Renormalization in Winter Model

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Abstract

We show that metastable states in Winter model can be related to the eigenstates
of a particle in a box by means of renormalization and mixing.

Key words: quantum mechanics, metastable state, perturbation theory, renormaliza-
tion, mixing.
1 Introduction

The theory of unstables states in quantum mechanics [1, 2, 3, 4, 5, 6] has applications virtually in any branch of physics: statistical and condensed matter physics [7], atomic and molecular physics [8], nuclear physics [9, 10], quantum field theory and particle physics [11], and so on.

In this note we show that the particle states inside the cavity of the Winter model [12, 13, 14, 15, 16, 17] can be obtained from the states of a particle in a box by means of renormalization and mixing. Winter model describes the coupling of a cavity with the outside and is given, in the notation [17], by the Hamiltonian

$$\hat{H} = -\frac{\partial^2}{\partial x^2} + \frac{1}{\pi g} \delta(x - \pi)$$  \hspace{1cm} (1)

in the half-line $0 \leq x < \infty$, and with vanishing boundary condition at zero: $\psi(x = 0, t) = 0$. The distribution $\delta(y)$ is the Dirac $\delta$-function. Eq. (1) describes a model with one parameter, $g \in \mathbb{R}$, the inverse of the area of the potential barrier in $x = \pi$ (up to a factor $\pi$). The metastable states are nothing but wave packets initially (i.e., for example at $t = 0$) concentrated inside the cavity, i.e., in the interval $0 < x < \pi$. The time evolution of metastable states is controlled by wave propagation and imperfect multiple reflections on the right cavity wall, in $x = \pi$, leading to a leakage of the wave amplitude outside it. For $0 < g \ll 1$ (high barrier), there is weak coupling of the cavity with the outside and resonant long-lived states come into play. The idea is that, by means of them, we can describe the dynamics of the particles initially inside the cavity as if the outside did not exist.

Our results involve a second-order computation in $g$ extending the $O(g)$ results in [17], in which we repeated the original Winter’s computation, finding additional contributions in the time evolution of unstable states, which were absent in [12]. These new terms have a small strength $O(g) \ll 1$ compared to the old ones, but decay generally slower in time, with the smallest decay width. These contributions therefore dominate the evolution of all unstable states except the lowest one at large times and cannot be neglected. In this note we show that such contributions can be in principle “rotated away” by means of a linear transformation in the infinite-dimensional vector space of the resonances.

To show the relevance of such interpretation, let us briefly discuss the importance of renormalization in understanding the dynamics of many physical theories. It is common practise in physics to compare a given physical system with a simplified one in which some interactions are omitted. These interactions can be related to control parameters in experiments involving, for example, external electric or magnetic fields, or can be treated theoretically as variable quantities. The idea of renormalization is that switching on an interaction in a physical system has, as one of the main dynamical effects, that of modifying the parameters of the starting, non-interacting, system; once renormalization has been made, the residual effects of the interaction are substantially weaker than before renormalization. When the coupling setting the strength of the interaction gets too large, there is usually little connection between the free and the interacting system and renormalization often loses its meaning.

In condensed matter physics, renormalization is related to the so called “adiabatic continuity principle” [7]: by adiabatically (i.e., slowly) turning on an interaction, the free states
of the system go 1 − 1 onto the interacting states by means of a flow of the parameters such as masses, couplings, etc. A typical example is the normal Fermi liquid, i.e. a system with a repulsive interaction among electrons in the Fermi sphere\(^1\). In quantum field theory\([18,19,20,21]\), the relation between free parameters and interacting ones (masses, couplings and field normalizations) is often singular because of the lack of intrinsic energy scales cutting off the quantum fluctuations at large energies: that leads to the well-known ultraviolet infinities. One also encounters renormalization in non-relativistic quantum mechanics with \(\delta\)-functions potentials\([22]\).

It is remarkable that solving the non-relativistic Schrödinger equation for nuclei (which are strongly-interacting many-body systems), in order to obtain the low-energy excitations and the scattering cross sections, can be greatly simplified by implementing renormalization-group ideas, as recently discovered\([10]\). One finds the phenomenon of generation of many-body operators by RG flow, the problem of the stability under change of the ultraviolet cutoff, etc., which are typical of perturbative quantum field theory computations, in a completely different framework.

In classical physics, renormalization is usually implemented by the method of “multiple scales”\([23,24]\). In the case of a free anharmonic oscillator, for example, renormalization amounts to the “absorption” of secular terms into a shift of the harmonic frequency. These terms are formally resonances produced by forcing terms occurring in the perturbative expansion, are incompatible with energy conservation and spoil the convergence of the perturbative expansion at large times. After renormalization, such strong-coupling effects completely disappear; only a small coupling between the harmonics is left and a uniform approximation in time is obtained.

Let us remark that the adiabatic continuity principle is subjected to relevant violations. Let us quote, for example, the cases of the energy gap in the Bardeen-Cooper-Schriffer (BCS) theory of classical superconductors\([25]\) or the mass gap in massless Quantum-Chromo-Dynamics (QCD)\([26]\). These phenomena are typically characterized by functions which have an essential singularity when the interaction coupling \(g\) goes to zero, of the form \(e^{1/g}\) for \(g < 0\), making non smooth the connection between the interacting system and the related free one. In these cases, the relation between the free system and the interacting one is highly non trivial and the residual interaction is of nonperturbative character.

Let us end the introduction by observing that, even though renormalization is implemented and interpreted in quite different ways in different contexts, it is an ubiquitous phenomenon in physics — like the unstable states cited above — a thing which certainly could not be expected \(a\ priori\).

