New surfaces with $K^2 = 7$ and $p_g = q \leq 2$

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Abstract

We construct smooth minimal complex surfaces of general type with $K^2 = 7$ and: $p_g = q = 2$, Albanese map of degree 2 onto a (1, 2)-polarized abelian surface; $p_g = q = 1$ as a double cover of a quartic Kummer surface; $p_g = q = 0$ as a double cover of a numerical Campedelli surface with 5 nodes.

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1 Introduction

Despite the efforts of several authors in past years, surfaces of general type with the lowest possible value of the holomorphic Euler characteristic $\chi = 1$ are still not classified. For these surfaces the geometric genus $p_g$, the irregularity $q$ and the self-intersection of the canonical divisor $K$ satisfy:

$$1 + p_g \leq K^2 \leq 9 \text{ if } p_g \leq 1,$$

$$4 \leq K^2 \leq 9 \text{ if } p_g = 2,$$

$$K^2 = 6 \text{ or } 8 \text{ if } p_g = 3,$$

$$K^2 = 8 \text{ if } p_g = 4,$$

from the Bogomolov-Miyaoka-Yau and Debarre inequalities, [HP] and the Beauville Appendix in [De] (cf. also [CCM], [Pi]).

According to Sai-Kee Yeung, the case with $p_g = q = 2$ and $K^2 = 9$ does not occur (see Section 6 in the revised version of the paper [Ye], available at http://www.math.purdue.edu/~yeung/).

So there are examples for all possible values of the invariants except for one mysterious case:

$$K^2 = 7, p_g = q = 2.$$

The cases $K^2 = 7, p_g = q = 1$ or 0 are also intriguing:

- $p_g = q = 1$. Lei Zhang [Zh] has shown that one of three cases occur: a) the bicanonical map is birational; b) the bicanonical map is of degree 2 onto a rational surface; c) the bicanonical map is of degree 2 onto a Kummer surface. The author has given examples for a) [Ri2] and b) [Ri3], but so far it is not known if c) can occur.
• $p_g = q = 0$. Yifan Chen [Ch2] considers the case when the automorphism group of the surface $S$ contains a subgroup isomorphic to $\mathbb{Z}_2^2$. He shows that three different families of surfaces may exist:
  a) $S$ is an Inoue surface [In];
  b) $S$ belongs to the family constructed by him in [Ch1];
  c) a third case, in particular $S$ is a double cover of a surface with $p_g = q = 0$ and $K^2 = 2$ with 5 nodes.
The existence of this last case is an open problem.

In this paper we show the existence of the above three open cases. We give constructions for surfaces with $K^2 = 7$ and:

• $p_g = q = 2$, Albanese map of degree 2 onto a $(1, 2)$-polarized abelian surface;
• $p_g = q = 1$, bicanonical map of degree 2 onto a Kummer surface;
• $p_g = q = 0$ as a double cover of a numerical Campedelli surface with 5 nodes.

In all cases the surface can be seen as a double cover with branch locus as in the result below. In particular we show that a construction for the case $p_g = q = 2$ as suggested by Penegini and Polizzi [PP, Remark 2.2] does exist.

**Proposition 1.** Let $X$ be an Abelian, $K3$ or Enriques surface containing $n$ disjoint $(-2)$-curves $A_1, \ldots, A_n$, $n = 0, 16$ or 8, respectively. Assume that $X$ contains a reduced curve $B$ and a divisor $L$ such that

$$B + \sum_1^n A_i \equiv 2L,$$

$B$ is disjoint from $\sum_1^n A_i$, $B^2 = 16$ and $B$ contains a $(3, 3)$-point and no other singularity. Let $S$ be the smooth minimal model of the double cover of $X$ with branch locus $B + \sum_1^n A_i$. Then $\chi(O_S) = 1$ and $K^2_S = 7$.

