Fractal distributions of dark matter and gas in the MareNostrum Universe
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ABSTRACT

Context. Large simulations of dark matter and gas structure formation allow us to separate the pure gravitational dynamics from other processes and, hence, to better compare them with observations.
Aims. Our objective is to analyse the recent MareNostrum simulation from a new perspective, regarding the geometry of the dark matter and gas distributions in it. We intend to find the fractal geometry of the dark matter and to determine if the gas distribution is fractal as well. If it is, the question is whether or not the gas distribution is nonetheless biased with respect to the dark matter.
Methods. We use the methods of multifractal geometry, in particular, an improved method of coarse multifractal analysis based on counts in cells. To determine the gas biasing, we use statistical methods: we use the cross-correlation coefficient and we develop a Bayesian analysis connected with information theory. We also employ entropic measures to characterize both distributions further than the multifractal analysis.
Results. Both distributions are multifractals, with equal spectra. The analysis of cross-correlations is inconclusive, but the Bayesian analysis clearly demonstrates gas biasing. The main feature of the distribution of gas is that it is less concentrated in the high density regions (massive halos). The entropic measures show that this gas bias is small, such that both distributions have the same singularities.
Conclusions. Gravity determines a unique distribution of singularities in the gas and the dark matter and, therefore, determines a universal multifractal spectrum. Nevertheless, gas biasing exists and, in general, should be model dependent.
Key words. cosmology: large-scale structure of Universe – galaxies: clusters:general – methods: statistical

1. Introduction

N-body cosmological simulations have been of great help in testing theories of structure formation. They have been complementary to observations, since observations are biased towards the luminous matter, while simulations have fully considered the evolution of the dark matter. In fact, many simulations only consider (cold) dark matter, whose dynamics is simplest to simulate. This situation has changed recently, due to the advances in parallel computing, the development of efficient codes, and the availability of more powerful computers. Now it is possible to simulate the combined dynamics of dark matter and gas in large cosmological volumes and with good resolution.

We analyse here the data output of a recent large cosmological simulation of the combined gas and dark matter dynamics, namely, a simulation of 1024^3 dark-matter particles and an equal number of gas particles in the MareNostrum supercomputer at Barcelona. This dataset has already been analysed by the researchers in charge of the MareNostrum Universe project (Gottlöber et al, 2006; Faltenbacher et al, 2007; Gottlöber & Yepes, 2007). Here, we are interested in a particular aspect of the MareNostrum dark matter and gas distributions: their geometry and, in particular, their fractal geometry.

Fractal geometry (Mandelbrot, 1983) is related to scale invariance and indeed appears in nonlinear systems in which the dynamics is characterized by the absence of reference scales. This is the case of the dynamics of collision-less cold dark matter (CDM), only subjected to the gravitational interaction. Thus, a reasonable model of the CDM structure consists in a multifractal attractor of its gravitational dynamics. Such a model agrees well with the results of CDM simulations and, in addition, is adequate to describe two essential components of the large-scale structure of matter, namely, halos and voids (Gaite, 2005, 2007, 2008). In contrast, the gas dynamics is not only due to gravitation and is more complex (due to its pressure, etc.). However, the gas also takes part in the nonlinear dynamics of structure formation and, arguably, the polytropic gas dynamics does not introduce any scale. Therefore, the gas can also be led to a multifractal attractor. Indeed, scaling laws in the distribution of galaxies have a long history (Jones et al, 2004; Sylos Labini, & Pietronero, 2007). However, the distribution of gas does not have to follow the same scaling laws that the dark matter follows.

We begin with a description of a method of coarse multifractal analysis by counts in cells, based on the method of Gaite (2007) but with a significant improvement (Sect. 2). In Subsect. 2.1, we show how to obtain the main features of the multifractal spectrum. Then, we apply our method to the zero-redshift particle distributions of the MareNostrum Universe, obtaining the mass functions and the multifractal spectra at several scales (Sect. 3). The similarity of the
results corresponding to the gas and the dark matter shows that we need precise statistical methods to discriminate between both distributions. In other words, we need to rigorously determine if the gas distribution (for example) is biased. Since cross-correlations cannot give a definite answer (Sub-sect. 4.1), we develop an effective Bayesian analysis (Sub-sect. 4.2). This analysis connects with the thermodynamic entropy of mixing (Sub-sect. 4.3). Therefore, we study the application of entropic measures to discriminating between mass distributions, and the connection of entropies in the continuum limit with the multifractal spectrum (Sect. 5). Finally, we discuss our results (Sect. 6).

A note on notation: we use frequently the asymptotic signs $\sim$ and $\approx$; for example, $f(x) \sim g(x)$ or $f(x) \approx g(x)$ (often without making explicit the independent variable $x$). The former means that $\lim f(x)/g(x)$ when $x$ goes to zero or infinity, as the case may be, is finite, while the latter means, in addition, that the limit is one. We also use the sign $\simeq$, which only refers to imprecise numerical values (with unspecified errors).

2. Method of data analysis

The MareNostrum cosmological simulation is described by Gottlöber et al. (2006). It assumes a spatially flat concordance model with parameters $\Omega_m = 0.7$, $\Omega_b = 0.3$, $\Omega_{bar} = 0.045$, and Hubble parameter $h = 0.7$, in a comoving cube of 500 $h^{-1}$ Mpc edges. The Gadget-2 code (Springel, 2005) simulated the evolution of dark matter and gas from redshift $z = 40$ to $z = 0$. Both dark matter and gas are resolved by $1024^3$ particles, respectively, which results in a mass of $8.24 \times 10^9 h^{-1}$ $M_\odot$ per dark-matter particle and a mass of $1.45 \times 10^9 h^{-1}$ $M_\odot$ per gas particle. The Gadget-2 code implements polytropic (adiabatic) evolution of the gas, but it can also include dissipation due to radiation or conduction. However, these effects have not been included in the MareNostrum simulation. Nevertheless, the code always includes an artificial viscosity to take care of shock waves.

The MareNostrum Universe consists of 135 evenly spaced snapshots. We only need the $z = 0$ snapshot, in which the structures are most developed. The large size of a MareNostrum Universe snapshot makes it unwieldy, so it is convenient (and almost necessary) to analyse it in terms of compound structures, namely, clusters (or halos), rather than analysing the full particle distributions. The MareNostrum Universe researchers (Gottlöber et al, 2006; Faltenbacher et al, 2007; Gottlöber & Yepes, 2007) use a friends-of-friends algorithm to define halos, and then they study the distribution and features of these halos. However, we prefer the method of counts in cells, more suitable for studying scaling properties of the particle distributions. Hence, our objects (clusters or halos) are cells with constant size but variable mass. The definition of elementary objects is arbitrary to a high degree, as explained before, and this definition is very convenient (Gaite, 2007).

In the method of counts in cells, the statistical (fractional) $q$-moments are defined as

$$M_q = \sum_i \left( \frac{n_i}{N} \right)^q = \sum N(n) \left( \frac{n}{N} \right)^q,$$

where the index $i$ refers to non-empty cells, $n_i$ is the number of points (particles) in the cell $i$, $N$ is the total number of points, and $N(n)$ is the number of cells with $n$ points. The second expression involves a sum over cell populations and it is more useful than the sum over individual cells, because the range of $n$ is much smaller (if the cell size is small). $M_0$ is the number of non-empty cells and $M_1 = 1$. We understand the latter as a mass normalization, namely, the mass in cell $i$ is $n_i/N$ and the total mass is one, such that the mass distribution can be interpreted as a probability distribution (the physical masses of dark or dark-matter particles play no role in our analysis). An alternate definition of $q$-moments is

$$\mu_q = \langle \rho^q \rangle = \frac{M_q}{V^q M_0} = \sum_n \frac{N(n)}{M_0} \left( \frac{n}{NV} \right)^q,$$

where $V$ is the cell’s volume (the cube’s total volume is normalized to one). In this definition, $N(n)/M_0$ is the fraction of non-empty cells that contain $n$ points, and $\rho = n/(NV)$ is the density in those cells. With this definition, $\mu_0 = 1$ while $\mu_1$ is not fixed. We notice that the moments with positive integer $q$ ($M_q$ or $\mu_q$, $q \in \mathbb{N}$) are sufficient for regular distributions, but we do not impose this restriction here ($q \in \mathbb{R}$).

