A 6-dimensional simply connected complex and symplectic manifold with no Kähler metric

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We construct a simply connected compact manifold which has complex and symplectic structures but does not admit Kähler metric, in the lowest possible dimension where this can happen, that is, dimension 6. Such a manifold is automatically formal and has even odd-degree Betti numbers but it does not satisfy the Lefschetz property for any symplectic form.

1. Introduction

A Kähler manifold \((M, J, \omega)\) is a smooth manifold \(M\) of dimension \(2n\) endowed with an integrable almost complex structure \(J\) and a symplectic form \(\omega\) such that \(g(X, Y) = \omega(X, JY)\) defines a Riemannian metric, called Kähler metric. In order to check that a compact manifold does not carry any Kähler metric, one can use a collection of known topological obstructions to the existence of such a structure: theory of Kähler groups, evenness of odd-degree Betti numbers, Lefschetz property or the formality of the rational homotopy type (see [1, 7, 25]).

If \(M\) is a compact Kähler manifold, then it has a complex and a symplectic structure. However, the converse is not true. The first example of a compact manifold admitting complex and symplectic structures but no Kähler metric is the Kodaira-Thurston manifold [16, 23]. This 4-manifold is not simply connected (it is actually a nilmanifold) hence the fundamental group plays a key role in this property. The classification of complex and symplectic nilmanifolds of dimension 6 was given by Salamon in [22]. Generalizations to higher dimension \(2n \geq 6\) of the Kodaira-Thurston manifold are the generalized Iwasawa manifolds considered in [6]. Such manifolds have complex and symplectic structures but carry no Kähler metric. Note that, in dimension 2, every oriented surface admits a Kähler metric.
If one restricts attention to manifolds with trivial fundamental group, then every complex manifold of real dimension 4 admits a Kähler structure. Indeed, by the Enriques-Kodaira classification [16], if $M$ is a complex surface whose first Betti number $b_1$ is even (this holds in particular when $b_1 = 0$), then $M$ is deformation equivalent to a Kähler surface (see also [2, Theorem 3.1, page 144] for a direct proof of this fact). We point out that Gompf [13] has constructed the first examples of simply connected compact symplectic but not complex 4-manifolds. Also Fintushel and Stern [12] have given a family of simply connected symplectic 4-manifolds not admitting complex structures (the latter was proved by Park [21]).

In dimensions higher than 4, we have the following results. The first examples of simply connected compact symplectic non-Kählerian manifolds were given in dimension 6 by Gompf in the aforementioned paper [13] and in dimension $\geq 10$ by McDuff in [18] (these examples are not known to admit complex structures). Fine and Panov in [10] (see also [11]) have produced simply connected symplectic 6-manifolds with $c_1 = 0$ which do not have a compatible complex structure (but it is not known if they admit Kähler structures). Furthermore, Guan in [14] constructed the first family of simply connected, compact and holomorphic symplectic non-Kählerian manifolds of (real) dimension $4n \geq 8$. On the other hand, the first and third authors have proved [3] that the 8-dimensional manifold $X$ constructed in [9] is an example of a simply connected, symplectic and complex manifold which does not admit a Kähler structure (since it is not formal). For higher dimensions $2n = 8 + 2k$, $k \geq 1$, one can take $X \times \mathbb{C}P^k$. This is simply connected, complex and symplectic but not Kähler. Thus, a natural question arises:

**Does there exist a 6-dimensional simply connected, compact, symplectic and complex manifold which does not admit Kähler metrics?**

In this paper we answer this question in the affirmative by proving the following result:

**Theorem 1.1.** There exists a 6-dimensional, simply connected, compact, symplectic and complex manifold which carries no Kähler metric.

In order to construct such an example, we start with a 6-dimensional nilmanifold $M$ admitting both a complex structure $J$ and a symplectic structure $\omega$. Then we quotient it by a finite group preserving $J$ and $\omega$ to obtain a simply connected, 6-dimensional orbifold $\hat{M}$ with an orbifold complex structure $\hat{J}$ and an orbifold symplectic form $\hat{\omega}$. By Hironaka Theorem [15], there
A complex, symplectic and non-Kähler 6-manifold is a complex resolution $(\tilde{M}_c, \tilde{J})$ of $(\hat{M}, \hat{J})$. As in [5], we resolve symplectically the singularities of $(\hat{M}, \hat{\omega})$ to obtain a smooth symplectic 6-manifold $(\tilde{M}_s, \tilde{\omega})$. However, in our situation, the singular locus of the orbifold $\hat{M}$ does not consist only of a discrete set of points, in contrast with [5]. For a complex and symplectic orbifold, we provide conditions under which the complex and the symplectic resolution of singularities are diffeomorphic (Theorem 3.1). Using this we prove that the resolutions $\tilde{M}_c$ and $\tilde{M}_s$ are diffeomorphic. Thus, $\tilde{M} = \tilde{M}_c$ is not only a complex manifold but also a symplectic one.

To prove that $\tilde{M}$ satisfies the conditions of Theorem 1.1 we show that $\hat{M}$ is simply connected (Proposition 6.1), this resulting from the careful choice of the action of the finite group on $\hat{M}$. Then, we have that $\tilde{M}$ is also simply connected because any desingularization of a complex analytic variety with quotient singularities has the same fundamental group as the original variety [17, Theorem 7.8.1]. Since $\hat{M}$ is a 6-dimensional simply connected compact manifold, then $b_1(\hat{M}) = 0$, and $b_3(\hat{M})$ is even by Poincaré duality. Also $\tilde{M}$ is automatically formal by [8, Theorem 3.2]. Therefore, to ensure that $\tilde{M}$ does not carry any Kähler metric, we use the Lefschetz property; more precisely, we prove that the map $L[\Omega] : H^2(\tilde{M}) \to H^4(\tilde{M})$ given by the cup product with $[\Omega]$ is not an isomorphism for any possible symplectic form $\Omega$. Again the choice of nilmanifold $M$ and finite group action makes possible to have a non-zero $[\beta] \in H^2(\tilde{M})$ such that $[\beta] \wedge [\alpha_1] \wedge [\alpha_2] = 0$ for every $[\alpha_1], [\alpha_2] \in H^2(\tilde{M})$, which gives the result.

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2. Orbifolds

Definition 2.1. A (smooth) $n$-dimensional orbifold is a Hausdorff, paracompact topological space $X$ endowed with an atlas $\mathcal{A} = \{(U_p, \tilde{U}_p, \Gamma_p, \varphi_p)\}$
of orbifold charts, that is $U_p \subset X$ is a neighbourhood of $p \in X$, $\tilde{U}_p \subset \mathbb{R}^n$ an open set, $\Gamma_p \subset \text{GL}(n, \mathbb{R})$ a finite group acting on $\tilde{U}_p$, and $\varphi_p: \tilde{U}_p \rightarrow U_p$ is a $\Gamma_p$-invariant map with $\varphi_p(0) = p$, inducing a homeomorphism $\tilde{U}_p/\Gamma_p \cong U_p$.

