Concentration Bounds for Co-occurrence Matrices of Markov Chains

Jiezhong Qiu
Tsinghua University
qiujz16@mails.tsinghua.edu.cn

Chi Wang
Microsoft Research, Redmond
wang.chi@microsoft.com

Ben Liao
Tencent
bliao@tencent.com

Richard Peng
Georgia Tech
rpeng@cc.gatech.edu

Jie Tang
Tsinghua University
jietang@tsinghua.edu.cn

August 7, 2020

Abstract

Co-occurrence statistics for sequential data are common and important data signals in machine learning, which provide rich correlation and clustering information about the underlying object space. We give the first bound on the convergence rate of estimating the co-occurrence matrix of a regular (aperiodic and irreducible) finite Markov chain from a single random walk trajectory. Our work is motivated by the analysis of a well-known graph learning algorithm DeepWalk by [Qiu et al. WSDM ’18], who study the convergence (in probability) of co-occurrence matrix from random walk on undirected graphs in the limit, but left the convergence rate an open problem.

We prove a Chernoff-type bound for sums of matrix-valued random variables sampled via an ergodic Markov chain, generalizing the regular undirected graph case studied by [Garg et al. STOC ’18]. Using the Chernoff-type bound, we show that given a regular Markov chain with \( n \) states and mixing time \( \tau \), we need a trajectory of length \( O(\tau \log n + \log \tau / \epsilon^2) \) to achieve an estimator of the co-occurrence matrix with error bound \( \epsilon \). We conduct several experiments and the experimental results are consistent with the exponentially fast convergence rate from theoretical analysis. Our result gives the first sample complexity analysis in graph representation learning.

1 Introduction

Co-occurrence statistics are common and important data signals in machine learning. They provide rich correlation and clustering information about the underlying object space, such as the word co-occurrence in natural language processing [26,28,22,29], vertex co-occurrence in graph learning [30,35,14,15,9,31], item co-occurrence in recommendation system [35,24,3], action co-occurrence in reinforcement learning [38], and emission co-occurrence of hidden Markov models [17]. Given a sequence of objects, the co-occurrence statistics are computed by moving a sliding window of fixed size \( T \) over the sequence and recording the frequency of objects’ co-occurrence within the sliding window. A pseudocode of the above procedure is listed in Algorithm 1, which produces an \( n \) by \( n \) co-occurrence matrix where \( n \) is the size of the underlying object space.

A common assumption when building such co-occurrence matrices is that the sequential data are long enough to provide an accurate estimation. For instance, Mikolov et al. [27] use word sequences
Algorithm 1: The computation of the co-occurrence matrix.

1. **Input** window size $T$; sequence $(v_1, \cdots, v_L)$ such that each $v_i \in [n]$;
2. **Output** co-occurrence matrix $C$
3. $C \leftarrow 0_{n \times n}$
4. for $i = 1, 2, \ldots, L - T$ do
5.   for $r = 1, \ldots, T$ do
6.     $C_{v_i, v_i+r} \leftarrow C_{v_i, v_i+r} + 1/T$
7.     $C_{v_i+r, v_i} \leftarrow C_{v_i+r, v_i} + 1/T$
8. $C \leftarrow \frac{1}{2(L-T)} C$
9. Return $C$

from a news article dataset with one billion words in their Skip-gram model; Tennenholtz and Mannor [33] train their Act2vec model with action sequences from over a million StarCraft II game replays, which are equivalent to 100 years of consecutive gameplay; Perozzi et al. [30] samples large amounts of random walk sequences from graphs to capture the vertex co-occurrence. A recent work by Qiu et al. [31] studies the convergence of co-occurrence matrices of random walk on undirected graphs in the limit (i.e., when the length of random walk goes to infinity), but left the convergence rate an open problem. It remains unknown whether the co-occurrence statistics are sample efficient, and how efficient they are.

In this paper, we study the situation where the sequential data are sampled from a regular finite Markov chain (i.e., an aperiodic and irreducible finite Markov chain), and derive bounds on the sample efficiency of co-occurrence matrix estimation. We prove a Chernoff-type bound for sums of matrix-valued random variables sampled via an ergodic Markov chain, generalizing the undirected regular graph case studied by Garg et al. [12]. We then utilize this matrix Chernoff bound to characterize the convergence rate of co-occurrence matrices.

**Organisation** The rest of the paper is organized as follows. We discuss our contributions formally in Section 1.1 and review previous work in Section 1.2. In Section 2 we provide preliminaries, followed by the proof of matrix Chernoff bound in Section 3 and the proof of convergence rate of co-occurrence matrices in Section 4. In Section 5 we conduct experiments on both synthetic and real-world datasets. Finally, we conclude this work in Section 6.

1.1 **Our Contributions**

Our paper revolves around providing upper bounds on the sample complexity for estimating co-occurrence matrices, specifically on the length of the trajectory needed in the sampling algorithm shown in Algorithm 1. To give a formal statement, we first translate Algorithm 1 to linear algebra language. Given a trajectory $(v_1, \cdots, v_L)$ from state space $[n]$ and step weight coefficients $(\alpha_1, \cdots, \alpha_T)$, the co-occurrence matrix is defined to be

$$C \triangleq \frac{1}{L - T} \sum_{i=1}^{L-T} C_i, \text{ where } C_i \triangleq \sum_{r=1}^{T} \frac{\alpha_r}{2} \left( e_{v_i} e_{v_i+r}^\top + e_{v_i+r} e_{v_i}^\top \right).$$

1Please note that regular Markov chains are Markov chains which are aperiodic and irreducible, while an undirected regular graph is an undirected graph where each vertex has the same number of neighbors. In this work, the term “regular” may have different meanings depending on the context.
Here $C_i$ accounts for the co-occurrence within sliding window $(v_i, \cdots, v_{i+T})$, and $e_{vi}$ is a length-$n$ vector with a one in its $v_i$-th entry and zeros elsewhere. Thus $e_v^i e^T_{v_{i+r}}$ is a $n$ by $n$ matrix with its $(v_i, v_{i+r})$-th entry to be one and other entries to be zero, which records the co-occurrence of $v_i$ and $v_{i+r}$. Note that Algorithm 1 is a special case when step weight coefficients are uniform, i.e., $\alpha_r = 1/T, r \in [T]$, and the co-occurrence statistics in all the applications mentioned above can be formalized in this way. When trajectory $(v_1, \cdots, v_L)$ is a random walk from a regular Markov chain $P$ with stationary distribution $\pi$, the asymptotic expectation of the co-occurrence matrix within sliding window $(v_i, \cdots, v_{i+L})$ (i.e., $C_i$) is:

$$AE[C_i] \triangleq \lim_{T \to \infty} E(C_i) = \frac{1}{L-T} \sum_{i=1}^{L-T} E(C_i) = \sum_{r=1}^{T} \frac{\alpha_r}{2} \left( \Pi P^r + (\Pi P^r)^T \right),$$

where $\Pi \triangleq \text{diag}(\pi)$. Thus the asymptotic expectation of the co-occurrence matrix (i.e., $C$) is

$$AE[C] \triangleq \lim_{L \to \infty} E[C] = \lim_{L \to \infty} \frac{1}{L-T} \sum_{i=1}^{L-T} E(C_i) = \sum_{r=1}^{T} \frac{\alpha_r}{2} \left( \Pi P^r + (\Pi P^r)^T \right). \quad (1)$$

Our main result regarding the estimation of the co-occurrence matrix is the following convergence bound related to the length of the walk sampled.

**Theorem 1** (Convergence Rate of Co-occurrence Matrices). Let $P$ be a regular Markov chain with state space $[n]$, stationary distribution $\pi$ and mixing time $\tau$. Let $(v_1, \cdots, v_L)$ denote a $L$-step random walk on $P$ starting from a distribution $\phi$ on $[n]$. Given step weight coefficients $(\alpha_1, \cdots, \alpha_T)$ s.t. $\sum_{r=1}^{T} \alpha_r = 1$, and $\epsilon \in (0, 1)$, the probability that the co-occurrence matrix $C$ deviates from its asymptotic expectation $AE[C]$ (in 2-norm) is bounded by:

$$P[\|C - AE[C]\|_2 \geq \epsilon] \leq 2(\tau + T) \|\phi\|_\pi n^2 \exp \left( \frac{-c^2(L - T)}{576(\tau + T)} \right).$$

Specially, there exists $L = O\left((\tau + T)(\log n + \log \tau)/\epsilon^2 + T\right)$ such that $P[\|C - AE[C]\|_2 \geq \epsilon] \leq \frac{1}{n^{\Omega(1)}}$.

Assuming $T = O(1)$ gives $L = O\left(\tau(\log n + \log \tau)/\epsilon^2\right)$.

In the above theorem, $\|\cdot\|_\pi$ is the $\pi$-norm (which we define formally later in Section 2) measuring the distance between the initial distribution $\phi$ and the stationary distribution $\pi$. We arrive at this result by proving a concentration bound for matrix-valued quantities generated from such walks. The formal statement of our concentration bound is:

**Theorem 2** (A Real-Valued Matrix Chernoff Bound for Ergodic Markov Chains). Let $P$ be an ergodic Markov chain with state space $[N]$, stationary distribution $\pi$ and spectral norm $\lambda$. Let $f : [N] \to \mathbb{R}^{d \times d}$ be a function such that (1) $\forall v \in [N], f(v)$ is symmetric and $\|f(v)\|_2 \leq 1$; (2) $\sum_{v \in [N]} \pi_v f(v) = 0$. Let $(v_1, \cdots, v_k)$ denote a $k$-step random walk on $P$ starting from a distribution $\phi$ on $[N]$. Then given $\epsilon \in (0, 1)$,

$$P\left[\lambda_{\text{max}} \left( \frac{1}{k} \sum_{j=1}^{k} f(v_j) \right) \geq \epsilon \right] \leq \|\phi\|_\pi d^2 \exp \left( -\epsilon^2(1 - \lambda)k/72 \right)$$

$$P\left[\lambda_{\text{min}} \left( \frac{1}{k} \sum_{j=1}^{k} f(v_j) \right) \leq -\epsilon \right] \leq \|\phi\|_\pi d^2 \exp \left( -\epsilon^2(1 - \lambda)k/72 \right).$$
Our results in Theorem 1 gives the first sample complexity analysis for many graph representation learning algorithms. Given a graph, these algorithms aim to learn a function from the vertex space to a low dimensional vector space. Most of the graph representation learning algorithms (e.g., DeepWalk [30], node2vec [14], metapath2vec [9], GraphSAGE [15]) consist of two steps. The first step is to draw random sequences from a stochastic process defined on the graph and then count co-occurrence statistics from the sampled sequences, where the stochastic process is usually defined to be first-order or higher-order random walk on the graph. The second step is to train a model to fit the co-occurrence statistics. For example, DeepWalk can be viewed as factorizing a point-wise mutual information matrix [22, 31] which is a transformation of the co-occurrence matrix; GraphSAGE fits the co-occurrence statistics with a graph neural network [19]. These models usually assume that the number of samples is sufficiently large so that the co-occurrence statistics are accurately estimated.

We are the first work to study the sample complexity of the aforementioned algorithms. Theorem 1 implies that these algorithms need \( O(\tau (\log n + \log \tau)/\epsilon^2) \) samples to achieve a good estimator of the co-occurrence matrix.

1.2 Previous Work

Our work contributes to the literature of Chernoff-type bounds. The Chernoff Bound [7] is one of the most important probabilistic results in computer science, which gives exponentially decreasing bounds on tail distributions of sums of independent scalar-valued random variables. Later then, a series of works [18, 13, 23, 20, 41, 16, 8, 32, 40] relax the independent assumption to Markov dependency. In particular, these works suppose a Markov chain and a bounded function \( f \) on its state space, and study the tail distribution of \( \frac{1}{k} \sum_{j=1}^{k} f(v_j) \) where \( (v_1, \ldots, v_k) \) is a random walk sampled from the Markov chain. Another series of works [33, 1, 39] relax scalar-valued random variables to matrix-valued random variables. In particular, they characterize the tail distributions of the largest eigenvalue of a sum of independent random matrices. Recently, Garg et al. [12] proves Chernoff-type bound for matrix-valued random variables sampled via random walks on undirected regular graphs. Our work (Theorem 2) extends the undirected regular graph condition in [12] to general ergodic Markov chains. We want to emphasize that random walk on regular undirected graphs only covers a very small subset of general Markov chains, i.e., Markov chains that are reversible and have uniform stationary distribution. Our generalization allows for studying more practical problems such as random walk on social and information networks which are usually directed and have skewed stationary distributions [2].

