Non–Abelian, Self–Dual Chern–Simons Vortices Coupled To Gravity

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ABSTRACT

In this article we consider $SU(2)$ Chern–Simons/Higgs theory coupled to gravity in three dimensions. It is shown that for a cylindrically symmetric vortex both the Einstein equations and the field equations can be reduced to a set of first–order Bogomol’nyi equations provided, that we choose a specific eighth–order potential.

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1. Introduction

Vortices, monopoles and other topological defects are regular, classical solutions to gauge field theories which arise when a symmetry of the theory is spontaneously broken. In particular, vortices may be regarded as cosmic strings generated during phase transitions in the early universe \([1]\) and they can provide the seeds which are required for the formation of galaxies \([2]\). Vortices arise whenever a gauge group, \(G\) is spontaneously broken to a disconnected unbroken subgroup, \(H\). The simplest gauge theory that admits cosmic string (vortex) solutions is Abelian Higgs theory. Here a complex Higgs scalar field, \(\phi\) self-interacts via a fourth-order gauge invariant Higgs potential. After the \(U(1)\) symmetry is spontaneously broken, the vacuum is invariant under the action of \(I\) (which is a subgroup of \(U(1)\)), and it is characterised by a non-vanishing expectation value for the Higgs field. Mathematically, this situation corresponds to a topologically non-trivial vacuum with first homotopy group, \(\pi_1(U(1)/I) = \mathbb{Z}\).

Recently there has been a considerable amount of interest in Maxwell/Chern–Simons/Higgs theory in three dimensional Minkowski space \([3]\), due to its similarities with the theory of high-\(T_c\) superconductors. It was first remarked in \([4]\) that at large distances the Chern–Simons term dominates the Maxwell term and so it is reasonable to consider the simpler Abelian Chern–Simons/Higgs theory. Consequently, it was shown in \([4]\) that there exist vortex solutions to three dimensional Abelian Chern–Simons/Higgs theory and moreover, by choosing a specific sixth-order potential the field equations reduce to a set of first-order self-dual (Bogomol’nyi \([5]\)) equations. This potential has a symmetric and an antisymmetric vacuum. Solutions approaching the antisymmetric vacuum at infinity describe topologically stable vortices, whereas solutions which approach the symmetric phase at infinity are non-topological solutions \([6]\).

The most natural generalisation of the Abelian theories above is to examine their non–Abelian counterparts. In \([7]\) it was demonstrated that there exist self-dual vortex solutions to \(SU(N)\) Chern–Simons/Higgs theory in three dimensional
Minkowski space. As is to be expected from our knowledge of the Abelian case, a sixth–order potential is required in order to obtain a set of Bogomol’nyi equations, but in contrast to the Abelian case there are no topologically stable solutions, only stable non–topological solutions in both the symmetric and antisymmetric phases. In [8] and [9], flat space Yang–Mills/Chern–Simons/Higgs theories were investigated for the gauge groups $SU(2)$ and $SU(N)$ respectively. By including several Higgs multiplets in the theory (to ensure the maximal breaking of the gauge symmetry) topologically stable vortex solutions were found. However, since the models considered contained only a fourth–order potential these solutions were not shown to be self–dual. Self–dual vortex solutions to $SU(N)$ Yang–Mills/Higgs theory with $N$ Higgs multiplets and a sixth–order potential are found in [10]. An interesting feature of these self–dual vortices is that the Bogomol’nyi bound upon the mass of the configuration in terms of its topological charge is not a topologically invariant quantity, in contrast to the Abelian case. The difference in the values of the Bogomol’nyi bound associated with members of the same homotopy class is related to the fact that the topological charge of $SU(N)$ vortices is defined modulo $N$, whereas physical quantities may depend upon the actual value of the magnetic flux associated with this topological charge [10]. In [11], $SU(2)$ Chern–Simons/Higgs theory was examined and, by using a particular choice of sixth–order potential, self–dual vortex solutions were obtained. As in [10], these vortices obey a Bogomol’nyi bound which is not a topologically invariant quantity.

Another interesting generalisation of [3, 4] is to couple the vortex solutions to gravity. In [12], Abelian Chern–Simons/Higgs theory was coupled to three dimensional gravity. It was demonstrated that there exist vortex solutions such that, by choosing a non–renormalisable eighth–order potential whose constant parameters were precisely chosen, both the Einstein equations and the equations of motion can be reduced to a set of first order Bogomol’nyi equations. In this article we will search for self–dual vortex solutions to $SU(2)$ Chern–Simons/Higgs theory coupled to three dimensional gravity. We will show that both the mass and the angular momentum of configurations belonging to the same homotopy class can take dif-
ferent values due to the mathematical considerations explained below. This means
that neither the mass nor the angular momentum are topologically invariant.

