Strong times and first hitting.

F. Manzo, *
E. Scoppola †

June 24, 2016

Abstract

We generalize the notion of strong stationary time and we give a representation formula for the hitting time to a target set in the general case of non-reversible Markov processes.

1 Introduction

This work was originally motivated by the study of the first hitting to rare sets for ergodic Markov chains. Our aim was to provide a unifying language for different approaches to the problem, focusing on the link between rarity and exponentiality, in particular for metastable systems.

Central results in this approach, concern the distance between $e^{-t}$ and the tail distribution of the ratio between the hitting time $\tau_G$ and its mean starting from the invariant measure $\pi$

$$\mathbb{P}_\pi\left(\frac{\tau_G}{\mathbb{E}_\pi \tau_G} > t \right).$$

(1.1)

---

*Supported by Dipartimento di Matematica e Fisica, Università di Roma Tre
†Dipartimento di Matematica e Fisica, Università di Roma Tre, Largo S. Leonardo Murialdo 1, 00146 Rome, Italy
Explicit, time-uniform, bounds of this distance were given, starting from [2], in terms of the ratio between a “local relaxation time” $R$ and $\mathbb{E}_\pi (\tau_G)$. In the literature, the role of this local relaxation time $R$ was played by many different times such as the mixing time [2], the relaxation time (i.e., the inverse of the smallest non-zero eigenvalue of the generator of the dynamics) in [3], the inverse of the spectral radius of the sub-markovian generator restricted outside $G$ [8] and some other, sometimes model-dependent, choices. Heuristically, when $R \ll \mathbb{E}_\pi \tau_G$, the system relaxes to a local equilibrium before attempting to reach $G$. Extensions to other initial measures (in particular to the conditional equilibrium measure $\pi(\cdot|\mathcal{X}\setminus G)$ and to the quasi-stationary measure [3]), or to non-reversible settings were given over the years.

Rather independently, in the early eighties the statistical mechanics community began to study metastability as a dynamical phenomenon, in the framework of discrete models with Glauber dynamics. The regimes studied in this context were not as general as that in the above mentioned papers but, on the other hand, the physical meaning of the exponential behavior was very transparent. In this case the target set $G$ is often the “basin of attraction” of the stable state and the role of the relaxation time $\mathbb{E}_\pi \tau_G$ is taken by $\mathbb{E}_\mu \tau_G$ where $\mu$ is some “metastable measure” concentrated outside this basin. We refer to [1] and [14] for a discussion and comparison of the different approaches to prove exponential behavior in metastable system. We emphasize the fact that a characterization of “good metastable” initial measures, that give rise to the exponential behavior of the relaxation time, is of primary interest in this series of works.

Let us give a very quick heuristic. In the easiest asymptotic regime, the system is trapped into an energy well and spends in the bottom of this well most of the time before reaching the boundary and exit. If this bottom is represented by a single point, each return time the process looses memory of its past. This renewal property gives rise to the exponential behavior. Large-deviation methods, renormalization ideas, coupling, simulated annealing techniques and, more recently, potential theory and martingales have been used to extend this picture to physically more interesting settings, provided the renewal properties of a metastable point are strong enough.

Here we generalize this language, by “changing the renewal point for a renewal measure”. The obvious candidate for this role is the quasi-stationary measure (see [11]), namely:

$$
\mu^*(\cdot) := \lim_{t \to \infty} \mathbb{P}(X_t = \cdot \mid t < \tau_G).
$$

(1.2)
Indeed, it is easy to see that the evolution starting from this measure is exponential, in the sense that
\[
P_{\mu^*} (X_t = y) =: \mu_t^\alpha (y) = \begin{cases} 
\lambda^t \mu^* (y) & \text{if } y \notin G \\
(1 - \lambda^t) \omega (y) & \text{if } y \in G 
\end{cases}
\] (1.3)

where \( \lambda \) is the largest eigenvalue of the sub-markovian matrix obtained from \( P \) by canceling out the entries in \( G \), and where \( \omega (y) := P (\tau_{G^c}^\mu = \tau_y) \) is the probability that \( y \) is the hitting point.

We will refer to the measure \( \mu_t^\mu^* \) as the *squeasing quasi-stationary measure*.

The idea is then to control how close is \( \mu_t^\alpha (y) := P (X_t^\alpha = y) \) to \( \mu_t^\mu^* (y) \) when \( \alpha \) is some other starting measure.

In order to proceed in this direction, we introduce a sort of “hitting time to a measure” by generalizing the notion of *strong stationary time*, introduced in [6] (under the name of *strong uniform time*).

Using the strong time language, we are able to give a representation formula for the probability \( P (\tau_{G^c}^\alpha > t ; X_t^\alpha = y) \) in terms of events concerning these strong times. This representation formula gives a probabilistic interpretation of the errors in the exponential approximation and is very explicit about the role of the initial measure \( \alpha \).

Let us recall, from [6], the following

**Definition 1.1** A randomized stopping time \( \tau_{\pi}^\alpha \) is a Strong Stationary Time (SST in the following) for the Markov chain \( X_t^\alpha \) with starting distribution \( \alpha \) and stationary measure \( \pi \), if
\[
P (X_t^\alpha = y, \tau_{\pi}^\alpha = t) = \pi (y) P (\tau_{\pi}^\alpha = t).
\]

SSTs were introduced in [6], where their existence was proved. The proof also shows that the fastest SST is distributed according to the separation between the measure at time \( t \) and the stationary measure.

Explicit constructions of SSTs can be done in very particular cases, in one dimension or in very symmetric systems ([5, 12]), where these constructions were used e.g. to show cutoff behavior.

Separation itself is not the easiest notion of distance between measures to compute, but the separation between the measure at time \( t \) and the stationary measure has the remarkable property of being submultiplicative and makes it usable to give exponential bounds.
In our point of view, strong times provide a new language to describe the approach to equilibrium or, in our case, to quasi-stationarity. Actually, inspired by metastability, we will consider the hitting time $\tau_G$ as the decay time of the metastable state. It is then natural to assume $G$ an absorbing state and ergodicity on $\mathcal{X}\setminus G$. We are not interested in finding explicit constructions: all we want to do is to use exponential bounds on the separation in order to estimate the tail distribution of the hitting time.

Let us mention that the idea of using a strong time that somehow catches the arrival to the quasi-stationary measure is not new in the literature; in [13], for a birth-and-death process starting from 0, in a particular regime, the authors construct what they call a “strong quasi-stationary time” for this purpose.

Our approach is different under two fundamental aspects:

1. Our notion of Conditionally Strong Quasi Stationary Time is completely general and its existence does not require any additional assumptions besides ergodicity of the stochastic matrix outside $G$. The prize to pay is that, in general, we cannot construct explicitly such times.

2. Our target is not a fixed measure but a family of measures indexed by the time $t$.

The reason behind our choice is that while the measure $\mu^*$ is concentrated outside $G$, in general the evolved measure $\mu_t^G$ is not. On the contrary, when $G$ is absorbing, $\mu_t^G$ concentrates on $G$. Therefore, for general models and general starting states, there is no hope to reach $\mu^*$ at a positive time.

A natural candidate for the role of “target measure” is instead the “squeazing measure” $\mu_t^\mu$ or, more in general, a family of measures $\mu_t$, with the property $\mu_{t+1}(x) = \sum_{y \in \mathcal{X}} \mu_t(y) P_{x,y}$.

This choice allows to define properly a strong time $\tau^{\alpha}_\mu$ such that

$$\mathbb{P}(X^\alpha = y, \tau^{\alpha}_\mu = t) = \mu_t(y) \mathbb{P}(\tau^{\alpha}_\mu = t).$$

Unfortunately, as we will see, this time decays in a time of order $\mathbb{E}(\tau^G_G)$ and it is too large for the applications we have in mind. The reason is, $\tau^{\alpha}_\mu$ gives the same role to the points in $G$ and outside $G$. A good “local relaxation time” instead, should regard only what happens outside $G$.

