The equivalence principle in classical mechanics and quantum mechanics

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Abstract

We discuss our understanding of the equivalence principle in both classical mechanics and quantum mechanics. We show that not only does the equivalence principle hold for the trajectories of quantum particles in a background gravitational field, but also that it is only because of this that the equivalence principle is even to be expected to hold for classical particles at all.

1 The Equivalence Principle in Classical Mechanics

While the equivalence principle stands at the very heart of general relativity, there appear to be some aspects of it that are not quite as secure as they might be. In particular, while there is no disputing the fact that classical geodesics in a background gravitational field exhibit the equivalence principle, the question of whether the motions of real physical systems can explicitly be associated with such geodesics is actually a logically independent issue. Moreover, so also is the further question of what is supposed to happen when the physical systems are to be described by quantum mechanics, a situation which has actually been explored experimentally in the landmark Colella-Overhauser-Werner (COW) study [1, 2, 3, 4] of a quantum-mechanical beam of neutrons traversing an interferometer located in an external gravitational
field. In this paper we shall examine both of these issues to show that not only is the equivalence principle actually found to hold for quantum-mechanical particles, but that classical-mechanical particles actually inherit the classical-mechanical equivalence principle from them.

While the standard road to the $\hbar = 0$ classical-mechanical equivalence principle is of course completely familiar (see e.g. [5]), it is nonetheless pedagogically instructive to quickly recall the steps. Suppose we begin with a standard, free, spinless, relativistic, classical-mechanical Newtonian particle of non-zero kinematic mass $m$ moving in flat spacetime according to the special relativistic generalization of Newton’s second law of motion

$$m \frac{d^2 \xi^\alpha}{d\tau^2} = 0 \quad , \quad R_{\mu\nu\sigma\tau} = 0$$

where $d\tau = (-\eta_{\alpha\beta}d\xi^\alpha d\xi^\beta)^{1/2}$ is the proper time and $\eta_{\alpha\beta}$ is the flat spacetime metric, and where we have indicated explicitly that the Riemann tensor is (for the moment) zero. Now let us transform to an arbitrary coordinate system $x^{\mu}$. Using the definitions

$$\Gamma^\lambda_{\mu\nu} = \frac{\partial x^\lambda}{\partial x^\alpha} \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} \quad , \quad g_{\mu\nu} = \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta}$$

we find directly (see e.g. Ref. [3, 4]) that the invariant proper time takes the form $d\tau = (-g_{\mu\nu}dx^\mu dx^\nu)^{1/2}$, while the equation of motion of Eq. (1) gets rewritten as

$$m \left( \frac{D^2 x^\lambda}{D\tau^2} \right) \equiv m \left( \frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right) = 0 \quad , \quad R_{\mu\nu\sigma\tau} = 0$$

which serves to define $D^2 x^\lambda/D\tau^2$. As derived, Eq. (3) so far only holds in a strictly flat spacetime with zero Riemann curvature tensor, and indeed Eq. (3) is only a covariant rewriting of the special relativistic Newtonian second law of motion, i.e. it covariantly describes what an observer with a non-uniform velocity in flat spacetime sees, with the $\Gamma^\lambda_{\mu\nu}$ term emerging as an inertial, coordinate dependent force. While the four-velocity $dx^\lambda/d\tau$ is a general contravariant vector, its ordinary derivative $d^2 x^\lambda/d\tau^2$ (which samples adjacent points and not merely the point where the four-velocity itself is calculated) is not, and it is only $D^2 x^\lambda/D\tau^2$ which transforms as a general
contravariant four-acceleration, and it is thus only this particular four-vector on whose meaning all (accelerating and non-accelerating) observers can agree.

As regards the generalization of Eq. (3) to include a coupling of the non-zero mass particle to gravity, the great insight of Einstein was to then realize that in a non-flat spacetime if the gravitational field emerged purely from the Christoffel symbol $\Gamma_{\mu\nu}^\lambda$, Eq. (3) would then be replaced by the non-flat

$$m \left( \frac{d^2x^\lambda}{d\tau^2} + \frac{\Gamma^\lambda_{\mu\nu}}{m} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right) = 0, \quad R_{\mu\nu\sigma\tau} \neq 0. \quad (4)$$

Equation (4) achieves two things - it establishes the metric as the gravitational field in the first place, and, further, it specifies how a classical, relativistic, non-zero mass test particle is to couple to gravity. Moreover, given this Eq. (4), no less than three forms of the classical-mechanical equivalence principle then become apparent: (i) since both of the two terms on the left-hand side of Eq. (4) have the same coefficient $m$, the equality of the inertial and (passive) gravitational masses is automatically secured; (ii) since the mass parameter $m$ only appears as an irrelevant overall multiplier in Eq. (4), the trajectories associated with the integration of Eq. (4) are then independent of the masses of test particles; (iii) precisely because the Christoffel symbol is not a general coordinate tensor, it is always possible to find some general coordinate system in which all the components of $\Gamma_{\mu\nu}^\lambda$ can be made to vanish at some particular point, with it then being possible (cf. Eqs. (1) and (3)) to simulate the gravitational field at such a point by an accelerating coordinate system in flat spacetime. Since this same geodesic equation of motion can directly be obtained as the stationarity condition $\delta I_T/\delta x^\lambda = 0$ on the relativistically covariant action

$$I_T = -mc \int d\tau \quad (5)$$

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1 Collectively these three complementary forms constitute (see e.g. [7]) what is known as the weak equivalence principle, the principle which is usually invoked in order to fix the laws of motion for particles undergoing free fall in an external gravitational field.

2 Under the transformation $x'^\lambda = x^\lambda + \frac{1}{2} x^\mu x^\nu (\Gamma_{\mu\nu}^\lambda)_P$, the primed coordinate Christoffel symbols $(\Gamma_{\mu\nu}^\lambda)_P$ will be forced to vanish at the point $P$, independent in fact of how curved the Riemann tensor at that same point $P$ might be, i.e. regardless of how strong the gravitational field at $P$ might actually be, and even regardless of what particular covariant gravitational equation of motion is actually used to fix the Christoffel symbols in the first place.
under variation with respect to the coordinates of the particle itself, we thus see that a particle whose action is in fact the point test particle action $I_T$ will then automatically move geodesically in a background classical gravitational field and will then necessarily obey (all three of the above forms of) the classical-mechanical equivalence principle.