This article is organised as follows: in Section 2 we describe the mathematical
basis of the maximal breaking of an $SU(N)$ gauge symmetry, as well as describing
several unusual features of non–Abelian gauge theories. In Section 3 we present a
review of Einstein’s theory of gravity in three dimensions which will serve to define
the notation that will be used throughout this article. In Section 4 we derive a set
of first–order Bogomol’nyi equations from the Einstein equations and the equations
of motion of three dimensional Einstein/Chern–Simons/Higgs theory. In Section 5
we present our conclusions.

2. Charged Vortices In $SU(N)$ Gauge Theories

In this article we will consider classical gauge theories with gauge group $G$
(this will subsequently chosen to be either $SU(2)$ or $SU(N)$). Upon spontaneous
symmetry breaking the symmetry group is reduced to an unbroken group $H$ which
is a subgroup of $G$. This spontaneous symmetry breaking is achieved via the Higgs
mechanism using a symmetry breaking potential, $V$ whose zeros can be identified
with the coset space $G/H$. In $(d + 1)$–dimensions, in order to have topologically
stable solutions the $(d − 1)$th homotopy group of $G/H$, $\pi_{d−1}(G/H)$ must be non–
trivial, i.e. it must have more than one element. This is because the connected
components of the space of non–singular, finite energy solutions are in 1–to–1
correspondence with the homotopy classes of mappings from the $(d − 1)$–sphere,
$S^{d−1}$ to $G/H$.

For example, in the Abelian case $G = U(1)$ and $H = I$ (in fact the vacuum
manifold is topologically equivalent to $S^1$). Thus $\pi_1(G/H) \equiv \pi_1(U(1)) = Z$
and so there exists an infinity of topologically stable vortices labelled by an integer,
n ($n = 0$ labels the vacuum). We are interested in the non–Abelian case where
$G = SU(N)$. We consider theories for which the Higgs fields are in the adjoint re-
presentation of $SU(N)$ and we will assume maximal symmetry breaking of $SU(N)$. 
This means that the vacuum is only invariant under the unit matrix in the adjoint representation. There exist \( N \) elements with this property, namely the matrices at the centre of \( SU(N) \)

\[ I_N e^{\frac{2\pi i n}{N}}, \quad n = 0, 1, \ldots, N - 1, \]  

(2.1)

where \( I_N \) is the \( N \times N \) unit matrix. Hence, since the roots of unity \( e^{\frac{2\pi i n}{N}} \) provide a representation of the Abelian group, \( \mathbb{Z}_N \) we see that \( H = \mathbb{Z}_N \). Thus

\[ \pi_1 (G/H) \equiv \pi_1 (SU(N)/\mathbb{Z}_N) = \mathbb{Z}_N. \]  

(2.2)

This implies that there exist \((N - 1)\) topologically stable vortex solutions (recall that \( N = 0 \) labels the vacuum solution). An element of each of the non–trivial classes above can be obtained from the vacuum class by a non–trivial gauge rotation, \( g_n \in SU(N) \). Since we are concerned with three dimensional theories, the direction at infinity is characterised by an angle, \( \theta \). A map belonging to the \( n^{\text{th}} \) homotopy class satisfies, upon performing one turn around a closed contour

\[ g_n (2\pi) = e^{\frac{2\pi i n}{N}} g_n (0), \quad n = 0, 1, \ldots, N - 1, \]  

(2.3)

where \( g_n \) is an element of the Cartan subgroup of \( SU(N) \). A peculiar feature of these theories with maximal symmetry breaking is that, in order to obtain topologically stable vortex solutions, the theories must contain \( N \) Higgs multiplets [8, 9, 13].

Another unusual property of non–Abelian Chern–Simons theories is that if one wishes to have a consistent path integral formulation of the theory, the Chern–Simons coupling constant, \( \kappa \) is quantised. Under ‘small’ gauge transformations, \( i.e. \) gauge transformations which are connected to the identity, the Chern–Simons, \( S_{\text{CS}} \) is gauge invariant. However, under ‘large’ gauge transformations (those gauge
transformations that are not connected to the identity) $S_{CS}$ is not gauge invariant. In fact

$$S_{CS} \rightarrow S_{CS} - \frac{8\pi^2 \kappa}{e^2} \omega(U),$$

where $e$ is the gauge field coupling constant and $\omega(U)$ is the integer–valued winding number of the gauge transformation, $U \in SU(N)$. Explicitly, $\omega(U)$ is given by

$$\omega(U) = \frac{1}{24\pi^2} \int d^3x \varepsilon^{\mu\nu\lambda} \text{tr} \left( U^{-1} \partial_\mu U U^{-1} \partial_\nu U U^{-1} \partial_\lambda U \right).$$

