Seasonal fractional long-memory processes. A semiparametric estimation approach

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Abstract

This paper explores seasonal and long-memory time series properties by using the seasonal fractional ARIMA model when the seasonal data has one and two seasonal periods and short-memory counterparts. The stationarity and invertibility parameter conditions are established for the model studied. To estimate the memory parameters, the method given in Reisen, Rodrigues and Palma (2006 a,b) is generalized here to deal with a time series with two seasonal fractional long-memory parameters. The asymptotic properties are established and the accuracy of the method is investigated through Monte Carlo experiments. The good performance of the estimator indicates that it can be an alternative competitive procedure to estimate seasonal long-memory time series data. Artificial and PM$_{10}$ series were considered as examples of applications of the proposed estimation method.

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1 Introduction

Time series exhibiting seasonal or cyclical characteristics are very common in economics, hydrology, and many other disciplines. As a consequence, several methodologies have been developed to deal with these features. One of the most well known of these tools is the class of seasonal autoregressive integrated moving average (SARIMA) process. This model can describe many time series containing a mixture of seasonal phenomena of different periods. It is well known that many series may contain a persistent seasonal structure along with a long run trend. However, the case of more than one seasonal (non-zero) structures has not very much studied. In this direction, the study of the ARFIMA process with seasonal periods becomes an interesting research topic, and this is the main motivation of this work.

Let $X_t \equiv \{X_t\}_{t \in \mathbb{Z}}$ be a time series with a zero mean and a constant variance. A multiple seasonal ARIMA model can be written as follows:

$$M \prod_{j=1}^{M} \phi_j(B) \nabla^d(B) X_t = \prod_{\ell=1}^{N} \theta_\ell(B) \varepsilon_t,$$

where $\nabla^d(B) \equiv \prod_{i=1}^{k} (1 - B^{s_i})^{d_i}$, $M$ and $N$ are, respectively, the number of factors of the AR and MA components, $d_i \in \mathbb{N}$, $i = 1, ..., k$, is the differencing parameter and $k$ is the number of differencing factors, $s_i$ is the $i$-seasonal period, $BX_t = X_{t-1}$, and $\varepsilon_t$ is a white noise process with zero-mean and variance $\sigma^2_\varepsilon$. $\phi_j(B), j = 1, ..., M$, and $\theta_\ell(B), \ell = 1, ..., N,$ can also be polynomials with seasonal effects. The stationarity and invertibility properties of model (1) are established based on certain parameter conditions. The multiple seasonal ARIMA model belongs to a class of models with a
general difference operator given by

\[ \nabla^d(B) \equiv \prod_{\ell=1}^{L} [(1 - Be^{i\lambda_\ell})(1 - Be^{-i\lambda_\ell})]^{d_\ell}, \quad (2) \]

where now \( d_\ell \in \mathbb{R} \) \( (d_\ell > -1) \) and it is defined as the \textit{fractionally differencing parameter} and \( \lambda_\ell, \ell = 1, \ldots, L \), are fixed frequencies in the range \([-\pi, \pi]\). For suitable choices of the fractional parameters, time series models with filter given in (2) may have a finite number of zeros or singularities of order \( d_1, \ldots, d_L \) on the unit circle which allows the modeling of long and short memory data containing seasonal periodicities. In the time domain, the usual definition of long memory is the non-summability

\[ \sum_{h=0}^{\infty} |\gamma(h)| = \infty, \]

where \( \gamma(h) \) is the autocovariance at lag \( h \) of the process, whereas, in the frequency domain, this property is defined by the fact that the spectral density of the process becomes unbounded at some frequency in \([0, \pi]\).

A time series with both seasonal and non-seasonal fractional differencing parameters has a spectral density specified by

\[ f(\lambda) = f^*(\lambda)|\lambda|^{-2d} \prod_{i=1}^{k} \prod_{j=1}^{\xi_i} |\lambda - \lambda_{ij}|^{-2d_i}, \quad (3) \]

where \( d_i \in \mathbb{R} \) \( (d_i > -1) \), \( \lambda \in (-\pi, \pi] \), \( f^*(\lambda) \) is a continuous function, bounded above and away from zero and \( \lambda_{ij} \neq 0 \) are poles for \( j = 1, \ldots, \xi_i, \ i = 1, \ldots, k \). Processes with a spectral density given by (3) have been discussed by Arteche and Robinson (1999), Giraitis and Leipus (1995), Leipus and Viano (2000), Palma and Chan (2005) and Palma (2007, Ch.12), among others, to model time series with seasonal and cyclical long-memory behavior.

The main interest in models which have filter (2) and spectral density of the form (3) is related to the estimation of fractional memory parameters \( d_1, \ldots, d_L \). Ray (1993) used IBM product revenues to illustrate the usefulness
of modeling seasonal fractionally differenced ARMA models by allowing two seasonal fractional differencing parameters in the model, one at lag three and the other at lag twelve. Other papers related to this topic are, for example, Hassler (1994), Gray et al. (1989,1994), Giraitis and Leipus (1995), Ooms and France (2001), Woodward et al. (1998), Arteche and Robinson (2000), Palma and Chan (2005) and Reisen et al. (2006a,b), Gil-Alana (2001) among others. Hassler (1994) introduced the rigid and flexible filters and an application of this methodology to the economic activities in the Euro area is discussed in Ferrara and Guegan (2006). Arteche and Robinson (2000), Arteche (2002) and Arteche and Velasco (2005) dealt with robust semiparametric estimators and testing procedures for the seasonal fractional memory parameter. Woodward et al. (1998) extended the Gegenbauer ARMA process (GARMA). Independently of these works, a time series model for fitting long or short-memory data containing seasonal periodicities was introduced by Giraitis and Leipus (1995) which is called the Fractionally Autoregressive Unit Circle Moving Average model (ARUMA). These authors discussed the asymptotic properties of the ARUMA model and the estimation of its parameters.

Another equally relevant publication related to the asymptotic properties of seasonal and periodic time series is the work by Viano et al. (1995). Reisen et al. (2006a,b) dealt with the estimation of the seasonal ARFIMA model with long-memory innovations (SARFIMA (0, d, 0) × (0, D, 0) s) by using different estimation procedures for the seasonal and non-seasonal memory parameters, that is, for D and d respectively. The estimators are based on the multilinear regression equation of log f(·), where f(·) is the spectral density of the process satisfying $f(\lambda) \sim C^*|\lambda|^{-2(d+D)}$ as $\lambda \to 0$, where $C^*$ is a positive constant. Necessary conditions that guarantee the stationary and invertibility of the model were also established. Through Monte Carlo experiments, they compared their proposed methodology with other well-known parametric estimation procedures such as the Whittle and the maximum likelihood methods. The empirical evidence showed that the multilinear regression es-
timators are very promising.

Most of the works referred to above deal with the estimation of one seasonal long-memory parameter. However, in many practical situations the time series exhibits more than one seasonal component. In order to explore these more complex situations, this paper focuses on the estimation of models containing one and two seasonal periods which encompass long and short-memory dependence structures. Specifically, an ordinary least squares (OLS) procedure, based on a log-periodogram regression, is proposed to estimate all fractional parameters simultaneously.

Let now \( X_t \equiv \{X_t\}_{t \in \mathbb{Z}} \) be a zero-mean time series defined by

\[
\nabla^d (B) X_t \equiv (1 - B^{s_1})^{d_1}(1 - B^{s_2})^{d_2} X_t = \nu_t, \tag{4}
\]

where the vector \( d = (d_1, d_2)' \), \( s_1 \) and \( s_2 \) are seasonal periods and \( d_1, d_2 \in \mathbb{R} \) \((d_i > -1)\) are their seasonal memory parameters, respectively, and \( \nu_t \) has a spectral density that satisfies the following assumption.