For this reason, it is natural to consider a conditional time:
Definition 1.2. A randomized stopping time $\tau^*_\alpha$ is a conditionally-strong quasi-stationary time (CSQST in the following) if for any $y \not\in G$, $$P(X^\alpha_t = y, \tau^\alpha_* = t \mid t < \tau^\alpha_G) = \mu^*(y)P(\tau^\alpha_* = t \mid t < \tau^\alpha_G)$$ (1.5) or, in other words, $$P(X^\alpha_t = y, \tau^\alpha_* = t) = \mu^*(y)P(\tau^\alpha_* = t < \tau^\alpha_G).$$ (1.6)

The idea is to use this CSQST in the decomposition

$$P(\tau^\alpha_G > t) = P(\tau^\alpha_G > t; \tau^\alpha_* \leq t) + P(\tau^\alpha_G > t),$$ (1.7)

where $\tau^\alpha_*,G = \tau^\alpha_G \land \tau^\alpha_*$. The event in the first term in the r.h.s. of (1.7) can be read as the probability that the process reaches $G$ after reaching the “metastable equilibrium”. Since in our setting, the metastable equilibrium is related to the quasi-stationary measure $\mu^*$, we easily get exponential bounds.

Its counterpart is the event that the process stays away from $G$ without reaching the “metastable equilibrium”. From (1.7), we obtain a probabilistic interpretation of the error term in the exponential bound. The role of the local relaxation time, in our approach, is being played by $\tau^\alpha_*,G$.

Both terms in r.h.s of (1.7) have exponential decay for large $t$. The exponential behavior of $P(\tau^\alpha_G > t)$ emerges when the last term decays faster than the other and can be neglected.

The role of the initial measure $\alpha$ can be further clarified by the introduction of a “time-shift”: different starting measures can help or hinder achieving $G$. Asymptotically, this fact results in a time-shift $\delta_\alpha$, and our choice of CSQST corresponds to $\rho_t = \mu^\alpha_{t+\delta_\alpha}$ in (1.4).

Indeed, just like fastest SSTs are related to the separation between the measure at a given time and the stationary measure, “minimal” CSQSTs are related to the separation

$$\tilde{s}^\alpha(t) := \max_{y \not\in G} 1 - \frac{\mu^\alpha_t(y)}{\mu^\mu_{t+\delta_\alpha}(y)},$$ (1.8)

which quantifies a sort of distance between $\mu^\alpha_t$ and $\mu^\mu_{t+\delta_\alpha}$ by taking care only of the points outside $G$.

In subsection 1.2 we will show that $\tilde{s}^\alpha$ is the separation between the evolution of an auxiliary Markov chain in $\mathcal{X}\backslash G$ and its stationary measure. Therefore $\max_\alpha \tilde{s}^\alpha$ it is submultiplicative and decays exponentially in time.
By using $\tilde{s}^\alpha$, (1.7) can be rephrased in a more usable representation formula as:

$$P(\tau_G^\alpha > t) = \lambda^t + \delta^\alpha (1 - \tilde{s}^\alpha(t)) + P(\tau_{x,G}^\alpha > t). \quad (1.9)$$

We refer to subsection 1.3 for more precise statements.

Outline of the paper. In subsection 1.1 the general setting and definitions are fixed. In subsection 1.2 we introduce a local chain on $\mathcal{X}\setminus G$ which will be crucial in our discussion, while in subsection 1.3 our main results are stated. Section 2 is devoted to the introduction of the central object in this paper: the generalization of strong stationary times to other target evolving measures. Subsection 2.1 contains the proof of Theorem 1.4 and subsection 2.2 the construction of these strong times with an auxiliary chain. Section 3 contains the proof of Theorem 1.8 and Section 4 the proof of Theorem 1.9. Concluding remarks and future perspectives are discussed in Section 5.

1.1 General setting and definitions

We collect in this subsection definitions and notations used in the paper.

- **Process:** we will consider discrete time Markov chains $\{X_t\}_{t \in \mathbb{N}}$ on a countable state space $\mathcal{X}$. We denote by $P(x, y)$ the transition matrix and by $\mu^t_x(\cdot)$ the measure at time $t$, starting at $x$, i.e., $\mu^t_x(y) \equiv \mathbb{P}(X^t_x = y) = P^t(x, y)$, for any $y \in \mathcal{X}$. More generally given an initial distribution $\alpha$ on $\mathcal{X}$

$$\mu^\alpha_t(y) = \mathbb{P}(X^\alpha_t = y) = \sum_{x \in \mathcal{X}} \alpha(x) P^t(x, y)$$

Starting conditions (starting state $x$ or starting measure $\alpha$) will be denoted by a superscript in random variables (i.e., $X^x_t$, $X^\alpha_t$, $\tau^x$, $\tau^\alpha$, ...).

Let $G \subset \mathcal{X}$ be a target set and $\tau_G$ its first hitting time

$$\tau_G := \min\{t \geq 0 \; ; \; X_t \in G\}.$$ 

We will study the process $\{X_t\}_{t \in \mathbb{N}}$ up to time $\tau_G$, so it is not restrictive to assume that states in $G$ are absorbing. Let $A := \mathcal{X}\setminus G$, we assume ergodicity on $A$. More precisely, denoting by $[P]_A$ the sub-stochastic matrix obtained by $P$ by restriction to $A$

$$[P]_A(x, y) = P(x, y) \geq 0 \quad \forall x, y \in A, \quad \sum_{y \in A} [P]_A(x, y) \leq 1,$$
we suppose $[P]_A$ a primitive matrix, i.e., there exists an integer $n$ such that $([P]_A)^n$ has strictly positive entries.

• Quasi-stationary measure on $A$: by the Perron-Frobenius theorem it can be proved that there exists $\lambda < 1$ such that $\lambda$ is the spectral radius of $[P]_A$ and there exists a unique non negative left eigenvector of $[P]_A$ corresponding to $\lambda$, i.e.,

$$\mu^* [P]_A = \lambda \mu^*.$$ (1.10)

so that we get immediately

$$P^t \mu^*_G > t = \lambda^t.$$

• Evolving measures: we call evolving measure any family of measures $\{\mu_t\}_{t \in \mathbb{N}}$, on $\mathcal{X}$, such that $\mu_{t+1}(y) = \sum_x \mu_t(x) P(x, y)$.

Note that $\{\mu_t^\alpha\}_{t \in \mathbb{N}}$ is a particular evolving measure with $\mu_0 = \alpha$ and also $\{\mu_{t+t_0}^\alpha\}_{t \in \mathbb{N}}$ is an evolving measure, for fixed $t_0 \in \mathbb{N}$.

• Squeezing measure and first hitting distribution: as introduced in (1.3) a special role will be played by the squeezing-quasi-stationary measure on $\mathcal{X}$:

$$\mu^*_t(y) = \sum_{z \in A} \mu^*(z) P^t(z, y) = \begin{cases} 
\lambda^t \mu^*(y) & \text{if } y \in A \\
(1 - \lambda^t) \omega(y) & \text{if } y \in G
\end{cases}$$ (1.11)

where the probability measure $\omega$ on $G$ is the first hitting distribution defined, for $y \in G$, by:

$$\omega(y) = \mathbb{P}(X^*_1 = y \mid X^*_1 \in G) = \frac{\sum_{z \in A} \mu^*(z) P(z, y)}{1 - \lambda}$$ (1.12)

Clearly $\{\mu^*_t\}_{t \in \mathbb{N}}$ is an evolving measure.

