State Realizations of 2-Periodic Convolutional Codes: a switching system approach *

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Abstract: In this work we investigate the realization problem of periodic convolutional codes. As convolutional codes are discrete linear systems over a finite field we use systems theory techniques to address our problem. In particular, we aim at deriving and studying state-space realizations of 2-periodic convolutional codes. Although one cannot expect, in general, to obtain a periodic state-space realization of a periodic convolutional code by means of the individual realizations of each of the associated time-invariant codes, we show that one can implement the periodic system switching periodically the output in each state system. Comments on the minimality of this realization are given.

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Keywords: Periodic systems, convolutional codes, realizations.

1. INTRODUCTION

The two classes of error control codes are block codes and convolutional codes. Convolutional codes consider the data as a sequence in contrast with block codes that operate with fixed message blocks. Even though they split the data into blocks as block codes do, they do not encode the different blocks independently, and previous nodes in the sequence make an effect over the next encoded node. Because of this, convolutional codes have memory and can be represented by linear systems. Mathematically speaking, convolutional codes are discrete Linear Time-Invariant (LTI) systems over a finite field. This close relation between convolutional codes and LTI systems is known since the sixties and many fundamental results have been derived from such strong connection Rosenthal [1973], Massy and Saïn [1967], Forney [2011], Forney, Jr. [1975]. In particular, this connection has led to many fundamental results to address one of the main tasks of convolutional coding: the construction of state-space and trellis representations of convolutional codes. These representations are essential for the implementation and decoding of convolutional codes.

The realization theory of (Linear Time-Invariant) convolutional codes is highly developed and has produced many sophisticated results. On the other hand, the mathematical realization theory of Linear Time-Varying (LTV) convolutional codes is still in its infancy and only preliminary or partial results are known Climent et al. [2009], Napp et al. [2017, 2019]. LTV convolutional codes form an important class of convolutional codes that process the information through several different encoders in contrast to LTI convolutional codes that use one single encoder. This class of codes has attracted much attention after Costello conjectured in Costello [1974], see also Mooser [1983], Palazzo [1993], that time-varying convolutional codes can attain larger free distance, and therefore better error-correction capabilities, than their time-invariant counterparts. Yet, the state of the art of the realization problem of these codes is totally different for these two classes of codes.

In this work we study a subclass of time-varying convolutional codes, namely, periodic convolutional codes, and
focus on two main goals. First we investigate the structural properties of these codes. For the sake of simplicity we shall restrict ourselves to period $p = 2$.

The second part of the paper is devoted to investigate a state-space representation. The first observation is that one cannot expect, in general, to obtain a periodic input-state-output representation of a periodic convolutional code by means of the individual realizations of each of the associated time-invariant codes, see Napp et al. [2017, 2019]. To overcome this difficulty we introduce periodic encoding maps in such a way that their image are periodic convolutional codes. This will allow to construct a state-space realization that produce a 2-periodic convolutional code. This is achieved realizing each of the associated time-invariant codes and switching periodically the output in each state system, see figure 1. This approach provides an effective procedure to implement 2-periodic convolutional codes. We provide some comments on the minimality of such representation.

2. PRELIMINARIES

In this section we recall some basic facts about LTI convolutional codes and periodic convolutional codes. We also introduce an associated LTI code, called the lifted code, that will play an important role in the remaining part of the paper.

2.1 Time-invariant convolutional codes

Let $\mathbb{F}$ be a finite field and let $\mathbb{F}[d]$ be the ring of polynomials with coefficients in $\mathbb{F}$.

We introduce first the notion of LTI convolutional code as follows (Rosenthal and York [1999], Fornasini and Pinto [2004], Gluesing-Luerssen and Schneider [2007], Gluesing-Luerssen et al. [2006], McEliece [1998b]).

