On the predictivity of the non-renormalizable quantum field theories

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Following a Four Dimensional Renormalization approach to ultraviolet divergences (FDR), we extend the concept of predictivity to non-renormalizable quantum field theories at arbitrarily large perturbative orders. The idea of topological renormalization is introduced, which keeps a finite value for the parameters of the theory by trading the usual order-by-order renormalization procedure for an order-by-order redefinition of the perturbative vacuum. One additional measurement is then sufficient to systematically compute quantum corrections at any loop order, with no need of absorbing ultraviolet infinities in the Lagrangian.

1 Introduction

Renormalizability [1–3] provides a very powerful guideline in high-energy particle physics. For instance, the existence of intermediate vector bosons in the Electroweak Standard Model (SM) can be directly inferred from the non-renormalizability of the low-energy four-fermion interaction. A second example is the pattern of the SM fermions, which is very strongly constrained by the need of canceling the gauged axial anomaly. A renormalizable Quantum Field Theory (QFT) is complete. After fixing its free parameters, any observable can be predicted at any perturbative order and scale. The ultraviolet (UV) infinities occurring in the intermediate steps of the calculation [4–7], being universal, are re-absorbed in the parameters of the theory and any UV cutoff disappears, leaving a dependence on the renormalization scale at perturbative orders higher than the computed one.

On the contrary, in non-renormalizable theories new infinities are generated, order by order in the loop expansion, which cannot be re-absorbed into the original Lagrangian. Such theories are usually interpreted as effective ones [8], and the traditional renormalization program works untouched only if the original Lagrangian is modified, at any given order, by adding a new set of interactions, where the new generated infinities can be accommodated. The ignorance of the UV completion coming from the true theory is parametrized by a change in the original Lagrangian. However, the new interactions need to be fixed by experiment, which leads, in principle, to a lack of predictivity of the theory at arbitrarily large perturbative orders.

In this work, we use the FDR approach introduced in [9] to discuss a possible way out to this problem. The basic idea is as follows: the type and number of UV infinities occurring at any loop order is reduced and kept under control once they are classified in terms of their topology at the integrand level\(^1\), rather than according to their occurrence in the interactions. Thus, they can be subtracted from the physical spectrum without the need of re-absorbing them in the Lagrangian's parameters, which are left untouched. This subtraction can be interpreted as an order-by-order redefinition of the perturbative vacuum, dubbed topological renormalization, in which non-physical configurations are discarded.

In a non-renormalizable QFT this separation between physical and non-physical degrees of freedom may cause the appearance of an arbitrary logarithmic parameter at the same perturbative level one is calculating, that can be interpreted as the scale at which the UV divergent configurations are subtracted. To pull such a dependence back to higher perturbative orders one additional measurement is necessary, which links it to the parameters of the theory. In this way, the (unknown) high-energy UV behavior can be completely decoupled from the low-energy regime. The advantages are clear. Even though non-renormalizable theories remain – technically – effective, the original symmetries are kept (since the Lagrangian is never touched), and finiteness and predictivity are both restored.

\(^1\) We work in momentum space.

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Following this interpretation, interactions must be introduced in a non-renormalizable bare Lagrangian based solely on physical motivation, and not on the need of absorbing UV divergences. Thus, any effective Lagrangian [10] containing only a subset of all possible higher dimensional operators represents a legitimate QFT, whose validity has to be judged a-posteriori by comparing its predictions with data.

The outline of the paper is as follows. Section 2 reviews the aspects of FDR which are relevant to the present discussion. In Sec. 3 a classification of the UV infinities occurring in scalar and tensor loop integrals is presented and the topological renormalization is discussed. Finally, Sec. 4 analyzes the use of FDR in non-renormalizable theories compared to the renormalizable case.

2 The FDR integration

In FDR [9, 11–13] the UV infinities are subtracted at the integrand level by judiciously splitting the original integrand \( J(q_1, \ldots, q_k) \) of an \( \ell \)-loop function\(^2\) in two parts, \( J_{\text{INF}}(q_1, \ldots, q_k) \) and \( J_{F,\ell}(q_1, \ldots, q_k) \):

\[
J(q_1, \ldots, q_k) = J_{\text{INF}}(q_1, \ldots, q_k) + J_{F,\ell}(q_1, \ldots, q_k). \tag{1}
\]

