Backreaction in Acoustic Black Holes

Roberto Balbinot, Serena Fagnocchi
Dipartimento di Fisica dell’Università di Bologna and INFN sezione di Bologna, Via Irnerio 46, 40126 Bologna, Italy

Alessandro Fabbri†
Departamento de Fisica Teorica, Facultad de Fisica, Universidad de Valencia, Burjassot-46100, Valencia, Spain

Giovanni P. Procopio‡
DAMTP, Centre for Mathematical Sciences, University of Cambridge Wilberforce road, Cambridge CB3 0WA, UK

The backreaction equations for the linearized quantum fluctuations in an acoustic black hole are given. The solution near the horizon, obtained within a dimensional reduction, indicates that acoustic black holes, unlike Schwarzschild ones, get cooler as they radiate phonons. They show remarkable analogies with near-extremal Reissner-Nordström black holes.

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One of the most surprising and far reaching result for its implications in modern theoretical physics is the prediction made by Hawking [1] that black holes emit thermal radiation at a temperature \( T_H \) proportional to the surface gravity \( k \) of the horizon. For a Schwarzschild black hole of mass \( M \), \( k = (4M)^{-1} \) and \( T_H = \hbar (8\pi M)^{-1} \) (we have set the velocity of light and Boltzman constant equal to one). Hawking obtained this result using quantum field theory in curved space, a scheme for dealing with the matter-gravity system where matter is quantized according to quantum field theory whereas gravity is treated classically according to Einstein General Relativity. The scale at which this framework becomes unreliable is the Planck length where the description of spacetime as a continuous differentiable manifold probably breaks down. Coming back to black holes, because of the quantum emission, they are unstable. Extrapolating Hawking’s result (which is strictly valid only for stationary or static black holes) one can conjecture that as the mass decreases, the hole gets hotter and hotter (being the temperature inversely proportional to the mass) and eventually disappears in a time scale of the order of the initial mass to the third power. A more quantitative analysis can be performed by looking at the first order (in \( \hbar \)) corrections \( g^{(1)}_{\alpha\beta} \) to a classical black hole metric \( g^{(0)}_{\alpha\beta} \) induced by the quantum emission. These can be calculated using the semiclassical Einstein equations [2]

\[
G_{\mu\nu}(g^{(0)}_{\alpha\beta} + g^{(1)}_{\alpha\beta}) = 8\pi(T_{\mu\nu}(g^{(0)}_{\alpha\beta})).
\]

Here \( G_{\mu\nu} \) is the Einstein tensor evaluated for the quantum corrected metric \( g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + g^{(1)}_{\alpha\beta} \) and linearized in the perturbation \( g^{(1)}_{\alpha\beta} \) (of order \( \hbar \)). The r.h.s. represents the expectation value of the stress tensor for the quantum matter field evaluated in the classical background \( g^{(0)}_{\alpha\beta} \).

In a very interesting paper, appeared in 1981, Unruh [3] showed that a thermal radiation similar to the one predicted by Hawking for black holes is expected in a completely (at first sight) different physical scenario, namely a fluid undergoing hypersonic motion. This opened the way for the study of condensed matter analogues of Hawking radiation [4], a rather promising field of research where the connections to the experimental side do not seem so remote, compared to gravity.

The Eulerian equations of motion for an irrotational and homentropic fluid flow can be derived from the action [5], [6]

\[
S = -\int d^4x \left[ \rho \dot{\psi} + \frac{1}{2} \rho (\nabla \psi)^2 + u(\rho) \right],
\]

where \( \rho \) is the mass density, \( \psi \) the velocity potential, i.e. \( \nabla \psi = \nabla^\mu \psi \), \( u \) the internal energy density and a dot means time derivative.

