(v) the second smallest eigenvalue of $L(A)$, which is called the algebraic connectivity of $G(A)$ and denoted by $\lambda_2(L(A))$, is larger than zero;
(vi) the algebraic connectivity of $G(A)$ is equal to $
abla \nabla \xi^T L(A) \xi \xi^T \xi$, and thus, if $1^T \xi = 0$, then $\xi^T L(A) \xi \geq \lambda_2(L(A)) \xi^T \xi$.

Corollary 1: Suppose $G(A)$ is strongly connected and $\omega$ is positive column vector such that $w^T L(A) = 0$. Then \( \text{diag}(\omega) L(A) + L(A)^T \text{diag}(\omega) \) is the graph Laplacian of the undirected weighted graph $G(\text{diag}(\omega) A + A^T \text{diag}(\omega))$. And it is positive semi-definite, 0 is its algebraically simple eigenvalue and 1 is the associated eigenvector.

Corollary 2: For any nonnegative column vector $b$ with compatible dimensions, if $b \neq 0$ and $G(A)$ is undirected and connected, then $L(A) + \text{diag}(b)$ is positive definite.

Lemma 3 [14], Lemma 1): If $\xi_1, \xi_2, \ldots, \xi_n \geq 0$ and $0 < p \leq 1$, then
$$\sum_{i=1}^{n} \xi_i^p \leq \left( \sum_{i=1}^{n} \xi_i \right)^p.$$  

Lemma 3 (cf. [6], Theorem 1): Suppose that function $V(t) : [0, \infty) \rightarrow [0, \infty)$ is differentiable (the derivative of $V(t)$ at 0 is in fact its right derivative) and
$$\frac{dV(t)}{dt} \leq -K V(t)^\alpha$$  

where $K > 0$ and $0 < \alpha < 1$. Then $V(t)$ will reach zero at finite time $t^* \leq V(0)^{1-\alpha}/(K(1-\alpha))$ and $V(t) = 0$ for all $t \geq t^*$.  

II. BRIEF INTRODUCTION TO MAJORIZATION

Monotone Quantum Dynamics  

Haidong Yuan

Abstract—In this technical note, the author studies the dynamics of open quantum system in Markovian environment. The author gives necessary and sufficient conditions for such dynamics to be majorization monotone, which are those dynamics always mixing the states.

Index Terms—Majorization, open quantum dynamics.

I. INTRODUCTION

In the last two decades, control theory has been applied to an increasingly wide number of problems in physics and chemistry whose dynamics are governed by the time-dependent Schrödinger equation (TDSE), including control of chemical reactions [1]–[8], state-to-state population transfer [9]–[13], shaped wavepackets [14], NMR spin dynamics [15]–[19], Bose-Einstein condensation [20]–[22], quantum computing [23]–[27], oriented rotational wavepackets [28]–[30], etc. More recently, there has been vigorous effort in studying the control of open quantum systems which are governed by Lindblad equations, where the central object is the density matrix, rather than the wave function [31]–[37]. The Lindblad equation is an extension of the TDSE that allows for the inclusion of dissipative processes. In this article, the author will study those dynamics governed by Lindblad equations and give necessary and sufficient conditions for the dynamics to be majorization monotone, which are those dynamics always mixing the states. This study suggests that majorization may serve as time arrow under these dynamics in analog to entropy in second law of thermal dynamics.

The article is organized as follows: Section II gives a brief introduction to majorization; Section III gives the definition of majorization monotone quantum dynamics; and in Section IV, necessary and sufficient conditions for majorization monotone quantum dynamics are given.

