SIMULTANEOUS AVERAGING TO ZERO BY UNITARY MIXING OPERATORS

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text

1. Introduction

Let $A$ be a unital $C^*$-algebra. Let $U(A)$ denote its unitary group. We call a linear operator $T: A \to A$ a unitary mixing (or averaging) operator if it is a convex combination of inner automorphisms of $A$. That is,

$$T = \sum_{j=1}^{n} t_j \text{Ad}_{u_j},$$

where $u_j \in U(A)$ and $t_j \geq 0$ for $1 \leq j \leq n$, $\sum_{j=1}^{n} t_j = 1$, and where $\text{Ad}_{u}(x) = u x u^*$ for all $x \in A$. There is a large literature, going back to Dixmier’s approximation theorem for von Neumann algebras, around the averaging of elements of a $C^*$-algebra by means of unitary mixing operators. The question that we investigate here is that of simultaneously averaging a collection of elements of a $C^*$-algebra towards the zero element.

Let $S(A)$ denote the states space of $A$. Let $\text{Max}(A)$ denote the maximal (two-sided) ideals space of $A$. We prove the following theorem:

**Theorem 1.** Let $A$ be a unital $C^*$-algebra. Let $V \subseteq A$ be a linear subspace. The following are equivalent

(i) $V \subseteq [A, A]$ and for each maximal ideal $M \in \text{Max}(A)$ there exists $\rho_M \in S(A)$ that factors through $A/M$ and such that $V \subseteq \ker \rho_M$.

(ii) For each $v \in V$ there exists a sequence of unitary mixing operators $(T_n)_{n=1}^{\infty}$ such that $T_nv \to 0$.

(iii) There exists a net of unitary mixing operators $(T_\lambda)_{\lambda}$ such that $T_\lambda v \to 0$ for all $v \in V$.

It is not difficult to derive the equivalence of (i) and (ii) from [NRS18, Theorem 4.7]. Our main contribution here is that (i) implies (iii). Observe that in (i), the selection of states $M \mapsto \rho_M$ may be done only on the maximal ideals $M$ such that
$A/M$ has no bounded traces, for if $A/M$ has a trace, then we are guaranteed the existence of a $\rho_M$ vanishing on $V$ by the fact that $V \subseteq [A, A]$.

Recall that a C*-algebra is said to have the Dixmier property if for each $a \in A$ there exist unitary mixing operators $(T_n)_{n=1}^\infty$ such that $T_n a \to b \in Z(A)$, where $Z(A)$ denotes the center of $A$. The above mentioned approximation theorem of Dixmier states that every von Neumann algebra has the Dixmier property. This is not the case, however, for every C*-algebra (see [ART17]). As an immediate corollary of the previous theorem, we obtain the following:

**Corollary 2.** Let $A$ be a unital C*-algebra with the Dixmier property. Let $H: A \to Z(A)$ be a $Z(A)$-linear, positive, unital map such that

(a) $\tau \circ H = \tau$ for all bounded traces $\tau$ on $A$,
(b) $H(M) \subseteq M \cap Z(A)$ for all $M \in \text{Max}(A)$.

Then there exists a net of unitary mixing operators $(T_\lambda)_\lambda$ such that $T_\lambda \to H$ in the point-norm topology.

This result is possibly well known when $A$ is a von Neumann algebra, although we are not aware of a reference for this case. A related theorem of Magajna for weakly central C*-algebras replaces unitary mixing operators by C*-convex combinations ([Mag08, Corollary 1.2]). The special case of Corollary 2 when $A$ is simple and traceless is obtained by Zsido in [Zsi00]. We rely on Zsido’s result to prove Theorem 1 above.

2. Preliminaries on mixing operators and Dixmier sets

Let $A$ be a unital C*-algebra. Fix $n \in \mathbb{N}$. Consider the C*-algebra $A^n$, i.e., the direct sum of $n$ copies of $A$. (The norm in $A^n$ is the maximum norm: $\|a\| = \max \|a_j\|$.) We refer to the elements of $A^n$ as $n$-tuples. We regard $A$ embedded in $A^n$ as the constant $n$-tuples.

