TATE-SHAFAREVICH GROUPS OF ELLIPTIC CURVES WITH NONTRIVIAL 2-TORSION SUBGROUPS

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Abstract. For any number field, we prove that there exists an elliptic curve defined over this field such that its Shafarevich-Tate group has a nontrivial 2-torsion subgroup.

1. Introduction

Throughout this paper, let $K$ be a number field, and let $\Omega_K$ be the set of all nontrivial places of $K$. For each $v \in \Omega_K$, let $K_v$ be the completion of $K$ at $v$. Let $\mathcal{O}_K$ be the ring of adeles of $K$. We fix an algebraic closure $\overline{K}$ of $K$. Let $X$ be a proper algebraic variety defined over $K$. Let $\overline{X}$ be the base change of $X$ to $\overline{K}$.

We say that $X$ violates the local-global principle if $X(K_v) \neq \emptyset$ for all $v \in \Omega_K$, whereas $X(K) = \emptyset$.

As a consequence of the Hasse-Minkowski theorem for quadratic forms, the local-global principle holds for smooth, projective and geometrically connected curves of genus-0 over every number field. The first examples of varieties violating the local-global principle are due to Lind [Lin40] and independently, but a bit later, to Reichardt [Rei42]. For example, they proved that the genus-1 curve, given by the smooth projective model of plane curve $2y^2 = 1 - 17x^4$ over $\mathbb{Q}$, violating the local-global principle. Shortly thereafter, Selmer [Sel51] gave many genus-1 curves violating the local-global principle, of which the simplest one is defined over $\mathbb{Q}$ by $3w_0^3 + 4w_1^3 + 5w_2^3 = 0$ in $\mathbb{P}^2$ with homogeneous coordinates $(w_0 : w_1 : w_2)$. Poonen [Poo01] proved that there exist curves over every number field violating the local-global principle. Clark [Cla09, Section 5 Conjecture 1] conjectured that genus-1 curve will suffice, i.e. for any number field, there exists a genus-1 curve over this field violating the local-global principle. The author [Wu22] gave an explicit construction to prove that Clark’s conjecture [Cla09, Section 5 Conjecture 1] holds for any number field not containing $\mathbb{Q}(i)$.

Our goal is to prove that Clark’s conjecture [Cla09, Section 5 Conjecture 1] holds for any number field. More exactly, we will prove the following theorem.

Theorem 1.1 (Theorem 4.1). For any number field $K$, there exists an elliptic curve $E$ defined over $K$ such that $\Sha(K,E)[2] \neq 0$. Here $\Sha(K,E)[2]$ is the 2-torsion subgroup of the Shafarevich-Tate group of $E$.

The way to prove this theorem is to prove the existence of some genus-1 curve violating the local-global principle, and this curve has a 0-cycle of degree 4. In order to find this curve, we start with a smooth intersection of two quadrics in $\mathbb{P}^4$, which is a del Pezzo surfaces of degree 4, violating the local-global principle in Section 2. Then we analysis intersections of this surface with hyperplanes in $\mathbb{P}^4$, and find a pencil in this surface in Section 3. The genus-1 curve needed is a intersection of this surface with a hyperplane in the chosen pencil. The existence of local points of this curve is from the fibration method, cf. Section 4. Then the Jacobian of the chosen curve will meet the needs of Theorem 4.1.

2020 Mathematics Subject Classification. Primary 14H45; Secondary 11G05, 14G12, 14G05.

Key words and phrases. rational points, local-global principle., genus one curves, Brauer-Manin obstruction.
In this section, we briefly recall some facts on Del Pezzo surfaces of degree 4, for simplicity, over a number field. We refer to [Kol], [Poo] and [BKT89] for this topic.

