MULTIPLE SOLUTIONS FOR NONLINEAR CONE DEGENERATE ELLIPTIC EQUATIONS

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Dedicated to the 80th birthday of Professor Shuxing Chen

ABSTRACT. The present paper is concerned with the Dirichlet boundary value problem for nonlinear cone degenerate elliptic equations. First we introduce the weighted Sobolev spaces, inequalities and the property of compactness. After the appropriate energy functional established, we obtain the existence of infinitely many solutions in the weighted Sobolev spaces by applying the variational methods.

1. Introduction and main results. Write $B = [0,1) \times X$ as a local model of stretched cone-manifold (i.e. manifold with conical singularities) with dimension $N \geq 3$. Here $X \subset S^{N-1}$ is bounded open set in the unit sphere of $\mathbb{R}^{N-1}$, and $x' = (x_2, \ldots, x_N) \in X$. Let $\text{int} B$ be the interior of $B$ and $\partial B := \{0\} \times X$ be the boundary of $B$. The cone type degenerate quasi-linear elliptic operator is defined as follows,

$$-\Delta_{p,B} u := -x_1^{-p} \text{div}_B (|\nabla_B u|^{p-2} \nabla_B u),$$

where $\nabla_B = (x_1 \partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_N})$ is cone type gradient operator and for $F = (F_1, F_2, \ldots, F_N)$, the cone type divergence operator $\text{div}_B F = \nabla_B \cdot F = x_1 \partial_{x_1} F_1 + \partial_{x_2} F_2 + \ldots + \partial_{x_N} F_N$.

The present paper is concerned with the following Dirichlet boundary value problem involving the cone degenerate quasi-linear operator,

$$\begin{cases}
-\Delta_{p,B} u := -x_1^{-p} \text{div}_B (|\nabla_B u|^{p-2} \nabla_B u) = \lambda |u|^{q-2} u, & \text{in } \text{int} B \\
u = 0, & \text{on } \partial B.
\end{cases}$$

where $\lambda > 0$, $2 \leq p < N$ and $p < q < p^* = \frac{Np}{N-p}$.

The Dirichlet boundary value problems for nonlinear elliptic equations involving the standard $p$-Laplacian,

$$-\Delta_p u = -\text{div}(|\nabla u|^{p-2}\nabla u)$$

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have been widely studied. In Riemannian geometry the critical case for \( p = 2 \) and \( q = 2^* \) is the Yamabe problem in [21]. The model of Frank-Kamenetstki for solid ignition also has strong relationship with this kind of problem, see [14]. The existence and non-uniqueness for the \( p \)-Laplacian have been verified in [15]. More interesting and important results about the \( p \)-Laplacian can be found in [3, 4, 12, 16] and references therein.

The degenerate \( p \)-Laplacian

\[
-\text{div}(a(x)|\nabla u|^{p-2}\nabla u)
\]

with the set \( \{x | a(x) = 0\} \) being non-empty has been studied in [1, 5, 6]. The nonlinear equations with cone degenerate Laplacian have been investigated in [7, 8, 9, 10].

The motivation for the cone degenerate quasi-linear elliptic operators in (1.1) comes from the calculus on manifolds with conical singularities. A finite dimensional manifold \( B \) with conical singularities is a topological space with a finite subset \( B_0 = \{b_1, ..., b_M\} \subset B \) of conical singularities, with the two following properties:

1. \( B \setminus B_0 \) is a \( C^\infty \) manifold.
2. Every \( b \in B_0 \) has an open neighbourhood \( U \) in \( B \), such that there is a homeomorphism

\[
\varphi : U \to X^\Delta,
\]

and \( \varphi \) restricts a diffeomorphism

\[
\varphi' : U \setminus \{b\} \to X^\wedge.
\]

Here \( X \) be a bounded subset of the unit sphere \( S^{N-1} \) of \( \mathbb{R}^{N-1} \), and set

\[
X^\Delta = \mathbb{R}_+ \times X/(\{0\} \times X),
\]

which is the local model interpreted as a cone with the base \( X \). Since the analysis is formulated off the singularity it makes sense to pass to

\[
X^\wedge = \mathbb{R}_+ \times X,
\]

which is the open stretched cone with the base \( X \). Here we take the simplest case that

\[
\mathbb{B} = [0, 1) \times X, \quad \text{and} \quad \partial \mathbb{B} = \{0\} \times X.
\]

The typical linear differential operators on a manifold with conical singularities, are called Fuchs type, if the operators in a neighborhood of \( x_1 = 0 \) are of the following form

\[
A = x_1^{-m} \sum_{k=0}^{m} a_k(x_1) \left( -x_1 \frac{\partial}{\partial x_1} \right)^k \quad (1.3)
\]

with the coefficients \( a_k(x_1) \in C^\infty(\mathbb{R}_+, \text{Diff}^{m-k}(X)) \). More examples of this kind of operators are expressed in [20]. Furthermore, in [11, 13, 17, 19] and references therein, one can find more information about operators on manifolds with singularities.

For the case \( p = 2 \) in (1.2), the problem involving the linear Fuchs type operator

\[
-x_1^{-2}\text{div}_B(\nabla_B u)
\]

possesses infinity many weak solutions in weighted Sobolev spaces \( \mathcal{H}^{1,\frac{2}{m}}_{2,\theta}(\mathbb{B}) \) (the definition of \( \mathcal{H}^{1,\frac{2}{m}}_{p,\theta}(\mathbb{B}) \), please see Definition 2.4 below) by applying similar procedure in [8, 9].

Instead of the linear operators in (1.3), here in the present paper we study a kind of nonlinear degenerate operator (1.1). For the Dirichlet boundary value problem
we say \(u \in H^{1,\frac{N}{p}}_{p,0}(B)\) is a weak solution, if

\[
\int_B |\nabla_B u|^{p-2} \nabla_B u \cdot \nabla_B \varphi \frac{dx_1}{x_1} \, dx' = \lambda \int_B x_1^q |u|^{q-2} u \varphi \frac{dx_1}{x_1} \, dx',
\]

for any \(\varphi \in C_0^\infty(\text{int}B)\). Then we have the following results.

**Theorem 1.1.** For \(2 < p < N\) and \(p < q < p^*\), the Dirichlet boundary value problem (1.2) has infinitely many weak solutions in \(H^{1,\frac{N}{p}}_{p,0}(B)\).

