Integrable systems and symmetric products of curves

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Abstract

We show how there is associated to each non-constant polynomial $F(x, y)$ a completely integrable system with polynomial invariants on $\mathbb{R}^{2d}$ and on $\mathbb{C}^{2d}$ for each $d \geq 1$; in fact the invariants are not only in involution for one Poisson bracket, but for a large class of polynomial Poisson brackets, indexed by the family of polynomials in two variables. We show that the complex invariant manifolds are isomorphic to affine parts of $d$-fold symmetric products of a deformation of the algebraic curve $F(x, y) = 0$, and derive the structure of the real invariant manifolds from it. We also exhibit Lax equations for the hyperelliptic case (i.e., when $F(x, y)$ is of the form $y^2 + f(x)$) and we show that in this case the invariant manifolds are affine parts of distinguished (non-linear) subvarieties of the Jacobians of the curves. As an application the geometry of the Hénon-Heiles hierarchy — a family of superimposable integrable polynomial potentials on the plane — is revealed and Lax equations for the hierarchy are given.

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1. Introduction

Finite-dimensional integrable systems first appeared in the works of Euler (1758), Lagrange (1766), Jacobi (1836), Liouville (1846) and Kowalewski (1889). They were given as systems of (non-linear) differential equations describing the motion of a mechanical system, having a sufficient number of integrals. Their investigation was based on the fact that the equations and the integrals were polynomials (in some coordinates) and led, in all cases considered, to an explicit integration of these equations in terms of (hyperelliptic) theta functions, well-known in algebraic geometry. Their work clearly showed the rich interplay between the theory of Riemann surfaces/algebraic curves (which was in that time thought of as a chapter in complex analysis) and mechanics. During the first half of the present century however, algebraic geometry was refounded and became ever more abstract, while in the theory of mechanical systems, generic smooth dynamical systems were gaining interest (as opposed to integrable ones). So both theories got separated, and integrable systems — which were at the core of this intimate relationship — faded away from the picture.

The interest in both integrable systems and their connection to algebraic geometry revived in the early seventies; many integrable systems were found as finite-dimensional solutions of certain (integrable) partial differential equations (such as the well-known Korteweg-de Vries equation) and they were again integrated in terms of theta functions. Their study lead in particular to the concept of an algebraic completely integrable system (a.c.i. system). Shortly, such a system is an integrable system which has a complexification for which the invariant manifolds (the smooth level sets of the integrals) are (open subsets of) complex algebraic tori, and the flow (run with complex time) is linear on these tori (see [AvM1], [Mu]). Algebraic geometry has been shown to be a useful tool for the study of a.c.i. systems and a solution to some problems in algebraic geometry was found by using an a.c.i. system.

The present paper deals with a (new) class of integrable systems which (apart from an exceptional case, specified below) do not fall in the class of a.c.i. systems, but yet they have a natural complexification and their geometry is most naturally described by using algebraic geometry. Apart from their construction, their geometry will be analysed in detail and we will show how these systems can be used to explain the geometry of several known integrable systems which are not a.c.i.

1.1 Poisson structures on $\mathbb{R}^{2d}$

On $\mathbb{R}^{2d}$ with coordinates $(u_1, \ldots, u_d, v_1, \ldots, v_d)$ we show in Section 2.2 that there corresponds in a natural way to any non-zero polynomial $\varphi(x, y) \in \mathbb{R}[x, y]$ a Poisson bracket $\{\cdot, \cdot\}_d^\varphi$, which is given by

\[
\{u(\lambda), u_j\}_d^\varphi = \{v(\lambda), v_j\}_d^\varphi = 0, \\
\{u(\lambda), v_j\}_d^\varphi = \{u_j, v(\lambda)\}_d^\varphi = \varphi(\lambda, v(\lambda)) \left[ \frac{u(\lambda)}{\lambda^{d-j+1}} \right]_+ \mod u(\lambda), \quad 1 \leq j \leq d,
\]

where $u(\lambda) = \lambda^d + u_1 \lambda^{d-1} + \cdots + u_d$ and $v(\lambda) = v_1 \lambda^{d-1} + \cdots + v_d$; also $[R(\lambda)]_+$ denotes the polynomial part of a rational function $R(\lambda)$ and $f(\lambda) \mod g(\lambda)$ is the rest obtained when dividing $f(\lambda)$ by $g(\lambda)$. The map $\varphi \mapsto \{\cdot, \cdot\}_d^\varphi$ is clearly a linear map, which is moreover injective, since the Poisson structures obtained are of maximal rank except for $\varphi = 0$. If $\varphi(x, y)$ is a constant,
say \( \phi(x, y) = 1 \), then the bracket \( \{\cdot, \cdot\}_d = \{\cdot, \cdot\}_d^1 \) is given by the following matrix \( P \) of Poisson brackets:

\[
P = \begin{pmatrix} 0 & U \\ -U & 0 \end{pmatrix}
\]

where

\[
U = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & u_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & u_{d-3} & u_{d-2} \\ 1 & u_1 & \cdots & u_{d-2} & u_{d-1} \end{pmatrix}
\]

Thus, (1) provides us with a large class of Poisson structures on \( \mathbb{R}^{2d} \), which are in fact polynomial, i.e., all brackets of the coordinates \( u_i \) and \( v_j \) are polynomials; moreover they are all compatible in a sense explained in the text.

### 1.2 Integrable systems on \( \mathbb{R}^{2d} \)

What is remarkable is that these Poisson structures have a very large class of integrable systems in common, namely one corresponding to every polynomial \( F(x, y) \in \mathbb{R}[x, y] \setminus \mathbb{R}[x] \). To describe these, let \( F(x, y) \) be such a polynomial and expand \( F(\lambda, v(\lambda)) \mod u(\lambda) \) as a polynomial in \( \lambda \) (of degree \( d - 1 \)):

\[
F(\lambda, v(\lambda)) \mod u(\lambda) = H_1\lambda^{d-1} + H_2\lambda^{d-2} + \cdots + H_d;
\]

Remark that \( H_1, \ldots, H_d \) are polynomials in \( u_i \) and \( v_j \). The main result, established in Section 2.3, is that these polynomials Poisson commute for all brackets \( \{\cdot, \cdot\}_d^\mathbb{R} \) on \( \mathbb{R}^{2d} \), that is

\[
\{H_i, H_j\}_d = 0 \quad \text{for all } 1 \leq i, j \leq d \text{ and } \varphi(x, y) \in \mathbb{R}[x, y].
\]

Since \( H_1, \ldots, H_d \) are independent, the conclusion is that for any polynomial \( F(x, y) \in \mathbb{R}[x, y] \setminus \mathbb{R}[x] \) we have an integrable system on the Poisson manifold \( (\mathbb{R}^{2d}, \{\cdot, \cdot\}_d^\mathbb{R}) \), where \( d \geq 1 \) is arbitrary and \( 0 \neq \varphi(x, y) \in \mathbb{R}[x, y] \) is an arbitrary polynomial dictating the Poisson structure, and our construction is totally explicit.

Since everything in our construction is polynomial, these systems have a natural complexification as complex integrable systems on the Poisson manifold \( (\mathbb{C}^{2d}, \{\cdot, \cdot\}_d^\mathbb{C}) \), where the Poisson structure \( \{\cdot, \cdot\}_d^\mathbb{C} \) is now a holomorphic one.

### 1.3 The geometry of the systems

The meaning of the polynomial \( F(x, y) \) and the need for considering the complexified system becomes apparent in Section 3, when we study (for generic values of \( c_i \)) the level sets \( A_{F, d} = \{P \in \mathbb{R}^{2d} \mid H_i(P) = c_i\} \), which are preserved by the flows of the vector fields associated to all \( H_i \). Namely we will show in Section 3.2 that the complex invariant set (lying over 0)

\[
A_{F, d}^\mathbb{C} = \{(u(\lambda), v(\lambda)) \in \mathbb{C}^{2d} \mid H_{F, d}(u(\lambda), v(\lambda)) = 0\}
\]

is (biholomorphic to) an affine part of the \( d \)-fold symmetric product of the plane algebraic curve \( \Gamma_F \subset \mathbb{C}^2 \), defined by \( F(x, y) = 0 \) (\( \Gamma_F \) is supposed generic here, i.e., smooth); a similar description of the structure of the other complex invariant sets (lying over \( (c_1, \ldots, c_d) \)) follows at once. The real invariant sets being the fixed points on \( A_{F, d}^\mathbb{C} \) of the complex conjugation map, we obtain in Section 3.3 a description of \( A_{F, d} = A_{F, d}^\mathbb{C} \cap \mathbb{R}^{2d} \) as the set of all \( d \)-tuples in \( A_{F, d}^\mathbb{C} \), consisting only of real points and points which appear in complex conjugated pairs. We will show how this leads to an explicit description of the topology of the invariant manifolds \( A_{F, d} \), which are in general neither
tori nor cylinders. The compactification of the complex invariant manifolds, of major interest in several studies in this field, is discussed in Section 3.4: it turns out that in general a smooth compactification of the complex level manifolds, such that the vector fields of the system extend in a holomorphic way to them, does not exist.

1.4 The hyperelliptic case

The special case where \( F(x, y) \) is of the form \( F(x, y) = y^2 + f(x) \) will be considered in more detail in Section 4. Then the vector fields \( X^{ϕ}_H \) of the integrable system can be written as Lax equations

\[
X^{ϕ}_H A(λ) = [A(λ), [B_1(λ)]_+] + A(λ),
\]

where

\[
A(λ) = \left( -\left[ \frac{v(λ)}{F(λ, u(λ))} \right]_+ \frac{u(λ)}{u(λ)} \right) - v(λ) \quad \text{and} \quad B_1(λ) = \frac{φ(λ, v(λ))}{u(λ)} \left[ \frac{u(λ)}{λ^{d-i+1}} \right] + A(λ);
\]

see Section 4.1. The geometry of these systems can be related to that of the Jacobian of the curve \( Γ_F \). In particular, in the very special case that \( φ(x, y) = 1 \) and \( d = \text{genus} (Γ_F) \) the manifold \( A^{\mathbb{C}}_{F, d} \) is an affine part of the Jacobian of \( Γ_F \), the flow of the vector fields is linear and the system is algebraic completely integrable. If \( d < \text{genus} (Γ_F) \) then \( A^{\mathbb{C}}_{F, d} \) is interpreted as a very special non-linear subvariety of the Jacobian of \( Γ_F \).

The geometry of several integrable systems, such as the Hénon-Heiles hierarchy and its generalisations (in different aspects), the (generalised) Gaudin magnet, the discrete self-trapping timer, ..., can be described in very much detail by using our systems. We will show in Section 4.3 quite detailed how this is done for the Hénon-Heiles hierarchy, which consists of a family of (superimposable) integrable potentials on the plane. For the other examples one proceeds in a completely analogous way.
2. The systems and their integrability

In this section we describe our basic construction, which associates to a pair of polynomials \( F(x, y) \) and \( \varphi(x, y) \), an integrable system on \( \mathbb{R}^{2d} \) for any \( d \geq 1 \).

2.1. Notation

\( \mathbb{R}^{2d} \) is throughout viewed as the space of pairs of polynomials \((u(\lambda), v(\lambda))\), with \( u(\lambda) \) monic of degree \( d \) and \( v(\lambda) \) of degree less than \( d \), via

\[
\begin{align*}
  u(\lambda) &= \lambda^d + u_1 \lambda^{d-1} + \cdots + u_{d-1} \lambda + u_d, \\
  v(\lambda) &= v_1 \lambda^{d-1} + \cdots + v_{d-1} \lambda + v_d,
\end{align*}
\]

so the coefficients \( u_i \) and \( v_i \) serve as coordinates on \( \mathbb{R}^{2d} \). Some formulas below are simplified by denoting \( u_0 = 1 \).

For any rational function \( r(\lambda) \), we denote by \([r(\lambda)]_\pm\) its polynomial part and we let \([r(\lambda)]_- = r(\lambda) - [r(\lambda)]_+\). If \( f(\lambda) \) is any polynomial and \( g(\lambda) \) is a monic polynomial, then \( f(\lambda) \mod g(\lambda) \) denotes the polynomial of degree less than \( \deg g(\lambda) \), defined by

\[
f(\lambda) \mod g(\lambda) = g(\lambda) \left[ \frac{f(\lambda)}{g(\lambda)} \right]_-, \]

so \( f(\lambda) = f(\lambda) \mod g(\lambda) + h(\lambda)g(\lambda) \) for a unique polynomial \( h(\lambda) \), and \( f(\lambda) \mod u(\lambda) \) is easy computed as the rest obtained by the Euclidean division algorithm.

