The single-particle density of states and a resonance in the Aharonov-Bohm potential

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ABSTRACT

The single-particle density of states (DOS) for the Pauli and the Schrödinger Hamiltonians in the presence of an Aharonov-Bohm potential is calculated for different values of the particle magnetic moment. The DOS is a symmetric and periodic function of the flux. The Krein-Friedel formula can be applied to this long-ranged potential when regularized with the zeta function. We have found that whenever a bound state is present in the spectrum it is always accompanied by a resonance. The shape of the resonance is not of the Breit-Wigner type. The differential scattering cross section is asymmetric if a bound state is present and gives rise to the Hall effect. As an application, propagation of electrons in a dilute vortex limit is considered and the Hall resistivity is calculated.

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1 Introduction

In this letter, nonrelativistic physics described by the Schrödinger and the Pauli equations is considered in the presence of an Aharonov-Bohm (AB) potential $A(r)$ \([1]\). We shall use the regular radial gauge, in which

$$A_r = 0, \quad A_\varphi = \frac{\Phi}{2\pi r} = \frac{\alpha}{2\pi r} \Phi_0.$$  \hspace{1cm} (1)

Usually, $\Phi = \alpha \Phi_0$ is the total flux through the flux tube and $\alpha \geq 0$ is the total flux $\Phi$ in the units of the flux quantum $\Phi_0$, $\Phi_0 = \hbar c/|e|$. However, the AB potential can be considered in a more general sense, since, formally, the same potential (of nonmagnetic origin) is generated around a cosmic string. The parameter $\Phi$ is then $1/Q_{\text{Higgs}}$, $\alpha = e/Q_{\text{Higgs}}$, and $\Phi_0 = 2\pi/e$ (in the units $\hbar = c = 1$) with $e$ and $Q_{\text{Higgs}}$ being respectively the charge of a test particle and the charge of the Higgs particle \([2]\). In what follows $\alpha$ will be written as $\alpha = n + \eta$, where $n = [\alpha]$ is the nearest integer smaller than or equal to $\alpha$ and $\eta$ being the fractional part. The case of a nonsingular flux tube of finite radius $R$ will be discussed, too, as it is important from the experimental point of view. Indeed, flux tubes realized in experiments such as vortices in superconductor of type II are never of zero radius. Our main results are:

- the validity of the Krein-Friedel formula \([3, 4]\) for the density of states (DOS) is for the first time established for a singular potential and the change $\Delta \rho_\alpha(E)$ over all space of the DOS induced by the AB potential is calculated;

- a resonance is predicted to occur whenever a bound state is present in the spectrum;

- in contrast to zero modes \([5]\), the number of bound states does depend on the regularization of the interior of a flux tube;

- in the presence of a bound state, the differential scattering cross section is asymmetric and the Hall effect occurs.

Details of our calculations and complete proofs are given elsewhere \([6, 7, 10]\). Here, main ideas are presented and some of the proofs are outlined.
The Krein-Friedel formula (16) gives the DOS as the sum over phase shifts and thereby relates the DOS directly to the scattering properties. It is therefore very useful to have its extension in the direction of singular (especially Coulomb) potentials. Here the Krein-Friedel formula is applied directly and consistency with previous results is shown. For example, in the particular case without bound states we confirm the anticipation of Comtet, Georgelin, and Ouvry [8] that the change of the DOS is concentrated at zero energy. The DOS provides an important link between different physical quantities: the partition function, virial coefficients, effective action, and in the relativistic case between induced fermion number and anomaly [9, 10]. We have already used its knowledge to calculate the persistent current of free electrons induced in the plane by the AB potential [11].

The discovery of a resonance was quite unexpected. Rather surprisingly, the shape of the resonance [can be read from Eq. (22)] is not of the Breit-Wigner form. Since the latter is a direct consequence of analyticity, it poses an interesting question on the analytic structure of scattering amplitudes for singular potentials. The resonance can have a profound influence on the transport properties of electrons in an experimental setup where electrons can penetrate the interior of the flux tube provided that the latter is prepared in such a way that its interior is not isolated from the system under consideration. A realistic physical realization of the penetrable flux tube is that suggested originally by Rammer and Shelankov [12] and later realized experimentally by Bending, Klitzing, and Ploog [13], i.e., to put a type II superconducting gate on top of the heterostructure containing the two-dimensional electron gas (2DEG) (see Fig. 1). When a magnetic field is switched on the conventional superconductor is penetrated by vortices of flux with $\alpha = 1/2$. Therefore, electrons from the heterostructure do not move in the homogeneous magnetic field but in the field of a penetrable flux tube.

