Maximum size of reverse-free sets of permutations *

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Abstract

Two words have a reverse if they have the same pair of distinct letters on the same pair of positions, but in reversed order. A set of words no two of which have a reverse is said to be reverse-free. Let $F(n,k)$ be the maximum size of a reverse-free set of words from $[n]^k$ where no letter repeats within a word. We show the following lower and upper bounds in the case $n \geq k$: $F(n,k) \in n^{k-k/2+O(k/\log k)}$.

As a consequence of the lower bound, a set of $n$-permutations each two having a reverse has size at most $n^{n/2+O(n/\log n)}$.

1 Introduction

Let $[n]$ be the set of integers from 1 to $n$. A word $w$ of length $k$ over the alphabet $A$ is a sequence $w_1, \ldots, w_k$ of elements from $A$. The set of all words of length $k$ over $[n]$ is $[n]^k$. A word is repetition-free if it contains at most one occurrence of each symbol. The set of all repetition-free words of length $k$ over $[n]$ is $[n]_k$. Notice that when $n = k$, the set $[n]_k$ is the set $S_n$ of permutations on $n$ elements. A code $F$ of length $k$ is a subset of $[n]^k$. The size of $F$ is the number of words in $F$. Codes are usually defined to be sets of words that in some sense significantly differ from each other in order to be distinguishable when transmitted over a noisy channel. We study reverse-free codes introduced by Füredi, Kantor, Monti and Sinaimeri [9]. Two words $w$ and $x$ have a reverse if for some pair $(i, j)$ of positions, we have $w_i \neq w_j$, $w_i = x_j$ and $w_j = x_i$. If $w$ and $x$ do not have a reverse, they are reverse-free. A code is reverse-free if its words are pairwise reverse-free. Let $\overline{F}(n,k)$ be the size of the largest reverse-free code over $[n]$ of length $k$. Let $F(n,k)$ be the size of the largest reverse-free code over $[n]$ of length $k$ containing only repetition-free words. Let

$$f(k) \overset{\text{def}}{=} \lim_{n \to \infty} \frac{F(n,k)}{k! \binom{n}{k}}.$$

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The limit exists for every $k \geq 1$ [9]. We will use the following equivalent definitions of the limit:

$$f(k) = \lim_{n \to \infty} \frac{F(n, k)}{n^k} = \lim_{n \to \infty} \frac{n^k}{F(n, k)}.$$  

The first equality follows from the fact that $\lim_{n \to \infty} \frac{n^k}{k!n^{-k}} = 1$. The second equality is a consequence of the observation that for every fixed $k$, we have $F(n, k) \leq \overline{F}(n, k) \leq F(n, k) + O(n^{k-1})$ [9]. The only exact values of the limit known are $f(1) = 1$, $f(2) = 1/2$ and $f(3) = 5/24$ [9].

We tighten the bounds on the maximum size of reverse-free codes of length greater or equal to the size of the alphabet.

**Theorem 1.1.** For every $n \geq k$, we have

$$n^k k^{-k/2 - O(k/\log k)} \leq F(n, k) \leq \overline{F}(n, k) \leq n^k k^{-k/2 + O(k/\log k)}.$$  

The first inequality is proven in Section 2 as Corollary 2.5 and the last inequality is proven as Claim 3.4 in Section 3. As an immediate consequence, we obtain the following bounds for permutation codes:

$$n^{n/2 - O(n/\log n)} \leq F(n, n) \leq \overline{F}(n, n) \leq n^{n/2 + O(n/\log n)}$$

and for the limit for codes of fixed length $k$:

$$f(k) \in k^{-k/2 + O(k/\log k)}.$$  

A set of words is full of flips if each two words from the set have a reverse. Let $G(n, k)$ be the size of the largest code full of flips with elements in $[n]^k$. Let $G(n, k)$ be the size of the largest code full of flips with elements in $[n]_{(k)}$. By $G(n, n) F(n, n) \leq n!$ [9], we obtain the following corollary.

