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Branching Law for the Finite Subgroups of $\text{SL}_4\mathbb{C}$

Frédéric BUTIN

Abstract
In the framework of McKay correspondence we determine, for every finite subgroup $\Gamma$ of $\text{SL}_4\mathbb{C}$, how the finite dimensional irreducible representations of $\text{SL}_4\mathbb{C}$ decompose under the action of $\Gamma$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\text{sl}_4\mathbb{C}$ and let $\varpi_1, \varpi_2, \varpi_3$ be the corresponding fundamental weights. For $(p, q, r) \in \mathbb{N}^3$, the restriction $\pi_{p,q,r}|r$ of the irreducible representation $\pi_{p,q,r}$ of highest weight $p\varpi_1 + q\varpi_2 + r\varpi_3$ of $\text{SL}_4\mathbb{C}$ decomposes as $\pi_{p,q,r}|r = \bigoplus_{i=0}^{+\infty} m_i(p, q, r)\gamma_i$. We determine the multiplicities $m_i(p, q, r)$ and prove that the series $P_i(t, u, w) = \sum_{p=0}^{+\infty} \sum_{q=0}^{+\infty} \sum_{r=0}^{+\infty} m_i(p, q, r)t^pu^qw^r$ are rational functions.

This generalizes results from Kostant for $\text{SL}_4\mathbb{C}$ and our preceding works about $\text{SL}_3\mathbb{C}$.

Keywords: McKay correspondence; branching law; representations; finite subgroups of $\text{SL}_4\mathbb{C}$.

Mathematics Subject Classifications (2000): 20C15; 17B10; 15A09; 17B67.

1 Introduction and results

• Let $\Gamma$ be a finite subgroup of $\text{SL}_4\mathbb{C}$ and $\{\gamma_0, \ldots, \gamma_l\}$ the set of equivalence classes of irreducible finite dimensional complex representations of $\Gamma$, where $\gamma_0$ is the trivial representation. The character associated to $\gamma_i$ is denoted by $\chi_i$.

Consider $\gamma : \Gamma \rightarrow \text{SL}_4\mathbb{C}$ the natural $4$–dimensional representation, and $\gamma^*$ its contragredient representation. The character of $\gamma$ is denoted by $\chi$. By complete reducibility we get the decompositions

$$\forall j \in [0, l], \quad \gamma_j \otimes \gamma = \bigoplus_{i=0}^{l} a_{ij}^{(1)} \gamma_i, \quad \gamma_j \otimes (\gamma \wedge \gamma) = \bigoplus_{i=0}^{l} a_{ij}^{(2)} \gamma_i \quad \text{and} \quad \gamma_j \otimes \gamma^* = \bigoplus_{i=0}^{l} a_{ij}^{(3)} \gamma_i.$$

This defines the three following square matrices of $\mathbf{M}_{l+1}\mathbb{N}$:

$$A^{(1)} := \left( a_{ij}^{(1)} \right)_{i,j \in [0, l]^2}, \quad A^{(2)} := \left( a_{ij}^{(2)} \right)_{i,j \in [0, l]^2} \quad \text{and} \quad A^{(3)} := \left( a_{ij}^{(3)} \right)_{i,j \in [0, l]^2}.$$

• Let $\mathfrak{h}$ be a Cartan subalgebra of $\text{sl}_4\mathbb{C}$ and let $\varpi_1, \varpi_2, \varpi_3$ be the corresponding fundamental weights, and $V(p\varpi_1 + q\varpi_2 + r\varpi_3)$ the simple $\text{sl}_4\mathbb{C}$–module of highest weight $p\varpi_1 + q\varpi_2 + r\varpi_3$ with $(p, q, r) \in \mathbb{N}^3$. Then we get an irreducible representation $\pi_{p,q,r} : \text{SL}_4\mathbb{C} \rightarrow \text{GL}(V(p\varpi_1 + q\varpi_2 + r\varpi_3))$. The restriction of $\pi_{p,q,r}$ to the subgroup $\Gamma$ is a representation of $\Gamma$, and by complete reducibility, we get the decomposition

$$\pi_{p,q,r}|r = \bigoplus_{i=0}^{l} m_i(p, q, r)\gamma_i,$$

where the $m_i(p, q, r)$’s are non negative integers. Let $E := (e_0, \ldots, e_l)$ be the canonical basis of $\mathbb{C}^{l+1}$, and

$$v_{p,q,r} := \sum_{i=0}^{l} m_i(p, q, r)e_i \in \mathbb{C}^{l+1}.$$

We have in particular $v_{0,0,0} = e_0$ as $\gamma_0$ is the trivial representation. Let us consider the vector

$$P_i(t, u, w) := \sum_{p=0}^{+\infty} \sum_{q=0}^{+\infty} \sum_{r=0}^{+\infty} v_{p,q,r}t^pu^qw^r \in (\mathbb{C}[t, u, w])^{l+1}.$$

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and denote by $P_T(t, u, w)_j$ its $j$-th coordinate in the basis $E$, which is an element of $\mathbb{C}[t, u, w]$. Note that $P_T(t, u, w)$ can also be seen as a formal power series with coefficients in $\mathbb{C}^{l+1}$. The aim of this article is to prove the following theorem.