**Definition 2.1.** A time-invariant convolutional code $C$ of rate $k/n$ is a submodule of $\mathbb{F}^n[d]$ of rank $k$. A full column rank matrix $G(d) \in \mathbb{F}^{n \times k}[d]$ such that

$$C = \{v(d) \in \mathbb{F}^n[d] : v(d) = G(d)u(d); u(d) \in \mathbb{F}^k[d]\} = \text{Im} G(d)$$

is called an encoder of $C$. $u(d)$ is the information vector (the message) and $v(d)$ is the corresponding codeword.

Thus, an encoder of $C$ yields an injective map between the set of messages, $\mathbb{F}^k[d]$, and the set of codewords, $C$.

The encoders of a given code $C$ are not unique; however they only differ by right multiplication by unimodular matrices over $\mathbb{F}[d]$.

$G(d)$ is called column reduced if the sum of its column degrees attains the minimal possible value among all the encoders of the same code. If $G(d) \in \mathbb{F}^{n \times k}[d]$ has column degrees $\nu_1, \ldots, \nu_k$, it can be written as

$$G(d) = G_{\text{hc}} \begin{bmatrix} d^{\nu_1} & \cdots & d^{\nu_k} \\ \vdots & \ddots & \vdots \\ d^{\nu_k} & \cdots & d^{\nu_1} \end{bmatrix} + G_{\text{rem}}(d)$$

where $G_{\text{rem}}(d)$ is a polynomial matrix such that the degree of column $i$ is less than $\nu_i$, $i = 1, \ldots, k$, and $G_{\text{hc}} \in \mathbb{F}^{n \times k}$ is a matrix whose $i$-th column contains the coefficients of the term $d^{\nu_i}$ in the $i$-th column of $G(d)$. $G_{\text{hc}}$ is called the leading column coefficient matrix and $G(d)$ is column reduced if and only if $G_{\text{hc}}$ has full column rank. Basic encoders (i.e., encoders represented by right prime matrices) that are column reduced are also called minimal. An encoder $G(d)$ is delay-free if $G(0)$ has full column rank. See also Solomon and Tilborg [1979], McEliece [1998a] for more details.

We define the degree $\delta$ of a convolutional code as the sum of the column degrees of one, and hence any, column reduced encoder. Note that the list of column degrees (also known as Forney indices) of a column reduced encoder is unique up to a permutation. A code $C$ of rate $k/n$ and degree $\delta$ is said to be an $(n, k, \delta)$ code.

2.2 Periodic convolutional codes

We focus now on convolutional codes $C$ with periodic encoding maps. A periodic convolutional code can be viewed as the set of codewords generated by a finite family of convolutional encoders that periodically switch in coding the messages. More precisely, the definition of periodic encoders (or encoding maps) together with the definition of the corresponding periodic (time-varying) convolutional codes (see Costello [1974], Palazzo [1993], Truhachev et al. [2010]) is the following:

**Definition 2.2.** Given $r$ full column rank polynomial matrices $G^0(d), G^1(d), \ldots, G^{r-1}(d) \in \mathbb{F}^{n \times k}[d]$, the periodic encoding map induced by $G^0(d), G^1(d), \ldots, G^{r-1}(d)$ is defined as

$$\Phi(G^0, G^1, \ldots, G^{r-1}) : \mathbb{F}^k[d] \to \mathbb{F}^n[d]$$

$$u(d) \mapsto v(d)$$

where the coefficients of $v(d) = \sum_{i=-\infty}^{+\infty} v_id^i$ are given by

$$v_{rt+t} = (G^t(d)u(d))_{rt+t}, \ t = 0, 1, \ldots, r - 1, \ \ell \in \mathbb{N}_0,$$

and $(G^t(d)u(d))_{rt+t}$ represents the $(rt + t)$-coefficient of the polynomial $G^t(d)u(d)$.

The corresponding periodic convolutional code $C_p$ is

$$C_p = \{v(d) \in \mathbb{F}^n[d] : \exists u(d) \in \mathbb{F}^k[d] \text{ s.t. (1) holds} \} = \text{Im} \Phi(G^0, G^1, \ldots, G^{r-1})$$

$$v(d) = \Phi(G^0, G^1, \ldots, G^{r-1})(u(d)). \quad (1)$$

Such codes will be called $r$-periodic convolutional codes of rate $k/n$.

