A CRITERION FOR THE EXPPLICIT RECONSTRUCTION OF A
HOLOMORPHIC FUNCTION FROM ITS RESTRICTIONS ON
LINES

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Abstract. We deal with a problem of the explicit reconstruction of any holomorphic function $f$ on $\mathbb{C}^2$ from its restrictions on a union of complex lines. The validity of such a reconstruction essentially depends on the mutual repartition of these lines, condition that can be analytically described. The motivation of this problem comes also from possible applications in mathematical economics and medical imaging.

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1. Introduction

1.1. Presentation of the problem. In this paper we deal with a problem of the reconstruction of a holomorphic function from its restrictions on analytic subvarieties. The general presentation is the following: let $f$ be a holomorphic function on a domain $\Omega \subset \mathbb{C}^n$ and $\{Z_j\}_{j=1}^{N}$ a family of analytic subvarieties of $\Omega$. We assume that we just know the data $f|_{\{Z_j\}_{j=1}^{N}} := \{f|_{Z_j}\}_{j=1}^{N}$ and we want to find $f$. One can give interpolation functions $f_N \in \mathcal{O}(\Omega)$ that satisfy $f_N|_{\{Z_j\}_{j=1}^{N}} = f|_{\{Z_j\}_{j=1}^{N}}$ (see [2]), but generally $f_N \neq f$ (since there exist nonzero holomorphic functions that vanish on a finite union of analytic subvarieties). Then a natural way is to consider an infinite family of subvarieties $\{Z_j\}_{j=1}^{\infty}$ and construct the associate interpolating $f_{\infty}$ as $\lim_{N \to \infty} f_N$. In this case the uniqueness of the interpolating function will certainly be made sure but now without any guaranty of the convergence of the sequence $(f_N)_{N=1}^{\infty}$. Moreover, in case of positive result, we are motivated to give explicit reconstruction formula.

In this paper we will deal with the special case of $\mathbb{C}^2$ and a family of distinct complex lines that cross the origin. Such a family can be described as

$$\left\{ \{z \in \mathbb{C}^2, z_1 - \eta_j z_2 = 0\} \right\}_{j \geq 1},$$

(1.1)
with \( \eta_j \in \mathbb{C} \) all different, that we will simply denote by \( \eta = \{\eta_j\}_{j \geq 1} \) (without loss of generality we can forget the line \( \{z_2 = 0\} \) that is associate to \( \eta_0 = \infty \). On the other hand, we consider an entire function \( f \in \mathcal{O}(\mathbb{C}^2) \) and assume that we know all its restrictions \( f_j \in \mathcal{O}(\mathbb{C}), \forall j \geq 1 \), where

\[
(1.2) \quad \{f_j\}_{j \geq 1} := \{f_{\{z_1=\eta_j z_2\}}\}_{j \geq 1}.
\]

A way to give the interpolation function \( f_N \) is the one that uses the following relation that is proved in [7] by using residues and principal values:

\[
\lim_{\epsilon \to 0} \frac{1}{(2\pi)^2} \int_{\mathbb{C}^2} \mathcal{O}(\mathbb{C}^2) \omega(\zeta) \mathcal{O}(\mathbb{C}^2) \omega(\zeta) - \lim_{\epsilon \to 0} \frac{1}{(2\pi)^2} \int_{\mathbb{C}^2} \mathcal{O}(\mathbb{C}^2) \omega(\zeta) \mathcal{O}(\mathbb{C}^2) \omega(\zeta),
\]

with

\[
\omega'(\zeta) = \zeta_1 d\zeta_2 - \zeta_2 d\zeta_1,
\]

\[
\omega(\zeta) = d\zeta_1 \wedge d\zeta_2,
\]

and \( P_\zeta f_N(\zeta, z) \in \mathcal{O}(\mathbb{C}^2)^2 \) satisfying, for all \((\zeta, z) \in \mathbb{C}^2 \times \mathbb{C}^2, \]

\[
< P_\zeta f_N(\zeta, z) - \zeta_1 z_2 > = \prod_{j=1}^{N} (\zeta_1 - \eta_j \zeta_2) - \prod_{j=1}^{N} (\zeta_1 - \eta_j z_2).
\]

Both integrals can be explicit and yield to (Section 2 Lemma 1): \( \forall z \in \mathbb{C}^2, \]

\[
(1.3) \quad f(z) = E_N(f; \eta)(z) - R_N(f; \eta)(z) + \sum_{k+l \geq N} a_{k,l} z_1^k z_2^l,
\]

where \( \sum_{k+l \geq N} a_{k,l} z_1^k z_2^l \) is the Taylor expansion of \( f, \)

\[
E_N(f; \eta)(z) := \sum_{p=1}^{N} \left( \prod_{j=p+1}^{N} (z_1 - \eta_j z_2) \right) \sum_{q=p}^{N} \frac{1 + \eta_q z_2}{1 + |\eta_q|^2} \frac{1}{\prod_{j=p,q}^{N} (\eta_q - \eta_j)} \times
\]

\[
\sum_{m \geq N-p} \frac{1}{m!} \frac{\partial^m}{\partial \eta_q^m} \bigg|_{\eta = 0} \left| f(\eta_q v, v) \right|.
\]

(1.4)

and

\[
R_N(f; \eta)(z) := \sum_{p=1}^{N} \left( \prod_{j=1,j \neq p}^{N} (z_1 - \eta_j z_2) \right) \sum_{k+l \geq N} a_{k,l} \eta_q^k \left( \frac{z_2 + \eta_q z_2}{1 + |\eta_q|^2} \right)^{k+l-N+1}.
\]

(1.5)

The formula \( E_N(f; \eta) \) is an explicit interpolation formula constructed with the data \( (f_1, \ldots, f_N) \): it is an entire function that coincides with \( f \) on the first \( N \) lines, i.e \( \forall p = 1, \ldots, N, \)

\[
(E_N(f; \eta))_{\{z_1 = \eta_p z_2\}} = f_{\{z_1 = \eta_p z_2\}}.
\]

As \( N \to \infty \), the function \( f - E_N(f; \eta) \) will be a holomorphic function that will vanish on an increasing number of lines. If \( E_N(f; \eta) \) is uniformly bounded on any
compact subset $K \subset \mathbb{C}^2$ (in particular if it converges to some function), then by the Stieltjes-Vitali-Montel Theorem, there will be a subsequence of $f - E_N(f; \eta)$ that will converge to a holomorphic function that will vanish on an infinite number of lines. So this limit will be 0. In fact, any subsequence will have another subsequence that will also converge to 0. It will follow that the sequence $f - E_N(f; \eta)$ will converge to 0, i.e $E_N(f; \eta)$ will converge to $f$ (see [1] and [8] for examples of positive results of reconstruction).

The problem is that we do not have any control a priori of the function $E_N(f; \eta)$ and do not have any idea if $E_N(f; \eta)$ is always uniformly bounded for any $f \in \mathcal{O}(\mathbb{C}^2)$. In fact, this is false and one can construct special subsets $\{\eta_j\}_{j \geq 1}$ (see Section 5, Proposition 2) for which $E_N(f; \eta)$ is not bounded for any $f$ (i.e. there exists $f \in \mathcal{O}(\mathbb{C}^2)$ such that $E_N(f; \eta)$ does not converge). Conversely, if we choose for $f$ a polynomial function, the formula $E_N(f; \eta)$ will converge for any subset $\{\eta_j\}_{j \geq 1}$. Finally, if we choose special subsets for $\{\eta_j\}_{j \geq 1}$ like real lines or circles of $\mathbb{C}$, the formula $E_N(f; \eta)$ will always converge to $f$ (Theorem 3). Therefore we want to reduce the problem of the convergence of $E_N(f; \eta)$ to the one of classifying the good subsets $\{\eta_j\}_{j \geq 1}$ (i.e. the ones for which $E_N(f; \eta)$ converges for every $f$) and the others (the ones for which there is at least a function $f$ such that $E_N(f; \eta)$ does not converge).

On the other hand, since for any compact subset $K \subset \mathbb{C}^2$,
\[
\sup_{z \in K} \left| \sum_{k+l \geq N} a_{k,l}z_1^k z_2^l \right| \xrightarrow{N \to \infty} 0,
\]
the relation (1.3) allows us to reduce the problem of the convergence of $E_N(f; \eta)$ to the one of the formula $R_N(f; \eta)$. A preliminary property about the Lagrange interpolation polynomials (Section 2, Lemma 4) yields to
\[
R_N(f; \eta)(z) = \sum_{p=0}^{N-1} z_2^{N-1-p} \prod_{j=1}^p (z_1 - \eta_j z_2) \times \Delta_{p,\eta_1,\ldots,\eta_{p+1}} \left[ \sum_{m \geq N} a_{k,l}z_1^k \left( \frac{z_2 + \zeta z_1}{1 + |\zeta|^2} \right)^{m-N+1} \sum_{k+l=m} a_{k,l}z_1^k \right] (\eta_{p+1}),
\]
where the function $\Delta_p$ is defined as follows:
\[
\begin{align*}
\Delta_0,\eta(g)(\eta_1) &:= g(\eta_1), \\
\Delta_{p+1,\eta_1,\ldots,\eta_{p+1}}(g)(\eta_{p+2}) &:= \frac{\Delta_{p,\eta_1,\ldots,\eta_{p+1}}(g)(\eta_{p+2}) - \Delta_{p,\eta_1,\ldots,\eta_{p+1}}(g)(\eta_{p+1})}{\eta_{p+2} - \eta_{p+1}}.
\end{align*}
\]
$\Delta_p$ means the discrete derivative of $g$. A control of these $\Delta_p$ will allow us to get a control of $R_N(f; \eta)$. For example, the $\Delta_p$ of a holomorphic function looks like its usual iterative derivative (Section 2, Lemma 5). Nevertheless, a non holomorphic function, even $C^\infty$, can have a $\Delta_p$ without any control (Section 5, Lemma 20). It follows that, in the case of $R_N(f; \eta)$, the non-validity of the convergence for every $f$ is linked to the fact that, in its above expression (1.4), there is the $\Delta_p$ of a non holomorphic function with respect to $\zeta \in \mathbb{C}$ (although it is $C^\infty$, bounded on $\mathbb{C}$ and uniformly converges to 0 as $N \to \infty$). This should allow us to reduce the problem of the control of $R_N(f; \eta)$ to the one of the $\Delta_p$ of the function $\zeta$, that will be specified in the following subsection.
1.2. Results. In this part we give the results of this paper. The first gives an equivalent criterion about the convergence of $E_N(f; \eta)$ for every $f$, for the case when the subset $\{\eta_j\}_{j \geq 1}$ is bounded.

**Theorem 1.** Let $\{\eta_j\}_{j \geq 1}$ be bounded. Then the interpolation formula $E_N(f; \eta)$ converges to $f$ uniformly on any compact $K \subset \mathbb{C}^2$ and for all $f \in \mathcal{O}(\mathbb{C}^2)$, if and only if $\{\eta_j\}_{j \geq 1}$ satisfies the following condition: $\exists R_x \geq 1$ (that only depends on $\{\eta_j\}_{j \geq 1}$), $\forall p, q \geq 0$,

$$
\left| \Delta_{p, (\eta_p, \ldots, \eta_1)} \left[ \left( \frac{\zeta}{1 + |\zeta|^2} \right)^q \right] (\eta_{p+1}) \right| \leq R_x^{p+q}.
$$

First, we see that the convergence of $E_N(f; \eta)$ is linked to the control of the $\Delta_p$ of a non holomorphic function. Next, this is an analytic criterion that only depends on the configuration of the points $\eta_j$, $j \geq 1$, and does not involve any function $f \in \mathcal{O}(\mathbb{C}^2)$.

We also give an extension of this result for the case when $\{\eta_j\}_{j \geq 1}$ can be not bounded but is not dense in $\mathbb{C}$. Let be $\eta_\infty \in \mathbb{C} \setminus \{\eta_j\}_{j \geq 1}$ and set, for all $j \geq 1$,

$$
\theta_j := \frac{1 + \eta_\infty \eta_j}{\eta_j - \eta_\infty},
$$

Then the set $\{\theta_j\}_{j \geq 1}$ is bounded and an application of Theorem 1 with the $\theta_j$ yields to the following more general result.

**Theorem 2.** Let assume that $\{\eta_j\}_{j \geq 1}$ is not dense in $\mathbb{C}$. Let be any $\eta_\infty \notin \{\eta_j\}_{j \geq 1}$ and the associate set $\{\theta_j\}_{j \geq 1}$. Then for all $f \in \mathcal{O}(\mathbb{C}^2)$ the interpolation formula $E_N(f; \eta)$ converges to $f$ uniformly on any compact $K \subset \mathbb{C}^2$, if and only if $\{\theta_j\}_{j \geq 1}$ satisfies the condition (1.8) of Theorem 1, i.e $\exists R_\theta \geq 1$, $\forall p, q \geq 0$,

$$
\left| \Delta_{p, (\theta_p, \ldots, \theta_1)} \left[ \left( \frac{\zeta}{1 + |\zeta|^2} \right)^q \right] (\theta_{p+1}) \right| \leq R_\theta^{p+q}.
$$

Now we will give another condition for the set $\{\eta_j\}_{j \geq 1}$ to make converge any $E_N(f; \eta)$, that is more natural to be expressed. First, we give the following definition.

