It is argued that an electrically charged dilaton black hole can support a long range field of a Nielsen-Olesen string. Combining both numerical and perturbative techniques we examine the properties of an Abelian-Higgs vortex in the presence of the black hole under consideration. Allowing the black hole to approach extremality we found that all fields of the vortex are expelled from the extreme black hole. In the \textit{thin string} limit we obtained the metric of a \textit{conical} electrically charged dilaton black hole. The effect of the vortex can be measured from infinity justifying its characterization as black hole hair.
I. INTRODUCTION

The no hair conjecture of black holes was attributed to Wheeler, who motivated by the earlier researches on the uniqueness theorems for black holes (see for a historical account, e.g., [1]) stated that all exteriors of black hole solutions are characterized by at most three conserved parameters: mass, angular momentum and electric charge. However, nowadays we are faced with the discoveries of black hole solutions in many theories in which Einstein’s equations are coupled with self-interacting matter fields. The surprising discovery of Bartnik and McKinnon [2] of a nontrivial particle like structure in Einstein-Yang-Mills systems opens new realms of nontrivial solutions to Einstein-Abelian-gauge systems. Black holes can be colored [2] - [5], support a long-range Yang-Mills hair, but these solutions are unstable [6] - [7]. Nevertheless they exist.

Mavromatos stated [8] that, the balance between the non-Abelian gauge field repulsion and the gravitational attraction allowed for dressing black hole solutions by a non-trivial outside horizon configurations. It is true not only for non-Abelian gauge fields, obeying the Gauss law constraint, but also for scalar fields which are not bound with the Gauss law. In Ref. [9] it was analytically proved the existence of genuine hairy black hole solutions to the coupled Einstein-Yang-Mills Eqns. for an $su(N)$ gauge field, for every $N$. By a genuine $su(N)$ black hole, the authors mean a solution which is not simply the result of embedding a smaller group in $su(N)$.

The other sort of the problems is the extension of the no hair conjecture when the topologies of some field configurations are not trivial. One asks if the topological defects can act as hair for black holes. Aryal et al. [10] derived the metric which described a Schwarzschild black hole with a cosmic string passing through it. The extension of this problem was to show [11] the existence of the Euclidean Einstein equations corresponding to a vortex sitting on the horizon of a black hole. The vortex cut out a slice out of the Schwarzschild geometry. Achucarro et al., in Ref. [12], presented the numerical and analytical evidence that the Abelian Higgs vortex is not just dressing for the Schwarzschild black hole, but the vortex is truly hair, property of which can be detected by asymptotic observers. They argued, that the Vilenkin’s solution [10] is the thin vortex limit of the physical vortex- black hole configuration. Chamblin et al. [13] considered the problem of an Abelian Higgs vortex in a Reissner-Nordström background. They found that in the extreme limit all of the fields connected with vortex are expelled from the black hole. They showed numerically the evidence that the magnetic field wrapped around the horizon.

Superstring theories are taken into account as consistent quantum theories of gravity. Numerical studies of the solutions to the low-energy string theory, i.e., the Einstein-dilaton black holes in the presence of a Gauss-Bonnet type term, revealed that they were endowed with a nontrivial dilaton hair [14]. This dilaton hair is expressed in term of its ADM mass [15]. The extended moduli and dilaton hair and their associated axions for a Kerr-Newmann black hole background were computed in Ref. [16]. The problem of an Abelian Higgs vortex in the background of an Euclidean electrically charged black hole was studied in Ref. [17]. It was shown that this kind of the Euclidean black hole can support a vortex solution on a horizon of the black hole. The vortex effect was to cut out a slice out of the considered black hole geometry.

In this paper, we shall try to provide some continuity with our previous work [17]. Now, we shall consider the problem of an Abelian Higgs vortex in an electrically charged dilaton black hole background. In our considerations
we allow for the presence of two $U(1)$ gauge fields, one is originated from the low-energy string action, the other is the symmetry spontaneously broken in the ground state. We shall also be interested in finding the metric for the dilaton black hole-vortex configuration, in the thin string limit.

The outline of the paper is as follows. Section II is devoted to the general analytic considerations of the Nielsen-Olesen vortex in the presence of an electrically charged dilaton black hole. In Sec.III, we presented the numerical analysis of the problem. In Sec.IV, using the iterative method of solving equations and assuming the thin string limit, we find the metric for an electrically charged dilaton black hole with a cosmic string passing through it. In Sec.V we finish with general conclusions concerning our investigations.

II. THE NIELSEN-OLESEN VORTEX IN THE PRESENCE OF AN ELECTRICALLY CHARGED DILATON BLACK HOLE

In this section we shall study the Nielsen-Olesen Eqns. for an Abelian Higgs vortex in the presence of an electrically charged dilaton black hole background. In our considerations, we assume the black hole-string vortex configuration which involves the separation between the degrees of freedom of each of the considered objects. Namely, the system is described by the action

$$S = S_1 + S_2,$$

where $S_1$ is the low-energy string action, see e.g., [8]

$$S_1 = \int \sqrt{-g} d^4x \left[ \frac{R}{16\pi G} - 2(\nabla \phi)^2 - e^{-2\phi} F^2 \right],$$

where $F_{\alpha \beta} = 2\nabla_{[\alpha} A_{\beta]}$, $\phi$ is the dilaton field and $S_2$ is the action for an Abelian Higgs system minimally coupled to gravity and be subject to spontaneous symmetry breaking. It has the form

$$S_2 = \int \sqrt{-g} d^4x \left[ - D_\mu \Phi^\dagger D^\mu \Phi - \frac{1}{4} \hat{B}_{\mu \nu} \hat{B}^{\mu \nu} - \frac{\lambda}{4} (\Phi^4 - \eta^2)^2 \right],$$

where $\Phi$ is a complex scalar field, $D_\mu = \nabla_\mu + i e B_\mu$ is the gauge covariant derivative. $\hat{B}_{\mu \nu}$ is the field strength associated with $B_\mu$. One can define the real fields $X, P_\mu, \chi$ by the relations [12]

$$\Phi(x^\alpha) = \eta X(x^\alpha) e^{i\chi(x^\gamma)},$$

$$P_\phi(x^\alpha) = \frac{1}{e} [P_\mu(x^\alpha) - \nabla_\mu \chi(x^\alpha)].$$