## 2 The spectrum

For a positive coupling, \(g > 0\), i.e. for a repulsive potential, the Hamiltonian of Winter model in eq.(1) has a continuum spectrum only, with eigenfunctions of the form

\[
\psi(x; k, g) \propto \left[ -\frac{i}{2} \exp(ikx) + \frac{i}{2} \exp(-ikx) \right] \theta(\pi - x) +
\]

\(^1\) See footnote \[6\]
\[ a(k, g) \exp(ikx) + b(k, g) \exp(-ikx) \right \} \theta(x - \pi), \tag{2} \]

and eigenvalues
\[ \epsilon(k) = k^2. \tag{3} \]

The step function \( \theta(x) = 1 \) for \( x > 0 \) and 0 otherwise and the coefficients \( a(k, g) \) and \( b(k, g) \) have the following expressions:
\[ a(k, g) = -\frac{i}{2} + \frac{1}{4\pi gk} \left[ \exp(-2i\pi k) - 1 \right]; \tag{4} \]
\[ b(k, g) = +\frac{i}{2} + \frac{1}{4\pi gk} \left[ \exp(+2i\pi k) - 1 \right]. \tag{5} \]

These coefficients have the following two symmetries (which will be relevant in the discussion of the spectrum as well as of the time evolution):
\[ a(-k, g) = -b(k, g), \tag{6} \]
\[ a(k, g)^* = b(k^*, g^*), \tag{7} \]

where the star (\(^*\)) denotes complex conjugation. In general, \( k \) is a real quantum number but, because of eq. (6), the eigenfunctions are odd functions of \( k \) so one can assume \( k > 0 \), implying there is no energy degeneracy (trivial S-matrix). By normalizing the eigenfunctions as:
\[ \int_0^\infty \psi^*(x; k', g) \psi(x; k, g) dx = \delta(k - k'), \tag{8} \]

where \( \delta(g) \) is the Dirac \( \delta \)-function, the normalization factor reads:
\[ N(k, g) = \frac{1}{[2\pi a(k, g)b(k, g)]^{1/2}}. \tag{9} \]

The final expression for the eigenfunctions therefore can be written as:
\[
\psi(x; k, g) = \frac{1}{\sqrt{2\pi}} \left[ -\frac{i}{2} \left( \frac{\exp(ikx)}{\sqrt{a(k, g)b(k, g)}} \right) \theta(\pi - x) + \left( \frac{\sqrt{a(k, g)}}{\sqrt{b(k, g)}} \right)^{1/2} \exp(ikx) + \left( \frac{\sqrt{b(k, g)}}{\sqrt{a(k, g)}} \right)^{1/2} \exp(-ikx) \right] \theta(x - \pi). \tag{10} \]

Note that, because of continuum normalization, the amplitude of the eigenfunctions outside the wall is always \( \mathcal{O}(1) \), no matter which values are chosen for \( k \) and \( g \), while inside the cavity the amplitude has a non-trivial dependence on \( k \) and \( g \). For \( |g| \ll 1 \), the amplitude of \( \psi(x; k, g) \) inside the cavity shows marked peaks for \( k \approx n - gn \), where \( n \ll 1/|g| \) is a positive integer, because
\[ |a(n - gn, g)| \approx |b(n - gn, g)| \approx |g|n \ll 1. \tag{11} \]
3 Temporal evolution of unstable states

The eigenfunctions of a particle in a box of length $L = \pi$ with Hamiltonian

$$\hat{H}_0 = -\frac{\partial^2}{\partial x^2}$$

are given, as well known, by

$$\psi^{(l)}(x, t) = \sqrt{\frac{2}{\pi}} \sin(lx)e^{-il^2t},$$

where $l = 1, 2, 3, \cdots$ is a positive integer and $0 \leq x \leq \pi$. We study the time evolution of wavefunctions $\psi^{(l)}(x; t; g)$ which coincide at $t = 0$ with the free eigenfunctions in eq. (13) in the interval $x \in [0, \pi]$ (the cavity) and vanish outside it:

$$\psi^{(l)}(x, 0) = \begin{cases} \sqrt{\frac{2}{\pi}} \sin(lx) & \text{for } 0 \leq x \leq \pi; \\ 0 & \text{for } \pi < x < \infty. \end{cases}$$

The initial conditions above make the limit $g \to 0$ easy, because the wavefunctions $\psi^{(l)}(x; t; g)$ become eigenfunctions of Winter Hamiltonian in that limit. For $g \neq 0$, however, we will see in the next section that there are more natural initial conditions to consider.

The spectral representation in eigenfunctions of the wavefunction of the unstable state at time $t$ has the explicit expression:

$$\psi^{(l)}(x; t; g) = \left(\frac{2}{\pi}\right)^{3/2} \int_0^\infty p^{(l)}(k; x, g) e^{-ik^2t} dk, \quad 0 \leq x \leq \pi, \quad g > 0,$$

where

$$p^{(l)}(k; x, g) = (-1)^l \frac{\sin(k\pi)}{k^2 - l^2} \frac{\sin(kx) b(k, g)}{4a(k, g)}.$$  

The integral on the r.h.s. of eq. (15) can be exactly evaluated with numerical methods for $t$ not too large, because high-frequency oscillations occur in the factor $e^{-ik^2t}$ in the integrand for $t \to +\infty$. In order to study the large-time behavior we therefore have to develop analytic techniques.

3.1 Small-time Expansion

For small times, $t \ll 1$, the wavefunction in eq. (15) exhibits a power behavior that is not relevant to our discussion and will not be treated further.

3.2 Asymptotic Expansion for large times

To obtain explicit analytic formulae, we expand the integral for large $t$. The steepest descent method suggests to replace the integral on the r.h.s. of eq. (15) by the integral over the steepest descent ray $(0, \infty e^{-\pi/4})$, on which the fast oscillations of the integrand are absent (see fig. 1). Therefore the state $\psi^{(l)}(x; t; g)$ is decomposed in a natural way into the sum of
Figure 1: Rotation of the integration contour in complex $k$-plane and (simple) zeroes of the function $b(k, g)$ for $g = 0.1$ lying in the fourth quadrant.

two quite different contributions:

$$
\psi^{(l)}(x, t; g) = \psi^{(l)}_{\text{ex}}(x, t; g) + \psi^{(l)}_{\text{pw}}(x, t; g),
$$

where

$$
\psi^{(l)}_{\text{ex}}(x, t; g) \equiv -2\pi i \left(\frac{2}{\pi}\right)^{3/2} \sum_{n=1}^{\infty} \text{Res} \left[ p^{(l)}(k; x, g) e^{-ik^2 t}, k^{(n)}(g) \right];
$$

$$
\psi^{(l)}_{\text{pw}}(x, t; g) \equiv e^{-i\pi/4} \left(\frac{2}{\pi}\right)^{3/2} \int_{0}^{\infty} p^{(l)}(k e^{-i\pi/4}; x, g) e^{-k^2 t} dk.
$$

$k^{(n)}(g)$ is a simple pole of the integrand lying in the last octant of the complex $k$-plane for $n \in \mathbb{N}_+$ (see fig. [1]), to be evaluated in the next section.

In general, the contribution $\psi^{(l)}_{\text{ex}}(x, t; g)$ exhibits an exponential decay, while the contribution $\psi^{(l)}_{\text{pw}}(x, t; g)$ exhibits a power decay as $t \gg 1$. Let us consider the above contributions in turn.

