A∞ structures and Massey products

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Abstract

We show how to detect and recover higher Massey products on the cohomology $H$ of a differential graded algebra using $A_\infty$ structures induced on $H$ via homotopy transfer techniques.

1 Introduction

Higher order Massey products were introduced in [16] and generalized in [18]. These are of fundamental importance not only in the study of differential graded algebras (DGA’s, henceforward) per se, but also in those geometrical contexts where DGA’s play a role. Classical instances of this fact are the detection of linking numbers of knots [17] or the obstructions to formality of Kähler manifolds [3]. Recently, Massey products have proved to be useful in a wide range of applications which go from symplectic geometry [1] to algebraic geometry [25]. To name a few, in homotopy theory for instance, Massey products have been used to prove that in general, the homotopy type of a manifold $M$ does not determine that of its configuration space in $k \geq 2$ points, $F_k(M)$ [14]. In group cohomology, the authors of [22] show that 2-groups of maximal nilpotency class are determined by their mod 2 group cohomology algebra and iterated Massey products. In number theory, these represent obstructions for solving certain Galois embedding problems [20] [21].

On the other hand, given any DGA $A$, there is a structure of minimal $A_\infty$ algebra on its cohomology $H$, unique up to $A_\infty$ isomorphism, for which $A$ and $H$ are quasi-isomorphic $A_\infty$ algebras. This structure can be inherited from $A$ by exhibiting $H$ as a homotopy retract of it, but there is no canonical way of doing it. Hence, different homotopy retraction give rise to different, although isomorphic, $A_\infty$ structures on $H$. This is recalled in detail in Section 1.1.

In this paper, we show how to detect and recover, whenever it is possible, higher Massey products in the cohomology $H$ of a DGA using $A_\infty$ structures induced on $H$ via homotopy transfer techniques. In what follows $\langle x_1, \ldots, x_n \rangle$ and $m_n(x_1, \ldots, x_n)$ denote, respectively, the Massey product set of the cohomology classes $x_1, \ldots, x_n$.
and the $n$th multiplication of these classes induced by the chosen homotopy retract of $A$.

In general, and even up to sign, $m_n(x_1, \ldots, x_n)$ is not an $n$th Massey product, and the most general result one can be attained is the following:

**Theorem A** (Theorem 2.6) If $\langle x_1, \ldots, x_n \rangle \neq \emptyset$, then, for any homotopy retract, and for any Massey product $x \in \langle x_1, \ldots, x_n \rangle$,

$$\varepsilon m_n(x_1, \ldots, x_n) = x + \Gamma, \quad \Gamma \in \sum_{j=1}^{n-1} \text{Im}(m_j),$$

where $\varepsilon = (-1)^{\sum_{j=1}^{n-1} (n-j) |x_j|}$.

However, if the homotopy retract is adapted to a given Massey product (see Section §2 for a precise definition), then this is detected by the $n$th multiplication:

**Theorem B** (Theorem 2.2) Let $x \in \langle x_1, \ldots, x_n \rangle$. Then, for any homotopy retract adapted to $x$,

$$\varepsilon m_n(x_1, \ldots, x_n) = x,$$

where $\varepsilon = (-1)^{1+|x_{n-1}|+|x_{n-3}|+\cdots}$.

On the other hand, there is always a homotopy retract whose induced $n$th multiplication produces a Massey product:

**Theorem C** (Theorem 3.2) If $\langle x_1, \ldots, x_n \rangle \neq \emptyset$, then there exists an $A_\infty$ structure on $H$ such that,

$$\varepsilon m_n(x_1, \ldots, x_n) \in \langle x_1, \ldots, x_n \rangle,$$

with $\varepsilon$ as in Theorem B.

For this to hold for any $A_\infty$ structure on $H$, extra assumptions are needed:

**Theorem D** (Theorem 3.3) Let $\langle x_1, \ldots, x_n \rangle \neq \emptyset$ with $n \geq 3$. If for some (and hence for any) homotopy retract of $A$ onto $H$, the induced higher multiplications $m_k = 0$ for $k \leq n - 2$, then

$$\varepsilon m_k(x_1, \ldots, x_k) \in \langle x_1, \ldots, x_k \rangle,$$

with $\varepsilon$ as in Theorem A.

We remark that results relating higher Massey products and higher multiplications on an $A_\infty$ algebra already appear in [23, 15, 13]. Dually, in the sense of Eckmann-Hilton, results relating higher Whitehead products and higher brackets on an $L_\infty$ algebra were proved by the authors in [2]. In fact, we were puzzled by observing that Theorem [13] Eckmann-Hilton dual of Theorem 3.3 in [2] was proved in the very interesting and remarkable paper [15] in full generality. That is, without imposing any assumption on the considered homotopy retract. After a deep reading we found a gap in the proof of Theorem 3.1 of op. cit. and arrived to the conclusion that, as stated in Theorem [13] in the $A_\infty$ setting it is also necessary to work with retracts which are adapted to the considered higher Massey product.

Scattered along the paper we present examples and other minor statements which illustrate the above results, show the necessity of the imposed hypothesis on each of them, and connect them with topology via classical rational homotopy theory.
1.1 Preliminaries

In this paper graded objects are always assumed over \( \mathbb{Z} \) and all algebraic structures are considered over a field \( \mathbb{K} \).

