ππ scattering: theory is ahead of experiment

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Abstract. I draw attention to a recent breakthrough in the field of low energy pion physics: the consequences of the hidden symmetry of the QCD Hamiltonian have successfully been incorporated in the general dispersive framework for the ππ scattering amplitude, which is due to Roy. The meagre experimental information about the imaginary parts at and above 0.8 GeV suffices to unambiguously and accurately pin down the scattering amplitude at lower energies. The recent Brookhaven data on the reaction $K \to \pi \pi e\bar{\nu}$ provide a significant test of the theory. They imply that the Gell-Mann-Oakes-Renner relation is approximately valid – the bulk of the pion mass indeed originates in the quark condensate.

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PRELUDE

Arnulfo Zepeda published a pioneering paper in 1972, together with M.A.B. Beg [1]. In this paper, it is shown that, in the chiral limit, the charge radii of pions and nucleons contain infrared singularities. In particular, the pion radius diverges logarithmically if the pion mass is sent to zero,

$$\langle r^2 \rangle_{\text{em}} = \frac{1}{(4\pi F_\pi)^2} \ln \frac{\Lambda^2}{M_\pi^2} + O(M_\pi^2).$$

The result contains what is called a chiral logarithm, whose coefficient is determined by the pion decay constant, $F_\pi \simeq 92.4$ GeV. In the terminology of Chiral Perturbation Theory (ChPT), the formula gives the leading term in the chiral expansion. In that framework, momentum independent quantities such as the pion mass or the charge radii are expanded in powers of the quark masses $m_u$ and $m_d$. As is well-known, the leading term in the chiral expansion of $M_\pi$ is proportional to $\sqrt{m_u + m_d}$, while the expansion of $F_\pi$ starts with a constant term. Since the scale $\Lambda$ of the logarithm in eq. (1) is independent of the quark masses, the formula states that the charge radius of the pion diverges logarithmically if the quark masses $m_u, m_d$ are sent to zero. The scale $\Lambda$ is related to the effective coupling constant $\tilde{\ell}_6$:

$$\ln \frac{\Lambda^2}{M_\pi^2} = \tilde{\ell}_6 - 1.$$

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In my talk on the *Electromagnetic form factor of the pion*, I discussed the progress made in this field since 1972. The recent interest in this quantity originates in the fact that a very accurate knowledge of the pion form factor is needed if the uncertainty in the Standard Model prediction for the magnetic moment of the muon is to match the fabulous experimental precision.

The final state interaction theorem states that, in the elastic region, the phase of the electromagnetic form factor of the pion coincides with the P-wave phase shift of the scattering process $\pi^+\pi^- \rightarrow \pi^+\pi^-$. For this reason, $\pi\pi$ scattering plays an essential role in the analysis of the form factor. In the following, I restrict myself to a discussion of the state of the art in $\pi\pi$ scattering. Concerning the application of that knowledge to the form factor, I refer to [2].

**ROY EQUATIONS**

In ChPT, the scattering amplitude is expanded in powers of the momenta as well as in the quark masses. The perturbation series has explicitly been worked out to two loops, that is to next-to-next-to-leading order [3]. The masses $m_u$ and $m_d$ are very small, so that the expansion in these variables converges rapidly. The expansion in powers of the momenta may be viewed as an expansion around the center of the Mandelstam triangle, that is a power series in $s - \frac{4}{3}M^2_\pi$ and $t - \frac{4}{3}M^2_\pi$. In view of the strong, attractive interaction in the channel with $I = \ell = 0$, the higher orders of that expansion are sizeable already at threshold, $s = 4M^2_\pi, t = 0$. The chiral representation does account for the threshold singularities generated by two-pion states, but it accounts for resonances only indirectly, through their contributions to the effective coupling constants. Dispersive methods are needed to extend the range of the chiral representation beyond the immediate vicinity of the Mandelstam triangle.

The method we are using to implement analyticity, unitarity and crossing symmetry is by no means new. As shown by Roy more than 30 years ago [4], these properties of the $\pi\pi$ scattering amplitude subject the partial waves to a set of coupled integral equations. The equations involve two subtraction constants, which may be identified with the two S–wave scattering lengths $a^0_0, a^2_0$. If these two constants are given, the Roy equations allow us to calculate the scattering amplitude in terms of the imaginary parts above the "matching point" $E_m = 0.8$ GeV. The available experimental information suffices to evaluate the relevant dispersion integrals, to within small uncertainties [5, 6]. In this sense, $a^0_0, a^2_0$ represent the essential parameters in low energy $\pi\pi$ scattering.

