Commutators of weighted Hardy operator on weighted 
λ-central Morrey space

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Abstract. In this paper, the authors prove the boundedness of commutators generated by the weighted Hardy operator on weighted λ-central Morrey space with the weight \( \omega \) satisfying the doubling condition. Moreover, the authors give the characterization for the weighted λ-central Campanato space by introducing a new kind of operator which is related to the commutator of weighted Hardy operator.

\section{Introduction}

To study the local behavior of solutions to second order elliptical partial differential equations, Morrey [12] introduced Morrey space, which is defined as

\[ M^{p,\lambda}(\mathbb{R}^n) = \left\{ f \in M^{p,\lambda}(\mathbb{R}^n) : \|f\|_{M^{p,\lambda}(\mathbb{R}^n)} := \sup_B \frac{1}{|B|^\lambda} \left( \frac{1}{|B|} \int_B |f(x)|^p \, dx \right)^{1/p} < \infty \right\} \]

with the exponents \( p \) and \( \lambda \) satisfying \( p \geq 1 \) and \( -\frac{1}{p} < \lambda < 0 \).

For the extension of Morrey space \( M^{p,\lambda}(\mathbb{R}^n) \), the classical Campanato space \( C^{p,\lambda}(\mathbb{R}^n) \) is defined by

\[ C^{p,\lambda}(\mathbb{R}^n) = \left\{ f \in C^{p,\lambda}(\mathbb{R}^n) : \|f\|_{C^{p,\lambda}(\mathbb{R}^n)} := \sup_B \frac{1}{|B|^\lambda} \left( \frac{1}{|B|} \int_B |f(x) - f_B|^p \, dx \right)^{1/p} < \infty \right\}, \]

where \( p \in [1, \infty) \), \( \lambda \in (-\frac{1}{p}, \frac{1}{n}) \), \( f_B = \frac{1}{|B|} \int_B f(x) \, dx \) and \( B \subset \mathbb{R}^n \) denotes any ball in \( \mathbb{R}^n \).

It is well known that \( M^{p,\lambda}(\mathbb{R}^n) \subset C^{p,\lambda}(\mathbb{R}^n) \). For the studies of such two function spaces and the action of various operators on them, one may see [14-16] et al. for more details.

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In [10, 11], Lu and Yang studied a new kind of homogeneous Hardy type space $HA_q$ with $q > 1$ and they found that the dual space of $HA_q$ can be defined by the following norm.

$$\| f \|_{CBMO^q} := \sup_{R > 0} \left( \frac{1}{|B(0, R)|} \int_{B(0, R)} |f(x) - f_{B(0, R)}|^q \, dx \right)^{1/q} < \infty.$$  

Obviously, $CBMO^q$ is the homogeneous central bounded mean oscillation depending on $q$. Moreover, the famous John-Nirenberg inequality no longer holds in such space. Thus, it can be regarded as an extension of the classical BMO space.

In 2000, Alvarez, Lakey and Guzmán-Partida [1] introduced the $\lambda$-central Campanato space and $\lambda$-central Morrey space respectively.

**Definition 1.1.** ([1]) Let $-\frac{1}{p} < \lambda < \frac{1}{n}$ with $1 < p < \infty$. Then, a function $f \in L^p_{loc}(\mathbb{R}^n)$ is said to belonged to the $\lambda$-central Campanato space $C^{p,\lambda}(\mathbb{R}^n)$ if

$$\| f \|_{C^{p,\lambda}(\mathbb{R}^n)} := \sup_{R > 0} \left( \frac{1}{|B(0, R)|^{1 + \lambda p}} \int_{B(0, R)} |f(x) - f_{B(0, R)}|^p \, dx \right)^{1/p} < \infty.$$  

**Definition 1.2.** ([1]) Let $-\frac{1}{p} < \lambda < \frac{1}{n}$ and $1 < p < \infty$. Then, the $\lambda$-central Morrey space $M^{p,\lambda}(\mathbb{R}^n)$ is defined by

$$M^{p,\lambda}(\mathbb{R}^n) = \left\{ f \mid \| f \|_{M^{p,\lambda}(\mathbb{R}^n)} = \sup_{R > 0} \left( \frac{1}{|B(0, R)|^{1 + \lambda p}} \int_{B(0, R)} |f(x)|^p \, dx \right)^{1/p} < \infty \right\}.$$  

If $0 < \lambda < 1/p$, the $\lambda$-central-Campanato space becomes the $\lambda$-central bounded mean oscillation space $CBMO^{p,\lambda}(\mathbb{R}^n)$. Moreover, it is easy to check that $C^{p,\lambda}(\mathbb{R}^n) \subset \dot{C}^{p,\lambda}(\mathbb{R}^n)$ and $\dot{C}^{p,0}(\mathbb{R}^n) = CBMO^{p}(\mathbb{R}^n)$. For the case $-1/p < \lambda < 0$, there is $M^{p,\lambda}(\mathbb{R}^n) \subset \dot{C}^{p,\lambda}(\mathbb{R}^n)$.

Suppose that $T$ is an integral operator and $b$ is a local integrable function. Then, the commutator of $T$ is defined by

$$T_b(f)(x) = b(x)T(f)(x) - T(b)(x).$$

For the actions of commutators on $M^{p,\lambda}(\mathbb{R}^n)$, one may see [4,8,17] et al. to find more details with $b \in \dot{C}^{p,\lambda}(\mathbb{R}^n)$ and $0 < \lambda < \frac{1}{n}$. For the case $b \in \dot{C}^{p,\lambda}(\mathbb{R}^n)$ with $-\frac{1}{p} < \lambda < 0$, Shi and Lu [16] studied the boundedness of commutators generated by the Hardy operators.

Next, we give some definitions about Hardy operators.

For $f \in L^p(\mathbb{R}^+) \setminus [0, \infty)$ with $1 < p < \infty$, the classical Hardy operator is defined by

$$Hf(x) = \frac{1}{x} \int_0^x f(t) \, dt, \quad x \neq 0.$$  

In 1920, Hardy [7] proved the $L^p(\mathbb{R}^+)$ boundedness of $H$ and showed the constant $\frac{p}{p-1}$ of (1) is the best possible.

$$\| Hf \|_{L^p(\mathbb{R}^+)} \leq \frac{p}{p-1} \| f \|_{L^p(\mathbb{R}^+)}. \quad (1)$$

In 1995, Christ and Grafakos [2] introduced the following $n$-dimensional Hardy operator $\mathcal{H}$ defined by

$$\mathcal{H}f(x) = \frac{1}{|x|^n} \int_{|y| < |x|} f(y) \, dy, \quad x \in \mathbb{R}^n \setminus \{0\},$$
Christ and Grafakos [2] also showed the operator $\mathcal{H}$ satisfies the analogue results of (1).

The dual operator of $\mathcal{H}$ is $\mathcal{H}^*$, which is defined by

$$\mathcal{H}^* f(x) = \int_{|y| \geq |x|} \frac{f(y)}{|y|^n} dy, \quad x \in \mathbb{R}^n \setminus \{0\}.$$ 

It is easy to check that

$$\int_{\mathbb{R}^n} g(x) \mathcal{H} f(x) dx = \int_{\mathbb{R}^n} f(x) \mathcal{H}^* g(x) dx.$$

Thus, the commutator of Hardy type operator is defined as

$$\mathcal{H}_b f(x) = b(x) \mathcal{H} f(x) - \mathcal{H}(fb)(x) \quad \text{and} \quad \mathcal{H}_b^* f(x) = b(x) \mathcal{H}^* f(x) - \mathcal{H}^*(fb)(x).$$

The operators $\mathcal{H}_b$ and $\mathcal{H}_b^*$ were first studied in [5] where Fu et al. gave the characterization of $\mathcal{C}BM\Omega^p(\mathbb{R}^n)$ with $1 < p < \infty$ via the boundedness of $\mathcal{H}_b$ and $\mathcal{H}_b^*$ on $L^p(\mathbb{R}^n)$. For more studies about the operator $\mathcal{H}_b$ and $\mathcal{H}_b^*$, one may see [16,19] to find more details about $\mathcal{H}_b$ and $\mathcal{H}_b^*$ with $b \in \mathcal{C}^{p,\lambda}(\mathbb{R}^n)$.

On the other hand, the weighted norm inequalities for integral operators was first studied in the last 70s and one may see [3,13] et. al. for more details. In 2009, Komori and Shirai [9] defined the weighted Morrey space and they showed the boundedness of some classical integral operators and their commutators on the weighted Morrey spaces.

