In 2004, H. Fan. and S. Ghosh et al found that any $N > d$ maximally entangled states cannot be distinguished by local operations and classical communication (LOCC) in $\mathbb{C}^d \otimes \mathbb{C}^d$. An interesting problem arise: whether there is any $d$ mutually orthogonal maximally entangled states which can not be distinguished by LOCC in $\mathbb{C}^d \otimes \mathbb{C}^d$. In 2012, N. Yu et al gave the first such example when $d = 4$. More recently, A. Cosentino generalized it to the cases $d = 2^n$. In this letter, we show that there exists $d$ mutually orthogonal maximally entangled states which cannot be distinguished by PPT measurement in $\mathbb{C}^d \otimes \mathbb{C}^d$ for any $d \geq 4$.

PACS numbers: 03.67.-a

Introduction  Quantum entanglement plays an important role in quantum computation and quantum information. A fundamental question is to consider the distinguishability of entangled states. In the bipartite case, Alice and Bob share a quantum system which is chosen from one of a known set of mutually orthogonal quantum states. Their goal is to determine the state using only local operations and classical communication (LOCC). Walgate et al. showed that any two orthogonal pure states can be distinguished by LOCC. In this letter, we further use the semidefinite program to construct a finite program having a feasible solution of special form. We apply the theorem to construct $d$-states set satisfying this condition. We give a construction of $d$ PPT-indistinguishable maximally entangled states in $\mathbb{C}^d \otimes \mathbb{C}^d$ for any $d \geq 4$. This gives an answer to the conjecture proposed by S. Bandyopadhyay and N. Yu et al. The author gave an example when $d = 4$. In this letter, we are mainly dealing with $d$ orthogonal maximally entangled states $\{|\psi_i\rangle\}_{i=1}^d$ in $\mathbb{C}^d \otimes \mathbb{C}^d$. We can suppose $|\psi_i\rangle = (I \otimes U_i)|\psi_1\rangle$, where $|\psi_1\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |i\rangle$, and $U_i$ are unitary matrices. Then the orthogonal conditions $\langle \psi_i | \psi_j \rangle = \delta_{ij}$ are equivalent with $Tr(U_i U_j^\dagger) = d \delta_{ij}$. Because there is an one-one correspondence between the maximally entangled state and the unitary matrix, we can use the unitary matrices $U_i$ to be the defining unitary matrices of the maximally entangled states $\{|\psi_i\rangle\}_{i=1}^d$. In this paper we use $\rho_i$ to denote the density matrix of the pure state $|\psi_i\rangle$, that is $\rho_i = |\psi_i\rangle \langle \psi_i|$. We call a set of states $\{|\psi_i\rangle\}_{i=1}^d$ is PPT-distinguishable if there exists a PPT measurement $\{P_i\}_{i=1}^d$ such that $\frac{1}{d} \sum_{i=1}^d \langle P_i, \rho_i \rangle = 1$. Otherwise, we call the set $\{|\psi_i\rangle\}_{i=1}^d$ is PPT-indistinguishable. Our task is to find the maximal probability of success of distinguishing the set $\{|\psi_i\rangle\}_{i=1}^d$ with a PPT measurement. According to the paper [10], this is equivalent with the following semidefinite program.
As it has been pointed out in [10] that its dual problem is as follows:

Dual program

\[
\text{minimize: } \frac{1}{d} \text{Tr}(\gamma) \\
\text{subject to: } \gamma - \rho_j \geq T_A(Q_j), \quad j = 1, \ldots, d, \\
\gamma \in \text{Herm}(A \otimes B), \\
Q_1, \ldots, Q_d \in \text{Pos}(A \otimes B).
\]

Theorem 1. For any \( d \geq 4 \), there exists \( d \) PPT-indistinguishable mutually orthogonal maximally entangled states in \( \mathbb{C}^d \otimes \mathbb{C}^d \).

Lemma 1. Suppose \( |\psi_i\rangle = (I \otimes U_i)|\psi_i\rangle \), \( \rho_i = |\psi_i\rangle \langle \psi_i| \), then \( T_A(\rho_i) = (I \otimes U_i)T_A(\rho_i)(I \otimes U_i) \).

