Ornstein–Uhlenbeck Type Processes on Wasserstein Spaces.

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Abstract

Let $\mathcal{P}_2$ be the space of probability measures on $\mathbb{R}^d$ having finite second moment, and consider the Riemannian structure on $\mathcal{P}_2$ induced by the intrinsic derivative on the $L^2$-tangent space. By using stochastic analysis on the tangent space, we construct an Ornstein–Uhlenbeck (OU) type Dirichlet form on $\mathcal{P}_2$ whose generator is formally given by the intrinsic Laplacian with a drift. The log-Sobolev inequality holds and the associated Markov semigroup is $L^2$-compact. Perturbations of the OU Dirichlet form are also studied.

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1 Introduction

To construct the Ornstein–Uhlenbeck process on the Wasserstein space, we introduce an inherent Gauss measure on the Wasserstein space, which together with the intrinsic derivative provides an Ornstein–Uhlenbeck type Dirichlet form. Up to the quasi-regularity, the Dirichlet form is associated with a diffusion process on the Wasserstein space, which will be addressed in the forthcoming paper \cite{37}.

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Let $\mathcal{P}_2$ be the space of all probability measures on $\mathbb{R}^d$ having finite second moment. It is a complete and separable space under the quadratic Wasserstein distance

$$W_2(\mu, \nu) := \inf_{\pi \in \mathcal{C}(\mu, \nu)} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \pi(dx, dy) \right)^{\frac{1}{2}} , \quad \mu, \nu \in \mathcal{P}_2,$$

where $\mathcal{C}(\mu, \nu)$ is the set of all couplings of $\mu$ and $\nu$. This space has been equipped with a natural Riemannian structure and becomes an infinite-dimensional Riemannian manifold, which is non-standard as explained in Remark 2.1(1) below. This structure is induced by the intrinsic derivative introduced by Albeverio, Kondratiev and Röckner [6], and is consistent with Otto’s structure [33] defined for probability measures having smooth and positive density functions, see Remark 2.1 below for details. The Wasserstein space is a fundamental research object in the theory of optimal transport and related analysis, see [5, 42] and reference therein.

In this paper we aim to study stochastic analysis on the Wasserstein space. To this end, we first recall some existing literature concerning measure-valued diffusion processes.

Firstly, to construct measure-valued diffusion processes using the Dirichlet-form theory, the integration by parts formula has been established for the intrinsic and extrinsic derivatives with respect to a reference distribution $\Xi$ on the space of Radon measures, see [4, 30, 34, 14, 39, 40] and references therein. Moreover, functional inequalities have been derived for measure-valued processes, see [15, 19, 20, 21, 22, 36, 41, 45, 47]. A key point in the construction of a Dirichlet form is to establish the integration by parts formula of the reference measure for derivatives in measure. The stationary distributions in these references are chosen as either the entropic-type measures supported on the space of singular distributions without discrete part, or the Dirichlet/Gamma type measures concentrated on the space of discrete distributions, which have reasonable backgrounds from physics, population genetics and Bayesian non-parametrics. Along a different direction, a Rademacher type theorem is established in [13] for a class of reference measures satisfying the integration by parts formula for the intrinsic derivative.

Next, corresponding to Dean-Kawasaki type SPDEs, local Dirichlet forms have been constructed on the Wasserstein space over the real line induced by increasing functions (see [27, 28] and references therein), and it is proved in [29] that the associated diffusion process is given by the empirical measure of independent particle systems.

Moreover, by solving a conditional distribution-dependent SDE, [46] constructed a diffusion process on $\mathcal{P}_2$ with generator given by a second-order differential operator in intrinsic derivative, and establish the Feynman–Kac formula for the underline measure-valued PDE. Since the SDE is driven by finite-dimensional Brownian motion, the measure-valued diffusion process is highly degenerate. Note that the measure-valued diffusion processes constructed in [46] extends that generated by the partial Laplacian investigated in [11]. See also [17] for an extension to the Wasserstein space over a compact Riemannian manifold.

In this paper, we construct and study the Ornstein–Uhlenbeck (OU) type Dirichlet form on the Wasserstein space $\mathcal{P}_2$, whose stationary distribution is a fully supported Gaussian measure, and the generator is the Laplacian with a drift, where the Laplacian is induced by the Riemannian structure and is hence crucial in geometric analysis. In view of the fundamental role played by Ornstein–Uhlenbeck process in Malliavin calculus on the Wiener space, the present study should be crucial for developing stochastic analysis on the Wasserstein space.
Recall that the Brownian motion on a $d$-dimensional Riemannian manifold can be constructed by using the flat Brownian motion on the tangent space $\mathbb{R}^d$. In the same spirit, we will introduce the tangent space on $\mathcal{P}_2$ which is a separable Hilbert space, then recall the Gaussian measure and Ornstein–Uhlenbeck process on the Hilbert space, and finally construct the corresponding objects on $\mathcal{P}_2$ as projections from the tangent space. The Ornstein–Uhlenbeck type Dirichlet form we construct on $\mathcal{P}_2$ shares nice properties of the original process on the tangent space: it satisfies the log-Sobolev inequality and the generator has purely discrete spectrum.

The remainder of the paper is organized as follows. In Section 2, we recall the Riemannian structure induced by the intrinsic derivative due to [6], and calculate the Laplacian operator $\Delta_{\mathcal{P}_2}$. Since different structures and Laplacians exist in the literature, to distinguish we call ours intrinsic Riemannian structure and intrinsic Laplacian. In Section 3, we construct the Gaussian measure $N_{\mu_0,Q}$ determined by a reference measure $\mu_0 \in \mathcal{P}_2$ together with an unbounded positive definite linear operator $Q$ on the tangent space $T_{\mu_0} := L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu_0)$, and study the corresponding OU type Dirichlet form on $\mathcal{P}_2$. In Section 4, we try to formulate the generator of the OU process as

$$Lf(\mu) = \Delta_{\mathcal{P}_2} f(\mu) - \langle b(\mu), Df(\mu) \rangle_{T_{\mu}},$$

where $T_{\mu} := L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu)$ is the tangent space at $\mu$, and $b(\mu) \in T_{\mu}$ is induced by the linear operator $Q$ on $T_{\mu_0}$. This formulation is consistent with that of the OU process on a separable Hilbert space, but it is in the moment “either formal computation or non-rigorous formula” due to the lack of a reasonable class of functions making the right hand side meaningful. Finally, in Section 5 we study symmetric diffusion processes on $\mathcal{P}_2$ as perturbations of the OU process.

## 2 Intrinsic Riemannian structure on the Wasserstein space

By using the gradient flow of density functions arising from Monge’s optimal transport, Otto [33] constructed the Riemannian structure on $\mathcal{P}_2^{\text{ac}}$, the space of measures in $\mathcal{P}_2$ having strictly positive smooth density functions with respect to the Lebesgue measure, see also [42, Chapter 13]. Under Otto’s structure, the tangent space at $\mu \in \mathcal{P}_2^{\text{ac}}$ is the $L^2(\mu)$-closure of $\{\nabla f : f \in C_0^\infty(\mathbb{R}^d)\}$. The Ricci curvature was calculated in [31], while the Levi-Civita connection and parallel displacement have been studied in [16].

To build up a Riemannian structure, one needs to introduce the tangent space at each point $\mu \in \mathcal{P}_2$, and define an inner product (or metric) on the tangent space. From different point of views, several Riemannian structures on the Wasserstein space have been studied in the literature, see [23] for the tangent space induced by all germs of geodesic curves, and see [33] for the tangent space induced by germs of geodesic curves induced by optimal transport maps (the closure of gradient-type vector fields). In this paper, we adopt the $L^2$-tangent space introduced in [6] (see [33, Appendix]) to define the intrinsic derivative. This structure fits well to the Gâteaux derivative in infinite-dimensional analysis, and it works for the space of general Radon measures as well.
2.1 Intrinsic derivative

We will simply denote $\mu(f) = \int f \, d\mu$ for a measure $\mu$ and a function $f \in L^1(\mu)$. For any $\mu \in \mathcal{P}_2$ and measurable $\phi : \mathbb{R}^d \to \mathbb{R}^d$, let $\mu \circ \phi^{-1}$ be the image of $\mu$ under $\phi$, i.e.

$$(\mu \circ \phi^{-1})(A) := \mu(\phi^{-1}(A))$$

for measurable sets $A \subset \mathbb{R}^d$. It is easy to see that $\mu \circ \phi^{-1} \in \mathcal{P}_2$ if and only if

$$\phi \in T_\mu := L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu),$$

where $L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu)$ is the space of measurable maps $\phi$ from $\mathbb{R}^d$ to $\mathbb{R}^d$ with

$$\|\phi\|_{L^2(\mu)} := (\mu(|\phi|^2))^{\frac{1}{2}} < \infty.$$

So, it is natural to take $T_\mu$ as the tangent space at $\mu$, which is a separable Hilbert space with inner product

$$\langle \phi_1, \phi_2 \rangle_{T_\mu} := \mu(\langle \phi_1, \phi_2 \rangle) = \int_{\mathbb{R}^d} \langle \phi_1, \phi_2 \rangle \, d\mu, \quad \phi_1, \phi_2 \in T_\mu.$$

Let $id \in T_\mu$ be the identity map, i.e. $id(x) = x$.

**Definition 2.1.** Let $f \in C(\mathcal{P}_2)$, the class of continuous functions on $\mathcal{P}_2$.

1. We call $f$ intrinsically differentiable, if for any $\mu \in \mathcal{P}_2$,

$$T_\mu \ni \phi \mapsto D_\phi f(\mu) := \lim_{\varepsilon \downarrow 0} \frac{f(\mu \circ (id + \varepsilon \phi)^{-1}) - f(\mu)}{\varepsilon} \in \mathbb{R}$$

is a bounded linear functional. In this case, the intrinsic derivative $Df(\mu)$ is the unique element in $T_\mu$ such that

$$\langle Df(\mu), \phi \rangle_{T_\mu} := \mu(\langle \phi, Df(\mu) \rangle) = D_\phi f(\mu), \quad \phi \in T_\mu.$$

2. We write $f \in C^1(\mathcal{P}_2)$, if $f$ is intrinsically differentiable such that $Df(\mu)(x)$ has a jointly continuous version; i.e. each $Df(\mu)$ has a $\mu$-version $x \mapsto Df(\mu)(x)$ such that

$$(\mu, x) \in \mathcal{P}_2 \times \mathbb{R}^d \mapsto Df(\mu)(x)$$

is continuous. We denote $f \in C^1_b(\mathcal{P}_2)$ if moreover $f$ and $Df$ are bounded.

3. We denote $f \in C^2(\mathcal{P}_2)$, if $f \in C^1(\mathcal{P}_2)$, the jointly continuous version $Df(\mu)(x)$ is intrinsically differentiable in $\mu$ and differentiable in $x$, such that

$$D^2 f(\mu)(x,y) := D(Df(\cdot)(x))(\mu)(y), \quad \nabla Df(\mu)(x) := \nabla(Df(\mu)(\cdot))(x)$$

have versions jointly continuous in all arguments in the same sense of item (2). We write $f \in C^2_b(\mathcal{P}_2)$ if moreover $f, Df, D^2 f$ and $\nabla Df$ are bounded.
When \( f \in C^1(\mathcal{P}) \), we automatically take \( Df(\mu)(x) \) to be the jointly continuous version of \( Df \), which is unique. Indeed, by the continuity, \( Df(\mu)(\cdot) \) is unique for each \( \mu \in \mathcal{P}_2 \) with full support, so that it is unique for all \( \mu \in \mathcal{P}_2 \) since the set of fully supported measures is dense in \( \mathcal{P}_2 \). Under the Riemannian metric given by (2.1), the space \( \mathcal{P}_2 \) becomes an infinite-dimensional “Riemannian manifold with different level boundaries,” see Remark 2.1(1) below for an explanation.