Since, for a consistent path integral formulation $\exp(iS_{CS})$ is required to be gauge invariant we see that

$$-\frac{8\pi^2 \kappa}{e^2} = 2\pi m, \quad m \in \mathbb{Z},$$

which implies that $\kappa$ is quantised in units of $-\frac{e^2}{4\pi}$ [14], i.e.

$$\kappa = -\frac{me^2}{4\pi}.$$  

3. Einstein’s Theory Of Gravity In Three Dimensions

Einstein’s theory of gravity in three dimensions exhibits some rather interesting behaviour which makes its analysis as important as that of the more usual four dimensional case. In the absence of any matter sources it turns out that, in a topologically trivial three dimensional spacetime, the theory is trivial. However, upon the introduction of a point (or line) matter source the spacetime acquires a global conical structure and non–trivial gravitational effects can occur in the framework of both classical and quantum theories.

Throughout this article we will use the notation adopted in [12, 15]. Let us consider a three dimensional spacetime with metric $g_{\mu\nu}$ and coordinates $x^\mu = (\rho, t, \theta)$ where $t$ is the timelike coordinate. The metric has signature $(-, +, -)$. A
stationary, cylindrically symmetric three dimensional spacetime can be described
by the line element

\[ ds^2 = -d\rho^2 + g_{ij}(\rho) \, dx^i dx^j, \quad \rho \geq 0, \quad 0 \leq \theta < 2\pi, \quad i, j = (t, \theta) , \quad (3.1) \]

In what follows we will choose the coordinates such that \( x^t = t, \ x^\theta = \theta \). Note that
the inverse metric, \( g^{\mu \nu} \) is given by

\[ g^{\mu \nu} = \frac{1}{g} \begin{pmatrix}
-g & 0 & 0 \\
0 & g_{\theta \theta} & -g_{t \theta} \\
0 & -g_{t \theta} & g_{tt}
\end{pmatrix} , \quad (3.2) \]

where the determinant of the metric is given by

\[ \det g_{\mu \nu} = g^2 t_{\theta \theta} - g_{tt} g_{\theta \theta} = -g \equiv -\det g_{ij} . \quad (3.3) \]

Note that in order to obtain a metric of the required (pseudo–Riemannian) signa-
ture we impose the conditions \( g_{tt} > 0 \) and \( g_{\theta \theta} < 0 \). As was explained in \([12, 15]\),
the metric is required to have the following asymptotic behaviour

\[ ds^2 \sim -d\rho^2 + dt^2 - 2\omega \rho^2 dt d\theta - \rho^2 d\theta^2 \quad \text{as } \rho \to 0 \]

\[ ds^2 \sim -d\rho^2 + \left( Adt + \frac{4GJ}{B} d\theta \right)^2 - B^2 \rho^2 d\theta^2 \quad \text{as } \rho \to \infty , \quad (3.4) \]

where \( A, B, G, J \) and \( \omega \) are constants\(^*\). The second of the metrics (3.4) (as a
vacuum metric) describes a particle of mass \( M \) and spin \( J \) located at \( \rho = 0 \), where

\(^*\) Note that in three dimensions there is a certain ambiguity over the sign of Newton’s con-
tant, \( G \). For simplicity we will assume that \( G \) is positive.
$M$ is given by $B = 1 - 4GM$. Furthermore, note that for this metric we have

$$g = -(AB\rho)^2.$$  \hfill (3.5)

We now define the quantities

$$\chi^i_j \equiv g^{ik} \frac{d}{d\rho} g_{kj},$$  \hfill (3.6)

such that

$$\text{tr} \chi^i_j \equiv \chi^i_i = \frac{2}{\sqrt{-g}} \frac{d}{d\rho} \sqrt{-g}.$$  \hfill (3.7)

It is well known that in three dimensions the Einstein equations ($G_{\mu\nu} = 8\pi G T_{\mu\nu}$) can be expressed in terms of the $\chi^i_j$ as

$$\frac{1}{\sqrt{-g}} \frac{d}{d\rho} \left( \sqrt{-g} \chi^i_j \right) = 16\pi G \left( T^i_j - T \delta^i_j \right),$$  \hfill (3.8)

and

$$G^{\rho \rho} = -\frac{1}{4} \det \chi^i_j = 8\pi G T^{\rho \rho}.$$  \hfill (3.9)

