Bright solitons in a 2D spin-orbit-coupled dipolar Bose-Einstein condensate

Yong Xu\textsuperscript{1}, Yongping Zhang\textsuperscript{2}, and Chuanwei Zhang\textsuperscript{1}\textsuperscript{*}

\textsuperscript{1}Department of Physics, The University of Texas at Dallas, Richardson, Texas 75080, USA and \textsuperscript{2}Quantum Systems Unit, OIST Graduate University, Onna, Okinawa 904-0495, Japan

We study a two-dimensional spin-orbit-coupled dipolar Bose-Einstein condensate with repulsive contact interactions by both the variational method and the imaginary time evolution of the Gross-Pitaevskii equation. The dipoles are completely polarized along one direction in the 2D plane so as to provide an effective attractive dipole-dipole interaction. We find two types of solitons as the ground state arising from such attractive interaction: a plane wave soliton with a spatially varying phase and a stripe soliton with a spatially oscillating density for each component. Both types of solitons possess smaller size and higher anisotropy than the soliton without spin-orbit coupling. Finally, we discuss the properties of moving solitons, which are nontrivial because of the violation of Galilean invariance.

PACS numbers: 03.75.Lm, 03.75.Mn, 71.70.Ej

I. INTRODUCTION

Ever since the first achievement of Bose-Einstein condensates (BECs) in ultracold atomic gases \cite{1}, matter wave solitons have been the central focus of many experimentalists and theorists \cite{2}. Solitons are the result of the interplay between nonlinearity and dispersion and keep their shape while traveling. In BECs, nonlinearity originates from collisional interactions between atoms, which can be readily tuned via Feshbach resonances \cite{3}. In general, there are two types of solitons: a bright soliton with a density bump for attractive interactions and a dark soliton with a density notch and a phase jump for repulsive interactions. Both bright and dark solitons have been experimentally observed in cold atoms with contact interactions \cite{4–12}. However, for such contact attractive interactions, bright solitons can only exist in one dimension (1D), but not in two dimension (2D) where the states either collapse or expand \cite{13}.

Different from the local nonlinearity resulting from contact interactions, the non-local nonlinearity can stabilize a 2D bright soliton \cite{14, 15}, in particular, the nonlinearity introduced by the dipole-dipole interaction. This interaction is long ranged and anisotropic with the strength and sign (i.e. repulsive or attractive) depending on the dipole orientation. When an external rotating magnetic field is applied to reverse the sign of the interaction \cite{16}, or the dipoles are completely polarized in a 2D plane \cite{17}, the dipolar interaction can become attractive and 2D bright solitons can be, therefore, stabilized under appropriate conditions. It is essential to note that although the relevant interaction in common experiments with cold atom gases is contact, increasing interest has been focused on the atoms with large magnetic moments that possess dipole-dipole interactions \cite{15, 18, 19}. In fact, the Bose-Einstein condensation of several dipolar atoms such as Chromium \cite{20–22}, Dysprosium \cite{23}, and Erbium \cite{24}, as well as the degeneracy of a dipolar Fermi gas \cite{25, 26} have been observed in experiments.

Recently, the spin-orbit coupling between two hyperfine states of cold atom gases has been experimentally engineered \cite{27–32}. And this achievement has ignited tremendous interest in this field because of the dramatic change in the single particle dispersion (induced by spin-orbit coupling) which in conjunction with the interaction leads to many exotic superfluids \cite{33–43} (also see \cite{44–51} for review). Such change in dispersion also results in exotic solitons even when the interaction is contact, including bright solitons \cite{52–58}, dark solitons \cite{59, 60}, and gap solitons \cite{61–63} for BECs, as well as dark solitons for Fermi superfluids \cite{64, 65}. These solitons exhibit unique features that are absent without spin-orbit coupling, for instance, the plane wave profile with a spatially varying phase and the stripe profile with a spatially oscillating density for BECs, as well as the presence of Majorana fermions inside a soliton for Fermi superfluids. Also, the violation of Galilean invariance \cite{53, 66, 67} by spin-orbit coupling dictates that the structure of solitons changes with their velocities.

On the other hand, spin-orbit-coupled BECs with dipole-dipole interactions \cite{68–71} have also been explored, and intriguing quasicrystals \cite{72} as well as meron states \cite{73} have been found. However, whether a soliton can exist in such BECs in 2D with long ranged dipole-dipole interactions and spin-orbit dispersion has not yet been investigated.

