April 2002

Gluon propagators and the choice of the gauge field in $SU(2)$ theory on the lattice

I.L. Bogolubsky$^a$ and V.K. Mitrjushkin$^{a,b}$

$^a$ Joint Institute for Nuclear Research, 141980 Dubna, Russia
$^b$ Institute of Theoretical and Experimental Physics, Moscow, Russia

Abstract

We study numerically magnetic $G_M(p)$ and electric $G_E(p)$ gluon propagators and their dependence on the choice of the lattice gauge field $A_{x\mu}$ in $SU(2)$ gauge theory, especially, in the low–momentum limit. We find that two different $A_{x\mu}$ definitions are equivalent up to a trivial renormalization of the propagator, at least, in the main approximation.

1 Introduction

Gauge variant Green functions are supposed to contain an important information about the large distance physics and mechanism(s) of confinement. For example, at zero temperature the expected infrared suppression of the gluon propagator in the Lorentz (or Landau) gauge is believed to be connected to the Gribov’s confinement scenario [1, 2] (see also [3]). In the high temperature (chromoplasma) phase the large distance and low–momentum dependence of the gluon propagator may provide valuable information about the electric and magnetic screening mechanisms [4, 5].

On the lattice the definition of gauge variant Green functions, e.g. gluon propagators, is not free from the ambiguity. Indeed, the lattice gluon propagator $G_{\mu\nu}$ is given by

$$G_{\mu\nu}(x - y) \sim \langle \text{Tr} (A_{x\mu} A_{y\nu}) \rangle,$$

(1.1)

where $A_{x\mu}$ are gauge fields on the lattice and $\langle \ldots \rangle$ means the statistical average with some gauge fixing condition. There is no unique definition of $A_{x\mu}$ on the lattice in terms of the link variables $U_{x\mu}$. This problem has been addressed recently in [4, 5, 6, 7] and some possible choices of $A_{x\mu}$ have been discussed.

* Work supported by the grant INTAS-00-00111 and RFBR grant 02-02-17308.
A similar ambiguity exists in the choice of the gauge which is the Lorentz gauge in our case. On the lattice there are different ways to choose the Lorentz gauge fixing condition, all of them being the same in the continuum limit.

In this paper we study the space–space (magnetic) $G_M(p)$ and time–time (electric) $G_E(p)$ correlators in the high temperature plasma phase with a special attention paid to their low–momentum behaviour. The main point in our study is the dependence of the correlators on the choice (definition) of lattice gauge field $A_{x\mu}$ as well as on the specific lattice implementation of Lorentz gauge fixing condition.

In what follows the periodic boundary conditions are used, the lattice size is $V_4 = N_4 \times N^3_s$ and $\bar{\partial}_\mu$ is a backward lattice derivative. We use also $K_\mu(p) = \frac{2}{a} \sin \frac{ap_\mu}{2}$ and $K^2(p) = \sum_\mu K^2_\mu(p)$, $a$ being a lattice spacing.

2 Gauge fields and Lorentz gauge conditions

2.1 Definitions of the gauge fields

The standard Wilson action \[10\] with $SU(2)$ gauge group is

$$S(U) = \beta \sum_x \sum_{\mu > \nu} \left[ 1 - \frac{1}{2} \text{Tr} \left( U_{x\mu} U_{x+\mu;\nu} U_{x+\nu;\mu}^\dagger U_{x\nu}^\dagger \right) \right] ; \quad \beta = \frac{4}{g^2} ,$$  \hspace{1cm} (2.1)

where $g$ is a bare coupling constant and $U_{x\mu} \in SU(2)$ are link variables. Under gauge transformations $\Omega$ field variables $U_{x\mu}$ transform as follows

$$U_{x\mu} \xrightarrow{\Omega} U_{x\mu}^\Omega = \Omega_x U_{x\mu} \Omega_{x+\mu}^\dagger ; \quad \Omega_x \in SU(2) .$$ \hspace{1cm} (2.2)

A standard definition \[11\] of the lattice gauge field $A_{x\mu}$ in terms of link variables $U_{x\mu}$ is

$$A_{x\mu}^{(1)} = \frac{1}{2iag} \left( U_{x\mu} - U_{x\mu}^\dagger \right) ,$$ \hspace{1cm} (2.3)

