CHARGE ON TABLEAUX AND THE POSET OF $k$-SHAPES

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Abstract. A poset on a certain class of partitions known as $k$-shapes was introduced in [4] to provide a combinatorial rule for the expansion of a $k-1$-Schur functions into $k$-Schur functions at $t = 1$. The main ingredient in this construction was a bijection, which we call the weak bijection, that associates to a $k$-tableau a pair made out of a $k-1$-tableau and a path in the poset of $k$-shapes. We define here a concept of charge on $k$-tableaux (which conjecturally gives a combinatorial interpretation for the expansion coefficients of Hall-Littlewood polynomials into $k$-Schur functions), and show that it is compatible in the standard case with the weak bijection. In particular, we obtain that the usual charge of a standard tableau of size $n$ is equal to the sum of the charges of its corresponding paths in the poset of $k$-shapes, for $k = 2, 3, \ldots, n$.

1. Introduction

For each integer $k \geq 1$, a family of symmetric functions $s_\mu^{(k)}(x; t)$ (now called $k$-Schur functions) were introduced in [9] in connection with Macdonald polynomials. To be more precise, computer evidence suggested, for each positive integer $k$, the existence of a family of symmetric polynomials defined by certain sets of tableaux $A_\mu^{(k)}$ as:

$$s_\mu^{(k)}(x; t) = \sum_{T \in A_\mu^{(k)}} e^{\text{charge}(T)} s_{\text{shape}(T)}(x)$$

(1.1)

with the property that any (plethystically modified) Macdonald polynomial $H_\lambda(x; q, t)$ indexed by a partition $\lambda$ whose first part is not larger than $k$ (a $k$-bounded partition), can be decomposed as:

$$H_\lambda(x; q, t) = \sum_{\mu: \mu_1 \leq k} K_{\mu\lambda}^{(k)}(q, t) s_\mu^{(k)}(x; t), \quad K_{\mu\lambda}^{(k)}(q, t) \in \mathbb{N}[q, t].$$

(1.2)

Moreover, given that in this setting $s_\mu^{(k)}(x; t) = s_\mu(x)$ for large $k$, the coefficients $K_{\mu\lambda}^{(k)}(q, t)$ reduce to the usual $q, t$-Kostka coefficients when $k$ is large enough [9] [15]. The study of the $s_\lambda^{(k)}(x; t)$ led to several conjecturally equivalent characterizations [9] [10] [6], but for the purpose of this article only the characterization of [6] will be relevant. Note that from now on, we will always index $k$-Schur functions by $k+1$-cores rather than by $k$-bounded partitions (see for instance [11] for the connection between the two concepts).

It was shown that the $k$-Schur functions at $t = 1$ (in the characterization [6]) provide the natural basis to work in the quantum cohomology of the Grassmannian just as the Schur functions do for the usual cohomology [12]. Another key development was T. Lam’s proof [5] that the Schubert basis of the homology of the affine Grassmannian is given by the $k$-Schur functions, and that the Schubert basis of the cohomology of the affine Grassmannian is given by functions dual to the $k$-Schur functions, called dual $k$-Schur functions or affine Schur functions [4] [12].

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Many combinatorial conjectures about $k$-Schur functions have been formulated, but the conjecture especially relevant to this article is that the $k$-Schur functions expand positively into $k'$-Schur functions for $k' > k \[7\]:

$$s^{(k)}(\lambda; t) = \sum_{\mu} b^{(k \to k')}_{\lambda, \mu}(t) s^{(k')}_{\mu}(x; t), \quad \text{for } b^{(k \to k')}_{\lambda, \mu}(t) \in \mathbb{Z}_{\geq 0}[t]. \quad (1.3)$$

When $k'$ is large enough, this conjecture says that $k$-Schur functions are Schur positive (partial results in this direction have been obtained in \[1\]). The conjecture also states in particular that $(k-1)$-Schur functions expand positively into $k$-Schur functions. We refer to the $b^{(k-1, k)}_{\lambda, \mu}(t)$ coefficients as the $k$-branching coefficients. We have defined in \[7\] a poset called the poset of $k$-shapes, whose maximal elements are $(k+1)$-cores and whose minimal elements are $k$-cores, and have a conjecture for the $k$-branching coefficients as enumerating maximal chains in the poset of $k$-shapes modulo an equivalence (see Section \(8\)). Using the duality between the ungraded $k$-Schur and dual $k$-Schur functions \[7\], this conjecture has been shown to be valid when $t = 1$ by proving that dual $k'$-Schur functions expand positively into dual $k'$-Schur functions for $k' > k \[7\].

One of the obstructions to the generalization of the the results of \[7\] to a generic value of $t$ was the lack of an explicit definition for the graded version of the dual $k$-Schur functions. In this article we introduce a charge on $k$-tableaux that provides such a definition. To be more precise, the dual $k$-Schur functions are the generating function of certain combinatorial objects called $k$-tableaux. We define the graded version of the dual $k$-Schur functions as\[4\]

$$\mathcal{G}^{(k)}_{\lambda}(x; t) = \sum_{Q} t^{\text{ch}(Q)} x^{Q} \quad (1.4)$$

where the sum is over all $k$-tableaux $Q$ of shape $\lambda$ (where for simplicity the dual $k$-Schur function is indexed by a $k+1$-core), and where $\text{ch}(Q)$ is a certain generalization (see Section \[4\]) of the charge of a tableau \[14\]. We conjecture that the charge also provides the $k$-Schur expansion of a Hall-Littlewood polynomial indexed by a $k$-bounded partition

$$H_{\lambda}(x; t) = \sum_{Q^{(k)}} t^{\text{ch}(Q^{(k)})} s^{(k)}_{\text{sh}(Q^{(k)})}(x; t) \quad (1.5)$$

where the sum is over all $k$-tableaux of weight $\lambda$, and where $\text{sh}(Q^{(k)})$ is the shape of the $k$-tableau $Q^{(k)}$. This formula, which would prove the case $q = 0$ of \[12\], generalizes a well-known result of Lascoux and Schützenberger providing a $t$-statistic on tableaux for the Kostka-Foulkes polynomials \[14\].

We give evidence of the validity of the definition of charge by proving that the concept of charge on $k$-tableaux is compatible in the standard case (the complications pertaining to the extension to the non-standard case are discussed in the conclusion) with the weak bijection introduced in \[7\].

**Theorem 1.** The weak bijection in the standard case

$$\text{SWTab}^{k}_{\lambda} \rightarrow \bigsqcup_{\mu \in \mathbb{C}^{k}} \text{SWTab}^{k-1}_{\mu} \times \mathcal{P}^{k}(\lambda, \mu)$$

$$Q^{(k)} \mapsto (Q^{(k-1)}, [p]) \quad (1.6)$$

where $\text{SWTab}^{k}_{\lambda}$ is the set of standard $k$-tableau (or standard weak tableau) of shape $\lambda$, and where $[p]$ is a certain equivalence class of paths in the poset of $k$-shapes, is such that

$$\text{ch}(Q^{(k)}) = \text{ch}(Q^{(k-1)}) + \text{ch}(p) \quad (1.7)$$

with $\text{ch}(p)$ the charge of the path $p$ (see Definition \[13\]).

\[1\] Another version of the graded dual $k$-Schur functions can be obtained from the $k$-Schur functions by duality with respect to the Hall-Littlewood scalar product (see \[8\]). This version is however not monomial positive.

\[2\] A different definition of graded dual $k$-Schur functions has been proposed in \[2\].
As discussed in Section 8, Theorem 1 is one of the key ingredients in our approach to prove the $k$-Schur expansion (1.5) of the Hall-Littlewood polynomials.

A simple consequence of the compatibility between the charge and the weak bijection is that the charge of a usual standard tableau $T$ of $n$ letters is equal to the sum of the charges of its corresponding paths in the poset of $k$-shapes, for $k = 2, 3, \ldots, n$. In effect, iterating the weak bijection starting from $T$

$$T \mapsto (T^{(n-1)}, [p_n]), \quad T^{(n-1)} \mapsto (T^{(n-2)}, [p_{n-1}]), \quad \ldots, \quad T^{(2)} \mapsto (T^{(1)}, [p_2])$$

we obtain a bijection that puts in correspondence $T$ and $(T^{(1)}, [p_n], [p_{n-1}], \ldots, [p_2])$. Given that there is a unique standard 1-tableau $T^{(1)}$ we have that $T$ is in correspondence with the equivalence of paths $(p_n, p_{n-1}, \ldots, p_2)$. Moreover, the charge of $T^{(1)}$ being 0, the compatibility between the charge and the weak bijection in the standard case implies

$$\text{ch}(T) = \text{ch}(p_n) + \cdots + \text{ch}(p_2)$$

Here is the outline of the article. For the article to be self-contained, a good deal of results and definitions of [7] must be introduced or specialized to the standard case. These include for instance moves, covers, the poset of $k$-shapes, $k$-shape tableaux, the pushout algorithm, and the weak bijection. This is essentially the content of Sections 2, 3, 5, 6 and 7. The charge of a $k$-tableau is defined in Section 4. In Section 8 we describe the general context of this work, such as the Schur positivity of $k$-Schur functions, the $k$-Schur positivity of Hall-Littlewood polynomials, and finally the connection with the atoms of [9]. In Section 9 we define the charge and cocharge of a $k$-shape tableau and derive a relation between the two concepts (Proposition 47). Section 10 contains the proof of the main result of this article, namely the compatibility between (co)charge and the weak bijection in the standard case (Theorem 1). Finally, we discuss in the conclusion why the non-standard case is still out of reach and how fundamental is the problem of defining a Lascoux-Schützenberger type action of the symmetric group on $k$-shape tableaux.

2. Preliminaries

For a fixed positive integer $k$, the object central to our study is a family of "$k$-shape" partitions that contains both $k$ and $k+1$-cores. The formula for $k$-branching coefficients counts paths in a poset on $k$-shapes. As with Young order, we will define the order relation in terms of adding boxes to a given vertex $\lambda$, but now the added boxes must form a sequence of "strings". Here we introduce $k$-shapes, strings, and moves – the ingredients for our poset.

2.1. Partitions. A partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ of degree $|\lambda| = \sum \lambda_i$ is a vector of non-negative integers such that $\lambda_i \geq \lambda_{i+1}$ for $i = 1, 2, \ldots$. The length $\ell(\lambda)$ of $\lambda$ is the number of non-zero entries of $\lambda$. Each partition $\lambda$ has an associated Ferrers’ diagram with $\lambda_i$ lattice squares in the $i$th row, from bottom to top (French notation). For instance, if $\lambda = (6, 5, 5, 4, 3, 2, 2, 1, 1, 1)$, the associated Ferrers’ diagram is

Any lattice square in the Ferrers diagram is called a cell (or simply a square), where the cell $(i, j)$ is in the $i$th row and $j$th column of the diagram. Given a cell $b = (i, j)$, we let $\text{row}(b) = i$ and
col(b) = j. The conjugate $\lambda'$ of a partition $\lambda$ is the partition whose diagram is obtained by reflecting the diagram of $\lambda$ about the main diagonal. Given a cell $b = (i, j)$ in $\lambda$, we let

$$a_\lambda(b) = \lambda_i - j, \quad \text{and} \quad l_\lambda(b) = \lambda'_j - i. \quad (2.1)$$

The quantities $a_\lambda(b)$ and $l_\lambda(b)$ are respectively called the arm-length and leg-length. The hook-length of $b = (i, j) \in \lambda$ is then defined by $h_\lambda(b) = a_\lambda(b) + l_\lambda(b) + 1$. A $p$-core is a partition without cells of hook-length equal to $p$. We let $C^p$ be the set of $p$-cores.

We say that the diagram $\mu$ is contained in $\lambda$, denoted $\mu \subseteq \lambda$, if $\mu_i \leq \lambda_i$ for all $i$. We also let $\lambda + \mu$ be the partitions whose entries are $(\lambda + \mu)_i = \lambda_i + \mu_i$, and $\lambda \cup \mu$ be the partition obtained by reordering the entries of the concatenation of $\lambda$ and $\mu$. The dominance ordering on partitions is such that $\lambda \geq \mu$ iff $|\lambda| = |\mu|$ and $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for all $i$.

A cell $b$ is $\lambda$-addable (or $b$ is an addable corner of $\lambda$) if adding $b$ to $\lambda$ produces a partition $\mu$. Similarly, a cell of $b$ is $\lambda$-removable (or $b$ is a removable corner of $\lambda$) if removing $b$ from $\lambda$ leads to a partition $\mu$. The diagonal index of $b = (i, j)$ is $d(b) = j - i$. From it we define the distance between cells $x$ and $y$ as $|d(x) - d(y)|$.

Let $D = \mu/\lambda$ be a skew shape, the difference of Ferrers diagrams of partitions $\mu \supset \lambda$. Although such a set of cells may be realized by different pairs of partitions, unless specifically stated otherwise, we shall use the notation $\mu/\lambda$ with the fixed pair $\lambda \subset \mu$ in mind. A horizontal (resp. vertical) strip is a skew shape that contains at most one cell in each column (resp. row).

2.2. $k$-shapes. The $k$-interior of a partition $\lambda$ is the subpartition made out of the cells with hook-length larger than $k$:

$$\text{Int}^k(\lambda) = \{ b \in \lambda \mid h_\lambda(b) > k \}.$$  

The $k$-boundary of $\lambda$ is the skew shape of cells with hook-length bounded by $k$:

$$\partial^k(\lambda) = \lambda/\text{Int}^k(\lambda).$$

We define the $k$-row shape $rs^k(\lambda) \in \mathbb{Z}_{\geq 0}^\infty$ (resp. $k$-column shape $cs^k(\lambda) \in \mathbb{Z}_{\leq 0}^\infty$) of $\lambda$ to be the sequence giving the number of cells in the rows (resp. columns) of $\partial^k(\lambda)$.

**Definition 2.** Let $k \geq 2$ be an integer. A partition $\lambda$ is a $k$-shape if $rs^k(\lambda)$ and $cs^k(\lambda)$ are partitions. We let $\Pi^k$ denote the set of $k$-shapes and $\Pi^k_N = \{ \lambda \in \Pi^k : |\partial^k(\lambda)| = N \}$.

