Hyperstability and Stability of a Logarithm-type Functional Equation

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Abstract In 2001, Maksa and Páles [12] introduced a new type’s stability: hyperstability for a class of linear functional equation \( f(x) + f(y) = \frac{1}{n} \sum_{i=1}^{n} f(x \varphi_i(y)) \). Riedel and Sahoo [14] have generalized a functional equation associated with the distance between the probability distributions, which is \( f(pr,qs) + f(ps,qr) = 2M(rs)f(p,q) + 2M(pq)f(r,s) \). Elfen etc. [7] obtained the solution of the functional equation \( f(pr,qs) + f(ps,qr) = 2f(p,q) + 2f(r,s) \) on semigroup \( G \). The aim of this paper is to investigate the hyperstability and the Hyers-Ulam stability for the above Logarithm-type functional equation considered by Elfen, etc. Namely, if \( f \) is an approximative equation related to the above equation, then it is a solution of this equation which exists within \( \varepsilon \) - bound of a given approximative function \( f \).

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1 Introduction

The following stability problem is well-known as Ulam’s stability problem [16]:

\[ f(1) + f(-1) = 2f(0) \]

Let \( G_1 \) be a group and let \( G_2 \) be a metric group with a metric \( d(\cdot,\cdot) \). Given \( \varepsilon > 0 \), does there exist a \( \delta > 0 \) such that if a mapping \( h : G_1 \to G_2 \) satisfies the inequality \( d(h(xy), h(x)h(y)) < \delta \) for all \( x, y \in G_1 \), then there exists a homomorphism \( H : G_1 \to G_2 \) with \( d(h(x), H(x)) < \varepsilon \) for all \( x \in G_1 \)?

In next year, Hyers [11] proved a first partial answer to Ulam’s problem for an additive mapping on a Banach space. D. G. Bourgin obtained many excellent results for the stability ([3], [4]). Hyers’ theorem was generalized by Aoki [1] for the case bounded by variables, and their results are improved by Rassias [13] to the case of the linear mapping and by Ger [9]. Găvruta [8] proved a further generalization of the Rassias’ theorem by using a general control function.

The superstability phenomenon of the exponential equation \( f(x + y) = f(x)f(y) \) was discovered by Baker, Lawrence, and Zorzitto [2] in 1979. The superstability for asymptotic phenomenon of the exponential equation was discovered by Ger [9].

In 2001, Maksa and Páles [12] proved a new type’s stability for a class of linear functional equation

\[ f(x) + f(y) = \frac{1}{n} \sum_{i=1}^{n} f(x \varphi_i(y)), \]

where \( f \) is a real-valued mapping defined on a semigroup \( S \), and the mappings \( \varphi_1, \varphi_2, \cdots, \varphi_n : S \to S \) are pairwise distinct automorphisms. That is as following:

Let \( \varepsilon : S \times S \to \mathbb{R} \) be a function such that there exists a sequence \( u_k \) that satisfies

\[ \lim_{k \to \infty} \varepsilon(u_k s, t) = 0 \quad (s, t \in S). \]

Assume that \( f : S \to X \) satisfies the stability inequality

\[ \left| \left| f(s) + f(t) - \frac{1}{n} \sum_{i=1}^{n} f(s \varphi_i(t)) \right| \right| \leq \varepsilon(s, t) \quad (s, t \in S), \]
where $X$ is a real normed space. Then, $f$ is a solution of (1).

Such a phenomenon is called the hyperstability of the functional equation. Gselmann [10], Brazdk and Cieplinski [5] investigated the hyperstability of functional equations. A similar concept was introduced by Sirouni and Kabbaj [15].

Riedel and Sahoo [14] solved a functional equation associated with the distance between the probability distributions. Let $M : (0, 1) \rightarrow \mathbb{C}$ be a given multiplicative function. Then, if $f : (0, 1)^2 \rightarrow \mathbb{C}$ satisfies the functional equation

$$f(pr, qs) + f(ps, qr) = 2M(rs)f(p, q) + 2M(pq)f(r, s)$$

if and only if

$$f(p, q) = M(p)M(q)\left[L(p) + L(q) + l\left(\frac{p}{q}\right)\right],$$

where $M : (0, 1) \rightarrow \mathbb{C}$ is an arbitrary logarithmic function and $l : (0, 1)^2 \rightarrow \mathbb{C}$ is a bilogarithmic function. Thus, we will call it a logarithm-type functional equation

In addition, Elfen, Riedel and Sahoo [7] solved a functional equation

$$f(pr, qs) + f(sp, rq) = 2f(p, q) + 2f(r, s)$$
on semigroup $G$. Its solution type of $f$ on $G$ is given by

$$f(p, q) = A(p) + A(q) + \psi(pq^{-1}, pq^{-1}),$$

where $A : G \rightarrow \mathbb{C}$ is a homomorphism and $\psi : G \rightarrow \mathbb{C}$ is a symmetric bi-homomorphism.

