Deforming Maps for Lie Group Covariant Creation & Annihilation Operators

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Abstract

Any deformation of a Weyl or Clifford algebra $A$ can be realized through a ‘deforming map’, i.e. a formal change of generators in $A$. This is true in particular if $A$ is covariant under a Lie algebra $g$ and its deformation is induced by some triangular deformation $U_h g$ of the Hopf algebra $U g$. We propose a systematic method to construct all the corresponding deforming maps, together with the corresponding realizations of the action of $U_h g$. The method is then generalized and explicitly applied to the case that $U_h g$ is the quantum group $U_h sl(2)$. A preliminary study of the status of deforming maps at the representation level shows in particular that ‘deformed’ Fock representations induced by a compact $U_h g$ can be interpreted as standard ‘undeformed’ Fock representations describing particles with ordinary Bose or Fermi statistics.

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I Introduction

In recent years the idea of noncocommutative Hopf algebras \([1]\) (in particular quantum groups \([2]\)) as candidates for generalized symmetry transformations in quantum physics has raised an increasing interest. One way to implement this idea in quantum field theory or condensed matter physics would be to deform the canonical commutation relations (CCR) of some system of mode creators/annihilators, covariant under the action of a Lie algebra \(g\), in such a way that they become covariant under the action of a noncocommutative deformation \(U_h g\) (with deformation parameter \(h\)) of the cocommutative Hopf algebra \(U g\), as it has been done e.g. in Ref. \([3, 4]\) for the \(U_h sl(N)\) covariant Weyl algebra in \(N\) dimensions.

As a toy model for these deformations one can consider the deformed Weyl algebra \(A_h\) in 1 dimension \([5]\) with generators fulfilling the ‘quantum’ commutation relation (QCR)

\[
\tilde{A} \tilde{A}^+ = 1 + q^2 \tilde{A}^+ \tilde{A}
\]  

(I.1)

with \(q = e^h\). When \(q = 1\) the above reduces to the classical Weyl algebra \(A\)

\[
a a^+ = 1 + a^+ a.
\]  

(I.2)

If we define \(n := a^+ a\), \((x)_z = \frac{x^z - 1}{z - 1}\) and \([3]\)

\[
A := a \sqrt{\frac{(n)_q^2}{n}}; \quad A^+ := \sqrt{\frac{(n)_q^2}{n}} a^+,
\]  

(I.3)

we find out that \(A, A^+\) fulfil the QCR (I.1); hence we can define an algebra homomorphism \(f : A_h \to A[[h]]\), or “deforming map” (in the terminology of Ref. \([7, 8]\)), starting from

\[
f(\tilde{A}) = A \quad f(\tilde{A}^+) = A^+.
\]  

(I.4)

The RHS(I.3) have to be understood as formal power series in the deformation parameter \(h\).

We are interested in deformed multidimensional Weyl or Clifford algebras \(A_h\) where the QCR:

1. keep a quadratic structure as in eq. (I.1), so that one can represent the generators as creation or annihilation operators;

2. are covariant under the action \(\tilde{g}_h\) of \(U_h g\).
More precisely, the generators \( \tilde{A}_i^+ \), \( \tilde{A}_i \) should transform linearly under the action of \( \tilde{\varnothing}_h \),

\[
x \tilde{\varnothing}_h \tilde{A}_i^+ = \tilde{\rho}_i^j (x) A_j^+
\]

(I.5)

\[
x \tilde{\varnothing}_h \tilde{A}_i = \tilde{\rho}^{ij}_i (x) A_j
\]

(I.6)

with \( \tilde{\rho}^\vee \) being the the contragradient representation of the representation \( \tilde{\rho} \) of \( U_h g \) \(( \tilde{\rho}^\vee = \tilde{\rho}^T \circ S_h \), where \( S_h \) is the antipode of \( U_h g \) and \( T \) is the operation of matrix transposition).

In this work we essentially stick to the case that \( U_h g \) is \textit{triangular}; we treat the general quasitriangular case in Ref. [9]. In the former case one can show easily that, for arbitrary \( \tilde{\rho} \), \( U_h g \)-covariant QCR are given by

\[
\tilde{A}_i \tilde{A}_j = \pm R_{ij}^{vu} \tilde{A}_u \tilde{A}_v
\]

(I.7)

\[
\tilde{A}_i^+ \tilde{A}_j^+ = \pm R_{ij}^{vu} \tilde{A}_u^+ \tilde{A}_v^+
\]

(I.8)

\[
\tilde{A}_i^+ \tilde{A}_j^+ = \delta_j^1 \delta_A + \pm R_{ij}^{vu} \tilde{A}_u^+ \tilde{A}_v^+;
\]

(I.9)

here the sign \( \pm \) refers to the Weyl/Clifford case respectively, and \( R \) is the corresponding `\( R \)-matrix' of \( U_h g \) \[4\].

\( A_h \) is a left-module algebra of \( U_h g \): the ‘quantum’ action \( \tilde{\varnothing}_h \) is extended to products of the generators as a left-module algebra map \( \tilde{\varnothing}_h : U_h \times A_h \rightarrow A_h \) \( (i.e. \) consistently with the QCR) using the coproduct \( \Delta_h (x) = \sum \mu x_\mu^1 \otimes x_\mu^2 \) of \( U_h g \),

\[
x \tilde{\varnothing}_h (a \cdot b) = \sum \mu (x_\mu^1 \tilde{\varnothing}_h a) \cdot (x_\mu^2 \tilde{\varnothing}_h b),
\]

(I.10)

because the (I.8) are covariant under \( (i.e. \) compatible with) \( \tilde{\varnothing}_h \).

The existence of deforming maps for arbitrary \( (i.e. \) not necessarily of the kind described above) deformations of Weyl (or Clifford) algebras is a consequence \[10\] of a theorem \[11\] asserting the triviality of the cohomology groups of the latter (see Ref. \[12, 9\] for an effective and concise presentations of these results. See also Ref. \[13\], where the problem of stability of quantum mechanics under deformations was addressed for the first time.). However, no general method for their explicit construction is available. Actually, using cohomological arguments, one can also easily show that deforming maps are unique up to a inner automorphism,

\[
f \rightarrow f_\alpha := \alpha f (\cdot) \alpha^{-1} \quad \quad \alpha = 1_A + O(h); \quad (I.11)
\]

\footnote{One just has to note that \( R = \tilde{\rho} \otimes \tilde{\rho} R \), where \( R \) is the universal triangular structure of \( H_h \), and that \( \tau \circ \Delta_h (x) = R \Delta_h (x) R^{-1} \) (\( \tau \) denotes the flip operator).}
therefore it is enough to construct one to find all of them.

In this work we present a general method which allows, given a triangular Hopf algebra \( U_h g \) and any \( U_h g \)-covariant deformed Weyl or Clifford algebra, to explicitly construct the corresponding deforming maps \( f \) and the corresponding realizations \( \rho_h \) of \( \rho_h \) is defined by \( \rho_h := (\text{id} \otimes f) \circ \rho_h \circ (\text{id} \otimes f^{-1}) \). In a first attempt to generalize our construction procedure to \emph{quasitriangular} \( U_h g \), we also generalize the construction to the case that \( U_h g \) is the quantum group \( \mathcal{U}_h \text{sl}(2) \) and \( \rho \) is its fundamental representation. Finally we investigate on the status of deforming maps at the representation-theoretic level.

The construction method is based (Sect. II.1) on use of the Drinfel’d-Reshetikhin twist \( \mathcal{F} \) \cite{14,15}, intertwining between the coproducts of \( U g \) and \( U_h g \), and on the fact that within \( \mathcal{A}[[h]] \) one can realize both the action \( \triangleright \) of \( U_h g \) (Section II.1) and the action \( \triangleright_h \) in an ‘adjoint-like’ way. We show first (Section II) that \( \mathcal{F} \) can be used in a \emph{universal} way to construct, within \( \mathcal{A}_h \), \( U_h g \)-tensors out of \( U g \)-tensors, and in particular out of \( a_i^+, a_i \) objects \( A_i^+, A_i \) that transform under \( \triangleright_h \) as in formula (II.6). Then (Section IV) we verify that the objects \( A_i^+, A_i \) really satisfy the QCR (II.8). In Section V we generalize our construction (by means of the Drinfel’d twist \cite{16}) to the case of deformed Weyl & Clifford algebras with generators belonging to the fundamental representation of the quantum group \( \mathcal{U}_h \text{sl}(2) \); the deforming map is again completely explicit thanks to the semuniversal expression \cite{8} for \( \mathcal{F} \). We compare our deforming map with the one previously found in Ref. \cite{17}. At the representation-theoretic level it would be natural to interpret deforming maps as “operator maps”, in other words as intertwiners between the representations of \( \mathcal{A} \) and \( \mathcal{A}_h \). However we have to expect that, in the role of intertwiners, deforming maps may become singular at \( h = 0 \), because the representation theories of \( \mathcal{A} \), \( \mathcal{A}_h \) are in general rather different. In Section VI we show that there is always a \( * \)-representation of \( \mathcal{A}_h \) which is intertwined by \( \rho \) with the Fock representation of \( \mathcal{A} \); this allows to interpret \( \tilde{A}^i, \tilde{A}_i^+ \) as ‘composite’ operators on a classical Fock space describing ordinary Bosons and Fermions. We also explicitly show that \( f^{-1}_\alpha \) is ill-defined as an intertwiner from the remaining (if any) unitarily inequivalent \( * \)-representations of \( \mathcal{A}_h \).