We use the cell-counts statistics as a part of a method of coarse multifractal analysis (Gaite, 2007). In a multifractal, the $q$-moments must be power laws of the scale (given by the cell size $V$) with exponents $\tau(q)$. Thus, the coarse function $\tau(q)$ can be defined by

$$\tau(q) = 3 \frac{\log(M_q/V_0^{q-1})}{\log(V/V_0)},$$

where $V_0$ is the homogeneity scale, such that the density is homogeneous and $M_q \approx V^{q-1}$ for $V > V_0$. This definition improves the definition used by Gaite (2007) for the GIF2 simulation, which did not set any homogeneity scale ($V_0 = 1$). The MareNostrum Universe cube has 500 $h^{-1}$ Mpc edges, much longer than the 110 $h^{-1}$ Mpc edges of the GIF2 simulation cube. Therefore, it is important now to take the transition to homogeneity into account in the definition of scaling functions. We define homogeneity in the MareNostrum Universe by requiring that the mass fluctuations are smaller than $10\%$, that is to say, $\mu_2 < 1.1$. Thus, we find that the transition to homogeneity takes place over the scale of $1/16$th of the edge of the cube, namely, $31 h^{-1}$ Mpc. It is similar to the GIF2 homogeneity scale, which was found a posteriori. Since we work with normalized units, we set $V_0 = 2^{-12}$.

In a multifractal, the local dimension at one point says how the mass grows from that point outwards. Every set of points with a given local dimension $\alpha$ constitutes a fractal set with dimension $f(\alpha)$. The spectrum of local dimensions is given by

$$\alpha(q) = \tau(q), \quad q \in \mathbb{R},$$

and the spectrum of fractal dimensions $f(\alpha)$ is given by the Legendre transform

$$f(\alpha) = q \alpha - \tau(q).$$

The fractal dimension fulfills $f(\alpha) \leq \alpha$ and it coincides with the local dimension at $q = 1$ [note that Eq. (3) gives

1 Central moments are defined by subtracting from $n/N$ its average. In the strongly nonlinear regime, central moments are less convenient.

2 The value quoted by Gaite (2007), $r_0 \simeq 14 h^{-1}$ Mpc, is equivalent to half the edge of the cube such that $\mu_2 < 1.1$.}
The set of singularities with \( f(\alpha_1) = \alpha_1 := \alpha(1) \) contains the bulk of the mass and is called the “mass concentrate.”

Besides the multifractal spectrum \( f(\alpha) \), it is useful to define the spectrum of coarse Rényi dimensions (Harte, 2001)

\[
D(q) = \frac{\tau(q)}{q-1} = \frac{3}{\log(V/V_0)} \log(M_q/V_0^{q-1})/q-1.
\]

They have an information-theoretic meaning, on which we will dwell in Sect. 5. In particular, the dimension of the mass concentrate \( \alpha_1 = f(\alpha_1) = D(1) \) is also called the entropy dimension. In the homogeneous regime \( V > V_0 \), \( M_q \approx V^{q-1} \) and \( D(q) = 3 \) for any \( q \). In a uniform fractal (a unifractal or monofractal) \( D(q) \) is a constant smaller than three. In general, \( D(q) \) is a non-increasing function of \( q \).

### 2.1. Features of the multifractal spectrum

In a multifractal, the size of the cell \( V \) in the definition of \( \tau(q) \) by Eq. (3) is irrelevant, as long as it is sufficiently smaller than the homogeneity scale \( V_0 \). However, the intrinsic discreteness of a particle distribution sets another scale, namely, the size of the cell such that there is one particle per cell on average \( (V = N^{-1}) \). This coarse-graining scale is such that it produces the largest variety of clusters of particles. Thus, it provides a master cell distribution that characterizes the multifractal. Since the numbers of dark-matter or gas particles in the MareNostrum Universe are perfect cubes and, indeed, powers of cells, the master cell distribution is easily obtained. Gaite (2007) shows that the mass function of clusters in the master cell distribution follows the power law \( N(m) \sim m^{-2} \) (except at the large mass end, where it decays faster). This power law represents the mass concentration of the multifractal. In contrast, the master cell distribution contains no information of the matter distribution in voids (sets of points with \( \alpha > 3 \)), because they are empty (Gaite, 2007, 2008).

For any coarse-graining scale, we deduce from Eq. (1) and Eqs. (3–5) that the low end of the multifractal spectrum is given by the limit \( q \rightarrow -\infty \), that is to say, by the cell(s) with maximal number of particles:

\[
\alpha_{\min} = \lim_{q \rightarrow -\infty} \alpha(q) = -3 \log[M_{max}/(N(V_0))] \log(V/V_0), \tag{7}
\]

\[
f(\alpha_{\min}) = -3 \log[N(n_{max}) V_0] \log(V/V_0). \tag{8}
\]

Since \( \alpha_{\min} \) denotes the local dimension of the strongest singularity, it changes little with the scale, unless \( V \rightarrow V_0 \), which implies that \( \alpha_{\min} \rightarrow 3 \). Usually, \( N(n_{max}) = 1 \). Therefore, the setting \( V_0 = 1 \) would imply that \( f(\alpha_{\min}) = 0 \). However, our current setting \( V_0 = 1/4096 \) implies that the fractal dimension \( f(\alpha_{\min}) \) is negative!

Intuitively, negative fractal dimensions are meaningless, but they have already arisen in the study of random multifractals. The origin of negative fractal dimensions has been discussed recently by Mandelbrot (2003). In brief, the coarse fractal dimension of a set of singularities in a random multifractal is proportional to the logarithm of their number, but the expected value of this number can be smaller than one. Therefore, sets of singularities with negative fractal dimension are almost surely empty. In our case, by setting \( V_0 \) to be a fraction of the total volume, the mass in cubes of size \( V_0 \) fluctuates and is indeed largest in cubes that contain the cell with \( n_{max} \) points, which is the strongest singularity. Let us assume that the total cube is divided into \( V_0^{-1} = 4096 \) non-overlapping cubes of size \( V_0 \).

If \( N(n_{max}) \) were equal to \( V_0^{-1} \), namely, if we had one singularity with \( \alpha_{\min} \) per cube of size \( V_0 \), then \( f(\alpha_{\min}) = 0 \). Thus, it is convenient to “average” over the \( V_0^{-1} = 4096 \) cubes and consider at once the 4096 singularities with smallest \( \alpha \), truncating the negative values of the fractal dimension spectrum.

In analogy with the low end of the spectrum of local dimensions, we can deduce that the high end of the spectrum of local dimensions is given by the limit \( q \rightarrow -\infty \), that is, by the cells with one particle (as long as \( V \) is not so large that there are none). In fact,

\[
\alpha_{\max} = \lim_{q \rightarrow -\infty} \alpha(q) = -3 \frac{\log[N(V_0)]}{\log(V/V_0)}, \tag{9}
\]

\[
f(\alpha_{\max}) = -3 \frac{\log[N(n_{max}) V_0]}{\log(V/V_0)}. \tag{10}
\]

Therefore, the master cell distribution has \( \alpha_{\max} = 3 \), and its spectrum is limited to non-void regions (\( \alpha \leq 3 \)). Furthermore, \( 0 \leq f(\alpha_{\max}) < 3 \), and its actual value depends on the ratio of cells with one particle, \( N(1)/N \), such that \( 0 < N(1)/N < 1 \). The truncation of the multifractal spectrum at \( \alpha_{\max} = 3 \) is removed by considering larger coarse-graining scales, namely, \( V > 1/N \). Thus, \( \alpha_{\max} \) increases. For sufficiently large \( N \), \( N(1) \) decreases and approaches \( 1/V_0 = 4096 \) (only one cell with one particle per each cube of size \( V_0 \)). Then, \( f(\alpha_{\max}) \) decreases to zero. At this point, the distribution can be considered continuous, and we have the full positive multifractal spectrum in the region \( \alpha > 3 \), corresponding to voids. We truncate again the negative values of \( f(\alpha) \).

