Absence of Breakdown of the Poisson Hypothesis

I. Closed Networks at Low Load

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Abstract

We prove that the general mean-field type networks at low load behave in accordance with the Poisson Hypothesis. That means that the network equilibrates in time independent of its size. This is a “high-temperature” counterpart of our earlier result, where we have shown that at high load the relaxation time can diverge with the size of the network (“low-temperature”). In other words, the phase transitions in the networks can happen at high load, but cannot take place at low load.

Keywords: coupled dynamical systems, non-linear Markov processes, stable attractor, phase transition, long-range order.

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1 Introduction

The Poisson Hypothesis is a device to predict the behavior of large queuing networks. It was formulated first by L. Kleinrock, and concerns the following situation.
Suppose we have a large network of servers, through which many customers are traveling, being served at different nodes of the network. If the node is busy, the customers wait in the queue. Customers are entering into the network from the outside via some nodes, and these external flows of customers are Poissonian, with constant rates. The service time at each node is random, depending on the node, and the customer.

We are interested in the stationary distribution $\pi_N$ at a given node $N$: what is the distribution of the queue at $N$, what is the average waiting time, etc.

Except for a very few special cases, when the service times are exponential, the distributions $\pi_N$ in general can not be computed. The recipe of the Poisson Hypothesis for approximate computation of $\pi_N$ and the prediction for the (long-time, large-size) behavior of the network is the following:

- Consider the total flow $F$ of customers to a given node $N$. Then $F$ is approximately equal to a Poisson flow, $P$, with a time dependent rate function $\lambda_N(T)$.

- The exit flow from $N$ – not Poissonian in general! – has a rate function $\gamma_N(T)$, which is smoother than $\lambda_N(T)$ (due to averaging, taking place at the node $N$).

- As a result, the flows $\lambda_N(T)$ at various nodes $N$ should converge to constant limits $\bar{\lambda}_N \approx \frac{1}{T} \int_0^T \lambda(t) dt$, as $T \to \infty$, the flows to different nodes being almost independent.

- The above convergence is uniform in the size of the network.

- Compute the stationary distribution $\hat{\pi}_N$ at $N$, corresponding to the inflow $P(\bar{\lambda}_N)$. (These computations are the subject of classical queuing theory and usually provide explicit formulas.) The claim is that $\hat{\pi}_N \approx \pi_N$.

The Poisson Hypothesis is supposed to give a good estimate if the internal flow to every node $N$ is a sum of flows from many other nodes, and each of these flows constitute only a small fraction of the total flow to $N$.

Clearly, the Poisson Hypothesis can not be literally true. It can hopefully hold only after some kind of “thermodynamic” limit is taken. Its meaning is that in the long run the different nodes become virtually independent, i.e.
propagation of chaos takes place. The reason for that should be that any synchronization of the nodes, if initially present, dissolves with time, due to the randomness of the service times.

For some time it was believed that the Poisson Hypothesis behavior is a general characteristic of all large highly connected networks. It was proven in some special cases in [St1] and [RSh1]. However, the counterexamples were also found recently (see [RSh2], where some special service times were considered, and especially [RShV1], where the network with exponential service times is constructed, for which the PH breaks down at high load). We think that the situation here resembles the one of statistical mechanics, where all the models behave alike at high temperatures, while at low temperatures some of them exhibit phase transition behavior. In the network context the role of the temperature is played by the average load per server, \( \rho \), and the example described in [RShV1] is the example of such (first-order) phase transition at high load, where the corresponding infinite system has multiple equilibrium states. In the present paper we pursue this analogy further, by showing that under very general conditions the Poisson Hypothesis holds for general mean-field type closed networks described below in the low load regime. In the forthcoming paper [RShV2] we will prove similar results for open networks.

1.1 The elementary network

Let \( G = (V, E) \) be a finite graph. We want to think about the graph \( G \) as a network of servers, serving clients. So we suppose that \( G \) is endowed with some extra structures. First, at every server \( v \in V \) there might be clients of different nature, so we associate to every \( v \) a (finite) set of types – or colors – \( \{ c \in \mathcal{C}(v) \} \). So we introduce the disjoint union \( \tilde{V} = \cup_{v \in V} \mathcal{C}(v) \) of all possible types of clients. We connect a pair \((v_1, c_1)\) to a pair \((v_2, c_2)\), where \( c_i \in \mathcal{C}(v_i) \), by a directed bond, \( e \in \tilde{E} \), if \((v_1, v_2)\) is an edge in \( \tilde{E} \), and moreover if it can happen that a client of type \( c_1 \) is served by the server \( v_1 \) and as a result is sent to the server \( v_2 \) as a type \( c_2 \) client.

Thus far we just described another (bigger, and directed) graph, \( \tilde{G} = (\tilde{V}, \tilde{E}) \). Next, we have to specify the service times. We suppose them to have exponential distribution, with the rates \( \gamma(v, c) > 0 \).

The last piece of information we want to have on \( G \) is the transition probability matrix, \( P = ||P[(v_1, c_1), (v_2, c_2)]|| \). The number \( P[(v_1, c_1), (v_2, c_2)] \) is the probability that the client of type \( c_1 \) at the node \( v_1 \) will go to the node...
as a type \( c_2 \) client.

The condition we want to impose on our elementary network \((\bar{G}, \gamma)\) is that of connectedness:

**Condition 1** Let us consider a continuous time Markov process on \( G \), corresponding to the Case of a Single Client. That means that we have just one client in our network. Initially it is sitting at some server \( v \), having some color \( c \in C(v) \). As time goes, the client is changing his location and color randomly, with the rates given by the rate function \( \gamma(\cdot, \cdot) \), and transition probabilities \( P \). We want this (continuous time finite state) Markov process to be ergodic. (This is equivalent to the ergodicity of the Markov chain defined by \( P \).)