**Theorem 1**

The coefficients of $P_T(t, u, w)$ are rational functions in $t, u, w$, i.e. the formal power series $P_T(t, u, w)_i$ are rational functions

$$P_T(t, u, w)_i = \frac{N_T(t, u, w)_i}{D_T(t, u, w)}, \quad i \in [0, l],$$

where the $N_T(t, u, w)_i$'s and $D_T(t, u, w)$ are elements of $\mathbb{Q}[t, u, w]$.

- The proof of this theorem uses a key-relation satisfied by $P_T(t, u, w)$ as well as a so-called inversion formula. Two essential ingredients are the decomposition of the tensor product of $\text{SL}_2 \mathbb{C}$ and the simultaneous diagonalizability of certain matrices. The effective calculation of $P_T(t, u, w)$ then reduces to matrix multiplication.

In [BP09] we applied a similar method for $\text{SL}_2 \mathbb{C}$ — recovering thereby in a quite easy way the results obtained by Kostant in [Kos85], [Kos06], and by Gonzales-Sprinberg and Verdier in [GSV83] — and for $\text{SL}_4 \mathbb{C}$ in order to get explicit computations of the series for every finite subgroup of $\text{SL}_4 \mathbb{C}$.

The general framework of that study is the construction of a minimal resolution of singularities of the orbifold $\mathbb{C}^n / \Gamma$. It is related to the McKay correspondence (see [BKR01], [GSV83] and [GNS04]). For example, Gonzalez-Sprinberg and Verdier use in [GSV83] a Poincaré series to construct explicitly minimal resolutions for singularities of $V = \mathbb{C}^2 / \Gamma$ when $\Gamma$ is a finite subgroup of $\text{SL}_2 \mathbb{C}$. To go further in this approach, our results for $\text{SL}_4 \mathbb{C}$ could be used to construct an explicit synthetic minimal resolution of singularities for orbifolds of the form $\mathbb{C}^4 / \Gamma$ where $\Gamma$ is a finite subgroup of $\text{SL}_4 \mathbb{C}$.

## 2 Properties of the matrices $A^{(1)}$, $A^{(2)}$, $A^{(3)}$

In order to compute the series $P_T(t, u, w)$, we first establish here some properties of the matrices $A^{(1)}$, $A^{(2)}$, $A^{(3)}$. The first proposition essentially follows from the uniqueness of the decomposition of a representation as sum of irreducible representations.

**Proposition 2**

- $A^{(3)} = {}^t A^{(1)}$.
- $A^{(2)}$ is a symmetric matrix.
- $A^{(1)}$, $A^{(2)}$ and $A^{(3)}$ commute. In particular, $A^{(1)}$ is a normal matrix.

**Proof:**

Since $a_{ij}^{(1)} = (\chi_i | \chi_{\gamma \gamma_i}) = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \chi_i(g) \chi_{\gamma_i}(g)$, we have $\gamma \otimes \gamma_j = \bigoplus_{i=0}^l a_{ij}^{(1)} \gamma_i$. In the same way,

$$(\gamma \wedge \gamma) \otimes \gamma_j = \bigoplus_{i=0}^l a_{ij}^{(2)} \gamma_i \text{ and } \gamma^* \otimes \gamma_j = \bigoplus_{i=0}^l a_{ij}^{(3)} \gamma_i.$$  

Then

$$a_{ij}^{(3)} = (\chi_i | \chi_{\gamma_j \gamma^*}) = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \chi_i(g) \chi_{\gamma_j}(g) \chi_{\gamma^*}(g) = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \chi_i(g) \chi_{\gamma_j}(g) \chi(g^{-1})$$  

$$= \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \chi_i(g^{-1}) \chi_{\gamma_j}(g) \chi(g) = a_{ji}^{(1)},$$

hence $A^{(3)} = {}^t A^{(1)}$.

We also have $(\gamma \gamma_j \otimes \gamma^*) = \bigoplus_{k=0}^l \sum_{i=0}^l a_{ki}^{(1)} \gamma_k \otimes \gamma^* = \bigoplus_{i=0}^l a_{ij}^{(1)} \left( \bigoplus_{k=0}^l a_{ki}^{(3)} \gamma_k \right) = \bigoplus_{k=0}^l \left( \sum_{i=0}^l a_{ki}^{(1)} a_{ij}^{(1)} \right) \gamma_k$ and $\gamma \otimes (\gamma_j \otimes \gamma^*) = \gamma \otimes \left( \bigoplus_{i=0}^l a_{ij}^{(3)} \gamma_i \right) = \bigoplus_{i=0}^l a_{ij}^{(3)} \left( \bigoplus_{k=0}^l a_{ki}^{(1)} \gamma_k \right) = \bigoplus_{k=0}^l \left( \sum_{i=0}^l a_{ki}^{(1)} a_{ij}^{(3)} \right) \gamma_k$, hence $A^{(3)} A^{(1)} = A^{(1)} A^{(3)}$. The proofs of the other statements are the same.