The former piece collects integrands which would produce UV divergences upon integration (divergent integrands) and is discarded, while the latter is kept and generates the physical finite contribution. To regulate the spurious infrared (IR) divergences caused by this break-up\(^3\), the \(+i\theta\) propagator prescription is made explicit by identifying it with a vanishing mass \( -\mu^2 \) and taking the limit \( \mu \to 0 \) outside integration. The rationale for this separation is that the divergent integrands in \( J_{\text{INF}}(q_1, \ldots, q_k) \) are allowed to depend on \( \mu \), but not on physical scales, so that physics is entirely contained in \( J_{F,\ell}(q_1, \ldots, q_k) \). As a two-loop example consider the rank-2 irreducible tensor

\[
J_{\alpha\beta}(q_1, q_2) = \frac{q_1^\alpha q_2^\beta}{D_1 D_2 D_{12}}, \tag{2}
\]

with

\[
D_1 = q_1^2 - m_1^2, \quad D_2 = q_2^2 - m_2^2, \quad D_{12} = q_{12}^2 - m_{12}^2.
\]

\[
q_{12} = q_1 + q_2, \quad q_{12}^2 = q_2^2 - \mu^2. \tag{3}
\]

The desired splitting can be obtained by means of a repeated use of the identities

\[
\frac{1}{D_1} = \frac{1}{q_1^2} + \frac{m_1^2}{q_1^2 D_1}, \quad \frac{1}{D_{12}} = \frac{1}{q_{12}^2} - \frac{q_2^2}{q_{12}^2 D_{12}},
\]

\[
\frac{1}{q_{12}^2} = \frac{1}{q_1^2} - \frac{q_{12}^2}{q_{12}^2 D_{12}}, \tag{4}
\]

and reads

\[
J_{\alpha\beta}(q_1, q_2) = q_1^\alpha q_2^\beta \left\{ \left[ \frac{1}{q_1^2 q_{12}^2} \right] + \left( \frac{1}{D_1} - \frac{1}{q_1^2} \right) \right\}
\]

\[
\times \left[ \left( \frac{1}{D_1} - \frac{1}{q_1^2} \right) - \frac{q_2^2}{q_{12}^2 D_{12}} \left( \frac{m_1^2}{D_1} + \frac{m_2^2}{q_{12}^2} \right) \right], \tag{5}
\]

where divergent integrands are written between square brackets. Therefore

\[
J_{\text{INF}}^{\alpha\beta}(q_1, q_2) = q_1^\alpha q_2^\beta \left\{ \left[ \frac{1}{q_1^2 q_{12}^2} \right] + \left( \frac{1}{D_1} - \frac{1}{q_1^2} \right) \left[ \frac{1}{D_1} \right] \right\}
\]

\[
\text{and}
\]

\[
J_{F,\ell}^{\alpha\beta}(q_1, q_2) = q_1^\alpha q_2^\beta \left\{ \frac{1}{D_1 D_2 D_{12}} \left( \frac{m_1^2}{D_2} + \frac{m_2^2}{q_{12}^2} \right) \right.
\]

\[
- \left( \frac{1}{D_1} - \frac{1}{q_1^2} \right) \frac{q_2^2}{q_{12}^2 D_{12}} \left( q_1^2 - \mu^2 \right) \right\}. \tag{6}
\]

The FDR integral over the original integrand \( J(q_1, \ldots, q_k) \) is defined, through Eq. (1), as\(^4\)

\[
\int [d^3 q_1] \cdots [d^3 q_k] J(q_1, \ldots, q_k) = \lim_{\mu \to 0} \int [d^3 q_1] \cdots [d^3 q_k] J_{F,\ell}(q_1, \ldots, q_k). \tag{7}
\]

and the expansion needed to extract \( J_{F,\ell}(q_1, \ldots, q_k) \) is called the FDR defining expansion of \( J(q_1, \ldots, q_k) \). For instance, from Eq. (6)

\[
\int [d^3 q_1] [d^3 q_2] q_1^\alpha q_2^\beta \left[ \frac{q_1^2 q_2^2}{D_1 D_2 D_{12}} \right] = \lim_{\mu \to 0} \int [d^3 q_1] [d^3 q_2] J_{F,\ell}^{\alpha\beta}(q_1, q_2). \quad (8)
\]

Notice that, as anticipated, the IR behavior \( J_{F,\ell}^{\alpha\beta}(q_1, q_2) \sim \frac{1}{q_1^2} \) is regulated by \( \mu^2 \). In convergent

---

\(^2\) \( q_1, \ldots, q_k \) are integration momenta and \( J(q_1, \ldots, q_k) \) can be a tensor.

\(^3\) The sum \( J_{\text{INF}} + J_{F,\ell} \) is free of spurious IR poles while \( J_{F,\ell} \) alone is not.

\(^4\) FDR integration is denoted by the symbol \([d^3 q_1]\).
integrals \( f_{\text{INF}}(q_1, \ldots, q_k) = 0 \), thus FDR integration and normal integration coincide. Conversely, polynomials in the integration variables represent a limiting case of Eq. (1), in which \( f_{\nu,\ell}(q_1, \ldots, q_k) = 0 \). As a consequence

\[
\int [d^4 q_i] (\vec{q}_i^2)^\alpha = 0,
\]

for any integer \( \alpha \geq 0 \).

