Exact bound state solution of the Klein Gordon equation with a position-energy dependent mass and a Coulomb-like energy dependent potential energy

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Abstract

In this manuscript, we investigate the exact bound state solution of the Klein Gordon equation with an energy-dependent Coulomb-like potential energy in the presence of position-energy dependent mass. First, we examine the case where the mixed vector and scalar potential energy possess equal magnitude and equal sign. Then, we extend the investigation with the cases where the mixed potential energies have equal magnitude and opposite sign. Furthermore, we study pure scalar and pure vector cases. In each case, we derive an analytic expression of the energy spectrum by employing the asymptotic iteration method. We obtain a non-trivial relation between the tuning parameters which lead the examined problem to a constant mass one. Finally, we employ the Secant method to calculate the energy spectra. We use the calculated spectra and show that the unnormalized wave functions satisfy the boundary conditions. We conclude the manuscript with a comparison of the calculated energy spectra versus tuning parameters.

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One of the important goals in relativistic or non-relativistic quantum mechanics is to obtain an exact solution of a physical problem by employing potential energy. It is found that among the analytically solvable potential energies, only a few of them are very appropriate to represent the physical systems. For example, Coulomb potential energy is used to describe the Hydrogen atom. Bound state solution of the Coulomb-like potential energy is investigated in details in the relativistic and non-relativistic equations.

Klein Gordon (KG) equation is one of the relativistic equation that is introduced to describe the dynamics of the bosonic particles. Although it was first introduced about a century ago, it still attracts the scientists’ interest. Especially, researchers focus on the solutions of the KG equation with equal and unequal vector and scalar potential energies. Among a large number of articles, we would like to underline the investigations that employ Woods-Saxon, Hulthén, Morse, Rosen-Morse, Eckart, Manning-Rosen, Pöschl-Teller, Kratzer, Hylleraas, multiparameter, exponential-type molecule, pseudoharmonic oscillator, Hartmann, double ring-shaped oscillator potential energies.

In the last decades, position-dependent mass (PDM) are gained popularity to interpret the properties of several quantum systems such as quantum liquids, quantum wells, quantum rings, quantum dots, etc. More applications about quantum semiconductor structures can be found in the book of Weisbuch and Vinter.

On the other hand, energy-dependent potential (EDP) energies are being used to describe the physical systems for a while. In recent times, the number of studies on the solutions of EDP energies in both relativistic and non-relativistic wave equations is increased.

In this manuscript, our motivation is to investigate the effects of the EDP Coulomb-like mixed vector and scalar potential energy in the presence of PDM on the bound state energy eigenvalues and corresponding wave functions for the KG particle in terms of following cases: mixed vector and scalar potential energies with equal magnitude and equal sign (EMES), mixed vector and scalar potential energies with equal magnitude
and opposite sign (EMOS), pure vector (PV) potential energy, pure scalar (PS) potential energy and constant mass energy limits.

We prepared the manuscript as follows. In sect. II we gave a brief outline of the AIM method. In sect. III we introduced the radial KG equation with mixed vector and scalar equation. Then, in subsection III A we declared the position and energy dependent mass function. Then, in its subsections, we solved the radial KG equation by employing the AIM method in different limits. In sect. III B we obtained the wave function solution by using the boundary conditions in terms of confluent hypergeometric functions. Then, in sect. IV we employed the numerical methods and calculated energy spectra for a neutral pion particle. We verified the spectra results by plotting their corresponding wave functions in means of obeying the boundary conditions. Before we concluded the manuscript in sect. V we analyzed the role of the tuning parameters on the energy spectrum in the pure scalar limits.

II. ASYMPTOTIC ITERATION METHOD

The AIM has been proposed for solving the homogeneous linear second-order differential equation of the form

\[ \phi''(x) = \lambda_0(x)\phi'(x) + s_0(x)\phi(x), \quad (1) \]

where \( \lambda_0 \) function has a non-zero expression. Here, the prime is used to denote the derivative of the function with respect to \( x \). The functions \( \lambda_0(x) \) and \( s_0(x) \) are assumed to have sufficiently differentiable. The first derivative of Eq. (1) gives

\[ \phi'''(x) = \lambda_1(x)\phi'(x) + s_1(x)\phi(x), \quad (2) \]

where

\[ \lambda_1(x) = \lambda_0'(x) + \lambda_0^2(x) + s_0(x), \quad (3) \]
\[ s_1(x) = s_0'(x) + s_0(x)\lambda_0(x). \quad (4) \]

