FRACTIONAL HARDY–TYPE INEQUALITIES IN DOMAINS WITH PLUMP COMPLEMENT

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Abstract. We establish fractional Hardy-type inequalities in a bounded domain with plump complement. In particular our results apply in bounded $C^\infty$ domains and Lipschitz domains.

1. Introduction

Let $\Omega$ be a proper subdomain in $\mathbb{R}^n$, $n \geq 2$. Let $s \in (0,1)$ and let $p, q \in (1, \infty)$ be given such that $0 < 1/p - 1/q < s/n$. We investigate the inequality

\[ \int_\Omega \frac{|u(x)|^q}{\text{dist}(x, \partial \Omega)^q(s+n(1/q-1/p))} dx \leq c \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{ps+n}} dy dx \right)^{q/p} \]

for every $u \in W^{s,p}(\mathbb{R}^n)$ with $\text{spt} u \subset \overline{\Omega}$; here the finite constant $c$ depends only on $s, n, p, q, \Omega$. Our work was motivated by the following fractional order inequality

\[ \int_\Omega \frac{|u(x)|^p}{\text{dist}(x, \partial \Omega)^p} dx \leq c \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{ps+n}} dy dx \]

for all $u \in C_0(\Omega)$ with a finite constant $c$ which depends only on $s, n, p, \Omega$. B. Dyda proved that inequality (1.2) holds in $\Omega$ with $p > 0$, if one of the following conditions is valid:

1. if $\Omega$ is a bounded Lipschitz domain and $sp > 1$,
2. if $\Omega$ is a complement of a bounded Lipschitz domain and $sp \in (0, \infty) \setminus \{1, n\}$,
3. if $\Omega$ is a complement of a point and $sp \in (0, \infty) \setminus \{n\},$

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(4) if $\Omega$ is a domain above the graph of a Lipschitz function $\mathbb{R}^{n-1} \to \mathbb{R}$ and $sp \in (0, \infty) \setminus \{1\},$

[D, Theorem 1.1]. He showed also that inequality (1.2) is false if $\Omega$ is a bounded Lipschitz domain with $sp \leq 1$ and $s < 1.$ Inequality (1.2) was proved for convex domains when $1 < p < \infty$ and $1/p < s < 1$ by M. Loss and C. A. Sloane, [LS, Theorem 1.2]. Inequality (1.2) holds in a half-space whenever $0 < s < 1, sp \neq 1, 1 \leq p < \infty,$ by R. L. Frank and R. Seiringer [FS, Theorem 1.1]; the $p = 2$-case was considered in [BD, Theorem 1.1].

We prove fractional Hardy-type inequalities (1.1) in a bounded domain whose complement is plump in the sense of the following definition. The open and closed $n$-dimensional Euclidean balls, centered at a point $x$ and with radius $r > 0,$ are denoted by $B_n(x, r)$ and $\overline{B}_n(x, r),$ respectively.

1.3. **Definition.** Let $n \geq 2$ and $\eta \geq 1.$ A set $A$ in $\mathbb{R}^n$ is **$\eta$-plump** if for all $x \in \overline{A}$ and all $r \in (0, \text{diam}(A))$ there is a point $z$ in $B_n(x, r)$ with $B_n(z, r/\eta) \subset A.$

The following is our main theorem.

1.4. **Theorem.** Suppose that $\Omega$ is a bounded domain in $\mathbb{R}^n,$ $n \geq 2,$ with an $\eta$-plump complement $\mathbb{R}^n \setminus \Omega, \eta \geq 1.$ Let $s \in (0, 1)$ and let $p, q \in (1, \infty).$ If $0 < 1/p - 1/q < s/n,$ then

$$
\left( \int_{\Omega} \frac{|u(x)|^q}{\text{dist}(x, \partial \Omega)^{q(s+n(1/q-1/p))}} dx \right)^{1/q} \\
\leq c_{s,n,p,q} \eta^{2n/q+s-n/p} \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{sp+n}} dy \ dx \right)^{1/p}
$$

for every $u \in W^{s,p}(\mathbb{R}^n)$ with $\text{spt} \ u \subset \overline{\Omega}.$

Examples of bounded domains with $\eta$-plump complement include Lipschitz domains and convex domains. More examples are obtained by using $K$-quasiconformal mappings $f : \mathbb{R}^n \to \mathbb{R}^n.$ If $\Omega$ in $\mathbb{R}^n$ is a bounded domain with an $\eta$-plump complement, then the image $f\Omega$ is also bounded and has a $\mu$-plump complement, where $\mu$ depends on $n, K$ and $\eta$ only, see e.g. [V, Theorem 6.6].