The FDR integration in Eq. (7) encodes the UV subtraction directly into its definition and satisfies, at the same time, the two mathematical properties required for (regulated) divergent integrals to maintain the symmetries of the QFT at hand (including gauge invariance), namely:

i) invariance under shift of any integration variable [16]

\[
\int [d^4 q_i] \cdots [d^4 q_k] J(q_1, \ldots, q_k) = \int [d^4 q_i] \cdots [d^4 q_k] J(q_i + p_i, \ldots, q_k + p_k) \quad \forall p_i \text{, with } i = 1, \ldots, \ell;
\]

ii) preservation of the cancellations between numerators and denominators [17]

\[
\int [d^4 q_i] \cdots [d^4 q_k] \frac{\vec{q}_i^2 - m_i^2}{(\vec{q}_i^2 - m_i^2)^m} \cdots = \int [d^4 q_i] \cdots [d^4 q_k] \frac{1}{(\vec{q}_i^2 - m_i^2)^{m-1}} \cdots.
\]

The first property can be proven by rewriting FDR integrals as a finite difference of UV divergent integrals regulated in Dimensional Regularization (DR) [7]

\[
\int [d^4 q_i] \cdots [d^4 q_k] J(q_1, \ldots, q_k) = \lim_{\mu \to 0} \mu^{-\epsilon \ell} \left( \int [d^4 q_i] \cdots [d^4 q_k] J(q_1, \ldots, q_k) - \int [d^4 q_i] \cdots [d^4 q_k] J_{\text{INF}}(q_1, \ldots, q_k) \right).
\]

and it follows from the fact that the r.h.s. of Eq. (12) is shift invariant. The second property holds if a replacement

\[
q_i^2 \rightarrow q_i^2 = q_i^2 - \mu_i^2
\]

is performed for any \( q_i^2 \) generated by Feynman rules. For instance, from the defining expansions of the two integrands

\[
\int [d^4 q_i][d^4 q_i] \frac{\vec{q}_i^2 - m_i^2|1}{D_1 D_2 D_1 2} = \int [d^4 q_i][d^4 q_i] \frac{1}{D_1 D_2 2 1 2},
\]

as long as the integral containing \( \mu_i^2|1 \) is defined as

\[
\int [d^4 q_i][d^4 q_i] \frac{\mu_i^2|1}{D_1 1 D 2 2} = \lim_{\mu_i \to 0} \mu_i^2 \int d^4 q_i d^4 q_i \left[ \frac{1}{D_1 1} + \frac{m_i^2}{D_2} \left( \frac{m_i^2}{D_2} + \frac{m_i^2}{D_1 2} \right) \right]
\]

where the denominator expansion in the r.h.s. is the same one performed in front of \( q_i^2 q_i^2 \) in Eq. (6). An explicit computation [13] gives

\[
\int [d^4 q_i][d^4 q_i] \frac{\mu_i^2|1}{D_1 D_2 D_1 2} = -\pi^4 \left( \frac{1}{2} + \frac{2}{3} f \right)
\]

with

\[
f = \frac{i}{\sqrt{3}} (\text{Li}_2(e^{i\frac{1}{2}}) - \text{Li}_2(e^{-i\frac{1}{2}})).
\]

Integrals with powers of \( \mu_i^2|1 \) in the numerators are called extra integrals and automatically generate the constants needed to preserve gauge invariance when decomposing

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5 We rely here on the existence of graphical proofs of the Ward-Slavnov-Taylor identities [14], in which the correct relations among Green’s functions are demonstrated diagrammatically [15] – at any loop order – by means of algebraic manipulations of the integrands of the loop functions.

6 See Appendix A of [9] for more details.

7 Here and in the following \( n = 4 + \epsilon \) and \( \mu_\alpha \) is the arbitrary scale of DR.

8 Only one kind of \( \mu^2 \) exists. The index \( i \) in \( \mu_i^2|1 \) only denotes that the denominator expansion in front of \( \mu^2 \) should be the same one used for \( q_i^2 \) when it appears in the numerator of an integral, as in Eq. (15).

9 The replacement in Eq. (13) is called global prescription.
tensors. For example\textsuperscript{10}

\[
\int [d^4 q_1][d^4 q_2] \frac{q_1^\alpha q_2^\beta}{D_1 D_2 D_{12}} = \frac{g^{\alpha\beta}}{4} \int [d^4 q_1][d^4 q_2] \frac{q_1^2}{D_1 D_2 D_{12}} \\
= \frac{g^{\alpha\beta}}{4} \left( \int [d^4 q_1][d^4 q_2] \frac{1}{D_1 D_2 D_{12}} \right) \\
+ \int [d^4 q_1][d^4 q_2] \frac{m^2}{D_1 D_2 D_{12}} + \int [d^4 q_1][d^4 q_2] \frac{\mu^2}{D_1 D_2 D_{12}} \right). 
\]

