Multi-Model Cantor Sets

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Abstract

In this paper we define a new class of metric spaces, called multi-model Cantor sets. We compute the Hausdorff dimension and show that the Hausdorff measure of a multi-model Cantor set is finite and non-zero. We then show that a bilipschitz map from one multi-model Cantor set to another has constant Radon-Nikodym derivative on some clopen. We use this to obtain an invariant up to bilipschitz homeomorphism.

Introduction

A multi-model Cantor set is a metric space which has the following property. There is a partition of $C$ into finitely many clopens $A_1, A_2, \ldots, A_n$ (called models) so that given any point $x$ in $C$ and $\epsilon > 0$ there is a neighborhood $U$ of $x$ with $\text{diam}(U) < \epsilon$ and a metric similarity mapping $U$ onto one of the models.

Such a Cantor set is described by a map $\tau : C \to C$ which is a piecewise metric similarity (an expanding map). One may consider the Cantor set as determined by the dynamical system $\tau$. The middle third Cantor set is an example of a multi-model Cantor set where one model suffices.

The middle third Cantor set is self-similar. At every scale it contains identical copies of itself. In a sense a multi-model Cantor set has finitely many local pictures which are replicated, at different scales, everywhere. This behaviour is similar to certain fractals, such as Julia Sets, which have a compact family of local geometries, (rather than a finite family).

We associate to $C = C(\tau)$ and a constant $d > 0$, an $n \times n$ matrix $M_d$ with non-negative entries. Roughly speaking the $i,j$ entry of $M_d$ is the sum
of the $d$-powers of the inverse scale factors of the similarities of the clopens (later called clones) of type $i$ contained in model $j$. This matrix determines the Hausdorff measure and dimension as follows. We use $H_d$ to denote the $d$-dimensional Hausdorff measure.

**Theorem 1.** Suppose $d > 0$ and $C$ is a multi-model Cantor set with matrix $M_d$ and $\lambda_d$ is the Frobenius eigenvalue for each $M_d$. Let $d$ be chosen such that $\lambda_d = 1$. Then the Hausdorff measure of $C$ is finite and non-zero in this dimension. Therefore $d$ is the Hausdorff dimension of $C$. Also, let $\vec{v}$ be the Frobenius eigenvector of $M_d$ such that $\sum_{i=1}^{n} v_i = H_d(C)$ then $v_i = H_d(A_i)$.

This result is given at the end of section 4. Then in section 5 we investigate bilipschitz maps between multi-model Cantor sets. We prove that every bilipschitz map from one multi-model Cantor set to another is measure linear (has constant Radon-Nikodym derivative) on some clopen. This is a generalization of the results of Cooper [3], Pignataro [2] and Vu’o’ng [5, 6] to this wider class of Cantor set. We use this in corollary 4 to provide an invariant up to bilipschitz homeomorphism of such Cantor sets. These results are from the author’s Master’s thesis [1].

1 Definitions

We are concerned, in this paper, with the study of Cantor sets with particular metrics. In the following sections we shall show some results about the Hausdorff measure of these Cantor sets. In order to make these results clearer to the reader we include the definitions of similarity map, $K$-bilipschitz map, and diameter and state two elementary results concerning Hausdorff dimension.

**Definition.** A surjective map $f : X \rightarrow Y$ between metric spaces is a similarity map if there is a constant $K > 0$ such that for all $x_1, x_2$ in $X$

$$d_Y(f(x_1), f(x_2)) = K d_X(x_1, x_2).$$

**Definition.** A map $f : X \rightarrow Y$ between metric spaces is $K$-bilipschitz if there is a constant $K > 0$ such that for all $x_1, x_2$ in $X$

$$\frac{1}{K} d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2).$$
Definition. For a set $U$ contained in a metric space $X$, the **diameter** of $U$ is given by

$$diam(U) = \sup_{x,y \in U} d_X(x, y).$$

Lemma 1. Suppose $X$ is a metric space and $d > 0$ and the $d$-dimensional Hausdorff measure of $X$ satisfies $0 < \mathcal{H}_d(X) < \infty$. Then $d$ is the Hausdorff dimension of $X$.

*Proof.* This follows directly from the definition. 

Lemma 2. Let $X$, $Y$ be metric spaces. Let $f : X \rightarrow Y$ be a surjective similarity map with scale factor $K > 0$, then

$$\mathcal{H}_d(Y) = K^d \mathcal{H}_d(X).$$

*Proof.* This follows directly from the definition. 

2 The Middle Third Cantor Set

Recall that a topological space $C$ is a Cantor set if it is compact, totally disconnected, perfect and non-empty. It is a fact that any two Cantor sets are homeomorphic. The situation becomes more interesting when one puts a metric on a Cantor set and studies bilipschitz homeomorphisms, as shall be seen in section 5.

In this paper we are interested in considering Cantor sets which arise from dynamical systems. We illustrate this by exhibiting a dynamical system which generates the middle third Cantor set. This dynamical system is determined by a clone structure which we shall now describe.

Let $I = [0,1]$, $A_1 = [0, \frac{1}{3}]$ and $A_2 = [\frac{2}{3}, 1]$ and define the clone maps $\tau_i : A_i \rightarrow I$ by

$$\tau_1 = 3x \quad \text{and} \quad \tau_2 = 3x - 2.$$ 

Then this clone structure is either denoted as $(A_1, \tau_1, A_2, \tau_2)$ or, because the domain $A_i$ is implicit in the definition of $\tau_i$, it can also be denoted $(\tau_1, \tau_2)$.

To construct the middle third Cantor set from the dynamical system we define the map $\tau = \tau_1 \cup \tau_2 : A_1 \cup A_2 \rightarrow I$ by

$$\tau(x) = \begin{cases} 
3x & \text{if } x \in A_1; \\
3x - 2 & \text{if } x \in A_2.
\end{cases}$$
Observe that \( \tau \) maps each of the \( A_i \) linearly onto \( I \). Then

\[
\tau^{-1}(I) = A_1 \cup A_2 = C_1.
\]

In general

\[
\tau^{-n}(I) = C_n
\]

is a disjoint union of \( 2^n \) intervals, each of length \( 3^{-n} \). The middle third Cantor set is

\[
C(\tau) = \bigcap_{n=0}^{\infty} \tau^{-n}(I).
\]

We must note though that the clone structure is not unique. At each level any clone can be replaced by a finite collection of clones which partition it. Therefore we usually refer to a Cantor set \( C \), rather than \( C(\tau) \). In figures 1 and 2 we show two different clone structures for the middle third Cantor set.