**Assumption 1:** The spectral density of \( \nu_t \) satisfies as \( \lambda \to 0 \)

\[
f_{\nu} \left( \frac{2\pi k}{s'} + \lambda \right) = f_{\nu,k} + c_k |\lambda|^{\alpha_k} + O(|\lambda|^{\alpha_k + \iota})
\]

for some \( \iota > 0, \ f_{\nu,k} \equiv f_{\nu} \left( \frac{2\pi k}{s'} \right), \ k = 0, 1, ..., \lfloor s'/2 \rfloor, \ s' = \max(s_1, s_2) = s_1 \) (without loss of generality) and \( \alpha_k = \alpha_1 \) for \( k = 0, s'/2 \) (if \( s' \) even) and \( \alpha_k = \alpha_2 \) otherwise. If \( \nu_t \) is a stationary and invertible ARMA process then \( \alpha_1 = 2, \ \alpha_2 = 1 \) and \( \iota = 1 \). In this case, the process \( X_t \) is usually defined as Seasonal ARFIMA (SARFIMA) model.

In the next section some properties of the model given by (4) are discussed. In particular, the stationarity and invertibility conditions of Model (4) are established in Proposition 1. The estimation of these models is discussed in Section 3, where the proposed ordinary least squares (OLS) estimator is introduced. Some asymptotic properties of these estimators are established in Theorems 1 and 2. For example, the proposed OLS estimator is shown to be asymptotically unbiased and normally distributed. For comparison purposes, the quasi-likelihood Fox-Taqqu estimator (Fox and Taqqu
is adapted here for Gaussian seasonal long-memory processes. The comparison between parametric and semiparametric approaches may appear to be unfair for the former class, in the case of a correct and complete parametric specification. So, the misspecification problem of the FT method is also included to be a part of the simulation section. The finite sample performance of the proposed estimator is investigated in Section 4 while Section 5 discuss some applications. Final remarks are presented in Section 6.

2 Model properties

Let $X_t$ be a time series process defined by (4). For simplicity, it is assumed that $s_1$ and $s_2$ are even numbers. The fractional $d_i$ difference is a generalization of the binomial expression $(1 - B)^d$ and it can be written as

$$(1 - B^{s_i})^{d_i} = \sum_{k=0}^{\infty} \pi_k B^{ks_i},$$

where

$$\pi_0 = 1, \quad \pi_k = \frac{\Gamma(k - d_i)}{\Gamma(k + 1)\Gamma(-d_i)}, \quad i = 1, 2,$$

and $\Gamma(\cdot)$ is the Gamma function.

In the literature of the seasonal long-memory process, there are some specific time series models of interest obtained from the solution of the general fractional operator (2) and the spectral density of form (3). The specific filters and their models are: (a) $(1 - B)^d$ is the filter of the fractional integrated I(d) process see, for example, Hosking (1981), among others; (b) $(1 - B)^{d_1}(1 - B^{s_2})^{d_2}$ is the filter in the SARFIMA process that has been explored in the literature of seasonal fractional ARMA model, see for example, Porter-Hudak (1990), Hassler (1993), Arteche (2002), Arteche and Robinson (2000) and Reisen et al. (2006a,b); (c) $(1 - v_1 B - \cdots - v_l B^d)$ is the filter that belongs to the ARUMA and k-GARMA processes, proposed by giraitis.
and Leipus (1995) and independently by Woodward et al. (1998), respectively; (d) \((1 - B^3)^{d_1}(1 - B^{12})^{d_2}\) is the filter used by Ray (1993) to model and forecast a monthly IBM revenue data under the restriction \(d_1 + d_2 = 1\).

Returning to our specific model of interest for \(X_t\) given in (4), the filter may be written as follows:

\[
(1 - B^{s_1})^{d_1}(1 - B^{s_2})^{d_2} = \prod_{i=1}^{2} \prod_{j=0}^{\xi_i} ((1 - B e^{i\lambda_{ij}})(1 - B e^{-i\lambda_{ij}}))^{d_{ij}}, \tag{5}
\]

where \(\lambda_{ij} = \frac{2\pi j}{s_i} (j = 0, 1, \ldots, \xi_i)\) are the frequencies of the period \(i\), \(\xi_i = \frac{s_i}{2}\) \((i = 1, 2)\), and

\[
\begin{align*}
\lambda_{ij} &= \frac{2\pi j}{s_i} (j = 0, 1, \ldots, \xi_i) \\
d_{ij} &= \begin{cases} d_1 & \text{ when } \lambda_{1j} \neq \lambda_{2j}, 0, \pi; \\
d_1/2 & \text{ when } \lambda_{1j} \neq \lambda_{2j} \text{ and } \lambda_{1j} = 0, \pi; \\
d_{1j} + d_{2j} = \frac{d_1 + d_2}{2} & \text{ when } \lambda_{1j} = \lambda_{2j} = 0, \pi; \\
d_{1j} + d_{2j} = d_1 + d_2 & \text{ when } \lambda_{1j} = \lambda_{2j} \neq 0, \pi;
\end{cases} \tag{6}
\end{align*}
\]

and similarly when \(i = 2\)

It is easy to show that the filter (5) is a particular case of the operator \((2)\) by using the equality

\[
1 - z^s = (1 - z)(1 + z)^{\frac{s-1}{2}} \prod_{k=1}^{\frac{s-1}{2}} (1 - ze^{2\pi ik/s})(1 - ze^{-2\pi ik/s}), \text{ for } s \text{ even.}
\]

When \(s\) is an odd number, the term \((1 + z)\) does not appear in the above equation. From the expression of \(1 - z^s\), the following proposition is reached:

**Proposition 1.** Let the process \(X_t\) be a solution of equation

\[
X_t = (1 - B^{s_1})^{-d_1}(1 - B^{s_2})^{-d_2}\nu_t, \tag{7}
\]

where \(\nu_t\) is a covariance stationary ARMA process \((\nu_t = \frac{\Theta(B)}{\Phi(B)}\epsilon_t)\), \(\epsilon_t\) is an i.i.d Gaussian sequence with zero mean and variance \(\sigma^2\), and \(d_i \in \mathbb{R}\) is the fractional parameter at seasonal period \(s_i\) for \(i = 1, 2\). Then,
(a) The process $X_t$ is stationary and invertible if $|d_1 + d_2| < 1/2$ and $|d_i| < 1/2$, $i = 1, 2$.

(b) The spectral density of $X_t$ is given by

$$ f(\lambda) = f_\nu(\lambda) \prod_{i=1}^{2} \prod_{j=0}^{\xi_i} |2 \sin(\frac{\lambda_i}{2})| \sin(\frac{\lambda_i + \lambda_j}{2})|^{-2d_{ij}} $$

$$ = f_\nu(\lambda) \left( \frac{2 \sin \frac{\lambda_i}{2}}{2 \sin \frac{\lambda_j}{2}} \right)^{-2d_{ij}} \left( \frac{2 \sin \frac{\lambda_j}{2}}{2 \sin \frac{\lambda_i}{2}} \right)^{-2d_{ij}}, $$

where $f_\nu(\lambda) \ (0 \leq \lambda \leq \pi)$ is the spectral density of $\nu_t$, $\lambda_{ij} = \frac{2\pi j}{s_i}$, $i = 1, 2$ and $j = 0, 1, ..., \frac{s_i}{2}$, and $d_{ij}$ are given by (6).

(c) Assuming that $\max\{d_{ij}\} > 0$, the asymptotic autocovariance of $X_t$, $\gamma(h) = E(X_h X_0)$, is given by

$$ \gamma(h) = \sum_{i=1}^{2} \sum_{j=1}^{\xi_i} a_{ij} \ |h|^{2d_{ij}-1} (\cos h\lambda_{ij} + o(1)) \quad \text{as} \quad k \to \infty, $$

where

$$ a_{ij} = \begin{cases} a'_{ij} & \lambda_{ij} = 0, \pi \\
2a_{ij} & 0 < \lambda_{ij} < \pi, \end{cases} $$

$d_{ij}$ is specified as in (6) and

$$ a'_{ij} = \left| \frac{\Theta(e^{-2\pi \lambda_{ij}})}{\Phi(e^{-2\pi \lambda_{ij}})} \right|^2 \frac{\sigma^2}{\pi} \Gamma(1 - 2d_{ij}) \sin(d_{ij}\pi) D_{ij}^2, $$

where

$$ D_{ij} = \begin{cases} 2 \sin \lambda_{ij} & -d_{ij} \prod_{\ell \neq j} (\cos \lambda_{ij} - \cos \lambda_{i\ell}) |^{-d_{ij}}, \quad 0 < \lambda_{ij} < \pi, \\
\prod_{\ell \neq j} (\cos \lambda_{ij} - \cos \lambda_{i\ell}) |^{-d_{ij}}, & \lambda_{ij} = 0, \pi. \end{cases} $$

