan very closely and, as is clear, one has to make low-height assumptions as in Theorem 3.1.

The method of [2] follows the one in [3] with some technical and conceptual changes. One chooses an embedding $G \to SL(n)$ and a representation $W$ for $SL(n)$ such that (1) $G$ is the stabilizer of some $w_0 \in W$. (2) $W$ is a “low separable index representation” of $SL(n)$, i.e., all stabilizers are reduced and $W$ is low-height over $SL(n)$. The semistable reduction theorem is proved using the theory of Bruhat-Tits. Here also suitable low-height assumptions have to be made.

4. Singularities and specialization of the moduli spaces

We first discuss the singularities of $M_G$, assuming throughout that $G$ is simply connected. In char 0, $M_G$ has rational singularities, this follows from Boutot’s theorem. In char $p$, the following theorem due to Hashimoto [8] is relevant:
**Theorem 4.1:** Let $V$ be a representation of $G$ such that all the symmetric powers $S^n(V)$ have a good filtration. Then the ring of invariant $[S(V)]^G$ is strongly $F$-regular.

Strongly $F$-regularity is a notion in the theory of tight closure in commutative algebra. We just note that if a geometric domain is strongly $F$-regular then it is normal, Cohen-Macaulay, $F$-split and has “rational-like” singularities. Now let $t \in M_G$ be the “worst point”, i.e., the trivial $G$-bundle on $X$.

The local ring $(\mathcal{O}_{M_G,t})^\wedge$ is isomorphic to $(S(W)/G)^\wedge$, where $W =$ direct sum of $g$ copies of $g$, with $G$ acting diagonally. If $p$ is a good prime for $G$, then Hashimoto’s theorem implies that $\mathcal{O}_{M_G,t}$ is strongly $F$-regular. The other points of $M_G$ are not so well understood. This would require a detailed study of the automorphism groups of polystable bundles, both in char 0 and $p$, and of their invariants of the slice representations. This is necessary also to study the specialization problem, i.e., when $M_G$ in char 0 specializes to $M_G$ in char $p$. One has to show that the invariants of the slice representations in char 0 specialize to the invariants in char $p$. However for $G=\text{SL}(n)$, the situation is much simpler. One can write down the automorphism group of a polystable bundle and its representation on the local moduli space. Consequently, one expects the moduli spaces to specialize correctly and that the local rings of $M_G$ are strongly $F$-regular in all positive characteristics.

We briefly discuss Pic $M_G$ in char 0. It follows from [4,10] that $M_G$ has the following properties in char 0:

1. Pic $M_G \simeq \mathbb{Z}$.
2. $M_G$ is a normal, projective, Gorenstein variety with rational singularities and with $K$ negative ample.

Now let $X$ be a normal, Cohen-Macaulay variety in char 0. It is proved in [13], in response to a conjecture of Karen Smith, that if $X$ has rational singularities, then the reduction of $X$ mod $p$ is $F$-rational for all large $p$. This result together with 1 and 2 above imply that $M_G$ reduced mod $p$ is $F$-split for all large $p$. We cannot give effective bounds on the primes involved. One partial result is known in this direction [12].

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**References**

[1] M.F. Atiyah, R. Bott, The Yang-Mills equations over Riemann Surfaces, *Phil. Trans. R. Soc.* London A 308 (1982), 523–615.

[2] V. Balaji, A.J. Parameswaran Semistable Principal Bundles-II (in positive characteristics) to appear in *Transformation Groups*.

[3] V. Balaji, C.S. Seshadri Semistable Principal Bundles-I (in characteristic zero), to appear in *Journal of Algebra*.

[4] A. Beauville, Y. Laszlo, C. Sorger, The Picard Group of the moduli of $G$-bundles over curves, *Compositio Math.* 112, (1998), No.2, 183–216.
[5] K. Behrend, Semistability of reductive group schemes over curves, *Math. Ann.* 301 (1995), 281–305.

[6] G. Faltings, Stable $G$-bundles and projective connections, *J. Algebraic Geom.* 2 (1993) No.3, 507–568.

[7] D. Gieseker, Stable Vector Bundles and the Frobenius morphism, *Ann. Sci. Ecol. Nor. Sup.* 6, (1973).

[8] M. Hashimoto, Good filtrations of symmetric algebras and strong $F$-regularity of invariant subrings, *Math. Z.* 236 (2001), No.3, 605–623.

[9] S. Ilangovan, V.B. Mehta, A.J. Parameswaran, Semistability and Semisimplicity in representations of low-height in positive characteristics, preprint.

[10] S. Kumar, M.S. Narasimhan, Picard group of the moduli spaces of $G$-bundles, *Math. Ann.* 308, (1997), No.1, 155–173.

[11] V.B. Mehta, A.J. Parameswaran, Geometry of low-height representations, *Proceedings of the International Colloquium on Algebra, Arithmetic and Geometry*, (ed. R. Parimala), TIFR Mumbai 2000.

[12] V.B. Mehta, T.R. Ramadas, Moduli of vector bundles, Frobenius splitting and invariant theory, *Ann. of Math.* (2) 144, (1996), 269–313.

[13] V.B. Mehta, V. Srinivas, A characterization of rational singularities, *Asian J. Math.*, Vol.1, (1997), No.2, 249–271.

[14] V.B. Mehta, S. Subramanian, On the Harder-Narasimhan Filtration of Principal Bundles, *Proceedings of the International Colloquium on Algebra, Arithmetic and Geometry*, (ed. R. Parimala), TIFR Mumbai 2000.

[15] V.B. Mehta, S. Subramanian, Moduli of Principal $G$-bundles on curves in positive characteristic, in preparation.

[16] A. Ramanathan, Moduli for principal bundles over algebraic curves I, *Proc. Indian Acad. Sci. Math. Sci.*, 106, (1996), No.3, 301–328.

[17] A. Ramanathan, Moduli for principal bundles over algebraic curves II, *Proc. Indian Acad. Sci. Math. Sci.*, 106 (1996), No.4, 421–449.

[18] A. Ramanathan, Moduli for principal bundles, in: *Algebraic Geometry*, Proceedings, Copenhagen 1978, 527–533, Lecture Notes in Mathematics vol. 732, Springer.

[19] S. Ramanan, A. Ramanathan, Some remarks on the instability flag, *Tohoku Math. Journal* 36, (1984), 269–291.

[20] J-P. Serre, *Moursund Lectures*, University of Oregon Mathematics Department, notes by W.E. Duckworth (1998).