To make calculus on \( \mathcal{P}_2 \), we introduce the displacement of the tangent space. For any \( \phi \in T_\mu \), consider the displacement of measures along \( \phi \) from \( \mu \):

\[
[0, \infty) \ni s \mapsto \mu \circ (id + s\phi)^{-1} \in \mathcal{P}_2.
\]

Then the tangent space is shifted as

\[(2.2)\quad T_{\mu(\mu+\phi)^{-1}} = T_\mu \circ (id + s\phi)^{-1} := \{ h \circ (id + s\phi)^{-1} : h \in T_\mu \}, \quad s \geq 0,
\]

where \( h \circ (id + s\phi)^{-1} \in T_{\mu(\mu+\phi)^{-1}} \) is uniquely determined by

\[(2.3)\quad \langle h \circ (id + s\phi)^{-1}, \psi \rangle_{T_{\mu(\mu+\phi)^{-1}}} := \langle h, \psi \circ (id + s\phi) \rangle_{T_\mu}, \quad \psi \in T_{\mu\circ(\mu+\phi)^{-1}},
\]

by noting that \( \psi \circ (id + s\phi) \in T_\mu \) is due to

\[(2.4)\quad \|\psi \circ (id + s\phi)\|_{T_\mu}^2 = \mu(|\psi \circ (id + s\phi)|^2)
\]

\[= (\mu \circ (id + s\phi)^{-1})(|\psi|^2) < \infty, \quad \psi \in T_{\mu\circ(\mu+\phi)^{-1}}.
\]

Obviously, \( T_{\mu(\mu+\phi)^{-1}} \supset T_\mu \circ (id + s\phi)^{-1} \). On the other hand, for any \( \psi \in T_{\mu\circ(\mu+\phi)^{-1}} \), (2.4) implies \( \tilde{\psi} := \psi \circ (id + s\phi) \in T_\mu \) and

\[
\psi = \tilde{\psi} \circ (id + s\phi)^{-1} \in T_\mu \circ (id + s\phi)^{-1}.
\]

Therefore, (2.2) holds.

The following result implies that a function \( f \in C^1_b(\mathcal{P}_2) \) is \( L \)-differentiable, i.e. it is intrinsically differentiable and

\[(2.5)\quad \lim_{\|\phi\|_{L^2(\mu)} \to 0} |f(\mu \circ (id + \phi)^{-1}) - f(\mu) - D\phi f(\mu)| = 0.
\]

In this case, the intrinsic derivative is also called the \( L \)-derivative, which coincides with Lions’ derivative introduced in [10].

**Proposition 2.1.** Let \( f \in C^1(\mathcal{P}_2) \) such that

\[(2.6)\quad \lim_{N \to \infty} \limsup_{\|\phi\|_{L^2(\mu)} \to 0} \left\| |Df(\mu \circ (id + \phi)^{-1})(id + \phi)| - N \right\|_{L^2(\mu)} = 0,
\]

then (2.5) holds, i.e. \( f \) is \( L \)-differentiable.
Proof. Let $\mu \in \mathcal{P}_2$ and $\phi \in T_\mu$. By (2.2) we have $\phi \circ (id + s\phi)^{-1} \in T_{\mu_0(id + s\phi)^{-1}}$ for $s \in [0, 1]$, and

$$
\frac{d}{ds}f(\mu \circ (id + s\phi)^{-1}) = \lim_{\varepsilon \downarrow 0} \frac{f(\mu \circ (id + (s + \varepsilon)\phi)^{-1}) - f(\mu \circ (id + s\phi)^{-1})}{\varepsilon} 
= \lim_{\varepsilon \downarrow 0} \frac{f((\mu \circ (id + s\phi)^{-1}) \circ (id + \varepsilon\phi \circ (id + s\phi)^{-1})^{-1}) - f(\mu \circ (id + s\phi)^{-1})}{\varepsilon} 
= D_{\phi(id + s\phi)^{-1}}f(\mu \circ (id + s\phi)^{-1}) = \mu((\phi, (Df(\mu \circ (id + s\phi)^{-1})) \circ (id + s\phi))).
$$

Combining this with $f \in C^1(\mathcal{P}_2)$, we arrive at

$$
\limsup_{\|\phi\|_{L^2(\mu)} \downarrow 0} \frac{|f(\mu \circ (id + \phi)^{-1}) - f(\mu) - D\phi f(\mu)|}{\|\phi\|_{L^2(\mu)}} 
\leq \limsup_{\|\phi\|_{L^2(\mu)} \downarrow 0} \int_0^1 \frac{1}{\|\phi\|_{L^2(\mu)}} |\mu((\phi, (Df(\mu \circ (id + s\phi)^{-1})) \circ (id + s\phi) - Df(\mu)))| \, ds 
\leq \limsup_{\|\phi\|_{L^2(\mu)} \downarrow 0} \int_0^1 \|Df(\mu \circ (id + s\phi)^{-1}) \circ (id + s\phi) - Df(\mu)\|_{L^2(\mu)} \, ds = 0,
$$

where the last step follows from the continuity of $Df$, (2.6) and the dominated convergence theorem. Indeed, if the last step does not hold, then there exist a constant $\varepsilon > 0$ and a sequence $\{\phi_n\}_{n \geq 1}$ with $\|\phi_n\|_{L^2(\mu)} \leq \frac{1}{n}$ such that

$$
\xi_n(s) := Df(\mu \circ (id + s\phi_n)^{-1})(id + s\phi_n), \quad n \geq 1, s \in [0, 1]
$$

satisfies

$$
(2.7) \quad \int_0^1 \|\xi_n(s) - Df(\mu)\|_{L^2(\mu)} \, ds \geq \varepsilon, \quad n \geq 1.
$$

Up to a subsequence, we may also assume that $\mu(\{\phi_n \to 0\}) = 1$, which together with

$$
\sup_{s \in [0, 1]} \|C_{\phi(id + s\phi_n)^{-1}, \mu} \leq \|\phi_n\|_{L^2(\mu)} \leq \frac{1}{n}, \quad n \geq 1
$$

and $f \in C^1(\mathcal{P}_2)$ implies

$$
\mu\left(\lim_{n \to \infty} \xi_n(s) = Df(\mu), \quad s \in [0, 1]\right) = 0.
$$

So, by the dominated convergence theorem, we obtain

$$
(2.8) \quad \lim_{n \to \infty} \int_0^1 \|\xi_n(s) - Df(\mu)\|_{L^2(\mu)} \, ds = 0, \quad N \geq 1.
$$
On the other hand, (2.6) implies
\[ \lim_{N \to \infty} \limsup_{n \to \infty} \int_0^1 \left\| \left( |\xi_n(s)| + |Df(\mu)| \right) \mathbf{1}_{\{|\xi_n(s)| \geq 2N\}} \right\|_{L^2(\mu)} \, ds \leq \lim_{N \to \infty} \limsup_{n \to \infty} \int_0^1 \left\| \left( |\xi_n(s)| + |Df(\mu)| - N \right)^+ \right\|_{L^2(\mu)} \, ds \leq \lim_{N \to \infty} \limsup_{n \to \infty} 2 \left\| \left( Df(\mu \circ (id + \phi)^{-1})(id + \phi) \right) + |Df(\mu)| - N \right)^+ \right\|_{L^2(\mu)} = 0. \]

Combining this with (2.8) leads to
\[ \limsup_{n \to \infty} \int_0^1 \left\| \xi_n(s) - Df(\mu) \right\|_{L^2(\mu)} \, ds \leq \limsup_{n \to \infty} \int_0^1 \left\| (\xi_n(s) - Df(\mu)) \mathbf{1}_{\{|\xi_n(s)| < 2N\}} \right\|_{L^2(\mu)} \, ds + \limsup_{n \to \infty} \int_0^1 \left\| (|\xi_n(s)| + |Df(\mu)| \mathbf{1}_{\{|\xi_n(s)| \geq 2N\}} \right\|_{L^2(\mu)} \, ds = 0, \]
which contradicts to (2.7).

We are ready to introduce the chain rule for the intrinsic derivative in the distribution of random variables. Let \( L_\xi \) be the law of a random variable under a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). Note that (2.9) implies
\[ (|Df(\mu \circ (id + \phi)^{-1})(id + \phi)| - N)^+ \leq \left( c(1 + |x| + |\phi(x)|) - N \right)^+ \leq c|\phi(x)| + (c + c|x| - N)^+, \]
so that (2.6) holds for any \( \mu \in \mathcal{P}_2 \). Then the following result follows from Proposition 2.1 and [8 Theorem 2.1(2)] for \( p = 2 \), see also [26 Lemma A.8] for an earlier result.

**Proposition 2.2.** Let \( (\xi_s)_{s \in [0,1]} \) be a family of \( \mathbb{R}^d \)-valued random variables on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), such that \( \mu_s := L_{\xi_s} \in \mathcal{P}_2 \) and
\[ \xi_0 := \lim_{s \to 0} \frac{\xi_s - \xi_0}{s} \]
exists in \( L^2(\mathbb{P}) \). Then for any \( f \in C^1(\mathcal{P}_2) \) such that
\[ |Df(\mu)(x)| \leq c(1 + |x|), \quad x \in \mathbb{R}^d, \quad \mu \in \mathcal{P}_2, \quad \mathbb{W}_2(\mu, \mu_0) \leq 1 \]
holds for some constant \( c > 0 \), we have
\[ \frac{d}{ds} \bigg|_{s=0} f(L_{\xi_s}) := \lim_{s \to 0} \frac{f(L_{\xi_s}) - f(L_{\xi_0})}{s} = \mathbb{E} \left[ (Df(\mu_0)(\xi_0), \dot{\xi}_0) \right]. \]
Remark 2.1. Below we make some comments on the above Riemannian structure.

(1) Since the dimension of the tangent space $T_\mu$ is $d$ times the number of points contained in the support of $\mu$, tangent spaces at different points may have different dimensions. This is different from a standard Riemannian manifold, but fits well to the feature of Riemannian manifolds with different level boundaries (sub-manifolds). For instance, consider a half ball in $\mathbb{R}^3$. It is a 3D manifold with 2D boundary, and the 2D boundary itself is a manifold with 1D boundary. At a point on the boundary outside the edge, the tangent space only consists of vectors tangent to the boundary, since the geodesic along the normal vector may go beyond the manifold, so geodesics from this point are only defined along tangent vectors of the boundary under the induced Riemannian metric, so that the Laplacian becomes degenerate comparing to 3D, and the Brownian motion starting from this point stays on the 2D boundary. The same happens to points on the 1D boundary.

(2) The intrinsic derivative coincides with Otto’s derivative when it exists, where the latter is only defined for absolutely continuous measures in $\mathcal{P}_2$. See [23] for an extension of Otto’s structure to the whole space $\mathcal{P}_2$.

