Quantum Dynamical Entropy of Spin Systems

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Abstract

We investigate a quantum dynamical entropy of one-dimensional quantum spin systems. We show that the dynamical entropy is bounded from above by a quantity which is related with group velocity determined by the interaction and mean entropy of the state.

Key words: quantum dynamical system, quantum dynamical entropy, quantum spin system
1 Introduction

The classical dynamical entropy, formulated by Kolmogorov and Sinai, is a powerful and sophisticated tool to classify dynamical systems by characterizing their chaotic property. It can be regarded as a quantity which represents an optimal rate of entropy production by successive observations of a dynamical system. As for quantum dynamical systems, there have been several attempts to define quantum versions of dynamical entropy [1, 2, 3, 4, 5, 6, 7]. The differences and similarities among them were discussed in [8, 9]. In spite of a lot of efforts, because of their difficulties to calculate, only a few models, like solvable models in quantum optics [7], non-commutative shift [9], noncommutative Cat map [10, 11] or quasi free Fermionic systems [4] have been discussed in each formulation, and to the authors’ knowledge, few nontrivial physical models have been ever treated. In the present paper, we discuss the quantum dynamical entropy defined by Alicki and Fannes [3, 4] for one-dimensional quantum spin systems. We give an upper bound for the quantum dynamical entropy of quantum spin systems, which is related with mean entropy and group velocity. The upper bound does not depend upon the details of the Hamiltonian, and is considered to be rather general one. The paper is organized as follows.

In section 2, we briefly review the quantum dynamical entropy with stress on its physical interpretation. In section 3, the one-dimensional quantum spin system for which we estimate the dynamical entropy is introduced. In section 4, we introduce a norm dense $*$-subalgebra on the spin system to define the quantum dynamical entropy. In section 5, the main theorem, an upper bound for the quantum dynamical entropy of the spin systems is obtained.

2 Quantum Dynamical Entropy

In this section we briefly review what the quantum dynamical entropy defined by Alicki and Fannes [3, 4] is. Suppose there exists a $C^*$ dynamical system $(\mathcal{A}, \alpha, \omega)$ where $\mathcal{A}$ is a $C^*$ algebra, $\alpha$ is a $*$-automorphism on $\mathcal{A}$ and $\omega$ is an $\alpha$-invariant state. Let us remind that the classical dynamical entropy can be regarded as an entropy production rate by successive observations procedure. The kind of the observations is to measure in what partition a trajectory locates. The philosophy of defining a quantum version of dynamical entropy is similar. We consider successive quantum measurements on the system and discuss how fast the entropy can grow. In quantum case, a measurement (observation) process is nothing but an interaction between the system and an apparatus. The process is described mathematically by a transition expectation [12]:

\[ E : \mathcal{B} \otimes \mathcal{A} \rightarrow \mathcal{A}. \] (1)

Here $\mathcal{B}$ represents an observable algebra of an apparatus. Since it is not natural to assume that we can perform any kinds of measurements, the forms of the above transition expectation should be restricted. First, since we can not count infinity, the dimension of the observable algebra of the apparatus should be finite. Thus $\mathcal{B}$ must be $d$-dimensional.
matrix algebra \((d = 1, 2, \cdots)\). Moreover, the interaction process cannot be too sharp. (If we allow too sharp observations, we always obtain infinite dynamical entropy for type III and type I\(_\infty\) von Neumann algebra \([11]\).) This restriction is realized by introducing a dense subalgebra \(A_0\) of \(A\). That is, the restriction of \(E\) on \(B \otimes A_0\) must have its image in \(A_0\),

\[
E : B \otimes A_0 \to A_0.
\]

The successive measurements procedure can be described by the following quantum Markov chain \([13]\),

\[
\alpha \circ E \circ (\text{id} \otimes \alpha \circ E) \circ (\text{id} \otimes \text{id} \otimes \alpha \circ E) \circ \cdots (\text{id} \otimes \cdots \otimes \text{id} \otimes \alpha \circ E) : B^{\otimes n} \otimes A \to A.
\]

