STRONG CLOSED RANGE ESTIMATES: NECESSARY CONDITIONS AND APPLICATIONS

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Abstract. The $L^2$ theory of the $\bar{\partial}$ operator on domains in $\mathbb{C}^n$ is predicated on establishing a good basic estimate. Typically, one proves not a single basic estimate but a family of basic estimates that we call a family of strong closed range estimates. Using this family of estimates on $(0,q)$-forms as our starting point, we establish necessary geometric and potential theoretic conditions.

The paper concludes with several applications. We investigate the consequences for compactness estimates for the $\bar{\partial}$-Neumann problem, and we also establish a generalization of Kohn’s weighted theory via elliptic regularization. Since our domains are not necessarily pseudoconvex, we must take extra care with the regularization.

1. Introduction

Since Hörmander’s pivotal work on the $L^2$-theory of the $\bar{\partial}$-problem [14], there has been a tremendous effort to characterize the regularity properties of the $\bar{\partial}$-Neumann operator in terms of estimates, geometry, and potential theory. It has been known since the 1960s that pseudoconvexity is both necessary and sufficient for the range of the $\bar{\partial}$-operator to be closed at every form level $1 \leq q \leq n$ and for the absence of nontrivial harmonic forms at every form level $1 \leq q \leq n$ [14, 1]. The primary tool that analysts use to prove closed range and other related properties is to establish an appropriate basic estimate or family of basic estimates. For example, the basic estimate

\begin{equation}
    c_\varphi \|f\|_{L^2(\Omega, \varphi)}^2 \leq (\|\bar{\partial} f\|_{L^2(\Omega, \varphi)}^2 + \|\bar{\partial}^{*} f\|_{L^2(\Omega, \varphi)}^2) + C_{\varphi} \|f\|_{W^{-1}(\Omega, \varphi)}^2
\end{equation}

for all $f \in L^2_{0,q}(\Omega, \varphi) \cap \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*)$ suffices to show that the space of harmonic forms is finite dimensional and $\bar{\partial}$ has closed range in $L^2_{0,q}(\Omega)$ and $L^2_{0,q+1}(\Omega)$. It turns out that in every case where (1.1) is known to hold, we can actually prove a family of estimates — namely, instead of (1.1) holding for a single function $\varphi$, we have (1.1) for every $\varphi = t\phi$ where $t$ is sufficiently large and $\phi$ is some fixed function and (typically) $c_{t\phi} = tC_{\phi}$ and $C_{t\phi} \leq O(t^2)$. This is an example of what we call a family of strong closed range estimates.

In this paper, we take strong closed range estimates as our starting point and explore the consequences for a domain admitting such a family. Our main result establishes a certain quantitative condition on the number of nonnegative/nonpositive eigenvalues of the Levi form (a geometric condition) as well as the number of positive/negative eigenvalues of the complex Hessian of the weight function restricted to
$T^{1,0}(\partial \Omega) \times T^{0,1}_p(\partial \Omega)$ (a potential theoretic condition). Our main result has an application to compactness estimates for the $\bar{\partial}$-problem, and the existence of a family of strong closed range estimates allows us to establish a generalization of Kohn’s weighted theory for solving the weighted $\bar{\partial}$-Neumann operator in $L^2$ Sobolev spaces. To prove this extension on non-pseudoconvex domains, we need to account for the possibility of non-trivial harmonic forms and the boundary condition induced by Dom($\partial^*\rho$) – typically, the weighted theory has only one of these issues (non-trivial harmonic forms for $\bar{\partial}$ on CR manifolds without boundary, and the boundary condition induced by Dom($\partial^*\rho$) for $\partial$ on pseudoconvex domains in $\mathbb{C}^n$), and we are particularly careful to avoid pitfalls that sometimes appear in the literature.

Surprisingly, since Hörmander’s work, the only results regarding closed range of $\bar{\partial}$ for $(0,q)$-forms on not-necessarily pseudoconvex domains have been to establish sufficient conditions and until the present work, none have attempted to find necessary conditions. For example there are several papers on the annulus or annular regions between two pseudoconvex domains $[19, 21, 15]$, and we have investigated very general sufficient conditions for both $\partial$ and $\partial_b$ to have closed range. In fact, in the language of this paper, we prove the existence of a family of strong closed range estimates and also establish a generalization of Kohn’s weighted theory $[6, 9, 10, 12, 11, 5]$.

The format of the paper is the following: we state the Main Results at the end of this section, define our notation and operators in Section 2, prove the main theorem regarding strong closed range estimates in Section 4, and present the applications in Section 5.

1.1. Statements of the Main Results.

**Theorem 1.1.** Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with $C^2$ boundary admitting a family of strong closed range estimates for some $1 \leq q \leq n - 1$ and some weight function $\varphi \in C^2(\overline{\Omega})$, as in Definition 2.1 below. Then for each connected component $S$ of $\partial \Omega$, one of the following two cases holds:

1. For every $z \in S$, the Levi form for $\partial \Omega$ has at least $n - q$ nonnegative eigenvalues and the restriction of $i\partial \bar{\partial} \varphi$ to $T^{1,0}_z(\partial \Omega) \times T^{0,1}_z(\partial \Omega)$ has at least $n - q$ positive eigenvalues bounded below by $C_\varphi \frac{z}{q}$.

2. For every $z \in S$, the Levi form for $\partial \Omega$ has at least $q + 1$ nonpositive eigenvalues and the restriction of $i\partial \bar{\partial} \varphi$ to $T^{1,0}_z(\partial \Omega) \times T^{0,1}_z(\partial \Omega)$ has at least $q + 1$ negative eigenvalues bounded above by $-\frac{C_\varphi \frac{z}{n-q}}{q}$.

If $\Omega$ admits a family of strong closed range estimates near some $p \in \partial \Omega$, then either (1) or (2) holds at $z = p$.

**Remark 1.2.** We have stated our result in a form which avoids technical details about the relationship between the Levi form and the complex hessian of $\varphi$, but we actually prove a much stronger statement. Let $\rho$ be a defining function for $\Omega$ normalized so that $|\nabla \rho| = 1$ on $\partial \Omega$. For a constant $s \geq 0$ and $z \in \partial \Omega$, let $\mathcal{L}_s(z)$ denote the linear combination $i\partial \bar{\partial} \varphi + s\partial \bar{\partial} \varphi$ restricted to $T^{1,0}_z(\partial \Omega) \times T^{0,1}_z(\partial \Omega)$, and let $\{\lambda_1^s(z), \ldots, \lambda_{n-1}^s(z)\}$ denote the eigenvalues of $\mathcal{L}_s(z)$ in nondecreasing order.

Then for each connected component $S$ of $\partial \Omega$, we either have $\lambda_1^s(z) \geq \frac{sC_\varphi}{q}$ for all $z \in S$ and $s \geq 0$ or we have $\lambda_{q+1}^s(z) \leq -\frac{sC_\varphi}{n-q}$ for all $z \in S$ and $s \geq 0$. If, for example, we are in the first case but $\lambda_0^q(z) = 0$ for some $z$, then this more refined
result can be used to deduce information about the restriction of $i\partial\bar{\partial}\varphi$ to the kernel of the Levi form at $z$.

**Corollary 1.3.** Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with $C^2$ boundary, and write $\Omega = \Omega_0 \cup \bigcup_{j \in J} \overline{\Omega}_j$ where $\Omega_0$ is a bounded domain with connected $C^2$ boundary and $\{\Omega_j\}_{j \in J}$ is a collection of domains with connected $C^2$ boundaries that are relatively compact in $\Omega$ such that $\{\overline{\Omega}_j\}_{j \in J}$ is disjoint. If $\Omega$ admits a family of strong closed range estimates for $q = 1$, then $\Omega_0$ is pseudoconvex and the restriction of $i\partial\bar{\partial}\varphi$ to $T_p^{1,0}(\partial\Omega_0) \times T_p^{0,1}(\partial\Omega_0)$ is positive definite. If $\Omega$ admits a family of strong closed range estimates for $q = n - 2$, then each $\Omega_j$ is pseudoconvex and the restriction of $i\partial\bar{\partial}\varphi$ to $T_p^{1,0}(\partial\Omega_j) \times T_p^{0,1}(\partial\Omega_j)$ is negative definite. If $\Omega$ admits a family of strong closed range estimates for $q = n - 1$, then $\Omega = \Omega_0$.

We will see that Definition 2.1 involves a family of smooth, compactly supported functions $\{\chi_t\}$ satisfying the growth condition $\lim_{t \to \infty} \frac{\|\chi_t\|^2_{C^1(\Omega)}}{t^3} = 0$. This may seem to be a technical convenience, but in fact this distinguishes strong closed range estimates, which require a non-trivial weight function $\varphi$, from stronger families of estimates, which hold with no weight function. For example, we have

**Proposition 1.4.** Let $\Omega \subset \mathbb{C}^n$ be a domain with $C^2$ boundary such that for some $1 \leq q \leq n - 1$ and some $\eta > 0$, $\Omega$ admits a subelliptic estimate of the form

$$\|u\|^2_{W^q(\Omega)} \leq C \left( \|\partial u\|^2_{L^2(\Omega)} + \|\bar{\partial} u\|^2_{L^2(\Omega)} \right)$$

for all $u \in L^2_{(0,q)}(\Omega) \cap \text{Dom} \bar{\partial} \cap \text{Dom} \bar{\partial}^*$. Then $\Omega$ admits a family of estimates of the form (2.1) for $\varphi = 0$ and a family of smooth, compactly supported functions $\{\chi_t\}$ such that

$$0 < \limsup_{t \to \infty} \frac{\|\chi_t\|^2_{C^1(\Omega)}}{t^{(1+\eta)/\eta}} < \infty.$$

On strictly pseudoconvex domains, we have subelliptic estimates for $\eta = \frac{1}{2}$, and hence the family of cutoff functions $\{\chi_t\}$ given by Proposition 1.4 satisfies

$$0 < \limsup_{t \to \infty} \frac{\|\chi_t\|^2_{C^1(\Omega)}}{t^{3}} < \infty.$$  

This is the sense in which the growth condition in Definition 2.1 is sharp: if we relax this growth condition, then we have a large class of examples admitting a family of estimates of the form (2.1) such that the conclusions of Theorem 1.1 do not hold.

As an immediate consequence of our main theorem, we have the following application to the compactness theory for the $\bar{\partial}$-Neumann problem.

**Theorem 1.5.** Let $\Omega \subset \mathbb{C}^n$ be a domain with $C^2$ boundary. Suppose that $\Omega$ admits a family of compactness estimates for some $1 \leq q \leq n - 1$, as in Definition 2.4 below. If $C_\varepsilon$ denotes the constant in (2.2) below, then

$$\limsup_{\varepsilon \to 0^+} \varepsilon^2 C_\varepsilon > 0.$$  

Our second and final application is to establish the weighted $L^2$-theory for the $\bar{\partial}$-problem in the presence of a family of strong closed range estimates.
Theorem 1.6. Let $\Omega \subset \mathbb{C}^n$ be a smooth domain which admits the family of strong closed range estimates (3.2) for some smooth function $\varphi$. Then for every $k \geq 1$ there exists $T_k$ so that if $t \geq T_k$, the following operators are continuous for all $0 \leq s \leq k$:

i. The $\bar{\partial}$-Neumann operator

$$N^q_{t\varphi} : \mathcal{L}^{2,s}_{0,q}(\Omega, t\varphi) \to \mathcal{L}^{2,s}_{0,q}(\Omega, t\varphi);$$

ii. The weighted canonical solution operators for $\Bar{\partial}$ and $\bar{\partial}^*_{t\varphi}$:

$$\partial N^q_{t\varphi} : \mathcal{L}^{2,s}_{0,q}(\Omega, t\varphi) \to \mathcal{L}^{2,s}_{0,q}(\Omega, t\varphi);$$

$$\bar{\partial} N^q_{t\varphi} : \mathcal{L}^{2,s}_{0,q}(\Omega, t\varphi) \to \mathcal{L}^{2,s}_{0,q}(\Omega, t\varphi);$$

iii. The projections:

$$\bar{\partial} \bar{\partial}^* N^q_{t\varphi} : \mathcal{L}^{2,s}_{0,q}(\Omega, t\varphi) \to \mathcal{L}^{2,s}_{0,q}(\Omega, t\varphi),$$

$$\bar{\partial} N^q_{t\varphi} \bar{\partial} : \mathcal{L}^{2,s}_{0,q}(\Omega, t\varphi) \to \mathcal{L}^{2,s}_{0,q}(\Omega, t\varphi);$$

iv. The harmonic projection $H^q_{t\varphi} : \mathcal{L}^{2,s}_{0,q}(\Omega, t\varphi) \to \mathcal{L}^{2,s}_{0,q}(\Omega, t\varphi).$

Remark 1.7. Note that we also obtain estimates for the weighted Bergman projections $P^q_{t\varphi} = I - \bar{\partial}^*_{t\varphi} \partial N^q_{t\varphi}$ and $P^{q-1}_{t\varphi} = I - \bar{\partial}^*_{t\varphi} N^q_{t\varphi}$, as well as the combined projections $P_q^\# = I - \bar{\partial}^*_{t\varphi} N^q_{t\varphi}$ and $P^{q+1}_{t\varphi} = I - \bar{\partial}^*_{t\varphi} N^q_{t\varphi}$.

Remark 1.8. We can also obtain estimates for the projection $N^q_{t\varphi} \bar{\partial} \bar{\partial}^*_{t\varphi}$ (resp. $N^q_{t\varphi} \partial \partial^*_{t\varphi}$), but note that this is equal to the restriction of $\bar{\partial} \bar{\partial}^*_{t\varphi} N^q_{t\varphi}$ (resp. $\partial \partial^*_{t\varphi} N^q_{t\varphi}$) to the space of forms $u \in \text{Dom} \bar{\partial}^*_{t\varphi}$ such that $\bar{\partial}^*_{t\varphi} u \in \text{Dom} \bar{\partial}$ (resp. $u \in \text{Dom} \partial$ such that $\partial u \in \text{Dom} \bar{\partial}^*_{t\varphi}$). The argument in [6, (18)-(20)] proves this for the complex Green operator, but the argument is the same.