Our work is also related to the research about random-walk matrix polynomial sparsification when the Markov chain \( P \) is a random walk on an undirected graph. In this case, we can rewrite \( P = D^{-1}A \) where \( D \) and \( A \) is the degree matrix and adjacency matrix of an undirected graph with \( n \) vertices and \( m \) edges, and the expected co-occurrence matrix in Equation 1 can be simplified as \[ \mathbb{A}E[C] = \frac{1}{\text{vol}(G)} \sum_{r=1}^{T} \alpha_r D(D^{-1}A)^r \] which is known as random-walk matrix polynomials [5, 6]. Cheng et al. [6] propose an algorithm which needs \( O(T^2 m \log n/\epsilon^2) \) steps of random walk to construct an \( \epsilon \)-approximator for the random-walk matrix polynomials. Our bound in Equation 3 is stronger than the bound proposed by Cheng et al. [6] when the Markov chain \( P \) mixes fast. Moreover, Cheng et al. [6] require \( \alpha_r \) to be non-negative, while our bound can handle negative step weight coefficients.

\[ \text{volume of a graph } G \text{ is defined to be } \text{vol}(G) \triangleq \sum_i \sum_j A_{ij}. \]
2 Preliminaries

In this paper, we denote $P$ to be a finite Markov chain on $n$ states. $P$ could refer to either the chain itself or the corresponding transition probability matrix — an $n$ by $n$ matrix such that its entry $P_{ij}$ indicates the probability that state $i$ moves to state $j$. A Markov chain is called an ergodic Markov chain if it is possible to eventually get from every state to every other state with positive probability. A Markov chain is regular if some power of its transition matrix has all strictly positive entries. A regular Markov chain must be an ergodic Markov chain, but not vice versa. An ergodic Markov chain has unique stationary distribution, i.e., there exists a unique probability vector $\pi$ such that $\pi^\top = \pi^\top P$. For convenience, we denote $\Pi \triangleq \text{diag}(\pi)$.

The time that a regular Markov chain needs to be “close” to its stationary distribution is called mixing time. Let $x$ and $y$ be two probability vectors. The total variation distance between them is $\|x - y\|_{TV} \triangleq \frac{1}{2} \|x - y\|_1$. For $\delta > 0$, the $\delta$-mixing time of regular Markov chain $P$ is $\tau(P) \triangleq \min \{ t : \max_{x} \| (x^\top P^t)^\top - \pi \|_{TV} \leq \delta \}$, where $x$ is an arbitrary probability vector.

The stationary distribution $\pi$ also defines a inner product space where the inner product (under $\pi$-kernel) is defined as $\langle x, y \rangle_\pi \triangleq y^\top \Pi^{-1} x$ for $\forall x, y \in \mathbb{C}^N$, where $y^*$ is the conjugate transpose of $y$. A naturally defined norm based on the above inner product is $\|x\|_\pi \triangleq \sqrt{\langle x, x \rangle_\pi}$. Then we can define the spectral norm $\lambda(P)$ of an ergodic Markov chain $P$ \cite{25, 11, 8} as $\lambda(P) \triangleq \max_{\langle x, \pi \rangle_\pi = 0, x \neq 0} \frac{\| (x^\top P)^* \|_x}{\|x\|_\pi}$.

The spectral norm $\lambda(P)$ is known to be a measure of mixing time of a Markov chain. The smaller $\lambda(P)$ is, the faster a Markov chain converges to its stationary distribution \cite{22}. If $P$ is reversible, $\lambda(P)$ is simply the second absolute eigenvalue of $P$ (the largest is always 1). The irreversible case is more complicated, since $P$ may have complex eigenvalues. In this case, $\lambda(P)$ is actually the square root of the second largest absolute eigenvalue of the multiplicative reversibilization of $P$ \cite{11}. When $P$ is clear from the context, we will simply write $\tau$ and $\lambda$ for $\tau(P)$ and $\lambda(P)$, respectively. We shall also refer $1 - \lambda(P)$ as the spectral gap of $P$.

3 Matrix Chernoff Bounds for Ergodic Markov Chains

This section provides a brief overview of our proof of the real-valued matrix Chernoff bound for ergodic Markov chains as stated in Theorem 2 above, as well as a general complex-valued version in Theorem 4. Due to space constraints, we defer the full proof to Section B in the supplementary material and instead present a sketch here. By symmetry, we only discuss on bounding $\lambda_{\text{max}}$ here. Using the exponential method, the probability in Theorem 2 can be upper bounded for any $t > 0$ by:

$$\mathbb{P} \left[ \lambda_{\text{max}} \left( \frac{1}{k} \sum_{j=1}^{k} f(v_j) \right) \geq \epsilon \right] \leq \mathbb{P} \left[ \text{Tr} \left[ \exp \left( t \sum_{j=1}^{k} f(v_j) \right) \right] \geq \exp (t \epsilon) \right] \leq \mathbb{E} \left[ \frac{\text{Tr} \left[ \exp \left( t \sum_{j=1}^{k} f(v_j) \right) \right]}{\exp (t \epsilon)} \right]$$

where the first inequality follows by the tail bounds for eigenvalues (See Proposition 3.2.1 in Tropp \cite{39}) which controls the tail probabilities of the extreme eigenvalues of a random matrix by producing a bound for the trace of the matrix moment generating function, and the second inequality follows by Markov’s inequality. The RHS of the above equation is the expected trace of the exponential of a sum of matrices (i.e., $tf(v_j)$’s). When $f$ is a scalar-valued function, we can easily write exponential

\footnote{Please note that we need the Markov chain to be regular to make the mixing-time well-defined. For a general ergodic Markov chain which could be periodic, the mixing time may be ill-defined.}
of a sum to be product of exponentials (since \( \exp(a + b) = \exp(a) \exp(b) \) for scalars). However, this is not true for matrices. To bound the expectation term, we invoke the following multi-matrix Golden-Thompson inequality from [12], by letting \( H_j = tf(v_j), j \in [k] \).

**Theorem 3** (Multi-matrix Golden-Thompson Inequality, Theorem 1.5 in [12]). Let \( H_1, \cdots H_k \) be \( k \) Hermitian matrices, then for some probability distribution \( \mu \) on \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \):

\[
\log \left( \text{Tr} \left[ \exp \left( \sum_{j=1}^{k} H_j \right) \right] \right) \leq \frac{4}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log \left( \text{Tr} \left[ \prod_{j=1}^{k} \exp \left( \frac{e^{i\phi}}{2} H_j \right) \prod_{j=k}^{1} \exp \left( e^{-i\phi} H_j \right) \right] \right) d\mu(\phi).
\]

The key point of this theorem is to relate the exponential of a sum of matrices to a product of matrix exponentials and their adjoints, whose trace can be further bounded via the following lemma by letting \( e^{i\phi} = \gamma + ib \).

**Lemma 1** (Analogous to Lemma 4.3 in [12]). Let \( P \) be an ergodic Markov chain with state space \( [N] \) with spectral norm \( \lambda \). Let \( f \) be a function \( f : [N] \to \mathbb{R}^{d \times d} \) such that (1) \( \sum_{v \in [N]} \pi_v f(v) = 0 \); (2) \( \|f(v)\|_2 \leq 1 \) and \( f(v) \) is symmetric, \( v \in [N] \). Let \( (v_1, \cdots, v_k) \) denote a \( k \)-step random walk on \( P \) starting from a distribution \( \phi \) on \( [N] \). Then for any \( t > 0, \gamma \geq 0, b > 0 \) such that \( t^2(\gamma^2 + b^2) \leq 1 \) and \( t\sqrt{\gamma^2 + b^2} \leq \frac{1-\lambda}{4\lambda} \), we have

\[
\mathbb{E} \left[ \text{Tr} \left[ \prod_{j=1}^{k} \exp \left( \frac{tf(v_j)(\gamma + ib)}{2} \right) \prod_{j=k}^{1} \exp \left( \frac{tf(v_j)(\gamma - ib)}{2} \right) \right] \right] \leq \|\phi\|_\pi \|d\| \exp \left( kt^2(\gamma^2 + b^2) \left( 1 + \frac{8}{1-\lambda} \right) \right).
\]

Proving Lemma 1 is the technical core of our paper. The main idea is to write the expected trace expression in LHS of Lemma 1 in terms of the transition probability matrix \( P \), which allows for a recursive analysis to track how much the expected trace expression changes as a function of \( k \). The analysis relies on incorporating the concentration of matrix-valued functions from [12] into the study of ergodic Markov chains from [3], which was originally for scalars. Key to this extension is the definition of an inner product related to the stationary distribution \( \pi \) of \( P \), and a spectral norm from such inner products. In contrast, the undirected regular graph case studied in [12] can be handled using the standard inner products, as well as the second largest eigenvalues of \( P \) instead of spectral norms. Detailed proofs of Theorem 2 and Lemma 1 can be found in Section B.2 and Section B.3 of the supplementary material, respectively.

### 3.1 A Complex-Valued Matrix Chernoff Bound

In this section, we show that Theorem 2 can be generalized to complex-valued matrices.

**Theorem 4** (A Complex-Valued Matrix Chernoff Bound for Ergodic Markov Chains). Let \( P \) be an ergodic Markov chain with state space \( [N] \), stationary distribution \( \pi \) and spectral norm \( \lambda \). Let \( f : [N] \to \mathbb{C}^{d \times d} \) be a function such that (1) \( \forall v \in [N], f(v) \) is Hermitian and \( \|f(v)\|_2 \leq 1 \); (2) \( \sum_{v \in [N]} \pi_v f(v) = 0 \). Let \( (v_1, \cdots, v_k) \) denote a \( k \)-step random walk on \( P \) starting from a distribution \( \phi \) on \( [N] \). Then given \( \epsilon \in (0,1), \)

\[
\mathbb{P} \left[ \lambda_{\max} \left( \frac{1}{k} \sum_{j=1}^{k} f(v_j) \right) \geq \epsilon \right] \leq 4 \|\phi\|_\pi d^2 \exp \left( -\epsilon^2 (1 - \lambda) k / 72 \right)
\]

\[
\mathbb{P} \left[ \lambda_{\min} \left( \frac{1}{k} \sum_{j=1}^{k} f(v_j) \right) \leq -\epsilon \right] \leq 4 \|\phi\|_\pi d^2 \exp \left( -\epsilon^2 (1 - \lambda) k / 72 \right).
\]
Proof. Our strategy is to adopt complexification technique \cite{10}. For any $d \times d$ complex Hermitian matrix $X$, we may write $X = Y + iZ$, where $Y$ and $iZ$ are the real and imaginary parts of $X$, respectively. Moreover, the Hermitian property of $X$ (i.e., $X^* = X$) implies that (1) $Y$ is real and symmetric (i.e., $Y^T = Y$); (2) $Z$ is real and skew symmetric (i.e., $Z = -Z^T$). The eigenvalues of $X$ can be found via a $2d \times 2d$ real symmetric matrix $H \triangleq \begin{bmatrix} Y & Z \\ -Z & Y \end{bmatrix}$, where the symmetry of $H$ follows by the symmetry of $Y$ and skew-symmetry of $Z$. Note the fact that, if the eigenvalues (real) of $X$ are $\lambda_1, \lambda_2, \ldots, \lambda_d$, then those of $H$ are $\lambda_1, \lambda_1, \lambda_2, \ldots, \lambda_d, \lambda_d$. I.e., $X$ and $H$ have the same eigenvalues, but with different multiplicity.

Using the above technique, we can formally prove Theorem 4. For the complex matrix function $f : [N] \to \mathbb{C}^{d \times d}$ in Theorem 4, we can separate its real and imaginary parts by $f(v) = f_1(v) + i f_2(v), \forall v \in [N]$. Then we construct a real matrix function $g : [N] \to \mathbb{R}^{2d \times 2d}$ such that $\forall v \in [N], g(v) = \begin{bmatrix} f_1(v) & f_2(v) \\ -f_2(v) & f_1(v) \end{bmatrix}$. According to the complexification technique, we know that (1) $\forall v \in [N], g(v)$ is real symmetric and $\|g(v)\|_2 = \|f(v)\|_2 \leq 1$; (2) $\sum_{v \in [N]} \pi_v g(v) = 0$. Then

$$
\mathbb{P} \left[ \lambda_{\max} \left( \frac{1}{k} \sum_{j=1}^{k} f(v_j) \right) \geq \epsilon \right] = \mathbb{P} \left[ \lambda_{\max} \left( \frac{1}{k} \sum_{j=1}^{k} g(v_j) \right) \geq \epsilon \right] \leq 4 \|\phi\|_\pi d^2 \exp \left(-c^2(1-\lambda)k/72\right),
$$

where the first step follows by the fact that $\frac{1}{k} \sum_{j=1}^{k} f(v_j)$ and $\frac{1}{k} \sum_{j=1}^{k} g(v_j)$ have the same eigenvalues (with different multiplicity), and the second step follows by Theorem 2\footnote{The additional factor 4 is because the constructed $g(v)$ has shape $2d \times 2d.$}. The bound on $\lambda_{\min}$ also follows similarly. \qed

## 4 Convergence Rate of Co-occurrence Matrices

In this section, we apply the matrix Chernoff bound for ergodic Markov chains from Theorem 2 to obtain our main result on the convergence of co-occurrence matrix estimation, as stated in Theorem 1. Informally, this result states that if the mixing time of $P$ is $\tau$, then the length of a trajectory needed to guaranteed an additive error (in 2-norm) of $\epsilon$ is roughly $O((\tau + T)(\log n + \log \tau)/\epsilon^2 + T)$, where $T$ is the co-occurrence window size. However, we cannot directly apply the matrix Chernoff bound because the co-occurrence matrix is not a sum of matrix-valued functions sampled from the original Markov chain $P$. The main difficulty is to construct the proper Markov chain and matrix-valued function as desired by Theorem 2. We formally give our proof as follows:

Proof. (of Theorem 1) Our proof has three main steps: the first two construct a Markov chain $Q$ according to $P$, as well as a corresponding matrix-valued function $f$ such that the sums of matrix-valued random variables sampled via $Q$ is exactly the error matrix $C = A \mathbb{E}[C]$. Then we invoke Theorem 2 to the constructed Markov chain $Q$ and matrix-valued function $f$ to bound the convergence rate. We give details on the three steps below.