The charts are compatible in the following sense: if $q \in U_q \cap U_p$, then there exist a connected neighbourhood $V \subset U_q \cap U_p$ and a diffeomorphism $f: \varphi^{-1}_p(V)_0 \rightarrow \varphi^{-1}_q(V)$, where $\varphi^{-1}_p(V)_0$ is the connected component of $\varphi^{-1}_p(V)$ containing $q$, such that $f(\sigma(x)) = \rho(\sigma)(f(x))$, for any $x$, and $\sigma \in \text{Stab}_{\Gamma_p}(q)$, where $\rho: \text{Stab}_{\Gamma_p}(q) \rightarrow \Gamma_q$ is a group isomorphism.

For each $p \in X$, let $n_p = \#\Gamma_p$ be the order of the orbifold point (if $n_p = 1$ the point is smooth, also called non-orbifold point). The singular locus of the orbifold is the set $S = \{p \in X \mid n_p > 1\}$. Therefore $M - S$ is a smooth $n$-dimensional manifold. The singular locus $S$ is stratified: if we write $S_k = \{p \mid n_p = k\}$, and consider its closure $\overline{S_k}$, then $\overline{S_k}$ inherits the structure of an orbifold. In particular $S_k$ is a smooth manifold, and the closure consists of some points of $S_{kl}$, $l \geq 2$.

We say that the orbifold is locally oriented if $\Gamma_p \subset \text{GL}_+(n, \mathbb{R})$ for any $p \in X$. As $\Gamma_p$ is finite, we can choose a metric on $\tilde{U}_p$ such that $\Gamma_p \subset \text{SO}(n)$. An element $\sigma \in \Gamma_p$ admits a basis in which it is written as

$$\sigma = \text{diag} \left( \begin{pmatrix} \cos \theta_1 & -\sin \theta_1 \\ \sin \theta_1 & \cos \theta_1 \end{pmatrix}, \ldots, \begin{pmatrix} \cos \theta_r & -\sin \theta_r \\ \sin \theta_r & \cos \theta_r \end{pmatrix} \right),$$

for $\theta_1, \ldots, \theta_r \in (0, 2\pi)$. In particular, the set of points fixed by $\sigma$ is of codimension $2r$. Therefore the set of singular points $S \cap U_p$ is of codimension $\geq 2$, and hence $X - S$ is connected (if $X$ is connected). Also we say that the orbifold $X$ is oriented if it is locally oriented and $X - S$ is oriented.

A natural example of orbifold appears when we take a smooth manifold $M$ and a finite group $\Gamma$ acting on $M$ effectively. Then $\tilde{M} = M/\Gamma$ is an orbifold. If $M$ is oriented and the action of $\Gamma$ preserves the orientation, then $\tilde{M}$ is an oriented orbifold. Note that for every $\tilde{p} \in \tilde{M}$, the group $\Gamma_{\tilde{p}}$ is the stabilizer of $p \in M$, with $\tilde{p} = \tilde{\pi}(p)$ under the natural projection $\tilde{\pi}: M \rightarrow \tilde{M}$.

**Definition 2.2.** A complex orbifold is a $2n$-dimensional orbifold $X$ whose orbifold charts have $\tilde{U}_p \subset \mathbb{C}^n$, $\Gamma_p \subset \text{GL}(n, \mathbb{C})$, and in the compatibility of charts the maps $f$ are biholomorphisms. Note that $X$ is automatically oriented.

If $M$ is a complex manifold and $\Gamma$ is a finite group acting effectively on $M$ by biholomorphisms, then $\tilde{M} = M/\Gamma$ is a complex orbifold.
The complex structure of a complex orbifold \( X \) can be given by the orbifold \((1,1)\)-tensor \( J \) with \( J^2 = -\text{id} \). This is given by tensors \( J_p \) on each \( \tilde{U}_p \) defining the complex structure, which are \( \Gamma_p \)-equivariant, for each \( p \in X \), and which agree under the functions \( f \) defining the compatibility of charts.

**Definition 2.3.** A complex resolution of a complex orbifold \((X,J)\) is a complex manifold \( \tilde{X} \) together with a holomorphic map \( \pi: \tilde{X} \to X \) which is a biholomorphism \( \tilde{X} - E \to X - S \), where \( S \subset X \) is the singular locus and \( E = \pi^{-1}(S) \) is the exceptional locus.

Let \( X \) be an orbifold. An orbifold \( k \)-form \( \alpha \) consists of a collection of \( k \)-forms \( \alpha_p \) on each \( \tilde{U}_p \) which are \( \Gamma_p \)-equivariant and that match under the compatibility maps between different charts.

**Definition 2.4.** A symplectic orbifold \((X,\omega)\) consists of a \( 2n \)-dimensional oriented orbifold \( X \) and an orbifold 2-form \( \omega \) such that \( d\omega = 0 \) and \( \omega^n > 0 \) everywhere.

If \( M \) is a symplectic manifold and \( \Gamma \) is a finite group acting effectively on \( M \) by symplectomorphisms, then \( \tilde{M} = M/\Gamma \) is a symplectic orbifold.

**Definition 2.5.** A symplectic resolution of a symplectic orbifold \((X,\omega)\) consists of a smooth symplectic manifold \((\tilde{X},\tilde{\omega})\) and a map \( \pi: \tilde{X} \to X \) such that:

- \( \pi \) is a diffeomorphism \( \tilde{X} - E \to X - S \), where \( S \subset X \) is the singular locus and \( E = \pi^{-1}(S) \) is the exceptional locus.
- \( \tilde{\omega} \) and \( \pi^*\omega \) agree in the complement of a small neighbourhood of \( E \).

### 3. Desingularization of orbifold points

In this section we suppose that \( X \) is an oriented orbifold whose singular locus \( S \) consists of a discrete set of points. Assume that \( X \) admits a complex structure \( J \) and a symplectic structure \( \omega \). Therefore we have a complex orbifold \((X,J)\) and a symplectic orbifold \((X,\omega)\).

It is well-known that \((X,J)\) admits a complex resolution \((\tilde{X}_c,\tilde{J})\) by Hironaka’s desingularization [15]. Also, the symplectic orbifold \((X,\omega)\) admits a symplectic resolution \((\tilde{X}_s,\tilde{\omega})\) by Theorem 3.3 in [5]. We want to compare the two resolutions.

First, let us look at the complex resolution of \((X,J)\). Consider \( p \in S \), and let \( U_p = \tilde{U}_p/\Gamma_p \) be an orbifold neighbourhood. Recall that we denote...
f \_p: \tilde{U}_p \to U_p \) the quotient map. By definition of complex orbifold, \( \tilde{U}_p \subset \mathbb{C}^n = \mathbb{R}^{2n} \) and \( \Gamma_p \subset \text{GL}(n, \mathbb{C}) \). As \( \Gamma_p \) is a finite group, we can choose a Kähler metric invariant by \( \Gamma_p \). With a linear change of variables, we can transform the Kähler metric into standard form. That is, we can suppose that there is an inclusion

\[(3.1) \quad \iota: \Gamma_p \hookrightarrow U(n).\]

Shrinking \( \tilde{U}_p \) if necessary, we can assume that \( \tilde{U}_p = B(0) \), for some \( \epsilon > 0 \).

