Mirror matter admixtures in $K_L \to \mu^+\mu^-$

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Abstract

Our previous analysis on the contributions of mirror matter admixtures in ordinary hadrons to $K_L \to \gamma\gamma$ is extended to study the relevance of such contributions to the $K_L \to \mu^+\mu^-$ rare decay. The mixing angle of the admixtures previously determined to describe the enhancement phenomenon in two body non-leptonic decays of strange hadrons is used, along with recent results for the description of the strong and electromagnetic interaction parts of the transition amplitudes. We find that these admixtures give a significant contribution with a small SU(3) breaking of only 2.8%, we also find a value of $\sim -17.9^\circ$ for the $\eta-\eta'$ mixing angle consistent with some of its determinations in the literature and a preferred negative value around $-16$ for the local counter-term contribution $\chi_1 + \chi_2$ consistent with the existence of a unique counter-term assuming lepton universality. We conclude that those mixings may be relevant in low energy physics and should not be ignored.

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Recently, we studied whether contributions of new matter may be relevant in the rare decay $K_L \rightarrow \gamma\gamma$ [1]. This study cannot be performed in a general way and one is required to use specific models. In particular, we applied to this decay a model we have referred to as manifest mirror matter admixtures in ordinary hadrons [2]. Our conclusion was, that indeed contributions of new physics via admixtures of this type may be relevant in this decay and, accordingly, they must be kept in mind when studying the description of the Standard Model (SM) of this decay.

As is well known the two photon decay mode of $K_L$ is closely related to the also rare decay mode $K_L \rightarrow \mu^+\mu^-$ and, because of this, we are required to extend our analysis of Ref. [1] to this latter mode. This we shall do in this paper.

Before proceeding, let us first review the current situation on this decay. Its branching ratio has been gradually measured reaching very recently quite a substantial precision, currently, $\text{Br}(K_L \rightarrow \mu^+\mu^-) = (7.27 \pm 0.14) \times 10^{-9}$ [3]. On the theoretical side, the short distance contributions due in the Standard Model to $W$ and $Z$ exchange are quite small and cannot explain the experimental amplitude. Then, the long distance contribution from the $2\gamma$ intermediate state is dominant. This long distance amplitude has a large absorptive part that almost saturates the total $K_L \rightarrow \mu^+\mu^-$ rate (in principle, the absorptive amplitude receives additional contributions from real intermediate states other than two photons, such as two- and three-pion cuts, but these are completely negligible [4]). However, the dispersive part of the $2\gamma$ contribution cannot be calculated in a model-independent way and it is subject to various uncertainties [5, 6].

Let us now proceed to apply our phenomenological model to $K_L \rightarrow \mu^+\mu^-$. As we mentioned above, we introduce parity and flavour admixtures of mirror matter in ordinary mesons [2], and the $K_L \rightarrow \mu^+\mu^-$ amplitude is assumed to be enhanced by parity and flavour conserving amplitudes arising from the matrix elements of the ordinary strong and electromagnetic parts of the Hamiltonian between states with such admixtures.

The ordinary physical mesons $K^0_{ph}$ and $\bar{K}^0_{ph}$ with parity and SU(3)-flavor violating admixtures are given by [2]

$$K^0_{ph} = K^0_p - \frac{1}{\sqrt{2}}\sigma\pi^0_p + \sqrt{\frac{3}{2}}\sigma\eta^0_{sp} + \sqrt{\frac{2}{3}}\delta\eta_{ss} - \frac{1}{\sqrt{3}}\delta\eta_{ts} - \frac{1}{\sqrt{2}}\delta\eta_{pt} + \frac{1}{\sqrt{6}}\delta\eta_{pt} + \frac{1}{\sqrt{3}}\delta\eta_{ts},$$

$$\bar{K}^0_{ph} = \bar{K}^0_p - \frac{1}{\sqrt{2}}\sigma\pi^0_p + \sqrt{\frac{3}{2}}\sigma\eta^0_{sp} - \sqrt{\frac{2}{3}}\delta\eta_{ss} + \frac{1}{\sqrt{3}}\delta\eta_{ts} + \frac{1}{\sqrt{2}}\delta\eta_{pt} - \frac{1}{\sqrt{6}}\delta\eta_{pt} - \frac{1}{\sqrt{3}}\delta\eta_{ts}. \quad (1)$$

We have used the SU(3)-phase conventions of Ref. [7]. We recall that the mixing angles $\sigma, \delta$, and $\delta'$ are the parameters of the model, which have been determined previously [3]. The subindices $s$ and $p$ refer to positive and negative parity eigenstates, respectively. Notice that the physical mesons satisfy $\text{CP} K^0_{ph} = -\bar{K}^0_{ph}$ and $\text{CP} \bar{K}^0_{ph} = -K^0_{ph}$.

