A NOTE ON GENERIC CLIFFORD ALGEBRAS OF BINARY CUBIC FORMS

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Abstract. We study the representation theoretic results of the binary cubic generic Clifford algebra $C$, which is an Artin-Schelter regular algebra of global dimension five. In particular, we show that $C$ is a PI algebra of PI degree three and compute its point variety and discriminant ideals. As a consequence, we give a necessary and sufficient condition on a binary cubic form $f$ for the associated Clifford algebra $C_f$ to be an Azumaya algebra.

1. Introduction

Let $f$ be a form of degree $m$ in $n$ variables over a base field $k$. The Clifford algebra $C_f$ associated to the form $f$ is defined to be an associative algebra $k \langle x_1, \ldots, x_n \rangle$ subject to the relations $(a_1 x_1 + \cdots + a_n x_n)^m - f(a_1, \ldots, a_n)$ for all $a_1, \ldots, a_n \in k$. In [10], Chan-Young-Zhang defined the generic Clifford algebra (or more precisely universal Clifford algebra) $C_{m,n} = k \langle x_1, \ldots, x_n \rangle$ subject to the relations $(a_1 x_1 + \cdots + a_n x_n)^m x_j - x_j (a_1 x_1 + \cdots + a_n x_n)^m$ for all $a_1, \ldots, a_n \in k$ and $1 \leq j \leq n$. The work of [2,6] implies that the Clifford algebra $C_f$ is a homomorphic image of the generic Clifford algebra $C_{m,n}$, which paves a way to understand the representations of the former through the representations of the latter. Chan-Young-Zhang further showed that ([10, Lemma 3.8]) when $m = 2, n = 2$, the binary quadratic generic Clifford algebra belongs to the nice family of connected graded algebras that are Artin-Schelter regular (see [2]).

In this note, we focus on the binary cubic generic Clifford algebra $(m = 3, n = 2)$ and denote it by $C$. In [19], all Artin-Schelter regular algebras of global dimension five that are generated by two generators with three generating relations are classified. The algebra $C$ is listed as one of the type $A$ algebras there, which is also proved to be strongly noetherian, Auslander regular and Cohen-Macaulay. In this note, we further show that $C$ is indeed a polynomial identity (PI) algebra of PI degree three (see Theorem 4.4). This and other known properties allow us to fully understand the representations of $C$ by the well-developed representation theory of PI algebras (e.g. [7, 8, 11]). As a consequence, we derive some ring-theoretic results of the binary cubic
Clifford algebra $C_f$, whose properties have been studied by Heerema [14] and Haile [12].

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2. Background

2.1. Let $k$ be an algebraically closed base field of characteristic not 2 or 3. The reader is referred (e.g.) to [11] and [15] for further background on the theory of PI rings. Recall that a ring $R$ with center $Z(R)$ is called Azumaya over $Z(R)$ if $R$ is a finitely generated projective $Z(R)$-module and the natural map $R \otimes_{Z(R)} R^{op} \to \text{End}_{Z(R)}(R)$ is an isomorphism. Let $\Lambda$ be a prime noetherian affine $k$-algebra finitely generated as a module over its center $Z$. Define

$A_\Lambda := \{m \in \text{maxSpec}(Z) \mid \Lambda_m \text{ is Azumaya over } Z_m\}$,

$S_\Lambda := \{m \in \text{maxSpec}(Z) \mid Z_m \text{ is not regular}\}$.

The set $A_\Lambda$ is called the Azumaya locus of $\Lambda$ over $Z$, and $S_\Lambda$ is the singular locus of $\Lambda$ (or of $Z$).

2.2. Theorem. [6, Theorem III.1.7][7, Proposition 3.1, Lemma 3.3] Let $\Lambda$ be a prime noetherian affine $k$-algebra that is module-finite over its center $Z$.

(a) The maximum $k$-dimension of irreducible $\Lambda$-modules equals the PI degree of $\Lambda$.

(b) Let $S$ be an irreducible $\Lambda$-module, $P = \text{Ann}_\Lambda(S)$, and $m = P \cap Z$. Thus $\dim_k(S) = \text{PI-deg}(\Lambda)$ if and only if $\Lambda_m$ is Azumaya over $Z_m$.

Moreover if $\text{gldim}(\Lambda) < \infty$, then $A_\Lambda$ is a nonempty open (and hence dense) subset of $\text{maxSpec}(Z) \setminus S_\Lambda$.

