L-AXIOMATIZABILITY IN INTERMEDIATE AND NORMAL MODAL LOGICS

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Abstract. A set $F$ of formulas is complete relative to a given class of logics, if every logic from this class can be axiomatized by formulas from $F$. A set of formulas $F$ is $L$-complete relative to a given class of logics, if every logic of this class can be $L$-axiomatized by formulas from $F$, that is, every of these logics can be defined by an $L$-deductive system with axioms and anti-axioms from $F$ and inference rules modus ponens, modus tollens, substitution and reverse substitution. We prove that every complete relative to $\text{ExtInt}$ (or $\text{ExtK4}$) set of formulas is $L$-complete. In particular, every logic from $\text{ExtInt}$ (or $\text{ExtK4}$) can be $L$-axiomatized by Zakharyaschev’s canonical formulas.

1. Introduction

Canonical formulas were introduced by M. Zakharyaschev (for details and references see [2]). They have been instrumental in studying intermediate and normal modal logic. The canonical formulas form a complete set of formulas, meaning that any intermediate logic or any normal extension of $\text{K4}$ can be axiomatized over intuitionistic propositional calculus ($\text{IPC}$) or, respectively, over $\text{K4}$ by canonical formulas. Our goal is to demonstrate that canonical formulas form the complete set not only for proving formulas but also for deriving, while using a Łukasiewicz-style calculi ($L$-deductive system), the rejection of formulas. We will prove a stronger statement: one can construct $L$-axiomatization of every logic from $\text{ExtInt}$ or $\text{ExtK4}$ using any given complete set of formulas.

The refutation system for various intermediate and normal modal logics were extensively studied by T. Skura, V. Goranko (see, for instance, [7, 3]). In [6] T. Skura observed that in case of finitely approximated logics the Jankov formulas give the complete set of anti-axioms, that is the additional axioms that can be used to prove refutation of a formula. The canonical formulas are, in a way, the modified Jankov or, more precisely, frame formulas. It turned out that we can effectively use the canonical formulas for refutation.

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In Section 2, we give the background information regarding L-deductive systems. In Section 3, we prove that every intermediate logic has an L-deductive system defining it and having axioms from a given complete set of formulas. And in Section 4, we extend this result to the normal extensions of K4.

2. L-Deductive Systems

2.1. Refutation Systems. Commonly, we use a deducting system in order to prove a formula and we use semantical means in order to disprove a formula. But the rejection of a formula can also be established syntactically. For instance, by Modus Tollens we can derive that a formula $A$ is refutable if we prove $A \rightarrow B$ and disprove $B$.

The idea, to include the rejected propositions into proofs can be traced back to R. Carnap\footnote{And in traditional logic even to Aristotle and the Stoics.}. But J. Lukasiewicz was the first who constructed a deductive system for proving refutability\footnote{And in traditional logic even to Aristotle and the Stoics.}.

In general, there are two ways of handling the refutation syntactically: direct and indirect. To determine weather a formula $A$ is refutable we can do one of the following

- derive in a meta-logic a statement about refutability of $A$ (L-proof - Lukasiewicz-style proof)
- derive from $A$ a formula $B$ that we already know is refutable (an anti-axiom) and then apply Modus Tollens (i-proof - indirect proof, Carnap’s way)

An existence of an L-proof entails the existence of i-proof. The converse is true under some assumptions (some weak form of the deduction theorem\footnote{And in traditional logic even to Aristotle and the Stoics.}).