**Proof:**
This follows from the double cover formulas (see e.g. [BHPV, V.22]) and the fact that a $(3, 3)$-point decreases both $\chi$ and $K^2$ by 1 (see e.g. [Pol, p. 185]):

$$\chi(O_S) = 2\chi(O_X) + \frac{1}{2}L(K_X + L) - 1 = 1,$$

$$K^2_S = 2(K_X + L)^2 + n - 1 = 7.$$

\[\square\]

**Notation**

We work over the complex numbers. All varieties are assumed to be projective algebraic. A $(-n)$-curve on a surface is a curve isomorphic to $\mathbb{P}^1$ with self-intersection $-n$. An $(m_1, m_2)$-point of a curve, or point of type $(m_1, m_2)$, is a singular point of multiplicity $m_1$ which resolves to a point of multiplicity $m_2$ after one blow-up. Linear equivalence of divisors is denoted by $\equiv$. The rest of
the notation is standard in Algebraic Geometry.

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2 Bidouble covers

A bidouble cover is a finite flat Galois morphism with Galois group $\mathbb{Z}_2^2$. Following [Ca] or [Pa], to define a bidouble cover $\pi : V \to X$, with $V, X$ smooth surfaces, it suffices to present:

- smooth effective divisors $D_1, D_2, D_3 \subset X$ with pairwise transverse intersections and no common intersection;
- line bundles $L_1, L_2, L_3$ such that $L_g + D_g \equiv L_j + L_k$ for each permutation $(g, j, k)$ of $(1, 2, 3)$.

One has

$$\chi(\mathcal{O}_V) = 4\chi(\mathcal{O}_X) + \frac{1}{2} \sum_{i=1}^{3} L_i(K_X + L_i),$$

$$p_g(V) = p_g(X) + \sum_{i=1}^{3} h^0(X, \mathcal{O}_X(K_X + L_i))$$

and

$$2K_V \equiv \pi^* \left( 2K_X + \sum_{i=1}^{3} L_i \right),$$

which implies

$$K^2_V = \left( 2K_X + \sum_{i=1}^{3} L_i \right)^2.$$

3 Example with $p_g = q = 2$

Step 1

Let $T_1, \ldots, T_4 \subset \mathbb{P}^2$ be distinct lines through a point $p_0$, let $p_1, p_2 \neq p_0$ be points in $T_1, T_2$, respectively, and $C_1, C_2$ be distinct smooth conics tangent to $T_1, T_2$ at $p_1, p_2$. Consider the map

$$\mu : X \to \mathbb{P}^2$$

which resolves the singularities of the divisor $C_1 + C_2 + T_1 + \cdots + T_4$. Then $\mu$ is given by blow-ups at

$$p_0, p_1, p_1', p_2, p_2', p_3, \ldots, p_{10},$$
where $p_i'$ is the point infinitely near to $p_i$ corresponding to the line $T_i$, and $p_3, \ldots, p_{10}$ are nodes of $C_1 + C_2 + T_3 + T_4$. Let $E_0, E_1, E_1', E_2, E_2', \ldots, E_{10}$ be the corresponding exceptional divisors (with self-intersection $-1$) and let

$$\pi : V \rightarrow X$$

be the bidouble cover defined by the divisors

$$D_1 := (\overline{T_1 + T_2} - 2E_0 - 2E_1' - 2E_2') + \sum_{i=3}^{10} E_i,$$

$$D_2 := T_3 + T_4 - 2E_0 - \sum_{i=3}^{10} E_i,$$

$$D_3 := C_1 + C_2 - 2E_1 - 2E_2 - 2E_2' - \sum_{i=3}^{10} E_i,$$

where the notation $\overline{\cdot}$ stands for the total transform $\mu^\ast(\cdot)$.

Notice that $D_1$ is the union of $\sum_{i=3}^{10} E_i$ with four $(-2)$-curves contained in the pullback of $T_1 + T_2$, and $D_2, D_3$ are just the strict transforms of $T_3 + T_4$, $C_1 + C_2$, respectively.

One can easily see that the divisors $D_1, D_2$ and $D_3$ have pairwise transverse intersections and no common intersection.

