Cosmological constraints for a two branes system in a vacuum bulk

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Received: 18 January 2014 / Accepted: 5 December 2014 / Published online: 10 January 2015
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Abstract We study a two 3-branes system embedded in a 5-dimensional spacetime with the fifth dimension compactified on a $S^1/Z_2$ orbifold. Assuming isotropic and homogeneous branes, we show that the dynamics of the visible brane affects the hidden brane through gravitational interaction. This influence is described by equations that relate the matter contents on each brane, and arise from the Israel’s junction conditions. We analyse the case in which both branes are composed of a single matter field with a vacuum bulk, and explore the different possibilities allowed for the cosmologies of the branes.

Keyword Braneworld cosmology · Dynamical branes · Bouncing cosmologies

1 Introduction

The proposal that our Universe is a (3+1)-dimensional hypersurface (brane) embedded in a higher dimensional spacetime (bulk) can be traced back to the Regge and
Teitelboim work in [1]. This seminal idea was largely ignored due to the lack of a phenomenological support. In recent years, this concept was reintroduced for brane-world models, which have been motivated by string theory and M-theory [2–5], where our Universe can be seen as a 3D manifold immersed in a spacetime of more than three spatial dimensions. Usually, the particles and fields of the Standard Model of Particle Physics are trapped in the brane, with the singular exception of gravity that can escape into the bulk.

Brane-worlds models have engaged a lot of interest in order to describe the dynamics of the Universe since they add new degrees of freedom that can help us to solve the problems of dark matter and dark energy. Originally, the brane-worlds models attempted to solve the hierarchy problem, namely, the large difference in magnitudes between the Planck and electroweak scales, $M_{pl}/M_{EW} \approx 10^{16}$. In fact, a novel feature of some brane models is that the energy scale $M_{pl} = M_4$, which is the 4D Planck scale, is no more the fundamental scale, a place now taken by a 5D energy scale $M_5$ that is in the range of the electroweak scale. In this scheme, the reason why gravity is weaker than other fundamentals interactions is because it dilutes into the extra spatial dimensions.

Extra dimensions can be introduced in different ways. For example, compact extra dimensions imply every multi-dimensional field corresponds to a Kaluza-Klein tower of four-dimensional particles with increasing masses. At low energies, only massless particles with energies $E \ll 1/R$, where $R$ is the compactification scale, can be produced, whereas at $E \approx 1/R$, extra dimensions begin to be detectable. This is the starting point of string theory which, trying to reconcile quantum mechanics and general relativity, postulate a spacetime with extra dimensions.

Recent research in string theory and its generalization to $M$-theory have suggested that the number of dimensions is 11. Inherited in these models are the $p$-branes ($0 < p < 9$), which are its fundamental constituents. Our Universe can be a very large brane extending over three spatial dimensions. Material objects, made of open strings, are confined to the brane, while gravity and other exotic matter such as the dilaton can propagate in the bulk. This is the fundamental concept behind brane-world cosmology.

The strong coupling limit of the $E_8 \times E_8$ heterotic string theory at low energy is described by 11D supergravity with the eleventh dimension compactified on an $S_1/Z_2$ orbifold. The two boundaries of the spacetime are two 10-branes on which gauge fields are confined. Witten argued that 6 of the 11 dimensions can be consistently compactified on a Calabi-Yau threefold, and that the size of the Calabi-Yau manifold can be substantially smaller than the space between the two boundary branes. Thus, in that limit, the spacetime looks 5-dimensional. A realization of this model and the corresponding brane-world cosmology is given in [6–8]. In these models the extra dimension can be large relative to the fundamental scale, providing the basis for the brane models of Arkani–Dimopoulos–Dvali (ADD) [9], Randall–Sundrum (RS) [10,11], and Dvali–Gabadadze–Porrati (DGP) [12,13].

For example, using large extra dimensions, Arkani et al. [9] assumed a $N$-dimensional bulk with the Planck scale $M_N$ and two four-dimensional branes. The corresponding four-dimensional Planck mass $M_{pl}$ is given by $M_{pl}^2 = V_{N-4} M_N^{N-2}$, where $V_{N-4}$ is the volume of the $(N-4)$-dimensional space. If extra dimensions are large enough, even $M_N$ is of order of electroweak scale $M_N \approx M_{EW} \approx TeV$, one can
get the correct order of $M_{pl} \simeq 10^{16} \text{TeV}$, whereby the hierarchy problem is resolved. In the Randall Sundrum I (RS1) model, the mechanism is completely different [10]. Instead of employing large dimensions, RS used a warped factor $\sigma(y) = k|y|$. In this model the mass $m_0$ measured on the invisible (Planck) brane is related to the mass $m$ measured on the visible (TeV) brane by $m = e^{-kyc}m_0$. Clearly, by properly choosing the distance $y_c$ between the two branes, one can lower $m$ to the order of TeV, even if $m_0$ is still in the order of $M_{pl}$. It should be noted that the five-dimensional Planck mass $M_5$ in the RS1 scenario is still of the order of $M_{pl}$, and the two are related by $M_{pl}^2 = M_5^3k^{-1}(1 - e^{-2kyc}) \simeq M_5^2$ for $k \simeq M_5$.