However, this definition is not unique. For example, instead of field $A_{x\mu}^{(1)}$ one can define gauge field $A'_{x\mu}$ \[6\]

$$A'_{x\mu} = \frac{1}{4iag} \left( U_{x\mu}^2 - (U_{x\mu}^\dagger)^2 \right) ,$$ \hspace{1cm} (2.4)

or another gauge field $A''_{x\mu}$, \[7\]

$$A''_{x\mu} = \frac{4}{3} A_{x\mu}^{(1)} - \frac{1}{3} A'_{x\mu} ,$$ \hspace{1cm} (2.5)

e tc. On the other hand, the proof of the (naive) continuum limit existence of the lattice action $S(U)$ as well as the construction of the lattice perturbation theory is based on the representation of the link variable $U_{x\mu}$ in the form
\[ U_{x\mu} \equiv \exp\left\{ iagA_{x\mu} \right\}, \quad |\vec{A}_{x\mu}| \leq \frac{2\pi}{ag}, \quad (2.6) \]

and the expansion in series in powers of coupling constant \( g \). Therefore, from the point of view of the analytical (e.g., perturbative) studies, there is a natural definition of the lattice gauge field, \( A_{x\mu} \)

\[ A_{x\mu}^{(2)} = A_{x\mu}(U_{x\mu}), \quad (2.7) \]

\( A_{x\mu} \) being defined in eq. (2.6). Given any group element \( U = c_0 \hat{1} + i\vec{c} = \exp\left\{ \frac{i}{2} \vec{\theta} \vec{\sigma} \right\}, \quad (2.8) \)

equation (2.8), one can easily find that

\[ \vec{\theta} = \frac{2\vec{c}}{|\vec{c}|} \arccos c_0, \quad (2.8) \]

unless \( c_0 \neq -1 \) (the case \( c_0 = -1 \) has measure zero and can be discarded). A formal expansion in powers of spacing \( a \) gives

\[ A_{x\mu}^{(1)} = A_{x\mu}^{(2)} + O(a^2), \quad (2.9) \]

so that both definitions are equivalent in the naive continuum limit.

In the rest of this paper the lattice spacing \( a \) is chosen to be unity.

### 2.2 Choice of the Lorentz gauge fixing condition

In lattice calculations the usual choice of the Lorentz gauge condition is [11]

\[ \sum_{\mu=1}^{4} \bar{\partial}_\mu A_{x\mu}^{(1)} = 0, \quad (2.10) \]

which is equivalent to finding an extremum of the functional \( F_U^{(1)}(\Omega) \),

\[ F_U^{(1)}(\Omega) = \frac{1}{4V_4} \sum_{x\mu} \frac{1}{2} \text{Tr} U_{x\mu}^\Omega, \quad (2.11) \]

with respect to gauge transformatins \( \Omega_x \). In what follows this gauge is referred to as \( LG_0 \). Equally, one can choose another form of the Lorentz gauge fixing condition (see also [3]) which will be referred to as \( LG_1 \):

\[ \sum_{\mu=1}^{4} \bar{\partial}_\mu A_{x\mu}^{(2)} = 0. \quad (2.12) \]

Evidently, both \( LG_0 \) and \( LG_1 \) are the same in the continuum limit. The \( LG_1 \) gauge condition is equivalent to finding an extremum of the functional \( F_U^{(2)}(\Omega) \)

\[ F_U^{(2)}(\Omega) = \frac{1}{4V_4} \sum_{x\mu} \frac{1}{2} \text{Tr} (A_{x\mu}^\Omega)^2. \quad (2.13) \]
The field $A_{x\mu} = A_{x\mu}(U_{x\mu})$ shows the nonanalytic dependence on the link variable $U_{x\mu}$. However, the functional $F_U^{(2)}(\Omega)$ still remains a continuous function of $\Omega$. Under the infinitesimal gauge transformation $\delta\Omega_x = \exp\{i\delta\omega_x\}$, the variation of the functional $F_U^{(2)}(\Omega)$ is

$$
\delta F_U^{(2)}(\delta\Omega) = -\sum_a \varphi_x^a \cdot \delta\omega_x^a + \ldots , \tag{2.14}
$$

where

$$
\varphi_x^a = \sum_\mu \bar{\partial}_\mu A_{x\mu}^a(U_{x\mu}) . \tag{2.15}
$$