A del Pezzo surface is a smooth, projective, and geometrically connected surface such that its anticanonical divisor is ample. For a del Pezzo surface $X$, let $K_X$ be the its canonical divisor. The self-intersection number $K_X^2$ on $X$ defines the degree of $X$, denoted by $d_X$. Since $-K_X$ is ample, the number $K_X^2 = (-K_X)^2$ is positive. Hence the degree $d_X$ is a positive integer. The classification of del Pezzo surfaces ([Poo, Theorem 9.4.4]) implies that del Pezzo surfaces are geometrically rational. When $d_X = 4$, by [Kol] Proposition III 3.4.3, the anticanonical divisor $-K_X$ is very ample, so it embeds $X$ as a degree 4 surface in $\mathbb{P}^4$. Geometrically, del Pezzo surfaces of degree 4 are the blowing up of $\mathbb{P}^2$ at 5 points in general position.

The following lemma states the equivalence between a del Pezzo surface of degree 4 and a smooth intersection of two quadrics in $\mathbb{P}^4$.

**Lemma 2.1.** Let $X$ be a $K$-scheme. Then $X$ is a del Pezzo surface of degree 4 if and only if $X$ can be embedded in $\mathbb{P}^4$ as a smooth intersection of two quadrics.

**Proof.** By [Kol] Theorem III 3.5.4, a del Pezzo surface of degree 4 embedding in $\mathbb{P}^4$ by $-K_X$, is a intersection of two quadrics. So $X$ can be embedded in $\mathbb{P}^4$ as a smooth intersection of two quadrics. Conversely, since every irreducible component of a intersection of two different quadrics in $\mathbb{P}^4$ is of codimension 1 or 2, by [Hmr97] Theorem 7.2, $X$ is geometrically connected. Hence $X$ is a smooth, projective, and geometrically connected surface. According to the adjunction formula, a smooth intersection of two quadrics in $\mathbb{P}^4$ gives a del Pezzo surface of degree 4. \qed

The local-global principle can fail for degree-4 del Pezzo surfaces. Birch and Swinnerton-Dyer [BSD75, Theorem 3] firstly gave an example of a del Pezzo surface of degree 4 over $\mathbb{Q}$, which violates the local-global principle. It is defined by the following two equations:

$$\begin{align*}
uv &= x^2 - 5y^2 \\
(u + v)(u + 2v) &= x^2 - 5z^2
\end{align*}$$

in $\mathbb{P}^4$ with homogeneous coordinates $(x : y : z : u : v)$. Viray [Vir10, Theorem 5.0.3] generalized their counterexample to the following theorem.

**Theorem 2.1.** (Compare to [Vir10, Theorem 5.0.3]) Let $K$ be a number field. There exists a del Pezzo surface of degree 4 over $K$, violating the local-global principle.

**Remark 2.2.** By choosing constants $a, b, c$ in a given field $K$, Viray [Vir10, Theorem 5.0.3] constructed a del Pezzo surface of degree 4 over $K$, violating the local-global principle. The surface is defined by the following two equations:

$$\begin{align*}
uv &= x^2 - ay^2 \\
(u - b^2cv)(cu + (1 - b^2c^2)v) &= x^2 - az^2
\end{align*}$$

in $\mathbb{P}^4$ with homogeneous coordinates $(x : y : z : u : v)$. Jahnel and Schindler [JS17] studied a family of degree-4 del Pezzo surfaces over $K$, defined by the following two equations:

$$\begin{align*}
x_0x_1 &= x_2^2 - Dx_3^2 \\
(x_0 + Ax_1)(x_0 + Bx_1) &= x_2^2 - Dx_4^2
\end{align*}$$

in $\mathbb{P}^4$ with homogeneous coordinates $(x_0 : x_1 : x_2 : x_3 : x_4)$ and triple parameters $(A, B, D)$ in $K$. They proved that there are infinite many del Pezzo surfaces in this family violating the local-global principle. Indeed, they are Zariski dense in the coarse moduli scheme given in [HKT13, Section 5].
3. Projectively dual varieties of del Pezzo surfaces

Let $X$ be a del Pezzo surface over $K$. By Lemma 2.1, we assume that $X = Q_0 \cap Q_\infty$ is the intersection of two quadrics $Q_0$ and $Q_\infty$ in $\mathbb{P}^4$. Let $(\mathbb{P}^4)^*\text{ and the dual projective space of } \mathbb{P}^4$, cf. [Lev05]. Let $Z_X \subset X \times (\mathbb{P}^4)^*$ be the algebraic subset defined by $Z_X = \{(x, H) \in X \times (\mathbb{P}^4)^* | X \cap H \text{ is singular at } x\}$.