**Step One** Given a random walk $(v_1, \ldots, v_L)$ on Markov chain $P$, we construct a sequence $(X_1, \ldots, X_{L-T})$ where $X_i \triangleq (v_i, v_{i+1}, \ldots, v_{i+T})$, i.e., each $X_i$ is a size-$T$ sliding window over $(v_1, \ldots, v_L)$. Meanwhile, let $S$ be the set of all $T$-step walks on Markov chain $P$, we define a new Markov chain $Q$ on $S$ such that $\forall (u_0, \ldots, u_T), (w_0, \ldots, w_T) \in S$:

$$
Q(u_0, \ldots, u_T), (w_0, \ldots, w_T) \triangleq \begin{cases} P_{u_T, w_T} & \text{if } (u_1, \ldots, u_T) = (w_0, \ldots, w_{T-1}); \\
0 & \text{otherwise.}
\end{cases}
$$
The following claim characterizes the properties of Markov chain $Q$. Its proof is deferred to Section A of the supplementary material.

Claim 1 (Properties of $Q$). If $P$ is a regular Markov chain, then $Q$ satisfies:

1. $Q$ is a regular Markov chain, with stationary distribution $\sigma$ such that $\sigma(u_0, \ldots, u_T) = \pi_{u_0} P_{u_0, u_1} \cdots P_{u_{T-1}, u_T}$;
2. The sequence $(X_1, \ldots, X_{L-T})$ is a random walk on $Q$ starting from a distribution $\rho$ such that $\rho(u_0, \ldots, u_T) = \phi_{u_0} P_{u_0, u_1} \cdots P_{u_{T-1}, u_T}$, and $\|\rho\|_{\sigma} = \|\phi\|_{\pi}$.
3. $Q$ is symmetric and $\|\|_{\sigma} = \|\phi\|_{\pi}$.

Proof.

Claim 2. The proof of Claim 2 is deferred to Section A of the supplementary material.

Step Two Defining a matrix-valued function $f : S \to \mathbb{R}^{n \times n}$ such that $\forall X = (u_0, \ldots, u_T) \in S$:

$$ f(X) \triangleq \frac{1}{2} \left( \sum_{r=1}^{T} \frac{\alpha_r}{2} \left( e_{u_r} e_{u_r}^T + e_{u_r} e_{u_0}^T \right) - \sum_{r=1}^{T} \frac{\alpha_r}{2} \left( \Pi P^r + (\Pi P^r)^T \right) \right). \tag{2} $$

With this definition of $f(X)$, the difference between the co-occurrence matrix $C$ and its Asymptotic expectation $\mathbb{E}[C]$ can be written as: $C - \mathbb{E}[C] = 2(\frac{1}{L-T} \sum_{i=1}^{L-T} f(X_i))$. We can further show the following properties of this function $f$:

Claim 2 (Properties of $f$). The function $f$ in Equation 2 satisfies (1) $\sum_{X \in S} \sigma_X f(X) = 0$; (2) $f(X)$ is symmetric and $\|f(X)\|_2 \leq 1, \forall X \in S$.

This claim verifies that $f$ in Equation 2 satisfies the two conditions of matrix-valued function in Theorem 2. The proof of Claim 2 is deferred to Section A of the supplementary material.

Step Three The construction in step two reveals the fact that the error matrix $C - \mathbb{E}[C]$ can be written as the average of matrix-valued random variables (i.e., $f(X_i)$’s), which are sampled via an ergodic Markov chain $Q$ (Recall that $Q$ is a regular Markov chain, and the fact that a regular Markov chain must be ergodic). This encourages us to directly apply Theorem 2. However, note that (1) the error probability in Theorem 2 contains a factor of spectral gap $(1 - \lambda)$; and (2) Part 4 of Claim 1 allows for the existence of a Markov chain $P$ with $\lambda(P) < 1$ while the induced Markov chain $Q$ has $\lambda(Q) = 1$. So we cannot directly apply Theorem 2 to $Q$. To address this issue, we utilize the following tighter bound on sub-chains.

Claim 3. (Claim 3.1 in Chung et al. [8]) Let $Q$ be an ergodic Markov chain with $\delta$-mixing time $\tau(Q)$, then $\lambda(Q^{\tau(Q)}) \leq \sqrt{2\delta}$. In particular, setting $\delta = \frac{1}{2}$ implies $\lambda(Q^{\tau(Q)}) \leq \frac{1}{2}$.

The above claim reveals the fact that, even though $Q$ could have zero spectral gap (Part 4 of Claim 1), we can bound the spectral norm of $Q^{\tau(Q)}$. We partition $(X_1, \ldots, X_{L-T})$ into $\tau(Q)$ groups\footnote{Consider random walk on the unweighted undirected graph $\overline{s}$ and $T = 2$. In this example, $\lambda(P) = 2/3$ but $\lambda(Q) = 1$. Detailed computation can be found in the supplementary material.} such that the $i$-th group consists of a sub-chain $(X_i, X_{i+\tau(Q)}, X_{i+2\tau(Q)}, \ldots)$ of length

\footnote{Without loss of generality, we assume $L - T$ is a multiple of $\tau(Q)$.}
The sub-chain can be viewed as generated from a Markov chain \( Q^{r(Q)} \). Apply Theorem 2 to the \( i \)-th sub-chain, whose starting distribution is \( \rho_i \triangleq (Q^T)^{i-1} \rho \), we have

\[
\Pr \left[ \lambda_{\max} \left( \frac{1}{k} \sum_{j=1}^{k} f(X_{i+(j-1)\tau(Q)}) \right) \geq \epsilon \right] \leq \| \rho_i \|_{\sigma} n^2 \exp \left( -\epsilon^2 \left( 1 - \lambda \left( Q^{r(Q)} \right) \right) k/72 \right)
\]

\[
\leq \| \rho_i \|_{\sigma} n^2 \exp \left( -\epsilon^2 k/144 \right) \leq \| \phi \|_{\pi} n^2 \exp \left( -\epsilon^2 k/144 \right),
\]

where that last step follows by \( \| \rho_i \|_{\sigma} \leq \| \rho_{i-1} \|_{\sigma} \leq \cdots \| \rho_1 \|_{\sigma} = \| \rho \|_{\sigma} \) and \( \| \rho \|_{\sigma} = \| \phi \|_{\pi} \) (Part 2 of Claim 1). Together with a union bound across each sub-chain, we can obtain:

\[
\Pr \left[ \lambda_{\max} \left( C - \lambda \| C \|_{2} \right) \geq \epsilon \right] = \Pr \left[ \lambda_{\max} \left( \frac{1}{L - T} \sum_{j=1}^{L-T} f(X_j) \right) \geq \frac{\epsilon}{2} \right]
\]

\[
= \Pr \left[ \lambda_{\max} \left( \frac{1}{\tau(Q)} \sum_{i=1}^{\tau(Q)} \frac{1}{k} \sum_{j=1}^{k} f(X_{i+(j-1)\tau(Q)}) \right) \geq \frac{\epsilon}{2} \right]
\]

\[
\leq \sum_{i=1}^{\tau(Q)} \Pr \left[ \lambda_{\max} \left( \frac{1}{k} \sum_{j=1}^{k} f(X_{i+(j-1)N}) \right) \geq \frac{\epsilon}{2} \right] \leq \tau(Q) \| \phi \|_{\pi} n^2 \exp \left( -\epsilon^2 k/576 \right).
\]

The bound on \( \lambda_{\min} \) also follows similarly. As \( C - \lambda \| C \|_{2} \) is a real symmetric matrix, its 2-norm is its maximum absolute eigenvalue. Therefore, we can use the eigenvalue bound to bound the overall error probability:

\[
\Pr \left[ \| C - \lambda \| C \|_{2} \geq \epsilon \right] = \Pr \left[ \lambda_{\max} \left( C - \lambda \| C \|_{2} \right) \geq \epsilon \lor \lambda_{\min} \left( C - \lambda \| C \|_{2} \right) \leq -\epsilon \right]
\]

\[
\leq 2 \tau(Q) n^2 \| \phi \|_{\pi} \exp \left( -\epsilon^2 k/576 \right)
\]

\[
\leq 2 \left( \tau(P) + T \right) \| \phi \|_{\pi} n^2 \exp \left( -\epsilon^2 (L - T) / 576 (\tau(P) + T) \right)
\]

where the first inequality follows by union bound, and the second inequality is due to \( \tau(Q) < \tau(P) + T \) (Part 3 of Claim 1).

This bound implies that the probability that \( C \) deviates from \( \mathbb{E}[C] \) could be arbitrarily small by increasing the sampled trajectory length \( L \). Specially, if we want the event \( \| C - \mathbb{E}[C] \|_{2} \geq \epsilon \) happens with probability smaller than \( 1/n^{O(1)} \), we need

\[
L = O \left( \left( \tau(P) + T \right) \left( \log n + \log \tau(P) \right) / \epsilon^2 + T \right).
\]

If we assume \( T = O(1) \), we can achieve \( L = O \left( \tau(P) \left( \log n + \log \tau(P) \right) / \epsilon^2 \right) \).

5 Experiments

In this section, we show experiments to illustrate the exponentially fast convergence rate of estimating co-occurrence matrices of regular Markov chains. We conduct experiments on three synthetic Markov chains (Barbell graph, winning streak chain, and random graph) and one real-world Markov chain (BlogCatalog). For each Markov chain and each trajectory length \( L \) from the set \( \{10, 10^2, \cdots, 10^7\} \), we measure the approximation error of the co-occurrence matrix \( C \) constructed by Algorithm 1 from a \( L \)-step random walk sampled from the chain. We performed 64 trials for each experiment, and the results are aggregated as an error-bar plot. We set \( T = 2 \) and \( \alpha_r \) to be uniform
unless otherwise mentioned. The relationship between trajectory length $L$ and approximation error is shown in Figure 1 (in log-log scale). Across all the four datasets, the observed exponentially fast convergence rates match what our bounds predict in Theorem 1. Below we discuss our observations for each of these datasets.

![Figure 1](image)

**Figure 1:** The convergence rate of co-occurrence matrices on Barbell graph, winning streak chain, BlogCatalog graph, and random graph (in log-log scale). The $x$-axis is the trajectory length $L$ and the $y$-axis is the approximation error $\|C - E[C]\|_2$. Each experiment contains 64 trials, and the error bar is presented.

**Barbell Graphs** [34] The Barbell graph is an undirected graph with two cliques connected by a single path. Such graphs' mixing times vary greatly: two cliques with size $k$ connected by a single edge have mixing time $\Theta(k^2)$; and two size-$k$ cliques connected by a length-$k$ path have mixing time about $\Theta(k^3)$. We evaluate the convergence rate of co-occurrence matrices on the two graphs mentioned above, each with 100 vertices. According to our bound that require $L = O(\tau(\log n + \log \tau)/\epsilon^2)$, we shall expect the approximate co-occurrence matrix to converge faster when the path bridging the two cliques is shorter. The experimental results are shown in Figure 1a, and indeed display faster convergences when the path is shorter (since we fix $n = 100$, a Barbell graph with clique size 50 has a shorter path connecting the two cliques than the one with clique size 33).

**Winning Streak Chains (Section 4.6 of [21])** A winning streak Markov chain has state space $[n]$, and can be viewed as tracking the number of consecutive ‘tails’ in a sequence of coin flips. Each state transits back to state 1 with probability 0.5, and the next state with probability 0.5. The $\delta$-mixing time of this chain satisfies $\tau \leq \lfloor \log_2(1/\delta) \rfloor$, and is independent of $n$. This prompted us to choose this chain, as we should expect similar rates of convergence for different values of $n$ according to our bound of $L = O(\tau(\log n + \log \tau)/\epsilon^2)$. In our experiment, we compare between $n = 50$ and $n = 100$ and illustrate the results in Figure 1b. As we can see, for each trajectory length $L$, the approximation errors of $n = 50$ and $n = 100$ are indeed very close.

