Revealing subgroup structure in ranked data using a Bayesian WAND

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Abstract

Ranked data arise in many areas of application ranging from the ranking of up-regulated genes for cancer to the ranking of academic statistics journals. Complications can arise when rankers do not report a full ranking of all entities; for example, they might only report their top–$M$ ranked entities after seeing some or all entities. It can also be useful to know whether rankers are equally informative, and whether some entities are effectively judged to be exchangeable. Recent work has focused on determining an aggregate (overall) ranking but such summaries can be misleading when there is important subgroup structure in the data. In this paper we propose a flexible Bayesian nonparametric model for dealing with heterogeneous structure and ranker reliability in ranked data. The model is a Weighted Adapted Nested Dirichlet (WAND) process mixture of Plackett-Luce models and inference proceeds through a simple and efficient Gibbs sampling scheme for posterior sampling. The richness of information in the posterior distribution allows us to infer many details of the structure both between ranker groups and between entity groups (within ranker groups), in contrast to many other (Bayesian) analyses. The methodology is illustrated using several simulation studies and two real data examples.

Keywords: Dirichlet process; Gibbs sampling; mixture models; Plackett-Luce model.
1 Introduction

Ranked data are central to many applications in science and social science and arise when \( n \) rankers (individuals) provide a ranking or ordering for a set of \( K \) entities subject to some criterion. A ranking or ordering is therefore equivalent to a specific permutation of a set; this perspective has proved useful for the development of models for such data. Many types of models exist including parametric, stagewise and distance-based; a detailed overview of such models is provided by Marden (1995). The majority of these models rely on strong assumptions about the homogeneity of ranked data; the idea that there is an overall consensus view is one such example. However, as we shall see, this is often not the case and our aim is to explore the latent subgroup structure within ranked data.

We consider the popular Plackett-Luce model (Luce, 1959; Plackett, 1975) which is an extension to multiple comparison (ranked) data of the model for paired comparisons proposed by Bradley and Terry (1952). Complete and partial rankings (to be defined in Section 2) are commonplace, however, and we shall detail how the Plackett-Luce model can be modified to allow for a much richer class of rankings; namely top–\( M \) and top–\( M \) partial rankings, which we shall define in Section 2. Throughout the majority of the literature it is assumed that a ranking provided by a particular individual is no more (or less) likely to share similar beliefs to a consensus group than any another individual. We question this assumption and propose that some individuals may be significantly more informed about the entities they are ranking, and thus, their opinion should hold more weighting. Ranker reliability is introduced into the model by means of a latent binary indicator within the Plackett-Luce likelihood. In related work, Deng et al. (2014) considered ranker reliability in the context of the rank aggregation problem and proposed the method “Bayesian Aggregation of Ranked Data” (BARD). They presented an example involving the ranking of NBA teams which we revisit in Section 6.2.

By definition the rank aggregation problem aims to form a single consensus ranking and thus the heterogeneity that exists within ranked data is often ignored and all individuals are assumed to form a single homogeneous group. The main focus of this paper is to uncover any latent group structures which exist within ranked data. We aim to identify homogeneous groups of individuals who share similar beliefs along with discovering how some, or indeed all, of these groups may struggle to distinguish between certain entities. Appealing to mixtures to model potentially heterogeneous populations is commonplace. Gormley and Murphy (2008a,b).
Mollica and Tardella (2014) and Mollica and Tardella (2016) propose finite mixtures of Plackett-Luce and related models to allow for different preferences between rankers. This approach was also taken by Vitelli et al. (2015), however they adopted a distance-based model — namely that by Mallows (1957) — rather than the Plackett-Luce model.

Here we aim to increase modelling flexibility by appealing to Bayesian nonparametrics (Hjort et al., 2010). Specifically, we adopt an infinite mixture of Plackett-Luce models by means of a Dirichlet process prior, thus allowing for full generality and allowing the number of mixture components to be inferred from the data. Furthermore we aim to investigate the latent grouping of entities. We propose that, within each group of rankers, there may be some underlying belief that some of the entities are homogeneous. That is to say, a certain group of rankers may not be able to distinguish between all of the entities and therefore we should allow for some to be judged as equivalent. This aspect is somewhat less explored within the literature and approaches that allow for this have been limited to distance based models (Marden, 1992) and to the best of our knowledge no such method exists for parametric models. To allow for the entity clustering within ranker groups we introduce a further set of Dirichlet processes; we refer to these as the “low level” Dirichlet processes and there shall be a unique one for each of the ranker clusters. This results in an infinite set of “skill parameters” for each ranker group from which \( K \) are sampled. The construction is similar to that of the Nested Dirichlet Process (NDP) (Rodriguez et al., 2008), however adaptations to the NDP are required in order to handle ranked data, and these are described in Section 4. Our Adapted Nested Dirichlet process (ANDP) construction differs from the nonparametric Bayesian model proposed by Caron et al. (2014) in that they focus solely on top–\( M \) rankings and allow for a potentially infinite number of “unobserved entities” (which have not yet appeared within any top–\( M \) ranking) to appear within a future ranking. Our Adapted Nested Dirichlet process prior, on the other hand, considers the situation where the number of entities from which a ranking is produced is finite and known, but utilises the nonparametric nature of the prior to facilitate the entity grouping structure.

Our full model specification, which we call WAND (for Weighted Adapted Nested Dirichlet process), provides a highly flexible framework for analysing heterogeneous ranked data: the ANDP prior allows us to cluster both rankers and entities and the weighted Plackett-Luce likelihood allows for variable ranker reliability. Furthermore we adopt a Bayesian approach to inference and propose an efficient Markov chain Monte Carlo (MCMC) scheme for sampling from the posterior distribution of the model parameters. The first step in the construction of the
computational scheme is to introduce an appropriate version of the latent variables proposed by Caron and Doucet (2012) for inference in the Plackett-Luce model. For their suggested choice of prior distribution these latent variables enable a semi-conjugate update for our skill parameters as part of a Gibbs sampling algorithm. To further increase computational efficiency we appeal to marginal sampling methods for infinite mixtures; namely Algorithm 8 of Neal (2000). We require a nested version of this algorithm; the top level to sample the ranker cluster allocations followed by the lower level which is used to sample entity cluster allocations within each ranker cluster. Marginal methods also have the additional appeal of integrating out the infinite-dimensional aspect of the distribution and thus avoid any approximations (and problems with label switching; Papaspiliopoulos and Roberts (2008); Hastie et al. (2015)) which would be obtained if we implemented an algorithm based on the stick-breaking construction (Ishwaran and James, 2001).