Using the form of the metric near $\rho = 0$ given by the first of equations (3.4) we obtain the relations (for regular matter fields)

$$\sqrt{-g} \chi^t_\theta = 16\pi G \int_0^\rho \sqrt{-g} T^t_\theta d\rho , \quad \sqrt{-g} \chi^\theta_t = 2\omega + 16\pi G \int_0^\rho \sqrt{-g} T^\theta_t d\rho.$$  \hfill (3.10)

Using the second of the metrics (3.4) we find that as $\rho$ tends to infinity the quantities on the right–hand side of equations (3.10) are given by

$$\sqrt{-g} \chi^t_\theta = -8GJ , \quad \sqrt{-g} \chi^\theta_t = 0,$$  \hfill (3.11)

and hence we obtain the following expressions for $J$ and $\omega$

$$J = -2\pi \int_0^\infty \sqrt{-g} T^t_\theta d\rho , \quad \omega = -8\pi G \int_0^\infty \sqrt{-g} T^\theta_t d\rho.$$  \hfill (3.12)
4. $SU(2)$ Chern–Simons/Higgs Vortices Coupled To Gravity

In this section we search for vortex solutions to $SU(2)$ Chern–Simons/Higgs theory coupled to gravity. To ensure maximal symmetry breaking we consider a theory containing two Higgs multiplets, $\Phi$ and $\Psi$ with vacuum expectation values $\eta$ and $\xi$ respectively. We see from equation (2.2) that here (since $G = SU(2)$) $\pi_1 (G/H) = \mathbb{Z}_2$ and so there exists only one class of topologically stable vortex solutions. Furthermore, an element $g$ of the Cartan subgroup of $SU(2)$ can be represented by

$$g = e^{i\sigma_3 \Omega(\theta)},$$

where $\sigma_3$ is the third Pauli matrix and $\Omega (\theta)$ obeys

$$\Omega (2\pi) - \Omega (0) = (n + 2s) \pi, \quad n = 0, 1, \quad s \in \mathbb{Z}.$$ (4.2)

In order to obtain solutions with finite energy we require that the gauge and Higgs fields have the following asymptotic behaviour

$$\lim_{\rho \to \infty} D_{\mu} \Phi = \lim_{\rho \to \infty} D_{\mu} \Psi = 0$$

$$\lim_{\rho \to \infty} A_{\mu} = \lim_{\rho \to \infty} A_{\mu}^a \sigma_a = \frac{1}{i} g^{-1} \partial_{\mu} g,$$

where $a$ is an $SU(2)$ index and the $\sigma_a$ are the Pauli matrices. Note that the second of these conditions implies that $A_{\mu}$ is pure gauge at infinity and so, at infinity, $F_{\mu \nu}$ vanishes. Upon combining the second of the above conditions with the form of $g$ given by equation (4.1) we find that

$$\lim_{\rho \to \infty} A_{3}^3 = \frac{d\Omega (\theta)}{d\theta},$$

whilst all the other components of $A_{\mu}^a$ vanish at infinity.
The action describing this theory is given by

\[
S = \int d^3x \sqrt{-g} \left( -\frac{1}{16\pi G} R + \frac{1}{2} [D_\mu \Phi \cdot D_\nu \Phi] + \frac{1}{2} [D_\mu \Psi \cdot D_\nu \Psi] + \frac{\kappa}{4} \sqrt{-g} \epsilon^{\mu\nu\lambda} \left[ F_{\mu\nu} \cdot A_\lambda - \frac{2e}{3} A_\mu \cdot (A_\nu \times A_\lambda) \right] - V(\Phi, \Psi) \right) \tag{4.5}
\]

where\(^*\) the SU(2) covariant derivative and field strength are respectively given by

\[
D_\mu = \partial_\mu + e A_\mu \times \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + e A_\mu \times A_\nu .
\tag{4.6}
\]

Note that the vector notation in the above quantities refers to the internal SU(2) vectors. The Einstein equations of the theory are

\[
G_{\mu\nu} = 8\pi T_{\mu\nu} .
\tag{4.7}
\]

where

\[
T_{\mu\nu} = \left( D_\mu \Phi \cdot D_\nu \Phi - \frac{1}{2} g_{\mu\nu} D_\alpha \Phi \cdot D_\beta \Phi g^{\alpha\beta} \right) + \left( D_\mu \Psi \cdot D_\nu \Psi - \frac{1}{2} g_{\mu\nu} D_\alpha \Psi \cdot D_\beta \Psi g^{\alpha\beta} \right) + g_{\mu\nu} V(\Phi, \Psi) .
\tag{4.8}
\]

