Strategy to find the two $\Lambda(1405)$ states from lattice QCD simulations

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Theoretical studies within the chiral unitary approach, and recent experiments, have provided evidence of the existence of two isoscalar states in the region of the $\Lambda(1405)$. In this paper we use the same chiral approach to generate energy levels in a finite box. In a second step, assuming that these energies correspond to lattice QCD results, we devise the best strategy of analysis to obtain the two states in the infinite volume case, with sufficient precision to distinguish them. We find out that using energy levels obtained with asymmetric boxes and/or with a moving frame, with reasonable errors in the energies, one has a successful scheme to get the two $\Lambda(1405)$ poles.

I. INTRODUCTION

The history of the $\Lambda(1405)$ as a composite state of meson baryon, dynamically generated from the meson baryon interaction, is rather long, starting from the works of Refs. \textsuperscript{1,2}. Early works using the cloudy bag model also reached similar conclusions \textsuperscript{3}. The advent of chiral unitary theory, combining chiral dynamics and unitarity in coupled channels, brought new light into this issue and the $\Lambda(1405)$ was one of the cleanest examples of states dynamically generated within this approach \textsuperscript{4,5,6}. Hints that there could be two states rather than one had also been reported using the cloudy bag model \textsuperscript{7} and the chiral unitary approach \textsuperscript{8,9,10}. A qualitative step forward was done in Ref. \textsuperscript{11}, where two different versions of the approach were used, the two poles remained, and their origin was investigated. It was found that in an SU(3) symmetric theory there were two degenerate octets and a singlet of dynamically generated resonances, but with the breaking of SU(3) the degeneracy was removed, one octet with isospin $I = 0$ moved to become the $\Lambda(1670)$ and the other one moved close to the singlet, producing two poles close by in the region of the $\Lambda(1405)$. One of the poles appears at energies around 1420 MeV, couples mostly to $K\Lambda$ and has a small width of around 30 MeV. The other pole is around 1395 MeV, couples mostly to $\pi\Sigma$ and is much wider, around 120 or 250 MeV depending on the model. After the work of Ref. \textsuperscript{11}, all further works on the chiral unitary approach have corroborated the two poles, with remarkable agreement for the pole at higher energy and larger variations for the pole at lower energies \textsuperscript{12,13,14,15,16,17,18,19,20,21}. Suggestions of experiments to confirm this finding were made, and it was shown that one should not expect to see two peaks in the cross sections, but rather different shapes in different reactions. In this sense, a suggestion was made to look for the $\Lambda(1405)$ peak in the $K^+p \rightarrow \gamma\pi\Sigma$ reaction \textsuperscript{22}, where the $\gamma$ would be radiated from the initial state, making the $K^+p$ system to lose energy and go below threshold and then excite the high energy state of the $\Lambda(1405)$, to which it couples most strongly. This reaction was not made, although it is planned for JPARC \textsuperscript{21}, but a similar one, where the photon was substituted by a pion, was implemented in Ref. \textsuperscript{22} studying the $K^+d \rightarrow \pi^0\pi^0\Sigma^0$ reaction at $p_K = 514$ MeV/c - 750 MeV/c. A neat and narrow peak was seen at $\sqrt{s} = 1420$ MeV, which was analyzed in Ref. \textsuperscript{23} and interpreted in terms of the high energy pole of the $\Lambda(1405)$. More recently it was noticed that old data on the $K^+d \rightarrow \pi^-\pi^0\Sigma^0$ reaction from Ref. \textsuperscript{24} produced a peak in the $\pi\Sigma$ spectrum around $\sqrt{s} = 1420$ MeV, with also a small width. These data were well reproduced in Ref. \textsuperscript{25} within the chiral unitary approach and multiple scattering, and once again it was shown that it gave support to the existence of the second pole of the $\Lambda(1405)$. It was shown there that the reaction proceeded with kaons in flight but not for stopped kaons, because the background from single scattering was too large in this latter case, obscuring the signal of the resonance that stems from double scattering. Even then, it was shown in Ref. \textsuperscript{26} that kaons from the DAFNE facility, coming from the decay of the $\phi$, would also be suited to search for this resonance if neutrons were measured in coincidence in order to reduce the background. The search for reactions where the $\Lambda(1405)$ is produced has continued, showing that, as predicted, different reactions have different shapes. In this sense there have been recent photoproduction experiments \textsuperscript{27,28} and proton induced experiments \textsuperscript{29,30} where the shapes are indeed different and the peaks appear at lower energies, around 1405 MeV, as the nominal mass. There are also theoretical studies for these reactions where the peaks appear around these energies, and the larger contribution of the lower energy state that couples mostly to $\pi\Sigma$ is mostly responsible for it \textsuperscript{31,32}.