**Example 3.** The partition $\lambda = (8, 4, 3, 2, 1, 1, 1) \in \Pi^2_{12}$, since $rs^4(\lambda) = (4, 2, 2, 1, 1, 1, 1)$ and $cs^4(\lambda) = (3, 2, 2, 1, 1, 1, 1)$ are partitions and $|\partial^4(\lambda)| = 4 + 2 + 2 + 1 + 1 + 1 + 1 = 12$. The partition $\mu = (3, 3, 1) \notin \Pi^4$ since $rs^4(\mu) = (2, 3, 1)$ is not a partition.

The set of $k$-shapes includes both the $k$-cores and $k + 1$-cores.

**Proposition 4** ([3]). $C^k \subset \Pi^k$ and $C^{k+1} \subset \Pi^k$.

**Remark 5.** Since $k$ remains fixed throughout, we shall often for simplicity suppress $k$ in the notation, writing $\partial \lambda$, $rs(\lambda)$, $cs(\lambda)$, $\Pi$, and so forth.
2.3. Strings. The primary notion to define our order on \(k\)-shapes is a string of cells lying at a diagonal distance \(k\) or \(k + 1\) from one another. To be precise, let \(b\) and \(b'\) be contiguous cells when \(|d(b) - d(b')| \in \{k, k + 1\}\).

Remark 6. Since \(\lambda\)-addable cells cannot occur on consecutive diagonals, a \(\lambda\)-addable corner \(x\) is contiguous with at most one \(\lambda\)-addable corner above (resp. below) it.

Definition 7. A string of length \(\ell\) is a skew shape \(\mu/\lambda\) which consists of cells \(\{a_1, \ldots, a_\ell\}\), where \(a_i\) and \(a_{i+1}\) are contiguous (with \(a_{i+1}\) below \(a_i\)) for each \(1 \leq i < \ell\).

Example 8. Let \(k = 3\), \(\lambda = (4, 2, 1)\) and \(\mu = (5, 3, 1, 1)\). Then \(\mu/\lambda\) is a string of length 3. If we denote each element of \(\mu/\lambda\) by a \(\bullet\), the string can be represented as:

\[
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & & \\
\end{array}
\]

Given a string \(s = \mu/\lambda = \{a_1, \ldots, a_\ell\}\), of particular importance are certain columns and rows called modified rows. Define \(\Delta_{rs}(s) = rs(\mu) - rs(\lambda) \in \mathbb{Z}^\infty\). The positively (resp. negatively) modified rows of \(s\) are those corresponding to positive (resp. negative) entries in \(\Delta_{rs}(s)\). We define in a similar way \(\Delta_{cs}(s)\), and the positively (resp. negatively) modified columns of \(s\). Any string \(s = \mu/\lambda\) can be categorized into one of four types

- A row-type string if \(s\) does not have positively or negatively modified rows
- A column-type string if \(s\) does not have positively or negatively modified columns.
- A cover-type string if \(s\) has a positively modified row and a positively modified column
- A cocover-type string if \(s\) has a negatively modified row and a negatively modified column

It is helpful to depict a string \(s = \mu/\lambda\) by its diagram, defined by the following data: cells of \(s\) are represented by the symbol \(\bullet\), cells of \(\partial \lambda \setminus \partial \mu\) are represented by \(\circ\), and cells of \(\partial \mu \cap \partial \lambda\) in the same row (resp. column) as some \(\bullet\) or \(\circ\) are collectively depicted by a horizontal (resp. vertical) line segment. The four possible string diagrams are shown in Figure 1.

2.4. Moves. The covering relations in the poset of \(k\)-shapes will be defined by letting a \(k\)-shape \(\lambda\) be larger than the \(k\)-shape \(\mu\) when the skew diagram \(\mu/\lambda\) is a particular succession of strings (called a move). To this end, define two strings to be translates when they are translates of each
other in $\mathbb{Z}^2$ by a fixed vector, and their corresponding modified rows and columns agree in size. Equivalently, two strings are translate if their diagrams have the property that $\bullet$'s and $\circ$'s appear in the same relative positions with respect to each other and the lengths of each corresponding horizontal and vertical segment are the same. We will also refer to cells $a_j$ and $b_j$ as translates when strings $s_1 = \{a_1, \ldots, a_\ell\}$ and $s_2 = \{b_1, \ldots, b_\ell\}$ are translates.

**Definition 9.** A row move $m$ of rank $r$ and length $\ell$ is a chain of partitions $\lambda = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^r = \mu$ that meets the following conditions:

1. $\lambda \in \Pi$
2. $s_i = \lambda^i/\lambda^{i-1}$ is a row-type string consisting of $\ell$ cells for all $1 \leq i \leq r$
3. The strings $s_i$ are translates of each other
4. The top cells of $s_1, \ldots, s_r$ occur in consecutive columns from left to right
5. $\mu \in \Pi$.

We say that $m$ is a row move from $\lambda$ to $\mu$ and write $\mu = m \ast \lambda$ or $m = \mu/\lambda$. A column move is the transpose analogue of a row move. A move is a row move or column move. It should be noted that it can be shown that moves are at most of rank $k - 1$.

**Example 10.** For $k = 5$, a row move of length 1 and rank 3 with strings $s_1 = \{A\}$, $s_2 = \{B\}$, and $s_3 = \{C\}$ is pictured below. The lower case letters are the cells that are removed from the $k$-boundary when the corresponding strings are added.

For $k = 3$, a row move of length 2 and rank 2 with strings $s_1 = \{A_1, A_2\}$ and $s_2 = \{B_1, B_2\}$ is:

The rationale for the “move” terminology is the following. Suppose that $m$ is a row move from $\lambda$ to $\mu$. By definition, $\text{rs}(\mu) = \text{rs}(\lambda)$ (since the strings are all of row-type). Hence $\partial \mu$ can be viewed as a right-shift (or move) of certain rows of $\partial \lambda$.

### 3. The poset of $k$-shapes

We now define a poset structure, called the poset of $k$-shapes, on the set $\Pi_N$ of $k$-shapes of fixed size $N$. We say that $\lambda$ dominates $\mu$ in the poset of $k$-shapes if there is a sequence of moves $m_1, \ldots, m_r$ such that $\mu = m_r \ast \cdots \ast m_1 \ast \lambda$. We let $\mathcal{P}^k(\lambda, \mu)$ denote the set of paths in the poset of $k$-shapes from $\lambda$ to $\mu$.

**Proposition 11.** An element of the poset of $k$-shapes is maximal (resp. minimal) if and only if it is a $(k + 1)$-core (resp. $k$-core).
Example 12. The poset of 2-shapes of size 4 is pictured below. Only the cells of the k-boundaries are shown. Row moves are indicated by \( r \) and column moves by \( c \).

\[
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}
\]

The poset of 3-shapes of size 5 is pictured below.

\[
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}
\]

Definition 13. Given a move \( m \), the charge of \( m \), written \( \text{ch}(m) \), is 0 if \( m \) is a row move and \( r\ell \) if \( m \) is a column move of length \( \ell \) and rank \( r \). Notice that in the column case, \( r\ell \) is simply the number of cells in the move \( m \) when viewed as a skew shape. The charge of a path \( p = (m_1, \ldots, m_n) \) in \( \Pi_N \) is \( \text{ch}(m_1) + \cdots + \text{ch}(m_n) \), the sum of the charges of the moves that constitute the path.

Definition 14. Given a move \( m \), the cocharge of \( m \), written \( \text{coch}(m) \), is 0 if \( m \) is a column move and \( r\ell \) if \( m \) is a row move of length \( \ell \) and rank \( r \). The cocharge of a path is again the sum of the cocharges of the moves forming the path.

Let \( \equiv \) be the equivalence relation on paths in \( \Pi_N \) generated by the following diamond equivalences:

\[
\tilde{M}m \equiv \tilde{m}M
\]

where \( m, M, \tilde{m}, \tilde{M} \) are moves (possibly empty) between \( k \)-shapes such that the diagram

\[
\begin{array}{ccc}
m & \lambda & M \\
\mu & & \nu \\
\tilde{M} & \gamma & \tilde{m} \\
\end{array}
\]

commutes and the charge is the same on both sides of the diamond:

\[
\text{ch}(m) + \text{ch}(\tilde{M}) = \text{ch}(M) + \text{ch}(\tilde{m}).
\]

The commutation is equivalent to the equality \( \tilde{M} \cup m = \tilde{m} \cup M \) where a move is regarded as a set of cells. Observe that the charge is by definition constant on equivalence classes of paths. We will let \( \mathcal{P}^k(\lambda, \mu) \) be the set of equivalences classes in \( \mathcal{P}^k(\lambda, \mu) \), that is, the set of equivalences classes of paths in the poset of \( k \)-shapes from \( \lambda \) to \( \mu \). It is easy to see that

\[
\text{coch}(m) + \text{coch}(\tilde{M}) = \text{coch}(M) + \text{coch}(\tilde{m}) \iff \text{ch}(m) + \text{ch}(\tilde{M}) = \text{ch}(M) + \text{ch}(\tilde{m}).
\]

and thus (3.5) can be replaced by the left hand side of (3.6) in the definition of diamond equivalence.
Example 15. Continuing Example 12, the two paths in the poset of 2-shapes from $\lambda = (3, 1, 1)$ to $\mu = (4, 3, 2, 1)$ have charge 2 and 3 respectively, and so are not equivalent.

\[(3.7)\]

The two paths in the poset of 3-shapes from $\lambda = (3, 2, 1)$ to $\nu = (4, 2, 1, 1)$ are diamond equivalent, both having charge 1.

\[(3.8)\]

4. (Co)charge of a $k$-tableau

A $k$-tableau (or weak tableau) is a special type of tableau originally introduced to describe the Pieri-type rules that the $k$-Schur functions satisfy. The dual $k$-Schur functions are the generating series of $k$-tableaux of a given shape.

Definition 16. A $k$-tableau of weight $(\alpha_1, \ldots, \alpha_N)$ is a sequence of $k+1$-cores $\emptyset = \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(N)} = \lambda$ such that, for all $i$, $\text{rs}(\lambda^{(i)})/\text{rs}(\lambda^{(i-1)})$ is a horizontal strip and $\text{cs}(\lambda^{(i)})/\text{cs}(\lambda^{(i-1)})$ is a vertical strip, both of size $\alpha_i$.

Example 17. Let $\alpha = (2, 3, 1, 2)$. An example of a 3-tableau of weight $\alpha$ is

\[
\begin{array}{cccc}
1 & 3 & 5 \\
3 & 4 \\
2 & 2 & 1 & 5 & 5 \\
1 & 1 & 2 & 2 & 1 & 3 & 5 \\
\end{array}
\]

The corresponding sequence of 4-cores (represented by $\partial(\lambda^{(i)})$) to view more easily $\text{rs}(\lambda^{(i)})$ and $\text{cs}(\lambda^{(i)})$ is

\[
\begin{array}{|c|c|c|c|c|}
\hline
\quad & \quad & \quad & \quad & \quad \\
\quad & \quad & \quad & \quad & \quad \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|}
\hline
\lambda^{(1)} & \lambda^{(2)} & \lambda^{(3)} & \lambda^{(4)} & \lambda^{(5)} \\
\hline
\end{array}
\]

It can be checked that $\text{rs}(\lambda^{(i)})/\text{rs}(\lambda^{(i-1)})$ is a horizontal strip and $\text{cs}(\lambda^{(i)})/\text{cs}(\lambda^{(i-1)})$ is a vertical strip both of size $\alpha_i$ for $i = 1, 2, 3, 4, 5$.

The $k+1$-residue (or simply residue if the value of $k$ is obvious from the context) of a cell $b = (i, j)$ is equal to $j - i \mod k + 1$. It can be shown that a $k$-tableau of weight $(\alpha_1, \ldots, \alpha_N)$ is such that the cells occupied by letter $i$ in the $k$-tableau have exactly $\alpha_i$ distinct residues.
For this article, we will mostly need \( k \)-tableaux of weight \((1, 1, \ldots, 1)\), that is, standard \( k \)-tableaux.

To be more specific:

**Definition 18.** A standard \( k \)-tableau is a sequence of \( k + 1 \)-cores \( \emptyset = \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(N)} = \lambda \) such that, for all \( i \), \( \lambda^{(i)}/\lambda^{(i-1)} \) is a vertical and a horizontal strip and \( |\partial(\lambda^{(i)})| = |\partial(\lambda^{(i-1)})| + 1 \).

Our previous observation then says that the cells occupied by letter \( i \) in a standard \( k \)-tableau all have the same residue.

