On the Complexity of finding Stopping Distance in Tanner Graphs

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Abstract— Two decision problems related to the computation of stopping sets in Tanner graphs are shown to be NP-complete. NP-hardness of the problem of computing the stopping distance of a Tanner graph follows as a consequence.

I. INTRODUCTION

Stopping sets were introduced in [1] for the analysis of erasure decoding of LDPC codes. It was shown that the iterative decoder fails to decode to a codeword if and only if the set of erasure positions is a superset of some stopping set in the Tanner graph [8] used in decoding. Considerable analysis has been carried out on the distribution of stopping set sizes in LDPC code ensembles, giving valuable insight into the asymptotic performance of message-passing decoding on LDPC ensembles — see for example [2], [3]. Since small stopping sets are directly responsible for poor performance of iterative decoding algorithms, it is of interest to determine the size of the smallest stopping set in a Tanner graph, called the stopping distance of the graph. Construction of codes for which there are Tanner graphs that do not contain small stopping sets has been studied — see for example [4], [5]. The stopping distance of the graph, is of interest as it gives the minimum number of erasures that can cause iterative decoding to fail.

The relationship between stopping distance and other graph parameters like girth has been explored in [6] where it is shown that large girth implies high stopping distance. Pishro-Nik and Fekri [12] showed that by adding a suitable number of parity checks the stopping distance of a Tanner graph for a code can be increased to the maximum possible, viz., the minimum distance of the code. Schwartz and Vardy [7] defines the stopping redundancy of a code as the minimum number of rows in a parity check matrix for the code such that the stopping distance of the corresponding Tanner graph is equal to the minimum distance of the code and proves some bounds on the stopping redundancy for various classes of codes. Further investigations on stopping redundancy may be found in [13].

In this correspondence, we show that the computational problems of determining whether a given Tanner graph has a stopping set of a given size or at most a given size are NP-complete. These are shown by reductions from the well known NP-complete problems of determining whether a given graph contains a vertex cover of a given size (respectively at most a given size) to the above problems. NP-hardness of the problem of finding the stopping distance of a Tanner graph follows as a consequence of the latter result.

II. BACKGROUND

Given a parity check matrix \( H = [h_{ij}] \in GF(2)\times n \), \( 1 \leq k \leq n \) for an \((n,k)\) binary linear code, the Tanner graph is the undirected bipartite graph \( G = (L,R,E) \) where \( L = \{x_i, 1 \leq i \leq n\} \), \( R = \{c_j, 1 \leq j \leq n-k\} \) and \( E = \{(x_i, c_j) : h_{ji} = 1, 1 \leq i \leq n, 1 \leq j \leq n-k\} \). The set \( L \) corresponds to the set of codeword elements and \( R \) corresponds to the set of parity checks. We refer to the set \( L \) and \( R \) as the set of left and right vertices respectively. For \( S \subseteq L \cup R \), we define \( N(S) = \{y : (x,y) \in E, x \in S\} \). \( S \subseteq L \) is a stopping set if for all \( c_j \in N(S) \), \( |N(\{c_j\}) \cap S| \geq 2 \) i.e., every vertex connected to some vertex in a stopping set must have at least two neighbours in the stopping set. The stopping distance of a Tanner graph is the size of the smallest stopping set in the graph. We define two decision problems concerning stopping sets:

Problem 1: STOPPING SET: Given a Tanner graph \( G \) and positive integer \( t \), does \( G \) have a stopping set of size \( t \).

Problem 2: STOPPING DISTANCE: Given a Tanner graph \( G \) and positive integer \( t \), does \( G \) have a stopping set of size at most \( t \).

Note that the corresponding decision problems arising out of the problem of finding the minimum distance of a code were shown to be NP-complete in [15] and [14].

It is clear that if either STOPPING SET or STOPPING DISTANCE can be solved in polynomial time, then evoking the algorithm at most \( |L| \) times, the problem of actually finding the stopping distance of a Tanner graph can be solved. Conversely, if there is a polynomial time algorithm for finding the stopping distance of a given Tanner graph \( G \), then we can use the algorithm to solve STOPPING DISTANCE since \( G \) has stopping distance less than or equal to \( t \) if and only if \( G \) contains a stopping set of size less than or equal to \( t \). Note that it is not immediately clear how to solve STOPPING SET in polynomial time even if a polynomial time algorithm for computing the stopping distance of a Tanner graph is known.