Let us denote by $\text{Mix}(A)$ the set of unitary mixing operators $T: A \to A$, as defined in the introduction. We consider unitary mixing operators in $\text{Mix}(A)$ acting on $n$-tuples coordinatewise: given $T \in \text{Mix}(A)$ and $a = (a_1, \ldots, a_n) \in A^n$, we write

$$T(a) = (T a_1, \ldots, T a_n).$$

Given an $n$-tuple $a = (a_1, \ldots, a_n)$ in $A^n$, let us define

$$D_A(a) = \{T a : T \in \text{Mix}(A)\},$$

which we call the Dixmier set generated by $a$ relative to $A$.

The next two lemmas are essentially obtained in [Arc77], but we establish them here in the form that will be needed later on.

**Lemma 3** (Cf. [Arc77, Proposition 2.4]). Each operator in $\text{Mix}(A)$ is a limit in the point-norm topology of a net of unitary mixing operators whose unitaries are exponentials, i.e., operators of the form

$$\sum_{j=1}^n t_j \text{Ad}_{e^{ih_j}},$$

(1)
where \( h_j \in A \) is selfadjoint for all \( j \).

**Proof.** It suffices to approximate \( \text{Ad}_u \), with \( u \in U(A) \), in the point-norm topology by unitary mixing operators of the form (1). Let \( \{x_1, \ldots, x_n\} \subseteq A \). By the Borel functional calculus in \( A^{**} \), there exists a selfadjoint \( h \in A^{**} \) such that \( u = e^{ih} \).

By Kaplansky’s density theorem, there exists a bounded net \( (h_\lambda) \) of selfadjoint elements in \( A \) such that \( h_\lambda \to h \) in the ultrastrong* topology. Set \( u_\lambda = e^{ih_\lambda} \). Then \( u_\lambda x_i u_\lambda^* \to ux_iu^* \) in the ultrastrong* topology, and thus also in the \( \sigma(A^{**}, A^*) \) topology for all \( i \). Thus, \( u_\lambda x u_\lambda^* \to xu u^* \) in the \( \sigma((A')^{**}, (A^*)^*) \)-topology, where we have set \( x = (x_1, \ldots, x_n) \). It follows, by a standard application of the Hahn-Banach Theorem, that \( xu u^* \) belongs to the norm closure of \( co\{u_\lambda x u_\lambda^*: \lambda \} \). This yields, for each \( \varepsilon > 0 \), an operator \( T \in \text{Mix}(A) \) of the form (1) and such that \( \|T(x) - xu u^*\| < \varepsilon \), as desired.

Recall that \( A \) is a said to have the Dixmier property if \( D_A(a) \cap Z(A) \neq \emptyset \) for all \( a \in A \), where \( Z(A) \) denotes the center of \( A \).

**Lemma 4.** Let \( A \) be a \( C^* \)-algebra with the Dixmier property. Let \( I \) be a closed two-sided ideal of \( A \). Let \( a \in A^n \). Then \( D_A(a) \cap Z(A)^n \) is mapped densely onto \( D_{A/I}(\pi(a)) \cap Z(A/I)^n \) by the quotient map \( \pi: A \to A/I \).

**Proof.** We follow arguments from [Arc77] adapted to \( n \)-tuples.

Let \( z \in D_{A/I}(\pi(a)) \cap Z(A/I)^n \). Let \( \varepsilon > 0 \). Then \( \|\tilde{T}a - z\| < \varepsilon \) for some \( \tilde{T} \in \text{Mix}(A/I) \). Moreover, by the previous lemma, we may choose \( \tilde{T} \) of the form (1). Since the unitaries in \( \tilde{T} \) are exponentials, they lift to unitaries in \( A \). In this way we get \( T \in \text{Mix}(A) \) such that \( \pi T = \tilde{T} \), where \( \pi: A \to A/I \) denotes the quotient map. Set \( b = T(a) \). Then \( \|\pi(b) - z\| < \varepsilon \). By a process of successive averagings by unitary mixing operators, we can find \( T_n \in \text{Mix}(A) \) such that \( T_n b \to z \in D_A(b) \cap Z(A)^n \).

This is [KR97, Lemma 8.3.4], stated for the von Neumann algebra case, but the proof applies without change to any \( C^* \)-algebra with the Dixmier property. Then \( z \in D_A(a) \) and \( \|\pi(z) - z\| < \varepsilon \), as desired.

**Lemma 5.** Let \( A \) be a von Neumann algebra. Then \( D_A(a) \) has the following central convexity property: \( zb + (1 - z)c \in D_A(a) \) for all \( b, c \in D_A(a) \) and \( 0 \leq z \leq 1 \) in \( Z(A) \).