Varying the action with respect to \( \psi \) one gets the continuity equation

\[
\dot{\rho} + \nabla \cdot (\rho \nabla \psi) = 0 ,
\]

whereas variation with respect to \( \rho \) gives Bernoulli’s equation (\( \mu = \frac{d}{d\rho} \))

\[
\dot{\psi} + \frac{\nabla^2 \psi}{2} + \mu(\rho) = 0 .
\]

Expanding the fields around a solution \( (\rho_0, \psi_0) \) of the...
FIG. I: A Laval nozzle. The waist of the nozzle represents the sonic horizon \(|v_0| = c\). In the region on the right of the waist \(|v_0| < c\) and on the left \(|v_0| > c\) (sonic hole).

classical equations of motion (3,4), Unruh showed that the quadratic action \(S_2\) for the fluctuations of the velocity potential can be written in a simple and elegant geometrical form, namely

\[
S_2 = -\frac{1}{2} \int d^4x \sqrt{g} (\partial_\mu \psi_1 \partial^\mu \psi_1) ,
\]

(5)

where \(\psi_1\) is the fluctuation and \(g^{(0)}_{\mu\nu}\) is the so-called acoustic metric

\[
g^{(0)}_{\mu\nu} = -\frac{\rho_0}{c} \left( c^2 - v_0^2 \frac{v_i v_i}{v_0^2} - I \right) \]

(6)

expressed in terms of the background quantities. \(c\) is the sound speed, i.e. \(c^2 = \rho_0 \mu / \rho_0\), and \(I\) is the three-dimensional identity matrix.

As it can be seen, \(S_2\) has exactly the same form of an action for a massless, minimally coupled, scalar field propagating in a “curved spacetime” whose line element is

\[ds^2 = g^{(0)}_{\mu\nu} dx^\mu dx^\nu.\]

The region of the fluid for which \(v_0^2 > c^2\) is called sonic black hole: its boundary \(|v_0| = c\) defines the sonic horizon. Sound waves cannot escape from this region, since they are dragged by the fluid.

A typical example is the Laval nozzle of Fig.1. The fluid flows from right to left. At the waist of the nozzle the fluid velocity reaches the speed of sound: this is the location of the sonic horizon for free fluid motion.

Using Hawking’s arguments Unruh, quantizing the field \(\psi_1\), showed that in the formation of a sonic hole one expects a thermal emission of phonons at a temperature \(T_U = \hbar k / (2\pi c)\), where \(k\) is the surface gravity of the sonic horizon [7]

\[k = \frac{1}{2} \frac{d}{d\eta} (c^2 - \bar{v}_0^2)\bigg|_H ,\]

(7)

\(n\) is the normal to the horizon.

In this paper we will give a first qualitative analysis of the effects this emitted radiation has on the fluid dynamics, i.e., the backreaction of the linearized phonons in sonic black holes.

Using standard background field formalism we write the fundamental quantum fields \(\hat{\rho}, \hat{\psi}\) as the sum of background fields \(\rho, \psi\) (not necessarily satisfying the classical equations of motion), plus quantum fluctuations, i.e.

\[\hat{\rho} = \rho + \delta \rho, \quad \hat{\psi} = \psi + \delta \psi.\]

Integrating out the quantum fluctuations one obtains the one loop effective action formally defined as \[\Gamma = S + \frac{1}{2} \hbar \ln \left[ \Box g(p,v) + O(h^2) \right],\]

where \(S\) is the action (2) and \(\Box g(p,v)\) is the covariant \(D^2\) Alamberian calculated from an acoustic metric \(g_{\mu\nu}(p,v)\) of the functional form of eq.(6) with \(\rho_0 \) and \(v_0\) replaced by \(\rho\) and \(v\) respectively [8]. We assume that divergences in the determinant of the above effective action are removed employing a covariant regularization scheme. This is the key hypothesis of our work. We will comment on it later.