![Figure 1: Middle third Cantor Set with clone structure \((\tau_1, \tau_2)\).](image1)

![Figure 2: Middle third Cantor Set with clone structure \((\tau_1 \circ \tau_1, \tau_2 \circ \tau_1, \tau_2)\).](image2)

Having chosen a clone structure we shall call a clopen a *clone* if it is the domain of a composition of clone maps in the clone structure. The empty composition is the identity map so the entire Cantor set \( C \) is also a clone. The *level* of the Cantor set is defined to be 0. The *level* of any other clone is defined to be \( n \) if the smallest clone which properly contains it has level \( n-1 \).
Hence the middle third Cantor set has two level-1 clones: \( A_1 = C \cap [0, \frac{1}{3}] \) and \( A_2 = C \cap [\frac{2}{3}, 1] \). Note that we will abuse notation and use \( A_1 \) to represent both the interval \([0, 1/3]\) and also the subset of the Cantor set contained in \([0, 1/3]\). A clone \( B \) is said to be at relative level 1 to a clone \( D \) if given the level of clone \( D \) as \( n \) then the level of clone \( B \) is \( n + 1 \).

The middle third Cantor set is generated by 2 similarity maps, each with a similarity factor of 3. Cantor sets can be generated by two or more maps with various similarity factors. They can also be generated in higher dimensions in Euclidean space \( \mathbb{R}^2 \) or in a general metric space. Below is a pictorial representation of a Cantor set in \( \mathbb{R}^2 \). This Cantor set is given by a dynamical system; a clone structure.

![Figure 3: Cantor set in \( \mathbb{R}^2 \) with clone structure \((\tau_1, \tau_2, \tau_3)\).](image)

Figure 3 is a Cantor set in \( \mathbb{R}^2 \). It has 3 level-1 clones, \( A_1, A_2, A_3 \). Below we describe how each level-1 clone is mapped onto the largest rectangle. Each clone contains a scaled down copy of the entire Cantor set. A clone structure on this Cantor set can be described by three similarity maps. If the origin is considered to be the top left vertex of the rectangle \( A \), then the map \( \tau_1 : A_1 \to A \) is a stretch by a factor of 2, \( \tau_2 : A_2 \to A \) is a rotation by \( \frac{\pi}{2} \) radians clockwise, a stretch by a factor of 2 and a translation and \( \tau_3 : A_3 \to A \) is a rotation by \( \pi \) radians, a stretch by a factor of 4 and a translation.

## 3 Multi-Model Cantor Sets

In section 2 we discussed clone Cantor sets in metric spaces arising from dynamical systems, where each clone can be mapped onto the whole Cantor
set by using a similarity clone map. In this section we shall be looking at a new class of examples, called \textit{multi-model Cantor sets} which also arise from dynamical systems.

In Vu'ong's thesis \cite{Vuong2001} and paper \cite{Vuong2002}, he studied clone Cantor sets which can be determined by collections \( \{\tau_1, \tau_2, \ldots, \tau_q\} \) where the \( A_i \) are level-1 clones of \( C \) and the \( \tau_i : A_i \to C \) are level-1 clone maps of \( C \). Vu'ong was considering Cantor sets where each clone is mapped onto the unit cube in \( \mathbb{R}^n \) by a clone map. In other words he was considering Cantor sets which look the same everywhere. In a sense a multi-model Cantor set has finitely many local pictures which are replicated, at different scales, everywhere.

\textbf{Definition.} Let \((C, d)\) be a metric space. Let \( A_1, A_2, \ldots, A_n \) be a partition of \( C \) by clopens. The \( A_j \) are called \textit{models}. Let \( E_1, E_2, \ldots, E_m \) be another partition of \( C \) by clopens such that each \( E_i \) is contained in some \( A_j \). The \( E_i \) are called \textit{level-1 clones}. We require that each \( A_j \) contains at least 2 of the \( E_i \) hence \( m \geq 2n \). For each \( E_i \) there is a surjective similarity map \( \tau_i : E_i \to A_j \) for some \( j \). We say that \( (\tau_1, E_1, \tau_2, E_2, \ldots, \tau_m, E_m) \) is a \textit{clone structure} of a \textit{multi-model Cantor set}, \( C \).

As before we define the piecewise metric similarity

\[
\tau = \bigcup_{i=1}^{m} \tau_i : \bigcup_{i=1}^{m} E_i \to \bigcup_{j=1}^{n} A_j.
\]

We may construct examples in Euclidean space by using similarities of \( \mathbb{R}^n \). For example if \( A \) is a hypercube in \( \mathbb{R}^n \) we obtain a nested set of clopens \( \{C_n\} \) where

\[
C_n = \tau^{-n}(A).
\]

Then the multi-model Cantor set is the intersection

\[
C(\tau) = \bigcap_{n=0}^{\infty} \tau^{-n}(A).
\]

Each model and each clone in a multi-model Cantor set has a \textit{type}. By definition each model \( A_j \) is of type \( j \). Each clone \( E_i \) is defined to be the same type as the model it is mapped to by \( \tau_i \). Clones at higher levels are defined to be of the same type as the model they are mapped to by compositions of the similarity maps, \( \tau_i \).
Each $\tau_i$ is a similarity map from a level-1 clone to a model. There is a lot of information we need to convey with the label $\tau_i$, such as which model the clone $E_i$ is mapped to and which it is contained in. For the sake of clarity we use only one subscript throughout most of this paper. In the next two paragraphs we include an explanation of the rules for compositions of these similarity maps. To do this we introduce three indices for each map which we shall subsequently omit in later sections.

We define $\tau_i = \tau_{\alpha,\beta,r} : E_i \rightarrow A_\beta$ where $1 \leq \alpha \leq n$, $1 \leq \beta \leq n$ and $1 \leq r \leq R_{\alpha,\beta}$. Here $\alpha$ is the type of model the level-1 clone $E_i$ is contained in, $\beta$ is the type of the level-1 clone $E_i$, and $R_{\alpha,\beta}$ is the number of level-1 clones of type $\beta$, contained in model $\alpha$.

The compositions of these similarity maps are called clone-model maps. They carry the restriction that if $\tau_{\alpha,\beta,r}$ is composed with $\tau_{\gamma,\delta,l}$ to form $\tau_{\gamma,\delta,l} \circ \tau_{\alpha,\beta,r}$ then $\alpha$ must be equal to $\delta$. For example $\tau_{1,3,1} \circ \tau_{1,2,2}$ is not a possible composition, but $\tau_{2,3,1} \circ \tau_{3,2,2}$ is. Also, the index in the $\beta$ position of the first map and the index in the $\alpha$ position on the last map must be the same. For example, $\tau_{2,3,1} \circ \tau_{3,1,1} \circ \tau_{1,3,1}$ is not defined but $\tau_{3,3,1} \circ \tau_{3,1,1} \circ \tau_{1,3,1}$ is defined.

