The distribution of tilt angles in newly born NSs: role of interior viscosity and magnetic field

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\textbf{ABSTRACT}

We study how the viscosity of neutron star (NS) matter affects the distribution of tilt angles ($\chi$) between the spin and magnetic axes in young pulsars. Under the hypothesis that the NS shape is determined by the magnetically-induced deformation, and that the toroidal component of the internal magnetic field exceeds the poloidal one, we show that the dissipation of precessional motions by bulk viscosity can naturally produce a bi-modal distribution of tilt angles, as observed in radio/$\gamma$-ray pulsars, with a low probability of achieving $\chi \sim (20^\circ - 70^\circ)$ if the interior B-field is $\sim (10^{11} - 10^{13})$ G and the birth spin period is $\sim 10 - 300$ ms. As a corollary of the model, the idea that the NS shape is solely determined by the poloidal magnetic field, or by the centrifugal deformation of the crust, is found to be inconsistent with the tilt angle distribution in young pulsars. When applied to the Crab pulsar, with $\chi \sim 45^\circ - 70^\circ$ and birth spin $\gtrsim 20$ ms, our model implies that: (i) the magnetically-induced ellipticity is $\epsilon_B \gtrsim 3 \times 10^{-6}$; (ii) the measured positive $\dot{\chi} \sim 3.6 \times 10^{-12}$ rad s$^{-1}$ requires an additional viscous process, acting on a timescale $\lesssim 10^4$ yrs. We interpret the latter as crust-core coupling via mutual friction in the superfluid NS interior. One critical implication of our model is a GW signal at (twice) the spin frequency of the NS, due to $\epsilon_B \sim 10^{-6}$. This could be detectable by Advanced LIGO/Virgo operating at design sensitivity.

\textbf{Key words:} --

\section{1 INTRODUCTION}

The interior structure of neutron stars (NS) can affect their rotational dynamics in measurable ways. Discontinuous exchanges of angular momentum between normal matter and the superfluid components in the crust and/or core can lead to sudden timing irregularities, \textit{i.e.} glitches. Anisotropic stresses in the NS crust/core cause small deviations from sphericity in the NS shape, typically measured in terms of the ellipticity $\epsilon$ (\textit{e.g.} Alpar & Pines 1985). In turn, such deviations induce a precessional motion if they are not perfectly aligned with the NS spin axis. The precessional motion is also sensitive to the dynamics of the superfluid interior, and to its coupling to the normal matter (\textit{e.g.}, Shaham 1977, Alpar & Sauls 1988, Sedrakian et al. 1999, Link 2003, 2006, Andersson et al. 2006).

The inertia associated to the dipole magnetic field provides a minimum “effective” ellipticity for a magnetized NS, $\epsilon_{\text{min}} \sim 10^{-13} B_{12}^2$ (Zanazzi & Lai 2015). Crustal deformations are limited by the maximum breaking strain of the crustal lattice, implying an ellipticity $\epsilon_c < 10^{-6}$ (Horowitz & Kadau 2009); however, for isolated NS, realistic sources of strain likely produce much smaller values of $\epsilon_c < 10^{-8}$, unless special circumstances occur (\textit{e.g.} Jones 2012 and references therein). A major source of anisotropic stress in the core is the NS magnetic field, which produces an ellipticity $\epsilon_B \sim 10^{-10} B_{13}^3$ for normal NS matter, or a factor $\sim 100$ times larger when protons in the core are superconducting (Baym et al. 1969, Easson & Pethick 1977, Cutler 2002). As such, magnetic deformations may even dominate a NS ellipticity provided that the core magnetic field is $\gtrsim 10^{13}$ G.

NS precession is hard to detect due to its small amplitude and long characteristic timescale (typically $\sim$ yrs). However, due to its diagnostic potential, it has been searched for decades: to date, its detection has been claimed in a couple of objects (Stairs et al. 2000, Kramer et al. 2006), plus a handful of additional candidates (Lyne et al. 2010), showing periodic modulations in timing properties well correlated with pulse profile changes. Their natural interpretation in terms of freebody precession has been challenged by detailed studies, that revealed a complex pattern in the periodic modulations not easily related to simple precession (Lyne et al. 2010). However Jones (2012) and, more extensively, Ashton et al. (2016) have revived the case for a precession interpretation of these objects.

\footnote{We will write $Q_n$ for a quantity $Q$ in units of $10^n$.}
Precessional motion might also be damped by viscosity in NS interiors, in particular early in a NS life, with possible implications for the distribution of tilt angles $\chi$, i.e. the angle between the spin and magnetic axes (Jones 1976). For a biaxial ellipsoid, depending on whether it is oblate or prolate, viscous dissipation will cause the spin and symmetry axes to become aligned or orthogonal, respectively (Mestel & Takhar 1972). Consequently, the tilt angle $\chi$ will also decay or grow on the viscous timescale, much shorter than the spindown time in young NS (Jones 1976).

Observationally, pulsar tilt angles have been studied by various authors. Tauris & Manchester (1998) first noted a preference for tilt angles to be either $\lesssim 40^\circ$ or $\gtrsim 80 - 90^\circ$, with fewer objects at intermediate values. They also found hints of alignment of the magnetic and spin axes, with an estimated timescale $\tau_{\text{align}} \sim 10^7$ yrs. The latter conclusion, with a somewhat shorter timescale, was reached by Young et al. (2010), while Rookyard et al. (2015a) found a similar bi-modality in the tilt angle distribution in a sample of young, gamma-ray emitting radio pulsars. The apparent alignment might be consistent with the effect of the electromagnetic torque (e.g., Goldreich 1970, Jones 1976). The lack of pulsars with intermediate tilt angles, and the abundance of small tilt angles even in pulsars with spindown ages $< \tau_{\text{align}}$ (Rookyard et al. 2015a,b), are very hard to reconcile with the hypothesis that they reflect a random tilt distribution at birth, as frequently assumed in the literature. Alternatively, the magnetic axis of the NS could be oriented close to the direction of its spin, soon after formation. This might be the case if, e.g., the NS fast rotation had an important influence on the helicity of the birth magnetic field (Braithwaite & Nordlund 2006), or if the NS magnetic field resulted from a dynamo in the proto-NS phase, during which differential rotation plays a key role (e.g., Braithwaite 2006). As the observed distribution of tilt angles is not consistent with either of these hypotheses, it appears likely that it rather reflects some evolutionary process. In particular, the bi-modalities seen in the young pulsars of Rookyard et al. (2015a) suggests that, along with long-term alignment, some faster process is also at work.

Recently, Lyne et al. (2013) measured an increase of the tilt angle in the Crab pulsar, at the rate $\dot{\chi} \sim 0.62^\circ$ per century. The latter was shown to be possibly consistent with freebody precession of the pulsar, for particular combinations of the NS ellipticity and tilt angle (Philippov et al. 2014, Zanazzi & Lai 2015). In this interpretation, the measured positive $\dot{\chi}$ is a transient effect, associated with half of the precession cycle, the secular term being an alignment driven by the magnetic dipole torque. An alternative explanation for the positive $\dot{\chi}$ of the Crab pulsar could be the dissipation of precession energy, if the NS shape is distorted into a prolate ellipsoid. This requires that the magnetic field in NS interiors is dominated by a toroidal component - such that precessional dynamics causes $\chi$ to grow over time - and that the magnetically-induced deformation dominates over crustal or other types of stress.