On the basis of the above result we conclude that also at the quantization-of-field level noncommutative Hopf algebra symmetries are not necessarily incompatible with Bose or Fermi statistics (contrary to what is often claimed). We arrived at
the same conclusion at the first-quantization level in Ref. [18, 19], where the initial
motivation for the present work has originated. The connection between the two
approaches through second quantization will be described elsewhere.

II Preliminaries and notation

II.1 Twisting groups into quantum groups

Let \( H = (U_g, m, \Delta, \varepsilon, S) \) be the cocommutative Hopf algebra associated to the
universal enveloping (UE) algebra \( U_g \) of a Lie algebra \( g \). The symbol \( m \) denotes
the multiplication (in the sequel it will be dropped in the obvious way \( m(a \otimes b) \equiv ab \),
unless explicitly required), whereas \( \Delta, \varepsilon, S \) the comultiplication, counit and antipode
respectively.

Let \( F \in U_g[[h]] \otimes U_g[[h]] \) (we will write \( F = F^{(1)} \otimes F^{(2)} \), in a Sweedler’s
notation with \( \text{upper} \) indices; in the RHS a sum \( \sum_i F_i^{(1)} \otimes F_i^{(2)} \) of many terms is
implicitly understood) be a ‘twist’, \( i.e. \) an element satisfying the relations

\[
(\varepsilon \otimes \text{id})F = 1 = (\text{id} \otimes \varepsilon)F \quad (\text{II.1})
\]

\[
F = 1 \otimes 1 + O(h) \quad (\text{II.2})
\]

\( (h \in \mathbb{C} \) is the ‘deformation parameter’, and \( 1 \) the unit in \( U_g \); from the second
condition it follows that \( F \) is invertible as a power series). It is well known \([14]\) that
if \( F \) also satisfies the relation

\[
(F \otimes 1)[(\Delta \otimes \text{id})(F)] = (1 \otimes F)[(\text{id} \otimes \Delta)(F)], \quad (\text{II.3})
\]

and \( (U_h g, m_h) \) is an algebra isomorphic to \( U_g[[h]] \) with isomorphism, say, \( \varphi_h : U_h g \to U_g[[h]] \) [in particular, if \( U_h g = U_g[[h]] \) and \( \varphi_h = \text{id} \) (mod \( h \)], or even
\( \varphi_h = \text{id} \)], then one can construct a triangular non-cocommutative Hopf algebra
\( H_h = (U_h g, m_h, \Delta_h, \varepsilon_h, S_h, R) \) having an isomorphic algebra structure \( [m_h = \varphi_h^{-1} \circ m \circ (\varphi_h \otimes \varphi_h)] \), an isomorphic counit \( \varepsilon_h := \varepsilon \circ \varphi_h^{-1} \), comultiplication and antipode
defined by

\[
\Delta_h(a) = (\varphi_h^{-1} \otimes \varphi_h^{-1})\{F\Delta[\varphi_h(a)]F^{-1}\}, \quad S_h(a) = \varphi_h^{-1}\{\gamma^{-1}S[\varphi_h(a)]\gamma\}, \quad (\text{II.4})
\]

where

\[
\gamma := SF^{-1(1)} \cdot F^{-1(2)}, \quad \gamma^{-1} = F^{(1)} \cdot SF^{(2)}, \quad (\text{II.5})
\]
and (triangular) universal R-matrix

\[ \mathcal{R} := [\varphi_h^{-1} \otimes \varphi_h^{-1}] (\mathcal{F}_{21}\mathcal{F}^1), \quad \mathcal{F}_{21} := \mathcal{F}^{(2)} \otimes \mathcal{F}^{(1)}. \] (II.6)

Condition (II.3) ensures that \( \Delta_h \) is coassociative as \( \Delta \). The inverse of \( S_h \) is given by

\[ S^{-1}_h (a) = \varphi^{-1}_h \{ \gamma S[\varphi_h(a)] \gamma'^{-1} \}, \]

where

\[ \gamma' := \mathcal{F}^{(2)} \cdot S \mathcal{F}^{(1)} \quad \gamma'^{-1} = S \mathcal{F}^{(2)} \cdot \mathcal{F}^{(1)}; \] (II.7)

\( \gamma^{-1} \gamma' \in \text{Centre}(U_g) \), and \( S \gamma = \gamma'^{-1} \).

Conversely, given a \( h \)-deformation \( H_h = (U_h g, m_h, \Delta_h, \varepsilon_h, S_h, \mathcal{R}) \) of \( H \) in the form of a triangular Hopf algebra, one can find \([14]\) and an isomorphism \( \varphi_h : U_h \rightarrow U_g [[h]] \) an invertible \( \mathcal{F} \) satisfying conditions (II.1), (II.2), (II.3) such that \( H_h \) can be obtained from \( H \) through formulae (II.4),(II.5),(II.7).

Examples of \( \mathcal{F} \)'s satisfying conditions (II.3), (II.1), (II.2) are provided \( e.g. \) by the so-called ‘Reshetikhin twists’ \([15]\)

\[ \mathcal{F} := e^{h \omega_{ij} h_i \otimes h_j}, \] (II.8)

where \( \{ h_i \} \) is a basis of the Cartan subalgebra of \( g \) and \( \omega_{ij} = -\omega_{ji} \in \mathbb{C} \). A less obvious example is for instance the ‘Jordanian’ deformation of Ref. \([20]\).

A similar result to the above holds for genuine quantum groups. A well-known theorem by Drinfel’d, Proposition 3.16 in Ref. \([16]\) proves, for any quasitriangular deformation \( H_h = (U_h g, m_h, \Delta_h, \varepsilon_h, S_h, \mathcal{R}) \) \([2, 21]\) of \( U_g \), with \( g \) a simple finite-dimensional Lie algebra, the existence of an algebra isomorphism \( \varphi_h : U_h g \rightarrow U_g [[h]] \) and an invertible \( \mathcal{F} \) satisfying condition (II.1) such that \( H_h \) can be obtained from \( H \) through formulae (II.4),(II.5),(II.7), as well, after identifying \( h = \ln q \). This \( \mathcal{F} \) does not satisfy condition (II.18), however the (nontrivial) coassociator

\[ \phi := [(\Delta \otimes \text{id})(\mathcal{F}^{-1})](\mathcal{F}^{-1} \otimes 1)(1 \otimes \mathcal{F})[(\text{id} \otimes \Delta)(\mathcal{F})] \] (II.9)

still commutes with \( \Delta^{(2)}(U_g) \),

\[ [\phi, \Delta^{(2)}(U_g)] = 0, \] (II.10)

thus explaining why \( \Delta_h \) is coassociative in this case, too. The corresponding universal (quasitriangular) R-matrix \( \mathcal{R} \) is related to \( \mathcal{F} \) by

\[ \mathcal{R} = [\varphi_h^{-1} \otimes \varphi_h^{-1}] (\mathcal{F}_{21}q^\frac{d}{d} \mathcal{F}^{-1}), \] (II.11)
where \( t := \Delta(\mathcal{C}) - 1 \otimes \mathcal{C} - \mathcal{C} \otimes 1 \) is the canonical invariant element in \( U\mathfrak{g} \otimes U\mathfrak{g} \) (\( \mathcal{C} \) is the quadratic Casimir). The twist \( \mathcal{F} \) is defined (and unique) up to the transformation

\[
\mathcal{F} \rightarrow \mathcal{F} T, \tag{II.12}
\]

where \( T \) is a \( \mathfrak{g} \)-invariant \([i.e. \text{ commuting with } \Delta(U\mathfrak{g})]\) element of \( U\mathfrak{g} [[h]] \otimes^2 \) such that

\[
T = 1 \otimes 1 + O(h), \quad (\varepsilon \otimes \text{id})T = 1 = (\text{id} \otimes \varepsilon)T. \tag{II.13}
\]

A function

\[
T = T(1 \otimes C_i, C_i \otimes 1, \Delta(C_i)) \tag{II.14}
\]

of the Casimirs \( C_i \in U\mathfrak{g} \) of \( U\mathfrak{g} \) and of their coproducts clearly is \( \mathfrak{g} \)-invariant.

In general, as a consequence of the existence of an isomorphism \( \varphi_h \), representations \( \tilde{\rho}, \rho \) of deformed and undeformed algebras are in one-to-one correspondence (except for special values of \( h \) making it singular) through

\[
\rho = \tilde{\rho} \circ \varphi_h. \tag{II.15}
\]

A special case of interest is when \( U\mathfrak{g} \) is a \( \ast \)-Hopf algebra and \( \mathcal{F} \) is unitary,

\[
\mathcal{F}^{\ast \otimes \ast} = \mathcal{F}^{-1}; \tag{II.16}
\]

note that in this case

\[
\gamma' = \gamma^\ast. \tag{II.17}
\]

One can show \([22]\) that \( \mathcal{F} \) can always be made unitary if \( \mathfrak{g} \) is compact.