Denote by $T$ a general line of $\mathbb{P}^2$ and let

$$L_1 := 3\overline{T} - E_0 - E_1 - E_1' - E_2 - E_2' - \sum_{i=3}^{10} E_i,$$

$$L_2 := 3\overline{T} - E_0 - E_1 - 2E_1' - E_2 - 2E_2',$$

$$L_3 := 2\overline{T} - 2E_0 - E_1' - E_2.'$$

Then

$$K_X + L_1 \equiv 0,$$

$$K_X + L_2 \equiv -E_1' - E_2' + \sum_{i=3}^{10} E_i,$$

$$K_X + L_3 \equiv -\overline{T} - E_0 + E_1 + E_2 + \sum_{i=3}^{10} E_i$$

and

$$\chi(\mathcal{O}_V) = 4 + \frac{1}{2}(0 - 4 - 4) = 0,$$

$$p_g(V) = 0 + 1 + 0 + 0 = 1.$$

Let $X_1$ be the surface given by the double covering $\phi : X_1 \rightarrow X$ with branch locus $D_2 + D_3$. The divisor $\phi^\ast(\overline{T_1 + T_2} - 2E_0 - 2E_1' - 2E_2')$ is a disjoint union of $8$ $(-2)$-curves, and the divisor $\phi^\ast(\sum_{i=3}^{10} E_i)$ is also a disjoint union of $8$ $(-2)$-curves. Hence $\phi^\ast(D_1)$ is a disjoint union of $16$ $(-2)$-curves. The canonical divisor $K_V$ of $V$ is the support of the pullback of $D_1$, a disjoint union of $16$ $(-1)$-curves. So the minimal model $V'$ of $V$ is an abelian surface, with Kummer surface $X_1$. Notice that the lines $T_1, \ldots, T_4$ give rise to elliptic fibres of type $I_0^*$ in $X_1$ (four disjoint $(-2)$-curves plus an elliptic curve with multiplicity 2).

**Step 2**

Now let $R$ be the tangent line to $C_1$ at $p_3 \in C_1 \cap T_3$. We claim that the strict transform $\overline{R} \subset V'$ of $R$ is a curve with a tacnode (singularity of type $(2, 2)$) at the pullback of $p_3$ and with self-intersection $\overline{R}^2 = 8$. In fact, the covering $\pi$ factors as

$$V \xrightarrow{\varphi} X_1 \xrightarrow{\phi} X.$$
The strict transforms $R', C'_1 \subset X$ of $R, C_1$ meet at a point in the $(-1)$-curve $E_3$. Since $C'_1$ is contained in the branch locus of the covering $\phi$, then the curve $R'' := \phi^*(R')$ is tangent to the $(-2)$-curve $\phi^*(E_3)$. This curve is in the branch locus of $\varphi$, hence the curve $R''' := \varphi^*(R'')$ has a node at a point $p$ in the $(-1)$-curve 
$$
E_3 := \frac{1}{2}(\phi \circ \varphi)^*(E_3).
$$

So the image of $R'''$ in the minimal model $V'$ of $V$ is a curve $\widehat{R}$ with a tacnode.

The reduced strict transform of the conic $C_1$ passes through $p$, hence its image $\widehat{C}_1 \subset V'$ is tangent to $\widehat{R}$ at the tacnode. So the divisor $\widehat{R} + \widehat{C}_1$ is reduced and has a singularity of type $(3, 3)$. We want to show that it is even, i.e. there is a divisor $L$ such that 
$$
\widehat{R} + \widehat{C}_1 \equiv 2L,
$$

and that 
$$
(\widehat{R} + \widehat{C}_1)^2 = 16.
$$

**Step 3**

The pencil of lines through the point $p_0$ induces an elliptic fibration of the surface $V$. For $i = 1, 2$, the line $T_i$ gives rise to a fibre (counted twice) which is the union of disjoint $(-1)$-curves $\xi'_1, \ldots, \xi'_4$ with an elliptic curve $T'_i$ such that $\xi'_i T'_i = 1$. These curves can be labeled such that $\xi'_1, \xi'_2$ correspond to the strict transform of $T_i$ and $\xi'_3, \xi'_4$ correspond to the $(-2)$-curve $E_i - E'_i$. The curve $R'''$ meets $\xi'_1, \xi'_2, \xi'_3, \xi'_4$, thus $R^3 = 4 + 4 = 8$ and then $(\widehat{R} + \widehat{C}_1)^2 = 8 + 0 + 2 \times 4 = 16$.

