SUMS OF SQUARES IN PSEUDOCONVEX HYPERSURFACES AND TORSION PHENOMENA FOR CATLIN’S BOUNDARY SYSTEMS

ALEXANDER BASYROV, ANDREEA C. NICOARA, AND DMITRI ZAITSEV

ABSTRACT. Given a pseudoconvex hypersurface in $\mathbb{C}^n$ and an arbitrary weight, we show the existence of local coordinates in which the polynomial model contains a particularly simple sum of squares of monomials. Our second main result provides a normalization of a part of any Catlin boundary system. We illustrate by an example that this normalization cannot be extended to the rest of the boundary system due to the existence of what we refer to as torsion.

1. Introduction

The goal of this paper is to gain new geometric insight into the tools developed for establishing global regularity and subelliptic estimates for the $\overline{\partial}$-Neumann problem. We are focusing on the approach by Catlin in [C84a, C87]. However, we expect our methods to also shed light on Kohn’s multiplier ideal technique initiated in [K79] and continued more recently in [CD10, S10, N14, KiZ17, S17] as well as on other research related to the $\overline{\partial}$-equation (see [Z17, 1.3]) and potentially more general PDE’s, as evidenced by the program pioneered by Siu in [S17].

Recall that Catlin established global regularity and subelliptic estimates for the $\overline{\partial}$-Neumann problem as consequences of his Property (P) type conditions. The only known proofs of Property (P) type conditions for general smooth pseudoconvex finite type domains in the sense of D’Angelo [D82] rely on the techniques of multitype, polynomial models, and boundary systems introduced in [C84b].

One of the motivations in this paper is reducing the complexity in Catlin’s techniques by a more explicit use of pseudoconvexity and a precise normalization of the geometry. Our first result is showing the existence of so-called positive balanced terms in polynomial models of pseudoconvex hypersurfaces. We call a monomial $z_1^{\alpha_1}z_1^{\beta_1} \cdots z_n^{\alpha_n}z_n^{\beta_n}$ balanced if $\alpha_j = \beta_j$ for all $j$. We shall use the (standard) lexicographic order for multiweights and the reverse lexicographic order for multidegrees. We prove the following:

Theorem 1.1. Let $M \subset \mathbb{C}^n$ be a pseudoconvex real smooth hypersurface with $0 \in M$. Let $z = (z_1, \ldots, z_n)$ be local holomorphic coordinates in a neighborhood of $0$ and

$$\mu = (\mu_1, \mu_2, \mu_3, \ldots, \mu_n), \quad 1 = \mu_1 > \mu_2 \geq \mu_3 \geq \ldots \geq \mu_n > 0,$$

Date: June 6, 2018.

2010 Mathematics Subject Classification. Primary 32T27, 41A10; Secondary 32F17.

Key words and phrases. multi-weight, inverse weight, normal form, Catlin multitype, Catlin boundary system, pseudoconvex domains in $\mathbb{C}^n$. 

1
be a (multi-)weight such that $M$ is given by
\[ r(z, \bar{z}) = 0, \quad dr \neq 0, \quad r = O_\mu(1), \]
i.e. the expansion of $r$ contains only terms of weight greater or equal to 1.

Then after a weighted homogenous polynomial change of coordinates, the defining function $r$ of $M$ admits a decomposition
\[ r(z, \bar{z}) = -2 \Re z_1 + p(z_2, \ldots, z_n, \bar{z}_2, \ldots, \bar{z}_n) + o_\mu(1), \]
where $p$ is a weighted homogeneous polynomial of weight 1 containing, as part of its expansion, the sum of squares
\[ A_2|z_2|^{2k_{22}} + A_3|z_2|^{2k_{32}}|z_3|^{2k_{33}} + \ldots + A_n|z_2|^{2k_{n2}} \cdots |z_n|^{2k_{nn}} \]
with $A_j \geq 0$, $k_{jj} > 0$ for all $j = 2, \ldots, n$, such that the (total) degree of $p$ in each $(z_j, \bar{z}_j)$ is not greater than $2k_{jj}$, and $o_\mu(1)$ stands for a smooth function in $(\Im z_1, z_2, \ldots, z_n, \bar{z}_2, \ldots, \bar{z}_n)$ whose formal Taylor series expansion contains only terms of weight greater than 1. In addition, for each $j$, the multidegree of each term of (1.2) in $(z_2, \bar{z}_2), \ldots, (z_j, \bar{z}_j)$ is maximal among all balanced monomials in $p$ in the reverse lexicographic order.

Furthermore, either all $A_j$ can be chosen positive, or the weight $\mu$ can be lowered lexicographically.

The following example illustrates that pseudoconvexity is an essential assumption:

**Example 1.1.** Let $M$ be the non-pseudoconvex domain given by $r < 0$ with
\[ r = -2 \Re z_1 + 2 \Re (z_2 \bar{z}_3). \]
Then no weighted polynomial change of variables transforms $M$ into a form satisfying the conclusion of Theorem 1.1. In fact, no terms as in (1.2) can be obtained (even with $A_j$ negative).

On the other hand, when $M$ is pseudoconvex, a natural question arises whether the statement could be improved by reducing the polynomial $p$ to just a sum of the terms in (1.2). It is not possible in general, however, as the following simple example illustrates:

**Example 1.2.** Let $M \subset \mathbb{C}^2$ be the tube given by $r = 0$ with
\[ r = -2 \Re z_1 + (\Re z_2)^2. \]
Then the polynomial $(\Re z_2)^2$ is invariant under weighted homogeneous polynomial coordinate changes, and hence cannot be reduced to contain the terms (1.2) only.

Furthermore, it is generally not possible to reduce $r$ to the sum (1.2) only, even when $r$ can be written as a sum of squares of holomorphic functions:

**Example 1.3.** Let $p, q \in \mathbb{N}$, $p, q \geq 2$. Consider
\[ r_0 = -2 \Re z_1 + |z_2|^{2p} + |z_3|^{2q} + 2\epsilon \Re z_2^p \bar{z}_3^q \]
with $|\epsilon| < 1$. Then $r_0$ determines a real-algebraic pseudoconvex hypersurface that can be written as a sum of squares of holomorphic functions:
\[ r_0 = -2 \Re z_1 + |z_2^p + \epsilon z_3^q|^2 + (1 - \epsilon^2)|z_3|^{2q}. \]
By a direct computation (or using e.g. [Ko10, Theorem 4.1]), it can be seen, however, that no biholomorphic change of variables can transform \( r_0 \) into a sum of squares of the form
\[
|a_2 z_2|^{2k_{j2}} + A_3 |z_2|^{2k_{j3}} z_3^{2k_{j3}}.
\]

The following example illustrates that the terms (1.2) cannot be in general reduced to sums of single powers \(|z_j|^{2k_{jj}}\).

**Example 1.4.** Let \( M \subset \mathbb{C}^3 \) be given by \( r = 0 \) with
\[
(1.4) \quad r = -2 \text{Re } z_1 + |z_2|^8 + |z_3|^4 |z_3|^6,
\]
a weighed homogeneous polynomial in \((z_1, z_2, z_3)\) and conjugates with corresponding weights \((1, 8, 12)\). Then the only weighted homogeneous changes of coordinates are the linear dilations \((z_1, z_2, z_3) \mapsto (a_1 z_1, a_2 z_2, a_3 z_3)\) that clearly preserve (1.4) up to a change of coefficients. In particular, it is not possible to obtain a conclusion similar to Theorem 1.1 with (1.2) consisting of sums of single powers \(|z_j|^{2k_{jj}}\) only.

Applying Theorem 1.1 to the Catlin multitype yields the following:

**Corollary 1.2.** Let \( M \) be a pseudoconvex smooth real hypersurface in \( \mathbb{C}^n \) with \( 0 \in M \) and of the Catlin multitype
\[
\Lambda = (1, \lambda_2, \lambda_3, \ldots, \lambda_n), \quad \lambda_n < +\infty
\]
at \( 0 \).

Then there exists a holomorphic change of coordinates at \( 0 \) preserving the multitype so that the defining function for \( M \) in the new coordinates is given by
\[
(1.5) \quad r = -2 \text{Re } z_1 + p(z_2, \ldots, z_n, \bar{z}_2, \ldots, \bar{z}_n) + o_{\Lambda^{-1}}(1),
\]
where \( p \) is a weighed homogeneous polynomial of weight 1 that contains the sum of squares
\[
(1.6) \quad |z_2|^{2k_{j2}} + |z_2|^{2k_{j3}} |z_3|^{2k_{j3}} + \ldots + |z_2|^{2k_{jn}} \ldots |z_n|^{2k_{jn}},
\]
where \( k_{jj} > 0 \) for all \( j \) and the (total) degree of \( p \) in each \((z_j, \bar{z}_j)\) is not greater than \( k_{jj} \).

**Remark 1.5.** It follows from the weighed homogeneity of (1.6) that
\[
(1.7) \quad \sum_{l=2}^j \frac{2k_{jl}}{\lambda_l} = 1, \quad j = 1, \ldots, n.
\]

In particular, since \( k_{jj} \neq 0 \), the weights \( \lambda_l \) are uniquely determined from (1.7).

A better understanding of the Catlin multitype has been obtained when \( M \) is a boundary of a convex domain, see [BS92, M92, Y92]. However, even in this case, Thereom 1.1 seems to be new.

For a pseudoconvex hypersurface, the multitype and the commutator multitype coincide as Catlin established in [C84b]. Without pseudoconvexity, however, it might not be possible to obtain terms of the type
\[
|z_2|^{2k_{j2}} \ldots |z_n|^{2k_{jj}}
\]
via a change of variables, and the two multitypes may differ, as the following example due to Bloom shows:

**Example 1.6.** Consider

\[ r_0 = \text{Re} z_1 + (\text{Re} z_2 + |z_3|^2)^2. \]

The multitype at 0 is given by \( \mathcal{M} = (1, 2, 4) \), whereas the commutator multitype at 0 is lexicographically strictly larger: \( \mathcal{C} = (1, 2, +\infty) \). Catlin proved that \( \mathcal{M} \leq \mathcal{C} \) for any domain, and this example shows nothing more can be expected to hold in the absence of pseudoconvexity.