• Separation “distance”: given two measures $\nu_1$ and $\nu_2$ on $\mathcal{X}$ their separation is defined by

$$sep(\nu_1, \nu_2) := \max_{y \in \mathcal{X}} \left[ 1 - \frac{\nu_1(y)}{\nu_2(y)} \right]$$

1.2 The local chain $\tilde{X}_t$ on $A$

In this subsection we construct an ergodic Markov chain $\tilde{X}_t$ on $A$, that we call the local chain.
To describe the local behavior of the process $X_t$ on $A$, many different dynamics have been used in the literature.

The restriction of the transition matrix to the set $A$, $[P]_A$, is a sub-stochastic matrix, by adding to it a diagonal matrix $D$ with the escape probabilities, $D(x, y) = \mathbb{1}_{x=y} \sum_{z \in G} P(x, z)$, one obtains the reflected process (see for instance [8] and [19]) as a local dynamics. Another frequently used local process is the conditioned process, defined by the original process $X_t$ on $\mathcal{X}$ but conditioned to remain in $A$. This conditioned process has obviously a crucial role in the study of the local behavior of the process $X_t$ before absorption in $G$. However the main problem in dealing with it, for instance to estimate the hitting time to $G$, is that this conditioned process is no more a Markovian process.

We use here a different local chain $\tilde{X}_t$ constructed by means of the right eigenvector of $[P]_A$ corresponding to $\lambda$. This construction is related to the Doob h-transform of $[P]_A$ (see for instance [18]). This chain $\tilde{X}_t$ is also related to the “reversed chain” in Darroch-Seneta, introduced in [11] while considering the large time asymptotics. Our process $\tilde{X}_t$ is the time reversal of this Darroch-Seneta “reversed chain”.

The construction is the following: by the Perron-Frobenius theorem there exists a unique non negative right eigenvector $\gamma$ of $[P]_A$ corresponding to $\lambda$, i.e.,

$$[P]_A \gamma = \lambda \gamma \quad \text{with} \quad (\mu^\ast, \gamma) = 1. \quad (1.13)$$

This eigenvector is related to the asymptotic ratios of the survival probabilities (see eg [10])

$$\lim_{t \to \infty} \frac{\mathbb{P}(\tau^x_G > t)}{\mathbb{P}(\tau^y_G > t)} = \frac{\gamma(x)}{\gamma(y)} \quad x, y \in A.$$

For any $x, y \in A$ define the stochastic matrix

$$\tilde{P}(x, y) := \frac{\gamma(y)}{\gamma(x)} \frac{P(x, y)}{\lambda}. \quad (1.14)$$

Let $\nu$ be its invariant measure

$$\sum_{x \in A} \nu(x) \tilde{P}(x, y) = \nu(y) = \sum_{x \in A} \nu(x) \frac{\gamma(y)}{\gamma(x)} \frac{P(x, y)}{\lambda}$$

so that

$$\gamma(x) = \frac{\nu(x)}{\mu^\ast(x)}$$
For the chain $\tilde{X}_t$ we define

$$\tilde{s}^x(t, y) := 1 - \frac{\tilde{P}^t(x, y)}{\nu(y)}$$

$$\tilde{s}^x(t) = \text{sep}(\tilde{\mu}^x_t, \nu) = \sup_{y \in A} \tilde{s}^x(t, y), \quad \tilde{s}(t) := \sup_{x \in A} \tilde{s}^x(t).$$

Note that $\tilde{s}^x(t) \in [0, 1]$. Moreover, since $\tilde{P}$ is a primitive matrix, it is well known (see for instance \cite{6}, Lemma 3.7) that $\tilde{s}(t)$ has the sub-multiplicative property:

$$\tilde{s}(t + u) \leq \tilde{s}(t)\tilde{s}(u).$$

This implies in particular an exponential decay in time of $\tilde{s}(t)$.

The relation between the local chain and the original chain $X_t$ on $\mathcal{X}$ is given by the definition (1.14) and more generally by

$$\tilde{P}^t(x, y) = \frac{\gamma(y) P^t(x, y)}{\gamma(x)} \lambda^t.$$  \hfill (1.15)

We can use this relation to obtain a rough estimate on the absorption time $\tau_{\alpha}$. We give here this simple calculation in order to point out the dependence on the initial distribution $\alpha$ of the distribution of $\tau_{\alpha}$ by means of a time shift.

As it will be clear in what follows, it is natural to associate to every initial measure $\alpha$ the following measure $\tilde{\alpha}$ for the local chain $\tilde{X}_t$:

$$\tilde{\alpha}(x) = \frac{\alpha(x)\gamma(x)}{\sum_{y \in A} \alpha(y)\gamma(y)}.$$

Indeed

$$\mathbb{P}(\tau_{\alpha} > t) = \sum_{y \in A} \sum_{x \in A} \alpha(x) P^t(x, y) =$$

$$\sum_{y \in A} \sum_{x \in A} \alpha(x) \gamma(x) \lambda^t \mu^*(y) \frac{\tilde{P}^t(x, y)}{\nu(y)} =$$

$$\sum_{y \in A} \sum_{x \in A} \alpha(x) \gamma(x) \lambda^t \mu^*(y)(1 - \tilde{s}^x(t, y)).$$

Since $\tilde{s}^x(t, y) \leq \tilde{s}(t)$ we get

$$\mathbb{P}(\tau_{\alpha} > t) \geq (1 - \tilde{s}(t)) \sum_{y \in A} \sum_{x \in A} \alpha(x) \gamma(x) \lambda^t \mu^*(y) = \lambda^{t + \delta_\alpha}(1 - \tilde{s}(t))$$
with
\[ \delta_\alpha := \log_\lambda \left( \sum_{x \in A} \alpha(x) \gamma(x) \right) \]

On the other side we can consider the minimal strong stationary time \( \tilde{\tau}_\nu^x \) such that
\[ \mathbb{P}(\tilde{X}_t^x = y, \tilde{\tau}_\nu^x = t) = \nu(y) \mathbb{P}(\tilde{\tau}_\nu^x = t) \]
with
\[ \mathbb{P}(\tilde{\tau}_\nu^x > t) = \tilde{s}^x(t). \]

We have immediately
\[ \mathbb{P}(\tau_G^\alpha > t) = \sum_{y \in A} \sum_{x \in A} \alpha(x) \gamma(x) \lambda^t \frac{\mu^x(y)}{\nu(y)} \mathbb{P}(\tilde{X}_t^x = y, \tilde{\tau}^x \leq t) + \]
\[ \sum_{y \in A} \sum_{x \in A} \alpha(x) \gamma(x) \lambda^t \frac{\mu^x(y)}{\nu(y)} \mathbb{P}(\tilde{X}_t^x = y, \tilde{\tau}^x > t) \]
\[ = \lambda^t + \delta_\alpha \left[ 1 + \tilde{s}^\alpha(t) \left( \sum_{y \in A} \frac{\mu^x(y)}{\nu(y)} - 1 \right) \right] \]
with
\[ \tilde{s}^\alpha(t) := \frac{\sum_{x \in A} \alpha(x) \gamma(x) \tilde{s}^x(t)}{\sum_{x \in A} \alpha(x) \gamma(x)} \]

Note that \( \sum_{y \in A} \frac{\mu^x(y)}{\nu(y)} \geq 1 \). This quantity could be much larger that 1 and so this estimate from above on the distribution of \( \tau_G^\alpha \) is quite rough due to the factor \( \left( \sum_{y \in A} \frac{\mu^x(y)}{\nu(y)} - 1 \right) \).

However, we have to note that this factor is independent of time so that, for large \( t \), due to the exponential decay of \( \tilde{s}(t) \), and so of \( \tilde{s}^\alpha(t) \), the estimate is not trivial. Similar results can be found in the literature.

There are some interesting points to note after these estimates, that will be important in our discussion especially for applications to metastability.