Note that periodic codes are not necessarily $\mathbb{F}[d]$-submodules of $\mathbb{F}^n[d]$.

Two sequences of polynomial matrices $G^0(d), \ldots, G^{r-1}(d)$ and $G^0(d), \ldots, G^{r-1}(d)$ are said to be equivalent if the corresponding periodic encoding maps have the same image (i.e., if the corresponding periodical convolutional codes coincide).

In the sequel, for simplicity we will consider $r = 2$ and denote $G^0(d) = G(d)$ and $G^1(d) = J(d)$.
3. STATE-SPACE REALIZATIONS

In this section we aim to introduce a state-space representation of periodic convolutional codes. State-space descriptions of time-invariant codes via linear systems have been first adopted by Massey and Sain Massey and Sain [1967] and studied later on by many researchers Rosenthal and York [1999], Gluesing-Luerssen and Schneider [2007], Fornasini and Pinto [2004]. Some basic facts on these realizations and a simple state-space realization for LTI convolutional codes allow to introduce a state-space realization of the periodic encoding map \( \Phi_{(C,J)} \).

3.1 State-space realizations of time-invariant convolutional codes

In the sequel, we identify a polynomial \( a(d) = \sum_{i \in \mathbb{N}_0} a_i d^i \in \mathbb{F} \) with the sequence of its coefficients \( a_0 = (a(d))_0, a_1 = (a(d))_1, \ldots \). The same applies for vectors with components in \( \mathbb{F} \).

A state-space system

\[
\begin{align*}
    x(\ell + 1) &= Ax(\ell) + Bu(\ell), \quad \ell \in \mathbb{N}_0, \\
    v(\ell) &= Cx(\ell) + Du(\ell),
\end{align*}
\]

denoted by \((A, B, C, D)\), where \(A \in \mathbb{F}^{m \times m}, B \in \mathbb{F}^{m \times k}, C \in \mathbb{F}^{n \times m}, D \in \mathbb{F}^{n \times k}\), with state \( x \), input \( u \) and output \( y \), is said to be an \( m \)-dimensional state-space realization of a time-invariant \((n, k, \delta)\) convolutional code \( C \) if \( C \) is the set of codewords \( v(d) \in \mathbb{F}^n[d] \) (identified with the finite support output sequences \( v \)) corresponding to finite support input sequences \( u \) (i.e., to information sequences \( u(d) \in \mathbb{F}[d] \)) and to zero initial conditions, i.e., \( x(0) = 0 \).

State-space realizations of convolutional codes can be obtained as state-space realizations of their encoders. If \( G(d) \in \mathbb{F}^{n \times k}[d] \) is an encoder of the code \( C \), \((A, B, C, D)\) is a state-space realization of \( G(d) \) if

\[
G(d) = C(I - Ad)^{-1}Bd + D. \quad (2)
\]

If the encoder is represented by a power series \( G(d) = \sum_{i \in \mathbb{N}_0} G_i d^i \), with \( G_i \in \mathbb{F}^{n \times k} \), (2) implies the following identities:

\[
G_0 = D, \quad G_i = CA^{i-1}B, \quad i \geq 1.
\]

It is well-known that a given \( G(d) \) admits many realizations. Moreover, a state-space realization \((A, B, C, D)\) of \( G(d) \) has minimal dimension among all the realizations of \( G(d) \) if \((A, B)\) is controllable and \((A, C)\) is observable, i.e., the polynomial matrices \( [d^{-1}I - A \mid B] \) and \( [d^{-1}I - A \mid C] \) have, respectively, right and left polynomial inverses (in \( d^{-1} \)). The minimal dimension of a state-space realization of \( G(d) \) is called the McMillan degree of \( G(d) \) and is denoted as \( \mu(G) \), Kailath [1980].