Next, we introduce the weighted central-Campanato space $\dot{C}^{p,\lambda}_w(\mathbb{R}^n)$ and weighted $\lambda$-central Morrey space $\dot{M}^{p,\lambda}_w(\mathbb{R}^n)$ respectively by the following norms.

$$\|f\|_{\dot{M}^{p,\lambda}_w(\mathbb{R}^n)} := \sup_{R > 0} \left( \frac{1}{\omega(B(0,R))^{1+\lambda p}} \int_{B(0,R)} |f(x)|^p \omega(x) dx \right)^{1/p}$$

and

$$\|f\|_{\dot{C}^{p,\lambda}_w(\mathbb{R}^n)} := \sup_{R > 0} \left( \frac{1}{\omega(B(0,R))^{1+\lambda p}} \int_{B(0,R)} |f(x) - f_{B,\omega(B(0,R))}|^p \omega(x) dx \right)^{1/p},$$

where the definition of $f_{B,\omega}$ is $f_{B,\omega} = \frac{1}{|B|} \int_B f(x) \omega(x) dx$ and the exponents of $p, \lambda$ are the same as in the definition of $\mathcal{C}^{p,\lambda}(\mathbb{R}^n)$ and $\dot{M}^{p,\lambda}(\mathbb{R}^n)$.

For the boundedness of integral operators on $\dot{M}^{p,\lambda}_w(\mathbb{R}^n)$ with $\lambda < 0$, one may see [18] et al. for more details.

Suppose that $\omega$ is a non-negative and locally integrable function. If for every cube $Q$, there exists a constant $D$ independent of $Q$, such that $\omega(2Q) \leq D \omega(Q)$. Then, we say $\omega$ satisfy the doubling condition and we simply denote $\omega \in \Delta_2$.

We would like to mention that in [18], the restriction of $\omega$ is $\omega \in A_p$ where $A_p$ denotes the Muckenhoupt weight classes (see [13]). From [6], we know that if $\omega \in A_p$, then $\omega$ satisfies the doubling condition (i.e. $\omega \in \Delta_2$). However, the converse is not true. Throughout this paper, we only assume that $\omega \in \Delta_2$.

Now, we are interested in the following weighted Hardy operator $\mathcal{H}_\omega$ with the weight $\omega \in \Delta_2$.

$$\mathcal{H}_\omega(f)(x) = \frac{1}{\omega(B(0,|x|))} \int_{|y| < |x|} f(y) \omega(y) dy.$$
For any $g \in L^1_{\text{loc}}(\omega)$ with $\omega \in \Delta_2$, we have

$$\langle \mathcal{H}_\omega(f)(x), g \rangle_{\omega} = \int_{\mathbb{R}^n} \frac{1}{\omega(B(0, |x|))} \int_{|y| < |x|} f(y) \omega(y) dy g(x) \omega(x) dx$$

$$= \int_{\mathbb{R}^n} \int_{|y| > |x|} \frac{g(x) \omega(x)}{\omega(B(0, |y|))} dx f(y) \omega(y) dy.$$ 

Thus, we can define the dual operator of $\mathcal{H}_\omega$ as

$$\mathcal{H}^*_\omega(g)(x) = \int_{|y| \geq |x|} \frac{g(y) \omega(y)}{\omega(B(0, |y|))} dy.$$ 

Then, the commutators of $\mathcal{H}_{\omega,b}$ and $\mathcal{H}^*_{\omega,b}$ can be stated as follows.

$$\mathcal{H}_{\omega,b}(f)(x) = b(x)\mathcal{H}_\omega(f)(x) - \mathcal{H}_\omega(fb)(x)$$

and

$$\mathcal{H}^*_{\omega,b}(f)(x) = b(x)\mathcal{H}^*_\omega(f)(x) - \mathcal{H}^*_\omega(fb)(x).$$

In this paper, we will give the boundedness of $\mathcal{H}_{\omega,b}$ and $\mathcal{H}^*_{\omega,b}$ on $\dot{\mathcal{M}}^{p,\lambda}(\mathbb{R}^n)$ with $\dot{C}^{p,\lambda}(\mathbb{R}^n)$ and $\lambda < 0$.

**Theorem 1.1.** Let $1 < p < \infty$, $-\frac{1}{p} < \lambda < 0$, $-\frac{1}{p_i} < \lambda_i < 0$ with $i = 1, 2$, $\frac{1}{p} = \sum_{i=1}^{2} \frac{1}{p_i}$ and $\lambda = \sum_{i=1}^{2} \lambda_i$. Then, both $\mathcal{H}_{\omega,b}$ and $\mathcal{H}^*_{\omega,b}$ are bounded from $\dot{\mathcal{M}}^{p,\lambda}_\omega(\mathbb{R}^n)$ to $\dot{\mathcal{M}}^{p,\lambda}_\omega(\mathbb{R}^n)$ with $b \in \dot{C}^{p,\lambda}(\mathbb{R}^n)$.

Moreover, we have

**Theorem 1.2.** Let $2 < p < \infty$ and $-\frac{1}{2p} < \lambda < 0$. Then, Both $\mathcal{H}_{\omega,b}$ and $\mathcal{H}^*_{\omega,b}$ are bounded from $\mathcal{M}^{p,\lambda}_\omega(\mathbb{R}^n)$ to $\mathcal{M}^{p,2\lambda}_\omega(\mathbb{R}^n)$ with $b \in \dot{C}^{p,\lambda}(\mathbb{R}^n)$.

In order to give the characterization of $\dot{C}^{p,\lambda}(\mathbb{R}^n)$, we introduce the operators $\mathcal{H}_{\omega,|b|}$ and $\mathcal{H}^*_{\omega,|b|}$ as follows.

$$\mathcal{H}_{\omega,|b|} = \frac{1}{\omega(B(0, |x|))} \int_{|y| < |x|} f(y)|b(x) - b(y)| \omega(y) dy$$

and

$$\mathcal{H}^*_{\omega,|b|}(f)(x) = \int_{|y| > |x|} \frac{f(y) \omega(y)}{\omega(B(0, |y|))} |b(x) - b(y)| dy.$$ 

By checking the proofs of Theorems 1.1 and 1.2, we know that the above two theorems still hold if we replace $\mathcal{H}_{\omega,b}$ by $\mathcal{H}_{\omega,|b|}$ and $\mathcal{H}^*_{\omega,b}$ by $\mathcal{H}^*_{\omega,|b|}$. Moreover, we have the following theorems.

**Theorem 1.3.** Let $1 < p < \infty$, $-\frac{1}{p} < \lambda < 0$, $-\frac{1}{p_i} < \lambda_i < 0$ with $i = 1, 2$, $\frac{1}{p} = \sum_{i=1}^{2} \frac{1}{p_i}$ and $\lambda = \sum_{i=1}^{2} \lambda_i$. Moreover, we assume that $b$ satisfies

$$\sup_{B(0,R) \ni x} |b(x) - b_{B,\omega}(B(0,R))| \leq \frac{C}{\omega(B(0,R))} \int_{B(0,R)} |b(x) - b_{B,\omega}(B(0,R))| \omega(x) dx,$$

for some constant $C > 0$. Then, the following two conditions are equivalent.

(a) Both $\mathcal{H}_{\omega,|b|}$ and $\mathcal{H}^*_{\omega,|b|}$ are bounded from $\dot{\mathcal{M}}^{p,\lambda}_\omega(\mathbb{R}^n)$ to $\dot{\mathcal{M}}^{p,\lambda}_\omega(\mathbb{R}^n)$.

(b) Both $\mathcal{H}_{\omega,|b|}$ and $\mathcal{H}^*_{\omega,|b|}$ are bounded from $\mathcal{M}^{p,\lambda}_\omega(\mathbb{R}^n)$ to $\mathcal{M}^{p,2\lambda}_\omega(\mathbb{R}^n)$. 
(b₄) \( b \in \mathcal{C}_w^{p,\lambda}(\mathbb{R}^n) \).

In order to give up the condition (2), we have

**Theorem 1.4.** Let \( 2 < p < \infty \) and \( -\frac{1}{p} < \lambda < 0 \). Then, the following two conditions are equivalent.

(a₄) Both \( \mathcal{H}_{\omega,b} \) and \( \mathcal{H}^*_{\omega,b} \) are bounded from \( \mathcal{M}^p_{\omega,\lambda}(\mathbb{R}^n) \) to \( \mathcal{M}^p_{\omega,2\lambda}(\mathbb{R}^n) \).