An important consequence is

\[
\int [d^4 q_1][d^4 q_2] \frac{4q_1^\alpha q_2^\beta - \tilde{q}_1^2 g^{\alpha\beta}}{D_1 D_2 D_{12}} = -\pi^4 g^{\alpha\beta} \left( \frac{1}{2} + \frac{2}{3} f \right) \neq 0.
\]

It is interesting to investigate how FDR integrals depend on $\mu$.\textsuperscript{11} The first term in the r.h.s. of Eq. (12) does not depend on $\mu$, because $\lim_{\mu \to 0}$ can be moved inside integration. On the other hand, any polynomially divergent integral in $\int_{\text{INF}}(q_1, \ldots, q_k)$ cannot contribute either, being proportional to positive powers of $\mu$, which vanish when $\mu \to 0$. Therefore, the $\mu$ dependence of the l.h.s. is entirely due to powers of $\ln(\mu/\mu_\circ)$ generated by the logarithmically divergent subtracted integrals. Therefore:

i) FDR integrals depend on $\mu$ logarithmically;

ii) if all powers of $\ln(\mu/\mu_\circ)$ are moved to the l.h.s. of Eq. (12), $\lim_{\mu \to 0}$ can be taken by formally trading $\ln(\mu)$ for $\ln(\mu_\circ)$.

Thus, FDR integrals do not depend on any cutoff but only on $\mu_\circ$, which is interpreted as the renormalization scale.\textsuperscript{12}

In summary, higher-order calculations can be performed by interpreting the loop integrals as FDR ones\textsuperscript{14}. FDR directly produces renormalized Green’s functions with no need of an order-by-order renormalization.\textsuperscript{15} The reason is that algebraic manipulations in FDR integrals are allowed as if they where normal convergent loop integrals\textsuperscript{16} and, at the same time, no cutoff remains to be re-absorbed in the bare parameters of the Lagrangian, which is left untouched. This is in contrast with any renormalization procedure based on counterterms, such as DR, where the presence of the cutoff at the intermediate stages of the calculation forces an iterative subtraction à la BPHZ [1–3]. More in particular, FDR differs from the Zimmermann’s definition of loop integration in three aspects:

i) the FDR subtraction is obtained by a formal expansion of the original loop integrands around poles in $q_0^2$, and not via a Taylor expansion in the external momenta;

ii) poles in $q_0^2$ giving rise to UV divergences are subtracted without any attempt of re-introducing them into the Lagrangian;

iii) gauge invariance is automatically respected via Eqs. (10) and (11), while it must be enforced by hand in BPHZ.

Explicit examples of FDR calculations in renormalizable QFTs have been presented in [11–13]. In particular, the two-loop photon self-energy at the leading log and the two-loop $O(\alpha)$ corrections to $H \to \gamma \gamma$ – mediated by an infinitely heavy top loop – have been computed in [13] without using order-by-order counterterms. It is precisely the absence of counterterms, together with the independence of the UV cutoff, which makes it appealing the use of FDR in non-renormalizable theorems, as will be discussed in Sec. 4.2.

3 Classifying the UV divergences

We devote this Section to a classification of the UV divergences contained in $\int_{\text{INF}}(q_1, \ldots, q_k)$. By defining integrals as in Eq. (7), and using the global prescription of Eq. (13), loop calculations can be carried out without making any reference to $\int_{\text{INF}}(q_1, \ldots, q_k)$ [18]. Nevertheless, our classification provides an interesting physical interpretation of the discarded integrands. Thus, we adopt here the

\textsuperscript{10} The global prescription is not applied to the r.h.s. of the first line in Eq. (18) because $q_0^2$ is generated by tensor decomposition and not by Feynman rules. Rather, the relation $q_0^2 = \tilde{q}_0^2 + \mu^2 |_{\text{1}}$ is used to achieve the reduction given in the second line.

\textsuperscript{11} In the absence of IR divergences.

\textsuperscript{12} This is equivalent to a redefinition of FDR integration in which the powers of $\ln(\mu/\mu_\circ)$ are not subtracted. We assume this in the following.

\textsuperscript{13} See Sec. 3.3.

\textsuperscript{14} The connection between FDR and DR at one loop is discussed in Appendix A.

\textsuperscript{15} We emphasize that, at this stage, the QFT (although finite) is not predictive. It becomes so only when the free parameters in the Green’s function are linked to experimental observables (see Sec. 4).

\textsuperscript{16} As already observed, this preserves the symmetries of the QFT.
definition of FDR integration given in Eq. (12) and study the part subtracted from \( J(q_1, \ldots, q_k) \).