**Corollary 1.2.** The size of a set of permutations full of flips is at most

$$G(n, n) \leq n^{n/2 + O(n/\log n)}.$$  

A position of an entry of a matrix is represented by a pair $(r, c)$ of the row number $r$ and the column number $c$. A $\{0, 1\}$-matrix is a matrix whose each entry is either 0 or 1. Every matrix in this paper is a $\{0, 1\}$-matrix, even when it is not explicitly mentioned.

All logarithms in this paper are of base 2.

2 Lower Bound

A submatrix of a matrix $B$ is a matrix that can be obtained from $B$ by the removal of some columns and rows. An $m \times n$ $\{0, 1\}$-matrix $B$ contains a $k \times l$ $\{0, 1\}$-matrix $Q$ if $B$ has a $k \times l$ submatrix $T$ that can be obtained from $Q$ by changing some (possibly none) 0-entries to 1-entries. Otherwise $B$ avoids $Q$.

Füredi and Hajnal [8] studied the following problems from the extremal theory of $\{0, 1\}$-matrices. Given a matrix $Q$ (the forbidden matrix), what is the maximum number of 1-entries in an $n \times n$ matrix that avoids $Q$?
We restrict our attention on forbidding the $2 \times 2$ matrix with each entry equal to 1 and we call this matrix $S$. Maximizing the number of 1’s in a matrix avoiding $S$ is closely related to maximizing the number of edges in an $n$-vertex graph without a 4-cycle as a subgraph \cite{8}. The maximum number of edges in a 4-cycle-free graph is known precisely for infinitely many values of $n$ \cite{7}. We will use a classical construction of a bipartite 4-cycle-free graph (see for example the book of Matoušek and Nešetřil \cite{10}). We reproduce the construction here in the matrix setting since we need some of its additional properties.

The construction of a matrix avoiding $S$ builds the matrix using a finite projective plane. Let $X$ be a finite set and let $L$ be a family of subsets of $X$. The set system $(X, L)$ is a finite projective plane if

(P0) There is a 4-tuple $F$ of elements of $X$ such that $|F \cap L| \leq 2$ for every $L \in L$.

(P1) For every $L_1, L_2 \in L$, $|L_1 \cap L_2| = 1$.

(P2) For every $x, y \in X$ there exists exactly one $L \in L$ containing both $x$ and $y$.

For every finite projective plane, we can find a number $r$, called the order of the projective plane, satisfying:

(P3) For every $L \in L$, $|L| = r + 1$.

(P4) Every $x \in X$ is contained in exactly $r + 1$ sets from $L$.

(P5) We have $|X| = |L| = r^2 + r + 1$. This value is the size of the projective plane.

It is known that for every number $r$ that is a power of a prime number, we can find a finite projective plane of order $r$ \cite{10}.

**Claim 2.1.** If $n$ is of the form $r^2 + r + 1$, where $r$ is a power of a prime, then

$$F(n, n) \geq n^{n/2 - O(n/\log n)}.$$ 

**Proof.** We fix a projective plane $(X, L)$ of size $n$. We order the elements of $X$ and the sets of $L$ arbitrarily. The incidence matrix of a finite projective plane of size $n$ is the $n \times n$ matrix $A$ with 1 on position $(i, j)$ exactly if the $i$-th set of $L$ contains the $j$-th element of $X$. Let $A$ be the incidence matrix of $(X, L)$.

An $n$-permutation matrix is an $n \times n$ matrix with exactly one 1-entry in every column and every row. An $n$-permutation is a permutation on $n$ elements. The following is a bijection between the set of $n$-permutations and the set of $n$-permutation matrices. A permutation $\pi$ is matched with the matrix $P$ with 1 on position $(i, j)$ exactly if $\pi_i = j$. Let $P$ be the set of $n$-permutation matrices contained in $A$ and let $\Pi$ be the set of $n$-permutations matched to the matrices from $P$. By (P3) and (P4), $A$ has exactly $r + 1$ 1’s in every row and every column. Thus by the van der Waerden conjecture proved independently by Falikman \cite{6} and Egorychev \cite{3},

$$|\Pi| = |P| \geq \left( \frac{r + 1}{n} \right)^n n! \geq \left( \frac{r + 1}{e} \right)^n \geq \left( \frac{n^{1/2}}{e} \right)^n \geq n^{n/2 - O(n/\log n)}.$$ 