The Roy equations possess an entire family of solutions, covering a rather broad spectrum of physically quite different scattering amplitudes, because analyticity, unitarity and crossing symmetry alone do not determine the subtraction constants, and the experimental information about these is consistent with a broad range of values. For this reason, previous Roy equation analyses invariably came up with a family of representations for the scattering amplitude rather than a specific one – the main result established

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1 Bose statistics forbids two neutral pions to occur in a configuration with angular momentum $\ell = 1$, so that the reaction $\pi^+\pi^- \rightarrow \pi^0\pi^0$ is irrelevant here.
on the basis of this framework was the resolution of the ambiguities inherent in phase shift analyses [7, 8]. The missing ingredient in the traditional Roy equation analysis is the fact that the Hamiltonian of QCD is almost exactly invariant under the group SU(2)$_R \times$SU(2)$_L$ of chiral rotations among the two lightest quark flavours. As will be discussed in some detail in the next section, the values of the two subtraction constants can be predicted very sharply on this basis. In effect, this turns the Roy equations into a framework that fully determines the low energy behaviour of the $\pi\pi$ scattering amplitude. As an example, I mention that the P-wave scattering length and effective range are predicted very accurately: $a^1_0 = 0.0379(5)M_\pi^{-2}$ and $b^1_0 = 0.00567(13)M_\pi^{-4}$. The manner in which the P-wave phase shift passes through $90^\circ$ when the energy reaches the mass of the $\rho$ is specified within the same framework, as well as the behaviour of the two S-waves. The analysis reveals, for instance, that the iso-scalar S-wave contains a pole on the second sheet and the position can be calculated rather accurately: the pole occurs at $E = M_\sigma - \frac{1}{2}i\Gamma_\sigma$, with $M_\sigma = 470 \pm 30$ MeV and $\Gamma_\sigma = 590 \pm 40$ MeV [9], etc.

Many papers based on alternative approaches can be found in the literature. Padé approximants, for instance, continue to enjoy popularity and the ancient idea that the $\sigma$ pole represents the main feature in the iso-scalar S-wave also found new adherents recently. Crude models such as these may be of interest in connection with other processes where the physics yet remains to be understood, but for the analysis of the $\pi\pi$ scattering amplitude, they cannot compete with the systematic approach based on analyticity and chiral symmetry. In view of the precision required in the determination of the pion form factor, ad hoc models are of little use, because the theoretical uncertainties associated with these are too large.

**PREDICTION FOR THE $\pi\pi$ SCATTERING LENGTHS**

Goldstone bosons of zero momentum do not interact: if the quark masses $m_u, m_d$ are turned off, the S-wave scattering lengths disappear, $a^0_0, a^2_0 \rightarrow 0$. Like the mass of the pion, these quantities represent effects that arise from the breaking of the chiral symmetry generated by the quark masses. In fact, as shown by Weinberg [10], $a^0_0$ and $a^2_0$ are proportional to the square of the pion mass

$$a^0_0 = \frac{7M_\pi^2}{32\pi F^2_\pi} + O(M_\pi^4) \quad , \quad a^2_0 = -\frac{M_\pi^2}{16\pi F^2_\pi} + O(M_\pi^4) .$$

The corrections of order $M_\pi^4$ contain chiral logarithms. In the case of $a^0_0$, the logarithm has an unusually large coefficient

$$a^0_0 = \frac{7M_\pi^2}{32\pi F^2_\pi} \left\{ 1 + \frac{9}{2} \frac{M_\pi^2}{(4\pi F_\pi)^2} \ln \frac{\Lambda_0^2}{M_\pi^2} + O(M_\pi^4) \right\} .$$

This is related to the fact that in the channel with $I = 0$, current algebra predicts a strong, attractive, final state interaction. The scale $\Lambda_0$ is determined by the coupling constants.
of the effective Lagrangian of $O(p^4)$:

$$\frac{9}{2} \ln \frac{\Lambda_0^2}{M_\pi^2} = \frac{20}{21} \ell_1 + \frac{40}{21} \ell_2 - \frac{5}{14} \ell_3 + 2 \ell_4 + \frac{5}{2}.$$ 

The same coupling constants also determine the first order correction in the low energy theorem for $a_0^2$.

