FRAMES BY ORBITS OF TWO OPERATORS THAT COMMUTE

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Abstract. Frames formed by orbits of vectors through the iteration of a bounded operator have recently attracted considerable attention, in particular due to its applications to dynamical sampling. In this article, we consider two commuting bounded operators acting on some separable Hilbert space $H$. We completely characterize operators $T$ and $L$ with $TL = LT$ and sets $\Phi \subset H$ such that the collection $\{T^kL^j\phi : k \in \mathbb{Z}, j \in J, \phi \in \Phi\}$ forms a frame of $H$. This is done in terms of model subspaces of the space of square integrable functions defined on the torus and having values in some Hardy space with multiplicity. The operators acting on these models are the bilateral shift and the compression of the unilateral shift (acting pointwisely). This context includes the case when the Hilbert space $H$ is a subspace of $L^2(\mathbb{R})$, invariant under translations along the integers, where the operator $T$ is the translation by one and $L$ is a shift-preserving operator.

1. Introduction

Let $H$ be a separable Hilbert space, and $\{g_i\}_{i \in I}$ a countable set in $H$. In sampling theory, every function $f$ belonging to $H$ needs to be recovered from its samples $\{\langle f, g_i \rangle\}_{i \in I}$. If we impose a stability condition on the reconstruction, this problem is equivalent to the fact that the set $\{g_i\}_{i \in I}$ is a frame of $H$.

In dynamical sampling, it is assumed that the signal evolves in time under an evolution operator $L$ defined on $H$ and that we are able to sample the functions $L^tf$ for $t = 0, 1, 2, ..., \text{where } L^tf$ represents the evolved signal at time $t$. Thus, if the samples $\{\langle f, g_i \rangle\}_{i \in I}$ are insufficient to reconstruct every function $f$ in the space, we hope to compensate the sparse data by sampling the evolved signal as well, that is, considering space-time samples.

In other words, the problem consists in trying to recover any function $f$ from the samples $\{\langle L^tf, g_i \rangle\}_{i \in I, t = 0, 1, 2, ...}$ in a stable way, which is equivalent to the fact that $\{(L^*)^tg_i\}_{i, t}$ is a frame of $H$. This motivates the study of the structure of orbits of operators in Hilbert spaces, i.e. to give conditions on a Hilbert space $H$, a bounded operator $L$ defined on $H$ and a set of functions $\Phi \subset H$ in order that the system $\{L^j\varphi : j \in J, \varphi \in \Phi\}$ forms a frame of $H$, for some index set $J$.

The dynamical sampling problem was first presented in [6, 7], taking inspiration from the research of Lu, Vetterli, and their colleagues in [25, 27, 32]. In [27], the
authors analyzed a linear diffusion (heat) equation where the initial state evolves over time. They proposed a trade-off between spatial and temporal sampling, with the goal of reconstructing the initial state using only a few spatial sensors (samples) and compensating with temporal samples. This can be considered as an inverse problem in differential equations and has potential applications in various fields, such as weather forecasting, ecology, and medical imaging between others.

Recently, there has been a lot of interest in this problem. See [4, 5, 6, 7, 10, 11, 15, 16, 19, 30] for foundations and theoretical developments, and [8, 9, 12, 18, 26, 28, 33] for applications.

In [3], dynamical sampling was considered for shift-invariant spaces: Let $V \subset L^2(\mathbb{R})$ be a closed subspace, $T$ the translation operator by one on $V$, i.e. $Tf(x) = f(x - 1)$, and $L : V \to V$ a bounded operator. We assume that $TV = V$ (that is, $V$ is shift invariant) and that $TL = LT$ (that is, $L$ is shift preserving). The main question here is to characterize $L$ and $\Phi \subset V$ such that $\{L^j \varphi : j \in J, \varphi \in \Phi \}$ is a set of frame generators by integer translations of $V$, which means that $\{T^k (L^j \varphi) : k \in \mathbb{Z}, j \in J, \varphi \in \Phi \}$ forms a frame of $V$.

For the case where $L$ is normal and $V$ is finitely generated, necessary and sufficient conditions were given in [3]. The conditions were obtained there defining a special diagonalization for $L$, carefully developed in [1], and then applying a result for the finite dimensional dynamical sampling problem. The approach works for $L^2(\mathbb{R}^d)$ and even when replacing $\mathbb{R}^d$ with an LCA group.

In this paper we study a generalization of this case. We consider the set of orbits $\mathcal{F} = \{T^k L^j w_i : k \in \mathbb{Z}, j \in J, i \in I\}$ where $\mathcal{H}$ is an abstract separable Hilbert space, $T$ and $L$ are commuting bounded operators acting on $\mathcal{H}$, and $\{w_i\}_{i \in I}$ is an at most countable set of vectors in $\mathcal{H}$. We do not assume that $T$ is unitary or that $L$ is normal. We do assume, however, that $T$ is invertible. Then, we ask whether the set $\mathcal{F}$ forms a frame of $\mathcal{H}$.

With this purpose, we define an equivalence relation between tuples of the form $(\mathcal{H}, T, L, \{w_i\}_{i \in I})$. We say that two tuples are similar if the operators from one tuple are similar to the operators in the other via a bounded isomorphism between the Hilbert spaces involved. The isomorphism also maps the set of vectors of one into the set of vector of the other. See Section 3.1 for a precise definition.

Our main result asserts that every tuple which generates a frame is similar to a tuple of a particular class, the class of basic tuples. See Definition 3.4. These basic tuples are simpler to study because of their structure and they provide a rich platform to prove properties of these systems. In this way, properties preserved by similarity can be obtained on a basic tuple and then the results can be translated to the abstract setting of the general Hilbert space.

The Hilbert space component of a basic tuple is a closed subspace of $L^2(\mathbb{T}, H^2_\mathcal{K})$ where $\mathbb{T}$ denote the unit circle, $\mathcal{K}$ a separable Hilbert space, and $H^2_\mathcal{K}$ the Hardy space of $\mathcal{K}$-valued functions defined on the unit circle. See Section 2.1.

The main idea comes from the observation that the kernel of the synthesis operator of the system $\mathcal{F}$ can be seen as a closed subspace $\mathcal{M}$ of $L^2(\mathbb{T}, H^2_\mathcal{K})$ that turns out to be reducing for $U$ (the bilateral shift) and also invariant for $\hat{S}$, an operator that acts pointwisely as the unilateral shift in $H^2_\mathcal{K}$. We then provide a functional representation for all the systems $\mathcal{F}$ that form a frame, in terms of the model space
\( N_M = L^2(T, H^2) \ominus M \), with the bilateral shift and the compression of \( \hat{S} \) on \( N_M \) as the commuting operators.

We want to remark that our approach was used previously in [19] for the case of just one operator \( L \) acting on \( H \) on one function, and in [17] for several functions.

Next we consider the following question: assume that \( H, T \) and \( L \) are given and satisfy the hypothesis of boundedness and commutativity. Which are all vectors \( w \in H \) such that the iterations form a frame? We were able to answer this question completely and the characterization is similar to the one obtained before in [19] for the iteration of only one operator. However, in our case, the proof is much more involved and strongly uses the Helson’s characterization of reducing subspaces for the bilateral shift. We finally show that this characterization does not work when we consider iterations of multiple generators.

All the results are proved for the two cases: first when we consider forward iterations of the operator \( L \), and second, when we allow negative powers of \( L \), in which case, \( L \) is invertible. The difference between these cases is that the corresponding basic tuples live in different vector-valued \( L^2 \)-spaces.

The paper is organized as follows. We first introduce in Section 2 the notation and the setting that we need in the paper. In Section 3 we study the properties of basic tuples and prove the characterization of tuples that give frames, splitting our analysis into two cases: \emph{unilateral tuples}, that is, when we consider forward iterations of the operator \( L \), and \emph{bilateral tuples} when we allow integer iterations of \( L \). Finally, in Section 4 we consider the problem of characterization of all vectors that give frames by iterations.

2. Preliminaries

Let us begin by introducing some notation and definitions. All the Hilbert spaces considered in this paper are separable. We will use the letters \( K \) and \( H \) to denote complex Hilbert spaces. As usual, \( \mathcal{B}(H, K) \) will denote the set of linear bounded operators from \( H \) into \( K \) and \( \mathcal{B}(H) := \mathcal{B}(H, H) \). We will write \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \) and \( \mathbb{T} \) for the complex unit circle. For a closed subspace \( N \) of \( H \), we will write \( N^\perp \) to denote the orthogonal complement of \( N \) in \( H \) and \( P_N \) to denote the orthogonal projection of \( H \) onto \( N \).

Let \( \mathcal{M} \) be a closed subspace of a Hilbert space \( H \) and \( A \in \mathcal{B}(H) \). The subspace \( \mathcal{M} \) is said to be \emph{invariant} for \( A \) (or \( A \)-invariant) if \( A(\mathcal{M}) \subseteq \mathcal{M} \). Furthermore, \( \mathcal{M} \) is called \emph{reducing} for \( A \) (or \( A \)-reducing) if \( \mathcal{M} \) and \( \mathcal{M}^\perp \) are \( A \)-invariant, which is equivalent to say that \( \mathcal{M} \) is invariant for \( A \) and \( A^* \). It is easy to check that \( \mathcal{M} \) is \( A \)-invariant if and only if \( P_M A P_M = A P_M \) and \( \mathcal{M} \) is \( A \)-reducing if and only if \( P_M A = A P_M \).