(3) To see that $\mathbb{W}_2$ is the intrinsic distance of the Riemannian structure, let $\mu_0 \in \mathcal{P}_2$ be absolutely continuous with respect to the Lebesgue measure. Then for any $\mu_1, \mu_2 \in \mathcal{P}_2$, there exists an optimal coupling $(h_1, h_2) \in T_{\mu_0} \times T_{\mu_0}$, in the sense of random variables on $\mathbb{R}^d$ under the probability measure $\mu_0$, such that

$$\nu_i = \mu_0 \circ h_i^{-1} (i = 1, 2), \quad \mathbb{W}_2(\mu_1, \mu_2)^2 = \mu_0(|h_1 - h_2|^2),$$

so that

$$\nu_t := \mu_0 \circ (th_1 + (1 - t)h_2)^{-1}, \quad t \in [0, 1]$$

is the geodesic linking $\mu_1$ and $\mu_2$, i.e.

$$\nu_0 = \mu_2, \quad \nu_1 = \mu_1, \quad \mathbb{W}_2(\nu_s, \nu_t) = |t - s|\mathbb{W}_2(\mu_1, \mu_2) \text{ for } t, s \in [0, 1].$$

Indeed, for any $0 \leq s \leq t \leq 1$, (2.10) implies

$$\mu_0 \circ \psi_{s,t}^{-1} \in \mathcal{C}(\nu_s, \nu_t)$$

for

$$\psi_{s,t} := (sh_1 + (1 - s)h_2, th_1 + (1 - t)h_2) : \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d,$$

so that by the definition of $\mathbb{W}_2$, we have

$$\mathbb{W}_2(\nu_s, \nu_t) \leq \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 (\mu_0 \circ \psi_{s,t}^{-1})(dx, dy) \right)^{\frac{1}{2}}$$

$$= \left( \int_{\mathbb{R}^d} |th_1 + (1 - t)h_2 - sh_1 - (1 - s)h_2|^2(x)\mu_0(dx) \right)^{\frac{1}{2}}.$$
We write \( f \exists \) exists. Note that the partial Laplacian considered in [11] is given by Proposition 2.3. we have \( f \) not depend on the choice of the ONB \( \{ \langle \cdot, \cdot \rangle \} \) where \( \{ \phi_i \}_{i=1}^{\infty} \) are orthonormal in \( T \). Let \( m \in \mathbb{N} \) \( \mathbb{N} \) in the definition of \( \Delta_P \) on \( P \). Recall that on a \( 2 \)-dimensional Riemannian manifold, the Laplacian is defined as the trace of the second-order derivative \( \nabla^2 \) (i.e. Hessian operator). Below we define the intrinsic Laplacian.

Let \( \mu \in \mathcal{P}_2 \) and let \( \{ \phi_m \}_{m \geq 1} \) be an ONB (orthonormal basis) of \( T_\mu := L^2(\mathbb{R}^d \rightarrow \mathbb{R}^d, \mu) \). Note that the number of \( \{ \phi_m \}_{m \geq 1} \) is a finite family if and only if \( \mu \) has finite support. For any \( \phi \in T_\mu \), let

\[
D^2_\phi f(\mu) := \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} (D_{\phi(\varepsilon)\phi^{-1}} f)(\mu \circ (id + \varepsilon \phi)^{-1}), \ f \in C^2_b(\mathcal{P}_2).
\]

We write \( f \in \mathcal{D}_\mu(\Delta_{\mathcal{P}_2}) \), the domain of \( \Delta_{\mathcal{P}_2} \) at point \( \mu \), if \( f \in C^2_b(\mathcal{P}_2) \) is such that

\[
\Delta_{\mathcal{P}_2} f(\mu) := \sum_{m \geq 1} D^2_{\phi_m} f(\mu)
\]

exists. Note that the partial Laplacian considered in [11] is given by

\[
\Delta_{\mathcal{P}_2} f(\mu) := \sum_{m \geq 1} D^2_{\phi_m} f(\mu),
\]

where \( \{ e_i \}_{1 \leq i \leq d} \) is the standard orthonormal basis of \( \mathbb{R}^d \), which is thus a partial sum from that in the definition of \( \Delta_{\mathcal{P}_2} f(\mu) \). Since \( \mu \) is a probability measure, the constant vectors \( \{ e_i \}_{1 \leq i \leq d} \) are orthonormal in \( T_\mu \mathcal{P}_2 \). We have the following formulation of \( \Delta_{\mathcal{P}_2} \).

**Proposition 2.3.** For any \( \mu \in \mathcal{P}_2 \) and \( f \in \mathcal{D}_\mu(\Delta_{\mathcal{P}_2}) \),

\[
\Delta_{\mathcal{P}_2} f(\mu) = \sum_{m \geq 1} \int_{\mathbb{R}^d \times \mathbb{R}^d} \left( \langle D^2 f(\mu)(x, y), \phi_m(x) \otimes \phi_m(y) \rangle_{HS} + \langle \nabla D f(\mu)(x), \phi_m(x) \otimes \phi_m(x) \rangle_{HS} \right) \mu(dx) \mu(dy),
\]

where \( \langle \cdot, \cdot \rangle_{HS} \) is the Hilbert–Schmidt inner product for matrices, and the right-hand side does not depend on the choice of the ONB \( \{ \phi_m \}_{m \geq 1} \). Consequently, for any \( f \in C^2_b(\mathcal{P}_2) \) and \( \mu \in \mathcal{P}_2 \), we have \( f \in \mathcal{D}_\mu(\Delta_{\mathcal{P}_2}) \) if and only if the following series exists:

\[
\text{tr}(\nabla D f(\mu)) := \sum_{m \geq 1} \int_{\mathbb{R}^d} \langle \nabla D f(\mu)(x), \phi_m(x) \otimes \phi_m(x) \rangle_{HS} \mu(dx).
\]
Proof. Noting that
\[
(D_{\phi_m}(\mu) \circ (id + \varepsilon \phi_m))' \mathbf{f}(\mu \circ (id + \varepsilon \phi_m)^{-1})
\]
\[
= (\mu \circ (id + \varepsilon \phi_m)^{-1})((\phi_m \circ (id + \varepsilon \phi_m)^{-1}, Df(\mu \circ (id + \varepsilon \phi_m)^{-1}))
\]
\[
= \mu((\phi_m, Df(\mu \circ (id + \varepsilon \phi_m))\mu(id + \varepsilon \phi_m)),
\]
we obtain
\[
D_{\phi_m}^2 f(\mu) = \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \mu((\phi_m, Df(\mu \circ (id + \varepsilon \phi_m)^{-1}(id + \varepsilon \phi_m)))
\]
\[
= \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle D^2 f(\mu)(x,y), \phi_m(x) \otimes \phi_m(y)\rangle_{HS} \mu(dx)\mu(dy)
\]
\[
+ \int_{\mathbb{R}^d} \langle \nabla Df(\mu)(x), \phi_m(x) \otimes \phi_m(x)\rangle_{HS} \mu(dx),
\]
where the first term comes from the derivative $\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} Df(\mu \circ (id + \varepsilon \phi_m)^{-1})$ by Proposition 2.22 for $\xi = id + \varepsilon \phi_m$, and the other term follows from the derivative $\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} Df(\mu)(id + \varepsilon \phi_m)$. Therefore, the desired formula holds by (2.11).

Next, it is easy to see that $\Delta_{\mathcal{H}} f(\mu)$ does not depend on the choice of the ONB $\{\phi_m\}_{m \geq 1}$. Indeed, for another ONB $\{\tilde{\phi}_m\}_{m \geq 1}$ in $T_\mu$, we have
\[
\tilde{\phi}_m = \sum_{l \geq 1} \langle \tilde{\phi}_m, \tilde{\phi}_l \rangle_{T_\mu} \phi_l, \quad m \geq 1,
\]
\[
\sum_{m \geq 1} \langle \tilde{\phi}_m, \phi_l \rangle_{T_\mu} \langle \tilde{\phi}_m, \phi_k \rangle_{T_\mu} = 1_{\{l=k\}}, \quad k, l \geq 1.
\]
So,
\[
\sum_{m \geq 1} \left(\langle D^2 f(\mu)(x,y), \tilde{\phi}_m(x) \otimes \tilde{\phi}_m(y)\rangle_{HS} + \langle \nabla Df(\mu)(x), \tilde{\phi}_m(x) \otimes \tilde{\phi}_m(x)\rangle_{HS}\right)
\]
\[
= \sum_{k,l,m \geq 1} \langle \tilde{\phi}_m, \phi_k \rangle_{T_\mu} \langle \tilde{\phi}_m, \phi_l \rangle_{T_\mu}
\]
\[
\times \left(\langle D^2 f(\mu)(x,y), \phi_k(x) \otimes \phi_l(y)\rangle_{HS} + \langle \nabla Df(\mu)(x), \phi_k(x) \otimes \phi_l(x)\rangle_{HS}\right)
\]
\[
= \sum_{k,l \geq 1} 1_{\{k=l\}} \left(\langle D^2 f(\mu)(x,y), \phi_k(x) \otimes \phi_l(y)\rangle_{HS} + \langle \nabla Df(\mu)(x), \phi_k(x) \otimes \phi_l(x)\rangle_{HS}\right)
\]
\[
= \sum_{m \geq 1} \left(\langle D^2 f(\mu)(x,y), \phi_m(x) \otimes \phi_m(y)\rangle_{HS} + \langle \nabla Df(\mu)(x), \phi_m(x) \otimes \phi_m(x)\rangle_{HS}\right).
\]

Finally, let $\{e_i\}_{1 \leq i \leq d}$ be the standard ONB in $\mathbb{R}^d$. For any $f \in C^2_b(\mathcal{H})$, by the Cauchy–Schwarz inequality we obtain
\[
\sum_{m \geq 1} \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle D^2 f(\mu)(x,y), \phi_m(x) \otimes \phi_m(y)\rangle_{HS} \mu(dx)\mu(dy) \right|
\]
Remark 2.2. We present some comments on \( \mathcal{D}_\mu(\Delta_{\mathcal{P}_2}) \).

(1) If \( \mu \) has finite support, then \( T_\mu \) is finite-dimensional so that \( C^2_b(\mathcal{P}_2) \subset \mathcal{D}_\mu(\Delta_{\mathcal{P}_2}) \).

(2) Let \( f_i(\mu) := \mu(\cdot, e_i) = \int_{\mathbb{R}^d} x_i \mu(dx), 1 \leq i \leq d \), where \( \{e_i\}_{1 \leq i \leq d} \) is the standard ONB in \( \mathbb{R}^d \) so that \( \langle x, e_i \rangle = x_i \). Then for any \( \mu \in \mathcal{P}_2 \) and \( g \in C^2_b(\mathbb{R}^d) \),

\[
f := g(f_1, \ldots, f_d) \in \mathcal{D}_\mu(\Delta_{\mathcal{P}_2}).
\]

Indeed, this case we have

\[
Df(\mu) = (\nabla g)(f_1(\mu), \ldots, f_d(\mu)),
\]

\[
D^2f(\mu) = (\nabla^2 g)(f_1(\mu), \ldots, f_d(\mu)),
\]

so that \( \nabla Df = 0 \) and for the standard ONB \( \{e_i\}_{1 \leq i \leq d} \) in \( \mathbb{R}^d \),

\[
\Delta_{\mathcal{P}_2}f(\mu) = \sum_{m \geq 1} \sum_{i,j=1}^{d} (\partial_i \partial_j g)(f_1(\mu), \ldots, f_d(\mu)) \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi_i^m(x)\phi_j^m(y)\mu(dx)\mu(dy)
\]

\[
= \sum_{i,j=1}^{d} (\partial_i \partial_j g)(f_1(\mu), \ldots, f_d(\mu)) \sum_{m=1}^{d} \mu(\langle \phi_m, e_i \rangle) \mu(\langle \phi_m, e_j \rangle)
\]

\[
= \sum_{i,j=1}^{d} (\partial_i \partial_j g)(f_1(\mu), \ldots, f_d(\mu)) \mu(\langle e_i, e_j \rangle)
\]

\[
= (\Delta g)(f_1(\mu), \ldots, f_d(\mu)), \quad \mu \in \mathcal{P}_2.
\]

(3) In general, non-constant cylindrical functions of type

\[
f(\mu) := g(\mu(f_1), \ldots, \mu(f_n)), \quad n \geq 1, f_i \in C^2_b(\mathbb{R}^d), g \in C^2_b(\mathbb{R}^n)
\]

are not in \( \mathcal{D}(\Delta_{\mathcal{P}_2}) := \cap_{\mu \in \mathcal{P}_2} \mathcal{D}_\mu(\Delta_{\mathcal{P}_2}) \), although they are in \( C^2_b(\mathcal{P}_2) \), see [223, Remark 5.6] for explanations. So, the domain of \( \Delta_{\mathcal{P}_2} \) might be too small to define a Brownian motion type diffusion process on \( \mathcal{P}_2 \). In the next section, we construct OU type Dirichlet forms on \( \mathcal{P}_2 \), so that the associated generators have dense domains in the \( L^2 \)-space, although the structure of functions in the domains is unknown, see Remark 4.1 below.