To assure this procedure is also the type \(B^\otimes n \otimes A_0 \to A_0, A_0\) should be chosen as an \(\alpha\)-invariant subalgebra. In addition, Alicki and Fannes impose externality on the map \(E\), that is, the map admits no nontrivial decompositions in sums of completely positive maps. Thanks to Kraus representation theorem, such an extreme map has a representation by the corresponding operational partition of unity \(\{x_i\}_{i=1}^Z \subset A_0\) such that \(\sum_i x_i^* x_i = 1\), as

\[
E(A) = \sum_{i,j} x_i^* A_{ij} x_j,
\]

where \(A_{ij}\) is an \(i, j\) component of \(M_Z(A_0) = M_Z(C) \otimes A_0\). On this setting, a finitely correlated state \([14]\) \(\rho_n\) over \(B^{\otimes n}\) is given by

\[
\rho_n(B_1 \otimes B_2 \otimes \cdots B_n) := \omega \circ \alpha \circ E \circ (\text{id} \otimes \alpha \circ E) \circ \cdots (\text{id} \otimes \cdots \otimes \text{id} \otimes \alpha \circ E)(B_1 \otimes B_2 \otimes \cdots B_n \otimes 1).
\]

The dynamical entropy is defined by the supremum of mean von Neumann entropy \(S(\rho_n)\) of \(\rho_n\) for all the possible transition expectations within the restriction. For \(M = 1, 2, \cdots,\), let us define \(Z^M \times Z^M\) matrix \(\rho(\{x_i\}, M)\) by its components

\[
\omega(x_{i_0}^* \alpha(x_{i_1}^*) \cdots \alpha^{M-1}(x_{i_{M-1}}^*) \alpha^{M-1}(x_{j_{M-1}}^*) \cdots \alpha(x_{j_1}^*) x_{j_0})\text{ for } i_s, j_s = 1, \cdots, Z, s = 1, \cdots, M.
\]

By use of this expression, one can write the dynamical entropy as

\[
h(\omega, A_0) := \sup \left\{ \limsup_M \frac{1}{M} S(\rho(\{x_i\}, M)) \; | \; \{x_i\}_{i=1}^Z \subset A_0; Z = 1, 2, \cdots; \sum_{i=1}^Z x_i^* x_i = 1 \right\}.
\]

## 3 One-dimensional Spin Systems

We deal with one-dimensional two-way infinite quantum spin systems. To each site \(x \in \mathbf{Z}\) a Hilbert space \(\mathcal{H}_x\) which is isomorphic to \(\mathbf{C}^{N+1}\) is attached and the observable algebra at site \(x\) is a matrix algebra on \(\mathcal{H}_x\) which is denoted by \(A(\{x\})\). The observable algebra on a finite set \(\Lambda \subset \mathbf{Z}\) is a matrix algebra on \(\otimes_{x \in \Lambda} \mathcal{H}_x\) and denoted by \(A(\Lambda)\). Natural identification can be used to derive an inclusion property \(A(\Lambda_1) \subset A(\Lambda_2)\) for \(\Lambda_1 \subset \Lambda_2\). The total observable algebra is a norm completion of sum of the finite region observable
algebra, $\mathcal{A} := \bigcup_{\Lambda \text{ finite}} \mathcal{A}(\Lambda)$, which becomes a $C^*$ algebra. We write the translation automorphism as $\tau_x$ ($x \in \mathbb{Z}$). (For detail, see [15].)