In as much as chiral dynamics is a good representation of QCD at low energies, the predictions of the chiral unitary approach on the $\Lambda(1405)$ stand on firm ground.
Yet, it would also be very interesting to have these predictions confirmed with lattice QCD simulations. In this sense, the determination of hadron spectra is one of the challenging tasks of Lattice QCD and many efforts are being devoted to this problem \cite{66,67,68,69}, some of them in particular to the search of the Λ(1405) \cite{70,71,72,73}. A review on the Lambda(1405) and attempts to see it from different points of view is given in Ref. \cite{74}. In some works the “avoided level crossing” is usually taken as a signal of a resonance, but this criteria has been shown insufficient for resonances with a large width \cite{75,76,77,78,79}. Sometimes, the lattice spectra at finite volumes is directly associated to the energies of the states in infinite volume invoking a weak volume dependence of the results, as done recently searching for the Λ(1405) resonance \cite{78}. A more accurate method consists on the use of Lüscher’s approach, for resonances with one decay channel. The method allows to reproduce the phase shifts for the decay channel from the discrete energy levels in the box \cite{71,72}. This method has been recently simplified and improved in Ref. \cite{75} by keeping the full relativistic two body propagator (Lüscher’s approach keeps the imaginary part of this propagator exactly but makes approximations on the real part) and extending the method to two or more coupled channels. The method has also been applied in Ref. \cite{75} to obtain finite volume results from the Jülich model for meson baryon interaction, including spectra for the Λ(1405) with finite volume, and in Ref. \cite{76}, to study the interaction of the DK and ηD, system where the D∗(2317) resonance is dynamically generated from the interaction of these particles \cite{77,78}. The case of the κ resonance in the Kπ channel is also addressed along the lines of Ref. \cite{77} in Ref. \cite{78}.

In the work of Ref. \cite{78}, the inverse problem of getting phase shifts and resonances from lattice results using two channels was addressed, paying special attention to the evaluation of errors and the precision needed on the lattice results to obtain phase shifts and resonance properties with a desired accuracy. Further work along these lines is done in Ref. \cite{78}. The main problem encountered is that the levels obtained from the box of a certain size range do not cover all the desired energy region that one would like to investigate. Several suggestions are given in order to produce extra levels, like using twisted boundary conditions or asymmetric boxes \cite{79}. These are, however, not free of problems since it is unclear whether a full twisting can be done in actual QCD simulations including sea quarks, and the asymmetric boxes have the problem of the possible mixing of different partial waves. Another alternative is to evaluate levels for a system in a moving frame as done in Ref. \cite{73}, but this also poses problems of mixing in principle. The generalization of Lüscher’s approach to the moving frame is done in Refs. \cite{73,74}, and it provides a convenient framework for lattice calculations since new levels can be obtained without enlarging the size of the box, with an economy in computational time. It is then quite convenient to carry out simulations using effective theories in a finite volume, preparing the grounds for future lattice calculations, trying to find an optimal strategy on which configurations to evaluate in order to obtain the desired observables in the infinite volume case.