Given \( V \) a graded vector space define the tensor coalgebra on \( V \) as the graded coalgebra \( TV = \bigoplus_{n \geq 0} T^n V = \bigoplus_{n \geq 0} V^\otimes n \), where \( 1 \in T^0 (V) = \mathbb{K} \) is the counit, \( \Delta (1) = 1 \otimes 1 \), and \( \Delta (v_1 \otimes \cdots \otimes v_n) = \sum_{i=0}^n (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_n) \).

Recall that an \( A_\infty \) algebra is a graded vector space \( A = \{ A^n \}_{n \in \mathbb{Z}} \) together with linear maps \( m_k : A^\otimes k \to A \) of degree 2 - \( k \), for \( k \geq 1 \), satisfying the Stasheff identities for every \( i \geq 1 \):

\[
\sum_{k=1}^i \sum_{n=0}^{i-k} (-1)^{k+n+kn} m_{i-k+1} (\text{id}^\otimes n \otimes m_k \otimes \text{id}^\otimes i-k-n) = 0.
\]

A differential graded algebra (DGA), is an \( A_\infty \) algebra for which \( m_k = 0 \) for all \( k \geq 3 \). An \( A_\infty \) algebra is minimal if \( m_1 = 0 \). Observe that \( A_\infty \) algebra structures on \( A \) are in bijective correspondence with codifferentials on the tensor coalgebra \( T(sA) \) on the suspension of \( A \), \( (sA)^n = A^{n-1} \). Indeed, a codifferential \( \delta \) on \( T(sA) \) is determined by a degree 1 linear map \( T^+ (sA) \to sA \) which is written as the sum of linear maps \( g_k : T^k (sA) \to sA, k \geq 1 \). In fact, \( \delta \) is written as the sum of coderivations,

\[
\delta = \sum_{k \geq 1} \delta_k, \quad \delta_k : T(sA) \to T(sA),
\]

each of which being the extension as coderivation of the corresponding \( g_k \),

\[
\delta_k (sa_1 \otimes \cdots \otimes sa_p) = \sum_{i=1}^{p-k} \pm sa_1 \otimes \cdots \otimes sa_{i-1} \otimes sg_k (a_i \otimes \cdots \otimes a_{i+k-1}) \otimes sa_{i+k} \otimes \cdots \otimes sa_p.
\]

Observe, that each \( \delta_k \) decreases the word length by \( k - 1 \), that is, \( \delta_k (T^p (sA)) \subset T^{p-k+1} (sA) \) for any \( p \).

Then, the operators \( \{ m_k \}_{k \geq 1} \) on \( A \) and the maps \( \{ g_k \}_{k \geq 1} \) (and hence \( \delta \)) uniquely determine each other as follows:

\[
m_k = s^{-1} \circ g_k \circ s^\otimes k : A^\otimes k \to A, \quad g_k = (-1)^{\frac{k(k-1)}{2}} s \circ m_k \circ (s^{-1})^\otimes k : T^k (sA) \to sA.
\]

Note that if \( A \) is a DGA, then the corresponding codifferential \( \delta \) on \( T(sA) \) has only linear and quadratic part, \( \delta = \delta_1 + \delta_2 \), determined by

\[
g_1 sa = -sda, \quad g_2 (sa_1 \otimes sa_2) = -(-1)^{|a_1|} s(a_1 a_2).
\]

In other words, \( (T(sA), \delta) \) is the bar construction of \( A \).

An \( A_\infty \) morphism \( f : A \to A' \) between two \( A_\infty \) algebras is a family of linear maps (components) \( f_k : A^\otimes k \to A' \) of degree 1 - \( k \) such that the following equation holds for every \( k \geq 1 \):

\[
\sum_{k=r+s+t \geq 1} \sum_{r,t \geq 0} (-1)^{r+s+t} f_{r+1+t} (\text{id}^\otimes r \otimes m_s \otimes \text{id}^\otimes t) = \sum_{1 \leq r \leq k} \sum_{k \geq i_1 + \cdots + i_r} (-1)^s m_r (f_{i_1} \otimes \cdots \otimes f_{i_r})
\]
being \( s = \sum_{r=1}^{r-1} \ell(i_r-i-1) \). It is said to be an \( A_\infty \) quasi-isomorphism if \( f_1: (A, m_1) \to (A', m'_1) \) is a quasi-isomorphism of cochain complexes. Observe that \( A_\infty \) morphisms from \( A \) to \( A' \) are in bijective correspondence with differential graded coalgebra (DGC) morphisms \( (T(sA), \delta) \to (T(sA'), \delta') \) being \( \delta \) and \( \delta' \) the codifferentials determining the \( A_\infty \) algebra structures. Indeed, a DGC morphism

\[
f: (T(sA), \delta) \to (T(sA'), \delta')
\]

is determined by \( \pi f: T(sA) \to sA' \) (\( \pi \) denotes the projection onto the indecomposables) which can be written as \( \sum_{k \geq 1} (\pi f)_k \), where \( (\pi f)_k: T^k(sA) \to sA' \). Note that the collection of linear maps \( \{ (\pi f)_k \}_{k \geq 1} \) is in one-to-one correspondence with a system \( \{ f_k \}_{k \geq 1} \) of linear maps \( f_k: A^\otimes k \to A' \) of degree \( 1-k \) defining an \( A_\infty \) morphism. Each \( f_k \) and \( (\pi f)_k \) determines the other by:

\[
\begin{align*}
f_k &= s^{-1} \circ (\pi f)_k \circ s^\otimes k, \\
(\pi f)_k &= (-1)^{k(k-1)} s \circ f_k \circ (s^{-1})^\otimes k.
\end{align*}
\]