In the networks that we are going to consider, there will be many clients, so we need to describe what happens if several clients come for the service to the same server. Here we can treat a fairly general situation. Suppose that at the server \( v \) at time \( t \) we have a queue \( x_v = \{c_1, c_2, ..., c_k\} \) of clients, \( c_i \in C(v) \). That means that these clients came to \( v \) before \( t \), and are still there at the moment \( t \), waiting to be served. They are listed in the order of their arrival. The number \( k = k(x_v) \) is called the queue length (at \( v \) at the moment \( t \)). The protocol \( R(v) \) is a rule \( i_v(\cdot) \) for the server \( v \), which assigns to every non-empty queue \( x_v \) the index \( 1 \leq i_v(x_v) \leq k \), which is the index of the client in \( x_v \) who is served at the moment \( t \). Once this client finishes its service and leaves the server, the queue \( x_v \) turns into

\[
x_v \ominus \{i_v(x_v), c_{i_v(x_v)}\} \equiv \{c_1, c_2, ..., c_{i_v(x_v)-1}, c_{i_v(x_v)+1}, ..., c_k\},
\]

so \( k(x_v \ominus \{i_v(x_v), c_{i_v(x_v)}\}) = k(x_v) - 1 \).

For example, the server can just serve the clients in the order they arrive, i.e. \( i_v(x_v) \equiv 1 \). This protocol is called FIFO – First-In-First-Out. We will treat fairly general protocols, with two restrictions.

The first is that the server can not be idle if there are clients waiting for the service. This is called conservative discipline.

The second is the following monotonicity property. Let \( c_1^1, c_2^1, ..., c_k^1, ... \) be the schedule of arrival of customers to the node \( v \), with \( t_i \) being the moment of arrival of the \( i \)-th customer. Suppose we know the service time needed for every client. Then the protocol \( R(v) \) allows us to define the function \( N(t) \), which is the length of the queue at the moment \( t \geq 0 \). Consider now another arrival schedule, which differs by exactly one extra client \( c \):
... $c_{t_i}, c_{t_{i+1}}, \ldots, c_{t_k}, \ldots$, where $t_i < t < t_{i+1}$. Then the queue length function would change to a different one, $N^c (t)$. We need that $N^c (t) \geq N (t)$ for all $t > 0$. This property holds for most of the natural disciplines.

(Note that it is allowed that the service of the client $c$ is interrupted once a client with higher priority arrives. The service of $c$ is then resumed according to the priority rule $R(v)$.)

1.2 Mean-field type graphs

Now we associate to the graph $G = (V, E)$ the sequence $\mathcal{G}_1 (G) = G \subset \mathcal{G}_2 (G) \subset \ldots \subset \mathcal{G}_M (G) \subset \ldots$ of graphs, which is constructed as follows. For every $M$ consider a disjoint union of $M$ copies $G_i = (V_i, E_i)$ of the graph $G$, $i = 1, \ldots, M$. Then the graph $G_M = (\mathcal{V}_M, \mathcal{E}_M)$ has for its vertices the set $\mathcal{V}_M = \bigcup_{i=1}^{M} V^i$ of all the vertices of these $M$ copies of $G$. We declare a pair $(v^i_1, v^j_2) \subset \mathcal{V}_M$ to be a bond in $\mathcal{E}_M$, $v^i_1 \in V^i, v^j_2 \in V^j, i, j = 1, \ldots, M$ iff two vertices $v_1, v_2 \in V$ are connected by an edge $e \in E$. Thus, every bond of $G$ produces exactly $M^2$ bonds of $\mathcal{E}_M$. There are no other bonds in $\mathcal{E}_M$.

For example, if $G$ consists of just one vertex $v$, connected to itself by a loop, then $\mathcal{G}_M (G)$ will be a complete graph with $M$ vertices (plus to every vertex there is a loop attached).

We now define the mean field graphs $\mathcal{G}_M (\bar{G})$ in precisely the same way as above. Of course, these graphs have natural orientations, inherited from $\bar{G}$. We keep the rate functions the same, and we define the transition probability matrix $P_M$ by

$$P_M [(v^i_1, c_1), (v^j_2, c_2)] = \frac{1}{M} P [(v_1, c_1), (v_2, c_2)].$$

In what follows we will fix the elementary network $(\bar{G}, \gamma)$, and we will be interested in the corresponding mean-field networks $\mathcal{G}_M (\bar{G})$, populated by $N = \lfloor \rho M \rfloor$ clients. We will call the parameter $\rho$ the load. We thus have for every $M$ the ergodic Markov process on the network $\mathcal{G}_M (\bar{G})$ with $N$ clients; let us denote this process by $\nabla^\rho_M$, while $\pi^\rho_M$ will denote its invariant measure. We will be interested in the asymptotic properties of the measures $\pi^\rho_M$ as $M \to \infty$, as well as in the character of the convergence $\nu^\rho_M (t) \to \pi^\rho_M$ of the state $\nu^\rho_M (t)$ of the process $\nabla^\rho_M$ at time $t$ to its limit $\pi^\rho_M$.

Informally speaking, the Poisson Hypothesis for the mean-field networks $\mathcal{G}_M (\bar{G})$ is the following statement about the behavior of the measures $\nu^\rho_M (t)$ for large $M$: if the initial state $\nu^\rho_M (0)$ is chosen reasonably – which means
that the initial queues at every node do not exceed some constant $K$ – then after some time $T(K)$, *independent of $M$*,

- the $M |V|$ servers of our network become almost independent, i.e. the measure $\nu_M^\rho(t)$ is close to a product measure over the set $V_M$;
- at every node $v \in V_M$ the process $\nu_M^\rho(t)$ looks as if the inflows of all possible types of customers to $v$ are independent Poisson flows $P_{c,v}$, $c \in C(v)$, with *constant* rates $\lambda_c$ (depending only on $\rho$, the graph $G$ and the function $\gamma$, *but not on $M$ or $t$*), which customers are then queuing at $v$ and are served according to the service rule $R(v)$ at $v$, and leaving the node after being served. Let us denote the stationary state of such a node by $\chi^{\{\lambda_c, c \in C(v)\}}$.