We can form the CP-eigenstates $K_1$ and $K_2$ as

$$K^0_{1ph} = \frac{1}{\sqrt{2}}(K^0_{ph} - \bar{K}^0_{ph}) \quad \text{and} \quad K^0_{2ph} = \frac{1}{\sqrt{2}}(K^0_{ph} + \bar{K}^0_{ph}),$$

the $K^0_{1ph}$ ($K^0_{2ph}$) is an even (odd) state with respect to CP. Here, we shall not consider CP-violation and therefore, $|K^0_{S,L}| = |K^0_{1,2}|$.

Substituting the expressions given in Eqs. (1), we obtain,
Given that to the usual definitions $K_{ph} = K_{Sp} + \frac{1}{\sqrt{3}}(2\delta + \delta')\eta_{8s} - \delta'\pi^0_s - \sqrt{\frac{2}{3}}(\delta - \delta')\eta_{1s},$

$$K_{L_{ph}} = K_{Lp} - \sigma\pi^0_p + \sqrt{3}\sigma\eta_{8p},$$

where the usual definitions $K_{Lp} = (K_{p}^0 - K_{p}^0)/\sqrt{2}$ and $K_{2p} = (K_{p}^0 + K_{p}^0)/\sqrt{2}$ were used.

As pointed out above, mirror matter admixtures in the physical mesons will contribute to the $K_{S,L} \to \mu^+\mu^-$ amplitudes via the parity and flavour-conserving part $\mathcal{H}_0$ of the full Hamiltonian $\mathcal{H}$, which contains the ordinary strong and electromagnetic interactions. The transition amplitudes will be given by the matrix elements $\langle \mu^+\mu^- | \mathcal{H}_0 | K_{S,L_{ph}} \rangle$. Using the above mixings, Eqs. (2), these amplitudes will have the form $\langle \mu^+\mu^- | \mathcal{H}_0 | K_{S,L_{ph}} \rangle = \bar{u}_{\mu^-}(p_{\mu^-})F_{K_{S,L},\mu^+\mu^-}v_{\mu^+}(p_{\mu^+})$, where,

$$F_{K_{S\mu^+\mu^-}} = \frac{1}{\sqrt{3}}(2\delta + \delta')F_{\eta_{8s},\mu^+\mu^-} - \delta'F_{\pi^0_s,\mu^+\mu^-} - \sqrt{3}(\delta - \delta')F_{\eta_{1s},\mu^+\mu^-}$$

and

$$F_{K_{L\mu^+\mu^-}} = -\sigma F_{\pi^0_s,\mu^+\mu^-} + \sqrt{3}\sigma F_{\eta_{8p},\mu^+\mu^-}.\quad (4)$$

Given that $K_S$ and $K_L$ are CP = +1 and CP = −1 pure states, respectively, and because the $\mu^+\mu^-$ state is a C = +1 state, then $K_S \to \mu^+\mu^-$ must go through a so-called parity-violating transition while $K_L \to \mu^+\mu^-$ goes through a parity-conserving transition. In the first case the $\mu^+\mu^-$ final state is $P = +1$ while in the second one, $P = -1$. $F_{K_{S\mu^+\mu^-}}$ and $F_{K_{L\mu^+\mu^-}}$ contribute, respectively, along the corresponding strangeness changing and parity violating and strangeness changing but parity conserving amplitudes of the SM mediated by $W^{\pm}$. However, as we can see from Eqs. (3) and (4), the contributions of the mirror matter admixtures are all flavor and parity conserving. The additive terms on the right-hand side of these equations involve only mirror mesons in $F_{K_{S\mu^+\mu^-}}$ and only ordinary mesons in $F_{K_{L\mu^+\mu^-}}$. However, notice that these states must carry the mass of the physical kaon $m_K$. The effective coupling constants $F_{\pi^0_s,\mu^+\mu^-}$, $F_{\eta_{8s},\mu^+\mu^-}$, and $F_{\eta_{1s},\mu^+\mu^-}$ correspond to the parity and flavour conserving decay processes $\pi^0_{s,p}, \eta_{8s,p}, \eta_{1s} \to \mu^+\mu^-$, of the $\pi^0_{s,p}$, $\eta_{8s,p}$ and $\eta_{1s}$ parity and flavour eigenstates present in the decaying physical $K_{S,L_{ph}}$.