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We claim that the set $\Pi$ is pairwise reverse-free. For contradiction, we take $\pi \in \Pi$ and $\rho \in \Pi$ with a reverse on positions $i$ and $j$. That is, $\pi_i = k$, $\pi_j = l$, $\rho_i = l$, $\rho_j = k$ for some $k$ and $l$. Since $P_{\pi}$ and $P_{\rho}$ are contained in $A$, this implies that $A$ contains the matrix $S$ on rows $i,j$ and columns $k$ and $l$; a contradiction with (P1).

By the prime number theorem, the gaps between two consecutive prime numbers in proportion to the primes tend to zero. There has been a significant progress in tightening the gap between two consecutive primes. Most recent is the following result of Baker, Harman and Pintz [1].

**Theorem 2.2** (Baker, Harman, Pintz, 2001). For every large enough $n$, the interval $[n - n^{0.525}, n]$ contains a prime number.

**Lemma 2.3.** For every $n$,

$$F(n, n) \geq n^{n/2 - O(n/\log n)}.$$  

**Proof.** For an arbitrary $n$ we take the largest $n'$ smaller than $n$ and expressible as $p^2 + p + 1$ for some prime $p$. The interval $[n^{1/2} - 1 - n^{0.525/2}, n^{1/2} - 1]$ contains a suitable prime number $p$. Thus

$$p \geq n^{1/2} - 1 - n^{0.525/2} \quad \text{and} \quad n' \geq n - O(n^{1.525/2}).$$

We take the set $\Pi'$ of $(n')^{n'/2 - O(n'/\log n')}$ pairwise reverse-free $n'$-permutations from Claim 2.1. We append the sequence $(n' + 1, n' + 2, \ldots, n)$ to the end of each $\pi' \in \Pi'$. Let the resulting set of $n$-permutations be $\Pi$. The set $\Pi$ of permutations is pairwise reverse-free and has size at least $n^{n/2 - O(n/\log n)}$. 

**Lemma 2.4.** For every $n \geq k$,

$$F(n, k) \geq n^k \binom{n}{k} F(k, k).$$

**Proof.** Let $\Pi$ be a reverse-free set of $k$-permutations of size $F(k, k)$. Given a word $u = (u_1, u_2, \ldots, u_k) \in [n]_k$, we call the word $(u_1 \mod k, u_2 \mod k, \ldots, u_k \mod k)$ the compression of $u$. Let $F$ be a set of all the words in $[n]_k$ whose compression is in $\Pi$. The size of $F$ is at least $n^k \binom{n}{k} |\Pi|$. It remains to show that $F$ is reverse-free. For contradiction, assume that some pair of words $(u_1, \ldots, u_k)$ and $(v_1, \ldots, v_k)$ has a reverse on the pair $(i, j)$ of positions. That is, $u_i = v_j$ and $u_j = v_i$ and, in particular, $u_i \mod k = v_j \mod k$ and $u_j \mod k = v_i \mod k$. Because the compression of $u$ is a permutation, $u_i \mod k \neq u_j \mod k$. This is a contradiction, because the compressions of $u$ and $v$ are in the reverse-free set $\Pi$.

**Corollary 2.5.** For every $n \geq k$,

$$F(n, k) \geq n^k k^{-k/2 - O(k/\log k)}.$$

**Proof.** Since $n \geq k$, we have $|n/k| \geq n/(2k)$. Therefore, by Lemmas 2.3 and 2.4

$$F(n, k) \geq \left(\frac{n}{2k}\right)^k F(k, k) \geq \frac{n^k}{(2k)^k} k^{-k/2 - O(k/\log k)} \geq n^k k^{-k/2 - O(k/\log k)}.$$

\[\square\]
3 Upper Bound

We use a result claiming that a matrix with many 1-entries contains many occurrences of the matrix $S = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. This corresponds to counting the occurrences of $K_{2,2}$ in a bipartite graph. Erdős and Simonovits \[5\] proved that an arbitrary graph $G$ with $e(G)$ edges and $v(G)$ vertices contains at least $e(G)^4/(2v(G))^4 - e(G)^2/(2v(G))$ copies of $K_{2,2}$. Sidorenko \[12\] proves a general result that also gives a lower bound on the number of occurrences of $K_{2,2}$ (and several other bipartite graphs) in a graph with many edges. It follows from \[12\] (Condition B) that a bipartite graph with parts of size $n$ and $k$ contains $e(G)^4/(4n^2k^2) - O(nk^2 + n^2k)$ copies of $K_{2,2}$.