In this paper, we examine the existence and properties of a bright soliton in a two species spin-orbit-coupled dipolar BEC in 2D with repulsive contact interactions via both the variational method and the imaginary time evolution of the Gross-Pitaevskii equation (GPE). The dipoles are completely oriented along the y direction in the 2D plane in order to provide an effective attractive dipole-dipole interaction. Thanks to such attractive interactions, we find two types of solitons: a plane wave soliton (when the repulsive intraspecies contact interaction is larger than the repulsive interspecies one) and a stripe soliton (when the interspecies one is larger). These

\textsuperscript{*} chuanwei.zhang@utdallas.edu
2D solitons as the ground states cannot exist without the dipolar interaction and spin-orbit coupling. Such solitons are highly anisotropic and their size is also reduced by spin-orbit coupling. Finally, we study the moving solitons, which are nontrivial because of the lack of Galilean invariance. The size of a soliton first increases and then decreases with the rise of the velocity and this change is anisotropic. The moving soliton also tends to be plane wave even when its stationary counterpart has the stripe structure.

The paper is organized as follows. In Sec. II, we introduce the energy functional and the time-dependent GPE, which are used to describe a spin-orbit-coupled dipolar BEC. In Sec. III, we calculate the bright soliton by performing the minimization of the energy of the variational ansatz wave functions and an imaginary time evolution of the GPE. The properties of such soliton are also explored by the former method. Then, we study the nontrivial moving solitons in Sec. IV. Finally, we conclude in Sec. V.

II. MODEL

We consider a Rashba-type spin-orbit-coupled BEC and write its single particle Hamiltonian as

$$H_s = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_z^2\rho^2 + \frac{1}{2}m\omega_{\perp}^2z^2 + \lambda(\hat{\rho} \times \sigma) \cdot \hat{e}_z,$$  \hspace{1cm} (1)

where $\hat{p} = -i/\hbar \nabla$ is the momentum operator, $m$ is the atom mass, $\lambda$ is the spin-orbit coupling strength, and $\sigma$ are Pauli matrices. $\omega_{\perp} (\omega_z)$ is the trap frequency in the $(x, y)$ plane (along the $z$ direction). Here, we assume that $\hbar\omega_z$ is much larger than $\hbar\omega_{\perp}$ and the mean-field interaction so that the atoms are frozen to the ground state in the $z$ direction. Given that a soliton is studied, we thus set $\omega_{\perp} = 0$.

When the $s$-wave contact and dipole-dipole interactions are involved, the energy functional of a 2D condensate can be written as

$$E = \int dr \left[ E(|\Psi|^4) + E_{dd} \right],$$  \hspace{1cm} (2)

where the condensate wave function $\Psi(r) = [\Psi^\dagger(r), \Psi^\dagger(r)]^T$ with two pseudo-spin components $\Psi^\dagger(r) = [\Psi^\dagger \Psi^\dagger]^T$ and $g_{12} = 4\pi \hbar^2/(\sqrt{2}\pi l_z m)$ and $g_{12} = 4\pi \hbar^2/(\sqrt{2}\pi l_z m)$ are the intraspecies and interspecies contact interaction strength respectively with the intraspecies and interspecies $s$-wave scattering length being $a$ and $a_{12}$ and the characteristic length being $l_z = \sqrt{\hbar/(2m\omega_z)}$. Here, $H_s = -\hbar^2(\partial_x^2 + \partial_y^2)/(2m) - i/\hbar(\partial_z \sigma_y - \partial_y \sigma_z)$ is the 2D single particle Hamiltonian, and the third dimension has been integrated out. For dipole-dipole interactions, we only consider the density-density interaction which is dominant when a two subspace (i.e. two pseudo-spin states) of a large spin atom (e.g. dysprosium) is considered. We also assume that the dipoles are all oriented along the $y$ direction, thus

$$E_{dd} = \frac{g_{dd}}{2} \int dkd\rho k^2 U_d(kl_z),$$  \hspace{1cm} (3)

where the Fourier transform of the total density is $\rho_k = \int d\gamma e^{-i\kappa \cdot r}(|\Psi|^2 + |\Psi|^2)$ and $U(k)$ is given by

$$U_d(kl_z) = -\sqrt{2\pi} + 3\pi l_z^2 k^2 \lambda^2/2 \text{erfc}(kl_z/\sqrt{2})/k,$$  \hspace{1cm} (4)

with erfc being the complementary error function. Here, $g_{dd} = \mu_0 \mu^2/(6\pi l_z)$ characterizes the strength of the dipole-dipole interaction where $\mu$ is the magnetic dipolar moment and $\mu_0$ is the permeability of the free space.