Our second main result is a normalization of any boundary system for a pseudoconvex domain via a change of variables:

**Theorem 1.3.** Let \( M \) be a pseudoconvex smooth real hypersurface in \( \mathbb{C}^n \) with \( 0 \in M \), Levi rank \( s_0 \) at 0, and of the Catlin multitype

\[ \Lambda = (1, 2, \ldots, 2, \lambda_{s_0+2}, \ldots, \lambda_n) \]

at 0, where

\[ 2 < \lambda_{s_0+2} = \cdots = \lambda_{s_0+s_1+1} < \lambda_{s_0+s_1+2} < +\infty. \]

Then for any boundary system at 0,

\[ \mathcal{B}_n(0) = \{ r_1, r_{s_0+2}, \ldots, r_n; L_2, \ldots, L_n \}, \]

there exists a holomorphic change of coordinates at 0 preserving the multitype and transforming \( \mathcal{B}_n(0) \) into

\[ \tilde{\mathcal{B}}_n(0) = \{ \tilde{r}_1, \tilde{r}_{s_0+2}, \ldots, \tilde{r}_n; \tilde{L}_2, \ldots, \tilde{L}_n \}, \]

satisfying the normalization

\[ \tilde{r}_j = \text{Re} z_j + o(\lambda_j^{-1}), \quad s_0 + 2 \leq j \leq s_0 + s_1 + 1, \]

\[ \tilde{L}_k = \partial z_k + o(\lambda_k^{-1}), \quad 2 \leq k \leq s_0 + s_1 + 2, \]

where the partial derivatives \( \partial z_j \) are counted with weight \( -\lambda_j^{-1} \).

Therefore, the first function in the boundary system beside the defining function can always be brought to the simplest possible form. In case several entries of the Catlin multitype are equal to the first \( \lambda_j > 2 \), all of their corresponding functions and vector fields in the boundary system can be normalized. This normalization process cannot be carried out on the subsequent functions, however. In fact, we provide a counterexample in the final section of the paper. This non-existence of a complete normalization provides a very important insight into the behavior of Catlin’s boundary systems that we call a torsion phenomenon.

In particular, for \( M \) of Levi rank 0 in \( \mathbb{C}^3 \), this flattening result yields a simplified geometric picture, which we state as a corollary:
Corollary 1.4. Let \( M \) be a pseudoconvex smooth real hypersurface in \( \mathbb{C}^3 \) with \( 0 \in M \), Levi rank 0, and of the Catlin multitype
\[
\Lambda = (1, \lambda_2, \lambda_3), \quad \lambda_3 < +\infty
\]
at 0. After a holomorphic change of variables, any boundary system at 0
\[
\mathcal{B}_3(0) = \{r_1, r_2, r_3; L_2, L_3\}
\]
becomes
\[
\tilde{\mathcal{B}}_3(0) = \{\tilde{r}_1, \tilde{r}_2, \tilde{r}_3; \tilde{L}_2, \tilde{L}_3\}
\]
with
\[
\tilde{r}_2 = \text{Re} \ z_2 + o(\lambda_2^{-1}), \\
\tilde{L}_j = \partial z_j + o(\lambda_j^{-1}), \quad j = 2, 3,
\]
\[(1.9)\]

The paper is organized as follows: Section 2 provides additional examples illustrating various phenomena. Section 3 gives the relevant definitions and notation. Section 4 defines Catlin’s multitype and boundary systems. Section 5 presents some elementary auxiliary results such as one-dimensional estimates for non-negative homogeneous polynomials and a several variables version proven via scaling using the Newton polygon. Section 6 contains the proofs of Theorem 1.1 and Corollary 1.2 carried out in a sequence of lemmas. In the same section, it is shown that the first function in the boundary system can always be normalized, thus establishing Theorem 1.3 and Corollary 1.4. Section 7 demonstrates by example that the same type of normalization cannot be carried out on subsequent functions in the boundary system.

2. Further motivation and examples

Example 2.1. The grandfather of all examples is a strongly pseudoconvex hypersurface given by \( r = 0 \) with
\[
\begin{align*}
\Lambda &= (1, 2, \ldots, 2), \\
r &= -2 \text{Re} \ z_1 + |z_2|^2 + \ldots + |z_n|^2 + o_{\Lambda^{-1}}(1),
\end{align*}
\]
where the whole leading polynomial (the quadric), is already of the form (1.2), once diagonalized. Of course, the diagonalization here is a special property of quadrics that does not extend to higher degree polynomials.

More generally, it was shown by the third author in [Z17] that any pseudoconvex hypersurface admits the form \( r = 0 \) with
\[(2.1)\]
\[
\begin{align*}
r &= -2 \text{Re} \ z_1 + |z_2|^2 + \ldots + |z_{q-1}|^2 + p_4(z_{[q+2,n]}, \bar{z}_{[q+2,n]}) + o_{\Lambda^{-1}}(1),
\end{align*}
\]
with the inverse weights \( \Lambda = (1, 2, \ldots, 2, 4, \ldots, 4) \), where the number of 2’s equals the Levi form rank \( q \), and \( p_4 \) represents the CR invariant quartic tensor defined in [Z17]. In view of this general fact, it suffices to only study terms arising from \( p_4 \) and \( o_{\Lambda^{-1}}(1) \).

Example 2.2. Any real hypersurface \( M \subset \mathbb{C}^2 \) of finite type \( m \) admits the form \( r = 0 \) with
\[
\begin{align*}
r &= -2 \text{Re} \ z_1 + p_m(z_2, \bar{z}_2) + o_{\Lambda^{-1}}(1), \\
p_m &= \sum_{j+k=m} a_{jk} z_2^j \bar{z}_2^k,
\end{align*}
\]
where $\Lambda = (1, m)$ and $p_m$ is not harmonic. In this (well-known) case, if $M$ is pseudoconvex, Theorem 1.1 implies that $m = 2l$ is even and $p_m$ contains a nontrivial term $a_l |z_2|^{2l}$, $a_l \neq 0$. Since $p_m$ is a (tensor) invariant, it is clear that other terms $a_{jk} z_2^j \bar{z}_2^k$ with $j \neq k$ cannot be eliminated.

**Example 2.3.** For a general pseudoconvex hypersurface $M \subset \mathbb{C}^3$ of the form (1.1) in its multitype coordinates, Corollary 1.2 implies that after a linear change of coordinates, $p$ contains nontrivial terms

\[(2.2) \quad p = |z_2|^{2k_{22}} + |z_2|^{2k_{32}} |z_3|^{2k_{33}} + \tilde{p}(z_2, z_3, \bar{z}_2, \bar{z}_3), \quad k_{22}, k_{33} \geq 1,\]

where $\tilde{p}$ consists of all remaining terms. In particular, the multitype $\Lambda = (1, m_2, m_3)$ is uniquely determined from the identities

\[m_2 = 2k_{22}, \quad \frac{2k_{32}}{m_2} + \frac{2k_{33}}{m_3} = 1,\]

expressing the property that the first two terms in (2.2) are of weight 1.

Furthermore, the additional degree property in Theorem 1.1 asserts that the degree of $p$ in $(z_3, \bar{z}_3)$ equals $2k_{33}$. This property puts additional restrictions on the terms in $p$ and makes the choice of the first two terms in (2.2) canonical. For instance, if

\[p = |z_2|^4 + |z_2 z_3|^2 + |z_3|^4,\]

the degree condition forces us to choose the terms $|z_2|^4 + |z_3|^4$ rather than $|z_2|^4 + |z_2 z_3|^2$, because the latter choice would violate the property that the degree of $p$ in $(z_3, \bar{z}_3)$ is 2.

**Example 2.4.** Let $M \subset \mathbb{C}^4$ be given by $r = 0$ with

\[r = -2 \Re z_1 + |z_2|^4 + |z_2|^2 |z_3|^2 + (|z_2|^2 + |z_3|^2) |z_4|^2.\]

Then the degree property in Theorem 1.1 implies that the sum of squares in (1.6) must be

\[(2.3) \quad |z_2|^4 + |z_2|^2 |z_3|^2 + |z_3|^2 |z_4|^2,\]

since the remaining square term $|z_2|^2 |z_4|^2$ has the same degree in $(z_4, \bar{z}_4)$ as $|z_3|^2 |z_4|^2$ but lower degree in $(z_3, \bar{z}_3)$. That last restriction again determines uniquely the terms in (2.3).

**Example 2.5.** Corollary 1.2 allows us to estimate how many inverse weights can arise as multitypes $\mathfrak{M} = (m_1, \ldots, m_n)$ at 0 of a pseudoconvex hypersurface $M \subset \mathbb{C}^n$ of finite 1-type $m$. We first recall that by part (4) of Catlin’s Main Theorem in [C84b], $m_n \leq m$. It is always the case that $m_1 = 1$, $m_2$ is an integer, and $2 \leq m_2 \leq \cdots \leq m_n$.

