The compactification of initial data on constant mean curvature time slices in spherically symmetric spacetimes

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Conformal mappings of surfaces of constant mean curvature onto compact bounded background spaces are constructed for Minkowski space-time and for Schwarzschild black hole spacetimes. In the black hole example, it is found that initial data on these CMC surfaces can be regular on the compact background space only when a certain condition is satisfied. That condition implies that the shift vector points inward from all parts of the boundary of the compact background. It also implies that the second fundamental form of these surfaces can never be isotropic when black holes are present.

I. INTRODUCTION

With the commissioning of several gravitational wave observatories, the calculation of the gravitational wave ‘signatures’ of astrophysical events has become a matter of urgency. Here, I focus on one aspect of this calculation: The meshing of the near-field, strong curvature, part of the calculation with the far-field region where the gravitational radiation is actually detected.

A relatively old idea for solving this problem has recently been revived: Use constant mean curvature (CMC) time slices, which are bounded by future null infinity where outgoing radiation can be identified unambiguously. The resulting constant time surfaces are spacelike everywhere so that the standard Cauchy 3+1 formulation of the initial value data can be used. One reason that this promising idea was abandoned for so long is the suspicion that generic radiating spacetimes do not admit regular CMC surfaces. For example, Eardley noticed that these surfaces do not seem to be compatible with the expansions in powers of $1/r$ of fields near future null infinity. Goddard’s conjecture appears to have been answered recently by Andersson and Iriondo who show that the surfaces exist under quite general assumptions. I interpret Eardley’s observation as a warning against expecting CMC time-slicing to match smoothly to the Characteristic formulation at future null infinity. Instead, the boundary conditions that are appropriate in the CMC description of a space-time need to be examined directly on their own, as has been done, for example, by Friedrich. The CMC description, and its generalization, the hyperboloidal initial value problem, have begun to be incorporated into numerical methods for solving Einstein’s equations.

Part of the difficulty with using either the CMC description or the hyperboloidal generalization is a lack of familiarity with this way of presenting space-time. This paper provides a small step toward solving that difficulty by analyzing two specific examples, the CMC slicings of Minkowski space and the Schwarzschild black-hole space-time. As a preceding paper pointed out, one of the advantages of a CMC time-slicing is that it promises to express all outgoing wave solutions in terms of smooth functions on a finite conformal background. This paper finds that both the CMC time-slicing of flat Minkowski space-time and some of the recently found CMC time-slicings of a Schwarzschild black hole space-time admit conformal compactifications that are simple and regular. The necessary and sufficient condition for a CMC time-slicing of a Schwarzschild space-time to admit a compactification in which waves are regular functions is found to be exactly the same as the “horizon hugging” condition noted by A.P. Gentle et al. The condition has two distinct geometrical consequences: (1) The shift vector field on the compact conformal background space is required to point inward from all parts of the boundary, including those places where black hole event horizons have been excised. (2) The second fundamental form is not isotropic.

Section II of this paper reviews the hyperbolic slicing of Minkowski space-time by surfaces of constant mean curvature and obtains the wave phase coordinates and the compact conformal coordinates on these surfaces. Section III uses the recently discovered CMC foliation of a Schwarzschild black hole to obtain the wave phase coordinates and a conformal compactification for this more complex and physically interesting case and discusses the condition for the phase coordinate functions to be bounded so that the compactification will be useful. Section IV discusses some of the ways in which these specific examples may provide insights into the general case.
II. MINKOWSKI SPACE

A. Hyperbolic Slicing and the Wave Phase Coordinates

Consider Minkowski space-time with $c = 1$ units with $T$ for the standard Minkowski time coordinate and $r$ for the spatial radius coordinate. The asymptotically retarded time surfaces are taken to be the future-facing spacelike hyperbolas of radius $A$ defined by the time function

$$ t = T - \left[ r^2 + A^2 \right]^{1/2}. $$

Thus, space-time is foliated by a set of time-translated spacelike hyperbolas, all with the same negative mean curvature, $K = -3/A$.

