We give a pedagogical introduction to dynamical invariant formalism of shortcuts to adiabaticity. For a given operator form of the Hamiltonian with undetermined coefficients, the dynamical invariant is introduced to design the coefficients. We discuss how the method allows us to mimic adiabatic dynamics and describe a relation to the counterdiabatic formalism. The equation for the dynamical invariant takes a familiar form and is often used in various fields of physics. We introduce examples of Lax pair, quantum brachistochrone and flow equation.

This article is part of the theme issue ‘Shortcuts to adiabaticity: theoretical, experimental and interdisciplinary perspectives’.

1. Introduction

The method of shortcuts to adiabaticity is a series of techniques controlling dynamical systems in efficient ways [1–4]. The word ‘adiabaticity’ assumes that the system is operated very slowly. However, by using some techniques, we can mimic adiabatic dynamics under fast operations.

One of the prominent methods is the counterdiabatic driving [5–7]. We introduce an additional term to the Hamiltonian to prevent non-adiabatic transitions. The method is reviewed in an article of this issue [8].

The main aim of this article is to discuss one of the other useful techniques of shortcuts to adiabaticity. We discuss a quantity called dynamical invariant [9]. Keeping the original form of the Hamiltonian unchanged, we can design the protocol so that the state gives a desired time evolution. The method of control by the dynamical invariant is called invariant-based inverse engineering [10]. The dynamical invariant was also used as a method to treat quantum computations [11].
The organization of this article is as follows. First, we discuss fundamental properties of the dynamical invariant and the idea of inverse engineering in §2. Next, we give several simple applications in §3. Then, we discuss a relation to the counterdiabatic formalism in §4 and several applications using the dynamical invariant in §5. The last §6 is devoted to summary.

2. Dynamical invariant formalism

(a) Dynamical invariant

We consider a quantum system described by a time-dependent Hamiltonian $H(t)$. A Hermitian operator $I(t)$ is called a dynamical invariant or a Lewis–Riesenfeld invariant when it satisfies

$$i\hbar \frac{dI(t)}{dt} = [H(t), I(t)].$$

(2.1)

As we discuss in the following sections, this type of operators is familiar in quantum mechanics and is found in many different contexts. Here, we describe the fundamental properties of the dynamical invariant.

The formal solution of equation (2.1) is written as

$$I(t) = U(t)I(0)U(t)^\dagger,$$

(2.2)

where $U(t)$ is the unitary time-evolution operator satisfying

$$i\hbar \dot{U}(t) = H(t)U(t),$$

(2.3)

with $U(0) = 1$. Equation (2.2) shows that the eigenvalues of $I(t)$ are time independent. We write the spectral representation

$$I(t) = \sum_n \lambda_n |\phi_n(t)\rangle \langle \phi_n(t)|.$$  

(2.4)

$\{\lambda_n\}_{n=1,2,\ldots}$ represents the set of eigenvalues and each element takes a real constant value. $\{|\phi_n(t)\rangle\}_{n=1,2,\ldots}$ is the corresponding set of eigenstates. Comparing this spectral representation with equation (2.2), we find that $|\phi_n(t)\rangle$ is equivalent to $|\chi_n(t)\rangle = U(t)|\phi_n(0)\rangle$ up to a phase. We can write

$$|\chi_n(t)\rangle = e^{i\alpha_n(t)}|\phi_n(t)\rangle,$$

(2.5)

where $\alpha_n(t)$ is real. We apply $i\hbar \dot{\chi}_n - H(t)$ on both sides of this equation. Since $|\chi_n(t)\rangle$ satisfies the Schrödinger equation, we have

$$0 = (i\hbar \dot{\chi}_n - H(t) - \hbar \dot{\theta}_n(t))|\phi_n(t)\rangle,$$

(2.6)

where the dot symbol denotes the time derivative. Then, we obtain

$$\alpha_n(t) = \frac{1}{\hbar} \int_0^t \langle \phi_n(s)|(i\hbar \dot{\theta}_n - H(s))|\phi_n(s)\rangle \, ds.$$  

(2.7)