Note: This lemma is true in any \( C^* \)-algebra, but we only prove here the von Neumann algebra case, as it is all that will be needed later on.

**Proof.** For \( n = 1 \) and a selfadjoint, this is [NRS18, Lemma 3.4]. The same proof holds in this case with the obvious modifications.

Let \( 0 \leq z \leq 1 \) be a central. Using Borel functional calculus, we can write \( z \) as a norm limit of elements of the form \( \sum_{k=1}^N t_k e_k \), where the \( (e_k)_{k=1}^N \) are central pairwise orthogonal projections adding up to 1, and \( t_k \in [0, 1] \) for all \( k \). It thus suffices to assume that \( z \) is exactly of this form.

We have \( b \in D_A(a) \) if and only \( e_k b \in D_{e_k A}(e_k a) \) for all \( k \). This holds since unitary mixing operators on \( A \cong \bigoplus_{k=1}^N e_k A \) are of the form \( \bigoplus_{k=1}^N T_k \), with \( T_k \in \text{Mix}(e_k A) \) for all \( k \). Since \( e_k z \) is a scalar multiple of \( e_k \) (the unit of \( e_k A \)), cutting down by each
$e_k$ reduces the proof to the case that $z$ is a scalar. In this case, the result is simply the convexity of $D_A(a)$.

In the next section we make essential use of the Zsido's Approximation Lemma from [Zsi00]:

**Lemma 6.** Let $A$ be a simple C*-algebra without bounded traces. Let $\rho \in S(A)$. Then the operator $A \ni a \mapsto \rho(a)1 \in A$ is in the point-norm closure of $\text{Mix}(A)$.

### 3. Proofs of Theorem 1 and Corollary 2

Let us introduce some notation: We denote by $T(A)$ the set of tracial states of $A$. Given an ideal $I \subseteq A$, we denote by $S(A)_I$ the states on $A$ that vanish on $I$, i.e., that factor through $A/I$. Given a state $\rho \in S(A)$ and $a \in A^n$ we evaluate $\rho$ on $a$ coordinatewise:

$$\rho(a) = (\rho(a_1), \ldots, \rho(a_n)) \in \mathbb{C}^n.$$ 

We regard $\mathbb{C}^n$ endowed with the maximum norm. In this way, $A^n \ni a \mapsto \rho(a) \in \mathbb{C}^n$ has norm 1.

We shall deduce the results stated in the introduction from the following result, of independent interest:

**Theorem 7.** Let $a_1, \ldots, a_m \in A^n$. Let $r \geq 0$. The following are equivalent:

(i) $\inf \{\|\sum_{i=1}^m T_i(a_i)\| : T_i \in \text{Mix}(A)\} \leq r$.

(ii) The following conditions hold:

(a) $\|\sum_{i=1}^m \tau(a_i)\| \leq r$ for all $\tau \in T(A)$,

(b) for each ideal $M \in \text{Max}(A)$ there exist states $\rho_1, \ldots, \rho_m \in S(A)_M$ such that $\|\sum_{i=1}^m \rho_i(a_i)\| \leq r$.

**Proof of (i) $\Rightarrow$ (ii).** Let $\tau \in T(A)$. Since $\tau \circ T = \tau$ for all $T \in \text{Mix}(A)$, it follows that

$$\sum_{i=1}^m \tau(a_i) = \tau(\sum_{i=1}^n T_i(a_i))$$

for all $T_i \in \text{Mix}(A)$. Since $\tau$ is a state, it is clear that (i) implies that the left hand side of the equation above has norm at most $r$. (Recall that we’ve endowed $\mathbb{C}^n$ with the maximum norm.) This proves (a).

Let $M \in \text{Max}(A)$ and $\rho \in S(A)_M$. Then $\rho \circ T \in S(A)_M$ for all $T \in \text{Mix}(A)$. Also,

$$\sum_{i=1}^m (\rho \circ T_i)(a_i) = \rho(\sum_{i=1}^n T_i(a_i)).$$

Passing to the infimum over all $T_1, \ldots, T_n \in \text{Mix}(A)$ and using (i), we deduce that

$$\inf \{\|\sum_{i=1}^m \rho_i(a_i)\| : \rho_i \in S(A)_M\} \leq r.$$ 

By the compactness of $S(A)_M$ in the weak* topology, this infimum is attained. This proves (b).
Before proving \((ii) \Rightarrow (i)\) of Theorem 7, we use a standard Hahn-Banach/Kaplansky density argument to reduce the proof to the von Neumann algebra case.