Let $X^* \subset (\mathbb{P}^4)^*$ be the projectively dual of $X$. Then $X^* = Pr_2(Z_X)$. We have the following commutative diagram:

\[
\begin{array}{ccc}
Z_X & \xrightarrow{pr_1} & X \times (\mathbb{P}^4)^* \\
\downarrow{pr_2} & & \downarrow{pr_2} \\
X^* & \xrightarrow{Pr_2} & (\mathbb{P}^4)^*
\end{array}
\]

Since $X$ is a smooth surface, the first projection $pr_1 : Z_X \to X$ makes $Z_X$ into a $\mathbb{P}^1$ bundle over $X$. Hence $Z_X$ is smooth, and $\dim Z_X = \dim X + 1 = 3$. Since $Z_X$ is smooth, projective and geometrically connected, and $X^* = Pr_2(Z_X)$, $X^*$ is a projective and geometrically integral variety. The following proposition states that the second projection $pr_2$ is a birational map between $Z_X$ and $X^*$.

**Proposition 3.1.** Let $X$ be a smooth intersection of two quadrics $Q_0$ and $Q_\infty$ in $\mathbb{P}^4$. Then the map $pr_2 : Z_X \to X^*$ is a birational map.

**Proof.** Since this is a geometric argument, it will be sufficient to check that $pr_2 : \overline{Z_X} \to \overline{X^*}$ is a birational map. By [CV03 Lemma 2.7], we only need to show that $\dim \overline{X^*} = 3$. By [Rei72 Proposition 2.1], we assume that $Q_0 = \sum_{i=0}^4 x_i^2$ and $Q_\infty = \sum_{i=0}^4 c_i x_i^2$ with $c_i \neq c_j$ for all $0 \leq i \neq j \leq 4$. Let $H \subset \mathbb{P}^4$ be a hyperplane, also a point in $(\mathbb{P}^4)^*$ with coordinates $(y_0 : \cdots : y_4)$. For some integer $i \in [0, 4]$, let $U_i \subset (\mathbb{P}^4)^*$ be the affine open variety defined by $y_i \neq 0$. Since $(\mathbb{P}^4)^*$ is covered by $U_i$, without loss of generality, we may assume $H \in U_4(\overline{K})$, and $H$ is defined by $x_4 + \sum_{i=0}^3 x_i y_i = 0$ and $c_4 = 0$. Hence $H \cap \overline{X^*}$ is isomorphic to the intersection of $Q_0' = \sum_{i=0}^3 x_i^2 + (\sum_{i=0}^3 x_i y_i)^2$ and $Q_\infty' = \sum_{i=0}^3 c_i x_i^2$ in $\mathbb{P}^3$ with coordinates $(x_0 : \cdots : x_3)$. Consider the pencil of quadrics: $Q_\lambda' = Q_0' + \lambda Q_\infty'$, $\lambda \in \overline{K}$. Let $P(\lambda) = \det(Q_\lambda')$. Then $P(\lambda)$ is a polynomial in $\lambda$ of degree 4, and the leading coefficient is $\prod_{i=0}^3 c_i \neq 0$. By the definition of $\overline{Z_X}$, $H \in \overline{X^*}$ if and only if $H \cap \overline{X^*}$ is singular. By [Rei72 Proposition 2.1], the intersection of $Q_0'$ and $Q_\infty'$ is nonsingular if and only if all roots of $P(\lambda)$ are distinct. Let $P'(\lambda)$ be the derivative of $P(\lambda)$. Then $H \cap \overline{X^*}$ is singular if and only if the polynomials $P(\lambda)$ and $P'(\lambda)$ have a root in common. Let $\text{Res}(P(\lambda), P'(\lambda))$ be the resultant of $P(\lambda), P'(\lambda)$. By [Lan02 Chapter IV, Corollary 8.4], the polynomials $P(\lambda)$ and $P'(\lambda)$ have a root in common if and only if $\text{Res}(P(\lambda), P'(\lambda)) = 0$. Then $U_4 \cap \overline{X^*}$ is the zero locus of $\text{Res}(P(\lambda), P'(\lambda))$ in $U_4$. Hence $\dim \overline{X^*} = 3$. \hfill \square