In many instances where we can establish a closed range estimate (e.g., [14, 20], [9, 2]), there is also sufficient information to prove that the space of harmonic $(0,q)$-forms $\mathcal{H}_{0,q}(\Omega) = \{0\}$ (the $q = n - 1$ case on the annulus being a notable exception, as $\bar{\partial}$ is closed but the space of harmonic forms is infinite dimensional [15]), hence the hypothesis in the next corollary is well-motivated.

Corollary 1.9. Let $\Omega \subset \mathbb{C}^n$ be a bounded smooth domain which admits the family of strong closed range estimates (3.2) for some smooth function $\varphi$. Then

1. $\mathcal{L}^{2,s}_{0,q}(\Omega) \cap \text{ker} \bar{\partial}$ is dense in $\mathcal{L}^{2,s}_{0,q}(\Omega) \cap \text{ker} \bar{\partial}$ for any $m > s \geq 0$.
2. If, in addition, $\mathcal{H}_{0,q}(\Omega) = \mathcal{H}_{0,q}(\Omega, t\varphi) = \{0\}$, then the $\bar{\partial}$-problem is solvable in $C_{0,q}(\Omega)$ if $\bar{\varphi} = q$ or $q - 1$. Namely, if $f \in C_{0,q+1}(\Omega)$ is $\bar{\partial}$-closed, then there exists $u \in C_{0,q}(\Omega)$ so that $\bar{\partial} u = f$. 

2. Notation

2.1. $L^2$ spaces. Let $\Omega \subset \mathbb{C}^n$ be a bounded, $C^m$ domain with $C^m$ defining function $\rho$, $m \geq 2$. Let $\varphi$ be a $C^2$ function defined near the closure of $\Omega$. We denote the $L^2$-inner product on $L^2(\Omega, e^{-\varphi})$ by

$$(f, g)_{L^2(\Omega, \varphi)} = \int_{\Omega} f \bar{g} e^{-\varphi} dV.$$

We denote the induced surface measure on $\partial \Omega$ by $d\sigma$. Also $\|f\|^2_{L^2(\Omega, \varphi)} = \int_{\Omega} |f|^2 e^{-\varphi} dV$ and if $\varphi = 0$, we suppress the $\varphi$ in the norm.

2.2. The $\bar{\partial}$ operator. Let $I_q = \{(i_1, \ldots, i_q) \in \mathbb{N}^q : 1 \leq i_1 < \cdots < i_q \leq n\}$. For $I \in I_{q-1}$, $J \in I_q$, and $1 \leq j \leq n$, let $\epsilon(I, J) = (-1)^{|\sigma|}$ if $\{j\} \cup I = J$ as sets and $|\sigma|$ is the length of the permutation that takes $\{j\} \cup I$ to $J$. Set $\epsilon(I, J) = 0$ otherwise. We use the standard notation that if $u = \sum_{J \in I_q} u_J d\bar{z}_J$, then

$$u_{J,k} = \sum_{J \in I_q} \epsilon(I, J) u_J.$$

The $\bar{\partial}$-operator on $(0,q)$-forms is defined as follows: $\bar{\partial} : L^2_{0,q}(\Omega, e^{-|z|^2}) \to L^2_{0,q+1}(\Omega, e^{-|z|^2})$ and if $f = \sum_{J \in I_q} f_J d\bar{z}_J$, then

$$\bar{\partial} f = \sum_{J \in I_q} \sum_{k=1}^n \epsilon^{k,J} \frac{\partial f_J}{\partial \bar{z}_k} d\bar{z}_k.$$

We let $\bar{\partial}_\varphi$ denote the $L^2$-adjoint of $\bar{\partial}$ in $L^2_{0,q}(\Omega, \varphi)$ and denote the weighted $\bar{\partial}$-Neumann Laplacian by $\square_\varphi = \bar{\partial} \bar{\partial}_\varphi + \bar{\partial}_\varphi \bar{\partial}$. If it exists, the inverse to $\square_\varphi$ on $(0,q)$-forms on the orthogonal complement to ker $\square_\varphi$ is called the $\bar{\partial}$-Neumann operator and is denoted by $N_{\varphi}^q$.

We use the notation $H_{0,q}(\Omega, \varphi)$ for the space of $L^2_{0,q}(\Omega, \varphi)$-harmonic forms, that is, $H_{0,q}(\Omega, \varphi) = \ker(\bar{\partial}) \cap \ker(\bar{\partial}_\varphi^*)$. We also let $H^\sharp_{\varphi} : L^2_{0,q}(\Omega, \varphi) \to H_{0,q}(\Omega, \varphi)$ denote the orthogonal projection.

2.3. CR geometry. The induced CR-structure on $\partial \Omega$ at $z \in \partial \Omega$ is

$$T^{0,1}_z(\partial \Omega) = \{ L \in T^{1,0}(\mathbb{C}) : \partial \rho(L) = 0 \},$$

where $\rho$ is an arbitrary $C^1$ defining function for $\Omega$. We denote the exterior algebra generated by these spaces by $T^{p,q}(\partial \Omega)$ and its dual by $\Lambda^{p,q}(\partial \Omega)$. If we normalize $\rho$ so that $|d\rho| = 1$ on $\partial \Omega$, then the normalized Levi form $\mathcal{L}$ is the real element of $\Lambda^{1,1}(\partial \Omega)$ defined by

$$\mathcal{L}(-iL \wedge \bar{L}) = i\partial \partial \rho(-iL \wedge \bar{L})$$

for any $L \in T^{1,0}(\partial \Omega)$.

In the case that $U$ is a small neighborhood of (say) 0, and we write $\Omega \cap U$

$$\Omega \cap U = \{(z', x_n + iy_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : y_n > \rho_1(z', x_n)\},$$

where $\rho_1$ is a $C^2$ function satisfying $\rho_1(0) = 0$ and $\nabla \rho_1(0) = 0$, then we can identify the normalized Levi form at 0 with the $(n-1) \times (n-1)$ matrix $\left(\frac{\partial^2 \rho_{1}}{\partial z_i \partial \bar{z}_k}(0)\right)$ (see (4.1) below).
2.4. *L² Sobolev spaces.* We define a Sobolev *W¹* norm that is adapted to the theory for the weighted ∂-Neumann operator. For *f* ∈ *C∞₀(Ω)* we define

\[ \| f \|_{W¹(Ω,ϕ)}² = \| f \|_{L²(Ω,ϕ)}² + \sum_{j=1}^{n} \left| \frac{\partial f}{\partial z_j} \right|_{L²(Ω,ϕ)}² + \sum_{j=1}^{n} \left| e^{ϕ} \frac{\partial}{\partial z_j} (e^{-ϕ} f) \right|_{L²(Ω,ϕ)}². \]

As usual, we define *W¹₀(Ω,ϕ)* to be the completion of *C∞₀(Ω)* with respect to this norm. Note that if we integrate by parts in the *L²*(Ω,ϕ) norm, we obtain the adjoint relation

\[ \left( \frac{\partial}{\partial z_j} \right)^* ϕ = -e^{ϕ} \frac{\partial}{\partial z_j} (e^{-ϕ}) = -\frac{\partial}{\partial z_j} + \frac{\partial ϕ}{\partial z_j}. \]

This motivates the decomposition used in our definition of *W¹(Ω,ϕ).* On bounded domains (or, more generally, domains on which *ϕ* and ∇*ϕ* are uniformly bounded), *W¹₀(Ω) = W¹₀(Ω,ϕ).* On unbounded domains, the theory for such norms has been studied extensively in [8] and [13], for example. We now define *W⁻¹(Ω,ϕ)* to be the dual of *W¹₀(Ω,ϕ)* with respect to *L²(Ω,ϕ).*

We let *L²,*k*(Ω,ϕ)* denote the usual weighted *L²*-Sobolev spaces, namely,

\[ \| f \|_{L²,*k(Ω,ϕ)}² = \sum_{|α|≤k} \| D^α ϕ \|_{L²(Ω,ϕ)}². \]

It is the case that *W¹(Ω,ϕ) = L²¹(Ω,ϕ).* It is convenient to use *L²,*k*(Ω,ϕ)* in the elliptic regularization and hence in the proof of Theorem 5.2, however we prefer to use *W¹(Ω,ϕ)* in Lemma 3.1 because it produces the most refined estimates.

2.5. *Estimates for the ∂-operator.*

**Definition 2.1.** Let Ω ⊂ ℂⁿ be a domain with *C²* boundary. We say that Ω admits a family of strong closed range estimates for some 1 ≤ *q* ≤ *n* − 1 if there exists a weight function *ϕ* ∈ *C²*(Ω) and constants *C_q* > 0 and *t₀* > 0 such that for every *t* ≥ *t₀* there exists a cutoff function *χ_t* ∈ *C∞₀*(Ω) such that

\[ \lim_{t→∞} \frac{∥χ_t∥_{C∞₀(Ω)}∥t^q∥}{t} = 0 \]

and

\[ \| \overline{∂} f \|_{L²(Ω,tϕ)}² + \| t∂_{tϕ} f \|_{L²(Ω,tϕ)}² + \| χ_t f \|_{L²(Ω,tϕ)}² ≤ tC_q^′ \| f \|_{L²(Ω,tϕ)}² \]

for all *f* ∈ *L²*(Ω) ∩ Dom ∂ ∩ Dom ∂ₚ. For *p* ∈ ∂Ω, we say that Ω admits a family of strong closed range estimates near *p* if, in addition to the above, there exists a family of open neighborhoods *U_t* of *p* such that

\[ \lim_{t→∞} t \sup_{z∈U_t} |z − p|^q = ∞ \]

and (2.1) holds for all *f* ∈ *L²*(Ω) ∩ Dom ∂ ∩ Dom ∂ₚ, supported in *U_t ∩ Ω*.

**Remark 2.2.** We could also define a family of strong closed range estimates for (p,q)-forms with 1 ≤ *p* ≤ *n*, but the presence of *p* > 0 does not impact the theory in any way, so we omit this case.

Closed range, in general, is not a local property. However, we note that strong closed range estimates localize in the following sense:

**Lemma 2.3.** Let Ω ⊂ ℂⁿ be a bounded domain with *C²* boundary. For some 1 ≤ *q* ≤ *n* − 1, Ω admits a family of strong closed range estimates if and only if Ω admits a family of strong closed range estimates for every *p* ∈ ∂Ω.
Proof. To see that global estimates imply local estimates, we simply let \( U_t \) be a neighborhood of \( \Omega \) that is independent of \( t \). For the converse, let \( \psi \in C^\infty(\mathbb{R}) \) be a non-decreasing function such that \( \psi(x) = 0 \) for all \( x \leq 0 \) and \( \psi(x) = 1 \) for all \( x \geq 1 \). For \( r > 0 \) and \( p \in \mathbb{C}^n \), set \( \tilde{\xi}(z) = \psi(\frac{2(|z^2|-|p|^2)}{r^2}) \). Then \( \text{supp} \tilde{\xi} = B(p, r) \), \( \tilde{\xi} \equiv 1 \) in a neighborhood of \( p \), and \( |\nabla \tilde{\xi}| \leq O(r^{-1}) \).

Cover \( \partial \Omega \) with a finite collection of neighborhoods \( U_{j,t} \) satisfying the local definition of strong closed range estimates with cutoff functions \( \chi_{j,t} \). We may assume that \( U_{j,t} = B(p_{j,t}, r_{j,t}) \) where \( r_{j,t}^{-2} \leq o(t) \). If we let \( \bar{\xi}_{j,t} \) denote the cutoff function defined in the previous paragraph for \( B(p_{j,t}, r_{j,t}) \), then \( \xi_{j,t} = \frac{\bar{\xi}_{j,t}}{\sum_j \bar{\xi}_{j,t}} \) defines a partition of unity in some neighborhood of \( \partial \Omega \) satisfying \( |\nabla \xi_{j,t}| \leq o(t) \). Hence, if we set \( f_{j,t} = f \xi_{j,t} \), then

\[
|\partial f_{j,t}|^2 \leq 2 |\xi_{j,t} \partial f|^2 + o(t |f|^2).
\]

Thus, we may decompose \( f = \sum_j f_j \), apply (2.1) to each \( f_j \), and patch the resulting estimates with error terms that can be absorbed by taking \( t \) sufficiently large. We complete the partition of unity of \( \bar{\Omega} \) using \( \xi_{0,t} = 1 - \sum_j \xi_{j,t} \), and note that we can choose \( \chi_t \) to be a constant multiple of \( \sqrt{\sum_j \chi_{j,t}^2 \xi_{j,t}^2 + tC_q \xi_0^2} \).

**Definition 2.4.** Let \( \Omega \subset \mathbb{C}^n \). We say that \( \Omega \) admits a compactness estimate for some \( 1 \leq q \leq n \) if for every \( \varepsilon > 0 \) there exists a constant \( C_\varepsilon > 0 \) such that

\[
\varepsilon ( \| \partial f \|^2_{L^2(\Omega)} + \| \partial^* f \|^2_{L^2(\Omega)} ) + C_\varepsilon \| f \|^2_{W^{-1}(\Omega)} \geq \| f \|^2_{L^q(\Omega)}
\]

for all \( f \in L^q(\Omega) \cap \text{Dom} \partial \cap \text{Dom} \partial^* \).

We call this a compactness estimates because (2.2) is equivalent to compactness of the \( \partial \)-Neumann operator (see Proposition 4.2 in [22]).

3. **Sufficient Conditions for Strong Closed Range Estimates**

In many settings, it is more natural to replace the term \( \| \chi_t f \|^2_{L^2(\Omega, \tau)} \) with a large multiple of the Sobolev norm \( \| f \|^2_{W^{-1}(\Omega, \tau)} \). The families of estimates in Lemma 3.1 are all candidates for our definition of strong closed range estimates; this lemma shows that the family we have chosen ((4) in Lemma 3.1) is a priori the weakest.

**Lemma 3.1.** Let \( \Omega \subset \mathbb{C}^n \) be a bounded domain with Lipschitz boundary and Lipschitz defining function \( \rho \). Let \( 1 \leq q \leq n - 1 \) and \( \varphi \in C^2(\overline{\Omega}) \). For the following families of estimates, we have (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) \( \Rightarrow \) (4) and (4) \( \Rightarrow \) (3).