Motivated by these findings, we reconsider and expand (Sec. 2) the idea first proposed by Jones (1976), that viscous dissipation of freebody precession in newly born NS can produce large tilt angles at ages $\lesssim 10^3$ yrs. With respect to previous discussions of the subject we (i) update the microphysics description, including effects of a realistic NS EoS and a detailed treatment of fluid motions in the precessing NS core; (ii) consider the effect of shear viscosity at late times, when the core temperature is $\lesssim 10^5$ K and protons are superconducting; (iii) explore the implications of a wide range of initial conditions on the final tilt angle distribution, and show that a bi-modal distribution of tilt angles at early age - as is observed - may be expected (Sec. 3).

We then generalize our model to explain the measured positive value of $\chi$ in the Crab pulsar, and propose an interpretation (Sec. 4) in which viscous dissipation of precession energy is provided, in this object, by crust-core coupling via mutual friction. We then conclude (Sec. 5) that mutual friction might affect, on longer timescales, the tilt angle distribution in NS before alignment kicks in.

## 2 GENERAL SCENARIO

Our work is based on a number of observation- and theory-driven assumptions, which we briefly summarize in the following in order to clarify their validity and scope.

### 2.1 Observed distribution of tilt angles

Our starting point is provided by the following observational facts and their interpretation:

(i) The distribution of pulsar tilt angles is not consistent with a random distribution at birth (Tauris & Manchester 1998, Rookyard et al. 2015a). While several caveats might affect the estimated tilt angles (Rookyard et al. 2015b), we assume that the overabundance of low-tilt pulsars, and the paucity of intermediate-tilt ones, are real effects.

(ii) In the sample of young (spindown age $< 10^6$ yrs), $\gamma$-ray emitting radio pulsars of Rookyard et al. (2015a), truly orthogonal rotators ($\chi > 80^\circ$) are, if anything, over-represented with respect to a flat distribution. The striking feature, in this relatively small sample, is the lack of pulsars with $40^\circ \lesssim \chi \lesssim 80^\circ$: based on this, it seems reasonable to consider the tilt angle distribution of at least these young pulsars as double-peak, or bimodal.

(iii) Several authors have found hints of a long-term alignment in the pulsar population on a timescale $\sim 10^6 - 10^7$ yrs (Tauris & Manchester 1998, Weltevrede & Johnston 2008, Young et al. 2010). We assume that such effect is real, and that it can be accounted for by the electromagnetic torque.

(iv) Long-term alignment driven by the electromagnetic torque can explain, at least in part, the over-abundance of low tilt angles in the pulsar population. However, the fact that a similar effect is found in young radio pulsars, where orthogonal rotators are also relatively abundant, suggests that some other process might already be favoring either low or large tilt angles, on a significantly shorter timescale.

### 2.2 Model assumptions

Our proposed theoretical framework is summarized here:

(i) The alignment time due to the electromagnetic torque is $\gtrsim 10^6$ yrs, $\chi^2 = P/cos \chi > 0.7$ “at birth” (Jones 1976).

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2. The numerical value also depends on the NS magnetic dipole.
For plausible values of pulsar birth spins, this requires \( \chi \) to be very close to 90°. Then, either all NSs are born nearly orthogonal or their tilt angle grows rapidly to \( \approx 90° \), before alignment kicks in. We will focus on the latter idea, originally proposed by Jones (1976).

(ii) The tilt angle in the Crab pulsar is indeed measured to increase at a rate of \( \dot{\chi} \approx 0.62°/\text{century} \) (Lyne et al. 2013). This measurement could provide support to the above idea, although the interpretation of \( \dot{\chi} \) is not unique (Sec. 6).

(iii) Tilt angles grow quickly to \( \approx 90° \) by viscous damping of precession in an oblate rotator (Mestel & Takhar 1972, Jones 1976). For this to work, the NS must have nonspherical shape with the largest axis of inertia (almost) aligned to the magnetic axis. The latter requires that (a) the deviation from sphericity is dominated by the magnetic field and (b) the magnetic field in the NS interior is predominantly toroidal, distorting the NS into a prolate ellipsoid (e.g., Cutler 2002, Dall’Osso et al. 2009). Note that, if (b) is not met, magnetic stresses will produce an oblate ellipsoid, in which case viscous dissipation drives \( \chi \) towards zero.

(iv) A generic stable configuration for a NS magnetic field is that of a twisted-torus, in which a toroidal-poloidal B-field is contained in a torus-shaped region in the NS core, threaded by the large scale dipole (Braithwaite & Nordlund 2006). Stability arguments suggest that the toroidal component can exceed the poloidal one even by very large factors (e.g., Reisenegger 2009, Braithwaite 2009, Akgin et al. 2013), although, for barotropic equations of state, it has been proven that the opposite may also be true (Lander & Jones 2009, Lander 2013).

(v) Tilt angles at birth might be very small, if the mechanism that amplifies the NS magnetic field inherits the direction of its spin. This seems plausible, if the magnetic field in the NS core has the twisted-torus shape discussed above. We will focus on this case, aiming at explaining the bi-modal distribution found in the young pulsar sample of Rookyard et al. (2015a). At the end of Sec. 5, in the light of our results, we will also discuss the possibility that the SN explosion produces a wider range of tilt angles at birth.

(vi) The newly formed NS is completely fluid as long as its temperature is \( > T_{\text{cray}} \sim 4 \times 10^{10} \text{ K} \). At such temperatures, dissipation is dominated by bulk viscosity. Below \( T_{\text{cray}} \), the crust starts to form and new dissipative processes become possible: in this work, we will only consider bulk (and shear) viscosity in the NS fluid core, thus focusing on a “minimal dissipation” scenario. Possible effects of the crust will be discussed in Sec. 6, in relation to the Crab pulsar.

### 3 VISCOSITY OF NEUTRON STAR MATTER

In this section we discuss both bulk and shear viscosity in the NS fluid core, deriving expressions for the associated energy dissipation rate and the corresponding dissipation timescales (cf. Eq. 2). The effects of NS cooling and the role of baryon condensation will be discussed in Sec. 4.

In full generality, we can write the energy dissipation rate due to bulk viscosity as (Friedmann & Seregiou 2013)

\[
\dot{E}_{\text{diss}}^{(\text{bulk})} \equiv \int \left( \nabla \cdot \delta v \right)^2 = \omega^2 \int \zeta (\rho, T, x) \left[ \frac{\Delta \rho}{\rho} \right]^2 \, dV, \tag{7}
\]

where \( \zeta \) is the bulk viscosity coefficient and \( x \) the charged particle fraction. The second step derives from \( \nabla \cdot \delta v = i\omega \Delta \rho / \rho \) (Lindblom & Owen 2002), where \( \Delta \rho \) is the Lagrangian compression accompanying fluid motions: its maximum value is obtained when \( \Delta \rho \approx \delta \rho \), where \( \delta \rho \) is the non-spherical component of the density perturbation due to the NS spin (Mestel & Takhar 1972, Lander & Jones 2017). Later (sec. 6.1.3) we will discuss a general relation between \( \Delta \rho \) and \( \delta \rho \), identifying a regime in which they are approximately equal.

The corresponding expression for shear viscosity is

\[
\dot{E}_{\text{diss}}^{(\text{shear})} \equiv 2 \int \eta \delta \sigma_{ab} \delta \sigma_{ac} \, dV, \tag{8}
\]

where \( \delta \sigma_{ab} = \nabla_a \delta v_b + \nabla_b \delta v_a - \frac{2}{3} \delta_{ab} \nabla_c \delta v^c \), and \( \eta \) represents
the shear viscosity coefficient. To zeroth order, the ratio between the two dissipation rates is mostly determined by the ratio between $\zeta$ and $\eta$ (e.g., Cutler & Lindblom 1987): this will be discussed further in the next subsections.