Eqs. (2.14), (2.15) define the numerical algorithm of gauge fixing. To maximize $F_U^{(2)}(\Omega)$ one can choose the gauge transformation matrix $\Omega_x = \exp\{i\omega_x\}$,

$$
\omega_x^a = -b \cdot \varphi_x^a ; \quad b > 0 , \tag{2.16}
$$

successfully at all lattice sites $x$. After a certain number of gauge fixing sweeps a local maximum $F_{max}^{(2)}(U)$ of the functional $F_U^{(2)}(\Omega)$ is reached with given accuracy. The value of the parameter $b$ in Eq. (2.16) should be tuned to optimize the convergence.

### 2.3 Lorentz gauge and Gribov copies

It is well–known that gauge fixing on the lattice as well as in the continuum is affected by the existence of Gribov copies [1]. If one repeatedly subjects a configuration $\{U_{x\mu}\}$ to a random gauge transformation as in Eq. (2.2) and subsequently applies to it the Lorentz gauge fixing procedure, one usually obtains Gribov (or gauge) copies with different values of $F_{max}^{(2)}(U)$. It is a long–standing believe [12] that the “true” gauge copy corresponds to the absolute maximum of $F_U(\Omega)$.

In our case the question of interest is the dependence of lattice quantities in the Lorentz gauge(s) on the choice of the gauge copy. This dependence has been well studied in a number of papers (see, e.g. [13, 14, 15, 16, 17]). It has been found that for different choices of gauge copies variation of the gluon propagator is small and comparable with statistical error.

Our study confirms this statement. For every equilibrium configuration $\{U_{x\mu}\}$ we performed 10 random gauge transformations and then applied a gauge fixing procedure. For the calculation of the gluon propagator we have used either the first gauge copy of the equilibrium configuration $\{U_{x\mu}\}$ or the “best” one, i.e. the copy with the maximal value of $F_{max}^{(2)}(U)$. The difference between the (averaged) propagators appeared to be very small and negligible in comparison with statistical error.

---

1It is worth noting that in some of the above mentioned papers many more gauge copies have been used.
errors. Therefore, we conclude that there is no in fact Gribov copy problem, at
least, for the SU(2) lattice gluon propagator in the Lorentz gauge.

Note that this situation is quite different from what one observes in the case of
$U(1)$ lattice gauge theory. Indeed, the 4d photon $U(1)$ propagator in the Lorentz
gauge exhibits an extremely strong dependence on gauge copies [18, 19].

3 Numerical results

We calculated magnetic $G_M(p)$ and electric $G_E(p)$ correlators,

$$G_M(p) = \frac{1}{2}(G_{11}(p) + G_{22}(p)) ; \quad G_E(p) = G_{44}(p) ,$$

where $G_{\mu\nu}(p)$ is given by

$$G_{\mu\nu}(p) = \frac{1}{2V_4} \left\langle \text{Tr} \left( A_\mu(p) A_\nu(-p) \right) \right\rangle .$$

Field $A_\mu(p)$ is a Fourier transform

$$A_\mu(p) = \sum_x e^{-ipx - \frac{i}{2}p\mu} \cdot A_{x\mu} ,$$

where $A_{x\mu}$ are defined in Eqs. (2.3) and (2.6) with unit spacing. Momenta $p$
have been chosen to be directed along the third axis, i.e. $p = (0,0,p_3,0)$ with
$p_3 = 2\pi n_3/N_s$ and $n_3 = 0, 1, \ldots$

Most of our calculations have been performed on $4 \times N_s^3$ lattices with $N_s = 16; 24; 32; 40$ and on the $N_4 \times 24^4$ lattices with $N_4 = 6; 8$. The following
values of $\beta$ have been chosen: $\beta = 2.512; 2.645$ and 2.74. The choice of $\beta = 2.512$
corresponds to the temperature $T = 2T_c$ on the lattice with $N_4 = 4$ as well as the
choice of $\beta = 2.645$ on the lattice with $N_4 = 6$ and $\beta = 2.74$ on the lattice with
$N_4 = 8$ [20].