**Definition 1.** $\{\eta_j\}_{j \geq 1}$ is real-analytically interpolated if, for all $\zeta \in \{\eta_j\}_{j \geq 1}$, there exist a neighborhood $V$ of $\zeta$ and $g \in \mathcal{O}(V)$ such that, $\forall \eta_j \in V$,

$$
\eta_j = g(\eta_j).
$$

Although the conjugate function $\overline{\zeta}$ is not holomorphic, this condition means that the closure of $\{\eta_j\}_{j \geq 1}$ can be embedded in some subset on which $\overline{\zeta}$ locally coincides with a certain holomorphic function. It follows that any $\eta_j$ is locally in the zero set of some real-analytic function: $(x, y) \in V' \subset \mathbb{R}^2 \mapsto x - iy - g(x + iy)$.

Then we can give the following result that is a sufficient condition for the set $\{\eta_j\}_{j \geq 1}$ to make converge $E_N(f; \eta)$ for every function $f$.

**Theorem 3.** If $\{\eta_j\}_{j \geq 1}$ is real-analytically interpolated, then for all $f \in \mathcal{O}(\mathbb{C}^2)$, the interpolation formula $E_N(f; \eta)$ converges to $f$ uniformly on any $K \subset \mathbb{C}^2$.
The condition for \( \{ \eta_j \}_{j \geq 1} \) to be real-analytically interpolated is also linked to the uniform control of its \( \Delta_p(\zeta) \) (Section 4, Proposition 1). This is an analogous condition to (1.8), but with a uniform estimation on the subsequences \( (\eta_{j_k})_{k \geq 1} \) and the same constant \( R_{\eta} \) (Section 4, Definition 2).

On the other hand, we understand why the convergence of \( E_N(f; \eta) \) is always true for subsets like lines. For example, the subset \( \mathbb{R} \) is the set of \( z \in \mathbb{C} \) such that \( \overline{z} = z \). In the same way, \( i \mathbb{R} = \{ z \in \mathbb{C} \} \), and more generally, any real line \( D \) of \( \mathbb{C} \) can be written as \( \{ z \in \mathbb{C}, az + b \mathbb{R} + c = 0 \} \), with \( (a, b) \in \mathbb{R} \setminus \{ (0, 0) \} \), then

\[
D = \left\{ \frac{z + \overline{z}}{2} + b \frac{z - \overline{z}}{2i} + c = 0 \right\} = \left\{ \overline{z} = -\frac{a+ib}{a+ib} z + c \right\}.
\]

In the same way, any circle \( C \) of \( \mathbb{C} \) can be written as \( \{ z \in \mathbb{C}, |z - z_0| = r \} \), with \( z_0 \in \mathbb{C}, r > 0 \), then

\[
C = \left\{ (z - z_0)(\overline{z} - \overline{z_0}) = r^2 \right\} = \left\{ \overline{z} = \frac{z_0 + r^2}{z - z_0} \right\}.
\]

Another fact is that the real-analytical condition for \( \{ \eta_j \}_{j \geq 1} \) can also be reformulated as a real-analytical dependence of the family \( \{ z_1 - \eta_j z_2 = 0 \}_{j \geq 1} \), formulation that can be extended in the case of any family of analytic sub varieties \( \{ Z_j \}_{j \geq 1} \) of any domain \( \Omega \subset \mathbb{C}^n \).

This also yields to another natural question: is this condition necessary? We think that it should not be the case since the formula \( E_N(f; \eta) \) still converges for a lots of subsets \( \{ \eta_j \}_{j \geq 1} \) that are in greater number than the ones that are real-analytically interpolated.

Nevertheless, we will see that this condition cannot be completely forgotten. Indeed, we will construct in Section 5 (Proposition 2) a bounded sequence \( (\eta_j)_{j \geq 1} \subset \mathbb{R} \cup i \mathbb{R} \) that converges to 0 without staying in \( \mathbb{R} \) or \( i \mathbb{R} \), and does not satisfy the condition (1.8) of Theorem 1 (then \( E_N(f; \eta) \) cannot converge for every function \( f \)).

Another fact that we want to specify is that the formula \( E_N(f; \eta) \) is a canonical interpolation formula in the meaning that it is essentially the most simple interpolation formula that fixes the polynomials of degree at most \( N - 1 \) (Section 2, Lemma 8), ie

\[
\forall P \in \mathbb{C}_{N-1}[z_1, z_2], \quad E_N(P; \eta) \equiv P.
\]

This problem of explicit reconstruction is also motivated by possible applications in mathematical economics and medical imaging. We want to reconstruct any given function \( F \) with compact support \( K \subset \mathbb{R}^n \) from the knowledge of its Radon transforms \( (RF)(\theta_j, s), (\theta_j, s) \in S^{n-1} \times \mathbb{R} \), on a finite number of directions \( j = 1, \ldots, N \) (see [4] and [5]).

I would like to thank G. Henkin for having introduced me this interesting problem and J. Ortega-Cerda for all the rewarding ideas and discussions about it.
2. Some preliminary results

2.1. Equality of residues/principal values. We remind from Introduction ([7], Proposition 1) that, for all \( f \in \mathcal{O}(\mathbb{B}_2) \), one has

\[
f(z) = \lim_{\varepsilon \to 0} \frac{1}{(2\pi)^2} \int_{|\zeta_j - \eta_j z_2| = \varepsilon} f(\zeta) \det(\zeta, P_N(\zeta, z)) \frac{\omega(\zeta)}{\Pi_{j=1}^N(\zeta_j - \eta_j \zeta_2)(1 - \zeta, z >)}
\]

\[
- \lim_{\varepsilon \to 0} \frac{1}{(2\pi)^2} \int_{|\zeta_j - \eta_j z_2| > \varepsilon} f(\zeta) \omega'(\zeta) \frac{\omega(\zeta)}{\Pi_{j=1}^N(\zeta_j - \eta_j \zeta_2)(1 - \zeta, z >)^2},
\]

the integration being on the unit sphere \( S_2 = \{ \zeta \in \mathbb{C}^2, |\zeta_1|^2 + |\zeta_2|^2 = 1 \} \), with

\[
\begin{aligned}
\omega'(\zeta) &= \zeta_1 d\zeta_2 - \zeta_2 d\zeta_1, \\
\omega(\zeta) &= d\zeta_1 \wedge d\zeta_2,
\end{aligned}
\]

and \( P_N(\zeta, z) \in (\mathcal{O}(\mathbb{C}^2))^2 \) is such that, for all \((\zeta, z) \in \mathbb{C}^2 \times \mathbb{C}^2\),

\[
< P_N(\zeta, z), \zeta - z > = \prod_{j=1}^N(\zeta_j - \eta_j \zeta_2) - \prod_{j=1}^N(z_j - \eta_j z_2)
\]

(see [7], Lemma 7 for an explicit expression of \( P_N \)).

We also have the analogous following relation ([7], Proposition 2)

\[
f(z) = \lim_{\varepsilon \to 0} \frac{1}{(2\pi)^2} \int_{\partial S_2} f(\zeta) \det(\zeta, P_N(\zeta, z)) \frac{\omega(\zeta)}{\Pi_{j=1}^N(\zeta_j - \eta_j \zeta_2)(1 - \zeta, z >)}
\]

\[
- \lim_{\varepsilon \to 0} \frac{1}{(2\pi)^2} \int_{\partial S_2} f(\zeta) \omega'(\zeta) \frac{\omega(\zeta)}{\Pi_{j=1}^N(\zeta_j - \eta_j \zeta_2)(1 - \zeta, z >)^2},
\]

where

\[
\Sigma_\varepsilon := \bigcup_{j=0}^N \{ \zeta \in S_2, \alpha_j + \varepsilon < |\zeta_1| < \alpha_{j+1} - \varepsilon \}
\]

and for all \( p = 1, \ldots, N \)

\[
\alpha_p := \frac{|\eta_p|}{\sqrt{1 + |\eta_p|^2}},
\]

with \( \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_N \) (that one can assume without loss of generality) and the convention that \( \alpha_0 = 0 \) and \( \alpha_{N+1} = 1 \). This allows us to explicit both above integrals and get the following relation: for all \( f \in \mathcal{O}(\mathbb{B}_2) \) and \( z \in \mathbb{B}_2 \),

\[
f(z) = E_N(f; \eta)(z) - R_N(f; \eta)(z) + \sum_{k+l \geq N} a_{k,l} z_1^k z_2^l,
\]

where

\[
f(z) = \sum_{k,l \geq 0} a_{k,l} z_1^k z_2^l,
\]
is the Taylor expansion of \( f \) (that absolutely converges on \( \mathbb{B}_2 \)),

\[
E_N(f; \eta)(z) = \sum_{p=1}^{N} \prod_{j=p+1}^{N} (z - \eta_j z_2) \sum_{q=p}^{N} \prod_{j=p,j\neq q}^{N} \frac{1}{1 + |\eta_q|^2} \prod_{j=p,j\neq q}^{N} \frac{1}{(\eta_q - \eta_j)} \times \\
\times \sum_{m \geq N-p} \frac{z_2 + \eta_p \eta_j}{1 + |\eta_j|^2}^{m-N+p} \frac{1}{m! \partial^m} v=0[f(\eta_q v, v)]
\]

(2.6)

\[
= \sum_{p=1}^{N} \prod_{j=1,j\neq p}^{N} (z_1 - \eta_j z_2) \sum_{q=p}^{N} \prod_{j=p,j\neq q}^{N} \frac{1}{1 + |\eta_q|^2} \prod_{j=p,j\neq q}^{N} \frac{1}{(\eta_q - \eta_j)} \times \\
\times \sum_{k+l \geq N-p} a_{k,l} \eta_p^k \left( \frac{z_2 + \eta_p \eta_j}{1 + |\eta_j|^2} \right)^{k+l-N+p},
\]

and

\[
R_N(f; \eta)(z) = \sum_{p=1}^{N} \prod_{j=1,j\neq p}^{N} (z_1 - \eta_j z_2) \sum_{q=p}^{N} \prod_{j=p,j\neq q}^{N} \frac{1}{1 + |\eta_q|^2} \prod_{j=p,j\neq q}^{N} \frac{1}{(\eta_q - \eta_j)} \times \\
\times \frac{1}{m! \partial^m} v=0[f(\eta_q v, v)]
\]

(2.7)

\[
= \sum_{p=1}^{N} \prod_{j=1,j\neq p}^{N} (z_1 - \eta_j z_2) \sum_{k+l \geq N} a_{k,l} \eta_p^k \left( \frac{z_2 + \eta_p \eta_j}{1 + |\eta_j|^2} \right)^{k+l-N+1}.
\]

This equality holds if \( f \in \mathcal{O}(\mathbb{C}^2) \). Since by Cauchy-Schwarz

\[
\frac{|z_2 + \eta_p \eta_j|}{1 + |\eta_p|^2} \leq \sqrt{|z_1|^2 + |z_2|^2 \sqrt{1 + |\eta_p|^2}} \leq \|z\|,
\]

for all \( z \in \mathbb{C}^2 \) and \( p = 1, \ldots, N \), \( E_N(f; \eta) \) and \( R_N(f; \eta) \) are still absolutely convergent series then are still well-defined for \( z \in \mathbb{C}^2 \). Then by analytic extension, (2.4) holds for all \( z \in \mathbb{C}^2 \).

Now we will prove the following preliminar result.