Since $m$ is rational in general, consider its floor (or integral part) $\lfloor m \rfloor$. By equation (1.6), $m_2 = 2k_{22} \leq m$, so there are at most $\lfloor \frac{m}{2} \rfloor$ values for

\[m_2 = 2, 4, \ldots, \lfloor \frac{m}{2} \rfloor.\]

Once again from equation (1.6) we obtain

\[(2.4) \quad \frac{2k_{32}}{m_2} + \frac{2k_{33}}{m_3} = 1.\]
with $k_{33} \neq 0$, implying

$$0 \leq k_{32} < \frac{m}{2}.$$  

If $\frac{m}{2} \not\in \mathbb{Z}$, then there are $\lfloor \frac{m}{2} \rfloor + 1$ choices for $k_{32}$, namely the integers from 0 to $\lfloor \frac{m}{2} \rfloor$. If $\frac{m}{2} \in \mathbb{Z}$, then $0 \leq k_{32} \leq \frac{m}{2} - 1$, so there are $\frac{m}{2}$ choices for $k_{32}$. In both cases, we get at most $\lfloor \frac{m}{2} \rfloor + 1$ choices for $k_{32}$. As for $k_{33}$, (2.4) implies

$$0 < 2k_{33} \leq m,$$

so there are at most $\lfloor \frac{m}{2} \rfloor$ choices for $k_{33}$. Since the choice of $k_{32}$ and $k_{33}$ determines $m_3$, we have $(\lfloor \frac{m}{2} \rfloor + 1) \lfloor \frac{m}{2} \rfloor$ choices for $m_3$, without accounting for different equations yielding the same solution. To determine the number of choice for $m_4$, we use the equation

$$\frac{2k_{42}}{m_2} + \frac{2k_{43}}{m_3} + \frac{2k_{44}}{m_4} = 1.$$

The same analysis gives us $\lfloor \frac{m}{2} \rfloor + 1$ choices for each of $k_{42}$ and $k_{43}$ and $\lfloor \frac{m}{2} \rfloor$ choices for $k_{44}$ due to the condition $k_{44} \neq 0$. We thus have at most $(\lfloor \frac{m}{2} \rfloor + 1)^2 \lfloor \frac{m}{2} \rfloor$ choices for $m_4$. In general, there are $(\lfloor \frac{m}{2} \rfloor + 1)^{j-2} \lfloor \frac{m}{2} \rfloor$ choices for $m_j$, where $2 \leq j \leq n$. Altogether, we have obtained

$$\left(\left\lfloor \frac{m}{2} \right\rfloor + 1\right)^{0+1+\cdots+(n-2)} \left(\left\lfloor \frac{m}{2} \right\rfloor\right)^{n-1} = \left(\left\lfloor \frac{m}{2} \right\rfloor + 1\right) \frac{(n-2)(n-1)}{2} \left(\left\lfloor \frac{m}{2} \right\rfloor\right)^{n-1}$$

possible multitypes at 0 of a pseudoconvex hypersurface $M \subset \mathbb{C}^n$ of finite 1-type $m$. This estimate significantly improves the one in [N14].

3. Notation

For an $n$-tuple $z = (z_1, \ldots, z_n)$, we shall use the short-hand notation

$$z_{[k,m]} := (z_k, z_{k+1}, \ldots, z_m), \quad 1 \leq k \leq m \leq n.$$

We use the extended sets of nonnegative rationals and reals

$$\mathbb{Q}_{\geq 0} := \mathbb{Q}_{\geq 0} \cup \{+\infty\}, \quad \mathbb{R}_{\geq 0} := \mathbb{R}_{\geq 0} \cup \{+\infty\},$$

and consider real nonnegative $n$-tuples of weights, or simply weights

$$\mu = (1, \mu_2, \ldots, \mu_n), \quad 1 = \mu_1 > \mu_2 \geq \ldots \geq \mu_n \geq 0, \quad \mu_i \in \mathbb{R}_{\geq 0}.$$  

(3.1)

Following Catlin’s notation [C84b], we also consider inverse weights

$$\Lambda = (1, \lambda_2, \lambda_3, \ldots, \lambda_n), \quad \lambda_i \in \mathbb{Q}_{\geq 0}.$$  

For every weight $\mu = (1, \mu_2, \mu_3, \ldots, \mu_n)$, we have its associated inverse weight given by reciprocals

$$\Lambda = (1, \lambda_2, \ldots, \lambda_n) = (1, \mu_2^{-1}, \ldots, \mu_n^{-1})$$

with the convention that $0^{-1} = +\infty$, $(+\infty)^{-1} = 0$. Let

$$\langle \alpha | \mu \rangle := \alpha_1 \mu_1 + \alpha_2 \mu_2 + \cdots + \alpha_n \mu_n$$

for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ a multi-index.
Given a smooth real function $r(z, \bar{z})$ defined in a neighborhood of 0 in $\mathbb{C}^n$, a weight $\mu$ as in (3.1) and nonnegative constant $C \geq 0$ we write
\[
r = O_{\mu}(C), \quad \text{resp.} \quad r = o_{\mu}(C),
\]
whenever $(\alpha + \beta|\mu) \geq C$, resp. $(\alpha + \beta|\mu) > C$ holds for any nonzero monomial $r_{\alpha\beta}z^\alpha\bar{z}^\beta$ in the Taylor expansion of $r$ at 0.

Let $M \subset \mathbb{C}^n$ be an oriented smooth real hypersurface defined in a neighborhood of a point $p = 0$ by $r = 0$ with $dr \neq 0$, such that $r < 0$ is the negative side with respect to the orientation. Recall that $M$ is pseudoconvex if and only if the restriction of the complex Hessian of $r$ to the complex tangent space of $M$ is positive semidefinite.

We have the following elementary properties, provided with short proofs for the reader’s convenience.

**Lemma 3.1.** Let
\[
r = -2 \text{Re} \, z_1 + f(z_{[2,n]}, \bar{z}_{[2,n]}),
\]
where $f$ is any smooth function. Then the domain given by $\{r < 0\}$ is pseudoconvex if and only if
\[
\sum_{j,k=2}^{n} f_{z_j\bar{z}_k} a_j\bar{a}_k \geq 0
\]
for all $(a_2, \ldots, a_n) \in \mathbb{C}^{n-1}$.

**Proof.** The Levi form of the boundary $M := \{r = 0\}$ is given by the restriction of the complex Hessian of $r$, given by the left-hand side of (3.2), to the complex tangent subbundle of $M$. Since the latter projects surjectively to $\{0\} \times \mathbb{C}^{n-1}$, the Levi form is positive semidefinite if and only if (3.2) holds, proving the statement. \(\square\)

**Lemma 3.2.** Given a weight $\mu$ as in (3.1), let
\[
r = -2 \text{Re} \, z_1 + p(z_{[2,n]}, \bar{z}_{[2,n]}) + o_{\mu}(1),
\]
where $p$ is a weighted homogeneous polynomial of weight 1. Assume that the domain given by $\{r < 0\}$ is pseudoconvex. Then the model domain given by $\{r_0 < 0\}$, where
\[
r_0 := -2 \text{Re} \, z_1 + p(z_{[2,n]}, \bar{z}_{[2,n]}),
\]
is also pseudoconvex.

**Proof.** The proof is obtained by a simple weighted scaling argument. Consider the weighted dilation
\[
T_t(z_1, \ldots, z_n) := (z_1, t^{\mu_2}z_2, \ldots, t^{\mu_n}z_n).
\]
Then $r_0$ is invariant under composition with $T_t$, whereas for any function $f(z, \bar{z}) = o_{\mu}(1)$, the rescaled function $f(T_t(z), T_t(\bar{z}))$ converges to 0 uniformly on compacta as $t \to 0$. The statement follows from the continuity of the complex tangent bundles and the Levi form under the limit $t \to 0$. \(\square\)
Finally, we shall write
\[(3.3) \quad A \sim B \]
whenever there is a nonzero constant \( c \) with \( A = cB \).

4. Catlin Multitype and Boundary Systems

We devote this section to defining the multitype notion as Catlin introduced in [C84b] in order to characterize the vanishing order of the defining function in different directions.

In the previous section, we defined completely general weights and inverse weights, but Catlin restricts the inverse weights he considers to only those that could represent the vanishing of the defining function. We restrict the set of weights via two natural definitions:

**Definition 4.1.** An inverse weight \( \Lambda = (1, \lambda_2, \lambda_3, \ldots, \lambda_n) \) is called admissible if for every \( i, 1 \leq i \leq n \), either \( \lambda_i = +\infty \) or there exists a set of non-negative integers \( a_1, \ldots, a_i \) with \( a_i > 0 \) such that \( \sum_{j=1}^{i} a_j \lambda_j^{-1} = 1 \). Let \( \Gamma_n \) be the set of all admissible inverse weights ordered lexicographically.

**Definition 4.2.** Consider a smooth domain \( \Omega \subset \mathbb{C}^n \) with defining function \( r \). An admissible inverse weight \( \Lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma_n \) is called distinguished at \( z_0 \in \partial \Omega \) if there exist holomorphic coordinates \( (z_1, \ldots, z_n) \) about \( z_0 \) with \( z_0 \) mapped to the origin such that if \( \sum_{i=1}^{n} \frac{\alpha_i + \beta_i}{\lambda_i} < 1 \), then \( D^\alpha \bar{D}^\beta r(0) = 0 \), where \( D^\alpha = \frac{\partial^{\alpha}}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}} \) and \( \bar{D}^\beta = \frac{\partial^{\beta}}{\partial \bar{z}_1^{\beta_1} \cdots \partial \bar{z}_n^{\beta_n}} \). Let \( \tilde{\Gamma}_n(z_0) \) be the set of distinguished weights at \( z_0 \).

**Definition 4.3.** The multitype \( \mathcal{M}(z_0) \) is defined to be lexicographically the smallest admissible weight \( \mathcal{M}(x_0) = (m_1, \ldots, m_n) \) such that \( \mathcal{M}(z_0) \geq \Lambda \) for every admissible distinguished weight \( \Lambda \in \tilde{\Gamma}_n(z_0) \).

Definitions 4.2 and 4.3 together prompt the following natural question:

**Question:** Let the multitype \( \mathcal{M}(z_0) = (m_1, \ldots, m_n) \) be such that \( m_n < +\infty \). What are the multi-indices \( \alpha \) and \( \beta \) satisfying \( \sum_{i=1}^{n} \frac{\alpha_i + \beta_i}{\lambda_i} = 1 \) such that \( D^\alpha \bar{D}^\beta r(0) \neq 0 \) after a holomorphic change of variables mapping \( z_0 \) to the origin?