With this proposition, we have the following corollary.

**Corollary 3.2.** Let $X$ be a smooth intersection of two quadrics $Q_0$ and $Q_\infty$ in $\mathbb{P}^4$. There exists a Zariski open subset $U \subset X^*$ such that for any hyperplane $H \in U(\overline{K})$, the intersection $H \cap \overline{X^*}$ has exactly one singular point, and this singular point is an ordinary double point on $H \cap \overline{X^*}$.

**Proof.** By Proposition 3.1, there exists a nonempty Zariski subset $U' \subset \overline{X^*}$ such that $pr_2^{-1}(U') \to U'$ is an isomorphism. By the definition of $\overline{Z_X}$, the intersection $H \cap \overline{X^*}$ has exactly one singular point. By [DK73 Chapter XVII 3.7], this singular point is an ordinary double point on $H \cap \overline{X^*}$. \hfill \square
The following lemma will be used to analyze intersections of del Pezzo surfaces with hyperplanes in $\mathbb{P}^4$.

**Lemma 3.3.** Let $Q_0$ and $Q_\infty$ be two quadrics in $\mathbb{P}^4_\mathbb{K}$. Let $Q_\lambda = Q_0 + \lambda Q_\infty$, $\lambda \in \mathbb{K}$. Let $P(\lambda) = \det(Q_\lambda)$, and let $P'(\lambda)$ be the derivative of $P(\lambda)$. Assume that $P(\lambda)$ is of degree 4, and the greatest common factor of polynomials $P(\lambda)$ and $P'(\lambda)$ is of degree 1. Then the intersection $Q_0 \cap Q_\infty$ is integral or has two singular points.

**Proof.** Let $V$ be a vector space of dimension 4 over $\mathbb{K}$. By $\mathbb{P}^4 \cong \mathbb{P}(V)$, we view $Q_0$, $Q_\infty$ as two quadrics in $\mathbb{P}(V)$. Since there exists a 1-1 correspondence between quadratic forms and symmetric bilinear forms, we denote both them by the same letter $\phi$. Let $\phi_0$, $\phi_\infty$ be two quadratic forms, which determine quadrics $Q_0$ and $Q_\infty$ respectively. Since $P(\lambda)$ is of degree 4, the quadratic form $\phi_\infty$ is nondegenerate. Let $\phi_1 = \phi_0 + \lambda \phi_\infty$. Since the greatest common factor of polynomials $P(\lambda)$ and $P'(\lambda)$ is of degree 1, without loss of generality, we assume $(P(\lambda), P'(\lambda)) = \lambda$. Let $\lambda_1$, $\lambda_2$ be the other two different nonzero roots of $P(\lambda)$, and $P(\lambda) = c\lambda^2(\lambda - \lambda_1)(\lambda - \lambda_2)$ for some nonzero constant $c \in \mathbb{K}^\times$. For $i = 1, 2$, let $e_i \in \text{Ker} \phi_i$ be a nonzero vector in $V$. Then $e_1$ and $e_2$ are orthogonal for $\phi_{1,2}$, and hence for all $\phi_\lambda$. Since $P(\lambda)$ has the factor $\lambda^2$, we have $\text{Ker} \phi_0 \neq 0$. There are the following two cases.

