CONSTRUCTION OF SEMI - FREE $S^3$ ACTIONS ON HOMOTOPY SPHERE WITH UNTWISTED FIXED POINT SET

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ABSTRACT. In his paper "Surgery and the theory of differentiable transformation groups", William Browder developed surgery techniques to study semi-free actions of $S^1$ on homotopy spheres, under the additional assumption that the fixed point set is a homotopy sphere. He used this surgery to show how to construct such actions. In this paper, I discuss a similar theory about semi-free actions of $S^3$ on homotopy spheres. An open problem is raised at the end of the paper.

1. Introduction and preliminaries

Throughout this paper, $R^n$ denotes the Euclidean $n$-space, $S^n$ denotes the unit $n$-sphere in $R^{n+1}$ and $QP(n)$ the quaterionic projective space, with the usual differentiable structures. A homotopy $n$-sphere is denoted by $\sum^n$, it means a closed differentiable $n$-manifold having the homotopy type of $S^n$. A homotopy quaterionic projective $n$-space is denoted by $HQP(n)$, it means a closed differentiable $4n$-manifold having the homotopy type of $QP(n)$.

$\pi_n(M)$ denotes the $n^{th}$-homotopy group of $M$, $H_i(M, G)$ and $H^i(M, G)$ denote the homology and cohomology of a space $M$, with coefficients in the group $G$ under the assumption that it satisfies the Eilenberg–Steenrod axioms, See [12],page 6. If $G = Z$, we write $H_i(M)$ and $H_i(M)$ for $H^i(M, Z)$ and $H^i(M, Z)$ respectively. It is well known that $S^1$ and $S^3$ are the only compact connected Lie groups which have free differentiable actions on homotopy spheres [11]. It follows from Gleason’s lemma [5] that such an action is always a principal fibration which is homotopically equivalent to the classical Hopf fibration.

In fact, there are always infinitely many differentiably distinct free actions of $S^3$ on $\sum^{4n+3}$ for $n \geq 2$, see [7].

2. Construction of semi-free actions:

An action $(G, M, \phi)$ is called semi-free, if it is free outside the fixed point set. That is, for an action $\phi : G \times M \rightarrow M$, let $F$ be the fixed point set of the action,
\( \phi \) is semi-free if \( \phi(g, x) = x \), for some \( x \in M - F \) then \( g = e \) which is the identity of \( G \). Notice that there are only two types of orbits, fixed points and \( G \).

**Lemma 2.1.** ([16]) Let \( \phi : S^i \times M \to M \ (i = 1, 3) \), be a semi free differentiable action. let \( F^k \) denote the union of the \( k \)-dimensional components of the set of all fixed points of \( \phi \). Then the normal bundle of an imbedding \( F^k \subset M \) has naturally a complex structure for \( i = 1 \) and a quaternionic structure for \( i = 3 \) and that the induced \( S^i \)-action on the normal bundle is a scalar multiplication.

It follows from Lemma 2.1 that the codimension of each component of \( F \) in \( M \) is even for \( i = 1 \) and is divisible by 4 for \( i = 3 \). We shall study the situation where \( (S^3, \sum^m, \phi) \) is a semi-free differentiable action on a homotopy sphere \( \sum^m \) and the fixed point set is a homotopy sphere \( \sum^r \). Let \( (S^3, \sum^m, \phi) \) be a semi-free action with fixed point set \( \sum^r \subset \sum^m \), then \( S^3 \) acts freely outside \( \sum^r \) and \( S^3 \) acts freely and linearly on the normal space to \( \sum^r \) at each point of \( \sum^r \). See [3], page 58]. By Lemma 2.1, the normal bundle of \( \sum^r \) has a quaternionic structure and \( m - r = 4k, \ k \geq 1 \). Let \( \mu \) be the (quaternionic) bundle over \( \sum^r \) that is defined by the action. We prove the following:

**Theorem 2.2.** If \( (S^3, \sum^m, \phi_1) \) and \( (S^3, \sum^m, \phi_2) \) are equivalent, then \( F_1 \) is diffeomorphic to \( F_2 \) and \( \mu_1 \) is equivalent to \( \mu_2 \), where \( F_1 \) and \( F_2 \) are the fixed point sets of \( \phi_1 \) and \( \phi_2 \) respectively, \( \mu_1 \) and \( \mu_2 \) are the normal bundles of \( F_1 \) and \( F_2 \) respectively.