The notion of NP-completeness was introduced in [11], and is well established in the computer science literature for the
analysis of the computational complexity of problems (see \cite{9}, \cite{10} for a detailed account). Typically, a problem is posed as a decision problem, i.e., one where the solution consists of answering it with a \textit{yes} or a \textit{no}. All inputs for which the answer is a \textit{yes} from a set. We identify this set with the problem. A decision problem \( A \) belongs to the class NP if there exists a polynomial time algorithm \( \Pi \) such that, for all \( x \in A \), there exists a string \( y \) (called a \textit{certificate} for membership of \( x \) in \( A \)), with \( |y| \) polynomially bounded in \( |x| \), such that \( \Pi \) accepts \((x, y)\), whereas, for all \( x \notin A \), \( \Pi \) rejects \((x, y)\) for any string \( y \) presented to the algorithm. In other words, problems in NP are precisely those for which membership verification is polynomially solvable. We say a decision problem \( A \) is \textit{polynomial time many-one reducible} to a decision problem \( B \) if there exists a polynomial time algorithm \( \Pi' \) such that, given an instance \( x \) of \( A \), \( \Pi' \) produces an instance \( z \) of \( B \) satisfying \( z \in B \) if and only if \( x \in A \). In such case, we write \( A \leq_p B \). A problem \( A \in \text{NP} \) is NP-complete if for every \( X \in \text{NP} \), \( X \leq_p A \). It is generally believed that NP-complete problems have no polynomial time algorithms.

Given an undirected graph (not necessarily bipartite) \( G = (V, E) \), \( S \subseteq V \) is a \textit{vertex cover} in \( G \) if for all \((u, v) \in E\) either \( u \in S \) or \( v \in S \) or both. We will be using in our reductions the following decision problems associated with the computation of vertex covers in a graph.

\textbf{Problem 3: VERTEX COVER:} Given a graph \( G \) and a positive integer \( t \) does \( G \) contain a vertex cover of size at most \( t \).

The above problem is shown to be \text{NP}-complete in \cite{10, p. 190}. A variant of this problem referred to by the same name and shown to be \text{NP}-complete in \cite[pp. 949–950]{9} will be referred to here as the following:

\textbf{Problem 4: VERTEX COVER(\(=\)):} Given a graph \( G \) and a positive integer \( t \) does \( G \) contain a vertex cover of size equal to \( t \).

In the following section we show that both \text{STOPPING DISTANCE} and \text{STOPPING SET} are \text{NP}-complete by establishing polynomial time many-one reductions from \text{VERTEX COVER} and \text{VERTEX COVER(\(=\))} respectively to the above problems.

\section*{III. HARDNESS OF STOPPING DISTANCE}

Let \((G = (V, E), t)\) be an instance of the \text{VERTEX COVER} problem. Let \(|V| = n\), \(|E| = m\). Excluding trivial cases of the problem we may assume \( 1 \leq t \leq n - 1 \). We shall make the further assumption that \( G \) is connected. It is not hard to show that both \text{VERTEX COVER} and \text{VERTEX COVER(\(=\))} remain \text{NP}-complete even when restricted to connected graphs.

The vertex-edge incidence graph of \( G \) is the undirected bipartite graph \( G' = (L, R, E') \) with \( L = V \), \( R = E \) and edges \((e, u)\) and \((e, v)\) in \( E' \) for each \( e = (u, v) \in E \). Figure 1 shows the vertex-edge incidence graph for a graph \( G \) with \( n = 4 \) and \( m = 3 \).

The advantage of assuming that \( G \) is connected arises out of the following lemma:

\textbf{Lemma 1:} Let \( G' = (L, R, E') \) be the vertex-edge incidence graph of a connected graph \( G = (V, E) \). Let \( S \) be a stopping set in \( G' \). Then \( S = L \).

\textit{Proof:} Let \( L \setminus S \neq \emptyset \). Then, as \( G \) is connected there exists \( v \in L \setminus S \) and \( u \in S \) such that \((u, v) \in E \). Let \( e = (u, v) \). Then \( e \in N(S) \). Since \( S \) is a stopping set \(|N(e) \cap S| \geq 2 \). But the only neighbours of \( e \) in \( G' \) are \( u \) and \( v \). Hence \( v \in S \) contradicting \( v \in L \setminus S \).