2.3. In [10], Chan-Young-Zhang defined the generic Clifford algebra and established some properties in the quadratic case. Recall that the binary cubic generic Clifford algebra $C$ is defined as the quotient algebra of the free algebra $k\langle x, y \rangle$ subject to the relations:

(1) $x^3y - yx^3, \ x^2y^2 + xyxy - yxyx - y^2x^2, \ xy^3 - y^3x$.

Moreover, $C$ is one of the type $A$ algebras studied in [19] (parameters $t = -1, l_2 = 1$).

2.4. Theorem. [19, Theorem 5] The connected graded algebra $C$ is Artin-Schelter regular of global dimension five. Moreover, $C$ is an Auslander regular, Cohen-Macaulay, and strongly noetherian domain.
2.5. By the diamond lemma [5], we have that
\[ \{ y^i(x^2y)^j(x^2y)^l x^m \mid i, j, k, l, m \in \mathbb{N} \} \]
is a \( k \)-linear basis for \( \mathcal{C} \). Consequently, the Hilbert series of \( \mathcal{C} \) is
\[ \frac{1}{(1-t)^5(1+t)(1+t+t^2)^2}. \]

2.6. Let \( f(u, v) = au^3 + 3bu^2v + 3cuv^2 + dv^3 \) be a cubic form with \( a, b, c, d \in k \). Its discriminant is given by
\[ (2) \quad D := \frac{1}{4} (ad - bc)^2 - (ac - b^2)(bd - c^2). \]
The Clifford algebra \( \mathcal{C}_f \) of the cubic form \( f \) is defined as
\[ \mathcal{C}_f := k\langle x, y \rangle / I, \]
where \( I \) is the ideal generated by the elements \((ux + vy)^3 - f(u, v)\) for all \( u, v \in k \). It is easy to see that
\[ (3) \quad I = (x^3 - a, y^3 - d, x^2y + yxy + yx^2 - 3b, y^2x + yxy + xy^2 - 3c). \]

There is a natural surjection \( \mathcal{C} \twoheadrightarrow \mathcal{C}_f \) by sending \( x \mapsto x, y \mapsto y \) since the ideal generated by the defining relations in Eq.(1) for \( \mathcal{C} \) is contained in \( I \).

2.7. Theorem. [12, Theorem 1.1', Corollary 1.2'] The Clifford algebra \( \mathcal{C}_f \) is an Azumaya algebra of PI degree three if \( D \neq 0 \). In this case, its center is a Dedekind domain, isomorphic to the coordinate ring of the affine elliptic curve \( u^2 = v^3 - 27D \).

3. The center and singular locus of \( \mathcal{C} \)
3.1. Consider the projective space \( \mathbb{P}^3 \) with homogenous coordinates \( [z_0 : z_1 : z_2 : z_3] \).
For cubic binary forms, the discriminant projective variety is defined by \( \Delta = 0 \), where (see Eq.(2))
\[ \Delta = \frac{1}{4} (z_0z_3 - z_1z_2)^2 - (z_0z_2 - z_1^2)(z_1z_3 - z_2^2). \]

3.2. Lemma. The reduced variety of the singular locus of the discriminant projective variety is given by the twisted cubic curve \( v : \mathbb{P}^1 \to \mathbb{P}^3 \) via
\[ v : [x_0 : x_1] \mapsto [x_0^3 : x_0^2x_1 : x_0x_1^2 : x_1^3] = [z_0 : z_1 : z_2 : z_3]. \]
\textbf{Proof.} It is well known (e.g. \cite{13} Chapter I Theorem 5.1) that the singular locus of $\Delta = 0$ is the zero locus of the polynomials

\begin{align*}
\frac{\partial \Delta}{\partial z_0} &= -\frac{3}{2}z_1z_2z_3 + \frac{1}{2}z_0z_3^2 + z_2^3 = 0, \\
\frac{\partial \Delta}{\partial z_1} &= -\frac{3}{2}z_0z_2z_3 - \frac{3}{2}z_1z_2^2 + 3z_2z_3 = 0, \\
\frac{\partial \Delta}{\partial z_2} &= -\frac{3}{2}z_0z_1z_3 - \frac{3}{2}z_1^2z_2 + 3z_0z_2^2 = 0, \\
\frac{\partial \Delta}{\partial z_3} &= -\frac{3}{2}z_0z_1z_2 + \frac{1}{2}z_0^2z_3 + z_1^3 = 0.
\end{align*}