The case of extracting the Λ(1405) parameters is specially challenging, particularly because two resonance must be found which are not too far from each other, which means that extra precision will be demanded to the lattice results. Furthermore, the two poles are not to be seen in the πΣ phase shifts, since, as mentioned before, different amplitudes give different weight to the two poles and the πΣ phase shifts provide insufficient information. The other reason is that the chiral unitary approach tells us that the two states couple strongly to K N and πΣ, so the use of the two channels in the analysis is mandatory and the use of one channel like in Luscher approach is bound to produce incorrect results. In view of this we face the problem using the two channels explicitly in the analysis and produce amplitudes in the coupled channels from where we can extract the pole positions in the complex plane by means of an analytical continuation of these amplitudes. Even then, the problem is subtle because using standard periodical boundary conditions, and a wide range of lattice volumes, there is a gap of energies in the levels of the box precisely for the energies where one finds the poles. Because of this problem one is then forced to use either asymmetric boxes or discretization in the moving frame in order to find eigenvalues of the box in the desired region. In the present paper we face all these problems and come out with some strategies that we find better suited to determine the position of the two Λ(1405) poles.

II. FORMALISM

In the chiral unitary approach the scattering matrix in coupled channels is given by the Bethe-Salpeter equation in its factorized form

\[ T = [1 - VG]^{-1}V = [V^{-1} - G]^{-1}, \]

where \( V \) is the matrix for the transition potentials between the channels and \( G \) is a diagonal matrix with the \( i \)th element, \( G_i \), given by the loop function of two propagators, a pseudoscalar meson and a baryon, which is defined as

\[ G_i = i2M_i \int \frac{d^4p}{(2\pi)^4} \frac{1}{(P - p)^2 - M_i^2 + i\epsilon} \frac{1}{p^2 - m_i^2 + i\epsilon}, \]

where \( m_i \) and \( M_i \) are the masses of the meson and the baryon, respectively, and \( P \) the four-momentum of the global meson-baryon system.

The loop function in Eq. (2) needs to be regularized and this can be accomplished either with dimensional regularization or with a three-momentum cutoff. The equivalence of both methods was shown in Refs. \cite{79,81}.
In dimensional regularization the integral of Eq. (2) is evaluated and gives for meson-baryon systems [8, 82]

\[ G_i(s, m_i, M_i) = \frac{2M_i}{(4\pi)^2} \left\{ a_i(\mu) + \log \frac{m_i^2}{\mu^2} \right. \]

\[ \left. + \frac{M_i^2 - m_i^2 + s}{2s} \log \frac{M_i^2}{m_i^2} \right\} \frac{Q_i(\sqrt{s})}{\sqrt{s}} \left[ \log \left( s - (M_i^2 - m_i^2) + 2\sqrt{s}Q_i(\sqrt{s}) \right) \right. \]

\[ \left. + \log \left( s + (M_i^2 - m_i^2) + 2\sqrt{s}Q_i(\sqrt{s}) \right) \right. \]

\[ \left. - \log \left( -s + (M_i^2 - m_i^2) + 2\sqrt{s}Q_i(\sqrt{s}) \right) \right. \]

\[ \left. - \log \left( -s - (M_i^2 - m_i^2) + 2\sqrt{s}Q_i(\sqrt{s}) \right) \right\}, \tag{3} \]

where \( s = E^2 \), with \( E \) the energy of the system in the center of mass frame, \( Q_i \) the on shell momentum of the particles in the channel \( i \), \( \mu \) a regularization scale and \( a_i(\mu) \) a subtraction constant (note that there is only one degree of freedom, not two independent parameters).