The next proposition, adapted from Fornasini and Pinto [2004], Gluesing-Luerssen and Schneider [2007], provides a state-space realization for a given encoder.

Proposition 3.1. Let \( G(d) \in \mathbb{F}^{n \times k}[d] \) be a polynomial matrix with rank \( k \), column degrees \( \nu_1, \ldots, \nu_k \), and let \( m := \sum_{i=1}^k \nu_i \). Let \( G(d) \) have columns \( g_i(d) = \sum_{i=0}^{m} g_{i,i} d^i \), \( i = 1, \ldots, k \) where \( g_{i,i} \in \mathbb{F}^m \). For \( i = 1, \ldots, k \) define the matrices

\[
\begin{align*}
    A_i &= \begin{bmatrix} 0 & \cdots & 0 \\ 1 & \cdots & \vdots \\ \vdots & \ddots & 1 \\ 0 & \cdots & 0 \end{bmatrix} \in \mathbb{F}^{m \times m}, \\
    B_i &= \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{F}^m, \\
    C_i &= \begin{bmatrix} g_{1,i} \cdots g_{m,i} \end{bmatrix} \in \mathbb{F}^{m \times 1}.
\end{align*}
\]

Then a state-space realization of \( G(d) \) is given by the matrix quadruple \((A, B, C, D)\) in \( \mathbb{F}^{m \times m} \times \mathbb{F}^{m \times k} \times \mathbb{F}^{n \times m} \times \mathbb{F}^{n \times k} \) where

\[
\begin{align*}
    A &= \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & A_k \end{bmatrix}, \\
    B &= \begin{bmatrix} B_1 \\ \vdots \\ B_k \end{bmatrix}, \\
    C &= \begin{bmatrix} C_1 & \cdots & C_k \end{bmatrix}, \\
    D &= \begin{bmatrix} g_{0,1} \cdots g_{0,k} \end{bmatrix} = G(0).
\end{align*}
\]

In the case where \( \nu_i = 0 \) the \( i \)th block of \( A \) and \( C \) are void and in \( B \) a zero column occurs.

In this realization the pair \((A, B)\) is controllable and if \( G(d) \) is column reduced, \((A, C)\) is observable. Thus, the McMillan degree of a column reduced encoder is equal to the sum of its column degrees.

3.2 State-space realizations of 2-periodic convolutional codes

The theory of linear time-invariant systems allows to obtain state-space realizations for periodic convolutional codes.

Let \( G(d), J(d) \in \mathbb{F}^{n \times k}[d] \) be two full column rank matrices, \( C_p \) the 2-periodic convolutional code with encoding map \( \Phi_{(G,J)} \). If \( \Sigma_1 = (A_1, B_1, C_1, G_0) \) and \( \Sigma_2 = (A_2, B_2, C_2, J_0) \) are realizations of \( G(d) \) and \( J(d) \), respectively, consider the system \( \Sigma = (A, B, C, D) \) given by

\[
\begin{align*}
    A &= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \\
    B &= \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \\
    C &= \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \\
    D &= \begin{bmatrix} G_0 \\ J_0 \end{bmatrix}.
\end{align*}
\]

Clearly, the dimension of \( \Sigma \) is equal to the sum of the dimensions of \( \Sigma_1 \) and \( \Sigma_2 \).

Let

\[
\begin{align*}
    v^{(1)}(d) &= \sum_{i \in \mathbb{N}} v_i^{(1)} d^i \in \mathbb{F}^n[d] \\
    v^{(2)}(d) &= \sum_{i \in \mathbb{N}} v_i^{(2)} d^i \in \mathbb{F}^n[d]
\end{align*}
\]

be the output of \( \Sigma \) corresponding to the input \( u(d) \in \mathbb{F}[d] \). Consider now the output sequence \( w(d) \in \mathbb{F}[d] \) defined as \( w_0 = v_1^{(1)} \) and \( w_{2j+1} = v_2^{(2)}, j \in \mathbb{N} \). Note that this new output may be obtained from the input-output “machine”, \( \Sigma_p \) in Figure 1, as follows:
Only one switch is on at each time instant, and the switch status goes from on to off and vice versa at each time instant. The initial position is on for the switch corresponding to \((A_1, B_1, C_1, G_0)\) and off for the corresponding to \((A_2, B_2, C_2, J_0)\). The initial state is zero both for \(\Sigma_1\) and \(\Sigma_2\). Physical implementation of the linear system \(\Sigma_i, i = 1, 2\), is possible by means of shift-registers.

