Quasi-linearized B-spline collocation method for coupled nonlinear boundary value problems

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Abstract. A collocation method with B-splines as shape functions is introduced to solve a coupled system of nonlinear boundary value problems in four unknowns and four equations. The proposed method uses the quasilinearization technique to linearize the nonlinear problems. Based on the order of derivative of each unknown in the given system, the approximation of each unknown is expressed as linear combination of B-spline functions of different degree. Upon imposing the boundary conditions to these approximations the shape functions take a new form. With the revised approximation collocation method is implemented for the linearized system. To test the efficiency of the method a real time problem from literature is considered and solved by the proposed method. Results obtained by the proposed method are in good agreement with the actual one.

Keywords : Nonlinear boundary value problems, Quasilinearization, B-spline, Collocation method.

1. Introduction

Existence of coupled nonlinear boundary value problems (BVP) is quite common in the areas of Fluid dynamics, Applied Science and Engineering. Specially in fluid dynamics when there are studies of certain effects on different fluids viz. MHD, micro polar, Nano fluid etc. over a geometry of infinite dimension leads to the modelling of coupled system of nonlinear boundary value problems [1],[2],[3]. However finding solution of those problems using analytical methods is not so easy. Hence it is must to use numerical methods for the solution of such problems. Currently majority of the researchers in CFD, use a conventional FEM technique to solve such type of problems. Some of them use methods like Sinc-collocation, perturbation technique etc. Every method has its own advantage and disadvantages. The conventional FEM technique takes more time for obtaining a better solution and had lot of computations. In the current paper, we propose a numerical method that uses collocation method with B-splines along with quasi linearization technique to solve such problems. B-splines are very well known for best approximation. Several researchers are working on with different numerical techniques with B-splines [4],[5],[6]. The proposed method works with a limited number of computations and gives the result in a short time. Also it is very easy to implement and adoptable.

In this paper we consider a general coupled nonlinear boundary value problem in four unknowns with four equations

\[ N_i(t, f, f', f'', g, g', g'', h, h', h'', x, x', x'') = 0, \quad i = 1, 2, 3, 4 \quad t_l \leq t \leq t_r \]  (1)
along with boundary conditions \( f(t_i) = F_0, f'(t_i) = F_1, f''(t_i) = F_2, \) \( g(t_i) = G_0, g(t_r) = G_1, h(t_i) = H_0, h(t_r) = H_1, x(t_i) = X_0, x(t_r) = X_1 \) where \( F_i, i = 0, 1, 2 \) and \( G_i, H_i, X_i : i = 0, 1 \) are real constants. Here \( t_l \) is the left boundary point and \( t_r \) is the right boundary point.

The given nonlinear BVP’s are converted into a sequence of linear BVP’s by quasilinearization technique [7]. The convergence of quasilinearization technique is proved by Bellman and Kallaba [7] and it is of second order convergent.

Consider the linear form of BVP along with given boundary conditions

\[
L_i(t, f, f', f'', g, g', h, h', x, x') = l_i, \quad i = 1, 2, 3, 4 \quad t_l \leq t \leq t_r
\]

where \( l_i \) are function of \( t \) only.

Now these linear BVP’s are solved using a collocation method with B-splines as shape functions. In the next section a brief introduction about B-splines is presented. The proposed method is described in section 3. An application of the proposed method along with the results is discussed in section 4. The last section is concluded with conclusion of the findings.

2. Understanding of B-Splines

B-splines are introduced by Schoenberg [8] and developed by Cox and De Boor [9],[10]. A conventional notation for B-splines over a uniform grid is present in the book written by P.M. Prenter [11]. Understanding of B-splines is quite easy and the definition of each B-spline curve is as follows.

A \( k \)-th degree B-splines curve \( \phi_{3,i} \) is a piecewise polynomial defined over an interval \([t_{i-k+1}, t_{i+k-1}]\) if \( k \) is odd and is defined over an interval \([t_{i-k+2}, t_{i+k-1}]\) if \( k \) is even. In particular a cubic B-splines function \( \phi_{3,i} \) is defined as

\[
\phi_{3,i}(t) = \begin{cases} 
\sum_{r=i-2}^{r=i+2} \frac{(t-t_r)^3}{\Pi(t)} & , \quad t \in [t_{i-2}, t_{i+2}] \\
0 & , \quad \text{otherwise}
\end{cases}
\]

where \( \Pi(t) = (t - t_{i-2})(t - t_{i-1}) \cdots (t - t_{i+2}) \) and \((.)^+ \) is the positive part function.