The remainder of the paper is structured as follows. In Section 2 we outline the Plackett-Luce model and describe modifications for various types of rankings (complete, partial, top–M). We also describe the binary latent variable structure for modelling ranker reliability. Sections 3 and 4 describe the two-way nested cluster structure and our proposed ANDP prior for dealing with this structure. The weighted likelihood and the ANDP prior are combined to give the WAND (Weighted Adapted Nested Dirichlet process) and details of the prior specification, latent variables and the Gibbs sampling algorithm for posterior sampling are given in Section 5. Two simulation studies illustrating the use of the WAND model in various scenarios are given in Section 1 of the Supplementary Materials. Section 6 in the paper illustrates the use of the WAND model on two real datasets and compares its results to those using other methods. Section 7 offers some conclusions.

2 Model for the ranking process

We assume our data (rankings) are observations from the Plackett-Luce model (Luce (1959); Plackett (1975)). We define the set $\mathcal{K} = \{1, \ldots, K\}$ which contains $K = |\mathcal{K}|$ entities to be ranked. All entities have a “skill rating” $\lambda_k > 0$ for $k = 1, \ldots, K$; let $\lambda = (\lambda_1, \ldots, \lambda_K)$. Each ranking need not contain every entity; let $n_i \leq K$ be the number of entities contained within ranking $i$. Thus, a typical observation from this model is $x_i = \{x_{i1}, \ldots, x_{in_i}\}$. The probability
of such an observation under the Plackett-Luce model is

$$\Pr(X_i = x_i | \lambda) = \prod_{j=1}^{n_i} \frac{\lambda_{x_ij}}{\sum_{m=j}^{n_i} \lambda_{x_{im}}}$$  \hspace{1cm} (1)$$

Suppose we have \(n\) rankings. Let the data \(x_i\) for \(i = 1, \ldots, n\) be collectively denoted \(D\). The likelihood is

$$\pi(D | \lambda) = \prod_{i=1}^{n} \prod_{j=1}^{n_i} \frac{\lambda_{x_ij}}{\sum_{m=j}^{n_i} \lambda_{x_{im}}},$$

which is invariant to scalar multiplication of the parameters \(\lambda\). This leads to a parameter identifiability issue and potential mixing problems for MCMC algorithms. The identifiability issue is not of great concern as the parameters can be normalised as required. It is, however, desirable to resolve the mixing issue and this can be achieved through suitable rescaling (Caron and Doucet, 2012); details of which are provided within Section 5.1.4.

A limitation of model (1) is that the probability is only suitably defined for certain types of ranking. The model requires each ranker to report a position for each of the entities they consider. This allows for two types of ranking: (a) complete rankings, in which a ranker considers and ranks all possible entities and (b) partial rankings, in which a ranker considers a subset of all the entities and reports back a position for each, and so \(n_i < K\) in this scenario. A paired comparison, that is, an ordered list of two entities is equivalent to a partial ranking subject to \(n_i = 2\). Special cases known commonly as top–\(M\) rankings in which individuals report a rank only for those entities which they classify as being positioned 1 to \(M\) (where they have considered more than \(M\) entities) also exist. The current model does not fully capture the information in data of this type. If we naively chose to include top–\(M\) rankings with \(n_i = M\) in (1) the entity that is in position \(M\) is treated as if it was ranked last. The additional information provided by the ranker that the entities not given a position are ranked at least \((M + 1)\)th is therefore neglected. Section 2.1 provides details of how the model can be extended to include this additional information and thus cope with top–\(M\) rankings.

### 2.1 Top–\(M\) rankings

In real world scenarios we often encounter a much broader class of rankings which can be classified as top–\(M\) and top–\(M\) partial rankings. A top–\(M\) ranking is obtained via a particular individual only reporting a ranking for those entities they consider to be positioned 1 to \(M\)
having taken all $K$ entities in to account. Furthermore, a top–$M$ partial ranking is classified as a ranking obtained from an individual who only considers $K_i < K$ entities and reports positions for entities they consider to be ranked 1 to $M$ out of the $K_i$ entities considered.

A modification to the probability under the Plackett-Luce model is required in order to allow for these additional ranking types. Recall the set of all entities is defined as $\mathcal{K} = \{1, \ldots, K\}$. Now suppose ranker $i$ considers $K_i \leq K$ entities, and denote the set of these entities $\mathcal{K}_i \subseteq \mathcal{K}$. We introduce $\mathcal{U}_i = \mathcal{K}_i \setminus \{x_i\}$ as the collection of entities considered by ranker $i$ that did not feature in their ranking; by definition entities contained within this set are considered to be ranked at least $(n_i + 1)$th. The probability under the Plackett-Luce model is then modified to

$$\Pr(X_i = x_i | \lambda) = \frac{\prod_{j=1}^{n_i} \lambda_{x_{ij}}}{\sum_{m=j}^{n_i} \lambda_{x_{im}} + \sum_{m \in \mathcal{U}_i} \lambda_m}.$$  

(2)

Note that if we have only complete or partial rankings then $\mathcal{U}_i = \emptyset \forall i$ and we recover (1).

### 2.2 Ranker reliability

Until now we have assumed that each ranking is equally informative. That is, a particular ranking is no more (or less) likely to be coherent with a certain group of rankings in comparison to any other. This is a rather strong assumption; we could easily find ourselves in a situation where certain rankers are significantly more informed about the entities in comparison to fellow rankers.

We choose to model this potential heterogeneity between rankings via a mixture model with two components: one for the informative rankings and the other for uninformative rankings. We introduce a latent binary indicator variable $W_i \in \{0, 1\}$ for each ranking $(i = 1, \ldots, n)$ such that $W_i = 0$ if ranking $i$ is uninformative and $W_i = 1$ otherwise.

We take the probability under our modified Plackett-Luce model (2) and make a further modification to incorporate our latent variables $W = (W_1, \ldots, W_n)$. The probability of the $i$th observed ranking conditional on $\lambda$ and the unobserved indicator variable is

$$\Pr(X_i = x_i | \lambda, W_i = w_i) = \prod_{j=1}^{n_i} \frac{\lambda_{w_{x_{ij}}}}{\sum_{m=j}^{n_i} \lambda_{w_{x_{im}}} + \sum_{m \in \mathcal{U}_i} \lambda_m}. $$

(3)

Note that $\Pr(X_i = x_i | \lambda, W_i = 0) = (K_i - n_i)! / K_i!$, that is, the reciprocal of the number of ordered permutations of $n_i$ entities from a set of $K_i$. Therefore $W_i = 0$ corresponds to there
being no information in ranking \(i\); essentially ranker \(i\) has picked \(x_i\) uniformly at random from all possible rankings of \(n_i\) out of \(K_i\) entities. Otherwise ranker \(i\) is deemed as informative \((W_i = 1)\) and the probability of the observed ranking is given in (2). We use the notation \(X_i | \lambda, w_i \sim \text{PL}_W(\lambda, w_i)\) to denote that the probability of observation/ranking \(i\) is given by (3).