2.2. The compatible Poisson structures \( \{\cdot, \cdot\}^\varphi_d \)

Any polynomial \( \varphi(x, y) \) specifies a Poisson bracket on \( \mathbb{R}^2 \) by \( \{y, x\} = \varphi(x, y) \), which induces a polynomial bracket on the cartesian product \( \mathbb{R}^2 \times \cdots \times \mathbb{R}^2 \) by

\[
\{y_i, x_j\} = \delta_{ij} \varphi(x_j, y_i), \quad \{x_i, x_j\} = \{y_i, y_j\} = 0.
\]

Let \( \Delta \) denote the closed subsets of \( \mathbb{R}^2 \times \cdots \times \mathbb{R}^2 \) defined by

\[
\Delta = \{(x_1, y_1), (x_2, y_2), \ldots, (x_d, y_d) \mid \exists i, j : i \neq j \text{ and } x_i = x_j\},
\]

and consider the map \( S: (\mathbb{R}^2)^d \setminus \Delta \to \mathbb{R}^{2d} \), given by

\[
((x_1, y_1), (x_2, y_2), \ldots, (x_d, y_d)) \mapsto (u(\lambda), v(\lambda)) = \left( \prod_{i=1}^d (\lambda - x_i), \sum_{i=1}^d y_i \prod_{j \neq i} \frac{\lambda - x_j}{x_i - x_j} \right).
\]

\( S \) is invariant for the obvious action of the permutation group \( S_d \) on \( (\mathbb{R}^2)^d \) and is a \( d! : 1 \) covering map onto an open subset of \( \mathbb{R}^{2d} \). Since the Poisson structure is also invariant for the action of \( S_d \), i.e.,

\[
\{f, g\} \circ \sigma = \{f \circ \sigma, g \circ \sigma\}, \quad f, g \in C^\infty \left( (\mathbb{R}^2)^d \right), \sigma \in S_d,
\]

a \( C^\infty \) Poisson bracket \( \{\cdot, \cdot\}^\varphi_d \) is defined on the image of \( S \) by requiring that \( S \) is a Poisson map, i.e., that for any \( f, g \in C^\infty(\mathbb{R}^{2d}) \), one has \( \{f, g\}^\varphi_d \circ S = \{f \circ S, g \circ S\} \). The following proposition
hence it defines a symplectic structure \( \{ \cdot, \cdot \}_d \) on \( \mathbb{R}^d \).

**Proposition 1**  
The Poisson bracket \( \{ \cdot, \cdot \}_d \) is given in terms of the coordinates \( u_i, v_i \) by
\[
\{ u(\lambda), u_j \}_d = \{ v(\lambda), v_j \}_d = 0, \\
\{ u(\lambda), v_j \}_d = \{ u_j, v(\lambda) \}_d = \varphi(\lambda, v(\lambda)) \left[ \frac{u(\lambda)}{\lambda^{d-j+1}} \right] \mod u(\lambda), \quad 1 \leq j \leq d, 
\]

hence all brackets of the coordinate functions \( u_i \) and \( v_j \) are polynomials and \( \{ \cdot, \cdot \}_d \) is defined on all of \( \mathbb{R}^d \). Except for the trivial bracket \( \{ \cdot, \cdot \}_0 \), all Poisson bracket \( \{ \cdot, \cdot \}_d \) are of rank \( 2d \) on a dense subset of \( \mathbb{R}^d \) whose complement is a (possibly empty) algebraic hypersurface; moreover they are all compatible, i.e., the sum of two such Poisson brackets is again a Poisson bracket.

As a special and most important case, if \( x \) and \( y \) are canonical variables, i.e., \( \varphi(x, y) = 1 \), then the Poisson structure \( \{ \cdot, \cdot \}_d \), also denoted by \( \{ \cdot, \cdot \}_d \), is of maximal rank at every point of \( \mathbb{R}^d \), hence it defines a symplectic structure \( \omega_d \) on \( \mathbb{R}^d \); the Poisson bracket (6) reduces in this case to
\[
\{ u(\lambda), v_j \}_d = \{ u_j, v(\lambda) \}_d = \left[ \frac{u(\lambda)}{\lambda^{d-j+1}} \right] \mod u(\lambda), \\
\]

and its matrix of Poisson brackets with respect to the coordinate functions \( u_i \) and \( v_j \), takes the form
\[
P = \begin{pmatrix} 0 & U \\ -U & 0 \end{pmatrix} \text{ where } U = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & u_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & u_{d-3} & u_{d-2} \\
1 & u_1 & \cdots & u_{d-2} & u_{d-1} \end{pmatrix}.
\]

In terms of \( \{ \cdot, \cdot \}_d \), the Poisson structure associated to a polynomial \( \varphi(x, y) \) is given by
\[
\{ u(\lambda), f \}_d = \varphi(\lambda, v(\lambda)) \{ u(\lambda), f \}_d \mod u(\lambda), \\
\{ v(\lambda), f \}_d = \varphi(\lambda, v(\lambda)) \{ v(\lambda), f \}_d \mod u(\lambda).
\]

**Proof**

Clearly \( \{ u(\lambda), u(\mu) \}_d = 0 \). The bracket \( \{ v(\lambda), v(\mu) \}_d \) is a polynomial of degree at most \( d-1 \) in \( \lambda \) and in \( \mu \), which vanishes for the \( d^2 \) values \( (\lambda, \mu) = (x_i, x_j) \), where \( x_i \) and \( x_j \) are roots of \( u(\lambda) \), hence \( \{ v(\lambda), v(\mu) \}_d \) vanishes on the image of \( S \). If \( 1 \leq j \leq d \), then
\[
\{ u_j, v(\lambda) \}_d = (-1)^j \left\{ \sum_{i_1 < i_2 < \cdots < i_j} x_{i_1} x_{i_2} \cdots x_{i_j}, \sum_{l=1}^{d} y_l \prod_{k \neq l}^{d} \frac{\lambda - x_k}{x_i - x_k} \right\}_d \varphi, \\
= (-1)^{j-1} \sum_{i_1 < i_2 < \cdots < i_{j-1}} \sum_{l=1}^{j} x_{i_1} x_{i_2} \cdots x_{i_{j-1}} \varphi(x_{i_j}, y_l) \prod_{k \neq i_j}^{d} \frac{\lambda - x_k}{x_i - x_k}, \\
= (-1)^{j-1} \sum_{l \in \{ i_1 < i_2 < \cdots < i_{j-1} \}} x_{i_1} x_{i_2} \cdots x_{i_{j-1}} \varphi(x_{i_j}, y_l) \prod_{k \neq l}^{d} \frac{\lambda - x_k}{x_i - x_k}, \\
= (-1)^{j-1} \sum_{l=1}^{d} \varphi(x_{i_l}, y_l) \prod_{k \neq l}^{d} \frac{\lambda - x_k}{x_i - x_k} (-1)^{j-1} \sum_{m=0}^{j-1} x_i^m u_{j-m-1}, \\
= \sum_{l=1}^{d} \sum_{m=0}^{j-1} x_i^m u_{j-m-1} \varphi(x_{i_l}, y_l) \prod_{k \neq l}^{d} \frac{\lambda - x_k}{x_i - x_k}.
\]
Since \( y_l = v(x_l) \) this shows that \( \{ u_j, v(\lambda) \}_d^\varphi \) is the (unique) polynomial in \( \lambda \) of degree less than \( d \), which takes at \( \lambda = x_l \) the value \( \sum_{m=0}^{j-1} \lambda^m u_{j-m-1} \varphi(x_l, v(x_l)) \), for \( l = 1, \ldots, d \). As the \( x_l \) are the zeros of \( u(\lambda) \), the same is true for \( \sum_{m=0}^{j-1} \lambda^m u_{j-m-1} \varphi(\lambda, v(\lambda)) \mod u(\lambda) \), and we find

\[
\{ u_j, v(\lambda) \}_d^\varphi = \sum_{m=0}^{j-1} \lambda^m u_{j-m-1} \varphi(\lambda, v(\lambda)) \mod u(\lambda),
\]

\[
= \varphi(\lambda, v(\lambda)) \left[ \frac{u(\lambda)}{\lambda^{d-j+1}} \right] + \mod u(\lambda),
\]

which proves the second equality in (6). For the first equality in (6), remark that

\[
\text{Choosing } \text{determinants one finds that for any values } x_1, \ldots, x_d, \text{ also denoted by } \{ \cdot, \cdot \}_d^\varphi. \text{ Compatibility of the brackets derives from the obvious formula } \{ \cdot, \cdot \}_d^\varphi + \{ \cdot, \cdot \}_d^\psi = \{ \cdot, \cdot \}_d^{\varphi+\psi}.
\]

For \( \varphi = 1 \) one obtains (7), because the degree of \( \left[ \frac{u(\lambda)}{\lambda^{d-j+1}} \right] + \mod u(\lambda) \) is less than \( d \) for any \( j = 1, \ldots, d \), which also leads at once to the matrix representation of \( \{ \cdot, \cdot \}_d \) — since its determinant equals \((-1)^d\), it is of rank \( 2d \) everywhere. Remark also that \( \{ \cdot, \cdot \}_d \) is not compatible with the standard structure \( \sum du_i \wedge dv_i \) on \( \mathbb{R}^{2d} \).

To see where the rank of the Poisson structure \( \{ \cdot, \cdot \}_d^\varphi \) fails to be maximal, we need to investigate the determinant of the matrix of Poisson brackets \( \{ u_i, v_j \}_d^\varphi \). By some elementary properties of determinants one finds that for any values \( x_1, \ldots, x_d \),

\[
\det (\{ u_i, v(x_j) \}_d^\varphi)_{1 \leq i, j \leq d} = \det (\{ u_i, v_j \}_d^\varphi)_{1 \leq i, j \leq d} \prod_{k<l}(x_k - x_l).
\]

Choosing \( x_1, \ldots, x_d \) to be the roots of \( u(\lambda) \) (which may be complex), we get from (6)

\[
\det (\{ u_i, v(x_j) \}_d^\varphi)_{1 \leq i, j \leq d} = \det (\{ u(x_j), v(x_j) \}_d^\varphi)_{1 \leq i, j \leq d},
\]

\[
= \det \left( \frac{u(\lambda)}{\lambda^{d-i+1}} \right)_{1 \leq i, j \leq d} \prod_{m=1}^{d} \varphi(x_m, v(x_m)),
\]

\[
= \det (\{ u_i, v(x_j) \}_d \prod_{m=1}^{d} \varphi(x_m, v(x_m)),
\]

\[
\overset{(i)}{=} (-1)^{[d/2]} \prod_{k<l}(x_k - x_l) \prod_{m=1}^{d} \varphi(x_m, v(x_m)),
\]
where in (i) we used (9) for \( \varphi = 1 \). It follows that (even if \( u(\lambda) \) has multiple roots)

\[
\det \{ \{ u_i, v_j \}_d \}_{1 \leq i, j \leq d} = (-1)^{[d/2]} \prod_{m=1}^{d} \varphi(x_m, v(x_m)),
\]
on all of \( \mathbb{R}^{2d} \), hence the Poisson structure is of lower rank on the locus \( \prod_{j=1}^{d} \varphi(x_j, v(x_j)) = 0 \), which for given \( \varphi \) is easy written as the equation of an algebraic hypersurface in \( \mathbb{R}^{2d} \).

Finally, (8) follows immediately from the Leibniz property of Poisson brackets.

**Amplification 1**

The condition that \( \varphi(x, y) \) is a polynomial is not essential: if \( \varphi(x, y) \) is any smooth function, then all the above formulas remain valid, yielding yet more examples of compatible Poisson structures. In this more general case, for \( f(\lambda) \) any smooth function and \( g(\lambda) \) a monic polynomial as before, \( f(\lambda) \mod g(\lambda) \) denotes the unique polynomial of degree less than \( \deg g(\lambda) \) which takes at the roots \( x_i \) of \( g(\lambda) \) the value \( f(x_i) \). The Poisson brackets \( \{ u_i, v_j \}_d \) are no longer polynomial and can not be computed by the Euclidean division algorithm.

Of interest is also the case that \( \varphi(x, y) \) is rational, in which all brackets \( \{ u_i, v_j \}_d \) are rational functions of the coordinates \( u_i \) and \( v_j \). Obviously, if \( \varphi(x, y) \) has poles on \( \mathbb{R}^2 \), the bracket \( \{ \cdot, \cdot \}_d \) will also have poles on \( \mathbb{R}^{2d} \), and is in this case only a Poisson bracket on a dense subset of \( \mathbb{R}^{2d} \).

### 2.3. Polynomials in involution for \( \{ \cdot, \cdot \}_d \)

We now show how there is associated, for fixed \( d \), to each polynomial \( F(x, y) \) which depends explicitly on \( y \), a set of \( d \) independent polynomials, which are in involution for all the brackets \( \{ \cdot, \cdot \}_d \), that is, the Poisson bracket of any pair of these polynomials vanishes (such polynomials are also said to Poisson-commute).