Consider now an algebraic resolution of the singularity of \( Y = \mathbb{C}^n / \Gamma_p \), provided by \([15]\). Denote it \( \pi: \tilde{Y} \to Y \) and let \( E = \pi^{-1}(p) \) be the exceptional locus. Write \( B = B(0) / \Gamma_p \) and \( \tilde{B} = \pi^{-1}(B) \). The complex resolution is defined as the smooth manifold

\[ \tilde{X}_c = (X - \{ p \}) \cup \pi \tilde{B}, \]

where the identification uses the map \( \pi: \tilde{B} - E \to B - \{ p \} = U_p - \{ p \} \). This has a natural complex structure since \( \pi \) is a biholomorphism.

Now we move to the construction of the symplectic resolution of \( (X, \omega) \), as done in \([5]\). For \( p \in S \), take an orbifold neighbourhood \( U'_p = \tilde{U}'_p / \Gamma'_p \), with \( \varphi'_p: \tilde{U}'_p \to U'_p \). By the equivariant Darboux theorem (see \([20]\) Theorem 7.3.1), there is a \( \Gamma'_p \)-equivariant symplectomorphism \( (\tilde{U}'_p, \omega_p) \cong (V, \omega_0) \), where \( V \subset \mathbb{R}^{2n} \) is an open set, and \( \omega_0 \) is the standard symplectic form (shrinking \( \tilde{U}'_p \) if necessary). So without loss of generality, we can assume that \( \tilde{U}'_p \subset (\mathbb{R}^{2n}, \omega_0) \), where \( \omega_0 \) is the standard symplectic form, and \( \Gamma'_p \subset \text{Sp}(2n, \mathbb{R}) \). As \( \Gamma'_p \) is a finite group, and \( U(n) \subset \text{Sp}(2n, \mathbb{R}) \) is the maximal compact subgroup, we can choose a complex structure \( J \) on \( \mathbb{R}^{2n} \) such that the pair \( (J, \omega_0) \) determines a Kähler metric, which is invariant by \( \Gamma'_p \). We perform a linear change of variables, which transforms the complex structure into standard form (so \( \tilde{U}'_p \) has the standard Kähler structure). Equivalently, we can suppose that there is an inclusion

\[(3.2) \quad \iota': \Gamma'_p \hookrightarrow U(n).\]

Shrinking \( \tilde{U}'_p \) if necessary, we can assume that \( \tilde{U}'_p = B_c(0) \), for some \( \epsilon' > 0 \).

Consider an algebraic resolution of singularities of \( Y' = \mathbb{C}^n / \Gamma'_p \), call it \( \pi': \tilde{Y}' \to Y' \), and let \( E' = (\pi')^{-1}(p) \) be the exceptional locus. Write \( B' = B_c(0) / \Gamma'_p \) and \( \tilde{B}' = (\pi')^{-1}(B') \). The symplectic resolution is defined as the
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\[ \tilde{X}_s = (X - \{p\}) \cup_{\pi'} \tilde{B}', \]

where \( \tilde{B}' - E' \) and \( B' - \{p\} = U'_p - \{p\} \) are identified by \( \pi' \). This has a symplectic structure that is constructed by gluing the symplectic structure of \( X - \{p\} \) and the Kähler form of \( \tilde{B}' \) by a cut-off process, as done in Theorem 3.3 of [5].

Now we are going to compare \( \tilde{X}_c \) and \( \tilde{X}_s \). First note that for \( p \in S \), we have \( \Gamma_p \sim \gamma' \), which follows from \( \Gamma_p \sim \pi_1(B - \{p\}) \) and \( \Gamma'_p \sim \pi_1(B' - \{p\}) \), and the fact that \( B, B' \) are homeomorphic. So we shall denote \( \Gamma'_p = \Gamma_p \) henceforth. We have the following result.

**Theorem 3.1.** If one can arrange that the inclusions \( \iota \) and \( \iota' \), given by (3.1) and (3.2), respectively, are such that \( \iota = \iota' \) for every singular point \( p \in S \), then there is a diffeomorphism \( \tilde{X}_c \cong \tilde{X}_s \), which is the identity outside a small neighbourhood of the exceptional loci. In particular, \( \tilde{X}_c \) admits both complex and symplectic structures.

**Proof.** The key point is obviously that if \( \iota = \iota' \), then \( Y' = \tilde{Y} \), so we can take \( \tilde{Y}' = \tilde{Y} \) and \( \pi' = \pi \) in the constructions above.

We fix a point \( p \in S \), and construct the required isomorphism in a neighbourhood of the exceptional locus over that point. Consider the map (reducing \( \epsilon > 0 \) if necessary)

\[ f = (\varphi'_p)^{-1} \circ \varphi_p : B_{\epsilon}(0) = \tilde{U}_p \rightarrow B_{\epsilon'}(0) = \tilde{U}'_p; \]

\( f \) is \( \Gamma_p \)-equivariant and an open embedding (it might fail to be surjective) with \( f(0) = 0 \). We shall construct a map \( F : B_{\epsilon}(0) \rightarrow B_{\epsilon'}(0) \) such that

- \( F = \text{id} \) in a small ball \( B_{0.2\epsilon}(0) \),
- \( F = f \) outside a slightly bigger ball \( B_{0.9\epsilon}(0) \),
- \( F \) is a \( \Gamma_p \)-equivariant diffeomorphism onto its image.

This gives a diffeomorphism \( F : \tilde{X}_c \rightarrow \tilde{X}_s \), defined by \( F \) on \( B_{\epsilon}(0)/\Gamma_p - \{p\} \), extended by the identity on \( \pi^{-1}(B_{0.2\epsilon}(0)/\Gamma_p) \), and also by the identity on \( X - \pi^{-1}(B_{0.9\epsilon}(0)/\Gamma_p) \).

Write \( f(x) = L(x) + R(x) \), where \( L \) is the linear part and \( |R(x)| \leq C|x|^2 \), for some constant \( C > 0 \). Both these maps are \( \Gamma_p \)-equivariant. Take a smooth, non-decreasing function \( \rho_1 : [0, \epsilon] \rightarrow [0, 1] \) such that \( \rho_1(t) = 0 \) for \( t \in [0, 0.8\epsilon] \) and \( \rho_1(t) = 1 \) for \( t \in [0.9\epsilon, 1] \). Consider \( g(x) = L(x) + \rho_1(|x|)R(x) \). Then,
$g(x) = L(x)$ for $|x| \leq 0.8\epsilon$, $g(x) = f(x)$ for $|x| \geq 0.9\epsilon$, and $g(x)$ is $\Gamma_p$-equivariant because $\Gamma_p \subset \text{SO}(2n)$. Also
\[
dg(x) - L = \rho'_1(|x|)R(x)d|x| + \rho_1(|x|)dR(x).
\]
Using that $|\rho'_1(t)| \leq C/\epsilon$ and $|dR(x)| \leq C|x|$ (we denote by $C > 0$ uniform constants, that can vary from line to line) we have that $|dg(x) - L| \leq C|x|$. For $\epsilon > 0$ small enough, we have that $g$ is a diffeomorphism onto its image.