Now we consider the logarithm-type functional equation given by

$$\frac{1}{2}[f(pr, qs) + f(ps, qr)] = f(p, q) + f(r, s). \tag{2}$$

For example, if $f(x, y) = \ln xy$, then $f$ is a solution of the equation (2). In this paper, we investigate the hyperstability and stability of the functional equation (2). Namely, we prove that if $f$ satisfies a stability inequality for the equation (2), then it is also a solution of this equation and also we can find an another solution of it which has an $\varepsilon$-error bound for $f$.

## 2 Hyperstability of the logarithm-type functional equation

In this section, we investigate the hyperstability of the equation (2). Throughout this section, let $(G, \cdot)$ denote a noncommutative semigroup, $X$ a real normed space, and $\mathbb{R}$ the set of real numbers. And let $\mathbb{R}_+$ denote the set of positive real numbers.

**Theorem 1.** Let $\varepsilon : G^2 \times G^2 \rightarrow \mathbb{R}$ be a function such that there exists a sequence $u_k \in G$ that satisfies

$$\lim_{k \rightarrow \infty} \varepsilon(u_k(p, q), (r, s)) = 0$$

for all $p, q, r, s \in G$. Assume that $f : G \times G \rightarrow X$ satisfies the stability inequality

$$\left\| \frac{1}{2} \left[f(pr, qs) + f(ps, qr)\right] - f(p, q) - f(r, s) \right\| \leq \varepsilon((p, q), (r, s)) \tag{3}$$

for all $p, q, r, s \in G$. Then,

$$\frac{1}{2} \left[f(pr, qs) + f(ps, qr)\right] = f(p, q) + f(r, s).$$

**Proof.** Define a function $F : G^2 \times G^2 \rightarrow X$ by

$$F((p, q), (r, s)) = f(p, q) + f(r, s) - \frac{1}{2} \left[f(pr, qs) + f(ps, qr)\right].$$

Then, for all $p, q, r, s, v, w \in G$, we have

$$F((p, q), (r, s)) = f(p, q) + f(r, s) + f(v, w)$$

and

$$F((v, w), (p, q)) = f(p, q) + f(r, s) + f(v, w).$$

Then the stability inequality (3) holds.

$$F((p, q), (r, s)) = f(p, q) + f(r, s) - \frac{1}{2} \left[f(pr, qs) + f(ps, qr)\right].$$
And also, for all \( p, q, r, s \in G \), we have

\[
F((r, s), (v, w)) + \frac{1}{2} \left[ F((p, q), (rv, sw)) + F((p, q), (rw, sv)) \right] = f(p, q) + f(r, s) + f(v, w)
\]

Thus, \( F \) satisfies the following functional equation

\[
F((p, q), (r, s)) + \frac{1}{2} \left[ F((pr, qs), (v, w)) + F((ps, qr), (v, w)) \right] = f((r, s), (v, w)) + \frac{1}{2} \left[ F((p, q), (rv, sw)) + F((p, q), (rw, sv)) \right].
\]

By (3), we get

\[
||F((p, q), (r, s))|| \leq \varepsilon((p, q), (r, s)),
\]

and with the assumed sequence \( \{u_k\} \), we obtain

\[
\lim_{k \to \infty} F(u_k(p, q), (r, s)) \leq \lim_{k \to \infty} \varepsilon(u_k(p, q), (r, s))
\]

for all \( p, q, r, s \in G \).