We distinguish between scalars and irreducible tensors and show that the UV behavior of any QFT is completely parametrized in terms of a well defined set of logarithmically divergent scalar integrands depending only on \( \mu \). We assume that polynomially divergences in \( J_{\text{INF}}(q_1, \ldots, q_k) \) do not contribute when \( \mu \to 0 \). This is trivially true if DR is used in the r.h.s. of Eq. (12), but it always holds\(^{17}\), because polynomials in any cutoff scale drop in the difference.

### 3.1 Scalars

At the one-loop level just one logarithmically divergent integrand is responsible for all possible occurring UV infinities, independently of the number of external legs and of the degree of divergence of the diagram. Consider, in fact, the quadratically divergent integrand

\[
\frac{1}{D_{\mu}}
\]

where

\[
D_{\mu} = (q + p_\mu)^2 - M_\mu^2 - \mu^2 = \bar{q}^2 - d_i \quad \text{and} \quad p_\mu = 0,
\]

which also defines \( d_i \). It can be FDR expanded as follows

\[
\frac{1}{D_{\mu}} = \left[ \frac{1}{\bar{q}^2} \right] + \frac{d_i}{\bar{q}^2 \cdot D_{\mu}} + \frac{d_i^2}{\bar{q}^2 \cdot D_{\mu} \cdot D_{\mu}}.
\]

The first term does not contribute, being quadratically divergent. Thus, the divergence is entirely generated by the integrand

\[
\left[ \frac{1}{\bar{q}^2} \right],
\]

which is also responsible for the UV behavior of

\[
\frac{1}{D_{\mu} \cdot D_{\mu}}.
\]

In fact

\[
\frac{1}{D_{\mu} \cdot D_{\mu}} = \left[ \frac{1}{\bar{q}^2} \right] + \frac{d_i}{\bar{q}^2 \cdot D_{\mu}} + \frac{d_i^2}{\bar{q}^2 \cdot D_{\mu} \cdot D_{\mu}}.
\]

When the number of loops and legs increases, the situation is similar. A two-loop example is given by the fundamental\(^{18}\) scalar integrand

\[
J(q_1, q_2) = \frac{1}{D_1 D_2 D_{12}},
\]

with denominators written in Eq. (3). Its FDR expansion reads

\[
J(q_1, q_2) = \left[ \frac{1}{\bar{q}_1^2 \bar{q}_2^2 \bar{q}_{12}^2} \right] + \frac{1}{\bar{q}_1^2 \bar{q}_2^2 D_{12}} + \frac{1}{\bar{q}_1^2 D_{12} \bar{q}_{12}^2} + \frac{1}{D_{12} \bar{q}_1^2 \bar{q}_{12}^2} + \frac{1}{D_{12} \bar{q}_2^2 \bar{q}_{12}^2} + \frac{1}{D_{12} D_{12} \bar{q}_{12}^2}.
\]

and since the first term in the r.h.s. is quadratically divergent, the only new logarithmic infinity is

\[
\left[ \frac{1}{\bar{q}_1^2 \bar{q}_2^2 \bar{q}_{12}^2} \right].
\]

At three loops, one obtains five additional logarithmic divergent integrands [9]

\[
\left[ \frac{1}{\bar{q}_1^2 \bar{q}_2^2 \bar{q}_3^2 \bar{q}_{12}^2 \bar{q}_{13}^2} \right], \quad \left[ \frac{1}{\bar{q}_1^2 \bar{q}_2^2 \bar{q}_3^2 \bar{q}_{13}^2} \right], \quad \left[ \frac{1}{\bar{q}_1^2 \bar{q}_3^2 \bar{q}_3^2 \bar{q}_{12}^2} \right], \quad \left[ \frac{1}{\bar{q}_1^2 \bar{q}_2^2 \bar{q}_3^2 \bar{q}_{13}^2} \right], \quad \left[ \frac{1}{\bar{q}_1^2 \bar{q}_2^2 \bar{q}_3^2 \bar{q}_{23}^2} \right],
\]

and so on.

### 3.2 Irreducible tensors

The divergent integrands subtracted from tensors can be immediately read from their scalarized form. For

\(^{17}\) Any regulator can be used to evaluate Eq. (12).

\(^{18}\) All scalars of the type \( 1/(D_1 D_2 D_{12} \bar{q}_{12}^2) \) are obtained by differentiating with respect to \( m_1^2, m_2^2 \) and \( m_{12}^2 \).
The topologies in Fig. 1 produce IR divergent singularities when at no price \( \mu \) between the UV and the IR part and does \( 137 = \alpha \beta \). Notice that, owing to the same logarithm appears, the dependence on \( i \) is replaced by a dependence on \( i \). Therefore, the UV part of the integrand reads

\[
\int \left[ \frac{d^4 q}{(q^2 - M^2)^3} \right]_{\text{INF}} = \frac{g^{\alpha \beta}}{4} \left[ \frac{1}{q^4} \right].
\]

where the scalar in Eq. (23) appears again.

