Braid Groups and Right Angled Artin Groups

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Abstract

In this article we prove a special case of a conjecture of A. Abrams and R. Ghrist about fundamental groups of certain aspherical spaces. Specifically, we show that the \( n \)-point braid group of a linear tree is a right angled Artin group for each \( n \).

1 Introduction. Statement of Results.

A graph is a connected one dimensional compact polyhedron. Ghrist and Abrams (1, 2, 5) have recently called attention to the \( n \)-point unordered configuration space of a graph \( X \), denoted here as \( U^\text{top}_n(X) \). This is the space of \( n \)-element subsets of \( X \) (see Definition 1.3, (1)). It is an aspherical space with the homotopy type of a finite polyhedron, for each \( n \) and \( X \) (see 3). Its fundamental group is the \( n \)-string braid group of \( X \), denoted \( B_n(X, c) \), if \( c \) is a base point of \( U^\text{top}_n(X) \). The group \( B_n(X, c) \) is therefore torsion free; it can have arbitrarily high finite cohomological dimension. Abrams and Ghrist (2, 5) have put forward the following striking conjecture:

Conjecture 1.1. If \( X \) is a planar graph, then the \( n \)-string braid group \( B_n(X, c) \) is a right angled Artin group for each \( n \).

A right angled Artin group is a group having a presentation in which the only relations are commutators between generators. It is known (2) that the planar condition cannot be removed from the above conjecture.

The purpose of this paper is to prove:

Theorem 1.2. For each \( n \), Conjecture 1.1 is true if \( X \) is a linear tree.
A linear tree is a contractible graph $X$ containing an interval $I$ (a homeomorphic copy of $[0, 1]$) such that each node of $X$ is in $I$.

(Recall that the nodes in a graph $X$ are the points of degree $\geq 3$. The degree of any point $x \in X$ is the number of points in $\text{Link}_X(x)$.)

Theorem 1.2 is a direct consequence of our main theorem (1.5) below. Before stating it we give a more careful definition of the objects just mentioned.

**Definition 1.3.** A subset $c$ with $n$ elements in a space $X$ is called an $n$–point configuration in $X$. For any $n \geq 0$ and any topological space $X$, the unordered $n$–point configuration space of $X$ is:

$$U_n^{\text{top}}(X) = \{c \subset X \mid |c| = n\}. \quad (1)$$

If $(X, d)$ is a metric space, then $U_n^{\text{top}}(X)$ is topologized by using the Hausdorff metric on closed sets of $X$. Explicitly, then:

$$d(c, c') = \max\{d(x, c'), d(y, c) \mid x \in c, y \in c'\} \quad \forall c, c' \in U_n^{\text{top}}(X). \quad (2)$$

Equivalently, one can give $U_n^{\text{top}}(X)$ the quotient topology of the map

$$(X^n - \Delta) \xrightarrow{\pi} U_n^{\text{top}}(X) : \pi(x_1, x_2, \ldots x_n) = \{x_1, x_2, \ldots x_n\}$$

where $\Delta = \{(x_1, x_2, \ldots x_n) \mid x_i = x_j$ for some $i \neq j\}$.

Note $U_0^{\text{top}}(X) = *$, and $U_1^{\text{top}}(X) = X$.

The $n$–string braid group of a space $X$ is:

$$B_n(X, c) = \pi_1(U_n^{\text{top}}(X), c), \quad (3)$$

where $c \in U_n^{\text{top}}(X)$ is a base point.

$B_n$ is a functor from the category of topological spaces $X$ with fixed base configurations $c \in U_n^{\text{top}}(X)$ and isotopy classes of injective continuous maps that preserve base configurations.