### 3.1 Bulk viscosity

The coefficient $\zeta$ can be expressed in terms of fundamental physical properties of the NS (e.g., Lindblom & Owen 2002)

$$\zeta = \frac{\partial \rho}{\partial x} \left[ 1 + (\omega \tau_\beta) \right],$$

where $n$ is the baryon number density, $p$ the pressure,

$$\tau_\beta = \frac{6.9}{T_{10}} \left( \frac{\rho}{\rho_0} \right)^{2/3} \text{s}$$

is the $\beta$-reaction equilibrium timescale for pure npe matter (Reisenegger & Goldreich 1992), and $\rho_0 \approx 2.7 \times 10^{14} \text{ g cm}^{-3}$ is the nuclear saturation density.

#### 3.1.1 The $\beta$-equilibrium timescale

Expression (10) for pure npe matter neglects interactions among the baryons, that determine the NS EoS, and considers only the neutron branch of modified Urca reactions. In a more realistic model of NS matter, three factors contribute to increase the $\beta$-reaction rate, thus decreasing $\tau_\beta$ (cf. Dall’Oso & Stella 2017): a) the nuclear symmetry energy, $S_s(n)$, which describes baryon interactions at supra-nuclear density; b) the appearance of more particles, e.g., muons at density $\gtrsim 2.2 \times 10^{14} \text{ g cm}^{-3}$, which adds new channels for Urca reactions; c) the proton branches of all modified Urca reactions, that provide a non-negligible neutrino emissivity (e.g., Yakovlev et al. 2001).

The net effect of all this, for typical values of $S_s(n)$, is to give a $\beta$-equilibration timescale $\tau_\beta' \sim \tau_\beta/3$. From now on, we will use $\tau_\beta'$ and omit the prime.

#### 3.1.2 Bulk viscosity regimes

Defining the variable $z \equiv \omega \tau_\beta'$, Eq. (9) implies two regimes of $\zeta$ as a function of $z$, as sketched in Fig. 3 (left panel):

i) $z \ll 1$, “low frequency” limit: $\beta$-reactions are much slower than the perturbation and chemical equilibrium is maintained almost instantaneously during one oscillation. Deviations from equilibrium are thus tiny, and energy losses are small. Accordingly, bulk viscosity is weak: $\zeta \propto z$.

ii) $z \gg 1$, “high frequency” limit: $\beta$-reactions are much faster than the perturbation and, during one cycle, deviations from chemical equilibrium grow almost unperturbed. The effect of $\beta$-reactions builds up slowly, eventually damping the perturbation over a large number of cycles: $\zeta \propto z^{-3}$.

**Pure npe matter** – In the high-frequency limit we have$^5$

$$\zeta_{\text{high}} = 60 \frac{T_{10}^6 \rho^2}{\Omega^2}.$$  \hspace{1cm} (11)

$^5$ Treating the NS as a collection of non-interacting, fully degenerate, fermion gases.

The high-frequency limit holds as long as

$$\cos \chi \gtrsim \frac{P}{2\pi} \approx 0.023 \frac{\tau_\beta'}{T_{10}} \frac{\rho_0}{\rho} \frac{317}{\rho_0 \beta} \approx 0.023 \frac{\tau_\beta'}{T_{10}} \frac{\rho_0}{\rho} \frac{317}{\rho_0} \left( \frac{\rho_0}{\rho_0} \right)^{2/3}.$$  \hspace{1cm} (12)

For large ellipticities and millisecond spins, i.e., newborn magnetars, condition (12) is always met unless the tilt angle is $\approx \pi/2$. For smaller values of $\epsilon$ and longer spin periods, expected for most NS, (12) is only satisfied at sufficiently large angles and late times. Therefore, ordinary NS start their life in the “low-frequency” regime, switching to high-frequency as they cool.

**Realistic NS matter** – Adopting a more realistic EoS and chemical composition, the bulk viscosity coefficient (11) can increase by a factor $N \sim 1.5 - 4$ (e.g. Haensel et al. 2001, Dall’Oso & Stella 2017). Accounting for this factor, and further multiplying Eq. (11) by $z^2/(1 + z^2)$, we derive a general expression for $\zeta$ as a function of $z$, valid in any regime

$$\zeta = \frac{60N}{1 + z^2} \frac{T_{10}^6 \rho^2}{\Omega^2} \approx \frac{317N \rho_0^3}{1 + 5.3 \rho_0^3 \Omega^2 \cos^2 \chi} \left( \frac{\rho_0}{\rho_0} \right)^{2/3}.$$  \hspace{1cm} (13)

In the following, the EoS-dependence of the bulk viscosity coefficient will be simply parametrized by the value of $N$.

#### 3.1.3 Compressibility of fluid motions

We turn now to the relation between $\Delta \rho$ and $\delta \rho$, needed to calculate the energy dissipation rates (Eqs. 7, 8). Let us first recall that $\Delta \rho \equiv \rho_0 + \xi \cdot \nabla \rho$ and $\delta \rho \equiv - (\xi \cdot \nabla \rho)$, $\xi$ being the fluid displacement due to the perturbation. When fluid motions are adiabatic, $\nabla \cdot \xi = \Delta \rho \equiv 0$: thus, one obtains $\delta \rho = - \xi \cdot \nabla \rho$, which was used to calculate $\delta \rho$ (Mestel & Takhar 1972, Mestel et al. 1981).

It can be argued that, due to the periodically changing pressure in the fluid, a field of motions with the magnitude calculated in the adiabatic approximation will always be excited (e.g., Mestel & Takhar 1972, Jones 1976, Lander & Jones 2017). However, the compressibility of such motions will change with different physical regimes. In the limit of highly dissipative fluid motions, for example, the density fluctuation $\delta \rho$ will occur mostly through a fluid compression, giving $\delta \rho \approx \rho \nabla \cdot \xi$ and, thus, $\xi \cdot \nabla \rho \approx 0$. From this, we deduce $\Delta \rho \approx \rho \nabla \cdot \xi \approx \delta \rho$.

The relation between $\delta \rho$ and $\Delta \rho$ can thus be summarized as

(i) **Low frequency, $\omega \tau_\beta \ll 1$**: Because particle reactions are faster than the oscillation, density fluctuations are accompanied by strong bulk compression of the fluid. The relation $\Delta \rho \approx \delta \rho$ for highly dissipative motions can be used in (7).

This is the regime considered by Dall’Oso et al. (2009).

(ii) **High frequency, $\omega \tau_\beta \gg 1$**: When particle reactions are slower than the perturbation, fluid motions are almost adiabatic. Thus, $\Delta \rho < \delta \rho = - (\xi \cdot \nabla \rho)$, and the factor by which they differ will be determined by the ratio between the two relevant timescales. In this regime we will adopt the relation (Dall’Oso & Stella 2017) $\Delta \rho \approx \delta \rho (T_{10}/\tau_\beta)$, where $T_\rho = 2\pi/\omega$ is the precession period. Note that the ratio of timescales is a strongly decreasing function of time, since $T_{10}$ can only decrease (following the decrease of $\cos \chi$) while $\tau_\beta \propto T^{-6}$ is rapidly growing as the NS cools. Therefore, our expression describes the transition between the
Following these arguments, we derive the general relation for the low-frequency and high-frequency regime and, in the limit of a sufficiently low temperature, it tends to the condition $\Delta \rho \approx 0$ assumed by Lasky & Glampedakis (2016).

Following these arguments, we derive the general relation

$$\Delta \rho \approx \delta \rho \left[ \hat{\Theta}(T_p - \tau_\beta) + \frac{T_p}{\tau_\beta} \hat{\Theta}(\tau_\beta - T_p) \right] \equiv \delta \rho \tilde{G}(T_p, \tau_\beta),$$  \hspace{1cm} (14)

$\hat{\Theta}$ being the Heaviside function.