The domain of each clone-model map is by definition a clone. A level-$k$ clone is the domain of a composition of $k$ of the clone-model maps.

Below are three pictorial representations of a single multi-model Cantor set in $\mathbb{R}^2$. This multi-model Cantor set is given by the dynamical system $\tau = (\tau_1, E_1, \tau_2, E_2, \ldots, \tau_8, E_8)$.

Figure 4 is a multi-model Cantor set in $\mathbb{R}^2$. The models, seen in figure 5, are $A_1$ - a square, $A_2$ - a circle and $A_3$ - a rectangle. The level-1 clones $E_1, E_2, \ldots, E_8$ are contained in the models. These can be seen more clearly in figure 6.

Figure 4: Models, level-1 clones and similarity maps. (Type 1 is dark, type 2 is light.)
Figure 5: The models of a multi-model Cantor set (shown fully in figure 6).

Figure 6: The models and level-1 clones of figure 5.
Figure 7: A multi-model Cantor set in $\mathbb{R}^2$ with clone structure $(\tau_1, E_1, \tau_2, E_2, \ldots, \tau_8, E_8)$.

We can see that the similarity maps from the level-1 clones onto the models are quite simple. If we assume the top left vertex to be the origin then clone $E_1$ is mapped onto $A_1$ by a stretch factor of 2. Clone $E_2$ maps onto $A_2$ by a stretch factor of 2 and a translation. Clone $E_3$ maps onto model $A_3$ by a stretch factor of 2 and a translation. Clone $E_4$ maps onto $A_1$ by a stretch factor of 4 and a translation. Clone $E_5$ maps onto $A_3$ by a stretch of factor 4 and a translation. Clone $E_6$ maps onto $A_1$ by a stretch of factor 2 and a translation. Clone $E_7$ maps onto $A_2$ by a rotation by $\pi$ and a translation and $E_8$ maps onto $A_1$ by a stretch of factor 2 and a translation.

Definition. Given a multi-model Cantor set $C = C(\tau)$ with $n$ models, we define for each $d > 0$ an $n \times n$ matrix $M_d$ as follows. The $i, j$ entry is the sum of the $d$-powers of the inverse scale factors of the clone maps of the level-1 clones of type $i$, contained in model $j$. Recall, $R_{ij}$ is the number of type $i$ clones contained in model $j$. So each entry is of the form $(M_d)_{i,j} = \sum_{r=1}^{R_{ij}} ((a_{ij})_r)^d$ where $0 < (a_{ij})_r < 1$ is the inverse of the scale factor of the relevant clone map.

Figure 8 gives an example of a multi-model Cantor set and its corresponding matrix. The scale factors of the similarity maps are given on the diagram.
Figure 8: A multi-model Cantor set and its corresponding matrix.

\[
M_d = \begin{pmatrix}
(\frac{1}{3})^d & (\frac{1}{5})^d \\
(\frac{1}{4})^d & (\frac{1}{6})^d + (\frac{1}{7})^d
\end{pmatrix}.
\]

**Definition.** Given a multi-model Cantor set \( C \) with \( n \) models and a finite collection of disjoint clones \( J = \{J_{i,r_i}\} \), where \( i \) is the type of clone and \( 1 \leq r_i \leq R_i \) where \( R_i \) is the number of type \( i \) clones in \( J \). Then given \( d > 0 \) we define a vector \( \vec{v} = (v_1, \ldots, v_n)^T \) by

\[
v_i = \sum_{r_i=1}^{R_i} \text{diam}(J_{i,r_i})^d.
\]

We say \( \vec{v} = \vec{v}(J) \) is the \( d \)-quantity of the collection of clones \( J \).

**Definition.** Given a finite collection of disjoint clones \( J \) and \( k \geq 1 \) let \( J^{(k)} \) be the collection of clones obtained by subdividing, \( k \) times, each clone in \( J \). (By subdividing once we mean replacing each clone by the clones contained in it at relative level 1.)

**Lemma 3.** Given a multi-model Cantor set \( C = C(\tau) \) and given \( d > 0 \) let \( M = M_d \) be the the multi-model Cantor set matrix defined above. Given a finite collection of clones \( J \) then

\[
\vec{v}(J^{(k)}) = M^k \vec{v}(J).
\]

**Proof.** First observe that if \( J = J_1 \cup J_2 \) and \( J_1 \cap J_2 = \emptyset \) then \( \vec{v}(J) = \vec{v}(J_1) + \vec{v}(J_2) \). Thus it suffices to prove the result when \( J \) is a single clone.
The case \( k = 1 \) follows from the definition of \( M_d \) (refer to figure 8). The inductive step follows from the observation that

\[ \mathcal{J}^{(k+1)} = (\mathcal{J}^{(k)})^{(1)} \]

and applying the case \( k = 1 \) to the collection \( \mathcal{J}^{(k)} \).

\[ \square \]

**Corollary 1.** Given a multi-model Cantor set \( C(\tau) \) with multi-model Cantor set matrix \( M_d \), then \( (M_d)^k \) is the multi-model Cantor set matrix for \( C(\tau^k) \).

(Observe that the Cantor sets \( C(\tau) \) and \( C(\tau^k) \) are identical.)

**Definition.** A square matrix \( M \) is **irreducible** if all its entries are non-negative and there exists some \( k > 0 \) such that all entries of \( M^k \) are positive.

(Note this is not the same definition of irreducible used in representation theory; namely that there are no invariant subspaces).

In this paper we study multi-model Cantor sets with irreducible matrices. The geometric significance of irreducibility, by corollary \( \text{II} \), is that a clone of each type is contained at some level within each model. Here are two figures which illustrate this. The first is irreducible, the second is not.

**Figure 9:** A multi-model Cantor set with irreducible matrix. (Each clone is contained in each model.)

Figure 9 shows a multi-model Cantor set with an irreducible matrix. There are no zeros in the corresponding matrix, \( (M_d)^k \), for any \( k > 0 \).

In figure 10 we see a multi-model Cantor set which does not have an irreducible matrix. The corresponding matrix \( (M_d)^k \) has zero entries in the 3rd row of both the 1st and 2nd columns, for every \( k > 0 \).
Figure 10: A multi-model Cantor set with non-irreducible matrix. (Type 3 clones are not contained in models $A_1$ or $A_2$.

4 The Hausdorff Measure Of Multi-Model Cantor Sets

In this section we prove theorem [1].

Theorem (Frobenius). An $n \times n$ irreducible matrix $M$ always has a real positive eigenvalue $\lambda$ that is a simple root of the characteristic equation and is larger than the moduli of all the other eigenvalues. To this ‘maximal’ eigenvalue $\lambda$ there corresponds an eigenvector with strictly positive coordinates.