Therefore, \( f \in \mathcal{D}_\mu(\Delta_{\mathcal{P}_2}) \) if and only if \( \text{tr}(\nabla Df(\mu)) \) exists.

\[\square\]
3 Ornstein–Uhlenbeck process on Wasserstein space

We will start from the OU process on the tangent space $T_{\mu_0}$ for a reference measure $\mu_0 \in \mathcal{P}_2$, then transform to the Wasserstein space.

To make sure that any $\mu \in \mathcal{P}_2$ is the distribution of some $h \in T_{\mu_0}$ under the probability $\mu_0$, i.e. $\mu = \mu_0 \circ h^{-1}$, we assume that $\mu_0$ is absolutely continuous with respect to the Lebesgue measure. In this case, for any $\mu \in \mathcal{P}_2$, there exists a unique $h \in T_{\mu_0}$ such that

$$\Psi(h) := \mu_0 \circ h^{-1} = \mu, \quad \mathbb{W}_2(\mu_0, \mu)^2 = \mu_0(|id - h|^2).$$

This $h$ is called the optimal map as solution of the Monge problem for $\mathbb{W}_2$, see [42, Theorem 10.41] or [5]. The map $\Psi : T_{\mu_0} \to \mathcal{P}_2$ is a Lipschitz surjection, i.e. $\Psi(T_{\mu_0}) = \mathcal{P}_2$ and

$$\mathbb{W}_2(\Psi(h), \Psi(\tilde{h})) \leq \mu_0(|h - \tilde{h}|^2)^{\frac{1}{2}} = \|h - \tilde{h}\|_{T_{\mu_0}}, \quad h, \tilde{h} \in T_{\mu_0}.$$

In the following, we first introduce some facts for the OU process on the Hilbert space $T_{\mu_0}$, then construct the corresponding one on $\mathcal{P}_2$.

3.1 OU process on tangent space

Since the tangent space $T_{\mu_0}$ is a separable Hilbert space, it has a natural flat Riemannian structure. Let $\{h_n\}_{n \geq 1}$ be a complete orthonormal basis of $T_{\mu_0}$. The Laplacian is given by

$$\Delta := \text{tr} (\nabla^2) = \sum_{n=1}^{\infty} \nabla^2_{h_n},$$

where $\nabla_{h_n}$ is the directional derivative along $h_n$, and the gradient operator $\nabla$ is determined by

$$\langle \nabla f(h), h_n \rangle_{T_{\mu_0}} := \nabla_{h_n} f(h), \quad n \geq 1, \quad h \in T_{\mu_0}, \quad f \in C^1(T_{\mu_0}).$$

Let $Q$ be a positive definite unbounded self-adjoint operator in $T_{\mu_0}$ with discrete spectrum $\{q_n\}_{n \geq 1}$ and eigenbasis $\{h_n\}_{n \geq 1}$ such that $q_n \uparrow \infty$ as $n \uparrow \infty$ and

$$\sum_{n=1}^{\infty} q_n^{-1} < \infty.$$

So, $Q$ has trace-class inverse $Q^{-1}$ and the centred Gaussian measure on $T_{\mu_0}$ with covariance $Q^{-1}$ is given by (see [9])

$$G_Q(dh) := \prod_{n=1}^{\infty} \left( \frac{q_n}{2\pi} \right)^{\frac{1}{2}} \exp \left[ - \frac{q_n \langle h, h_n \rangle_{T_{\mu_0}}^2}{2} \right] d\langle h, h_n \rangle_{T_{\mu_0}}$$

under the coordinates $\{(\langle h, h_n \rangle_{T_{\mu_0}})_{n \geq 1}\}$ referring to the expansion

$$h = \sum_{n=1}^{\infty} \langle h, h_n \rangle_{T_{\mu_0}} h_n, \quad h \in T_{\mu_0}.$$
The associated OU process can be constructed as (see [12, (5.2.9) or (6.2.1)])

\[ h_t = e^{-tQ}h_0 + \sqrt{2} \int_0^t e^{-(t-s)Q} dW_s, \quad t \geq 0, \]

where \( W_t \) is the cylindrical Brownian motion on \( T_{\mu_0} \), i.e.

\[ W_t = \sum_{n=1}^{\infty} B^n_t h_n, \quad t \geq 0 \]

for independent one-dimensional Brownian motions \( \{B^n_t\}_{n \geq 1} \).

Let \( \tilde{L}, \mathcal{D}(\tilde{L}) \) be generator of the OU process associated with \( G_Q \), which is a negative definite self-adjoint operator in \( L^2(G_Q) \) with domain \( \mathcal{D}(\tilde{L}) \) including the class of cylindrical functions \( \mathcal{F} C^2_b(T_{\mu_0}) \) consisting of

\[ h \mapsto \tilde{f}(h) := F(\langle h, h_1 \rangle_{T_{\mu_0}}, \ldots, \langle h, h_n \rangle_{T_{\mu_0}}), \quad n \geq 1, F \in C^2_b(\mathbb{R}^n), \]

and satisfying, for any such function,

\[
\tilde{L} \tilde{f}(h) = \Delta \tilde{f}(h) - \langle Q \nabla \tilde{f}(h), h \rangle_{T_{\mu_0}} \\
= \sum_{i=1}^{n} (\partial_i^2 - q_i \langle h, h_i \rangle_{T_{\mu_0}} \partial_i) F(\langle h, h_1 \rangle_{T_{\mu_0}}, \ldots, \langle h, h_n \rangle_{T_{\mu_0}}),
\]

where \( \Delta \) and \( \nabla \) are the Laplacian and gradient operators on \( T_{\mu_0} \) respectively. Moreover, the integration by parts formula yields

\[ \tilde{\mathcal{E}}(\tilde{f}, \tilde{g}) := \int_{T_{\mu_0}} \langle \nabla \tilde{f}, \nabla \tilde{g} \rangle_{T_{\mu_0}} dG_Q = - \int_{T_{\mu_0}} (\tilde{f} \tilde{L} \tilde{g}) dG_Q, \quad \tilde{f}, \tilde{g} \in \mathcal{F} C^2_b(T_{\mu_0}). \]

Consequently, \( (\tilde{\mathcal{E}}, \mathcal{F} C^2_b(T_{\mu_0})) \) is closable in \( L^2(G_Q) \) and the closure \( (\tilde{\mathcal{E}}, \mathcal{D}(\tilde{\mathcal{E}})) \) is a symmetric conservative Dirichlet form, see for instance [32]. Moreover, it satisfies the log-Sobolev inequality (see [24][25])

\[ G_Q(\tilde{f}^2 \log \tilde{f}^2) \leq \frac{2}{q_1} \tilde{\mathcal{E}}(\tilde{f}, \tilde{f}), \quad \tilde{f} \in \mathcal{D}(\tilde{\mathcal{E}}), \ G_Q(\tilde{f}^2) = 1. \]

The generator \( \tilde{L}, \mathcal{D}(\tilde{L}) \) of \( (\tilde{E}, \mathcal{D}(\tilde{E})) \) has purely discrete spectrum, i.e. its essential spectrum is empty. Indeed, consider the following one-dimensional OU operators \( \{L_i\}_{i \geq 1} \):

\[ L_i \varphi (r) = \varphi''(r) - q_i r \varphi'(r), \quad r \in \mathbb{R}. \]

It is well known that each \(-L_i\) has purely discrete spectrum consisting of simple eigenvalues

\[ \sigma(-L_i) = \{\lambda_{i,k} : k \geq 0\}, \]

where \( \lambda_{i,0} = 0, \lambda_{i,1} = q_i \) and \( \lambda_{i,k} \uparrow \infty \) with linear growth as \( k \uparrow \infty \). Since \( q_i \uparrow \infty \) as \( i \uparrow \infty \) and \( L \) is the sum of these operators, i.e.

\[ \tilde{L} \tilde{f}(h) = \sum_{i=1}^{\infty} L_i f_{i,h}(\langle h, h_i \rangle_{T_{\mu_0}}), \quad f_{i,h}(r) := \tilde{f}(h - \langle h, h_i \rangle_{T_{\mu_0}} h_i + rh_i), \]

\[ \tilde{\mathcal{E}}(\tilde{f}, \tilde{g}) := \int_{T_{\mu_0}} \langle \nabla \tilde{f}, \nabla \tilde{g} \rangle_{T_{\mu_0}} dG_Q = - \int_{T_{\mu_0}} (\tilde{f} \tilde{L} \tilde{g}) dG_Q, \quad \tilde{f}, \tilde{g} \in \mathcal{F} C^2_b(T_{\mu_0}). \]
the compactness in \( L \) is purely discrete with eigenvalues

\[
\sum_{i=1}^{n} \lambda_{i,k_i}, \quad n \geq 1, k_i \geq 0.
\]

According to the spectral theory, the pure discreteness of the spectrum for \( \tilde{L} \) is equivalent to the compactness in \( L^2(G_Q) \) of the associated Markov semigroup \( \tilde{P}_t := e^{Lt} \) for \( t > 0 \), they are also equivalent to the compactness in \( L^2(G_Q) \) of the set

\[
\{ \tilde{f} \in \mathcal{D}(\mathcal{E}) : \tilde{\mathcal{E}}_1(\tilde{f}) := \tilde{\mathcal{E}}(\tilde{f}, \tilde{f}) + G_Q(\tilde{f}^2) \leq 1 \}.
\]

Let \( C^1_b(T_{\mu_0}) \) be the class of all bounded functions on \( T_{\mu_0} \) with bounded and continuous Fréchet derivative. By an approximation argument, see the proof of Lemma 5.2 below for \( F = 0 \), we have \( \mathcal{D}(\tilde{\mathcal{E}}) \supset C^1_b(T_{\mu_0}) \) and (3.2) implies

\[
\tilde{\mathcal{E}}(\tilde{f}, \tilde{g}) = G_Q(\langle \nabla \tilde{f}, \nabla \tilde{g} \rangle_{T_{\mu_0}}) := \int_{T_{\mu_0}} \langle \nabla \tilde{f}, \nabla \tilde{g} \rangle_{T_{\mu_0}} dG_Q, \quad \tilde{f}, \tilde{g} \in C^1_b(T_{\mu_0}).
\]

### 3.2 OU process on \( \mathcal{P}_2 \)

We first introduce the Gaussian measure and the corresponding OU Dirichlet form on \( \mathcal{P}_2 \).

**Definition 3.1.** Let \( \Psi : T_{\mu_0} \to \mathcal{P}_2, \Psi(h) := \mu_0 \circ h^{-1} \).

1. \( N_{\mu_0,Q} := G_Q \circ \Psi^{-1} \) is called the Gaussian measure on \( \mathcal{P}_2 \) with parameter \((\mu_0, Q)\).

2. Define the following OU bilinear form on \( L^2(N_{\mu_0,Q}) \):

\[
\mathcal{D}(\mathcal{E}) := \{ f \in L^2(N_{\mu_0,Q}) : f \circ \Psi \in \mathcal{D}(\tilde{\mathcal{E}}) \},
\]

\[
\mathcal{E}(f,g) := \tilde{\mathcal{E}}(f \circ \Psi, g \circ \Psi), \quad f, g \in \mathcal{D}(\mathcal{E}).
\]

It is easy to see that \( L^2(N_{\mu_0,Q}) \) consists of measurable functions \( f \) on \( \mathcal{P}_2 \) such that \( f \circ \Psi \in L^2(G_Q) \), so that

\[
L^2(N_{\mu_0,Q}) = \{ \mu \mapsto G_Q(\tilde{f}|\Psi)|_{\Psi^{-1}(\mu)} : \tilde{f} \in L^2(G_Q) \},
\]

where \( G_Q(\tilde{f}|\Psi) \) is the conditional expectation of \( \tilde{f} \) with respect to \( G_Q \) given the sigma-algebra \( \sigma(\Psi) \) induced by \( \Psi \), which is constant on the atom \( \Psi^{-1}(\mu) := \{ h \in T_{\mu_0} : \Psi(h) = \mu \} \) of \( \sigma(\Psi) \) for each \( \mu \). It is easy to see that \( N_{\mu_0,Q} \) is shift-invariant in the following sense.