**Lemma 8.** Let \(a_1, \ldots, a_m \in A^n\). Regard \(A\) as a \(C^*\)-subalgebra of its bidual \(A^{**}\). Then

\[
\inf \{ \| \sum_{i=1}^{m} T_i(a_i) \| : T_i \in \text{Mix}(A) \} = \inf \{ \| \sum_{i=1}^{m} T_i(a_i) \| : T_i \in \text{Mix}(A^{**}) \}.
\]

**Proof.** Clearly, the right side is dominated by the left side. Let \(r\) denote the number on the left side. Let \(\varepsilon > 0\). Suppose that \(b = \sum_{i=1}^{m} b_i\) is such that \(\|b\| < r + \varepsilon\), with

\[
b_i = \sum_{k=1}^{n_i} t_{i,k} u_{i,k} a_i u_{i,k}^*.
\]

where the sum is a convex combination, and where \(u_{i,k} \in U(A^{**})\) for all \(i, k\). By Kaplansky’s density theorem for unitaries ([GK60, Theorem 2]), there exist (commonly indexed) nets of unitaries \((u_{i,k,\lambda}) \lambda \in U(A)\) such that \(u_{i,k,\lambda} \to u_{i,k}\) in the ultrastrong* topology for \(i = 1, \ldots, m\). For each \(i = 1, \ldots, m\) and \(\lambda\), set

\[
b_{i,\lambda} = \sum_{k=1}^{n_i} t_{i,k} u_{i,k,\lambda} a_i u_{i,k,\lambda}^* \in D_A(a_i).
\]

Then \(b_{i,\lambda} \to b_i\) coordinatewise in the weak* topology \(\sigma(A^{**}, A)\), equivalently, in the \(\sigma((A^n)^{**}, (A^n)^*)\) topology. Consider the set

\[
S = \text{co}\{ \sum_{k=1}^{m} b_{i,\lambda} : \lambda \}.
\]

This is a convex subset of \(A^n\) whose \(\sigma((A^n)^{**}, (A^n)^*)\) closure in \((A^{**})^n\) contains \(b\). It follows that \(S\) must intersect the ball \(\{ x \in A^n : \| x \| < r + \varepsilon \}\). For suppose that this is not the case. Then, by the Hahn-Banach theorem, there exists \(\rho \in (A^n)^*\) such that \(\text{Re}(\rho, x) < r + \varepsilon\) for all \(\| x \| < r + \varepsilon\) while \(\text{Re}(\rho, y) > r + \varepsilon\) for all \(y \in S\). The first inequality implies that \(\|\rho\| \leq 1\) and the second one that \(\text{Re}(\rho, b) \geq r + \varepsilon\). This contradicts that \(\| b \| < r + \varepsilon\). Thus, there exists a convex combination of sums of the form \(\sum_{k=1}^{m} b_{i,\lambda}\) with norm \(< r + \varepsilon\). This yields an element of norm \(< r + \varepsilon\) and of the desired form. \(\Box\)

**Proof of Theorem 7 (ii) \Rightarrow (i).** The proof proceeds in stages: We first obtain the result in the case that \(A\) is a simple \(C^*\)-algebra without bounded traces. Next, we deal with the case that \(A\) is a von Neumann algebra, which we break-up into the finite and the properly infinite case. Finally, relying on the previous lemma, we deal with the general case.

\(A\) is simple and traceless. Let \(a_1, \ldots, a_m \in A^n\). Let \(\varepsilon > 0\). By assumption, there exist \(\rho_1, \ldots, \rho_m \in S(A)\) such that

\[
\| \sum_{i=1}^{m} \rho_i(a_i) \| \leq r.
\]
By Lemma 6, for each \( i \) the map \( A \ni a \mapsto \rho_i(a)1 \in A \) is a point-norm limit of unitary mixing operators. Then, letting \( T_{i,\lambda} \in \text{Mix}(A) \) be such that \( \lim_{\lambda} T_{i,\lambda} = \rho_i1 \), we deduce at once that

\[
\left\| \sum_{i=1}^{m} T_{i,\lambda}(a_i) \right\| < r + \varepsilon,
\]

for some \( \lambda \). This, (ii) implies (i) in this case.