Definition. The Frobenius eigenvalue of an irreducible matrix is the eigenvalue of largest modulus. From the Frobenius theorem we see that it is positive, real and has multiplicity 1. A Frobenius eigenvector is a corresponding positive, real eigenvector.

The first step in the proof of theorem [1] is to show that the Frobenius eigenvalue $\lambda_d$, of the multi-model Cantor set matrix $M_d$, is a strictly decreasing continuous function of $d$. From this we shall see that there is a unique $d > 0$ such that $\lambda_d = 1$, and that this is the Hausdorff dimension of the Cantor set. We state some easily proved results on matrices without proof.

Definition. The $L_1$-norm on $\mathbb{R}^n$ is $\|(v_1, \ldots, v_n)\|_1 = \sum_{i=1}^{n} |v_i|$. 
Lemma 4. Let $M$ be an $n \times n$ irreducible matrix with Frobenius eigenvalue $\lambda$. Then there exists $Q > 0$ such that for all $k$ and for all vectors $\vec{v}$,

$$\|M^k \vec{v}\|_1 \leq Q \lambda^k \|\vec{v}\|_1.$$ 

Furthermore when $\lambda = 1$ there is a strictly positive matrix $M^\infty$ such that $M^k$ converges to $M^\infty$ as $k$ tends to $\infty$.

Definition. We put a partial ordering on real matrices. We say matrix $A$ is less than matrix $B$ if each entry in $A$ is less than the corresponding entry in $B$. So $A < B$ if $A_{ij} < B_{ij}$ for all $i$ and $j$.

Lemma 5. If $\lambda$ is an eigenvalue of $M$ with eigenvector $\vec{v}$ then $\lambda^k$ is an eigenvalue of $M^k$ with eigenvector $\vec{v}$.

Note, if $\lambda$ is the Frobenius eigenvalue then it has largest modulus of all eigenvalues for $M$ so $\lambda^k$ has largest modulus of all eigenvalues for $M^k$. Therefore $\lambda^k$ is the Frobenius eigenvalue for $M^k$.

Lemma 6. Suppose $0 < M_1 < M_2$, and $\lambda_i$ is the Frobenius eigenvalue for $M_i$, then $\lambda_1 < \lambda_2$.

Corollary 2. Suppose $M_d$ is an $n \times n$ strictly positive matrix such that each entry is a strictly monotonically decreasing continuous function of $d$. Let $\lambda_d$ be the Frobenius eigenvalue of $M_d$ for each $d$. Then as $d$ increases, $\lambda_d$ decreases.

Proof. Let $0 < d' < d$. Each entry of $M_d$ is a strictly monotonically decreasing function of $d$ so $M_d > M_{d'} > 0$. Then by lemma 5 we have $\lambda_{d'} > \lambda_d$. \hfill $\square$

Lemma 7. Suppose $M_d$ is an $n \times n$ strictly positive matrix where each entry is a strictly monotonically decreasing continuous function of $d$, and $\lambda_d$ is the Frobenius eigenvalue of $M_d$ for each $d$. Then $f : [0, \infty) \to \mathbb{R}^+$ such that $f(d) = \lambda_d$ is continuous.

Proof. Let $g_d(x)$ be the characteristic polynomial of $M_d$. The coefficients of $g_d(x)$ are polynomial functions of the entries of $M_d$. Since polynomials are continuous and the entries of $M_d$ are continuous functions of $d$, it follows that the coefficients of $g_d(x)$ vary continuously with $d$. 

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Frobenius gives us $\lambda_d$ which is a real, positive root of $g_d(x)$ with multiplicity 1. Without loss of generality we can assume $g_d(\lambda_d + \epsilon) > 0$ and $g_d(\lambda_d - \epsilon) < 0$ for some small $\epsilon > 0$. Now consider $g_{d'}(x)$ where $d' = d + \delta$ for some small $\delta > 0$, then $g_{d'}(\lambda_d + \epsilon) > 0$ and $g_{d'}(\lambda_d - \epsilon) < 0$ by continuity of coefficients of the characteristic polynomial. Then by the intermediate value theorem $g_{d'}(x_{d'}) = 0$ for some $\lambda_d - \epsilon < x_{d'} < \lambda_d + \epsilon$.

We have shown that $g_{d'}(x)$ has a root $x_{d'}$ in the interval $(\lambda_d - \epsilon, \lambda_d + \epsilon)$. We now complete the proof that $f(d) = \lambda_d$ is continuous. We argue by contradiction. If as $d' \to d$ we have $\lambda_{d'} \not\to \lambda_d$ then, since $\lambda_d$ is monotonic in $d$, there is an $\eta > 0$ and a sequence $\{d_n\}$ such that $d_n \to d$ with $|\lambda_{d_n} - \lambda_d| > \eta$. Since $\lambda_{d_n}$ is the largest root we know that $\lambda_{d_n} \geq x_{d_n}$ thus $\lambda_{d_n} > \lambda_d + \eta$. For $n$ large we have $d - 1 \leq d_n \leq d + 1$ so by corollary 2 we have $\lambda_{d+1} \leq \lambda_{d_n} \leq \lambda_{d-1}$. Therefore the $\lambda_{d_n}$ are contained in a compact interval and so there exists a convergent subsequence. Re-label the convergent subsequence, then

$$
\lim_{n \to \infty} \lambda_{d_n} = \lambda' \geq \lambda_d + \eta.
$$

**Claim.** The limit $\lambda'$ is a root of $g_d(x)$.

**Proof.** The sequence $d_n \to d$ and so $g_{d_n}(x) \to g_d(x)$ because the coefficients of $g_d(x)$ are continuous functions of $d$. The characteristic equation $g_{d_n}(x)$ has a root $\lambda_{d_n}$ for all $n$ so

$$
g_d(\lambda') = \lim_{n \to \infty} g_{d_n}(\lambda_{d_n}) = \lim_{n \to \infty} (0) = 0.
$$

This proves the claim. The claim contradicts $\lambda_d$ being the largest root of $g_d(x)$. The contradiction implies that $x_{d'}$ is the largest root of $g_{d'}(x)$ and so $x_{d'} = \lambda_{d'}$. Therefore $\lambda_d$ varies continuously with $d$ and hence $f(d) = \lambda_d$ is continuous. \[\square\]

In the following lemmas we shall suppose that the irreducible matrix $M_d$ is strictly positive. This can be justified as follows. Given $\tau$ and an integer $k > 0$ the multi-model Cantor sets $C(\tau)$ and $C(\tau^k)$ are identical. However their clone structures differ. The corresponding multi-model Cantor set matrices are respectively $M_d$ and $(M_d)^k$ by corollary 3. The hypothesis that $M_d$ is irreducible ensures that we may choose $k$ sufficiently large so that $(M_d)^k$ is strictly positive. Therefore replacing $\tau$ by $\tau^k$ justifies the supposition.