1. There exist \( C_q > 0 \) and \( t_0 > 0 \) such that for every \( t \geq t_0 \) there exists a cutoff function \( \chi_t \in C_0^\infty(\Omega) \) such that \( \lim_{t \to \infty} \frac{\| \chi_t \|^2_{L^2(\Omega)}}{t^\alpha} = 0 \) and

\[
\| \partial f \|^2_{L^2(\Omega, \tau, \varphi)} + \| \partial^* f \|^2_{L^2(\Omega, \tau, \varphi)} + \chi_t f \|^2_{L^2(\Omega, \tau)} \geq tC_q \| f \|^2_{L^2(\Omega, \tau)}
\]

for all \( f \in L^q(\Omega) \cap \text{Dom} \partial \cap \text{Dom} \partial^* \).

2. There exist \( C_q > 0 \) and \( t_0 > 0 \) such that for every \( t \geq t_0 \) there exists a constant \( C_t > 0 \) satisfying \( \lim_{t \to \infty} \frac{C_t}{t} \varphi \) = 0 and

\[
\| \partial f \|^2_{L^2(\Omega, \tau, \varphi)} + \| \partial^* f \|^2_{L^2(\Omega, \tau, \varphi)} + C_t \| f \|^2_{W^{-1}(\Omega, \tau, \varphi)} \geq tC_q \| f \|^2_{L^2(\Omega, \tau, \varphi)}
\]

for all \( f \in L^q(\Omega) \cap \text{Dom} \partial \cap \text{Dom} \partial^* \).
Remark 3.2. A careful analysis of the proof reveals that the condition on \( \chi_t \) in (1) can be relaxed to

\[
\lim_{t \to \infty} \frac{\| \chi_t \|^4_{L^\infty(\Omega)}}{t^3} = 0, \quad \sup_{\{t \geq t_0\}} t \| \nabla \chi_t \|^2_{L^\infty(\Omega)} < \infty,
\]

and the condition on \( \chi_t \) in (4) can be relaxed to

\[
\lim_{t \to \infty} \frac{\| \chi_t \|^4_{L^\infty(\Omega)}}{t^3} = 0, \quad \sup_{\{t \geq t_0\}} t \| \nabla \chi_t \|^2_{L^\infty(\Omega)} < \infty.
\]

This requires replacing \( \| \chi_t \|^4_{C^1(\Omega)} \) in (3.5) with \( \| \chi_t \|^4_{L^\infty(\Omega)} \).

Proof. To see that (1) implies (2), we will need to use the interior regularity for the \( \partial \)-Neumann problem. Let \( f \in C^1_{(0,q)}(\Omega) \) \( \cap \) \( \text{Dom} \bar{\partial} \cap \text{Dom} \bar{\partial} \ast \). By definition,

\[
\| \chi_t f \|^2_{L^2(\Omega,\bar{\partial} \varphi)} \leq \| \chi_t^2 f \|^2_{W^1(\Omega,\bar{\partial} \varphi)} \| f \|^2_{W^{-1}(\Omega,\bar{\partial} \varphi)}.
\]

For \( \varepsilon > 0 \) to be chosen later, a small constant/large constant estimate gives us

\[
(3.5) \quad \| \chi_t f \|^2_{L^2(\Omega,\bar{\partial} \varphi)} \leq \frac{\varepsilon}{2} \frac{\| \chi_t \|^4_{C^1(\Omega)}}{\| \chi_t \|^4_{C^1(\Omega)}} \| \chi_t^2 f \|^2_{W^1(\Omega,\bar{\partial} \varphi)} + \frac{\| \chi_t \|^4_{C^1(\Omega)}}{2\varepsilon} \| f \|^2_{W^{-1}(\Omega,\bar{\partial} \varphi)}.
\]

To estimate \( \| \chi_t^2 f \|^2_{W^1(\Omega,\bar{\partial} \varphi)} \), we observe that integration by parts gives us

\[
\left( \frac{\partial}{\partial \bar{z}_j} \right) \ast (\chi_t^2 f)_{L^2(\Omega,\bar{\partial} \varphi)} = \int_{\Omega} \left( \frac{\partial}{\partial \bar{z}_j} \left( \frac{\partial}{\partial \bar{z}_j} \right) \ast (\chi_t^2 f), \chi_t^2 f \right) e^{-t\varphi} dV.
\]

Since

\[
\left[ \frac{\partial}{\partial \bar{z}_j} \left( \frac{\partial}{\partial \bar{z}_j} \right) \ast (\chi_t^2 f) \right] = t \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_j} \chi_t^2 f,
\]

a second integration by parts will give us

\[
\left( \frac{\partial}{\partial \bar{z}_j} \right) \ast (\chi_t^2 f)_{L^2(\Omega,\bar{\partial} \varphi)} = \left( \frac{\partial}{\partial \bar{z}_j} \chi_t^2 f \right)_{L^2(\Omega,\bar{\partial} \varphi)} + t \int_{\Omega} \left( \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_j} \chi_t^2 f, \chi_t^2 f \right) e^{-t\varphi} dV.
\]
Hence,
\[
\|\chi_\sigma^2 f\|_{W^1(\Omega, t\varphi)}^2 = \|\chi_\sigma^2 f\|_{L^2(\Omega, t\varphi)}^2 + 2 \sum_{j=1}^n \left\| \frac{\partial}{\partial z_j} (\chi_\sigma^2 f) \right\|_{L^2(\Omega, t\varphi)}^2 + t \int_{\Omega} \sum_{j=1}^n \frac{\partial^2 \varphi}{\partial z_j^2} \chi_\sigma^2 f, \chi_\sigma^2 f \right\| e^{-t\varphi} dV.
\]
Since \(\chi_\sigma^2 f\) is compactly supported, we can use the Morrey-Kohn-Hörmander identity (see Proposition 4.3.1 in [3], for example) with no boundary term to show
\[
\|\chi_\sigma^2 f\|_{W^1(\Omega, t\varphi)}^2 \leq 2 \left( \|\tilde{\partial}(\chi_\sigma^2 f)\|_{L^2(\Omega, t\varphi)}^2 + \|\tilde{\partial}_t^*\varphi(\chi_\sigma^2 f)\|_{L^2(\Omega, t\varphi)}^2 \right) + O \left( (1 + t \|\varphi\|_{C^2(\Omega)}) \|\chi_\sigma^2 f\|_{L^2(\Omega, t\varphi)}^2 \right).
\]
Calculating \(\tilde{\partial}(\chi_\sigma^2 f)\) and \(\tilde{\partial}_t^*\varphi(\chi_\sigma^2 f)\) and using the inequality \((a + b)^2 \leq 2(a^2 + b^2)\) yields the inequality
\[
\|\chi_\sigma^2 f\|_{W^1(\Omega, t\varphi)}^2 \leq 4 \left( \|\chi_\sigma^2 \tilde{\partial} f\|_{L^2(\Omega, t\varphi)}^2 + \|\chi_\sigma^2 \tilde{\partial}_t^* f\|_{L^2(\Omega, t\varphi)}^2 \right) + O \left( \|\nabla \chi_\sigma\|_{L^{\infty}(\Omega)} \|\chi_\sigma f\|_{L^2(\Omega, t\varphi)} + (1 + t \|\varphi\|_{C^2(\Omega)}) \|\chi_\sigma^2 f\|_{L^2(\Omega, t\varphi)} \right),
\]
or
\[
\|\chi_\sigma^2 f\|_{W^1(\Omega, t\varphi)}^2 \leq 4 \|\chi_\sigma^4 \|_{L^{\infty}(\Omega)} \left( \|\tilde{\partial} f\|_{L^2(\Omega, t\varphi)}^2 + \|\tilde{\partial}_t^* f\|_{L^2(\Omega, t\varphi)}^2 \right) + O \left( \|\nabla \chi_\sigma\|_{L^{\infty}(\Omega)} \|\chi_\sigma^2 f\|_{L^2(\Omega, t\varphi)} + (1 + t \|\varphi\|_{C^2(\Omega)}) \|\chi_\sigma^2 f\|_{L^2(\Omega, t\varphi)} \right) f_{L^2(\Omega, t\varphi)}^2.
\]
Substituting this into (3.5) and repeatedly using \(\|\chi_\sigma\|_{L^{\infty}(\Omega)} \leq \|\chi_\sigma\|_{C^1(\Omega)}\) gives us
\[
\|\chi_t f\|_{L^2(\Omega, t\varphi)}^2 \leq \varepsilon \left( \|\tilde{\partial} f\|_{L^2(\Omega, t\varphi)}^2 + \|\tilde{\partial}_t^* f\|_{L^2(\Omega, t\varphi)}^2 \right) + O \left( \frac{\varepsilon}{2} \left( 1 + t \|\varphi\|_{C^2(\Omega)} \right) \|f\|_{L^2(\Omega, t\varphi)}^2 + \frac{\|\chi_t\|_{C^4(\Omega)}}{2\varepsilon} \|f\|_{W^{-1}(\Omega, t\varphi)}^2 .
\]
We may choose \(\varepsilon > 0\) sufficiently small so that
\[
\|\chi_t f\|_{L^2(\Omega, t\varphi)}^2 \leq \|\tilde{\partial} f\|_{L^2(\Omega, t\varphi)}^2 + \|\tilde{\partial}_t^* f\|_{L^2(\Omega, t\varphi)}^2
\]
\[
+ \frac{1}{2} t C_q \|f\|_{L^2(\Omega, t\varphi)}^2 + \frac{\|\chi_t\|_{C^4(\Omega)}}{2\varepsilon} \|f\|_{W^{-1}(\Omega, t\varphi)}^2 .
\]
Substituting this in (3.1) gives us (3.2) with \(C_t = \frac{\|\chi_t\|_{C^4(\Omega)}}{4\varepsilon}\) and a new constant \(C_q = \frac{1}{4} C_q\). When \(f \in \text{Dom} \tilde{\partial} \cap \text{Dom} \tilde{\partial}_t^*\), we use a standard density result (e.g., Proposition 2.3 in [22]).

To see that (2) implies (3), we first recall that there exists a constant \(C_{\Omega} > 0\) such that \(\|(-\rho)^{-1} g\|_{L^2(\Omega, t\varphi)} \leq C_{\Omega} \|g\|_{W^{-1}(\Omega, t\varphi)}\) for all \(g \in W^{-1}_0(\Omega, t\varphi)\) (see Theorem 1.4.4.3 in [4]). Now note that for any \(f \in L^2(\Omega, t\varphi)\) such that \((-\rho) f \in L^2(\Omega, t\varphi)\), we have
\[
\|f\|_{W^{-1}(\Omega, t\varphi)} = \sup_{g \in W^{-1}_0(\Omega, t\varphi), g \neq 0} \frac{\langle f, g \rangle_{L^2(\Omega, t\varphi)}}{\|g\|_{W^{-1}(\Omega, t\varphi)}} \leq C_{\Omega} \|(-\rho) f\|_{L^2(\Omega, t\varphi)} .
\]
To see that (3) implies (4), we may assume that \(\rho\) is a defining function for \(\Omega\) that is smooth in the interior of \(\Omega\), even if the boundary of \(\Omega\) is only \(C^2\). Let
\( \psi \in C^\infty(\mathbb{R}) \) denote a non-decreasing function satisfying \( \psi(x) = 0 \) for all \( x \leq 0 \) and \( \psi(x) = 1 \) for all \( x \geq 1 \). Set

\[
\chi_t(z) = \sqrt{C_t} \psi(-t\rho(z) - 1)(-\rho(z)).
\]

Then

\[
\nabla \chi_t(z) = -\sqrt{C_t} (\psi'(-t\rho(z) - 1)t(-\rho(z)) + \psi(-t\rho(z) - 1)) \nabla \rho(z).
\]

Since \( \psi'(-t\rho(z) - 1) = 0 \) whenever \( -\rho(z) \geq \frac{1}{t} \), we have \( \|\chi_t\|_{C^1(\Omega)} \leq O(\sqrt{C_t}) \), and so

\[
\lim_{t \to \infty} \frac{\|\chi_t\|^2_{C^1(\Omega)}}{t^3} \leq \lim_{t \to \infty} O \left( \frac{C_t}{t^3} \right) = 0.
\]

Since \( \psi(-t\rho(z) - 1) \neq 1 \) only when \( -\rho(z) \leq \frac{2}{t} \), we have

\[
C_t(-\rho(z))^2 = (\chi_t(z))^2 + C_t \left( 1 - (\psi(-t\rho(z) - 1))^2 \right) (-\rho(z))^2
\]

\[
\leq (\chi_t(z))^2 + \frac{4C_t}{t^2},
\]

so (3.3) implies

\[
\|\overline{\partial} f\|^2_{L^2(\Omega,t\varphi)} + \|\overline{\partial}_t \varphi f\|^2_{L^2(\Omega,t\varphi)} + \|\chi_t f\|^2_{L^2(\Omega,t\varphi)} \geq \left( tC_q - \frac{4C_t}{t^2} \right) \|f\|^2_{L^2(\Omega,t\varphi)}.
\]

For \( \tilde{t}_0 > t_0 \) sufficiently large, \( \frac{1}{2}C_q \geq \frac{4C_t}{t^2} \) for all \( t \geq \tilde{t}_0 \), so \( \tilde{C}_q = \frac{1}{2}C_q \) satisfies \( \tilde{C}_q \leq C_q - \frac{4C_t}{t^2} \) whenever \( t \geq \tilde{t}_0 \), and (3.4) follows with these new constants \( \tilde{C}_q \) and \( \tilde{t}_0 \).

To see that (4) implies (3), we note that since \( \chi_t = 0 \) on \( \partial \Omega \), we have

\[
\chi_t(z) \leq \|\chi_t\|_{C^1(\Omega)} \text{dist}(z, \partial \Omega)
\]
on \( \Omega \). Since \( \text{dist}(z, \partial \Omega) \leq -C_\rho \rho(z) \) on \( \Omega \) for some constant \( C_\rho > 0 \), we can let \( C_t = \|\chi_t\|^2_{C^1(\Omega)} C_\rho^2 \) and obtain (3.3) from (3.4).