### 3.1.4 Density perturbation

Mestel & Takhar (1972) derived a general expression for the density perturbation associated to the freebody precession of an oblique, fluid rotator

$$\delta \rho(r, r, \Phi, \Omega, \chi) = \frac{1}{2} f(r) \tilde{K}(\Theta, \Phi, \Omega, \chi).$$ \hspace{1cm} (15)

The angular part $\tilde{K}$, in which angles are defined with respect to the magnetic pole, is a complicated function to be discussed later. The radial part is set uniquely by the NS EoS. Realistic NS EoS can be approximated by piecewise polynomials with index $n \approx 0.5 - 1$, stiffening towards the center (Read et al. 2009). We verified that the volume integral in Eq. (7) has a very weak dependence on $n$, slightly increasing for stiffer EoS. For simplicity, we will assume $n = 1$. The adimensional density profile is $\hat{\rho}(\xi) = \rho(\xi)/\rho_c$, where $\xi = r/\alpha$ is the radial coordinate, $\alpha = R_s/\pi$ and $\rho_c = M/(4\pi^2 \alpha^3)$ is the central density.

The density profile $\hat{\rho}(\xi)$ of a rotating polytrope can be expressed, with respect to its non-rotating counterpart $\hat{\rho}_0(\xi)$, in terms of the velocity parameter $v = \Omega^2/(2\pi \rho_c G)$ (Chandrasekhar 1933)

$$\hat{\rho}(\xi) = \hat{\rho}_0(\xi) + v \left[ \psi_0(\xi) + A_2 \psi_2(\xi) P_2(\cos \tilde{\Theta}) \right],$$ \hspace{1cm} (16)

where $P_2(\cos \tilde{\Theta})$ is a Legendre polynomial, and $\tilde{\Theta}$ the latitude with respect to the spin pole. The second term in square brackets is the required non-spherical part of the rotational perturbation. The function $\psi_2(\xi)$ can be calculated numerically following Chandrasekhar (1933), and $A_2 \approx -0.54833$.

Recently, Lander & Jones (2017) studied the same problem to a higher perturbative order, including the effects on the magnetic field structure in the fluid NS. The density perturbation that they derived is perfectly consistent with the one adopted here, both in amplitude, radial and angular dependence.

#### 3.1.5 The energy dissipation rate

Inserting Eqs. (13) and (14) in Eq. (7), we eventually derive the energy dissipation rate due to bulk viscosity

$$\dot{E}_{\text{diss}} = \frac{317N\alpha^3 A_2^2}{16\pi^2 G^2} \left( \frac{\rho_c}{\rho} \right)^{4/3} \Omega^6 T_\beta^2 \tilde{K}^2(\Theta, \Phi, \Omega, \chi) \sin \Theta,$$

$$\times \int_0^\pi d\xi \frac{\xi^2 \psi_2(\xi) \theta(\xi)^{1/3} \tilde{G}^2(T_p, \tau_\beta)}{1 + 5.3(\xi^2/\chi^2)(\rho_c/\rho)\cos^2 \xi \theta(\xi)^{1/3}}$$

$$\times \int d\Phi d\Theta \tilde{K}^2(\Theta, \Phi, \Omega, \chi) \sin \Theta,$$ \hspace{1cm} (17)

where Eqs. (13), (15), (16) were used. The angular integral, averaged over one precession period, is $2\pi/5 \sin^2 \chi (1 + 3\cos^2 \chi)$.

Expression (17) follows the energy dissipation rate as the NS switches from one regime of bulk viscosity to the other. The number 1 in the denominator of Eq. (17) corresponds to the low-frequency regime while the second term, which grows as the temperature drops (although $\cos \chi$ and $\Omega$ decrease), represents dissipation in the high-frequency limit. Note that Eq. (17) depends on the NS EoS through $N$, as well as through $\alpha$ and $\rho_c$ (and hence mass and radius), appearing both in the normalization and inside the integral.

Writing the denominator in Eq. (17) as $1 + A(\theta(\xi)^{4/3})$, we calculated the integral numerically for a wide range of values of $\log A$: results are shown in the right panel of Fig. 1.

For given NS parameters, and since $T(t)$ can be calculated independently (Sec. 3), our result gives the integral in Eq. (17) as a function of $\cos \chi$ and $\Omega$. Finally, the damping timescale $\tau_\beta$ is obtained by combining this expression for $\dot{E}_{\text{diss}}$ with Eq. (3), thus inheriting a dependence on $M$ and $R$, as well as on the parameter $N$. These results will be
used later (Sec. 3) to solve numerically Eq. (8), for specific choices of the NS parameters.
The asymptotes of Eq. (17) have the expressions
\[
\dot{E}_1 = \frac{9510^5 A_p^2}{10 G^2} \left( \frac{\rho_0}{\rho_n} \right)^{4/3} \Omega^6 R_\Lambda^3 \sin^2(\cos \chi \sin \chi)^2 (1 + 3 \cos^2 \chi) I_1 \\
\dot{E}_2 = \frac{3170^5 A_p^2}{10 G^2} \frac{\tau_0^6 \Omega^4}{10^4} \sin^2(1 + 3 \cos^2 \chi) I_3
\]
where \( I_1 \approx 17.6087 \), \( I_2 \approx 148.815 \) for the \( n = 1 \) polytrope.

3.2 Shear viscosity

The coefficient of shear viscosity in NS interiors was calculated in detail by Shternin & Yakovlev (2008). Unlike \( \zeta \), the coefficient \( \eta \) grows as the temperature drops, and is further increased by baryon condensation in the NS core. Therefore, since shear viscosity dominates later stages of the NS life, we will consider here only the expression for \( \eta \) in a regime in which protons are strongly superconducting while neutrons are still in a normal state (see Sec. 3).

\[
\eta \approx 10^{19} \left( \frac{\rho_0}{T_0^2} \right)^2 \text{erg cm}^{-1},
\]
where \( \rho_0 \) is the proton fraction in units of 0.1.

The corresponding energy dissipation rate is obtained by integrating Eq. (8). To this aim, we recall the discussion summarized in Eq. (14). The pressure/density fluctuations produced by free body precession are achieved, in the low-frequency limit (\( z < 1 \)), mostly via fluid compression. In the opposite regime, compression is very limited and density fluctuations must be achieved by an almost adiabatic fluid circulation (e.g., Mestel & Takhar 1972). The latter is the regime relevant here: we can thus assume an almost adiabatic fluid circulation. In this limit, the expression for \( \delta \sigma \) in Eq. (8) must be, to order of magnitude, \( \sim \omega^2 \delta \rho/\rho \), as it was for \( \nabla \cdot \delta \mathbf{v} \) in the opposite limit. We will thus write
\[
\dot{E}_{\text{diss}}^{(\text{visc})} \approx 2 \times 10^{-11} \frac{\sigma^2}{T_0^2} \int x_0 \delta \rho^2 dV \sim 10^{-7} \frac{\sigma^2}{T_0^2} \dot{E}_{\text{diss}}^{(\text{bulk})}.
\]
(20)

As a matter of fact, our substitution implies that the shear-to-bulk viscosity dissipation timescales ratio is mostly determined by the ratio \( \zeta/\eta \), a conclusion already obtained by, e.g., Cutler & Lindblom (1987) in a different context. A general conclusion from Eq. (20) is that shear viscosity can only affect the evolution of the tilt angle on timescales longer than \( 10^7 - 10^8 \) yrs, for temperatures \( T < 2 \times 10^8 \) K, ellipticities \( \epsilon_B < 10^{-4} \) and spin periods \( \gtrsim 10 \) ms. Therefore, shear viscosity cannot affect the growth of \( \chi \) before alignment driven by the electromagnetic torque kicks in.