Given \( T \in \mathcal{B}(H) \) and a closed subspace \( \mathcal{M} \) of \( H \), the operator defined by \( P_M T|_\mathcal{M} \) is called the \emph{compression} of \( T \) to the subspace \( \mathcal{M} \).

Particular attention has been put over the years in the study of invariant and reducing subspaces under the action of sequential bilateral and unilateral shift operators, that is

\[
U : \{\ldots, a_{-1}, (a_0), a_1, \ldots\} \mapsto \{\ldots, a_{-2}, (a_{-1}), a_0, \ldots\}, \quad \{a_i\}_{i \in \mathbb{Z}} \subseteq \ell^2(\mathbb{Z}),
\]

\[
S : \{a_0, a_1, a_2, \ldots\} \mapsto \{0, a_0, a_1, a_2 \ldots\}, \quad \{a_i\}_{i \in \mathbb{N}} \subseteq \ell^2(\mathbb{N}).
\]
Via the Fourier transform, these can be naturally represented as operators acting over the functional spaces $L^2(\mathbb{T})$ and the Hardy space $H^2$, i.e., the subspace of $L^2(\mathbb{T})$ consisting of all $f \in L^2(\mathbb{T})$ whose Fourier coefficients vanish for $n < 0$, i.e.,

$$H^2 := \left\{ f \in L^2(\mathbb{T}) : \int_{\mathbb{T}} f(z)z^{-n}dz = 0 \text{ for } n < 0 \right\}.$$

Similarly, shift operators can be defined with multiplicity higher than 1. This means that instead of shifting a sequence of complex numbers, the operators shift a sequence with values in a Hilbert space $\mathcal{K}$, where $\dim(\mathcal{K})$ is the multiplicity. The representation of these operators over functional spaces requires the introduction of vector-valued functions.

### 2.1. Vector-valued functions and Hardy spaces with multiplicity.

Let $(\Omega, \mu)$ be a $\sigma$-finite measure space such that $L^2(\Omega) := L^2(\Omega, \mu)$ is a separable Hilbert space. For our purposes along this paper, we will be interested in the cases when $\Omega = \mathbb{T}$ or $\mathbb{R}^d$ and $\mu$ is the normalized Lebesgue measure ($\mu(\mathbb{T}) = 1$ or $\mu(\mathbb{R}^d) = 1$, respectively).

Let $\mathcal{K}$ be a separable Hilbert space. A vector-valued (or $\mathcal{K}$-valued) function $f : \Omega \to \mathcal{K}$ is said to be measurable if for each $x \in \mathcal{K}$, the complex-valued function $\omega \mapsto (f(\omega), x)_{\mathcal{K}}$ is measurable on $\Omega$. We denote by $L^2(\Omega, \mathcal{K})$ the space of all measurable $\mathcal{K}$-valued functions $f$ such that $\int_{\Omega} \|f(\omega)\|_{\mathcal{K}}^2 d\omega < \infty$, which is a Hilbert space equipped with the inner product

$$(f, g) = \int_{\Omega} \langle f(\omega), g(\omega) \rangle_{\mathcal{K}} d\omega, \quad f, g \in L^2(\Omega, \mathcal{K}).$$

Notice that the space $\mathcal{K}$ is naturally embedded in $L^2(\Omega, \mathcal{K})$ when identified with the subset of constant vector-valued functions of $L^2(\Omega, \mathcal{K})$. That is, given $x \in \mathcal{K}$ define $\tilde{x} : \Omega \to \mathcal{K}$ as the function in $L^2(\Omega, \mathcal{K})$ which is constantly $x$. For the sake of simplicity, we will write $x$ instead of $\tilde{x}$, when it is clear from the context that we mean the constant function. Given an orthonormal basis $\mathcal{B} = \{\varepsilon_i\}_{i \in I}$ of $\mathcal{K}$ and a vector-valued function $f \in L^2(\mathbb{T}, \mathcal{K})$, we can write

$$f(\lambda) = \sum_{i \in I} \langle f(\cdot), \varepsilon_i \rangle_{\mathcal{K}} \varepsilon_i, \quad \text{a.e. } \lambda \in \mathbb{T}.$$

We call $f_i := \langle f(\cdot), \varepsilon_i \rangle_{\mathcal{K}}$ the $i$th coordinate function of $f$ with respect to $\mathcal{B}$. It is easily seen that $f_i$ belongs to $L^2(\mathbb{T})$ for every $i \in I$.

The Hardy space with multiplicity $\alpha = \dim(\mathcal{K})$ is defined as the closed subspace of $L^2(\mathbb{T}, \mathcal{K})$ consisting of all functions $f \in L^2(\mathbb{T}, \mathcal{K})$ whose coordinate functions $f_i$ belong to the Hardy space $H^2$. Note that this is equivalent to say that $H^2_{\mathcal{K}}$ is the set of all functions $f \in L^2(\mathbb{T}, \mathcal{K})$ such that $\langle f(\cdot), u \rangle_{\mathcal{K}} \in H^2$ for every $u \in \mathcal{K}$. We will denote it by $H^2_{\mathcal{K}} := H^2(\mathbb{T}, \mathcal{K})$.

We are ready to formally define the shift operators with multiplicity acting over these spaces.

**Definition 2.1.** The operator $U : L^2(\mathbb{T}, \mathcal{K}) \to L^2(\mathbb{T}, \mathcal{K})$ defined by

$$(Uf)(\lambda) = \lambda f(\lambda), \quad \text{a.e. } \lambda \in \mathbb{T}, f \in L^2(\mathbb{T}, \mathcal{K}),$$

is called the bilateral shift on $L^2(\mathbb{T}, \mathcal{K})$ with multiplicity $\alpha = \dim(\mathcal{K})$.

**Definition 2.2.** The operator $S : H^2_{\mathcal{K}} \to H^2_{\mathcal{K}}$ given by the restriction of $U$ to $H^2_{\mathcal{K}}$ is called the unilateral shift on $H^2_{\mathcal{K}}$ with multiplicity $\alpha = \dim(\mathcal{K})$. 
The bilateral shift \( U \) is unitary and its adjoint operator is given by \( (U^* f)(\lambda) = f(\lambda) \), for a.e. \( \lambda \in \mathbb{T} \) and \( f \in L^2(\mathbb{T}, \mathcal{K}) \). Moreover, since \( H^2_{\mathcal{K}} \) is invariant under \( U \), then the operator \( S \) is an isometry.

There is a simple way to construct an orthonormal basis of \( L^2(\mathbb{T}, \mathcal{K}) \) or \( H^2_{\mathcal{K}} \) through iterations of the shift operators over an orthonormal basis of \( \mathcal{K} \) that we will sketch out now. Let \( \mathcal{B} = \{ \varepsilon_i \}_{i \in I} \) be an orthonormal basis of \( \mathcal{K} \). It is easy to see that \( \{ U^k \varepsilon_i : k \in \mathbb{Z}, i \in I \} \) is an orthonormal basis of \( L^2(\mathbb{T}, \mathcal{K}) \). Indeed, the system is orthogonal since for every \( k, k' \in \mathbb{Z} \) and \( i, i' \in I \)

\[
\langle U^k \varepsilon_i, U^{k'} \varepsilon_{i'} \rangle = \int_{\mathbb{T}} \langle \lambda^k \varepsilon_i, \lambda^{k'} \varepsilon_{i'} \rangle_{\mathcal{K}} d\lambda = \langle \varepsilon_i, \varepsilon_{i'} \rangle_{\mathcal{K}} \int_{\mathbb{T}} \lambda^{k-k'} d\lambda = \delta_{i,i'} \delta_{k,k'}.
\]

For the completeness, note that if \( f \in L^2(\mathbb{T}, \mathcal{K}) \) and \( f \) is orthogonal to each element in the basis \( \mathcal{B} \), we have

\[
0 = \langle f, U^k \varepsilon_i \rangle = \int_{\mathbb{T}} \langle f(\lambda), \varepsilon_i \rangle_{\mathcal{K}} \lambda^{-k} d\lambda, \quad \forall k \in \mathbb{Z}, \forall i \in I.
\]

Thus, the coordinate functions of \( f \) respect to the basis \( \mathcal{B} \), \( f_i(\lambda) = \langle f(\lambda), \varepsilon_i \rangle_{\mathcal{K}} \), are zero for a.e. \( \lambda \in \mathbb{T} \), implying that \( f = 0 \).

Analogously, we can see that the system \( \{ S^j \varepsilon_i : j \in \mathbb{N}_0, i \in I \} \) is an orthonormal basis of \( H^2_{\mathcal{K}} \).