Thus, the equations of motion derived from the action $S_2$ are as follows:

$$\nabla_\mu \nabla^\mu X - P_\mu P^\mu X - \frac{\lambda \eta^2}{2} (X^2 - 1) X = 0,$$

$$\nabla_\mu \hat{B}^{\mu \nu} - 2 e^2 \eta^2 X^2 P^\nu = 0.$$  

It turns out, that the field $\chi$ is not dynamical. The vortices of the Nielsen-Olesen type, in the flat spacetime, have the cylindrically symmetric solutions of the forms

$$\Phi = X(\tilde{\rho}) e^{iN\phi},$$

$$P_\phi = NP(\tilde{\rho}),$$

$$P_\mu = NP_\mu(\tilde{\rho}).$$

(8)

(9)
where \( \hat{r} \) is the cylinder radial coordinate, \( N \) is the winding number.

On the other hand, static, spherically symmetric solution of the Eqs. of motion derived from the action \( S_1 \) determines the metric of an electrically charged dilaton black hole. It is given by

\[
ds^2 = - \left(1 - \frac{2GM}{\hat{r}}\right)dt^2 + \frac{d\hat{r}^2}{\left(1 - \frac{2GM}{\hat{r}}\right)} + \hat{r} \left(\hat{r} - \frac{Q^2}{GM}\right) (d\theta^2 + \sin^2 \theta d\phi^2).
\]

(10)

One can define \( \tilde{r}_+ = 2GM \) and \( \tilde{r}_- = \frac{Q^2}{GM} \) which are related to the mass \( M \) and charge \( Q \) by the relation \( Q^2 = \frac{\tilde{r}_+ - \tilde{r}_-}{\tilde{r}_+} e^{2\phi_0} \).

The charge of the dilaton black hole \( Q \) couples to the field \( F_{\alpha\beta} \), is unrelated to the Abelian gauge field \( B_{\mu\nu} \) associated with the vortex. The dilaton field is given by \( e^{2\phi} = \left(1 - \frac{\tilde{r}}{\tilde{r}_-}\right) e^{-2\phi_0} \), where \( \phi_0 \) is the dilaton’s value at \( r \rightarrow \infty \). The event horizon is located at \( \tilde{r} = \tilde{r}_+ \). For \( \tilde{r} = \tilde{r}_- \) is another singularity, one can however ignore it because \( \tilde{r}_- < \tilde{r}_+ \).

The extremal black hole occurs when \( \tilde{r}_- = \tilde{r}_+ \), when \( Q^2 = 2M^2 e^{2\phi_0} \).

It is convenient to work with the rescaled radial coordinates and the black hole parameters by the Higgs wavelength, i.e., with the non-dimensional variables

\[
(r, E, q) = \sqrt{\lambda} \eta (\tilde{r}, GM, Q),
\]

(11)

where \( \eta \) parameter is the energy scale of symmetry breaking, \( \lambda \) is the Higgs coupling. These parameters can be related to the Higgs mass \( m_{Higgs} = \sqrt{\lambda} \eta \) and the mass of the vector field in the broken phase \( m_{vect} = \sqrt{2} e \eta \). Returning to the Eqs. of motion, we can consistently assume that

\[
X = X(r, \theta),
\]

(12)

\[
P_\phi = NP(r, \theta).
\]

(13)

Then, Eqs. of motion yield

\[
\frac{1}{r(r - \frac{\tilde{r}}{E})} \frac{\partial_r}{(r - \frac{\tilde{r}}{E})^2} (r - 2E) \partial_r X + \frac{1}{r(r - \frac{\tilde{r}}{E})} \partial_\theta [\sin \theta \partial_\theta X] - \frac{NP^2 X}{r(r - \frac{\tilde{r}}{E})^2} \sin \theta \partial_\theta P = 0,
\]

(14)

\[
= 0,
\]

(15)

where \( \beta = \frac{\lambda}{\sqrt{2}} = m_{Higgs}^2/m_{vect}^2 \) is the Bogomolny parameter. When \( \beta \rightarrow \infty \), the Higgs field decouples and like in the Reissner-Nordström case (13), one can study a free Maxwell field in the electrically charged black hole spacetime. The other situation will arise when \( P = 1 \). It will be the case of a global string in the presence of the electrically charged dilaton black hole.

Noting as in Refs. (13) and (20), that in normal spherically symmetric coordinates \( X \) is a function of \( \sqrt{\beta \tilde{r}} \), we shall try with the coordinates \( R = \sqrt{r(r - \frac{\tilde{r}}{E})} \sin \theta \), namely \( X = X(R) \) and \( P_\phi = P_\phi(R) \). Using the \( r \)-coordinates and taking into account the thin string limit, i.e., \( E \gg 1 \), Eqs. (14) and (15) can be rewritten

\[
\frac{(r + O(\frac{1}{E}))}{(r^2 + O(\frac{1}{E}))} \sin^2 \theta \frac{d}{dR} \left[ R(r - 2E) + O(\frac{1}{E}) \right] \frac{d}{dR} X = \frac{1}{2} (X - \frac{2E}{r} \sin^2 \theta) \left[ X'(R) + X''(R) \right] = 0,
\]

(16)

\[
\frac{1}{r(r + O(\frac{1}{E}))} \frac{d}{dR} \left[ (1 - \frac{2E}{r}) \sin^2 \theta (r + O(\frac{1}{E})) \right] P = \frac{1}{2} (X^2 - 1) X = 0,
\]

(17)

\[
= \frac{P''(R)}{R} - \frac{X^2 P'}{\beta} + \frac{2E \sin^2 \theta}{r} \left[ \frac{P'}{R} - \frac{P''}{R} \right] = 0,
\]
where $\dot{}$ is the derivative $\frac{d}{dt}$. The above Eqns. for $X$ and $P$ are of the Nielsen-Olesen type, up to the errors of the order $\frac{2E \sin^2 \theta}{r}$ multiplied by the other terms in the adequate Eqns. Since $R = \sqrt{r(r - \frac{q^2}{E})} \sin \theta \sim \mathcal{O}(1)$ in the core of the string, $\sin \theta = \mathcal{O}\left(\frac{1}{r}\right)$, the errors are of the order of $\mathcal{O}(\frac{E}{r}) < \mathcal{O}(\frac{1}{r^2}) \ll 1$. Hence, the Nielsen-Olesen vortex can sit on the electrically charged dilaton black hole. Of course, the above considerations have only approximate forms and do not prove the existence of a solution to the problem. Nevertheless, they substitute a good approximation.