Let \( F(x, y) \in \mathbb{R}[x, y] \setminus \mathbb{R}[x] \) and let us view \( \mathbb{R}^d \) as the space of polynomials (say in \( \lambda \)) of degree less than \( d \). Then there is a natural map \( \tilde{H}_{F, d} \) from \( (\mathbb{R}^2)^d \setminus \Delta \) to \( \mathbb{R}^d \), which assigns to a \( d \)-tuple \((x_1, y_1), \ldots, (x_d, y_d)\) the unique polynomial in \( \mathbb{R}[\lambda] \) of degree less than \( d \), which takes for \( \lambda = x_i \) the value \( F(x_i, y_i) \) (for \( i = 1, \ldots, d \)). Since \( \tilde{H}_{F, d} \) is invariant under the action of \( S_d \), there is defined on the image of \( S \) a map \( H_{F, d} \) such that \( \tilde{H}_{F, d} = H_{F, d} \circ S \), namely \( H_{F, d} \) is given by

\[
H_{F, d}(u(\lambda), v(\lambda)) = (F(\lambda, v(\lambda))) \mod u(\lambda).
\] (10)

The \( d \) components of the map \( H_{F, d} \) define \( d \) functions on \( \mathbb{R}^{2d} \), which will be simply denoted by \( H_1, \ldots, H_d \) (omitting the dependence on \( F \) and \( d \) in the notation), i.e., \( H_{F, d}(u(\lambda), v(\lambda)) = H_1\lambda^{d-1} + H_2\lambda^{d-2} + \cdots + H_d \). As \( u(\lambda) \) is a monic polynomial, these functions \( H_i \) are polynomial in our coordinates on \( \mathbb{R}^{2d} \) hence are defined on all of \( \mathbb{R}^{2d} \). The main result of this section is the following.

**Theorem 2**  

The coefficients \( H_1, \ldots, H_d \) of \( H(\lambda) = F(\lambda, v(\lambda)) \mod u(\lambda) \) define for any non-zero polynomial \( \varphi(x, y) \) a completely integrable system on the Poisson manifold \( (\mathbb{R}^{2d}, \{ \cdot, \cdot \}_d) \) with polynomial invariants, that is, \( \{ H_1, \ldots, H_d \} \) forms a set of \( d \) functional independent polynomials on \( \mathbb{R}^{2d} \), which are in involution for all brackets \( \{ \cdot, \cdot \}_d \).

---

1 If \( g(\lambda) \) has multiple roots, then \( f(\lambda) \mod g(\lambda) \) is not unique; since in this paper \( g(\lambda) = u(\lambda) \) depends on the coordinates \( u_i \), it is (as a function on \( \mathbb{R}^{2d} \)) uniquely defined on a dense subset of \( \mathbb{R}^{2d} \), hence its extension to \( \mathbb{R}^{2d} \) is also unique.
Before proving this theorem we prove a key lemma and write down explicit equations for the Hamiltonian vector fields $X^\varphi_{H_i}$, which — by the above theorem — commute as differential operators, in view of the identity (see [AM]) $[X^\varphi_{H_i}, X^\varphi_{H_j}] = X^\varphi_{[H_i,H_j]}$.

**Lemma 3.** Let $p(\lambda)$, $q(\lambda)$ and $r(\lambda)$ be polynomials, with $\deg q(\lambda) \geq \deg r(\lambda)$ and let $i \in \mathbb{N}$.

\begin{align*}
(1) & \quad r(\lambda) \left[\lambda^{-i} q(\lambda)\right]_+ \mod q(\lambda) = r(\lambda) \left[\lambda^{-i} q(\lambda)\right]_+ - q(\lambda) \left[\lambda^{-i} r(\lambda)\right]_+ , \\
(2) & \quad \sum_{l=1}^{\deg q} \mu^{l-1} p(\lambda) \left[\lambda^{-l} q(\lambda)\right]_+ \mod q(\lambda) = \sum_{l=1}^{\deg q} \lambda^{l-1} p(\mu) \left[\mu^{-l} q(\mu)\right]_+ \mod q(\mu). \tag{11}
\end{align*}

**Proof**

For the proof of (1) remark that if $\deg r(\lambda) \leq \deg q(\lambda)$ then the right hand side of the identity

$$r(\lambda) \left[\lambda^{-i} q(\lambda)\right]_+ - q(\lambda) \left[\lambda^{-i} r(\lambda)\right]_+ = -r(\lambda) \left[\lambda^{-i} q(\lambda)\right]_+ + q(\lambda) \left[\lambda^{-i} r(\lambda)\right]_+$$

is of degree less than $\deg q(\lambda)$, hence also the left hand side. To show (2) we may assume that $\deg p(\lambda) < \deg q(\lambda)$ because the identity depends only on $p(\lambda) \mod q(\lambda)$. Then

\begin{align*}
\sum_{l=1}^{\deg q} \lambda^{l-1} p(\mu) \left[\mu^{-l} q(\mu)\right]_+ \mod q(\mu) = & \sum_{l=1}^{\deg q} \lambda^{l-1} \left(p(\mu) \left[\mu^{-l} q(\mu)\right]_+ - q(\mu) \left[\mu^{-l} p(\mu)\right]_+\right), \\
= & \sum_{l=1}^{\deg q} \mu^{l-1} \left(p(\lambda) \left[\lambda^{-l} q(\lambda)\right]_+ - q(\lambda) \left[\lambda^{-l} p(\lambda)\right]_+\right), \\
= & \sum_{l=1}^{\deg q} \mu^{l-1} p(\lambda) \left[\lambda^{-l} q(\lambda)\right]_+ \mod q(\lambda).
\end{align*}

In (i) we applied part (1) of this lemma; the exchange property in (ii) is proven at once by expanding the polynomials or by induction on $\deg q(\lambda)$.

**Proposition 4** The coefficients $H_i$ of $F(\lambda, v(\lambda)) \mod u(\lambda)$ determine $d$ independent polynomial vector fields $X^\varphi_{H_i}$ on $\mathbb{R}^{2d}$, which are explicitly given by

\begin{align*}
X^\varphi_{H_i} u(\lambda) = & \varphi(\lambda, v(\lambda)) \left(\frac{\partial F}{\partial y}(\lambda, v(\lambda)) \left[\frac{u(\lambda)}{\lambda^{d-i+1}}\right]_+\right) \mod u(\lambda), \\
X^\varphi_{H_i} v(\lambda) = & \varphi(\lambda, v(\lambda)) \left[\frac{F(\lambda, v(\lambda))}{u(\lambda)}\right]_+ \left[\frac{u(\lambda)}{\lambda^{d-i+1}}\right]_+ \mod u(\lambda). \tag{12}
\end{align*}

Moreover, the following remarkable identities hold for all $1 \leq i, j \leq d$:

\begin{align*}
\{u_i, H_j\}_d^\varphi = & \{u_j, H_i\}_d^\varphi \text{ and } \{v_i, H_j\}_d^\varphi = \{v_j, H_i\}_d^\varphi. \tag{13}
\end{align*}
Proof

Writing $X_H$ as a shorthand for $X_H^1$, we first compute $X_H, u(\lambda) = \{u(\lambda), H_i\}_d$, which we obtain as the coefficients of $\mu^{d-i}$ in $\{u(\lambda), H_{F,d}(u(\mu), v(\mu))\}_d$.

$$\{u(\lambda), H_{F,d}(u(\mu), v(\mu))\}_d = \sum_{j=1}^{d} \{u(\lambda), v_j\}_d \frac{\partial H_{F,d}}{\partial v_j}(u(\mu), v(\mu)),$$

$$= \sum_{j=1}^{d} \left[ \frac{u(\lambda)}{\lambda^d - j + 1} \right] + \frac{\partial H_{F,d}}{\partial v_j}(u(\mu), v(\mu)),$$

$$= \sum_{j=1}^{d} \sum_{k=0}^{j-1} u_k \lambda^{j-k-1} \frac{\partial F}{\partial y}(\mu, v(\mu)) \mu^{d-j} \text{ mod } u(\mu),$$

$$= \sum_{l=1}^{d} \sum_{j=1}^{d} u_j \lambda^{j-1} \frac{\partial F}{\partial y}(\mu, v(\mu)) \mu^{d-j} \text{ mod } u(\mu),$$

$$= \sum_{l=1}^{d} \lambda^{l-1} \frac{\partial F}{\partial y}(\mu, v(\mu)) \left[ \frac{u(\mu)}{\mu^l} \right]_+ \text{ mod } u(\mu),$$

$$= \sum_{l=1}^{d} \mu^{l-1} \frac{\partial F}{\partial y}(\lambda, v(\lambda)) \left[ \frac{u(\lambda)}{\lambda^l} \right]_+ \text{ mod } u(\lambda),$$

where we used the exchange property (11) in the last step. Since $H_i$ is the coefficient of $\mu^{d-i}$ in $H(\lambda)$ this gives equation (12) for $X_H, u(\lambda)$ in case $\varphi(x, y) = 1$. In a similar way $X_H, v(\lambda)$ is found, the computation of $\frac{\partial}{\partial u_j} H_{F,d}(u(\mu), v(\mu))$ is however more involved: let $1 \leq j \leq d$ then

$$\frac{\partial}{\partial u_j} (F(\mu, v(\mu)) \text{ mod } u(\mu)) = \frac{\partial}{\partial u_j} \left( u(\mu) \left[ \frac{F(\mu, v(\mu))}{u(\mu)} \right]_+ \right),$$

$$= -\frac{\partial}{\partial u_j} \left( u(\mu) \left[ \frac{F(\mu, v(\mu))}{u(\mu)} \right]_+ \right),$$

$$= -u(\mu) \left[ \frac{\mu^{d-j} F(\mu, v(\mu))}{u(\mu)} \right]_+ - \left[ \frac{\mu^{d-j} F(\mu, v(\mu))}{u(\mu)} \right]_+, $$

$$(i) = -u(\mu) \left[ \frac{\mu^{d-j} F(\mu, v(\mu))}{u(\mu)} \right]_+, $$

$$= -\mu^{d-j} \left[ \frac{F(\mu, v(\mu))}{u(\mu)} \right]_+ \text{ mod } u(\mu).$$

In $(i)$ we used that if $R = R(\mu)$ and $P = P(\mu)$ are rational functions, with $[R]_+ = 0$, then

$$R[P]_+ - [RP]_+ = R[P]_+ - [R[P]_+]_+ = [R[P]_+]_-. $$
Granted this we obtain as above
\[
\{v(\lambda), H_{F,d}(u(\mu), v(\mu))\}_d = \sum_{j=1}^{d} \mu^{d-j} \left[ \frac{u(\lambda)}{\lambda^{d-j+1}} \right] + \left[ \frac{F(\mu, v(\mu))}{u(\mu)} \right] \mod u(\mu),
\]

which leads at once to the expression (12) for \(X_H, v(\lambda)\) in case \(\varphi(x, y) = 1\). Having obtained the formulas (12) for \(X_H, u(\lambda)\) and \(X_H, v(\lambda)\), the formulas for \(X^\varphi_H, u(\lambda)\) and \(X^\varphi_H, v(\lambda)\), are obtained at once upon using (8).

Finally, the exchange property (11) implies that \(\lambda\) and \(\mu\) are everywhere interchangeable in the above computations so we get \(\{u(\lambda), H_{F,d}(u(\mu), v(\mu))\}_d^\varphi = \{u(\mu), H_{F,d}(u(\lambda), v(\lambda))\}_d^\varphi\), which is tantamount to the identity \(\{u_i, H_j\}_d^\varphi = \{u_j, H_i\}_d^\varphi\). The second formula in (13) follows in the same way.

**Proof of Theorem 2**
We first prove that \(\{H_i, H_{F,d}(u(\lambda), v(\lambda))\}_d^\varphi = 0\) for \(1 \leq i \leq d\). To make the proof more transparent, we use the following abbreviations:

\[
F_y = \frac{\partial F}{\partial y}(\lambda, v(\lambda)), \quad F_u = \frac{F(\lambda, v(\lambda))}{u(\lambda)} \quad \text{and} \quad U_i = \frac{\varphi(\lambda, v(\lambda))}{u(\lambda)} \left[ \frac{u(\lambda)}{\lambda^{d-i+1}} \right]_+,
\]

so that (12) is rewritten as \(X^\varphi_{H_i} u(\lambda) = u(\lambda) [U_i F_y]_-\) and \(X^\varphi_{H_i} v(\lambda) = u(\lambda) [U_i F_u]_+\). Then

\[
\{H_i, H_{F,d}(u(\lambda), v(\lambda))\}_d^\varphi = X^\varphi_{H_i} \left( u(\lambda) \left[ \frac{F(\lambda, v(\lambda))}{u(\lambda)} \right]_+ \right),
\]

\[
= X^\varphi_{H_i} u(\lambda) [F_u]_- + u(\lambda) \left[ \frac{X^\varphi_{H_i} F(\lambda, v(\lambda))}{u(\lambda)} \right]_- - u(\lambda) \left[ \frac{F(u) X^\varphi_{H_i} u(\lambda)}{u(\lambda)} \right]_-,
\]

\[
= u(\lambda) \left( [U_i F_y]_- [F_u]_- + [F_y [U_i [F_u]_+]_-]_- - [F_u [U_i F_y]_-]_- \right),
\]

\[
= u(\lambda) \left( [U_i F_y]_- [F_u]_- + F_y [U_i [F_u]_+]_- - F_u [U_i F_y]_- \right),
\]

\[
\overset{(i)}{=} u(\lambda) \left( [U_i F_y]_- [F_u]_- + F_y U_i [F_u]_+ \right),
\]

\[
= u(\lambda) \left( [U_i F_y]_- [F_u]_+ \right),
\]

\[
= 0.
\]

In (i) we used the fact that \(F_y\) is a polynomial, i.e., \([F_y]_- = 0\).