The motivation for our formulation of strong closed range estimates is the family of estimates that arise naturally in the study of domains with disconnected boundaries (e.g., annuli). In the estimates constructed in, for example, [19], [9], or [2], different weight functions must be used in a neighborhood of each connected component of the boundary, so a cutoff function must be used to patch these functions together and obtain a global weight function. This leads to estimates of the form

\[
\|\overline{\partial} f\|^2_{L^2(\Omega,t\varphi)} + \|\overline{\partial}_t \varphi f\|^2_{L^2(\Omega,t\varphi)} + tC_\chi \|\varphi\|_{C^2(\Omega)} \|\chi f\|^2_{L^2(\Omega,t\varphi)} \geq tC_q \|f\|^2_{L^2(\Omega,t\varphi)}
\]

for all \( f \in \text{Dom} \overline{\partial} \cap \text{Dom} \overline{\partial}_t \varphi \) for some \( \chi \in C_0^\infty(\Omega) \) and \( C_\chi > 0 \). If we let \( \chi_t = \sqrt{tC_\chi \|\varphi\|_{C^2(\Omega)} \chi} \), then we clearly obtain strong closed range estimates. However, we also obtain the stronger formulation given by (1) in Lemma 3.1, so in fact all of the families of estimates considered in Lemma 3.1 can be obtained in this case.
4. Necessary Conditions for Strong Closed Range Estimates

Proof of Theorem 1.1. The beginning of our argument is an adaptation of the argument of Theorem 3.2.1 in [14]. By Lemma 3.1, we may assume that we have estimates of the form (3.3).

Fix $p \in \partial \Omega$. After a translation and rotation, we may assume that $p = 0$ and there exists some neighborhood $U$ of $p = 0$ such that

$$\Omega \cap U = \{ z \in U : y_n > \rho_1(z', x_n) \},$$

where $z' = (z_1, \ldots, z_{n-1})$, $z_n = x_n + iy_n$, and $\rho_1$ is a $C^2$ function in some neighborhood of the origin that vanishes to second order at the origin. Let $\tilde{\delta}$ denote the signed distance function for $\partial \Omega$. By [7, (2.9)], since $|\nabla(\rho_1(z', \Re z_n) - \Im z_n)| = 1 + O(|z|)$, we have

$$\frac{\partial^2 \tilde{\delta}}{\partial z_j \partial \bar{z}_k}(0) = \frac{\partial^2 \rho_1}{\partial z_j \partial \bar{z}_k}(0)$$

for all $1 \leq j, k \leq n - 1$.

Fix $s > 0$ and define

$$L_s(z') := \sum_{j,k=1}^{n-1} \frac{\partial^2 \rho_1}{\partial z_j \partial \bar{z}_k}(0) z_j \bar{z}_k + s \sum_{j,k=1}^{n-1} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k}(0) z_j \bar{z}_k.$$

After a unitary change of coordinates, we may assume that

$$L_s(z') = \sum_{j=1}^{n-1} \lambda_j^2 |z_j|^2$$

for some increasing sequence of real numbers $\{\lambda_j^2\}_{1 \leq j \leq n-1}$.

Let $\psi_1 \in C^\infty_0(\mathbb{C}^{n-1})$ and $\psi_2 \in C^\infty_0(\mathbb{R})$ satisfy $\psi_1 \equiv 1$ in a neighborhood of 0 and $\int_{\mathbb{R}} |\psi_3|^2 = 1$. For $x + iy \in \mathbb{C}$, if we define

$$\psi_2(x + iy) = \psi_3(y) \left( \psi_3(x) + iy \psi_3'(x) - \frac{y^2}{2} \psi_3''(x) \right),$$

then $\psi_2$ is a smooth, compactly supported function on $\mathbb{C}$ satisfying

$$\frac{\partial}{\partial z} \psi_2(z) \bigg|_{z=x} = 0,$$

$$\frac{\partial^2}{\partial z^2} \psi_2(z) \bigg|_{z=x} = 0,$$

and

$$\int_{\mathbb{R}} |\psi_2(x)|^2 dx = 1.$$

Let $A(z)$ and $B(z)$ be the holomorphic polynomials

$$A(z) = \sum_{j,k=1}^{n-1} \frac{\partial^2 \rho_1}{\partial z_j \partial \bar{z}_k}(0) z_j \bar{z}_k + \sum_{j=1}^{n-1} 2 \frac{\partial^2 \rho_1}{\partial z_j \partial x_n}(0) z_j z_n + \frac{1}{2} \frac{\partial^2 \rho_1}{\partial x^2_n}(0) z_n^2.$$
and
\[
B(z) = \frac{1}{2} \varphi(0) + \sum_{j=1}^{n} \frac{\partial \varphi}{\partial z_j}(0) z_j + \sum_{j,k=1}^{n-1} \frac{1}{2} \frac{\partial^2 \varphi}{\partial z_j \partial z_k}(0) z_j z_k + \sum_{j=1}^{n-1} \frac{\partial^2 \varphi}{\partial z_j \partial x_n}(0) z_j + \frac{1}{4} \frac{\partial^2 \varphi}{\partial x_n^2}(0) z_n^2.
\]

Then we have
\[
\left| \rho_1(z) - \operatorname{Re} A(z) - \sum_{j,k=1}^{n-1} \frac{\partial^2 \rho_1}{\partial z_j \partial \bar{z}_k}(0) z_j \bar{z}_k \right| \leq O(|z|^3 + |y_n||z| + |y_n|^2),
\]
and
\[
\left| \varphi(z) - 2 \operatorname{Re} B(z) - \sum_{j,k=1}^{n-1} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k}(0) z_j \bar{z}_k \right| \leq O(|z|^3 + |y_n||z| + |y_n|^2).
\]

For any \( \tau > 0 \) we define a form in \( C^\infty(0,q)(\overline{\Delta}) \cap \operatorname{Dom} \tilde{\partial}^* \) by
\[
f^\tau(z) = \psi_1(\tau z') \psi_2(\tau z_n) e^{\tau^2 (A(z) + 2sB(z) + iz_n)} \sum_{j=1}^{q-1} \left( d\bar{z}_j - \left( \frac{\partial \delta}{\partial z_n} \right)^{-1} \frac{\partial \delta}{\partial z_j} d\bar{z}_n \right).
\]

As in Hörmander’s construction, we note that
\[
\left| \sum_{j=1}^{q-1} \left( d\bar{z}_j - \frac{\partial \delta}{\partial z_n} \right)^{-1} \frac{\partial \delta}{\partial z_j} d\bar{z}_n \right|_{z=0} = \sum_{j=1}^{q-1} d\bar{z}_j,
\]
so the term involving \( \frac{\partial \delta}{\partial z_j} \) will vanish in our asymptotic computations. We introduce the change of coordinates \( z_j(\tau) = \tau^{-1} w_j \) for \( 1 \leq j \leq n-1 \) and \( z_n(\tau) = \tau^{-1} \operatorname{Re} w_n + i\tau^{-2} \operatorname{Im} w_n \). Using (4.7), we have
\[
\lim_{\tau \to \infty} \tau^2 (\rho_1(z(\tau), x_n(\tau)) - y_n(\tau)) = \operatorname{Re} A(w', \operatorname{Re} w_n) + \sum_{j,k=1}^{n-1} \frac{\partial^2 \rho_1}{\partial z_j \partial \bar{z}_k}(0) w_j \bar{w}_k - \operatorname{Im} w_n,
\]
so as \( \tau \to \infty \) in our special coordinates, we will be working on the domain
\[
\Omega_w = \left\{ w \in \mathbb{C}^n : \operatorname{Im} w_n > \sum_{j,k=1}^{n-1} \frac{\partial^2 \rho_1}{\partial z_j \partial \bar{z}_k}(0) w_j \bar{w}_k + \operatorname{Re} A(w', \operatorname{Re} w_n) \right\}.
\]

Furthermore, we may use (4.8) to check
\[
\lim_{\tau \to \infty} \tau^2 (\varphi(z(\tau)) - 2 \operatorname{Re} B(z(\tau))) = \sum_{j,k=1}^{n-1} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k}(0) w_j \bar{w}_k.
\]
Motivated by this, we set \( t(\tau) = 2s\tau^2 \). For such a value of \( t \), we may use (4.9) and (4.12) to show

\[
\lim_{\tau \to \infty} |f^\tau(z(\tau))|^2 e^{-t(\tau)\varphi(z(\tau))} = |\psi_1(w')|^2 |\psi_2(\text{Re } w_n)|^2
\]

\[
\exp \left( 2 \text{Re } A(w', \text{Re } w_n) - 2 \text{Im } w_n - 2s \sum_{j,k=1}^{n-1} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k}(0) w_j \bar{w}_k \right)
\]

so

\[
\lim_{\tau \to \infty} \tau^{2n+1} \|f^\tau\|^2_{L^2(\Omega, t(\tau)\varphi)} = \int_{\Omega_{w}} |\psi_1(w')|^2 |\psi_2(\text{Re } w_n)|^2
\]

\[
\exp \left( 2 \text{Re } A(w', \text{Re } w_n) - 2 \text{Im } w_n - 2s \sum_{j,k=1}^{n-1} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k}(0) w_j \bar{w}_k \right) dV_w.
\]

Since \( \int_{a}^{\infty} e^{-2x} dx = \frac{1}{2} e^{-2a} \) for any \( a \in \mathbb{R} \), we may use (4.11) to evaluate this integral with respect to \( \text{Im } w_n \) and then use (4.6) to evaluate this integral with respect to \( \text{Re } w_n \) to obtain

\[
\lim_{\tau \to \infty} \tau^{2n+1} \|f^\tau\|^2_{L^2(\Omega, t(\tau)\varphi)} = \int_{\mathbb{C}^{n-1}} \frac{1}{2} |\psi_1(w')|^2 e^{-2L_s(w')} dw'.
\]

By the same reasoning, we may use (4.9) to obtain

\[
\lim_{\tau \to \infty} \tau^{2n+1} \int_{\Omega} \sum_{j,k=1}^{n} \sum_{K \in \mathbb{Z}_{q-1}^n} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} f_{jK}^\tau f_{jK} \bar{\varphi} e^{-t(\tau)\varphi} dV = \int_{\mathbb{C}^{n-1}} \frac{1}{2} |\psi_1(w')|^2 \sum_{j=1}^{q} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_j}(0) e^{-2L_s(w')} dw'.
\]

If instead we integrate (4.13) over the boundary, we may use (4.11) and (4.6) directly to obtain

\[
\lim_{\tau \to \infty} \tau^{2n-1} \int_{\partial \Omega} |f^\tau|^2 e^{-t(\tau)\varphi} d\sigma = \int_{\mathbb{C}^{n-1}} |\psi_1(w')|^2 e^{-2L_s(w')} dw'.
\]

Similarly, using (4.1) and (4.9), we also have

\[
\lim_{\tau \to \infty} \tau^{2n-1} \int_{\partial \Omega} \sum_{j,k=1}^{n} \sum_{K \in \mathbb{Z}_{q-1}^n} \frac{\partial^2 \rho_1}{\partial z_j \partial \bar{z}_k} f_{jK}^\tau f_{jK} \bar{\rho}_1 e^{-t(\tau)\varphi} d\sigma = \int_{\mathbb{C}^{n-1}} |\psi_1(w')|^2 \sum_{j=1}^{q} \frac{\partial^2 \rho_1}{\partial z_j \partial \bar{z}_j}(0) e^{-2L_s(w')} dw'.
\]
Using (4.2) and (4.3) to add (4.15) and (4.16), we obtain

\[
\lim_{\tau \to \infty} \tau^{2n-1} \left( \int_{\Omega} \sum_{j,k=1}^{n} \sum_{K \in \mathcal{I}_{q-1}} \frac{\partial^2 \delta}{\partial z_j \partial \bar{z}_k} f^\tau_j K K \tilde{f}^{\tau}_{j'} K K e^{-t(\tau)\varphi} \, d\sigma \right) + t \int_{\Omega} \sum_{j,k=1}^{n} \sum_{K \in \mathcal{I}_{q-1}} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} f^\tau_j K K \tilde{f}^{\tau}_{j'} K K e^{-t(\tau)\varphi} \, dV \right) = \int_{\mathbb{C}^{n-1}} |\psi_1(w')|^2 \sum_{j=1}^{q} \lambda_j^2 e^{-2L_s(w')} \, dw'.
\]

For \(1 \leq k \leq n-1\), we compute

\[
\frac{\partial f^\tau}{\partial \bar{z}_k}(z) = \tau \frac{\partial \psi_1}{\partial \bar{z}_k}(\tau z') \psi_2(\tau z_n) e^{\tau^2(A(z) + 2sB(z) + iz_n)} \sum_{j=1}^{q} \left( d\bar{z}_j - \left( \frac{\partial \delta}{\partial z_n} \right)^{-1} \frac{\partial \delta}{\partial \bar{z}_j} d\bar{z}_n \right) \\
+ \psi_1(\tau z') \psi_2(\tau z_n) e^{\tau^2(A(z) + 2sB(z) + iz_n)} \frac{\partial}{\partial \bar{z}_k} \sum_{j=1}^{q} \left( d\bar{z}_j - \left( \frac{\partial \delta}{\partial z_n} \right)^{-1} \frac{\partial \delta}{\partial \bar{z}_j} d\bar{z}_n \right).
\]

Furthermore,

\[
\frac{\partial f^\tau}{\partial \bar{z}_n}(z) = \tau \psi_1(\tau z') \frac{\partial \psi_2}{\partial \bar{z}_n}(\tau z_n) e^{\tau^2(A(z) + 2sB(z) + iz_n)} \sum_{j=1}^{q} \left( d\bar{z}_j - \left( \frac{\partial \delta}{\partial z_n} \right)^{-1} \frac{\partial \delta}{\partial \bar{z}_j} d\bar{z}_n \right) \\
+ \psi_1(\tau z') \psi_2(\tau z_n) e^{\tau^2(A(z) + 2sB(z) + iz_n)} \frac{\partial}{\partial \bar{z}_n} \sum_{j=1}^{q} \left( d\bar{z}_j - \left( \frac{\partial \delta}{\partial z_n} \right)^{-1} \frac{\partial \delta}{\partial \bar{z}_j} d\bar{z}_n \right).
\]