The total span of the spectrum is

\[
\alpha_{\max} - \alpha_{\min} = -3 \frac{\log(n_{max}/n_{min})}{\log(V/V_0)},
\]

where \( n_{min} = 1 \). Naturally, the largest span is reached with cell volumes larger than \( V = 1/N \), as \( \alpha_{\max} \) grows and the spectrum for voids is included. In the transition to homogeneity, \( n_{max} \approx n_{min} \) and the span of the spectrum shrinks.

Regarding the master cell distribution, the simple mass function \( N(n) = N(1)/n^2 \) allows us to say more about its multifractal spectrum. For example, according to Eq. (1),

\[
M_0 = \sum_{n=1}^{n_{max}} N(n) \approx N(1) \sum_{n=1}^{\infty} \frac{1}{n^2} = N(1) \frac{\pi^2}{6}.
\]

Since this sum is just the number of non-empty cells, we deduce that the fraction of cells with one particle is \( N(1)/M_0 \approx 6/\pi^2 \approx 0.61 \). Thus, the distribution \( N(n) \) is determined by just the number of empty cells. Furthermore, from the expression

\[
M_1 = \sum_{n=1}^{n_{max}} N(n) \frac{n}{N} \approx \frac{N(1)}{N} \ln n_{max}, \tag{11}
\]
and the condition $M_1 = 1$ we can determine $n_{\text{max}}$. Then, the dimension of the mass concentrate $\alpha_1 = f(\alpha_1)$ is

$$\alpha_1 = \tau'(1) = \frac{3}{\ln(V/V_0)} \left( \frac{dM_q}{dq} \bigg|_{q=1} - \ln V_0 \right)$$

$$= \frac{3}{\ln(V/V_0)} \left( \sum_{n=1}^{n_{\text{max}}} N(n) \frac{n}{N} \ln \frac{n}{N} - \ln V_0 \right)$$

$$\approx \frac{3}{\ln(V/V_0)} \left( \frac{\ln n_{\text{max}}}{2} - \ln(NV_0) \right).$$

This dimension is the arithmetic mean of the general values of $\alpha_{\text{min}}$ in Eq. (7) and $\alpha_{\text{max}}$ in Eq. (9).

### 3. Multifractal distributions of dark matter and gas

We now present the results of the multifractal analysis of the MareNostrum Universe. We select the $z = 0$ snapshot, starting with the results of the counts of the master cell distribution (with one dark-matter particle and one gas particle per cell on average). The mass functions of dark-matter and gas clusters are plotted in Fig. 1. One can check that both mass functions follow the power law $N(m) \sim m^{-\alpha}$ over a considerable range of masses.

There are 170546782 cells with one gas particle and 156272463 cells with one dark-matter particle in the master cell distribution. According to Eq. (10), the fractal dimensions of the sets with $\alpha_{\text{max}} = 3$ are 2.56 and 2.54, for the gas and dark matter, respectively. The most massive gas cluster has 19200 particles and the most massive dark-matter halo has 20658 particles (they are actually the same object). The corresponding values of $\alpha_{\text{min}}$, according to Eq. (7), are 0.63 and 0.61, respectively. However, Eq. (8) yields negative values of $f(\alpha_{\text{min}})$, which we do not consider. We compute directly from Eqs. (3–5) that the values of $\alpha$ such that $f(\alpha) = 0$ are 0.95 (gas) and 0.91 (dark matter).

We have seen in the preceding section that the value of $\alpha_1$ corresponding to the master cell distribution can be estimated by the arithmetic mean of $\alpha_{\text{max}}$ and $\alpha_{\text{min}}$. Whether we use $\alpha_{\text{min}} \approx 0.6$ or $\alpha_{\text{min}} \approx 0.9$, this estimation yields smaller values than the actual values, which are 2.29 (gas) and 2.22 (dark matter). On the other hand, the estimation $m_{\text{max}} = \exp[N/N(1)]$ yields 542 and 967, respectively, well below the real values (see Fig. 1). The problem is that the power law is modified at the large mass end, as we can perceive in Fig. 1. On the one hand, at the large mass end, the values of $N(m)$ are so small that there are many values of $m$ for each value of $N$; on the other hand, $N$ as a function of the average of the values of $m$ decays faster than a power law. In fact, the above estimated values of $m_{\text{max}}$ actually mark the ends of the power laws, instead of the ends of the large masses.

We can improve the fit of the mass function by modelling the end of the power law. For this, we can take inspiration from the Press-Schechter mass function,

$$N(m) \propto \left( \frac{m}{m_*} \right)^{n/6-3/2} \exp\left[ -\left( \frac{m}{m_*} \right)^{3+1/n} \right],$$

(12)

where $n > -3$ is the spectral index of the initial power spectrum and $m_*$ stands for the large-mass cutoff. In fact, agreement with the power-law part of the spectrum demands $n \to -3$. Therefore, we take

$$N(m) \approx N(1) m^{-2} \exp\left[ -\left( \frac{m}{m_*} \right)^{\epsilon} \right],$$

(13)

where $\epsilon > 0$ is to be fitted, as well as $m_*$. The latter can be deduced from the condition $M_1 = 1$; namely,

$$M_1 \approx \frac{N(1)}{N} \sum_{m=1}^{\infty} \frac{\exp[-(m/m_*)^{\epsilon}]}{m} \approx \frac{N(1)}{N} \ln m_*.$$

It is independent of $\epsilon$, and coincides with the value given by Eq. (11) if we identify $m_{\text{max}}$ there with $m_*$. This identification is natural, because the exponential form is just a more adequate way of introducing a mass cutoff. It explains why the above quoted values of $m_{\text{max}}$, below 1000, mark the end of the power laws. The new values of $m_{\text{max}}$ are obtained from expression (13) by requiring $N(m_{\text{max}}) = 1$. This model raises the estimations of $m_{\text{max}}$, but the new values depend on $\epsilon$. For $\epsilon = 1$, $m_{\text{max}}$ is equal to 2022 (gas) or 2849 (dark matter). Naturally, better estimations are obtained by taking smaller $\epsilon$. In fact, the Press-Schechter mass function must be substituted by a lognormal mass function (Gaite, 2007), in which the power $(m/m_*)^\epsilon$ becomes $\ln(m/m_*)^2$.

The power-law mass function $N(m) \sim m^{-2}$ for the master cell distributions is consistent with multifractal distributions of the dark matter and gas, but it does not really prove that these distributions are multifractal. A definitive proof would be provided by the scaling of correlation functions. The large number of particles prevents us from attempting a direct calculation of correlation functions, but we could test the scaling of the counts-in-cells moments.
$M_\alpha$. However, we do not expect a good scaling of these moments in a multifractal, unless a very large range of scales is available (Gaite, 2007). In other words, the exponent $\tau(q)$ defined in Eq. (3) is not expected to be constant as a function of $V$.

One way to prove multifractality is to calculate halo-halo correlations of uniform halo populations (cells with given $\alpha$). Another is to calculate the full coarse multifractal spectrum at several scales. We choose the second way, because it gives a more condensed information. We plot in Fig. 2 the multifractal spectra of the dark-matter and gas distributions, for increasing scales, from the master cell distribution up to a cell edge equal to $2^{-7}$. We stop at this scale because we already have the full spectrum, and larger scales begin to show signs of approaching the transition to homogeneity. For comparison, we also plot the spectra corresponding to the distributions at scale $2^{-3}$, which are homogeneous [we have computed them using Eq. (3) with $V_0 = 1$].

The multifractal spectra in Fig. 2 are similar to the multifractal spectra of the GIF2 simulation obtained by Gaite (2007), but they span a slightly larger range of local dimensions. By increasing the reference scale $V_0$, we observe that the span of $\alpha$ at a given scale shrinks. Thus, we deduce that the slightly smaller spans obtained from the GIF2 simulation are due to having set there no homogeneity scale ($V_0 = 1$) The dimension of the mass concentrate in the spectra of Fig. 2 slightly rises as the coarse-graining length grows; we estimate $\alpha_1 \simeq 2.4$. This value agrees with the value obtained from the GIF2 simulation. It is a remarkably high value, which makes the mass concentrate relatively homogeneous. In particular, as regards scaling, the real cosmic structure differs from the structure produced by the adhesion model, which concentrates in Dirac-delta-type singularities, with $\alpha_1 = 0$. It seems to also differ from the fractal structure of galaxies according to standard determinations of the fractal dimension of the galaxy distribution, which yield a value close to two (Jones et al, 2004; Sylos Labini, & Pietronero, 2007). However, this dimension is determined from the two-point correlation function and, therefore, it corresponds to the correlation dimension $\tau(2) = D(2)$, which is always smaller than $\alpha_1 = D(1)$. Indeed, $\tau(2) \simeq 1.8$ for the dark-matter or gas distributions (its value varies between 1.6 and 1.9).

