Consistency Analysis and Priority Weights for Pythagorean Fuzzy Preference Relations

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ABSTRACT The Pythagorean fuzzy set (PFS), introduced by Yager, is a generalization of the intuitionistic fuzzy set. The PFS is of great significance for the decision-making problem because it promotes the domain of intuitionistic fuzzy set. This paper aims to conduct the consistency analysis and priority weights for Pythagorean fuzzy preference relations (PFPRs). First, a consistency index to measure the multiplicative consistency of Pythagorean fuzzy preference relation (PFPR) is presented. Then, for the non-consistent PFPR, an iteration algorithm is put forward to adjust it to achieve acceptable consistency. Furthermore, an absolute deviation model is built to derive the priority weighting vector of PFPR. Finally, an illustrative example is provided to test and verify the effectiveness and applicability of the proposed method.

INDEX TERMS Pythagorean fuzzy sets, Pythagorean fuzzy preference relation, multiplicative consistency, priority weighting vector.

I. INTRODUCTION
Decision making is a universal process in human daily life. How to properly express the preference information to alternatives is an important topic. On the basis of intuitionistic fuzzy set [1], [2], Yager [3] proposed a novel fuzzy set called the Pythagorean fuzzy set (PFSs). Further, Yager [5]–[7] developed a series of properties and applications of PFSs, such as the relationship between Pythagorean membership grades and complex numbers, the Pythagorean fuzzy multi-criteria decision making (MCDM), fuzzy relations between Dempster-Shafer belief structures, etc. In addition, many scholars developed several approaches and applications of PFSs. Considering Pythagorean fuzzy number (PFN) as variables, Gou [8] defined the change values of PFNs, the sequences, and investigated the convergence of sequences of PFNs. Yang [9] and Zhang [10] applied PFSs to extending the method of TOPSIS in MCDM. Zeng [11] developed the Pythagorean fuzzy ordered weighted averaging weighted average distance operator and a hybrid TOPSIS method to solve Pythagorean fuzzy MCDM problem. Zhang [12] proposed a particular similarity measure for PFN and discussed some desirable properties for Pythagorean fuzzy multiple criteria group decision making. Zhang [13] developed a new hierarchical multi-criteria Pythagorean fuzzy QUALIFLEX method. Garg [14]–[16] introduced some aggregation operators for MCDM problem, and then, a novel correlation coefficient for PFS is introduced. For interval-valued PFN, Peng [17]–[19] discussed the property of several aggregation operators, such as boundedness, idempotency, and monotonicity. Furthermore, Peng [20] also proposed Pythagorean fuzzy Choquet integral based on multi-attributive border approximation area comparison method. Garg [21] proposed a novel accuracy function under interval-valued PFSs to solve MCDM problem. Liang [22] provided the maximizing deviation method for multiple criteria group decision analysis. Liu [23], [24] developed some Pythagorean fuzzy uncertain linguistic aggregation operators and Pythagorean uncertain linguistic partitioned Bonferroni mean operators. Ma [25] explored the symmetric Pythagorean fuzzy weighted geometric averaging operator. Ren [26] presented Pythagorean fuzzy TODIM approach to solve the MCDM problems. Wu [27]

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applied hesitant Pythagorean fuzzy sets in multiple attribute group decision making. Ejegwa [28]–[30] has creatively combined PFSs with medical diagnosis, Personnel Appointments and career placements and other fields, expanding the application scope of PFSs.

In practical decision making processes, decision makers (DMs) need to provide their preferences for each pair of alternatives, and then a specific preference relation is constructed to derive the priority weights and ranking order. Preference relations can be usually divided into the following common types: multiplicative preference relation [31]–[33], fuzzy preference relation [34]–[38], linguistic preference relation [39]–[48], and hesitant fuzzy preference relation [49], [50], etc. Measuring the consistency of preference relationships is an important part in the decision-making process [51].

Intuitionistic fuzzy set (IFS) was first proposed by Atanassov [1] and then Xu [52] applied the IFS to represent the preference information. Furthermore, some scholars have also extended the research to intuitionistic fuzzy preference relation (IFPR) [53]–[58]. PFS can not only express the preference information but also represent the non-preference information. In [59], Ilbahar put forward a Pythagorean fuzzy proportional risk assessment and a Pythagorean fuzzy failure modes and effects analysis to solve the application problem for occupational health and safety based on Pythagorean fuzzy analytic hierarchy process. Mandal [60] introduced the definition of PFPR. In PFPR, the sum of squares of membership degree and non-membership degree must be less than 1 that DMs can handle uncertain information more flexibly and effectively than intuitionistic fuzzy preference relations when they compare the two alternatives in the process of decision-making. It will be interesting to further study PFPR consistency and its adjustment methods.

The structure of this paper is provided as follows. In Section 2, the definitions of IFSs and PFSs are reviewed. In section 3, a new Pythagorean fuzzy preference relation (PFPR) consistency is introduced and an algorithm is designed to modify non-consistent PFPR. In section 4, an absolute deviation model is proposed to obtain the priority weights of alternatives. In section 5, an example is given to illustrate the validity and practicality of the proposed models. In section 6, conclusion is achieved.

II. PRELIMINARIES

In decision making process, a DM needs to provide his or her preference to each pair of alternatives, and then constructs a preference relation. In order to express the preference information of DMs more reasonable, using fuzzy information is an appropriate choice.

A. INTUITIONISTIC FUZZY SETS

Atanassov [1] primitively introduced the intuitionistic fuzzy set (IFS), which is defined as:

**Definition 1:** Suppose X is a universe of discourse. An intuitionistic fuzzy sets (IFS) A in X is denoted as

\[ A = \{x, \mu_A(x), \nu_A(x) > |x \in X\}, \]

where \( \mu_A(x) : X \to [0, 1] \), \( \nu_A(x) : X \to [0, 1] \), \( x \in X \). \( \mu_A(x) \) and \( \nu_A(x) \) are defined as the degree of membership and non-membership functions, respectively. Meanwhile, \( \mu_A(x) \) and \( \nu_A(x) \) satisfy

\[ 0 \leq \mu_A(x) + \nu_A(x) \leq 1. \]

When the IFS A satisfies \( \mu_A(x) < 1 - \nu_A(x), x \in X \), Atanassov [1] defined the hesitancy degree \( \tau_A(x) \), which is called the degree of non-determinacy of the membership of element \( x \in X \) to set A, then

\[ \tau_A(x) = 1 - \mu_A(x) - \nu_A(x). \]

For calculation convenience, \( \alpha = (\mu, \nu) \) is called an intuitionistic fuzzy number (IFN).