Lemma 1. For all \( N \geq 1 \) and \( z \in \mathbb{B}_2 \),

\[
\lim_{\epsilon \to 0} \frac{1}{(2\pi)^2} \int_{\Omega} f(\zeta) \det(\zeta, P_N(\zeta, z)) \omega(\zeta) = \\
= \lim_{\epsilon \to 0} \frac{1}{(2\pi)^2} \int_{\partial \Omega} f(\zeta) \det(\zeta, P_N(\zeta, z)) \omega(\zeta)
\]

and

\[
\lim_{\epsilon \to 0} \frac{1}{(2\pi)^2} \int_{\Omega} f(\zeta) \omega'(\zeta) \wedge \omega(\zeta) = \\
= \lim_{\epsilon \to 0} \frac{1}{(2\pi)^2} \int_{\Omega} f(\zeta) \omega'(\zeta) \wedge \omega(\zeta)
\]

Before giving the proof of the lemma, we begin with this first result.
Lemma 2. We define for all $p = 1, \ldots, N$

\[(2.8) \quad C_p = \prod_{j=1,j\neq p}^{N} \frac{|\eta_j - \eta_p|}{\sqrt{1 + |\eta_j|^2}}.\]

Then

\[
\lim_{\varepsilon \to 0} \frac{\prod_{j=1}^{N}(z_1 - \eta_j \bar{z}_2)}{(2\pi)^2} \int_{|\Pi_{j=1}^{N}(\zeta_1 - \eta_j \bar{\zeta}_2)| > \varepsilon} \frac{f(\zeta \omega(\bar{\zeta}) \wedge \omega(\zeta))}{\prod_{j=1}^{N}(z_1 - \eta_j \bar{z}_2)(1 - \langle \zeta, z \rangle)^2} =
\]

\[
= \lim_{\varepsilon \to 0} \frac{\prod_{j=1}^{N}(z_1 - \eta_j \bar{z}_2)}{(2\pi)^2} \int_{\cap_{j=1}^{N}(|\zeta_1 - \eta_j \bar{\zeta}_2| > 2\varepsilon/C_j \cap \Pi_{j=1}^{N}(z_1 - \eta_j \bar{z}_2)(1 - \langle \zeta, z \rangle)^2} \frac{f(\zeta \omega(\bar{\zeta}) \wedge \omega(\zeta))}{\prod_{j=1}^{N}(z_1 - \eta_j \bar{z}_2)(1 - \langle \zeta, z \rangle)^2}.\]

Proof. First, $z \in \mathbb{B}_2$ being fixed, one has to integrate the following 3-differential form

\[
\frac{\varphi(\zeta)}{\prod_{j=1}^{N}(\zeta_1 - \eta_j \bar{\zeta}_2)},
\]

with $\varphi$ a 3-form that is $C^\infty$ on $\mathbb{S}_2$ (since $|\langle \zeta, z \rangle| \leq \|\zeta\| \|z\| < 1$).

On the other hand, one has

\[(2.9) \quad \mathbb{S}_2 \cap \{\zeta_1 = \eta_j \bar{\zeta}_2\} = \{\zeta_1 = \eta_j \bar{\zeta}_2, |\zeta_1| = \alpha_p\}.
\]

For all $\varepsilon > 0$ small enough, if $\prod_{j=1}^{N}|\zeta_1 - \eta_j \bar{\zeta}_2| \leq \varepsilon$, then $\exists p, |\zeta_1 - \eta_p \bar{\zeta}_2| \leq \varepsilon^{1/N}$ and

\[
\varepsilon \geq \prod_{j=1}^{N}|\zeta_1 - \eta_j \bar{\zeta}_2| \sim |\zeta_1 - \eta_p \bar{\zeta}_2| \prod_{j\neq p}|\eta_p - \eta_j||\bar{\zeta}_2| \sim |\zeta_1 - \eta_p \bar{\zeta}_2| \prod_{j\neq p} \frac{|\eta_p - \eta_j|}{\sqrt{1 + |\eta_p|^2}}
\]

\[
\sim C_p|\zeta_1 - \eta_p \bar{\zeta}_2|,
\]

thus

\[
A_\varepsilon := \left\{ \prod_{j=1}^{N}|\zeta_1 - \eta_j \bar{\zeta}_2| \leq \varepsilon \right\} \subset \bigcup_{j=1}^{N}\{|\zeta_1 - \eta_j \bar{\zeta}_2| \leq 2\varepsilon/C_j\} := B_\varepsilon
\]

(the union being disjoint for $\varepsilon$ small enough since the $\eta_j$ are all different). Since

\[
\int_{\mathbb{S}_2 \setminus A_\varepsilon} \frac{\varphi(\zeta)}{\prod_{j=1}^{N}(\zeta_1 - \eta_j \bar{\zeta}_2)} = \int_{\mathbb{S}_2 \setminus B_\varepsilon} \frac{\varphi(\zeta)}{\prod_{j=1}^{N}(\zeta_1 - \eta_j \bar{\zeta}_2)} + \int_{B_\varepsilon \setminus A_\varepsilon} \frac{\varphi(\zeta)}{\prod_{j=1}^{N}(\zeta_1 - \eta_j \bar{\zeta}_2)},
\]

in order to prove the lemma it is sufficient to prove that

\[(2.10) \quad \int_{B_\varepsilon \setminus A_\varepsilon} \frac{\varphi(\zeta)}{\prod_{j=1}^{N}(\zeta_1 - \eta_j \bar{\zeta}_2)} \xrightarrow{\varepsilon \to 0} 0.\]
One has
\[
\left| \int_{B_r \setminus A_r} \frac{\varphi(\zeta)}{\prod_{j=1}^{N}(\zeta_j - \eta_j \zeta_2)} \right| \leq \sum_{j=1}^{N} \int_{|\zeta_j - \eta_j \zeta_2| \leq 2\varepsilon/C_j} \left| \varphi(\zeta) \right| \left| \prod_{j=1}^{N}(\zeta_j - \eta_j \zeta_2) \right|
\]
\[
\leq \frac{1}{\varepsilon} \sum_{j=1}^{N} \int_{|\zeta_j - \eta_j \zeta_2| \leq 2\varepsilon/C_j} \left| \varphi(\zeta) \right| \left| \prod_{j=1}^{N}(\zeta_j - \eta_j \zeta_2) \right|
\]
\[
\leq \frac{1}{\varepsilon} \sum_{j=1}^{N} \int_{|\zeta_j - \eta_j \zeta_2| \leq 2\varepsilon/C_j} \left| \varphi(\zeta) \right|
\]
\[
= \frac{1}{\varepsilon} \sum_{j=1}^{N} \int_{|\zeta_j - \eta_j \zeta_2| \leq 2\varepsilon/C_j, ||\zeta_j - \alpha_j| \leq b_j \varepsilon} \left| \varphi(\zeta) \right|,
\]
the last equality coming from (2.21). Since \( \varphi \) is bounded on \( S_2 \), for \( j = 1, \ldots, N \) one has
\[
\int_{|\zeta_j - \eta_j \zeta_2| \leq 2\varepsilon/C_j, ||\zeta_j - \alpha_j| \leq b_j \varepsilon} \left| \varphi(\zeta) \right| \leq M \int_{|\zeta_j - \alpha_j| \leq 2\varepsilon/C_j} 2\varepsilon \int_{|\zeta_j| = \sqrt{1 - r^2}} d\theta_2 \int_{|\zeta_j| = r, |\zeta_j - \eta_j \zeta_2| \leq 2\varepsilon/C_j} d\theta_1.
\]
For all \( \varepsilon \) small enough and \( \zeta_2 \) fixed,
\[
\theta_1 = \arg(\zeta_1) = \arg(\eta_j \zeta_2) + \arg \left( \frac{\zeta_1}{\eta_j \zeta_2} \right) = \arg(\eta_j \zeta_2) + \arg \left( 1 + \frac{\eta_j \zeta_2}{\eta_j \zeta_2} \right) = \arg(\eta_j \zeta_2) + O(\varepsilon),
\]
then
\[
\int_{|\zeta_j - \alpha_j| \leq 2\varepsilon/C_j} 2\varepsilon \int_{|\zeta_j| = \sqrt{1 - r^2}} d\theta_2 \int_{|\zeta_j| = r} d\theta_1 = \int_{|\zeta_j - \alpha_j| \leq 2\varepsilon/C_j} 2\varepsilon \int_{|\zeta_j| = \sqrt{1 - r^2}} d\theta_2 O(\varepsilon) = O(\varepsilon^2).
\]
Finally
\[
\left| \int_{B_r \setminus A_r} \frac{\varphi(\zeta)}{\prod_{j=1}^{N}(\zeta_j - \eta_j \zeta_2)} \right| = \frac{1}{\varepsilon} N O(\varepsilon^2) = O(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0.
\]

Now we can prove Lemma [1]

**Proof.** By relations (2.11) and (2.22), in order to prove the lemma it is sufficient to prove the second equality. By the previous lemma it is sufficient to prove that
\[
\lim_{\varepsilon \rightarrow 0} \frac{\prod_{j=1}^{N}(z_1 - \eta_j z_2)}{(2\pi)^2} \int_{\Gamma_{\varepsilon}} \frac{f(\zeta) \omega(\zeta) \wedge \omega(\zeta)}{\prod_{j=1}^{N}(z_1 - \eta_j z_2)(1 - \zeta, \zeta, > 2\varepsilon/C_j)} =
\]
\[
= \lim_{\varepsilon \rightarrow 0} \frac{\prod_{j=1}^{N}(z_1 - \eta_j z_2)}{(2\pi)^2} \int_{\Sigma_{\varepsilon}} \frac{f(\zeta) \omega(\zeta) \wedge \omega(\zeta)}{\prod_{j=1}^{N}(z_1 - \eta_j z_2)(1 - \zeta, \zeta, > 2\varepsilon/C_j)}.
\]
If \( \exists \eta_{j_0} = 0 \) then \( \{|\zeta_1 - \eta_{j_0} \zeta_2| > 2\varepsilon/C_{j_0} = \{|\zeta_1| > 2\varepsilon/C_{j_0} \}. \) Since \( 1/\zeta_1 \) is defined in the neighborhood of any \( |\zeta_1 - \alpha_j| \leq b_j \varepsilon \), one just has to consider the case of
\( \eta_j \neq 0, j \neq j_0 \) with \( \varphi(\zeta)/\zeta_1 \). So one can assume that \( \eta_j \neq 0, \forall j = 1, \ldots, N \) (then \( \alpha_j > 0 \)). Without loss of generality, one has

\[
B_\varepsilon \subset \bigcup_{j=1}^{N} \{||\zeta_1| - \alpha_j| \leq b_j \varepsilon \} = \bigcup_{k=1}^{K} \{||\zeta_1| - \alpha_{jk}| \leq b_{jk} \varepsilon \} =: C_{\varepsilon},
\]

with \( \alpha_1 = \cdots = \alpha_{j_1} \neq \alpha_{j_1+1} = \cdots = \alpha_{j_2} \neq \cdots \neq \alpha_{j_{K-1}+1} = \cdots = \alpha_{j_K} \). Then for all \( \varepsilon \) small enough,

\[
C_{\varepsilon} \setminus B_\varepsilon = \bigcup_{k=1}^{K} \{||\zeta_1| - \alpha_{jk}| \leq c_{jk} \varepsilon \} \setminus B_\varepsilon
\]

\[
= \bigcup_{k=1}^{K} \bigcap_{\alpha_j = \alpha_{jk}} \{||\zeta_1| - \alpha_{jk}| \leq c_{jk} \varepsilon \} \cap \{|\zeta_1 - \eta_j \zeta_2| \geq 2\varepsilon/C_j \},
\]

thus

\[
\int_{C_\varepsilon \setminus B_\varepsilon} \frac{\varphi(\zeta)}{\prod_{j=1}^{N} (\zeta_1 - \eta_j \zeta_2)} = \sum_{k=1}^{K} \int_{\{|\zeta_1| - \alpha_{jk}| \leq c_{jk} \varepsilon \} \cap \{|\zeta_1 - \eta_j \zeta_2| \geq 2\varepsilon/C_j \}} \frac{\varphi(\zeta)}{\prod_{j=1}^{N} (\zeta_1 - \eta_j \zeta_2)}.
\]

Since for all \( j = 1, \ldots, N \)

\[
\frac{\varphi(\zeta)}{\prod_{j=1}^{N} (\zeta_1 - \eta_j \zeta_2)} = \frac{\varphi_j(\zeta)}{\zeta_1 - \eta_j \zeta_2} - \frac{\varphi_j(\eta_j \zeta_2, \zeta_2)}{\zeta_1 - \eta_j \zeta_2} + \frac{\varphi_j(\eta_j \zeta_2, \zeta_2)}{\zeta_1 - \eta_j \zeta_2} = \frac{\psi_{j,1}(\zeta_2)}{\zeta_1 - \eta_j \zeta_2} + \psi_{j,2}(\zeta),
\]

with \( \psi_{j,1}, \psi_{j,2} \) locally integrable and bounded in the neighborhood of \( \{\zeta_1 - \eta_j \zeta_2 = 0\} \). On the other hand, for all \( k = 1, \ldots, K \)

\[
\{|\zeta_1| - \alpha_{jk}| \leq c_{jk} \varepsilon \} = \bigcup_{j=jk-1+1}^{jk} (U_j \cap \{|\zeta_1| - \alpha_{jk}| \leq c_{jk} \varepsilon \}),
\]

where \( U_{jk-1+1}, \ldots, U_{jk} \) is a partition such that \( \varphi_j, \psi_j \) are bounded in \( U_j \) (for example, \( U_j = \{|\zeta_1| - \eta_j \zeta_2| \leq \varepsilon_0 \}, \) for all \( j = j_{k-1} + 1, \ldots, j_k \), with \( \varepsilon_0 \) small enough so that \( U_j \cap U_i \cap S_2 = \emptyset \) if \( j \neq i \); finally one replaces \( U_{jk-1+1} \) with \( U_{jk-1+1} \cup (S_2 \setminus \bigcup_{j=jk-1+2}^{jk} U_j) \). Then

\[
\int_{C_\varepsilon \setminus B_\varepsilon} \frac{\varphi(\zeta)}{\prod_{j=1}^{N} (\zeta_1 - \eta_j \zeta_2)} = \sum_{k=1}^{K} \sum_{j=jk-1+1}^{jk} \int_{U_j \cap \{|\zeta_1| - \alpha_{jk}| \leq c_{jk} \varepsilon \} \cap \{|\zeta_1 - \eta_j \zeta_2| \geq 2\varepsilon/C_j \}} \left( \frac{\psi_{j,1}(\zeta_2)}{\zeta_1 - \eta_j \zeta_2} + \psi_{j,2}(\zeta) \right).
\]