2.1.1. Examples of $i$-complete systems. Let $\text{Fm}$ be a set of (propositional) formulas and $\Sigma$ be a set of all simultaneous substitutions of formulas for (propositional) variables. Let $\vdash$ be a structural consequence relation, that is, for any finite set of formula $\Gamma \subseteq \text{Fm}$ and any formulas $A, B \in \text{Fm}$ the following holds

$\vdash A \rightarrow A$

$\vdash \varepsilon$ if $\Gamma \vdash A$, then $\Gamma, B \vdash A$

$\vdash \varepsilon$ if $\Gamma, A \vdash B$ and $\Gamma \vdash A$, then $\Gamma \vdash B$

$\vdash \varepsilon$ if $\Gamma \vdash A$, then $\sigma(\Gamma) \vdash \sigma(A)$

Given a consequence relation $\vdash$, we say that a pair of sets of formulas $\langle A^+, A^- \rangle$ is an $i$-complete system for $\vdash$ if

$\vdash A$ if and only if
2.2. Definitions. By $\text{Fm}$ we denote the set of all (propositional) formulas in a given language containing $\to$ among connectives.

A logic is a subset $L \subseteq \text{Fm}$ closed under rules Modus Ponens and Substitution, i.e. for any $A, B \in \text{Fm}$ and any $\sigma \in \Sigma$

$$A, (A \to B) \in L \text{ entails } B \in L \text{ and } \sigma(A) \in L.$$ 

We will assume that there is a class of models (algebras, matrices, etc.) $M$ and for every formula $A \in \text{Fm}$ it is defined whether $A$ is valid in a given model $M$ (in written $M \models A$), or not (in written $M \not\models A$).

2.3. $L$-deductive Systems. If $A \in \text{Fm}$ is a formulas, than $\oplus A$ and $\ominus A$ are (atomic) statements. $\oplus A$ is a positive statement (assertion) and $\ominus A$ is a negative statement (rejection). The set of all positive statements we denote by $\text{St}^+$, the set of all negative statements we denote by $\text{St}^-$, and $\text{St}$ denotes the set of all statements, that is, $\text{St} := \text{St}^+ \cup \text{St}^-$. 

By $L$-deductive system we understand a couple $S := \langle Ax, R \rangle$, where $Ax := Ax^+ \cup Ax^-$ and $Ax^+ \subseteq \text{St}^+$ is a set of axioms, $Ax^- \subseteq \text{St}^-$ is a set of anti-axioms, and $R$ is a set of the following rules:

- Modus Ponens $\ominus(A \to B), \ominus A \vdash B$ (MP)
- Substitution $\ominus A \vdash \ominus \sigma(A)$, for all $\sigma \in \Sigma$ (Sb)
- Modus Tolens $\ominus(A \to B), \ominus B \vdash \ominus A$ (MT)
- Reverse Substitution $\ominus \sigma(A) \vdash \ominus A$, for all $\sigma \in \Sigma$ (RS)

2.4. $L$-Inference. In a natural way, every deduction system $S := \langle Ax, R \rangle$ defines an inference: if $\Gamma$ is a set of statements and $\alpha$ is a statement, a sequence $\alpha_1, \ldots, \alpha_n$ of statements is an inference of $\alpha$ from $\Gamma$ if $\alpha_n$ is $\alpha$ and for each $i \in \{1, \ldots, n\}$ one of the following holds

(a) $\alpha_i \in Ax$
(b) $\alpha_i \in \Gamma$
(c) $\alpha_i$ can be obtained from the preceding statements by one of the rules.

If there exists an inference of $\alpha$ from $\Gamma$, we say that $\alpha$ is derivable in $S$ from $\Gamma$, and we denote this by $\Gamma \vdash_S \alpha$ (and we will omit index if no confusion arises). The length of an inference is a number of statements in it.

If $\odot \in \{\oplus, \ominus\}$, then $\odot A$ is a statement with the sign opposite to $\odot$, that is, if $\odot = \oplus$, then $\odot = \ominus$ and if $\odot = \ominus$, then $\odot = \oplus$.

Proposition 2.1. For any $L$-deductive system $S$, if $\Gamma \subseteq \text{St}^+$, $\alpha \in \text{St}^+$ and $I := \alpha_1, \ldots, \alpha_n, \alpha$ is an inference of $\alpha$ from $\Gamma$, then, omitting from $I$ all negative statements, the obtained sequence $I^+$ still will be an inference of $\alpha$ from $\Gamma$.