Next, take the linear map $L: \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. We can choose orthonormal (oriented) basis in both origin and target so that $L = \text{diag}(\lambda_1, \ldots, \lambda_{2n})$, where $\lambda_i > 0$ are real numbers (the first vector of the basis is a unitary vector $e_1$ such that $|L(e_1)|$ is maximized; then $L$ maps $\langle e_1 \rangle^\perp$ to $\langle L(e_1) \rangle^\perp$, and we proceed inductively). Consider the map
\[
h(x) = \begin{cases} 
  x, & |x| \leq 0.4\epsilon, \\
  x + \rho_2\left(\left(\frac{|x| - 0.4\epsilon}{0.3\epsilon}\right)^\alpha\right)(L(x) - x), & 0.4\epsilon \leq |x| \leq 0.7\epsilon, \\
  g(x), & |x| \geq 0.7\epsilon,
\end{cases}
\]
where $\rho_2: [0, 1] \to [0, 1]$ is smooth non-decreasing with $\rho_2(t) = 0$ for $t \in [0, \frac{1}{3}]$, and $\rho_2(t) = 1$ for $t \in [\frac{2}{3}, 1]$. Here $\alpha > 0$ is a constant to be fixed soon.

Clearly $h$ is $\Gamma_p$-equivariant, $h(x) = f(x)$ off $B_{0.9\epsilon}(0)$, and $h(x) = x$ in $B_{0.4\epsilon}(0)$ (but beware, we have chosen different coordinates on the origin $\mathbb{R}^{2n}$ and the target $\mathbb{R}^{2n}$, so $h$ is not the identity in the ball). The map $h$ is $C^\infty$ because for $0.4\epsilon \leq |x| \leq 0.5\epsilon$ we have also $h(x) = x$. Let us see that $h$ is a diffeomorphism onto its image. It only remains to see this for $0.5\epsilon \leq |x| \leq 0.7\epsilon$. Write $y = h(x)$, so in our coordinates $y_i = x_i + \rho_2(u)(\lambda_i - 1)x_i$, with $u = \left(\frac{|x| - 0.4\epsilon}{0.3\epsilon}\right)^\alpha$. Then,
\[
dy_i = (1 + (\lambda_i - 1)\rho_2(u))dx_i + (\lambda_i - 1)\rho_2'(u)\frac{\alpha}{0.3\epsilon}\left(\frac{|x| - 0.4\epsilon}{0.3\epsilon}\right)^{\alpha - 1}x_i\gamma
\]
with $\gamma = d|x| = \frac{1}{|x|^2} \sum x_j dx_j$. Write $\delta_i = (1 + (\lambda_i - 1)\rho_2(u))$, so $\delta_i$ takes values between 1 and $\lambda_i$. We compute
\[
\begin{align*}
dy_1 \wedge \ldots \wedge dy_n \\
= \delta_1 \ldots \delta_n \, dx_1 \wedge \ldots \wedge dx_n \\
+ \sum_{i=1}^n \delta_1 \ldots \delta_{i-1} \delta_i \ldots \delta_n \frac{(\lambda_i - 1)\rho_2'(u)x_i}{0.3\epsilon} \\
\times \left(\frac{|x| - 0.4\epsilon}{0.3\epsilon}\right)^{\alpha - 1} \, dx_1 \wedge \ldots \wedge \gamma \wedge \ldots \wedge dx_n
\end{align*}
\]
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\[
= \delta_1 \ldots \delta_n \left( 1 + \alpha \sum \frac{(\lambda_i - 1)\rho_2(u)(|x| - 0.4\epsilon)^{\alpha - 1/2}x_i^2}{|x|\delta_1(0.3\epsilon)^\alpha} \right) dx_1 \wedge \ldots \wedge dx_n.
\]

In the sum, the numerator is bounded above by \(C(0.3\epsilon)^{\alpha + 1}\) and the denominator is bounded below by \(C^{-1}(0.3\epsilon)^{\alpha + 1}\), for some uniform (independent of \(\alpha\)) constant \(C > 0\). Hence choosing \(\alpha > 0\) small enough, we get that the above quantity does not vanish, hence \(h\) is a diffeomorphism onto its image.

After this step is done, recall that we have taken coordinates given by an orthonormal basis \(\{e_i\}\) on the origin \(\mathbb{R}^{2n}\), and by the orthonormal basis \(\{L(e_i)/\lambda_i\}\) on the target \(\mathbb{R}^{2n}\). Written with respect to the same coordinates, we have an orthogonal transformation \(M : \mathbb{R}^{2n} \to \mathbb{R}^{2n}\) so that \(h(x) = M\) on \(B_{0,4\epsilon}(0)\). The final step is to change the isometry \(M \in SO(2n)\) by the identity. Take a smooth path \(M_t\) of matrices joining \(M_0 = id\) with \(M_1 = M\). Take a smooth non-decreasing \(\rho_3 : [0,\epsilon] \to [0,1]\) with \(\rho_3(t) = 0\) for \(t \in [0,0.2\epsilon]\), and \(\rho_3(t) = 1\) for \(t \in [0.3\epsilon,\epsilon]\). The map \(F(x) = M_{\rho_3(|x|)}(x), |x| \leq 0.4\epsilon\), and \(F(x) = h(x)\) for \(|x| \geq 0.4\epsilon\), is the required map. \(\square\)

**Remark 3.2.** Let \(F : (\widetilde{X}_c, \widetilde{J}) \to (\widetilde{X}_c, \widetilde{\omega})\) be the diffeomorphism provided by Theorem 3.1. Then if we denote \(\widetilde{\omega}' = F^*\widetilde{\omega}\), we have that \(\widetilde{X}_c\) admits a symplectic structure \(\widetilde{\omega}'\) and a complex structure \(\widetilde{J}\). These are not compatible in general, but they are compatible on a neighbourhood of the exceptional locus, and give a Kähler structure there.

**Remark 3.3.** The condition \(i = i'\) in Theorem 3.1 is not vacuous. Consider for instance the unit ball \(B = B(0,1) \subset \mathbb{C}^2\) with the standard complex structure and the symplectic form \(\omega = -i(dz_1 \wedge dz_2 - d\bar{z}_1 \wedge d\bar{z}_2)\). Let \(i : \Gamma_p = \mathbb{Z}_m \to U(2), m > 2, \zeta = e^{2\pi i/m}\), with the action given by \(\zeta \cdot (z_1, z_2) = (\zeta z_1, \zeta z_2)\). Then \((B, \omega) \cong (B', \omega_0)\), with the symplectomorphism given by \(w_1 = z_1, w_2 = \zeta z_2\), and \(\omega_0 = -i(dw_1 \wedge dw_2 + dw_2 \wedge d\bar{w}_2)\) the standard symplectic form. The inclusion \(i' : \mathbb{Z}_m \to U(2)\) is now given by the action \(\zeta \cdot (w_1, w_2) = (\zeta w_1, \zeta^{-m-1} w_2)\). Therefore \(i \neq i'\), for \(m > 2\).

### 4. A complex and symplectic 6-orbifold

Consider the complex Heisenberg group \(G\), that is, the complex nilpotent Lie group of (complex) dimension 3 consisting of matrices of the form

\[
\begin{pmatrix}
1 & u_2 & u_3 \\
0 & 1 & u_1 \\
0 & 0 & 1
\end{pmatrix}.
\]
In terms of the natural (complex) coordinate functions \((u_1, u_2, u_3)\) on \(G\), we have that the complex 1-forms \(\mu = du_1, \nu = du_2\) and \(\theta = du_3 - u_2 du_1\) are left invariant, and
\[
d\mu = d\nu = 0, \quad d\theta = \mu \wedge \nu.
\]
Let \(\Lambda \subset \mathbb{C}\) be the lattice generated by 1 and \(\zeta = e^{2\pi i/6}\), and consider the discrete subgroup \(\Gamma \subset G\) formed by the matrices in which \(u_1, u_2, u_3 \in \Lambda\). We define the compact (parallelizable) nilmanifold
\[
M = \Gamma \backslash G.
\]
We can describe \(M\) as a principal torus bundle
\[
T^2 = \mathbb{C}/\Lambda \hookrightarrow M \twoheadrightarrow T^4 = (\mathbb{C}/\Lambda)^2
\]
by the projection \((u_1, u_2, u_3) \mapsto (u_1, u_2)\).