## 3 Clustering

Until now we have assumed that the dataset can be adequately summarised by using a unique parameter vector \(\lambda\). Furthermore we have also made the assumption that each value \(\lambda_i\) \((i = 1, \ldots, K)\) is unique. We suppose that there may be groups of rankers; each of which has their own beliefs about the true ranking. To implement this structure we require a parameter vector, \(\lambda_i\), to be associated with each ranker. Furthermore, we also require some rankers to share the same parameter vector, that is to say, they share the same beliefs about the entities, whence we require \(\Pr(\lambda_i = \lambda_j) \neq 0\) for some \(i \neq j\). We further propose that particular ranker groups may not be able to distinguish between certain entities, that is, they believe them to be homogeneous (tied in strength). To allow for this we require the values within each parameter vector, \(\lambda_i\), to not necessarily be unique and thus \(\Pr(\lambda_{ij} = \lambda_{il}) \neq 0\), for all \(i\) and some \(j \neq l\). We shall appeal to Bayesian nonparametric clustering methods in order to implement this structure; namely Dirichlet processes.

We use the notation \(G \sim \text{DP}(\alpha, G_0)\) to denote that a distribution \(G\) is a Dirichlet process, where \(\alpha\) and \(G_0\) are the concentration parameter and base distribution respectively. Each Dirichlet process has a corresponding stick-breaking representation (Sethuraman [1994]), whence the distribution is given by \(G(\cdot) = \sum_{j=1}^{\infty} \psi_j \delta_{\lambda_j}(\cdot)\), where \(\psi_j = v_j \prod_{\ell<j}(1-v_\ell)\), \(v_j \overset{\text{indep}}{\sim} \text{Beta}(1, \alpha)\), \(\lambda_j \overset{\text{indep}}{\sim} G_0\) and \(\delta_x(\cdot)\) denotes the Dirac probability measure concentrated at \(x\). This defines a discrete distribution with probabilities (weights) \(\psi_j\) at atoms \(\lambda_j\).

### 3.1 Two-way clustering

Using a single Dirichlet process is sufficient to cluster rankers or entities, however, we desire a model that allows for clustering on both. We therefore appeal to two-way clustering methods and focus on the Nested Dirichlet Process (Rodriguez et al. [2008]). The Nested Dirichlet
Process (NDP) is a process whose atoms are Dirichlet processes. This model has the desirable feature of having two sets of skill parameters, say $\lambda_1$ and $\lambda_2$, that are either drawn from a distribution over the same atoms with the same weights, or alternatively, from a distribution over different atoms with different weights.

The NDP can be used when all data points, $x_{ij}$ ($i = 1, \ldots, n$, $j = 1, \ldots, n_i$), are observations in their own right. As discussed in Section 2, a typical observation under the Plackett-Luce model is $x_i = \{x_{i1}, \ldots, x_{in_i}\}$, whence each element $x_{ij}$ only forms part of a single observation. The construction of the NDP dictates that a Dirichlet process would first be assigned to ranker $i$ before the skill parameters are then drawn from the DP with probability proportional to the likelihood of the ranking. However, there is no information contained within a single ranking in order to cluster entities within $x_i$. We require information from numerous rankers in order to be able to group entities together; therefore we need to alter the definition of the NDP in order to make it suitable for ranked data. We propose the Adapted Nested Dirichlet Process (ANDP), the full details of which are given in Section 4.

### 4 Adapted Nested Dirichlet Process (ANDP)

An adaptation to the Nested Dirichlet Process is required in order to build a model which allows for clustering on both the rankers and entities contained within these data. Under the standard NDP, rankers are assigned to a distribution before a sample is then drawn for each of them. The adaptation we make dictates that we first draw samples from the distributions (DPs) based on the information contained within numerous rankings before then assigning these samples to rankers via another Dirichlet process. We let $\Lambda$ denote the collection of all skill parameters in the model and use the notation $Q \sim \text{ANDP}(\alpha, \gamma, G_0)$ to refer to the distribution specified by the stick-breaking representation $G(\Lambda) = \sum_{s=1}^{\infty} \psi_s \delta_{\lambda_s^*}(\lambda^*)$, $G'(\lambda_s^*) = \sum_{t=1}^{\infty} \omega_{st} \delta_{\lambda_{st}}(\lambda)$, where $\omega_{st} = u_{st} \prod_{t' \leq t} (1 - u_{st})$, $\psi_s = v_s \prod_{t' \leq s} (1 - v_{t'})$, $u_{st} \overset{\text{iid}}{\sim} \text{Beta}(1, \gamma_s)$, $v_s \overset{\text{iid}}{\sim} \text{Beta}(1, \alpha)$ and $\lambda_{st} \sim G_0$ for $s, t \in \mathbb{N}$. A graphical representation of the NDP and the ANDP, based on Figure 1 in [Rodriguez et al., 2008], is given in the supplementary material.

It is worth noting that the NDP (and thus ANDP) models are usually specified by two concentration parameters; one controls the top level clustering, in our case rankers, the second corresponds to the lower level clustering, that is, the entity clustering. However, here we
introduce an infinite dimensional space for our low level concentration parameters, resulting in the vector $\gamma = (\gamma_1, \gamma_2, \ldots)$. This allows more flexibility within the model as we can handle scenarios where different ranker groups have different levels of entity clustering.

5 The Bayesian WAND

We are now in a position to define our complete Bayesian model — the Weighted Adapted Nested Dirichlet Process (WAND) — the main components of which are

$$X_i | \lambda_i, w_i \sim \text{PL}_W(\lambda_i, w_i), \quad i = 1, \ldots, n,$$

$$(\lambda_1, \ldots, \lambda_n) | Q \sim Q$$

$$Q | \alpha, \gamma, G_0 \sim \text{ANDP}(\alpha, \gamma, G_0),$$

that is, we use the weighted Plackett-Luce model \textup{[3]} for our data and the ANDP as the prior distribution.

5.1 Computation

There are numerous ways to perform inference for Dirichlet process mixture models. In our model the low-level Dirichlet processes are independent, hence allowing standard methods to be used in order to perform inference under the WAND. The majority of methods can be classified as taking either a conditional or marginal approach, as summarised in, for example, \cite{Papaspiliopoulos2008}. The conditional approaches typically use truncation in order to approximate the infinite-dimensional aspect of the stick-breaking prior, as pioneered by \cite{Ishwaran2001}. Avoiding approximations would be beneficial; the slice and retrospective samplers of \cite{Walker2007} and \cite{Papaspiliopoulos2008} provide methods for achieving this. However, these methods can suffer from poor mixing and convergence as they attempt to adequately sample multimodal posterior distributions. One solution is the addition of appropriate label switching moves \cite{Papaspiliopoulos2008, Hastie2015} though, in general, further empirical work is needed to determine the number and types of move that give an effective solution.