### 3. Systems of iterations

Let \( \mathcal{H} \) be a separable Hilbert space, \( T, L \in \mathcal{B}(\mathcal{H}) \) such that \( T \) is invertible and \( TL = LT \), and \( \{ w_i \}_{i \in I} \subset \mathcal{H} \) an at most countable collection of vectors. In this section, we provide a characterization of all systems of iterations \( \{ T^k L^j w_i : k \in \mathbb{Z}, j \in J, i \in I \} \) that form a frame, Parseval frame or Riesz basis of \( \mathcal{H} \). We will see that these systems are associated to a particular class of systems of iterations called basic, which have much more structure and consist of two shift operators acting on a vector-valued \( L^2 \)-subspace.

For this purpose, let us consider the family of all tuples \( (\mathcal{H}, T, L, \{ w_i \}_{i \in I}) \) with \( \mathcal{H}, T, L \) and \( \{ w_i \}_{i \in I} \) as above. We will say that two tuples \( (\mathcal{H}_1, T_1, L_1, \{ v_i \}_{i \in I}) \) and \( (\mathcal{H}_2, T_2, L_2, \{ w_i \}_{i \in I}) \) are similar if there exists a bounded isomorphism \( C : \mathcal{H}_1 \to \mathcal{H}_2 \) that satisfies \( C(v_i) = w_i \) for every \( i \in I \), and has the following intertwining properties:

\[
T_2 C = C T_1 \quad \text{and} \quad L_2 C = C L_1.
\]

It can be easily seen that the similarity is an equivalence relation. Moreover, two similar tuples are said to be unitarily equivalent if the associate isomorphism \( C \) is unitary.

Furthermore, we will say that a tuple \( (\mathcal{H}, T, L, \{ w_i \}_{i \in I}) \) is a frame-tuple (Parseval-tuple or Riesz-tuple) if the collection

\[
\{ T^k L^j w_i : k \in \mathbb{Z}, j \in J, i \in I \}
\]

forms a frame (a Parseval frame or a Riesz basis, respectively) of \( \mathcal{H} \). In this case, the vectors \( \{ w_i \}_{i \in I} \) will be called generators. The index set \( J \) will be \( \mathbb{N}_0 \) or \( \mathbb{Z} \).

When the iterations of \( L \) are taken over \( \mathbb{N}_0 \), we call the tuple \( (\mathcal{H}, T, L, \{ w_i \}_{i \in I}) \) a unilateral tuple. Observe that when the iterations are taken over \( \mathbb{Z} \), the operator \( L \) must be invertible. In that case we call \( (\mathcal{H}, T, L, \{ w_i \}_{i \in I}) \) a bilateral tuple.
The following two lemmas hold regardless of the type of tuple (unilateral or bilateral).

**Lemma 3.1.** The similarity relation between tuples preserves the frame property.

*Proof.* Assume that the tuples \((H_1, T_1, L_1, \{v_i\}_{i \in I}) \) and \((H_2, T_2, L_2, \{w_i\}_{i \in I})\) are similar via the isomorphism \(C \in \mathcal{B}(H_1, H_2)\). Let \(g \in H_1\). It is sufficient to observe that

\[
\sum_{k,j,i} |\langle g, T_2^j L_2^k v_i \rangle|^2 = \sum_{k,j,i} |\langle g, (CT_1^k C^{-1})(CL_1^j C^{-1})Cv_i \rangle|^2 = \sum_{k,j,i} |\langle C^* g, T_1^j L_1^k v_i \rangle|^2
\]

where the sum is indexed over \(k \in \mathbb{Z}, j \in J\), where \(J = \mathbb{N}_0 \) or \(\mathbb{Z}\) depending on the case (unilateral or bilateral), and \(i \in I\). Since \(C\) is a bounded isomorphism, \((H_1, T_1, L_1, \{v_i\}_{i \in I})\) is a frame tuple if and only if so is \((H_2, T_2, L_2, \{w_i\}_{i \in I})\). \(\square\)

**Lemma 3.2.** If two Parseval-tuples are similar, then they are unitarily equivalent.

*Proof.* Let \((H_1, T_1, L_1, \{v_i\}_{i \in I})\) and \((H_2, T_2, L_2, \{w_i\}_{i \in I})\) be similar Parseval-tuples via the isomorphism \(C \in \mathcal{B}(H_1, H_2)\). Then, by the intertwining properties of \(C\), we have that \(C(T_1^k L_1^j v_i) = T_2^k L_2^j w_i\), for every \(k \in \mathbb{Z}, j \in J\), where \(J = \mathbb{N}_0 \) or \(\mathbb{Z}\), and \(i \in I\). In particular, \(C\) sends a Parseval frame into another Parseval frame, thus, it must be unitary. To see this, assume that \(C \in \mathcal{B}(H_1, H_2)\) is an isomorphism and \(C(f_k) = g_k\), where \(\{f_k\}_k\) and \(\{g_k\}_k\) are Parseval frames of the Hilbert spaces \(H_1\) and \(H_2\), respectively. Set \(\tilde{C} := C^{-1}\). Then we have:

\[
\|f\|^2 = \sum_k |\langle f, f_k \rangle|^2 = \sum_k |\langle f, \tilde{C} C f_k \rangle|^2 = \sum_k |\langle \tilde{C}^* f, g_k \rangle|^2 = \|\tilde{C}^* f\|^2.
\]

Thus, \(\tilde{C}^*\) is unitary, and therefore so is \(C\). This completes the proof. \(\square\)

Our goal is to identify all frame-tuples, Parseval-tuples and Riesz-tuples, up to similarity. Let us begin by presenting a particular class of frame-tuples which will be crucial for our purpose.

### 3.1. Basic tuples.

In this subsection we define two types of tuples, the *unilateral basic tuple* and the *bilateral basic tuple*.

For the unilateral case, we will consider subspaces of \(L^2(\mathbb{T}, \mathcal{K})\) with \(\mathcal{K} = H^2_{\ell^2(I)}\), where \(I\) is an at most countable index set. So, for \(f \in L^2(\mathbb{T}, H^2_{\ell^2(I)})\), we have \(f(\lambda) \in H^2_{\ell^2(I)}\) and \(f(\lambda)(z) \in \ell^2(I)\) for a.e. \(\lambda, z \in \mathbb{T}\). Let us define a new operator acting on \(L^2(\mathbb{T}, H^2_{\ell^2(I)})\) which will play the role of the pointwise unilateral shift operator.

**Definition 3.3.** Define \(\hat{S} : L^2(\mathbb{T}, H^2_{\ell^2(I)}) \rightarrow L^2(\mathbb{T}, H^2_{\ell^2(I)})\) as

\[
\hat{S}f(\lambda) = S(f(\lambda)), \quad \text{a.e. } \lambda \in \mathbb{T}, f \in L^2(\mathbb{T}, H^2_{\ell^2(I)}).
\]

More precisely, for \(f \in L^2(\mathbb{T}, H^2_{\ell^2(I)})\) and for a.e. \(\lambda, z \in \mathbb{T}\)

\[
\hat{S}f(\lambda)(z) = S(f(\lambda))(z) = zf(\lambda)(z).
\]
From the definition of the bilateral shift $U \in \mathcal{B}(L^2(\mathbb{T}, H^2_\ell(I)))$ and $\hat{S}$, it is easy to see that they commute with each other. Moreover, $\hat{S}$ is an isometry, since so is $S$. Observe that with this operator in hand, we can construct an orthonormal basis of $L^2(\mathbb{T}, H^2_\ell(I))$ from an orthonormal basis $\{\varepsilon_i\}_{i \in I}$ of $\ell^2(I)$. Indeed, as discussed in the previous section, $\{S^j\varepsilon_i : j \in \mathbb{N}_0, i \in I\}$ is an orthonormal basis of $H^2_\ell(I)$ and it is clear that the same system can be seen inside of $L^2(\mathbb{T}, H^2_\ell(I))$ as $\{\hat{S}^j\varepsilon_i : j \in \mathbb{N}_0, i \in I\}$. Hence, the system
\begin{equation}
\{U^k\hat{S}^j\varepsilon_i : k \in \mathbb{Z}, j \in \mathbb{N}_0, i \in I\}
\end{equation}
is an orthonormal basis of $L^2(\mathbb{T}, H^2_\ell(I))$. Observe that (1) is a Fourier basis, $(U^k\hat{S}^j\varepsilon_i)(\lambda)(z) = \lambda^k z^j\varepsilon_i$ for a.e. $\lambda, z \in \mathbb{T}$ and $i \in I$.

**Definition 3.4.** A tuple $(\mathcal{N}, U|_{\mathcal{N}}, A, \{\varphi_i\}_{i \in I})$ is called a unilateral basic tuple if $\mathcal{N}$ is a closed subspace of $L^2(\mathbb{T}, H^2_\ell(I))$ that is $U$-reducing and $\hat{S}$-invariant, $A : \mathcal{N} \to \mathcal{N}$ is the compression of $\hat{S}$ to $\mathcal{N}$, i.e., $A := P_\mathcal{N}\hat{S}|_{\mathcal{N}}$, and $\varphi_i := P_\mathcal{N}\varepsilon_i$, where $\{\varepsilon_i\}_{i \in I}$ is the canonical orthonormal basis of $\ell^2(I)$.