Neither of these results is applicable in cases when $k$ is much smaller than $n$ and the number of edges is of the order $O(nk^{1/2})$. Thus, we follow the approach used in the above mentioned papers to prove the following lemma which gives a more precise bound in such cases. This approach appeared already in 1964 in a proof of a similar result of Erdős and Moon \[4\].

**Lemma 3.1.** Let $k$ and $n$ be integers such that $n \geq k \geq 1$ and let $m$ be a real number from the closed interval $[1, k^{1/2}]$. Let $A$ be a $k \times n \{0, 1\}$-matrix with at least $mnk^{1/2}$ 1-entries. The number of occurrences of $S$ in $A$ is at least

$$\frac{n^2(m^2 - 1)^2}{4} - m^3nk^{1/2}.$$  

**Proof.** If $k = 1$, then $A$ has at most $n$ 1’s and so $m = 1$ and the claim is trivially satisfied. So we assume that $k \geq 2$. We also assume that $A$ has no empty rows. If $A$ has empty rows, we remove them and use the claim for the matrix with no empty rows. Since the removal increases $m$, and does not change $mnk^{1/2}$ and $n$, we obtain at least the required number of occurrences of $S$.

We first count the number $q$ of pairs of 1’s that are in the same row. Let $d_i$ be the number of 1’s in the $i$-th row of $A$. We have

$$q = \sum_{i, d_i \geq 2} \left( \frac{d_i}{2} \right) \geq \sum_{i=1}^{k} \left( \frac{d_i - 1}{2} \right).$$

Let $d$ be the average number of 1’s in a row, that is,

$$d \overset{\text{def}}{=} \frac{\sum_{i=1}^{k} d_i}{k} \geq \frac{mnk^{1/2}}{k} = mnk^{-1/2}.$$

By the convexity of the function $f(x) = (x - 1)^2/2$, we have

$$q \geq k \left( \frac{d - 1}{2} \right)^2 \geq k \left( \frac{mnk^{-1/2} - 1}{2} \right)^2 \geq \frac{(mn - k^{1/2})^2}{2} \geq \frac{m^2n^2}{2} - mnk^{1/2}.$$  

Let $r_{i,j}$ be the number of rows that have a 1-entry in columns $i$ and $j$. Let $\mathcal{R}$ be the set of pairs $\{i, j\}$ of column indices satisfying $1 \leq i < j \leq n$ and $r_{i,j} \geq 2$.

First, we consider the case $q \leq n^2/2$. From the estimate $q \geq (mn - k^{1/2})^2/2$, we obtain $m \leq 1 + k^{1/2}/n$. Therefore $n^2(m^2 - 1)^2 \leq (m + 1)^2 k \leq 4m^2k$ and the result holds trivially because $m^2k \leq m^3nk^{1/2}$.
Now, we assume \( q > n^2/2 \), which implies \( |\mathcal{R}| > 0 \). By double counting,

\[
q = \sum_{1 \leq i < j \leq n} r_{i,j} \leq \sum_{\{i,j\} \in \mathcal{R}} r_{i,j} + \binom{n}{2} - |\mathcal{R}|.
\]

Let

\[
r \overset{\text{def}}{=} \frac{\sum_{\{i,j\} \in \mathcal{R}} r_{i,j}}{|\mathcal{R}|} \geq \frac{q - \left( \binom{n}{2} - |\mathcal{R}| \right)}{|\mathcal{R}|} = q - \binom{n}{2} + 1.
\]