In order to distinguish the above three forms of the equivalence principle (something necessary for the discussion of the quantum-mechanical case which we give below), we note that the very writing of $I_T$ in the form $-mc \int d\tau$ entails that $I_T$ only involves one mass parameter $m$. Thus there are not two independent mass scales available to even permit independent parameterizations of the inertial and gravitational masses in the first place. With this one mass parameter also only acting as an overall multiplier in $I_T$, both the first and second forms of the classical-mechanical equivalence principle are thus seen to be explicit consequences of using the action $I_T$. However, the third form of the equivalence principle given above, namely our ability to remove the Christoffel symbols at any chosen point, is actually a property of the geometry itself independent of the existence or otherwise of any such $I_T$, and thus needs to be considered in and of itself.

To explicitly illustrate this specific point it is convenient to consider the geometry near the surface of a static, spherically symmetric source of (active) gravitational mass $M$ and radius $R$ such as the earth or a star. For such sources the geometrical line element can be written as $d\tau^2 = B(r)c^2 dt^2 - dr^2/B(r) - r^2 d\Omega$ where $B(r) = 1 - 2MG/c^2r$. If we erect a Cartesian coordinate system $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta - R$ at the surface of the earth, then, with $z$ being normal to the earth’s surface, to lowest order in $x/R, y/R, z/R, MG/c^2R (= gR/c^2)$ the Schwarzschild line element is then found [8] to take the form

$$d\tau^2 = [1 - a(z)]c^2 dt^2 - dx^2 - dy^2 - [1 + a(z)]dz^2 - b(xdx + ydy)dz$$

(6)

where $a(z) = 2g(R - z)/c^2$ and $b = 4g/c^2$. For the metric of Eq. (6) we can then calculate the Christoffel symbols near the surface of the earth with Eq. (4) then yielding

$$\ddot{x} = 0, \quad \ddot{y} = 0, \quad \ddot{z} + (2g/c^2)\dot{z} = 0, \quad \ddot{\tau} + (g/c^2)(2\dot{x}^2 + 2\dot{y}^2 - \dot{z}^2) + g\dot{\tau}^2 = 0,$$

(7)

where the dot denotes differentiation with respect to the proper time $\tau$. For trajectories for which the initial velocity $v$ is in the horizontal $x$ direction, to
lowest order in \(x/R, y/R, z/R, gR/c^2\) Eq. (7) then reduces to

\[
\ddot{x} = 0, \quad \ddot{y} = 0, \quad \ddot{t} = 0, \quad \ddot{z} + 2gv^2/c^2 + g = 0, \tag{8}
\]

(on using \(\ddot{t} = 0\) to set \(\dot{t} = 1\)), with the motion thus being equivalent to that of an acceleration in flat spacetime. Consequently, we anticipate that it must be possible to transform the line element of Eq. (6) to the flat coordinate

\[
dx'^2 = c^2 dt'^2 - dx'^2 - dy'^2 - dz'^2,
\]

something which we indeed readily achieve via the transformation

\[
x' = x, \quad y' = y, \quad t' = t(1 - gR/c^2 + gz/c^2), \quad z' = z(1 + gR/c^2 - gz/2c^2) + gt^2/2 + g(x^2 + y^2)/c^2. \tag{9}
\]

While we thus show that near its surface the earth’s gravity indeed acts the same way as an acceleration in flat spacetime, some caution is needed in trying to interpret this result. Specifically, since the metric of Eq. (4)

\footnote{The \(x = 0, y = 0, z = 0\) origin of the unprimed system obeys \(z' = gt^2/2\) in the primed system, with the Cartesian primed system straight line \(x' = vt'\), \(y' = 0\), \(z' = 0\) taking the unprimed form \(x = vt(1 - gR/c^2)\), \(y = 0\), \(z = -g(1 + 2v^2/c^2)t^2/2\) in accord with Eq. (5).}

\footnote{In passing we note that while this same analysis could of course also be made for other metrics as well, it turns out to give an instructive surprise when applied to the specific metric \(d\tau^2 = B(r)c^2 dt^2 - dx^2 - dy^2 - dz^2\) where now \(B(r) = 1 - 2MG/c^2r + \gamma r\), viz. the specific metric obtained in the alternate conformal gravity theory described in \cite{2} and references therein. Specifically, in this case in the weak gravity limit near the surface of the earth (i.e. \(\gamma R\) also small), the metric is found to reduce to the same generic form as given in Eq. (4), where now \(a(z) = 2g(R - z)/c^2 - \gamma(R + z)\) and \(b = 4g/c^2 - 2\gamma\). The resulting metric is also flat and it can be directly brought to the Cartesian form by the transformation \(x' = x, \quad y' = y, \quad z' = z(1 + gR/c^2 - gz/2c^2 - \gamma R/2 - \gamma z/4) + (g + \gamma c^2/2)t^2/2 + g(z^2 + y^2), \quad t' = t(1 - gR/c^2 + gz/c^2 + \gamma R/2 + \gamma z/2), \quad \gamma' = (g + \gamma c^2/2)t^2/2\) and under which the primed system straight line \(x' = vt\), \(y' = 0\), \(z' = 0\) takes the form \(x = vt(1 - gR/c^2 + \gamma R/2), \quad y = 0, \quad z = -g(1 + 2v^2/c^2)t^2/2 - \gamma c^2(1 - 2v^2/c^2)t^2/4\) in the unprimed system. In the unprimed coordinate system the metric yields trajectories of the form \(\ddot{x} = 0, \quad \ddot{y} = 0, \quad \ddot{t} = 0, \quad \ddot{z} + 2(g - \gamma c^2/2)v^2/c^2 + g + \gamma c^2/2 = 0\). Thus in the non-relativistic \(v = 0\) case the motion is described by \(\ddot{z} + 3g - \gamma c^2/2 = 0\), while in the relativistic \(v = c\) case we instead obtain \(\ddot{z} + 3g - \gamma c^2/2 = 0\). Thus, as already noted in Refs. [10, 11], the effect of the \(\gamma\) term is opposite in these two limits. Thus the fact that a potential may be attractive for non-relativistic motions does not in and of itself mean that it must therefore also be attractive for light, with the \(v^2/c^2\) type terms not only modifying the magnitude of the effect of gravity (something already found by Einstein following his various attempts to calculate the gravitational bending of light), but even being able to modify the sign of the effect
can be transformed to a flat metric, despite its explicit dependence on $g$ the metric of Eq. 3 must therefore be flat, with explicit calculation directly confirming the vanishing of the Riemann tensor associated with Eq. 3 in this low order. Thus we need to ascertain what happened to the non-trivial curvature which the full Schwarzschild metric is known to possess, a curvature which is associated with an explicitly non-zero Riemann tensor even in lowest order in $g$. The answer to this puzzle is that while the Christoffel symbols are first order derivative functions of the metric, the Riemann tensor is a second order derivative function. Thus to get the lowest non-trivial term in the Riemann tensor we need to expand the metric to second order in $x/R$, $y/R$, $z/R$. Since a first order expansion suffices for the Christoffel symbols, we thus see that there is a mismatch between orders of expansion of the Christoffel symbols and the Riemann tensor. Hence a first order study of the geodesics is simply not sensitive to the curvature, and thus we see why the equivalence principle not only works for weak gravity near the surface of the earth, but why in fact it even has to do so.