**Proposition 3.1.** Let \( \tilde{h} \in T_{\mu_0} \) be a homeomorphism on \( \mathbb{R}^d \). Then \( N_{\mu_0,Q} = N_{\mu_0 \circ \tilde{h}^{-1}, Q \circ \tilde{h}^{-1}} \) for \( Q \circ \tilde{h}^{-1} \) being the linear operator on \( T_{\mu_0 \circ \tilde{h}^{-1}} \) determined by

\[
(Q \circ \tilde{h}^{-1})\tilde{h}_n := q_n \tilde{h}_n, \quad n \geq 1,
\]

where \( \{ \tilde{h}_n \}_{n \geq 1} := \{ h_n \circ \tilde{h}^{-1} \}_{n \geq 1} \) is an ONB of \( T_{\mu_0 \circ \tilde{h}^{-1}} \).

We have the following result for the OU process on \( \mathcal{P}_2 \).
Theorem 3.2. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be defined above. Then the following assertions hold.

1. $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a conservative symmetric Dirichlet form on $L^2(N_{\mu_0,Q})$ with $\mathcal{D}(\mathcal{E}) \supset C^1_b(\mathcal{P}_2)$ and

$$\mathcal{E}(f,g) = \int_{\mathcal{P}_2} \langle Df(\mu), Dg(\mu) \rangle_{T_{\mu}N_{\mu_0,Q}} \, d\mu, \quad f, g \in C^1_b(\mathcal{P}_2).$$

Moreover, the following log-Sobolev inequality holds:

$$N_{\mu_0,Q}(f^2 \log f^2) \leq \frac{2}{q_1} \mathcal{E}(f,f), \quad f \in \mathcal{D}(\mathcal{E}), \quad N_{\mu_0,Q}(f^2) = 1.$$  

2. The generator $(L, \mathcal{D}(L))$ of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ has discrete spectrum, and satisfies

$$\mathcal{D}(L) \supset \tilde{\mathcal{D}}(L) := \{ f \in L^2(N_{\mu_0,Q}) : f \circ \Psi \in \mathcal{D}(\tilde{L}) \}, \quad Lf(\mu) = G_Q(\tilde{L}(f \circ \Psi)\mid \Psi = \mu) := G_Q(\tilde{L}(f \circ \Psi)\mid \Psi = \mu), \quad f \in \tilde{\mathcal{D}}(L).$$

3. Let $P_t$ be the associated Markov semigroup of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. Then $P_t$ is compact in $L^2(N_{\mu_0,Q})$ for any $t > 0$, $P_t$ converges exponentially to $N_{\mu_0,Q}$ in entropy:

$$N_{\mu_0,Q}((P_t f) \log P_t f) \leq e^{-2q_1 t} N_{\mu_0,Q}(f \log f), \quad t \geq 0, \quad 0 \leq f, \quad N_{\mu_0,Q}(f) = 1,$$

and it is hypercontractive:

$$\|P_t\|_{L^p(N_{\mu_0,Q}) \to L^{p_t}(N_{\mu_0,Q})} := \sup_{\|f\|_{L^p(N_{\mu_0,Q})} \leq 1} \|P_t f\|_{L^{p_t}(N_{\mu_0,Q})} \leq 1,$$

$$t > 0, \quad p > 1, \quad p_t := 1 + (p-1)e^{2q_1 t}.$$  

**Proof.** (1) We first prove that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is closed and holds, so that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a symmetric conservative Dirichlet form in $L^2(N_{\mu_0,Q})$.

Let

$$\mathcal{E}_1(f) := \mathcal{E}(f,f) + \|f\|_{L^2(N_{\mu_0,Q})}^2, \quad \tilde{\mathcal{E}}_1(f) := \tilde{\mathcal{E}}(f,f) + \|f\|_{L^2(G_Q)}^2.$$  

Let $\{f_n\}_{n \geq 1} \subset \mathcal{D}(\mathcal{E})$ such that

$$\lim_{m,n \to \infty} \mathcal{E}_1(f_n - f_m) = 0.$$  

Then $f := \lim_{n \to \infty} f_n$ exists in $L^2(N_{\mu_0,Q})$ and by definition, $\{f_n \circ \Psi\}_{n \geq 1} \subset \mathcal{D}(\tilde{\mathcal{E}})$ with

$$\lim_{m,n \to \infty} \tilde{\mathcal{E}}_1(f_n \circ \Psi - f_m \circ \Psi) = 0.$$  

Thus, the closed property of $(\tilde{\mathcal{E}}, \tilde{\mathcal{D}}(\tilde{\mathcal{E}}))$ implies

$$f \circ \Psi = \lim_{n \to \infty} f_n \circ \Psi \in \mathcal{D}(\tilde{\mathcal{E}}),$$

so that $f \in \mathcal{D}(\mathcal{E})$ by definition. Thus, $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is closed.
On the other hand, let \( f \in C^1_b(\mathcal{P}_2) \). By Proposition 2.2 for the reference probability \( \mu_0 \), we see that for any \( h \),

\[
\nabla_\phi(f \circ \Psi)(h) := \frac{d}{ds} \bigg|_{s=0} (f \circ \Psi)(h + s\phi) = \langle \phi, (Df)(\Psi(h))(h) \rangle_{T_{\mu_0}}, \quad \phi \in T_{\mu_0},
\]

so that

\[
(3.8) \quad \nabla(f \circ \Psi)(h) = (Df)(\Psi(h)) \circ h, \quad f \in C^1_b(\mathcal{P}_2), \quad h \in T_{\mu_0}.
\]

Hence, \( f \in C^1_b(\mathcal{P}_2) \) implies \( f \circ \Psi \in C^1_b(T_{\mu_0}) \subset \mathcal{D}(\hat{\mathcal{L}}) \), so that \( f \in \mathcal{D}(\mathcal{E}) \) by definition.

It remains to verify (3.4), which together with (3.3) implies (3.5). By (3.8) and \( N_{\mu_0,Q} = G_Q \circ \Psi^{-1} \),

\[
\mathcal{E}(f,g) := \hat{\mathcal{E}}(f \circ \Psi, g \circ \Psi) = \int_{T_{\mu_0}} \langle \nabla(f \circ \Psi), \nabla(g \circ \Psi) \rangle_{T_{\mu_0}} dG_Q
\]

is compact in \( L^2(G_Q) \). By the definition of \( N_{\mu_0,Q} \) and \( (\mathcal{E}, \mathcal{D}(\mathcal{E})) \), this implies that the set

\[
\{ f \in \mathcal{D}(\mathcal{E}) : \mathcal{E}_1(f) := \mathcal{E}(f,f) + \|f\|^2_{L^2(N_{\mu_0,Q})} \leq 1 \}
\]

is compact in \( L^2(N_{\mu_0,Q}) \). So, \( L \) has purely discrete spectrum.

Next, let \( f \in L^2(N_{\mu_0,Q}) \) such that \( f \circ \Psi \in \mathcal{D}(\hat{\mathcal{L}}) \). By the definition of \( (\mathcal{E}, \mathcal{D}(\mathcal{E})) \), we obtain

\[
\int_{\mathcal{P}_2} g(\mu)G_Q(\hat{\mathcal{L}}(f \circ \Psi)|\Psi = \mu)N_{\mu_0,Q}(d\mu) = \int_{T_{\mu_0}} (g \circ \Psi)\hat{\mathcal{L}}(f \circ \Psi)dG_Q
\]

Thus, \( f \in \mathcal{D}(L) \) and \( Lf(\mu) = G_Q(\hat{\mathcal{L}}(f \circ \Psi)|\Psi = \mu) \).

(3) The log-Sobolev inequality (3.5) is equivalent to each of (3.6) and (3.7), see [25]. Moreover, by the spectral theory, since \( L \) has purely discrete spectrum, \( P_t \) is compact in \( L^2(N_{\mu_0,Q}) \) for \( t > 0 \).
Remark 3.1. (1) Theorem 3.2 implies that \((\mathcal{E}, C^1_b(\mathcal{P}_2))\) is closable in \(L^2(N_{\mu_0,Q})\). We wonder if the closure coincides with \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\) or not.

(2) In general, \(\mathcal{D}(L) \neq \tilde{\mathcal{D}}(L)\). For any \(f \in \mathcal{D}(L)\), we have

\[
\tilde{\mathcal{E}}(g \circ \Psi, f \circ \Psi) = \mathcal{E}(g, f) = -N_{\nu_0,Q}(g(Lf)) = G_Q((g \circ \Psi)(Lf) \circ \Psi), \quad g \in \mathcal{D}(\mathcal{E}).
\]

If \(\{g \circ \Psi : g \in \mathcal{D}(\mathcal{E})\}\) is dense in \(\mathcal{D}(\mathcal{E})\), this would imply that \(f \circ \Psi \in \mathcal{D}(\tilde{L})\) and \(\tilde{L}(f \circ \Psi) = (Lf) \circ \Psi\). If so, the OU process on \(\mathcal{P}_2\) starting at \(\nu_0\) could be constructed as \(\nu_t = \Psi(h_t)\) for \(h_t\) in (3.1) with \(\nu_0 = \Psi(h_0)\). However, in general this is not true, as there might be different \(h_0\) satisfying \(\nu_0 = \Psi(h_0)\) and the corresponding \(\Psi(h_t)\) may have different distributions. In the next section, we formulate \(L\) as Laplacian with a drift on \(\mathcal{P}_2\).

4 Generator as Laplacian with drift

We shall introduce a subclass of \(\mathcal{D}(L)\), such that for functions in this class the generator is formulated as

\[
Lf(\mu) = \Delta_{\mathcal{P}_2}f(\mu) - \langle b(\mu), D_f(\mu) \rangle_{T_{\mu}}
\]

for a drift \((b, \mathcal{D}(b)):\)

\[
b : \mathcal{P}_2 \supset \mathcal{D}(b) \ni \mu \mapsto b(\mu) \in T_{\mu}.
\]

This is compatible with the \(d\)-dimensional case where the OU process has a generator of type

\[
L_0f(x) = \Delta f(x) - (Ax) \cdot \nabla f(x)
\]

for a positive definite \(d \times d\)-matrix \(A\).

Definition 4.1. Let \(\mathcal{D}\) be the space of functions \(f \in C^2_b(\mathcal{P}_2)\) such that for \(G_Q\)-a.e. \(h\),

\[
(Df(\mu_0 \circ h^{-1})) \circ h \in \mathcal{D}(Q), \quad f \in \mathcal{D}(\Delta_{\mu_0 \circ h^{-1}}),
\]

and

\[
\int_{T_{\mu_0}} \left| \Delta_{\mathcal{P}_2}f(\mu_0 \circ h^{-1}) - \langle h, Q[(Df(\mu_0 \circ h^{-1})) \circ h]\rangle_{T_{\mu_0}} \right|^2 G_Q(dh) < \infty.
\]

Remark 4.1. (1) As explained in Remark 2.2(3) that \(\mathcal{D}\) seems too small to be dense in \(L^2(N_{\mu_0,Q})\), so the following characterization (4.2) on the generator is only formal. It would be interesting if one could make this formula meaningful in a weak sense for a dense class of functions.