Proof:

\[
T_A(\rho_i) = T_A((I \otimes U_i)|\psi_i\rangle \langle \psi_i|(I \otimes U_i)) \\
= \frac{1}{d} T_A(\sum_{mn}(I \otimes U_i)|m\rangle \langle n|(I \otimes U_i)) \\
= \frac{1}{d} T_A(\sum_{mn}(|m\rangle \langle n|)(U_i|m\rangle \langle n|U_i)) \\
= \frac{1}{d} \sum_{mn}(|m\rangle \langle m|(I \otimes U_i)|n\rangle \langle n|U_i) \\
= \frac{1}{d} (I \otimes U_i) \sum_{mn}(|m\rangle \langle n|)(I \otimes U_i) \\
= (I \otimes U_i)T_A(\rho_i)(I \otimes U_i).
\]

After an easy calculation, we can find that the eigenvalues of \( T_A(\rho_i) \) are \( \frac{1}{d} \) or \( -\frac{1}{d} \). Moreover, we can present its eigen subspaces concisely.

Lemma 2. \( V_{-\frac{1}{d}}(T_A(\rho_1)) = \text{span}\{|kl\} - |lk\}, 1 \leq k < l \leq d \) and \( V_{\frac{1}{d}}(T_A(\rho_1)) = \text{span}\{|kl\} + |lk\}, 1 \leq k \leq l \leq d \). (Here if \( A \) is an \( n \times n \) matrix, \( \lambda \) is an eigenvalue of \( A \), we use the notation \( V_\lambda(A) \) to denote the set of all eigenvectors of \( A \) corresponding to the eigenvalue \( \lambda \).)

We notice that the number of linear independent eigenvectors of \( T_A(\rho_1) \) corresponding to the eigenvalue \( -\frac{1}{d} \) and \( \frac{1}{d} \) are \( L = \frac{(d-1)d}{2} \) and \( M = \frac{(d+1)d}{2} \) respectively.

Theorem 2. Suppose \( |\psi_i\rangle = (I \otimes U_i)|\psi_i\rangle \), where \( |\psi_i\rangle = \frac{1}{{d\choose i}} \sum_{l=1}^{d} |li\rangle \) and \( U_i \) are unitary matrices. Then \( \bigcap_{i=1}^{d} V_{-\frac{1}{d}}(T_A(\rho_i)) \neq \{0\} \), if and only if there is a feasible solution of semidefinite program (3) satisfying

\[
\gamma \leq \frac{1}{d} I_A \otimes I_B, \quad \text{with } \gamma \neq \frac{1}{d} I_A \otimes I_B.
\]

Proof:

Suppose \( 0 \neq |v\rangle \in \bigcap_{i=1}^{d} V_{-\frac{1}{d}}(T_A(\rho_i)) \) for \( i = 1, \ldots, d \), that is, \( T_A(\rho_i)|v\rangle = -\frac{1}{d}|v\rangle \) for \( i = 1, \ldots, d \). Because \( T_A(\rho_i) \) is a Hermitian unitary matrix, by the singular value decomposition, there exists orthogonal normal vectors \( |v_1\rangle, |w_m\rangle \), s.t.

\[
T_A(\rho_i) = -\frac{1}{d}|v\rangle \langle v| - \frac{1}{d} \sum_{l=2}^{L} |v_l\rangle \langle v_l| + \frac{1}{d} \sum_{m=1}^{M} |w_m\rangle \langle w_m|,
\]

\[
I_A \otimes I_B = |v\rangle \langle v| + \sum_{l=2}^{L} |v_l\rangle \langle v_l| + \sum_{m=1}^{M} |w_m\rangle \langle w_m|.
\]

So we have

\[
\frac{1}{d} I_A \otimes I_B - \frac{2}{d} |v\rangle \langle v| - T_A(\rho_i) = \frac{2}{d} \sum_{l=2}^{L} |v_l\rangle \langle v_l| \geq 0.
\]

Clearly, \( \gamma = \frac{1}{d} I_A \otimes I_B - \frac{2}{d} |v\rangle \langle v| \in \text{Herm}(A \otimes B) \), hence \( \gamma \) is a feasible solution of semidefinite program (3) which satisfies \( \gamma \leq \frac{1}{d} I_A \otimes I_B \) and \( \gamma \neq \frac{1}{d} I_A \otimes I_B \).