\( A \) is a properly infinite von Neumann algebra. Let \( a_1, \ldots, a_m \in A^n \) be \( n \)-tuples satisfying condition (b) of Theorem 7 for a given \( r \geq 0 \). Let \( \varepsilon > 0 \). Let \( M \) be a maximal ideal of \( A \). Passing to the quotient \( A/M \), which is simple and traceless, we know, as established in the previous paragraph, that

\[
\left( \sum_{i=1}^{m} D_{A/M}(\pi_M(a_i)) \right) \cap Z(A/M)^n
\]

contains an element of norm less than \( r + \varepsilon \) (where \( \pi_M: A \to A/M \) denotes the quotient map). Since \( D_A(a_i) \cap Z(A)^n \) maps densely onto \( D_{A/M}(\pi_M(a_i)) \cap Z(A/M) \) (Lemma 4), there exist \( z_i \in D_A(a_i) \cap Z(A)^n \) such that

\[
\left\| \sum_{i=1}^{m} \pi_M(z_i) \right\| < r + \varepsilon.
\]

Recall that von Neumann algebras are weakly central, i.e., \( \text{Max}(A) \) is homeomorphic to the spectrum of \( Z(A) \) via the map \( \text{Max}(A) \ni M \mapsto M \cap Z(A) \in \widehat{Z(A)} \). Identifying in this way \( \text{Max}(A) \) with \( \widehat{Z(A)} \), we can rephrase the inequality above as

\[
\left\| \sum_{i=1}^{m} \hat{\pi}_M(z_i) \right\| < r + \varepsilon,
\]

where \( \hat{\pi}: \widehat{Z(A)} \to \mathbb{C} \) denotes the Gelfand transform of \( z \in Z(A) \). This inequality is valid in neighborhood \( U_M \) of \( M \):

\[
\left\| \sum_{i=1}^{m} \hat{\pi}_M(z_i(M')) \right\| < r + \varepsilon,
\]

for all \( M' \in U_M \). Since the spectrum of \( Z(A) \) is totally disconnected, we can refine the cover \( (U_M)_M \) to a finite partition of \( \widehat{Z(A)} \) by clopen sets. We thus obtain central projections \( (e_k)_{k=1}^{N} \), corresponding to these clopen sets, such that \( \sum_{k=1}^{N} e_k = 1 \), and for each \( k = 1, \ldots, N \) there exists \( z_{i,k} \in D_A(a_i) \cap Z(A)^n \) such that

\[
(2) \quad \left\| e_k \sum_{i=1}^{m} z_{i,k} \right\| < r + \varepsilon.
\]

Now let

\[
z_i = e_1 z_{i,1} + \ldots + e_N z_{i,N} \text{ for } i = 1, \ldots, m.
\]
By Lemma 5, $z_i \in D_A(a_i) \cap Z(A)^n$. Further,
\[
\| \sum_{i=1}^m z_i \| = \max_{1 \leq k \leq N} \| e_k \sum_{i=1}^m z_{i,k} \| < r + \varepsilon.
\]
This proves that (ii) implies (i) in the case that $A$ is a properly infinite von Neumann algebra.

*Case of an arbitrary $C^*$-algebra*. Let us argue that if $A$ is unitally embedded in some $C^*$-algebra $B$, then conditions (a) and (b) are verified relative to $B$, as well. Since every bounded trace on $B$ restricts to a trace on $A$, we immediately deduce that (a) holds in $B$. Let $M \subseteq B$ be a maximal ideal of $B$. By condition (b) applied in $A$, there exist states $\rho_1, \ldots, \rho_m \in S(A)$ that vanish on $A \cap M$ and such that
\[
\| \sum_{i=1}^m \rho_i(a_i) \| \leq r.
\]
By the Hahn-Banach theorem, the state induced by $\rho_i$ on $A/(A \cap M)$ extends to a state on $B/M$. We thus obtain states $\tilde{\rho}_1, \ldots, \tilde{\rho}_m \in S(B)_M$ extending $\rho_1, \ldots, \rho_m$. This yields condition (b) relative to $B$. Let us now regard $A$ as a $C^*$-subalgebra of $A^{**}$. Since the latter is a von Neumann algebra, we already know that (ii) implies (i) in this case. Applying Lemma 8, we deduce (i).
Let us now prove the results stated in the introduction:

**Proof of Theorem 1.** Let us prove that (i) implies (iii). Let $F = \{a_1, \ldots, a_n\}$ be a finite subset of $V$, and let $\varepsilon > 0$. Set $a = (a_1, \ldots, a_n)$. Then $a$ satisfies conditions (a) and (b) of Theorem 7 (ii), with $m = 1$ and $r = 0$. We conclude that $0 \in D_A(a)$. Thus, there exists $T_{F, \varepsilon} \in \text{Mix}(A)$ such that $\|T_{F, \varepsilon} a_i\| < \varepsilon$ for all $i$. The net of operators $(F, \varepsilon) \mapsto T_{F, \varepsilon}$ is then as desired.