In the context of two-brane models with matter, one can naturally ask if the parameters which determines the evolution of matter fields in both branes are related. Binetruy et al. [14] have shown that there exists an equation which relate the fields in both branes assuming a mutual gravitational interaction between them, through topological constrains. This effect can have interesting physical consequences, for example, in [15,16] the authors assume that the hidden brane is dominated by a scalar field, trying to reproduce the dark matter effect in the visible brane.

Based on the RS models, and in the previous results found by [17–19], this paper generalizes previous solutions for a five-dimensional vacuum bulk for which the metric coefficients have a particular mathematical structure. This formalism generates a dynamical equation for the Hubble parameters of the branes; in other words, the fields immersed in the visible brane affects the dynamics of the other through gravitational effects.

The present paper is organized as follows. In Sec. 2 we explore the mathematical properties of the five-dimensional Einstein equations with a compact dimension for a system with two branes, together with the boundary conditions that must be simultaneously accomplished. Then, we solve the aforementioned equations and find a family of exact solutions for the metric coefficients. In Sec. 3, we use these solutions and give the equations that relates the cosmology of both branes; particular cases are studied for a single matter component. Finally, the conclusions are presented in Sec. 4.

2 Mathematical background

To start with, we write the action for two branes embedded in a five-dimensional manifold as:

$$S = -\frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g(5)} R(5) + \int d^4x \sqrt{-g(4)}(\mathcal{L}_0 + \mathcal{L}_c),$$

(1)

where $g(5)$, $\kappa(5)$, and $R(5)$ are, respectively, the determinant of the metric, the gravitational constant, and the Ricci scalar of the five-dimensional setting\(^1\). The branes are located at $y = 0$, and $y = y_c$, respectively, and their corresponding matter field

\(^1\) Notice that we are using units with $c = h = 1$, and that indices are $\mu, \nu = 0, 1, 2, 3$, $i, j = 1, 2, 3$, and $A, B = 0, 1, 2, 3, 4$. 
lagrangians are $\mathcal{L}_0$ and $\mathcal{L}_c$. Let us consider the most general metric in the form

$$ds^2 = -n^2(t, |y|)dt^2 + a^2(t, |y|)g_{ij}dx^idx^j + b^2(t, |y|)dy^2.$$  

(2)

For all functions henceforth, we impose the following symmetries:

1. $(x^\mu, y) \rightarrow (x^\mu, -y)$. Reflection.
2. $(x^\mu, y) \rightarrow (x^\mu, y + 2iy_c)$, $i = 1, 2, \ldots$ Compactification.

Thus, we demand that each metric coefficients $a(t, |y|), n(t, |y|)$ and $b(t, |y|)$ to be subjected to the conditions

$$F' \bigg|_{y=0^+} = 2F' \bigg|_{y=y_c^-},$$

$$F'' = \frac{d^2F(t, |y|)}{d|y|^2} + \left[F'\right]_0 \delta(y) + \left[F'\right]_c \delta(y - y_c),$$

(3a) (3b)

where the prime denotes derivative with respect to coordinate $y$, the square brackets denote the discontinuity in the first derivative at the positions $y = 0$ and $y = y_c$ and $F$ is a generic function which fulfill the last conditions. Equation (3b) is obtained if we demand that $d|y|/dy = 1$, and $d^2|y|/dy^2 = 2\delta(y) - 2\delta(y - y_c)$, for $y \in [0, y_c]$. The subindex 0 will be used for quantities valued at $y = 0$, whereas a subindex $c$ will be used for quantities valued at $y = y_c$. The above properties are a result of having a $\mathbb{Z}_2$ symmetry. The delta distribution in the second derivatives of the metric coefficients is necessary because the five-dimensional Einstein equations contain a similar function in the energy momentum tensor (see [20]).

On the other hand, the five-dimensional energy-momentum tensor is of the form:

$$\tilde{T}^A_B = -\frac{\Lambda_5}{\kappa(5)}g_{AB} + \tilde{T}^A_B + \frac{\delta(y)}{b_0} \text{diag}(-\rho_0, p_0, p_0, p_0, 0)$$

$$+ \frac{\delta(y - y_0)}{b_c} \text{diag}(-\rho_c, p_c, p_c, p_c, 0),$$

(4)

where the first term corresponds to a five-dimensional cosmological constant, and the third and fourth terms correspond to a perfect fluid embedded in the four-dimensional branes.

