Examples of Berezin-Toeplitz Quantization: Finite sets and Unit Interval

J.-P. Gazeau, T. Garidi, E. Huguet, M. Lachièze Rey, and J. Renaud

In memory of Bob Sharp

Abstract. We present a quantization scheme of an arbitrary measure space based on overcomplete families of states and generalizing the Klauder and the Berezin-Toeplitz approaches. This scheme could reveal itself as an efficient tool for quantizing physical systems for which more traditional methods like geometric quantization are uneasy to implement. The procedure is illustrated by (mostly two-dimensional) elementary examples in which the measure space is a $N$-element set and the unit interval. Spaces of states for the $N$-element set and the unit interval are the 2-dimensional euclidean $\mathbb{R}^2$ and hermitian $\mathbb{C}^2$ planes.

1. Quantum processing of a measure space

Quantum Physics and Signal Analysis have many aspects in common. As a departure point of their respective formalism, one finds a raw set $X = \{x\}$ of basic parameters or data. This set may be a classical phase space in the former case whereas it might be a temporal line or a time-frequency half-plane in the latter one. In reality it can be any set of data accessible to observation. The minimal significant structure one requires of it is the existence of a measure $\mu(dx)$, together with a $\sigma$-algebra of measurable subsets. As a measure space, $X$ will be given the name of an observation set in the present context, and the existence of a measure provides us with a statistical reading of the set of measurable real or complex valued functions $f(x)$ on $X$: computing for instance average values on subsets with bounded measure. Actually, both approaches deal with quadratic mean values and correlation/convolution involving signal pairs, and the natural frameworks of studies are the real (Euclidean) or complex (Hilbert) spaces, $L^2(X, \mu) \equiv L^2_{\mathbb{R}}(X, \mu)$ or $L^2_{\mathbb{C}}(X, \mu)$ of square integrable functions $f(x)$ on the observation set $X$: $\int_X |f(x)|^2 \mu(dx) < \infty$. One will speak of finite-energy signal in Signal Analysis and of quantum state in Quantum Mechanics. However, it is precisely at this stage that “quantum processing” of $X$ differs from signal processing on at least three points:

1. not all square integrable functions are eligible as quantum states,
(2) a quantum state is defined up to a nonzero factor,
(3) those ones among functions \( f(x) \) that are eligible as quantum states with unit norm, \( \int_X |f(x)|^2 \mu(dx) = 1 \), give rise to a probability interpretation: \( X \supset \Delta \to \int_\Delta |f(x)|^2 \mu(dx) \) is a probability measure interpretable in terms of localisation in the measurable \( \Delta \). This is inherent to the computing of mean values of quantum observables, (essentially) self-adjoint operators with domain included in the set of quantum states.

The first point lies at the heart of the quantization problem: what is the more or less canonical procedure allowing to select quantum states among simple signals? In other words, how to select the right (projective) Hilbert space \( \mathcal{H} \), a closed subspace of \( L^2(X, \mu) \), or equivalently the corresponding orthogonal projecteur \( I_\mathcal{H} \)?

In various circumstances, this question may be answered through the selection, among elements of \( L^2(X, \mu) \), of an orthonormal set \( \mathcal{S}_N = \{ \phi_n(x) \}_{n=1}^N \), \( N \) being finite or infinite, which spans, by definition, the separable Hilbert subspace \( \mathcal{H} \equiv \mathcal{H}_N \). Furthermore, and this is a crucial assumption [1, 2, 3], we require that

\[
N(x) \equiv \sum_n |\phi_n(x)|^2 < \infty \quad \text{almost everywhere.}
\]

Of course, if \( N \) is finite the above condition is trivially checked.

