Mobile impurity probing a two-dimensional superfluid phase transition

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The use of atomically sized quantum systems as highly sensitive measuring devices represents a new and exciting research direction with great technological potential. A main goal is to develop atomically sized probes with maximal sensitivity and minimal back-action on the environment \[1\]. Impurity atoms are promising candidates for this, and they have already been used experimentally to measure the temperature \[2\]-\[4\] and density \[5\] of a surrounding quantum degenerate gas, as well as to detect induced interactions \[6\]. So far, the vast majority of investigations into mobile impurities in atomic gases concerned cases where the environment is either a weakly interacting Bose-Einstein condensate (BEC) or an ideal Fermi gas, and the impurity forms a quasiparticle called the Bose or Fermi polaron respectively \[7\]-\[15\].

Much less attention has been payed to mobile impurities in environments with correlations between the particles. The properties of an impurity in a fermionic superfluid across the strongly correlated BCS-BEC cross-over were examined \[16\]-\[17\], but a general description turns out to be complicated by the presence of ultraviolet divergencies related to three-body physics \[18\]. A particularly interesting case concerns two-dimensional (2D) systems, where quantum and thermal fluctuations are more pronounced than in 3D and true long range order is prohibited at a non-zero temperature \[19\]-\[20\]. Nevertheless, 2D systems can exhibit a so-called Berezinskii-Kosterlitz-Thouless (BKT) phase transition to a superfluid phase with quasi-long range order \[21\]-\[24\], which has been observed in a range of bosonic systems including $^4$He films \[24\], magnetic layers \[25\], and atomic Bose gases \[26\]-\[30\]. A 2D superfluid in a two-component strongly interacting atomic fermi gas was observed only recently \[31\]-\[33\], and the underlying discontinuity of the phase transition has so far not been seen unambiguously in this system. In general, our understanding of 2D fermionic superfluids is less developed as compared to their bosonic counterparts.

Here, we investigate a mobile impurity immersed in a 2D fermionic superfluid. Interactions between the impurity and the surrounding fermions lead to the formation of a quasiparticle, i.e. a polaron, and we show that its energy is lowered due to an increase in the compressibility of the environment caused by superfluid pairing. We furthermore demonstrate that the abrupt vanishing of the superfluid density at the critical temperature, characteristic for a superfluid to normal phase transition in 2D \[34\], gives rise to a rapid increase in the polaron energy. The increase depends non-monotonically on the Fermi-Fermi interaction strength exhibiting a maximum when the size of the Cooper pairs is comparable to the interparticle spacing. Our results show how a mobile impurity can serve as a sensitive probe for thermal and quantum correlations of a 2D fermionic system thereby providing important guides for improving our understanding of its non-trivial superfluid to normal phase transition.

II. SYSTEM

Consider an impurity of mass $m_I$ immersed in a two-component ($\sigma = \uparrow, \downarrow$) gas of fermions of mass $m$ in 2D. The $\uparrow$ and $\downarrow$ fermions interact attractively and form a superfluid below a critical temperature. Using BCS theory...
The superfluid gap $\Delta$ at temperature $T$ where $m$ temperature $T$ between a superfluid and a normal phase at the critical temperature gives rise to a discontinuous phase transition between a superfluid and a normal phase at the critical temperature $T_{\text{BKT}}$, which is suppressed from the mean-field BCS prediction $T_c$ by phase fluctuations. The vertical lines indicate the coupling strengths for which we plot the polaron energy in Fig. 1.

FIG. 1: We consider a mobile impurity (green ball without arrow) forming a quasiparticle by interacting with a two-component Fermi gas (blue and red balls with arrows) in 2D. An attractive interaction with strength $-\ln(k_F a_F)$ between the fermions gives rise to a discontinuous phase transition between a superfluid and a normal phase at the critical temperature $T_{\text{BKT}}$, which is suppressed from the mean-field BCS prediction $T_c$ by phase fluctuations. The vertical lines indicate the coupling strengths for which we plot the polaron energy in Fig. 1.