Proof. (a) As previously noted, filter (5) is a particular case of (2) which is the operator of the ARUMA($p, d_1, ..., d_L, q$) model where $p$ and $q$ are the
polynomial orders of a stationary and invertible ARMA\((p, q)\) process. From Theorem 1 of Giraitis and Leipus (1995), the ARUMA \((0, d_1, ..., d_L, 0)\) process is stationary and invertible if the fractional parameters \(d_\ell, \ell = 1, ..., L\), in (2) satisfy \(|d_\ell| < 1/2\) when \(\lambda_\ell \neq 0, \pi\) and \(|d_\ell| < 1/4\) otherwise. From this fact and by means of equations (5) and (6), it is straightforward to establish the stationary and invertibility properties. The proof of (b) is immediately obtained from (3) and Theorem 2 in Giraitis and Leipus (1995). This theorem is also used to prove the asymptotic covariance given in (c) where \(d_{ij}\) is defined by (6).

\[\]

3 Seasonal fractional parameter estimators

This section deals with the estimation method based on the regression equation of \(\log f(\lambda)\) to obtain the estimates of model (7). Since the procedure proposed here provides simultaneous estimates for multiple seasonal memory parameters, the method is a more general approach than those discussed in Reisen et al. (2006a,b) and related references. Let \(n\) be the sample size and let \(X_1, \ldots, X_n\) be a realization of the process defined by (7), where \(\nu_t\) is a Gaussian ARMA process. The well-known periodogram function \(I(\lambda) = (2\pi n)^{-1} |\sum_{t=1}^{n} X_t e^{i\lambda t}|^2\) is an asymptotic unbiased and inconsistent estimator the spectral density and it is the standard estimator used in time series modeling.

3.1 The OLS regression estimators

The fractional memory OLS estimators are the slope estimators of the multiple regression equation

\[
\log I_{k,j} = a_k - 2d_1 \log X_{1,k,j} - 2d_2 \log X_{2,k,j} + V_{k,j}, \quad k = 0, 1, ..., \lfloor s'/2 \rfloor, \quad (8)
\]
where \( s' = \max(s_1, s_2) \), \( j = 1, \ldots, m \) ( \( m \in \mathbb{N}^* \)) if \( k = 0 \), \( j = -1, \ldots, -m \)
if \( k = s'/2 \) ( \( s \) even) and \( j = \pm 1, \ldots, \pm m \) otherwise, \( I_{k,j} = I(2\pi k/s' + \lambda_j) \),
\( \lambda_j = 2\pi j/n \) is the Fourier frequency, \([ \cdot ]\) means the integer part and
\[
a_k = \log f_{\nu,k} + E \left( \log \frac{I_{k,j}}{f_{k,j}} \right)
\]
\[
V_{k,j} = U_{k,j} + \varepsilon_{k,j}
\]
\[
U_{k,j} = \log \frac{I_{k,j}}{f_{k,j}} - E \left( \log \frac{I_{k,j}}{f_{k,j}} \right)
\]
\[
\varepsilon_{k,j} = \log \frac{f_{\nu, k}^{\left( \frac{2\pi k}{s'} + \lambda_j \right)}}{f_{\nu, k}} = b_k \lambda_j^\alpha + O(\lambda_j^{\alpha+\iota})
\]
\[
X_{1,k,j} = 2 \sin \left( \frac{s_1}{2} \left[ \frac{2\pi k}{s'} + \lambda_j \right] \right)
\]
\[
X_{2,k,j} = 2 \sin \left( \frac{s_2}{2} \left[ \frac{2\pi k}{s'} + \lambda_j \right] \right)
\]
for \( f_{k,j} = f \left( \frac{2\pi k}{s'} + \lambda_j \right) \) and \( b_k = c_k / f_{\nu,k} \). The regression equation (8) is easily
derived from the expression of the \( \log f(\lambda) \) where \( f(\lambda) \) is the spectral density
given in Proposition 1. To avoid the estimation of the constants \( a_k \), the
variables are locally centered such that the estimates are obtained by least
squares in the regression model
\[
Y_{k,j} = d_1 Z_{1,k,j} + d_2 Z_{2,k,j} + V^*_{k,j}
\]
where \( V^*_{kj} = V_{k,j} - \frac{1}{m_k} \sum^*_{j} V_{k,j} \) for \( m_k = \delta_k m \) with \( \delta_k = 1 \) for \( k = 0, s'/2 \)
and \( \delta_k = 2 \) otherwise and the sum \( \sum^* \) runs for \( j = 1, \ldots, m \) if \( k = 0 \), \( j = -1, \ldots, -m \)
if \( k = s'/2 \) and \( j = \pm 1, \ldots, \pm m \) otherwise. \( Y_{k,j}, Z_{1,k,j} \) and \( Z_{2,k,j} \)
are the locally centered dependent variable and regressors in (8) similarly
defined. The local centering is needed here because the regression model in
(8) has different constants depending on the frequency bandwidth. A global
centering can be used only if \( a_1 = \ldots = a_{\lfloor s'/2 \rfloor} \) which holds for example if \( \nu_t \)
is a white noise process with a constant spectral density function.

The estimation procedure based on the above regression equation is
motivated by the pioneer regression estimator proposed by Geweke and Porter-
Hudak (1983) for the ARFIMA model. Since the introduction of the method,
it has become one of the most popular estimation procedures and its empirical
and asymptotic properties have been well established. Robinson (1995) and
Hurvich, Deo and Brodsky (1998) proved that the GPH-estimator is consis-
tent and asymptotically normal for Gaussian time series processes. Hurvich
et al. (1998) also established that the optimal bandwidth is of order \( O(n^{4/5}) \).

When the model is a SARFIMA(0, d, 0) \times (0, d_s, 0)\_s process, Reisen et
al. (2006a,b) proposed different estimation methods for \( d_s \) and \( d \). Basically,
the regression estimators considered in their study are distinguished by the
choice of the bandwidth when regressing \( \log[I(\lambda)] \) on \( \log[2 \sin(\lambda s/2)] \) and
\( \log[2 \sin(\lambda/2)] \). Following the same direction, their study is generalized here
in the case where the model has two seasonal fractional parameters \( d_1 \) and
\( d_2 \) for the seasonal periods \( s_1 \) and \( s_2 \), respectively.

**Assumption 2:** \( s_1 \) is a multiple of \( s_2 \).

**Assumption 3:** Let \( m = m(n) \) is a sequence satisfying
\[
\left( \frac{m}{n} \right) \log m + \frac{1}{m} \to 0 \quad \text{as} \quad n \to \infty
\]
for some \( \iota > 0 \).

**Assumption 4:**
\[
\frac{m^{\alpha^* + 0.5}}{m^{\alpha^*}} \to K \quad \text{as} \quad n \to \infty
\]
where \( \alpha^* = \min(\alpha_1, \alpha_2) \).

**Theorem 1.** Under assumptions 1, 2 and 3, as \( n \to \infty \),
\[
E(\hat{d}) - d = Q^{-1}b_n(1 + o(1))
\]
\[
\text{Var}(\hat{d}) = m^{-1} \frac{\pi^2}{6} Q^{-1}(1 + o(1))
\]
where
\[
b_n = -2 \left( \sum_{k=0}^{[s'/2]} b_k \delta_k (2\pi)^{\alpha_k} \frac{\alpha_k}{(\alpha_k + 1)^2} \left( \frac{m}{n} \right)^{\alpha_k} \right) \]
where $I_k = \{0, k \text{ such that } ks_2 \text{ is a multiple of } s' \}$, $\delta_k = 1$ for $k = 0,s'/2$ and $\delta_k = 2$ otherwise. In consequence, $\hat{d}$ is consistent.

**Theorem 2.** Under assumptions 1, 2, 3 and 4, as $n \to \infty$,

$$\sqrt{m}(\hat{d} - d) \xrightarrow{d} N \left( Q^{-1}b, \frac{\pi^2}{6}Q^{-1} \right)$$

for

$$b = -2(2\pi)^{\alpha^*} \frac{\alpha^*}{(\alpha^* + 1)^2} K \left( \frac{\sum_{k \in J_k} b_k \delta_k}{\sum_{k \in I_k \cap J_k} b_k \delta_k} \right),$$

where $J_k = \{k \text{ such that } \alpha_k = \alpha^* \}$.