To discuss the dynamics, we need, $\{\alpha_t\}_{t \in \mathbb{R}}$, a one-parameter $*$-automorphism group on $\mathcal{A}$, which we assume is induced by a local interaction. That is, we assume that for each finite subset $\Lambda \subset \mathbb{Z}$, a potential $\Phi(\Lambda) \in \mathcal{A}(\Lambda)$ is defined. The translational invariance $\tau_x(\Phi(\Lambda)) = \Phi(\Lambda + x)$ for all $x \in \mathbb{Z}$ and finite $\Lambda \in \mathbb{Z}$ is assumed. The potential satisfies the locality condition, that is, there exists $\lambda > 0$ satisfying the following inequality,

$$\|\Phi\|_\lambda := \sup \left\{ |X| (N + 1)^2 |e^{iD(X)}\| \Phi(X)\|\ |	ext{X is a finite subset of Z} \right\} < \infty,$$

where $D(X) := \sup_{x,y \in X} |x - y|$. (This condition is used later to define a group velocity.)

Local Hamiltonian with respect to a finite region $\Lambda \in \mathbb{Z}$ is defined as

$$H_\Lambda := \sum_{V \subset \Lambda} \Phi(V),$$

which induces a one-parameter $*$-automorphism group $\alpha^\Lambda_t$ ($t \in \mathbb{R}$) by

$$\alpha^\Lambda_t(A) := e^{iH_\Lambda t} Ae^{-iH_\Lambda t}$$

for each local element $A \in \mathcal{A}$. Thanks to the locality of the interaction (7), the infinite volume limit for $\Lambda$ converges to a $*$-automorphism $\alpha_t$ in norm topology, i.e.,

$$\lim_{\Lambda \to \mathbb{Z}} \|\alpha_t(A) - \alpha^\Lambda_t(A)\| = 0$$

holds [15].

4 Dynamical Entropy of Quantum Spin Systems

Let us consider the dynamical entropy of the spin systems. We fix a time-invariant state $\omega$ and express simply $\alpha := \alpha_{t=1}$. We investigate the dynamical entropy for discrete dynamical system $(\mathcal{A}, \alpha, \omega)$. To calculate dynamical entropy, we must choose a natural time-invariant subalgebra $\mathcal{A}_0 \subset \mathcal{A}$ for partitions of unity. Although the most natural choice in our spin system case seems to be a norm dense subalgebra which is composed of strictly local objects,

$$\mathcal{A}_{\text{loc}} := \bigcup_{\Lambda \subset \mathbb{Z} \text{ finite}} \mathcal{A}(\Lambda).$$

This subalgebra, however, is not time invariant and therefore cannot be a candidate. Let us define a slightly larger algebra composed of exponentially localized objects. For a quasilocal object $A \in \mathcal{A}$, how strong it lives on a site $x$ can be measured by a quantity,

$$F_x(A) := \sup_{a \in \mathcal{A}(\{0\}), a \neq 0} \left( \frac{\|[A, \tau_x(a)]\|}{\|a\|} \right).$$
The set of exponentially localized objects is defined as
\[ \mathcal{A}_{\text{exp}} := \{ A \in \mathcal{A} \mid \exists \mu > 0 \text{ such that } \lim_{|x| \to \infty} e^{\mu |x|} F_x(A) = 0 \} \] (13)

This set becomes a *-subalgebra. In fact, one can easily check that for any \( c_1, c_2 \in \mathbb{C} \) and \( A_1, A_2 \in \mathcal{A}_{\text{exp}} \),
\[ F_x(c_1 A_1 + c_2 A_2) \leq |c_1| F_x(A_1) + |c_2| F_x(A_2) \] (14)
and hence \( \mathcal{A}_{\text{exp}} \) is closed with linear combination. In addition, for \( A, B \in \mathcal{A}_{\text{exp}} \), the equation,
\[ F_x(AB) = \sup_{a \in \mathcal{A}(\{y\}), a \neq 0} \left( \frac{||[A, a]|B|}{||a||} \right) \]
\[ = \sup_{a \in \mathcal{A}(\{y\}), a \neq 0} \left( \frac{||[A, a]|B + A[B, a]|}{||a||} \right) \]
\[ \leq F_x(A)||B|| + ||A||F_x(B) \] (15)
leads that \( AB \in \mathcal{A}_{\text{exp}} \). Finally, obviously \( F_x(A) = F_x(A^*) \) holds. Moreover one can show the following theorem:

**Theorem 1** The set of exponentially localized objects \( \mathcal{A}_{\text{exp}} \) is an \( \alpha_t \)-invariant *-subalgebra.