The system \(\Sigma_p\) described above corresponds to the following periodic state-space equations

\[
\begin{align*}
\dot{x}(\ell + 1) &= A(\ell)x(\ell) + B(\ell)u(\ell) \\
\dot{w}(\ell) &= C(\ell)x(\ell) + D(\ell)u(\ell)
\end{align*}
\]

with

\[
A(\ell) := \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B(\ell) := \begin{bmatrix} B_1 \\ 0 \end{bmatrix},
\]

\[
C(2\ell) := \begin{bmatrix} C_1 & 0 \end{bmatrix}, \quad D(2\ell) := G_0,
\]

\[
C(2\ell + 1) := \begin{bmatrix} 0 & C_2 \end{bmatrix}, \quad D(2\ell + 1) := J_0, \quad \ell \in \mathbb{N}_0.
\]

For short, we write \(\Sigma_p = (A(\ell), B(\ell), C(\ell), D(\ell)).\)

\(\Sigma_p\) is a realization of a periodic encoding map \(\Phi_{G,J}\) if the output \(w(d)\) of \(\Sigma_p\) that corresponds to an input \(u(d)\) is equal to \(\Phi_{G,J}(u(d))\), for all messages \(u(d) \in \mathbb{F}^k[d]\). We say that \(\Sigma_p\) has dimension equal to the dimension of \(\Sigma\) defined in (3).

The next result provides necessary and sufficient conditions for a system \(\Sigma_p\) given by (4) to be a realization of \(\Phi_{G,J}\). These conditions can be easily verified using the well-known realization theory for linear time-invariant systems.

**Theorem 3.1.** The periodic system \(\Sigma_p\) (4) is a realization of the periodic encoding map \(\Phi_{G,J}\) if and only if the time-invariant system \(\Sigma\) described in (3) is a realization of

\[
\begin{bmatrix} G(d) \\ J(d) \end{bmatrix}.
\]

**Proof.** Clearly, if \(\Sigma\) is a realization of \(\begin{bmatrix} G(d) \\ J(d) \end{bmatrix}\) then \(\Sigma_p\) is a realization of \(\Phi_{G,J}\).

Conversely, assume that the time varying system \(\Sigma_p\) given by (4) is a realization of \(\Phi_{G,J}\). Then, for every input \(u(d)\) and every \(m \in \mathbb{N}_0\) we have:

\[
\begin{align*}
(w(d))_{2m} &= \left(\sum_{j=1}^{2m} C_1 A_1^{j-1} B_1 d^j + G_0\right) u(d)_{2m}, \quad \text{and} \\
(w(d))_{2m+1} &= \left(\sum_{j=1}^{2m+1} C_2 A_2^{j-1} B_2 d^j + J_0\right) u(d)_{2m+1}.
\end{align*}
\]

In particular, for \(u(d) = e_i, i = 1, \ldots, k\), where \(e_i\) denotes the \(i\)-th vector of the canonical basis of \(\mathbb{F}^k\), we obtain

\[
\begin{align*}
(w(d))_{2m} &= \left(\sum_{j=1}^{2m} C_1 A_1^{j-1} B_1 d^j + G_0\right) e_i_{2m} = G_0 e_i, \quad m = 0 \\
&= C_1 A_1^{2m-1} B_1 e_i, \quad m \geq 1
\end{align*}
\]