We now show that the \(d\) coefficients of \(H_{F,d}(u(\lambda), v(\lambda)) = F(\lambda, v(\lambda)) \mod u(\lambda)\) are functional independent. Clearly the last \(d\) coefficients \(H_1, \ldots, H_d\) of \(F(\lambda, v(\lambda))\) are independent because \(v_i\) appears only in \(H_1, \ldots, H_i\) (it does appear since \(F(x, y) \notin R[x]\)). Reducing \(F(\lambda, v(\lambda)) \mod u(\lambda)\)...
amounts to substracting from $\tilde{H}_i$ polynomials of lower degree in the variables $v_j$, so it cannot make these functions dependent and the independence of $\{H_1, \ldots, H_d\}$ follows.

Amplification 2

If $F(x, y)$ and $F'(x, y)$ differ only by a polynomial which is independent of $y$ and is of degree less than $d$ in $x$, then clearly the $d$-dimensional integrable systems which are associated to $F$ and $F'$ are the same; in this sense, for $\varphi(x, y)$ fixed, a system is associated to a coset

$$\tilde{F}(x, y) = \left\{ F(x, y) + \sum_{i=0}^{d-1} c_i x^i \mid c_i \in \mathbb{R} \right\}.$$ 

If a (differentiable) deformation family $M$ of classes $\tilde{F}(x, y)$ is given (rather than a single class) then our construction is easy adapted to give (for each non-zero $\varphi(x, y) \in \mathbb{R}[x, y]$) a $d$-dimensional integrable system on a Poisson manifold, which is the product of the deformation manifold $M$ and $\mathbb{R}^{2d}$. Namely let the brackets (4) on $(\mathbb{R}^2)^d$ be extended trivially to $(\mathbb{R}^2)^d \times M$, i.e., if $\pi_M$ denotes the projection map $(\mathbb{R}^2)^d \to M$ then the annihilator of the Poisson bracket is chosen as $\{ f \circ \pi_M \mid f \in C^\infty(M) \}$. Also the map $S$ given by (5) is extended to the map

$$S \times \text{Id}_M : \left((\mathbb{R}^2)^d \setminus \Delta\right) \times M \to \mathbb{R}^{2d} \times M,$$

which is the identity map $\text{Id}_M$ on the second component. As both this Poisson structure and these maps are again invariant for the action of $S_d$ (on the first component) we obtain a Poisson structure $\{ \cdot, \cdot \}_d, M$ on the image of $S \times \text{Id}_M \subset \mathbb{R}^{2d} \times M$, which extends to all of $\mathbb{R}^{2d} \times M$, again because all brackets are polynomial. The commuting vector fields $\{ \cdot, H_i \}_d, M$ are tangent to the (linear) Poisson submanifolds $\{ \tilde{F} \} \times \mathbb{R}^{2d}$, $(\tilde{F} \in M)$, to which $\{ \cdot, \cdot \}_d, M$ restricts as $\{ \cdot, \cdot \}_d$. Therefore, these commuting vector fields restricts to these submanifolds and give there the vector fields $\{ \cdot, H_i \}_d$ (as given by (12)) of the integrable system associated to $\tilde{F}$ (i.e., to $F$).

Amplification 3

In all the above definitions, $\mathbb{R}$ can be replaced by $\mathbb{C}$; our construction then associates to each complex polynomial in two variables, a maximal set of holomorphic functions (polynomials), defined on $\mathbb{C}^{2d}$, which are in involution with respect to a holomorphic Poisson bracket, itself determined by an arbitrary non-zero polynomial in two variables.
3. The geometry of the invariant manifolds

The integrable systems introduced in Section 2 provide us (for each $d \geq 1$ and $F(x, y) \in \mathbb{R}[x, y] \setminus \mathbb{R}[x]$) with a surjective map defined by $H_{F,d}(u(\lambda), v(\lambda)) = F(\lambda, v(\lambda)) \mod u(\lambda)$. The fibers of $H_{F,d}$ are preserved by the flows of the $d$ vector fields $X_{F,d}^i$, which correspond via $\{\cdot, \cdot\}_d$ to the components of this map. By Sard’s Theorem, the generic fiber of this map is smooth. These smooth fibers are called the invariant manifolds of the system; they are Lagrangian submanifolds of $(\mathbb{R}^d, \{\cdot, \cdot\}_d)$, i.e., the restriction of $\{\cdot, \cdot\}_d$ to these $d$-dimensional submanifolds vanishes. In this section we investigate the geometry of these invariant manifolds and discuss their compactification.

3.1. The invariant manifolds $A_{F,d}$ and $A_{F,d}^C$

Since $H_{F,d}(u(\lambda), v(\lambda))$ is defined as $F(\lambda, v(\lambda)) \mod u(\lambda)$, the fiber over $h(\lambda) \in \mathbb{R}_{d-1}[\lambda]$ is the same as the fiber over 0 for $H_{F',d}$, where $F'(x, y) = F(x, y) - h(x)$. Therefore we may restrict ourselves to the fiber lying over 0, denoted by $A_{F,d}$; thus, by definition, $A_{F,d}$ is given by

$$A_{F,d} = \left\{ (u(\lambda), v(\lambda)) \in \mathbb{R}^{2d} \mid \left[ \frac{F(\lambda, v(\lambda))}{u(\lambda)} \right] = 0 \right\}. \quad (14)$$

Sard’s Theorem says now that this fiber is smooth if $F(x, y)$ is generic. Clearly if $F(x, y)$ is generic then the complex algebraic curve $\Gamma_F \subset \mathcal{C}^2$, defined by $F(x, y) = 0$, is smooth. We show now that in fact the latter suffices for $A_{F,d}$ to be smooth.

**Proposition 5** If the algebraic curve $\Gamma_F \subset \mathcal{C}^2$ defined by $F(x, y) = 0$ is smooth, then the fiber $A_{F,d} \subset \mathbb{R}^{2d}$ is also smooth.

**Proof**

$A_{F,d}$ will be smooth if and only if $H_{F,d}$ is submersive at each point of $A_{F,d}$, i.e., iff

$$\text{rank} \left( \frac{\partial H_i}{\partial u_1}, \ldots, \frac{\partial H_i}{\partial u_d}, \frac{\partial H_i}{\partial v_1}, \ldots, \frac{\partial H_i}{\partial v_d} \right)_{1 \leq i \leq d} = d \text{ along } A_{F,d}.$$

From the proof of Theorem 2 and the definition (14) of $A_{F,d}$, the $j$-th and $d+j$-th columns of this matrix are respectively given by

$$\lambda^{d-j} \frac{F(\lambda, v(\lambda))}{u(\lambda)} \mod u(\lambda) \quad \text{and} \quad \lambda^{d-j} \partial F \partial y (\lambda, v(\lambda)) \mod u(\lambda).$$

It is therefore sufficient to show that if $\Gamma_F$ is smooth then the dimension of the linear space

$$\left( R_1(\lambda) \frac{F(\lambda, v(\lambda))}{u(\lambda)} + R_2(\lambda) \partial F \partial y (\lambda, v(\lambda)) \right) \mod u(\lambda), \deg R_i(\lambda) < d, \quad (15)$$

equals $d$. Let $\lambda_1, \ldots, \lambda_r$ be the distinct roots of $u(\lambda)$, $\lambda_i$ having multiplicity $s_i$. We claim that

$$\frac{F(\lambda_i, v(\lambda_i))}{u(\lambda_i)} = 0 \quad \text{and} \quad \partial F \partial y (\lambda_i, v(\lambda_i)) = 0 \quad (16)$$

cannot hold simultaneously if $\Gamma_F$ is smooth. For otherwise $(x_i, y_i) = (\lambda_i, v(\lambda_i))$ would be a singular point of $\Gamma_F$: if (16) holds then clearly $\partial F \partial y (x_i, y_i) = 0$, but also $F(x_i, y_i) = \partial F \partial x (x_i, y_i) = 0$ because in this case $F(x, y_i)$ has a double zero at $x = x_i$. 

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The dimension of (15) is now investigated by remarking that for any polynomial \( p(\lambda) \), the value of \( p(\lambda) \mod u(\lambda) \) at \( \lambda_i \) is just \( p(\lambda_i) \), and the values of the first \( s_i - 1 \) derivatives of \( p(\lambda) \mod u(\lambda) \) at \( \lambda_i \) are given by the values of the corresponding derivatives of \( p(\lambda) \) at \( \lambda_i \). Let us suppose that the different roots of \( u(\lambda) \) are ordered such that \( \lambda_1, \ldots, \lambda_t \) are also zeros of \( \frac{\partial F}{\partial y}(\lambda, v(\lambda)) \), while \( \lambda_{t+1}, \ldots, \lambda_r \) are not. As a first restriction, let \( R_1(\lambda) \) (resp. \( R_2(\lambda) \)) be such that its first \( s_i - 1 \) derivatives vanish at \( \lambda_i \) for \( t + 1 \leq i \leq r \) (resp. \( 1 \leq i \leq t \)). As a further restriction it is (by the first restriction and as (16) cannot happen) now easy to see that \( R_1(\lambda) \) (resp. \( R_2(\lambda) \)) can be determined such that the polynomial given by (15) and the first \( s_i - 1 \) derivatives of (15) take any given values at \( \lambda_i \) for \( 1 \leq i \leq t \) (resp. \( t + 1 \leq i \leq r \)). These \( d \) conditions are independent, hence the dimension of (15) equals \( d \) and \( \mathcal{A}_{F,d} \) is smooth.

We aim at a more precise description of the structure of the invariant manifolds \( \mathcal{A}_{F,d} \), which will be useful for describing their topological structure. If the fixed point set of the complex conjugation map \( \tau: \mathbb{C}^{2d} \to \mathbb{C}^{2d}; z \mapsto \bar{z} \) is denoted as \( \text{Fix}(\tau) \), then clearly \( \mathcal{A}_{F,d} \) is given by

\[
\mathcal{A}_{F,d} = \text{Fix}(\tau) \cap \mathcal{A}_{F,d}^\mathbb{C}, \quad \text{where} \quad \mathcal{A}_{F,d}^\mathbb{C} = \left\{ (u(\lambda), v(\lambda)) \in \mathbb{C}^{2d} \mid \frac{F(\lambda, v(\lambda))}{u(\lambda)} = 0 \right\}. \tag{17}
\]

Therefore, \( \mathcal{A}_{F,d} \) is called a real algebraic variety (see [S]). Remark that \( \mathcal{A}_{F,d}^\mathbb{C} \) is the complex invariant manifold lying over 0 of the integrable system on \( \mathbb{C}^{2d} \) associated to \( F \) (see Amplification 3). The following proposition is the complex analog of Proposition 5.

**Proposition 6** The curve \( \Gamma_F \subset \mathbb{C}^2 \) is smooth if and only if the fiber \( \mathcal{A}_{F,d}^\mathbb{C} \subset \mathbb{C}^{2d} \) is smooth.

**Proof**

If \( \Gamma_F \) has a singular point \( P_1 = (x_1, y_1) \), choose \( d - 1 \) different points \( P_i = (x_i, y_i) \) on \( \Gamma_F \) and define \( (u(\lambda), v(\lambda)) = S((x_1, y_1), \ldots, (x_d, y_d)) \in \mathcal{A}_{F,d}^\mathbb{C} \). All polynomials given by (15) vanish for \( \lambda = x_1 \), hence they span a linear space of dimension less than \( d \). Thus \( H_{F,d} \) is not submersive at \( (u(\lambda), v(\lambda)) \) and \( \mathcal{A}_{F,d} \) is singular at this point. This shows the if part of the proposition; the only if part is proven verbatim as in the real case (Proposition 5).