We will often use a ‘tensor notation’ for our formulae; eq. (II.3) will read

\[
\mathcal{F}_{12,3} = \mathcal{F}_{23,1}, \tag{II.18}
\]

and definition (II.9) \( \phi \equiv \phi_{123} = \mathcal{F}_{12,3}^{-1} \mathcal{F}_{12}^{-1} \mathcal{F}_{1,23}, \) for instance; the comma separates the tensor factors \text{not} stemming from the coproduct. For practical purposes it will be often convenient in the sequel to use the Sweedler’s notation with \textit{lower} indices \( \Delta(x) \equiv x_{(1)} \otimes x_{(2)} \) for the cocommutative coproduct (in the RHS a sum \( \sum_i x_{(1)}^i \otimes x_{(2)}^i \) of many terms is implicitly understood); similarly, we will use the Sweedler’s notation \( \Delta^{(n-1)}(x) \equiv x_{(1)} \otimes \ldots \otimes x_{(n)} \) for the \((n-1)\)-fold coproduct. For the non-cocommutative coproducts \( \Delta_h \), instead, we will use a Sweedler’s notation with barred indices: \( \Delta_h(x) \equiv x_{(1)} \otimes x_{(2)}. \)
II.2 Classical $Ug$-covariant creators and annihilators

Let $\mathcal{A}$ be the unital algebra generated by $1_\mathcal{A}$ and elements $\{a^+_i\}_{i \in I}$ and $\{a^j\}_{j \in I}$ satisfying the (anti)commutation relations
\[
[a^i, a^j]_+ = 0 \\
[a^+_i, a^+_j]_+ = 0 \\
[a^i, a^+_j]_+ = \delta^i_j 1_\mathcal{A}
\] (II.19)
(the $\pm$ sign denotes commutators and anticommutators respectively), belonging respectively to some representation $\rho$ and to its contragradient $\rho^\vee = \rho^T \circ S$ of $H$ ($T$ is the transpose):
\[
x \triangleright a^+_i = \rho(x)_i^j a^+_j \\
x \triangleright a^j = \rho(Sx)_i^j a^i
\] (II.20)

Equivalently, one says that $a^+_i, a^j$ are “covariant” under $\triangleright$, or that they span two (left) modules of $Ug$:
\[
(xy) \triangleright a = x \triangleright (y \triangleright a), \quad x, y \in Ug, \quad \rho(x)_j^i \in \mathbb{C}.
\] (II.21)

with either $a = a^i$ or $a^+_i$.

$\mathcal{A}$ is a (left) module algebra of $(H, \triangleright)$, if the action $\triangleright$ is extended on the whole $\mathcal{A}$ by means of the (cocommutative) coproduct:
\[
x \triangleright (ab) = (x(1) \triangleright a)(x(2) \triangleright b).
\] (II.22)

Then property (II.21) holds for all $a \in \mathcal{A}$.

Setting
\[
\sigma(X) := \rho(X)^i_j a^+_i a^j
\] (II.23)
for all $X \in g$, one finds that $\sigma : g \to \mathcal{A}$ is a Lie algebra homomorphism, so that $\sigma$ can be extended to all of $Ug$ as an algebra homomorphism $\sigma : Ug \to \mathcal{A}$; on the unit element we set $\sigma(1_{Ug}) := 1_\mathcal{A}$. $\sigma$ can be seen as the generalization of the Jordan-Schwinger realization of $g = su(2)$ [23] [formula (V.8)].

Then it is easy to check the following

**Proposition 1** The (left) action $\triangleright : Ug \times \mathcal{A} \to \mathcal{A}$ can be realized in an ‘adjoint-like’ way:
\[
x \triangleright a = \sigma(x(1)) a \sigma(Sx(2)), \quad x \in Ug, \quad a \in \mathcal{A}.
\] (II.24)
In the specially interesting case of a compact section $g$ (with *-structure “$\star$”)
one can introduce in $\mathcal{A}$ a *-structure, the ‘hermitean conjugation’ (which we will
denote by $\ast$), such that

$$(a^i)^* = a_i^\pm.$$  \hspace{1cm} (II.25)

Correspondingly, $\rho$ is a $\ast$-representation ($\rho \circ \ast = \ast \circ \rho^T$) and $\sigma$ becomes a $\ast$-homomorphism, i.e. $\sigma \circ \ast = \ast \circ \sigma$.

### III Quantum covariant creators and annihilators

Let $H_h$ and $\varphi_h$ be as in section [II.1]. Clearly, $\sigma_{\varphi_h} := \sigma \circ \varphi_h$ is an algebra homomorphism $\sigma_{\varphi_h} : U_h g \to \mathcal{A}[\llbracket h \rrbracket]$. Inspired by proposition 3 we are led to define

$$x \triangleright_h a := \sigma_{\varphi_h} x(1)a\sigma_{\varphi_h} S_h x(2).$$  \hspace{1cm} (III.1)

Using the Hopf algebra axioms it is straightforward to prove the relations [cfr. relations (I.10)]

$$(xy) \triangleright_h a = x \triangleright_h (y \triangleright_h a)$$

$$(ab) \triangleright_h = (x(1) \triangleright_h a)(x(2) \triangleright_h b), \quad \forall x, y \in U_h \mathbb{g} \llbracket h \rrbracket, \quad \forall a, b \in \mathcal{A}[\llbracket h \rrbracket].$$  \hspace{1cm} (III.2)

In other words

**Proposition 2** The definition (III.1) realizes $\tilde{\triangleright}_h$ (the left action of $H_h$) on the algebra $\mathcal{A}[\llbracket h \rrbracket]$.

However, $a_i^+, a^j$ are not covariant w.r.t. to $\triangleright_h$. One may ask whether there exist some objects $A_i^+, A^j \in \mathcal{A}$ that are covariant under $\triangleright_h$ and transform as in eq. (I.8).

The answer comes from the crucial

**Proposition 3** The elements

$$A_i^+ := \sigma(F^{(1)}) a_i^+ \sigma(SF^{(2)}\gamma) \in \mathcal{A}[\llbracket h \rrbracket]$$

$$A^i := \sigma(\gamma' SF^{-1(2)}) A^i \sigma(F^{-1(1)}) \in \mathcal{A}[\llbracket h \rrbracket]$$  \hspace{1cm} (III.3,4)

are “covariant” under $\triangleright_h$, more precisely belong respectively to the representation $\tilde{\rho}$ and to its quantum contragredient $\tilde{\rho}^\vee = \tilde{\rho}^T \circ S_h$ of $U_h g$ acting through $\triangleright_h$:

$$x \triangleright_h A_i^+ = \tilde{\rho}(x)_i^j A_i^+ \quad x \triangleright_h A^i = \tilde{\rho}(S_h x)^i_m A^m.$$  \hspace{1cm} (III.5)
Proof. Due to relation (II.4), $F$ is an intertwiner between $\Delta_h$ and $\Delta$ (in this proof we drop the symbol $\varphi_h$):

$$x_{(1')}F^{(1)} \otimes x_{(2')}F^{(2)} = F^{(1)}x_{(1')} \otimes F^{(2)}x_{(2')}.$$  \hspace{1cm} (III.6)

Applying $\text{id} \otimes S$ on both sides of the equation and multiplying the result by $1 \otimes \gamma$ from the right we find [with the help of relation (II.5)]

$$x_{(1')}F^{(1)} \otimes (SF^{(2)})\gamma S_h x_{(2')} = F^{(1)}x_{(1')} \otimes (Sx_{(2')})(SF^{(2)})\gamma.$$  \hspace{1cm} (III.7)

Applying $\sigma \otimes \sigma$ to both sides and sandwiching $a^+_i$ between the two tensor factors we find

$$\sigma(x_{(1')})A^+_i \sigma(S_h x_{(2')}) = \sigma(F^{(1)})\sigma(x_{(1')})a^+_i \sigma(Sx_{(2')})\sigma[(SF^{(2)})\gamma],$$  \hspace{1cm} (III.8)

which, in view of formula (III.4), proves the first relation.