Consider the action of the finite group \(\mathbb{Z}_6\) on \(G\) given by the generator
\[
\rho: G \to G
\]
\[
(u_1, u_2, u_3) \mapsto (\zeta^4 u_1, \zeta u_2, \zeta^5 u_3).
\]
This action satisfies that \(\rho(p \cdot q) = \rho(p) \cdot \rho(q)\), for \(p, q \in G\), where \(\cdot\) denotes the natural group structure of \(G\). Moreover, \(\rho(\Gamma) = \Gamma\). Thus, \(\rho\) induces an action on the quotient \(M = \Gamma \backslash G\). Denote by \(\rho: M \to M\) the \(\mathbb{Z}_6\)-action. The action on 1-forms is given by
\[
\rho^\ast \mu = \zeta^4 \mu, \quad \rho^\ast \nu = \zeta \nu, \quad \rho^\ast \theta = \zeta^5 \theta.
\]

**Proposition 4.1.** \(\hat{M} = M/\mathbb{Z}_6\) is a 6-orbifold admitting complex and symplectic structures.

*Proof.* The nilmanifold \(M\) is a complex manifold whose complex structure \(J\) is the multiplication by \(i\) at each tangent space \(T_pM, p \in M\). Then one can check that \(J\) commutes with the \(\mathbb{Z}_6\)-action \(\rho\) on \(M\), that is, \((\rho_s)_p \circ J_p = J_{\rho(p)} \circ (\rho_s)_p\), for any point \(p \in M\). Hence, \(J\) induces a complex structure on the quotient \(\hat{M} = M/\mathbb{Z}_6\).

Now we define the complex 2-form \(\omega\) on \(M\) given by
\[
(4.1) \quad \omega = -i \mu \wedge \bar{\mu} + \nu \wedge \theta + \bar{\nu} \wedge \bar{\theta}.
\]
Clearly, \(\omega\) is a real closed 2-form on \(M\) which satisfies \(\omega^3 > 0\), that is, \(\omega\) is a symplectic form on \(M\). Moreover, \(\omega\) is \(\mathbb{Z}_6\)-invariant. Indeed, \(\rho^\ast \omega = -i \mu \wedge
We denote by \( \hat{\pi} : M \to \hat{M} \) the natural projection. The orbifold points of \( \hat{M} \) are the following:

1) The points \((\frac{1}{2}a(1 + \zeta), \frac{1}{2}b(1 + \zeta), \frac{1}{2}c(1 + \zeta) + \frac{3}{3}ab(1 + \zeta^2)) \in M\), with \( a, b, c \in \{0, 1, 2\} \) and \((b, c) \neq (0, 0)\), are points of order 3; their isotropy group is \( K = \{\text{id}, \rho^2, \rho^4\}\). These points are mapped in pairs by \( \mathbb{Z}_6 \), so they define 12 orbifold points in \( \hat{M} = M/\mathbb{Z}_6 \), with models \( \mathbb{C}^3/K \).

2) The surfaces \( S_{(p,q)} = \{(u_1, p, pu_1 + q) \mid u_1 \in \mathbb{C}/\Lambda\} \subset M \), where \( p, q \in \{0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}, (p, q) \neq (0, 0)\). These are 15 tori, which consist of points of order 2, with isotropy \( H = \{\text{id}, \rho^3\}\). These surfaces are permuted by the group \( \mathbb{Z}_6 \), so they come in 5 groups of three tori each. Thus they define 5 tori in the orbifold \( \hat{M} \), formed by orbifold points of order 2.

3) The surface \( S_0 = \{(u_1, 0, 0) \mid u_1 \in \mathbb{C}/\Lambda\} \subset M \) is a torus consisting generically of points of order 2, with isotropy \( H \). Here \( \rho : S_0 \to S_0 \) and it is a map of order 3, with three fixed points \((\frac{1}{2}a(1 + \zeta), 0, 0)\), \( a = 0, 1, 2\). These points have isotropy \( \mathbb{Z}_6 \). The quotient \( S_0/\langle \rho \rangle \subset \hat{M} \) is homeomorphic to a sphere (with three orbifold points of order 3).

## 5. Resolution of the 6-orbifold

Now we want to desingularize the orbifold \( \hat{M} \). We shall treat each of the connected components of the singular locus determined before independently. Recall that \( K = \{\text{id}, \rho^2, \rho^4\} \cong \mathbb{Z}_3 \) and \( H = \{\text{id}, \rho^3\} \cong \mathbb{Z}_2 \). There is a natural isomorphism \( \langle \rho \rangle = \mathbb{Z}_6 \cong K \times H \).

### 5.1. Resolution of the isolated orbifold points

We know that there are 12 isolated orbifold points in \( \hat{M} \). Let \( \hat{p} \in \hat{M} \) be one of them. The preimage of \( \hat{p} \) under \( \hat{\pi} \) consists of two points, \( \hat{\pi}^{-1}(\hat{p}) = \{p_1, p_2\} \). The isotropy group of \( p_1 \) is \( K \). Consider a \( K \)-invariant neighbourhood \( U \) of \( p_1 \) in \( M \). Then,

\[
\hat{U} = \hat{\pi}(U) \cong U/K
\]

is an orbifold neighbourhood of \( \hat{p} \) in \( \hat{M} \). This has complex and symplectic resolutions as in Section 3. In order to apply Theorem 3.1 we check that \( i =
\]
For the complex resolution, we have \( i(\zeta^2) = \text{diag}(\zeta^2, \zeta^2, \zeta^4) \).

For the symplectic resolution, the symplectic form \( \omega = -i\, du_1 \wedge d\bar{u}_1 + du_2 \wedge d\bar{u}_2 + d\bar{u}_3 \wedge du_3 \).

We have to do a change of variables to transform \( K \subset \text{Sp}(6, \mathbb{R}) \) into a subgroup of \( \text{U}(3) \). This is obtained with

\[
\begin{align*}
v_1 &= u_1 \\
v_2 &= \frac{1}{\sqrt{2}}(u_2 - i\bar{u}_3) \\
v_3 &= \frac{1}{\sqrt{2}}(\bar{u}_2 - iu_3).
\end{align*}
\]

This transforms \( \omega \) into

\[
\omega = -i\, dv_1 \wedge d\bar{v}_1 - i\, dv_2 \wedge d\bar{v}_2 - i\, dv_3 \wedge d\bar{v}_3,
\]

the standard Kähler form. In the new coordinates the \( K \)-action is given by

\[
(\zeta^2 v_1, \zeta^2 v_2, \zeta^4 v_3), \text{ so } i'(\zeta^2) = \text{diag}(\zeta^2, \zeta^2, \zeta^4), \text{ and } i = i'.
\]

### 5.2. Resolution of the singular sets \( \hat{\pi}(S_{(p,q)}) \)

Now we consider a connected component of the singular set which is homeomorphic to a 2-torus. There are 5 such components in \( \hat{M} \), all of them are images by \( \hat{\pi} \) of the sets \( S_{(p,q)} = \{(u_1, p, pu_1 + q) \mid u_1 \in \mathbb{C}/\Lambda \} \), where \( (p, q) \in I = \left( \{0, \frac{1}{2}, \frac{\zeta}{2}, \frac{1+\zeta}{2}\} \right)^2 - \{(0, 0)\} \).