We give applications of Theorem 1.4 in Section 4.
2. Notation and auxiliary results

The Lebesgue measure of a measurable set $E$ in $\mathbb{R}^n$ is written as $|E|$. For a measurable set $E$, with a finite and positive measure, we write
\[ \int_E f(x) \, dx = \frac{1}{|E|} \int_E f(x) \, dx. \]

We write $\chi_E$ for the characteristic function of a set $E$.

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, and let $\mathcal{W}$ be its Whitney decomposition. For the properties of Whitney cubes $Q \in \mathcal{W}$. We refer to E. M. Stein's book, [S]. In particular, we need the inequalities
\[ \text{diam}(Q) \leq \text{dist}(Q, \partial \Omega) \leq 4 \text{diam}(Q), \quad Q \in \mathcal{W}. \]

We let $Q \in \mathcal{W}$ be a cube with center $x_Q$ and side length $\ell(Q)$. By $tQ$, $t > 0$, we mean a cube with sides parallel to those of $Q$ that is centered at $x_Q$ and whose side length is $t\ell(Q)$.

We recall definition of the fractional order Sobolev spaces in a domain $\Omega$ in $\mathbb{R}^n$. For $1 \leq p < \infty$ and $s \in (0, 1)$ we let $W^{s,p}(\Omega)$ be the collection of all functions $f$ in $L^p(\Omega)$ with
\[ |f|_{W^{s,p}(\Omega)} := \left( \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{sp+\eta}} \, dx \, dy \right)^{1/p}, \]
where $\eta = sp_n/n$. The support of a function $f : \mathbb{R}^n \to \mathbb{C}$ is denoted by $\text{spt} f$, and it is the closure of the set $\{x : f(x) \neq 0\}$ in $\mathbb{R}^n$.

The notation $a \lesssim b$ mean that an inequality $a \leq cb$ holds for some constant $c > 0$ whose exact value is not important. We use subscripts to indicate the dependence on parameters, for example, a quantity $c_d$ depends on a parameter $d$.

We state fractional Sobolev–Poincaré inequalities for a cube.

2.2. Lemma. Let $Q$ be a cube in $\mathbb{R}^n$, $n \geq 2$. Suppose that $p, q \in [1, \infty)$, and $s \in (0, 1)$ satisfy $0 \leq 1/p - 1/q < s/n$. Then, for every $u \in L^p(Q)$,
\[ \frac{1}{|Q|} \int_Q |u(x) - u_Q|^q \, dx \leq c|Q|^{sp/q - q/p} \left( \int_Q \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dy \, dx \right)^{q/p}. \]
Here the constant $c > 0$ is independent of $Q$ and $u$.

Proof. The inequality follows from [H-SV, Remark 4.14], when $Q = [-1/2, 1/2]^n$. A change of variables gives the general case. \qed

Let $0 < \sigma < d$. The Riesz potential of a function $f$ is given by
\[ I_\sigma f(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x - y|^{d-\sigma}} \, dy. \]

The following theorem is from [He, Theorem 1].
2.3. **Theorem.** Suppose that $0 < \sigma < d$ and let $p, q \in (1, \infty)$. If

$$0 < 1/p - 1/q = \sigma/d,$$

then there is a constant $c > 0$ such that inequality $\|I_\sigma(f)\|_q \leq c\|f\|_p$ holds for every $f \in L^p(\mathbb{R}^d)$.

Recall from [A] that the fractional maximal function of a locally integrable function $f : \mathbb{R}^d \to [-\infty, \infty]$ is

$$M_\sigma f(x) = \sup_{r>0} \frac{r^\sigma}{|B^d(x, r)|} \int_{B^d(x, r)} |f(y)| dy.$$

If $Q$ is a cube in $\mathbb{R}^d$ and $x \in Q$, then

$$\frac{\ell(Q)^\sigma}{|Q|} \int_Q |f(y)| dy \leq c_d M_\sigma f(x).$$

Since $0 < \sigma < d$, there is a constant $c_d > 0$ such that

$$M_\sigma f(x) \leq c_d I_\sigma |f|(x)$$

for every $x \in \mathbb{R}^d$.