The dynamical behavior of a BEC can be described by the time-dependent GPE

$$i\hbar \frac{\partial \Psi(r)}{\partial t} = H_s \Psi(r) + gG \Psi(r) + g_d U_d(r) \Psi(r),$$  \hspace{1cm} (5)

where the contact interaction matrix is

$$G = \begin{pmatrix} |\Psi|^2 + \frac{2g_{dd}}{\gamma} & 0 \\ 0 & |\Psi|^2 + \frac{2g_{dd}}{\gamma} \end{pmatrix},$$  \hspace{1cm} (6)

and the dipolar interaction potential is

$$U_d(r) = \frac{1}{(2\pi)^2} \int dke^{-ik \cdot r} \rho(k) U_d(kl_z).$$  \hspace{1cm} (7)

For numerical simulation, we choose $\omega_{\perp}, l_z$, and $1/\omega_z$ as the units of energy, length, and time, respectively, and the dimensionless energy per atom hence reads

$$E = \int d\gamma \left[ \Phi^\dagger \mathcal{H}_s \Phi(r) + \frac{1}{2}\sum (|\Phi|^4 + |\Phi|^4) + \gamma_{d12} |\Phi|^2 |\Phi|^2 + \gamma_{d2} \int d\kappa \kappa \Phi(r) \right],$$  \hspace{1cm} (8)

where $\mathcal{H}_s = -(\partial_x^2 + \partial_y^2)/2 - i\alpha (\partial_x \sigma_y - \partial_y \sigma_x)$, $\alpha = \lambda/(\omega_{\perp} l_z)$, $\gamma = \sqrt{2\gamma_{d}} N\gamma_{d}/l_z$, $\gamma_{d12} = \sqrt{2\gamma_{d}} N\gamma_{d12}/l_z$ with the total particle number $N_0$, $\gamma_{d} = 2N_0 a_{d}/l_z$ with $a_d = \hbar/\mu_0 \mu^2/(12\pi \hbar^2)$, and $\Phi_{\gamma} = \int d\gamma e^{-i\kappa \cdot r} (|\Phi|^2 + |\Phi|^2)$. The wave function is normalized to 1 (i.e. $\int d\gamma (|\Phi|^2 + |\Phi|^2) = 1$).

The dimensionless time-dependent GPE reads

$$i\hbar \frac{\partial \Phi(r)}{\partial t} = \mathcal{H}_s \Phi(r) + \gamma G \Phi(r) + \gamma_{d} \int d\kappa \kappa \Phi(r) \Phi(r),$$  \hspace{1cm} (9)

where

$$\mathcal{G} = \begin{pmatrix} |\Phi|^2 + \frac{2g_{dd}}{\gamma} |\Phi|^2 & 0 \\ 0 & |\Phi|^2 + \frac{2g_{dd}}{\gamma} |\Phi|^2 \end{pmatrix}. $$  \hspace{1cm} (10)
FIG. 1. (Color online) Profiles of the density $n_{\uparrow \downarrow} = |\Phi_{\uparrow \downarrow}|^2$ of spin $\uparrow (\downarrow)$ in (a)(b), the total density $n_\uparrow + n_\downarrow$ in (c), the phase of spin $\uparrow (\downarrow)$ in (d)(e) for a plane wave soliton (the first two panels) with $\gamma_{12} = 6$ and a stripe soliton (the last two panels) with $\gamma_{12} = 10$. The solitons in the first and third panels are obtained by the variational method, while the solitons in the second and forth panels are calculated by the imaginary time evolution of the GP Eq. (9). The dashed white line labels the $x = 0$ line. Here, $\gamma = 8$, $\gamma_d/\gamma = 0.67$, and $\alpha = 2$.

III. STATIONARY BRIGHT SOLITONS

To shed light on the structure of a soliton, we start from the homogeneous noninteracting single particle scenario and write its momentum space dispersion as

$$E(k) = \frac{k^2}{2} \pm \alpha k,$$

with two branches labeled by the helicity $\pm$. Clearly, the ground state is degenerate with the energy being $-\alpha^2/2$ when the momenta lie in the $k = |\alpha|$ ring. This is different from the case without spin-orbit coupling where the ground state only occurs at $k = 0$. In this single particle case, any superposition of the states in the ring is also its ground state. Yet, this is not the case when the repulsive contact interaction is involved. The ground state either possesses a single momentum (i.e. plane wave phase) when $\gamma_{12}/\gamma < 1$ or two opposite momenta (i.e. stripe phase) when $\gamma_{12}/\gamma > 1$ [34]. When the dipolar interaction is turned on, one may expect that this effective long ranged attractive interaction along with contact repulsive interaction could support two types of solitons: plane wave and stripe solitons.

To examine whether a soliton can indeed exist in
In writing down the ansatz (12), we have assumed that the wave vector $J_p$ is in the $y$ direction. The prerequisite of this assumption is that the rotation symmetry [38, 39] about the $z$ axis has been broken by the dipole-dipole interaction. Indeed, without the dipole-dipole interaction, this state with $J_p$ along $y$ is not special and other states with $J_p$ along other directions are degenerate with it. For example, the state with $J_p$ along $y$ has the same energy as a state with $J_p$ along $x$. Yet, with the specific dipole-dipole interaction arising from the dipoles entirely oriented along $y$, the symmetry is broken and the ground state should be elongated along $y$ ($a_x > a_y$) so as to provide an effective attractive interaction because of the head-to-tail configuration of polarized dipoles. This elongated configuration allows the existence of a 2D soliton [17] and also requires the wave vector to be along $y$ [75].