The dark-matter and gas distributions are both multifractal and, remarkably, neither the mass functions (Fig. 1) nor the multifractal spectra (Fig. 2) allow us to differentiate them. To be precise, the numerical differences between them seem to be too small to attribute them statistical significance. This conclusion may seem counterintuitive, since the dark-matter and gas dynamics are clearly different. In this regard, we have seen, for example, that the respective numbers of cells with one particle in the master cell distribution are somewhat different. We have seen as well that the respective number of particles in the most populated cell differ, but by little, and other populated cells have similar numbers of both types of particles. To assess the statistical significance of all the numerical data, we carry out next a detailed study.

![Fig. 2. Multifractal spectra at length scales $2^{-10,9,8,7}$, with solid lines in successively lighter tones of grey, and diagonal lines, which touch the spectra lines at the mass concentrate dimension $\alpha \simeq 2.4$. The light dashed lines are the spectra of the homogeneous distributions at scale $2^{-3}$. All the corresponding dark matter (top) and gas spectra (bottom) are very similar.](image)

4. Relation between the gas and dark-matter distributions

We see that the dark-matter and gas distributions are very similar multifractals, which could actually be identical. In general, one may ask if two finite samples of continuous distributions can come from the same continuous distribution. In particular, it is possible that the differences in the distributions of gas and dark matter particles are only due to statistical sample variance, while there is no bias in the continuous gas distribution with respect to the total mass distribution. We know that the gas dynamics is different from the collisionless dark-matter dynamics, with the likely result of bias, but we need to ascertain the existence of bias from the actual particle distributions.

We can use the cross-correlation function of gas and dark-matter particles to measure the similarity of both distributions. Indeed, this test confirms that both distributions are very similar, as we show in sub-section 4.1. However, this test cannot prove that, actually, the samples come from the same continuous distribution. In fact, it is easy to see that there is no way to prove it and we must satisfy ourselves with obtaining a probability of its being true. Rather, assuming a Bayesian point of view, we can quantify the “degree of belief” in the hypothesis that there is a common continuous distribution. The application of this method in sub-section 4.2 allows us to confidently conclude that the gas distribution is biased. Then, we study the nature of that bias in sub-section 4.3.

To simplify the analysis, independently of the chosen method, we use counts-in-cells, like in the multifractal analysis. Thus, we assume that the continuous distribution de-
rem tells us how to adjust probabilities in regard to new ability as a measure of a state of knowledge. Bayes’ theo-

4.2. Bayesian analysis

is stably above 0.99, that is to say, the correlation between the two vectors

We can define the cross-correlation coefficient of (g) and nonlinear regime, where

The autocorrelation of a mass distribution coarse-grained over a volume scale \( V \) can be measured by its second order cumulant, namely, by

\[
\xi_2 = \frac{1}{V^2} \int_V d^3x_1 d^3x_2 \xi_2(x_1, x_2),
\]

where \( \xi_2(r) \) is the two-point correlation function. In the nonlinear regime,

\[
\xi_2 = \frac{\langle \rho^2 \rangle}{\langle \rho \rangle^2} - 1 \approx \frac{\langle \rho^2 \rangle}{\langle \rho \rangle^2} \frac{\mu_2}{\mu_1^2} \gg 1.
\]

We can define the cross-correlation coefficient of (g) and dark-matter (m) as

\[
e_{gm} = \frac{\xi_{gm}}{\langle \xi_{gm} \rangle^{1/2}} = \frac{\langle \rho_g \rho_m \rangle}{\left( \langle \rho_g^2 \rangle \langle \rho_m^2 \rangle \right)^{1/2}} = \frac{\sum_i n_{g,i} n_{m,i}}{\left( \sum_i n_{g,i}^2 \sum_i n_{m,i}^2 \right)^{1/2}},
\]

where the last expression refers to counts in cells. This coefficient can be viewed as the cosine of the angle formed by the two vectors \( \{n_{g,i}\} \) and \( \{n_{m,i}\} \). It can depend on the size of the cells (V) but this dependence should be mild.

In Fig. 3, we plot the cross-correlation coefficient of the gas and dark-matter in massive halos (taken from the master cell distribution), ranked by their mass. This coefficient is stably above 0.99, that is to say, the correlation between both distributions is very strong. However, we do not know if it is sufficiently strong to affirm that both samples come from the same multinomial distribution.

4.2. Bayesian analysis

Bayes’ theory of probability interprets the concept of probability as a measure of a state of knowledge. Bayes’ theorem tells us how to adjust probabilities in regard to new evidence; namely,

\[
P(H|E) = \frac{P(E|H) P(H)}{P(E)},
\]

where \( H \) is a hypothesis with prior probability \( P(H) \), \( E \) is an event that provides new evidence for \( H \), and \( P(E|H) \) is the conditional probability of having \( E \) if the hypothesis \( H \) happens to be true. \( P(E) \) is the a priori probability of observing the event \( E \) under all possible hypotheses. \( P(H|E) \) adjusts \( P(H) \) and is called the posterior probability of \( H \) given \( E \). Bayesian analysis is routinely employed for model selection in many scientific areas.

In Bayes’ theorem, the hypothesis \( H \) can belong to a continuum of possibilities. For example, if we are given the results of \( N \) trials of a binomial experiment, we can analyse the information gained from them on the probability \( p \) of “success” (one of the two possible outcomes). This probability is a number \( 0 < p < 1 \) (while the probability of “failure” is \( 1 - p \)). For a given value of \( p \), the probability of \( n \) successes in \( N \) trials is given by the binomial distribution

\[
P(N, n|p) = \binom{N}{n} p^n (1-p)^{N-n}.
\]

Since \( N \) and \( n \) are fixed, and \( p \) is unknown, we can apply Bayes’ theorem in the form

\[
P(p|N, n) = \frac{P(N, n|p) P(p)}{\int_0^1 P(N, n|p) P(p) \, dp} = \frac{p^n (1-p)^{N-n} P(p)}{\int_0^1 p^n (1-p)^{N-n} P(p) \, dp},
\]

in terms of the prior probability of \( p \). If no prior information about \( p \) is available, we must assume that \( P(p) = 1 \) (the principle of insufficient reason). Then, the posterior probability \( P(p|N, n) \) is the beta distribution with parameters \( n + 1 \) and \( N - n + 1 \). It is trivial to check that it reaches its maximum at \( p = n/N \) (mode value) and that its variance is proportional to \( 1/N \) (for fixed \( n/N \) and large \( N \)).

This example is, in fact, relevant for our problem, namely, for estimating the probability that the given gas and dark-matter samples belong to the same distribution. If we choose one cell, with \( n_m \) dark-matter particles, say, the probability that the mass fraction in that cell is \( p_m \) is given by the beta distribution with parameters \( n_m + 1 \) and \( N_m - n_m + 1 \) (\( N_m \) being the total number of dark-matter particles in the sample). Analogously, the probability of a gas mass fraction \( p_g \) in that cell is given by the beta distribution with parameters \( n_g + 1 \) and \( N_g - n_g + 1 \). We can obtain the probability of the difference \( p_g - p_m \) by performing a change of variables and integrating

\[
P(p_m|n_m, N_m) P(p_g|n_g, N_g) \bigg|_{p_m + p_g} \bigg|_{2} \bigg( \text{within the appropriate limits} \bigg).
\]

Nevertheless, we expect to get some information from the value of the probability density at \( p_m = p_g \). Thus, we calculate

\[
\int_{p_m}^{p_g} P(p|m, N_m) P(p|n_g, N_g) \, dp = \frac{B(n_m + n_g + 1, N_m - n_m + N_g - n_g + 1)}{B(n_m + 1, N_m - n_m + 1) B(n_g + 1, N_g - n_g + 1)},
\]

where \( B(\cdot, \cdot) \) is the beta function.
where \( B(x, y) = \Gamma(x) \Gamma(y)/\Gamma(x+y) \) is the Euler beta function. The value of the integral is enhanced when the maxima of \( P(p|n_m, N_m) \) and \( P(p|n_g, N_g) \) coincide, namely, when \( n_m/N_m = n_g/N_g \). For fixed \( N_m = N_g \), the function on the right-hand side of Eq. (14) is a symmetric function of \( \{n_m, n_g\} \). Therefore, for a fixed value of \( n_m + n_g \), it is just a symmetric function of the difference \( n_m - n_g \) and has its maximum when \( n_m = n_g \).