Proofs of the above theorems are in Appendix A.

As a particular case of Model 7, the statistical properties of the SARFIMA(0, $d$, 0)$_s$ model, with $\nu_t \equiv \epsilon_t$, are now discussed.

The OLS estimator of $d$ is given by

$$\hat{d} = (-0.5) \frac{\sum_{k=0}^{[s'/2]} \sum_{j=1}^{m} (X_{1,k,j} - \bar{X}_1) \log I_{k,j}}{\sum_{k=0}^{[s'/2]} \sum_{j=1}^{m} (X_{1,k,j} - \bar{X}_1)^2}, \quad (10)$$

where $X_{1,k,j} = \log \{2 \sin((s\lambda_{k,j}/2))\}$. By simple algebra, the following expression is reached.

$$\hat{d} - d \approx -\frac{1}{2S_{X_1,X_1}} \sum_{k=0}^{[s'/2]} \sum_{j=1}^{m} (X_{1,k,j} - \bar{X}_1)U_{k,j}, \quad (11)$$

where $S_{X_1,X_1} = \sum_{k=0}^{[s'/2]} \sum_{j=1}^{m} (X_{1,k,j} - \bar{X}_1)^2$ and $j$ is defined as in (8).

**Proposition 2.** Let $X_t$ be a SARFIMA(0, $d$, 0)$_s$ model and $\hat{d}$ is the OLS estimator of $d$ provided by (10). Under assumptions 1 to 4, as $n \to \infty$,

(a) $E(\hat{d}) \approx d.$
(b) The variance of the estimator is given by
\[ \text{Var}(\hat{d}) \approx \frac{\pi^2}{24sm}. \] (12)

(c) The estimate \( \hat{d} \) satisfies
\[ \sqrt{m}(\hat{d} - d) \overset{d}{\rightarrow} N\left(0, \frac{\pi^2}{24s}\right). \]

\[ \begin{proof} \] The above results are particular cases of Theorem 1 and coincide with Theorems 1 and 2 in Hurvich, Deo and Brodsky (1998). Note that the variance of the estimator suggested in Porter-Hudak (1990) and Ray (1993) is approximately \( 4s\text{Var}(\hat{d}) \).
\end{proof} \]

4 Finite sample investigation

The finite sample performance of the estimator discussed previously is investigated in this section through Monte Carlo experiments for different structures of Model 7 where \( \nu_t \) follows a SARMA model. To generate the models, the procedure used is the one suggested in Hosking (1984) with i.i.d innovations from a \( N(0,1) \) distribution. The models are: SARFIMA(\( p, d_1, q_1 \)),(\( P, d_2, Q_2 \)) with \( p = P = 0, 1, s_1 = 1, 4, s_2 = 4, 12 \) and the AR non-seasonal (\( \phi_1 \)) and seasonal (\( \phi_s \)) parameters with values \( \phi_1 = \phi_s = 0.0, 0.3 \) and 0.8. The parameters are also displayed in the tables. The empirical investigations were based on sample size \( n = 1080 \), and the sample quantities mean, correlation and mean squared error (mse) of the estimators were calculated over 2,000 replications. The calculations were carried out by means of an Ox program in an AMD Athlon XP 1800 computer.

Since the models also involve short-range dynamics, the regression estimators were obtained by using different bandwidths. In the case where the model has not AR contribution, the bandwidth \( m = \left\lfloor \frac{n-1}{\max(s_1,s_2)} \right\rfloor \) was fixed.
In this context, the regression estimator \((GPH_T)\) becomes a parametric procedure. For the models with short-memory dynamics, the two bandwidths \(m = n^{\alpha_i}, \alpha_1 = 0.5\) and \(\alpha_2 = 0.3\) were used, and the estimators are denoted as \(GPH_1\) and \(GPH_2\), respectively. The bandwidth \(n^{0.5}\) is here considered because this specification has been widely used in the case of ARFIMA models with short-memory components, while the choice of \(n^{0.3}\) is based on the empirical investigation discussed below.

For a comparison purpose between semiparametric and parametric approaches, the procedure due to Fox and Taqqu (1986) (FT), which possesses good asymptotic properties, is also adapted here for Gaussian seasonal long-memory processes. This estimator is obtained by using all harmonic frequencies between the seasonal frequencies. It is calculated by minimizing the approximate Gaussian log-likelihood

\[
\mathcal{L}_W(\theta) = \frac{1}{2n} \sum_j \left\{ \ln f(\lambda_j) + \frac{I(\lambda_j)}{f(\lambda_j)} \right\}, \tag{13}
\]

where \(f(\lambda)\) is the spectral density, \(\theta\) denotes the vector of unknown parameters and \(\sum_j\) is the sum over \(j = 1, \ldots, n - 1\), excluding those values \(\lambda_j\) coinciding with the seasonal frequencies. Under some conditions, the FT estimator for non-seasonal ARFIMA models, is asymptotically normal and consistent, and for Gaussian process, the estimator is also asymptotically efficient (Giraitis and Surgailis, 1990, Fox and Taqqu, 1986 and others). By assuming that the poles of the spectral density \(f(\lambda_j)\) are known, the asymptotic theory for the FT method can be extended to the SARFIMA model (see the discussion in Arteche and Robinson (2000), Section 2). It should be noted that the focus of this paper is to estimate the seasonal fractional parameters only even though the parametric FT method also provides estimates for the AR parameters.

Table \[\text{II}\] summarizes the results for the SARFIMA model with \(d_1 = 0.3\) \((s_1 = 4)\). The first part of this table shows the performance of the regression methods when there is no seasonal AR contribution. For the case of \(\phi_1 = 0\), \(GPH_T\) has the best performance among the GPH based ones, which
is an expected result since the method uses all non-seasonal frequencies in the regression equation and in this sense it is parametric and comparable to the FT. The effect of the bandwidth is also a motivation of this study. The reduction of the bandwidth causes an increase in the \( \text{mse} \), especially when \( \phi_1 = 0 \). This is a not surprising result, since the AR contribution is mainly concentrated at zero frequency. The absence of short-memory component allows a wider bandwidth because the bias is quite controlled. Thus, a reduction of \( m \) implies a larger variance and does not reduce the bias.

In the second part of the table, the estimates were computed when the model has the AR contribution at the seasonal period \( s = 4 \). From this, the GPH estimates are more affected by the AR component than the previous case, the bias is strongly positive and the \( \text{mse} \) also increases. In this case the AR component has spectral power not only at frequency zero but also at the seasonal ones, affecting to a greater extent the estimation of \( d_1 \). The small value of the bandwidth mitigates the effect of the short-memory parameters. This is clearer when \( \phi_4 \) changes from 0.3 to 0.8. Hence, in this context, the decrease of the bandwidth produces reduction on the size of the bias and the \( \text{mse} \).

In general, the FT estimates outperform the GPH estimates in terms of the \( \text{mse} \), which is not surprising considering the parametric nature of the FT method in a correctly specified model. However, the FT estimates also present a significant increase of the bias when the model has AR seasonal components. The bias of the estimates also increases substantially when there is model misspecification, as can be seen from the examples presented in Tables 4 and 5. The misspecification problem will be discussed in the end of this section.

The following tables present the estimates when the models have more than one fractional parameter. Thus, the sample correlations between the estimates were also calculated.