In order to prove Theorem 1.1, variational methods and the abstract theory for critical points will be employed. The weak solutions for the Dirichlet boundary value problem (1.2) are critical points of the energy functional

\[
J(u) = \frac{1}{p} \int_B |\nabla_B u|^{p-2} \nabla_B u \cdot \nabla_B \varphi \frac{dx_1}{x_1} \, dx' - \lambda \int_B x_1^q |u|^{q-2} u \varphi \frac{dx_1}{x_1} \, dx'.
\]

Furthermore, the functional \(J(\cdot)\) possesses unbounded critical value sequence, the precise behavior could be expressed as follows.

**Corollary 1.1.** If \(\{c_m\}_{m \in \mathbb{N}}\) is the critical value sequence obtained in Theorem 1.1, then we have

\[
c_m \to \infty \quad \text{as} \quad m \to \infty.
\]

This paper is organized as follows. In section 2, some preliminaries are given here, such as definition of the weighted Sobolev spaces, cone type Sobolev inequality, cone type Poincaré inequality and the property of compact embedding. Afterwards, the abstract theory for critical points is introduced, which will be applied for the proof of Theorem 1.1. In section 3, we first prove the (PS) condition of the energy functional, and then use the variational methods to investigate the existence of infinitely many solutions for the Dirichlet problem (1.2). Finally, we derive the precise behavior of the critical value sequence obtained in Theorem 1.1.

2. Preliminaries. In order to express the weak solutions for Dirichlet problem (1.2), we need the adequate distribution spaces. To define the weighted Sobolev spaces on the stretched cone \(B\), we first introduce the weighed Sobolev spaces and weighted \(L_p^\gamma\) spaces on \(\mathbb{R}^N_+\).

**Definition 2.1.** For the weight data \(\gamma \in \mathbb{R}\) and \(x \in \mathbb{R}_+ \times \mathbb{R}^{N-1}\), we say that \(u(x) \in L^\gamma_p(\mathbb{R}^N_+)\) if \(u \in D'(\mathbb{R}^N_+)\) and

\[
\|u\|_{L^\gamma_p(\mathbb{R}^N_+)} = \left\{ \int_{\mathbb{R}_+} \int_{\mathbb{R}^{N-1}} |x_1^\frac{N}{p} u(x)|^p \, dx \, d\sigma \right\}^{\frac{1}{p}} < +\infty,
\]

here and after we simplify the notation as \(d\sigma = \frac{dx_1}{x_1} \, dx'\).

**Definition 2.2.** For \(m \in \mathbb{N}\), and \(\gamma \in \mathbb{R}\), the spaces

\[
\mathcal{H}^m_{\gamma}(\mathbb{R}^N_+) := \left\{ u \in D'(\mathbb{R}^N_+) : (x_1 \partial_{x_1})^\alpha \partial_\beta u \in L^\gamma_p(\mathbb{R}^N_+) \right\},
\]

for arbitrary \(\alpha \in \mathbb{N}\), \(\beta \in \mathbb{N}^{N-1}\), and \(\alpha + |\beta| \leq m\).

On the weighted Sobolev spaces, we have the following cone-type Sobolev inequality. According to Section 2.1 in [20], we can generalize the definitions of the weighted Sobolev spaces on \(\mathbb{R}^N_+\) to \(X^\gamma\).
**Definition 2.3.** Let $X$ be a closed compact $C^\infty$ manifold, and $\mathcal{U} = \{U_1, \ldots, U_M\}$ an open covering of $X$ by coordinate neighborhoods. Fix a subordinate partition of unity $\{\varphi_1, \ldots, \varphi_M\}$ and charts $\chi_j : U_j \to \mathbb{R}^{N_j}$, $j = 1, \ldots, M$, then $u \in \mathcal{H}^{m,\gamma}_p(X^\omega)$ if and only if $u \in \mathcal{D}'(X^\omega)$ with the norm

$$
\|u\|_{\mathcal{H}^{m,\gamma}_p(X^\omega)} = \left\{ \sum_{j=1}^M \| (1 \times \chi_j^*)^{-1} \varphi_j u \|_{\mathcal{H}^p(X^\omega)}^p \right\}^{\frac{1}{p}} < +\infty.
$$

Here $1 \times \chi_j : C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^{N_j}) \to C_c^\infty(\mathbb{R}_+ \times U_j)$ is the pull-back function with respect to $1 \times \chi_j : \mathbb{R}_+ \times U_j \to \mathbb{R}_+ \times \mathbb{R}^{N_j}$. Denote $\mathcal{H}^{m,\gamma}_p(X^\omega)$ as the subspace of $\mathcal{H}^{m,\gamma}_p(X^\omega)$ which is defined as the closure of $C_c^\infty(X^\omega)$ with respect to the norm $\| \cdot \|_{\mathcal{H}^{m,\gamma}_p(X^\omega)}$.

**Definition 2.4.** Let $W^{m,p}_{\text{loc}}(\operatorname{int}B)$ denote the classical local Sobolev space (where $\operatorname{int}B$ is interior of $B$). For $1 \leq p < \infty$, $m \in \mathbb{N}$ and the weighted data $\gamma \in \mathbb{R}$, then $\mathcal{H}^{m,\gamma}_p(B)$ denotes the subspace of all $u \in W^{m,p}_{\text{loc}}(\operatorname{int}B)$, such that

$$
\mathcal{H}^{m,\gamma}_p(B) = \{ u \in W^{m,p}_{\text{loc}}(\operatorname{int}B) \mid \omega u \in \mathcal{H}^{m,\gamma}_p(X^\omega) \}
$$

for any cut-off function $\omega$, supported by a collar neighborhood of $[0,1) \times \partial B$. Moreover, the subspace $\mathcal{H}^{m,\gamma}_{p,0}(B)$ of $\mathcal{H}^{m,\gamma}_p(B)$ can be defined by

$$
\mathcal{H}^{m,\gamma}_{p,0}(B) := [\omega] \mathcal{H}^{m,\gamma}_p(X^\omega) + [1 - \omega] W^{m,p}_{0,\text{loc}}(\operatorname{int}B),
$$

where the cut-off functions $\omega$ defined as before, and $W^{m,p}_{0,\text{loc}}(\operatorname{int}B)$ denotes the closure of $C_c^\infty(\operatorname{int}B)$ in Sobolev space $W^{m,p}(\tilde{X})$ when $\tilde{X}$ is a closed compact $C^\infty$ manifold of dimension $N$ that containing $B$ as a submanifold with boundary. Also, we have $L^\gamma_p(B) := \mathcal{H}^{0,\gamma}_p(B)$ and $x_1^{\gamma_1} \mathcal{H}^{m,\gamma_2}_p(B) = \mathcal{H}^{m,\gamma_1+\gamma_2}_p(B)$.