The conservation of the energy-momentum tensor, $\nabla_A \tilde{T}^A_B = 0$, immediately yields the known results for each brane:

$$\dot{\rho}_0 + 3(p_0 + \rho_0) \frac{\dot{a}_0}{a_0} = 0,$$

$$\dot{\rho}_c + 3(p_c + \rho_c) \frac{\dot{a}_c}{a_c} = 0,$$

(5a) (5b)

where a dot denotes derivative with respect to time. We will further assume that the perfect fluid components satisfy equations of state in the form $\rho_0 = w_0 \rho_0$, and $p_c = w_c \rho_c$. At this point, both equations of state may cover any type of matter, but some particular cases will be discussed in detail in Sec. 3.
According to the Israel’s junction conditions [21], we describe the presence of an energy density in terms of a discontinuity of the metric across the origin in the extra coordinate \(y\). We then find that the metric coefficients satisfy the following boundary conditions:

\[
\frac{[a']_0}{a_0 b_0} = -\frac{\kappa^2_5}{3} \rho_0, \tag{6a}
\]

\[
\frac{[n']_0}{n_0 b_0} = \frac{\kappa^2_5}{3} (3p_0 + 2\rho_0). \tag{6b}
\]

These relations imply that the discontinuity in the first derivative of the metric coefficients is proportional to the energy density across the brane at \(y = 0\). Similarly, at \(y = y_c\), we have

\[
\frac{[a']_c}{a_c b_c} = -\frac{\kappa^2_5}{3} \rho_c, \tag{7a}
\]

\[
\frac{[n']_c}{n_c b_c} = \frac{\kappa^2_5}{3} (3p_c + 2\rho_c). \tag{7b}
\]

In order to obtain exact dynamical solutions, we write explicitly the five-dimensional Einstein tensor, which for the metric (2) reads

\[
\tilde{G}_{00} = 3\frac{\dot{a}}{a} \left( \frac{\dot{a}}{a} + \frac{\dot{b}}{b} \right) - 3\frac{n^2}{b^2} \left[ \frac{a''}{a} + \frac{a'}{a} \left( \frac{a'}{a} - \frac{b'}{b} \right) \right] + 3k\frac{n^2}{a^2}, \tag{8a}
\]

\[
\tilde{G}_{ij} = \frac{a^2}{b^2} \delta_{ij} \left\{ \frac{a'}{a} \left( \frac{a'}{a} + 2\frac{n'}{n} \right) - \frac{b'}{b} \left( \frac{n'}{n} + 2\frac{a'}{a} \right) \right\} + \frac{a^2}{b^2} \delta_{ij} \left[ 2\frac{a''}{a} + \frac{n''}{n} - \frac{a''}{n} \delta_{ij} \right] \left( \frac{\dot{a}}{a} - \frac{\dot{a}^2}{a} + \frac{\dot{n}}{n} \right) - k\delta_{ij}, \tag{8b}
\]

\[
\tilde{G}_{04} = 3\left( \frac{\dot{a} \dot{n'}}{a n} + \frac{\dot{b} a'}{b a} - \frac{\dot{a'}}{a} \right), \tag{8c}
\]

\[
\tilde{G}_{44} = 3\frac{a'}{a} \left( \frac{a'}{a} + \frac{n'}{n} \right) - 3\frac{b^2}{n^2} \left[ \frac{\ddot{a}}{a} + \frac{\dot{a}}{a} \left( \frac{\dot{a}}{a} - \frac{\dot{n}}{n} \right) \right] - 3k\frac{b^2}{a^2}. \tag{8d}
\]

2.1 Exact solutions

Now we are going to describe the procedure for obtaining exact solutions of the equations of motion. First of all, we take Eq. (8c): \(\tilde{G}_{04} = \kappa^2_5 T_{04} = 0\) (which physically
means that there is no flow of matter along the fifth dimension), and find that
\[
\frac{\dot{b}}{b} = \frac{n}{a'} \left[ \frac{\dot{a}}{n} \right].
\] (9)

This last equation can be easily solved if one looks for static solutions, like in [17] where it was chosen that \( \dot{b} = 0 \) and then \( b = 1 \). We want to emphasize that there are non-static exact solutions under a more general ansatz in which the metric coefficient \( b \) is a general function of the scale factor \( a \). In fact Eq. (9) can be integrated up in general terms giving as a result,
\[
b = f(a) \quad \Rightarrow \quad \dot{a} = n b \alpha(t),
\] (10)

where \( f(a) \) is an arbitrary function of the scale factor \( a \), and \( \alpha(t) \) is an arbitrary function of time. As we mentioned before, in order to have a stabilized bulk the lapse function \( b \) is usually taken to be unity, and therefore, \( f(a) = 1 \) (see for instance the known cases studied in [17,22]). Thus, a general \( f(a) \) will describe the cases of non-static internal dimensions [23].