Let \( s \) be the number of occurrences of \( S \) in \( A \), that is, \( s = \sum_{\{i,j\} \in \mathcal{R}} r_{i,j} \). By the convexity of \( f(x) = (x - 1)^2/2 \) and since \( r > 1 \), we have

\[
s \geq \frac{|\mathcal{R}|(r-1)^2}{2} \geq \frac{|\mathcal{R}|}{2} \left( q - \binom{n}{2} \right)^2
\]

\[
\geq \frac{(m^2n^2/2 - mnk^{1/2} - n^2/2)^2}{4|\mathcal{R}|}
\]

\[
\geq \frac{(n^2(m^2 - 1)/2 - mnk^{1/2})^2}{n^2}
\]

\[
\geq \frac{n^2(m^2 - 1)^2}{4} - m^3nk^{1/2}.
\]

\[\square\]

We first give some definitions and outline the proof of the upper bound in Theorem 1.1 without mentioning precise values used. We use a modification of a method of Raz [11], that was used for proving upper bounds in another extremal problem on sets of permutations [11, 2].

A \( k \times n \) word matrix is a \( k \times n \) matrix with exactly one 1-entry in every row. A \( k \times n \) word \( u \) is a sequence \( u_1, u_2, \ldots, u_k \) of \( k \) letters from the alphabet \([n] \). The following is a bijection between the set of \( k \times n \) words and the set of \( k \times n \) word matrices. A word \( u \) is matched with the matrix \( U \) having 1 on position \((i, j)\) exactly if \( u_i = j \). A set \( \mathcal{U} \) of \( k \times n \) word matrices is reverse-free if the set of corresponding words is reverse-free.

Given a set \( \mathcal{U} \) of \( k \times n \) word matrices, we let the overall matrix \( A_{\mathcal{U}} \) be the \( k \times n \) matrix having 1-entries on those positions where at least one matrix of \( \mathcal{U} \) has a 1-entry. The basic idea is to design a procedure that shrinks the set \( \mathcal{U} \) in order to decrease the number of 1’s in the overall matrix. When the overall matrix has few 1’s, we use a trivial estimate on the size of what remained in \( \mathcal{U} \). By analyzing the procedure, we then deduce that the original size of \( \mathcal{U} \) was small.

The shrinking procedure uses the result of Lemma 3.1 applied on the overall matrix. Assume that the overall matrix contains \( S \) on the intersection of rows \( r_1 \) and \( r_2 \) and columns \( c_1 \) and \( c_2 \). Let an avoided pair be a pair of 1-entries of the overall matrix that do not appear together in any matrix in \( \mathcal{U} \). By the reverse-free property of \( \mathcal{U} \), we know that at least one of the two pairs \( \{(r_1, c_1), (r_2, c_2)\} \) and \( \{(r_1, c_2), (r_2, c_1)\} \) is avoided. When the overall matrix contains many occurrences of \( S \), we find a 1-entry \((r, c)\) occurring in many avoided pairs. If the 1-entry \((r, c)\) is not present in enough matrices from \( \mathcal{U} \), we remove from \( \mathcal{U} \) all the matrices containing \((r, c)\), thus removing \((r, c)\) from the overall
matrix. Otherwise, we keep only the matrices that contain \((r, c)\), thus removing all the matrices containing any of the 1-entries that appear in some avoided pair together with \((r, c)\).

Given a reverse-free set \(\mathcal{U}\) of \(k \times n\) word matrices, let \(A_{\mathcal{U}}\) be the overall matrix of \(\mathcal{U}\). Let the weight \(|A_{\mathcal{U}}|\) of the overall matrix be the number of its 1-entries. The density of the overall matrix is \(m_{\mathcal{U}} = |A_{\mathcal{U}}|/(nk^{1/2})\). The 1-entry of the overall matrix \(A_{\mathcal{U}}\) on the position \((r, c)\) is light if the number of matrices \(U \in \mathcal{U}\) having 1 on position \((r, c)\) is at most \(|\mathcal{U}|/n\). Let the emptiness \(z_{\mathcal{U}}\) of \(\mathcal{U}\) be the number of rows of \(A_{\mathcal{U}}\) with at most one 1-entry.

