Markovian Entanglement Dynamics under Locally Scrambled Quantum Evolution

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We study the time evolution of quantum entanglement for a specific class of quantum dynamics, namely the locally scrambled quantum dynamics, where each step of the unitary evolution is drawn from a random ensemble that is invariant under local (on-site) basis transformations. In this case, the average entanglement entropy follows Markovian dynamics that the entanglement property of the future state can be predicted solely based on the entanglement properties of the current state and the unitary operator at each step. We introduce the entanglement feature formulation to concisely organize the entanglement entropies over all subsystems into a many-body wave function, which allows us to describe the entanglement dynamics using an imaginary-time Schrödinger equation, such that various tools developed in quantum many-body physics can be applied. The framework enables us to investigate a variety of random quantum dynamics beyond Haar random circuits and Brownian circuits. We perform numerical simulations for these models and demonstrate the validity and prediction power of the entanglement feature approach.

I. INTRODUCTION

Quantum entanglement dynamics[1–5] is an emerging field that ties several interesting topics together, including non-equilibrium and driven quantum systems[6–8], many-body localization and thermalization[9–14], quantum chaos and holography[15–19]. The central theme is to understand the production and propagation of quantum entanglement in quantum many-body systems. For pure states, the amount of quantum entanglement between a subsystem $A$ and its environment $\bar{A}$, which could be of experimental relevance. In this work, we further develop the Ising formulation by encoding the EF as a fictitious spin Ising models. The key idea is to label each entanglement region $A$ by a set of Ising variables $\sigma = (\sigma_1, \sigma_2, \cdots)$, such that $\sigma_1 = \downarrow$ (or $\uparrow$) corresponds to $i \in A$ (or $i \in \bar{A}$) for each site $i$. Then the EE $S^{(2)}(A) \equiv S^{(2)}[\sigma]$ can be treated as a free energy associated to the Ising configuration $\sigma$, and the entanglement feature (EF) refers to the corresponding Boltzmann weight $W[\sigma] = e^{-S^{(2)}[\sigma]} = \text{Tr} \rho_A^2$, which is simply the purity for the 2nd Rényi case. Its time evolution can be related to the Loschmidt echo on the duplicated system,$[5]$ which could be of experimental relevance. This rewriting packs the exponentially many entanglement data into a single EF state (as a many-body wave function). This conceptual simplification enables us to formulate the entanglement dynamics in a concise form of imaginary-time Hamiltonian evolution of the EF state

$$\frac{\partial_t W}{W} = -\hat{H}_{\text{EF}} W,$$  

which can be further analyzed using powerful tools that have been developed in quantum many-body physics. Our development is along the line of mapping entanglement dynamics to statistical mechanical problems, as discussed in a few recent works $[21, 23, 25, 26, 32–35]$. The given equivalence between statistical mechanics and imaginary-time quantum mechanics, it is not surprising that the entanglement dynamics could admit a quantum mechanical formulation as Eq. (1).

Treating the EF $W[\sigma]$ as an (unnormalized) probability distribution of entanglement regions $\sigma$, the proposed dynamic equation in Eq. (1) could be interpreted as a Markov equation. The assumption behind this equation is that the future EF of a many-body state can be entirely determined based on the current EF without the need to know about the past EF or about other information beyond the EF, which is actually not true. In fact,
the entanglement dynamics is generally non-Markovian, meaning that knowing the present EE’s even for all possible regions is still insufficient to determine their evolution in the future,\cite{36} so we should not expect Eq. (1) to hold in general. In this work, we propose a specific yet rich enough class of quantum dynamics, called the \textit{locally scrambled} quantum dynamics, whose entanglement dynamics can be describe by Eq. (1) (or some discrete version of it). Quantum dynamics can always be formulated as a unitary evolution \( U = \prod_t U_t \) that can be chopped up into products of simpler unitaries \( U_t \) at each time slice \( t \) following a time ordering. A quantum dynamics is said to be locally scrambled, if for every time step, the unitary \( U_t \) is drawn from a random unitary ensemble that is invariant under local (on-site) basis transformations, and \( U_t \) at different time \( t \) are sampled independently. Such dynamics can be constructed by inserting local scramblers (product of on-site Haar random unitaries) between every time step, as if the system constantly forget about the choice of local basis from one time step to another. It can be used to model those quantum many-body systems with fast and random dynamics on each site, such that the quantum information is scrambled on each site quickly and sufficiently during each step of the time evolution. One famous example in this class is the Haar random unitary circuit\cite{21, 24, 25, 33}. We will provide more examples of locally scrambled quantum dynamics in this work.

The reason that the future EE can be uniquely determined by the present EE under the locally scrambled quantum dynamics is related to the fact that the EE is a local-basis-independent quantity. As the local scramblers constantly remove the local-basis-dependent information in the quantum many-body state, only the local-basis-independent information can survive in time to govern the future evolution. Such local-basis-independent information can be captured by EE’s over all possible entanglement regions, which are summarized as the EF of the quantum many-body state. In this work, we develop the theoretical framework to derive the dynamic equation governing the evolution of the EF under locally scrambled quantum dynamics. We establish a systematic approach to construct the EF Hamiltonian \( \mathcal{H}_{\text{EF}} \) based on the entanglement properties of the physical Hamiltonian or unitary operators that describe the quantum dynamics. We also carry out numerical simulations to justify the assumptions made in the theoretical development, and demonstrate the prediction power of the EF approach.

The paper is organized as follows. In Sec. II, we first develop the theoretical framework for the EF and its application to the locally scrambled quantum dynamics. We start with the definition of EF for both quantum many-body state and quantum unitary circuits in Sec. II A. We then promote these notions to their quantum mechanical versions, introducing the EF states and EF operators in Sec. II B. With this setup, in Sec. II C, we prove an important relation between the the state and the unitary EF’s, thereby mapping the unitary evolution of the quantum state to the dissipative evolution of the EF state under the corresponding entanglement dynamics in Sec. II D. Taking the continuum limit, we obtain the Schrödinger equation for EF state and derived the most generic form of the EF Hamiltonian in Sec. II E. We analyze the spectral properties of the EF Hamiltonian and their consequences on the universal behavior of entanglement dynamics in Sec. II F. We investigate the excitation spectrum of the EF Hamiltonian and obtain the quasiparticle dispersion in Sec. II G, which allows us to predict the long-time saturation behavior of the entanglement. We will provide numerical evidences in Sec. III to demonstrate the validity of the EF approach. We first introduce two models of locally scrambled quantum dynamics in Sec. III A, namely the locally scrambled quantum circuit and the locally scrambled Hamiltonian dynamics, which are further discussed in details in Sec. III B and Sec. III C separately. We sum up in Sec. IV making connections to related topics and potential future development.

II. THEORETICAL FRAMEWORK

A. Definition of Entanglement Features

Let us consider a quantum many-body system consisting of \( L \) qudits, where each qudit (\( d \)-dimensional quantum system) has a \( d \)-dimensional physical Hilbert space, such that the total Hilbert space dimension is \( d^L \). To define the 2nd Rényi entropy, we will need to duplicate the system and evaluate the expectation value of swap operators within a subsystem \( A \) of interest. There are altogether \( 2^L \) possible choices of a subsystem \( A \), as each qudit can independently decide to be included in \( A \) or not. To label the \( 2^L \) different bipartitions of the system, we introduce a set of classical Ising variables \( \sigma = (\sigma_1, \sigma_2, \cdots, \sigma_L) \), such that the Ising variable \( \sigma_i \) determines if the \( i \)-th qudit belongs to region \( A \) or its complement \( \bar{A} \), following

\[ \sigma_i = \begin{cases} \uparrow & i \in \bar{A}, \\ \downarrow & i \in A. \end{cases} \quad (2) \]

These Ising variables do not correspond to any degrees of freedom of the underlying quantum many-body system. Instead, they represent the identity or swap operator supported on the duplicated system, which are used to define the 2nd Rényi entropy. To be more specific, we define a permutation operator \( \mathcal{X}_{\sigma_i} \) acting on the duplicated Hilbert space of the \( i \)-th qudit,

\[ \mathcal{X}_{\sigma_i} = \begin{cases} |1_i \rangle \equiv \sum_{\alpha, \beta = 1}^{d} |\alpha \beta_i \rangle_i & \text{if } \sigma_i = \uparrow, \\ \chi_i \equiv \sum_{\alpha, \beta = 1}^{d} |\alpha \beta_i \rangle_i & \text{if } \sigma_i = \downarrow, \end{cases} \quad (3) \]

which is assigned to the identity operator \( |1_i \rangle \) or the swap operator \( \chi_i \) depending on the Ising variable \( \sigma_i \). Assembling these permutation operators together, we define \( \mathcal{X}_{\sigma} = \bigotimes_{i=1}^{L} \mathcal{X}_{\sigma_i} \) for the duplicated \( L \)-qudit system, which
implies swap operations in the region $A$ specified by the Ising configuration $\sigma$.

\[
\begin{array}{c}
\text{(a)} \quad \Psi \otimes 2 \\
\text{(b)} \quad \text{Tr} \\
\end{array}
\]

FIG. 1. Diagrammatic representation of (a) the state EF $W_{\Psi} |\sigma\rangle$ and (b) the unitary EF $W_U |\sigma, \tau\rangle$. The Tr operator contracts the dangling bottom legs with the corresponding dangling top legs.

With these notation setup, we can define the entanglement feature (EF) of quantum many-body states and time-evolution unitary circuits\cite{31, 37}. The EF of a many-body pure state $|\Psi\rangle$ is defined as

\[
W_{\Psi} |\sigma\rangle \equiv e^{-S^{(2)} |\sigma\rangle} = \text{Tr} \left( \mathcal{X}_{\sigma} |\Psi\rangle \langle \Psi| \otimes 2 \right),
\]

which resembles Boltzmann weights for Ising configurations $\sigma$ labeling different entanglement regions. In terms of the tensor network representation, the state EF can be depicted as Fig. 1(a). Not only for quantum states, the EF can also be defined for unitary circuits under the state-operator correspondence.\cite{16, 38, 39} The EF of a unitary circuit $U$ is defined as

\[
W_U |\sigma, \tau\rangle = \text{Tr} \left( \mathcal{X}_{\sigma} U \otimes 2 \mathcal{X}_{\tau} U^\dagger \otimes 2 \right),
\]

which depends on two sets of Ising configurations $\sigma$ and $\tau$ that separately specifies the entanglement regions on the past (input) and the future (output) sides of the unitary circuit, as illustrated in Fig. 1(b). The state EF $W_{\Psi} |\sigma\rangle$ provides a comprehensive description of the entanglement properties of the pure state $|\Psi\rangle$, which contains the information about EE, mutual information and multipartite information among different subsystems. Similarly, the unitary EF $W_U |\sigma, \tau\rangle$ characterizes the entanglement properties of the unitary circuit $U$, including the EE and mutual information between past and future degrees of freedoms, which are also closely related to the operator-averaged out-of-time ordered correlator (OTOC)\cite{16, 40, 41} under the quantum dynamics $U$.

It worth mention that entanglement features are invariant under local basis transformations. A generic local basis transformation takes the form of $V = \bigotimes_{i=1}^L V_i$ with $V_i$ being a unitary operator acting on the $i$th qudit. It is easy to see that both the state EF and the unitary EF are independent of the choice of local basis, i.e.

\[
W_{V|\Psi} = W_{|\Psi\rangle}, \quad W_{V^* U V} = W_U.
\]

In this way, the EF forgets about the local basis dependent information in quantum states or unitary circuits, and only captures the entanglement properties that are universal to local basis choices.

B. Operator Formalism of Entanglement Features

To make our notation more concise, let us introduce a set of Ising basis $|\sigma\rangle$, then we can pack $W_{\Psi} |\sigma\rangle$ to an entanglement feature state (EF state) $|W_{\Psi}\rangle$ as

\[
|W_{\Psi}\rangle = \sum_\sigma W_{\Psi} |\sigma\rangle |\sigma\rangle,
\]

and $W_U$ to an entanglement feature operator (EF operator) $\hat{W}_U$ as

\[
\hat{W}_U = \sum_{\sigma, \tau} |\sigma\rangle \langle \sigma| W_U |\sigma, \tau\rangle \langle \tau|.
\]

The Ising basis $|\sigma\rangle$ span a $2^L$-dimensional Hilbert space of $L$ qubits, called the entanglement feature Hilbert space (EF Hilbert space). It should not be confused with the $d^L$-dimensional physical Hilbert space of the underlying quantum many-body system. Each Ising basis state $|\sigma\rangle$ in the EF Hilbert space simply corresponds to a bipartition of the $L$ physical qudits following Eq. (2).

Given the EF state $|W_{\Psi}\rangle$, the EE $S^{(2)} |\sigma\rangle$ over all regions can be retrieved from the inner product of $|W_{\Psi}\rangle$ with the corresponding Ising basis state

\[
e^{-S^{(2)} |\sigma\rangle} = W_{\Psi} |\sigma\rangle \langle \sigma| W_{\Psi}.
\]

In particular, a product state $|\Psi_{\text{prod}}\rangle = \bigotimes_i |\psi_i\rangle$ has zero EE in any region ($\forall |\sigma\rangle : S^{(2)} |\sigma\rangle = 0$), so its EF state is therefore an equal weight superposition of all Ising configurations,

\[
|W_{\text{prod}}\rangle = \sum_\sigma |\sigma\rangle \quad (\text{product state}),
\]

which corresponds to the (ideal) paramagnetic state of Ising spins. On the other hand, a Page state\cite{42} $|\Psi_{\text{Page}}\rangle$ exhibits the maximal volume-law EE, whose EF state is given by

\[
|W_{\text{Page}}\rangle = \sum_\sigma \frac{\cosh(\eta L)}{\cosh(\eta L_n)} |\sigma\rangle \quad (\text{Page state}),
\]

where $\eta = \frac{1}{2} \log d$ and we have adopted $\sigma_i = \pm 1$ in the formula to represent $\uparrow, \downarrow$ spins. This result follows from the definition. Its detailed derivation can be found in Appendix A. The state $|W_{\text{Page}}\rangle$ contains extensive ferromagnetic correlations among Ising spins. In this picture, the process of quantum state thermalization corresponds to the process of building up ferromagnetic correlations in the EF state (until saturation to the Page state).
Let us also provide some examples for the EF of unitary gates which will be useful later. The EF of a single-qudit identity operator is straightforward to calculate based on the definition in Eq. (5),

\[
\hat{W}_I = d^2(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|) + d(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|),
\]

where \( X \) denotes the Pauli-X operator acting on the qudit site (acting in the EF Hilbert space, not in the qudit Hilbert space), and \( d \) is the qudit dimension. A more non-trivial example is the EF of a two-qudit Haar random unitary gate \( \hat{U}_{ij} \) (averaged over Haar ensemble) that acts on qudits \( i \) and \( j \),

\[
\hat{W}_{\text{Haar}} = d^2(d + X_i)(d + X_j)
\]

\[
- d^2(\frac{d^2 - 1}{2d^2 + 1})(1 - Z_iZ_j)(d^2 - X_iX_j),
\]

where \( X_i \) and \( Z_i \) are Pauli-X and Z operators acting on site \( i \). The derivation can be found in Appendix B.

![Diagram](image)

**FIG. 2.** The mapping from the unitary operator in the physical Hilbert space to the corresponding EF operator in the EF Hilbert space. Locality is preserved under the mapping, enabling us to factorize the operators in the same manner on both sides.

Unitary gates are the building blocks to construct more complicated unitary circuits. One nice property of the EF operator is that it preserves the locality in space, meaning that if a unitary \( U \) operator can be factorized to smaller unitaries \( U_x \) over the space \( x \), its corresponding EF operator \( \hat{W}_U \) is also factorized in the same manner

\[
U = \bigotimes_x U_x \implies \hat{W}_U = \bigotimes_x \hat{W}_{U_x},
\]

as exemplified in Fig. 2. This property allows us to assemble the local EF operators together. For example, the EF operator \( \hat{W}_I \) of the identity operator for a \( L \) qudit system be obtained by assembling the single-qudit result in Eq. (12) together

\[
\hat{W}_I = \prod_{i=1}^{L} d(d + X_i) = (\coth \delta \text{csch } \delta)^L \prod_{i=1}^{L} e^{\delta X_i},
\]

where we have introduced the constant \( \delta \)

\[
\delta = \text{arccoth } d = \frac{1}{2} \log \frac{d + 1}{d - 1}
\]

to exponentiate the operator. The exponential form allows us to take the operator inverse easily, such that

\[
\hat{W}_I^{-1} = \prod_{i=1}^{L} \frac{1}{d} \prod_{i=1}^{L} e^{-\delta X_i}.
\]

These results will be useful in later discussions. In the following, we will show how the evolution of the EF state can be inferred from the EF operator of the unitary circuit.

**C. Relation between State and Unitary Entanglement Features**

Suppose \( U \) describes a unitary circuit that evolves an initial quantum many-body state \( |\Psi\rangle \) to the final state \( U|\Psi\rangle \). This quantum dynamics will induce a corresponding entanglement dynamics, under which the EF of the initial state \( \hat{W}_{|\Psi\rangle} \) evolves to that of the final state \( \hat{W}_{U|\Psi\rangle} \). Can we predict the final state EF \( \hat{W}_{U|\Psi\rangle} \) based on our knowledge about the initial state EF \( \hat{W}_{|\Psi\rangle} \) and the EF \( W_U \) of the unitary evolution?