**Lemma 8.** Suppose $M_d$ is a multi-model Cantor set matrix and $\lambda_d$ is the Frobenius eigenvalue for each $d$. Then there is a unique $d > 0$ such that $\lambda_d = 1$. 

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Proof. Recall that the inverse scale factors of clone maps satisfy $0 < (a_{ij})_r < 1$. As $d$ approaches $\infty$, we have that $(M_d)_{ij} = \sum_{r=1}^{R_{ij}} ((a_{ij})_r)^d$ gets close to 0, for all $i, j$. This implies $M_d$ approaches the zero matrix and so $\lambda_d$ tends to 0 by lemma 6.

When $d = 0$, the sum $\sum_{r=1}^{R_{ij}} ((a_{ij})_r)^0 = R_{ij}$ for all $i, j$. Thus each of the entries of the matrix $M_0$ are at least 1. Let $\vec{v}$ be the eigenvector for $\lambda_0$ then

$$
\lambda_0 \vec{v} = M_0 \vec{v} \geq \begin{pmatrix}
1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 1 
\end{pmatrix} \vec{v} = \begin{pmatrix}
\sum_{i=1}^{n} v_i \\
\sum_{i=1}^{n} v_i \\
\vdots \\
\sum_{i=1}^{n} v_i 
\end{pmatrix}.
$$

Note, $\vec{v}$ is the Frobenius eigenvector of $M_0$ and hence $v_i > 0$ for all $i$, therefore $\sum_{i=1}^{n} v_i > v_j$, for all $j$. This implies $\lambda_0 \vec{v} > \vec{v}$ and hence $\lambda_0 > 1$.

The function $f(d) = \lambda_d$ is continuous, so by the intermediate value theorem $\lambda_d = 1$ for some value of $d > 0$. The function $f(d) = \lambda_d$ is strictly monotonic by lemma 6. Therefore this $d$ is unique.

Lemma 9. Suppose $M$ is an irreducible non-negative $n \times n$ matrix with Frobenius eigenvalue 1. Then there is a $Q > 0$ such that for all $p > 0$ and for all vectors $\vec{v}$ in $\mathbb{R}^n$ we have

$$
||M^p \vec{v}||_1 \leq Q ||\vec{v}||_1.
$$

The following gives the Hausdorff measure of every model in a multi-model Cantor set. A clone is a scaled down copy of a model and a clopen is a finite union of clones. Thus the Hausdorff measure of every clopen can be determined from the following result.

Theorem 1. Suppose $d > 0$ and $C$ is a multi-model Cantor set with matrix $M_d$ and $\lambda_d$ is the Frobenius eigenvalue for each $M_d$. Let $d$ be chosen such that $\lambda_d = 1$. Then the Hausdorff measure of $C$ is finite and non-zero in this dimension. Therefore $d$ is the Hausdorff dimension of $C$. Also, let $\vec{v}$ be the Frobenius eigenvector of $M_d$ such that $\sum_{i=1}^{n} v_i = \mathcal{H}_d(C)$ then $v_i = \mathcal{H}_d(A_i)$. 

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Proof. By lemma 1, if we show that $0 < \mathcal{H}_d(C) < \infty$ then $d$ is the Hausdorff dimension of $C$.

To show $\mathcal{H}_d(C) < \infty$ it suffices to show there is a $K > 0$ such that for all $\epsilon > 0$ there is an open cover $U$ of $C$ such that $\text{diam}(U) < \epsilon$ for all $U$ in $U$ and

$$\sum_{U \in U} (\text{diam}(U))^d < K.$$ 

Given $\epsilon > 0$ there is a $k > 0$ such that the diameter of every level-k clone is less than $\epsilon$. Let $U$ be the set of level-k clones. Then $U$ is an open $\epsilon$-cover of $C$. If $M = M_d$ and if $\{A_j\}_{1 \leq j \leq n}$ are the level-0 models then by lemma 3 the $d$-quantity of $U$ is

$$\bar{v}(U) = \sum_{U \in U} (\text{diam}(U))^d = \sum_{i,j=1}^{n} (M^k)_{ij}(\text{diam}(A_j))^d.$$ 

Now $M^k$ converges to $M^\infty$ as $k \to \infty$ by lemma 4. Hence

$$\lim_{k \to \infty} \sum_{i,j=1}^{n} (M^k)_{ij}(\text{diam}(A_j))^d = \sum_{i,j=1}^{n} (M^\infty)_{ij}(\text{diam}(A_j))^d = K'.$$

If we take $K = K' + 1$ then we are done.

To show $\mathcal{H}_d(C) > 0$ it suffices to show there is an $\eta > 0$ such that for every open cover $U$ of $C$ we have

$$\sum_{U \in U} (\text{diam}(U))^d > \eta.$$ 

Given an open set $U$, let $J_U$ be the smallest clone containing $U$. By lemma 12 there is a constant $\beta = \beta(\tau)$ depending only on the Cantor set and $\tau$ such that $\text{diam}(U) \geq \beta^{-1}\text{diam}(J_U)$. Hence

$$\sum_{U \in U} (\text{diam}(U))^d \geq \beta^{-d} \sum_{U \in U} (\text{diam}(J_U))^d.$$ 

Therefore it suffices to show that there is an $\eta' > 0$ such that for every cover $U$ of $C$ by clones we have

$$\sum_{U \in U} (\text{diam}(U))^d > \eta'.$$ (1)
Then $\eta = \beta^{-d}\eta'$ satisfies the above.

Given a cover of $C$ by clones, since $C$ is compact there is a finite subcover. We may choose the subcover so that no clone in the subcover is contained in any other clone in the subcover. If two clones intersect then one is contained in the other. Hence we may choose a subcover consisting of finitely many disjoint clones. Therefore it suffices to prove inequality (1) for $\mathcal{U}$ a finite cover of $C$ by disjoint clones.

The clones in $\mathcal{U}$ may be at different levels. Let $\mathcal{U}_k$ be the cover obtained by subdividing the clones in $\mathcal{U}$ so that they are all at the same level, $k$.

**Claim.** There is a $Q > 0$ such that for all $k$

$$\sum_{U \in \mathcal{U}} (\text{diam}(U))^d > Q^{-1} \sum_{U \in \mathcal{U}_k} (\text{diam}(U))^d.$$  

Given the claim we see that the right hand side is the sum over all level-$k$ clones. We have seen above that as $k \to \infty$ this sum converges to $K' > 0$. Thus for $k$ large we have

$$\sum_{U \in \mathcal{U}_k} (\text{diam}(U))^d > \frac{K'}{2}.$$  

Thus we have shown that $\eta' = Q^{-1}(K'/2)$ satisfies (1). It remains to prove the claim.

