Toward a Compositional Theory of Leftist Grammars and Transformations

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Abstract. Leftist grammars [Motwani et al., STOC 2000] are special semi-Thue systems where symbols can only insert or erase to their left. We develop a theory of leftist grammars seen as word transformers as a tool toward rigorous analyses of their computational power. Our main contributions in this first paper are (1) constructions proving that leftist transformations are closed under compositions and transitive closures, and (2) a proof that bounded reachability is NP-complete even for leftist grammars with acyclic rules.

1 Introduction

Leftist grammars were introduced by Motwani et al. to study accessibility and safety in protection systems [MPSV00]. In this framework, leftist grammars are used to show that restricted accessibility grammars have decidable accessibility problems (unlike the more general access-matrix model).

Leftist grammars are both surprisingly simple and surprisingly complex. Simplicity comes from the fact that they only allow rules of the form “a → ba” and “cd → d” where a symbol inserts, resp. erases, another symbol to its left while remaining unchanged. But the combination of insertion and deletion rules makes leftist grammars go beyond context-sensitive grammars, and the decidability result comes with a high complexity-theoretical price [Jur08]. Most of all, what is surprising is that apparently leftist grammars had not been identified as a relevant computational formalism until 2000.

The known facts on leftist grammars and their computational and expressive power are rather scarce. Motwani et al. show that it is decidable whether a given word can be derived (accessibility) and whether all derivable words belong to a given regular language (safety) [MPSV00]. Jurdziński and Loryś showed that leftist grammars can define languages that are not context-free [JL07] while leftist grammars restricted to acyclic rules are less expressive since they can only recognize regular languages. Then Jurdziński showed a PSPACE lower bound for accessibility in leftist grammars [Jur07], before improving this to a nonprimitive-recursive lower bound [Jur08].

Jurdziński’s results rely on encoding classical computational structures (linear-bounded automata [Jur07] and Ackermann’s function [Jur08]) in leftist grammars. Devising such encodings is difficult because leftist grammars are very hard to control. Thus,
for computing Ackermann’s function, devising the encoding is actually not the hardest part: the harder task is to prove that the constructed leftist grammar cannot behave in unexpected ways. In this regard, the published proofs are necessarily incomplete, hard to follow, and hard to fully acknowledge. The final results and intermediary lemmas cannot easily be adapted or reused.

**Our Contribution.** We develop a compositional theory of leftist grammars and leftist transformations (i.e., operations on strings that are computed by leftist grammars) that provides fundamental tools for the analysis of their computational power. Our main contributions are effective constructions for the composition and the transitive closure of leftist transformations. The correctness proofs for these constructions are based on new definitions (e.g., for greedy derivations) and associated lemmas.

A first application of the compositional theory is given in Section 6 where we prove the NP-completeness of bounded reachability questions, even when restricted to acyclic leftist grammars.

A second application, and the main reason for this paper, is our forthcoming construction proving that leftist grammars can simulate lossy channel systems and “compute” all multiply-recursive transformations and nothing more (based on [CS08b]), thus providing a precise measure of their computational power. Finally, after our introduction of Post’s Embedding Problem [CS07,CS08a], leftist grammars are another basic computational model that will have been shown to capture exactly the notion of multiply-recursive computation.

As further comparison with earlier work, we observe that, of course, the complex constructions in [Jur07,Jur08] are built modularly. However, the modularity is not made fully explicit in these works, the interfacing assumptions are incompletely stated, or are mixed with the details of the constructions, and correctness proofs cannot be given in full.

**Outline of the Paper.** Basic notations and definitions are recalled in Section 2. Section 3 defines leftist grammars and proves a generalized version of the completeness of greedy derivations. Sections 4 introduces leftist transformers and their sequential compositions. Section 5 specializes on the “simple” transformers that we use in Section 6 for our encoding of 3SAT. Finally Section 7 shows that so-called “anchored” transformers are closed under the transitive closure operation, this in an effective way.

## 2 Basic Definitions and Notations

**Words.** We use $x, y, u, v, w, \alpha, \beta, \ldots$ to denote words, i.e., finite strings of symbols taken from some alphabet. Concatenation is denoted multiplicatively with $\varepsilon$ (the empty word) as neutral element, and the length of $x$ is denoted $|x|$. The congruence on words generated by the equivalences $a \approx aa$ (for all symbols $a$ in the alphabet) is called the *stuttering equivalence* and is also denoted $\approx$: every word $x$ has a minimal and canonical stuttering-equivalent $x'$ obtained by repeatedly eliminating symbols in $x$ that are adjacent to a copy of themselves.

We say that $x$ is a *subword* of $y$, denoted $x \sqsubseteq y$, if $x$ can be obtained by deleting some symbols (an arbitrary number, at arbitrary positions) from $y$. We further write $x \sqsubseteq^* y$...
when all the symbols deleted from \( y \) belong to \( \Sigma \) (NB: we do not require \( y \in \Sigma^* \)), and let \( \equiv \) denote the inverse relation \( \subseteq^{-1} \).

Relations and Relation Algebra. We see a relation \( R \) between two sets \( X \) and \( Y \) as a set of pairs, i.e., some \( R \subseteq X \times Y \). We write \( x \mathbin{R} y \) rather than \( (x, y) \in R \). Two relations \( R \) and \( R' \) can be composed, denoted multiplicatively with \( R \cdot R' \), and defined by \( x (R \cdot R') y \iff \exists z (x R z \land z R' y) \).

The union \( R + R' \), also denoted \( R \cup R' \), is just the set-theoretic union. \( R^n \) is the \( n \)-th power \( R \ldots R \) of \( R \) and \( R^{-1} \) is the inverse of \( R \) such that \( x R^{-1} y \iff y \mathbin{R} x \). The transitive closure \( \bigcup_{n=1}^{\infty} R^n \) of \( R \) assumes \( Y = X \) and is denoted \( R^* \), while its reflexive-transitive closure is \( R^+ \cup \text{Id}_X \), denoted \( R^\ast \).

Below we often use notations from relation algebra to state simple equivalences. E.g., we write \( "R = R" \) and \( "R \subseteq S" \) rather than \( "x R y \iff x R' y" \) and \( "x R y \iff x S y" \). Our proofs often rely on well-known basic laws from relation algebra, like \( (R \cdot R')^{-1} = R^{-1} \cdot R^{-1} \), or \( (R + R').R'' = R.R'' + R'.R'' \), without explicitly stating them.

3 Leftist Grammars

A leftist grammar (an LGr) is a triple \( G = (\Sigma, P, g) \) where \( \Sigma \cup \{g\} = \{a, b, \ldots\} \) is a finite alphabet, \( g \not\in \Sigma \) is a final symbol (also called “axiom”), and \( P = \{r, \ldots\} \) is a set of production rules that may be insertion rules of the form \( a \to ba \), and deletion rules of the form \( cd \to d \). For simplicity, we forbid rules that insert or delete the axiom \( g \) (this is no loss of generality [JL07, Prop. 3]).

Leftist grammars are not context-free (deletions are contextual), or even context-sensitive (deletions are not length-preserving). For our purposes, we consider them as string rewrite systems, more precisely semi-Thue systems. Writing \( \Sigma_n \) for \( \Sigma \cup \{g\} \), the rules of \( P \) define a 1-step rewrite relation in the standard way: for \( u, u' \in \Sigma_n^* \), we write \( u \Rightarrow u' \) whenever \( r \) is some rule \( \alpha \to \beta \), \( u \) is some \( u_1 \alpha u_2 \) with \( |u_1 \alpha| = p \) and \( u' = u_1 \beta u_2 \). We often write shortly \( u \Rightarrow u' \), even if \( u \nRightarrow u' \), when the position or the rule involved in the step can be left implicit. On the other hand, we sometimes use a subscript, e.g., writing \( u \Rightarrow_G v \), when the underlying grammar has to be made explicit.

A derivation is a sequence \( \pi \) of consecutive rewrite steps, i.e., is some \( u_0 \Rightarrow^{r_1} u_1 \Rightarrow^{r_2} u_2 \Rightarrow^{r_3} \ldots \Rightarrow^{r_n} u_n \), often abbreviated as \( u_0 \Rightarrow^n u_n \), or even \( u_0 \Rightarrow^* u_n \). A subsequence \( (u_{i-1} \Rightarrow^{r_i} u_i)_{i=m+1, \ldots, l} \) of \( \pi \) is a subderivation. As with all semi-Thue systems, steps (and derivations) are closed under adjunction: if \( u \Rightarrow u' \) then \( uvw \Rightarrow vu'w \).

Two derivations \( \pi_1 = (u \Rightarrow^* u') \) and \( \pi_2 = (v \Rightarrow^* v') \) can be concatenated in the obvious way (denoted \( \pi_1 \pi_2 \) if \( u' = v \). They are equivalent, denoted \( \pi_1 \equiv \pi_2 \), if they have same extremities, i.e., if \( u = v \) and \( u' = v' \).

We say that \( u \in \Sigma^* \) is accepted by \( G \) if there is a derivation of the form \( ug \Rightarrow^* g \) and we write \( L(G) \) for the set of accepted words, i.e., the language recognized by \( G \).

We say that \( I \subseteq \Sigma^* \) is an invariant for an LGr \( G = (\Sigma, P, g) \) if \( u \in I \) and \( ug \Rightarrow vg \) entail \( v \in I \). Knowing that \( I \) is an invariant for \( G \) is used in two symmetric ways: (1) from \( u \in I \) and \( ug \Rightarrow vg \) one deduces \( v \in I \), and (2) from \( ug \Rightarrow vg \) and \( v \not\in I \) one deduces \( u \not\in I \).
3.1 Graphs and Types for Leftist Grammars

When dealing with LGr’s, it is convenient to write insertion rules under the simpler form “\(a \rightarrow b\)”, and deletion rules as “\(d \cdots c\)”, emphasizing the fact that \(a\) (resp. \(d\)) is not modified during the insertion of \(b\) (resp. the deletion of \(c\)) on its left. For \(a \in \Sigma_g\), we let
\[
\text{ins}(a) \overset{\text{def}}{=} \{ b \mid P \ni (a \rightarrow b) \}
\]
denote the set of symbols that can be inserted (respectively, deleted) by \(a\). We write \(\text{ins}^+(a)\) for the smallest set that contains \(b\) and \(\text{ins}^+(b)\) for all \(b \in \text{ins}(a)\), while \(\text{del}^+(b)\) is defined similarly. We say that \(a\) is inactive in a LGr if \(\text{del}(a) \cup \text{ins}(a) = \emptyset\).

It is often convenient to view LGr’s in a graph-theoretical way. Formally, the graph of \(G = (\Sigma, P, g)\) is the directed graph \(\tau_G\) having the symbols from \(\Sigma_g\) as vertices and the rules from \(P\) as edges (coming in two kinds, insertions and deletions). Furthermore, we often decorate such graphs with extra bookkeeping annotations.