4 SUPERFLUIDITY AND SUPERCONDUCTIVITY

The strong temperature dependence of viscous effects requires that we model the NS cooling in order to calculate the long-term evolution of the tilt angle. In particular, we must account for the transition to superfluidity of baryons in the NS core. In addition to the effect discussed in Sec. 3.2, superfluidity will reduce the rate of \( \beta \)-reactions, implying (1) a decrease of the bulk viscosity coefficient; (2) a decrease of the neutrino cooling rate, which will keep the NS hotter than it would be otherwise. Neutron superfluidity can also affect the precessional dynamics in significant ways (Shaham 1977, Sedrakian et al. 1999, Andersson et al. 2006). So, before proceeding further, we must specify the superfluid parameters that we assume, based on observational constraints derived from the cooling of the NS in Cas A (Page et al. 2011, Shternin et al. 2011).

(i) Neutron condensation (triplet state) occurs at a critical temperature \( T_{\text{on}} \approx (5 - 6) \times 10^9 \) K, which is reached at an age \( \sim 300 \) yrs. Given that our calculations will extend up to \( t \gtrsim 300 \) yrs, we will consistently neglect neutron superfluidity: in particular, this implies that the neutrons making up most of the NS will precess as a “normal” fluid.

(ii) The proton energy gap (singlet state), \( \Delta_p \approx (0.5 - 1) \) MeV, implies a critical temperature \( T_{\text{cp}} \approx (3.5 - 7) \times 10^9 \) K. Thus, proton superconductivity occurs early in a NS life, and will be included in our model.

Reduction of bulk viscosity – Haensel et al. (2001) provide analytical fits to numerical calculations of the reduction coefficient of bulk viscosity in NS matter due to baryon superfluidity. We will consider their case with superconducting protons and normal neutrons and, for definiteness, we will set \( T_{\text{cp}} = 5 \times 10^9 \) K (\( \Delta_p = 0.75 \) MeV). Writing \( T = T/T_{\text{cp}} \), the fitting formulae are \( b \)

\[
R_p^{(n)} = \frac{a^{5.5} + b^{3.5}}{2} \exp \left[ 3.245 - \sqrt{(3.245)^2 + v} \right] \\
R_p^{(p)} = c^5 \exp \left[ 5.033 - \sqrt{(5.033)^2 + (2v)^2} \right],
\]
(21)

for the neutron (\( n \)) and proton (\( p \)) branch of the modified-Urca reactions, respectively. The coefficients are

\[
a = 0.1863 + \sqrt{(0.8137)^2 + (0.1310v)^2} \\
b = 0.8137 + \sqrt{(0.8137)^2 + (0.1437)^2} \\
c = 0.3034 + \sqrt{(0.6966)^2 + (0.1437v)^2} \\
v = \sqrt{1 - \frac{1}{\sqrt{1.456 - 0.157(1.764/r)}}}.
\]
(22)

Effect on NS cooling – Haensel et al. (2001) also provide analytical fits to the reduction factor for the neutrino emissivity \( R_{\nu}^{(n)} \) in superfluid NS cores. This is slightly different from the coefficients in Eq. (21).

To model neutrino cooling of the NS we will consider three main factors: (1) modified Urca reactions, the main emission process; (2) proton superconductivity, which reduces the modified Urca reaction rate by the factor \( R_p^{(p)} < 1 \) at \( T < T_{\text{cp}} \); (3) neutrino bremsstrahlung, a weaker emission process that might become dominant once proton superconductivity has suppressed modified Urca reactions.

Therefore, we will write

\[
\frac{dT}{dt} = - \left[ R_p^{(n)}(T) \frac{N^S}{C} + N^{(br)}(T) \right] T^7 \Rightarrow \int \frac{dT}{f(T) T^7} = -(t - t_0),
\]
(23)

where \( C_V(T) = C \cdot T \) and the coefficients on the r.h.s. are, in c.g.s. units, \( N^S = 10^{-32} \) and \( C = 10^{30} \) for modified Urca reactions.

6 The subscript \( p \) indicates that protons are superfluid.
5 TILT ANGLE DISTRIBUTION OF NEWBORN PULSARS: EFFECT OF NS VISCOSITY

As already stated in Sec. 2.2, the pulsar population should be characterized by large tilt angles at birth, given the estimated alignment time \( t_{\text{align}} \sim 10^6 - 10^7 \) yrs. With the classical dipole formula, the alignment timescale is \( \tau_{\text{align}} \) (Jones 1976)

\[
\tau_{\text{align}} = \frac{2 \tau_{\text{dip},i}}{\cos^2 \chi_i},
\]

where \( \tau_{\text{dip},i} = \Omega_i/(2 \Omega) \) is the NS spindown timescale at birth, and \( \chi_i \) the initial tilt angle. This relation implies that the alignment time can be \( \gg \tau_{\text{dip},i} \sim 10^3 - 10^7 \) yrs (for typical NS birth parameters), only if \( \chi_i \approx 90^\circ \). Recently, Philippov et al. (2014) have shown that (a) the alignment time is somewhat shorter when plasma effects in NS magnetospheres are accounted for, obtaining \( \tau_{\text{align}} = 2 \tau_{\text{dip},i} \sin^2 \chi_i / \cos^4 \chi_i \), and (b) at late times, \( t \gg \tau_{\text{align}} \), alignment further slows down, scaling as \( \sim t^{-1+2/\beta} \), where the structure constant \( k_2 \approx 1 \). Even in this case, a characteristic alignment time \( \sim 10^6 - 10^7 \) yrs would require, at least, \( \chi_i \geq 60^\circ - 70^\circ \).

It is therefore of great importance to be able to determine whether tilt angles can grow enough in a timescale \( < \tau_{\text{align}} \), since this appears to be a general requirement of a long alignment time. If, for example, the tilt angles were to remain relatively small in some NSs, their alignment would be faster, potentially producing an over-abundance of small tilt angles already at a young age (cf. Rookyard et al. 2015a,b).

Note that \( \tau_{\text{align}} \) is a constant, since \( \Omega \cos \chi \) is a conserved quantity.

5.1 Study of the parameter space

In order to explore the possible outcomes of the tilt angle evolution on timescales shorter than \( \tau_{\text{align}} \), we solve Eq. (6) for a range of initial conditions. We include the effects of bulk and shear viscosity and the onset of proton superconductivity, and integrate the evolution equations up to an age \( t = 10^{10} \) s. Additional dissipative processes, which might affect the growth of \( \chi \) on longer times (see Sec. 6 and 7), will be considered in future work, along with the long-term alignment driven by the electromagnetic torque.

Results of the evolution turn out to be quite sensitive to the choice of the NS EoS, and hence on mass, radius and \( N \). Therefore, we chose two cases that encompass the range of uncertainty in NS parameters: 1) a relatively low-mass, large-radius NS (1.4 \( M_\odot \), 12 km) with a relatively large value of the bulk viscosity coefficient \( (N = 3.5) \); 2) a relatively high-mass NS (1.9 \( M_\odot \), 10.5 km), with a smaller value of the bulk viscosity coefficient \( (N = 1.75) \).