Therefore one can write \[\Gamma = S + S_q(g_{\mu\nu}(\rho, v)),\]

where the quantum part of the action \(\Gamma\) depends on the dynamical variables \(\rho, v\) via the acoustic metric only, and coincides with the effective action for a massless scalar field propagating in a spacetime whose metric is \(g_{\mu\nu}(\rho, v)\). Using the chain rule we write \[\delta S_{\rho} = \delta S_{\rho} \delta g_{\rho\rho}^{(0)} + \text{and similarly for} \delta S_{\rho};\]

We write \(\rho = \rho_q + \rho_0\) and \(v = v_q + v_0\), where \(\rho_q, v_q\) satisfy the classical equations of motion (3,4) and \(\rho_q, v_q\) are the corrections of order \(O(\hbar)\) induced by the quadratic fluctuations of the phonons field. The linearized backreaction equations, analogues of semiclassical Einstein eq.(1), read (assuming for simplicity sake a constant velocity of sound \(c\))

\[\rho_q + \nabla_i (\rho_q v_i) + \nabla_i \left( \rho_0 (v_q v_i \cdot \frac{T_{ij}}{c^2} - \frac{v_0}{c^2} (T_{ij})) \right) = 0 ;
\]

(8)

\[\psi_q + v_q \cdot \psi_q + c^2 \rho_q \rho_q - \frac{1}{2} \frac{\rho_0}{c} (T) = 0 ,
\]

(9)

where \(\langle T_{\rho\rho} \rangle \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\rho\rho}} \big|_{g^{(0)}_{\mu\nu}}\) is the expectation value of the quantum version of the so called “pseudo energy momentum tensor” \(T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\rho\rho}}\) [6] and \(T = g^{(0)\mu\nu}(T^*_\mu)_\nu\). From the above construction it follows that \(\langle T_{\rho\rho} \rangle = \delta g_{\rho\rho}^{(0)}(\rho_0, v_0)\). These expectation values should be taken in the sonic analogue of the Unruh state [9] (the quantum state appropriate to describe black hole evaporation at late times), in which, in the remote past prior to the time dependent formation of the sonic hole, the quantum field is in its vacuum state.

Inspection of the backreaction equations reveals that eq.(8) is the first order in \(\hbar\) conservation equation for the Noether current associated to the \(c^2\)-term of the symmetry of the effective action \(\Gamma\). Note that the phonons contribution in the Noether flux coming from \(S_q(g_{\mu\nu})\) is just the pseudo momentum density \(\sqrt{g^{(0)}}(T^*_\mu)^\rho_\nu\). Eq.(9) is the first order Bernoulli equation modified by the presence of the trace term which represents an additional contribution to the chemical potential induced by the fluctuations which, as we shall see, causes the fluid to slow down. These equations reflect the underlying two-component structure of the system as in the Landau-Khalatnikov theory of superfluidity [10]. One can also rewrite the equation (8) as the usual fluid continuity
equation by a simple redefinition of the velocity field $v_{q} \rightarrow v_{q} - \left(\frac{T_{01}}{c^2} + \frac{v_{0}}{c} \langle T_{g1} \rangle\right)$ which incorporates the phonon momentum density. Consequently the Bernoulli equation (9) rewritten in terms of this redefined velocity contains, besides the trace, also other terms. Unfortunately no explicit solutions of eqs.(8,9) can be given since $\langle T_{\mu \nu} \rangle$ is unknown.

In the black hole case, where similar difficulties arise, a qualitative insight in the evaporation process can be given using 2D dimensional models [11]. The most popular is the one proposed by Callan, Giddings, Harvey and Strominger (CGHS) [12] in which the 4D quantum stress tensor is replaced by a 2D one associated to a minimally coupled massless scalar field described at the quantum level by the Polyakov action [13]. With the same spirit a qualitative description of the backreaction in a hypersonic fluid can be obtained assuming a one dimensional flow for the fluid, let’s say along the axis of the nozzle, the $z$ direction. So all physical fields will depend only on $t$ and $z$.

The effective action for the CGHS-like model for the fluid quantum dynamics can be then given as

$$\Gamma^{(2)} = S^{(2)} + S_{pol}$$

where

$$S^{(2)} = -\int d^{2}x A \left[ \rho \dot{\psi} + \frac{1}{2} \rho (\partial_{z} \psi)^{2} + u(\rho) \right]$$

is obtained integrating $S$ over the transverse coordinates $x,y$ and $A$ is the area of the transverse section of the nozzle. $S_{pol}$ is the Polyakov action. The backreaction equations following from $\Gamma^{(2)}$ are

$$\dot{\rho}_{q} + \partial_{z} \left[ A(\rho_{q} v_{0} + \rho_{0} v_{q}) \right] - \frac{3}{c^{4}} \left[ (\langle T_{zz}^{(2)} \rangle + \rho_{0} \langle T_{z}^{(2)} \rangle) \right] = 0$$

$$A \left( \dot{\psi}_{q} + v_{0} v_{q} + \frac{c^{2}}{\rho_{q}} \partial_{z} \dot{\psi} - \frac{\langle \psi^{2} \rangle}{2} \right) = 0.$$