**Proof:**
According to Proposition 6.2.9. in [15], for any \( \Lambda \subset \mathbb{Z} \), for all \( a_y \in \mathcal{A}(\{y\}) \) and \( B \in \mathcal{A}_{\text{exp}} \),
\[ ||[a_y, \alpha_t(B)]|| \leq ||a_y|| \sum_{x \in \mathbb{Z}} \sup_{c \in \mathcal{A}(\{y\})} \left( \frac{||[\tau_{x+y} c], B||}{||c||} \right) e^{-|x|\lambda+2|t|\|\Phi\|_{\lambda}} \] (16)
holds. By letting \( \Lambda \) to \( \mathbb{Z} \), we obtain
\[ \frac{||[a_y, \alpha_t(B)]||}{||a_y||} \leq \sum_{x \in \mathbb{Z}} F_{x+y}(B) e^{-|x|\lambda+2|t|\|\Phi\|_{\lambda}}. \] (17)
Thus
\[ F_y(\alpha_t(B)) \leq \sum_{x \in \mathbb{Z}} F_{x+y}(B) e^{-|x|\lambda+2|t|\|\Phi\|_{\lambda}} \]
\[ \leq \sum_{z \in \mathbb{Z}} F_z(B) e^{-|z-y|\lambda+2|t|\|\Phi\|_{\lambda}} \]
\[ = \sum_{z \geq y} F_z(B) e^{-(z-y)\lambda+2|t|\|\Phi\|_{\lambda}} + \sum_{z < y} F_z(B) e^{-(y-z)\lambda+2|t|\|\Phi\|_{\lambda}} \] (18)
holds. Since we are interested in asymptotic behavior for large \( |y| \), let us consider the case when \( y \) is positively large. For any \( \epsilon_0 > 0 \), there exists \( N > 0 \) such that for \( |z| > N \),
F_{z}(B) < \epsilon_{0}e^{-\mu|z|} holds. By use of these relations we decompose the above equation. Let us choose y which is larger than N, then the first term is bounded as

$$\sum_{z \geq y} F_{z}(B)e^{-(z-y)\lambda+2t||\Phi||_{\lambda}} < \epsilon_{0}\sum_{z \geq y} e^{-\mu z}e^{-(z-y)\lambda+2t||\Phi||_{\lambda}} = \epsilon_{0}\frac{e^{-\mu y}}{1 - e^{-(\mu+\lambda)}}e^{2t||\Phi||_{\lambda}}. \quad (19)$$

The second term is also decomposed and bounded as

$$\sum_{z < y} F_{z}(B)e^{-(y-z)\lambda+2t||\Phi||_{\lambda}} = \sum_{z \in (-N,N)} F_{z}(B)e^{-(y-z)\lambda+2t||\Phi||_{\lambda}} + \sum_{z \leq -N} F_{z}(B)e^{-(y-z)\lambda+2t||\Phi||_{\lambda}}$$

$$\leq e^{-\lambda y} \sum_{z \in (-N,N)} F_{z}(B)e^{z\lambda+2t||\Phi||_{\lambda}} + \epsilon_{0}e^{-\lambda y} \frac{e^{N(\lambda-\mu)}}{1 - e^{-(\lambda+\mu)}}e^{2t||\Phi||_{\lambda}}$$

$$+ \epsilon_{0}e^{-\lambda y} \frac{e^{N(\lambda-\mu)}}{1 - e^{\lambda-\mu}}e^{2t||\Phi||_{\lambda}} - \epsilon_{0}e^{-\mu y} \frac{1}{1 - e^{\lambda-\mu}}e^{2t||\Phi||_{\lambda}}. \quad (20)$$

Thus for sufficiently small \(\epsilon'\),

$$\lim_{y \to \infty} e^{\epsilon'y}F_{y}(\alpha_{t}(B)) = 0$$

holds. With respect to the asymptotic behavior for \(y \to -\infty\), the proof goes similarly. Q.E.D.