Let us now concentrate on $K_L \to \mu^+\mu^-$, Eq. (4). As a working hypothesis we shall assume that the experimental branching ratio is saturated by the contribution of such admixtures, and we shall neglect the SM contributions. This is the same assumption we have used in our previous work. Of course, this is an extreme assumption. However, the reason for adopting it is that it provides a stringent test on the above admixtures. Clearly, if a poor or even a wrong prediction is obtained, then severe constraints on those admixtures are imposed.

To the leading order (the fourth order) in electromagnetic interaction and to all orders in strong interaction, the decay amplitudes of the non-strange mesons $P = \pi^0, \eta_8$ into $l^+l^-$ are given in terms of the purely real couplings to two on-shell photons $F_{P\gamma\gamma}$

$$F_{P^{l^+l^-}} = 2\alpha^2 m_lF_{P\gamma\gamma}R_P.\quad (5)$$

The relevant dynamics is contained in the reduced amplitudes $R_P$.

Assuming the obvious dominance of the two photon contribution, the reduced amplitudes $R_P = R(q^2)$ can be written as.
\[ R(q^2) = \frac{2}{i\pi^2 q^2} \int d^4k \frac{q^2 k^2 - (q \cdot k)^2}{[k^2 + i\epsilon][(q - k)^2 + i\epsilon][(p - k)^2 - m^2 + i\epsilon]} F(k^2, (q - k)^2) \]

where \( q^2 = m^2_p \) and \( F \) is a generic and model dependent form factor, with \( F(0, 0) = 1 \) for on-shell photons. The absorptive parts of \( R_P \) are finite and model independent and are given by \([10]\),

\[ \text{Im} R_P = \frac{\pi}{2\beta_p} \ln \left( \frac{1 - \beta P}{1 + \beta_p} \right), \]

(6)

with \( \beta_P = \sqrt{1 - 4m^2_\mu/m^2_K} \). By contrast, their real parts contain an \textit{a priori} divergent \( \gamma\gamma \) loop (if a constant \( F(k^2, (q - k)^2) = 1 \) form factor is assumed). The cure to this problem is model dependent and proceeds either through the inclusion of non-trivial form factors -which depend on the hadronic physics governing the \( P \to \gamma^*\gamma^* \) transition- or, in a more modern ChPT language, the inclusion of local counter-terms to render the result finite \([11, 12]\), namely,

\[ \text{Re} R(q^2 = m^2_p) = -\frac{\chi_1(\Lambda) + \chi_2(\Lambda)}{4} - \frac{5}{2} + 3 \ln \left( \frac{m_\mu}{\Lambda} \right) + \frac{1}{4\beta_p} \ln^2 \left( \frac{1 - \beta_P}{1 + \beta_p} \right) + \frac{\pi^2}{12\beta_p} + \frac{1}{\beta_p} \text{Li}_2 \left( \frac{\beta_P - 1}{\beta_P + 1} \right). \]

(7)

The explicit logarithmic dependence on \( \Lambda \) reflects the ultraviolet divergence of the loop and cancels with the inclusion of the local counter-terms contribution \( \chi_1(\Lambda) + \chi_2(\Lambda) \).

From Eqs. (4)-(7), the magnitude of the \( K_L \to \mu^+\mu^- \) amplitude in the mirror matter admixtures context, is given in terms of the \( F_{\pi^0\gamma\gamma} \) and \( F_{\eta_8\gamma\gamma} \) decay amplitudes, the mixing angle \( \sigma \), and the local contribution \( \chi_1(\Lambda) + \chi_2(\Lambda) \). It is explicitly given by

\[ |F_{K_L \mu^+\mu^-}| = 2\alpha^2 m_\mu - \sigma F_{\pi^0\gamma\gamma} + \sqrt{3}\sigma F_{\eta_8\gamma\gamma} \times \left\{ \left[ -\frac{\chi_1(\Lambda) + \chi_2(\Lambda)}{4} - \frac{5}{2} + 3 \ln \left( \frac{m_\mu}{\Lambda} \right) + \frac{1}{4\beta_K} \ln^2 \left( \frac{1 - \beta_K}{1 + \beta_K} \right) \right. \right. \]

\[ \left. \left. + \frac{\pi^2}{12\beta_K} + \frac{1}{\beta_K} \text{Li}_2 \left( \frac{\beta_K - 1}{\beta_K + 1} \right) \right] ^2 \right\} ^{1/2}, \]