We say that \(G \text{ has type } \tau\) when \(\tau_G\) is a sub-graph of \(\tau\). Thus a “type” is just a restriction on what are the allowed symbols and rules between them. Types are often given schematically, grouping symbols that play a similar role into a single vertex. For example, Fig. 1 displays schematically the type (parametrized by the alphabet) observed by all LGr’s.

3.2 Leftmost, Pure and Eager Derivations

We speak informally of a “letter”, say \(a\), when we really mean “an occurrence of the symbol \(a\)” (in some word). Furthermore, we follow letters along steps \(u \Rightarrow v\), identifying the letters in \(u\) and the corresponding letters in \(v\). Hence a “letter” is also a sequence of occurrences in consecutive words along a derivation.

A letter \(a\) is a \(n\)-th descendant of another letter \(b\) (in the context of a derivation) if \(a\) has been inserted by \(b\) (when \(n = 1\)), or by a \((n - 1)\)-th descendant of \(b\).

Given a step \(u \Rightarrow v\), we say that the \(p\)-th letter in \(u\), written \(u[p]\), is the active letter: the one that inserts, or deletes, a letter to its left. This is often emphasized by writing the step under the form \((u =) u_1au_2 \Rightarrow u_1'au_2' (= v)\) (assuming \(u[p] = a\)).

A letter is inert in a derivation if it is not active in any step of the derivation. A set of letters is inert if it only contains inert letters. A derivation is leftmost if every step \(u_1au_2 \Rightarrow u_1'au_2'\) in the derivation is such that \(u_1\) is inert in the rest of the derivation.

A letter is useful in a derivation \(\pi = (u \Rightarrow v)\) if it belongs to \(u\) or \(v\), or if it inserts or deletes a useful letter along \(\pi\). This recursive definition is well-founded: since letters only insert or delete to their left, the “inserts-or-deletes” relation between letters is acyclic. A derivation \(\pi\) is pure if all letters in \(\pi\) are useful. Observe that if \(\pi\) is not pure,
it necessarily inserts at some step some letter $a$ (called a useless letter) that stays inert and will eventually be deleted.

A derivation is eager if, informally, deletions occur as soon as possible. Formally, $\pi = (u_0 \Rightarrow r_1 u_1 \Rightarrow r_2 u_2 \cdots \Rightarrow r_n u_n)$ is not eager if there is some $u_{i-1}$ of the form $w_1 b w_2$ where $b$ is inert in the rest of $\pi$ and is eventually deleted, where $P$ contains the rule $a \to b$, and where $r_i$ is not a deletion rule.\(^1\)

A derivation is greedy if it is leftmost, pure and eager. Our definition generalizes [Jur07, Def. 4], most notably because it also applies to derivations $u g \Rightarrow^* v g$ with nonempty $v$. Hence a subderivation $\pi'$ of $\pi$ is leftmost, eager, pure, or greedy, when $\pi$ is.

The following proposition generalizes [Jur07, Lemma 7].

**Proposition 3.1 (Greedy derivations are sufficient).** Every derivation $\pi$ has an equivalent greedy derivation $\pi'$.

**Proof.** With a derivation $\pi$ of the form $u_0 \Rightarrow r_1 u_1 \Rightarrow r_2 u_2 \cdots \Rightarrow r_n u_n$, we associate its measure $\mu(\pi) \overset{\text{def}}{=} \langle n, p_1, \ldots, p_n \rangle$, a $(n+1)$-tuple of numbers. Measures are linearly ordered with the lexicographic ordering, giving rise to a quasi-ordering, denoted $\preceq$, between derivations. A derivation is called $\mu$-minimal if any equivalent derivation has greater or equal measure.

We can now prove Prop. 3.1 along the following lines (see Appendix A for full details): first prove that every derivation has a $\mu$-minimal equivalent (Lemma A.1), then show that $\mu$-minimal derivations are greedy (Lemma A.2). \(\Box\)

Observe that $\preceq$ is compatible with concatenation of derivations: if $\pi_1 \preceq \pi_2$ then $\pi_1 \pi_1' \preceq \pi_1 \pi_2' \pi_2'$ when these concatenations are defined. Thus any subderivation of a $\mu$-minimal derivation is $\mu$-minimal, hence also greedy.

$\mu$-minimality is stronger than greediness, and is a powerful and convenient tool for proving Prop. 3.1. However, greediness is easier to reason with since it only involves local properties of derivations, while $\mu$-minimality is “global”. These intuitions are reflected by, and explain, the following complexity results.

**Theorem 3.2.** 1. Greediness (deciding whether a given derivation $\pi$ in the context of a given LGr $G$ is greedy) is in L.
2. $\mu$-Minimality (deciding whether it is $\mu$-minimal) is coNP-complete, even if we restrict to acyclic LGr’s.

**Proof.** 1. Being leftmost or eager is easily checked in logspace (i.e., is in L). Checking non-purity can be done by looking for a last inserted useless letter, hence is in L too.

2. $\mu$-minimality is obviously in coNP. Hardness is proved as Coro. 6.9 below, as a byproduct of the reduction we use for the NP-hardness of Bounded Reachability. \(\Box\)

\(^1\) Eagerness does not require that $r_i$ deletes $b$: other deletions are allowed, only insertions are forbidden.
4 Leftist Grammars as Transformers

Some leftist grammars are used as computing devices rather than recognizers of words. For this purpose, we require a strict separation between input and output symbols and speak of leftist transformers, or shortly LTr’s.

4.1 Leftist Transformers

Formally, an LTr is a LGr $G = (\Sigma, P, g)$ where $\Sigma$ is partitioned as $A \sqcup B \sqcup C$, and where symbols from $A$ are inactive in $P$ and are not inserted by $P$ (see Fig. 2). This is denoted $G : A \vdash C$. Here $A$ contains the input symbols, $B$ the temporary symbols, and $C$ the output symbols, and $G$ is more conveniently written as $G = (A, B, C, P, g)$. When there is no need to distinguish between temporary and output symbols, we write $G$ under the form $G = (A, D, P, g)$, where $D \equiv B \cup C$ contains the “working” symbols.

A consequence of the restrictions imposed on LTr’s is the following:

**Fact 4.1** $A^*D^*$ is an invariant in any LTr $G = (A, D, P, g)$.

With $G = (A, B, C, P, g)$, we associate a transformation (a relation between words) $R_G \subseteq A^* \times C^*$ defined by

$$ u R_G v \iff u \Rightarrow^* \ast g \land u \in A^* \land v \in C^* $$

and we say that $G$ realizes $R_G$. Finally, a leftist transformation is any relation on words realized by some LTr. By necessity, a leftist transformation can only relate words written using disjoint alphabets (this is not contradicted by $\varepsilon R_G \varepsilon$).

Leftist transformations respect some structural constraints. In this paper we shall use the following properties:

**Proposition 4.2** (Closure for leftist transformations, see App. B). If $G : A \vdash C$ is a leftist transformer, then $R_G = (\sqsubseteq_A \cdot \approx \cdot .R_G \cdot \approx)$.

4.2 Composition

We say that two leftist transformations $R_1 \subseteq A_1^* \times C_1^*$ and $R_2 \subseteq A_2^* \times C_2^*$ are chainable if $C_1 = A_2$ and $A_1 \cap C_2 = \emptyset$. Two LTr’s are chainable if they realize chainable transformations.

**Theorem 4.3.** The composition $R_1 \cdot R_2$ of two chainable leftist transformations is a leftist transformation. Furthermore, one can build effectively a linear-sized LTr realizing $R_1 \cdot R_2$ from LTr’s realizing $R_1$ and $R_2$. 
For a proof, assume $G_1 = (A_1, B_1, C_1, P_1, g)$ and $G_2 = (A_2, B_2, C_2, P_2, g)$ realize $R_1$ and $R_2$. Beyond chainability, we assume that $A_1 \cup B_1$ and $B_2 \cup C_2$ are disjoint, which can be ensured by renaming the intermediary symbols in $B_1$ and $B_2$. The composed LTr $G_1.G_2$ is given by

$$G_1.G_2 \overset{\text{def}}{=} (A_1, B_1 \cup C_1 \cup B_2, C_2, P_1 \cup P_2, g).$$

This is indeed a LTr from $A_1$ to $C_2$. See Fig. 3 for a schematics of its type. Since $G_1.G_2$

has all rules from $G_1$ and $G_2$ it is clear that $(\Rightarrow_{G_1} + \Rightarrow_{G_2}) \subseteq \Rightarrow_G$, from which we deduce $R_{G_1}.R_{G_2} \subseteq R_{G_1.G_2}$. Furthermore, the inclusion in the other direction also holds:

**Lemma 4.4 (Composition Lemma, see Appendix C).** $R_{G_1.G_2} = R_{G_1}.R_{G_2}$.

**Remark 4.5 (Associativity).** The composition $(G_1.G_2).G_3$ is well-defined if and only if $G_1.(G_2.G_3)$ is. Furthermore, the two expressions denote exactly the same result. □

5 Simple Leftist Transformations

As a tool for Sections 6 and 7, we now introduce and study restricted families of leftist grammars (and transformers) where deletion rules are forbidden (resp., only allowed on $A$).

An *insertion grammar* is a LGr $G = (\Sigma, P, g)$ where $P$ only contain insertion rules. See Fig. 4 for a graphic definition. For an arbitrary leftist grammar $G$, we denote with $G^{\text{ins}}$ the insertion grammar obtained from $G$ by keeping only the insertion rules.

The insertion relation $I_G \subseteq \Sigma^* \times \Sigma^*$ associated with an insertion grammar $G = (\Sigma, P, g)$ is defined by $u I_G v \overset{\text{def}}{=} \exists g \Rightarrow_G^* vg$. Obviously, $I_G \subseteq \subseteq \Sigma$. Observe that $I_G$ is not necessarily a leftist transformation since it does not require any separation between input and output symbols.

A *simple* leftist transformer is an LTr $G = (A, B, C, P, g)$ where $B = \emptyset$ and where no rule in $P$ erases symbols from $C$. See Fig. 4 for a graphic definition. We give, without proof, an immediate consequence of the definition:

**Lemma 5.1.** Let $G = (A, B, C, P, g)$ be a simple LTr and assume $u \Rightarrow_G^* v$ for some $u \in A^*$ and $v \in C^*$. Then $k = |u| + |v|$. 
Given a simple LTr \( G = (A, \emptyset, C, P, g) \) and two words \( u = a_1 \cdots a_n \in A^+ \) and \( v = c_1 \cdots c_m \in C^+ \), we say that a non-decreasing map \( h : \{1, \ldots, n\} \to \{1, \ldots, m\} \) is a \( G \)-witness for \( u \) and \( v \) if \( P \) contains the rules \( c_{h(i)} \to a_i \) and \( c_{j+1} \to c_j \) (for all \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \), with the convention that \( c_{m+1} = g \)). Finally, we write \( u \not\sqsubseteq v \) when such a \( G \)-witness exists. Clearly, \( \sqsubseteq_G \subseteq R_G \). Indeed, when \( G \) is a simple transformer, \( \sqsubseteq_G \) can be used as a restricted version of \( R_G \) that is easier to control and reason about.