In such a case we clearly have the relations

$$
\lambda_{c'} = \sum_{v \in V} \sum_{c \in C(v)} \lambda_c P\left((v, c), (v', c')\right),
$$

(2)

where $c' \in C(v')$. They define the set $\bar{\lambda} = \{\lambda_c, c \in \bar{V}\}$ of the rates of the Poisson inflows up to a common factor, $\alpha$. Note that the expected number of customers, $N = N(\bar{\lambda})$, present in the network $G$ in the state $\prod_{v \in V} \chi^{\{\lambda_c, c \in C(v)\}}$, considered as a function of $\alpha$, i.e. $N(\alpha \bar{\lambda})$, is continuous strictly increasing in $\alpha$ once $\bar{\lambda} \neq 0$. (Here we need the monotonicity property of the service discipline $R(v)$). Therefore the rates $\bar{\lambda}^\rho$ are uniquely defined by the relations (2), supplemented by the equation $N(\bar{\lambda}) = \rho$. The stationary distribution of the queues, corresponding to the above Poisson inflows $\bar{\lambda}^\rho$ will be denoted by $\chi^\rho$:

$$
\chi^\rho = \prod_{v \in V} \chi^{\{\lambda_c^\rho, c \in C(v)\}}.
$$

(3)

To save on notation, we will consider below the case when the service discipline is FIFO with priorities. To describe it we need to endow every set of colors $C(v)$ with priority relation $\succ$, which is a linear order, and we say that the client of type $c_1$ has higher priority than $c_2$ iff $c_2 \succ c_1$. At every moment when there are several clients waiting at a server for the service, the client with the highest priority is served. If there are several such clients, they are served in the order they came to the server. The priorities are respected to such an extent that even when a high priority client arrives at the time when a low priority client is under the service, then the service of the latter is
interrupted, and is resumed only after the former one leaves the node. The (notational) advantage of such discipline is that the queue can be described by just a vector in $\Omega = \mathbb{Z}^{\lvert C(v) \rvert}$. But otherwise our proof can be carried out literally.

Due to the symmetry of our network we can consider all the measures $\nu^\rho_M (t)$ and $\pi^\rho_M$ to be the measures on the same space $\mathcal{M} (\Omega)$, interpreting them as the frequency of a given state of a given server in the network $G_M (\bar{G})$. Thus, $\nu^\rho_M (t)$ and $\pi^\rho_M$ are elements of $\mathcal{M} (\mathcal{M} (\Omega))$. However, the limiting measure $\nu^\rho_\infty (t)$, unlike all the other measures above, is again the element of $\mathcal{M} (\Omega)$. The process $\nu^\rho_\infty (t)$ is a Non-Linear Markov Process (NLMP). See [RSh1] for more details.

Using these notations, the above claims mean that the convergence $\nu^\rho_M (t) \to \pi^\rho_M$ is uniform in $M$, and that $\pi^\rho_M \to \chi^\rho$ as $M \to \infty$. The measure $\chi^\rho$ is called “the Poisson Hypothesis behavior”.

> **2 The main result**

Suppose we are given the elementary network $(\bar{G}, \gamma)$, which is connected in the sense of Condition 1. We suppose that every server $v \in V$ is supplied with the conservative service rule $R(v)$ (which allows the server to decide the order in which the queuing clients are served). Consider now the sequence $\nabla^\rho_M$ of Markov process on the networks $G_M (\bar{G})$, $M = 1, 2, \ldots$, populated by $N = \lfloor \rho M \rfloor$ clients. Let $\pi^\rho_M$ stands for its invariant measure.

**Theorem 2** There exists the value $\rho^\star = \rho (\bar{G}, \gamma, P)$, such that for any $\rho < \rho (\bar{G}, \gamma, P)$ the weak limit $\lim_{M \to \infty} \pi^\rho_M$ exists and is equal to the measure $\chi^\rho$.

Our next result deals with the convergence to the stationary states $\pi^\rho_M$. For that, we have to specify the class of the initial states of our Markov processes.

**Theorem 3** Suppose we start the Markov process $\nabla^\rho_M$ in the state $\nu^\rho_M (0)$, which has the property that at every node $v \in V_M$ the queue length $k_v = k(v, t = 0)$ satisfies

$$\int \exp \{\kappa k_v\} \; dv^\rho_M (0) \leq K,$$  \hfill (4)
where \( \kappa > 0, K \) are some constants, defined below. Suppose that \( \rho < \rho (\bar{G}; \gamma) \). Then for every local function \( f \) we have

\[
\left| \int f \, d\nu^\rho_M (t) - \int f \, d\pi^\rho_M \right| \leq \|f\| \exp \{-\tau t\},
\]

where the constant \( \tau = \tau (\kappa, K) \) does not depend on \( M \).

3 Proof

The plan of the proof is the following. First, we establish the exponential convergence of the limiting process, \( \nu^\rho_\infty (t) \), which is NLMP, to its limiting state. We will see that the limiting state \( \chi^\rho \) is unique, and coincides with the Poisson point. Since by the Khasminsky theorem – see Theorem 1.2.14 in [L] – any accumulation point of the sequence \( \pi^\rho_M \) is a stationary measure of \( \nu^\rho_\infty (t) \), while the family \( \pi^\rho_M, M = 1, 2, \ldots \) is compact, that will prove all our claims.

In this paper we present a part of the proof of our theorems for a concrete network studied in [RShV1]. The generalization to other networks is straightforward.

The graph \( G \) in [RShV1] has three vertices, \( \bar{O}, \bar{A} \) and \( \bar{B} \). The clients at \( \bar{O} \) have no priorities, i.e. \( \mathcal{C} (\bar{O}) = 1 \), while the clients at \( \bar{A} \) or \( \bar{B} \) are of two types, i.e. \( \mathcal{C} (\bar{A}) = 2 \). In notations of [RShV1], \( \mathcal{C} (\bar{A}) = \{A, BA\} \), \( \mathcal{C} (\bar{B}) = \{B, AB\} \), and \( BA \)-clients have priority over \( A \)-clients (at \( \bar{A} \)), while \( AB \)-clients - over \( B \)-clients (at \( \bar{B} \)). The rates \( \gamma (\cdot, \cdot) \) are the following:

\[
\begin{align*}
\gamma (\bar{O}) &\equiv \gamma_O = 3, \\
\gamma (\bar{A}, BA) &\equiv \gamma_{BA} = \gamma (\bar{B}, AB) \equiv \gamma_{AB} = 2, \\
\gamma (\bar{A}, A) &\equiv \gamma_A = \gamma (\bar{B}, B) \equiv \gamma_B = 10. 
\end{align*}
\]

The matrix \( P \) is given by

\[
P = \begin{bmatrix}
O & A & BA & B & AB \\
O & o & \frac{1}{2} & o & \frac{1}{2} & o \\
A & o & o & o & o & 1 \\
BA & 1 & o & o & o & o \\
B & o & o & 1 & o & o \\
AB & 1 & o & o & o & o
\end{bmatrix}.
\]

It is clearly ergodic.
3.1 Compactness

We first prove the following compactness statement.