The equation (4) implies

\[
F((r, s), (v, w)) = F((p, q), (r, s)) + \frac{1}{2} \left[ F((pr, qs), (v, w)) + F((ps, qr), (v, w)) \right] - \frac{1}{2} \left[ F((p, q), (rv, sw)) + F((p, q), (rw, sv)) \right].
\]

Let \( r, s, v, w, p_0, q_0 \) be fixed. Applying the norm and substituting \( p = u_kp_0, q = u_kq_0 \) in (6), and as \( k \to \infty \), respectively, we obtain

\[
|| \lim_{k \to \infty} F(r, s), (v, w) || = || \lim_{k \to \infty} \left[ F(u_k(p_0, q_0), (r, s)) + \frac{1}{2} F(u_k(p_0 r, q_0 s), (v, w)) + F(u_k(p_0 s, q_0 r), (v, w)) \right] - \frac{1}{2} \left[ F(u_k(p_0, q_0), (rv, sw)) + F(u_k(p_0, q_0), (rw, sv)) \right] ||.
\]

By applying of (5) and the triangle inequalities, we obtain

\[
|| F(r, s), (v, w) || \leq \left[ \lim_{k \to \infty} \varepsilon(u_k(p_0, q_0), (r, s)) + \frac{1}{2} \left[ \varepsilon(u_k(p_0 r, q_0 s), (v, w)) + \varepsilon(u_k(p_0 s, q_0 r), (v, w)) \right] \right] + \left[ \lim_{k \to \infty} \frac{1}{2} \left[ \varepsilon(u_k(p_0, q_0), (rv, sw)) + \varepsilon(u_k(p_0, q_0), (rw, sv)) \right] \right].
\]

Hence, we obtain from the assumed sequence \( \{u_k\} \) the required result

\[
F((r, s), (v, w)) = 0
\]

for any \( r, s, v, w \in G \).

\[\square\]

**Corollary 2.** Assume that \( f : R_+ \times R_+ \to X \) satisfies the stability inequality

\[
\left| \frac{1}{2} \left[ f(pr, qs) + f(ps, qr) \right] - f(p, q) - f(r, s) \right| \leq \frac{rs}{pq} \quad \text{or} \quad pqr
\]

for all \( p, q, r, s \in R_+ \). Then,

\[
\frac{1}{2} \left[ f(pr, qs) + f(ps, qr) \right] = f(p, q) + f(r, s).
\]

**Proof.** Let \( \varepsilon((p, q), (r, s)) = \frac{rs}{pq} \) and \( u_k = a^k \) for \( a > 1 \), or \( \varepsilon((p, q), (r, s)) = pqr \) and \( u_k = a^k \) for \( 0 < a < 1 \). Then, we obtain

\[
\lim_{k \to \infty} \varepsilon(u_k(p, q), (r, s)) = 0,
\]

so the result holds. \[\square\]
3 Stability of the logarithm-type functional equation

In this section, we investigate the stability of the equation (2). Throughout this section, let \((G, \cdot)\) denote a commutative semigroup, \(N\) the set of natural numbers, and \(X\) a Banach space.

**Theorem 3.** Let \(\varepsilon > 0\). Assume that \(f : G \times G \to X\) satisfies the stability inequality

\[
\left\| \frac{1}{2} \left[ f(pr, qs) + f(ps, qr) \right] - f(p, q) - f(r, s) \right\| \leq \varepsilon
\]

for all \(p, q, r, s \in G\). Then there exists a function \(F : G \times G \to X\) such that

\[
\frac{1}{2} \left[ F(pr, qs) + F(ps, qr) \right] = F(p, q) + F(r, s)
\]

and \(\| F(p, q) - f(p, q) \| \leq \frac{3\varepsilon}{40} \) for any \(p, q \in G\), where \(F\) is defined by

\[
F(p, q) := \lim_{n \to \infty} \left[ \frac{1}{2^{2n}} f(p^{2n}, q^{2n}) + \left( \sum_{i=0}^{n-2} \frac{1}{2^i} \cdot \frac{1}{2^{2n-1}} + \frac{1}{2^{n+1}} \right) f((pq)^{2n-1}, (pq)^{2n-1}) \right]
\]

for any \(p, q, r, s \in G\).