**Example 19.** Let \( k = 3 \). An example of a 3-tableau is

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 & 13 & 14 & 15 & 16
\end{array}
\]

The corresponding sequence of 4-cores (with the residue of each letter) is

\[
\begin{array}{cccccccc}
\emptyset & \subseteq & \lambda^{(1)} & \subseteq & \lambda^{(2)} & \subseteq & \lambda^{(3)} & \subseteq & \lambda^{(4)}
\end{array}
\]

We now introduce concepts of charge and cocharge for \( k \)-tableaux. The cell corresponding to the lowermost (resp. uppermost) occurrence of a given letter \( n \) will be denoted \( n_\downarrow \) (resp. \( n_\uparrow \)). For instance, the cell \( 6_\downarrow \) is the marked one in the tableau \( T \). We will use \( n_\uparrow - 1 \) and \( n_\uparrow + 1 \) to denote respectively \((n - 1)_\uparrow \) and \((n + 1)_\uparrow \) (and similarly for \( n_\downarrow - 1 \) and \( n_\downarrow + 1 \)). Given two cells \( b_1 \) and \( b_2 \) of a \((k + 1)\)-core \( \lambda \) such that \( b_2 \) is weakly below \( b_1 \), we let \( \text{diag}_{e}(b_1, b_2) \) be the number of diagonals of residue \( e \) strictly between \( b_1 \) and \( b_2 \). The charge of a standard \( k \)-tableau \( T \) on \( N \) letters is

\[
\text{ch}(T) = \sum_{n=1}^{N} \text{ch}(n)
\]

where \( \text{ch}(1) = 0 \), and where \( \text{ch}(n) \) for \( n > 1 \) is defined recursively in the following way. Suppose that \( n_\uparrow \) and \( n_\downarrow \) have residues \( e \) and \( e_\downarrow \) respectively. Then \( \text{ch}(n) \) is defined as

\[
\text{ch}(n) = \begin{cases} 
\text{ch}(n-1) + \text{diag}_{e_\downarrow}(n_\uparrow, n_\downarrow) + 1 & \text{if } n_\downarrow \text{ is weakly above } n_\uparrow \\
\text{ch}(n-1) - \text{diag}_{e}(n_\uparrow, n_\downarrow) & \text{if } n_\downarrow \text{ is below } n_\uparrow
\end{cases}
\]

**Example 20.** Consider the 4-tableau with the cells \( n_\uparrow \) marked

\[
\begin{array}{cccccccc}
11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\
19 & 20 & 21 & 22 & 23 & 24 & 25 & 26
\end{array}
\]

Adding the residues to the core

\[
\begin{array}{cccccccc}
8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
16 & 17 & 18 & 19 & 20 & 21 & 22 & 23
\end{array}
\]

we obtain

\[
\begin{align*}
\text{ch}(1) &= 0; & \text{ch}(2) &= \text{ch}(1) + 1 = 1; & \text{ch}(3) &= \text{ch}(2) + 1 = 2; & \text{ch}(4) &= \text{ch}(3) = 2 \\
\text{ch}(5) &= \text{ch}(4) = 2; & \text{ch}(6) &= \text{ch}(5) + 1 = 3; & \text{ch}(7) &= \text{ch}(6) = 3; & \text{ch}(8) &= \text{ch}(7) = 3; \\
\text{ch}(9) &= \text{ch}(8) + \text{diag}_{2}(8_\uparrow, 9_\uparrow) + 1 = 3 + 1 + 1 = 5; & \text{ch}(10) &= \text{ch}(9) - \text{diag}_{1}(10_\uparrow, 9_\uparrow) + 1 = 5 - 1 = 4
\end{align*}
\]

Hence \( \text{ch}(T) = 0 + 1 + 2 + 2 + 3 + 3 + 3 + 5 + 4 = 25 \).
Similarly, the cocharge of a standard $k$-tableau $T$ on $N$ letters is
\[
\text{coch}(T) = \sum_{n=1}^{N} \text{coch}(n)
\]
where $\text{coch}(1) = 0$, and where $\text{coch}(n)$ for $n > 1$ is defined recursively as (supposing that $n^+$ and $n^-$ have residues $e$ and $e$ respectively)
\[
\text{coch}(n) = \begin{cases} 
\text{coch}(n-1) - \text{diag}_{e-}(n^+, n^-) & \text{if } n^+ \text{ is weakly above } n^- \\
\text{coch}(n-1) + \text{diag}_{e}(n^+, n^-) + 1 & \text{if } n^- \text{ is below } n^+
\end{cases}
\]
(4.3)
(4.4)

\textbf{Example 21.} Consider the 4-tableau in Example 20 with this time the cells $n^+$ marked
\[
T = \begin{array}{cccc}
10 & 8 & 5 & 7 \\
4 & 6 & 10 & 12 \ \\
3 & 2 & 9 & 11
\end{array}
\]
We have $\text{coch}(1) = 0$; $\text{coch}(2) = \text{coch}(1) - \text{diag}_4(1^+, 2^+) = 0$; $\text{coch}(3) = \text{coch}(2) - \text{diag}_3(2^+, 3^+) = 0$; $\text{coch}(4) = \text{coch}(3) + \text{diag}_4(4^+, 3^+) = 1$; $\text{coch}(5) = \text{coch}(4) - \text{diag}_4(4^+, 5^+) = 1$; $\text{coch}(6) = \text{coch}(5) + \text{diag}_6(6^+, 5^+) = 2$; $\text{coch}(7) = \text{coch}(6) - \text{diag}_4(6^+, 7^+) = 2$; $\text{coch}(8) = \text{coch}(7) + \text{diag}_1(8^+, 7^+) = 4$; $\text{coch}(9) = \text{coch}(8) - \text{diag}_1(8^+, 9^+) = 3$; $\text{coch}(10) = \text{coch}(9) - \text{diag}_4(9^+, 10^+) = 3$. Hence $\text{coch}(T) = 0 + 0 + 0 + 1 + 1 + 2 + 2 + 4 + 3 + 3 = 16$.

\textbf{Remark 22.} That the charge and cocharge are given by nonnegative integers follows from their compatibility with the weak bijection. In effect, iterating the weak bijection (as was done for instance in the introduction) puts in correspondence a $k$-tableau $Q^{(k)}$ with a sequence of paths $([p_1], [p_{k-1}], \ldots, [p_2])$ such that $\text{ch}(Q^{(k)}) = \text{ch}(p_k) + \text{ch}(p_2)$ and $\text{coch}(Q^{(k)}) = \text{coch}(p_k) + \text{coch}(p_2)$, from which the non-negativity is immediate.

The remainder of this section is concerned with the definition of charge in the non-standard case. Our initial goal was to show that charge was also compatible with the weak bijection in the non-standard case, but as is discussed in the conclusion, we were unfortunately not able to reach that goal.

The definition of charge is given for $k$-tableaux of dominant weights, and as in the usual case, a Lascoux-Schützenberger type action of the symmetric group on $k$-tableaux allows to send a $k$-tableau of any weight into a $k$-tableau of dominant weight. We do not prove here that the definition of charge and of the Lascoux-Schützenberger type action of the symmetric group on $k$-tableaux are well-defined.

We first define the charge of a $k$-tableau $T$ in the non-standard case. First suppose that the weight $(\alpha_1, \ldots, \alpha_N)$ of the $k$-tableau is dominant, that is, that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_N$. As mentioned earlier, the letter $n$ in $T$ occupies exactly $\alpha_n$ residues $i_1, \ldots, i_{\alpha_n}$. We can thus denote the letters $n$ in $T$ as $n_{i_1}, \ldots, n_{i_{\alpha_n}}$. We now form $\alpha_1$ words in the following way. Start with $1_{j_1} = 1_{\alpha_1-1}$ (the rightmost 1 in $T$) and construct a word recursively by appending to $1_{j_1}2_{j_2} \cdots n_{j_n}$ the letter $(n + 1)_{j_{n+1}}$, where $j_{n+1}$ is the largest element in the total order $j_n + 1 < j_n + 2 < \cdots < j_n - 1$ (the residues are taken modulo $k + 1$). Note that it can be shown that there is such a letter. Once $w_1 = 1_{j_1}2_{j_2} \cdots \alpha_{jn}^{(N)}$ has been constructed, remove all the letters $1_{j_1}, 2_{j_2}, \ldots, N_{j_n}$ from $T$, and construct $w_2$ in the same manner starting this time with $1_{\alpha_1-2}$ (the second rightmost 1 in $T$) and stopping at the largest letter. After all the words $w_1, \ldots, w_{\alpha_1}$ have been constructed, the charge of $T$ is the sum of the charges of
the \( w_i \)'s, where the charge of \( w_i \) is the charge of the subtableau of \( T \) obtained by considering only the letters in \( w_i \). For instance, let \( k = 4 \) and consider the 4-tableau of evaluation \((2, 2, 2, 2, 2, 2, 1)\)

\[
T = \begin{array}{cccccccc}
7 & 6 \\
5 & 6 \\
3 & 4 & 7 \\
2 & 3 & 5 & 6 \\
1 & 1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}
\]

The letters are \( 1, 1, 2, 2, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 7, 0 \), from which we extract \( w_1 = 1_42_43_44_05_26_17_0 \) and \( w_2 = 1_02_30_45_16_0 \). The charge of \( w_1 \) is computed using the cells

\[
\begin{array}{cccccccc}
7 & 6 \\
5 & 6 \\
3 & 4 & 7 \\
2 & 3 & 5 & 6 \\
1 & 1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}
\]

Hence \( \text{ch}(w_1) = 0 + 0 + 0 + 2 + 1 + 1 + 1 = 5 \). Similarly, the charge of \( w_2 \) is computed using the cells

\[
\begin{array}{cccccccc}
7 & 6 \\
5 & 6 \\
3 & 4 & 7 \\
2 & 3 & 5 & 6 \\
1 & 1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}
\]

and is such that \( \text{ch}(w_2) = 0 + 1 + 1 + 1 + 2 + 2 = 7 \). The charge of \( T \) is therefore equal to \( 5 + 7 = 12 \).

If the weight of \( T \) is not dominant, we define the charge of \( T \) to be the charge of \( \sigma(T) \), where \( \sigma(T) \) is the unique \( k \)-tableau of dominant weight obtained by the following (conjectural) Lascoux-Schützenberger type action of the symmetric group on \( k \)-tableaux. Consider the elementary transposition \( \sigma_i \) which sends a tableau of weight \( (\alpha_1, \ldots, \alpha_N) \) to a tableau of weight \( (\alpha_1, \ldots, \alpha_{i-1}, \alpha_i, \alpha_{i+1}, \ldots, \alpha_N) \). Let \( i \) and \( i + 1 \) be denoted respectively as \( a \) and \( b \). Suppose that the \( a \)'s and \( b \)'s in \( T \) occupy residues \( i_1, \ldots, i_r \) and \( j_1, \ldots, j_s \), respectively. We say that \( b_j \) lies on the floor if there is a \( b_j \) in \( T \) without an \( a \) below it (that is, one unit downward). If \( b_j \) does not lie on the floor then let \( b_j = \tilde{b}_{j+1} \), otherwise let \( b_j = \tilde{b}_j \). Order the \( a \)'s and \( b \)'s using the total order

\[
\tilde{b}_0 < a_0 < \tilde{b}_1 < a_1 < \cdots < \tilde{b}_k < a_k
\]

and let the word corresponding to the the ordered \( a \)'s and \( b \)'s be \( w \). Then do the usual pairing of the letters \( a \)'s and \( b \)'s followed by a permutation of their weight to obtain a word \( w' \). That is, pair every factor \( ba \) of \( w \), and let \( w_1 \) be the subword of \( w \) made out of the unpaired letters. Pair every factor \( \tilde{b}a \) of \( w_1 \), and let \( w_2 \) be the subword made out of the unpaired letters. Continue in this fashion as long as possible. When all factors \( \tilde{b}a \) are paired and unpaired letters of \( w \) are of the form \( a^i b^j \), send \( a^i b^j \) to \( a^i \tilde{b}^j \) (keeping track of the residues). Now the \( a \)'s in \( w' \) will have residue \( i'_1, \ldots, i'_r \). Take \( T'_i = T_{i-1} \) and let \( T'_i \) be the unique tableau such that letter \( i \) occupies residues \( i'_1, \ldots, i'_r \). Let \( T_{i+1} \) be the unique tableau of the same shape as \( T_{i+1} \) with subtableau \( T'_i \). Finally, let \( T' \) be obtained by filling the rest of the tableau as in \( T \). We then define \( \sigma_i(T) = T' \). Since the \( \sigma_i \)'s generate the symmetric group, this defines an action of the symmetric group on \( k \)-tableaux. It is proven in [17] (in a more general context) that \( \sigma_i \) is an involution that sends a \( k \)-tableau into a \( k \)-tableau. The fact that the \( \sigma_i \)'s obey the Coxeter relations was always in our mind a consequence of the yet unproven compatibility of the Lascoux-Schützenberger type action of the symmetric group on \( k \)-tableaux with the weak bijection (in which case the Coxeter relations would follow from the known Coxeter relations when \( k \) is large), and as such we never intended to prove them.
5. Standard $k$-shape tableaux

An generalization of $k$-tableaux (fillings of $k+1$-cores) to certain fillings of $k$-shapes called $k$-shape tableaux was introduced in [7]. For the purposes of this article, we will only need to describe explicitly the standard case.

**Definition 23.** We say that a string $s = \mu/\lambda = \{a_1, \ldots, a_\ell\}$ can be continued below (resp. above) if there is an addable corner of $\lambda$ below (resp. above) the string $s$ that is contiguous to $a_\ell$ (resp. $a_1$). We say that a cover-type string is maximal if it cannot be continued above or below.

**Example 24.** Let $k = 5$ and consider $\lambda = \begin{array}{|c|c|c|c|c|}
\hline
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\end{array}$

$\mu = \begin{array}{|c|c|c|c|c|}
\hline
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\end{array}$, $\nu = \begin{array}{|c|c|c|c|c|}
\hline
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\end{array}$ and $\gamma = \begin{array}{|c|c|c|c|c|}
\hline
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\end{array}$

In this case $\mu/\lambda$ (indicated by framed boxes in the diagram) is a string that can be continued below, $\nu/\lambda$ is a string that can be continued above and $\gamma/\lambda$ is a maximal cover-type string. We denoted by $\bullet$ the contiguous adadble corner below or above.

**Definition 25.** We say that a string $s = \mu/\lambda = \{a_1, \ldots, a_\ell\}$ can be reverse-continued below (resp. above) if there is a removable corner of $\mu$ below (resp. above) the string $s$ that is contiguous to $a_\ell$ (resp. $a_1$). We say that a cover-type string is reverse-maximal if it cannot be reverse-continued above or below.

**Example 26.** Let $k = 5$ and consider $\lambda = \begin{array}{|c|c|c|c|c|}
\hline
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\end{array}$

$\mu = \begin{array}{|c|c|c|c|c|}
\hline
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\end{array}$, $\nu = \begin{array}{|c|c|c|c|c|}
\hline
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\end{array}$ and $\gamma = \begin{array}{|c|c|c|c|c|}
\hline
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\end{array}$

We have that $\mu/\lambda$ is a string that can be reverse-continued below, $\nu/\lambda$ is a string that can be reverse-continued above and $\gamma/\lambda$ is a reverse-maximal cover-type string. We denoted by $\ast$ the contiguous removable corners below or above.

**Definition 27.** We say that $\mu/\lambda$ is a cover if $\lambda$ and $\mu$ are $k$-shapes and $\mu/\lambda$ is a cover-type string. It is maximal (resp. reverse-maximal) if $\mu/\lambda$ is a maximal string (resp. reverse-maximal string).

A standard $k$-shape tableau of shape $\lambda$ is a sequence of $k$-shapes

$\emptyset = \lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(n-1)}, \lambda^{(n)} = \lambda$

such that $\lambda^{(i)}/\lambda^{(i-1)}$ is a cover for all $i = 1, \ldots, n$. A standard $k$-shape tableau is maximal (resp. reverse-maximal) if every cover composing the tableau is maximal (resp. reverse-maximal). A standard $k$-shape tableau is naturally associated to a filling of $\lambda$ such that letter $i$ occupies the cells $\lambda^{(i)}/\lambda^{(i-1)}$. Given a $k$-shape tableau $T$ of $n$ letters, we let $T_i$ be the subtableau of $T$ obtained by removing all letter $i + 1, \ldots, n$ from $T$.

