Metric on Quantum Spaces

Andrzej Sitarz

Department of Field Theory
Institute of Physics
Jagiellonian University
Reymonta 4, 30-059 Kraków, Poland

Abstract

We introduce the analogue of the metric tensor in case of $q$-deformed differential calculus. We analyse the consequences of the existence of the metric, showing that this enforces severe restrictions on the parameters of the theory. We discuss in detail the examples of the Manin plane and the $q$-deformation of $SU(2)$. Finally we touch the topic of relations with the Connes’ approach.
1 Introduction

Quantum groups and quantum spaces are an interesting non-trivial generalization of Lie groups and manifolds [1]. The deformation parameter $q$ allows to recover the latter in the continuous limit $q \to 1$, which suggests that the noncommutativity of space could possibly provide a regularization mechanism [2]. Therefore one may expect that $q$-deformations could be an interesting basis for physical theories, in particular for gravity and gauge theories [3]. The natural language for such studies is the $q$-deformed differential calculus (see [4] for a review) constructed within the framework of noncommutative differential geometry [7]. The construction, however is not unique, and even after imposing bicovariance in the situation of quantum groups, in general we are left with many choices of possible theories.

The additional problem, which arises in the course of constructing physical $q$-deformed theories is the question of metric. In the classical situation, $q = 1$, the metric is given by the metric tensor. The latter could be equivalently defined as a bilinear functional from $\Omega^1 \times \Omega^1$ to $\mathcal{A}$, where $\mathcal{A}$ is the original commutative algebra, and $\Omega^1$ is the bimodule of one-forms over $\mathcal{A}$. Now, we may generalize it to the case of noncommutative geometry and define metric as a middle-linear functional $\eta : \Omega^1 \times \Omega^1 \to \mathcal{A}$, i.e. $\eta$ is linear with respect to addition in $\Omega^1$ and satisfies:

$$\eta(a\omega b, \rho c) = a\eta(\omega, b\rho)c,$$

(1)

for every $a, b, c \in \mathcal{A}$ and $\omega, \rho \in \Omega^1$. The middle linearity naturally replaces the bilinearity condition in this case, however, we shall see that it is far more restrictive. The above definition proved to be suitable in our studies of discrete geometries [5].

We may also introduce the notion of hermitian metric, which could be defined on the differential algebra with involution. We say that metric $\eta$ is hermitian, if for all one-forms $u, v$ the following identity holds:

$$\eta(u, v) = (\eta(v^*, u^*))^*.$$

(2)

In this paper we shall briefly discuss the consequences of the introduction of the metric to the analysis of $q$-deformed theories. We shall concentrate on two simple examples, leaving the general case for future studies [6]. Finally, we shall discuss the relations between the above definition of the metric and the approach of Connes [7].
2 Metric on the Manin Plane

Let us remind that the Manin plane is defined by replacing the commutativity of the generators of \([x, y]\) by the relation:

\[ xy = qyx, \tag{3} \]

where \(q\) is a complex unitary number, \(q\bar{q} = 1\). We restrict our considerations here to the algebra obtained as a quotient of the free algebra generated by \(y\) and \(y\) by the ideal set by the relation (3).

Now, let us consider a \(GL(2)_q\) invariant differential calculus [8]. It has one free parameter \(s\) and its multiplication rules are as follows:

\[
\begin{align*}
xdx & = sdx, \\
xdy & = (s - 1)dy + qdy, \\
ydx & = sq^{-1}dx, \\
ydy & = sdy.
\end{align*}
\tag{4-7}
\]

The metric, due to the middle-linearity, is completely determined by its values on the forms \(dx\) and \(dy\). If we call them \(\eta^{xx}\) for \(\eta(dx, dx)\), and \(\eta^{xy}, \eta^{yx}, \eta^{yy}\) for other combinations respectively, we find the following set of constraints:

\[
\begin{align*}
x\eta^{xx} & = s^2 \eta^{xx}x, \\
y\eta^{xx} & = s^2 q^{-2} \eta^{xy}y, \\
x\eta^{xy} & = s(s - 1) \eta^{xx}y + sq \eta^{xy}x, \\
y\eta^{xy} & = s^2 q^{-1} \eta^{xy}y, \\
x\eta^{yx} & = s(s - 1)q^{-1} \eta^{xx}y + sq \eta^{yx}x, \\
y\eta^{yx} & = s^2 q^{-1} \eta^{yx}y, \\
x\eta^{yy} & = s(s - 1) \eta^{xy}y + (s - 1)q \eta^{yx}y + q^2 \eta^{yy}x, \\
y\eta^{yy} & = s^2 \eta^{yy}y. \\
\end{align*}
\tag{8-15}
\]