(b₄) \( b \in \mathcal{C}_w^{p,\lambda}(\mathbb{R}^n) \).

§2 Some useful lemmas.

In this section, we give some lemmas that will be used throughout this paper. For simplicity, we denote \( B = B(0,R) \), \( B_i = \{ x \in \mathbb{R}^n : |x| \leq 2^i \} \), \( C_i = B_i \setminus B_{i-1} \) with \( i \in \mathbb{Z} \) and \( C \) may represents different constants in different places.

**Lemma 2.1.** ([6]) If \( \omega \in \Delta_2 \), then there exists a constant \( D_1 : D_1 > 1 \) independent of \( Q \) such that \( \omega(2Q) \leq D_1 \omega(Q) \), for any cube \( Q \).

**Lemma 2.2.** ([9, Lemma 4.1]) If \( \omega \in \Delta_2 \), then there exists a constant \( D_2 : D_2 > 1 \) independent of \( Q \) such that \( \omega(2Q) \geq D_2 \omega(Q) \).

**Remark 2.1.** By checking the proof of [9, Lemma 4.1], we know that the constant \( D_2 \) is strictly less than 2. Moreover, there exists a constant \( r > 1 \) independent of \( Q \), such that \( r \leq D_2 < 2 \).

**Lemma 2.3.** For \( \forall i, j \in \mathbb{Z} \) and \( \forall \omega \in \Delta_2 \), we have the following inequalities.

(i) If \( i \geq j \), there is \( D_2^{i-j} \omega(B_j) \leq \omega(B_i) \leq D_1^{i-j} \omega(B_j) \).

(ii) For the case \( i \leq j \), we have \( D_1^{i-j} \omega(B_j) \leq \omega(B_i) \leq D_2^{i-j} \omega(B_j) \).

**Proof.** Lemma 2.3 is a simple derivation of Lemmas 2.1 and 2.2 and we omit the proof process.

**Lemma 2.4.** For \( \forall \omega \in \Delta_2 \) and \( x \in C_i \) with \( i \in \mathbb{Z}^+ \), there is \( \frac{1}{D_1} \omega(B_i) \leq \omega(B(0,|x|)) \leq \omega(B_i) \).

**Proof.** As \( i \in \mathbb{Z} \) and \( x \in C_i \), it is easy to see \( \omega(B_{i-1}) \leq \omega(B(0,|x|)) \leq \omega(B_i) \). Since \( \omega \in \Delta_2 \), we get \( \frac{1}{D_1} \omega(B_i) \leq \omega(B_{i-1}) \).

Thus, we obtain

\[
\frac{1}{D_1} \omega(B_i) \leq \omega(B(0,|x|)) \leq \omega(B_i).
\]

**Lemma 2.5.** Let \( 1 < p < \infty \), \( -\frac{1}{p} < \lambda < 0 \) and \( b \in \mathcal{C}_w^{p,\lambda}(\mathbb{R}^n) \). Then, for any \( i, j \in \mathbb{Z} \) with \( i < j \) and \( \forall \omega \in \Delta_2 \), we have

\[
|b_{B_i,\omega} - b_{B_j,\omega}| \leq C \|b\|_{\mathcal{C}_w^{p,\lambda}(\mathbb{R}^n)} \omega(B_i)^\lambda.
\]
Proof. As \( i < j \), using Lemma 2.3 and the Hölder inequality with \( 1 < p < \infty \) and \(-\frac{1}{p} < \lambda < 0\), we obtain

\[
|b_{B_i, \omega} - b_{B_j, \omega}| \leq \sum_{k=i}^{j-1} |b_{B_k, \omega} - b_{B_{k+1}, \omega}|
\]

\[
\leq \sum_{k=i}^{j-1} \frac{1}{\omega(B_k)} \int_{B_k} |b(x) - b_{B_{k+1}, \omega}| \omega(x) dx
\]

\[
\leq \sum_{k=i}^{j-1} \frac{1}{\omega(B_k)} \left( \int_{B_{k+1}} |b(x) - b_{B_{k+1}, \omega}|^p \omega(x) dx \right)^{\frac{1}{p}} \left( \int_{B_{k+1}} \omega(x) dx \right)^{\frac{1}{p}}
\]

\[
\leq C \sum_{k=i}^{j-1} \frac{1}{\omega(B_k)} (\omega(B_{k+1}))^{\lambda + \frac{1}{p}} ||b||_{\mathcal{M}^p_\lambda(\mathbb{R}^n)} (\omega(B_{k+1}))^{\frac{1}{p}}
\]

\[
\leq CD_1 ||b||_{\mathcal{M}^p_\lambda(\mathbb{R}^n)} \omega(B_i)^{\lambda} \sum_{k=i}^{j-1} D_2^{(k+1-i)\lambda}
\]

\[
\leq CD_1 ||b||_{\mathcal{M}^p_\lambda(\mathbb{R}^n)} \omega(B_i)^\lambda,
\]

here \( C \) is a positive constant independent of \( j \) and \( i \).

**Lemma 2.6.** Let \( 1 < p < \infty, -\frac{1}{p} < \lambda < 0 \) and \( f \in L^p_{\omega}(\mathbb{R}^n) \) with \( \omega \in \Delta_2 \). If \( \int_{B_k} |f(x)|^p \omega(x) dx \leq C \omega(B_k)^{1+\lambda p} \) holds for any \( k \in \mathbb{Z} \), then for any \( R > 0 \), we have

\[
\int_{B(0, R)} |f(x)|^p \omega(x) dx \leq C \omega(B(0, R))^{\lambda p + 1}.
\]

Proof. For any \( R > 0 \), there exists a \( k \in \mathbb{Z} \), such that \( 2^{k-1} < R \leq 2^k \). Thus, we get \( B_{k-1} \subset B(0, R) \subset B_k \) and \( \omega(B_{k-1}) \leq \omega(B(0, R)) \leq \omega(B_k) \). Then, using Lemmas 2.3-2.4, we get

\[
\int_{B(0, R)} |f(x)|^p \omega(x) dx \leq \int_{B_k} |f(x)|^p \omega(x) dx
\]

\[
\leq C \omega(B_k)^{1+\lambda p} \leq C (D_1 \omega(B_{k-1}))^{\lambda p + 1}
\]

\[
\leq C \omega(B(0, R))^{\lambda p + 1}.
\]

**Lemma 2.7.** Let \( 1 < p < \infty \) and \(-\frac{1}{p} < \lambda < 0\). Then for \( \forall R > 0 \) and \( \forall \omega \in \Delta_2 \), there is

\[
||\chi_B||_{\mathcal{M}^p_\lambda(\mathbb{R}^n)} \leq \omega(B)^{-\lambda}.
\]

Proof. By the definition of \( \mathcal{M}^p_\lambda(\mathbb{R}^n) \), we have

\[
||\chi_B||_{\mathcal{M}^p_\lambda(\mathbb{R}^n)} = \sup_{r > 0} \left( \frac{1}{(\omega(B(0, r)))^{1+\lambda p}} \int_{B(0, r)} |\chi_B(x)|^p \omega(x) dx \right)^{1/p}
\]

\[
= \sup_{r > 0} \left( \frac{1}{(\omega(B(0, r)))^{1+\lambda p}} \int_{B(0, r) \cap B(0, r)} \omega(x) dx \right)^{1/p},
\]
For the case $r \leq R$, as since $1 < p < \infty$ and $-\frac{1}{p} < \lambda < 0$, there is
\[
\left( \frac{1}{(\omega(B(0, r))^{1+\lambda p}} \int_{B(0,r) \cap B(0,R)} \omega(x)dx \right)^{1/p} = \left( \frac{1}{(\omega(B(0, r))^{1+\lambda p}} \int_{B(0,r)} \omega(x)dx \right)^{1/p}
\]
\[
= \left( \frac{1}{\omega(B(0, r))^{1}} \right)^{1/p} \leq \left( \frac{1}{\omega(B(0, R))^{1}} \right)^{1/p} = \omega(B)^{-\lambda},
\]

For the case $r > R$, we get
\[
\left( \frac{1}{(\omega(B(0, r))^{1+\lambda p}} \int_{B(0,r) \cap B(0,R)} \omega(x)dx \right)^{1/p} = \left( \frac{1}{(\omega(B(0, r))^{1+\lambda p}} \int_{B(0,R)} \omega(x)dx \right)^{1/p}
\]
\[
\leq \left( \frac{1}{\omega(B(0, R))^{1+\lambda p}} \int_{B(0,R)} \omega(x)dx \right)^{1/p} = \left( \frac{1}{\omega(B(0, R))^{1}} \right)^{1/p} = \omega(B)^{-\lambda},
\]

Thus, for any $r > 0$, we have
\[
\sup_{r > 0} \left( \frac{1}{(\omega(B(0, r))^{1+\lambda p}} \int_{B(0,r) \cap B(0,R)} \omega(x)dx \right)^{1/p} \leq \omega^{-\lambda}(B),
\]
which finishes the proof of Lemma 2.7.

