TOPOLOGICAL PHASE TRANSITIONS
IN LOW-DIMENSIONAL SYSTEMS

S.A.Bulgadaev

L.D.Landau Institute for Theoretical Physics
Kosyghin Str.2, Moscow, 117334, RUSSIA

Abstract

A general theory of the Berezinsky-Kosterlitz-Thouless (BKT) type phase transitions in low-dimensional systems is proposed. It is shown that in d-dimensional case the necessary conditions for it can take place are 1) conformal invariance of kinetic part of model action and 2) vacuum homotopy group \( \pi_{d-1} \) must be nontrivial and discrete. It means a discrete vacuum degeneracy for 1\( d \) systems and continuous vacuum degeneracy for higher \( d \) systems. For such systems topological excitations have logarithmically divergent energy and they can be described by corresponding effective field theories generalizing two-dimensional euclidean sine-Gordon theory, which is an effective theory of the initial \( XY \)-model. In particular, the effective actions for two-dimensional chiral models on maximal abelian tori \( T_G \) of simple compact groups \( G \) and for one-dimensional models with periodic potentials are found. In general case the sufficient conditions for existence of the BKT type phase transition are 1) constraint \( d \leq 2 \) and 2) \( \pi_{d-1} \) must have some crystallographic symmetries. Critical properties of possible low-dimensional effective theories are determined and it is shown that in two-dimensional case they are characterized by the Coxeter numbers \( h_G \) of lattices from the series \( A, D, E, Z \) and can be interpreted as those of conformal field theories with integer central charge \( c = r \), where \( r \) is a rank of groups \( \pi_1 \) and \( G \).

In one-dimensional case analogous critical properties have ferromagnetic Dyson chains with discrete Cartan-Ising spins. In contrast, critical properties of one-dimensional models with periodic potentials have a weak dependence on group \( G \).

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E-mail: bulgad@itp.ac.ru
A discovery of the possibility of phase transition (PT) in two-dimensional XY-model \[55\] from very beginning attracts a great attention of theorists due to its unusual properties. First of all it seems that such PT contradicts to the well-known theorems by Peierls - Landau \[17, 18\] and Bogolyubov - Goldstone \[1, 24\] telling us that spontaneous magnetization and spontaneous breaking of continuous symmetry cannot exist in low-dimensional \((d \leq 2)\) systems \[14, 27\]. Secondly, due to the absence of spontaneous magnetization, correlation functions in low-temperature phase must fall off algebraically \[51, 30\], what means that the whole low-temperature phase have to be massless.

All these controversies were brilliantly resolved in series of papers by Berezinsky \[8\], Popov \[50\] and by Kosterlitz and Thouless \[36, 37\], who have proven for the first time an important role of topological excitations - vortices. Existence of vortices is connected with the fact that the manifold of values of XY-model \(M = S^1\) being the simplest compact manifold with nontrivial topology has homotopy group \(\pi_1(M) = \mathbb{Z}\).

Such important role of vortices has reborn an interest to the topological excitations in quantum field theory, solid state physics and brings a discovery of monopoles \[48, 28\], instantons \[5, 6\], and others topological excitations \[56, 57, 41\]. But, the main efforts were devoted to the discovery of topological excitation with finite energy. All such excitations give finite contribution to the partition function, but cannot induce PT similar to the BKT PT, since the latter is induced by topological excitations with logarithmically divergent energy. Instead, topological excitations with finite energy induce a mass generation through so-called "dimensional transmutation" mechanism and, consequently, absence of PT in such models, similar to all other \(\sigma\)-models on compact manifolds \[49\].

Importance of topological excitations with logarithmically divergent energies for such PT has been earlier discovered by Anderson, Yuval and Hamann in their epochal paper \[3\], devoted to the Kondo problem. It was shown there that PT take place in one-dimensional Ising system with long-range \(1/r^2\) interaction \[20\] due to the presence of logarithmically interacting domain walls. Though the authors of \[36\] have noted a similarity between this model and two-dimensional XY-model, but obvious external differences between these two models (discrete symmetry and nonlocality of one-dimensional model and continuous symmetry and locality of two-dimensional model) has not allowed to clear up this similarity completely.

For more deep understanding of the unifying properties of this two models...
it is useful to note that domain walls are also topological excitations, corresponding to discrete set of vacuum configurations and consequently can be related with homotopic group $\pi_0$ of vacuum configuration manifold $\mathcal{M}$, which in this case simply means a number of connectivity components of $\mathcal{M}$. Starting from these facts, one can show, that in general case the condition of existence of topological excitations with logarithmic energy puts over the following constraints on non-linear $\sigma$-models: 1) conformal invariance at classical level, 2) their homotopical group $\pi_{d-1}(\mathcal{M})$ must be nontrivial and discrete. They define $\sigma$-models almost uniquely in arbitrary dimensions. Both abovementioned models satisfy these conditions.