(2) Next, we show that \(\mathcal{D}\) may contain functions given in Remark 2.2(2) of type

\[
f = g(f_1, \ldots, f_d), \quad g \in C^\infty_b(\mathbb{R}^d), f_i(\mu) := \mu(\langle \cdot, e_i \rangle).
\]

Indeed, by Remark 2.2(2) we have \(f \in \mathcal{D}(\Delta_{\mathcal{P}_2})\) and \(\|\Delta_{\mathcal{P}_2}f\|_\infty < \infty\). Moreover, noting that

\[
f_i(\mu_0 \circ h^{-1}) = (\mu_0 \circ h^{-1})(\langle e_i, \cdot \rangle) = \mu_0(\langle e_i, h \rangle) = \langle e_i, h \rangle_{T_{\mu_0}},
\]
we have
\[
\left| \langle h, Q[(Df(\mu_0 \circ h^{-1})) \circ h] \rangle_{T_{\mu_0}} \right|^2
\]
\[
= \left| \sum_{i=1}^{d} \sum_{m \geq 1} q_m \langle h, m \rangle_{T_{\mu_0}} (\partial_i g)(\langle h, e_1 \rangle_{T_{\mu_0}}, \ldots, \langle h, e_d \rangle_{T_{\mu_0}}) \langle h, m, e_i \rangle_{T_{\mu_0}} \right|^2
\]
\[
\leq \left( \sum_{i=1}^{d} \sum_{m \geq 1} \langle h, m \rangle_{T_{\mu_0}}^2 \right) \sum_{i=1}^{d} \sum_{m \geq 1} q_m^2 (\partial_i g)(\langle h, e_1 \rangle_{T_{\mu_0}}, \ldots, \langle h, e_d \rangle_{T_{\mu_0}}) \langle e_i, h, m \rangle_{T_{\mu_0}}^2.
\]
Since
\[
\int_{T_{\mu_0}} \left( \sum_{i=1}^{d} \sum_{m \geq 1} \langle h, m \rangle_{T_{\mu_0}}^2 \right) G_Q(dh) = d \sum_{m \geq 1} q_m^{-1} < \infty,
\]
we conclude that \( f \in \mathcal{D} \) provided
\[
(4.1) \quad \sum_{m \geq 1} q_m^2 \langle e_i, h, m \rangle_{T_{\mu_0}}^2 < \infty, \quad 1 \leq i \leq d,
\]
which may be verified by showing that \( e_i \) is in a Sobolev type space. For instance, let \( d = 1 \) (the higher dimension cases can be discussed in the same way) and let \( \mu_0 \) be the standard Gaussian measure such that \( L_0 := \Delta - x \cdot \nabla \) in \( L^2(\mu_0) \) has discrete spectrum \( \{-\lambda_m := -m-1\}_{m \geq 1} \) and eigenbasis \( \{h_m\}_{m \geq 1} \) with \( \mu_0(h_m) = 0 \) for \( m \geq 1 \). In this case \( e_1 = 1 \) so that \( \langle e_1, h, m \rangle_{T_{\mu_0}} = 0 \) for \( m \geq 1 \) hence \( (4.1) \) holds.

**Theorem 4.1.** We have \( \mathcal{D} \subset \mathcal{D}(L) \) and for \( U_Q^f(h) := \langle h, Q[(Df(\mu_0 \circ h^{-1})) \circ h] \rangle_{T_{\mu_0}} \),
\[
Lf(\mu) = \Delta \Psi f(\mu) - G_Q(U_Q^f|\Psi = \mu), \quad f \in \mathcal{D}.
\]
Formally, we may write \( Lf(\mu) = \Delta \Psi f(\mu) - (b(\mu), Df(\mu))_{T_\mu} \) where the drift is given by
\[
b(\mu) = G_Q((Qh \circ h^{-1}|\Psi(h) = \mu).
\]
Proof. (a) Let \( h \in T_{\mu_0} \) and \( \mu := \mu_0 \circ h^{-1} \). Recall that for any \( \tilde{h} \in T_{\mu_0} \), \( \tilde{h} \circ h^{-1} \in T_\mu \) is determined by
\[
(4.3) \quad \langle \tilde{h} \circ h^{-1}, \phi \rangle_{T_\mu} = \langle \tilde{h}, \phi \circ h \rangle_{T_{\mu_0}}, \quad \phi \in T_\mu.
\]
Then by Proposition 2.2 for the reference probability \( \mathbb{P} = \mu_0 \), we obtain
\[
\nabla_{\tilde{h}}(f \circ \Psi)(h) := \lim_{\varepsilon \downarrow 0} \frac{(f \circ \Psi)(h + \varepsilon \tilde{h}) - (f \circ \Psi)(h)}{\varepsilon}
\]
\[
= \lim_{\varepsilon \downarrow 0} \frac{f(\mu_0 \circ (h + \varepsilon \tilde{h})^{-1}) - f(\mu_0 \circ h^{-1})}{\varepsilon} = \int_{\mathbb{R}^d} \langle Df(\mu)(h), \tilde{h} \rangle d\mu_0
\]
\[
= \langle Df(\mu), \tilde{h} \circ h^{-1} \rangle_{T_\mu} = D_{\tilde{h} \circ h^{-1}}f(\mu), \quad \tilde{h}, h \in T_{\mu_0}, \quad \mu = \mu_0 \circ h^{-1}.
\]
(b) By definition, for any \( f \in \mathcal{D} \), \( Lf \) given in (4.2) is a well-defined function in \( L^2(N_{\mu_0,Q}) \). It suffices to prove the integration by parts formula

\[
\mathcal{E}(f, g) = - \int_{\mathcal{P}_2} g(\mu)Lf(\mu)N_{\mu_0,Q}(d\mu), \quad g \in C^1_b(\mathcal{P}_2).
\]

Simply denote \( \tilde{f} = f \circ \Psi \) and \( \tilde{g} = g \circ \Psi \). By (3.4) and the integration by parts formula for \( G_Q \), we obtain

\[
\mathcal{E}(f, g) = \int_{T_{\mu_0}} \langle \nabla \tilde{f}, \nabla \tilde{g} \rangle dG_Q = \sum_{n=1}^\infty \int_{T_{\mu_0}} (\nabla_{h_n} \tilde{f}) (h) (\nabla_{h_n} \tilde{g}) (h) G_Q(dh)
\]

\[
= \sum_{n=1}^\infty \int_{T_{\mu_0}} [\nabla_{h_n} (\langle \nabla_{h_n} \tilde{f} \rangle \tilde{g}) (h) - \tilde{g}(h) \nabla_{h_n} \nabla_{h_n} \tilde{f} (h)] G_Q(dh)
\]

\[
= \sum_{n=1}^\infty \int_{T_{\mu_0}} \tilde{g}(h) \left[ q_n \langle h_n, h_n \rangle_{T_{\mu_0}} (\nabla_{h_n} \tilde{f}) (h) - \nabla_{h_n} \nabla_{h_n} \tilde{f} (h) \right] G_Q(dh).
\]

By (3.4),

\[
\sum_{n=1}^\infty q_n \langle h_n, h_n \rangle_{T_{\mu_0}} (\nabla_{h_n} \tilde{f}) (h) = \sum_{n=1}^\infty q_n \langle h_n, h_n \rangle_{T_{\mu_0}} \langle h_n, \nabla \tilde{f}(h) \rangle_{T_{\mu_0}}
\]

\[
= \sum_{n=1}^\infty q_n \langle h_n, h_n \rangle_{T_{\mu_0}} \langle h_n, (Df(\mu_0 \circ h^{-1}) \circ h) \rangle_{T_{\mu_0}} = \langle h, Q[(Df(\mu_0 \circ h^{-1}) \circ h)] \rangle_{T_{\mu_0}},
\]

so that

\[
\sum_{n=1}^\infty \int_{T_{\mu_0}} \tilde{g}(h) q_n \langle h_n, h_n \rangle_{T_{\mu_0}} \langle h_n, \nabla \tilde{f}(h) \rangle_{T_{\mu_0}} G_Q(dh)
\]

\[
= \int_{\mathcal{P}_2} g(\mu) G_Q \left( \langle h, Q[(Df(\mu_0 \circ h^{-1}) \circ h)] \rangle_{T_{\mu_0}} \bigg| \Psi(h) = \mu \right) N_{\mu_0,Q}(d\mu).
\]

By Proposition 2.2 (4.1) also implies

\[
\nabla_{h_n} \nabla_{h_n} \tilde{f}(h) = \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \mu_0 \langle ((Df)(\mu_0 \circ (h + \varepsilon h_n)^{-1})(h + \varepsilon h_n), h_n) \rangle
\]

\[
= \int \langle (D^2 f)(\mu_0 \circ h^{-1})(h(x), h(y)), h_n(x) \otimes h_n(y) \rangle_{HS} \mu_0(dx) \mu_0(dy)
\]

\[
+ \int \langle (\nabla Df)(\mu_0 \circ h^{-1})(h(x)), h_n(x) \otimes h_n(x) \rangle_{HS} \mu_0(dx)
\]

\[
= I_1(n) + I_2(n),
\]

where \( \mu = \mu_0 \circ h^{-1} \).

\[
I_1(n) := \int \langle (D^2 f)(\mu_0 \circ h^{-1})(x, y), (h_n \circ h^{-1})(x) \otimes (h_n \circ h^{-1})(y) \rangle_{HS} \mu(dx) \mu(dy),
\]
\[ I_2(n) := \int_{\mathbb{R}^d} \langle (\nabla Df)(\mu_0 \circ h^{-1})(x), (h_n \circ h^{-1})(x) \otimes (h_n \circ h^{-1})(x) \rangle_{HS\mu}(dx). \]

Let \(\{\phi_m\}_{m \geq 1}\) be an ONB of \(T_\mu := L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu)\). By (4.3),

\[
(h_n \circ h^{-1}) = \sum_{m \geq 1} \langle h_n \circ h^{-1}, \phi_m \rangle_{T_\mu} \phi_m = \sum_{m \geq 1} \langle h_n, \phi_m \circ h \rangle_{T_{\mu_0}} \phi_m,
\]

so that

\[
\sum_{n=1}^\infty I_1(n) = \sum_{m,l \geq 1} \sum_{n=1}^\infty \mu_0(\langle h_n, \phi_m \circ h \rangle) \mu_0(\langle h_n, \phi_l \circ h \rangle)
\cdot \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle (D^2 f)(\mu)(x, y), \phi_m(x) \otimes \phi_l(y) \rangle_{HS\mu}(dx) \mu(dy)
\]

\[
= \sum_{m,l \geq 1} \mu(\langle \phi_m \circ h, \phi_l \circ h \rangle) \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle (D^2 f)(\mu)(x, y), \phi_m(x) \otimes \phi_l(y) \rangle_{HS\mu}(dx) \mu(dy)
\]

\[
= \sum_{m,l \geq 1} \mu(\langle \phi_m, \phi_l \rangle) \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle (D^2 f)(\mu)(x, y), \phi_m(x) \otimes \phi_m(y) \rangle_{HS\mu}(dx) \mu(dy)
\]

Similarly,

\[
\sum_{n=1}^\infty I_2(n) = \sum_{m,l \geq 1} \sum_{n=1}^\infty \mu_0(\langle h_n, \phi_m \circ h \rangle) \mu_0(\langle h_n, \phi_l \circ h \rangle)
\cdot \int_{\mathbb{R}^d} \langle (\nabla Df)(\mu)(x), \phi_m(x) \otimes \phi_l(x) \rangle_{HS\mu}(dx)
\]

\[
= \sum_{m \geq 1} \int_{\mathbb{R}^d} \langle (\nabla Df)(\mu)(x), \phi_m(x) \otimes \phi_m(x) \rangle_{HS\mu}(dx).
\]

These together with (4.8) and Proposition 2.3 yield

\[
\sum_{n=1}^\infty \nabla h_n \nabla h_n \tilde{f}(h) = \Delta_{\mathcal{P}_2} f(\mu_0 \circ h^{-1}).
\]

Combining this with (4.6) and (4.7), we prove (4.2).

\[ \square \]

### 5 Perturbation of the OU process

Let \(V\) be a measurable function on \(\mathcal{P}_2\) such that

\[
N_{\mu_0, Q}^V(d\mu) := e^{V(\mu)} N_{\mu_0, Q}(d\mu)
\]
is a probability measure on $\mathcal{P}_2$. We consider the pre-Dirichlet form

$$E^V(f, g) := \int_{\mathcal{P}_2} \langle Df(\mu), Dg(\mu) \rangle_{T_\mu} N^V_{\mu_0, Q}(d\mu), \quad f, g \in C^1_b(\mathcal{P}_2).$$

If this form is closable in $L^2(N^V_{\mu_0, Q})$, then its closure $(E^V, \mathcal{D}(E^V))$ is a symmetric conservative Dirichlet form, whose generator can be formally written as

$$LVf(\mu) = Lf(\mu) + \langle DV(\mu), Df(\mu) \rangle_{T_\mu}.$$

We call the associated Markov process a perturbation of the OU process.