The Eqns. of motion are rather intractable in the exact form, so one can use an approximate method of solving them. Now, we try to describe an analytic solution to these Eqns. for the black hole size which is small comparing to the vortex size. One will require the limit of $\sqrt{N} \gg E$, using the radius of a flux tube of the order $r \sim \sqrt{2N} \beta^+$ for $N \gg 1$. The large $N$-limit was first considered in Ref. [21].

Inside a core of the vortex the gauge symmetry will be unbroken, then one can expect that $\frac{N^2}{\beta} \approx 0$. Taking into consideration Eqn. (15), we can see that the solution is provided by

$$P \approx 1 - pR \sin^2 \theta, \quad (18)$$

where $p$ is an integration constant equal to twice the magnetic field strength at the center of the core [12]. Using the analysis performed in Ref. [21], one can show that far from the electrically charged dilaton black hole but still inside the vortex $p \approx \frac{1}{2N \sqrt{\beta}}$. Then, large $N$ means small $p$. Following the method used in [12] and [21], we set $X = \xi N$ and try to solve Eqns. for the Higgs fields $X$, expanding in powers of $\frac{1}{N}$ and taking the limit $\mathcal{O}\left(\frac{1}{N^2}\right)$. Then, one has

$$\left(1 - \frac{2E}{r}\right) \left(\frac{\partial_r \xi}{\xi}\right)^2 + \frac{1}{r(r - \frac{q^2}{E})} \left(\frac{\partial_r \xi}{\xi}\right)^2 - \frac{P^2}{r(r - \frac{q^2}{E}) \sin^2 \theta} + \mathcal{O}\left(\frac{1}{N^2}\right) = 0. \quad (19)$$

Assuming that, $\xi = b(r) \sin \theta$, one gets the following Eqn. for $b(r)$:

$$\frac{1}{b} \frac{db}{dr} = \frac{1 - pr(r - \frac{q^2}{E})}{\sqrt{(r - 2E)(r - \frac{q^2}{E})}}, \quad (20)$$

which is solved by the following expression:

$$b(r) = k \left[r - \left(E + \frac{q^2}{E}\right) + \sqrt{(r - 2E)(r - \frac{q^2}{E})}\right]^{1 - \frac{2}{E}\left(6E - \frac{q^2}{E}\right)(2E + \frac{q^2}{E}) +pq^2} \left[\frac{6E - \frac{q^2}{E}}{2E + \frac{q^2}{E}}\right] \left[r + \frac{1}{2}\left(6E - \frac{q^2}{E}\right)\right]^{pq^2}$$

$$e^{\left[\frac{6E - \frac{q^2}{E}}{2E + \frac{q^2}{E}}\right] \left[\frac{6E - \frac{q^2}{E}}{2E + \frac{q^2}{E}}\right]^{pq^2}}, \quad (21)$$

where $k$ is an integration constant. We can now establish the following:

$$X \approx b^N(r) \sin^N \theta. \quad (22)$$

Eqns. (18) and (21) consist the approximate solutions for the case of an electrically charged dilaton black hole that sits inside the vortex core. The thickness of the vortex, i.e., the distance at which $P \approx 0$, is roughly defined as

$$r \sin \theta = \frac{q^2}{2E} + \frac{1}{\sqrt{p}} \left(1 + \frac{pq^4}{8E^2}\right). \quad (23)$$

Comparing this result to the outcome obtained in the case of the Abelian Higgs vortex in the Schwarzschild background [12], one can see that the vortex is thicker on the equatorial region when we consider the electrically charged dilaton.
Following the notation from Ref. [12], the finite difference scheme for Eqs. (14) and (15) has the form

\[ \tilde{B}_{\theta \phi} |_{r_+} = -p r_+ \left( r_+ - \frac{q^2}{E} \right) \sin^2 \theta, \]  

(24)

we see that it decreases when the charge increases. For the extreme electrically dilaton black hole, i.e., \( r_+ = r_- \), one has the expelling of Higgs fields from the horizon. In the extreme Reissner-Nordström black hole case Chamblin et al. get the similar result, see Ref. [13]. Both these results are of \( p \sim \frac{1}{N} \) order and are relatively small due to the small value of the charge \( q \) comparing to \( E \).

### III. Numerical Solutions

In this section we shall analyze Eqs. (14) and (15) numerically, outside and on the horizon of the black hole. The Eqns. under considerations are elliptic in the electrically charged dilaton black hole background, while on the horizon of the black hole they are parabolic. At large radii, one wants to obtain the Nielsen-Olesen solutions, then the asymptotic values of the functions \( X \) and \( P \) are

\[ (X, P) = \begin{cases} (1, 0), & r \to \infty \\ (0, 1), & r \geq 2E, \theta = 0, \pi. \end{cases} \]  

(25)

On the horizon Eqns. (14) and (15) become parabolic and have the forms

\[ \frac{1}{2E} \partial_r X |_{r=2E} = -\frac{1}{2E} \frac{\partial \theta [\sin \theta \partial_\theta X]}{\sin \theta} + \frac{NP^2 X}{2E (2E - \frac{q^2}{E}) \sin^2 \theta} + \frac{1}{2} \sin^2 \theta (X^2 - 1), \]  

(26)

\[ \frac{1}{2E} \partial_r P |_{r=2E} = -\frac{1}{2E} \frac{\partial \theta [\csc \theta \partial_\theta P]}{\csc \theta} + \frac{X^2 P}{\beta}. \]  

(27)

In order to solve numerically Eqs. (14) and (15) we use the simultaneous over-relaxation method [24] modified in the way described in [12] to handle boundary conditions [26] and [27] on the horizon. Eqns. are solved on a uniformly spaced polar grid \((r_i, \theta_i)\), with boundaries at \( r_{in} = 2E \), outer radius \( r_{out} \gg 2E \) (we usually choose \( r_{out} = 10E \)), and \( \theta \) ranging from 0 to \( \pi \). The mesh is divided into 200 × 200 cells (only for extreme black holes we use 300 × 300 cells).