To approximate a function over interval \([a, b]\) using cubic B-splines we need to introduce a partition over \([a, b]\) along with additional grid points outside of given interval. Consider a partition \( a = t_0 < t_1 < \ldots < t_n = b \) over \([a, b]\). Introduce six additional points \( t_{-3}, t_{-2}, t_{-1}\) to the left side of \( t_0 \) and \( t_{n+1}, t_{n+2}, t_{n+3}\) to the right side of \( t_n \). Then the set of functions \( \{\phi_{3,-1}, \phi_{3,0}, \ldots, \phi_{3,n+1}\} \) forms a basis for the space of all cubic spline polynomials over the interval \([a, b]\). In a similar way a fourth degree(quartic) B-spline \( \phi_{4,i} \) is defined on the interval \([t_{i-2}, t_{i+3}]\), given by (4) and the complete basis set for the space of quartic spline polynomials is \( \{\phi_{4,-2}, \phi_{4,-1}, \phi_{4,0}, \ldots, \phi_{4,n+1}\} \).

\[
\phi_{4,i}(t) = \begin{cases} 
\sum_{r=i-2}^{i+3} \frac{(t_t-t_r)^4}{\Pi(t)} & , \quad t \in [t_{i-2}, t_{i+3}] \\
0 & , \quad \text{otherwise}
\end{cases}
\]

3. QBSCM

The equations (2) can be considered in the following form.

\[
\sum_{j=1}^{4} a_{ij}(t) f^{(4-j)} + \sum_{j=1}^{3} b_{ij}(t) g^{(3-j)} + \sum_{j=1}^{3} c_{ij}(t) h^{(3-j)} + \sum_{j=1}^{3} d_{ij}(t) x^{(3-j)} = l_i(t) \quad \text{where} \quad i = 1, 2, 3, 4.
\]
Consider a a uniform partition of grid size \( h, t_l = t_0 < t_1 < \cdots < t_n = t_r \) over the interval \([t_l, t_r]\). Introduce four additional grid points \( t_{-4} < t_{-3} < t_{-2} < t_{-1} \) to left side of \( t_0 \) and another four grid points \( t_{n+1} < t_{n+2} < t_{n+3} < t_{n+4} \) to the right of \( t_n \). Now we can define the complete set of quartic and cubic B-splines over these grid points. Each unknown \( f, g, h \) and \( x \) is approximate as linear combination of B-splines as follows.

\[
f(t) = \sum_{j=-2}^{n+1} f_j \phi_{4,j}, \quad g(t) = \sum_{j=-1}^{n+1} g_j \phi_{3,j}, \quad h(t) = \sum_{j=-1}^{n+1} h_j \phi_{3,j}, \quad x(t) = \sum_{j=-1}^{n+1} x_j \phi_{3,j} \tag{6}
\]

where \( f_j, g_j, h_j \) and \( x_j \) are parameters in the approximation of respective function.

Imposing the boundary conditions in (1), to the respective unknown, each of these approximations in (6) changes into the form given by (7)-(10)

\[
f(t) = fw(t) + \sum_{j=0}^{n} f_j Q_j(t) \tag{7}
\]

\[
g(t) = gw(t) + \sum_{j=0}^{n} g_j R_j(t) \tag{8}
\]

\[
h(t) = hw(t) + \sum_{j=0}^{n} h_j R_j(t) \tag{9}
\]

\[
x(t) = xw(t) + \sum_{j=0}^{n} x_j R_j(t) \tag{10}
\]

where

\[
Q_j(t) = \begin{cases} 
  P_j(t) - \left[ \frac{P'_j(t_l)}{P'_{j-1}(t_l)} \right] P_{j-1}(t), & \text{for } j = 0, 1; \\
  P_j(t), & \text{for } j = 2, 3, \ldots, n-3; \\
  P_j(t) - \left[ \frac{P'_j(t_r)}{P'_{j+1}(t_r)} \right] P_{j+1}(t), & \text{for } j = n-2, n-1, n.
\end{cases}
\]

\[
w_1(t) = \frac{F_0}{\phi_{4,-2}(t)} \phi_{4,-2}(t)
\]

\[
P_j(t) = \begin{cases} 
  \phi_{4,j}(t) - \left[ \frac{\phi_{4,2}(t)}{\phi_{4,2}(t)} \right] \phi_{4,-2}(t), & \text{for } j = -1, 0, 1; \\
  \phi_{4,j}(t), & \text{for } j = 2, 3, \ldots, n + 1.
\end{cases}
\]

\[
R_j(t) = \begin{cases} 
  \phi_{3,j}(t) - \left[ \frac{\phi_{3,2}(t)}{\phi_{3,2}(t)} \right] \phi_{3,-1}(t), & \text{for } j = 0, 1; \\
  \phi_{3,j}(t), & \text{for } j = 2, 3, \ldots, n - 2; \\
  \phi_{3,j}(t) - \left[ \frac{\phi_{3,2}(t)}{\phi_{3,2}(t)} \right] \phi_{3,n+1}(t), & \text{for } j = n - 1, n.
\end{cases}
\]

\[
gw(t) = \frac{G_0}{\phi_{3,-1}(t)} \phi_{3,-1}(t) + \frac{G_1}{\phi_{3,n+1}(t)} \phi_{3,n+1}(t)
\]

\[
hw(t) = \frac{H_0}{\phi_{3,-1}(t)} \phi_{3,-1}(t) + \frac{H_1}{\phi_{3,n+1}(t)} \phi_{3,n+1}(t),
\]

\[
xw(t) = \frac{X_0}{\phi_{3,-1}(t)} \phi_{3,-1}(t) + \frac{X_1}{\phi_{3,n+1}(t)} \phi_{3,n+1}(t)
\]
Now the unknowns $f,g,h$ and $x$ are approximated as (7), (8), (9) and (10) respectively. With these approximations, we collocate the coupled linear equations (5) at the collocated points. Here we are considering the partition points as collocation points.