The first property defines form of action $S$ and the second one defines a dimension and form of $\mathcal{M}$

$$S = \frac{1}{2\alpha} \int d^dxd^dx' \psi_a(x) \Box^{(d)}_{ab}(x - x')\psi_b(x'), \ a, b = 1, 2, ..., n,$$

where $\psi \in \mathcal{M}$ and $n$ is dimension of $\mathcal{M}$ and form of kernel $\Box$ depends on dimension of space $d$. Further, for simplicity, it will be supposed that internal space is decoupled from physical space. Then $\Box$ can be decomposed

$$\Box^{(d)}_{ab}(x) = g_{ab}\Box_{d}(x),$$

where $g_{ab}$ is, in general, the Euclidean metric of the space $\mathbb{R}^{N(n)}$, in which a manifold $\mathcal{M}$ can be embedded. It is more convenient to write $\Box_{d}$ in the momentum space. For small $k$

$$\Box_{d}(k) \simeq |k|^d(1 + a_1(ka) + ...),$$

where $a$ is a UV cut-off parameter. Action (1) can be named $d$-dimensional conformal nonlinear $\sigma$-model. The kernel $\Box_{d}$ generalizes an usual local and conformal kernel of two-dimensional $\sigma$-model

$$\Box(k) \equiv \Box_{2}(k) = k^2$$

For local models an expression for $\Box$ can be defined in terms of manifold $\mathcal{M}$ only

$$\Box(x) = g_{ab}(\phi)\Box_{d}(x)$$

In odd dimensions $\Box_{d}$ is nonlocal

$$\Box_{d}(x) \sim 1/|x|^{2d},$$
but models with such kernels are used often in physics.

Further development of the theory of the BKT PT [31, 52] shows that in long-wave limit the partition sum $Z$ of $XY$-model can be approximated by product of partition functions of dilute logarithmic gas (LG) of topological excitations $Z_{LG}$ and free “spin-waves” $Z_{sw}$

$$Z_{XY} \simeq Z_{sw} Z_{LG}$$

(6)

where $Z_{LG}$ in its turn can be represented in the form of effective field theory with sine-Gordon action $S_{SG}$ [31, 58]. If one assumes that analogous approximate factorization of partition function takes place for $d$-dimensional conformal $\sigma$-models with $\pi_{d-1} = \mathbb{Z}$

$$Z_{\sigma} \simeq Z_{sw} Z^{(d)}_{LG}$$

(7)

then $Z^{(d)}_{LG}$ can be represented in the form of effective field theory with $S^{(d)}_{eff}$

$$S^{(d)}_{eff} = S^{d}_{0} + S_{I}, \ S^{d}_{0} = \frac{1}{2} \int dx dx' \phi(x) \Box^{d}_{d}(x - x') \phi(x')$$

(8)

$$S_{I} = \int dx V(\phi), \ V(\phi) = \mu^2 \cos(\beta \phi)$$

where an operator $\Box^{d}_{d}$ is inverse to the logarithmic interaction function, which can be equalized to free correlation function

$$D(x) =< \phi(x) \phi(0) >= D(x) - D(0) \sim -\frac{1}{2\pi} \log |x/a|,$$

(9)

and for this reason its Fourier components at small $k$ have the form

$$\Box^{d}_{d}(k) \simeq B_{d} |k|^{d}(1 + b_{1}(ak) + ...).$$

(10)

It follows from (10) that form of $\Box^{d}_{d}$ coinsides (up to the coefficients) with that of conformal invariant kernel $\Box^{d}_{d}$. As a result one gets that conformal $\sigma$-models on compact spaces with $\pi_{d-1} = \mathbb{Z}$ in long-wave limit are equivalent to non-compact field theories with effective action, free part of which has the same form as initial one and potential term has a corresponding periodicity. The latter theories can be named linear conformal $\sigma$-models, associated with conformal non-linear $\sigma$-models. This approximate equivalence can be considered as some kind of duality relation between compact conformal and noncompact theories.
Similar representation takes place not only for models with \( \pi_{d-1} \) of rank \( r(\pi_{d-1}) \equiv r = 1 \), but also for models with \( r > 1 \) and with \( \pi_{d-1} \) not equal to the direct sum \( \pi_{d-1} \not= \bigoplus_1^r \mathbb{Z}_i \).