To prove the second relation, let us note that relation (II.4) implies an analogous relation

$$\Delta_h(a)\tilde{F} = \tilde{F}\Delta(a), \quad \text{with} \quad \tilde{F} := [\gamma' SF^{-1} (2) \otimes \gamma' SF^{-1} (1)]\Delta(S\gamma).$$ \hspace{1cm} (III.9)

This can be shown by applying in the order the following operations to both sides of eq. (II.4): multiplying by $F^{-1}$ from the left and from the right, applying $S \otimes S$, multiplying by $\gamma' \otimes \gamma'$ from the left and by $\Delta(S\gamma)$ from the right, replacing $a \rightarrow S_h x$, using the properties (II.4) and $(S_h \otimes S_h) \circ \Delta_h = \tau \circ \Delta_h \circ S_h$. Next, we observe that $A^i$ can be rewritten as

$$A^i = \sigma(\tilde{F}^{(1)}(S(\gamma^{-1})(1))a^i \sigma[(\gamma^{-1})(2)S\tilde{F}^{(2)}(2)]) = \sigma(\tilde{F}^{(1)})a^i \sigma(S\tilde{F}^{(2)}(2))\rho(\gamma^{-1})^i;$$  \hspace{1cm} (III.10)

whence, reasoning as for the first relation,

$$\sigma(x_{(1')})A^i \sigma(S_h x_{(2')}) \quad \text{(III.10)} \quad \sigma(F^{(1)})\sigma(x_{(1')})a^i \sigma(Sx_{(2')})\sigma[(SF^{(2)}(2))\gamma]\rho(\gamma^{-1})^i$$

$$\sigma(\tilde{F}^{(1)})a^i \sigma[(SF^{(2)}(2))\gamma]\rho(\gamma^{-1}Sx)^i$$

$$\sigma(\tilde{F}^{(1)})a^i \sigma[(SF^{(2)}(2))\gamma]\rho(S_h x \cdot \gamma^{-1})^i$$

$$\rho(S_h x)^i A^i$$

which proves the second relation. \(\Box\)

Remark 1 The proposition clearly holds even if one chooses in formulae (III.3), (III.4) two $F$'s differing by a transformation (II.12). We shall exploit this freedom when $U_h g$ is genuinely quasitriangular.
Remark 2 Note that in the $\ast$-Hopf algebra case eq. (II.25), (II.16), (II.17) imply
\[
(A^i)^\dagger = A_i^+.
\] (III.11)

Remark 3 Under the right action $\triangleleft_h (a \triangleleft_h x := (S_h^{-1}x)\triangleright_h a$ with $a \in \mathcal{A}$, $x \in U\mathbf{g}$) the covariance properties of $A^i, A_i^+$ read
\[
A^i \triangleleft_h x = \rho(x) A^i \quad \quad A_i^+ \triangleleft_h x = \rho(S_h^{-1}x)^m A_i^+.
\] (III.12)

Remark 4 $\sigma_{\varphi_h}$ is not the only algebra homomorphism $U_h\mathbf{g} \to \mathcal{A}[\lbrack h \rbrack]$. For any $\alpha = 1_A + O(h)$ we find a new one by setting
\[
\sigma_{\varphi_h, \alpha}(x) = \alpha \sigma_{\varphi_h}(x)\alpha^{-1};
\] (III.13)
correspondingly, we can define a new realization of $\triangleright_h$ by
\[
x \triangleright_h \alpha a := \sigma_{\varphi_h, \alpha}(x(1)) a \sigma_{\varphi_h, \alpha}(S_h x(2)).
\] (III.14)
Covariant objects under $\triangleright_h \alpha$ will be given by
\[
A_i^+ := \alpha A_i^+ \alpha^{-1}, \quad A^i := \alpha A^i \alpha^{-1}.
\] (III.15)

If relations (III.11) hold, we can preserve them by choosing $\alpha^* = \alpha^{-1}$.

To conclude this section, let us give useful alternative expressions for $A_i^+, A^i$ by ‘moving’ to the right/left past $a_i^+, a^i$ the expressions $\sigma(\cdot)$ lying at their left/right in definitions (III.3), (III.4).

Lemma 1 If $\mathcal{T} \in U\mathbf{g} \lbrack[h\rbrack]^\otimes 3$ is $\mathbf{g}$-invariant (i.e. $[\mathcal{T}, U\mathbf{g} \lbrack[h\rbrack]^\otimes 3] = 0$) then $m_{ij} S_i \mathcal{T}$, $m_{ij} S_j \mathcal{T}$ ($i, j = 1, 2, 3, i \neq j$) are $\mathbf{g}$-invariants belonging to $U\mathbf{g} \lbrack[h\rbrack]^\otimes 2$.

(Here $S_i$ denotes $S$ acting on the $i$-th tensor factor, and $m_{ij}$ multiplication of the $i$-th tensor factor by the $j$-th from the right.)

Proof. For instance,
\[
\mathcal{T}^{(1)} x_{(1)} \otimes \mathcal{T}^{(2)} x_{(2)} \otimes \mathcal{T}^{(3)} x_{(3)} \otimes x_{(4)} = x_{(1)} \mathcal{T}^{(1)} \otimes x_{(2)} \mathcal{T}^{(2)} \otimes x_{(3)} \mathcal{T}^{(3)} \otimes x_{(4)} \Rightarrow \mathcal{T}^{(1)} x_{(1)} \otimes \mathcal{T}^{(2)} x_{(2)} S x_{(3)} S \mathcal{T}^{(3)} x_{(4)} = x_{(1)} \mathcal{T}^{(1)} \otimes x_{(2)} \mathcal{T}^{(2)} S \mathcal{T}^{(3)} S x_{(3)} x_{(4)}
\] for any $x \in U\mathbf{g} \lbrack[h\rbrack]$, whence (because of $a_{(1)} S a_{(2)} = \varepsilon(a) = S a_{(1)} a_{(2)}$)
\[
\mathcal{T}^{(1)} x_{(1)} \otimes \mathcal{T}^{(2)} S \mathcal{T}^{(3)} x_{(2)} = x_{(1)} \mathcal{T}^{(1)} \otimes x_{(2)} \mathcal{T}^{(2)} S \mathcal{T}^{(3)},
\] (III.16)
so that $\mathcal{T}^{(1)} \otimes \mathcal{T}^{(2)} S \mathcal{T}^{(3)} \in U\mathbf{g} \lbrack[h\rbrack]^\otimes 2$ is $\mathbf{g}$-invariant. ☐
Proposition 4 If $H_h$ is triangular the definitions (III.3), (III.4) amount to

\[
\begin{align*}
A_i^+ &= a_i^+ \sigma(F^{-1(2)}(2)) \rho(F^{-1(1)}(1))_i \\
A_i^- &= \rho(SF^{(2)}(2)) \sigma(F^{(1)}(1)) a_i^- \\
A_i^0 &= \rho(F^{(1)}(1)) a_i^0 \\
A_i^- &= a_i^0 \sigma(F^{-1(2)}(2)) \rho(\gamma^{-1} SF^{(1)}(1))_i.
\end{align*}
\]  

(III.17)

If $H_h$ is quasitriangular the same formulae hold with in general four different $F$’s [related to each other by transformations (II.12)].

Proof. Observing that

\[
\begin{align*}
\sigma(x)a &= \sigma(x_1) a \sigma(Sx_2 \cdot x_3) \quad \text{(III.18)} \\
a \sigma(x) &= \sigma(x_3) Sx_2 \sigma(x_1) \quad \text{(III.19)}
\end{align*}
\]

for all $x \in Ug$, $a \in A$, we find from relations (III.3), (III.4) and (II.20)

\[
\begin{align*}
A_i^+ &= a_i^+ \sigma(F^{(1)} S F^{(2)}(2)) \rho(F^{(1)}(1))_i \\
A_i^0 &= \sigma(\gamma SF^{-1(2)}(1)) F^{-1(1)}(1) a_i^0.
\end{align*}
\]  

(III.20)

(III.21)

On the other hand, applying the previous lemma to $T = \phi$ [formula (II.9)] we find that

\[
T := m_{23}(id \otimes id \otimes S) \phi (F^{(1)}(1) \otimes F^{-1(1)}(2)) F^{-1}(1) \otimes \gamma^{-1} SF^{-1(2)}(2)
\]

is $g$-invariant, whence one easily finds the relation

\[
(F^{(1)}(1) \otimes F^{(1)}(2)) S(F^{(2)}(2)) \gamma = T^{-1} F^{-1} \equiv F^{-1},
\]  

(III.22)

after noting that $[T, F^{\pm 1(1)}(1) \otimes F^{\pm 1(1)}(2)] = 0$. Replacing in eq. (III.21) one finds relation (III.17)$_1$. In the triangular case $\phi \equiv 1^{\otimes 3}$, implying $T \equiv 1^{\otimes 2}$ and $F' = F$. Similarly one proves the other relations. Relations (III.17)$_3$, (III.17)$_4$, can be found also more directly starting from relations (III.17)$_1$, (III.17)$_2$ by observing that in the unitary-$F$ case they follow from the latter two by applying the $*$-conjugation.

\[\square\]

IV Quantum commutation relations: the triangular case
**Theorem 1** If the nonco-commutative Hopf algebra $H_h$ is triangular [i.e. the twist $\mathcal{F}$ satisfies equation (II.1)], then $A^i, A_j^+$ close the quadratic commutation relations

$$
A^i A_j^+ = \delta^i_j A_A^+ \pm R^{ui}_{uj} A_u^+ A_v^u
$$

(IV.1)

$$
A^i A^j = \pm R_{ui}^{ij} A_u^v A_v^u
$$

(IV.2)

$$
A_j^+ A_j^+ = \pm R^{ui}_{uj} A_u^+ A_v^u
$$

(IV.3)

where $R$ is the (numerical) quantum $R$-matrix of $U\Gamma$ in the representation $\rho$,

$$
R^{ij}_{hk} := [(\rho \otimes \rho)(\mathcal{R})]_{hk}^{ij}.
$$

(IV.4)