**Lemma 5.2 (See App. D).** Let \( G = (A, \emptyset, C, P, g) \) be a simple LTr. Then \( R_G = \sqsubseteq_G \sqsubseteq_G \). Combining Lemma 5.2 with \( \sqsubseteq_{C^*} \subseteq I_{\text{Ins}} \subseteq C \), we obtain the following weaker but simpler statement.

**Corollary 5.3.** Let \( G = (A, \emptyset, C, P, g) \) be a simple LTr. Then \( \sqsubseteq_G \subseteq R_G \subseteq \sqsubseteq_G \sqsubseteq C \).

### 5.1 Union of Simple Leftist Transformers

We now consider the combination of two simple LTr’s \( G_1 = (A, \emptyset, C_1, P_1, g) \) and \( G_2 = (A, \emptyset, C_2, P_2, g) \) that transform from a same \( A \) to disjoint output alphabets, i.e., with \( C_1 \cap C_2 = \emptyset \). We define their union with \( G_1 + G_2 \) |\((=) (A, \emptyset, C_1 \cup C_2, P_1 \cup P_2, g) \). This is clearly a simple LTr with \( (R_{G_1} + R_{G_2}) \subseteq R_{G_1+G_2} \). It further satisfies:

**Lemma 5.4.** If \( u R_{G_1+G_2} v \) then \( u (R_{G_1} + R_{G_2}) v' \) for some \( v' \sqsubseteq v \).

**Proof.** Assume \( u R_{G_1+G_2} v \). With Cor. 5.3, we obtain \( u \not\sqsubseteq_{G_1+G_2} v' \) for some \( v' = c_1 \cdots c_m \sqsubseteq v \). Hence \( G_1 + G_2 \) has insertion rules \( c_{j+1} \to c_j \) for all \( j = 1, \ldots, m \), and deletion rules of the form \( c_{h(i)} \to u[i] \). Since \( C_1 \) and \( C_2 \) are disjoint, either all these rules are in \( G_1 \) (and \( u \not\sqsubseteq_{G_1} v' \)), or they are all in \( G_2 \) (and \( u \not\sqsubseteq_{G_2} v' \)). Hence \( u (R_{G_1} + R_{G_2}) v' \).

### 6 Encoding 3SAT with Acyclic Leftist Transformers

This section proves the following result.

**Theorem 6.1.** Bounded Reachability and Exact Bounded Reachability in leftist grammars are NP-complete, even when restricting to acyclic grammars.

(Exact) Bounded Reachability is the question whether there exists a \( n \)-step derivation \( u \Rightarrow^n v \) (respectively, a derivation \( u \Rightarrow^{\leq n} v \) of non-exact length at most \( n \)) between given \( u \) and \( v \). These questions are among the simplest reachability questions and, since we consider that the input \( n \) is given in unary,\(^2\) they are obviously in NP for leftist grammars (and all semi-Thue systems).

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\(^2\) It is natural to begin with this assumption when considering fundamental aspects of reachability since writing \( n \) more succinctly would blur the complexity-theoretical picture.
Consequently, our contribution in this paper is the NP-hardness part. This is proved by encoding 3SAT instances in leftist grammars where reaching a given final ν amounts to guessing a valuation that satisfies the formula. While the idea of the reduction is easy to grasp, the technicalities involved are heavy and it would be difficult to really prove the correctness of the reduction without relying on a compositional framework like the one we develop in this paper. It is indeed very tempting to “prove” it by just running an example.

Rather than adopting this easy way, we shall describe the reduction as a composition of simple leftist transformers and use our composition theorems to break down the correctness proof in smaller, manageable parts. Once the ideas underlying the reduction are grasped, a good deal of the reasoning is of the type-checking kind: verifying that the conditions required for composing transformers are met.

Throughout this section we assume a generic 3SAT instance Φ = \bigwedge_{i=1}^{m} C_i with m 3-clauses on n Boolean variables in X = \{x_1, \ldots, x_n\}. Each clause has the form C_i = \bigvee_{k=1}^{3} \varepsilon_{i,k} x_{i,k} for some polarity ε_{i,k} ∈ \{+, −\} and x_{i,k} ∈ X. (There are two additional assumptions on Φ that we postpone until the proof of Coro. 6.5 for clarity.) We use standard model-theoretical notation like \models Φ (validity), or \models Φ (entailment) when σ is a Boolean formula or a Boolean valuation of some variables.

We write σ[x → b] for the extension of a valuation σ with \(x, b\), assuming \(x \not\in Dom(σ)\). Finally, for a valuation θ : X → \{⊤, ⊥\} and some \(j = 0, \ldots, n\), we write \(θ_j\) to denote the restriction \(θ_{\{x_1, \ldots, x_j\}}\) of θ on the first \(j\) variables.

### 6.1 Associating an LTr \(G_Φ\) with Φ

For the encoding, we use an alphabet Σ = \{\(T^j_i, U^j_i, T'^j_i, U'^j_i\) | \(i = 1, \ldots, m \land j = 0, \ldots, n\}\}, i.e., \(4(n + 1)\) symbols for each clause. The choice of the symbols is that a \(U\) means “Undetermined” and a \(T\) means “True”, or determined to be valid.

For \(j = 0, \ldots, n\), let \(V_j \defeq \{U^j_1, \ldots, U^j_m, T^j_1, \ldots, T^j_m\}\), \(V'_j \defeq \{U'^j_1, \ldots, U'^j_m, T'^j_1, \ldots, T'^j_m\}\), and \(W_j \defeq V_j ∪ V'_j\), so that Σ is partitioned in levels with Σ = \(\bigcup_{j=0}^{n} W_j\). With each \(x_j \in X\) we associate two intermediary LTr’s:

\[G^T_j \defeq (W_{j-1}, \emptyset, V_j, P_j, g), \quad G^\perp_j \defeq (W_{j-1}, \emptyset, V'_j, P'_j, g)\]

with sets of rules \(P_j\) and \(P'_j\). The rules for \(G^T_j\) are given in Fig. 5: some deletion rules are conditional, depending on whether \(x_j\) appears in the clauses \(C_1, \ldots, C_m\). The rules for \(G^\perp_j\) are obtained by switching primed and unprimed symbols, and by having conditional rules based on whether \(¬x_j\) appears in the \(C_i\)’s. One easily checks that \(G^T_j\) and \(G^\perp_j\) are indeed simple transformers. They have same inputs and disjoint outputs so that the union \((G^T_j + G^\perp_j) : W_{j-1} \vdash W_j\) is well-defined. Hence the following composition is well-formed:

\[G_Φ \defeq (G^T_1 + G^\perp_1) \circ (G^T_2 + G^\perp_2) \circ \cdots \circ (G^T_n + G^\perp_n)\]

We conclude the definition of \(G_Φ\) with an intuitive explanation of the idea behind the reduction. \(G_Φ\) operates on the word \(u_0 = U^0_1 \cdots U^0_m\) where each \(U^0_i\) stands for “the validity of clause \(C_i\) is undetermined at step 0 (i.e., at the beginning)”. At step \(j\), \(G^T_j + G^\perp_j\) picks
a valuation for \( x_j \); \( G^j \) picks “\( x_j = \top \)” while \( G_j^+ \) picks “\( x_j = \bot \).” This transforms \( U_{i-1}^j \) into \( U_i^j \), and \( T_{i-1}^j \) into \( T_i^j \), moving them to the next level. Furthermore, an undetermined \( U_{i-1}^j \) can be transformed into \( T_i^j \) if \( C_i \) is satisfied by \( x_j \). In addition, and because \( G_j^+ \) and \( G_j^- \) must have disjoint output alphabets, the symbols in the \( V_j^+ \)’s come in two copies (hence the \( V_j^- \)’s) that behave identically when they are input in the transformer for the next step.

The reduction is concluded with the following claim that we prove by combining Corollaries 6.5 and 6.8 below.

\[
\Phi \text{ is satisfiable iff } U_1^0 U_2^0 \cdots U_m^0 \Rightarrow_{G_\Phi} T_1^m T_2^m \cdots T_m^m
\]

\[
\text{if } U_1^0 U_2^0 \cdots U_m^0 \Rightarrow_{G_\Phi} T_1^m T_2^m \cdots T_m^m \quad \text{(Correctness)}
\]

Observe finally that \( G_\Phi \) is an acyclic grammar in the sense of [JL07], that is to say, its rules define an acyclic “may-act-upon” relation between symbols. Such grammars are much weaker than general LGr’s since, e.g., languages recognized by LGr’s with acyclic deletion rules (and arbitrary insertion rules) are regular [JL07].

**Remark 6.2.** The construction of \( G_\Phi \) from \( \Phi \), mostly amounting to copying operations for the \( G_j^+ \)’s and \( G_j^- \)’s, to type-checking and sets-joining operations for the composition of the LTr’s, can be carried out in logarithmic space. \( \square \)

### 6.2 Correctness of the Reduction

We say that a word \( u \) is \( j \)-clean if it has exactly \( m \) symbols and if \( u[i] \in \{ T_i^j, T_i^{j+1}, U_i^j, U_i^{j+1} \} \) for all \( i = 1, \ldots, m \). It is \( \top \)-homogeneous (resp. \( \bot \)-homogeneous) if it does not contain any (resp., only contains) primed symbols.

Let \( 0 \leq j \leq n \) and \( \Theta_j \) be a Boolean valuation of \( x_1, \ldots, x_j \); we say that a \( j \)-clean \( u \) respects (\( \Phi \) under) \( \Theta_j \) when, for all \( i = 1, \ldots, m \), \( \Theta_j \models C_i \) when \( u[i] \) is determined (i.e., \( \in T_i^j + T_i^{j+1} \)). Finally \( u \) codes (\( \Phi \) under) \( \Theta_j \) if additionally each \( u[i] \) is determined when \( \Theta_j \models C_i \). Thus, a word \( u \) that codes some \( \Theta_j \) exactly lists (via determined symbols) the clauses of \( \Phi \) made valid by \( \Theta_j \), and the only flexibility in \( u \) is in using the primed or the unprimed copy of the symbols. Hence there is only one \( j \)-clean \( u \) coding \( \Theta_j \) that is \( \top \)-homogeneous, and only one that is \( \bot \)-homogeneous. If \( u \) respects \( \Theta_j \) instead of coding it, more latitude exists since symbols may be undetermined even if the corresponding
that no clause is valid under $\theta_j$.

Assume that, for some $j \in \{1, \ldots, n\}$, $u_{j-1}$ codes $\theta_{j-1}$ and $u_j$ codes $\theta_j$. Write $b$ for $\theta(x_j)$ (NB: $b \in \{\top, \bot\}$).

**Lemma 6.3.** If $u_j$ is $b$-homogeneous then $u_{j-1} \nabla_{G^j_b} u_j$.

**Proof.** Let $h \overset{\text{def}}{=} \text{Id}_{\{1, \ldots, m\}}$. We claim that $h$ is a $G^b_j$-witness for $u_{j-1}$ and $u_j$, i.e., that $G^b_j$ contains the required insertion and deletion rules.