1. Suppose that $\dim \text{Ker} \phi_0 = 2$. By the same argument as in the previous sentence, the vector spaces $\text{Ker} \phi_0$, $\mathbb{K} e_1$ and $\mathbb{K} e_2$ are orthogonal for all $\phi_\lambda$. Hence $V = \text{Ker} \phi_0 \oplus \mathbb{K} e_1 \oplus \mathbb{K} e_2$. By normalizing $e_1$, $e_2$ with respect to $\phi_\infty$, we carefully choose two linearly independent vectors $e_0^1, e_0^2 \in \text{Ker} \phi_0$ so that $\phi_0(x_0 e_0^1 + x_1 e_1^1 + x_2 e_1 + x_3 e_2) = \sum_{i=0}^3 x_i^2$. And $\phi_0(x_0 e_0^1 + x_1 e_0^2 + x_2 e_1 + x_3 e_2) = -\lambda_1 x_2^2 - \lambda_2 x_3^2$. Hence $Q_0 \cap Q_\infty$ has two irreducible component, and has exactly two singular points $(x_0 : x_1 : x_2 : x_3) = (\pm \sqrt{-1} : 1 : 0 : 0)$.

2. Suppose that $\dim \text{Ker} \phi_0 = 1$. Since the quadratic form $\phi_\infty$ is nondegenerate, we choose a basis $\xi_0, \cdots, \xi_3$ such that $\phi_\infty(\sum_{i=0}^3 x_i \xi_i) = \sum_{i=0}^3 x_i^2$. Then $\phi_0(\sum_{i=0}^3 x_i \xi_i) = \sum_{i=0}^3 x_i^2$. For some symmetric matrix $A = (a_{ij})_{0 \leq i,j \leq 3}$ with coefficients in $\mathbb{K}$. The matrix $A$ gives a linear map $V \to V$, denoted by $A$. Let $P_A(\lambda)$ be the characteristic polynomial of $A$. Then $P_A(\lambda) = \lambda^2(\lambda + 1)(\lambda + 2)$, $	ext{Ker} \phi_0 = \text{Ker} A$, and $-\lambda_1$ and $-\lambda_2$ are eigenvalues of $A$ belonging to the eigenvectors $e_1$ and $e_2$ respectively. Since $P_A(\lambda)$ has the factor $\lambda^2$, by Jordan normal form theorem [Lan04] Theorem 6.2], $\dim \text{Ker} A^2 = 2$. The symmetric matrix $A^2$ gives a quadratic form $\phi_{A^2}$. By the same argument applies to quadratic forms $\phi_{A^2}$ and $\phi_\infty$, the vector spaces $\text{Ker} A^2$, $\mathbb{K} e_1$ and $\mathbb{K} e_2$ are orthogonal for $\phi_{A^2}$ and $\phi_\infty$. Hence $V = \text{Ker} A^2 \oplus \mathbb{K} e_1 \oplus \mathbb{K} e_2$. Let $e_0^1 \in \text{Ker} A$ be a nonzero vector. Since $\text{Ker} A \subseteq \text{Ker} A^2$, there exists a vector $e_0^2$ such that $e_0^2 = A(e_0^1)$. Then $\phi_\infty(e_0^1) = \phi_\infty(A(e_0^1)) = \phi_\infty(e_0^2, A^2(e_0^1)) = 0$. Since $\phi_\infty$ is nondegenerate, we can choose a nonzero vector $e_0^1 \in \text{Ker} A^2$ such that $\phi_\infty(e_0^1, e_0^2) = 1$ and $\phi_\infty(e_0^1, e_0^2) = 0$. By normalizing $e_1$, $e_2$ with respect to $\phi_\infty$, we have $\phi_\infty(x_0 e_0^1 + x_1 e_0^2 + x_2 e_1 + x_3 e_2) = x_0 x_1 + x_2^2 + x_3^2$, and $\phi_\infty(x_0 e_0^1 + x_1 e_0^2 + x_2 e_1 + x_3 e_2) = c_0 x_2^2 - \lambda_1 x_3^2 - \lambda_2 x_3^2$ for some nonzero constant $c_0 \in \mathbb{K}^\times$. Hence $Q_0 \cap Q_\infty$ is integral, and has exactly one singular point $(x_0 : x_1 : x_2 : x_3) = (1 : 0 : 0 : 0)$.