A homotopy retract (of \( M \) onto \( N \)) is a diagram of the form

\[
K \xrightarrow{q} M \xrightarrow{i} N,
\]

where \( M \) and \( N \) are cochain complexes and \( q \) and \( i \) are cochain maps such that \( q_i = \text{id}_N \) and \( iq \simeq \text{id}_M \) via a chain homotopy \( K \) which satisfies \( K^2 = K i = q K = 0 \). We often denote it simply by \((M, N, i, q, K)\).

We will be using the following particular instance of the homotopy transfer theorem \([9, 11, 12, 13, 19]\), also known as the homological perturbation lemma \([6, 7, 8, 10]\):

**Theorem 1.1** Let \((A, H, i, q, K)\) be a homotopy retract of the DGA \((A, d)\) onto its cohomology \((H, 0)\). Then, there exists a minimal \( A_\infty \) algebra structure \( \{ m_k \} \) on \( H \), unique up to isomorphism, and \( A_\infty \) algebra quasi-isomorphisms

\[
(A, d) \xleftarrow{Q} (H, \{ m_k \}),
\]

such that \( I_1 = i \), \( Q_1 = q \) and \( Q I = \text{id}_H \). In other words, there are DGC quasi-isomorphisms extending \( i \) and \( q \)

\[
(T(sA), \delta) \xrightarrow{Q} (T(sH), \delta),
\]

which make \((T(sH), \delta)\) a quasi-isomorphic retract of the bar construction on \( A \).

Moreover, the transferred higher products \( \{ m_k \} \) and the components \( \{ I_k \} \) of \( I \) are given inductively as follows \([9, 19]\): formally, set \( K \lambda_1 = -i \), and define \( \lambda_k: H^\otimes k \to A, k \geq 2 \), recursively by

\[
\lambda_k = m \left( \sum_{s=1}^{k-1} (-1)^{s+1} K \lambda_s \otimes K \lambda_{k-s} \right).
\]

Then,

\[
m_k = q \circ \lambda_k \quad \text{and} \quad I_k = K \circ \lambda_k \quad \text{for all } k \geq 2.
\]
Remark 1.2 Theorem 1.1 above remains valid if we do not require the chain homotopy \( K \) to satisfy the vanishing equations \( K^2 = Ki = qK = 0 \). These are assumed to have a bijective correspondence between homotopy retracts \((M, H, i, q, K)\) and decompositions of \( M \) of the form

\[
M = B \oplus dB \oplus C,
\]

where \( B \) is a complement of \( \text{Ker} \, d \) (and thus \( d: B \xrightarrow{\cong} dB \)) and \( C \cong H \). Indeed, Let \( M = B \oplus dB \oplus C \) be such a decomposition. Define \( i: H \cong C \hookrightarrow M, q: M \to C \cong H \) and \( K(B) = K(C) = 0 \). \( K: dB \xrightarrow{\cong} B \). Then, \((M, H, i, q, K)\) is a homotopy retract. Conversely, given \((M, H, i, q, K)\) a homotopy retract of \( M \) onto \( H \) define \( B = KdM \) and \( C = \text{Im} \, i \) to have a decomposition \( M = B \oplus dB \oplus C \), Nevertheless, if one does not assume the vanishing conditions for \( K \) in the definition of the homotopy retract, the new chain homotopy \( GdG \) where \( G = (Kd + dK)K(Kd + dK) \) does satisfy these conditions.

Given \( A \) a DGA define the Massey product of two cohomology classes \( x_1, x_2 \in H(A) \) as the usual product \( x_1x_2 \). We next define third order Massey products, and then recursively we also define higher order Massey products. For it, in what follows, we write \( H = H(A) \) and \( \sigma = (-1)^{|a|+1}a \) where \( a \) is a homogeneous element of \( A \) and \( |a| \) denotes its degree.

Definition 1.3 Let \( x_1, x_2, x_3 \in H \) be such that \( x_1x_2 = x_2x_3 = 0 \). A defining system (for the triple Massey product) is a set \( \{a_{ij}\}_{0 \leq i < j \leq 3} \subseteq A \) defined as follows:

For \( i = 1, 2, 3 \) choose a cocycle \( a_{i-1} \), representative of \( x_i \). These are \( \{a_{01}, a_{12}, a_{23}\} \).

For \( 0 \leq i < j \leq 3 \) and \( j - i = 2 \), choose \( a_{ij} \in A \) with the property that \( d(a_{ij}) = \sigma_{i,j}a_{i+1,j} \). These are \( \{a_{02}, a_{13}\} \), with differential given by:

\[
d(a_{02}) = \sigma_{01}a_{12} \quad \text{and} \quad d(a_{13}) = \sigma_{12}a_{23}.
\]

The existence of such elements is guaranteed by the condition \( x_1x_2 = x_2x_3 = 0 \). Then, the triple Massey product is defined as the set

\[
\{x_1, x_2, x_3\} = \{[\sigma_{01}a_{13} + \sigma_{02}a_{23}] \mid \{a_{ij}\} \text{ is a defining system}\} \subseteq H^{s-1},
\]

where \( s = |x_1| + |x_2| + |x_3| \). If the condition \( x_1x_2 = x_2x_3 = 0 \) is not satisfied, define the triple Massey product \( \{x_1, x_2, x_3\} \) as the empty set.