**BlogCatalog Graph** [37] is widely used to benchmark graph representation learning algorithms [30, 14, 31]. It is an undirected graph of social relationships of online bloggers with 10,312 vertices and 333,983 edges. The random walk on the BlogCatalog graph has spectral norm $\lambda \approx 0.57$. Following Levin and Peres [21], we can upper bound its $\frac{1}{\delta}$-mixing time by $\tau \leq 36$. We choose $T$ from $\{2, 4, 8\}$

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7 The $\delta$-mixing time of a reversible and irreducible Markov chain is upper bounded by $\tau \leq \log \left( \frac{1}{\delta \tau_{\min}} \frac{1}{1-\tau} \right)$ where
and illustrate the results in Figure 1c. The convergence rate is robust to different values of $T$. Moreover, the variance in BlogCatalog is much smaller than that in other datasets.

**Random Graph** The small variance observed on BlogCatalog leads us to hypothesize that it shares some traits with random graphs. To gather further evidence for this, we estimate the co-occurrence matrices of an Erdős–Rényi random graph for comparison. Specifically, we take a random graph on 100 vertices where each undirected edge is added independently with probability 0.1, aka. $G(100, 0.1)$. The results Figure 1d show very similar behaviors compared to the BlogCatalog graph: small variance and robust convergence rates.

6 Conclusion and Future Work

In this paper, we analyze the convergence rate of estimating the co-occurrence matrix of a regular Markov chain. The main technical contribution of our work is to prove a Chernoff-type bound for sums of matrix-valued random variables sampled via an ergodic Markov chain, and we show that the problem of estimating co-occurrence matrices is a non-trivial application of the Chernoff-type bound. Our results show that, given a regular Markov chain with $n$ states and mixing time $\tau$, we need a trajectory of length $O(\tau(\log n + \log \tau)/\epsilon^2)$ to achieve an estimator of the co-occurrence matrix with error bound $\epsilon$. Our work leads to some natural future questions:

- **Is it a tight bound?** Our analysis on convergence rate of co-occurrence matrices relies on union bound, which probably gives a loose bound. It would be interesting to shave off the leading factor $\tau$ in the bound, as the mixing time $\tau$ could be large for some Markov chains.

- **Can we generalize the regular Markov chain condition to ergodic Markov chains?** We need the regular Markov chain condition to make the mixing-time well defined. We believe that with a more general definition of mixing time (e.g., Definition 4.1 in [4]), we can extend our bound of co-occurrence matrices to ergodic Markov chains.

- **What if the construction of the co-occurrence matrix is coupled with a learning algorithm?** For example, in word2vec [27], the co-occurrence in each sliding window outputs a mini-batch to a logistic matrix factorization model. This problem can be formalized as the convergence of stochastic gradient descent with non-i.i.d but Markovian random samples.

- **Can we find more applications of the matrix Chernoff bound for ergodic Markov chains?** We believe Theorem 2 could have further applications, e.g., in reinforcement learning which also involves Markov chains.

$\pi_{\text{min}} = \min_{i \in [n]} \pi_i$. See Theorem 12.3 of Levin and Peres [21].
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Supplementary Material

A Convergence Rate of Co-occurrence Matrices

Claim 1 (Properties of $Q$). If $P$ is a regular Markov chain, then $Q$ satisfies:

1. $Q$ is a regular Markov chain, with stationary distribution $\sigma$ such that $\sigma(u_0, \ldots, u_T) = \pi_{u_0} P_{u_0, u_1} \cdots P_{u_{T-1}, u_T}$.

2. The sequence $(X_1, \ldots X_{L-T})$ is a random walk on $Q$ starting from a distribution $\rho$ such that $\rho(u_0, \ldots, u_T) = \phi_{u_0} P_{u_0, u_1} \cdots P_{u_{T-1}, u_T}$, and $\|\rho\|_\sigma = \|\phi\|_\pi$.

3. $\forall \delta > 0$, the $\delta$-mixing time of $P$ and $Q$ satisfies $\tau(Q) < \tau(P) + T$;

4. $\exists \rho$ with $\lambda(P) < 1$ such that the induced $Q$ has $\lambda(Q) = 1$, i.e. $Q$ may have zero spectral gap$^8$.

Proof. We prove the fours parts of this Claim one by one.

Part 1 To prove $Q$ is regular, it is sufficient to show that $\exists N_1, \forall n_1 > N_1, (v_0, \ldots, v_T)$ can reach $(u_0, \ldots, u_T)$ at $n_1$ steps. We know $P$ is a regular Markov chain, so there exists $N_2 \geq T$ s.t., for any $n_2 \geq N_2$, $v_T$ can reach $u_0$ at exact $n_2$ step, i.e., there is a $n_2$-step walk s.t. $(v_T, w_1, \ldots, w_{n_2-1}, u_0)$ on $P$. This induces an $n_2$-step walk from $(v_0, \ldots, v_T)$ to $(w_{n_2-T+1}, \ldots, w_{n_2-1}, u_0)$. Take further $T$ step, we can reach $(u_0, \ldots, u_T)$, so we construct a $n_1 = n_2 + T$ step walk from $(v_0, \ldots, v_T)$ to $(u_0, \ldots, u_T)$. Since this is true for any $n_2 \geq N_2$, we then claim that any state can be reached from any other state in any number of steps greater than or equal to a number $N_1 = N_2 + T$. Next to very $\sigma$ s.t. $\sigma(u_0, \ldots, u_T) = \pi_{u_0} P_{u_0, u_1} \cdots P_{u_{T-1}, u_T}$ is the stationary distribution,

$$\sum_{(u_0, \ldots, u_T) \in S} \sigma(u_0, \ldots, u_T) Q(u_0, \ldots, u_T, (u_0, \ldots, u_T)) = \sum_{u_0: (u_0, w_0, \ldots, w_{T-1}) \in S} \pi_{u_0} P_{u_0, u_0} P_{u_0, w_1, \ldots, P_{u_{T-2}, w_{T-1}} P_{u_{T-1}, w_T}}$$

$$= \left( \sum_{u_0} \pi_{u_0} P_{u_0, u_0} \right) P_{u_0, w_1} \cdots, P_{u_{T-2}, w_{T-1}} P_{u_{T-1}, w_T} = \pi_{u_0} P_{u_0, w_1, \ldots, P_{u_{T-2}, w_{T-1}} P_{u_{T-1}, w_T} = \sigma_{u_0, \ldots, u_T}.$$  

Part 2 Recall $(v_1, \ldots, v_T)$ is a random walk on $P$ starting from distribution $\phi$, so the probability we observe $X_1 = (v_1, \ldots, v_{T+1})$ is $\phi_{v_1} P_{v_1, v_2} \cdots P_{v_T, v_T} = \rho(v_1, \ldots, v_{T+1})$, i.e., $X_1$ is sampled from the distribution $\rho$. Then we study the transition probability from $X_i = (v_i, \ldots, v_{i+T})$ to $X_{i+1} = (v_{i+1}, \ldots, v_{i+T+1})$, which is $P_{v_{i+T}, v_{i+T+1}} = Q_{X_i, X_{i+1}}$. Consequently, we can claim $(X_i, \cdots, X_{L-T})$ is a random walk on $Q$. Moreover,  

$$\|\rho\|_\sigma^2 = \sum_{(u_0, \ldots, u_T) \in S} \rho^2(u_0, \ldots, u_T) = \sum_{(u_0, \ldots, u_T) \in S} \left( \phi_{u_0} P_{u_0, u_1} \cdots P_{u_{T-1}, u_T} \right)^2$$

$$= \sum_{u_0} \phi_{u_0}^2 \sum_{u_0, (u_0, u_1, \ldots, u_T) \in S} P_{u_0, u_1} \cdots P_{u_{T-1}, u_T} = \sum_{u_0} \phi_{u_0}^2 \pi_{u_0} = \|\phi\|_\pi^2,$$

which implies $\|\rho\|_\sigma = \|\phi\|_\pi$.

$^8$Consider random walk on the unweighted undirected graph $G$ and $T = 2$. In this example, $\lambda(P) = 2/3$ but $\lambda(Q) = 1$. Detailed computation can be found in the supplementary material.
Part 3  For any distribution $\mathbf{y}$ on $\mathcal{S}$, define $\mathbf{x} \in \mathbb{R}^n$ such that $x_i = \sum_{(v_1, \ldots, v_T-1, i) \in \mathcal{S}} y_{v_1, \ldots, v_T-1, i}$. Easy to see $\mathbf{x}$ is a probability vector, since $\mathbf{x}$ is the marginal probability of $\mathbf{y}$. For convenience, we assume for a moment the $\mathbf{x}, \mathbf{y}, \sigma, \pi$ are row vectors. We can see that:

\[
\|yQ^{*P+T-1} - \sigma\|_{TV} = \frac{1}{2} \sum_{(u,v)} \left| yQ^{*P+T-1 - \sigma} \right|_{v_1, \ldots, v_T} = \frac{1}{2} \sum_{(u,v)} \left| xP^{*P} - \pi \right|_{v_1, \ldots, v_T} = \frac{1}{2} \sum_{v_1} \left| xP^{*P} - \pi \right|_{v_1} \leq \frac{1}{2} \| yQ^{*P} - \pi \|_1 = \| xQ^{*P} - \pi \|_{TV} \leq \delta.
\]

which indicates $\tau(Q) \leq \tau(P) + T - 1 < \tau(P) + T$.

Part 4 This is an example showing that $\lambda(Q)$ cannot be bounded by $\lambda(P)$ — even though $P$ has $\lambda(P) < 1$, the induced $Q$ may have $\lambda(Q) = 1$. We consider random walk on the unweighted undirected graph $\mathbb{S}$ and $T = 2$. The transition probability matrix $P$ is:

\[
P = \begin{bmatrix}
0 & 1/3 & 1/3 & 1/3 \\
1/2 & 0 & 1/2 & 0 \\
1/3 & 1/3 & 0 & 1/3 \\
1/2 & 0 & 1/2 & 0
\end{bmatrix}
\]

with stationary distribution $\pi = [0.3 \ 0.2 \ 0.3 \ 0.2]^\top$ and $\lambda(P) = \frac{2}{3}$. When $T = 2$, the induced Markov chain $Q$ has stationary distribution $\sigma_{u,v} = \pi_u P_{u,v} = \frac{d_u}{2m} \cdot \frac{1}{2m} = \frac{1}{2m}$ where $m = 5$ is the number of edges in the graph. Construct $\mathbf{y} \in \mathbb{R}^{|\mathcal{S}|}$ such that

\[
y_{(u,v)} = \begin{cases}
1 & (u,v) = (0,1), \\
-1 & (u,v) = (0,3), \\
0 & \text{otherwise}.
\end{cases}
\]

The constructed vector $\mathbf{y}$ has norm

\[
\|y\|_\sigma = \sqrt{\langle y, y \rangle_\sigma} = \sqrt{\sum_{(u,v) \in \mathcal{S}} \frac{y_{(u,v)} y_{(u,v)}}{\sigma_{(u,v)}}} = \sqrt{\frac{y_{(0,1)} y_{(0,1)}}{\sigma_{(0,1)}} + \frac{y_{(0,3)} y_{(0,3)}}{\sigma_{(0,3)}}} = 2\sqrt{m}.
\]

And it is easy to check $\mathbf{y} \perp \sigma$, since $\langle y, \sigma \rangle_\sigma = \sum_{(u,v) \in \mathcal{S}} \frac{\sigma_{(u,v)} y_{(u,v)}}{\sigma_{(u,v)}} = y_{(0,1)} + y_{(0,3)} = 0$. Let $x = (y^*Q)^*$, we have for $(u,v) \in \mathcal{S}$:

\[
x_{(u,v)} = \begin{cases}
1 & (u,v) = (1,2), \\
-1 & (u,v) = (3,2), \\
0 & \text{otherwise}.
\end{cases}
\]
This vector has norm:
\[
\|x\|_2 = \sqrt{(x, x)_2} = \sqrt{\left(\sum_{(u, v) \in S} \frac{x_{(u,v)}^* x_{(u,v)}}{\sigma(u,v)}\right)} = \sqrt{\frac{y(1,2)\bar{y}(1,2)}{\sigma(1,2)} + \frac{y(3,2)\bar{y}(3,2)}{\sigma(3,2)}} = 2\sqrt{m}
\]

Thus we have \(\|y Q^* y\|_2 = 1\). Taking maximum over all possible \(y\) gives \(\lambda(Q) \geq 1\). Also note that fact that \(\lambda(Q) \leq 1\), so \(\lambda(Q) = 1\). \(\square\)

**Claim 2** (Properties of \(f\)). The function \(f\) in Equation 2 satisfies (1) \(\sum_{X \in S} \sigma_X f(X) = 0\); (2) \(f(X)\) is symmetric and \(\|f(X)\|_2 \leq 1, \forall X \in S\).