It will be seen that a clear understanding of the structure of the complex manifolds \( \mathcal{A}_{F,d}^\mathbb{C} \) (for \( \Gamma_F \) smooth), leads also to a precise description of the real manifolds \( \mathcal{A}_{F,d} \).

### 3.2. The structure of the complex invariant manifolds \( \mathcal{A}_{F,d}^\mathbb{C} \)

We will show that \( \mathcal{A}_{F,d}^\mathbb{C} \) is an affine part of the \( d \)-fold symmetric product \( \text{Sym}^d \Gamma_F \subset \mathbb{R}^2 \). Recall (e.g. from [Gu]) that \( \text{Sym}^d \Gamma_F \) is defined as the orbit space of the obvious action of the permutation group \( S_d \) on the cartesian product \( \Gamma_F^d = \Gamma_F \times \cdots \times \Gamma_F \ (d \text{ factors}) \), i.e.,

\[
\text{Sym}^d \Gamma_F = \Gamma_F^d / S_d.
\]

\( \text{Sym}^d \Gamma_F \) inherits its structure as a complex algebraic variety from the algebraic structure of \( \Gamma_F \). Moreover the smoothnes of \( \Gamma_F \) implies smoothnes of \( \text{Sym}^d \Gamma_F \): namely each point \( P = \langle P_1^{m_1}, \ldots, P_r^{m_r} \rangle \in \text{Sym}^d \Gamma_F \) (with all \( P_i \) different; \( m_i \) is the multiplicity of \( P_i \) in \( P \)) has a neighborhood which is isomorphic to a neighborhood of \( \langle P_1^{m_1}, \ldots, P_r^{m_r} \rangle \) in \( \text{Sym}^{m_1} \Gamma_F \times \cdots \times \text{Sym}^{m_r} \Gamma_F \), and a point \( \langle P_i^{m_i} \rangle \) on the diagonal of \( \text{Sym}^{m_i} \Gamma_F \) has coordinates given by the \( m_i \) elementary symmetric functions of the \( m_i \) coordinate functions on \( \Gamma_F^{m_i} \).
Theorem 7  If the algebraic curve $\Gamma_F$ in $\mathbb{C}^2$, defined by $F(x, y) = 0$ is smooth, then $\mathcal{A}_{F,d}^C$ is biholomorphic to the (Zariski) open subset of $\text{Sym}^d \Gamma_F$, obtained by removing from it the divisor

$$\mathcal{D}_{F,d} = \left\{ (P_1, \ldots, P_d) \mid \exists i, j: 1 \leq i < j \leq d, \left( x(P_i) = x(P_j) \text{ with } P_i \neq P_j, \text{ or } P_i = P_j \text{ is a ramification point of } x \right) \right\}.$$ 

Proof

- **Construction of the map $\phi_{F,d}: \mathcal{A}_{F,d}^C \to \text{Sym}^d \Gamma_F \setminus \mathcal{D}_{F,d}$**

Given a point $(u(\lambda), v(\lambda)) \in \mathcal{A}_{F,d}^C$, a point in $\text{Sym}^d \Gamma_F$ is associated to it as follows: for every root $\lambda_i$ of $u(\lambda)$ one has $F(\lambda_i, v(\lambda_i)) = 0$, because $\left[ \frac{F(\lambda, v(\lambda))}{u(\lambda)} \right]_\lambda = 0$, so each root $\lambda_i$ of $u(\lambda)$ determines a point $(\lambda_i, v(\lambda_i))$ on $\Gamma_F$. Thus there corresponds to $(u(\lambda), v(\lambda)) \in \mathcal{A}_{F,d}^C$ an unordered set of $d$ points $(P_1, \ldots, P_d) \in \text{Sym}^d \Gamma_F$, where $P_i$ is defined by $(x(P_i), y(P_i)) = (\lambda_i, v(\lambda_i))$. Clearly, if $x(P_i) = x(P_j)$ then $P_i = P_j$; therefore, to show that $(P_1, \ldots, P_d)$ stays away from $\mathcal{D}_{F,d}$ we only need to prove that $P_i = P_j$ cannot occur for $i \neq j$ if $P_i$ is a ramification point for $x$, i.e., if $y(P_i)$ is a multiple root of $F(x(P_i), y)$ (as a polynomial in $y$). As $P_i = P_j$ ($i \neq j$) implies that $u(\lambda)$ has a multiple root $x(P_i)$, in such a case $F(x, y(P_i))$ would have a multiple root $x = x(P_i)$, again because $\left[ \frac{F(\lambda, v(\lambda))}{u(\lambda)} \right]_\lambda = 0$. If moreover $P_i$ is a ramification point of $x$ then also $\frac{\partial F}{\partial y}(x(P_i), y(P_i)) = 0$ and it follows that $(x(P_i), y(P_i))$ is a singular point of $\Gamma_F$, a contradiction.

- **$\mathcal{D}_{F,d}$ is a divisor on $\text{Sym}^d \Gamma_F$**

This means that $\mathcal{D}_{F,d}$ is given locally as the zero locus of a holomorphic function. If $(P_1, \ldots, P_d) \in \mathcal{D}_{F,d}$ let the set of indices $\{1, \ldots, d\}$ be decomposed as $S_1 \cup \cdots \cup S_n$, such that all points $P_i$ where $i$ runs through one of the subsets $S_j$ have the same $x$-coordinate, which is disjoint from the $x$-coordinates of the points which correspond to the other subsets. For each $P_i$ ($i = 1, \ldots, d$) let $x_i$ denote the lifting of $x$ to a small neighborhood of $(P_1, \ldots, P_d)$ (corresponding to the factor $P_i$). Then a local defining equation of $\mathcal{D}_{F,d}$ is given by

$$\prod_{i=1}^n \prod_{j,k \in S_i, j < k} (x_j - x_k) = 0.$$ 

- **$\phi_{F,d}$ is a biholomorphism**

We first construct the inverse of $\phi_{F,d}$, which is closely related to the map $S$, as given by (5). Let $(P_1, \ldots, P_d) \in \text{Sym}^d \Gamma_F \setminus \mathcal{D}_{F,d}$. Clearly $u(\lambda)$ is taken as

$$u(\lambda) = \prod_{i=1}^d (\lambda - x(P_i)). \quad (18)$$

If all $x(P_i)$ are different then $v(\lambda)$ is uniquely determined as the polynomial of degree $d - 1$ whose value at $\lambda = x(P_i)$ is $y(P_i)$, i.e., $v(\lambda)$ is given by

$$v(\lambda) = \sum_{l=1}^d y_l \prod_{k \neq l} \frac{\lambda - x_k}{x_l - x_k}. \quad (19)$$

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and is holomorphic there. If two values coincide, say \( x(P_1) = x(P_2) \), then \( P_1 = P_2 \) is not a ramification point (since we stay away from \( \mathcal{D}_{F,d} \)), hence the equation \( F(x, y) = 0 \) can be solved uniquely as \( y = f(x) \) in a neighborhood of \( P_1 = P_2 \). For \( P'_1 \) and \( P'_2 \) in this neighborhood, substitute

\[
f(x(P'_i)) = f(x(P_i)) + (x(P'_i) - x(P_i)) \frac{df}{dx}(x(P_1)) + O((x(P'_i) - x(P_1))^2), \quad (i = 1, 2)
\]

for \( y_1 \) and \( y_2 \) in (19), to obtain that \( v(\lambda) \) has no poles as \( P'_1, P'_2 \to P_1 \), hence extends to a holomorphic function on the larger subset where at most two points coincide. Since the complement of this larger subset in \( \text{Sym}^d \Gamma_F \setminus \mathcal{D}_{F,d} \) is of codimension at least two, \( v(\lambda) \) extends to a holomorphic function on \( \text{Sym}^d \Gamma_F \setminus \mathcal{D}_{F,d} \). It also follows that this holomorphic function is the inverse of \( \phi_{F,d} \) on all of \( \text{Sym}^d \Gamma_F \setminus \mathcal{D}_{F,d} \): if the point \( P_i \) has multiplicity \( s_i \), then the first \( s_i - 1 \) derivatives of \( v(\lambda) \) at \( x(P_i) \) coincide with those of \( f(\lambda) \) at \( x(P_i) \), hence \( F(\lambda, y(P_i)) \) has a zero of order \( s_i \) at \( \lambda = x(P_i) \). Finally, the inverse of a holomorphic bijection between complex manifolds is always holomorphic (see [GH]), hence \( \phi_{F,d} \) is a biholomorphism.

\[\Box\]

3.3. The structure of the real invariant manifolds \( \mathcal{A}_{F,d} \)

Since \( \mathcal{A}_{F,d} \) is given as \( \mathcal{A}_{F,d}^C \cap \text{Fix}(\tau) \), it consists of those polynomials \((u(\lambda), v(\lambda)) \in \mathcal{A}_{F,d} \) whose coefficients are all real. We figure out what this means for the corresponding point in \( \text{Sym}^d \Gamma_F \).

**Proposition 8** Under the biholomorphism \( \phi_{F,d} \), the real invariant manifolds \( \mathcal{A}_{F,d} \) correspond to the set of all unordered \( d \)-tuples of points \((P_1, \ldots, P_d) \) on \( \Gamma_F \), consisting only of real points \( P_i \in \mathbb{R}^2 \cap \Gamma_F \) and complex conjugated pairs \( P_i = \bar{P}_j \), each ramification point \((x) \) occurring at most once, and \( x(P_i) = x(P_j) \) only if \( P_i = P_j \). Moreover its manifold structure derives from the structure of the \( d \)-fold symmetric product of \( \Gamma_F \).

**Proof**

\( u(\lambda) \) is real if and only if its roots consist only of real roots and roots which occur in complex conjugate pairs. Obviously, if \( v(\lambda) \) is real, then at each root \( x_i \) of \( u(\lambda) \), with multiplicity \( s_i \), \( v(\lambda) \) and the first \( s_i - 1 \) derivatives of \( v(\lambda) \) take complex conjugate values when evaluated at complex conjugate points (in particular, real values at real points). It is checked that this is also a sufficient condition for \( v(\lambda) \) to be real. Since \( v(x_i) = y_i \), this means that the real polynomials \((u(\lambda), v(\lambda)) \) on \( \mathcal{A}_{F,d}^C \) correspond to those points \((P_1, \ldots, P_d) \) in \( \text{Sym}^d \Gamma_F \) consisting of real points \( P_i = (x(P_i), y(P_i)) \in \mathbb{R}^2 \) and complex conjugated pairs \( P_j = (x(P_j), y(P_j)) = (\overline{x(P_i)}, \overline{y(P_i)}) \neq \bar{P}_k \), but not belonging to \( \mathcal{D}_{F,d} \), i.e., the multiplicity of each ramification point \((x) \) is at most one, and \( x(P_i) = x(P_j) \) only if \( P_i = P_j \).

\[\Box\]

**Proposition 8** can be used to obtain a precise description of the topology of the real invariant manifolds \( \mathcal{A}_{F,d} \), as we show now for \( d = 2 \) (for \( d = 1 \), \( \mathcal{A}_{F,d} \) is just \( \Gamma_F \cap \mathbb{R}^2 \), the real part of \( \Gamma_F \)). For a fixed \( F \) such that \( \Gamma_F \) is smooth, let the connected components of \( \Gamma_F \cap \mathbb{R}^2 \) (if any) be denoted by \( \Gamma_1, \ldots, \Gamma_s \) and define for \( 1 \leq i, j, k \leq s, i < j \)

\[
\begin{align*}
\Gamma_{00} &= \{ (P, \bar{P}) \mid P \in \Gamma_F, x(P) \notin \mathbb{R} \}, \\
\Gamma_{ij} &= \{ (P_1, P_2) \in \Gamma_i \times \Gamma_j \mid x(P_1) = x(P_2), P_1 = P_2 \}, \\
\Gamma_{kk} &= \left\{ (P_1, P_2) \in \frac{\Gamma_k \times \Gamma_k}{S_2} \mid x(P_1) = x(P_2) \Rightarrow (P_1 = P_2 \text{ and is not a ramification point of } x) \right\}
\end{align*}
\]
Then the union of $\Gamma_{00}$ with all the sets $\Gamma_{ij}$ and $\Gamma_{kk}$ is easy identified with $\phi_{F,2}(A_{F,2})$, the surface to be described. It is remarked that the only paths in it which are not contained in $\mathbb{R}^2$, are in $\Gamma_{00}$, and $\Gamma_{00}$ connects exactly the surfaces $\Gamma_{kk}$. Therefore, if $i \neq j$ then $\Gamma_{ij}$ is not connected to any other $\Gamma_{mn}$, $\Gamma_{kk}$, nor to $\Gamma_{00}$.