**Lemma 4** Suppose all the initial states \( \nu_M^\rho (0) \) of our processes satisfy the condition (4), \( M = 1,2,... \). Suppose that the load \( \rho \) is small enough. Then there exist two values \( \kappa' \) and \( K' \), such that for all nodes \( v \), all \( t \) and \( M \)

\[
\int \exp \{ \kappa k_v \} \ dv_M^\rho (t) \leq K'.
\] (7)

**Proof.** Note that the rate of the flow of the clients of type \( c_1 \) from the server \( v_1 \) to the server \( v_2 \), where they turn into the clients of type \( c_2 \), is bounded from above by \( \rho \gamma (v,c) P [(v_1,c_1),(v_2,c_2)] \). That implies (7) for \( \rho \) small enough.

Suppose we know the convergence of the generators of the Markov processes \( \nabla_M^\rho \) to that of the limiting Non-Linear Markov Process \( \nabla_\infty^\rho \). That implies the convergence of the processes \( \nabla_M^\rho \) to \( \nabla_\infty^\rho \) on any finite time interval. Suppose also that we can show that the NLMP \( \nabla_\infty^\rho \) is ergodic – i.e. for every its initial state \( \nu_\infty^\rho (0) \) satisfying (4) we have the convergence to the (unique) limiting state, \( \nu_\infty^\rho (t) \to \pi_\infty^\rho \), which is uniform in the choice of \( \nu_\infty^\rho (0) \). Then, first of all, we have the convergence \( \pi_M^\rho \to \nu_\infty^\rho (\infty) \) as \( M \to \infty \). Indeed, the family \( \pi_M^\rho \) is compact, due to (7). Since by the Khasminsky theorem – see Theorem 1.2.14 in [L] – any accumulation point of the sequence \( \pi_M^\rho \) is a stationary measure of \( \nabla_\infty^\rho \), while the latter is unique, the convergence follows.

Let us check that the convergence \( \nu_M^\rho (t) \to \pi_M^\rho \) is uniform in \( M \). Let \( \varepsilon > 0 \), and let \( T = T(\varepsilon) \) be the time for which the estimate \( \rho_{KROV} (\nu_M^\rho (T), \pi_M^\rho) < \varepsilon \) holds for any initial condition \( \nu_\infty^\rho (0) \) satisfying (4). We will show that for all \( M \) large enough and for all \( t > T(\varepsilon) \) we have

\[
\rho_{KROV} (\nu_M^\rho (t), \pi_M^\rho) < 3\varepsilon.
\] (8)

Indeed, the state \( \nu_M^\rho (t - T) \) satisfies (7), and therefore the evolution \( \nabla_\infty^\rho \), applied to it for time \( T \), results in a state \( \nu_M^\rho (t) \), satisfying \( \rho_{KROV} (\nu_M^\rho (t), \pi_M^\rho) < \varepsilon \). But for \( M \) large enough \( \rho_{KROV} (\nu_M^\rho (t), \nu_M^\rho (t)) < \varepsilon \), \( \rho_{KROV} (\pi_M^\rho, \pi_\infty^\rho) < \varepsilon \), so (8) follows.

So to prove our theorems it remains to show that for any initial state \( \nu_\infty^\rho (0) \) of the NLMP satisfying (4) we have the convergence \( \nu_\infty^\rho (t) \) to \( \chi^\rho \).
3.2 The derived process

We want to study our particle system with many nodes $M$, or even with infinite number of them, in the limit as the number of particles, $N$, or the ratio $\rho = \frac{N}{M}$ goes to zero. Understood literally, the limiting object is trivial. We want a non-trivial version of it. The object we construct is very similar to the derivative of the function at the point where it vanishes, so we look not on the function, but on the properties of its increments.

We will construct this derived process for the case of $M = \infty$, i.e., for the NLMP. It can be defined in the following way. Let first $M$ is finite, and $N$ has some value. Let $\mu_{M,\rho}$ be a state of our Markov process. For every configuration $\sigma$ of our process we can define a measure $v_\sigma$ on $\mathcal{Z} = \bigcup_{v \in V} \left( \mathbb{Z}_+^{C(v)} \setminus 0 \right)$ – the disjoint union of lattices without the origins – which is the distribution of “the queue of the average customer”. Namely, for every client $c$, forming $\sigma$, we consider a point $x_c$ in $\mathcal{Z}$, by first taking the server $v(c)$, corresponding to $c$, and then marking the point $x_c \in \mathbb{Z}_+^{C(v)} \setminus 0$, describing the queue at this server. Then $v_\sigma = \frac{1}{N} \sum_{c \in \sigma} \delta_{x_c}$. We define the derived process $\mu'_{M,\rho}$ by

$$\mu'_{M,\rho} = \int v_\sigma \, d\mu_{M,\rho}(\sigma).$$

Clearly, we can take the limit of $\mu'_{M,\rho}$ as $M \to \infty$, keeping $\rho$ fixed. The limiting process – the derivative – $\mu'_{\infty,\rho}$ will be also NLMP.

It can be described alternatively as follows. The state of the NLMP on the graph $G = (V, E)$ is represented by the collection of $|V|$ probability measures $\mu_v, v \in V$, defined, correspondingly, on lattices $\mathbb{Z}_+^{C(v)}$. For every $x \in \mathcal{Z}$ we put $|x| = x_1 + \ldots + x_d(x)$, where $d(x)$ is the dimension of $x$. The derivative is a single probability measure $\mu'$ on the disjoint union $\mathcal{Z} = \bigcup_{v \in V} \left( \mathbb{Z}_+^{C(v)} \setminus 0 \right)$ of lattices. It is defined by its density with respect to various $\mu_v$-s: for every $x \in \mathbb{Z}_+^{C(v)} \setminus 0 \subset \mathcal{Z}$ we put

$$\mu'(x) = \frac{1}{N(\mu)} |x| \mu_v(x),$$

where $N(\mu) = \sum_{x \in \mathcal{Z}} \mu(x) \equiv \sum_{v \in V} \sum_{x \in \mathbb{Z}_+^{C(v)} \setminus 0} |x| \mu_v(x)$.