In general, this problem is not tractable. Because \( U \) and \( |\Psi\rangle \) contain many “non-universal” features that are specific to the choice of local basis, such features may affect the final state entanglement, but they are not captured by the EF, as the EF is invariant under local basis transformations. Therefore, the final state EF can not be inferred from the initial state EF and the unitary EF in general. However, instead of dealing with a specific unitary circuit \( U \), we consider an ensemble of unitary circuits \( U' = V^iUV \) related to \( U \) by local basis transformations \( V \), denoted by

\[
\mathcal{E}_U = \left\{ V^iUV \middle| V = \bigotimes_{i=1}^{L} V_i, V_i \in \text{Haar} \right\},
\]

where each \( V_i \) is independently drawn from the Haar random unitary ensemble defined on the \( i \)th qudit. We will call \( \mathcal{E}_U \) the **locally scrambled** unitary ensemble associated with \( U \). According to Eq. (6), one immediately see that all unitary operators \( U' \in \mathcal{E}_U \) in the ensemble share the same entanglement feature as that of \( U \), i.e. \( \hat{W}_{U'} = \hat{W}_U \). Rather than asking about the EF of a specific final state \( U|\Psi\rangle \), if we are able to consider the ensemble average of the EF over all final states \( U'|\Psi\rangle \) with \( U' \in \mathcal{E}_U \), the final state EF \( \hat{W}_{U'|\Psi\rangle} \) will indeed be constructable from the initial state EF \( \hat{W}_{|\Psi\rangle} \) and the unitary EF \( \hat{W}_U \) on the average level. Using the operator formalism, the relation can be written in a concise form as

\[
\mathbb{E}_{U' \in \mathcal{E}_U} |W_{U'|\Psi\rangle} = \hat{W}_U \hat{W}_I^{-1} |W_{\Psi}\rangle,
\]

where \( \hat{W}_I \) is the EF operator for the identity evolution \( I \) and \( \hat{W}_I^{-1} \) is its inverse, which was given in Eq. (17) explicitly. One can derive Eq. (19) using tensor network methods.
diagrams, see Appendix C for details. To simplify the notation, we may suppress spelling out the ensemble average $\mathbb{E}_{U \in E_U}$ explicitly in later discussions, with the understanding that in this work any unitary operator appeared in the subscript of the EF operator will be implicitly averaged over local basis transformations. Eq. (19) establishes an important relation between the state and the unitary EF’s, which enables us to compute the evolution of the state EF induced by the underlying quantum dynamics, given the EF of the corresponding unitary evolution $U$. A special case of Eq. (19) has been discussed in Ref. [31, 41], where the initial state is restricted to product states.

As a side remark, we would like to provide some justifications for the use of locally scrambled unitary ensembles $E_U$. Technically speaking, working with these ensembles enables us to predict the future evolution of EE’s purely based on their current data, because the local-basis-dependent features of a quantum state are removed by local scrambling and the remaining local-basis-independent features are captured by the EF[43]. This setup allows us to make progress in understanding the entanglement dynamics with a tractable theoretical limit. Physically speaking, we can imagine systems with separating time scales between the on-site and the inter-site quantum dynamics. Suppose the on-site dynamics is much faster, then the quantum information would be sufficiently scrambled on every site, before it can spread out to other sites at a longer time scale. So the overall unitary evolution will constantly be interrupted by the insertion of local scramblers $V_i \in \text{Haar}$, making the evolution effectively local-basis-independent. In fact, many well explored random unitary ensembles in the field of entanglement dynamics are local-basis-independent (or “locally scrambled” in our language), including random unitary dynamics[21], random Hamiltonian dynamics[31, 44, 45] and random Floquet dynamics[46, 47]. This strategy has also been adopted in the discussion of operator dynamics[23–25, 28, 29] and random tensor networks[32, 35, 37, 48]. Historically, the study of these models has advanced our understanding about the universal behavior of entanglement dynamics, so we would like to carry on this line of research.

D. Markovian Entanglement Dynamics

As long as we know how to construct the EF operator $W_U$ for any unitary evolution $U$ of interest, we can apply the operator formalism in Eq. (19) to compute the entanglement dynamics. However, calculating the EF for a large and deep unitary circuit is a difficult many-body problem, hence the relation Eq. (19) is still hard to apply. But if all unitary gates in the unitary circuit are independently drawn from locally scrambled unitary ensembles, they will be decoupled in time, such that we can apply the EF operator iteratively to drive the evolution of the EF state.

To be more concrete, let us consider the case where the full unitary evolution can be broken up into discrete time steps (or layers), and each single-step unitary evolution at time $t$ is described by $U_t$, as illustrated on the left of Fig. 3. Then the quantum many-body state $|\Psi_t\rangle$ evolves from step to step following

$$|\Psi_{t+1}\rangle = U_t|\Psi_t\rangle. \quad (20)$$

Suppose $U_t$ at different time $t$ are independently drawn from random unitary ensembles (not necessary Haar random) which are invariant under local basis transformation, the full unitary evolution

$$U = \prod_t U_t = U_t U_{t-1} \cdots U_1 U_0 \quad (21)$$

will form a random unitary circuit that defines a locally scrambled quantum dynamics. If we spell out the local basis transformations $V_i$ that has been made at each time step, i.e. $U_t = V_i^\dagger U_t' V_i$, then

$$U = V_{t}^\dagger U_{t}' V_{t-1}^\dagger U_{t-1}' V_{t-1} \cdots, \quad (22)$$

we can see that the neighboring transformations $V_i V_{i-1}^\dagger$ can merge into a single layer of local scramblers. Therefore a locally scrambled quantum dynamics can also be viewed as repeatedly applying the on-site scrambling $V_i V_{i-1}^\dagger$ followed by the inter-site unitary $U_t'$. In this way, the quantum many-body state is always sufficiently scrambled on each qudit and the scrambling is uncorrelated in time, such that the information about local basis choice does not pass on from step to step. Separating each step of the unitary evolution by local scramblers is our key assumption about the quantum dynamics, which enables us to proceed.
The entanglement dynamics induced by the locally scrambled quantum dynamics is Markovian, and admits a simple transfer matrix description. To see this, we evaluate the final state EF averaging over all locally scrambled unitary ensembles at different steps

$$|W_{\Psi_{t+1}}\rangle = \mathbb{E}_{U_t, U_{t-1}} \cdots |W_{U_t, U_{t-1}} \cdots \Psi_0\rangle.$$  

(23)

Applying Eq. (19), we arrive at the recurrent equation for the ensemble averaged EF state

$$|W_{\Psi_{t+1}}\rangle = \tilde{T}_t |W_{\Psi_t}\rangle,$$  

(24)

where we have introduced the transfer matrix

$$\tilde{T}_t = \tilde{W}_{U_t} \tilde{W}_t^{-1},$$  

(25)

to evolve the EF state $|W_{\Psi_t}\rangle$ according to the EF of the single-step unitary $U_t$. As summarized in Fig. 3, Eq. (25) is the key equation that bridges the quantum dynamics and entanglement dynamics, allowing us to predict the evolution of entanglement properties of a quantum state based on the entanglement properties of the unitary operator applied at each time step. If we further assume locality of the quantum dynamics such that $U_t = \bigotimes_x U_{t,x}$ can be decomposed into products of non-overlapping local unitary gates $U_{t,x}$ (each gate only acts on a few qudits and its spatial position is labeled by $x$), the EF operator $\tilde{W}_U$ can be factorized in the same manner following Eq. (14)

$$\tilde{W}_{U_{t,x}} = \bigotimes_x \tilde{W}_{U_{t,x}},$$  

(26)

where $\tilde{W}_{U_{t,x}}$ is the EF operator for each local unitary gate, which can be easily computed (as it only involves a few qudits). Along this line, the transfer matrix $\tilde{T}_t$ can be constructed purely based on our knowledge about the EF of each unitary gate involved in the quantum dynamics.

Using Eq. (24), we can evolve the EF of any initial quantum state in time, given the locally scrambled quantum dynamics. The time evolution of the (2nd Rényi) EE can be read out from the EF by

$$S^{(2)}(\sigma)(t) = - \log \langle \sigma | W_{\Psi_t} \rangle,$$  

(27)

following Eq. (9). Strictly speaking, there is a subtle issue about exchanging the order of the logarithm with all the ensemble average in Eq. (23). We are typically more interested in the ensemble average of the EF other than the EF. So the correct average for the EF should be the geometric mean $\exp(\mathbb{E} \log W(\sigma))$, but we are replacing it by the algebraic mean $\mathbb{E} W(\sigma)$ in Eq. (23), which always overestimates the EF and hence underestimates the EE. So the EE obtained in Eq. (27) only serves as a lower bound of the ensemble averaged EE. We may treat this lower bound as an approximation, but we can not claim that it is always a good approximation, because there are known scenarios where this approximation is problematic. For example, near the entanglement transition[35, 49, 50] where critical fluctuation is important, this approximate treatment gives wrong answers about the universality class and critical exponents. There have been more rigorous treatments developed in Ref. [35, 51] using replica tricks, but we will not pursue that direction in this paper. For thermalizing dynamics and volume-law states, we believe that the lower bound estimation in Eq. (27) will provided a decent approximation, because the EF of thermalizing state contains strong ferromagnetic correlation to suppress the spin fluctuation, which allows us to replace the geometric mean by the algebraic mean as the fluctuation is small. We will rely on numerical simulations in Sec. III to justify this assumption.

To conclude, the EF formalism provides a concise description for the entanglement dynamics, when the underlying quantum dynamics is locally scrambled. However, there are also several limitations of locally scrambled quantum dynamics. First of all, the dynamics is not translation invariant in time, because the local scrambles at each step must be sampled independently. As a result, energy is not conserved under such dynamics. Secondly, global symmetry[28, 52] can not be implemented in the current scheme, because symmetry representations on each site will all be scrambled together, such that the symmetry can not be preserved. Finally, in lack of the local-basis-specific information, we can not discuss the operator dynamics for specific local operators[53] (but we can discuss operator averaged behaviors). To go beyond the local scrambling assumption, one idea could be to gradually introduce the correlation of unitary gates in time. But we will leave that for future study. We believe that our discussion of the locally scrambled quantum dynamics will set a cornerstone for future developments.

E. Entanglement Feature Hamiltonian

In the previous section, we have derived the dynamic equation Eq. (24) for EF states under discrete time dynamics. We can also consider the continuum limit of the dynamics, where we refine the time step and take $U_t$ to be close to identity (up to local basis transformation).

For example, we can consider generating $U_t$ by a local Hamiltonian for a short amount of “time” $\epsilon \ll 1$ with the local basis scrambled

$$U_t = V_t^\dagger e^{-i\epsilon H} V_t,$$  

(28)

where $V_t = \bigotimes_{i=1}^L V_{t,i}$ is a layer of local scramblers and each scrambler $V_{t,i}$ is an on-site unitary operator independently drawn from Haar random ensemble. The full unitary evolution $U = \prod_t U_t$ is given by the time-order product. The onsite scrambling does not generate entanglement (among different sites). The entanglement generation and propagation all depend on the inter-site couplings in the Hamiltonian $H$. As $\epsilon$ is small, the entanglement dynamics will be slow (smooth) enough that admits a continuum time description. We will study this model in more details later, but the goal here is to first
establish a Hamiltonian formulation for the evolution of EF state in the continuum limit.

When $U_t$ is close to an identity operator (up to local basis transformations), its EF operator $\hat{W}_t$ will approach $\hat{W}_1$, hence the transfer matrix $\hat{T}_t = \hat{W}_t \hat{W}_1^{-1}$ will also be close to the identity operator $\hat{1}$ (in the EF Hilbert space). It turns out that the difference between $\hat{T}_t$ and $\hat{1}$ is of the order $\epsilon^2$ (not $\epsilon$ as one might expect). A general argument for this property is as follows. Given $U_t$ in Eq. (28), its EF is described by

$$W_{U_t}[\sigma, \tau] = \mathcal{T}(\chi_{\sigma} \mathbb{e}^{-i\mathcal{H}} \chi_{\tau} \mathbb{e}^{i\mathcal{H}}),$$

with $\mathcal{H} = H \otimes \mathbb{1} + \mathbb{1} \otimes H$. It can be shown that $W_{U_t}[\sigma, \tau]$ must be even in $\epsilon$, because it is real by definition but $\epsilon$ comes with the imaginary unit in Eq. (29), thus the odd-power expansions of $W_{U_t}[\sigma, \tau]$ in $\epsilon$ could only be imaginary, and must therefore vanish altogether. So the operators $\hat{W}_t$ and $\hat{T}_t$ are even in $\epsilon$, hence the leading order deviation of $\hat{T}_t$ from $\hat{1}$ is of the order $\epsilon^2$.

Given this, we expand $\hat{T}_t$ around the identity operator $\hat{1}$ and define the entanglement feature Hamiltonian (EF Hamiltonian)

$$\hat{H}_{\text{EF}} = \frac{1}{\epsilon^2}(\mathbb{1} - \hat{T}_t) = \frac{1}{\epsilon^2}(\mathbb{1} - \hat{W}_t \hat{W}_1^{-1}),$$

such that the recurrent equation Eq. (24) transforms to an imaginary-time Schrödinger equation in the continuum limit of $\epsilon \ll 1$,

$$\partial_t |W_{\Psi_i}\rangle = -\hat{H}_{\text{EF}}|W_{\Psi_i}\rangle.$$  

The differentiation $\partial_t |W_{\Psi_i}\rangle$ should be considered as the limit of $(|W_{\Psi_{i+2}}\rangle - |W_{\Psi_i}\rangle)/\epsilon^2$, where $\epsilon^2$ serves as the infinitesimal time step. In general, $\hat{H}_{\text{EF}}$ can be time-dependent, but let us omit the explicit time dependence for simplicity. The locality of the EF operator $\hat{W}_t$, as discussed in Eq. (26) translates to the locality of the EF Hamiltonian $\hat{H}_{\text{EF}}$, which allow us to write $\hat{H}_{\text{EF}} = \sum_x \hat{H}_x$ as sum of local terms. In principle, the specific form of these local terms $\hat{H}_x$ can be derived from the terms in the quantum many-body Hamiltonian $\hat{H}$ that drives the quantum dynamics, which we will demonstrate later in Sec. III C. However, even if we have no specific knowledge about $H$, we can already learn a lot about $\hat{H}_{\text{EF}}$ based on the general properties of entanglement dynamics. In the following, we will show how the physical constraint of entanglement dynamics can pin down the general form of the EF Hamiltonian.

Let us consider the two-local EF Hamiltonian, meaning that the local terms $\hat{H}_x$ span over two sites at most. We find that the most general two-local EF Hamiltonian should take the following form

$$\hat{H}_{\text{EF}} = \sum_{i,j} g_{ij} \frac{1}{2} e^{-\beta_{ij} X_i X_j - \delta(X_i + X_j)},$$

where $g_{ij} \geq 0$ and $\beta_{ij} \in \mathbb{R}$ are model parameters and the constant $\delta$ is fixed by the qudit dimension $d$ via $\coth \delta = d$. Here $X_i, Z_i$ are Pauli operators acting on the $i$th Ising spin (that labels the entanglement region). Each local term in the Hamiltonian consists of a term $e^{-\beta_{ij} X_i X_j - \delta(X_i + X_j)}$ that fluctuates Ising spins, followed by a ferromagnetic projection operator $(1 - Z_i Z_j)/2$. Although we call $\hat{H}_{\text{EF}}$ a Hamiltonian, it is not a Hermitian operator as expected in conventional quantum mechanics, because fluctuation term and the projection term do not commute. As a result, the left- and the right-eigenstates of $\hat{H}_{\text{EF}}$ could be different. The coupling strength $g_{ij}$ describes the entangling power of the quantum dynamics, i.e. the velocity that the entanglement builds up between sites $i$ and $j$ if initialized from a product state.

The postulated form of $\hat{H}_{\text{EF}}$ in Eq. (32) is constrained by the following physical requirements (or assumptions).

- Pure state remains pure under quantum dynamics (i.e. a $\mathbb{Z}_2$ Ising symmetry),

$$[H_{\text{EF}}, \prod_i X_i] = 0.$$  

An important entanglement property of pure states is that the EE of a region $A$ should be the same as that of its complement $A$, therefore the pure state EF must be invariant under Ising symmetry, i.e. $W_{[\sigma]} = W_{[-\sigma]}$, which can be equivalently written as $|W_{\Psi}\rangle = \prod_i X_i |W_{\Psi}\rangle$. Since any quantum dynamics (described by a unitary evolution) will preserve the purity of the quantum state, the entanglement dynamics should also respect this Ising symmetry, such that the EF Hamiltonian $\hat{H}_{\text{EF}}$ must commute with the symmetry operator $\prod_i X_i$ as asserted in Eq. (33).

- EE must vanish for empty entanglement regions,

$$\langle \cdots | \hat{H}_{\text{EF}} | \hat{0} \rangle = 0.$$  

By empty entanglement region, we mean $A = \emptyset$ is an empty set, which correspond to the Ising configuration $\sigma = \cdots \uparrow \cdots \uparrow \uparrow$ (i.e. $\forall i : \sigma_i = +1$). Henceafter we use the symbol $\uparrow$ to denote the all-up configuration to simplify the notation. When the entanglement region is empty, the EE must be zero, i.e. $S_{\{\uparrow\}} = 0$. This requires $\langle \uparrow | W_{\Psi}\rangle = W_{\Psi}|\uparrow\rangle = e^{-S_{\{\uparrow\}}}/8 = 1$ to remain at constant under any entanglement dynamics. Now suppose $|W_{\Psi_i}\rangle$ is time dependent under the entanglement dynamics. Taking the time derivative on both sides of $\langle \uparrow | W_{\Psi_i}\rangle = 1$ and apply the dynamic equation Eq. (31), we can see that $\langle \uparrow | \partial_t |W_{\Psi_i}\rangle = -\langle \uparrow | \hat{H}_{\text{EF}}|W_{\Psi_i}\rangle = 0$ must hold for any EF state $|W_{\Psi_i}\rangle$, therefore we must require $\langle \uparrow | \hat{H}_{\text{EF}} | \hat{0} \rangle = 0$ as claimed in Eq. (34).

- Statistical time-reversal symmetry of random unitary ensembles,

$$\hat{W}_1 \hat{H}_{\text{EF}} = \hat{H}_{\text{EF}} \hat{W}_1.$$  


We assume that the random unitary gates in the circuit are statistically invariant under time-reversal, meaning that \( U_t \) and \( U_t^\dagger \) will appear with equal probability in the unitary ensemble. Then according to the definition of unitary EF in Eq. (5), the time-reversal symmetry implies to \( W_U[\sigma, \tau] = W_U[\sigma, \tau] \), i.e. \( W_U^\dagger = W_U \). As a special case, we also have \( W_U^\dagger = W_U \) by definition. Transposing both sides of \( W_U W_U^{-1} = \mathbb{I} - c^2 H_{\text{EF}} \), we obtain \( W_U^{-1} W_U = \mathbb{I} - c^2 H_{\text{EF}}^\dagger \). Therefore \( H_{\text{EF}}^\dagger \) and \( H_{\text{EF}} \) must be related by \( W_U H_{\text{EF}}^\dagger W_U^\dagger = H_{\text{EF}}^\dagger W_U^\dagger \) as stated in Eq. (35). One known scenario that the statistical time-reversal symmetry is broken is that the unitary operators cyclically permute the qudit along one direction, which describes a quantum dynamics that has dynamic anomaly.[54, 55] We conjecture that the statistical time-reversal symmetry effectively restricts the quantum dynamics to be anomaly-free.