But in order to finesse the many changes of base configuration required, we employ the following artifice. Let $(X, A)$ be a pair where $A$ is a nonempty simply connected subspace of $X$. The fundamental group of $(X, A)$ denoted $\pi_1(X, A)$ is the set of homotopy classes of maps $(I, \partial I) \xrightarrow{\sigma} (X, A)$

The multiplication in this group is:

$$[\sigma][\tau] = [\sigma \cdot \rho \cdot \tau]$$

where $\rho$ is any path in $A$ from $\sigma(1)$ to $\tau(0)$.
This group is functorial in such \((X, A)\) and isomorphic to \(\pi_1(X, x_0)\) if \(x_0 \in A\). In particular, if \(I\) denotes an interval (a subspace homeomorphic to \([0, 1]\)) in \(X\), then \(U^\text{top}_n(I)\) is a contractible subset of \(U^\text{top}_n(X)\). We define the \(n\)-point braid group of \((X, I)\) by:

\[
B_n(X; I) = \pi_1(U^\text{top}_n(X), U^\text{top}_n(I)).
\]

\(\text{(4)}\)

**Definition 1.4.** (The Endpoint Inclusion Map). Let \(X\) be a graph. Let \(p\) be an endpoint of \(X\) (that is, \(\text{degree}(p) = 1\)). For each \(n \geq 1\), we define a map

\[
B_{n-1}(X, c) \xrightarrow{i_p} B_n(X, c')
\]

as follows. Choose an isotopy of \(1_X\), say \(\{r_t : X \to X \mid 0 \leq t \leq 1\}\), which is stationary outside a small neighborhood of \(p\) and which satisfies \(r_1(X) \subset X - \{p\}\). Then \(r_1\) induces a map

\[
U^\text{top}_n(X) \xrightarrow{i_p} U^\text{top}_n(X) : i_p(d) = \{p\} \cup r_1(d) \quad \forall d \in U^\text{top}_{n-1}(X).
\]

In turn, \(i_p\) induces a map of fundamental groups, denoted:

\[
B_{n-1}(X, c) \xrightarrow{i_p} B_n(X, c'), \quad \text{if } c' = i_p(c)
\]

(or \(i_{p, X}\) when such explicitness is needed).

Now if \(I\) is an interval in \(X\) containing \(p\) and if the isotopy fixes \(X - I\), then \(i_p\) induces

\[
B_{n-1}(X; I) \xrightarrow{i_p} B_n(X; I)
\]

which is independent of the isotopy chosen. Write \(i_p^k : B_{n-k}(X; I) \to B_n(X; I)\) for the \(k\)-fold iteration of this map. Abrams (\cite{Abrams}, Lemma 3.4, p.24) shows that \(i_p\) is injective if \(X\) is any graph.

(Note: Abrams proves this for the *pure* braid groups. These have finite index in our braid groups \(B_n(X; I)\), which are torsion free. This implies that \(i_p\) is injective).

Our main theorem will say that the groups \(B_n(X; I)\), for \(n = 0, 1, 2, \ldots\), admit right angled Artin presentations which are all related by the maps \(i_p\) above. A right angled Artin presentation of a group \(G\), denoted \(\langle \beta, \mathcal{R} \rangle\), consists of a subset \(\beta\) of \(G\), and a subset \(\mathcal{R}\) of \(F(\beta)\), the free group on \(\beta\), such that \(\mathcal{R}\) consists of elements of the form \((xyx^{-1}y^{-1})\), where \(x, y \in \beta\), and the following sequence is exact

\[
1 \to N \to F(\beta) \xrightarrow{j} G \to 1,
\]
where \( j \) denotes the natural homomorphism, and \( N \) denotes the smallest normal subgroup containing \( \mathcal{R} \).

Here is the main theorem.

**Theorem 1.5.** Let \( X \) be a tree. Let \( p \) be an endpoint of \( X \). Let \( I \subset X \) be an interval containing \( p \) and every node of \( X \). Then for each integer \( n \geq 0 \) there is a right angled Artin presentation, \( \langle \beta(n), \mathcal{R}(n) \rangle \) for \( B_n(X; I) \) such that

\[
\iota_p(\beta(n - 1)) \subset \beta(n) \quad \text{and} \quad \iota_p(\mathcal{R}(n - 1)) \subset \mathcal{R}(n) \quad \forall n \geq 1. \tag{5}
\]

Here \( \iota_p : F(\beta(n - 1)) \to F(\beta(n)) \) is the homomorphism induced by the function \( \iota_p : \beta(n - 1) \to \beta(n) \).

It is easy to see that Theorem 1.2 follows from Theorem 1.5 because every interval in a tree \( X \) lies in a bigger interval \( I \) containing an endpoint of \( X \).