**Example 28.** For $k = 3$, and $n = 8$ the standard 3-shape tableau $T = \begin{array}{|c|c|c|c|c|}
\hline
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\end{array}$
corresponds to the sequence of 3-shapes

\[
\begin{align*}
\lambda^{(1)} &= \begin{array}{c}
\cdot
\end{array} & \lambda^{(2)} &= \begin{array}{c|c}
\cdot & \cdot \\
\end{array} & \lambda^{(3)} &= \begin{array}{c}
\cdot
\end{array} & \lambda^{(4)} &= \begin{array}{c}
\cdot
\end{array} & \lambda^{(5)} &= \begin{array}{c|c}
\cdot & \cdot \\
\end{array} \\
\lambda^{(6)} &= \begin{array}{c|c|c|c}
\cdot & \cdot & \cdot & \cdot \\
\end{array} & \lambda^{(7)} &= \begin{array}{c|c|c|c}
\cdot & \cdot & \cdot & \cdot \\
\end{array} & \lambda^{(8)} &= \begin{array}{c|c|c|c|c}
\cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\end{align*}
\]

It can be checked easily that \(\lambda^{(i)}/\lambda^{(i-1)}\) (represented by \(\bullet\) in the diagrams) is a cover for all \(i\) from 1 to 8.

For a given \(k\), the following proposition allows to connect sequences of maximal and reverse-maximal covers to standard \(k-1\)-tableaux and \(k\)-tableaux respectively.

**Proposition 29 ([7]).** A sequence \(\emptyset = \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \ldots \subseteq \lambda^{(N)} = \lambda\) is a standard \(k\)-tableau if and only if \(\lambda\) is a \(k+1\)-core and \(\lambda^{(i)}/\lambda^{(i-1)}\) is a reverse-maximal cover for all \(i\). Similarly, a sequence \(\emptyset = \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \ldots \subseteq \lambda^{(N)} = \lambda\) is a \(k-1\)-tableau if and only if \(\lambda^{(i)}/\lambda^{(i-1)}\) is a maximal cover for all \(i\).

6. **Pushout algorithm in the standard case**

The main result of [7] is the construction of a bijection between pairs \((S, [p])\) and \((\tilde{S}, [\tilde{p}])\), where \(S = \mu/\lambda\) (resp. \(\tilde{S} = \omega/\delta\)) is a certain reverse-maximal strip (resp. maximal strip), where \(p\) (resp. \(\tilde{p}\)) is a path in the poset of \(k\)-shapes from \(\lambda\) to \(\delta\) (resp. \(\mu\) to \(\omega\)), and where \([p]\) denotes the equivalence class of the path \(p\). The bijection can be described diagrammatically by the commuting diagram

\[
\begin{array}{ccc}
\lambda & \xrightarrow{[p]} & \delta \\
S & \downarrow & \tilde{S} \\
\mu & \xrightarrow{[\tilde{p}]} & \omega
\end{array}
\]  
(6.1)

In the standard case, that is, when the strips \(S\) and \(\tilde{S}\) are of rank 1, the bijection is between pairs \((c, [p])\) and \((\tilde{c}, [\tilde{p}])\) where \(c = \mu/\lambda\) (resp. \(\tilde{c}\)) is a reverse-maximal cover (resp. maximal cover)

\[
\begin{array}{ccc}
\lambda & \xrightarrow{[p]} & \delta \\
c & \downarrow & \tilde{c} \\
\mu & \xrightarrow{[\tilde{p}]} & \omega
\end{array}
\]  
(6.2)

The map \((c, [p]) \rightarrow (\tilde{c}, [\tilde{p}])\) is given by a certain pushout algorithm. We will now describe the canonical form of the pushout algorithm, which given the pair \((c, p)\) outputs a pair \((\tilde{c}, \tilde{p})\). Note that the inverse algorithm associates to \((\tilde{c}, \tilde{p})\) a pair \((c, p')\), where \([p'] = [p]\). This explains the need to work modulo equivalences.

The basic ingredients of the algorithm are the maximization (below and above) of a cover \(c\) and the pushout of the pair \((c, m)\) when \(c\) is a maximal cover and \(m\) is a move. Repeated application of the following three steps will produce the pair \((\tilde{c}, \tilde{p})\) out of \((c, p)\).

**Step 1 (maximization below).** If the cover \(c = \mu/\lambda = \{a_1, \ldots, a_t\}\) can be continued below, then let \(c' = c \cup m = (\mu \cup m)/\lambda\), where \(m = \{a_{t+1}, \ldots, a_{t+n}\}\) is the longest sequence of contiguous addable corners of \(\lambda\) such that \(a_{t+1}\) is contiguous to \(a_t\) and below it. In this case, it is immediate
that $c'$ is a cover and that $m$ is a row-type string. Moreover, it can be shown [7] that $m$ is in fact a row move from $\mu$ to $\mu \cup m$. Diagrammatically, this gives

![Diagram](image)

$$\lambda \rightarrow^0 \lambda$$  \hspace{1cm} (6.3)

**Example 30.** Here is an example of maximization below when $k = 4$ (the cover and its maximization below are highlighted with dots):

![Diagram](image)

**Step 2 (maximization above).** If the cover $c$ cannot be continued below but can be continued above, then let $c' = c \cup m = (\mu \cup m) / \lambda$, where $m = \{a_{-n}, \ldots, a_0\}$ is the longest sequence of contiguous addable corners of $\lambda$ such that $a_0$ is contiguous to $a_1$ and above it. It is again immediate that $c'$ is a cover and that $m$ is a column-type string. Moreover, it is shown [7] that $m$ is in fact a column move from $\mu$ to $\mu \cup m$. Diagrammatically, this gives

![Diagram](image)

$$\lambda \rightarrow^0 \lambda$$  \hspace{1cm} (6.4)

**Example 31.** Here is an example of a maximization above when $k = 4$ (the cover and its maximization above are highlighted with dots):

![Diagram](image)

**Step 3 (maximal pushout).** If $c$ is a maximal cover and $m$ is any move then the maximal pushout of $(c, m)$ produces the pair $(\tilde{c}, \tilde{m})$ (which we will describe explicitly below), where $\tilde{m}$ is a move:

![Diagram](image)

$$\lambda \rightarrow^m \nu$$  \hspace{1cm} (6.5)

We say that the cover $c = \mu / \lambda$ and the row move $m$ from $\lambda$ to $\mu$ are interfering if $c$ and $m$ do not intersect and $\mu \cup m$ is not a $k$-shape (that is, if $cs(\mu) + \Delta cs(m)$ is not a partition). Similarly, we say that the cover $c = \mu / \lambda$ and the column move $m$ from $\lambda$ to $\mu$ are interfering if $c$ and $m$ do not intersect and $\mu \cup m$ is not a $k$-shape (that is, if $rs(\mu) + \Delta rs(m)$ is not a partition).
For a set of cells $A$, we let $A_U$ and $A_R$ be the set of cells obtained by translating the cells of $A$ by one unit respectively upward and to the right.

When $m$ is a row move, the pushout is of one of the 4 possible types [7].

I If $m$ and $c$ do not intersect and are not interfering then $\tilde{m} = m$ and $\tilde{c} = c$.
II If $m$ and $c$ do not intersect but are interfering, then $\tilde{c} = c \cup m_{\text{comp}}$ and $\tilde{m} = m \cup m_{\text{comp}}$, where $m_{\text{comp}}$ (the completion of $m$) is the string obtained by translating to the right (by one unit) the rightmost string of $m$.
III If $c$ and $m$ intersect, and $c$ continues above $m$ (but not below), then $\tilde{c} = c \setminus (c \cap m)$ and $\tilde{m} = m \setminus (c \cap m)$.
IV If $c$ and $m$ intersect and $c$ continues above and below $m$, then $\tilde{c} = (c \setminus (c \cap m)) \cup (c \cap m)_U$ and $\tilde{m} = (m \setminus (c \cap m)) \cup (c \cap m)_U$.

Observe that to the exception of type II, $\eta$ always corresponds to the union of the cells of $\lambda$, $c$ and $m$ (there is interference when this union is not a $k$-shape).

**Example 32.** Here are examples of the 4 possible types of pushout when $k = 4$. We indicate cells of the moves with $\bullet$ and cells of the covers by framed boxes.

I $m$ and $c$ do not intersect and are not interfering

II $m$ and $c$ do not intersect but are interfering
III  $c$ and $m$ intersect, and $c$ continues above $m$ (but not below)

IV  $c$ and $m$ intersect and $c$ continues above and below $m$

Similarly, if $m$ is a column move the possible types are:

I  If $m$ and $c$ do not intersect and are not interfering, then $\tilde{m} = m$ and $\tilde{c} = c$.

II  If $m$ and $c$ do not intersect but are interfering, then $\tilde{c} = c \cup m_{\text{comp}}$ and $\tilde{m} = m \cup m_{\text{comp}}$, where $m_{\text{comp}}$ is the string obtained by translating by one unit upward the uppermost string of $m$.

III  If $c$ and $m$ intersect, and $c$ continues below $m$ (but not above), then $\tilde{c} = c \setminus (c \cap M)$ and $\tilde{M} = M \setminus (c \cap M)$.

IV  If $c$ and $m$ intersect and $c$ continues above and below $m$, then $\tilde{c} = (c \setminus (c \cap m)) \cup (c \cap m)_R$ and $\tilde{m} = (m \setminus (c \cap m)) \cup (c \cap m)_R$.

Using the three steps repeatedly, the pushout algorithm produces from any pair $(c, m)$ a pair $(\tilde{c}, q)$, where $\tilde{c}$ is a maximal cover and $q$ is a path

\[
\begin{array}{c}
\lambda \\
\downarrow \\
\mu
\end{array} \xrightarrow{m} \begin{array}{c}
\nu \\
\downarrow \\
\delta \\
\downarrow \\
\gamma
\end{array} \quad (6.6)
\]

Applying this algorithm for every move in a given path $p$, the pushout algorithm thus produces from any pair $(c, p)$ a pair $(\tilde{c}, \tilde{p})$, where $\tilde{c}$ is a maximal cover and $\tilde{p}$ is a path

\[
\begin{array}{c}
\lambda \\
\downarrow \\
\mu
\end{array} \xrightarrow{p} \begin{array}{c}
\delta \\
\downarrow \\
\epsilon \\
\downarrow \\
\omega
\end{array} \quad (6.7)
\]
In the special case where $c$ is a reverse-maximal cover, this corresponds to the bijection in \((6.2)\) when equivalences of paths are considered.

### 7. Weak bijection in the standard case

The main reason to construct bijection \((6.2)\) is to obtain the following bijection (weak bijection in the standard case):

\[
\text{SWTab}_k^\lambda \rightarrow \bigsqcup_{\mu \in \mathbb{C}_k^k} \text{SWTab}_k^{k-1} \times \prod^k(\lambda, \mu)
\]

\[
Q^{(k)} \rightarrow (Q^{(k-1)}, [p]) \tag{7.1}
\]

where SWTab$_k^\lambda$ is the set of standard $k$-tableau (or standard weak tableau) of shape $\lambda$. This bijection proceeds as follows. From Proposition 29, a $k$-tableau $Q^{(k)}$ of shape $\lambda$ is a sequence of reverse-maximal covers $c_1 = \lambda/(1)/\emptyset, \ldots, c_n = \lambda/\lambda^{(n-1)}$. Starting with the pair $(c_1, [\emptyset])$, where $\emptyset$ is the empty move from the empty partition to itself, the bijection \((6.2)\) gives a pair $(\tilde{c}_1, [p_1])$ with $\tilde{c}_1$ a maximal cover. Then, the pair $(c_2, [p_1])$ leads to the pair $(\tilde{c}_2, [p_2])$, where $\tilde{c}_2$ is again a maximal cover. Continuing this way we obtain that $Q^{(k)}$ is in correspondence with a sequence of maximal covers $\tilde{c}_1, \ldots, \tilde{c}_n$ and an equivalence class of paths $[p]$. This is illustrated in the following diagram:

\[
\begin{aligned}
&\emptyset \xrightarrow{c_1} \emptyset \\
&\quad \uparrow \quad \quad \quad \uparrow \\
&\lambda^{(1)} \xrightarrow{[p_1]} \mu^{(1)} \\
&\quad \downarrow \quad \quad \quad \downarrow \\
&\lambda^{(2)} \xrightarrow{[p_2]} \mu^{(2)} \\
&\quad \vphantom{\quad \downarrow} \quad \vphantom{\quad \downarrow} \\
&\vdots \\
&\lambda^{(n-1)} \xrightarrow{[p_{n-1}]} \mu^{(n-1)} \\
&\quad \downarrow \quad \quad \quad \downarrow \\
&\lambda \xrightarrow{[p]} \mu
\end{aligned}
\tag{7.2}
\]

8. Missing bijections and what they would entail

To give some perspective to the present work, we explain our general approach to prove the Schur positivity of $k$-Schur functions and dual $k$-Schur functions. We also describe how this approach would provide a combinatorial formula for the $k$-Schur expansion of Hall-Littlewood polynomial indexed by $k$-bounded partitions, and how it relates to the atoms of \([9]\).
Certain weak and strong tableaux (a.k.a. $k$-tableaux and dual $k$-tableaux) related respectively to the weak and strong order on Grassmannian permutations of the affine symmetric group $\tilde{S}_{k+1}$ were introduced in [11] [3]. These tableaux have a certain weight (just as the usual tableaux do) which tells how many times a given letter appears in the tableau.

The graded $k$-Schur functions (depending on a parameter $t$ and indexed by $k+1$-cores) are defined as [6]

$$s^{(k)}_\lambda(x; t) = \sum_P t^{\text{spin}(P)} x^P$$

(8.1)

where the sum is over all strong tableaux $P$ of shape $\lambda$, where spin is a certain statistic on strong tableaux, and where $x^P = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ if the weight of $P$ is $(\alpha_1, \ldots, \alpha_n)$.

We now provide a similar definition for the graded dual $k$-Schur functions (indexed again by $k+1$-cores). Let

$$\Theta^{(k)}_\lambda(x; t) = \sum_Q t^{\text{ch}(Q)} x^Q$$

(8.2)

where the sum is over all weak tableaux $Q$ of shape $\lambda$, and where we recall that the charge of a $k$-tableau was defined in Section 4.