Corollary 1.2, which we shall prove, not only gives an answer to this question but also identifies balanced terms in the defining function responsible for the condition \( D^\alpha \bar{D}^\beta r(0) \neq 0 \).

A priori, Definition 4.3 gives no indication how to compute the multitype \( \mathcal{M}(z_0) \). To achieve that, Catlin introduced in [C84b] the commutator multitype \( \mathcal{C}(z_0) \) computed by differentiating the Levi form along certain lists of vector fields arising from a geometric object called a boundary system. He was then able to prove that this commutator multitype \( \mathcal{C}(z_0) \) equals the multitype for a pseudoconvex domain.

Recall from [C84b] that a boundary system is a collection of vector fields and real-valued functions
\[ \mathcal{B}_\nu(z_0) = \{ r_1, r_{p+2}, \ldots, r_\nu; L_2, \ldots, L_\nu \} \]
for some $\nu \leq n$. The first function in the boundary system is $r_1 = r$, the defining function. Let $p$ be the rank of the Levi form of $b\Omega$ at $z_0$. Since $r = 0$ defines a manifold, we can choose the vector field $L_1$ such that $L_1(r) = 1$. Recall from the beginning of this section that the multitype seeks to capture the vanishing order of the defining function in different directions. From the information we have so far, if $C(z_0) = (1, c_2, c_3, \ldots, c_n)$, then the first entry comes from the condition $L_1(r) = 1$, and subsequently, $c_2 = \cdots = c_{p+1} = 2$, an entry of 2 for every non-zero eigenvalue of the Levi form. We choose vector fields of type $(1, 0)$ $L_2, \ldots, L_{p+1}$ such that $L_i(r) = \partial r(L_i) \equiv 0$ and the $p \times p$ Hermitian matrix $\partial \partial r(L_i, L_j)(x_0)$ is nonsingular for $2 \leq i, j \leq p + 1$. We have kept Catlin’s notation of round parentheses for the evaluation of forms on vector fields. If $p + 1 = n$, our construction is finished; otherwise, we need to make sense of vanishing orders higher than two. Let us denote by $T_{p+2}^{(1,0)}$ the bundle composed of $(1, 0)$ vector fields $L$ such that $\partial r(L) = 0$ and $\partial \partial r(L, \bar{L}_j) = 0$ for $j = 2, \ldots, p + 1$. For $l \geq 3$ we denote by $L$ a list of vector fields $L = \{L^1, \ldots, L^l\}$ and by $\mathcal{L} \partial r$ the function

$$
\mathcal{L} \partial r(z) = L^1 \cdots L^{l-2} \partial r ([L^{l-1}, L^l])(z)
$$

for $z \in b\Omega$. We are interested in lists $L$ such that $\mathcal{L} \partial r(z_0) \neq 0$ that are chosen in the most natural way possible. For every $j$ such that $p + 2 \leq j \leq n$, we will pick a $(1, 0)$ vector field $L_j \in T_{p+2}^{(1,0)}$ and a corresponding real-valued function $r_j$ such that $L_j r_j \neq 0$ but $L_j r_i = 0$ whenever $i < j$. The process is inductive. When $j = p + 2$, the simplest possible list $L$ of $(1, 0)$ vectors in $T_{p+2}^{(1,0)}$ that can yield $\mathcal{L} \partial r(z_0) \neq 0$ consists of a smooth $(1, 0)$ vector field $L_{p+2}$ and its conjugate $\bar{L}_{p+2}$. If no such list exists, we set $c_{p+2} = \cdots = c_n = +\infty$, and we have finished our construction of the commutator multitype. If such a list exists, however, we choose a list $L_{p+2} = \{L^1, \ldots, L^l\}$ of minimal length $l$ and set $c_{p+2} = l$. Let $L'_{p+2} = \{L'^1, \ldots, L'^l\}$. Set $r_{p+2} = \text{Re}(L'_{p+2} \partial r)$ or $r_{p+2} = \text{Im}(L'_{p+2} \partial r)$ so that the condition $L_{p+2} r_{p+2} \neq 0$ holds. Define $S_{p+2} = \{L_{p+2}, \bar{L}_{p+2}\}$.

Now assume that for some integer $j - 1$ with $p + 2 \leq j - 1 < n$ we have already constructed finite positive numbers $c_1, \ldots, c_{j-1}$ as well as real-valued functions $r_1, r_{p+2}, \ldots, r_{j-1}$; linearly independent smooth $(1, 0)$ vector fields $L_2, \ldots, L_{j-1}$; and lists $L_{p+2}, \ldots, L_{j-1}$ such that the following properties hold:

1. $\mathcal{L}_i \partial r(z_0) \neq 0$ for every $i$, $p + 2 \leq i \leq j - 1$;
2. If $\mathcal{L}_i = \{L^1, \ldots, L^l\}$, then $\mathcal{L}'_i = \{L^2, \ldots, L^l\}$ and $r_i = \text{Re}(\mathcal{L}'_i \partial r)$ or $r_i = \text{Im}(\mathcal{L}'_i \partial r)$ in order that the condition $L_i r_i \neq 0$ holds;
3. $L_i r_k = 0$ for $p + 2 \leq k < i \leq j - 1$;
4. Each of the lists $\mathcal{L}_i = \{L^1, \ldots, L^l\}$ is
   (a) $i$-admissible (in Catlin’s terminology) meaning $L^1 \in S_i = \{L_i, \bar{L}_i\}$ and if $l_k$ is the number of times a vector from $S_k$ occurs in the list, then
   $$
   \sum_{k=p+2}^{i-1} \frac{l_k}{c_k} < 1
   $$
   and
   (b) ordered meaning $L^k \in S_{\alpha_k}$ for every $1 \leq k \leq l$ and $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_l$. 


(5) If \( l_k^i \) equals the number of times \( L_k \) and \( \bar{L}_k \) occur in \( \mathcal{L}_i \), then \( l_k^i = 0 \) whenever \( k > i \) and \[
\sum_{k=p+2}^{j-1} \frac{l_k^i}{c_k} = 1; \]

(6) All lists \( \mathcal{L}_i \) are of minimal length, namely if \( \mathcal{L} = \{L_1, \ldots, L^l\} \) is any ordered list, \( l_k \)
eq 0 whenever \( k > i \) and \( \sum_{k=p+2}^{j-1} \frac{l_k^i}{c_k} < 1 \), then \( \mathcal{L} \partial r(z_0) = 0 \).

We will show that we can chose a positive rational number \( c_j \) such that properties (1)-(6) are fulfilled with \( j \) replacing \( j - 1 \). Let \( T_j^{(1,0)} \) denote the set of \((1,0)\) smooth vector fields \( L \) such that \( \partial \partial r(L, L_i) = 0 \) for \( i = 2, \ldots, p + 1 \) and \( L(r_k) = 0 \) for \( k = 1, p + 2, \ldots, j - 1 \). For any smooth vector field \( L_j \in T_j^{(1,0)} \), we consider all ordered \( j \)-admissible lists \( \mathcal{L} \). If for all such lists, \( \mathcal{L} \partial r(z_0) = 0 \), then we set \( c_j = \cdots = c_n = +\infty \); otherwise, there exists at least one such list \( \mathcal{L} \) for which \( \mathcal{L} \partial r(z_0) \neq 0 \). We choose one of minimal length and denote it \( \mathcal{L}_j \). The vector field \( L_j \) used in its construction gets added to the collection \( \mathcal{L}_2, \ldots, \mathcal{L}_{j-1} \). For \( p + 2 \leq k \leq j \) let \( l_k \) be the number of times a vector from \( S_k = \{L_k, \bar{L}_k\} \) occurs in the list \( \mathcal{L}_j \), and let \( c(\mathcal{L}) \) denote the solution to the equation

\[
\sum_{k=p+2}^{j-1} \frac{l_k}{c_k} + \frac{l_j}{c(\mathcal{L})} = 1.
\]

Since \( \mathcal{L}_j \) is \( j \)-admissible, \( c(\mathcal{L}) \in \mathbb{Q}^+ \). Let \( c_j = c(\mathcal{L}) \) and let \( r_j = Re(\mathcal{L}_j' \partial r) \) or \( r_j = Im(\mathcal{L}_j' \partial r) \) so that the condition \( L_j r_j \neq 0 \) holds. All properties (1)-(6) are thus fulfilled. We continue this process until we have generated \( \mathcal{C}(z_0) = (1, c_2, c_3, \ldots, c_n) \), the commutator multitype at \( z_0 \). Let \( \nu \leq n \) be the highest index for which the entry \( c_\nu \) is finite. The collection

\[
\mathcal{B}_\nu(x_0) = \{r_1, r_{p+2}, \ldots, r_\nu; L_2, \ldots, L_\nu\}
\]

of functions and vector fields that we have generated in the process of computing \( \mathcal{C}(z_0) \) is called a boundary system of rank \( p \) and codimension \( n - \nu \).

5. Estimates for Non-negative Homogeneous Polynomials

5.1. One Variable Case.

Lemma 5.1. Let \( P(z, \bar{z}) \geq 0 \) be a real homogeneous polynomial of degree \( 2m \) in \( \mathbb{C} \),

\[
P(z, \bar{z}) = \sum_{k=-m}^{m} C_k z^{m+k} \bar{z}^{m-k}.
\]

Then \( C_0 \geq 0 \) and \( |C_k| \leq C_0 \) for all \( k \). Furthermore, if \( P(z, \bar{z}) \neq 0 \), then \( C_0 > 0 \).