Observe that in Definition 3.4 we are assuming that $U|_{\mathcal{N}}$ commutes with $A$. Indeed, for every $f \in \mathcal{N}$
\[ U|_{\mathcal{N}}Af = U|_{\mathcal{N}}P_\mathcal{N}\hat{S}f = P_\mathcal{N}U|_{\mathcal{N}}\hat{S}f = P_\mathcal{N}\hat{S}U|_{\mathcal{N}}f = AU|_{\mathcal{N}}f. \]
Here we used that $U$ and $P_\mathcal{N}$ commute because $\mathcal{N}$ is $U$-reducing, and that $U$ and $\hat{S}$ also commute.

We now turn to the bilateral case. These tuples will be constructed from subspaces belonging to $L^2(\mathbb{T}^2, \ell^2(I))$ that are reducing for the bilateral shift in each variable, defined as follows. For $i = 1, 2$, let $U_i : L^2(\mathbb{T}^2, \ell^2(I)) \to L^2(\mathbb{T}^2, \ell^2(I))$ be the operator given by
\[ (U_i f)(z_1, z_2) = z_i f(z_1, z_2), \quad \text{a.e. } (z_1, z_2) \in \mathbb{T}^2. \]
Observe that $U_1$ and $U_2$ commute and $\{U_1^kU_2^j\varepsilon_i : k, j \in \mathbb{Z}, i \in I\}$ is an orthonormal basis of $L^2(\mathbb{T}^2, \ell^2(I))$.

**Definition 3.5.** A tuple $(\mathcal{N}, U_1|_{\mathcal{N}}, U_2|_{\mathcal{N}}, \{\varphi_i\}_{i \in I})$ is called a bilateral basic tuple if $\mathcal{N}$ is a closed subspace of $L^2(\mathbb{T}^2, \ell^2(I))$ that is reducing for $U_1$ and $U_2$ and $\varphi_i := P_\mathcal{N}\varepsilon_i$, where $\{\varepsilon_i\}_{i \in I}$ is the canonical orthonormal basis of $\ell^2(I)$.

Basic tuples are always frame-tuples. In fact, they are Parseval-tuples as we show next.

**Proposition 3.6.** Every basic tuple is a Parseval-tuple.

**Proof.** Unilateral case. We first note that the restriction of $U$ to $\mathcal{N}$, $U|_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}$, is unitary. To prove the proposition we have to see that the system
\begin{equation}
\{U^k|_{\mathcal{N}}A^j\varphi_i : k \in \mathbb{Z}, j \in \mathbb{N}_0, i \in I\}
\end{equation}
is a Parseval frame of $\mathcal{N}$. We observe that (2) is the orthogonal projection onto $\mathcal{N}$ of the orthonormal basis $\{U^k\hat{S}^j\varepsilon_i : k \in \mathbb{Z}, j \in \mathbb{N}_0, i \in I\}$. Indeed, since $\mathcal{N}$ is
\( \hat{S}^* \)-invariant, we have \( \hat{S}^* P_N = P_N \hat{S}^* P_N \) and then \( P_N \hat{S} = P_N \hat{S} P_N \). As \( \mathcal{N} \) is also \( U \)-reducing it follows that

\[
(3) \quad P_N U^k \hat{S}^j \varepsilon_i = U^k P_N \hat{S}^j \varepsilon_i = U^k P_N \hat{S}^j P_N \varepsilon_i = U^k A^j \varepsilon_i.
\]

Thus, the proposition for the unilateral case is proved.

**Bilateral case.** This is very similar to the unilateral case. We just need to see that 
\( \{U_1^k U_2^j |_{\mathcal{N}} \varphi_i : k \in \mathbb{Z}, j \in \mathbb{Z}, i \in I \} \) is a Parseval frame of \( \mathcal{N} \). Since \( \{U_1^k U_2^j |_{\mathcal{N}} \varphi_i : k \in \mathbb{Z}, j \in \mathbb{Z}, i \in I \} \) is an orthonormal basis of \( L^2(\mathbb{T}^2, \ell^2(I)) \), then \( \{P_N U_1^k U_2^j |_{\mathcal{N}} \varphi_i : k, j \in \mathbb{Z}, i \in I \} \) is a Parseval frame for \( \mathcal{N} \). But \( P_N U_1^k U_2^j \varepsilon_i = U_1^k U_2^j |_{\mathcal{N}} P_N \varphi_i \) for every \( k, j \in \mathbb{Z}, i \in I \). Therefore, \( (\mathcal{N}, U_1 |_{\mathcal{N}}, U_2 |_{\mathcal{N}}, \{\varphi_i\}_{i \in I}) \) is a Parseval-tuple. \( \Box \)

Since basic tuples are Parseval-tuples, then two similar basic tuples must be unitarily equivalent by Lemma 3.2. Even more, they must be equal, as show next.

**Theorem 3.7.** If two basic tuples are similar, then they are equal.

**Proof.** We will just prove the unilateral case, as the bilateral case will be analogous. Let \( (\mathcal{N}_1, U_1 |_{\mathcal{N}_1}, A_1, \{P_{\mathcal{N}_1} \varepsilon_i\}_{i \in I}) \) and \( (\mathcal{N}_2, U_1 |_{\mathcal{N}_2}, A_2, \{P_{\mathcal{N}_2} \varepsilon_i\}_{i \in I}) \) be two similar basic tuples. Then, there exists a unitary isomorphism \( C : \mathcal{N}_1 \to \mathcal{N}_2 \) such that \( C A_1 = A_2 C \), \( C U_1 |_{\mathcal{N}_1} = U_1 |_{\mathcal{N}_2} C \) and \( C P_{\mathcal{N}_1} \varepsilon_i = P_{\mathcal{N}_2} \varepsilon_i \) for every \( i \in I \).

Moreover, by the intertwining properties of \( C \) and (3) we have that

\[
C(P_{\mathcal{N}_1}(U^k \hat{S}^j \varepsilon_i)) = C(U^k A_1^j P_{\mathcal{N}_1} \varepsilon_i) = U^k A_2^j C(P_{\mathcal{N}_1} \varepsilon_i) = U^k A_2^j P_{\mathcal{N}_2} \varepsilon_i
\]

for every \( k \in \mathbb{Z}, j \in \mathbb{N}_0 \) and \( i \in I \). Since \( \{U^k \hat{S}^j \varepsilon_i\}_{k,j,i} \) is an orthonormal basis of \( L^2(\mathbb{T}, H^2_H(I)) \), then [23, Proposition 2.6] implies that \( P_{\mathcal{N}_1} = P_{\mathcal{N}_2} \), and so \( \mathcal{N}_1 = \mathcal{N}_2 \) and the tuples must be equal. \( \Box \)

In what follows we will show that, in fact, any frame-tuple \( (\mathcal{H}, T, L, \{w_i\}_{i \in I}) \) must be similar to a unique basic tuple.

Since the \( U \)-reducing and \( \hat{S}^* \)-invariant subspaces of \( L^2(\mathbb{T}, H^2_H(I)) \) serve as models for frames of iterations of two commuting operators -one of them acting unilaterally-the study of their structure, that is very rich, is of independent interest. This is done in great detail in [2].

On the other hand, for the bilateral case, the model subspaces are in \( L^2(\mathbb{T}, \ell^2(I)) \) and they are reducing for \( U_1 \) and \( U_2 \). This implies that they are multiplication-invariant subspaces, and their structure has been very well studied, see for instance [13, 14].

3.2. **Unilateral frame-tuples.** Let \( \mathcal{H} \) be a separable Hilbert space. The main purpose of this subsection is to characterize the unilateral frame-tuples in \( \mathcal{H} \). That is, those tuples \( (\mathcal{H}, T, L, \{w_i\}_{i \in I}) \) where \( T, L \in \mathcal{B}(\mathcal{H}), T \) is invertible, \( TL = LT \), \( \{w_i\}_{i \in I} \subset \mathcal{H} \) is an at most countable set, and the system

\[
(4) \quad \{T^k L^j w_i : k \in \mathbb{Z}, j \in \mathbb{N}_0, i \in I \}
\]

forms a frame for \( \mathcal{H} \).
Assume the system (4) is a Bessel sequence. We then consider its associated synthesis operator defined on \( L^2(\mathbb{T}, H_{\ell^2(I)}^2) \), that is

\[
C : L^2(\mathbb{T}, H_{\ell^2(I)}^2) \to \mathcal{H}, \quad C f = \sum_{k,j,i} f^i_{kj} T^k L^j w_i,
\]

where

\[
f^i_{kj} := \langle f, U^k \hat{S}^j \varepsilon_i \rangle = \int_{\mathbb{T}} \int_{\mathbb{T}} (f(\lambda)(z), \varepsilon_i) \lambda^{-k} z^{-j} \, dz \, d\lambda,
\]

for \( k \in \mathbb{Z}, j \in \mathbb{N}_0, i \in I \) (the Fourier coefficients in the basis (1)). Note that, since the system (4) is a Bessel sequence, the series converges unconditionally.