Computational evidence suggests the conjecture that the $k$-Schur functions expand positively into $k'$-Schur functions for $k' > k$:

$$s^{(k)}_\mu(x; t) = \sum_\lambda b^{(k-1 \rightarrow k')}_{\mu \lambda}(t) s^{(k')}_\lambda(x; t), \quad \text{for } b^{(k-1 \rightarrow k')}_{\mu \lambda}(t) \in \mathbb{Z}_{\geq 0}[t].$$

(8.3)

For sufficiently large $k'$, it is known that $k'$-Schur functions and Schur functions coincide. The conjecture thus implies in particular the Schur positivity of $k$-Schur functions.

In order to prove the previous conjecture, it is sufficient to understand the case when $k$ and $k'$ differ by one. An explicit conjecture concerning this case was stated in [7] (and proven in the case $t = 1$). It relates the coefficients $b^{(k-1 \rightarrow k')}_{\mu \lambda}(t)$ (the branching coefficients) to certain paths in the poset of $k$-shapes weighted by charge. Let

$$b^{(k)}_{\mu \lambda}(t) := \sum_{[p] \in P^k(\lambda, \mu)} t^{\text{ch}(p)}.$$  

(8.4)

**Conjecture 33.** For all $\lambda \in C^{k+1}$ and $\mu \in C^k$, the special case $k \mapsto k - 1$ and $k' \mapsto k'$ of (8.3) is given by

$$b^{(k-1 \rightarrow k')}_{\mu \lambda}(t) = b^{(k)}_{\mu \lambda}(t)$$

(8.5)

It is also conjectured that the dual $k$-Schur functions expand positively into $k'$-Schur functions, but this time for $k' < k$:

$$\Theta^{(k)}_\mu(x; t) = \sum_\lambda b^{(k' \rightarrow k)}_{\lambda \mu}(t) \Theta^{(k')}_{\lambda}(x; t) \mod I_{k'}$$

(8.6)

where $I_{k'}$ is the ideal generated by the monomial symmetric functions $m_{\rho}$ such that $\rho_1 > k'$. We stress that the conjecture makes the stronger claim that the coefficients are the same (up to transposition) as those in the decompositions (8.3). This conjecture, also shown to hold when $t = 1$ in [7], would follow again from the case where $k$ and $k'$ differ by one.

**Conjecture 34.** For all $\mu \in C^{k+1}$ and $\lambda \in C^k$, the special case $k' \mapsto k - 1$ of (8.3) is given by

$$b^{(k-1 \rightarrow k')}_{\lambda \mu}(t) = b^{(k)}_{\lambda \mu}(t)$$

(8.7)
We now describe an approach to prove Conjectures 33 and 34. Let \( \text{WTab}_k^\lambda \) and \( \text{STab}_k^\lambda \) stand respectively for the set of \( k \)-tableaux and dual \( k \)-tableaux of shape \( \lambda \). As we will see, it would be immediate that the conjectures hold if one could find two weight-preserving bijections (respectively the weak and strong bijections)

\[
\text{WTab}_k^\lambda \rightarrow \bigsqcup_{\mu \in \mathbb{C}^k} \text{WTab}_{k-1}^\mu \times \mathbb{P}^k(\lambda, \mu) \\
Q^{(k)} \mapsto (Q^{(k-1)}, [p])
\]

and

\[
\text{STab}_{k-1}^\lambda \rightarrow \bigsqcup_{\mu \in \mathbb{C}^{k+1}} \text{STab}_k^\mu \times \mathbb{P}^k(\mu, \lambda) \\
P^{(k-1)} \mapsto (P^{(k)}, [p])
\]

such that

\[
\text{ch}(Q^{(k)}) = \text{ch}(p) + \text{ch}(Q^{(k-1)})
\]

and

\[
\text{spin}(P^{(k-1)}) = \text{ch}(p) + \text{spin}(P^{(k)})
\]

We should add that in the weak bijection the weight of \( Q^{(k)} \) in (8.8) needs to be \( k-1 \)-bounded (that is no letter can occur more than \( k-1 \) times). The weak bijection was constructed in [7]. The aim of this article is to show (Theorem 1) that it satisfies (8.10) in the standard case (we will discuss in the conclusion why the semi-standard case is still out of reach). The obtention of the strong bijection (8.9) satisfying (8.11) is a wide open problem.

We now show how the weak bijection implies Conjecture 34 if it satisfies (8.10) (the proof that the strong bijection implies Conjecture 33 if it satisfies (8.11) is practically identical and will thus be omitted). We have from (8.8) and (8.10) that

\[
\sum_{Q^{(k)} \in \text{WTab}_k^\lambda} t^{\text{ch}(Q^{(k)})} x^{Q^{(k)}} = \sum_{\mu \in \mathbb{C}^k} \sum_{[p] \in \mathbb{P}^{k}(\lambda, \mu)} \sum_{Q^{(k-1)} \in \text{WTab}_{k-1}^\mu} t^{\text{ch}(Q^{(k-1)}) + \text{ch}(p)} x^{Q^{(k-1)}} \mod I_{k-1}
\]

where the equality only holds modulo \( I_{k-1} \) since the leftmost sum is only over \( k \)-tableaux \( Q^{(k)} \) of \( k-1 \)-bounded weight. Therefore, it immediately follows that

\[
\mathcal{S}_{\lambda}^{(k)}(x; t) = \sum_{\mu} \sum_{[p] \in \mathbb{P}^k(\lambda, \mu)} t^{\text{ch}(p)} \mathcal{S}_{\mu}^{(k-1)}(x; t) \mod I_{k-1}
\]

which is equivalent to Conjecture 34.

Another interesting consequence of the weak and strong bijections would be the following explicit expression for the \( k \)-Schur expansion of Hall-Littlewood polynomials indexed by \( k \)-bounded partitions:

\[
H_{\lambda}(x; t) = \sum_{Q^{(n)}} t^{\text{ch}(Q^{(k)})} s_{\lambda}^{(k)}(x; t)
\]

where the sum is over all \( k \)-tableaux of weight \( \lambda \). This formula generalizes a well-known result of Lascoux and Schützenberger providing a \( t \)-statistic on tableaux for the Kostka-Foulkes polynomials [14].

This is shown in the following way. Let \( n = |\lambda| \). If we iterate the weak bijection

\[
Q \mapsto (Q^{(n-1)}, [p_n]), \quad Q^{(n-1)} \mapsto (Q^{(n-2)}, [p_{n-1}]), \quad \ldots, \quad Q^{(k+1)} \mapsto (Q^{(k)}, [p_{k+1}])
\]

We now show that the weak bijection satisfies the conjectures.
which implies that

\[ \text{Moreover, the previous correspondence satisfies} \]

\[ \sum_{\mu \in C^{k+1}} \text{WTab}^\mu_{\lambda} \rightarrow \bigsqcup_{Q \in P^{n\rightarrow k}(\lambda, \mu)} \text{WTab}^k_{\mu} \times P^{n\rightarrow k}(\lambda, \mu) \]

\[ Q \rightarrow (Q^{(k)}, [p_n], \ldots, [p_{k+1}]) \quad (8.16) \]

where \( P^{n\rightarrow k}(\lambda, \mu) \) is the set of all sequences \([p_n], \ldots, [p_{k+1}]\) such that

\[ p_n \in P^n(\lambda, \mu^{(n-1)}), p_{n-1} \in P^{n-1}(\mu^{(n-1)}, \mu^{(n-2)}), \ldots, p_{k+1} \in P^{k+1}(\mu^{(k+1)}, \mu) \]

Moreover, the previous correspondence satisfies

\[ \text{ch}(Q) = \text{ch}(Q^{(k)}) + \text{ch}(p_n) + \cdots + \text{ch}(p_{k+1}) \]

which implies that

\[ H_\lambda(x; t) = \sum_Q t^{\text{ch}(Q)} s_{\text{sh}(Q)}(x) = \sum_{Q^{(k)}} t^{\text{ch}(Q^{(k)})} \sum_{\mu \in P^{n\rightarrow k}(\mu, \text{sh}(Q^{(k)}))} \sum_{[p] \in \text{WTab}^k_{\mu}} t^{\text{ch}(p)} s_{\mu}(x) \quad (8.17) \]

where for short we use \([p]\) for \([p_n], \ldots, [p_{k+1}]\), and \( \text{ch}(p) = \text{ch}(p_n) + \cdots + \text{ch}(p_{k+1}) \). Equation \(8.14\) then follows since repeated applications of Conjecture \(33\) gives

\[ \sum_{\mu \in P^{n\rightarrow k}(\mu, \text{sh}(Q^{(k)}))} \sum_{[p] \in \text{WTab}^k_{\mu}} t^{\text{ch}(p)} s_{\mu}(x) = s^{(k)}_{\mu}(x; t) \quad (8.18) \]

Finally, we discuss the connection between the set \( A^{(k)}_\mu \) (called atoms in \(9\)) and the weak bijection \(8.25\). When \( Q \mapsto (Q^{(k)}, [p_n], \ldots, [p_{k+1}]) \), we say that \( Q^{(k)} \) is the \( k \)-tableau associated to \( Q \) or that \( Q \) maps to \( Q^{(k)} \). It was conjectured in \(7\) that the set \( A^{(k)}_\mu \) appearing in \(11\) is given by the set of tableaux that map to a certain \( k \)-tableaux \( T^{(k)}_{\mu} \):

**Conjecture 35.** Let \( \rho \) be the unique element of \( C^{k+1} \) such that \( rs(\rho) = \mu \), and let \( T^{(k)}_{\mu} \) be the unique \( k \)-tableau of weight \( \mu \) and shape \( \rho \) (see \(11\)). Then

\[ A^{(k)}_\mu = \left\{ T \text{ of weight } \mu \mid T^{(k)}_{\mu} \text{ is the } k \text{-tableau associated to } T \right\} . \quad (8.19) \]

Our results would imply that if \( Q^{(1)}_1 \) and \( Q^{(2)}_2 \) are two \( k \)-tableaux of the same shape, then the sets of tableaux \( A \) and \( B \) that respectively map to \( Q^{(1)}_1 \) and \( Q^{(2)}_2 \) are each in correspondence with the elements \([p_n], \ldots, [p_{k+1}]\) of \( P^{n\rightarrow k}(\lambda, \mu) \). Hence

\[ t^{-\text{ch}(Q^{(1)}_1)} \sum_{T \in A} t^{\text{ch}(T)} s_{\text{shape}(T)} = t^{-\text{ch}(Q^{(2)}_2)} \sum_{T' \in B} t^{\text{ch}(T')} s_{\text{shape}(T')} \quad (8.20) \]

given that

\[ \text{ch}(T) - \text{ch}(Q^{(1)}_1) = \text{ch}(Q^{(2)}_2) = \text{ch}(p_n) + \cdots + \text{ch}(p_{k+1}) \quad (8.21) \]

The two symmetric functions \( \sum_{T \in A} t^{\text{ch}(T)} s_{\text{shape}(T)} \) and \( \sum_{T' \in B} t^{\text{ch}(T')} s_{\text{shape}(T')} \) thus only differ by a power of \( t \). In the language of \(9\), these are instances of copies of atoms (which in \(9\) where only conjectured to exist).

### 9. (Co)charge of a Standard \( k \)-shape Tableau

We now generalize the notions of charge and cocharge to standard \( k \)-shape tableau. We will then establish a relation between charge and cocharge (see Proposition \(17\)) that will prove very useful in the proof of the compatibility between (co)charge and the weak bijection.
9.1. *k*-connectedness. Let \( r \) and \( r' \) (with \( r > r' \)) be rows of the \( k \)-shape \( \lambda \) that each have an addable corner. We say that \( r' \) is the \( k \)-connected row below row \( r \) (or simply that \( r \) and \( r' \) are \( k \)-connected rows) if \( r' \) is the lowest row such that the distance between the addable corners in row \( r \) and \( r' \) is not larger than \( k + 1 \). If the distance between the addable corners in rows \( r \) and \( r' \) is \( k \) or \( k + 1 \) then \( r \) and \( r' \) are said to be contiguously connected. We say that \( r_1, \ldots, r_m \) form a sequence of \( k \)-connected rows of length \( m \) if \( r_{i+1} \) is the \( k \)-connected row below row \( r_i \) for all \( i = 1, \ldots, m - 1 \).

**Example 36.** Let \( k = 5 \) and \( \lambda = \begin{array}{ccc} \\
\end{array} \). The pairs of 5-connected rows are: 7 and 5; 5 and 3; 3 and 2; 2 and 1; 6 and 4; 4 and 2. Moreover rows 4 and 2 are contiguously connected, as are rows 5 and 3. Rows 7, 5, 3, 1 form a sequence of 5-connected rows of length 4. Observe that two distinct rows can have the same \( k \)-connected row below: for instance row 2 is the 5-connected row below rows 3 and 4.

The \( r \) and \( r' \) (with \( r > r' \)) be rows of the \( k \)-shape \( \lambda \) that each have an addable corner. Define \([r, r']_k\) to be equal to the length of the longest sequence of \( k \)-connected rows \( r_1, r_2, \ldots, r_m \) such that \( r_1 = r \) and \( r_m \geq r' \). We define \((r, r')_k\) and \((r', r)_k\) in the same fashion (not counting row \( r \) or \( r' \) according to whether the interval is closed or open).

9.2. **Definition of charge and cocharge.** The charge of a \( k \)-shape tableau \( T \) on \( N \) letters is

\[
\text{ch}(T) = \sum_{n=1}^{N} \text{ch}(n) \tag{9.1}
\]

where \( \text{ch}(1) = 0 \), and where \( \text{ch}(n) \) for \( n > 1 \) is defined recursively in the following way. Let \( r \) be the row above that of \( n \) and \( r' \) be the row of \( n' \). Then

\[
\text{ch}(n) = \begin{cases} 
\text{ch}(n - 1) + [r, r']_k & \text{if } r \geq r' \\
\text{ch}(n - 1) - (r', r]_k & \text{if } r < r'
\end{cases} \tag{9.2}
\]

where \([r, r']_k\) and \((r', r]_k\) are calculated using the \( k \)-shape corresponding to the shape of \( T_{n-1} \). We also stress that \([r, r']_k = 0 \) if \( r = r' \).