- First of all we are able to consider arbitrary initial measures and to determine the effect of the initial condition on the distribution of the first hitting time to \( G \). Indeed we can associate to every initial measure \( \alpha \) a corresponding time shift \( \delta_\alpha \).

- We are interested in the application of first hitting results to metastability. In metastable situations the chain \( \tilde{X}_t \) has typically a relaxation time much smaller than
the mean absorption time of the chain $X_t$. This fast convergence to equilibrium will be given by a fast exponential decay of the separation distance

$$s_{\tilde{\alpha}}(t) = \max_{y \in A} \tilde{s}_{\alpha}(t, y).$$

Our control on the process $X_t$ with the local chain $\tilde{X}_t$ given by (1.15) is really strong when looking at convergence to equilibrium in separation distance, see Proposition 1.6. This implies that we can use the good convergence to equilibrium of $\tilde{X}_t$ to obtain better estimates on the absorption time of the chain $X_t$.

### 1.3 Main results

We first extend the notion of Strong Stationary Time (SST) to strong time w.r.t. evolving measures, different from the stationary one, with the following.

**Definition 1.3** For any initial distribution $\alpha$ and for any evolving measure $\{\mu_t\}_{t \in \mathbb{N}}$ on $X$, we call strong time w.r.t. $\{\mu_t\}_{t \in \mathbb{N}}$ a randomized stopping time, $\tau_{\mu}$, for $X_{\mu}$ such that for any $y \in X$ we have

$$\mathbb{P}(X_{\tau_{\mu}} = y, \tau_{\mu} = t) = \mu_t(y)\mathbb{P}(\tau_{\mu} = t)$$

(1.16)

Note that:

$$\mathbb{P}(X_{\tau_{\mu}} = y, \tau_{\mu} \leq t) = \mu_t(y)\mathbb{P}(\tau_{\mu} \leq t).$$

(1.17)

Indeed

$$\mathbb{P}(X_{\tau_{\mu}} = y, \tau_{\mu} \leq t) = \sum_{u=0}^{t} \mathbb{P}(X_{\tau_{\mu}} = y, \tau_{\mu} = u) = \sum_{u=0}^{t} \sum_{z \in X} \mathbb{P}(X_{\tau_{\mu}} = z, \tau_{\mu} = u)\mathbb{P}(X_{\tau_{\mu}} = y) = \sum_{u=0}^{t} \sum_{z \in X} \mathbb{P}(\tau_{\mu} = u)\mu_u(z)\mathbb{P}(\tau_{\mu} = t)$$

For these strong times we have a result similar to what is proved in [7] for strong stationary times and their relation with separation distance between the evolution and the stationary measure.

**Theorem 1.4** For any initial distribution $\alpha$ and for any reference evolving measure $\{\mu_t\}_{t \in \mathbb{N}}$ and any strong time $\tau_{\mu}$ w.r.t. $\{\mu_t\}_{t \in \mathbb{N}}$ we have

$$\mathbb{P}(\tau_{\mu} > t) \geq \text{sep}(\mu_{\alpha}, \mu_t).$$

Moreover there exists a minimal strong time w.r.t. $\{\mu_t\}_{t \in \mathbb{N}}$ such that

$$\mathbb{P}(\tau_{\mu} > t) = \text{sep}(\mu_{\alpha}, \mu_t).$$
The proof of Theorem 1.4 is given in Section 2.

By using the time shift associated to the initial measure $\alpha$, obtained in the rough estimate on the absorption time given in the previous section, we are going to define the \textit{squeezed-quasi-stationary reference measure} $\rho_t$.

Recall that for any initial distribution $\alpha$ we define the \textit{time shift}

$$\delta_\alpha := \log \frac{\lambda}{\gamma(x)}$$

with $\gamma$ defined in (1.13).

\textbf{Definition 1.5} The following reference evolving measure $\{\rho_t\}_{t \in \mathbb{N}}$ depending on $\alpha$:

$$\rho_t := \mu^\ast_{t+\delta_\alpha}$$

is a probability measure if $t + \delta_\alpha \geq 0$.

Moreover for any $y \in A$ and $t + \delta_\alpha \geq 0$ define:

$$s^\alpha(t, y) = 1 - \frac{\mu^\ast_{t}(y)}{\rho_t(y)} + 1 - \frac{\mu^\ast_{t}(y)}{\mu^\ast_{t+\delta_\alpha}(y)}$$

and

$$\tilde{s}^\alpha(t, y) = 1 - \frac{\sum_{x \in A} \gamma(x) \tilde{P}^t(x, y)}{\nu(y)}$$

\textbf{Proposition 1.6} For any $y \in A$ we have:

$$s^\alpha(t, y) = \tilde{s}^\alpha(t, y)$$

The proof is immediate since

$$s^\alpha(t, y) = 1 - \frac{\sum_{x \in A} \alpha(x) P^t(x, y)}{\lambda^{t+\delta_\alpha} \mu^\ast_{\rho}(y)} = 1 - \frac{\sum_{x \in A} \alpha(x) \gamma(x) \lambda^t \frac{\mu^\ast_{\rho}(y)}{\nu(y)} \tilde{P}^t(x, y)}{\lambda^{t+\delta_\alpha} \mu^\ast_{\rho}(y)} =$$

$$1 - \frac{\sum_{x \in A} \alpha(x) \gamma(x) \lambda^t \frac{\mu^\ast_{\rho}(y)}{\nu(y)} \tilde{P}^t(x, y)}{\sum_{x \in A} \alpha(x) \gamma(x)} = \tilde{s}^\alpha(t, y).$$

By Theorem 1.4 from the separation $s^\alpha(t)$ we can define a minimal strong time w.r.t. the reference measure $\rho_t$, say $\tau^\alpha_{\rho}$, such that

$$\mathbb{P}(X^\alpha_t = y, \tau^\alpha_{\rho} = t) = \rho_t(y) \mathbb{P}(\tau^\alpha_{\rho} = t) \quad \text{with} \quad \mathbb{P}(\tau^\alpha_{\rho} > t) = s^\alpha(t).$$
With a simple argument we have
\[
\mathbb{P}(\tau_G^\alpha > t) = \sum_{s \leq t} \sum_{y \in A} \mathbb{P}(X_t^\alpha = y, \tau_\rho^\alpha = s) + \mathbb{P}(\tau_G^\alpha > t, \tau_\rho^\alpha > t) = \\
\lambda^{t+\delta_\alpha} (1 - s^\alpha(t)) + \mathbb{P}(\tau_G^\alpha > t, \tau_\rho^\alpha > t). 
\] (1.18)

If \(s^\alpha(t)\) decays in time faster than \(\lambda^{t+\delta_\alpha}\), we can obtain from (1.18) good estimates from above and from below, i.e., we get
\[
\mathbb{P}(\tau_G^\alpha > t) = \lambda^{t+\delta_\alpha} (1 + o(1)).
\]

Notice that, while \(\tilde{s}^\alpha(t)\) decays exponentially in time, in general we don’t have such a good long time behavior for \(s^\alpha(t)\) = \(\max_{y \in A} s^\alpha(t, y) = \max_{y \in A} \tilde{s}^\alpha(t, y) \lor \max_{y \in G} s^\alpha(t, y)\).

Indeed, even in metastable situations, where we expect a decay of \(\tilde{s}^\alpha(t)\) faster do than \(\lambda^{t+\delta_\alpha}\), in general we cannot control the term \(\max_{y \in G} s^\alpha(t, y)\).

To solve this problem, we define a new random time by looking at the conditioned process by means of the separation \(\tilde{s}^\alpha(t)\) instead of \(s^\alpha(t)\).