The previous paper [16] began with a particular mapping of these hyperbolas onto the three-sphere and obtained a compactification of the wave equation. That particular mapping was not the best choice and it is better to start with the final result and work backwards to the mapping that best realizes it. The essential result of the previous paper was to introduce a dimensionless space coordinate $\sigma$ such that Minkowski retarded time takes the form

$$ T - r = t - (3/K) \sigma. $$

This form guarantees that outgoing waves have constant phase velocity in terms of the new variables $t, \sigma$. I will refer to $\sigma$ as the outgoing wave phase coordinate and denote it by $\sigma_+$. Notice that outgoing waves propagate in the negative $\sigma$ direction and $\sigma = 0$ corresponds to infinite $R$. For the variable $\sigma$ to have a finite range, it is necessary for the Minkowski advanced time to take a singular form near $\sigma = 0$. The form implicit in the previous paper was

$$ T + r = t - (3/K) \sigma^{-1} $$

from which the relation between $r$ and $\sigma$ is

$$ r = \frac{3}{2K} \left( \sigma - \sigma^{-1} \right). $$

I will refer to $\sigma_- = \sigma^{-1}$ as the ingoing phase coordinate.

Suppose that a high frequency oscillating source of scalar waves is located on a spherical shell at $r = r_0$. The wave fronts moving outward from the shell are evenly spaced in the outgoing phase coordinate $\sigma_+$ while the waves moving inward from the shell are evenly spaced in the ingoing phase coordinate $\sigma_-$. If $\sigma_+$ takes on a finite range of values as $r$ increases from $r_0$ to infinity and $\sigma_-$ takes on a finite range of values as $r$ decreases from $r_0$, then there will, on any given constant-$t$ surface, be a finite number of wavefronts. If a computer program is used to construct a numerical simulation of these waves, a finite number of mesh points will take the waves all the way to infinite $r$. A conformal mapping of each constant-$t$ surface onto a compact, bounded space is one efficient way to implement such a finite mesh. Adaptive mesh refinement is another. In any event, the key requirement for a finite mesh to work is that the phase coordinates be bounded in their respective directions. In the hyperbolically sliced Minkowski example, $\sigma_+$ ranges from a maximum value of 1 at $r = 0$ to a minimum value of 0 at infinite $r$ while $\sigma_-$ decreases from a value larger than 1 at $r = r_0$ to a value of 1 at $r = 0$. Thus, the phase coordinates are indeed bounded in this simple example.

Notice that interchanging the outgoing and ingoing phase coordinates $\sigma$ and $\sigma^{-1}$ corresponds to reversing the sign of $r$ as one would expect. The solution of Eq. (4) for $\sigma$ will be needed later. Of the two solutions for $\sigma$, the one that yields positive values is

$$ \sigma_+ = \sigma = \frac{1}{3}rK + \frac{1}{3}\sqrt{r^2K^2 + 9}. $$

Similarly, the outgoing phase coordinate is

$$ \sigma_- = \sigma^{-1} = -\frac{1}{3}rK + \frac{1}{3}\sqrt{r^2K^2 + 9}. $$

It is helpful here, and later in this paper to define the scaled inverse radius coordinate

$$ s = \frac{3}{|K|} \frac{1}{r}. $$

The relation between the wave phase coordinates $\sigma_{\pm}$ and the scaled inverse radius coordinate can then be put into the form

$$ \sigma_{\pm} = \frac{s}{\pm 1 + \sqrt{1 + s^2}}. $$
B. Conformal Compactification

Because waves propagating on these time-slices have just a finite number of wavefronts, it is possible to map each slice into a finite background space and represent the waves by regular functions on that space. One of several reasons for insisting on a conformal mapping is that the resulting representation is easily combined with the York conformal decomposition of the full gravitational initial value problem. Since all of the space geometries considered here are spherically symmetric, it is a straightforward matter to construct conformal mappings between them.