**Example 37.** Let \( k = 4 \) and \( T = \begin{array}{cccccccccc}
0 & & & & & & & & & \\
7 & & & & & & & & & \\
1 & 6 & 9 & & & & & & & \\
3 & 5 & 7 & & & & & & & \\
1 & 2 & 4 & 6 & 8 & 9 & & & & \\
\end{array} \)

The sequence of \( k \)-shapes with the corresponding rows \( r \) and \( r' \) are

\[
\begin{array}{cccccc}
\begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} \\
\begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} \\
\begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} \\
\begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} \\
\begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} & \begin{array}{c}
\end{array} \\
\end{array}
\]

Therefore \( \text{ch}(1) = 0; \) \( \text{ch}(2) = \text{ch}(1) + [2, 1]_4 = 1; \) \( \text{ch}(3) = \text{ch}(2) - (2, 2]_4 = 1; \) \( \text{ch}(4) = \text{ch}(3) - (3, 3]_4 = 1; \) \( \text{ch}(5) = \text{ch}(4) + [4, 2]_4 = 2; \) \( \text{ch}(6) = \text{ch}(5) - (3, 3]_4 = 2; \) \( \text{ch}(7) = \text{ch}(6) - (4, 4]_4 = \)
\[ \text{ch}(8) = \text{ch}(7) + [5,1]_4 = 4; \quad \text{ch}(9) = \text{ch}(8) - (5,2)_4 = 3. \] This gives \( \text{ch}(T) = \sum_{n=1}^{9} \text{ch}(n) = 0 + 1 + 1 + 2 + 2 + 2 + 4 + 3 = 16. \)

Similarly, the cocharge of a \( k \)-shape tableau \( T \) on \( N \) letters is

\[ \text{coch}(T) = \sum_{n=1}^{N} \text{coch}(n) \quad (9.3) \]

where \( \text{coch}(1) = 0 \), and where \( \text{coch}(n) \) for \( n > 1 \) is defined recursively in the following way. Let \( r \) be the row above that of \( n^- \) and \( r' \) be the row of \( n^+ \). Then

\[
\text{coch}(n) = \begin{cases} 
\text{coch}(n-1) - (r, r')_k & \text{if } r > r' \\
\text{coch}(n-1) + [r', r]_k & \text{if } r \leq r' 
\end{cases} \quad (9.4)
\]

where \((r, r')_k\) and \([r', r]_k\) are again calculated using the \( k \)-shape corresponding to the shape of \( T_{n-1} \).

\textbf{Example 38.} Using the same tableau as in Example 37, we get the following sequence of \( k \)-shapes (with rows \( r \) and \( r' \) identified)

\[
\begin{array}{cccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

We thus have \( \text{coch}(1) = 0; \) \( \text{coch}(2) = \text{coch}(1) - (2,1)_4 = 0; \) \( \text{coch}(3) = \text{coch}(2) + [2,2]_4 = 1; \) \( \text{coch}(4) = \text{coch}(3) - (3,1)_4 = 1; \) \( \text{coch}(5) = \text{coch}(4) + [2,2]_4 = 2; \) \( \text{coch}(6) = \text{coch}(5) - (3,1)_4 = 2; \) \( \text{coch}(7) = \text{coch}(6) + [2,2]_4 = 3; \) \( \text{coch}(8) = \text{coch}(7) - (3,1)_4 = 3; \) \( \text{coch}(9) = \text{coch}(8) - (2,1)_4 = 3. \) Therefore, \( \text{coch}(T) = \sum_{n=1}^{9} \text{coch}(n) = 0 + 0 + 1 + 1 + 2 + 2 + 3 + 3 + 3 = 15. \)

We now show that the definition of (co)charge of a \( k \)-shape tableau actually extends that of (co)charge of a \( k \)-tableau.

\textbf{Proposition 39.} Let \( T \) be a standard \( k \)-shape tableau that is also a \( k \)-tableau. Then definitions \( (4.2) \) and \( (9.2) \) of charge coincide (and similarly for cocharge).

\textbf{Proof.} Let \( r \) and \( r' \) be respectively the row above that of \( n^- \) and the row of \( n^+ \). Let also \( e \) and \( e' \) be the residues of \( n^- \) and \( n^+ \) respectively.

Suppose that \( r \geq r' \). Then we need to show that

\[ [r, r']_k = \text{diag}_{e'}(n^+, n^-)_k + 1 \quad (9.5) \]

or more simply, that

\[ (r, r')_k = \text{diag}_{e'}(n^+, n^-) \quad (9.6) \]

It is an elementary fact of \( k \)-tableaux (see for instance [11]) that \( e \neq e' \). This implies that \( \text{diag}_{e'}(n^+, n^-) = \text{diag}_{e'-1}(n^+, n^-) \). Since there is a residue \( e' - 1 \) in row \( r \), \( (9.6) \) follows from Lemma 10.

Now suppose that \( r < r' \). We need to show that

\[ (r', r)_k = \text{diag}_{e}(n^+, n^-) \quad (9.7) \]

or equivalently, that

\[ (r', r - 1)_k = \text{diag}_{e}(n^+, n^-) \quad (9.8) \]

Since there is a residue \( e \) in row \( r' \), the equation follows again from Lemma 10.

The proof when charge is replaced by cocharge is similar. \( \square \)
Lemma 40. Let $\lambda$ be a $k+1$-core. Let $r'$ be the $k$-connected row below a certain row $r$ of $\lambda$. If there is a residue $e$ on the ground in row $r$ of $\lambda$ (that is, there is a cell $b = (i, j)$ of residue $e$ in row $r$ such that $(i, j) \not\in \lambda$ and $(i - 1, j) \in \lambda$), then there is also a residue $e$ on the ground in row $r'$. Moreover, there is no diagonal of residue $e$ between rows $r$ and $r'$.

Proof. Let $b$ be the position of the cell of residue $e$ on the ground in row $r$. Let $c$ be the position of the uppermost cell of $\text{Int}^k(\lambda)$ in the column of $b$. It is easy to see that $c$ lies in row $r - 1$. Since $h_c(\lambda) > k$, there is a cell $a$ of residue $e$ in the row of $c$ (see picture).

Observe that by definition of $c$, the cell $b'$ in the picture does not belong to $\lambda$. It thus suffices to show that the cell $b'$ has a cell of $\lambda$ immediately below. But this has to be the case since $\lambda$ is a $k+1$-core (otherwise we would have $h_c(\lambda) = k+1$).

9.3. Relation between charge and cocharge. Before establishing the relation between charge and cocharge given in Proposition 47 below, we need to prove a series of technical lemmas that will culminate with Lemma 46 (which is essentially equivalent to Proposition 47).

Lemma 41. Let $r_1$ and $r_2$ be two contiguously connected rows of a given $k$-shape $\lambda$, with $r_1 > r_2$. If $s_1$ is a row of $\lambda$ such that $s_1 > r_1$, then the $k$-connected row $s_2$ below $s_1$ is such that $s_2 > r_2$.

Proof. The proof can be easily visualized with the following diagram:

Obviously $h_b'(\lambda) \geq h_b(\lambda) + 2$. By hypothesis, $h_b \geq k - 1$, which implies $h_b'(\lambda) \geq k + 1$. Hence, from the definition of $k$-connected rows, we have $s_1 > r_2$.

The next lemma is immediate.

Lemma 42. Let $r_1$ and $r_2$ be two $k$-connected rows of a given $k$-shape $\lambda$, with $r_1 > r_2$. If $s_1$ is a row of $\lambda$ such that $s_1 \leq r_1$, then the $k$-connected row $s_2$ below $s_1$ is such that $s_2 \leq r_2$.

Lemma 43. Let $T$ be any $k$-shape tableau, and let $\text{coch}'(n + 1)$ be the hypothetical cocharge in $T$ of the letter $n + 1$ if $n^3_+$ were located in the position of $n^1_+$. Then $\text{coch}'(n + 1) - \text{coch}(n + 1) = \ell$, where $\ell = |e_{n+1}| - 1$. 23
Proof. Let $T_n$ be the shape of $\lambda$. Let $r_1, \ldots, r_{\ell+1}$ be the rows where the letter $n+1$ occurs in $T$ ($r_1$ and $r_{\ell+1}$ are thus the rows of $n^+=1$ and $n^+=1$ respectively). These correspond by definition of $c_{n+1}$ to a sequence of contiguously connected rows of $\lambda$.

Suppose that $s_1$ is the row of $\lambda$ above that of $n^+$. We will only prove the case where $s_1$ lies between $r_1$ and $r_{\ell+1}$ (the other cases are simplified versions of that case). We thus suppose that $s_1$ lies between $r_1$ and $r_{\ell+1}$. Let $r_i$ be the lowest among $r_1, \ldots, r_{\ell+1}$ such that $r_i \geq s_1$. Let also $s_1, \ldots, s_{\ell-i+1}$ be the sequence of $k$-connected rows weakly below row $s_1$. From Lemma 41 and 42, we have

$$r_i \geq s_1 > r_{i+1} \geq s_2 > \cdots > s_{\ell-i} > r_{\ell+1} \geq s_{\ell-i+1}$$

and it is then immediate that $(s_1, r_{\ell+1}) = (r_i, r_{\ell+1})_k$.

We can now proceed with the proof of the lemma. We have $\text{coch}'(n+1) = \text{coch}(n) + [r_1, s_1]_k$ and $\text{coch}(n+1) = \text{coch}(n) - (s_1, r_{\ell+1})$. Thus $\text{coch}'(n+1) - \text{coch}(n+1) = [r_1, s_1]_k + (s_1, r_{\ell+1})$. But $[r_1, s_1]_k = [r_1, r_i]_k$ and, as we have seen, $(s_1, r_{\ell+1}) = (r_i, r_{\ell+1})_k$. This implies that $\text{coch}'(n+1) - \text{coch}(n+1) = (r_i, r_{\ell+1})_k = \ell$.

The next lemma, which is similar to the previous one, is stated without proof.

Lemma 44. Let $T$ be any $k$-shape tableau, and let $\text{ch}'(n)$ be the hypothetical charge in $T$ of the letter $n$ if $n^+$ were located instead in the position of $n^k$. Then $\text{ch}'(n) - \text{ch}(n) = \ell$, where $\ell = |c_n| - 1$.

Lemma 45. Let $T$ be any $k$-shape tableau, and let $\text{ch}'(n)$ and $\text{ch}'(n+1)$ be the hypothetical charge in $T$ of the letters $n$ and $n+1$ respectively if $n^+$ were located instead in the position of $n^+$. Then $\text{ch}(n+1) = \text{ch}'(n+1)$.

Proof. Let $T_n$ be the shape of $\lambda$. Let also $r_1, \ldots, r_{\ell+1}$ be the rows where the letter $n$ occurs in $T_n$, and define $r_i = r_{i-1}+1$ for $i = 1, \ldots, \ell+1$. Because of the presence of the letter $n$ in rows $r_1, \ldots, r_{\ell+1}$, the $k$-shape $\lambda$ has addable corners in rows $r_1, \ldots, r_{\ell+1}$. Furthermore, since by definition of $c_n$ the rows $r_1, \ldots, r_{\ell+1}$ form a sequence of contiguously connected rows, we get that $r_1, \ldots, r_{\ell+1}$ are $k$-connected rows below row $r_1$. Observe also that $r_i$ is the row above that of $n^+$. Now, let $s_1$ be the row of $\lambda$ where $n^+$ lies, and let $s_1, \ldots, s_{\ell-i+1}$ be the sequence of $k$-connected rows weakly below row $s_1$. We will again only treat the case where $s_1$ is between $r_1$ and $r_{\ell+1}$. Define $r_i$ to be the lowest among $r_1, \ldots, r_{\ell+1}$ such that $r_i \geq s_1$. Then from Lemmas 41 and 42, we have

$$r_i \geq s_1 > r_{i+1} \geq s_2 > \cdots > s_{\ell-i} > r_{\ell+1} \geq s_{\ell-i+1}$$

which is equivalent to

$$\bar{r}_i > s_1 \geq \bar{r}_{i+1} \geq s_2 \geq \cdots > s_{\ell-i} \geq \bar{r}_{\ell+1} \geq s_{\ell-i+1}$$

Observe that $(s_1, \bar{r}_{\ell+1})_k = (\bar{r}_{i+1}, \bar{r}_{\ell+1})_k$.

We can now finalize the proof of the lemma. We have $\text{ch}'(n+1) = \text{ch}'(n) - (s_1, \bar{r}_{\ell+1})_k$ and $\text{ch}(n+1) = \text{ch}(n) + [\bar{r}_1, s_1]_k$. Thus $\text{ch}(n+1) - \text{ch}'(n+1) = \text{ch}(n) - \text{ch}'(n) + [\bar{r}_1, s_1]_k + (s_1, \bar{r}_{\ell+1})_k$. But $[\bar{r}_1, s_1]_k = [\bar{r}_1, r_i]_k$ and, as we have seen, $(s_1, \bar{r}_{\ell+1})_k = (\bar{r}_{i+1}, \bar{r}_{\ell+1})_k$. This implies from Lemma 44 that $\text{ch}(n+1) - \text{ch}'(n+1) = \text{ch}(n) - \text{ch}'(n) + [\bar{r}_1, r_i]_k + (\bar{r}_{i+1}, \bar{r}_{\ell+1})_k = -\ell + \ell = 0$.