Proof. Proof by induction on the length of $I$.

Basis. If $I$ contain a single statement $\alpha$, the inference already consists of only positive statements.

Inductive Hypothesis. Assume that for all inferences of the length at most $m$ the statement is true.
Step. Let \( J \) be an inference of \( \alpha \) from \( \Gamma \) of the length \( m + 1 \). By the definition of inference, either \( \alpha \in Ax \cup \Gamma \), or \( \alpha \) is obtained by some rule from the preceding members of \( J \). If \( \alpha \in Ax \) or \( \alpha \in \Gamma \), then the single-element sequence \( \alpha \) is an inference.

Suppose \( \alpha \) is obtained by one of the rules. Since \( \alpha \) is a positive statement, it can be obtained only by (MP) or (Sb). Let us consider these two cases.

A Case of (MP). Let \( I = \alpha_1, \ldots, \alpha_m, \alpha \). Suppose \( \alpha \) is obtained by (MP) and \( \alpha = \oplus A \) for some \( A \in Fm \). Then, for some formula \( B \in Fm \), the statements \( \oplus (B \rightarrow A) \) and \( \oplus B \) occur in \( J \). Assume \( \oplus B = \alpha_i \) and \( \oplus (B \rightarrow A) = \alpha_j \) members of \( J \). Let \( 1 \leq k \leq m \) be the greatest index such that \( \alpha_k \in J \) and \( \alpha_k \) is a positive statement (that is, all statements \( \alpha_{k+1}, \ldots, \alpha_m \) are negative).

Clearly, \( 1 \leq i, j \leq k \). Then, the first \( k \) elements \( J \) form an inference \( J_k \) and \( J_k \) contains both of statements \( \oplus B \) and \( \oplus (B \rightarrow A) \). By the inductive hypothesis, we can omit in \( J_k \) all negative statements and obtain a new inference \( J_k^+ \) that contains only positive statements. It is easy to see that the statements \( \oplus B \) and \( \oplus (A \rightarrow B) \) are members of \( J_k^+ \). Hence, we can add to \( J_k^+ \) the statement \( \oplus A \) and obtain an inference of \( \alpha \) from \( \Gamma \). Note, that obtained inference is exactly an inference obtained from \( J \) by omitting all negative statements.

A Case of (Sb). This case can be considered in the way similar to the case of (MP). \( \square \)

Corollary 2.2. For any L-deductive system \( S := (Ax, R) \), if \( \Gamma \subseteq St^+ \), \( \alpha \in St^+ \) and \( \Gamma \vdash_S \oplus \alpha \), then there is an inference of \( \alpha \) from \( \Gamma \) containing only the positive statements.

2.5. Coherent and Full L-deductive Systems.

Definition 2.3. L-deductive system \( S := (Ax, R) \) we will call coherent if for no \( A \in Fm \)

\[ \vdash_S \oplus A \text{ and } \vdash_S \ominus A. \]

And we will call \( S \) full if for every \( A \in Fm \)

\[ \vdash_S \oplus A \text{ or } \vdash_S \ominus A. \]

A coherent and full system will be called standard.

If \( A \in Fm \) is a formula and \( M \) is a model we let

\[ M \models \oplus A \Leftrightarrow M \models A \text{ and } M \models \ominus A \Leftrightarrow M \not\models A. \]

If \( M \models \ominus A \) we say that the statement \( \ominus A \) is valid in \( M \).

We say that a model \( M \) is an adequate regular model for an L-deductive system \( S \), if for every \( A \in Fm \)

\[ \vdash_S \ominus A \text{ if and only if } M \not\models \ominus A. \]

It is not hard to see that the following holds.

Proposition 2.4. If a given L-deductive system \( S \) has an adequate regular model, then the system \( S \) is standard.
In this paper, we consider only regular models.