We note that $|\chi_n(t)\rangle$ is independent of the phase choice of $|\phi_n(t)\rangle$, except the one at the initial time. $|\chi_n(t)\rangle$ is invariant under the replacement $|\phi_n(t)\rangle \rightarrow e^{i\theta_n(t)}|\phi_n(t)\rangle$ where $\theta_n(t)$ represents an arbitrary real function with $\theta_n(0) = 0$. The phase $\alpha_n(t)$ takes a familiar form known in the adiabatic approximation of dynamical systems. The first term in equation (2.7) is known as the geometric phase and the second term as the dynamical phase. In fact, the general solution of the Schrödinger equation is written as

$$|\psi(t)\rangle = \sum_n c_n e^{i\alpha_n(t)}|\phi_n(t)\rangle,$$  

(2.8)

where $c_n$ represents a constant determined from the initial condition. This representation denotes that the solution of the Schrödinger equation is given by the ‘adiabatic state’ of the dynamical invariant.
The term ‘invariant’ indicates that the eigenvalues of $I(t)$ are time independent. In the classical limit, the commutation relation is replaced by the Poisson bracket and the operation in equation (2.1) is interpreted as the total time derivative:

$$\frac{\partial}{\partial t} - \frac{i}{\hbar} [H, \cdot] \rightarrow \frac{\partial}{\partial t} + \sum_i p_i \frac{\partial}{\partial p_i} + \sum_i \dot{x}_i \frac{\partial}{\partial x_i} = \frac{d}{dt},$$

where $x_i$ and $p_i$ represent canonical coordinates. Then, the classical analogue of $I(t)$ becomes a constant of motion.

A remarkable feature of this method is that the time-evolved state can be obtained by solving the eigenvalue problem of the dynamical invariant. Of course, we must find the explicit operator form of the dynamical invariant before attacking the eigenvalue problem. For our aim to realize an ideal control of the system, we see in the following that it is not necessary to solve the eigenvalue problem.

(b) Invariant-based inverse engineering

The authors in [10] proposed to use the dynamical invariant for a control of dynamical systems. The term ‘shortcut to adiabaticity’ was coined there. In this section, we review the method of inverse engineering briefly.

To understand the fundamental idea, it is convenient to use a basis-operator representation as discussed in [11–14]. We assume that the Hilbert space of the system has finite dimension $N$. We write the Hamiltonian

$$H(t) = \sum_{\mu} h_{\mu}(t) X_{\mu},$$

and the dynamical invariant

$$I(t) = \sum_{\mu} b_{\mu}(t) X_{\mu}.$$  

(2.11)

Each component of $\{h_{\mu}(t)\}$ and $\{b_{\mu}(t)\}$ is real. Greek indices generally run from 1 to $N^2$. $\{X_{\mu}\}$ represents a set of basis operators. Their operators are Hermitian and satisfy the orthonormal condition

$$\frac{1}{N} \text{Tr} X_{\mu} X_{\nu} = \delta_{\mu,\nu},$$

(2.12)

and the commutation relation

$$[X_{\mu}, X_{\nu}] = i \sum_{\lambda} f_{\mu\nu\lambda} X_{\lambda},$$

(2.13)

where $f_{\mu\nu\lambda}$ represents the structure constant. $f_{\mu\nu\lambda}$ is real and totally antisymmetric under permutation of indices. The simplest example is when $N = 2$. Then, the basis operators are given by the unit operator and the three Pauli operators, for example.

By using the basis-operator representation, we find that equation (2.1) is written as

$$\dot{b}_{\mu}(t) = \sum_{\nu,\lambda} f_{\mu\nu\lambda} h_{\nu}(t) b_{\lambda}(t).$$

(2.14)

To obtain the dynamical invariant, we solve this set of equations for a given set of $\{h_{\mu}(t)\}$. Although these equations are linear ones, it is generally a difficult task even when the dimension of the Hilbert space is not so large.

In the inverse engineering, as the name suggests, we obtain $\{h_{\mu}(t)\}$ for a given set of $\{b_{\mu}(t)\}$. Then, equation (2.14) is interpreted as a simple algebraic relation. There is no need to solve
differential equations. We can introduce a vector representation

\[ A[b(t)] \begin{pmatrix} h_1(t) \\ h_2(t) \\ \vdots \\ b_1(t) \\ b_2(t) \\ \vdots \end{pmatrix} = \begin{pmatrix} \dot{h}_1(t) \\ \dot{h}_2(t) \\ \vdots \\ \dot{b}_1(t) \\ \dot{b}_2(t) \\ \vdots \end{pmatrix}, \]

(2.15)

where \( A \) represents an antisymmetric real matrix. Each component of \( A \) is a linear combination of \( b_\mu(t) \). Since \( A \) is not invertible, we must be careful in handling this matrix equation [14]. Here, we do not give general discussions on the formal solution of equation (2.15).

In principle, the solution \([h_\mu(t)]\) in equation (2.14) can be obtained for various choices of \([b_\mu(t)]\). The original aim of shortcuts to adiabaticity is to prevent non-adiabatic transitions in systems under control. Then, the problem is to find an appropriate choice of \([b_\mu(t)]\) that meets the purposes of system control.