Proof. Considering the values of \( z \) on \( |z| = 1 \), we observe that \( \bar{z} = 1/z \), and

\[
P = \sum_{k=-m}^{m} C_k z^{2k} \geq 0, \quad |z| = 1.
\]
We parametrize $|z| = 1$ as $\gamma(\theta) = e^{i\theta}$ on $[-\pi, \pi]$ and observe that

$$0 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} P(\theta) d\theta = C_0,$$

whence $C_0 \geq 0$. Note that if $P > 0$ for some $z$ in a neighbourhood of 0, then $\int_{-\pi}^{\pi} P(\theta) d\theta > 0$ so $C_0 > 0$.

Since for any $k \neq 0$ and $z$ with $|z| = 1$,

$$0 \leq (2 \text{Re } z^k)^2 = (z^k + \bar{z}^k)^2 = (z^k + 1/z^k)^2 = z^{2k} + \bar{z}^{-2k} + 2$$

and

$$0 \geq (2i \text{Im } z^k)^2 = (z^k - \bar{z}^k)^2 = (z^k - 1/z^k)^2 = z^{2k} + \bar{z}^{-2k} - 2,$$

we have

$$0 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} (e^{i\theta k} + e^{-i\theta k})^2 P(\theta) d\theta = C_{-k} + C_k + 2C_0$$

and

$$0 \geq \frac{1}{2\pi} \int_{-\pi}^{\pi} (e^{i\theta k} - e^{-i\theta k})^2 P(\theta) d\theta = C_{-k} + C_k - 2C_0,$$

which immediately gives $|C_{-k} + C_k| \leq 2C_0$. Since $C_k = \bar{C}_{-k}$, we have

$$(5.1) \quad |\text{Re } C_{-k}| = |\text{Re } C_k| \leq C_0.$$

Furthermore, by rotating $z$ we may assume $C_k \in \mathbb{R}$, and hence the above inequality yields $|C_k| \leq C_0$ as desired. \hfill \Box

Inspired by these estimates, we seek to divide the terms of a non-negative homogenous polynomial $P(z, \bar{z})$ in $\mathbb{C}^n$ into the terms that control others and those that are controlled.

**Definition 5.1.** Let a universal constant $M > 0$ be given, and let $P(z, \bar{z})$ be a non-negative homogeneous polynomial of degree $2m$ in $\mathbb{C}^n$. If

$$P(z, \bar{z}) = \sum_{|\alpha| + |\beta| = 2m} C_{\alpha\beta} z^\alpha \bar{z}^\beta,$$

then a coefficient $C_{\alpha\beta}$ is called $M$-dominant if $|C_{\alpha'\beta'}| \leq M |C_{\alpha\beta}|$ for all $\alpha', \beta'$ such that $|\alpha'| + |\beta'| = 2m$.

**5.2. Newton Polygon Lemma.**

**Lemma 5.2.** Let $P(x, y)$ be a non-negative homogeneous polynomial of degree $2m$ in $\mathbb{R}^p \times \mathbb{R}^q$ for $x \in \mathbb{R}^p$ and $y \in \mathbb{R}^q$. If

$$P(x, y) = \sum_{p+q=2m} P_{pq},$$

where $P_{pq}(\lambda x, \mu y) = \lambda^p \mu^q P(x, y)$, then $P_{p_0q_0} \geq 0$ if either $p_0 = \max p$ or $q_0 = \max q$. 


Proof: We use a scaling argument. \( P(x, y) \geq 0 \) for all \( (x, y) \in \mathbb{R}^p \times \mathbb{R}^q \) implies for any \( t \in \mathbb{R}, t \neq 0 \), we have that \( P(tx, t^{-1}y) \geq 0 \). Letting \( t \to \infty \) shows \( P_{p_0q_0} \geq 0 \) when \( p_0 = \max p \). Letting \( t \to 0 \) shows \( P_{p_0q_0} \geq 0 \) when \( q_0 = \max q \). \( \square \)

Remark 5.2. The statement of Lemma 5.2 also holds for weighted homogeneous polynomials.

6. Normal forms in the pseudoconvex case

6.1. Setup. Consider a smooth real hypersurface \( M \) passing through 0, defined near 0 by \( r = 0 \), where

\[
r = -2 \operatorname{Re} z_1 + p(z_{[2,n]}, \bar{z}_{[2,n]}) + o_\mu(1),
\]

\( p \) is a weighted homogeneous polynomial of weight 1, and where the components \( z_1, \ldots, z_n \) and their conjugates are assigned the corresponding weights \( 1, \mu_2, \mu_3, \ldots, \mu_n \). In other words, we can write

\[
p(z_{[2,n]}, \bar{z}_{[2,n]}) = \sum_{(\alpha + \beta)_{\mu} = 1} C_{\alpha\beta} z^\alpha \bar{z}^\beta.
\]

Together with \( M \), we consider its model hypersurface \( M_0 \) defined by the weighted homogeneous part

\[
r_0 = 0, \quad r_0 := -2 \operatorname{Re} z_1 + p(z_{[2,n]}, \bar{z}_{[2,n]}).
\]

Note that by elementary scaling argument, if \( M \) is pseudoconvex, then so is its model \( M_0 \) as seen in Lemma 3.2.

As customary, we shall assume that \( p(z_{[2,n]}, \bar{z}_{[2,n]}) \) does not contain any pure (harmonic) monomials of the form \( z^\alpha \) or \( \bar{z}^\beta \). Otherwise, they can be always eliminated by a holomorphic transformation

\[
(z_1, z_{[2,n]}) \mapsto (z_1 + h(z_{[2,n]}), z_{[2,n]}).
\]

In what follows, homogeneity will be gauged with respect to a regular weight \( \mu \)

6.2. First Step.

Lemma 6.1. Let \( M_0 \subset \mathbb{C}^n, n \geq 2 \), be a pseudoconvex model hypersurface defined by

\[
r_0 = 0, \quad r_0 := -2 \operatorname{Re} z_1 + p(z_{[2,n]}, \bar{z}_{[2,n]}),
\]

where \( p \) is a weighted homogeneous polynomial of weight 1 with respect to some weights

\[
\mu = (\mu_1, \mu_2, \mu_3, \ldots, \mu_n), \quad 1 = \mu_1 > \mu_2 \geq \mu_3 \geq \ldots \geq \mu_n \geq 0.
\]

Let \( s \) be such that

\[
\mu_2 = \ldots = \mu_s > \mu_{s+1},
\]

and assume that

\[
p_{[2,s]}(z_{[2,s]}, \bar{z}_{[2,s]}):= p(z_{[2,s]}, 0, \bar{z}_{[2,s]}, 0) \neq 0.
\]

Then after a unitary change of the variables \( (z_2, \ldots, z_s) \), we have

\[
p_2(z_2, \bar{z}_2) := p(z_2, 0, \bar{z}_2, 0) \neq 0,
\]
which is a plurisubharmonic homogeneous polynomial of even degree \(2k_{22}\), where \(k_{22} := \frac{1}{2\mu_2}\).

Furthermore, \(p_2\) has the form

\[
p_2 = \sum_{j=-k_{22}+1}^{k_{22}-1} C_{2,j} z_2^{k_{22}+j} \bar{z}_2^{k_{22}-j},
\]

where \(C_{2,0} > 0\) is \(\frac{1}{2\mu_2}\)-dominant among all the coefficients of \(p_2\).

**Proof.** By our assumptions, \(p_{[2,s]}\) is a nonzero weighted homogeneous polynomial in \((z_{[2,s]}, \bar{z}_{[2,s]})\).

Then clearly after a unitary transformation of \(z_{[2,s]}\), we may assume that (6.2) holds with \(p_2\) of the form (6.3).

Using pseudoconvexity of \(M_0\) we obtain

\[
\partial z_2 \bar{z}_2 p_2 \geq 0,
\]

where \(p_2\) is defined by (6.2). Since \(p_2\) does not contain any harmonic terms by our assumption, each monomial in \((z_2, \bar{z}_2)\) will contribute to \(\partial z_2 \bar{z}_2 p_2\). We compute

\[
\partial z_2 \bar{z}_2 p_2 = \sum_{j=-k_{22}+1}^{k_{22}-1} C_{2,j} (k_{22} + j)(k_{22} - j) z_2^{k_{22}+j} \bar{z}_2^{k_{22}-j-1}.
\]

By Lemma 5.1 applied to \(\partial z_2 \bar{z}_2 p_2\), we conclude \(C_{2,0} > 0\), and

\[
|C_{2,j}| \leq \frac{k_{22}^2}{k_{22}^2 - j^2} C_{2,0} \leq \frac{k_{22}^2}{2k_{22} - 1} C_{2,0} < k_{22} C_{2,0}, \quad j \neq 0.
\]

Note that \(C_{2,0}\) is the coefficient of the balanced term \(C_{2,0} z_2^{k_{22}} \bar{z}_2^{k_{22}}\) in \(p_2\). Since \(r_0\) has weight 1 with respect to the weight \(\mu\), we have \(k_{22} = \frac{1}{2\mu_2}\) and \(2k_{22}\) is the degree of \(p_2\). \(\Box\)

We shall now apply Lemma 6.1 with the weights given by the Catlin multitype. Using the previous lemma, we can normalize the first non-trivial function in the Catlin boundary system:

**Lemma 6.2.** Let \(M\) be a pseudoconvex hypersurface defined by \(r = 0\), where

\[
r = -2 \text{Re} \, z_1 + p(z_{[2,n]}, \bar{z}_{[2,n]}) + o_\mu(1),
\]

such that \(p\) is a weighted homogeneous polynomial of weight 1 with respect to the Catlin multitype at 0,

\[
\Lambda = (1, \lambda_2, \lambda_3, \ldots, \lambda_n).
\]

Assume the Levi rank at 0 is \(s_0\) and \(2 < \lambda_{s_0+2} = \cdots = \lambda_{s_0+s_1+1} < \lambda_{s_0+s_1+2} < +\infty\). Let

\[
\mathfrak{B}_n(0) = \{r_1, r_{s_0+2}, \ldots, r_n; L_2, \ldots, L_n\}
\]

be any boundary system at 0. Then there exists a holomorphic change of coordinates at 0 preserving the multitype so that the boundary system in the new coordinates

\[
\tilde{\mathfrak{B}}_n(0) = \{\tilde{r}_1, \tilde{r}_{s_0+2}, \ldots, \tilde{r}_n; \tilde{L}_2, \ldots, \tilde{L}_n\},
\]
has the simplest possible first functions \( r_{s_0+2}, \ldots, r_{s_0+s_1+1} \) given by

\[
(6.4) \quad r_j = \text{Re} z_j + o\left(\frac{1}{\lambda_j}\right)
\]

for \( s_0 + 2 \leq j \leq s_0 + s_1 + 1 \).