**Proof of claim.** It suffices to show this when $\mathcal{U}$ contains only one clone $J$. By lemma $\PageIndex{4}$, there is a $Q > 0$ such that $\|M^k\vec{v}\|_1 \leq Q\|\vec{v}\|_1$ for all positive vectors $\vec{v}$ in $\mathbb{R}^n$. Let $J$ be a clone of type $i \in \{1, \ldots, n\}$. Then $\vec{v} = (\text{diam}(J))^d\vec{e}_i$ is the $d$-quantity of the clone. The clone $J$ subdivided $k$ times forms the collection of clones $J^{(k)}$ with $d$-quantity $\vec{v}(J^{(k)}) = M^k\vec{v}(J)$. Then by lemma $\PageIndex{4}$ we have

$$\sum_{U \in J^{(k)}} (\text{diam}(U))^d = \sum_{i=1}^n |(M^k\vec{v})_i| = \|M^k\vec{v}\|_1 \leq Q\|\vec{v}\|_1 = Q(\text{diam}(J))^d.$$  

This proves the claim.

This proves that the Hausdorff measure of $C$ is finite and non-zero in this dimension. Therefore, by lemma $\PageIndex{4}$, we see that $d$ is the Hausdorff dimension.

Let $\vec{v}$ be the vector where $v_i$ is the Hausdorff measure of the level-0 model $A_i$. The $i, j$ entry of the transposed matrix $(M_d)^T$ is the sum of the $d$-powers
of the inverse scale factors of the clone-model maps from the type \( j \) clones to the type \( i \) models. Then the \( i \) entry of \((M_d)^T \vec{v}\) is

\[
\sum_{j=1}^{n} (M_d)^T_{ij} v_j.
\]

By lemma \(2\), this is the sum of the Hausdorff \( \delta \)-measures of all the level-1 clones contained in model \( i \). Hausdorff measure is \( \sigma \)-additive so the Hausdorff \( \delta \)-measure of model \( A_i \) must equal the measure of the level-1 clones contained in \( A_i \). Therefore

\[
(M_d)^T \vec{v} = \vec{v}.
\]

This implies \( \vec{v} \) is an eigenvector with eigenvalue 1. \( \square \)

5 Measure Linearity on Multi-Model Cantor Sets

In this section we shall generalize some results by Cooper [3] on clone Cantor sets to the case of multi-model Cantor sets. The main result is theorem 2 which states that given a bilipschitz map between multi-model Cantor sets then the map is measure linear on some clone.

**Definition.** Given a subset \( A \) of a clone Cantor set \( C \), the *separation of \( A \)* is the minimum distance between \( A \) and \( C \setminus A \).

\[
sep(A) = \inf \{ d_C(x, y) : x \in A, y \in C \setminus A \}.
\]

By convention, \( sep(C) = \infty \). The *relative separation of \( A \)* is the quotient of the separation of \( A \) by the diameter of \( A \):

\[
rel(A) = \frac{sep(A)}{diam(A)}.
\]

As we discuss concepts related to Cantor sets we will often want to consider clones after “rescaling”. Applying a similarity map with stretch factor \( K \) to clones increases the diameter of the clones by a factor \( K \) and also increases the diameter of the gaps between clones by the same factor. Thus the quotient of these values remains constant. Relative separation is therefore
an invariant factor under rescaling.

**Definition.** The *minimum separation* of \( C \) is denoted

\[
\alpha(C) = \min \{ \text{sep}(A) : A \text{ is a level-1 clone of } C \}.
\]

We now generalize some of the results from a paper written by Cooper [3] on clone Cantor sets. We have made the necessary changes and additions to make these work in the case of the multi-model Cantor set. We have rewritten the statements of the lemmas and the main theorem from this paper and have summarized each. We have outlined the main points of the proofs. The reader should refer to [3] for further details.

Lemma 11 includes a change of wording to signify that \( \tau \) is a map from a clone to the level-0 model of the same type. Here we use the definition of clone-model map given in section 3. Lemmas 12, 13 and 14 now involve an extra case where a clopen is not contained in any model and theorem 2 follows from these.

We refer to two quantities, dependent on certain parameters, as being *approximately equal* if their ratio is bounded above and below by positive constants independent of those parameters.

**Lemma 10 (Lemma 10.1 from [3]).** Every clopen in a multi-model Cantor set is a finite union of clones.

**Lemma 11 (Lemma 10.2 from [3]).** Given a multi-model Cantor set \( C \) there is a constant \( \xi(C) > 1 \) such that for every clone \( A \) of \( C \) with level of \( A \geq 1 \) we have

\[
\frac{1}{\xi(C)} \leq \text{rel}(A) \leq \xi(C).
\]

This lemma tells us that the separation of a clone approximates its diameter. The replacement of the clone map by a clone-model map suffices to make the proof from [3] apply to the multi-model case. We now sketch this.

There are only a finite number of clones contained in the Cantor set of more than a certain diameter. Two points are chosen, of minimal distance apart, one inside clone \( A \) and the other in \( C - A \). Rescaling by a clone-model map does not change relative separation, so we can assume the two points are contained in different level-1 clones and that the distance between the two points is no less than the minimum separation of \( C \). This now gives a lower
bound on the diameter of $D$, the smallest clone which properly contains $A$. There are only a finite number of possibilities for $D$ due to the lower bound on its diameter. Therefore there are a finite number of options for $A$, as $A$ is at relative level 1 to $D$. Since there are only a finite number of choices for $A$ then there are only a finite number of possibilities for $\text{rel}(A)$. □

Lemma 12 (Lemma 10.3 from [3]). Given a multi-model Cantor set $C$ there is a constant $\beta$ with the following property. Let $B$ be any subset of $C$ of positive diameter and let $D$ be the smallest clone of $C$ containing $B$ (or choose $D=C$ if there is no such clone). Then

$$\text{diam}(B) \leq \text{diam}(D) \leq \beta \text{diam}(B).$$

This lemma says that the diameter of any subset is approximately the same as the diameter of the smallest clone which contains it.

Given a multi-model Cantor set $C$, we note that not every subset of $C$ is contained in a clone. However, if $B$ is not contained in any clone then $\text{diam}(B) \geq \alpha(C)$. Taking $D$ to be the whole Cantor set then we already have

$$\frac{\text{diam}(D)}{\text{diam}(B)} \leq \frac{\text{diam}(C)}{\alpha(C)}$$

and the result follows as below.