§3 Proof of Theorem 1.1.

By the definitions of $\hat{M}_\omega^{p,\lambda}(\mathbb{R}^n)$ and Lemma 2.6, it suffices to show the following estimates with $k \in \mathbb{Z}$.

\[
\int_{B_k} |\mathcal{H}_{\omega,b}(f)(x)|^p \omega(x)dx \leq C\omega(B_k)^{1+\lambda p} ||b||_{C^{p+1,1}(\mathbb{R}^n)}^p ||f||_{\hat{M}_\omega^{p,\lambda}(\mathbb{R}^n)}^p
\tag{3}
\]
\[
\int_{B_k} |\mathcal{H}_{\omega,b}^*(f)(x)|^p \omega(x)dx \leq C\omega(B_k)^{1+\lambda p} ||b||_{C^{p+1,1}(\mathbb{R}^n)}^p ||f||_{\hat{M}_\omega^{p,\lambda}(\mathbb{R}^n)}^p.
\tag{4}
\]

We begin with the proof of (3). Using Lemma 2.4, we get
\[
\int_{B_k} (\mathcal{H}_{\omega,b}(f)(x))^p \omega(x)dx
\]
\[
\leq \sum_{j=-\infty}^k \int_{C_j} \left( \frac{1}{\omega(B(0,|x|))} \int_{B(0,|x|)} |b(x) - b(y)||f(y)|\omega(y)dy \right)^p \omega(x)dx
\]
\[
\leq \sum_{j=-\infty}^k \frac{1}{\omega(B_j)^p} \int_{B_j} \left( \int_{B_j} |b(x) - b(y)||f(y)|\omega(y)dy \right)^p \omega(x)dx
\]
\[
\leq \sum_{j=-\infty}^k \frac{1}{\omega(B_j)^p} \int_{B_j} \left( \sum_{i=-\infty}^j \int_{B_i} |b(x) - b(y)||f(y)|\omega(y)dy \right)^p \omega(x)dx
\]
\[
\leq \sum_{j=-\infty}^k \frac{1}{\omega(B_j)^p} \int_{B_j} \left( \sum_{i=-\infty}^j \int_{B_i} |b(x) - b_{B_j,B_i}||f(y)|\omega(y)dy \right)^p \omega(x)dx
\]
\[
+ \sum_{j=-\infty}^k \frac{1}{\omega(B_j)^p} \int_{B_j} \left( \sum_{i=-\infty}^j \int_{B_i} |b(y) - b_{B_j,B_i}||f(y)|\omega(y)dy \right)^p \omega(x)dx
\]
\[
=: I_1 + I_2.
\]

For \(I_1\), we prove the fact.
\[
\sum_{i=-\infty}^j \int_{B_i} |f(y)|\omega(y)dy \leq C\|f\|_{M^{p_2,\lambda_2}(\mathbb{R}^n)} (\omega(B_j))^{\lambda_2+1}
\]  
with \(1 < p_2 < \infty\) and \(-1 < \lambda_2 < 0\).

By Lemma 2.3 and the Hölder inequality, there is
\[
\sum_{i=-\infty}^j \int_{B_i} |f(y)|\omega(y)dy
\]
\[
\leq C \sum_{i=-\infty}^j \left( \int_{B_i} (|f(y)|\omega^{p_2}(y))^{p_2} dy \right)^{\frac{1}{p_2}} \left( \int_{B_i} (\omega(y))^{(1-\frac{1}{p_2})p_2} dy \right)^{1/(\frac{1}{p_2})}
\]
\[
\leq C\|f\|_{M^{p_2,\lambda_2}(\mathbb{R}^n)} \sum_{i=-\infty}^j (\omega(B_i))^{\lambda_2+\frac{1}{p_2}} (\omega(B_i))^{(1-\frac{1}{p_2})}
\]
\[
\leq C\|f\|_{M^{p_2,\lambda_2}(\mathbb{R}^n)} \sum_{i=-\infty}^j (\omega(B_i))^{\lambda_2+1}
\]
\[
\leq C\|f\|_{M^{p_2,\lambda_2}(\mathbb{R}^n)} \sum_{i=-\infty}^j (D_2(i-j)\omega(B_j))^{\lambda_2+1}
\]
\[
\leq C\|f\|_{M^{p_2,\lambda_2}(\mathbb{R}^n)} (\omega(B_j))^{\lambda_2+1}
\]
and we finish the proof of (5).

Then, using Lemma 2.3, the Hölder inequality and the conditions of Theorem 1.1, we get
where \( x \) and we finish the proof of (6). For \( s \) Denote \[ \sum_{j=-\infty}^\infty |b(y) - b_{B_j, \omega}| \omega(y) dy \leq C \|b\|_{\mathcal{C}^{p_1, 1}}(\mathbb{R}^n) \|f\|_{\mathcal{M}^{p_2, 2}}(\mathbb{R}^n) \omega(B_j)^{1+\lambda}. \] where \( x \in C_j, 1 < p_1 < \infty, -\frac{1}{p_1} < \lambda_i < 0 \) with \( i = 1, 2 \) and \( \lambda = \sum_{i=1}^2 \lambda_i. \)

Denote \( s \) by \( 1/s = 1 - 1/p_1 - 1/p_2. \) Using Lemma 2.3 and the Hölder inequality, we get

\[
\sum_{i=-\infty}^j \int_{B_i} |b(y) - b_{B_i, \omega}| |f(y)| \omega(y) dy \leq C \|b\|_{\mathcal{C}^{p_1, 1}}(\mathbb{R}^n) \|f\|_{\mathcal{M}^{p_2, 2}}(\mathbb{R}^n) \omega(B_j)^{1+\lambda}. \tag{6}
\]
Thus, we obtain

\[
I_2 \leq C \sum_{j=-\infty}^{k} \int_{B_j} \frac{1}{\omega(B_j)^p} \left( \sum_{i=-\infty}^{j} \int_{B_i} |b(y) - b_{B_j, \omega}| |f(y)| \omega(y)dy \right)^p \omega(x)dx
\]

\[
\leq C \|b\|_{L_p, \lambda_1(\mathbb{R}^n)}^p \|f\|_{M^{p, \lambda_2}(\mathbb{R}^n)}^p \sum_{j=-\infty}^{k} \int_{B_j} \frac{1}{\omega(B_j)^{1+\lambda p}} \omega(B_j)^{1+\lambda p} \omega(x)dx
\]

\[
\leq \|b\|_{L_p, \lambda_1(\mathbb{R}^n)}^p \|f\|_{M^{p, \lambda_2}(\mathbb{R}^n)}^p \omega(B_k)^{1+\lambda p} \sum_{j=-\infty}^{k} D_2(b, \lambda)^{(1+\lambda p)}
\]

\[
\leq C \|b\|_{L_p, \lambda_1(\mathbb{R}^n)}^p \|f\|_{M^{p, \lambda_2}(\mathbb{R}^n)}^p \omega(B_k)^{1+\lambda p}.
\]

Combing the estimates of \( I_1 \) and \( I_2 \), we finish the proof of (3).