Conversely, if \( \gamma \) is a feasible solution satisfies \( \gamma \leq \frac{1}{d} I_A \otimes I_B \), then we can suppose \( \frac{1}{d} I_A \otimes I_B - \gamma = \sum_{k=1}^{K} \mu_k |v_k\rangle \langle v_k| \) with \( \mu_k \geq 0 \) for all \( k = 1, \ldots, K \) and at least one strictly positive (suppose \( \mu_1 > 0 \)). Because \( \gamma \) is a feasible solution, we have the following inequalities for all \( i = 1, \ldots, d \):

\[
\frac{1}{d} I_A \otimes I_B - \sum_{k=1}^{K} \mu_k |v_k\rangle \langle v_k| \geq T_A(\rho_i).
\]

These imply

\[
\frac{1}{d} I_A \otimes I_B - T_A(\rho_i) \geq \mu_1 |v_1\rangle \langle v_1|.
\]

For each \( 1 \leq i \leq d \) \( T_A(\rho_i) \) has the following singular value decomposition

\[
T_A(\rho_i) = -\frac{1}{d} \sum_{i=1}^{L} |v_i\rangle \langle v_i| + \frac{1}{d} \sum_{m=1}^{M} |w_m\rangle \langle w_m|,
\]

where \( \{|v_i\rangle\}_{i=1}^{L} \cup \{|w_m\rangle\}_{m=1}^{M} \) form an orthogonal normal base of \( \mathbb{C}^d \otimes \mathbb{C}^d \). So we have the following identity:

\[
I_A \otimes I_B = \sum_{i=1}^{L} |v_i\rangle \langle v_i| + \sum_{m=1}^{M} |w_m\rangle \langle w_m|.
\]
By (4), (5), (6) we have

$$\sum_{l=1}^{L} |v_i^l\rangle \langle v_i^l| \geq \mu_1 |v^1\rangle \langle v^1|.$$ 

We must have $|v^1\rangle \in \text{span}\{ |v_i^l\rangle \}_{l=1}^{L}$. By the singular value decomposition equation (5), we know that span$\{ |v_i^l\rangle \}_{l=1}^{L}$ is just the set of the eigenvectors of $T_A(\rho_1)$ corresponding to the eigenvalue $-\frac{1}{2}$. Hence $|v^1\rangle$ must be an eigenvector of $T_A(\rho_1)$ corresponding to the eigenvalue $-\frac{1}{2}$. That is, $T_A(\rho_1)|v^1\rangle = -\frac{1}{2} |v^1\rangle$. Hence $|v^1\rangle \in \bigcap_{i=1}^{d} V_{-\frac{1}{2}}(T_A(\rho_1))$. This gives a complete proof of the theorem.

**Theorem 3.** $\bigcap_{i=1}^{d} V_{-\frac{1}{2}}(T_A(\rho_1)) \neq \{0\}$ if and only if $\bigcap_{i=1}^{d} (I \otimes U_i^d)V_{-\frac{1}{2}}(T_A(\rho_1)) \neq \{0\}$.

**Proof:** If $0 \neq |v\rangle \in \bigcap_{i=1}^{d} (I \otimes U_i^d)V_{-\frac{1}{2}}(T_A(\rho_1)) \neq \{0\}$, that is, $T_A(\rho_1)|v\rangle = -\frac{1}{2} |v\rangle$, $i = 1, 2, \ldots, d$. By lemma 1, we obtain $(I \otimes U_i)T_A(\rho_1)(I \otimes U_i)|v\rangle = -\frac{1}{2} |v\rangle$. By the unitarity of $U_i$, $T_A(\rho_1)(I \otimes U_i)|v\rangle = -\frac{1}{2} (I \otimes U_i)|v\rangle$. Hence $|v\rangle \in \bigcap_{i=1}^{d} (I \otimes U_i^d)V_{-\frac{1}{2}}(T_A(\rho_1))$. The converse is straightforward.