Note that, since \( \tau_3 \) depends on the NS spin for either type of viscosity considered here, Eq. (6) is coupled to the spin evolution of the NS. The latter is determined by the dipole formula (Spitkovsky 2006):

\[
L_{\text{dip}} = \mu^2 / c^3 \Omega^3 (1 + \sin^2 \chi),
\]

where \( \mu = B_p/2R^3 \) is the magnetic dipole moment and \( B_p \) the dipole field strength at the magnetic pole. Once the microphysics is specified (Sec. 3 and 4), the relevant parameters of the model are the NS birth spin period, \( P \), and the magnetically-induced ellipticity, \( \epsilon_B \). We cover a wide range of \( P \) and \( \epsilon_B \), inclusive of plausible NS initial spin and ellipticity values that would make the magnetic deformation dominant over other possible sources. The magnetic dipole is fixed at a typical value \( B \sim 3 \times 10^{12} \) G, since NS spindown occurs on timescales longer than the viscous effects considered here. This is true as long as \( B < 10^{13} \) G and \( P > 10 \) ms: these values set the limit of validity of our study. Cases not included here, with stronger B and/or faster spin, have a more rapid spindown, which slows down viscous dissipation (cf. Eq. 17) and accelerates the alignment due to the electromagnetic torque. Both effects make it more likely that the tilt angle of a fast spinning, highly magnetized NS remains small; it can grow to large values, though, if the B-field in the NS core is particularly strong (cf. Dall’OssO et al. 2009, Dall’OssO & Stella 2017).

Fig. 3 shows the value of the tilt angle, as a function of \( P_i \) and \( \epsilon_B \), at two different ages. In the left panel we consider a very young age, \( \sim 10^9 \) s, where little evolution of the tilt angle occurs apart from a small region in parameter space. In the right panel the tilt angle is shown at a much later age, \( \sim 300 \) yrs, and we can appreciate the dominant effect of viscous evolution. Note that the effect of the magnetic dipole on the tilt angle is still negligible at this age. The effect of viscosity, on the other hand, is manifested much earlier: even for the slowest-evolving NS in our grid, the tilt angle stops growing at \( t \lesssim 10 \) yrs, since bulk viscosity is quenched after that time.

It is important to notice that there is only a small fraction of the parameter space where the tilt angle has intermediate values. In general, viscous evolution appears to prefer either small angles or full orthogonalization. This bimodality is mainly a consequence of the strong temperature-
mediate angles (is particularly relevant, as is the lack of objects with inter-
comparisons of pulsar tilt angles (Tauris & Manchester 1998, Young et al. 2010, Rookyard et al. 2015a) show an
indication for a bimodal distribution of tilt angles, with two
peaks at $\sim 40^\circ$ and $\gtrsim 80^\circ$. As these authors discuss, observational biases favour the detection of highly inclined objects: thus, the abundance of pulsars with tilt angles $\lesssim 40^\circ$ is particularly relevant, as is the lack of objects with intermediate angles ($\sim 40^\circ - 80^\circ$), given that orthogonal rotators ($\alpha > 80^\circ$) are fairly numerous.

A comparison with the tilt angles of magnetars is less direct, since magnetars have been seen to emit in radio only following an outburst (Camilo et al. 2010), and the location of the outburst may not be the same as that of the dipolar field. Using quiescent X-ray data, constraints have been made on the viewing angle (i.e. the angle between the line of sight and the hottest region on the star) in several objects (i.e. Dedeo et al. 2000; Pernia & Gotthelf 2008; Bernardini et al. 2011; Guillot et al. 2015). However, if toroidal fields are largely dominant in the NS crust (Thompson & Duncan 1995; Pernia & Pons 2011), then the location of the hottest point on the surface of the star may not be coincident with that of the magnetic pole (Viganò et al. 2013; Perna et al. 2013), and hence inferences of the viewing geometry may not yield the correct value of the tilt angle. Therefore, here we only consider the observational data set of the ‘standard’ pulsars, whose observations in radio and gamma-rays allows a more reliable inference of the distribution of the tilt angles.

5.2 Pulsar observations

Compilations of pulsar tilt angles (Tauris & Manchester 1998, Young et al. 2010, Rookyard et al. 2015a) show an indication for a bimodal distribution of tilt angles, with two peaks at $\lesssim 40^\circ$ and $\gtrsim 80^\circ$. As these authors discuss, observational biases favour the detection of highly inclined objects: thus, the abundance of pulsars with tilt angles $\lesssim 40^\circ$ is particularly relevant, as is the lack of objects with intermediate angles ($\sim 40^\circ - 80^\circ$), given that orthogonal rotators ($\alpha > 80^\circ$) are fairly numerous.

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5.3 Corollary: oblate ellipsoids

We briefly comment on a crucial assumption made in Sec. 2.2: Viscous damping of free precession will work even in the absence of a strong toroidal B-field in the NS core. In this case, the dipole B-field would cause an oblate distortion, which would be dominant as long as $B_p > 3 \times 10^{11} (P_{\text{ms}}/10)^{-2}$ G, for superconducting protons (Cutler 2002). In an oblate ellipsoid, minimization of the rotational energy at constant angular momentum will align the NS symmetry axis with the spin axis and, as a result, will cause a decrease of the tilt angle towards zero. Alignment will still occur on the viscous timescale, according to Eq. 4. For the typical magnetic field of pulsars, the ellipticity would be $\sim 10^{-10} - 10^{-9}$, implying a particularly short timescale for damping through bulk viscosity. We solved Eq. 4 for a few values of the initial tilt angle and birth spin, and verified that

Figure 3. The NS tilt angle $\chi$ at an age of $10^{10}$ s, using $M=1.4M_\odot$, $R=12$ km, $N=3.5$ (left panel) or $M=1.9M_\odot$, $R=10.5$ km, $N=1.75$ (right panel), as a function of the initial spin period $P_\text{i}$ and the magnetically-induced ellipticity $\epsilon_B$. The angle evolution occurs under the effect of the bulk and sheer viscosities, and it saturates after an age $\lesssim$ a few years for all parameter choices ($< 1$ yr in most cases). A remarkable result is the narrowness of the parameter space $(P_\text{i},\epsilon_B)$ which leads to intermediate values of the tilt angle, $20^\circ \lesssim \chi \lesssim 70^\circ$. Growth saturation is achieved at either small or large angles, hence resulting in a bimodal distribution of tilt angles.
The Crab pulsar has a spin period $P_{\text{6 CRAB}} \approx 33.7$ ms and period derivative $P_{\text{dot}} \approx 4.23 \times 10^{-13}$ (Abbott et al. 2008). With the dipole formula (24) and $\chi_C \approx 60^\circ$ (Harding et al. 2008; Watters et al. 2009; Du, Qiao & Wang 2012), we estimate $B_p \approx (3.3 - 5.2) \times 10^{12}$ G for the mass/radius range of Sec. 6. The tilt angle is observed to be growing at the rate $\chi_{C} \approx 0.62^\circ$/century or $3.6 \times 10^{-12}$ rad s$^{-1}$ (Lyne et al. 2013).

The measured growth rate for the tilt angle has been interpreted in terms of freebody precession of the NS, which would require its symmetry axis to be almost aligned with either the spin or the magnetic axis (Phillippov et al. 2014, Zanazzi & Lai 2015), since $\chi < \Omega t$ even for the smallest possible deformation of the NS. In this interpretation, the growth of $\chi$ is only an “apparent” effect, associated to one half of the precession cycle (period $\approx 100 - 200$ yrs): the secular trend can only be an alignment, driven by the electromagnetic torque.

### 6 CRAB PULSAR

The Crab pulsar has a spin period $P_{C} \approx 33.7$ ms and period derivative $P_{\text{dot}} \approx 4.23 \times 10^{-13}$ (Abbott et al. 2008). With the dipole formula (24) and $\chi_C \approx 60^\circ$ (Harding et al. 2008; Watters et al. 2009; Du, Qiao & Wang 2012), we estimate $B_p \approx (3.3 - 5.2) \times 10^{12}$ G for the mass/radius range of Sec. 6. The tilt angle is observed to be growing at the rate $\chi_{C} \approx 0.62^\circ$/century or $3.6 \times 10^{-12}$ rad s$^{-1}$ (Lyne et al. 2013).