The solution of the original Schrödinger equation is represented by the adiabatic state of the dynamical invariant. Since the dynamical invariant is not an observable quantity, this property is not a convenient one. When we consider a time evolution from \( t = 0 \) to \( t = t_f \), we set the condition that the dynamical invariant and the Hamiltonian commute with each other at \( t = 0 \) and \( t = t_f \):

\[ [H(0), I(0)] = [H(t_f), I(t_f)] = 0. \]

(2.16)

Then, it becomes possible to find a time evolution from an eigenstate of \( H(0) \) to that of \( H(t_f) \). These conditions can be written as

\[ \sum_{\nu, \lambda} f_{\mu \nu \lambda} h_\nu(0)b_\lambda(0) = \sum_{\nu, \lambda} f_{\mu \nu \lambda} h_\nu(t_f)b_\lambda(t_f) = 0. \]

(2.17)

By using equation (2.14), we can also write equation (2.17) as

\[ \dot{b}_\mu(0) = \dot{b}_\mu(t_f) = 0, \]

(2.18)

which means that we start and finish the time evolution of \( b_\mu(t) \) slowly. In the inverse engineering, we determine \([h_\mu(t)]\) for a given \([b_\mu(t)]\). When we choose \([b_\mu(t)]\), we must be careful for the boundary conditions of \([b_\mu(t)]\) at \( t = 0 \) and \( t = t_f \) so that the Hamiltonian at those times can take proper forms. We treat several examples in the next section.

We summarize the invariant-based inverse engineering as follows. First, we find an operator form of \( I(t) \) satisfying equation (2.1) for a given operator form of \( H(t) \). The time-dependent coefficients of \( I(t) \) and \( H(t) \) are related with each other. Second, we choose a specific form of the coefficients of \( I(t) \). Third, the coefficients of \( H(t) \) are determined from equation (2.14). The coefficients are chosen so that the boundary conditions in equation (2.16) are satisfied.

In practical applications, the most important point in the first step is that \( H(t) \) and \( I(t) \) are expanded in terms of a small number of operators. The equation can always be solved if we use all kinds of operators \( \{X_\mu\}_{\mu = 1,2,\ldots,N^2} \) defined in the \( N \)-dimensional Hilbert space. However, this formal property does not any help for our aim to implement the protocol. The exact compact solution is known for limited cases as we discuss in the following sections. In the second step, we have many possible choices of the coefficients of \( I(t) \). They are determined so that \( H(t) \) obtained in the third step has a physically feasible form. Furthermore, they are required to satisfy the boundary conditions at \( t = 0 \) and \( t = t_f \). These constraints restrict possible forms of the coefficients significantly.

The advantage of this method is that the original form of the Hamiltonian is unchanged. We do not need to introduce additional operators to the Hamiltonian in contrast with the counterdiabatic driving. Furthermore, once if we can find a set of operators satisfying equation (2.1), the procedure becomes simple. We do not need to solve difficult problems such as eigenvalue problems and differential equations. On the other hand, finding a possible operator form of the dynamical invariant becomes a formidable task except several known examples. We also find a difficulty when the Hamiltonian is restricted to a specific form. In that case, \([b_\mu(t)]\) required from the condition of \([h_\mu(t)]\) takes an infeasible form.
3. Examples

(a) Two-level system

As the simplest application, we treat the case where the dimension of the Hilbert space is equal to two [15]. This example is used to drive a single spin-1/2 particle. The magnetic field is applied to control the spin state.

In this case, the standard basis operators are given by the Pauli operators \( \sigma = (\sigma_x, \sigma_y, \sigma_z) \). The Hamiltonian of the system is generally written as

\[
H(t) = \frac{\hbar}{2} h(t) n(t) \cdot \sigma ,
\]

where \( h(t) \) is non-negative and \( n(t) \) is a unit vector. The dynamical invariant is also written by using a unit vector as

\[
I(t) = e(t) \cdot \sigma .
\]

The eigenvalues of \( I(t) \) are \( \pm 1 \) since \( I^2(t) = 1 \). We note that the dynamical invariant generally has the ambiguity of a multiplicative constant. A part proportional to the unit operator is an irrelevant constant and can be dropped out without losing generality. Calculating the commutation relations, we obtain from equation (2.14)

\[
\dot{e}(t) = h(t) n(t) \times e(t).
\]

As a simple example, we parametrize \( e(t) \) as

\[
e(t) = \begin{pmatrix} \sin \theta(t) \\ 0 \\ \cos \theta(t) \end{pmatrix}.
\]