The intuitionistic fuzzy preference relation (IFPR) is introduced as follows [52]:

**Definition 2:** Let \( X = \{x_1, x_2, \ldots, x_n\} \) be a set of alternatives, then \( R = (r_{ij})_{n \times n} \) is called the IFPR where \( r_{ij} = (\mu_{ij}, \nu_{ij}) \) satisfies:

\[ \mu_{ij} + \nu_{ij} \leq 1, \mu_{ij} = \nu_{ij}, \mu_{ij} = \nu_{ij} = 0.5, i, \]

\[ j = 1, 2, \ldots, n. \]

where \( \mu_{ij} \) represents the degree of importance on which alternative \( x_i \) is preferred to \( x_j \), and \( \nu_{ij} \) indicates the degree of importance on which alternative \( x_i \) is non-preferred to \( x_j \). Furthermore, the indeterminacy degree between \( x_i \) and \( x_j \) is:

\[ \tau_{ij} = 1 - \mu_{ij} - \nu_{ij}, i, j = 1, 2, \ldots, n. \]

B. PYTHAGOREAN FUZZY SET (PFS)

In [3], Yager extended the IFS and put forward the Pythagorean fuzzy set, which is defined as follows.

**Definition 3:** Let \( X \) be a universe of discourse, then a PFS \( K \) is shown as:

\[ K = \{x, \mu_K(x), v_K(x) > |x \in X\}, \]

where \( \mu_K : X \to [0, 1] \) is the degree of membership and \( v_K : X \to [0, 1] \) is the degree of non-membership. In the case of \( x \in X \) satisfy

\[ (\mu_K(x))^2 + (v_K(x))^2 \leq 1, \]

then,

\[ \tilde{\pi}_K = \sqrt{1 - \mu_K^2(x) - v_K^2(x)} \]

is called the Pythagorean fuzzy indeterminacy degree.

For convenience, \( (\mu, \nu) \) is called a Pythagorean fuzzy number (PFN). Throughout this paper, we use \( \Phi \) to denote the set of all PFNs.

Zhang and Xu [10] investigated the following operations on PFNs:

Let \( \alpha = (\mu_{\alpha}, v_{\alpha}) \in \Phi \) and \( \beta = (\mu_{\beta}, v_{\beta}) \in \Phi \), then
(1) Complementary operation:
\[\tilde{\alpha}^c = (v_\tilde{\alpha}, \mu_\tilde{\alpha});\]

(2) Addition operation:
\[\tilde{\alpha} \oplus \tilde{\beta} = (\sqrt{\mu_\tilde{\alpha}^2 + \mu_\tilde{\beta}^2 - \mu_\tilde{\alpha}^2 \mu_\tilde{\beta}^2}, v_\tilde{\alpha}v_\tilde{\beta});\]

(3) Multiplication operation:
\[\tilde{\alpha} \otimes \tilde{\beta} = (\mu_\tilde{\alpha}\mu_\tilde{\beta}, \sqrt{v_\tilde{\alpha}^2 + v_\tilde{\beta}^2 - v_\tilde{\alpha}^2 v_\tilde{\beta}^2});\]

(4) Scalar multiplication operation:
\[\lambda \cdot \tilde{\alpha} = (\sqrt{1 - (1 - \mu_\tilde{\alpha}^2)\lambda^2}, v_\tilde{\alpha}^\lambda);\]

(5) Power operation:
\[\tilde{\alpha}^\lambda = (\mu_\tilde{\alpha}^\lambda, \sqrt{1 - (1 - v_\tilde{\alpha}^2)^\lambda}).\]

In order to rank the PFNs, Zhang and Xu [10] developed the score function \(\tilde{S}(\tilde{\alpha})\) and accuracy function \(\tilde{H}(\tilde{\alpha})\) as
\[\tilde{S}(\tilde{\alpha}) = \mu_\tilde{\alpha}^2 - v_\tilde{\alpha}^2, \quad (9)\]
and
\[\tilde{H}(\tilde{\alpha}) = \mu_\tilde{\alpha}^2 + v_\tilde{\alpha}^2. \quad (10)\]

According to Eqs. (9) and (10), we can obtain the Pythagorean indeterminacy index:
\[\tilde{\pi}(\alpha) = \sqrt{1 - \tilde{H}(\tilde{\alpha})}. \quad (11)\]

Let \(\tilde{\alpha} = (\mu_\tilde{\alpha}, v_\tilde{\alpha}) \in \Phi\) and \(\tilde{\beta} = (\mu_\tilde{\beta}, v_\tilde{\beta}) \in \Phi\), then

1. If \(\tilde{S}(\tilde{\alpha}) > \tilde{S}(\tilde{\beta})\) then \(\tilde{\alpha} > \tilde{\beta}\);
2. If \(\tilde{S}(\tilde{\alpha}) < \tilde{S}(\tilde{\beta})\) then \(\tilde{\alpha} < \tilde{\beta}\);
3. If \(\tilde{S}(\tilde{\alpha}) = \tilde{S}(\tilde{\beta})\) then
   a. If \(\tilde{H}(\tilde{\alpha}) > \tilde{H}(\tilde{\beta})\) then \(\tilde{\alpha} > \tilde{\beta}\);
   b. If \(\tilde{H}(\tilde{\alpha}) < \tilde{H}(\tilde{\beta})\) then \(\tilde{\alpha} < \tilde{\beta}\);
   c. If \(\tilde{H}(\tilde{\alpha}) = \tilde{H}(\tilde{\beta})\) then \(\tilde{\alpha} = \tilde{\beta}\).

III. PYTHAGOREAN FUZZY PREFERENCE RELATIONS

Based on PFNs, the definition of the Pythagorean fuzzy preference relation (PFPR) is introduced by Mandal [60].