Next, we will prove (4). First, we can decompose \( \int_{B_k} |\mathcal{H}_{\omega, b}(f)(x)|^p \omega(x)dx \) as follows.

\[
\int_{B_k} |\mathcal{H}_{\omega, b}(f)(x)|^p \omega(x)dx = \int_{B_k} \left( \int_{|y| \geq |x|} \frac{|b(x) - b(y)|}{\omega(B(0, |y|))} |f(y)| \omega(y)dy \right)^p \omega(x)dx
\]

\[
\leq \int_{B_k} \left( \int_{2^k \geq |y| \geq |x|} \frac{|b(x) - b(y)|}{\omega(B(0, |y|))} |f(y)| \omega(y)dy \right)^p \omega(x)dx
\]

\[
+ \int_{B_k} \left( \int_{|y| > 2^k} \frac{|b(x) - b(y)|}{\omega(B(0, |y|))} |f(y)| \omega(y)dy \right)^p \omega(x)dx
\]

\[
= J_1 + J_2.
\]

Similar to the estimates of \( \mathcal{H}_{\omega, b} \), we have

\[
J_1 \leq \int_{B_k} \left( \int_{|y| \leq 2^k} \frac{|b(x) - b(y)|}{\omega(B(0, |y|))} |f(y)| \omega(y)dy \right)^p \omega(x)dx
\]

\[
\leq \int_{B_k} \left( \sum_{i=-\infty}^{k} \int_{B_i} \frac{|b(x) - b(y)|}{\omega(B(0, |y|))} |f(y)| \omega(y)dy \right)^p \omega(x)dx
\]

\[
\leq \int_{B_k} \left( \sum_{i=-\infty}^{k} \frac{1}{\omega(B_i)} \int_{B_i} |b(x) - b(y)| |f(y)| \omega(y)dy \right)^p \omega(x)dx
\]

\[
\leq \|b\|_{L_p, \lambda_1(\mathbb{R}^n)}^p \|f\|_{M^{p, \lambda_2}(\mathbb{R}^n)}^p \omega(B_k)^{1+\lambda p}.
\]
Thus, it remains to give the estimates of $J_2$. By Lemma 2.4, we can decompose $J_2$ as

$$J_2 \leq C \int_{B_k} \left( \sum_{i=k}^{\infty} \frac{1}{\omega(B_i)} \int_{C_i} |b(x) - b(y)||f(y)| \omega(y) dy \right)^p \omega(x) dx$$

$$\leq C \int_{B_k} \left( \sum_{i=k}^{\infty} \frac{1}{\omega(B_i)} \int_{B_i} |b(x) - b(B_k, \omega)||f(y)| \omega(y) dy \right)^p \omega(x) dx$$

$$+ C \int_{B_k} \left( \sum_{i=k}^{\infty} \frac{1}{\omega(B_i)} \int_{B_i} |b(y) - b(B_i, \omega)||f(y)| \omega(y) dy \right)^p \omega(x) dx$$

$$+ C \int_{B_k} \left( \sum_{i=k}^{\infty} \frac{1}{\omega(B_i)} \int_{B_i} |b(B_i, \omega) - b(B_k, \omega)||f(y)| \omega(y) dy \right)^p \omega(x) dx$$

$$=: CJ_{21} + CJ_{22} + CJ_{23}.$$ 

For $J_{21}$, we show the following fact.

$$\sum_{i=k}^{\infty} \frac{1}{\omega(B_i)} \int_{B_i} |f(y)| \omega(y) dy \leq C \| f \|_{\mathcal{M}^p_{\lambda_2}(\mathbb{R}^n)} (\omega(B_k))^{\lambda_2}$$

(7)

with $y \in B_k$, $1 < p_2 < \infty$ and $\lambda_2 < 0$.

Then, using Lemma 2.3 and the Hölder inequality, we get

$$\sum_{i=k}^{\infty} \frac{1}{\omega(B_i)} \int_{B_i} |f(y)| \omega(y) dy \leq C \| f \|_{\mathcal{M}^p_{\lambda_2}(\mathbb{R}^n)} \sum_{i=k}^{\infty} \omega(B_i)^{\lambda_2}$$

$$\leq C \| f \|_{\mathcal{M}^p_{\lambda_2}(\mathbb{R}^n)} \sum_{i=k}^{\infty} \left( D_2^{i-k} \omega(B_k) \right)^{\lambda_2}$$

$$\leq C \| f \|_{\mathcal{M}^p_{\lambda_2}(\mathbb{R}^n)} (\omega(B_k))^{\lambda_2}$$

and we finish the proof of (7).

Using Lemma 2.3, the Hölder inequality and the conditions of Theorem 1.1, we have
\[ J_{21} \leq \int_{B_k} |b(x) - b_{B_k, \omega}|^p \omega(x) dx \left( \sum_{i=k}^{\infty} \frac{1}{\omega(B_i)} \int_{B_i} \omega(y) |f(y)| dy \right)^{\frac{1}{p}} \]
\[ \leq C \left( \int_{B_k} |b(x) - b_{B_k, \omega}|^p \omega(x) dx \right)^{\frac{p}{p_1}} \left( \int_{B_k} \omega(x) (1 - \frac{1}{p}) \left( \frac{\omega(y)}{\omega(B_k)} \right)^{\frac{1}{1+p}} dx \right)^{\frac{1}{p_1}} \]
\[ \times \left( \| f \|_{M^{\mathcal{P}_1, \mathcal{L}_2}(\mathbb{R}^n)} (\omega(B_k))^\lambda \right)^p \]
\[ \leq C \| b \|_{c^{P_1, \lambda_1}(\mathbb{R}^n)}^p \| f \|_{M^{\mathcal{P}_1, \mathcal{L}_2}(\mathbb{R}^n)} \omega(B_k)^{1+\lambda_1} \left( \omega(B_k) \right)^\lambda \]
\[ \leq C \| b \|_{c^{P_1, \lambda_1}(\mathbb{R}^n)}^p \| f \|_{M^{\mathcal{P}_1, \mathcal{L}_2}(\mathbb{R}^n)} \omega(B_k)^{1+\lambda}. \]  

For \( J_{22} \), we need to show the following inequality.
\[ \sum_{i=k}^{\infty} \frac{1}{\omega(B_i)} \int_{B_i} |b(y) - b_{B_i, \omega}| |f(y)| \omega(y) dy \leq C \| b \|_{c^{P_1, \lambda_1}(\mathbb{R}^n)} \| f \|_{M^{\mathcal{P}_1, \mathcal{L}_2}(\mathbb{R}^n)} (\omega(B_k))^\lambda. \]  

with \( x \in B_k, 1 < p_i < \infty, -\frac{1}{p_i} < \lambda_i < 0 \) and \( \lambda = \sum_{i=1}^{n} \lambda_i \).

Denote \( s \) by \( 1/s = 1 - 1/p_1 - 1/p_2 \). Then, using Lemma 2.3 and the Hölder inequality, we get

\[ \sum_{i=k}^{\infty} \frac{1}{\omega(B_i)} \int_{B_i} |b(y) - b_{B_i, \omega}| |f(y)| \omega(y) dy \]
\[ \leq C \sum_{i=k}^{\infty} \frac{1}{\omega(B_i)} \left( \int_{B_i} |b(y) - b_{B_i, \omega}|^p \omega(y) dy \right)^{1/p_1} \left( \int_{B_i} |f(y)|^p \omega(y) dy \right)^{1/p_2} \left( \int_{B_i} \omega(y)^{\frac{1}{s}} dy \right)^{1/s} \]
\[ \leq C \| b \|_{c^{P_1, \lambda_1}(\mathbb{R}^n)} \| f \|_{M^{\mathcal{P}_1, \mathcal{L}_2}(\mathbb{R}^n)} \sum_{i=k}^{\infty} \omega(B_i)^\lambda \]
\[ \leq C \| b \|_{c^{P_1, \lambda_1}(\mathbb{R}^n)} \| f \|_{M^{\mathcal{P}_1, \mathcal{L}_2}(\mathbb{R}^n)} \sum_{i=k}^{\infty} \left( D_2 (1 - k) \omega(B_k) \right)^\lambda \]
\[ \leq C \| b \|_{c^{P_1, \lambda_1}(\mathbb{R}^n)} \| f \|_{M^{\mathcal{P}_1, \mathcal{L}_2}(\mathbb{R}^n)} \omega(B_k)^\lambda \]

and we finish the proof of (8).