Let us focus on one such component \( \hat{T} = \hat{\pi}(T), T \cong \mathbb{C}/\Lambda \). Then \( H \) fixes \( S_{(p,q)} \), and its orbit under \( K \) is given by \( S_{(p,q)} \), for three elements \( (p_1, q_1) = (p, q), (p_2, q_2), (p_3, q_3) \in I \). Consider a neighbourhood \( U \) of \( T \subset M \) via

\[
T \times B_\epsilon(0) \rightarrow U \\
(u_1, u_2, u_3) \mapsto (u_1, u_2 + p, u_3 + pu_1 + q),
\]

where \( B_\epsilon(0) \subset \mathbb{C}^2 \). The image is

\[
\hat{U} = \hat{\pi}(U) \cong U/H \cong T \times (B_\epsilon(0)/H),
\]

where \( H \cong \mathbb{Z}_2 \) acts as \( (u_2, u_3) \mapsto (-u_2, -u_3) \).
We see that the complex structure on (5.2) is the product complex structure. Also, the symplectic structure
\[ \omega = \text{id} u_1 \wedge d\bar{u}_1 + du_2 \wedge d\bar{u}_2 + d\bar{u}_3 \wedge d\bar{u}_3 \]
is the product of the natural symplectic structure of \( \mathbb{C}/\Lambda \) with an orbifold symplectic structure on \( B_\epsilon(0)/H \). Using the construction of Section 3, we have a desingularization
\[ \tilde{Y} \to B_\epsilon(0)/H \]
which is a smooth manifold endowed with both a complex structure and a symplectic structure coinciding with the given ones outside a small neighbourhood of the exceptional locus \( E \). The condition \( \iota = \iota' \) of Theorem 3.1 is trivially satisfied, since \( \iota(\rho^3) = \iota'(\rho^3) = -\text{id} \). Multiplying by \( T = \mathbb{C}/\Lambda \), we have that
\[ \tilde{U} = T \times \tilde{Y} \]
is a smooth manifold endowed with a complex structure \( \tilde{J} \), and a symplectic structure \( \tilde{\omega} \), which coincide with those of \( \tilde{U} \) outside a small neighbourhood of the exceptional locus \( T \times E \subset \tilde{U} \).

The complex and the symplectic resolutions of \( \hat{M} \) in a neighbourhood of \( \hat{T} \) are obtained by replacing \( \hat{U} \subset \hat{M} \) with \( \tilde{U} \). The two resolutions are diffeomorphic by the considerations above.

### 5.3. Resolution of the singular set \( \hat{\pi}(S_0) \)

Finally we consider the connected component of the singular set which is homeomorphic to a 2-sphere. This is \( \hat{S}_0 = \hat{\pi}(S_0) \), where \( S_0 = \{ (u_1, 0, 0) \mid u_1 \in \mathbb{C}/\Lambda \} \). As before, a neighbourhood of \( S_0 \) in \( M \) is of the form
\[ U_0 = (\mathbb{C}/\Lambda) \times B_\epsilon(0), \]
where \( B_\epsilon(0) \subset \mathbb{C}^2 \). The action of \( H = \mathbb{Z}_2 \) is trivial on \( \mathbb{C}/\Lambda \) and as \( \pm 1 \) on \( \mathbb{C}^2 \). The action of \( K = \mathbb{Z}_3 \) is of the form \( \rho^3(u_1, u_2, u_3) = (\zeta^2 u_1, \zeta^2 u_2, \zeta^4 u_3) \).

Let us focus on \( B_\epsilon(0)/H \). By the construction of Section 3, we have a complex desingularization \( (\hat{Y}_c, \hat{J}) \to B_\epsilon(0)/H \). The holomorphic action of \( K \) on \( B_\epsilon(0) \) induces an action on \( (\hat{Y}_c, \hat{J}) \). Also, there is a symplectic desingularization \( (\hat{Y}_s, \hat{\omega}) \to B_\epsilon(0)/H \). The action of \( K \) on \( B_\epsilon(0) \) induces an action on \( (\hat{Y}_s, \hat{\omega}) \). This follows by taking an orbifold chart of the singular point that is \( (H \times K) \)-equivariant, using the equivariant Darboux theorem.

By Theorem 3.1, there is a diffeomorphism \( F : (\hat{Y}_c, \hat{J}) \to (\hat{Y}_s, \hat{\omega}) \). Let us see that \( F \) can be taken to be \( K \)-equivariant. This follows by the arguments in the proof of Theorem 3.1 by using that \( \iota : H \times K \to U(2) \) and \( \iota' : H \times K \to U(2) \) are equal. For the complex case, \( \iota \) is given by the representation
\[(u_2, u_3) \mapsto (\zeta u_2, \zeta^5 u_3), \text{ so } \iota(\zeta) = \text{diag}(\zeta, \zeta^5). \]

For the symplectic case, we have to do a change of variables to transform \(H \times K \subset \text{Sp}(4, \mathbb{R})\) into a subgroup of \(U(2)\). This is given by

\[
v_2 = \frac{1}{\sqrt{2}}(u_2 - i\bar{u}_3), \quad v_3 = \frac{1}{\sqrt{2}}(\bar{u}_2 - iu_3),
\]

which transforms \(\omega = du_2 \wedge du_3 + d\bar{u}_2 \wedge d\bar{u}_3\) into the standard Kähler form 

\[-i dv_2 \wedge d\bar{v}_2 - i dv_3 \wedge d\bar{v}_3.\]

As \((u_2, u_3) \mapsto (\zeta v_2, \zeta^5 v_3)\), we have that \(\iota'(\zeta) = \text{diag}(\zeta, \zeta^5)\). Hence \(\iota = \iota'\).

This produces a desingularization \(\tilde{Y} \rightarrow B_e(0)/H\) with a symplectic and a complex structure, which match the given ones outside a small neighbourhood of the exceptional set \(E \subset Y\), which are compatible (they give a Kähler structure) in a smaller neighbourhood of \(E\), by Remark 3.2 and which have an action of \(K\) preserving both the complex and symplectic structures. A desingularization of

\[U_0/H = (\mathbb{C}/\Lambda) \times (B_e(0)/H)\]

is given by substituting a neighbourhood of \(\hat{S}_0 = (\mathbb{C}/\Lambda) \times \{0\}\) by \((\mathbb{C}/\Lambda) \times \hat{Y}\). The fixed points of action of \(K\) in \(U_0/H\) lie on \(\hat{S}_0\), hence the fixed points of the action of \(K\) on the desingularization of \(U_0/H\) lie in the exceptional divisor. In this part of the manifold, we have a Kähler structure, so the symplectic and complex desingularization are the same.