Since

\[
\left| \int_{U_j \cap \{|\zeta_1| - \alpha_{jk}| \leq c_{jk} \varepsilon \} \cap \{|\zeta_1 - \eta_j \zeta_2| \geq 2\varepsilon/C_j \}} \psi_{j,2}(\zeta) \right| \leq M \int_{S_2 \cap \{|\zeta_1| - \alpha_{jk}| \leq c_{jk} \varepsilon \}} d\lambda \quad \varepsilon \to 0,
\]
one has
\[ \lim_{\varepsilon \to 0} \int_{C_{\varepsilon} \setminus B_{\varepsilon}} \frac{\varphi(\zeta)}{\prod_{j=1}^{N} |\zeta_1 - \eta_j \zeta_2|} = \]
\[ = \lim_{\varepsilon \to 0} \sum_{k=1}^{K} \sum_{j_{k,j_{k-1}+1}} \int_{U_j \cap \{ |\zeta_1 - \alpha_{j_{k}}| \leq \varepsilon \}} \frac{\psi_{j_1}(\zeta_1)}{\zeta_1 - \eta_j \zeta_2} \]
\[ = \lim_{\varepsilon \to 0} \sum_{k=1}^{K} \sum_{j_{k,j_{k-1}+1}} \int_{|\zeta_1| = \sqrt{1 - r^2}} \frac{\psi_j(r, \zeta_2) d\zeta_2}{\zeta_1 (\zeta_1 - \eta_j \zeta_2)}. \]

because
\[ \omega(\zeta) \wedge \omega(\zeta) = \left( \frac{r^2}{\zeta_1} \left( \frac{1 - r^2}{\zeta_2} \right) - \frac{1 - r^2}{\zeta_2} \right) \wedge d\zeta_1 \wedge d\zeta_2 \]
\[ = -2\pi r dr \wedge \frac{d\zeta_1}{\zeta_1} \wedge \frac{d\zeta_2}{\zeta_2}. \]

Now for all \( r \neq \alpha_j \) and \( |\zeta_2| = \sqrt{1 - r^2} (\neq \alpha_j/|\eta_j|) \), one has (since \( r \varepsilon \) small enough, \( \{ |\zeta_1 - \eta_j \zeta_2| \leq 2\varepsilon/C_j \} \subset U_j \)
\[ \int_{|\zeta_1| = r, \zeta \in U_j, |\zeta_1 - \eta_j \zeta_2| \geq 2\varepsilon/C_j} d\zeta_1 \]
\[ = \left[ \int_{|\zeta_1| = r} - d\zeta_1 \right] - \left[ \int_{|\zeta_1| = r, |\zeta_1 - \eta_j \zeta_2| \leq 2\varepsilon/C_j} \right] \frac{d\zeta_1}{\zeta_1 (\zeta_1 - \eta_j \zeta_2)}. \]

First, since \( |\zeta_1| > |\eta_j \zeta_2| \) if and only if \( r > \alpha_j \), one has
\[ \left| \int_{|\zeta_1| = r} \frac{d\zeta_1}{\zeta_1 (\zeta_1 - \eta_j \zeta_2)} \right| = \left| (2\pi) \left( -1 - \frac{1}{\eta_j \zeta_2} \right) \right| \]
\[ = \frac{2\pi |\eta_j \zeta_2|}{|1 - r > \alpha_j|} \leq \frac{2\pi}{|\eta_j| \sqrt{1 - r^2}}. \]

Next, \( \forall j = j_{k-1} + 1, \ldots, j_k, U_j \supset \{ |\zeta_1 - \eta_j \zeta_2| \leq \varepsilon_0 \} \), then
\[ \left| \int_{|\zeta_1| = r, \zeta \notin U_j} \frac{d\zeta_1}{\zeta_1 (\zeta_1 - \eta_j \zeta_2)} \right| \leq \int_{|\zeta_1| = r, \zeta \notin U_j} \left| \frac{d\zeta_1}{r \varepsilon_0} \right| \leq \int_{|\zeta_1| = r} \frac{d\zeta_1}{r \varepsilon_0} = \frac{2\pi}{\varepsilon_0}. \]

Thus
\[ \left| \int_{|\zeta_1| = r} \frac{d\zeta_1}{\zeta_1 (\zeta_1 - \eta_j \zeta_2)} \right| \leq \sum_{k=1}^{K} \sum_{j_{k,j_{k-1}+1}} \int_{|\zeta_1| = \sqrt{1 - r^2}} \frac{\psi_j(r, \zeta_2) d\zeta_2}{\zeta_1 (\zeta_1 - \eta_j \zeta_2)} \leq \]
\[ \leq \sum_{j=1}^{N} \int_{|\zeta_1| = r, |\zeta_1 - \eta_j \zeta_2| \leq 2\varepsilon/C_j} \frac{d\zeta_1}{\zeta_1 (\zeta_1 - \eta_j \zeta_2)}. \]

Finally, we consider the following integral
\[ \int_{|\zeta_1| = r, |\zeta_1 - \eta_j \zeta_2| \leq 2\varepsilon/C_j} \frac{d\zeta_1}{\zeta_1 (\zeta_1 - \eta_j \zeta_2)}. \]
(in the case where the measure of \( \{ |\zeta_1| = r, |\zeta_1 - \eta_j \zeta_2| \leq 2\varepsilon/C_j \} \) is not zero). We set \( \theta_j := \text{Arg}(\eta_j \zeta_2) \), \( \zeta_1 = re^{i\theta} \) so

\[
\frac{2\pi}{C_j} \geq |\zeta_1 - \eta_j \zeta_2| = |re^{i\theta} - |\eta_j \zeta_2|e^{i\theta_j}| = |re^{i(\theta - \theta_j)} - |\eta_j \zeta_2|
\geq 3 \left( |e^{i(\theta - \theta_j)} - |\eta_j \zeta_2| \right) = |r \sin(\theta - \theta_j)|.
\]

Since for all \( \varepsilon \) small enough, \( r \sim \alpha_j > 0 \), one has \( |\sin(\theta - \theta_j)| \leq O(\varepsilon) \), then \( |\theta - \theta_j| \leq O(\varepsilon) \), i.e. \( \theta \in [\theta_j - \varepsilon, \theta_j + \varepsilon] \) with \( \varepsilon, \theta'_\varepsilon = O(\varepsilon) \). Since

\[
|e^{i((\theta_j - \theta_j) - \theta_j)} - |\eta_j \zeta_2|| = |re^{-i\theta} - |\eta_j \zeta_2|| = |re^{i\theta} - |\eta_j \zeta_2|
= |re^{i((\theta_j + \theta_j) - \theta_j)} - |\eta_j \zeta_2|
\]

then \( \theta'_\varepsilon = \theta \). So

\[
\int_{|\zeta_1| = r, |\zeta_1 - \eta_j \zeta_2| \leq 2\varepsilon/C_j} \frac{d\zeta_1}{\zeta_1(\zeta_1 - \eta_j \zeta_2)} = \int_{\theta_j - \varepsilon}^{\theta_j + \varepsilon} \frac{i e^{i\theta} d\theta}{r e^{i\theta} - |\eta_j \zeta_2|e^{i\theta_j}} = \int_{-\theta_j}^{\theta_j} r \cos \theta - |\eta_j|/\sqrt{1 - r^2} + ir \sin \theta
\]

With the change of variables \( t = \tan(\theta/2) \), one has \( t \in [-t_\varepsilon, t_\varepsilon] \), with \( t_\varepsilon = \tan(\theta_\varepsilon/2) = O(\varepsilon) \), then

\[
\int_{-\theta_j}^{\theta_j} \frac{d\theta}{r \cos \theta - |\eta_j|/\sqrt{1 - r^2} + ir \sin \theta} = \int_{-t_\varepsilon}^{t_\varepsilon} \frac{2 \sqrt{1 + t^2}}{1 + \sqrt{1 - r^2}} dt
\]

\[
= \int_{-t_\varepsilon}^{t_\varepsilon} \frac{2 \sqrt{1 + t^2}}{r + |\eta_j|/\sqrt{1 - r^2}} dt
= \int_{-t_\varepsilon}^{t_\varepsilon} \frac{2 \sqrt{1 + t^2}}{(t - t_+)(t - t_-)} dt
\]

with

\[
\begin{aligned}
t_+ &= \frac{ir + i|\eta_j|/\sqrt{1 - r^2}}{r + |\eta_j|/\sqrt{1 - r^2}} = i,
\end{aligned}
\]

\[
\begin{aligned}
t_- &= \frac{ir - i|\eta_j|/\sqrt{1 - r^2}}{r + |\eta_j|/\sqrt{1 - r^2}} = i \frac{r - |\eta_j|/\sqrt{1 - r^2}}{r + |\eta_j|/\sqrt{1 - r^2}}.
\end{aligned}
\]

Since

\[
\frac{1}{(t - t_+)(t - t_-)} = \frac{r + |\eta_j|/\sqrt{1 - r^2}}{2i|\eta_j|/\sqrt{1 - r^2}} \left( \frac{1}{t - t_+} - \frac{1}{t - t_-} \right),
\]

one has

\[
\int_{-\theta_j}^{\theta_j} \frac{d\theta}{r \cos \theta - |\eta_j|/\sqrt{1 - r^2} + ir \sin \theta} = \frac{2 \sqrt{1 + t^2}}{r + |\eta_j|/\sqrt{1 - r^2}} \left( \frac{1}{t - t_+} - \frac{1}{t - t_-} \right).
\]
Log being the principal determination of the logarithm on \( \mathbb{C} \setminus \mathbb{R}^+ \) (that is well-defined since \( t_+ , t_- \neq 0 \) for all \( r \neq \alpha_j \)), \( \Im(t_+ - t_-) = \Im(-t_+ - t_-) \) then \( \Log(t_+ - t_-) = \Log(-t_+ - t_-) \), so

\[
\Log \left( \frac{t_+ - t_-}{t_+ - t_-} \right) = \Log \left( \frac{t_+ - t_-}{-t_+ - t_-} \right) = \Log(1 + O(\varepsilon)) = O(\varepsilon).
\]

On the other hand, one can write \( r = \alpha_j + r' \) with \( r' \in [-c_j \varepsilon, c_j \varepsilon] \) (and still keep the variable \( r \)) so

\[
t_- = \frac{r + \alpha_j - |\eta_j| \sqrt{1 - \alpha_j^2} \sqrt{1 - \frac{2\alpha_j}{r - \alpha_j}}}{r + \alpha_j + |\eta_j| \sqrt{1 - \alpha_j^2}} \sqrt{1 + O(r)} + O(r^2)
\]

\[
= \frac{r + \alpha_j - \alpha_j \left( 1 - \frac{\alpha_j}{1 - \alpha_j^2} \right)}{r + \alpha_j + \alpha_j + O(r)} + O(r^2) = \frac{i}{2\alpha_j + O(r)} + O(r^2)
\]

Then

\[
|t_+ - t_-| \geq |t_-| = \left| \frac{i}{2\alpha_j(1 - \alpha_j^2)} r + O(r^2) \right| \geq \frac{|r|}{4\alpha_j(1 - \alpha_j^2)},
\]

for all \( \varepsilon \) small enough and \( r \in [-c_j \varepsilon, c_j \varepsilon], r \neq 0 \). Therefore

\[
|\Log(t_+ - t_-)| \leq |\Log(t_+ + t_-)| + |\Arg(t_+ - t_-)|
\]

\[
\leq \left| \Log \left( \frac{|r|}{4\alpha_j(1 - \alpha_j^2)} \right) \right| + \pi \leq |\Log|r| + O(1),
\]

thus

\[
\left| \int_{-\theta}^{\theta} \frac{d\theta}{r \cos \theta - |\eta_j| \sqrt{1 - r^2} + i r \sin \theta} \right| \leq \frac{1}{|\eta_j| \sqrt{1 - r^2 + O(\varepsilon)}} \leq \frac{1}{|\eta_j| \sqrt{1 - (r + \alpha_j)^2 + O(\varepsilon)}} (O(\varepsilon) + 2|\Log|r| + O(1))
\]

\[
= \frac{2}{\alpha_j} |\Log|r| + O(1) = O(\Log|r|).
\]

Finally, we get for all \( \varepsilon \) small enough

\[
\sum_{k=1}^{N} \int_{\eta_j c_{j-1}}^{\eta_j c_j} (-2rdr) \int_{|\zeta_2| = \sqrt{1 - r^2}} \tilde{\psi}_j(r, \zeta_2) d\zeta_2 \int_{|\zeta_1| = \Sigma} \frac{d\zeta_1}{\zeta_1(\zeta_1 - \eta_j \zeta_2)} \leq \sum_{j=1}^{N} \int_{-c_j \varepsilon}^{c_j \varepsilon} O(|r| |\Log|r|)dr \xrightarrow{\varepsilon \to 0} 0.
\]

This result yields to the following consequence that will be usefull in Section 4 Subsection 4.2 (Lemma [17]).
Corollary 1. One has, for all \( f \in O(\mathbb{C}^2) \), all \( N \geq 1 \) and all \( z \in \mathbb{B}_2 \),
\[
\lim_{\varepsilon \to 0} \frac{\prod_{j=1}^N (z_1 - \eta_j z_2)}{(2\pi)^2} \int \frac{f(\zeta) \omega' (\zeta) \wedge \omega(\zeta)}{\prod_{j=1}^N (\zeta_1 - \eta_j \zeta_2) (1 - < \zeta, z >)^2} = R_N(f; \eta)(z) - \sum_{k+l \geq N} a_{k,l} z_1^k z_2^l.
\]

2.2. Some properties of \( \Delta_p \). In this part, \( \{\eta_j\}_{j \geq 1} \) will be any set of points all different, and \( h \) will be any function defined on the \( \eta_j \), \( j \geq 1 \).