The second derivative of Eq. (1) gives

\[ \phi''''(x) = \lambda_2(x)\phi'(x) + s_2(x)\phi(x), \quad (5) \]
where
\[ \lambda_2(x) = \lambda'_1(x) + \lambda_0(x)\lambda_1(x) + s_1(x), \quad (6) \]
\[ s_2(x) = s'_1(x) + s_0(x)\lambda_1(x). \quad (7) \]

The repetition of the derivatives up to \((k + 1)\)th and \((k + 2)\)th order, ends up with
\[ \varphi^{(k+1)}(x) = \lambda_{k-1}(x)\varphi'(x) + s_{k-1}(x)\varphi(x) \quad (8) \]
\[ \varphi^{(k+2)}(x) = \lambda_k(x)\varphi'(x) + s_k(x)\varphi(x) \quad (9) \]

where
\[ \lambda_{k-1}(x) = \lambda'_{k-2}(x) + \lambda_0(x)\lambda_{k-2}(x) + s_{k-2}(x), \quad (10) \]
\[ s_{k-1}(x) = s'_{k-2}(x) + s_0(x)\lambda_{k-2}(x), \quad (11) \]
\[ \lambda_k(x) = \lambda'_{k-1}(x) + \lambda_0(x)\lambda_{k-1}(x) + s_{k-1}(x), \quad (12) \]
\[ s_k(x) = s'_{k-1}(x) + s_0(x)\lambda_{k-1}(x), \quad (13) \]

Division of Eq.(9) with Eq.(8) yields
\[ \frac{\varphi^{(k+2)}(x)}{\varphi^{(k+1)}(x)} = \frac{\lambda_k(x)}{\lambda_{k-1}(x)} \left[ \varphi'(x) + \frac{s_k(x)}{\lambda_k(x)}\varphi(x) \right] \]
\[ \frac{\varphi^{(k+2)}(x)}{\varphi^{(k+1)}(x)} = \frac{\lambda_{k-1}(x)}{\varphi'(x) + \frac{s_{k-1}(x)}{\lambda_{k-1}(x)}\varphi(x)} \]

(14)

For sufficiently large integer values of \(k\), the ratio
\[ \frac{s_k(x)}{\lambda(k)} = \frac{s_{k-1}(x)}{\lambda_{k-1}(x)} \]
converges to an \(\alpha(x)\) function. This is known as the quantization condition and leads to the calculation of the energy spectrum.

III. KLEIN-GORDON EQUATION

In \((3+1)\) dimensions, in the presence of spherical symmetric vector, \(V_v(r)\), and scalar, \(V_s(r)\), potential energies the time independent KG equation with a PDM is given as
\[ \left[ \hbar^2c^2\nabla^2 + \left( \left( E - V_v(r) \right)^2 - \left( m(r,E)c^2 + V_s(r) \right)^2 \right) \right] \phi(r) = 0. \quad (16) \]
Here, $\hbar$ and $c$ represent the reduced Planck constant and the speed of light, respectively. $E$ is the energy of the bosonic particle. The mass term, $m(r, E)$, depends on the energy in addition to the spatial coordinate. The decomposition of the spatial wave function into radial wave function, $R(r)$ and angular-dependent spherical harmonics, $Y^l_m(\theta, \varphi)$, ends up with

$$
\left[ \hbar^2 c^2 \left( \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} \right) + \left( E - V_v(r) \right)^2 - \left( m(r, E)c^2 + V_s(r) \right)^2 \right] u(r) = 0, \quad (17)
$$

where $u(r) \equiv rR(r)$.

### A. Bound state solution

In this manuscript, we investigate bound state solutions of the KG equation with a position and energy dependent mass function that is defined by

$$
m(r, E) = m_0 \left( 1 - \lambda b A \frac{(1 + \delta E)}{r} \right). \quad (18)
$$

Here, $\lambda$ and $m_0$ are the reduced Compton wavelength and the rest mass of the spin-0 particle, respectively. $b$ is the coupling constant. The term $-\frac{A(1+\delta E)}{r}$ is a Coulomb-like energy-related potential energy function that will be called ”vector potential energy” in the rest of the article. $A$ represents the strength of the potential well, where $\delta$ determines the energy contribution. On the other hand, $b$ and $\delta$ can be seen as two tuning parameters that allow projecting the obtained analytic solutions on various physical problems. For instances, $(b = 0)$, $(b = 0, \ \delta = 0)$, and $(b \neq 0, \ \delta = 0)$ cases represent the constant mass with energy dependent Coulomb-like potential energy, Coulomb-like potential, and the position dependent as well as energy independent solutions, respectively.