The space-time metric is

\[ ds^2 = -dT^2 + dr^2 + r^2d\Omega^2 \]

where \( d\Omega^2 \) is the two-sphere metric. Use Eq. (1) to express the metric in terms of the hyperbolic time parameter \( t \)

\[ ds^2 = -dt^2 - \frac{2kr}{\sqrt{(r^2k^2 + 9)}}dtdr + \frac{9}{r^2k^2 + 9}dr^2 + r^2d\Omega^2 \tag{9} \]

so that the intrinsic metric or first fundamental form \( ^3ds^2 \) of each constant-\( t \) surface is given by

\[ ^3ds^2/A^2 = \frac{9}{r^2k^2 + 9}dr^2 + r^2d\Omega^2. \tag{10} \]

Consider conformal mappings of each constant-\( t \) surface into the unit ball \( B^3 \) in Euclidean three-space. In order to focus attention on the boundary at infinity, use an inward directed radial coordinate \( \lambda \) that is zero on the boundary and takes the value \( \lambda = 1 \) at the center. The metric on \( B^3 \) then takes the form

\[ d\ell^2 = d\lambda^2 + (1 - \lambda)^2d\Omega^2 \]

For the mapping defined by \( r = f(\lambda) \) to be conformal, these metrics must be related by

\[ ^3ds^2 = \Phi^4d\ell^2. \]

This condition reduces to a first order ordinary differential equation for \( f \), which is readily integrated. The solutions that satisfy \( f(1) = 0, f(0) = \infty \) are most easily expressed in terms of the inverse radius coordinate \( s \) defined by Eq. (7). The relationship between \( s \) and the conformal coordinate \( \lambda \) is

\[ s = \frac{\lambda(2 - \lambda)}{2(1 - \lambda)} \tag{11} \]

and the conformal factor becomes

\[ \Phi^2 = -\left(\frac{6}{K}\right)[\lambda(2 - \lambda)]^{-1} \tag{12} \]

The resulting form of the space-time metric can be obtained by expressing the space-time metric (See Eq. (9).) in terms of \( \lambda = x^1 \) with the result

\[ ds^2 = -dt^2 - \frac{24}{K} \frac{1 - \lambda}{\lambda^2(2 - \lambda)^2}dtd\lambda + \frac{36}{K^2} \frac{1}{\lambda^2(2 - \lambda)^2} \left( d\lambda^2 + (1 - \lambda)^2d\Omega^2 \right) \tag{13} \]

The main reason for suspecting that the conformal map would be useful is the bounded nature of the wave phase coordinates. Thus it is not too surprising that these are simply related to the conformal coordinate. From equations (8) and (11) the relation is

\[ \sigma_+ = \frac{\lambda}{2 - \lambda}, \quad \sigma_- = \frac{2 - \lambda}{\lambda} \]
C. 3+1 Initial Data for Flat Space-time

The usual procedure for generating a curved space-time begins with Cauchy initial data on a spacelike surface. Thus, it is useful to summarize the 3+1 initial data that corresponds to this hyperbolic slicing time-slicing of flat space-time. The Arnowitt, Deser, and Misner 3+1 notation will be used. This data consists of the second fundamental form or extrinsic curvature $K_{ij}$ split into its trace $K$, and its trace-free part $A_{ij} = K_{ij} - (1/3) Kg_{ij}$, the three dimensional metric tensor $g_{ij}$, the lapse function $N$ and the shift vector $\vec{N}$. Solutions to the initial value constraints are usually found by using the York conformal rescaling procedure which expresses the actual space metric in terms of a conformal background metric $\tilde{g}_{ij}$ and solves the Hamiltonian constraint for the conformal factor $\Phi^4$.

The second fundamental form or extrinsic curvature of a hyperbolic time slice is readily found to be

$$K_{ij} = -A^{-1} g_{ij} = \frac{K}{3} g_{ij}$$

where $g_{ij}$ is the three-metric on the slice. The trace-free part $A_{ij}$ of $K_{ij}$ is zero in this case. The background space metric $\tilde{g}_{ij}$ is simply the metric on Euclidean space.

