Lifshitz-like transition and enhancement of correlations in a rotating bosonic ring lattice

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We study the effects of rotation on one-dimensional ultra-cold bosons confined to a ring lattice. For commensurate systems, at a critical value of the rotation frequency, an infinitesimal interatomic interaction energy opens a gap in the excitation spectrum, fragments the ground state into a macroscopic superposition of two states with different circulation and generates a sudden change in the topology of the momentum distribution. These features are reminiscent of the topological changes in the Fermi surface that occurs in the Lifshitz transition in fermionic systems \cite{11}. The entangled nature of the ground state induces a strong enhancement of quantum correlations and decreases the threshold for the Mott insulator transition. In contrast to the commensurate case, the incommensurate lattice is rather insensitive to rotation. Our studies demonstrate the utility of noise correlations as a tool for identifying new physics in strongly correlated systems.

Ultra-cold gases loaded in an optical lattices are becoming one of the most exciting platforms for exploring complex phases of strongly correlated systems. Flexible experimental control over the parameters of the lattice and the strength of the interatomic interactions have led to the realization of fermionization of bosons, \textit{i.e.} the Tonks-Girardeau (TG) regime \cite{1,2}, and a one-dimensional Mott insulator (MI) state \cite{11}. Recent experimental advances \cite{3} are opening a new arena for the investigation of persistent currents and various novel physics that emerge when cold atoms are trapped in ring-shaped optical lattice. In these systems it is now possible to create an artificial magnetic field by rotating the atomic cloud \cite{4,5,6,7,8} and to study the exotic phenomena that emerge when interacting particles are subjected to strong gauge fields.

In this paper, we investigate the effects of rotation on the ground state properties of bosonic atoms confined in a one-dimensional ring lattice. When the lattice is rotated with an angular frequency \( \Omega \), the Coriolis force generates an effective vector potential, and hence a net circulation or non-zero vorticity. The physics of the interacting quantum many-body system differs considerably between the commensurate case (CO: where the number of atoms, \( N \), is an integral multiple of the number of lattice sites, \( L \)) and the incommensurate case, ICO. Recent studies\cite{9} have shown that at a critical frequency of rotation, \( \Omega_c \), the ground state of CO lattices becomes a macroscopic superposition (cat state) of two states of opposite circulation \cite{10}. Here we focus on the role of the interatomic interaction. For CO lattices, we show that a maximally entangled state exists at an infinitesimal value of the on-site interaction parameter, accompanied by a discontinuous rearrangement of momentum distribution, the opening of a gap in the energy spectrum and non-analytic behavior of the compressibility. These effects are analogous to the sudden change in the topology of the Fermi surface in the conventional Lifshitz transition in fermionic systems \cite{11}, in which a gap and a discontinuous change in the momentum distribution are induced as the pressure approaches a critical value. We show that the entangled nature of the ground state as the system approaches \( \Omega_c \), is accompanied by a large depletion of the condensate population, which in turn drives the system to the MI phase at lower interaction strengths and by a strong enhancement of the intrinsic quantum noise signaled in the noise correlations. Our analysis demonstrates that these higher order corrections, which can be measured experimentally in time of flight images \cite{12,13}, provide a unique experimental probe for detecting different quantum phases in rotating lattices.

We consider a system of \( N \) ultra-cold bosons with mass \( M \) confined in a uniform 1D ring lattice of \( L \) sites with lattice constant \( d \). The ring is rotated about its axis (\( z \) axis) with angular velocity \( \Omega \). In the rotating frame of the ring the many-body Hamiltonian is given by

\begin{equation}
\mathcal{H} = \int dx \hat{\Phi} \left[ -\frac{\hbar^2}{2M} \nabla^2 + V(x) + \frac{4\pi\hbar^2 a}{2M} \hat{\Phi} \cdot \Phi - \Omega \hat{\Phi} \right] \dot{\Phi}
\end{equation}

where \( a \) is the \( s \)-wave scattering length, \( V(x) \) the lattice potential and \( \hat{L}_z \) the angular momentum. Assuming that the lattice is deep enough to restrict tunneling between nearest-neighbor sites and the band gap is larger than the rotational energy, the bosonic field operator, \( \hat{\Phi} \) can be expanded in Wannier orbitals confined to the first band \( \hat{\Phi} = \sum_j \hat{\alpha}_j W_j(x) \), \( W_j(x) = \exp \left[ \frac{-iM}{\hbar} \int_0^\pi A(x') \cdot dx' \right] W_j(x) \). Here \( W_j(x) \) are the Wannier orbitals of the stationary lattice centered at the site \( j \), \( A(x) = \Omega \hat{\Phi} \times x \) an effective vector potential and \( \hat{\alpha}_j \) the bosonic annihilation operator of a particle at site \( j \).