$\langle T_{ab}^{(2)} \rangle$ is the quantum stress tensor for a massless scalar field minimally coupled to the $(t,z)$ section $g_{ab}^{(2)}$ of the acoustic metric $g_{\mu \nu}^{(0)}$ of eq. (6). The trace $\langle T^{(2)} \rangle$ is completely anomalous and given by

$$\langle T^{(2)} \rangle = \frac{h}{24\pi} R^{(2)},$$

where $R^{(2)}$ is the Ricci scalar for the metric $g_{ab}^{(2)}$. The phonons expectation values appearing in the conservation equation (12) can be easily expressed by a coordinate transformation in terms of $\langle T_{\pm \pm} \rangle$ where $x^{\pm} = t \pm z$ and $z^{\pm}_{\mu} = \int dz/c + v_{0})^{-1}$, for which the Polyakov approximation gives

$$\langle T_{\pm \pm} \rangle = -\hbar (12\pi)^{-1} C^{1/2} C_{-1/2} + \Delta_{\pm}.$$

Here $C$ is the conformal factor for the 2D metric, $C = \rho_{0}(c^{2} - v_{0}^{2})c^{-3}$, and for the Unruh state $\Delta_{+} = 0$, $\Delta_{-} = h \kappa^{2}/48\pi c^{4}$. Assuming as profile for the Laval nozzle $A = \bar{A} + \beta z^{2}$, with $\bar{A}, \beta$ constant, the classical solution reads

$$\rho_{0} = \bar{\rho} e^{-\frac{3}{4}\kappa^{2} z^{2} \bar{A}}, \quad \bar{\rho} = \text{const}$$

$$z^{2} = \frac{\bar{A} \beta}{\bar{\rho} |v_{0}|^{c^{2}}/4c^{4} - 1},$$

where the sound velocity $c$ is taken to be constant. The location of the sonic horizon is $z = 0$, where $v_{0} = -c$, and the region $z < 0$ is the sonic black hole for which $|v_{0}| > c$. These expressions should be regarded as describing the classical would be asymptotic configuration of the fluid resulting from the (time dependent) formation of a sonic hole [14].

Expanding the stress tensor and the background quantities near the horizon $z = 0$ one eventually arrives [15] to the following solution for the velocity

$$v = v_{0} + v_{q} \simeq -c + c k z - \frac{c}{6} \kappa^{2} z^{2} + \epsilon(b_{1} + c_{1} k z) \kappa t,$$

where $\epsilon = h/(\bar{A}^{2} \bar{\rho} c^{-1/2} c)$ is the dimensionless expansion parameter and $b_{1} = 9\gamma/2$, $c_{1} = -304\gamma/15$, where $\gamma = \bar{A}^{2} \kappa^{2}/24\pi$, and $\kappa = \sqrt{\beta}/\bar{A}$ has dimension $L^{-1}$ and is related to the surface gravity $k = c/\kappa^{2}$.

The solution is valid for $\kappa z \ll 1$ and $c k t \ll 1$.

The boundary conditions imposed on the backreaction equations have been chosen so that at some given time (say $t = 0$) the evaporation is switched on starting from the classical configuration $(v_{0}, \rho_{0})$, i.e. $\rho_{q}(z,t = 0) = v_{q}(z,t = 0) = v_{0}(z,t = 0) = 0$.

Inspection of eq.(17) shows the net effect of the backreaction. Being $\kappa z \ll 1$, from eq.(17) we have $v_{q} > 0$, i.e. the fluid is slowing down. This goes with an decrease of the density ($\rho_{q} < 0$) in the same limit [15].