Secondly, we integrate Eq. (8a) in the coordinate \( y \), and obtain that
\[
\left( \frac{a'}{ab} \right)^2 - \left( \frac{\dot{a}}{an} \right)^2 = ka^{-2} - \frac{\Lambda_5}{6} + C_{DR} a^{-4},
\] (11)

where \( C_{DR} \) is an integration constant and parameterizes the so called dark radiation contribution [24].

For purposes of simplicity, we shall restrict ourselves to the cases in which the curvature \( k \), the dark radiation term \( C_{DR} \), and the five-dimensional cosmological constant are all negligible, i.e. \( k = C_{DR} = \Lambda_5 = 0 \). It should be noticed that for this particular case \( a' = b^2 \alpha(t) \).

In this paper, we take a power-law function of the form \( f(a) = a^m \), and then the general solution for the scale factor is
\[
a(t, y) = a_0 \left[ 1 + (2m - 1)y \alpha a_0^{2m-1} \right]^{1/(1-2m)},
\] (12)

where \( a_0 \) is the time-dependent scale factor at \( y = 0 \). Similar expressions are obtained for the other metric functions \( b \) and \( n \).

2.2 Matter contents on the branes

Taking into account the mathematical properties of the metric functions in the branes [as summarized in Eq. (3)], we can find the general solutions that satisfy the boundary

\footnote{Equation (11) can be analytically solved if \( b = a^m \) and \( m = 0, -1, 1 \). The case \( m = 0 \) can actually be found in [17], whereas the other two cases can be solved in terms of elliptic functions.}
conditions (6). For instance, for the scale factor $a$ the boundary conditions (6a) can be rewritten in the form

$$\left[\frac{a'}{a_0 b_0}\right]_0 = -\frac{2 b_0^2}{a_0 b_0} = -2 \alpha a_0^{2m-1} b_0^{-1} = -\frac{\kappa_5^2}{3} \rho_0,$$  \hspace{1cm} (13)$$

Using similar results for the other metric functions, we can then write the full solutions in terms of the matter content of the brane located at $y = 0$:

$$a(t, y) = a_0 \left[ 1 + (2m - 1) \frac{\kappa_5^2}{6} \rho_0 a_0^m y \right]^{1/(1-2m)},$$  \hspace{1cm} (14a)$$

$$n(t, y) = n_0 \left[ 1 + (m + 2 + 3 \omega_0) \frac{\kappa_5^2}{6} \rho_0 a_0^m y \right] \times \left[ 1 + (2m - 1) \frac{\kappa_5^2}{6} \rho_0 a_0^m y \right]^{m/(1-2m)},$$  \hspace{1cm} (14b)$$

$$b(t, y) = a_0^m \left[ 1 + (2m - 1) \frac{\kappa_5^2}{6} \rho_0 a_0^m y \right]^{m/(1-2m)}.$$  \hspace{1cm} (14c)$$

for all $m \neq 1/2$ and any $\alpha(t)$.

Equation (14) then generalize previous results in the literature. For instance, if we take $m = 0$, we recover the linear solutions of [14]:

$$a(t, y) = a_0 \left[ 1 - \frac{\kappa_5^2}{6} \rho_0 y \right],$$  \hspace{1cm} (15a)$$

$$n(t, y) = n_0 \left[ 1 + (2 + 3 \omega_0) \frac{\kappa_5^2}{6} \rho_0 y \right]$$  \hspace{1cm} (15b)$$

$$b(t, y) = 1.$$  \hspace{1cm} (15c)$$

Another possibility is the case $m = 1$ and $\omega_0 = -1$, where we find a conformal RS metric:

$$a(t, y) = a_0 \left[ 1 + \frac{\kappa_5^2}{6} \rho_0 a_0 y \right]^{-1} = \frac{a_0}{n_0} n = b.$$  \hspace{1cm} (16)$$

Even though it is similar to the metric for a RS cosmology, in general the function $a_0$ is now time-dependent. Note that in this case, from Eq. (17a), we also have $\rho_0 = -\rho_c$. To exactly recover the RS solutions we should have also kept $\Lambda_5 \neq 0$ and $k = 0$ in Eq. (11).

It is also necessary to satisfy the boundary conditions at $y = y_c$, and then we must apply the boundary conditions (7) to the solutions given in Eq. (14), which must translates in a relationship between the energy densities $\rho_0$ and $\rho_c$ located on each brane.
After straightforward calculations, the resulting expressions for the energy densities and the equations of state for each fluid in the branes are:

\[
\rho_c = -\rho_0 \left[ 1 + (2m - 1) \frac{\kappa^2}{6} \rho_0 a^m_0 y_c \right]^{(m-1)/(1-2m)},
\]

(17a)

\[
w_c = \frac{w_0 + (m + 2 + 3w_0)(m/3 - 1) \frac{\kappa^2}{6} \rho_0 a^m_0 y_c}{1 + (m + 2 + 3w_0) \frac{\kappa^2}{6} \rho_0 a^m_0 y_c}.
\]

(17b)

The above equations imply that the cosmology of the visible brane at \( y = y_c \) is constrained by the cosmology of both the hidden brane and the bulk geometry. Recall that we have not specified the form of the equations of state, but it is clear that we cannot impose upon them more physical conditions unless they are consistent with Eq. (17). However, from Eq. (17a), we can see that the energy densities on the branes have opposite signs due to the \( Z_2 \) symmetry, i.e. the negative effective energy density is a topological effect due to the mirror symmetry when using the boundary terms in Eq. (3) (see for instance the RS case [10]).