**Definition 1.7** For any initial distribution \(\alpha\) on \(A\) we call conditionally strong quasi stationary time (CSQST) a randomized stopping time \(\tau_*^\alpha\) for \(X_t^\alpha\) such that for any \(y \in A\) we have
\[
\mathbb{P}(X_t^\alpha = y, \tau_*^\alpha = t | t < \tau_G^\alpha) = \mu^*(y) \mathbb{P}(\tau_*^\alpha = t | t < \tau_G^\alpha) 
\]
which is equivalent to
\[
\mathbb{P}(X_t^\alpha = y, \tau_*^\alpha = t) = \mu^*(y) \mathbb{P}(\tau_*^\alpha = t < \tau_G^\alpha) 
\] (1.19)

We note that the analogous of the equation (1.17) holding for strong times, does not hold for CSQST. Due to the conditioning, we have:
\[
\mathbb{P}(X_t^\alpha = y, \tau_*^\alpha \leq t) = \mu^*(y) \sum_{u \leq t} \lambda^{t-u} \mathbb{P}(\tau_*^\alpha = u < \tau_G^\alpha) \neq \mu^*(y) \mathbb{P}(\tau_*^\alpha \leq t < \tau_G^\alpha) 
\] (1.20)

indeed
\[
\sum_{u \leq t} \sum_{z \in A} \mathbb{P}(X_u^\alpha = z, \tau_*^\alpha = u) P^{t-u}(z, y) = \sum_{u \leq t} \sum_{z \in A} \mu^*(z) \mathbb{P}(\tau_*^\alpha = u < \tau_G^\alpha) P^{t-u}(z, y) = \\
\mu^*(y) \sum_{u \leq t} \lambda^{t-u} \mathbb{P}(\tau_*^\alpha = u < \tau_G^\alpha). 
\]

This remark actually suggests a new notion of minimality as given in the following.
Theorem 1.8 For any initial distribution $\alpha$ on $A$ and for any $\tau_\alpha^*$ conditionally strong quasi stationary time (CSQST) for $X_t^\alpha$ and for all $t \geq 0$ we have
\[
\sum_{u \leq t} \lambda^{-u} \mathbb{P}(\tau_\alpha^* = u < \tau_G^\alpha) \leq \lambda^\delta (1 - \tilde{s}(t)).
\]
Moreover there exists a minimal conditionally strong quasi stationary time $\tau_\alpha^*$ such that
\[
\sum_{u \leq t} \lambda^{-u} \mathbb{P}(\tau_\alpha^* = u < \tau_G^\alpha) = \lambda^\delta (1 - \tilde{s}(t)).
\]
with
\[
\mathbb{P}(\tau_\alpha^* = t < \tau_G^\alpha) = \lambda^t + \delta \tilde{s}(t - 1) - \tilde{s}(t).
\]
Note that in particular for a minimal conditionally strong quasi stationary time we have
\[
\mathbb{P}(\tau_\alpha^* > t, \tau_\alpha^* < \tau_G^\alpha) = \sum_{u > t} \lambda^{u-\delta} (\tilde{s}(u - 1) - \tilde{s}(u)) \leq \lambda^t + \delta \tilde{s}(t).
\]

The interest of this minimal conditionally strong quasi stationary time is given by the following:

Theorem 1.9 For any initial distribution $\alpha$ on $A$, if $\tau_\alpha^*$ is a minimal conditionally strong quasi stationary time and $t + \delta \geq 0$ we have
\[
\mathbb{P}(\tau_G^\alpha > t) = \lambda^t + \delta (1 - \tilde{s}(t)) + \mathbb{P}(\tau_\alpha^* < \tau_G^\alpha, \tau_\alpha^* > t)
\]
with $\tau_{\alpha,G}^\alpha = \tau_G^\alpha \wedge \tau_\alpha^*$.
Moreover for any $y \in G$ we have
\[
\mathbb{P}(X_G^\alpha = y) = \mathbb{P}(\tau_G^\alpha < \tau_\alpha^*, X_G^\alpha = y) + \omega(y) \mathbb{P}(\tau_G^\alpha > \tau_\alpha^*).
\]

This theorem provides a quantitative control on the convergence to an exponential distribution for the hitting time $\tau_G$ and on the exit distribution. Note that with this CSQST, $\tau_\alpha^*$, we are obtaining conditioning benefits without explicitly using the conditioned process.

As far as the exit distribution is concerned in the metastable case the quantity
\[
\mathbb{P}(\tau_G^\alpha < \tau_\alpha^*) =: \varepsilon
\]
should be small. This is the case in which the results of Theorem 1.9 are relevant. If, indeed, $\varepsilon$ small implies that the distribution of the first hitting to $G$ is well approximated by the measure $\omega$ since equation (1.21) gives
\[
\omega(y)(1 - \varepsilon) \geq \mathbb{P}(X_G^\alpha = y) \leq \varepsilon + \omega(y)(1 - \varepsilon)
\]
2 Strong time w.r.t. evolving measures

In this section we extend the results obtained in [7], relating strong stationary times and separation distance, to strong times w.r.t. evolving measures. The ideas of the proof are simple.

2.1 Proof of Theorem 1.4

For any \( y \in \mathcal{X} \) and \( t \geq 0 \) and any evolving measure \( \mu_t \), define

\[
\sigma_t(y) := \mu_t(y)[s^\alpha(t - 1) - s^\alpha(t)] \quad \theta_t(y) := \mu_t(y)[s^\alpha(t - 1) - s^\alpha(t, y)]
\]

We have for any \( y \in \mathcal{X} \) and \( t \geq 0 \)

\[
0 \leq \sigma_t(y) \leq \theta_t(y)
\]

and more precisely

\[
\theta_t(y) - \sigma_t(y) = \mu_t(y)[s^\alpha(t) - s^\alpha(t, y)]
\]

so that the vectors \( \sigma_t \) and \( \theta_t \) satisfy the iterative equation

\[
(\theta_t - \sigma_t)P = \theta_{t+1} \quad \forall t \geq 0 \tag{2.22}
\]

with \( \theta_0 = \alpha \), and \( \sigma_t = \left( \min_{z \in \mathcal{X}} \frac{\theta_t(z)}{\mu_t(z)} \right) \mu_t \) for all \( t \geq 0 \).
Define a randomized stopping time $\tau^\alpha$ by imposing
\begin{equation}
\mathbb{P}\left(\tau^\alpha = t \mid \tau^\alpha \geq t, X^\alpha_t = y, X_s, s < t \right) = \frac{\sigma_t(y)}{\theta_t(y)} = \frac{s^\alpha_{\mu}(t-1) - s^\alpha_{\mu}(t)}{s^\alpha_{\mu}(t-1) - s^\alpha_{\mu}(t, y)} \tag{2.23}
\end{equation}

It is easy to prove by induction that for any $t \geq 0$
\begin{equation}
\mathbb{P}(\tau^\alpha = t, X^\alpha_t = y) = \sigma_t(y), \quad \mathbb{P}(\tau^\alpha \geq t, X^\alpha_t = y) = \theta_t(y) \tag{2.24}
\end{equation}
since also these probabilities satisfy the iterative equation (2.22). Indeed if (2.24) holds for $t$ then by (2.22) we obtain the statement for $\theta_{t+1}$ and by (2.23) the same for $\sigma_{t+1}$. We can immediately conclude that $\tau^\alpha$ is a strong time w.r.t. $\mu_t$ with $\mathbb{P}(\tau^\alpha > t) = s^\alpha_{\mu}(t)$. Thus, it is minimal.

### 2.2 Construction of the strong time w.r.t. $\{\mu_t\}_{t \in \mathbb{N}}$ with an auxiliary chain

We give here a construction of the minimal strong time $\tau^\alpha$ inspired by [5, 13]. We define an auxiliary chain so that the strong time can be seen as an hitting time for this new process.

Consider the initial distribution $\alpha$ as a parameter and define an auxiliary process $Y^\alpha_t$ with state space $Y := \mathcal{X} \times \{0, 1\}$, so that on $\{0\}$ the process is like $X^\alpha_t$ but with a rate jump to $\{1\}$ given by $J^\alpha$.