Since $A^{(1)}$, $A^{(2)}$, $A^{(3)}$ are normal, we know that they are diagonalizable with eigenvectors forming an orthogonal basis. Now we will diagonalize these matrices by using the character table of the group $\Gamma$. Let
us denote by \{C_0, \ldots, C_l\} the set of conjugacy classes of \(\Gamma\), and for any \(j \in [0, l]\), let \(g_j\) be an element of \(C_j\). So the character table of \(\Gamma\) is the matrix \(T_\Gamma \in M_{l+1}(\mathbb{C})\) defined by \((T_\Gamma)_{i,j} := \chi_i(g_j)\).

**Proposition 3**
- For \(k \in [0, l]\), set \(w_k := (\chi_0(g_k), \ldots, \chi_l(g_k)) \in \mathbb{C}^{l+1}\). Then \(w_k\) is an eigenvector of \(A^{(3)}\) associated to the eigenvalue \(\chi(g_k)\). Similarly, \(w_k\) is an eigenvector of \(A^{(1)}\) associated to the eigenvalue \(\frac{1}{2} (\chi(g_k)^2 + \chi(g_k^2))\).

**Proof:**
From the relation \(\gamma_i \otimes \gamma = \sum_{j=0}^l a_{ij}^{(1)} \gamma_j\), we get \(\chi_i \otimes \gamma = \sum_{j=0}^l a_{ij}^{(1)} \chi_j\). By evaluating this on \(g_k\), we obtain \(\chi_i(g_k) \chi(g_k) = \sum_{j=0}^l a_{ij}^{(1)} \chi_j(g_k) = \sum_{j=0}^l a_{ij}^{(1)} \chi_j(g_k)\) according to Proposition 2. So \(w_k\) is an eigenvector of \(A^{(3)}\) associated to the eigenvalue \(\chi(g_k)\). The method is similar for the other results. 

As the \(w_i\)s are the column of \(T_\Gamma\), which are always orthogonal, the matrix \(T_\Gamma\) is invertible and the family \(W := (w_0, \ldots, w_l)\) is a common basis of eigenvectors of \(A^{(1)}, A^{(2)}\) and \(A^{(3)}\). Then \(A^{(1)} := T^{-1}_\Gamma A^{(1)} T_\Gamma, A^{(2)} := T^{-1}_\Gamma A^{(2)} T_\Gamma\) and \(A^{(3)} := T^{-1}_\Gamma A^{(3)} T_\Gamma\) are diagonal matrices, with \(A^{(1)}_{jj} = \chi(g_j)\), \(A^{(2)}_{jj} = \frac{1}{2}(\chi(g_j)^2 - \chi(g_j^2))\) and \(A^{(3)}_{jj} = \chi(g_j)\).

Now, we make use of the Clebsch-Gordan formula

\[
\begin{align*}
\pi_{0,0,0} \otimes \pi_{p,q,r} &= \pi_{p+1,q,r} \oplus \pi_{p,q,r-1} \oplus \pi_{p-1,q+1,r} \oplus \pi_{p,q-1,r+1}, \\
\pi_{1,0,0} \otimes \pi_{p,q,r} &= \pi_{p+1,q,r} \oplus \pi_{p,q-1,r+1} \oplus \pi_{p-1,q+1,r} \oplus \pi_{p,q-1,r+1} \oplus \pi_{p+1,q,r-1} \oplus \pi_{p+1,q,r-1} \oplus \pi_{p,q,r-1} \oplus \pi_{p,q,r-1}. \\
\end{align*}
\]

**Proposition 4**
The vectors \(v_{m,n}\) satisfy the following recurrence relations

\[
\begin{align*}
A^{(1)}v_{p,q,r} &= v_{p+1,q,r} + v_{p,q,r-1} + v_{p-1,q+1,r} + v_{p,q-1,r+1}, \\
A^{(2)}v_{p,q,r} &= v_{p,q+1,r} + v_{p,q-1,r+1} + v_{p+1,q-1,r+1} + v_{p,q+1,r+1} + v_{p,q+1,r-1} + v_{p,q+1,r-1}, \\
A^{(3)}v_{p,q,r} &= v_{p,q+1} + v_{p-1,q,r} + v_{p,q+1} + v_{p+1,q,r-1} + v_{p+1,q,r-1}. \\
\end{align*}
\]