Here is an outline of the rest of this paper. In Section 2, we study the case of a star (a tree with one node). If \( X \) is a star and \( I \) is an interval whose endpoints are endpoints of \( X \), we show that \( B_n(X; I) \) is a free group admitting a basis \( \beta(n) \), for each \( n \), such that \( \iota_p(\beta(n - 1)) \subset \beta(n) \) if \( p \) is either endpoint. In Section 3 we prove a theorem computing the braid group \( B_n(X; I) \) if \( X \) is the one point union of two graphs along a common endpoint. In section 4 we use the previous results to prove the main theorem, 1.5.

We wish to thank Aaron Abrams here for suggesting this problem to us, in 2003. His comments about it were encouraging and helpful.

## 2 Braid Groups of Stars.

A star \( S \) is a tree with no more than one node. The node is denoted \( v \). If \( S \) is a tree with no nodes (an interval), then we assume further that a fixed interior point \( v \) of \( S \) is given. It turns out that, for any star \( S \) and any integer \( n \geq 0 \), the \( n \)-point configuration space \( \mathcal{U}^{\text{top}}_n(S) \) contains a compact one dimensional polyhedron \( D_n(S) \) which is a deformation retract of \( S \). This is constructed in [4]. We review the construction here. We then use \( D_n(S) \) to prove the following:
Proposition 2.1. Let $S$ be a star. Let $p$ and $q$ be two distinct endpoints of $S$, and let $I$ be the interval $[p, q]$. Then for each $n \geq 0$, $B_n(S; I)$ is a free group. Moreover, there is a basis $\beta(n)$ for $B_n(S; I)$ such that, if $n \geq 1$,

$$\iota_p(\beta(n-1)) \subset \beta(n) \text{ and } \iota_q(\beta(n-1)) \subset \beta(n).$$

Note: The rank of this free group $B_n(S; I)$ is

$$1 + (k - 1)\left(\frac{n + k - 2}{k - 1}\right) - \left(\frac{n - k - 1}{k - 1}\right)$$

where $k$ is the number of endpoints of $S$. This is proved by Doig in [4], but an equivalent formula for the corresponding pure, braid group of $S$ appears earlier in Ghrist [5], Prop. 4.1.

Before beginning the proof of 2.1 we now choose a fixed metric $d$ on the star $S$. To this end, fix the integer $n \geq 0$. The metric $d$ will be a constant multiple of the standard simplicial metric $\rho$ on $S$. The star $S$ has a canonical simplicial structure in which the vertices are the points of degree different from 2. For the corresponding simplicial metric $\rho$, $[p, q]$ is isometric to $[0, 2]$ and $v$ is the midpoint of $[p, q]$. Define the metric $d$ by:

$$d(x, y) = C \rho(x, y) \quad \forall x, y \in S$$

where $C$ is a fixed constant such that $C \geq \max(1, n - 1)$.

Construction 2.2. (of $D_n(S)$; compare [4]).

Let $S$ be a star. Let $c \in \mathcal{U}_n^{\text{top}}(S)$ be any $n$–point configuration in the star $S$. For each endpoint $p$ of $S$, we define

$$A_p(c) = c \cap [p, v].$$

We call this an arm of $c$ if $A_p(c) \neq \emptyset$.

We say $c$ is regular if it satisfies the following rules:

1. For all $x, y \in c$ with $x \neq y$, $d(x, y) \geq 1$. Also $d(x, y) = 1$ if $x$ and $y$ lie in a single arm of $c$ and $[x, y] \cap c = \{x, y\}$.

2. If $v \notin c$, and if $A_p(c)$ is an arm of $c$, then there is a point $x \in c$ such that $d(x, A_p(c)) = 1$. 
In English: successive points in a single arm of $c$ are one unit apart; if $v \notin c$, the innermost point of each arm of $c$ has distance one from some other arm, and has distance at least one from every other arm.

Therefore $c$ has at least one arm $A_p(c)$ such that $d(v, A_p(c)) \leq \frac{1}{2}$. There is at most one arm of $c$ that satisfies: $0 < d(v, A_p(c)) < \frac{1}{2}$ (the governing arm in the language of \cite{4}). When $A_p(c)$ is the unique governing arm of $c$, every other arm, $A_q(c)$, satisfies: $d(v, A_q(c)) = 1 - d(v, A_p(c))$.