One could criticize the preceding approach for only focusing on the value of the probability density at \( p_m = p_g \), while values of \( p_m - p_g \) close to zero might also be relevant. We can avoid the problem of having to deal with a continuous probability by singling out the value \( p_m = p_g \) from the outset. Thus, we formulate a Bayesian analysis with this hypothesis and the event \( E = \{n_m, N_m, n_g, N_g\} \):

\[
P(p_m = p_g|E) = \frac{P(p_m = p_g) P(E|p_m = p_g)}{P(p_m = p_g) P(E|p_m = p_g) + P(p_m \neq p_g) P(E|p_m \neq p_g)}.
\]

Here, \( P(E|p_m \neq p_g) \) is just the probability of \( E \) given any values of \( p_m \) and \( p_g \), because the event \( p_m = p_g \) has probability zero; namely,

\[
P(E|p_m \neq p_g) = \binom{N_m}{n_m} \binom{N_g}{n_g} \times \frac{1}{1 - p} \int_0^1 dp_m \int_0^1 dp_g p_m^{n_m}(1 - p_m)^{N_m - n_m} p_g^{n_g}(1 - p_g)^{N_g - n_g}.
\]

On the other hand,

\[
P(E|p_m = p_g) = \binom{N_m}{n_m} \binom{N_g}{n_g} \times \frac{1}{1 - p} \int_0^1 dp p_m^{n_m+n_g}(1 - p)^{N_m - n_m + N_g - n_g}.
\]

Computing the integrals and substituting, we obtain

\[
P(p_m = p_g|E) = \frac{P(p_m = p_g) b(n_m, N_m; n_g, N_g)}{P(p_m = p_g) b(n_m, N_m; n_g, N_g) + P(p_m \neq p_g) b(n_m, N_m; n_g, N_g)},
\]

where

\[
b(n_m, N_m; n_g, N_g) = \frac{P(E|p_m = p_g)}{P(E|p_m \neq p_g)} = \frac{B(n_m + n_g + 1, N_m - n_m + N_g - n_g + 1)}{B(n_m + 1, N_m - n_m + 1) B(n_g + 1, N_g - n_g + 1)}.
\]

This function of \( \{n_m, N_m, n_g, N_g\} \) coincides with the value of the probability density of \( p_m = p_g \) at 0 in Eq. (14). Thus, this approach is consistent with the preceding one: if \( b(n_m, N_m; n_g, N_g) \) is large, then \( P(p_m = p_g|E) \) tends to one, independently of the prior probability \( P(p_m = p_g) \). On the other hand, we have no way to estimate this prior probability.

The assignment of prior probabilities is a usual problem in Bayesian analyses, to the extent that Bayes’ theory of probability has been deemed subjective. However, there is no subjective element if we indeed understand Bayes’ theory as a way to adjust probabilities in regard to new evidence. The Bayes factor defined in Eq. (15) is such that

\[
\log \frac{P(p_m = p_g|E)}{P(p_m \neq p_g|E)} = \log b(n_m, N_m; n_g, N_g) + \log \frac{P(p_m = p_g)}{P(p_m \neq p_g)}.
\]

Hence, we can endow this equation with an information theory meaning: the prior information on the odds of our hypothesis is updated by the information provided by the event \( E = \{n_m, N_m, n_g, N_g\} \). The prior information is null if \( P(p_m = p_g) = P(p_m \neq p_g) \), but the information provided by the event is independent of it. This information can be positive or negative according to whether the Bayes factor is larger or smaller than one. The addition of information is independent of the (common) base of the logarithms, but it is convenient to use base two and measure the information in bits. If the Bayes factor is larger than one half and smaller than two, the information provided by \( E \) is smaller than one bit and can hardly be considered. For example, with \( N_m = N_g = 200 \), \( \log_2 b(100, 200; 100, 200) = 3.00 \) bits, \( \log_2 b(100, 200; 70, 200) = -3.61 \) bits, and only the first case or the last case provide evidence for or against \( p_m = p_g \), respectively.

Since we actually divide the sample into many cells, we need to generalize the above method of comparing binomial distributions to the case of multinomial distributions. This generalization is straightforward, except that we now have to take care of normalizing the \( P(E|\cdot) \) such that \( \sum_E P(E|\cdot) = 1 \). The resulting Bayes factor is

\[
b(n_m_1, \ldots, n_m_k; n_g_1, \ldots, n_g_k) = \frac{B(n_m_1 + n_g_1 + 1, \ldots, n_m_k + n_g_k + 1)}{B(n_m_1 + 1, \ldots, n_m_k + 1) B(n_g_1 + 1, \ldots, n_g_k + 1) (k-1)!},
\]

where \( \{n_m_i\}_{i=1}^k \) and \( \{n_g_i\}_{i=1}^k \) are the vectors denoting the numbers of dark-matter and gas particles, respectively, in the \( k \) cells and \( B(x_1, \ldots, x_k) = \Gamma(x_1) \cdots \Gamma(x_k)/\Gamma(x_1 + \cdots + x_k) \) is the generalized Euler beta function. We can write this Bayes factor as follows:

\[
b(n_m_1, \ldots, n_m_k; n_g_1, \ldots, n_g_k) = \frac{(n_m_1 + n_g_1)! \cdots (n_m_k + n_g_k)!}{(n_m_1)! \cdots (n_m_k)! (n_g_1)! \cdots (n_g_k)!} \times \frac{(N_m + k - 1)! (N_g + k - 1)!}{(N_m + N_g + k - 1)! (k-1)!},
\]

where \( N_m = n_m_1 + \cdots + n_m_k \) and \( N_g = n_g_1 + \cdots + n_g_k \) are the total numbers of dark-matter and gas particles, respectively (which are equal, in our case). The latter form has the advantage of being the product of \( k \) binomial numbers, each one corresponding to one cell, times an overall factor. Each binomial number expresses the number of ways of dividing the total number of particles in the corresponding cell between the gas and dark-matter particles. We can associate the (base-two) logarithm of that binomial number with a “cell entropy”. This entropy is maximal when the dark-matter and gas particles are evenly divided in the cell and vanishes when there are no particles of one type in the cell.

Let us take \( N_m = N_g = N \). We could compute the Bayes factor at once but we follow instead a more elaborate procedure. We begin by noticing that the cells implied by the multinomial Bayes factor are of logical nature and, therefore, we can group several physical cells into one of them. In particular, we group the less significant cells, namely, the ones with small numbers of particles. A systematic procedure consists in ordering the cells by decreasing total number of particles and taking the most populated ones firstly into account. Thus, we choose the first one and compare
we notice that there are more fluctuations indicating the same pattern that the number of dark-matter particles is above. In other words, the massive halos concentrate less but the line that corresponds to the dark-matter particles in the figure that both distributions approximately follow distributions (ranked by halo mass) in Fig. 5. We observe gas particles is clearly observed in the log-log plots of both particle distributions in massive halos reveals that these generally have less gas particles than dark-matter particles. An analogous difference has been described by these generally have less gas particles than dark-matter and gas particle distributions in massive halos.

4.3. Basic difference between the gas and dark-matter distributions

An inspection of the differences between the dark-matter and gas particle distributions in massive halos reveals that these generally have less gas particles than dark-matter particles. An analogous difference has been described by Faltenbacher et al (2007). The smaller average number of gas particles is clearly observed in the log-log plots of both distributions (ranked by halo mass) in Fig. 5. We observe in the figure that both distributions approximately follow linear log-log laws (sort of Zipf’s laws), with common slope, but the line that corresponds to the dark-matter particles is above. In other words, the massive halos concentrate less gas, although the number of gas particles decreases according to the same pattern that the number of dark-matter particles. We also notice that there are more fluctuations in the number of gas particles, due to their smaller physical mass.