We then consider the family of states \( \{ |x \rangle \}_{x \in X} \) through the following linear superpositions:

\[
|x \rangle = \frac{1}{\sqrt{N(x)}} \sum_n \phi_n(x)|n \rangle,
\]

in which the ket \( |n \rangle \) designates the element \( \phi_n(x) \) in a “Fock” notation. This defines an injective map

\[
X \ni x \to |x \rangle \in \mathcal{H}_N \in \mathcal{H},
\]

where \( \mu(dx) = N(x) \mu(dx) \) is another measure on \( X \), absolutely continuous with respect to \( \mu(dx) \). The coherent states [1, 2] form in general an overcomplete (continuous) basis of \( \mathcal{H} \).

The resolution of the unity in \( \mathcal{H}_N \) can alternatively been understood in terms of the scalar product \( \langle x | x' \rangle \) of two states of the family. Indeed, [4, 5] implies that, to any vector \( |\phi \rangle \) in \( \mathcal{H}_N \) one can (anti-)isometrically associate the function

\[
\phi^*(x) \equiv \sqrt{N(x)} \langle x | \phi \rangle
\]

in \( L^2(X, \mu) \), and this function obeys

\[
\phi^*(x) = \int_X \sqrt{N(x)N(x')} \langle x | x' \phi^*(x') \mu(dx').
\]
Hence, $\mathcal{H}_N$ is (anti-) isometric to a reproducing Hilbert space with kernel
\begin{equation}
K(x, x') = \sqrt{N(x)N(x')} \langle x | x' \rangle,
\end{equation}
and the latter assumes finite diagonal values (a.e.), $K(x, x) = N(x)$, by construction.

A classical observable is a function $f(x)$ on $X$ having specific properties in relationship with some supplementary structure allocated to $X$, topology, geometry or something else. Its quantization $\mathcal{A}_f$ simply consists in associating to $f(x)$ the operator
\begin{equation}
A_f := \int_X f(x) \langle x | \nu(dx).
\end{equation}
In this context, $f(x)$ is said upper (or contravariant) symbol of the operator $A_f$ and denoted by $f = \hat{A}_f$, whereas the mean value $\langle x | f(x) | x \rangle$ is said lower (or covariant) symbol of $A_f$ and denoted by $\tilde{A}_f$. Through this approach, one can say that a quantization of the observation set is in one-to-one correspondence with the choice of a frame in the sense of (1.4) and (1.5). To a certain extent, a quantization scheme consists in adopting a certain point of view in dealing with $X$. This frame can be discrete, continuous, depending on the topology furthermore allocated to the set $X$, and it can be overcomplete, of course. The validity of a precise frame choice is asserted by comparing spectral characteristics of quantum observables $A_f$ with data issued from a predefined experimental protocol. Of course, operators acting in $\mathcal{H}_N$ are not all of them of the “diagonal” type $A_f$, and many different classical $f(x)$’s can give rise to the same operator $A_f$. The frame should be complete or rich enough in order to meet all experimental possibilities determined by the protocol.