Therefore we first concentrate on such a subset $\Gamma_{ij}$, say on $\Gamma_{12}$. If the intervals $x(\Gamma_1)$ and $x(\Gamma_2)$ are disjoint, then $\Gamma_{12} = \Gamma_1 \times \Gamma_2$, so $\Gamma_{12}$ is either homeomorphic to a torus, a cylinder or a disc, depending on whether the components $\Gamma_1$ and $\Gamma_2$ are closed or open. If $x(\Gamma_1)$ and $x(\Gamma_2)$ have a point $P$ in common, then one finds again these surfaces, but with a number of punctures (holes), equal to

$$\prod_{i=1}^{2} \# \{ Q \in \Gamma_i \ | \ x(Q) = P \}.$$

If $x(\Gamma_1)$ and $x(\Gamma_2)$ have an interval in common, $\Gamma_{12}$ may even disconnect in different pieces. The structure of these pieces is easy determined from a picture of the real part of the curve. Namely, on a square representing $\Gamma_1 \times \Gamma_2$, the divisor $\{(P_1, P_2) \in \Gamma_1 \times \Gamma_j \ | \ x(P_1) = x(P_2)\}$ is drawn by counting points on the vertical lines $x = \text{constant}$, the only care one needs to take is that if $\Gamma_1$ (or $\Gamma_2$) is closed, then an origin should be marked on it, and if one passes this origin, one needs to pass over the corresponding edge of the rectangle.

In the same way $\Gamma_{kk}$ is investigated by drawing the divisor

$$\left\{ (P_1, P_2) \in \Gamma_i \times \Gamma_i \ | \ x(P_1) = x(P_2) \right\} \text{ and } \left\{ P_1 \neq P_2 \text{ or, } P_1 = P_2 \text{ is a ramification point of } x \right\},$$

on a rectangle representing $\Gamma_i \times \Gamma_i$. Either triangle cut off from the rectangle by its main diagonal then represents $\frac{\Gamma_i \times \Gamma_i}{S_2}$ and $\Gamma_{ii}$ is the complement of the divisor in the triangle. For every $\Gamma_i$ such a piece is found and will be glued to $\Gamma_{00}$ precisely along the part of its boundary which comes from the diagonal in the rectangle.

In order to explain how $\Gamma_{00}$ is described, we recall the classical picture of a (smooth, complete) algebraic curve $\Gamma$. An equation $F(x, y) = 0$ of such a curve defines an $m:1$ ramified covering map to $\mathbb{P}^1$ by $(x, y) \mapsto x$, when $m$ is the degree of $F(x, y)$ in $y$. This may be visualised by drawing concentric spheres (called sheets), on which there are marked some non-intersecting intervals (called cuts, every cut is equally present on all sheets). The topology is such that if you are walking on a sheet $i$ and pass a cut $j$ (from a fixed side) then you move to a sheet $p_j(i)$, each $p_j$ being a permutation of $\{1, \ldots , m\}$. It is clear that the datum of cuts and their corresponding permutations determines the topology of the curve completely. Since each cut connects two ramification points (of $x$), these cuts may, for a real curve, be taken on the real axis and orthogonal to it.

$\Gamma_{00}$ is now given as follows. Consider the described picture for the smooth completion $\bar{\Gamma}_F$ of $\Gamma_F$. Clearly the conjugation map interchanges the upper and lower hemispheres and is fixed on the equator(s) $\{ P \in \bar{\Gamma}_F \ | \ x(P) \in \mathbb{R} \cup \infty \}$. It follows that the open upper (lower) hemispheres give precisely $\Gamma_{00}$. A convenient way to represent them is by drawing a disc for each upper hemisphere and labelling the different parts of the boundary which correspond to the horizontal and vertical cuts. A moment’s thought reveals that the different sheets are to be connected along those lines which correspond to the vertical cuts, while the pieces $\Gamma_{kk}$ are to be connected to the corresponding horizontal cuts. This gives a topological model of $\Gamma_{00} \cup \bigcup_{i=1}^{2} \Gamma_{kk}$ as a disc with holes. See the complete version of this paper for an example (Pub. IRMA Lille 33 (1993)); it has been suppressed because it contains several figures.
3.4. Compactification of the complex invariant manifolds $\mathcal{A}_{F,d}^C$

We now discuss the (smooth) compactification of the manifolds $\mathcal{A}_{F,d}^C$. There is one obvious and natural compactification, namely the compact manifold $\text{Sym}^d \Gamma_F$, defined in a similar way as $\text{Sym}^d \Gamma_F$; as above $\Gamma_F$ denotes the smooth compactification of $\Gamma_F$. However $\mathcal{A}_{F,d}^C$ has the disadvantage that none of the vector fields $X_H$ extends holomorphically to it — a compactification such that at least one of these vector fields extends in a holomorphic way to it, will simply be called good. The interest in good compactifications is that it allows one to integrate the corresponding vector fields in terms of theta functions, or degenerations of theta functions, which are analytic, quasi-periodic functions on $\mathbb{C}^d$. The purpose of this paragraph is to show that even for very simple choices of $F(x,y)$, a good compactification of $\mathcal{A}_{F,d}^C$ does not exist. We believe that this is true for almost all choices of $F(x,y)$. A class of examples for which a good compactification does exist is considered in the next section.

At first we compute, for fixed $F(x,y)$ how the vector fields $X_H$, behave on the compact manifold $\text{Sym}^d \Gamma_F$, which relates to $\mathcal{A}_{F,d}^C = \text{Sym}^d \Gamma_F \setminus \mathcal{D}_{F,d}$ as follows:

$$\text{Sym}^d \Gamma_F = \mathcal{A}_{F,d}^C \cup \mathcal{D}_{F,d} \cup \mathcal{E}_{F,d}.$$ 

Here $\mathcal{D}_{F,d}$ is the closure of $\mathcal{D}_{F,d}$ in $\text{Sym}^d \Gamma_F$ and $\mathcal{E}_{F,d}$ is a divisor whose irreducible components $\mathcal{E}_{F,d}(\infty_i)$ correspond to the points $\infty_i$ in $\Gamma_F \setminus \Gamma_F$, namely

$$\mathcal{E}_{F,d}(\infty_k) = \{ (\infty_k, P_2, \ldots, P_d) \mid P_k \in \Gamma_F \text{ for } 2 \leq k \leq d \}.$$ 

Each vector field $X_H$, being a polynomial vector field on $\mathbb{C}^{2d}$, it is holomorphic on $\mathcal{A}_{F,d}^C$. We determine its behaviour along the irreducible components of $\mathcal{D}_{F,d}$ and $\mathcal{E}_{F,d}$, which may be done by computing the order of vanishing of $X_H$ at a generic point of each component, which in turn is done by using local coordinates at such a point (see [GH]).

**Proposition 9** Every vector field $X_H$, has a simple pole along all irreducible components of $\mathcal{D}_{F,d}$. It has a zero of order $\rho_k$ along $\mathcal{E}_{F,d}(\infty_k)$ (i.e., a pole of order $-\rho_k$ if $\rho_k < 0$), where

$$\rho_k = \mu_k - \nu_k + d + 1, \quad \nu_k < 0,$$

$$\rho_k = \mu_k - \nu_k + 1, \quad \nu_k > 0;$$

$\mu_k$ is the order (of vanishing) of $\frac{\partial F}{\partial y}(x,y)$ at $\infty_k$, and $\nu_k$ is the order of $x$ at $\infty_k$ (resp. the order of $x - x(\infty_k)$ if $x$ is finite at $\infty_k$).

**Proof**

We first write down the vector field $X_H$, at a generic point $(u(\lambda), v(\lambda)) \in \mathcal{A}_{F,d}^C$; the genericity condition taken here is that for $\phi_{F,d}(u(\lambda), v(\lambda)) = \langle (x_1, y_1), \ldots, (x_d, y_d) \rangle$ all $x_i$ are different and none of the points $(x_i, y_i)$ is a ramification point of $x$. Varying the point $(u(\lambda), v(\lambda))$ in a small neighborhood, each $x_i$ gives a local coordinate on a neighborhood $U_i \subset \Gamma_F$ of $(x_i, y_i)$ as well as a local coordinate on a neighborhood $U \subset \mathcal{A}_{F,d}^C$ of $\langle (x_1, y_1), \ldots, (x_d, y_d) \rangle$. Since on the one hand the derivative of $u(\lambda) = \prod_{k=1}^d (\lambda - x_k)$ at $\lambda = x$ is $X_H, u(x_j) = -\prod_{j \neq k} (x_j - x_k) X_H, x_j$, while at the other hand, direct substitution in (12) gives

$$X_H, u(x_j) = \frac{\partial F}{\partial y}(x_j, y_j) \sum_{k=0}^{i-1} u_k x_j^{i-k-1}.$$ 

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we find that
\[
X_{H, x_j} = -\prod_{l \neq j} (x_j - x_l)^{-1} \frac{\partial F}{\partial y}(x_j, y_j) \sigma_{i-1}(\hat{x}_j),
\]
where \( \sigma_{i-1}(\hat{x}_j) \) is the \( i - 1 \)-th symmetric function in \( x_1, \ldots, x_d \), evaluated at \( x_j = 0 \).

The right hand side of (20) has at a generic point of \( \bar{D}_{F, d} \) a simple pole, hence each vector field \( X_{H, x_j} \) has a simple pole on (every component of) \( \partial \bar{D}_{F, d} \). The behaviour of \( X_{H, x_j} \) along \( \partial \bar{D}_{F, d} \) is slightly more complicated since it depends on \( F(x, y) \), and may even behave differently on each components \( \bar{E}_{F, d}(\infty) \). For a generic point in a neighborhood of a point of \( \bar{E}_{F, d}(\infty) \), let us introduce coordinates \( x_i \) as above. If we denote by \( \mu_k \) and \( \nu_k \) the integers introduced in the Proposition, then clearly \( x_1 \) is given on a neighborhood of \( \infty \) in terms of a local parameter \( t_1 \) at \( \infty \) as \( x_1 = t_1^{\nu_k} (\nu_k < 0) \), or as \( x_1 = c_1 + t_1^{\nu_k} (\nu_k > 0) \), depending on whether \( x \) is infinite in a neighborhood of \( \infty \) or has a finite value \( c_1 \in \mathbb{C} \) at \( \infty \); also \( \frac{\partial F}{\partial y}(x_1(t), y_1(t)) = t^{\nu_k} (f_1 + \mathcal{O}(t)) \) with \( f_1 \neq 0 \). We define for \( 2 \leq j \leq d \) local parameters \( t_j \) (centered at \( P_j \), which may be assumed to be generic) by
\[
x_j = x(P_j) + t_j.
\]
Elementary substitution in (20) yields
\[
\begin{align*}
X_{H, t_1} &= t_1^{\nu_k} (c_1 + \mathcal{O}(t_1)), \\
X_{H, t_j} &= c_j + \mathcal{O}(t_1), \quad (j = 2, \ldots, d);
\end{align*}
\]
where \( \rho_k \) is defined as above. We conclude that \( X_{H, x_j} \) has a zero (resp. pole) of order \( |\rho_k| \) along \( \bar{E}_{F, d}(\infty) \) if \( \rho_k \geq 0 \) (resp. \( \rho_k < 0 \)).

Thus we have shown that \( \text{Sym}^d \bar{\Gamma}_F \) is not a good compactification, since all vector fields \( X_{H, x_j} \) have at least a pole along \( \partial \bar{D}_{F, d} \). This divisor can be contracted in some cases, as we will show in the next section. The following example shows that a good compactification does not exist in general.

**Example**

Let \( F(x, y) = y^3 + f(x) \), where the degree of \( f \) is at least three, and let \( d = 2 \). To show that \( \mathcal{A}_{F, 2}^\mathcal{V} \) has no good compactification we use some results about algebraic surfaces which can be found in [Ha]. Suppose that \( \mathcal{A} \) is a good compactification of \( \mathcal{A}_{F, 2}^\mathcal{V} \) then \( \mathcal{A} \) and \( \text{Sym}^2 \bar{\Gamma}_F \) are birational; for surfaces this means that there exists a finite series of monoidal transformations (also known as blow-up’s) which transforms \( \mathcal{A} \) into \( \text{Sym}^2 \bar{\Gamma}_F \). Then there exist (Zariski) open subsets \( \mathcal{U} \subset \mathcal{A} \) and \( \mathcal{V} \subset \text{Sym}^2 \bar{\Gamma}_F \) to which all these monoidal transformations restrict as isomorphisms and the vector fields on \( \mathcal{U} \) and \( \mathcal{V} \) correspond exactly under this isomorphism. In particular \( \bar{D}_{F, 2} \) is entirely contained in the complement of \( \mathcal{V} \) and must be contracted by one of the monoidal transformations, so at least we know that the genus of \( \bar{D}_{F, 2} \) must be 0 (only \( \mathbb{P}^1 \)'s can be contracted).