Although the wave vector $J_p$ of the ground state is along $y$, there are still two options: negative and positive directions in terms of the time-reversal symmetry $\mathcal{T}$ (i.e. $-i\sigma_y \mathcal{K}$ with the complex conjugate operator $\mathcal{K}$). Specifically, the state $\Phi_{PS} = \mathcal{T} \Phi_P$ is degenerate with $\Phi_P$. In the absence of interactions, any superposition state of $\Phi_P$ and $\Phi_{PS}$,

$$\Phi_{PS} = |\cos \theta| \Phi_P + |\sin \theta| e^{i\varphi} \Phi_{PS},$$  

is degenerate. This degeneracy may be broken by the interaction so that their superposition state has lower energy than either of them or the opposite is true. But this degeneracy breaking should not happen at $\gamma_{12}/\gamma = 1$ since the interaction energy only depends on the total density which is independent of $\theta$ and $\varphi$. This gives us an intuitive understanding that $\gamma_{12}/\gamma = 1$ may separate the plane wave soliton ($|\cos \theta| = 0$ or 1) and the stripe soliton ($|\sin \theta| = |\sin \theta|$), similar to the homogenous spin-orbit-coupled BEC [34] without dipole-dipole interactions. For the stripe soliton, we note that $\varphi = 0, \pi$ corresponds to the ground state as the energy contributed by $\varphi$ is $-\gamma_{12}\sqrt{a_x a_y} e^{-(a_x x^2 + 2a_y y^2)}/(\mathcal{P} \frac{x^2}{a_x^2} + \frac{y^2}{a_y^2})$ [76].

To evaluate the variational parameters $\alpha_x, a_y, x_0, J_p$, and $\theta$, we minimize the energy $\mathcal{E}$ after substituting $\Phi_{PS}$ in Eq. (14) to Eq. (8). Indeed, the calculated variational solutions reveal that there are two types of soliton solutions: plane wave solitons when $\gamma_{12}/\gamma < 1$ and stripe solitons when $\gamma_{12}/\gamma > 1$. We present the density and phase profiles of a typical plane wave soliton (we choose $\theta = \varphi = 0$) in the first panel of Fig. 1, where the stripe structure of the phase of both two components reveals the plane wave feature. The soliton is highly elongated along the $y$ direction and the centers of two components are spatially separated along the $x$ direction because of nonzero $x_0$. To confirm that this variational solution can qualitatively characterize the ground state of the system, we numerically compute the ground state by an imaginary time evolution of the GP Eq. (9). This exact numerical solution also concludes that $\gamma_{12}/\gamma < 1$ yields the plane wave soliton while $\gamma_{12}/\gamma > 1$ the stripe soliton. In the second panel of Fig. 1, we also plot the corre-
sponding density and phase profiles of the GP obtained plane wave soliton. The variational ansatz is in qualitative agreement with it given the separated centers and the plane wave varying phase that both states possess. Yet, the shape of the soliton obtained by the imaginary time evolution deviates slightly from the Gaussian and its phase is also distorted at the region where the density of the soliton is extremely low.

When \( \theta = \pi/4 \) and \( \varphi = 0 \), \( \Phi_{PS} \) is a stripe state with a density oscillation along the \( y \) direction for each component. And there is no stripe for the total density. Along the \( x \) direction, two components are not spatially separated, and the phase for the spin \( \uparrow \) reverses suddenly across \( x = 0 \). Following these properties by replacing \( \Phi_0(x_0/2) + \Phi_0(-x_0/2)/\sqrt{2} \) with \( \cos(J_xx)\Phi_0(0) \) and \( \Phi_0(x_0/2) - \Phi_0(-x_0/2)/\sqrt{2} \) with \( \sin(J_xx)\Phi_0(0) \) in Eq. (14), we obtain another better variational ansatz for the stripe soliton

\[
\Phi_S = \Gamma\Phi_0(0), \tag{15}
\]

where

\[
\Gamma = \left( \begin{array}{cc}
\cos(J_yy)\cos(J_xx) - i\sin(J_yy)\sin(J_xx) \\
\cos(J_yy)\sin(J_xx) + i\sin(J_yy)\cos(J_xx)
\end{array} \right), \tag{16}
\]

with the variational parameters \( J_x \) and \( J_y \). The period of the stripe along the \( y \) direction is \( \pi/J_y \). Interestingly, this stripe soliton corresponds to four points \((-J_x,B)\) in momentum space instead of traditional two points \([53]\) when \( J_x = 0 \), if we do not consider the Gaussian profile \( \Phi_0 \).