Hence, using (4.4) and observing that the second term in each derivative is uniformly bounded in \(\tau\), we have

\[
\lim_{\tau \to \infty} \tau^{-2} \sum_{j=1}^{n} \left| \frac{\partial f^\tau}{\partial \bar{z}_j}(z) \right|^2 _{z=z(\tau)} e^{-t(\tau)\varphi(z(\tau))} = \sum_{j=1}^{n-1} \left| \frac{\partial \psi_1}{\partial w_j}(w') \right|^2 |\psi_2(\text{Re } w_n)|^2 \exp \left( 2 \text{Re } A(w', \text{Re } w_n) - 2 \text{Im } w_n - 2s \sum_{j,k=1}^{n-1} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k}(0) w_j w_k \right).
\]

Integrating (4.19) as before, we obtain

\[
\lim_{\tau \to \infty} \tau^{2n-1} \sum_{j=1}^{n} \left\| \frac{\partial}{\partial \bar{z}_j} f^\tau \right\|^2 _{L^2(\Omega \setminus t(\tau)\varphi)} = \int_{\mathbb{C}^{n-1}} \frac{1}{2} \sum_{j=1}^{n-1} \left| \frac{\partial \psi_1}{\partial w_j}(w') \right|^2 e^{-2L_s(w')} \, dw'.
\]
Recall the Morrey-Kohn-Hörmander identity:

\[
(4.21) \quad \|\bar{\partial}f\|_{L^2(\Omega, t\varphi)}^2 + \|\partial_t f\|_{L^2(\Omega, t\varphi)}^2 = \sum_{j=1}^{n} \left\| \frac{\partial}{\partial z_j} f \right\|_{L^2(\Omega, t\varphi)}^2 + \int_{\partial \Omega} \sum_{j,k=1}^{n} \sum_{K \in K_{q-1}} \frac{\partial^2 \bar{\partial}}{\partial z_j \partial z_k} f_{jK} \overline{f_{kK}} e^{-t\varphi} \, d\sigma
\]

\[
+ t \int_{\Omega} \sum_{j,k=1}^{n} \sum_{K \in K_{q-1}} \frac{\partial^2 \bar{\partial}}{\partial z_j \partial z_k} f_{jK} \overline{f_{kK}} e^{-t\varphi} \, dV
\]

Note that if \( \rho \) is an arbitrary defining function for \( \Omega \), then \( \rho(z) = h(z)(\rho_1(z) - \text{Im} \, z_n) \) for a bounded function \( h \) that is uniformly bounded away from zero, so (4.10), the fact that \( t(\tau) = 2s\tau^2 \), and the hypothesis that \( C_{t(\tau)} \to 0 \) imply that

\[
\lim_{\tau \to \infty} \frac{C_{t(\tau)}(-\rho(z(\tau)))^2}{t(\tau)} = \lim_{\tau \to \infty} \frac{C_{t(\tau)}(-2s\tau^2 \rho(z(\tau)))^2}{(t(\tau))^3} = 0.
\]

As a result, (4.22)

\[
\lim_{\tau \to \infty} \tau^{2n-1} C_{t(\tau)} \frac{\|(-\rho)f\|_{L^2(\Omega, t(\tau)\varphi)}^2}{t(\tau)} = \lim_{\tau \to \infty} \tau^{2n+1} \frac{2sC_{t(\tau)} \|(-\rho)f\|_{L^2(\Omega, t(\tau)\varphi)}^2}{t(\tau)} = 0.
\]

Hence, combining (3.3) with (4.21) gives us

\[
t(\tau)C_{q} \|f\|_{L^2(\Omega, t(\tau)\varphi)}^2 \leq \sum_{j=1}^{n} \left\| \frac{\partial}{\partial z_j} f \right\|_{L^2(\Omega, t(\tau)\varphi)}^2 + \int_{\partial \Omega} \sum_{j,k=1}^{n} \sum_{K \in K_{q-1}} \frac{\partial^2 \bar{\partial}}{\partial z_j \partial z_k} f_{jK} \overline{f_{kK}} e^{-t(\tau)\varphi} \, d\sigma
\]

\[
+ t(\tau) \int_{\Omega} \sum_{j,k=1}^{n} \sum_{K \in K_{q-1}} \frac{\partial^2 \bar{\partial}}{\partial z_j \partial z_k} f_{jK} \overline{f_{kK}} e^{-t(\tau)\varphi} \, dV + C_{t(\tau)} \|(-\rho)f\|_{L^2(\Omega, t(\tau)\varphi)}^2 .
\]

Multiplying this by \( \tau^{2n-1} \) and taking a limit using (4.14), (4.17), (4.20), and (4.22), we obtain

\[
sC_{q} \int_{\mathbb{C}^{n-1}} |\psi_1(w')|^2 e^{-2L_{s}(w')} \, dw' \leq \int_{\mathbb{C}^{n-1}} \left( \frac{1}{2} \sum_{j=1}^{n} \left| \frac{\partial}{\partial w_j} \psi_1(w') \right|^2 + |\psi_1(w')|^2 \sum_{j=1}^{q} \lambda_j^s \right) e^{-2L_{s}(w')} \, dw'.
\]

Rearranging terms, we obtain

\[
\left( sC_{q} - \sum_{j=1}^{q} \lambda_j^s \right) \int_{\mathbb{C}^{n-1}} |\psi_1(w')|^2 e^{-2L_{s}(w')} \, dw' \leq \frac{1}{2} \int_{\mathbb{C}^{n-1}} \sum_{j=1}^{n-1} \left| \frac{\partial}{\partial w_j} \psi_1(w') \right|^2 e^{-2L_{s}(w')} \, dw'.
\]
From Lemma 3.2.2 in [14], we immediately obtain

(4.23) \[ sC_q - \sum_{j=1}^{q} \lambda_j^s \leq \sum_{j=1}^{n-1} \max(-\lambda_j^s, 0). \]

Now, let \( n_- \) denote the number of \( j \) for which \( \lambda_j^s < 0 \) and let \( n_+ \) denote the number of \( j \) for which \( \lambda_j^s > 0 \). Since we have assumed that the \( \{\lambda_j^s\} \) are arranged in increasing order, (4.23) implies

(4.24) \[ sC_q \leq \sum_{j=1}^{q} \lambda_j^s - \sum_{j=1}^{n_-} \lambda_j^s. \]

If \( n_- \leq q \) and \( n_+ \leq n - q - 1 \), then \( sC_q \leq 0 \), so we obtain an immediate contradiction. Hence, either \( n_- \geq q + 1 \) or \( n_+ \geq n - q \).

We now think of \( s \) as a free parameter rather than a fixed constant. Note that \( n_- (z) \) and \( n_+ (z) \) are lower semicontinuous functions for \( (s, z) \in \mathbb{R}^+ \times \partial \Omega \). Hence, if \( n_- (z) > q \) at a point in \( \mathbb{R}^+ \times \partial \Omega \), then this condition also holds for a neighborhood of that point, with the analogous statement for \( n_+ (z) > n - q - 1 \). We conclude that for every connected component \( S \subset \partial \Omega \), we must have \( n_- (z) \geq q + 1 \) for all \( z \in S \) and \( s > 0 \) or \( n_+ \geq n - q \) for all \( z \in S \) and \( s > 0 \).

We first consider the case in which \( n_+ (z) \geq n - q \) for all \( z \in S \subset \partial \Omega \) and \( s > 0 \). Then \( n_- (z) < q \), so (4.24) gives us

\[ sC_q \leq \sum_{j=n_- (z)+1}^{q} \lambda_j^s (z) \leq (q - n_- (z)) \lambda_q^s (z) \leq q \lambda_q^s (z), \]

for all \( z \in S \) and \( s > 0 \), where the final inequality relies on the fact that the previous inequality guarantees \( \lambda_q^s (z) > 0 \). Letting \( s \to 0 \), we see that \( \lambda_q^s (z) \geq 0 \), i.e., the Levi form has at least \( n - q \) nonnegative eigenvalues at \( z \). Since

\[ \lim_{s \to \infty} \frac{\lambda_q^s (z)}{s} \geq \frac{C_q}{q}, \]

we see that the restriction of \( i\partial \bar{\partial} \phi \) to \( T^{1,0} (\partial \Omega) \times T^{0,1} (\partial \Omega) \) must have at least \( n - q \) eigenvalues bounded below by \( \frac{C_q}{q} \).

We next suppose \( n_- (z) \geq q + 1 \) for all \( z \in S \subset \partial \Omega \) and \( s > 0 \). Then (4.24) gives us

\[ sC_q \leq - \sum_{j=q+1}^{n_-} \lambda_j^s \leq -(n_- (z) - q) \lambda_q^s (z) \leq -(n - 1 - q) \lambda_{q+1}^s (z), \]

for all \( z \in S \) and \( s > 0 \), where the final inequality relies on the fact that the previous inequality guarantees \( \lambda_{q+1}^s (z) < 0 \). Letting \( s \to 0 \), we see that \( \lambda_{q+1}^0 (z) \leq 0 \), i.e., the Levi form has at least \( q + 1 \) nonpositive eigenvalues at \( z \). Since

\[ \lim_{s \to \infty} \frac{\lambda_{q+1}^s (z)}{s} \leq - \frac{\lambda_{q+1}^0 (z)}{n - 1 - q}, \]

we see that the restriction of \( i\partial \bar{\partial} \phi \) to \( T^{1,0} (\partial \Omega) \times T^{0,1} (\partial \Omega) \) must have at least \( q + 1 \) eigenvalues bounded above by \( \frac{C_q}{n - q - 1} \). We have now proved the global version of Theorem 1.1.

If we only have local estimates, it suffices to note that the support of \( f^r \) is contained in a neighborhood of radius \( O(\tau^{-1}) \), and so \( |z - p|^2 \leq O(\tau^{-1}) \) when
\( f^r(z) \neq 0 \). Hence, for \( t \) sufficiently large, \( f^r \) is supported in \( U_t \), and the rest of the proof follows to obtain pointwise information at \( p \in \partial \Omega \).

**Proof of Corollary 1.3.** It suffices to note that since \( \Omega \) is bounded, if we write \( \Omega = \Omega_0 \setminus \bigcup_{j \in J} \Omega_j \), then \( \partial \Omega_0 \) and \( \partial \Omega_j \) must each admit at least one strictly convex point for all \( j \in J \). Since \( \Omega_0 \) admits a strictly convex point, \( \partial \Omega_0 \) must satisfy (1) in Theorem 1.1. For \( j \in J \), \( \partial \Omega_j \) admits at least one strictly convex point, so \( \partial \Omega_j \) viewed as a component of \( \partial \Omega \) admits at least one strictly concave point, and hence satisfies (2) in Theorem 1.1. \( \square \)

5. Applications

5.1. Subelliptic and Compactness Estimates.

**Proof of Proposition 1.4.** Suppose that for some \( \eta > 0 \), \( \Omega \) admits a subelliptic estimate of the form

\[
\| u \|_{W^{\eta}(\Omega)}^2 \leq C \left( \| \partial u \|_{L^2(\Omega)}^2 + \| \partial^* u \|_{L^2(\Omega)}^2 \right)
\]

for all \( u \in L^2_{(\alpha, \beta)}(\Omega) \cap \text{Dom} \partial \cap \text{Dom} \partial^* \). By definition,

\[
\| u \|_{L^2(\Omega)}^2 \leq \| u \|_{W^{\eta}(\Omega)} \| u \|_{W^{-\eta}(\Omega)}
\]

so for any \( t > 0 \) we may use a small constant/large constant inequality to obtain

\[
\| u \|_{L^2(\Omega)}^2 \leq \frac{1}{2t} \| u \|_{W^{\eta}(\Omega)}^2 + \frac{t}{2} \| u \|_{W^{-\eta}(\Omega)}^2.
\]

If \( \rho \) is a defining function for \( \Omega \), then there exists a constant \( C_{\Omega, \eta} > 0 \) such that

\[
\| u \|_{W^{-\eta}(\Omega)} \leq C_{\Omega, \eta} \| (-\rho)^\eta u \|_{L^2(\Omega)}.
\]

This follows from Theorem 1.4.4.3 in [4] by a duality argument, as in (3.6). Hence,

\[
\| u \|_{L^2(\Omega)}^2 \leq \frac{1}{2t} \| u \|_{W^{\eta}(\Omega)}^2 + \frac{tC_{\Omega, \eta}^2}{2} \| (-\rho)^\eta u \|_{L^2(\Omega)}^2.
\]

Substituting our subelliptic estimate yields

\[
(5.1) \quad \| u \|_{L^2(\Omega)}^2 \leq \frac{C}{2t} \left( \| \partial u \|_{L^2(\Omega)}^2 + \| \partial^* u \|_{L^2(\Omega)}^2 \right) + \frac{tC_{\Omega, \eta}^2}{2} \| (-\rho)^\eta u \|_{L^2(\Omega)}^2.
\]

We may assume that \( \rho \) is a defining function for \( \Omega \) that is smooth in the interior of \( \Omega \), even if the boundary of \( \Omega \) is only \( C^2 \). Let \( \psi \in C^\infty(\mathbb{R}) \) denote a non-decreasing function satisfying \( \psi(x) = 0 \) for all \( x \leq 0 \) and \( \psi(x) = 1 \) for all \( x \geq 1 \). Fix \( b > a > 0 \) and set

\[
\chi_t(z) = \frac{tC_{\Omega, \eta}}{\sqrt{C}} \psi \left( \frac{-t^{1/2}\rho(z)}{b-a} \right) (-\rho(z))^\eta.
\]

Since \( \psi \left( \frac{-t^{1/2}\rho(z)}{b-a} \right) \neq 1 \) only when \( -\rho(z) \leq \frac{b}{t^{1/2}b} \), we have

\[
\frac{t^2C_{\Omega, \eta}^2}{C} (-\rho(z))^{2\eta} = (\chi_t(z))^2 + \frac{t^2C_{\Omega, \eta}^2}{C} \left( 1 - \psi \left( \frac{-t^{1/2}\rho(z)}{b-a} \right) \right)^2 (-\rho(z))^{2\eta}
\]

\[
\leq (\chi_t(z))^2 + \frac{tC_{\Omega, \eta}^2}{C} b^{2\eta}.
\]