For the proof of the main result, the following inequality is requisite.

**Proposition 2.1** (Cone type Poincaré Inequality). Let $1 \leq p < \infty$ and $\gamma \in \mathbb{R}$. If $u(x) \in \mathcal{H}^{1,\gamma}_{p,0}(B)$, then

$$
\|u(x)\|_{L^p_\omega(B)} \leq c \|\nabla \chi u(t,x)\|_{L^p_\omega(B)},
$$

where the constant $c$ depends only on $B$ and $p$.

**Proof.** Follow the same process of Theorem 2.5 in [8].

**Remark 2.1.** The cone type Poincaré inequality implies that the norm $\|u\|_{\mathcal{H}^{1,\gamma}_{p,0}(B)}$ is equivalent to the norm $\|\nabla u\|_{L^p_\omega(B)}$.

**Lemma 2.1.** For $1 < p_2 < N$ and $1 \leq p_1 < p_2^* = \frac{Np_2}{N-p_2}$, the embedding

$$
\mathcal{H}^{1,\gamma_2}_{p_2,0}(B) \hookrightarrow L^{\gamma_1}_{p_1}(B)
$$

is compact, if $\frac{N}{p_1} - \gamma_1 > \frac{N}{p_2} - \gamma_2$.

**Proof.** According to Definition 2.4, we write

$$
\mathcal{H}^{1,\gamma_2}_{p_2,0}(B) := [\omega] \mathcal{H}^{1,\gamma_2}_{p_2,0}(X^\omega) + [1 - \omega] W^{1,p_2}_{0,\text{loc}}(\operatorname{int}B),
$$

$$
\mathcal{H}^{0,\gamma_1}_{p_1,0}(B) := [\omega] \mathcal{H}^{0,\gamma_1}_{p_1,0}(X^\omega) + [1 - \omega] W^{0,p_1}_{0,\text{loc}}(\operatorname{int}B).
$$

Observe that the embedding $\mathcal{H}^{0,\gamma_1}_{p_1,0}(B) \hookrightarrow L^{\gamma_1}_{p_1}(B)$ is continuous, thus we only need to prove that $\mathcal{H}^{1,\gamma_2}_{p_2,0}(B) \hookrightarrow \mathcal{H}^{0,\gamma_1}_{p_1,0}(B)$ is compact. To verify this result, we employ the classical embedding in the following

$$
[1 - \omega] W^{1,p_2}_{0,\text{loc}}(\operatorname{int}B) \hookrightarrow [1 - \omega] W^{0,p_1}_{0,\text{loc}}(\operatorname{int}B)
$$
is compact for $1 \leq p_1 < p_2^*$. It remains to prove that the embedding
\[ [\omega]H^{1,\gamma_2}_{p_2,0}(X^\wedge) \hookrightarrow [\omega]H^{0,\gamma_1}_{p_1,0}(X^\wedge) \]
is compact.
To this end, we introduce a map as follows. Set $1 \leq q < \infty$. For any $v(x) \in H^m_{q,0}(X^\wedge)$ we define
\[ (S_q^{\gamma,\gamma})'(x_1, x') = e^{-r\left(\frac{p}{p_1} - \gamma\right)} v(e^{-r} x, x'). \]
Then $S_q^{\gamma,\gamma}$ induces an isomorphism as follows,
\[ S_q^{\gamma,\gamma} : [\omega]H^m_{q,0}(X^\wedge) \rightarrow [\hat{\omega}]W^{m,q}_0(\mathbb{R} \times X), \]
with $\hat{\omega}(r) = \omega(e^{-r})$, where $W^{m,q}_0(\mathbb{R} \times X)$ is the classical Sobolev space.
For $u_1(x) \in H^{1,\gamma_1}_{p_1,0}(X^\wedge)$, one has
\[ S_q^{\gamma,\gamma}(\omega(x_1) u_1(x)) = \omega(e^{-r}) e^{-r\left(\frac{p}{p_1} - \gamma\right)} u_1(e^{-r} x, x'), \]
and it induces an isomorphism
\[ S_q^{\gamma,\gamma} : [\omega]H^{1,\gamma_1}_{p_1,0}(X^\wedge) \rightarrow [\hat{\omega}]W^{0,p_1}_0(\mathbb{R} \times X). \]
On the other hand, for $u_2(x) \in H^{1,\gamma_1}_{p_2,0}(X^\wedge)$,
\[ S_q^{\gamma,\gamma}(\omega(x_1) u_2(x)) = \omega(e^{-r}) e^{-r\left(\frac{p}{p_2} - \gamma\right)} u_2(e^{-r} x, x') \]
\[ = e^{-r\left(\frac{p}{p_1} - \gamma\right) - \left(\frac{p}{p_2} - \gamma\right)} \omega(e^{-r}) e^{-r\left(\frac{p}{p_2} - \gamma\right)} u_2(e^{-r} x, x'), \]
and it also induces an isomorphism
\[ S_q^{\gamma,\gamma} : [\omega]H^{1,\gamma_2}_{p_2,0}(X^\wedge) \rightarrow [\hat{\omega}]e^{-r\delta}W^{1,p_2}_0(\mathbb{R} \times X), \]
with $\delta := (\frac{p}{p_1} - \gamma_1) - (\frac{p}{p_2} - \gamma_2) > 0$. The following embedding
\[ [\omega]e^{-r\delta}W^{1,p_2}_0(\mathbb{R} \times X) \hookrightarrow [\omega]W^{0,p_1}_0(\mathbb{R} \times X) \]
is compact, since the function $\varphi(r) = e^{-r\delta} r^s$ and all derivatives in $r$ are uniformly bounded on supp $\hat{\omega}$ for every $s > 0$. This completes the proof. \qed

**Remark 2.2.** With the same idea, for $1 < p_2 < N$ and $1 \leq p_1 < p_2^*$, the embedding
\[ H^{1,\gamma_2}_{p_2,0}(\mathbb{B}) \hookrightarrow H^{0,\gamma_1}_{p_1,0}(\mathbb{B}) \]
is continuous, if $\frac{N}{p_1} - \gamma_1 \geq \frac{N}{p_2} - \gamma_2$.

For investigating the existence of solutions for Dirichlet problem (1.2), some important concepts in variational methods are presented in the following. Let $E$ be a Banach space.