Definition 4: [60] Let \(X = \{x_1, \ldots, x_n\}\) be a set of alternatives. \(\tilde{R} = (\tilde{r}_{ij})_{n \times n}\) is called a PFPR, where \(\tilde{r}_{ij} = (r_{ij}^\mu, r_{ij}^\nu)\) is a PF-N, indicating the preference of \(x_i\) over \(x_j\), \(r_{ij}^\mu\) and \(r_{ij}^\nu\) are the preference degree and non-preference degree of \(x_i\) is preferred to \(x_j\), respectively, and
\[(r_{ij}^\mu(x))^2 + (r_{ij}^\nu(x))^2 \leq 1, \quad r_{ij}^\nu = r_{ij}^\nu(x), \quad r_{ij}^\mu = \sqrt{2}/2, \quad i, j = 1, 2, \ldots, n. \quad (12)\]

Next, we discuss some properties of a PFPR and propose a consistency index.

Property 1: If \(\tilde{R} = (\tilde{r}_{ij})_{n \times n}\) is a PFPR, then the following properties hold:

1. (Weak Transitivity):
   If \(\tilde{r}_{ik} \geq (\sqrt{2}/2, \sqrt{2}/2), \tilde{r}_{kj} \geq (\sqrt{2}/2, \sqrt{2}/2)\), then
   \[\tilde{r}_{ij} \geq (\sqrt{2}/2, \sqrt{2}/2)\] for all \(i, j, k = 1, 2, \ldots, n\).

2. (Max-Min Transitivity):
   \[\tilde{r}_{ij} \geq \min(\tilde{r}_{ik}, \tilde{r}_{kj})\] for all \(i, j, k = 1, 2, \ldots, n\).

3. (Max-Max Transitivity):
   \[\tilde{r}_{ij} \geq \max(\tilde{r}_{ik}, \tilde{r}_{kj})\] for all \(i, j, k = 1, 2, \ldots, n\).

4. (Restricted Max-Min Transitivity):
   If \(\tilde{r}_{ik} \geq (\sqrt{2}/2, \sqrt{2}/2), \tilde{r}_{kj} \geq (\sqrt{2}/2, \sqrt{2}/2)\), then
   \[\tilde{r}_{ij} \geq \min(\tilde{r}_{ik}, \tilde{r}_{kj})\] for all \(i, j, k = 1, 2, \ldots, n\).

5. (Restricted Max-Max Transitivity):
   If \(\tilde{r}_{ik} \geq (\sqrt{2}/2, \sqrt{2}/2), \tilde{r}_{kj} \geq (\sqrt{2}/2, \sqrt{2}/2)\), then
   \[\tilde{r}_{ij} \geq \max(\tilde{r}_{ik}, \tilde{r}_{kj})\] for all \(i, j, k = 1, 2, \ldots, n\).

6. (Triangle Condition):
   \[\tilde{r}_{ik} + \tilde{r}_{kj} \geq \tilde{r}_{ij}\] for all \(i, j, k = 1, 2, \ldots, n\).

This property can be explained geometrically as follows. The importance degrees of \(x_i, x_k, x_j\) over \(x_i\) and \(x_k\) over \(x_j\) are equal to or greater than \((\sqrt{2}/2, \sqrt{2}/2)\), then the importance degree of \(x_i\) over \(x_j\) is not less than the minimum of the importance degrees of \(\tilde{r}_{ik}\) and \(\tilde{r}_{kj}\).

\[\tilde{r}_{ik} + \tilde{r}_{kj} \geq \tilde{r}_{ij}\] for all \(i, j, k = 1, 2, \ldots, n\).

The values of preference relations are obtained from the pairwise comparison between alternatives. Because of the problem complexity or limitations of knowledge, DMs may give unreasonable PFPRs. In this situation, how to measure and modify the consistency level of PFPR is very important.

In the following, we introduce consistency of the PFPR.

Definition 5: Let \(\tilde{R} = (\tilde{r}_{ij})_{n \times n}\) be a PFPR, where \(\tilde{r}_{ij} = (r_{ij}^\mu, r_{ij}^\nu)\), 
\[i, j = 1, 2, \ldots, n. \quad (13)\]
then \(\tilde{R}\) is called a multiplicative consistent PFPR.

Obviously, \(\tilde{R}\) is multiplicative consistent if and only if
\[\ln(r_{ij}^\mu) + \ln(r_{jk}^\mu) = \ln(r_{ki}^\mu) + \ln(r_{ji}^\mu), \quad i, j = 1, 2, \ldots, n. \quad (14)\]

Refer to (14), the consistency index of the PFPR can be defined as follows.

Definition 6: Let \(\tilde{R} = (\tilde{r}_{ij})_{n \times n}\) be a PFPR, \(\tilde{r}_{ij} = (r_{ij}^\mu, r_{ij}^\nu), \quad i, j = 1, 2, \ldots, n. \quad (15)\]
1, 2, \ldots, n$, then,
\[ CI(\tilde{R}) = \sqrt{\frac{1}{n^3} \sum_{i=1}^{n} \sum_{k=1}^{n} \sum_{j=1}^{n} (\delta_{ijk})^2} \]  \hfill (15)

is called the multiplicative consistency index of a PFPR and 
\[ \delta_{ijk} = \ln(r_{ij}^\mu) + \ln(r_{kj}^\mu) + \ln(r_{ki}^\mu) - \ln(r_{ij}^\mu) - \ln(r_{kj}^\mu) - \ln(r_{ki}^\mu), \]
i, j = 1, 2, \ldots, n.

**Theorem 1:** Let $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ as before, and $CI(\tilde{R})$ be a multiplicative consistency index, then $\tilde{R}$ is a perfectly multiplicative consistent if and only if $CI(\tilde{R}) = 0$.

**Proof:**
(1) Let $\tilde{R}$ is a perfectly multiplicative consistent. According to the definition 5, we have 
\[ r_{ij}^\mu r_{jk}^\mu r_{ki}^\mu = r_{ij}^\mu r_{jk}^\mu r_{ki}^\mu, \]
i, j, k = 1, 2, \ldots, n.