We begin with this first result that follows from the definition of \( \Delta_p \).

Lemma 3. For all \( p \geq 0 \) and \( 0 \leq q \leq p \),
\[
\Delta_{p,(\eta_{p-1},\ldots,\eta_1)}(h)(\eta_{p+1}) = \Delta_{1,\eta_p} [\zeta \mapsto \Delta_{p-1,(\eta_{p-1},\ldots,\eta_1)}(h)(\zeta)](\eta_{p+1})
\]
\[
\Delta_{p-1,(\eta_{p-1},\eta_2)} [\zeta \mapsto \Delta_{1,\eta_1}(h)(\zeta)](\eta_{p+1})
\]
\[
\Delta_{p-1,(\eta_{p-1},\eta_2)} [\zeta \mapsto \Delta_{q,(\eta_{p-1},\ldots,\eta_1)}(h)(\zeta)](\eta_{p+1})
\]
\[
\Delta_{1,\eta_p} [\zeta \mapsto \Delta_{1,\eta_{p-1}} [\ldots [\zeta \mapsto \Delta_{1,\eta_1}(h)(\zeta_1)]\ldots](\zeta_p)](\eta_{p+1}).
\]

Now we will prove the two following results that will be useful in Section 3

Lemma 4. Let \( h \) be a function that is defined on every \( \eta_j \), \( j \geq 1 \). Then for all \( N \geq 1 \), one has
\[
\sum_{p=1}^N \prod_{j=1 \atop j \neq p}^N \frac{X - \eta^j}{\eta_p - \eta_j} h(\eta_p) = \sum_{p=0}^{N-1} \prod_{j=1}^p (X - \eta_j) \Delta_{p,(\eta_{p-1},\ldots,\eta_1)}(h)(\eta_{p+1})
\]

Proof. This relation can be proved by induction on \( N \geq 1 \). It is obvious for \( N = 1 \).

If it is true for \( N \geq 1 \), then
\[
\sum_{p=1}^N \prod_{j=1 \atop j \neq p}^N \frac{X - \eta^j}{\eta_p - \eta_j} h(\eta_p) = (X - \eta_1) \sum_{p=2}^{N+1} \prod_{j=2 \atop j \neq p}^N \frac{X - \eta_j}{\eta_p - \eta_j} h(\eta_p) \pm h(\eta_1) \prod_{j=2}^{N+1} \frac{X - \eta_j}{\eta_p - \eta_j}
\]
\[
= \sum_{p=2}^{N+1} \prod_{j=2 \atop j \neq p}^N \frac{X - \eta_j}{\eta_p - \eta_j} h_{1,X}(\eta_p) + h(\eta_1) \sum_{p=1}^{N+1} \prod_{j=1 \atop j \neq p}^N \frac{X - \eta_j}{\eta_p - \eta_j},
\]

where
\[
h_{1,X}(u) := (X - \eta_1) \frac{h(u) - h(\eta_1)}{u - \eta_1}.
\]

By induction,
\[
\sum_{p=2}^{N+1} \prod_{j=2 \atop j \neq p}^N \frac{X - \eta_j}{\eta_p - \eta_j} h_{1,X}(\eta_p) = \sum_{p=1}^N \prod_{j=1}^p (X - \eta_j) \Delta_{p-1,(\eta_{p-2},\ldots,\eta_1)}(h_{1,X})(\eta_{p+1})
\]
\[
= \sum_{p=1}^N (X - \eta_1) \prod_{j=2}^p (X - \eta_j) \Delta_{p-1,(\eta_{p-2},\ldots,\eta_1)}(u \mapsto \Delta_{1,\eta_1}(h)(u))(\eta_{p+1})
\]
\[
= \sum_{p=1}^N \prod_{j=1}^p (X - \eta_j) \Delta_{p,(\eta_{p-1},\ldots,\eta_1)}(h)(\eta_{p+1}).
\]
On the other hand,
\[ \sum_{p=1}^{N+1} \prod_{j=1,j\neq p}^{N+1} \frac{X - \eta_j}{\eta_p - \eta_j} = 1, \]
since 1 is the unique polynomial of degree at most \(N\) that coincides with 1 on \(N+1\) points. Then
\[ \Delta \]
\[ \text{Lemma 6.} \]

For all \(p\), \(h\) functions defined on the \(\eta_j\),
\[ \Delta_{p,(\eta_p,\eta_{p+1})}(gh)(\eta_{p+1}) = \sum_{q=0}^{p} \Delta_{p-q,(\eta_{p+1},\eta_{q+2})}(g)(\eta_{p+1}) \Delta_{q,(\eta_{p+1},\eta_{q+2})}(h)(\eta_{q+1}). \]

**Proof.** We prove this result by induction on \(p \geq 0\). It is obvious for \(p = 0\). If \(p \geq 0\) one has
\[ \Delta_{p+1,(\eta_{p+1},\eta_{p+2})}(gh)(\eta_{p+2}) = \Delta_{p,(\eta_{p+1},\eta_{p+2})} \left[ \zeta \mapsto \frac{(gh)(\zeta) - (gh)(\eta_{p+1})}{\zeta - \eta_{p+1}} \right](\eta_{p+2}) \]
\[ = \Delta_{p,(\eta_{p+1},\eta_{p+2})} \left[ g(\zeta)\Delta_{1,\eta_{p+1}}(h)(\zeta) + h(\eta_{p+1})\Delta_{1,\eta_{p+1}}(g)(\zeta) \right](\eta_{p+2}) \]
\[ = \sum_{q=0}^{p} \Delta_{p-q,(\eta_{p+1},\eta_{q+2})}(g)(\eta_{p+2}) \Delta_{q,(\eta_{p+1},\eta_{q+2})} \left[ \Delta_{1,\eta_{p+1}}(h)(\zeta) + h(\eta_{p+1})\Delta_{1,\eta_{p+1}}(g)(\zeta) \right](\eta_{q+2}) \]
and this achieves the induction.

The following result is an example of explicit calculation of the \(\Delta_p\) of an holomorphic function.

**Lemma 6.** Let \(h \in O(D(w_0,r))\) and \(h(w) = \sum_{n \geq 0} a_n (w - w_0)^n\) its Taylor expansion for all \(|w - w_0| < r\). Assume that \(\forall j \geq 1, \eta_j \in D(w_0,r)\). Then for all
\[ p \geq 0, \]
\[ \Delta_{p,(\eta_p, \ldots, \eta_1)}(h)(\eta_{p+1}) = \sum_{n \geq p} a_n \sum_{l_1 = 0}^{n-p} (\eta_1 - w_0)^{n-p-l_1} \sum_{l_2 = 0}^{l_1} (\eta_2 - w_0)^{l_1-l_2} \cdots \sum_{l_{p-2} = 0}^{l_{p-1}} (\eta_{p-1} - w_0)^{l_{p-2}} \sum_{l_{p-1} = 0}^{l_{p-2}} (\eta_p - w_0)^{l_{p-1}} \sum_{l_p = 0}^{l_{p-1}} (\eta_p - w_0)^{l_p}(\eta_{p+1} - w_0)^{l_p}. \]

(2.12)

In particular,
\[ \lim_{\eta_1, \ldots, \eta_p, \eta_{p+1} \rightarrow w_0} \Delta_{p,(\eta_p, \ldots, \eta_1)}(h)(\eta_{p+1}) = a_p = \frac{h(p)(0)}{p!}. \]

On the other hand, if \( h \in \mathbb{C}[w] \), then for any subset \( \{\eta_j\}_{j \geq 1} \subset \mathbb{C} \) and all \( p > \deg h \),
\[ \Delta_{p,(\eta_p, \ldots, \eta_1)}(h)(\eta_{p+1}) = 0. \]

**Proof.** The second relation is a consequence of (2.12) and the third with the choice of \( r = +\infty \).

By translation we can assume that \( w_0 = 0 \). The lemma will be proved by induction on \( p \geq 0 \). This is true for \( p = 0 \) if we admit that \( l_0 = n - p = n \) and
\[ \prod_{j=1}^{0} \sum_{l_j = 0}^{l_j-1} \eta_j^{l_j-1} \eta_1^{l_1} = \eta_1^n. \]

Now if it is true for \( p \geq 0 \) then
\[ \Delta_{p+1,(\eta_{p+1}, \ldots, \eta_1)}(h)(\eta_{p+2}) = \sum_{n \geq p} a_n \sum_{l_1 = 0}^{n-p} \eta_1^{n-p-l_1} \cdots \sum_{l_{p+1} = 0}^{l_{p}+1} \eta_{p+1}^{l_{p+1}-l_{p+1}} \sum_{l_{p+1} = 0}^{l_{p+1}} \eta_p^{l_{p+1}} \sum_{l_p = 0}^{l_{p+1}} \eta_p^{l_p} \sum_{l_{p-1} = 0}^{l_p} \eta_p^{l_{p-1}} \sum_{l_{p-1} = 0}^{l_{p-1}} \eta_p^{l_{p-1}} \sum_{l_{p-1} = 0}^{l_{p-1}} \eta_p^{l_{p-1}} \eta_{p+1}^{l_{p+1}} \eta_{p+2}, \]

and the induction is achieved.

\( \checkmark \)

The following result will be usefull in Section 4 Subsection 4.4 (Proposition 4).

**Lemma 7.** For all \( p \geq 0 \), \( h \) function defined on the \( \eta_j \),
\[ \Delta_{p,(\eta_1(p), \ldots, \eta_{1(1)})}(h)(\eta_{p(p+1)}) = \Delta_{p,(\eta_p, \ldots, \eta_1)}(h)(\eta_{p+1}). \]
Proof. We prove the lemma by induction on \( p \geq 0 \). It is obvious for \( p = 0 \) and \( p = 1 \). Let be \( p \geq 2 \) and \( \sigma \in \mathcal{S}_{p+1} \). First, assume that \( \sigma(1) = 1 \). Then by induction
\[
\Delta_{p,(\eta_{(p)},...,\eta_{(1)})}(h) (\eta_{(p+1)}) = \Delta_{p,(\eta_{(p)},...,\eta_{(2)},\eta_{(1)})}(h) (\eta_{(p+1)})
\]
\[
= \Delta_{p-1,(\eta_{(p)},...,\eta_{(2)})} [\zeta \mapsto \Delta_{1,\eta_{(1)}}(h)(\zeta)] (\eta_{(p+1)})
\]
\[
= \Delta_{p-1,(\eta_{(p)},...,\eta_{(2)})} [\zeta \mapsto \Delta_{1,\eta_{(1)}}(h)(\zeta)] (\eta_{(p+1)})
\]
\[
= \Delta_{p,(\eta_{(p)},...,\eta_{(1)})}(h) (\eta_{(p+1)})
\]

Now one can assume that \( \sigma(1) \neq 1 \) and consider the transposition \( \tau = (1 \ \sigma(1)) \). Then \((\tau \sigma)(1) = 1\) and
\[
\Delta_{p,(\eta_{(\tau \sigma(p))},...,\eta_{(\tau \sigma(1))})}(h) (\eta_{(\tau \sigma(p+1))}) = \Delta_{p,(\eta_{(p)},...,\eta_{(1)})}(h) (\eta_{(p+1)}).
\]

Now let assume that the lemma is also proved for all transposition \((1 \ j), \ 2 \leq j \leq p + 1\). Then
\[
\Delta_{p,(\eta_{(p)},...,\eta_{(1)})}(h) (\eta_{(p+1)}) = \Delta_{p,(\eta_{(p)},...,\eta_{(1)},\eta_{(p+1)})}(h) (\eta_{(p+1)})
\]
\[
= \Delta_{p-1,(\eta_{(p)},...,\eta_{(1)},\eta_{(p+1)})} [\zeta \mapsto \Delta_{1,\eta_{(p)}}(h)(\zeta)] (\eta_{(p+1)})
\]
\[
= \Delta_{p-1,(\eta_{(p)},...,\eta_{(1)},\eta_{(p+1)})} [\zeta \mapsto \Delta_{1,\eta_{(p)}}(h)(\zeta)] (\eta_{(p+1)})
\]
\[
= \Delta_{p,(\eta_{(p)},...,\eta_{(1)})}(h) (\eta_{(p+1)})
\]
and the lemma will be proved. So it is sufficient to prove it for any \( \tau = (1 \ j), \ 2 \leq j \leq p + 1 \). On the other hand,
\[
\Delta_{p,(\eta_{(p)},...,\eta_{(1)})}(h) (\eta_{p+1}) = \frac{\Delta_{p-1,(\eta_{p-1},...,\eta_{(1)})}(h) (\eta_{p}) - \Delta_{p-1,(\eta_{p-1},...,\eta_{(1)})}(h) (\eta_{p+1})}{\eta_{p} - \eta_{p+1}}
\]
\[
= \Delta_{p,(\eta_{p+1},\eta_{p-1},...,\eta_{(1)})}(h) (\eta_{p}),
\]
then it is also true for \( \tau_{p} := (p \ p + 1) \).