Let us illustrate the above construction with the well-known Klauder-Glauber- Sudarshan coherent states $\mathcal{A}_f$ and the subsequent so-called canonical quantization. The observation set $X$ is the classical phase space $\mathbb{R}^2 \simeq \mathbb{C} = \{ x \equiv z = \frac{1}{\sqrt{2}} (q + ip) \}$ (in complex notations) of a particle with one degree of freedom. The measure on $X$ is Gaussian, $\mu(dx) = \frac{1}{\pi} e^{-|z|^2} d^2z$ where $d^2z$ is the Lebesgue measure of the plane. The functions $\phi_n(x)$ are the normalised powers of the complex variable $z$, $\phi_n(x) \equiv z^n \sqrt{n!}$, so that the Hilbert subspace $\mathcal{H}$ is the so-called Fock-Bargmann space of all entire functions that are square integrable with respect to the Gaussian measure. Since $\sum_n |z|^2 = c|z|^2$, the coherent states read
\begin{equation}
|z\rangle = e^{-\frac{|z|^2}{2}} \sum_n \frac{z^n}{\sqrt{n!}} |n\rangle,
\end{equation}
and one easily checks the normalisation and unity resolution:
\begin{equation}
\langle z | z \rangle = 1, \quad \frac{1}{\pi} \int_{\mathbb{C}} |z\rangle \langle z | d^2z = 1_{\mathcal{H}},
\end{equation}
Note that the reproducing kernel is simply given by $e^{zz'}$. The quantization of the observation set is hence achieved by selecting in the original Hilbert space $L^2(\mathbb{C}, \frac{1}{\pi} e^{-|z|^2} d^2z)$ all holomorphic entire functions, which geometric quantization specialists would call a choice of polarization. Quantum operators acting on $\mathcal{H}$ are yielded by using (1.4). We thus have for the most basic one,
\begin{equation}
\frac{1}{\pi} \int_{\mathbb{C}} z |z\rangle \langle z | d^2z = \sum_n \sqrt{n+1} |n\rangle \langle n+1 | \equiv a,
\end{equation}
which is the lowering operator, \( a|n\rangle = \sqrt{n}|n-1\rangle \). Its adjoint \( a^\dagger \) is obtained by replacing \( z \) by \( \bar{z} \) in \( (2.2) \), and we get the factorisation \( N = a^\dagger a \) for the number operator, together with the commutation rule \([a, a^\dagger] = i\mathcal{H} \). Also note that \( a^\dagger \) and \( a \) realize on \( \mathcal{H} \) as multiplication operator and derivation operator respectively, \( a^\dagger f(z) = z f(z), \ a f(z) = df(z)/dz \). From \( q = \frac{1}{2}(z + \bar{z}) \) et \( p = \frac{1}{2i}(z - \bar{z}) \), one easily infers by linearity that \( q \) and \( p \) are upper symbols for \( \frac{1}{2}(a + a^\dagger) \equiv Q \) and \( \frac{1}{2i}(a - a^\dagger) \equiv P \) respectively. In consequence, the self-adjoint operators \( Q \) and \( P \) obey the canonical commutation rule \([Q, P] = i\mathcal{H} \), and for this reason fully deserve the name of position and momentum operators of the usual (galilean) quantum mechanics, together with all localisation properties specific to the latter.

The next examples which are presented in this paper are, although elementary, rather unusual. In particular, we start with observation sets which are not necessarily phase spaces, and such sets are far from having any physical meaning in the common sense. We first consider a two-dimensional quantization of a \( N \)-element set which leads, for \( N \geq 4 \), to a Pauli algebra of observables. We then study two-dimensional (and higher-dimensional) quantizations of the unit segment. In the conclusion, we shall mention some questions of physical interest which are currently under investigation.

## 2. Quantum processing of a \( N \)-element set

An elementary (but not trivial!) exercise for illustrating the quantization scheme introduced in the previous section involves an arbitrary \( N \)-element set \( X = \{x_i\} \) as observation set. An arbitrary non-degenerate measure on it is given by a sum of Dirac measures:

\[
\mu(dx) = \sum_{i=1}^{N} a_i \delta_{\{x_i\}}, \quad a_i > 0.
\]

The Hilbert space \( L^2(X, \mu) \) is simply isomorphic to \( \mathbb{C}^N \). An obvious orthonormal basis is given by \( \left\{ \sqrt{a_i} \chi_{\{x_i\}}(x), \ i=1, \cdots, N \right\} \), where \( \chi_{\{a\}} \) is the characteristic function of the singleton \( \{a\} \). We now consider the two-element orthonormal set \( \{\phi_1 \equiv \phi_\alpha \equiv |\alpha\rangle, \phi_2 \equiv \phi_\beta \equiv |\beta\rangle\} \) defined in the most generic way by:

\[
\phi_\alpha(x) = \sum_{i=1}^{N} \alpha_i \frac{1}{\sqrt{a_i}} \chi_{\{x_i\}}(x), \quad \phi_\beta(x) = \sum_{i=1}^{N} \beta_i \frac{1}{\sqrt{a_i}} \chi_{\{x_i\}}(x),
\]

where complex coefficients \( \alpha_i \) and \( \beta_i \) obey

\[
\sum_{i=1}^{N} |\alpha_i|^2 = 1 = \sum_{i=1}^{N} |\beta_i|^2, \quad \sum_{i=1}^{N} \alpha_i \beta_i \bar{\alpha}_i = 0.
\]