Then, we can obtain 
\[ \delta_{ijk} = \ln(r_{ij}^\mu) + \ln(r_{jk}^\mu) + \ln(r_{ki}^\mu) - \ln(r_{ij}^\mu) - \ln(r_{jk}^\mu) - \ln(r_{ki}^\mu) = 0. \]

According to the Eqs. (15), we have 
\[ CI(\tilde{R}) = \sqrt{\frac{1}{n^3} \sum_{i=1}^{n} \sum_{k=1}^{n} \sum_{j=1}^{n} (\delta_{ijk})^2} = 0. \]

(2) Let $CI(\tilde{R}) = 0$. we have $(\delta_{ijk})^2 = 0$, then 
\[ \delta_{ijk} = \ln(r_{ij}^\mu) + \ln(r_{jk}^\mu) + \ln(r_{ki}^\mu) - \ln(r_{ij}^\mu) - \ln(r_{jk}^\mu) - \ln(r_{ki}^\mu) = 0. \]

Moreover, $r_{ij}^\mu r_{jk}^\mu r_{ki}^\mu = r_{ij}^\mu r_{jk}^\mu r_{ki}^\mu$, According to the definition 5, we know $\tilde{R}$ is a perfectly multiplicative consistent.

This completes the proof of Theorem 1.

The acceptable consistency index of the PFPR $\tilde{R}$ is defined as:

**Definition 7:** Let $\tilde{R}$ and $CI(\tilde{R})$ be as before. If $CI(\tilde{R}) \leq CI$, then $\tilde{R}$ is an acceptable multiplicative consistent PFPR. Otherwise, $\tilde{R}$ is an unacceptable consistent one, where $CI$ is the acceptable consistency threshold.

When a PFPR does not satisfy an acceptable consistency, it is necessary to modify it. In this paper, we develop a new method to obtain an acceptable multiplicative consistent PFPR based on the multiplicative consistency index.

DMs often have certain consistency tendency in making pairwise comparisons [61]. Motivated by Dong et al. [39], if $\delta_{ijk}$ are independent normally distributed with the mean 0 and standard deviation $\sigma$, $\delta_{ijk} \sim N(0, \sigma^2)$. Then, the following statistical property of $\tilde{R}$ is reached.

**Theorem 2:** Let $\tilde{R}$ and $CI(\tilde{R})$ be as before. If $\delta_{ijk} \sim N(0, \sigma^2)$, then 
\[ n^3 \left( \frac{1}{\sigma} \times CI(\tilde{R}) \right)^2 \sim \chi^2(n^3). \]  \hfill (16)

**Proof:**
Since 
\[ n^3 \left( \frac{1}{\sigma} \times CI(\tilde{R}) \right)^2 = \sum_{i=1}^{n} \sum_{k=1}^{n} \sum_{j=1}^{n} (\delta_{ijk}/\sigma)^2 \]
and $\delta_{ijk}/\sigma$ are independent normally distributed with mean 0 and standard deviation 1, then we have 
\[ n^3 \left( \frac{1}{\sigma} \times CI(\tilde{R}) \right)^2 \sim \chi^2(n^3). \]

This completes the proof of Theorem 2.

Consideration of alternative hypothesis $H_1$: 
\[ H_0 : \sigma^2 \geq \sigma_0^2; \quad H_1 : \sigma^2 < \sigma_0^2. \]

For a significance level $\alpha$, if the critical value of $\chi^2$ distribution is $\lambda_{\alpha}$, then we obtain 
\[ CI(\tilde{R}) = \sqrt{\frac{1}{n^3} \lambda_{1-\alpha}}. \]  \hfill (17)

Refer to (17), different threshold values can be obtained with different parameters $\alpha$ and $\sigma$. In this paper, following [39] we assume that $\alpha = 0.1$ and $\sigma = 0.2$. Table 1 provides the threshold values of $CI(\tilde{R})$ with different $n$.

| $n$ | $CI(\tilde{R})$ |
|-----|----------------|
| 3   | 0.1638         |
| 4   | 0.1768         |
| 5   | 0.1835         |
| 6   | 0.1875         |

**Table 1. Consistency index threshold.**

For an unacceptable multiplicative consistent PFPR $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$, it is necessary to revise it to achieve an acceptable multiplicative consistent PFPR. Let $\tilde{F} = (\tilde{f}_{ij})_{n \times n}$ be an ideal acceptable multiplicative consistent PFPR. For a given consistency index threshold $CI$, we can minimize the deviation between the original PFPR $\tilde{R}$ and the ideal acceptable multiplicative consistent PFPR $\tilde{F}$ as follows.

**Model (M-1)**

\[ \min \frac{1}{n^3} \sum_{i=1}^{n} \sum_{j=1}^{n} (\epsilon_{ij}^\mu + \epsilon_{ij}^\nu) \]
subject to
\[ \sqrt{\frac{1}{n^3} \sum_{i=1}^{n} \sum_{k=1}^{n} \sum_{j=1}^{n} (\delta_{ijk})^2} \leq CI; \]
\[ |\ln(r_{ij}^\mu) - \ln(f_{ij}^\mu)| = \epsilon_{ij}^\mu; \]
\[ |\ln(r_{ij}^\nu) - \ln(f_{ij}^\nu)| = \epsilon_{ij}^\nu; \]
\[ |\ln(f_{ij}^\mu) + \ln(f_{jk}^\nu) + \ln(f_{ki}^\nu) - \ln(f_{ij}^\mu) - \ln(f_{jk}^\nu) - \ln(f_{ki}^\nu)| = \delta_{ijk}; \]
\[ (f_{ij}^\mu)^2 + (f_{ij}^\nu)^2 \leq 1, \]
i, j = 1, 2, \ldots, n.

By solving (M-1), the ideal acceptable multiplicative consistent PFPR $\tilde{F}$ can be obtained.

Let
\[ \epsilon_{ij}^\mu + = (\epsilon_{ij}^\mu + \epsilon_{ij}^\nu) / 2, \epsilon_{ij}^\mu - = (|\epsilon_{ij}^\mu| - \epsilon_{ij}^\mu) / 2, \]

\[ \epsilon_{ij}^\mu = \epsilon_{ij}^\mu + - \epsilon_{ij}^\mu - . \]
\[ e_{ij}^{v+} = \left( \left| e_{ij}^{v} \right| + e_{ij}^{v-} \right) / 2, \ e_{ij}^{v-} = \left( \left| e_{ij}^{v} \right| - e_{ij}^{v-} \right) / 2. \]

\[ i, j = 1, 2, \ldots, n. \]

And \( e_{ij}^{v+} \cdot e_{ij}^{v-} = 0, e_{ij}^{v+} + e_{ij}^{v-} = 0 \), then (M-1) is rewritten by Model (M-2).