**Proof.** Letting \(r = p, s = q\) in (8) and dividing it by 2, we have

\[
\left\| \frac{1}{2^2} \left[ f(p^2, q^2) + f(pq, pq) \right] - f(p, q) \right\| \leq \frac{\varepsilon}{2}.
\]

And also, letting \(p = q = r = s\) in (8) and dividing it by 2, we have

\[
\left\| \frac{1}{2} f(p^2, p^2) - f(p, p) \right\| \leq \frac{\varepsilon}{2}.
\]

Let us show that the following inequality holds for every \(n \in N\):

\[
\left\| \frac{1}{2} f(p^{2n}, p^{2n}) - f(p^{2n-1}, p^{2n-1}) \right\| \leq \frac{\varepsilon}{2}.
\]

Replacing \(p\) by \(p^2\) and \(q\) by \(q^2\) in (9) respectively, and dividing \(2^2\), we have

\[
\left\| \frac{1}{2^2} \left[ f(p^{2^2}, q^{2^2}) + f((pq)^2, (pq)^2) \right] - \frac{f(p^2, q^2)}{2^2} \right\| \leq \frac{\varepsilon}{2 \cdot 2^2}.
\]

Thus by (9),(10), and (12), we have

\[
\left\| \frac{1}{2^{2n}} f(p^{2n}, q^{2n}) + \left( \frac{1}{2^{2n-2}} + \frac{1}{2} \cdot \frac{1}{2^{2n-2}} \right) f((pq)^{2n-1}, (pq)^{2n-1}) - f(p, q) \right\|
\leq \left\| \frac{1}{2^{2n}} \left[ f(p^{2n}, q^{2n}) + f((pq)^{2n-1}, (pq)^{2n-1}) \right] - \frac{f(p^2, q^2)}{2^{2n}} \right\|
\leq \frac{\varepsilon}{2 \cdot 2^{2n}} + \frac{\varepsilon}{2} + \frac{\varepsilon}{2 \cdot 2^{2n-2}} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2^{2n}}.
\]

In addition, by letting \(p\) by \(p^{2^{n-1}}\) and \(q\) by \(q^{2^{n-1}}\) in (9), and dividing \(2^{2^{n-1}}\), the following inequality holds for every \(n \in N\):

\[
\left\| \frac{1}{2^{2n}} \left[ f(p^{2^{n}}, q^{2^{n}}) + f((pq)^{2^{n-1}}, (pq)^{2^{n-1}}) \right] - \frac{f(p^{2^{n-1}}, q^{2^{n-1}})}{2^{2n-1}} \right\|
\leq \frac{\varepsilon}{2 \cdot 2^{2n-1}}.
\]
By (10), (12), and (14), we have

\[
\left\| \frac{1}{2^{n+1}} f(p^{2^n}, q^{2^n}) + \left( \frac{1}{2^{2^n}} + \frac{1}{2^{2^n+1}} \right) f((pq)^{2^{n-1}}, (pq)^{2^{n-1}}) - f(p, q) \right\|
\leq \left\| \frac{1}{2^{2^n}} \left[ f(p^{2^n}, q^{2^n}) + f((pq)^{2^{n-1}}, (pq)^{2^{n-1}}) \right] - f(p^{2^{n-1}}, q^{2^{n-1}}) \right\|
+ \left\| \frac{1}{2^{2^n}} \left[ f(p^{2^n}, q^{2^{n-1}}) + \left( \frac{1}{2^{2^n}} + \frac{1}{2^{2^n+1}} \right) f((pq)^{2^n}, (pq)^{2^n}) - f(p, q) \right] \right\|
+ \left( \frac{1}{2^{2^n}} + \frac{1}{2^{2^n+1}} \right) \left\| f((pq)^{2^{n-1}}, (pq)^{2^{n-1}}) - f((pq)^{2^{n-1}}, (pq)^{2^{n-1}}) \right\|
\leq \frac{\varepsilon}{2^{2^n}} + \frac{\varepsilon}{2^{2^n+2}} + \left( \frac{\varepsilon}{2^{2^n}} + \frac{\varepsilon}{2^{2^n+2}} \right) + \frac{\varepsilon}{2^{2^n+1}} \left( \frac{1}{2^{2^n}} + \frac{1}{2^{2^n+1}} + \frac{1}{2^{2^n+2}} + \cdots \right)
+ \cdots + \frac{\varepsilon}{2^{n-1}} \left( \frac{1}{2^{2^n}} + \frac{1}{2^{2^n+1}} + \frac{1}{2^{2^n+2}} + \cdots \right)
+ \frac{\varepsilon}{2^{2^n}} + \frac{\varepsilon}{2^{2^n+2}} + \frac{\varepsilon}{2^{2^n+3}} \left( \frac{1}{2^{2^n}} + \frac{1}{2^{2^n+1}} + \frac{1}{2^{2^n+2}} + \cdots \right)
+ \cdots + \frac{\varepsilon}{2^{n-1}} \left( \frac{1}{2^{2^n}} + \frac{1}{2^{2^n+1}} + \frac{1}{2^{2^n+2}} + \cdots \right)
+ \frac{\varepsilon}{2^{2^n}} + \frac{\varepsilon}{2^{2^n+2}} + \frac{\varepsilon}{2^{2^n+3}} \left( \frac{1}{2^{2^n}} + \frac{1}{2^{2^n+1}} + \frac{1}{2^{2^n+2}} + \cdots \right)