In a Hermitian geometry language, our choice of \( \{\phi_\alpha, \phi_\beta\} \) amounts to selecting in \( \mathbb{C}^N \) the two orthonormal vectors \( \alpha = \{\alpha_i\}, \beta = \{\beta_i\} \), and this justifies our notations for indices.

It follows the expression for the coherent states:

\[
|x\rangle = \frac{1}{\sqrt{N(x)}} [\phi_\alpha(x) \ |\alpha\rangle + \phi_\beta(x) \ |\beta\rangle],
\]
in which $\mathcal{N}(x)$ is given by

$$\mathcal{N}(x) = \sum_{i=1}^{N} \frac{|\alpha_i|^2 + |\beta_i|^2}{a_i} \chi_{\{x_i\}}(x).$$  

The resolution of unity (2.5) here reads as:

$$\mathbb{I} = \sum_{i=1}^{N} \left(|\alpha_j|^2 + |\beta_j|^2\right) |x_i\rangle\langle x_i|$$

The overlap between two coherent states is given by the following kernel:

$$\langle x_i|x_j \rangle = \frac{\bar{\alpha}_i \alpha_j + \bar{\beta}_i \beta_j}{\sqrt{|\alpha_i|^2 + |\beta_i|^2 \sqrt{|\alpha_j|^2 + |\beta_j|^2}}}$$

To any real-valued function $f(x)$ on $X$, i.e., to any vector $f \equiv (f(x_i))$ in $\mathbb{R}^N$, there corresponds the following hermitian operator $A_f$ in $\mathbb{C}^2$, expressed in matrix form with respect to the orthonormal basis $|22\rangle$:

$$A_f = \int_X \mu(dx) \mathcal{N}(x) f(x)|x\rangle\langle x|$$

$$= \left( \sum_{i=1}^{N} |\alpha_i|^2 f(x_i) \sum_{i=1}^{N} \bar{\alpha}_i \beta_i f(x_i) \right) \equiv \left( \langle F|\alpha \rangle \langle \beta|F|\alpha \rangle \right),$$

where $F$ holds for the diagonal matrix $\{(f(x_i))\}$. It is clear that, for a generic choice of the complex $\alpha_i$'s and $\beta_i$'s, all possible hermitian $2 \times 2$-matrices can be obtained in this way if $N \geq 4$. By generic we mean that the following $4 \times N$-real matrix

$$C = \begin{pmatrix}
|\alpha_1|^2 & |\alpha_2|^2 & \cdots & |\alpha_N|^2 \\
|\beta_1|^2 & |\beta_2|^2 & \cdots & |\beta_N|^2 \\
\Re(\alpha_1 \beta_1) & \Re(\alpha_2 \beta_2) & \cdots & \Re(\alpha_N \beta_N) \\
\Im(\alpha_1 \beta_1) & \Im(\alpha_2 \beta_2) & \cdots & \Im(\alpha_N \beta_N)
\end{pmatrix}$$

has rank equal to 4. The case $N = 4$ with $\det C \neq 0$ is particularly interesting since then one has uniqueness of upper symbols of Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_0 = \mathbb{I}$, which form a basis of the four-dimensional Lie algebra of complex Hermitian $2 \times 2$-matrices. As a matter of fact, the operator (2.9) decomposes with respect to this basis as:

$$A_f = \langle f \rangle + \sigma_0 + (\langle f \rangle - \sigma_3 + \Re(\langle \beta|F|\alpha\rangle)) \sigma_1 - \Im(\langle \beta|F|\alpha\rangle) \sigma_2,$$

where the symbols $\langle f \rangle$ stand for the following averagings:

$$\langle f \rangle = \frac{1}{2} \sum_{i=1}^{N} \left(|\alpha_i|^2 \pm |\beta_i|^2\right) f(x_i) = \frac{1}{2} \left(\langle F|\alpha \rangle \pm \langle F|\beta\rangle\right).$$