Model (M-2)

Minimize \[ \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \varepsilon_{ij}^{v+} + \varepsilon_{ij}^{v-} + e_{ij}^{v+} + e_{ij}^{v-} \right) \]

subject to \[ \sum_{i=1}^{n} \sum_{j=1}^{n} \delta_{jk}^2 \leq C^T; \]

\[ \ln(r_{ij}^{u}) - \ln(f_{ij}^{u}) - \varepsilon_{ij}^{v+} + \varepsilon_{ij}^{v-} = 0; \]

\[ \ln(r_{ij}^{v}) - \ln(f_{ij}^{v}) - \varepsilon_{ij}^{v+} + \varepsilon_{ij}^{v-} = 0; \]

Example 1. Let \( \tilde{X} = \{ \tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4 \} \) be a set of alternatives, and \( \tilde{R} = (\tilde{r}_{ij})_{4 \times 4} \) be a PPFR, where

\[ \tilde{R} =
\begin{pmatrix}
(1/\sqrt{2}, 1/\sqrt{2}) & (0.7, 0.4) & (0.4, 0.5) & (0.9, 0.4) \\
(0.4, 0.7) & (1/\sqrt{2}, 1/\sqrt{2}) & (0.3, 0.8) & (0.4, 0.6) \\
(0.5, 0.4) & (0.8, 0.3) & (1/\sqrt{2}, 1/\sqrt{2}) & (0.9, 0.2) \\
(0.4, 0.9) & (0.6, 0.4) & (0.2, 0.9) & (1/\sqrt{2}, 1/\sqrt{2})
\end{pmatrix}.
\]

Refer to (15) and Table 1, the consistency level of \( \tilde{R} \) is derived and it follows that \( CI(\tilde{R}) = 0.2731 > 0.1768 \), which means that \( \tilde{R} \) is unacceptable multiplicative consistent. By using (M-1) and referring to Eqs. (21)-(22), the acceptable consistent PPFR \( \tilde{F} \) is obtained as

\[ \tilde{F} =
\begin{pmatrix}
(1/\sqrt{2}, 1/\sqrt{2}) & (0.7, 0.4) & (0.4, 0.5) & (0.9, 0.4) \\
(0.4, 0.7) & (1/\sqrt{2}, 1/\sqrt{2}) & (0.3, 0.8) & (0.4, 0.6) \\
(0.5, 0.4) & (0.8, 0.3) & (1/\sqrt{2}, 1/\sqrt{2}) & (0.9, 0.2) \\
(0.4, 0.9) & (0.6, 0.4) & (0.2, 0.9) & (1/\sqrt{2}, 1/\sqrt{2})
\end{pmatrix}.
\]

Moreover, the CI of \( \tilde{F} \) is computed again, and \( CI(\tilde{F}) = 0.1768 \), which indicates \( \tilde{F} \) is acceptable consistent.

IV. AN ABSOLUTE DEVIATION MODEL TO DERIVING PRIORITY WEIGHTS OF PPFR BY USING MULTIPlicative CONSISTENCY

In order to derive priority weights of PPFR with its multiplicative consistency, inspired by [62]-[64], we first introduce the normalized Pythagorean fuzzy weights as:

Definition 8: Let \( \tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_n)^T \) be a Pythagorean fuzzy vector where \( \tilde{\omega}_i^{v+} = (\omega_i^{v+}, \omega_i^{v-}) \in \Phi \) for \( i = 1, 2, \ldots, n \). Then \( \tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_n \) are called normalized Pythagorean fuzzy weights if they satisfy the following conditions:

\[ \sum_{j=1,j \neq i}^{n} \omega_i^{v+} + \sqrt{1 - (\omega_i^{v-})^2} \leq 1, \sum_{j=1,j \neq i}^{n} \sqrt{1 - (\omega_j^{v-})^2} + \omega_i^{v+} \geq 1, \]

\[ \omega_i^{v+}, \omega_i^{v-} \in [0, 1], (\omega_i^{v+})^2 + (\omega_i^{v-})^2 \leq 1, \]

For a Pythagorean fuzzy vector \( \tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_n)^T \), if \( \bar{v}_i \) is denoted as the importance degree of alternative \( \tilde{x}_i \) in alternative set \( X = \{ \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n \} \), then \( \tilde{\omega} \) is also called the Pythagorean fuzzy priority weighting vector of PPFR \( \tilde{R} = (\tilde{r}_{ij})_{n \times n} \), where \( \tilde{R} \) is the pairwise comparison matrix on \( X \).

In the following, we can construct a Pythagorean fuzzy characteristic matrix of a PPFR by using its normalized Pythagorean fuzzy priority weighting vector.

Definition 9: Let \( \tilde{R} = (\tilde{r}_{ij})_{n \times n} = ((r_{ij}^{u}, r_{ij}^{v}))_{n \times n} \) be a PPFR, and \( \tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_n)^T = ((\omega_1^{v+}, \omega_1^{v-}), (\omega_2^{v+}, \omega_2^{v-}), \ldots, (\omega_n^{v+}, \omega_n^{v-}))^T \) be the Pythagorean fuzzy priority weighting vector. Then the Pythagorean fuzzy matrix \( \tilde{R} = (\tilde{r}_{ij})_{n \times n} = ((r_{ij}^{u}, r_{ij}^{v}))_{n \times n} \) is called a characteristic matrix of \( \tilde{R} \), if it satisfies:

\[ r_{ij}^{\mu} = \frac{2(\omega_i^{v+})^2}{(\omega_i^{v+})^2 + 1 - (\omega_i^{v-})^2 + (\omega_j^{v+})^2 + 1 - (\omega_j^{v-})^2}, \]

\[ r_{ij}^{v} = \frac{2(\omega_i^{v+})^2}{(\omega_i^{v+})^2 + 1 - (\omega_i^{v-})^2 + (\omega_j^{v+})^2 + 1 - (\omega_j^{v-})^2}, \]

for all \( i, j = 1, 2, \ldots, n \).