If $z_0 = 0$, then $z_1 = z_2 = 0$ and $z_3 \neq 0$, which corresponds to the point $v([0 : 1])$. Now let $z_0 = 1$. From Eq.\((7)\), we have $z_3 = 3z_1z_2 - 2z_1^3$. Substituting it into Eq.\((6)\), we obtain $(z_1^2 - z_2)^2 = 0$ which implies that $z_2 = z_1^2$ and $z_3 = z_1^3$. One can check that $[1 : z_1 : z_1^2 : z_1^3] = v([1 : z_1])$ satisfies Eq.\((4)\)-Eq.\((7)\). Hence the solutions are exactly given by the twisted cubic curve. \hfill $\Box$

3.3. In the following, we denote by $Z$ the center of the generic Clifford algebra $C$. We will describe the center $Z$ according to \cite{18}. Consider the following central elements of $C$, where $\omega \in k$ is a primitive third root of unity.

\begin{align*}
z_0 &= x^3, \\
z_1 &= \frac{1}{3}(x^2y + yxy + yx^2), \\
z_2 &= \frac{1}{3}(y^2x + yxy + yx^2), \\
z_3 &= y^3, \\
z_4 &= (xy - \omega yx)^3 - \frac{3}{2}\omega(1 - \omega)x^3y^3 - \frac{9}{2}(1 + 2\omega^2)z_1z_2 \\
z_5 &= (xy)^2 - y^2x^2 = (yx)^2 - x^2y^2.
\end{align*}

Define the formal discriminant element $\Delta$ in $C$ as

\begin{align*}
\Delta = \frac{1}{4}(z_0z_3 - z_1z_2)^2 - (z_0z_2 - z_1^2)(z_1z_3 - z_2^2).
\end{align*}

Under the natural surjection $C \to C_f$, it follows that $(z_0, z_1, z_2, z_3) \mapsto (a, b, c, d)$. In particular, the formal discriminant element $\Delta$ maps to the discriminant $D \in k$ of the binary cubic form $f$ under the surjection $C \to C_f$.

\textbf{3.4. Theorem.} \cite{18} The generic Clifford algebra $C$ is finitely generated as a module over its center $Z$. Moreover, the center $Z$ is generated by $(z_i)_{0 \leq i \leq 5}$ subject to one relation $z_1^2 = z_0^3 - 27\Delta$. Consequently, $\text{maxSpec}(Z)$ is isomorphic to the coordinate ring of a relative quasiprojective curve over the 4-dimensional affine space $\mathbb{A}^4 = \text{maxSpec}(k[z_0, z_1, z_2, z_3])$ that is elliptic over an open subset of $\mathbb{A}^4$. 


3.5. Corollary. The reduced variety of the singular locus $S_C$ of $Z$ is an affine twisted cubic curve in $A^4 = \text{maxSpec}(k[z_0, z_1, z_2, z_3])$ defined by

$$S_C = \mathbb{V}(z_4, z_5, z_0z_3 - z_1z_2, z_0z_2 - z_1^2, z_1z_3 - z_2^2).$$

Proof. Since the center $Z$ generated by $(z_i)_{0 \leq i \leq 5}$ is subject to one relation $R := z_4^2 - z_5^2 + 27\Delta$, the singular locus $S_C$ of $Z$ is the zero locus of the derivatives $\partial R/\partial z_i$ for $0 \leq i \leq 5$, or equivalently (if ignoring multiplicity) $z_4 = z_5 = 0$ and $\partial \Delta/\partial z_0 = \partial \Delta/\partial z_1 = \partial \Delta/\partial z_2 = \partial \Delta/\partial z_3 = 0$.

Thus, the result follows from Lemma 3.2. \hfill $\square$

4. Irreducible representations and Azumaya locus of $C$

4.1. It is clear that the set of isomorphism classes of one-dimensional representations of $C$ is bijective to $\mathbb{A}^2$ via the one-to-one correspondence $\text{Ann}_k(S) = (x - a, y - b) \leftrightarrow (a, b) \in \mathbb{A}^2$ for any $C$-module $S$ with $\text{dim}_k(S) = 1$. The following result is crucial to our work.