**Proof.** By a Chern-Moser type argument [ChM74], we may assume that at 0 the Levi rank \( s_0 = 0 \). Furthermore, it is easy to see this lemma reduces to proving that at the level of the model hypersurface \( r_0 = 0 \), where

\[
r_0 = -2\text{Re} z_1 + p(z_{[2,n]}, \bar{z}_{[2,n]}),
\]

for \( s_0 + 2 \leq j \leq s_0 + s_1 + 1 \) we can bring \( r_j \) to the form \( r_j = \text{Re} z_j \) via a holomorphic polynomial change of the variables. We start by noting that the assumption (6.1) must hold for the Catlin’s multitype of the model hypersurface, which is the same as that of the original hypersurface. Indeed, otherwise we could increase the inverse weights

\[
\lambda_2 = \ldots = \lambda_{s_1+1}
\]

and decrease the remaining bigger inverse weights, still keeping the total weighted degrees of the terms in \( p \) greater or equal 1, which would contradict that assumption that \((1, \lambda_2, \lambda_3, \ldots, \lambda_n)\) is the Catlin multitype at 0. For the moment, assume \( s_1 = 1 \).

We can apply Lemma 6.1 to obtain a decomposition

\[
r_0 = -2\text{Re} z_1 + p_2(z_2, \bar{z}_2) + q_2(z_{[2,n]}, \bar{z}_{[2,n]}),
\]

where \( p_2 \) satisfies (6.3) and

\[
q_2(z_2, 0, \bar{z}_2, 0) \equiv 0.
\]

We would like to make another change of variables that would ensure the function \( r_2 \) in the boundary system has the simplest possible expression, namely \( r_2 = \text{Re} z_2 \). Note that the boundary system contains a function \( r_2 \) due to the assumption \( 2 < \lambda_2 < +\infty \). To determine what change of variables needs to be made, we notice that regardless of what form \( r_2 \) initially assumes, after the change of variables mandated by the application of Lemma 6.1,

\[
(6.5) \quad \frac{1}{(k' - 1)!k''!} \partial^{k'-1}_{z_2} \partial^{k''}_{\bar{z}_2} r_0 = C_{2,0} z_2 + C_{2,-1} \bar{z}_2 + T(z_{[3,n]}, \bar{z}_{[3,n]}),
\]

where \( k' + k'' = 2k_{22} = \lambda_2 \) and \( T(z_{[3,n]}, \bar{z}_{[3,n]}) \) is the sum of terms coming from differentiation of \( q_2 \) in the expression for \( r_0 \). Note that \( T \) cannot depend on \( z_2 \) due to the weight restriction.

It is beneficial to split \( q_2 \) as

\[
q_2 = \sum_{k+l=2k_{22}-1} z_2^k \bar{z}_2^l T_{kl}(z_{[3,n]}, \bar{z}_{[3,n]}) + S_2(z_{[2,n]}, \bar{z}_{[2,n]}),
\]

where all monomials in \( S_2 \) have their total degree in \((z_2, \bar{z}_2)\) less than \( 2k_{22} - 1 \).

By the pseudoconvexity of \( M_0 \), we see that for \( j \geq 3 \),

\[
\partial_{z_2} r_0 = \sum_{k+l=2k_{22}-1} z_2^k \bar{z}_2^l \partial_{z_2} \partial_{\bar{z}_2} T_{kl}(z_{[3,n]}, \bar{z}_{[3,n]}) + \partial_{z_2} S_2 \geq 0.
\]
Choosing $|z_2| >> |z_j|$ for all $j \geq 3$, we conclude
\[
\sum_{k+l=2k_{22}-1} z_2^k z_j^l \partial_{z_j \bar{z}_j} T_{kl}(z_{[3,n]}, \bar{z}_{[3,n]}) \geq 0, \quad j \geq 3.
\]
Since the left-hand side changes the sign when $z_2$ does, we obtain
\[
\partial_{z_j \bar{z}_j} T_{kl} = 0, \quad j \geq 3.
\]
Since the $z_j$-direction can be rotated arbitrarily by a change of coordinates, by a similar argument, the whole Levi form of each $T_{kl}$ must vanish identically, which means that each $T_{kl}$ is harmonic, i.e. a sum of holomorphic and antiholomorphic functions in $z_{[3,n]}$.

In particular, after the change of variables mandated by the application of Lemma 6.1, in the boundary system
\[
L_2 := \partial_{z_2} + p_{z_2} \partial_{z_1}, \quad \mathcal{L} = \{ \bar{L}_2, L_2, \bar{L}_2, \ldots, L_2, \bar{L}_2 \},
\]
where $L_2$ and $\bar{L}_2$ appear $k' - 1$ and $k''$ times respectively. Note that if $\lambda_2 = \lambda_3 = \cdots = \lambda_{s_1+1}$, another linear change of variables to gather all terms coming from $z_3, \ldots, z_{s_1+1}$ into $z_2$ may be required in order to achieve the expression for $L_2$ claimed above.

Since $p$ does not depend on $z_1$, we can ignore differentiations in that direction in the expressions of $L \partial r$. Hence we can compute $r_2$ up to a constant as
\[
r_2 = \mathcal{L} \partial r \sim \Re \partial_{z_2}^{k' - 1} \partial_{\bar{z}_2}^{k''} r_0 \sim \Re(C_{2,0} z_2 + C_{2,-1} \bar{z}_2 + T),
\]
where
\[
T = \phi_2(z_{[3,n]}) + \psi_2(z_{[3,n]})
\]
with $\phi_2$ and $\psi_2$ holomorphic. By a rotation in $z_2$, we can assume $C_{2,-1}$ real and nonnegative. Since $C_{2,0} > 0$, we can consider the change of variables,
\[
(6.6) \quad z_2 = \frac{1}{(C_{2,0} + C_{2,-1})} \bar{z}_2' - (\phi_2 + \psi_2)
\]
leading to $r_2 \sim \Re z_2$ and finally by scaling to
\[
r_2 = \Re z_2.
\]
Now let $s_1 > 1$. For every $j$ such that $3 \leq j \leq s_1 + 1$, we carry out the same procedure noting that due to weight considerations the change of variables required to transform $r_j$ into $r_j = \Re z_j$ will not affect variables $z_1, \ldots, z_{j-1}$.

Remark 6.1. A similar argument cannot be carried out to normalize the next boundary system function $r_{s_0 + s_1 + 2}$ if $\lambda_{s_0 + s_1 + 1} < \lambda_{s_0 + s_1 + 2}$. A counterexample is given at the end of the paper in Section 7.

Proof of Theorem 1.3: At the level of the model hypersurface, $r_0 = 0$, with
\[
r_0 = -2 \Re z_1 + p(z_{[2,n]}, \bar{z}_{[2,n]}),
\]
Lemma 6.2 shows that after a change of variables $\tilde{r}_j = \Re z_j$ for $s_0 + 2 \leq j \leq s_0 + s_1 + 1$. Therefore, $\tilde{L}_j = \frac{\partial}{\partial z_j}$ and $\tilde{L}_{s_0+s_1+2} = \frac{\partial}{\partial z_{s_0+s_1+1}} + o\left(-\frac{1}{\lambda_{s_0+s_1+2}}\right)$ for the model hypersurface as $\tilde{L}_{s_0+s_1+2}$ is chosen so that $\tilde{L}_{s_0+s_1+2} \tilde{r}_{s_0+s_1+1} = 0$, and by Catlin own normalization result in [C84b], Proposition 5.3,

$$r_0 = 2 \Re z_1 + \sum_{j=2}^{s_0+1} |z_j|^2 + f_1(z_{s_0+2}, \ldots, z_n) + 2 \Re \left(\sum_{j=2}^{s_0+1} z_j f_j(z_{s_0+2}, \ldots, z_n)\right).$$

The proposition follows. □

Proof of Corollary 1.4: We apply Proposition 1.3 with $s_0 = 0$ and $n = 3$. □

### 6.3. Second Step.

**Lemma 6.3.** Let $M_0$ be a pseudoconvex hypersurface with polynomial defining function

$$r_0 = -2 \Re z_2 + p(z_{[2,n]}, \tilde{z}_{[2,n]}),$$

such that $p$ is a weighted homogeneous polynomial in $z_2, \ldots, z_n$ of total weight 1 with respect to the weights

$$\mu = (1, \mu_2, \mu_3, \ldots, \mu_n).$$

Let $s$ be such that

$$\mu_3 = \ldots = \mu_s > \mu_{s+1},$$

and assume that

$$p(z_{[2,s]}, 0, \tilde{z}_{[2,s]}, 0) \neq p_2(z_2, \tilde{z}_2),$$

where $p_2$ is given by (6.2).