A simple situation is that $V$ is bounded. In this case, the closability of $(E^V, C^1_b(\mathcal{P}_2))$ follows from that of $(E, C^1_b(\mathcal{P}_2))$, and (3.3) implies the log-Sobolev inequality (see [18])

$$N^V_{\mu_0, Q}(f^2 \log f^2) \leq \frac{2}{q_1} e^{\sup V - \inf V} E^V(f, f), \quad f \in \mathcal{D}(E^V), \quad N^V_{\mu_0, Q}(f^2) = 1.$$

Consequently, the associate Markov semigroup $P^V_t$ is hypercontractive and exponentially convergent in entropy. Moreover, the compactness of $\{f \in \mathcal{D}(E) : E_1(f) \leq 1\}$ in $L^2(N_{\mu_0, Q})$ implies that of $\{f \in \mathcal{D}(E^V) : E^V_1(f) \leq 1\}$ in $L^2(N^V_{\mu_0, Q})$, so that the generator $LV$ has empty essential spectrum and $P^V_t$ is compact in $L^2(N^V_{\mu_0, Q})$ for $t > 0$. In the following, we intend to extend these to unbounded perturbation $V$.

In the framework of local Dirichlet forms, unbounded perturbations have been studied in many papers, where the key points are to prove the closability of the pre-Dirichlet form and to see which functional inequalities of the original Dirichlet form can be kept under the perturbation, see for instance [3, 7, 38] and references therein. However, in all of related references one needs an algebra of bounded measurable functions $\mathcal{A} \subset \mathcal{D}(L)$ which is dense in $\mathcal{D}(L)$ such that the square field is given by

$$(5.1) \quad \Gamma(f, g) = \frac{1}{2} (L(fg) - fLg - gLf), \quad f, g \in \mathcal{A}.$$

In the present situation, the square field reads

$$\Gamma(f, g)(\mu) = \langle Df(\mu), Dg(\mu) \rangle_{T_\mu}, \quad \mu \in \mathcal{P}_2, \quad f, g \in C^1_b(\mathcal{P}_2).$$

But we do not have explicit choice of the algebra $\mathcal{A}$ such that (5.1) holds. Therefore, we again come back to the tangent space $T_{\mu_0}$ by considering the following probability measure on $T_{\mu_0}$:

$$G^V_Q(dh) := e^{(V \circ \Psi)(h)} G_Q(dh),$$

and the corresponding bilinear form

$$(5.2) \quad \tilde{E}^V(\tilde{f}, \tilde{g}) := \int_{T_{\mu_0}} \langle \nabla \tilde{f}, \nabla \tilde{g} \rangle_{T_{\mu_0}} dG^V_Q, \quad \tilde{f}, \tilde{g} \in C^1_b(T_{\mu_0}).$$

For any $r \in \mathbb{R}$, let $r^+ := \max\{0, r\}$ and $r^- := (-r)^+$. By studying properties of $\tilde{E}^V$, we obtain the following result under assumption
Theorem 5.1. Assume according to the proof of Theorem 3.2, Theorem 5.1 is a consequence of the following Lemma

(A) $V \in C^1(\mathcal{P}_2)$ such that $dN_{\mu_0,Q}^V := e^V dN_{\mu_0,Q}$ is a probability measure on $\mathcal{P}_2$, and there exists $p > 1$ such that

$$\int_{\mathcal{P}_2} \left( \|DV(\mu)\|_{T_\mu} e^{V(\mu)^+} + \|DV(\mu)\|^p_{T_\mu} \right) N_{\mu_0,Q}(d\mu) < \infty.$$

Theorem 5.1. Assume (A). Then the following assertions hold.

1. $(\mathcal{E}^V, C^1_b(\mathcal{P}_2))$ is closable in $L^2(N_{\mu_0,Q}^V)$, and the closure $(\mathcal{E}^V, \mathcal{D}(\mathcal{E}^V))$ is a symmetric conservative Dirichlet form.

2. If there exists $\lambda > \frac{1}{2p} \varepsilon$ such that $N_{\mu_0,Q}(e^{\lambda \|DV\|^2}) < \infty$, where $\|DV\|(\mu) := \|DV(\mu)\|_{T_\mu}$, then the associated Markov semigroup $P_t^V$ is compact in $L^2(N_{\mu_0,Q}^F)$ for $t > 0$.

2. If there exists $\varepsilon > 0$ such that

$$\int_{\mathcal{P}_2} \left( e^{\frac{1}{2p}\|DV\|^2} + e^{V^+ + \varepsilon V^-} \right) dN_{\mu_0,Q} < \infty,$$

then there exists a constant $c > 0$ such that

$$N_{\mu_0,Q}^V(f^2 \log f^2) \leq c \mathcal{E}^V(f, f), \quad f \in \mathcal{D}(\mathcal{E}^V), \quad N_{\mu_0,Q}^V(f^2) = 1.$$

Consequently, for any $t > 0$,

$$\|P_t^V\|_{L^p(N_{\mu_0,Q}^V) \to L^p(N_{\mu_0,Q}^F)} \leq 1, \quad p > 1, \quad p_t = 1 + (p-1)e^{4t/c},$$

$$N_{\mu_0,Q}^V((P_t^V f) \log P_t^V f) \leq e^{-4t/c} N_{\mu_0,Q}^V(f \log f), \quad f \geq 0, \quad N_{\mu_0,Q}^V(f) = 1.$$

Noting that $N_{\mu_0,Q} = G_Q \circ \Psi^{-1}, \quad dG_Q^V := e^{V \circ \Psi} dG_Q$ is a probability measure on $T_{\mu_0}$, and

$$\mathcal{E}^V(f, g) = \int_{T_{\mu_0}} \langle \nabla \tilde{f}, \nabla g \rangle_{T_{\mu_0}} dG_Q^V, \quad \tilde{f}, \tilde{g} \in C^1_b(T_{\mu_0}),$$

according to the proof of Theorem 3.2. Theorem 5.1 is a consequence of the following Lemma 5.2 for $F = V \circ \Psi$, where the condition $\|\nabla F\|_{T_{\mu_0}} e^{F^+} + \|\nabla F\|^p_{T_{\mu_0}} \in L^1(G_Q)$ for some $p > 1$ is much weaker than $e^{\|\nabla F\|^2_{T_{\mu_0}} + |F|} \in \cap_{p > 1} L^p(G_Q)$ used in [3] Proposition 3.2. We will use the dimension-free Harnack inequality and Bismut formula for $P_t$ to prove the closability under this weaker condition. Moreover, the condition (5.3) for the log-Sobolev inequality is slightly better that than in [2] Lemma 4.1 where $F^+ + \varepsilon F^-$ is replaced by $(1 + \varepsilon)|F|$.

Lemma 5.2. Let $F \in C^1(T_{\mu_0})$ such that $G_Q^F(dh) := e^{F(h)} G_Q(dh)$ is a probability measure on $T_{\mu_0}$ and $\|\nabla F\|_{T_{\mu_0}} e^{F^+} + \|\nabla F\|^p_{T_{\mu_0}} \in L^1(G_Q)$ for some constant $p > 1$. Then:

1. The bilinear form

$$\tilde{\mathcal{E}}^F(f, g) := \int_{T_{\mu_0}} \langle \nabla \tilde{f}, \nabla g \rangle_{T_{\mu_0}} dG_Q^F, \quad \tilde{f}, \tilde{g} \in \mathcal{F} C^1_b(T_{\mu_0})$$

is closable in $L^2(G_Q^F)$, and the closure $(\tilde{\mathcal{E}}^F, \mathcal{D}(\tilde{\mathcal{E}}^F))$ is a symmetric conservative Dirichlet form. Moreover, $\mathcal{D}(\tilde{\mathcal{E}}^F) \supset C^1_b(T_{\mu_0})$. 

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(2) If there exists $\lambda > \frac{1}{2\nu}$ such that $G_Q(e^{\lambda \|\nabla F\|^2_{T_{t_0}}}) < \infty$, then the associated Markov semi-group $\tilde{P}_t^F$ is compact in $L^2(G_Q^F)$ for $t > 0$.

(3) If there exists $\varepsilon > 0$ such that
\begin{equation}
\int_{T_{t_0}} e^{\frac{1+\varepsilon}{\nu}\|\nabla F\|^2_{T_{t_0}} + F^+ + \varepsilon F^-} \, dG_Q < \infty,
\end{equation}
then there exists a constant $c > 0$ such that
\begin{equation}
G_Q(\tilde{f}^2 \log \tilde{f}^2) \leq c \tilde{e}^F(\tilde{f}, \tilde{f}), \quad \tilde{f} \in \mathcal{D}(\tilde{e}^F), \quad G_Q^F(\tilde{f}^2) = 1.
\end{equation}

Proof. (1) We will establish the integration by parts formula
\begin{equation}
\tilde{e}^F(\tilde{f}, \tilde{g}) = -\int_{T_{t_0}} \tilde{g}(\tilde{V} \tilde{f}) \, dG_Q^F, \quad \tilde{f}, \tilde{g} \in \mathcal{F}C^2_b(T_{t_0})
\end{equation}
for $\tilde{V} \tilde{f} := \tilde{L} \tilde{f} + \langle \nabla F, \nabla \tilde{f} \rangle_{T_{t_0}}$, so that $(\tilde{e}^F, \mathcal{F}C^1_b(T_{t_0}))$ is closable. Since a function in $\mathcal{F}C^1_b$ can be approximated by functions in $\mathcal{F}C^2_b(T_{t_0})$ under the $C^1_b$-norm, this also implies that $(\tilde{e}^V, \mathcal{F}C^1_b(T_{t_0}))$ is closable.