In Eq. (17), we have chosen to express \( \rho_c \) and \( \omega_c \) as functions of their counterparts in the brane at \( y = 0 \). But the inverse process is also possible, and a short calculation reveals that

\[
\rho_0 = -\rho_c \left[ 1 + (2m - 1) \frac{\kappa^2}{6} \rho_c a^m_0 y_c \right]^{(m-1)/(1-2m)},
\]

(18a)

\[
w_0 = \frac{w_c + (m + 2 + 3w_c)(m/3 - 1) \frac{\kappa^2}{6} \rho_c a^m_0 y_c}{1 + (m + 2 + 3w_c) \frac{\kappa^2}{6} \rho_c a^m_0 y_c}.
\]

(18b)

It is then clear that the branes are exactly symmetric with respect to their matter contents. Hereafter, we will be working with Eq. (17), under the assumption that we can only be sure about the matter contents of our visible brane.

We now determine the expansion rates in each brane by calculating their corresponding Friedmann parameter. From Eq. (11), we find that

\[
H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \left( \frac{a' n}{a b} \right)^2.
\]

(19)

Because of the discontinuities induced by the branes on the metric functions, we see that the Hubble parameters must be calculated from

\[
H_0^2 = \left( \frac{\dot{a}_0}{a_0} \right)^2 = \left( \frac{[a']_0 n_0}{a_0 b_0} \right)^2, \quad H_c^2 = \left( \frac{\dot{a}_c}{a_c} \right)^2 = \left( \frac{[a']_c n_c}{a_c b_c} \right)^2.
\]

(20)
and then

\[ H_0^2 = \frac{k_0^2}{36} n_0^2 \rho_0^2, \quad H_c^2 = \frac{k_0^2}{36} n_c^2 \rho_c^2. \]  

(21)

Using Eqs. (14b) and (17a), and imposing the gauge condition \( n_0 = 1 \), the final expressions for the Hubble parameters are

\[ H_0^2 = \frac{k_0^2}{36} \rho_0^2, \]  

(22a)

\[ H_c^2 = \frac{k_0^2}{36} \rho_0^2 \left[ 1 + (m + 2 + 3\omega_0) \frac{k_0^2}{6} \rho_0 a_0^m y_c \right]^2 \times \left[ 1 + (2m - 1) \frac{k_0^2}{6} \rho_0 a_0^m y_c \right]^2. \]  

(22b)

Notice that our gauge choice \( n_0 = 1 \) implies that \( \alpha(t) = \dot{a}_0 a_0^{-m} \) in Eq. (10), and then \( n_c = (\dot{a}_c/\dot{a}_0)(a_0/a_c)^m \). Hence, it is only necessary to find the explicit form of \( a_0(t) \) in order to have a complete solution of the metric functions.

### 3 Cosmological analysis between the branes

Before proceeding further with the analysis of the cosmology in the two branes, we discuss the high and low energy limits for the exact solutions found above.

#### 3.1 High and low energy limits

For the low energy limit in our brane, we have

\[ \left| \frac{k_0^2}{6} \rho_0 a_0^m y_c \right| \ll 1. \]  

(23)

The effects coming from the extra dimension are negligible, and then both branes evolve similarly as \( \rho_c \simeq \rho_0 \) and \( w_c \simeq w_0 \); in fact, we also obtain

\[ H_0^2 = H_c^2 = \frac{k_0^2}{36} \rho_0^2. \]  

(24)

This limit resembles the traditional high energy limit in brane-world models, in which the square of the Hubble parameter in the 4D Universe is directly related to the square of the energy density [24].

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3 The following inequality is valid for \( m < 3(1 + \omega) \) otherwise it is reversed except for \( m = 3(1 + \omega) \) where the expression inside the absolute value is a constant.
For the high energy limit in the visible brane, we have the opposite approximation,

\[
\left| \frac{\kappa^2_{(5)}}{6} \rho_0 a_0^m y_c \right| \gg 1. \tag{25}
\]

The effects of the radion are present, and the energy density in the hidden brane becomes a complicated function of the energy density in the visible brane. Curiously enough, the equation of state parameter in the hidden brane approaches to an asymptotic value given by

\[
\omega_c \rightarrow \frac{m}{3} - 1, \tag{26}
\]

and the Hubble parameters in both branes would be related through

\[
H_c = \frac{m + 2 + 3 \omega_0}{2m - 1} H_0. \tag{27}
\]

It should be noted that the asymptotic values of the equation of state and Hubble parameters are obtained in the case when \(a_0\) is small enough in order to satisfy the relation (25).