**Proof.** Since \( (S^3, \sum^m, \phi_1) \) and \( (S^3, \sum^m, \phi_2) \) are equivalent then there exists an equivariant diffeomorphism \( f \) such that the following diagram commutes:

\[
\begin{array}{ccc}
S^3 \times \sum^m & \xrightarrow{\phi_1} & \sum^m \\
I \times f \downarrow & & \downarrow f \\
S^3 \times \sum^m & \xrightarrow{\phi_2} & \sum^m
\end{array}
\]

Let \( x \in F_1 \), i.e. \( \phi_1(q, x) = x \ \forall \ q \in S^3 \) then \( f \circ \phi_1(q, x) = \phi_2 \circ (I \times f)(q, x) \ \forall \ q \in S^3 \), thus \( f(x) = \phi_2(q, f(x)) \ \forall \ q \in S^3 \). See [3], page 58]. Let \( \mu_1 \) be the (quaternionic) bundle over \( \sum^r \) that is defined by the action.

Now, assume that \( y \in F_2 \), i.e. \( \phi_2(q, y) = y \ \forall \ q \in S^3 \), but \( f \) is an equivariant diffeomorphism, then \( \exists \ x \in \sum^m \) such that \( y = f(x) \) and \( f \circ \phi_1(q, x) = \phi_2 \circ (I \times f)(q, x) \ \forall \ q \in S^3 \), then \( \phi(f(q, x)) = \phi_2(q, f(x)) = f(x) \), but \( f \) is 1-1, we get \( \phi_1(q, x) = x \ \forall \ q \in S^3 \), hence \( x \in F_1 \) and \( f(x) \in f(F_1) \), therefore \( f(F_1) = F_2 \). Moreover, the equivalence \( f : \sum^m \to \sum^m \) defines a quaternionic map of the normal bundles \( \mu_1 \) and \( \mu_2 \) of \( F_1 \) and \( F_2 \) respectively, so they are equivalent.

Now, \( \sum^r \) is a closed submanifold of \( \sum^m \) and is invariant under the action of \( S^3 \), therefore there exists a tubular neighbourhood \( E \) of \( \sum^r \) which is invariant under the action of \( S^3 \). So we may consider \( \mu : E \to \sum^r \), the normal bundle to \( \sum^r \) in \( \sum^m \) and \( S^3 \) acts differentiably on \( E \), see [3], page 58]. Let \( S^{4k-1} \) be the boundary of a fibre of \( E \), then we can prove the following:

**Lemma 2.3.** \( S^{4k-1} \subset \sum^m - \sum^r \) is a homotopy equivalence.
Proof. Consider the exact cohomology sequence of the pair \((\sum^m, \sum^r)\):

\[
\rightarrow H^{i-1}(\sum^r) \rightarrow H^i(\sum^m, \sum^r) \rightarrow H^i(\sum^m) \rightarrow H^i(\sum^r) \rightarrow H^{i+1}(\sum^m, \sum^r) \rightarrow H^{i+1}(\sum^m) \rightarrow
\]

At \(i = m\) : \(0 \rightarrow H^m(\sum^m, \sum^r) \rightarrow Z \rightarrow 0\), then \(H^m(\sum^m, \sum^r) \cong Z\).

At \(i = r\) : \(0 \rightarrow Z \rightarrow H^{r-1}(\sum^m, \sum^r) \rightarrow 0\), hence \(H^{r+1}(\sum^m, \sum^r) \cong Z\).

And for \(i \neq m, r + 1 : 0 \rightarrow 0 \rightarrow H^i(\sum^m, \sum^r) \rightarrow 0\), thus \(H^i(\sum^m, \sum^r) \cong 0\) for \(i \neq m, r + 1\). Finally, we obtain

\[
H^i(\sum^m, \sum^r) = \begin{cases} Z & \text{for } i = m, r + 1 \\ 0 & \text{otherwise} \end{cases}
\]

It is clear that \(\sum^m - \sum^r\) is simply connected \([9], page 3\) and using Lefshetz duality \([12], page 32\),

\[
H_i(\sum^m - \sum^r) \cong H^{m-i}(\sum^m, \sum^r),
\]

we finally deduce that

\[
H_i(\sum^m - \sum^r) = \begin{cases} Z & \text{for } i = 0, 4k - 1 \\ 0 & \text{otherwise} \end{cases}
\]

hence, the inclusion map \(i : S^{4k-1} \rightarrow \sum^m - \sum^r\) induces

\[i_* : \pi_j(S^{4k-1}) \rightarrow \pi_j(\sum^m - \sum^r)\] an isomorphism \(\forall j\), see \([14], page 283\). Therefore \(S^{4k-1} \subset \sum^m - \sum^r\) is a homotopy equivalence.

