A note on geodesics on ellipsoid

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Abstract

The purpose of this note is to show that the Jacobi problem of geodesics on ellipsoid [1], [2], [3] may be reduced to a more simple case, namely, to the Clebsch problem [4]. The last one is the problem with quadratic nonlinearity and may be solved directly by using Weber’s approach [5] in terms of multi-dimensional theta functions.

1. The problem of geodesics on ellipsoid is a classical one. For two-dimensional ellipsoid, its solution was announced by Jacobi on 28 December, 1838 [1] (see Appendix). Using the remarkable substitution, he reduced this problem to quadratures. These results were published in the paper [2] and then considered in details with the use of elliptic coordinates in his lectures at Königsberg University in 1842/43. They can be found in the book [3].

Later Weierstrass described another approach and succeeded in integrating the equations for geodesics explicitly in terms of two-dimensional theta functions [6].

Another solution of this problem was given by Moser ([7], [8]) who included it to the modern scheme of isospectral deformations and described the relation of this problem with the geometry of quadrics and algebraic geometry of spectral curve. For n-dimensional case, the algebro-geometrical approach was given by Knörer [9].

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Here we show that this problem may be considered without using elliptic coordinates and algebraic geometry as a projection of a simpler system (for a general description of this method, see [10], [11]). Namely, it may be considered as a projection of the system with a larger number of degrees of freedom and with the simplest (quadratic) nonlinearity, the so-called Clebsch system [4]. The equations of motion for this system were integrated in terms of multi-dimensional theta functions by Weber [5] (for modern exposition, see [12]).

2. The \((n - 1)\)-dimensional ellipsoid in \(n\)-dimensional Euclidean space \(\mathbb{R}^n\) is defined by the equation

\[
Q_0(x) = \sum_{j=1}^{n} \frac{x_j^2}{a_j} = 1. \tag{1}
\]

The standard metric in Euclidean space induces the metric on ellipsoid. The free motion of a point particle of unit mass on such an ellipsoid is described by the Hamiltonian

\[
H = \frac{1}{2} \sum_{j=1}^{n} y_j^2. \tag{2}
\]

The equations of motion have the form

\[
\dot{x}_j = y_j, \quad \dot{y}_j = \ddot{x}_j = -\nu(x, y) \frac{x_j}{a_j}, \quad j = 1, \ldots, n. \tag{3}
\]

To obtain \(\nu(x, y)\), we shall differentiate the equation (1) twice with respect to time. As a result, we get

\[
\sum_{j=1}^{n} \frac{x_j y_j}{a_j} = 0 \tag{4}
\]

and

\[
\nu(x, y) = \frac{B(y)}{A(x)}. \tag{5}
\]

where

\[
B(y) = \sum_{j=1}^{n} \frac{y_j^2}{a_j}, \quad A(x) = \sum_{j=1}^{n} \frac{x_j^2}{a_j^2}. \tag{6}
\]
Theorem. Equations (3), (5), (6) for geodesics on ellipsoid are equivalent to the equations of motion for the Clebsch system,

\[ \frac{d}{d\tau} y_j = - \sum_k \omega_{jk} y_k, \]

\[ \frac{d}{d\tau} l_{jk} = \sum_m (l_{jm} \omega_{mk} - \omega_{jm} l_{mk}) - (a_j^{-1} - a_k^{-1}) y_j y_k. \] (7)

Here we introduced additional variables related to angular momentum tensor

\[ l_{jk} = x_j y_k - x_k y_j \quad \text{and} \quad \omega_{jk} = \frac{1}{a_j a_k} l_{jk} \] (8)

and \( \tau \) is a local time defined by the formula

\[ d\tau = A^{-1} dt, \quad \frac{d}{d\tau} = A \frac{d}{dt}. \] (9)

Proof. After the change of a standard time \( t \) by a local time \( \tau \), the right-hand side of equations (3) takes a polynomial form

\[ \frac{d}{d\tau} x_j = A(x) y_j, \quad \frac{d}{d\tau} y_j = - B(y) \frac{x_j}{a_j}. \] (10)

One can see that there are following identities:

\[ B(y) \frac{x_j}{a_j} = \sum_k \omega_{jk} y_k, \quad B(y) = \sum_{j<k} \frac{1}{a_j a_k} l_{jk}^2, \] (11)

\[ (a_j^{-1} - a_k^{-1}) B(y) x_j x_k = \sum_n (l_{jm} \omega_{mk} - \omega_{jm} l_{mk}) - (a_j^{-1} - a_k^{-1}) y_j y_k. \]

From here, it follows equations (7). They are the equations with the simplest, namely, quadratic nonlinearity and they are exactly equations of motion for the Clebsch system [4].