**Observation 3.2.** Let \(\mathcal{U}\) be a reverse-free set of \(k \times n\) word matrices such that \(A_{\mathcal{U}}\) has a light 1-entry. Then the set \(\mathcal{U}'\) of word matrices of \(\mathcal{U}\) not containing the light 1-entry satisfies

\[
|\mathcal{U}'| \geq \left(1 - \frac{1}{n}\right)|\mathcal{U}|,
\]

\[
|A_{\mathcal{U}'}| \leq |A_{\mathcal{U}}| - 1 \quad \text{and} \quad z_{\mathcal{U}'} \geq z_{\mathcal{U}}.
\]

Let \(n_0\) be a constant such that for every \(n \geq n_0\), \(k \leq n\), and \(m \geq 5\), every matrix \(A\) with \(mnk^{1/2}\) 1’s contains \(n^2m^4/5\) occurrences of \(S\). The existence of \(n_0\) follows from Lemma 3.1.

**Claim 3.3.** Let \(n \geq n_0\) and let \(k \leq n\). Let \(\mathcal{U}\) be a reverse-free set of \(k \times n\) word matrices with \(m_{\mathcal{U}} \geq 5\) and such that \(A_{\mathcal{U}}\) has no light 1-entry. Then there exists a set \(\mathcal{U}' \subset \mathcal{U}\) satisfying

\[
|\mathcal{U}'| \geq \frac{|\mathcal{U}|}{n},
\]

\[
|A_{\mathcal{U}'}| \leq |A_{\mathcal{U}}| - \frac{2nm_{\mathcal{U}}^3}{5k^{1/2}} \quad \text{and} \quad z_{\mathcal{U}'} \geq z_{\mathcal{U}} + 1.
\]

**Proof.** The overall matrix \(A_{\mathcal{U}}\) contains \(n^2m_{\mathcal{U}}^4/5\) occurrences of \(S\). So at least \(n^2m_{\mathcal{U}}^4/5\) pairs of 1-entries of \(A_{\mathcal{U}}\) are avoided. Thus there is a 1-entry of \(A_{\mathcal{U}}\) such that the number of avoided pairs containing this 1-entry is at least

\[
\frac{2n^2m_{\mathcal{U}}^4}{5|A_{\mathcal{U}}|} = \frac{2n^2m_{\mathcal{U}}^4}{5nk^{1/2}m_{\mathcal{U}}} = \frac{2nm_{\mathcal{U}}^3}{5k^{1/2}}.
\]

Let \((r, c)\) be the position of this 1-entry. Let \(\mathcal{U}'\) be the set of those matrices from \(\mathcal{U}\) that have 1 at position \((r, c)\). We consider a position \((r', c')\) such that \{\((r, c), (r', c')\)\} is an avoided pair. Every matrix \(U' \in \mathcal{U}'\) has 0 at position \((r', c')\). So also \(A_{\mathcal{U}'}\) has 0 at position \((r', c')\). Therefore \(|A_{\mathcal{U}'}| \leq |A_{\mathcal{U}}| - 2nm_{\mathcal{U}}^3/(5k^{1/2})\). Because \((r, c)\) is not a light 1-entry, \(|\mathcal{U}'| \geq |\mathcal{U}|/n\). Since \(\mathcal{U}'\) contains only word matrices, the matrix \(A_{\mathcal{U}'}\) contains only one 1-entry in row \(r\). On the other hand, the 1-entry at position \((r, c)\) is contained in at least one occurrence of \(S\) in \(A_{\mathcal{U}}\), so \(A_{\mathcal{U}'}\) contains more than one 1-entry in row \(r\). Thus \(z_{\mathcal{U}'} \geq z_{\mathcal{U}} + 1\). \(\square\)
Claim 3.4. Let $U$ be a set of $n \times k$ word matrices, where $n \geq k$. If $U$ is reverse-free, then

$$|U| \leq n^k k^{-k/2 + O(k/\log k)}$$

Proof. We first consider the case that the density $m_U$ of the overall matrix is smaller than 10. Since the number of $k \times n$ word matrices contained in $A_U$ is maximized when each of its rows has the same number of 1-entries, we obtain

$$|U| \leq (10n k^{-1/2})^k$$

and the result holds.