(8)

where use has been made of \( \beta_{\pi^0_\mu} = \beta_{\eta_8} = \beta_K = \sqrt{1 - 4m^2_\mu/m^2_K} \), because \( \pi^0_\mu \) and \( \eta_8 \) share the mass \( m_K \), as mentioned before.

To be able of perform a numerical application, we shall include in our analysis the experimentally observed processes \( K_L, \eta, \eta' \to \gamma\gamma \). \( K_L \to \gamma\gamma \) in the mirror matter admixtures context is also related to \( \pi^0 \to \gamma\gamma \) and \( \eta_8 \to \gamma\gamma \) by a relation analogous to Eq. (4)

\[ F_{K_L\gamma\gamma} = -\sigma F_{\pi^0\gamma\gamma} + \sqrt{3}\sigma F_{\eta_8\gamma\gamma}. \]

(9)
Notice that in this context, $F_{K_L\gamma\gamma}$ and in consequence $F_{K_L\mu^+\mu^-}$ vanish in the strong flavour SU(3) symmetry limit (U-spin invariance) \[1\]. Thus, if we define

$$\Delta = \frac{F_{\eta\gamma\gamma}}{F_{\pi^0\gamma\gamma}/\sqrt{3}}, \quad (10)$$

then in the symmetry limit one has $\Delta = 1$.

Concerning the $\eta, \eta' \to \gamma\gamma$ processes, it has been established that in general two angles are necessary to describe the $\eta$-$\eta'$ mixing scheme. One cannot assume that the same rotation applies to the octet-singlet states and to their decay constants. For a review see Ref. \[13\]. However, we shall use this mixing scheme only at the amplitude level and in this case only one mixing angle appears \[14\]. In this respect, it should be clear that we are not making the questionable assumption that only one mixing angle is used both for the states and the decay constants. Then, following Ref. \[14\], we introduce the rotation

$$\left( \begin{array}{c} \eta_p \\ \eta'_p \end{array} \right) = \left( \begin{array}{cc} \cos \theta_p & -\sin \theta_p \\ \sin \theta_p & \cos \theta_p \end{array} \right) \left( \begin{array}{c} \eta_8p \\ \eta_1p \end{array} \right)$$

and this leads at the amplitude level to

$$F_{\eta\gamma\gamma} = \cos \theta_p F_{\eta\eta'\gamma\gamma} - \sin \theta_p F_{\eta\eta'\gamma\gamma}, \quad (11)$$

$$F_{\eta'\gamma\gamma} = \sin \theta_p F_{\eta\eta'\gamma\gamma} + \cos \theta_p F_{\eta\eta'\gamma\gamma}. \quad (12)$$

The experimental data we shall use come from Ref. \[3\]. The corresponding experimental values for $|F_{K_L\mu^+\mu^-}|$ and $F_{P\gamma\gamma}$ ($P = K_L, \pi^0, \eta, \eta'$) are displayed in Table \[I\]. They are obtained using

$$|F_{Pl+l-}| = \left[ \frac{8\pi}{m_P\beta_P} \Gamma(P \to l^+l^-) \right]^{1/2} \quad (13)$$

and

$$F_{P\gamma\gamma} = \pm \frac{2}{\alpha} \left[ \frac{1}{\pi m_P^2} \Gamma(P \to \gamma\gamma) \right]^{1/2}. \quad (14)$$

Also, the SU(3) symmetry relation, Eq. \([10]\), should be obeyed within a reasonable breaking of SU(3). We shall assume in what follows that Eq. \([10]\) is obeyed within an uncertainty of about 10%.

Of the values for the mixing angles of the mirror matter admixtures obtained previously \[8\], we shall only need $\sigma = (4.9 \pm 2.0) \times 10^{-6}$. We do not quote the values of the other two mixing angles because we shall not use them here.