Now, let \(N = \sum^m - E_0\) where \(E_0\) is the interior of an equivariant tubular neighbourhood of \(\sum^r\) with \(E_0 \subset \text{int}(E)\). Then \(S^3\) acts freely on \(N\) and \(S^{4k-1} \subset N\). Notice that \(S^{4k-1}\) is a homotopy equivalence to \(N\), it follows from the exact homotopy sequence of the fibre maps, using the diagram :

\[
\begin{array}{ccc}
\rightarrow S^3 & \rightarrow & S^{4k-1} \\
\downarrow & & \downarrow \\
\rightarrow S^3 & \rightarrow & N \\
\end{array} \rightarrow \frac{N}{S^3} \rightarrow
\]

that is \(S^{4k-1}/S^3 \rightarrow N/S^3\) is a homotopy equivalence.

Set \(N^1 = N/S^3\) and \(S^{4k-1}/S^3 = QP(k - 1)\). Notice that the region between \(\partial N^1\) and \(QP(k - 1) \times S^r\) is an \(h\)--cobordism, so if \(m \geq 6\), then by the \(h\)--cobordism theorem of Smale \([15]\), \(N^1\) is diffeomorphic to \(QP(k - 1) \times D^{r+1}\) and \(N \rightarrow N^1\) is equivalent to

\[h \times I : S^{4k-1} \times D^{r+1} \rightarrow QP(k - 1) \times D^{r+1}\], where \(h : S^{4k-1} \rightarrow QP(k - 1)\) is the hopf map. Hence, we have proved the following theorem:

**Theorem 2.4.** Let \((S^3, \sum^m, \phi)\) be a semi–free \(S^3\) action on \(\sum^m\) with fixed point set \(\sum^r\), \(m - r = 4k, k \geq 1, m \geq 6\). If \(N\) is the complement of an open tubular neighbourhood of \(\sum^r\) in \(\sum^m\), then \(N\) is equivariantly diffeomorphic to \(S^{4k-1} \times D^{r+1}\), with the standard action on \(S^{4k-1}\) and trivial action on \(D^{r+1}\).
Now, we will describe how to construct smooth semi–free $S^3$ actions on a homotopy $m$–sphere $\sum^m$. Let $\sum^r$ be a homotopy $r$–sphere and $\mu$ a (quaternionic) normal bundle over $\sum^r$ given by $\mu : E(\mu) \to \sum^r$ where $E(\mu)$ is the total space of $\mu$ such that $E(\mu) \cong D^{4k} \times \sum^r$, i.e., $E(\mu)$ is the trivial bundle and suppose that $h : S^{4k-1} \times \sum^r \to S^{4k-1} \times \sum^r$ is an equivariant diffeomorphism.

**Theorem 2.5.** There is a semi free action $(S^3, \sum^m, \phi)$ with a fixed point set $\sum^r$ and $\sum^m = E(\mu) \cup_h (S^{4k-1} \times D^{r+1})$ where $\cup_h$ means that we identify $S^{4k-1} \times \sum^r \subset E(\mu)$ with $S^{4k-1} \times \sum^r \subset S^{4k-1} \times D^{r+1}$ via the diffeomorphism $h$.

**Proof.** Consider the semi–free action on the total space of $\mu : E(\mu) \to \sum^r$ defined by the quaternionic structure and the free $S^3$–action on $S^{4k-1} \times D^{r+1}$ defined by the free action on $S^{4k-1}$, i.e., the standard action and $h : S^{4k-1} \times \sum^r \to S^{4k-1} \times \sum^r$ is an equivariant diffeomorphism, then $M = E(\mu) \cup_h (S^{4k-1} \times D^{r+1})$ has a semi–free action of $S^3$ with fixed point set $\sum^r$ and normal bundle $\mu$, it is enough to show that $M$ is a homotopy sphere.