Then after a unitary change of variables $z_3, \ldots, z_s$, the polynomial $p$ admits a decomposition

$$p = p_2(z_2, \tilde{z}_2) + p_3(z_{[2,3]}, \tilde{z}_{[2,3]}) + q_3(z_{[2,n]}, \tilde{z}_{[2,n]}),$$

where $p_3$ is of degrees $2k_{23}$ and $2k_{33}$ in $(z_2, \tilde{z}_2)$ and $(z_3, \tilde{z}_3)$ respectively, $q_3$ has only terms of degree less than $2k_{33}$ in $(z_3, \tilde{z}_3)$, and $p_3$ contains a non-zero term

$$C|z_2|^{2k_{23}}|z_3|^{2k_{33}} \geq 0, \quad k_{33} > 0.$$

**Proof:** Just like at the beginning of the proof of Lemma 6.2, we write $r_0$ as

$$r_0 = -2 \Re z_2 + p_2(z_2, \tilde{z}_2) + q_2(z_{[2,n]}, \tilde{z}_{[2,n]}).$$

After a possible unitary change of variables in $z_{[3,s]}$, we can assume that

$$q_2(z_{[2,3]}, 0, \tilde{z}_{[2,3]}, 0) \neq 0.$$

Using Lemma 5.2 for guidance, we next identify the non-zero terms in $q_2$ of the highest (total) degree $d_3$ in $(z_3, \tilde{z}_3)$ and denote their sum by $p_3(z_{[2,3]}, \tilde{z}_{[2,3]})$. Then $p$ can be decomposed as

$$p = p_2(z_2, \tilde{z}_2) + p_3(z_{[2,3]}, \tilde{z}_{[2,3]}) + q_3(z_{[2,n]}, \tilde{z}_{[2,n]}),$$

where all monomials in $q_3(z_{[2,3]}, 0, \tilde{z}_{[2,3]}, 0)$ have degree less than $d_3$ in $(z_3, \tilde{z}_3)$. 
We shall consider the inequality
\[(\partial_{z_2} + t\partial_{\bar{z}_2})(\partial_{\bar{z}_2} + t\partial_{z_2})(p_2(z_2, \bar{z}_2) + p_3(z_{[2,3]}, \bar{z}_{[2,3]}) + q_3(z_{[2,3]}, 0, \bar{z}_{[2,3]}, 0)) \geq 0,\]
that follows from the pseudoconvexity of \(M_0\), where \(t\) is an arbitrary parameter. Identifying terms of the highest degree in \((z_3, \bar{z}_3)\) we obtain
\[\partial_{z_2}\bar{z}_3 p_3 \geq 0.\]

We first assume that
\[(6.10) \quad \partial_{z_2}\bar{z}_3 p_3 \equiv 0,\]
i.e. all terms in \(p_3\) are harmonic in \(z_2\). Then again identifying terms of the highest degree in \((z_3, \bar{z}_3)\) under this assumption, we obtain
\[2t \Re \partial_{z_2}\bar{z}_3 p_3 + \partial_{z_2}\bar{z}_3 \tilde{q}_3 \geq 0,\]
where \(\tilde{q}_3\) is the sum of certain terms from \(q_3\). Since \(t\) is arbitrary, we must have
\[\partial_{z_2}\bar{z}_3 p_3 \equiv 0.\]

Since \(p_3\) has no harmonic terms and all terms are harmonic in \(z_2\), the only possibility remaining is that \(p_3\) is independent of \(z_2\). But then, since \(p_3\) is nonzero and has no harmonic terms, we must have
\[(6.11) \quad \partial_{z_2}\bar{z}_3 p_3 \not\equiv 0.\]

On the other hand, if (6.10) does not hold, we obtain the polynomial \(\partial_{z_2}\bar{z}_3 p_3 \geq 0\), which for any generic fixed \(z_2\), is non-constant and homogeneous in \((z_3, \bar{z}_3)\), which again implies (6.11).

Thus in all cases, we must have (6.11). Applying Lemma 5.1 to \(\partial_{z_3}\bar{z}_3 p_3\) for fixed \(z_2\) (when it does not identically vanish), we conclude that the degree \(d_3\) in \((z_3, \bar{z}_3)\) is even, \(d_3 = 2k_{33}\), and \(\partial_{z_3}\bar{z}_3 p_3\) contains nonzero terms of the form
\[\tilde{p}(z_2, \bar{z}_2)z_{k_{33}}^{k_{33}}\bar{z}_3^{k_{33}} \geq 0.\]

Applying again Lemma 5.1, this time to \(\tilde{p}\), we conclude that \(p_3\) contains a nonzero term
\[Cz_2^{k_{33}}z_3^{k_{33}}\bar{z}_2^{k_{33}}\bar{z}_3^{k_{33}} \geq 0,\]
as desired. \(\square\)

6.4. Inductive Step. The general inductive step will be obtained from the following result.

**Lemma 6.4.** Let \(M_0\) be a pseudoconvex (model) hypersurface through 0 with polynomial defining function
\[r = -2\Re z_1 + p(z_{[2,n]}, \bar{z}_{[2,n]}),\]
such that \(p\) is a weighted homogeneous polynomial in \(z_2, \ldots, z_n\) of weight 1 with respect to the weights
\[\mu = (1, \mu_2, \ldots, \mu_n), \quad 1 > \mu_2 \geq \ldots \geq \mu_n.\]
Assume that we have already shown that
\[ p(z_{[2,n]}, \bar{z}_{[2,n]}) = p_2(z_2, \bar{z}_2) + p_3(z_{[2,3]}, \bar{z}_{[2,3]}) + \cdots \]
\[ + p_{m-1}(z_{[2,m-1]}, \bar{z}_{[2,m-1]}) + q_{m-1}(z_{[2,n]}, \bar{z}_{[2,n]}). \]
where
\[ q_{m-1}(z_{[2,m-1]}, 0, \bar{z}_{[2,m-1]}, 0) \equiv 0. \]
Let \( s \) be such that
\[ \mu_m = \ldots = \mu_s > \mu_{s+1}, \]
and assume that
\[ (6.12) \quad q_{m-1}(0, z_{[m,s]}, 0, 0, \bar{z}_{[m,s]}, 0) \neq 0. \]
Then after a unitary change of the variables \( (z_m, \ldots, z_s) \), \( q_{m-1} \) admits the decomposition
\[ (6.13) \quad q_{m-1} = p_m(z_{[2,m]}, \bar{z}_{[2,m]}) + q_m(z_{[2,n]}, \bar{z}_{[2,n]}), \]
where \( p_m \) is a homogeneous polynomial of degree \( 2k_{mn} > 0 \) in \( (z_m, \bar{z}_m) \) whose expansion contains a term
\[ C |z_2|^{2k_{m2}} |z_3|^{2k_{m3}} \cdots |z_{m}|^{2k_{m,m}}, \quad C > 0, \]
and \( q_m \) has only terms of degree less than \( 2k_{mn} \) in \( (z_m, \bar{z}_m) \).

**Proof:** After a possible unitary change of variables in \( z_{[m,s]} \), we can assume that
\[ q_{m-1}(0, z_m, 0, 0, \bar{z}_m, 0) \neq 0. \]
We next identify the non-zero terms in \( q_{m-1} \) of the highest degree \( d_m > 0 \) in \( (z_m, \bar{z}_m) \) and denote their sum by \( p_m(z_{[2,m]}, \bar{z}_{[2,m]}) \). We thus write
\[ q_{m-1} = p_m(z_{[2,m]}, \bar{z}_{[2,m]}) + q_m(z_{[2,n]}, \bar{z}_{[2,n]}), \]
where all monomials in \( q_m(z_{[2,m]}, 0, \bar{z}_{[2,m]}, 0) \) have degree less than \( d_m \) in \( (z_m, \bar{z}_m) \) by construction.
We shall now use the pseudoconvexity assumption on \( M_0 \). For any \( j, 2 \leq j < m \), and any arbitrary real parameter \( t \), consider
\[ (6.14) \quad (\partial_{z_j} + t \partial_{z_m})(\partial_{\bar{z}_j} + t \partial_{\bar{z}_m}) \left( p_2(z_2, \bar{z}_2) + p_3(z_{[2,3]}, \bar{z}_{[2,3]}) + \cdots \right. \]
\[ + p_{m-1}(z_{[2,m-1]}, \bar{z}_{[2,m-1]}) + p_m(z_{[2,m]}, \bar{z}_{[2,m]}) \]
\[ \left. + q_m(z_{[2,n]}, \bar{z}_{[2,n]}) \right) \geq 0. \]
Identifying the highest degree terms in \( (z_m, \bar{z}_m) \), we obtain that
\[ (6.15) \quad \partial_{z_j} \partial_{\bar{z}_j} p_m \geq 0, \quad j < m. \]
We first assume \( \partial_{z_j} \partial_{\bar{z}_j} p_m \equiv 0 \) for all \( j < m \), i.e. all terms of \( p_m \) are harmonic in \( z_j \). Then looking at the highest degree terms in \( (z_m, \bar{z}_m) \) in (6.14) yields
\[ 2t \text{ Re } \partial_{z_j} \partial_{\bar{z}_m} p_m + \partial_{z_j} \partial_{\bar{z}_j} q_m \geq 0, \]
where $q_m$ consists of the sum of the terms of degree $d_m - 1$ in $(z_m, \bar{z}_m)$ from $p_2 + \cdots + p_m + q_m$. Given that $t$ is arbitrary, we conclude
\[
\partial z_j \bar{z}_m p_m \equiv 0.
\]
Note that $p_m$ contains no harmonic terms and by our assumption, all terms of $p_m$ are harmonic in $z_j$ for all $j < m$. Hence any nonzero term of $p_m$ must have both $z_m$ and $\bar{z}_m$, that is we must have that
\[
\partial z_m \bar{z}_m p_m \neq 0.
\]
Now, on the contrary, assume that $\partial z_j \bar{z}_m p_m \neq 0$ for some $j < m$. For any generic fixed $z_2, \ldots, z_{m-1}$, the polynomial $\partial z_j \bar{z}_m p_m$ is non-constant, non-negative by (6.15), and homogeneous in $(z_m, \bar{z}_m)$ of degree $d_m > 0$. Clearly, (6.16) must hold in this case as well. Therefore, regardless of the case, (6.16) holds.