By considering the case of a regular flux tube with a finite radius $R$ we shall show that a bound state can only occur if the gyromagnetic ratio $g_m$ is anomalous and greater than two. If $g_m$ equals exactly to two, then according to Aharonov-Casher theorem [4] zero modes occur. If $g_m < 2$, the coupling with magnetic field is not sufficiently strong enough to form neither zero modes nor bound states. In the region $g_m > 2$, i.e., exactly where the gyromagnetic ratio of electron ($g_m = 2.00232$) lies, the coupling with the magnetic field is enhanced and zero modes turn out to be bound states. However, the number of bound
Figure 1: Black layer is a superconductor of type II put on top of the heterostructure containing the two-dimensional electron gas (dotted region). When this sample is put in a homogeneous magnetic field, magnetic field penetrates the superconductor in Abrikosov vortices. Therefore, electrons from the heterostructure do not move in the homogeneous magnetic field but in the field of a (penetrable) flux tube.

state is generally higher than the number of zero modes and the number of bound states does not only depend on the total flux but also on the energy of the magnetic field.

The differential scattering cross section is a periodic function of the flux $\alpha$ and asymmetric with respect to $\varphi \rightarrow -\varphi$, where $\varphi$ is the scattering angle [7]. The asymmetry of the differential scattering cross section is easy to understand as for $\alpha \geq 0$ bound states occur only for $l \leq 0$, $l$ being the orbital angular momentum. The asymmetry of the differential scattering cross section has direct experimental consequences since it leads to the Hall effect. The Hall resistivity is calculated in the dilute vortex limit.

One has the unitary equivalence between a spin 1/2 charged particle in a 2D magnetic field and a spin 1/2 neutral particle with an anomalous magnetic moment in a 2D electric field [14] and our results apply to this case as well.

2 The Pauli and the Schrödinger Hamiltonian in the Aharonov-Bohm potential and scattering phase shifts

Let us consider the Pauli Hamiltonian,

$$H = \frac{(p - e\hbar A)^2}{2m} - \hat{\mu} \cdot B,$$

where $\hat{\mu} = \mu \hat{s}/s$ is the magnetic moment operator, $\hat{s}$ is the spin operator, and $s$ is the magnitude of the particle spin. For electron $\mu_e = -g_m|e|h/4mc = -\mu_B g_m/2$, $\mu_B$ being the Bohr magneton, and $g_m$ is the gyromagnetic ratio that characterizes the strength
of the magnetic moment \[15\]. By separating the variables, assuming \( e = -|e| \), \( H \) is written as a direct sum, \( H = \bigoplus_l H_l \), of channel radial Hamiltonians \( H_l \) in the Hilbert space \( L^2([0, \infty), rdr) \) \[1, 16\],

\[
H_l = -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{\nu^2}{r^2} + g_m \frac{\alpha}{r} s_z \delta(r). \tag{3}
\]

Here \( \nu = |l + \alpha| \) and \( s_z = \pm 1 \) is the projection of the spin on the direction of the flux tube \[1, 16\]. The Schrödinger equation is recovered upon setting \( s_z = 0 \). For positive (negative) energies the eigenvalue equation in the \( l \)-th channel reduces to the (modified) Bessel equation of the order \( \nu = |l + \alpha| \),

\[
H_l \psi_l = k^2 \psi_l \tag{4}
\]

with \( k = \sqrt{2mE/\hbar} \). In case of the impenetrable flux tube the spectra of both the Pauli and the Schrödinger equations are identical. There are neither zero modes nor bound states in this case \[7\]. The boundary condition selects only regular solutions at the origin and the “spectrum” is given by

\[
\psi_l(r, \varphi) = J_{|l+\alpha|}(kr)e^{il\varphi}. \tag{5}
\]

Phase shifts \[1\],

\[
\delta_l = \frac{1}{2} \pi (|l| - |l + \alpha|), \tag{6}
\]

are in general singular: they do not decay to zero in the limit \( E \to \infty \).