to describe this superfluid phase, the Hamiltonian of the system is

$$
\hat{H} = \sum_{k, \sigma} \xi_k \hat{a}_{k, \sigma}^\dagger \hat{a}_{k, \sigma} + \Delta \sum_k (\hat{a}_{-k, 1} \hat{a}_{k, 1}^\dagger + \hat{a}_{-k, 1}^\dagger \hat{a}_{k, 1}) + g_F \sum_{k, k', \sigma} \hat{a}_{k' \sigma}^\dagger \hat{a}_{k' \sigma} \hat{a}_{k \sigma}^\dagger \hat{a}_{k, \sigma}.$$

(1)

Here, $\hat{a}_{k \sigma}^\dagger$ is the creation operator of a fermion with momentum $k$, spin $\sigma$, and kinetic energy $\xi_k = k^2 / 2m - \mu$ with $\mu$ the chemical potential, and $\hat{c}_k$ creates an impurity with momentum $k$ and kinetic energy $\epsilon_k = k^2 / 2m_I$, $g_F$ is the interaction strength for momenta less than a cutoff $\Lambda$, it can be eliminated in favor of a 2-body bound state with energy $\epsilon_B$ using 35 39:

$$\frac{1}{g_F} = \frac{1}{V} \sum_{|q| < \Lambda} \frac{1}{E_B + q^2 / 2m_r}.$$

(2)

where $m_r = mm_i / (m + m_i)$ and $E_B = 1 / 2m_r a_F^2$. The superfluid gap $\Delta$ at temperature $T$ is determined from

$$\Delta = -\frac{g_F}{\beta V} \sum_{k, n} G_{12}(k, i\omega_n).$$

(3)

where $G_{12}$ is the anomalous Green’s function. This leads to the following gap equation

$$\int dk \left( \frac{\tanh(\epsilon_B / 2T)}{2E_k} - \frac{1}{\epsilon_B + k^2 / 2m_r} \right) = 0,$$

(4)

where $E_k = \sqrt{\xi_k^2 + \Delta^2}$ and we have renormalised the gap equation by replacing the Fermi-Fermi interaction strength by the energy of a bound state of two fermions $\epsilon_B$, which is always present for an attractive interaction. We work in units where $\hbar$, $k_B$, and the system volume are all unity.

III. THE BKT TRANSITION

The 2D superfluid with quasi-long range order melts into a normal phase when vortex and anti-vortex pairs unbind and proliferate. This occurs at the critical temperature determined by the condition 22 37

$$T_{\text{BKT}} = \frac{\pi}{8m} n_s(T_{\text{BKT}}),$$

(5)

where $n_s$ is the superfluid density given by

$$n_s(T) = 1 + \frac{1}{2\pi m} \int_0^\infty d k \epsilon_k^F \frac{\partial f(E_k)}{\partial E_k}.$$

(6)

Here $f(E) = [\exp(E / T) + 1]^{-1}$ is the Fermi-Dirac distribution and $n = k_F^2 / 2\pi$ is the total density from both spin states of the Fermi gas. The integral is always negative and vanishes at $T = 0$ ensuring that the superfluid density is always equal to or smaller than the total density. It follows from Eq. (5) that the superfluid density of the Fermi gas exhibits a universal jump $\Delta n_s / m T_{\text{BKT}} = 8 / \pi$ at the BKT transition. This jump has not yet been observed in the experiments exploring two-dimensional atomic fermi gases, and a main goal here is to demonstrate that the discontinuity of the phase transition can be detected by looking at the properties of the impurity.

Fig. (1) shows the phase diagram of the Fermi gas as a function of the Fermi-Fermi interaction length strength parametrised by $-\ln(k_F a_F)$ and temperature. Here, $a_F$ is a scattering length defined by writing the energy of the bound Fermi-Fermi dimer as $-1 / ma_F^2$. The Fermi gas is in a superfluid phase below a critical temperature $T_{\text{BKT}}$ obtained by solving Eqs. (1)-4 self-consistently.

We vary the chemical potential $\mu$ to keep the density $n = 2 \sum_k \left[ u_k^2 (1 - f_k) + v_k^2 f_k \right]$ fixed, where $v_k^2 = 1 - u_k^2 = (1 - \xi_k / E_k) / 2$ are the coherence factors.