We calculate the variational parameters of stripe solitons by performing the minimization of the energy Eq. (8) where \( \Phi \) is replaced with \( \Phi_S \). The density and phase profiles of a calculated typical stripe soliton is displayed in the third panel of Fig. 1. Evidently, the density of each component exhibits the stripe structure while the total density does not. The phase of spin \( \uparrow \) along the \( y \) direction varies like a plane wave, but reverses across \( x = 0 \) due to the presence of \( \sin(J_xx) \) in the imaginary part. The phase of spin \( \downarrow \) along the \( y \) direction exhibits the phase rotation like vortices around \( x = 0 \) and \( y = n\pi/J_y \) with integer \( n \); around these points, the wave function \( \Phi_{SL} \) is proportional to \((-1)^n(J_xx + i(J_yy - n\pi)) \) and the corresponding density of spin \( \downarrow \) is extremely low. Moreover, in the last panel of Fig. 1, we plot the density and phase profiles of the corresponding stripe soliton obtained by the imaginary time evolution of the GPE; comparing this figure with the third panel of Fig. 1 implies that the stripe variational ansatz is qualitatively consistent with the GP results.

To study the properties of a soliton with respect to dipole-dipole interactions, we evaluate the variational parameters of both the plane wave and stripe solitons by the variational method and plot them in Fig. 2 as the dipole-dipole interaction \( \gamma_d/\gamma \) varies. Clearly, with increasing \( \gamma_d/\gamma \), \( a_x \) and \( a_y \) increase monotonously because of the enhanced effective attractive interaction, indicating that the size \( l_x \) and \( l_y \) of the soliton decrease monotonously.

![FIG. 3. (Color online) Plot of \( x_0 \) (dash-dot blue line) and \( J_p \) (dashed blue line) for the plane wave variational ansatz, and \( J_s \) (solid green line) and \( J_s \) (dotted green line) for the stripe variational ansatz with respect to \( a_x \) by the minimization of the energy \( E^{PW}_s \) and \( E^{s}_{\text{stripe}} \) in (a). (b) shows the minimum energy of \( E^{PW}_s \) (dotted blue line) and \( E^{s}_{\text{stripe}} \) (solid green line) as a function of \( a_x \). Here, \( \alpha = 2 \).](image)
It is important to note that although $E_{\text{PW}}^\text{strip}$ is not a function of $a_y$, other energy such as the kinetic energy (when $x_0 = J_y = 0$), the contact and dipolar interaction energy depends on it.

For the stripe soliton, the single particle energy due to the presence of $J_x$ and $J_y$ is

$$E_{\text{strip}} = 1/2(J_x^2 + J_y^2) - \alpha(J_x + J_y e^{-J^2/a_x}).$$  \hspace{1cm} (20)

Similar to the plane wave case, this energy is independent of $a_y$. For fixed $a_x$, the minimization of this energy yields both $J_x$ and $J_y$ as a function of $a_x$ as shown in Fig. 3(a). When $a_x$ moves towards zero, the solution approaches $(J_x = \alpha, J_y = 0)$ or $(J_x = 0, J_y = \alpha)$; when it moves away from zero, there is only one solution where $J_y$ decreases from $\alpha$ while $J_x$ increases from zero with the rise of $a_x$. Also, the energy $E_{\text{strip}}$ decreases as $a_x$ increases. Analogous to the plane wave soliton, the total energy decrease resulted from spin-orbit coupling and dipole-dipole interactions as $a_x$ and $a_y$ increase from the value without spin-orbit coupling exceeds the energy gain of the kinetic (when $J_x = 0$ and $J_y = 0$) and contact interaction; this leads to the increased $a_x$ and $a_y$ compared with the soliton without spin-orbit coupling. This picture is also consistent with Fig. 2(c) where $J_x$ increases while $J_y$ decreases with respect to $\gamma_d/\gamma$.

To explicitly demonstrate the effect of the spin-orbit coupling on the properties of a soliton, we plot the variational parameters as a function of the spin-orbit coupling strength $\alpha$ for both the plane wave and stripe solitons in Fig. 4. Consistent with the aforementioned feature that spin-orbit coupling can suppress the size of the soliton, both Fig. 4(a) and Fig. 4(b) display a monotonic increasing behavior of $a_x$ and $a_y$ as a function of $\alpha$. Also, the aspect ratio $\sqrt{a_y/a_x}$ is decreased by spin-orbit coupling. Similar to Fig. 2(a) and Fig. 2(b), $a_x$ and $a_y$ for the plane wave soliton are slightly larger than the stripe soliton in that the former has a smaller contact interaction. For the plane wave soliton, $J_P$ (determined mainly by the spin-orbit coupling strength) increases with respect to $\alpha$ while $x_0$ decreases; for the stripe soliton, both $J_x$ and $J_y$ increase.