3. It is easy to check that equations (7) have \( n \) independent integrals of motion,

\[ F_j = y_j^2 + \frac{1}{a_j - a_k} l_{jk}^2, \quad j, k = 1, \ldots, n, \] (12)
in involution, \( \{ F_j, F_k \} = 0 \). So, the system under consideration is completely integrable.

The generating function of integrals of motion has the form
\[
G_\lambda(y, l) = \sum_{j=1}^{n} \frac{1}{a_j - \lambda} F_j = \sum_{j=1}^{n} \frac{y_j^2}{a_j - \lambda} - \sum_{j=1}^{n} \frac{l_{jk}^2}{(a_j - \lambda)(a_k - \lambda)},
\]
and we have
\[
\{ G_\lambda, G_\mu \} = 0.
\]

Let us rewrite these equations in the Hamiltonian form. For this, we use the Poisson structure related to the Lie algebra \( e(n) \) of motion of Euclidean space \( \mathbb{R}^n \),
\[
\{ l_{ij}, l_{km} \} = \delta_{ik} l_{jm} - \delta_{im} l_{jk} - \delta_{jk} l_{im} + \delta_{jm} l_{ik},
\]
\[
\{ l_{ij}, y_k \} = \delta_{jk} y_i - \delta_{ik} y_j, \quad \{ y_j, y_k \} = 0.
\]
So, the quantities \( y_j, l_{km} \) generate the Lie algebra \( e(n) \) of motion in \( n \)-dimensional Euclidean space.

The Hamiltonian \( H \) should be the function of \( F_j \) and in fact, due to the quadratic form of equations of motion, it should be a linear combination of \( F_j \),
\[
H = \frac{1}{2} b_j F_j.
\]
The simple calculations show that \( b_j = a_j^{-1} \). Hence
\[
\dot{y}_j = \{ H, y_j \}, \quad \dot{l}_{jk} = \{ H, l_{jk} \},
\]
\[
H = \frac{1}{2} \left( \sum_j \frac{y_j^2}{a_j} - \sum_{i,j} \frac{l_{ij}^2}{a_i a_j} \right) = G_0(y, l).
\]
Eqs. (15) are the Hamiltonian equations related to the Poisson structure for the Lie algebra \( e(n) \).

For two-dimensional case these equations were integrated explicitly by Weber [5] in terms of theta functions of two variables. For multi-dimensional case, see [9] and [12].
Note that as it was shown by Joachimsthal [13], the quantity

\[ I(x, y) = A(x) B(y) \]

is an integral of motion, \( I(x, y) = c^2 \).

Now we may obtain the expressions for the quantities \( x_j \) and \( A(x) \):

\[
x_j(\tau) = -B^{-1}(y) a_j \frac{d}{d\tau} y_j, \quad A(x) = \sum_j \frac{x_j^2}{a_j^2} = c^2 B^{-1}(y).
\]  \( (16) \)

Finally, the expression for \( t \) as the function of \( \tau \) is given by a quadrature

\[
t(\tau) = \int_0^\tau A(\tau) d\tau.
\]  \( (17) \)

So, we get the formulae for all dynamical variables.
Appendix

Lettre de M. Jacobi à M. Arago, concernant les lignes géodésiques tracées sur un ellipsoide à trois axes [1].

Königsberg, le 28 décembre 1838

Monsieur,

Je suis parvenu à ramener aux quadratures la ligne géodésique sur un ellipsoïde à trois axes inégaux. Ce sont des formules extrêmement simples, des intégrales abéliennes qui se changent dans les intégrales elliptiques connues, en égalant entre eux deux de ces trois axes. Ce problème m’ayant paru long-temps très difficile, je crois que sa solution pourra intéresser peut-être quelques-uns des illustres membres de l’Académie des Sciences.

Acknowledgements I am grateful to Max–Planck–Institut für Mathematik in Bonn, where this paper was prepared, for the hospitality.

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