The lapse and shift functions, $N$ and $\tilde{N}$, can be obtained from Eq. (13), which yields the radial shift form component

$$N_1 = -\frac{24}{K} \frac{1 - \lambda}{\lambda^2 (2 - \lambda)^2}$$

and radial shift vector component

$$N^1 = g^{11} N_1 = \Phi^{-4} N_1 = -\frac{2}{3} K (1 - \lambda)$$

The lapse function is then

$$N = \sqrt{1 + N_1 N^1} = \sqrt{1 + 16 (1 - \lambda)^2 \lambda^2 (2 - \lambda)^2}.$$ 

Notice that it is the well-behaved shift vector given by Eq. (15) that actually enters the dynamical equations through the Lie derivative of the background metric $\tilde{g}$ and not the singular shift form $N_i$ described by Eq. (14). The positive value of the radial shift vector component corresponds to a shift vector that points inward as one would expect because the local inertial frames of these hyperbolic surfaces are expanding outward.

The lapse function, as a scalar quantity on the three-surface, diverges at the boundary of the three-ball just as it diverges at infinite $r$. A conformally rescaled lapse function that remains finite everywhere on the three-ball can be defined as follows:

$$\tilde{N} = \Phi^{-2} N = -\frac{K}{6} \sqrt{(2 - \lambda)^2 \lambda^2 + 16 (1 - \lambda)^2}.$$ 

This rescaled lapse corresponds to applying the conformal factor, $\Phi^4$, to the entire space-time metric. This procedure is similar to the Penrose compactification procedure but with one important difference: Here the conformal factor is time independent and the resulting boundary at future null infinity is a symmetric $S^2 \times R$ cylinder.

III. THE SCHWARZSCHILD BLACK HOLE

A. CMC slicings and the naked shift-reversal condition

A remarkably simple static form of the Schwarzschild metric, found by Iriondo, Malec, and O’Murchadha and analyzed recently by Adrian P. Gentle et. al. displays the most general spherically symmetric CMC time-slicing of a black hole space-time. The metric tensor takes the form

$$ds^2 = -(1 - 2m/r) dt^2 + 2v P^{-1/2} dt dr + r^4 P^{-1} dr^2 + r^2 d\Omega^2$$

where $v, P$ are polynomials
\[ P = v^2 + (1 - 2m/r) r^4 \]  
(17)

\[ v = Kr^3/3 - H \]  
(18)

\( K \) is the mean curvature, \( m \) is the mass of the black hole, and \( H \) is an integration constant. The radius coordinate \( r \) that is used here is just the usual two-sphere area radius. A straightforward computation of the second fundamental form of the \( t = \text{constant} \) surfaces yields the components

\[
  K_{11} = g_{11} \left( K/3 + 2H/r^3 \right), \quad K_{22} = g_{22} \left( K/3 - H/r^3 \right), \quad K_{33} = g_{33} \left( K/3 - H/r^3 \right)
\]

where the coordinates are labeled in the usual way with \( r = x^1, \theta = x^2, \varphi = x^3 \). The integration constant \( H \) measures the anisotropic, trace-free curvature. Since these surfaces are conformally flat, the case \( H = 0 \) corresponds to initial data that is completely isotropic everywhere and only this case leads to surfaces that resemble hyperbolas at infinity. The value of \( H \) is determined by boundary conditions at the ‘center’ of the system. For the \( m = 0 \) case, regularity at \( r = 0 \) requires that \( H \) vanish. For \( m > 0 \), the value of \( H \) must be adjusted to produce the best foliation of the black hole near its horizon.