REFERENCES

[1] W. Ren, R. W. Beard, and E. M. Atkins, “Information consensus in multivehicle cooperative control,” IEEE Control Syst. Mag., vol. 27, no. 2, pp. 71–82, Apr. 2007.
[2] R. Olfati-Saber and R. M. Murray, “Consensus problems in networks of agents with switching topology and time-delays,” IEEE Trans. Autom. Control, vol. 49, no. 9, pp. 1520–1533, Sep. 2004.
[3] Y. Kim and M. Mesbahi, “On maximizing the second smallest eigenvalue of a state-dependent graph laplacian,” IEEE Trans. Autom. Control, vol. 51, no. 1, pp. 116–120, Jan. 2006.
[4] L. Xiao and S. Boyd, “Fast linear iterations for distributed averaging,” Syst. Control Lett., vol. 53, pp. 65–78, 2006.
[5] R. Olfati-Saber, “Ultrafast consensus in small-world networks,” in Proc. American Control Conf., Jun. 2005, pp. 2371–2378.
[6] S. P. Bhat and D. S. Bernstein, “Finite-time stability of continuous autonomous systems,” SIAM J. Control and Optimiz., vol. 38, no. 3, pp. 751–766, 2000.
[7] S. Sundaram and C. N. Hadjicostis, “Finite-time distributed consensus in graphs with time-invariant topologies,” in Proc. American Control Conf., Jul. 2007, pp. 711–716.
[8] J. Cortés, “Finite-time convergent gradient flows with applications to network consensus,” Automatica, vol. 42, no. 11, pp. 1993–2000, 2006.
[9] V. T. Haimo, “Finite time controllers,” SIAM J. Control and Optimiz., vol. 24, no. 4, pp. 760–770, 1986.
[10] W. Ren and R. W. Beard, “Consensus seeking in multiagent systems under dynamically changing interaction topologies,” IEEE Trans. Autom. Control, vol. 50, no. 5, pp. 655–661, May 2005.
[11] P. Hartman, Ordinary Differential Equations, 2nd ed. Basel, Switzerland: Birkhäuser, 1982.
[12] T. Chu, L. Wang, T. Chen, and S. Mu, “Complex emergent dynamics of anisotropic swarms: Convergence vs oscillation,” Chaos, Solitons & Fractals, vol. 30, no. 4, pp. 875–885, 2006.
[13] F. Xiao, “Consensus Problems in Networks of Multiple Autonomous Agents,” Ph.D. thesis, Peking Univ., Beijing, China, 2008.
[14] F. Xiao and L. Wang, “Reaching agreement in finite time via continuous local state feedback,” in Proc. 26th Chinese Control Conf., Jul. 2007, pp. 711–715.
Proposition 1: $x \prec y$ iff $x$ lies in the convex hull of all $P_y$, where $P_i$ are permutation matrices.

Proposition 2: $x \prec y$ if and only if $x = D y$ where $D$ is a doubly stochastic matrix.

Remark 1: A doubly stochastic matrix $D$ is a matrix with nonnegative entries and every column and row sum to 1, i.e., $d_{ij} \geq 0, \sum_i d_{ij} = 1, \sum_j d_{ij} = 1$.

Proposition 3: Suppose $f$ is a convex function on $\mathbb{R}$, and $x \prec y$ in $\mathbb{R}^n$, then
\[
\sum_i^n f(x_i) \leq \sum_i^n f(y_i).
\]

Proposition 4: For a vector $\lambda = (\lambda_1, \ldots, \lambda_n)^T$, denote $D_\lambda$, a diagonal matrix with $(\lambda_1, \ldots, \lambda_n)$ as its diagonal entries, let $a = (a_1, \ldots, a_n)^T$ be the diagonal entries of matrix $A = K^T D_\lambda K$, where $K \in SO(n)$. Then $a \prec \lambda$. Conversely for any vector $a \prec \lambda$, there exists a $K \in SO(n)$, such that $(a_1, \ldots, a_n)^T$ are the diagonal entries of $A = K^T D_\lambda K$.

Remark 2: $SO(n)$ is the group of special orthogonal matrices, $K \in SO(n)$ means $K^T K = I$ and $\det(K) = 1$.