In one-dimensional case there are such models with finite \( \pi_0 \) and with infinite \( \pi_0 = \mathbb{L} \), where \( \mathbb{L} \) is some \( r \)-dimensional lattice. To the first type belong the Dyson spin chains with Cartan-Ising spins \([17, 13]\), generalizing Ising spins, which are the weights of the fundamental representation of group \( SU(2) \), on the Cartan weights of the irreducible representations of other simple compact groups \( G \). The latter form the discrete sets of classical spins \( \{s_a\}, a = 1, ..., q \), where \( q \) is dimension of representation, invariant under some point symmetry groups, so called Weyl groups \( W_G \). For example, \( q \)-state Potts model spins are a particular case of weights of fundamental representation of \( G = SU(q) \). In these cases the elements of the corresponding \( \pi_0 \) are simply spin states. These spin chains correspond to the conformal non-linear \( \sigma \)-models. To the second type belong conformal non-compact linear \( \sigma \)-models with periodic potentials. Under abovementioned duality transformation they transform into non-compact models with periodicity of the dual lattice \([13, 14]\).

In two-dimensional case they include models of crystall melting \([14, 15]\) and \( \sigma \)-models on maximal abelian tori \( T_G \) of the simple compact groups \( G \), generalizing \( XY \)-model, with homotopical group \( \pi_1(T_G) = \mathbb{L}_{r^v} \) (for simply connected \( G \)), where \( \mathbb{L}_{r^v} \) is a dual root lattice of the Lie algebra \( \mathfrak{g} \) of the group \( G \) \([12]\). The corresponding topological excitations - vortices - have isovectorial topological charges \( Q \in \mathbb{L}_{r^v} \), interacting through logarithmic law

\[
\sim (Q_1 Q_2) \ln |x_1 - x_2 / a|
\]

And in three-dimensional case they can be conformal (or van der Vaals, since \( \Box_3(x) \sim 1/|x|^6 \)) \( \sigma \)-models on the maximal flag spaces \( F_G = G/T_G \) of the simple compact groups \( G \), with \( \pi_2(F_G) = \mathbb{L}_{r^v} \) \([12]\). Flag spaces include as a particular case sphere \( S^2 = SU(2)/U(1) \). The corresponding topological excitations - instantons - also have isovectorial topological charges \( Q \in \mathbb{L}_{r^v} \) and logarithmic energy.

In all these cases an effective noncompact field theory will have a potential of the form

\[
V(\vec{\phi}) = \mu^2 \sum_{\{r_a^v\}} \exp(i \beta (r_a^v \vec{\phi})) \tag{11}
\]

where \( \{r_a^v\} \) is a set of the minimal dual roots of the Lie algebra \( \mathfrak{g} \), which
characterize minimal topological charges of the possible topological excitations with logarithmic energies. All these roots can belong only to four different root lattices $\mathbb{L}_G$ connected with corresponding groups $G$: $\mathbb{A}_n \iff G = SU(n + 1), \mathbb{D}_n \iff G = O(2n), \mathbb{E}_n \iff G = E_n(n = 6, 7, 8), \mathbb{Z}_n \iff G = O(2n + 1)$. To the exceptional groups $G_2, F_4$ correspond lattices $\mathbb{A}_2$ and $\mathbb{D}_4$ respectively. The lattices $\mathbb{L}_G$ are invariant under corresponding affine or crystallographic symmetry groups $E_l = W_G \rtimes L_G$.

The BKT type PT can be investigated by renormalization of the corresponding effective field theories $[58, 1, 44, 11, 12]$. One can show that, in general, PT of BKT type can take place only for $d \leq 2$, since conformal symmetry in $d > 2$ is finite-dimensional and is broken by renormalization $[11]$. For this reason the following discussion will be concentrated on models in space with $d \leq 2$, where conformal group is infinite-dimensional. Just the effective field theories connected with lattices $\mathbb{L}_G$ are renormalizable $[11]$.

As is well known critical singularities at BKT type PT are essential instead of algebraic ones at II order PT. For example, a correlation length $\xi$

$$\xi \sim a \exp(A\tau^{-\nu}), \quad \tau = \frac{T - T_c}{T_c}$$

where $a$ is some UV cut-off parameter and $A$ is a nonuniversal constant $\sim O(1)$.

It appears that in two-dimensional case all critical properties of models with $\pi_1 = \mathbb{L}_{\pi^\nu}$ are determined by the Coxeter numbers $h_G$ of the corresponding lattices $\mathbb{L}_{\pi^\nu}$ and groups $G$

$$h_G = \frac{\text{(number of roots)}}{\text{(rank of group)}}$$

For groups from series $A, D, E$ $h_G$ is connected with second Casimir operator in adjoint representation $K_2 = 2h_G$.