Let us also observe that in order to prove that a model \( M \) is adequate for a given \( L \)-deductive system \( S := (Ax, R) \) as long as all axioms and anti-axioms are valid in \( M \).

**Proposition 2.5.** Let \( M \) be a model and \( S := (Ax, R) \) be an \( L \)-deductive system. If

\[
\text{for every } A \in Ax, M \vDash \Diamond A,
\]

then \( M \) is adequate for \( S \).

**Proof.** The proof can be done by induction on the length of inference. Indeed, all rules preserve the validity of statements, i.e. if the premisses of a rule are valid in \( M \), then the conclusion is valid too. \( \square \)

2.6. Logics Defined by \( L \)-deductive Systems. Every given deductive system \( S \) defines the pair

\[
\langle L^+, L^- \rangle, \text{ where } L^+ := \{ A \in \text{Fm} \mid \vdash S \odot A \} \text{ and } L^- := \{ A \in \text{Fm} \mid \vdash S \ominus A \}
\]

that we call a logic. The logic defined by a given \( L \)-deductive system \( S \) we will denote by \( L(S) \).

We say that a logic \( L = \langle L^+, L^- \rangle \) is coherent, full or standard if the defining \( L \)-deductive system is coherent, full or, respectively, standard. It is easy to see that a logic \( L \) is coherent if and only if \( L^+ \cap L^- = \emptyset \); logic \( L \) is full if and only if \( L^+ \cup L^- = \text{Fm} \); and logic \( L \) is standard if and only if \( L^- = \text{Fm} \setminus L^+ \).

A logic is said to be finitely \( L \)-axiomatizable if it can be defined by an \( L \)-deductive system with the finite set of axioms.

Any pair \( L = \langle L^+, L^- \rangle \), where \( L^+, L^- \subseteq \text{Fm} \) and \( L^+ \) is closed under \((\text{MP})\) and \((\text{SB})\) and \( L^- \) is closed under \((\text{MT})\) and \((\text{RS})\), is a logic. Indeed, \( L = \langle L^+, L^- \rangle \) can be defined by an \( L \)-deductive system in which

\[
Ax = \{ \ominus A \mid A \in L^+ \} \cup \{ \odot A \mid A \in L^- \}.
\]

Recall that a couple \( M(L) := \langle \text{Fm}, L^+ \rangle \), where \( L^+ \) is a set of designated values, is a Lindenbaum matrix of a logic \( L = \langle L^+, L^- \rangle \).

**Proposition 2.6.** If \( S := (Ax, R) \) is a standard \( L \)-deductive system, the Lindenbaum matrix of \( M(L(S)) \) is an adequate model of \( S \).

**Proof.** Due to Proposition 2.5 it suffices to check that all axioms of \( S \) are valid in \( M(L(S)) \).

Suppose \( \ominus A \in Ax \). Then \( A \in L^+ \). Recall that \( L^+ \) is closed under substitutions. Hence, \( M(L(S)) \vdash \ominus A \).

Suppose \( \odot A \in Ax \). Then \( A \in L^- \). Hence, \( M(L(S)) \vdash \odot A \). \( \square \)

2.7. The Theorem about Symmetry in ExtInt. From this point forward we consider only the deductive systems \( S \) in which \( \vdash S \odot (A \rightarrow (B \rightarrow A)) \) for all \( A, B \in \text{Fm} \).

The meaning of the following theorem is very straightforward: if we cannot derive a formula \( A \) in a given regular deductive system, but can derive it
from the some set of formulas \( \Gamma \), then \( \Gamma \) contains a formula \( B \) not derivable in the system and, moreover, \( \varnothing B \) and be \( \mathcal{L} \)-derived from \( \varnothing A \). In a way, the following theorem can be regarded as a strengthening of Modus Tollens.