Thus, we obtain
\[ J_{22} = \int_{B_k} \left( \sum_{i=k}^{\infty} \frac{1}{\omega(B_i)} \int_{B_i} |b(y) - b_{B_i, \omega}| |f(y)| \omega(y) dy \right)^p \omega(x) dx \]
\[ \leq C \int_{B_k} \left( \| b \|_{c^{P_1, \lambda_1}(\mathbb{R}^n)} \| f \|_{M^{\mathcal{P}_1, \mathcal{L}_2}(\mathbb{R}^n)} \omega(B_k)^\lambda \right)^p \omega(x) dx \]
\[ \leq C \| b \|_{c^{P_1, \lambda_1}(\mathbb{R}^n)}^p \| f \|_{M^{\mathcal{P}_1, \mathcal{L}_2}(\mathbb{R}^n)} (\omega(B_k))^{1+\lambda}. \]
To estimate $J_{23}$, using the conditions of Theorem 1.1, Lemmas 2.3 and 2.5, we get

$$J_{23} \leq C\|b\|_{p, \lambda_1}^{p} \int_{B_k} \left( \sum_{i=k}^{\infty} \omega(B_i)^{-1} \int_{B_i} \omega(B_k)^{\lambda_1} |f(y)| |\omega(y)| dy \right)^{p} \omega(x) dx$$

$$\leq C\|b\|_{p, \lambda_1}^{p} \int_{B_k} \left( \sum_{i=k}^{\infty} \omega(B_i)^{-1} \omega(B_k)^{\lambda_1} \left( \int_{B_i} |f(y)|^{p_2} \omega(y) dy \right)^{1/p_2} \right)^{p} \omega(x) dx$$

$$\times \left( \int_{B_k} \omega(y)^{\left(1-\frac{1}{p_2}\right)\frac{p_2}{p_2-1}} dy \right)^{\frac{p_2}{p_2-1}}$$

$$\leq C\|b\|_{p, \lambda_1}^{p} \int_{B_k} \left( \sum_{i=k}^{\infty} \omega(B_i)^{-1} \omega(B_k)^{\lambda_1} \|f\|_{M_{p, 2}^{\lambda_2}(\mathbb{R}^n)} \omega(B_k)^{1+\lambda_2} \right)^{p} \omega(x) dx$$

$$\leq C\|b\|_{p, \lambda_1}^{p} \|f\|_{M_{p, 2}^{\lambda_2}(\mathbb{R}^n)} \int_{B_k} \omega(B_k)^{\lambda_1 p} \left( \sum_{i=k}^{\infty} \omega(B_i)^{\lambda_2} \right)^{p} \omega(x) dx$$

$$\leq C\|b\|_{p, \lambda_1}^{p} \|f\|_{M_{p, 2}^{\lambda_2}(\mathbb{R}^n)} \int_{B_k} \omega(B_k)^{\lambda_1 p} \left( \sum_{i=k}^{\infty} D_{p, 2}^{(i-k)\lambda_2} \omega(B_k)^{\lambda_2} \right)^{p} \omega(x) dx$$

$$\leq C\|b\|_{p, \lambda_1}^{p} \|f\|_{M_{p, 2}^{\lambda_2}(\mathbb{R}^n)} \omega(B_k)^{1+\lambda_2} \int_{B_k} \left( \sum_{i=k}^{\infty} D_{p, 2}^{(i-k)\lambda_2} \right)^{p} \omega(x) dx$$

$$\leq \|b\|_{p, \lambda_1}^{p} \|f\|_{M_{p, 2}^{\lambda_2}(\mathbb{R}^n)} \omega(B_k)^{1+\lambda_2}.$$

Combining the estimates of $J_{1}$, $J_{21}$, $J_{22}$ and $J_{23}$, we get (4) and finish the proof of Theorem 1.1.

§4 Proof of Theorem 1.2

By Lemma 2.6, it suffices to show that for any $k \in \mathbb{Z}$, the following estimates hold.

$$\int_{B_k} |\mathcal{H}_{\omega, b}(f)(x)|^{p} \omega(x) dx \leq C\omega(B_k)^{1+2\lambda p} \|b\|_{p, \lambda_1}^{p} \|f\|_{M_{p, 2}^{\lambda_2}(\mathbb{R}^n)}^{p},$$

$$\int_{B_k} |\mathcal{H}_{\omega, b}^{*}(f)(x)|^{p} \omega(x) dx \leq C\omega(B_k)^{1+2\lambda p} \|b\|_{p, \lambda_1}^{p} \|f\|_{M_{p, 2}^{\lambda_2}(\mathbb{R}^n)}^{p}.$$
To prove (9), using Lemma 2.4, we get
\[ \int_{B_k} |\mathcal{H}_{\omega,b} f(x)|^p \omega(x) dx \leq \int_{B_k} \left( \frac{1}{\omega(\{0,|x|\})} \int_{|y|<|x|} |b(x) - b(y)||f(y)|\omega(y) dy \right)^p \omega(x) dx \]
\[
\leq C \sum_{j=-\infty}^k \int_{C_j} \left( \frac{1}{\omega(\{0,|x|\})} \sum_{i=-\infty}^j \int_{B_i} |b(x) - b_{B_j}\omega||f(y)|\omega(y) dy \right)^p \omega(x) dx \]
\[+ C \sum_{j=-\infty}^k \int_{C_j} \left( \frac{1}{\omega(\{0,|x|\})} \sum_{i=-\infty}^j \int_{B_i} |b(y) - b_{B_j}\omega||f(y)|\omega(y) dy \right)^p \omega(x) dx \]
\[\leq C \sum_{j=-\infty}^k \int_{C_j} \frac{1}{\omega(B_j)^p} \left( \sum_{i=-\infty}^j \int_{B_i} |b(x) - b_{B_j}\omega||f(y)|\omega(y) dy \right) \omega(x) dx \]
\[+ C \sum_{j=-\infty}^k \int_{C_j} \frac{1}{\omega(B_j)^p} \left( \sum_{i=-\infty}^j \int_{B_i} |b(y) - b_{B_j}\omega||f(y)|\omega(y) dy \right) \omega(x) dx \]
\[=: CL_1 + CL_2. \]

For \(L_1\), recall that \(2 < p < \infty\) and \(-\frac{1}{2p} < \lambda < 0\). Moreover, in this case, \(p_2 = p\), \(\lambda_2 = \lambda\). Then, using (5) in Section 3, we have \(1 < p < \infty\) and \(-1 < \lambda < 0\). Thus, we obtain
\[\sum_{i=-\infty}^j \int_{B_i} |f(y)|\omega(y) dy \leq C \|f\|_{\mathcal{H}_{\omega,b}^p,\lambda,R^n}(\omega(B_j))^{\lambda+1}, \]

From Lemma 2.3 and the Hölder inequality, there is
\[L_1 \leq C \sum_{j=-\infty}^k \frac{1}{\omega(B_j)^p} \int_{B_j} |b(x) - b_{B_j}\omega|^p \omega(x) dx \left( \sum_{i=-\infty}^j \int_{B_i} |f(y)|\omega(y) dy \right)^p \]
\[\leq C \sum_{j=-\infty}^k \frac{1}{\omega(B_j)^p} \left( \omega(B_j)^{1+\lambda} \|b\|_{\mathcal{H}_{\omega,b}^{p,\lambda},R^n} \|f\|_{\mathcal{H}_{\omega,b}^{p,\lambda},R^n}(\omega(B_j))^{\lambda+1} \right)^p \]
\[\leq C \|b\|_{\mathcal{H}_{\omega,b}^{p,\lambda},R^n}^p \|f\|_{\mathcal{H}_{\omega,b}^{p,\lambda},R^n}^p \sum_{j=-\infty}^k (\omega(B_j))^{(1+2\lambda p)} \]
\[\leq C \|b\|_{\mathcal{H}_{\omega,b}^{p,\lambda},R^n}^p \|f\|_{\mathcal{H}_{\omega,b}^{p,\lambda},R^n}^p \sum_{j=-\infty}^k \left( D_2(j-k) \omega(B_k) \right)^{(1+2\lambda p)} \]
\[\leq C \|b\|_{\mathcal{H}_{\omega,b}^{p,\lambda},R^n}^p \|f\|_{\mathcal{H}_{\omega,b}^{p,\lambda},R^n}^p \omega(B_k)^{(1+2\lambda p)} \]

For \(L_2\), using (6) in Section 3 and it is easy to see
\[\sum_{i=-\infty}^j \int_{B_i} |b(y) - b_{B_j}\omega||f(y)|\omega(y) dy \leq C \|b\|_{\mathcal{H}_{\omega,b}^{p,\lambda},R^n} \|f\|_{\mathcal{H}_{\omega,b}^{p,\lambda},R^n}(\omega(B_j))^{1+2\lambda}. \]

Then, using Lemma 2.3, the Hölder inequality and the conditions of Theorem 1.2, we have
\[ L_2 = \sum_{j = -\infty}^{k} \int_{C_j} \frac{1}{\omega(B_j)^p} \left( \sum_{i = -\infty}^{j} \int_{B_i} |b(y) - b_{B_j, \omega}||f(y)||\omega(y)dy \right)^p \omega(x)dx \]

\[ \leq C\|b\|_{L_2}^p \|f\|_{L_2}^p \sum_{j = -\infty}^{k} \int_{C_j} \frac{1}{\omega(B_j)^p} (\omega(B_j)^{1+2\lambda})^p \omega(x)dx \]

\[ \leq C\|b\|_{L_2}^p \|f\|_{L_2}^p \sum_{j = -\infty}^{k} \omega(B_j)^{1+2\lambda_p} \]

\[ \leq C\|b\|_{L_2}^p \|f\|_{L_2}^p \omega(B_k)^{1+2\lambda_p} \]

Combining the estimates of \( L_1, L_2 \), we find that (9) is true.