For these reasons we avoid conditional methods and instead implement a marginal sam-
pler. These samplers typically involve a Pólya urn scheme and marginalise over the infinite-dimensional distribution (Escobar and West [1995], MacEachern and Müller [1998]), and thereby avoid the need for approximations. Algorithm 8 of Neal (2000) is one such sampler; this algorithm has been shown to be one of the most efficient sampling methods for Dirichlet Process mixtures; see, for example, Papaspiliopoulos and Roberts (2008). Also there is no need for additional label switching moves. Efficiency is achieved by the algorithm only performing updates for the unique components which are currently assigned to an observation. Each observation is then assigned to either a component which is currently in use or to one of \( m \) auxiliary components which are drawn from the base distribution. This algorithm is designed to sample from a single DP mixture; we therefore propose a nested version which will enable inference to be performed under the WAND.

5.1.1 Prior specification

Specifying suitable prior distributions is a problem well discussed within the Bayesian literature (Bernardo and Smith [1994]). Here our choice of priors is mainly for mathematical convenience, however, we believe they are sufficiently general to portray most prior beliefs. The skill parameters \( \lambda_k > 0 \) must be strictly positive, therefore a suitable choice for the base distribution is a gamma distribution with mean \( a/b \), that is \( G_0 = \text{Ga}(a,b) \). Without loss of generality we set \( b = 1 \), since \( b \) is not likelihood identifiable. The latent ranker reliability variables, \( w = (w_1, \ldots, w_n) \), are binary indicators and thus we choose \( w_i \sim \text{Bern}(p_i) \) with \( p_i \in (0,1] \) for \( i = 1, \ldots, n \).

The DP concentration parameters can of course be fixed constants but making such choices can be difficult. Instead we take \( \alpha \sim \text{Ga}(a_\alpha, b_\alpha) \) and \( \gamma_s \sim \text{Ga}(a_\gamma, b_\gamma) \) for \( s \in \mathbb{N} \) a priori which allows posterior realisations for the concentration parameters to be sampled as in Escobar and West (1995); full details are provided within Section 5.1.3.

5.1.2 Latent variables

Before we can detail how posterior computation is achieved we must first introduce some latent cluster indicators. We introduce \( c = (c_1, \ldots, c_n) \) where \( c_i = j \) denotes that ranker \( i \) is associated with parameter vector \( \lambda_j \). Furthermore we require additional indicators to denote
the clustering within each parameter vector; we let \( d_{ij} = \ell \) denote that entity \( j \) within parameter vector \( i \) is allocated to entity cluster \( \ell \). We shall let \( D \) refer to the full collection of the entity cluster indicators. The value of the skill parameter assigned to entity \( j \) from ranking \( i \) is therefore given by \( \lambda_{c_i,d_{ij}} \).

The form of the Plackett-Luce likelihood does not admit conjugate Bayesian inference. However using latent variables proposed by Caron and Doucet (2012) gives a semi-conjugate update of the skill parameters and thereby a Gibbs sampler. This algorithm benefits from having no reliance on tuning parameters and, in our experience, is highly efficient. The latent variables required here are

\[
\begin{align*}
    z_{ij} | D, \Lambda, c, D, w_{indep} &\sim \text{Exp} \left( \sum_{m=j}^{n_i} \lambda_{c_i,d_{ij},x_{im}} + \sum_{m \in U_i} \lambda_{c_i,d_{ij},m} \right), \\
\end{align*}
\]

for \( i = 1, \ldots, n, j = 1, \ldots, n_i \). These latent variables result in the contribution to the complete data likelihood from ranker \( i \) being

\[
f(x_i, z_i | \Lambda, c, D, w) = \prod_{j=1}^{n_i} \lambda_{c_i,d_{ij},x_{ij}} \exp \left\{ - \left( \sum_{m=j}^{n_i} \lambda_{c_i,d_{ij},x_{im}} + \sum_{m \in U_i} \lambda_{c_i,d_{ij},m} \right) z_{ij} \right\}.
\]

### 5.1.3 Algorithm outline

We are now in a position to detail the algorithm used for sampling from the posterior distribution \( \pi(\Lambda, Z, c, D, w, \alpha, \gamma | D) \) under the WAND model. We first define \( N_r^r = |\{c_i\}_{i=1,\ldots,n}| \) to be the current number of ranker clusters and \( N_s^e = |\{d_{sj}\}_{j=1,\ldots,K}| \) as the number of entity clusters within ranker cluster \( s \). The state of the Markov chain then consists of \( c = (c_i), D = (d_{sl}), \Lambda = (\lambda_{sl}), Z = (z_{ij}), w = (w_i), \gamma = (\gamma_s) \) and \( \alpha \) for \( s = 1, \ldots, N_r^r, t = 1, \ldots, N_s^e, i = 1, \ldots, n, j = 1, \ldots, n_i \) and \( l = 1, \ldots, K \). The updates then proceed as follows.

- For \( i = 1, \ldots, n \): Let \( q^r \) be the number of distinct \( c_j \) for \( j \neq i \) and \( h^r = q^r + m^r \). Label these \( c_j \) values in \( \{1, \ldots, q^r\} \). If \( c_i = c_j \) for some \( j \neq i \), draw \( \lambda_{c_i} \sim \text{DP}(\gamma_c, G_0) \) for \( q^r < c \leq h^r \). If \( c_i \neq c_j \forall j \neq i \), let \( c_i \) have the label \( q^r + 1 \), and draw \( \lambda_{c_i} \sim \text{DP}(\gamma_c, G_0) \) for \( q^r + 1 < c \leq h^r \).

Draw a new value for \( c_i \) from \( \{1, \ldots, h^r\} \) using the probabilities

\[
\Pr(c_i = c | D, \Lambda, Z, c_{-i}, D, w) = \begin{cases} 
    b \frac{n_{i,c}^r}{n^r} f(x_i, z_i | \Lambda, c_{-i}, c_i = c, D, w), & 1 \leq c \leq q^r, \\
    b \frac{\alpha}{m^r} f(x_i, z_i | \Lambda, c_{-i}, c_i = c, D, w), & q^r < c \leq h^r,
\end{cases}
\]

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where $n^c_{i,c}$ is the number of $c_j$ for $j \neq i$ that are equal to $c$, and $b$ is the appropriate normalising constant. Change the state to contain only those $\lambda_c$ that are now associated with one or more observations. i.e. let $\Lambda = (\lambda_c : c \in \{c_1, \ldots, c_n\})$.

- Relabel $c$ such that $c_i \in \{1, \ldots, N^r\}$ for $i = 1, \ldots, n$.