For weak coupling $-\ln(k_F a_F) \ll -1$ corresponding to the so-called BCS regime with large Cooper pairs, the superfluid transition temperature is close to that obtained
from mean-field BCS theory, which predicts a smooth decrease of the superfluid density to zero at the critical temperature $T_c$. It follows that the jump in the superfluid density at $T_{BKT}$ is small for weak coupling. For stronger coupling however, phase fluctuations significantly suppress the critical temperature below the BCS prediction leading to a large jump in the superfluid density at the phase transition. We obtain $T_{BKT} = T_f/8$ in the strong coupling regime with $-ln(k_F a_F) \gtrsim -1$ reflecting that the superfluid density equals the total density $n = k_F^2/2\pi$ just below the transition giving rise to a maximal jump $\Delta n_s = n$ \cite{49, 51}. We note however that the gas eventually enters the BEC regime with increasing $-ln(k_F a_F) \gtrsim 1$, where it can be described as a Bose gas of tightly bound Cooper pairs with a BKT critical temperature that decreases slowly \cite{51, 52, 53, 54}. 

IV. PERTURBATION THEORY

We now turn to the properties of the impurity in the Fermi gas. Since the case of general interaction strengths between the impurity and the fermions as well as between the fermions is very complicated, we focus on the regime of weak impurity-fermion interactions where a reliable perturbative theory can be developed.

To do this, consider the scattering matrix between the impurity and a fermion. As detailed in appendix A, it can in the ladder approximation be written as

$$T(k) = \frac{g}{1 - g\Delta \Pi(k)} \simeq g + g^2 \Delta \Pi(k) + \ldots$$  \hspace{1cm} (7)

where $k = (k, \omega_n)$ denotes the center-of-mass momentum $k$ of the colliding pair with $\omega_n$ a Matsubara frequency, and an expression for the pair propagator $\Delta \Pi(k)$ is given in appendix B. It follows from Eq. (7) that

$$g = - \frac{\pi}{m_r} \frac{1}{\ln(k_F a_F)}$$  \hspace{1cm} (8)

is an effective 2D interaction strength between the impurity and the fermions \cite{44, 45}. We have thus eliminated the bare impurity-Fermi coupling strength $g_{IF}$ in favour of an effective interaction strength, which is a function of the energy $-1/2m_ra_{IF}^2$ of the bound impurity-fermion dimer with $a_{IF}$ the impurity-fermion scattering length and $m_r = mm_I/(m + m_f)$ the reduced mass \cite{23, 24}. When $|\ln(k_F a_{IF})| \gg 1$, the effective interaction is weak and the impurity properties can be calculated reliably using perturbation theory in $g$ as used in Eq. (7).

To first order in $g$, the energy shift of the impurity is simply given by the mean-field expression $\Sigma_1(p) = gn$. The second order term is

$$\Sigma_2(p) = 2Tg^2 \sum_k [G_{11}(k)\Delta \Pi(p + k) + G_{12}(k)\Pi_{21}(p + k)] = Tg^2 \sum_k G_0(k)\chi(p - k)$$ \hspace{1cm} (9)

where

$$G_{11}(k, \omega_n) = \frac{u_k^2}{\omega_n - E_k} + \frac{v_k^2}{\omega_n + E_k},$$  \hspace{1cm} (10)

$$G_{12}(k, \omega_n) = u_kv_k \left( \frac{1}{\omega_n + E_k} - \frac{1}{\omega_n - E_k} \right)$$ \hspace{1cm} (11)

are the normal and anomalous propagators for the superfluid and $G_0^{-1}(k) = \omega_n - E_k$ is the impurity Green’s function in the absence of interactions. The first term in the first line of Eq. (9) describes the coupling of the impurity to particle-hole excitations in the superfluid whereas the second term describes coupling to pair breaking excitations. In the second line of Eq. (9), we write the impurity self-energy in terms of the density-density correlation function $\chi(k)$ of the superfluid, which gives the compressibility for long wave lengths. Expressions for $\Pi_{21}(k)$ and $\chi(k)$ are given in the appendix B.