Since all our measures have exponential moments, the above normalization is possible. The recovering of the probability measures $\mu_v$ from $\mu'$ can be done only “up to a constant”, as usual. We will take care of it below.
In accordance with our choice of variables we introduce the norm
\[ \|\mu\| = \sum_{x \in \mathbb{Z}} \frac{\mu'(x)}{|x|} e^{|x|}. \] (9)

In case of the signed measure \( \mu \) on \( \mathbb{Z} \), we define \( \|\mu\| \) by the same relation, with \( \mu(x) \) replaced by \( |\mu(x)| \). Below we are assuming that all the measures \( \mu \) considered are elements of the space \( Y \) of measures on \( \mathbb{Z} \) with finite norm \( \|\mu\| \).

The process \( \mu'_{\infty, \rho} \) is again a NLMP. Note, however, that the limit \( \mu'_{\infty,0} \equiv \lim_{\rho \to 0} \mu'_{\infty, \rho} \) is a usual (linear) Markov process on \( \mathbb{Z} \). It is defined by the following jump rates. Let \( v \) be a server, and \( x \in \mathbb{Z}^{[C(v)]} \subset \mathbb{Z}, x = (x_1, ..., x_{|C(v)|}) \). We will suppose that the coordinates are listed in the priority order, so if \( x = (x_1, ..., x_{k-1}, x_k, 0, ..., 0) \), with \( x_k > 0 \), then after a single act of service the queue will be \( \bar{x} = (x_1, ..., x_{k-1}, x_k - 1, 0, ..., 0) \). The rate \( c_{x \to \bar{x}} \) of the jump \( x \to \bar{x} \) is then
\[ c_{x \to \bar{x}} = \frac{|x| - 1}{|x|} \sum_{i,j} \gamma(v, k) P[(v, k), (v_i, c_j)], \] (10)
where the sum is taken over all possible outcomes of the service at \( v \) of the client of \( k \)-th priority. Another possible jump for the process \( \mu'_{\infty,0} \) from \( x \) is to the cite \( 1_{ij} \equiv (0, ..., 0, 1_j, 0, ..., 0) \in \mathbb{Z}^{[C(v_i)]} \subset \mathbb{Z} \); it happens with the rate
\[ c_{x \to 1_{ij}} = \frac{1}{|x|} \gamma(v, k) P[(v, k), (v_i, c_j)]. \] (11)

From the definitions (10) – (11) and our Condition 1 it follows that the process \( \mu'_{\infty,0} \) is ergodic, and its stationary distribution coincides with that of the Markov process on \( G \), corresponding to the Case of a Single Client.

3.3 Dynamics

The NLMP dynamics in \( ' \)-coordinates is given by a differential equation
\[ \dot{\mu}(t) = F(\mu(t)), \quad \mu(0) = \mu_0. \] (12)

The right-hand side of (12) is the sum of two terms; the first one is linear in \( \mu \), and the second one is quadratic:
\[ F(\mu) = G\mu + \rho H(\mu). \] (13)
To convince the reader, we will write down these equations for the system studied in [RShV1].

The description of the NLMP in [RShV1] was done via probability measure \( \nu_t = \{ \nu_t(x), \ x \in \mathbb{Z}^5 \} \). Its evolution was given by the equation

\[
\frac{d\nu_t(x)}{dt} = -\nu_t(x) \sum_{a=O,AB,BA} \lambda_a(t) \\
-\nu_t(x) \left( \sum_{a=O,AB,BA} \gamma_a (1-\delta_{xa}) + \gamma_A (1-\delta_{x_A}) \delta_{x_{BA}} + \gamma_B (1-\delta_{x_B}) \delta_{x_{AB}} \right) \\
+ \nu_t(x - \Delta_O) (1-\delta_{x_O}) (\lambda_{AB}(t) + \lambda_{BA}(t)) + \sum_{a=A,B} \nu_t(x - \Delta_a) (1-\delta_{xa}) \frac{\lambda_O(t)}{2} \\
+ \nu_t(x - \Delta_{AB}) (1-\delta_{x_{AB}}) \lambda_A(t) + \nu_t(x - \Delta_{BA}) (1-\delta_{x_{BA}}) \lambda_B(t) \\
+ \sum_{a=O,AB,BA} \nu_t(x + \Delta_a) \gamma_a + \nu_t(x + \Delta_A) \gamma_A \delta_{x_{BA}} + \nu_t(x + \Delta_B) \gamma_B \delta_{x_{AB}},
\]

where

\[
\lambda_a(t) = \gamma_a \sum_{x:x_a>0} \nu_t(x), \text{ for } a = O, AB, BA, \\
\lambda_A(t) = \gamma_A \sum_{x:x_A>0,x_{BA}=0} \nu_t(x), \quad \lambda_B(t) = \gamma_B \sum_{x:x_B>0,x_{AB}=0} \nu_t(x),
\]

and the 5D vectors \( \Delta_a \) are the basis vectors of the lattice \( \mathbb{Z}^5 \). In fact, \( \nu_t = \nu_t^A \times \nu_t^B \times \nu_t^O \), where the probability measures \( \nu_t^A, \nu_t^B \) are defined on \( \mathbb{Z}^5_{+} \), while \( \nu_t^O \) is on \( \mathbb{Z}^1_{+} \), and their evolution is given by the relations