We may however compute the genus of \( \bar{D}_{F, 2} \) directly. Recall that it consists of the points \( \langle P_1, P_2 \rangle \) with \( x(P_1) = x(P_2) \) for which \( P_1 \neq P_2 \) or \( P_1 = P_2 \) is a ramification point, so its smoothness is easy checked. However the map \( x \) expresses \( \bar{D}_{F, 2} \) as a 3:1 cover of \( \mathbb{P}^1 \), ramified at the \( n = \deg f \) points \( (x_i, y_i) \) for which \( f(x_i) = 0 \) (and at infinity if \( n \) is not divisible by 3). So it has the same ramification divisor as \( \bar{\Gamma}_F \), hence genus(\( \bar{D}_{F, 2} \)) = genus(\( \bar{\Gamma}_F \)) > 0, a contradiction.

**Amplification 4**

Summing up (20) over all \( j \) (and for any \( \varphi \)) we find that for any fixed integer \( r < d \),
\[
\sum_{j=1}^d x_j^r \frac{X_{H, x_j}}{\varphi(x_j, y_j)} \frac{\partial F}{\partial y}(x_j, y_j) = -\sum_{k=0}^{j-1} u_k \sum_{j=1}^d x_j^{r+i-k-1} \prod_{l \neq j} \frac{1}{x_j - x_l} = -\delta_{i+r,d}.
\]
Therefore the $d$ variables

$$\chi_r = \sum_{i=1}^{d} \frac{x_j^r \chi^\varphi_i x_j}{\varphi(x_j, y_j) \frac{\partial F}{\partial y}(x_j, y_j)}, \quad r = 0, \ldots, d - 1, \quad (21)$$

have linear dynamics in time and lead to the explicit integration of the vector fields $X_{H_i}$ along the real manifolds $A_{F,d}$.

**Amplification 5**

In the one-dimensional case ($d = 1$) the invariant manifolds are punctured Riemann surfaces and have a unique compactification. If the genus of such a Riemann surface exceeds one, then it supports no holomorphic vector fields, so for $d = 1$ good compactifications of $A_{F,d}$ rarely exist.
4. The hyperelliptic case

If \( F(x, y) \) is of the form \( F(x, y) = y^2 + f(x) \), for some polynomial \( f(x) \), then some simplifications occur; we call this the hyperelliptic case, because \( \Gamma_F \) is then a hyperelliptic curve. We derive Lax equations for this case and discuss a hierarchy of potentials on the plane, the so-called Hénon-Heiles hierarchy, which is intimately related to the \( d = 2 \) hyperelliptic case.

4.1. Lax equations

If \( F(x, y) = y^2 + f(x) \), then
\[
\frac{\partial F}{\partial y}(\lambda, v(\lambda)) = 2v(\lambda),
\]
and equations (12) can be written as Lax equations, i.e., they can be written as a commutator in some Lie algebra (see e.g. [Gr]), as given by the following theorem.

**Theorem 10** The differential equations describing the vector fields for the hyperelliptic case are written in the Lax form (with spectral parameter \( \lambda \))
\[
X^\varphi_{H_i} A(\lambda) = [A(\lambda), [B_i(\lambda)]_+],
\]
where
\[
A(\lambda) = \begin{pmatrix} v(\lambda) & u(\lambda) \\ w(\lambda) & -v(\lambda) \end{pmatrix}, B_i(\lambda) = \varphi(\lambda, v(\lambda)) \frac{u(\lambda)}{\lambda^{d-i+1}} + A(\lambda) \text{ and } w(\lambda) = -\left[ \frac{F(\lambda, v(\lambda))}{u(\lambda)} \right]_+.
\]
The spectral curve \( \text{det}(A(\lambda) - \mu \text{Id}) = 0 \), preserved by the flow of the vector fields \( X^\varphi_{H_i} \), is given by \( \mu^2 + f(\lambda) = H_{F,d}(u(\lambda), v(\lambda)) \).

**Proof**

If we define the polynomial \( w(\lambda) \) as stated above, then equations (12) are easy rewritten as
\[
X^\varphi_{H_i} u(\lambda) = 2\varphi(\lambda, v(\lambda)) v(\lambda) \left[ \frac{u(\lambda)}{\lambda^{d-i+1}} \right]_+ - 2u(\lambda) \left[ \varphi(\lambda, v(\lambda)) \frac{v(\lambda)}{u(\lambda)} \left[ \frac{u(\lambda)}{\lambda^{d-i+1}} \right]_+ \right]_+,
\]
\[
X^\varphi_{H_i} v(\lambda) = -\varphi(\lambda, v(\lambda)) w(\lambda) \left[ \frac{u(\lambda)}{\lambda^{d-i+1}} \right]_+ + u(\lambda) \left[ \varphi(\lambda, v(\lambda)) \frac{w(\lambda)}{u(\lambda)} \left[ \frac{u(\lambda)}{\lambda^{d-i+1}} \right]_+ \right]_+.
\]
upon using (22). From (23) let us also compute \( X^\varphi_{H_i} w(\lambda) \) and remark that the explicit dependence on \( F \) disappears completely!

\[
X^\varphi_{H_i} w(\lambda) = -X^\varphi_{H_i} \left[ \frac{F(\lambda, v(\lambda))}{u(\lambda)} \right]_+,
\]
\[
= -2 \left[ \frac{v(\lambda)}{u(\lambda)} X^\varphi_{H_i} v(\lambda) \right]_+ + \left[ \frac{F(\lambda, v(\lambda))}{u(\lambda)} X^\varphi_{H_i} u(\lambda) \right]_+, \]
\[
= 2 \left[ v(\lambda) \left[ \frac{w(\lambda) \varphi(\lambda, v(\lambda))}{u(\lambda)} \left[ \frac{u(\lambda)}{\lambda^{d-i+1}} \right]_+ \right]_+ - w(\lambda) \left[ X^\varphi_{H_i} u(\lambda) \right]_+ \right], \]
\[
= -2v(\lambda) \left[ \varphi(\lambda, v(\lambda)) \left[ \frac{u(\lambda)}{\lambda^{d-i+1}} \right]_+ \right]_+ + 2w(\lambda) \left[ \varphi(\lambda, v(\lambda)) \frac{v(\lambda)}{u(\lambda)} \left[ \frac{u(\lambda)}{\lambda^{d-i+1}} \right]_+ \right]_+.
\]
This leads at once to the above Lax equations. The associated spectral curve is computed as follows:

\[
\det(A(\lambda) - \mu \text{Id}) = \mu^2 - v^2(\lambda) + u(\lambda) \left[ \frac{f(\lambda) + v^2(\lambda)}{u(\lambda)} \right]_+, \\
= \mu^2 + f(\lambda) - u(\lambda) \left[ \frac{f(\lambda) + v^2(\lambda)}{u(\lambda)} \right]_-, \\
= \mu^2 + f(\lambda) - H_{F,d}(u(\lambda), v(\lambda)).
\]

For example, if we restrict ourselves to \(d = 1\) (i.e., one degree of freedom), then \(u(\lambda) = \lambda + u_1, v(\lambda) = v_1\) and

\[
H_{F,1}(u_1, v_1) = (v_1^2 + f(\lambda)) \mod u(\lambda), \\
= (v_1^2 + f(\lambda)) \mod (\lambda + u_1), \\
= v_1^2 + f(-u_1),
\]

and \(\{\cdot, \cdot\}_1\) is the standard bracket on \(\mathbb{R}^2\), so we find that for \(\varphi = 1\) the hyperelliptic case in one degree of freedom corresponds exactly to the case of polynomial potentials on the line.

**Amplification 6**

In the special case that \(\varphi = 1\) and \(d = \text{genus}(\Gamma_F)\) one obtains the so-called odd or even master systems, according to whether the degree of \(f(x)\) is odd or even. The odd master system was introduced by Mumford in [Mu] and his construction was adapted by us to obtain the even master system (see [V]). Both systems are known to be algebraic completely integrable. Indeed, the Abel map (recalled in the next paragraph) linearises the vector fields on the Jacobian of the spectral curve \(\det(A(\lambda) - \mu \text{Id}) = 0\), as follows from (21). In this special case we may rewrite the matrix \([B_i(\lambda)]_+\) as

\[
[B_i(\lambda)]_+ = \left[ \frac{A(\lambda)}{\lambda^{d-i+1}} \right]_+ + \begin{pmatrix} 0 & 0 \\ b_i & 0 \end{pmatrix},
\]

where \(b_i = -u_i\) or \(b_i = -u_i\lambda + 2u_1u_i - u_{i+1}\), according to whether the degree of \(f(x)\) is odd or even, showing that these systems coincide indeed with the master systems in [V].

#### 4.2. \(A^\Gamma_{F,d}\) as strata of hyperelliptic Jacobians

Recall from [GH] that a complex (algebraic) torus is associated to any (complex, smooth, complete) algebraic curve \(\Gamma\) as follows: choose a base \(\{\omega_1, \ldots, \omega_d\}\) for the holomorphic differentials and a symplectic base \(\{A_1, \ldots, A_d, B_1, \ldots, B_d\}\) for \(H^1(\Gamma, \mathbb{Z})\), symplectic meaning here that \(A_iA_j = B_iB_j = 0\) and \(A_iB_j = \delta_{ij}\). If we denote \(\tilde{\omega} = (\omega_1, \ldots, \omega_d)\) then the lattice

\[
\Lambda_\Gamma = \text{span} \left\{ \int_{A_i} \tilde{\omega}, \int_{B_i} \tilde{\omega} \mid 1 \leq i \leq d \right\}
\]

is of rank \(2d\) and a complex torus (which can be shown to be independent of the choices made) is defined by \(\text{Jac}(\Gamma) = \mathbb{C}^g/\Lambda_\Gamma\), the so-called *Jacobians* or \(\Gamma\). Fixing any base point \(P_0 \in \Gamma\) there is for each \(d \in \mathbb{N}\) a well-defined holomorphic map \(A_d: \text{Sym}^d\tilde{\Gamma} \to \text{Jac}(\Gamma)\) defined by

\[
A_d(P_1, \ldots, P_d) = \frac{1}{d!} \sum_{\sigma \in S_d} \int_{P_{\sigma(1)}}^{P_{\sigma(d)}} \tilde{\omega} \mod \Lambda_\Gamma,
\]

\footnote{We killed in the latter case the coefficient of \(x^{g+1}\) in \(f(x)\), precisely as we did in [V].}
classically known as *Abel's map*; Abel's Theorem says that $A_d((P_1, \ldots, P_d)) = A_d((Q_1, \ldots, Q_d))$ if and only if there exists a meromorphic function on $\Gamma$ with zeros at the points $P_1, \ldots, P_d$ and poles at $Q_1, \ldots, Q_d$.

The complex invariant manifolds $A_{\mathbb{C},d}$ behave well with respect to this map in the hyperelliptic case, as is shown in the following proposition. For a smooth curve $\Gamma \subset \mathbb{C}^2$ we will denote its completion (i.e., smooth compactification) by $\bar{\Gamma}$. In the case of hyperelliptic curves $y^2 + f(x) = 0$ this completion $\bar{\Gamma}$ is obtained by adding to $\Gamma$ one or two points, depending on whether the degree of $f(x)$ is odd or even; these points will be denoted by $\infty$, resp. $\infty_1$ and $\infty_2$. Remark that if $F(x, y) = y^2 + f(x)$ then $\Gamma_F$ is smooth (or, equivalently, $A_{\mathbb{C},d}$ is smooth) if and only if $f(x)$ has no multiple roots.