Note that $\langle f \rangle$ alone has a meanvalue status, precisely with respect to the probability distribution

$$p_i = \frac{1}{2} \left(|\alpha_i|^2 + |\beta_i|^2\right).$$
Also note the appearance of these average terms in the spectral values of the quantum observable $A_f$:

\begin{equation}
\text{Sp}(f) = \left\{ \langle f \rangle_{+} \pm \sqrt{(\langle f \rangle_{-})^2 + |\langle \mathbf{F}|\mathbf{\alpha} \rangle|^2} \right\}.
\end{equation}

Just remark that if vector $\mathbf{\alpha} = (1, 0, \cdots, 0)$ is part of the canonical basis and $\mathbf{\beta} = (0, \beta_2, \cdots, \beta_n)$ is unit vector orthogonal to $\mathbf{\alpha}$, then $A_f$ is diagonal and Sp$(f)$ is trivially reduced to $(f(x_1), \langle \mathbf{F}\rangle_{\mathbf{\beta}})$. The upper symbols for Pauli matrices read in vector form as

\begin{equation}
\hat{\sigma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{\sigma}_1 = C^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = C^{-1} \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{\sigma}_3 = C^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{equation}

On the other hand, and for any $\mathbf{N}$, components of the lower symbol of $A_f$ are given in terms of another probability distribution in which the importance of each one is precisely doubled relatively to its counterpart in (2.12):

\begin{equation}
\langle x_i | A_f | x_i \rangle = \hat{A}_f(x_i) = \sum_{i=1}^{N} \omega_{il} f(x_i),
\end{equation}

with

\begin{equation}
\omega_{il} = |\alpha_{il}|^2 + |\beta_{il}|^2, \quad \omega_{li} = \frac{|\alpha_{il} + \overline{\beta}_{il}|^2}{|\alpha_{li}|^2 + |\beta_{li}|^2}, \ i \neq l.
\end{equation}

Note that the matrix $(\omega_{il})$ is stochastic. As a matter of fact, components of lower symbols of Pauli matrices are given by:

\begin{equation}
\hat{\sigma}_0(x_i) = 1, \quad \hat{\sigma}_1(x_i) = \frac{2 \Re (\overline{\alpha_{il}} \beta_{il})}{|\alpha_{li}|^2 + |\beta_{li}|^2},
\end{equation}

\begin{equation}
\hat{\sigma}_2(x_i) = \frac{2 \Im (\overline{\alpha_{il}} \beta_{il})}{|\alpha_{li}|^2 + |\beta_{li}|^2}, \quad \hat{\sigma}_3(x_i) = \frac{|\alpha_{li}|^2 - |\beta_{li}|^2}{|\alpha_{li}|^2 + |\beta_{li}|^2}.
\end{equation}

Hidden behind this formal game lies an interpretation resorting to Hermitian geometry probability []. For instance, consider $X = \{x_i\}$ as a set of $N$ real numbers. One then can view the real-valued function $f$ defined by $f(x_i) = x_i$ as the position observable, the measurement of which on the quantum level determined by the choice of $\mathbf{\alpha} = \{\alpha_i\}, \mathbf{\beta} = \{\beta_i\}$ has the two possible outcomes given by (2.13). Moreover, the position $x_i$ is privileged to a certain (quantitative) extent in the expression of the average value of the position operator when computed in state $|x_i\rangle$.