Otherwise, we apply the following procedure on $U$. We proceed in several steps. Let $U_i \subset U$ be the set of word matrices before the step $i$. Let $U_1 = U$. If the overall matrix at the beginning of the step $i$ has a light 1-entry, we obtain $U_{i+1}$ from $U_i$ by applying Observation 3.2; otherwise by applying Claim 3.3. Let $m_i \defeq m_{U_i}$ and $A_i \defeq A_{U_i}$. Light steps are the steps when Observation 3.2 is applied and heavy steps are the remaining ones. The steps are further grouped into phases. Phase 1 starts with step $p_1 = 1$. For every $j \geq 2$, phase $j$ starts with step $p_j$ chosen as the smallest index such that $m_{p_j} \leq m_{p_{j-1}} / 2$. The last phase is the first phase $\ell$ that decreases the density of the overall matrix below 10. So at the beginning of the last phase, we have

$$m_{p_\ell} \geq 10.$$

Because each light step decreases the number of 1’s in the overall matrix by 1, only at most $nk$ light steps are done during the whole procedure.

It remains to count the heavy steps. At the beginning of a heavy step $i$ of phase $j$, we have

$$m_i \geq m_{p_j} / 2.$$

By Claim 3.3, the heavy step decreases the number of 1-entries in the overall matrix by

$$|A_i| - |A_{i+1}| \geq \frac{2nm_i^3}{5k^{1/2}} \geq \frac{nm_{p_j}^3}{20k^{1/2}}.$$

Since the phase $j$ ends at the moment when at least $|A_{p_j}| / 2$ 1-entries are removed, the number of heavy steps of phase $j$ is at most

$$\left\lceil \frac{nk^{1/2} m_{p_j} / 2}{nm_{p_j}^3 / (20k^{1/2})} \right\rceil = \left\lceil \frac{10k}{m_{p_j}^2} \right\rceil.$$

Each phase shrinks the weight of the overall matrix by a factor of at least 2, so $m_{p_j} \geq m_{p_\ell} 2^{4-j}$ for every $j \in \{1, \ldots, \ell\}$. We also have for every such $j$

$$10 \leq m_{p_j} \leq k^{1/2}.$$

Let $t$ be the total number of heavy steps. We have

$$t \leq \sum_{j=1}^\ell \left\lceil \frac{10k}{m_{p_j}^2} \right\rceil \leq \sum_{j=1}^\ell \frac{11k}{(m_{p_{p_\ell}} 2^{4-j})^2} \leq \frac{11k}{m_{p_{p_\ell}}^2} \sum_{j=0}^\infty 2^{-2j} \leq \frac{11k}{m_{p_{p_\ell}}^2} \cdot \frac{4}{3} \leq \frac{k}{6}.$$
Let $U'$ be the set of word matrices after phase $\ell$. During the whole procedure, at most $nk$ light steps and $t \leq k/6$ heavy steps were made. We have

$$|U'| \geq |U| \left(1 - \frac{1}{n}\right)^{nk} \left(\frac{1}{n}\right)^{t} \geq |U| \frac{1}{e^{2k}n^{-t}}. \quad (3.1)$$

The overall matrix $A_{U'}$ has at most $10nk^{1/2}$ 1-entries and at least $t$ rows with a single 1-entry. The number of $k \times n$ word matrices contained in $A_{U'}$ is maximized when each of its rows with at least 2 1-entries has the same number of 1-entries. Thus,

$$|U'| \leq \left(\frac{10nk^{1/2}}{k-t}\right)^{k-t} \leq n^{k-t} \left(\frac{12}{k^{1/2}}\right)^{k} \quad \text{since } t \leq k/6. \quad (3.2)$$

By combining Equations (3.1) and (3.2), we conclude that

$$|U| \leq n^{k-t}(12k^{-1/2})^k e^{2k} n^{t} \leq n^{k-k/2 + O(k/\log k)}.$$

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