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### 6.1 Viscous damping of precession

Here we consider the alternative, that $\chi_C$ effectively represents a growth of the tilt angle, driven by a slow dissipative process in the NS core. This of course requires that the NS has a predominantly prolate shape, hence a toroidal magnetic field in its core. From Fig. 5, and for the likely birth spin $\chi_{\text{birth}} \gtrsim 20$ ms (e.g. Haensel et al. 2007), we see that $\epsilon_B \sim (3 - 10) \times 10^{-5}$ in order for the tilt angle to be in the narrow range $\sim (45^\circ - 75^\circ)$. Note that the low-end of the range corresponds to the more massive NS, having a smaller radius and a lower $N$-value, while the high-end is reached in the opposite case. In the superconducting NS core, Maxwell stresses are enhanced by a factor $H_{s1}/B$, where the critical field $H_{c1} \sim 10^{15}$ G. This yields the scaling $\epsilon_B \sim 5 \times 10^{-8} H_{s1,15} B_{12}$ (e.g. Cutler 2002), from which the (volume-averaged) toroidal magnetic field is estimated, $B_T \sim (6 - 20) \times 10^{14}$ G, for $M = 1.4 M_\odot$ and $R = 12$ Km. In this interpretation, the measured $\chi_C$ implies, via Eq. (6), a dissipation time $\tau_d \sim 5 \times 10^{11} \cos \chi/\sin \chi \approx (0.5 - 3) \times 10^{11}$ s, too short for either bulk or shear viscosity. This is also too fast, and of the wrong sign, to be consistent with the effects of the electromagnetic torque.

An additional viscous process consistent with the above estimate could be crust-core coupling via mutual friction in the superfluid core if the NS has a triaxial shape. The mutual friction coupling time is estimated to be (e.g., Jones 2012 and references therein)

$$\tau_{\text{sf}} = \frac{1}{R \Omega \cos \chi} \frac{I_{\text{pre}}}{I_{\text{el}}} \approx 5 \times 10^{10} \frac{P_{\text{ms}}/33}{\epsilon_{\text{el}} \cos \chi} \frac{I_{\text{pre}}/I_{\text{sf}}}{0.03} \text{ s},$$

(26)

where we have used $R \approx 5 \times 10^{-5}$ (e.g. Ashton et al. 2017) and $I_{\text{pre}}/I_{\text{el}}$ is the crust-core ratio of moments of inertia. An ellipticity $\epsilon \sim 10^{-10} - 10^{-9}$ matches well the number expected from the elastic deformation of the NS crust, if either set by the centrifugal force

$$\epsilon_{\text{el}}^{(\text{break})} \sim b \epsilon_{\text{break}} \sim 10^{-7} \epsilon_0 \approx 5 \times 10^{-11} \left( \frac{P_{\text{ms}}}{33} \right)^{-2},$$

(27)

or by the crustal breaking strain (e.g., Jones 2012)

$$\epsilon_{\text{el}}^{(\text{break})} \sim b \epsilon_{\text{break}} \approx 10^{-9} \left( \frac{H_{\text{break}}}{10^{-2}} \right).$$

(28)

How does this make the tilt angle grow, given that the elastic deformation produces an oblate ellipsoid? Let us first consider the fluid interior: here, the centrifugal deformation is always aligned with the $\Omega$-axis, while the magnetic deformation is tilted by the angle $\chi$: it is the latter that excites freebody precession. The NS crust inherits the same magnetically-induced deformation, from the time when it first crystallized. Thus, if there was only the magnetic deformation, the NS core and crust would precess together, at the frequency $\omega = \epsilon_B \Omega \cos \chi$, and no friction would occur. However, because of the magnetic tilt and of crustal elasticity, a small part of the centrifugal deformation in the crust is misaligned with respect to the $\Omega$-axis: this gives rise to $\epsilon_{\text{el}}$, an extra ellipticity specific to the crust. Therefore, in the frame of the NS core, the crust will have an extra “periodic” motion associated to $\epsilon_{\text{el}}$; it is this extra motion that induces mutual friction, and dissipation on the timescale (26). If, however, the NS has a prolate magnetic ellipticity $\epsilon_{B} \gg \epsilon_{\text{el}}$, then the minimum energy state will have the magnetic axis orthogonal to the spin axis, a condition that will also guarantee alignment of the centrifugal deformation in the crust with the $\Omega$-axis. It is interesting to mention here the suggestion (Jones et al. 2016) that a triaxial star, with both crustal and magnetic deformation (and, possibly, a prolate shape), might help resolve a tension between glitches and precession in PSR B1828-11.

Two comments are in order concerning the constrain on $\epsilon_B$ implied by our model. First, the required toroidal magnetic field in the Crab is two orders of magnitude stronger than the poloidal one. As already stated (Sec. 2.2), the question of magnetic equilibria in NS interiors is an open issue in current research, with results pointing towards possible poloidal-dominated states (e.g., Colaiuda et al. 2008, Lander & Jones 2009, Lander 2013) or toroidal-dominated ones.

At the age of the Crab, both protons and neutrons are expected to be in a condensed state.

The motion of the triaxial crust is, in general, more complicated. Our argument should be considered valid as an order of magnitude estimate.

---

8 Obtained from self-consistent solutions to Eqs. 6 and 25.
(Spruit & Braithwaite 2004, Braithwaite 2009, Pons & Perna 2011, Akgun et al. 2013). Our result is consistent with the stability limit of toroidal fields derived for a stably stratified NS interior (Akgun et al. 2013, Dall’Osso et al. 2015). From this point of view, the tilt angle of the Crab pulsar and its measured growth rate might represent a signature of specific physical conditions existing in the NS interior, not accessible to direct observation.

Second, we note that the constraint on $\epsilon_B$ found above is a factor ~ 3-10 lower than the most recent upper limits placed by the first science run of Advanced LIGO (Abbott et al. 2017). This makes it possible that the Crab pulsar may become an interesting source of GWs for current laser interferometers, once they operate at design sensitivity. The orientation-averaged, instantaneous strain for a NS spinning down at frequency $\nu$, with an ellipticity $\epsilon$ and at a distance $d$ is (Ushomirsky, Cutler & Bildsten 2000)

$$h_a = 4\pi^2 \sqrt{\frac{f}{5c^3}} \frac{G I_e}{d} \sqrt{\sin^2 \chi (1 + 15 \sin^2 \chi)} \approx 1.4 \times 10^{-27} \frac{\epsilon - 6}{(d/2\text{kpc})^2} \left(\frac{P_{\text{ms}}}{33}\right)^{-2}$$

for a tilt angle $\chi = \pi/3$, where the GW signal frequency is $f = 2\nu$. For a total observing time $T$, the minimum detectable amplitude for a single template search with $D$ detectors at frequency $f$ is $h^0_{\text{(min)}} = 11.4 \sqrt{S_n(f)/DF}$ (Andersson et al. 2011), where $S_n$ is the one-sided noise spectral density of the detector at frequency $f$. Therefore, the required observing time $T$ in order for the GW signal to be detectable can be estimated roughly by setting $h_a > h^0_{\text{(min)}}$. Plugging in the numbers for the Crab, and adopting the design sensitivity curve of Advanced LIGO at $70\text{kHz}$, we obtain

$$T > \frac{6 \times 10^7}{D} \frac{s}{(P_{\text{ms}}/33)^4(d/2\text{kpc})^2} \left(\frac{\epsilon - 6}{5\text{kHz}}\right)^2 .$$