With these conditions, we can start from a generic two-local Hamiltonian \( H_{\text{EF}} = \sum_{i,j} \hat{H}_{ij} \) and derive the generic form of Eq. (32). First of all, the Ising symmetry in Eq. (33) restricts \( \hat{H}_{ij} \) to be a linear combination of the following operators \( \hat{H}_{ij} = x_1 + x_2 X_j + x_3 X_i + x_4 X_i X_j + x_5 Y_j + x_6 Y_i Z_j + x_7 Z_j Y_j + x_8 Z_i Z_j \), which contains all the two-locators that commute with \( X_i, X_j \). Then the left-null-state requirement in Eq. (34) further requires \( x_1 = x_2 = i x_7, x_3 = i x_6, x_4 = x_5 \), which reduce \( \hat{H}_{ij} \) to \((1 - Z_i Z_j)(x_1 + x_2 X_j + x_3 X_i + x_4 X_i X_j) \). Finally, the statistical time-reversal symmetry in Eq. (35) requires

\[
x_2 = x_3 = -\frac{d(x_1 + x_4)}{d^2 + 1}, \tag{36}
\]

leaving only two independent parameters \( x_1 \) and \( x_4 \). This relation can be resolved by introducing another two parameters \( g \) and \( \beta \) to parametrize \( x_1 + x_2 X_j + x_3 X_i + x_4 X_i X_j = \frac{g}{d} e^{-\beta X_i X_j \delta(X_i + X_j)} \) with \( \coth \delta = d \) fixed, such that

\[
x_1 = \frac{g(d^2 \cosh \beta - \sinh \beta)}{2(d^2 - 1)}, \]
\[
x_2 = x_3 = -\frac{gde^{-\beta}}{2(d^2 - 1)}, \tag{37}
\]
\[
x_4 = \frac{g(cosh \beta - d^2 \sinh \beta)}{2(d^2 - 1)},
\]

automatically satisfies Eq. (36). The resulting local term reads \( \hat{H}_{ij} = g \frac{(1 - Z_i Z_j)}{2} e^{-\beta X_i X_j \delta(X_i + X_j)} \), which matches the form of Eq. (32).

**F. Universal Behaviors of Entanglement Dynamics**

The generic form of the EF Hamiltonian \( \hat{H}_{\text{EF}} \) in Eq. (32) is already useful to illustrate several universal behaviors about the entanglement dynamics. Suppose the EF Hamiltonian admits the following spectral decomposition

\[
\hat{H}_{\text{EF}} = \sum_a \langle R_a | \sigma \rangle \lambda_a \langle L_a |,
\]

where \( |R_a\rangle \) and \( \langle L_a| \) are respectively the right- and left-eigenstate of the eigenvalue \( \lambda_a \). The right-eigenstate is related to the corresponding left-eigenstate by \( |R_a\rangle \propto \langle L_a| W_{\text{Page}} \rangle \), which follows from Eq. (35). Then the Schrödinger equation for EF state Eq. (31) can be formally solved as

\[
|W_{\Psi_t}\rangle = \sum_a e^{-\lambda_a t} |R_a\rangle \langle L_a| W_{\Psi_0}\rangle. \tag{39}
\]

The dynamics of the EE can be inferred from Eq. (27) as

\[
S^{(2)}[\sigma](t) = -\log \langle \sigma \| W_{\Psi_t} \rangle = -\log \sum_a e^{-\lambda_a t} \langle \sigma \| R_a \rangle \langle L_a| W_{\Psi_0}\rangle. \tag{40}
\]

Independent of the choice of model parameters \( g_{ij}, \beta_{ij} \), the EF Hamiltonian \( \hat{H}_{\text{EF}} \) has the following spectral properties:

- \( \hat{H}_{\text{EF}} \) is positive semi-definite (all its eigenvalues \( \lambda_a \geq 0 \) are real and non-negative),
- \( \hat{H}_{\text{EF}} \) always has (at least) a zero eigenvalue \( \lambda_0 = 0 \) in the \( \mathbb{Z}_2 \) (Ising parity) even sector, whose left- and right-eigenstates are

\[
\langle L_0| = \frac{\langle \uparrow \| + \langle \downarrow \| }{2}, \tag{41}
\]
\[
| R_0 \rangle = | W_{\text{Page}} \rangle.
\]

The left zero mode \( \langle L_0| \) is the Ising symmetric superposition of the all-up and the all-down states. The right zero mode \( |R_0\rangle \) is the Page EF state given in Eq. (11).

The proof can be found in Appendix D. With these results, we can obtain several universal behaviors of entanglement dynamics with local scrambling in the short-time and long-time limit.

In the short-time limit \( (t \to 0) \), expanding the solution of EF state in Eq. (40) to first order in \( t \), we can show that the EE grows linearly in time,

\[
S^{(2)}[\sigma](t) = S^{(2)}[\sigma](0) + v_E^{(2)}[\sigma] \times t + \mathcal{O}(t^2), \tag{42}
\]

where the linear-time coefficient \( v_E^{(2)}[\sigma] \) is defined to be the entanglement velocity

\[
v_E^{(2)}[\sigma] = \partial_t S^{(2)}[\sigma](0) = \frac{\langle \sigma \| \hat{H}_{\text{EF}}| W_{\Psi_0}\rangle}{\langle \sigma \| W_{\Psi_0}\rangle}. \tag{43}
\]

The entanglement velocity \( v_E^{(2)}[\sigma] \) characterizes how fast the EE grows in a given entanglement region specified by
It is proportional to the matrix element of the EF Hamiltonian $H_{\text{EF}}$, as can be seen in Eq. (43), because $H_{\text{EF}}$ is the time-evolution generator that drives the entanglement dynamics. In particular, if the initial state is a generic product state, i.e. $|W_{\psi_0}\rangle = |W_{\text{prod}}\rangle = \sum_{\sigma} |\sigma\rangle$ as given in Eq. (10), the entanglement velocity $v^{(2)}_E[\sigma]$ admits an explicit formula

\[ v^{(2)}_E[\sigma] = \sum_{\langle ij \rangle} \tilde{g}_{ij} \frac{1 - \sigma_i \sigma_j}{2}, \tag{44} \]

where $\tilde{g}_{ij} = g_{ij} e^{-\beta_{ij} - 25} \geq 0$ is the effective coupling.

Eq. (44) describes how the entanglement velocity $v^{(2)}_E[\sigma]$ depends on the choice of the entanglement region $\sigma$. It is obvious that the entanglement velocity $v^{(2)}_E[\sigma] \geq 0$ is non-negative for all choices of entanglement regions, because the EE can only grow from an unentangled product state. If $\tilde{g}_{ij} = \tilde{g}$ is uniform throughout the system, $v^{(2)}_E[\sigma]$ will simply be proportional to the number of domain walls in the Ising configuration $\sigma$, which is also the area $|A|$ of the entanglement region $A$. Therefore the entanglement velocity follows the area-law scaling,

\[ v^{(2)}_E = \tilde{g} |\partial A|, \tag{45} \]

which can be expected from the locality of the entanglement dynamics in our setup.

In the long-time limit ($t \to \infty$), the EF state is dominated by the zero mode and all the other modes decay exponentially with time. The positive semi-definite property of the EF Hamiltonian, i.e. $\lambda_0 \geq 0$, ensures that all modes (except the zero mode) will decay exponentially in time. As $t \to \infty$, Eq. (39) reduces to

\[ |W_{\psi_\infty}\rangle = |R_0\rangle \langle L_0 |W_{\psi_0}\rangle, \tag{46} \]

with the left and right zero modes given by Eq. (41). Given that the EE vanishes in trivial regions, $\langle \uparrow | W_{\psi} \rangle = \langle \downarrow | W_{\psi} \rangle = 1$, so $\langle L_0 | W_{\psi} \rangle = 1$ for any EF state $|W_{\psi}\rangle$. Then Eq. (46) results in

\[ |W_{\psi_\infty}\rangle = |R_0\rangle = |W_{\text{Page}}\rangle, \tag{47} \]

meaning that the EF always converge to that of the Page state in the long-time limit regardless what the initial state is. All states are doom to thermalize under the quantum dynamics with local scrambling. The Page state will be their final destiny, whose EE reads

\[ S^{(2)}[\sigma] = -\log \frac{\cosh(\eta \sum_{i=1}^{L} \sigma_i)}{\cosh(\eta L)}, \tag{48} \]

which follows from Eq. (11). For $|A| \ll L$, the EE exhibits the volume-law scaling

\[ S^{(2)}(A) = 2\eta |A|, \tag{49} \]

with the volume law coefficient given by $2\eta = \log d$.

\section{Excitation Spectrum of the Entanglement Feature Hamiltonian}

Having discussed the ground state property of the EF Hamiltonian $H_{\text{EF}}$, let us turn to the low-lying excited states of $H_{\text{EF}}$. According to Eq. (39), every eigenmode with finite eigenenergy $\lambda_0$ will decay exponentially in time as $e^{-\lambda_0 t}$. Eventually, only the ground state with zero eigenenergy ($\lambda_0 = 0$) would survive, and the system thermalizes to Page states. Hence the low-energy excitation spectrum determines how the EE approaches to its thermal limit in the late-time regime. Here we will focus on the spectrum of two kinds of excitations, namely the two-domain-wall excitation and the single spin-flip excitation, which dominate the low-energy excitations. We obtain the analytical expression of their dispersion relations, from which we can estimate the excitation gap and determine the relaxation time. In Sec. III.C, we further compare the analytically estimated relaxation time with the numerical fitted one, and find good consistency.

For simplicity, we assume the parameters $g_{ij}, \beta_{ij}$ in the EF Hamiltonian $H_{\text{EF}}$ are spatially homogeneous (i.e. $g_{ij} = g, \beta_{ij} = \beta$). For the parameter $\beta$, any unitary evolution generated from Hamiltonian $e^{-iHt}$ would have nonzero $\beta$ only at the order of $O(\epsilon^2)$ in small $\epsilon$ limit (see Appendix G for details). Hence, we will take $\beta = 0$ in the following. More general results for $\beta \neq 0$ can be found in Appendix F. To first gain some intuitions about the excitation spectrum, we start with the exact diagonalization (ED) of EF Hamiltonian. The result is shown in Fig. 4. Apart from the eigenenergy $\lambda_0$, every state $|R_{\alpha}\rangle$ is also label by its crystal momentum $k_0$, which is defined through its translation eigenvalue as $T|R_{\alpha}\rangle = e^{ik_0} |R_{\alpha}\rangle$, where the translation operator $T$ is defined by its action on the Ising basis $T|\sigma_L \sigma_{L-1} \cdots \sigma_1\rangle = |\sigma_L \sigma_{L-1} \cdots \sigma_{-1}\rangle$. One can see that above the ground state at $\lambda_0 = 0$ and $k_0 = 0$, there is a continuum of excited states.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{We perform exact diagonalization for the EF Hamiltonian $H_{\text{EF}}$ with $\beta = 0, L = 10$. Each small circle represents an eigenstate label by its eigenenergy $\lambda_0$ and its crystal momentum $k$. The blue curve is the analytical result of two-domain-wall ansatz Eq. (52). The red curve is the analytical result of single spin-flip ansatz Eq. (54).}
\end{figure}
To better understand these excited states, we look into their wave function. We realize that the excitation can be classified based on the number of domain walls in the left-eigenstate. For instance, $|\uparrow \cdots \uparrow \downarrow \cdots \downarrow \uparrow |$ is an example of two-domain-wall states. As mentioned in Eq. (41), the left ground state $|L_0\rangle = ((\langle \uparrow | + \langle \downarrow |)/2 contains no domain wall and hence no excitation. Other excited left-eigenstate will be a superposition of states of different domain-wall number. Note that the corresponding right eigenstate can be obtained from $|R_k\rangle = (\langle L_0| W_k)^T$. Fig. 5 shows the weights of different domain-wall states in the lowest-energy excited state of various momenta. The ED result indicates that the lowest-energy excited state mainly consists of two-domain-wall states, so we will focus on them in the following.

Based on the numerical observation, we approximate low energy excitation by the two-domain-wall (2DW) ansatz state as follows,

$$|k\rangle \propto \sum_{i_1, i_2} e^{ik_{i_1}+ik_{i_2}} \phi_{i_2-i_1}(i_1, i_2),$$

(50)

where $|i_1, i_2\rangle = |\uparrow \rangle \prod_{i=1}^{l-1} X_i$ is a two-domain-wall state with domain walls located at $i_1$ and $i_2$. $l$ labels the center of mass momentum of the pair of domain walls. $\phi_{i_2-i_1}$ is an optimizable wave function that describes the relative motion between the domain walls. We can then evaluate the energy expectation value $\lambda(k)$ on the ansatz state $|k\rangle$,

$$\lambda_{2DW}(k) = \frac{\langle k| H_{EF} W_1 |k\rangle}{\langle k| W_1 |k\rangle},$$

(51)

where $W_1|k\rangle$ is understood as the corresponding right-state of the ansatz left-state $|k\rangle$. Two assumptions are made to derive the analytical expression of the dispersion relation. The first assumption is that these domain walls have no interaction with each other and thus $\phi_{i_2-i_1}$ can be approximated by plane waves. The second assumption is the thermodynamic limit $L \to \infty$, which would simplify the calculation but suppress the contribution from short two-domain-wall states (see Appendix E for details). Based on these assumptions, the dispersion relation for $\beta = 0$ can be derived as,

$$\lambda_{2DW}(k) = 2g \left(1 + \frac{1}{d^2}\right) - 4g \frac{k}{2} \cos \frac{k}{2} + O(d^{-3}).$$

(52)

The band minimum is at $k = 0$, which defines the excitation gap

$$\Delta = \min_k \lambda(k) = 2g \left(1 - \frac{1}{d}\right)^2 + O(d^{-3}).$$

(53)

It turns out that the gap remains open (i.e. $\Delta > 0$) for any finite $g > 0$.

The comparison between ED result (black circles) and our analytical expression (blue curve) is shown in Fig. 4. The lower-edge of the excitation spectrum is pretty well captured by the two-domain-wall ansatz. The comparison also reveals a finite-size-effect in the spectrum. In Fig. 6, we show how the gap at $k = 0$ (from ED) approaches the analytic result of Eq. (53) with increasing system size $L$. We also observe a systematic deviation of our analytical result from the excitation edge near $k = \pi$. The reason is that the eigenstate around $k = \pi$ is dominated by single-site excitations, where the domain-walls are next to each other such that their interaction can not be ignored. To capture the interaction effect, we switch to another ansatz state, which describes the motion of a tightly-bound domain-wall pair, or equivalently a single spin-flip (SSF) excitation (see Appendix F for details). The dispersion of the SSF excitation reads

$$\lambda_{SSF}(k) = 2g,$$

(54)

which turns out to be independent of the qudit dimension $d$ and the momentum $k$. This dispersion relation basically passes a series of points in Fig. 4 and only becomes the lowest excited state around $k = \pi$.
III. APPLICATIONS AND NUMERICS

A. Models of Locally Scrambled Quantum Dynamics

In the following, we will apply the entanglement feature formalism to several scenarios of locally scrambled quantum dynamics. We will consider two types of models: random circuit models with discrete time as in Fig. 7(a), and Hamiltonian generated evolutions with local scramblers in the limit of continuous time as in Fig. 7(b). For the discrete time models, namely locally scrambled random circuits, we will adopt the transfer matrix method to study the entanglement dynamics. For the continuous time models, namely locally scrambled Hamiltonian dynamics, we will apply the EF Hamiltonian approach.

The random circuit we consider will be of the “brick wall” structure as shown in Fig. 7(a). The entire unitary circuit \( U = \prod_{t} U_{t} \) is constructed by stacking layers of unitary gates. Each layer \( U_{t} \) is described by

\[
U_{t} = \left\{ \begin{array}{ll} \bigotimes_{x} U_{t;2x-1,2x} & t \in \text{odd}, \\
\bigotimes_{x} U_{t;2x,2x+1} & t \in \text{even}, \end{array} \right.
\]

(55)

where \( U_{t;ij} \) denotes the two-qudit unitary gate acting on sites \( i \) and \( j \) at time \( t \). Each gate \( U_{t;ij} \) is independently sampled from a locally scrambled unitary ensemble, so the quantum circuit \( U \) will be dubbed as a locally scrambled random circuit. In fact, any gate can be made locally scrambled by symmetrizing over local basis transformations as constructed in Eq. (18). The construction here is more general than the Haar random unitary circuit[21], as the unitary gate here does not need to be Haar random. As the quantum state evolves by \( |\Psi_{t+1}\rangle = U_{t}|\Psi_{t}\rangle \), the corresponding EF state evolves by \( |W_{\Psi_{t+1}}\rangle = \hat{T}_{t}|W_{\Psi_{t}}\rangle \). The transfer matrix \( \hat{T}_{t} \) follows the same structure as \( U_{t} \),

\[
\hat{T}_{t} = \left\{ \begin{array}{ll} \bigotimes_{x} \hat{T}_{2x-1,2x} & t \in \text{odd}, \\
\bigotimes_{x} \hat{T}_{2x,2x+1} & t \in \text{even}. \end{array} \right.
\]

(56)

According to Eq. (25), \( \hat{T}_{ij} \) is fully determined by the EF of \( U_{t;ij} \) via

\[
\hat{T}_{ij} = \hat{W}_{U_{t;ij}} W_{i,j}^{-1}. \tag{57}
\]

Here we have assumed that \( U_{t;ij} \) are drawn from identical unitary ensembles, such that \( \hat{T}_{ij} \) is time-independent (despite of the time-dependence in \( U_{t;ij} \)). In the following, we will provide examples of the locally scrambled two-qudit unitary ensemble. We will use the transfer matrix approach to calculate the entanglement dynamics. The result will be compared with exact numerics by explicitly constructing the random circuit and average the final state EE over random realizations.