It is also classical to define

\[
\text{In}(x_1, x_2, x_3) = x_1H^{[x_2]+|x_3|-1} + H^{[x_1]+|x_2|-1}x_3
\]

as the indeterminacy subgroup, and see triple Massey products as elements in the quotient \( H^{s-1}/\text{In}(x_1, x_2, x_3) \), but we do not use this here.

The definition of higher Massey products is inductively given as follows.

Definition 1.4 Let \( x_1, \ldots, x_n \in H \) be such that for \( 1 \leq i < j \leq n \) and \( j - i \leq n - 2 \), \( \{x_i, \ldots, x_j\} \) exists and contains the zero class. A defining system (for the \( n \)th order Massey product) is a set \( \{a_{ij}\}_{0 \leq i < j \leq n, 1 \leq j - i \leq n - 1} \subseteq A \) defined as follows.
• For \( i = 1, \ldots, n \) choose a cocycle \( a_{i-1,i} \) representative of \( x_i \).

• For \( 0 \leq i < j \leq n \) and \( 2 \leq j - i \leq n - 1 \), choose \( a_{ij} \in A \) with the property that

\[
d(a_{ij}) = \sum_{0 \leq i < k < j \leq n} a_{ik} a_{kj}.
\]

Their existence follows from the condition imposed on \( \langle x_i, \ldots, x_j \rangle \).

The \( n \)th order Massey product is then the set

\[
\langle x_1, \ldots, x_n \rangle = \left\{ \left[ \sum_{0 \leq i < k < j \leq n} a_{ik} a_{kj} \right] \mid \{a_{ij}\} \text{ is a defining system} \right\}.
\]

Observe that \( \langle x_1, \ldots, x_n \rangle \subseteq H^{s+2-n} \) where \( s = \sum_{i=1}^{n} |x_i| \). If the condition on \( \langle x_i, \ldots, x_j \rangle \) is not satisfied, define \( \langle x_1, \ldots, x_n \rangle \) as the empty set.

To finish, we fix some notation to be used henceforth: whenever we choose an element \( x \in \langle x_1, \ldots, x_n \rangle \), we are implicitly assuming that we have chosen classes \( x_1, \ldots, x_n \in H \) in the cohomology of a DGA \( A \) such that \( x \in \langle x_1, \ldots, x_n \rangle \neq \emptyset \).

When talking about an \( A_\infty \) structure on \( H \) we always mean that such a structure has been inherited on \( H \) via Theorem 1.1 through a given homotopy retract. Given a Massey product set \( \langle x_1, \ldots, x_n \rangle \), we say that the \( A_\infty \) structure \( \{m_n\} \) detects Massey products if \( \pm m_n(x_1, \ldots, x_n) \in \langle x_1, \ldots, x_n \rangle \). On the other hand, given a Massey product element \( x \in \langle x_1, \ldots, x_n \rangle \), we say that the \( A_\infty \) structure \( \{m_n\} \) recovers \( x \) if, up to sign, \( m_n(x_1, \ldots, x_n) = x \).

### 2 Recovering Massey products

Recall from Remark 1.2 that homotopy retracts \((A, H, i, q, K)\) correspond to decompositions

\[
A = B \oplus dB \oplus C,
\]

where \( B \) is a complement of \( \text{Ker } d \) (and thus \( d: B \cong dB \)) and \( C \cong H \). The next definition is essential in what follows.

**Definition 2.1** Let \( x \in \langle x_1, \ldots, x_n \rangle \). A homotopy retract \((A, H, i, q, K)\) is adapted to \( x \) if there exists a defining system \( \{a_{ij}\} \) for \( x \) such that \( i(x_j) = a_{j-1,j} \) for every \( j \) and \( \{a_{ij}\}_{j-i \geq 2} \subseteq B \) with \( B \) as above.

**Theorem 2.2** Let \( x \in \langle x_1, \ldots, x_n \rangle \). Then, for any homotopy retract adapted to \( x \),

\[
\varepsilon m_n(x_1, \ldots, x_n) = x,
\]

where \( \varepsilon = (-1)^{1+|x_{n-1}|+|x_{n-2}|+\ldots} \).
Proof: Let \( x \in \langle x_1, \ldots, x_n \rangle \) and let \( \{a_{ij}\} \) be a defining system for which \( (A, H, i, q, K) \) is a homotopy retract adapted to \( x \). Consider the map \( \lambda_n: H^\otimes n \to A, \quad n \geq 2 \), in formula (4) for which \( m_n = q \circ \lambda_n \). First, we prove by induction on \( s \), for \( 2 \leq s \leq n - 1 \), that

\[
K\lambda_s(x_1, \ldots, x_s) = (-1)^{bs}a_{0s},
\]

(5)

where \( b_s = |x_{s-1}| + |x_{s-3}| + \cdots + 1 \). For \( s = 2 \), it is straightforward. Assume this equation holds for every \( p \leq s - 1 \), and prove it for \( p = s \):

\[
K\lambda_s(x_1, \ldots, x_s) = K\lambda_2 \left( \sum_{i=1}^{s-1} (-1)^{i+1} K\lambda_i \otimes K\lambda_{s-i} \right) (x_1, \ldots, x_s)
\]

\[
= K\lambda_2 \left( \sum_{i=1}^{s-1} (-1)^{i+1 + |x_1| + \cdots + |x_i|} (-1)^{i-s+1} K\lambda_i(x_1, \ldots, x_i) \otimes K\lambda_{s-i}(x_{i+1}, \ldots, x_s) \right)
\]

apply the induction hypothesis and note that, recursively,

\[
1 + |a_{0k}| = |x_1| + \cdots + |x_k| - (k - 2)
\]

for every \( k = 1, \ldots, i \).