The shift vector field, \( \mathbf{N} \) describes the three-velocity of constant-coordinate lines relative to the local inertial frame defined by the time-slice. From the form of the space time metric given by Eq. (16), the radial shift vector component is

\[ N^1 = r^{-4} v \sqrt{P} = r^{-4} \left( Kr^3/3 - H \right) \sqrt{P}. \]  
(19)

So long as \( K \) is negative and \( r \) is sufficiently large, the shift vector field points inward, just as in the case of hyperbolically sliced Minkowski space. An inward shift vector means that the local inertial frames of the surface are expanding outward. However, unlike the Minkowski space case, the local inertial frames that are sufficiently near a black hole can be falling inward instead of expanding outward. The shift vector then reverses. For the reversal to happen, \( H \) and \( K \) must have the same sign. From the above expression, this reversal happens at

\[ r_0 = \left( \frac{3H}{K} \right)^{1/3}. \]

For the reversal to happen outside the event horizon, we need the condition

\[ r_0 > 2m. \]

We will see this same combination of requirements again later, so I will give it a name: the \text{\textit{naked shift reversal condition}}. The condition implies

\[ |H| > \frac{8}{3} m^3 |K|, \]

which for negative \( K \), becomes

\[ H < H_+ \]

where

\[ H_+ = \frac{8}{3} m^3 K \]

and is identical to one used by A.P. Gentle \textit{et.al.} to describe the CMC foliations that they characterize as useful. Their paper points out the presence of the shift reversal outside the event horizon in these CMC foliations.

\[ B. \text{\textit{Bounded wave phase coordinates}} \]

As in the case of hyperbolically sliced Minkowski space, outgoing waves can be represented on CMC time-slices as smooth functions of an outgoing phase coordinate \( \sigma_+(r) \) and an ingoing phase coordinate \( \sigma_- (r) \). If these phase coordinates are \textit{bounded} in their respective directions, then a conformal mapping can represent waves as regular functions on a compact background space. Thus, our first task is to seek the phase coordinates. As in the Minkowski
space example, take $U_\pm = t - \frac{3}{2} \sigma_\pm (r)$ to be null coordinates on space-time so that ingoing and outgoing waves move at constant velocities $d\sigma_\pm / dt$ in their respective directions. The corresponding general requirement is the null condition $dU \cdot dU = 0$ which, for the CMC sliced Schwarzschild metric, yields

$$
\left( 1 - \frac{2m}{r} \right) \frac{3}{K} \frac{d\sigma_\pm}{dr} = v P^{-1/2} + b_\pm
$$

where $b = \pm 1$. The choice of $b$ determines whether $\sigma_\pm$ is the outgoing phase coordinate or the ingoing phase coordinate.

The outgoing phase coordinate $\sigma_+$ must be bounded as $r$ increases which means that $d\sigma_+ / dr$ should go to zero in that limit. The choices that meet this requirement are:

$$
b_+ = \begin{cases} 
-1 & \text{for } K > 0 \\
+1 & \text{for } K < 0
\end{cases}
$$

The opposite choices must then correspond to the ingoing phase coordinate so that

$$
b_- = \begin{cases} 
+1 & \text{for } K > 0 \\
-1 & \text{for } K < 0
\end{cases}
$$

The ingoing phase coordinate $\sigma_-$ and therefore its derivative $d\sigma_- / dr$ needs to be bounded as $r$ decreases past the Schwarzschild limit at $r = 2m$. Write Eq. (20) in the form

$$
(1 - 2m/r) \frac{3}{K} \frac{d\sigma_-}{dr} = P^{-1/2} \left( v + b_- \sqrt{P} \right)
$$

and note that

$$
( -b_- v + \sqrt{P} ) \left( b_- v + \sqrt{P} \right) = P - v^2 = (1 - 2m/r) r^4
$$

so that the expression for the derivative becomes

$$
\frac{d\sigma_-}{dr} = \frac{(K/3) b_-}{\sqrt{P} - b_- v + \sqrt{P}} \frac{r^4}{-b_- v + \sqrt{P}}
$$

Because the polynomial $P$ is positive for all $r \geq 2m$, this expression remains finite at $r = 2m$ if and only if $b_- v < 0$. From Eq. (22), the necessary and sufficient condition for $b_- v < 0$ is that $v$ and $K$ have opposite signs at $r = 2m$. From the above expression, it can be seen that $d\sigma_- / dr$ then has bounded magnitude and is bounded away from zero for any values of $r$ in the range $2m \leq r < R$ where $R$ is any number larger than $2m$. Thus, a bounded ingoing phase coordinate $\sigma_-$ can extend across the event horizon at $r = 2m$ if and only if $v$ and $K$ have opposite signs at $r = 2m$.