Let $H$ be the line through the points $p_1$ and $p_2$. We have 
$$
\pi^*(R + H) = R'''' + H' + 2E'_3 + \sum_{i=1}^{2} (T'_i + 2\xi'_3 + 2\xi'_4),
$$

where $H' \subset V$ is the strict transform of $H$. Denote by $\widehat{R}, \widehat{H}, \widehat{T}_1$ and $\widehat{T}_2$ the projections of $R''''$, $H'$, $T'_1$ and $T'_2$ into the minimal model $V'$ of $V$. Then there is a divisor $L'$ such that 
$$
\widehat{R} + \widehat{H} + \widehat{T}_1 + \widehat{T}_2 \equiv 2L'.
$$
The pencil of conics tangent to the lines $T_1, T_2$ at $p_1, p_2$ induces another elliptic fibration of the surface $V'$. The curves $\hat{C}_1$ and $\hat{H}$ are fibres of this fibration. We have

$$\hat{R} + \hat{C}_1 + \hat{H} + \hat{C}_1 + \hat{T}_1 + \hat{T}_2 \equiv 2(L' + \hat{C}_1).$$

Since the above fibrations have elliptic bases, the sums $\hat{H} + \hat{C}_1$ and $\hat{T}_1 + \hat{T}_2$ are even, thus there exists a divisor $L$ such that $\hat{R} + \hat{C}_1 \equiv 2L$.

**Step 4**

Finally, consider the double cover

$$\rho : S' \to V'$$

with branch locus $\hat{R} + \hat{C}_1$, determined by $L$. It follows from Proposition 1 that the smooth minimal model $S$ of $S'$ is a surface of general type with $\chi = 1$ and $K^2 = 7$. It is known that there is no smooth minimal surface of general type with $\chi = 1$, $K^2 = 7$ and $q > 2$ (see [HP] and the Beauville Appendix in [De]). Since $q(S) \geq q(V') = 2$, we conclude that $p_g(S) = q(S) = 2$.

Recall that $p_3 \in C_1 \cap T_3$ and assume that $p_4 \in C_2 \cap T_4$. The branch curve $C_1 + C_2 + T_1 + \cdots + T_4$ is determined by the points $p_0, \ldots, p_4$. Since any two sequences of 4 points in $\mathbb{P}^2$, in general position, are projectively equivalent, we can fix $p_0, \ldots, p_3$. This implies that our family of examples is parametrized by a 2-dimensional open subset of $\mathbb{P}^2$.

**4 Example with $p_g = q = 1$**

Let $T_1, T_2, T_3 \subset \mathbb{P}^2$ be distinct lines through a point $p_0$ and $p_1, p_2, p_3 \neq p_0$ be non-collinear points in $T_1, T_2, T_3$, respectively. For the construction of an example with $p_g = q = 0$ and $K^2 = 7$, Y. Chen has shown that for a general point $p_4 \neq p_0, \ldots, p_3$, there exist:

- an irreducible sextic curve $C_6$ with a node at $p_0$, a tacnode at $p_i$ with tangent line $T_i, i = 1, 2, 3$, and having a triple point at $p_4$;
- an irreducible quintic curve $C_5$ through $p_0, p_4$ and with a tacnode at $p_i$ with tangent line $T_i, i = 1, 2, 3$.

The curves $C_5, C_6$ correspond to the curves $\hat{B}_2, \hat{B}_3$ given in [Ch1] Proposition 2.5.

Let $T$ be a general line through $p_0$. Keeping a notation analogous to the one in Section 3 consider the map

$$\mu : X \to \mathbb{P}^2$$

which resolves the singularities of the curve $C_6$ and let

$$\pi : V \to X$$

be the bidouble cover defined by the divisors

$$D_1 := \widetilde{T} - E_0 + E_4,$$

$$D_2 := \widetilde{T}_1 + \widetilde{T}_2 + \widetilde{T}_3 - 3E_0 - \sum_{i=1}^3 2E_i' + \left( \widetilde{C}_6 - 2E_0 - \sum_{i=1}^3 (2E_i + 2E_i') - 3E_4 \right),$$

$$D_3 := \widetilde{C}_5 - E_0 - \sum_{i=1}^3 (2E_i + 2E_i') - E_4.$$
Notice that $D_2$ is the union of the strict transform of $C_6$ with six $(-2)$-curves contained in the pullback of $T_1 + T_2 + T_3$, and $D_3$ is the strict transform of $C_5$.