In particular, it was found that the critical exponent

$$\nu_G = 2/(2 + h_G)$$

In Table 1 are deduced all possible values of exponent $\nu_G$

| $h_G$ | $\nu_G$ |
|-------|---------|
| 0     | 1       |
| 1     | 0.5     |
| 2     | 0.333   |
| 3     | 0.25    |
| 4     | 0.2     |

Table 1
The exponents $\nu_{A_1} = \nu_{D_2} = \nu_{B_n}$ correspond to the initial KT exponent $\nu = 1/2$. For $V(\vec{\phi})$ containing the set of minimal roots $\{r_a\}$ the exponents $\nu_G$ for groups $B_n$ and $C_n$ pass into one another, since their root sets are mutually dual. For other groups exponents remain the same. Since $\nu_G$ depends only on $h_G$, they can coincide for different groups having different rank and acting in different spaces. This fact could be important when potential $V(\vec{\phi})$ is composed of the characters of different representations of different groups.

The series $A_n$ possesses the largest set of possible values of $\nu_G$, since besides exponents of the form $1/k$ (where $k$ are integers $\geq 2$), it also contains exponents of the form $2/(2k+1)$.

In the low-temperature phase the correlation functions of the fields exponentials equal to the free correlation functions with a renormalized "temperature" $\bar{\beta}$ which depends on initial values $\beta_0$, $\bar{\beta} = \lim_{l \to \infty} \beta(l)$ [11]:

$$\left\langle \prod_{s=1}^{t} \exp(i(r_s \vec{\phi}(x_s))) \right\rangle = \prod_{i \neq j}^{t} \left| \frac{x_i - x_j}{a} \right|^{\beta(r_i r_j)/2\pi}, \sum_{i=1}^{t} r = 0. \quad (13)$$

At the PT point (where $\bar{\beta} = \beta^* = 8\pi/r^2 = 4\pi$) an additional logarithmic factor, related with the "null charge" behaviour of $g = (a\mu)^d$ and $\delta = ((\beta|r|)^2 - 4d\pi)/4d\pi$ on the critical separatrix – the phase separation line, appears in them:

$$\prod_{i \neq j}^{t} \left( \ln \left| \frac{x_i - x_j}{a} \right| \right)^{\delta^*(r_i r_j)/2\pi A_G} = \prod_{i \neq j}^{t} \left( \ln \left| \frac{x_i - x_j}{a} \right| \right)^{h_G \cos(r_i r_j)}, \quad (14)$$

where $A_G = 4/h_G$ is a coefficient in RG equations for $\delta$ on the critical separatrix (schematic phase diagram see on Fig.1).
Fig. 1. Schematic phase diagram of two-dimensional models.

Free-like behaviour of correlation functions in low-temperature phase (except logarithmic corrections) gives one a possibility to use for their description conformal field theories with integer central charges $C = r$, like $C = 1$ theories in [59] and instead of PT points of two-dimensional systems with discrete symmetries, which are described by conformal theories with rational central charges [7, 23, 19, 4, 29]. The BKT type PT can be considered as some degeneration of II order PT. In this relation it is interesting that $\nu_G$ coincides with ”screening” factor in formulas for central charges of affine Lie algebras $\hat{G}$ at level $k = 2$ [32]

$$C_k = \frac{k}{k + h_G} \dim G$$

and of coset realization of minimal unitary conformal models at level $k = 1$ [24]

$$C_k = r \left(1 - \frac{h_G(h_G + 1)}{(k + h_G)(k + h_G + 1)}\right).$$

In one-dimensional case a situation is different. There are two possibilities. For models with finite $\pi_0(\mathcal{M})$ (they include ferromagnetic Dyson chains with Cartan-Ising spins [17, 13], double-well systems with dissipation [14, 40]) critical exponents $\nu$ coincide with corresponding two-dimensional ones [17, 13].
For non-compact models in periodic potential with infinite discrete \( \pi_0(\mathcal{M}) = \mathbb{L}_i, \ i = A, D, E, Z \) critical exponents \( \nu \) of the essential singularity weakly depend on type of lattice due to non-renormalization of the kernel \( \Box_1 \) and can take only one value \( \nu = 1 \) for all lattices [11, 14]. Their schematic phase diagrams are depicted on Fig. 2.

As a consequence of absence of the kernel renormalization the correlation functions in low-temperature phase do not have logarithmic corrections [11].

Last time have appeared many low-dimensional models where topological PT can take place. Effective field theories discussed above can be applied for their description due to their universality. For example, two-dimensional models on \( T_G \) can be applied for discussion of properties of strings, compactified on \( T_G \) [26] (see case of \( S^1 \) in [35, 34] and of the “decompactification” transitions in perturbed chiral models on compact groups [15].

One-dimensional models are applied for description of many solid state systems. Among them are quantum macroscopic systems with “ohmic” dissipation [16, 33, 2, 10, 18, 53, 14, 54], quantum wires [33], tunneling between edge states in systems with quantum Hall effect [13], single electron boxes [24] and many others. Their number is constantly growing.

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