**Theorem 2.7** (about symmetry in \( \text{ExtInt} \)). For any \( \mathcal{L} \)-deductive system \( S := (Ax, R) \) and any \( A_1, \ldots, A_n, B \in \text{Fm} \) if

\[
\vdash_S \ominus B \quad \text{and} \quad \ominus A_1, \ldots, \ominus A_n \vdash_S \ominus B,
\]

then

\[
\ominus B \vdash_S \ominus A_i
\]

for some \( 1 \leq i \leq n \).

**Proof.** We will prove the claim by induction on the length of inference of \( \ominus B \) from \( \ominus A_1, \ldots, \ominus A_n \). By virtue of the Proposition 2.1 we can safely assume that the inference consists of only positive statements.

**Basis.** Suppose there is an inference of \( \ominus B \) from \( \ominus A_1, \ldots, \ominus A_n \) of the length 1. Then, by the definition of inference, \( \ominus B = \ominus A_i \) for some \( 1 \leq i \leq n \), for \( \ominus B \in Ax \), due to \( \vdash_S \ominus B \). Hence, \( \ominus B \vdash_S \ominus A_i \).

**Inductive Hypothesis.** Assume that if there is an inference of the length at most \( m \) of \( \ominus B \) from \( \ominus A_1, \ldots, \ominus A_n \), then \( \ominus B \vdash_S \ominus A_i \) for some \( 1 \leq i \leq n \).

**Inductive Step.** Let \( \ominus B_1, \ldots, \ominus B_m, \ominus B \) be an inference of \( \ominus B \) from \( \ominus A_1, \ldots, \ominus A_n \). The cases (a) and (b) from the definition of inference can be considered in the basis of induction. Let us assume that the statement \( \ominus B \) is obtained by one of the rules. Due to this statement is positive, it can be obtained only by (MP) or (Sb).

**The case of (MP).** Suppose \( \vdash_S \ominus (C \rightarrow B) \) and \( B_k = \ominus C \), where \( 1 \leq j, k \leq m \). There are two possible subcases:

(a) \( \vdash_S \ominus (C \rightarrow B) \);

(b) \( \vdash_S \ominus (C \rightarrow B) \).

**Subcase (a).** Suppose \( \vdash_S \ominus (C \rightarrow B) \). Then, \( \vdash_S \ominus C \), for \( \vdash_S \ominus B \). Note, that the sequence \( \ominus B_1, \ldots, \ominus B_k \) is an inference of \( \ominus C \) from \( \ominus A_1, \ldots, \ominus A_n \) and \( 1 \leq k \leq m \). Hence, by the induction hypothesis,

\[ \ominus C \vdash_S \ominus A_i, \quad \text{for some} \quad 1 \leq i \leq n. \]  

(3)

On the other hand, we can apply (MT) to \( \vdash_S \ominus (C \rightarrow B) \) and \( \ominus B \) and obtain

\[ \ominus B \vdash_S \ominus C. \]

(4)

And from (3) and (4) we can derive

\[ \ominus B \vdash_S \ominus A_i, \quad \text{for some} \quad 1 \leq i \leq n. \]

(5)

**Subcase (b).** Suppose \( \vdash_S \ominus (C \rightarrow B) \). Then, we observe that \( \ominus B_1, \ldots, \ominus B_j \) is an inference of \( \ominus (C \rightarrow B) \) from \( \ominus A_1, \ldots, \ominus A_n \) and \( 1 \leq i \leq m \). So, we can apply the induction hypothesis and get

\[ \ominus (C \rightarrow B) \vdash_S \ominus A_i \quad \text{for some} \quad 1 \leq i \leq n. \]

(6)
On the other hand, we can apply (MT) to \( \vdash \oplus(B \rightarrow (C \rightarrow B)) \) and \( \ominus B \) and obtain
\[
\ominus B \vdash \ominus(C \rightarrow B).
\] (7)
And from (6) and (7) we can derive
\[
\ominus B \vdash \ominus(\ominus A_i), \text{ for some } 1 \leq i \leq n.
\] (8)