III. MAJORIZATION IN OPEN QUANTUM DYNAMICS

The state of an open quantum system of $N$-level can be represented by a $N \times N$ positive semi-definite, trace 1 matrix, called density matrix. Let $\rho$ denote the density matrix of a quantum system, its dynamics in markovian environment is governed by the Lindblad equation, which takes the form
\[
\dot{\rho} = -i[H, \rho] + L(\rho)
\]

where $-i[H, \rho]$ is the unitary evolution of the quantum system and $L(\rho)$ is the dissipative part of the evolution. The term $L(\rho)$ is linear in $\rho$ and is given by the Lindblad form [38], [40]
\[
L(\rho) = \sum_{\alpha, \beta} a_{\alpha \beta} \left( F_{\alpha} \rho F_{\beta}^\dagger - \frac{1}{2} \left( F_{\alpha}^\dagger F_{\beta} \rho + \rho F_{\beta} F_{\alpha}^\dagger \right) \right)
\]
where $F_{\alpha}, F_{\beta}$ are the Lindblad operators, which form a basis of $N \times N$ trace 0 matrices (we have $N^2 - 1$ of them) and $[A, B] = AB + BA$. We use the notation $a_{\alpha \beta}$, it is known as the Gorini, Kossakowski, and Sudarshan (GKS) matrix [39], which needs to be positive semi-definite.

Equation (2) has the following three well-known properties: 1) $Tr(\rho)$ remains unity for all time; 2) $\rho$ remains a Hermitian matrix; and 3) $\rho$ stays positive semi-definite, i.e., $\rho$ never develops negative eigenvalues.

Definition 2: Suppose $\rho_1$ and $\rho_2$ are two states of a quantum system, we say $\rho_1$ is majorized by $\rho_2 (\rho_1 \prec \rho_2)$ if the eigenvalues of $\rho_1$ majorized by the eigenvalues of $\rho_2 (\lambda(\rho_1) \prec \lambda(\rho_2))$.

Basically majorization gives an order of mixed-ness of quantum states, i.e., if $\rho_1 \prec \rho_2$, then $\rho_1$ is more mixed than $\rho_2$, which can be seen from the following propositions.

Definition 3 (Von Neumann Entropy): The Von Neumann entropy of a density matrix is given by
\[
S(\rho) = -Tr[\rho \log \rho]
\]

Proposition 5: If $\rho_1 \prec \rho_2$, then $S(\rho_1) \geq S(\rho_2)$.

Proposition 6: If $\rho_1 \prec \rho_2$, then $Tr(\rho_1^2) \leq Tr(\rho_2^2)$.

Remark 3: The above two propositions can be easily derived from Proposition 3.

The entropy and trace norm are usually used to quantify how mixed quantum states are. But majorization is a more strong condition than these two functions, and in some sense it gives a more proper order of mixed-ness as we can see from the following proposition.

Proposition 7 [44]: $\rho_1 \prec \rho_2$ if and only if $\rho_1(t) = \sum p_i U_i \rho_2 U_i^\dagger$, where $p_i > 0, \sum p_i = 1$ and $U_i$ are unitary operators.

IV. NECESSARY AND SUFFICIENT CONDITION OF MAJORIZATION MONOTONE QUANTUM DYNAMICS

Definition 4 (Majorization Monotone Dynamics): An open quantum dynamics governed by (2) is majorization monotone if and only if $\rho(t_2) \prec \rho(t_1)$ when $t_2 > t_1, \forall t_1, t_2$.

Intuitively majorization monotone dynamics are those kind of dynamics which always mixing the states. As we can see from Proposition 5, these kind of dynamics always increase the entropy of the system. One can immediately see a necessary condition for a dynamics to be majorization monotone: the state $\rho_t = (1/N)I$ has to be a steady state of such dynamics, where $I$ is identity matrix. As $(1/N)I$ is the most mixed state, any state $\rho_t \prec (1/N)I$ would imply $\rho_t = (1/N)I$. The question now is whether this condition is also sufficient.

Let us first look at a simple system: a single spin in a Markovian environment.