**Proof.** Beside eq.’s (II.17), we will need their ‘inverse’ relations:

$$
a_i^+ = A_i^+ \sigma(\mathcal{F}^{(2)}) \rho(\mathcal{F}^{(1)})^i_l
$$

$$
a_i^+ = \rho(\gamma S \mathcal{F}^{-1(1)})^i_j \sigma(\mathcal{F}^{-1(2)}) A_i^+
$$

$$
A_l^i = \rho(\mathcal{F}^{-1(1)})^i_j \sigma(\mathcal{F}^{-1(2)}) A_l^i
$$

(IV.5)

$$
A_l^{i} = \rho(\mathcal{F}^{(2)}) \rho(S \mathcal{F}^{(1)} \cdot \gamma)_l^i.
$$

Using eq.’s (II.17),

$$
A_i^{j} A_j^+ \equiv (III.17) \quad \rho(\mathcal{F}^{(1)})^i_l \sigma(\mathcal{F}^{(2)}) \alpha^l_m \sigma(\mathcal{F}^{-1(2')}) \rho(\mathcal{F}^{-1(1')})^m_j
$$

(II.19)

$$
\rho(\mathcal{F}^{(1)})^i_l \sigma(\mathcal{F}^{(2)})[\delta^l_m \pm \alpha^l_m \alpha^i_j] \sigma(\mathcal{F}^{-1(2')}) \rho(\mathcal{F}^{-1(1')})^m_j
$$

(III.19)

$$
\rho(\mathcal{F}^{(1)})^i_l \sigma(\mathcal{F}^{(2)})[\delta^l_m \pm \alpha^l_m \alpha^i_j] \sigma(\mathcal{F}^{-1(2')}) \rho(\mathcal{F}^{-1(1')})^m_j
$$

(III.20)

$$
\rho(\mathcal{F}^{(1)})^i_l \sigma(\mathcal{F}^{(2)})[\delta^l_m \pm \alpha^l_m \alpha^i_j] \sigma(\mathcal{F}^{-1(2')}) \rho(\mathcal{F}^{-1(1')})^m_j
$$

(IV.3)

where $G := \mathcal{F}_{12} \mathcal{F}_{23,1} \mathcal{F}_{12}^{-1} \mathcal{F}_{32}^{-1}$. On the other hand, applying to eq. (II.18) the permutations $\tau_{23} \circ \tau_{12}$ and $\tau_{23}$ we obtain respectively

$$
\mathcal{F}_{12} \mathcal{F}_{23,1} = \mathcal{F}_{31} (\mathcal{F}^{(1)}_{(1)} \otimes \mathcal{F}^{(2)}_{(2)})
$$

(IV.7)

$$
\mathcal{F}_{12}^{-1} \mathcal{F}_{32}^{-1} = (\mathcal{F}^{(1)}_{(1)} \otimes \mathcal{F}^{(2)}_{(2)})^{-1} \mathcal{F}_{13}^{-1},
$$

(IV.8)

which replaced in the definition of $G$ [using the definition (II.4)] give $G = \mathcal{R}_{13}$; this proves eq. (IV.1).

As for relation (IV.2),

$$
A^i A^j \equiv (III.17) \quad \rho(\mathcal{F}^{(1)})^i_l \sigma(\mathcal{F}^{(2)}) \alpha^l_m \sigma(\mathcal{F}^{-1(2')}) \rho(\gamma^{-1} S \mathcal{F}^{-1(1')})^j_m
$$

(II.13)

$$
\pm \rho(\mathcal{F}^{(1)})^i_l \sigma(\mathcal{F}^{(2)}) \alpha^l_m \alpha^i_j \sigma(\mathcal{F}^{-1(2')}) \rho(\gamma^{-1} S \mathcal{F}^{-1(1')})^j_m
$$

(IV.2)
II. 20

\[ \pm \rho(F_0^{1(2)}a^m \sigma(F_0^{2(2)}F_0^{-1(2)})a^i \rho(\gamma^{-1}SF_0^{-1(1)}), SF_0^{2(2)}j_m) \]

IV. 3

\[ \pm \rho(F_0^{1(2)}F_0^{-1(1)}j_i A^m \sigma(F_0^{2(2)}F_0^{-1(2)})F_0^{-1(2)}) \times \rho(\gamma^{-1}S(F_0^{1(2)}F_0^{2(2)}F_0^{-1(1)})) \gamma j_m \]

II. 4

\[ \pm \rho(G^{-1(1)})j_i A^m \sigma(G^{-1(2)})A^i \rho(S_h(G^{-1(3)}))j_m. \] (IV.9)

But we have already shown that \( G = R_{13} \); by recalling that \((\text{id} \otimes S_h)R^{-1} = R\), relation (IV.2) follows.

Similarly one can prove relation (IV.3), which can be found also more directly by observing that in the unitary-\( F \) case it follows from the previous one by applying the \( \ast \)-conjugation and by noting that \( \bar{R} = R_{21} \). \( \square \)

Remark 5. It is interesting to ask how the invariants which can be constructed from \( A^i, A^+_j \) are related to the ones which can be constructed from \( a^i, a^+_j \). It is straightforward to prove that e.g. any invariant of the form \( I^n_h := A^+_i A^+_{i_1} \ldots A^+_{i_n} \ldots A^{+i} \) coincides with the invariant of the form \( I^n := a^+_i a^+_{i_1} a^+_{i_2} \ldots a^{+i} \) (in particular, so is \( I^2 \equiv N := A^+_i A^i = n \)). One can show \[ 3 \] the equality \( P(A^i, A^+_j) = P(a^i, a^+_j) \) for all polynomial invariants \( P(\tilde{A}^i, \tilde{A}^+_j) \). This is no more true if \( H \) is genuinely quasitriangular.

V Quantum commutation relations: the \( \mathcal{U}_h sl(2) \) case

It is now tempting to consider the quasitriangular case and ask whether a transformation such as in Remark 1 can map \( a^i, a^+_j \) into \( U_h g - \text{covariant } A^i, A^+_j \) satisfying relations of the type found in Ref. \[ 3, 4 \]. We will stick here to the case \( g = sl(2, \mathbb{C}) \), \( \rho \equiv \text{fundamental representation, addressing the reader to Ref. } [9] \) for the general case.

We fix our conventions as follows. As ‘classical’ generators of \( Usl(2) \) we choose \( j_0, j_+, j_- \in g \),

\[ [j_0, j_\pm] = \pm j_\pm \quad [j_+, j_-] = 2j_0, \] (V.1)

\[ \Delta(x) = 1 \otimes x + x \otimes 1 \quad x = j_0, j_+, j_-; \] (V.2)
as generators of $U_h sl(2)$ we choose $J_0, J^+, J^-$ satisfying

\[ [J_0, J_\pm] = \pm J_\pm \quad [J_+, J_-] = \frac{q^{2j_0} - q^{-2j_0}}{q - q^{-1}} \]  \tag{V.4}

\[ \Delta_h(J_0) = 1 \otimes J_0 + J_0 \otimes 1 \quad \Delta_h(J_\pm) = J_\pm \otimes q^{-j_0} + q^{j_0} \otimes J_\pm. \] \tag{V.5}

At the coalgebra level, the universal $\mathcal{F}$ [connecting $\Delta$ to $\Delta_h$ in formula (II.4)] is not explicitly known; however the $U sl(2)[[h]]$-valued matrix $F := (\rho \otimes \text{id})F$ has been determined in Ref. [8] and reads

\[
F = \begin{pmatrix}
    a(j, j_0) & b(j, j_0)j- & -j+b(j, j_0) \\
    -j, b(j, j_0) & a(j, j_0-1) & \end{pmatrix}; \quad F^{-1} = \begin{pmatrix}
    a(j, j_0) & -b(j, j_0)j- & j+b(j, j_0) \\
    -j, b(j, j_0) & a(j, j_0-1) & \end{pmatrix}, \tag{V.6}
\]

where

\[
a(j, j_0) := \frac{q^{j-j_0}}{\sqrt{(1+2j)(1+2j_0)}} \left[ (1+j+j_0) \left( \frac{1}{q} \right) + q^{-\frac{(j+j_0)}{2}} \sqrt{(j-j_0)(j-j_0)_q} \right], \tag{V.7}
\]

\[
b(j, j_0) := \frac{q^{j-j_0}}{\sqrt{(1+2j)(1+2j_0)}} \left[ \frac{1}{1+j+j_0} - q^{-\frac{(j+j_0)}{2}} \sqrt{\frac{(j-j_0)(j-j_0)_q}{j-j_0}} \right].
\]