In terms of these quantities, the many-body Hamiltonian can be written, up to onsite diagonal terms which we neglect for simplicity, as \cite{8,14}:

\begin{equation}
\mathcal{H} = - J \sum_j e^{-i\theta} \hat{\alpha}_{j+1}^\dagger \hat{\alpha}_{j+1} + e^{i\theta} \hat{\alpha}_j^\dagger \hat{\alpha}_j + \frac{U}{2} \sum_j \hat{n}_j (\hat{n}_j - 1)
\end{equation}

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Here \( \hat{n}_j = \hat{a}_j^\dagger \hat{a}_j \), \( \theta \) is the effective phase twist induced by the Gauge field, \( \theta = \int_{x_i}^{x_{i+1}} A(x') \cdot dx' = \frac{\Omega L^2}{\hbar} \), and \( J \) is the hopping energy: 
\[
J = \int dx W_i^0 \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(x) \right] W_{i+1},
\]
and \( U \) the on-site interaction energy: 
\[
U = \frac{4\pi^2\hbar^2}{M} \int dx |W_i|^4.
\]

We first carry out a Bogoliubov analysis (BA) \([15]\) to understand the effects of rotation on the bosonic ring in the weakly interacting limit. The BA approximates the field operator by a c-number plus small fluctuations \( \hat{a}_n = \bar{a}_n + \delta_n \). Replacing it in the many-body Hamiltonian and including terms up to second order in \( \delta \) yields a quadratic Hamiltonian which can be diagonalized by the canonical transformation, 
\[
\delta_n = \sum q u^q_n \hat{a}_q - n^q_n \hat{a}^\dagger_n.
\]
For the Hamiltonian described by Eq. (2) the diagonalization procedure yields: 
\[
\bar{a}_n = \frac{1}{\sqrt{L}} e^{i n \phi} u_q, \quad u_q = \frac{1}{\sqrt{L}} e^{i n \phi} v_q,
\]
with \( \phi_k = \frac{2\pi}{L} k, k \) and q integers and
\[
\mu = -2J \cos[\phi_k - \theta] + n_0 U,
\]
and
\[
v_q^2 = u^2_q - 1 = \frac{\epsilon_q + U\bar{n}}{(\sqrt{\epsilon^2 + 2U\bar{n}\epsilon} - \frac{1}{2})}.
\]

where \( \mu \) is the chemical potential of the system and \( \epsilon_q = 4J \sin^2 \left[ \frac{\phi_k}{2} \right] \cos \left[ \phi_k - \theta \right] \). \( n_0 = \bar{n} - \frac{1}{L} \sum q \neq 0 v^2_q \) is the condensate density and \( \bar{n} \) and \( \bar{n} \) the condensate depletion.

If we write \( \theta = \frac{2\pi}{L} m + \frac{2\pi}{L} \) with \( m \) an integer and \( 0 \leq \Delta \theta < 2\pi \), the ground state energy corresponds to \( k = m \) if \( 0 \leq \Delta \theta \) and \( k = m + 1 \) if \( \pi < \Delta \theta \). In order words, to lower the energy gained by rotation, the condensate acquires a phase and becomes a current carrying state with quasi-momentum (QM) \( 2\pi/k \). The quantum number \( k \) can also be viewed as a measure of the circulation or vorticity of the gas. Therefore, a change of \( k \) from \( m \) to \( m + 1 \) at the critical angular velocity \( \Omega_c = \frac{\pi}{2\rho(L+1)} \) (i.e. \( \Delta \theta = \pi \)) implies the nucleation of a vortex in the system. For \( \Omega \neq \Omega_c \) there is always a unique ground configuration corresponding to a macroscopically occupied state with QM equal to \( 2\pi/k \). On the contrary, at \( \Omega_c \), the ground state is not unique as the \( k = m \) and \( k = m + 1 \) solutions are degenerate. At this critical rotation, we expect the BA to break down as the underlying assumption of the existence of a single macroscopically occupied mode is no longer valid. Explicit comparison between the BA and numerical results, discussed later, shows that the BA provides a fair description of the system for small \( U \), and away from \( \Omega_c \).