Hence the system of equations (5) will be rearranged in matrix form as below.

\[
\begin{align*}
K_{11}f + K_{12}g + K_{13}h + K_{14}x &= L_1 \\
K_{21}f + K_{22}g + K_{23}h + K_{24}x &= L_2 \\
K_{31}f + K_{32}g + K_{33}h + K_{34}x &= L_3 \\
K_{41}f + K_{42}g + K_{43}h + K_{44}x &= L_4
\end{align*}
\]

where each matrix of $K_{11}, K_{21}, K_{31}$ and $K_{41}$ are 4 band matrices and all other $K_{ij}$ are 3 band matrices. Also each matrix is of size $n + 1$.

Here for each $r = 1, 2, 3, 4$

\[
[K_r]_{ij} = \begin{cases} 4 \sum_{k=1}^{r} a_{rk}(t_i) Q_j^{(4-k)}(t_i) & i, j = 0 \\
3 \sum_{k=1}^{r} b_{rk}(t_i) R_j^{(3-k)}(t_i) & i, j = 0 \text{ to } n \\
3 \sum_{k=1}^{r} c_{rk}(t_i) T_j^{(3-k)}(t_i) & i, j = 0 \text{ to } n \\
\end{cases}
\]

\[
[L_r]_i = l_i(t_i) - \left[ 4 \sum_{k=1}^{r} a_{rk}(t_i) f w^{(4-k)}(t_i) + \sum_{k=1}^{r} b_{rk}(t_i) g w^{(3-k)}(t_i) + \sum_{k=1}^{r} c_{rk}(t_i) h w^{(3-k)}(t_i) + \sum_{k=1}^{r} d_{rk}(t_i) x w^{(3-k)}(t_i) \right] \quad i = 0 \text{ to } n
\]

Solving these system of equations we get the unknown parameter vectors $f, g, h$ and $x$.

### 4. Application of the proposed method

To test the efficiency of the proposed method we consider a modelled problem in [1]. The coupled system of nonlinear problems along with boundary conditions is given by

\[
\begin{align*}
f'''' + B_1 h' + f f'' + G_r \theta + G_m \varphi - K f' - M f' &= 0 \\
\lambda h'' - \frac{2}{G_1} \left(2h + f''\right) + f' h + h' &= 0 \\
\theta'' + Pr f' \theta' + Pr D w \varphi'' + Pr Ec (f'')^2 &= 0 \\
\varphi'' + Sc f \varphi' + Sc Sr \theta'' - \tau (\theta' \varphi' + \theta'' \varphi) &= 0
\end{align*}
\]

The corresponding boundary conditions are

\[
\begin{align*}
f' &= 1, f = V_0, h = -sf''', \theta = 1, \varphi = 1 \quad \text{at} \quad \eta = 0 \\
f' &= 0, h = 0, \theta = 0, \varphi = 0 \quad \text{at} \quad \eta \to \infty
\end{align*}
\]

This problem was solved by the proposed method and we obtained the solution of each unknown of the above problem. In the paper [1], the solution of the considered problem is to find the variables $f', h, \theta$ and $\varphi$. The velocity which is denoted by $f'$, from the obtained solution it was approximated by $f' = f w'(t) + \sum_{j=0}^{\infty} f_j Q_j(t)$. The other unknown variables are approximated by as like in (8), (9) and (10). The graphical representation for the obtained solutions is presented below.
These are the graphical representation of effect of $V_0$ on the velocity $f'$, temperature $\theta$ and concentration $\phi$ with a specific constant values. Fig:1-3 are the graphs obtained by the proposed method whereas Fig:4-6 are the original graphs which are in [1]. These graphs tell us that the proposed method is working well. Also we can understand that the implementation of the proposed method is quite easier than any other existing methods.

5. Conclusion

The current article addresses a numerical method namely Quasi-linearized B-spline collocation method(QBSCM) for coupled system of nonlinear boundary value problems. In the paper understanding and defining the B-splines functions of cubic and quartic(fourth) degree is described. Implementation of QBSCM along with different degree B-splines functions is explained. Results obtained by the proposed method on a real time problem are shown in graphs. We conclude that the proposed method will be useful for researchers, engineers and scientists who deal with the coupled system of non linear boundary value problems.

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