Then,

\[
= K\lambda_2 \left( \sum_{i=1}^{s-1} (-1)^{i+1 + |x_1| + \cdots + |x_i|(i-s+1) + b_i} a_{0k} \otimes a_{is} \right)
\]

\[
= K\lambda_2 \left( \sum_{i=1}^{s-1} (-1)^{i+1 + |x_1| + \cdots + |x_i|(i-s+1) + b_i + b^{i-1} + 1} a_{0k} \otimes a_{is} \right)
\]

\[
= (-1)^{bs} K \left( \sum_{0 < i < s} a_{0k} a_{is} \right) = (-1)^{bs} Kd(a_{0s}) = (-1)^{bs} a_{0s}.
\]

An analogous argument proves that

\[
K\lambda_{n-s}(x_{s+1}, \ldots, x_n) = (-1)^{b^{n-s}} a_{sn},
\]

(6)

where \( b^{n-s} = |x_{n-1}| + |x_{n-3}| + \cdots + 1 \). Applying equations (5) and (6) we have,

\[
\lambda_n(x_1, \ldots, x_n) = \lambda_2 \left( \sum_{s=1}^{n-1} (-1)^{s+1} K\lambda_s \otimes K\lambda_{n-s} \right) (x_1, \ldots, x_n)
\]

\[
= \lambda_2 \left( \sum_{s=1}^{n-1} (-1)^{s+1 + |x_1| + \cdots + |x_s|} K\lambda_s(x_1, \ldots, x_s) \otimes K\lambda_{n-s}(x_{s+1}, \ldots, x_n) \right)
\]

\[
= \sum_{s=1}^{n-1} (-1)^{bs} a_{0s} a_{sn},
\]

where \( b = |x_{n-1}| + |x_{n-3}| + \cdots + 1 \). Hence, \( m_n(x_1, \ldots, x_n) = \varepsilon x \)

The next example shows that if a retract is not adapted, then the associated higher multiplication might not recover a given Massey product.
Example 2.3 Let \((\Lambda V, d)\) be the commutative DGA over \(\mathbb{Q}\), where

\[
V = \text{Span}\{a_{01}, a_{12}, a_{23}, a_{34}, \ a_{02}, a_{13}, a_{24}, z_1, z_2, \ a_{03}, a_{13}\},
\]

\(a_{i-1,i}\) and \(z_1, z_2\) are cocycles, and for the rest of the elements \(da_{ij} = \sum_{i<k<j} a_{ik}a_{kj}\).

It is straightforward to check that, fixing the cohomology classes \(x_i = [a_{i-1,i}]\) for \(i = 1, 2, 3, 4\), there is a unique defining system \(\{a_{ij}\}\) (given by the obvious elections) which gives rise to a unique non trivial Massey product \(x = [a_{01}a_{14} + a_{02}a_{24} + a_{03}a_{34}]\). That is, \((x_1, x_2, x_3, x_4) = \{x\} \subseteq H^{10}\) and \(x \neq 0\). It is also easy to see that the cohomology in degree 10 admits the basis \(\{x, [z_1 z_2]\}\).

Next, fix the decomposition of \((\Lambda V, d)\) as \(B \oplus dB \oplus C\) like in Remark 1.2 given in the table below. On it, elements appearing in \((dB)^s\) come from differentiating the elements of \(B^s\) in the order written, and a dot \(\cdot\) indicates that the corresponding subspace is the trivial one. The decomposition above degree 10 is irrelevant for our purposes.

| degree | \(B\) | \(dB\) | \(C\) |
|--------|-------|--------|--------|
| 3      | \(-\) | \(-\)  | \(a_{01}, a_{12}, a_{23}, a_{34}\) |
| 4      | \(-\) | \(-\)  | \(-\)  |
| 5      | \(a_{02} + z_1, a_{13}, a_{24} + z_2\) | \(-\)  | \(z_1, z_2\) |
| 6      | \(a_{01}a_{12}, a_{12}a_{23}, a_{23}a_{34}\) | \(a_{01}a_{23}, a_{01}a_{34}, a_{12}a_{34}\) | \(-\)  |
| 7      | \(a_{03}, a_{14}\) | \(-\)  | \(-\)  |
| 8      | \(a_{01}a_{13}, a_{01}a_{24}, a_{01}a_{13} + a_{02}a_{23}, a_{02}a_{34}, a_{12}a_{24} + a_{13}a_{34}\) | \(a_{13}a_{23}, a_{23}a_{24}, a_{23}a_{34}, a_{01}z_1, a_{12}z_1, a_{23}z_1, a_{34}z_1, a_{01}z_2, a_{12}z_2, a_{23}z_2, a_{34}z_2\) | \(-\)  |
| 9      | \(-\) | \(a_{01}a_{13}a_{23}, a_{01}a_{23}a_{34}, a_{01}a_{12}a_{34}, a_{12}a_{23}a_{34}\) | \(-\)  |
| 10     | \(-\) | \(a_{01}a_{14} + a_{02}a_{24} + a_{03}a_{34}\) | \(z_1 z_2\) |

Hence, a straightforward computation for the homotopy retract associated to this decomposition and the induced \(A_\infty\) structure on \(H^*(\Lambda V, d)\) gives the cohomology class

\[
m_4(x_1, x_2, x_3, x_4) = -x - [z_1 z_2],
\]

which is not a Massey product by the discussion above.