Following the notation from Ref. [12], the finite difference scheme for Eqs. (14) and (15) has the form

\[ X_{00} = \frac{1 - \frac{2E}{r} X_{\theta \phi} + X_{\theta \phi} \frac{X_{\theta \phi} + X_{\theta \phi}}{(\Delta \tau)^2} + \left( 1 + \frac{1}{r} \frac{\partial X_{\theta \phi}}{\partial \theta} \right) \frac{X_{\theta \phi} - X_{\theta \phi}}{2 \Delta \tau} + \frac{\cot \theta \cdot X_{\theta \phi} - X_{\theta \phi}}{2 \Delta \theta}}{\frac{1}{1 - \frac{2E}{r} \frac{X_{\theta \phi}}{(\Delta \tau)^2} + \frac{2}{r (r - \frac{q^2}{E}) (\Delta \theta)^2} + \frac{1}{2} \left( \frac{X_{\theta \phi}^2}{2} - 1 \right)}} \]  

(28)

\[ P_{00} = \frac{1 - \frac{2E}{r} P_{\theta \phi} + P_{\theta \phi} \frac{P_{\theta \phi} + P_{\theta \phi}}{(\Delta \tau)^2} + \left( 1 + \frac{1}{r} \frac{\partial P_{\theta \phi}}{\partial \theta} \right) \frac{P_{\theta \phi} - P_{\theta \phi}}{2 \Delta \tau} + \frac{\cot \theta \cdot P_{\theta \phi} - P_{\theta \phi}}{2 \Delta \theta}}{\frac{1}{1 - \frac{2E}{r} \frac{P_{\theta \phi}}{(\Delta \tau)^2} + \frac{2}{r (r - \frac{q^2}{E}) (\Delta \theta)^2} + \frac{1}{2} \left( \frac{P_{\theta \phi}^2}{2} \right)}} \]  

(29)

while on the horizon we have

\[ X_{00} = \frac{1}{\Delta \tau} X_{\theta \phi} + \frac{\cot \theta \cdot X_{\theta \phi} - X_{\theta \phi}}{2 \Delta \theta} + \frac{\frac{1}{(2E - \frac{q^2}{E})(\Delta \theta)^2}}{\frac{1}{\Delta \tau} + E (X_{\theta \phi}^2 - 1) + \frac{1}{(2E - \frac{q^2}{E})(\Delta \tau)^2} \left( \frac{P_{\theta \phi}}{\sin \theta} \right)^2 + \frac{1}{(2E - \frac{q^2}{E})(\Delta \theta)^2} \left( \frac{2 E}{(2E - \frac{q^2}{E})(\Delta \theta)^2} \right)^2} \]  

(30)

\[ P_{00} = \frac{1}{\Delta \tau} P_{\theta \phi} + \frac{\cot \theta \cdot P_{\theta \phi} - P_{\theta \phi}}{2 \Delta \theta} + \frac{\frac{1}{(2E - \frac{q^2}{E})(\Delta \theta)^2}}{\frac{1}{\Delta \tau} + \frac{2 E X_{\theta \phi}^2}{(2E - \frac{q^2}{E})(\Delta \tau)^2} + \frac{1}{(2E - \frac{q^2}{E})(\Delta \theta)^2} \left( \frac{2 E}{(2E - \frac{q^2}{E})(\Delta \theta)^2} \right)^2} \]  

(31)
Note that for \( q = 0 \) we reproduce Eqns. obtained by Achucarro et al. in Ref. \[12\] and we use their results to check our numerical code obtaining the excellent agreement. In order to begin the numerical calculations, we first set the boundary conditions according to \( (25) \). On the horizon, we initially set \( X = 1 \) and \( P = 0 \). Then, the grid is over relaxed, i.e., values of \( P \) and \( X \) at each grid point are updated with new values \( X_{00} = \omega X_{new} + (1 - \omega)X_{00} \). The over relaxation parameter \( \omega \), \( (1 < \omega < 2) \) is found by trying some values and choosing the optimal one (we don’t use Chebyshev acceleration in our calculations). After each iteration, values of the fields on the horizon are updated according to \( (30) \) and \( (31) \). The whole step is repeated until reaching the convergence.

The examples of our numerical investigations are depicted in Figs.1-14. We also performed panels which enable to see cuts around the horizon and along the equator for the \( X \) and \( P \) fields in the electrically charged dilaton black hole background. These results confirm our analytical considerations given above.

Figs.1-6 presented calculations for the winding number \( N = 1 \) and different values of \( E \) and \( q \) parameters. In Figs.1-2 the Bogomolnyi parameter \( \beta \) is set equal to unity, i.e., the magnetic and the Higgs flux tubes are of the same size. Figs.7-8 are performed for the increasing winding number, we set \( N = 100 \). The string is still noticeably pinched. For all these parameters the \( X \) and \( P \) fields are passing right through the electrically charged dilaton black hole.

