Fields of locally compact quantum groups: continuity and pushouts

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Abstract

We prove that (a) discrete compact quantum groups (or more generally locally compact, under additional hypotheses) with coamenable dual are continuous fields over their central closed quantum subgroups, and (b) the same holds for free products of discrete quantum groups with coamenable dual amalgamated over a common central subgroup. Along the way we also show that free products of continuous fields of $C^*$-algebras are again free via a Fell-topology characterization for $C^*$-field continuity, recovering a result of Blanchard’s in a somewhat more general setting.

Key words: $C^*$-algebra; continuous field; weak containment; Fell topology; locally compact quantum group; discrete quantum group; pushout; free product with amalgamation

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Introduction

The initial motivation for the present note was the desire to extend one of the main results of [10] (Theorem 3.2 therein) in two ways: from plain (“classical”) to quantum groups, and from discrete to locally compact. The result appears below as Theorem 2.1:

Theorem 0.1 For

- a locally compact quantum group $G$ with coamenable dual
- with a central closed quantum subgroup $H \leq G$
- such that $G/H$ has coamenable dual (automatic if $G$ is discrete)

the group $C^*$-algebra $C^*_0(\hat{G})$ forms a continuous field over the group algebra $C^*_0(\hat{H})$ of any central closed quantum subgroup $H \leq G$.

We recall the terminology and notation below in §1.1 (e.g. $\hat{G}$ for the dual $\hat{G}$ of $G$), pausing here only to remind the reader that a reduced locally compact quantum group $G$ in the sense of [18] (see also [22, Definition 8.1.17]) consists of a generally non-unital $C^*$-algebra $C^*_0(G)$ (thought of as a algebra of continuous functions vanishing at infinity on $G$) equipped with

- a coassociative comultiplication morphism $\Delta : C^*_0(G) \to C^*_0(G)^{\otimes 2}$ (minimal $C^*$ tensor product) in the sense of Definition 1.1 (i.e. landing in the multiplier algebra);

- left and right-invariant Haar weights (on which we do not elaborate);

- and hence a von Neumann algebra $L^\infty(G)$ ( [22, Definition 8.1.4] or [19]) attached to one of these invariant weights via the GNS construction.
Since furthermore [10, Theorem 3.2] handles pushouts of amenable groups $G_i$, $i = 1, 2$ over a common central subgroup $H$, it seemed desirable to have an analogous extension here (Theorem 2.2):

**Theorem 0.2** Let $G_i$, $i \in I$ be a family of discrete quantum groups with coamenable duals and a common central closed quantum subgroup $H \subseteq G_i$. Then, the $C^*$ pushout

$$
\ast_{C^*(\tilde{H})}^u \left( \tilde{G}_i \right)
$$

is a continuous field over the commutative $C^*$-algebra $C^u\left( \tilde{H} \right)$.

Note that as opposed to Theorem 0.1, where $G$ is locally compact, here the $G_i$ are discrete (equivalently, $C_0^u(G)$ are unital, hence the missing ‘0’ superscript in $C_0^u$): this is to avoid the unpleasantness of working with non-unital pushouts.

One natural path to Theorem 0.2 (or something like it, perhaps covering pushouts of only two quantum groups) would be to start with Theorem 0.1 and apply [7, Theorem 3.7], to the effect that a pushout $A \ast C B$ of fields of $C^*$-algebras continuous over a central $C^*$-algebra $C$ is again continuous over $C$. This was the initial intention, but in the process of unwinding that cited result the proof appeared to contain a gap. For that reason, it seemed worthwhile to try to recover [7, Theorem 3.7] here via a different approach (Theorem 3.4):

**Theorem 0.3** Let $X$ be a compact Hausdorff space and $A_i$, $i \in I$ a family of unital $C(X)$-algebras. If all $A_i$ are continuous then so is the pushout

$$
A := \ast_{C(X)}A_i.
$$

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1 Preliminaries

$C^*$-algebras are not assumed unital unless we do so explicitly, and morphisms are defined as is customary for generally-non-unital $C^*$-algebras (e.g. [18, Notations and conventions]):

**Definition 1.1** Let $A$ and $B$ be two possibly-non-unital $C^*$-algebras. A morphism $A \to B$ is a linear, bounded, multiplicative and $*$-preserving map $f : A \to M(B)$ (the multiplier algebra of $B$; [25, §2.2] or [4, II.7.3]) that is non-degenerate in the sense that

$$
f(A)B := \text{span}\{f(a)b \mid a \in A, \ b \in B\}
$$

is dense in $B$.