**Remark 3.8.** Usually, the synthesis operator of a system like in (4) would be defined on the sequence space \( \ell^2(\mathbb{Z} \times \mathbb{N}_0 \times I) \). Here, we precomposed the usual synthesis operator with the isometric isomorphism between \( L^2(\mathbb{T}, H_{\ell^2(I)}^2) \) and \( \ell^2(\mathbb{Z} \times \mathbb{N}_0 \times I) \) defined by \( f \mapsto \{ \langle f, U^k \hat{S}^j \varepsilon_i \rangle \}_{k,j,i} \).

Now we are ready to prove the characterization of unilateral frame-tuples.

**Theorem 3.9.** A tuple \( (\mathcal{H}, T, L, \{ w_i \}_{i \in I}) \) is a unilateral frame-tuple if and only if it is similar to a unilateral basic tuple. Moreover, this basic tuple is unique.

**Proof.** Assuming that \( (\mathcal{H}, T, L, \{ w_i \}_{i \in I}) \) is a unilateral frame-tuple we will find a similar unilateral basic tuple \( (\mathcal{N}, U, A, \{ \varphi_i \}_{i \in I}) \).

Let \( C \) be the synthesis operator defined as in (5). If \( f \in L^2(\mathbb{T}, H_{\ell^2(I)}^2) \) we have:

\[
\langle U f, U^k \hat{S}^j \varepsilon_i \rangle = f^i_{k-1,j} \quad \text{and} \quad \langle U^* f, U^k \hat{S}^j \varepsilon_i \rangle = f^i_{k+1,j},
\]

for every \( (k, j, i) \in \mathbb{Z} \times \mathbb{N}_0 \times I \). Analogously, we also have \( \langle \hat{S} f, U^k \hat{S}^j \varepsilon_i \rangle = f^i_{k,j-1} \).

Thus, it is easy to see that the following intertwining relations hold:

\[
TC = CU, \quad T^{-1} C = CU^* \quad \text{and} \quad LC = C \hat{S}.
\]

From here we deduce that \( \ker(C) \subseteq L^2(\mathbb{T}, H_{\ell^2(I)}^2) \) is reducing for \( U \) and invariant for \( \hat{S} \) or equivalently, \( \mathcal{N} := \ker(C)^\perp \) is reducing for \( U \) and invariant for \( \hat{S}^* \).

Given that the system (4) is a frame of \( \mathcal{H} \), \( C \) is a bounded surjective operator. Then, the restriction of \( C \) to \( \mathcal{N} \)

\[
C|_\mathcal{N} : \mathcal{N} \to \mathcal{H},
\]

is a bounded isomorphism. On the other hand, let \( \varphi_i := P_{\mathcal{N}} \varepsilon_i \) for every \( i \in I \) and observe that \( C(P_{\mathcal{N}} \varepsilon_i) = 0 \) for every \( i' \in I \), and that \( \varepsilon_{i'} \) is orthogonal to \( U^k \hat{S}^j \varepsilon_i \) for every \( (k, j, i) \in \mathbb{Z} \times \mathbb{N}_0 \times I \) except for \( k = j = 0 \) and \( i = i' \) in which case \( \langle \varepsilon_{i'}, \varepsilon_{i'} \rangle = 1 \). Then, we obtain

\[
C|_\mathcal{N}(\varphi_{i'}) = C \varepsilon_{i'} = \sum_{k,j,i} \langle \varepsilon_{i'}, U^k \hat{S}^j \varepsilon_i \rangle T^k L^j w_i = w_{i'}, \quad \text{for every} \ i' \in I.
\]

Since relations in (6) still hold when restricting \( C \) to \( \mathcal{N} \), we conclude that \( (\mathcal{H}, T, L, \{ w_i \}_{i \in I}) \) is similar to \( (\mathcal{N}, U, A, \{ \varphi_i \}_{i \in I}) \).

The converse is a consequence of Lemma 3.1 and Proposition 3.6. The unicity follows from Theorem 3.7. That is, if there exists another basic tuple similar to \( (\mathcal{H}, T, L, \{ w_i \}_{i \in I}) \), then by transitivity it must be similar to \( (\mathcal{N}, U, A, \{ \varphi_i \}_{i \in I}) \), and therefore equal. \( \square \)
If \( \#I = 1 \) in Theorem 3.9, then \( \ell^2(I) = \mathbb{C} \) and \( H_{\ell^2(I)}^2 = H^2 \). Thus, we have the following:

**Corollary 3.10.** A tuple \((\mathcal{H}, T, L, w)\) is a unilateral frame-tuple if and only if there exists a \( U \)-reducing and \( \hat{S}^* \)-invariant subspace \( N \subseteq L^2(\mathbb{T}, H^2) \) such that \((\mathcal{H}, T, L, w)\) is similar to \((N, U, A, \varphi)\), where \( A \) is the compression of \( \hat{S} \) to \( N \) and \( \varphi = P_N 1 \).

Another implication of Theorem 3.9 is the characterization of Parseval-tuples.

**Corollary 3.11.** A tuple \((\mathcal{H}, T, L, \{w_i\}_{i \in I})\) is a unilateral Parseval-tuple if and only if it is unitarily equivalent to a unilateral basic tuple.

**Proof.** Let us assume that \((\mathcal{H}, T, L, \{w_i\}_{i \in I})\) is a unilateral Parseval-tuple. Then, from Theorem 3.9, it is similar to a basic tuple, that by Proposition 3.6 is a Parseval-tuple. Thus, by Lemma 3.2, they must be unitarily equivalent. The other implication is immediate since Parseval frames are preserved under unitary operators. \(\square\)

In the next result, we show that any unilateral Riesz-tuple must be similar to a basic tuple where the Hilbert space is the whole \( L^2(\mathbb{T}, H_{\ell^2(I)}^2) \).

**Proposition 3.12.** A tuple \((\mathcal{H}, T, L, \{w_i\}_{i \in I})\) is a unilateral Riesz-tuple if and only if it is similar to \((L^2(\mathbb{T}, H_{\ell^2(I)}^2)), U, \hat{S}, \{\varepsilon_i\}_{i \in I})\), where \( \{\varepsilon_i\}_{i \in I} \) is the canonical basis of \( \ell^2(I) \).

**Proof.** If \((\mathcal{H}, T, L, \{w_i\}_{i \in I})\) is a unilateral Riesz-tuple, the kernel of the synthesis operator is the trivial subspace \( \{0\} \). Thus, its orthogonal complement is \( L^2(\mathbb{T}, H_{\ell^2(I)}^2) \) and by the proof of Theorem 3.9 we obtain the claim in the proposition. The converse is straightforward. \(\square\)

### 3.3. Bilateral frame-tuples

When the operator \( L \) is invertible, it is possible to consider integer iterations of it. Given a tuple \((\mathcal{H}, T, L, \{w_i\}_{i \in I})\) with \( L \) being invertible, we seek for the frame condition on the set

\[
\{T^k L^j w_i : k, j \in \mathbb{Z}, i \in I\}.
\]

As well as in the previous subsection, we obtain that the set of iterations of \( T \) and \( L \) generated by \( \{w_i\}_{i \in I} \) is a frame for \( \mathcal{H} \) if and only if the tuple \((\mathcal{H}, T, L, \{w_i\}_{i \in I})\) is similar to a unique bilateral basic tuple. This is proved by following the same arguments as in the proof of Theorem 3.9 replacing the synthesis operator by the corresponding one for this case, that is, \( C : L^2(\mathbb{T}^2, \ell^2(I)) \to \mathcal{H} \)

\[
C f = \sum_{k,j,i} f_{ij}^k T^k L^j w_i
\]

with \( f_{ij}^k = \langle f, U^k_1 U^j_2 \varepsilon_i \rangle \), for \( k, j \in \mathbb{Z}, i \in I \). Hence, we just state the following theorem without proof.

**Theorem 3.13.** A tuple \((\mathcal{H}, T, L, \{w_i\}_{i \in I})\) is a bilateral frame-tuple if and only if it is similar to a bilateral basic tuple. Moreover, this basic tuple is unique.

As in the case of unilateral iterations, we have the following particular case of Theorem 3.13 when \( \#I = 1 \).
Corollary 3.14. A tuple \((\mathcal{H}, T, L, w)\) is a bilateral frame-tuple if and only if there exists a subspace \(\mathcal{N} \subseteq L^2(\mathbb{T}^2)\) which is reducing for \(U_1\) and \(U_2\) such that \((\mathcal{H}, T, L, w)\) is similar to \((\mathcal{N}, U_1|_{\mathcal{N}}, U_2|_{\mathcal{N}}, P_{\mathcal{N}}1)\), where \(1 \in L^2(\mathbb{T}^2)\) is the function which is constantly one.

Also, we have the next results regarding Parseval-tuples and Riesz-tuples.

Corollary 3.15. A tuple \((\mathcal{H}, T, L, \{w_i\}_{i \in I})\) is a bilateral Parseval-tuple if and only if it is unitarily equivalent to a bilateral basic tuple.

Proposition 3.16. A tuple \((\mathcal{H}, T, L, \{w_i\}_{i \in I})\) is a bilateral Riesz-tuple if and only if it is similar to \((L^2(\mathbb{T}^2, \ell^2(I)), U_1, U_2, \{\varepsilon_i\}_{i \in I})\), where \(\{\varepsilon_i\}_{i \in I}\) is the canonical basis of \(\ell^2(I)\).