**Proof.** Note that Equation 2 is indeed a random value minus its expectation, so naturally Equation 2 has zero mean, i.e., \(\sum_{X \in S} \sigma_X f(X) = 0\). Moreover, \(\|f(X)\|_2 \leq 1\) because
\[
\|f(X)\|_2 \leq \frac{1}{2} \left( \sum_{r=1}^{T} \frac{\alpha_r}{2} \left( \|e_{v_0} e_{v_0}^\top\|_2 + \|e_{v_r} e_{v_0}^\top\|_2 \right) + \sum_{r=1}^{T} \frac{\alpha_r}{2} \left( \|\Pi\|_2 \|P\|_2 + \|P^\top\|_2 \|\Pi\|_2 \right) \right)
\]
\[
\leq \frac{1}{2} \left( \sum_{r=1}^{T} \alpha_r + \sum_{r=1}^{T} \alpha_r \right) = 1.
\]
where the first step follows triangle inequality and submultiplicativity of 2-norm, and the third step follows by (1) \(\|e_{i} e_{j}^\top\|_2 = 1\); (2) \(\|\Pi\|_2 = \|\text{diag}(\pi)\|_2 \leq 1\) for distribution \(\pi\); (3) \(\|P\|_2 = \|P^\top\|_2 = 1\).

\(\square\)

## B Matrix Chernoff Bounds for Ergodic Markov Chains

### B.1 Preliminaries

**Kronecker Products** If \(A\) is an \(M_1 \times N_1\) matrix and \(B\) is a \(M_2 \times N_2\) matrix, then the Kronecker product \(A \otimes B\) is the \(M_2 M_1 \times N_2 N_1\) block matrix such that
\[
A \otimes B = \begin{bmatrix}
A_{1,1} B & \cdots & A_{1,N_1} B \\
\vdots & \ddots & \vdots \\
A_{M_1,1} B & \cdots & A_{M_1,N_1} B
\end{bmatrix}.
\]

Kronecker product has the mixed-product property. If \(A, B, C, D\) are matrices of such size that one can from the matrix products \(AC\) and \(BD\), then \((A \otimes B)(C \otimes D) = (AC) \otimes (BD)\).

**Vectorization** For a matrix \(X \in \mathbb{C}^{d \times d}\), vec\((X) \in \mathbb{C}^{d^2}\) denote the vectorization of the matrix \(X\), s.t. vec\((X) = \sum_{i \in [d]} \sum_{j \in [d]} X_{i,j} e_i \otimes e_j\), which is the stack of rows of \(X\). And there is a relationship between matrix multiplication and Kronecker product s.t. vec\((AXB) = (A \otimes B^\top)\) vec\((X)\).

**Matrices and Norms** For a matrix \(A \in \mathbb{C}^{N \times N}\), we use \(A^\top\) to denote matrix transpose, use \(\overline{A}\) to denote entry-wise matrix conjugation, use \(A^*\) to denote matrix conjugate transpose \((A^* = \overline{A}^\top)\). The vector 2-norm is defined to be \(\|x\|_2 = \sqrt{x^* x}\), and the matrix 2-norm is defined to be \(\|A\|_2 = \max \{x \in \mathbb{C}^N : x \neq 0\} \frac{\|Ax\|_2}{\|x\|_2}\).

We then recall the definition of inner-product under \(\pi\)-kernel in Section 2. The inner-product under \(\pi\)-kernel for \(\mathbb{C}^N\) is \((x, y)_{\pi} = y^* \overline{\Pi}^{-1} x\) where \(\overline{\Pi} = \text{diag}(\pi)\), and its induced \(\pi\)-norm \(\|x\|_{\pi} = \sqrt{(x, x)_{\pi}}\). The above definition allow us to define a inner product under \(\pi\)-kernel on \(\mathbb{C}^{Nd^2}\):
Definition 1. Define inner product on $\mathbb{C}^{Nd^2}$ under $\pi$-kernel to be $\langle x, y \rangle_\pi = y^* (\Pi^{-1} \otimes I_{d^2}) x$.

Remark 1. For $x, y \in \mathbb{C}^N$ and $p, q \in \mathbb{C}^{d^2}$, then inner product (under $\pi$-kernel) between $x \otimes p$ and $y \otimes q$ can be simplified as

$$\langle x \otimes p, y \otimes q \rangle_\pi = (y \otimes q)^* (\Pi^{-1} \otimes I_{d^2}) (x \otimes p) = (y^* \Pi^{-1} x) \otimes (q^* p) = \langle x, y \rangle_\pi (p, q).$$

Remark 2. The induced $\pi$-norm is $\|x\|_\pi = \sqrt{\langle x, x \rangle_\pi}$. When $x = y \otimes w$, the $\pi$-norm can be simplified to: $\|x\|_\pi = \sqrt{\langle y \otimes w, y \otimes w \rangle_\pi} = \sqrt{\langle y, y \rangle_\pi (w, w)} = \|y\|_\pi \|w\|_2$.

Matrix Exponential The matrix exponential of a matrix $A \in \mathbb{C}^{d \times d}$ is defined by Taylor expansion $\exp(A) = \sum_{j=0}^{+\infty} \frac{A^j}{j!}$. And we will use the fact that $\exp(A) \otimes \exp(B) = \exp(A \otimes I + I \otimes B)$.

Golden-Thompson Inequality We need the following multi-matrix Golden-Thompson inequality from from Garg et al. [12].

Theorem 3 (Multi-matrix Golden-Thompson Inequality, Theorem 1.5 in [12]). Let $H_1, \cdots, H_k$ be $k$ Hermitian matrices, then for some probability distribution $\mu$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

$$\log \left( \text{Tr} \left[ \exp \left( \sum_{j=1}^{k} H_j \right) \right] \right) \leq \frac{4}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log \left( \text{Tr} \left[ \prod_{j=1}^{k} \exp \left( \frac{\epsilon_i \phi}{2} H_j \right) \prod_{i=k}^{1} \exp \left( \frac{-\epsilon_i \phi}{2} H_j \right) \right] \right) d\mu(\phi).$$

B.2 Proof of Theorem

Theorem 2 (A Real-Valued Matrix Chernoff Bound for Ergodic Markov Chains). Let $P$ be an ergodic Markov chain with state space $[N]$, stationary distribution $\pi$ and spectral norm $\lambda$. Let $f : [N] \to \mathbb{R}^{d \times d}$ be a function such that (1) $\forall v \in [N], f(v)$ is symmetric and $\|f(v)\|_2 \leq 1$; (2) $\sum_{v \in [N]} \pi_v f(v) = 0$. Let $(v_1, \cdots, v_k)$ denote a $k$-step random walk on $P$ starting from a distribution $\phi$ on $[N]$. Then given $\epsilon \in (0, 1),$

$$\mathbb{P} \left[ \lambda_{\max} \left( \frac{1}{k} \sum_{j=1}^{k} f(v_j) \right) \geq \epsilon \right] \leq \|\phi\|_\pi d^2 \exp \left( -\epsilon^2 (1 - \lambda) k/2 \right)$$

$$\mathbb{P} \left[ \lambda_{\min} \left( \frac{1}{k} \sum_{j=1}^{k} f(v_j) \right) \leq -\epsilon \right] \leq \|\phi\|_\pi d^2 \exp \left( -\epsilon^2 (1 - \lambda) k/2 \right).$$

Proof. Due to symmetry, it suffices to prove one of the statements. Let $t > 0$ be a parameter to be chosen later. Then

$$\mathbb{P} \left[ \lambda_{\max} \left( \frac{1}{k} \sum_{j=1}^{k} f(v_j) \right) \geq \epsilon \right] = \mathbb{P} \left[ \lambda_{\max} \left( \sum_{j=1}^{k} f(v_j) \right) \geq k\epsilon \right] \leq \mathbb{P} \left[ \text{Tr} \left[ \exp \left( t \sum_{j=1}^{k} f(v_j) \right) \right] \geq \exp(tk\epsilon) \right] \leq \frac{\mathbb{E}_{v_1, \cdots, v_k} \left[ \text{Tr} \left( \exp \left( t \sum_{j=1}^{k} f(v_j) \right) \right) \right]}{\exp(tk\epsilon)}.$$
Next to bound \( E_{v_1, \ldots, v_k} \left[ \text{Tr} \left[ \exp \left( t \sum_{j=1}^{k} f(v_j) \right) \right] \right] \). Using Theorem 3, we have:

\[
\log \left( \text{Tr} \left[ \exp \left( t \sum_{j=1}^{k} f(v_j) \right) \right] \right) \leq \frac{4}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log \left( \text{Tr} \left[ \prod_{j=1}^{k} \exp \left( \frac{e^{i\phi}}{2} tf(v_j) \right) \prod_{j=k}^{1} \exp \left( \frac{e^{-i\phi}}{2} tf(v_j) \right) \right] \right) d\mu(\phi)
\]

\[
\leq \frac{4}{\pi} \log \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \text{Tr} \left[ \prod_{j=1}^{k} \exp \left( \frac{e^{i\phi}}{2} tf(v_j) \right) \prod_{j=k}^{1} \exp \left( \frac{e^{-i\phi}}{2} tf(v_j) \right) \right] d\mu(\phi),
\]

where the second step follows by concavity of log function and the fact that \( \mu(\phi) \) is a probability distribution on \( [-\frac{\pi}{2}, \frac{\pi}{2}] \). This implies

\[
\text{Tr} \left[ \exp \left( t \sum_{j=1}^{k} f(v_j) \right) \right] \leq \left( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \text{Tr} \left[ \prod_{j=1}^{k} \exp \left( \frac{e^{i\phi}}{2} tf(v_j) \right) \prod_{j=k}^{1} \exp \left( \frac{e^{-i\phi}}{2} tf(v_j) \right) \right] d\mu(\phi) \right)^{\frac{1}{t}}.
\]

Note that \( \|x\|_p \leq d^{1/p-1} \|x\|_1 \) for \( p \in (0, 1) \), choosing \( p = \pi/4 \) we have

\[
\left( \text{Tr} \left[ \exp \left( \frac{\pi}{4} \sum_{j=1}^{k} f(v_j) \right) \right] \right)^{\frac{1}{t}} \leq d^{\frac{2}{\pi}-1} \text{Tr} \left[ \exp \left( t \sum_{j=1}^{k} f(v_j) \right) \right].
\]

Combining the above two equations together, we have

\[
\text{Tr} \left[ \exp \left( \frac{\pi}{4} t \sum_{j=1}^{k} f(v_j) \right) \right] \leq d^{\frac{2}{\pi}-\frac{2}{\pi}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \text{Tr} \left[ \prod_{j=1}^{k} \exp \left( \frac{e^{i\phi}}{2} tf(v_j) \right) \prod_{j=k}^{1} \exp \left( \frac{e^{-i\phi}}{2} tf(v_j) \right) \right] d\mu(\phi).
\]

Write \( e^{i\phi} = \gamma + ib \) with \( \gamma^2 + b^2 = |\gamma + ib|^2 = |e^{i\phi}|^2 = 1 \):

**Lemma 1** (Analogous to Lemma 4.3 in [12]). Let \( P \) be an ergodic Markov chain with state space \([N]\) with spectral norm \( \lambda \). Let \( f \) be a function \( f : [N] \to \mathbb{R}^{d \times d} \) such that (1) \( \sum_{v \in [N]} \pi_v f(v) = 0 \); (2) \( \|f(v)\|_2 \leq 1 \) and \( f(v) \) is symmetric, \( v \in [N] \). Let \((v_1, \cdots, v_k)\) denote a \( k \)-step random walk on \( P \) starting from a distribution \( \phi \) on \([N]\). Then for any \( t > 0, \gamma \geq 0, b > 0 \) such that \( t^2(\gamma^2 + b^2) \leq 1 \) and \( t\sqrt{\gamma^2 + b^2} \leq \frac{1-\lambda}{4\lambda} \), we have

\[
E \left[ \text{Tr} \left[ \prod_{j=1}^{k} \exp \left( \frac{tf(v_j)(\gamma + ib)}{2} \right) \prod_{j=k}^{1} \exp \left( \frac{tf(v_j)(\gamma - ib)}{2} \right) \right] \right] \leq \|\phi\|_\pi d \exp \left( kt^2(\gamma^2 + b^2) \left( 1 + \frac{8}{1 - \lambda} \right) \right).
\]
Assuming the above lemma, we can complete the proof of the theorem as:

\[
\mathbb{E}_{v_1, \ldots, v_k} \left[ \text{Tr} \left[ \exp \left( \frac{\pi}{4} t \sum_{j=1}^{k} f(v_j) \right) \right] \right] \\
\leq d^{-\frac{t^2}{4}} \mathbb{E}_{v_1, \ldots, v_k} \left[ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \text{Tr} \left[ \prod_{j=1}^{k} \exp \left( \frac{e^{i\phi}}{2} tf(v_j) \right) \prod_{j=k}^{1} \exp \left( \frac{e^{-i\phi}}{2} tf(v_j) \right) \right] \right) d\mu(\phi) \right] \\
= d^{-\frac{t^2}{4}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \|\phi\|_\pi d\exp \left( kt^2 |e^{i\phi}|^2 \left( 1 + \frac{8}{1 - \lambda} \right) \right) d\mu(\phi) \\
= \|\phi\|_\pi d^{2^{-\frac{t^2}{4}}} \exp \left( kt^2 \left( 1 + \frac{8}{1 - \lambda} \right) \right) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\mu(\phi) \\
= \|\phi\|_\pi d^{2^{-\frac{t^2}{4}}} \exp \left( kt^2 \left( 1 + \frac{8}{1 - \lambda} \right) \right)
\]

where the first step follows Equation \[5\] the second step follows by swapping \[\mathbb{E}\] and \[\int\], the third step follows by Lemma \[1\] the forth step follows by \(|e^{i\phi}| = 1\), and the last step follows by \(\mu\) is a probability distribution on \([-\frac{\pi}{2}, \frac{\pi}{2}]\) so \(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\mu(\phi) = 1\)