The case of (Rs). Suppose \( B = \sigma(B_j) \), where \( 1 \leq j \leq m \). Then \( \not\vdash \ominus B_j \), for \( \not\vdash \ominus B \). Also, note that \( \ominus B_1, \ldots, \ominus B_j \) is an inference of \( B_j \) from \( \ominus A_1, \ldots, \ominus A_n \) and \( 1 \leq j \leq m \). Hence, by the induction hypothesis,
\[
\ominus B_j \vdash \ominus A_i \text{ for some } 1 \leq i \leq n.
\] (9)
On the other hand, \( \ominus B = \ominus \sigma(B_j) \) and from \( \ominus \sigma(B_j) \), by (RS), we have
\[
\ominus B \vdash \ominus B_j.
\] (10)
From (9) and (10) we have
\[
\ominus B \vdash \ominus A_i \text{ for some } 1 \leq i \leq n.
\] (11)

3. Refutation in ExtInt

If \( \Gamma \subseteq \text{Fm} \) and \( A \in \text{Fm} \), then by \( \Gamma \vdash A \) we denote that \( A \) is derivable from \( \Gamma \) in Intuitionistic Propositional Calculus (IPC) with substitution (e.g., [4, Section 7.1.3]). \( \text{Int} + \Gamma \) will denote a logic axiomatized over \( \text{Int} \) by \( \Gamma \), that is \( \text{Int} + \Gamma := \{ A \in \text{Fm} \mid \Gamma \vdash A \} \). And \( \Gamma + A \) means the same as \( \Gamma + \{ A \} \).

A set \( F \) of formulas is said to be complete [10] (or sufficiently rich [9]) if every logic from ExtInt can be axiomatized over \( \text{Int} \) by some formulas from \( F \). An obvious characterization of completeness can be given by the following Proposition:

**Proposition 3.1.** A set of formulas \( F \) is complete if and only if for each formula \( A \) such that \( \text{Int} \not\vdash A \) there are formulas \( A_1, \ldots, A_n \in F \) and
\[
A_1, \ldots, A_n \vdash A \text{ and } A \vdash A_i \text{ for all } i = 1, \ldots, n.
\] (12)

**Proof.** Clearly, if (12) holds, every logic from ExtInt can be axiomatized over \( \text{Int} \) by some formulas from \( F \).

Conversely, if \( F \) is a complete set, we can consider a logic \( L := \text{Int} + A \) axiomatized over \( \text{Int} \) by formula \( A \). By the definition of completeness, for some \( A_1, \ldots, A_n \in F \) we have \( L = \text{Int} + \{ A_1, \ldots, A_n \} \), from which (12) immediately follows.

Perhaps, the best known complete set of formulas is a set of canonical formulas introduced by M. Zakharyaschev (cf. [2] for definitions, references and history). For our purposes it is important only that canonical formulas satisfy (12) (cf. [2, Theorem 9.44(i)]) and, thus, they form a complete set.

By \( \text{IPL} \) we denote the intuitionistic propositional logic, that is, \( \text{IPL} := \{ A \in \text{Fm} \mid \vdash A \} \).
We say that \( L = \langle L^+, L^- \rangle \) is a **standard intermediate logic** if \( \text{IPL} \subseteq L^+ \subset \text{Fm} \), and \( L \) is closed under rules Modus Ponens and Substitution.

### 3.1. Completeness Theorem

By \( Ax^i \) we will denote the set of positive statements obtained from the axioms of IPC. And by \( \text{Fm}^c \) we denote a given complete set of all formulas (for instance, a set of all canonical formulas).

Let us note the following.