In order to underscore this last point, we note that the essence of Eq. 4 is that it asserts that the coupling of test particles to gravity is purely inertial, with there being no direct coupling of the particle to any non-inertial, coordinate independent quantities such as the Riemann curvature. However, in principle, Eq. 3 admits of covariant generalizations other than that given in Eq. 4. For instance, equations of motion such as the fully covariant

$$m \left( \frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right) = -\kappa_1 R^\beta_\beta \left( \frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right) \left( \kappa_1 R^\gamma_{\beta,\alpha} + \kappa_2 R_{\alpha,\beta} \frac{dx^\beta}{d\tau} + \kappa_3 R_{\alpha,\gamma} \frac{dx^\gamma}{d\tau} \right) \left( g^{\lambda,\alpha} + \frac{dx^\lambda}{d\tau} \frac{dx^\alpha}{d\tau} \right)$$

(10)
as well. Thus in general we see that even after fixing the sign of the numerical coefficient of a gravitational potential term once and for all, such a potential is then not necessarily always attractive.

In fact the Christoffel symbols are the only non-trivial first order derivative functions of the metric which are available since the covariant first order derivative of the metric just happens to vanish identically in a Riemannian geometry. Since the Christoffel symbols themselves are not covariant tensors, the identical vanishing of $g^{\mu\nu} \nu$ entails that there is no non-trivial covariant first order derivative function of the metric at all. Consequently lowest order study of the inertial properties of geodesics cannot be sensitive to any non-inertial, coordinate independent, covariant properties of the metric such as its curvature.
(κ₁, κ₂ and κ₃ are appropriate constants and Sₖ is an appropriate spin vector) also reduce back to the flat spacetime Eq. (3) in the absence of curvature; with their departure from Eq. (4) in the presence of curvature being due to terms which enable the particle to exchange energy and momentum with the gravitational field as it propagates in that field, terms which can thus be regarded as a gravitational analog of the electromagnetic Lorentz force. Moreover, in the presence of these additional terms, the motion of a test particle would then depend on its mass, with it being impossible to ever remove these additional terms at any point by any clever choice of coordinates. Despite the fact that Eq. (10) would lead to a non-null result for an Eotvos type experiment performed in a strong enough gravitational field such as, for instance, the Ricci non-flat one found near the surface of a charged or a radiating black hole, we note that since the κᵢ dependent terms in Eq. (10) are all proportional to the Riemann tensor, they are all second order in x/R, y/R, z/R. Consequently, weak gravity tests of the equivalence principle for non-zero mass particles in the small x/R, y/R, z/R, MG/c²R limit are somewhat insensitive to their possible presence (at least for some range of values of the κᵢ), with such weak gravity tests of the equivalence principle thus not requiring the explicit use of Eq. (4) where such curvature dependent terms are explicitly assumed to be absent. This may be just as well, since no proof appears to have ever been given in the standard gravity literature which would actually enable us to explicitly exclude equations of motion such as that of Eq. (10) for real, non-zero mass classical particles in the real world.

In order to emphasize the full import of this last remark, we note that for many people the equivalence principle is actually regarded as an axiom or as a postulate which is considered to be separate from general relativity itself and which is thus to be imposed as an addition to general relativity. Indeed, with test particles being defined as particles which move on geodesics, the (weak) equivalence principle postulate may be regarded as the assumption (or requirement) that for gravitational purposes real parti-

{\footnote{The κ₁ dependent term, for instance, can be derived as the variation with respect to xₙ of the action I = −κ₁ \int dτ R^{β}_{\ β}, and can thus be considered as a curvature dependent analog of the electromagnetic action I = e \int dτ A_{μ}(dx^μ/dτ).}}

{\footnote{For actual practical purposes we note that since all three of the κᵢ dependent terms just happen to vanish for spinless particles propagating in a Ricci flat geometry, weak gravity tests near the surface of the earth would anyway be completely insensitive to their possible presence.}}
cles can be treated as test particles. Now since the equivalence principle predates full general relativity, and since the equivalence principle was so instrumental in its eventual development, historically there was thus a time when the equivalence principle did have an independent existence. However, now that general relativity is so well established, we need to ask whether the equivalence principle actually is or is not a consequence of general relativity; and in a sense, one would only consider the equivalence principle to be logically independent of general relativity if one was prepared to contemplate that results such as the (weak equivalence principle establishing) null Eotvos result could be valid even while the full (strong equivalence principle) covariant apparatus of general relativity might somehow not be. Moreover, there even appears to be a shortcoming in how the weak equivalence principle is ordinarily formulated in the first place. Specifically, it is generally viewed as a "covariantization" prescription, viz. take the equations of motion of special relativity and simply replace everything that appears in those equations (such as the metric, ordinary derivatives and ordinary accelerations) by their general covariant analogs (such as the contravariant four-acceleration $D^2x^\lambda/D\tau^2$ of Eqs. (3) and (4)). Thus, since point particles, and equally light rays, move on straight lines in gravity free flat spacetime, covariantization then requires particles and rays to be just as geodesic when they propagate in a background gravitational field. However, as explicitly exhibited in Eq. (10), such a prescription is not the most general one that is permitted by the requirement that the equations of motion which are to describe the motions of particles in a background gravitational field be general coordinate invariant. Thus the central question for real (as opposed to test) particles is to what degree their dynamics is constrained by the requirement that they are to couple covariantly to gravity, and whether such a constraint in and of itself does or does not lead to their having to obey the weak equivalence principle (viz. Eq. (4)) without further assumption. It is this apparently not yet explored issue then which we shall specifically study in this paper, and as we shall see, the question actually relates to the dynamics of particles rather

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8While a great deal of attention has of course been given to the weak equivalence principle in the literature, nonetheless the primary thrust has generally been to ask whether there might be any possible non-covariant departure from the geodesic motion exhibited in Eq. (4) (the issue of preferred frames being a typical example), rather than whether there could or should be any possible covariant departure such as that exhibited in Eq. (10).
than to any constraints that geometry might impose on their kinematics.