Before ending this section, let us examine the lower-dimensional cases $N = 2$ and $N = 3$. When $N = 2$ the basis change (2.2) reduces to a $U(2)$ transformation with $SU(2)$ parameters $\alpha = \alpha_1, \beta = -\overline{\beta}_1, |\alpha|^2 - |\beta|^2 = 1$, and some global phase factor. The operator (2.13) simplifies as

\begin{equation}
A_f = f_+ \mathbb{I} + f_- \begin{pmatrix} |\alpha|^2 - |\beta|^2 & -2\alpha\beta \\ -2\overline{\alpha}\overline{\beta} & |\beta|^2 - |\alpha|^2 \end{pmatrix},
\end{equation}

with $f_\pm := (f(x_1) \pm f(x_2))/2$. We now have a two-dimensional commutative algebra of “observables” $A_f$, generated by the identity matrix $\mathbb{I} = \sigma_0$ and the $SU(2)$
transform of $\sigma_3$: $\sigma_3 \rightarrow g\sigma_3g^\dagger$ with $g = \begin{pmatrix} \alpha \\ \beta \\ \beta \end{pmatrix} \in SU(2)$. As is easily expected in this case, lower symbols reduce to components:

$$(2.20) \quad \langle x_l | A_f | x_l \rangle = \tilde{A}_f(x_l) = f(x_l), \ l = 1, 2.$$ 

Finally, it is interesting to consider the $N = 3$ case when all considered vector spaces are real. The basis change (2.2) involves four real independent parameters, say $\alpha_1, \alpha_2, \beta_1,$ and $\beta_2$, all with modulus $< 1$. The counterpart of (2.9) reads here as

$$(2.21) \quad C_3 = \begin{pmatrix} (\alpha_1)^2 & (\alpha_2)^2 & 1 - (\alpha_1)^2 - (\alpha_2)^2 \\ (\beta_1)^2 & (\beta_2)^2 & 1 - (\beta_1)^2 - (\beta_2)^2 \\ \alpha_1\beta_1 & \alpha_2\beta_2 & -\alpha_1\beta_1 - \alpha_2\beta_2 \end{pmatrix}$$

If $\det C_3 = (\alpha_1\beta_2 - \alpha_2\beta_1)(\beta_1\beta_2 - \alpha_1\alpha_2) \neq 0$, then one has uniqueness of upper symbols of Pauli matrices $\sigma_1, \sigma_3$, and $\sigma_0 = I$ which form a basis of the three-dimensional Jordan algebra of real symmetric $2 \times 2$-matrices. These upper symbols read in vector form as

$$(2.22) \quad \sigma_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \sigma_1 = C_3^{-1} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \sigma_3 = C_3^{-1} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}.$$ 

Finally, the extension of this quantization formalism to $N'$-dimensional subspaces of the original $L^2(X, \mu) \simeq \mathbb{C}^N$ appears as being straightforward on a technical if not interpretational level.

3. Quantum processing of the unit interval

3.1. Quantization with finite subfamilies of Haar wavelets. Further simple examples of quantization are provided when we deal with the unit interval $X = [0, 1]$ of the real line and its associated Hilbert space $L^2[0, 1]$.

Let us start out by simply selecting the two first elements of the orthonormal Haar basis [?], namely the characteristic function $1(x)$ of the unit interval and the Haar wavelet:

$$(3.1) \quad \phi_1(x) = 1(x), \ \phi_2(x) = 1(2x) - 1(2x - 1).$$

Then we have,

$$(3.2) \quad N(x) = \sum_{n=1}^2 |\phi_n(x)|^2 = 2 \ \text{a.e.}.$$ 

The corresponding coherent states read as

$$(3.3) \quad |x\rangle = \frac{1}{\sqrt{2}} [\phi_1(x) |1\rangle + \phi_2(x) |2\rangle].$$