A. An Example on Single Spin

Take the general expression of the master equation
\[
\dot{\rho} = -i[H, \rho] + L(\rho)
\]

where
\[
L(\rho) = \sum_{\alpha, \beta} a_{\alpha \beta} \left( F_{\alpha} \rho F_{\beta}^\dagger - \frac{1}{2} \left( F_{\alpha}^\dagger F_{\beta} \rho + \rho F_{\beta} F_{\alpha}^\dagger \right) \right)
\]

For the single spin, we can take the basis $\{ F_{\alpha} \}$ as normalized Pauli spin operators $(1/\sqrt{2})\{ \sigma_x, \sigma_y, \sigma_z \}$, where $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, and $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

The coefficient matrix
\[
A = \begin{pmatrix} a_{xx} & a_{xy} & a_{xz} \\ a_{yx} & a_{yy} & a_{yz} \\ a_{zx} & a_{zy} & a_{zz} \end{pmatrix}
\]

is positive semi-definite.

If identity state is a steady state, the right-hand side of (3) should be 0 when $\rho = (1/N)I$. As $-i[H, (1/N)I] = 0$, so the condition reduces to $L(I) = 0$ (since $L(\rho)$ is a linear map, we can ignore the constant $1/N$). This is equivalent to
\[
\sum_{\alpha, \beta} a_{\alpha \beta} \left( F_{\alpha} \rho F_{\beta}^\dagger \right) = 0.
\]

In the single-spin case, substitute $F$ by Pauli matrices and it is easy to see that the above condition reduces to
\[
a_{\alpha \beta} = a_{\beta \alpha}
\]
i.e., the GKS matrix should be real symmetric, positive semi-definite matrix [41], while the general GKS matrix is Hermitian, positive semi-definite. We want to see whether the dynamics of single spin under this condition is majorization monotone.

As majorization monotone is defined by the eigenvalues of density matrices, we are going to focus on the dynamics of the eigenvalues of density matrix. Let $\Lambda$ be its associated diagonal form of eigenvalues of density matrix $\rho$, i.e., $\Lambda$ is a diagonal matrix with eigenvalues of $\rho$ as its diagonal entries. At each instant of time, we can diagonalize the density matrix $\rho(t) = U(t) \Lambda(t) U(t)^\dagger$ by a unitary matrix $U(t)$. 

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Substitute \( \rho(t) = U(t) \Lambda(t) U^\dagger(t) \) into (2), we get
\[
\dot{\rho}(t) = U(t) \Lambda(t) U^\dagger(t) + U(t) \dot{\Lambda}(t) U^\dagger(t) + U(t) \Lambda(t) U^\dagger(t) \\
= -i H'(t) U(t) \Lambda(t) U^\dagger(t) + U(t) \Lambda(t) U^\dagger(t) \\
+ U(t) \Lambda(t) U^\dagger(t) i H'(t) \\
= -i \left[ H'(t), U(t) \Lambda(t) U^\dagger(t) \right] + U(t) \Lambda(t) U^\dagger(t) \\
= -i \left[ H(t), U(t) \Lambda(t) U^\dagger(t) \right] + L \left[ U(t) \Lambda(t) U^\dagger(t) \right]
\]
(5)

where \( H'(t) \) is defined by \( U(t) = -i H'(t) U(t) \), which is Hermitian. We obtain
\[
\dot{\Lambda}(t) = U(t) \left\{ -i \left[ H(t) - H'(t), U(t) \Lambda(t) U^\dagger(t) \right] \\
+ L \left[ U(t) \Lambda(t) U^\dagger(t) \right] \right\} U(t) \\
= -i \left[ U^\dagger(t) \left( H(t) - H'(t) \right) U(t), \Lambda(t) \right] \\
+ U(t) L \left[ U^\dagger(t) \Lambda(t) U^\dagger(t) \right] U(t).
\]
(6)