Now recall that $v$ is related to the radial shift vector component $N_1$ by Eq. (19) and, from Eq. (18) has the same sign as $K$ for sufficiently large values of $r$. The condition that $v$ and $K$ have opposite signs at $r = 2m$ is then equivalent to requiring a naked shift reversal and the following result has been established:

**Theorem 1** The naked shift reversal condition is necessary and sufficient for the existence of bounded wave phase coordinates on a constant mean curvature slice in the exterior region of a Schwarzschild space-time. The resulting phase coordinates are monotonic functions of the luminosity radius $r$.

As an example of the value of considering a specific example, notice that the naked shift reversal condition rules out the simplifying assumption, $K_{ij} = (1/3) Kg_{ij}$ that underlies many of the results obtained by Friedrich. [8,9] For these black hole space times, that assumption ($H = 0$) leads to constant-time surfaces on which the ingoing phase coordinate is not bounded. Thus the $K_{ij} = (1/3) Kg_{ij}$ assumption contradicts the assumption that these constant-time surfaces can be mapped into a compact manifold where waves may be described by regular functions.

**C. Conformal Compactification**

If the wave phase coordinates exist and behave properly, we expect that it should be possible to map all of the outgoing wave action in one of these CMC sliced spacetimes onto $B^3$. Thus, we seek a function $\lambda (r)$ that corresponds to a conformal map of the CMC spatial metric.
\[ \frac{3}{P} ds^2 = \frac{r^4}{dr^2} + r^2 d\Omega^2 \]

into the metric
\[ \Phi^4 \left( d\lambda^2 + (1 - \lambda)^2 d\Omega^2 \right) \]

The resulting condition on \( \lambda \) is
\[ \frac{d\lambda}{1 - \lambda} = -\frac{rdr}{\sqrt{(H - \frac{1}{3}Kr^3)^2 + (1 - \frac{2m}{r^4})r^4}} \]  
(23)

and can be integrated to
\[ \lambda = 1 - e^{-F(m,H,K,s)} \]  
(24)

where \( s \) is the inverse radius coordinate given by Eq. (7) and \( F \) is the integral
\[ F(m,H,K,s) = \int_{s}^{0} \frac{dx}{\sqrt{(1 + (1/9)HK^2x^3)^2 + x^2(1 + \frac{2mKx}{3})}}. \]  
(25)

The coordinate transformation from the luminosity radius \( r \) to the three-ball coordinate \( \lambda \) is only available in an implicit form, given by equations (24) and (25). However, it is easy to calculate the derivative \( d\lambda/dr \) from this form and then the combination \( P^{-1/2} \frac{dr}{d\lambda} \) that is needed to transform the metric. The result is found to be amazingly simple.

\[ P^{-1/2} \frac{dr}{d\lambda} = -\frac{1}{r} \frac{1}{1 - \lambda} \]

The resulting form of the space-time metric is then
\[ ds^2 = -\left(1 - 2m/r\right) dt^2 - 2\frac{Kr^2/3 - H/r}{1 - \lambda} dt d\lambda + \frac{r^2}{(1 - \lambda)^2} d\lambda^2 + r^2 d\Omega^2 \]  
(26)

and the conformal factor is
\[ \Phi^2 = \frac{r}{1 - \lambda} \]

while the shift vector (for the coordinate \( \lambda \)) is
\[ N^1 = -\left(\frac{K}{3} - \frac{H}{r^3}\right) (1 - \lambda). \]

Just as for the Minkowski space example, we discover a compactified solution that is built entirely from rational polynomials.