To any integrable function $f(x)$ on the interval there corresponds the linear operator $A_f$ on $\mathbb{R}^2$ or $\mathbb{C}^2$:

$$A_f = 2 \int_0^1 dx f(x) |x\rangle \langle x|$$

$$(3.4) \quad = \left[ \int_0^1 dx f(x) \right] |1\rangle \langle 1| + |2\rangle \langle 2| + \left[ \int_0^1 dx f(x) \phi_2(x) \right] |1\rangle \langle 2| + |2\rangle \langle 1|.$$
or, in matrix form with respect to the orthonormal basis \( \xi_{j} \),

\begin{equation}
A_{f} = \begin{pmatrix}
\int_{0}^{1} dx f(x) & \int_{0}^{1} dx f(x) \phi_{2}(x) \\
\int_{0}^{1} dx f(x) \phi_{2}(x) & \int_{0}^{1} dx f(x)
\end{pmatrix}.
\end{equation}

In particular, with the choice \( f = \phi_{1} \) we recover the identity whereas for \( f = \phi_{2} \),
\( A_{\phi_{2}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_{1} \), the first Pauli matrix. With the choice \( f(x) = x^{p}, \Re p > -1 \),

\begin{equation}
A_{x^{p}} = \frac{1}{p+1} \begin{pmatrix} 1 & 2^{-p} - 1 \\ 2^{-p} - 1 & 1 \end{pmatrix}.
\end{equation}

For an arbitrary coherent state \( |x_{0}\rangle \), \( x_{0} \in [0,1] \), it is interesting to evaluate the average values (lower symbols) of \( A_{x^{p}} \). This gives

\begin{equation}
\langle x_{0}|A_{x^{p}}|x_{0}\rangle = \begin{cases}
\frac{2^{-p}}{p+1} & 0 \leq x_{0} \leq \frac{1}{2}, \\
\frac{2^{-p}}{2^{p+1}} & \frac{1}{2} \leq x_{0} \leq 1,
\end{cases}
\end{equation}

the two possible values being precisely the eigenvalues of the above matrix. Note the average values of the “position” operator: \( \langle x_{0}|A_{x}|x_{0}\rangle = 1/4 \) if \( 0 \leq x_{0} \leq \frac{1}{2} \) and \( 3/4 \) if \( \frac{1}{2} \leq x_{0} \leq 1 \).

Clearly, like in the \( N = 2 \) case of the previous section, all operators \( A_{f} \) commute, since they are linear combinations of the identity matrix and the Pauli matrix \( \sigma_{1} \). The procedure is easily generalized to higher dimensions. Let us add to the previous set \( \{ \phi_{1}, \phi_{2} \} \) other elements of the Haar basis, say up to “scale” \( J \):

\begin{equation}
\{ \phi_{1}(x), \phi_{2}(x), \phi_{3}(x) = \sqrt{2}\phi_{2}(2x), \phi_{4}(x) = \sqrt{2}\phi_{2}(2x - 1), \\
\cdots, \phi_{s}(x) = 2^{J/2}\phi_{2}(2x - k), \phi_{N}(x) = 2^{J/2}\phi_{2}(2x - 2^{J} + 1) \},
\end{equation}

where, at given \( j = 1, 2, \cdots, J \), the integer \( k \) assumes its values in the range \( 0 \leq k \leq 2^{j} - 1 \). The total number of elements of this orthonormal system is \( N = 2^{J+1} \). The expression of \( 1_{J} \) is also given by \( \mathcal{N}(x) = 2^{J+1} \), and this clearly diverges at the limit \( J \to \infty \). Then, it is remarkable if not expected that spectral values as well as average values of the “position” operator are given by \( \langle x_{0}|A_{x^{p}}|x_{0}\rangle = (2k + 1)/2^{J+1} \) for \( k/2^{J} \leq x_{0} \leq (k + 1)/2^{J} \) where \( 0 \leq k \leq 2^{J} - 1 \). Our quantization scheme in the present case achieves a dyadic discretization of the localization in the unit interval.