3.2 The case of static extra dimensions

To illustrate the new possible cosmic characteristics of this brane model, we consider here the static case with \(m = 0\), together with a constant equation of state parameter \(w_0\) on the visible brane, then \(\rho_0 = \rho_v a_0^{-3(1+w_0)}\), where \(\rho_v\) is a constant. This case was first presented in Ref. [18], but here we provide more details about the solutions. Equations (14a), (17b) and (22b) explicitly read

\[
a_c = a_0 \left[ 1 - (a_0/a_v)^{-3(1+w_0)} \right], \tag{28a}
\]

\[
w_c = \frac{w_0 - (2 + 3 w_0)(a_0/a_v)^{-3(1+w_0)}}{1 + (2 + 3 w_0)(a_0/a_v)^{-3(1+w_0)}}, \tag{28b}
\]

\[
H_c^2 = H_0^2 \left[ 1 + (2 + 3 w_0)(a_0/a_v)^{-3(1+w_0)} \right]^2 \left[ 1 - (a_0/a_v)^{-3(1+w_0)} \right]^2, \tag{28c}
\]

where \(H_0^2 = \frac{\kappa^4_{(5)}}{36} \rho_v^2 a_0^{-6}\).

If we impose the condition that the scale factor in both branes must be positive definite, then we find the following condition imposed on the scale factor of our visible brane [see Eq. (28a) above]

\[
a_0 > a_v = \left( \frac{\kappa^2_{(5)} \rho_v y_c}{6} \right)^{1/(1+w_0)}, \tag{29}
\]

which means that our scale factor is bounded from below, i.e., our Universe has a minimum size which is directly related to the separation of the branes along the fifth
dimension. The critical value $a_c$ exists as long as $w_0 \neq -1$, but notice that if $w_v = -1$ then quantities in the two branes evolve equally in time as the term $(a_0/a_v)$ disappears from Eq. (28). Notice that the critical value in Eq. (29) implies also the divergence of the energy density $\rho_c$ and of the Hubble parameter $H_c$, see Eqs. (17a) and (28c), as $a_0 \rightarrow a_v$.

An illustration of the behavior of the scale factor $a_c$, the equation of state $w_c$, and the Hubble parameter $H_c$, in the hidden brane is shown in Fig. 1 for $w_0 = 0$. As mentioned before, the scale factor of the visible brane is bounded from below, $a_0/a_v \leq 1$, whereas for the scale factor of the hidden brane $a_c > 0$. In other words, the visible Universe starts with a finite size, whereas the hidden Universe witnesses the occurrence of a Big Bang. However, both scale factors evolve in the same way once $a_0 \gg a_v$.

This is also manifest in the behavior of the Hubble parameters: that of the hidden brane diverges as $a_0 \rightarrow a_v$, but the two evolve equally $H_c = H_0$ for $a_0 \gg a_v$. On the other hand, the equation of state of the visible brane is always of the dust type, $w_0 = 0$, but that of the hidden brane is always negative in the range $-1/2 < w_c \leq 0$.

It must be noticed that for this case the usual energy limits are in agreement with the standard nomenclature: the high energy limit, as discussed in the previous section, corresponds to small values of the scale factor, $a_0 \rightarrow a_v$, whereas the low energy limit corresponds to large values $a_0 \gg a_v$. That is to say, the evolution of both branes coincide in the low energy limit.

Much the same happens for other values of the equation of state $w_0$, which indicates that the general features of the solutions are determined mainly by the static property of the fifth dimension. For instance, in the case of radiation when $w = 1/3$ there is a similar behavior for the scale factor $a_c$, the equation of state $w_c$, and the Hubble parameter $H_c$ to the depicted in Fig. 1, but with some small differences. The asymptotic values for $a_c$, $w_c$, $H_c$ for $a_0 \gg a_v$ are obtained faster than for the non-relativistic matter case because they depend on $(a_0/a_v)^{-4}$ instead of $(a_0/a_v)^{-3}$. On the other side, the minimum value of the scale factor is smaller than in the dust case.