In Fig. 2(d) and Fig. 4(d), for both plane wave and stripe solitons, we compare their energy obtained by the variational procedure with the one obtained by the imaginary time evolution of the GPE. Both figures show that the energy calculated by the imaginary time evolution is lower as expected. Yet, the difference between these two energy is not large (no more than 10%), suggesting that the variational ansatz can qualitatively characterize the solitons. We note that in Fig. 4(d), the energy is shifted by $\alpha^2/2$ in order to clearly present the different results of the two methods, which could be smeared by the large value of $\alpha^2/2$.

**IV. MOVING BRIGHT SOLITONS**

Generally, the wave function of a moving soliton with the velocity $v$ can be simply written as $\exp(i v \cdot r) \Phi_s(r - vt)$ where $\Phi_s$ is the wave function of a stationary soliton. But this is only valid for a system respecting Galilean transform invariance. In fact, Galilean invariance is broken in a spin-orbit-coupled BEC [66], and this violation dictates that the shape of a soliton depends on its velocity strength [53]. Here, for a soliton in a spin-orbit-coupled dipolar BEC in 2D, we assume that a moving soliton can be written as

$$\Phi_M(r, t) = \Phi_{s}(r - vt, t) \exp(i v \cdot r - \frac{1}{2} v^2 t),$$ \hspace{1cm} (21)

where $\Phi_{s}$ is a localized function. Plugging $\Phi_M(r, t)$ into Eq. (9) yields

$$i \frac{\partial \Phi_{s}(r)}{\partial t} = \mathcal{H}_{\text{s}}(v) \Phi_{s}(r) + \gamma \mathcal{Q} \Phi_{s}(r) + \gamma_d \int d\mathbf{k} e^{i \mathbf{k} \cdot \mathbf{r}} a(k) U_d(k) \Phi_{s}(r),$$ \hspace{1cm} (22)
where $\mathcal{H}_s(v) = \mathcal{H}_s + \alpha(v \times \sigma) \cdot e_z$. Compared to Eq. (9), this dynamical equation has an additional term $\alpha(v \times \sigma) \cdot e_z$, acting as a Zeeman field; this additional term implies the violation of Galilean invariance. This violation means that it is no longer a trivial task to find a moving bright soliton for a BEC with spin-orbit coupling; we need to perform an imaginary time evolution of the Eq. (22), but not Eq. (9). Furthermore, such a 2D moving soliton should be different for different velocity directions even if their amplitude is the same, in contrast to a 1D soliton which can only move in one direction.

To examine how the shape of a soliton changes with respect to the velocities along $x$ and $y$ directions, we plot the imbalance $Imb$ and the width $l_x$ of a soliton of spin $\uparrow$ as a function of the velocities $v_x$ and $v_y$ in Fig. 5. Here, the imbalance for spin $\uparrow$ is defined as

$$Imb = \left| \frac{\Phi_\uparrow(0)}{\Phi_\uparrow(\pi/2\alpha)} \right|^2 = \frac{|\Phi_\uparrow(0)|^2 - |\Phi_\uparrow(\pi/2\alpha)|^2}{|\Phi_\uparrow(0)|^2 + |\Phi_\uparrow(\pi/2\alpha)|^2}.$$  

(23)

which characterizes a stripe soliton (as shown in Fig. 5(c) and Fig. 5(d)) when it approaches one and a plane wave soliton (as shown in Fig. 5(e) and Fig. 5(f)) when it approaches zero. Fig. 5(a) and Fig. 5(b) demonstrate that $Imb$ suffers a sharp decline from one to near zero as $v_x$ and $v_y$ increase, indicating that a moving soliton tends to be a plane wave state. The reason is the broken rotation symmetry of the single particle Hamiltonian by the velocity induced Zeeman field, giving rise to a ground state of the single particle system lying at one momentum point located along the $x$ ($y$) direction when the velocity is along that direction. This also explains why the phase of a moving plane wave soliton with the velocity along the $x$ ($y$) direction varies along that direction.

Furthermore, Fig. 5(a) demonstrates that the width of the soliton gradually grows when the velocity along the $y$ direction is enlarged. To explain the growth, we consider the plane wave ansatz in Eq. (12) which yields an additional term $-\alpha v_y e^{-\alpha x^2/4}$ for the single particle energy when a soliton moves; this energy decrease enlarges exponentially with the decline of $a_\sigma$ (i.e. increase of the width), leading to an expanded soliton with the rise of the velocity. However, this is not a monotonous behavior and the soliton begins shrinking when the velocity goes larger, due to the enlarged $J_y$ by the velocity induced Zeeman field, similar to increasing spin-orbit coupling. On the other hand, when the velocity is along the $x$ direction, the width of the soliton gains a sudden rise as the velocity varies, as shown Fig. 5(b). This corresponds to a change from a stripe soliton with the wave vector along the $y$ direction to a plane wave soliton with the wave vector along the $x$ direction. For the stationary solitons, the soliton with the wave vector mainly along the $y$ direction has lower energy than the one with the wave vector mainly along the $x$ direction as the dipoles are completely oriented along $y$. But the Zeeman field induced by the presence of a velocity along the $x$ direction gives rise to the single particle ground state that possesses the wave vector along $x$. The two states with the wave vector along these two directions compete and change from the former to the latter (i.e. first order phase transition). For the decrease of the width when $v_x$ goes even larger, the reason is the same as the case for $v_y$.