**Corollary 1.** If $\bigcap_{i=1}^{d} (I \otimes U_i^d)V_{-\frac{1}{2}}(T_A(\rho_1)) \neq \{0\}$, then the set $\{|\psi_1\rangle\}_{d}^{d}$ defined by $\{|U_i\rangle\}_{d}^{d}$ is PPT indistinguishable. Particularly, $\{|\psi_1\rangle\}_{d}^{d}$ is LOCC indistinguishable.

**Remark:** If $|v\rangle \in \bigcap_{i=1}^{d} (I \otimes U_i^d)V_{-\frac{1}{2}}(T_A(\rho_1))$, we have $|v\rangle \in V_{-\frac{1}{2}}(T_A(\rho_1))$ for all $U_1 = I_B$. So all the matrices $I \otimes U_i^d$ transfer the same $|v\rangle \in V_{-\frac{1}{2}}(T_A(\rho_1))$ to some eigenvector of $T_A(\rho_1)$ with eigenvalue $-\frac{1}{2}$. Lemma 2 has given a precise description of the set $T_A(\rho_1)$, so it is helpful for us to find the unitary matrices satisfying the above conditions.

Because we cannot find an unified set of $d$ PPT-indistinguishable states, we separate it in the following four parts: $d = 4n$, $d = 2n$, $d = 2n + 1$ and some exceptional cases $d = 5, 7, 11$.

**Case I: $d=4n$.** First, we present the four PPT indistinguishable states in $\mathbb{C}^4 \otimes \mathbb{C}^4$ which has been found by Yu [1].

The corresponding defined unitary matrices are as follows:

$$V_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, V_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

$$V_3 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, V_4 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$ 

It’s not difficult to check the following identities:

$$(I \otimes V_1^d)(|12\rangle - |34\rangle + |42\rangle - |24\rangle) = |12\rangle - |34\rangle + |43\rangle - |21\rangle,$$

$$(I \otimes V_2^d)(|12\rangle - |34\rangle + |42\rangle - |24\rangle) = |12\rangle - |34\rangle - |43\rangle - |21\rangle,$$

$$(I \otimes V_3^d)(|12\rangle - |34\rangle + |42\rangle - |24\rangle) = -|14\rangle - |32\rangle + |41\rangle + |23\rangle.$$

By lemma 2, we notice that all the vectors on the right hand side of the equalities belong to $V_{-\frac{1}{2}}(T_A(\rho_1))$. Hence we have

$$\bigcap_{i=1}^{d} (I \otimes V_i^d)V_{-\frac{1}{2}}(T_A(\rho_1)).$$

By Corollary 1, we can conclude that the above four states are PPT-indistinguishable just the same as the proof in [10].

**Case II: $d=2n$.** We present a construction which is different with the Case I. That is, even $n = 2m$ for some $m$, the $4m$ states constructed below do not coincide with the $4m$ states constructed in Case I. Set $w = e^{\frac{2\pi i}{4m}}$. We construct the $2n$ orthogonal unitary matrices as follows:

$$U_1 = diag(1, w^0, \ldots, w^{0(n-1)}, 1, w^0, \ldots, w^{0(n-1)}),$$

$$U_2 = diag(1, w^1, w^n, \ldots, w^{n(n-1)}),$$

$$\vdots$$

$$U_n = diag(1, w^{n-1}, \ldots, w^{(n-1)^2}),$$

$$U_{n+2} = diag(1, w^{n-1}, w^{n-2}, 1, w^{n-1}, \ldots, w^{n-2})U_{n+1},$$

$$\vdots$$

$$U_{2n} = diag(1, w^{(n-1)x1}, \ldots, w^{(n-1)^2}),$$

$$U_{n+1} = \begin{bmatrix} S & 0 \\ 0 & S^T \end{bmatrix},$$

where $S^T = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \end{bmatrix}$. 

Then by Corollary 1, we conclude that the $4n$ states defined by $\{|\psi_i\rangle = (I \otimes U_i)|\psi_1\rangle\}_{i=1}^{d}$ are PPT-indistinguishable.