**Insertions.** $G^b_j$ has all insertion rules $g \rightarrow u_j[m] \rightarrow u_j[m-1] \rightarrow \ldots \rightarrow u_j[1]$ (leftmost rules in Fig. 5) since $u_j$ is $b$-homogeneous.

**Deletions.** $G^b_j$ has all deletion rules $u_j[i] \rightarrow \ldots \rightarrow u_j[1]$. Firstly, both undetermined symbols $U^*_{j-1}$ and $U^*_{j}$ may delete their counterparts $U_{j-1}$ and $U_{j}$, and similarly for the determined symbols (the unconditional deletion rules in Fig. 5). This is used if $C_i$ is not more valid under $\theta_j$ than under $\theta_{j-1}$. Secondly, if $C_i$ is valid under $\theta_j$ but not under $\theta_{j-1}$, then $x_i \models C_i$ (or $\neg x_i \models C_i$, depending on $b$) and the conditional rules in Fig. 5 allow a determined $T^i_j$ (or $T^i_j$ depending on $b$) to delete $U^*_{j-1}$ or $U^*_{j}$. \qed

**Lemma 6.4.** If $u_j$ is $b$-homogeneous, then $u_{j-1} \Rightarrow^{2m} u_j g$. \[ \hspace{1cm} \]

**Proof.** From $u_{j-1} \nabla_{G^j_b} u_j$ (Lemma 6.3) we deduce $u_{j-1} \Rightarrow^{2m} u_j g$, i.e., $u_{j-1} \Rightarrow^{*} u_j g$, by Lemma 5.2, and then $u_{j-1} \Rightarrow^{2m} u_j g$ by Lemma 5.1. \qed

**Corollary 6.5.** If $\Phi$ is satisfiable, then $U^0_{1} \cdots U^0_{m} \Rightarrow^{2mn} T^1_n \cdots T^m_n g$.

**Proof.** Since $\Phi$ is satisfiable, $\theta \models \Phi$ for some valuation $\theta$. For $j = 1, \ldots, m$, we write $b_j$ for $\theta(x_j)$ and let $u_j$ be the only $j$-clean $b_j$-homogeneous word that codes for $\theta_j$.

We now make two assumptions on $\Phi$ that are no loss of generality. First we require that no clause $C_i$ contains both a literal and its negation, hence no $C_i$ is tautologically valid. Then $u_0 \overset{\text{def}}{=} U^0_{1} \cdots U^0_{m}$ codes the empty valuation $\theta_0$. Second, we require that $\Phi$ is only satisfiable with $b_n = \top$ (which can be easily ensured by adding a few extra variables). Then necessarily $u_n = T^1_n \cdots T^m_n$.

Lemma 6.4 gives $u_0 g \Rightarrow^{2m} u_1 g \Rightarrow^{2m} u_2 g \cdots \Rightarrow^{2m} u_n g$, since $\Rightarrow^{2m}_G \subseteq \Rightarrow^{2m}_H \subseteq \Rightarrow^{2m}_G$ for all $b$ and $j$, we deduce $u_0 g \Rightarrow^{2mn} u_n g$ as claimed. \qed

Fix some $\theta$, some $j \in \{1, \ldots, n\}$ and let $b = \theta(x_j)$.

**Lemma 6.6.** If $u$ respects $\theta_{j-1}$ and $u \nabla_{G^j_b} v$, then $v$ respects $\theta_j$.

**Proof.** Write $l$ for $|v|$. From $u \nabla_{G^j_b} v$ (witnessed by some $h$) we deduce that $G^j_b$ has insertion rules $g \rightarrow v[l] \rightarrow v[l-1] \rightarrow \ldots \rightarrow v[1]$. Inspecting Fig. 5, we conclude that necessarily $l \leq m$. Since deletion rules $v[h(i)] \rightarrow u[i]$ are required for all $i = 1, \ldots, m$, we further see from Fig. 5 that $h$ is injective, so that $l \geq m$. Finally $l = m$, $h = \text{Id}_{\{1, \ldots, m\}}$, $v$ is $j$-clean and $b$-homogeneous.
Now, knowing that \( G_\Phi^j \) contains the rules \( v[i] \rightarrow u[i] \), we show that \( v \) respects \( \theta_j \). Suppose, by way of contradiction, that it does not. Thus there is some \( i \in \{1, \ldots, m\} \) with \( v[i] = T_i^j \) (assuming \( b = T \) w.l.o.g.) while \( \theta_j \not\models C_i \) (so that \( \theta_{j-1} \not\models C_i \)). From \( \theta_j \not\models C_i \) we deduce that \( x_j \not\models C_i \). Hence \( G_\phi^j \) does not have the conditional rules \( T_i^j \rightarrow U_i^{j-1} \) and \( T_i^j \rightarrow U_i^{j-1} \). Thus \( u[i] \not\in \{ U_i^{j-1}, U_i^{j-1} \} \). But then \( u \) does not respect \( \theta_{j-1} \), contradicting our assumption. \( \square \)

We immediately deduce:

**Lemma 6.7.** If \( x R_{G_\phi^j} y \) and there is some \( u \sqsubseteq x \) that respects \( \theta_{j-1} \), then there is some \( v \sqsubseteq y \) that respects \( \theta_j \).

**Proof.** From the Closure Property 4.2, we get \( u R_{G_\phi} u \). Then, from \( R_{G_\phi^j} \subseteq \nabla_{G_\phi^j} \subseteq \subseteq \) (Coro. 5.3) we deduce \( u \nabla_{G_\phi^j} v \) for some \( v \sqsubseteq y \). Now \( v \) respects \( \theta_j \) thanks to Lemma 6.6. \( \square \)

**Corollary 6.8.** If \( U_1^0 \cdots U_m^0 \Rightarrow^* G_{\Phi} T_1^n \cdots T_m^n \), then \( \Phi \) is satisfiable.

**Proof.** Write \( u_0 \) for \( U_1^0 \cdots U_m^0 \) and \( u_n \) for \( T_1^n \cdots T_m^n \). From the definition of \( G_\Phi \) and the Composition Lemma 4.4, we deduce that there exist some words \( u_1, \ldots, u_{n-1} \) such that \( u_{j-1} R_{G_\phi^j} u_j \) for all \( j = 1, \ldots, n \).

With Lemma 5.4, we further deduce that there exist some words \( u_1', \ldots, u_n' \) and Boolean values \( b_1, \ldots, b_n \) such that \( u_j' \subseteq u_j \) and \( u_{j-1} R_{G_\phi^j} u_j' \) for all \( j = 1, \ldots, n \). Hence also \( u_j' R_{G_\phi^j} u_j' \) by Prop. 4.2 (and letting \( u_0' = u_0 \)).

Write \( \theta \) for \( [x_1 \mapsto b_1, \ldots, x_n \mapsto b_n] \). With Lemma 6.7, induction on \( j \), and since \( u_0' \) respects \( \theta_0 \), we further deduce that there exists some words \( u_1', \ldots, u_n' \) such that, for all \( j = 1, \ldots, n \), \( u_j' \subseteq u_j \) and \( u_j' \) respects \( \theta_j \). From \( |u_n'| = m \) (it respects \( \theta \)) and \( u_n' \subseteq u_n \), we deduce that \( u_n' = u_n \). Finally, \( \theta \models \Phi \) since \( u_n' \) respects \( \theta \) and \( u_n' = u_n = T_1^n \cdots T_m^n \). \( \square \)

**Corollary 6.9.** \( \mu \)-Minimality of a derivation is \( \text{coNP} \)-hard.

**Proof (Sketch).** We define \( G_\Phi' \) by taking \( G_\Phi \), adding \( k \) extra symbols \( a_1, \ldots, a_k \), and adding the following two sets of rules:

1. all \( a_{i-1} \rightarrow a_i \) and \( a_{i-1} \cdots a_i \) for \( i = 1, \ldots, k \) (with the convention that \( a_0 \) is \( T_i^n \));
2. all \( a_k \cdots U_i^0 \) for \( i = 1, \ldots, m \).

Observe that \( G_\Phi' \) is acyclic. It has a derivation \( \pi : U_1^0 \cdots U_m^0 \Rightarrow 2m+2k T_1^n \cdots T_m^n \) of the following form:

\[
U_1^0 \cdots U_m^0 \Rightarrow 2m T_1^n \cdots T_m^n \Rightarrow k U_1^0 \cdots U_m^0 a_k a_{k-1} \cdots a_1 T_1^n \cdots T_m^n \Rightarrow 2m+2k T_1^n \cdots T_m^n.
\]

This derivation uses the extra symbols to bypass the normal behaviour of \( G_\Phi \). If \( k \) is large enough, i.e., \( k > m(n-1) \), \( \pi \) is \( \mu \)-minimal if, and only if, \( \Phi \) is not satisfiable. \( \square \)
7 Anchored Leftist Transformers and Their Transitive Closure

When \( b_1, b_2 \in B \) are two different working symbols, and \((A, B, C, P, g)\) is a LTr, we call \( G = (A, B, C, b_1, b_2, P, g)\) an anchored LTr, or shortly an ALTr. With an ALTr \( G \) we associate an anchored transformation \( S_G \subseteq A^* \times C^* \) defined by

\[
\forall S \vdash \exists^* G \exists v G \cup b_1 G \Rightarrow^*_G b_2 G.
\]

Here the anchors \( b_1, b_2 \) are used to control what happens at the left-hand end of transformed words. Mostly, they ensure that the derivation \( b_1 G \Rightarrow^*_G b_2 G \) goes all the way to the left and erases \( b_1 \) rather than stopping earlier. One intuitive way of seeing \( S_G \) is that it is a variant of \( R_G \) restricted to derivations that replace the anchors.

A first difficulty for building the transitive closure of an anchored transformation \( S_G \subseteq A^* \times C^* \) is that the input and output sets are disjoint (a requirement that allowed the developments of Sections 4 and 5). To circumvent this, we assume w.l.o.g. that \( A \) and \( C \) are two different copies of a same set, equipped with a bijective renaming \( h : C^* \to A^* \). Then, the closure \( S_G, (h, S_G)^+ \) behaves like we would want \( S_G^+ \) to behave.

For the rest of this section, we assume \( h \) is a bijection between \( C \) and \( A \). W.l.o.g., we write \( A \) and \( C \) under the forms \( A = \{a_1, \ldots, a_n\} \) and \( C = \{c_1, \ldots, c_n\} \) so that \( h(c_i) = a_i \) for all \( i = 1, \ldots, n \). Then \( h \) is lifted as a (bijective) morphism \( h : C^* \to A^* \) that we sometimes see as a relation between words.

The exact statement we prove in this section is the following:

**Theorem 7.1 (Transitive Closure).** Let \( G : A \vdash C \) be an ALTr such that \( S_G = S_G^C \subseteq C \). Then there exists an ALTr \( G^+ : A \vdash C \) such that \( S_G^+ = S_G, (h, S_G)^+ \).

Furthermore, it is possible to build \( G^+ \) from \( G \) using only logarithmic space.