This means that \((U_0/H)/K \cong U_0/(H \times K)\) admits a desingularization \(\tilde{V}\) with a complex and a symplectic structure. The resolution of \(\tilde{M}\) in a neighbourhood of \(\hat{S}_0\) is obtained by substituting \(\tilde{\pi}(U_0) = U_0/(H \times K) \subset \tilde{M}\) with \(\tilde{V}\).

All together, we get a smooth 6-manifold \(\tilde{M}\) with a complex structure and a symplectic structure, and with a map

\[
\pi: \tilde{M} \rightarrow \tilde{M},
\]

which is simultaneously a complex and a symplectic resolution.

### 6. Topological properties of \(\tilde{M}\)

In this section, we are going to complete the proof of Theorem 1.1 by proving that \(\tilde{M}\) is simply-connected and that it does not admit a Kähler structure.

**Proposition 6.1.** \(\tilde{M}\) is simply connected.
Proof. By [17, Theorem 7.8.1], it is sufficient to prove that \( \tilde{M} \) is simply connected.

We fix base points \( p_0 = (0, 0, 0) \in M \) and \( \hat{p}_0 = \tilde{\pi}(p_0) \in \tilde{M} \). There is an epimorphism of fundamental groups

\[
\Gamma = \pi_1(M, p_0) \to \pi_1(\tilde{M}, \hat{p}_0),
\]

since the \( \mathbb{Z}_6 \)-action has a fixed point [4, Chapter II, Corollary 6.3]. Now the nilmanifold \( M \) is a principal 2-torus bundle over the 4-torus \( T^4 \), so we have an exact sequence

\[
\mathbb{Z}^2 \to \Gamma \to \mathbb{Z}^4.
\]

The group \( \Gamma = \pi_1(M, p_0) \) is thus generated by the images of the fundamental groups of the surfaces \( \Sigma_1 = \{(u_1, 0, 0)\}, \Sigma_2 = \{(0, u_2, 0)\} \) and \( \Sigma_3 = \{(0, 0, u_3)\} \) in \( M \). The image \( \tilde{\pi}(\Sigma_1) \) is a 2-sphere, since \( \tilde{\pi} : \Sigma_1 \to \tilde{\pi}(\Sigma_1) \) is a degree 3 map with three ramification points of order 3 (namely \( \frac{1}{2}a(1 + \zeta), 0, 0 \), with \( a = 0, 1, 2 \)). The image of \( \Sigma_2 \) is also a 2-sphere, since \( \tilde{\pi} : \Sigma_2 \to \tilde{\pi}(\Sigma_2) \) is a degree 6 map with one point of order 6, \( (0, 0, 0) \), two of order 3, \( (0, \frac{1}{2}b(1 + \zeta), 0) \), \( b = 1, 2 \), and three of order 2 (namely \( (0, p, 0) \), \( p = \frac{1}{2}, \frac{1}{2}, \frac{1}{2} + \zeta \)). Analogously, \( \tilde{\pi}(\Sigma_3) \) is a 2-sphere. This proves that \( \pi_1(\tilde{M}, \hat{p}_0) = \{1\} \).

Now we look at the resolution process. As mentioned before, the desingularisation process does not change the fundamental group [17, Theorem 7.8.1]. However, for simplicity, we give a direct proof of this result in the case at hand. Let \( S \subset \tilde{M} \) be the singular locus and suppose \( p \in S \) is an isolated orbifold point. The resolution replaces a neighbourhood \( B = B_p(0)/\Gamma_p \) of \( p \) with a smooth manifold \( \tilde{B} \), such that \( \pi : \tilde{B} \to B \) is a complex resolution of singularities. The manifold \( \tilde{B} \) is simply connected by [24, Theorem 4.1]. A Seifert-Van Kampen argument gives that \( \pi_1(\tilde{M}) \) is the amalgamated sum of \( \pi_1(M - \{p\}) \) and \( \pi_1(B) \) along \( \pi_1(\partial B) \). Also \( \pi_1(M) \) is the amalgamated sum of \( \pi_1(M - E) \) and \( \pi_1(B) \) along \( \pi_1(\partial B) \). As \( \pi_1(B) = \pi_1(\tilde{B}) = \{1\} \), we have that \( \pi_1(\tilde{M}) = \pi_1(M) \).

Suppose now that we have a connected component \( S' \) of the singular locus \( S \) of positive dimension. Let \( E' = \pi^{-1}(S') \) be the corresponding exceptional locus. The invariance of the fundamental group under resolution is proved along the same lines as before if we know that the map \( \pi : E' \to S' \) induces an isomorphism \( \pi_1(E') \to \pi_1(S') \). In our case, we have two possibilities: if \( S' = \hat{\pi}(S_{(p,q)}) \cong T^2 \), then \( E' = T^2 \times E \), where \( E \) is the exceptional divisor of the resolution \( \tilde{Y} \to B_\epsilon(0)/H \), which is clearly simply connected, and the result follows.
The second possibility is $S' = \tilde{\pi}(S_0)$. In this case, the exceptional divisor over $S'$ is the exceptional divisor of the resolution of

$$((C/\Lambda) \times (C^2/H))/K.$$  

The resolution of $C^2/H$ is done by blowing-up $C^2$ at the origin,

$$\tilde{C}^2 = \{(a, b, [u : v]) \in C^2 \times \mathbb{C}P^1 \mid av = bu\},$$

and then quotienting by $H = \{ \pm \mathrm{id} \}$. Clearly, the fundamental groups of $(C/\Lambda) \times (C^2/H)$ and $(C/\Lambda) \times (C^2/H)$ coincide. The action of $K$ is given by $(a, b, [u : v]) \mapsto ((\zeta^2 a, \zeta^4 b), [u : \zeta^2 v])$, with fixed points $(0, 0, [1 : 0])$ and $(0, 0, [0 : 1])$ The fixed points of $K$ on $(C/\Lambda) \times (\tilde{C}^2/H)$ occur when $K$ fixes both factors. Therefore, all fixed points are isolated, and the second resolution does not alter the fundamental group. □

In order to prove that $\tilde{M}$ does not admit a Kähler structure, we are going to check that it does not satisfy the Lefschetz condition for any symplectic form. For this, it is necessary to understand the cohomology $H^*(\tilde{M})$.

We start by computing the cohomology of $\tilde{M}$. By Nomizu theorem [19], the cohomology of the nilmanifold $M$ is:

- $H^0(M, \mathbb{C}) = \langle 1 \rangle$,
- $H^1(M, \mathbb{C}) = \langle [\mu], [\tilde{\mu}], [\nu], [\tilde{\nu}] \rangle$,
- $H^2(M, \mathbb{C}) = \langle [\mu \wedge \tilde{\mu}], [\mu \wedge \tilde{\nu}], [\tilde{\mu} \wedge \nu], [\nu \wedge \tilde{\mu}], [\mu \wedge \tilde{\theta}], [\nu \wedge \tilde{\theta}], [\tilde{\nu} \wedge \tilde{\theta}] \rangle$,
- $H^3(M, \mathbb{C}) = \langle [\mu \wedge \tilde{\mu} \wedge \theta], [\mu \wedge \tilde{\nu} \wedge \theta], [\nu \wedge \tilde{\mu} \wedge \theta], [\nu \wedge \tilde{\nu} \wedge \theta], [\mu \wedge \nu \wedge \theta], [\tilde{\mu} \wedge \nu \wedge \theta], [\tilde{\mu} \wedge \nu \wedge \tilde{\theta}] \rangle$,
- $H^4(M, \mathbb{C}) = \langle [\mu \wedge \tilde{\mu} \wedge \nu \wedge \theta], [\mu \wedge \tilde{\mu} \wedge \tilde{\nu} \wedge \theta], [\mu \wedge \tilde{\nu} \wedge \nu \wedge \theta], [\mu \wedge \nu \wedge \nu \wedge \theta], [\tilde{\mu} \wedge \nu \wedge \nu \wedge \theta], [\tilde{\nu} \wedge \nu \wedge \nu \wedge \theta] \rangle$,
- $H^5(M, \mathbb{C}) = \langle [\mu \wedge \tilde{\mu} \wedge \nu \wedge \theta \wedge \tilde{\theta}], [\mu \wedge \tilde{\mu} \wedge \tilde{\nu} \wedge \theta \wedge \tilde{\theta}], [\mu \wedge \tilde{\nu} \wedge \nu \wedge \theta \wedge \tilde{\theta}], [\mu \wedge \nu \wedge \nu \wedge \theta \wedge \tilde{\theta}], [\tilde{\mu} \wedge \nu \wedge \nu \wedge \theta \wedge \tilde{\theta}], [\tilde{\nu} \wedge \nu \wedge \nu \wedge \theta \wedge \tilde{\theta}] \rangle$,
- $H^6(M, \mathbb{C}) = \langle [\mu \wedge \tilde{\mu} \wedge \nu \wedge \nu \wedge \theta \wedge \tilde{\theta}] \rangle$.

The cohomology of $\tilde{M}$ is $H^*(\tilde{M}, \mathbb{C}) = H^*(M, \mathbb{C})^{2\varepsilon}$:

- $H^0(\tilde{M}, \mathbb{C}) = \langle 1 \rangle$,
- $H^1(\tilde{M}, \mathbb{C}) = 0$,
- $H^2(\tilde{M}, \mathbb{C}) = \langle [\mu \wedge \tilde{\mu}], [\nu \wedge \tilde{\nu}], [\nu \wedge \theta], [\tilde{\nu} \wedge \tilde{\theta}] \rangle$,
A complex, symplectic and non-Kähler 6-manifold

\[ H^3(\hat{M}, \mathbb{C}) = 0, \]
\[ H^4(\hat{M}, \mathbb{C}) = \langle [\mu \wedge \bar{\mu} \wedge \nu \wedge \theta], [\mu \wedge \bar{\mu} \wedge \bar{\nu} \wedge \bar{\theta}], [\nu \wedge \bar{\nu} \wedge \theta \wedge \bar{\theta}] \rangle, \]
\[ H^5(\hat{M}, \mathbb{C}) = 0, \]
\[ H^6(\hat{M}, \mathbb{C}) = \langle [\mu \wedge \bar{\mu} \wedge \nu \wedge \bar{\nu} \wedge \theta \wedge \bar{\theta}] \rangle. \]

**Proposition 6.2.** \( \hat{M} \) does not admit a Kähler structure since it does not satisfy the Lefschetz property for any symplectic form on \( \hat{M} \).

**Proof.** Let \( \Omega \) be a symplectic form on \( \hat{M} \). The Lefschetz map \( L_{[\Omega]} : H^2(\hat{M}) \to H^4(\hat{M}) \) is given by the cup product with \( [\Omega] \). We show that there is a class \( [\hat{\beta}] \in H^2(\hat{M}) \) which is in the kernel of \( L_{[\Omega]} \). We prove this by checking that \( [\Omega] \wedge [\hat{\beta}] \wedge [\alpha] = 0 \), for any 2-form \( [\alpha] \in H^2(\hat{M}) \).

We need to determine the cohomology \( H^2(\hat{M}) \). For this, the first step is to construct a map \( H^2(\hat{M}) \to H^2(\hat{M}) \). Let \( h : M \to M \) be a map which:

- is the identity outside small neighbourhoods of each point with non-trivial isotropy,
- contracts a neighbourhood of each of the 24 isolated points whose isotropy is \( K \) onto the corresponding point,
- contracts a neighbourhood of each \( S_{(p,q)} \) onto \( S_{(p,q)} \) (fixing \( S_{(p,q)} \) point-wise),
- in a neighbourhood of \( S_0 \), is the composition of a contraction onto \( S_0 \) with a map that contracts neighbourhoods (in \( S_0 \)) of the 3 fixed points to the points, and
- is \( \mathbb{Z}_6 \)-equivariant.

\( h \) induces a map \( \hat{h} : \hat{M} \to \hat{M} \). Note that for any closed form \( \alpha \in \Omega^*(\hat{M}) \), \( \hat{h}^* \alpha \in \Omega^*(\hat{M}) \) is cohomologous to \( \alpha \) and can be lifted to a form \( \pi^* \hat{h}^* \alpha \in \Omega^*(\hat{M}) \), where \( \pi : M \to \hat{M} \) is the resolution map. This induces a well-defined map

\[ \Psi = \pi^* \circ \hat{h}^* : H^*(\hat{M}) \to H^*\hat{M}. \]

Now consider \( U = \hat{M} - S \), where \( S \subset \hat{M} \) is the singular locus and \( V \subset \hat{M} \) is a small neighbourhood of \( S \). Let also \( \hat{U} = \pi^{-1}(U) \) and \( \hat{V} = \pi^{-1}(V) \subset \hat{M} \).
Using compactly supported de Rham cohomology, we have a diagram

\[
H_2^c(U) \oplus H_2^c(V) \to H_2^c(\hat{M}) \to H_2^c(U \cap V) \to H_2^c(U) \oplus H_2^c(V)
\]

\[
H_2^c(\tilde{U}) \oplus H_2^c(\tilde{V}) \to H_2^c(\tilde{M}) \to H_3^c(\tilde{U} \cap \tilde{V}) \to H_3^c(\tilde{U}) \oplus H_3^c(\tilde{V})
\]

Since \( V \) retracts onto a set of dimension 2, \( H_3^c(V) = 0 \). By Poincaré duality, \( H_3^c(V) = 0 \) as well. Now a simple diagram chasing proves that \( H_2^c(\hat{M}) = H_2^c(\tilde{M}) \) is generated by \( H_2^c(\hat{M}) = H_2^c(\tilde{M}) \) and \( H_2^c(\tilde{V}) \).

Consider the closed form \( \nu \wedge \bar{\nu} \in \Omega^2(\hat{M}) \). Since \( \nu \wedge \bar{\nu} | S_{(p,q)} = 0 \) for any surface \( S_{(p,q)} \) and \( \nu \wedge \bar{\nu} | S_0 = 0 \) as well, the 2-cohomology class

\[
[\beta] = \Psi([\nu \wedge \bar{\nu}])
\]

vanishes on \( \tilde{V} \). Clearly \([\beta] \wedge [\alpha_1] \wedge [\alpha_2] = 0\) if either \([\alpha_1], [\alpha_2] \in H_2^c(\tilde{V})\). Moreover, one can check that \([\beta] \wedge [\alpha_1] \wedge [\alpha_2] = 0\), for \([\alpha_1], [\alpha_2] \in H^2(\hat{M})\), which completes the proof.

\[\square\]

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