Finally, putting it all together:

\[
P \left[ \lambda_{\text{max}} \left( \frac{1}{k} \sum_{j=1}^{k} f(v_j) \right) \geq \epsilon \right] \\
\leq \frac{\mathbb{E} \left[ \text{Tr} \left[ \exp \left( t \sum_{j=1}^{k} f(v_j) \right) \right] \right]}{\exp(kt\epsilon)} \\
= \frac{\mathbb{E} \left[ \text{Tr} \left[ \exp \left( \frac{\pi}{4} (\frac{1}{k} t) \sum_{j=1}^{k} f(v_j) \right) \right] \right]}{\exp(kt\epsilon)} \\
\leq \|\phi\|_\pi d^{2^{-\frac{t^2}{4}}} \exp \left( \left( \frac{4}{\pi} \right) \frac{k}{\lambda} \frac{1}{36^2} \frac{1}{1 - \lambda} - k \frac{(1 - \lambda)\epsilon}{36} \right) \\
\leq \|\phi\|_\pi d^{2^{-\frac{t^2}{4}}} \exp(-k\epsilon^2(1 - \lambda)/72).
\]

where the first step follows by Equation \[4\] the second step follows by Equation \[6\] the third step follows by choosing \(t = (1 - \lambda)\epsilon/36\). The only thing to be check is that \(t = (1 - \lambda)\epsilon/36\) satisfies \(t\sqrt{\gamma^2 + b^2} = \epsilon \leq \frac{1 - \lambda}{4\lambda}\). Recall that \(\epsilon < 1\) and \(\lambda < 1\), we have \(t = \frac{(1 - \lambda)\epsilon}{36} \leq \frac{1 - \lambda}{4\lambda}\).

\[\Box\]

B.3 Proof of Lemma \[1\]

Lemma 1 (Analogous to Lemma 4.3 in \[12\]). Let \(P\) be an ergodic Markov chain with state space \([N]\) with spectral norm \(\lambda\). Let \(f\) be a function \(f : [N] \rightarrow \mathbb{R}^{d \times d}\) such that (1) \(f(v) = 0\); (2) \(\|f(v)\|_2 \leq 1\) and \(f(v)\) is symmetric, \(v \in [N]\). Let \((v_1, \ldots, v_k)\) denote a \(k\)-step random walk on \(P\) starting from a distribution \(\phi\) on \([N]\). Then for any \(t > 0\), \(\gamma > 0\), \(b > 0\) such that \(t^2(\gamma^2 + b^2) \leq 1\)
and \(t\sqrt{\gamma^2 + b^2} \leq \frac{1-\lambda}{4\lambda}\), we have
\[
E \left[ \prod_{j=1}^{k} \exp \left( \frac{tf(v_j)(\gamma + ib)}{2} \right) \prod_{j=k}^{1} \exp \left( \frac{tf(v_j)(\gamma - ib)}{2} \right) \right] \leq \|\phi\|_\pi d \exp \left( kt^2(\gamma^2 + b^2) \left( 1 + \frac{8}{1 - \lambda} \right) \right).
\]

**Proof.** Note that for \(A, B \in \mathbb{C}^{d \times d}\), \((A \otimes B)\text{vec}(I_d)\), vec(\(I_d\)) = Tr \([AB]_\top\]. By letting \(A = \prod_{j=1}^{k} \exp \left( \frac{tf(v_j)(\gamma + ib)}{2} \right)\) and \(B = \left[ \prod_{j=k}^{1} \exp \left( \frac{tf(v_j)(\gamma - ib)}{2} \right) \right]_\top = \prod_{j=1}^{k} \exp \left( \frac{tf(v_j)(\gamma - ib)}{2} \right)\). The trace term in LHS of Lemma 1 becomes
\[
\text{Tr} \left[ \prod_{j=1}^{k} \exp \left( \frac{tf(v_j)(\gamma + ib)}{2} \right) \prod_{j=k}^{1} \exp \left( \frac{tf(v_j)(\gamma - ib)}{2} \right) \right] = \left\langle \left( \prod_{j=1}^{k} \exp \left( \frac{tf(v_j)(\gamma + ib)}{2} \right) \right) \otimes \left( \prod_{j=1}^{k} \exp \left( \frac{tf(v_j)(\gamma - ib)}{2} \right) \right) \right\rangle \text{vec}(I_d), \text{vec}(I_d) \rightangle.
\]

By iteratively applying \((A \otimes B)(C \otimes D) = (AC) \otimes (BD)\), we have
\[
\prod_{j=1}^{k} \exp \left( \frac{tf(v_j)(\gamma + ib)}{2} \right) \otimes \prod_{j=1}^{k} \exp \left( \frac{tf(v_j)(\gamma - ib)}{2} \right) = \prod_{j=1}^{k} \left( \exp \left( \frac{tf(v_j)(\gamma + ib)}{2} \right) \otimes \exp \left( \frac{tf(v_j)(\gamma - ib)}{2} \right) \right) = \prod_{j=1}^{k} M_{v_j},
\]

where we define
\[
M_{v_j} \triangleq \exp \left( \frac{tf(v_j)(\gamma + ib)}{2} \right) \otimes \exp \left( \frac{tf(v_j)(\gamma - ib)}{2} \right).
\]

Plug it to the trace term, we have
\[
\text{Tr} \left[ \prod_{j=1}^{k} \exp \left( \frac{tf(v_j)(\gamma + ib)}{2} \right) \prod_{j=k}^{1} \exp \left( \frac{tf(v_j)(\gamma - ib)}{2} \right) \right] = \left\langle \left( \prod_{j=1}^{k} M_{v_j} \right) \text{vec}(I_d), \text{vec}(I_d) \right\rangle.
\]

Next, taking expectation on Equation 7 gives
\[
E_{v_1, \ldots, v_k} \left[ \text{Tr} \left[ \prod_{j=1}^{k} \exp \left( \frac{tf(v_j)(\gamma + ib)}{2} \right) \prod_{j=k}^{1} \exp \left( \frac{tf(v_j)(\gamma - ib)}{2} \right) \right] \right] = E_{v_1, \ldots, v_k} \left[ \left\langle \left( \prod_{j=1}^{k} M_{v_j} \right) \text{vec}(I_d), \text{vec}(I_d) \right\rangle \right] = E_{v_1, \ldots, v_k} \left[ \left( \prod_{j=1}^{k} M_{v_j} \right) \text{vec}(I_d), \text{vec}(I_d) \right].
\]

We turn to study \(E_{v_1, \ldots, v_k} \left[ \prod_{j=1}^{k} M_{v_j} \right]\), which is characterized by the following lemma:

**Lemma 2.** Let \(E \triangleq \text{diag}(M_1, M_2, \ldots, M_N) \in \mathbb{C}^{Nd^2 \times Nd^2}\) and \(\bar{P} \triangleq P \otimes I_{d^2} \in \mathbb{R}^{Nd^2 \times Nd^2}\). For a random walk \((v_1, \ldots, v_k)\) such that \(v_1\) is sampled from an arbitrary probability distribution \(\phi\) on \([N]\), \(E_{v_1, \ldots, v_k} \left[ \prod_{j=1}^{k} M_{v_j} \right] = (\phi \otimes I_{d^2})_\top \left( (E \bar{P})^{k-1} E \right) (1 \otimes I_{d^2})\), where \(1\) is the all-ones vector.
Proof. (of Lemma 2) We always treat $E\tilde{P}$ as a block matrix, s.t.,

$$E\tilde{P} = \begin{bmatrix} M_1 & \cdots & P_{1,1}I_{d^2} & \cdots & P_{1,N}I_{d^2} \\ \vdots & \ddots & \vdots & & \vdots \\ M_N & \cdots & P_{N,1}I_{d^2} & \cdots & P_{N,N}I_{d^2} \end{bmatrix} = \begin{bmatrix} P_{1,1}M_1 & \cdots & P_{1,N}M_1 \\ \vdots & \ddots & \vdots \\ P_{N,1}M_N & \cdots & P_{N,N}M_N \end{bmatrix}.$$ 

I.e., the $(u,v)$-th block of $E\tilde{P}$, denoted by $(E\tilde{P})_{u,v}$ is $P_{u,v}M_u$.

$$E_{v_1,\ldots,v_k} \prod_{j=1}^k M_{v_j} = \sum_{v_1,\ldots,v_k} \phi_{v_1}P_{v_1,v_2} \cdots P_{v_{k-1},v_k} \prod_{j=1}^k M_{v_j}$$

$$= \sum_{v_1} \phi_{v_1} \prod_{j=2}^k (P_{v_1,v_2}M_{v_1}) \cdots \sum_{v_k} (P_{v_{k-1},v_k}M_{v_{k-1}}) M_{v_k}$$

$$= \sum_{v_1} \phi_{v_1} \prod_{j=2}^k (E\tilde{P})_{v_1,v_2} \cdots \sum_{v_k} (E\tilde{P})_{v_2,v_3} \cdots \sum_{v_k} (E\tilde{P})_{v_{k-1},v_k}$$

$$= \sum_{v_1} \phi_{v_1} \sum_{v_k} ((E\tilde{P})^{k-1}E)_{v_1,v_k} = (\phi \otimes I_{d^2})^\top ((E\tilde{P})^{k-1}E) (1 \otimes I_{d^2})$$

Given Lemma 2 Equation 9 becomes:

$$E_{v_1,\ldots,v_k} \left[ \prod_{j=1}^k \exp \left( \frac{t f(v_j)(\gamma + ib)}{2} \right) \prod_{j=k}^1 \exp \left( \frac{t f(v_j)(\gamma - ib)}{2} \right) \right]$$

$$= \left\langle (E\tilde{P})^{k-1}E (1 \otimes \vec{I}_d), \vec{I}_d \right\rangle$$

$$= (\phi \otimes I_{d^2})^* ((E\tilde{P})^{k-1}E) (1 \otimes \vec{I}_d)$$

$$= (\phi \otimes \vec{I}_d)^* ((E\tilde{P})^{k-1}E) ((P\Pi^{-1}\pi) \otimes (I_{d^2}I_{d^2} \vec{I}_d))$$

$$= (\phi \otimes \vec{I}_d)^* (E\tilde{P})^k (\Pi^{-1} \otimes I_{d^2}) (\pi \otimes \vec{I}_d) \triangleq \langle \pi \otimes \vec{I}_d, z_k \rangle_{\pi},$$

where we define $z_0 = \phi \otimes \vec{I}_d$ and $z_k = (z_0^* (E\tilde{P})^k)^* = (z_{k-1}^* E\tilde{P})^*$. Moreover, by Remark 2 we have $\|\pi \otimes \vec{I}_d\|_{\pi} = \|\pi\|_\pi \|\vec{I}_d\|_2 = \sqrt{d}$ and $\|z_0\|_\pi = \|\phi \otimes \vec{I}_d\|_\pi = \|\phi\|_\pi \sqrt{d}$.
Definition 2. Define linear subspace \( \mathcal{U} = \{ \pi \otimes w, w \in \mathbb{C}^{d^2} \} \).

Remark 3. \( \{ \pi \otimes e_i, i \in [d^2] \} \) is an orthonormal basis of \( \mathcal{U} \). This is because \( \langle \pi \otimes e_i, \pi \otimes e_j \rangle_{\pi} = \langle \pi, \pi \rangle \langle e_i, e_j \rangle = \delta_{ij} \) by Remark 1, where \( \delta_{ij} \) is the Kronecker delta.