**Definition 2.5.** The functional $I$ satisfies the $(PS)_c$ condition, if for any sequence $\{u_k\} \subset E$ with the properties:
\[ I(u_k) \rightarrow c \quad \text{and} \quad I'(u_k) \parallel E \rightarrow 0, \]
there exists a subsequence which is convergent, where $I'(\cdot)$ is the Fréchet differentiation of $I$ and $E'$ is the dual space of $E$. If it holds for any $c \in \mathbb{R}$, we say that $I$ satisfies $(PS)$ condition.

Define the class of closed symmetric subsets in $E$
\[ \sigma(E) = \{ A \subset E | A \text{ is closed, and } A = -A \}. \]
Definition 2.6. For $A \in \sigma(E)$, define the genus of $A$, denoted by $\gamma(A)$, as
\[
\gamma(A) = \begin{cases} 
0, & \text{if } A = \emptyset \\
\infty, & \text{if } \{m \in \mathbb{N}_+; \exists h \in C(A, \mathbb{R}^m \setminus \{0\}), h(-x) = -h(x)\} = \emptyset
\end{cases}
\] (2.2)

Proposition 2.2. Let $A, B \in \sigma(E)$, the genus $\gamma$ possesses the following properties.

1. If $\psi \in C(A, B)$ is odd, then $\gamma(A) \leq \gamma(B)$.
2. If $\psi \in C(A, B)$ is an odd homeomorphism, then $\gamma(A) = \gamma(B) = \gamma(\psi(A))$.
3. If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.
4. If $\gamma(B) < \infty$, $\gamma(A - B) \geq \gamma(A) - \gamma(B)$.
5. $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.
6. If $S^{n-1}$ is the sphere in $\mathbb{R}^n$, then $\gamma(S^{n-1}) = n$.
7. If $A$ is compact, then $\gamma(A) < \infty$.
8. If $A$ is compact, there exists $\delta > 0$ such that for $N_\delta(A) = \{x \in X : d(x, A) < \delta\}$ we have $\gamma(A) = \gamma(N_\delta(A))$.

Proof. The proof can be found in section 3 of [18].

The abstract theory in [2] will be employed to investigate the existence of solutions for Dirichlet problem (1.2). We recall it in the following. Let $E$ be an infinite dimensional Banach space over $\mathbb{R}$. Let the functional $I \in C^1(E, \mathbb{R})$ and $B_r = \{u \in E; \|u\|_E \leq r\}$. For convenience, set $B := B_1$. Assume $I$ satisfies $I(0) = 0$ and the following five properties,

(I) the functional $I$ satisfies that $I(u) = I(-u)$ for all $u \in E$;
(II) the functional $I$ verifies the Palais-Smale condition;
(III) there exists a $\rho > 0$ such that $I > 0$ in $B_\rho \setminus \{0\}$ and $I \geq \alpha > 0$ on $\partial B_\rho$;
(IV) there exists $v \in E$ such that $\|v\|_E > \rho$ and $I(v) < \alpha$;
(V) for any finite dimensional subspaces $E_m \subset E$, it holds $E_m \cap A_0$ is bounded, where $A_0 = \{u \in E; 0 \leq I(u) < +\infty\}$.

Let
\[
\Gamma := \{h \in C(E, E) \mid h(0) = 0; h \text{ is odd homeomorphism}; h(B) \subset A_0\}
\]
and
\[
\Gamma_m = \{K \subset E \mid K \text{ compact}; K = -K; \gamma(K \cap h(\partial B)) \geq m, \forall h \in \Gamma\}
\]

Lemma 2.2. Suppose $I$ satisfies (I')-(I_5). For each $m \in N$, Let
\[
b_m = \inf_{K \in \Gamma_m} \max_{u \in K} I(u).
\] (2.3)

Then $0 < \alpha \leq b_m \leq b_{m+1}$ and $b_m$ is a critical value of $I$. Moreover, if $b_{m+1} = \cdots = b_{m+r} = b$, then $\gamma(K_b) \geq r$, where $K_b = \{u \in E \mid I'(u) = 0, I(u) = b\}$.

Proof. See Theorem 2.8 in [2].

Let $\{E_m\}_{m \in N}$ be a sequence of subspaces of $E$, such that

1. $\dim(E_m) = m$,
2. $E_m \subset E_{m+1}$,
3. $\mathcal{L}(\cup_{m \in N} E_m)$, linear manifold generated by $\cup_{m \in N} E_m$ is dense in $E$.

By $E'_m$ we denote the algebraically and topologically complementary of $E_m$. 



Lemma 2.3. Let $I$ satisfies $(I_1)-(I_5)$. For each $m \in \mathbb{N}$, let
\[ c_m = \sup_{h \in \Gamma} \inf_{u \in \partial B \cap E_{m-1}} I(h(u)). \] (2.4)
Then $0 < \alpha \leq c_m \leq b_m \leq \infty$, $c_m \leq c_{m+1}$, and $c_m$ is a critical value of $I$.

**Proof.** See Theorem 2.13 in [2].

3. **Proof of Theorem 1.1.** In this section, we will carry out the proof of Theorem 1.1, by verifying the functional $J$ to satisfy the condition $(I_1)-(I_5)$ in Lemma 2.2 and Lemma 2.3. To this end, we need the following Brezis-Lieb result in the weighted Sobolev spaces.

**Lemma 3.1** (Brezis-Lieb type result). Let $1 \leq p < \infty$ and $\{u_k\} \subset L_p^\gamma(\mathbb{B})$. If the following conditions are satisfied
(i) $\{u_k\}$ is bounded in $L_p^\gamma(\mathbb{B})$,
(ii) $u_k \to u$ a.e in $\mathbb{B}$, as $k \to \infty$,
then
\[ \lim_{k \to \infty} (\|u_k\|_{L_p^\gamma(\mathbb{B})}^p - \|u_k - u\|_{L_p^\gamma(\mathbb{B})}^p) = \|u\|_{L_p^\gamma(\mathbb{B})}^p. \] (3.1)