Let \( b_1, b_2 \not\in A \cup C \). The ALTr \( R_{b_2, b_1} \) defined by

\[
P_R = \left\{ g \to a_i, a_i \to a_j, a_i \to b_1 \right\}_{i,j} \text{ for all } i, j = 1, \ldots, n
\]

is called a renamer (of \( C \) to \( A \)), and often simply written \( R \). Observe that \( R : C \vdash A \) is indeed an ALTr. It further satisfies \( S_R = \approx \subseteq \check{h} \).

We shall now glue an ALTr \( G : A \vdash C \) with the renamer \( R : C \vdash A \) into some larger LGr \( H \). But before this can be done we need to put some wrapping control on \( G \) (and on \( R \)) that will let us track what comes from \( G \) inside \( H \)’s derivations.

Formally, given an ALTr \( G = (A, B, C, b_1, b_2, P, g) \) and two new anchor symbols \( \Box_1, \Box_2 \not\in \Sigma_g \), we let \( \Sigma_\Box \) defined by \( \{\Box_1, \Box_2\} \) and define a new ALTr \( F_{G, \Box_1, \Box_2} \) (or shortly just \( F_G \)) for “wrapping \( G \) with \( \Box_1, \Box_2 \)”, and given by \( F_{G, \Box_1, \Box_2} \) defined by \( \left(\overline{A}, \overline{B}, \overline{C}, \Box_1, \Box_2, P, g\right) \) where

\[
\begin{align*}
\overline{A} & \equiv A \cup A' \cup \{b_1, b_1'\}, A', b_1' \text{ being a copy of } A, b_1, \\
\overline{B} & \equiv \{\Box_1, \Box_2\} \cup B \setminus \{b_1\}, \\
\overline{C} & \equiv C \cup \{b_2\} \cup C' \cup B' \setminus \{b_1'\}, B' \text{ and } C' \text{ being copies of } B \text{ and } C.
\end{align*}
\]

Finally, let \( D \equiv C \cup B' \) and \( D' \equiv C' \cup B' \). (The copies are denoted by priming the original symbols, and a primed set like \( A' = \{a' | a \in A\} \) is just the set of corresponding primed symbols.) The rules in \( P' \) are derived from the rules of \( P \) in the following way. (See Fig. 6 in App. E for a schematic type.)
kept: \( \mathcal{P}' \) retains all rules of \( \mathcal{P} \) that do not erase a letter in \( A \cup \{ b_1 \} \),
replace: \( \mathcal{P}' \) has a rule \( d' \cdot \Rightarrow a \) for each rule \( d \cdot \Rightarrow a \) in \( \mathcal{P} \) that erases a letter in \( A \cup \{ b_1 \} \),
mirror: \( \mathcal{P}' \) has a rule \( d \rightarrow d' \) for each \( d \in D \).
clean: \( \mathcal{P}' \) has all rules \( \square_2 \rightarrow e' \) and \( \square_2 \rightarrow d' \) for \( d', e' \in D' \setminus \{ b'_1 \} \) and \( a' \in A' \cup \{ b'_1 \} \),
b-rules: \( \mathcal{P}' \) has the rules \( \square_2 \rightarrow \square_1 \) and all rules \( d' \rightarrow \square_2 \) for \( d' \in D' \setminus \{ b'_1 \} \).

We now relate the derivations in \( G \) and the derivations in \( F_G \). For this, assume \( u \in (A + b_1)^* \) and \( v \in (C + b_2)^* \).

Lemma 7.2 (See App. E.1). 1. If \( u.g \Rightarrow_G^+ v.g \) then for all words \( \alpha \in (A' + b'_1)^* \) there exists a symbol \( \beta \in C' \cup \{ b'_2 \} \) such that \( \square_1.\alpha.u.g \Rightarrow_{F_G}^+ \square_1.\alpha.\beta.v.g \).
2. Reciprocally, for all \( \alpha \in (A' + b'_1)^* \), for all \( \beta \in (C' + b'_2)^* \) if \( \square_1.\alpha.u.g \Rightarrow_{F_G}^+ \square_2.\beta.v.g \) then \( u.g \Rightarrow_G^+ v.g \).

Thus we can relate anchored derivations in \( F_G \) with anchored derivations in \( G \) via:

Corollary 7.3. Let \( u \in (A + b_1)^* \) and \( v \in (C + b_2)^* \). Then \( b_1.u.g \Rightarrow_G^+ b_2.v.g \) if and only if there exists \( \beta \in (C' \cup \{ b'_2 \}) \) such that \( \square_1.\alpha.b_1.u.g \Rightarrow_{F_G}^+ \square_2.b_2.v.g \). In other words, \( u.S_G.v \) iff \( \square_1.\alpha.b_1.S_{F_G}^+ b_2.v \) for some \( \beta \in (C' \cup \{ b'_2 \}) \).

We may now glue the wrapped versions of \( G \) and its associated \( R \). Recall that \( F_G = (\mathcal{A}, \mathcal{B}, \mathcal{C}, \square_1, \square_2, \mathcal{P}, g) \). We denote the set of new symbols with \( \Sigma \equiv \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \) and observe that \( F_R \) (short for \( F_{R(\mathcal{A}, \mathcal{B}, \mathcal{C}, \square_1, \square_2, \mathcal{P}, g)} \)), being some \( \langle \mathcal{C} \cup \mathcal{C}' \cup \{ b_2, b'_2 \}, \Sigma, \mathcal{A}, \mathcal{B}, \square_2, \square_1, P_R, g \rangle \), does not use more symbols. Let \( H \equiv \langle \Sigma, P_H, g \rangle \) be the LGr such that and \( P_H = \mathcal{P} \cup \mathcal{P}' \).

Essentially, \( H \) is a union of the two wrapping ALTr's. (See Fig. 7 in App. E.2 for a schematic description). Note that \( H \) is not a LTr since it does not respect any distinction between input, intermediary, and output symbols.

Lemma 7.4 (See App. E.3). Let \( \alpha, \beta \in A^{*'} \) and \( u, v \in A^* \). If \( \square_1.\alpha.u.g \Rightarrow_{H}^+ \square_1.\beta.v.g \) and \( S_G = \{ \square_A . S_G . \square_C \} \) then \( u \subseteq_A . (S_G . \hat{h})^* . v \).

We now extend \( H \) to turn it into an ALTr \( H' : \hat{A} \vdash A \cup A' \), introducing again new copies, denoted \( \hat{a}, . . . \), of previously used symbols and writing \( \hat{a} = a_1 a_2 . . . a_n \) for the dotted copy of some \( a = a_1 a_2 . . . a_n \). Formally,

\[
H' \equiv (\hat{A}, B \cup B' \cup C \cup C' \cup \{ \square_1, \square_2, \square_1, \square_2 \}, A \cup A', \square_1, \square_2, P', g)
\]

where \( P' \) extends \( P_H \) by the rules \( \square_2 \cdot \square_1 \Rightarrow \square_1 \), \( \square_1 \Rightarrow \square_2 \), and all \( a \cdot \hat{a} \Rightarrow \hat{a} \) for \( a \in A \).

The anchored transformation \( S_{H'}.h \) computed by \( H' \) is captured by the following:

Lemma 7.5 (See App. E.4). Let \( u, v \in A^* \). Then \( \hat{u} S_{H'} \square_1.\beta.v \) for some \( \beta \in A^{*'} \) iff \( u \subseteq_A . (S_{H'} . \hat{h})^* . v \).

We are nearly done. There only remains to compose \( H' \) with a LTr that checks for the presence of \( \square_1.\beta \) (and then erases it). For this last step, we shall use further dotted copies \( \Sigma, \Sigma', . . . \), of the previously used symbols.

Formally, we define two new ALTr's \( T_1 \) and \( T_2 \): see App. E.5. The rules of \( T_1 \) ensure that it satisfies

\[
u S_{T_1} v \text{ iff } u = \square_1.\alpha.b_1.u' \text{ and } \hat{u} L_{I_{\text{ins}}} \hat{v} \text{. (T}_1\text{-spec)}
\]
Regarding $T_2$, let $u \in (\bar{A} \cup \bar{A}' \cup \{\bar{b}_1, \bar{b}_1'\})^*$ and $v \in \bar{A}^*$. If $\bar{u}'$ is the largest subword of $u$ such that $\bar{u}' \in \bar{A}^*$, then

$$u \xrightarrow{T_2} v \text{ if } \bar{u}' \subseteq \bar{A} v.$$  \hfill (T_2\text{-spec})

Combining ($T_1\text{-spec}$) and ($T_2\text{-spec}$) we obtain

$$u \xrightarrow{T_1 . T_2} v \text{ if } u = \square_1.\alpha.\bar{b}_1.\bar{u}' \text{ and } \bar{u}' \subseteq \bar{A} v.$$  

Composing these LTr’s as $H'.T_1 . T_2$ yields a resulting $G^{(+)} : \bar{A} \mapsto \bar{A}$, which, up to a bijective change of symbols, is what we need to build to prove Theorem 7.1.

## 8 Conclusion

In this paper we introduce a notion of transformations computed by leftist grammars and define constructions showing how these transformations are effectively closed under sequential composition and transitive closure.

These operations require that some “typing” assumptions are satisfied (e.g., we only know how to build a transitive closure on leftist transformers that are “anchored”) which may be seen as a lack of elegance and generality of the theory, but which we see as an indication that leftist grammars are very hard to control and reason about.

Anyway, the restrictive assumptions are not a problem for our purposes: we intend to rely on the compositional foundations for building, in a modular way, complex leftist grammars that are able to simulate lossy channel systems. Here the modularity is essential not so much for building complex grammars. Rather, it is essential for proving their correctness by a divide-and-conquer approach, in the way we proved the correctness of our encoding of 3SAT instances in Section 6.

As another direction for future work, we would like to mention that the proof that accessibility is decidable for LGr’s (see [MPSV00]) has to be fixed and completed.

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A Greedy derivations are sufficient

The lexicographic ordering between derivations is denoted \( \leq_{\text{lex}} \).

**Lemma A.1.** Every derivation has a \( \mu \)-minimal equivalent.

**Proof.** Direct from observing that \( \leq_\mu \) is a well-founded quasi-ordering over derivations. Indeed, while \( \leq_{\text{lex}} \) is not well-founded over the set of tuples of natural numbers, it is well-founded over the set \( \bigcup_{n \in \mathbb{N}} \{n\} \times \mathbb{N}^n \) to which measures of derivations belong. \( \square \)

The proof of Proposition 3.1 is concluded with the following:

**Lemma A.2.** A \( \mu \)-minimal derivation is greedy.

**Proof.** By combining lemmas A.4, A.5 and A.6 below. \( \square \)

**Lemma A.3.** Assume \( \pi = u_0 \Rightarrow^{r_1 \cdot p_1} u_1 \Rightarrow^{r_2 \cdot p_2} u_2 \) is a two-step derivation. If \( p_2 < p_1 - 1 \), or if \( p_2 = p_1 - 1 \) and \( r_1 \) is an insertion rule, then \( \pi \) is not \( \mu \)-minimal.