Now, let us focus on the proof of (10). First, we decompose \( \int_{B_k} \|H_{\omega,b}(f)(x)\|_p \omega(x)dx \) as follows.

\[ \int_{B_k} \|H_{\omega,b}(f)(x)\|_p \omega(x)dx = \int_{B_k} \left( \int_{|y| \geq |x|} \frac{|b(x) - b(y)|}{\omega(B(0,|y|))} f(y) \omega(y)dy \right)^p \omega(x)dx \]

\[ \leq \int_{B_k} \left( \int_{2^k \geq |y| \geq |x|} \frac{|b(x) - b(y)|}{\omega(B(0,|y|))} f(y) \omega(y)dy \right)^p \omega(x)dx \]

\[ + \int_{B_k} \left( \int_{|y| > 2^k} \frac{|b(x) - b(y)|}{\omega(B(0,|y|))} f(y) \omega(y)dy \right)^p \omega(x)dx \]

\[ =: M_1 + M_2. \]

Similar to the estimates of \( H_{\omega,b} \), we have

\[ M_1 \leq \int_{B_k} \left( \int_{|y| \leq 2^k} \frac{|b(x) - b(y)|}{\omega(B(0,|y|))} f(y) \omega(y)dy \right)^p \omega(x)dx \]

\[ \leq \int_{B_k} \left( \sum_{i = -\infty}^{k} \int_{C_i} \frac{|b(x) - b(y)|}{\omega(B(0,|y|))} f(y) \omega(y)dy \right)^p \omega(x)dx \]

\[ \leq C \int_{B_k} \left( \sum_{i = -\infty}^{k} \frac{1}{\omega(B_i)} \int_{C_i} |b(x) - b(y)||f(y)||\omega(y)dy \right)^p \omega(x)dx \]

\[ \leq C\|b\|_{L_2}^p \|f\|_{L_2}^p \omega(B_k)^{1+2\lambda_p}. \]
Thus, it remains to give the estimates of $M_2$. Using Lemma 2.4, we can decompose $M_2$ as

$$M_2 \leq C \int_{B_k} \left( \sum_{i=k}^{\infty} \frac{1}{\omega(B_i)} \int_{B_i} |b(x) - b(y)| |f(y)| \omega(y)dy \right)^p \omega(x)dx$$

$$\leq C \int_{B_k} \left( \sum_{i=k}^{\infty} \frac{1}{\omega(B_i)} \int_{B_i} |b(x) - b_{B_k, \omega}| |f(y)| \omega(y)dy \right)^p \omega(x)dx$$

$$+ C \int_{B_k} \left( \sum_{i=k}^{\infty} \frac{1}{\omega(B_i)} \int_{B_i} |b(y) - b_{B_k, \omega}| |f(y)| \omega(y)dy \right)^p \omega(x)dx$$

$$+ C \int_{B_k} \left( \sum_{i=k}^{\infty} \frac{1}{\omega(B_i)} \int_{B_i} |b_{B_k, \omega} - b_{B_i, \omega}| |f(y)| \omega(y)dy \right)^p \omega(x)dx$$

$$=: CM_{21} + CM_{22} + CM_{23}.$$

For $M_{21}$, using (7) in Section 3 and the conditions of Theorem 1.2, we may easily get

$$\sum_{i=k}^{\infty} \frac{1}{\omega(B_i)} \int_{B_i} |f(y)| \omega(y)dy \leq \|f\|_{\mathcal{M}_{\omega}^{p, \lambda}(\mathbb{R}^n)} (\omega(B_k))^\lambda.$$

Then, from Lemma 2.3 and the Hölder inequality, we obtain

$$M_{21} \leq \int_{B_k} |b(x) - b_{B_k, \omega}| \omega(x)dx \left( \sum_{i=k}^{\infty} \frac{1}{\omega(B_i)} \int_{B_i} |f(y)| \omega(y)dy \right)^p$$

$$\leq \left( \omega(B_k)^{(\lambda + \frac{1}{p})} \|b\|_{\mathcal{M}_{\omega}^{p, \lambda}(\mathbb{R}^n)} \right)^p \left( \|f\|_{\mathcal{M}_{\omega}^{p, \lambda}(\mathbb{R}^n)} (\omega(B_k))^\lambda \right)^p$$

$$\leq C \|b\|_{\mathcal{M}_{\omega}^{p, \lambda}(\mathbb{R}^n)} \|f\|_{\mathcal{M}_{\omega}^{p, \lambda}(\mathbb{R}^n)} \omega(B_k)^{1+2\lambda p}.$$

For $M_{22}$, using (8) in Section 3 and we can easily obtain

$$\sum_{i=k}^{\infty} \frac{1}{\omega(B_i)} \int_{B_i} |b(y) - b_{B_i, \omega}| |f(y)| \omega(y)dy \leq C \|b\|_{\mathcal{M}_{\omega}^{p, \lambda}(\mathbb{R}^n)} \|f\|_{\mathcal{M}_{\omega}^{p, \lambda}(\mathbb{R}^n)} (\omega(B_k))^2\lambda.$$

Then, applying Lemma 2.3 and the Hölder inequality, there is

$$M_{22} \leq C \int_{B_k} \left( \|b\|_{\mathcal{M}_{\omega}^{p, \lambda}(\mathbb{R}^n)} \|f\|_{\mathcal{M}_{\omega}^{p, \lambda}(\mathbb{R}^n)} (\omega(B_k))^{2\lambda} \right)^p \omega(x)dx$$

$$\leq C \|b\|_{\mathcal{M}_{\omega}^{p, \lambda}(\mathbb{R}^n)}^p \|f\|_{\mathcal{M}_{\omega}^{p, \lambda}(\mathbb{R}^n)}^p \omega(B_k)^{1+2\lambda p}.$$

Finally, we give the estimates of $M_{23}$. Using Lemma 2.5, the fact $i \geq k$ and $\lambda < 0$, there is

$$|b_{B_k, \omega} - b_{B_i, \omega}| \leq C \|b\|_{\mathcal{M}_{\omega}^{p, \lambda}(\mathbb{R}^n)} \omega(B_k)^\lambda.$$
Thus, we have
\[
M_{23} \leq C \|b\|_{L^p_1(\mathbb{R}^n)}^p \omega(B_k)^{\lambda p} \int_{B_k} \left( \sum_{i=k}^{\infty} \frac{1}{\omega(B_i)} \int_{B_i} |f(y)| \omega(y) dy \right)^p \omega(x) dx \\
\leq C \|b\|_{L^p_1(\mathbb{R}^n)}^p \omega(B_k)^{\lambda p} \times \int_{B_k} \left( \sum_{i=k}^{\infty} \frac{1}{\omega(B_i)} \left( \int_{B_i} |f(y)|^p \omega(y) dy \right)^{1/p} \left( \int_{B_i} \omega(y)^{(1-\frac{1}{p'})} dy \right)^{1/p'} \right)^p \omega(x) dx \\
\leq C \|b\|_{L^p_1(\mathbb{R}^n)}^p \|f\|_{L^p_1(\mathbb{R}^n)}^p \omega(B_k)^{\lambda p} \int_{B_k} \left( \sum_{i=k}^{\infty} \omega(B_i)^{1} \right)^p \omega(x) dx \\
\leq C \|b\|_{L^p_1(\mathbb{R}^n)}^p \|f\|_{L^p_1(\mathbb{R}^n)}^p \omega(B_k)^{2\lambda p} \left( \sum_{i=k}^{\infty} D_2(i-k)^{\lambda} \right)^p \int_{B_k} \omega(x) dx \\
\leq C \|b\|_{L^p_1(\mathbb{R}^n)}^p \|f\|_{L^p_1(\mathbb{R}^n)}^p \omega(B_k)^{1+2\lambda p}.
\]

Combing the estimates of $M_1, M_{21}, M_{22}, M_{23}$, we complete the proof of (10) and the proof of Theorem 1.2 has been finished.