The time dependence of \( \theta(t) \) is determined below. Solving equation (3.3) with respect to \( n(t) \), we obtain

\[
n(t) = \begin{pmatrix} \sqrt{1 - \left( \frac{\dot{\theta}(t)}{h(t)} \right)^2} \sin \theta(t) \\ \frac{\dot{\theta}(t)}{h(t)} \sin \theta(t) \\ \sqrt{1 - \left( \frac{\dot{\theta}(t)}{h(t)} \right)^2} \cos \theta(t) \end{pmatrix}.
\]

We see that the difference between \( e \) and \( n \) represents non-adiabatic effects. When the magnitude of \( \dot{\theta}(t)/h(t) \) takes a small value, \( n \) is close to \( e \), which is consistent with the property that the adiabaticity condition is determined by \( |\dot{\theta}(t)/h(t)| \ll 1 \).

We examine the boundary conditions

\[
\dot{\theta}(0) = \dot{\theta}(t_f) = 0.
\]

There are many possible choices of \( \theta(t) \) satisfying these boundary conditions. The simplest choice is a polynomial function

\[
\theta(t) = \theta(0) + (\theta(t_f) - \theta(0)) \left[ 3 \left( \frac{t}{t_f} \right)^2 - 2 \left( \frac{t}{t_f} \right)^3 \right].
\]

It is required that the resulting \( n(t) \) takes a physically feasible form. In the present example, we see that the condition \( |\dot{\theta}(t)|/h(t) \leq 1 \) is required. It gives the relation

\[
h(t_f) t_f \geq 6 |\dot{\theta}(t_f) - \theta(0)| \left[ \frac{t_f}{t_f} - \left( \frac{t_f}{t_f} \right)^2 \right].
\]

We need to take a large value of \( h(t_f) t_f \) so that this relation holds for any \( t \) with \( 0 \leq t \leq t_f \).
Figure 1. Inverse engineering for a two-level system. (a) A trajectory of $e(t)$ for the dynamical invariant is denoted by the dashed (black) curve. The corresponding trajectories of $n(t)$ are denoted by solid curves. We take $h = 2h_0$ (blue) and $h = h_0$ (red) where $h_0 = 3\pi/4$ represents the threshold value determined from equation (3.8). We note that all the trajectories are on the unit sphere. (b) The direction of the magnetic field is fixed to the $y$-direction as $n(t) = (0,1,0)$. We plot the magnetic field $h(t)$ to be applied when $e(t)$ takes a trajectory presented in (a). (Online version in colour.)

We show a trajectory of $e(t)$ and the corresponding $n(t)$ in figure 1a. We consider the case
\[ \theta(0) = 0, \quad \theta(t_f) = \frac{\pi}{2}, \]  
and set $h(t)$ to a time-independent value. We note that $e(t)$ represents the Bloch vector and $n(t)$ is the direction of the magnetic field to be applied. We find a singular behaviour of $n(t)$ at the point where the equality holds in equation (3.8). The adiabaticity condition $h(t) t_f \gg 1$ is required to obtain a smooth trajectory close to $e(t)$.

It is also possible to keep the vector $n(t)$ in the $y$-direction. We choose the magnitude of the magnetic field as
\[ h(t) = \|\dot{\theta}(t)\| = \frac{6|\theta(t_f) - \theta(0)|}{t_f} \left[ \frac{t}{t_f} - \left( \frac{t}{t_f} \right)^2 \right]. \]  
This is plotted in figure 1b. Then, we find from equation (3.5) that $n(t)$ is independent of $\theta(t)$. The magnetic field is applied to the axis perpendicular to the plane where the Bloch vector lies. This protocol is easily understood without using the present technique.

When the Hamiltonian takes a restricted form, finding $b_\mu(t)$, $e(t)$ in the present example, with the required boundary conditions becomes a cumbersome task. For example, suppose one of the components of $n(t)$ is set to zero:
\[ n(t) = \begin{pmatrix} \sin \Theta(t) \\ 0 \\ \cos \Theta(t) \end{pmatrix}. \]  
Then, parametrizing $e(t)$ as
\[ e(t) = \begin{pmatrix} \sin \theta(t) \cos \phi(t) \\ \sin \theta(t) \sin \phi(t) \\ \cos \theta(t) \end{pmatrix}, \]  
we obtain the relation between $h(t)n(t)$ and $e(t)$
\[ h(t) \begin{pmatrix} \cos \Theta(t) \\ \sin \Theta(t) \end{pmatrix} = \begin{pmatrix} -\frac{\dot{\theta}(t)}{\tan \theta(t) \tan \phi(t)} + \dot{\phi}(t) \\ -\frac{\dot{\theta}(t)}{\sin \phi(t)} \end{pmatrix}. \]
Since each component of the right-hand side goes to infinity at $\theta \to 0$ and $\varphi \to 0$, a careful choice is required for $e(t)$.