In other works one uses regularization with a cutoff in three momentum, once the \( p^0 \) integration is analytically performed [82], and one gets

\[ G_i = \int_{|\vec{p}| < p_{\text{max}}} \frac{d^3\vec{p}}{(2\pi)^3 \left( \omega_1(\vec{p})\omega_2(\vec{p}) \right)} \frac{\omega_1(\vec{p}) + \omega_2(\vec{p})}{E^2 - (\omega_1(\vec{p}) + \omega_2(\vec{p}))^2 + i\epsilon}, \]

\[ \omega_{1,2}(\vec{p}) = \sqrt{m_{1,2}^2 + \vec{p}^2}, \tag{4} \]

with \( m_1, m_2 \) corresponding to \( m_i \) and \( M_i \) of Eq. (2).

When one wants to obtain the energy levels in the finite box, instead of integrating over the energy states of the continuum, with \( p \) being a discrete variable as in Eq. (4), one must sum over the discrete momenta allowed in a finite box of side \( L \) with periodic boundary conditions. We then have to replace \( G \) by \( \tilde{G} = \text{diag}(G_1, G_2) \) (in two channels), where

\[ \tilde{G}_j = \frac{2M_j}{L^3} \sum_{\vec{p}} \frac{1}{2\omega_1(\vec{p})\omega_2(\vec{p})} \frac{\omega_1(\vec{p}) + \omega_2(\vec{p})}{E^2 - (\omega_1(\vec{p}) + \omega_2(\vec{p}))^2 + i\epsilon}, \]

\[ \vec{p} = \frac{2\pi}{L} \vec{n}, \quad \vec{n} \in \mathbb{Z}^3 \tag{5} \]

This is the procedure followed in Ref. [64]. The eigenenergies of the box correspond to energies that produce poles in the \( T \) matrix, Eq. (4), which correspond to zeros of the determinant of \( 1 - V\tilde{G} \),

\[ \det(1 - V\tilde{G}) = 0. \tag{6} \]

For the case of two coupled channels Eq. (6) can be writen as

\[ \det(1 - V\tilde{G}) = 1 - V_{11}\tilde{G}_1 - V_{22}\tilde{G}_2 \]

\[ + (V_{11}V_{22} - V_{12}^2)\tilde{G}_1\tilde{G}_2 = 0. \tag{7} \]

The problem of the \( \bar{K}N \) interaction with its coupled channels and the \( \Lambda(1405) \) was addressed in Ref. [6] using the cut off method, but more recently it has been addressed using dimensional regularization [8, 82]. For this reason we will also use the dimensional regularization method for the finite box, which was developed in Ref. [64]. The change to be made is also very simple, the \( G \) function of dimensional regularization of Eq. (3) has to be substituted by

\[ \tilde{G}(E) = G^D(E) + \lim_{p_{\text{max}} \to \infty} \int_{\vec{p}} \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{L}} \sum_{\vec{p}_i} I(p_i) \]

\[ - \int_{-p_{\text{max}}}^{p_{\text{max}}} \frac{d^3p}{(2\pi)^3} I(p) \tag{8} \]

where \( I(p) \) is given by

\[ I(p) = \frac{2M_i}{2\omega_1(\vec{p})\omega_2(\vec{p})} \left( \frac{\omega_1(\vec{p}) + \omega_2(\vec{p})}{E^2 - (\omega_1(\vec{p}) + \omega_2(\vec{p}))^2 + i\epsilon} \right). \tag{9} \]

We will also consider the case where the meson-baryon system moves with a fourmomentum \( P = (P^0, \vec{P}) \) in the box. In this case we still have to define the integrals and the sums in the CM frame, where \( p_{\text{max}} \) is defined, but the momenta of the two particles must be discretized in the box, where the system moves with momentum \( P \). We follow the approach of Refs. [74, 80] and use the boost transformation from the moving frame, with the variables \( \vec{p}_{1,2} \), to the CM frame with the variables \( \vec{p}_1^* \),