The energy $\epsilon_P$ of the polaron can now be found from the pole of the impurity Green’s function $G(k)$ analytically continued to real frequencies. Using the Dyson equation $G^{-1}(k) = G_0^{-1}(k) - \Sigma(k)$ with $\Sigma(k) = \Sigma_1(k) + \Sigma_2(k)$ to second order in $g$ then yields

$$\epsilon_P = gn + \Sigma_2(0, \epsilon_P + i0^+),$$ \hspace{1cm} (12)

where we take zero momentum and $m_I = m$ in the following for simplicity.

V. NORMAL PHASE

Strictly speaking, it is only the superfluid density that exhibits a discontinuity at $T_{BKT}$. This is because the long range phase coherence of the gap is lost for $T > T_{BKT}$ whereas its amplitude $|\Delta|$ remains continuous. In the following, we will nevertheless assume that the pairing gap jumps to zero at the temperature $T_{BKT}$ for the BKT phase transition. That is, we will use BCS theory for $T \leq T_{BKT}$ and set $\Delta = 0$ for $T > T_{BKT}$, corresponds to assuming that the Fermi gas is non-interacting. The reasons for this are the following. First, vortices with a vanishing gap in their centers proliferate at the transition temperature, which will significantly decrease the average gap for $T > T_{BKT}$. Second, even if the gap is non-zero in the normal phase it does not lead to a perfect vanishing of the density of states around the Fermi level as opposed to in the superfluid phase. Indeed, the gap in the normal phase is often refereed to as a pseudo-gap for this reason, and its description requires inclusion of fluctuation effects beyond BCS theory. A central feature of the theories for the pseudo-gap region is that predict a suppressed but non-zero density of states at the Fermi level \cite{46, 47}. This means that impurity can scatter on low energy excitations in the surrounding bath even for a non-zero pseudo-gap in the normal phase above $T_{BKT}$. Using a non-zero $\Delta$ in the BCS Green’s functions Eq. (11) above $T > T_{BKT}$ would on the other
The impurity-fermion interaction is constant density to find the properties of the Fermi bath. This is found by first solving Eqs. (4)-(6) numerically for interaction, which are shown by vertical lines in Fig. (1). The temperature for different values of the Fermi-Fermi temperature in analogy with what is found in 3D [54, 55]. In the normal phase, which in turn decreases with temperature increases, the superfluid gap decreases (inset in Fig. (2)), and the polaron energy approaches the value in the normal phase transition in a 2D ideal Fermi gas [49]. First, we note that the polaron has a lower energy when the Fermi gas is in the superfluid phase as compared to when it is in the normal phase (dashed lines). Physically, this is because pairing correlations increase the compressibility of the Fermi gas [50–53] so that the impurity more easily can perturb its surroundings thereby lowering the energy. As the temperature increases, the superfluid gap decreases (inset in Fig. (2)), and the polaron energy approaches the value in the normal phase, which in turn decreases with temperature in analogy with what is found in 3D [54,55]. In particular, the polaron energy exhibits a discontinuity at the critical temperature $T_{BKT}$, since we assume that the superfluid gap in the surrounding medium jumps to zero at the transition. It is strictly only the superfluid density that is discontinuous at the transition due to the loss of phase coherence, whereas as the amplitude of the gap and therefore the polaron energy is continuous. Nevertheless, as we argued above the gap must be expected to exhibit a steep decrease in a narrow region around $T_{BKT}$. The results shown in Fig. (2) should in this sense be understood as a limiting form of a continuous but rapid increase in $\epsilon_P$ on the scale of the mean-field (BCS) transition temperature. This should be contrasted to BCS theory predicting a gap that goes to zero at a much higher transition temperature $T_c$ (except for weak coupling), giving rise to a much more smooth behaviour of polaron energy. Thus, the abrupt change of the Fermi gas at $T_{BKT}$ is reflected in the polaron energy, which exhibits a sizeable and steep increase in its energy in a narrow temperature region. Importantly, this should be observable with the spectral resolution of current Fermi polaron experiments [11,15]. These results show that the polaron can be used as a probe of abrupt nature of the superfluid to normal phase transition in a 2D fermionic superfluid. In particular, a measurement of the polaron energy and its behaviour around $T_{BKT}$ should provide valuable information regarding the thermal and quantum fluctuations in the critical region.