Eq.(17) allows also to follow the evolution of the acoustic horizon. Being the horizon defined by $v = -c$, this yields

$$z_{H} \simeq -\frac{c b_{1} t}{c},$$

i.e. the horizon is moving to the left with respect to the nozzle: the hypersonic region shrinks in size. The coefficient $b_{1}$ determining the quantum correction to the velocity (17) and hence the evolution of the horizon is just the gradient of the additional chemical potential related to the expectation value of the trace evaluated at $z = 0$. This should be compared to the black hole case where the evolution of the horizon is determined by the energy flux ($M \propto \langle T_{t}^{t} \rangle$ in spherical symmetry [2]). While in the latter case Hawking radiation occurs at the expense of the gravitational energy of the black hole, in the fluid phonons emission takes away kinetic energy from the system.

From eqs.(17, 18) one can evaluate the correction to the emission temperature

$$T_{U} = \frac{h}{2\pi} \left. \frac{\partial v}{\partial z} \right|_{z_{H}} = \frac{h c}{2\pi} \left[ 1 - \frac{563\pi}{720\pi} \kappa^{3} c \bar{A} \right].$$


Using particular values for liquid helium, the fractional change of the temperature per unit time is of order $10^{-8}$ s$^{-1}$.

The expression we have obtained is rather significant. Unlike a Schwarzschild black hole the emission temperature for a sonic black hole decreases in time as the radiation emission proceeds. This behaviour is reminiscent of the near-extremal Reissner-Nordström black hole ($M > |Q|$) where $Q$ is the conserved charge of the hole. As the mass decreases because of Hawking evaporation, the Hawking temperature decreases as well, vanishing when $M = |Q|$. This ground state is approached in an infinite time (third law of black hole thermodynamics).

The basic question which remains open is to what extent our results do depend on the dispersion relation assumed (free scalar field), which ignores short distance corrections due to the molecular structure of the fluid, and on the covariant regularization scheme used to subtract ultraviolet divergences. These two aspects are deeply connected. For the Polyakov theory we have used in the 2D backreaction equations (12, 13), Jacobson [16] has argued that, within a covariant regularization scheme, no significant deviation from the usual expression for the trace anomaly and the components of $\langle T_{\mu \nu} \rangle$ are expected if one introduces a cutoff at high frequencies. However, for the hydrodynamical system we have considered, covariance is a symmetry of the phonons low energy effective theory only, which is broken at short distance. Hence non covariant terms depending on the microscopic physics are expected to show up in the effective action and are crucial for a correct description of the unperturbed quantum vacuum of the fluid. However the expectation values $\langle T_{\mu \nu} \rangle$ entering the backreaction equations (8, 9) do not represent the energy momentum of the fluid quantum vacuum. They describe instead the perturbation of the stationary vacuum (whose energy is strictly zero [17]) induced by inhomogeneities and by the time dependent formation of the sonic hole which triggers the phonons emission. In this paper we have assumed that these deviations can be computed within the low energy theory. This situation is not unusual. Casimir effects are well known examples of vacuum disturbances caused by the presence of boundaries. It happens that the Casimir energy is often (but not always, see G. Volovik in Ref. [4]) independent on the microscopic physics and can be calculated within the framework of the low energy theory. This happens because, while low frequency modes are reflected by the boundaries, for the high energy ones the wall is transparent. They produce a divergent contribution to the vacuum energy which is canceled by a proper regularization scheme and does not affect the finite result. We have assumed that a similar decoupling happens for the acoustic black hole. The check of our hypothesis would require an analysis of the quantum system within the microscopic theory which takes into account the time dependent non homogeneous formation of the sonic hole. This is for the moment beyond computational capability. Anyway, it has been shown that modifications of the dispersion relations, to take into account short-distance behaviour of the high-energy modes, basically do not affect the spectrum of the emitted phonons [18]. This is not a proof that observables like $\langle T_{\mu \nu} \rangle$ are also unaffected by short-distance physics. However, it can be an illuminating hint taking in mind what $\langle T_{\mu \nu} \rangle$ does really represent and the indications coming from the Casimir effect.

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