**Proof:**
The definition of \(v_{p,q,r}\) reads \(v_{p,q,r} = \sum_{i=0}^l m_i(p, q, r)e_i\), thus \(A^{(1)}v_{p,q,r} = \sum_{i=0}^l \left(\sum_{j=0}^l m_j(p, q, r)a_{ij}^{(1)}\right) e_i\). Now

\[
(\pi_{1,0,0} \otimes \pi_{p,q,r})|\Gamma = \pi_{p,q,r}|\Gamma \otimes \gamma = \sum_{j=0}^l m_j(p, q, r)\gamma_j \otimes \gamma = \sum_{j=0}^l \left(\sum_{i=0}^l m_i(p, q, r)a_{ij}^{(1)}\right) \gamma_i,
\]

and

\[
\begin{align*}
\pi_{p+1,q,r}|\Gamma + \pi_{p,q,r-1}|\Gamma + \pi_{p-1,q+1,r}|\Gamma + \pi_{p,q-1,r+1}|\Gamma \\
&= \sum_{i=0}^l (m_i(p + 1, q, r) + m_i(p, q, r - 1) + m_i(p - 1, q + 1, r) + m_i(p, q - 1, r + 1)) \gamma_i. \\
\end{align*}
\]

By uniqueness,

\[
\sum_{j=0}^l m_j(p, q, r)a_{ij}^{(1)} = m_i(p + 1, q, r) + m_i(p, q, r - 1) + m_i(p - 1, q + 1, r) + m_i(p, q - 1, r + 1). \\
\]

**3 The series** \(P_\Gamma(t, u, w)\) **is a rational function**

This section is mainly devoted to the proof of Theorem 1.
3.1 A key-relation satisfied by the series $P_t(t, u, w)$

Proposition 5

Set

\[ J(t, u, w) := (1 - u^2)((1 + ut^2)(1 + uw^2) - tw(1 + u^2))I_n + twu(1 - u^2)A(2) - tu(1 + uw^2)(A(3) - uA(1)) - wu(1 + ut^2)(A(1) - uA(3)). \]

Then the series $P_t(t, u, w)$ satisfies the following relation

\[ J(t, u, w) v_{0,0,0} = \left\{ (1 - tA(1) + t^2A(2) - t^3A(3) + t^4) \left( 1 - wA(3) + w^2A(2) - w^3A(1) + w^4 \right) \right\} \left( 1 + u^2(1 - u^2)^2 - u(1 - u^2)^2A(2) + u^2(A(1) - uA(3))(A(3) - uA(1)) \right) P_t(t, u, w). \]

Proof:

- Set $x := P_t(t, u, w)$. Set also $v_{p,q,-1} := 0$, $v_{p,-1,r} := 0$ and $v_{-1,q,r} := 0$ for $(p, q, r) \in \mathbb{N}^3$, such that, according to the Clebsch-Gordan formula, the formulae of the preceding corollary are still true for $(p, q, r) \in \mathbb{N}^3$. So we have (by denoting $\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty}$ by $\sum_{pqr}$)

\[
(1 - wA(3) + w^2A(2) - w^3A(1) + w^4)x = (1 - tw + uw^2 - t^{-1}uw) \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} v_{p,q,0} t^p u^q + t^{-1}uw \sum_{q=0}^{\infty} v_{0,q,0} u^q. \tag{2}
\]

- In the same way (by denoting $\sum_{p=0}^{\infty} \sum_{q=0}^{\infty}$ by $\sum_{pq}$)

\[
(1 - tA(1) + t^2A(2) - t^3A(3) + t^4) \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} v_{p,0,q} t^p u^q = (1 - tu) \sum_{q=0}^{\infty} v_{0,q,0} u^q - tu \sum_{q=0}^{\infty} v_{0,q,1} u^q. \tag{3}
\]

Moreover, we have

\[
(1 - tA(1) + t^2A(2) - t^3A(3) + t^4) \sum_{q=0}^{\infty} v_{0,q,0} u^q = \sum_{q=0}^{\infty} v_{0,q,0} u^q - \sum_{q=0}^{\infty} (v_{1,q,0} + v_{0,q-1,1})tu^q + \sum_{q=0}^{\infty} (v_{0,q+1,0} + v_{0,q-1,0} + v_{1,q-1,1})t^2u^q - \sum_{q=0}^{\infty} (v_{0,q,1} + v_{1,q-1,0})t^3u^q + \sum_{q=0}^{\infty} v_{0,q,0} t^4u^q,
\]
By combining Equations (2), (3) and (4), we get

\[ (1 - tA^{(1)} + t^2 A^{(2)} - t^3 A^{(3)} + t^4) (1 - wA^{(3)} + w^2 A^{(2)} - w^3 A^{(3)} + w^4) x = \]

\[ (1 - tA^{(1)} + t^2 A^{(2)} - t^3 A^{(3)} + t^4) \left( (1 - tw + uw^2 - t^{-1}uw) \sum_{pq} v_{p,q,0} t^p u^q + t^{-1}uw \sum_{q=0}^{\infty} v_{0,q,0} u^q \right) \]