We verify that the divisors $D_1$, $D_2$ and $D_3$ have pairwise transverse intersections and no common intersection. Let $\tilde{C}_5$, $\tilde{C}_6$ be the strict transforms of $C_5$, $C_6$. These curves are disjoint from the $(-2)$-curves contained in the pullback of $T_1 + T_2 + T_3$. It is shown in \[Ch1\] Proposition 2.5 that the divisor $\tilde{C}_5 + \tilde{C}_6 + E_4$ has at most nodal singularities. Since the line $T$ through $p_0$ is generic, the result follows.

We have
\[
\begin{align*}
L_1 & := 7\tilde{T} - 3E_0 - \sum_1^3(2E_i + 3E_i') - 2E_4, \\
L_2 & := 3\tilde{T} - E_0 - \sum_1^3(E_i + E_i'), \\
L_3 & := 5\tilde{T} - 3E_0 - \sum_1^3(E_i + 2E_i') - E_4, \\
K_X + L_1 & \equiv 4\tilde{T} - 2E_0 - \sum_1^3(E_i + 2E_i') - E_4, \\
K_X + L_2 & \equiv E_2, \\
K_X + L_3 & \equiv 2\tilde{T} - 2E_0 - \sum_1^3 E_i'
\end{align*}
\]
and
\[
2K_X + \sum_{i=1}^3 L_i \equiv 9\tilde{T} - 5E_0 - \sum_{i=1}^3 (2E_i + 4E_i') - E_4.
\]
Thus
\[
\chi(O_V) = 4 + \frac{1}{2}(-4 + 0 - 2) = 1,
\]
\[
p_g(V) = 0 + 0 + 1 + 0 = 1
\]
and
\[
K_V^2 = -5.
\]
Since the minimal model $V'$ of $V$ is obtained contracting the 12 $(-1)$-curves contained in $\pi^*(\tilde{T}_1 + \tilde{T}_2 + \tilde{T}_3)$, then $K_{V'}^2 = 7$.

Notice that the minimal smooth resolution of the double plane $Q \rightarrow X$ with branch locus $D_1 + D_3$ is a $K_3$ surface with 16 disjoint $(-2)$-curves, and one can obtain the surface $V$ as a double cover of $Q$ with a branch curve $B$ as in Proposition [1]. It can be shown that the bicanonical map of $V$ factors through this double covering. In fact, it follows from [Zh] that the bicanonical map is of degree 2 onto a Kummer surface.

Finally we can see, as in [Ch1] Section 3, that this family of examples is parametrized by a 3-dimensional open subset of $\mathbb{P}^2 \times \mathbb{P}^1$: the point $p_4$ moves in an open subset of $\mathbb{P}^2$ and $\tilde{T} - E_0$ moves in a pencil.

5 Example with $p_g = q = 0$

In [Ri2] §4.6, the author has computed points $p_0, \ldots, p_5 \in \mathbb{P}^2$ such that there exist:

- an irreducible curve $C_7$ of degree 7 with triple points at $p_0, p_5$ and tacnodes at $p_1, \ldots, p_4$ with tangent line the line $T_i$ through $p_0, p_i$, $i = 1, \ldots, 4$;
- an irreducible curve $C_6$ of degree 6 with a node at $p_0$, tacnodes at $p_1, \ldots, p_4$ with tangent line the line $T_i$ through $p_0, p_i$, $i = 1, \ldots, 4$, and passing through $p_5$ such that the singularity of $C_6 + C_7$ at $p_5$ is ordinary.
For the readers convenience, we give in the Appendix the equations of the curves $C_6, C_7$ computed in \cite{Ri2} §4.6 (but with a different choice of $p_0, \ldots, p_5$ in order to get shorter equations) and we verify that the curves are exactly as stated above.