As we have so far seen, our above discussion actually raises no less than three separate issues: (i) to what extent are real physical systems actually describable by the test particle action $I_T$ - and if not to what extent are they actually then geodesic (i.e. to what extent do we need to guard against the possible presence of explicit curvature dependent terms such as those exhibited in Eq. (10)); (ii) how do we demonstrate that light waves propagate geodesically (for massless particles $mc f d\tau$ vanishes); and (iii) to what extent does the discussion carry over to quantum-mechanical systems, systems which exhibit both particle and wave aspects. Interestingly, through the use of the quantum-mechanical study we make below, we will find that we will then be able to establish an explicit equivalence principle for massive particles in the classical limit.

In order to specifically address the issue of the extension of the equivalence principle to the massless light wave case, we note that, unlike the particle case, this time we have to begin with a classical wave equation. While we shall see below that our analysis will actually need modifying in the presence of waves with polarization, for simplicity we shall initially restrict our discussion to massless scalar classical waves, viz. waves which obey the covariant minimally coupled curved space scalar field wave equation (viz. the covariantized form of the flat space massless Klein-Gordon equation)

$$S^\mu_{;\mu} = 0 , \quad R_{\mu\nu\alpha\beta} \neq 0$$

where $S^\mu$ denotes the contravariant derivative $\partial S/\partial x^\mu$ of the scalar field $S(x)$. If we set $S(x) = \exp(it(x))$, the eikonal phase $T(x)$ is then found to obey the equation $T^\mu T^\mu - iT^\mu_{;\mu} = 0$, a condition which reduces to $T^\mu T^\mu = 0$ in the short wavelength limit. From the associated condition $T^\mu T^\mu_{;\nu} = 0$ it then follows that $T^\mu T^\nu_{;\mu} = 0$. Since normals to the wavefronts obey the eikonal relation $T^\mu = dx^\mu/dq = k^\mu$ where $q$ is a convenient affine parameter which measures distance along the normals and where $k^\mu$ is the wave vector of the wave, we thus obtain (see e.g. [6]) $k^\mu k^\nu_{;\mu} = 0$, a condition which we recognize as being exactly the massless particle geodesic equation, with rays then precisely being found to be geodesic in the eikonal limit. Thus we see that once given the standard minimally coupled Klein-Gordon equation, then not only do we indeed obtain geodesic motion, but, additionally, we explicitly find that no additional curvature dependent terms such as the ones exhibited in Eq.
are in fact then generated. Thus unlike the non-zero mass case, we see that for the massless wave case the very imposition of the minimally coupled Klein-Gordon equation eliminates any possible ambiguity in how massless waves might couple to gravity, and forces them to be strictly geodesic (in weak or strong gravity and both near to or far from a gravitational source for that matter). Since the discussion given earlier of the coordinate dependence of the Christoffel symbols was purely geometric, we thus see that once rays such as light rays are geodesic, they immediately obey the third form of the equivalence principle given earlier, with phenomena such as the gravitational bending of light then immediately following. Thus we see that the third form of the equivalence principle has primacy over the first and second ones for light (indeed light has no inertial mass or gravitational mass to begin with), and that geodesic motion need not be intimately tied to the classical test particle action $I_T$ at all. As we thus can anticipate, since quantum-mechanical systems obey wave equations, the discussion of the quantum-mechanical equivalence principle should be expected to be closer in spirit to the discussion associated with light rather than with that associated with test particles, a point which we explore below.

However, before leaving the classical-mechanical equivalence principle, it is instructive to note that not only is there an Eq. (10) type ambiguity in the response of non-zero mass particles to an external gravitational field, whenever these same particles act as gravitational sources there again is an analogous such ambiguity, one which then affects the specific gravitational fields these sources are capable of producing. Specifically, we note that it is conventional in applications of general relativity to classical macroscopic non-zero mass sources to simply take the energy-momentum tensor of typical gravitational sources to be kinematic perfect fluids, viz. of the form $T_{\mu\nu}^{\text{kin}} = (\rho + p)U^\mu U^\nu + p g^{\mu\nu}$ where $\rho$ and $p$ are the fluid energy density and pressure. The motivation for doing this is that (i) this form provides a covariant generalization of the flat spacetime perfect fluid form which is known to work extremely well for non-gravitational interactions, and (ii) the covariant conservation of this same $T_{\mu\nu}^{\text{kin}}$ precisely leads to free fall for the particles in

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9According to Eqs. (7) and (8), an observer in Einstein’s elevator would not be able to tell if a light ray (for light the dot symbol in Eq. (7) denotes differentiation with respect to the affine parameter $q$) is falling downwards under gravity or whether the elevator is accelerating upwards.
the fluid in the pressure free case, i.e. precisely to geodesic motion. While this emergence of geodesic motion is of course highly desirable, it has led to the assertion that macroscopic sources then are in fact perfect fluids. However, this quite widespread assertion is unwarranted, since while the above discussion does show that the condition \( (T_{\text{kin}}^{\mu\nu})_{;\nu} = 0 \) is indeed sufficient to give geodesic motion, it does not follow that \( T_{\text{kin}}^{\mu\nu} \) is then necessarily the complete gravitational source \( T^{\mu\nu} \). Indeed, \( T_{\text{kin}}^{\mu\nu} \) need not be the full covariant source \( T^{\mu\nu} \) of gravity, since the covariant conservation of \( T^{\mu\nu} = T_{\text{kin}}^{\mu\nu} + T_{\text{extra}}^{\mu\nu} \) will also give geodesic motion in the presence of any \( T_{\text{extra}}^{\mu\nu} \) which is itself separately covariantly conserved. Moreover, since \( T_{\text{kin}}^{\mu\nu} \) only involves the excitations of the one-particle sector out of the vacuum, it is the only piece of the full gravitational \( T^{\mu\nu} \) which is measurable in non-gravitational physics, a regime which (in contrast to gravity) is only sensitive to changes in energy and not to their zero. Thus just as we noted in our discussion of Eq. \( \text{(10)} \), in the presence of gravity it is perfectly reasonable to anticipate the emergence of explicit curvature dependent terms in \( T^{\mu\nu} \) in gravitational sources in which gravitational binding plays a direct dynamical role. And even while such explicit curvature dependent effects may be negligible in weak gravity sources such as normal stars, our ability to neglect them there in no way entails their absence in the strong gravity limit associated with collapsed stars or black holes. Thus, without a demonstration that such explicit curvature dependent terms are absent (or negligible), it is not yet warranted to assert that macroscopic sources are describable by perfect fluids at all. Hence not only is the