If $B$ is contained in a clone then rescale $D$ onto the appropriate level-0 model $A_j$, using a clone-model map $\tau$. After rescaling, $B$ must contain points in at least two level-1 clones by choice of $D$. Therefore the diameter of $B$ must be at least as great as the minimum separation of $C$. Now rescaling multiplies the diameter of all clones by the same constant so the ratio of the diameter of $D$ to $B$ is the same as the ratio of the diameters after rescaling. This is then no greater than the ratio of the diameter of $A_j$ to the minimum separation of $C$. Therefore

$$\frac{\text{diam}(D)}{\text{diam}(B)} = \frac{\text{diam}(\tau(D))}{\text{diam}(\tau(B))} \leq \frac{\text{diam}(A_j)}{\alpha(C)} \leq \frac{\text{diam}(C)}{\alpha(C)}.$$ 

So if we let $\beta = \frac{\text{diam}(C)}{(\alpha(C))^{-1}}$ we get the desired result. □

Lemma 13 (Lemma 10.4 in [3]). Given $K > 1$, $\epsilon > 0$ and multi-model Cantor sets $C$, $C'$ there is a constant $M$ with the following property. Let $f$ be any $K$-bilipschitz map of $C$ onto a clopen in $C'$, and suppose that $\text{rel}(f(C)) \geq$
Let $A$ be any clone of $C$. If $P'$ is the smallest clone of $C'$ containing $f(A)$ and $\tau'$ is the clone map taking $P'$ onto $C'$ then there are at most $M = M(C, C', K, \epsilon)$ possibilities for the image $\tau'(f(A))$ as a subset of $C'$. This is also true if $f(A)$ is not contained in any clone of $C'$ with $\tau'$ taken to be the identity map.

This lemma says that given a $K$-bilipschitz map from $C$ to $C'$ and a clone $A$ in $C$, up to rescaling in $C'$, there are only a finite number of possibilities for the image of $A$.

The proof given in [3] does not consider the case where the clopen $f(A)$ is not contained in any clone. Suppose the clopen $f(A)$ is not contained in any clone. Then there exist points $x$ and $y$ in $f(A)$ such that $x$ and $y$ are in different models, i.e. level-0 clones. This implies $\text{diam}(f(A)) \geq \alpha(C')$. The rest of the proof in [3] then applies but with $\tau'$ replaced by the identity map. We now sketch this proof.

The relative separation is not affected by rescaling. Given a clone $A$ in $C$, then the diameter of $f(A)$ is bounded above by a factor of $K$ times the diameter of $A$ because $f$ is $K$-bilipschitz. The separation of $f(A)$ is bounded below by a factor of $K^{-1}$ times the separation of $A$ for the same reason. Hence the relative separation of $f(A)$ is bounded below by some positive constant. We then rescale $f(A)$ using a clone-model map $\tau'$ and find that the diameter of $\tau'f(A)$ is bounded below by the minimum separation of models in $C'$ and as relative separation is unchanged by rescaling we find the separation of $\tau'f(A)$ is also bounded below. We then choose $B$ to be the biggest clone in $\tau'f(A)$ and let $D$ be the smallest clone which properly contains $B$ (or all of $C'$). The diameter of $D$ is bounded below by the separation of $\tau'f(A)$ and so there are only finitely many possibilities for $D$. Clone $B$ is one level higher than clone $D$ and hence there are only finitely many possibilities for $B$. The number of maximal clones in $\tau'f(A)$ is bounded above because the diameter of each is bounded below. Since there are a finite number of possibilities for the image of each of these clones we have a finite number of possibilities for $\tau'f(A)$. \hfill \Box

**Lemma 14 (Corollary 10.5 in [3])**. Given $K > 1$, $\epsilon > 0$ and multi-model Cantor sets $C$, $C'$ there is a finite set $S$ of positive real numbers with the following property. Suppose that $f$ is any $K$-bilipschitz map of $C$ onto a clopen in $C'$, and suppose that $\text{rel}(f(C)) \geq \epsilon$. If $B$ is any clone of level $n$ in $C$...
containing a clone $D$ of level $n+1$ then

\[ \frac{\mathcal{H}_\delta(f(D))}{\mathcal{H}_\delta(f(B))} \in S. \]

This may be interpreted as saying that non-linearity, at the level of measure theory, comes in discrete “packets”. A bilipschitz map need not distort the measure of all clones by the same multiplicative factor but, if $D$ is a clone contained in $B$ of relative level 1, then the ratio of the masses of the images of $B$ and $D$ can be one of only finitely many possible numbers.

In generalizing the proof to the case of multi-model Cantor sets we need to consider the possibility that $f(B)$ is not contained in any clone. By lemma \[\text{13}\] we already know there are a finite number of possibilities for $f(B)$. We need to show there are only finitely many possibilities for $f(D)$ and the result will follow.

We consider two cases below. If $f(D)$ is contained in a clone then case 1 applies. If $f(D)$ is not contained in any clone then case 2 applies.

**Case 1.** Suppose $f(D)$ is contained in some clone $Q'$. We know that

\[ \frac{\text{diam}(D)}{\text{diam}(B)} = \frac{\text{diam}(E_i)}{\text{diam}(A_j)}, \]

where $A_j$ is a model and $E_i$ is a level-1 clone. Hence this quotient is equal to one of a finite number of possible positive scale factors and so is bounded away from 0. Now $f(B)$ is not contained in a model so $\text{diam}(f(B)) \geq \alpha(C')$ and hence $\text{diam}(B) \geq K^{-1}\alpha(C')$ as $f$ is $K$-bilipschitz. We have that $\text{diam}(Q')$ approximates $\text{diam}(f(D))$ by lemma \[\text{12}\] and so approximates $\text{diam}(D)$ because $f$ is $K$-bilipschitz. Hence

\[ \frac{\text{diam}(Q')}{K^{-1}\alpha(C')} \geq \text{approximately } \frac{\text{diam}(D)}{\text{diam}(B)}, \]

and so is bounded away from 0 and thus $\text{diam}(Q')$ is bounded away from 0. There are only a finite number of clones this large so we have only finitely many choices for $Q'$. We map $Q'$ onto the appropriate level-0 model $A_j$ by a clone-model map $\phi$, (using $Q' = A_j$ and $\phi = \text{identity if level } Q' = 0$). Then $\phi(f(D))$ is one of only finitely many possibilities by lemma \[\text{13}\] and since there are only a finite number of choices for $\phi$ then there are only finitely many possibilities for the clopen $f(D)$.  

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Case 2. Suppose \( f(D) \) is not contained in any clone. Then by lemma [13] we have only finitely many possibilities for \( f(D) \).

If \( f(B) \) is contained in a clone then the rest of the proof for lemma [14] follows from the observation that there are a finite number of possibilities for the rescaled image of clone \( B \), and the rescaled image of clone \( D \), where \( D \) contains \( B \) at relative level 1. Therefore there are only a finite number of values for

\[
\frac{\mathcal{H}_\delta(\tau'f(D))}{\mathcal{H}_\delta(\tau'f(B))}.
\]

We take \( \tau' \) to be the identity map if \( f(B) \) is not contained in any clone.