We, hereafter, consider the dynamical entropy \(h(\omega, \alpha, A_{\exp})\) for a translationally invariant stationary state \(\omega\).

## 5 Bound for Dynamical Entropy of Spin Systems

In this section we bound the dynamical entropy of quantum spin systems. It seems natural to imagine that the range of the interaction has relationship with the dynamical entropy. For instance, if the potential \(\Phi(X)\) vanishes for \(|X| \geq 2\) and the partition of unity is strictly local, the state can be disturbed only for the fixed finite region. Thus the dynamical entropy is vanishing in such a case. Such an observation guides us to a conjecture that in general case the quantum dynamical entropy is related with the range of the interaction through its corresponding group velocity.

First let us begin with a lemma approximating an element \(A \in A_{\exp}\) by strictly local objects. As was introduced in [11], we introduce a conditional expectation on \(A([-L,L]) \quad (L > 0)\) by

$$id_{[-L,L]} \otimes \tau_{[-L,L]}\quad (22)$$
where \( \tau_{[-L,L]}^c \) is normalized trace on \([-L,L]^c = \mathbb{Z} \setminus [-L,L] \). For \( A \in \mathcal{A}_{\exp} \), an estimate,
\[
\| A - id_{[-L,L]} \otimes \tau_{[-L,L]^c}(A) \| \leq \sum_{x \in [-L,L]^c} F_x(A) \tag{23}
\]
holds [11]. Now for any \( \epsilon_0 \), there exists \( M > 0 \) such that for all \( |x| > M \), \( F_x(A) < \epsilon_0 e^{-\mu|x|} \) holds. If we take \( L \) as \( L > M \) we obtain
\[
\sum_{x \in [-L,L]^c} F_x(A) < \frac{2\epsilon_0}{1 - e^{-\mu}} e^{-\mu L}.
\tag{24}
\]
Thus the following lemma holds.

**Lemma 1** For any \( A \in \mathcal{A}_{\exp} \) and \( \epsilon_1 > 0 \), there exists \( M > 0 \) such that the following condition is satisfied. For any \( L > M \), there exists a strictly local object \( A_L \in \mathcal{A}([-L,L]) \) such that \( \| A - A_L \| < \epsilon_1 e^{-\mu L} \) and \( \| A_L \| \leq \| A \| \) holds.

Thus we obtained a good approximation of exponentially localized object by local ones. Although the strictly local object does not remain strictly local as time elapses, we can prove the following lemma.

**Lemma 2** For any strictly local object \( A \in \mathcal{A}([-L,L]), t \in \mathbb{R} \) and \( \epsilon_2 > 0 \), if \( R \in \mathbb{Z} \) satisfies
\[
R > L + \frac{2\|\Phi\|\lambda |t|}{\lambda} + \frac{1}{\lambda} \left\{ \log \left( \frac{2e^{-\lambda}}{1 - e^{-\lambda}} \right) + 4(2L + 1) \log(N + 1) + \log(2L + 1) - \log \epsilon_2 \right\}, \tag{25}
\]
an inequality
\[
\| \alpha_t(A) - \alpha_t^{[-R,R]}(A) \| \leq \epsilon_2 \| A \| \tag{26}
\]
holds.