**Proof.** Due to Fatou Lemma, it yields
\[ \|u\|_{L_p^\gamma}^p = \int |x_1^{\frac{N}{p}} - \gamma u|^p d\sigma \leq \liminf_{k \to \infty} \int |x_1^{\frac{N}{p}} - \gamma u_k|^p d\sigma = \liminf_{k \to \infty} \|u_k\|_{L_p^\gamma}^p < \infty. \]
For simplicity, we set here $\tilde{u}_k = x_1^{\frac{N}{p}} - \gamma u_k$ and $\tilde{u} = x_1^{\frac{N}{p}} - \gamma u$. Since $p > 1$, then $j(t) = t^p$ is convex. For any fixed $\varepsilon > 0$, there exists a constant $c_\varepsilon$, such that
\[ |\tilde{u}_k - \tilde{u} + \tilde{u}|^p + |\tilde{u}_k - \tilde{u}|^p \leq \varepsilon |\tilde{u}_k - \tilde{u} + c_\varepsilon |\tilde{u}|^p, \]
and then
\[ |\tilde{u}_k - \tilde{u} + \tilde{u}|^p + |\tilde{u}_k - \tilde{u}|^p \leq \varepsilon |\tilde{u}_k - \tilde{u} + (1 + c_\varepsilon)|\tilde{u}|^p. \]
Therefore, we obtain that
\[ f_k^\varepsilon := (|\tilde{u}_k|^p - |\tilde{u}_k - \tilde{u}|^p - |\tilde{u}|^p - \varepsilon |\tilde{u}_k - \tilde{u}|^p)^+ \leq (1 + c_\varepsilon)|\tilde{u}|^p. \]
Then Lebesgue dominate theorem induces
\[ \lim_{k \to \infty} \int_{\mathbb{B}} f_k^\varepsilon(x) d\sigma = \int_{\mathbb{B}} \lim_{k \to \infty} f_k^\varepsilon(x) d\sigma = 0. \]
Since
\[ |x_1^{\frac{N}{p}} - \gamma u_k|^p - |x_1^{\frac{N}{p}} - \gamma u_k - x_1^{\frac{N}{p}} - \gamma u|^p - |x_1^{\frac{N}{p}} - \gamma u|^p | \leq f_k^\varepsilon + \varepsilon |x_1^{\frac{N}{p}} - \gamma u_k - x_1^{\frac{N}{p}} - \gamma u|^p, \]
then it follows that
\[ \limsup_{k \to \infty} \int_{\mathbb{B}} |x_1^{\frac{N}{p}} - \gamma u_k|^p - |x_1^{\frac{N}{p}} - \gamma (u_k - u)|^p - |x_1^{\frac{N}{p}} - \gamma u|^p | d\sigma \leq c \cdot \varepsilon, \]
where
\[ c := \sup_{x \in \mathbb{B}} |x_1^{\frac{N}{p}} - \gamma (u_k - u)|^p d\sigma. \]
Let $\varepsilon \to 0$, then it verifies the result. \qed
By a direct calculation, one can derive that the energy functional
\[ J(u) = \frac{1}{p} \int_B |\nabla_B u|^p d\sigma - \frac{\lambda}{q} \int_B x^p|u|^q d\sigma \in C^1(\mathcal{H}^{1,\frac{N}{q}}_p(B), \mathbb{R}), \]
satisfies \( J(0) = 0 \) and \( J(u) = J(-u) \) for any \( u \in \mathcal{H}^{1,\frac{N}{q}}_p(B) \).

**Proposition 3.1.** Let \( p < q < p^* \), then the functional
\[ J(u) = \frac{1}{p} \int_B |\nabla_B u|^p d\sigma - \frac{\lambda}{q} \int_B x^p|u|^q d\sigma \]
verifies the (PS) condition.

**Proof.** Let \( \{u_k(x)\} \in \mathcal{H}^{1,\frac{N}{q}}_p(B) \) be a (PS) sequence. Then
\[ J(u_k) - \frac{1}{q} < J'(u_k), u_k \geq (\frac{1}{p} - \frac{1}{q}) \int_B |\nabla_B u|^p d\sigma < \infty, \]
which implies that \( \{\|u_k\|_{\mathcal{H}^{1,\frac{N}{q}}_p(B)}\} \) is bounded. Hence
\[ u_k \rightharpoonup u \text{ in } \mathcal{H}^{1,\frac{N}{q}}_p(B), \text{ as } k \to \infty, \]
and together with Lemma 2.1, it follows
\[ u_k \to u \text{ in } L^q(B), \text{ as } k \to \infty, \]
for \( 1 < q < p^* \) and \( 0 < \gamma_1 < \frac{N}{q} \). Let us calculate that
\[ o(1) = \langle J'(u_k) - J'(u), u_k - u \rangle = \int_B (|\nabla_B u_k|^{p-2}\nabla_B u_k - |\nabla_B u|^{p-2}\nabla_B u)(\nabla_B u_k - \nabla_B u)d\sigma - \lambda \int_B x^p(|u_k|^{q-2}u_k - |u|^{q-2}u)(u_k - u)d\sigma =: I_1 - I_2. \]
Choose some \( \gamma_1 \), such that \( \frac{N}{q} - p < \gamma_1 < \frac{N}{q} \). Due to Hölder inequality, we can derive that
\[ I_2 = \lambda \int_B x^p(|u_k|^{q-2}u_k - |u|^{q-2}u)(u_k - u)d\sigma \leq \lambda \left( \int_B x^{\frac{N}{q}-\gamma_1} u_k - u \right)^{\frac{q}{q-\gamma_1}} \left( \int_B x^{p-\left(\frac{N}{q}-\gamma_1\right)} (u_k - u)^\frac{q}{q-\gamma_1} d\sigma \right)^{\frac{q-\gamma_1}{q}} := T_1 \cdot T_2. \]
Combining Lemma 2.1 and the fact that \( \{u_k\} \) is bounded in \( \mathcal{H}^{1,\frac{N}{q}}_p(B) \) (i.e. \( \{u_k\} \) is bounded in \( L^p(B) \)), we derive that \( T_1 \to 0 \) and \( T_2 \) is bounded which implies
\[ I_2 \to 0, \text{ as } k \to \infty. \]
Set \( P_k(x) = (|\nabla_B u_k|^{p-2}\nabla_B u_k - |\nabla_B u|^{p-2}\nabla_B u)(x)(\nabla_B u_k - \nabla_B u)(x) \), we arrive that
\[ I_1 = \int_B P_k(x)d\sigma \to 0. \quad (3.2) \]
Here, denote the \( i^{th} \) component of \( \nabla_B u \) by \( (\nabla_B u)_i \). We have the following conditions that
\[ P_k(x) \geq 0; \text{ and } P_k(x) > 0, \text{ if } \nabla_B u_k \neq \nabla_B u. \quad (3.3) \]
In fact, for any \( x_0 \in \text{int} \mathbb{B} \), without loss of generality, we assume \((\nabla_\mathbb{B} u_k)_i(x_0) > (\nabla_\mathbb{B} u)_i(x_0)\), we have
\[
(|\nabla_\mathbb{B} u_k|^{p-2}\nabla_\mathbb{B} u_k - |\nabla_\mathbb{B} u|^{p-2}\nabla_\mathbb{B} u)_i(x_0)(\nabla_\mathbb{B} u_k - \nabla_\mathbb{B} u)_i(x_0) > 0.
\]
This shows (3.3). In the following, we verify that
\[
(\nabla_\mathbb{B} u_k)_i \to (\nabla_\mathbb{B} u)_i \quad \text{for } 1 \leq i \leq N, \text{ as } k \to \infty,
\]
a.e in \( \text{int} \mathbb{B} \), which can be deduced by contradiction. Assume, there exists a point \( x_1 \in \text{int} \mathbb{B} \), and its neighborhood \( U_{x_1} \), such that for any \( x_0 \in U_{x_1} \),
\[
\lim_{k \to \infty} \nabla_\mathbb{B} u_k(x_0) \neq \nabla_\mathbb{B} u(x_0).
\]
Since \((\nabla_\mathbb{B} u_k|^{p-2}\nabla_\mathbb{B} u_k - |\nabla_\mathbb{B} u|^{p-2}\nabla_\mathbb{B} u)(\nabla_\mathbb{B} u_k - \nabla_\mathbb{B} u)\) is bounded, then it holds
\[
(|\nabla_\mathbb{B} u_k|^{p-2}\nabla_\mathbb{B} u_k - |\nabla_\mathbb{B} u|^{p-2}\nabla_\mathbb{B} u)_i(x_0)(\nabla_\mathbb{B} u_k - \nabla_\mathbb{B} u)_i(x_0) \leq c.
\]
It follows that
\[
(|\nabla_\mathbb{B} u_k|^{p-2}\nabla_\mathbb{B} u_k)_i(x_0)(\nabla_\mathbb{B} u_k)(x_0)
\leq c + (|\nabla_\mathbb{B} u_k|^{p-2}\nabla_\mathbb{B} u_k)_i(x_0)(\nabla_\mathbb{B} u)_i(x_0) + (|\nabla_\mathbb{B} u|^{p-2}\nabla_\mathbb{B} u)_i(x_0)(\nabla_\mathbb{B} u_k - \nabla_\mathbb{B} u)_i(x_0)
\leq c + (|\nabla_\mathbb{B} u_k|^{p-2} + |\nabla_\mathbb{B} u|^{p-2} + |\nabla_\mathbb{B} u|^{p-2})(\nabla_\mathbb{B} u_k)_i(x_0)(\nabla_\mathbb{B} u)_i(x_0),
\]
which indicates that \( |\nabla_\mathbb{B} u_k(x_0)|^p = \sum_{i=1}^N ((|\nabla_\mathbb{B} u_k|^{p-2}\nabla_\mathbb{B} u_k)_i(x_0)(\nabla_\mathbb{B} u_k)_i(x_0)\) is bounded. There exists a subsequence, here still denoted by \( \{u_k\} \) such that
\[
(\nabla_\mathbb{B} u_k)_i(x_0) \to \xi' \neq \xi = \nabla_\mathbb{B} u(x_0), \text{ as } k \to \infty.
\]
This induces that
\[
P_k(x_0) = (|\nabla_\mathbb{B} u_k|^{p-2}\nabla_\mathbb{B} u_k - |\nabla_\mathbb{B} u|^{p-2}\nabla_\mathbb{B} u)(\nabla_\mathbb{B} u_k - \nabla_\mathbb{B} u)(x_0) \to c_0 > 0,
\]
for any \( x_0 \in U_{x_1} \), as \( k \to \infty \). It follows that
\[
I_1 = \int_{\mathbb{B}} P_k(x) d\sigma \to c \neq 0, \text{ as } k \to \infty,
\]
which contradicts to (3.2).