The present method can be used for the case of mixed states [16]. The result for the two-state systems was also used to describe spin state in a quantum dot [17] and many-spin systems with mean-field interactions [18,19]. Furthermore, it can be applied to a generating function of full counting statistics in a classical stochastic system [20]. A similar analysis is possible when the dimension of the Hilbert space is not so large. We can find some applications to three-level systems [21] and four-level/two-qubit systems [12,22].

(b) Harmonic oscillator

In the general discussion and the example of two-level systems, we treated the case where the dimension of the Hilbert space is finite. It is possible to apply the same idea to systems with infinite-dimensional Hilbert space. We next consider a harmonic oscillator whose angular frequency changes as a function of time. This system was first discussed in [9] to solve the Schrödinger equation with the time-dependent Hamiltonian. The result was used to implement the inverse engineering in [10].

We consider the one-dimensional Hamiltonian

$$H(t) = \frac{1}{2m} p^2 + \frac{m}{2} \omega^2(t) x^2, \quad (3.14)$$

with the position and momentum operators, $x$ and $p$. The particle mass $m$ represents a positive constant and the angular frequency $\omega(t)$ is a time-dependent function. Then, it was found in [9] that the following form of $I(t)$ satisfies equation (2.1):

$$I(t) = \frac{1}{2m} (b(t)p - mb(t)x)^2 + \frac{m\omega^2(0)}{2} \left( \frac{x}{b(t)} \right)^2, \quad (3.15)$$

provided that $b(t)$ obeys the Ermakov equation

$$\ddot{b}(t) + \omega^2(t)b(t) = \frac{\omega^2(0)}{b^3(t)}. \quad (3.16)$$

For a given $b(t)$, $\omega(t)$ is determined from the relation

$$\omega^2(t) = \frac{1}{b(t)} \left( -\ddot{b}(t) + \frac{\omega^2(0)}{b^3(t)} \right). \quad (3.17)$$

The boundary conditions are given by $\dot{I}(0) = I(t_f) = 0$, which give

$$\dot{b}(0) = \ddot{b}(t_f) = 0. \quad (3.18)$$

The simplest polynomial function is

$$b(t) = 1 + \left( \sqrt{\frac{\omega(0)}{\omega(t)}} - 1 \right) \left[ 10 \left( \frac{t}{t_f} \right)^3 - 15 \left( \frac{t}{t_f} \right)^4 + 6 \left( \frac{t}{t_f} \right)^5 \right]. \quad (3.19)$$

In figure 2a, we plot trajectories of $b(t)$ for several values of $\omega(t_f)/\omega(0)$. The resulting $\omega(t)$ depends not only on $b(t)$ but also on $\omega(0)t_f$. We plot $\omega^2(t)$ in figure 2(b–d). We see that $\omega^2(t)$ strongly depends on the value of $\omega(0)t_f$. For a large $\omega(0)t_f$, the adiabaticity condition is satisfied and the corresponding result of $\omega(t)$ has a smooth trajectory. In the opposite limit of small $\omega(0)t_f$, we find that $\omega^2(t)$ shows a rapid change and goes negative in some cases. These properties are basically the same as the previous example of two-level systems. Generally speaking, the inverse engineering becomes problematic when the adiabaticity condition is not satisfied.

The described procedure above is enough to find protocols to be implemented. As a supplementary calculation, we demonstrate the diagonalization of the dynamical invariant in the
present example. Finding the explicit forms of the eigenstates is instructive since we can discuss
how the quantum state changes as a function of $t$. We introduce an operator

$$a(t) = \sqrt{\frac{\hbar m\omega_0}{2}} \frac{x}{b(t)} + \frac{i}{\sqrt{2\hbar m\omega_0}}(b(t)p - mb(t)x).$$

It satisfies the commutation relation $[a(t), a^+(t)] = 1$ and the dynamical invariant is written as

$$I(t) = \left(a^+(t)a(t) + \frac{1}{2}\right)\hbar\omega_0.$$ 

Thus, the dynamical invariant can easily be diagonalized by the standard procedure of harmonic
oscillators.