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$$\vdots$$

$$U_n = diag(1, w^{n-1}, \ldots, w^{(n-1)^2}),$$

$$U_{n+2} = diag(1, w, w^n, \ldots, w^{n-2})U_{n+1},$$

$$\vdots$$

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$$U_{n+1} = \begin{bmatrix} S & 0 \\ 0 & S^T \end{bmatrix},$$

where $S^T = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \end{bmatrix}$. 

Then by Corollary 1, we conclude that the $4n$ states defined by $\{|\psi_i\rangle = (I \otimes U_i)|\psi_1\rangle\}_{i=1}^{d}$ are PPT-indistinguishable.
By lemma 2, we notice that the right hand side of the observation of (7) are very helpful when we consider the case $d= 4n+1$ which might be a bit difficult to construct.

**Case III: $d=2n+1$.** Now we give a construction of 2$n+1$ PPT-indistinguishable states. We need to separate the odd numbers case into two classes: Case (i) $d=4n+1$ and Case (ii) $d=4n+3$.

Case (i): we construct $U_j$ to be block unitary matrix of the form

$$
\begin{bmatrix}
V_j & 0 \\
0 & W_j
\end{bmatrix}
$$

where $V_j$ are $(2n+2) \times (2n+2)$ matrix and $W_j$ are $(2n-1) \times (2n-1)$ matrix. Suppose $V_j$ are chosen from the $(n+1)^2$ matrices define above in the case $d=2(n+1)$. And $W_j$ are chosen from the $(2n-1)^2$ generalized Pauli matrices of $(2n-1) \times (2n-1)$. If $n \geq 2$, we have $(n+1)^2 \geq 4n+1$ and $(2n-1)^2 \geq 4n+1$. So we can really construct $4n+1$ orthogonal unitary matrices $\{U_j\}$. 

Case (ii): Similarly with the above construction, we block $U_j$ into the form

$$
\begin{bmatrix}
V_j & 0 \\
0 & W_j
\end{bmatrix}
$$

where $V_j$ are $(2n+2) \times (2n+2)$ matrix and $W_j$ are $(2n+1) \times (2n+1)$ matrix. Suppose $V_j$ are chosen from the $(n+1)^2$ matrices define above in the case $d=2(n+1)$. And $W_j$ are chosen from the $(2n+1)^2$ generalized Pauli matrices of $(2n+1) \times (2n+1)$. If $n \geq 3$, we have $(n+1)^2 \geq 4n+3$ and $(2n+1)^2 \geq 4n+3$. Then we can construct $4n+3$ orthogonal unitary matrices $\{U_j\}$. 

In these two cases, the orthogonality of $\{U_j\}$ derived from the orthogonality of $\{V_j\}$ and the orthogonality of the generalized Pauli matrices $\{V_j\}$. If we let $|v\rangle = \sum_{k=1}^{n+1} |(k)\rangle |n+1+k\rangle |k\rangle$, then we can easily check that

$$
|v\rangle \in \bigcap_{i=1}^{d} (I \otimes U_i)V_{-\frac{4}{2}}(T_A(\rho_i)).
$$

So we conclude that when $n \geq 2, d=4n+1$ or $n \geq 3, d=4n+3$, the $d$ states $\{|\psi_i\rangle = (I \otimes U_i)\rangle\rangle\}_{i=1}^{2n+1}$ are PPT-indistinguishable. We notice that the exceptional case of the odd numbers are just $5, 7, 11$.

**Case IV: $d=5, 7, 11$.** First, when $d=5$, Cosentine had presented 5 PPT indistinguishable states [10].

Now suppose $d=7$, we let $U_i = \begin{bmatrix} V_i & 0 \\
0 & W_i \end{bmatrix}$ where $V_1, V_2, V_3, V_4$ are the matrices defined in **Case I** and

$$
V_5 = \begin{bmatrix} 1 & 1 \\
1 & -1 \end{bmatrix}, V_6 = \begin{bmatrix} 1 & 0 \\
0 & 1 \end{bmatrix},
$$