Remark 4. Given \( x = y \otimes w \). The projection of \( x \) on to \( \mathcal{U} \) is \( x^\parallel = (1^*y)(\pi \otimes w) \). This is because

\[
x^\parallel = \sum_{i=1}^{d^2} \langle y \otimes w, \pi \otimes e_i \rangle_{\pi} (\pi \otimes e_i) = \sum_{i=1}^{d^2} \langle y, \pi \rangle_{\pi} (\pi \otimes e_i) = (1^*y)(\pi \otimes w).
\]

We want to bound

\[
\langle \pi \otimes \text{vec}(I_d), z_k \rangle_{\pi} = \left\langle \pi \otimes \text{vec}(I_d), z_k^\parallel + z_k^\perp \right\rangle_{\pi} = \left\langle \pi \otimes \text{vec}(I_d), z_k^\parallel \right\rangle_{\pi} \\
\leq \| \pi \otimes \text{vec}(I_d) \|_{\pi} \| z_k^\parallel \|_{\pi} = \sqrt{d} \| z_k^\parallel \|_{\pi}.
\]

As \( z_k \) can be expressed as recursively applying operator \( E \) and \( \tilde{P} \) on \( z_0 \), we turn to analyze the effects of \( E \) and \( \tilde{P} \) operators.

Definition 3. The spectral norm of \( \tilde{P} \) is defined as \( \lambda(\tilde{P}) \triangleq \max_{x, \bot \mathcal{U}, x \neq 0} \frac{\| (x^* \tilde{P}^*)^* \|_{\pi}}{\| x \|_{\pi}} \).

Lemma 3. \( \lambda(P) = \lambda(\tilde{P}) \).

Proof. First show \( \lambda(\tilde{P}) \geq \lambda(P) \). Suppose the maximizer of \( \lambda(P) \triangleq \max_{y, \bot \pi, y \neq 0} \frac{\| (y^*P)^* \|_{\pi}}{\| y \|_{\pi}} \) is \( y \in \mathbb{C}^n \), i.e., \( \| (y^*P)^* \|_{\pi} = \lambda(P) \| y \|_{\pi} \). Construct \( x = y \otimes o \) for arbitrary non-zero \( o \in \mathbb{C}^{d^2} \). Easy to check that \( x \perp \mathcal{U} \), because \( \langle x, \pi \otimes w \rangle_{\pi} = \langle y, \pi \rangle_{\pi} (o, w) = 0 \), where the last equality is due to \( y \perp \pi \).

Then we can bound \( \left\| (x^* \tilde{P}^*)^* \right\|_{\pi} \) such that

\[
\left\| (x^* \tilde{P}^*)^* \right\|_{\pi} = \| \tilde{P}^* x \|_{\pi} = \| (P^* \otimes I_{d^2}) (y \otimes o) \|_{\pi} = \| (P^* y) \otimes o \|_{\pi} \\
= \| (y^*P)^* \|_{\pi} \| o \|_2 = \lambda(P) \| y \|_{\pi} \| o \|_2 = \lambda(P) \| x \|_{\pi},
\]

which indicate for \( x = y \otimes o \), \( \left\| (x^* \tilde{P}^*)^* \right\|_{\pi} = \lambda(P) \). Taking maximum over all \( x \) gives \( \lambda(\tilde{P}) \geq \lambda(P) \).

Next to show \( \lambda(P) \geq \lambda(\tilde{P}) \). For \( \forall x \in \mathbb{C}^{d^2} \) such that \( x \perp \mathcal{U} \) and \( x \neq 0 \), we can decompose it to be

\[
x = \begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_{N \cdot d^2}
\end{bmatrix} = \begin{bmatrix}
x_1 \\
x_{d^2+1} \\
\vdots \\
x_{(N-1) \cdot d^2+1}
\end{bmatrix} \otimes e_1 + \begin{bmatrix}
x_2 \\
x_{d^2+2} \\
\vdots \\
x_{(N-1) \cdot d^2+2}
\end{bmatrix} \otimes e_2 + \cdots + \begin{bmatrix}
x_{d^2} \\
x_{2d^2} \\
\vdots \\
x_{N \cdot d^2}
\end{bmatrix} \otimes e_{d^2} = \sum_{i=1}^{d^2} x_i \otimes e_i,
\]

where we define \( x_i \triangleq \begin{bmatrix} x_1 & \cdots & x_{(N-1) \cdot d^2+i} \end{bmatrix}^\top \) for \( i \in [d^2] \). We can observe that \( x_i \perp \pi, i \in [d^2] \), because for \( \forall j \in [d^2] \), we have

\[
0 = \langle x, \pi \otimes e_j \rangle_{\pi} = \sum_{i=1}^{d^2} \langle x_i \otimes e_i, \pi \otimes e_j \rangle_{\pi} = \sum_{i=1}^{d^2} \langle x_i, \pi \rangle_{\pi} \langle e_i, e_j \rangle = \sum_{i=1}^{d^2} \lambda_i \langle e_i, e_j \rangle = 0.
\]

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which indicates \( x_j \perp \pi, j \in [d^2] \). Furthermore, we can also observe that \( x_i \otimes e_i, i \in [d^2] \) is pairwise orthogonal. This is because for \( \forall i, j \in [d^2] \), \( \langle x_i \otimes e_i, x_j \otimes e_j \rangle_\pi = \langle x_i, x_j \rangle_\pi \langle e_i, e_j \rangle = \delta_{ij} \), which suggests us to use Pythagorean theorem such that \( \| x_i \|^2_\pi = \sum_{i=1}^{d^2} \| x_i \otimes e_i \|^2_\pi = \sum_{i=1}^{d^2} \| x_i \|^2_\pi \| e_i \|^2_2 \).

We can use similar way to decompose and analyze \( (x^* \tilde{P})^* \):

\[
(x^* \tilde{P})^* = \tilde{P}^* x = \sum_{i=1}^{d^2} (P^* \otimes I_{d^2}) (x_i \otimes e_i) = \sum_{i=1}^{d^2} (P^* x_i) \otimes e_i.
\]

where we can observe that \( (P^* x_i) \otimes e_i, i \in [d^2] \) is pairwise orthogonal. This is because for \( \forall i, j \in [d^2] \), we have \( \langle (P^* x_i) \otimes e_i, (P^* x_j) \otimes e_j \rangle_\pi = \langle P^* x_i, P^* x_j \rangle_\pi \langle e_i, e_j \rangle = \delta_{ij} \). Again, applying Pythagorean theorem gives:

\[
\| (x^* \tilde{P})^* \|^2_\pi = \sum_{i=1}^{d^2} \| (P^* x_i) \otimes e_i \|^2_\pi = \sum_{i=1}^{d^2} \| (x_i^* P)^* \|^2_\pi \| e_i \|^2_2 \\
\leq \sum_{i=1}^{d^2} \lambda(P)^2 \| x_i \|^2_\pi \| e_i \|^2_2 = \lambda(P)^2 \left( \sum_{i=1}^{d^2} \| x_i \|^2_\pi \| e_i \|^2_2 \right) = \lambda(P)^2 \| x \|^2_\pi,
\]

which indicate that for \( \forall x \) such that \( x \perp \mathcal{U} \) and \( x \neq 0 \), we have \( \| (x^* \tilde{P})^* \|^2_\pi \leq \lambda(P) \), or equivalently \( \lambda(\tilde{P}) \leq \lambda(P) \).

Overall, we have shown both \( \lambda(\tilde{P}) \geq \lambda(P) \) and \( \lambda(\tilde{P}) \leq \lambda(P) \). We conclude \( \lambda(\tilde{P}) = \lambda(P) \). \( \square \)

Lemma 4. (The effect of \( \tilde{P} \) operator) This lemma is a generalization of lemma 3.3 in [5].

1. \( \forall y \in \mathcal{U}, \) then \( (y^* \tilde{P})^* = y \).

2. \( \forall y \perp \mathcal{U}, \) then \( (y^* \tilde{P})^* \perp \mathcal{U}, \) and \( \| (y^* \tilde{P})^* \|_\pi \leq \lambda \| y \|_\pi. \)

Proof. First prove the Part 1 of lemma 3 \( \forall y = \pi \otimes w \in \mathcal{U}: \)

\[
y^* \tilde{P} = (\pi^* \otimes w^*) (P \otimes I_{d^2}) = (\pi^* P) \otimes (w^* I_{d^2}) = \pi^* \otimes w^* = y^*,
\]

where third equality is because \( \pi \) is the stationary distribution. Next to prove Part 2 of lemma 3. Given \( y \perp \mathcal{U}, \) want to show \( (y^* \tilde{P})^* \perp \pi \otimes w, \) for every \( w \in \mathbb{C}^{d^2} \). It is true because

\[
\langle \pi \otimes w, (y^* \tilde{P})^* \rangle_\pi = y^* \tilde{P} (\Pi^{-1} \otimes I_{d^2}) (\pi \otimes w) = y^* ((P \Pi^{-1} \pi) \otimes w) = y^* ((\Pi^{-1} \pi) \otimes w)
\]

\[
= y^* (\Pi^{-1} \otimes I_{d^2}) (\pi \otimes w) = (\pi \otimes w, y)_\pi = 0.
\]

The third equality is due to \( P \Pi^{-1} \pi = P 1 = 1 = \Pi^{-1} \pi \). Moreover, \( \| (y^* \tilde{P})^* \|_\pi \leq \lambda \| y \|_\pi \) is simply a re-statement of definition 3. \( \square \)

Remark 5. Lemma 3 implies that \( \forall y \in \mathbb{C}^{d^2} \)

1. \( \| (y^* \tilde{P})^* \| = \| (y^* \tilde{P})^* \| + \| (y^* \tilde{P})^* \| = y^* + 0 = y^* \)

2. \( (y^* \tilde{P})^* = (y^* \tilde{P})^* + (y^* \tilde{P})^* = 0 + (y^* \tilde{P})^* = (y^* \tilde{P})^* \).

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Lemma 5. (The effect of $E$ operator) Given three parameters $\lambda \in [0,1], \ell \geq 0$ and $t > 0$. Let $P$ be an ergodic Markov chain on state space $[N]$, with stationary distribution $\pi$ and spectral norm $\lambda$. Suppose each state $i \in [N]$ is assigned a matrix $H_i \in \mathbb{C}^{d^2 \times d^2}$ s.t. $\|H_i\|_2 \leq \ell$ and $\sum_{i \in [N]} \pi_i H_i = 0$. Let $\widetilde{P} = P \otimes I_d$ and $E$ denotes the $Nd^2 \times Nd^2$ block matrix where the $i$-th diagonal block is the matrix $\exp(t H_i)$, i.e., $E = \text{diag}(\exp(t H_1), \ldots, \exp(t H_N))$. Then for any $\forall z \in \mathbb{C}^{Nd^2}$, we have:

1. $\left\| \left( z^* E \widetilde{P} \right)^{\dagger} \right\|_\pi \leq \alpha_1 \|z\|_\pi$, where $\alpha_1 = \exp(t \ell) - t \ell$.
2. $\left\| \left( z^* E \widetilde{P} \right)^{\dagger} \right\|_\pi \leq \alpha_2 \|z\|_\pi$, where $\alpha_2 = \lambda (\exp(t \ell) - 1)$.
3. $\left\| \left( z^{1/2} E \widetilde{P} \right)^{\dagger} \right\|_\pi \leq \alpha_3 \|z^{1/2}\|_\pi$, where $\alpha_3 = \exp(t \ell) - 1$.
4. $\left\| \left( z^{1/2} E \widetilde{P} \right)^{-1} \right\|_\pi \leq \alpha_4 \|z^{1/2}\|_\pi$, where $\alpha_4 = \lambda \exp(t \ell)$.

Proof. (of Lemma 5) We first show that, for $z = y \otimes w$,

$$(z^* E)^* = E^* z = \begin{bmatrix} \exp(t H_1^*) & \cdots & \exp(t H_N^*) \\
\vdots & \ddots & \vdots \\
\exp(t H_1^*) & \cdots & \exp(t H_N^*) \\
\end{bmatrix} \begin{bmatrix} y_1 w \\
y_2 w \\
\vdots \\
y_N w \\
\end{bmatrix} = \begin{bmatrix} y_1 \exp(t H_1^*) w \\
y_2 \exp(t H_2^*) w \\
\vdots \\
y_N \exp(t H_N^*) w \\
\end{bmatrix} = \sum_{i=1}^N y_i (e_i \otimes (\exp(t H_i^*) w)).$$

Due to the linearity of projection,

$$(z^* E)^* = \sum_{i=1}^N y_i (e_i \otimes (\exp(t H_i^*) w))^* = \sum_{i=1}^N y_i (1^* e_i) (\pi \otimes (\exp(t H_i^*) w)) \pi = \pi \otimes \left( \sum_{i=1}^N y_i \exp(t H_i^*) w \right),$$

where the second inequality follows by Remark 4.