The charge and cocharge of the letter $n$ are related in the following way:
Lemma 46. Let $c_n$ be the cover corresponding to letter $n$ in the $k$-shape tableau $T$. Then the charge and cocharge of the letter $n$ satisfy the relation

$$\text{ch}(n) = n - \text{coch}(n) - |c_n|$$  \hspace{1cm} (9.9)

Proof. We proceed by induction. The case $n = 1$ is easily seen to hold. We need to establish that

$$\text{ch}(n + 1) = n + 1 - \text{coch}(n + 1) - |c_{n+1}|$$  \hspace{1cm} (9.10)

Using the induction hypothesis, it suffices to prove that

$$\text{ch}(n + 1) - \text{ch}(n) + \text{coch}(n + 1) - \text{coch}(n) = |c_n| - |c_{n+1}| + 1$$  \hspace{1cm} (9.11)

We first consider the case where $n^\uparrow_+ \text{ and } n^\downarrow_+$ are above $n^\uparrow$ and $n^\downarrow$. We can visualize this case with a diagram. The symbol $\star$ denotes the rows that are $k$-connected with the row of $n^\uparrow_+$, while $\circ$ denote the rows above the letters $n$.

\[
\begin{array}{ccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & \star & \ldots & 1 & \star & \circ & \circ \\
\end{array}
\]

Let $r_1, \ldots, r_\ell$ be the rows of the $\circ$’s in the picture ($r_1$ and $r_\ell$ are the rows of the $\circ$ above $n^\uparrow$ and $n^\downarrow$ respectively). Let also $s_1$ be the row of the lowest $\star$ such that $s_1 \geq r_1$. By Lemma 42, the sequence $s_1, \ldots, s_{\ell+1}$ of connected row starting from $s_1$ is such that

$$s_1 \geq r_1 > s_2 \geq r_2 > \cdots > s_\ell \geq r_\ell > s_{\ell+1}$$

We then get

$$(n^\uparrow_+, n^\downarrow_+)_k + (n^\uparrow, n^\downarrow)_k = (n^\uparrow_+, n^\downarrow_+)_k + [n^\uparrow_+, n^\downarrow_+]_k$$

where we identified the cells with their rows. Therefore, the definition of charge and cocharge imply

$$\text{ch}(n) - \text{ch}(n + 1) + |c_n| - 1 = |c_{n+1}| - 2 + \text{coch}(n + 1) - \text{coch}(n)$$  \hspace{1cm} (9.12)

which is equivalent to (9.11).

We have established that the result holds when $n^\uparrow_+$ and $n^\downarrow_+$ are above $n^\uparrow$ and $n^\downarrow$. In all the other cases we have either:

1) $n^\uparrow_+$ or $n^\downarrow_+$ lies between $n^\uparrow$ and $n^\downarrow$

2) $n^\uparrow$ or $n^\downarrow$ lies between $n^\uparrow_+$ and $n^\downarrow_+$

Consider case 1). From Lemmas 44 and 45, if we hypothetically shift the position of $n^\uparrow$ to that of $n^\downarrow$, we obtain new $\text{ch}'(n)$ and $\text{ch}'(n + 1)$ such that $\text{ch}'(n) - \text{ch}(n) = |c_n| - 1$ and $\text{ch}'(n + 1) = \text{ch}(n + 1)$. From

$$\text{ch}'(n + 1) - \text{ch}'(n) + \text{coch}'(n + 1) - \text{coch}'(n) = |c'_n| - |c'_{n+1}| + 1$$  \hspace{1cm} (9.13)

we get using the extra relations $|c'_n| = 1$, $|c'_{n+1}| = |c_{n+1}|$, $\text{coch}'(n + 1) = \text{coch}(n + 1)$ and $\text{coch}'(n) = \text{coch}(n)$, that (9.11) holds if (9.13) holds. Note that shifting $n^\uparrow$ changes its position relative to either $n^\uparrow_+$ or $n^\downarrow_+$.

Consider case 2). From Lemma 43, if we hypothetically shift the position of $n^\downarrow_+$ to that of $n^\uparrow_+$, we obtain a new $\text{coch}'(n + 1)$ such that $\text{coch}'(n + 1) - \text{coch}(n + 1) = |c_{n+1}| - 1$. Using (9.13) with $|c'_n| = |c_n|$, $|c'_{n+1}| = 1$, $\text{coch}'(n) = \text{coch}(n)$, $\text{ch}'(n) = \text{ch}(n)$, and $\text{ch}'(n + 1) = \text{ch}(n + 1)$, we obtain again that (9.11) holds if (9.13) holds. Observe that shifting $n^\downarrow_+$ changes its position relative to either $n^\uparrow$ or $n^\downarrow$. 


Applying cases \textbf{I} and \textbf{II}) again and again, we get that the general case follows from the previously established case where $n_{-1}^+$ and $n_{-1}^+$ are above $n^+$ and $n^-$.

An immediate consequence of Lemma \[\text{46}\] is the following relation between the charge and the cocharge of a standard $k$-shape tableau, which generalizes the usual relation between charge and cocharge of a standard tableau.

**Proposition 47.** Let $T$ be a standard $k$-shape tableau of shape $\lambda$. Then
\[
\text{ch}(T) = \frac{n(n - 1)}{2} - \text{coch}(T) - |\text{Int}^k(\lambda)|
\]
(9.14)
where we recall that $\text{Int}^k(\lambda)$ was defined at the beginning of Subsection 2.2.

**Proof.** Summing (9.9) from 1 to $n$, we get
\[
\text{ch}(T) = \frac{n(n + 1)}{2} - \text{coch}(T) - (|c_1| + \cdots + |c_n|)
\]
(9.15)
Adding the cover $c_i$ increases by $|c_i| - 1$ the number of hooks larger than $k$. Hence $|c_1| + \cdots + |c_n| - n = \text{Int}^k(\lambda)$, and the corollary follows.

**Example 48.** Using the standard $k$-shape tableau $T$ in Examples \[\text{39}\] and \[\text{38}\] we get
\[
\text{ch}(T) = \frac{9(9 - 1)}{2} - \text{coch}(T) - |\text{Int}^k(\lambda_0)| = 36 - 15 - 5 = 16
\]
as wanted.

10. **Compatibility between (co)charge and the weak bijection**

Our goal is to show that the weak bijection \[\text{8.8}\] satisfies the conditions
\[
\text{coch}_k(Q^{(k)}) = \text{coch}(p) + \text{coch}_{k-1}(Q^{(k-1)}) \quad \text{and} \quad \text{ch}_k(Q^{(k)}) = \text{ch}(p) + \text{ch}_{k-1}(Q^{(k-1)})
\]
(10.1)
when $Q^{(k)}$ (resp. $Q^{(k-1)}$) is standard $k$-tableau (resp. $k-1$-tableau). We have emphasized that the (co)charge of $Q^{(k)}$ and $Q^{(k-1)}$ are computed considering that they are sequences of $k$ and $k-1$-shapes respectively. The next proposition shows that this distinction is not necessary.

**Proposition 49.** The cocharge (resp. charge) of a standard $k$-tableau $T$ is the same whether it is considered as a sequence of $k$-shapes or as a sequence of $k+1$-shapes. That is,
\[
\text{coch}_k(T) = \text{coch}_{k+1}(T) \quad \text{and} \quad \text{ch}_k(T) = \text{ch}_{k+1}(T)
\]
in $T$ is a standard $k$-tableau.

**Proof.** Since a $k+1$-core does not have hooks of length $k+1$, two rows of a $k+1$-core are $k+1$-connected if and only if they are $k$-connected. In the case of a $k$-tableau, the corresponding sequence of $k$-shapes is a sequence of $k+1$-cores. Hence the result follows immediately from the definition of cocharge (resp. charge).

It thus suffices to show that the compatibility between the (co)charge and the weak bijection (in the standard case) is true when the $k-1$-tableau is considered as a sequence of $k$-shapes. The remainder of this section will be devoted to showing the next proposition, which from Proposition \[\text{49}\] implies Theorem \[\text{II}\].

**Proposition 50.** Let $Q^{(k)}$ and $Q^{(k-1)}$ be standard $k$ and $k-1$-tableaux respectively such that $Q^{(k)} \leftrightarrow (Q^{(k-1)}, [p])$ in the weak bijection \[\text{8.8}\]. Then
\[
\text{coch}(Q^{(k)}) = \text{coch}(p) + \text{coch}(Q^{(k-1)}) \quad \text{and} \quad \text{ch}(Q^{(k)}) = \text{ch}(p) + \text{ch}(Q^{(k-1)})
\]
(10.2)
where the (co)charge is computed considering that both $Q^{(k)}$ and $Q^{(k-1)}$ are sequences of $k$-shapes.
In order to show Proposition 50, we will proceed locally using the pushout algorithm. We say that the standard \( k \)-shape tableaux \( T \) and \( U \) differ by moves if there exists a sequence of row moves (resp. column moves) \( m_1, \ldots, m_n \) such that the following diagram commutes:

\[
\begin{array}{ccc}
\emptyset & \xrightarrow{c_1} & \lambda^{(1)} \\
& \downarrow & \downarrow \\
& \mu^{(1)} & \\
\end{array}
\begin{array}{ccc}
\emptyset & \xrightarrow{\tilde{c}_1} & \lambda^{(1)} \\
& \downarrow & \downarrow \\
& \mu^{(1)} & \\
\end{array}
\begin{array}{ccc}
\lambda^{(2)} & \xrightarrow{m_1} & \mu^{(2)} \\
& \downarrow & \downarrow \\
& \tilde{c}_2 & \\
\end{array}
\begin{array}{ccc}
\lambda^{(2)} & \xrightarrow{\tilde{c}_2} & \mu^{(2)} \\
& \downarrow & \downarrow \\
& \tilde{c}_2 & \\
\end{array}
\begin{array}{ccc}
\vdots & \vdots & \vdots \\
& \vdots & \vdots \\
& \vdots & \\
\end{array}
\begin{array}{ccc}
\lambda^{(N-1)} & \xrightarrow{m_{N-1}} & \mu^{(N-1)} \\
& \downarrow & \downarrow \\
& \tilde{c}_N & \\
\end{array}
\begin{array}{ccc}
\lambda^{(N-1)} & \xrightarrow{\tilde{c}_N} & \mu^{(N-1)} \\
& \downarrow & \downarrow \\
& \tilde{c}_N & \\
\end{array}
\begin{array}{ccc}
\lambda & \xrightarrow{m_N} & \mu \\
& \downarrow & \downarrow \\
& \tilde{c}_N & \\
\end{array}
\end{array}
\]

(10.3)

where \( c_1, \ldots, c_N \) and \( \tilde{c}_1, \ldots, \tilde{c}_N \) correspond respectively to \( T \) and \( U \), and where every commutative square in the diagram corresponds to one of the 3 steps described in Section 6. The pushout algorithm ensures that one can obtain a sequence

\[
Q^{(k)} = T^{(0)}, T^{(1)}, \ldots, T^{(r-1)}, T^{(r)} = Q^{(k-1)}
\]

(10.4)

where, for all \( i \), \( T^{(i)} \) and \( T^{(i+1)} \) either differ by row moves or column moves. Therefore, in order to prove Proposition 50 it suffices to show that \( T \) and \( U \) described in (10.3) are such that

\[
\text{coch}(T) = \text{coch}(m_N) + \text{coch}(U) \quad \text{and} \quad \text{ch}(T) = \text{ch}(m_N) + \text{ch}(U)
\]

(10.5)

Using the definition of charge and cocharge, it is straightforward to see that this is equivalent to proving that

\[
\text{coch}(N) - \text{coch}(\bar{N}) = \text{coch}(m_N) - \text{coch}(m_{N-1})
\]

(10.6)

and

\[
\text{ch}(N) - \text{ch}(\bar{N}) = \text{ch}(m_N) - \text{ch}(m_{N-1})
\]

(10.7)

where for simplicity we denote the charge of the letter \( N \) in \( T \) by \( \text{ch}(N) \), and the charge of the letter \( N \) in \( U \) by \( \text{ch}(\bar{N}) \) (and similarly for cocharge).

The next lemma shows that both problems are equivalent.

**Lemma 51.** We have

\[
\text{coch}(N) - \text{coch}(\bar{N}) = \text{coch}(m_N) - \text{coch}(m_{N-1}) \iff \text{ch}(N) - \text{ch}(\bar{N}) = \text{ch}(m_N) - \text{ch}(m_{N-1})
\]

(10.8)

**Proof.** Suppose that \( T \) and \( U \) differ by a row move. We have to show that

\[
\text{coch}(N) - \text{coch}(\bar{N}) = |m_N| - |m_{N-1}| \iff \text{ch}(N) = \text{ch}(\bar{N})
\]

(10.9)
From Lemma 46, we get
\[
\coch(N) - \coch(\bar{N}) = (N - \text{ch}(N) - |c_N|) - (N - \text{ch}(\bar{N}) - |\bar{c}_N|) = \text{ch}(\bar{N}) + |\bar{c}_N| - \text{ch}(N) - |c_N| \tag{10.10}
\]
which leads to
\[
\coch(N) - \coch(\bar{N}) = |m_N| - |m_{N-1}| \iff \text{ch}(\bar{N}) + |\bar{c}_N| - \text{ch}(N) - |c_N| = |m_N| - |m_{N-1}| \tag{10.11}
\]
By inspection of Step 1 and Step 3 (in the row case), it is easy to deduce that in all cases
\[
|\bar{c}_N| - |c_N| = |m_N| - |m_{N-1}| \tag{10.12}
\]
and thus (10.9) follows from (10.11). The proof when \(T\) and \(U\) differ by a column move is identical. \(\square\)

It thus suffices to prove (10.6) when \(T\) and \(U\) differ by a column move, and (10.7) when \(T\) and \(U\) differ by a row move. Observe that in both cases the right hand side of the equations is then equal to zero. We will proceed by induction. We will suppose that the relations hold for all \(N\) up to \(N = n\) and then show that the case \(N = n + 1\) also holds, that is, that

- \(\coch(n + 1) = \coch(\bar{n} + 1)\) when \(T\) and \(U\) differ by a column move
- \(\text{ch}(n + 1) = \text{ch}(\bar{n} + 1)\) when \(T\) and \(U\) differ by a row move

The proof will rely on commutative diagrams of the type
\[
\begin{array}{ccc}
T_n & \xrightarrow{m_n} & U_n \\
\downarrow^{c_{n+1}} & & \downarrow^{\bar{c}_{n+1}} \\
T_{n+1} & \xrightarrow{m_{n+1}} & U_{n+1}
\end{array}
\tag{10.13}
\]
where \(T_n\) and \(U_n\) denote standard \(k\)-shape tableaux of \(n\) letters, and where, for simplicity, we use \(T_{n+1}\) instead of \(\text{sh}(T_{n+1})\). We will also keep denoting by \(\bar{n}\) the letter \(n\) in \(U\). For instance \(\bar{n}^\dagger\) denotes the highest occurrence of the letter \(n\) in \(U\), while \(n^\dagger\) denotes the highest occurrence of the letter \(n\) in \(T\).

We now proceed to analyze all the possible cases.