4.2. Lemma. There are no two-dimensional irreducible representations over $C$.

Proof. Suppose $S$ is any two-dimensional irreducible representation over $C$. This implies that there is a surjective algebra map $\varphi : C \to \text{End}_k(S) = M_2(k)$. One sees easily that $\varphi(x) \neq 0$; otherwise $S$ can be viewed as a module over $C/(x) \cong k[y]$, where all irreducible representations are one-dimensional. The same argument shows that $\varphi(x)$ is not a scalar multiple of the identity matrix in $M_2(k)$. Since $x^3$ is central in $C$, $\varphi(x^3) = \varphi(x)^3$ is a scalar multiple of the identity matrix. Therefore by a linear transformation of $S$ and a possible rescaling of the variables $x, y$ of $C$, we can assume that $\varphi(x) = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}$ where $\omega$ is a primitive third root of unity. Now write

$$\varphi(y) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for some $a, b, c, d \in k$. When we apply $\varphi$ to the relations in $C$, we obtain

$$\varphi(x^2y^2 + (xy)^2 - (y^2x^2) = \begin{pmatrix} 0 & 3b(a - \omega^2d) \\ -3c(a - \omega^2d) & 0 \end{pmatrix} = 0.$$ 

If $a \neq \omega^2d$, then we have $b = c = 0$. So $\varphi(x)$ and $\varphi(y)$ are both diagonal. This shows that $S$ is not irreducible, which is a contradiction. Hence we have $a = \omega^2d$. Thus, $\varphi$ sends the central element $y^3$ to

$$\varphi(y^3) = \begin{pmatrix} a^3 + (2 + \omega)abc \\ bc^2 \\ a^3 + (1 + 2\omega)abc \end{pmatrix}.$$ 

Since $\varphi(y^3)$ is a scalar multiple of the identity matrix, we obtain $bc = 0$ and $\varphi(y)$ is either upper or lower triangular. Again, it is a contradiction since $S$ is irreducible. \hfill $\square$

4.3. We denote by $\text{Irr}_n C$ the isomorphism classes of all irreducible representations over $C$. For each possible integer $n \geq 1$, $\text{Irr}_n C$ denotes the isomorphism subclasses of all
Corollary 3.5. The result follows.

According to [8, Theorem 3.1(f)],

\[ \chi \text{ via } S \to \text{Ann}_C(S) \cap Z \text{ for any irreducible representation } S \text{ over } C. \]

4.4. Theorem. The following hold for the binary cubic generic Clifford algebra \( C \).

(a) \( C \) is a PI algebra of PI degree three.

(b) \( \text{Irr}_1 C = \text{Irr}_1 C \sqcup \text{Irr}_3 C. \)

(c) The map \( \text{Irr}_1 C \to S_C \) is three to one.

(d) The map \( \text{Irr}_3 C \to \text{maxSpec}(Z) \setminus S_C \) is one to one.

(e) The Azumaya locus of \( C \) coincides with the smooth locus of \( C \).

Proof. (a) By Theorem 3.4, we know \( C \) is module-finite over its center \( Z \) and hence is PI by [15 Corollary 13.1.13(iii)]. Now suppose \( C \) has PI degree \( n \). Regarding Eq. (10), we know \( \chi(\text{Irr}_n C) \) is an open dense subset of \( Y \) by Theorem 2.2. Take the formal discriminant \( \Delta \) of \( C \) as in Eq. (9). Note that \( Y \setminus \mathcal{V}(\Delta) \) is another open dense subset of \( Y \) since \( Y \) is irreducible. So there exists a maximal ideal

\[ m = (z_0 - a, z_1 - b, z_2 - c, z_3 - d, z_4 - e, z_5 - f) \in \chi(\text{Irr}_n C) \setminus \mathcal{V}(\Delta), \]

for some \( a, b, c, d, e, f \in k \). Choose any irreducible representation \( S \) over \( C \) such that \( \chi(S) = m \) and \( \text{dim}_k(S) = n \). Let \( C_f \) be the Clifford algebra associated to the binary cubic form \( f(u, v) = au^3 + 3bu^2v + 3cuv^2 + dv^3 \). Therefore, there are algebra surjections

\[ C \twoheadrightarrow C_f \twoheadrightarrow C/mC, \]

where the first surjection sends \( \Delta \) to a nonzero discriminant \( D \) in \( C_f \) (see Eq. (2)). By Theorem 2.7, \( C_f \) is an Azumaya algebra of PI degree three. So every irreducible representation over \( C_f \) is three-dimensional and the same holds for its image \( C/mC \) as well. Since \( mS = 0 \), we can view \( S \) as an irreducible representation over \( C/mC \). This implies that \( \text{dim}_k(S) = n = 3 \).