Remark 2.4 Observe that, by basic facts on rational homotopy theory \([4]\), the commutative DGA \((\Lambda V, d)\) in the example above is the minimal model of a simply connected elliptic complex \(X\) which is of the form

\[
X = S^5 \times S^5 \times Y
\]
where $Y$ lies as the total space of a fibration of this sort

$$(S^5)^3 \times (S^7)^2 \longrightarrow Y \longrightarrow (S^3)^4.$$ 

Hence, the example above shows that in $H^*(X; \mathbb{Q})$ the set $(x_1, x_2, x_3, x_4)$ reduces to the single element $x$ which is not recovered by the $A_\infty$ structure induced by the given decomposition.

The following is an example of a DGA with infinitely many triple Massey products that are never recovered by any $A_\infty$ structure on its cohomology.

**Example 2.5** Let $(\Lambda V, d)$ be the commutative DGA over $\mathbb{Q}$ where

$$V = \text{Span}\{a_{01}, a_{12}, a_{23}, a_{02}, a_{13}\}$$

and

$$da_{01} = da_{12} = da_{23} = 0, \quad da_{02} = a_{01}a_{12}, \quad da_{13} = a_{12}a_{23}.$$ 

Let

$$A = (\Lambda V, d)/J$$

where $J$ is the differential ideal generated by $\{a_{01}a_{12}, a_{12}a_{23}\}$. We denote the elements of the quotient algebra as in the original without confusion.

Fixed $x_1 = [a_{01}], x_2 = [a_{12}]$ and $x_3 = [a_{23}]$, the possible defining systems $\{b_{ij}\}$ for the triple Massey product $\langle x_1, x_2, x_3 \rangle$ are of the following form, where $\alpha_k, \beta_k \in \mathbb{Q}$:

$$b_{01} = a_{01}, \quad b_{12} = a_{12}, \quad b_{23} = a_{23},$$

$$b_{02} = \alpha_1 a_{02} + \alpha_2 a_{13} \quad \text{and} \quad b_{13} = \beta_1 a_{02} + \beta_2 a_{13}.$$ 

This implies that the triple Massey product set is

$$\langle x_1, x_2, x_3 \rangle = \{\alpha_1 [a_{01}a_{02}] + \alpha_2 [a_{01}a_{13}] + \beta_1 [a_{02}a_{23}] + \beta_2 [a_{13}a_{23}] \mid \alpha_k, \beta_k \in \mathbb{Q}\}.$$ 

The zero class belongs to the set, but there are infinitely many other non trivial Massey product elements. It is a straightforward computation now to check that for any homotopy retract, one has that $m_3(x_1, x_2, x_3) = 0$. Therefore, one never recovers a non trivial Massey product element. \hfill \Box

Without imposing any extra condition on a given Massey product, and/or on a considered $A_\infty$ structure, the most general result is the following, Eckmann-Hilton dual of [2] Prop. 3.1:

**Theorem 2.6** For any homotopy retract and any $x \in \langle x_1, \ldots, x_n \rangle$,

$$\varepsilon m_n(x_1, \ldots, x_n) = x + \Gamma, \quad \Gamma \in \sum_{j=1}^{n-1} \text{Im}(m_j),$$

where $\varepsilon = (-1)^{\sum_{j=1}^{n-1} (n-j)\lvert x_j \rvert}$. 

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Proof: Recall (see for instance [24, III.6]) that the Eilenberg-Moore spectral sequence of $A$ is the coalgebra spectral sequence obtained by filtering the bar construction $(T(sA), \delta)$ by the ascending filtration $F_p = (sA)^{\otimes \leq p}$. Consider the DGC quasi-isomorphisms of Theorem 1.1, $(T(sA), \delta) \xrightarrow{Q} (T(sH), \delta)$, and choose the same filtration on $T(sH)$. Observe that at the $E_1$ level the induced morphisms of spectral sequences are both the identity on $T(sH)$. By comparison, all the terms in both spectral sequences are also isomorphic. Now, translating [24, Thm. V.7(6)] to the spectral sequence on $\Lambda sH$ we obtain that if $\langle x_1, \ldots, x_n \rangle$ is nonempty, then the element $sx_1 \otimes \cdots \otimes sx_n$ survives to the $n - 1$ page $(E^{n-1}, \delta^{n-1})$. Moreover, given any $x \in \langle x_1, \ldots, x_n \rangle$, one has

$$\delta^{n-1}(sx_1 \otimes \cdots \otimes sx_n) = sx.$$ 

Here $\overline{sx}$ denotes the class in $E^{n-1}$. In other words, there exists $\Phi \in T^{\leq n-1}(sH)$ such that

$$\delta(sx_1 \otimes \cdots \otimes sx_n + \Phi) = sx.$$ 

Write $\delta = \sum_{i \geq 2} \delta_i$ with each $\delta_i$ as in equation (1), and decompose $\Phi = \sum_{i=2}^{n-1} \Phi_i$ with $\Phi_i \in T^i(sH)$. By a word length argument,

$$\delta_k(sx_1 \otimes \cdots \otimes sx_n) + \sum_{i=2}^{n-1} \delta_i(\Phi_i) = sx.$$ 