When a stationary soliton is plane wave, the moving behavior is similar except that the moving soliton is always the plane wave soliton.

V. CONCLUSION

We have studied the bright solitons as the ground states in a spin-orbit-coupled dipolar BEC in 2D with dipoles completely polarized along one direction in the 2D plane. It is important to note that the solitons are the ground states in 2D, but they are the metastable states in quasi-2D where the true ground state would collapse and there is an energy barrier between the soliton state and this ground state. Two types of solitons have been found: a plane wave soliton and a stripe soliton. The former has the plane wave phase variation and its two components are slightly spatially separated; while for the latter, the density of each component is spatially oscillating and the variational ansatz suggests that four points in momentum space are involved. Both plane wave and stripe solitons are highly anisotropic and their size is decreased by spin-orbit coupling. These solitons cannot exist in 2D without the dipole-dipole interaction and spin-orbit coupling.
Moreover, the shape of solitons changes with velocity due to the absence of Galilean invariance and this change is anisotropic.

The 2D bright soliton, albeit mainly plane wave soliton, can also exist when equal Rashba and Dresselhaus spin-orbit coupling is considered. In experiments, this type of spin-orbit coupling has been engineered by coupling two hyperfine states of atoms through two counter-propagating Raman laser beams [27–32] and such setup could be employed to realize this spin-orbit coupling in Dysprosium with large dipole-dipole interactions. Also, the large magnetic moment in Dysprosium atoms may permit the realization of Rashba spin-orbit coupling [77].

ACKNOWLEDGMENTS

We would like to thank L. Jiang, K. Sun, C. Qu, and Z. Zheng for helpful discussions. Y. Xu and C. Zhang are supported by ARO(W911NF-12-1-0334) and AFOSR (FA9550-13-1-0045). Y. Zhang is supported by Okinawa Institute of Science and Technology Graduate University. We also thank Texas Advanced Computing Center as parts of our numerical calculations were performed there.