Substituting in (5.1) and rearranging terms, we obtain

\[
\frac{t}{C} \left( 2 - C_{\Omega, \eta} b^{2\eta} \right) \| u \|_{L^2(\Omega)}^2 \leq \| \partial u \|_{L^2(\Omega)}^2 + \| \partial^* u \|_{L^2(\Omega)}^2 + \| \chi_t u \|_{L^2(\Omega)}^2.
\]
If we choose \( b \) sufficiently small, then we may set \( C_q = \frac{1}{\epsilon} (2 - C_2^2 \frac{1}{(2\epsilon)^2} b^{2n}) \) and obtain \( C_q > 0 \). Thus we have an estimate of the form (2.1) with \( \varphi \equiv 0 \), but without the growth condition on \( \|\chi_t\|_{C^1(\Omega)} \). Since \( \Omega \) is bounded, we have \( \|\chi_t\|_{L^\infty(\Omega)} \leq O(t) \). We compute

\[
\nabla \chi_t(z) = -\frac{\eta t C_{\Omega, \eta} \psi}{\sqrt{C}} \left( -\frac{t^{1/(2n)} \rho(z) - a}{b - a} \right) (-\rho(z))^{n-1} \nabla \rho(z) - \frac{t^{1 + 1/(2n)} C_{\Omega, \eta} \rho(z) - a}{(b - a) \sqrt{C}} (-\rho(z))^{n-1} \nabla \rho(z).
\]

Since \( \psi \left( -\frac{t^{1/(2n)} \rho(z) - a}{b - a} \right) \neq 0 \) only when \( -\rho(z) \geq \frac{a}{1 + 1/(2n)} \), the first term is bounded by \( O(t \cdot t^{(1-n)/(2n)}) = O(t^{(1+n)/(2n)}) \). Since \( \psi' \left( -\frac{t^{1/(2n)} \rho(z) - a}{b - a} \right) \neq 0 \) only when \( \frac{b}{t^{1/(2n)}} \geq -\rho(z) \), the second term is bounded by \( O(t^{1 + 1/(2n)} \cdot t^{-1/2}) = O(t^{(1+n)/(2n)}) \). Consequently,

\[
\limsup_{t \to \infty} \frac{\|\chi_t\|_{C^1(\Omega)}}{t^{(1+n)/n}} < \infty.
\]

Since \( \varphi \equiv 0 \), the conclusions of Theorem 1.1 does not follow, and hence

\[
\limsup_{t \to \infty} \frac{\|\chi_t\|_{C^1(\Omega)}}{t^3} > 0.
\]

\[\square\]

**Proof of Theorem 1.5.** Suppose that (1.2) fails. Then

\[\lim_{\varepsilon \to 0^+} \varepsilon^2 C_\varepsilon = 0.\]

For any \( t > 0 \), let \( \varepsilon = \frac{1}{t} \). Then we may set \( C_t = t C_\varepsilon \), \( C_q = 1 \), and \( \varphi \equiv 0 \) to show that (2.2) implies (3.2) (observe that \( \frac{1}{C_t} = \varepsilon^2 C_\varepsilon \)). By Lemma 3.1, this also implies (2.1). Hence, Theorem 1.1 implies that \( i \partial \bar{\partial} \varphi \) has nontrivial eigenvalues, contradicting the fact that \( \varphi \) is constant. \[\square\]

### 5.2. Sobolev Estimates

For the remainder of this note, we concentrate on the implications of (3.2). Note that the following arguments do not require \( C_t \) to depend on \( t \) in a prescribed way.

The following lemma appears in [16], though it is well-known.

**Lemma 5.1.** Let \( 1 \leq q \leq n - 1 \). Suppose that \( \Omega \subset \mathbb{C}^n \) is a bounded domain. Then the following are equivalent:

1. The space of harmonic forms \( \mathcal{H}_{0,q}(\Omega, \phi) \) is finite dimensional and the \( L^2 \)-basic estimate

\[
\|u\|_{L^2(\Omega, \phi)}^2 \leq C (\|\bar{\partial}u\|_{L^2(\Omega, \phi)}^2 + \|\bar{\partial}^* u\|_{L^2(\Omega, \phi)}^2)
\]

holds for all \( u \in \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*) \cap (\mathcal{H}_{0,q}(\Omega, \phi)) \perp \).

2. The \( L^2 \)-basic estimate

\[
\|u\|_{L^2(\Omega, \phi)}^2 \leq C (\|\bar{\partial}u\|_{L^2(\Omega, \phi)}^2 + \|\bar{\partial}^* u\|_{L^2(\Omega, \phi)}^2 + \|\partial\phi\|_{L^2(\Omega, \phi)}^2 - ^{(n-1)}(\Omega, \phi)) \]

holds for all \( u \in \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*) \).
We now use an elliptic regularization argument to analyze the regularity of the \(\bar{\partial}\)-Neumann operator and harmonic forms. Let \(\rho\) be a defining function for \(\Omega\) normalized so that \(|d\rho| = 1\). In real coordinates for \(\mathbb{R}^{2n}\), we define the tangential gradient

\[
(\nabla_T u, \nabla_T v)_\phi = \sum_{j=1}^{2n} \left( \frac{\partial}{\partial x_j} - \frac{\partial \rho}{\partial x_j} \sum_{k=1}^{2n} \frac{\partial \rho}{\partial x_k} \frac{\partial}{\partial x_k} \right) u, \left( \frac{\partial}{\partial x_j} - \frac{\partial \rho}{\partial x_j} \sum_{k=1}^{2n} \frac{\partial \rho}{\partial x_k} \frac{\partial}{\partial x_k} \right) v \right)_\phi,
\]

for \(u, v \in L^{2,1}(\Omega, \phi)\) with the corresponding tangential Laplacian

\[
\Delta_{T,\phi} u = \sum_{j=1}^{2n} \left( \frac{\partial}{\partial x_j} - \frac{\partial \rho}{\partial x_j} \sum_{k=1}^{2n} \frac{\partial \rho}{\partial x_k} \frac{\partial}{\partial x_k} \right) u
\]

for \(u \in L^{2,2}(\Omega, \phi)\). Define the quadratic forms

\[
Q^{\delta,\nu}_\phi(u, v) = (\bar{\partial} u, \bar{\partial} v)_\phi + (\bar{\partial}^*_\delta u, \bar{\partial}^*_\delta v)_\phi + \delta(\nabla_T u, \nabla_T v)_\phi + \nu(u, v)_\phi
\]

for \(u, v \in \text{Dom}(\bar{\partial}^*_\delta) \cap L^{2,1}_0(\Omega, \phi)\) when \(\delta > 0\), or \(u, v \in \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*_\delta)\) when \(\delta = 0\). When \(\delta > 0\), we may use, for example, Lemma 2.2 in [22] to show that there exists a constant \(c_{\delta,\nu} > 0\) such that

\[
\|u\|^2_{L^{2,1}_1(\Omega, \phi)} \leq c_{\delta,\nu} Q^{\delta,\nu}_\phi(u, u) \quad \text{holds for all } u \in L^{2,1}_0(\Omega, \phi) \cap \text{Dom}(\bar{\partial}^*_\delta).
\]

When \(\delta = 0\) or \(\nu = 0\), we omit the corresponding superscript, so, for example, \(Q^{0,0}(u, v) = Q(u, v)\). We may use standard techniques to construct the corresponding Laplacians

\[
\Box^{\delta,\nu}_\phi = \Box_{\phi} + \delta \Delta_{T,\phi} + \nu I
\]

with appropriate domains \(\text{Dom}(\Box^{\delta,\nu}_\phi)\) (see, for example, Section 2.8 in [22] when \(\delta = 0\) or Section 3.3 in [22] when \(\delta > 0\)). Since (5.4) implies that \(\Box^{\delta,\nu}_\phi\) is elliptic when \(\delta > 0\), we have \(u \in L^{2,2}_0(\Omega, \phi)\) whenever \(\Box^{\delta,\nu}_\phi u \in L^2(\Omega, \phi)\), and hence \(\text{Dom}(\Box^{\delta,\nu}_\phi) \subset L^{2,2}_0(\Omega, \phi)\). Since the term with the coefficient of \(\delta\) involves only tangential derivatives, we have

\[
\text{Dom}(\Box^{\delta,\nu}_\phi) = L^{2,2}_0(\Omega, \phi) \cap \text{Dom}(\Box_{\phi}).
\]

**Theorem 5.2.** Let \(\Omega \subset \mathbb{C}^n\) be a bounded domain and \(1 \leq q \leq n\). Assume that

1. There is a constant \(c > 0\) so that for any \(u \in \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*_\delta) \cap L^{2,q}_0(\Omega, \phi)\), the following \(L^2\)-basic estimate holds:

\[
\|u\|^2_{L^{2,2}(\Omega, \phi)} \leq c \left( \|\bar{\partial} u\|^2_{L^{2,2}(\Omega, \phi)} + \|\bar{\partial}^*_\delta u\|^2_{L^{2,2}(\Omega, \phi)} + \|u\|^2_{L^{2,2,1}(\Omega, \phi)} \right)
\]

2. For some fixed \(s_0 \in \mathbb{N}\) and all \(0 \leq s \leq s_0\), there exists a constant \(c_s > 0\) so that

\[
\|u\|^2_{L^{2,2,s}(\Omega, \phi)} \leq c_s \left( \|\Box^{\delta,\nu}_\phi u\|^2_{L^{2,2,s}(\Omega, \phi)} + \|u\|^2_{L^{2,2}(\Omega, \phi)} \right)
\]

for any \(u \in L^{2,q}_0(\Omega, \phi)\) so that \(u \in \text{Dom}(\Box_{\phi})\) and \(\Box^{\delta,\nu}_\phi u \in L^{2,2,s}_0(\Omega, \phi)\).

Then \(\mathcal{H}_{0,q}(\Omega, \phi) \subset L^{2,q,s}_0(\Omega, \phi)\) and the \(\bar{\partial}\)-Neumann operator \(N^\delta_{\phi}\) is exactly regular on \(L^{2,q,s}_0(\Omega, \phi)\) for \(0 \leq s \leq s_0\).
Proof. Suppose $s \leq s_0$ is a positive integer. Then from (5.6), there is an $\nu_{s_0}$ such that for any $\nu < \nu_{s_0}$ the following estimate

\begin{equation}
\|u\|^2_{L^2,\nu;L^2(\Omega,\phi)} \leq 2c_s \left( \|\Box^{\delta,\nu}_\phi u\|^2_{L^2,\nu;L^2(\Omega,\phi)} + \|u\|^2_{L^2(\Omega,\phi)} \right)
\end{equation}

holds for any $u \in L^{2,\nu}_{0,q}(\Omega,\phi) \cap \text{Dom}(\Box^{\delta,\nu}_\phi)$ satisfying $\Box^{\delta,\nu}_\phi u \in L^{2,\nu}_{0,q}(\Omega,\phi)$. By construction, for any $\nu > 0$ and $\delta \geq 0$ we also have

\begin{equation}
\|u\|^2_{L^2(\Omega,\phi)} \leq \frac{1}{\nu} Q^{\delta,\nu}_\phi(u,u)
\end{equation}

for all $u \in L^{2,\nu}_{0,q}(\Omega,\phi) \cap \text{Dom}(\Box^{\delta,\nu}_\phi)$.

Consequently, $\Box^{\delta,\nu}_\phi$ has closed range and a trivial kernel. This means $\Box^{\delta,\nu}_\phi$ has a continuous inverse on $L^2_{0,q}(\Omega,\phi)$ that we denote by $N^{\delta,\nu}_{0,q}$. Also, for each $\nu > 0$, the inverse $N^{\delta,\nu}_{\phi}$ satisfies

\begin{equation}
\|N^{\delta,\nu}_{\phi}\|^2_{L^2(\Omega,\phi)} \leq \frac{1}{\nu} \|\alpha\|^2_{L^2(\Omega,\phi)} \quad \text{for all } \alpha \in L^2_{0,q}(\Omega,\phi).
\end{equation}

**Step 1:** We will first show that if $\alpha \in L^{2,s}_{0,q}(\Omega,\phi)$ then $N^{0,\nu}_{\phi} \alpha \in L^{2,s}_{0,q}(\Omega,\phi) \cap \text{Dom}(\Box^{\delta,\nu}_\phi)$. By (5.4), it follows that $\Box^{\delta,\nu}_\phi$ is elliptic which means that if $\alpha \in L^{2,s}_{0,q}(\Omega,\phi)$, then $N^{\delta,\nu}_{0,q} \alpha \in L^{2,s+2}_{0,q}(\Omega,\phi) \cap \text{Dom}(\Box^{\delta,\nu}_\phi)$. Moreover, $L^{2,s+2}_{0,q}(\Omega,\phi) \cap \text{Dom}(\Box^{\delta,\nu}_\phi) \subset \text{Dom}(\Box^{\delta,\nu}_\phi)$. We can therefore use (5.7) with $u = N^{\delta,\nu}_{\phi} \alpha$ and estimate

\begin{equation}
\|N^{\delta,\nu}_{\phi} \alpha\|^2_{L^2(\Omega,\phi)} \leq 2c_s \left( \|\Box^{\delta,\nu}_\phi N^{\delta,\nu}_{\phi} \alpha\|^2_{L^2,\nu;L^2(\Omega,\phi)} + \|N^{\delta,\nu}_{\phi} \alpha\|^2_{L^2,\nu;L^2(\Omega,\phi)} \right)
\end{equation}

\begin{equation}
= 2c_s \left( \|\alpha\|^2_{L^2(\Omega,\phi)} + \|N^{\delta,\nu}_{\phi} \alpha\|^2_{L^2(\Omega,\phi)} \right) \leq 2c_s \|\alpha\|^2_{L^2(\Omega,\phi)} + c_s \|\alpha\|^2_{L^2(\Omega,\phi)}
\end{equation}

for any positive integer $s \leq s_0$. The equality in (5.10) follows from the identity $\Box^{\delta,\nu}_\phi N^{\delta,\nu}_{\phi} = I$ and the fact that $\text{ker}(\Box^{\delta,\nu}_\phi) = \{0\}$. The (second) inequality follows by (5.9) and the independence of the constants $c_s, c_{\delta,\nu}$ on $\delta > 0$.