Note also that $\delta_n = g_n$ for elements of word length $n$, with $g_n$ as in equation (2). Therefore,

$$g_n(sx_1 \otimes \cdots \otimes sx_n) + \sum_{i=2}^{n-1} g_i(\Phi_i) = sx.$$ 

To finish, apply the identities (3) and write each $g_i$ in terms of the corresponding $m_i$ for all $i = 1, \ldots, n$. In particular, the sign $\varepsilon$ appears when writing

$$m_n(x_1, \ldots, x_n) = s^{-1} \circ g_n \circ s \otimes n(x_1, \ldots, x_n) = \varepsilon s^{-1} g_n(sx_1 \otimes \cdots \otimes sx_n).$$

□

Corollary 2.7 Let $A$ be a DGA such that for some (and hence for any) homotopy retract of $A$ into $H$, the induced higher multiplications $m_k = 0$ for $1 \leq k \leq n - 1$. Then, for any cohomology classes $x_1, \ldots, x_k \in H$, the Massey product set $\langle x_1, \ldots, x_n \rangle = \{x\}$ consists of a single class which is recovered by the $n$th multiplication, that is, $\varepsilon m_n(x_1, \ldots, x_n) = x$, with $\varepsilon$ as in Theorem 2.6.

Proof: By induction on $s = j - i$, we have that

$$\{0\} = \langle x_1, \ldots, x_j \rangle \neq \emptyset \quad \text{for all} \quad 1 \leq i < \cdots < j \leq n, \quad \text{and} \quad j - i \leq n - 1.$$ 

Now given $x \in \langle x_1, \ldots, x_n \rangle$, the result follows from a direct application of Theorem 2.6. □
Remark 2.8 Note that the least $k$ for which the $k$th multiplication $m_k$ is non trivial is an invariant of a given $A_\infty$ structure and therefore, it is independent of the chosen homotopy retract. Hence the result above holds for all of them. The same applies for Theorem 3.3.

3 Detecting Massey products

In Theorem 3.1 of the very interesting paper [15], the authors prove that higher multiplications always detect Massey products: for any cohomology classes $x_1, \ldots, x_n$ in the cohomology $H$ of a given DGA $A$ such that $\langle x_1, \ldots, x_n \rangle$ is non empty, and for any $A_\infty$ structure on $H$ induced by a homotopy retract, $\varepsilon m_n(x_1, \ldots, x_n) \in \langle x_1, \ldots, x_n \rangle$, where $\varepsilon$ is as in Theorem 2.2.

Unfortunately, as stated, this result is only valid for $n = 3$ which is the first inductive step in its proof, and also, our Corollary 3.4 below. For $n \geq 4$, Example 2.3 is a clear counterexample. The small gap in the proof occurs when assuming that the elements

$$\{a_{ij} = K\lambda_{j-i+1}(x_i, \ldots, x_j) \mid 2 < j - i < n - 1\}$$

together with suitable representatives $a_{i-1,i}$ of the classes $x_i$ form a defining system. This is not the case, for instance, in the homotopy retract chosen in Example 2.3. If one imposes this extra assumption, then the inductive proof in [15, Thm. 3.1] works and it shows the following:

**Theorem 3.1** Let $A$ be a DGA and assume $\langle x_1, \ldots, x_n \rangle \neq \emptyset$, $n \geq 3$. Then, for any homotopy retract of $A$ such that the elements $\{a_{ij} = K\lambda_{j-i+1}(x_i, \ldots, x_j) \mid 2 < j - i < n - 1\}$ assemble into a defining system,

$$\varepsilon m_n(x_1, \ldots, x_n) \in \langle x_1, \ldots, x_n \rangle$$

with $\varepsilon$ as in Theorem 2.2. □

Nevertheless, provided that the Massey product set $\langle x_1, \ldots, x_n \rangle$ is non empty, we may construct a particular homotopy retract such that the assumption in this result holds and therefore, the $n$th multiplication induced by this retract $m_n(x_1, \ldots, x_n)$ detects a Massey product.