Note that, we can express the investigated mass energy term as follows:

$$
m(r, E)c^2 = m_0c^2 + bhV_v(r, E). \quad (19)
$$
1. Mixed vector and scalar potential energies with equal magnitude and equal sign case

In this subsection, we use attractive scalar potential energy that has EMES with the vector potential energy.

\[ V(r, E) \equiv V_v(r, E) = V_s(r, E) = -\frac{A(1 + \delta E)}{r} \]  

(20)

The EMES potential energies condition turns Eq. (17) into

\[ \left( \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} \right) + \frac{1}{\hbar^2 c^2} \left( E^2 - m(r, E)^2 c^4 - 2V(r, E) \left( E + m(r, E)c^2 \right) \right) u(r) = 0. \]  

(21)

Then, we employ Eqs. (18) and (20) in Eq. (21) and we obtain

\[ u''(r) + \left[ E^2 - m_0^2 c^4 \frac{2\lambda b A m_0^2 c^4}{\hbar^2 c^2} + 2A(1 + \delta E) \right] (1 + \delta E) \frac{l(l+1)}{r^2} u(r) = 0. \]  

(22)

We introduce a new coordinate transformation, \( z = (1 + \delta E)r \), hence, \( u(r) \rightarrow v(z) \). We get

\[ \frac{d^2 v}{dz^2} + \left( -\tau^2 + \frac{\beta^2}{z} - \frac{\eta(\eta + 1)}{z^2} \right) v = 0. \]  

(23)

where

\[ \tau^2 \equiv -\frac{E^2 - m_0^2 c^4}{\hbar^2 c^2 (1 + \delta E)^2}, \]  

(24)

\[ \beta^2 \equiv \frac{2A(1 + \delta E) + \lambda b m_0^2 c^4}{\hbar^2 c^2}, \]  

(25)

\[ \eta(\eta + 1) \equiv \left( \frac{m_0^2 c^4 \lambda b^2 A^2}{\hbar^2 c^2} + \frac{2\lambda b A^2 m_0 c^2}{\hbar^2 c^2} + \frac{l(l+1)}{(1 + \delta E)^2} \right) (1 + \delta E)^2. \]  

(26)

We put forward an ansatz by examine the asymptotic behaviours

\[ v(z) = e^{-\tau z} z^{\eta+1} f(z) \]  

(27)

and we find

\[ f''(z) = \frac{2\left[ \tau z - (\eta + 1) \right]}{z} f'(z) + \frac{2\tau (\eta + 1) - \beta^2}{z} f(z). \]  

(28)
We compare Eq. (28) with Eq. (1) to adopt the AIM method. We get
\[ \lambda_0(z) = \frac{2(\tau z - (\eta + 1))}{z}, \]
\[ s_0(z) = \frac{2\tau (\eta + 1) - \beta^2}{z}. \] (29) (30)

After several consecutive iterations
\[ \frac{s_0}{\lambda_0} = \frac{s_1}{\lambda_1} \Rightarrow \tau_0 = \frac{\beta^2}{2(\eta_0 + 1)}, \quad n = 0. \] (31)
\[ \frac{s_1}{\lambda_1} = \frac{s_2}{\lambda_2} \Rightarrow \tau_1 = \frac{\beta^2}{2(\eta_1 + 2)}, \quad n = 1. \] (32)
\[ \frac{s_2}{\lambda_2} = \frac{s_3}{\lambda_3} \Rightarrow \tau_2 = \frac{\beta^2}{2(\eta_2 + 3)}, \quad n = 2. \] (33)

We obtain the quantization condition as follows
\[ \tau_n = \frac{\beta^2}{2(\eta_n + n + 1)}. \] (34)

We substitute \( \tau, \beta \) and \( \eta \) parameters and find
\[ \sqrt{\frac{m_0^2 c^4 - E_{nl}^2}{(1 + \delta E_{nl})^2}} = \sqrt{\frac{4}{h^2 c^2} \left( E_{nl} + m_0 c^2 + \lambda bm_0^2 c^4 \right)} \] (35)

where \( n = 0, 1, 2, 3, \ldots \).