As noted in [18, Notations and conventions], morphisms $A \to B$ in this sense extend uniquely to unital morphisms $M(A) \to M(B)$ that are strictly continuous on bounded subsets. Recall that strict continuity means continuity with respect to the seminorms

$$
M(A) \ni x \mapsto \|ax\| + \|xa\|, \ a \in A.
$$
1.1 Locally compact quantum groups

We will need some background on these (abbreviated as LCQGs) as introduced in [18]. Additional sources include the excellent textbook [22] as well as various other papers cited in the process of (very briefly) recalling some of the relevant notions.

In addition to the structure reviewed briefly in the introduction, one can define, for an LCQG $G$,

- the *universal* version $C_0^u(G)$ [17, §11] that is again a $C^*$-algebra equipped with a coassociative morphism $\Delta : C_0^u(G) \rightarrow C_0^u(G)^{\otimes 2}$ and a surjection $C_0^u(G) \rightarrow C_0^r(G)$ intertwining the comultiplications;

- the *Pontryagin dual* $\hat{G}$ of $G$ ([22, Definition 8.3.14]), whose underlying universal $C^*$-algebra $C_0^u(\hat{G})$ analogizes the universal group algebra of $G$ (in particular, representations of $G$ on Hilbert spaces are precisely representations of $C_0^u(\hat{G})$ as a $C^*$-algebra [17, §5]).

The following version of the notion of discreteness will be most directly applicable below (see also [22, §3.3] or [24]).

**Definition 1.2** $G$ is *discrete* if $C_0^v(G)$ is unital.

Regarding the relationship between $C_0^u$ and $C_0^r$, recall [1, Definition 3.1, Theorem 3.1]:

**Definition 1.3** A locally compact quantum group $G$ is *coamenable* if either of the two following equivalent conditions holds:

- there is a character $\varepsilon : C_0^r(G) \rightarrow \mathbb{C}$ such that $(id \otimes \varepsilon)\Delta = id_{C_0^r(G)}$;

- the surjection $C_0^u(G) \rightarrow C_0^r(G)$ is an isomorphism.

As in the classical setting, we can talk about closed quantum subgroups ([23, Definition 2.5] and [12, Definitions 3.1 and 3.2]):

**Definition 1.4** Let $G$ be a locally compact quantum group.

(a) A *Vaes-closed quantum subgroup* $H \leq G$ is a locally compact quantum group $H$ equipped with a normal embedding

$$L^\infty(\hat{H}) \rightarrow L^\infty(\hat{G})$$

intertwining the comultiplications.

(b) This then induces a surjection $C_0^u(G) \rightarrow C_0^u(H)$, thus realizing $H$ as a *Woronowicz-closed quantum subgroup* of $G$ ([12, Definition 3.2, Theorems 3.5 and 3.6]).

(c) The (Vaes-)closed $H \leq G$ is *central* [16, §1.1] if

$$L^\infty(\hat{H}) \subseteq L^\infty(\hat{G})$$

is contained in the center.
1.2 Fields of $C^*$-algebras

Denoting, as usual, by $C_0(X)$ the algebra of continuous functions vanishing at infinity on a locally compact Hausdorff space $X$, we work with $C_0(X)$-algebras in the sense of [9, Introduction] or [8, Definition 2.1] (see also [5, Definition 1.1] and [7, Definition 2.2] for the case of compact $X$):

**Definition 1.5** A $C_0(X)$-algebra is a (possibly non-unital) $C^*$-algebra $A$ equipped with a non-degenerate morphism from $C_0(X)$ to the center of the multiplier algebra $M(A)$.

One can form, for every point $x \in X$, the fiber $A_x$ of $A$ at $x$ as

$$A_x = A/\{\text{ideal generated by } ma \text{ and } am \}$$

where $a \in A$ and $m \in M(A)$ ranges over the image through $C_0(X) \to M(A)$ of the ideal of functions vanishing at $x$.