In general case, Hamiltonians \( H_l \) for which \( |l + \alpha| < 1 \) admit a one-parametric family of self-adjoint extensions \[16, 17\]. They correspond to different physics inside the flux tube. The situation will be considered when bound states

\[
B_l(r, \varphi) = K_{|l+\alpha|}(\kappa_l r)e^{il\varphi} \tag{7}
\]

of energy \( E_l = -(h^2/2m)\kappa_l^2 \) are present in the \( l = -n, -n - 1 \) channels, with \( n = [\alpha] \).

In the presence of the bound states the scattering states \( 3 \) in these channels have to be modified. They become

\[
\psi_l(r, \varphi) = [J_{|l+\alpha|}(kr) - A_l J_{-|l+\alpha|}(kr)]e^{il\varphi}. \tag{8}
\]
This is because $H_l$ has necessarily to be a symmetric operator what already determines $A_l$ to be

$$A_l = (k/\kappa_l)^{2\nu},$$

i.e., energy dependent. The radial part of the general solution (8) behaves for ($r \to \infty$) as

$$R_l(r) \sim \text{const} \left( e^{-ikr} + \frac{1 - A_l e^{i\pi|l+\alpha|}}{1 - A_l e^{-i\pi|l+\alpha|}} e^{-i\pi(|l+\alpha|+1/2)} e^{ikr} \right)$$

Therefore,

$$\delta_l = \frac{1}{2\pi}(|l| - |l + \alpha|) + \arctan \left( \frac{\sin(|l + \alpha|\pi)}{\cos(|l + \alpha|\pi) - A_l^{-1}} \right),$$

which determines S matrices, $S_i = e^{2i\delta_i}$, in these two channels.

Note that a bound state has the most profound influence on phase shifts in the limit $E_b \uparrow 0$. In this limit

$$\delta_l \to \frac{1}{2\pi}(|l| + |l + \alpha|)$$

and the phase-shift flip occurs (cf. [18]). On contrary, in the limit $E_b \downarrow -\infty$,

$$\delta_l \to \frac{1}{2\pi}(|l| - |l + \alpha|).$$

### 3 The Krein-Friedel formula and the DOS

The DOS in the presence of the AB potential is defined to be

$$\rho_\alpha(E) \equiv -\frac{1}{\pi} \text{ImTr} G_\alpha(x,x,E+i\epsilon),$$

where $G_\alpha(x,y,E+i\epsilon)$ the resolvent (the Green function) of $H$. The integrated density of states $N_\alpha(E)$ is then as usual given by

$$N_\alpha(E) \equiv \int_{-\infty}^{E} \rho_\alpha(E') \, dE'.$$

The DOS in two dimensions when all interactions are switched-off is $\rho_0(E) = (m/2\pi \hbar^2) V$, with $V = \int d^2r$ being the (infinite) volume. To calculate the change of the integrated density of states (IDOS) in the whole space we shall make use of the Krein-Friedel formula that gives the change $\Delta N_\alpha(E)$ of the IDOS induced by the the presence of a scatterer of a finite range directly by summing over phase shifts,

$$\Delta N_\alpha(E) \equiv N_\alpha(E) - N_\alpha(E) = \frac{1}{\pi} \sum_l \delta_l(E) = (2\pi i)^{-1} \ln \det S,$$
with $S$ the total on-shell S-matrix. The fact that phase shifts can be rather easily calculated without any care of the proper normalization of wave functions greatly facilitates the calculation. Moreover, by means of the Krein-Friedel formula it is rather easy to calculate the change of the IDOS for all possible self-adjoint extensions. In the case of the long-ranged AB potential we have found that the Krein-Friedel formula when combined with the $\zeta$-function regularization can still be used despite the fact that phase shifts (6) are in general singular [6, 7]. In the absence of bound states,\[ \ln \det S = \sum_{l=-\infty}^{\infty} 2i\delta_l = i\pi \sum_{l=-\infty}^{\infty} (|l| - |l + \alpha|) \]

\[ = i\pi \left[ 2\sum_{l=1}^{\infty} l^{-s} - \sum_{l=0}^{\infty} (l + \eta)^{-s} - \sum_{l=1}^{\infty} (l - \eta)^{-s} \right]_{s=1}^{s=-1} \]

\[ = i\pi \left[ 2\zeta_R(s) - \zeta_H(s, \eta) - \zeta_H(s, 1 - \eta) \right]_{s=1}^{s=-1} = -i\pi\eta(1 - \eta), \tag{17} \]

where $\zeta_R$ and $\zeta_H$ are the Riemann and the Hurwitz $\zeta$-function. Thus, using (16), the change of the DOS is

\[ \Delta \rho_\alpha(E) = \rho_\alpha(E) - \rho_0(E) = -\frac{1}{2} \eta(1 - \eta) \delta(E), \tag{18} \]

and $\Delta \rho_\alpha(E)$ is only the function of a distance from the nearest integer.