\[ \vec{p}_{1,2} = \vec{p}_{1,2} + \left[ \frac{M_i}{P^0} - 1 \right] \left( \frac{\vec{p}_{1,2} \cdot \vec{P}}{|\vec{P}|^2} - \frac{\vec{p}_{1,2}^0}{P^0} \right) \vec{P}. \tag{10} \]

where \( M_i^2 = P^2 = P^0^2 - |\vec{P}|^2 \), the subindexes 1, 2, represent the meson, baryon particles and \( \vec{p}_{1,2}^0 \) are the CM energies of the particles given by

\[ p_{1,2}^0 = \frac{M_i^2 + m_{1,2}^2 - m_{2,1}^2}{2M_i}. \tag{11} \]

Then we must do the substitution in Eq. (8) for the evaluation of the energies in the box,

\[ \lim_{p_{\text{max}} \to \infty} \int_{\vec{p}} \frac{d^3p}{(2\pi)^3} \sum_{\vec{p}_i} I(p_i) \to \int_{\vec{p}} \frac{d^3p}{(2\pi)^3} \sum_{\vec{p}_i} \frac{M_i}{P^0} I(p_i^*), \]

\[ \vec{p} = \frac{2\pi}{L} \vec{n}, \quad \vec{n} \in \mathbb{Z}^3 \tag{12} \]
and thus we can only use values of \( \vec{p} \) given in terms of \( \vec{p}_i \) by means of Eq. (10).

Since \( \vec{p}_1 \) and \( \vec{p}_2 = \vec{P} - \vec{p}_1 \) must both satisfy the periodic boundary conditions, this forces \( \vec{P} \) to be also discretized and thus we can only use values of \( \vec{P} \) such that

\[
\vec{P} = \frac{2\pi}{L} \vec{N}, \quad \vec{N} \in \mathbb{Z}^3.
\]

III. RESULTS

A. Energy levels in the box

In this section we show the energy levels obtained from the solution of Eq. (6) as a function of the side length of the box, \( L \), and for different physical cases: using periodical boundary conditions in a: (1) symmetric box, (2) asymmetric box and (3) symmetric box but in a moving frame, i.e., with non-zero value for the total center of mass momentum \( \vec{P} \) (Eq. (13)).

1. Periodical boundary conditions in a symmetric box

In Fig. 1 we show the first six energy levels related to the system formed by the coupled channels \( KN, \pi \Sigma, \eta \Lambda \) and \( K \Xi \), which generate a double pole structure for the \( \Lambda(1405) \) and a pole for the \( \Lambda(1670) \) [11]. These levels are obtained by solving Eq. (6) using the chiral model of Ref. [11] and imposing periodical boundary conditions in a symmetric box of side length \( L \) (measured in units of \( m^{-1} \)).

As it can be seen in Fig. 1, the gap between the levels 0, 1 and specially between levels 1 and 2 is considerable, giving rise to the presence of only two levels in the energy region of interest, i.e., the energy range in which the two poles of the \( \Lambda(1405) \) are found (1390 – 1430 MeV). This fact shows the difficulty that one can face to extract information about the poles of the \( \Lambda(1405) \) in an infinite volume considering these energy levels as reference.

2. Periodical boundary conditions in an asymmetric box

To see if we can obtain more energy levels in the region of the \( \Lambda(1405) \), it is also possible to solve Eq. (6) but in an asymmetric box. To do this we just need to substitute \( L^3 \) by \( L_x L_y L_z \) and the momentum \( \vec{p} \) of Eq. (12) by \( \vec{p} = (2\pi)\left(\frac{n_x}{L_x}, \frac{n_y}{L_y}, \frac{n_z}{L_z}\right) \). In Fig. 2 we show the first three energy levels determined in a box of side lengths \( L_x = L_y = L \) and \( L_z = zL \), with \( z = 0.5 – 2.5 \) in steps of 0.5.