\[ = (1 - tw + uw^2 - t^{-1}uw) \left( (1 + t^2 u) \sum_{q=0}^{\infty} v_{0,q,0} u^q - tu \sum_{q=0}^{\infty} v_{0,q,1} u^q \right) \]

\[ + (1 + t^4 + t^2 u - t^3 u) \sum_{q=0}^{\infty} v_{0,q,0} u^q - twv_{0,0,0} - (1 + t^2 u)uw \sum_{q=0}^{\infty} v_{1,q,0} u^q \]

\[ - (u + t^2)uw \sum_{q=0}^{\infty} v_{0,q,1} u^q + tu^2 \sum_{q=0}^{\infty} v_{1,q,1} u^q, \]

hence

\[ (1 - tA^{(1)} + t^2 A^{(2)} - t^3 A^{(3)} + t^4) (1 - wA^{(3)} + w^2 A^{(2)} - w^3 A^{(3)} + w^4) x \]

\[ = (1 + ut^2)(1 + uw^2) \sum_{q=0}^{\infty} v_{0,q,0} u^q - tu(1 + uw^2) \sum_{q=0}^{\infty} v_{0,q,1} u^q \]

\[ - uw(1 + ut^2) \sum_{q=0}^{\infty} v_{1,q,0} u^q - twv_{0,0,0} + tu^2 \sum_{q=0}^{\infty} v_{1,q,1} u^q. \]

Besides, we have the two following equations

\[ A^{(1)} \sum_{q=0}^{\infty} v_{0,q,0} u^q = \sum_{q=0}^{\infty} v_{1,q,0} u^q + u \sum_{q=0}^{\infty} v_{0,q,1} u^q, \]

and

\[ A^{(3)} \sum_{q=0}^{\infty} v_{0,q,0} u^q = \sum_{q=0}^{\infty} v_{0,q,1} u^q + u \sum_{q=0}^{\infty} v_{1,q,0} u^q. \]

From these two equations, we deduce

\[ \sum_{q=0}^{\infty} v_{0,q,1} u^q = (1 - u^2)^{-1} A^{(3)} - u A^{(1)} \sum_{q=0}^{\infty} v_{0,q,0} u^q. \]

Now, we have

\[ A^{(1)} \sum_{q=0}^{\infty} v_{0,q,1} u^q = \sum_{q=0}^{\infty} v_{1,q,1} u^q + \sum_{q=0}^{\infty} v_{0,q,0} u^q + u \sum_{q=0}^{\infty} v_{0,q,2} u^q, \]

and

\[ A^{(3)} \sum_{q=0}^{\infty} v_{0,q,1} u^q = \sum_{q=0}^{\infty} v_{0,q,2} u^q + u^{-1} \sum_{q=0}^{\infty} v_{0,q,0} u^q + u \sum_{q=0}^{\infty} v_{1,q,1} u^q - w^{-1} v_{0,0,0}, \]
By using Equation (11), we may write Equation (5) as

\[ \sum_{q=0}^{\infty} v_{1,q,1} u^q = (1 - u^2)^{-1}(A^{(1)} - uA^{(3)}) \sum_{q=0}^{\infty} v_{0,q,1} u^q - (1 - u^2)^{-1} v_{0,0,0}. \]

So, according to Equation (8), we deduce

\[ \sum_{q=0}^{\infty} v_{1,q,1} u^q = (1 - u^2)^{-2}(A^{(1)} - uA^{(3)})(A^{(3)} - uA^{(1)}) \sum_{q=0}^{\infty} v_{0,q,0} u^q - (1 - u^2)^{-1} v_{0,0,0}. \]  \hspace{1cm} \text{(11)}

By using Equation (11), we may write Equation (5) as

\[
(1 - tA^{(1)} + t^2A^{(2)} - t^3A^{(3)} + t^4)(1 - wA^{(3)} + w^2A^{(2)} - w^3A^{(1)} + w^4)x
\]

\[
= \left( (1 + ut^2)(1 + uw^2) + tu^2 w(1 - u^2)^{-2}(A^{(1)} - uA^{(3)})(A^{(3)} - uA^{(1)}) \right) \sum_{q=0}^{\infty} v_{0,q,0} u^q
\]

\[-tu(1 + uw^2) \sum_{q=0}^{\infty} v_{0,q,1} u^q - wu(1 + ut^2) \sum_{q=0}^{\infty} v_{1,q,0} u^q - (tw + tu^2 w(1 - u^2)^{-1}) v_{0,0,0}. \]  \hspace{1cm} \text{(12)}

From Equations (6) and (7), we also deduce

\[ \sum_{q=0}^{\infty} v_{1,q,0} u^q = (1 - u^2)^{-1}(A^{(1)} - uA^{(3)}) \sum_{q=0}^{\infty} v_{0,q,0} u^q. \]  \hspace{1cm} \text{(13)}