We can express the pattern of the gas distribution in massive halos in terms of their Bayes information. Notice that we have a very large number of particles, namely, $N_m = N_g = N = 2^{30}$, and an equal number of cells. As long as we consider a number of massive halos $h$ such that the total number of particles in them is small in comparison with $N$, we can take a suitable approximation of the Bayes information. Indeed, under that condition, the largest contributions to the Bayes information come from the cell with the remaining particles and from the overall factor; namely,

$$
\text{log}_2 \left( 2N - \sum_{i=1}^{h} (n_{m,i} + n_{g,i}) \right) = 2N - \sum_{i=1}^{h} (n_{m,i} + n_{g,i}) - \frac{\log_2 (\pi N)}{2} + O(N^{-1})
$$

and

$$
\text{log}_2 \left( \frac{(N+h)!^2}{(2N+h)! h!} \right) = -2N + h \log_2 \frac{N}{2} + \frac{\log_2 (\pi N)}{2} + O(N^{-1}) - \log_2 h!,
$$

both having a first term proportional to $N$ (we have used Stirling’s approximation). However, these large terms cancel one another. Therefore,

$$
\text{log}_2 b = \sum_{i=1}^{h} \left[ \log_2 \left( \frac{n_{m,i} + n_{g,i}}{n_{g,i}} \right) - (n_{m,i} + n_{g,i}) \right] + h \log_2 \frac{N}{2} - \log_2 h! + O(N^{-1}),
$$

which is a sum of halo contributions plus a global contribution. Each halo contribution is negative, because the “cell entropy” is bounded above by the number of particles in the cell $(n_{m,i} + n_{g,i})$, as is easily proved. Since each halo contribution is larger in absolute value than $\log_2 (N/2) = 29$ bits, on average, the total information is negative (as displayed in Fig. 4). For illustration, we calculate the contribution of the most massive halo, with $n_{g,i} = 19200$ and $n_{m,i} = 20658$, namely, $\log_2 (19200) \approx 11.2$, $\log_2 (20658) \approx 14.6$, $\log_2 (39858) \approx -38.5$ bits.

The information contribution of a massive halo can be given a more familiar expression by using again Stirling’s approximation. When $n_{m,i}, n_{g,i} \gg 1$, the “cell entropy” can
be written as

$$\log_2 \left( \frac{n_m + n_g}{n_g} \right) \approx (n_m + n_g) \log_2(n_m + n_g) - n_g \log_2 n_g - n_m \log_2 n_m$$

$$= -(n_m + n_g) [x_g \log_2 x_g + (1-x_g) \log_2(1-x_g)],$$

where we have introduced the fraction of gas particles

$$x_g = \frac{n_g}{n_m + n_g}.$$

In this form, the “cell entropy” can be identified with the familiar entropy of mixing. The maximal entropy of mixing per particle is one bit and it corresponds to the most mixed distribution, with $x_g = 1/2$. We observe in Fig. 5 that the ratio $n_{g,i}/n_{m,i}$, $i = 1, \ldots, M$ is positive and almost constant on average. In fact, $n_{g,i}/n_{m,i} \simeq 0.81$. Hence, $x_g \simeq 0.45$, and the entropy of mixing per particle is

$$-x_g \log_2 x_g - (1-x_g) \log_2(1-x_g) \simeq 0.992.$$ 

Therefore, each halo contribution is roughly proportional to the total number of particles in it. The proportionality constant is the entropy of mixing per particle minus one, namely, $0.992 - 1 = -0.008$, which yields about $-250$ bits for the contribution per halo in Eq. (16). Then, the absolute value of each halo contribution is larger than 29 bits, making negative the total Bayes information per halo in Eq. (16). However, the value of the entropy per particle is very close to one, telling us that the distributions are very mixed, even though not completely mixed.

In equilibrium thermodynamics, the entropy of mixing is, of course, only one part of the total entropy. Therefore, it is not the entropy considered by Faltenbacher et al. (2007). When other thermodynamic parameters are equal, the entropy of mixing indeed determines the equilibrium configuration to be the most mixed distribution. In our case, the thermal states of the gas and of the dark matter in massive halos do not need to coincide, because they are not coupled and they follow different dynamics. In a mixture of ideal gases, the chemical potential of each gas can be expressed as

$$\mu = -T \log_2 \left( \frac{\zeta(T)}{n} \right)$$

(Reif, 1965), where we have used units consistent with measuring the entropy in bits. The function $\zeta(T)$ increases with $T$ and is characteristic of the gas. It is calculated from the possible states of its particles (translational and internal states); for a monoatomic gas, $\zeta(T) \propto T^{3/2}$. Thus, a condition of “chemical” equilibrium of gas and dark matter would imply

$$\frac{\zeta_g(T)}{n_g} = \frac{\zeta_m(T)}{n_m} \Rightarrow \frac{\zeta_g(T)}{\zeta_m(T)} = \frac{n_g}{n_m} \simeq 0.81.$$

Hence, assuming that both $\zeta_g$ and $\zeta_m$ correspond to monoatomic gases, we deduce that $T_g/T_m \simeq 0.87$.

This derivation of a relation between the temperatures of the gas and the dark matter in massive halos relies on assuming that they are in local thermodynamical equilibrium at different but well-defined temperatures. We can assign a local temperature to dark matter in terms of its (local) velocity dispersion. However, the assumption of thermodynamical equilibrium is questionable. In particular, it should imply that the dark-matter gas has pressure and, therefore, its dynamics should be like the gas dynamics.

5. Entropic comparison of multifractals

In the comparison of the gas and dark-matter distributions, we have found it useful to introduce the “cell entropies” and, hence, the entropy of mixing. In general, for a discrete probability distribution $\{p_i\}_{i=1}^M$ its statistical entropy is defined as

$$S(\{p_i\}) = -\sum_{i=1}^M p_i \log_2 p_i ,$$

and it represents the uncertainty or lack of information of the result of an experiment with that probability distribution. The entropy has some desirable properties, such as the bounds $0 \leq S(\{p_i\}) \leq \log_2 M$, and the property of additivity, in particular, additivity for independent sets of events (Wherl, 1978). This property and the bounds are shared by a uniparametric class of functions, the Rényi entropies

$$S_q(\{p_i\}) = \frac{\log_2(\sum_{i=1}^M p_i^q)}{1-q}, \quad q \in \mathbb{R}.$$ 

The value of $S_1$ is obtained as the limit $q \to 1$ and it coincides with the standard entropy defined by Eq. (19).

We can apply the definition of entropy to the probability distribution $p_i = n_i/N$ in $M$ cells. The entropy measures the uncertainty of the cell in which a particle is located (or a group of $q$ particles). In particular, we can interpret Eq. (19) as follows. According to Boltzmann, we should weight a macroscopic state, given by the occupation numbers $\{n_i\}_{i=1}^M$, with the number of microscopic states compatible with it. Then, the entropy is the logarithm of this weight. Since the $N$ particles (of each type) are multinomially distributed, their entropy is given by the logarithm of the corresponding multinomial number (the Boltzmann weight), namely,

$$\log_2 \left( \frac{N}{n_1 \cdots n_M} \right) \approx -N \sum_{i=1}^M p_i \log_2 p_i ,$$

where $M$ is the number of (non-empty) cells and we have assumed that $n_i \gg 1$. The entropy per particle $S(\{p_i\})$ is positive and bounded above by $\log_2 M$. If the distribution is uniform, the bound is reached (in particular, the bound is $\log_2 M = -\log_2 V$). Then, the distribution contains the largest uncertainty or, equivalently, the smallest information. Moreover, all the Rényi entropies reach the same bound.

Naturally, it is important to know the behaviour of the entropies in the continuum limit $V \to 0$. Not surprisingly, the entropies diverge in the continuum limit: one needs an infinite amount of information to locate a point in a continuum. However, there is information in the way in which the entropies diverge; in particular, the Rényi entropies fulfill

$$\lim_{V \to 0} \frac{3S_q(\{p_i\})}{-\log_2 V} = D(q),$$

where the finite quantities $D(q)$ are called the Rényi dimensions of the continuous distribution. The coarse Rényi dimensions have been already defined in Eq. (6) (with respect to a reference scale $V_0$). The most important Rényi dimension is $D(1)$, which characterizes the divergence of the standard entropy and is the dimension of the mass concentrate.
We deduce that all the continuous distributions with the same set of Rényi dimensions appear equivalent in regard to their information content. In particular, every continuous distribution with \( D(q) = 3 \), \( q \in \mathbb{R} \) appears equivalent to a homogeneous and uniform distribution, in which the Rényi entropies reach their upper bound \( (D(q) = 3 \) is also the upper bound to the Rényi dimensions). Indeed, only part of the information contained in a continuous distribution is preserved in its Rényi dimensions.