Another type of locally scrambled quantum dynamics that we will consider is generated by a local Hamiltonian \( H = \sum_{(ij)} H_{ij} \), which is a sum of local terms \( H_{ij} \) defined on nearest neighboring bonds \( \langle ij \rangle \) along a 1D chain. Each step of the unitary evolution \( U_{t} \) is independently drawn from the locally scrambled unitary ensemble \( \mathcal{E}_{\text{EE}-\text{U}} \) generated by the Hamiltonian \( H \),

\[
\mathcal{E}_{\text{EE}-\text{U}} = \{ V \hat{e}^{-i\epsilon H} V | V = \bigotimes_{i=1}^{L} V_{i} \} \in \text{Haar}, \tag{58}
\]

which may be simply denoted by \( U_{t} = V_{t}^{\dagger} \hat{e}^{-i\epsilon H} V_{t} \), as in Eq. (28). Combining the adjacent local scramblers following Eq. (22), the unitary evolution can be considered as repeatedly applying a short-time unitary evolution \( e^{-i\epsilon H} \) followed by a layer of local scramblers, as illustrated in Fig. 7(b). Such dynamics will be called the locally scrambled Hamiltonian dynamics. It is similar to the Brownian random circuit model[56] in that each step of the evolution is driven by a different random Hamiltonian, but our construction is more general in that the random Hamiltonian ensemble only needs to be invariant under local basis transformations other than the full basis transformation of the many-body Hilbert space. For small \( \epsilon \), we can take the continuous time approach to calculate the entanglement dynamics by solving the imaginary-time Schrödinger equation \( \partial_{t} |W_{\Psi_{t}}\rangle = -\hat{H}_{\text{EE}} |W_{\Psi_{t}}\rangle \) in Eq. (31). It worth mentioning that the locally scrambled quantum dynamics we considered here should be distinguished from Trotterizing a Hamiltonian dynamics. Here, the short-time evolutions \( e^{-i\epsilon H} \) are interrupted by local scramblers, such that they do not combine to a coherent long-time evolution generated by the same Hamiltonian \( H \). The local scramblers destroy the original notion of time. In the quantum dynamics, \( e^{-i\epsilon H} \) advances the quantum state by \( \epsilon \) in time, but after the insertion of layers of local scramblers, the entanglement dynamics...
only progress by $\hat{\tau}^2$, which is much slower. This phenomenon is analogous to the quantum Zeno effect due to the insertion of measurement. We conjecture that the local scramblers play a similar role as random local measurement in implementing random local basis transformations, such that the quantum dynamics is no longer coherent.

B. Locally Scrambled Random Circuits

Let us first consider the locally scrambled random circuit as in Fig. 7(a). The building blocks of the random circuit are two-qudit unitary gates. Each gate is independently drawn from local basis independent random ensembles. The EF of a two-qudit unitary operator $U_{ij}$ is completely characterized by two parameters: the cross channel mutual information $I_{ij}^\times$ and the tripartite information $I_{ij}^\gamma$. Let us label the input and output channels of the two-qudit unitary by $A, B, C, D$ as shown in Fig. 8(a), then $I_{ij}^\times$ and $I_{ij}^\gamma$ are defined as follows

$$I_{ij}^\times = I^{(2)}(A : D) = I^{(2)}(B : C),$$

$$I_{ij}^\gamma = I^{(2)}(A : C) + I^{(2)}(A : D) - I^{(2)}(A : CD).$$

The mutual information, such as $I^{(2)}(A : D) = S^{(2)}_A + S^{(2)}_B - S^{(2)}_{AB}$, is understood by treating the unitary gate as a quantum state by bending the input and output legs to the same side, and calculating the operator EE following the definition in Ref. [16, 38].

In terms of these information measures $I_{ij}^\times$ and $I_{ij}^\gamma$ of the unitary gate $U_{ij}$, the EF operator $\hat{W}_{U_{ij}}$ is given by

$$\hat{W}_{U_{ij}} = d^2(d + X_i)(d + X_j)$$

$$- A_{ij} \frac{1 - Z_i Z_j}{2} + B_{ij} \frac{1 - Z_i Z_j}{2} X_i X_j,$$

$$A_{ij} = d^4 (1 - e^{i\hat{I}_{ij}^\times})^2,$$

$$B_{ij} = d^2 (e^{i\hat{I}_{ij}^\gamma} - 1).$$

The cross channel mutual information $I_{ij}^\times \geq 0$ is nonnegative by the subadditivity of entropy. It describes the entanglement propagation, as it measures the amount of information transferred between site $i$ and $j$. The tripartite information $I_{ij}^\gamma$ can be negative, nevertheless $I_{ij}^\times - I_{ij}^\gamma \geq 0$ holds because of the bound $I^{(2)}(A : C) \leq 2 \log d = I^{(2)}(A : CD)$ for unitary channels. The negative tripartite information $(-I_{ij}^\gamma)$ is proposed [16] to be a description of information scrambling, since it measures the amount of information about $A$ that is encoded in $C$ and $D$ jointly but can not be told by local measurements exclusively performed on $C$ or $D$.

To gain more intuition about $I_{ij}^\times$ and $I_{ij}^\gamma$, let us provide a few examples of local basis independent ensembles of two-qudit gates, as pictured in Fig. 8(b-d).

- Identity gate with local scrambling, i.e. two on-site Haar random unitary gates direct product together, as Fig. 8(b). In this rather trivial case, we have

$$I_{ij}^\times = I_{ij}^\gamma = 0,$$

such that the EF operator in Eq. (60) reduces to $\hat{W}_I = d^2(d + X_i)(d + X_j)$, consistent with the previous result in Eq. (15) by direct evaluation.

- Swap gate with local scrambling, i.e. two on-site Haar random unitary gates followed by an intersite swap operator, as Fig. 8(c). In this case,

$$I_{ij}^\times = 2 \log d, \quad I_{ij}^\gamma = 0,$$

such that the EF operator in Eq. (60) reduces to

$$\hat{W}_{\text{swap}} = d^2(d + X_i)(d + X_j)$$

$$- d^2(d^2 - 1) \frac{1 - Z_i Z_j}{2} (1 - X_i X_j).$$

The swap gate can generate and propagate quantum entanglement due to the non-vanishing cross channel information $I_{ij}^\times$. But there is no information scrambling happening between the qudits (despite of the sufficient on-site scrambling), because the qubits are simply interchanged by the swap gate, such that local operators do not spread out other than being moved around in the space. The zero scrambling power of the swap gate is reflected in the zero tripartite information $I_{ij}^\gamma$. 

FIG. 8. (a) A generic two-qudit gate acting on qudits $i$ and $j$. The input channels are labeled by $A$ and $B$, and the output channels are labeled by $C$ and $D$. The EF of the gate will be labeled by the Ising configuration $\sigma = (\sigma_i, \sigma_j)$ on the input side and $\tau = (\tau_i, \tau_j)$ on the output side. (b-d) Examples of local basis independent ensembles of two-qudit gates: (b) identity gate with local scrambling, (c) swap gate with local scrambling, (d) Haar random unitary gate acting on both qudit (local basis automatically scrambled).
• Haar random unitary gate acting on the two qudits, as Fig. 8(d). In this case,

\[ I_{ij}^X = \log \frac{2d^2}{d^2 + 1}, \quad I_{ij}^Y = \log \frac{4d^2}{(d^2 + 1)^2}, \]  

such that the EF operator in Eq. (60) reduces to \( \hat{W}_{\text{Haar}} \) given in Eq. (13), see Appendix B for derivation. The Haar random unitary gate not only propagates quantum entanglement, but also scrambles the quantum information efficiently, as it has a negative tripartite information \( I_{ij}^T \) (as long as \( d > 1 \)).

The above are examples of locally scrambled random unitary ensembles. Unitary gates drawn from such ensembles serve as the building block of locally scrambled random circuits. The entanglement dynamics of locally scrambled random circuits can be universally described by the transfer matrix approach as has been discussed in Sec. III A. On the level of EF, the formulation is exact: the evolution of the average state EF can be precisely calculated from \( |W_{\Psi_{t+1}}\rangle = \hat{W}_{t}^\dagger |W_{\Psi_{t}}\rangle \) given the EF of the unitary. However, when applying the result to predict the EE, we rely on the assumption that the average EE can be approximated by the negative log of average EF following Eq. (27), where we effectively switch the order between the ensemble average and the logarithm. One major goal of the following is to provide numerical evidences to check this assumption in various different cases. It turns out that the negative log of EF generally provides a good estimate of the averaged EE, which makes our EF formulation useful in describing the entanglement dynamics for a broad class of random unitary circuits.

Our first example is the standard Haar random unitary circuit, where each two-qudit gate is drawn from Haar random unitary ensemble independently. The model has been extensively studied in the literature[21, 24, 25, 28] and the statistical mechanical model description has been developed by Zhou and Nahum in their pioneering work Ref. [33]. We revisit this model to show that our formalism is equivalent to the Zhou-Nahum approach and can reproduce the known behaviors. Let us first calculate the transfer matrix \( \hat{T}_{ij} \) of a single Haar random unitary gate \( U_{ij} \) from its EF. Based on Eq. (13) and Eq. (17), we obtain

\[ \hat{T}_{ij} = \hat{W}_{\text{Haar}} \hat{W}^{-1}_i = \left(1 + \frac{d(X_i + X_j)}{d^2 + 1}\right) \frac{1 + Z_i Z_j}{2}. \]  

Using the Ising basis \( |\sigma, \sigma_j\rangle \), Eq. (65) can be expressed as

\[ \hat{T}_{ij} = |\uparrow \uparrow\rangle \langle \uparrow \uparrow | + \frac{d}{d^2 + 1} \left(|\uparrow \downarrow\rangle \langle \uparrow \downarrow | + |\downarrow \uparrow\rangle \langle \downarrow \uparrow | + \right), \]  

which is equivalent to the triangle weights \( \nabla = 1 \) and \( \Psi = \Psi = d/(d^2 + 1) \) that defines the Ising model in Ref. [33]. Plugging Eq. (65) to Eq. (56), we obtain the transfer matrix \( \hat{T}_t \) that describes the EF state evolution under the quantum dynamics of the Haar random circuit. We assume the initial state is a product state, s.t. \( |W_0\rangle = |W_{\text{prod}}\rangle \). We evolve the EF state by Eq. (24). We can then compute the EE following Eq. (27) and compare the result with the numerical simulation. In the simulation, we applied randomly sampled unitary gates to an initial product state and measure the final state EE, then perform the ensemble average of the EE. As shown in Fig. 9, the EF approach provides pretty good prediction of the EE that matches the numerical result.

![FIG. 9](image_url) The final state EE of the Haar random circuit on a 10-site system for different choices of the entanglement regions: (a) single site, (b) half-system, (c) alternating[58]. The qudit dimension is \( d = 2 \) and the entropy is measured in unit of bit (= log 2).

![FIG. 10](image_url) (a) Swap gate circuit. Gray blocks mark out the swap gates. (b) Locally scrambled fractional swap gate circuit. Each swap gate is powered by the fraction \( 0 < \alpha < 1 \).
Now let us turn to a new example of locally scrambled random circuits, namely the swap gate circuit, which is designed to mimic the entanglement dynamics in integrable conformal field theories (CFT) where entanglement spreads with the propagation of quasiparticles.\cite{1, 38, 39} The circuit takes the architecture of the brick wall circuit in Fig. 7(a) with gates drawn from the locally scrambled swap gate ensemble in Fig. 8(c), the resulting circuit is equivalent to an interweaving network as shown in Fig. 10(a). The local scramblers in different layers can commute through the swap gates and combine to a single scrambling layer acting on the initial state, which can further be dropped as long as the initial state ensemble is already local basis invariant. For this model, we use a different initial state other than the product state. As illustrated in Fig. 10(a), the initial state is chosen to be a product of Einstein-Podolsky-Rosen (EPR) pairs arranged along a one-dimensional chain, whose EF can be described by

\[ |W_0\rangle = \prod_x \left( 1 + \frac{1}{d} X_{2x-1} + \frac{1}{d} X_{2x} + X_{2x-1} X_{2x} \right) \uparrow \uparrow. \]  

For each EPR pair, the qudit labeled by \( L \) (or \( R \)) will travel to the left (or right) in the swap gate circuit, which mimics the behaviors of left (or right) moving quasiparticles in an integrable CFT. In this way, entanglement spread out along the chain as EPR pairs stretch out, following the steps depicted in Fig. 11. On a finite-sized chain with periodic boundary condition, we expect to observe the half-system entanglement entropy to first grow and then decrease in time, and continue to oscillate like this. This recurrent behavior can be perfectly produced by the EF formulation, because, based on Eq. (63), the transfer matrix for a single swap gate turns out to be

\[ \hat{T}_{ij} = \hat{W}_{\text{swap}} \hat{W}_1^{-1} = \frac{1}{2} (1 + X_i X_j + Y_i Y_j + Z_i Z_j), \]

which is precisely the swap operator for Ising spins. In this way, the permutation of entangled qudits under the quantum dynamics is equivalently modeled as the permutation of correlated Ising spins in the EF formulation.

The periodic recurrence of the low-entanglement state in the swap gate circuit seems to contradict with our previous conclusion in Sec. II F that locally scrambled quantum dynamics generally thermalize. The swap gate circuit evades thermalization because its corresponding EF transfer matrix admits more than one leading eigenstate. Let \( \hat{T} = \bigotimes_x \hat{T}_{2x-1,2x} \bigotimes_x \hat{T}_{2x,2x+1} \) be two steps (one period) of the transfer matrix that translates the \( L \) (or \( R \)) sublattice to the left (or right) by one unit-cell. On a chain of \( 2n \) sites, the operator \( \hat{T} \) has \( n^{-1} \sum_{d|n} \varphi(d) 4^{n/d} \) fold degenerated eigenstates of eigenvalue 1, with \( \varphi(d) \) being the Euler totient function and \( d \) running over all divisors of \( n \). These eigenstates can be constructed by taking any Ising basis state and symmetrizing over the cyclic group generated by \( \hat{T} \). Their degeneracy can be counted by mapping the problem to the number of \( n \)-bead necklaces with four colors,\cite{59} where the four colors correspond to the four choices of \( \uparrow \downarrow \uparrow \downarrow \downarrow \uparrow \downarrow \downarrow \uparrow \downarrow \downarrow \downarrow \uparrow \downarrow \downarrow \) configurations in each unit-cell. Therefore the Page state is not the unique state that can survive in the long-time limit, and thermalization is not the ultimate fate.

![FIG. 11. Evolution of EPR pairs under the swap gate circuit on a 12-site chain with periodic boundary condition. The entanglement entropy between the left- and right-half system is proportional to the EPR pairs across the cut (indicated by dotted vertical line).](image)

The swap gate circuit model can be generalized by introducing the *fractional swap gate* that interpolates be-
between the identity gate and the swap gate. The fractional swap gate can be written as a fractional power \( \alpha \) of the swap gate with \( 0 < \alpha < 1 \)

\[
\text{SWAP}^\alpha = \frac{1 + e^{i \alpha \pi}}{2} I + \frac{1 - e^{i \alpha \pi}}{2} X.
\]

The fractional swap gate reduces to the identity gate (or the swap gate) at \( \alpha = 0 \) (or \( \alpha = 1 \)). But unlike both identity and swap gates which do not scramble quantum information between the two qudits, the fractional swap gate does have finite scrambling power. We can construct a locally scrambled fractional swap gate circuit by starting from the architecture of the random circuit in Fig. 7(a) and sampling every gate independently from local basis invariant fractional swap gate ensemble, as illustrated in Fig. 10(b). The EF operator of the fractional swap gate follows the general form of Eq. (60) with parameters \( A_{ij} \) and \( B_{ij} \) given by

\[
A_{ij} = d^2(d^2 - 1) \left( \frac{3}{2} \cos \alpha \frac{\pi}{2} \right) \sin^2 \frac{\alpha \pi}{2},
\]

\[
B_{ij} = d^2(d^2 - 1) \sin^2 \frac{\alpha \pi}{2}.
\]

Based on this result, the corresponding transfer matrix \( T_{ij} \) can be constructed by Eq. (57) and the evolution of the EF state can be calculated following the transfer matrix approach described in Eq. (56). In Fig. 12(b-d), we compare the EE calculated based on the EF approach with the ensemble averaged EE from numerical simulation. They match perfectly for different values of \( \alpha \). Because the fractional swap gate has finite scrambling power, the recurrence behavior no longer persist and the system can now thermalize. The entanglement dynamics is somewhat between that of the swap gate circuit and the Haar random circuit, in that the EE grows mostly linearly in time with small oscillations, until the EE eventually saturates to the thermal limit. As \( \alpha \) becomes small, the system will take longer time (more steps) to thermalize. As shown in Fig. 12(d), the oscillation of EE is suppressed and its growth curve is more smooth. In the \( \alpha \to 0 \) limit, the entanglement dynamics approaches the continuum limit that can be described by the EF Hamiltonian, which is the topic of the following discussion.

### C. Locally Scrambled Hamiltonian Dynamics

Now we turn to the locally scrambled Hamiltonian dynamics as illustrated in Fig. 7(b). We consider the local Hamiltonian \( H = \sum_{\langle ij \rangle} H_{ij} \) and assume that \( H_{ij} \) on every bond is drawn from a local-basis-independent ensemble of two-qudit Hermitian operators. Equivalently, we can choose \( H \) to be a fixed Hamiltonian and construct a locally scrambled unitary ensemble \( \mathcal{E}_{-iH} \) by applying local basis transformations following Eq. (58). The quantum dynamics is described by the unitary

\[
U = \prod_t \left( V_t e^{-i H} \right),
\]

where \( V_t \) describe the layer of local scramblers at time \( t \), as illustrated in Fig. 7(b). The corresponding entanglement dynamics is described by the imaginary-time Schrödinger equation Eq. (31), where the EF Hamiltonian takes the form of

\[
\hat{H}_{\text{EF}} = \sum_{\langle ij \rangle} g_{ij} \left( 1 - \frac{Z_i Z_j}{2} e^{-\beta_{ij} X_i X_j - \delta(X_i + X_j)} \right).
\]

It turns out that the parameters \( \beta_{ij} \sim \mathcal{O}(\epsilon^2) \) always vanish in the \( \epsilon \to 0 \) limit. The parameters \( g_{ij} \) are the only non-trivial parameters to the leading order of \( \epsilon \), which are determined by the local terms \( H_{ij} \) in the Hamiltonian

\[
g_{ij} = \frac{2}{d(d^2 - 1)} \left( (\Tr H_{ij})^2 + d^2 \Tr (H_{ij}^2) \right) - d \Tr (\Tr_j (H_{ij})^2). \tag{73}
\]

The detailed derivation of these results can be found in Appendix G.

One well-studied example of the locally scrambled Hamiltonian dynamics is the Brownian random circuit,[56] where each step of the time evolution is generated by a random Hamiltonian drawn from the Gaussian unitary ensemble (GUE). The Hamiltonian can be written as a random \( U(d) \) spin model,

\[
H_t = \sum_{\langle ij \rangle} J^{ab}_{t,ij} T^{a}_i T^{b}_j, \tag{74}
\]

where \( T^a_i \) (for \( a = 1, 2, \cdots, d^2 \)) are \( U(d) \) generators on site \( i \) with \( \Tr T^a_i T^b_i = \delta^{ab} \). The coupling \( J^{ab}_{t,ij} \) are independently drawn for each time \( t \) and indices \( i, j, a, b \) from the Gaussian distribution with zero mean and \( d^{-2} \) variance. The quantum dynamics is described by \( U = \prod_t e^{-i H_t} \). The operator growth dynamics and the spectral form factor of the Brownian random circuit has been investigated in Ref. [60–63] recently, where differential equations governing the evolution of operator weight distribution were derived. Our approach also applies to the Brownian circuit model and results in similar differential equations for the evolution of EF state, whose EF Hamiltonian reads (see Appendix G for derivation)

\[
\hat{H}_{\text{EF}} = \frac{2(d^2 - 1)}{d^2} \sum_{\langle ij \rangle} \frac{1 - Z_i Z_j}{2} e^{-\delta(X_i + X_j)}. \tag{75}
\]

We will not discuss this model in further details, given the extensive study of Brownian circuits in the literature. Instead, we will consider a new type of locally scrambled Hamiltonian dynamics.