It is clear that $\pi_1(\partial E(\mu)) \cong \pi_1(S^{4k-1} \times \sum^r) \cong \pi_1(S^{4k-1}) \cong \pi_1(\sum^r)$ and $\pi_0(\partial E(\mu)) \cong \pi_0(\sum^r) \cong 0$, i.e., $E(\mu)$ and $S^{4k-1} \times D^{r+1}$ are simply connected and $E(\mu) \cap S^{4k-1} \times D^{r+1}$ is simply connected, hence by VanKampen’s theorem [2], $M$ is simply connected.

Now, we consider the Mayer -Vietoris sequence for $M$ [4]:

$$
\to H_{s+1}(M) \to H_s(\partial E(\mu)) \to H_s(E(\mu)) \oplus H_s(S^{4k-1} \times D^{r+1}) \to H_s(M) \to
$$

By the K"unneth formula ([6], page 98), since $H_s(\partial E(\mu))$, $H_s(E(\mu))$, $H_s(S^{4k-1} \times D^{r+1})$ are torsion free for $0 < s < 4k + r - 1$, we obtain

$$H_s(\partial E(\mu)) = H_s(S^{4k-1} \times \sum^r) \cong \bigoplus_{i=0}^s H_i(S^{4k-1}) \otimes H_{s-i}(\sum^r).$$

If $i = 0$, then $H_0(S^{4k-1}) \otimes H_0(S^r) \cong Z \otimes H_0(\sum^r)$.
If $i = s$, then $H_s(S^{4k-1}) \otimes H_0(S^r) \cong H_s(S^{4k-1}) \otimes Z$ and for $i \neq 0$

$H_i(S^{4k-1}) \otimes H_{n-i}(S^r) \cong 0$ therefore $H_s(\partial E(\mu)) \cong Z \otimes H_s(S^r) \oplus H_s(S^{4k-1}) \otimes Z$.

Again, we compute

$H_s(S^{4k-1} \times D^{r+1}) = \bigoplus_{i=0}^s H_i(S^{4k-1}) \otimes H_{s-i}(D^{r+1}) \cong H_s(S^{4k-1}) \otimes Z.$

Similarly $H_s(\partial E(\mu)) \cong H_s(E(\mu)) \oplus H_s(S^{4k-1} \times D^{r+1})$,

hence $H_s(M) \cong H_{s+1}(M) \cong 0, \forall 0 < s < 4k + r - 1$.

For, $s = 4k + r - 1 : H_{4k+r-1}(E(\mu)) \cong 0, H_{4k+r-1}(S^{4k-1} \times D^{r+1}) \cong 0$ and $H_{4k+r-1}(\partial E(\mu)) \cong Z \otimes Z \cong Z$.

Substituting in the Mayer -Vietoris sequence for $M : 0 \to H_{4k+r}(M) \to Z \to 0$.

Finally, we obtain

$$H_s(M) \cong \begin{cases} 
  z & \text{for } s = 0, 4k + r, \\
  0 & \text{otherwise}
\end{cases}$$
3. Applying surgery to construct semi free $S^3$–actions:

In this section, we used surgery techniques as Browder [1] to create a diffeomorphism of $QP(k-1) \times \sum$ with $QP(k-1) \times S^r$, then we apply Theorem 2.5

**Theorem 3.1.** Let $\sum^{4n-1}$ be a homotopy sphere which bounds a parallelizable manifold, $n \geq 1$. Then for each even $k \geq 2$, there is a semi–free action of $S^3$ on a homotopy sphere $\sum^{4(n+k)-1}$ with $\sum^{4n-1}$ as untwisted fixed point set.

**Proof.** Let $\sum^{4n-1} = \partial W^{4n}$, $W$ is a parallelizable manifold. We may consider $W_0 = W - int(D^{4n})$ as a parallelizable cobordism between $\sum^{4n-1}$ and $S^{4n-1}$ thus we may define a normal map

$$f : (W_0, \sum^{4n-1} \cup S^{4n-1}) \to (S^{4n-1} \times I, S^{4n-1} \times \{0\} \cup S^{4n-1} \times \{1\})$$

with $f|_{S^{4n-1}} = $ Identity. Since $W$ is a parallelizable manifold [[10], page 514], we may assume that $W_0$ is $(2n-1)$ connected.