Experimental values for the local contribution $\chi_1(\Lambda) + \chi_2(\Lambda)$ can be determined from Eq. \([7]\) by using Eq. \([6]\) and the “reduced” ratio

$$\frac{\text{Br}(P \to l^+l^-)}{\text{Br}(P \to \gamma\gamma)} = 2\beta_P \left( \frac{\alpha m_{\mu}}{\pi m_P} \right)^2 (\text{Re}^2 R_P + \text{Im}^2 R_P). \quad (15)$$

The present experimental $\eta \to \mu^+\mu^-$ branching ratio \[3\] requires a counter-term (for $\Lambda = m_{\rho} = 775.8 \pm 0.5 \text{ MeV}$) given by

$$\chi_1(m_{\rho}) + \chi_2(m_{\rho}) = -6.8 \pm 3.5, \quad (16a)$$
\begin{equation}
\chi_1(m_\rho) + \chi_2(m_\rho) = -31.9 \pm 3.5.
\end{equation}

In turn, the less precise \( \pi^0 \to e^+e^- \) available experimental data translate correspondingly into

\begin{equation}
\chi_1(m_\rho) + \chi_2(m_\rho) = 69.5 \pm 6.6,
\end{equation}

\begin{equation}
\chi_1(m_\rho) + \chi_2(m_\rho) = -10.0 \pm 6.6.
\end{equation}

Values (16a) and (17a) correspond to the negative root of Re\( R_P \) in Eq. (15) and values (16b) and (17b) to the positive one. The values (16a) and (17b) are favored because they are consistent with the existence of a unique counter-term assuming lepton universality and they are also in coincidence with the predictions of the resonance saturation hypothesis \cite{12} and Lowest Meson Dominance in large-\( N_c \) QCD models \cite{6}. On these grounds, the values (16b) and (17a) should be discarded. In our analysis we shall consider each value of Eqs. (16) and (17) separately.

From the above formulation of the problem, we are now in a position to make a prediction for the \( K_L \to \mu^+\mu^- \). As we mentioned above, we shall use the \( K_L, \pi^0, \eta, \eta' \to \gamma\gamma \) decay amplitudes as constraints. The way to proceed is to form a \( \chi^2 \) function and fit it. This \( \chi^2 \) contains eight summands, corresponding to the five experimental amplitudes, the allowed range of SU(3) breaking, the range for \( \sigma \) determined from our analysis of strange hadron two-body non-leptonic decays, and the local contribution to the \( P \to l^+l^- \) decay amplitudes of Eqs. (16) and (17). Then the six quantities, \( F_{\pi^0\gamma\gamma}, F_{\eta\gamma\gamma}, F_{\eta'\gamma\gamma}, \sigma, \chi_1(m_\rho) + \chi_2(m_\rho), \) and \( \theta_p \), are allowed to minimize this \( \chi^2 \). All the two possible signs in front of each \( F_{P\gamma\gamma} \) \((P = K_L, \pi^0, \eta, \eta') \) amplitude must be explored. Our best results are displayed in Tables I and II.

Looking through Tables I and II one can observe several features. In all the four cases considered the experimental \( K_L \to \mu^+\mu^- \) amplitude is well reproduced, the other four experimental amplitudes are also well reproduced, the ranges obtained for \( \sigma \) overlap neatly with the one of Ref. \cite{3}, the \( \eta-\eta' \) mixing angle \( \theta_p \) is stable and consistent with values obtained recently \cite{14, 15}, and finally the SU(3) symmetry breaking \( \Delta \) parameter is also stable and corresponds to quite a small breaking (\( \sim 2.8\% \)). The important differences that can be observed in these tables come from the \( \chi_1(m_\rho) + \chi_2(m_\rho) \) local contribution. Its positive value of (17a) is clearly excluded. A negative value around \(-16\) is preferred; however, the range down to \(-25\) may still be acceptable. This last is consistent with a unique counter-term in ChPT.