The subspace $D_n(S)$ of $\mathcal{U}_n^{\text{op}}(S)$ can now be defined:

$$D_n(S) := \{ c \in \mathcal{U}_n^{\text{op}}(S) \mid c \text{ is regular} \}. \quad (8)$$

Note $D_0(S) = * = \mathcal{U}_0^{\text{op}}(S)$, and $D_1(S) = \{ v \}$.

$D_n(S)$ has the structure of a 1-dimensional compact polyhedron, as we now explain. Its vertices come in two types. The Type I vertices are those configurations $c \in D_n(S)$ such that $v \in c$. The Type II vertices are those $c \in D_n(S)$ such that $d(v, c) = \frac{1}{2}$. For Type II vertices, note that $d(v, A_p(c)) = \frac{1}{2}$ for each arm $A_p(c)$.

Each 1-simplex of $D_n(S)$ has a single Type I vertex $c$ and a single Type II vertex $c'$. We denote this 1-simplex $[c, c']$ or $[c', c]$. For each Type II vertex $c'$, there is a single 1-simplex $[c', c]$ for each endpoint $p$ of $S$ such that $A_p(c') \neq \emptyset$. The other endpoint $c$ of this 1-simplex $[c', c]$ is defined as the unique Type I vertex such that:

$$|A_q(c)| = |A_q(c')| \text{ for all endpoints } q \text{ of } S, \quad \text{and } 0 < d(v, A_p(e)) < \frac{1}{2}. \quad (9)$$

The points of the 1-simplex $[c', c]$ are $c'$, $c$, and those $e \in D_n(S)$ such that $|A_q(e)| = |A_q(c')| \text{ for all endpoints } q \text{ of } S, \text{ and } 0 < d(v, A_p(e)) < \frac{1}{2}. \quad (10)$

The rule $e \mapsto d(v, A_p(e))$ gives a homeomorphism from $[c', c]$ onto the interval $[0, \frac{1}{2}]$.

Each point $e \in D_n(S)$, other than a vertex, belongs to a unique 1-simplex $[c', c]$. The Type II vertex $c'$ is specified by the requirement that $|A_q(e)| = |A_q(c')| \text{ for all endpoints } q$. The endpoint $p$, and also (by \cite{9}) the Type I vertex $c$, is specified by the requirement that $0 < d(v, A_p(e)) < \frac{1}{2}$.

This completes the construction of the 1-dimensional polyhedron $D_n(S)$.

Doig proves in \cite{4} that $D_n(S)$ is a strong deformation retract of $\mathcal{U}_n^{\text{op}}(S)$. 

Incidentally, if we change the metric $d$ on $S$ to $d'$, by changing the constant $C$ to $C'$, then $D_n(S, d)$ is isometric to $D_n(S, d')$. But the two are not identical.

Before beginning the proof of Proposition 2.1 we need to relate $i_p$ to the deformation retract $D_n(S)$.

If $p$ and $q$ are two endpoints of $S$ and $I = [p, q]$, then $D_n(I) \subset D_n(S)$ and $D_n(I)$ is an interval.

Analogous to the map $i_p$ of Definition 1.4 is a simplicial inclusion map $	ilde{i}_p : D_{n-1}(S) \to D_n(S)$, defined by:

$$\tilde{i}_p(c) = \{x\} \cup c \quad \forall c \in D_{n-1}(S),$$

where $x$ is the unique point of $[p, v]$ such that $\{x\} \cup c \in D_n(S)$. (We have chosen the constant $C$ above, so that $d(p, c) \geq 1$. This ensures that there is such a point $x$). It is elementary to see that $\tilde{i}_p(D_n(I)) \subset D_n(S)$, and that the following diagram commutes up to homotopy:

$$\begin{array}{ccc}
(D_{n-1}(S), D_{n-1}(I)) & \xrightarrow{\tilde{i}_p} & (D_n(S), D_n(I)) \\
\text{incl.} & & \text{incl.} \\
(U_{n-1}^{\text{top}}(S), U_{n-1}^{\text{top}}(I)) & \xrightarrow{i_p} & (U_n^{\text{top}}(S), U_n^{\text{top}}(I))
\end{array}$$

Therefore we can identify the group $B_n(S; I)$ with $\pi_1(D_n(S), D_n(I))$, and we can identify the homomorphism $\iota_p : B_{n-1}(S; I) \to B_n(S; I)$ with the map $(\tilde{i}_p)_* : \pi_1(D_{n-1}(S), D_{n-1}(I)) \to \pi_1(D_n(S), D_n(I))$.