As it has been already mentioned, at zero temperature one expects the infrared
suppression of the gluon correlator, i.e. the nonperturbatively calculated gluon cor-
relator is expected to be less singular than the perturbative one. It has been shown
that Lorentz (and Coulomb) gauge fixed gluon correlators on the infinite lattice
vanish in the zero momentum limit due to the proximity of the Gribov horizon [4].
(This has been confirmed recently in the framework of Dyson–Schwinger equation
formalism [3]). Recently this study has been extended to the finite temperature
case [21]. It has been shown that in the infinite volume limit magnetic correlation
functions vanish in the infrared unlike the electric correlation functions [21].

In Figure [1] one can see the momentum dependence of the magnetic correlator
$G_M^{(1)}(p)$ for different lattice sizes. At larger values of the momentum $(K^2 \gtrsim 0.2)$
the finite volume dependence is practically absent. On the other hand, at values

\[^4\text{We are grateful to F. Karsch and J. Engels for providing us the } \beta\text{-value for } N_4 = 6\]
\( K^2 \sim 0 \) this dependence is rather strong. For our largest lattice with \( N_s = 40 \) the momentum dependence acquires a local maximum at some nonzero value of \( p \). A similar structure in the high temperature phase has been observed recently in \[22\] (compare with the low temperature case \[17, 23, 24\]). In Figure 2 we show the dependence of the zero momentum correlator \( G_M^{(1)}(0) \) on \( N_s \) at \( N_4 = 4 \) and \( \beta = 2.512 \). One can see a monotonic decrease of \( G_M(0) \) with increase of the lattice size \( N_4 \). However, one needs much larger lattices to judge on the infinite volume limit.

The momentum dependence of the electric correlator \( G_E^{(1)}(p) \) shown in Figure 3 differs strongly from that of the magnetic correlator, in agreement with \[21\]. All curves are monotonic and there is no any local maximum at nonzero momentum \( p \). Note also that at small momenta \( (K^2 \sim 0) \) the finite size dependence of \( G_E^{(1)}(p) \) is rather weak, in contrast to \( G_M^{(1)}(p) \) case.

To see the dependence of the correlators on the choice of the lattice gauge fields we defined ratios \( R_M \) and \( R_E \):

\[
R_M(p; g^2; T) = \frac{G_M^{(1)}(p)}{G_M^{(2)}(p)}; \quad R_E(p; g^2; T) = \frac{G_E^{(1)}(p)}{G_E^{(2)}(p)}.
\]

These ratios are shown in Figures 4a,b for different lattices at \( \beta = 2.512 \), \( LG_0 \) and \( LG_1 \). Up to “second order” corrections (not visible in these Figures) both ratios are equal and momentum independent:

\[
R_M(p; g^2; T) \approx R_E(p; g^2; T) \approx C(g^2; T).
\]

A more detailed structure is shown in Figure 5 for the ratio of magnetic correlators under \( LG_0 \) and \( LG_1 \) at fixed temperature \( T = 2T_c \) and different \( \beta \)'s. One can see rather weak \((\sim 1\%)\) dependence on \( \beta \) and even weaker dependence on the choice of the gauge condition (i.e. \( LG_1 \) vs. \( LG_0 \)). Another observation which can be made in Figure 5 is a very weak momentum dependence of the ratios. At the moment, we cannot exclude that this dependence is a numerical artifact. This point needs more detailed study.

## 4 Conclusions

We studied numerically the transverse magnetic \( G_M(p) \) and electric \( G_E(p) \) gluon propagators in the high temperature plasma phase \((T > T_c)\) in the \( SU(2) \) lattice gauge theory.

The transverse magnetic propagator \( G_M(p) \) exhibits very strong volume dependence in the infrared region. The value \( G_M(0) \) tends to decrease with volume increase in agreement with the prediction by Zahed and Zwanziger \[21\]. It is worthwhile to note that one needs rather large lattices to study this effect in details.
The momentum dependence of the electric correlator is very much different from that of the magnetic correlator. There is no any local maximum at nonzero momentum $p$ and in the infrared region the finite size dependence of $G_E(p)$ is rather weak, in contrast to $G_M(p)$ case.

We studied the dependence of the propagators on the choice of the lattice gauge field $A_{x\mu}$, one of them being standard and another one being “natural” from the viewpoint of the perturbation theory, as well as their dependence on the choice of the lattice Lorentz gauge. We found that two different $A_{x\mu}$ definitions are equivalent up to a trivial renormalization of the propagator, at least, in the main approximation. Most probably, this proportionality factor can be explained (at least, partially) by the tadpole renormalization (see also the discussion in [25]).