Note that the left side of the above equation is a diagonal matrix, so for the right side we only need to keep the diagonal part. It is easy to see that the diagonal part is zero for the first term, thus we get
\[
\dot{\Lambda}(t) = \text{diag} \left( U^\dagger(t) L \left[ U^\dagger(t) \Lambda(t) U^\dagger(t) \right] U(t) \right)
\]
(7)

where we use \( \text{diag}(M) \) to denote a diagonal matrix whose diagonal entries are the same as matrix \( M \)
\[
\dot{\Lambda}(t) = \text{diag} \left( U^\dagger(t) L \left[ U^\dagger(t) \Lambda(t) U^\dagger(t) \right] U(t) \right) \\
= \text{diag} \left( \sum_{\alpha,\beta} a_{\alpha,\beta} \left( U^\dagger F_{\alpha} U \Lambda U^\dagger F_{\beta} U \right. \right.
\left. \left. + \frac{1}{2} \left( U^\dagger F_{\alpha} U \Lambda U^\dagger F_{\beta} \right) \right) \right) \\
= \text{diag} \left( \sum_{\alpha,\beta} a_{\alpha,\beta} \left( U^\dagger F_{\alpha} U \Lambda U^\dagger F_{\beta} U \right. \right.
\left. \left. + \frac{1}{2} \left( U^\dagger F_{\alpha} U \Lambda U^\dagger F_{\beta} \right) \right) \right)
\]
(8)

For the last step, we just used the fact that \( F_{\beta} \) is a Pauli matrix which is Hermitian. Now
\[
U^\dagger F_{\alpha} U = C_{\alpha,\gamma} F_{\gamma}
\]
where \( C = \begin{pmatrix} c_{x,x} & c_{x,y} & c_{x,z} \\ c_{y,x} & c_{y,y} & c_{y,z} \\ c_{z,x} & c_{z,y} & c_{z,z} \end{pmatrix} \in SO(3) \) is the adjoint representation of \( U \). Substituting these expressions into (8), we obtain
\[
\dot{\Lambda}(t) = \text{diag} \left( \sum_{\alpha,\beta} a_{\alpha,\beta} \left( F_{\alpha} F_{\beta} - \frac{1}{2} \left( F_{\beta} F_{\alpha} \right) \right) \right)
\]
(9)

where
\[
a'_{\alpha,\beta} = c_{\alpha} a_{\beta,\rho} c_{\rho,\beta}
\]
are entries of transformed GKS matrix
\[
A' = C^T A C.
\]

We can write \( \Lambda(t) = \begin{pmatrix} (1/2) + \lambda(t) & 0 \\ 0 & (1/2) - \lambda(t) \end{pmatrix} \in [0, 1/2] \) and \( \Lambda(t) \in [0, 1/2] \). We substitute it into (9), we obtain the dynamics for \( \lambda(t) \)
\[
\dot{\lambda}(t) = -\left( a'_{11} + a'_{22} \right) \lambda(t).
\]

From Proposition 4, we know
\[
\mu_3 + \mu_2 \leq a'_{11} + a'_{22} \leq \mu_2 + \mu_1
\]
where \( \mu_1 \geq \mu_2 \geq \mu_3 \) are eigenvalues of the GKS matrix. From this it is easy to see that at time \( T \), the value of \( \lambda(T) \) lies in the following interval:
\[
\left[ e^{-(\nu_1+\nu_2)T} \lambda(0), e^{-(\nu_1+\nu_3)T} \lambda(0) \right],
\]
which is always less or equal to \( \lambda(0) \). And this is sufficient for the dynamics to be majorization monotone in the single-spin case, as \( (1/2) + \lambda(T) \prec (1/2 + \lambda(0)) \)
when \( \lambda(T) \leq \lambda(0) \). So \( L(I) = 0 \) is necessary and sufficient for the dynamics to be majorization monotone in the single-spin case. And this condition holds even when we have coherent control on the spin, as the identity state remains as steady state in the presence of control as we can see from the following controlled dynamics:
\[
\rho = -i \left[ H(t) + \sum_i u_i H_i, \rho \right] + L(\rho)
\]
where \( \sum_i u_i H_i \) are our coherent controls. Suppose the controllers are allowed to generate any unitary operations on the spin fast compare to the dissipative rate, then \( \lambda(T) \) can actually take any value in the interval \( \left[ e^{-(\nu_1+\nu_2)T} \lambda(0), e^{-(\nu_1+\nu_3)T} \lambda(0) \right] \), i.e., the reachable set for the single spin under the controlled Lindblad dynamics is
\[
\rho(T) = \left\{ U \left( \begin{array}{cc} \frac{1}{2} + \lambda(T) & 0 \\ 0 & \frac{1}{2} - \lambda(T) \end{array} \right) U^\dagger | \lambda(T) \in \left[ e^{-(\nu_1+\nu_2)T} \lambda(0), e^{-(\nu_1+\nu_3)T} \lambda(0) \right], U \in SU(2) \right\}.
\]