- For $s = 1, \ldots, N^r$, $i = 1, \ldots, K$: Let $q^e_s$ be the number of distinct $d_{sj}$ for $j \neq i$ and $h^e_s = q^e_s + m^e$. Label these $d_{sj}$ values in $\{1, \ldots, q^e_s\}$. If $d_{si} = d_{sj}$ for some $j \neq i$, draw $\lambda_d \overset{\text{indep}}{\sim} G_0$ for $q^e_s < d \leq h^e_s$. If $d_{si} \neq d_{sj} \forall j \neq i$, let $d_{si}$ have the label $q^e_s + 1$, and draw $\lambda_d \overset{\text{indep}}{\sim} G_0$ for $q^e_s + 1 < d \leq h^e_s$.

Draw a new value for $d_{si}$ from $\{1, \ldots, h^e_s\}$ using the probabilities

$$\Pr(d_{si} = d|D, \Lambda, Z, c, D_{-si}, w) =
\begin{cases} 
    b \frac{n^e_{s,-i,d}}{n^e_{s,-i}} \prod_{i \in R} f(x_i, z_i|\Lambda, c, D_{-si}, D_{si} = d, w), & 1 \leq d \leq q^e_s, \\
    b \frac{\gamma^e_s}{m^e} \prod_{i \in R} f(x_i, z_i|\Lambda, c, D_{-si}, D_{si} = d, w), & q^e_s < d \leq h^e_s,
\end{cases}$$

where $n^e_{s,-i,d}$ is the number of $d_{sj}$ for $j \neq i$ that are equal to $d$, $R = \{i : c_i = s\}$ and $b$ is the appropriate normalising constant. Change the state to contain only those $\lambda$ that are now associated with one or more observations. i.e. let $\Lambda = (\lambda_{st} : s = 1, \ldots, N^r)$, $t \in d_s = \{d_{s1}, \ldots, d_{sK}\}$.

- For $s = 1, \ldots, N^r$ relabel $d_s$ such that $d_{sj} \in \{1, \ldots, N^e_s\}$ for $j = 1, \ldots, K$.

- For $s = 1, \ldots, N^r$, $t = 1, \ldots, N^e_s$ sample

$$\lambda_{st}|D, Z, c, D, w \overset{\text{indep}}{\sim} \text{Ga} \left( a + \beta_{st}, 1 + \sum_{i=1}^{n} w_i \sum_{j=1}^{n_i} \zeta_{ij}(s,t) z_{ij} \right),$$

where

$$\beta_{st} = \sum_{i=1}^{n} w_i \sum_{j=1}^{n_i} \mathbb{I}(c_i = s) \sum_{j=1}^{n_i} \mathbb{I}(d_{ci,xij} = t) \quad \text{and} \quad \zeta_{ij}(s,t) = \mathbb{I}(c_i = s) \sum_{m=j}^{n_i} \mathbb{I}(d_{ci,xim} = t),$$

is the number of times that the element $\lambda_{st}$ represents an entity in the rankings which are deemed informative and the number of times an entity associated with parameter $\lambda_{st}$ is ranked no higher than $j$th in the $i$th ranking, respectively. The indicator function $\mathbb{I}(A)$ takes value 1 if $A$ is true and 0 otherwise.
For $i = 1, \ldots, n$, $j = 1, \ldots, n_i$ sample

$$z_{ij}|\mathcal{D}, \Lambda, c, D, w \overset{i}{\sim} \text{Exp} \left( \sum_{m=j}^{n_i} \lambda_{ci,d_i,x_{im}}w_i + \sum_{m \in U_i} \lambda_{ci,d_i,x_{im}}w_i \right).$$

For $i = 1, \ldots, n$, sample $w_i$ from the discrete distribution given by

$$\Pr(w_i = 1|\mathcal{D}, \Lambda, Z, c, D, w_{-i}) \propto p_i f(x_i, z_i|\Lambda, c, D, w_{-i}, w_i = 1),$$
$$\Pr(w_i = 0|\mathcal{D}, \Lambda, Z, c, D, w_{-i}) \propto (1 - p_i) f(x_i, z_i|\Lambda, c, D, w_{-i}, w_i = 0) \propto (1 - p_i) \exp \left\{- \sum_{j=1}^{n_i} z_{ij}(K_i - j + 1) \right\}.$$

Conditional on the prior distribution discussed within Section 5.1.1, the concentration parameters can be sampled from mixtures as follows.

- Sample

$$\alpha|\cdots \sim \pi \text{Ga}(a_\alpha + N^r, b_\alpha - \log \eta) + (1 - \pi) \text{Ga}(a_\alpha + N^r - 1, b_\alpha - \log \eta),$$

where

$$\frac{\pi}{1 - \pi} = \frac{a_\alpha + N^r - 1}{n(b_\alpha - \log \eta)},$$

and

$$\eta|\cdots \sim \text{Beta}(\alpha + 1, n).$$

- For $s = 1, \ldots, N^r$ sample

$$\gamma_s|\cdots \overset{i}{\sim} \pi_s \text{Ga}(a_\gamma + N^e_s, b_\gamma - \log \eta_s) + (1 - \pi_s) \text{Ga}(a_\gamma + N^e_s - 1, b_\gamma - \log \eta_s),$$

where

$$\frac{\pi_s}{1 - \pi_s} = \frac{a_\gamma + N^e_s - 1}{K(b_\gamma - \log \eta_s)},$$

and

$$\eta_s|\cdots \overset{i}{\sim} \text{Beta}(\gamma_s + 1, K).$$

### 5.1.4 Rescaling

Let us consider $\lambda^\dagger = \sum_{s=1}^{N^r} \sum_{t=1}^{N^e_s} \lambda_{st}$, that is, the sum of all the $N$ unique parameters within the Markov chain. As discussed in Section 2 the Plackett-Luce likelihood is invariant to scalar multiplication of the parameters, hence $\lambda^\dagger$ is not likelihood identifiable. Indeed, if we let $\Lambda^* = \Lambda/\lambda^\dagger$, it is true that $\pi(\Lambda^*, \lambda^\dagger|\mathcal{D}) = \pi(\Lambda^*|\mathcal{D})\pi(\lambda^\dagger)$.

Caron and Doucet (2012) noted that without the addition of a rescaling step MCMC schemes for Plackett-Luce models can suffer from poor mixing. The idea is to rescale the parameters
so that the posterior distribution of $\lambda^\dagger$ is the same as its $\text{Ga}(Na, 1)$ prior distribution. This is achieved at each iteration by taking $\Lambda \rightarrow k\Lambda$, where $k = \lambda^\dagger / \sum_{s=1}^{N^r} \sum_{t=1}^{N^e} \lambda_{st}$ and $\lambda^\dagger$ is a realisation from a $\text{Ga}(Na, 1)$ distribution.

### 5.1.5 Simulation study

Two simulation studies are given in Section 1 of the Supplementary Materials which illustrate the use of the WAND model in various scenarios.

### 6 Real data analyses

We now consider two real datasets that have been analysed in the literature, and compare their conclusions with those obtained from fitting the WAND model.