For $a \in A$ we follow [7] in denoting by $a_x$ the image of $a$ through $A \to A_x$, and when the need arises to distinguish between the norms of the various $A_x$ we write \|$ \cdot \|_x$ for the latter. Recall [6, Définition 3.1] (see also [8, Definition 2.2] or [7, discussion following Definition 2.2]):

**Definition 1.6** The $C_0(X)$-algebra $A$ is continuous as such, or a continuous field over $X$ if

$$X \ni x \mapsto \|a_x\|$$

is continuous for every $a \in A$.

**Remark 1.7** Note that (1-1) is always upper semicontinuous [21, Proposition 1.2], so it is lower semicontinuity that is the core issue motivating Definition 1.6.

When working with pushouts (e.g. [7, 20]) we specialize to the unital case: $A_i$ will typically be unital algebras equipped with unital morphisms $C(X) \to A_i$, thus allowing the formation of the pushout

$$\oplus_{C(X)} A_i.$$  

(1-2)

To make sense of Lemma 1.8 below we need some background on the Fell topology on (unitary isomorphism classes of) representations and on weak containment. We refer to [15, 14], [13, Chapter 3] and [3, Appendix F] for material on these topics.

In the statement of Lemma 1.8, for a point $x \in X$

$$p_x : C_0(X) \to \mathbb{C}$$

denotes the evaluation character, regarded as a $C_0(X)$-representation.

**Lemma 1.8** A $C_0(X)$-algebra $A$ is a continuous field if and only if for every convergent net

$$x_\alpha \to x \in X$$

(1-3)

every representation of $A$ that factors through $A \to A_x$ is a Fell limit a net of representations that factor through $A \to A_{x_\alpha}$.

**Proof** We prove the two implications separately.

($\Rightarrow$) Suppose $A$ is continuous over $X$. To prove the desired conclusion we have to show ([13, Theorem 3.4.4]) that for a net (1-3) and a representation

$$\rho : A \to A_x \to B(H)$$

(1-4)
where $C_0(X)$ acts via $p := p_x$ we can find representations

$$\rho_\alpha : A \to B(H_\alpha)$$

such that

- $\rho_\alpha$ restricts to $C_0(X)$ as the character $p_\alpha$, and
- the intersection of the kernels of $\rho_\alpha$ is contained in $\ker \rho$.

Now choose a net of representations

$$\rho_\alpha : A \to A_\alpha \to B(H_\alpha),$$

faithful on $A_\alpha$ respectively. By the continuity of the field $A$ over $X$, any element annihilated by all $\rho_\alpha$ must also be annihilated by $A \to A_\alpha$, and hence by (1-4).

(\Rightarrow) Conversely, to prove that $A$ is continuous over $X$ we have to argue that for every convergent net (1-3) and every $a \in A$ we have

$$\|a_x\| \leq \limsup_{\alpha} \|a_{\alpha x}\|. \quad (1-6)$$

The hypothesis says that we can find a net of representations (1-5) Fell-converging to some representation (1-5) faithful on $A_x$. Since the Fell topology does not distinguish between sums of copies of a given representation (regardless of the cardinality of the set of summands), we may as well assume that

$$\rho \cong \rho^{\otimes \aleph_0}$$

and similarly for all $\rho_\alpha$. But in that case [15, Lemma 2.4] shows that

- we can realize $\rho$ concretely and non-degenerately on some large Hilbert space $H$
- which also houses (possibly degenerate) copies of $\rho_\alpha$
- so that for every $a \in A$ and every $\xi \in H$ we have

$$\lim_{\alpha} \|\rho(a)\xi - \rho_\alpha(a)\xi\| = 0.$$ 

Since $\rho$ is assumed faithful on $A_x$, (1-6) follows.

\section{Fields over central locally compact quantum groups}

We begin with

\textbf{Theorem 2.1} Let $G$ be an LCQG with coamenable dual and $H \leq G$ a central closed quantum subgroup.

(a) If $G/H$ also has coamenable dual then $C^u_0(\hat{H}) \to C^u_0(\hat{G})$ is a continuous field.

(b) The hypothesis in (a) is automatic if $G$ is discrete.
Proof We treat the two clauses separately.