The field equations associated with the variations of \(\Phi, \Psi\) and \(A_\mu\) are given by

\[
D_\mu \left( \sqrt{-g} g^{\mu
u} D_\nu \Phi \right) = -\sqrt{-g} \frac{\partial V}{\partial \Phi}
\] \[
D_\mu \left( \sqrt{-g} g^{\mu\nu} D_\nu \Psi \right) = -\sqrt{-g} \frac{\partial V}{\partial \Psi}
\] \[
\frac{\kappa}{2} \epsilon^{\mu\nu\lambda} F_{\nu\lambda} = -e \sqrt{-g} g^{\mu\delta} \left( D_\delta \Phi \times \Phi + D_\delta \Psi \times \Psi \right) .
\tag{4.9}
\]

Guided by work on self–dual \(U(1)\) vortices coupled to gravity [12] and flat space self–dual \(SU(2)\) vortex solutions [11], we search for solutions to the Einstein

\(*\) Note that \(R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} R^\lambda_{\mu\lambda\nu}\) and \(\epsilon^{\mu\nu\lambda}\) is the totally antisymmetric Levi–Civita tensor density.
equations and the field equations with a potential of the form

\[ V(\Phi, \Psi) = \alpha \Phi^2 (\Phi^2 - \eta^2)^2 + \beta (\Phi^2 - \eta^2)^4 \\
+ \gamma \Psi^2 (\Psi^2 - \xi^2)^2 + \delta (\Psi^2 - \xi^2)^4 \\
+ \lambda (\Phi \cdot \Psi)^2 , \]  

(4.10)

where \( \alpha, \beta, \gamma, \delta \) and \( \lambda \) are constants to be determined\(^\dagger\). It has been shown in flat space [8, 11] that \( SU(2) \) vortices can be obtained if one assumes that the only rôles played by one of the Higgs fields is to ensure maximal symmetry breaking, i.e. the Higgs field is taken to be constant over all of the spacetime and hence has no dynamical rôles. Since we are searching for vortex solutions that correspond to a set of Bogomol’nyi equations (i.e. a minimal energy configuration) it is reasonable to assume (as in flat space [8, 11]) that any configuration with non–constant \( \Psi \) has greater energy than a configuration with constant \( \Psi \). Hence, we impose the following conditions (valid over all of the spacetime) upon \( \Psi \)

\[
\Psi^2 = \xi^2 \\
D_\mu \Psi = 0 \\
\Phi \cdot \Psi = 0 .
\]

(4.11)

The last of these conditions ensures that \( \Phi \) and \( \Psi \) are not parallel in the internal space. Upon imposing these conditions we find that the second of the field equations (4.9) is automatically satisfied, whereas the remaining equations reduce to

\[
D_\mu (\sqrt{-g} g^{\mu \nu} D_\nu \Phi) = -\sqrt{-g} \frac{\partial V}{\partial \Phi} \\
\frac{\kappa}{2} \varepsilon^{\mu \nu \lambda \rho} F_{\nu \lambda} = -\varepsilon \sqrt{-g} g^{\mu \delta} (D_\delta \Phi \times \Phi) .
\]

(4.12)

\(^\dagger\) Note that we can of course write a more general eighth–order potential but upon imposing the conditions described by equations (4.11) we see that these terms vanish. Hence, for clarity we include only the first few possible terms.
Furthermore, the energy–momentum tensor and the potential are given by
\[
T_{\mu\nu} = \left( D_\mu \Phi \cdot D_\nu \Phi - \frac{1}{2} g_{\mu\nu} D_\alpha \Phi \cdot D_\beta \Phi g^{\alpha\beta} \right) + g_{\mu\nu} V(\Phi) ,
\]
and
\[
V(\Phi) = \alpha \Phi^2 (\Phi^2 - \eta^2)^2 + \beta (\Phi^2 - \eta^2)^4 ,
\]
respectively.

As we remarked above, for gauge group \( SU(2) \) there is only one class of topologically stable vortex solutions \( n = 1 \). Thus, we search for a cylindrically symmetric vortex solution of the form
\[
A_\rho = 0 , \quad A_t = \epsilon \frac{W(\rho)}{e} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} , \quad A_\theta = \frac{(P(\rho) - n - 2s)}{e} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} ,
\]
\[
\Phi = R(\rho) \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix} , \quad \Psi = \xi \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} ,
\]
where \( \epsilon = \pm 1 \). Note that the presence of the integer \( s \) in the expression for \( A_\theta \) is necessary for the ansatz to be consistent with equation (4.4). We also require that the fields obey the boundary conditions at the origin
\[
P(0) = n , \quad R(0) = 0 ,
\]
which ensures that the fields are single valued. The finite energy boundary conditions given by equations (4.3) imply that we must impose the following boundary conditions at infinity
\[
P(\infty) = 0 , \quad R(\infty) = \eta , \quad W(\infty) = 0 .
\]
Furthermore, to enable us to obtain a solution we henceforth assume that \( g_{tt} = 1 \) and thus in equation (3.4), \( A = 1 \).