When we start the time evolution from the ground state of $H(0)$, the wave function, the
solution of the Schrödinger equation, is obtained from $a(t)|\psi(t)\rangle = 0$ up to a phase. It is written
in a coordinate representation as

$$\langle x|\psi(t)\rangle = e^{i\omega(t)} \left(\frac{m\omega(0)}{\pi\hbar b^2(t)}\right)^{1/4} \exp \left[-\frac{m\omega(0)}{2\hbar} \left(1 - i \frac{b(t)b(t)}{\omega(0)} \right) \left(\frac{x}{b(t)}\right)^2\right].$$

In the adiabatic approximation, the wave function is represented by a real Gaussian form except
the phase $e^{i\omega(t)}$. We see that the non-adiabatic effect in this case is represented by the imaginary
part in the second exponential function. It gives an oscillating behaviour of the wave function.

**Figure 2.** Inverse engineering for a harmonic oscillator system. In (a), we plot $b(t)$ in equation (3.19) for several values of
$\omega(t_f)/\omega(0)$. The corresponding results of $\omega^2(t)$ from equation (3.17) are plotted in (b–d). We set $\omega(0)t_f = 2$ in (b), $\omega(0)t_f = 1$
in (c) and $\omega(0)t_f = 0.5$ in (d). (Online version in colour.)
At $t = 0$ and $t = t_f$, the imaginary part vanishes and the wave function coincides with that by the adiabatic approximation.

We note that equation (3.15) is not the only possible form of the dynamical invariant. For example, the following linear form satisfies equation (2.1):

$$I(t) = b(t)p - mb(t)x, \quad (3.23)$$

provided that $b(t)$ satisfies the equation for classical harmonic oscillators

$$\ddot{b}(t) = -\omega^2(t)b(t). \quad (3.24)$$

This linear invariant was discussed in [23] and we can find some application to quantum field theory [24]. This solution restricts possible protocols in the inverse engineering because the boundary conditions $b(0) = b(t_f) = 0$ and $\dot{b}(0) = \dot{b}(t_f) = 0$ give $\omega(0) = \omega(t_f) = 0$. This linear invariant was used for momentum or position scaling [25] and for coupled/multi-dimensional harmonic oscillators [26–28].

The harmonic oscillator Hamiltonian only involves quadratic operators and we can construct a dynamical invariant which is a homogeneous polynomial of $x$ and $p$. This property is due to commutation relations

$$[(k\text{-th order polynomial of } x \text{ and } p), (\text{quadratic polynomial of } x \text{ and } p)] = (k\text{-th order polynomial of } x \text{ and } p). \quad (3.25)$$

We can construct a closed algebra within a limited space of operators. We note a general property of the dynamical invariant that the product of dynamical invariants also represents a dynamical invariant. It is not evident whether we can find higher-order invariants that cannot be factorized.

As a non-trivial generalization, it is known that the following set of operators satisfies equation (2.1):

$$H(t) = \frac{1}{2m}p^2 - F(t)x + \frac{m}{2}\omega^2(t)x^2 + \frac{1}{b^2(t)}U\left(\frac{x - x_c(t)}{b(t)}\right) \quad (3.26)$$

and

$$I(t) = \frac{1}{2m}[b(t)(p - mx_c(t)) - mb(t)(x - x_c(t))]^2 + \frac{m\omega^2(0)}{2}\left(\frac{x - x_c(t)}{b(t)}\right)^2 + U\left(\frac{x - x_c(t)}{b(t)}\right), \quad (3.27)$$

where $F(t)$ and $U(x)$ represent arbitrary functions. Here, $b(t)$ satisfies the Ermakov equation (3.16) and $x_c(t)$ satisfies the equation for forced oscillators:

$$\ddot{x}_c(t) + \omega^2(t)x_c(t) = \frac{1}{m}F(t). \quad (3.28)$$

This type of the Hamiltonian was discussed in [23] and was used for an inverse engineering in [29]. The potential $U((x - x_c(t))/b(t))/b^2(t)$ has a scale-invariant form and is also known as an example that the explicit form of the counterdiabatic term is available [30,31].

### 4. Dynamical invariant and counterdiabatic driving

The dynamical invariant is introduced as an auxiliary object to treat the solution of the Schrödinger equation by an eigenvalue problem. If we can find a pair of operators $I(t)$ and $H(t)$ satisfying equation (2.1), we can use it to construct a counterdiabatic driving.