This is to say that although controls cannot reverse the direction of mixing, it can change the rate within some region.

Remark 4: From the single-spin case, we can see that a dynamical system being majorization monotone does not imply that the states of the system always converge to identity state, as identity being a steady state does not exclude the possible existence of other steady states, for example, the dynamics given by
\[
\dot{\rho} = -i [\sigma_z, \rho] + \gamma [\sigma_z, [\sigma_z, \rho]]
\]
which describes the transverse relaxation mechanism in NMR, satisfies the majorization monotone condition in the single-spin case, and it is easy to see that the state of this system does not necessary converge to identity matrix, in fact it can well be converged to any state of the form \( (1/2) I + \alpha \sigma_z \), where \( \alpha \in [0, 1/2] \).

B. General Case

In this section, we will show that \( L(I) = 0 \) is also sufficient for the dynamics to be majorization monotone in the general case. Suppose we solved the Lindblad equation
\[
\dot{\rho} = -i [H(t), \rho] + L(\rho)
\]
(10)

integrated this equation from \( t_1 \) to \( t_2 \), where \( t_2 > t_1 \), and get a map
\[
\Psi : \rho(t_1) \rightarrow \rho(t_2).
\]
Such a map has a Kraus operator sum representation [40], [43]
is straightforward to show that, by using the two conditions in (14) entries, substitute them into (11), we get

\[
\rho(t_2) = \Psi(\rho(t_1)) = \sum_i K_i \rho(t_1) K_i^\dagger
\]

where in our case \(\{K_i\}\) are \(N \times N\) matrices, which depend on the dynamical (10) and the time difference between \(t_1\) and \(t_2\). Also the Kraus operator sum has to be trace preserving as the trace of density matrix is always 1, which implies that

\[
\sum_i K_i^\dagger K_i = I.
\]

If we have additional condition that identity state is a steady state of this dynamics, which means if \(\rho(t_1) = (1/N)I\) then \(\rho(t_2)\) remains at \((1/N)I\), substitute them into the Kraus operator sum representation, we will get an extra condition

\[
\sum_i K_i K_i^\dagger = I.
\]

We will show these two conditions are enough to ensure the dynamics to be majorization monotone. First, let us diagonalize \(\rho(t_1)\) and \(\rho(t_2)\)

\[
\rho(t_1) = U_1 \Lambda(\rho(t_1)) U_1^\dagger,
\]

\[
\rho(t_2) = U_2 \Lambda(\rho(t_2)) U_2^\dagger
\]

where \(\Lambda(\rho)\) are diagonal matrix with eigenvalues of \(\rho\) as its diagonal entries, substitute them into (11), we get

\[
U_2 \Lambda(\rho(t_2)) U_2^\dagger = \sum_i K_i U_1 \Lambda(\rho(t_1)) U_1^\dagger K_i^\dagger,
\]

\[
\Lambda(\rho(t_2)) = U_2 \Lambda(\rho(t_1)) U_1^\dagger U_2^\dagger.
\]

Let \(V_i = U_2^\dagger K_i U_1\), then

\[
\Lambda(\rho(t_2)) = \sum_i V_i \Lambda(\rho(t_1)) V_i^\dagger
\]

and it is easy to check that

\[
\sum_i V_i V_i^\dagger = U_2 (\sum_i K_i K_i^\dagger) U_2^\dagger = I
\]

\[
\sum_i V_i^\dagger V_i = U_1 (\sum_i K_i K_i^\dagger) U_1^\dagger = I.
\]