Applying Lemma 3.1 to \((\nabla_\mathbb{B} u_k)_i\), for \( 1 \leq i \leq N \), we have
\[
\lim_{k \to \infty} \left( \frac{\|\nabla_\mathbb{B} u_k\|_{L^p(\mathbb{B})}^p}{\|\nabla_\mathbb{B} u\|_{L^p(\mathbb{B})}^p} - \frac{\|\nabla_\mathbb{B} u_k - \nabla_\mathbb{B} u\|_{L^p(\mathbb{B})}^p}{\|\nabla_\mathbb{B} u\|_{L^p(\mathbb{B})}^p} \right) \to 0.
\]
(3.4)

To the end, what left is to show that
\[
\int_{\mathbb{B}} |\nabla_\mathbb{B} u_k|^p d\sigma \to \int_{\mathbb{B}} |\nabla_\mathbb{B} u|^p d\sigma, \text{ as } k \to \infty.
\]
Due to Egorov Theorem, we obtain that for any \( \delta > 0 \), there exists a subset \( E \subset \text{int} \mathbb{B} \) with the measure \( m(E) < \delta \), such that
\[
(\nabla_\mathbb{B} u_k)_i \to (\nabla_\mathbb{B} u)_i \quad \text{for } 1 \leq i \leq N, \text{ as } k \to \infty,
\]
uniformly on \( \text{int} \mathbb{B} \setminus E \). It follows that
\[
\int_{\mathbb{B} \setminus E} |\nabla_\mathbb{B} u_k|^p d\sigma \to \int_{\mathbb{B} \setminus E} |\nabla_\mathbb{B} u|^p d\sigma, \text{ as } k \to \infty.
\]
(3.5)

Now we claim that for any \( \varepsilon_k > 0 \) and \( \varepsilon_k \to 0 \), there is \( \delta(\varepsilon_k) > 0 \), such that for subset \( E_k \subset \mathbb{B} \) with the measure \( m(E_k) < \delta(\varepsilon_k) \), we have
\[
\int_{E_k} |\nabla_\mathbb{B} u_k|^p d\sigma < \varepsilon_k.
\]
(3.6)
Thus from (3.2),

\[ o(1) = I_1 = \int_{\mathbb{B}} |\nabla_B u_k|^{p-2} \nabla_B u_k - |\nabla_B u|^{p-2} \nabla_B u)(\nabla_B u_k - \nabla_B u) \, d\sigma, \]

which implies that

\[ \int_{\mathbb{B}} |\nabla_B u|^{p} \, d\sigma = \left( \int_{\mathbb{B}} |\nabla_B u_k|^{p-2} + |\nabla_B u|^{p-2}(\nabla_B u_k \cdot \nabla_B u) \, d\sigma - \int_{\mathbb{B}} |\nabla_B u_k|^{p} \, d\sigma + o(1) \right). \]