Let assume that it is also true for any permutation that fixes \( p + 1 \). Then it will be true for all \((1 \ j), \ 2 \leq j \leq p\), also for \((1 \ p + 1) = \tau_{p}(1 p) \tau_{p}\) and the proof will be achieved.

Finally let be \( \sigma \in \mathcal{S}_{p+1} \) such that \( \sigma(p + 1) = p + 1 \). Then \( \prod_{j=1}^{p} (X - \eta_{(j)}) = \prod_{j=1}^{p} (X - \eta_{j}) \) and
\[
\sum_{q=0}^{p} \prod_{j=1}^{q} (X - \eta_{(j)}) \Delta_{q,(\eta_{(q)},...,\eta_{(1)})}(h) (\eta_{(q+1)})
\]
\[
= \sum_{q=0}^{p-1} \left( \prod_{j=1}^{q} (X - \eta_{(j)}) \Delta_{q,(\eta_{(q)},...,\eta_{(1)})}(h) (\eta_{(q+1)}) \right)
\]
\[
+ \prod_{j=1}^{p} (X - \eta_{j}) \Delta_{p,(\eta_{(p)},...,\eta_{(1)})}(h) (\eta_{(p+1)})
\]

Since for all \( q = 0, \ldots, p - 1 \), the family \( \{1, X - \eta_{1}, \ldots, (X - \eta_{1}) \cdots (X - \eta_{q})\} \) is a basis of \( \mathbb{C}_{q}[X] = \{ P \in \mathbb{C}[X], \ \deg P \leq q \} \), one has
\[
\prod_{j=1}^{q} (X - \eta_{(j)}) = \sum_{l=0}^{q} c_{q,l} \prod_{j=1}^{l} (X - \eta_{j}),
\]
then
\[
\sum_{q=0}^{p} \prod_{j=1}^{q} (X - \eta_{\sigma(j)}) \Delta_{q, (\eta_{\sigma(q)}, \ldots, \eta_{\sigma(1)})} (h)(\eta_{\sigma(q+1)}) = \sum_{q=0}^{p-1} C_q \prod_{j=1}^{q} (X - \eta_j) + \prod_{j=1}^{p} (X - \eta_j) \Delta_{p, (\eta_{\sigma(p)}, \ldots, \eta_{\sigma(1)})} (h)(\eta_{\sigma(p+1)}) .
\]

On the other hand, one has by Lemma 4
\[
\sum_{q=0}^{p} \prod_{j=1}^{q} (X - \eta_j) \Delta_{q, (\eta_{\sigma(q)}, \ldots, \eta_{\sigma(1)})} (h)(\eta_{\sigma(q+1)}) = \sum_{q=1}^{p+1} \left( \prod_{j=1, j \neq q}^{p+1} X - \eta_j \right) h(\eta_q) = \sum_{j=1}^{p+1} (X - \eta_j) \Delta_{p, (\eta_{\sigma(p)}, \ldots, \eta_{\sigma(1)})} (h)(\eta_{p+1}) + \prod_{j=1}^{p} (X - \eta_j) \Delta_{p, (\eta_{\sigma(p)}, \ldots, \eta_{\sigma(1)})} (h)(\eta_{p+1}).
\]

Since the family \( \{1, X - \eta_1, (X - \eta_1)(X - \eta_2), \ldots, (X - \eta_1) \cdots (X - \eta_p) \} \) is a basis of \( \mathbb{C}_p[X] \), it follows that
\[
\Delta_{p, (\eta_{\sigma(p)}, \ldots, \eta_{\sigma(1)})} (h)(\eta_{p+1}) = \Delta_{p, (\eta_{\sigma(p)}, \ldots, \eta_{\sigma(1)})} (h)(\eta_{p+1})
\]
and the lemma is proved.

2.3. About the formula \( E_N(\cdot, \eta) \). In this part we want to justify what we mean when we claim that the formula \( E_N(f; \eta) \) is the canonical interpolation formula for any \( f \in \mathcal{O}(\mathbb{C}^2) \). We set, for all \( p \geq 1 \),
\[
(2.13) \quad w_p(z) := \frac{z_2 + \eta_p z_1}{1 + |\eta_p|^2}.
\]

Then the formula \( E_N(f; \eta) \) can be written as
\[
E_N(f; \eta)(z) = \sum_{p=1}^{N} \prod_{j=p+1}^{N} (z_1 - \eta_{q_2}) \sum_{q=p}^{N} \frac{1 + \eta_{q}^* z_1}{1 + |\eta_q|^2} \frac{1}{|\eta_q - \eta_j|} \frac{f(\eta_q w_q(z), w_q(z))]_{N-p}}{w_q(z)^{N-p}} ,
\]
where \([h]_{N-p} \) is the truncation of \( h \) at order \( N-p \), ie \([h]_{N-p}(w) = \sum_{m \geq N-p} \frac{w^m}{m!} h^{(m)}(0) \).

On the other hand, notice that the point \((\eta_q w_q(z), w_q(z)) \) is the orthogonal projection of \( z \) on the line \( \{z_1 - \eta_q z_2 = 0 \} \) with respect to the hermitian scalar product on \( \mathbb{C}^2 \), since \((\eta_q w_q(z), w_q(z)) = \langle z, u_q \rangle > u_q \), with \( u_q := (\eta_q, 1)/\sqrt{1 + |\eta_q|^2} \) being a normalized director vector of \( \{z_1 - \eta_q z_2 = 0 \} \). In particular, \( z \in \mathbb{C}^2 \) being given, \( f(\eta_q w_q(z), w_q(z)) \) is a natural way to use the restriction \( f|_{\{z_1 = \eta_q z_2 \}} \).

Finally, on the expression of \( E_N(f; \eta) \) appear derivatives of the restrictions of \( f \) of order at most \( \max(N-2, 0) \) since \( p \geq 1 \) and
\[
[f(\eta_q w_q(z), w_q(z))]_{N-p} = f(\eta_q w_q(z), w_q(z)) - \sum_{p=0}^{N-p-1} \frac{w_q(z)^m}{m!} \frac{\partial^m}{\partial v^m} \big|_{v=0} (f(\eta_q v, v)) .
\]

Now we can give the following result about the fact that the formula \( E_N(f; \eta) \) is a canonical interpolation formula.
Lemma 8. For all $N \geq 1$, $E_N (f; \eta)$ is essentially the unique interpolation formula between the interpolation formula that fix $\mathbb{C}_{N-1} [z_1, z_2]$ and have the following expression

$$F = \sum_{q=1}^{N} F_q (z),$$

where $F_q (z) = \sum_{k+l \leq N-1} C_{q,k,l} z_1^k \bar{z}_2^l$, with $C_{q,k,l}$ being an operator on the space $\mathcal{O} (\mathbb{C})_q := (\mathcal{O} (\mathbb{C}^2))_{\{ z_1 = \eta_q z_2 \}}$, of order at most $\max (N-2, 0)$.

Proof. First, in the expression of such an operator $F_q$ on $\mathcal{O} (\mathbb{C})_q$ and any $f \in \mathcal{O} (\mathbb{C}^2)$, there must appear parts of $f (\eta_q v_q (z), v_q (z))$ where $v_q (z) = a_q z_1 + b_q z_2$ is such that $\| z \| \geq \| (\eta_q v_q (z), v_q (z)) \| = \| v_q (z) \| \sqrt{1+|\eta_q|^2}$, i.e. $|v_q (z)| \leq \| z \| / \sqrt{1+|\eta_q|^2}$. This yields to

$$\sup_{\| z \| \leq 1} |a_q z_1 + b_q z_2| = \sqrt{|a_q|^2 + |b_q|^2} \leq \frac{1}{\sqrt{1+|\eta_q|^2}}.$$

On the other hand, the condition of interpolation must satisfy $F(f)_{\{ z_1 = \eta_q z_2 \}} = f_{\{ z_1 = \eta_q z_2 \}}$, then

$$1 = |v_q (\eta_q, 1)| = |\eta_q a_q + b_q| \leq \sqrt{|a_q|^2 + |b_q|^2} \sqrt{1+|\eta_q|^2}.$$

It follows from (2.14) and (2.15) that $\sqrt{|a_q|^2 + |b_q|^2} = 1 / \sqrt{1+|\eta_q|^2}$ and $(a_q, b_q) = \lambda (\eta_q z_1 + z_2)$, $\lambda \in \mathbb{C}$. Then $|\lambda| = 1 / (1 + |\eta_q|^2)$ and

$$v_q (z) = \omega_q \frac{\eta_q z_1 + z_2}{1 + |\eta_q|^2}, \ |\omega_q| = 1.$$

Finally, the condition $(\eta_q v_q (\eta_q, 1), v_q (\eta_q, 1)) = (\eta_q, 1)$ yields to $\omega_q = 1$, then

$$v_q (z) = \frac{\eta_q z_1 + z_2}{1 + |\eta_q|^2} = w_q (z).$$

Next, for any $P \in \mathbb{C}_{N-1} [z]$, in particular, for all $z_2 \in \mathbb{C} \setminus \{ 0 \}$, $P(\cdot, z_2) \in \mathbb{C} [z_1]$. It follows by Lemma 4 that

$$P(z) = \sum_{p=1}^{N} \prod_{j=1, j \neq p}^{N} \frac{\eta_p z_2 - \eta_j z_2}{\eta_p z_2 - \eta_j z_2} \Delta^{-p} \eta_p z_2 = \Delta^{-p} (\eta_p z_2) \Delta^{-p} \eta_p z_2.$$

On the other hand, by Lemma 5 for all $p = 1, \ldots, N$, $\Delta^{-p} \eta_p z_2 \in \mathbb{C} [z_2]$. Moreover,

\begin{align*}
\deg \Delta^{-p} \eta_p z_2 \Delta^{-p} \eta_p z_2 (P(\cdot, z_2)) (\eta_p z_2) & \leq \deg_z P - (N - p) + \deg_z P \\
& \leq p - 1.
\end{align*}
It follows from (2.17), (2.18) and (2.19) that one can write $F$ as

$$F = \sum_{q=1}^{N} \sum_{p=1}^{N} \prod_{j=p+1}^{N} (z_1 - \eta_j z_2) \Delta_{N-p,(\eta_{p+1} z_2, \ldots, \eta_N z_2)} (F_q(\cdot, z_2)) (\eta_j z_2)$$

(2.20)$$= \sum_{q,p=1}^{N} Q_{q,p}(z_2) \left( \prod_{j=p+1}^{N} (z_1 - \eta_j z_2) \right) F_{q,p},$$

where $Q_{q,p} \in \mathbb{C}[z_2]$ and, for all $f \in \mathcal{O}(\mathbb{C}^2)$,

$$F_{q,p}(f) = \sum_{l=0}^{d_{q,p}} c_{N,q,p,l} w_q(z)^l \sum_{m \geq r_{q,p}} \frac{w_q(z)^{m-r_{q,p}}}{m!} \frac{\partial^m}{\partial v^m}|_{v=0} [f(\eta_q v, v)],$$

with $r_{q,p} \leq N - 1$, $\forall l = 0, \ldots, d_{q,p}$ (since $F_{q,p}$ is of order at most $N - 2$).