### 3.2.1 Exponential Contributions

The explicit expression of the exponential part of the unstable wavefunction at time $t \geq 0$ reads:

$$
\psi^{(l)}_{\text{ex}}(x, t; g) = -2\pi i \left(\frac{2}{\pi}\right)^{3/2} \sum_{n=1}^{\infty} \text{Res} \left[ (-1)^l l \frac{\sin(k\pi) \sin(kx) \exp(-ik^2 t)}{k^2 - l^2} \frac{1}{4a(k, g)b(k, g)}, k^{(n)}(g) \right].
$$

The Hamiltonian is hermitian (physical case) for real $g$ only, which we assume from now on. The integrand (the first argument in the square bracket above), as a function of the complex $k$ variable, has removable singularities at the positive integers, $k = l$, and pole singularities corresponding to the zeroes of the functions $a(k, g)$ and $b(k, g)$ constrained by conditions:

$$
\text{Im} k^{(n)}(g) < 0; \quad \text{Re} k^{(n)}(g) > |\text{Im} k^{(n)}(g)|.
$$
The transcendental equation

\[ b(k, g) = 0 \]  \hspace{1cm} (22)

has simple zeroes for \(|g| \ll 1\) of the form

\[ k^{(n)}(g) = n - ng + ng^2 - i\pi n^2 g^2 + O(g^3), \]  \hspace{1cm} (23)

where \(n\) is a nonzero integer. All these zeroes lie in the lower half of the complex \(k\)-plane, i.e. have \(\text{Im} \, k^{(n)}(g) < 0\), and satisfy also the second condition in eq. (21) for \(n > 0\). In general, the function \(k^{(n)}(g)\) is the branch with \(k^{(n)}(0) = n\) of the multi-valued analytic function \(k(g)\) satisfying \(b(k(g), g) = 0\). Numerical computation actually shows that conditions (21) remain satisfied up to values of \(|g|\) of order one. The zeroes leave the last octant \((-\pi/4 < \theta < 0)\) for very large values of \(|g|\), where the unstable-state description becomes irrelevant.

Because of eq. (21), which for real \(g\) reads

\[ a(k, g) = b(k^*, g)^*, \]  \hspace{1cm} (24)

the zeroes of the equation \(a(k, g) = 0\) are complex conjugates of the ones of eq. (22), therefore lie in the upper half \(k\)-plane and consequently do not enter the residue sum.

The only non-trivial residue to evaluate is therefore:

\[
\text{Res} \left[ \frac{1}{b(k, g)}; k^{(n)}(g) \right] = \lim_{k \to k^{(n)}(g)} \frac{k - k^{(n)}(g)}{b(k, g)} = \frac{1}{(\partial b/\partial k)(k, g)|_{k=k^{(n)}(g)}} = \frac{-2igk^{(n)}(g)}{1 + g[1 - 2\pi i k^{(n)}(g)]}, \]  \hspace{1cm} (25)

where, after the evaluation of the derivative, we have simply replaced \(k \to k^{(n)}(g)\) and used the relation

\[ \exp[2\pi ik^{(n)}(g)] = 1 - 2\pi igk^{(n)}(g), \]  \hspace{1cm} (26)

which is true for any solution of eq. (22). We then have the following exact expression in terms of the zero set \(\{k^{(n)}(g)\}\):

\[
\psi_{\overline{\text{e}} l}^{(l)}(x, t; g) = -2\pi i \left( \frac{2}{\pi} \right)^{3/2} (-1)^l \sum_{n=1}^{\infty} \frac{1}{4a[k^{(n)}(g), g]} \frac{1}{[\partial b/\partial k][k^{(n)}(g), g]} \frac{\sin[k^{(n)}(g)\pi]}{[k^{(n)}(g)]^2 - l^2} \times
\]

\[
\times \sin[k^{(n)}(g)x] E^{(n)}(t; g) \times \sin[k^{(n)}(g)x] E^{(n)}(t; g), \]  \hspace{1cm} (27)

where we have defined the time evolution factors

\[ E^{(n)}(t; g) \equiv \exp[-i\varphi^{(n)}(g) t] = \exp \left[ -i \omega^{(n)}(g) t - \frac{1}{2} \Gamma^{(n)}(g) t \right]. \]  \hspace{1cm} (28)

Since the energies are complex for \(g \neq 0\), on the last member we have split them into real and imaginary parts as:

\[ \varphi^{(n)}(g) = \left( k^{(n)}(g) \right)^2 = \omega^{(n)}(g) - \frac{i}{2} \Gamma^{(n)}(g), \]  \hspace{1cm} (29)
where $\omega^{(n)}(g)$ is the frequency and $\Gamma^{(n)}(g)$ is the decay width of the pole state $n$. Note that $E^{(n)}(0; g) = 1$, as it should. In eq.(27) we have chosen the principal branch of the complex square root, $-\pi < \arg z \leq \pi$ ($1^{1/2} = 1$). In deriving the last member in eq.(27) we have also used a relation obtained by taking the square root of eq.(26):

$$\exp \left[ i\pi k^{(n)}(g) \right] = (-1)^n \left[ 1 - 2\pi i g k^{(n)}(g) \right]^{1/2}. \quad (30)$$

The sign in front of the square root is fixed by taking the limit $g \to 0$ on both sides, i.e. by setting $g = 0$ and replacing $k^{(n)}(g) \to n$. Once the poles $\{k^{(n)}(g)\}$ have been exactly evaluated (with numerical methods), the last member in eq.(27) allows an exact computation of the exponential part of the wavefunction. In the next sections, however, we present an expansion for $g \ll 1$ which allows for explicit analytic expressions.

Equation (27) is conveniently rewritten as:

$$\psi_{\text{ex}}^{(l)}(x, t; g) = \sum_{n=1}^{\infty} V(g)_{ln} \theta^{(n)}(x, t; g), \quad (31)$$

where the entries of the mixing matrix $V(g)$ read

$$V(g)_{ln} \equiv g(-1)^{l+n} 2i k^{(n)}(g) \left[ 1 - 2\pi i g k^{(n)}(g) \right]^{1/2} \left\{ 1 + \left[ 1 - 2\pi i k^{(n)}(g) \right] g \right\}. \quad (32)$$

We have defined the pole wavefunctions (which evolve diagonally with time):

$$\theta^{(n)}(x, t; g) \equiv \sqrt{\frac{2}{\pi}} \sin \left[ k^{(n)}(g)x \right] E^{(n)}(t; g). \quad (33)$$

For $|g| \ll 1$ there is a similarity between the pole wavefunctions above and the eigenfunctions in eq.(10) for $k \simeq n - gn \in \mathbb{R}$. Let us stress however the differences between the “true”, exact eigenstates, having real energies and lying in the continuum spectrum, and the resonance states, normalizable states with complex energy describing dynamics in a simple but approximate way for a finite amount of time only [7].