**Proof.** (of Proposition 2.1):

Suppose a maximal tree $\mathcal{T}$ of $D_n(S)$ is chosen, containing $D_n(I)$. Because $\text{dim}(D_n(S)) = 1$, $B_n(S; I)$ is a free group, and a basis for $B_n(S; I)$ is given by those 1-simplices of $D_n(S)$ which are not in $\mathcal{T}$. Therefore, it is enough to exhibit, for each $n$, a maximal tree $\mathcal{T}(n)$ for $D_n(S)$ such that

$$\mathcal{T}(n) \supset D_n(I) \quad \text{and} \quad \mathcal{T}(n - 1) = \tilde{i}_p^{-1}(\mathcal{T}(n)) = \tilde{i}_q^{-1}(\mathcal{T}(n)). \quad (11)$$

We do this now.

Let $c^{(n)}$ be the unique point of $D_n(S)$ such that $c^{(n)} \subset [p, v]$. The configuration $c^{(n)}$ is a Type I vertex. For each vertex $c$ of $D_n(S)$ except $c^{(n)}$, we are going to construct a successor vertex $s(c)$ such that $[c, s(c)]$ is a 1-simplex and $s^k(c) = c^{(n)}$ for some $k > 0$. Then $\mathcal{T}(n)$ will consist of the union of these $[c, s(c)]$. 

Number the endpoints of $S : p_1, p_2, \ldots, p_m$. Ensure that $p = p_1$ and $q = p_2$. Define $s(c)$ as follows. If $c$ is a Type I vertex, then $s(c)$ is the unique Type II vertex satisfying:

$$|A_{p_1}(s(c))| = |A_{p_1}(c)|; \quad |A_{p_j}(s(c))| = |A_{p_j}(c)| - 1 \text{ if } j \neq 1$$

(12)

If $c$ is a Type II vertex, let $r = r(c)$ be the biggest index for which $A_{p_r(c)} \neq \emptyset$. Note $r \geq 2$ (by 2.2.2.). Define $s(c)$ as the unique Type I vertex satisfying:

$$|A_{p_r}(s(c))| = |A_{p_r}(c)|; \quad |A_{p_j}(s(c))| = |A_{p_j}(c)| + 1 \text{ if } j \neq r.$$  

(13)

For any vertex $c$ of $D_{n-1}(S)$ except $c^{(n-1)}$, we have, by (12) and (13)

$$\tilde{i}_{p_j}(s(c)) = s(\tilde{i}_{p_j}(c)),$$  

for $j \leq 2$.  

(14)

By (9), $c$ and $s(c)$ span a 1-simplex for each vertex $c \neq c^{(n)}$. It is also clear from (12) and (13) that, for each vertex $c$, there is an integer $k \geq 0$ such that $s^k(c) = c^{(n)}$.

Therefore we define:

$$T(n) = \bigcup \{ [c, s(c)] \mid c \text{ is a vertex of } D_n(S) \text{ other than } c^{(n)} \}.$$  

(15)

$T(n)$ is a tree containing every vertex of $D_n(S)$; therefore it is maximal.

By (14), $\tilde{i}_{p_j}(T(n-1)) \subset T(n)$, for $j = 1$ and 2. But since $T(n-1)$ is a maximal tree in $D_{n-1}(S)$, and $T(n)$ contains no cycles, this implies:

$$T(n - 1) = \tilde{i}_{p}^{-1}(T(n)) = \tilde{i}_{q}^{-1}(T(n)).$$

Finally we must show that $D_n(I) \subset T(n)$. The key is to note that each Type II vertex $c$ of $D_n(I)$ belongs to exactly two 1-simplices. One of these is $[c, s(c)]$. The other is $[c, d]$, where $d$ is that Type I vertex of $D_n(I)$ such that $c = s(d)$. It follows that $D_n(I) \subset T(n)$. This completes the proof of Proposition 2.1.