The counterdiabatic driving is formulated by introducing an additional counterdiabatic term $H_1(t)$ for a given original Hamiltonian $H_0(t)$ [5–8]. The total Hamiltonian is given by

$$\hat{H}(t) = H_0(t) + H_1(t). \quad (4.1)$$

When the original Hamiltonian is written by the spectral representation

$$H_0(t) = \sum_n E_n(t)|n(t)\rangle\langle n(t)|, \quad (4.2)$$
the counterdiabatic term is written as
\[ H_1(t) = i\hbar \sum_{m,n(m \neq n)} |m(t)\rangle\langle m(t)|\dot{n}(t)\rangle\langle n(t)|. \] (4.3)

As a special case, when the eigenvalues \( E_1(t), E_2(t), \ldots \) are independent of \( t \), we can identify \( H_0(t) \) as a dynamical invariant and \( H_1(t) \) as the corresponding Hamiltonian:
\[ H_0(t) = \epsilon I(t) \] (4.4)
and
\[ H_1(t) = H(t). \] (4.5)

Since the dimension of the dynamical invariant is arbitrary, we introduce a constant \( \epsilon \) such that the dimension of \( \epsilon I(t) \) coincides with the dimension of energy. When the eigenvalues of \( H_0(t) \) are time-dependent, \( H_0(t) \) and \( H_1(t) \) satisfy the relation
\[ [H_0(t), i\hbar \partial_t H_0(t)] - [H_1(t), H_0(t)] = 0. \] (4.6)
Thus, the dynamical invariant is interpreted as the special case where the second entry of the commutator vanishes. Equation (4.6) is recognized as a method for obtaining approximate counterdiabatic terms [32,33]. For a given \( H_0(t) \), we seek \( H_1(t) \) such that the norm of the left-hand side takes a minimum value.

5. Another views of dynamical invariant

As we mentioned before, the equation for the dynamical invariant (2.1) takes a familiar form. For example, the Liouville–von Neumann equation takes the same form as equation (2.1). Then, the density operator represents a dynamical invariant. This property shows that the dynamical invariant is not a special quantity but is ubiquitous in any quantum systems.

In this section, we present several problems that use an equivalent object to the dynamical invariant. We expect that those examples offer another views on the method of shortcuts to adiabaticity.

(a) Lax pair

A relation between quantum shortcuts to adiabaticity and classical nonlinear integrable systems was pointed out in [34]. In classical nonlinear integrable systems, we treat non-trivial nonlinear equations. The integrability denotes that the system has an infinite number of conserved quantities. There are highly sophisticated techniques on such systems. The Lax formalism was used to describe the integrable systems in a unified way [35].

A set of two operators \((L(t), M(t))\) is called a Lax pair when it satisfies
\[ \frac{\partial L(t)}{\partial t} = [M(t), L(t)]. \] (5.1)
We see from the comparison to equation (2.1) that \( L \) is equivalent to the dynamical invariant. The existence of the Lax pair represents the integrability of the corresponding classical nonlinear system. We can define \( \psi \) satisfying
\[ L\psi = \lambda \psi \] (5.2)
and
\[ \frac{\partial}{\partial t} \psi = M\psi. \] (5.3)

\( \psi \) is used to construct the solution of the corresponding nonlinear equation by the inverse scattering method. The existence of the dynamical invariant implies an infinite numbers of conserved quantities represented by the time-independent eigenvalues of \( L, \lambda \). Instead of giving general discussions, we here introduce several examples that are relevant to the present problems.
The most familiar example is given by the following form of the Lax pair:

\[ L = -\frac{\partial^2}{\partial x^2} + u(x,t) \]  

(5.4)

and

\[ M = -4\frac{\partial^3}{\partial x^3} + 3\frac{\partial}{\partial x} u(x,t) + 3u(x,t) \frac{\partial}{\partial x}. \]  

(5.5)

Here, \( u(x,t) \) is real and satisfies the Korteweg–de Vries (KdV) equation \[36\]

\[ \frac{\partial u(x,t)}{\partial t} = 6u(x,t) \frac{\partial u(x,t)}{\partial x} - \frac{\partial^3 u(x,t)}{\partial x^3}. \]  

(5.6)

This equation is known to have multi-soliton solutions. The simplest solution is the single soliton

\[ u(x,t) = -\frac{2\kappa^2}{\cosh^2(\kappa x - 4\kappa^3 t)}, \]  

(5.7)

where \( \kappa \) is a positive constant. This form of the Lax pair is practically useful since \( L \) is interpreted as a one-dimensional Hamiltonian with a moving soliton potential. To prevent the non-adiabatic transitions, we need to introduce the counterdiabatic term obtained from the form of \( M \). It involves a cubic term in momentum operator and is difficult to implement. However, we can discuss a deformation of the counterdiabatic term to a simple implementable form \[34\].

(b) Quantum brachistochrone equation

Various optimal control of systems may be obtained in shortcuts to adiabaticity. The word ‘optimal’ is somewhat ambiguous and its meaning strongly depends on the problems to be solved. We can consider optimizations in many different ways. The method of quantum brachistochrone is one of the optimization methods and can be a prominent method by its
generality [39]. For a quantum trajectory in a Hilbert space, we define an action to determine the optimal Hamiltonian and the corresponding state by a variational principle. The main part of the action is determined from the geometric structure of quantum states. It is well known that the overlap between quantum states is characterized by the Fubini–Study metric.