References

[1] V.N. Gribov, Nucl. Phys. B139 (1978) 1.

[2] D. Zwanziger, Phys. Lett. B257 (1991) 168; Nucl. Phys. B364 (1991) 127.

[3] L. von Smekal, R. Alkofer and A. Hauck, Phys. Rev. Lett. 79 (1997) 3591; Ann. Phys. 267 (1998) 1.

[4] A.D. Linde, Phys. Lett. B96 (1980) 289.

[5] D.J. Gross, R.D. Pisarski and L.G. Yaffe, Rev. Mod. Phys. 53 (1981) 43.

[6] L. Giusti, M.L. Paciello, S. Petrarca, B. Taglienti and M. Testa, Phys. Lett. B432 (1998) 196.

[7] A. Cucchieri and T. Mendes, [hep-lat/9902024].

[8] A. Cucchieri and F. Karsch, Nucl.Phys.Proc.Suppl. 83 (2000) 357.

[9] H. Nakajima and S. Furui, Nucl.Phys.Proc.Suppl. 73 (1999) 865; H. Nakajima, S. Furui and A. Yamaguchi, (2000) [hep-lat/0007001].

[10] K. Wilson, Phys. Rev. D10 (1974) 2445.

[11] J.E. Mandula and M. Ogilvie, Phys. Lett. B185 (1987) 127.

[12] D. Zwanziger, Nucl. Phys. B364 (1991) 127; Nucl. Phys. B378 (1992) 525.

[13] M.L. Paciello, C. Parrinello, S. Petrarca, B. Taglienti and A. Vladikas, Phys. Lett. B289 (1992) 405.

[14] M.L. Paciello, S. Petrarca, B. Taglienti and A. Vladikas, Phys. Lett. B341 (1994) 187.
[15] U.M. Heller, F. Karsch and J. Rank, Phys. Lett. B355 (1995) 511.

[16] A. Cucchieri, Nucl. Phys. B507 (1997) 353.

[17] K. Langfeld, H. Reinhardt and J. Gattnar, Nucl. Phys. B621 (2002) 131.

[18] A. Nakamura and M. Plewnia, Phys. Lett. B255 (1991) 274.

[19] V.G. Bornyakov, V.K. Mitrjushkin, M. Mueller–Preussker and F. Pahl, Phys. Lett. B317 (1993) 596;
    V.K. Mitrjushkin, Phys. Lett. B389 (1996) 713;
    I.L. Bogolubsky, V.K. Mitrjushkin, M. Mueller–Preussker and P. Peter, Phys. Lett. B458 (1999) 102.

[20] U.M. Heller, F. Karsch and J. Rank, Phys. Rev. D57 (1998) 1438.

[21] I. Zahed and D. Zwanziger, Phys. Rev. D61 (2000) 037501.

[22] A. Cucchieri, F. Karsch and P. Petreczky, Phys. Lett. B497 (2001) 80;
    Phys. Rev. D64 (2001) 036001.

[23] D.B. Leinweber, J.I. Skullerud, C. Parrinello and A.G. Williams,
    Phys. Rev. D60 (1999) 094507; Phys. Rev. D61 (2000) 079901.

[24] F.D.R. Bonnet, P.O. Bowman, D.B. Leinweber, A.G. Williams and J.M. Zanotti,
    Phys. Rev. D64 (2001) 034501.

[25] M. Testa, JHEP 04 (1998) 002.
Figure 1: $G^{(1)}_{M}(p)$ at $\beta = 2.512$ for $LG_0$. The line is to guide the eye.
Figure 2: The dependence of $G_M^{(1)}(0)$ on $N_s$ at $\beta = 2.512$ for $LG_0$. The line is to guide the eye.
Figure 3: $G_E^{(1)}(p)$ at $\beta = 2.512$ for $LG_0$. The line is to guide the eye.
Figure 4: Ratios of magnetic (a) and electric (b) correlators at $\beta = 2.512$ for $LG_0$. Lines are to guide the eye.
Figure 5: Ratios of magnetic correlators at $\beta = 2.512$