**Proof.** The hypothesis ensures that the two steps do not interfere. Thus they can be swapped, yielding an equivalent derivation \( \pi' = u_0 \Rightarrow^{r_2} \Rightarrow^{r_1} u_2 \). Clearly, \( \pi' <_\mu \pi \). \( \square \)

In the rest of this section, we consider a generic transformation \( \pi \) of the form \( u_0 \Rightarrow^{r_1 \cdot p_1} u_1 \Rightarrow^{r_2 \cdot p_2} u_2 \cdots \Rightarrow^{r_n \cdot p_n} u_n \) in the context of some LGr \( G = (\Sigma, P, g) \).

**Lemma A.4.** A \( \mu \)-minimal derivation is leftmost.

**Proof.** Assume \( \pi \) is not leftmost. Then it contains a step \( u_{i-1} = w_1aw_2 \Rightarrow w_1'aw_2 = u_i \) where \( w_1 \) is not inert in the rest of \( \pi \). Let \( j > i \) be the first step after \( i \) where a letter of \( w_1 \) is active: the subderivation \( u_{j-2} \Rightarrow^{r_j \cdot p_j} u_{j-1} \Rightarrow^{r_j \cdot p_j} u_j \) has \( p_j < p_{j-1} \), and even \( p_j < p_{j-1} - 1 \) if \( r_{j-1} \) is a deletion rule. Lemma A.3 applies and entails that this subderivation, hence also \( \pi \), is not \( \mu \)-minimal. \( \square \)

**Lemma A.5.** A \( \mu \)-minimal derivation is eager.

**Proof.** Assume \( \pi \) is not eager. Let \( u_{i-1} \Rightarrow^{r_i \cdot p_i} u_i \) be the first step that violates eagerness: then \( u_{i-1} \) is some \( w_1baw_2, w_1b \) will remain inert in the rest of \( \pi \), and \( b \) will eventually be deleted at some step \( j > i \), but this is not done right now even though \( P \) contains \( a \cdots \cdot b \).

We now consider several cases for step \( i \). If the active letter occurs to the right of \( a \), then one obtains a new derivation \( \pi' \) by deleting \( b \) (using \( a \cdots \cdot b \) right now, continuing like \( \pi \), and skipping step \( j \) since \( b \) has already been deleted. This produces an equivalent derivation, with \( \mu(\pi') = \langle n, p_1, \ldots, p_{i-1}, l, \ldots \rangle \) where \( l = |w_1b| < p_i \). Hence \( \pi \) is not \( \mu \)-minimal. If \( a \) is the active letter, the step must be an insertion \( a \Rightarrow c \) and \( p_{i+1} \geq p_i \): we obtain, as in the previous case, an equivalent \( \pi' \) with \( \mu(\pi') = \langle n, p_1, \ldots, p_{i+1}, p_i - 1, \ldots \rangle \leq_{\text{lex}} \mu(\pi) \). Finally, the active letter cannot be to the left of \( a \) since \( w_1b \) remains inert. \( \square \)

**Lemma A.6.** A \( \mu \)-minimal derivation is pure.

**Proof.** Assume \( \pi \) is not pure. Then it inserts at some step a useless letter \( a \) that stays inert and is eventually deleted. By not inserting \( a \) and not deleting it later, one obtains an equivalent but shorter derivation. \( \square \)
B Proof of the Closure Property (Prop. 4.2)

Let \( G = (\Sigma, P, g) \) be some arbitrary LGr. The following two observations are easy.

**Fact B.1** Assume \( uau'g \Rightarrow v_g \) is a derivation where the letter \( a \) is inert and eventually erased. Then \( uau^ng \Rightarrow v_g \) for all \( n \in \mathbb{N} \).

**Fact B.2** Assume \( a \) does not occur in \( u \). Then \( ug \Rightarrow v_g \) iff \( ug \Rightarrow v_g \).

Now, in the case where \( G \) is an LTr \((A, D, P, g)\), we deduce \( \sqsubseteq_{A, D, P, g} R_G \subseteq R_G \) from Fact B.1, and \( R_G \approx \subseteq R_G \) from Fact B.2. This entails \( R_G = \sqsubseteq_{A, D, P, g} R_G \approx \).

C Proof of the Composition Lemma (Lemma 4.4)

There only is to prove that \( R_{G_1, G_2} \subseteq R_{G_1} R_{G_2} \). For this we consider a greedy derivation \( \pi = (ug \Rightarrow v_g) \) with \( u \in A_1^* \) and \( v \in C_2^* \), and consider two cases:

1. If \( \pi \) never uses a rule from \( G_2 \), then no symbols from \( C_2 \) are inserted and necessarily \( v = \varepsilon \). We obtain \( uR_{G_1} R_{G_2} \varepsilon \) by observing that \( uR_{G_1} \varepsilon \) (as witnessed by \( \pi \)) and that \( \varepsilon R_{G_2} \varepsilon \) (true of all leftist transformations).

2. Otherwise, we isolate the first \( G_2 \) step and write \( \pi \) under the form

\[
\begin{array}{c}
\pi_1 \\
ug \Rightarrow v_g \Rightarrow w \Rightarrow \pi_2
\end{array}
\]

Necessarily, \( w \in A_1^* D_1^* \) (Fact 4.1) and, since symbols from \( A_1 \cup D_1 \) are inactive in \( G_2 \), the first \( G_2 \) step is an insertion by \( g \), i.e., some \( wg \Rightarrow wg = w'g \) with \( e \in D_2 = B_2 \cup C_2 \). Since \( \pi \) is greedy, \( w \) is inert in \( \pi_2 \).

Now, every word along \( \pi_2 \) is some \( xyg \) with \( x \) an inert prefix of \( w \) and \( y \in D_2^* = (B_2 \cup C_2)^* \). This claims holds at the first step (since \( e \in D_2 \)) and is proved by induction for the next steps. Assume that the \( k \)-th step is some \( xyg \Rightarrow zg \) since \( x \) is inert, the active letter is in \( y \), hence in \( D_2 \) (by ind. hyp.) and the step is a \( G_2 \) step. If the step is a deletion step, of the last letter in \( x \) or of some letter in \( y \), \( zg \) satisfies the claim. If the step is an insertion step, the claim is satisfied again since \( G_2 \) can only insert letters from \( D_2 \) and to the right of \( x \).

Finally, \( \pi \) must have the form \( u \Rightarrow w \Rightarrow v_g \). Since symbols from \( A_1 \cup B_1 \) cannot be erased by \( G_2 \) rules, then necessarily \( w \in C_1^* \). Hence \( uR_{G_1} w \) and \( wR_{G_2} v \), proving \( uR_{G_1}R_{G_2} \).

D Proof of Lemma 5.2

The inclusion \( \nabla_G I_{\mathcal{L}_{\text{ins}}} \subseteq R_G \) is clear in view of \( \nabla_G \subseteq R_G \) and since \( u \in \mathcal{L}_{\text{ins}} v \) implies \( u, v \in C^* \) and \( u \Rightarrow v \).

\( I_{\mathcal{L}_{\text{ins}}} = R_{\mathcal{L}_{\text{ins}}} \subseteq R_G \cap (C^* \times C^*) \) and

For the other inclusion, \( R_G \subseteq \nabla_G I_{\mathcal{L}_{\text{ins}}} \), we consider a greedy derivation

\[
u = w_0 g \Rightarrow w_1 g \Rightarrow g \cdots \Rightarrow w_n g = v_g.
\]
Every $w_i$ is some $u_i v_i$ with $u_i \in A^*$ and $v_i \in C^*$ (Fact 4.1) and, since $A$ is inert in $G$, $u_i$ is a prefix of $u$ (so that we can write $u$ under the form $u_i u_i'$). Let $k$ be the first index s.t. $u_k = \varepsilon$, so that $w_i \in C^*$ for all $i = k, \ldots, l$ and $w_k I_{G_{\text{fin}}} v$. There remains to show $u \nabla_G w_k$, i.e., $u_k' \nabla_G v_k$.

For this we show more generally that $u_i' \nabla_G v_i$ for all $i = 0, \ldots, k$. We proceed by induction on $i$. The base case clearly holds since $u_0' = v_0 = \varepsilon$. For the induction step, we assume $u_i' \nabla_G v_i$ and a witness $h_i : \{1, \ldots, |u_i'|\} \rightarrow \{1, \ldots, |v_i|\}$ for some $i < k$. Consider the step $u_i v_i g \Rightarrow u_{i+1} v_{i+1} g$. There are two cases.

1. If $r = c \rightarrow c'$ is an insertion rule, then the insertion must take place in front of $v_i$ otherwise the derivation is not leftmost (the first letter in $v_i$ cannot be inactive since it cannot be deleted and $u_i \neq \varepsilon$ must be deleted). Hence $v_{i+1} = c' v_i$, $u_{i+1}' = u_i'$, and a witness $h'$ for $u_i' \nabla_G v_i$ is obtained from $h$ with $h'(i) \overset{\text{def}}{=} h(i) + 1$.

2. If $r = c \rightarrow a$ is a deletion rule, then $u_i = u_{i+1} a$, i.e., $u_{i+1}' = a u_i'$, and $u_{i+1}' \nabla_G v_{i+1}$ is witnessed by $h'$ defined as $h'(1) \overset{\text{def}}{=} 1$ and $h'(i + 1) \overset{\text{def}}{=} h(i)$.

\[ \square \]

\section*{E Proofs for Section 7}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig6}
\caption{A schematic type for $F_{G, \square_1, \square_2}$.}
\end{figure}
E.1 Proof of Lemma 7.2

1. We first prove, by induction on the length of a derivation $u G \Rightarrow^* v_1 v_2 g$, with $v_1 \in (A + b_1)^*$ and $v_2 \in (D + b_2)^*$, that it can be mimicked as $\Box_1, \alpha, u \Rightarrow^{\text{mirror}}_{F_G} \Box_1, \alpha, v_1, \gamma, v_2, g$ for some $\gamma \in D' \setminus \{b_1'\}$. First, and since $F_G$ contains rules “kept” from $G$, it can mimic any $G$-step that does not delete a letter from $A + b_1$. For steps $v_1, a d, v_2 \Rightarrow G v_1, d, v_2$ using $d \rightarrow a$ with $a \in A + b_1$ and $d \in D$ then, for all $\gamma \in (D' \setminus \{b_1'\})^*$, the following derivation exists:

$$\Box_1, \alpha, v_1, a \cdot \gamma, d, v_2 \Rightarrow^{\text{mirror}}_{F_G} \Box_1, \alpha, v_1, a \cdot \gamma, d', v_2 \Rightarrow^{\text{clean}}_{F_G} \Box_1, \alpha, v_1, a \cdot d', v_2 \Rightarrow^{\text{replace}}_{F_G} \Box_1, \alpha, v_1, a \cdot d, v_2.$$

Once $v_1 = \varepsilon$, there remains to show that $\Box_1, \alpha, \gamma, v, g \Rightarrow^{\text{mirror}}_{F_G} \Box_1, \alpha, \beta, v, g$ for some $\beta \in C' + b_2'$. This means erasing $\alpha$, and replacing $\gamma$ (that may belong to $D' \setminus C'$), using a primed version of the first symbol of $v$. Formally, we use

$$\Box_1, \alpha, \gamma, v, g \Rightarrow^{\text{mirror}}_{F_G} \Box_1, \alpha, \beta, v, g \Rightarrow^{\text{clean}}_{F_G} \Box_1, \alpha, \beta, v, g \Rightarrow^{\text{b-rules}}_{F_G} \Box_1, \alpha, \beta, v, g \Rightarrow^{\text{clean}}_{F_G} \Box_1, \alpha, \beta, v, g.$$

2. If $F_G$ uses a keep rule, then the same rule exists in $G$ and is also usable. If a mirror, clean, or b-rule is used then $G$ can mimic by doing nothing. If a replace rule occurs in a step of the form $\Box_1, \alpha, v_1, a \cdot d', v_2 \Rightarrow^{\text{replace}}_{G} \Box_1, \alpha, v_1, d', v_2, \alpha, a \in A, d' \in D', v_1 \in (A + b_1)^*, v_2 \in (D \setminus \{b_1\})^+$ then in a previous step there was the letter $d$ at the head of the $(D \setminus \{b_1\})^+$ part of the word (in order to insert $d'$) and $a$ was not deleted. At that time $G$ could delete $a$ to finally reach $v_1, v_2$. \hfill \Box

E.2 Proofs for the Correctness of $H$

Let $\alpha, \alpha' \in (A' + b_1')^+$ and $u, v \in (A + b_1)^*$.