So, by using Equations (8) and (13), we obtain

\[
(1 - tA^{(1)} + t^2A^{(2)} - t^3A^{(3)} + t^4)(1 - wA^{(3)} + w^2A^{(2)} - w^3A^{(1)} + w^4)x
\]

\[
= \left( (1 + ut^2)(1 + uw^2) - tu(1 + uw^2)(1 - u^2)^{-1}(A^{(3)} - uA^{(1)})
\]

\[-wu(1 + ut^2)(1 - u^2)^{-1}(A^{(1)} - uA^{(3)}) + tu^2 w(1 - u^2)^{-2}(A^{(1)} - uA^{(3)})(A^{(3)} - uA^{(1)}) \right) \sum_{q=0}^{\infty} v_{0,q,0} u^q
\]

\[-(tw + tu^2 w(1 - u^2)^{-1}) v_{0,0,0}. \]  \hspace{1cm} \text{(14)}

i.e., by multiplying (14) by \((1 - u^2)^2\),

\[
(1 - u^2)^2(1 - tA^{(1)} + t^2A^{(2)} - t^3A^{(3)} + t^4)(1 - wA^{(3)} + w^2A^{(2)} - w^3A^{(1)} + w^4)x
\]

\[
= \left( (1 - u^2)^2(1 + ut^2)(1 + uw^2) - tu(1 + uw^2)(1 - u^2)(A^{(3)} - uA^{(1)})
\]

\[-wu(1 + ut^2)(1 - u^2)(A^{(1)} - uA^{(3)}) + tu^2 w(A^{(1)} - uA^{(3)})(A^{(3)} - uA^{(1)}) \right) \sum_{q=0}^{\infty} v_{0,q,0} u^q
\]

\[-(tw(1 - u^2)^2 + tw^2 w(1 - u^2)^{-1}) v_{0,0,0}. \]  \hspace{1cm} \text{(15)}

\[ \text{Consider now the following equation} \]

\[ A^{(2)} \sum_{q=0}^{\infty} v_{0,q,0} u^q = u^{-1} \sum_{q=0}^{\infty} v_{0,q,0} u^q + u \sum_{q=0}^{\infty} v_{0,q,0} u^q + u \sum_{q=0}^{\infty} v_{1,q,1} u^q - u^{-1} v_{0,0,0}. \]  \hspace{1cm} \text{(16)}

Then, according to Equation (11), we have

\[ A^{(2)} \sum_{q=0}^{\infty} v_{0,q,0} u^q = u^{-1} \sum_{q=0}^{\infty} v_{0,q,0} u^q + u \sum_{q=0}^{\infty} v_{0,q,0} u^q
\]

\[ + u(1 - u^2)^{-2}(A^{(1)} - uA^{(3)})(A^{(3)} - uA^{(1)}) \sum_{q=0}^{\infty} v_{0,q,0} u^q - u(1 - u^2)^{-1} v_{0,0,0} - u^{-1} v_{0,0,0}. \]
Now, by using Equations (15) and (18), we get
\[
(A^{(2)} - u^{-1} - u - u(1 - u^2)^{-2}(A^{(1)} - uA^{(3)})(A^{(3)} - uA^{(1)})) \sum_{q=0}^{\infty} v_{0,q,0} u^q = -(u(1 - u^2)^{-1} + u^{-1})v_{0,0,0}.
\]
This last equation reads
\[
(-u(1 - u^2)^2A^{(2)} + (1 + u^2)(1 - u^2)^2 + u^2(A^{(1)} - uA^{(3)})(A^{(3)} - uA^{(1)})) \sum_{q=0}^{\infty} v_{0,q,0} u^q = (1 - u^2)v_{0,0,0}.
\]
Now, by using Equations (15) and (18), we get
\[
(1 - u^2)^2(1 - tA^{(1)} + t^2A^{(2)} - t^3A^{(3)} + t^4)(1 - wA^{(3)} + w^2A^{(2)} - w^3A^{(1)} + w^4)\left(-u(1 - u^2)^2A^{(2)} + (1 + u^2)(1 - u^2)^2 + u^2(A^{(1)} - uA^{(3)})(A^{(3)} - uA^{(1)})\right)x
\]
\[
= -tw(1 - u^2)\left(-u(1 - u^2)^2A^{(2)} + (1 + u^2)(1 - u^2)^2 + u^2(A^{(1)} - uA^{(3)})(A^{(3)} - uA^{(1)})\right)v_{0,0,0}
\]
\[
\left((1 - u^2)^2(1 + u^2)(1 + uu^2) - tu(1 + uu^2)(1 - u^2)(A^{(3)} - uA^{(1)}) -wu(1 + u^2)(1 - u^2)(A^{(1)} - uA^{(3)}) + tu^2w(A^{(1)} - uA^{(3)})(A^{(3)} - uA^{(1)})\right)(1 - u^2)v_{0,0,0},
\]
i.e., after simplification by \((1 - u^2)^2\),
\[
(1 - tA^{(1)} + t^2A^{(2)} - t^3A^{(3)} + t^4)(1 - wA^{(3)} + w^2A^{(2)} - w^3A^{(1)} + w^4)\left((1 + u^2)(1 - u^2)^2 - u(1 - u^2)^2A^{(2)} + u^2(A^{(1)} - uA^{(3)})(A^{(3)} - uA^{(1)})\right)x
\]
\[
= \left((1 - u^2)((1 + uu^2)1 + uu^2) - tw(1 + uu^2)\right)\left(-tu(1 + uu^2)(A^{(3)} - uA^{(1)}) -wu(1 + uu^2)(A^{(1)} - uA^{(3)})\right)v_{0,0,0}.
\]
The proposition is proved. ■