Figure (2) moreover shows that the amplitude of the rapid energy increase is larger for $-\ln(k_F a_F) = -2.342$ than for $-\ln(k_F a_F) = -1.386$. This is as expected, since a stronger Fermi-Fermi coupling gives rise to a larger decrease in the superfluid gap (inset) at $T_{BKT}$. The discontinuity is however smaller again for even stronger coupling with $-\ln(k_F a_F) = -0.077$, even though the pairing energy is larger. To explore this further, we plot in Fig. (3) the value $\Delta \epsilon_P = \epsilon_P(T_{BKT}) - \epsilon_P(T)$ as a func-

![Figure 2: Polaron energy as a function of temperature for the Fermi-Fermi coupling strengths $-\ln(k_F a_F) = -2.342$ (green upper line), $-\ln(k_F a_F) = -1.386$ (red lower line), and $-\ln(k_F a_F) = -0.077$ (blue middle line). The dashed pink line gives polaron energy in the normal phase, and the dash-dotted lines the polaron energy assuming the Fermi gas remains superfluid up to the mean-field critical temperature $T_c$. The inset shows the superfluid gap as a function of temperature for the same coupling strengths with the dash-dotted lines giving the mean-field prediction for $T > T_{BKT}$. The colors blue (upper), red (middle), and green (lower) correspond to the scattering values of the main plot. The dotted lines illustrate the expected continuous behavior of the polaron energy and magnitude $|\Delta|$ of the gap around the BKT transition as explained in section VI.](image-url)
tion of the Fermi-Fermi coupling strength \(-\ln(k_F a_F)\), which corresponds to the change in the polaron energy in the limit where it occurs with infinite slope at \(T_{BKT}\). We see that \(\Delta \epsilon_P\) initially increases with the coupling strength in the BCS regime. It reaches a sizeable maximum of \(\Delta \epsilon_P \simeq 0.32 \epsilon_F\) in the cross-over region around \(\ln(k_F a_F) \sim -0.6\), after which it decreases as the BEC region is approached with increasing interaction, even though the gap continues to increase as shown in the inset. This should be contrasted to the gap, which increases monotonically as a function of the coupling both for \(T = 0\) and \(T = T_{BKT}\) as shown in the inset of Fig. 3. Interestingly, the maximum in \(\Delta \epsilon_P\) occurs when the size of the Cooper pairs is comparable to the interparticle spacing, where one has also observed a maximum in the critical temperature \([31, 32]\) and in the critical velocity \([33]\). As the coupling strength increases further with \(-\ln(k_F a_F) \gg 1\), the Cooper pairs shrink and the Fermi gas becomes a BEC of dimers with a transition temperature in a narrow region \(\propto 1/\ln[\ln(1/na_D^2)]\) below the mean-field prediction with \(a_D\) the dimer-dimer scattering length, and a correspondingly small discontinuity in the superfluid density \([31, 32, 42, 43]\). It follows that there should be a maximum in \(\Delta \epsilon_P\) somewhere in the cross-over region as indeed predicted here. Note however that our theory is unreliable in the BEC regime, since it does not include the Bogoliubov-Anderson mode, which becomes the dominant excitation compared to particle-hole and pair breaking excitations. A consistent description of the polaron in the whole BEC-BCS cross-over of the Fermi gas is remains an open and very challenging problem beyond the present scope.

Finally, Fig. 4 plots the limiting change \(\Delta \epsilon_P\) in the polaron energy at \(T_{BKT}\) in units of the jump \(\Delta n_s/m\) in the superfluid density at the critical temperature, which is shown in the inset. This shows that while the rapid change in the polaron energy is a direct consequence of the abrupt nature of the BKT phase transition of the surrounding medium, there is no simple proportionality between \(\Delta \epsilon_P\) and \(\Delta n_s\). Instead, the ratio \(\Delta \epsilon_P m/\Delta n_s\) increases with Fermi-Fermi interaction strength in the BCS regime reaching a maximum at \(\ln(k_F a_F) \sim -0.6\) after which is decreases. The jump in the superfluid density at the transition temperature on the other hand increases monotonically with the coupling strength, reaching the limiting value \(n\) in the BEC regime.

