On the number of connected components of complements to arrangements of subtori

I. N. Shnurnikov

Abstract

We consider the arrangements of subtori in a flat $d$–dimensional torus $T^d$. Let us consider an arrangement on $n$ subtori of codimension one, let $f$ be the number of connected components of the complement in $T^d$ to the union of subtori. We found the set of all possible numbers $f$ for given $n$ and $d$ and arbitrary arrangements of subtori.

MSC: 52C35

Keywords: arrangements of subtori, number of connected components.

Introduction

The theory of plane arrangements in affine or projective spaces has been investigated thoroughly, see the book of Orlic and Terao, [4] and Vassiliev’s review [8]. Inspired by the conjecture of Grünbaum [2], Martinov [3] found all possible pairs $(n, f)$ such that there is a real projective plane arrangement of $n$ pseudolines and $f$ regions. It turns out, that some facts concerning the arrangements of hyperplanes or oriented matroids could be generalized to arrangements of submanifolds, see Deshpande [1].

We study the sets $F(T^d, n)$ of connected components numbers of the complements in a flat $d$–dimensional torus $T^d$ to the unions of $n$ closed connected codimensional one subtori. Author in [7] found the sets $F(T^2, n)$ of region numbers in arrangements of $n$ closed geodesics in a two dimensional torus and a Klein bottle with locally flat metrics. The main result of the present paper is the following:

$$F(T^d, n) = \{l \in \mathbb{N} \mid n - d + 1 \leq l \leq n \text{ or } 2(n - d) \leq l\}$$

for $n > d \geq 2$. Also $F(T^d, n) = \mathbb{N}$ for $2 \leq n \leq d$. $F(T^d, 1) = \{1\}$.

Main part.

Let $\mathbb{R}^d$ be the real $d$–dimensional vector space and $u_1, \ldots, u_d$ are linear independant vectors in $\mathbb{R}^d$. Let $L$ be the set of all linear combinations $k_1u_1 + \cdots + k_du_d$ with integer coefficients $k_1, \ldots, k_d$. The set $L$ could be considered as the group of transitions, acting on $\mathbb{R}^d$. The quotient space of this action is torus, the induced metric (from euclidean metric in $\mathbb{R}^n$) is locally flat. So we call this quotient space flat torus $T^d$ and denote by $\mathbb{R}^d/L$. So the factorisation map $\varphi : \mathbb{R}^d \to \mathbb{R}^d/L$ sends point $x$ to equivalent class $x + L$. Let $(x_1, \ldots, x_d)$ are coordinates of $\mathbb{R}^d$ according to the basis $u_i$. Let us consider a hyperplane $K$ in $\mathbb{R}^d$ with equation $\sum a_ix_i = c$. The subset $\varphi(K) \subset T^d$ is closed iff the numbers $\frac{a_i}{a_j}$ are rational for all pairs $i, j$ such that $a_j \neq 0$. In the set $L$ one may choose another basis $v_1, \ldots, v_d$, then the matrix of basis transformation is a $SL(\mathbb{Z})$ matrix.

Let $A$ be the union of $n$ codimensional one subtori in the flat $d$–dimensional torus $T^d$. Consider the connected components of the complement $T^d \setminus A$; denote the number of connected components by $f = |\pi_0(T^d \setminus A)|$; let $F(T^d, n)$ be the set of all possible numbers $f$. 

*NRU HSE, Moscow
Lemma 1. Let $K$ be a hyperplane in $\mathbb{R}^d$ with equation $\sum a_i x_i = c$ in a basis $U = (u_i)$ of the lattice for integers $a_i$ such that $\gcd(a_1, \ldots, a_d) = 1$.

(a) There is a new basis $V = (v_i)$ of the lattice, according to which $K$ is given by equation $y_1 = 0$. The matrix $M$, $V = UM$ of basis transfer is a $SL(\mathbb{Z})$ matrix with integer coefficients and the determinant is 1.

(b) Let $L$ be such that $U$ is an orthogonal basis and $|u_i| = 1$ for all $i$. Then the distance between $K$ and the nearest point $x \in L$ of the lattice such that $x \notin K$ is $\sqrt{\sum_{i=1}^d a_i^2}$.

(c) Let $U$ be an orthogonal basis and $|u_i| = 1$ for all $i$. The volume of $T^d$ is 1. The subset $\varphi(K)$ is a subtorus in $T^d$ and the $(d-1)$-dimensional volume of $\varphi(K)$ is $\sqrt{\sum_{i=1}^d a_i^2}$.