3. Periodical boundary conditions in a moving frame

Another method to try to get more energy levels around the pole positions of the \( \Lambda(1405) \) and thus, different information about the dynamics of the system under consideration, consist in imposing periodical boundary conditions in a symmetric box of side length \( L \) but considering the system in a moving frame, i.e., with non-zero center of mass momentum \( \vec{P} \). In Fig. 3 we show the results found in this case for the first three levels obtained and for different values of the vector \( \vec{N} \) (see Eq. (13)). As it can be seen, the use of different values of \( \vec{P} \) gives rise to a splitting of the levels. In particular, the splitting of level 1 is precisely in the energy region of interest, 1390 – 1450 MeV. This is different to the case of the asymmetric box, where level 2 is required in order to have energy levels around 1420-1450 MeV, as it can be seen in Fig. 2.
Choosing a region of energies around a certain value of \( E, E_0 \), the inverse function of \( E \) can be expanded as a function of \( E - E_0 \) to a good extend. Particularizing \( E_0 \) to the value given by the sum of the kaon and nucleon masses, i.e., \( m_K + M_N \), we can write the potential in Eq. (16) as

\[
V_{ij} = a_{ij} + b_{ij}[E - (m_K + M_N)].
\]  

(17)

The value of the coefficients \( a_{ij} \) and \( b_{ij} \) can be obtained comparing Eq. (17) with Eq. (16) and substituting \( 1/E \) by its Taylor expansion around \( E_0 = m_K + M_N \).

To solve the inverse problem, we use the energy levels obtained from Eq. (6) with the potential of Eq. (17) but treating \( a_{ij} \) and \( b_{ij} \) as parameters which are determined by fitting the corresponding solutions for the energy levels to the “synthetic” lattice data considered. Since this potential has the same energy dependence than the chiral potential, the best fit we can perform will have as minimum value for the \( \chi^2 \) the result \( \chi^2_{min} = 0 \). However, other possible potentials, giving rise to solutions compatible with the error assumed in the data points, can be also found as an answer for the inverse problem. These solutions can be obtained by generating random numbers for the parameters \( a_{ij} \) and \( b_{ij} \) close to those of the minimum such that \( \chi^2 \leq \chi^2_{min} + 1 \).

It is important to notice that the loop function \( \tilde{G} \), used in Eq. (6), needs to be regularized and, thus, depends on a cut-off or a subtraction constant. Consequently, so do the fitted parameters, but the \( T \) matrix obtained from Eq. (11) and the observables related to it should be independent of this regularization parameter. This means that the inverse method cannot depend on the cut-off or subtraction constant assumed in the evaluation of the \( \tilde{G} \) function. For the case of one channel, it is possible to show analytically this independence in the choice of the cut-off or subtraction constant \( [64, 69] \), but if more channels are involved it can only be seen numerically by changing the cut-off or subtraction constant in a reasonable physical range \( [64, 69] \).

In the next sections we show the results found for the inverse problem. To accomplish this we have considered different sets of points extracted from the energy levels shown in Figs. 1 2 3 and fitted them from the solution that Eq. (6) produces with the potential of Eq. (17). To solve Eq. (6) we have taken into account two coupled channels, \( \pi \Sigma \) (which we named channel 1) and \( \bar{K}N \) (or channel 2), which are the most relevant channels to describe the properties of the \( \Lambda(1405) \). This implies, as it can be seen in Eq. (7), that we have to determine three parameters and, thus, the potentials, are known, we can use them to solve Eq. (11) and determine the pole positions of the \( \Lambda(1405) \) in an infinite volume.
the fit and the loop function \( G \) of the random choices of parameters satisfying the condition with the potential of Eq. (17). The band corresponds to different choices of parameters within errors.

1. Periodical boundary conditions in a symmetric box

In Fig. 1 we show the results of the energy levels reconstructed from the best fits to the “synthetic” lattice data considered in Fig. 4 in a range of \( L \) between 1.5 \( m_\pi^{-1} \) and 3.34 \( m_\pi^{-1} \), using the potential of Eq. (17). The band corresponds to different choices of parameters within errors.