A two-loop example is provided by Eq. (18). The first two terms of the second line contain the divergent scalar integrands dropped from its l.h.s.\(^{19}\). Therefore, the UV part of tensors is reducible to combinations of the same divergent scalar integrands classified in the previous section, that play the role of universal subtraction terms.

### 3.3 Topological renormalization

The classification of infinities given above can be rephrased in terms of vacuum topologies. The one-, two- and three-loop logarithmically divergent scalar integrands in Eqs. (23), (28) and (29) can be thought as the vacuum bubbles of Fig. 1.

An interpretation which legitimates their subtraction from the original integrands is that they represent universal unphysical vacuum configurations, which do not take part in the scattering process \([9]\). Differently stated, the subtraction of \( J_{\text{INF}} \) in Eq. (12) can be understood as an order-by-order redefinition of the perturbative vacuum, which we dub topological renormalization, and FDR is an operative way to choose, order by order in the loop expansion, the right vacuum. Thus, the problem with the UV divergences is not where they can be accommodated, but rather what is the price of subtracting them. As pointed out in Sec. 2, polynomially divergent integrands can be dropped at no price\(^{20}\), since they naturally decouple, without any observable effect, in the limit \( \mu \to 0 \). Conversely, logarithmic divergent integrands cannot be fully subtracted, because, by doing so, logarithmic IR\(^{21}\) divergences would be generated (when \( \mu \to 0 \)) in \( J_{F, F} \), which one interprets as the physical part of the interaction. Therefore, in order to keep \( J_{F, F} \) finite and independent on \( \mu \), one introduces an (arbitrary) separation scale \( \mu_s \) between the UV and the IR part and does not subtract the latter. This can be clearly exemplified in the context of the one-loop example of Eq. (25), regularized with a momentum cutoff \( \Lambda_{\text{UV}} \). The integral over the divergent contribution reads

\[
\int_{\Lambda_{\text{UV}}} \frac{1}{q^4} \left[ 1 + \ln \frac{\mu_s}{\Lambda_{\text{UV}}} \right] = -i \pi^2 \left( 1 + \ln \frac{\mu^2}{\mu_s^2} + \ln \frac{\mu^2}{\Lambda_{\text{UV}}^2} \right)
\]

and, if \( \ln \frac{\mu^2}{\mu_s^2} \) is not subtracted, the dependence on \( \mu \) of \( J_{F, F} \) is replaced by a dependence on \( \mu_s \), which plays the role of the renormalization scale, so that \( \lim \mu \to 0 \) can be taken. The DR version of Eq. (32) is given in Eq. (A5), where the same logarithm appears.\(^{22}\) Notice that, owing to the IR origin of \( \ln \mu^2 \), any UV regulator produces the same coefficient in front of the logarithm. Different constants generated by different UV regulators are immaterial because they are fully subtracted. This is again a consequence of the independence of FDR from any regularization procedure.

With all of that in mind, to renormalize a QFT at \( \ell \)-loops one simply drops the divergent integrands, computes the physical part \( J_{F, F} \) in the limit \( \mu \to 0 \) (to get its logarithmic dependence) and evaluate the result in \( \mu = \mu_s \), which corresponds to the definition of FDR \( \ell \)-loop integral.

### 4 Fixing renormalizable and non-renormalizable theories

In what follows, we consider the problem of fixing a QFT in the context of the topological renormalization approach driven by FDR. Consider a theory described by a

\[20\] They are obtained by differentiating Eq. (27) with respect to \( m_1^2 \).

\[21\] The topologies in Fig. 1 produce IR divergent singularities when \( \mu \to 0 \). Discarding such IR infinities has nothing to do with the UV subtraction.

\[22\] Now the arbitrary scale of DR becomes the renormalization scale.
Lagrangian
\[ \mathcal{L}(p_1, \ldots, p_m) \] (33)

dependent on \( m \) bare parameters \( p_i \), with \( i = 1 : m \). Before making any prediction, the \( p_i \) must be fixed in terms of \( m \) measurements

\[ O_i^{\text{EXP}} = O_i^{\text{TH}, \ell \text{-loop}}(p_1, \ldots, p_m), \] (34)

which determine them in terms of measured observables \( O_i^{\text{EXP}} \) and corrections computed at the loop level \( \ell \) one is working

\[ \bar{p}_i^{\ell \text{-loop}}(O_1^{\text{EXP}}, \ldots, O_m^{\text{EXP}}) \equiv \bar{p}_i. \] (35)

Both the \( p_i \) and the \( \bar{p}_i \) are finite in FDR. Since the UV divergences have already been dropped, the issue of fixing the QFT is separated from the UV subtraction. Thus, a global determination of the bare parameters is possible: the \( p_i \) do not need to be calculated iteratively in the perturbative expansion when inverting Eqs. (34).\(^{23}\) We refer to the procedure leading to Eqs. (35) as a global finite renormalization.