Thus, $\|N^{\delta,\nu}_{\phi} \alpha\|_{L^{2,s}(\Omega,\phi)}$ is uniformly bounded in $\delta > 0$. Therefore, there exists a sequence $\delta_k \downarrow 0$ such that $N^{\delta_k,\nu}_{\phi} \alpha \to u_\nu$ weakly in $L^{2,s_0}_{0,q}(\Omega,\phi)$. For any integer $0 \leq s \leq s_0$, if $f \in L^{2,s}_{0,q}(\Omega,\phi)$, then by the Riesz Representation Theorem there exists $\hat{f} \in L^{2,s_0}_{0,q}(\Omega,\phi)$ such that $(f,g)_{L^{2,s}(\Omega,\phi)} = (\hat{f},g)_{L^{2,s_0}(\Omega,\phi)}$ for all $g \in L^{2,s_0}_{0,q}(\Omega,\phi)$, and hence $N^{\delta_k,\nu}_{\phi} \alpha \to u_\nu$ weakly in $L^{2,s}_{0,q}(\Omega,\phi)$ for all integers $0 \leq s \leq s_0$. Thus, it follows that $u_\nu \in L^{2,s}_{0,q}(\Omega,\phi)$, $0 \leq s \leq s_0$. Additionally, $N^{\delta,\nu}_{\phi} \alpha \to u_\nu$ weakly in the $Q^{0,\nu}_{\phi}(\cdot,\cdot)^{1/2}$-norm. This means that if $v \in L^{2,2}_{0,q}(\Omega) \cap \text{Dom}(\Box^{\delta,\nu}_\phi)$, then

\[ \lim_{k \to \infty} Q^{0,\nu}_{\phi}(N^{\delta_k,\nu}_{\phi} \alpha, v) = Q^{0,\nu}_{\phi}(u_\nu, v). \]

On the other hand,

\[ Q^{0,\nu}_{\phi}(N^{0,\nu}_{\phi} \alpha, v) = (\alpha, v) = Q^{\delta,\nu}_{\phi}(N^{\delta,\nu}_{\phi} \alpha, v) = Q^{0,\nu}_{\phi}(N^{\delta,\nu}_{\phi} \alpha, v) + \delta (\nabla T N^{\delta,\nu}_{\phi} \alpha, \nabla T v) \]
for all \( v \in L^2_{0,q}(\Omega, \phi) \cap \text{Dom}(\partial^*_\phi) \). It follows that
\[
|Q^{0,\nu}_\phi((N^{0,\nu,q}_\phi - N^{0,\nu,q}_\phi)\alpha, v)| = |\langle \nabla_T \nabla_T N^{0,\nu,q}_\phi \alpha, \nabla_T v \rangle| \\
\leq \delta \|N^{0,\nu,q}_\phi \alpha\|_{L^2(\Omega,\phi)} \|v\|_{L^2(\Omega,\phi)} \leq \delta \|\alpha\|_{\text{Dom}(\Omega,\phi)} \|v\|_{L^2(\Omega,\phi)}
\]
where we again used the inequality \( \|N^{0,\nu,q}_\phi \alpha\|_{L^2(\Omega,\phi)} \leq c_\nu \|\alpha\|_{L^2(\Omega,\phi)} \) uniformly in \( \delta \geq 0 \). Thus,
\[
\lim_{\delta \to 0} Q^{0,\nu}_\phi((N^{0,\nu,q}_\phi - N^{0,\nu,q}_\phi)\alpha, v) = 0.
\]
Since \( \ker \Box^\delta = \{0\} \), it follows that \( N^{0,\nu,q}_\phi \alpha = u_\nu \) and hence \( N^{0,\nu,q}_\phi \alpha \in L^2_{0,q}(\Omega, \phi) \) for all integers \( 0 \leq s \leq s_0 \). Moreover, we may apply (5.7) with \( u = N^{0,\nu,q}_\phi \alpha \in L^2_{0,q}(\Omega, \phi) \cap \text{Dom}(\Box) \) and \( \delta = 0 \) and observe

\[
\|N^{0,\nu,q}_\phi \alpha\|_{L^2(\Omega,\phi)}^2 \leq 2c_s \left( \|\alpha\|_{L^2(\Omega,\phi)}^2 + \|N^{0,\nu,q}_\phi \alpha\|_{L^2(\Omega,\phi)}^2 \right),
\]
holds for all \( \alpha \in L^2_{0,q}(\Omega, \phi) \).

**Step 2:** We next show that \( \mathcal{H}_{0,q}(\Omega, \phi) \subset L^2_{0,q}(\Omega, \phi) \) for \( 0 \leq s \leq s_0 \).

By Lemma 5.1, the space of \( L^2 \)-harmonic forms \( \mathcal{H}_{0,q}(\Omega, \phi) \) is finite dimensional. Let \( \theta_0, \ldots, \theta_N \in \mathcal{H}_{0,q}(\Omega, \phi) \) be an orthonormal basis, and set \( \theta_0 = 0 \). We will prove \( \theta_j \in L^2_{0,q}(\Omega, \phi)(\Omega) \) for all \( j \) by induction. Certainly \( \theta_0 \in L^2_{0,q}(\Omega, \phi)(\Omega) \). Assume now that for some \( 0 \leq k < N, \theta_j \in L^2_{0,q}(\Omega, \phi)(\Omega) \) for all \( 0 \leq j \leq k \). We will construct \( \theta \in \mathcal{H}_{0,q}(\Omega, \phi) \cap L^2_{0,q}(\Omega, \phi)(\Omega) \) with \( \|\theta\|_{L^2(\Omega,\phi)} = 1 \) and \( (\theta, \theta_j)_{L^2(\Omega,\phi)} = 0 \) for \( j \leq k \). If we replace \( \theta_{k+1} \) with \( \theta \), we may proceed by induction to obtain a basis of \( \mathcal{H}_{0,q}(\Omega, \phi) \) which is contained in \( L^2_{0,q}(\Omega, \phi) \).

Let \( \alpha \in L^2_{0,q}(\Omega, \phi) \) be a form such that \( \alpha \) is orthogonal (in \( L^2_{0,q}(\Omega, \phi) \)) to \( \theta_j \) for \( j \leq k \) but not to \( \theta_{k+1} \). This can be obtained, for example, by regularizing \( \theta_{k+1} \) and projecting onto the orthogonal complement of the span of \( \{\theta_1, \ldots, \theta_k\} \). Then, for \( \nu > 0, N^{0,\nu,q}_\phi \alpha \in L^2_{0,q}(\Omega, \phi) \cap \text{Dom}(\Box) \) and satisfies (5.11). We claim that \( \{\|N^{0,\nu,q}_\phi \alpha\|_{L^2(\Omega,\phi)} : 0 < \nu < 1\} \) is unbounded. If it were bounded then by (5.11) we could find a subsequence converging (weakly) to a form \( u \in L^2_{0,q}(\Omega, \phi) \cap \text{Dom}(\Box) \) satisfying
\[
Q_\phi(u, \psi) = \langle \alpha, \psi \rangle
\]
for all \( \psi \in \text{Dom}(Q_\phi) \). By setting \( \psi = \alpha \), we see that \( u \neq 0 \), and if \( \psi = \theta_{k+1} \), the left-hand side is zero while the right-hand side is different from zero, a contradiction. Thus the set \( \{\|N^{0,\nu,m,q}_\phi \alpha\|_{L^2(\Omega,\phi)} : 0 < \nu < 1\} \) is unbounded and we can therefore find a subsequence \( \{\|N^{0,\nu,m,q}_\phi \alpha\|_{L^2(\Omega,\phi)} \} \) such that \( \lim_{m \to \infty} \|N^{0,\nu,m,q}_\phi \alpha\|_{L^2(\Omega,\phi)} = \infty \) and \( \nu_m \to 0 \). Set \( w_m = \frac{N^{0,\nu,m,q}_\phi \alpha}{\|N^{0,\nu,m,q}_\phi \alpha\|_{L^2(\Omega,\phi)}} \). Then \( w_m \in L^2_{0,q}(\Omega, \phi) \cap \text{Dom}(\Box) \), \( \|w_m\|_{L^2(\Omega,\phi)} = 1 \), and by (5.11)
\[
\|w_m\|_{L^2(\Omega,\phi)} \leq c_{s_0} \left( \|\alpha\|_{L^2(\Omega,\phi)} + 1 \right).
\]
Thus, there is a subsequence \( \{w_{m_j}\} \) converging weakly to \( \theta \in L^2_{0,q}(\Omega, \phi) \). The compact inclusion \( L^2_{0,q}(\Omega, \phi) \hookrightarrow L^2_{0,q}(\Omega, \phi) \) forces norm convergence of \( w_{m_j} \) to \( \theta \) in
$L^2_{0,q}(\Omega, \phi)$. Thus, $\|\theta\|_{L^2(\Omega, \phi)}^2 = 1$. To see that $\theta \in \mathcal{H}_{0,q}(\Omega, \phi)$, we use the inequality

$$Q_{\phi}(w_{m_j}, w_{m_j}) \leq Q_{\phi}^{0,\nu_{m_j}}(w_{m_j}, w_{m_j})$$

where

$$\frac{1}{\|N_{\phi}^{0,\nu_{m_j}}(\alpha)\|_{L^2(\Omega, \phi)}} (\alpha, w_{m_j}) \leq \frac{\|\alpha\|_{L^2(\Omega, \phi)}}{\|N_{\phi}^{0,\nu_{m_j},q}(\alpha)\|_{L^2(\Omega, \phi)}}.$$ 

Indeed, since $\{w_{m_j}\}$ converges weakly to $\theta$ in $L^2_{0,q}(\Omega, \phi)$, we have

$$Q_{\phi}(\theta, \theta) = \lim_{j \to \infty} \lim_{k \to \infty} Q_{\phi}(w_{m_j}, w_{m_k}) \leq \lim_{j \to \infty} \lim_{k \to \infty} Q_{\phi}(w_{m_j}, w_{m_j})^{1/2} Q_{\phi}(w_{m_k}, w_{m_k})^{1/2} = \left( \lim_{j \to \infty} Q_{\phi}(w_{m_j}, w_{m_j})^{1/2} \right)^2 = 0.$$

Hence $\theta \in \mathcal{H}_{0,q}(\Omega, \phi)$. Finally, to prove $(\theta, \theta_j) = 0$ for $j \leq k$, for any $0 \leq j \leq k$ we have

$$\nu_m(w_{m_j}, \theta_j) = Q_{\phi}^\nu_m(w_{m_j}, \theta_j) = \frac{1}{\|N_{\phi}^{0,\nu_m}(\alpha)\|_{L^2(\Omega, \phi)}} (\alpha, \theta_j) = 0.$$

This means $w_{m_j}$ is orthogonal to $\theta_k$ for $j = 1, \ldots, k$ and so $\theta$ is as well. Therefore, $\mathcal{H}_{0,q}(\Omega, \phi) \subset L^{2,\infty}_0(\Omega, \phi)$.

**Step 3:** Finally, we show that $N^{q}_\phi := N^{0,0,q}_\phi$ is exactly regular on $L^{2,s}_0(\Omega, \phi)$ for $0 \leq s \leq \infty$. We start this step by combining Lemma 5.1 and (5.5). In particular, for any $u \in \text{Dom}(\tilde{\partial}) \cap \text{Dom}(\tilde{\partial}_s^\nu) \cap \mathcal{H}^{1}_{0,q}(\Omega, \phi)$

$$\|u\|_{L^2(\Omega, \phi)}^2 \leq c \left( \|\tilde{\partial} u\|_{L^2(\Omega, \phi)}^2 + \|\tilde{\partial}_s^\nu u\|_{L^2(\Omega, \phi)}^2 \right) = cQ_{\phi}(u, u)$$

and hence

$$\|u\|_{L^2(\Omega, \phi)}^2 \leq c \left( Q_{\phi}(u, u) + \nu \|u\|_{L^2(\Omega, \phi)}^2 \right) = cQ_{\phi}^{0,\nu}(u, u)$$

where $c$ is independent of $\nu$.

By the definition of $Q_{\phi}^{0,\nu}$ and $N^{0,\nu}_\phi$, we have

$$Q_{\phi}(N^{0,\nu}_\phi(\alpha, \psi) + \nu(N^{0,\nu}_\phi(\alpha, \psi)) = Q_{\phi}^\nu(N^{0,\nu}_\phi(\alpha, \psi) = (\alpha, \psi)$$

for any $\alpha, \psi \in L^{2,q}_0(\Omega, \phi)$ with $\psi \in \text{Dom}(\tilde{\partial}) \cap \text{Dom}(\tilde{\partial}_s^\nu)$. Thus, if $\alpha \perp \mathcal{H}_{0,q}(\Omega, \phi)$ and $\psi \in \mathcal{H}_{0,q}(\Omega, \phi)$, then it follows that $N^{0,\nu}_\phi(\alpha) \perp \mathcal{H}_{0,q}(\Omega, \phi)$ since a consequence of the harmonicity of $\psi$ is that $Q_{\phi}(f, \psi) = 0$ for any $f \in \text{Dom}(\tilde{\partial} \cap \text{Dom}(\tilde{\partial}_s^\nu)$. Thus, if $u = N^{0,\nu}_\phi(\alpha)$ and $\alpha \perp \mathcal{H}_{0,q}(\Omega, \phi)$, then the uniformity of (5.12) (in $\nu > 0$) implies

$$\|N^{0,\nu}_\phi(\alpha)\|_{L^2(\Omega, \phi)} \leq c\|\alpha\|_{L^2(\Omega, \phi)}.$$ 

Combining this uniform $L^2$ estimate with (5.11) yields the uniform (in $\nu > 0$) $L^{2,s}$-estimate

$$\|N^{0,\nu}_\phi(\alpha)\|_{L^2(\Omega, \phi)} \leq c\|\alpha\|_{L^2(\Omega, \phi)} + \|\alpha\|_{L^2(\Omega, \phi)} \leq c\|\alpha\|_{L^2(\Omega, \phi)},$$

for any $\alpha \in L^{2,s}_0(\Omega, \phi) \cap \mathcal{H}_{0,q}^{1}(\Omega)$.