\[10\text{Thus if the particles in a composite object such as a planet are all in free fall in the gravitational field of the sun, then the center of mass of the planet will be in free fall too.}\]

\[11\text{Typical possible candidates for such a } T_{\text{extra}}^{\mu\nu} \text{ would be a tensor which transforms as } g^{\mu\nu} \text{ or one which transforms as the Einstein tensor } R^{\alpha\nu} - \frac{1}{2} g^{\alpha\nu} R_{\alpha}^{\alpha}, \text{ a tensor whose possible presence or absence in the full gravitational } T^{\mu\nu} \text{ is simply not ascertainable via studies of non-gravitational interactions.}\]

\[12\text{In passing we note that the notorious cosmological constant problem derives from the difficulty inherent in locating the position of none other than this very zero. Thus, absent a resolution of the cosmological constant problem, it is simply impossible to assess whether any } g^{\mu\nu} \text{ type term may or may not be present in the full gravitational } T^{\mu\nu}.\]

\[13\text{In passing we note that it turns out that if the full } T^{\mu\nu} \text{ source of gravity is locally conformal invariant, then even while the spontaneous breakdown of the conformal symmetry then explicitly induces a specific } T_{\text{extra}}^{\mu\nu}, \text{ no exchange of energy and momentum between it and } T_{\text{kin}}^{\mu\nu} \text{ is actually found to occur, with } T_{\text{kin}}^{\mu\nu} \text{ still being covariantly conserved in this particular case.}\]
response of matter to an external gravitational field not yet fully understood in classical physics, but neither is the mechanism by which classical matter sets up such gravitational fields in the first place.

2 The Equivalence Principle in Quantum Mechanics

While our above classical study of the response of non-zero mass particles to a gravitational field relied heavily on the use of the action

$$I_T = -mc \int d\tau,$$

and while this action even leads (in the presence of a gravitational source of mass $M$) to the non-relativistic Newtonian Lagrangian

$$L = T - V,$$

where $T = mi v^2 / 2$, where $V = -mgMG/r$, (and where $m_i = m_g = m$), it is important to note that, even though its variation would have led to the self-same non-relativistic Newtonian Law of Gravity, nonetheless, the Newtonian Hamiltonian $H = T + V$ was not actually explicitly encountered in our above discussion, with it actually being somewhat peripheral to it.\(^{14}\)

\(^{14}\)In passing we note that in general relativity even though the energy-momentum tensor is locally covariantly conserved, nonetheless its global integrals (such as $Q^\mu = \int d^3x (-g)^{1/2} T^{0\mu}$) are not necessarily constants of the motion in a general curved spacetime (and not even contravariant vectors in general), thus making it difficult to even define a curved spacetime Hamiltonian in the general case, or to know whether $Q^0$ might be able to serve as one in some specific one. Now, for the restricted case of the test particle action

$$I_T = -mc \int d\tau,$$

the associated $Q^\mu$ is readily calculated through use of the energy-momentum tensor $T^{\mu\nu} = 2(-g)^{-1/2} \delta I_T / \delta g_{\mu\nu} = mc(-g)^{-1/2} \int d\tau \delta^4(x - y(\tau)) (dy^\mu / d\tau)(dy^\nu / d\tau)$, and is found to take the simple, suggestive form

$$Q^\mu = mc dx^\mu / d\tau.$$

However, explicit evaluation of the non-relativistic limit of this particular $Q^\mu$ in the curved geometry where $g_{00} = -(1 - 2MG/c^2r)$ is then found to yield $cQ^0 \to mc^2 + mv^2/2 + mMG/r \sim mc^2 + T + V$, an expression which is not of our desired $T + V$ form. On the other hand, in this same geometry the quantity $-cQ_0$ does in fact reduce to $mc^2 + mv^2/2 - mMG/r$ and thus is nicely of the $T + V$ form. In order to determine which one, if either, of these two particular quantities is to be identified as the energy of the particle, we recall that in classical mechanics there is actually a second, entirely different way to define the energy, viz. via the quantities $R_\mu = \partial I_M^{ST} / \partial x^\mu$, where the derivatives act on the end point $x^\mu$ of the integral of some general matter action $I_M^{ST}$ as calculated in the specific stationary path which minimizes $I_M$. Since $I_M$ is always a general coordinate scalar, the $R_\mu$ always transform as a covariant four-vector no matter how complicated a function $I_M$ might be, so that $E = -cR_0$ can nicely serve as a well-defined energy even in curved spacetime. As we show in detail below, explicit evaluation of $R_\mu$ in the situation where $I_M$ is the test particle action $I_T$ then yields $R_\mu = \partial I_T^{ST} / \partial x^\mu = mc dx_\mu / d\tau$, i.e. just the one which does in fact lead to the requisite $E = mc^2 + T + V$. Now it is important to stress that in flat spacetime the two quantities $cQ^0$ and $-cQ_0$ actually do coincide, with
in the non-relativistic quantum case it is precisely the Hamiltonian which plays the most direct role, with the Schrödinger equation

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m_i} \nabla^2 \psi + V(r)\psi \tag{12} \]

following directly as the quantization of \( H = T + V = E \). Thus we immediately need to ask whether it is in fact legitimate to use this familiar and very tempting quantization prescription in the case where \( V \) is not just any arbitrary potential energy but is in fact precisely the gravitational \( V(r) = -m_gMG/r \). Thus we now have to ask all over again whether it is in fact legitimate to set \( m_i = m_g = m \) in the quantum-mechanical Eq. (12), and note immediately that even if we are allowed to do so, nonetheless, the mass parameter \( m \) no longer appears as the overall multiplier found in the classical Eq. (4). Consequently, we see that no matter what the quantum-mechanical status of the first form of the equivalence principle given above, the second form immediately fails in quantum mechanics, with the solutions to the Schrödinger equation very much depending on the mass \( m \). Consequently, at this point all three of our forms of the classical equivalence principle now become questionable in quantum mechanics, and so it is to this issue to that we now turn.