\[
\frac{d\nu_t^A(x)}{dt} = -\nu_t^A(x) \left( \frac{\lambda_O(t)}{2} + \lambda_B(t) + \gamma_{BA}(1-\delta_{x_{BA}}) + \gamma_A (1-\delta_{x_A}) \delta_{x_{BA}} \right) \\
+ \nu_t^A(x - \Delta_A) (1-\delta_{x_A}) \frac{\lambda_O(t)}{2} + \nu_t^A(x - \Delta_{BA}) (1-\delta_{x_{BA}}) \lambda_B(t) \\
+ \nu_t^A(x + \Delta_A) \gamma_A \delta_{x_{BA}} + \nu_t^A(x + \Delta_{BA}) \gamma_{BA},
\]
\[
\frac{d\nu^B_t(x)}{dt} = -\nu^B_t(x) \left( \frac{\lambda_0(t)}{2} + \lambda_A(t) + \gamma_{AB} (1 - \delta_{x,AB}) + \gamma_B (1 - \delta_{xB}) \delta_{x,AB} \right) \\
\quad + \nu^B_t(x - \Delta_B)(1 - \delta_{xB}) \frac{\lambda_0(t)}{2} + \nu^B_t(x - \Delta_{AB})(1 - \delta_{x,AB}) \lambda_A(t) \\
\quad + \nu^B_t(x + \Delta_B) \gamma_B \delta_{x,AB} + \nu^B_t(x + \Delta_{AB}) \gamma_{AB},
\]

\[
\frac{d\nu^O_t(x)}{dt} = -\nu^O_t(x) \left( \sum_{a = AB, BA} \lambda_a(t) + \gamma_{O} (1 - \delta_{x,O}) \right) + \nu^O_t(x - \Delta_O)(1 - \delta_{x,O}) (\lambda_{AB}(t) + \lambda_{BA}(t)) + \nu^O_t(x + \Delta_O) \gamma_{O}.
\]

The new \textquoteleft ν\textprime\textquotesingle -variables are introduced as follows: \( \nu^O_t(x) = \frac{|\nu^O_t(x)|}{\rho} \) (with \( x \in \mathbb{Z}^+_1 \)), \( \nu^A_t(x) = \frac{|\nu^A_t(x)|}{\rho} \), \( \nu^B_t(x) = \frac{|\nu^B_t(x)|}{\rho} \) (with \( x \in \mathbb{Z}^+_1 \)), \( |x| \geq 1 \), where \( \rho = \sum_{x \in \mathbb{Z}^+_1} x \nu_t(x) \) is the number of particles, (which is conserved). Substituting, we have for \( x \neq \Delta_{BA}, \Delta_A \)

\[
\frac{d\nu^A_t(x)}{dt} = -\rho \nu^A_t(x) \left( \frac{\gamma_{O}}{2} \sum_{x:|x|>0} \frac{\nu^O_t(x)}{|x|} + \gamma_B \sum_{x:xB>0, xAB=0} \frac{\nu^B_t(x)}{|x|} \right) \\
\quad + \frac{\rho \gamma_{O}}{|x - \Delta_A|} \nu^A_t(x - \Delta_A)(1 - \delta_{x,A}) \sum_{x:|x|>0} \frac{\nu^O_t(x)}{|x|} + \frac{\rho \gamma_B}{|x - \Delta_{BA}|} \nu^A_t(x - \Delta_{BA})(1 - \delta_{x,BA}) \sum_{x:xB>0, xAB=0} \frac{\nu^B_t(x)}{|x|} \\
\quad + \frac{|x|}{|x + \Delta_A|} \nu^A_t(x + \Delta_A) \gamma_A \delta_{x,BA} + \frac{|x|}{|x + \Delta_{BA}|} \nu^A_t(x + \Delta_{BA}) \gamma_{BA} \\
\quad - \nu^A_t(x) (\gamma_{BA}(1 - \delta_{x,BA}) + \gamma_A(1 - \delta_{x,A}) \delta_{x,BA}).
\]

Also,

\[
\frac{d\nu^A_t(\Delta_A)}{dt} = -\rho \nu^A_t(\Delta_A) \left( \frac{\gamma_{O}}{2} \sum_{x:|x|>0} \frac{\nu^O_t(x)}{|x|} + \gamma_B \sum_{x:xB>0, xAB=0} \frac{\nu^B_t(x)}{|x|} \right) \\
\quad - \rho \frac{\gamma_{O}}{2} \sum_{|x|>0} \frac{\nu^A_t(x)}{|x|} \sum_{x:|x|>0} \frac{\nu^O_t(x)}{|x|} \\
\quad + \frac{\gamma_{O}}{2} \sum_{x:|x|>0} \frac{\nu^O_t(x)}{|x|} + \frac{1}{2} \nu^A_t(2\Delta_A) \gamma_A + \frac{1}{2} \nu^A_t(\Delta_A + \Delta_{BA}) \gamma_{BA} - \nu^A_t(\Delta_A) \gamma_{A},
\]

13
\[
\frac{d\nu_t^A(\Delta_{BA})}{dt} = -\rho \nu_t^A(\Delta_{BA}) \left( \frac{\gamma_0}{2} \sum_{x:x_B>0} \frac{\nu_t^O(x)}{|x|} + \gamma_B \sum_{x:x_B>0,x_{AB}=0} \frac{\nu_t^B(x)}{|x|} \right) \\
- \rho \gamma_B \sum_{|x|>0} \frac{\nu_t^A(x)}{|x|} \sum_{x:x_B>0,x_{AB}=0} \frac{\nu_t^B(x)}{|x|} \\
+ \gamma_B \sum_{x:x_B>0,x_{AB}=0} \frac{\nu_t^B(x)}{|x|} + \frac{1}{2} \nu_t^A(2\Delta_{BA}) \frac{\gamma_{BA}}{2} - \nu_t^A(\Delta_{BA}) \frac{\gamma_{BA}}{2}
\]