2.6. **Lemma.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, and let $\mathcal{W}$ be its Whitney decomposition. Suppose that $1 < r < p < q < \infty$ and $\kappa \geq 1$. Then

$$\sum_{Q \in \mathcal{W}} |\kappa Q|^{2\beta} \left( \int_{\kappa Q} \int_{\kappa Q} |g(x, y)| \, dx \, dy \right)^t$$

$$\leq c_{n, r, p, q, \kappa} \left( \int_{\mathbb{R}^n \times \mathbb{R}^n} |g(x, y)|^s \, dx \, dy \right)^{t/s}$$

for every $g \in L^s(\mathbb{R}^n \times \mathbb{R}^n)$, where $s = p/r$, $t = q/r$ and $\beta = t/s = q/p$.

**Proof.** The fractional maximal function $M_\sigma$ and the Riesz potential $I_\sigma$ are both associated with $\mathbb{R}^d$. Throughout this proof $d = 2n$ and $\sigma = 2n(\beta - 1)/t$.

Let us rewrite the left hand side of inequality (2.7) as

$$LHS = \kappa^n \sum_{Q \in \mathcal{W}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_{\kappa Q}(z) \chi_{\kappa Q}(w)$$

$$\left( \frac{\ell(\kappa Q)^{2n(\beta - 1)/t}}{|\kappa Q|} \int_{\kappa Q} \frac{1}{|\kappa Q|} \int_{\kappa Q} |g(x, y)| \, dx \, dy \right)^t \, dz \, dw.$$
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By (2.4) with \((z, w) \in \kappa Q \times Q \subset \kappa Q \times \kappa Q \subset \mathbb{R}^d\) and by (2.5)

\[
\kappa^{-n}LHS \lesssim \sum_{Q \in \mathcal{W}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_{\kappa Q}(z)\chi_Q(w) [\mathcal{M}_\sigma g(z, w)]^t \, dz \, dw
\]

\[
\lesssim \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} [\mathcal{M}_\sigma g(z, w)]^t \, dz \, dw
\]

\[
\lesssim \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} [I_\sigma |g|(z, w)]^t \, dz \, dw.
\]

Since \(1 < s = p/r < t = q/r < \infty\) and

\[
\frac{r}{p} - \frac{r}{q} = \beta - 1 = \frac{\sigma}{2n},
\]

we obtain \(0 < 1/s - 1/t = \sigma/2n < 1\). Hence, Theorem 2.3 yields the right hand side of inequality (2.7). \(\square\)

3. A Proof of Theorem 1.4

We prove a fractional Hardy-type inequality in a domain \(\Omega\) whose complement is \(\eta\)-plump.

**Proof of Theorem 1.4.** By [V, Theorem 3.52] and inequalities (2.1) we see that for every \(Q \in \mathcal{W}\) there is a closed cube \(Q^*\) in \(\mathbb{R}^n\) such that

\(Q^* \subset \mathbb{R}^n \setminus \bar{\Omega}, \ \text{diam}(Q) = \text{diam}(Q^*), \ \text{dist}(Q, Q^*) \leq 15\eta \text{diam}(Q)\).

We write \(Q^* := \kappa Q\) for the dilated cube of \(Q\) having the same centre as \(Q\) and side length \(\kappa \ell(Q), \ \kappa = 40\eta \sqrt{n}\). The triangle inequality implies that \(Q^* \subset Q^*\). Let

\[
(3.1) \quad \alpha = s + n/q - n/p > 0.
\]

Suppose that \(u \in W^{s,p}(\mathbb{R}^n)\) has support in \(\bar{\Omega}\). By (2.1),

\[
\int_{\Omega} \frac{|u(x)|^q}{\text{dist}(x, \partial \Omega)^{\alpha q}} \, dx \leq \sum_{Q \in \mathcal{W}} \text{diam}(Q)^{-\alpha q} \int_{Q} |u(x) - u_{Q^*}|^q \, dx.
\]

For a given \(Q \in \mathcal{W}\) the inclusion \(Q \subset Q^*\) yields

\[
\int_{Q} |u(x) - u_{Q^*}|^q \, dx \lesssim \int_{Q^*} |u(x) - u_{Q^*}|^q \, dx + |Q||u_{Q^*} - u_{Q^*}|^q
\]