#### 6.1 Roskam’s data set

We begin by analysing a data set originally collected in 1968 by Roskam, more recently studied by [de Leeuw (2006)](deLeeuw2006), and available in the R package [homals](deLeeuw2009). The data consist of rankings obtained from $n = 39$ psychologists within the Psychology Department at the University of Nijmegen (Netherlands). Each ranker gives a complete ranking of $K = 9$ sub-areas (entities), listed according to how appropriate the sub-areas are to their work. The sub-areas are: SOC - Social Psychology, EDU - Educational and Developmental Psychology, CLI - Clinical Psychology, MAT - Mathematical Psychology and Psychological Statistics, EXP - Experimental Psychology, CUL - Cultural Psychology and Psychology of Religion, IND - Industrial Psychology, TST - Test Construction and Validation, and PHY - Physiological and Animal Psychology.

The heterogeneity within these data has been analysed by [de Leeuw (2006)](deLeeuw2006) using a non-linear principal component analysis to detect groupings within the rankings. Their analysis supported the idea that there are two groups of rankings: one group which favours the qualitative fields and the other favouring the quantitative fields of psychology. A homogeneity analysis was later performed by [de Leeuw and Mair (2009)](deLeeuwMair2009) which exposed groupings of entities within the
rankings. More recently Choulakian (2016) performed a Taxicab correspondence analysis to look at structure both between the rankings and the entities within ranker groups. Their results support the conclusions of de Leeuw (2006) and suggest that the psychologists comprise two homogeneous groups with 23 and 16 members respectively. Within the larger ranker group they obtain the entity clustering \( \{ \text{MAT, EXP} \} \succ \{ \text{IND, TST} \} \succ \{ \text{PHY, SOC, EDU} \} \succ \text{CLI} \succ \text{CUL} \), where \( \succ \) means “is preferred to”, and quantitative areas of psychology appear to be preferred. The corresponding clustering of entities for the other ranker group is \( \{ \text{EDU, CLI, SOC} \} \succ \{ \text{CUL, MAT, EXP} \} \succ \{ \text{TST, IND} \} \succ \text{PHY} \), and here qualitative areas of psychology appear to be preferred. They also conclude that the larger ranker group is somewhat more homogeneous than the smaller group.

We now use our WAND model to investigate subgroup structure in these data and take our prior specification for the base distribution and concentration parameters to be \( a = 1 \) and \( \alpha = \beta = 1 \), \( \gamma = 3 \), as suggested in the Supplementary Materials. These data contain orderings of individual preferences which we believe to be informative and so take \( p_i = 0.75 \). The posterior distribution is fairly robust to this choice; a sensitivity analysis is considered in the Supplementary Materials. We report the results from a typical run of our MCMC scheme initialised from the prior, with a burn-in of 10K iterations and then run for a further 1M iterations and thinned by 100 to obtain 10K (almost) un-autocorrelated realisations from the posterior distribution. Convergence was assessed by using multiple starting values, inspection of traceplots of parameters and the log-likelihood, and standard statistics available in the R package `coda` (Plummer et al., 2006). The MCMC scheme runs fairly quickly, with C code on a single thread of an Intel Core i7-4790S CPU (3.20GHz clock speed) taking around 5 minutes.

Table 1 shows both the prior and posterior distribution for the number of ranker clusters. The data clearly have been informative and suggest that it is likely that there are between two and four ranker groups, with two groups being most plausible. Note that there is almost no posterior support to suggest there is a single (homogeneous) ranker group and so an aggregate ranking from this dataset may be misleading. The posterior distribution of the allocation of rankers to ranker groups is, of course, quite complex. A common way to summarise ranker heterogeneity is by the maximum a posteriori (MAP) allocation conditional on the MAP number of ranker groups. However, this summary can be misleading unless the posterior probability of the MAP number of groups is fairly large. Instead we prefer to summarise ranker heterogeneity using probabilities \( \Delta_{ij} = \Pr(c_i \neq c_j|D) \), that is, the posterior probability that two rankers (i
Table 1: Prior and posterior distribution of the number of ranker clusters (to 2 d.p.).

|   | 1   | 2   | 3   | 4   | 5   | 6   | 7   | ≥ 8 |
|---|-----|-----|-----|-----|-----|-----|-----|-----|
| Posterior | 0.00 | 0.43 | 0.33 | 0.16 | 0.06 | 0.02 | 0.00 | 0.00 |
| Prior    | 0.20 | 0.18 | 0.16 | 0.13 | 0.10 | 0.08 | 0.05 | 0.10 |

Figure 1: Roskam’s dataset: Dendrogram (left) showing the ranker cluster structure along with the posterior probability, $Pr(w_i = 1 | D)$, for each ranker $i$ (right).

and $j$) are not allocated to the same cluster. The allocation of rankers to groups can then be determined by thresholding these probabilities. However this too can suffer from inconsistent allocations of say ranker triples, particularly when their $\Delta_{ij}$-values are near the threshold. Therefore we use a standard summary method from cluster analysis, namely a dendrogram calculated from the dissimilarity matrix $(\Delta_{ij})$ obtained using the complete linkage method, also known as furthest neighbour clustering, as this tends to produce more densely packed clusters and does not suffer from “chaining”. Figure 1 shows the dendrogram of rankers along with the posterior probability that each ranker is informative. The dendrogram suggests that there are two ranker groups (taking height > 0.60), and this is consistent with the posterior distribution in Table 1 and the conclusions of previous analyses. We note that the data are consistent with most rankers being informative (with $Pr(w_i = 1 | D) \geq 0.8$), an increase from their prior probabilities ($p_i = 0.75$). Also the rankers whose probabilities have decreased (rankers 1, 5, 8, 10, 13, 14, 15, 31) are those with (slightly) different preferences and hence late to join the right-hand cluster in the dendrogram.

We now turn to the subgroup structure of entities within the ranker clusters, and here we condition on there being two ranker clusters. Figure 2 shows the (marginal) posterior distribution for the number of entity clusters within each ranker cluster together with the prior distribution. Again we choose not to summarise the clusters using the MAP allocation conditional on the
Figure 2: Prior and marginal posterior densities for the number of entity clusters within each ranker cluster (conditional on two ranker clusters).

Figure 3: Roskam’s dataset: Dendrograms showing the entity clustering structure within ranker cluster 1 and 2 (left and right respectively) conditional on two ranker clusters.