\[
\text{‡ The sign of } \epsilon \text{ will be chosen depending upon the sign of } (n + 2s) \text{ to obtain a positive mass.}
\]
Substituting the *ansätze* (4.15) into the field equations (4.12) we obtain

\[
\begin{pmatrix}
\cos \theta \\
\sin \theta \\
0
\end{pmatrix}
\left[
\frac{1}{\sqrt{-g}} \frac{d}{d\rho} \left( \sqrt{-g} \frac{dR}{d\rho} \right) + \frac{R}{g} \left( P^2 + W^2 g_{\theta\theta} \right) - \frac{2\epsilon}{g} PRW g_{t\theta}
\right] = \frac{\partial V(\Phi)}{\partial \Phi}
\]

\[
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}
\left[
\frac{\kappa}{\sqrt{-g}} \frac{dP}{d\rho} + \frac{e^2 R^2}{g} (\epsilon W g_{\theta\theta} - P g_{t\theta})
\right] = 0
\]

\[
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}
\left[
\frac{\epsilon \kappa}{\sqrt{-g}} \frac{dW}{d\rho} - \frac{e^2 R^2}{g} (P - \epsilon W g_{t\theta})
\right] = 0 .
\]

Furthermore, it is easy to see from (4.14) that

\[
\frac{\partial V(\Phi)}{\partial \Phi} = 2 \begin{pmatrix}
\cos \theta \\
\sin \theta \\
0
\end{pmatrix} R \left( R^2 - \eta^2 \right) \left[ \alpha \left( R^2 - \eta^2 \right) + 2\alpha R^2 + 4\beta \left( R^2 - \eta^2 \right)^2 \right]
\]

\[
\equiv \begin{pmatrix}
\cos \theta \\
\sin \theta \\
0
\end{pmatrix} H(R) ,
\]

and thus, since the field equations are valid for all $\theta$, the field equations (4.18) become

\[
\frac{1}{\sqrt{-g}} \frac{d}{d\rho} \left( \sqrt{-g} \frac{dR}{d\rho} \right) + \frac{R}{g} \left( P^2 + W^2 g_{\theta\theta} \right) - \frac{2\epsilon}{g} PRW g_{t\theta} = H(R)
\]

\[
\frac{\kappa}{\sqrt{-g}} \frac{dP}{d\rho} + \frac{e^2 R^2}{g} (\epsilon W g_{\theta\theta} - P g_{t\theta}) = 0
\]

\[
\frac{\epsilon \kappa}{\sqrt{-g}} \frac{dW}{d\rho} - \frac{e^2 R^2}{g} (P - \epsilon W g_{t\theta}) = 0 .
\]

We now notice these field equations are identical to the field equations (3.9) of [12] for Abelian Chern–Simons vortices coupled to gravity. Hence, proceeding as
in [12] we assume that \( W(\rho) \) is given by

\[
W(\rho) = \frac{e^2}{2\kappa} (R^2 - \eta^2) .
\] (4.21)

Thus, the third of the field equations (4.20) implies that

\[
\sqrt{-g} \frac{dR}{d\rho} = RW g_{t\theta} - \epsilon RP .
\] (4.22)

In order to satisfy the first of the above field equations we find that (using the last two field equations)

\[
\alpha = \frac{e^4}{8\kappa^2} , \quad \frac{1}{\sqrt{-g}} \frac{dg_{t\theta}}{d\rho} = \frac{16\beta\kappa}{e^2} (R^2 - \eta^2)^2 .
\] (4.23)

To pursue this analysis any further, which will enable us to determine the values of \( \beta \) and \( \frac{dg_{t\theta}}{d\rho} \), we must examine the Einstein equations (4.7). Using equation (4.13) for the energy–momentum tensor we calculate \( T^\theta_t \) and \( T^t_\theta \). We simplify the resulting expressions using the third of the field equations (4.20) to obtain

\[
T^\theta_t = \frac{\kappa}{e^2 \sqrt{-g}} W \frac{dW}{d\rho} , \quad T^t_\theta = -\frac{\kappa}{e^2 \sqrt{-g}} P \frac{dP}{d\rho} .
\] (4.24)