**Proof:** \( A \in \mathcal{A}([-L,L]) \) can be decomposed into
\[
A = \sum_{\{i_x\},\{j_x\}} C(\{i_x\},\{j_x\}) \prod_{x \in [-L,L]} e(i_x,j_x) \tag{27}
\]
where \( \{e(i_x,j_x)\} \subset \mathcal{A}(\{x\}) \), \( i_x, j_x = 0, 1, \cdots, N \) is a set of matrix elements. Then we must bound
\[
\| \alpha_t(A) - \alpha_t^{[-R,R]}(A) \| \leq \sum_{\{i_x\},\{j_x\}} |C(\{i_x,j_x\})| \| \prod_{x \in [-L,L]} \alpha_t(e(i_x,j_x)) - \prod_{x \in [-L,L]} \alpha_t^{[-R,R]}(e(i_x,j_x)) \| \leq \sum_{\{i_x\},\{j_x\}} |C(\{i_x,j_x\})| \sum_{x \in [-L,L]} \| \alpha_t(e(i_x,j_x)) - \alpha_t^{[-R,R]}(e(i_x,j_x)) \|.
\]
Due to theorem 6.2.11 in [15], for \( x \in [-L,L] \), we can apply the following inequality,
\[
\| \alpha_t(e(i_x,j_x)) - \alpha_t^{[-R,R]}(e(i_x,j_x)) \| \leq (2L + 1) \left( \frac{2e^{-\lambda}}{1 - e^{-\lambda}} e^{-\lambda(R-L) + 2|t|\|\Phi\|\lambda} \right). \tag{28}
\]
We obtain,
\[
\sum_{\{i_x\}, \{j_x\}} |C(\{i_x, j_x\})| \sum_{x \in [-L, L]} \|\alpha_t(e(x, j_x)) - \alpha_t^{[-R,R]}(e(x, j_x))\|
\leq \sum_{\{i_x\}, \{j_x\}} |C(\{i_x, j_x\})|(2L + 1) \frac{2e^{-\lambda}}{1 - e^{-\lambda}} e^{-\lambda(R - L) + 2|t|\|\Phi\|_\lambda}
\leq \|A\|(N + 1)^{(2L + 1)(2L + 1)} \frac{2e^{-\lambda}}{1 - e^{-\lambda}} e^{-\lambda(R - L) + 2|t|\|\Phi\|_\lambda},
\]
(29)
where we used \( |C(\{i_x, j_x\})| \leq \|A\| \).
Q.E.D.

Now we can prove the main theorem.

**Theorem 2** For spin systems, the dynamical entropy with respect to a translationally invariant stationary state \( \omega \), is bounded from above by the following inequality,
\[
h(\omega, \alpha, \mathcal{A}_{\text{exp}}) \leq 2V(\Phi)(\sigma(\omega) + \log(N + 1)),
\]
(30)
where \( \sigma(\omega) := \lim_{\Lambda \rightarrow Z} \frac{S(\omega|\Lambda)}{|\Lambda|} \) is a mean entropy and \( V(\Phi) := \inf_\lambda \left( \frac{\|\Phi\|_\lambda}{\lambda} \right) \) is a quantity called group velocity.

**Proof:** Let us consider the dynamical entropy for a fixed partition of unity \( \{x_i\}_{i=1}^Z \subset \mathcal{A}_{\text{exp}} \). The strategy is to approximate \( \alpha_t(x_i) \) by strictly local object. For any \( \epsilon_1 > 0 \) and \( \epsilon_2 > 0 \), there exists \( M > 0 \) such that for \( L \geq M \) one can choose \( x_i^L \in \mathcal{A}([-L, L]) \) with \( \|x_i^L\| \leq \|x_i\| \) satisfying
\[
\|\alpha_t(x_i) - \alpha_t^{[-R,R]}(x_i^L)\| \leq \epsilon_1 e^{-\mu L} + \epsilon_2 \|x_i\|,
\]
(31)
where \( R \in \mathbb{Z} \) is fixed by
\[
R \geq L + \frac{2\|\Phi\|_\lambda |t|}{\lambda} + \frac{1}{\lambda} \left\{ \log \left( \frac{2e^{-\lambda}}{1 - e^{-\lambda}} \right) + 4(2L + 1) \log(N + 1) + \log(2L + 1) - \log \epsilon_2 \right\}
\geq R - 1.
\]
(32)
For any \( \epsilon > 0 \), we fix \( \epsilon_1 \) and \( \epsilon_2 \) by
\[
\epsilon_1 = \frac{\epsilon}{2}, \\
\epsilon_2 = \frac{\epsilon}{2t^2\|x_i\|^2}.
\]
(33)
Then there exists \( M > 0 \) such that for \( L \geq M \) one can find \( x_i^L \) satisfying
\[
\|\alpha_t(x_i) - \alpha_t^{[-R_i,R_i]}(x_i^L)\| \leq \frac{\epsilon}{2}(e^{-\mu L} + \frac{1}{t^2}),
\]
(34)
where $R = R_i$ is determined by the above condition (32) with (33). Thus if we fix $L$ by