Lemma 2. Let $A_1, \ldots, A_n$ be the set of $n$ codimensional one subtori in the flat $d$-dimensional torus $T^d$. Let $m$ be the maximal number of parallel tori among $A_1, \ldots, A_n$. Then $f \geq m(n - m - d + 2)$.

Proof. Induction on $d$, base $d = 1$ is trivial. Suppose the statement is true for $d - 1$ and every $n$. Let us prove the statement for $d$. Consider $m$ parallel subtori, they divide torus into $m$ components $U_i$, $i = 1, \ldots, m$ and each component is homeomorphic to prime product of a segment and $(d - 1)$-dimensional torus. For each $U_i$ let us consider a control $(d - 1)$-dimensional torus $R_i$ such that the other $n - d$ tori of $A_1, \ldots, A_n$ form on $R_i$ an arrangement of $n - m$ subtori and so by the induction assumption they divide $R_i$ into at least $n - m - d + 2$ components. So, each $U_i$ is divided by the other $n - d$ tori of $A_1, \ldots, A_n$ into at least $n - m - d + 2$ regions and the induction is over.

Lemma 3. Let $n > d \geq 2$. Then $f \geq 2n - 2d$ or: $f \leq n$ and there are at least $n - d + 1$ parallel tori.

Proof. Induction on $d$, base $d = 2$ is found in [7]. Suppose the statement is true for $d - 1$, let us prove it for $d$. Let $m$ be the maximal number of parallel tori in the arrangement $A = \{A_1, \ldots, A_n\}$. By lemma 2 we have $f \geq m(n - m - d + 2)$. So we over in case $2 \leq m \leq n - d$, because in this case $m(n - m - d + 2) \geq 2(n - d)$.

Let us consider the case $m = 1$. Then we consider a control torus $R_1$ which is parallel to torus $A_1$ and is in general position to other tori of arrangement. Let us consider arrangement $B = \{B_1, \ldots, B_{n-1}\}$, which is formed in $R_1$ by other tori of arrangement $A$. We know that $f(A) \geq f(B)$. If in $B$ there are at most $n - d$ parallel tori then by induction assumption we have $f(B) \geq 2(n - d)$. Otherwise in $B$ there is at least $n - d + 1$ parallel tori and this means that there exist a $(d - 2)$-dimensional torus $S^{d-2}$ which is parallel to at least $n - d + 2$ tori of $A$. Consider the two-dimensional subtori $W^2$ which is orthogonal to $S^{d-2}$. Then by induction assumption for $d = 2$ we have that $A$ divides $W^2$ into $f(W^2) \geq 2(n - d)$ regions and so $f(A) \geq f(W^2) \geq 2(n - d)$.

Let us consider the case $m \geq n - d + 1$. The subset of $m$ parallel tori divide $T^d$ into $m$ regions. If at least one of them is divided into more then one connected components by the other tori of $A$, then every of $m$ regions is divided in at least two connected components and so $f(A) \geq 2m \geq 2(n - d)$. In the other case the induction is over.

Theorem 1.

\[
F(T^d, n) = \{l \in \mathbb{N} \mid n - d + 1 \leq l \leq n \text{ or } 2(n - d) \leq l\}
\]

for $n > d \geq 2$. Also $F(T^d, n) = \mathbb{N}$ for $2 \leq n \leq d$. $F(T^d, 1) = \{1\}$.

Proof. The case $d = 2$ is proved in [7]. Now we assume $d \geq 3$. From the lemma 2 lemma 2 we see that the set $F(T^d, n)$ does not contain other integers.

We construct examples for $\leq n$ and $\geq 2n - 2d$ regions separately.

Let us consider $n$ hyperplanes in $\mathbb{R}^d$ (an equation corresponds to a hyperplane):

\[
x_i = 0, \quad 1 \leq i \leq k,
\]

\[
x_{k+1} = c_{i-k}, \quad k + 1 \leq i \leq n
\]
for some integer $k$, $0 \leq k \leq d-1$ and real $c_{i-k}$ with different fractional parts. By the factorization map $\mathbb{R}^d \to \mathbb{R}^d/Z^d$ we get a set $\{T_i^{d-1}, i = 1, \ldots, n\}$ of $n$ codimensional one subtori. And the complement is homeomorphic to the prime product

$$T^d \setminus \bigcup_i T_i^{d-1} \approx \mathbb{R}^k \times (S^1 \setminus \{p_1, \ldots, p_{n-k}\}) \times (S^1)^{d-k-1},$$

where $S^1 \setminus \{p_1, \ldots, p_{n-k}\}$ denotes a circle without $n-k$ points. Hence the number of complement regions equals $n-k$, for an integer $k$ such that $0 \leq k \leq d-1$.