It is a linear map from the eigenvalues of \(\rho(t_1)\) to eigenvalues of \(\rho(t_2)\), so we can find a matrix \(D\), such that

\[
\lambda(\rho(t_2)) = D \lambda(\rho(t_1))
\]

where \(\lambda(\rho)\) is a vector in \(\mathbb{R}^N\) with eigenvalues of \(\rho\) as its entries, which is arranged in the same order as the diagonal entries of \(\Lambda(\rho)\). The matrix \(D\) can be computed from (13) as

\[
D_{\alpha\beta} = \sum_i |(V_i)_{\alpha\beta}|^2
\]

where \(D_{\alpha\beta}\) and \((V_i)_{\alpha\beta}\) are the \(\alpha\beta\) entry of \(D\) and \(V_i\), respectively. It is straightforward to show that, by using the two conditions in (14)

\[
\sum_{\alpha} D_{\alpha\beta} = 1,
\]

\[
\sum_{\beta} D_{\alpha\beta} = 1
\]

i.e., \(D\) is a doubly stochastic matrix. From Proposition 2, we get

\[
\lambda(\rho(t_2)) \prec \lambda(\rho(t_1)),
\]

so

\[
\rho(t_2) \prec \rho(t_1), \quad \forall t_2 > t_1
\]

i.e. it is majorization monotone.

From Proposition 5 and 6, it is easy to see that majorization monotone implies entropy monotone and trace norm monotone, and they share the same necessary and sufficient condition: \(L(I) = 0\).

V. CONCLUSION

Understanding open quantum systems is an important problem for a wide variety of physics, chemistry, and engineering applications. This technical note analyzed the dynamics of open quantum systems and gives necessary and sufficient condition on majorization monotone dynamics, which are those dynamics always mixing the states. This suggests that for this class of dynamics, majorization defines an evolution arrow, which begs for the connection to the entropy arrow in the second law of thermal dynamics. The author hopes further investigation will reveal more on this connection.

REFERENCES

[1] D. J. Tannor and S. A. Rice, “Control of selectivity of chemical reaction via control of wave packet evolution,” J. Chem. Phys., vol. 83, p. 5015, 1985.
[2] D. J. Tannor, R. Kosloff, and S. A. Rice, “Coherent pulse sequence induced control of selectivity of reactions: Exact quantum mechanical calculations,” J. Chem. Phys., vol. 85, p. 5805, 1986.
[3] D. J. Tannor and S. A. Rice, “Coherent pulse sequence control of product formation in chemical reactions,” Adv. Chem. Phys., vol. 70, p. 441, 1988.
[4] R. Kosloff, S. A. Rice, P. Gaspard, S. Tersigni, and D. J. Tannor, “Wavepacket dancing: Achieving chemical selectivity by shaping light pulses,” Chem. Phys., vol. 139, p. 201, 1989.
[5] S. A. Rice and M. Zhao, Optical Control of Molecular Dynamics. New York: Wiley, 2000.
[6] M. Shapiro and P. Brumer, Principles of the Quantum Control of Molecular Processes. New York: Wiley, 2003.
[7] T. Brixner and G. Gerber, “Quantum control of gas-phase and liquid-phase femtochemistry,” Chem. Phys. Chem., vol. 4, p. 418, 2003.
[8] R. Mitrić, M. Hartmann, J. Pittner, and V. Bonačić-Koutecký, “New strategy for optimal control of femtosecond pump-dump processes,” J. Phys. Chem. A, vol. 106, p. 10477, 2002.
[9] A. P. Peirce, M. A. Dahleh, and H. Rabitz, “Optimal control of quantum-mechanical systems: Existence, numerical approximation, and applications,” Phys. Rev. A, vol. 37, p. 4950, 1988.
[10] S. Shi and H. Rabitz, “Optimal control of bond-selectivity in unimolecular reactions,” Comp. Phys. Commun., vol. 63, p. 71, 1991.
[11] S. E. Sklarz, D. J. Tannor, and N. Khaneja, “Optimal control of quantum dissipative dynamics: Analytic solution for cooling the three-level Lambda system,” Phy. Rev. A, vol. 69, p. 053408, 2004.
[12] W. Jakubetz and J. Manz, “Theory of optimal laser pulses for selective transitions between molecular eigenstates,” Chem. Phys. Lett., vol. 165, p. 100, 1990.
[13] H. Shen, J. P. Dussault, and A. D. Bandrauk, “Optimal pulse shaping for coherent control by the penalty algorithm,” Chem. Phys. Lett., vol. 221, p. 498, 1994.
[14] Y. J. Yan, R. E. Gillilan, R. M. Whitnell, K. R. Wilson, and S. Mukamel, “Optimal control of molecular dynamics: Liouville-space theory,” J. Phys. Chem., vol. 97, p. 2320, 1993.
[15] N. Khaneja, R. Brockeittel, and S. J. Glaser, “Time optimal control in spin systems,” Phys. Rev. A, vol. 63, p. 032308, 2001.
[16] H. Yuan and N. Khaneja, “Reachable set of bilinear control system with time varying drift,” Syst. Control Lett., vol. 55/6, pp. 501–507, 2006.
[17] H. Yuan and N. Khaneja, “Time optimal control of coupled qubits under nonstationary interactions,” Phys. Rev. A, vol. 72, p. 040301, 2005.
[18] H. Yuan, S. Glaser, and N. Khaneja, “Geodesics for efficient creation and propagation of order along Ising spin chains,” Phys. Rev. A, vol. 76, p. 012311, 2007.
On the Dynamic Response of a Saturating Static Feedback-Controlled Single Integrator Driven by White Noise