In summary, the intersection $Q_0 \cap Q_\infty$ is integral or has two singular points. ∎

**Remark 3.4.** One can check in both case that the singular points have ordinary singularity.

The following proposition states that intersections of $X$ with hyperplanes in $X^*$ are almost geometrically integral.

**Proposition 3.5.** Let $X$ be a smooth intersection of two quadrics $Q_0$ and $Q_\infty$ in $\mathbb{P}^4_\mathbb{K}$. There exists a nonempty Zariski open subset $U \subset X^*$ such that for any hyperplane $H \in U(\mathbb{K})$, the intersection $H \cap X$ is integral.
Proof. Since \(X^*\) is geometrically integral, by Galois decent, it will be suffice to find a nonempty Zariski open subset \(U \subset \overline{X}\) such that for any hyperplane \(H \in U(\overline{K})\), the intersection \(H \cap \overline{X}\) is integral. By the same argument as in the proof of Proposition 3.1, we assume that \(Q_0 = \sum_{i=0}^{3} x_i^2 \) and \(Q_\infty = \sum_{i=0}^{3} c_i x_i^2\) with nonzero coefficients \(c_i \in \overline{K}\) and \(c_i \neq c_j\) for all \(0 \leq i \neq j \leq 3\). By Proposition 3.3 there exists a nonempty Zariski subset \(U' \subset \overline{X}\) such that \(pr_2^{-1}(U') \to U'\) is an isomorphism. Let \((y_0 : \cdots : y_k)\) be the coordinates of \((\mathbb{P}^4)^*\), and let \(U_4 \subset (\mathbb{P}^4)^*\) be the affine open variety defined by \(y_k \neq 0\). Take a hyperplane \(H' \in U_4(\overline{K})\), and \(\tilde{H}\) is defined by \(x_4 + \sum_{i=0}^{3} x_i y_i = 0\). Hence \(\tilde{H} \cap \overline{X}\) is isomorphic to the intersection of \(Q'_0 = \sum_{i=0}^{3} x_i^2 + (\sum x_i y_i)^2\) and \(Q'_\infty = \sum_{i=0}^{3} c_i x_i^2\) in \(\mathbb{P}^3\) with coordinates \((x_0 : \cdots : x_3)\). Let \(Q'_\lambda = Q'_0 + \lambda Q'_\infty, \lambda \in \overline{K}\). Let \(P(\lambda) = \det(Q'_\lambda)\), and let \(P'(\lambda)\) be the derivative of \(P(\lambda)\). Let \(\text{Res}(P(\lambda), P'(\lambda))\) be the resultant of \(P(\lambda), P'(\lambda)\). From the proof of Proposition 3.4, \(U_4 \cap \overline{X}\) is the zero locus of \(\text{Res}(P(\lambda), P'(\lambda))\) in \(U_4\). The condition that the greatest common factor of polynomials \(P(\lambda)\) and \(P'(\lambda)\) is of degree 1, gives an open condition for the zero locus of \(\text{Res}(P(\lambda), P'(\lambda))\) in \(U_4\). Let \(U'' \subset U_4 \cap \overline{X}\) be a Zariski open subset such that for any \(\tilde{H} \in U''(\overline{K})\), the greatest common factor of polynomials \(P(\lambda)\) and \(P'(\lambda)\) is of degree 1. Let \(U = U' \cap U''\). Take an arbitrary \(H \in U(\overline{K})\). Since \(pr_2\) is an isomorphism over \(U\), \(H \cap \overline{X}\) has exactly one singular point. Because of \(H \in U''(K)\), the greatest common factor of polynomials \(P(\lambda)\) and \(P'(\lambda)\) is of degree 1. By Lemma 3.3 the intersection \(H \cap \overline{X}\) is integral. □

Next, we will choose a pencil in a del Pezzo surface \(X\) such that for every \(H\) in this pencil, the intersection \(H \cap X\) is geometrically integral.