Since \(|Q| = |Q^*|\) and \(Q^* \subset Q^*\), we obtain

\[
|Q||u_{Q^*} - u_{Q^*}|^q = \int_{Q^*} |u_{Q^*} - u_{Q^*}|^q \, dx
\]

\[
\lesssim \int_{Q^*} |u(x) - u_{Q^*}|^q \, dx + \int_{Q^*} |u(x) - u_{Q^*}|^q \, dx.
\]
Because \(0 < 1/p - 1/q < s/n\), there is a number \(r \in (1, p)\) such that
\[
\mu = n(1/p - 1/r) + s \in (0, s)
\]
and \(0 < 1/r - 1/q < \mu/n\). Application of Lemma 2.2 to the cubes \(Q^*\) and \(Q^s\) yields
\[
\int_Q |u(x) - u_{Q^s}|^q \, dx \lesssim |Q^*|^{1+\mu n/q - r/q} \left( \int_{Q^*} \int_{Q^*} \frac{|u(x) - u(y)|^r}{|x - y|^{n+\mu r}} \, dy \, dx \right)^{q/r}.
\]
Hence,
\[
\int_{\Omega} \frac{|u(x)|^q}{\text{dist}(x, \partial \Omega)^{\alpha q}} \, dx \lesssim \sum_{Q \in \mathcal{W}} \text{diam}(Q)^{-\alpha q} \int_Q |u(x) - u_{Q^s}|^q \, dx
\]
\[
\lesssim \eta^{\alpha q} \sum_{Q \in \mathcal{W}} |Q^*|^{1+\mu n/s - 1/r - \alpha/n} \left( \int_{Q^*} \int_{Q^*} \frac{|u(x) - u(y)|^r}{|x - y|^{n+\mu r}} \, dy \, dx \right)^{q/r}
\]
\[
\lesssim \eta^{\alpha q} \sum_{Q \in \mathcal{W}} |Q^*|^{1+\mu n/s + 1/r - \alpha/n} \left( \int_{Q^*} \int_{Q^*} \frac{|u(x) - u(y)|^r}{|x - y|^{n+\mu r}} \, dy \, dx \right)^{q/r}.
\]
Equations (3.1) and (3.2) imply that
\[
1 + q(\mu/n + 1/r - \alpha/n) = 2q/p.
\]
Hence, Lemma 2.6 yields
\[
\int_{\Omega} \frac{|u(x)|^q}{\text{dist}(x, \partial \Omega)^{\alpha q}} \, dx \lesssim \eta^{\alpha q} \sum_{Q \in \mathcal{W}} |Q^*|^{2q/p} \left( \int_{Q^*} \int_{Q^*} \frac{|u(x) - u(y)|^r}{|x - y|^{n+\mu r}} \, dy \, dx \right)^{q/r}
\]
\[
\lesssim \eta^{q+n} \left( \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{p\alpha + n}} \, dx \, dy \right)^{q/p}.
\]
Since \(\alpha q + n = 2n + q(s - n/p)\), the claim follows. \(\square\)

4. Applications of Theorem 1.4

Let us begin with certain function spaces. The usual Besov space \(B^s_{p,p}(\mathbb{R}^n)\) coincides with the Sobolev space \(W^{s,p}(\mathbb{R}^n)\), [T2, pp. 6–7]. Hence, we may define
\[
\tilde{B}^s_{p,p}(\Omega) = \{ u \in W^{s,p}(\mathbb{R}^n) : \text{spt } u \subset \overline{\Omega} \},
\]
\[
\|u\|_{\tilde{B}^s_{p,p}(\Omega)} = \|u\|_{W^{s,p}(\mathbb{R}^n)}.
\]

(4.1)
The following corollary follows immediately from Theorem 1.4.

4.2. **Corollary.** Suppose that $\Omega$ is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with an $\eta$-plump complement $\mathbb{R}^n \setminus \Omega$, $\eta \geq 1$. Let $s \in (0, 1)$ and $p, q \in (1, \infty)$. If $0 < 1/p - 1/q < s/n$, then

$$
\left( \int_{\Omega} \frac{|u(x)|^q}{\text{dist}(x, \partial \Omega)^{eq+n(1-q/p)}} \, dx \right)^{1/q} \leq c_{s,n,p,q}\eta^{2n/q+s-n/p}||u||_{\tilde{B}_{sp}(\Omega)}
$$

for every $u \in \tilde{B}_{sp}(\Omega)$.