The couplings $\ell_1$ and $\ell_2$ control the momentum dependence of the scattering amplitude at first non-leading order. Using the Roy equations, these constants can be determined very accurately [9]. The terms $\ell_3$ and $\ell_4$, on the other hand, describe the dependence of the scattering amplitude on the quark masses – since these cannot be varied experimentally, $\ell_3$ and $\ell_4$ cannot be determined on the basis of $\pi\pi$ phenomenology. The constant $\ell_3$ specifies the correction in the Gell-Mann-Oakes-Renner relation [11],

$$M_\pi^2 = M^2 \left\{ 1 - \frac{1}{2} \left( \frac{M^2}{4\pi F^2} \right)^2 \ell_3 + O(M^4) \right\}.$$  
(4)

Here $M^2$ stands for the term linear in the quark masses,

$$M^2 = (m_u + m_d) \langle 0 | \bar{u}u | 0 \rangle \frac{1}{F^2}.$$  
(5)

($F$ and $\langle 0 | \bar{u}u | 0 \rangle$ are the values of the pion decay constant and the quark condensate in the chiral limit, respectively). The coupling constant $\ell_4$ occurs in the analogous expansion for $F_\pi$,

$$F_\pi = F \left\{ 1 + \frac{M^2}{(4\pi F)^2} \ell_4 + O(M^4) \right\}.$$  
(6)

A low energy theorem relates it to the scalar radius of the pion [12],

$$\langle r^2 \rangle_s = \frac{6}{(4\pi F_\pi)^2} \left\{ \ell_4 - \frac{13}{12} + O(M^2) \right\},$$  
(7)

a formula that is very similar to one of Beg and Zepeda in eq. (1). The dispersive analysis of the scalar pion form factor in ref. [9] leads to

$$\langle r^2 \rangle_s = 0.61 \pm 0.04 \text{ fm}^2.$$  
(8)

The constants $\ell_1, \ldots \ell_4$ depend logarithmically on the quark masses:

$$\ell_i = \ln \frac{\Lambda_i^2}{M^2}, \quad i = 1, \ldots, 4$$

In this notation, the above value of the scalar radius amounts to

$$\Lambda_4 = 1.26 \pm 0.14 \text{ GeV}.$$  
(9)
Unfortunately, the constant $\bar{\ell}_3 \leftrightarrow \Lambda_3$ is not known with comparable precision. The crude estimate for $\bar{\ell}_3$ given in ref. [12] corresponds to

$$0.2 \text{ GeV} < \Lambda_3 < 2 \text{ GeV}.$$  \hfill (10)

It turns out, however, that the contributions from $\bar{\ell}_3$ are very small, so that the uncertainty in $\Lambda_3$ does not strongly affect the predictions for the scattering lengths. The result obtained in ref. [9] reads

$$a_0^0 = 0.220 \pm 0.005, \quad a_0^2 = -0.0444 \pm 0.0010.$$  \hfill (11)

The analysis of the $K_{e4}$ form factors reported in ref. [13] led to very similar results. Since that determination invokes an expansion not only in $m_u$ and $m_d$, but also in $m_s$, it does not quite reach the precision of the method underlying eq. (11).

In fig. 2, the "Universal Band" shows the region in the $(a_0^0, a_0^2)$-plane where the Roy equations at all admit solutions. The dot at the left represents the leading order result in eq. (3), while the small ellipse corresponds to the values in eq. (11).

**EXPERIMENTAL TEST**

Stern and collaborators [14] pointed out that "Standard ChPT" relies on a hypothesis that calls for experimental test. Such a test has now been performed and I wish to briefly describe this development.
The hypothesis in question is the assumption that the quark condensate represents the leading order parameter of the spontaneously broken chiral symmetry. More specifically, the standard analysis assumes that the term linear in the quark masses dominates the expansion of $M_\pi^2$. According to the Gell-Mann-Oakes-Renner relation (5), this term is proportional to the quark condensate, which in QCD represents the order parameter of lowest dimension. The dynamics of the ground state is not well understood. The question raised by Stern et al. is whether, for one reason or the other, the quark condensate might turn out to be small, so that the Gell-Mann-Oakes-Renner formula would fail – the "correction" might be comparable to or even larger than the algebraically leading term.

According to eq. (4), the correction is determined by the effective coupling constant $\bar{l}_3$. The estimate (10) implies that the correction amounts to at most 4% of the leading term, but this does not answer the question, because that estimate is based on the standard framework, where $\langle 0|\bar{u}u|0\rangle$ is assumed to represent the leading order parameter. If that estimate is discarded and $\bar{l}_3$ is treated as a free parameter ("Generalized ChPT"), the scattering lengths cannot be predicted individually, but the low energy theorem (7) implies that – up to corrections of next-to-next-to leading order – the combination $2a_0^0 - 5a_0^2$ is determined by the scalar radius:

$$2a_0^0 - 5a_0^2 = \frac{3M_\pi^2}{4\pi F_\pi^2} \left\{ 1 + \frac{M_\pi^2 \langle r^2 \rangle_s}{3} + \frac{41M_\pi^2}{192\pi^2 F_\pi^2} + O(M_\pi^4) \right\}.$$

The resulting correlation between $a_0^2$ and $a_0^0$ is shown as a narrow strip in fig. 2 (the strip is slightly curved because the figure accounts for the corrections of next-to-next-to leading order).