It is clear that (iii) implies (ii).

Let us prove that (ii) implies (i). Let $V$ be as in (ii). Observe first that $a - Ta \in [A, A]$ for any $T \in \text{Mix}(A)$. Hence, if $T_n a \to 0$ for some sequence $T_n \in \text{Mix}(A)$, then $a \in [A, A]$. This shows that $V \subseteq [A, A]$.

Let $M$ be a maximal ideal. Let $F = \{a_1, \ldots, a_n\}$ be a finite subset of $V$. Suppose, for the sake of contradiction, that $0 \notin \{\rho(a) : \rho \in S(A)_M\}$. Then, by the Hahn-Banach Theorem applied to the compact convex set $\{\rho(a) : \rho \in S(A)_M\}$ and $0 \in \mathbb{C}^n$, there exist $\alpha_1, \ldots, \alpha_n \in \mathbb{C}^n$ and $c > 0$ such that

$$\text{Re}(\sum_{i=1}^n \alpha_i \beta_i) \geq c$$

for all $(\beta_1, \ldots, \beta_n) \in \{\rho(a) : \rho \in S(A)_M\}$.

Let $a = \sum_{i=1}^n \alpha_i a_i$. Then, $0 \notin \{\rho(a) : \rho \in S(A)_M\}$, which in turn implies that $0 \notin D_A(a)$, by the equivalence of (i) and (ii) for the case $n = 1$ ([NRS18] Theorem 4.7]). This contradicts (ii). We have shown that for any finite set $F \subseteq V$ the set $\{\rho \in S(A)_M : \rho(F) = \{0\}\}$ is non-empty. By the compactness of $S(A)_M$ in the weak* topology, we obtain $\rho \in S(A)_M$ such that $\rho(V) = \{0\}$, as desired.

**Remark 9.** If every quotient of $A$ has a bounded trace, then Theorem 1 (i) simply asserts that $V \subseteq [A, A]$. In this case, it is not difficult to go from (i) to (ii) to (iii) by a process of successive averagings towards zero of a given collection of elements: starting with $a_1, a_2, \ldots, a_n$ in $[A, A]$, we argue by the equivalence of (i) and (ii) that there exists $T_1 \in \text{Mix}(A)$ such that $\|T_1 a_1\| < 1/2$. Since $T_1 a_2 \in [A, A]$, we can choose $T_2 \in \text{Mix}(A)$ such that $\|T_2 T_1 a_2\| < \frac{1}{2^2}$, etc. This simple strategy breaks down when the C*-algebra $A$ has traceless quotients, since after the first step there is no guarantee that $0 \in D_A(T_1 a_2)$.

**Proof of Corollary 2.** Let $V = \{a - H(a) : a \in A\}$. By the assumption that $H$ preserves traces, $\tau|_V = 0$ for any bounded trace $\tau$. Since $[A, A] = \bigcap_{\tau \in T(A)} \ker \tau$ (see [CP79]), it follows that $V \subseteq [A, A]$.

Let $M$ be a maximal ideal and denote by $\pi_M : A \to A/M$ the quotient map. Since $A/M$ is simple and unital, its center is isomorphic to $\mathbb{C}$. We may thus regard the restriction of $\pi_M$ to $Z(A)$ as a homomorphism onto $\mathbb{C}$. Set $\rho_M = \pi_M \circ H$. Then $\rho_M$ is a state factoring through $A/M$ and such that $\rho_M(a - H(a)) = 0$ for all $a \in A$, i.e., $V \subseteq \ker \rho_M$. It follows by Theorem 1 that there exists a net of unitary mixing operators $(T_\lambda)_\lambda$ such that $T_\lambda(a - H(a)) \to 0$ for all $a \in A$. Since the $T_\lambda$s fix the center, we obtain that $T_\lambda a \to Ha$ for all $a \in A$, as desired. \qed
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