Theorem 3: Let \( \tilde{R} = (\tilde{r}_{ij})_{n \times n} = ((r_{ij}^{u}, r_{ij}^{v}))_{n \times n} \) be a PPFR, and \( \tilde{R} = (\tilde{r}_{ij})_{n \times n} = ((r_{ij}^{u}, r_{ij}^{v}))_{n \times n} \) be the characteristic matrix of \( \tilde{R} \). Then \( \tilde{R} \) is a consistent PPFR.

Proof: Refer to (22) and (23), it follows that

\[ 0 \leq r_{ij}^{\mu} \leq 1, \]

\[ 0 \leq r_{ij}^{v} \leq 1, \]

Moreover,

\[ (r_{ij}^{\mu})^2 + (r_{ij}^{v})^2 \]

\[ = \left( \frac{2(\omega_i^{v+})^2}{(\omega_i^{v+})^2 + 1 - (\omega_i^{v-})^2 + (\omega_j^{v+})^2 + 1 - (\omega_j^{v-})^2} \right)^2 + \left( \frac{2(\omega_i^{v+})^2}{(\omega_i^{v+})^2 + 1 - (\omega_i^{v-})^2 + (\omega_j^{v+})^2 + 1 - (\omega_j^{v-})^2} \right)^2 \]
It is obvious that $\tilde{r}_{ij} \tilde{r}_{jk} \tilde{r}_{ki} = \tilde{r}_{ij} \tilde{r}_{jk} \tilde{r}_{ki}$, $i, j, k = 1, 2, \ldots, n$.

Based on definition 5, $\hat{R}$ is a multiplicative consistent PFPR. For convenience, the characteristic matrix of $\hat{R}$ is called the Pythagorean fuzzy characteristic preference relation (PFCPR) of $\hat{R}$.

Generally, the PFPR provided by DM does not satisfy consistency. In order to measure the consistency degree of a PFPR, we construct the deviations between PFPR and its PFCPR as follows:

$$\varepsilon_{ij} = \tilde{r}_{ij}^\mu - r_{ij}^\mu, \quad i, j = 1, 2, \ldots, n \quad (24)$$

and

$$\eta_{ij} = \tilde{r}_{ij}^\nu - r_{ij}^\nu, \quad i, j = 1, 2, \ldots, n. \quad (25)$$

where $\tilde{R} = (\tilde{r}_{ij})_{n \times n} = (r_{ij}^\mu, r_{ij}^\nu)_{n \times n}$ is a PFPR and $\hat{R} = (\bar{r}_{ij})_{n \times n} = (\hat{r}_{ij}^\mu, \hat{r}_{ij}^\nu)_{n \times n}$ is the PFCPR of $\tilde{R}$. $\varepsilon_{ij}$ and $\eta_{ij}$ are called the membership deviation and non-membership deviation between $r_{ij}^\mu$ and $\hat{r}_{ij}^\nu$, respectively. Thus, the following absolute deviation model is constructed to obtain the priority weights of PFPR $\tilde{R}$:

**Model (M-3)**

$$\xi = \min \frac{2}{n(n - 1)} \sum_{i=1}^{n} \sum_{j>i}^{n} (|\varepsilon_{ij}| + |\eta_{ij}|)$$

s.t.

$$\begin{align*}
\sum_{j=1, j \neq i}^{n} \omega_{ij}^\mu + \sqrt{1 - (\omega_{ij}^\nu)^2} &\leq 1, \\
\sum_{j=1, j \neq i}^{n} \omega_{ij}^\nu + \sqrt{1 - (\omega_{ij}^\mu)^2} &\geq 1,
\end{align*}$$

$$i = 1, 2, \ldots, n. \quad (26)$$

**Example 2. (continued with Example 1):** Let $\hat{F} = (1/\sqrt{2}, 1/\sqrt{2}), (0.7, 0.4), (0.4, 0.5), (0.9, 0.4), (0.4, 0.7), (1/\sqrt{2}, 1/\sqrt{2}), (0.3, 0.8), (0.4777, 0.6), (0.5, 0.4), (0.8, 0.3), (1/\sqrt{2}, 1/\sqrt{2}), (0.9, 0.2), (0.4, 0.9), (0.6, 0.4777), (0.2, 0.9), (1/\sqrt{2}, 1/\sqrt{2})$. be an acceptable multiplicative consistent PFPR.

Using (M-3), we obtain the following priority weights:

$$\tilde{\omega}_1 = (0.1665, 0.9839), \tilde{\omega}_2 = (0.0544, 0.9600), \tilde{\omega}_3 = (0.5085, 0.6656), \tilde{\omega}_4 = (0.0329, 0.9995).$$

Refer to (9), we have

$$\begin{align*}
\tilde{S}(\tilde{\omega}_1) &= -0.9403, \quad \tilde{S}(\tilde{\omega}_2) = -0.9186, \\
\tilde{S}(\tilde{\omega}_3) &= -0.1845, \quad \tilde{S}(\tilde{\omega}_4) = -0.9979.
\end{align*}$$

If follows that

$$\tilde{S}(\tilde{\omega}_3) > \tilde{S}(\tilde{\omega}_2) > \tilde{S}(\tilde{\omega}_1) > \tilde{S}(\tilde{\omega}_4).$$

It means that

$$x_3 \succ x_2 \succ x_1 \succ x_4.$$
Algorithm 1:
Step 1. Expert provides the PFPR \( \tilde{R} = (\tilde{r}_{ij})_{n \times n} \) for alternatives set \( X = \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n\} \) and acceptable consistency threshold \( \tilde{C}I \).

Step 2. Calculate the consistency index \( CI(\tilde{R}) \) of PFPR by using (15).

Step 3. If \( CI(\tilde{R}) \leq \tilde{C}I \), let \( \tilde{F} = \tilde{R} \) and go to step 5. Otherwise, go to next step.

Step 4. Adjust the original PFPR through (M-2) to obtain an acceptable consistency PFPR \( \tilde{F} = (\tilde{f}_{ij})_{n \times n} \).