We could also try using different levels than the ones employed in Figs. 3 and 5 to see if we can get more reliable information from them. In Fig. 6 we consider a “synthetic” data obtained from levels 1 and 2 of Fig. 1. The data points, as well as the results from the fits, are shown in Fig. 5. Similarly, if we use now the potentials associated with the band of solutions shown in Fig. 5 to solve Eq. (1) and calculate the \( T \) matrix in the unphysical sheet, we get again two poles in the complex energy plane associated with the \( \Lambda(1405) \): one in the region \( 1390-1433 \) MeV, with half width between 70-100 MeV, and other at \( 1410-1421 \) MeV with half width 17-30 MeV. Comparing them with the previous results, we find that the consideration of a bigger box has improved slightly the width associated with the first pole of the \( \Lambda(1405) \), however, we continue having a similar energy dispersion for the real part of the pole.

We could also try using different levels than the ones employed in Figs. 4 and 5 to see if we can get more reliable information from them. In Fig. 6 we consider a “synthetic” data obtained from levels 1 and 2 of Fig. 1. We have taken into account 5 points for level 1 in a range of \( L \) between 1.5 \( m_\pi^{-1} \) and 3.9 \( m_\pi^{-1} \) and 4 points for level 2 for values of \( L \) inside 2 \( m_\pi^{-1} \) to 3.9 \( m_\pi^{-1} \). This is because for level 2 the points for values of \( L \) below 2 \( m_\pi^{-1} \) are influenced by the \( \eta \Lambda \) and \( K \Xi \) channels and, thus, it is not possible to make a fit to them considering only the \( \pi \Sigma \) and \( K \Sigma \), channels, as we do. We can use now the potentials associated with the different fits shown in Fig. 5 to calculate the pole positions of the \( \Lambda(1405) \) in infinite volume by means of Eq. (1). In this case, we continue getting a double pole structure for the \( \Lambda(1405) \), but this time one pole is at \( (1375-1430) - i(70-85) \) MeV and the other one is at \( (1412-1427) - i(21-34) \) MeV. The position of the second pole remains basically the same as in the two previous cases. However, the use of “synthetic” points generated from levels 1 and 2 instead than from levels 0 and 1 has restricted more the imaginary part of the first pole, although we continue getting a similar energy dispersion for the real part of the pole position.

Finally, we could consider all the energy levels present
in Fig. 6 below 1600 MeV to generate data points to check if the consideration of more levels can restrict more the energy region at which the pole positions of the Λ(1405) are found. Following this idea, in Fig. 7 we consider a set of 14 points extracted from levels 0, 1 and 2 of Fig. 2. Similar to the previous results, the consideration of data points associated to three energy levels puts a restriction on the imaginary part of the first pole of the Λ(1405), which in this case is in the range 54-68 MeV (closer to the chiral solution, 66 MeV). However the dispersion on the real part continues basically equal, 1400-1428 MeV.

The solution of the Bethe-Salpeter equation in an infinite volume, Eq. (1), using the potentials related to the band of solutions plotted in Fig. 8 shows the presence of a double pole structure for the Λ(1405) with pole positions at (1383−1407)−i(57−69) MeV and (1425−1434)−i(25−35) MeV. Thus, using this new set of data points, there is an improvement in the determination of the first pole of the Λ(1405), which is now quite close to the chiral result (1390−i66 MeV). However, the second pole appears at higher energies as compared to the case of a symmetric box and sometimes is far from the chiral solution (1426−i16 MeV), being even close to the KN threshold.