Zheng Wen, Sandip Roy, and Ali Saberi

Abstract—In this technical note, we fully characterize the dynamic response of a saturating static feedback-controlled single integrator driven by Gaussian white noise by solving the derived Fokker-Planck equation.

Index Terms—Actuator saturation, dynamic response, white noise.

I. INTRODUCTION

Our effort here is motivated by the need for understanding the external (disturbance) responses of nonlinear and in particular saturating feedback systems. Actuator saturation nonlinearities, as well as numerous other static nonlinearities, are ubiquitous in control systems, and so characterizing both the internal stability and the disturbance response of control systems with such nonlinearities is of fundamental importance in numerous application areas. While the internal stability of feedback systems with actuator saturation is very well understood, much remains to be done in characterizing the responses of saturating control systems to disturbances and noise, see [1] and [2] for background. With this motivation in mind, several works aim to find the steady-state probability distribution and/or statistics of some special nonlinear stochastic systems driven by noise (e.g., [3] and [5]), including of feedback-controlled saturating systems (e.g., [6]). However, to the best of our knowledge, there have been no works so far that explicitly characterize the the transient response of feedback-controlled dynamic systems with actuator saturation. With the aim of better understanding the transient response, we here fully characterize the joint probability density functions (pdfs) of a canonical feedback-controlled saturating dynamic system, namely a single integrator with linear static feedback.

Specifically, we characterize the dynamic solution of the stochastic differential equation

\[
\begin{align*}
\dot{x}(t) &= \sigma [\mathcal{K} x(t)] + \omega(t) \\
\dot{x}(t_0) &= x_0
\end{align*}
\]

(1)

where \(t \in R^+, x(t) \) is the state, the feedback gain \( \mathcal{K} \) is a positive real number, \( \omega(t) \) is a Gaussian white noise process with zero mean and autocorrelation \( \mathbb{E}[\omega(t)\omega(t')]=\delta(t-t') \), the initial state \( x_0 \) is a random

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Z. Wen is with the Stanford University, Stanford, CA 94305-7519 USA (e-mail: zhengwen@stanford.edu; zhengwen_forever@hotmail.com).
S. Roy and A. Saberi are with the Washington State University, Pullman, WA 99164-2752 USA (e-mail: sroy@ece.wsu.edu; saberi@ece.wsu.edu).

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