Once the \( \bar{p}_i \) are known, the QFT becomes predictive. In the following two Sections, we discuss renormalizable and the non-renormalizable theories in turn.

4.2 Non-renormalizable QFTs

If \( \mathcal{L} \) is non-renormalizable, the global finite renormalization does not necessarily\(^{24}\) compensate the dependence on \( \mu_a \) of \( C_{m+1}^{\text{TH}, \ell \text{-loop}} \), which might depend on \( \ln \mu_a \) at the same perturbative order one is computing

\[ C_{m+1}^{\text{TH}, \ell \text{-loop}}(\bar{p}_1, \ldots, \bar{p}_m, \ln \mu_a). \] (41)

This is due to the fact that in this case different combinations of interactions with a different dependence on \( \mu_a \) appear in \( C_{m+1}^{\text{TH}, \ell \text{-loop}} \) and in the r.h.s. of Eqs. (34).

Since \( \mu_a \) is not calculable in the framework of the theory itself, the QFT has to be considered as an effective one. However, it is possible to restore its predictivity in the infinite loop limit by determining \( \mu_a \) from data.\(^{25}\) In fact, combinations of observables in which \( \mu_a \) disappears can still be unambiguously predicted. For instance, one computes, at one loop,

\[ C_{m+1}^{\text{TH}, 1 \text{-loop}}(\bar{p}_1, \ldots, \bar{p}_m, \ln \mu_a) = c_1 \ln \mu_a + d_1 \]

\[ C_{m+2}^{\text{TH}, 1 \text{-loop}}(\bar{p}_1, \ldots, \bar{p}_m, \ln \mu_a) = c_2 \ln \mu_a + d_2, \] (42)

\(^{24}\) It is interesting to note that the absence of UV infinities in \( I_{\text{INF}} \) is a sufficient but not necessary condition for the absence of \( \ln \mu_a \ln f_{E, \ell} \). Consider, for instance, the combination

\[ J(q_1, q_2) = \left( \frac{2}{D_1^2 D_2^2 D_1 D_2} - \frac{1}{D_1^2 D_2^2} + 4n_2^2 \right), \] (38)

with \( n_1^2 = n_2^2 = m_{12}^2 = n^2 \). One computes

\[ \int [d^4 q_1][d^4 q_2] J(q_1, q_2) = 2\pi^4 f, \] (39)

[with \( f \) given in Eq. (17)], which does not depend on \( \mu_a \). While integrating in \( n \) dimensions its infinite part gives

\[ \mu_a^{-2\epsilon} \int d^4 q_1 d^4 q_2 J(q_1, q_2)_{\text{INF}} \]

\[ = \pi^4 \left[ -2 \left( \frac{1}{\epsilon} + \ln \pi + \gamma_E + \ln \frac{m^2}{\mu_a^2} \right) - 1 + 2 f \right], \] (40)

which contains a \( 1/\epsilon \) pole.

\(^{25}\) Here the FDR replacement of \( \mu \rightarrow \mu_a \) discussed in Sec. 4.3 plays an essential role. The scale \( \mu_a \) can be adjusted, while \( \mu \rightarrow 0 \).

\(^{23}\) This is due to the absence of counterterms.
and a quantity independent of $\mu_a$ can be constructed as follows

$$
O_{\text{Predictable}}^{\text{EXP}}(\bar{p}_1, \ldots, \bar{p}_m) = \frac{O_{m+1}^{\text{EXP}}(\bar{p}_1, \ldots, \bar{p}_m, \ln \mu_a)}{c_1}
- \frac{O_{m+2}^{\text{TH}}(\bar{p}_1, \ldots, \bar{p}_m, \ln \mu_a)}{c_2}
= \frac{d_1}{c_1} - \frac{d_2}{c_2},  \tag{43}
$$

which is equivalent to extracting $\ln \mu_a$ from $O_{m+2}^{\text{EXP}} = O_{m+2}^{\text{TH}, 1-\text{loop}}$ and inserting the result in $O_{m+1}^{\text{TH}, 1-\text{loop}}$. This can be generalized to $\ell$-loops. One additional measurement\textsuperscript{26} fixes $\mu_a$ at any order

$$
O_{m+2}^{\text{EXP}} = O_{m+2}^{\text{TH}, \ell-\text{loop}}(\bar{p}_1, \ldots, \bar{p}_m, \ln \mu'_a), \tag{44}
$$

and $O_{m+1}^{\text{TH}, \ell-\text{loop}}(\bar{p}_1, \ldots, \bar{p}_m, \ln \mu'_a)$ is a prediction\textsuperscript{27} of the non-renormalizable QFT in terms of the $m$ input observables $O_{\text{EXP}}^{\text{EXP}}$ of Eq. (34) and $O_{\text{EXP}}^{\text{TH}}$, as pictorially illustrated in Fig. 2.\textsuperscript{28} Notice that the functional dependence on $\mu_a$ is predicted by the QFT at any given perturbative order.