2. Mixed vector and scalar potential energies with equal magnitude and opposite sign case

In this subsection, we use repulsive scalar potential energy that possesses equal magnitude with the vector potential energy but with an opposite sign.

\[ V(r, E) = V_v(r, E) = -V_s(r, E) = -\frac{A(1 + \delta E)}{r} \] (36)

We employ the EMOS condition in Eq. (17) and we derive
\[ \left[ \frac{d^2}{dr^2} - \frac{l(l + 1)}{r^2} \right] + \frac{1}{h^2 c^2} \left( E^2 - m(r, E)^2 c^4 - 2V(r, E) \left( E - m(r, E)c^2 \right) \right) u(r) = 0. \] (37)

8
We follow similar steps and find Eq. (23) with different parameters
\begin{align*}
\tau^2 &\equiv -\frac{E^2 - m_0^2c^4}{\hbar^2c^2(1 + \delta E)^2}, \\
\beta^2 &\equiv \frac{2A(E - m_0c^2 + \lambda m_0^2c^4)}{\hbar^2c^2}, \\
\eta(\eta + 1) &\equiv \left( \frac{m_0^2c^4\lambda^2b^2A^2}{\hbar^2c^2} - \frac{2\lambda bA^2m_0c^2}{\hbar^2c^2} + \frac{l(l + 1)}{(1 + \delta E)^2} \right) (1 + \delta E)^2.
\end{align*}

First, we employ the same ansatz and then, apply the AIM method, we get the quantization condition as follows
\begin{equation}
\sqrt{\frac{m_0^2c^4 - E_{nl}^2}{(1 + \delta E_{nl})^2}} = \frac{A}{\hbar c} \left( E_{nl} - m_0c^2 + \lambda m_0^2c^4 \right) + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \sqrt{\frac{m_0^2c^4\lambda^2b^2A^2}{\hbar^2c^2} - \frac{2\lambda bA^2m_0c^2}{\hbar^2c^2} + \frac{l(l + 1)}{(1 + \delta E_{nl})^2}} (1 + \delta E_{nl})^2
\end{equation}

where \( n \) is an integer.

3. Pure vector potential energy case

In this subsection, we examine the case where the scalar potential energy is equal to zero.

\begin{equation}
V_s(r) = 0,
\end{equation}
\begin{equation}
V(r, E) \equiv V_v(r, E) = -\frac{A(1 + \delta E)}{r}
\end{equation}

We would like to remind you that since the variable mass has vector potential energy coupling, even in the absence of scalar potential energy, the KG equation is not in the minimal coupling form. We use the pure vector condition in Eq. (21) and we find
\begin{equation}
\left[ \left( \frac{d^2}{dr^2} - \frac{l(l + 1)}{r^2} \right) + \frac{1}{\hbar^2c^2} \left( E^2 - m(r, E)^2c^4 - V(r, E) \left( 2E - V(r, E) \right) \right) \right] u(r) = 0
\end{equation}

We go through the same procedure given in details above and we obtain Eq. (23) with different parameters as
\begin{align*}
\tau^2 &\equiv -\frac{E^2 - m_0^2c^4}{\hbar^2c^2(1 + \delta E)^2}, \\
\beta^2 &\equiv \frac{2A(E + \lambda m_0^2c^4)}{(\hbar c)^2}, \\
\eta(\eta + 1) &\equiv \left( \frac{A^2}{\hbar^2c^2} \left( m_0^2c^4\lambda^2b^2 - 1 \right) + \frac{l(l + 1)}{(1 + \delta E)^2} \right) (1 + \delta E)^2.
\end{align*}
After, we use the same ansatz and the several iterations in the AIM method, we derive the quantization condition as given

\[ \sqrt{m_0^2c^4 - E_{nl}^2} = \frac{\hbar c (E_{nl} + \lambda bm_0^2c^4)}{(1 + \delta E_{nl})^2} \]

\[ n + \frac{1}{2} \pm \sqrt{\frac{A^2}{\hbar^2c^2} \left( m_0^2c^2 - 1 \right) + \frac{l(l+1)}{(1+\delta E_{nl})^2}} (1 + \delta E_{nl})^2 \]

Here, the quantum number \( n \) is an integer.

4. Pure scalar potential energy case

In this subsection, we investigate the case where the vector potential energy is equal to zero. Consequently, the mass term does not depend on either position or energy. On the other hand, the mass term couples with an attractive scalar potential energy.

\[ V_v(r) = 0, \]
\[ V(r, E) \equiv V_s(r, E) = -\frac{A(1 + \delta E)}{r}. \]

In this case the KG equation given in Eq. (17) turns to be

\[ \left[ \left( \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} \right) + \frac{1}{\hbar^2c^2} \left( E^2 - m_0^2c^4 - V_v(r, E) \right) \frac{2m_0c^2 + V_v(r, E)}{2} \right] u(r) = 0. \]

After basic algebra which is given in details in the previous subsections, we find the quantization condition as follows:

\[ \sqrt{m_0^2c^4 - E_{nl}^2} = \frac{\hbar c (E_{nl} + \lambda bm_0^2c^4)}{(1 + \delta E_{nl})^2} \]

\[ n + \frac{1}{2} \pm \sqrt{\frac{A^2}{\hbar^2c^2} \left( m_0^2c^2 - 1 \right) + \frac{l(l+1)}{(1+\delta E_{nl})^2}} (1 + \delta E_{nl})^2 \]