In either case the map \( \tau' \) is a similarity map with stretch factor \( L \) we have, using lemma [4]

\[
\frac{\mathcal{H}_\delta(\tau'f(D))}{\mathcal{H}_\delta(\tau'f(B))} = \frac{L^\delta \mathcal{H}_\delta(f(D))}{L^\delta \mathcal{H}_\delta(f(B))} = \frac{\mathcal{H}_\delta(f(D))}{\mathcal{H}_\delta(f(B))}.
\]

Thus there are only a finite number of possibilities for this ratio. \( \square \)

Definition. Given a \( K \)-bilipschitz map \( f : C \to C' \), the mass ratio (MR) of a clone \( A \) in \( C \) is defined to be

\[
MR(A) = \frac{\mathcal{H}_\delta f(A)}{\mathcal{H}_\delta(A)}.
\]

Thus if \( f \) is the identity then \( MR(A) = 1 \). The larger this number the “greedier” we consider the clone \( A \) to be.

Corollary 3 (Assertion in [3]). With the hypothesis of lemma [14] there is a finite set \( S \) of real numbers such that for every pair of clones \( E \subset D \) in \( C \) with level\((E) = \text{level}(D) + 1 \) then

\[
\frac{MR(E)}{MR(D)} \in S.
\]

Here we see that when a clone \( D \) and a clone \( E \) contained within it at relative level 1 are mapped into a multi-model Cantor set with a \( K \)-bilipschitz map then the quotient of the mass ratios of these clones is restricted to some finite set.
**Definition.** A map $f(X, \mu_X) \to (Y, \mu_Y)$ between metric spaces is *measure linear*, i.e. has constant Radon-Nikodym derivative, if for some constant $K$,

$$\mu_X(f(A)) = K\mu_Y(A)$$

for all measurable subsets $A$ of $X$.

This is a weak generalization of linearity. Linearity refers to metric whereas this concept refers to measure. For Lebesgue measure on the real line, linear and measure linear are the same for continuous functions.

**Theorem 2 (Theorem 10.6 in [3]).** Suppose that $C, C'$ are multi-model Cantor sets and that $f$ is a $K$-bilipschitz map of $C$ onto a clopen subset of $C'$. Then $C$ and $C'$ have the same Hausdorff dimension. Furthermore there is a clopen $A$ in $C$ such that the restriction $f|_A$ of $f$ to $A$ is measure linear with respect to Hausdorff measure.

In other words, any $K$-bilipschitz map from one multi-model Cantor set to another is measure linear on some clopen.

We sketch the proof from [3]. The map $f$ is $K$-bilipschitz so the mass ratio function is bounded above. Thus there exists a supremum for the possible mass ratios. Let $A$ be a clone contained in $C$ with mass ratio within $\epsilon$ of this supremum. First note that if the mass ratios of all clones contained in $A$ were equal, then we would already have a measure linear map on $A$. Also if each clone contained in $A$ had mass ratio smaller than that of $A$ then the union of these clones would also have smaller mass ratio which is a contradiction. So assume there is a clone $E$ of minimum level, with mass ratio greater than that of $A$. We call this a “greedy” clone. We then, by corollary 3, have only a finite number of possibilities for the quotient of mass ratios of $E$ and the smallest clone $D$ which contains it.

If the mass ratio of $D$ were larger than the mass ratio of $A$ we would have chosen this for $E$ as it has lower level. On the other hand the mass ratio of $D$ cannot be smaller than the mass ratio of $A$ else there exists a greedy clone at the same level as $D$. Therefore the mass ratio of $D$ must be equal to the mass ratio of $A$. This implies that the quotient of mass ratios of $E$ and $D$ must be greater than 1.

Thus $\text{MR}(E)/\text{MR}(A)$ is in a finite set of numbers bigger than 1. This implies that the mass ratio of $E$ is larger than the supremum of mass ratios, which is of course a contradiction and so proves that there could not have been a “greedy” clone. Therefore the map must have been measure linear on $A$. $\blacksquare$
In [3], Cooper defines various invariants of clone Cantor sets under bilipschitz maps. We say two sets of real numbers $A$ and $B$ are similar if there exist non-zero scalars $\alpha, \beta$ such that

$$\alpha A \subset B \quad \text{and} \quad \beta B \subset A.$$ 

Let $C$ be a multi-model Cantor set with Hausdorff measure $\mathcal{H}_\delta(C)$ where $\delta$ is the Hausdorff dimension. We define the clopen invariant of an arbitrary subset $B$ of $C$ as the similarity class of the countable set of real numbers \(\{\mathcal{H}_\delta(A) : A \text{ a clopen in } B\}\).

**Lemma 15.** Given a multi-model Cantor set with irreducible matrix and any two clones, $A$ and $B$, then the clopen invariants of $A$ and $B$ are equal.

**Proof.** The clone-model map from $A$ to the model $A_j$ shows that $A$ has the same clopen invariant as model $A_j$. Similarly $B$ has the same clopen invariant as $A_i$ for some $i$. A scaled down copy of $A_j$ is contained in $A_i$ for all $i$ hence the clopen invariant for the model $A_j$ is similar to that for every other model. \(\square\)

We can now define the clopen invariant of a multi-model Cantor set. Let $C$ be a multi-model Cantor set with Hausdorff measure $\mathcal{H}_\delta(C)$ where $\delta$ is the Hausdorff dimension. We define the clopen invariant of $C$ as the similarity class of the countable set of real numbers \(\{\mathcal{H}_\delta(A) : A \text{ a clopen in } A_j\}\) for some $j$. The choice of $j$ is irrelevant by lemma 15.

**Corollary 4 (Corollary 10.7 from [3]).** If $C, C'$ are multi-model Cantor sets and there is a bilipschitz map of $C$ onto a clopen in $C'$, then $C$ and $C'$ have the same clopen invariant.

By theorem 2, there is a clopen, hence a clone $A$ in $C$ with a measure linear map of $A$ onto a clopen in $C'$. The clone-model map from $A$ to $A_j$ for some $j$ shows that $A$ has the same clopen invariant as $A_j$ and hence $C$ by lemma 15. Let $\alpha$ be the Radon-Nikodym derivative of the measure linear map of $A$ into $C'$. The image of a clopen $B$ in $A$ is a clopen in $C'$ with $\alpha$ times the measure of $B$. By the same argument as above, this clopen invariant is the same as the clopen invariant for $C'$. Using that the inverse of a bilipschitz map is bilipschitz, we obtain the inverse relationship.
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