Proof of Lemma 5

Part 1
Firstly we can bound $\left\| \sum_{i=1}^N \pi_i \exp(t H_i^*) \right\|_2$ by

$$\left\| \sum_{i=1}^N \pi_i \exp(t H_i^*) \right\|_2 = \left\| \sum_{i=1}^N \pi_i \sum_{j=0}^{+\infty} \frac{t^j H_i^*}{j!} \right\|_2 = \left\| I + \sum_{i=1}^N \pi_i \sum_{j=2}^{+\infty} \frac{t^j H_i^*}{j!} \right\|_2$$

$$\leq 1 + \sum_{i=1}^N \pi_i \sum_{j=2}^{+\infty} \frac{\|H_i\|^j_2}{j!} \leq 1 + \sum_{i=1}^N \pi_i \sum_{j=2}^{+\infty} \frac{(t \ell)^j}{j!} = \exp(t \ell) - t \ell,$$

where the first step follows by $\|A\|_2 = \|A^*\|_2$, the second step follows by matrix exponential, the third step follows by $\sum_{i \in [N]} \pi_i H_i = 0$, and the fourth step follows by triangle inequality. Given the
where step one follows by Part 1 of Remark 5 and step two follows by Equation 10.

Proof of Lemma 5 Part 2 For \( \forall z \in \mathbb{C}^{Nd^2} \), we can write it as block matrix such that:

\[
z = \begin{bmatrix} z_1 \\ \vdots \\ z_N \end{bmatrix} = \begin{bmatrix} z_1 \\ \vdots \\ z_N \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \sum_{i=1}^{N} e_i \otimes z_i ,
\]

where each \( z_i \in \mathbb{C}^{d^2} \). Please note that above decomposition is pairwise orthogonal. Applying Pythagorean theorem gives \( \|z\|_2^2 = \sum_{i=1}^{N} \|e_i \otimes z_i\|_2^2 = \sum_{i=1}^{N} \|e_i\|_\pi^2 \|z_i\|_2^2 \). Similarly, we can decompose \( (E^* - I_{Nd^2})z \) such that

\[
(E^* - I_{Nd^2})z = \begin{bmatrix} \exp(tH_1^*) - I_{d^2} \\ \vdots \\ \exp(tH_N^*) - I_{d^2} \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_N \end{bmatrix} = \begin{bmatrix} (\exp(tH_1^*) - I_{d^2})z_1 \\ \vdots \\ (\exp(tH_N^*) - I_{d^2})z_N \end{bmatrix}.
\]

Note that above decomposition is pairwise orthogonal, too. Applying Pythagorean theorem gives

\[
\| (E^* - I_{Nd^2})z \|_\pi^2 = \sum_{i=1}^{N} \|e_i \otimes (\exp(tH_i^*) - I_{d^2})z_i \|_\pi^2 \leq \max_{i \in [N]} \| \exp(tH_i^*) - I_{d^2} \|_2 \sum_{i=1}^{N} \|e_i\|_\pi^2 \|z_i\|_2^2
\]

which indicates

\[
\| (E^* - I_{Nd^2})z \|_\pi = \max_{i \in [N]} \| \exp(tH_i) - I_{d^2} \|_2 z_i \|_\pi = \max_{i \in [N]} \| \exp(tH_i) - I_{d^2} \|_2 \|z\|_\pi^2 \leq \left( \sum_{j=1}^{\infty} \frac{t^j |H_j|}{j!} \right) \|z\|_\pi = (\exp (t \ell) - 1) \|z\|_\pi.
\]
Now we can formally prove Part 2 of Lemma 5 by:

\[
\left\| \left( (z^\perp E \mathcal{P}^*)^\perp \right)^\perp \right\|_\pi = \left\| \left( (E^* z)^\perp \mathcal{P}^* \right)^\perp \right\|_\pi \leq \lambda \left\| (E^* z)^\perp \right\|_\pi = \lambda \left\| (E^* z^\perp - z^\perp + z^\perp)^\perp \right\|_\pi \\
= \lambda \left\| \left( (E^* - I_{N^2}) z^\perp \right)^\perp \right\|_\pi \leq \lambda \left\| (E^* - I_{N^2}) z^\perp \right\|_\pi \leq \lambda (\exp (t\ell) - 1) \|z^\perp\|_\pi .
\]

The first step follows by Part 2 of Remark 5, the second step follows by Part 1 on Lemma 4 and the forth step is due to \((z^\perp)^\perp = 0\).

**Proof of Lemma 5. Part 3** Note that

\[
\left\| E^* z \right\|_\pi^2 = \left\| \sum_{i=1}^N (e_i \otimes (\exp(tH_v^*) z_i)) \right\|_\pi^2 = \left\| \sum_{i=1}^N e_i \|_\pi^2 \| \exp(tH_v^*) z_i \|_2^2 \right\|_\pi^2 \leq \sum_{i=1}^N \| e_i \|_\pi^2 \| \exp(tH_v^*) \|_2^2 \| z_i \|_2^2
\]

which indicates \( E^* z \|_\pi \leq \exp (t\ell) \| z \|_\pi \). Now we can prove Part 4 of Lemma 5: Note that

\[
\left\| \left( (z^{\perp \perp} E \mathcal{P})^* \right)^\perp \right\|_\pi = \left\| \left( (E^* z^{\perp \perp})^\perp \mathcal{P}^* \right)^\perp \right\|_\pi \leq \lambda \left\| (E^* z^{\perp \perp})^\perp \right\|_\pi \leq \lambda \left\| E^* z^{\perp \perp} \right\|_\pi \leq \lambda \exp (t\ell) \| z^{\perp \perp} \|_\pi .
\]

**Recursive Analysis** We now use Lemma 5 to analyze the evolution of \( z^\parallel \) and \( z^{\perp \perp} \). Let \( H_v = \frac{1}{2} f(v)(\gamma + ib) I_{d^2} + I_{d^2} \otimes f(v)(\gamma - ib) I_{d^2} \) in Lemma 5. We can see verify the following three facts: (1) \( \exp(tH_v) = M_v \); (2) \( \|H_v\|_2 \) is bounded (3) \( \sum_{v \in \mathcal{N}} \pi_v H_v = 0 \).

Firstly, easy to see that

\[
\exp(tH_v) = \exp \left( \frac{tf(v)(\gamma + ib)}{2} I_{d^2} + I_{d^2} \otimes \frac{tf(v)(\gamma - ib)}{2} I_{d^2} \right)
\]

\[
= \exp \left( \frac{tf(v)(\gamma + ib)}{2} \right) \otimes \exp \left( \frac{tf(v)(\gamma - ib)}{2} \right) = M_v,
\]

where the first step follows by definition of \( H_v \) and the second step follows by the fact that \( \exp(A \otimes I_d + I_d \otimes B) = \exp(A) \otimes \exp(B) \), and the last step follows by Equation 8.

Secondly, we can bound \( \|H_v\|_2 \) by:

\[
\|H_v\|_2 \leq \left\| \frac{f(v)(\gamma + ib)}{2} I_{d^2} \right\|_2 + \left\| I_{d^2} \otimes \frac{f(v)(\gamma - ib)}{2} I_{d^2} \right\|_2 \leq \frac{\sqrt{\gamma^2 + b^2}}{2} .
\]
where the first step follows by triangle inequality, the second step follows by the fact that $\|A \otimes B\|_2 = \|A\|_2 \|B\|_2$, the third step follows by $\|Id\|_2 = 1$ and $\|f(v)\|_2 \leq 1$. We set $\ell = \sqrt{\gamma^2 + \lambda^2}$ to satisfy the assumption in Lemma 5 that $\|H_v\|_2 \leq \ell$. According to the conditions in Lemma 1, we know that $\ell \leq 1$ and $\ell \leq \frac{1 - \lambda}{\alpha_3}$.

Finally, we show that $\sum_{v \in [N]} \pi_v H_v = 0$, because
\[
\sum_{v \in [N]} \pi_v H_v = \sum_{v \in [N]} \left( \frac{f(v)(\gamma + ib)}{2} \otimes I_d + \frac{f(v)(\gamma - ib)}{2} \otimes I_d \right)
= \frac{\gamma + ib}{2} \left( \sum_{v \in [N]} \pi_v f(v) \right) \otimes I_d + \frac{\gamma - ib}{2} \left( \sum_{v \in [N]} \pi_v f(v) \right) = 0,
\]
where the last step follows by $\sum_{v \in [N]} \pi_v f(v) = 0$.

**Claim 4.** $\|z_i^+\|_\pi \leq \frac{\alpha_2}{1 - \alpha_4} \max_{0 \leq j < i} \|z_j^+\|_\pi$.

**Proof.** Using Part 2 and Part 4 of Lemma 5 we have
\[
\|z_i^+\|_\pi = \left\| \left( \left( z_{i-1}^+ E \tilde{P} \right)^* \right)^\bot \right\|_\pi
\leq \left\| \left( \left( z_{i-1}^+ E \tilde{P} \right)^* \right)^\bot \right\|_\pi + \left\| \left( \left( z_{i-1}^+ E \tilde{P} \right)^* \right)^\bot \right\|_\pi
\leq \alpha_2 \|z_{i-1}^+\|_\pi + \alpha_4 \|z_{i-1}^+\|_\pi
\leq (\alpha_2 + \alpha_2 \alpha_4 + \alpha_2 \alpha_4^2 + \cdots) \max_{0 \leq j < i} \|z_j^+\|_\pi
\leq \frac{\alpha_2}{1 - \alpha_4} \max_{0 \leq j < i} \|z_j^+\|_\pi.
\]

**Claim 5.** $\|z_i^\parallel \pi \leq \left( \alpha_1 + \frac{\alpha_2 \alpha_3}{1 - \alpha_4} \right) \max_{0 \leq j < i} \|z_j^\parallel \pi$.

**Proof.** Using Part 1 and Part 3 of Lemma 5 as well as Claim 4 we have
\[
\|z_i^\parallel \pi = \left\| \left( \left( z_{i-1}^+ E \tilde{P} \right)^* \right)^\bot \right\|_\pi
\leq \left\| \left( \left( z_{i-1}^+ E \tilde{P} \right)^* \right)^\bot \right\|_\pi + \left\| \left( \left( z_{i-1}^+ E \tilde{P} \right)^* \right)^\bot \right\|_\pi
\leq \alpha_1 \|z_{i-1}^\parallel \pi + \alpha_3 \|z_{i-1}^\parallel \pi
\leq \alpha_1 \|z_{i-1}^\parallel \pi + \alpha_3 \frac{\alpha_2}{1 - \alpha_4} \max_{0 \leq j < i-1} \|z_j^\parallel \pi
\leq \left( \alpha_1 + \frac{\alpha_2 \alpha_3}{1 - \alpha_4} \right) \max_{0 \leq j < i} \|z_j^\parallel \pi.
\]

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Combining Claim 4 and Claim 5 gives

$$
\|z_k\|_\pi \leq \left(\alpha_1 + \frac{\alpha_2 \alpha_3}{1 - \alpha_4}\right) \max_{0 \leq j < k} \|z_j\|_\pi
$$

(because $\alpha_1 + \alpha_2 \alpha_3/(1 - \alpha_4) \geq \alpha_1 \geq 1$)

$$
= \|\phi\|_\pi \sqrt{d} \left(\alpha_1 + \frac{\alpha_2 \alpha_3}{1 - \alpha_4}\right)^k,
$$

which implies

$$
\langle \pi \otimes \text{vec}(I_d), z_k \rangle_\pi \leq \|\phi\|_\pi d \left(\alpha_1 + \frac{\alpha_2 \alpha_3}{1 - \alpha_4}\right)^k.
$$

Finally, we bound $\left(\alpha_1 + \frac{\alpha_2 \alpha_3}{1 - \alpha_4}\right)^k$. The same as [12], we can bound $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ by:

$$
\alpha_1 = \exp(t \ell) - t \ell \leq 1 + t^2 \ell^2 = 1 + t^2 (\gamma^2 + b^2),
$$

and

$$
\alpha_2 \alpha_3 = \lambda(\exp(t \ell) - 1)^2 \leq \lambda(2t \ell)^2 = 4 \lambda t^2 (\gamma^2 + b^2)
$$

where the second step is because $\exp(x) \leq 1 + 2x, \forall x \in [0, 1]$ and $t \ell < 1$,

$$
\alpha_4 = \lambda \exp(t \ell) \leq \lambda(1 + 2t \ell) \leq \frac{1}{2} + \frac{1}{2} \lambda
$$

where the second step is because $t \ell < 1$, and the third step follows by $t \ell \leq \frac{1 - \lambda}{4\lambda}$.

Overall, we have

$$
\left(\alpha_1 + \frac{\alpha_2 \alpha_3}{1 - \alpha_4}\right)^k \leq \left(1 + t^2 (\gamma^2 + b^2) + \frac{4 \lambda t^2 (\gamma^2 + b^2)}{\frac{1}{2} - \frac{1}{2} \lambda}\right)^k
$$

$$
\leq \exp\left(kt^2 (\gamma^2 + b^2) \left(1 + \frac{8}{1 - \lambda}\right)\right).
$$

This completes our proof of Lemma [1]. \qed