**D. Relation between phase coordinates**

In the Minkowski case it was seen that the phase coordinates \( \sigma_{\pm} \) and the coordinate \( \lambda \) on the compact conformal background are simply related. In the Schwarzschild geometry, we do not have the luxury of analytic forms for \( \sigma_{\pm} \) but we can seek relationships directly from the differential equations that they satisfy. From equations (24) and (25) for the case \( K < 0 \),
\[ \left(1 - \frac{2m}{r}\right) \frac{3}{K} \frac{d\sigma_+}{dr} = \frac{v}{P^{-1/2}} + 1 \]  
(27)
\[ \left(1 - \frac{2m}{r}\right) \frac{3}{K} \frac{d\sigma_-}{dr} = \frac{v}{P^{-1/2}} - 1. \]  
(28)

Subtract Eq. (28) from Eq. (27) and obtain a simple relation between the ingoing and outgoing phase coordinates
\[ \frac{3}{K} (\sigma_+ - \sigma_-) = r_0 \ln (r_0 - 2m) - r \ln (r - 2m) \]

where \( r_0 \) can be chosen to have any value greater than \( 2m \).
E. Location of the event horizon in the three-ball

The conformal coordinate $\lambda$ ranges from a value of zero at the outer boundary of the three-ball to a value of $\lambda = 1$ at the center. The event horizon at $r = 2m$ is located at the conformal coordinate value

$$\lambda_h (H, K, m) = 1 - e^{-F(1,H,K,3/(2m|K|))}$$

For the values $H = -1.25$, $K = -0.1$, $m = 1$ that A.P. Gentle et al. [4] use as an illustration of CMC foliation, the horizon is located at

$$\lambda_h (-1.25, -0.1, 1) = 0.98771.$$ 

The coordinate radius $1 - \lambda$ of the horizon surface would then be $0.01229$ and is thus quite small in this picture. A black hole with twice the mass and the same CMC time parameters, would have its event horizon at

$$\lambda_h (-1.25, -0.1, 2) = 0.94878.$$ 

Thus, it would have much more than twice the coordinate size in this description. The horizon size is not very sensitive to the anisotropic curvature parameter $H$. For example, with $m = 2$ the naked shift reversal condition requires that $H$ be less than $-2.133$. Changing the value of $H$ from $-1.25$ to $-3$ results in only a small change in the location of the event horizon.

$$\lambda_h (-3, -0.1, 2) = 0.9494.$$ 

The scale of the representation is set by the mean curvature parameter $K$. However, even with the choice $K = -1$, a unit mass black hole would still be quite close to the origin of the three-ball with

$$\lambda_h (-3, -1, 1) = 0.87113.$$ 

IV. DISCUSSION

Here, the constant mean-curvature initial conditions for Minkowski space and the Schwarzschild black hole spacetime, have been described in terms of regular functions on the unit three-ball. The condition that waves also be described by regular functions of the compact coordinates has been found to restrict these initial conditions in a natural way: If one thinks of the region outside the black hole event horizon as the active region, then the shift vector at the inner and outer boundaries of that region must point into the active region. A somewhat less obvious result is that requiring waves to be described by regular functions of the compact coordinates rules out the assumption that the second fundamental form is isotropic. This simplifying assumption cannot be used to analyze the initial value problem for spacetimes which contain black holes.

Both the Minkowski space and the Schwarzschild black hole metrics display a curious feature when expressed in compact CMC coordinates. In both cases, the metric functions are built from rational polynomials. In the Schwarzschild case, the polynomial fractions involve both the compact radius coordinate $\lambda$ and the area-radius $r(\lambda)$. It is generally understood that the metric functions are not analytic functions of $1/r$ so that expansions in powers of $1/r$ are, at best, only asymptotically convergent. Here, this nonanalyticity is concentrated in the area-radius function $r(\lambda)$. It is a straightforward exercise to use Equations 24 and 25 to expand $1/r$ in powers of $\lambda$. One learns that the series deviates from the Minkowski space solution (See Equations 7 and 11) for $1/r$ only in fourth order and above and is sharply divergent for all $r < 15m$. 

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