Now, if we analyse them we find restrictions on the parameters \(s\) and \(q\). Since only monomials satisfy the commutation relations of the type (3), we come to conclusion that in the relation (8) \(s^2\) must be equal to \(q^n\) for
some $n \geq 0$. Analogously, from (15) we see that $s^2$ must be $q^{-m}$ for $m \geq 0$. Therefore, either $n = m = 0$ and hence $s^2 = 1$ or $q^n m = 1$ and $s$ is any of the powers of $q$. In either case the possible values of $s$ are restricted to a finite set. In particular, we have found that out of infinitely many models of $q$-deformed, $GL(2)_q$ invariant differential calculus on the Manin plane, only two of them admit a metric for every value $q$.

3 Metric on $SU(2)_q$

The Hopf algebra of $SU(2)_q$ is generated by two elements (and their conjugates), satisfying the following relations [9]:

\[
\begin{align*}
    a^* a + b^* b &= 1 \\
    b^* b &= b b^* \\
    a b &= q b a \\
    a^* b^* &= q b^* a
\end{align*}
\]

where $q \in [-1, 1]$.

Following Woronowicz [9] we introduce the bimodule of one-forms, to be a free right-module generated by three elements $c^0, c^1, c^2$, with the following rules of left multiplication by the generator of $SU(2)_q$:

\[
\begin{align*}
    c^0 a &= q^{-1} ac^0 \\
    c^0 b &= q^{-1} bc^0 \\
    c^0 a^* &= qa^* c^0 \\
    c^0 b^* &= qb^* c^0 \\
    c^1 a &= q^{-2} ac^1 \\
    c^1 b &= q^{-2} bc^1 \\
    c^1 a^* &= q^2 a^* c^1 \\
    c^1 b^* &= q^2 b^* c^1 \\
    c^2 a &= q^{-1} ac^2 \\
    c^2 b &= q^{-1} bc^2 \\
    c^2 a^* &= qa^* c^2 \\
    c^2 b^* &= qb^* c^2.
\end{align*}
\]

Additionally, the involution is extended to the bimodule of one forms and we have:

\[
\begin{align*}
    (c^0)^* &= qc^0 \quad (c^1)^* = -c^1 \quad (c^2)^* = q^{-1} c^2.
\end{align*}
\]

Suppose now that we introduce the metric $\eta$, as proposed in the first section. Since the module of one-forms is free, the metric is completely determined by its values on the one-forms forming the basis. Let us call
η(c_i, c_j) by η^{ij}, i = 0, 1, 2. Then, if we impose the condition of middle-linearity, we obtain the following set of relations for the elements η^{ij}:

\[ a\eta^{ij} = q^{\phi(i,j)} \eta^{ij}a, \]  
\[ a^* \eta^{ij} = q^{-\phi(i,j)} \eta^{ij}a^*, \]  
\[ b\eta^{ij} = q^{\phi(i,j)} \eta^{ij}b, \]  
\[ b^* \eta^{ij} = q^{-\phi(i,j)} \eta^{ij}b^*, \]

where \( \phi(i, j) \) is defined as:

\[ \phi(i, j) = \begin{cases} 
4 & \text{if } i = j = 1 \\
3 & \text{if } i \neq j = 1 \text{ or } j \neq i = 1 \\
2 & \text{if } i \neq 1 \text{ and } j \neq 1 
\end{cases} . \]

One may easily verify that in the considered algebra such constraints may be satisfied only if \( q^2 = 1 \). Indeed, from the relations (18) and (19) we obtain that \( q^{2\phi(i,j)} = 1 \). By taking all possible values of \( i \) and \( j \) we recover the above condition \( q^2 = 1 \). In the only non-trivial case \( q = -1 \) we could have, for instance, the following metric:

\[ \eta^{ij} = ab, \text{ for } i \neq j = 1 \text{ or } j \neq i = 1, \]  
and the other components taken as constants. Of course, we could scale each component by an arbitrary element of the center of the algebra.