3 The Endpoint Union of Two Graphs.

This section is devoted to computing the $n$-point braid group of the union of two graphs which intersect at a single endpoint of each.

Our goal is Proposition 3.1 below.
We let $X$ be a graph of the form $X = S \cup T$ where $S$ and $T$ are graphs. Assume that $\{q\} = S \cap T$, a single point, and that $q$ is an endpoint of $S$ and of $T$. Let $j : S \to X$ and $j' : T \to X$, be inclusion maps. These induce maps of braid groups with the same names. Let $I$ and $J$ be intervals in $S$ and $T$ respectively so that $\{q\} = I \cap J$. Form the free product $B_n(S; I) \ast B_n(T; J)$. Let $N$ be the smallest normal subgroup of this product containing each of the commutator subgroups $[\iota_q B_{n-k}(S; I), \iota_q^n B_k(T; J)]$, $1 < k < n$. Then we have:

**Proposition 3.1.** With the hypotheses above, the following sequence is exact:

$$1 \to N \to B_n(S; I) \ast B_n(T; J) \xrightarrow{j \ast j'} B_n(X; I \cup J) \to 1.$$  

We will need:

**Lemma 3.2.** Let $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n$ and $B_0 \subseteq B_1 \subseteq \cdots \subseteq B_n$ be two increasing sequences of groups. Suppose that the following diagram is a pushout diagram:

\[
\begin{array}{ccc}
G & \xrightarrow{j} & B_n(X; I \cup J) \\
\nearrow & & \nearrow \\
A_n \times B_0 & \to & A_n \times B_1 \\
\searrow i \times 1 & \nearrow 1 \times i' & \searrow i \times 1 \\
&A_{n-1} \times B_0 & \to A_{n-1} \times B_1 \\
\searrow i \times 1 & \nearrow 1 \times i' & \searrow i \times 1 \\
&A_{n-2} \times B_2 & \to A_{n-2} \times B_1 \\
\searrow i \times 1 & \nearrow 1 \times i' & \searrow i \times 1 \\
& & \vdots
\end{array}
\]

(i.e., $G$ is the direct limit of the diagram obtained by deleting $G$ and the maps to $G$). Here $i$ and $i'$ denote inclusions. Then there is an exact sequence:

$$1 \to N \to A_n \ast B_n \xrightarrow{j \ast j'} G \to 1$$

where $N$ is the smallest normal subgroup of $A_n \ast B_n$ containing $[A_{n-k}, B_k]$ for each $k = 0, 1, 2, \ldots, n$. Here $j$ and $j'$ are restrictions of the limit maps $A_n \times B_0 \to G$, and $A_0 \times B_n \to G$ to $A_n$ and $B_n$ respectively.

**Proof.** If $n = 1$ this is clear. Working inductively, we let $G'$ be the limit of the diagram obtained by omitting $A_0$ and $B_n$. This gives us $G' = A_n \ast B_{n-1}/N'$ where $N'$ is the smallest normal subgroup containing all $[A_{n-k}, B_k]$ for $k = 0, 1, 2, \ldots, n-1$. It also gives a new diagram whose limit is obviously $A_n \ast B_n/N$ where $N$ is the smallest normal subgroup containing $N'$ and $[A_0, B_n]$.

\[\square\]
Proof. (of Proposition 3.1): For each \( k = 0, 1, \ldots, n \), let
\[
U(k) = \{ c \in U_n^{\text{top}}(X) \mid |c \cap S| \geq k, |c \cap T| \geq n - k \},
\]
and let \( j_k : U(k) \to U_n^{\text{top}}(X) \) denote the inclusion map. Since \( |S \cap T| = 1 \), we see that \( U(j) \cap U(k) = \emptyset \) if \( |k - j| > 1 \) and that \( U_n^{\text{top}}(X) = \bigcup_{k=0}^{n} U(k) \). Each \( c \in U(k) \) can be written uniquely in the form:
\[
c = c_S(k) \cup c_T(n - k)
\]
where:
(i) \( c_S(k) \subset S \);  (ii) \( c_T(n-k) \subset T \);  (iii) \( |c_S(k)| = k \);  (iv) \( |c_T(n-k)| = n-k \).
Necessarily, \( c_S(k) \cap c_T(n-k) = \emptyset \).