4. Frames of orbits of a single vector

Let \(\mathcal{H}\) be a separable Hilbert space, \(T, L \in \mathcal{B}(\mathcal{H})\) such that \(T\) is invertible and \(TL = LT\). Assume that \(\mathcal{H}, T, L\) and the index set \(I = \{1, \ldots, n\}\) are fixed and consider the set

\[\mathcal{V}_n := \{\{v_i\}_{i \in I} \subset \mathcal{H} : (\mathcal{H}, T, L, \{v_i\}_{i \in I})\text{ is a unilateral (bilateral) frame-tuple}\}.\]

Now, define an equivalent relation in \(\mathcal{V}_n\) given by \(\{v_i\}_{i \in I} \sim \{w_i\}_{i \in I}\) if and only if \((\mathcal{H}, T, L, \{v_i\}_{i \in I})\) is similar to \((\mathcal{H}, T, L, \{w_i\}_{i \in I})\).

In this section we will prove that when \#\(I = 1\) there is only one equivalence class (whenever \(\mathcal{V}_1\) is not empty). We will later show that this is not true for multiple generators (see Remark 4.9). Specifically, for the unilateral case the claim read as follows.

Theorem 4.1. Let \(\mathcal{H}\) be a Hilbert space, \(T, L \in \mathcal{B}(\mathcal{H})\) such that \(T\) is invertible and \(LT = TL\) and consider the set

\[\mathcal{V} := \{v \in \mathcal{H} : (\mathcal{H}, T, L, v)\text{ is a unilateral frame-tuple}\} .\]

Assume that \(w \in \mathcal{V}\). Then, \(v \in \mathcal{V}\) if and only if \((\mathcal{H}, T, L, v)\) is similar to \((\mathcal{H}, T, L, w)\), i.e.

\[\mathcal{V} = \{Bw : B \in \mathcal{B}(\mathcal{H}), B \text{ is invertible and commutes with } T \text{ and } L\} .\]

In order to prove Theorem 4.1, we will show that for \#\(I = 1\) a weaker condition than similarity is sufficient for two unilateral basic-tuples to be equal.

Theorem 4.2. For \(i = 1, 2\), let \(\mathcal{N}_i \subseteq L^2(\mathbb{T}, H^2)\) be a reducing subspace for \(U\) and invariant for \(\hat{S}^*\), and let \(A_i = P_{\mathcal{N}_i}\hat{S}\). If there exists an isomorphism \(\Psi : \mathcal{N}_1 \to \mathcal{N}_2\) such that

\[\Psi A_1 = A_2\Psi \quad \text{and} \quad \Psi U|_{\mathcal{N}_1} = U|_{\mathcal{N}_2}\Psi , \quad (7)\]

then \(\mathcal{N}_1 = \mathcal{N}_2\) and \(A_1 = A_2\).

Before we prove Theorem 4.2 we need to recall some notions on the structure of reducing subspaces of \(L^2(\mathbb{T}, \mathcal{K})\), where \(\mathcal{K}\) is a separable Hilbert space. For more details on this topic, we refer the reader to [14] where the \(U\)-reducing subspaces of \(L^2(\mathbb{T}, \mathcal{K})\) are the multiplicative-invariant subspaces of \(L^2(\mathbb{T}, \mathcal{K})\) with respect to the determining set \(\{\gamma^j\}_{j \in \mathbb{Z}}\), with \(\gamma(\lambda) = \lambda\) a.e \(\lambda \in \mathbb{T}\), see [14, Definition 2.2 and 2.3].
Helson proved in [24] that $U$-reducing subspaces of $L^2(\mathbb{T}, \mathcal{K})$ can be characterized in terms of range functions. Later, Bownik and Ross gave in [14, Theorem 2.4] an extended version of this result.

A range function in $\mathcal{K}$ is a mapping $J : \mathbb{T} \to \{\text{closed subspaces of } \mathcal{K}\}$. It is measurable if for each $x, y \in \mathcal{K}$, the complex-valued function $\lambda \mapsto \langle P_J(\lambda)x, y \rangle$ is measurable.

**Theorem 4.3.** [14, Theorem 2.4] Let $\mathcal{M}$ be a closed subspace of $L^2(\mathbb{T}, \mathcal{K})$. The following statements are equivalent:

i) $\mathcal{M}$ is $U$-reducing,

ii) there exists a measurable range function such that

$$\mathcal{M} = \{f \in L^2(\mathbb{T}, \mathcal{K}) : f(\lambda) \in J(\lambda) \text{ for a.e. } \lambda \in \mathbb{T}\}.$$ 

The correspondence between $U$-reducing subspaces and measurable range functions is one-to-one and onto, assuming that range functions which are equal almost everywhere are identified.

Moreover, when $\mathcal{M}$ is $U$-reducing, since $L^2(\mathbb{T}, \mathcal{K})$ is separable, there is a countable set $A \subset L^2(\mathbb{T}, \mathcal{K})$ such that $\mathcal{M} = \text{span}\{U^k f : f \in A, k \in \mathbb{Z}\}$. Then, the measurable range function $J$ associated to $\mathcal{M}$ is given by

$$J(\lambda) = \text{span}\{f(\lambda) : f \in A\}.$$ 

In [24], Helson also proved the following property in terms of projections: if $\mathcal{M}$ is $U$-reducing and $J$ is its range function, then

$$P_{\mathcal{M}} f(\lambda) = P_{J(\lambda)}(f(\lambda)), \text{ a.e. } \lambda \in \mathbb{T}, f \in L^2(\mathbb{T}, \mathcal{K}).$$ 

For proving Theorem 4.2 we need a characterization of subspaces of $L^2(\mathbb{T}, H^2)$ that are reducing for $U$ and invariant for $\hat{S}^*$, that we include below. Its proof can be found in [2, Corollary 3.10]. Recall that a function $h \in H^2$ is inner if $|h(z)| = 1$ a.e. $z \in \mathbb{T}$.

**Theorem 4.4.** Let $\mathcal{N} \subseteq L^2(\mathbb{T}, H^2)$ be a closed subspace. The following statements are equivalent:

i) $\mathcal{N}$ is $U$-reducing and $\hat{S}^*$-invariant,

ii) There exists $\phi \in L^2(\mathbb{T}, H^2)$ such that $\phi(\lambda)$ is an inner function for a.e. $\lambda \in $ \sigma($\mathcal{N}^\perp$) and $\mathcal{N}^\perp = \phi L^2(\mathbb{T}, H^2) := \{\phi f : f \in L^2(\mathbb{T}, H^2)\}$, where $\sigma(N^\perp) = \{\lambda \in \mathbb{T} : J_{\mathcal{N}^\perp}(\lambda) \neq \{0\}\}$.

Moreover, it can be proved that the measurable range function associated to a $U$-reducing subspace of the form $\mathcal{M} = \phi L^2(\mathbb{T}, H^2)$ for $\phi \in L^2(\mathbb{T}, H^2)$ is given by

$$J_{\mathcal{M}}(\lambda) = \phi(\lambda)H^2 \text{ for a.e. } \lambda \in \mathbb{T} \text{ (see [2, Proposition 3.11]).}$$

**Remark 4.5.** A note about the terminology. If $H^2_\mathcal{K}$ is the Hardy space of $\mathcal{K}$-valued functions, and $\mathcal{Q} \subset H^2_\mathcal{K}$ is a non-trivial closed subspace, invariant by $\hat{S}^*$, the adjoint of the unilateral shift $S$, then $\mathcal{Q}$ is called a model space. These spaces had proven to be very important in operator theory since the compression of the shift on a model space serves as a model for a certain class of contractions. (See [20] for a very comprehensive and detailed introduction to the theory of model spaces).

The Hilbert space $\mathcal{N}$ of a basic tuple is a subspace of $L^2(\mathbb{T}, H^2_\mathcal{K})$ that is $U$-reducing and $\hat{S}^*$-invariant. After Theorem 4.3, we know that $\mathcal{N}$ has a range function. The
invariance under $\hat{S}^*$ tells us that the values of the range function are subspaces of $H^2_0$ invariant under $S^*$, that is, they are model spaces.

Thus, $\mathcal{N}$ has a range function whose values are model spaces or, in the terminology of direct integrals, it is the direct integral over $\mathbb{T}$ of model spaces. This connection justifies calling the space $\mathcal{N}$, in the $L^2$-context, a model space.

Another ingredient that we need for the proof of Theorem 4.2 is the concept of operator-valued function. We give now the definition and some properties.

An operator-valued function in $\mathcal{K}$ is a function $F : \mathbb{T} \to \mathcal{B}(\mathcal{K})$. It is said to be measurable if for every $x \in \mathcal{K}$, $\lambda \mapsto F(\lambda)(x)$ is measurable. The norm of an operator-valued function in $\mathcal{K}$ is defined as $\|F\|_\infty = \text{ess sup}_{\lambda \in \mathbb{T}} \|F(\lambda)\|_{op}$.