To this end, we make approximations of $F$. Let $\varphi \in C^\infty(\mathbb{R})$ such that $\varphi(r) = r$ for $|r| \leq 1$, $1 \geq \varphi' \geq 0$ and $\varphi(r) = 0$ for $|r| \geq 2$. For any $m, n \geq 1$, let
\begin{equation}
F_m := m\varphi(F/m), \quad F_{m,n} := \tilde{P}_{t_0}^n F_m.
\end{equation}
We have
\begin{equation}
F_m \in C_b(T_{t_0}) \cap C^1(T_{t_0}), \quad \|\nabla F_m\| \leq \|\nabla F\|.
\end{equation}
Since $\tilde{P}_t f(h_0) = \mathbb{E}[f(h_t)]$ for $h_t$ in (5.1), by [13] Theorem 3.2.1 and Theorem 3.2.2] for $A = -Q, b = 0$ and $\sigma(t) = \sqrt{2}$, we have the Harnack inequality
\begin{equation}
(\tilde{P}_t \tilde{f}(h + v))^p \leq (\tilde{P}_t \tilde{f}^p(h))e^{\frac{p}{2} - \mu \frac{|v|^2}{2} \|\nabla F\|^2_{T_{t_0}}}, \quad \tilde{f} \geq 0, p > 1, h, v \in T_{t_0},
\end{equation}
and the Bismut formula
\begin{equation}
\nabla \tilde{P}_t \tilde{f}(h_0) = \sqrt{\frac{2}{t}} \mathbb{E} \left[ \tilde{f}(h_t) \int_0^t e^{-Qs} \, dW_s \right], \quad t > 0, \tilde{f} \in \mathcal{B}(T_{t_0}).
\end{equation}
By (5.10), we see that $F_{m,n} \in C^1_b(T_{t_0})$, and (5.9) together with (5.8) and $Q \geq 0$ implies
\begin{align*}
|\tilde{P}_t F_m(h_0 + \varepsilon v) - \tilde{P}_t F_m(h_0)| &\leq \mathbb{E}|F_m(\varepsilon e^{-Qt} v + h_t) - F_m(h_t)| \\
&\leq \mathbb{E} \int_0^\varepsilon \|\nabla F_m\|_{T_{t_0}} (re^{-Qt} v + h_t) \, dr \\
&\leq \int_0^\varepsilon (\tilde{P}_t \|\nabla F_m\|^2_{T_{t_0}})^{\frac{1}{2}} (h_0) e^{\frac{2|v|^2}{2}} \, dr.
\end{align*}
\[ \leq \varepsilon (\tilde{P}_n \| \nabla F \|^p_{\mu_0}) \frac{\varepsilon^2 |u|^2}{\| h_0 \| e^{2(p-1)}} \], \ \varepsilon > 0, h_0, v \in T_{\mu_0}. \]

By letting \( \varepsilon \downarrow 0 \) we derive

\[(5.11) \quad \| \nabla F_{m,n} \|_{\mu_0} = \| \tilde{P}_n F_m \|_{\mu_0} \leq (\tilde{P}_n \| \nabla F \|^p_{\mu_0})^{\frac{1}{p}}.\]

Since \( F_{m,n} \in C^1_b(T_{\mu_0}) \), the integration by parts formula for \( G_Q \) yields

\[
\int_{T_{\mu_0}} \langle \nabla \tilde{f}, \nabla \tilde{g} \rangle T_{\mu_0} e^{F_{m,n}} dG_Q = \int_{T_{\mu_0}} \left( \nabla \nabla \tilde{f}(e^{F_{m,n}} \tilde{g}) - e^{F_{m,n}} \tilde{g} \nabla \nabla F_{m,n} \right) dG_Q \\
= - \int_{T_{\mu_0}} e^{F_{m,n}} \tilde{g} (\tilde{L} \tilde{f} + \nabla \nabla F_{m,n} \tilde{f}) dG_Q \\
= - \int_{T_{\mu_0}} (\tilde{g}(\tilde{L} + \nabla \nabla F_{m,n}) \tilde{f}) e^{F_{m,n}} dG_Q, \quad \tilde{f}, \tilde{g} \in \mathscr{F} C^1_b(T_{\mu_0}).
\]

Noting that \( (5.11) \) implies

\[ |\tilde{g}(\tilde{L} + \nabla \nabla F_{m,n}) \tilde{f}| e^{F_{m,n}} \leq c_m (1 + \tilde{P}_n \| \nabla F \|^p_{\mu_0})^{\frac{1}{p}}, \quad n \geq 1 \]

for some constant \( c_m > 0 \), which are bounded in \( L^p(G_Q) \) since

\[ G_Q(\tilde{P}_n \| \nabla F \|^p_{\mu_0}) = G_Q(\| \nabla F \|^p_{\mu_0}) < \infty, \]

by the dominated convergence theorem we may let \( n \to \infty \) to derive

\[
\int_{T_{\mu_0}} \langle \nabla \tilde{f}, \nabla \tilde{g} \rangle T_{\mu_0} e^{F_m} dG_Q = - \int_{T_{\mu_0}} (\tilde{g}(\tilde{L} + \nabla \nabla F_m) \tilde{f}) e^{F_m} dG_Q.
\]

Since \( (5.8) \) and \( e^F + \| \nabla F \|_{T_{\mu_0}} e^{F^+} \in L^1(G_Q) \) implies

\[ |\tilde{g}(\tilde{L} + \nabla \nabla F_m) \tilde{f}| e^{F_m} \leq c (1 + \| \nabla F \|_{T_{\mu_0}}) e^{F^+} \in L^1(G_Q), \]

by using the dominated convergence theorem again, we may let \( m \to \infty \) to get \( (5.7) \).

Next, let \( \tilde{f} \in C^1_b(T_{\mu_0}) \), and, for any \( n \geq 1 \) let

\[ \tilde{f}_n := \tilde{f} \circ \pi_n, \quad \pi_n h := \sum_{i=1}^{n} \langle h_i, h_i \rangle_{T_{\mu_0}} h_i. \]

Then \( \{ \tilde{f}_n \}_{n \geq 1} \subset \mathscr{F} C^1_b(T_{\mu_0}) \subset \mathcal{D}(\mathcal{E}^F) \), and

\[
\lim_{n \to \infty} \int_{T_{\mu_0}} \left( |\tilde{f}_n - \tilde{f}|^2 + \| \nabla (\tilde{f}_n - \tilde{f}) \|^2_{T_{\mu_0}} \right) dG_Q^F = 0.
\]
where the last step follows from \( \nabla \tilde{f} \in C_b(T_{\mu_0}) \) and the dominated convergence theorem. So, \( \tilde{f} \in \mathcal{D}(\tilde{\mathcal{E}}_{1}) \).

(2) It suffices to prove that the set

\[
B_1^F := \{ \tilde{f} \in C_b^1(T_{\mu_0}) : \tilde{\mathcal{E}}_{1}^F(\tilde{f}) := \tilde{\mathcal{E}}_{1}^F(\tilde{f}, \tilde{f}) + G_Q^F(\tilde{f}^2) \leq 1 \}
\]

is relatively compact in \( L^2(G_Q^F) \). By the chain rule and Young’s inequality, for any \( \varepsilon > 0 \) and \( \tilde{f} \in B_1^F \), we have

\[
\tilde{\mathcal{E}}_{1}^F(\tilde{f}e^\frac{\varepsilon}{2}, \tilde{f}e^\frac{\varepsilon}{2}) \leq G_Q^F(\tilde{f}^2) + (1 + \varepsilon^{-1})\tilde{\mathcal{E}}_{1}^F(\tilde{f}, \tilde{f}) + \frac{1 + \varepsilon}{4} G_Q^F(\tilde{f}^2 e^{\varepsilon F} \|\nabla F\|^2_{T_{\mu_0}})
\]

\[
\leq 1 + \varepsilon^{-1} + \frac{1 + \varepsilon}{4 \lambda} G_Q^F \left( \tilde{f}^2 e^{\varepsilon F} \log \frac{\tilde{f}^2 e^{\varepsilon F}}{G_Q^F(\tilde{f}^2 e^{\varepsilon F})} \right) + \frac{1 + \varepsilon}{4 \lambda} G_Q^F(\tilde{f}^2 e^{\varepsilon F}) \log G_Q^F(e^{\lambda \|\nabla F\|^2_{T_{\mu_0}}}).
\]

Since \( G_Q^F(\|\nabla (\tilde{f}e^{\varepsilon F})\|^2_{T_{\mu_0}}) \leq \tilde{\mathcal{E}}_{1}^F(\tilde{f}e^\frac{\varepsilon}{2}, \tilde{f}e^\frac{\varepsilon}{2}) \) and \( G_Q^F(\tilde{f}^2 e^{\varepsilon F}) = G_Q^F(\tilde{f}^2) \leq 1 \), by combining this with \([3,3]\) we derive

\[
\tilde{\mathcal{E}}_{1}^F(\tilde{f}e^\frac{\varepsilon}{2}, \tilde{f}e^\frac{\varepsilon}{2}) \leq 1 + \varepsilon^{-1} + \frac{1 + \varepsilon}{2 \lambda q_1} \tilde{\mathcal{E}}_{1}^F(\tilde{f}e^\frac{\varepsilon}{2}, \tilde{f}e^\frac{\varepsilon}{2}) + \frac{1 + \varepsilon}{4 \lambda} \log G_Q^F(e^{\lambda \|\nabla F\|^2_{T_{\mu_0}}}).
\]

Since \( \lambda > \frac{1}{2q_1} \), we may take a small \( \varepsilon > 0 \) such that \( \frac{1 + \varepsilon}{2q_1 \lambda} < 1 \), so that this estimate and

\[
G_Q^F(e^{\lambda \|\nabla F\|^2_{T_{\mu_0}}} < \infty \text{ yield }
\]

\[
\tilde{\mathcal{E}}_{1}^F(\tilde{f}e^\frac{\varepsilon}{2}, \tilde{f}e^\frac{\varepsilon}{2}) \leq C, \quad \tilde{f} \in B_1^F
\]

for some constant \( C > 0 \). Since \( \tilde{L} \) has empty essential spectrum, this implies that the set

\[
\{ \tilde{f}e^\frac{\varepsilon}{2} : \tilde{f} \in B_1^F \}
\]

is relatively compact in \( L^2(G_Q^F) \). Equivalently, \( B_1^F \) is relatively compact in \( L^2(G_Q^F) \).

(3) The proof of \([3,6]\) is similar to that of \([2, \text{Lemma } 4.1]\), but we make a more careful estimate by separating \( F^+ \) and \( F^- \). Let \( \tilde{f} \in C_b^1(T_{\mu_0}) \) such that \( G_Q^F(\tilde{f}^2) = 1 \). By \([3,3]\) and Young’s inequality, we obtain

\[
G_Q^F(\tilde{f}^2 \log \tilde{f}^2) = G_Q^F(\tilde{f}^2 e^{\varepsilon F}(\log(\tilde{f}^2 e^{\varepsilon F})) - G_Q^F(\tilde{f}^2 e^{\varepsilon F})
\]

\[
\leq \frac{2}{q_1} G_Q^F(\|\nabla (\tilde{f}e^{\varepsilon F})\|^2_{T_{\mu_0}}) + G_Q^F(\tilde{f}^2 e^{\varepsilon F})
\]

\[
\leq \frac{2(1 + r_1^{-1})}{q_1} \tilde{\mathcal{E}}_{1}^F(\tilde{f}, \tilde{f}) + G_Q^F(\tilde{f}^2 \left[ \frac{1 + r_1}{2q_1} \|\nabla F\|^2_{T_{\mu_0}} + F^- \right])
\]

\[
\leq \frac{2(1 + r_1^{-1})}{q_1} \tilde{\mathcal{E}}_{1}^F(\tilde{f}, \tilde{f}) + r_2 G_Q^F(\tilde{f}^2 \log \tilde{f}^2) + r_2 \log G_Q^F \left( e^{\frac{1 + r_1}{2q_1} \|\nabla F\|^2_{T_{\mu_0}} + \frac{1}{r_2} F^-} \right)
\]

\[
= \frac{2(1 + r_1^{-1})}{q_1} \tilde{\mathcal{E}}_{1}^F(\tilde{f}, \tilde{f}) + r_2 G_Q^F(\tilde{f}^2 \log \tilde{f}^2) + r_2 \log G_Q^F \left( e^{\frac{1 + r_1}{2q_1} \|\nabla F\|^2_{T_{\mu_0}} + F^- + \frac{r_2}{r_1} F^-} \right)
\]
for any $r_1, r_2 \in (0, 1)$. By taking $r_1$ small enough and $r_2$ close enough to $r_2$ such that
\[
\frac{1-r_2}{r_2} \sqrt{\left(\frac{1+r_2}{r_2} - 1\right)} \leq \varepsilon,
\]
we deduce from this and (5.5) that the defective log-Sobolev inequality
\[
G_Q^F(\tilde{f}^2 \log \tilde{f}^2) \leq c_1 \tilde{E}^F(\tilde{f}, \tilde{f}) + c_2, \quad \tilde{f} \in \mathcal{D}(\tilde{E}^F), G_Q^F(\tilde{f}^2) = 1
\]
holds for some constants $c_1, c_2 > 0$. By [1, Theorem 1.1], the Dirichlet form $(\tilde{E}^F, \mathcal{D}(\tilde{E}^F))$ is irreducible. According to [44, Corollary 1.3] for $\phi(t) = 2 - t$, (5.12) implies the log-Sobolev inequality (5.6) for some constant $c > 0$.

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