In the case of SU(2)_q the existence of the metric is a very strong requirement, which practically determines the value of the deformation parameter \( q \) in the considered example of the differential calculus.

4 Conclusions

As we have shown in two previous sections, the existence of a non-trivial metric is, in general, a very strong assumption. We have demonstrated that the noncommutativeness of the original algebra as well as of the differential calculus, enforce severe restrictions on the possible metrics. They could not be satisfied in general and lead to the constraints on the free parameters of theory. Therefore some models of differential calculus seem to be selected in a natural way by admitting the existence of the metric. It should be therefore
interesting to determine such relations for other models, in particular for the general case of the bicovariant differential calculus on quantum groups.

Having defined the metric one could also use the construction to pursue the physical aspect of $q$-deformed theories. The natural next step should be the introduction of linear connections and $q$-deformed gravity, which is the topic of our current investigation [6].

5 Appendix

In Connes approach, the basic object is a $K$-cycle, defined by the algebra $\mathcal{A}$, its representation $\pi$ on a Hilbert space and the Dirac operator $D$. The differential algebra could be derived from this construction by extending the definition of $\pi$ to the universal differential algebra $\Omega(\mathcal{A})$:

$$\pi(a^0 da^1 \ldots da^n) = a^0[D, a^1] \ldots [D, a^n],$$

and by dividing $\Omega(\mathcal{A})$ by the differential ideal $\pi^{-1}(0) + d\pi^{-1}(0)$.

Now, introducing a Dixmier trace, we have the integration on $\mathcal{A}$:

$$\int a = \text{Tr} \pi(a),$$

as well as a complex valued functional on the bimodule of one-forms:

$$\langle u, v \rangle = \text{Tr}(\pi(u)\pi(v)).$$

Let us turn back to the situation we were analysing. If we have a hermitian metric $\eta$ and a positive trace (integration $\int$) on the algebra $\mathcal{A}$ (which could be equivalent to the Dixmier trace), we can recover the functional of the type (23) as follows:

$$\langle u, v \rangle = \int \eta(u, v),$$

Now, the interesting question is whether the existence of the metric for a given differential calculus over $\mathcal{A}$ is equivalent to the existence of the corresponding $K$-cycle over $\mathcal{A}$. If so, we could use the results of our studies of the metric tensor in noncommutative geometry also in the broader context. This should provide us with a link, which would enable to extend the discrete geometry formalism of the Standard model [10] to include also the gravitational component. Additionally, we could then proceed with the introduction of $q$-deformed spinors, attempting to deform physical models of fundamental matter fields.
References

[1] V.Drinfeld, Sov.Math.Dokl. 32 (1985) 254, 
M.Jimbo, Lett. Math.Phys. 10 (1985) 63, 
S.Majid, Int.J.Mod.Phys. A5 (1990) 1.

[2] S.Majid, Int.J.Mod.Phys. A5 (1990) 4689.

[3] L.Castellani, preprint DFTT 19/92, 
L.Castellani, preprint DFTT 74/92 
T.Brzezinski, S.Majid, preprint DAMPT/92-27 
T.Brzezinski, S.Majid, Phys.Lett. B298 (1993) 339,

[4] B.Zumino, preprint LBL-33249 
P.Aschieri,L.Castellani, preprint CERN-TH.6565/92

[5] A.Sitarz, preprint TPJU 7/92 
A.Sitarz, preprint TPJU 4/93, to appear in Phys.Lett.B,

[6] A.Sitarz, in preparation

[7] A.Connes, Publ.Math. IHES Vol. 62 (1986) 41, 
A.Connes, Géométrie non commutative (Inter Editions, Paris, 1990); 
Noncommutative Geometry, (Academic Press, in press),

[8] T.Brzezinski, H.Dabrowski, J.Rembielinski, J.Math.Phys. 33 (1992) 19, 

[9] S.L.Woronowicz, Publ. RIMS, Kyoto Univ., Vol.23, 1 (1987), 117,

[10] A.Connes, J.Lott, Nucl.Phys.B 18B (1990) 29,