### 3.2.2 Matrix Notation

To simplify formulae, it is convenient to introduce matrix notation. Let us define an infinite column vector containing all the pole states

$$\Theta(x, t; g) \equiv \begin{pmatrix} \theta^{(1)}(x, t; g) \\ \theta^{(2)}(x, t; g) \\ \vdots \\ \theta^{(n)}(x, t; g) \\ \vdots \end{pmatrix}. \quad (34)$$

and an infinite diagonal matrix representing the evolution of the pole states

$$\mathcal{E}(t; g) \equiv \text{diag} \left[ E^{(1)}(t; g), E^{(2)}(t; g), \ldots, E^{(n)}(t; g), \ldots \right]. \quad (35)$$
In more standard notation:
\[
E(t; g) = \begin{pmatrix}
E^{(1)}(t; g) & 0 & \cdots & 0 \\
0 & E^{(2)}(t; g) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & E^{(n)}(t; g)
\end{pmatrix}.
\] (36)

The temporal evolution of the pole states can be rewritten in matrix notation as:
\[
\Theta(x, t; g) = E(t; g) \Theta(x, 0; g).
\] (37)

Similarly, let us define an infinite column vector containing the metastable wavefunctions,
\[
\Psi(x, t; g) = \begin{pmatrix}
\psi^{(1)}(x, t; g) \\
\psi^{(2)}(x, t; g) \\
\vdots \\
\psi^{(n)}(x, t; g)
\end{pmatrix},
\] (38)
as well as the vectors \(\Psi_{ex}(x, t; g)\) and \(\Psi_{pw}(x, t; g)\) containing the exponential and power parts respectively.\(^2\) Eq. (17) now reads:
\[
\Psi(x, t; g) = \Psi_{ex}(x, t; g) + \Psi_{pw}(x, t; g).
\] (39)

Setting \(t = 0\) in eq. (38), one obtains a list of the initial conditions for all \(l = 1, 2, 3, \cdots\) (see eq. (14)):
\[
\Psi(x, 0) = \sqrt{\frac{2}{\pi}} \theta(\pi - x) \begin{pmatrix}
\sin(x) \\
\sin(2x) \\
\vdots \\
\sin(nx) \\
\end{pmatrix}.
\] (40)

Eq. (31) reads in new notation:
\[
\Psi_{ex}(x, t; g) = V(g) \Theta(x, t; g) = V(g) E(t; g) \Theta(x, 0; g).
\] (41)

Let us remark that since there are no power corrections in time, eq. (41) does not reproduce the initial value in eq. (40) for \(t = 0\).

### 3.3 Power Contributions

The integral \(\psi^{(l)}_{pw}(x, t; g)\), over the ray \((0, \infty e^{-i\pi/4})\) in complex \(k\)-plane, can be exactly evaluated with numerical methods without problems for any time \(t \geq 0\), as it does not involve any oscillation. It is also convergent at the initial time \(t = 0\); in other words, the decomposition in eq. (17) does not spoil the convergence at \(t = 0\). However, for large times,
$t \gg 1$, the integral takes the dominant contribution from a neighborhood of $k = 0$, where the integrand is analytic and can therefore be expanded in powers of $k$:

$$p^{(l)}(k; x, g) = \frac{g^2}{(1 + g)^2} \sum_{j=1}^{\infty} p_j^{(l)}(x, g) k^{2j}. \quad (42)$$

The first two coefficients explicitly read:

$$p_1^{(l)}(x, g) = \frac{(-1)^{l+1}}{l} \pi x; \quad (43)$$

$$p_2^{(l)}(x, g) = \frac{(-1)^{l+1}}{l} \pi x \left[ \frac{1}{l^2} + \frac{\pi^2}{6} + \frac{2}{3} \frac{\pi^2 g}{1 + g} - \frac{\pi^2 g^2}{(1 + g)^2} - \frac{x^2}{6} \right]. \quad (44)$$

Replacing this series into the integral and performing the change of variable $\nu = k^2 t$, one obtains the following asymptotic expansion:

$$\psi_{pw}^{(l)}(x, t, g) \approx \frac{\sqrt{2}}{\pi^{3/2}} \frac{e^{-i\pi/4} g^2}{(1 + g)^2} \sum_{j=1}^{\infty} \frac{(-i)^j p_j^{(l)}(x, g)}{t^{j+\frac{1}{2}}} \int_0^\infty d\nu \nu^{j-\frac{1}{2}} e^{-\nu} \quad (45)$$

$$= \frac{\sqrt{2}}{\pi} \frac{e^{-i\pi/4} g^2}{(1 + g)^2} \sum_{j=1}^{\infty} \frac{(-i)^j (2j - 1)!! p_j^{(l)}(x, g)}{2^j t^{j+\frac{1}{2}}}, \quad 0 \leq x \leq \pi, \quad t \gg 1, \quad (46)$$

whose first two terms read:

$$\psi_{pw}^{(l)}(x, t, g) \approx \frac{e^{i\pi/4} (-1)^l}{\sqrt{2}} \frac{g^2}{l} \frac{x}{(1 + g)^2 t^{3/2}} \times$$

$$\times \left\{ 1 - \frac{3i}{2t} \left[ \frac{1}{l^2} + \frac{\pi^2}{6} + \frac{2}{3} \frac{\pi^2 g}{1 + g} - \frac{\pi^2 g^2}{(1 + g)^2} - \frac{x^2}{6} \right] + O\left(\frac{1}{t^2}\right) \right\}. \quad (47)$$
Let us make a few remarks. The above asymptotic expansion is uniformly valid for all \( g \geq 0 \), since the coefficients \( p_j^{(l)}(x, g) \) are uniformly bounded in that region (see eq. (12)).

The exponent 3/2 controlling the power decay, \( \psi_{pw} \approx 1/t^{3/2} \), does not depend on \( l \) and \( g \); power corrections however vanish for \( g \to 0 \) (impermeable cavity).