To understand the behavior of the system at critical rotation, we write the many-body Hamiltonian in terms of QM operators, 
\[
\hat{H} = \sum_{j=0}^{N-1} E_q \hat{b}_j^\dagger \hat{b}_j + \frac{U}{2L} \sum_{q,s,l=0}^{N-1} \hat{b}_q^\dagger \hat{b}_s^\dagger \hat{b}_l \hat{b}_{q+s-l\|L}.
\]
Here \( E_q = -2J \cos[\phi_q - \theta] \) are single-particle energies and the notation \( \| \) \( \|L \) indicates modulo \( L \). In the non-interacting limit, at \( \Omega_c \), all the \( N + 1 \) states given by \( |n, N-n| = \)
\[
\frac{1}{\sqrt{n(N-n)}} \hat{b}_m^\dagger \hat{b}_{m+1}^\dagger (N-n)|0\rangle, \quad \text{with} \quad n = 0, \ldots, N \text{ are degenerate and span the ground state manifold. For small } U, \text{ we apply first order perturbation theory by diagonalizing the Hamiltonian in the degenerate subspace. In view of the absence of direct coupling between the different states, the matrix is diagonal and the energy shifts are given by:}
\[
E_n^{(1)} = E_n + \frac{U}{2L} [N + n]^2 - 3n^2 - N].
\]

The states with minimal energy are those corresponding to \( n = 0 \) and \( n = N \). To first order in \( U \), these two states remain degenerate and higher order perturbation theory is needed to break the degeneracy. To understand the effects of interaction on these degenerate states, one has to distinguish between CO and ICO cases.

In the ICO case, it has been shown \([9]\) that there is no coupling between \( |N, 0\rangle \) and \( |0, N\rangle \) and hence these states remain degenerate for all values of \( U \). Therefore, one can take one of these states as the macroscopically occupied mode and use the BA to take into account the depletion introduced by quantum fluctuations as \( U/J \) increases.

In contrast, in the CO system, there are many different paths that couple these two states and their number increases exponentially with the number of atoms and the sites. The coupling leads to the opening of a gap \( \Delta E \). If we write an effective Hamiltonian between the two states, we find that
\[
V_{12} = \langle 0, N | H_{eff} | N, 0 \rangle = \Delta E = \left( \frac{U}{2L} \right)^{\bar{n}(L-1)} A
\]
with \( A = \sum_{i,j,l=0}^{N} \frac{H_{0i} H_{1j} H_{2l}}{(E_{0i}^0 - \epsilon_i)(E_{1j}^1 - \epsilon_j)(E_{2l}^2 - \epsilon_l)} \) being \( H_{ij} \), the transition matrix elements introduced by the interaction term of the Hamiltonian and \( \epsilon_i \) the non-interacting many-body eigenergies of the intermediate states. The factor \( \bar{n}(L-1) \) corresponds to the minimum number of collision processes necessary to couple the states \( |N, 0\rangle \) and \( |0, N\rangle \) and the sum is over all the different paths that generate such couplings. The ground state of the system becomes:
\[
|\Psi\rangle = \frac{a_{1}|0, N\rangle + a_{2}|N, 0\rangle}{\sqrt{2}}
\]
with \( a_{2} = -\frac{1}{\sqrt{12}} \) (see \([9]\)). Note that any infinitesimal value of \( U \) is enough to generate the superposition. As the interaction increases, the states corresponding to other QMs also contribute to the ground state and at some point the system becomes a MI state. Therefore, at criticality (\( \Omega = \Omega_c \)), the ICO and CO respond very differently to the rotation.

Fig. 1 shows the variation in the spectral gap with interaction for different values of rotation. The critical rotation is characterized by the opening of a gap at small \( U \) with a power-law dependence on \( U \) as predicted by Eq. 6. Furthermore, the compressibility \( (d\mu/d\hbar) \) shows a singular behavior at integer fillings (see the inset in Fig.1). Away from criticality, the system undergoes the usual Mott transition at finite but reduced value of \( U/J \). The decrease in the MI threshold can be explained by means of a mean field analysis and will be discussed later in the paper.

The complexity inherent to the highly entangled nature of the ground state as \( \Omega \) approaches \( \Omega_c \), and the striking different behavior between the CO and ICO system
two modes are occupied, other modes become populated as $k/J = 0, 0.3, 0.48$ and 0.5 respectively. The grey line is proportional to $(U/J)^N$. The dots correspond to $U_c$ calculated from the maximum in the noise correlations (see Fig. 3). The inset shows the variation in the chemical potential ($\mu$) vs density ($n$) and $U/J$ at $U_c$.

can be best illustrated by the first and the second order correlations, namely the momentum distribution $\hat{n}(Q) = \sum_{m,n} e^{i2\pi Q (m-n)/L} \langle a_m^\dagger a_n \rangle$, $Q = 0, 1, \ldots, L - 1$ and the noise correlations, $\Delta(Q_1, Q_2) = \langle \hat{n}(Q_1)\hat{n}(Q_2) \rangle - \langle \hat{n}(Q_1) \rangle \langle \hat{n}(Q_2) \rangle$.