VII. DISCUSSION AND OUTLOOK

We investigated the properties of a quasiparticle formed by a mobile impurity in a fermionic superfluid. The characteristic discontinuity of the superfluid to normal phase transition of a 2D system was shown to give rise to a rapid increase in the quasiparticle energy around the transition temperature. We demonstrated that the amplitude of the increase depends non-monotonically on the pairing strength with a maximum in the cross-over region between the BCS and BEC limits, and that it is measurable with present experimental techniques. In particular, radio-frequency spectroscopy has proven to be a very powerful technique for measuring the polaron energy both in Fermi gases and in BECs \([7,15]\).

Our results show a way to probe the properties of the
superfluid as well as its transition to the normal phase for a 2D fermionic system. This is of particular interest since there is no quantitatively reliable theory for this challenging problem in the strong coupling regime. For instance, a non-zero width of the rapid increase of the polaron energy around $T_{\text{BKT}}$ would provide important guides for this challenging problem.

The magnitude of the jump in the polaron energy will clearly be even larger for stronger interactions between the impurity and the surrounding Fermi gas. Exploring this requires going beyond the perturbative approach used here, which is an interesting topic for future study.

Another fascinating but very challenging problem is to explore how the quasiparticle evolves smoothly from a Fermi to a Bose polaron as the Fermi gas changes from a BCS superfluid to a BEC of dimers.

From a broader perspective, our results illustrate how impurities entangled with their environment via particle collisions can be used as sensitive probes for non-trivial quantum and thermal correlations. This motivates further investigations into how coherent superpositions of internal spin states of the impurity can be used to enhance the sensitivity of the impurity probe while minimising the back-action on the environment. Another intriguing research direction is to investigate how impurities can be used to probe non-local correlations and order, as well as the geometric and topological properties of the environment.

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**Appendix A: $T$ matrix**

In order to get an expression for $g_{1F}$ in terms of the pair propagator we write the $T$ matrix in vacuum as

$$T_v(k, \omega) = \frac{1}{g_{1F}^{-1} - \Pi_v(k, \omega)},$$  \hspace{1cm} (A1)

where $\Pi_v$ is the pair propagator for two fermions in a vacuum

$$\Pi_v(k, \omega) = \frac{1}{V} \sum_{\Lambda q} \frac{1}{\omega - k^2/2M - q^2/2m_r},$$  \hspace{1cm} (A2)

with $M = m + m_t$ and $\Lambda$ is a UV cutoff that we send to infinity at the end of the calculation. We perform a variable change $q^2 = x \Rightarrow dq = dx/2\sqrt{x}$ and after a straightforward calculation we get

$$\Pi_v(k, \omega) = - \frac{m_r}{2\pi} \left( \ln(\frac{\omega + i0+ - k^2/2M - \Lambda}{\omega_n - k^2/2M + \mu}) + i\pi \right.
\left. - i \arg(\omega + i0+ - k^2/2M) \right).$$  \hspace{1cm} (A3)

Lastly, since the dimer bound state energy is a pole of the vacuum $\Pi$ matrix we can write from Eq. (A1) $g_{1F} = \Pi_v(0, \epsilon_B)$.

We can now write the scattering matrix in the medium as

$$\mathcal{T}(k, i\omega_n) = \frac{1}{\Pi_v(0, \epsilon_B) - \Pi(k, \omega_n)}$$  \hspace{1cm} (A4)

where

$$\Pi(k, i\omega_n) = \frac{1}{V} \sum_p \left( \frac{u_p^2(1-n_p)}{i\omega_n - E_p - \epsilon_{k-p}} + \frac{v_p^2n_p}{i\omega_n + E_p - \epsilon_{k-p}} \right).$$  \hspace{1cm} (A5)

Note that here we take $\epsilon_B = -1/2m_r a_{1F}^2 < 0$. In order to remove the divergence in both pair propagators, we add and subtract $\text{Re}[\Pi_v(0, \epsilon_F)]$ in the denominator and with this we can define

$$\frac{1}{g} = \Pi_v(0, \epsilon_B) - \text{Re}[\Pi_v(0, \epsilon_F)] \simeq - \frac{\pi}{m_r \ln(k_F a_{1F})},$$  \hspace{1cm} (A6)

where we left out $m_r/m$ since $k_F a_{1F} \gg m_r/m$.