Thus, for any \( E_k \subset \mathbb{B} \), we can deduce that

\[ \int_{E_k} |\nabla_B u|^{p} \, d\sigma \]
\[ \leq \int_{E_k} |\nabla_B u_k|^{p-1} |\nabla_B u| \, d\sigma + \int_{E_k} |\nabla_B u|^{p-1} |\nabla_B u_k| \, d\sigma + \int_{E_k} |\nabla_B u_k|^{p} \, d\sigma + o(1). \]

According to Hölder inequality, it follows that

\[ \int_{E_k} |\nabla_B u|^{p} \, d\sigma \leq (\int_{E_k} |\nabla_B u_k|^{p} \, d\sigma)^{\frac{p-1}{p}} \left( \int_{E_k} |\nabla_B u|^{p} \, d\sigma \right)^{\frac{1}{p}} + \left( \int_{E_k} |\nabla_B u_k|^{p} \, d\sigma \right)^{\frac{p-1}{p}} + \int_{E_k} |\nabla_B u_k|^{p} \, d\sigma + o(1), \]

which implies that \( \int_{E_k} |\nabla_B u|^{p} \, d\sigma \to 0 \) if \( \int_{E_k} |\nabla_B u_k|^{p} \, d\sigma \to 0 \). Hence, combining with (3.5) we have

\[ \int_{\mathbb{B}} |\nabla_B u_k|^{p} \, d\sigma \to \int_{\mathbb{B}} |\nabla_B u|^{p} \, d\sigma, \quad \text{as} \quad k \to \infty, \]

that means \( u_k \to u \) in \( H^{1,p} (\mathbb{B}) \). \( \square \)

The following two propositions verifies that the functional \( J(u) \) satisfies the conditions \( I_3, I_4, I_5 \) in Lemma 2.2 and Lemma 2.3.

**Proposition 3.2.** If \( p < q < p^* \), then there exists \( r > 0 \) such that

(i) \( J(u) > 0 \) if \( 0 < \|u\|_{H^{1,p}_0} < r \) and \( J(u) \geq \alpha > 0 \) if \( \|u\|_{H^{1,p}_0} = r \).

(ii) there exists \( v \in H^{1,p}_0 \) such that \( \|v\|_{H^{1,p}_0} > r \) and \( J(v) < \alpha \).

**Proof.** According to Lemma 2.1, it holds that

\[ H^{1,p}_0 (\mathbb{B}) \hookrightarrow L^{q^*} (\mathbb{B}). \]

We derive that

\[ J(u) = \frac{1}{p} \int_{\mathbb{B}} |\nabla_B u|^{p} \, d\sigma - \frac{\lambda}{q} \int_{\mathbb{B}} x^p_k |u|^q \, d\sigma = \frac{1}{p} \|u\|^{p}_{H^{1,p}_0} - \frac{\lambda}{q} \|u\|^{q}_{H^{1,p}_0} \left( 1 - \frac{c_1}{q} \|u\|^{q - p}_{H^{1,p}_0} \right). \]

Let \( r = \left( \frac{q}{2pc\lambda} \right)^{\frac{1}{p-q}} > 0 \), if \( \|u\|_{H^{1,p}_0} = r \), then \( J(u) \geq \alpha = \left( \frac{1}{2p} \right)^{\frac{1}{p}} > 0 \) and if \( 0 < \|u\|_{H^{1,p}_0} < r \), then \( J(u) > \alpha > 0 \). Then the condition (i) is proved. Set \( \|u\|_{H^{1,p}_0} = r \), and \( \theta > 0 \), it holds that

\[ J(\theta u) = \frac{\theta^p}{p} \int_{\mathbb{B}} |\nabla_B u|^{p} \, d\sigma - \frac{\lambda \theta^q}{q} \int_{\mathbb{B}} x^p_k |u|^q \, d\sigma \to -\infty, \]
4. Proof of Corollary 1.1. In this section, we will give the proof of Corollary 1.1, by using the following definition and lemma.

Definition 4.1. Define the manifold $M$ as follows

$$M = \{ u \in \mathcal{H}_{p,0}^{1,N} \setminus \{0\} : \| u \|_{\mathcal{H}_{p,0}^{1,N}}^p = \lambda \int_{\mathbb{B}} |x|^q |u|^q \, d\sigma \}.$$  \hspace{1cm} (4.1)

Lemma 4.1. For any $u \in \mathcal{H}_{p,0}^{1,N} \setminus \{0\}$, there exists a unique

$$\beta := \beta(u) \geq 0 \quad \text{such that} \quad (\beta u) \in M.$$  

The maximum of $J(\beta u)$ for $\beta \geq 0$ is achieved at $\beta = \beta(u) > 0$. The function

$$u \mapsto \beta = \beta(u)$$

is continuous and the map

$$u \mapsto \beta(u)u$$
defines a homeomorphism of the unit sphere in \( \mathcal{H}^{1,\frac{N}{p}}_{p,\beta}(\mathbb{B}) \) with manifold \( M \) in (4.1).

Proof. Let \( u \in \mathcal{H}^{1,\frac{N}{p}}_{p,0}(\mathbb{B}) \) be fixed, define \( g(\beta) := J(\beta u) \) on \([0, \infty)\). (4.2)

Then it follows that
\[
g'(\beta) = 0 \iff \beta u \in M \iff \|u\|_{\mathcal{H}^{1,\frac{N}{p}}_{p,\beta}} = \frac{1}{\beta^p} \int_{\mathbb{B}} x_1^p |\beta u|^q d\sigma. \tag{4.3}
\]

It is obvious that
\[
g(0) = 0; \quad g(\beta) > 0 \text{ for } \beta > 0 \text{ small enough}; \quad g(\beta) < 0 \text{ for } \beta > 0 \text{ large}.
\]

Therefore, \( \max_{[0, \infty)} g(\beta) \) is achieved at a unique \( \beta = \beta(u) \) such that \( g'(\beta) = 0 \) and \( \beta u \in M \).

To prove the continuity of \( \beta(u) \), let us assume that \( u_n \to u \) in \( \mathcal{H}^{1,\frac{N}{p}}_{p,0}(\mathbb{B}) \). Then \( \{\beta(u_n)\} \) is bounded. If a subsequence of \( \{\beta(u_n)\} \) converges to \( \beta_0 \), then it follows from the right side of (4.3) that \( \beta_0 = \beta(u) \).