We construct an undirected bipartite graph \( G'' \) as follows: \( L = \bigcup_{i=0}^{m+1} L_i, R = \bigcup_{j=0}^{n+1} R_j \), where, \( R_0 = \{z_1, ..., z_{m-1}\}, R_j = \{u_j, u \in V\} \) for \( 2 \leq j \leq m + 1 \), \( R_1 = L_0 = E, L_j = \{u_1, u \in V\} \) for \( 1 \leq i \leq m + 1 \). Edges in \( G'' \) are connected as the following:

- Connect \( u'_i \in L_i \) to \( u'_i \in R_i \), \( 2 \leq i \leq m + 1 \).
- Connect \( u'_i \in L_i \) to \( u'_{i-1} \in R_i-1 \), \( 1 \leq i \leq m \).
- For each \( e = (u, v) \in E \), connect \( e \in R_1 \) to \( u \) and \( v \) in \( L_1 \).
- For each \( e \in E \) Connect \( e \in L_0 \) to \( e \in R_1 \).
- For the purpose of defining the edges between \( R_0 \) and \( L_0 \), temporarily re-label vertices in \( L_0 \) as \( e_1, e_2, ..., e_m \) in some arbitrary way. Add the edges \((e_i, z_i)\) for \( 1 \leq i \leq m - 1 \) and the edges \((e_i, z_{i-1})\) for \( 2 \leq i \leq m \).

The example in figure 3 illustrates the construction of \( G'' \) for the graph in figure 1. The graph \( G'' \) consists of a copy of the vertex-edge incidence graph of \( G \) (vertex sets \( L_1 \) and \( R_1 \)). Additionally, there are \( m \) copies of the vertex set \( V \) on the left \((L_2, L_3, ..., L_{m+1})\) and right \((R_2, R_3, ..., R_{m+1})\). The connections between \( R_0 \) and \( L_0 \) ensure that any stopping set in \( G'' \) containing any one vertex in \( L_0 \) must contain the whole of \( L_0 \). The vertex \( u'_i \) in \( R_i \) has neighbours \( u'_{i-1} \) and \( u'_i \) for each \( 2 \leq i \leq m + 1 \) and each \( u \in V \). This ensures that if a stopping set \( S \) in \( G'' \) contains \( u'_i \) for some \( i \in \{1, 2, ..., m + 1\} \) then all the \( m + 1 \) vertices \( u'_1, u'_2, ..., u'_{m+1} \) must be present in \( S \). These observations summarized below play a crucial role in the arguments that follow.

\textbf{Observation 1:} A stopping set \( S' \) in \( G'' \) satisfies \( u'_i \in S' \) for some \( 1 \leq i \leq m + 1 \) if and only if it satisfies \( u'_i \in S'' \) for every \( 1 \leq i \leq m + 1 \). Moreover either \( L_0 \subseteq S' \) or \( L_0 \cap S' = \emptyset \).

The following two claims establish the connection between
vertex covers in $G$ and stopping sets in $G''$.

Lemma 2: If $G$ contains a vertex cover $S$ of size $t$ for some $1 \leq t \leq n-1$ then $G''$ contains a stopping set of size $t(m+1) + m$.

Proof: Consider the set $S' = L_0 \cup \{u_i' : u \in S, 1 \leq i \leq m+1\}$ in $G''$. Clearly $S'$ has $t(m+1) + m$ elements. Let $w \in N(S')$. Then either $w = u_i'$ for some $u \in S$, $i \in \{2, 3, ..., m+1\}$ or $w \in R_1$ or $w \in R_0$. In the first case, both $u_i'$ and $u_{i-1}'$ are neighbours of $w$. If $w \in R_1$, then by construction, $w$ must correspond to some edge $e = (u, v)$ in $E$. Since $L_0 \subseteq S'$, $e \in L_0$ is a neighbour of $w$. Since $S$ is a vertex cover in $G$, either $u$ or $v$ or both must belong to $S$. Hence one or both of $u_i'$ and $v_i'$ must be a neighbour of $w$ in $S'$. Finally if $w \in R_0$, then both the neighbours of $w$ are in $L_0$, and therefore in $S'$. Thus in all cases $w$ has at least two neighbours in $S'$. Consequently $S'$ is a stopping set.