Now, if $N \geq 1$ and $F_N$ is such an operator, the operator $F_N - E_N$ can still be written as the expression (2.20). Moreover, $(F_N - E_N)|_{C_{N-1}[z]} = 0$ since $E_N$ and $F_N$ fix $C_{N-1}[z]$. On the other hand, let consider the Lagrange monomials

$$L_q(z) := \prod_{j=1, j \neq q}^{N} \frac{z_1 - \eta_j z_2}{\eta_q - \eta_j}, \quad 1 \leq q \leq N.$$

In particular, $\deg L_q \leq N - 1$ and $L_q((z_1 - \eta_q z_2)) = 1_{p=q} z_2^{N-1}$. Then one has, for all $N \geq 1$ and $q = 1, \ldots, N$,

$$0 = (F_N - E_N)(L_q) = \sum_{p=1}^{N} Q_{q,p}(z_2) \prod_{j=p+1}^{N} (z_1 - \eta_j z_2) \times$$

$$\times \sum_{l=0}^{d_{q,p}} c_{N,q,p,l} w_q(z)^l \sum_{m \geq r_{q,p}} \frac{w_q(z)^{m-r_{q,p}}}{m!} \frac{\partial^m}{\partial v^m}|_{v=0} [v^{N-1}]$$

(2.23)$$= \sum_{p=1}^{N} Q_{q,p}(z_2) \prod_{j=p+1}^{N} (z_1 - \eta_j z_2) \sum_{l=0}^{d_{q,p}} c_{N,q,p,l} w_q(z)^l + (N-1-r_{q,p}).$$

Now we will prove the lemma by induction on $N \geq 1$. If $N = 1$, (2.23) becomes

$$0 = Q_{1,1}(z_2) \sum_{l=0}^{d_{1,1}} c_{1,1,l} w_1(z)^l, \quad \forall z \in \mathbb{C}^2,$$

then $Q_{1,1} = 0$ (in this case, $F_1 = E_1$) or $\sum_{l=0}^{d_{1,1}} c_{1,1,l} w_1(z)^l = 0, \forall z \in \mathbb{C}^2$ (in this case, $\sum_{l=0}^{d_{1,1}} c_{1,1,l} w_1 = 0, \forall w \in \mathbb{C}$, then $c_{1,1,l} = 0, \forall l = 0, \ldots, d_{1,1}$ and $F_1 = E_1$).

Now if the lemma is proved for $N \geq 1$ and we consider $N + 1$, then (2.23) becomes, for all $q = 1, \ldots, N + 1$,

$$0 = (z_1 - \eta_{N+1} z_2) \sum_{p=1}^{N} Q_{q,p}(z_2) \prod_{j=p+1}^{N} (z_1 - \eta_j z_2) \sum_{l=0}^{d_{q,p}} c_{N+1,q,p,l} w_q(z)^l + (N-r_{q,p})$$

(2.24)$$+ Q_{q,N+1}(z_2) \sum_{l=0}^{d_{q,N+1}} c_{N+1,q,N+1,l} w_q(z)^l + (N-r_{q,N+1}).$$
In particular, \((z_{1} - \eta_{N+1}z_{2})\) divides \(Q_{q,N+1}(z_{2})\) \(\sum_{l=0}^{d_{q,N+1}} c_{N+1,q,N+1,l} w_{q}(z)^{l+N-r_{q,N+1}}\). Then one (and only one) of these different cases can happen:

- \((z_{1} - \eta_{N+1}z_{2})\) divides \(Q_{q,N+1}(z_{2})\), then \(Q_{q,N+1}(z_{2}) = 0\) and \((2.24)\) becomes

\[
0 = \sum_{p=1}^{N} Q_{q,p}(z_{2}) \prod_{j=p+1}^{N} (z_{1} - \eta_{j}z_{2}) \sum_{l=0}^{d_{q,p}} c_{N+1,q,p,l} w_{q}(z)^{l+N-r_{q,p}} ,
\]

that yields by induction to \(F_{N+1} = E_{N+1}\).

- Otherwise (since \((z_{1} - \eta_{N+1}z_{1})\) is irreducible and \(\mathbb{C}[z_{1}, z_{2}]\) is factorial), \((z_{1} - \eta_{N+1}z_{2})\) divides \(\sum_{l=0}^{d_{q,p}} c_{N+1,q,p,l} w_{q}(z)^{l+N-r_{q,p}}\). If \(\sum_{l=0}^{d_{q,p}} c_{N+1,q,p,l} X' = 0\), then \((2.24)\) and the induction yield to \(F_{N+1} = E_{N+1}\).

- Else \(\sum_{l=0}^{d_{q,p}} c_{N+1,q,p,l} w_{q}(z)^{l+N-r_{q,p}} = w_{q}(z)^{m'} \sum_{l=0}^{d'} c'_{l} w_{q}(z)^{l}\), with \(c'_{0} \neq 0\), and \((z_{1} - \eta_{N+1}z_{2})\) divides \(\sum_{l=0}^{d'} c'_{l} w_{q}(z)^{l}\). Then \(\exists Q \in \mathbb{C}[z_{1}, z_{2}]\) such that

\[
\sum_{l=0}^{d'} c'_{l} w_{q}(z)^{l} = (z_{1} - \eta_{N+1}z_{2}) \bar{Q}(z)
\]

and \(z = 0\) yields to \(c'_{0} = 0\), which is impossible.

- Finally, \((z_{1} - \eta_{N+1}z_{2})\) must divide \(w_{q}(z)^{m'}\) then divide \(w_{q}(z) = \frac{w_{z_{1}+z_{2}}}{1 + |\eta_{q}|^{2}}\) that is irreducible too. It follows that they are proportional then \(1 + |\eta_{q}|^{2} = 0\).

In this case, the part

\[
\frac{1 + \eta_{N+1} \eta_{q}}{1 + |\eta_{q}|^{2}} \prod_{j=N+1,j \neq q}^{N+1} \left( \eta_{q} - \eta_{j} \right) \sum_{m=0}^{\infty} \left( \frac{z_{2} + \eta_{q} \eta_{z_{1}}}{1 + |\eta_{q}|^{2}} \right)^{m} \frac{1}{m!} \frac{\partial^{m}}{\partial v^{m}} \mid v=0 \left[ f(\eta_{q} v, v) \right]
\]

will disappear in the expression \((1.4)\) of \(E_{N}\). It follows that \(E_{N}\) will be the most natural choice for the interpolation formula and this achieves the induction.

\[
\check{\sqrt{}}
\]

3. Proof of Theorem 1

In this part we will assume that the set \(\{\eta_{j}\}_{j \geq 1}\) is bounded, what we will write as

\[
(3.1) \quad \|\eta\|_{\infty} := \sup_{j \geq 1} |\eta_{j}| < +\infty .
\]

**Remark 3.1.** We will see that the condition \((1.8)\) is equivalent to the existence of \(R_{\eta}\) such that, for all \(p, q, s \geq 0\) with \(s \leq q\),

\[
(3.2) \quad \left| \Delta_{p, (\eta_{p}, \ldots, \eta_{q})} \left( \frac{\zeta}{(1 + |\zeta|^{2})^{q}} \right) (\eta_{p+1}) \right| \leq R_{\eta}^{p+q}.
\]

In all the following, we will mean the Taylor expansion of any function \(f \in \mathcal{O}(\mathbb{C}^{2})\) (that absolutely converges in any compact subset \(K \subset \mathbb{C}^{2}\)) by

\[
(3.3) \quad f(z) = \sum_{k,l \geq 0} a_{k,l} z_{1}^{k} z_{2}^{l}.
\]
3.1. Condition (1.8) is necessary. We begin with this result.

**Lemma 9.** For all \( f \in \mathcal{O}(\mathbb{C}^2), \; N \geq 1 \) and \( k_1 \geq N, \)

\[
\frac{1}{k_1!} \frac{\partial^{k_1}}{\partial z_1^{k_1}} |_{z=0} [R_N(f; \eta)(z)] = \Delta_{N-1,(\eta_{N-1}, \ldots, \eta_1)} \left( \left( \frac{\zeta}{1 + |\zeta|^2} \right)^{k_1-N+1} \sum_{k+l=k_1} a_{k,l} \zeta^k \right) (\eta_N).
\]

**Proof.** First, we claim that

\[
(3.4) \quad R_N(f; \eta)(z) = \sum_{p=0}^{N-1} z_2^{N-p} \prod_{j=1}^{p} (z_1 - \eta_j z_2) \Delta_{p,(\eta_p, \ldots, \eta_1)} (\zeta \mapsto r_N(\zeta, z)) (\eta_{p+1}),
\]

with

\[
(3.5) \quad r_N(\zeta, z) := \sum_{k+l \geq N} a_{k,l} \zeta^k \left( \frac{z_2 + \zeta z_1}{1 + |\zeta|^2} \right)^{k+l-N+1} = \sum_{m \geq N} \left( \frac{z_2 + \zeta z_1}{1 + |\zeta|^2} \right)^{m-N+1} \sum_{k+l=m} a_{k,l} \zeta^k.
\]

Indeed, by Lemma 4

\[
R_N(f; z)(z) = z_2^{N-1} \sum_{p=0}^{N} \prod_{j=1, j \neq p}^{N} \frac{z_1/z_2 - \eta_j}{\eta_p - \eta_j} \sum_{k+l \geq N} a_{k,l} \eta_p^k \left( \frac{z_2 + \eta_p z_1}{1 + |\eta_p|^2} \right)^{k+l-N+1} = z_2^{N-1} \sum_{p=0}^{N} \prod_{j=1}^{p} (z_1/z_2 - \eta_j) \Delta_{p,(\eta_p, \ldots, \eta_1)} \left[ \sum_{k+l \geq N} a_{k,l} \zeta^k \left( \frac{z_2 + \zeta z_1}{1 + |\zeta|^2} \right)^{k+l-N+1} \right] (\eta_{p+1}).
\]

It follows that

\[
R_N(f; \eta)(z) = \sum_{p=0}^{N-1} z_2^{N-1-p} \prod_{j=1}^{p} (z_1 - \eta_j z_2) \Delta_{p,(\eta_p, \ldots, \eta_1)} (\zeta \mapsto r_N(\zeta, z)) (\eta_{p+1})
\]

\[
= z_2^{N-1} \sum_{p=0}^{N-1} \sum_{r=0}^{p} \left( -1 \right)^{p-r} z_2^{r} \sigma_{p-r}(\eta_1, \ldots, \eta_p) \Delta(p(r_N(\zeta, z)),
\]

with \( \sigma_r(\eta_1, \ldots, \eta_p) = \sum_{1 \leq j_1 < \cdots < j_r \leq p} \eta_{j_1} \cdots \eta_{j_r} \). Then

\[
\frac{1}{k_1!} \frac{\partial^{k_1}}{\partial z_1^{k_1}} |_{z=0} [R_N(f; \eta)(z)] = \Delta_{N-1,(\eta_{N-1}, \ldots, \eta_1)} \left( \zeta \mapsto \frac{1}{k_1!} \frac{\partial^{k_1}}{\partial z_1^{k_1}} |_{z=0} [z_1^{N-1} r_N(\zeta, z)] \right) (\eta_N).
\]
Since $k_1 \geq N$, one has for all $\zeta \in \mathbb{C}$
\[
\frac{1}{k_1!} \frac{\partial^{k_1}}{\partial z_1^{k_1}}|_{z=0} \left[ z_1^{N-1} r_N(\zeta, z) \right] =
\]
\[
= \sum_{s=0}^{k_1} \frac{1}{s!} \frac{\partial^s}{\partial z_1^s}|_{z=0} \left( z_1^{N-1} \right) \frac{1}{(k_1-s)!} \frac{\partial^{k_1-s}}{\partial z_1^{k_1-s}}|_{z=0} \left[ r_N(\zeta, z) \right]
\]
\[
= \frac{1}{(k_1 - N + 1)!} \frac{\partial^{k_1-N+1}}{\partial z_1^{k_1-N+1}}|_{z=0} \left[ r_N(\zeta, z) \right]
\]
\[
= \sum_{m \geq N} \sum_{k+l=m} a_{k,l} \zeta^k \frac{1}{(k_1 - N + 1)!} \frac{\partial^{k_1-N+1}}{\partial z_1^{k_1-N+1}}|_{z=0} \left[ \left( \frac{z_2 + \bar{\zeta} z_1}{1 + |\zeta|^2} \right)^m \right]
\]
\[
= \sum_{k+l=k_1} a_{k,l} \zeta^k \left( \frac{\zeta}{1 + |\zeta|^2} \right)^{k_1-N+1},
\]
and the proof is achieved.

\[ \checkmark \]

Now we can prove the first sense of Theorem I.