More precisely for every $z \in \mathcal{X}$ define the function
\begin{equation}
J^\alpha(t, z) := \frac{s^\alpha_{\mu}(t-1) - s^\alpha_{\mu}(t)}{s^\alpha_{\mu}(t-1) - s^\alpha_{\mu}(t, z)}, \tag{2.25}
\end{equation}
with the convention $0/0 = 0$. By the monotonicity of $s^\alpha(t)$, we have $J^\alpha(t, z) \in [0, 1]$ for any $z \in \mathcal{X}$ and any $t$.

Consider the following time dependent transition probabilities for the process $Y^\alpha_t$:
\begin{align*}
Q^\alpha_{(y, 0), (z, 0)} &= P(y, z) \left(1 - J^\alpha(t, z)\right), \quad Q^\alpha_{(y, 0), (z, 1)} = P(y, z) J^\alpha(t, z), \quad Q^\alpha_{(y, 1), (z, e)} = P(y, z) \delta_{1, e}.
\end{align*}

Note that the marginal distribution of $Y^\alpha_t$ on $\mathcal{X}$ corresponds to the distribution of $X^\alpha_t$ so that we can study each event defined for the process $X^\alpha_t$ in terms of set of paths of the process $Y^\alpha_t$. For this reason, with an abuse of notation, we denote with the same symbol $\mathbb{P}$ the probability of events defined in terms of the process $Y^\alpha_t$. Consider the hitting time:
\begin{equation}
\tau^{\alpha}_{1} := \tau^\alpha_{\mathcal{X} \times \{1\}} = \min\{t \geq 0 ; Y^\alpha_t = (y, 1) \text{ for some } y \in \mathcal{X}\},
\end{equation}
We want to show that $\tau_1^\alpha$ is a minimal strong time w.r.t. the evolving measure $\{\mu_t\}_{t \in \mathbb{N}}$, i.e.,

$$\mathbb{P}(X_t^\alpha = y, \tau_1^\alpha = t) = \mu_t(y)\mathbb{P}(\tau_1^\alpha = t) = \mu_t(y)\left(s_\mu^\alpha(t) - s_\mu^\alpha(t-1)\right),$$ (2.26)

We proceed by induction on $t$. For $t = 0$, by definition of $Y_t^\alpha$, we have

$$\mathbb{P}(X_0^\alpha = y, \tau_1^\alpha = 0) = \mathbb{P}(Y_0^\alpha = (y, 1)) = \alpha(y)J^\alpha(0, y) = \alpha(y)\frac{1 - s_\mu^\alpha(0)}{1 - s_\mu^\alpha(t, y)} = \mu_0(y)(1 - s_\mu^\alpha(0)).$$

To prove the induction step we use the following:

**Lemma 2.1** If for any $u \leq t$ we have

$$\mathbb{P}(X_u^\alpha = y | \tau_1^\alpha = u) = \mu_u(y)$$

then

$$\mathbb{P}(Y_t^\alpha = (z, 1)) = \mu_t(z)\mathbb{P}(\tau_1^\alpha \leq t)$$

**Proof.**

$$\mathbb{P}(Y_t^\alpha = (z, 1)) = \sum_{u \leq t} \sum_{y \in \mathcal{X}} \mathbb{P}(Y_t^\alpha = (z, 1) | Y_u^\alpha = (y, 1))\mathbb{P}(\tau_1^\alpha = u, X_u^\alpha = y) = \sum_{u \leq t} \sum_{y \in \mathcal{X}} P^{t-u}(y, z)\mu_u(y)\mathbb{P}(\tau_1^\alpha = u) = \mu_t(z)\mathbb{P}(\tau_1^\alpha \leq t) \quad \square$$

Suppose now that (2.26) holds for $u \leq t$. By using then Lemma 2.1 we get

$$\mathbb{P}(X_{t+1}^\alpha = y, \tau_1^\alpha = t + 1) = \sum_{z \in \mathcal{X}} \mathbb{P}(Y_{t+1}^\alpha = (y, 1) | Y_t^\alpha = (z, 0))\mathbb{P}(Y_t^\alpha = (z, 0)) =$$

$$\sum_{z \in \mathcal{X}} P(z, y)J^\alpha(t + 1, y)\left[\mu_t^\alpha(z) - \mathbb{P}(Y_t^\alpha = (z, 1))\right] = J^\alpha(t + 1, y)\left[\mu_{t+1}^\alpha(z) - \sum_{z \in \mathcal{X}} \mu_t(z)P(z, y)\mathbb{P}(\tau_1^\alpha \leq t)\right] =$$

$$J^\alpha(t + 1, y)\mu_{t+1}^\alpha(z)\left[1 - s_\mu^\alpha(t + 1, y) - (1 - \mathbb{P}(\tau_1^\alpha > t))\right] = \frac{\mu_{t+1}(y)\left(s_\mu^\alpha(t) - s_\mu^\alpha(t + 1)\right)}{s_\mu^\alpha(t) - s_\mu^\alpha(t + 1, y)}\left[s_\mu^\alpha(t) - s_\mu^\alpha(t + 1, y)\right]$$

and summing on $y$ we get $\mathbb{P}(\tau_1^\alpha = t + 1) = s_\mu^\alpha(t) - s_\mu^\alpha(t + 1, y) = \mathbb{P}(\tau_1^\alpha > t) - s_\mu^\alpha(t + 1)$ so that $\mathbb{P}(\tau_1^\alpha > t + 1) = s_\mu^\alpha(t + 1)$ and

$$\mathbb{P}(X_{t+1}^\alpha = y, \tau_1^\alpha = t + 1) = \mu_{t+1}(y)\mathbb{P}(\tau_1^\alpha = t + 1). \quad \square$$
3 Conditionally strong quasi stationary times (CSQST)

In this section we prove Theorem 1.8. The main idea is the following. As noted after Proposition 1.6, given a reference evolving measure $\rho_t$, obtained by the squeezing-quasi-stationary measure with time shift, we know how to construct from the corresponding separation distance a minimal strong time w.r.t. $\rho_t$. Proposition 1.6 opens the way to the construction of a faster strong time (with finite moments) if we take the supremum of $s^\alpha(t,y)$ only for $y$ in $A$, since this quantity coincides with $\tilde{s}^\alpha(t)$ that, being the separation from stationarity for the process $\tilde{X}$, decays exponentially in time. The idea is then to define a new random time $\tau^\alpha_*$ by using $\tilde{s}^\alpha(t)$ instead of $s^\alpha(t)$, following the construction given in the proof of Theorem 1.4. This time is not strong w.r.t. $\rho_t$, but it works like a strong time when looking at the process conditioned to $A$. This construction gives a conditionally strong-quasi-stationary time without working directly with the conditioned process.

We first prove that if $\tau^\alpha_*$ is a CSQST, i.e., if satisfies

$$\mathbb{P}(X^\alpha_t = y, \tau^\alpha_* = t) = \mu^\alpha(y)\mathbb{P}(\tau^\alpha_* = t < \tau^\alpha_G)$$

then for all $t \geq 0$ we have

$$\sum_{u \leq t} \lambda^{-u} \mathbb{P}(\tau^\alpha_* = u < \tau^\alpha_G) \leq \lambda^\delta\alpha(1 - \tilde{s}^\alpha(t)).$$

(3.27)

Indeed for any $y \in A$ we have

$$\mu^\alpha_t(y) \geq \mathbb{P}(\tau^\alpha_* \leq t, X^\alpha_t = y) = \sum_{u \leq t} \sum_{z \in A} \mathbb{P}(\tau^\alpha_* = u, X^\alpha_u = z)P^{t-u}(z, y) =$$

$$\lambda^\delta\alpha \sum_{u \leq t} \lambda^{-u} \mathbb{P}(\tau^\alpha_* = u < \tau^\alpha_G)\mu^\alpha(y)$$

so that

$$\frac{\mu^\alpha_t(y)}{\lambda^\delta\alpha(y)} = \lambda^\delta\alpha(1 - s^\alpha(t, y)) \geq \sum_{u \leq t} \lambda^{-u} \mathbb{P}(\tau^\alpha_* = u < \tau^\alpha_G)$$

since this holds for any $y \in A$ and we have $s^\alpha(t, y) = \tilde{s}^\alpha(t, y)$ for every $y \in A$ then (3.27) holds.