Table 2 displays the result when the models are SARFIMA \( (1, d_1, 0)_{s_1} (1, d_2, 0)_{s_2} \) with \( d_1 = 0.1 (s_1 = 1) \), \( d_2 = 0.3 (s_2 = 4) \), \( \phi_1 = 0.0, 0.3, 0.8 \) and \( \phi_4 = \)
Table 1: Results for the seasonal ARFIMA model with $d_1 = 0.3 \ (s_1 = 4)$ and $\phi_s, \ s = 1, 4, \ n = 1080$.

| $\phi_s$ | estimators | $d_1$ mean | $d_1$ mse | $\hat{\phi}$ mean | $\hat{\phi}$ mse |
|---------|------------|-------------|------------|---------------------|------------------|
| $\phi_1 = 0.0$ | GPH$_T$ | 0.3004 | 0.0012 | — | — |
|          | GPH$_1$ | 0.2988 | 0.0046 | — | — |
|          | GPH$_2$ | 0.2984 | 0.0311 | — | — |
|          | FT | 0.2885 | 0.0009 | — | — |
| $\phi_1 = 0.3$ | GPH$_1$ | 0.3002 | 0.0041 | — | — |
|          | GPH$_2$ | 0.3074 | 0.0255 | — | — |
|          | FT | 0.2888 | 0.0009 | 0.2980 | 0.0009 |
| $\phi_1 = 0.8$ | GPH$_1$ | 0.3097 | 0.0054 | — | — |
|          | GPH$_2$ | 0.3085 | 0.0242 | — | — |
|          | FT | 0.2828 | 0.0011 | 0.8011 | 0.0004 |
| $\phi_4 = 0.3$ | GPH$_1$ | 0.3459 | 0.0068 | — | — |
|          | GPH$_2$ | 0.3114 | 0.0278 | — | — |
|          | FT | 0.1792 | 0.0463 | 0.4144 | 0.0433 |
| $\phi_4 = 0.8$ | GPH$_1$ | 0.7281 | 0.1877 | — | — |
|          | GPH$_2$ | 0.4144 | 0.0426 | — | — |
|          | FT | 0.2519 | 0.0076 | 0.8141 | 0.0035 |

0.3, 0.8 whereas Table 3 shows the performance of the estimates when the SARFIMA model has seasonal periods $s_1 = 4$ and $s_2 = 12$. From Table 2 it should be noted that the contribution of the parameter $d_1$ is mainly at zero frequency. Hence, in general, the semiparametric estimators perform similarly to the previous case that is, the estimate of $d_1$ depends on the values of the bandwidth and of the AR counterpart. The memory parameters are estimated simultaneously, thus there is a balance effect between the two estimates $\hat{d}_1$ and $\hat{d}_2$ which justifies the negative correlation values between them. In addition, the estimates are balanced to have the value of $\hat{d}_1 + \hat{d}_2$ approximately equal to $d_1 + d_2$ which is the total memory at zero frequency. The correlations between the GPH estimates increases with the bandwidth and the AR coefficients. As was expected, the FT method presents superiority.
performance compared with the semiparametric approaches.

Table 2: Results for models $d_1 = 0.1$ ($s_1 = 1$), $d_2 = 0.3$ ($s_2 = 4$) and $\phi_s$, $s = 1, 4$, case $n = 1080$.

| $\phi_s$ | estimators | $d_1$ | corr. | $d_2$ | $\phi$ |
|----------|------------|-------|-------|-------|-------|
|          |            | mean  | mse   | mean  | mse   | mean  | mse   |
| $\phi_1 = 0.0$ | GPH$_r$ | 0.1043 | 0.0018 | -0.2228 | 0.3010 | 0.0013 | —     | —     |
|          | GPH$_1$   | 0.1135 | 0.0290 | -0.5144 | 0.2995 | 0.0053 | —     | —     |
|          | GPH$_2$   | 0.0818 | 0.1327 | -0.4164 | 0.3121 | 0.0388 | —     | —     |
|          | FT        | 0.1008 | 0.0006 | -0.1188 | 0.2868 | 0.0009 | —     | —     |
| $\phi_1 = 0.3$ | GPH$_1$ | 0.1166 | 0.0215 | -0.3397 | 0.3098 | 0.0045 | —     | —     |
|          | GPH$_2$   | 0.1463 | 0.1553 | -0.4927 | 0.3083 | 0.0308 | —     | —     |
|          | FT        | 0.0983 | 0.0050 | -0.4216 | 0.2893 | 0.0009 | 0.2997 | 0.0060 |
| $\phi_1 = 0.8$ | GPH$_1$ | 0.2208 | 0.0388 | -0.5415 | 0.3004 | 0.0062 | —     | —     |
|          | GPH$_2$   | 0.1257 | 0.1607 | -0.5190 | 0.3148 | 0.0375 | —     | —     |
|          | FT        | 0.1069 | 0.0148 | -0.0868 | 0.2819 | 0.0014 | 0.7857 | 0.0124 |
| $\phi_4 = 0.3$ | GPH$_1$ | 0.1074 | 0.0249 | -0.4751 | 0.3404 | 0.0083 | —     | —     |
|          | GPH$_2$   | 0.1107 | 0.1299 | -0.4233 | 0.3037 | 0.0348 | —     | —     |
|          | FT        | 0.1006 | 0.0060 | -0.0780 | 0.1828 | 0.0404 | 0.4135 | 0.0393 |
| $\phi_4 = 0.8$ | GPH$_1$ | 0.1045 | 0.0285 | -0.5011 | 0.7269 | 0.1874 | —     | —     |
|          | GPH$_2$   | 0.0445 | 0.1591 | -0.4773 | 0.4251 | 0.0549 | —     | —     |
|          | FT        | 0.1073 | 0.0011 | -0.2241 | 0.2422 | 0.0079 | 0.8183 | 0.0035 |

Although the model in Table 3 has fractional parameters at seasonality periods 4 and 12, similar conclusions of the performance of the estimates to the previous cases are observed.
Table 3: Results for the SARFIMA model with $d_1 = 0.1$ ($s_1 = 4$), $d_2 = 0.3$ ($s_2 = 12$), $\phi_s$, $s = 1, 4, 12$ and $n = 1080$.

| $\phi_s$ | estimators | $d_1$ | Corr. | $d_2$ | $\hat{\phi}$ |
|----------|------------|-------|-------|-------|--------------|
|          |            | mean  | mse   | mean  | mse          | mean  | mse    |
| $\phi_1 = 0.0$ | GPH$_T$   | 0.1047| 0.0018| −0.3194| 0.3065 | 0.0015| —     | —     |
|          | GPH$_1$    | 0.0994| 0.0052| −0.4405| 0.3071 | 0.0021| —     | —     |
|          | GPH$_2$    | 0.0723| 0.0442| −0.5529| 0.3095 | 0.0113| —     | —     |
|          | FT         | 0.1022| 0.0007| −0.2856| 0.2637 | 0.0021| —     | —     |
| $\phi_1 = 0.3$ | GPH$_1$   | 0.1063| 0.0061| −0.5230| 0.3017 | 0.0022| —     | —     |
|          | GPH$_2$    | 0.0797| 0.0433| −0.5446| 0.2954 | 0.0119| —     | —     |
|          | FT         | 0.1012| 0.0009| −0.3759| 0.2630 | 0.0023| 0.2992| 0.0008|
| $\phi_1 = 0.8$ | GPH$_1$   | 0.1468| 0.0067| −0.5212| 0.2861 | 0.0030| —     | —     |
|          | GPH$_2$    | 0.1177| 0.0374| −0.5767| 0.2861 | 0.0136| —     | —     |
|          | FT         | 0.1010| 0.0008| −0.1308| 0.2619 | 0.0023| 0.7996| 0.0004|
| $\phi_4 = 0.3$ | GPH$_1$   | 0.2043| 0.0160| −0.4707| 0.2813 | 0.0020| —     | —     |
|          | GPH$_2$    | 0.1146| 0.0338| −0.4223| 0.3019 | 0.0084| —     | —     |
|          | FT         | 0.0609| 0.0319| −0.1445| 0.2575 | 0.0028| 0.3430| 0.0296|
| $\phi_4 = 0.8$ | GPH$_1$   | 0.5962| 0.2528| −0.4792| 0.2618 | 0.0038| —     | —     |
|          | GPH$_2$    | 0.2634| 0.0625| −0.5317| 0.2912 | 0.0130| —     | —     |
|          | FT         | 0.0754| 0.0234| 0.1184 | 0.2343 | 0.0054| 0.8121| 0.0203|
| $\phi_{12} = 0.3$ | GPH$_{\alpha_1}$ | 0.1055| 0.0062| −0.5344| 0.5044 | 0.0439| —     | —     |
|          | GPH$_{\alpha_2}$ | 0.1255| 0.0481| −0.6629| 0.3480 | 0.0161| —     | —     |
|          | FT         | 0.1010| 0.0008| −0.0427| 0.2881 | 0.0031| 0.3341| 0.0311|
| $\phi_{12} = 0.8$ | GPH$_{\alpha_1}$ | 0.0863| 0.0063| −0.4587| 1.0064 | 0.5010| —     | —     |
|          | GPH$_{\alpha_2}$ | 0.1089| 0.0354| −0.5466| 0.7553 | 0.2187| —     | —     |
|          | FT         | 0.1071| 0.0017| −0.2963| 0.1859 | 0.0197| 0.8190| 0.0042|

As an additional illustrative form to observe the method’s performance, the box-plots in Figures 1 and 2 show the variation of the estimates for the model in Table 3 with $\phi_1 = 0.0$ and $\phi_1 = 0.3$, respectively.
The asymptotic distribution given in Theorem 2 is also empirically investigated for the model in Table 3 with $\phi = 0$, and the results are depicted in Figure 3 which presents the empirical densities of the standardized GPH estimates of a SARFIMA model. These figures are examples to support the claim given in Theorem 2. The empirical densities of the estimates appear to be fairly close to the density of $N(0,1)$ distribution.
Figure 3: Empirical densities of the standardized GPH estimates of $d_1$ (a) and $d_2$ (b) for the SARFIMA model with $d_1 = 0.1(s_1 = 4)$ and $d_2 = 0.3(s_2 = 12)$.