We are in complete agreement with [12] as far as the asymptotic power behavior in time is concerned; however we remark that our results for the power corrections in \( t \) are valid for any \( g \), i.e. do not involve any expansion in \( g \). In particular, one can take the limit \( g \to \infty \), in which the potential barrier disappears.

4 Resonances

For \( g \ll 1 \), i.e. for weak coupling, there is a large time slice between a pre-exponential small-\( t \) region [8, 12, 17] and a post-exponential one related to the power-like decay just discussed, 

\[
1 \ll t \lesssim \frac{\log(1/g)}{g^2}, \tag{48}
\]

in which the unstable wavefunctions \( \psi^{(l)}(x, t; g) \) evolve to a good approximation as a superposition of pole states, i.e. of resonances (see fig.2). Relation (48) is a consequence of the first-order results in the next section. It is clear that non-exponential contributions do not have a resonance interpretation: they constitute an intrinsic limit of the scheme.

4.1 First-order computation, \( \mathcal{O}(g) \)

By expanding in powers of \( g \) the mixing matrix

\[
V(g) = \sum_{k=0}^{\infty} g^k V^{(k)}, \tag{49}
\]

one obtains up to first order:

\[
V^{(0)} = Id; \tag{50}
\]

\[
V^{(1)} = -\frac{1}{2} Id + A, \tag{51}
\]

where \( A \) is the real antisimmetric matrix with entries

\[
A_{l,n} \equiv (-1)^{l+n} \frac{2ln}{l^2 - n^2} \quad \text{for} \quad l \neq n \tag{52}
\]

and \( A_{l,l} = 0 \). The frequencies and widths entering the pole wavefunctions have the following lowest-order expressions:

\[
\omega^{(n)}(g) \equiv + \left[ \text{Re} k^{(n)}(g) \right]^2 - \left[ \text{Im} k^{(n)}(g) \right]^2 = n^2 (1 - 2g) + \mathcal{O}(g^2); \tag{53}
\]

\[
\Gamma^{(n)}(g) \equiv -4 \text{Re} k^{(n)}(g) \text{Im} k^{(n)}(g) = 4\pi n^3 g^2 + \mathcal{O}(g^3). \tag{54}
\]
Since it is convenient to have some freedom in the normalization of the pole states, let us introduce the renormalization constants

\[ Z^{(n)}(g) = 1 + \sum_{k=1}^{\infty} g^k z_k^{(n)}, \]  

where \( z_k^{(n)} \) are (in general complex) coefficients, which depend in general on \( n \): they have to be determined by imposing chosen renormalization conditions. We define the renormalized pole states \( \xi^{(n)}(x, t; g) \) by dividing \( \theta^{(n)}(x, t; g) \) by \( Z^{(n)}(g) \):

\[ \xi^{(n)}(x, t; g) \equiv \frac{\theta^{(n)}(x, t; g)}{Z^{(n)}(g)} = \frac{1}{Z^{(n)}(g)} \sqrt{\frac{2}{\pi}} \sin \left[ k^{(n)}(g) x \right] E^{(n)}(t; g) \]  

and define the renormalized mixing matrix entries \( U(g)_{l,n} \) by multiplying \( V_{l,n}(g) \) by the same factor,

\[ U(g)_{l,n} = V(g)_{l,n} Z^{(n)}(g) \]  

(no sum over \( n \) is implied). In order to introduce a matrix notation, let us represent the renormalization constants through the diagonal matrix

\[ Z(g) \equiv \text{diag} \left[ Z^{(1)}(g), Z^{(2)}(g), \ldots, Z^{(n)}(g), \ldots \right]. \]  

In components, eq. (58) reads:

\[ Z(g)_{l,n} = \delta_{ln} Z^{(n)}(g), \]  

where \( \delta_{ln} = 1 \) for \( l = n \) and zero otherwise is the Kronecker \( \delta \). The matrix renormalization constant also possesses a power-series expansion:

\[ Z(g) = \text{Id} + \sum_{k=1}^{\infty} g^k Z^{(k)}, \]  

where \( Z^{(k)} \) are diagonal matrices. The renormalized mixing matrix then reads

\[ U(g) = V(g) Z(g). \]  

In terms of the renormalized pole states, eq. (41) reads:

\[ \Psi_{\text{ex}}(x, t; g) = U(g) \Xi(x, t; g) = U(g) \Xi(t; g) \Xi(x, 0; g). \]

By multiplying with each other the power serieses of \( V(g) \) and \( Z(g) \), one obtains the power expansion for \( U(g) \),

\[ U(g) = \sum_{k=0}^{\infty} g^k U^{(k)}, \]  

where

\[ U^{(n)} = \sum_{k=0}^{n} V^{(k)} Z^{(n-k)}. \]
Because of the change of wavelength due to the interaction, i.e. to $g \neq 0$, the pole states $\theta^{(n)}(x, t; g)$ are not normalized to one at the initial time $t = 0$, unlike the initial conditions. If we introduce normalized pole states, i.e. satisfying the condition

$$\int_0^\pi |\xi^{(n)}(x, 0; g)|^2 dx = 1,$$

we obtain

$$Z^{(n)}(g) = 1 + \frac{g}{2} + O\left(g^2\right).$$

Therefore, up to first order:

$$U^{(0)} = Id; \quad (67)$$

$$U^{(1)} = A. \quad (68)$$

It is remarkable that condition (65) has the effect of removing the diagonal contributions from the first-order mixing matrix $U^{(1)}$. Let us also note that the matrix $U(g) = Id + gA + \cdots$ represents an infinitesimal rotation in the infinite dimensional vector space of the normalized pole states. So the natural question, to be treated in the next section, is: “what happens in higher orders in $g$?”