To further define the information content of a continuous distribution, we focus on distributions with \( D(q) = 3 \), \( q \in \mathbb{R} \), for the moment. Any of these distributions is regular and has a well-defined density everywhere.\(^3\) In the limit \( V \to 0 \), any cell of size \( V \) only sees a uniform distribution around it (given by the density in it), so that the most divergent part of the entropy is insensitive to the variations of the density. Hence, the variations of the density are reflected in the finite part. Indeed,

\[
S\{x_i\} \approx - \sum_i p(x_i) V \log_2[p(x_i) V] = - \sum_i p(x_i) V \log_2[p(x_i)] - \log_2 V,
\]

where \( x_i \) is a point in the cell \( i \). Therefore, we can write the entropy as

\[
S[p(x)] \approx - \int p(x) d^3 x \log_2[p(x)] - \log_2 V.
\]

The finite part is not defined in an absolute way: according to how the continuum limit is taken, we can obtain the integrand \( \log_2[p(x)/q(x)] \), where \( q(x) \) is any positive function with \( \int d^3 x q(x) = 1 \), that is to say, any positive probability density.\(^4\) It is easy to prove that this finite part is always negative (in contrast with the total entropy \( S[p(x)] \)). Conventionally, the relative entropy is defined as

\[
S[p|q] = \int p(x) d^3 x \log_2 \frac{p(x)}{q(x)} \geq 0.
\]

It is also called the Kullback or Kullback-Leibler divergence (studied in detail by Kullback, 1968). Therefore, the absolute entropy of a coarse distribution gives rise, in the continuum limit, to an absolute part, the dimension, and a relative part, the relative entropy.\(^5\) Only the latter differentiates regular distributions. Notice that the entropy relative to the uniform distribution is only defined for distributions over a finite volume (in our case, the unit cube).\(^6\)

We have similar results for singular multifractal distributions with \( D(1) < 3 \). One singular distribution \( \nu \) can be relatively regular, that is to say, it can be regular with respect to another singular distribution \( \mu \).\(^7\) Their relative entropy is defined as

\[
S(\nu|\mu) = \int d\nu(x) \log_2 \frac{d\nu(x)}{d\mu(x)} \geq 0,
\]

where \( d\nu(x)/d\mu(x) \) is the density of \( \nu \) with respect to \( \mu \) at the point \( x \). This relative entropy differentiates one multifractal distribution \( \nu \) from another \( \mu \), when the former is regular with respect to the latter and, in particular, they have the same dimension \( D(1) \). In fact, \( S(\nu|\mu) = 0 \) if and only if \( \nu = \mu \).

The relative entropy differentiates distributions but has two shortcomings. First, \( S(\nu|\mu) \) is only defined when \( \nu \) is \( \mu \)-regular. Second, the relative entropy does not have the necessary properties to qualify as a distance between distributions: it fails to be symmetric or to fulfill the triangle inequality. However, it is possible to define a real distance between any two distributions in terms of their entropies (Endres & Schindelin, 2003). For discrete distributions, Endres & Schindelin (2003) define

\[
D^2_{PQ} = 2S(R) - S(P) - S(Q) = \sum_{i=1}^{M} \left( p_i \log_2 \frac{p_i}{p_i + q_i} + q_i \log_2 \frac{2q_i}{p_i + q_i} \right),
\]

where \( P = \{ p_i \}, Q = \{ q_i \} \) and \( R = \{ (p_i + q_i)/2 \} \). Then, they prove that \( D_{PQ} \) is a distance. Furthermore, Endres & Schindelin (2003) note that it can be applied to continuous distributions. This follows from

\[
D^2_{PQ} = S(P|R) + S(Q|R),
\]

namely, from its being a sum of relative entropies, and from taking into account that any two continuous distributions are both regular with respect to their mean. Indeed, in our case, the limit \( V \to 0 \) is well defined.

Thus we can measure the distance between the coarse distributions \( p_i = n_{g,i}/N \) and \( q_i = n_{m,i}/N \), and consider the limit \( V \to 0 \). The distribution \( R \) corresponds to the total particle distribution. We can write the squared distance between the coarse distributions as

\[
D^2_{PQ} = 2 + \frac{1}{N} \sum_{i=1}^{M} \left( n_{g,i} \log_2 n_{g,i} + n_{m,i} \log_2 n_{m,i} - (n_{m,i} + n_{g,i}) \log_2 (n_{m,i} + n_{g,i}) \right) \] (21)

\[
= \frac{1}{N} \sum_{i=1}^{M} \left( n_{g,i} \log_2 n_{g,i} + n_{m,i} \log_2 n_{m,i} + (n_{m,i} + n_{g,i}) [1 - \log_2 (n_{m,i} + n_{g,i})] \right). \] (22)

\(^3\) Let us express the regularity condition in rigorous mathematical terms. In general, the density exists almost everywhere, but it may not yield the right mass distribution. A suitable regularity condition is absolute continuity with respect to the Lebesgue measure, namely, the condition that every set with zero volume (null Lebesgue measure) contains no mass. It implies, by the Radon-Nikodym theorem, that the mass distribution is the integral of a density that is unique almost everywhere (Capinski & Kopp, 2004). Absolute continuity allows for the presence of some singularities. Indeed, we only need that \( D(1) = 3 \) in the following.

\(^4\) Here we incur a slight notational inconsistency, since we have been using \( q \) for the parameter in the Rényi entropies or dimensions. Hence, we leave it to the reader to discern from the context whether \( q \) means the probability functions \( q(x) \) or \( q_i \) or the number \( q \).

\(^5\) It is useful (but optional) to also define the relative entropy of discrete distributions.

\(^6\) The relative entropy with respect to the uniform distribution has been considered as a measure of the evolution of inhomogeneity in cosmology by Hosoya, Buchert & Morita (2004).

\(^7\) Again, regularity means absolute continuity, now with respect to the measure \( \mu \) (every set with null \( \mu \)-measure has null \( \nu \)-measure). By the Radon-Nikodym theorem, there is a density \( d\nu/d\mu \), unique except in a set of null \( \mu \)-measure.
Referring to the cell entropy found in Eq. (17), we deduce that, in the sum of terms (one per cell) given by Eq. (22), each term represents the gap between the maximal cell entropy of mixing (one bit per particle) and its actual value. Naturally, $D^p_{PQ}$ decreases with mixing and vanishes for the most mixed distribution $P = Q = R$. Conversely, it takes its maximum, $D^p_{PQ} = 2$, when \{n_{gi}\} and \{n_{mi}\} are disjoint, namely, when they are not mixed at all (as we deduce from Eq. [21]).

Let us notice that the above statistical distance is consistent with the Bayesian analysis but cannot replace it. On the one hand, it relies on treating the particle fractions $n_{i}/N$ as continuous variables $p_i$ or $q_i$. Of course, this assumption is reasonable for massive halos but not for cells with a few particles. Indeed, the entropy of mixing in the form given by Eq. (17) is only meaningful for populated cells $(n_g, n_m \gg 1)$. On the other hand, a statistical distance does not provide a sharp criterium to decide if two discrete distributions are samples from the same continuous distribution. Nevertheless, Endres & Schindelin’s distance can be connected with a standard statistical measure of discrimination as follows.