The identical relations hold for the measure \( \nu_t^B \). For the measure \( \nu_t^O \) we get

\[
\frac{d\nu_t^O(x)}{dt} = \rho \left( \frac{|x| \nu_t^O(x-1)}{|x-1|} - \nu_t^O(x) \right) \left[ \gamma_{AB} \sum_{x:x_{AB}>0} \frac{\nu_t^B(x)}{|x|} + \gamma_{BA} \sum_{x:x_B>0} \frac{\nu_t^A(x)}{|x|} \right] \\
+ \left[ -\nu_t^O(x) + \frac{|x|}{|x+1|} \nu_t^O(x+1) \right] \frac{\gamma_{O}}{2} \text{ for } |x| > 1,
\]

\[
\frac{d\nu_t^O(1)}{dt} = -\rho \left( \sum_{x \geq 1} \frac{\nu_t^O(x)}{|x|} + \nu_t^O(1) \right) \left[ \gamma_{AB} \sum_{x:x_{AB}>0} \frac{\nu_t^B(x)}{|x|} + \gamma_{BA} \sum_{x:x_B>0} \frac{\nu_t^A(x)}{|x|} \right] \\
+ \left[ -\nu_t^O(1) + \frac{1}{2} \nu_t^O(2) \right] \gamma_{O} + \left[ \gamma_{AB} \sum_{x:x_{AB}>0} \frac{\nu_t^B(x)}{|x|} + \gamma_{BA} \sum_{x:x_B>0} \frac{\nu_t^A(x)}{|x|} \right].
\]

As we see, the quadratic terms all have the factor \( \rho \) in front of them. Putting it to zero, we get the linear part of the evolution:

\[
\frac{d\nu_t^A(x)}{dt} = \frac{|x|}{|x|+1} \nu_t^A(x+\Delta) \frac{\gamma_A \delta_{x_B}}{2} + \frac{|x|}{|x|+1} \nu_t^A(x+\Delta_{BA}) \frac{\gamma_{BA}}{2} \\
- \nu_t^A(x) \left( \gamma_{BA} (1-\delta_{x_B}) + \gamma_A (1-\delta_{x_A}) \delta_{x_B} \right)
\]

for \( x \neq \Delta_{BA}, \Delta_A \). Also,

\[
\frac{d\nu_t^A(\Delta_A)}{dt} = \frac{\gamma_0}{2} \sum_{x:x_B>0} \frac{\nu_t^O(x)}{|x|} + \frac{1}{2} \nu_t^A(2\Delta_A) \frac{\gamma_A}{2} \\
+ \frac{1}{2} \nu_t^A(\Delta_A+\Delta_{BA}) \frac{\gamma_{BA}}{2} - \nu_t^A(\Delta_A) \frac{\gamma_{A}}{2},
\]

\[14\]
\[
\frac{d\nu^A_t(\Delta_{BA})}{dt} = \gamma_B \sum_{x:B>0,x_{AB}=0} \frac{\nu_t^B(x)}{|x|} + \frac{1}{2}\nu_t^A(2\Delta_{BA})\gamma_{BA} - \nu_t^A(\Delta_{BA})\gamma_{BA}.
\]

(21)

Exchanging the indices \( A \leftrightarrow B \), we get the equations for \( \nu^B_t \). As for \( \nu^O_t \), we have

\[
\frac{d\nu^O_t(x)}{dt} = \left[ -\nu^O_t(x) + \frac{|x|}{|x+1|}\nu^O_t(x+1) \right] \gamma_O \text{ for } |x| > 1,
\]

(22)

\[
\frac{d\nu^O_t(1)}{dt} = \left[ -\nu^O_t(1) + \frac{1}{2}\nu^O_t(2) \right] \gamma_O + \sum_{a=AB,BA} \left[ \gamma_a \sum_{x:x_a>0} \frac{\nu^a_t(x)}{|x|} \right] \text{ for } x = 1.
\]

(23)

We want to establish the contraction property of the evolution (13) – so, in particular, the evolution (14) – (18).

To exhibit it we will compare it with the linear evolution

\[
\dot{\mu}^l(t) = G\mu^l(t), \quad \mu^l(0) = \mu_0,
\]

(24)

with the same initial point. This is the Markov process with rates (10) – (11), discussed above. In our example the linear evolution is given by (19) – (23).

Its informal description is the following. Consider the infinite network \( G_\infty \), which is populated by infinitely many clients, having nevertheless zero density: \( \rho = 0 \). The initial distribution \( \mu_0 \) of the queue seen by an average client can be arbitrary. The dynamics is the following: after waiting in the queue and then being served, the client goes to a next server of corresponding type – but he finds it free with probability one! The reason is the vanishing density: it is improbable that a client will get to an occupied server. Therefore in the limit \( t \rightarrow \infty \) the measure \( \mu^l(t) \) is concentrated only on points \( x \in \mathcal{Z} \) with property \( |x| = 1 \).

This linear evolution is contracting in the following sense: for every \( K > 0 \) there exists the time \( T = T(K) < \infty \), such that for every two trajectories \( \mu^l_1(t) \) and \( \mu^l_2(t) \) with \( \|\mu^l_1(0)\| \leq K \) we have for all \( t \geq 0 \)

\[
\|\mu^l_1(t + T) - \mu^l_2(t + T)\| \leq \frac{1}{2}\|\mu^l_1(t) - \mu^l_2(t)\|.
\]

(25)

Let us define on the subspace \( Y_0 = \{\nu \in Y : \nu(\mathcal{Z}) = 0\} \) a new norm:

\[
\|\nu\|_1 = \int_0^\infty e^{\beta t} \|\nu^l(t)\| dt,
\]

(26)
where $\nu'(t)$ is the solution of (24) with $\nu'(0) = \nu$. If $\beta > 0$ is small enough, the norm $\|\cdot\|_1$ is finite on $Y_0$ and equivalent there to $\|\cdot\|$. Moreover (unlike the norm $\|\cdot\|$!), it satisfies the infinitesimal version of (25): for any $t \geq 0$

$$\frac{d}{dt} \|\mu(t) - \mu(t)\|_1 \leq -\beta \|\mu(t) - \mu(t)\|_1.$$ 

### 3.4 Non-linear part

The quadratic part of (12) can be written as

$$[H(\mu)](z) = \sum_{x,y \in \mathcal{Z}} v_{xy}(z) \frac{\mu(x) \mu(y)}{|x| |y|},$$

where $z \in \mathcal{Z}$, and for every pair $x, y \in \mathcal{Z}$ the function $v_{xy}(z)$ on $\mathcal{Z}$ has finite support, the size of which depends only on the structure of the network $\bar{G}$.