If $F : \mathbb{T} \to \mathcal{B}(\mathcal{K})$ is a measurable operator-valued function in $\mathcal{K}$ such that $\|F\|_\infty < \infty$, we denote by $\hat{F} : L^2(\mathbb{T}, \mathcal{K}) \to L^2(\mathbb{T}, \mathcal{K})$ the operator defined by

$$\hat{F}f(\lambda) = F(\lambda)f(\lambda), \quad \text{for a.e. } \lambda \in \mathbb{T}, \ f \in L^2(\mathbb{T}, \mathcal{K})$$

and by $\mathcal{F}$ the class of all measurable functions $F : \mathbb{T} \to \mathcal{B}(\mathcal{K})$ such that $\|F\|_\infty < \infty$, and $\hat{\mathcal{F}} = \{\hat{F} : F \in \mathcal{F}\}$. In [31, Theorem 3.17], it is shown that the correspondence $F \mapsto \hat{F}$ is an adjoint-preserving algebra isomorphism. Moreover, $\hat{F}$ is normal (self-adjoint, unitary or a projection), if and only if $F(\lambda)$ is normal (self-adjoint, unitary or a projection) for a.e. $\lambda \in \mathbb{T}$. Also, in [31, Corollary 3.19] the authors showed that the commutant of $U$ acting on $L^2(\mathbb{T}, \mathcal{K})$ is $\hat{\mathcal{F}}$.

We will make use of the following properties of operator-valued functions and reducing subspaces for $U$ acting on $L^2(\mathbb{T}, \mathcal{K})$ whose proofs follows from [13, Theorem 4.1 and Lemma 2].

**Lemma 4.6.** Let $\mathcal{M}, \mathcal{N} \subseteq L^2(\mathbb{T}, \mathcal{K})$ be reducing subspaces for $U$ with range functions $J_{\mathcal{M}}, J_{\mathcal{N}}$ respectively, and let $\hat{F} \in \hat{\mathcal{F}}$. Then, we have:

i) If $\hat{F}|_{\mathcal{M}} = 0$, then, $F(\lambda)|_{J_{\mathcal{M}}(\lambda)} = 0$ for a.e. $\lambda \in \mathbb{T}$.

ii) If $\hat{F}|_{\mathcal{M}} : \mathcal{M} \to \mathcal{N}$ is an isomorphism, then so is $F(\lambda)|_{J_{\mathcal{M}}(\lambda)} : J_{\mathcal{M}}(\lambda) \to J_{\mathcal{N}}(\lambda)$, for a.e. $\lambda \in \mathbb{T}$.

Now we are ready to prove Theorem 4.2.

**Proof of Theorem 4.2.** Since $\mathcal{N}_i \subseteq L^2(\mathbb{T}, H^2)$ for $i = 1, 2$ is $U$-reducing and $\hat{S}^*$-invariant, then Theorem 4.4 says that there exists $\phi_i \in L^2(\mathbb{T}, H^2)$ such that $\phi_i(\lambda)$ is an inner function for a.e. $\lambda \in \sigma(\mathcal{N}_i^\perp)$ and $\mathcal{N}_i = (\phi_i L^2(\mathbb{T}, H^2))^\perp$. Thus, its range function $J_i$ is given by

$$J_i : \lambda \mapsto J_i(\lambda) = (\phi_i(\lambda)H^2)^\perp.$$

To conclude that $\mathcal{N}_1 = \mathcal{N}_2$, and hence $A_1 = A_2$, it is sufficient to show that $\phi_1 = \phi_2$.

Observe first that if we extend $\Psi$ as zero on $\mathcal{N}_1^\perp$, from the second equation in (7) we get that this extension commutes with $U$ and therefore, by [31, Corollary 3.19], there is $\hat{F} \in \hat{\mathcal{F}}$ such that $\hat{F} = \Psi$ on $\mathcal{N}_1$ and $\hat{F} = 0$ on $\mathcal{N}_1^\perp$. Moreover, since $\hat{F}|_{\mathcal{N}_1} = \Psi : \mathcal{N}_1 \to \mathcal{N}_2$ is an isomorphism, ii) of Lemma 4.6 implies that $F(\lambda)|_{J_1(\lambda)} : J_1(\lambda) \to J_2(\lambda)$ is an isomorphism for a.e. $\lambda \in \mathbb{T}$.

On the other hand, the first equation in (7) can be rewritten in terms of $\hat{F}|_{\mathcal{N}_1}$ as

(9) $$\hat{F}|_{\mathcal{N}_1}A_1 = A_2\hat{F}|_{\mathcal{N}_1}.$$
We want to see now that \( A_1 \) and \( A_2 \) belong to \( \hat{F} \). To do this we define for a.e. \( \lambda \in \mathbb{T} \) and for \( i = 1, 2 \) the function \( G_i : \mathbb{T} \to \mathcal{B}(H^2) \) which for every \( \lambda \in \mathbb{T} \) is defined as the operator \( G_i(\lambda) := P_{J_i(\lambda)}S \). Then, \( \lambda \mapsto G_i(\lambda) \) is measurable so is \( \lambda \mapsto P_{J_i(\lambda)} \) and \( \|G_i\|_{\text{op}} = \text{ess sup}_{\lambda \in \mathbb{T}} \|G_i(\lambda)\|_{\text{op}} \leq 1 < \infty \). This proves that \( G_i \in \mathcal{F} \).

Besides, for \( i = 1, 2 \) we have that \( \hat{G}_i \) and \( A_i \) coincide. This is a consequence of (8) as the following computation shows:

\[
\hat{G}_i f(\lambda) = G_i(\lambda)(f(\lambda)) = P_{J_i(\lambda)}(S f(\lambda)) = P_{J_i(\lambda)}(S \hat{f})(\lambda) = A_i f(\lambda)
\]

for every \( f \in L^2(\mathbb{T}, H^2) \) and for a.e \( \lambda \in \mathbb{T} \).

Hence, for a.e. \( \lambda \in \mathbb{T} \) it is obtained from (9) that

\[
F(\lambda)|_{J_1(\lambda)} G_1(\lambda) = G_2(\lambda)|_{J_1(\lambda)} F(\lambda).
\]

The equation above says that \( F(\lambda)|_{J_1(\lambda)} \) intertwines \( G_1(\lambda) \) and \( G_2(\lambda) \) which are two compressions of the unilateral shift operator acting in the subspaces \( J_1(\lambda) = (\phi_1(\lambda) H^2)^{\perp} \) and \( J_2(\lambda) = (\phi_2(\lambda) H^2)^{\perp} \) respectively.

At this point we use some results about quasi-similar \( C_0 \) contractions and minimal functions which are developed in [29, Chapter III].

First, we observe that by [29, Proposition 4.3, Chapter III], for \( i = 1, 2 \) and for a.e. \( \lambda \in \mathbb{T} \), the operator \( G_i(\lambda) \) is a \( C_0 \) contraction whose minimal function is \( \phi_i(\lambda) \). Second, since \( G_1(\lambda) \) and \( G_2(\lambda) \) are quasi-similar, i.e, there exists a bounded isomorphism that intertwines \( G_1(\lambda) \) and \( G_2(\lambda) \) for a.e. \( \lambda \in \mathbb{T} \), then by [29, Proposition 4.6, Chapter III] we get that the inner functions \( \phi_1(\lambda) \) and \( \phi_2(\lambda) \) coincide a.e. \( \lambda \in \mathbb{T} \) and therefore \( \phi_1 = \phi_2 \). Consequently, from the definitions of \( N_i \) and \( A_i \) for \( i = 1, 2 \), the result follows.

Finally, we prove the main results of this section.

**Proof of Theorem 4.1.** Observe that if \( v \in \mathcal{H} \) is such that \((\mathcal{H}, T, L, v)\) is similar to \((\mathcal{H}, T, L, w)\), then by Lemma 3.1, \((\mathcal{H}, T, L, v)\) is a frame-tuple. Hence \( v \in \mathcal{V} \).

It remains to show that if \( v \in \mathcal{V} \), then it can be defined a linear bounded operator \( B : \mathcal{H} \to \mathcal{H} \) with \( BT = TB \) and \( BL = LB \) such that \( v = Bw \).