We note that for generic points $p_0, \ldots, p_5 \in \mathbb{P}^2$ there is no such curve $C_7$. This is because the dimension of the linear system of plane curves of degree 7 is 35, and the imposition of singularities as above puts 36 conditions. We don’t know how to construct $C_7$ without using computer algebra. Thus here we compute just one surface, and we make no considerations about the dimension of the family of surfaces.

Keeping a notation as above, consider the map

$$\mu : X \rightarrow \mathbb{P}^2$$

which resolves the singularities of the curve $C_7$ and let

$$\pi : V \rightarrow X$$

be the bidouble cover defined by the divisors

$$D_1 := (\overline{T}_1 - E_0 - 2E'_1) + E_5,$$
$$D_2 := (\overline{T}_4 - E_0 - 2E'_4) + (\overline{C}_6 - 2E_0 - \sum^4_1 (2E_i + 2E'_i) - E_5),$$
$$D_3 := (\overline{T}_2 + \overline{T}_3 - 2E_0 - 2E'_2 - 2E'_3) + (\overline{C}_7 - 3E_0 - \sum^4_1 (2E_i + 2E'_i) - 3E_5).$$

Notice that $D_1$ is the union of $E_5$ with two $(-2)$-curves contained in the pullback of $T_1$, the divisor $D_2$ is the union of the strict transform of $C_6$ with two $(-2)$-curves contained in the pullback of $T_4$, and $D_3$ is the union of the strict transform of $C_7$ with four $(-2)$-curves contained in the pullback of $T_2 + T_3$.

To show that the divisors $D_1, D_2$ and $D_3$ have pairwise transverse intersections and no common intersection, notice that the strict transforms $\overline{C}_6, \overline{C}_7$ of $C_6, C_7$ meet at an unique point, because the intersection number of $C_6$ and $C_7$ at the points $p_0, \ldots, p_5$ is $6 + 4 \times 8 + 3 = 41$. It suffices to show that this point is not in $E_5$. In the Appendix we compute that in fact the singularities of $C_6 + C_7$ at $p_0, \ldots, p_5$ are no worse than stated; there is an ordinary double point not in $\{p_0, \ldots, p_5\}.$

Let $T$ be a general line through $p_0$. We have

$$L_1 := 8\overline{T} - 4E_0 - (2E_1 + 2E'_1) - \sum^4_2 (2E_i + 3E'_i) - 2E_5,$$
$$L_2 := 5\overline{T} - 3E_0 - \sum^4_1 (E_i + 2E'_i) - (E_4 + E'_4) - E_5,$$
$$L_3 := 4\overline{T} - 2E_0 - (E_1 + 2E'_1) - \sum^4_1 (E_i + E'_i) - (E_4 + 2E'_4),$$

$$K_X + L_1 \equiv (\overline{T}_2 + \overline{T}_3 + \overline{T}_4 - 3E_0 - \sum^4_2 2E'_i) + \left(2\overline{T} - (E_1 + E'_1) - \sum^4_1 E_i\right),$$
$$K_X + L_2 \equiv 2\overline{T} - 2E_0 - \sum^4_1 E'_i,$$
$$K_X + L_3 \equiv \overline{T} - E_0 - E'_1 - E'_4 + E_5$$

and

$$2K_X + \sum^3_1 L_i \equiv 11\overline{T} - 7E_0 - \sum^4_1 (2E_i + 4E'_i) - E_5.$$
The divisor
\[ \tilde{T}_2 + \tilde{T}_3 + \tilde{T}_4 - 3E_0 - \sum_{2}^{4} 2E_i' \]
is a disjoint union of 6 \((-2)\)-curves, each meeting \(K_X + L_1\) with intersection number \(-1\). Hence \(K_X + L_1\) is effective only if
\[ 2\tilde{T} - (E_1 + E_1') - \sum_{2}^{5} E_i \]
is effective. This is not the case, we can verify that the conic through the points \(p_1, \ldots, p_5\) is not tangent to the line \(T_1\). Therefore \(h^0(X, O_X(K_X + L_1)) = 0\) and then
\[ p_9(V) = 0 + 0 + 0 + 0 = 0. \]
Also
\[ \chi(O_V) = 4 + \frac{1}{2}(-2 - 2 - 2) = 1 \]
and
\[ K_V^2 = -9. \]
Since the minimal model \(V'\) of \(V\) is obtained contracting the 16 \((-1)\)-curves contained in \(\pi^* (\tilde{T}_1 + \cdots + \tilde{T}_4)\), then \(K_{V'}^2 = 7\).