2. Periodical boundary conditions in an asymmetric box

We consider now the case of an asymmetric box of side lengths $L_x = L_y = L$ and $L_z = zL$ to solve the inverse problem. In this case, we generate a set of 20 data points extracted from levels 0, 1 and 2 shown in Fig. 8. In particular, we use 5 points for level 0 calculated with $z = 2.5$, 10 points for level 1 (5 for the case $z = 0.5$ and 5 more for $z = 2.0$) and 5 points for level 2 obtained with $z = 2.0$. In this way we ensure the presence of some energy levels in the region of the Λ(1405), as it can be seen in Fig. 8.

The solution of the Bethe-Salpeter equation in an infinite volume, Eq. (1), using the potentials related to the band of solutions plotted in Fig. 8 shows the presence of a double pole structure for the Λ(1405) with pole positions at $(1383 − 1407)−i(57 − 69)$ MeV and $(1425 − 1434)−i(25 − 35)$ MeV. Thus, using this new set of data points, there is an improvement in the determination of the first pole of the Λ(1405), which is now quite close to the chiral result $(1390−i66)$ MeV. However, the second pole appears at higher energies as compared to the case of a symmetric box and sometimes is far from the chiral solution $(1426−i16)$ MeV, being even close to the KN threshold.

3. Periodical boundary conditions in a moving frame

We can also study the information which can be extracted for the poles of the Λ(1405) using the levels ob-
tained when we consider the system in a symmetric box, but in a moving frame, to generate “synthetic” lattice data. In this case, we consider levels 0 and 1 of Fig. 3 determined for 5 different values of the center of mass momentum (the ones shown in the legend of Fig. 3) and two points in each of these curves. In particular, we take points at \( L = 1.757m^{-1} \) and \( L = 2.014m^{-1} \), obtaining then 20 data points. The results are shown in Fig. 9. From the solution of the best fits, we can use the potentials obtained to solve Eq. (1), getting then two poles for the Λ(1405): one at \((1388 - 1418) - i(59 - 77)\) MeV and other at \((1412 - 1427) - i(16 - 34)\) MeV.

In Fig. 10 we show the results for the pole positions of the Λ(1405) obtained from the different data set considered in this work. As it can be seen in Fig. 10 out of the different data sets considered to solve the inverse problem, the cases of an asymmetric box and of a symmetric box but in a moving frame seem to be more suited to get the two poles of the Λ(1405) with more precision.

IV. CONCLUSIONS

We have made a study of the \( \bar{K}N \) interaction with its coupled channels in a finite box and found the levels obtained as a function of the box size. We have done this for standard periodic conditions and symmetric boxes, for asymmetric boxes and for symmetric boxes but with the particles in a moving frame. The aim of the work has been to solve the inverse problem in which, assuming that the levels in the box would correspond to “QCD lattice results” we want to determine the pole positions in the complex plane for the two Λ(1405) states provided by the chiral unitary approach and supported by several experiments.

We found that the problem is not trivial, and even the use of a large number of energies of the box corresponding to different levels and volumes with standard periodic conditions cannot provide the mass and width of the states with the accuracy of the chiral unitary approach and present experiments. For this reason we investigated other possible strategies and found that the use of asymmetric boxes and levels coming from the particles in moving frames helped considerably to narrow down the uncertainties in the determination of the mass and width of these resonances. The choices of levels and energies made for this analysis should be a guiding tool for future QCD lattice evaluations, showing the number of levels needed, the errors that should be demanded in the determination of the energies of the box and the type of asymmetric boxes or total momenta of the pair of particles in the moving frames. Having this information before hand is of tremendous value given the time consuming runs of actual QCD lattice runs.

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FIG. 9. Fits to the “synthetic” data extracted from the energy levels 0 and 1 of Fig. 3, which correspond to the case of a symmetric box, but with the particles being in a moving frame.

FIG. 10. Pole positions of the \( \Lambda(1405) \) reconstructed from the different set of “synthetic” data generated for the different cases considered in this work. The shaded symbols corresponds to the positions obtained for the first pole of the \( \Lambda(1405) \), while the empty symbols are related to the second pole of the \( \Lambda(1405) \).
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