The matrix elements in eq. (V.6) are w.r.t. the orthonormal basis \{ |\uparrow\rangle, |\downarrow\rangle \} (spin up, spin down) of eigenstates of $j_0$ with eigenvalues $\pm \frac{1}{2}$. $a_\uparrow^+ |0\rangle = |\uparrow\rangle$ etc. All indices in the sequel will run over \{ |\uparrow\rangle, |\downarrow\rangle \}. $F$ is unitary w.r.t. the $su(2)$ *-structure $j_0^* = j_0$, $(j_+)^* = j_-$. The homorphism $\sigma$ in this case coincides with the well-known Jordan-Schwinger realization of $sl(2)$ \[23\] and reads

\[
\sigma(j_+) = a_\uparrow^+ a_\uparrow, \quad \sigma(j_-) = a_\downarrow^+ a_\downarrow, \quad \sigma(j_0) = \frac{1}{2} (a_\uparrow^+ a_\downarrow - a_\downarrow^+ a_\uparrow) \tag{V.8}
\]

\[\text{Let us recall incidentally that the mapping } \varphi_h : U_h sl(2) \rightarrow U sl(2)[[h]], \text{ up to an inner automorphism } \varphi_h \rightarrow \varphi_h v := v \varphi_h (\cdot) v^{-1} \text{ is given by } \]

\[
\varphi_h(J_0) = j_0 \quad \varphi_h(J_\pm) = \sqrt{\frac{j \pm j_0}{j_0}(1+j+j_0)} j_\pm. \tag{V.3}
\]

where $j$ is the positive root of the equation $j(j+1) - C = 0$, $C = j_j - j + j_0(j_0 + 1)$ is the Casimir, and $[x]_q := q^{x}-q^{-x}$. \[3\] See formulae (3.1), (3.30) in Ref. [8]. To match our conventions with theirs, one has to rescale $j_\pm$ by $\sqrt{2}$ and note that the right correspondence between our notation and theirs is $\mathcal{F} \equiv \mathcal{F}_q \leftrightarrow U_q^{-1}$, what is needed to match the coproducts.
implying in particular
\[ \sigma(j) = \frac{n}{2}, \]  
(V.9)
where \( n := a_+^i a^i \) is the ‘classical number of particles’ operator. The \( \mathcal{U}_h\mathfrak{su}(N) \)-covariant Weyl algebra (with \( N \geq 2, q \) real and positive) was introduced by Pusz and Woronowicz [3]; independently, Wess and Zumino [4] introduced its \( \mathcal{U}_h\mathfrak{sl}(N) \)-covariant generalization (arbitrary complex \( q \)) in R-matrix notation. One can consider also its Clifford version [24]. In the R-matrix notation [4] the QCR of the generators (which we will denote here by \( \tilde{A}_i, \tilde{A}_j^+ \)) read
\[ \tilde{A}_i \tilde{A}_j^+ = \pm q^{\mp 1} R_{ij}^{uv} \tilde{A}_u^+ \tilde{A}_v, \]  
(V.10)
\[ \tilde{A}_i^+ \tilde{A}_j = \pm q^{\mp 1} R_{ij}^{uv} \tilde{A}_u^+ \tilde{A}_v, \]  
(V.11)
\[ \tilde{A}_i \tilde{A}_j^+ = 1_A \delta_i^j \pm q^{\mp 1} R_{ij}^{uv} \tilde{A}_u^+ \tilde{A}_v, \]  
(V.12)
where \( i, j = 1, \ldots, N, R = (\rho_d \otimes \rho_d) \mathcal{R} \) is the \( R \)-matrix of \( \mathcal{U}_h\mathfrak{sl}(N) \) in the defining representation \( \rho_d \), and the sign \( \pm \) refers to Weyl and Clifford respectively. Both have been treated subsequently also by many other authors. They are related, but should not be confused, with the celebrated Biedenharn-Macfarlane-Hayashi \( q \)-oscillator (super)algebras [3, 26]. When \( q > 0 \) the QCR of the generators are compatible with the \( * \)-structure [3] of annihilators and creators,
\[ A_+^i = (A^i)^*. \]  
(V.13)
When \( N = 2 \) we will pick up \( i, j, \ldots = \uparrow, \downarrow \); the \( R \)-matrix will read
\[ R \equiv \| R_{ij}^{uv} \| := \begin{pmatrix} q & 1 \\ (q - q^{-1}) & 1 \\ q \end{pmatrix}, \]  
(V.14)
where the row and columns of the matrix (V.14) are ordered in the usual way: \( \uparrow, \downarrow \) from left to right and from up to down.

4The ‘braiding’ (V.12) between \( \tilde{A}_i, \tilde{A}_j^+ \) could be also replaced by the inverse one: \( \tilde{A}_i \tilde{A}_j^+ = 1_A \delta_i^j \pm q^{\mp 1} R_{ij}^{uv} \tilde{A}_u^+ \tilde{A}_v \). For a comprehensive introduction to braiding see Ref. [24].

5The generators \( \alpha^i, \alpha_+^i \) of the latter fulfill ordinary (anti)commutation relations, except for the \( q \)-(anti)commutation relations \( \alpha^i \alpha_+^i = q^{\mp 2} \alpha^i \alpha_+^i \) and are not \( \mathcal{U}_h\mathfrak{g} \)-covariant (in spite of the fact that they are usually used to construct a generalized Jordan-Schwinger realization of \( \mathcal{U}_h\mathfrak{g} \)). It is of interest to note that, however, the generators \( \alpha^i, \alpha_+^i \) can be tipically realized as algebraic ‘functions’ of \( \tilde{A}_i, \tilde{A}_i^+ \) [3], whereas the generators \( A^i, A_+^i \) can be tipically realized only as transcendental ‘functions’ of \( \tilde{A}_i, \tilde{A}_i^+ \).
More explicitly, the Weyl QCR (V.10) - (V.12) read

\[
\begin{align*}
\tilde{A}^+ \tilde{A}^+ &= 1_A + q^2 \tilde{A}^+_i \tilde{A}^+_i + (q^2 - 1) \tilde{A}^+_i \tilde{A}^+_i, \\
\tilde{A}^+ \tilde{A}^+_i &= 1_A + q^2 \tilde{A}^+_i \tilde{A}^+_i, \\
\tilde{A}^+ \tilde{A}^+_i &= q \tilde{A}^+_i \tilde{A}^+_i, \\
\tilde{A}^+ \tilde{A}^+_i &= q \tilde{A}^+_i \tilde{A}^+_i, \quad \text{(V.15)}
\end{align*}
\]

and the Clifford

\[
\begin{align*}
\tilde{A}^+ \tilde{A}^+_i &= 1_A - \tilde{A}^+_i \tilde{A}^+_i + (q^{-2} - 1) \tilde{A}^+_i \tilde{A}^+_i, \\
\tilde{A}^+ \tilde{A}^+_i &= 1_A - \tilde{A}^+_i \tilde{A}^+_i, \\
\tilde{A}^+ \tilde{A}^+_i &= -q^{-1} \tilde{A}^+_i \tilde{A}^+_i, \\
\tilde{A}^+ \tilde{A}^+_i &= -q^{-1} \tilde{A}^+_i \tilde{A}^+_i, \quad \text{(V.17)}
\end{align*}
\]

\[
\begin{align*}
\tilde{A}^+ \tilde{A}^+_i &= -q^{-1} \tilde{A}^+_i \tilde{A}^+_i, \\
\tilde{A}^+ \tilde{A}^+_i &= 0, \\
\tilde{A}^+ \tilde{A}^+_i &= 0; \quad \text{(V.18)}
\end{align*}
\]

\[
\begin{align*}
\tilde{A}^+ \tilde{A}^+_i &= -q^{-1} \tilde{A}^+_i \tilde{A}^+_i, \\
\tilde{A}^+ \tilde{A}^+_i &= 0, \\
\tilde{A}^+ \tilde{A}^+_i &= 0. \quad \text{(V.19)}
\end{align*}
\]

Now we try to construct the \(A^+_i, A^i\). The Ansatz of Proposition 4 is equivalent to

\[
\begin{align*}
A^+_i &= a^+_i \sigma(F^{-1(2)}) \rho(F^{-1(1)}) \rho(f(n)) \\
A^i &= g(n) \rho(F^{1(1)}) \sigma(F^{1(2)}) a^i. \quad \text{(V.20)}
\end{align*}
\]

with the same \(F\) and two invertible functions \(f, g\). The product \(fg\) is determined by requiring that the commutation relation between \(N := A^+_i A^i\) and \(A^i, A^+_j\) are the same as those between \(\tilde{N} := \tilde{A}^+_i \tilde{A}^i\) and \(\tilde{A}^i, \tilde{A}^+_j\):

\[
NA^+_i = A^+_i (1 + q^{\pm 2} N) \quad \quad NA^i = A^i q^{\mp 2} (N - 1); \quad \text{(V.21)}
\]

one easily finds

\[
gf(x) = \sqrt{\frac{(x + 1) q^{\pm 2}}{(x + 1)}}. \quad \text{(V.22)}
\]

\[6\text{In fact, in } T = K \mathbb{C} \otimes 1, 1 \otimes \mathbb{C}, \Delta(\mathbb{C}) \text{ the dependence on the last argument results only in a numerical factor, because of eq's (I.120), (I.24). The } \mathbb{C} \otimes 1 \text{- and } 1 \otimes \mathbb{C} \text{-dependences can be replaced by the } n \text{-dependence, since it is easy to prove that } \sigma(\mathbb{C}) = \frac{n}{2} \left( \frac{n}{2} + 1 \right); \text{ the latter can be concentrated either at the left or at the right of } a^i, a^+_i, \text{ upon using the commutation relations } [n, a^+_i] = a^+_i, [n, a^i] = -a^i.\]
If \( q \in \mathbb{R}^+ \) and we wish that \( A_i^+ = (A^i)^* \), we need to choose

\[
g(x) = f(x) = \sqrt{\frac{(x+1)q^{\pm 2}}{(x+1)}}.
\]

This is nothing else but the already encountered function \( [3] \) needed to transform the classical creation/annihilation operators in one dimension \( a^+, a \) into the quantum ones \( A^+, A \).