We define now a random time $\tau^\alpha_*$ which is not strong w.r.t. the reference evolving measure $\rho_t \equiv \mu^\alpha_{t+\delta_\alpha}$ on the hole space $X$ but which is constructed with similar ideas, by using Proposition 1.6 by means of the separation $\tilde{s}^\alpha$ in the following way. Define

$$\sigma_t(y) = \mathbb{1}_{y \in A}\rho_t(y)(\tilde{s}^\alpha(t) - \tilde{s}^\alpha(t))$$

18
\[ \theta_t(y) = \mathbf{1}_{y \in A} \rho_t(y)(\bar{s}(t - 1) - \bar{s}(t, y)) \]

Then for any \( y \in A \) we still can define \( \tau_{\star}^\alpha \) such that

\[
P(X_t^\alpha = y, \tau_{\star}^\alpha = t) = \sigma_t(y) = \rho_t(y)(\bar{s}(t - 1) - \bar{s}(t)) = \mu^*(y)P(\tau_{\star}^\alpha = t < \tau_G^\alpha)
\]

with

\[
P(\tau_{\star}^\alpha = t < \tau_G^\alpha) = \lambda^t \alpha \left( \bar{s}(t - 1) - \bar{s}(t) \right)
\]

\[
P(\tau_{\star}^\alpha = t \geq \tau_G^\alpha) = 0
\]

and

\[
P(\tau_{\star}^\alpha = +\infty) = 1 - \sum_{t \geq 0} \lambda^t \alpha \left( \bar{s}(t - 1) - \bar{s}(t) \right) = P(\tau_G^\alpha < \tau_{\star}^\alpha) > 0
\]

For such a \( \tau_{\star}^\alpha \) we have:

**Proposition 3.1** \( \tau_{\star}^\alpha \) is a conditionally strong quasi-stationary time, i.e.,

\[
P(X_t^\alpha = y, \tau_{\star}^\alpha = t \mid \tau_G^\alpha > t) = \mu^*(y)P(\tau_{\star}^\alpha = t \mid \tau_G^\alpha > t)
\]

Indeed

\[
P(X_t^\alpha = y, \tau_{\star}^\alpha = t \mid \tau_G^\alpha > t) = \frac{P(X_t^\alpha = y, \tau_{\star}^\alpha = t < \tau_G^\alpha)}{P(\tau_G^\alpha > t)} = \mu^*(y)\frac{P(\tau_{\star}^\alpha = t < \tau_G^\alpha)}{P(\tau_G^\alpha > t)} \quad \square
\]

**4 Representation formula for \( \tau_G^\alpha \) with \( \tau_{\star}^\alpha \)**

In this section we prove Theorem 1.9. We first prove that

\[
P(\tau_G^\alpha > t) = \lambda^t \alpha (1 - \bar{s}(t)) + P(\tau_{\star,G} > t)
\]

Indeed we have

\[
P(\tau_G^\alpha > t) = P(\tau_G^\alpha > t, \tau_{\star,G} \leq t) + P(\tau_G^\alpha > t, \tau_{\star,G} > t) = \sum_{y \in A} P(X_t^\alpha = y, \tau_{\star,G} \leq t) + P(\tau_G^\alpha \wedge \tau_{\star,G} > t) =
\]
\[ \sum_{y \in A} \sum_{z \in A} \sum_{u=0}^{t} \mathbb{P}(X^\alpha_u = z, \tau^\alpha_\ast = u, X^\alpha_t = y) + \mathbb{P}(\tau^\alpha_\ast > t) = \]
\[ \sum_{y \in A} \sum_{z \in A} \sum_{u=0}^{t} \mu^*(z)\lambda^{u+\delta}(s^\alpha(t-u) - s^\alpha(u))P^{t-u}(z, y) + \mathbb{P}(\tau^\alpha_\ast > t) = \]
\[ \sum_{y \in A} \sum_{u=0}^{t} \mu^*(y)\lambda^{u+\delta} + (t-u)(s^\alpha(t-u) - s^\alpha(u)) + \mathbb{P}(\tau^\alpha_\ast > t) = \]
\[ \lambda^{t+\delta}(1 - s^\alpha(t)) + \mathbb{P}(\tau^\alpha_\ast > t). \]

Moreover for any \( y \in G \) we have
\[ \mathbb{P}(X^\alpha_{\tau^\alpha_G} = y) = \mathbb{P}(\tau^\alpha_G < \tau^\alpha_\ast, X^\alpha_{\tau^\alpha_G} = y) + \mathbb{P}(\tau^\alpha_G > \tau^\alpha_\ast, X^\alpha_{\tau^\alpha_G} = y) \]

The second term in the r.h.s. can be written as
\[ \sum_{t=0}^{\infty} \sum_{z \in A} \mathbb{P}(\tau^\alpha_G > t = \tau^\alpha_\ast, X^\alpha_t = z) \mathbb{P}(X^z_{\tau^\alpha_G} = y) = \sum_{t=0}^{\infty} \sum_{z \in A} \mu^*(z) \mathbb{P}(\tau^\alpha_G > t = \tau^\alpha_\ast) \mathbb{P}(X^z_{\tau^\alpha_G} = y) = \mathbb{P}(\tau^\alpha_G > \tau^\alpha_\ast) \sum_{z \in A} \mu^*(z) \sum_{u=0}^{\infty} \sum_{w \in A} \mathbb{P}(X^w_{u+1} = w, \tau^\alpha_G = u+1) = \omega(y) \mathbb{P}(\tau^\alpha_G > \tau^\alpha_\ast) \]

so that (1.21) holds.

5 Concluding remarks and future perspectives

In this paper we describe the relation between rarity and exponentiality with the help of a new class of strong times. We give an exact representation formula (Theorem 1.9) that provides probabilistic interpretations for the leading exponential term as well as for the error term.

Our setting is completely general: we do not need reversibility and we only assume that \([P]_A\) is a primitive matrix. Our representation formula applies to any initial state \( \alpha \). To our knowledge, no other result is so general and so transparent about the role of the starting state. As discussed in the introduction, in the literature many results about exponentiality of the first hitting time are obtained with renewal arguments based on the idea of recurrence to a “basin of attraction” of the metastable state. The control that we have in this paper on the role of the starting state \( \alpha \) is such that we can obtain estimates on the distribution of \( \tau^\alpha_G \) without recurrence on a particular set but converging to a particular
evolving measure, that depends on the initial distribution $\alpha$, without error propagation. Indeed, we associate to each starting distribution $\alpha$ a time shift $\delta_\alpha$, in such a way that the dependence on the initial distribution of the distribution of $\tau_\alpha^G$ is described in terms of this time shift. The evolving measure associated to the starting distribution $\alpha$ with this time shift is a probability measure for every $t \geq 0 \lor (-\delta_\alpha)$. The set of states with $\delta_\alpha < 0$ can be seen as metastable basin.

The main novelty of this paper is the introduction of a new language to describe the hitting of a set in terms of strong times. Under very general conditions, the distribution of Conditionally strong quasi stationary times (see Def. 1.7) has a good asymptotic behavior. In many physical applications however, one is more interested in the short time behavior of the process, and our notion of strong time w.r.t. other evolving measures (see Def 1.3) may give interesting estimates for such small times.