### 10.1. Row and column maximization

We first consider the situation where \(c_{n+1}\) is not maximal. The maximization described in Step 1 and 2 is such that
\[
\begin{array}{ccc}
T_n & \xrightarrow{\emptyset} & T_n \\
\downarrow^{c_{n+1}} & & \downarrow^{\bar{c}_{n+1}} \\
T_{n+1} & \xrightarrow{m_{n+1}} & U_{n+1}
\end{array}
\tag{10.14}
\]
where \(m\) is a maximization below or above of the cover \(c_{n+1}\). Suppose that \(m\) is a row move (maximization below). We have that \(n^\dagger\) and \(\bar{n}^\dagger\) lie in the same position (above the move), as do obviously \(n^\dagger\) and \(\bar{n}^\dagger\) (given that \(T_n = U_n\)). Since \(\text{ch}(n + 1)\) and \(\text{ch}(\bar{n} + 1)\) are computed using the \(k\)-shape corresponding to the shape of \(T_n\), it is immediate that \(\text{ch}(n+1) - \text{ch}(n) = \text{ch}(\bar{n} + 1) - \text{ch}(\bar{n}) = \text{ch}(\bar{n} + 1) - \text{ch}(n)\). Therefore, we get \(\text{ch}(n + 1) = \text{ch}(\bar{n} + 1)\).

If \(m\) is a column move, we can use a similar analysis (this time \(n^\dagger\) and \(\bar{n}^\dagger\) lie in the same position below the move), to show that \(\coch(n + 1) = \coch(\bar{n} + 1)\).
10.2. **Maximal pushout (row case).** We now show that the charge is conserved in the pushout of a maximal cover \( c_{n+1} \) and a row move \( m \).

**Lemma 52.** Consider the following situation

\[
\begin{align*}
T_{n-1} \xrightarrow{m'} U_{n-1} \\
\downarrow^{c_n} \quad \downarrow^{c_n} \\
T_n \xrightarrow{m} U_n \\
\downarrow^{c_{n+1}} \quad \downarrow^{\tilde{c}_{n+1}} \\
T_{n+1} \xrightarrow{\tilde{m}} U_{n+1}
\end{align*}
\] (10.15)

where \( c_{n+1} \) is maximal, and where either \( c_n \) is maximal or \( m \) is the maximization below of \( c_n \) (in which case \( m' \) is empty). Then \( ch(n + 1) = ch(\bar{n} + 1) \) if \( ch(n) = ch(\bar{n}) \).

Before proceeding to the proof of the lemma, we need to establish a few elementary results. Let \( m \) be a row move from \( \lambda \) to \( \mu \) originating, as in Lemma 52, either from a maximization below of \( c_n \) or from the maximal pushout of the pair \((c_n, m')\). The following observations follow easily from the definition of maximal pushout in the row case.

(i) \( n_+ \) and \( \bar{n}_+ \) are in the same position

(ii) \( n_+ \) and \( \bar{n}_+ \) are in the same position

(iii) \( n_+ \) is never in a row that belongs to the move \( m \).

(iv) \( n_+ \) is never in a row that belongs to the move \( m \).

We also need to describe how the move \( m \) affects \( k \)-connectedness between rows. We will always consider that \( r_1 \) and \( r_2 \) (\( r_1 > r_2 \)) are two \( k \)-connected rows in \( \lambda \). There are essentially 3 cases to consider.

1. If the move \( m \) intersects row \( r_2 \) but does not continue above, then as seen in the picture, the row that corresponds to the negatively modified column of \( m \) is now \( k \)-connected to row \( r_2 + 1 \), while the other rows remain connected in the same way:

2. If both rows \( r_1 \) and \( r_2 \) belong to \( m \), then rows \( r_1 + 1 \) and \( r_2 + 1 \) are \( k \)-connected in \( \mu \), while row \( r_1 \) and \( r_2 \) remain connected (if there is still an addable corner in row \( r_1 \) of \( \mu \))

3. If row \( r_1 \) belongs to the move \( m \) but \( r_2 \) does not then \( r_1 + 1 \) now connects with \( r_2 \) in \( \mu \), while row \( r_1 \) and \( r_2 \) remain connected (if there is still an addable corner in row \( r_1 \) of \( \mu \))
We are now in a position to prove Lemma \[52\.

**Proof of Lemma [52]** Suppose that \( n \uparrow \) is above \( n_\downarrow \). Let \( r_1 \) be the row above that of \( n \uparrow \) in \( \lambda \), and let \( r_1, \ldots, r_{\ell+1} \) be the connected rows below row \( r_1 \) such that \( r_\ell > s \) and \( r_{\ell+1} \leq s \), where \( s \) is the row of \( n_\downarrow \) in \( \lambda \). From observations (i) and (ii), in \( \mu \) row \( r_1 \) is still the row above that of \( n_\uparrow \) and \( s \) is still the row of \( n_\downarrow \). From the analysis of the 3 cases considered above, in \( \mu \) the connected rows below \( r_1 \) will be \( r_1, \ldots, r_{i-1}, r_i + 1, \ldots, r_\ell + 1, r_{\ell+1} \) if rows \( r_i \) up to \( r_\ell \) belong to the move \( m \). Note that \( i > 1 \) since, as we just mentioned, \( r_1 \) did not change. We have to show that \( (r_1, s)_k \) is the same in \( \lambda \) and in \( \mu \). In \( \lambda \), we have \( (r_1, s)_k = \ell \). In \( \mu \) we will still have \( (r_1, s)_k = \ell \) unless \( r_{\ell+1} = s \) and \( s \) is a row of the move. But this is impossible from (iv).

Now suppose that \( n \uparrow \) is weakly below \( n_\downarrow \). Let \( r_1 \) be the row of \( n_\uparrow \) in \( \lambda \), and let \( r_1, \ldots, r_{\ell+1} \) be the connected rows below row \( r_1 \) such that \( r_\ell \geq s \) and \( r_{\ell+1} < s \), where \( s \) is the row above that of \( n \uparrow \) in \( \lambda \). From observation (i) and (ii), in \( \mu \) row \( r_1 \) is still the row of \( n_\downarrow \) and \( s \) is still the row above that of \( n \uparrow \). In \( \mu \), the connected rows below \( r_1 \) will again be \( r_1, \ldots, r_{i-1}, r_i + 1, \ldots, r_\ell + 1, r_{\ell+1}, \ldots, r_{\ell+1} \), with \( i > 1 \), if rows \( r_i \) up to \( r_\ell \) belong to the move \( m \). We have to show that \( (r_1, s)_k \) is the same in \( \lambda \) and in \( \mu \). In \( \lambda \), we have \( (r_1, s)_k = \ell - 1 \). In \( \mu \) we will still have \( (r_1, s)_k = \ell - 1 \) unless \( r_{\ell+1} = s - 1 \) and \( s - 1 \) is a row of the move. But this is impossible by (iii) since \( s - 1 \) is the row of \( n \uparrow \). \[ \square \]

10.3. **Maximal pushout (column case).** We need to show that the cocharge is conserved in the pushout of a maximal cover \( c_{n+1} \) and a column move \( m \).

**Lemma 53.** Consider the following situation

\[
\begin{array}{c}
T_{n-1} \xrightarrow{m'} U_{n-1} \\
\downarrow c_n \quad \downarrow \hat{c}_n \\
T_n \xrightarrow{m} U_n \\
\downarrow c_{n+1} \quad \downarrow \hat{c}_{n+1} \\
T_{n+1} \xrightarrow{m'} U_{n+1}
\end{array}
\]  

(10.16)

where \( c_{n+1} \) is maximal, and where either \( c_n \) is maximal or \( m \) is is the maximization above of \( c_n \) (in which case \( m' \) is empty). Then \( \text{coch}(n+1) = \text{coch}(\bar{n}+1) \) if \( \text{coch}(n) = \text{coch}(\bar{n}) \).

Before proceeding to the proof of the lemma, we need as in the previous subsection to establish a few elementary results. Let \( m \) be a row move from \( \lambda \) to \( \mu \) originating, as in Lemma \[53\] either from a maximization above of \( c_n \) or from the maximal pushout of the pair \((c_n, m')\). The following observations follow easily from the definition of maximal pushout in the column case.

(i) \( n \uparrow \) and \( \bar{n} \uparrow \) are in the same position
(ii) \( n_\downarrow \) and \( \bar{n}_\downarrow \) are in the same position
(iii) \( n \uparrow \) is never in a row that belongs to the move \( m \).
(iv) \( n_\downarrow \) is never in a row that belongs to the move \( m \).

We also need to describe how the move \( m \) affects \( k \)-connectedness between rows. The main claim is the following.
Lemma 54. Let $m$ be a column move from $\lambda$ to $\mu$. Suppose that $\lambda$ and $\mu$ have an addable corner in row $r_1$. Then the $k$-connected row $r_2$ below row $r_1$ is the same in $\lambda$ and in $\mu$.

Proof. We consider all possible cases and see that the result always hold.

(1) If neither $r_1$ nor $r_2$ belong to the move $m$, then the result is immediate.

(2) If only one of $r_1$ and $r_2$ belong to $m$, we have either

\begin{align*}
&\begin{array}{c}
\begin{array}{c}
\ast \\
\downarrow
\end{array}
\end{array}
m
\begin{array}{c}
\begin{array}{c}
\ast \\
\downarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\ast \\
\downarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\ast \\
\downarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\ast \\
\downarrow
\end{array}
\end{array}
\end{align*}

$h_b(\lambda) \leq k - 1$

or

\begin{align*}
&\begin{array}{c}
\begin{array}{c}
\ast \\
\downarrow
\end{array}
\end{array}
m
\begin{array}{c}
\begin{array}{c}
\ast \\
\downarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\ast \\
\downarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\ast \\
\downarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\ast \\
\downarrow
\end{array}
\end{array}
\end{align*}

$h_b(\lambda) = k$

(3) If $r_1$ and $r_2$ both belong to $m$

\begin{align*}
&\begin{array}{c}
\begin{array}{c}
\ast \\
\downarrow
\end{array}
\end{array}
m
\begin{array}{c}
\begin{array}{c}
\ast \\
\downarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\ast \\
\downarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\ast \\
\downarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\ast \\
\downarrow
\end{array}
\end{array}
\end{align*}

$h_b(\lambda) = k$ or $k - 1$

We now proceed to the proof of Lemma 53.

Proof of Lemma 53. Suppose that $n^\downarrow$ is above $n^\downarrow_+$. By (i) and (ii), we have that $n^\downarrow$ coincides with $\bar{n}^\downarrow$ and that $n^\downarrow_+$ coincides with $\bar{n}^\downarrow_+$. The row $r$ above that of $n^\downarrow$ still has an addable corner in $\mu$ from (i). Using Lemma 54 again and again we get that the string of $k$-connected rows below $r$ is the same in $\lambda$ and $\mu$. It is thus immediate that $\coch(\bar{n} + 1) = \coch(n + 1)$ if $\coch(\bar{n}) = \coch(n)$.

Suppose that $n^\downarrow$ is above $n^\downarrow_+$. By (i) and (ii), we still have that $n^\downarrow$ coincides with $\bar{n}^\downarrow$ and that $n^\downarrow_+$ coincides with $\bar{n}^\downarrow_+$. The row $r$ of $n^\downarrow_+$ thus has an addable corner in $\lambda$ and $\mu$ by definition. Using Lemma 54 again and again we get that the string of $k$-connected rows below $r$ is the same in $\lambda$ and $\mu$. It is then again immediate that $\coch(\bar{n} + 1) = \coch(n + 1)$ if $\coch(\bar{n}) = \coch(n)$. 

□
11. Conclusion

The compatibility between charge and the weak bijection was only established in the standard case. We discuss here briefly the obstruction to extending this compatibility to the semi-standard case. In order to extend the charge of a $k$-tableau of dominant weight given at the end of Section 4 to arbitrary $k$-shape tableaux of dominant weight, we would need a way to order the various occurrences of the letters in the tableau (such as is done when computing the charge). Finding this order is essentially equivalent to defining a Lascoux-Schützenberger-type action of the symmetric group on $k$-shape tableaux [15, 13] that would extend that on $k$-tableaux defined at the end of Section 4. Unfortunately, we have been able to define such an action only on maximal (and reverse-maximal) tableaux (see [17]), which does not appear sufficient to prove the compatibility with the weak bijection. One of the reasons why the extension to the non-maximal case is non-trivial is that the number of $k$-shape tableaux of a given shape and weight depends in general on the weight (the number does not depend on the weight only after a notion of equivalence classes on $k$-shape tableaux of a given weight and shape has been defined).

Our ultimate goal would be to show that our Lascoux-Schützenberger-type action of the symmetric group on $k$-tableaux commutes with the weak bijection. To be more precise, recall that the Lascoux-Schützenberger action of the symmetric group on words associates to every given permutation an operator $\sigma$ that permutes the weight of a word (or tableau) according to the permutation. We conjecture that the weak bijection is such that (using the language of (8.16))

$$T \leftrightarrow (T^{(k)}, [p]) \iff \sigma(T) \leftrightarrow (\sigma(T^{(k)}), [p])$$

where we still denote by $\sigma$ the corresponding operator in the Lascoux-Schützenberger action of the symmetric group on $k$-tableaux. That is, the pushout algorithm appears to commute with the action of $\sigma$ (observe that $[p]$ is left unchanged):

$$\begin{array}{ccc}
T & \longrightarrow & \sigma(T) \\
\downarrow & & \downarrow \\
T^{(k)} & \longrightarrow & \sigma(T^{(k)})
\end{array} \quad (11.1)$$

Apart from providing us a tool to demonstrate the compatibility of the charge and the weak bijection in the non-standard case, proving the commutativity (11.1) would imply that the pushout is only necessary in the standard case (which is technically much simpler than in the non-standard case). In effect, a natural standardization $\text{Std}$ follows from the Lascoux-Schützenberger action of the symmetric group on words. The standardization $\text{Std}$ would then immediately commute with the pushout:

$$T \leftrightarrow (T^{(k)}, [p]) \iff \text{Std}(T) \leftrightarrow (\text{Std}(T^{(k)}), [p])$$

Since the standardization has a left inverse $\text{Std}^{-1}$, to obtain the non-standard pushout $T \leftrightarrow (T^{(k)}, [p])$, one would simply need to compute the standard pushout $U = \text{Std}(T) \leftrightarrow (U^{(k)}, [p])$ and then get $T^{(k)}$ from the relation $T^{(k)} = \text{Std}^{-1} \circ U^{(k)}$.

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