In view of the correlation between $a_0^0$ and $a_0^2$, the data taken by the E865-collaboration at Brookhaven [15] allow a significant test of the Gell-Mann-Oakes-Renner relation. The final state interaction theorem implies that the phase of the form factors relevant for the decay $K^+ \rightarrow \pi^+ \pi^- e^+ \bar{\nu}_e$ is determined by the elastic $\pi\pi$ scattering amplitude. Conversely, the phase difference $\delta_0^0 - \delta_1^1$ can be measured in this decay. The analysis of the $4 \cdot 10^5$ events of this type collected by E865 leads to the round data points in fig. 3, taken from ref. [16] (the triangles represent the $K_{e4}$ data collected in the seventies of the last century).

The three bands show the result obtained for $a_0^0 = 0.18, 0.22, 0.26$, respectively. The width of the bands corresponds to the uncertainty in the prediction. A fit of the data that exploits the correlation between $a_0^0$ and $a_0^2$ yields

$$a_0^0 = 0.216 \pm 0.013 \text{ (stat)} \pm 0.004 \text{ (syst)} \pm 0.005 \text{ (th)} \ [15],$$

where the third error bar accounts for the theoretical uncertainties. The result thus beautifully confirms the prediction of ChPT in eq. (11). The agreement implies that more than 94% of the pion mass originate in the quark condensate, thus confirming that the Gell-Mann-Oakes-Renner relation is approximately valid [16]. May Generalized ChPT rest in peace.
FIGURE 3. Interpretation of the data on the phase difference $\delta_1 - \delta_0$ in Generalized ChPT.

CONCLUSION

In view of the fact that the pions are by far the lightest hadrons, they play a prominent role in low energy physics. Chiral Perturbation Theory offers a systematic approach for analyzing their properties. For the $\pi\pi$ scattering amplitude, the perturbation series has been worked out to two loops, but the resulting representation has two serious limitations: (1) it yields a decent approximation only at very low energies and (2) it contains quite a few effective coupling constants, which need to be determined in order to fully specify the scattering amplitude.

In the course of the last two years, a breakthrough was achieved here: we now have an accurate representation of the scattering amplitude that (1) is valid in a significantly wider energy range than the chiral representation and (2) does not contain any free parameters. In the vicinity of the center of the Mandelstam triangle, where the chiral perturbation series is rapidly convergent, the new representation agrees with the two-loop result of ChPT.

The new representation exploits the fact that the dependence of the $\pi\pi$ scattering amplitude on the momenta is very strongly constrained by general kinematics (unitarity, analyticity, crossing). Although the Roy equations do not fully exhaust these constraints, they do yield a very suitable framework for the low energy analysis. In this framework, the S-wave scattering lengths $a_0^0$ and $a_0^2$ represent the essential low energy parameters. They enter as subtraction constants of the partial wave dispersion relations – once these are known, the available experimental information suffices to accurately calculate the partial waves below 0.8 GeV. In this context, ChPT is needed exclusively to determine the two subtraction constants.

Indeed, ChPT does yield very sharp predictions for $a_0^0$ and $a_0^2$ [9]. These predictions are based on the standard framework, where it is assumed that the quark condensate is the leading order parameter. That hypothesis has now been confirmed by the E865 data on the decay $K \to \pi\pi\nu\bar{\nu}$. More precise data on this decay are forthcoming from experiment NA48/2 at CERN.
A beautiful and qualitatively quite different experiment is also under way at CERN [17]. There, charged pions are produced in abundance. Occasionally, a pair of these binds to a $\pi^+\pi^-$ atom, "pionium". The atoms almost instantly decay through the strong transition $\pi^+\pi^- \rightarrow \pi^0\pi^0$. Since the momentum of the pions circulating in pionium is very small, of order $\alpha M_\pi$, the transition amplitude is determined by the S-wave scattering lengths: the decay rate is proportional to $(a_0^0 - a_2^0)^2$. The interplay of the electromagnetic and strong interactions in bound state and decay is now very well understood [18]. A measurement of the pionium lifetime at the planned accuracy of 10% thus yields a measurement of $a_0^0 - a_2^0$ at the 5% level, thereby providing a very sensitive test of the prediction. DIRAC is a fabulous laboratory for low energy pion physics – it would be most deplorable if this beautiful project were aborted for financial reasons before its physics potential is tapped. In particular, pionium level splittings would offer a clean and direct measurement of the second subtraction constant. Data on $\pi K$ atoms would also be very valuable, as they would allow us to explore the role played by the strange quarks in the QCD vacuum.

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