4.1.1 Comparison with Winter results

We are in disagreement with [12] regarding the exponential behavior of the excited metastable states, i.e. of $\psi^{(l)}_{ex}(x, t; g)$ with $l > 1$. Let us show that in detail. Eq.(62) reads in components:

$$\psi^{(l)}_{ex}(x, t; g) = \xi^{(l)}(x, 0; g) \exp \left[ -i \omega^{(l)}(g) t - \frac{1}{2} \Gamma^{(l)}(g) t \right] +$$

$$+ \sum_{n \neq l}^{1, \infty} g \frac{(-1)^{l+n} 2ln}{l^2 - n^2} \xi^{(n)}(x, 0; g) \exp \left[ -i \omega^{(n)}(g) t - \frac{1}{2} \Gamma^{(n)}(g) t \right]. \quad (69)$$

As usual, $l$ labels the initial state and $n$ the pole state. The diagonal term on the r.h.s. of the above equation (the one with $n = l$) is in agreement with the r.h.s. of eq.(2a) in [12]. In [12] however, the non-diagonal contributions (the ones with $n \neq l$), entering the sum on the r.h.s., are not included. These terms have a coefficient suppressed by a power of $g \ll 1$ compared to the diagonal one, but have a slower exponential decay for $n < l$ ($\Gamma^{(n)} \propto n^3$, see eq.(54)), and therefore dominate in the exponential time region (i.e. before power-effects take over). For example, for $l = 2$ eq. (69) may be approximated by neglecting higher poles as

$$\psi^{(2)}_{ex}(x, t; g) \simeq \xi^{(2)}(x, 0; g) \exp \left[ -i \omega^{(2)}(g) t - 16\pi g^2 t \right] +$$

$$- \frac{4}{3} g \xi^{(1)}(x, 0; g) \exp \left[ -i \omega^{(1)}(g) t - 2\pi g^2 t \right]. \quad (70)$$

As a measure of the size of the above terms, let us take the square of the modulus integrated over the cavity ($0 < x < \pi$):

$$\int_0^\pi |\cdots|^2 dx. \quad (71)$$
As shown in fig. 3 for \( g = 0.1 \) there is a large temporal region, from \( t \simeq 5 \) up to \( t \simeq 160 \), where the non-diagonal contribution from the first pole,

\[
\frac{16}{9} g^2 \int_0^\pi |\xi^{(1)}(x,;g)|^2 dx \exp (-4\pi g^2 t),
\]

(72)
dominates over the diagonal one from the second pole,

\[
\int_0^\pi |\xi^{(2)}(x,0;g)|^2 dx \exp (-32\pi g^2 t),
\]

(73)
in the temporal evolution of the first excited state \( (l = 2) \). For very large times, \( t \gtrsim 160 \), the power contribution dominates over the exponential ones and we enter the asymptotic region. Neglecting the non-diagonal contributions is therefore a reasonable approximation only for the time-evolution of the lowest-lying state \( \psi^{(1)}(x,t;g) \). The presence of the non-diagonal terms shows that the evolution of general unstable states is far more complicated than as implied by the analysis in [12]. As far as we know, the occurrence and the relevance of such off-diagonal terms has been originally noted in [17]. A physical interpretation of such effect will be presented in the next section.

![Figure 3: Time evolution of the contributions to the \( l = 2 \), i.e. first excited, state for \( g = 0.1 \). Dotted line: second pole contribution; Dashed line: first pole contribution; Continuous line: power contribution.](image)

### 4.1.2 Physical Interpretation of Pole State Mixing

In order to express the metastable wavefunctions of Winter model \( \psi^{(l)}(x,t;g) \) in terms of the eigenfunctions of the particle in a box, one has to diagonalize the time evolution. That is achieved by “counter-rotating” the vector containing the initial conditions, i.e. by considering the evolution not of \( \Psi(x,0;g) \) but of

\[
\Phi(x,0;g) \equiv U^{-1}(g) \Psi(x,0;g).
\]

(74)

By using the first equality in eq. (62), it is immediate to show that

\[
\Phi(x,t;g) = \Xi(x,t;g) = \mathcal{E}(t;g) \Xi(x,0;g).
\]

(75)
By looking at the vector equation (74) component by component, the new initial conditions read:

\[
\phi^{(l)}(x, 0; g) = \sqrt{\frac{2}{\pi}} \theta(\pi - x) \sum_{n=1}^{\infty} \left( U^{-1}(g) \right)_{ln} \sin(nx),
\]

(76)
each evolving as a single pole wavefunction:

\[
\phi^{(l)}(x, t; g) = \xi^{(l)}(x, t; g).
\]

(77)

The “experimental meaning” of eq. (74) or eq. (76) is clear: in order to observe a diagonal time evolution as in the free case (13), one has to prepare the initial state as the coherent superposition of free eigenfunctions given by eq. (74) or by eq. (76). In the case of excited states, \( l > 1 \), the superposition in eq. (74) or (76) has also the effect of subtracting the contributions from smaller \( l \)'s, which decay slower in time and therefore tend to dominate the evolution, as discussed in [17]. If the matrix \( U(g)^{-1} \) is computed in an approximate way (as a truncated power series in \( g \) for example), there is a small residual contamination in the time evolution of the \( l \)-th state from the lower ones, which becomes substantial asymptotically in time. In other words, the problem of isolating the \( l \)-th mode for all times can in principle be solved only with an exact computation of \( U(g) \).

Let us now explicitly evaluate the initial wavefunction which evolves diagonally in \( t \), according to eq. (76). To order \( g \):

\[
U(g)^{-1} = 1 - gA + \mathcal{O}(g^2).
\]

(78)

The sum of the trigonometric series on the r.h.s. of eq. (76) reads:

\[
\phi^{(l)}(x, 0; g) = \sqrt{\frac{2}{\pi}} \left[ \left( 1 - \frac{g}{2} \right) \sin(lx) - glx \cos(lx) \right] \theta(\pi - x) + \mathcal{O}(g^2).
\]

(79)

The r.h.s. of the above equation is the expansion to \( \mathcal{O}(g) \) of

\[
\sqrt{\frac{2}{\pi}} \left( 1 - \frac{g}{2} \right) \sin \left[ l(1 - g) x \right] \theta(\pi - x).
\]

(80)

It is tempting to think that higher orders in \( g \) actually lead to the result (80); the second-order term \( g^2 A^2/2 \) in \( U^{(2)} \) actually confirms this guess (see next section).

The interpretation of eq. (80) is straightforward: the counter-rotation of the initial wavefunction in index space amounts to the shift of the wave-vector \( l \rightarrow k^{(l)}(g) = l(1 - g) + \cdots \) in momentum space, with a consequent change of normalization. In other words, in order to have a diagonal evolution in \( t \) of the initial wavefunction, the latter has to be prepared with the corrected wave-vector \( k^{(l)}(g) \), which is dynamically generated from \( l = k^{(l)}(0) \), the free one. Temporal evolution is then simply given by multiplication by the factor \( E^{(l)}(t, g) \) in eq. (28).