5. Constant mass energy limit

In this section, we discuss a limit where the vector and scalar potential energy couplings to the mass term vanish. Basically this limit is obtained by employing the definition of the non constant mass term, which is given in Eq. (19), in the KG equation, which is expressed in Eq. (17). We find

\[ \left[ \hbar^2c^2 \left( \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} \right) + \left( E - V_v(r, E) \right)^2 - \left( m_0c^2 + bhV_v(r, E) + V_s(r, E) \right)^2 \right] u(r) = 0. \]
Since \( b \) is an arbitrary parameter, it is always possible to obtain
\[
\left( m_0c^2 + bhV_v(r, E) + V_s(r, E) \right)^2 = m_0^2c^4. \tag{54}
\]
via \( b = -\frac{V_s(r, E)}{W_v(r, E)} \). For instance if we assume an attractive vector and a repulsive scalar potential energies with different magnitudes
\[
V_v(r, E) = -\frac{A}{r}(1 + \delta E), \tag{55}
\]
\[
V_s(r, E) = \frac{B}{r}(1 + \delta E) \tag{56}
\]
with position and energy dependence. Then, we find
\[
b = \frac{B}{\hbar A} \tag{57}
\]
Note that, the tuning parameter \( b \) does not need to be a positive number.

Before we conclude this subsection, we strongly advice to read papers that examine some classes of exactly-solvable KG equations within \( V_v(r) \equiv V_0 + \beta V_s(r) \) relation where \( V_0 \) and \( \beta \) are arbitrary constants \[8, 39, 40\].

### B. Wave function solution

In this section we obtain the wave function solution. We start with a new transformation, \( y = 2\tau z \), in Eq. (28). We find
\[
\frac{d^2g(y)}{dy^2} + \frac{2(\eta + 1)}{y} \frac{dg(y)}{dy} + \frac{\beta^2}{2y} - \frac{(\eta + 1)}{y} g(y) = 0. \tag{58}
\]
This equation is similar to the confluent hypergeometric equation \[41\]
\[
xy'' + (c - x)y' - ay = 0 \tag{59}
\]
which has the solution
\[
y = N_1 {}_1F_1(a, c, x) + N_2 {}_1U_1(a, c, x) \tag{60}
\]
Here, \(_1F_1(a, c, x) \) and \(_1U_1(a, c, x) \) are the first and second kind confluent hypergeometric functions, respectively. We obtain the coefficients with the match of the Eq. (58) and Eq. (59)
\[
a = \frac{2\tau(\eta + 1) - \beta^2}{2\tau} \tag{61}
\]
\[
c = 2(\eta + 1). \tag{62}
\]
and therefore the solution of Eq. (28)

\[
g(r) = N_1 {}_1 F_1 \left( \frac{2\tau(\eta + 1) - \beta^2}{2\tau}, 2(\eta + 1), 2\tau(1 + \delta E)r \right) + N_2 {}_1 U_1 \left( \frac{2\tau(\eta + 1) - \beta^2}{2\tau}, 2(\eta + 1), 2\tau(1 + \delta E)r \right)
\]  

(63)

Then the wave function \( u(r) \) is found to be

\[
u(r) = N_1 e^{-\tau(1+\delta E)r} (1 + \delta E) r^{\eta + 1} {}_1 F_1 \left( \frac{2\tau(\eta + 1) - \beta^2}{2\tau}, 2(\eta + 1), 2\tau(1 + \delta E)r \right).  
\]  

(64)

Note that second kind confluent hypergeometric function does not satisfy the boundary conditions. Hence \( N_2 \) is chosen to be zero.

IV. RESULTS AND DISCUSSIONS

In this section, we use the Secant method to calculate energy spectra in order to present concrete results. We employ a neutral pion as the examined spin-0 particle. The rest mass energy and Compton wavelength of the neutral pion are 134.977 MeV and 1.462 fm, respectively. We assume the strength of the potential energy parameter, \( A \), as 200 MeV \( \cdot \) fm. We assign negative and positive values to the tuning parameters \( \delta \) and \( b \) in addition to their vanishing values. Note that in all figures of unnormalized wave functions, we denote the mass function, and the potential energy with green dots, red dashdots, respectively. Moreover, we use blue color with dashed and orange solid lines to present the lower and upper wave functions.