Multiplying by $QP(k-1)$, we get $I \times f$:

$$QP(k-1) \times (W_0, \sum^{4n-1} \cup S^{4n-1}) \to QP(k-1) \times (S^{4n-1} \times I, S^{4n-1} \times \{0\} \cup S^{4n-1} \times \{1\})$$

with $I \times f|_{QP(k-1) \times S^{4n-1}} = $Identity.

The remainder of the proof is computing the obstruction $\sigma$ for this map to be a cobordism and using this to determine if $QP(k-1) \times \sum^{4n-1}$ is diffeomorphic to $QP(k-1) \times S^{4n-1}$.

**Claim:** $Ker(I \times f)_* = H_*(QP(k-1)) \times Ker(f_*)$.

By the Künneth formula, since $H_*(QP(k-1))$ is torsion free then,

$$H_*(QP(k-1) \times (W_0, \sum^{4n-1} \cup S^{4n-1})) \cong H_*(QP(k-1)) \otimes H_*(W_0, \sum^{4n-1} \cup S^{4n-1})$$

and $(I \times f)_* = I \otimes f_*$

therefore, $Ker(I \times f)_* = H_*(QP(k-1)) \times Ker(f_*)$.

Now, consider the commutative diagram induced by $f$:

$$\begin{array}{cccc}
\to & H_i(\partial W_0) & \to & H_i(W_0) & \to & H_i(W_0, \partial W_0) & \to \\
\downarrow & & & \downarrow & & f_* \downarrow & \\
\to & H_i(\partial S^{4n-1} \times I) & \to & H_i(S^{4n-1} \times I) & \to & H_i(S^{4n-1} \times I, \partial S^{4n-1} \times I) & \to
\end{array}$$
Notice that, $H_i(\partial S^{4n-1} \times I) \cong H_i(\partial W_0)$ and $H_i(W_0) \cong 0$ for $i \neq 0, 2n$. We get

\[ 0 \longrightarrow H_{2n}(W_0) \cong H_{2n}(W_0, \partial W_0) \longrightarrow H_{2n}(W_0, \partial W_0) \downarrow f_* \longrightarrow 0 \longrightarrow 0 \]

Hence $\text{Ker}(f_*) \cong H_{2n}(W_0)$.

But $\text{Ker}(I \times f_*) = \text{Ker}(I \otimes f_*) = H_s(QP(k-1)) \otimes \text{Ker}(f_*) \cong H_s(QP(k-1)) \otimes H_{2n}(W_0)$, and $\text{Ker}(I \times f)_{*2n+2k-2} = H_{2k-2}(QP(k-1)) \otimes H_{2n}(W_0)$.

Since $2k - 2 = 2(2s) - 2 \neq 0 \mod 4$, $k$ is even, then $H_{2k-2}(QP(k-1)) \cong 0$ and $\text{Ker}(I \times f)_{*2n+2k-2} \cong 0$.

Therefore, $\sigma(I \times f) = 0$ and there exists an $h$-cobordism between $QP(k-1) \times \sum_{4n-1}^m$ and $QP(k-1) \times S^{4n-1}$, but $k \geq 2$ and $n \geq 1$, then $(4k-4) + (4n-1) = 4(k+n) - 5 \geq 7$. Hence, Smale’s $h$-cobordism theorem can be applied and $QP(k-1) \times \sum_{4n-1}^m$ is diffeomorphic to $QP(k-1) \times S^{4n-1}$. Applying Theorem 2.5, it follows that there is a semi-free action of $S^3$ on some homotopy sphere $\sum_{4n-1}^m$ with $\sum_{4n-1}^m$ as untwisted fixed point set, where $m = 4(n+k) - 1$. □

Open Problem

Browder [1] showed how to construct semi-free $S^1$ actions, with $\sum_{4n-1}^m$ as untwisted fixed point set, i.e., its normal bundle is trivial. He stated that he did not know of any action with a twisted fixed point set. However Schultz [14] used complicated computations of homotopy groups, proved the following Theorem: Let $k \geq 2$ be a positive integer. Then there exist infinitely many values of $n$ for which $S^{2n}$ has a semi-free $S^1$ action with $S^{2(n-k+1)}$ as twisted fixed point set. In this work, as in the work of Browder, we consider $\sum_{4n-1}^m$ as untwisted fixed point set and $I$ raise the following question: Does there exist smooth semi free $S^3$ actions on homotopy spheres for which the fixed point set is twisted?

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