To solve practical optimization problems, we introduce constraints for the Hamiltonian to be obtained. Then, the minimization condition of the action gives a form in equation (2.1). For example, when we represent the constraints as

$$\text{Tr} \ H(t) X_\mu = h_\mu(t),$$

(5.12)

by using basis operators $$X_\mu$$ with $$\mu = 1, 2, \ldots, M$$, the corresponding dynamical invariant takes a form

$$I(t) = \sum_{\mu=1}^{M} \lambda_\mu(t) X_\mu.$$  

(5.13)

Here, $$\{\lambda_n(t)\}$$ is obtained by solving the quantum brachistochrone equation. It is not surprising that the system is characterized by a dynamical invariant. The point here is that the form of the dynamical invariant is determined from the constraints.

The use of the action integral allows us to study stabilities of the counterdiabatic driving [13]. The optimized solution is obtained by using the variation of the action up to the first order. The stability can be studied by expanding the action up to the second order.

(c) Flow equation

As a final application of the dynamical invariant, we point out that the flow equation takes the same form as the equation for the dynamical invariant. The flow equation is a method of diagonalizing a matrix by iterations [40–43]. For a given Hermitian matrix $$H$$, we consider a time evolution described by

$$\frac{\partial H(t)}{\partial t} = [\eta(t), H(t)],$$

(5.14)

with $$H(0) = H$$. $$\eta(t)$$ represents a generator of the time evolution. One of possible choices is given by Wegner [42]

$$\eta(t) = [H_{\text{diag}}(t), H(t)],$$

(5.15)

where $$H_{\text{diag}}(t)$$ is obtained by setting off-diagonal components of $$H(t)$$ to zero. That is, we have for a given set of base kets $$\{|n\rangle\}$$

$$\langle m | H_{\text{diag}}(t) | n \rangle = \delta_{m,n} \langle m | H(t) | n \rangle.$$  

(5.16)

Then, we can show

$$\frac{\partial}{\partial t} \sum_{m,n (m \neq n)} |\langle m | H(t) | n \rangle|^2 = -2 \sum_{m,n} (\epsilon_n(t) - \epsilon_m(t))^2 |\langle m | H(t) | n \rangle|^2 \leq 0,$$

(5.17)

where $$\epsilon_n(t) = \langle n | H(t) | n \rangle$$. This relation shows that the magnitude of each off-diagonal component of $$H(t)$$ gradually decreases as a function of $$t$$. Then, we expect

$$\lim_{t \rightarrow \infty} H(t) = \lim_{t \rightarrow \infty} H_{\text{diag}}(t).$$

(5.18)

Since equation (5.14) is equivalent to equation (2.1), the eigenvalues of $$H(t)$$ are independent of $$t$$, which means that the diagonal components at $$t \rightarrow \infty$$ represent the eigenvalues of the original matrix $$H$$.

From the aspect of shortcuts to adiabaticity, the generator $$i\eta(t)$$ is interpreted as a counterdiabatic term for $$H(t)$$. We specify the form of the counterdiabatic term instead of specifying the time dependence of the matrix $$H(t)$$. Then, the resulting dynamics is interpreted as a diagonalization process of the original matrix $$H$. 
6. Summary

We have presented a brief introduction to the dynamical invariant formalism of shortcuts to adiabaticity. After some of fundamental properties are summarized, we discussed the method of inverse engineering together with several simple examples. We also discussed the relation to the counterdiabatic driving and several different aspects of the dynamical invariant.

The most important property of the dynamical invariant is that we can understand the dynamical system in the same way as the static systems. If we can find the dynamical invariant operator, the problem is reduced to solving an eigenvalue equation. For our purposes of quantum control, it is not necessary to solve the eigenvalue problem. Owing to many possible choices of the coefficients of the dynamical invariant, the resulting protocol is not unique and is obtained in an adapted manner.

Although experimental implementations can be broadly covered by the examples of a two-level system and a harmonic oscillator, it is an interesting challenging problem to find a dynamical invariant for other systems. We expect that we can find a non-trivial use of the dynamical invariant by combining ideas from different fields using a similar quantity to the dynamical invariant.

Data accessibility. This article has no additional data.
Conflict of interest declaration. I declare I have no competing interests.
Funding. The author was supported by JSPS KAKENHI grant nos. JP20K03781 and JP20H01827.
Acknowledgements. The author is grateful to Gonzalo Muga and Mikio Nakahara for useful comments.

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