The next problem, will be to analyze the behavior of the \( X \) and \( P \) fields in the background of the extreme black hole. In Sec.II, we obtained the analytical results, assuming that the vortex size is large compared to the black hole size. Now, we shall deal with the problem using the numerical code. Figs.9-14 show our sample results and confirm the previous considerations. To begin with, we took into considerations the extreme black holes with \( E = 1, 10.0, 20.0 \) and the adequate values of \( q \). In Figs.13-14 we take \( \beta = 1 \), as in the previous non-extreme case. In all cases, the \( X \) field wraps around the black hole horizon. As far as the \( P \) field is concerned, one has the same situation. The \( P \) fields wrap around the black hole horizon and there is no penetration at all. Having in mind the relation between \( P \) and \( \tilde{B}_\theta \), it is obvious that no magnetic flux is crossing the horizon. The extreme electrically charged dilaton black hole behaves like a perfect diamagnet. The same situation was revealed in the case of the extreme Reissner-Nordström black hole, in Ref. \[13\].

One can conclude that, studying the behavior of a non-extreme electrically charged dilaton black hole and a black hole which is quite close to the extremal case, we have the situation that the \( X \) and \( P \) fields flow through the outer horizon of the black hole. On the other hand, in the case of the extreme electrically charged dilaton black hole there is an expulsion of the fields of the vortex from the horizon of the black hole.

IV. GRAVITATING STRINGS

In this Sec. our main task will be to find the metric of an Abelian Higgs vortex passing through an electrically charged dilaton black hole. To deal with the problem, we shall use the iterative procedure and assume the thin string limit, i.e., \( E \gg 1 \).

As in Ref. \[12\], to consider the gravitational effect of the Abelian Higgs vortex sitting on the electrically charged black hole, one needs to take into account a general axially symmetric line element of the form

\[
\text{d}s^2 = -e^{2\psi} \text{d}t^2 + \alpha^2 e^{2\psi} \text{d}\phi^2 + e^{-2\psi + 2\gamma} (d\tilde{\rho}^2 + dz^2).
\]  

\( (32) \)
We introduce the rescaled coordinates \[12\]
\[
\rho = \sqrt{\lambda} \hat{\rho},
\]
\[
\zeta = \sqrt{\lambda} \hat{\zeta},
\]
\[
\alpha = \sqrt{\lambda} \hat{\alpha}.
\]

In terms of them the Einstein Eqns. yield
\[
\alpha_{,\zeta} + \alpha_{,\rho\rho} = \epsilon \sqrt{-g} \left( \hat{T}_\zeta^{\zeta} + \hat{T}_\rho^{\rho} \right),
\]
\[
(\alpha \psi_{,\zeta})_{,\zeta} + (\alpha \psi_{,\rho})_{,\rho} = -\frac{1}{2} \epsilon \sqrt{-g} \left( \hat{T}_0^0 - \hat{T}_\zeta^{\zeta} - \hat{T}_\rho^{\rho} - \hat{T}_\phi^{\phi} \right),
\]
\[-\gamma_{,\rho} (\alpha_{,\rho} + \alpha_{,\zeta}) + \alpha_{,\rho} (\psi_{,\rho}^2 - \psi_{,\zeta}^2) + 2\alpha \psi_{,\rho} \psi_{,\zeta} + \alpha_{,\rho} \alpha_{,\rho\rho} + \alpha_{,\zeta} \alpha_{,\rho\zeta} = \epsilon \sqrt{-g} \left( \alpha_{,\rho} \hat{T}_\zeta^{\zeta} - \alpha_{,\zeta} \hat{T}_\rho^{\rho} \right),
\]
\[-\gamma_{,\zeta} (\alpha_{,\rho} + \alpha_{,\zeta}) - \alpha_{,\zeta} (\psi_{,\rho}^2 - \psi_{,\zeta}^2) - 2\alpha_{,\rho} \psi_{,\rho} \psi_{,\zeta} + \alpha_{,\rho} \alpha_{,\rho\rho} - \alpha_{,\rho} \alpha_{,\rho\zeta} = \epsilon \sqrt{-g} \left( \alpha_{,\zeta} \hat{T}_\rho^{\rho} - \alpha_{,\zeta} \hat{T}_\rho^{\rho} \right),
\]
\[-\gamma_{,\zeta} + \gamma_{,\rho\rho} + \psi_{,\rho}^2 + \psi_{,\zeta}^2 = \epsilon \sqrt{-g} \left( e^{2\gamma - 2\psi} \hat{T}_\phi^{\phi} \right),
\]
where \( \epsilon = 8\pi G \eta^2 \) which is assumed to be small. This assumption is well justified because, e.g., for the GUT string \( \epsilon \leq 10^{-6} \). The rescaled energy momentum tensor, i.e., \( \hat{T}_{\alpha \beta} = \frac{T_{\alpha \beta}}{\lambda \eta^4} \) is as follows:
\[
\hat{T}_0^0 = -e^{-2(\gamma - \psi)} \left( X_{,\rho}^2 + X_{,\zeta}^2 \right) - \frac{X_{,\rho}^2 P_{,\rho} e^{2\psi}}{\alpha^2} - \frac{\beta e^{-2\gamma + 4\psi} (P_{,\rho}^2 + P_{,\zeta}^2)}{\alpha^2} - V(X),
\]
\[
\hat{T}_{\phi}^{\phi} = -e^{-2(\gamma - \psi)} \left( X_{,\rho}^2 + X_{,\zeta}^2 \right) + \frac{X_{,\rho}^2 P_{,\rho} e^{2\psi}}{\alpha^2} + \frac{\beta e^{-2\gamma + 4\psi} (P_{,\rho}^2 + P_{,\zeta}^2)}{\alpha^2} - V(X),
\]
\[
\hat{T}_{\rho}^{\rho} = -e^{-2(\gamma - \psi)} \left( X_{,\rho}^2 - X_{,\zeta}^2 \right) - \frac{X_{,\rho}^2 P_{,\rho} e^{2\psi}}{\alpha^2} + \frac{\beta e^{-2\gamma + 4\psi} (P_{,\rho}^2 - P_{,\zeta}^2)}{\alpha^2} - V(X),
\]
\[
\hat{T}_{\zeta}^{\zeta} = -e^{-2(\gamma - \psi)} \left( X_{,\rho}^2 - X_{,\zeta}^2 \right) - \frac{X_{,\rho}^2 P_{,\rho} e^{2\psi}}{\alpha^2} + \frac{\beta e^{-2\gamma + 4\psi} (P_{,\rho}^2 - P_{,\zeta}^2)}{\alpha^2} - V(X),
\]
\[
\hat{T}_{\rho}^{\zeta} = 2e^{-2(\gamma - \psi)} \left( X_{,\rho} X_{,\zeta} + \frac{\beta}{\alpha^2} P_{,\rho} P_{,\zeta} \right),
\]
where \( V(X) = \frac{(X^2 - 1)^2}{4} \).