and

\[
\begin{align*}
(w(d))_{2m+1} &= \left(\sum_{j=1}^{2m+1} C_2 A_2^{j-1} B_2 d^j + J_0\right) e_i_{2m+1} = C_2 A_2^{2m} B_2 e_i, \quad m \geq 0
\end{align*}
\]

while for \(u(d) = d e_i\),

\[
\begin{align*}
(w(d))_{2m} &= \left(\sum_{j=1}^{2m} C_1 A_1^{j-1} B_1 d^j + G_0\right) de_i_{2m} = C_1 A_1^{2m-2} B_1 e_i, \quad m \geq 1
\end{align*}
\]

and

\[
\begin{align*}
(w(d))_{2m+1} &= \left(\sum_{j=1}^{2m+1} C_2 A_2^{j-1} B_2 d^j + J_0\right) de_i_{2m+1} = C_2 A_2^{2m-1} B_2 e_i, \quad m \geq 1
\end{align*}
\]

On the other hand, for \(u(d) = e_i\),

\[
\begin{align*}
(w(d))_{2m} &= (G(d)u(d))_{2m} = (G(d)e_i)_{2m} \quad \text{and}
\end{align*}
\]

\[
\begin{align*}
(w(d))_{2m+1} &= (J(d)u(d))_{2m+1} = (J(d)e_i)_{2m+1}
\end{align*}
\]

whereas for \(u(d) = de_i\),

\[
\begin{align*}
(w(d))_{2m} &= (G(d)u(d))_{2m} = (G(d)de_i)_{2m} = (G(d)e_i)_{2m-1}
\end{align*}
\]

and

\[
\begin{align*}
(w(d))_{2m+1} &= (J(d)u(d))_{2m+1} = (J(d)de_i)_{2m+1} = (J(d)e_i)_{2m}.
\end{align*}
\]

Now, from (5) and (9), and (7) and (11), we respectively get

\[
\begin{align*}
(G(d)e_i)_{2m} &= G_0 e_i, \quad m = 0 \\
&= C_1 A_1^{2m-1} B_1 e_i, \quad m \geq 1
\end{align*}
\]

which implies that

\[
G(d) = G_0 + \sum_{i=1}^{\infty} C_1 A_1^{i-1} B_1 d^i.
\]

Hence \((A_1, B_1, C_1, G_0)\) is a state-space realization of \(G(d)\). The proof that \((A_2, B_2, C_2, J_0)\) is a state-space realization of \(J(d)\) is analogous. Consequently, system (3) is a realization of \(\begin{bmatrix} G(d) \\ J(d) \end{bmatrix}\).

The next lemma characterizes the state-space realizations of linear periodic encoding maps \(\Phi_{G,J}\) with minimal dimension, called minimal realizations of \(\Phi_{G,J}\), and is a direct consequence of Theorem 3.1.

**Lemma 3.1.** Let \(C_p\) be a periodic convolutional code and \(\Phi_{G,J}\) a periodic encoding map of \(C_p\), for some \(G(d), J(d) \in \mathbb{F}^{n \times k}[d]\). A periodic system \(\Sigma_p\) is a minimal realization of \(\Phi_{G,J}\) if and only if the associated systems \(\Sigma_1\) and \(\Sigma_2\) are minimal realizations of \(G(d)\) and \(J(d)\), respectively. Moreover, the minimal dimension of a realization of \(\Phi_{G,J}\) is equal to \(\mu(G) + \mu(J)\).
4. CONCLUSIONS

In this paper we have studied the representations of periodic convolutional codes of period $r$ by means of state-space realizations. For simplicity we have presented the case in which $r = 2$ (but the result easily extends to $r > 2$). Namely, we considered two encoders of the associated time-invariant convolutional codes and obtained the periodic code by periodically switching the output in each state system. We also characterized the realizations of this type with minimal dimensional.

Note that, as the pair of encoders that generate the code $C_p$ is not unique, two different choices of encoder pairs can lead to different values of the minimal realization of this type. Therefore, an interesting problem that remains to investigate is the minimal state-space dimension of a periodic encoder that produces a periodic code $C_p$.

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