The above analysis shows that, in the framework of manifest mirror matter admixtures, new physics may give relevant contributions to the description of \( K_L \to \mu^+\mu^- \). At this point we should notice an important parallelism with \( K_L \to \gamma\gamma \). The angle \( \sigma \) previously determined from strange hadron two-body non-leptonic decays predicts an estimate for the \( K_L \to \mu^+\mu^- \) branching ratio larger than its experimental value, the SU(3) symmetry limit cancellation of the transition amplitude corresponding to Eq. (10) then requires the small symmetry breaking of 2.8% to reproduce the experimental value. This is the same mechanism that in the \( K_L \to \gamma\gamma \) case led to small symmetry breaking, too.

Let us conclude with an important remark. The real constraint one can obtain from studies such as the above one and our previous ones is on the existence of new forms of matter, specifically of mirror matter. Its contributions in low energy physics are relevant
only is as much as the SM contributions leave room for them to be observed. At present the uncertainties in the determination of the SM contributions in this realm of physics do allow for new physics to be observed there. However, if in the future it were to be the case that the SM leaves no room for other effects, then one should conclude that manifest mirror matter, if it exists, can only be found very far away. According to the lower bound established in Ref. [16], it would be found above $10^6 \text{ GeV}$.

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TABLE I: Experimentally observed, predicted values, and \( \Delta \chi^2 \) contributions of the \( K_L \to \mu^+ \mu^- \) and \( K_L, \pi^0, \eta, \eta' \to \gamma \gamma \) decay amplitudes for each one of the local counter terms values of Eq. (16). Only the magnitudes of the experimental values are displayed, the signs for the predictions of these amplitudes correspond to the ones obtained in our best fit. In each case, the values obtained for the parameters of the fit along with the constraints imposed (where applicable) and their \( \Delta \chi^2 \) contribution are also displayed. All the \( 2\gamma \) decay amplitudes are in MeV\(^{-1}\). \( \Delta \) gives the magnitude of SU(3) breaking and its corresponding \( \Delta \chi^2 \) is with respect to the assumed 10\% breaking. The total \( \chi^2 \) are displayed in parenthesis in the left column.

| Parameter | Constraint | Prediction | \( \Delta \chi^2 \) |
|-----------|------------|------------|-------------------|
| \( F_{\pi^0 \gamma \gamma} \) | \( (2.744 \pm 0.098) \times 10^{-4} \) | \( (-2.744 \pm 0.098) \times 10^{-4} \) | \( \sim 10^{-6} \) |
| \( F_{\eta \gamma \gamma} \) | \( (4.9 \pm 2.0) \times 10^{-6} \) | \( (5.0 \pm 2.0) \times 10^{-6} \) | \( \sim 10^{-3} \) |
| \( \chi_1(m_\rho) + \chi_2(m_\rho) \) | \( -6.8 \pm 3.5 \) | \( -15.7 \pm 1.1 \) | 6.47 |
| \( \theta_\rho \) | \( -17.9 \pm 1.5 \)\(^\circ\) | \( -17.9 \pm 1.5 \)\(^\circ\) | — |
| \( \Delta \) | \( 1.00 \pm 0.10 \) | \( 0.97 \) | 0.09 |

(\( \chi^2 = 7.32 \))
TABLE II: Experimentally observed, predicted values, and Δχ² contributions of the $K_L \rightarrow \mu^+\mu^-$ and $K_L, \pi^0, \eta, \eta' \rightarrow \gamma\gamma$ decay amplitudes for each one of the local counter terms values of Eq. (17). Only the magnitudes of the experimental values are displayed, the signs for the predictions of these amplitudes correspond to the ones obtained in our best fit. In each case, the values obtained for the parameters of the fit along with the constraints imposed (where applicable) and their Δχ² contribution are also displayed. All the $2\gamma$ decay amplitudes are in MeV⁻¹. Δ gives the magnitude of SU(3) breaking and its corresponding Δχ² is with respect to the assumed 10% breaking. The total χ² are displayed in parenthesis in the left column.