In order to bypass having to address the issue of the quantization of the non-relativistic Newtonian \( H \), one could instead start with the massive flat spacetime Klein-Gordon equation

\[ S_{\mu}^{\nu} - (mc/\hbar)^2 S = 0, \quad R_{\mu\nu\sigma\tau} = 0 \tag{13} \]

and make the general coordinate transformation of Eq. (9) from flat Cartesian coordinates to those associated with the (still flat) metric of Eq. (6). In this way Greenberger and Overhauser [3] found that a subsequent non-relativistic reduction of Eq. (13) in such coordinates then led precisely to it being only in curved spacetime that there is any difference between these covariant and contravariant quantities. Thus it is through the explicit use of the stationary action that we are able to identify the covariant \( mcdx_\mu /d\tau \) rather than the contravariant \( mcdx^{\mu} /d\tau \) as the appropriate energy-momentum vector (i.e. the energy is conjugate to the contravariant time \( t = x^0/c \) and can thus be defined as the derivative of the stationary action with respect to \( x^0 \)). Thus, as we see, our very ability to introduce an appropriate non-relativistic Hamiltonian at all requires us to first formulate an appropriate fully relativistic theory, an issue we explicitly take care of in the following.
the Schrödinger equation of Eq. (12) near the surface of the earth. Under such a procedure the presence of only one mass parameter \( m \) in the initial Klein-Gordon equation then entailed the equality \( m_i = m_g = m \) in the resulting Schrödinger equation (with the gravitational potential energy near the surface of the earth then being given by \( V = mgz \)). However, since all of this analysis was made in flat spacetime, it served only to show that quantization of \( H = T + V \) is equivalent to simulating gravity by an acceleration in flat spacetime, and thus did not address the question of whether in quantum mechanics it is actually legitimate to simulate true curvature by such an acceleration in the first place.

To address the specific issue of the actual quantum-mechanical status of the third form of the equivalence principle, we must instead look at a fully covariant analysis of the above non-zero mass Klein-Gordon equation

\[
S_{\mu}^{\mu} - (mc/\hbar)^2 S = 0, \quad R_{\mu\nu\sigma\tau} \neq 0
\]  

(14)

as covariantized to curved spacetime. And indeed we find \[12\] that in the background gravitational field of the earth, viz. \( d\tau^2 = B(r)c^2dt^2 - dr^2/B(r) - r^2d\Omega \) where \( B(r) = 1 - 2MG/c^2r \), setting \( S(x) = \exp(-imc^2t/\hbar)\psi(x) \) then explicitly leads under non-relativistic reduction to

\[
\frac{i\hbar}{\hbar} \frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 \psi = \frac{mc^2}{2} [B(r) - 1] \psi = -\frac{mMG}{r} \psi,
\]  

(15)

to thus not only lead us directly to Eq. (12) and to not only directly enforce \( m_i = m_g = m \), but also to show that on restricting the Schwarzschild metric to the weak gravity Eq. (6) near the surface of the earth, we are then able to directly recover the inertial result presented in Ref. [3]. Thus we show that our third (purely geometric) form of the equivalence principle does survive quantum mechanics, with a gravitational field still being simulatable by an acceleration in flat spacetime even for quantum-mechanical systems.

Since our above discussion recovers the equivalence principle without any reference to \( I_T = -mc \int d\tau \), we see that, just as in our earlier discussion of the classical-mechanical Klein-Gordon equation, the test particle action \( I_T \) is again found to be somewhat peripheral to the equivalence principle, with the heart of the issue being not the first or the second forms of the equivalence principle at all, but rather the primacy of the third form instead.\[15\]

\[15\]From the point of view of quantum mechanics, the equality of \( m_i \) and \( m_g \) should really
while we have just seen that we did not need to introduce $I_T$, it is nonetheless possible to make some contact with it. Specifically, if we make the substitution $S(x) = \exp(iP(x)/\hbar)$ in the quantum-mechanical curved spacetime Klein-Gordon equation given as Eq. (14), we then obtain

$$P^\mu P_\mu + m^2 c^2 = i\hbar P^\mu_{\mu}.$$  \hfill (16)

In the eikonal or in the small $\hbar$ approximation the $i\hbar P^\mu_{\mu}$ term can be dropped, so that the phase $P(x)$ is then seen to obey the purely classical condition

$$g_{\mu\nu}P^\mu P^\nu + m^2 c^2 = 0,$$  \hfill (17)

a condition which we immediately recognize as the covariant Hamilton-Jacobi equation of classical mechanics, an equation whose solution is known to be the stationary classical action $\int p_\mu dx^\mu$ between relevant end points.\(^{16}\) In the eikonal approximation then we can thus identify the wave phase $P(x)$ as $\int p_\mu dx^\mu$, with the phase derivative $P^\mu$ then being given as the particle momentum $p^\mu = mcdx^\mu/d\tau$, a four-vector momentum which obeys

$$p^\mu p_\mu + m^2 c^2 = 0,$$  \hfill (18)

viz. the familiar fully covariant particle energy-momentum relation.\(^{17}\) Co-

be regarded as an equality of the inertial $(h/mv)$ and gravitational $(h/m_g v)$ de Broglie wavelengths, with the quantum-mechanical equivalence principle being interpretable as the statement that neutron beams interfere in horizontal and vertical interferometers with one and the same de Broglie wavelength $h/mv$, and with the general rule being that rather than coupling primarily to mass, gravity (itself a field theory) couples first and foremost to wavelength (and then only subsequently - via second quantization - to mass).

\(^{16}\)Viz. the action as calculated in that particular path which minimizes it.