Substituting these two equations into equation (3.8) and integrating yields respectively

\[
\sqrt{-g} \chi^\theta_t = \frac{8\pi G\kappa}{e^2} W^2 + C , \quad \sqrt{-g} \chi^t_\theta = -\frac{8\pi G\kappa}{e^2} P^2 + D ,
\] (4.25)

where \( C \) and \( D \) are constants of integration. We now examine the equation for \( \chi^\theta_t \)
as $\rho \to \infty$. Using the metric and boundary conditions near infinity we find that

$$\sqrt{-g} \chi_t = \frac{8\pi G \kappa}{e^2} W^2.$$  \hfill (4.26)

Moreover, investigating the equation for $\chi_t \theta$ near the origin yields

$$\sqrt{-g} \chi_t \theta = -\frac{8\pi G \kappa}{e^2} (P^2 - n^2).$$  \hfill (4.27)

Hence, using the expressions for $J$ and $\omega$ given by equations (3.12) we obtain

$$J = -\frac{\pi \kappa (n + 2s)^2}{e^2}, \quad \omega = \frac{\pi Ge^2}{\kappa \eta^4}.$$  \hfill (4.28)

In general, $\chi_t = g^{-1} \frac{d\phi_s}{d\rho}$. Thus combining equations (4.23) and (4.26) yields

$$\beta = -\frac{\pi Ge^4}{8\kappa^2},$$  \hfill (4.29)

and so the potential is given by

$$V(\Phi) = \frac{e^4}{8\kappa^2} (\Phi^2 - \eta^2)^2 \left[ \Phi^2 - \pi G (\Phi^2 - \eta^2)^2 \right].$$  \hfill (4.30)

Without this specific form for $V(\Phi)$ one cannot obtain a set of Bogomol’nyi equations, which justifies the ansatz (4.10) for the potential. We note that, as anticipated in the introduction, this eighth–order potential is non–renormalisable and furthermore, the self–dual solutions with $\lim_{\rho \to \infty} \Phi^2 = \eta^2$ are locally stable and the potential is unbounded below as $\Phi \to \infty$. As in the Abelian case discussed in [12], one of the principal differences between this theory and the flat space theory discussed in [11] is that here the required potential is of eighth–order, whereas in flat space a sixth–order potential was required to obtain a set of Bogomol’nyi equations. Another distinguishing feature of the model considered here is the lack of non–topological solutions. In flat space it was demonstrated in [6, 7] that there exist
solutions that asymptotically approach the symmetric vacuum of the potential. These solutions are known as non-topological solutions since the topology of the symmetric vacuum is trivial. For the model under consideration here there do not exist any non-topological solutions since there no longer exists a symmetric vacuum, due to the presence of the eighth-order term in the potential. Note that there exists the possibility of a solution corresponding to the minimum of the potential at $\Phi^2 = 0$, but this is not of the type of solutions considered here.

Note that (using equation (4.8))

$$T^t_{\ t} - T^\theta_{\ \theta} = \frac{R^2}{g} \left( W^2 g_{\theta\theta} - P^2 \right) = -\frac{\epsilon \kappa}{e^2 \sqrt{-g}} \left( W \frac{dP}{d\rho} + P \frac{dW}{d\rho} \right),$$

(4.31)

where we have used equations (4.20) to obtain the second equality. Therefore, equation (3.8) yields (upon integrating)

$$\frac{1}{\sqrt{-g}} \frac{dg_{\theta\theta}}{d\rho} = -\frac{16\pi G \epsilon \kappa}{e^2} PW + E, \quad E \in \mathbb{R}.$$

(4.32)

Near $\rho = 0$, $g_{\theta\theta} \sim -\rho^2$. Consequently, it is straightforward to see that

$$\frac{1}{\sqrt{-g}} \frac{dg_{\theta\theta}}{d\rho} = -\epsilon 8\pi G \left[ P \left( R^2 - \eta^2 \right) + (n + 2s) \eta^2 \right] - 2.$$

(4.33)

As $\rho \to \infty$, $g_{\theta\theta} \sim -B^2 \rho^2$. Thus, using equation (4.33) (as $\rho \to \infty$) we obtain

$$B = 1 + \epsilon 4\pi G (n + 2s) \eta^2.$$

(4.34)

Since, as was stated in Section 3, $B = 1 - 4GM$, we find that

$$M = -\epsilon \pi (n + 2s) \eta^2.$$

(4.35)

There remains one component of Einstein’s equation to examine, namely equa-
tion (3.9). Using equations (4.21), (4.22) and (4.30) we see that

\[ T^{\rho}_{\rho} = -\frac{2\pi G \kappa^2}{e^4} W^4. \]  

(4.36)

Furthermore,

\[ \det \chi_{ij} = \chi^{\theta \phi} \chi^t_t - \chi^t_{\theta} \chi^\theta_t = -\frac{1}{g} \left( \frac{dg_{t\theta}}{d\rho} \right)^2, \]

(4.37)

where we have used equation (3.6) to obtain the second equality. Hence, using equations (4.23) we see that

\[ \det \chi_{ij} = \frac{64\pi^2 G^2 \kappa^2}{e^4} W^4, \]

(4.38)

and so equation (3.9) is satisfied.