$$L \geq \max\{\frac{2\log t}{\mu}, M\} \geq L - 1, \quad (35)$$

we obtain a bound,

$$\|\alpha_t(x_i) - \alpha_t[-R_i,R_i](x_i^L)\| < \frac{\epsilon}{t^2}. \quad (36)$$

As a result, for sufficiently large $t > 0$ we can approximate $\alpha_t(x_i)$ by a strictly local element $\alpha_t[-R,R](x_i^L) \in \mathcal{A}([-R,R])$ with

$$\|\alpha_t(x_i) - \alpha_t[-R,R](x_i^L)\| < \frac{\epsilon}{t^2} \quad (37)$$

where $R$ is determined by $R := \max\{R_i\}$ and

$$R_i \geq \frac{2\log t}{\mu} + \frac{2\|\Phi\|_\lambda}{\lambda}|t| + \frac{1}{\lambda}\left\{\log\left(\frac{2e^{-\lambda}}{1-e^{-\lambda}}\right) + 4\left(\frac{2\log t}{\mu} + 1\right)\log(N+1) + \log(\frac{2\log t}{\mu} + 1) - \log \epsilon + 2 \log |t| + 2 \log \|x_i\|\right\} > R_i - 1. \quad (38)$$

Thanks to the lemma 3.3 in [11], for sufficiently small $\epsilon > 0$, we obtain

$$|S(\rho(\{x_i\}, t, \alpha_1)) - S(\rho(\{x_i^L\}, t; \alpha_1[-R,R])| \leq 2Z \log(2Z\epsilon^2/6) - 2Z \left(\frac{\epsilon^2}{6}\right) \log \left(\frac{2\epsilon^2}{6}\right). \quad (39)$$

Now thanks to the lemma 3.2 in [11], $S(\rho(\{x_i^L\}, t; \alpha_1[-R,R])$ can be bounded as

$$S(\rho(\{x_i^L\}, t; \alpha_1[-R,R]) \leq S(\omega|_{-R,R}) + (2R + 1)\log(N+1), \quad (40)$$

where $\omega|_{-R,R}$ is a restriction of the state $\omega$ on $\mathcal{A}([-R,R])$ and $S(\omega|_{-R,R})$ is its von Neumann entropy. Combination of (39) and (40) and taking a limit of $\epsilon \to 0$ we finally obtain

$$h(\omega, \alpha, \mathcal{A}_{\exp}) \leq 2\frac{\|\Phi\|_\lambda}{\lambda} (\sigma(\omega) + \log(N + 1)), \quad (41)$$

where $\sigma(\omega) := \lim_{|\Lambda| \to \infty} \frac{1}{|\Lambda|} S(\omega|_\Lambda)$ is a mean entropy. Q.E.D.

6 Conclusion and Outlook

In this paper we investigated the quantum dynamical entropy of one-dimensional quantum spin systems. A generalization of our result to higher dimensional lattices is straightforward. The upper bound we found seems to be natural since the quantum dynamical
entropy roughly represents the rate of how large subalgebra of the system can be concerned as time elapses, and the group velocity bounds the region one can disturb. Our result does not depend upon the details of the interaction, and is rather general one. One might wonder whether it is possible to obtain a lower bound for the spin systems. It will depend upon the form of the interaction since the ergodic property will be strongly related. Estimations of the other dynamical entropies on the spin systems are interesting. We will address these problems elsewhere.

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