Now we use argument of Step 1 to send $\nu \to 0$ and establish that $N^{q}_\phi(\alpha) \in L^{2,s}_0(\Omega, \phi) \cap \mathcal{H}_{0,q}^{1}(\Omega, \phi)$ and (5.13) holds for $\nu = 0$. For $\alpha \in L^{2,s}_0(\Omega, \phi)$, we decompose $\alpha = (I - H^q_\phi)\alpha + H^q_\phi\alpha$ (recall that $H^q_\phi$ is the orthogonal projection $L^2_{0,q}(\Omega, \phi)$ onto
\( \mathcal{H}_{0,q}(\Omega, \phi) \). Since \( \alpha \in L^{2,q}_{0,q}(\Omega, \phi) \), it follows from Step 2 that \( (I - H^s_\phi)\alpha \in L^{2,q}_{0,q}(\Omega, \phi) \), and by using (5.13) for \( \nu = 0 \), we may conclude that

\[
\|N_\phi^q\alpha\|^2_{L^2(\Omega, \phi)} \leq c_s \| (I - H^s_\phi)\alpha \|^2_{L^2(\Omega, \phi)} \leq c_s \| \alpha \|^2_{L^2(\Omega, \phi)},
\]

for all \( \alpha \in L^{2,q}_{0,q}(\Omega, \phi) \). That means \( N_\phi^q \) is exactly regular in \( L^{2,q}_{0,q}(\Omega, \phi) \), \( 0 \leq s \leq s_0 \).

We turn to showing that a family of closed range estimates will suffice to satisfy the hypotheses of Theorem 5.2 for sufficiently large \( t \).

**Proposition 5.3.** Let \( \Omega \subset \mathbb{C}^n \) be a smooth domain which admits the family of strong closed range estimates (3.2) for some smooth function \( \varphi \). Then for every \( k \geq 1 \) there exists \( T_k \) so that if \( t \geq T_k \), then the hypotheses of Theorem 5.2 hold for \( s_0 = k \).

**Proof.** Suppose that \( X^k \) is a real order \( k \) differential operator that is tangential on \( \partial \Omega \). We define the action of \( X^k \) on differential forms by locally writing each form in a special boundary chart (see 2.2 in [22], for example) and applying \( X^k \) to the coefficients of the form in this chart. Hence, \( X^k \) will preserve the domain of \( \partial_t^\varphi \).

We first note that (5.3) in [22] holds in our case: for any \( k \geq 1 \), if \( X^k \) is a real order \( k \) differential operator that is tangential on \( \partial \Omega \), then we have

\[
\| \partial X^k f \|_{L^2(\Omega, t^\varphi)}^2 + \| \partial_t^\varphi X^k f \|_{L^2(\Omega, t^\varphi)}^2 + \| \nabla_T X^k f \|_{L^2(\Omega, t^\varphi)}^2 \\
\leq C \left( \| \partial_t^\varphi \|_{L^2(\Omega, t^\varphi)}^2 + \| f \|_{L^2(\Omega, t^\varphi)}^2 \right) + C_t \| f \|_{L^2(\Omega, t^\varphi)}^2,
\]

for all \( f \in W^{2k+1}_{0,q}(\Omega, t^\varphi) \cap \text{Dom}(\Box_t^\varphi) \), where \( C > 0 \) is a constant that is independent of \( f \) and \( t \), and \( C_t > 0 \) is a constant that is only independent of \( f \). If we make the substitution \( f = N^t_\varphi u \), then the only difference between (5.15) and (5.3) in [22] is the final term, which would be \( C_t \| \Box_t^\varphi f \|_{L^2(\Omega, t^\varphi)}^2 \) in our notation. This relies on the estimate \( \| f \|_{L^2(\Omega, t^\varphi)} \leq C \| \Box_t^\varphi f \|_{L^2(\Omega, t^\varphi)}^2 \), which is true in the pseudoconvex case studied by Straube, but not necessarily in our case. Since our domain is not necessarily pseudoconvex, we also note that (5.15) may fail when \( k = 0 \).

If \( f \in W^{2k+1}_{0,q}(\Omega, t^\varphi) \cap \text{Dom}(\Box_t^\varphi) \), then \( X^k f \in \text{Dom}(\partial_t^\varphi) \) so that (3.2) holds. Consequently,

\[
(tC_q)^{\| X^k f \|_{L^2(\Omega, t^\varphi)}^2} \leq \\
\| \partial X^k f \|_{L^2(\Omega, t^\varphi)}^2 + \| \partial_t^\varphi X^k f \|_{L^2(\Omega, t^\varphi)}^2 + \| \nabla_T X^k f \|_{L^2(\Omega, t^\varphi)}^2 + C_t \| f \|_{W^{k-1}(\Omega, t^\varphi)}^2.
\]

Plugging (5.15) into (5.16), we see that for any \( f \in W^{2k+1}_{0,q}(\Omega, t^\varphi) \cap \text{Dom}(\Box_t^\varphi) \)

\[
\| X^k f \|_{L^2(\Omega, t^\varphi)}^2 \leq \frac{C_t}{t} \left( \| \Box_t^\varphi f \|_{L^2(\Omega, t^\varphi)}^2 + \| f \|_{L^2(\Omega, t^\varphi)}^2 \right) + C_t \left( \| f \|_{W^{k-1}(\Omega, t^\varphi)}^2 + \| f \|_{L^2(\Omega, t^\varphi)}^2 \right).
\]

As noted in the proof of (5.3) in [22], we may use Sobolev interpolation to estimate \( \| f \|_{W^{k-1}(\Omega, t^\varphi)} \leq \varepsilon \| f \|_{W^k(\Omega, t^\varphi)} + C_t \| f \|_{L^2(\Omega, t^\varphi)}^2 \), so we have

\[
\| X^k f \|_{L^2(\Omega, t^\varphi)}^2 \leq \frac{C_t}{t} \left( \| \Box_t^\varphi f \|_{L^2(\Omega, t^\varphi)}^2 + \| f \|_{L^2(\Omega, t^\varphi)}^2 \right) + C_t \| f \|_{L^2(\Omega, t^\varphi)}^2.
\]
Using, for example, Lemma 2.2 in [22], we see that for forms $f \in \text{Dom}(\partial_{t\varphi}^*)$, normal derivatives of $f$ are controlled by $\partial f$, $\partial_{t\varphi}^* f$, tangential derivatives of $f$, and $f$ itself. For higher order normal derivatives, we may use, for example, (3.42) in [22] to reduce the order and proceed by induction on the number of normal derivatives. It follows that for $f \in W^{2k+1}(\Omega, t\varphi) \cap \text{Dom}(\partial_{t\varphi}^*)$

$$\|f\|_{L^2,2,k(\Omega,t\varphi)}^2 \leq \frac{C}{k} \left( \|\square_{t\varphi}^* f\|_{L^2,2,k(\Omega,t\varphi)}^2 + \|f\|_{L^2,2,k(\Omega,t\varphi)}^2 \right) + C \|f\|_{L^2(\Omega,t\varphi)}^2.$$ 

Thus, by choosing $t$ large enough,

$$\|f\|_{L^2,2,k(\Omega,t\varphi)}^2 \leq \frac{C}{k} \left( \|\square_{t\varphi}^* f\|_{L^2,2,k(\Omega,t\varphi)}^2 + C \|f\|_{L^2,2,k(\Omega,t\varphi)}^2 \right).$$

While this inequality holds for forms $f \in L^{2,2k+1}_0(\Omega, t\varphi) \cap \text{Dom}(\partial_{t\varphi}^*) \cap \text{Dom}(\square_{t\varphi}^*)$, this space of forms is dense in $L^{2,2k}_0(\Omega, t\varphi) \cap \text{Dom}(\square_{t\varphi}^*)$ for which $\square_{t\varphi}^* f \in W^k(\Omega, t\varphi)$. Thus, the proof of the proposition is complete. 

**Proof of Theorem 1.6.** We have already proven that $N^q_{t\varphi} : L^{2,s}_0(\Omega, t\varphi) \rightarrow L^{2,s}_0(\Omega, t\varphi)$ in Theorem 5.2 and Proposition 5.3, so (i) is proven. Additionally, in Theorem 5.2 we proved that $H^q_{t\varphi}$ is continuous on the same range of $s$ (this requires the Closed Graph Theorem and the fact that $H^q_{t\varphi}$ is a closed operator). With this in mind, estimates for $\tilde{\partial}_{t\varphi}^* N^q_{t\varphi}$, $\partial N^q_{t\varphi}$, $\tilde{\partial}_{t\varphi}^* \partial N^q_{t\varphi}$, and $\tilde{\partial}_{t\varphi}^* \partial^s N^q_{t\varphi}$ will follow from the proof of Theorem 5.1 in [22]. The key difference is that we will use the identity

$$\|f\|_{H^q_{t\varphi}}^2 = (1 - H^q_{t\varphi}) f \quad \text{on } \text{Dom}(\tilde{\partial}_{t\varphi}^*)$$

but we have already proven estimates for $H^q_{t\varphi}$.

For operators such as $N^q_{t\varphi} \tilde{\partial}_{t\varphi}$, observe its adjoint $(N^q_{t\varphi} \tilde{\partial}_{t\varphi})^* = \overline{\partial}_{t\varphi}^* N^q_{t\varphi}$ is a bounded operator from $L^2_{0,q}(\Omega, t\varphi) \rightarrow L^2_{0,q}(\Omega, t\varphi)$. Consequently, since $N^q_{t\varphi} \tilde{\partial}_{t\varphi}$ agrees with $(\tilde{\partial}_{t\varphi}^* N^q_{t\varphi})^*$ on $\text{Dom}(\tilde{\partial}_{t\varphi})$, we simply extend $N^q_{t\varphi} \tilde{\partial}_{t\varphi}$ to be the bounded operator from $L^2_{0,q}(\Omega, t\varphi) \rightarrow L^2_{0,q}(\Omega, t\varphi)$ that agrees with $(\tilde{\partial}_{t\varphi}^* N^q_{t\varphi})^*$. When $N^q_{t\varphi}$ exists, this extension satisfies $N^q_{t\varphi} \tilde{\partial}_{t\varphi} = \overline{\partial}_{t\varphi}^* N^q_{t\varphi}$. Similarly, we can extend $N^q_{t\varphi} \tilde{\partial}_{t\varphi}^*$ to be a well-defined operator on the appropriate weighted $L^2$-spaces. Note that on the space $W^1_{0,q}(\Omega, t\varphi)$, we have $N^q_{t\varphi} \partial_{t\varphi} = N^q_{t\varphi} \partial_{t\varphi}^*.$

To estimate $N^q_{t\varphi} \tilde{\partial}_{t\varphi}$ and $N^q_{t\varphi} \tilde{\partial}_{t\varphi}$, we observe that the proof of Theorem 5.1 in [22] concludes by proving estimates for $N^q_{t\varphi} \tilde{\partial}_{t\varphi}$ and $\overline{\partial}_{t\varphi}^* N^q_{t\varphi}$, and this proof is easily generalized to the $q > 1$ case. The same proof can be adapted to estimate $N^q_{t\varphi} \tilde{\partial}_{t\varphi}$ and $\overline{\partial}_{t\varphi}^* N^q_{t\varphi}$.

**Proof of Corollary 1.9.** Let $t$ be chosen sufficiently large so that we have estimates for $I - \partial_{t\varphi}^* \overline{\partial}_{t\varphi}^* N^q_{t\varphi}$ in $L^{2,m}_0(\Omega, t\varphi)$. If $f \in L^{2,s}_0(\Omega)$ is $\tilde{\partial}$-closed, then $f \in L^{2,s}_0(\Omega, t\varphi)$ since the spaces are equivalent on bounded domains. A $\tilde{\partial}$-closed approximation

\[ \text{Recall the following well-known fact (cf. Theorem 3.19 in [17], see also [18]).} \]

**Lemma 5.4.** Let $\Omega \subset \mathbb{C}^n$ be a domain satisfying (3.2) for some $1 \leq q \leq n$ and $\varphi : \Omega \rightarrow \mathbb{C}$ a (smooth) bounded function. Then for all $t \in \mathbb{R}$, $\dim_{\mathbb{C}} H_{0,q}(\Omega, t\varphi) = \dim_{\mathbb{C}} H_{0,q}(\Omega)$. 

**Proof.** This follows immediately from the observation that $H_{0,q}(\Omega, t\varphi)$ is the orthogonal complement of $\text{Range } \overline{\partial}$ in $\ker \partial$. Since $\text{Range } \overline{\partial}$ and $\ker \partial$ are independent of the weight $t\varphi$, the orthogonal complement of $\text{Range } \overline{\partial}$ in $\ker \partial$ has the same dimension, whether measured in the weighted or unweighted spaces. 

\[ \text{Proof of Corollary 1.9.} \] Let $t$ be chosen sufficiently large so that we have estimates for $I - \partial_{t\varphi}^* \overline{\partial}_{t\varphi}^* N^q_{t\varphi}$ in $L^{2,m}_0(\Omega, t\varphi)$. If $f \in L^{2,s}_0(\Omega)$ is $\tilde{\partial}$-closed, then $f \in L^{2,s}_0(\Omega, t\varphi)$ since the spaces are equivalent on bounded domains. A $\tilde{\partial}$-closed approximation
in $L_{0,q}^{2,m}(\Omega, t\varphi)$ is produced as follows: Let $\tilde{f}$ be an approximation in $L_{0,q}^{2,m}(\Omega, t\varphi)$. Then $f - \frac{\partial}{\partial t}\varphi^*N_{t\varphi}^q\tilde{f} \in L_{0,q}^{2,m}(\Omega, t\varphi)$ is a $\tilde{\partial}$-closed approximation of $f$ for $t$ sufficiently large and $\tilde{f}$ sufficiently close to $f$ in $L_{0,q}^{2,m}(\Omega, t\varphi)$. This will also be an approximation in $L_{0,q}^{2,m}(\Omega)$ since the norm on this space is equivalent to the norm on $L_{0,q}^{2,m}(\Omega)$ for fixed $\Omega$ when $\Omega$ is bounded.

Smooth solvability will follow from the proof of Theorem 6.1.1 in [3] (see also [6, Section 6.8]). It suffices to note that these proofs require Sobolev regularity for the weighted Bergman projection $I - \frac{\partial}{\partial t}\varphi^*N_{t\varphi}^q\tilde{\partial}$ and the weighted canonical solution operator $\frac{\partial}{\partial t}\varphi^*\varphi \varphi$ (when $\tilde{q} = q - 1$) or $I - \frac{\partial}{\partial t}\varphi^*\bar{\partial}N_{t\varphi}^q$ and $N_{t\varphi}^q\bar{\partial}t\varphi$ (when $\tilde{q} = q$).

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