Model Misspecification

As previously mentioned, the FT method was included in the study to compare the finite sample property between semiparametric and parametric approaches. This comparison would be unfairly biased against the semiparametric estimator proposed here if the empirical investigation was only based under correct model specification. Then, the next two tables deal with the estimation of the model under model misspecification of some models considered in Tables 2 and 3. The simulated models have AR parts whereas these short-memory parameters are omitted in the estimated models. Thus, the order misspecification is related to the non specification of the short-memory dynamics in the estimated model. Tables 4 and 5 give the FT estimates for the SARFIMA($0, d_1, 0)_{s_1}, (0, d_2, 0)_{s_2}$ models with periods $s_1 = 1$ and $s_2 = 4$ and $s_1 = 4$ and $s_2 = 12$, respectively.

In contrast to the study presented in Tables 2 and 3, an order misspecification, however, radically alters the performance of the parametric FT method and the semiparametric GPH here proposed tends to perform significantly better. The FT estimates are highly biased. It is not surprising that the significative increase of the bias and $mse$ are closely related to omitting the seasonal or non-seasonal AR value. For example, in the first two cases of
Table 4, the order misspecification is due to the non-seasonal short-dynamic part. It produces a significative positive bias of the non-seasonal memory parameter \( (d_1) \) whereas the estimate of \( d_2 \) is much less affected. The bias and the \( mse \) increase significantly when the seasonal or non-seasonal AR parameter is close to the non-stationary region. In the second part of Table 4, the results are on the contrary to the estimation performance of the vector \( d \) observed in the first part of the table. Because the short-term contributions are now at a seasonal period, the estimate of the memory parameter at \( s = 4 \), \( d_2 \), is much more affected than \( d_1 \).

In Table 5, the slowly decaying autocorrelations are at period lags \( s_1 = \) and \( s_2 = 12 \). The performance of the parametric FT method is very similar to that previously considered. The biases of the seasonal memory parameters are directly related to the period and magnitude of the AR seasonal and non-seasonal coefficients.

Table 4: Results of FT estimates of SARFIMA\((0, d_1, 0)_{s_1}(0, d_2, 0)_{s_2}\) models. The true SARFIMA models have the parameters \( d_1 = 0.1 \) \( (s_1 = 1) \), \( d_2 = 0.3 \) \( (s_2 = 4) \) and \( \phi_s \), \( s = 1, 4, n = 1080 \).

| \( \phi_s \) | \( d_1 \) | Corr. | \( d_2 \) |
|-------------|---------|-------|---------|
|             | mean    | mse   | mean    | mse     |
| \( \phi_1 = 0.3 \) | 0.3270  | 0.0523 | -0.1504 | 0.2404  | 0.0043 |
| \( \phi_1 = 0.8 \) | 0.8503  | 0.5638 | -0.0553 | 0.2193  | 0.0073 |
| \( \phi_4 = 0.3 \) | 0.0990  | 0.0008 | -0.1641 | 0.5193  | 0.0488 |
| \( \phi_4 = 0.8 \) | 0.1063  | 0.0014 | -0.4924 | 1.0398  | 0.5485 |
Table 5: Results of FT estimates of the SARFIMA($0, d_1, 0)_{s_1}(0, d_2, 0)_{s_2}$ models. The true SARFIMA models have the parameters $d_1 = 0.1$ ($s_1 = 4$), $d_2 = 0.3$ ($s_2 = 12$) and $\phi_s$, $s = 1, 4$ and $12$, $n = 1080$.

| $\phi_s$ | $d_1$ | Corr. | $d_2$ |
|----------|-------|-------|-------|
|          | mean  | mse   | mean  | mse   |
| $\phi_1 = 0.3$ | 0.1079 | 0.0010 | -0.1582 | 0.2600 | 0.0026 |
| $\phi_1 = 0.8$ | 0.4344 | 0.1146 | -0.0411 | 0.1718 | 0.0184 |
| $\phi_4 = 0.3$ | 0.3355 | 0.0562 | -0.2795 | 0.1968 | 0.0118 |
| $\phi_4 = 0.8$ | 0.8647 | 0.5855 | -0.1354 | 0.1748 | 0.0168 |
| $\phi_{12} = 0.3$ | 0.1059 | 0.0007 | -0.0739 | 0.5068 | 0.0435 |
| $\phi_{12} = 0.8$ | 0.1158 | 0.0018 | -0.5797 | 1.0035 | 0.4962 |

5 Examples of Application

This section illustrates the usefulness of the SARFIMA model and the semiparametric fractional estimator using three examples. The first two examples are artificial series and the third example consists of the analysis of daily average PM$_{10}$ concentrations.

5.1 Artificial data

Samples, with sizes $n = 1080$, from models SARFIMA($0, d_1, 0)_{s_1}(0, d_2, 0)_{s_2}$ (Model I) and SARFIMA($1, d_1, 0)_{s_1}(0, d_2, 0)_{s_2}$ (Model II), with $d_1 = 0.1$ ($s_1 = 4$), $d_2 = 0.3$ ($s_2 = 12$) and $\phi_4 = 0.3$, were simulated according to the data generating process described in the previous section. The non-zero AR parameter is the only factor that differentiates the two models. So, the influence of a short-memory parameter in the estimation of the fractional memory parameters of a single series is the main purpose of the analysis of these artificial data sets. The sample autocorrelation functions (ACF) are in Figures 4(a) and (b) for Models I and II, respectively, and the estimates of the models are
in Table 6.

Figure 4: (a) Sample ACF of Model I (b) Sample ACF of Model II.

Model I does not have the AR part. As expected, the ACF only has significant spike at lags which are multiples of 4 and 12, and they appear to have a slow decay pattern (see Figure 4a). The seasonal autocorrelations related to the fractional parameter $d_1$ ($s_1 = 4$) only become insignificant after lag 40. So, if the data had been analyzed with no prior information, inspection of the sample autocorrelation would indicate that the series has seasonal periods $s = 4$ and 12 with possible long-memory structure. As can be seen in Figure 4b, even though both models have the same seasonal long-memory parameters, an introduction of a positive AR coefficient may produce a significant impact on the correlation structure. The ACF of Model II (Figure 4b) also shows slow decaying behavior at seasonal periods, however, with a stronger correlation structure at and between the seasonal periods than Model I and, in general, the patterns of this ACF are more complicated.

Apart from the usual steps for the identification of a single long-memory time series, the estimation of the fractional parameter based on different choice of the bandwidth may be an additional tool when using semiparametric approaches to estimate the fractional parameters. This is exemplified in Table 6. The estimates of the memory parameters were calculated using different bandwidths $m = n^\alpha$, $\alpha = 0.35(0.3)..., max$, where $m$ satisfies the condition previously stated. For each $\alpha$, the corresponding total number
of frequencies used in the regression is given in parenthesis. Table 6 also displays the estimates and their empirical \( mse \)-the square of the bias plus the OLS variance. The smallest value of \( mse \) is given in bold.