\leq \frac{\varepsilon}{2^{2^n}} + \frac{\varepsilon}{2^{2^n+2}} + \frac{\varepsilon}{2^{2^n+3}} + \cdots + \frac{\varepsilon}{2^{2^n+1}} + \frac{\varepsilon}{2^{2^n}} + \frac{\varepsilon}{2^{2^n+2}} + \frac{\varepsilon}{2^{2^n+3}} + \cdots
\leq \frac{\varepsilon}{2^{2^n}} + \frac{\varepsilon}{2^{2^n+2}} + \frac{\varepsilon}{2^{2^n+3}} + \cdots + \frac{\varepsilon}{2^{2^n+1}} + \frac{\varepsilon}{2^{2^n}} + \frac{\varepsilon}{2^{2^n+2}} + \frac{\varepsilon}{2^{2^n+3}} + \cdots
\leq \frac{39\varepsilon}{40}.
\]

Suppose that the following inequality holds for \( n \geq 4 \) and for any \( p, q \in G \):

\[
\left\| \frac{1}{2^{2^n}} f(p^{2^n}, q^{2^n}) + \left( \sum_{i=0}^{n-2} \frac{1}{2^{2^{n-i}}} \right) f((pq)^{2^{n-1}}, (pq)^{2^{n-1}}) - f(p, q) \right\|
\leq \sum_{j=4}^{n} \sum_{i=2}^{j-2} \frac{\varepsilon}{2^{2^i} \cdot 2^{2^{j-i}}} + \sum_{i=1}^{n} \frac{\varepsilon}{2^{2^i-1}} + \sum_{i=3}^{n} \frac{\varepsilon}{2^{i+1}}.
\]

Note that

\[
\sum_{i=0}^{n-1} \frac{1}{2^{2^i} \cdot 2^{2^{n+1-i}}} = \frac{1}{2^{2^{n+1}}} + \frac{1}{2} \sum_{i=0}^{n-2} \frac{1}{2^{2^{n-i}}}.
\]
for all \( n \in N \). Then, for any \( p, q \in G \), based on (14) and (18), we obtain

\[
\left| \frac{1}{2 \cdot 2^n} f(p^{2n+1}, q^{2n+1}) \right| + \left( \sum_{i=0}^{n-1} \frac{1}{2^{i+1} \cdot 2^n} + \frac{1}{2^{n+1}} \right) \left| f((pq)^{2n}, (pq)^{2n}) - f(p, q) \right| \leq \left| \frac{1}{2 \cdot 2^n} f(p^{2n+1}, q^{2n+1}) \right| + \left( \sum_{i=0}^{n-1} \frac{1}{2^{i+1} \cdot 2^n} + \frac{1}{2^{n+1}} \right) \left| f((pq)^{2n}, (pq)^{2n}) - f(p, q) \right|
\]

(19)

Thus, by induction, inequality (17) holds for all \( n \geq 4 \) and for any \( p, q \in G \). Now for \( n \geq 4 \), we have

\[
\left| \frac{1}{2 \cdot 2^n} f(p^{2n}, q^{2n}) \right| + \left( \sum_{i=0}^{n-2} \frac{1}{2^{i+1} \cdot 2^n} + \frac{1}{2^{n+1}} \right) \left| f((pq)^{2n-1}, (pq)^{2n-1}) - f(p, q) \right| \leq \left| \frac{1}{2 \cdot 2^n} f(p^{2n}, q^{2n}) \right| + \left( \sum_{i=0}^{n-2} \frac{1}{2^{i+1} \cdot 2^n} + \frac{1}{2^{n+1}} \right) \left| f((pq)^{2n-1}, (pq)^{2n-1}) - f(p, q) \right|
\]