For example, the one-loop curves describing $O_{m+1}^{\text{TH}}$ and $O_{m+2}^{\text{TH}}$ as functions of $\ln(\mu_a)$ are straight lines with slopes $c_1$ and $c_2$ as given in Eq. (42).

It is important to realize that, within the proposed formulation, nothing can be said on the form of the bare Lagrangian. If $L$ does contain an infinite amount of interactions, they have to be fixed anyway, either by knowing the underlying theory (as usually done when matching HQET and NRQCD with QCD [22, 23]) or by experiment. Nevertheless, at fixed number of terms in $L$, all QFTs, including the non-renormalizable ones, can be tested experimentally at any loop order. Unlike in the customary effective approach to non-renormalizable QFTs [8, 24], there is no need, in FDR, to introduce new interactions to absorb UV infinities, and interactions other than those originally present in the bare Lagrangian are generated only by loops.

It is possible that QFTs exist for which an internal determination of their UV completion is intrinsically impossible or not at reach. The suggested procedure can then be used to rescue such theories in their minimal form. In other words, parametrizing unknown or uncomputable UV effects in terms of a single adjustable scale $\mu_a$ might be considered as a powerful and economic way to restore predictivity.

5 Conclusions

Based on the FDR classification of the UV infinities in terms of their topology, an alternative interpretation of the renormalization procedure can be formulated. Topological renormalization is introduced as an operative way to subtract the divergences directly at the level of the integrand of the loop functions, that, in turn, can be thought as an order-by-order redefinition of the perturbative vacuum.

This formulation is equivalent to the standard renormalization procedure for renormalizable theories - although technically simpler - and, in the case of non-renormalizable theories, it allows one to fit the scale at which the UV subtraction is performed. The net effect is that the bare Lagrangian is left untouched and one additional measurement fixes the theory, which becomes predictive in the infinite loop limit. In this context, the validity of a Quantum Field Theory should be judged a-posteriori by comparing predictions with data and not necessarily based on its renormalizability.

\textsuperscript{26} The role of this extra measurement is separating any UV effect from the physical degrees of freedom.

\textsuperscript{27} Requiring perturbativity implies the condition $|g^2 \ln \mu'_a| < 1$ on the solution $\mu'_a$, where $g$ is the coupling constant of the QFT.

\textsuperscript{28} A similar procedure has been studied by Bettinelli, Ferrari and Quadri [9] in the context of the nonlinear sigma model [20] and of the massive Yang-Mills theory [21]. They obtained a predictive version of such non-renormalizable QFTs in DR by throwing away poles in $\epsilon$ and fixing the renormalization scale with an additional measurement.
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1 Appendix: FDR versus Dimensional Regularization

In this appendix, we prove that one-loop FDR tensors [as defined in Eq. (1)] and DR tensors in $\overline{\text{MS}}$ coincide. As a consequence, FDR is equivalent, at one loop, to Dimensional Reduction in the $\overline{\text{MS}}$ scheme.

Consider, in fact, the integrand of a rank-two three-function point

$$\frac{q^\alpha q^\beta}{D_\alpha D_\beta D_\rho D_\sigma}$$

which does not match Eq. (19). This is due to the presence of the cutoff $\epsilon$. In DR physical results are obtained only after poles and constants created by tensor decomposition are consistently combined with lowest order counterterms required by the order-by-order renormalization. Conversely, FDR tensors are regulator free objects and, thanks to the absence of counterterms, physical constants are solely generated by extra integrals via tensor decomposition.

A corollary of this theorem is the equivalence, at one loop, between FDR and Dimensional Reduction$^{29}$ in the $\overline{\text{MS}}$ scheme.

At $\ell$ loops, with $\ell > 1$, DR tensors differ from their FDR counterpart. For example

$$\mu^{-2\ell} \int d^n q_1 d^n q_2 \frac{q_1^\alpha q_2^\beta - q_1^\beta q_2^\alpha}{D_1^\alpha D_1^\beta D_2^\alpha D_2^\beta}$$

which does not match Eq. (19). This is due to the presence of the cutoff $\epsilon$. In DR physical results are obtained only after poles and constants created by tensor decomposition are consistently combined with lowest order counterterms required by the order-by-order renormalization. Conversely, FDR tensors are regulator free objects and, thanks to the absence of counterterms, physical constants are solely generated by extra integrals via tensor decomposition.

**Key words.** quantum field theory, renormalization, effective theories, predictivity.

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$^{29}$ Where no explicit $\epsilon$ appears in the numerator of the loop integrals.

$^{30}$ DR and FDR coincide at one loop owing to the lack of lowest order counterterms.
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