6.2 Further implications

The measured value of $\dot{\chi}_C$ suggests that additional sources of viscosity, in addition to those considered in Sec. 3, may affect the tilt angle on timescales $> 10^5$ yrs. Interpreting $\dot{\chi}_C$ as due to mutual friction, the expected tilt angle evolution can be calculated by inserting (26) and (27) into (6), which gives $\dot{\chi} \propto \Omega^3 \cos^2 \chi / \sin \chi$. Thus, the growth rate of the tilt angle decreases rapidly with time, as the NS spins down. Because mutual friction is due to the interaction, in the NS core, between the charged particle superfluid neutron vortices, expression (26) will only hold if $T < T_{\text{cn}}$, i.e. at an age $> 300$ yrs if the cooling of the NS in Cas A can be taken as representative. Hence, starting from $t_i = 10^{10}$ s, one may estimate the overall effect of mutual friction in the Crab pulsar by the integral $\Delta \chi = \int_{t_i}^{t_f} \dot{\chi} dt$. Normalizing to the current value $\dot{\chi}_C$, and approximating $\cos^2 \chi / \sin \chi \approx \cos^2 \chi_C / \sin \chi_C \approx 0.3$, we obtain $\Delta \chi \lesssim 19^\circ$, for $t_f \gg \tau_{\text{ad}, i}$ and an initial spin period $\sim 20$ ms. Thus, mutual friction would have a limited, yet non negligible, effect on the long term evolution of the tilt angle.

11 It is $S_n(60\text{kHz}) \approx 2.5 \times 10^{-47}$ Hz$^{-1}$ (e.g. Martynov et al. 2016).
12 Coupled to the crust on a very short timescale.
13 This overestimates the integral given that $\chi$ actually grows.

The above estimate is an absolute upper limit to $\Delta \chi$. Indeed, for $\chi_C \sim 60^\circ$, the alignment time due to the electromagnetic torque is $\tau_{\text{ad}} \sim 24\tau_{\text{ad}, i} \sim 2 \times 10^4$ yrs (Sec. 5). Thus, this mechanism would give an important negative contribution to $\chi$, limiting the growth of $\chi$ already at an early age before causing its decrease on longer timescales.

We can draw a general lesson from the above argument. Slow dissipative processes, like e.g. mutual friction, can offset the tilt angles calculated in Sec. 5 by non-negligible amounts. However, they are unlikely to substantially alter the bi-modality in the tilt angle distribution shown in Fig. 3. Such slower processes, and the alignment due to the electromagnetic torque, depend on additional physical parameters that cannot be fully modelled at this stage and hence will have to be reserved to future investigations.

7 CONCLUSIONS

We have studied the effect of viscosity on the tilt angle evolution of newborn NSs. In particular, we have modelled bulk and shear viscosity of NS matter, following the neutrino cooling down to a temperature $T \sim 10^5$ K. Effects of proton superconductivity on both types of viscosity and on the NS cooling are accounted for, assuming $T_{\text{cp}} \sim 5 \times 10^5$ K. We have focused on a specific scenario in which (i) the NS has a non-spherical shape which is mostly determined by magnetic stresses, (ii) the internal magnetic field is dominated by a toroidal component, which causes a prolate deformation, and (iii) at birth, the magnetic axis has a small tilt angle with respect to the spin axis, which grows by viscous dissipation of free precession, the latter being excited by the magnetic distortion. We have solved the evolution equation for the tilt angle, coupled to the spin evolution of the NS, for a wide range of values of the initial spin $(P_i \sim 10 - 300$ ms) and the magnetically-induced ellipticity $(\epsilon_B \sim 10^{-8} - 10^{-5})$, up to an age of $10^{10}$ s. At this age, all viscous effects are already exhausted, while the magnetic dipole-driven alignment of the magnetic axis has not yet started and neutron superfluidity has not yet occurred (for $T_{\text{cn}} \lesssim 6 \times 10^6$ K).

Our results show that viscous evolution of the tilt angle can either lead to fast orthogonalization of the NS magnetic axis, or to very little evolution, depending on the combination of $P_i$ and $\epsilon_B$. For most parameter combinations, the tilt angle at the end of the integration is either $\gtrsim 80^\circ$ or $\lesssim 10^\circ$: the slower the NS spin, the larger the ellipticity required for $\chi$ to reach $\approx 90^\circ$. Parameter combinations that lead to intermediate values of the tilt angle occupy a relatively narrow strip in parameter space. This very pronounced bi-modality is a result of bulk viscosity dominating the evolution: NSs start their life in the low-frequency regime, where the bulk viscosity coefficient is very low, and later evolve towards the high-frequency regime as they cool. Most of the dissipation occurs around the turnover, where the bulk viscosity coefficient has a peak, which roughly sets the timescale for orthogonalization via the condition $\omega \tau \sim 1$. The latter is essentially a relation between $\epsilon_B$ and $P_i$, with temperature (time) as a parameter.

Our model does not include additional viscous processes, that might affect the tilt angle evolution on longer timescales. For example, crust-core coupling is an impor-
tant aspect that we have not addressed here. Processes of this type could help populate the region of intermediate tilt angles, on timescales $>10^{10}$ s but still short compared to the alignment time. As such, they may mitigate the pronounced bi-modality in tilt angle distribution expected from our results (Fig. 3), yet without removing it. This possibility, and its implications for the long-term alignment, will have to be addressed in future studies.

We have applied our model, in particular, to the Crab pulsar, where measurements of $\chi$ and $\dot{\chi}$ allow a more direct test of theoretical expectations. If due to viscous dissipation, the measured growth of the tilt angle implies that the NS has a predominantly prolate deformation, hence a toroidal magnetic field. The fact that the tilt angle is less than 90°, and still growing, implies that the magnetically-induced ellipticity should be $\gtrsim 3 \times 10^{-6}$ for a likely birth spin $\sim 20$ ms (Fig. 3). The corresponding toroidal magnetic field is $\gtrsim 6 \times 10^{14}$ G (assuming protons are superconducting), about 100 times larger than the large-scale dipole. The measured value of $\dot{\chi}$ on the other hand, points to a dissipation time $\sim (0.5 - 3) \times 10^{11}$ s, the physical interpretation of which is still open. With bulk and shear viscosities ruled out, crust-core coupling via mutual friction would be a natural candidate: the relative motion between core and crust might be due to the elastic deformation of the latter. In this case, mutual friction would work on a timescale set by $\epsilon_{\text{cf}} \sim 10^{-10} - 10^{-9}$, but the growth of $\chi$ is guaranteed by the larger, magnetically-induced distortion.

This scenario has two possible observational tests: one is the existence of modulations in the timing residuals of the Crab pulsar, with a period of $\sim (0.4 - 2) \times 10^5$ s or $\sim 3 \times 10^7$ s, associated to either $\epsilon_B$ or $\epsilon_{\text{cf}}$. Detection of the faster modulation might not be straightforward, though, if the radio beam is approximately aligned with the magnetic axis and, hence, with the precession axis. Given the geometrical constraints, the slower modulation might be more easily detectable. The second, even more direct test, is the detection of a periodic GW signal at $\nu \approx 60$ Hz, due to the magnetic deformation $\epsilon_B \gtrsim 10^{-6}$. In this case, the instantaneous strain would be $h \gtrsim 10^{-27}$ for a distance of 2 kpc, likely detectable by Advanced LIGO/Virgo once operating at design sensitivity.

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Tilt angle distribution in newly born NSs 11
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