(20)

as \( n \to \infty \). Thus, if we let

\[
Y_n = \frac{1}{2 \cdot 2^n} f(p^{2n}, q^{2n}) + \left( \sum_{i=0}^{n-2} \frac{1}{2^{i+1} \cdot 2^n} + \frac{1}{2^{n+1}} \right) f((pq)^{2n-1}, (pq)^{2n-1}),
\]

0
then \( \{ Y_n \} \) is a Cauchy sequence due to (20), and so we can define a function \( F : G \times G \to X \) by

\[
F(p, q) := \lim_{n \to \infty} \left[ \frac{1}{2n} f(p^{2n}, q^{2n}) + \left( \sum_{i=0}^{n-2} \frac{1}{2i, 2^{2n-i} + \frac{1}{2n+1}} \right) f((pq)^{2n-1}, (pq)^{2n-1}) \right].
\]

(21)

Then, due to (16), (17), and (21), we have

\[
\|F(p, q) - f(p, q)\| \leq \frac{39\varepsilon}{40} \quad \forall p, q \in G.
\]

Finally, the function \( F \) defined in (21) holds the required equation (2) as follows:

\[
\left| \frac{1}{2} \left[ F(pr, qs) + F(ps, qr) \right] - F(p, q) - F(r, s) \right| 
\leq \lim_{n \to \infty} \left| \frac{1}{2} \frac{1}{2n} f((pr)^{2n}, (qs)^{2n}) + \left( \sum_{i=0}^{n-2} \frac{1}{2i, 2^{2n-i} + \frac{1}{2n+1}} \right) f((pqrs)^{2n-1}, (pqrs)^{2n-1}) 
+ \frac{1}{2} \frac{1}{2n} f((ps)^{2n}, (qr)^{2n}) + \left( \sum_{i=0}^{n-2} \frac{1}{2i, 2^{2n-i} + \frac{1}{2n+1}} \right) f((pqrs)^{2n-1}, (pqrs)^{2n-1}) 
- \frac{1}{2n} f((pr)^{2n}, (q)^{2n}) - \left( \sum_{i=0}^{n-2} \frac{1}{2i, 2^{2n-i} + \frac{1}{2n+1}} \right) f((pq)^{2n-1}, (pq)^{2n-1}) 
- \frac{1}{2n} f((r)^{2n}, (s)^{2n}) - \left( \sum_{i=0}^{n-2} \frac{1}{2i, 2^{2n-i} + \frac{1}{2n+1}} \right) f((rs)^{2n-1}, (rs)^{2n-1}) \right| 
\leq \lim_{n \to \infty} \frac{1}{2n} \left| \frac{1}{2} \left[ f((pr)^{2n}, (qs)^{2n}) + f((pq)^{2n}, (qr)^{2n}) \right] 
- f(p^{2n}, q^{2n}) - f(r^{2n}, s^{2n}) \right| 
+ \lim_{n \to \infty} \left( \sum_{i=0}^{n-2} \frac{1}{2i, 2^{2n-i} + \frac{1}{2n+1}} \right) 
\times \left| \frac{1}{2} \left[ f((pqrs)^{2n-1}, (pqrs)^{2n-1}) + f((pqrs)^{2n-1}, (pqrs)^{2n-1}) \right] 
- f((pq)^{2n-1}, (pq)^{2n-1}) - f((pq)^{2n-1}, (pq)^{2n-1}) \right| 
\leq \lim_{n \to \infty} \frac{\varepsilon}{2n} + \lim_{n \to \infty} \left( \sum_{i=0}^{n-2} \frac{1}{2i, 2^{2n-i} + \frac{1}{2n+1}} \right) \varepsilon 
= 0.
\]

\[ \square \]

**Corollary 4.** Let \( \varepsilon > 0 \). Assume that \( f : G \times G \to X \) satisfies the stability inequality

\[
\left| \frac{1}{2^2} \left[ f(p^2, q^2) + f(pq, pq) \right] - f(p, q) \right| \leq \varepsilon
\]

(22)

for all \( p, q \in G \). Then there exists a function \( F : G \times G \to X \) such that

\[
\frac{1}{2^2} \left[ F(p^2, q^2) + F(pq, pq) \right] = F(p, q)
\]

and \( \|F(p, q) - f(p, q)\| \leq \frac{39\varepsilon}{40} \) for any \( p, q \in G \).
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