If we are interested in Fock space representations, eq. \( A_i^+|0\rangle = a_i^+|0\rangle \), stating that the quantum and classical one-particle state coincide, is automatically satisfied, because \( f(0) = 1 \).

Now one can express the RHS (V.20) thoroughly in terms of \( a^i, a_j^+ \). It is convenient to introduce the up-down ‘number of particle’ operators \( n^\uparrow, n^\downarrow \) \((n^\uparrow + n^\downarrow = n)\), by

\[
n^\uparrow := a_i^+a_i^\dagger \quad n^\downarrow := a_i^+a_i^\dagger.
\]

Using Eq.’s (V.6), (V.7), (V.8), (V.20), it is easy algebra\(^7\) to prove the following

**Proposition 5** Equations (V.20) amount, in the Weyl case, to

\[
A_i^+ = \sqrt{\frac{(n^\uparrow)^2}{n^\uparrow}} q^n a_i^\dagger, \quad A_i^+ = \sqrt{\frac{(n^\downarrow)^2}{n^\downarrow}} q^n a_i^\dagger, \quad (V.25)
\]

and in the Clifford one, to

\[
A_i^+ = q^{-n^\uparrow} a_i^\dagger, \quad A_i^+ = a_i^\dagger, \quad (V.26)
\]

We are ready for the main theorem of this section (the proof is a straightforward computation).

**Theorem 2** The elements \( A_i^+A_j^+ \in \mathcal{A} \) (\( \mathcal{A} \) Weyl or Clifford) defined in formulae (V.20), (V.23) satisfy the QCR (V.12) - (V.11).

Let us compare the map (V.25) (for the Weyl algebra) with the one found in Ref. [17]. That map, in our notation, would read

\[
A_i^+ = q^{n^\uparrow} a_i^\dagger, \quad A_i^+ = a_i^\dagger, \quad (V.27)
\]

\(^7\)In the fermionic case one has to fully exploit the nilpotency of \( a^i, a_j^+ \).
[clearly, it is not compatible with the $\ast$-structure $A^+_{i,\alpha} = (A^{i,\alpha})^\ast$. It is straightforward to check that the element $\alpha \in \mathcal{A}[[h]]$ needed to transform $A^+_i, A^i$ into $A^+_{i,\alpha}, A^{i,\alpha}$ [formulae (III.15)] is

$$\alpha := \sqrt{\frac{\Gamma(n^+ + 1)\Gamma(n^- + 1)}{\Gamma_q(n^+ + 1)\Gamma_q(n^- + 1)}}.$$  (V.28)

where $\Gamma$ is the Euler $\Gamma$-function and $\Gamma_q$ its $q$-deformation [27] satisfying the property

$$\Gamma_q(a + 1) = (a)_q \Gamma_q(a).$$  (V.29)

The corresponding realization of $\tilde{\mathcal{H}}_h$ is obtained through relation (III.14)

\textbf{VI Representation theory}

In this section we compare representations of $\mathcal{A}$ with representations of $\mathcal{A}_h$ and investigate whether the deforming maps found in the preceding sections can be interpreted as intertwiners between them.

We start with a general remark. At least perturbatively in $h$, we expect that, for any given representation $\pi$ of $\mathcal{A}$ on some space $V$, the objects $\pi(A^+_{i,\alpha}), \pi(A^{i,\alpha})$ are well-defined operators on $V$, since $A^+_{i,\alpha} = a^+_i + O(h), A^{i,\alpha} = a^i + O(h)$; consequently, $\pi \circ f_\alpha$ is a representation of $\mathcal{A}_h$ on $V$. Let us prove this statement more rigorously for some specific kind of representations.

If $g$ is compact, we can choose $\rho$ to be a unitary representation; the $g$-covariant CCR will admit the $\ast$-relations (II.25). Assuming the latter, Stone-Von Neumann theorem (or its Clifford counterpart) applies: there exists a ‘$\ast$-representation’ $\pi$ of eq. (II.19) and (II.25) on a separable Hilbert space $\mathcal{H}$, i.e. a $\pi$ fulfilling the following properties:

1. $\pi(a^i), \pi(a^+_i)$ are closed;
2. $\pi(a^+_i) \subset [\pi(a^i)]^\dagger$;
3. there exists a dense linear subset $\mathcal{D} \subset \mathcal{H}$ contained in the domain of the product of any two operators $\pi(a^i), \pi(a^+_i)$;
4. the CCR (II.19) hold on $\mathcal{D}$;
5. the nonnegative-definite operator $\pi(n) := \pi(a^+_i a^i)$ is essentially self-adjoint on $\mathcal{D}$. 18
Moreover, the irreducible components \((\pi_i, H_i)\) of \((\pi, H)\) are, up to a unitary transformation \(U_i : H \rightarrow H_i\), Fock space representations, i.e. there exist ‘ground states’ \(|0\rangle_i \in H_i\):

\[
\pi(a^i)|0\rangle_i = 0. \quad (VI.1)
\]

One can choose the dense set \(D\) as the linear span of all the analytic vectors of the form \(\pi(a_{i_1}^+ \cdots a_{i_k}^+)|0\rangle_i\).

In the case under consideration, choosing \(F\) unitary [so that relation (III.11) hold] and setting \(\Pi := \pi \circ f\), one can easily realize that \(\Pi\) is a \(*\)-representation of the QCR \((\text{III.8})\) or \((\text{V.10-V.12})\) on \(H\), i.e. it fulfills conditions analogous to 1. - 5., with \(\tilde{A}^i, \tilde{A}^+_i\) in the place of \(a^i, a^+_i\).

The reason is essentially the following. Assume first that \(\pi\) and \(\rho\) are irreducible, and let \(H^{(k)} := \text{Span}_C\{\pi(a_{i_1}^+ \cdots a_{i_k}^+)|0\rangle\}\); \(H^{(k)}\) is a finite dimensional eigenspace of \(\pi(n)\), because \(\rho\) is finite-dimensional and therefore, \(A\) has a finite number of generators. From equations (III.17) and the relation \([\sigma(U g), n] = 0\) it follows that \(\pi(A^i), \pi(A^+_i)\) differ from \(\pi(a^i), \pi(a^+_i)\) just by operators mapping each \(H^{(k)}\) into itself. Therefore all the properties 1. - 5. are inherited by \(\pi(A^i), \pi(A^+_i)\) as well. The same result holds for \(\Pi_\alpha := \pi \circ f\alpha\) if \(\alpha^* = \alpha^{-1}\), because \(\pi(\alpha)\) can be absorbed in the unitary transformation \(U\). Finally, the result extends by linearity and orthogonality also to the case that \(\pi\) and/or \(\rho\) are reducible.

Summing up, \(f\alpha\) is an intertwiner between the \(*\)-representation of the CCR and a \(*\)-representation of the QCR. In other words, we can represent the objects \(\tilde{A}^i, \tilde{A}^+_i\) [fulfilling the QCR and the \(*\)-relations (III.11)] as composite operators acting on the Fock space \(H\), as anticipated in the introduction. The latter can be used to describe ordinary Bosons and Fermions. This disproves the quite common belief that non-cocommutative Hopf algebra symmetries are necessarily incompatible with ordinary Bose and Fermi statistics.

Let us analyze now a different situation. Let \(g = sl(2), A\) be the corresponding two-dimensional Weyl algebra and \((\pi, V)\) the Fock space representation of the latter (with ground state \(|0\rangle\); let \(A_h\) be the deformation of \(A\) considered in section \(V\) with \(q^2 = e^{2\hbar}\) a root of unity, i.e. \(q^{2p} = 1, q^{2k} \neq 1\) with \(p, k \in \mathbb{N}\) and \(k < p\). It is easy to realize that \(\pi((n_i), \varphi(a_{i}^+)^{mp}(a_{i}^+)^n)|0\rangle = 0\) for \(l, m = 0, 1, \ldots\); consequently, \(A^i, A^k\) annihilate all the vectors of the form \(|m, n\rangle := \pi((a_{i}^+)^{mp}(a_{i}^+)^n)|0\rangle\) \((n, m = 0, 1, \ldots)\). Although \((\pi, V)\) was irreducible as a representation of \(A\), it is reducible.

\[\text{Incidentally, from relation (II.1) it follows in particular } \pi(A^i)|0\rangle = 0 \text{ and } \pi(A^+_i)|0\rangle = \pi(a^+_i)|0\rangle.\]
as a representation of the subalgebra of $\mathcal{A}$ generated by $A^i, A^{i+}$; the irreducible components $V_{m,n}$ are isomorphic and $p^2$-dimensional, and are obtained by applying $A^{i+}$'s to the cyclic vectors $|m,n\rangle$. $V_{m,n}$ is also a (reducible) representation of $U_h sl(2)$ and may be called (with an abuse of terminology, since we have not introduced any spatial degrees of freedom) an ‘anyonic space’. Thus, $f$ can be seen as an intertwiner from the classical Bosonic Fock representation onto a direct sum of ‘anyonic’ representations of $\mathcal{A}_h$.