**Proposition 3.2.** Assume \( A_1, \ldots, A_n, B \in \text{Fm} \) and \( S := (Ax, R) \) is such an \( L \)-deductive system that \( Ax^i \subseteq Ax \). Then

\[
A_1, \ldots, A_n \Vdash B, \text{ entails } \oplus A_1, \ldots, \oplus A_n \vdash S \oplus B.
\]

**Proof.** It is not hard to see that any inference of \( B \) from \( A_1, \ldots, A_n \) in IPC can be easily converted into an inference of \( \oplus B \) from \( \oplus A_1, \ldots, \oplus A_n \) in \( S \). □

**Theorem 3.3.** Every intermediate logic \( L \) can be defined by a standard deductive system \( S := (Ax, R) \), where every axiom \( \oplus A \in Ax \) is as statement obtained either from an axiom of IPC or from a formula \( A \in \text{Fm}^c \). In other words, given a complete set of formulas \( \text{Fm}^c \), every intermediate logic can be \( L \)-axiomatized over IPC by formulas from \( \text{Fm}^c \) as additional axioms and anti-axioms.

**Proof.** Let \( L = \langle L^+, L^- \rangle \) be an intermediate logic. Let us consider the \( L \)-deductive system \( S := (Ax, R) \), where

\[
Ax = Ax^i \cup \{ \oplus A \mid A \in L^+ \cap \text{Fm}^c \} \cup \{ \ominus A \mid A \in L^- \cap \text{Fm}^c \},
\]

i.e. axioms of \( S \) are statements obtained from the axioms of IPC and canonical formulas. We need to demonstrate that \( DS \) defines \( L \). We will show

(a) If \( A \in L^+ \), then \( \vdash S \oplus A \);

(b) If \( A \in L^- \), then \( \vdash S \ominus A \);

(c) \( S \) is coherent.

Note, that fullness of \( S \) immediately follows from (a) and (b). Thus, if \( S \) enjoys (a),(b) and (c), then \( S \) is standard. Also, it is not hard to see, that if \( S \) enjoys (a),(b) and (c), then \( S \) defines \( L \). So, all we need to do is to prove (a),(b) and (c).

First, we will establish coherence of the system \( S \).

**Proof of (c).** Let us take a Lindenbaum matrix \( M(L) := (\text{Fm}, L^+) \). By the definition of \( S \) all the axioms of \( S \) are valid in \( M(L) \). Hence, by the Proposition \( \ref{Lindenbaum} \) \( M(L) \) is an adequate model of \( S \) and, by virtue of the Proposition \( \ref{Lindenbaum} \) \( S \) is a standards \( L \)-deductive system and, thus, is coherent.

**Proof of (a).** Assume \( A \in L^+ \). If \( A \) is derivable in IPC, that is \( \Vdash A \), then, by virtue of the Proposition \( \ref{Proposition 3.2} \)

\[
\vdash S \oplus A.
\]

Assume \( A \in L^+ \) and \( A \) is not derivable in IPC. Then, by virtue of \( \ref{Proposition 3.2} \), there are such formulas \( C_1, \ldots, C_n \in \text{Fm}^c \) that

\[
C_1, \ldots, C_n \Vdash A.
\]
Then, by virtue of the Proposition 3.2,
\[ \bigoplus C_1, \ldots, \bigoplus C_n \vdash_S \bigoplus A. \]
Recall, that by the definition of \( S \), we have \( \bigoplus C_1, \ldots, \bigoplus C_n \in Ax. \) Hence,
\[ \vdash_S \bigoplus A. \]

**Proof of (b).** Assume \( A \in L^- \). Then, by virtue of (12), there are such formulas \( C_1, \ldots, C_n \in \text{Fm}^c \) that
\[ C_1, \ldots, C_n \vdash A \text{ and } A \vdash C_i \text{ for all } i = 1, \ldots, n. \quad (14) \]
Let us observe that, due to \( A \in L^- \), one of the formulas \( C_i, i = 1, \ldots, n \) is in \( L^- \). Suppose \( C_1 \in L^- \) and, hence,
\[ \ominus C_1 \in Ax. \quad (15) \]
We already proved that system \( S \) is coherent. Thus,
\[ \not\vdash_S \ominus C_1. \quad (16) \]
On the other hand, by (14)
\[ A \vdash C_1. \quad (17) \]
And, by virtue of the Proposition 3.2
\[ \bigoplus A \vdash_S \bigoplus C_1. \quad (18) \]
From (16) and (18), by virtue of the Theorem 2.7, we have
\[ \ominus C_1 \vdash_S \ominus A. \quad (19) \]
And from (15) and (19) we can conclude
\[ \vdash_S \ominus A. \]