To see this, let us fix a pair of points $x, y \in \mathcal{Z}$, and mark the equations $\frac{d\nu'(z)}{dt} = \ldots$ from those listed above, (14) – (18), containing the cross-term $\nu'_t(x) \nu'_t(y)$. Let us write down the vector $v_{xy} = \{v_{xy}(z), z \in \mathcal{Z}\}$. This vector is non-zero only if the two indices $x, y$ belong to two different lattices among the three present: $O$, $A$ or $B$. For example, if $x \in O$, $x \neq 1$ and $y \in A$, $y \neq \Delta_{BA}, \Delta_A$, then the following six coordinates of $v_{xy}$ are non-zero:

$$v_{xy}(y) = -\frac{\gamma_O}{2 |x|}, \quad v_{xy}(y + \Delta_A) = \frac{\gamma_O |y + \Delta_A|}{2 |x| |y|}, \quad v_{xy}(\Delta_A) = -\frac{\gamma_O}{2 |y| |x|},$$

$$v_{xy}(x) = -\frac{\gamma_{BA}}{|y|}, \quad v_{xy}(x + 1) = \frac{|x + 1| \gamma_{BA}}{|x| |y|}, \quad v_{xy}(1) = -\frac{\gamma_{BA}}{|x| |y|}$$

(the last three relations require $y_{BA} > 0$). The rest of them vanish.

It is also easy to see that for some constant $C > 0$ and for each pair $x, y \in \mathcal{Z}$ we have $\|v_{xy}(\cdot)\| \leq C e^{x+y}$ (see (9)).

### 3.5 Convergence

Let us estimate $H(\mu) - H(\nu)$. We have

$$\|H(\mu) - H(\nu)\| \leq \sum_{x,y \in \mathcal{Z}} \|v_{xy}\| \left| \frac{\mu_x \mu_y}{|x| |y|} - \frac{\nu_x \nu_y}{|x| |y|} \right| \leq$$
\[ \leq C \sum_{x,y \in X} e^{\|x| + \|y\|} \left( \left| \frac{\mu_x}{|x|} \right| \left| \frac{\mu_y}{|y|} \right| - \left| \frac{\nu_y}{|y|} \right| + \left| \frac{\nu_y}{|y|} \right| \left| \frac{\mu_x}{|x|} \right| - \left| \frac{\nu_x}{|x|} \right| \right) = \]
\[ = C \sum_{x,y \in X} \left[ \left( e^{\|x\|} \left| \frac{\mu_x}{|x|} \right| \right) \left( e^{\|y\|} \left| \frac{\mu_y - \nu_y}{|y|} \right| \right) + \left( e^{\|y\|} \left| \frac{\nu_y}{|y|} \right| \right) \left( e^{\|x\|} \left| \frac{\mu_x - \nu_x}{|x|} \right| \right) \right] \leq \]
\[ \leq C(\|\mu\| + \|\nu\|)\|\mu - \nu\|. \]  

(27)

By the equivalence of \(\|\cdot\|\) and \(\|\cdot\|_1\), we get the bound
\[ \|H(\mu) - H(\nu)\|_1 \leq C_1(\|\mu\| + \|\nu\|)\|\mu - \nu\|_1. \]  

(28)

For our NLMP we have
\[ \frac{d}{dt}\|\mu(t) - \nu(t)\|_1 \leq -\beta\|\mu(t) - \nu(t)\|_1 + \rho\|H(\mu(t)) - H(\nu(t))\|_1. \]  

(29)

It remains to notice that for every \(K > 0\) there exists a constant \(C_2 > 0\) such for any \(\mu(0) \in Y\) with \(\|\mu(0)\| < K\) we have \(\|\mu(t)\| \leq C_2K\) for all \(t\), so according to (28), the first term beats the second one, once \(\rho\) is small.

Now let us go back to our initial NLMP with low load \(\rho\). Let \(\kappa(t)\) be some trajectory of it, with \(\|\kappa(t)\| \leq C_2K\), while \(\chi^\rho\) be stationary “Poisson Hypothesis” trajectory. Then the derivative process \(\kappa'(t)\) satisfies (13), so by (29) we have
\[ \frac{d}{dt}\|\kappa'(t) - (\chi^\rho)'\|_1 \leq \left( -\beta + \rho C_1 \left( \tilde{C}_2K + \|(\chi^\rho)'\| \right) \right) \|\kappa'(t) - (\chi^\rho)'\|_1. \]

Since \(\kappa(t)\) is the only measure with the load \(\rho\), having derivative \(\kappa'(t)\), our claim is proven.

References

[L] Liggett, Thomas M. Interacting particle systems. Grundlehren der Mathematischen Wissenschaften, 276. Springer-Verlag, New York, 1985.

[RSh1] A.N. Rybko, S.B. Shlosman: Poisson Hypothesis for Information Networks, Sinai’s Festschrift, Moscow Math. J., v. 5, 679-704, 2005, Tsfasman’s Festschrift, Moscow Math. J., v.5, 927-959, 2005.
[RSh2] Rybko, A. N.; Shlosman, S.B.: *Phase Transitions in the queuing networks and the violation of the Poisson Hypothesis*, Moscow Math. Journal, v. 8, 159-180, 2008.

[RShV1] Rybko, A. N.; Shlosman, S.B. and Vladimirov A.: *Spontaneous Resonances and the Coherent States of the Queuing Networks*, [http://arxiv.org/PS_cache/arxiv/pdf/0708/0708.3073v2.pdf](http://arxiv.org/PS_cache/arxiv/pdf/0708/0708.3073v2.pdf) to appear in Journal of Statistical Physics.

[RShV2] Rybko, A. N.; Shlosman, S.B. and Vladimirov A.: *Absense of Breakdown of the Poisson Hypothesis. II. Open Networks*, in preparation.

[RSt] Rybko, A. N. and Stolyar, A. L.: *Ergodicity of stochastic processes describing the operation of open queuing networks*. Prob. Inf. Trans. 28, 199 –220, 1992.

[St1] Stolyar, A. L.: *The asymptotics of stationary distribution for a closed queueing system*. (Russian) Problemy Peredachi Informatsii 25 (1989), no. 4, 80–92; translation in Problems Inform. Transmission 25 (1989), no. 4, 321–331 (1990)

[St2] Stolyar, A. L.: *On the stability of multiclass queueing networks: a relaxed sufficient condition via limiting fluid processes*. Markov Process and Related Fields, 1:491–512, 1995.