In the presence of bound states, the contribution of scattering states to $\Delta N_\alpha$ for $E \geq 0$ is

\[ \Delta N_\alpha(E) = - \frac{1}{2} \eta(1 - \eta) + \frac{1}{\pi} \arctan \left( \frac{\sin(\eta\pi)}{\cos(\eta\pi) - (|E_{-n}|/E)^\eta} \right) \]

\[ - \frac{1}{\pi} \arctan \left( \frac{\sin(\eta\pi)}{\cos(\eta\pi) + (|E_{-n-1}|/E)^{(1-\eta)}} \right), \tag{19} \]

where $E_{-n}$ and $E_{-n-1}$ are the binding energies in $l = -n$ and $l = -n - 1$ channels. By repeating the same calculation for $\alpha \leq 0$ one finds that $\Delta N_\alpha(E)$ is a symmetric function of $\alpha$ [7],

\[ \Delta N_{-\alpha}(E) = \Delta N_\alpha(E). \tag{20} \]

### 3.1 The resonance

Note that for $0 < \eta < 1/2$ the resonance appears at

\[ E = \frac{|E_{-n}|}{[\cos(\eta\pi)]^{1/\eta}} > 0. \tag{21} \]
The phase shift $\delta_n(E)$ (11) changes by $\pi$ in the direction of increasing energy and the integrated density of states (19) has a sharp increase by one. The profile of the resonance [the argument of $\arctan$ in (11)] is given by
\[ \frac{E^n \tan \eta \pi}{E^n - E_{\text{res}}^n} = \frac{\Gamma}{E^n - E_{\text{res}}^n}, \tag{22} \]
where $\Gamma = E_{\text{res}}^n \tan \eta \pi$ is the width of the resonance. Note that its profile (22) is not of the Breit-Wigner form (see Ref. [19], & 145),
\[ \frac{\Gamma}{E - E_{\text{res}}} \tag{23} \]
For $1/2 < \eta < 1$ the resonance is shifted to the $l = -n - 1$ channel. $\eta = 1/2$ is a special point since resonances occur in both channels at infinity. Therefore the contribution of the $\arctan$ terms in (19) does not vanish as $E \to \infty$, but instead gives $-1$.

4 Regularization, $R \to 0$ limit, and the interpretation of self-adjoint extensions

Different self-adjoint extensions correspond to different physics inside the flux tube (see an example in Ref. [20], p. 144). To identify the physics which underlines them we have considered the situation when the AB potential is regularized by a uniform magnetic field $B$ within the radius $R$ and satisfies the constraint
\[ \int_{\Omega} B(r) d^2 r = \Phi = \text{const}. \tag{24} \]
One finds that in the absence of the magnetic moment ($g_m = 0$) or any other attractive interaction with the interior of the flux tube the matching equation for the exterior and interior solutions in the $l$-th channel is (see [21] for example),
\[ \frac{K'_{|l+\alpha|}(x)}{K_{|l+\alpha|}(x)} = -\alpha + |l| + \alpha \frac{|l| + l + 1 + (x^2/2\alpha) + (x^2/4\alpha), |l| + 2, \alpha}{1F_{1} \left( \frac{|l| + l + 1}{2} + (x^2/4\alpha), |l| + 1, \alpha \right)}, \tag{25} \]
Here $1F_{1}(a, b, c)$ is the Kummer hypergeometric function [22], and $x_l = \kappa_l R \neq 0$. However, since the l.h.s. decreases from $-|l + \alpha|$ to $-\infty$ as $x \to \infty$ and the r.h.s. is always greater than $-l + |\alpha|$ one finds that Eq. (25) does not have a solution unless it is an attractive potential $V(r)$ inside the flux tube,
\[ V(r)|_{r \leq R} = -\frac{\hbar^2}{2m} \frac{\alpha}{R^2} c(R), \tag{26} \]
where \( V(r) = 0 \) otherwise. Here, \( c(R) = 2(1 + \varepsilon(R)), \varepsilon(R) > 0, \) and \( \varepsilon(R) \to 0 \) as \( R \to 0 \) [21, 23]. This amounts to changing \( x^2/2\alpha \) to \( x^2/2\alpha - c/2 \) on the r.h.s. of (25). Note that in the limit \( R \to 0 \)

\[
V(r)|_{r \leq R} \to -\frac{\hbar^2}{m} \frac{\alpha}{r} \delta(r).
\]  