**Proposition 3.6.** Let \(X\) be a smooth intersection of two quadrics \(Q_0\) and \(Q_\infty\) in \(\mathbb{P}^4_K\). There exists a line \(D\) in \((\mathbb{P}^4)^*\) such that for every \(H\) in this pencil, the intersection \(H \cap X\) is geometrically integral.

Proof. Since \(\dim X^* = 3\), we take a hyperplane \(H_0 \in (\mathbb{P}^4)^* \setminus X^*\). Then \(H_0 \cap X\) is smooth. By [Har97, Chapter III, Corollary 7.9], \(H_0 \cap X\) is geometrically connected. (The existence of \(H_0\) is also from Bertini’s theorem [Har97, Chapter II, Theorem 8.18].) By Proposition 3.5, there exists a Zariski open subset \(U \subset X^*\) such that for any hyperplane \(H \in U(\overline{K})\), the intersection \(H \cap X\) is integral. Since \(\dim X^* = 3\), \(\dim(X^* \setminus U)\) is at most 2. Then the cone over \(X^* \setminus U\) with axis \(H_0\) has dimension at 3. So we can choose a hyperplane \(H_\infty\) not in this cone. Let \(D \subset (\mathbb{P}^4)^*\) be the line jointing \(H_0\) and \(H_\infty\). Hence \(D \cap X^* \subset U\). By the same argument as in the previous sentence and Proposition 3.5 for all \(H \in D(\overline{K})\), the intersection \(H \cap X\) is integral. □

**Remark 3.7.** The pencil that we chosen in the proof of this proposition, is a so-called Lefschetz pencil in \(X\).

With the help of Proposition 3.6, we can find a fibration having the following properties.

**Proposition 3.8.** Let \(X\) be a del Pezzo surface \(X\) over \(K\). Then there exist a smooth, projective and geometrically integral surface \(\tilde{X}\), a birational morphism \(\tilde{X} \to X\), and a dominant morphism \(\pi: \tilde{X} \to \mathbb{P}^4\) such that every geometric fiber is integral, and the generic fibre of \(\pi\) is a curve of genus 1.

Proof. By Lemma 2.1, we embed \(X\) in \(\mathbb{P}^4\) as a smooth intersection of two quadrics. By Proposition 3.6 let \(P_1 \subset (\mathbb{P}^4)^*\) be a line such that for every \(H \in P_1(\overline{K})\), the intersection \(H \cap X\) is geometrically integral. Let \(\tilde{X} \to X\) be the blowing up of \(X\) along the intersection \(\bigcap_{t \in P_1} H_t \cap X\), which is a reduced 0-dimensional scheme of degree 4. Let \(\pi: \tilde{X} \to \mathbb{P}^4\) be from this blowing up. Then the fiber over \(t \in \mathbb{P}^1(\overline{K})\) is \(H_t \cap \overline{X}\), which is integral. The generic fibre of \(\pi\) is a smooth intersection of two quadrics in \(\mathbb{P}^4\). Hence it is a curve of genus 1. □
Remark 3.9. In the proof of this theorem, the fiber over each point in \( \mathbb{P}^1(K) \) is a intersection of two quadrics in \( \mathbb{P}^3 \). Hence every rational fibre of \( \pi \) has a 0-cycle of degree 4.

4. Genus-1 curves violating the local-global principle

In this section, we find a genus-1 curve violating the local-global principle, and this curve has a 0-cycle of degree 4.

Proposition 4.1. For any number field \( K \), there exists a genus-1 curve \( C \) violating the local-global principle. Furthermore, this curve \( C \) has a 0-cycle of degree 4.