The covering \(\pi\) factors as
\[ V \rightarrow Y \rightarrow X, \]
where \(Y \rightarrow X\) is the double cover with branch locus \(D_2 + D_3\). Using the double cover formulas, one can verify that the smooth minimal model of \(Y\) is a numerical Campedelli surface \((p_g = q = 0, K^2 = 2)\). The double cover \(V \rightarrow Y\) is ramified over the pullback of \(D_1\) (which contains four \((-2)\)-curves) and over the node corresponding to the transverse intersection of \(D_2\) and \(D_3\).

**Appendix: Magma code**

Here we use the computer algebra system Magma [BCP] to show that the curves \(C_6\) and \(C_7\) referred in Section 5 are exactly as stated there. This code can be tested on the online Magma calculator [MC].

```magma
R<i>:=PolynomialRing(Rationals());
K<i>:=ext<Rationals()|i^2+1>;
P<x,y,z>:=ProjectiveSpace(K,2);
F6:=4*x^6-273*x^2*y^4-481*y^6+720*x^4*y*z+1740*x^2*y^3*z+4020*y^5*z-520*x^4*z^2-3190*x^2*y^2*z^2-12670*y^4*z^2+1200*x^2*y*z^3+17700*y^3*z^3+900*x^2*z^4-9225*y^2*z^4;
F7:=12*x^7+(8*i+420)*x^6*y+1611*x^5*y^2+(174*i+3060)*x^4*y^3+4086*x^3*y^4+(924*i+3360)*x^2*y^5+987*x*y^6+(-242*i+720)*y^7-560*x^6*z-4320*x^5*y*z+(-480*i+13580)*x^4*y^2*z+23940*x^3*y^3*z+97560*x^2*y^4*z+(1320*i+6960)*y^5*z+2760*x^5*z^2+(240*i+16200)*x^4*y*z^2+44970*x^3*y^2*z^2+9
```
\[(9780 \cdot i + 63900) \cdot x^2 \cdot y^3 \cdot z^2 + 39210 \cdot x \cdot y^4 \cdot z^2 + (-2460 \cdot i + 25200) \cdot y^5 \cdot z^2 - 4400 \cdot x^4 \cdot z^3 - 28800 \cdot x^3 \cdot y^3 \cdot z^3 + (-7200 \cdot i - 62300) \cdot x^2 \cdot y^2 \cdot z^3 - 60300 \cdot x \cdot y \cdot z^3 + (1800 \cdot i - 40400) \cdot y^4 \cdot z^3 - 4 \cdot z^4 + (1800 \cdot i + 16500) \cdot x^2 \cdot y^4 + 33075 \cdot x \cdot y^3 \cdot z^4 + (-450 \cdot i + 24000) \cdot y^3 \cdot z^4; \]

\[C6 := \text{Curve}(P,F6); \]
\[C7 := \text{Curve}(P,F7); \]
\[\text{IsAbsolutelyIrreducible}(C6); \]
\[\text{IsAbsolutelyIrreducible}(C7); \]
\[p := [P[0,0,1], P[-2,1,1], P[2,1,1], P[-1,2,1], P[1,2,1], P[3,2*i,1]]; \]
\[[\text{ResolutionGraph}(C6, p[i]) : i \in [1..5]]; \]
\[[\text{ResolutionGraph}(C7, p[i]) : i \in [1..6]]; \]
\[[\text{ResolutionGraph}(C6 \text{ join } C7, p[i]) : i \in [1..6]]; \]
\[\text{SingularPoints}(C6 \text{ join } C7); \]

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