In Fig. 2 we display the fractional number of atoms in the $Q = k$ and $Q = k \pm 1$ modes as a function of $\Delta \theta$ and $U/J$, with $\pm$ determined by $\Delta \theta \leq \pi$. In the ICO case there is a single macroscopically occupied mode even at $\Omega_c$, and the nature of the ground state is weakly dependent on $\theta$. In contrast, for the CO system there is an abrupt redistribution of the population of atoms as $\theta$ approaches $\Omega_c$. At criticality, for any $U/J > 0$, the population of the $Q = k$ and $Q = k \pm 1$ modes becomes identical. While for small $U$, only these two modes are occupied, other modes become populated as $U$ increases and for $U/J \gg 1$, irrespective of the $\theta$ value, the state of the system becomes an equal superposition of all the different QM components.

The sudden rearrangement of atoms in different QM states changes the topology of the $\hat{n}(Q = k, U, \theta)$ and $\hat{n}(Q = k \pm 1, U, \theta)$ surfaces: they touch at the critical rotation frequency illustrating the entanglement. This redistribution of momentum, accompanied by an opening of a gap in the spectrum and a non-analyticity in the compressibility (see Fig. 1) is reminiscent of the Lifshitz transition in fermionic systems where a change in the pressure beyond a critical value results in an abrupt change in the Fermi surface. In other words, rotation mimics the effect of pressure and induces a Lifshitz-like transition in bosonic atoms confined to a ring shaped geometry.

The existence of maximal entanglement at the onset to the transition can be effectively demonstrated by the noise correlations which maps the intrinsic quantum noise into an intensity signal. Away from the critical frequency, the onset of the superfluid to MI transition at $U_c$ is signaled by the appearance of a maximum in the intensity of $\Delta(k, k)$ [17]. However, at $\Omega_c$ (see Fig. 3), the maximal entanglement occurs at infinitesimal $U$, coinciding with the formation of the cat state. As a consequence, the usual peak at $U_c$ is transformed to an abrupt jump in the second-order correlations at $U > 0$. This jump is correctly predicted by Eqs. (7), which yields $\hat{n}(Q) = \frac{N}{2}(\delta_{m,Q} + \delta_{m+1,Q})$ and $\Delta(m, m) = \Delta(m + 1, m + 1) = -\Delta(m, m + 1) = \frac{N^2}{2}$ and zero otherwise, in excellent agreement with the numerical simulations. Fig. 3 also shows a faster convergence to the asymptotic behavior, $\Delta(Q, k) \rightarrow 2\delta_{Q,k}$ and $n(Q) \rightarrow \bar{n}$, characteristic of a system deep in the MI phase with suppressed number fluctuations [12,17].

In Fig. 2, $u(Q)/L$ vs. $U/J$ and $\Delta \theta$ for $L = 9$: Right $N = 9$, Left $N = 8$. Black: $Q = k$ (condensate mode), red: $Q = k \pm 1$ ($\pm$ depending if $\Delta \theta \leq \pi$).

FIG. 1: (color online) Gap for $U/J$. The solid (red), dashed (blue), dotted-dashed (green) and dotted (black) curves are for $\Delta \theta/2\pi = 0, 0.3, 0.48$ and 0.5 respectively. The solid (red), dashed (blue), dotted-dashed (green) and dotted (black) curves are for $U/J$ vs density ($n$) and $U/J$. The dots correspond to $U_c$ calculated from the maximum in the noise correlations (see Fig. 3). The inset shows the variation in the chemical potential ($\mu$) vs density ($n$) and $U/J$ at $U_c$.

FIG. 2: (color online) $n(Q)/L$ vs. $U/J$ and $\Delta \theta$ for $L = 9$: Right $N = 9$, Left $N = 8$. Black: $Q = k$ (condensate mode), red: $Q = k \pm 1$ ($\pm$ depending if $\Delta \theta \leq \pi$).

FIG. 3: (color online) (a-c) shows $\Delta(Q, k) \rightarrow 2\delta_{Q,k}$ and $n(Q) \rightarrow \bar{n}$, characteristic of a system deep in the MI phase with suppressed number fluctuations [12,17].