First, we use Eq. (35) to calculate the energy spectra in the EMES limit. We tabulate the obtained energy spectra in Table I. We illustrate the unnormalized wave functions, the vector potential energy and the mass function in Fig. I. We use blue color with dashed and orange solid lines to present the lower and upper wave functions.

Note that, when \( b \) is zero, \( \lambda b \) becomes zero and the mass loses both energy and position dependence. However, energy dependency on the potential energy continues. To comprehend the effect of energy dependence, we use three values in which the tuning parameter \( \delta \) has negative, \(-0.003 \text{ MeV}^{-1}\), and positive \(0.003 \text{ MeV}^{-1}\), values in addition to zero. In this case, the spectra consist only of the upper or lower energy eigenvalues. During the increase of delta parameter from the negative value to the positive value, we
observe that lower energy eigenvalues increase and upper energy eigenvalues decrease. When $b$ has negative value, we realize that lower energy eigenvalues do not occur. In this case, the spectra consist only of the upper energy eigenvalues, and as the delta increases, the values of the energy eigenvalues decrease. This decrease is relatively smaller as quantum numbers increase. When $b$ has a positive value, the upper and lower energy eigenvalues exist in the energy spectra. Unlike the previous case, the increase of delta parameter decreases the lower energy eigenvalues. On the other hand, like the previous case, the upper energy eigenvalues decreases.

Then, we examine the energy spectra of the EMOS limit by solving Eq. (41). We find that the energy spectra persist non zero values only in positive values of the tuning parameter $b$. We tabulate the calculated energy spectra in Table II. We plot the existing unnormalized wave functions in Fig. 2. One of the authors of this study, BCL, in a study published in 2018, examined the generalized symmetric Woods-Saxon potential energy with a constant mass term in the KG equation and showed that an energy spectrum at the EMOS limit cannot be obtained [42]. The results obtained in this study are in agreement with the results given by BCL. In addition to those results, we show that a spectrum can be calculated with upper energy eigenvalues for a specific value of the effective mass.

Then, we calculate the energy spectra in the PV limit, where $V_s(r) = 0$. We present the energy spectra in Tab. III. We observe that the ground state eigenvalue does not exist, instead the lowest eigenvalue is $E_{11}$. In all cases, the eigenvalues have positive values. The increase of $\delta$ parameter with the negative value of $b$ parameter creates a decrease in the value of the lowest eigenvalue. Furthermore, these decreases have an increment when $b$ becomes positive. We demonstrate some of the unnormalized wave functions in Fig. 3.

We study the pure scalar limit and tabulate the energy spectra in Tab. IV. We illustrate the unnormalized wave functions in the pure scalar limit in Fig. 4. In each particular sub-case, two energy eigenvalues, namely upper and lower eigenvalues are obtained. Similar to the pure vector case, a decrease between the energy eigenvalues is observed in the increasing values of $\delta$ for a fixed $b$ value. To demonstrate it, we plot Fig. 5. There, we illustrate the variation of particular upper and lower energy eigenvalues of
the spectra via $\delta$ parameter for the fixed negative value of $b$. When the potential energy is independent of the energy and the mass has a constant value, the energy eigenvalues become symmetrical as expected. In addition, when the mass and potential energy are independent of energy, a symmetry of upper and lower energy eigenvalues exist. Finally, in Fig. 6 we demonstrate the upper and lower energy eigenvalues of the spectra versus $\lambda b$ values for a negative constant value of the tuning parameter $\delta$. We see a higher differentiation of the eigenvalues for lower quantum numbers. We observe a decrease in upper and an increase in lower eigenvalues for small quantum numbers. As the quantum numbers for a fixed $lb$ increase, the difference between the two adjacent energy levels decreases.

V. CONCLUSION

In this paper, we obtained the bound state solution of a Klein Gordon particle whose mass varies via position and energy. We took into account an energy-dependent Coulomb-like potential energy and employed the asymptotic iteration method. We examined the solution in five different limits, i.e. mixed vector and scalar potential energies with equal magnitudes with equal/opposite signs, pure vector, pure scalar, and constant mass. Except for the trivial constant mass limit, in each cases, we derived a transcendental equation that yields to calculating the energy spectrum. We showed that the radial wave functions are occurring in terms of the confluent hypergeometric functions. Next, we considered a neutral pion particle and used the Secant numerical methods to obtain an energy spectrum in each limit. Although the potential energy parameters were chosen at random, we have set various tuning parameters to investigate all critical situations. We verified the tabulated energy eigenvalues by examining their corresponding wave function behavior which fulfills the boundary conditions. Finally, we discussed the role of the tuning parameters in the pure scalar limit. We believe that the results
obtained in this article would have applications in physics.