We start with a fixed Hamiltonian on the one dimensional chain of qudits

\[
H = - \sum_{\langle ij \rangle} T_i T_j, \tag{76}
\]

where \( T_i \) is one particular traceless Hermitian operator on site \( i \) that squares to identity (i.e. \( \Tr T_i = 0 \) and \( T_i^2 = 1 \)).
For the qubit case \((d = 2)\), Eq. (76) reduces to an Ising model. Note that there is no randomness in the Hamiltonian \(H\). The randomness will be introduced by the local scramblers, when we use \(H\) to generate the locally scrambled Hamiltonian dynamics following Eq. (71). The entanglement dynamics will be described by the following EF Hamiltonian

\[
\hat{H}_{\text{EF}} = \frac{2d^2}{d^2 - 1} \sum_{\langle ij \rangle} \frac{1 - Z_i Z_j}{2} e^{-\delta (X_i + X_j)},
\]

which takes the same form as Eq. (75) but with a different parameter \(g\). We can test the EF approach with numerical simulation on a 12-qubit system with the choice of \(g = 0.1\). We start with a product state \(|W_{\text{prod}}\rangle = |W_{\prod}\rangle\), evolve the EF state by Eq. (31) and calculate the EE from Eq. (27). The result is shown in Fig. 13. We can see that the averaged EE obtained from numerics matches well with the result of the EF approach over difference choices of the entanglement regions. These numerical evidences suggest that exchanging the order between taking ensemble average and taking logarithm does not seem to matter much, so the evolution equation we established for the EF in this work can provide reliable descriptions for the entanglement dynamics under locally scrambled quantum dynamics. Comparing Fig. 13 with Fig. 9, one can see that the entanglement dynamics of the locally scrambled Hamiltonian dynamics closely resembles that of the Haar random unitary circuit. Thus the former can be considered as a continuum limit of the later.

![Figure 13](image)

**FIG. 13.** The final state EE of the locally scrambled Hamiltonian dynamics on a 12-site system for different choices of the entanglement regions: (a) single site, (b) half-system, (c) alternating\[^{58}\]. The qudit dimension is \(d = 2\) and the entropy is measured in unit of bit (= \(\log 2\)).

We also notice that, in agreement with the imaginary time EF Schrödinger equation, the EE always approaches to its final thermalized value exponentially with the same relaxation time \(\tau\) independent of the choice of the entanglement region,

\[
S^{(2)}[\sigma](t) \to S^{(2)}[\sigma](\infty) - A[\sigma]e^{-t/\tau}.
\]

The relaxation time \(\tau\) is intrinsically related to the excitation gap \(\Delta\) of the EF Hamiltonian \(\hat{H}_{\text{EF}}\), which can be estimated by Eq. (53) in the thermodynamic limit,

\[
\tau^{-1} = \Delta = 2g \left(1 - \frac{1}{d}\right)^2 = \frac{4}{3},
\]

where the coupling \(g\), as inferred from Eq. (77), is given by \(g = 2d^2/(d^2 - 1) = 8/3\) for qubits \((d = 2)\). To check this prediction, we fit the numerical simulation data using Eq. (78) in the late-time regime to extract the excitation gap \(\Delta\). As shown in Fig. 14, the EE approaches to the thermal value with the same rate (within error bars) regardless of the different choice of entanglement regions. The numerically fitted gap is around \(\Delta = 1.48\), which is close to the thermodynamic-limit analytic prediction \(\Delta = 4/3 = 1.33\). The small discrepancy mainly arises from the finite-size effect. If we use the finite-size gap formula \(\Delta = 0.56g\) based on the ED result in Fig. 6 at the system size \(L = 12\), we will obtain a better prediction of the gap \(\Delta = 1.49\), which matches the simulation result perfectly.

![Figure 14](image)

**FIG. 14.** The difference between the EE and its final saturation value, i.e. \(A[\sigma]e^{-t/\tau}\), plot in the logarithmic scale vs time \(t\). Different colors correspond to different choices of entanglement region (labeled by \(\sigma\)). The shaded region indicates the error interval. The excitation gap \(\Delta = \tau^{-1}\) is extracted by fitting the decay rate.

### IV. SUMMARY AND DISCUSSIONS

In this work, we introduced the concept of locally scrambled quantum dynamics, where each step of the unitary evolution is randomized by local scramblers (on-site Haar random unitary gates). Surrounding each unitary gate in a quantum circuit by local scramblers effectively blocks the local-basis-specific quantum information from propagating in the circuit and decouples the gates.
from each other under ensemble average. In this way, the average EF of the entire circuit can be constructed piece-by-piece from the EF of each gate, which makes the entanglement dynamics Markovian and enables us to write down the evolution equation for the EF of quantum states. The framework provides us the freedom to design the EF for each gate, such that we can go beyond the conventional Haar random gates and build the random circuit with more general random gates as long as their ensemble is local-basis-independent. This enables us to define and explore the continuum limit of locally scrambled quantum dynamics, under which the evolution of the EF state will be governed by an EF Hamiltonian. We obtained the general form of the EF Hamiltonian on symmetry ground and discussed the implication of its spectral properties on the entanglement dynamics. When the EF Hamiltonian is \textit{gapped}, the excitation modes in the EF state will decay exponentially in time $W_t = e^{-S \sim e^{-t/\tau}}$, which corresponds to a linear growth of EE in time, i.e., $S \sim t/\tau$, as the system thermalizes. What has not been much discussed previously is the possibility that the EF Hamiltonian can become \textit{gapless} under fine-tuning, then the EF will decay in a power-law manner $W_t = e^{-S \sim t^{-\alpha}}$, which corresponds to a logarithmic growth of EE, i.e., $S \sim \alpha \log t$. Such scenario could happen at entanglement transitions,[35, 49, 50] where the entanglement scaling of the long-time final state switches from volume-law to area-law. The transition can be interpreted as an order to disorder phase transition of the EF Hamiltonian. One interesting future direction is to explore different models of the EF Hamiltonian and to map out the phase diagram using analytical and numerical tools developed in quantum many-body physics.

Although we focused on the entanglement dynamics of pure states in this work, the EF formulation can be easily generalized to describe mixed state or operator entanglement. Following Eq. (4), it is straightforward to define the EF $W_\rho[\sigma]$ for any many-body operator $O$ (including the density matrix $\rho$ as a special case),

$$W_\rho[\sigma] = \text{Tr}(\chi_\sigma O^{\otimes 2}),$$  \hspace{1cm} (80)

and quantize the EF to a state $|W_\rho\rangle = \sum_\sigma W_\rho[\sigma]|\sigma\rangle$. Suppose the operator evolves in time under a locally scrambled quantum dynamics $O' = UOU^\dagger$, the average EF will still be described by the same set of equation $|W_{O'}\rangle = W_t|W_{O'}\rangle$ as Eq. (19). Based on this, all the dynamic equation that we developed in this work applies directly, such that we do not need to derive a new set of equations for operator dynamics. The EF state $|W_\rho\rangle$ encodes the operator EE[22] over all possible regions, which can be used to construct various quantities characterizing the operator size. To name a few, let us first assume $O$ to be a traceless Hermitian operator normalized to $\text{Tr} O^2 = d$. We can decompose the operator $O = \sum_{[a]} O_{[a]} T_{[a]}$ in the operator basis $T_{[a]} = \prod_i T_{i[a]}$ (where $T_{i[a]}$ is the SU(d) generator on the $i$th qudit), and define the operator weight $p_{[a]} = O_{[a]}^2$.[25, 28, 60, 62] The fraction of the operator in a subsystem $A$ then reads

$$p_A = \sum_{[a] \in A} p_{[a]},$$  \hspace{1cm} (84)

which can be extracted from the EF state $|W_\rho\rangle$ by taking its inner product with a state $|\Psi_A\rangle$ that labels the subsystem $A$:

$$p_A = \langle \Psi_A | W_\rho | \Psi_A \rangle.$$  \hspace{1cm} (85)

The fraction $p_A$ can be further used to characterize the average operator size $\ell_O = \sum_A p_A |A|$. The evolution equation for $p_A$ under Brownian dynamics is recently discussed in Ref. [60, 62]. Another way to probe $|W_\rho\rangle$ is to consider the variance of the expectation value of $O$ on random mixed states $\rho$, i.e. $\text{var}_\rho(O)_\rho = \mathbb{E}_\rho(\text{Tr} \rho O)^2$. Suppose $\rho$ is drawn from a local-basis-independent ensemble characterized by its EF state $|W_\rho\rangle$, then the variance of operator expectation value is given by

$$\text{var}_\rho(O)_\rho = \langle W_\rho | \hat{W}_1^{-1} | W_\rho \rangle.$$  \hspace{1cm} (82)

It was recently pointed out by Ref. [65] that $\text{var}_\rho(O)_\rho = \sum_A p_A (d + 1)^{-|A|}$ can be expressed in terms of $p_A$, if $\rho$ is uniformly sampled from the ensemble of pure product states. The fact that $\text{var}_\rho(O)_\rho$ and $p_A$ are related to each other is less surprising in the EF formulation, because they are simply two different ways to probe the same EF state $|W_\rho\rangle$. The evolution equation of $|W_\rho\rangle$ under locally scrambled quantum dynamics is identical to that of $|W_\rho\rangle$, from which the evolution equations of $p_A$, $\ell_O$ or $\text{var}_\rho(O)_\rho$ follow automatically. In this way, the EF formulation developed in our work provides a unified framework to discuss various aspects of the operator dynamics.

Another immediate generalization of the framework is to extend the unitary evolution to generic quantum channels allowing measurements to take place. The recent observation of measurement-induced entanglement transition in random unitary circuits [51, 66, 67] has attracted much research interest.[49, 50, 68–71] In these models, the quantum circuit is doped with local measurements (which can be either weak measurements or projective measurements happened with probability), and the final state EE is measured conditioned on the measurement outcome. If each measurements basis is randomly chosen each time, or if the local measurement take place only after the local basis has been sufficiently scrambled by the unitary evolution, the whole quantum channel still falls in the scope of locally scrambled quantum dynamics, which can be described by the EF approach developed in this work. In this case, each measurement, described by the Kraus operator $M$, is also a local-basis-independent component in the quantum circuit, and has its own EF similar to Eq. (5)

$$W_M[\sigma, \tau] = \text{Tr}(\chi_\sigma M^{\otimes 2} \chi_\tau M^{\dagger \otimes 2}),$$  \hspace{1cm} (83)

from which the EF operator

$$\hat{W}_M = \sum_{\sigma, \tau} |\sigma\rangle W_M[\sigma, \tau]|\tau\rangle.$$  \hspace{1cm} (84)
can be constructed. The EF state will evolve under measurement by \( |W \rangle = W_M W_1^{-1} |W \rangle \), such that the approaches developed in this work seamlessly apply. The EF provides a finer resolution of the entanglement structure of a quantum many-body state beyond the single region scaling of EE, which turns out to be useful in diagnosing the error correction capacity\[70\] in the volume-law states prepared by the measurement-doped quantum circuits. We will leave this topic to future works\[72\]. More generally, the EF formulation can be further generalized to locally scrambled tensor networks, which does not even need to have a preferential time direction. As long as each tensor in the tensor network is independently drawn from local-basis-independent ensembles, the entanglement structured of the random tensor network can be described by the EF approach. The freedom to design the EF for each separate tensor in the tensor network opens up a large space of models to explore in the future. There are also a few more challenging future directions that worth further investigation. The first direction is to generalize the 2nd Rényi EF to arbitrary Rényi index. As a consequence, the Ising variable on each site will be promoted to a permutation group element \( \sigma_i \in S_n \). Such generalization will also allow us to access other measures of entanglement, such as Rényi negativity\[39, 73, 74\], as the moment of the partial transposed density matrix \( \rho^{\perp A} \)[75–78] can be expressed in terms of the \( n \)th Rényi EF,

\[
\text{Tr}(\rho^{\perp A})^n = W_\rho[g], \quad g_i = \{ (n \cdots 21) \quad i \in A, \\
(12 \cdots n) \quad i \in A. \} \tag{85}
\]

The \( n \)th Rényi generalization of EF states \( |W \rangle \) and EF operators \( W_{\rho} \) can still be defined, but it will be more difficult to perform explicit calculations given that the number of group elements \( n \) grows quickly with \( n \). Perhaps the most subtle issue is how to take the \( n \to 1 \) replica limit systematically, which has been identified\[35, 49, 50\] as an important step to understand the nature of entanglement transitions. The second direction is to include global symmetries and conservation laws\[28, 52\] into the discussion. This amounts to refining the generic local scramblers to symmetry-preserving local scramblers, which only performs basis transformations within each irreducible representations of the symmetry group. The formulation to describe the interference between the entanglement dynamics and the flow of symmetry representations in the quantum circuits still need to be developed. The third direction is to go beyond the locally scrambled quantum dynamics and to gradually introduce correlations among random gates in the spacetime. Can the current EF formulation serves as a good starting point to construct phenomenological descriptions for weakly correlated random gates? Can we eventually approach the limit of coherent quantum evolution for Hamiltonian or Floquet dynamics? There are many interesting open question awaiting us to explore.

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Appendix A: Entanglement Feature of Page State

The Page state can be considered as a single random tensor. According to Ref. [32], the 2nd Rényi entanglement feature of a random tensor network can be calculated as the partition function of an Ising model,

$$ W_{\text{RTN}}[\sigma] = \frac{1}{Z} \sum_{\tau} e^{-E_{\text{RTN}}[\sigma, \tau]}, $$

(A1)

where each random tensor is mapped to an Ising spin $\tau_i$ coupled together via the network, and the boundary condition pinned by external Zeeman field along the direction specified by $\sigma$. Applying this result to the Page state,

$$ W_{\text{Page}}[\sigma] = \frac{1}{Z} \sum_{\tau} e^{\eta \sum_{i=1}^L \sigma_i \tau_i}, $$

(A2)

where there is only one Ising spin $\tau$ because the Page state is only a single random tensor. The $\tau$ spin couples to the boundary condition $\sigma$ via uniform field strength $\eta = \frac{1}{2} \log d$, which is determined by the qudit Hilbert space dimension $d$ (see Ref. [32] for derivation). Complete the summation over Ising spin $\tau$, we obtain

$$ W_{\text{Page}}[\sigma] = \frac{2}{Z} \cosh \left( \eta \sum_{i=1}^L \sigma_i \right). $$

(A3)

The normalization constant $Z$ is determined by the condition that $W_{\text{Page}}[\sigma] \equiv 1$, such that $Z = 2 \cosh(\eta L)$, hence

$$ W_{\text{Page}}[\sigma] = \frac{\cosh(\eta \sum_{i=1}^L \sigma_i)}{\cosh(\eta L)}, $$

(A4)

which can be rewritten as the EF state $|W_{\text{Page}}\rangle$ in Eq. (11).

Appendix B: Entanglement Feature of Two-Qudit Haar Random Unitary

Here we derive the ensemble averaged EF operator for two-qudit Haar random unitary gate. We start with the definition

$$ W_{U_{ij}}[\sigma, \tau] = \text{Tr}(X_{\sigma} U_{ij}^{\otimes 2} X_{\tau} U_{ij}^{\dagger \otimes 2}). $$

(B1)

$U_{ij}$ is a two-qudit gate acting on qudits labeled by $i$ and $j$. Focusing on these two qudits, the Ising variables $\sigma = (\sigma_i, \sigma_j)$ and $\tau = (\tau_i, \tau_j)$ both contain only two components. Consider averaging the EF $W_{U_{ij}}$ over unitary gates $U_{ij}$ in the Haar random unitary ensemble,

$$ E_{U_{ij} \in \text{Haar}} W_{U_{ij}}[\sigma, \tau] = E_{U_{ij} \in \text{Haar}} \text{Tr}(X_{\sigma} U_{ij}^{\otimes 2} X_{\tau} U_{ij}^{\dagger \otimes 2}) = \sum_{g, h \in S_2} W_g(g^{-1} h, d^2) \text{Tr}(X_g X_{\sigma_1}) \text{Tr}(X_g X_{\sigma_2}) \text{Tr}(X_h X_{\tau_1}) \text{Tr}(X_h X_{\tau_2}), $$

(B2)

where $W_g$ is the Weingarten function[81, 82] and $g, h$ are $S_2$ group elements

$$ W_g(g^{-1} h, d^2) = \left\{ \begin{array}{ll} \frac{1}{d^2} & g^{-1} h = \| \| \\ \frac{1}{d^2(d^2-1)} & g^{-1} h = X \end{array} \right. $$

(B3)

The cycle counting function $\text{Tr}(X_g X_h)$ follows

$$ \text{Tr}(X_g X_h) = \left\{ \begin{array}{ll} d^2 & gh = \| \| \\ d & gh = X \end{array} \right. $$

(B4)

Substitute Eq. (B3) and Eq. (B4) into Eq. (B2), we can evaluate $E_{U_{ij} \in \text{Haar}} W_{U_{ij}}[\sigma, \tau]$ for all configurations of $\sigma, \tau$. In terms of Ising variables (following the identification $\| \leftrightarrow \uparrow$ and $X \leftrightarrow \downarrow$), we can summarize the result as the following matrix in the Ising basis $\sigma, \tau = \uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow$,

$$ E_{U_{ij} \in \text{Haar}} W_{U_{ij}} = \begin{bmatrix} \left[ \begin{array}{cccc} d^4 & d^3 & d^3 & d^2 \\ 2d^4 & 2d^3 & 2d^3 & d^2 \\ d^3 & d^3 & d^3 & d^2 \\ d^2 & d^3 & d^3 & d^2 \end{array} \right] \\ \begin{array}{cccc} \left[ \begin{array}{cccc} d^3 & d^3 & d^3 & d^2 \\ d^3 & d^3 & d^3 & d^2 \\ d^3 & d^3 & d^3 & d^2 \\ d^3 & d^3 & d^3 & d^2 \end{array} \right] \end{array} \end{bmatrix}, $$

(B5)
which is also the matrix representation of the (ensemble averaged) EF operator $\hat{W}_{U_{ij}}$. The matrix can as well be written in terms of Pauli operators as

$$E_{U_{ij}\in\text{Haar}}\hat{W}_{U_{ij}} = d^2(d + X_i)(d + X_j) - \frac{d^2(d^2 - 1)}{2(d^2 + 1)}(1 - Z_iZ_j)(d^2 - X_iX_j),$$  \hfill (B6)

as claimed in Eq. (13). For simplicity, we have omitted $E_{U_{ij}\in\text{Haar}}$ notation in Eq. (13), with the understanding that the EF for an ensemble of unitaries is implicitly averaged.