We now prove that every stopping set in $G''$ of size less than $n(m+1) + m$ must correspond to some vertex cover of size $t$ in $G$ for some $1 \leq t \leq n-1$ and must have size exactly $t(m+1) + m$.

Lemma 3: Let $S'$ be a stopping set in $G''$ of size less than $n(m+1) + m$. Then the following must hold:

- $L_0 \subseteq S'$,
- $|S'| = t(m+1) + m$ for some $1 \leq t \leq n-1$ and $|S' \cap L_1| = t$ for every $1 \leq t \leq m + 1$.
- $S = \{u \in V : u_i' \in S' \text{ for some } 1 \leq i \leq m + 1\}$ is a vertex cover of size $t$ in $G$.

Proof: Suppose $L_0$ is not contained in $S'$. Then by Observation 1, $L_0 \cap S' = \emptyset$. Since $S' \neq \emptyset$, there must be some $u \in V$ and $i \in \{1, 2, ..., m+1\}$ such that $u_i' \in S'$. By Observation 1, $u_i' \in S'$. Since vertices in the set $R_1$ are connected only to $L_1$ and $L_0$, every neighbour of $S'$ in $R_1$ must have two neighbours in $S' \cap L_1$ in order for $S'$ to satisfy the conditions of a stopping set. In other words, $S' \cap L_1$ must be a stopping set in the subgraph of $G''$ induced by the vertices $L_1 \cup R_1$. Note that this subgraph is the vertex-edge incidence graph of $G$. Applying Lemma 1 we get $S' \cap L_1 = L_1$. Hence by Observation 1, $S' = \bigcup_{t=1}^{m+1} L_t$. But in that case $|S'| = n(m+1) + m$, a contradiction. Hence $L_0 \subseteq S'$ and $|S' \cap L_1| = t$ for some $1 \leq t \leq n-1$. Applying Observation 1 once again, $|S' \cap L_i| = t$ for all $1 \leq i \leq m + 1$. Hence $|S'| = t(m+1) + m$.

To complete the proof of the lemma, it is sufficient to prove that $S = \{u \in V : u_i' \in S'\}$ is a vertex cover of $G$. Since $L_0 \subseteq S'$, $R_1 \subseteq N(S')$. Since every vertex $e$ in $R_1$ has only one neighbour in the set $L_0$, for $S'$ to satisfy the stopping set condition $e$ must have a neighbour in $L_1 \cap S'$. Then, by construction $\{u \in V : u_i' \in S'\}$ must be a vertex cover in $G$ as required.

As a consequence of Lemma 2 and Lemma 3 we have:

Corollary 1: $G$ has a vertex cover of size $t$ if and only if $G''$ has a stopping set of size $t(m+1) + m$, $1 \leq t \leq n-1$. Hence $(G, t) \in \text{VERTEX COVER}(=)$ if and only if $(G'', t(m+1) + m) \in \text{STOPPING SET}$.

Corollary 2: $G$ has a vertex cover of size at most $t$ if and only if $G''$ has a stopping set of size at most $t(m+1) + m$, $t \in \{1, 2, ..., n-1\}$. Hence $(G, t) \in \text{VERTEX COVER}$ if and only if $(G'', t(m+1) + m) \in \text{STOPPING DISTANCE}$.

We are now ready to prove:

Theorem 1: STOPPING DISTANCE and STOPPING SET are NP-complete.

Proof: We have proved that $(G, t) \in \text{VERTEX COVER}$ if and only if $(G'', t(m+1) + m) \in \text{STOPPING SET}$.

Since $G''$ can be constructed from $G$ in polynomial time (in time $O(nm)$ time suffices), it follows that $\text{VERTEX COVER}(=) \leq_p \text{STOPPING SET}$ and $\text{VERTEX COVER} \leq_p \text{STOPPING DISTANCE}$ from Corollary 1 and Corollary 2 respectively. It is easy to verify whether a given set of left vertices of a bipartite graph forms a stopping set in time linear in the size of the graph. Hence both STOPPING DISTANCE and STOPPING SET belong to the class NP.

As a consequence, we have:

Corollary 3: Computing stopping distance in a Tanner graph is NP-hard.

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