MAP number of groups (within each ranker group) as this is often misleading. Instead we form the entity clusters within each ranker cluster by examining a complete linkage dendrogram using dissimilarity probabilities $\Pr(d_{ij} \neq d_{ij}'|D)$ of entities $j$ and $j'$ within ranker cluster $i$; see Figure 3. We take entity clusters at heights in ranges (0.45,0.95) and (0.63,0.89) for rankers groups 1 and 2 respectively and form a preference ordering of these entity clusters by examining the marginal posteriors for the skill parameters $\lambda_{c_i|d_{ij}}$ within each ranker group $c_i$. Conditioning on these allocations to both ranker and entity groups and ordering by posterior mean, we obtain $\{\text{EXP, MAT}\} \succ \{\text{TST, PHY, IND}\} \succ \{\text{EDU, SOC, CLI}\} \succ \{\text{CUL}\}$ (with entity cluster means 3.02, 0.72, 0.22, 0.06) in ranker cluster 1 and $\{\text{SOC, EDU, CLI, MAT}\} \succ \{\text{CUL, IND, EXP, TST}\} \succ \{\text{PHY}\}$ (with entity cluster means 1.96, 0.82, 0.12) in ranker cluster two. These entity clusters (within ranker groups) are similar to those given by Choulakian (2016). Also if we use the average value of $\Pr(w_i = 1|D)$ as a measure of homogeneity within a ranker cluster then we obtain 0.68 and 0.56 for clusters 1 and 2 respectively, which again agrees with the Choulakian (2016) conclusion that ranker cluster 1 is more homogeneous than ranker cluster 2. Note that, for this data analysis, we obtain a very similar entity ordering using marginal posterior means of the skill parameters within each ranker group (marginal over the distribution of entity clusters);
Table 2: Roskam’s dataset: entity rankings by posterior mean within ranker cluster (conditional on two ranker clusters). Rank 1 corresponds to the entity most preferred within each cluster.

see Table 2. Indeed the table suggests that the ranker groups almost have opposite (reverse) preferences to each other.

We looked at the sensitivity of the posterior distribution (and inferences) to modest changes to the prior distribution. The posterior distribution was fairly insensitive to changes in the index ($a$) of the gamma base distribution and to changes in the parameters ($a_\alpha, b_\alpha, a_\gamma, b_\gamma$) of the gamma prior distributions for the concentration parameters. The posterior distribution was most sensitive to changes in the prior probabilities ($p_i$) of rankers being informative. Not surprisingly most affected by such changes were their posterior equivalents $\Pr(w_i = 1|\mathcal{D})$ though the conclusion of two ranker groups and the membership of these groups was robust.

The allocation of entities to groups (within each ranker cluster) was also fairly robust, with only a minor change in the allocation in the $p = 0.85$ case. The supplementary materials contains the (ranker and entity) dendrograms and plots of $\Pr(w_i = 1|\mathcal{D})$ for $p_i = 0.65$ and $p_i = 0.85$ in addition to the choice $p_i = 0.75$ used in this analysis.

6.2 NBA study

We now consider another dataset of ranks, studied by Deng et al. (2014) and involving rankings of NBA (National Basketball Association) teams. In their paper, Deng et al. propose a model (named “Bayesian Aggregation of Ranked Data”, BARD) which aims to aggregate rankings and identify the “relevant entities”. Their model also accommodates the possibility that rankings may not be equally reliable. One drawback of the BARD model is that it assumes that all rankings come from a single homogeneous group. We now investigate this assumption by using our model and also produce an aggregate ranking to compare with the BARD aggregate ranking.
In 2011/12 the NBA league contained $K = 30$ teams (entities) and the dataset we consider has a ranking of these teams from each of $n = 34$ rankers. The first six complete rankings were obtained from odds given at “professional” websites and the other top–8 rankings obtained from amateurs. Further, each amateur was asked to classify themselves into one of the following groups: “Avid fans” (never missed an NBA game), “Fans” (watched NBA games frequently), “Infrequent watchers” (occasionally watched NBA games) and “Not interested” (never watched NBA games). Each ranker considered all teams and so we have $K_i = K$ for $i = 1, \ldots, n$. The rankers are numbered as follows: Professionals (1–6), Avid fans (7–12), Fans (13–18), Infrequent watchers (19–25) and Not interested (26–34). Therefore we have $n_i = K = 30$ for $i = 1, \ldots, 6$ and $n_i = 8$ for $i = 7, \ldots, n$. The data are reproduced in the supplementary material. Further details on how these data were collected can be found in Deng et al. (2014).

We now analyse these data using our WAND model and see whether it is plausible that these rankers are homogeneous or whether the self-assessed groups behave differently. We take the same prior for the base distribution ($a = 1$) as in the previous example. However, to reflect weak prior beliefs that there are several ranker groups, we take $a_\alpha = b_\alpha = 3$ in addition to the previous choice for entities, $a_\gamma = b_\gamma = 3$. The prior we choose for each ranker’s ability is based on how much attention they reportedly pay to the NBA, with professional rankers likely to be most informative, followed by the Avid fans, then Fans and so on. We do this by giving the same $p_i$-value for each ranker in the same “ability” group, with $p_i = 0.9$ for professionals, $p_i = 0.7$ for Avid fans, $p_i = 0.5$ for Fans, $p_i = 0.3$ for Infrequent watchers and $p_i = 0.1$ for Not interested.

As in the previous analysis, we report the results from a typical run of our MCMC scheme initialised from the prior, with a burn-in of 10K iterations and then run for a further 1M iterations and thinned by 100 to obtain 10K (almost) un-autocorrelated realisations from the posterior distribution. Convergence was assessed by using multiple starting values, inspection of traceplots of parameters and the log-likelihood, and standard statistics available in the R package coda. Again the MCMC scheme runs reasonably quickly, with C code on a single thread of an Intel Core i7-4790S CPU (3.20GHz clock speed) taking just under 18 minutes.

Our analysis of the posterior realisations reveals very little posterior support for a single homogeneous group of rankers, with most support for two ranker groups ($\Pr(N^r = 1|\mathcal{D}) =$...
Figure 4: NBA dataset: Dendrogram (left) showing the clustering structure of rankers and highlighting those rankers with $\Pr(w_i = 1|D) < 0.25$. Plot (right) of the posterior probabilities $\Pr(w_i = 1|D)$ for each ranker, with vertical lines separating the self-certified groups.

$0.00, \Pr(N^r = 2|D) = 0.80$ and $\Pr(N^r = 3|D) = 0.17$). Figure 4 (left) shows a dendrogram of the posterior clustering structure of rankers and confirms the conclusion that there are two distinct groups of rankers: one with rankers 1–10, 12, 15 and the other with rankers 11, 14, 17–26, 28 and 32. Nearly all the other rankers are classed as uninformative, with $\Pr(w_i = 1|D) < 0.25$, except informative ranker 16 who is (roughly) equally likely to be allocated to each cluster; see Figure 4 (right). Note that obtaining a clustering by using the MAP allocation would be misleading as the MAP allocation occurs in only 60 of the 10K iterations in the MCMC chain. Unsurprisingly, uninformative rankers are typically those who pay less attention to the NBA, with average values of $\Pr(w_i = 1|D)$ for rankers in the self-certified groups (from professionals down to the not interested individuals) of 1, 1, 0.87, 0.88, 0.34 respectively. A similar conclusion was found under BARD through its ranking quality parameters; see Figure 8 in Deng et al. (2014).