(b) follows from Theorem 2.2 and Lemma 4.2 since \( C \) has PI degree three.

(c) Let \( S(a, b) = C/(x - a, y - b) \) be any one-dimensional representation over \( C \) for some \( a, b \in k \). One checks by Eq. (8) that

\[ \chi(S(a, b)) = (z_0 - a^3, z_1 - a^2b, z_2 - ab^2, z_3 - b^3, z_4, z_5) =: n. \]

Also we have \( \chi(S(\omega a, \omega b)) = \chi(S(\omega^2 a, \omega^2 b)) = n. \) Suppose \( \chi^{-1}(n) = \{ S_1, S_2, \ldots, S_t \}. \) According to [8, Theorem 3.1(f)],

\[ \text{dim}_k(S_1) + \text{dim}_k(S_2) + \cdots + \text{dim}_k(S_t) \leq \text{PI-deg}(C) = 3. \]

Hence \( t \leq 3 \) and \( \chi^{-1}(n) \) exactly contains three one-dimensional irreducible representations \( S(a, b), S(\omega a, \omega b) \) and \( S(\omega^2 a, \omega^2 b) \). Moreover, \( \chi \) maps \( \text{Irr}_1 C \) onto \( S_C \) by Corollary 3.35. The result follows.
(d) By (b), we have the surjection $\chi : C = \text{Irr}_1 C \sqcup \text{Irr}_3 C \twoheadrightarrow Y = S_C \sqcup (Y \setminus S_C)$. By (c), we have an induced surjection $\chi : \text{Irr}_3 C \twoheadrightarrow Y \setminus S_C$. Note that $C$ has PI degree three. By the same argument as in (c), $\chi$ yields a one-to-one correspondence between $\text{Irr}_3 C$ and $Y \setminus S_C$.

(e) is a consequence of (d) and (b) of Theorem 2.2

4.5. As recalled in Theorem 2.1, Haile showed that the Clifford algebra $C_f$ is an Azumaya algebra when the binary cubic form $f$ is nondegenerate or the discriminant $D \neq 0$ (see Eq. 2.4). In the following, we give a necessary and sufficient condition for $C_f$ to be Azumaya. It turns out that $C_f$ can be an Azumaya algebra even when $f$ is degenerate or $D = 0$.

4.6. Corollary. Let $k$ be a base field of characteristic different from 2 or 3. The Clifford algebra $C_f$ associated to the binary cubic form $f(u, v) = au^3 + 3bu^2v + 3cuv^2 + dv^3$ is an Azumaya algebra (of PI degree three) if and only if the point $(a, b, c, d)$ does not lie on the affine twisted cubic curve in $\mathbb{A}^4$ described in Lemma 3.2.

Proof. Without loss of generality, we can assume $k$ to be algebraically closed. Recall $Z = k[z_0, \ldots, z_5]$, where $z_4^2 = z_5^2 - 27\Delta$, is the center of $C$. It is easy to check that $Z$ has a $k$-basis

$$\{z_0^{i_0}z_1^{i_1}z_2^{i_2}z_3^{i_3}z_4^{i_4}z_5^{i_5} | 0 \leq i_0, i_1, i_2, i_3, i_4, i_5 \leq 2\}.$$

As a consequence, it induces a projection $\pi : \text{maxSpec}(Z) \rightarrow \mathbb{A}^4$ given by the natural inclusion $k[z_0, z_1, z_2, z_3] \hookrightarrow Z$. Therefore the natural surjection $C \twoheadrightarrow C_f$ factors through $C/\mathfrak{m}C$ for some $\mathfrak{m} \in \text{maxSpec}(Z)$ if and only if $\mathfrak{m} \in \pi^{-1}(a, b, c, d)$. By Theoem 3.4, we know

$$C/\mathfrak{m}C \cong \begin{cases} M_3(k) & \text{if } \mathfrak{m} \in \text{maxSpec}(Z) \setminus S_C, \\ \text{a local algebra} & \text{if } \mathfrak{m} \in S_C. \end{cases}$$

By Corollary 3.5, $\pi^{-1}(a, b, c, d) \cap S_C \neq \emptyset$ if and only if $(a, b, c, d)$ lies in the affine twisted cubic curve in $\mathbb{A}^4$ given by Lemma 3.2. In this case, we know $C_f$ has irreducible representations of both dimensions one and three. Otherwise, all the irreducible representations over $C_f$ are of dimension three. Thus the result follows by the Artin-Procesi theorem on polynomial identities (see [1] Theorem 8.3). □