**Appendix C: Relation Between State and Unitary Entanglement Features**

Here we prove Eq. (19). Consider a many-body state (multi-qudit) state $|\Psi\rangle$ and an unitary operator $U_t$ supported in the same Hilbert space. Suppose that $|\Psi'\rangle = U_t|\Psi\rangle$, our goal is to derive the time evolution of the corresponding EF state. In general, this is not tractable since the unitary operator $U_t$ contains many non-universal features that are specific to the choice of local basis. Such features may affect the entanglement of the final state but such features are not captured in EF formalism. By this property, we instead consider an ensemble of unitary operator $U$,

$$E_U = \left\{ V |U|V\right| = \bigotimes_{i=1}^L V_i, V_i \in \text{Haar} \right\},$$  \hfill (C1)

where each $V_i$ independently follows Haar random unitary distribution on the $i$th qudit. Our goal is to compute $E_{U' \in E_U} W_{U'|\Psi}[\sigma]$,

$$E_{U' \in E_U} W_{U'|\Psi}[\sigma] = E_{U' \in E_U} \text{Tr}[\mathcal{X}_\sigma(U'\Psi)\langle\Psi|U'^\dagger|\Psi\rangle\otimes^2]$$

$$= \int_{V \in \text{Haar}} \text{Tr}[\mathcal{X}_\sigma(V^\dagger UV|\Psi\rangle\langle\Psi|V^\dagger V^\dagger U)|\Psi\rangle\otimes^2]$$

$$= \int_{V \in \text{Haar}} \langle\Psi|\otimes^2 V^\dagger \otimes^2 U^\dagger \otimes^2 V^\perp \otimes^2 \mathcal{X}_\sigma V^\dagger \otimes^2 U \otimes^2 V^\perp \otimes^2 |\Psi\rangle\otimes^2$$

$$= \int_{V \in \text{Haar}} \langle\Psi|\otimes^2 V^\dagger \otimes^2 U^\dagger \otimes^2 \mathcal{X}_\sigma U \otimes^2 V^\perp \otimes^2 |\Psi\rangle\otimes^2$$

$$\sum_{\tau, \tau'} \text{Tr}(\mathcal{X}_\tau |\Psi\rangle\langle\Psi|\otimes^2 \text{Tr}(\mathcal{X}_\tau U^\dagger (t) \otimes^2 \mathcal{X}_\sigma U^\dagger t) \prod_i W_g(\tau_i^{-1} \tau_i, d)$$

$$= \sum_{\tau, \tau'} W_{|\Psi\rangle|\tau} |U|_{\sigma, \tau'} \prod_i W_g(\tau_i^{-1} \tau_i, d),$$

where $W_g$ denotes the Weingarten function\[81, 82\] originated from the Haar ensemble average of $V^\perp \otimes^2 V \otimes^2$, and $\tau, \tau'$ are new set of Ising variables. The derivation in Eq. (C2) can also be diagrammatically represented as Fig. 15.

By definition, the Weingarten function, when viewed as a matrix indexed by $\tau$ and $\tau'$, is the inverse of the Gram matrix $\text{Tr}(\mathcal{X}_\tau \mathcal{X}_{\tau'}) = \langle\tau'|\hat{W}_1|\tau\rangle$, which is simply the matrix representation of the EF operator $\hat{W}_1$ of the identity operator. So the Weingarten function is given by the matrix element of $\hat{W}_1^{-1}$ as

$$\prod_i W_g(\tau_i^{-1} \tau_i, d) = \langle\tau'|\hat{W}_1^{-1}|\tau\rangle.$$

(C3)

Therefore, in operator form, we have

$$E_{U' \in E_U} W_{U'|\Psi} = \sum_\sigma W_{U'|\Psi}[\sigma]|\sigma\rangle$$

$$= \sum_{\sigma, \tau, \tau'} (|\sigma\rangle W_{U|\sigma, \tau|\tau'})(|\sigma\rangle \hat{W}_1^{-1}(W_{\Psi}|\tau\rangle|\tau\rangle)$$

$$= \hat{W}_U \hat{W}_1^{-1} |\Psi\rangle,$$

as stated in Eq. (19).
Appendix D: Spectral Properties of Entanglement Hamiltonian

Let us start with the most general form of the EF Hamiltonian \( \hat{H}_{EF} \) given in Eq. (32) and investigate its spectral properties.

\[
\hat{H}_{EF} = \sum_{i,j} \hat{H}_{ij}, \quad \hat{H}_{ij} = g_{ij} \frac{1 - Z_i Z_j}{2} e^{-\beta_i X_i X_j - \delta (X_i + X_j)},
\]

with coth \( \delta = d \). Our first goal is to show that \( \hat{H}_{EF} \) is positive semi-definite. The trick is to first deform \( \hat{H}_{EF} \) to a Hermitian version \( \hat{H}'_{EF} \), given by

\[
\hat{H}'_{EF} = \hat{W}_1^{-1/2} \hat{H}_{EF} \hat{W}_1^{1/2}.
\]

Because \( \hat{H}_{EF} \) and \( \hat{H}'_{EF} \) are related by similar transformation, they share the same set of eigenvalues. So the positivity of the original EF Hamiltonian \( \hat{H}_{EF} \) is equivalent to the positivity of the transformed Hermitian version \( \hat{H}'_{EF} \). The later turns out to be easier to prove. By the way, to see that \( \hat{H}'_{EF} \) is Hermitian (or real symmetric to be more precise), we use \( \hat{W}_1 = \hat{W}_1^{\dagger} \) and Eq. (35) that \( \hat{W}_1 \hat{H}'_{EF} = \hat{H}_{EF} \hat{W}_1 \), then

\[
\hat{H}'_{EF} = \hat{W}_1^{-1/2} \hat{H}_{EF} \hat{W}_1^{-1/2} = \hat{W}_1^{-1/2} (\hat{W}_1 \hat{H}_{EF} \hat{W}_1^{\dagger}) \hat{W}_1^{-1/2} = \hat{W}_1^{-1/2} (\hat{H}_{EF} \hat{W}_1) \hat{W}_1^{-1/2} = \hat{W}_1^{-1/2} \hat{H}_{EF} \hat{W}_1^{1/2} = \hat{H}'_{EF},
\]

meaning that \( \hat{H}'_{EF} \) is transpose symmetric. Moreover \( \hat{H}'_{EF} \) is real by definition, so \( \hat{H}'_{EF} \) is real and symmetric and therefore Hermitian. As a real symmetric operator, \( \hat{H}'_{EF} \) admits the following spectral decomposition

\[
\hat{H}'_{EF} = \sum_{\alpha} |V_{\alpha}\rangle \lambda_{\alpha} \langle V_{\alpha}|,
\]

with \( |V_{\alpha}\rangle = (V_{\alpha})^\dagger \) being the eigenvector corresponding to the eigenvalue \( \lambda_{\alpha} \). If we can show that the expectation value \( \langle V|\hat{H}'_{EF}|V \rangle \geq 0 \) is non-negative on any state \( |V \rangle \) in the EF Hilbert space (including the eigenstates \( |V_{\alpha}\rangle \)), we will be able to prove that all eigenvalues \( \lambda_{\alpha} = \langle V_{\alpha}|\hat{H}'_{EF}|V_{\alpha}\rangle \geq 0 \) are non-negative, hence \( \hat{H}'_{EF} \) will be positive semi-definite.

We can show \( \langle V|\hat{H}'_{EF}|V \rangle \geq 0 \) by finding the Cholesky decomposition for each terms in \( \hat{H}'_{EF} \). A useful trick is to note that \( d(d + X_i) = e^{\delta X_i} / (\tanh \delta \sinh \delta) \) given \( d = \coth \delta \), so \( \hat{W}_1 \) can be rewritten as

\[
\hat{W}_1 = \prod_{i=1}^L d(d + X_i) = \prod_{i=1}^L \frac{e^{\delta X_i}}{\tanh \delta \sinh \delta} = \frac{1}{(\tanh \delta \sinh \delta)^L} \prod_{i=1}^L e^{\delta X_i},
\]

FIG. 15. Diagrammatic proof of Eq. (19)
such that any $\hat{W}_1^\alpha$ can be simply calculated,

$$\hat{W}_1^\alpha = (\tanh \delta \sinh \delta)^{-\alpha L} \prod_{i=1}^L e^{\alpha \beta X_i},$$

(D6)

With this, and substitute Eq. (D1) in Eq. (D2), we can show that

$$\hat{H}'_{\text{EF}} = \sum_{i,j} \hat{H}'_{ij},$$

$$\hat{H}'_{ij} = \hat{W}_1^{-1/2} \hat{H}_{ij} \hat{W}_1^{1/2}$$

$$= \prod_{i=1}^L e^{-\frac{1}{2} X_i} \hat{H}_{ij} \prod_{i=1}^L e^{\frac{1}{2} X_i}$$

$$= e^{-\frac{1}{4}(X_i+X_j)} \hat{H}_{ij} e^{\frac{1}{4}(X_i+X_j)}$$

$$= g_{ij} e^{-\frac{1}{4}(X_i+X_j)} \frac{1-Z_i Z_j}{2} e^{-\beta_i X_i X_j} e^{\frac{1}{4}(X_i+X_j)}$$

$$= g_{ij} e^{-\frac{1}{4}(X_i+X_j)} \frac{1-Z_i Z_j}{2} e^{-\beta_i X_i X_j} \frac{1-Z_i Z_j}{2} e^{-\frac{1}{4}(X_i+X_j)}.$$  

(D7)

In the last step, we use the fact that $\frac{1-Z_i Z_j}{2}$ is a projection operator, so $(\frac{1-Z_i Z_j}{2})^2 = \frac{1-Z_i Z_j}{2}$. Also $\frac{1-Z_i Z_j}{2}$ and $e^{-\beta_i X_i X_j}$ commute with each other, so we are free to move $e^{-\beta_i X_i X_j}$ through $\frac{1-Z_i Z_j}{2}$. The final form of $\hat{H}'_{ij}$ admits the following Cholesky decomposition explicitly

$$\hat{H}'_{ij} = \hat{A}_{ij}^\dagger \hat{A}_{ij}, \quad \hat{A}_{ij} = g_{ij}^{1/2} e^{-\beta_{ij}/2} X_i X_j \frac{1-Z_i Z_j}{2} e^{-\frac{1}{4}(X_i+X_j)}.$$  

(D8)

For any state $|V\rangle$ in EF Hilbert space, the expectation value $\langle V|\hat{H}'_{ij}|V\rangle = \langle V|\hat{A}_{ij}^\dagger \hat{A}_{ij}|V\rangle \geq 0$ is non-negative, therefore $\hat{H}'_{ij}$ is positive semi-definite. In consequence, the transformed EF Hamiltonian $\hat{H}'_{\text{EF}} = \sum_{i,j} \hat{H}'_{ij}$ is also positive semi-definite, as it is the sum of positive semi-definite terms $\hat{H}'_{ij}$. Recall that the similar transformation does not affect the eigenvalues, so $\hat{H}_{\text{EF}} = \hat{W}_1^{1/2} \hat{H}'_{\text{EF}} \hat{W}_1^{-1/2}$ is also positive semi-definite.

We can further show that $\hat{H}_{\text{EF}}$ always has two zero modes: one is even under $\mathbb{Z}_2$ Ising symmetry, and the other is odd. Using the left-null-state property $\langle \uparrow | \hat{H}_{\text{EF}} | \uparrow \rangle = 0$ given in Eq. (34), it is ensured that $\langle \uparrow |$ is an left-eigenstate of $\hat{H}_{\text{EF}}$ with zero eigenvalue. Since $\hat{H}_{\text{EF}}$ is $\mathbb{Z}_2$ symmetric, the $\mathbb{Z}_2$ related state $\langle \downarrow | = \langle \uparrow | \prod_{i=1}^L X_i$ is also a left zero mode. So by explicit construction, we have shown that $\hat{H}_{\text{EF}}$ has at least two zero eigenvalues. The left zero mode subspace is spanned by $\langle \uparrow |$ and $\langle \downarrow |$. Using the relation between left- and right-eigenstate $|R_0\rangle \propto (|L_0\rangle |W_1\rangle)^T$, the corresponding right zero mode subspace is spanned by $|W_1\rangle \langle \uparrow |$ and $|W_1\rangle \langle \downarrow |$.

Since we are most interested about the EF of pure states, we should focus on the $\mathbb{Z}_2$ even state in the zero mode subspace. In that regard, the left and right zero modes in the $\mathbb{Z}_2$ even sector are given by

$$\langle L_0 | = \frac{\langle \uparrow | + \langle \downarrow |}{2},$$

$$|R_0 \rangle \propto \frac{\hat{W}_1}{2}$$

$$= \frac{1}{2} \left( \prod_{i=1}^L d(d + X_i) \right) \langle \uparrow | + \langle \downarrow |$$

$$= \frac{1}{2} d^{3L/2} \left( \prod_{i=1}^L (e^\eta + e^{-\eta} X_i) \right) \langle \uparrow | + \langle \downarrow |$$

$$= \frac{1}{2} d^{3L/2} \sum_{\sigma} \left( \prod_{i=1}^L e^{\eta_{\sigma_i}} |\sigma\rangle + \prod_{i=1}^L e^{-\eta_{\sigma_i}} |\sigma\rangle \right)$$

$$= d^{3L/2} \sum_{\sigma} \cosh \left( \eta \sum_{i=1}^L \sigma_i \right) |\sigma\rangle.$$  

(D9)
The normalization of $|R_0\rangle$ is determined by the condition $\langle L_0| R_0\rangle = 1$, such that

$$\langle L_0| R_0\rangle = \sum_\sigma \frac{\cosh(\eta \sum_{i=1}^L \sigma_i)}{\cosh(\eta L)} |\sigma\rangle = |W_{\text{page}}\rangle.$$  \hspace{1cm} (D10)

In summary, we have shown that in the $\mathbb{Z}_2$ even sector, the EF Hamiltonian $\hat{H}_\text{EF}$ has at least one zero eigenvalue $\lambda_0 = 0$, whose left- and right-eigenstates are given by

$$\langle L_0| = \frac{\langle \uparrow | + \langle \downarrow |}{2}, \quad |R_0\rangle = |W_{\text{page}}\rangle,$$  \hspace{1cm} (D11)

as claimed in Eq. (41).

Appendix E: Derivation of the Dispersion Relation for two-domain-wall Ansatz

Here, we show the derivation of Eq. (52). Our goal here is to obtain the analytical expression of excited state energy, namely dispersion relation, $\omega(k)$. Note that the left and right eigenstates are not simply each other’s conjugate transpose due to the non-hermitian nature of EF Hamiltonian (Eq. (32)). For simplicity, we focus on the excitation of left eigenstates and construct the corresponding right eigenstate with $|R\rangle = (|L\rangle |W_1\rangle)^\dagger$. From the discussion in Sec. II F, the universal left ground state for any parameters $g_{ij}$, $\beta$ is the linear combination of all spin-up and spin-down states,

$$\langle L_0| = \frac{\langle \uparrow | + \langle \downarrow |}{2}.$$  \hspace{1cm} (E1)

Based on our ED result in Fig. 5, the low energy left excited state mainly consists of two-domain-wall states. The generic form of two-domain-wall state can be expressed as

$$\langle k| \equiv C_k \sum_{n,a} \langle k_{n,a} | a \rangle \equiv C_k \sum_{n,a} e^{ikn}\hat{\phi}^*(a)\langle \uparrow | \prod_{i=n}^{n+a} X_i$$

where $C_k$ is the normalization constant and $\langle k_{n,a} |$ represents the two-domain-wall state ranging from $n$ to $n + a$.