\(^{17}\)In passing we note that in a static, weak gravitational background metric with $g_{00} = -(1 - 2MG/c^2 r)$, use of the Hamilton-Jacobi equation thus enables us to identify the non-relativistic classical energy as $E = -c\partial P/\partial x^0 \rightarrow mc^2 + mv^2/2 - mMG/r$. Thus while we see that we can indeed identify the non-relativistic Hamiltonian as $H = T + V$, we are able to do so only after having first obtained Eqs. (17) and (18), i.e. only after the quantum-mechanical problem associated with the fully covariant Klein-Gordon equation has already been solved. Moreover, it is important to emphasize, that even while we thus are able to make contact with the classical Hamiltonian, we were only able to do so in the eikonal approximation where the $i\hbar P^\mu_{\mu}$ term is dropped. However, since it was only the non-relativistic reduction of the full Klein-Gordon equation which led to the Schrödinger equation of Eq. (12), we see that the $i\hbar P^\mu_{\mu}$ term does (in principle) contribute non-relativistically ($\nabla^2 \psi = \psi \nabla (ln \psi) \cdot \nabla (ln \psi) + \psi \nabla^2 (ln \psi)$) and thus can play a role in fixing the quantum-mechanical Hamiltonian, with the gravitational Schrödinger equation actually going beyond the eikonal approximation in the long wavelength limit.

\(^{15}\)
variant differentiation of Eq. (18) immediately leads to the classical massive particle geodesic equation \( p^\mu p^\nu_{\mu} = 0 \), to thus recover the well known (wave-particle duality) result that quantum-mechanical rays move on classical geodesics, i.e. that the center of a quantum-mechanical wave packet follows the classical trajectory. Further, since we may also reexpress the stationary \( \int p_\mu dx^\mu \) as \( -mc \int d\tau \), we see that we can also identify the quantum-mechanical eikonal phase as \( P(x) = -mc \int d\tau \), to thus nicely enable us to make contact with \( I_T \) after all. Though we see that the classical action \( I_T \) does thus play a role, it is important to realize that even though this classical action is indeed a solution to the classical Hamilton-Jacobi equation, we were only able to arrive at Eq. (17) after first imposing the equation of motion of Eq. (14), i.e. only after variation of the Klein-Gordon action had already been made, with only the stationary classical action (viz. \( I_T^{ST} \)) actually being a solution to the Hamilton-Jacobi equation. Thus in the quantum-mechanical case, just as noted for the Hamiltonian \( H = T + V \), we see that the action \( I_T = -mc \int d\tau \) is a part of the solution, i.e. the output, rather than being part of the input. Thus unlike the situation in the classical-mechanical case, in the quantum-mechanical case we never need to assume the existence of any point particle action \( I_T \) at all. Rather we need only assume the existence of equations such as the standard Klein-Gordon equation, with the eikonal approximation then precisely putting particles onto classical geodesics just as desired.

Now while we have just seen that use of the standard Klein-Gordon equation does indeed lead to geodesic motion, we still need to address the fact that the standard minimally coupled Klein-Gordon equation (viz. the one obtained by writing the flat spacetime one covariantly) is not in fact the most general one that could be used in curved spacetime because of possible direct non-inertial couplings to curvature once the Riemann tensor is non-zero, couplings that could potentially lead us to the kind of (weak) equivalence principle violating terms exhibited in Eq. (10). To explicitly address this issue, as well as the issue of how the discussion might need modifying in

\[18\] In Refs. [12, 13] this result was utilized to analyze the gravitationally induced quantum interference detected in the COW neutron beam interferometry experiment. The interested reader may find some further, complementary discussion of the implications of the COW experiment for the quantum-mechanical equivalence principle in Ref. [14].

\[19\] Thus we cannot appeal to \( I_T \) to put particles on geodesics, since we already had to put them on geodesics in order to get to \( I_T \) in the first place.
the presence of non-zero spin, we note first that with the fundamental fields of nature all being thought to be describable by renormalizable field theories, since such second quantized fields are to obey second quantized wave equations, the matrix elements of these second quantized fields between the vacuum and the one-particle state will then typically obey first quantized wave equations just like the first-quantized Klein-Gordon equation (and its spin non-zero analogs) which was studied above. Moreover, with all the fundamental field theories of nature even being thought to be local gauge field theories, since such theories have no dimensionful coupling constants (and no intrinsic mass scales either if all masses are to generated dynamically) these theories then contain no explicit fundamental dimensionful parameters which might serve as the $\kappa_i$ parameters exhibited in Eq. (10). Thus as far as the couplings of fundamental fields to curvature tensors such as the Riemann tensor are concerned, the only permissible such ones would have to involve dimensionless couplings, with the scalar field wave equation of Eq. (14) for instance then being generalized to the non-minimally coupled

$$S^\mu_{;\mu} - (mc/\hbar)^2 S + (\xi/6)R^\alpha_\alpha S = 0, \quad R_{\mu\nu\sigma\tau} \neq 0$$

(19)

where $\xi$ is a dimensionless parameter, a parameter which, incidentally, takes the specific value $\xi = 1$ should the theory possess an additional local conformal invariance above and beyond its local gauge invariance.\footnote{In fact once local conformal invariance is invoked, it alone is sufficient to unambiguously fix the couplings of all fields to gravity, with the $\xi = 1$ term being the only direct Riemann curvature tensor dependent one possible.}

Now even though (massless) fermions couple to gravity only through the fermion spin connection $\Gamma_\mu(x)$, so that the massless curved space Dirac equation $i\gamma^\mu(x)\nabla_\mu \psi(x) = 0$ (where $\nabla_\mu = \partial_\mu + \Gamma_\mu$) contains no explicit direct dependence on the Riemann tensor, nonetheless there is actually an indirect dependence on the curvature in this particular case, since the second order differential equation obtained from the covariant Dirac equation is found to take the form $\nabla_\mu \nabla^\mu \psi(x) + (1/4)R^\alpha_\alpha \psi(x) = 0$, with the standard covariant coupling of a fermion to gravity thus always containing a non-inertial piece.\footnote{In passing we note when $\xi = 1$ both the scalar and fermion wave equation operators can be written in the generic form $D_\mu D^\mu + (d/6)R^\alpha_\alpha$, where $d$ is the dimension of the field.} Likewise, even though the curved space Maxwell equations, viz. $F^\mu_{\nu,\nu} = 0$, $F_{\mu\nu,\lambda} + F_{\lambda\mu,\nu} + F_{\nu\lambda,\mu} = 0$ also possess no direct
coupling to curvature, in this case also there is a hidden dependence on curvature, with manipulation of the Maxwell equations leading to the second order equations \( g^{\alpha\beta} F_{\mu\nu} :_{\alpha;\beta} + F_{\mu\alpha} R_{\nu\alpha} - F_{\nu\alpha} R_{\mu\alpha} = 0 \) (or equivalently \( g^{\alpha\beta} A_{\mu;\alpha;\beta} - A_{\alpha;\mu} + A_{\alpha} R_{\mu\alpha} = 0 \) where \( A^\mu \) is the vector potential). Both the Dirac and Maxwell equations thus lead us\(^{22}\) to an implicit expressly non-inertial coupling to the Riemann curvature tensor.\(^{23}\) Thus rather than being quite unlikely, we see that an explicit coupling to curvature actually turns out to be the general rule in the field theoretic case, so that strictly geodesic behavior is not to ever be expected in general.\(^{24}\) However, it turns out that in practice we can come very close to geodesic, and in order to see just how close, it is sufficient to explore the generic scalar field wave equation of Eq. \((19)\), a subject to which we now turn.