Part (a) We mimic the proof of [10, Theorem 3.2]. Having chosen a convergent net
\[ x_\alpha \to x \in \tilde{H} \]
of characters, we have to argue that an arbitrary unitary representation \( \rho \) of \( G \) on which \( H \) acts via \( p_x \) is in the Fell closure of a family of unitary representations where \( H \) acts by \( p_{x_\alpha} \). The coamenability condition ensures that \( G/H \) has coamenable dual and hence its regular representation weakly contains its trivial representation \( 1_{G/H} \). We thus similarly have the weak containment
\[ 1_G \leq \text{Ind}_H^G(1_H), \]
and it follows that
\[ \rho \equiv \rho \otimes 1_G \leq \rho \otimes \text{Ind}_H^G(1_H); \]
the latter representation is a Fell-topology limit of
\[ \rho \otimes \text{Ind}_H^G(p_{x_\alpha x^{-1}}) \]
where \( H \) acts respectively by \( p_{x_\alpha} \), hence the conclusion.

Part (b) According to [11, Theorem 3.2] the amenability of \( G \) entails that of \( G \hat{\times} H \), whereas by [2, Corollary 9.6] amenability and dual coamenability are equivalent for discrete quantum groups. ■

Next, Theorem 2.1 extends to arbitrary pushouts.

Theorem 2.2 Let \( G_i, i \in I \) be a family of discrete quantum groups with coamenable duals and a common central closed quantum subgroup \( H \subseteq G_i \). Then, the \( C^* \) pushout
\[ \ast_{C^*} \left( \tilde{H} \right) C^u \left( H \right) \]
is a continuous field over the commutative \( C^* \)-algebra \( C^u \left( \tilde{H} \right) \).

Proof Once we have Theorem 2.1 we can conclude
• for pushouts of two CQGs by Theorem 3.4 (or [7, Theorem 3.7], but see Section 3 for an aside on its proof), stating that
\[ A_1 \ast_{C(X)} A_2 \]
is continuous whenever \( A_i \) are;
• for finite families \( \{G_i\} \) by induction;
• in general, by taking filtered colimits over the finite subsets of the index set \( I \) (Proposition 2.3).

This finishes the proof, modulo the last item regarding colimits. ■

It remains to address the filtered-colimit claim that the proof of Theorem 2.2 punts on:

Proposition 2.3 Let \( X \) be a locally compact Hausdorff space, \((I, \leq)\) a filtered poset, and
\[ \iota_{ji} : A_i \to A_j, \ \forall i \leq j \in I \]
a functor from \( I \) to the category of \( C_0 \left( X \right) \)-algebras.
If all \( A_i \) are continuous then so is
\[ A := \lim_{i \in I} A_i \]
Proof As noted before (Remark 1.7) we need lower semicontinuity, since upper semicontinuity is automatic. Concretely, having fixed an element \( a \in A \), a point \( x \in X \) and \( \varepsilon > 0 \), we have to argue that for \( y \) ranging over some neighborhood of \( x \) we have

\[
\|a_y\| > \|a_x\| - \varepsilon. \tag{2-1}
\]

Denote by \( \iota_i : A_i \to A \) the structure map of the colimit. The conclusion follows by

- approximating \( a \) (and hence \( a_x \) and \( a_y \)) arbitrarily well by elements \( \iota_i(a_i) \) for \( a_i \in A_i \)
- for which \( \|a_i\| \) (norm in \( A_i \)) is arbitrarily close to \( \|\iota_i(a_i)\| \) (norm in \( A \))
- and such that (2-1) holds for \( y \) in some neighborhood of \( x \) with \( a_i \) in place of \( a \) (possible by the continuity of \( A_i \)).

\[\blacksquare\]

3 A comment on the literature

The present side-note is devoted to an issue I believe is present in the proof of [7, Theorem 3.7]. Though apparently the problem is fixable, the proof as-is posed some difficulties (for this reader, at least). The setup is as follows: one considers an arbitrary element \( a \) in the dense purely algebraic pushout

\[ A_1 \ast_{C(X)} A_2 \subseteq A_1 \ast_{C(Y)} A_2 \]

and seeks to show that

\[ X \ni x \mapsto \|a_x\| \]

is lower semicontinuous. This is done for separable C*-algebras first, in [7, Lemma 3.5], and then generalized to the present setting by