One sees easily that there is a homotopy equivalence
\[
U(k) \xrightarrow{h_k} U_k^{\text{top}}(S) \times U_{n-k}^{\text{top}}(T) : h_k(c) = (c_S(k), c_T(n-k)).
\]
It sends \( U(k) \cap U_n^{\text{top}}(I \cup J) \) to \( U_k^{\text{top}}(I) \times U_{n-k}^{\text{top}}(J) \). Similarly
\[
U(k) \cap U(k - 1) = \{ c \in U_n^{\text{top}}(X) \mid q \in c, |c \cap S| = k, |c \cap T| = n - k + 1 \}.
\]
Therefore there is a homotopy equivalence for each \( k \geq 1 \):
\[
U(k) \cap U(k - 1) \xrightarrow{j_k'} U_{k-1}^{\text{top}}(S) \times U_{n-k}^{\text{top}}(T) : j_k'(c) = (c_S(k-1), c_T(n-k)).
\]

Passing to fundamental groups of these spaces, we obtain the pushout diagram below by using the version of Van Kampen’s Theorem in Hatcher (6.1.20, p.43):
Now, by Lemma 3.2, where $j = j_n$, $j' = j_0$, we get the exact sequence
\[1 \to N \to B_n(S; J) \ast B_n(T; J) \to B_n(X; I \cup J) \to 1.\]
where $j_S$ and $j_T$ are induced by the inclusions $S \subset X$ and $T \subset X$ and $N$ is the smallest normal subgroup of the free product containing each of the sets:

$$[\iota_q^k \beta_S(n - k), \iota_q^{n-k} \beta_T(k)].$$

Set

$$\beta_X(n) = j_S(\beta_S(n)) \cup j_T(\beta_T(n)) \quad (18)$$

$$\mathcal{R}_X(n) = j_{T*}(\mathcal{R}_T(n)) \cup \left[j_{S*} \iota_q^k \beta_S(n - k), j_{T*} \iota_q^{n-k} \beta_T(k)\right] \quad (19)$$

where $j_{S*} : F(\beta_S(n)) \to F(\beta_X(n))$ and $j_{T*} : F(\beta_T(n)) \to F(\beta_X(n))$ are induced by $j_S$ and $j_T$.

Clearly $\langle \beta_X(n); \mathcal{R}_X(n) \rangle$ is an Artin presentation of $B_n(X; I)$ for each $n$. To complete the argument, we must prove (5). First note that

$$\iota_p X \circ j_T = j_T \circ \iota_q T : B_{n-1}(T; K) \to B_n(X; I) \quad (20)$$

$$\iota_p X \circ j_S = j_S \circ \iota_p S : B_{n-1}(S; J) \to B_n(X; I) \quad (21)$$

$$\iota_p S \circ \iota_q S = \iota_q S \circ \iota_p S : B_{n-2}(S; J) \to B_n(S; J) \quad (22)$$

because the corresponding diagrams of spaces commute up to homotopy. It follows that

$$\iota_p (\beta_X(n - 1)) \subset \beta_X(n) \quad (23)$$

by (18). Also, by (17), (19) and (23) we have

$$(\iota_p X)_{T*}(\mathcal{R}_T(n-1)) = j_{T*}(\iota_q T)_{T*}(\mathcal{R}_T(n-1)) \subset j_{T*}(\mathcal{R}_T(n)) \subset \mathcal{R}_X(n). \quad (24)$$

Finally by (20), (21) and (22) we have:

$$\iota_p X [j_{S*}(\iota_q S)_* \beta_S(n - 1 - k), j_{T*}(\iota_q T)_* \beta_T(k)] \subset [j_{S*}(\iota_q S)_* \beta_S(n - k), j_{T*}(\iota_q T)_* \beta_T(k)]$$

which, with (23) and (24) implies

$$(\iota_p X)_* \mathcal{R}_X(n - 1) \subset \mathcal{R}_X(n).$$

This completes the proof of Theorem 1.3. \qed
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