**Proposition 11** In the hyperelliptic case $F(x, y) = y^2 + f(x)$, the complex invariant manifold $A_{\mathbb{C},d}$ is for $d \leq g$ biholomorphic to a (smooth) affine part of a distinguished $d$-dimensional subvariety $W_d$ of $\text{Jac}(\Gamma_F)$, namely

$$A_{\mathbb{C},d} \cong W_d \setminus W_{d-1} \quad \text{deg } f(x) \text{ odd},$$

$$A_{\mathbb{C},d} \cong W_d \setminus (W_{d-1} \cup (\bar{e} + W_{d-1})) \quad \text{deg } f(x) \text{ even},$$

where $\bar{e} \in \text{Jac}(\bar{\Gamma}_F)$ is given by $\bar{e} = A_1(\infty_1) - A_1(\infty_2) = \int_{\infty_2}^{\infty_1} \bar{\omega} \mod \Lambda_{\bar{\Gamma}_F}$. Also

$$W_g = \text{Jac}(\bar{\Gamma}_F),$$

$$W_{g-1} = \text{theta divisor } \Theta \subset \text{Jac}(\bar{\Gamma}_F),$$

$$\vdots$$

$$W_1 = \text{curve } \bar{\Gamma}_F \text{ embedded in } \text{Jac}(\bar{\Gamma}_F),$$

$$W_0 = \text{origin of } \text{Jac}(\bar{\Gamma}_F).$$

**Proof**

We prove the proposition only for the case in which $\text{deg } f(x)$ is odd. We choose $\infty$ as the base point for the Abel map and define $W_k$ for $k = 1, \ldots, g$ as $W_k = A_k(\text{Sym}^k \bar{\Gamma}_F)$. By a theorem due to Jacobi $W_g = \text{Jac}(\bar{\Gamma}_F)$ and by Riemann’s Theorem, $W_{g-1}$ is (a translate of) the Riemann theta divisor (see [GH]). Clearly for each $k \leq g$, $W_{k-1}$ is a divisor in $W_k$ and, by another theorem of Riemann, $W_k \setminus W_{k-1}$ is smooth. We claim that

$$A_d(\text{Sym}^d \bar{\Gamma}_F \setminus D_{F,d}) = W_d \setminus W_{d-1},$$

more precisely $A_d$ realises a holomorphic bijection between these smooth varieties. Namely,

$$\langle P_1, \ldots, P_d \rangle \in \text{Sym}^d \bar{\Gamma}_F \setminus D_{F,d}$$

iff $\forall i \ P_i \neq \infty$ and $\exists i \neq j : x(P_i) = x(P_j) \Rightarrow \left( P_i = P_j \text{ and } P_i \text{ is not a ramification point of } x \right) \left( P_i \neq P_j \text{ and } P_i \text{ is not a ramification point of } x \right)$

iff $A_d(P_1, \ldots, P_d) \notin W_d \setminus W_{d-1}$,

where we used Abel’s Theorem in the last step. It follows that $\text{Sym}^d \bar{\Gamma}_F \setminus D_{F,d}$ and $W_d \setminus W_{d-1}$ are biholomorphic, hence by Proposition 7, $A_d \circ \phi_{F,d}$ is a biholomorphism and the manifolds $A_{\mathbb{C},d}$ and $W_d \setminus W_{d-1}$ are biholomorphic. \hfill \blacksquare
Using the results (and notation) of Sect. 3.4 we can determine very precisely how the vector fields $X_{H_i}$ behave on $\text{Sym}^d \Gamma_F$. If $\deg f(x)$ is odd, then $\nu(\infty) = -2$ and $\mu(\infty) = -2g - 1$, so that $\rho(\infty) = 2(d - g)$. In the even case, we have that for $i = 1, 2$, $\nu(\infty_i) = -1$, $\mu(\infty) = -g - 1$ and $\rho(\infty) = d - g$. Recall that these vector fields have in both cases a simple pole along $\bar{\mathcal{D}}_{F,d}$. If $d = g$, Abel’s Theorem implies that the Abel map contracts $\bar{\mathcal{D}}_{F,d}$ into something lower dimensional; this fact, combined with the preceding computation and Proposition 9, yields a holomorphic vector field on the complex torus $A_g(\text{Sym}^d \Gamma_F)$ (it can have no poles on $A_g(\bar{\mathcal{D}}_{F,g})$ since this is of codimension two). This explains why the master systems are algebraic completely integrable (see Amplification 6). For $d > g$ we identify two $d$-tuples in $\text{Sym}^d \Gamma_F$ when both contain a pair of points of the form $(x_1, y_1), (x_1, -y_1)$ and have their other $d - 2$ points equal. When a smooth manifold is obtained, the vector fields $X_{H_i}$ are again holomorphic on them and they are integrated in terms of degenerations of theta functions.

### 4.3. The Hénon-Heiles hierarchy

It was found by Ramani (see [DGR]) that the integrable Hénon-Heiles potential $V_3 = 8q_2^3 + 4q_1^2 q_2$ is part of a hierarchy of integrable potentials

$$V_n = \sum_{k=0}^{[n/2]} 2^{n-2k} \binom{n-k}{k} q_1^{2k} q_2^{n-2k}.$$ 

Namely, the energy $E_n = (p_1^2 + p_2^2)/2 + V_n$ has an extra invariant given by

$$G_n = -q_2 p_1^2 + q_1 p_1 p_2 + q_2^2 V_n - 1,$$

as is checked immediately by direct computation. These potentials have moreover the special property that they can be superimposed freely in the sense that any linear combination of them gives an integrable potential. The case $n = 3$ was studied in [AvM4] and the case $n = 4$ in [V] (it was called the quartic potential there). In fact, in [V] we constructed a map which relates this quartic potential to the two-dimensional even master system. This map will prove useful to understand the geometry of the whole Hénon-Heiles hierarchy. Namely define a map $T: \mathbb{C}^4 \to \mathbb{C}^4$ by

$$T(q_1, q_2, p_1, p_2) = (\lambda^2 - 2q_2 \lambda - q_1^2, -2p_2 \lambda - 2q_1 p_1),$$

which is invariant for the action of $\mathbb{Z}_2$ on each complex invariant manifold

$$\mathcal{A}_{eg,n} = \{ P \in \mathbb{C}^4 \mid E_n(P) = e, G_n(P) = g \},$$

the action being given by $(q_1, q_2, p_1, p_2) \mapsto (-q_1, q_2, -p_1, p_2)$. It is fixed point free on $\mathcal{A}_{eg,n}$ if $g \neq 0$.

**Proposition 12** The map $T: \mathbb{C}^4 \to \mathbb{C}^4$ given by

$$T(q_1, q_2, p_1, p_2) = (\lambda^2 - 2q_2 \lambda - q_1^2, -2p_2 \lambda - 2q_1 p_1),$$

restricts to an unramified 2:1 covering map on each invariant manifold $\mathcal{A}_{eg,n}$ (with $g \neq 0$) and this restriction is onto $\mathcal{A}_{F,2}$, where $F$ is given by

$$F(x, y) = y^2 + 8(x^{n+2} - ex^2 - gx),$$

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where we used in the last step the recursion formula for the potentials which is the derivative of (25) with respect to \( q \) by using the relation

\[ \tilde{\Gamma} = \text{the complement of one (resp. two) curves, isomorphic to} \quad \Gamma \quad \text{and} \quad \bar{\Gamma} \]

Clearly, from the definition of \( \Gamma \) and \( \bar{\Gamma} \), the restriction \( \tilde{T} \) of \( T \) to \( \mathcal{A}_{c,g,n}^{\bar{\Gamma}} \) maps also the vector fields \( X_{E_2} \) and \( X_{G_{2g,n}} \) to (a multiple of) \( X_{H_1} \) and \( X_{H_2} \), and leads to the Lax equations

\[ X_{E_n} A(\lambda) = \frac{1}{2} [A(\lambda), B_n(\lambda)], \]

for the Hénon-Heiles hierarchy, where

\[ A(\lambda) = \begin{pmatrix} -2p_2 \lambda - 2q_1 p_1 & \lambda^2 - 2q_2 \lambda - q_1^2 \\ 4p_1^2 - \sum_{i=0}^n V_i \lambda^{n-i} & 2p_2 \lambda + 2q_1 p_1 \end{pmatrix}, \quad B(\lambda) = \begin{pmatrix} -8 \sum_{i=1}^{n-1} V_i^* \lambda^{n-i-1} & 1 \\ 0 & 0 \end{pmatrix} \]

and \( V_j^* = \frac{\partial V_j}{\partial q_2} (q_1, q_2) \).

**Proof**

Let us fix values \( e, g \) and denote by \( \tilde{T} \) the restriction of \( T \) to \( \mathcal{A}_{c,g,n}^{\bar{\Gamma}} \). We show that \( \tilde{T} \) maps \( \mathcal{A}_{c,g,n} \) in \( \mathcal{A}_{c,g,n}^{\bar{\Gamma}} \), when \( F(x, y) \) is defined as \( F(x, y) = y^2 + 8(x^{n+2} - e_n x^2 - g_n x) \). To show this, let \( (q_1, q_2, p_1, p_2) \in \mathcal{A}_{c,g,n} \) and let \( u(\lambda) = \lambda^2 - 2q_2 \lambda - q_1^2 \) and \( v(\lambda) = -2p_2 \lambda - 2q_1 p_1 \). Then the equality

\[ \frac{F(\lambda, v(\lambda))}{u(\lambda)} = 8 \sum_{i=0}^{n-1} V_i \lambda^{n-i} - 4p_1^2, \]

follows immediately from

\[
\left( \sum_{i=0}^{n-1} V_i \lambda^{n-i} \right) (\lambda^2 - 2q_2 \lambda - q_1^2) = \sum_{i=-2}^{n-2} V_{i+2} \lambda^{n-i} - 2q_2 \sum_{i=-1}^{n-2} V_{i+1} \lambda^{n-i} - q_1^2 \sum_{i=-1}^{n-1} V_i \lambda^{n-i},
\]

\[
= \lambda^{n+2} + \sum_{i=-1}^{n-2} (V_{i+2} - 2q_2 V_{i+1} - q_1^2 V_i) \lambda^{n-i} - V_n \lambda^2 - q_1^2 V_{n-1} \lambda,
\]

\[
= \lambda^{n+2} - V_n \lambda^2 - q_1^2 V_{n-1} \lambda,
\]

where we used in the last step the recursion formula

\[ V_{i+2} = 2q_2 V_{i+1} + q_1^2 V_i \quad (24) \]

for the potentials \( V_i \) (valid for \( i \geq -1; \ V_{-1} = 0 \)). It follows that \( \tilde{T} \) maps \( \mathcal{A}_{c,g,n}^{\bar{\Gamma}} \) indeed in \( \mathcal{A}_{c,g,n}^{\bar{\Gamma}} \).

Clearly \( \tilde{T} \) is onto and unramified.

To obtain a Lax pair, let \( \varphi(x, y) = 1 \) and compute the entries in \([B_i(\lambda)]_+\) as given by (22). The only non-trivial element in \( B_i(\lambda) \) is \[ \frac{u(\lambda)}{w(\lambda)} \], where \( w(\lambda) = 4p_1^2 - 8 \sum_{i=0}^{n} V_i \lambda^{n-i} \), as follows from the definition of \( w(\lambda) \) in (22) and (24). As in the preceding calculation we get

\[ (\lambda^2 - 2q_2 \lambda - q_1^2) \sum_{j=1}^{n-1} V_j^* \lambda^{n-j-1} = 2u(\lambda) \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} V_i V_j \lambda^{n-i} \quad (\text{polynomial of degree } \leq 1), \]

by using the relation

\[ V_{i+2} - 2q_2 V_{i+1} - q_1^2 V_i = 2V_{i+1}, \]

which is the derivative of (25) with respect to \( q_2 \). From this representation it is seen at once that \( \tilde{T}_* X_{E_n} = \frac{1}{2} X_{H_1} \). Similarly one shows that \( \tilde{T}_* X_{G_{2g,n}} \) is a multiple of \( X_{H_2} \).
Using the results of Paragraph 3.3, the topology of the real invariant manifolds as well as the bifurcations of the Hénon-Heiles hierarchy can be determined, in analogy with [Ga], where this is done for the case $n = 3$ (the Hénon-Heiles potential).

**Amplification 7**

As we learned from V. Kuznetsov, the Hénon-Heiles hierarchy has a higher dimensional generalisation, which consists of a family of potentials on $\mathbb{R}^d$, defined by a recursion relation which generalises (25), namely let $B$ and $A_1, \ldots, A_{d-1}$ be arbitrary parameters, the $A_i$ being all different. Then the potentials are defined by

$$V^{(d)}_{i+2} = 2(qd - B)V^{(d)}_{i+1} + \sum_{k=1}^{d-1} \sum_{j=0}^i (-1)^j q_k^2 V^{(d)}_{i-j} A_k^j;$$

the Hénon-Heiles hierarchy discussed above corresponds then to the case $d = 2$, $A_1 = B = 0$. Using the results obtained in [EEKL], it is easy to construct the generalisation of our map $T$ and to generalise Proposition 12, i.e., to prove that for the $n$-th member $V^n_d$ of the hierarchy ($n \geq 3$), the complex invariant manifolds are $2^{d-1} : 1$ unramified covers of (an affine part of) the $W_d$ stratum of the hyperelliptic Jacobian $\text{Jac}(\Gamma_F)$, where

$$F(x, y) = y^2 + \pi(x) \left( 16x^{n-2}(x + B)^2 + 8e_n + \sum_{i=1}^{d-1} f_i \frac{\pi(x)}{x + A_i} \right), \quad \pi(x) = \prod_{i=1}^{d-1} (x + A_i),$$

which defines a hyperelliptic curve of genus $\left[ \frac{n-3}{2} \right] + d$. It leads also in a natural way to Lax equations for this hierarchy.

**Acknowledgments.** I wish to thank P. Bueken, L. Haine, V. Kutznetsov and P. van Moerbeke for valuable and stimulating discussions which contributed to this paper. The support of the Max-Planck-Institut für Mathematik is also greatly acknowledged.
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