**Lemma E.1.** If the derivation $r = \Box_1, \alpha, u, g \Rightarrow^{*}_H \Box_1, \alpha', v, g$ is greedy, then there are no insertions of a $\Box_1$ to the right of another $\Box_1$ or of $a$ letter $\in (A' \cup b_1')$, and there also are no insertions of a $\Box_2$ to the right of another $\Box_2$ or $a$ of letter $\in (D' \setminus \{b_1\})^+$.

**Proof (Idea).** If there is that kind of insertion, then the $\Box_1$ is kept until the end of the derivation or deleted. Since that letter can’t insert or delete in that position, then if it is kept, there is always some letter on its left. There is only one $\Box_1$ at the end of the derivation, so the letter is deleted. That $\Box_1$ has no descendant, is not present at the end and did not delete anything, so it is useless, which contradicts the greediness hypothesis.

**Lemma E.2.** The following three languages are invariants of $H$:

$$\Sigma^*, (A + D) \cdot \Sigma \cdot \Sigma^*,$$

$$\Sigma^*, (A + b_1) \cdot (\Sigma \cdot A' + b_1') \cdot \Sigma^*,$$

$$\Sigma^*, (D \setminus b_1) \cdot (\Sigma \cdot D' \setminus b_1') \cdot \Sigma^*.$$

**Proof (Idea).**
Proof. We only prove the first invariant, the other two rely on similar arguments. Let 
\( w = u.l.\square.v \) with \( l \in A \cup D \) and \( \square \in \Sigma_\square \). The invariance of \( I_1 \) could be violated by
inserting a symbol between \( l \) and \( \square \), or deleting \( l \) or \( \square \). \( \square \) cannot delete \( l \) or insert letters, and \( \square \) can only be deleted by a letter from \( \Sigma_\square \), a situation where the invariant is preserved.

We say that a word \( a.w \) blocks a language \( L \subseteq \Sigma^* \) if for all \( v \in L \), for all \( u \in \Sigma^* \), for all derivations \( \pi = (u.a.w.v.g \Rightarrow^* xg) \) where \( uv \) is inert, \( u.a \) is not deleted. The definition means that no \( vg \) with \( v \in L \) can erase anything left of \( w \). Obviously, since we only consider derivations with \( uv \) inert, any \( w' \) with \( a.w \subseteq w' \) blocks \( L \) when \( a.w \) does.

The arguments used to prove Lemma E.2 can be reused to show the following:

**Lemma E.3.** For all \( a \in (A \cup \{b_1\}) \) and all \( d \in (D \setminus \{b_1\}) \),

- \( a \) blocks \( (A' + b'_1 + \square_1 + \square_2).\Sigma^* \),
- \( d \) blocks \( (D' \setminus \{b'_1\} + \square_1 + \square_2).\Sigma^* \),
- \( a.d \) blocks \( (A' + A' \cup \{b_1, b'_1, \square_1\}).\Sigma^* \),
- \( d.a \) blocks \( (D \cup D' \setminus \{b_1, b'_1\} \cup \{\square_2\}).\Sigma^* \),
- \( a.d' \) blocks \( (A' \cup \{b'_1\}).\Sigma^* \),
- \( d.a' \) blocks \( (D' \setminus \{b'_1\}).\Sigma^* \),
- \( a.d'.a \) blocks \( \Sigma^{*} \),
- \( d.a'.a.d \) blocks \( \Sigma^{*} \).
Lemma E.4. If there is a derivation

$\begin{align*}
\text{greedy one such that each step is in } L
\end{align*}$

Proof. Consider the greedy derivation with the fewest number of steps out of $L_{AC} \cup L_{CA}$.

We assume that this number is $> 0$ and obtain a contradiction, thus proving the Lemma. For this we consider the first step that goes out of $L_{AC} \cup L_{CA}$. Assume that this step is $w \Rightarrow_{H}$ ... for $w, g \in L_{AC}$ (hence $w$ can be written under the form $\Box_1, X, \Box_2, Y, Z, T, g$) and proceed by a case analysis of which letter is active in this step:

- $\Box_1$ has no rule that can be applied here.
- some letter $x \in A' \cup b'_1$ in $X$:
  * $x \rightarrow a', a' \in A' \cup b'_1$ stays in $L_{AC}$,
  * $x \rightarrow c$, $c \in C$ is not usable,
  * $x \rightarrow \Box_1$ forbidden by Lemma E.1 (whether the insertion is at the head of $X$ or inside it),
- $\Box_2$:
  * $\Box_2 \rightarrow \Box_1$ leads to $L_{CA}$ (since necessarily in this case $n_1 = 0$, $n_2 = 1$, $n_3 = 0$, $n_4 > 0$, and $n_5 > 0$),
  * $\Box_2 \rightarrow a'$, $a' \in A' \cup b'_1$ stays in $L_{AC}$,
- some letter $x \in A \cup b_1$ in $Y$:
  * $x \rightarrow a$ or $x \rightarrow a$, $a \in A \cup b_1$ stays in $L_{AC}$,
  * $x \rightarrow a'$, $a' \in A' \cup b'_1$ stays in $L_{AC}$ if inserted at the head of $Y$, is forbidden by Lemma E.1 if inside of $Y$,
- some letter $x \in D' \setminus b'_1$ in $Z$:
  * $x \rightarrow a$, $a \in A \cup b_1$ stays in $L_{AC}$,
  * $x \rightarrow a$ if $n_3 > 0$ or inserts inside $Z$, forbidden by Lemma E.2, else stays in $L_{AC}$ if $n_2 = 0$, else forbidden by Lemma E.1,
- some letter $x \in D \setminus b_1$ in $T$:
  * $x \rightarrow d$, or $x \rightarrow d$, $d \in D \setminus b_1$ stays in $L_{AC}$,
Technical appendix, not for the proceedings version.

- \( x \rightarrow d', \ d' \in D \smallsetminus b'_1 \) stays in \( L_{AC} \) if inserts at head of \( T \), else forbidden by Lemma E.2.

1. if \( n_2 = 0 \) then \( n_3 = 0 \) and \( n_4 > 0 \) so the step stays in \( L_{AC} \).

- \( g' \):
  - \( g \rightarrow d, \ d \in D \smallsetminus b_1 \) stays in \( L_{AC} \).
  - \( g \rightarrow a, \ a \in A \cup b_1 \):
    * if \( n_5 = 0 \) then \( n_4 = 0 \) and \( n_3 > 0 \) so the step stays in \( L_{AC} \).
    * if \( n_5 > 0 \):
      - if \( n_4 = 0 \) then \( n_3 > 0, n_2 = 0 \) and \( n_1 > 0 \):
        Then \( w = \square_1 X Y Z T \) with \( |T| > 0 \). We write \( T = yv \) with \( y \in D \smallsetminus b_1 \) and let \( w' = wa \). After the insertion of \( a, \ w \) is inert (by leftmost). Since \( y \) does not appear at the end of the derivation, it is deleted. If \( y \) is deleted by a descendant of a letter of \( A \cup b_1 \), then it is by some \( a' \in A' \cup b'_1 \). The word at the step just after the deletion would start with \( A \cup b_1 \) which is forbidden by Lemma E.2.

- The only other letters able to delete \( y \) are letters from \( D \smallsetminus b_1 \). When \( g \) inserts the first \( d \in D \smallsetminus b_1 \) after \( a \), the letter at its left is a \( a \). So there is \( d.a \) as subword of the inert letters. Since \( d.a \) blocks \( (D \smallsetminus b_1) \Sigma^* \), the \( d \) won’t be deleted.

  - if \( n_4 > 0 \)
    1. if \( n_3 > 0 \) then \( n_2 = 0 \) and \( w \) is some \( \square_1 X Y Z T \) with non-empty \( X \), \( Y \) and \( Z \). We write \( Z = yZ' \) with \( y \in D' \smallsetminus b'_1 \) and \( w' = wa \). \( \alpha u y' v' v, \alpha \in (A' + b'_1)^+, u \in (A + b_1)^+, y' \in D' \smallsetminus b'_1, v' \in (D' \smallsetminus b'_1)^+, v \in (D \smallsetminus b_1)^* \) and \( w' = wa \).
      After the insertion of \( a, \ w \) is inert (by leftmost). Since \( y \) does not appear at the end of the derivation, it is deleted. It cannot be deleted by descendants of \( A \cup b_1 \), which could be \( \square_1 \) because \( (A + b_1) \square_1 \) is a forbidden invariant (Lemma E.2). \( \square_1, \alpha u \) cannot be deleted since \( u y' v a \) blocks \( \Sigma^* \) (Lemma E.3). Hence letters from \( D' \smallsetminus b'_1 \) are eventually deleted by some \( \square_1 \) which is not the one on the left and all the accessible words are in \( \square_1, \alpha u \Sigma^*, (\square_1 + \square_2) \Sigma^*, g \), to which \( \square_1, \beta v \) does not belong.
    2. if \( n_3 = 0 \)
      (a) if \( n_2 = 0 \) then \( n_3 > 0 \) and \( w \) is some \( \square_1 X Y Z T \) with non-empty \( X \) and \( Z \). We write \( Z = yZ' \) with \( y \in D' \smallsetminus b'_1 \) and \( w' = wa \). The \( \square_1, \alpha \) prefix is eventually deleted: the letters from \( D' \smallsetminus b'_1 \) cannot be deleted without introducing a \( \square_1 \) to the right of the first \( \square_1 \). Since in the last word, there is only one \( \square_1 \), at least one of the two is deleted. Since \( \Sigma^*, (\square_1 + \square_2) \Sigma^* \) is an invariant, if the leftmost \( \square_1 \) is not deleted, the word will stay in \( \square_1, \Sigma^*, (\square_1 + \square_2) \Sigma^* \).
      So there is a subderivation of the form
      \[
      \square_1 X.Y.Z'.g \Rightarrow \square_1 X.Y.Z'.T.a.g \Rightarrow \square_1 x'.g \Rightarrow \square_1, \beta. T.g
      \]
such that $\Box_1 x.g$ is the step where the $\Box_2$ from $\Box_2 x.g$ is deleted
and $\Box_1 X.y.Z'' .g$ is the last step where $y$ is active or when it was
inserted.