Table 6: Estimated parameters of Models I and II.

| \( \phi_s \) | Estim. | \( \alpha \) | \( \hat{d}_1 \) | \( \hat{d}_2 \) | \( mse(\hat{d}_1) \) | \( mse(\hat{d}_2) \) |
|---|---|---|---|---|---|---|
| 0.35  | 0.38  | 0.41  | 0.44  | 0.47  | 0.50  | 0.54 (max.) |
| (132) | (174) | (210) | (258) | (318) | (390) | (534) |
| \( \phi_4 = 0.0 \) | | | | | | |
| \( d_1 \) | 0.1483 | 0.1407 | 0.1185 | 0.1208 | 0.1084 | 0.1112 | 0.1241 |
| \( d_2 \) | 0.4117 | 0.3700 | 0.3518 | 0.3284 | 0.3188 | 0.3021 | 0.3133 |
| \( mse(d_1) \) | 0.0257 | 0.0189 | 0.0135 | 0.0102 | 0.0076 | 0.0058 | 0.0023 |
| \( mse(d_2) \) | 0.0205 | 0.0108 | 0.0073 | 0.0043 | 0.0032 | 0.0022 | 0.0017 |
| \( \phi_4 = 0.3 \) | | | | | | |
| \( d_1 \) | 0.1492 | 0.1396 | 0.1914 | 0.0997 | 0.0887 | 0.1449 | 0.2535 |
| \( d_2 \) | 0.3138 | 0.3715 | 0.3139 | 0.3187 | 0.3199 | 0.3144 | 0.2758 |
| \( mse(\hat{d}_1) \) | 0.0293 | 0.0229 | 0.0241 | 0.0115 | 0.0084 | 0.0086 | 0.0255 |
| \( mse(\hat{d}_2) \) | 0.0094 | 0.0125 | 0.0057 | 0.0044 | 0.0035 | 0.0028 | 0.0023 |

Figure 5: (a) \( mse \)'s estimates of the fractional parameters of Model I (b) \( mse \)'s estimates of the fractional parameters of Model II.

As was expected, the presence of seasonal and nonseasonal short-memory components may bias the estimates of the fractional parameters, which is
in accordance with the simulation results presented in Section 4. The bias may be reduced by an appropriate choice of the bandwidth, however, it is not easy task in a real practical application. So, looking at the behavior of the estimates across different bandwidth values may, at least, indicate the unfavorable estimates. In the samples here considered, it can be observed that when there is no short-memory part (Model I), the estimates are, in general, very stable across the bandwidth. The reduction of the variance and, also, the $mse$ is obtained by increasing $m$.

In Model II the seasonal period of the short-memory is $s = 4$, so the corresponding fractional estimate is more affected, whereas the fractional estimate of period $s = 12$ remains more stable in a wide range of bandwidths. Now, the smallest $mse$ of the estimates are not at the same bandwidths. Since the estimate of $d_1$ is more affected with the AR part, its estimate has the smallest $mse$ with a smaller number of regression than the estimate of $d_2$. To have a better understanding of the behavior of the $mse$ across the size of the bandwidth, these values are displayed in Figure 5. So, from this simple example it can be seen that the bias of each fractional estimate may be substantially affected, if at the same seasonal period, there is a short-memory component. As an example of a stronger correlation AR seasonal structure, a SARFIMA model with $\phi_4 = 0.8$ was also considered, but it is not presented here to save space. As expected, the smallest $mse$ of both seasonal fractional estimates were achieved for bandwidths smaller than the case where $\phi_4 = 0.3$.

To conclude, in Table 6 the estimates based on the smallest $mse$ were used to estimate the fractional parameters of the artificial series. The model adequacies for the adjusted models were carried out, for example, the residual analysis. These evidenced that the estimated models fit the series well, that is, no anomaly of the residuals were found (residual analysis of the artificial data are available upon-request).

In the same direction of the above exercises, other single series were also analyzed with different short-memory parameters and seasonal periods, how-
ever, in general, the estimates presented similar patterns of those here presented. These are available upon-request.

5.2 Daily average PM$_{10}$ concentration

The daily average Particulate Matter (PM$_{10}$) concentration is expressed in µg/m$^3$ and it was observed in the Metropolitan Region of Greater Vitória (RGV) in Brazil. RGV is comprised of five cities with a population of approximately 1.7 million inhabitants in an area of 1,437 km$^2$. The region is situated on the South Atlantic coast of Brazil (latitude 20°19S, longitude 40°20W) and has a tropical humid climate, with average temperatures ranging from 24°C to 30°C. The rainfall is fairly distributed throughout the entire year (average precipitation of 98.3 mm per month during the period of study), but with drier periods from June to August (average precipitation of 60.8 mm per month) and more heavier precipitation from October to January (average precipitation of 158.3 mm per month).

The raw series has a sample size of 2037 observations, measured from the 1st of January 2001 to 2nd of August 2006, and it is shown graphically in Figure 6. The sample autocorrelation (ACF) and partial autocorrelation (PACF) functions are shown in Figures 7 (a) and (b), respectively. From these plots a strong seasonal component in the series is evident, which was an expected property due to the characteristic of such a physical phenomena. It is also observed that the seasonality behavior has period $s = 7$, which is also an expected data behavior since the series was observed daily.

An interesting feature observed from the sample ACF is the slow decay of the correlations in the first lags, in the lags multiple of 7 and in the lags between the seasonal periods. The ACF plot strongly indicates that the process has fractional memory parameters in the lung-run and in the seasonal periods. This empirical evidence indicates the use of a particular case of the SARFIMA model defined previously (Model (7)) with $s_1 = 1$, $s_2 = 7$. The modeling strategy follows the same steps suggested in Hosking (1981) and investigated empirically by Reisen (1994), Reisen & Lopes (1999)
among others. Firstly, the fractional parameters are estimated by using the GPH semiparametric tool described in the previous section. This was carried out by using different sizes of bandwidth $m$. Secondly, the truncated filter $(1 - B)^{d_1}(1 - B^s)^{d_2}$ is used to filter the observation and obtain a new series which approximately follows an ARMA model. This new series is used to achieve the complete short-memory model structure.

Figure 6: $PM_{10}$ series.

Figure 7: (a) Sample ACF of $PM_{10}$ (b) Sample PACF of $PM_{10}$.
Table 7 displays the results of the memory estimates obtained from different values of the bandwidth $m = n^\alpha$, $0 < \alpha < 1$ and the values in parenthesis are the corresponding number of the frequency used in the regression. From this table, it can be seen that the values of the estimates of $d_1$ and $d_2$ are stable for $0.52 < \alpha < 0.65$ (max.), and they are in the range $0 < d_1, d_2 < 0.5$. The estimated standard errors of $\hat{d}_2$ are relatively small and two-sided confidence intervals for $d_1$ and $d_2$ are correspondingly tight. Therefore, for $\alpha > 0.52$ the null hypotheses that $H_0: d_1 = 0$ and $H_0: d_2 = 0$ are rejected. Also, for all values of the bandwidths given in the table, F test was performed for the null hypothesis $H_0 : \mathbf{d} = \mathbf{0}$, and it indicated that at least one fractional parameter is different from zero. The stable value of the estimate of $d_1$ in the range $0.52 < \alpha < 0.65$ (max.) gives an empirical evidence that if there is any non-seasonal short-memory part in the model, the parameter is not large enough to make a significant contribution in the regression estimators. A similar conclusion is also observed in the case of the seasonal fractional estimate $\hat{d}_2$. Therefore, $\alpha = 0.54$ was chosen to estimate the memory parameters. The vector $\mathbf{\hat{d}} = (0.1918, 0.1798)$ shows that data presents the stationarity, invertibility and long-memory properties. The choice of these estimates was also confirmed by the Akaike Criterion (AIC), which gave the smallest value for $\alpha = 0.54$.