In order to solve the Einstein Eqns. we shall use the iterative procedure expanding Eqns. of motion in terms of the quantity \( \epsilon \). Our starting point is to rewrite the line element of a charged dilaton black hole into \( \rho \) and \( z \) coordinates defined as
\[
\hat{\rho}^2 = (\hat{r} - 2GM) \left( \hat{r} - \frac{Q^2}{GM} \right) \sin^2 \theta,
\]
\[
z = \left( \hat{r} - GM - \frac{Q^2}{2GM} \right) \cos \theta.
\]
In the above coordinates the metric of the electrically charged dilaton black hole takes the form
\[
ds^2 = -e^{2\psi} dt^2 + \hat{\rho}^2 e^{-2\psi} d\phi^2 + e^{-2\psi} (d\hat{\rho}^2 + dz^2),
\]
where
\[ e^{-2\psi_0} = \frac{\tilde{r}}{\tilde{r} - 2GM}, \] (49)
\[ e^{-2\psi_0 + 2\gamma_0} = \frac{(\tilde{r} - \frac{Q^2}{GM})}{A}, \] (50)
\[ \tilde{\rho}^2 = \tilde{\alpha}_0^2, \] (51)
\[ A = (\tilde{r} - 2GM) \left( \tilde{r} - \frac{Q^2}{GM} \right) \cos^2 \theta + \sin^2 \theta \left[ (\tilde{r} - 2GM) + \left( \tilde{r} - \frac{Q^2}{GM} \right) \right]^2. \] (52)

To the zeroth order, they will constitute our background solution.

We proceed to check if the energy momentum tensor admits a geodesic shear free event horizon. Taking into account the relation
\[ R_{\alpha\beta} l^\alpha l^\beta = 0, \] (22),
we reach to the conclusion that it is equivalent to the condition
\[ T_{00} - T_{rr} = 0. \] (53)

Using the coordinates
\[ R = \sqrt{r(r - \frac{q^2}{E}) \sin \theta} = \rho e^{-\psi_0} \] (54)
and the exact form of the energy momentum tensor, we finally get
\[ \hat{T}_{00} - \hat{T}_{\rho\rho} = -2e^{-2(\gamma - \psi)} \left( \frac{dR}{d\rho} \right)^2 \left[ \frac{\beta}{R^2} \left( \frac{d}{dR} P \right)^2 + \left( \frac{d}{dR} X \right)^2 \right]. \] (55)

In Eqn. (53) all terms remain finite and non-zero as \( \rho \to 0 \), except \( R_{\rho\rho} \). Using the exact form of \( R_{\rho\rho} \) and \( R_{\zeta\zeta} \) derivatives, namely
\[ R_{\rho\rho} = \frac{\partial r}{\partial \rho} \frac{\partial R}{\partial r} + \frac{\partial \theta}{\partial \rho} \frac{\partial R}{\partial \theta} = \frac{\rho}{R} \left[ \frac{2r - \frac{q^2}{E}}{(r - 2E) + (r - \frac{q^2}{E})} + \frac{r}{r - 2E} \right], \] (56)
and
\[ R_{\zeta\zeta} = \frac{\partial r}{\partial \zeta} \frac{\partial R}{\partial r} + \frac{\partial \theta}{\partial \zeta} \frac{\partial R}{\partial \theta} = \frac{(2r - \frac{q^2}{E}) \sin \theta}{2 \sqrt{r(r - \frac{q^2}{E}) \cos \theta}} - \frac{\sqrt{r(r - \frac{q^2}{E}) \cos \theta}}{2 \sin \theta (2r - 2E - \frac{q^2}{E})}. \] (57)

One can see that, on the horizon where \( \rho \to 0, \frac{dR}{d\rho} \to 0 \) which implies that \( \hat{T}_{00} - \hat{T}_{\rho\rho} = 0 \). Then, we draw the conclusion that, at least in a linearized method of considering the problem of an Abelian Higgs vortex on an electrically charged dilaton black hole, there is no gravitational obstacles of painting the vortex on the horizon of the black hole under consideration. Near the core of the string where \( \sin \theta \approx \mathcal{O}(E^{-1}) \), we have the following relation:
\[ R_{\rho\rho}^2 + R_{\zeta\zeta}^2 \sim e^{2\gamma_0 - 2\psi_0}. \] (58)

This relation implies that near the core of the string, to the zeroth order, the energy momentum tensor reads as follows:
\[
\begin{align*}
\dot{T}_{(0)0} &= -V(X_0) - \left( \frac{d}{dR} X_0 \right)^2 - \frac{X_0 P_0^2}{R^2} - \frac{\beta}{R^2} \left( \frac{d}{dR} P_0 \right)^2, \\
\dot{T}_{(0)\phi\phi} &= -V(X_0) - \left( \frac{d}{dR} X_0 \right)^2 + \frac{X_0 P_0^2}{R^2} + \frac{\beta}{R^2} \left( \frac{d}{dR} P_0 \right)^2, \\
\dot{T}_{(0)\phi\psi} &= -V(X_0) - \left( \frac{d}{dR} P_0 \right)^2 + e^{-2(\gamma_0 - \psi_0)} \left[ \frac{\beta}{R^2} \left( \frac{d}{dR} P_0 \right)^2 + \left( \frac{d}{dR} X_0 \right)^2 \right] (R_{\psi}^2 - R_{\phi}^2), \\
\dot{T}_{(0)\phi\zeta} &= -V(X_0) - \left( \frac{d}{dR} P_0 \right)^2 - e^{-2(\gamma_0 - \psi_0)} \left[ \frac{\beta}{R^2} \left( \frac{d}{dR} P_0 \right)^2 + \left( \frac{d}{dR} X_0 \right)^2 \right] (R_{\psi}^2 - R_{\phi}^2), \\
\dot{T}_{(0)\zeta\psi} &= 2e^{-2(\gamma_0 - \psi_0)} \left[ \frac{\beta}{R^2} \left( \frac{d}{dR} P_0 \right)^2 + \left( \frac{d}{dR} X_0 \right)^2 \right] R_{\psi} R_{\zeta}.
\end{align*}
\]