Figure 5 shows the marginal posterior distribution for the number of entity clusters within each ranker cluster (conditional on there being two ranker clusters) together with the prior distribution. The posterior mean number of entity clusters for ranker clusters 1 and 2 is 8.88 and 4.58 respectively, with corresponding standard deviations 1.55 and 1.29. These distributions suggest that rankers within cluster 1 are able to distinguish between many more entities than those in cluster 2. Again this should come as no surprise as ranker cluster 2 mainly consists of rankers who typically pay little attention to the NBA. The dendrograms in Figure 5 show the entity clustering in each ranker cluster, and suggest that there are six distinct entity clusters within ranker cluster 1 (taking height $> 0.81$) and three entity clusters in ranker cluster 2 (taking height $> 0.61$). We note that the MAP clustering gives six and two entity clusters respectively,
Figure 5: Prior and marginal posterior densities for the number of entity clusters within each ranker cluster (conditional on two ranker clusters).

Figure 6: NBA dataset: Dendrograms showing the entity cluster structure within ranker clusters 1 and 2 (left and right respectively) conditional on two ranker clusters.

though there are relatively few MCMC iterations contributing to the MAP allocation for cluster 2.

It is also of interest to look at the differences in preferences between the two ranker clusters by examining the within-cluster aggregate rankings; see Table 7 in the supplementary materials. As before, these are determined by the marginal posterior mean for each entity (within each ranker cluster). The horizontal lines in this table show the MAP entity clustering described above and the (quite small) number of occurrences of the MAP is also given. So that our results can be compared to those of Deng et al. (2014), the table also includes the overall aggregate ranking, determined by ordering the mean of the (fully) marginal posterior distribution for each entity (marginalised over ranker clusters).

The entity rankings in ranker cluster 1 strongly favour the Heat (entity 1) and Thunder (2), with the Bulls (10) as the 3rd most preferred team. Ranker cluster 2 also favours the Heat but differs in their preferences for second and third positions – here being the Lakers (6) and Celtics (4). There are many differences in preference orderings between the ranker clusters, for example, the Thunder and Bulls appear in positions 11 and 5 in ranker cluster 2.
The analysis given in Deng et al. (2014) looks at “relevant entities”, defined to be those entities within the top–16, and concludes that these are \{1, 2, \ldots, 15, 18\}. The overall aggregate ranking under our WAND model gives the top–16 as \{1, 2, \ldots, 6, 8, \ldots, 12, 14, 15, 18, 20, 26\}; see Table 7 in the supplementary materials. Perhaps surprisingly, despite the BARD analysis assuming only a single ranker cluster, there is considerable overlap between the WAND and BARD top–16 lists – the differences being that entities 20 and 26 feature in our list whereas entities 7 and 13 are omitted, with entities 7 and 13 just missing out from our top–16 and appearing in positions 18 and 17. However, this can be explained by the WAND overall aggregate ranking being formed by a consensus between the very discriminating ranker cluster 1 and the much less discriminating ranker cluster 2.

If we now compare the BARD top–16 with the rankings in the WAND ranker clusters, we see that the entity rankings in ranker cluster 1 are consistent with the BARD results, with the only differences being that entity 18 is ranked 17th and entity 17 moves into the top–16. The entity rankings in ranker cluster 2 are much less consistent with the BARD results, and this is partially explained by the larger uncertainty on entity positions within this cluster. The closeness of the entity posterior means (in ranker cluster 2) helps to explain this level of rank uncertainty as these rankers clearly struggle to distinguish between entities.

The BARD analysis also reports a probability for each entity being a relevant entity, that is, the probability \(P_{16}\) that each entity is in the top–16. The values for these probabilities under both BARD and WAND models are shown in Figure 7; the vertical dashed line separates entities 1–16 from the remainder, that is, separates the teams that actually reached the top–16 playoffs that season from the others. It is interesting to see that the WAND model places much more uncertainty on many top–16 teams than under BARD, in the sense that their \(P_{16}\) values are smaller – BARD values are essentially zero or one.

7 Conclusion

In this paper, we have introduced the Weighted Plackett–Luce model – an extension to the standard Plackett–Luce model which includes a quality parameter for each ranker which can be inferred from data. This quality parameter not only gives us a tool for quantifying the ability of each ranker but also allows the model to be used in other purposes such as spam ranker
Figure 7: The probability $P_{16}$ that each entity is in the top–16 relevant entities under BARD (×) or WAND (○) models. The vertical line separates out the teams that actually reached the top–16 playoffs.

detection. We have found that inferences under the new Weighted Plackett–Luce model are reasonably robust to the addition of such (spam) rankings whereas the standard Plackett–Luce model is heavily effected (not reported here).

The Weighted Plackett–Luce model does cater for heterogeneity in ranker ability but cannot deal with rankers who express quite different preferences. To handle such scenarios we introduced a flexible non-parametric prior distribution, the Adapted Nested Dirichlet Process (ANDP), and modelled the ranked data using an infinite mixture of Weighted Plackett–Luce models with the ANDP as the conjugate prior distribution (the WAND model). The nested structure of this Dirichlet process prior results in our WAND model being capable not only of allowing for heterogeneity between rankers but also for a flexible grouping structure within the entities themselves. To the best of our knowledge the grouping of entities under the Plackett–Luce model has not been considered previously within the literature. The (marginal) posterior sampling scheme we outline allows for fast and efficient inference under our WAND model.

The modelling framework described in this paper also allows for inferences to be made using only partial information on ranks, such as top–$M$ or partial rankings. We saw through the simulation studies (in the supplementary materials) that reasonable inferences can be made under the WAND model even when only limited (partial) information is available. Although not considered here, ties within rankings can easily be accounted for within our simulation based inference approach.
The richness of information in the posterior distribution allows us to infer many details of the structure both between ranker groups and between entity groups (within ranker groups), in contrast to many other (Bayesian) analyses. The high dimension of the posterior distribution can make the production of insightful but simple summaries quite difficult and we have explored different approaches, ranging from conditioning on modal number of groups to adopting a classification based on calculations from a dissimilarity matrix summary. However our analyses have made one thing very clear: misleading inferences will routinely happen if no account is taken for (actual) heterogeneity in the data.

A fundamental assumption of the Weighted Plackett–Luce model is that rankers form their rankings under a forward ranking process. In certain situations this assumption may not be tenable. Of course, our methods can be adjusted easily to fit a Weighted Reverse Plackett–Luce model but this model is only appropriate to scenarios where the data contain only complete rankings. Currently we are studying how to relax the assumption of a ranking process a priori by considering an infinite mixture of (perhaps weighted) Extended Plackett–Luce models (Mollica and Tardella, 2014).

A copy of the Supplementary Materials can be obtained by emailing the authors.

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