The inset of Fig. 3 illustrates the enhancement of quantum correlations as the system approaches $\Omega_c$. For a given $U/J$, we plot the intensity of $\Delta(k, k)$ as $U/J$ is varied. Furthermore, we also show the actual $U/J$ value at which the maximum is reached. As stated earlier, this value for $\Omega \neq \Omega_c$ is related to the onset of the MI transition, and so the plot demonstrates a shift in $U_c$ as the the rotation approaches the critical frequency. The Gutzwiller ansatz (GA) [18] provides further insight into the MI transition in the rotating ring lattice. The generic GA is based on the assumption that the wave function can be approximated as a product of the wave functions at the
For the ICO system, correlations can be computed analytically in both small and the large $U$ limit. For small $U$, using Eq. (3) and (4), we can determine the $n(Q)$ and $\Delta(Q, Q')$. The resulting momentum distribution $\hat{n}(Q) = \hat{L}_{m0} \delta_{Q,Q} + v_{k,-Q}$ implies a sharp interference peak at the QM of the condensate whose width increases with interactions due to quantum depletion. On the other hand, noise correlations are directly related to the quantity $\hat{v}_Q \hat{v}_Q$, the so-called anomalous average $\langle \hat{v}_Q \hat{v}_Q \rangle$ which is a measure of the many-body scattering matrix and accounts for the modification of binary scattering properties due to the presence of other surrounding atoms. The only non-vanishing correlations are $\Delta(k, k)$, $\Delta(k, Q)$, $\Delta(Q, Q)$ and $\Delta(2k - Q, Q)$ with $Q \neq k$. $\Delta(k, Q) = -2u_{k-Q}v_{k-Q}$ probes pair excitations of the condensate and has a negative intensity. The anti-correlation can be understood as a consequence of the destructive interference between the excited pair generated when two condensate atoms collide. $\Delta(Q, Q) = \Delta(Q, 2k - Q) = u_{k-Q}v_{k-Q}$ probe collisions between atoms out-of the condensate; these processes are less frequent than condensate collisions and that is why they have half of the intensity of $\Delta(k, Q)$. $\Delta(k, k) = 2 \sum_{Q \neq 0} u_{Q}^2 v_{Q}^2$, has the larger intensity as its value is twice the sum of the intensities of the other $\Delta(Q, Q)$ peaks. Fig. 4 shows a comparison between analytic results obtained within BA approximation and the exact numerical diagonalization of Eq. (2) for a finite ICO system. We plot $\Delta(m, m)$, $\Delta(m + 1, m + 1)$ and $\Delta(m, m + 1)$ as a function of the ratio $U/J$ for different rotation velocities. The agreement in the small $U$ for a system of only 9-lattice sites is reassuring that our analysis based on a finite lattice with few atoms captures relevant features.

In the strongly interacting limit, the behavior of the system can be described by mapping the repulsive bosons into a gas of ideal fermions [19]. For simplicity we will assume $n < 1$, although the mapping to fermions can be generalized to larger filling factors [20]. In this limit, the single-particle energies are given by $E_q = -2J \cos(\pi/L(2q - 1) - \theta)$ when $N$ is even, and $E_q = -2J \cos(\pi/L(2q) - \theta)$ for odd $N$ [22]. These equations clearly show that for an ICO system at $\Omega_c$, there is always one empty energy level with the same energy as the Fermi level. Therefore, the two-fold degeneracy of the ICO ground state can be seen even in the $U \rightarrow \infty$ limit.

In summary, our study demonstrates the effectiveness of experimentally-accessible noise correlations in capturing the complexity of highly entangled states, and in heralding a novel type of phase transition. Our analysis also shows the appeal of noise correlation spectroscopy as a probe of the onset of the MI transition in finite systems, where the condensate population decreases gradually and thus the momentum distribution does not provide a distinctive signature. Our studies raise important open questions regarding the onset and nature of the MI at $\Omega_c$ as instead of the standard transition from a superfluid to a MI here we have a transition from a gapped, fragmented and incompressible state to a MI. We hope that our suggestion of Lifshitz-like transition in bosonic system will stimulate a new line of research in condensed matter community.

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\[ \text{Ref.}[1-8]\]
While preparing this manuscript we have learned of a recent report by Q. Zhou, cond-mat/0610485 which partially overlaps with our present work.

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The different energy spectrum for even and odd number of particles comes from the fact that when N is even the mapping to fermions requires to use anti-periodic boundary condition