5. Point variety of $C$

5.1. In noncommutative projective algebraic geometry (e.g. see [3]), a point module $M = \bigoplus_{i \geq 0} M_i$ over $C$ is a cyclic graded module with Hilbert series $\frac{1}{1 - x z}$, or namely $M$ is generated by $M_0$ and $\dim_k M_i = 1$ for all $i \geq 0$. Note that every point module $M$ over $C$ can be denoted by the following diagram

$$M(p_0, p_1, p_2, \ldots) : \bullet \cdots \bullet \ x_0 \cdots \bullet \ x_1 \cdots \bullet \ x_2 \cdots \bullet \ x_3 \cdots \bullet \ x_4 \cdots \bullet \ \cdots$$
where $M = \bigoplus_{i \geq 0} k e_i$ and $0 \neq p_i = (x_i, y_i) \in k^2$ with the $C$-action on $M$ given by

\[ xe_i = x_{i+1} e_{i+1}, \quad ye_i = y_{i+1} e_{i+1}. \]

(11)

It is straightforward to check that two point modules $M(p_0, p_1, \ldots)$ and $M(q_0, q_1, \ldots)$ are isomorphic as graded $C$ modules if and only if $p_i = q_i$ in $\mathbb{P}^1$ for all $i$ or $(p_0, p_1, \ldots) = (q_0, q_1, \ldots)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \cdots$. Therefore, the isomorphism classes of all point modules over $C$ can be parametrized by a subvariety in $\mathbb{P}^\infty$, which is called the point variety of $C$. Sometimes, the point variety in $\mathbb{P}^\infty$ is determined by a variety $X$ in $\mathbb{P}^r$ for some $r \geq 1$ and an automorphism $\sigma : X \to X$, where the point variety is determined by the natural embedding $X \hookrightarrow \mathbb{P}^\infty$ via $x \mapsto (x, \sigma(x), \sigma^2(x), \ldots)$ for any $x \in X$. In such a case for simplicity, we say the point variety is given by the pair $(X, \sigma)$.

5.2. **Theorem.** The point variety of $C$ is given by the pair $(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \text{id})$.

**Proof.** Let $M(p_0, p_1, \ldots)$ be a point module. Taking any relation $f$ in Eq.(1) of $C$, we have $f e_0 = 0$. Therefore we obtain

\[
\begin{align*}
x_3 x_2 x_1 y_0 - y_3 x_2 x_1 x_0 &= 0 \\
x_3 x_2 y_1 y_0 + x_3 y_2 x_1 y_0 - y_3 x_2 y_1 x_0 - y_3 y_2 x_1 x_0 &= 0 \\
x_3 y_2 y_1 y_0 - y_3 y_2 y_1 x_0 &= 0
\end{align*}
\]

Consider the above equations as a system of linear equations in terms of $x_3, y_3$ and rewrite it in the matrix form.

\[
\begin{pmatrix}
x_1 x_2 y_0 \\
x_2 y_0 y_1 + x_1 y_0 y_2 \\
y_0 y_1 y_2
\end{pmatrix}
\begin{pmatrix}
x_3 \\
y_3
\end{pmatrix}
= 0.
\]

It is easy to check that it has a solution $(x_3, y_3) = (x_0, y_0)$. We claim that it is the only solution up to a scalar multiple by showing that the coefficient matrix can not be identically zero. Suppose it is not true, then all the entries appearing in the $3 \times 2$ matrix must be zero. Hence

\[ x_1 x_2 y_0 = x_0 x_1 x_2 = x_2 y_0 y_1 + x_1 y_0 y_2 = x_0 x_2 y_1 + x_1 x_0 y_2 = y_0 y_1 y_2 = x_0 y_1 y_2 = 0. \]

One can check the equations have no solution.

By repeating the above argument regarding the basis vector $e_i$ for all $i$, we see that $M(p_0, p_1, \ldots)$ is well defined if and only if $p_i = p_{i+3}$ for all $i$. This yields our result. \qed

5.3. Let $\underline{p} = (p_0, p_1, p_2)$ be a point in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. According to Theorem 5.2, the point variety over $C$ is given by all $M(\underline{p})$ for some $\underline{p} \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Denote by $\Gamma = \{(p, p, p) \mid p \in \mathbb{P}^1\} \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ the diagonal part.

5.4. **Theorem.** Let $M(\underline{p})$ be any point module over $C$, and $S$ be a simple quotient of $M(\underline{p})$.