- first embedding \( a \) into a pushout \( D_1 \ast_{C(Y)} D_2 \) where \( C(Y) \subseteq C(X) \) and \( D_i \subseteq A_i \) are separable C*-subalgebras;
- citing [20, Theorem 4.2] to conclude that there is an embedding

\[ D_1 \ast_{C(Y)} D_2 \subseteq A_1 \ast_{C(X)} A_2. \tag{3-1} \]

The problem is with this last step: [20, Theorem 4.2] proves inclusions of the form

\[ D_1 \ast_{C} D_2 \subseteq A_1 \ast_{C} A_2 \]

given inclusions

\[ C \subseteq D_i \subseteq A_i, \ i = 1, 2. \]

Note that the amalgam \( C \) is the same on both sides. On the other hand, given that on the right-hand side of (3-1) one amalgamates over a (generally-speaking) larger algebra \( C(X) \supset C(Y) \), it is not at all obvious that the natural map (3-1) is indeed an inclusion. Indeed, it certainly will not be in full generality:

Example 3.1 Consider the case where \( D_i = A_i \), but \( C(Y) \) is trivial (i.e. \( C \)) whereas \( C(X) \) is not. (3-1) is then a surjection but not an injection.
Note also that one cannot exhaust $A_i$ and $C(X)$ respectively by separable $D_i$ and $C(Y)$ and naively hope for a continuity permanence property under filtered colimits

$$A_1 *_{C(X)} A_2 = \lim_{D_i, C(Y)} D_1 *_{C(Y)} D_2 :$$

**Example 3.2** Consider the space $X = \mathbb{Z}_p$ (the $p$-adic integers), expressed as a limit

$$X = \lim_{n} (X_n := \mathbb{Z}/p^n).$$

This affords us a filtered-colimit description

$$C(X) = \lim_{n} C(X_n),$$

but if $A \supset C(X)$ denotes the algebra of all (possibly discontinuous) bounded functions on $X$, then

- all $C(X_n) \subset A$ are continuous simply because $X_n$ are discrete, while
- $C(X) \subset A$ isn’t,
- despite the fact that the inclusion $C(X) \subset A$ is the filtered colimit of the inclusions

$$C(X_n) \subset A.$$ ♦

**Remark 3.3** Contrast Example 3.2 with Proposition 2.3, where the base space for the fields $A_i$ whose colimit is being considered is fixed. Such problems arise precisely when the base space changes, much as in the initial observation that (3-1) need not be an embedding. ♦

For all of these reasons, it would seem worthwhile to have a proof of field-continuity permanence under pushouts that is independent of the separable case. We sketch such a proof here.

**Theorem 3.4** Let $X$ be a compact Hausdorff space and $A_i, i \in I$ a family of unital $C(X)$-algebras. If all $A_i$ are continuous then so is the pushout

$$A := \star_{C(X)} A_i.$$ 

**Proof** As in the proof of Theorem 2.2, once we have the statement for *pairs* of algebras $A_1$ and $A_2$ the general conclusion follows via Proposition 2.3 by taking a filtered colimit. We thus focus, for the duration of the proof, on the case of just two algebras $A_i, i = 1, 2$.

The proof will mimic that of [10, Theorem 3.2]. Fix a convergent net (1-3) in $X$ and consider a representation $\rho$ of $A = A_1 *_{C(X)} A_2$ that

- factors through $A_x$ for some $x \in X$ fixed throughout the proof, and
- induces a faithful representation of $A_x$.

Then, by Lemma 1.8 (and as in its proof), the continuity of $A_i, i = 1, 2$ implies that the restrictions $\rho_i$ of $\rho$ to $A_i$ can be Fell-approximated by representations $\rho_{i, \alpha}$ factoring respectively through $(A_i)_{x, \alpha}$, in the sense that

$$\lim_{\alpha} \|\rho_i(a_i)\xi - \rho_{i, \alpha}(a_i)\xi\| = 0, \ i = 1, 2$$

for $a_i \in A_i$. Since $C(X)$ acts via the character $p_{x, \alpha}$ in both both $\rho_{i, \alpha}$, one can form the amalgamated free product of the $\rho_{i, \alpha}, i = 1, 2$; by (3-2) that free product will Fell-converge to $\rho$, hence the conclusion by Lemma 1.8.
References

[1] E. Bédos and L. Tuset. Amenability and co-amenability for locally compact quantum groups. Internat. J. Math., 14(8):865–884, 2003.