Whenever some class $\mathcal{P}$ of representations of $\mathcal{A}_h$ is ‘larger’ than the corresponding class $\mathcal{p}$ of representations of the CCR, then $f^{-1}_a$ can of course be well-defined (as an intertwiner between $\mathcal{P}$ and $\mathcal{p}$) only on some proper subset of $\mathcal{P}$; on its complement it must be singular.

It is instructive to see whether and how this phenomenon occurs in some concrete example, e.g. for the class of $*$-representations considered above. Both a deformed algebra $\mathcal{A}_h$ covariant under some triangular Hopf algebra $H_h$ of the type (I.8) (with $\mathcal{F}$ unitary) and the deformed Clifford algebra of section IV are not good for this purpose, because $\mathcal{P}$ is as large as $\mathcal{p}$. On the contrary, it was proved in Ref. [3] that there are many unitarily non-equivalent $*$-representations on separable Hilbert spaces of the $U_h su(2)$-covariant ($q \in \mathbb{R}^+$) deformed Weyl algebra.

9For the latter algebra this was shown in Ref. [24]. For the former this can be understood as follows. In the present case $\mathcal{R} = \mathcal{F}^{-2}$. Given any representation $\rho$ of $Ug$ and the corresponding representation $\tilde{\rho}$ of $U_h g$, let us choose a basis of eigenvectors $|l\rangle$ of the Cartan subalgebra. In this basis $R := (\tilde{\rho} \otimes \tilde{\rho}) \mathcal{R}$ will be diagonal, and in particular it will be $R_{ii} = 1$ as a consequence of the antisymmetry of $\omega_{ij} (h_i \otimes h_j)$. The QCR (I.8) will imply in particular

$$\tilde{A}^i \tilde{A}^+_i = 1_A \pm \tilde{A}^+_i \tilde{A}^i \quad \text{(no sum over } i), \quad (VI.2)$$

and, setting $\tilde{N}_i := A^+_i \tilde{A}^i \quad \text{(no sum over } i), \quad (VI.3)$

$$[\tilde{N}_i, \tilde{N}_j] = 0 \quad [\tilde{N}_i, \tilde{A}^+_j] = \delta_{ij} \tilde{A}^+_j, \quad [\tilde{N}_i, \tilde{A}^j] = -\delta_{ij} \tilde{A}^i.$$  

Let us denote by $\mathcal{A}_h^{(i)}$ the subalgebra generated by $\tilde{A}^i, \tilde{A}^{i+}$. Each $\mathcal{A}_h^{(i)}$ separately is isomorphic to a classical one-dimensional Weyl/Clifford algebra, and therefore admits, up to unitary equivalences, a unique $*$-representation in the form of a Fock representation with level $N_i$. Because of the relations (VI.3), we can choose $\{N_i\}_{i \in I}$ as a complete set of commuting observables, whence the uniqueness, up to unitary equivalences, of the $*$-representation of $\mathcal{A}_h$ follows immediately.

10Incidentally, even 1-dimensional deformed Heisenberg algebras may have more unitarily inequivalent representations [28]. Moreover, within each representation one has still some freedom in the ‘physical’ interpretation of the observables, e.g. what are the ‘right’ momentum/position observables, see e.g. Ref. [29].
Hence, setting \( \tilde{\eta} \) are eigenvectors of the following operators: \( E, r, s \) parameters unitarily inequivalent irreducible \(*\)-representations of this algebra by three parameters \( E, r, s \), where \( q^2 < E < 1 \), and \( r, s \) are nonnegative integers with \( r + s \leq 2 \). We shall denote the corresponding Hilbert space by \( \mathcal{H}_{E,r,s} \). We divide them in the following classes for clarity:

1. In the representation \( s = 2 \) (and \( r = 0 \), the value \( E \) is irrelevant) one parametrizes the vectors of an orthonormal basis of \( \mathcal{H}_{E,0,2} \) by \( \{ |q^{2m_1}E, \frac{q^{2m_2}}{q^2-1}\rangle \} \).

2. In the representations with \( s = 1 = r \), one parametrizes the vectors of an orthonormal basis of \( \mathcal{H}_{E,1,1} \) by \( \{ |q^{2n_1}E, \frac{q^{2n_2}}{q^2-1}\rangle \} \).

3. In the representations with \( s = 1, r = 0 \), one parametrizes the vectors of an orthonormal basis of \( \mathcal{H}_{E,0,1} \) by \( \{ 0, \frac{q^{2n_1}}{q^2-1}\} \) (the value \( E \) is irrelevant).

4. In the representations with \( s = 0, r = 2 \), one parametrizes the vectors of an orthonormal basis of \( \mathcal{H}_{E,2,0} \) by \( \{ |q^{2n_1}E, q^{2n_2}E\rangle \} \).

5. Finally, in the representations with \( s = 0, r = 1 \), one parametrizes the vectors of an orthonormal basis of \( \mathcal{H}_{E,1,0} \) by \( \{ |q^{2n_1}E, 0\rangle \} \).

Here \( n_1, n_2; m_1, m_2 \) denote integers, with \( m_1 \geq m_2 \geq 0, n_1 < n_2 \). Only representations 1, 3 have a ground state; but representation 3 is degenerate.

In the rest of this section we drop the symbols \( \Pi \) to avoid a too heavy notation.

On the vectors \( |\eta_\uparrow, \eta_\downarrow\rangle \) of the above basis of any \( \mathcal{H}_{E,r,s} \), \( \tilde{A}_\uparrow, \tilde{A}_\downarrow; i = \uparrow, \downarrow \), are defined (modulo a possible but here irrelevant phase in the case \( r + s < 2 \)) by

\[
\begin{align*}
\tilde{A}_\uparrow |\eta_\uparrow, \eta_\downarrow\rangle &= \sqrt{\eta - \eta_\downarrow} |q^{-2}\eta, \eta_\downarrow\rangle \\
\tilde{A}_\downarrow |\eta_\uparrow, \eta_\downarrow\rangle &= \sqrt{q^2\eta - \eta_\uparrow} |q^2\eta_\downarrow, \eta_\downarrow\rangle
\end{align*}
\]

Hence, setting \( \tilde{\tilde{N}} := \tilde{A}_\uparrow \tilde{A}_\downarrow \), \( \tilde{\tilde{N}} := \tilde{A}_\downarrow \tilde{A}_\uparrow \) and \( \tilde{N} := \tilde{N}_\uparrow + \tilde{N}_\downarrow \), we find that \( |\eta, \eta_\downarrow\rangle \) are eigenvectors of the following operators:

\[
\begin{align*}
\tilde{N}_\uparrow |\eta, \eta_\downarrow\rangle &= (\eta - \eta_\downarrow) |\eta, \eta_\downarrow\rangle \\
\tilde{N}_\downarrow |\eta, \eta_\downarrow\rangle &= (\eta_\downarrow - \frac{1}{q^2-1}) |\eta, \eta_\downarrow\rangle, \\
[1 + (q^2 - 1)\tilde{N}_\downarrow] |\eta, \eta_\downarrow\rangle &= (q^2 - 1) \eta_\downarrow |\eta, \eta_\downarrow\rangle, \\
[1 + (q^2 - 1)\tilde{N}_\uparrow] |\eta, \eta_\downarrow\rangle &= (q^2 - 1) \eta |\eta, \eta_\downarrow\rangle.
\end{align*}
\]

(VI.5)
Formally, the inverse of the transformation (VI.25) reads

\[
f^{-1}(\tilde{A}_\uparrow) := \tilde{a}_\uparrow = \sqrt{\frac{\log[1 + (q^2 - 1)\tilde{N}_\downarrow]}{2\tilde{N}_\downarrow \log q}} \tilde{A}_\uparrow
\]

\[
f^{-1}(\tilde{A}_\downarrow) := \tilde{a}_\downarrow = \tilde{A}_\downarrow \sqrt{\frac{\log[1 + (q^2 - 1)\tilde{N}_\downarrow]}{2\tilde{N}_\downarrow \log q}} \tilde{A}_\uparrow
\]

\[
f^{-1}(\tilde{A}_\uparrow) := \tilde{a}_\uparrow = \sqrt{\frac{1 + (q^2 - 1)\tilde{N}_\downarrow}{2\tilde{N}_\downarrow \log q}} \log \left[ \frac{1 + (q^2 - 1)\tilde{N}_\uparrow}{1 + (q^2 - 1)\tilde{N}_\downarrow} \right] \tilde{A}_\uparrow
\]

A glance at formula (VI.3) shows that the arguments of both logarithms in the inverse transformation (VI.6) are positive-definite on representation 1, whereas at least one of the arguments of the two logarithms is negative on any representation 2, 3, 4 or 5 (in the representation 3 the other argument vanishes). This makes \( f^{-1} \) ill-defined on all representations 2, 3, 4 or 5.

We conclude that representation 1 is the one intertwined by \( f^{-1} \) to the standard bosonic Fock representation of the \( su(2) \)-covariant Weyl algebra \( \mathcal{A} \), whereas the representations of the classes 2, 4, 5 have no classical analog, and representation 3 reduces to the representation of a 1-dimensional Weyl algebra.

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