Finally, the continuous map from the unit sphere of \( \mathcal{H}^{1,\frac{N}{p}}_{p,0}(\mathbb{B}) \) to \( M \), \( u \mapsto \beta(u) \), is inverse to the retraction \( u \mapsto u/\|u\|_{\mathcal{H}^{1,\frac{N}{p}}_{p,0}(\mathbb{B})} \). \( \square \)

By Definition 4.1, there exists \( r > 0 \), such that
\[
\int_{\mathbb{B}} x_1^p |u|^q d\sigma > r, \quad \text{for any } u \in M.
\]
Indeed, if \( u \in M \), then by Remark 2.2, it follows
\[
\|u\|_{\mathcal{H}^{1,\frac{N}{p}}_{p,\beta}} = \lambda \|u\|_{L^q_{\mathcal{H}^{1,\frac{N}{p}}_{p,\beta}}} \leq c \lambda \|u\|_{\mathcal{H}^{1,\frac{N}{p}}_{p,\beta}}. \tag{4.4}
\]

For \( q > p \), then it follows
\[
\|u\|_{\mathcal{H}^{1,\frac{N}{p}}_{p,\beta}} \geq \left( \frac{1}{c \lambda} \right)^{\frac{1}{q-p}}
\]
Set \( r = \frac{1}{2}(\frac{1}{c \lambda})^{\frac{1}{q-p}} \), it holds that
\[
\int_{\mathbb{B}} x_1^p |u|^q d\sigma > r, \quad \text{for each } u \in M.
\]

Let \( d_m = \inf\{\|u\|_{\mathcal{H}^{1,\frac{N}{p}}_{p,\beta}} : u \in M \cap E_m^c\} \), then we claim that
\[
d_m \to \infty \quad \text{as} \quad m \to \infty. \tag{4.5}
\]
In fact, if there exist \( d > 0 \) and \( u_m \in M \cap E_m^c \), such that
\[
\|u_m\|_{\mathcal{H}^{1,\frac{N}{p}}_{p,\beta}} \leq d \quad \text{for all} \quad m \in \mathbb{N}_+.
\]
Then there exists \( u \in \mathcal{H}^{1,\frac{N}{p}}_{p,0}(\mathbb{B}) \), such that
\[
u_m \to u \quad \text{in} \quad \mathcal{H}^{1,\frac{N}{p}}_{p,0}(\mathbb{B}).
Since \( u_m \in E_m^c \), and \( L(\cup E_m) \) is dense in \( \mathcal{H}^{1,\frac{N}{p}}_{p,0}(\mathbb{B}) \), then we have \( u = 0 \). According to Lemma 2.1, it follows that

\[ u_m \to 0 \quad \text{in} \quad L^{\frac{N}{p-q}}(\mathbb{B}). \]

This is a contradiction to \( \|u_m\|_{L^{\frac{N}{p-q}}(\mathbb{B})} > r > 0 \) for any \( u_m \in M \). That means \( d_m \) will be unbounded as \( m \to \infty \), which proves the claim (4.5).

Next, for some \( R > 1 \), we define a homeomorphism

\[ h_m = R^{-1}d_m u : E_m^c \to E_m^c. \]  

By Lemma 4.1, let \( \beta := \beta(u) \) such that \( \beta u \in M \). Set

\[ B = \{ u \in \mathcal{H}^{1,\frac{N}{p}}_{p,0}(\mathbb{B}) \mid \|u\|_{\mathcal{H}^{1,\frac{N}{p}}_{p,0}(\mathbb{B})} \leq 1 \}. \]

For \( u_1 \in E_m^c \cap B \), \( u_1 \neq 0 \) and \( R > 1 \), we have

\[ R^{-1}d_m \beta u \in \mathcal{H}^{1,\frac{N}{p}}_{p,0}(\mathbb{B}), \]

so we can define

\[ h_m(E_m^c \cap B) \subset A_0 := \{ u \in \mathcal{H}^{1,\frac{N}{p}}_{p,0}(\mathbb{B}) \mid 0 \leq J(u) < +\infty \}. \]

In fact, if \( u \in E_m^c \cap B \), then

\[ J(h_m(u)) = J(R^{-1}d_m u) = \frac{1}{p} \|R^{-1}d_m u\|_{\mathcal{H}^{1,\frac{N}{p}}_{p,0}(\mathbb{B})}^p - \frac{1}{q} \left| \frac{R^{-1}d_m u}{\beta} \right|_{L^q(\mathbb{B})}^q \]

\[ = \frac{1}{p} \left( \frac{R^{-1}d_m u}{\beta} \right)^p - \frac{1}{q} \left| \frac{R^{-1}d_m u}{\beta} \right|_{L^q(\mathbb{B})}^q \]

\[ = \frac{1}{p} \left( \frac{R^{-1}d_m u}{\beta} \right)^p - \frac{1}{q} \left( \frac{R^{-1}d_m u}{\beta} \right) \parallel u \parallel_{\mathcal{H}^{1,\frac{N}{p}}_{p,0}(\mathbb{B})}^{q-p}. \]

where \( \beta \) is chosen as above, such that \( \beta u \in M \) and \( d_m \leq \beta \). Then let \( R \) be large enough, it follows that \( J(h_m(u)) \geq 0 \), which proves that \( h_m(E_m^c \cap B) \subset A_0 \).

Therefore, we can define

\[ \tilde{h}_m(u) = \begin{cases} h_m(u) & \text{if } u \in E_m^c, \\ \varepsilon e_j, j = 1, 2, ..., m & \text{and } \{e_j\}_{j=1}^m \text{ is basis of } E_m \text{ if } u \in E_m. \end{cases} \]

for \( \varepsilon \) small enough. In this way, it is shown that for \( R \) large enough, the mapping \( h_m \) in (4.6) defined on \( E_m^c \) admits an extension \( \tilde{h}_m \in \Gamma \) for each \( m \).

Finally, we take \( u \in \partial B \cap E_m^c \), then

\[ J(\tilde{h}_m(u)) = J(R^{-1}d_m u) = (R^{-1}d_m u)^p \left( \frac{1}{p} - \frac{1}{q} \left( \frac{R^{-1}d_m u}{\beta} \right) \parallel u \parallel_{\mathcal{H}^{1,\frac{N}{p}}_{p,0}(\mathbb{B})}^{q-p} \right). \]

where the calculus in (4.9) is the same as that in (4.8). Since \( d_m \leq \beta := \beta(u) \) proved in (4.7), then we choose \( R \) large enough to deduce that

\[ J(\tilde{h}_m(u)) \geq \frac{1}{2p} (R^{-1}d_m)^p \to \infty \quad \text{as} \quad m \to \infty. \]
Since \( \{c_m\} \) is critical value sequence of \( J \) (as defined by (2.4)), thus we have
\[
c_m \to \infty \quad \text{as} \quad m \to \infty.
\]
Corollary 1.1 is proved.

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