Let us note that the wavefunction in eq. (80) has a finite jump \( \mathcal{O}(g) \) at the right border of the cavity, in \( x = \pi \). The Fourier series in eq. (76) exhibits indeed the Gibbs phenomenon in \( x = \pi \), as the coefficients decay asymptotically \( \approx 1/n \) for \( n \to \infty \). It is remarkable

\[\text{3The related "vertical slope" in } x = \pi \text{ can be derived by differentiating eq. (76) with respect to } x \text{ and then setting } x = \pi.\]
that we obtain a discontinuous initial wavefunction, while the eigenfunctions only have a discontinuous first derivative \[12, 13, 17\]. Let us remark however that the results above are obtained by means of power series in \(g\) which we have not shown to be convergent, and are probably only asymptotic. The Fourier series therefore should be truncated to some finite order in \(g\), regularizing the discontinuity.

4.2 Second-Order Computation, \(\mathcal{O}(g^2)\)

In this section we push the perturbative expansion for \(g \ll 1\) up to \(\mathcal{O}(g^2)\) included, in order to obtain more accurate results and to get some insight into the general structure of the expansion, if any. The exact expression in eq.(32) for the mixing matrix in terms of the exact solutions \(k(n)(g)\) of the equation \(b(k, g) = 0\) indeed is not very illuminating. By inserting the small-\(g\) expansion for the poles pushed one order further with respect to previous section,

\[
k(n)(g) = n - ng + (n - i\pi n^2)g^2 + \left(\frac{4}{3}\pi^2 n^3 + 3i\pi n^2 - n\right)g^3 + \mathcal{O}\left(g^4\right), \tag{81}
\]

we obtain for the mixing matrix:

\[
V^{(0)} = Id; \tag{82}
\]

\[
V^{(1)} = A - \frac{1}{2} Id; \tag{83}
\]

\[
V^{(2)} = \frac{1}{2} A^2 - A + \frac{3}{8} Id + i\pi AH - \frac{3}{2} i\pi H, \tag{84}
\]

where for convenience we have repeated the lowest-order results and \(H\) is the real diagonal matrix

\[H \equiv \text{diag} \left(1, 2, 3, \cdots, n, \cdots\right). \tag{85}\]

The coefficients entering the pole wavefunctions read:

\[
\omega^{(n)}(g) = n^2 \left(1 - 2g + 3g^2\right) + \mathcal{O}\left(g^3\right); \tag{86}
\]

\[
\Gamma^{(n)}(g) = 4\pi n^3 g^2 \left(1 - 4g\right) + \mathcal{O}\left(g^4\right). \tag{87}
\]

Eq.(84) has been obtained by using the explicit expression

\[
V^{(2)}_{ln} = \delta_{ln} \left(\frac{1}{4} - \frac{\pi^2}{6} l^2 - i\pi \frac{3}{2} l\right) + (1 - \delta_{ln}) \left[\frac{(-1)^{l+n} 2ln}{l^2 - n^2} (i\pi n - 1) + \frac{(-1)^{l+n+1} 2ln}{(l^2 + n^2)^2} \right], \tag{88}
\]

together with the formula

\[
\frac{1}{2} (A^2)_{ln} = (1 - \delta_{ln}) \left(-1\right)^{l+n+1} \frac{2ln (l^2 + n^2)}{(l^2 - n^2)^2} - \delta_{ln} \left(\frac{\pi^2}{6} l^2 + \frac{1}{8}\right). \tag{89}
\]
Note that the squared matrix is symmetric, as it should, being the square of an antisymmetric matrix. The last equation has been derived by means of the identities

\[
\sum_{k \neq m}^{1, \infty} \frac{1}{k^2 - m^2} = \frac{3}{4m^2} \quad \text{and} \quad \sum_{k \neq m}^{1, \infty} \frac{k^2}{(k^2 - m^2)^2} = \frac{\pi^2}{12} + \frac{1}{16m^2},
\]

which hold for \( m \) a positive integer. Let us remark that it is not trivial that the matrix \( A^2 \) does exist, as its entries involve the summation of infinite series, which in effect turn out to be (absolutely) convergent. By looking at the asymptotic form of the coefficients of \( A \) and \( A^2 \) given above, it is straightforward to show that \( A^3 \) and \( A^4 \) also exist. We expect that all the positive powers of \( A \) do exist.

Let us now discuss renormalization at second order. The explicit expansion of \( U(g) \) up to \( \mathcal{O}(g^2) \) reads:

\[
U^{(0)} = \text{Id};
\]

\[
U^{(1)} = V^{(1)} + Z^{(1)} = A - \frac{1}{2} \text{Id} + Z^{(1)};
\]

\[
U^{(2)} = V^{(2)} + V^{(1)} Z^{(1)} + Z^{(2)} = \frac{1}{2} A^2 - A + \frac{3}{8} \text{Id} + i\pi AH - \frac{3}{2} i\pi H + \left( A - \frac{1}{2} \text{Id} \right) Z^{(1)} + Z^{(2)}. \tag{93}
\]

As shown in previous section, if we set

\[
Z^{(1)} = \frac{1}{2} \text{Id},
\]

we obtain at first order

\[
U^{(1)} = A, \tag{95}
\]

so that at second order we get:

\[
U^{(2)} = \frac{1}{2} A^2 - \frac{1}{2} A + \frac{1}{8} \text{Id} - \frac{3}{2} i\pi H + i\pi AH + Z^{(2)}. \tag{96}
\]

We may set for example

\[
Z^{(2)} = -\frac{1}{8} \text{Id} + \frac{3}{2} i\pi H, \tag{97}
\]

to give

\[
U^{(2)} = \frac{1}{2} A^2 - \frac{1}{2} A + i\pi AH. \tag{98}
\]

The renormalized mixing matrix then finally reads:

\[
U(g) = \text{Id} + gA + \frac{1}{2} g^2 A^2 - \frac{1}{2} g^2 A + i\pi g^2 AH + \mathcal{O}(g^3). \tag{99}
\]

\footnote{These identities are obtained from \( \sum_{k \neq m}^{-\infty, +\infty} 1/(k - m) = 0 \) and \( \sum_{k \neq m}^{-\infty, +\infty} 1/(k - m)^2 = \pi^2/3 \) respectively, by splitting the sums into positive and negative indices and rearranging them in order to have a single sum. The first identity can be found in \cite{28}.}
4.2.1 Exponentiation

The first three terms on the r.h.s. of eq. (99) are actually the expansion of

$$\exp[gA] = \mathbb{I} + gA + \frac{1}{2} g^2 A^2 + \mathcal{O}(g^3),$$

so it is not difficult to conjecture that higher orders in $g$ will lead to the exponential above. We can also absorb the fourth term on the r.h.s. of eq. (99) by means of a modified exponent:

$$\exp \left[ g \left( 1 - \frac{g}{2} \right) A \right] = \mathbb{I} + gA + \frac{1}{2} g^2 A^2 - \frac{1}{2} g^2 A + \mathcal{O}(g^3).$$