Step 5. Obtain Pythagorean fuzzy priority weighting vector \( \tilde{\omega} = (\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_n)^T \) of the acceptable consistent PFPR \( \tilde{F} = (\tilde{f}_{ij})_{n \times n} \) by using (M-3).

Step 6. Calculate \( \tilde{S}(\tilde{\omega}_i) \) and \( \tilde{H}(\tilde{\omega}_i) \) according to (9) and (10), respectively.

Step 7. Rank the alternatives and select the best one(s) based on \( \tilde{S}(\tilde{\omega}_i) \) and \( \tilde{H}(\tilde{\omega}_i) \).

The flow chart of Algorithm 1 is shown in Figure 1.

V. ILLUSTRATIVE EXAMPLE

With the development of the economy, it will inevitably bring water pollution, which will eventually endanger humans themselves. Water pollution treatment has become an important social topic. In particular, from Figure 2, Anhui province, a province in central China, is paying more and more attention to the treatment of industrial wastewater, and funds for treating industrial wastewater are increasing year by year, and the effect is getting better and better. On the one hand, industrial wastewater discharge in Anhui province has gradually reached emission standards. On the other hand, the facilities and scales of wastewater treatment are also increasingly optimized. In order to maintain the stability of wastewater treatment capacity, the factories need to update the facilities periodically.

Recently, an electroplating factory needs to renew a batch of wastewater treatment facilities. Taking into account a variety of relevant factors, five alternatives \( C_1, C_2, C_3, C_4, C_5 \) can be selected. Due to the existence of complexity and uncertainty in decision making course, PFS is more convenient for decision maker expressing uncertain information, DMs compare them with each other and provide their Pythagorean fuzzy preference information. Consequently, they achieve a consensus and obtain the following PFPR \( \tilde{R} = (\tilde{r}_{ij})_{n \times n} \), where

\[
\tilde{R} = \begin{pmatrix}
(1/\sqrt{2}, 1/\sqrt{2}) & (0.31, 0.9) & (0.4, 0.7) & (0.7, 0.4) & (0.8, 0.2) \\
(0.9, 0.31) & (1/\sqrt{2}, 1/\sqrt{2}) & (0.7, 0.4) & (0.6, 0.29) & (0.99, 0.1) \\
(0.7, 0.4) & (0.4, 0.7) & (1/\sqrt{2}, 1/\sqrt{2}) & (0.8, 0.3) & (0.78, 0.2) \\
(0.4, 0.7) & (0.29, 0.6) & (0.3, 0.8) & (1/\sqrt{2}, 1/\sqrt{2}) & (0.7, 0.4) \\
(0.2, 0.8) & (0.1, 0.99) & (0.2, 0.78) & (0.4, 0.7) & (1/\sqrt{2}, 1/\sqrt{2})
\end{pmatrix}
\]

Next, the priority weights of \( \tilde{R} \) will be derived by the proposed method. Let \( \tilde{C}I = 0.1835 \) be the acceptable consistency threshold based on Table 1. Then, refer to (15), the consistency index is obtained:

\[
CI(\tilde{R}) = 0.3865
\]

According to Model(M-2), we have,

\[
\begin{align*}
\min \frac{1}{25} \sum_{i=1}^{5} \sum_{j=1}^{5} (\varepsilon^{u+}_{ij} + \varepsilon^{l-}_{ij} + \varepsilon^{v+}_{ij} + \varepsilon^{v-}_{ij}) \\
\sqrt{\sum_{i=1}^{5} \sum_{j=1}^{5} \sum_{k=1}^{5} (\delta_{ik})^2} \leq \tilde{C}I; \\
\ln(f^{u+}_{ij}) - \ln(f^{l-}_{ij}) - \varepsilon^{u+}_{ij} + \varepsilon^{l-}_{ij} = 0; \\
s.t. \quad \ln(f^{v+}_{ij}) - \ln(f^{v-}_{ij}) - \varepsilon^{v+}_{ij} + \varepsilon^{v-}_{ij} = 0; \\
\ln(f^{v+}_{kj}) + \ln(f^{v-}_{kj}) - \ln(f^{v+}_{ki}) - \ln(f^{v-}_{ki}) = \delta_{ik}; \\
(f^{u+}_{ij})^2 + (f^{l-}_{ij})^2 \leq 1, f^{u+}_{ij} = f^{l-}_{ij}, i, j = 1, 2, \ldots, 5.
\end{align*}
\]
\[ \tilde{F} = \begin{pmatrix} (1/\sqrt{3}, 1/\sqrt{3}) & (0.31, 0.9) & (0.4, 0.7) & (0.7, 0.4) & (0.8, 0.2) \\ (0.9, 0.31) & (1/\sqrt{3}, 1/\sqrt{3}) & (0.7, 0.4) & (0.6, 0.1965) & (0.99, 0.1) \\ (0.7, 0.4) & (0.4, 0.7) & (1/\sqrt{3}, 1/\sqrt{3}) & (0.8, 0.3) & (0.78, 0.2) \\ (0.4, 0.7) & (0.1956, 0.6) & (0.3, 0.8) & (1/\sqrt{3}, 1/\sqrt{3}) & (0.7, 0.4) \\ (0.2, 0.8) & (0.1, 0.99) & (0.2, 0.78) & (0.4, 0.7) & (1/\sqrt{3}, 1/\sqrt{3}) \end{pmatrix} \]

By solving (M-2), the optimal matrix \( \tilde{F} = (\hat{f}_{ij})_{5 \times 5} \) can be derived. We also obtain \( CI(\tilde{F}) = 0.1835 \leq CI \), which means that \( \tilde{F} \) satisfies acceptable consistency.