To summarise, we have demonstrated that the Einstein and field equations can be reduced to a set of first-order Bogomol’nyi equations given by

\[
\begin{align*}
\frac{dP}{d\rho} &= \pm \frac{e^4}{2\kappa^2 \sqrt{-g}} R^2 (R^2 - \eta^2) g_{\theta\theta} - \frac{e^2}{\kappa \sqrt{-g}} P R g_{t\theta} \\
\frac{dR}{d\rho} &= \frac{e^2}{2\kappa \sqrt{-g}} R (R^2 - \eta^2) g_{t\theta} \mp \frac{1}{\sqrt{-g}} P R \\
\frac{1}{\sqrt{-g}} \frac{dg_{t\theta}}{d\rho} &= -\frac{2e^2 \pi G}{\kappa} (R^2 - \eta^2)^2 \\
\frac{1}{\sqrt{-g}} \frac{dg_{\theta\theta}}{d\rho} &= \mp 8\pi G \left[ P (R^2 - \eta^2) + (n + 2s) \eta^2 \right] - 2.
\end{align*}
\]  

(4.39)

Solutions to these Bogomol’nyi equations have spin and mass given by

\[ J = -\frac{\pi \kappa (n + 2s)^2}{e^2}, \quad M = \mp \pi \eta^2 (n + 2s), \]

(4.40)

where we choose the upper (lower) sign for \((n + 2s)\) negative (positive) in order to obtain a positive value for the mass, \(M\). We believe that solutions to these
Bogomol’nyi equations exist and furthermore, guided by the flat space case [8, 11], we imagine that these solutions depend upon a single constant parameter that will be determined by requiring the correct behaviour of the fields and metric components as $\rho \to \infty$.

Note that, as claimed in the introduction, in contrast to the Abelian case [12] neither the angular momentum nor the mass are topologically invariant. For each topological class (labelled by $n$) one can obtain different values of $J$ and $M$ corresponding to different choices of $s$. This is a manifestation of the fact that the topological charge of $SU(N)$ vortices is defined modulo $N$, whereas physical quantities may depend upon the actual value of the magnetic flux associated with this topological charge [10]. This should not be overly surprising since although two solutions belonging to the same homotopy class are gauge equivalent at infinity, the gauge transformations connecting solutions with different $s$ (but the same $n$) cannot be well defined over all of the spacetime. Hence it is to be expected that $J$ and $M$ differ according to the value of $s$. The most stable vortex solutions are given by $n = 1$, $s = 0$ and $n = 1$, $s = -1$. Furthermore, due to the quantisation of the Chern–Simons coupling constant, $\kappa$ given by equation (2.7) we can express $J$ in the form

$$J = \frac{m(n + 2s)^2}{4},$$

where both $m$ and $s$ are integers. Thus, the angular momentum is quantised. Note that this result (for $s = 0$ and $s = -1$) agrees with the value for the angular momentum in flat space given in [8, 11].
5. Conclusions

In this article we have shown that, by considering an ansatz for a cylindrically symmetric vortex solution, both the field equations and the Einstein equations can be reduced to a set of four, first–order Bogomol’nyi equations. In the flat space limit \((G = 0, \ g_{\theta \theta} = 0)\) these Bogomol’nyi equations reduce to the flat space Bogomol’nyi equations given in [11]. In order to obtain this set of equations it was necessary to choose an eighth–order, non–renormalisable potential. Furthermore, we have demonstrated that both the mass and the angular momentum of the vortex solutions are not topologically invariant. It would be interesting to numerically search for solutions to the Bogomol’nyi equations (4.39). Hopefully, this would justify the claims concerning the solutions made above. Another open question is whether or not these vortex solutions are stable. One way of investigating this problem is to embed this theory in a supergravity theory. This would yield a set of supersymmetry transformations that could be used to define a supercovariant derivative. One could then perform a Witten–like positive energy proof to obtain a Bogomol’nyi bound on the energy. We imagine that solutions to the Bogomol’nyi equations above would saturate this bound and hence be stable. In principle this appears to be possible, but the construction of an appropriate supergravity theory and the resulting supersymmetry transformations is not immediately evident. This will be discussed in a future work.

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