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FIG. 1. Unnormalized wave functions in the EMES limit. The horizontal axis is the radial distance in \( fm \) unit.
|          | E_{00} | E_{10} | E_{11} | E_{20} | E_{21} | E_{22} |
|----------|--------|--------|--------|--------|--------|--------|
| $E_{00}$ | No Eigenvalue |        |        |        |        |        |
| $E_{10}$ | No Eigenvalue |        |        |        |        |        |
| $E_{11}$ | 133.472 |        |        |        |        |        |
| $E_{20}$ | 134.397 |        |        |        |        |        |
| $E_{21}$ | 134.717 |        |        |        |        |        |
| $E_{22}$ | 134.813 |        |        |        |        |        |

**FIG. 2.** Unnormalized wave functions in the EMOS limit. The horizontal axis is the radial distance in $fm$ unit.
FIG. 3. Unnormalized wave functions in the pure vector limit. The horizontal axis is the radial distance in $fm$ unit.
FIG. 4. Unnormalized wave functions in the pure scalar limit. The horizontal axis is the radial distance in $fm$ unit.
FIG. 5. The differentiation of particular upper and lower energy eigenvalues versus $\delta$ in the pure scalar case. Here, the other tuning parameter $\lambda b = -0.003 \text{ MeV}^{-1}$. 
FIG. 6. The differentiation of particular upper and lower energy eigenvalues versus $\lambda b$ in the pure scalar case. Here, the other tuning parameter $\delta = -0.003 \text{ MeV}^{-1}$. 
\begin{table}
\begin{tabular}{cccccccccccc}
0.00000 & 0.00000 & -1.81465 & None & None & None & None & None & None & None & None & None \\
 & None & 79.81538 & 107.32122 & 107.32122 & 118.69067 & 124.32234 & 118.69067 & 124.32234 & 127.48761 & 129.43379 \\
0.00000 & 0.00300 & -129.64146 & -133.19720 & -133.88866 & -134.07683 & -134.34733 & -134.54808 & -134.43223 & -134.56603 & -134.67698 & -134.75200 \\
 & 22.77861 & 83.76911 & 101.24833 & 106.48183 & 114.41062 & 120.62914 & 117.00422 & 121.19553 & 124.78258 & 127.26083 \\
0.00300 & -0.00300 & None & None & None & None & None & None & None & None & None \\
 & None & None & 98.91588 & None & 112.76280 & 121.45996 & None & 120.24283 & 125.36472 & 127.98363 \\
0.00300 & 0.00000 & -1.29096 & None & None & None & None & None & None & None & None \\
 & None & 63.05730 & 92.57467 & 92.57467 & 107.70124 & 116.20530 & 107.70124 & 116.20530 & 121.36605 & 124.69813 \\
0.00300 & 0.00300 & -132.17136 & -134.17013 & -134.55301 & -134.59518 & -134.73627 & -134.81915 & -134.75456 & -134.82187 & -134.86714 & None \\
 & 18.73778 & 71.69401 & 87.71693 & 95.21508 & 103.47799 & 111.34457 & 107.83252 & 112.63070 & 117.52188 & 121.28761 \\
\end{tabular}
\caption{Calculated energy spectrum in the EMES limit of the mixed potential energies. Energy eigenvalues are denoted with two quantum numbers \( n \) and \( l \) as \( E_{nl} \). The units of \( \delta \) and \( \lambda b \) are \( MeV^{-1} \). The energy eigenvalue is in \( MeV \) unit.}
\end{table}
| δ  | λb  | n  | l  | E  | E  | E  | E  | E  | E  | E  | E  |
|----|-----|----|----|----|----|----|----|----|----|----|----|
| 0.00000 | 0.00000 | None | None | None | None | None | None | None | None | None | None |
| None | None | None | None | None | None | None | None | None | None | None | None |
| 0.00000 | 0.00300 | None | None | None | None | None | None | None | None | None | None |
| None | None | 133.55348 | None | 134.19336 | 134.50388 | None | 134.48332 | 134.64992 | 134.74036 | None | None |
| 0.00300 | -0.00300 | None | None | None | None | None | None | None | None | None | None |
| None | None | None | None | None | None | None | None | None | None | None | None |
| 0.00300 | 0.00000 | None | None | None | None | None | None | None | None | None | None |
| None | None | None | None | None | None | None | None | None | None | None | None |
| 0.00300 | 0.00300 | None | None | None | None | None | None | None | None | None | None |
| None | None | 131.74999 | None | 133.23676 | 134.00740 | None | 133.89988 | 134.30896 | 134.50141 | None | None |