| $\chi_1(m_\rho) + \chi_2(m_\rho)$ | Decay | Experiment | Prediction | Δχ² |
|-----------------|-------|------------|------------|-----|
| 69.5 ± 6.6      | $K_L \rightarrow \mu^+\mu^-$ | $(2.270 \pm 0.024) \times 10^{-12}$ | $2.308 \times 10^{-12}$ | 2.51 |
|                 | $K_L \rightarrow \gamma\gamma$ | $(3.814 \pm 0.027) \times 10^{-11}$ | $3.785 \times 10^{-11}$ | 1.15 |
| (χ² = 166.02)   | $\pi^0 \rightarrow \gamma\gamma$ | $(2.744 \pm 0.098) \times 10^{-4}$ | $-2.744 \times 10^{-4}$ | ~10⁻⁵ |
|                 | $\eta \rightarrow \gamma\gamma$ | $(2.720 \pm 0.074) \times 10^{-4}$ | $-2.720 \times 10^{-4}$ | ~10⁻¹² |
|                 | $\eta' \rightarrow \gamma\gamma$ | $(3.41 \pm 0.18) \times 10^{-4}$ | $-3.41 \times 10^{-4}$ | ~10⁻¹³ |

| Parameter       | Constraint | Prediction (±1σ) | Δχ² |
|-----------------|------------|-----------------|-----|
| $F_{\pi^0\gamma\gamma}$ | $(2.744 \pm 0.098) \times 10^{-4}$ | $(-2.744 \pm 0.098) \times 10^{-4}$ | ~10⁻⁵ |
| $F_{\eta\gamma\gamma}$ | $(-1.540 \pm 0.060) \times 10^{-4}$ | $(-1.540 \pm 0.060) \times 10^{-4}$ | — |
| $F_{\eta'\gamma\gamma}$ | $(-4.08 \pm 0.16) \times 10^{-4}$ | $(-4.08 \pm 0.16) \times 10^{-4}$ | — |
| $\chi_1(m_\rho) + \chi_2(m_\rho)$ | $69.5 \pm 6.6$ | $-14.6 \pm 0.9$ | 162.37 |
| $\theta_p$ | $(-17.9 \pm 1.5)^{0}$ | $(-17.9 \pm 1.5)^{0}$ | — |
| $\Delta$ | $1.00 \pm 0.10$ | $0.97$ | 0.09 |

| $\chi_1(m_\rho) + \chi_2(m_\rho)$ | Decay | Experiment | Prediction | Δχ² |
|-----------------|-------|------------|------------|-----|
| -10.0 ± 6.6     | $K_L \rightarrow \mu^+\mu^-$ | $(2.270 \pm 0.024) \times 10^{-12}$ | $2.274 \times 10^{-12}$ | 0.03 |
|                 | $K_L \rightarrow \gamma\gamma$ | $(3.814 \pm 0.027) \times 10^{-11}$ | $3.811 \times 10^{-11}$ | 0.01 |
| (χ² = 1.13)     | $\pi^0 \rightarrow \gamma\gamma$ | $(2.744 \pm 0.098) \times 10^{-4}$ | $-2.744 \times 10^{-4}$ | ~10⁻⁵ |
|                 | $\eta \rightarrow \gamma\gamma$ | $(2.720 \pm 0.074) \times 10^{-4}$ | $-2.720 \times 10^{-4}$ | ~10⁻¹² |
|                 | $\eta' \rightarrow \gamma\gamma$ | $(3.41 \pm 0.18) \times 10^{-4}$ | $-3.41 \times 10^{-4}$ | ~10⁻¹³ |

| Parameter       | Constraint | Prediction (±1σ) | Δχ² |
|-----------------|------------|-----------------|-----|
| $F_{\pi^0\gamma\gamma}$ | $(2.744 \pm 0.098) \times 10^{-4}$ | $(-2.744 \pm 0.098) \times 10^{-4}$ | ~10⁻⁵ |
| $F_{\eta\gamma\gamma}$ | $(-1.540 \pm 0.060) \times 10^{-4}$ | $(-1.540 \pm 0.060) \times 10^{-4}$ | — |
| $F_{\eta'\gamma\gamma}$ | $(-4.08 \pm 0.16) \times 10^{-4}$ | $(-4.08 \pm 0.16) \times 10^{-4}$ | — |
| $\sigma$ | $(4.9 \pm 2.0) \times 10^{-6}$ | $(5.0 \pm 2.0) \times 10^{-6}$ | ~10⁻³ |
| $\chi_1(m_\rho) + \chi_2(m_\rho)$ | $-10.0 \pm 6.6$ | $-16.6 \pm 1.6$ | 1.00 |
| $\theta_p$ | $(-17.9 \pm 1.5)^{0}$ | $(-17.9 \pm 1.5)^{0}$ | — |
| $\Delta$ | $1.00 \pm 0.10$ | $0.97$ | 0.09 |