Now let us take a nonnegative integer $k$ and construct an arrangement with $2n-2d+k$ connected components of the complement. We determine the subtori by equations:

$$x_i = 0, \quad \text{for } 2 \leq i \leq d,$$
$$x_2 = kx_1 + \frac{1}{2},$$
$$x_1 = c_j \quad \text{for } j = 1, \ldots, n-d,$$

whereas numbers $kc_j + \frac{1}{2}$ are not integer for any $j$. (This means that the intersection of three subtori

$$x_2 = kx_1 + \frac{1}{2}, \quad x_1 = c_j, \quad x_2 = 0$$

is an empty set.) Therefore

$$T^d \setminus \bigcup_{i=3}^d \{x_i = 0\} \approx T^2 \times \mathbb{R}^{d-2}.$$

In the 2-dimensional torus the equations

$$x_2 = 0,$$
$$x_2 = kx_1 + \frac{1}{2},$$
$$x_1 = c_j \quad \text{for } j = 1, \ldots, n-d$$

produce the arrangement of $n-d+2$ closed geodesics. The geodesic union divides the torus into $2n-2d+k$ connected components (for more details on the arrangements of closed geodesics in a flat torus see [2]).

**Lemma 4.** Let $K_1$ and $K_2$ be hyperplanes in $\mathbb{R}^d$ given by the equations $x_1 = 0$ and $\sum_i a_ix_i = c$ where $a_i$ are integers. Then the intersection of subtori $\varphi(K_1)$ and $\varphi(K_2)$ in $T^d$ consists of $\gcd(a_2, \ldots, a_d)$ connected components, each of which is a $(d-2)$-dimensional subtorus.

**Lemma 5.** Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be integer vectors such that $\gcd(a_1, \ldots, a_n) = 1$ and $\gcd(b_1, \ldots, b_n) = 1$. Let two subtori in a flat $d$-dimensional torus are given by vectors $a$ and $b$. Let $f(a, b)$ be the number of connected components of the intersection of two subtori. Then

$$f(a, b) = \gcd\left(\frac{a_2b_1 - b_2a_1}{\gcd(a_1, a_2)}, \frac{a_3(b_1u_1^{(1)} + b_2u_2^{(1)}) - b_3 \gcd(a_1, a_2)}{\gcd(a_1, a_2, a_3)},
\ldots, \frac{a_n(b_1u_1^{(n-2)} + b_2u_2^{(n-2)} + \ldots + b_{n-1}u_{n-1}^{(n-2)}) - b_n \gcd(a_1, \ldots, a_{n-1})}{\gcd(a_1, \ldots, a_n)}\right)$$

where and $u_i^j$ are integers such that

$$\gcd(a_1, \ldots, a_{j+1}) = a_1u_1^{(j)} + a_2u_2^{(j)} + \ldots + a_{j+1}u_{j+1}^{(j)}$$

for each $j = 1, \ldots, n-2$. 

3
References

[1] P. Deshpande, Arrangements of Submanifolds and the Tangent Bundle Complement. *Electronic Thesis and Dissertation Repository*, Paper 154 (2011).

[2] B. Grünbaum, *Arrangements and Spreads*. AMS, Providence, Rhode Island, 1972.

[3] N. Martinov, Classification of arrangements by the number of their cells. *Discrete and Comput. Geometry* (1993) 9, N 1, 39–46.

[4] P. Orlic, H. Terao, Arrangements of Hyperplanes. Springer – Verlag, Berlin – Heidelberg, 1992. 329 pp.

[5] R.W. Shannon, A lower bound on the number of cells in arrangements of hyperplanes. *Jour. of combinatorial theory (A)*, 20, (1976) 327–335.

[6] I.N. Shnurnikov, Into how many regions do $n$ lines divide the plane if at most $n - k$ of them are concurrent? *Moscow Univ. Math. Bull., ser. 1* (2010) 65:5, 208 – 212.

[7] I.N. Shnurnikov, On the number of regions formed by arrangements of closed geodesics on flat surfaces, *Math. Notes* 90, N 3 – 4 (2011), 619 – 622.

[8] V.A. Vassiliev, Topology of plane arrangements and their complements. *Uspekhi Mat. Nauk*, 56, iss. 2(338), (2001), 167 — 203.