First, we start from deriving the normalization constant.

$$\langle k|k\rangle = |C_k|^2 \sum_{n,m,a,b} e^{i(kn-m\beta)}\hat{\phi}^*(a)\phi(b)\langle \uparrow | \prod_{i=n}^{n+a} X_i \prod_{j=m}^{m+b} X_j | \uparrow \rangle = 1.$$  \hspace{1cm} (E2)

Next step is to evaluate $\langle \uparrow | \prod_{i=n}^{n+a} X_i \hat{W}_1 \prod_{j=m}^{m+b} X_j | \uparrow \rangle$. The physical meaning is the transition amplitude between two Bethe string states with evolution as $\hat{W}_1$. There are two possibilities for each site. When both Bethe strings have/do not have excitation at site $i$, the answer would be $\langle \uparrow | (\hat{W}_1) \uparrow \rangle = d^2$. When either Bethe string has excitation on site $i$, the result becomes $\langle \uparrow | X_i (\hat{W}_1) \uparrow \rangle = d$. To evaluate this quantity, we perform perturbative expansion as $1/d$ series. To obtain analytical expression of $|C_k|^2$, we also approximate $\phi(a)$ as plane wave $\sim e^{-ika/2}$. Physical intuition is that we assume these domain walls have little interaction with each other and thus they can move through each other almost freely. Consequently, plane wave solution is assumed and $a/2$ represents the center location of domain wall. Let’s start evaluating the normalization constant up to the order of $1/d^2$,

$$\langle \uparrow | \prod_{i=n}^{n+a} X_i \hat{W}_1 \prod_{j=m}^{m+b} X_j | \uparrow \rangle = \delta_{n,m}\delta_{a,b}d^{2N} + (\delta_{n,m}\delta_{a,b+1} + \delta_{a,m}\delta_{a,b-1} + \delta_{n,m+1}\delta_{a,b-1} + \delta_{n,m-1}\delta_{a,b+1})d^{2N-1}$$

$$+ (\delta_{n,m}\delta_{a,b+2} + \delta_{n,m}\delta_{a,b-2} + \delta_{n,m-2}\delta_{a,b+2} + \delta_{n,m+2}\delta_{a,b-2} + \delta_{n,m+1}\delta_{a,b} + \delta_{n,m+1}\delta_{a,b} + \delta_{n,m-1}\delta_{a,b} + \delta_{n,m-1}\delta_{a,b+2})d^{2N-2}$$

$$+ \mathcal{O}(d^{2N-3})$$  \hspace{1cm} (E3)

For $b = 0, 1$, we would have different terms,

$$\langle \uparrow | \prod_{i=n}^{n+a} X_i \hat{W}_1 X_m | \uparrow \rangle = \delta_{n,m}\delta_{a,0}d^{2N} + (\delta_{n,m}\delta_{a,1} + \delta_{n,m-1}\delta_{a,1})d^{2N-1}$$

$$+ (\delta_{n,m}\delta_{a,2} + \delta_{n,m-2}\delta_{a,2} + \delta_{n,m+1}\delta_{a,0} + \delta_{n,m-1}\delta_{a,0} + \delta_{n,m-1}\delta_{a,2})d^{2N-2} + \mathcal{O}(d^{2N-3})$$  \hspace{1cm} (E4)

\textbf{For} $b = 0, 1$, \textbf{we would have different terms,}
\begin{align}
\langle \uparrow | \prod_{i=n}^{n+a} X_i \hat{W}_i X_{m+1} \uparrow \rangle &= \delta_{n,m} \delta_{n,1} d^{2N} + (\delta_{n,m} \delta_{a,2} + \delta_{n,m} \delta_{a,0} + \delta_{n,m+1} \delta_{a,0} + \delta_{n,m-1} \delta_{a,2}) d^{2N-1} \\
&+ (\delta_{n,m} \delta_{a,3} + \delta_{n,m-2} \delta_{a,3} + \delta_{n,m+1} \delta_{a,1} + \delta_{n,m-1} \delta_{a,1} + \delta_{n,m-1} \delta_{a,3}) d^{2N-2} + \mathcal{O}(d^{2N-3})
\end{align}

(E6)

Put them back to Eq. (E3) and we can obtain

\begin{align}
|C_k|^2 d^{2N} N^2 \left( \frac{N - 2}{N} \left[ 1 + \frac{4}{d} \cos \frac{k}{2} + \frac{1}{d^2} (2 + 6 \cos k) \right] + \frac{1}{N} \left[ 1 + \frac{2}{d} \cos \frac{k}{2} + \frac{1}{d^2} (1 + 4 \cos k) \right] \\
+ \frac{1}{N} \left[ 1 + \frac{4}{d} \cos \frac{k}{2} + \frac{1}{d^2} (1 + 4 \cos k) \right] + \mathcal{O}(d^{-3}) \right) = 1
\end{align}

(E7)

To simplify the whole calculation, the thermodynamics limit is taken \( N \to \infty \). The main effect of thermodynamics limit is that the contribution from short two domain wall states (e.g. single-site or two-site excitations) is fully suppressed. Thus, up to \( \mathcal{O}(\frac{1}{d^2}) \), we have

\begin{align}
|C_k|^2 d^{2N} N^2 [1 + \frac{4}{d} \cos \frac{k}{2} + \frac{1}{d^2} (2 + 6 \cos k)] = 1
\end{align}

(E8)

Now, we are ready to evaluate the energy expectation value of our two-domain-wall state, \( \langle k | \hat{H}_{EF} | k \rangle \). For simplicity, we assume \( g_{ij} = 1, \beta_{ij} = \beta \) and reorganize the EF Hamiltonian

\begin{align}
\hat{H}_{EF} &= \sum_{i,j} \frac{1}{2} - Z_i Z_j e^{-\beta X_i X_j - \delta(X_i + X_j)} \\
&= \sum_{i,j} \frac{1}{2} - Z_i Z_j \frac{d^2}{d^2 - 1} [\cosh \beta - \sinh \beta X_i X_j] \left[ 1 - \frac{1}{d} (X_i + X_j) + \frac{1}{d^2} X_i X_j \right] \\
&= \sum_{i,j} \frac{1}{2} - Z_i Z_j \frac{d^2}{d^2 - 1} [\cosh \beta - \sinh \beta \frac{d}{d^2} - \frac{1}{d} (\cosh \beta - \sinh \beta) (X_i + X_j) + \frac{\cosh \beta - \sinh \beta}{d^2} X_i X_j] \\
&= \sum_{i,j} \frac{1}{2} - Z_i Z_{i+1} \left[ a(\beta, d) + b(\beta, d) (X_i + X_{i+1}) + c(\beta, d) X_i X_{i+1} \right]
\end{align}

(E9)

where

\begin{align}
a(\beta, d) &= \frac{d^2}{d^2 - 1} (\cosh \beta - \frac{\sinh \beta}{d^2}) = \cosh \beta + \frac{\cosh \beta - \sinh \beta}{d^2} + \mathcal{O}(\frac{1}{d^4}) \\
b(\beta, d) &= - \frac{d}{d^2 - 1} (\cosh \beta - \sinh \beta) = - \frac{1}{d} (\cosh \beta - \sinh \beta) + \mathcal{O}(\frac{1}{d^2}) \\
c(\beta, d) &= \frac{d^2}{d^2 - 1} (\frac{\cosh \beta}{d^2} - \sinh \beta) = \frac{\cosh \beta - \sinh \beta}{d^2} - \sinh \beta + \mathcal{O}(\frac{1}{d^4}).
\end{align}

(E10)

The first term in \( \langle k | \hat{H}_{EF} | k \rangle \) is

\begin{align}
|C_k|^2 a(\beta, d) \sum_{n,m,a,b} e^{i(k(n-m))} e^{i(k(a-b))/2} \langle \uparrow | \prod_{i=n}^{n+a} X_i \sum_{l=1}^{m+b} W_l \prod_{j=m}^{m+b} X_j \uparrow \rangle \\
= 2a(\beta, d) |C_k|^2 \sum_{n,m,a,b} e^{i(k(n-m))} e^{i(k(a-b))/2} \langle \uparrow | \prod_{i=n}^{n+a} X_i \hat{W}_l \prod_{j=m}^{m+b} X_j \uparrow \rangle = 2a(\beta, d)
\end{align}

(E11)

As for the second term, since \( b(\beta, d) \) already contains \( 1/d \) power, we just compute the terms up to \( 1/d \) order and the result is

\begin{align}
|C_k|^2 b(\beta, d) \sum_{n,m,a,b} e^{i(k(n-m))} e^{i(k(a-b))/2} \langle \uparrow | \prod_{i=n}^{n+a} X_i \sum_{l=1}^{m+b} W_l \prod_{j=m}^{m+b} X_j \uparrow \rangle \\
= |C_k|^2 b(\beta, d) \sum_{n,m,a,b} e^{i(k(n-m))} e^{i(k(a-b))/2} \\
\times [h(n-1, m, a+1, b) + h(n+1, m, a-1, b) + h(n, m, a+1, b) + h(n, m, a-1, b)]
\end{align}

(E12)
where
\[ h(n, m, a, b) = \langle \hat{\eta} | \prod_{i=n}^{n+a} X_i \hat{W}_1 \prod_{j=m}^{m+b} X_j | \hat{\eta} \rangle. \] (E13)

For each \( h(n, m, a, b) \), the boundary terms would have different result. For example, the results of \( h(n - 1, m, a + 1, b) \) are as follows
\[
\begin{align*}
&h(n - 1, m, a + 1, 0) = (\delta_{n, m+1}\delta_{a, 0} + \delta_{n, m}\delta_{a, 0})d^{2N-1} + \mathcal{O}(d^{2N-2}) \\
&h(n - 1, m, a + 1, 1) = (\delta_{n, m+1}\delta_{a, 0}d^{2N} + (\delta_{n, m+1}\delta_{a, 1} + \delta_{n, m}\delta_{a, 1})d^{2N-1} + \mathcal{O}(d^{2N-2}) \\
&h(n - 1, m, a + 1, 2) = (\delta_{n, m+1}\delta_{a, 0}d^{2N} + (\delta_{n, m+1}\delta_{a, 2} + \delta_{n, m+1}\delta_{a, 0} + \delta_{n, m}\delta_{a, 0} + \delta_{n, m}\delta_{a, 2})d^{2N-1} + \mathcal{O}(d^{2N-2}) \\
&h(n - 1, m, a + 1, b \neq 0, 1, 2) = (\delta_{n, m+1}\delta_{a, b-1}d^{2N} + (\delta_{n, m+1}\delta_{a, b} + \delta_{n, m+1}\delta_{a, b-2} + \delta_{n, m+2}\delta_{a, b-2} + \delta_{n, m}\delta_{a, b})d^{2N-1} + \mathcal{O}(d^{2N-2}).
\end{align*}
\] (E14)

Since the thermodynamics limit would be taken \((N \to \infty)\), the "boundary effect" from short two-domain-wall states would be suppressed. Consequently, we only keep the last term in our calculation. Combine these four terms and compute the sum with thermodynamic limit,
\[
|C_k|^2 b(\beta, d)d^{2N}N^2[4\cos k/2 + \frac{8}{d}(1 + \cos k)] + \mathcal{O}(\frac{1}{d^2}) = b(\beta, d)[4\cos k/2 - \frac{16}{d}\cos^2 k/2 + \frac{4}{d}(2 + 2\cos k)] + \mathcal{O}(\frac{1}{d^2}) \] (E15)

For the third term, the following quantity is computed
\[
|C_k|^2 c(\beta, d) \sum_{n, m, a, b} e^{ik(n-m)}e^{i(k-a)b/2}(\hat{\eta} | \prod_{i=n}^{n+a} X_i \hat{W}_1 \prod_{j=m}^{m+b} X_j | \hat{\eta} \rangle. \] (E16)

The EF Hamiltonian would give extra \( X_iX_{i+1} \) term. In most two-domain-wall states (length > 1), the two-domain-wall structure would destroyed. However, for single site excitation, this \( X_iX_{i+1} \) term would only shift the position of excitation with one site. Due to the suppression of thermodynamic limit, we would also drop this term. Eventually, we can have
\[
c(\beta, d)[\frac{4}{d}\cos k/2 + \frac{1}{d^2}(8 + 8\cos k)](1 - \frac{4}{d}\cos k/2) + \mathcal{O}(\frac{1}{d^3}) = c(\beta, d)[\frac{4}{d}\cos k/2 + \frac{1}{d^2}(8 + 8\cos k) - \frac{16}{d^2}\cos^2 k/2] + \mathcal{O}(\frac{1}{d^3}). \] (E17)

Combining Eq. (E11), Eq. (E15) and Eq. (E17) and keeping terms up to \( \mathcal{O}(\frac{1}{d^3}) \), \langle k|H_{EF}|k \rangle would be
\[
\langle k|H_{EF}|k \rangle = 2[\cosh \beta + \frac{\cosh \beta - \sinh \beta}{d^2} - \frac{1}{d}(\cosh \beta - \sinh \beta)][4\cos k/2 - \frac{16}{d}\cos^2 k/2 + \frac{4}{d}(2 + 2\cos k)] - \sinh \beta[\frac{4}{d}\cos k/2 + \frac{1}{d^2}(8 + 8\cos k) - \frac{16}{d^2}\cos^2 k/2] + \mathcal{O}(\frac{1}{d^3}) \] (E18)

**Appendix F: Derivation of the Dispersion Relation for Single-Site Excitation ansatz**

This appendix is similar with the calculation in Appendix E. The only difference is the ansatz state we use. The single-site excitation ansatz is defined as
\[
\langle k | = C_k \langle \hat{\eta} | \sum_{n} X_n e^{i kn} | k \rangle = \hat{W}_1 \sum_{n} X_n e^{-ikn} | \hat{\eta} \rangle. \] (F1)

First, we start from the normalization condition \( \langle k | k \rangle = 1 \),
\[
\langle k | k \rangle = 1 = C_k \langle \hat{\eta} | \sum_{n, m} e^{i(k-n)m}X_n \hat{W}_1 X_m | \hat{\eta} \rangle = C_k[N d^{2(N-1)}(d^2 - 1) + N^2 \delta_{k,0} d^{2(N-1)}]. \] (F2)

Following the expression in Eq. (F3),
\[
\hat{H}_{EF} = \sum_i \frac{1 - Z_i Z_{i+1}}{2}[a(\beta, d) + b(\beta, d)(X_i + X_{i+1}) + c(\beta, d)X_i X_{i+1}] \] (F3)
where
\[ a(\beta, d) = \frac{d^2}{d^2 - 1} (\cosh \beta - \sinh \beta \frac{\beta}{d^2}), b(\beta, d) = -\frac{d}{d^2 - 1} (\cosh \beta - \sinh \beta), c(\beta, d) = \frac{d^2}{d^2 - 1} (\cosh \beta - \sinh \beta). \]  

(F4)

The first term is
\[ a(\beta, d) \langle k | \sum_l l \frac{1}{2} Z_l Z_{l+1} W_l | k \rangle = a(\beta, d) C_k \sum_{m,n} e^{i(k-m)} \langle n | \sum_l l \frac{1}{2} Z_l Z_{l+1} W_l | m \rangle \]
\[ = a(\beta, d) C_k \sum_{m,n} e^{i(k-m)} \langle n | Z_l \delta_{l,n-1} + \delta_{l,n} \rangle W_l | m \rangle \]
\[ = 2a(\beta, d) C_k \sum_{m,n} e^{i(k-m)} \langle n | W_l \rangle | m \rangle \]
\[ = 2a(\beta, d) C_k \sum_{m,n} e^{i(k-m)} [\delta_{n,m} d^2 + (1 - \delta_{n,m}) d^2 (N-1)] \]
\[ = 2a(\beta, d) C_k [N d^2 (N-1) (d^2 - 1) + N^2 \delta_{k,0} d^2 (N-1)] = 2a(\beta, d). \]  

(F5)

The second term is
\[ b(\beta, d) \langle k | \sum_l l \frac{1}{2} Z_l Z_{l+1} (X_l + X_{l+1}) W_l | k \rangle = b(\beta, d) C_k \sum_{m,n} e^{i(k-m)} \langle n | \sum_l l \frac{1}{2} Z_l Z_{l+1} (X_l + X_{l+1}) W_l | m \rangle \]
\[ = b(\beta, d) C_k \sum_{m,n} e^{i(k-m)} \langle n | (X_{n-1} + 2X_n + X_{n+1}) W_l | m \rangle \]
\[ = b(\beta, d) C_k \sum_{m,n} e^{i(k-m)} [\langle n, n - 1 | W_l | m \rangle + \langle n, n + 1 | W_l | m \rangle + 2 \langle \mathbb{1} | W_l | m \rangle] \]
\[ = b(\beta, d) C_k \sum_{m,n} e^{i(k-m)} [d^{2N-3}(1 - \delta_{m,n})(1 - \delta_{m,n-1}) + d^{2N-3}(1 - \delta_{m,n})(1 - \delta_{m,n+1}) + d^{2N-3}(\delta_{m,n} + \delta_{m,n-1}) + d^{2N-3}(\delta_{m,n} + \delta_{m,n+1}) + 2d^{2N-1}] \]
\[ = b(\beta, d) C_k \sum_{m,n} e^{i(k-m)} [2d^{2N-3} + 2d^{2N-1} + (d^{2N-1} - d^{2N-3})(\delta_{m,n-1} + 2\delta_{m,n} + \delta_{m,n+1})] \]
\[ = b(\beta, d) C_k [N^2 \delta_{k,0} d^{2N-3}(d^2 + 1) + 2d^{2N-3}(d^2 - 1) N (1 + \cos k)]. \]

For single-site excitation, we focus on the region where \( k \neq 0 \). The result would be
\[ 2b(\beta, d) \frac{1 + \cos k}{d} \times C_k [N d^2 (N-2) (d^2 - 1)] = \frac{2b(\beta, d)}{d} (1 + \cos k). \]  

(F7)

The third term is
\[ c(\beta, d) \langle k | \sum_l l \frac{1}{2} Z_l Z_{l+1} X_l X_{l+1} W_l | k \rangle = c(\beta, d) C_k \sum_{m,n} e^{i(k-m)} \langle n | \sum_l l \frac{1}{2} Z_l Z_{l+1} X_l X_{l+1} W_l | m \rangle \]
\[ = c(\beta, d) C_k \sum_{m,n} e^{i(k-m)} \langle n | Z_l \delta_{l,n-1} + \delta_{l,n} X_l X_{l+1} W_l | m \rangle \]
\[ = c(\beta, d) C_k \sum_{m,n} e^{i(k-m)} [(\delta_{l,n-1} + \delta_{l,n}) X_l X_{l+1} W_l | m \rangle \]
\[ = c(\beta, d) C_k \sum_{m,n} e^{i(k-m)} [(\delta_{l,n-1} + \delta_{l,n}) X_l X_{l+1} W_l \rangle | m \rangle \]
\[ = c(\beta, d) C_k \sum_{m,n} e^{i(k-m)} [2d^{2N-1} + (\delta_{m,n-1} + \delta_{m,n+1}) (d^{2N} - d^{2N-1})] \]
\[ = 2c(\beta, d) C_k [N d^2 (N-1) (d^2 - 1) \cos k + N^2 d^{2N} (N-1) \delta_{k,0}] = 2c(\beta, d) \cos k \]

The overall result would be
\[ \langle k | \hat{H}_{E,F} | k \rangle = 2a(\beta, d) + \frac{2b(\beta, d)}{d} (1 + \cos k) + 2c(\beta, d) \cos k = 2a(\beta, d) + \frac{2b(\beta, d)}{d} + \cos k [2c(\beta, d) + \frac{2b(\beta, d)}{d}] \]
\[ = \frac{2d^2}{d^2 - 1} (\cosh \beta - \frac{\sinh \beta}{d^2}) - \frac{2}{d^2 - 1} (\cosh \beta - \sinh \beta) + \cos k \left[ \frac{2d^2}{d^2 - 1} (\cosh \beta - \sinh \beta) - \frac{2}{d^2 - 1} (\cosh \beta - \sinh \beta) \right]. \]  

(F9)
Appendix G: Diagrammatic Expansion of Entanglement Feature Hamiltonian

In this appendix, we derive the EF Hamiltonian for the locally scrambled Hamiltonian dynamics. We start from the definition of the EF for $e^{-i\epsilon H}$ following Eq. (5),

$$W_{e^{-i\epsilon H}}[\sigma, \tau] = \text{Tr}(\mathcal{A}'(e^{-i\epsilon H}) \otimes \mathcal{A}'(e^{i\epsilon H})^\otimes) = \text{Tr}(\mathcal{A}'e^{-i\epsilon\mathcal{H}}\mathcal{A}'e^{i\epsilon\mathcal{H}}),$$