Table 7: Estimates of the fractional parameters of the $PM_{10}$.

| Estim. | $\alpha$     |
|--------|--------------|
|        | 0.52 | 0.54 | 0.56 | 0.58 | 0.6  | 0.65 (max.) |
|        | (361) | (424) | (494) | (578) | (669) | (1016) |
| $\hat{d}_1$ | 0.1004 | 0.1918 | 0.1787 | 0.2071 | 0.1645 | 0.2534 |
| $\hat{d}_2$ | 0.1954 | 0.1798 | 0.1575 | 0.1443 | 0.1548 | 0.0806 |
| Var $\hat{d}_1$ | 0.0110 | 0.0090 | 0.0072 | 0.0060 | 0.0049 | 0.0014 |
| Var $\hat{d}_2$ | 0.0016 | 0.0013 | 0.0011 | 0.0009 | 0.0008 | 0.0002 |
To obtain the approximation of the model \( \nu_t \), the observations were filtered by \( \nabla^d \) truncated at \( n=2037 \). The new series is \( \hat{\nu}_t = \sum_{j=0}^{n} \hat{\pi}_j^* X_{t-j} \), where \( \hat{\pi}_j^*, j = 1, 2, \ldots, 2037 \), are the estimated coefficients of the AR(\( \infty \)) representation of a SARFIMA(0, \( d_1 \), 0)(0, \( d_2 \), 0)\( 7 \) model. As an example to verify the impact of \( X_j \), for large \( j \), in the AR infinite representation, the \( \hat{\pi}_{731}^* \) is \( \approx 10^{-6} \), which is nearly zero. Since the observations are in scale of 10\(^2\), the contribution of \( X_j \) becomes negligible for large \( j \).

Figures 8(a) and (b) present the sample autocorrelation and partial autocorrelation functions of \( \hat{\nu}_t \), respectively. These plots possibly indicate that an MA(1) model may be adequate to describe \( \hat{\nu}_t \). However, the MA estimate does not seem to have significant value. This is in accordance with the stable values of the memory estimates given in Table 7.

![Figure 8](image)

Figure 8: (a) ACF of \( \hat{\nu}_t \) (b) PACF of \( \hat{\nu}_t \).

To identify the model’s order of \( \hat{\nu}_t \), the AIC Criterion was used which suggested an MA(1) model. Therefore, the model SARFIMA(0, \( d_1 \), 1) \times (0, \( d_2 \), 0)\( 7 \) with \( \hat{\theta} = -0.2673 \ (sd = 0.0214) \) was chosen for the \( PM_{10} \) average data. The standard residual analysis did not present anomaly of the residuals (the results are upon-request). Most of the correlations of the residuals fall inside the confidence boundaries (figures available upon-request).
6 Conclusions

The paper deals with the seasonal ARFIMA model with two seasonal fractional parameters. Properties and model estimation are discussed. To estimate the parameters, a multilinear regression method is used. A parametric estimator was also considered for empirical comparison. The Monte Carlo experiment evidenced that, in general, all methods gave good estimates and they were very competitive. The estimators presented very good accuracy for sample size equal to 1080. The method is very easy to be implemented and does not require sophisticated computer capacities. The usefulness of the SARFIMA model and the semiparametric fractional estimator was exemplified using artificial and a daily average PM$_{10}$ concentration series.

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APPENDIX A

Proof of Theorem 1: The asymptotic mean and variance of \( \hat{d} \) follows as in Theorem 1 in Hurvich et al (1998). Denote \( Z \) the \( \sum_k m_k \times 2 \) matrix with the regressors in \( \Pi \) and similarly the vector \( V \) for the disturbances. Then

\[
\hat{d} - d = (Z'Z)^{-1}Z'V.
\]

Note now that

\[
2 \log X_{1,k,j} = 2 \log \left\{ \frac{2 \sin \left( \pi k + \frac{\pi js'}{n} \right)}{n} \right\} = 2 \log |\lambda_{js'}| + O(|\lambda_{js'}|^2)
\]

\[
2 \log X_{2,k,j} = 2 \log \left\{ \frac{2 \sin \left( \frac{\pi k s_2}{s'} + \frac{\pi js}{n} \right)}{n} \right\} = 2 \log |\lambda_{js_2}| + O(|\lambda_{js'}|^2)
\]

if \( k \in I_k \) and

\[
2 \log X_{2,k,j} = 2 \log \left\{ \frac{2 \sin \left( \frac{\pi k s_2}{s'} \right)}{n} \left[ 1 + \frac{\cos(\pi k s_2 / s') \pi |j| s_2}{\sin(\pi k s_2 / s')} + O(\lambda_{js_2}^2) \right] \right\}
\]

if \( k \not\in I_k \). Then

\[
\sum_{k=0}^{[s'/2]} \sum_j Z_{1,k,j}^2 = 4 \sum_{k=0}^{[s'/2]} m_k (1 + o(1))
\]

\[
\sum_{k=0}^{[s'/2]} \sum_j Z_{2,k,j}^2 = 4 \sum_{k \in I_k} m_k (1 + o(1))
\]

\[
\sum_{k=0}^{[s'/2]} \sum_j Z_{1,k,j} Z_{2,k,j} = 4 \sum_{k \in I_k} m_k (1 + o(1))
\]

This result follows from the fact that for those \( k \not\in I_k \), \( \sum_j Z_{2,k,j}^2 = O(m^3 n^{-2}) = o(m) \) and \( \sum_j Z_{2,k,j} Z_{1,k,j} = O(m^2 n^{-1} \log m) = o(m) \). Then

\[
Z'Z = mQ(1 + o(1))
\]

Denoting now \( \varepsilon \) the vector with elements \( \varepsilon_{k,j} \) we have that

\[
Z'\varepsilon = mb_n(1 + o(1))
\]
since

\[
\sum_{k=0}^{[s'/2]} \sum_j^{*} Z_{1,k,j} \varepsilon_{k,j} = -2m \sum_{k=0}^{[s'/2]} b_k \delta_k (2\pi)^{\alpha_k} \frac{\alpha_k}{(\alpha_k + 1)^2} \left( \frac{m}{n} \right)^{\alpha_k} \left( 1 + O \left( \log m \left( \frac{m}{n} \right)^t \right) \right)
\]

\[
\sum_{k=0}^{[s'/2]} \sum_j^{*} Z_{2,k,j} \varepsilon_{k,j} = -2m \sum_{k \in I_k} b_k \delta_k (2\pi)^{\alpha_k} \frac{\alpha_k}{(\alpha_k + 1)^2} \left( \frac{m}{n} \right)^{\alpha_k} \left( 1 + O \left( \log m \left( \frac{m}{n} \right)^t \right) \right)
\]

because for \( k \notin I_k \)

\[
\sum_{k \notin I_k} \sum_j^{*} Z_{2,k,j} \varepsilon_{k,j} = O \left( \sum_j |\lambda_j|^{\alpha_k+1} \right) = o \left( m \left( \frac{m}{n} \right)^{\alpha_k} \right)
\]

The rest of the proof follows as in Hurvich et al (1998).

**Proof of Theorem 2:** The proof follows as in Hurvich et al (1998) applying Lemma 4 in Sun and Phillips (2003) which holds in our multiple log periodogram regression context. Since

\[
\sqrt{m} (\hat{d} - d) = (m^{-1} Z'Z)^{-1} m^{-1/2} Z' \varepsilon + (m^{-1} Z'Z)^{-1} m^{-1/2} Z' U
\]

by using (15) and (16) it only remains to show that \( m^{-1/2} v' Z' U \overset{d}{\rightarrow} N(0, \pi^2 v' Q v / 6) \) for any vector \( v = (v_1, v_2) \). As in Hurvich et al. (1998)

\[
\frac{1}{\sqrt{m}} v' Z' U = o_p(1) + \frac{1}{\sqrt{m}} \sum_{k=0}^{[s'/2]} \sum_{j \geq m^{0.5+\delta}}^{*} g_{k,j} U_{k,j}
\]

for some \( 0.5 > \delta > 0 \) and \( g_{k,j} = v_1 Z_{1,k,j} + v_2 Z_{2,k,j} \). Now \( \max_{j,k} |g_{k,j}| = O(\log m) \) and \( \sum_{j \geq m^{0.5+\delta}} |g_{k,j}|^p = O(m) \) for all \( p \geq 1 \) (see formula (A18) in Hurvich et al (1998) for \( Z_{1,k,j} \) and \( Z_{2,k,j} \) for \( ks_2 \in I_k \) and use (5) for \( ks_2 \notin I_k \)). Since by equation (15) \( \sum_{j \geq m^{0.5+\delta}} g_{k,j}^2 \overset{d}{\rightarrow} m v' Q v (1 + o(1)) \), we can apply Lemma 4 in Sun and Phillips (2003) to get the desired result.