**Corollary 3.4.** Every finitely \( L \)-axiomatizable intermediate logic \( L \) can be defined by a standard deductive system \( S := \langle Ax, R \rangle \) with finite number of axioms and every axiom \( \Box A \in Ax \) is as statement obtained from an axiom of IPC or from a canonical formula \( A \).

*Proof.* The proof immediately follows from the above Theorem and the finiteness of the relation \( \vdash_S \). \( \square \)

## 4. Refutation in NExtK4

From this point forward, \( \text{Fm} \) will denote the set of all formulas in the signature \( \wedge, \lor, \rightarrow, \neg, \Box, \Diamond \).

In order to consider normal modal logics, first, we need to extend the set \( R \) of rules by adding to (MP),(MT),(Sb) and (RS) the rules
\[
\text{Necessitation} \quad \bigoplus (A), \bigoplus \Box A \quad (NS) \\
\text{Reverse Necessitation} \quad \bigoplus \Box A/ \bigoplus A \quad (RN)
\]
Next, we need to establish that the Theorem about symmetry holds in NExtK4.
Theorem 4.1 (about symmetry in $\text{NExtK4}$). For any $L$-deductive system $S := \langle Ax, R \rangle$ and any $A_1, \ldots, A_n, B \in \text{Fm}$ if

$$\not\vdash_S \oplus B \text{ and } \oplus A_1, \ldots, \oplus A_n \vdash_S \oplus B,$$

then

$$\ominus B \vdash_S \ominus A_i$$

for some $1 \leq i \leq n$.

Proof. Similarly to Theorem 2.7, the proof can be done by induction on the length of inference. We can repeat the proof of the Theorem 4.1 and consider only the additional case for (NS).

The case of (NS). Suppose $B = \oplus \Box A_j$, where $1 \leq j < m$. Then $\not\vdash_S \oplus A_j$, for $\not\vdash_S \oplus \Box A_j$. Note, that $\oplus A_1, \ldots, \oplus A_{j-1}$ is an inference of $\oplus A_j$ from $\oplus A_1, \ldots, \oplus A_{j-1}$. Hence, by the induction hypothesis,

$$\ominus A_j \vdash_S \ominus A_i, \text{ for some } 1 \leq i < j. \quad (20)$$

By (RN), we also have

$$\ominus \Box A_j \vdash_S \ominus A_j. \quad (21)$$

And, combining (20) and (21), we have

$$\ominus \Box A_j \vdash_S \ominus A_i.$$

Recall, that $B = \Box A_j$, thus, we can conclude the proof of this case. $\square$

Given a complete set of formulas, for instance, the set of canonical formulas, one can prove the following theorem (using the same argument as in proof of the Theorem 3.3).

Theorem 4.2. Every logic $\mathcal{L} \in \text{NExtK4}$ can be defined by a standard deductive system $S := \langle Ax, R \rangle$, where every axiom $\ominus A \in Ax$ is as statement obtained from an axiom of $K4$ or from a canonical formula $A$. In other words, every logic from $\text{NExtK4}$ can be $L$-axiomatized by canonical formulas as additional axioms.

And, similarly to intermediate logics, the following holds.

Corollary 4.3. Every finitely $L$-axiomatizable logic $\mathcal{L} \in \text{NExtK4}$ can be defined by a standard deductive system $S := \langle Ax, R \rangle$ with finite number of axioms and every axiom $\ominus A \in Ax$ is as statement obtained from an axiom of $K4$ or from a canonical formula $A$.

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