(27)

The attractive potential can be either put in by hand or, if the Pauli Hamiltonian is considered, as arising from the magnetic moment coupling of the electrons with spin opposite to the direction of the magnetic field \( B \). In the latter case the critical potential corresponds to the case when gyromagnetic ratio \( g_m = 2 \mid \varepsilon(R) = (g_m - 2)/2 \equiv 0 \) (cf. Eq. (3)). Then the matching equation (25) has a solution in \( x = 0 \) for \( l = -n \). Since the magnetic field is not singular any more the Aharonov-Casher theorem [5] applies. It is known that there are \( \lfloor \alpha \rfloor - 1 \) zero modes in this case, \( \lfloor \alpha \rfloor \) the nearest integer larger than or equal to \( \alpha \). If \( \alpha \) is an integer then one has exactly \( n - 1 \) zero modes, if not, their number is \( n = \lfloor \alpha \rfloor \) [5]. The result only depends on the total flux \( \alpha \) and not on a particular distribution of a magnetic field \( B \).

Whenever \( g_m > 2 \) (and hence \( \varepsilon > 0 \)) or \( g_m = 2 \) with an attractive potential \( V(r) = -\varepsilon/R^2, \varepsilon > 0 \) arbitrary small, the bound states may occur in the spectrum in the channels \( l \leq 0 \). They correspond to solutions \( x_l > 0 \) of (25). In other words the coupling with the interior of the flux tube becomes sufficiently strong for the particle to be confined on the cyclotron orbit inside it. Note that the wave function (7) of bound state decays exponentially outside the flux tube. In contrast to the zero modes their number does depend on a particular distribution of the magnetic field \( B \). Using three different regularizations, uniform, regular, and cylindrical one finds that the number of bound states is less than or equals to [6, 7, 24]

\[
\#_b = 1 + n + \lfloor \alpha(g_m - 2)/4 \rfloor + \lfloor \alpha(g_m + 2)/4 - n \rfloor,
\]

(28)

with \( \lfloor \cdot \rfloor \) as above. Note that the number of bound states is generally higher than the number of zero modes [7]. The bound is saturated [6, 7] if the cylindrical shell regularization [18] of the AB potential is used. The physical origin of this difference can be understood in a simple way. In the latter case the energy \( E_B \) of magnetic field is infinite for any \( R \neq 0 \) and in this sense the magnetic field inside the flux tube is much stronger than, for
example, in the homogeneous field regularization when $E_B$, 

$$E_B = \pi B^2 R^2 / 2 = \Phi^2 / (2\pi R^2),$$

stays finite for any nonzero $R$.

## 5 The Hall effect

The S matrix, $s_\alpha(\varphi)$, in the AB potential was calculated according to

$$s_\alpha(\varphi) \equiv \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} e^{2i\delta_l+il\varphi}.$$ (30)

We have found that either in the absence or presence of bound states,

$$s_{-|\alpha|}(\varphi) = s_{|\alpha|}(-\varphi),$$ (31)

under the transformation $\alpha \to -\alpha$.

In the presence of bound states, the differential scattering cross section for $\varphi \neq 0$ was found to be

$$\left(\frac{d\sigma}{d\varphi}\right)(k, \varphi) = \left(\frac{d\sigma^0}{d\varphi}\right)(k, \varphi) + \frac{8\pi}{k} \sum_{l=-n-1}^{-n} \sin^2 \triangle l$$

$$+ \frac{4}{k \sin(\varphi/2)} \left[ \sin \triangle_{-n} \cos (\triangle_{-n} - \pi \alpha + \varphi/2) + \sin \triangle_{-n-1} \cos (\triangle_{-n-1} + \pi \alpha - \varphi/2) \right],$$ (32)