**TABLE II.** Calculated energy spectrum in the EMOS limit of the mixed potential energies. Energy eigenvalues are denoted with two quantum numbers $n$ and $l$ as $E_{nl}$. The units of $\delta$ and $\lambda b$ are $MeV^{-1}$. The energy eigenvalue is in $MeV$ unit.
| $\delta$  | $\lambda b$ | $E_{00}$ | $E_{10}$ | $E_{11}$ | $E_{20}$ | $E_{21}$ | $E_{22}$ | $E_{30}$ | $E_{31}$ | $E_{32}$ | $E_{33}$ |
|----------|----------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 0.00000  | 0.00000  | None    | None    | None    | None    | None    | None    | None    | None    | None    | None |
| None     | None     | 125.7963 | 129.9412 | 132.0477 | 131.8229 | 132.9521 | 133.5233 | None    | None    | None    | None |
| 0.00000  | 0.00300  | None    | None    | None    | None    | None    | None    | None    | None    | None    | None |
| None     | None     | 118.1035 | 125.5211 | 129.3188 | 128.9835 | 131.0480 | 132.1380 | None    | None    | None    | None |
| 0.00300  | -0.00300 | None    | None    | None    | None    | None    | None    | None    | None    | None    | None |
| None     | None     | 127.3731 | 130.8900 | 132.8447 | 132.4708 | 133.5083 | 133.9413 | None    | None    | None    | None |
| 0.00300  | 0.00000  | None    | None    | None    | None    | None    | None    | None    | None    | None    | None |
| None     | None     | 113.8281 | 122.9972 | 129.0045 | 127.5082 | 130.8328 | 132.0645 | None    | None    | None    | None |
| 0.00300  | 0.00300  | None    | None    | None    | None    | None    | None    | None    | None    | None    | None |
| None     | None     | 102.7271 | 115.5703 | 123.9710 | 122.3532 | 127.2193 | 129.4167 | None    | None    | None    | None |

TABLE III. Calculated energy spectrum in the pure vector limit of the mixed potential energies. Energy eigenvalues are denoted with two quantum numbers $n$ and $l$ as $E_{nl}$. The units of $\delta$ and $\lambda b$ are \textit{MeV}$^{-1}$. The energy eigenvalue is in \textit{MeV} unit.
| $\delta$ | $\lambda b$ | $E_{00}$ | $E_{10}$ | $E_{11}$ | $E_{20}$ | $E_{21}$ | $E_{22}$ | $E_{30}$ | $E_{31}$ | $E_{32}$ | $E_{33}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0000 | 0.0000 | -105.71706 | -124.55269 | -128.49478 | -129.60941 | -131.19239 | -132.38590 | -131.70348 | -132.49571 | -133.15986 | -133.61163 |
| 0.0000 | 0.00300 | -95.22484 | -118.90859 | -123.76368 | -126.17664 | -128.25226 | -130.15812 | -129.40852 | -130.48890 | -131.56907 | -132.36880 |
| 0.00300 | 0.00300 | -127.33455 | -132.98452 | -134.02566 | -134.07068 | -134.46021 | -134.63349 | -134.73678 | -134.80009 |
| 0.00300 | 0.00300 | -112.14923 | -127.07890 | 130.31170 | 130.97926 | 132.26062 | 133.14380 | 132.56605 | 133.20127 | 133.69216 | 134.01646 |
| 0.00300 | 0.00000 | -117.23009 | -129.98704 | -132.36231 | -132.63318 | -133.49989 | -134.01007 | -133.61571 | -134.02737 | -134.30500 | -134.47994 |
| 0.00300 | 0.00300 | -95.22484 | 119.53740 | 124.15598 | 126.39841 | 128.40409 | 130.24408 | 129.50566 | 130.55972 | 131.61308 | 132.39564 |
| 0.00300 | 0.00300 | -107.10831 | -126.36678 | -130.01599 | -130.78578 | -132.16035 | -133.09651 | -132.49330 | -133.15825 | -133.66895 | -134.00335 |
| 0.00300 | 0.00300 | -87.73377 | 105.71706 | 117.75088 | 121.77591 | 124.07708 | 126.64236 | 126.17297 | 127.44184 | 128.95466 | 130.21501 |

**TABLE IV.** Calculated energy spectrum in the pure scalar limit of the mixed potential energies. Energy eigenvalues are denoted with two quantum numbers $n$ and $l$ as $E_{nl}$. The units of $\delta$ and $\lambda b$ are $MeV^{-1}$. The energy eigenvalue is in $MeV$ unit.