(a) If $\underline{p} \in \Gamma$, then $\dim_k(S) = 1$. 
(b) If \( \underline{p} \not\in \Gamma \), then \( S \) is either trivial or \( \dim_k(S) = 3 \).

**Proof.** Let \( M = M(p_0, p_1, p_2) = \bigoplus_{i \geq 0} k e_i \). We have \( C \) acts on \( M \) as in (11).

(a) Suppose \( p_0 = p_1 = p_2 = (a, b) \in \mathbb{P}^1 \). It is direct to check that the central elements \( \{z_i\}_{0 \leq i \leq 5} \) act on \( M \) as follows.

\[
z_0 e_i = a^3 e_{i+3}, \quad z_1 e_i = a^2 b e_{i+3}, \quad z_2 e_i = ab^2 e_{i+3}, \quad z_3 e_i = b^3 e_{i+3}, \quad z_4 e_i = z_5 e_i = 0.
\]

Hence \( (z_0 z_3 - z_1 z_2, z_0 z_2 - z_1^2, z_1 z_3 - z_2^2, z_4, z_5) \subset \text{Ann}_C(M) \subset \text{Ann}_C(S) \). By Theorem 4.4 and Corollary 3.5, we have \( V(\text{Ann}_C(S) \cap Z) \subset S_C \). Hence \( \dim_k S = 1 \).

(b) According to Eq. (8), we obtain the following actions on \( M \).

\[
(z_0 z_3 - z_1 z_2)e_i = -\frac{a}{9} e_{i+6}, \quad (z_0 z_2 - z_1^2)e_i = -\frac{b}{18} e_{i+6}, \quad (z_1 z_3 - z_2^2)e_i = -\frac{c}{18} e_{i+6},
\]

where the coefficients \( a, b, c \) are given by

\[
a = x_2 y_2 (x_1 y_0 - x_0 y_1)^2 + x_1 y_1 (x_2 y_0 - x_0 y_2)^2 + x_0 y_0 (x_2 y_1 - x_1 y_2)^2,
\]

\[
b = x_0^2 (x_2 y_1 - x_1 y_2)^2 + x_1^2 (x_0 y_2 - x_2 y_0)^2 + x_2^2 (x_1 y_0 - x_0 y_1)^2,
\]

\[
c = y_0^2 (x_2 y_1 - x_1 y_2)^2 + y_1^2 (x_0 y_2 - x_2 y_0)^2 + y_2^2 (x_1 y_0 - x_0 y_1)^2.
\]

Here we write \( p_i = (x_i, y_i) \in \mathbb{P}^1 \) for all \( i \geq 0 \). Now suppose \( \dim_k S = 1 \) and it is not trivial. By Theorem 4.4 and Corollary 3.5, we have \( (z_0 z_3 - z_1 z_2, z_0 z_2 - z_1^2, z_1 z_3 - z_2^2) \subset \text{Ann}_C(S) \). If any of \( a, b, c \) is not zero, it implies that \( e_{i+6} \neq 0 \) in \( S \) for all \( i \). Thus \( x e_5 = y e_5 = 0 \) in \( S \). Hence \( e_5 = 0 \) in \( S \) otherwise \( S = k e_5 \) would be trivial. Repeating the argument, we obtain all \( e_i = 0 \) in \( S \). It is absurd. So \( a = b = c = 0 \). Now set the variables

\[
X := (x_1 y_0 - x_0 y_1)^2, \quad Y := (x_2 y_0 - x_0 y_2)^2, \quad Z := (x_2 y_1 - x_1 y_2)^2.
\]

We write the above conditions as a system of linear equations in terms of \( X, Y, Z \) as follows.

\[
(12) \quad \begin{bmatrix}
x_2 y_2 & x_1 y_1 & x_0 y_0 \\
x_1^2 & x_0^2 & y_0^2 \\
y_2^2 & y_1^2 & y_0^2
\end{bmatrix}
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix} = 0.
\]

We will show that Eq. (12) has no solution whenever \( \underline{p} \not\in \Gamma \), which yields that \( \dim_k S > 1 \) if \( S \) is not trivial. Hence part (b) follows from Theorem 4.4. Note that \( X = 0 \) implies that \( p_0 = p_1 \), \( Y = 0 \) implies that \( p_0 = p_2 \), and \( Z = 0 \) implies that \( p_1 = p_2 \). Since \( \underline{p} \not\in \Gamma \), \( (X, Y, Z) \) is a nonzero solution for Eq. (12). So

\[
0 = \det \begin{pmatrix}
x_2 y_2 & x_1 y_1 & x_0 y_0 \\
x_1^2 & x_0^2 & y_0^2 \\
y_2^2 & y_1^2 & y_0^2
\end{pmatrix} := \gamma.
\]