[1] C. J. Pethick and H. Smith, *Bose-Einstein Condensation in Dilute Gases*, 2nd ed. (Cambridge University Press, Cambridge, 2008).
[2] P. G. Kevrekidis, D. J. Frantzeskakis, and R. Carretero-González. Emergent Nonlinear Phenomena in Bose-Einstein Condensates, Springer, 2007.
[3] C. Chin, R. Grimm, P. Julienne, and E. Tiesinga, Rev. Mod. Phys. 82, 1225 (2010).
[4] S. Burger, K. Bongs, S. Dettmer, W. Ertmer, K. Sen- gstock, A. Sanpera, G. V. Shlyapnikov, and M. Lewenstein, Phys. Rev. Lett. 83, 5198 (1999).
[5] J. Denschlag, J. E. Simsarian, D. L. Feder, C. W. Clark, L. A. Collins, J. Cubizolles, L. Deng, E. W. Hagley, K. Helmerson, W. P. Reinhardt, S. L. Rolston, B. I. Schneider, and W. D. Phillips, Science 287, 97 (2000).
[6] B. P. Anderson, P. C. Haljan, C. A. Regal, D. L. Feder, L. A. Collins, C. W. Clark, and E. A. Cornell, Phys. Rev. Lett. 86, 2926 (2001).
[7] K. E. Strecker, G. B. Partridge, A. G. Truscott, and R. G. Hulet, Nature (London) 417, 150 (2002).
[8] L. Khaykovich, F. Schreck, G. Ferrari, T. Bourdel, J. Cubizolles, L. D. Carr, Y. Castin, and C. Salomon, Science 296, 1290 (2002).
[9] S. L. Cornish, S. T. Thompson, and C. E. Wieman, Phys. Rev. Lett. 96, 170401 (2006).
[10] B. Eiermann, Th. Anker, M. Albiez, M. Taglieber, P. Treutlein, K.-P. Marzlin, and M. K. Oberthaler, Phys. Rev. Lett. 92, 230401 (2004).
[11] C. Hammer, Y. Zhang, J. J. Chang, C. Zhang, and P. Engels, Phys. Rev. Lett. 111, 264101 (2013).
[12] H. V. Nguyen, P. Dyke, D. Luo, B. A. Malomed, and R. G. Hulet. Nat. Phys. 10, 918 (2014).
[13] Both the kinetic energy and the contact attractive interaction energy in 2D is proportional to $1/L^2$ with the size $L$ of a state; to lower energy, the state collapses (i.e. $L \rightarrow 0$) when the interaction energy is larger than the kinetic one, and the state expands (i.e. $L \rightarrow 0$) otherwise.
[14] W. Krollkowski, O. Bang, J. J. Rasmussen, and J. Wyller, Phys. Rev. E 64, 016612 (2001).
[15] T. Lahaye, C. Menotti, L. Santos, M. Lewenstein, and T. Pfau, Rep. Prog. Phys. 72, 126401 (2009).
[16] P. Pedri and L. Santos, Phys. Rev. Lett. 95, 200404 (2005).
[17] I. Tikhonenkov, B. A. Malomed, and A. Vardi, Phys. Rev. Lett. 100, 090406 (2008).
[18] C. Ticknor, R. M. Wilson, and J. L. Bohn, Phys. Rev. Lett. 106, 065301 (2011).
[19] M. A. Baranov, M. Dalmonte, G. Pulci, and P. Zoller, Chem. Rev. 112, 5012 (2012).
[20] A. A. Griesmaier, J. Werner, S. Hensler, J. Stuhler, and T. Pfau, Phys. Rev. Lett. 94, 160401 (2005).
[21] T. Koch, T. Lahaye, J. Metz, B. Fröhlich, A. Griesmaier, and T. Pfau, Nat. Phys. 4, 218 (2008).
[22] Q. Beaufils, R. Chicireanu, T. Zanon, B. Laburthe-Tolra, E. Maréchal, L. Vernac, J.-C. Keller, and O. Gorceix, Phys. Rev. A 77, 061601(R) (2008).
[23] M. Lu, N. Q. Burdick, S. H. Youn, and B. L. Lev, Phys. Rev. Lett. 107, 190401 (2011).
[24] K. Aikawa, A. Frisch, M. Mark, S. Baier, A. Rietzler, R. Grimm, and F. Ferlaino, Phys. Rev. Lett. 108, 210401 (2012).
[25] M. Lu, N. Q. Burdick, and B. L. Lev, Phys. Rev. Lett. 108, 215301 (2012).
[26] K. Aikawa, A. Frisch, M. Mark, S. Baier, R. Grimm, and F. Ferlaino, Phys. Rev. Lett. 112, 010404 (2014).
[27] Y.-J. Lin, K. Jiménez-García, and I. B. Spielman, Nature (London) 471, 83 (2011).
[28] P. Wang, Z.-Q. Yu, Z. Fu, J. Miao, L. Huang, S. Chai, H. Zhai, and J. Zhang, Phys. Rev. Lett. 109, 095301 (2012).
[29] L. W. Cheuk, A. T. Sommer, Z. Hadzibabic, T. Yefsah, W. S. Bakr, and M. W. Zwierlein, Phys. Rev. Lett. 109, 095302 (2012).
[30] J.-Y. Zhang, S.-C. Ji, Z. Chen, L. Zhang, Z.-D. Du, B. Yan, G.-S. Pan, B. Zhao, Y.-J. Deng, H. Zhai, S. Chen, and J.-W. Pan, Phys. Rev. Lett. 109, 115301 (2012).
[31] C. Qu, C. Hammer, M. Gong, C. Zhang, and P. Engels, Phys. Rev. A 88, 021604(R) (2013).
[32] R. A. Williams, M. C. Beeler, L. J. LeBlanc, K. Jiménez-García, and I. B. Spielman, Phys. Rev. Lett. 111, 095301 (2013).
[33] T. D. Stanescu, B. Anderson, and V. Galitski, Phys. Rev. A 78, 023616, (2008).
[34] C. Wang, C. Gao, C.-M. Jian, H. Zhai, Phys. Rev. Lett. 105, 160403 (2010).
[35] C. Wu, I. Mondragon-Shem, and X. F. Zhou, Chin. Phys. Lett. 28, 097102 (2011).
[36] T.-L. Ho and S. Zhang, Phys. Rev. Lett. 107, 150403 (2011).
[37] S. Sinha, R. Nath, and L. Santos, Phys. Rev. Lett. 107, 270401 (2011).
This elongated configuration along with the wave vector in the $y$ direction has the lower single particle energy contributed by the spin-orbit coupling than the case with the wave vector in the $x$ direction. This energy is generally so small that the states with different $\phi$ are nearly degenerate. That might be the reason why the stripe state for $\phi = 0$ which possesses a sharp phase change across the symmetric axis in a harmonically trapped spin-orbit-coupled BEC has not been noticed [38].