which is the purely function of \( R \). As in the Schwarzschild case \[12\], this strongly suggests to look for the metric perturbations as a function of \( R \).

Writing Eqns. of motion to the first order of \( \epsilon \), one obtains

\[
\alpha_{1,\rho\rho} + \alpha_{1,\zeta\zeta} = -\alpha_0 e^{2\gamma_0 - 2\psi_0} \left( 2V(X_0) + \frac{2X_0 P_0^2}{R^2} \right),
\]

\[
\alpha_{1,\psi_0,\rho} + (\rho \psi_1,\rho_0) \psi_0, \zeta + \rho \psi_1, \zeta = -\frac{1}{2} \alpha_0 e^{2\gamma_0 - 2\psi_0} \left( 2V(X_0) - \frac{2\beta}{R^2} (P_0')^2 \right),
\]

\[
\gamma_{1,\zeta\zeta} + \gamma_{1,\rho\rho} + 2\psi_0 \psi_1, \rho = e^{2\gamma_0 - 2\psi_0} \left( -(X_0')^2 + \frac{X_0^2 P_0^2}{R^2} + \frac{\beta}{R^2} (P_0')^2 - V(X_0) \right).
\]

Taking into account \[58\], one has that \[12\], \( \partial_\zeta^2 + \partial_\rho^2 = e^{2(\gamma_0 - \psi_0)} \left[ \frac{d^2}{dR^2} + \mathcal{O}\left( \frac{1}{R^2} \right) \right] \). The close inspection of Eqn. \[64\] reveals that, we can write \( \alpha_1 \) as a function of \( \rho \) and \( R \), namely \( \alpha_1 = \rho a(R) \). \( a(R) \) yields

\[
\frac{d^2}{dR^2} a(R) + \frac{2}{R} \frac{d}{dR} a = \dot{T}_{\rho} - \dot{T}_\zeta.
\]

Then, one reaches to the following expression for \( a(R) \):

\[
a(R) = \int_0^\infty \frac{1}{R^2} dR \int_0^R R'^2 \left( -2V(X_0) - \frac{2X_0^2 P_0^2}{R^2} \right) dR'.
\]

Eqn. \[68\] can be rewritten as

\[
a(R) \sim -\frac{A}{\epsilon} + \frac{B}{\epsilon R},
\]

where

\[
A = \epsilon \int_0^R R(\dot{T}_{(0)\rho} + \dot{T}_{(0)\zeta}) dR, \quad (70)
\]

\[
B = \epsilon \int_0^R R(\dot{T}_{(0)\rho} + \dot{T}_{(0)\zeta}) dR. \quad (71)
\]

Using the form of \( \alpha_1 \) and setting \( \psi_1 = \psi_1(R) \), we have

\[
\psi_1(R) = -\frac{1}{2} \int_0^\infty \frac{1}{R} dR \int_0^R R' \left( 2V(X_0) - \frac{2\beta}{R^2} \left( \frac{dP_0}{dR} \right)^2 \right) dR'.
\]
While for $\gamma_1(R)$, one arrives at the expression of the form
\[
\gamma_1(R) = \int_R^\infty dR \int_0^R \tilde{T}_\phi \phi dR' = 2\psi_1(R). \tag{73}
\]

Having in mind the above corrections to the metric functions, the asymptotic form of the metric yields
\[
ds^2 \to e^C \left[ -e^{2\psi_0} dt^2 + e^{2(\gamma_0 - 2\psi_0)} (dz^2 + d\tilde{\rho}^2) \right] + \rho^2 \left( 1 - A + \frac{B}{\sqrt{\lambda_\epsilon e^{\psi_0}}} \right)^2 e^{-C} e^{-2\psi_0} d\phi^2, \tag{74}
\]
or consistently with the coordinates (35) is determined by
\[
ds^2 \to e^C \left[ - \left( 1 - \frac{2GM}{\tilde{r}} \right) dt^2 + \frac{d\tilde{r}^2}{(1 - \frac{2GM}{\tilde{r}})} + \tilde{r} \left( \tilde{r} - \frac{Q^2}{GM} \right) d\theta^2 \right] + \tilde{r} \left( \tilde{r} - \frac{Q^2}{GM} \right) \left[ 1 - A + \frac{B}{\sqrt{\lambda_\epsilon e^{\psi_0}}} \right]^2 e^{-C} \sin^2 \theta d\phi^2, \tag{75}
\]
where $e^C = e^{2\psi_1}$.

The $B$-term represents the effect outside the range of the applicability of the considered approximation [12]. Therefore, one should drop this term. Rescaling the metric, $\hat{t} = e^{C/2} t$, $\hat{\rho} = e^{C/2} \tilde{\rho}$ and defining the quantities
\[
\hat{M} = e^{C/2} M, \tag{76}
\hat{Q} = e^{C/2} Q, \tag{77}
\]
one gets the metric of the electrically charged dilaton black hole with a string passing through it, the metric of the conical electrically charged dilaton black hole, namely
\[
ds^2 = - \left( 1 - \frac{2GM}{\tilde{r}} \right) dt^2 + \frac{d\tilde{r}^2}{(1 - \frac{2GM}{\tilde{r}})} + \tilde{r} \left( \tilde{r} - \frac{\hat{Q}^2}{GM} \right) d\theta^2 + \tilde{r} \left( \tilde{r} - \frac{\hat{Q}^2}{GM} \right) (1 - A)^2 e^{-2C} \sin^2 \theta d\phi^2. \tag{78}
\]

As in the Schwarzschild case [12], due to the presence of the radial pressure term $e^{-C}$, we have the modified Schwarzschild mass parameter at infinity (76) and the modified black hole charge (77).