We claim that by Lemma 5.1 inductively applied to $\partial z_m \bar{z}_m p_m$, the expansion of the polynomial $\partial z_m \bar{z}_m p_m$ contains a term
\[
C|z_2|^{2\alpha_2} |z_3|^{2\alpha_3} \cdots |z_m|^{2\alpha_m}, \quad \alpha_m > 0, \quad C > 0.
\]
Indeed, first keep $z_2, \ldots, z_{m-1}$ fixed and apply Lemma 5.1 to $\partial z_m \bar{z}_m p_m$, a non-constant, non-negative, and homogeneous polynomial in $(z_m, \bar{z}_m)$ of degree $d_m - 2$. We conclude that the sum of the terms in $p_m$ having equal degrees in $z_m$ and $\bar{z}_m$ is of the form
\[
P_m^{-1}(z_2, \ldots, z_{m-1}, \bar{z}_2, \ldots, \bar{z}_{m-1})|z_m|^{2\alpha_m} \geq 0, \quad \alpha_m = d_m/2 > 0,
\]
where $P_m^{-1}$ is a nonzero weighted homogeneous polynomial.

Let $l$ be the highest index among $2, \ldots, m-1$ for which $P_m^{-1}$ has degree $d_l > 0$ in $(z_l, \bar{z}_l)$. If no such $l$ exists, then $P_m^{-1}$ is constant and positive, and we are done; otherwise we extract the sum $P_m^{-1}$ of terms of the top degree $d_l$ in $(z_l, \bar{z}_l)$, which is not identically zero, and nonnegative in view of Lemma 5.2, keep $z_2, \ldots, z_{l-1}$ fixed and apply Lemma 5.1 to $P_m^{-1}$ viewed as a homogeneous polynomial in $(z_l, \bar{z}_l)$. Proceeding inductively, we see that $p_m$ contains a non-zero term (6.17) maximizing the multidegree in $(z_2, \bar{z}_2), \ldots, (z_m, \bar{z}_m)$ in the reversed lexicographic order as claimed.

\[
\square
\]

**Proof of Theorem 1.1:** Our Main Theorem is a consequence of Lemmas 6.1, 6.3, and 6.4. In fact, at each step, either the nonvanishing assumption in the Lemmas holds, and hence we obtain a positive $A_j$ or we can lower the weight $\mu$ lexicographically starting with $\mu_j$. In the latter case, either we regain the nonvanishing assumption for a lower $\mu_j$ and keep applying the Lemmas with lower weights, or no further term is left and the proof is complete.

**Proof of Corollary 1.2:** This result follows from Theorem 1.1.

\[
\square
\]

7. Counterexample to a Boundary System Normalization

We would like to show via an example that the kind of normalization of function $r_2$ in Catlin’s boundary system that we carried out in Theorem 1.3 fails for $r_3$. Let the defining function of
the domain be given by
\[ r_0 = -2 \text{Re} \, z_1 + p(z_{[2,4]}, \bar{z}_{[2,4]}), \]
where \( p \) is a weighted homogeneous polynomial chosen so that the weight of \( z_2^2 \) equals the weight of \( z_3^3 \). For example, the polynomial
\[
p(z_{[2,4]}, \bar{z}_{[2,4]}) = |z_2|^6 + |z_2|^2|z_3|^6 + |z_2|^4|z_3|^2|z_4|^2 + |z_2|^2|z_3|^4|z_4|^4
\]
\[ + 2\epsilon \text{Re}(|z_2|^2z_3^2\bar{z}_3^3|z_4|^2) + |z_3|^8|z_4|^2 \]
is homogeneous of weight 1 with respect to
\[
\Lambda = \left( \frac{1}{6}, \frac{1}{9}, \frac{1}{18} \right).
\]
Note that
\[
f := \partial_{z_2}\partial_{z_2}\partial_{z_3}\partial_{z_3}p = c_1z_3 + c_2|z_4|^2,
\]
and hence \( r_3 = \text{Re} \, f \) cannot be transformed into \( c \text{Re} \, z_3 \) by any holomorphic coordinate change.

We would like to show that \( p \) is plurisubharmonic when \( \epsilon \) is small, where 0 < \( \epsilon \) < 1. First, we observe that if \( z_2 = 0, \ z_3 = 0, \text{ or } z_4 = 0 \), the term \( 2\epsilon \text{Re}(|z_2|^2z_3^2\bar{z}_3^3|z_4|^2) \) vanishes, and thus \( p \) is a sum of squares, making it automatically plurisubharmonic. Therefore, without the loss of generality, we can assume simultaneously that \( z_2 \neq 0, \ z_3 \neq 0, \text{ and } z_4 \neq 0 \). As a result, we can compute the Levi form in terms of vectors fields
\[
X = \sum_{j=2}^{4} a_j z_j \frac{\partial}{\partial z_j}
\]
and \( X \), which keep the weight of each term of \( p \) unchanged.

Rather than writing out the Levi form in full in terms of \( X, \bar{X} \) as a quadratic form in \( a_j \), we observe that by Cauchy-Schwarz,
\[
|2 \text{Re}(|z_2|^2z_3^2\bar{z}_3^3|z_4|^2)| \leq (|z_2|^2|z_3|^6 + |z_2|^2|z_3|^4|z_4|^4)
\]
and
\[
|2 \text{Re}(|z_2|^2z_3^2\bar{z}_3^3|z_4|^2)| \leq (|z_2|^4|z_3|^2|z_4|^2 + |z_3|^8|z_4|^2).
\]
For the right-hand side expression in (7.1), its kernel is given by the simultaneous vanishing of \( \{a_2 + 3a_3 = 0\} \) and \( \{a_2 + 2a_3 + 2a_4 = 0\} \). For the right-hand side expression in (7.2), its kernel is given by the simultaneous vanishing of \( \{2a_2 + a_3 + a_4 = 0\} \) and \( \{4a_3 + a_4 = 0\} \). The intersection of these kernels is just the origin, so for small \( \epsilon \), \( p \) is indeed plurisubharmonic. Therefore, \( r_0 \) defines a pseudoconvex domain.

**References**

[BG77] T. Bloom and I. Graham. On “type” conditions for generic real submanifolds of \( \mathbb{C}^n \). *Invent. Math.*, 40(3):217–243, 1977.

[BS92] H. P. Boas and E. J. Straube. On equality of line type and variety type of real hypersurfaces in \( \mathbb{C}^n \). *J. Geom. Anal.* 2 (1992), no. 2, 95–98.
[C84a] D. W. Catlin. Global regularity of the $\bar{\partial}$-Neumann problem. Complex analysis of several variables (Madison, Wis., 1982), 39–49, Proc. Sympos. Pure Math., 41, Amer. Math. Soc., Providence, RI, 1984.

[C84b] D. W. Catlin. Boundary invariants of pseudoconvex domains. *Ann. of Math.* (2), 120(3):529–586, 1984.

[C87] D. W. Catlin. Subelliptic estimates for the $\bar{\partial}$-Neumann problem on pseudoconvex domains. *Ann. of Math.* (2), 126 (1): 131–191, (1987).

[CD10] D. W. Catlin and J. P. D’Angelo. Subelliptic estimates. In *Complex analysis*, Trends Math., pages 75–94. Birkhäuser/Springer Basel AG, Basel, 2010.

[ChM74] S. S. Chern and J. K. Moser. Real hypersurfaces in complex manifolds. *Acta Math.*, 133:219–271, 1974.

[D82] J. P. D’Angelo. Real hypersurfaces, orders of contact, and applications. *Ann. of Math.* (2), 115(3):615–637, 1982.

[Ko10] M. Kolar. The Catlin multitype and biholomorphic equivalence of models. *Int. Math. Res. Not. IMRN* 2010, no. 18, 3530–3548.

[KoM11] M. Kolar and F. Meylan. Chern-Moser operators and weighted jet determination problems. In *Geometric analysis of several complex variables and related topics*, volume 550 of Contemp. Math., pages 75–88. Amer. Math. Soc., Providence, RI, 2011.

[KoMZ14] M. Kolar, F. Meylan, and D. Zaitsev. Chern-Moser operators and polynomial models in CR geometry. *Adv. Math.*, 263:321–356, 2014.

[K79] J. J. Kohn. Subellipticity of the $\bar{\partial}$-Neumann problem on pseudo-convex domains: sufficient conditions. *Acta Math.*, 142(1-2):79–122, 1979.

[KiZ17] S.Y. Kim and D. Zaitsev. Jet vanishing orders and effectivity of Kohn’s algorithm in dimension 3. Preprint 2017. https://arxiv.org/abs/1702.06908.

[M92] J. D. McNeal. Convex domains of finite type. *J. Funct. Anal.* 108 (1992), no. 2, 361–373.

[N14] A. C. Nicoara. Direct Proof of Termination of the Kohn Algorithm in the Real-Analytic Case. https://arxiv.org/abs/1409.0963v1.

[S10] Y.-T. Siu. Effective termination of Kohn’s algorithm for subelliptic multipliers. *Pure Appl. Math. Q.*, 6(4, Special Issue: In honor of Joseph J. Kohn. Part 2):1169–1241, 2010.

[S17] Y.-T. Siu. New procedure to generate multipliers in complex Neumann problem and effective Kohn algorithm. *Sci. China Math.*, 60(6):1101–1128, 2017.

[Y92] J. Yu. Multitypes of convex domains. *Indiana Univ. Math. J.*, 41(3):837–849, 1992.

[Z17] D. Zaitsev. A geometric approach to Catlin’s boundary systems. Preprint 2017. https://arxiv.org/abs/1704.01808.

**Department of Mathematics, Statistics and Computer Science, University of Wisconsin-Stout, Menomonie, WI 54751**

*E-mail address:* abasyrov@gmail.com

**School of Mathematics, Trinity College Dublin, Dublin 2, Ireland**

*E-mail address:* anicoara@maths.tcd.ie, zaitsev@maths.tcd.ie