Related Hardy inequalities for a wider scale of Triebel–Lizorkin and Besov spaces $\tilde{F}_{pq}^s(\Omega)$ and $\tilde{B}_{pq}^s(\Omega)$, respectively, have been considered in [T1]. The novelty in our result is that we only require the complement of $\Omega$ in $\mathbb{R}^n$ to be $\eta$-plump.

Let us study the validity of an intrinsic Hardy-type inequality. We focus on bounded Lipschitz domains and $C^\infty$ domains in $\mathbb{R}^n$, [T3, p. 64]. In both cases, the complement of $\Omega$ in $\mathbb{R}^n$ is $\eta$-plump for some $\eta \geq 1$.

The following corollary applies to all $u \in W^{s,p}(\Omega)$ but is restricted to the case $0 < s < 1/p$.

4.3. **Corollary.** Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. Let $p, q \in (1, \infty)$ and $s \in (0, 1/p)$. If $0 < 1/p - 1/q < s/n$, then there is a constant $c > 0$ such that the inequality

$$
\left( \int_{\Omega} \frac{|u(x)|^q}{\text{dist}(x, \partial \Omega)^{eq+n(1-q/p)}} \, dx \right)^{1/q} \leq c||u||_{L^p(\Omega)} + c \left( \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{sp+n}} \, dx \, dy \right)^{1/p} = c||u||_{W^{s,p}(\Omega)}
$$

holds for all $u \in W^{s,p}(\Omega)$.

**Proof.** Since $B_{sp}(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n)$ the usual Besov space $B_{sp}(\Omega)$ can be defined by

$$
B_{sp}(\Omega) = \{ f \in L^p(\Omega) : f = g|\Omega \text{ for some } g \in W^{s,p}(\mathbb{R}^n) \}, ~ ~ ~ ||f||_{B_{sp}(\Omega)} = \inf ||g||_{W^{s,p}(\mathbb{R}^n)},
$$

where the infimum is taken over all functions $g \in W^{s,p}(\mathbb{R}^n)$, $g|\Omega = f$.

In the following two identifications we assume that $\Omega$ is a bounded Lipschitz domain. First,

$$
\tilde{B}_{sp}(\Omega) = B_{sp}(\Omega)
$$

with equivalent norms, (4.1) and [T3, p. 66]. The spaces $B_{sp}(\Omega)$ and $W^{s,p}(\Omega)$ coincide and the norms are equivalent, [DS, Theorem 6.7] and
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[T3, Theorem 1.118]. Inequality (4.4) is therefore a consequence of Corollary 4.2. □

The assumption $0 < s < 1/p$ can be relaxed if we restrict the boundary behavior of functions. We state the following corollary.

4.5. Corollary. Suppose that $\Omega$ is a bounded $C^\infty$ domain in $\mathbb{R}^n$, $n \geq 2$. Let $p, q \in (1, \infty)$ and $s \in (0, 1)$, $s \neq 1/p$. If $0 < 1/p - 1/q < s/n$, then there is a constant $c > 0$ such that the inequality

$$
\left( \int_{\Omega} \frac{|u(x)|^q}{\text{dist}(x, \partial \Omega)^{sq+n(1-q/p)}} \, dx \right)^{1/q} \leq c \|u\|_{L^p(\Omega)} + c \left( \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{sp+n}} \, dx \, dy \right)^{1/p} = c \|u\|_{W^{s,p}(\Omega)}
$$

holds for all $u \in W^{s,p}_0(\Omega) := C_0^\infty(\Omega)^{W^{s,p}(\Omega)}$.

Proof. Observe that $\Omega$ is also a bounded Lipschitz domain in $\mathbb{R}^n$. Hence, reasoning as in the proof of Corollary 4.3 yields $W^{s,p}(\Omega) = B^s_{pp}(\Omega)$ and, consequently,

$$
W^{s,p}_0(\Omega) = C_0^\infty(\Omega)^{B^s_{pp}(\Omega)} = \tilde{B}^s_{pp}(\Omega).
$$

Because $s \neq 1/p$,

$$
B^s_{pp}(\Omega) = \tilde{B}^s_{pp}(\Omega);
$$

we refer to [T3, pp. 66-67]. Inequality (4.6) follows from these facts and Corollary 4.2. □

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