Most of the bounds of the error term in the exponential approximation of hitting times known in the literature are function of the ratio between a “mean local relaxation time” and the mean hitting time. At heuristic level this time-scale comparison is a very popular characterization of metastability. One of our strongest motivations has been to give a rigorous base to this idea and to give a general characterization of metastability in terms of a time comparison. This is still the first point in our agenda. In Theorem 1.9 these two different time scales are given by the times $\tau_{\alpha,G}^\ast$ (that plays the role of local relaxation time) and $\tau_\alpha^G$.

The usability of our representation formula in Theorem 1.9 to get explicit error bounds relies on the possibility to estimate $\tilde{s}^\alpha(t)$. At first glance, this task seems rather difficult because this quantity is defined in terms of the matrix $\bar{P}$ and of the eigenvectors $\mu^*$ and $\gamma$ of $[P]_A$. Moreover, generally speaking, separation is not the most manageable notion of distance between measures. However, since $\tilde{s}^\alpha(t) \equiv \text{sep}(\tilde{\mu}_t^\alpha, \nu)$ is the separation for the chain $\tilde{X}_t$, it is positive; most important, it is bounded above by $\tilde{s}(t) := \sup_{\alpha} \tilde{s}^\alpha(t)$ which is submultiplicative. Therefore, it is sufficient to find a time $R$ for which $\tilde{s}(t)$ is bounded above by a constant $c$ smaller than 1 to get an exponential bound like $c^{t/R}$. Useful inequalities that relate separation from stationarity are known (see e.g. [6]) and can be used to find such a bound.

In order to control the effect of the initial distribution $\alpha$ on the distribution of $\tau_\alpha^G$, a crucial tool turns out to be the local chain $\tilde{X}_t$. The main feature of this local chain is given
by Proposition 1.6 which allows to see that \( \sup_{y \in A} s^\alpha(t, y) = \tilde{s}^\alpha(t) \) decays exponentially uniformly in \( \alpha \). Proposition 1.6 also allows to compute \( \tilde{s}^\alpha(t) \) without computing \( \tilde{P}, \mu^* \) and \( \gamma \). In metastable situations we expect that this exponential decay is much faster that the decay of \( s^\alpha(t) = \sup_{y \in X} s^\alpha(t, y) \). In the strong-time language, this means that \( \tau^{\alpha}_p \) is slower than \( \tau^{\alpha}_* \) for it triggers the arrival to \( \rho_t \) also for the points in \( G \). By using the notion of CSQST with the representation formula of Theorem 1.9 we can use the fast decay of \( \tilde{s}(t) \) in order to control the distribution of \( \tau^\alpha_{G} \) by means of the distribution of \( \tau^\alpha_{*,G} \). The exponential behavior emerges when the term \( \mathbb{P}\left( \tau^\alpha_{G} > t \right) \) has a decay strictly faster than \( \chi^{t+\delta_\alpha} \), a sort of time comparison that may be used to characterize metastability.

The statement of Theorem 1.9 has a strong analogy with the description of metastability in terms of recurrence \[14\], \[15\]. In the simple case of recurrence to a single state \( x_0 \), the main metastability hypothesis was on the decay in time of the quantity \( \sup_{x \in X} \mathbb{P}(\tau^{x}_{x_0 \cup G} > t) \), here replaced by a decay of \( \mathbb{P}\left( \tau^\alpha_{x_0,G} > t \right) \). Moreover, an analogous of the auxiliary chain given in subsection 2.2 can be defined to see \( \tau^\alpha_{*} \) as a hitting time. In this way we expect that exponential estimates from above can be obtained for the conditioned probability \( \mathbb{P}\left( \tau^\alpha_{*} > t \mid \tau^\alpha_{G} > t \right) \), with arguments similar to those used to estimate \( \sup_{x \in X} \mathbb{P}(\tau^{x}_{x_0 \cup G} > t) \) in some examples, see for instance \[14\].

**Acknowledgments:**

We thank Amine Asselah, Nils Berglund, Pietro Caputo, Frank den Hollander, Roberto Fernandez and Alexandre Gaudilliére for many fruitful discussions. This work was partially supported by the A*MIDEX project (n. ANR-11-IDEX-0001-02) funded by the “Investissements d’Avenir” French Government program, managed by the French National Research Agency (ANR).

**References**

[1] M. Abadi, A. Galves “Inequalities for the occurrence times of rare events in mixing processes. The state of the art” *Markov Process. Relat. Fields* 7, 97–112 (2001).

[2] D. Aldous, “Markov chains with almost exponential hitting times” *Sto.Proc.Appl* 13, 305–310 (1982).
[3] D. Aldous, M. Brown, “Inequalities for rare events in time reversible Markov chains I”, in Stochastic Inequalities, M. Shaked and Y.L. Tong eds., pp. 1–16, Lecture Notes of the Institute of Mathematical Statistics, vol. 22 (1992).

[4] D. Aldous, M. Brown, “Inequalities for rare events in time reversible Markov chains II”, Stochastic Proc. Appl 44, 15-25 (1993).

[5] D. Aldous, P. Diaconis, “Shuffling cards and stopping times”, Amer. Math. Monthly 93, 333-348 (1986).

[6] D. Aldous, P. Diaconis, “Strong uniform times and finite random walks I”, Adv. in Appl. Math. 8, 66-97 (1987).

[7] D. Aldous, J.A.Fill, “Reversible Markov Chains and Random Walks on Graphs”, Unfinished monograph, 2002, recompiled 2014, available at http://www.stat.berkeley.edu/~aldous/RWG/book.html

[8] A. Bianchi, A. Gaudilliere, “Metastable states, quasi-stationary and soft measures, mixing time asymptotics via variational principles”, arXiv:1103.1143, (2011).

[9] Brown 1999 “Interlacing eigenvalues in time reversible Markov chains” Math. Op. Res. 24, 847 - 864(1999)

[10] P. Collet, S. Martínez, J. San Martín, “Quasi-stationary distributions: Markov chains, diffusions and dynamical systems” Springer Science & Business Media 2012.

[11] J.N.Darroch, E. Seneta, “On quasi-stationary distributions in absorbing discrete-time finite Markov chains”, J. Appl. Prob. 2, 88-100 (1965).

[12] P. Diaconis, J. A. Fill. 1990. “Strong stationary times via a new form of duality”, Ann. Probab. 18, no. 4, 1483?1522.

[13] P. Diaconis, L. Miclo, “On Times to Quasi-Stationary for Birth and Death Processes”, Journal of Theoretical Probability, 22 (3) 558-586 (2009)

[14] R. Fernández, F. Manzo, F.R. Nardi, E. Scoppola, Asymptotically exponential hitting times and metastability: a pathwise approach without reversibility, Electronic Journal of Probability 20 (2015) 122, 1-37
[15] R. Fernández, F. Manzo, F.R. Nardi, E. Scoppola, J. Sohier, “Conditioned, quasi-
stationary, restricted measures and escape from metastable states”, *Ann.Appl.Prob.*, 26, 760-793 (2016).

[16] J.A. Fill, V. Lyzinski, “Hitting times and interlacing eigenvalues: a stochastic ap-
proach using intertwining”, *Journal of Theoretical Probability*, 28, Springer Sci-
ence+Business Media New York 201210.1007/s10959-012-0457-9 (2012).

[17] J. Keilson, *Markov Chain Models–Rarity and Exponentiality*, Springer-Verlag (1979).

[18] D.A. Levin, Y. Peres, E.L. Wilmer *Markov Chains and Mixing Times*, AMS (2009).

[19] E. Olivieri and M.E. Vares, *Large deviations and metastability* Encyclopedia of Math-
ematics and its Applications, 100. Cambridge University Press, Cambridge, (2005).