As regards Eq. \((19)\) we note immediately that if the scalar field has a non-zero mass, the ratio of the curvature term to the mass term in Eq. \((19)\) is given as \((L_C/L)^2\) where \(L_C = \hbar/mc\) is the Compton wavelength of the scalar particle and \(L\) is the scale on which the Ricci scalar varies \(\sim 1/L^2\). Thus if \(L\) is macroscopic the effect of the Ricci scalar term will be completely irrelevant, and thus to explore any possible consequences due to the Ricci scalar term, we need only consider the massless limit of Eq. \((19)\). For it, we can again set \(S(x) = \exp(iP(x)/\hbar)\), to obtain

\[
P_\mu P_\mu - (\xi \hbar^2/6) R_{\alpha\alpha} = i \hbar P^\mu_{;\mu} = 0 \tag{20}
\]

in the eikonal approximation. Equation \((20)\) admits of two types of solution depending on the strength of the Ricci scalar term. When the Ricci scalar

\(^{22}\)Because fields are defined at all spacetime points and not just along the trajectories associated with point particles, the associated second order field equations are sensitive to the values of the Christoffel symbols at differing points. And even though it is possible to remove the Christoffel symbols at any given point, it is nonetheless impossible to remove them from an entire region, with the curvature dependent terms we find in the above second order wave equations being the field theoretic generalization of the (covariantly describable) geodesic deviation found for pairs of nearby freely falling particles.

\(^{23}\)Since the only curvature dependence possible is in the form of couplings to the Ricci tensor, we note that any possible such terms are anyway immaterial to standard Ricci flat tests of the (weak) equivalence principle.

\(^{24}\)Thus we see that even if we do choose to generate general relativistic curved space field equations simply by covariantizing their special relativistic flat space forms, we still are led to non-inertial effects, and we are simply not free to postulate (viz. the weak equivalence principle) that those terms should be absent.
term is weak we may set \( P^\mu = \hbar dx^\mu / dq \), with \( dx^\mu / dq \) then being found to be close to but not quite on the light cone (with the four-acceleration, unusually, not then being orthogonal to the four-velocity), with trajectories being found to obey

\[
\frac{d^2 x^\lambda}{dq^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{dq} \frac{dx^\nu}{dq} = \frac{\xi}{12} g^{\lambda\sigma} (R^\alpha_\sigma)_\alpha. \tag{21}
\]

Since the right hand side of Eq. (21) behaves as \( 1/L^3 \), it will be negligible in the short wavelength limit unless \( L \) is microscopic. Thus in the short wavelength eikonal approximation we can nicely neglect the effects of any macroscopic Ricci scalar term. The second type of solution to Eq. (20) is obtained when the Ricci scalar term is strong, a limit in which we may set

\[
P^\mu = \hbar (-\xi R^\alpha_\alpha/6)^{1/2} dx^\mu / d\tau \tag{22}
\]

a limit where the (thus microscopic) Ricci scalar now sets the scale for the quantum-mechanical wavelength. Given Eq. (22), covariant differentiation of Eq. (20) then directly yields

\[
R^\beta_\beta \left( \frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right) = - \frac{1}{2} R^\beta_\beta (g^{\lambda\alpha} + \frac{dx^\lambda}{d\tau} \frac{dx^\alpha}{d\tau}). \tag{23}
\]

As we see, Eq. (23) bears some resemblance to Eq. (10), and indeed Eq. (23) can also be obtained via variation of the point particle action \( I = -\kappa \int (R^\alpha_\alpha)^{1/2} d\tau \), where the coefficient \( \kappa \) is actually dimensionless (in units of Planck’s constant).\(^{25}\) While we see that the motion associated with Eq. (23) is not geodesic, such non-geodesic behavior would only be detectable microscopically with \( L \) having to be of order the wavelength of the quantum-mechanical system.\(^{26}\) Thus if a large \( L \), macroscopic, Ricci scalar term is to contribute in Eq. (19) at all, it will only be able to do so in the long wavelength limit, a limit in which there would anyway be no eikonal approximation to begin with and in which we would (just as in the non-ray, \( iT^\mu_{\mu} \neq 0 \), regime of the standard classical Klein-Gordon equation) anyway

\(^{25}\)A point particle action in which the particle couples to the square root of the Ricci scalar is the only one available whose overall coefficient is dimensionless.

\(^{26}\)In passing we note that in the microscopic case the Ricci scalar term acts somewhat like a mass term, and explicitly as one in fact in a background de Sitter geometry where \( R^\alpha_\alpha \) takes the constant value \(-12k\) and generates a mass \( m = \hbar (2\xi k)^{1/2}/c \) to yield either a massive particle or a tachyon dependent on the sign of \( \xi k \).
have to use the full curved space Klein-Gordon or Schrodinger equations, with there then being no geodesic limit at all. Thus while the curvature term in Eq. (19) might possibly have to be included in some quantum-mechanical situations, in those (short wavelength) cases where it is possible to make a ray approximation in the first place, those rays will always be insensitive to the Ricci tensor dependent term, and will thus always lead to the geodesic equation of Eq. (4) to very high accuracy.

Thus, to conclude, we see that in this paper we have shown that the equivalence principle is indeed found to hold in quantum mechanics, that its primary characterization is as being purely geometric (our third version of the equivalence principle as given above), and that, moreover, it would appear that it is only through quantum mechanics that the classical-mechanical equivalence principle is even to be expected to hold at all. The author would like to thank W. Moreau, J. M. Bardeen and D. I. Santiago for helpful comments. This work has been supported in part by the Department of Energy under grant No. DE-FG02-92ER40716.00.

\textsuperscript{27}Thus in the presence of Eq. (19) we either produce Eq. (4) with no Eq. (14) type terms, or we have to stay quantum-mechanical and never eikonalize at all.
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