We can transform the sub-derivation such that $\Box_1 X.y.Z'' .g \Rightarrow
\Box_1 X.\Box_2 y.Z'' .g \Rightarrow (\Box_2 x.T.a) .\Box_2 x.g \Rightarrow r'_1 \Box_2 x.g \Rightarrow r'_1 .\Box_1 .T.g$.

That derivation is still greedy, and has strictly fewer steps out of
$L_{AC} \cup L_{CA}$ than the original.

(b) if $n_2 > 0$ and $n_1 > 0$ then the rule $\Box_2 \Rightarrow a', a' \in A' \cup b'_1$ has priority
over the insertion (by eagerness).

(c) if $n_2 > 0$ and $n_1 = 0$ then the rule $\Box_2 \Rightarrow \Box_1$ has priority over the
insertion (by eagerness).

The case where $w, g \in L_{CA}$ is symmetrical. The only differences are the reasons
why some letters must be deleted. In the $L_{AC}$ case, when a letter $a \in A \cup A' \cup \{b_1, b'_1\}$
is inert and has some letter from $D' \cup D$ at its left, it must be deleted to permit the
deletion of the letters from $D \cup D' \setminus \{b_1, b'_1\}$. In the $L_{CA}$ case, the equivalent letter is in
$D \cup D' \setminus \{b'_1, b_1\}$ so is not present at the end.

□

E.3 Proof of Lemma 7.4

Let $\alpha, \beta \in (A' + b'_1)^+$ and $u, v \in (A + b_1)^*$.

Lemma E.5. If there is a derivation $\Box_1 . \alpha . u . g \Rightarrow \Box_1 . \beta . v . g$, then there exists some
$n$ and some words $u_1, v_1, u_2, \ldots, u_n, v_n, u_{n+1}$ such that, for all $i \leq n$, $u_i \in \Box_1 . (A' + b'_1)^+ . (A + b_1)^+$, $v_i \in \Box_2 . (D' \setminus \{b_1\})^+ . (C + b_2)^+$ and

\[
\begin{cases}
\alpha \Rightarrow u_i.g \Rightarrow_F \beta \Rightarrow u_{i+1}.g, \\
\Box_1 . \alpha . u . g \Rightarrow_F u_1.g, u_{n+1}.g \Rightarrow \Box_1 . \beta . v . g.
\end{cases}
\]

Proof. Let $\alpha, \beta \in (A' + b'_1)^+, u, v \in (A + b_1)^*$ and a derivation $\Box_1 . \alpha . u . g \Rightarrow \Box_1 . \beta . v . g$. By Lemma E.4, we know that there is a greedy derivation, say $w_0 \Rightarrow w_1 \ldots w_n$, such that
every $w_i$ is in $L_{AC} \cup L_{CA}$. We first note that if $w_i, g \in L_{AC}$ and $w_{i+1}.g \in L_{CA}$, then

\[
w_{i+1}.g \in \Box_2 . (D' \setminus b'_1)^+ . (D \setminus b_1)^+ . g.
\]

Note also that if $w_i, g \in \Box_2 . (D' \setminus b'_1)^+ . (D \setminus b_1)^+ . g$ and $w_{i+1}.g \not\in \Box_2 . (D' \setminus b'_1)^+ . (D \setminus b_1)^+ . (A + b_1).g$, then $w_{i+1}.g \in \Box_2 . (D' \setminus b'_1)^+ . (D \setminus b_1)^+ . (A + b_1).g$.

We will choose as $v_i$ all such $w_j, g \in \Box_2 . (D' \setminus b'_1)^+ . (D \setminus b_1)^+ . A$. Similarly for $u_i$, $w_k, g \in \Box_2 . (A' + b'_1)^+ . (A + b_1)^+ . g$ and $w_{k+1} . \Box_2 . (D' \setminus b'_1)^+ . (D \setminus b_1)^+ . (A + b_1).g$.

We now see that $\forall i, j_i \leq k_i$ and $k_i \leq j_{i+1}$. This is directly implied by the fact that
there is no way to go from $\Box_2 . (D' \setminus b'_1)^+ . (D \setminus b_1)^+ . (A + b_1)$ to $\Box_2 . (D' \setminus b'_1)^+ . (D \setminus b_1)^+ . A$ (included) to a step $\Box_1 . (A' + b'_1)^+ . (A + b_1)^+ \Rightarrow \Box_1 . (A' + b'_1)^+ . (A + b_1)^+ . (D \setminus b_1)$ (excluded) there are only letters from $\Box_1 . A' + A + b_1 + b'_1 + g$ active and $g$ only
inserts letters from \(A + b_1\). Those are rules from \(F_R\), i.e., \(v_i, g \Rightarrow F_R u_{i+1}.g\). Conversely \(u_i.g \Rightarrow F_G v_i.g\) by the same arguments.

To conclude, we need to show that in fact \(v_i \in \square_2.(C' + b'_2)^+.(C + b_2)^+\). There are no rule in \(F_R\) deleting letters from \(B \cup B' \setminus \{b_1, b'_1\}\) so if \(v_i.g \Rightarrow F_R u_{i+1}.g\), since there are no letter from \(B \cup B' \setminus \{b_1, b'_1\}\) in \(u_{i+1}.g\), then there were none in \(v_i.g\).

We can now prove Lemma 7.4: consider a derivation \(\square_1.\alpha.u \Rightarrow F_R \square_1.\beta.v\). With Lemma E.5, we can find \(u_i, v_i\) such that \(u_i \in \square_1.A'^+.A^+, v_i \in \square_2.D'^+.C^+.\) Then for all \(i \leq n\),

\[
u_i.g \Rightarrow F_G v_i.g \Rightarrow F_R u_{i+1}.g,
\]

and \(\square_1.\alpha.u.g \Rightarrow F_R u_1.g, u_{n+1}.g \Rightarrow F_R \square_1.\beta.v.g\).

Let us write \(u_i, v_i\) as \(u_i = \square_1.\alpha.u_i', \alpha \in A'^+, u_i' \in A^+\) and \(v_i = \square_2.\beta.v_i', \beta \in D'^+, v_i' \in C^+\).

This means that \(u_i S_{F_G} v_i\) and \(v_i S_{F_R} u_{i+1}\). Hence \(u_i' S_{G} v_i'\) and \(v_i' S_{R} u_{i+1}'\) using Lemma 7.3. With \(S_R = \approx \subseteq H\) and \(\square_1.S_R, \subseteq C = S_G\) we have \(u_i'(S_G.h) u_{i+1}'\). So \(u_i'(S_G.h)^{\alpha} u_{n+1}'\).

Thus \(u_1\) and \(u_{n+1}\) are in \(\square_1.A'^+.A^+\) and

\[
\square_1.\alpha.u \Rightarrow F_R \square_1.\alpha'.u_1'\quad\text{and}\quad\square_1.\alpha'.u_{n+1}' \Rightarrow F_R v.
\]

Since \(R\) is a simple transformer, it has no rules \(a \rightarrow b\) for \(a, b \in A\) so there are only insertion on \(A\) between \(u\) and \(u_1'\) and between \(u_{n+1}'\) and \(v\). Thus \(u \subseteq u_1'\) and \(u_{n+1}' \subseteq v\).

This concludes the proof that \(u \subseteq u_1'(S_G.h)^{\alpha} u_{n+1}' \subseteq v\).

### E.4 Proof of Lemma 7.5

Let be a greedy derivation \(\square_1.\dot{u} \Rightarrow F_R \square_2.\dot{v}\).

First note that \(\dot{u}\) is deleted before the insertion of a letter from \(D\) since every letter from \(A\) blocks \(D\).

There could be a derivation where a word from \(\square_1.A'^+.A^+.D.\dot{g}\) is reached. But since \(\square_1\) will be eventually deleted by \(B_2\), it is possible to find another greedy derivation where \(\square_1\) is inserted which inserts \(B_2\) and delete \(\square_1\) before the insertion of \(d\).

The rules used before the first insertion of a letter of \(D\) define a renaming from \(\dot{A}\) to \(A\). So there is \(u_1'\) such that \(u \subseteq u_1'(S_G.h)^{\alpha} v\).

### E.5 Finals steps of the construction: \(T_1\) and \(T_2\)

\(T_1\) is defined as \((A \cup A', \{\square_1, b_1, b'_1\}, \{\square_2, o_1\}, \bar{A} \cup \bar{A}' \cup \{b_1, b'_1, \square_1\}, \square_2, o_1, P_1, g)\) with the following set of rules:

- all \(l \rightarrow \bar{l}\), for \(l \in \{\square_1, b_1, b'_1\} \cup A \cup A'\),
- all \(g \rightarrow \dot{a}_i\), for \(a_i \in \bar{A}\),
- all \(a_i \rightarrow \dot{a}_j\), for \(a_i, a_j \in \bar{A}\),
- all \(a_i \rightarrow b_1\), for \(a_i \in \bar{A}\),
- all \(b_1 \rightarrow a'\), for \(a' \in \bar{A}' \cup \bar{b}'_1\),
- all \(a' \rightarrow \bar{b}\), for \(a', b' \in \bar{A}' \cup \bar{b}'_1\).
all $\vec{a}' \rightarrow \Box_1$, for $\vec{a}' \in \vec{A}' \cup \vec{B}'_1$,
$\Box_1 \rightarrow o_1$,
$\rightarrow o_2$.

We let the reader check that these rules ensure the satisfaction of $(T_1\text{-spec})$.

We further define $T_2 \overset{\text{def}}{=} (\vec{A} \cup \vec{A}' \cup \{\vec{b}_1, \vec{B}_1, \Box_1\}, \{o_1, o_2\}, \vec{A}, o_1, o_2, P_2, g)$ with the following set of rules:

all $g \rightarrow \vec{a}_i$, for $\vec{a}_i \in \vec{A}$,
all $\vec{a}_i \rightarrow \vec{a}_j$, for $\vec{a}_i, \vec{a}_j \in \vec{A}$,
all $\vec{a}_i \rightarrow o_2$, for $\vec{a}_i \in \vec{A}$,
all $\vec{a}_i \rightarrow o_1$, for $\vec{a}_i \in \vec{A}$,
all $\vec{a}_i \rightarrow l$, for $\vec{a}_i \in \vec{A}$ and $l \in \vec{A}' \cup \{\vec{b}_1, \vec{B}_1, \Box_1\}$,
$\rightarrow o_1$.

We let the reader check that these rules ensure the satisfaction of $(T_2\text{-spec})$. 