Direct computation shows that \( \gamma^2 = XYZ \). So \( V(\gamma) = V(X) \cup V(Y) \cup V(Z) \). Without loss of generality, we assume that \( \underline{p} \in V(\gamma) \cap V(X) \). Since \( X = 0 \), we know \( (x_0, y_0) = (x_1, y_1) \in \mathbb{P}^1 \). Moreover, we can check that \( Y = Z \neq 0 \). But in such a
case, \((X, Y, Z)\) does not satisfy Eq. (12). We obtain a contradiction, which proves our result. 

\[\square\]

6. Discriminant ideals of \(C\)

6.1. Discriminant ideals of PI algebras play an important role in the study of a maximal orders \([16]\), the automorphism and isomorphism problems for noncommutative algebras \([9]\), the Zariski cancellation problem for noncommutative algebras \([4]\), and the description of dimensions of irreducible representations \([8]\).

6.2. Let \(A\) be an algebra and \(C \subseteq Z(A)\) be a central subalgebra. A trace map on \(A\) is a nonzero map \(\text{tr} : A \to C\) that is cyclic \((\text{tr}(xy) = \text{tr}(yx) \text{ for } x, y \in A)\) and \(C\)-linear. For a positive integer \(\ell\), the \(\ell\)-th discriminant ideal \(D_{\ell}(A/C)\) and the \(\ell\)-th modified discriminant ideal \(MD_{\ell}(A/C)\) of \(A\) over \(C\) are the ideals of \(C\) with generating sets

\[
\begin{align*}
\{ & \det([\text{tr}(y_i y_j)]_{i,j=1}^\ell) \mid y_1, \ldots, y_\ell \in A \} \\
\{ & \det([\text{tr}(y_i y'_j)]_{i,j=1}^\ell) \mid y_1, y'_1, \ldots, y_\ell, y'_\ell \in A \}.
\end{align*}
\]

6.3. Since we know the binary cubic generic Clifford algebra \(C\) is Auslander-regular, Cohen-Macaulay, and stably-free (it is connected graded), we can employ Stafford’s work \([17, \text{Theorem 2.10}]\) to conclude that \(C\) is a maximal order in a central simple algebra and admits the reduced trace map \(\text{tr} : C \to Z\) (e.g. \([16, \text{Section 9}]\)). The next theorem describes the zero sets of the discriminant ideals of \(C\) which has PI degree three.

6.4. Theorem. Let \(C\) be the binary cubic generic Clifford algebra of PI degree three with reduced trace map \(\text{tr} : C \to Z\). For all positive integers \(\ell\), the zero sets of the \(\ell\)-th discriminant and \(\ell\)-th modified discriminant ideals of \(C\) over its center \(Z\) coincide, \(\mathbb{V}(D_{\ell}(C/Z), \text{tr}) = \mathbb{V}(MD_{\ell}(C/Z), \text{tr})\); denote this set by \(\mathbb{V}_\ell \subseteq \mathcal{Y} := \text{maxSpec}(Z)\). The following hold:

1. \(\mathbb{V}_\ell = \emptyset\) for \(\ell \leq 3\).
2. \(\mathbb{V}_\ell = \mathcal{Y}^{\text{sing}} = \mathcal{S}_C\) for \(4 \leq \ell \leq 9\).
3. \(\mathbb{V}_\ell = \mathcal{Y}\) for \(\ell > 9\).

Proof. For any \(m \in \text{maxSpec}(Z)\), denote by \(\text{Irr}_m(C)\) the set of isomorphism classes of irreducible representations of \(C\) with central annihilator \(m\). By \([8, \text{Main Theorem (a),(e)}]\), we have

\[
\mathbb{V}(D_{\ell}(C/Z), \text{tr}) = \mathbb{V}(MD_{\ell}(C/Z), \text{tr}) = \left\{ m \in \text{maxSpec}(Z) \mid \sum_{V \in \text{Irr}_m(C)} (\dim_k V)^2 < \ell \right\}.
\]

Hence, our result follows directly from Theorem \([4,3]\). \(\square\)
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