Next, we can obtain the absolute deviation model (M-3):

\[
\xi = \min \frac{1}{10} \sum_{i=1}^{5} \sum_{j=1}^{5} |(e_{ij} + |n_{ij}|) |
\]

\[
s.t. \begin{cases} f_{ij}^\mu - \hat{f}_{ij}^\mu = e_{ij} = 0; \\ f_{ij}^\nu - \hat{f}_{ij}^\nu = n_{ij} = 0; \\ i, j = 1, 2, \ldots, 5; \ i \neq j; \\ \omega_{i}^\mu, \omega_{i}^\nu \in [0, 1]; (\omega_{i}^\mu)^2 + (\omega_{i}^\nu)^2 \leq 1; \\ \sum_{j=1, j \neq i}^{5} \omega_{j}^\mu + \sqrt{1 - (\omega_{i}^\nu)^2} \leq 1; \\ \sum_{j=1, j \neq i}^{5} \sqrt{1 - (\omega_{i}^\nu)^2} + \omega_{i}^\mu \geq 1; \\ i = 1, 2, \ldots, 5. \end{cases}
\]

By solving Model 3, the Pythagorean fuzzy priority weighting vector \( \tilde{\omega} \) is derived as

\[ \tilde{\omega}_1 = (0.0631, 0.9923), \tilde{\omega}_2 = (0.5008, 0.7798), \]

\[ \tilde{\omega}_3 = (0.1933, 0.9126), \tilde{\omega}_4 = (0.0206, 0.9987), \]

\[ \tilde{\omega}_5 = (0.0067, 0.9999). \]

Then, according (9), we have

\[ \tilde{S}(\tilde{\omega}_1) = -0.9807, \tilde{S}(\tilde{\omega}_2) = -0.3573, \tilde{S}(\tilde{\omega}_3) = -0.7955, \]

\[ \tilde{S}(\tilde{\omega}_4) = -0.9970, \tilde{S}(\tilde{\omega}_5) = -0.9999. \]

Therefore,

\[ \tilde{S}(\tilde{\omega}_2) > \tilde{S}(\tilde{\omega}_3) > \tilde{S}(\tilde{\omega}_1) > \tilde{S}(\tilde{\omega}_4) > \tilde{S}(\tilde{\omega}_5). \]

Thus, the ranking of alternatives is

\[ C_2 > C_3 > C_1 > C_4 > C_5, \]

which means that \( C_2 \) is the best choice. Compared to [52], [60] and [64], the characteristics of proposed method are listed as follows:

(1) The new approach proposed in this paper expands the range of problems that can be handled, because PFPR contains more decision information than IFPR, making the new method more flexible than IFPR-based methods. Our proposed method is used to solve example 2 provided in [64]. Then, the related results are as follows:

\[ \tilde{\omega}_1 = (0.2808, 0.6553), \tilde{\omega}_2 = (0.1880, 0.7746), \]

\[ \tilde{\omega}_3 = (0.0567, 0.9962). \]

Then, according (9), we have

\[ \tilde{S}(\tilde{\omega}_1) = -0.3506, \tilde{S}(\tilde{\omega}_2) = -0.5647, \tilde{S}(\tilde{\omega}_3) = -0.9892. \]

Therefore,

\[ \tilde{S}(\tilde{\omega}_1) > \tilde{S}(\tilde{\omega}_2) > \tilde{S}(\tilde{\omega}_3). \]

Thus, the ranking of alternatives is

\[ C_1 > C_2 > C_3, \]

which means that \( C_1 \) is the best choice. Compared to [52], [60] and [64], the characteristics of proposed method are listed as follows:

(1) The new approach proposed in this paper expands the range of problems that can be handled, because PFPR contains more decision information than IFPR, making the new method more flexible than IFPR-based methods. Our proposed method is used to solve example 2 provided in [64]. Then, the related results are as follows:

\[ \tilde{\omega}_1 = (0.2808, 0.6553), \tilde{\omega}_2 = (0.1880, 0.7746), \]

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Then, according (9), we have

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Therefore,

\[ \tilde{S}(\tilde{\omega}_1) > \tilde{S}(\tilde{\omega}_2) > \tilde{S}(\tilde{\omega}_3). \]

Thus, the ranking of alternatives is

\[ C_1 > C_2 > C_3, \]

which means that \( C_1 \) is the best choice. Compared to [52], [60] and [64], the characteristics of proposed method are listed as follows:

(1) The new approach proposed in this paper expands the range of problems that can be handled, because PFPR contains more decision information than IFPR, making the new method more flexible than IFPR-based methods. Our proposed method is used to solve example 2 provided in [64]. Then, the related results are as follows:

\[ \tilde{\omega}_1 = (0.2808, 0.6553), \tilde{\omega}_2 = (0.1880, 0.7746), \]

\[ \tilde{\omega}_3 = (0.0567, 0.9962). \]

(2) Adjusting algorithm make the final decision result reasonable, because algorithm transforms the PFPR with unacceptable consistency into acceptable one, and retains the DM’s decision information as much as possible during the adjustment process.

(3) Compared with [60], this paper proposes a new definition method of PFPR consistency, which effectively expands the application of PFPR. At the same time, based on the new definition of consistency, a new method of PFPR consistency level adjustment is also proposed in this paper. The adjustment of PFPR which does not satisfied the consistency level given by experts makes the final decision result more reasonable. The group decision making method based on this method will be further studied in the future. According to the method in [60], the following alternatives ranking results can be obtained for solving the PFPR problem in this paper as following:

\[ C_2 > C_3 > C_1 > C_4 > C_5. \]

It indicates that this method is reasonable and an effective extension of PFPR application.

VI. CONCLUSION

PFPR is an effective way to express DMs’ preferences among the alternatives. In this paper, the main contributes are as follows:

(1) We investigated the PFPR and its related properties. The consistent index of the PFPR was defined to measure multiplicative consistent.

(2) In order to obtain an acceptable consistent PFPR, we proposed an optimization model to modify the PFPR.
(3) Then, we used the new absolute deviation model to derive the weighting vector of the alternatives and to sort the alternatives. In the future, we will continue to consider the following issues:

(1) In the process of decision making, information transmission may cause partial decision information loss. In this case, there are two methods: the DM gives the decision information again and the technical method estimates the missing information. Therefore, it is necessary to study PFPR under incomplete information [65].

(2) Interval-valued PFN is an important extension of PFN, so interval-value PFPR will also be our focus in the future.

(3) In real-world decision making, alternatives are generally evaluated by a group of experts based on several different attributes, i.e. multiple attribute group decision making. It will be interesting to study the consensus reaching process [66]–[70] and compatibility measure [71] base on PFPR. Moreover, PFPR will be applied in more fields, such as venture investment evaluation [72], pattern recognition problems [73], competency-based test evaluation [74], etc. It would be an important expansion in future research to apply PFPR in those fields.

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