**Appendix B: Perturbative expansion**

Going back to the full expression for the scattering matrix, we can expand it in the weak coupling regime ($k_F a_{1F} \gg 1$) in a perturbative series up to second order in $g$

$$\mathcal{T}(k, i\omega_n) = \frac{g}{1 - g\Delta\Pi(k, i\omega_n)} \simeq g + g^2 \Delta\Pi(k, i\omega_n),$$  \hspace{1cm} (B1)

where $g$ is expressed by Eq. (A6) and

$$\Delta\Pi(k, i\omega_n) = \Pi(k, i\omega_n) - \text{Re}[\Pi_v(0, \epsilon_F)].$$  \hspace{1cm} (B2)
To first order this gives the mean-field contribution to the self-energy of the polaron

$$\Sigma_1(q, i\omega_n) = 2g \sum_k G_{11}(k)$$  \hspace{1cm} (B3)

with \( k = (k, i\omega_n) \) and \( \sum_k = \frac{1}{2\pi} \sum_k \sum_n \). After a straightforward calculation we find

$$\Sigma_1(0, \epsilon_p) = \frac{g}{\pi} \int_0^\infty dk \left( u_k^2 n_k + v_k^2 (1 - n_k) \right) .$$  \hspace{1cm} (B4)

The second order contribution is written in Eq. 9 of the main text with

$$\chi(k) = \sum_q \left( G_{11}(q) G_{11}(p - k + q) + G_{12}(q) G_{12}(p - k + q) \right) .$$  \hspace{1cm} (B5)

We can write this in a different way as \( \Sigma_2 = \Sigma_{2a} + \Sigma_{2b} \), where

$$\Sigma_{2a}(0, \epsilon_p) = \sum_p \sum_n G_{11}(k, i\omega_n) \Delta \Pi(k, i\omega_n + \epsilon_p) .$$  \hspace{1cm} (B6)

After a straightforward calculation we find

$$\Sigma_{2a}(0, \epsilon_p) = \frac{2g^2}{\beta V^2} \sum_{k,p} \sum_n \frac{v_k^2 u_k^2 n_k n_p}{\epsilon_p + E_k + E_p - \epsilon_{k-p}}$$

$$+ \frac{u_p^2 v_p^2 (1 - n_k)(1 - n_p)}{\epsilon_p - E_k - E_p - \epsilon_{k-p}} \frac{v_p^2 u_p^2 (1 - n_k) n_p}{\epsilon_p - E_k + E_p - \epsilon_{k-p}}$$

$$+ \frac{u_p^2 v_p^2 (1 - n_p)}{\epsilon_p + E_k - E_p - \epsilon_{k-p}} \frac{u_k^2 n_k + v_k^2 (1 - n_k)}{\epsilon_p - \epsilon_F(p) - \epsilon_F(p)} \right) ,$$  \hspace{1cm} (B7)

We also get

$$\Sigma_{2b}(0, \epsilon_p) = \frac{2g^2}{\beta V^2} \sum_k \sum_n \frac{G_{12}(k, i\omega_n) \Pi_{21}(k, i\omega_n + \epsilon_p) }{\epsilon_p + E_k + E_p - \epsilon_{k-p}} ,$$  \hspace{1cm} (B8)

where

$$\Pi_{21}(k, i\omega_n + \epsilon_p) = \frac{1}{\beta V} \sum_p \sum_m G_{12}(p, i\omega_m) G_{0,1}(k - p, i\omega_m + \epsilon_p) .$$  \hspace{1cm} (B9)

After a straightforward calculation we can write

$$\Sigma_{2b}(0, \epsilon_p) = \frac{2g^2}{\beta V^2} \sum_{k,p} \sum_n \frac{u_k v_k n_k n_p}{\epsilon_p + E_k + E_p - \epsilon_{k-p}}$$

$$+ \frac{(1 - n_k)(1 - n_p)}{\epsilon_p - E_k - E_p - \epsilon_{k-p}} \frac{(1 - n_k) n_p}{\epsilon_p - E_k + E_p - \epsilon_{k-p}}$$

$$+ \frac{n_k(1 - n_p)}{\epsilon_p + E_k - E_p - \epsilon_{k-p}} \right) .$$  \hspace{1cm} (B10)

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