[2] Erik Bédos, Roberto Conti, and Lars Tuset. On amenability and co-amenability of algebraic quantum groups and their corepresentations. Canad. J. Math., 57(1):17–60, 2005.

[3] Bachir Bekka, Pierre de la Harpe, and Alain Valette. Kazhdan’s property (T), volume 11 of New Mathematical Monographs. Cambridge University Press, Cambridge, 2008.

[4] B. Blackadar. Operator algebras, volume 122 of Encyclopaedia of Mathematical Sciences. Springer-Verlag, Berlin, 2006. Theory of C*-algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III.

[5] Etienne Blanchard. Tensor products of C(X)-algebras over C(X). Number 232, pages 81–92. 1995. Recent advances in operator algebras (Orléans, 1992).

[6] Étienne Blanchard. Déformations de C*-algèbres de Hopf. Bull. Soc. Math. France, 124(1):141–215, 1996.

[7] Etienne Blanchard. Amalgamated free products of C*-bundles. Proc. Edinb. Math. Soc. (2), 52(1):23–36, 2009.

[8] Etienne Blanchard and Eberhard Kirchberg. Global Glimm halving for C*-bundles. J. Operator Theory, 52(2):385–420, 2004.

[9] Etienne Blanchard and Simon Wassermann. Exact C*-bundles. Houston J. Math., 33(4):1147–1159, 2007.

[10] Alexandru Chirvasitu. Residual finiteness for central pushouts, 2020. arXiv:2002.11232.

[11] Jason Crann. On hereditary properties of quantum group amenability. Proc. Amer. Math. Soc., 145(2):627–635, 2017.

[12] Matthew Daws, Paweł Kasprzak, Adam Skalski, and Piotr M. Sołtan. Closed quantum subgroups of locally compact quantum groups. Adv. Math., 231(6):3473–3501, 2012.

[13] Jacques Dixmier. C*-algebras. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977. Translated from the French by Francis Jellett, North-Holland Mathematical Library, Vol. 15.

[14] J. M. G. Fell. C*-algebras with smooth dual. Illinois J. Math., 4:221–230, 1960.

[15] J. M. G. Fell. Weak containment and induced representations of groups. Canadian J. Math., 14:237–268, 1962.

[16] Paweł Kasprzak, Adam Skalski, and Piotr Mikołaj Sołtan. The canonical central exact sequence for locally compact quantum groups. Math. Nachr., 290(8-9):1303–1316, 2017.

[17] Johan Kustermans. Locally compact quantum groups in the universal setting. Internat. J. Math., 12(3):289–338, 2001.
[18] Johan Kustermans and Stefaan Vaes. Locally compact quantum groups. *Ann. Sci. École Norm. Sup. (4)*, 33(6):837–934, 2000.

[19] Johan Kustermans and Stefaan Vaes. Locally compact quantum groups in the von Neumann algebraic setting. *Math. Scand.*, 92(1):68–92, 2003.

[20] Gert K. Pedersen. Pullback and pushout constructions in $C^*$-algebra theory. *J. Funct. Anal.*, 167(2):243–344, 1999.

[21] Marc A. Rieffel. Continuous fields of $C^*$-algebras coming from group cocycles and actions. *Math. Ann.*, 283(4):631–643, 1989.

[22] Thomas Timmermann. *An invitation to quantum groups and duality*. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2008. From Hopf algebras to multiplicative unitaries and beyond.

[23] Stefaan Vaes. A new approach to induction and imprimitivity results. *J. Funct. Anal.*, 229(2):317–374, 2005.

[24] A. Van Daele. Discrete quantum groups. *J. Algebra*, 180(2):431–444, 1996.

[25] N. E. Wegge-Olsen. *K-theory and $C^*$-algebras*. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993. A friendly approach.

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