Proof. By Theorem 2.1, we choose a del Pezzo surface \( X \) of degree 4 over \( K \), violating the local-global principle. We choose a smooth, projective and geometrically integral surface \( \tilde{X} \) as in the proof of Proposition 3.8. Let \( \pi : \tilde{X} \to \mathbb{P}^1 \) be a dominant morphism such that every geometric fiber is integral, and the generic fibre of \( \pi \) is a curve of genus 1. Since \( X \) violates the local-global principle, \( X(\mathbb{A}_K) \neq \emptyset \) and \( X(K) = \emptyset \). By the implicit function theorem, \( \tilde{X}(\mathbb{A}_K) \neq \emptyset \). Since \( X \) and \( \tilde{X} \) are smooth, projective and geometrically integral surface, and they are birationally equivalent, by comparing the exceptional curves between \( X \) and \( \tilde{X} \), \( X(K) = \emptyset \) implies \( \tilde{X}(K) = \emptyset \). Hence \( \tilde{X} \) violates the local-global principle. Since every geometric fiber of \( \pi \) is integral, by the fibration method \( [\text{CTP}00] \) Lemma 3.1 (firstly used in \( [\text{CTSSD87}] \) and \( [\text{CTSSD87a}] \)), there is a finite set of places \( S \subset \Omega_K \) such that for every place \( v \notin S \), \( \pi(X(K_v)) = \mathbb{P}^1(K_v) \). Since the generic fibre of \( \pi \) is a curve of genus 1, by semicontinuity theorem \( [\text{Har97}] \) Theorem 12.8, there exists an open Zariski subset \( U \subset \mathbb{P}^1 \) such that every geometric fiber over points in \( U \) is a smooth curve of genus 1. For \( v \in S \), by the implicit function theorem, the set \( \pi(X(K_v)) \cap U(K_v) \), denoted by \( U_v \), is a nonempty open subset of \( U(K_v) \). For the nonempty open subset \( \prod_{v \in S} U_v \times \prod_{v \in S} \mathbb{P}^1(K_v) \subset \mathbb{P}^1(\mathbb{A}_K) \), since \( \mathbb{P}^1 \) satisfies weak approximation, we can choose a \( P \in \left( \prod_{v \in S} U_v \times \prod_{v \in S} \mathbb{P}^1(K_v) \right) \cap \mathbb{P}^1(K) \).

Let \( C = \pi^{-1}(P) \) be the fibre over \( P \). Then \( C \) is a genus-1 curve violating the local-global principle. By Remark 3.9, the curve \( C \) has a 0-cycle of degree 4. \( \square \)

Remark 4.2. The method to proof this proposition is mainly from \( [\text{Poo10}] \) Subsection 3.1.

Applying Proposition 4.1 we get our main theorem.

Theorem 4.1. For any number field \( K \), there exists an elliptic curve \( E \) defined over \( K \) such that \( \text{III}(K, E)[2] \neq 0 \).

Proof. By Proposition 4.1, we choose a genus-1 curve \( C \) violating the local-global principle, and \( C \) has a 0-cycle of degree 4. Let \( E \) be the Jacobian of \( C \). Then \( E \) is an elliptic curve. Consider the class \( [C] \in H^1(K, E) \). Since \( C \) violates the local-global principle, we have \( [C] \in \text{III}(K, E) \) and \( [C] \neq 0 \). Since \( C \) has a 0-cycle of degree 4, we have \( 4[C] = 0 \) in \( \text{III}(K, E) \). Hence \( [C] \) or \( 2[C] \) is a nonzero element of \( \text{III}(K, E)[2] \). \( \square \)

Acknowledgements. The author would like to thank X. Z. Wang, D. S. Wei, and C. Lv for many fruitful discussions. The author is grateful to anonymous referees for their valuable suggestions. The author was partially supported by NSFC Grant No. 12071448.

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