Since \( w, v \in \mathcal{V} \), \((\mathcal{H}, T, L, w)\) and \((\mathcal{H}, T, L, v)\) are unilateral frame-tuples. Therefore, by Corollary 3.10, there exist two basic tuples \((N_1, U, A_1, \varphi_1)\) and \((N_2, U, A_2, \varphi_2)\) that are similar to \((\mathcal{H}, T, L, w)\) and \((\mathcal{H}, T, L, v)\) respectively. Recall that the isomorphism \( C_i \in \mathcal{B}(N_i, \mathcal{H}) \) for \( i = 1, 2 \) given by the similarity relation satisfies that

\[
C_1 A_1 = L C_1, \quad C_1 U|_{N_1} = T C_1 \quad \text{and} \quad C_1 \varphi_1 = w
\]

and

\[
C_2 A_2 = L C_2, \quad C_2 U|_{N_2} = T C_2 \quad \text{and} \quad C_2 \varphi_2 = v.
\]

Now, from equations (10) and (11) we have that

\[
C_1 A_1 C_1^{-1} = C_2 A_2 C_2^{-1}, \quad C_1 U|_{N_1} C_1^{-1} = C_2 U|_{N_2} C_2^{-1}.
\]

which is equivalent to

\[
\Psi A_1 = A_2 \Psi, \quad \Psi U|_{N_1} = U|_{N_2} \Psi
\]

where \( \Psi = C_2^{-1} C_1 : N_1 \to N_2 \). Since \( \Psi \) is an isomorphism, Theorem 4.2 gives that \( N_1 = N_2, A_1 = A_2 \) and consequently \( \varphi_1 = \varphi_2 \).
Now, let \( B : \mathcal{H} \to \mathcal{H} \) be defined as \( B := C_2C_1^{-1} \). Observe that \( B \) is bounded, invertible and
\[
v = C_2\varphi_2 = C_2C_1^{-1}C_1\varphi_1 = Bw.
\]
Moreover,
\[
BL = C_2C_1^{-1}L = C_2A_1C_1^{-1} = C_2A_2C_1^{-1} = LC_2C_1^{-1} = LB,
\]
and then, \( B \) and \( L \) commute. Analogously, it can be seen that \( BT = TB \). This completes the proof. \( \square \)

The same characterization given in Theorem 4.1 can be obtained when we consider bilateral frame-tuples. More precisely, we have:

**Theorem 4.7.** Let \( \mathcal{H} \) be a Hilbert space, \( T, L \in \mathcal{B}(\mathcal{H}) \) such that \( LT = TL \) and consider the set
\[
\mathcal{V} := \{ v \in \mathcal{H} : (\mathcal{H}, T, L, v) \text{ is a bilateral frame-tuple} \}.
\]
Assume that \( w \in \mathcal{V} \). Then, \( v \in \mathcal{V} \) if and only if \( (\mathcal{H}, T, L, v) \) is similar to \( (\mathcal{H}, T, L, w) \), i.e.
\[
\mathcal{V} = \{ Bw : B \in \mathcal{B}(\mathcal{H}), B \text{ is invertible and commutes with } T \text{ and } L \}.
\]

The proof of Theorem 4.7 is based in Corollary 3.14 and the following proposition that characterize reducing subspaces of \( L^2(\mathbb{T}^2) \), and whose proof can be found in [21].

**Proposition 4.8.** A subspace \( \mathcal{M} \subset L^2(\mathbb{T}^2) \) is reducing for \( U_i, i = 1, 2 \) if and only if there exists a Borel set \( E \subset \mathbb{T}^2 \) such that \( \mathcal{M} = \chi_E L^2(\mathbb{T}^2) \).

**Proof of Theorem 4.7.** Let \( w \in \mathcal{V} \). Since \( (\mathcal{H}, T, L, w) \) and \( (\mathcal{H}, T, L, v) \) are bilateral frame-tuples, then by Corollary 3.14 they are similar to basic frame-tuples \( (\mathcal{M}, U_1|_\mathcal{M}, U_2|_\mathcal{M}, P_\mathcal{M}1) \) and \( (\mathcal{N}, U_1|_\mathcal{N}, U_2|_\mathcal{N}, P_\mathcal{N}1) \) respectively.

Analogously as in the proof of the case of unilateral iterations, the key is to show that the subspaces \( \mathcal{M} \) and \( \mathcal{N} \) are equal.

From the isomorphisms given by the similarity relations, we can define an isomorphism \( V : \mathcal{M} \to \mathcal{N} \) satisfying
\[
V U_1|_\mathcal{M} = U_1|_\mathcal{N} V \quad \text{and} \quad V U_2|_\mathcal{M} = U_2|_\mathcal{N} V.
\]

Since \( \mathcal{M} \) and \( \mathcal{N} \) are reducing for \( U_1 \) and \( U_2 \), by Proposition 4.8 we have that there exist Borel sets \( E_1, E_2 \subset \mathbb{T}^2 \) such that \( \mathcal{M} = \chi_{E_1} L^2(\mathbb{T}^2) \) and \( \mathcal{N} = \chi_{E_2} L^2(\mathbb{T}^2) \).

In order to show that \( \mathcal{M} = \mathcal{N} \), we will see that the characteristic functions \( \chi_{E_1} \) and \( \chi_{E_2} \) coincide a.e. on \( \mathbb{T}^2 \). To do this, take \( g = \chi_{E_1 \setminus E_2 \supseteq \mathcal{M} \subset L^2(\mathbb{T}^2)} \). Since \( \{U_1^kU_2^j : k, j \in \mathbb{Z}\} \) is an orthonormal basis of \( L^2(\mathbb{T}^2) \) we have the expansion
\[
g = \sum_{k \in \mathbb{Z}, j \in \mathbb{Z}} \beta_{kj} U_1^k U_2^j 1,
\]
whose coefficients satisfy \( \{\beta_{kj}\} \in \ell^2(\mathbb{Z}^2) \). By applying the isomorphism \( V \) and using the relations in (12) we get that
\[
V g = V P_{\mathcal{M}1} g = V \left( \sum_{k \in \mathbb{Z}, j \in \mathbb{Z}} \beta_{kj} U_1^k U_2^j P_{\mathcal{M}1} \right) = \sum_{k \in \mathbb{Z}, j \in \mathbb{Z}} \beta_{kj} U_1^k U_2^j 1 = \sum_{k \in \mathbb{Z}, j \in \mathbb{Z}} \beta_{kj} U_1^k |_\mathcal{N} U_2^j |_\mathcal{N} V P_{\mathcal{N}1}.
\]
Let \( h = V P \mathcal{N} 1 \) and observe that \( \text{supp}(g) \subseteq E_1 \setminus E_2 \) and \( \text{supp}(h) \subseteq E_2 \). Then, it is obtained
\[
V g = gh = 0.
\]
Since \( V \) is an isomorphism, we conclude that \( g = 0 \). A similar argument can be used to show that \( \mathcal{X}_{E_2 \setminus E_1} = 0 \). Thus, \( |E_2 \setminus E_1| = 0 = |E_1 \setminus E_2| \), and therefore \( \mathcal{X}_{E_1} \) coincide with \( \mathcal{X}_{E_2} \) a.e. \((z_1, z_2) \in \mathbb{T}^2\). As a consequence, \( \mathcal{M} = \mathcal{N} \).

The proof concludes in the same manner of Theorem 4.1.

\[\square\]

**Remark 4.9.**

(a) It is not true that in the case of more than one generator there exists only one equivalent class of vectors \( \{v_i\} \in \mathcal{V}_n \) as we show next. Suppose that \( \mathcal{V}_n \) is not empty for some \( n > 1 \). Let \( \{v_i\}_{i \in I} \in \mathcal{V}_n \), that is, \((\mathcal{H}, T, L, \{v_i\}_{i \in I})\) is a unilateral frame-tuple, and assume that \( \{v_1, v_2\} \) is linearly independent. Then, it is easy to check that \( \{v_i\}_{i \in I} \cup \{v_1 + v_2\} \) and \( \{v_i\}_{i \in I} \cup \{v_1 - v_2\} \) belong to \( \mathcal{V}_{n+1} \). Suppose that the corresponding tuples \((\mathcal{H}, T, L, \{v_i\}_{i \in I} \cup \{v_1 + v_2\})\) and \((\mathcal{H}, T, L, \{v_i\}_{i \in I} \cup \{v_1 - v_2\})\) are similar via an isomorphism \( C : \mathcal{H} \to \mathcal{H} \). Then, in particular, it must hold that
\[
C(v_1) = v_1, \quad C(v_2) = v_2, \quad \ldots, \quad C(v_1 + v_2) = v_1 - v_2
\]
which is not possible.

(b) As we already mentioned, if we fix the Hilbert space \( \mathcal{H} \) and the operator \( T \) and \( L \) in \( \mathcal{B}(\mathcal{H}) \), Theorems 4.1 and 4.7 say that all the frame-tuples with a single generator are similar (when there is at least one). Before, we saw that this is no longer true for more generators. At this point one could ask if there is some relation between \( \# I \) and the number of equivalent classes of the relation defined at the beginning of this section. What we know is that if \( \# I = 1 \) and \( \mathcal{V}_1 \neq \emptyset \), there is only one equivalence class. But a slight modification in part (a) shows that for \( \# I = n > 1 \) there are infinitely many equivalence classes. Indeed, suppose that \( n > 1 \) and \( \mathcal{V}_n \neq \emptyset \). Consider \( \{v_i\}_{i \in I} \in \mathcal{V}_n \) and assume that \( \{v_1, v_2\} \) is linearly independent. Then, for \( a, b \in \mathbb{C} \setminus \{0\} \), we have that \( \{v_i\}_{i \in I} \cup \{av_1 + bv_2\} \in \mathcal{V}_{n+1} \). As in (a), it can be seen that each generator set \( \{v_i\}_{i \in I} \cup \{av_1 + bv_2\} \) belongs to a different equivalence class. Then, there are infinitely many when we have 3 generators or more. The same idea works for 2 generators, since if \( v \in \mathcal{V}_1 \), \( \{v, av\} \in \mathcal{V}_2 \) for every \( a \in \mathbb{C} \setminus \{0\} \) and they all belong to different equivalence classes.

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