![Fig. 1 Evolution of the scale factor $a_c$, the equation of state $w_c$, and the Hubble parameter $H_c$, in the hidden brane as functions of the scale factor in the visible brane $a_0$, see Eq. (28); notice that the scale factor is normalized in terms of the critical value $a_v$, see Eq. (29). The case depicted here corresponds to a dust fluid in the visible Universe with $w_0 = 0$. The evolution of the visible and hidden Universes differ in the high energy limit, $a_0 \rightarrow a_v$, but they coincide in the low energy limit $a_0 \gg a_v$, see the text for more details](image-url)
3.3 The case of dynamic extra dimensions

We now turn our attention to the case in which the separation between the branes along the fifth dimension changes with time, with its scale factor evolving in the form \( b = a^m \). Again from Eqs. (14a) and (17b), the general expressions for the scale factor and the equation of state in the hidden brane are, respectively,

\[
\begin{align*}
    a_c &= a_0 \left[ 1 + (2m - 1)(a_0/a_v)^{m-3(1+w_0)} \right]^{1/(1-2m)}, \\
    w_c &= \frac{w_0 + (m + 2 + 3w_0)(m/3 - 1)(a_0/a_v)^{m-3(1+w_0)}}{1 + (m + 2 + 3w_0)(a_0/a_v)^{m-3(1+w_0)}},
\end{align*}
\]

where we have also defined a critical value of the scale factor as

\[
a_v = \left( \frac{\kappa^2_v}{6 \rho_v y_c} \right)^{1/(3(1+w_0)-m)}.
\]

There is a more involved interplay between the values of the parameters in the visible brane and the parameter \( m \). In particular, a non trivial dynamics appears for the scale factors whenever \( m - 3(1 + w_0) \neq 0 \) and \( m \neq 1/2 \), and then we can consider four general types of behavior as follows.

- We start with the case \( m - 3(1 + w_0) < 0 \) and \( m < 1/2 \). The first condition establishes that the differences in the evolution of the scale factors \( a_0 \) and \( a_c \) should disappear at late times when \( a_0/a_v \gg 1 \), as the term \((a_0/a_v)^{m-3(1+w_0)} \rightarrow 0\) in Eq. (30a). However, condition \( m < 1/2 \) establishes that the visible scale factor \( a_0 \) has to have a minimum value if the hidden scale factor \( a_c \) is to remain positive definite, but it still allows the occurrence of a Big Bang in the hidden brane: \( a_c \rightarrow 0 \) as \( a_0 \) takes on its minimum value. This whole picture resembles what we discussed in Sec. 3.2 in the case of static extra dimensions with \( m = 0 \). It can be seen that the two conditions above can be accomplished for any value of the equation of state \( w_0 \) just by choosing an appropriate value of \( m \).

If we now change the second condition to \( m > 1/2 \) a Big Bang singularity is allowed for the visible brane as now is permitted that \( a_0 \rightarrow 0 \). This same fact makes the hidden scale factor to vanish too at early times, \( a_c \rightarrow 0 \), whereas at late times we expect that \( a_c \simeq a_0 \). It must be said that this case does not exist for any value of the equation of state \( w_0 \), as the two conditions combined result into \( 1/2 < m < 3(1 + w_0) \); it is then necessary that \( w_0 > -5/6 \). For instance, the case could not be realized for a cosmological constant with \( w_0 = -1 \).

- Let us consider the case \( m - 3(1 + w_0) > 0 \) and \( m < 1/2 \). In contrast to the cases above, the scale factors evolve differently at late times when \( a_0/a_v \gg 1 \), and then

\[
a_c/a_v \sim (a_0/a_v)^{1+[m-3(1+w_0)]/(1-2m)}.
\]

At early times the visible scale factor \( a_0 \) is pretty similar to \( a_c \). This is also a restricted case, as the two conditions above can be combined into \( 1/2 > m > \)
3(1 + w₀), which can be satisfied for a given m only if w₀ < −5/6. The last case corresponds to m − 3(1 + w₀) > 0 and m > 1/2. At early times a₀ → 0 we find that a₀ ∼ a₀, whereas at late times they are related again through Eq. (32). This is a non-restricted case, as we can always satisfy the two conditions for any given value of w₀ by choosing an appropriate value of m.

An interesting feature is the appearance or not of a minimum scale factor in the visible brane, which is mainly decided by the condition m < 1/2 or m > 1/2, as long as we demand that both brane scale factors must be positive definite. As an example, we show different values of m for an equation of state of dust w₀ = 0 and of a cosmological constant w₀ = −1 in Fig. 2. The corresponding cases for the equations of state in the two branes can be seen in Fig. 3, which result from the solution (30b). In all cases we are assuming that the properties of the perfect fluid in the visible brane are fixed by physical considerations, and then the perfect fluid in the hidden brane must evolve according to the matching conditions in Eq. (18b).

![Fig. 2](image1.png) (Left) Evolution of the scale factor of the hidden brane aₗ in terms of that in the visible brane a₀, for different values of m in the case of an equation of state w₀ = 0, see Eq. (30a). Both branes correspond to expanding Universes, but the visible scale factor is bounded from below if m < 1/2. (Right) The same as before but for an equation of state w₀ = −1. This time again the visible scale factor is bounded from below if m < 1/2. Notice that both scale factors have been normalized in terms of the critical value aᵥ defined in Eq. (31). See the text for more details.