The final issue which we wish to consider is a few remarks concerning the thermodynamics of an electrically charged dilaton black hole with a string passing through it. The temperature of a static black hole one can obtain considering its behavior for imaginary value of time [23]
\[
T = \frac{1}{8\pi GM}, \tag{79}
\]
where the Boltzmann constant is set to unity.

The entropy may be inferred from the second law of thermodynamics, or alternatively computed from the interpretation of the black hole as a saddle-point contribution to the partition function. Then, it reads
\[
S = 2\pi \hat{M} \left( 2\hat{M} - \frac{\hat{Q}^2}{GM} \right) (1 - A) e^{-C}. \tag{80}
\]

The black hole temperature in unchanged comparing to the nonconical case. Its value is in terms of the modified gravitational mass measured at infinity. On the other hand, entropy of the black hole with a string is less than that of a black hole of the same temperature without a string. It happened because of the fact that the internal mass (which can be found by considering the black hole as formed by a spherical shell of matter falling from infinity) and gravitational one were no longer equal. The very similar situation took place in the Schwarzschild-vortex system [12].
V. CONCLUSIONS

In our work we deal with the problem if the nontrivial topology of some field configurations (such as strings, domain walls, textures) can act as a hair for black holes. Namely, we considered an electrically charged dilaton black hole originated in the low-energy string theory and an Abelian Higgs vortex. Assuming in our investigations a clear separation between the degrees of freedom of the black hole-vortex configuration, we justify that the effect of the vortex can be regarded from infinity as black hole hair. Our preliminary analytic studies in Sec.II, in the limit where the vortex is thick compared to the horizon radius of the black hole, reveal that the extreme electrically charged dilaton black hole expels from its horizon all the fields living in the core of a vortex. The extreme black hole behaves like a perfect diamagnet. The similar kind of a Meissner effect was revealed studying an extreme Reissner-Nordström-vortex configuration [13].

By means of the simultaneous over-relaxation method modified in order to handle the boundary conditions, we obtained the solutions of Eqns. of motion for an Abelian Higgs vortex living in the background of the electrically charged dilaton black hole. The Eqns. are elliptic outside the event horizon, while they are parabolic on it. The numerical results confirm the approximate calculations conducted in the previous section. Summing it all up, we conclude that the fields of a vortex are always expelled from the extreme black hole horizon. However, for the non-extreme electrically charged dilaton black holes the fields $X$ and $P$ are passing through the black hole horizon. One should be aware that we do not take into account the back reaction of the vortex on the geometry. This problem needs to be treated more carefully and we hope to return to this question elsewhere.

Starting with the background solution and the Nielsen-Olesen forms of $X$ and $P$, by means of an iterative procedure of solving the field Eqns., we found the metric of an electrically charged dilaton black hole with a string passing through it. The temperature of such a black hole is equal to the temperature of the black hole without a string. It is measured in terms of the modified Schwarzschild mass parameter measured at infinity. The entropy, in turn, is less than the entropy of the black hole at the same temperature without a string. The reason is caused by the fact that the inertial mass is not equal to the gravitational one.

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FIG. 1. Numerical solution of the Nielsen-Olesen Eqns. for the $X$ field with $N = 1$, $\beta = 1.0$, $E = q = 10.0$. The event horizon is indicated by a semicircle. Upper and lower panels show cuts around the horizon and along the equator for the $X$ field in the electrically charged dilaton black hole background.
FIG. 2. As in Fig.1 but for the $P$ field with $N = 1$, $\beta = 1.0$, $E = q = 10.0$. 
FIG. 3. As in Fig.1 for the $X$ field with $N = 1$, $\beta = 0.5$, $E = q = 1.0$. 
FIG. 4. As in Fig.2 for the $P$ field with $N = 1$, $\beta = 0.5$, $E = q = 1.0$. 
FIG. 5. As in Fig.1 for the $X$ field with $N = 1$, $\beta = 0.5$, $E = q = 4.0$. 

$X$-Contours (0.1,...,0.9)
FIG. 6. As in Fig. 2 for the $P$ field with $N = 1$, $\beta = 0.5$, $E = q = 4.0$. 

$P$-Contours $(0.9,\ldots,0.1)$
FIG. 7. As in Fig. 1 for the $X$ field with $N = 100$, $\beta = 0.5$, $E = q = 10.0$. 
FIG. 8. As in Fig. 2 for the $P$ field with $N = 100$, $\beta = 0.5$, $E = q = 10.0$. 
FIG. 9. Numerical solution of the Nielsen-Olesen Eqns. for the $X$ field with $N = 1$, $\beta = 0.05$, $E = 10.0$, $q = 14.142$ in background of the extreme electrically charged dilaton black hole. One can notice the expulsion of the $X$ field by the extreme black hole.
FIG. 10. As in Fig.7 but for the $P$ field with $N = 1$, $\beta = 0.05$, $E = 10.0$, $q = 14.142$. The same situation, one has the expulsion of the $P$ field from the extreme electrically charged dilaton black hole.
FIG. 11. As in Fig. 7 for the $X$ field with $N = 1$, $\beta = 0.05$, $E = 1.0$, $q = 1.4142$. 
FIG. 12. As in Fig. 8 for the \( P \) field with \( N = 1, \beta = 0.05, E = 1.0, q = 1.4142 \).
FIG. 13. As in Fig.9 for the $X$ field with $N = 1$, $\beta = 1$, $E = 20.0$, $q = 28.28$. 
FIG. 14. As in Fig.10 for the $P$ field with $N = 1$, $\beta = 1$, $E = 20.0$, $q = 28.28$. 