$W_{i+1} = diag(1, \omega^i, \omega^{2i}), W_{i+4} = diag(1, \omega^i, \omega^{2i})S$

for $i = 0, 1, 2$, where $S = \begin{bmatrix} 0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \end{bmatrix}$. If we set $|v\rangle = \frac{1}{\sqrt{2}}(|13\rangle - |31\rangle + |24\rangle - |42\rangle)$, then we have

$$
|v\rangle \in \bigcap_{i=1}^{6} (I \otimes U_i)V_{-\frac{4}{2}}(\rho_i).
$$

Now we set $\gamma = \frac{1}{\sqrt{2}}I_A \otimes I_B - \frac{1}{\sqrt{2}}|v\rangle\langle v|$, then $\gamma \geq T_A(\rho_i)$ where $|\psi_i\rangle = (I \otimes U_i)|\psi_1\rangle$ for $i = 1, 2, \cdots, 5$. But we need 7 unitary matrices, we choose $U_7$ to be the following matrix

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}.
$$

Easily, $U_7$ is orthogonal with the above six unitary matrices $\{U_i\}_{i=1}^{6}$, we define $|\psi_i\rangle = (I \otimes U_i)|\psi_1\rangle$. After some annoying calculations, we obtain $|v\rangle = \sqrt{\frac{2}{3}}|u\rangle + \sqrt{\frac{1}{3}}|w\rangle$, where $T_A(\rho_7)|u\rangle = -\frac{1}{\sqrt{2}}|u\rangle$ and $T_A(\rho_7)|w\rangle = \frac{1}{\sqrt{2}}|w\rangle$ with $\langle u|u\rangle = (|u|w\rangle) = 1$.

Unfortunately, we can’t find a feasible solution of programm (3) with the form given by theorem 2. Instead,
we find a feasible solution of program (3) with the form
$$\gamma = \frac{1}{T} \otimes I_B - \frac{\lambda}{T}|v\rangle\langle v| + \frac{\mu}{T}|w\rangle\langle w|, $$
then we must have
$$\frac{1}{T} I_A \otimes I_B - \frac{\lambda}{T}|v\rangle\langle v| + \frac{\mu}{T}|w\rangle\langle w| > T_A(\rho_T).$$ \(8\)

Suppose the singular value decomposition of \(T_A(\rho_T)\) as follows:
$$T_A(\rho_T) = \frac{1}{T}(-|u\rangle\langle u| - \sum_{l=2}^{L} |u_l\rangle\langle u_l| + |w\rangle\langle w| + \sum_{m=2}^{M} |w_m\rangle\langle w_m|),$$
then we have
$$I_A \otimes I_B = |w\rangle\langle w| + \sum_{l=2}^{L} |u_l\rangle\langle u_l| + |w\rangle\langle w| + \sum_{m=2}^{M} |w_m\rangle\langle w_m|.$$ 

After an easy calculation, we obtain that inequality (8) is equivalent with
$$1 - \frac{\frac{\lambda}{T} + \frac{\mu}{T}}{7} |u\rangle\langle u| - \frac{\lambda}{T} |w\rangle\langle w| + \frac{\mu}{T} |w\rangle\langle w| \geq 0.$$ 

This is satisfied if the below inequalities hold.
$$\begin{cases} 
1 - \frac{\lambda}{T} > 0 \\
(1 - \frac{\lambda}{T})(\mu - \frac{\lambda}{T}) - \frac{15}{64}\lambda^2 > 0
\end{cases}$$

Moreover, we want to find a feasible solution \(\gamma\) with \(\frac{i}{T} \text{Tr}(\gamma) < 1\). So we must have \(\mu < \lambda\), then we get \(\mu < \lambda < \frac{\mu}{\frac{\lambda}{T} + \frac{\mu}{T}}\).

Now choosing \(\mu = \frac{\lambda}{T}, \lambda = \frac{\lambda}{T}\) , then the corresponding \(\gamma\) satisfies \(\gamma \geq T_A(\rho_T)\). Moreover, the satisfying of \(\gamma \geq T_A(\rho_T)\) follows by \(\gamma \geq T_A(\rho_T)\) for \(i = 1, 2, \cdots, 6\). So \(\gamma\) is a feasible solution of the program with \(\frac{i}{T} \text{Tr}(\gamma) < 1\). Hence the 7 states defined above are PPT-indistinguishable.