3.2 An inversion formula

In order to inverse the relation obtained in Proposition 5 and get an explicit expression for \(P_T(t, u)\), we need the rational function \(f\) defined by
\[
f: \mathbb{C}^3 \quad \mapsto \quad \mathbb{C}(t, u, w) \quad \mapsto \quad (1 - td_1 + t^2d_2 - t^3d_3 + t^4)^{-1}(1 - wv_1 + w^2v_2 - w^3v_3 + w^4)^{-1}\left((1 + u^2)(1 - u^2)^2 - u(1 - u^2)^2A^{(2)} + u^2(A^{(1)} - uA^{(3)})(A^{(3)} - uA^{(1)})\right)^{-1}.
\]
According to Proposition 5, we may write
\[
J(t, u, w) v_{0,0,0} = T_T(1 - tA^{(1)} + t^2A^{(2)} - t^3A^{(3)} + t^4)(1 - wA^{(3)} + w^2A^{(2)} - w^3A^{(1)} + w^4)\left((1 + u^2)(1 - u^2)^2 - u(1 - u^2)^2A^{(2)} + u^2(A^{(1)} - uA^{(3)})(A^{(3)} - uA^{(1)})\right)T_T^{-1}P_T(t, u, w).
\]
We deduce that
\[
P_T(t, u, w) = T_T(1 - tA^{(1)} + t^2A^{(2)} - t^3A^{(3)} + t^4)(1 - wA^{(3)} + w^2A^{(2)} - w^3A^{(1)} + w^4)\left((1 + u^2)(1 - u^2)^2 - u(1 - u^2)^2A^{(2)} + u^2(A^{(1)} - uA^{(3)})(A^{(3)} - uA^{(1)})\right)T_T^{-1}P_T(t, u, w).
\]
where \(\Delta(t, u, w) \in \mathbf{M}_{q+1}\mathbb{C}(t, u, w)\) is the diagonal matrix defined by
\[
\Delta(t, u, w)_{i,j} = f(\Lambda^{(1)}_{i,j}, \Lambda^{(2)}_{i,j}, \Lambda^{(3)}_{i,j}) = f\left(\frac{1}{2}(\chi(g_j)^2 - \chi(g_j^2)), \frac{1}{2}(\chi(g_j)^2 - \chi(g_j^2))\right).
\]
This last formula proves Theorem 1.
Remark 6
The Poincaré series $\hat{P}_\Gamma(t)$ of the algebra of invariants $\mathbb{C}[z_1, z_2, z_3, z_4]^\Gamma$ is given by
$$\hat{P}_\Gamma(t) = P_\Gamma(t, 0, 0)_0 = P_\Gamma(0, 0, t)_0.$$ 

3.3 Remark for $\text{SL}_n \mathbb{C}$

In this section, we consider an integer $n \geq 2$ and a subgroup $\Gamma$ of $\text{SL}_n \mathbb{C}$. As in paragraph 1, let $\{\gamma_0, \ldots, \gamma_l\}$ be the set of equivalence classes of irreducible finite dimensional complex representations of $\Gamma$, where $\gamma_0$ is the trivial representation. The character associated to $\gamma_j$ is denoted by $\chi_j$.

Consider $\gamma : \Gamma \to \text{SL}_n \mathbb{C}$ the natural $n$-dimensional representation, and $\chi$ its character. By complete reducibility we get the decomposition $\gamma_j \otimes \gamma = \bigoplus_{i=0}^{l} a_{j, i}^{(1)} \gamma_i$ for every $j \in [0, l]$, and we set $A^{(1)} := (a_{j, i}^{(1)})_{(j, i) \in [0, l]} \in M_{l+1} \mathbb{N}$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{sl}_n \mathbb{C}$ and let $\varpi_1, \ldots, \varpi_{n-1}$ be the corresponding fundamental weights, and $V(p_1 \varpi_1 + \cdots + p_{n-1} \varpi_{n-1})$ the simple $\mathfrak{sl}_n \mathbb{C}$-module of highest weight $p_1 \varpi_1 + \cdots + p_{n-1} \varpi_{n-1}$ with $p := (p_1, \ldots, p_{n-1}) \in \mathbb{N}^{n-1}$. Then we get an irreducible representation $\pi_p : \text{SL}_n \mathbb{C} \to \text{GL}(V(p_1 \varpi_1 + \cdots + p_{n-1} \varpi_{n-1}))$. The restriction of $\pi_p$ to the subgroup $\Gamma$ is a representation of $\Gamma$, and by complete reducibility, we get the decomposition $\pi_p|_\Gamma = \bigoplus_{i=0}^{l} m_i(p) \gamma_i$, where the $m_i(p)$’s are non negative integers. Let $\mathcal{E} := (e_0, \ldots, e_l)$ be the canonical basis of $\mathbb{C}^{l+1}$, and
$$v_p := \sum_{i=0}^{l} m_i(p)e_i \in \mathbb{C}^{l+1}.$$ 