![Fig. 3](image2.png) Evolution of the equation of state in the hidden brane wₗ as a function of the normalized scale factor in the visible brane a₀/aᵥ, for w₀ = 0 (left) and w₀ = −1 (right), see Eq. (30b). The equation of state is negative (positive) or null for m ≥ −2(m < −2). See the text for more details.
In the case of dust, it can be seen that for values \( m \leq 3 \) both branes correspond to expanding Universes as the scale factor in the hidden brane \( a_c \) is a growing function of that in the visible brane \( a_0 \), but with \( a_c \) growing at a less rate than \( a_0 \) at early times, whereas for late times \( a_c \) grows faster than \( a_0 \). When \( m > 3 \) we observe the opposite behavior. All of this is related to the behavior of the equation of state in the hidden brane, as \( w_c \) is negative (positive) definite for \( m > -2 \) (\( m < -2 \)), whereas \( w_c = w_0 = 0 \) for the special cases \( m = -2, 3 \), see Fig. 3.

In the case of radiation-like equation of state in the visible brane, with \( w_0 = 1/3 \), there is a pretty similar evolution between the hidden and visible scale factor as described before for the dust case \( w_0 = 0 \). The equation of state is negative for \( m \leq 3 \) in the small scale factor region, but in all cases at late times becomes positive definite. For \( m < 0 \), there is a minimum value for the scale factor in the visible brane as was mentioned before in the text (because \( m < 1/2 \)).

As we said above, an interesting feature of this model is the existence of a minimum value for our universe \( a_0 \). This characteristic suggests a possible relation with a type of bouncing cosmology, as we are going to discuss now. The Hubble factor in the visible brane can be cast in the following form (using the gauge \( n_c = 1 \))

\[
H_0 = \frac{1}{y_c a_0^{m}} (a_0/a_v)^{-3(1+w_0)} \left[ 1 + (2m - 1) (a_0/a_v)^{m-3(1+w_0)} \right]^{m/(2m-1)} \left[ 1 + (m + 2 + 3w_0)(a_0/a_v)^{m-3(1+w_0)} \right]^{-m/(2m-1)},
\]

(33)

where we have used the same notation as in Eq. (30). The behavior of \( H_0 \) is shown in Fig. 4, where we can see that \( H_0 \) vanishes at positive \( a_0 = (1 - 2m)^{1/(3(1-w_0)-m)} a_v \) for \( m < 1/2 \), indicating the possible occurrence of a bouncing cosmology, although of the singular type, see for instance [25]. It is possible to consider other values of \( m \), for example, \( m = -1/2, -1/6, -1/10 \), where the behavior of \( H_0 \) is similar to the presented in Fig. 4.

These bouncing behaviors have been suggested as an alternative to inflation since, under some conditions, they produce a nearly scale invariant spectrum of cosmological
perturbations (see for example [26]). Quite recently [27], it has been argued the bounce universe is a viable model which is compatible with the latest results presented by BICEP2 [28] experiment because the ratio to tensor–scalar perturbation is high in some bouncing cosmologies that were excluded before by WMAP data.

4 Conclusions and remarks

We have worked out the details of the relationship between the cosmologies of two branes as given by the general solutions of Einstein’s equations in five dimensions and the Israel’s junction conditions. The solutions found suggest that the matter fields (in our case represented by perfect fluids) cannot evolve separately but their properties are connected by gravity along the fifth dimension.

For the case of a vacuum bulk, and when the visible brane (our Universe) is dominated by a single component, we found exact solutions for the relationship between the scale factors and the equations of state of the matter fields on the branes. The solutions then allow for the coexistence of two expanding Universes, or the coexistence of an expanding Universe with a contracting one.

An important characteristic which is present in the cosmology of the system is the existence of a minimum value for the scale factor of the visible brane. Besides of this property, the model also admits bouncing cosmologies. This last feature is interesting because bouncing cosmologies have been suggested as an alternative to inflation and, it turns out that, some of them are also compatible with the latest results presented by the BICEP2 experiment. We think a potential application of the model could be extended in this direction.

The expressions for the gravitational relationships of the matter fields have a more general application than for the single components studied here, and can be used for more general cases in which the visible brane contains more than one perfect fluid. This is an ongoing work that will be published elsewhere.

Acknowledgments We thank M. A. García-Aspeitia for useful comments. LAU-L thanks the Royal Observatory, Edinburgh, where part of this work was done during a sabbatical stay, for its kind hospitality. We acknowledge partial support by SNI-México, CONACyT research Grants J1-60621-I, 103478, COFAA-IPN and SIP-IPN Grant 20131541 and 20144150. This work was partially supported by PIFI, PROMEP, DAIP-UG, CONACyT México under Grants 167335 and 179881, the Fundación Marcos Moshinsky, and the Instituto Avanzado de Cosmología (IAC) collaboration.

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