\section{Proof of Theorem 1.3.}

We just give the proof of $(a_3) \Rightarrow (b_3)$. From (2) and the Hölder inequality, we have
\[
\frac{1}{\omega(B)^{1+p_1 \lambda_1}} \int_B |b(y) - b_{B, \omega}|^{p_1} \omega(y) dy \leq \frac{C}{\omega(B)^{p_1 \lambda_1}} \sup_{y \in B} |b(y) - b_{B, \omega}|^{p_1} \\
\leq \frac{C}{\omega(B)^{p_1 \lambda_1}} \left( \frac{1}{\omega(B)} \int_B |b(y) - b_{B, \omega}| \omega(y) dy \right)^{p_1} \\
\leq \frac{C}{\omega(B)^{p_1 \lambda_1}} \left[ \frac{1}{\omega(B)} \left( \int_B |b(y) - b_{B, \omega}|^{p_1} \omega(y) dy \right)^{1/p} \left( \int_B \omega(y)^{(1-\frac{1}{p'})} dy \right)^{1/p'} \right]^{p_1} \\
\leq \frac{C}{\omega(B)^{p_1 \lambda_1}} \left[ \frac{1}{\omega(B)} \left( \int_B |b(y) - b_{B, \omega}|^{p_1} \omega(y) dy \omega(B)^{1/p'} \right)^{1/p} \right]^{p_1} \\
= \frac{C}{\omega(B)^{p_1 \lambda_1}} \left( \frac{1}{\omega(B)} \int_B |b(y) - b_{B, \omega}|^{p_1} \omega(y) dy \right)^{p_1/p}.
\]
As
\[
\int_B |b(y) - b_{B, \omega}|^p \omega(y)dy \\
= \int_B \left| b(y) - \frac{1}{\omega(B)} \int_B b(z) \omega(z) dz \right|^p \omega(y)dy \\
= \int_B \left| \frac{1}{\omega(y)} \int_B (b(y) - b(z)) \omega(z) dz \right|^p \omega(y)dy \\
\leq \frac{1}{\omega(B)^p} \int_B \left[ \int_B |b(y) - b(z)| \omega(z) dz \right]^p \omega(y)dy \\
\leq \frac{1}{\omega(B)^p} \int_B \left[ \int_{|z \in B : |z| < |y|} \chi_B(z)|b(y) - b(z)| \omega(z) dz \right]^p \omega(y)dy \\
+ \frac{1}{\omega(B)^p} \int_B \left[ \int_{|z \in B : |z| \geq |y|} \chi_B(z)|b(y) - b(z)| \omega(z) dz \right]^p \omega(y)dy \\
=: A + B.
\]

For \(A\), as \(y \in B\) implies \(\omega(B(0, |y|)) \leq \omega(B(0, r)) =: \omega(B)\), there is
\[
A = \frac{1}{\omega(B)^p} \int_B \omega(B(0, |y|))^p \left| \frac{1}{\omega(B(0, |y|))} \int_{|z \in B : |z| < |y|} \chi_B(z)|b(y) - b(z)| \omega(z) dz \right|^p \omega(y)dy \\
= \frac{1}{\omega(B)^p} \int_B \omega(B(0, |y|))^p |\mathcal{H}_{\omega, |b|}(\chi_B)(y)|^p \omega(y)dy \\
\leq \int_B |\mathcal{H}_{\omega, |b|}(\chi_B)(y)|^p \omega(y)dy \\
\leq \omega(B)^{1+\lambda p} \|\mathcal{H}_{\omega, |b|}(\chi_B)\|_{\mathcal{M}_{\omega}^{p, \lambda}}^p \\
\leq \omega(B)^{1+\lambda p} \|\chi_B\|_{\mathcal{M}_{\omega}^{p, \lambda}}^p \leq \omega(B)^{1+\lambda p}.
\]

For \(B\), as \(z \in B\) implies \(\omega(B(0, |z|)) \leq \omega(B(0, r)) =: \omega(B)\), we obtain
\[
B = \frac{1}{\omega(B)^p} \int_B \left[ \int_{z \in B : |z| \geq |y|} \chi_B(z)|b(y) - b(z)| \omega(z) \frac{\omega(B(0, |z|))}{\omega(B(0, |y|))} \omega(B(0, |z|)) dz \right]^p \omega(y)dy \\
\leq \int_B |\mathcal{H}_{\omega, |b|}^*(\chi_B)(y)|^p \omega(y)dy \\
\leq \omega(B)^{1+\lambda p} \|\mathcal{H}_{\omega, |b|}^*(\chi_B)\|_{\mathcal{M}_{\omega}^{p, \lambda}}^p \\
\leq \omega(B)^{1+\lambda p} \|\chi_B\|_{\mathcal{M}_{\omega}^{p, \lambda}}^p \leq \omega(B)^{1+\lambda p}.
\]

Combining the estimates of \(A\) and \(B\), it is easy to see
\[
\frac{1}{\omega(B)^{1+p_1 \lambda_1}} \int_B |b(y) - b_{B, \omega}|^p \omega(y)dy \leq C,
\]
which implies \(b \in \hat{C}^{p_1, \lambda_1}(\mathbb{R}^n)\).
§6 Proof of Theorem 1.4.

We just give the proof of (a_4) ⇒ (b_4). For any ball $B = B(0, R)$, we have
\[
\frac{1}{\omega(B)^{1+\lambda p}} \int_B |b(y) - b_B, \omega|^p \omega(y) dy
\]
\[
= \frac{1}{\omega(B)^{1+\lambda p}} \int_B \left| \frac{1}{\omega(B)} \int_B (b(y) - b(z)) \omega(z) dz \right|^p \omega(y) dy
\]
\[
\leq \frac{1}{\omega(B)^{1+\lambda p}} \int_B \left| \int_{B: |z| < |y|} (b(y) - b(z)) \omega(z) dz \right|^p \omega(y) dy
\]
\[
+ \frac{1}{\omega(B)^{1+\lambda p}} \int_B \left| \int_{B: |z| \geq |y|} (b(y) - b(z)) \omega(z) dz \right|^p \omega(y) dy =: G + H.
\]

For $G$, by the boundedness of $\mathcal{H}_{\omega, |B|}$ from $\dot{M}^{p, \lambda}_\omega(\mathbb{R}^n)$ to $\dot{M}^{p, 2\lambda}_\omega(\mathbb{R}^n)$, there is
\[
G \leq \frac{1}{\omega(B)^{1+\lambda p}} \int_B \omega(B(0, |y|))^p |\mathcal{H}_{\omega, |B|}(\chi_B)(z)|^p \omega(y) dy
\]
\[
\leq \frac{C}{\omega(B)^{1+\lambda p}} \int_B |\mathcal{H}_{\omega, |B|}(\chi_B)(z)|^p \omega(y) dy
\]
\[
\leq \frac{C}{\omega(B)^{1+\lambda p}} \|\mathcal{H}_{\omega, |B|}(\chi_B)(\cdot)\|_{\dot{M}^{p, 2\lambda}_\omega(\mathbb{R}^n)}^p \omega(B)^{p(1+2\lambda)}
\]
\[
\leq \omega(B)^{p\lambda} \|\chi_B\|_{\dot{M}^{p, \lambda}_\omega(\mathbb{R}^n)}^p \leq C.
\]

For $H$, using the fact $\mathcal{H}_{\omega, |B|}^*$ is from $\dot{M}^{p, \lambda}_\omega(\mathbb{R}^n)$ to $\dot{M}^{p, 2\lambda}_\omega(\mathbb{R}^n)$, we obtain
\[
H \leq \frac{1}{\omega(B)^{1+\lambda p}} \int_B \left| \int_{B: |z| \geq |y|} \omega(B(0, |z|)) \frac{|b(y) - b(z) \omega(z)|}{\omega(B(0, |z|))} dz \right|^p \omega(y) dy
\]
\[
\leq \frac{1}{\omega(B)^{1+\lambda p}} \int_B |\mathcal{H}_{\omega, |B|}^*(\chi_B)(z)|^p \omega(y) dy
\]
\[
\leq \frac{1}{\omega(B)^{1+\lambda p}} \|\mathcal{H}_{\omega, |B|}^*(\chi_B)(\cdot)\|_{\dot{M}^{p, 2\lambda}_\omega(\mathbb{R}^n)}^p \omega(B)^{p(1+2\lambda)}
\]
\[
\leq \omega(B)^{p\lambda} \|\chi_B\|_{\dot{M}^{p, \lambda}_\omega(\mathbb{R}^n)}^p \leq C.
\]

Consequently, the proof of Theorem 1.4 has been finished.

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