(G1)

where we have introduced $\mathcal{H} = H \otimes \mathbb{1} + \mathbb{1} \otimes H$ to denote the double Hamiltonian. Given the locality of $H = \sum_x H_x$, the double Hamiltonian $\mathcal{H}$ is also a sum of local terms $\mathcal{H} = \sum_x \mathcal{H}_x$ with $\mathcal{H}_x = H_x \otimes \mathbb{1} + \mathbb{1} \otimes H_x$ being the doubled version of $H_x$. Expand around $\epsilon \to 0$ to the order of $\epsilon^2$, we obtain

$$W_{e^{-i\epsilon H}}[\sigma, \tau] = \text{Tr}(\mathcal{A}'\mathcal{X}_\tau) - \epsilon^2 \text{Tr}(\mathcal{A}'\mathcal{X}_\tau\mathcal{H}^2 - \mathcal{X}_\tau\mathcal{H}\mathcal{X}_\tau\mathcal{H}) + \mathcal{O}(\epsilon^4),$$

$$= W_1[\sigma, \tau] - \epsilon^2 \sum_{x,x'} \text{Tr}(\mathcal{X}_\tau\mathcal{X}_\tau^2 - \mathcal{X}_\tau\mathcal{H}\mathcal{X}_\tau) + \mathcal{O}(\epsilon^4),$$

(G2)

where the first order term in $\epsilon$ vanishes by the cyclic identity of trace, confirming the argument in Sec. II E that $W_U(\epsilon)$ will be even in $\epsilon$. The last equality in Eq. (G2) relies on the fact that $\text{Tr}(\mathcal{X}_\tau\mathcal{X}_\tau^2) = \text{Tr}(\mathcal{X}_\tau\mathcal{X}_\tau\mathcal{H}\mathcal{X}_\tau) = 0$ as long as $x \neq x'$. To prove this, we first consider the case when $x = (ij)$ and $x' = (kl)$ do not overlap,

$$\text{Tr} \mathcal{A}' \mathbb{H}_{(ij)} \mathcal{X}_\tau \mathbb{H}_{(kl)} = \text{Tr} \mathcal{A}' \mathbb{H}_{(ij)} \left( \mathcal{X}_\tau \mathcal{X}_\tau \mathcal{X}_\tau \mathcal{X}_\tau \right) \mathbb{H}_{(kl)}$$

$$= \text{Tr} \mathcal{A}' \mathcal{X}_\tau \mathcal{X}_\tau \mathcal{X}_\tau \mathcal{X}_\tau \mathbb{H}_{(ij)} \mathbb{H}_{(kl)}$$

$$= \text{Tr} \mathcal{A}' \mathcal{X}_\tau \mathcal{X}_\tau \mathcal{X}_\tau \mathcal{X}_\tau \mathbb{H}_{(ij)} \mathbb{H}_{(kl)}$$

$$(G3)$$

where we have used the fact that $[\mathbb{H}_{(ij)}, \mathcal{X}_\tau \mathcal{X}_\tau] = 0$ for $i, j \neq k, l$, and $[\mathcal{X}_\sigma, \mathcal{X}_\tau] = 0$ for any $i, j$ as the $S_2$ group is Abelian. We then consider the case when $x = (ij)$ and $x' = (jk)$ overlaps on a single site $j$,

$$\text{Tr} \mathcal{A}' \mathbb{H}_{(ij)} \mathcal{X}_\tau \mathbb{H}_{(jk)} = \text{Tr} \mathcal{A}' \mathbb{H}_{(ij)} \left( \mathcal{X}_\tau \mathcal{X}_\tau \mathcal{X}_\tau \mathcal{X}_\tau \right) \mathbb{H}_{(jk)}$$

$$= \text{Tr} \mathcal{A}' \mathcal{X}_\tau \mathcal{X}_\tau \mathcal{X}_\tau \mathcal{X}_\tau \mathbb{H}_{(ij)} \mathbb{H}_{(jk)}$$

$$(G4)$$

At this point, it seems that $\mathcal{X}_\tau$ is caught between $\mathbb{H}_{(ij)}$ and $\mathbb{H}_{(jk)}$. The solution is to make use of the property that $\mathbb{H}_{(jk)} = \mathcal{X}_\alpha \mathcal{X}_\alpha \mathbb{H}_{(jk)} \mathcal{X}_\alpha \mathcal{X}_\alpha$, for any $\alpha_j = \alpha_k \in S_2$, due to the permutation symmetry to exchange the two replicas of the double Hamiltonian. Now we choose $\alpha_j = \alpha_k = \tau$, such that $\mathcal{X}_\tau \mathcal{X}_\alpha^{-1} = 1$, then

$$\text{Tr} \mathcal{A}' \mathbb{H}_{(ij)} \mathcal{X}_\tau \mathbb{H}_{(jk)} = \text{Tr} \mathcal{A}' \mathcal{X}_\tau \mathcal{X}_\tau \mathcal{X}_\tau \mathcal{X}_\tau \mathbb{H}_{(ij)} \mathbb{H}_{(jk)}$$

$$= \text{Tr} \mathcal{A}' \mathcal{X}_\tau \mathcal{X}_\tau \mathcal{X}_\tau \mathcal{X}_\tau \mathbb{H}_{(ij)} \mathbb{H}_{(jk)}$$

$$(G5)$$
Hence we have shown that \( \text{Tr} \, \mathcal{H}_n \mathcal{H}_{(ij)} \mathcal{H}_{(kl)} = \text{Tr} \, \mathcal{X}_0 \mathcal{X}_0 \mathcal{H}_{(ij)} \mathcal{H}_{(kl)} \) as long as \( (ij) \neq (kl) \), meaning that \( \text{Tr}(\mathcal{X}_0 \mathcal{X}_0 \mathcal{H}_{(ij)} \mathcal{H}_{(kl)}) = \delta_{ij} \) \( \text{Tr}(\mathcal{X}_0 \mathcal{X}_0 \mathcal{H}_{(ij)} ^2 - \mathcal{X}_0 \mathcal{H}_{(ij)} \mathcal{X}_0 \mathcal{H}_{(kl)} \). Thus the derivation of Eq. (G2) is justified.

If we consider the difference between \( W_{e^{-i\mathcal{H}} \mathcal{I}} \) and \( W_\mathcal{I} \), denoted as \( \delta W \),

\[
\delta W[\sigma, \tau] \equiv W_{e^{-i\mathcal{H}}} [\sigma, \tau] - W_\mathcal{I} [\sigma, \tau] = -\epsilon^2 \sum_x \text{Tr}(\mathcal{X}_0 \mathcal{X}_0 \mathcal{H}_{x}^2 - \mathcal{X}_0 \mathcal{H}_{x} \mathcal{X}_0 \mathcal{H}_{x}) + O(\epsilon^4). \tag{G6}
\]

\( \delta W[\sigma, \tau] \equiv \sum_x \delta W_x[\sigma, \tau] \) can be expressed as a sum of terms on each bond \( x \) (at least to the order of \( \epsilon^2 \)). To carry out the \( \epsilon \) expansion more systematically, we choose to focus on a single bond, and define the EF difference

\[
\delta W_x[\sigma, \tau] \equiv W_{e^{-i\mathcal{H}_x}} [\sigma, \tau] - W_\mathcal{I} [\sigma, \tau] = \text{Tr}(\mathcal{X}_0 \mathcal{X}_0 \mathcal{H}_{x}^2 - \mathcal{X}_0 \mathcal{H}_{x} \mathcal{X}_0 \mathcal{H}_{x}) - \text{Tr}(\mathcal{X}_0 \mathcal{X}_0 \mathcal{H}_{x}), \tag{G7}
\]

where \( \sigma = (\sigma_i, \sigma_j) \) is restricted to the two sites \( i, j \) connected by the bond \( x \) and similarly for \( \tau \). \( \delta W_x[\sigma, \tau] = 0 \) vanishes as long as \( \sigma_i = \sigma_j \) or \( \tau_i = \tau_j \), because in that case, \( \mathcal{X}_0 \) or \( \mathcal{X}_0 \) will commute with \( \mathcal{H}_{x} \) and hence the two traces will cancel with each other. Therefore there are only two independent non-trivial components of \( \delta W_x[\sigma, \tau] \), which we denote as \( u \) and \( v \):

\[
u = \delta W_x[\sigma, \tau] = \delta W_x[\sigma, \tau], \tag{G8}
\]

So we only need to focus on these terms and perform the \( \epsilon \) expansion following the definition

\[
u = \epsilon^2 \left( -2\epsilon^2 + (2\epsilon^2\mathcal{I} + 2d\mathcal{I}) - \frac{1}{\pi} (4\epsilon^2\mathcal{I}) \right)
+ \epsilon^4 \left( \epsilon^2 + \frac{1}{2} \left( 2\epsilon^2\mathcal{I} - 4\epsilon^2\mathcal{I} - 4\epsilon^2\mathcal{I} + \frac{1}{(2\epsilon^2\mathcal{I})} \left( 2\epsilon^2\mathcal{I} + 2\epsilon^2\mathcal{I} + 2\epsilon^2\mathcal{I} \right) \right)
+ \frac{1}{(2\epsilon^2\mathcal{I})} \left( 4\epsilon^2\mathcal{I} - 4\epsilon^2\mathcal{I} - 4\epsilon^2\mathcal{I} + \frac{1}{(2\epsilon^2\mathcal{I})} \left( 4\epsilon^2\mathcal{I} + 4\epsilon^2\mathcal{I} + 4\epsilon^2\mathcal{I} \right) \right) \right) \right) + O(\epsilon^6)
\]

\[
u = \epsilon^2 \left( 0 + \frac{1}{\pi} \left( 2\epsilon^2\mathcal{I} - 4\epsilon^2\mathcal{I} + 4\epsilon^2\mathcal{I} + \frac{1}{(2\epsilon^2\mathcal{I})} \left( 2\epsilon^2\mathcal{I} + 2\epsilon^2\mathcal{I} + 2\epsilon^2\mathcal{I} \right) \right)
+ \frac{1}{(2\epsilon^2\mathcal{I})} \left( 4\epsilon^2\mathcal{I} - 4\epsilon^2\mathcal{I} - 4\epsilon^2\mathcal{I} + \frac{1}{(2\epsilon^2\mathcal{I})} \left( 4\epsilon^2\mathcal{I} + 4\epsilon^2\mathcal{I} + 4\epsilon^2\mathcal{I} \right) \right) \right) \right) + O(\epsilon^6)
\]

In the diagrams, each small red block represents a copy of the bond Hamiltonian \( H_x \). Their legs are contracted according to the assignment of the permutations \( \sigma \) and \( \tau \). The result can be expressed in terms of the generalized spectral form factor \( R^{(ij)}_{g_i} \), labeled by two permutations \( g_i, g_j \in S_n \) acting separately on sites \( i \) and \( j \),

\[
R^{(ij)}_{g_i} = \text{Tr}(\mathcal{H}_n \mathcal{H}_{g_i} g_j). \tag{G11}
\]

where \( \mathcal{X}_{g_i, g_j} = \mathcal{X}_{g_i} \mathcal{X}_{g_j} \) is the representation of \( g_i \) and \( g_j \) in the \( n \)-replicated Hilbert space. For example, \( R^{(ij)}_{(ij)} = (\text{Tr} H_{ij}^2) \), \( R^{(ij)}_{(ij)(kl)} = \text{Tr}(\text{Tr} H_{ij}^2) \) (where \( \text{Tr}_i \) denotes the partial trace over site \( i \)), and \( R^{(ij)}_{(kl)} = \text{Tr}(H_k^2) \).

Given the components \( u \) and \( v \), we can rewrite \( \delta W_x[\sigma, \tau] \) in the operator form

\[
\delta \tilde{W}_x = \frac{1}{2} (1 - \frac{Z_i Z_j}{2}) (u + v X_i X_j) \frac{1}{2} (1 - \frac{Z_i Z_j}{2}), \tag{G12}
\]

therefore the EF operator reads

\[
\tilde{W}_{e^{-i\mathcal{H}}} = \tilde{W}_I + \sum_x \delta \tilde{W}_x \otimes \tilde{W}_{1_x} + \sum_{ij} \frac{1}{2} (1 - \frac{Z_i Z_j}{2}) (u + v X_i X_j) \frac{1}{2} (1 - \frac{Z_i Z_j}{2}) \otimes \tilde{W}_{1_i, 1_j}, \tag{G13}
\]
The EF Hamiltonian is therefore given by

\[
\hat{H}_\text{EF} = \frac{1}{c^2} (1 - W e^{-i n \tilde{W}^{-1}})
\]

\[
= -\frac{1}{c^2} \sum_{ij} \frac{1}{2} \left( 1 - Z_i Z_j \right) (u + v X_i X_j) \frac{1 - Z_i Z_j}{2} \tilde{W}_{ij}
\]

\[
= -\frac{1}{c^2} \sum_{ij} \frac{1}{2} \left( 1 - Z_i Z_j \right) (u + v X_i X_j) \frac{1 - Z_i Z_j}{2} \frac{1}{d^2 (d^2 - 1)} e^{-\delta(X_i, X_j)}
\]

\[
= -\sum_{ij} \frac{1}{2} \frac{1}{c^2d^2 (d^2 - 1)} e^{-\delta(X_i, X_j)}
\]

Therefore the EF Hamiltonian generally take the form of

\[
\hat{H}_\text{EF} = \sum_{ij} g \frac{1}{2} \frac{1}{c^2d^2 (d^2 - 1)} e^{-\beta X_i X_j} \delta(X_i, X_j),
\]

consistent with the general form in Eq. (32). Comparing Eq. (G14) with Eq. (G15), we should identify

\[
ge^{-\beta X_i X_j} = \frac{u + v X_i X_j}{c^2d^2 (d^2 - 1)},
\]

which indicates

\[
g \cosh \beta = -\frac{u}{c^2d^2 (d^2 - 1)} = \frac{1}{d^2 (d^2 - 1)} (u_2 - u_4 \epsilon^2 + O(\epsilon^4))
\]

\[
g \sinh \beta = \frac{v}{c^2d^2 (d^2 - 1)} = \frac{1}{d^2 (d^2 - 1)} (v_4 \epsilon^2 + O(\epsilon^4)),
\]

where the coefficients \(u_2, u_4, v_4\) are defined in terms of generalized spectral form factors \(R^{ij}_{2i}\) as

\[
\begin{align*}
 u_2 &= 2 R^{(1)(2)}_{(1)(2)} - 2 d (R^{(12)}_{(12)} - R^{(12)}_{(12)}) + 2 d^2 R^{(12)}_{(12)} \\
 u_4 &= R^{(13)(24)}_{(12)(34)} + 2 R^{(12)(3)}_{(12)(4)} - R^{(123)(34)}_{(12)(34)} - R^{(123)(4)}_{(12)(34)} + 1 = \frac{1}{2} \left( R^{(12)(34)}_{(12)(34)} + d R^{(12)(34)}_{(12)(34)} + d R^{(12)(34)}_{(12)(34)} \right) \\
 v_4 &= R^{(1432)}_{(12)(34)} - 2 R^{(12)(34)}_{(12)(34)} + R^{(12)(34)}_{(12)(34)}
\end{align*}
\]

For specific model of \(H_{ij}\), we can evaluate the generalized spectral form factors, then we can determined the parameters \(g\) and \(\beta\) as well as the EF Hamiltonian. In the following, we will perform the calculation for random \(U(d)\) spin model and the locally scrambled Ising model.

For two-qudit GUE Hamiltonians, the generalized spectral form factors, defined in Eq. (G11), can be evaluated under the GUE average using the basic property that

\[
\mathbb{E}_{\text{GUE}} \left[ H_{ij}^{\otimes^2} \right] = \mathbb{E}_{\text{GUE}} \left[ \frac{1}{d^2} \left( h_i h_j \right) \right] = \frac{1}{d^2} \mathcal{X}_{(12).i.(12)}
\]

the GUE average of \(n\)-replicated Hamiltonian \(H_{ij}\) can be obtained by summing over Wick contractions

\[
\mathbb{E}_{\text{GUE}} \left[ H_{ij}^{\otimes^n} \right] = \left\{ \begin{array}{ll}
 d^{-n} \sum_{h_i=h_j \in P_n} \mathcal{X}_{h_i h_j} & n \text{ even,} \\
 0 & n \text{ odd,}
\end{array} \right.
\]

where \(P_n\) denotes all possible pair-wise exchange of \(n\) replicas. Then the generalized spectral form factor reads

\[
\mathbb{E}_{\text{GUE}} \left[ R^{ij}_{2i} \right] = \frac{1}{d^n} \sum_{h \in P_n} \text{Tr}(\mathcal{X}_{g_i} \mathcal{X}_h) \text{Tr}(\mathcal{X}_{g_j} \mathcal{X}_h),
\]
whose results are enumerated in Tab. I. Substitute these results to Eq. (G18), we find $u_2 = 2(d^2 - 1)^2$, $u_4 = \frac{11}{2}(d^2 - 1)^2$, and $v_4 = 2(d^2 - 1)^2/d^2$. By solving Eq. (G17), we can determine the parameters $g$ and $\beta$ to the order of $\epsilon^2$:

$$g = 2(1 - d^{-2})(1 - \frac{11}{12} \epsilon^2 + O(\epsilon^4)),$$

$$\beta = \epsilon^2/d^2 + O(\epsilon^4).$$

(G22)

In conclusion, as we consider the locally scrambled quantum dynamics by alternatively applying the small unitary $e^{-itH}$ and the local scramblers, the evolution of the corresponding EF state will be governed by $\partial_t |W_{\Psi_t}\rangle = -\hat{H}_{EF} |W_{\Psi_t}\rangle$, with the EF Hamiltonian $\hat{H}_{EF}$ given by Eq. (G15). The random U($d$) spin model $H$ in Eq. (74) corresponds to the set of parameters in Eq. (G22) for $\hat{H}_{EF}$.

### TABLE I. Spectral form factor of two-qudit GUE Hamiltonian

| $R_{1(2)}^{(1)(2)}$ | $d$ | $R_{1(12)}^{(1)(2)}$ | $d$ | $R_{1(12)}^{(1)(2)}$ | $d$ |
|-------------------|-----|---------------------|-----|---------------------|-----|
| $R_{1(123)(4)}^{(1)(2)}$ | $d^2 + 2$ | $R_{1(123)(4)}^{(123)(4)}$ | $3$ | $R_{1(123)(4)}^{(123)(4)}$ | $2d + \frac{1}{3}$ |
| $R_{1(123)(4)}^{(123)(4)}$ | $d^2 + 2$ | $R_{1(123)(4)}^{(123)(4)}$ | $2d + 1$ | $R_{1(123)(4)}^{(123)(4)}$ | $2d^2 + 1$ |

| $R_{1(12)(34)}^{(1)(2)}$ | $d^4 + 2$ | $R_{1(12)(34)}^{(1)(2)}$ | $d^4 + 2$ | $R_{1(12)(34)}^{(1)(2)}$ | $d^4 + 2$ |
|-------------------|-----|---------------------|-----|---------------------|-----|
| $R_{1(12)(34)}^{(12)(34)}$ | $d^4 + 2$ | $R_{1(12)(34)}^{(1)(2)}$ | $d^4 + 2$ | $R_{1(12)(34)}^{(1)(2)}$ | $d^4 + 2$ |