The problem is that we are not sure that the conjectured exponentiation is “legitimate”, i.e. that it includes all the leading terms order by order in $g$. To $\mathcal{O}(g^2)$ we found indeed the term $AH$, which has a large size,

$$(AH)_{l,n} = \frac{(-1)^{l+n} 2ln^2}{l^2 - n^2} \text{ for } l \neq n \text{ and 0 otherwise,}$$

and will presumably produce iterates of similar size in higher orders, which we are unable to control. Since $Z^{(2)}$ is a diagonal matrix, whatever value is chosen for it, we cannot cancel the term $AH$ in $U^{(2)}$ with an ad-hoc renormalization condition. This term also produces a highly singular behavior in the counter-rotated initial states considered in the previous section, $U(g)^{-1}\Psi(x,0;g)$, because

$$(AH)_{l,n} = \mathcal{O}(1)$$

for $n \to \infty$ (at fixed $l$). The detailed investigation of such effects requires the study of the convergence properties of the series in $g$ involved, which is beyond the scope of the present paper. A third-order computation in $g$ could probably reveal further structure of the perturbative expansion.

5 Discussion

Let us now discuss the renormalized wavefunctions $\phi^{(l)}(x,t;g)$. The main qualitative difference between the free case and the interacting one is that in the latter case there are non-zero widths. The appearance of an imaginary part in the ab initio real energy is a second order effect in $g$. Once a non-zero width is allowed, the key point is that the $\phi^{(l)}(x,t;g)$’s have a similar form to the eigenfunctions of the free system $\psi_0^{(l)}(x,t)$ in eq. (13). The differences between the free case and the interacting one, as long as $0 < g \ll 1$, can be relegated to small modifications of the parameters entering the free wavefunctions $\psi_0^{(l)}(x,t)$. In other words, switching on the interaction, i.e. going from $g = 0$ to $0 < g \ll 1$, produces finite renormalizations only. Let us discuss these renormalizations in turn:

5 Non zero widths are clearly not in contradiction with the unitarity of the fundamental theory because we are looking at a subsystem, an “open” system.
1. the normalization coefficient \( Z^{(l)}(g) \) has a modulus greater than one for \( 0 < g \ll 1 \) and reduces to 1 in the free case \( g = 0 \); it has a first-order correction in \( g \) and is the analog of the field renormalization constant \( Z \) in quantum field theory \([17]\). Unlike the most common cases (QED for example), \( Z^{(l)}(g) \) is not real because the one-particle states are unstable;

2. the wave-vector \( k^{(n)}(g) \) is renormalized to first order in \( g \) by the interaction and reduces to the free case for \( g \to 0 \): \( k^{(l)}(0) = l \). It acquires an imaginary part at second order in \( g \), related to the decay width. That implies the disappearance of the node of the wavefunction around \( x = \pi \) and a (small) exponential growth of \( \phi^{(l)}(x, t; g) \) by going from the impermeable wall in \( x = 0 \) toward the permeable one in \( x = \pi \);

3. the real part of the energy \( \omega^{(l)}(g) \) is also renormalized to first order in \( g \) by the interaction and reduces to the free case for \( g \to 0 \): \( \omega^{(l)}(0) = l^2 \). Note that the free dispersion relation \( \omega = k^2 \) is not renormalized at first order.

Let us make a few remarks.

- We do not expand in powers of \( g \) the wavefunctions \( \phi^{(l)}(x, t; g) \), but only the parameters \( k^{(n)}(g), \omega^{(n)}(g) \), etc. entering them through the functions appearing in \( \psi_0^{(l)}(x, t) \). That implies that we are resumming classes of higher order corrections in \( g \) in the wave function, in the spirit of renormalization in quantum field theory \([18, 19, 20, 21]\) and statistical mechanics \([27]\), or the method of multiple scales in classical physics \([23, 24]\);

- The decay widths grow faster with increasing \( n \) than the frequencies:

\[
\Gamma^{(n)}(g) \propto n^3, \quad \omega^{(n)}(g) \propto n^2.
\]

Since our renormalized theory has meaning only for

\[
\Gamma^{(n)}(g) \ll \omega^{(n)}(g),
\]

we cannot take \( n \) too large. Therefore, while in principle the state vectors and the evolution/mixing matrices are infinite, in practice for any fixed \( g \) one has to make a truncation in \( n \) according to the condition \([105]\). This limitation is also reasonable from physics viewpoint: high energy particles pass through the barrier in \( x = \pi \) without difficulty and therefore there is no sense in including them to describe the dynamics inside the cavity. By restricting on \( n \) one is also cutting off small wavelengths \( \lambda < 2\pi/n \) and therefore is limiting space resolution.

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6 In general, in the theory of elementary excitations, one only considers states lying slightly above the ground state \([7]\). In the case of normal Fermi liquids, for example, one only considers electrons slightly above the Fermi surface, i.e. with \( \epsilon_k \equiv k^2/(2m) - E_F \ll E_F \), where \( E_F \) is the Fermi energy. An electron slightly above the Fermi surface is an unstable state because it can hit an electron slightly below it creating a pair. In this case \( \Gamma_k \propto \epsilon_k^2 \), i.e. the width increases with the square of the excitation energy.
6 Conclusions

We have shown that the resonant states of the Winter model can be related to the states of a particle inside an impenetrable box by means of renormalization of the parameters entering the “free” eigenfunctions and mixing of the states, after allowing for non-zero widths. The less trivial aspect of this renormalization procedure is related to a term which was overlooked in [12] and has been found in [17]: the initial unstable states (at $t = 0$) have to be properly “counter-rotated” by means of the infinite matrix $U(g)^{-1}$, the inverse of the state mixing matrix $U(g)$, in order to have a diagonal, exponential time evolution. We have explicitly computed $U(g)$ to second order in $g \ll 1$ and we have conjectured a resummed form for $U(g)$, which however does not seem to contain all the leading terms order by order in $g$.

The counter-rotation $U^{-1}(g)$ of the initial wavefunction $\sqrt{2/\pi}\sin(lx)\theta(\pi - x)$ in index space amounts to the shift $l \to k^{(l)}(g)$ in momentum space. That implies that, in order to have a diagonal time evolution, the initial wavefunction must contain the renormalized wave-vector $k^{(l)}(g) = l - gl + \cdots$ in place of the free-theory one $l$.

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