where

$$\left(\frac{d\sigma^0}{d\varphi}\right)(k, \varphi) = \frac{1}{2\pi k \sin^2(\varphi/2)} \sin^2(\pi \alpha)$$ (33)

is the differential scattering cross section in the absence of bound states. The periodicity of the differential cross section with respect to the substitution $\alpha \to \alpha \pm 1$ then follows from Eq. (32). Note that in the presence of bound states, the differential cross section becomes asymmetric with regard to $\varphi \to -\varphi$ (what is equivalent, with regard to $\alpha \to -\alpha$). The origin of the asymmetry is easy to understand since bound states for $\alpha \geq 0$ are only formed in channels with $l \leq 0$.

Now, if one considers a random distribution of flux tubes with the density $n_v$, then the Hall effect is induced. The Hall resistivity $\rho_{xy}$ can be calculated in the dilute vortex limit, i.e., when the multiple-scattering effects are ignored. The quantity that
measures the fraction of the electrons moving in a transverse direction is \( \sin(\varphi) \frac{d\sigma}{d\varphi}(k_F, \varphi) \).

Therefore, if the density of vortices is \( n_v \), the Hall current in the dilute vortex limit is proportional to

\[
n_v \int_{-\pi}^{\pi} d\varphi \sin \varphi \frac{d\sigma}{d\varphi}(k_F, \varphi).
\]  

(34)

By inverting the conductivity tensor one finds that the Hall resistivity, \( \rho_{xy} \), is

\[
\rho_{xy} = \rho_0 \frac{k_F}{\alpha} \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi} \sin \varphi \frac{d\sigma}{d\varphi}(k_F, \varphi),
\]

which was also obtained by Nielsen and Hedegaard [25]. Substituting (32) to (35) then gives

\[
\rho_{xy} = \frac{4n_v}{n_e} \frac{h}{e^2} \sin(\pi \alpha) \left[ \sin \Delta_n \cos(\Delta_n - \pi \alpha) + \sin \Delta_{n-1} \cos(\Delta_{n-1} + \pi \alpha) \right].
\]  

(36)

The result shows that the Hall resistivity is proportional to the density of vortices and depends on their vorticity via trigonometrical functions. In particular, as a self-consistency check, the Hall resistivity (36) vanishes for \( \alpha \) an integer. The Hall resistivity also vanishes whenever \( \Delta_n = -\Delta_{n-1} \) modulo \( \pi \).

6 Discussion of the results

The single-particle density of states \( \rho_\alpha \) induced by the AB potential was calculated. It was shown that \( \rho_\alpha \) is a symmetric and periodic function of the flux. Existence of the resonance in the AB potential was proven and the phase-shift flip was discussed. In addition to the flux, the number of bound states for a nonsingular flux tube was also shown to depend on the energy of the magnetic field. The Hall resistivity in the dilute vortex limit was calculated.

Our results are not only of academic but also of practical interest [13] thanks to the recent developments in the fabrication of microstructures and in mesoscopic physics (see [20] for a recent review). In particular, challenging is observation of the resonance and the Hall resistivity. They occur only in the case if a bound state is present or, in the latter case, if the phase-shift flip occurs. It will be interesting to consider an application of our results in the set-up (see Fig. [1]) proposed by Rammer and Shelankov [12], especially because recent measurements on yttrium-barium-copper oxide (YBCO) delta rings with three
grain-boundary Josephson junction [27], reported the observation of vortices that curry a flux $\alpha = 1/4$ which is smaller than the standard flux quantum $\hbar c/2e$ (corresponding to $\alpha = 1/2$) in the superconductor. Therefore, when the high-$T_c$ YBCO film is used as a gate on top of the heterostructure containing 2DEG, the resonance is at some finite energy and could in principle be observed. The same experimental set-up is also promising for detecting the Hall effect for the random distribution of vortices.

Our results have been presented in the mathematical language of self-adjoint extensions. A self-adjoint extension is actually the $R \to 0$ limit, where $R$ is the radius of a flux tube. Experimentally, infinitely thin means nothing but that the radius of the flux tube is negligibly small when compared to any other length, such as a wavelength of particles, in the system. Therefore, this is the regime in which our results can be applied. Parameters $\Delta_{-n,-n-1}$ of self-adjoint extensions are then determined by bound state energies in the $l = -n$ and $l = -n - 1$ channels.

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