**Theorem 3.2** If $\langle x_1, \ldots, x_n \rangle \neq \emptyset$, then there exists an $A_\infty$ structure on $H$ such that, $\varepsilon m_n(x_1, \ldots, x_n) \in \langle x_1, \ldots, x_n \rangle$, with $\varepsilon$ as in Theorem 2.2.
Proof: Fix an $A_\infty$ algebra structure on $H$. For $n = 3$, if $a_{01}, a_{12}$ and $a_{23}$ are cocycle representatives of $x_1, x_2$ and $x_3$, respectively, then the elements $a_{02} = K(a_{01}a_{12})$ and $a_{13} = K(\bar{a}_{12}a_{23})$ are valid elections. Assume that for every $p \leq n - 2$ we have found a decomposition $A = B \oplus dB \oplus C$ (possibly different to the fixed one at the beginning) and elements $\{a_{ij}\}_{j-i\leq n-2}$ with the property that

$$a_{ij} = K\lambda_{j-i+1}(x_i, \ldots, x_j),$$
so that

$$da_{ij} = \sum_{0\leq s<k<j<n} \bar{a}_{ik}a_{kj} \in dB.$$ 

Define the elements

$$a_{0,n-1} = K\lambda_{n-1}(x_1, \ldots, x_{n-1})$$
and

$$a_{1,n} = K\lambda_{n-1}(x_2, \ldots, x_n).$$

Then,

$$da_{0,n-1} = (dK)\lambda_{n-1}(x_1, \ldots, x_{n-1})$$

$$= (dK) \left( \sum_{s=1}^{n-2} \pm K\lambda_s(x_1, \ldots, x_s)K\lambda_{n-1-s}(x_{s+1}, \ldots, x_{n-1}) \right)$$

$$= (dK) \left( \sum_{s=1}^{n-2} \bar{a}_{0s}a_{s,n-2} \right) = \sum_{s=1}^{n-2} \bar{a}_{0s}a_{s,n-2} \in dB.$$ 

Similarly, $da_{1,n} = \sum_{s=1}^{n-2} \bar{a}_{1s}a_{s,n-1}$. Therefore, the hypothesis of Theorem 3.1 hold and the proof is complete. 

To ensure that any $A_\infty$ structure on $H$ detects Massey product, extra assumptions are needed:

**Theorem 3.3** Let $\langle x_1, \ldots, x_n \rangle \neq \emptyset$ with $n \geq 3$. If for some (and hence for any) homotopy retract of $A$ onto $H$, the induced higher multiplications $m_k = 0$ for $k \leq n - 2$, then

$$\varepsilon m_k(x_1, \ldots, x_k) \in \langle x_1, \ldots, x_k \rangle,$$

with $\varepsilon$ as in Theorem 2.6.

As observed in Remark 2.8, this result holds for any homotopy retract.

Proof: Recall that $\langle x_1, \ldots, x_k \rangle \neq \emptyset$ implies that $0 \in \langle x_i, \ldots, x_j \rangle$, for any $3 \leq j - i \leq k - 1$. Therefore, we apply Theorem 2.6 taking into account that $m_i = 0$ for $i \leq k - 2$ to deduce,

$$m_{k-1}(x_i, \ldots, x_j) = 0 \quad \text{for any} \quad j - i = k - 1. \quad (8)$$

Let $A = B \oplus dB \oplus C$ be the decomposition equivalent to the chosen homotopy retract. By induction on $p$, with $2 \leq p \leq k - 1$, we will construct a set of elements $\{a_{ij}\}_{2 \leq j-i \leq p} \subseteq B$ with the property that $d(a_{ij}) = \sum_{1 \leq i \leq j} \bar{a}_{ij}a_{ij}$. For each $i$, we denote by $x'_i$ a cocycle representing $x_i$. Let $p = 2$. As $\langle x_1, \ldots, x_k \rangle \neq \emptyset$, we can (and do) define $a_{ij} = Kdb_{ij}$, being $b_{ij}$ any election such that $d(b_{ij}) = x'_ix'_j$. 

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Then, \( a_{ij} \in B \) by construction, and the differential behaves as expected. Assume the assertion true for \( p \leq k - 2 \). Then, there exists a family of elements of \( B \), \( \{a_{ij}\}_{2j-i \leq k-2} \), such that \( d(a_{ij}) = \sum_{i < l < j} \bar{a}_{il}a_{lj} \). Now, as the homotopy retract is adapted to the defining system we are building, the same argument as in the proof of Theorem 2.2 proves that

\[
m_p(x_1, \ldots, x_j) = q \left( \sum_{i < l < j} \bar{a}_{il}a_{lj} \right) \quad \text{for any} \quad 3 \leq p = j - i \leq k - 2.
\]

By equation (8),

\[
q \left( \sum_{i < l < j} \bar{a}_{il}a_{lj} \right) = 0 \quad \text{for} \quad j - i = k - 1.
\]

Hence, there exists some \( \Psi \) with \( d\Psi = \sum_{i < l < j} \bar{a}_{il}a_{lj} \). Finally, define

\[
a_{ij} = K \left( \sum_{i < l < j} \bar{a}_{il}a_{lj} \right) \quad \text{for} \quad j - i = k - 1,
\]

which belongs to \( B \) and satisfies our claim, proving the result. \( \square \)

**Corollary 3.4** Let \( A \) be a DGA and let \( x_1, x_2, x_3 \in H \) be such that \( \langle x_1, x_2, x_3 \rangle \neq \emptyset \). Then, for any homotopy retract of \( A \) onto \( H \),

\[
\varepsilon m_3(x_1, x_2, x_3) \in \langle x_1, x_2, x_3 \rangle.
\]

\( \square \)

**Corollary 3.5** Let \( A \) be a DGA such that the product on \( H \) is trivial. If \( \langle x_1, x_2, x_3, x_4 \rangle \) is non empty, then, for any homotopy retract,

\[
\varepsilon m_4(x_1, x_2, x_3, x_4) \in \langle x_1, x_2, x_3, x_4 \rangle.
\]

\( \square \)

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