As $\gamma_0$ is the trivial representation, we have $v_0 = e_0$. Let us consider the vector (with elements of $\mathbb{C}[t_1, \ldots, t_{n-1}] = \mathbb{C}[t]$ as coefficients)
$$P_\Gamma(t) := \sum_{p \in \mathbb{N}^{n-1}} v_p t^p \in (\mathbb{C}[t])^{l+1},$$
and denote by $P_\Gamma(t)_j$ its $j$-th coordinate in the basis $\mathcal{E}$.

Given the results from Kostant ([Kos85] and [Kos06]) for $\text{SL}_2 \mathbb{C}$ and our results ([BP09]) about $\text{SL}_3 \mathbb{C}$, we then formulate the following conjecture:

**Conjecture 7**

The coefficients of the vector $P_\Gamma(t)$ are rational fractions in $t$, i.e. the formal power series $P_\Gamma(t)_i$ are rational functions
$$P_\Gamma(t)_i := \frac{N_\Gamma(t)_i}{D_\Gamma(t)}, \ i \in [0, l],$$
where the $N_\Gamma(t)_i$’s and $D_\Gamma(t)$ are elements of $\mathbb{Q}[t]$.

4 An example of explicit computation

The classification of finite subgroups of $\text{SL}_4 \mathbb{C}$ is given in [HH01]. It consists in infinite series and 30 exceptional groups (types $I, II, \ldots, XXX$). We give here an explicit computation of $P_\Gamma(t, u, w)$ for one of these exceptional groups. Consider the matrices
$$F_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & j & 0 \\ 0 & 0 & 0 & j^2 \end{pmatrix}, \ F_2' = \frac{1}{3} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 2 & 2 \\ 0 & 2 & -1 & 2 \\ 0 & 2 & 2 & -1 \end{pmatrix}, \ F_3' = \frac{1}{4} \begin{pmatrix} -1 & \sqrt{15} & 0 & 0 \\ \sqrt{15} & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \end{pmatrix},$$
and the subgroup $\Gamma = \langle F_1, F_2, F_3 \rangle$ of $\text{SL}_4 \mathbb{C}$ (type $II$ in [HH01]).

Here $l = 4$,

$$A^{(1)} = A^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 1 \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & 2 & 2 \\ 0 & 2 & 2 & 2 & 2 \end{pmatrix}.$$ 

$\text{rank}(A^{(1)}) = \text{rank}(A^{(2)}) = 4$, and the eigenvalues of $A^{(1)}$, $A^{(2)}$, $A^{(3)}$ are

$$\Theta^{(1)} = \Theta^{(3)} = (4, 0, -1, 1, -1), \quad \Theta^{(2)} = (6, -2, 1, 0, 1),$$ 

$p = 4$, and $\tau_0 = s_0 s_1, \tau_1 = s_2, \tau_2 = s_3, \tau_3 = s_4$.

According to formula 21, we get

$$D_T(t, u, w) = (w - 1)^4 (u + 1)^3 (u - 1)^5 (t - 1)^7 (t^2 + t + 1) (w^4 + w^3 + w^2 + w + 1) (w + 1)^2 (w^2 + w^3 + u^2 + u + 1) (w^2 + u + 1)^2 (t^2 + t + 1) (t^4 + t^3 + t^2 + t + 1) (t + 1)^2$$

$$= (u - 1) (u + 1) (u^2 + u + 1) \tilde{D}_T(t) \tilde{D}_T(u) \tilde{D}_T(w),$$

with $\tilde{D}_T(t) = (t - 1)^4 (t + 1)^2 (t^2 + t + 1) (t^4 + t^3 + t^2 + t + 1)$. Moreover,

$$\tilde{P}_T(t) = \frac{t^8 - t^6 + t^4 - t^2 + 1}{t^{12} - 2 t^{10} - t^9 + t^8 + t^7 + t^6 + t^5 + t^3 + 2 t^2 + 1}.$$

Because of the too big size of the numerators $N_T(t, u, w)$’s, only the denominator is given in the text: all the numerators may be found on the web (http://math.univ-lyon1.fr/~butin/).

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