At last, \(d = 11\), we also block \(U_j\) into \[
\begin{pmatrix}
V_j & 0 \\
0 & W_j
\end{pmatrix}
\]
where \(V_j\) are 6 \(\times\) 6 unitary matrices and \(W_j\) are 5 \(\times\) 5 unitary matrices. Suppose the 6 \(\times\) 6 unitary matrices are chosen from the following matrices:
$$V_i = (11|1\rangle + |2\rangle|2\rangle + w^i(|3\rangle|3\rangle + |4\rangle|4\rangle) + w^{2i}(|5\rangle|5\rangle + |6\rangle|6\rangle),$$
$$V_{3+i} = (13|2\rangle - |1\rangle|4\rangle) - w^i(|4\rangle|6\rangle + |5\rangle|3\rangle) + w^{2i}(|2\rangle|5\rangle - |6\rangle|1\rangle),$$
$$V_{6+i} = (11|3\rangle + |4\rangle|2\rangle + w^i(|5\rangle|1\rangle - |2\rangle|6\rangle) + w^{2i}(|3\rangle|5\rangle + |6\rangle|4\rangle),$$
$$V_{9+i} = (5|2\rangle - |1\rangle|5\rangle) - w^i(|2\rangle|4\rangle + |3\rangle|1\rangle) + w^{2i}(|4\rangle|6\rangle - |6\rangle|3\rangle),$$

where \(i = 1, 2, 3\). If we choose 12 unitary matrices \(W_j\) from generalized 5 \(\times\) 5 Pauli matrices, then the twelve unitary matrices \(\{U_j\}\) are orthogonal with each others. Moreover, if we set \(|v\rangle = (12|21\rangle) + (34|43\rangle) + (56|65\rangle),\) then we have
$$|v\rangle \in \bigcap_{j=1}^{12}(I \otimes U_j)W_{\frac{i}{T}}(T_A(\rho_1)).$$

Particularly,
$$|v\rangle \in \bigcap_{j=1}^{11}(I \otimes U_j)W_{\frac{i}{T}}(T_A(\rho_1)).$$

This implies that the 11 states defined by \(\{|\psi_i\rangle = (I \otimes U_{i+1})|\psi_1\rangle\}_{i=1}^{11}\) are PPT-indistinguishable.

By summarising the Case I,II,III,IV, we can obtain the theorem 1.

**Conclusion** In this letter, we show an explicit method to generate \(d\) mutually orthogonal maximally entangled states which are PPT-indistinguishable (hence LOCC indistinguishable) in \(\mathbb{C}^d \otimes \mathbb{C}^d\) for any \(d \geq 4\). This gives an answer to the conjecture proposed by S. Bandyopadhyay in \(9\).

**Acknowledgments** This work is supported by the NSFC 11475178 and NSFC 11275131.

[1] C.H. Bennett, D.P. DiVincenzo, C.A. Fuchs, T. Mor, E. Rains, P.W. Shor, J.A. Smolin, and W.K. Wootters. Phys. Rev. A, 59:1070-1091, (1999).
[2] J. Walgate, A. J. Short, L. Hardy, and V. Vedral. Phys. Rev. Letters 85, 4972 (2000)
[3] M. Nathanson. J. Math. Phys. 46, 062103 (2005)
[4] S. Ghosh, G. Kar, A. Roy, and D. Sarkar. Phys. Rev. A 70, 022304, (2004).
[5] H. Fan. Phys. Rev. Letters 92, 177905 (2004).
[6] S. Bandyopadhyay, S. Ghosh, and G. Kar. New J. Phys. 13 123013, (2011)
[7] N. Yu, R. Duan, and M. Ying. Phys. Rev. Letters 109, 020506 (2012).
[8] N. Yu, R. Duan, and M. Ying. IEEE Transactions on Information Theory Vol.60, No.4 (2014).
[9] M. Nathanson. Phys. Rev. A, 88, 062316, (2013).
[10] A. Cosentino. Phys. Rev. A, 87, 012321, (2013).
[11] A. Cosentino and V. Russo. Quantum Information & Computation, 14, 1098–1106, (2014).