### 3.2. A two-dimensional non-commutative quantization of the unit interval.

Now we choose another orthonormal system, in the form of the two first elements of the trigonometric Fourier basis,

\begin{equation}
\phi_{1}(x) = 1(x), \ \phi_{2}(x) = \sqrt{2}\sin 2\pi x.
\end{equation}

Then we have,

\begin{equation}
\mathcal{N}(x) = \sum_{n=1}^{2} |\phi_{n}(x)|^{2} = 1 + 2\sin^{2} 2\pi x,
\end{equation}

and corresponding coherent states read as

\begin{equation}
|x\rangle = \frac{1}{\sqrt{1 + 2\sin^{2} 2\pi x}} \left[ |1\rangle + \sqrt{2}\sin 2\pi x |2\rangle \right].
\end{equation}
To any integrable function $f(x)$ on the interval, corresponds the linear operator $A_f$ on $\mathbb{R}^2$ or $\mathbb{C}^2$ (in its matrix form),

\begin{equation}
A_f = \begin{pmatrix}
\int_0^1 dx f(x) & \sqrt{2} \int_0^1 dx f(x) \sin 2\pi x \\
\sqrt{2} \int_0^1 dx f(x) \sin 2\pi x & 2 \int_0^1 dx f(x) \sin^2 2\pi x
\end{pmatrix}
\end{equation}

Like in the previous case, with the choice $f = \phi_1$ we recover the identity whereas for $f = \phi_2$, $A\phi_2 = \sigma_1$, the first Pauli matrix.

We now have to deal with a non-commutative Jordan algebra of operators $A_f$, like in the $N = 3$ real case of the previous section. It is generated by the identity matrix and the two real Pauli matrices $\sigma_1$ and $\sigma_3$.

In this context, the position operator is given by:

$$A_x = \begin{pmatrix}
\frac{1}{2} & -\frac{\sqrt{2\pi}}{2} \\
-\frac{\sqrt{2\pi}}{2} & \frac{1}{2}
\end{pmatrix},$$

with eigenvalues $\frac{1}{2} \pm \frac{\sqrt{2\pi}}{2}$. Note its average values in function of the coherent state parameter $x_0 \in [0, 1]$:

$$\langle x_0 | A_x | x_0 \rangle = \frac{1}{2} - \frac{2}{\pi} \frac{\sin 2\pi x_0}{1 + 2 \sin^2 2\pi x_0}$$

In Fig.1 we give the curve of $\langle x_0 | A_x | x_0 \rangle$ in function of $x_0$. It is interesting to compare with the two-dimensional Haar quantization presented in the previous subsection.
4. Conclusion

The examples we have given in this contribution are mainly of pedagogical nature. Other examples, specially devoted to Euclidean and pseudo-euclidean spheres will be presented elsewhere, having in view possible connections with objects of noncommutative geometry (like fuzzy spheres, see for instance [7]). They show the extreme freedom we have in analyzing a set $X$ of data or possibilities just equipped with a measure by following a quantumlike procedure. The crucial step lies in the choice of a countable orthonormal subset in $L^2(X, \mu)$ obeying (1.1). A $\mathbb{C}^N$ (or $l^2$ if $N = \infty$) unitary transform of this original subset would actually lead to the same specific quantization, and the latter could as well be obtained by using unitarily equivalent continuous orthonormal distributions defined within the framework of some Gel’fand triplet. Of course, further structure like symplectic manifold combined with spectral constraints imposed to some specific observables will considerably restrict that freedom and will lead hopefully to a unique solution, like Weyl quantization, deformation quantization, or geometric quantization are able to achieve in specific situations. Nevertheless, we believe that the generalization of Berezin quantization which has been described here, and which goes far beyond the context of Classical and Quantum Mechanics, not only will shed light on the specific nature of the latter, but also will help to solve in a simpler way some quantization problems.

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