It is useful to note that $D^p_{PQ}$ adopts a simplified form when $P$ and $Q$ are close, namely,

$$D^p_{PQ} \approx \frac{1}{2 \ln 2} \sum_{i=1}^{M} \left( \frac{p_i - q_i}{2(p_i + q_i)} \right)^2 = \frac{1}{2 \ln 2} \chi^2_{PQ}$$

(Endres & Schindelin, 2003). This is just a quadratic (Gaussian) approximation. In our case, we have defined the master cell distribution (at the scale of homogeneity equal to one sixteenth of the cube edge, namely, about $30 \ h^{-1} \ Mpc$, similar to the scale of homogeneity in the GIF2 simulation by Gaite (2007). We have defined the master cell distribution (at the scale of $1/1024$ of the cube edge, with one particle per cell), and we have worked with a scaling range spanning a factor of 16 from the master cell distribution. This scale range is sufficient to demonstrate multifractality. The introduction of the scale of homogeneity in the definition of scaling functions produces an anomalous extension of the multifractal spectrum: it gives rise to negative fractal dimensions. They can be understood as referring to improbable matter fluctuations that can be ignored.

Therefore, we could also apply Pearson’s chi-square test to discriminate between the two distributions. This test has the advantage of highlighting that the expected fluctuations of $[n_{mi} - n_{gi}]$ in a common distribution are of the order of $(n_{mi} + n_{gi})^{1/2}$. At any rate, it is an approximation of $D^2_{PQ}$ and it cannot replace the Bayesian analysis either.

The connection with the notion of relative entropy and with Endres & Schindelin’s distance opens a new perspective of the notion of biasing, with regard to the multifractal nature of the gas and dark-matter distributions. Let us recall that we have found, through the multifractal analysis in Sect. 3, that both the gas and dark-matter distributions have essentially the same coarse multifractal spectra, in particular, the same entropy dimension (see Fig. 2). However, the multifractal spectrum only gives the “size” (the dimension) of every set of singularities with common strength (local dimension), but tells us nothing about the precise geometry (location or shape) of those sets. In this regard, the study in Sect. 4 of the cross-correlation coefficient in massive halos (strong singularities) from the master cell distributions shows that the gas and dark-matter distributions are highly correlated. In contrast, the Bayesian comparison of the respective master cell distributions in Sect. 4.2 has clearly ruled out a common continuous distribution. Nevertheless, the entropy of mixing per particle is 0.992 bits, very close to the maximum of one bit. Therefore, the two distributions are indeed very mixed, in accord with the high cross-correlation coefficient.

Furthermore, the closeness of the gas and dark matter distributions suggests that their individual singularities coincide and, therefore, the two distributions are mutually regular. In the coarse formalism that we use, the local dimension of cell $i$ is

$$\alpha_i = 3 \frac{\log[n_i/(NV_0)]}{\log(V/V_0)}.$$ 

Therefore, the difference between the strengths of gas and dark matter singularities is

$$\alpha_g - \alpha_m = 3 \frac{\log(n_g/n_m)}{\log(V/V_0)}.$$ 

We deduce that this difference vanishes if $n_g/n_m$ stays bounded (above and below) while the cell volume $V$ shrinks. Although we have found that the ratio $n_g/n_m$ is not unity in populated cells, its logarithm is small (in absolute value) with respect to $- \log(V/V_0)$. Therefore, the singularities of the gas and dark matter distributions are essentially identical on the available scales.

6. Discussion and Conclusions

We have analysed the gas and dark matter distributions in the MareNostrum Universe at redshift $z = 0$ with the method of coarse multifractal analysis based on counts in cells (Gaite, 2007). We have improved the method by primarily calculating the scale of homogeneity and explicitly introducing it in the formalism. We find a scale of transition to homogeneity equal to one sixteenth of the cube edge, namely, about $30 \ h^{-1} \ Mpc$, similar to the scale of homogeneity found in the GIF2 simulation by Gaite (2007). We have defined the master cell distribution (at the scale of $1/1024$ of the cube edge, with one particle per cell), and we have worked with a scaling range spanning a factor of 16 from the master cell distribution. This scale range is sufficient to demonstrate multifractality. The introduction of the scale of homogeneity in the definition of scaling functions produces an anomalous extension of the multifractal spectrum: it gives rise to negative fractal dimensions. They can be understood as referring to improbable matter fluctuations that can be ignored.

Separate analyses of the gas and dark matter distributions show that both are multifractal and that their mass functions and multifractal spectra agree with the results of a previous analysis of cosmological simulations (Gaite, 2007). In fact, the multifractal analysis does not reveal noticeable differences between the gas and dark matter distributions. We can conclude that both distributions are equivalent as regards their scaling properties.

To determine if the equivalence of distributions goes beyond scaling properties, we have undertaken a detailed

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8 In general, Endres & Schindelin’s distance is maximal for mutually singular distributions, denoted $\mu \perp \nu$, namely, for distributions that concentrate in disjoint sets (as defined by Capinski & Kopp, 2004). A particular case is that they have disjoint support, but this is not necessary: for example, the uniform distributions in the Cantor set and in the unit interval are mutually singular.

9 The same conclusion follows by using only the relative entropy (Kullback, 1968).

10 Naturally, two distributions are mutually regular if they are absolutely continuous with respect to one another. For example, Kullback (1968) always works within an equivalence class of mutually regular distributions.
statistical study of the relation between the gas and dark matter distributions. The major objective of this study is to discriminate one distribution from another. A natural measure of discrimination is the cross-correlation coefficient of the cell distributions, which we have computed for the more populated cells (massive halos). The value of the coefficient for massive halos, above 0.99, seems to indicate that the two distributions are identical, within statistical errors. In other words, the gas distribution in the MareNostrum Universe does not seem to be significantly biased with respect to the dark-matter distribution. However, we cannot establish if the correlation coefficient constitutes sufficient evidence against bias.

Therefore, we have applied a Bayesian analysis to find the evidence that both distributions are two independent samples of a common multinomial probability distribution in the cells. We have shown that Bayes’ theory provides the best approach to our problem. The analysis demonstrates that the gas distribution in massive halos is clearly biased with respect to the dark matter distributions. Then, an inspection of massive halos shows that the gas is less concentrated in them. Undoubtedly, the physical cause of this bias is the gas pressure. A convenient measure of bias is the entropy of mixing.

Since the Bayesian analysis leads us to an entropic formulation (in terms of the entropy of mixing), we have developed an entropic method of comparison of multifractals. In particular, we have exploited the relation between multifractal analysis and Rényi entropies to connect with our multifractal spectrum results. Fortunately, the understanding of this connection allows us to reconcile the identity of gas and dark-matter multifractal spectra with the actual difference in their distributions (the gas bias). We have remarked two aspects of the entropic comparison of the gas and the dark-matter. The multifractal spectrum only characterizes the singularity structure of a singular distribution in terms of sizes of singular sets, where “size” stands for dimension. Thus, the Rényi dimensions preserve only a minor part of the information, namely, of the entropy of the coarse distributions. The major part of the information is of relational nature and can be expressed as a relative entropy or, better, as a statistical entropic distance. This distance is a neg-entropy of mixing, which goes from zero (identical distributions) to two (mutually singular distributions).

Hence, we have shown that there is a closer relationship between the gas and dark-matter distributions, manifested by the high correlation coefficient and, more definitely, by the high entropy of mixing: namely, the singularities of the gas and dark-matter distributions actually coincide. In general, the singularities coincide if the ratio of gas to dark-matter particle number densities is bounded as the coarse-graining length shrinks. This ratio is close to one in massive halos. On the other hand, this ratio is given by the mass fraction of gas, which can be called the (point-dependent) gas bias factor. Therefore, the condition for common gas and dark-matter singularities is mild: the gas bias factor must not approach zero or one at any point.

The appearance of common singularities in the gas and in the dark matter probably has a physical origin, despite the different dynamics of either component. It is natural to conjecture that the common multifractal structure is a consequence of the fact that the gas and the dark matter are both subjected to the long-range interaction that leads to power-law singularities, namely, the gravitational interaction. In contrast, the short-range interactions in the gas alter its distribution with respect to the dark matter (produce bias), but they do not interfere with the features of the distribution that are due to gravity (the singularities). In fact, the MareNostrum Universe is not based on a very realistic model of gas dynamics, insofar as it does not consider thermal radiation or conduction. Nevertheless, if the singularity structure is due to gravity only, the analysis of future simulations will corroborate that the gas biasing does not interfere with it. Then, we can speak of a type of universality: the gravitational dynamics has a unique multifractal attractor, independent of the initial conditions. This attractor is characterized by its multifractal spectrum, as obtained here from the MareNostrum Universe or before from the GIF2 simulation (Gaite, 2007).

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