**Proof.** We assume that, for every $f \in \mathcal{O}(\mathbb{C}^2)$, $R_N(f; \eta)$ is uniformly bounded on any compact subset $K \subset \mathbb{C}^2$, i.e.
\[
\sup_{N \geq 1} \sup_{z \in K} |R_N(f; \eta)(z)| \leq M(f, K) < +\infty.
\]

In particular, $\forall \ p \geq 0,$
\[
\sup_{z \in D_2(0,1,1)} |R_{p+1}(f; \eta)(z)| \leq M(f).
\]

Then for all $p \geq 0$ and $q \geq 1,$
\[
\left| \frac{1}{(p+q)!} \frac{\partial^{p+q}}{\partial z_1^{p+q}}|_{z=0} [R_{p+1}(f; \eta)(z)] \right| = \left| \frac{1}{(2\pi i)^2} \int_{|\zeta_1|=|\zeta_2|=1} R_{p+1}(f; \eta)(\zeta_1, \zeta_2) d\zeta_1 \wedge d\zeta_2 \right|
\]
\[
\leq \sup_{z \in D_2(0,1,1)} |R_{p+1}(f; \eta)(z)| \leq M(f).
\]

Since $p + q \geq p + 1$, one can deduce from the above lemma that
\[
\left| \Delta_{p,\{\eta_p,\ldots,\eta_1\}} \left( \frac{\zeta}{1 + |\zeta|^2} \right)^q \sum_{k+l=p+q} a_{k,l} \zeta^k (\eta_{p+1}) \right| \leq M(f).
\]

Now consider any function entire on $\mathbb{C}$, $h(w) = \sum_{n \geq 0} a_n w^n$ and set $f_h(z) := h(z_2)$. Then $f_h \in \mathcal{O}(\mathbb{C}^2)$ and for all $p \geq 0$, $q \geq 1$,
\[
\sum_{k+l=p+q} a_{k,l}(f_h) \zeta^k = a_{0,p+q}(f_h) = a_{p+q}.
\]

It follows that for all $p \geq 0$, $q \geq 1,$
\[
\left| \Delta_{p,\{\eta_p,\ldots,\eta_1\}} \left( \frac{\zeta}{1 + |\zeta|^2} \right)^q \sum_{k+l=p+q} a_{k,l}(f_h) \zeta^k (\eta_{p+1}) \right| =
\]
the proof will be achieved.

Since \( h \in \mathcal{O}(\mathbb{C}) \), \( \limsup_{n \to \infty} |a_n|^{1/n} = 0 \). Conversely, if \( (\varepsilon_n)_{n \geq 1} \) is any sequence that converges to 0, the function \( h_\varepsilon(w) := \sum_{n \geq 1} \varepsilon_n w^n \) is entire on \( \mathbb{C} \) and

\[
\text{(3.6)} \quad \sup_{p \geq 0, q \geq 1} \left\{ |\varepsilon_{p+q}| \left| \Delta_{p,(\eta_{p}, \ldots, \eta_{1})} \left[ \left( \frac{\zeta}{1 + |\zeta|^2} \right)^q \right] (\eta_{p+1}) \right|^{\frac{1}{p+q}} \right\} < +\infty.
\]

Now we claim that

\[
\text{(3.7)} \quad R_n := \sup_{p \geq 0, q \geq 1} \left\{ \left| \Delta_{p,(\eta_{p}, \ldots, \eta_{1})} \left[ \left( \frac{\zeta}{1 + |\zeta|^2} \right)^q \right] (\eta_{p+1}) \right|^{\frac{1}{p+q}} \right\} < +\infty.
\]

If it is true, we will have proved the lemma for \( p \geq 0, q \geq 1 \). On the other hand, since for all \( q = 0, p \geq 1 \), one has

\[
|\Delta_{p,(\eta_{p}, \ldots, \eta_{1})}(\zeta \to 1)(\eta_{p+1})| = 0,
\]

and

\[
|\Delta_{q}(\zeta \to 1)(\eta_{1})| = 1 = R_n^q,
\]

the proof will be achieved.

Assume that \( \text{(3.7)} \) is not true. For all \( n \geq 1 \), there exist \( p_n \) and \( q_n \) such that

\[
\left| \Delta_{p_n,(\eta_{p_n}, \ldots, \eta_{1})} \left[ \left( \frac{\zeta}{1 + |\zeta|^2} \right)^{q_n} \right] (\eta_{p_n+1}) \right|^{\frac{1}{p_n+q_n}} \geq n.
\]

One can choose \( (p_n)_{n \geq 1} \) and \( (q_n)_{n \geq 1} \) such that the sequence \( (p_n + q_n)_{n \geq 1} \) is strictly increasing. Indeed, let be \( p_1 \) and \( q_1 \) for \( n = 1 \) and assume that there are \( p_1, \ldots, p_n \) and \( q_1, \ldots, q_n \) such that \( \forall j = 1, \ldots, n - 1, p_j + q_j < p_{j+1} + q_{j+1} \). Since

\[
+\infty = \sup_{p \geq 0, q \geq 1} \left\{ \left| \Delta_p \left[ \left( \frac{\zeta}{1 + |\zeta|^2} \right)^q \right] \right|^{\frac{1}{p+q}} \right\} = \max \left\{ \sup_{p+q \leq p_n + q_n} \left| \Delta_p \left[ \left( \frac{\zeta}{1 + |\zeta|^2} \right)^q \right] \right|^{\frac{1}{p+q}}, \sup_{p+q > p_n + q_n} \left| \Delta_p \left[ \left( \frac{\zeta}{1 + |\zeta|^2} \right)^q \right] \right|^{\frac{1}{p+q}} \right\}
\]

and the set \( \{p \geq 0, q \geq 1, p + q \leq p_n + q_n\} \) is finite, it follows that

\[
\sup_{p+q > p_n + q_n} \left| \Delta_p \left[ \left( \frac{\zeta}{1 + |\zeta|^2} \right)^q \right] \right|^{\frac{1}{p+q}} = +\infty.
\]

In particular, there are \( p_{n+1}, q_{n+1} \) such that \( p_{n+1} + q_{n+1} > p_n + q_n \) and

\[
\left| \Delta_{p_{n+1},(\eta_{p_{n+1}}, \ldots, \eta_{1})} \left[ \left( \frac{\zeta}{1 + |\zeta|^2} \right)^{q_{n+1}} \right] (\eta_{p_{n+1}+1}) \right|^{\frac{1}{p_{n+1}+q_{n+1}}} \geq n + 1.
\]
This allows us to construct by induction on \( n \geq 1 \) the sequences \((p_n)_{n \geq 1}\) and \((q_n)_{n \geq 1} \).

Now we define \((\varepsilon_n)_{n \geq 1}\) by

\[
\varepsilon_n := \begin{cases} 1/\sqrt{j}, & \text{if } \exists j \geq 1, p_j + q_j = n, \\ 0 & \text{otherwise.} \end{cases}
\]

(3.8)

Since \((p_n + q_n)_{n \geq 1}\) is strictly increasing, if such a \( j \) exists, it is unique then \((\varepsilon_n)_{n \geq 1}\) is well-defined.

On the other hand, \( \lim_{n \to \infty} \varepsilon_n = 0 \). Indeed, \( \forall \varepsilon > 0, \exists J \geq 1, \forall j \geq J, 1/\sqrt{j} < \varepsilon \). We set \( N_\varepsilon := p_j + q_j \) and let be any \( n \geq N_\varepsilon \). If there is no \( j \) such that \( p_j + q_j = n \), then \( \varepsilon_n = 0 < \varepsilon \); otherwise \( \exists j \geq 1, n = p_j + q_j \) and one has \( p_j + q_j = n \geq N_\varepsilon = p_j + q_j \). In particular, \( j \geq J \) (else \( n = p_j + q_j < p_j + q_j = N_\varepsilon \)), then \( \varepsilon_n = 1/\sqrt{j} \leq 1/\sqrt{J} < \varepsilon \).

It follows that for all \( j \geq 1 \),

\[
\varepsilon_{p_j+q_j} \Delta_{p_j,(\eta_{p_j}, \ldots, \eta_{j})} \left[ \left( \frac{\zeta}{1 + |\zeta|^2} \right)^{q_j} \right] (\eta_{p_j+1}) \geq \frac{1}{\sqrt{j}} j = \sqrt{j},
\]

then

\[
\sup_{p \geq 0, q \geq 1} \left\{ \varepsilon_{p+q} \Delta_{p,(\eta_{p}, \ldots, \eta_{1})} \left[ \left( \frac{\zeta}{1 + |\zeta|^2} \right)^{q} \right] (\eta_{p+1}) \right\} \geq \sup_{j \geq 1} \{ \sqrt{j} \} = +\infty,
\]

which is in contradiction with (3.6), thus (3.7) follows.

\[ \sqrt{ } \]

3.2. **Condition (1.8) is sufficient.** In this part we will prove the inverse sense of Theorem 1. We assume that the (bounded) set \( \{ \eta_j \}_{j \geq 1} \) satisfies (1.8). We begin with the following result that is a little stronger consequence (see Remark 3.1).

**Lemma 10.** There is \( R^p_\eta \geq 1 \) such that for all \( p, q, s \geq 0 \) with \( 0 \leq s \leq q \), one has

\[
\left| \Delta_{p,(\eta_p, \ldots, \eta_1)} \left[ \zeta \mapsto \frac{\zeta^s}{(1 + |\zeta|^2)^q} \right] (\eta_{p+1}) \right| \leq R^p_\eta^{p+q}.
\]

**Proof.** Set

\[
\begin{cases} R = \max (1, R_\eta), \\ Q = \max (3, R_\eta), \\ S = 3 \max (1, \|\eta\|_\infty). \end{cases}
\]

(3.9)

In order to prove the lemma, we want to prove the following estimation: \( \forall p, q, s \geq 0 \) with \( s \leq q \), one has

\[
\left| \Delta_{p,(\eta_p, \ldots, \eta_1)} \left[ \zeta \mapsto \frac{\zeta^s}{(1 + |\zeta|^2)^q} \right] (\eta_{p+1}) \right| \leq R^p Q^q S^{q-s}.
\]

(3.10)
This will be proved by induction on $p + q - s \geq 0$. If $p + q - s = 0$ then since $p, q - s \geq 0$, necessarily $p = 0$ and $s = q \geq 0$. Thus

$$\left| \Delta_0 \left( \frac{\zeta}{(1 + |\zeta|^2)^q} \right) (\eta_1) \right| = \left| \frac{\eta_1}{(1 + |\eta_1|^2)^q} \right| = \left( \frac{0.1 + 1, |\eta_1|}{1 + |\eta_1|^2} \right)^q \leq \left( \frac{1, \sqrt{1 + |\eta_1|^2}}{1 + |\eta_1|^2} \right)^q \text{ (Cauchy-Schwarz inequality)}$$

$$= \frac{1}{(\sqrt{1 + |\eta_1|^2})^q} \leq 1 \leq R^0 Q^q S^0.$$ 

If $p + q - s = 1$, then $p = 1$ and $q = s \geq 0$, or $p = 0$ and $0 \leq s = q - 1$. In the first case, since (1.8) is satisfied, one has

$$\left| \Delta_{1, \eta_1} \left( \frac{\zeta}{1 + |\zeta|^2} \right)^q (\eta_2) \right| \leq R_1^{1+q} \leq R^1 Q^q S^0.$$ 

In the second case, one has for all $q \geq 1$

$$\left| \Delta_0 \left( \frac{\zeta^{1+q}}{(1 + |\zeta|^2)^q} \right) (\eta_1) \right| = \left| \frac{\eta_1^{1+q}}{(1 + |\eta_1|^2)^q} \right| = \frac{1}{1 + |\eta_1|^2} \left| \frac{\eta_1}{1 + |\eta_1|^2} \right|^q \leq 1 \leq R^0 Q^q S^{q-1}.$$ 

Now let be $m \geq 1$ and assume that (3.10) is true for all $p, q, s \geq 0$ with $s \leq q$ and such that $p + q - s \leq m$. Now let be $p, q, s \geq 0$ with $s \leq q$ and such that $p + q - s = m + 1$. One has different cases.

If $p = 0$ then

$$\left| \Delta_0 \left( \frac{\zeta}{(1 + |\zeta|^2)^q} \right) (\eta_1) \right| = \left| \frac{\eta_1}{(1 + |\eta_1|^2)^q} \right| = \frac{1}{(1 + |\eta_1|^2)^q} \leq 1 \leq R^0 Q^q S^{q-1}.$$ 

If $s = q$ then by (1.8)

$$\left| \Delta_p.(\eta_p, ..., \eta_1) \left( \zeta \mapsto \frac{\zeta^q}{(1 + |\zeta|^2)^q} \right) (\eta_{p+1}) \right| \leq R_0^{p+q} \leq R^p Q^q S^0.$$ 

Otherwise $p \geq 1$ and $0 \leq s \leq q - 1$ (in particular $q \geq 1$). On one hand, one has by Lemmas 5 and 6

$$\Delta_p.(\eta_p, ..., \eta_1) \left( \frac{\zeta^{1+q}}{(1 + |\zeta|^2)^q} \right) (\eta_{p+1}) =$$

$$= \sum_{r=0}^{p} \Delta_r.(\eta_r, ..., \eta_1) \left( \frac{\zeta^{1+q}}{(1 + |\zeta|^2)^q} \right) (\eta_{r+1}) \Delta_{p-r}.(\eta_p, ..., \eta_{r+1}) \left( \zeta \mapsto \zeta \right) (\eta_{p+1})$$

$$= \Delta_{p-1}.(\eta_{p-1}, ..., \eta_1) \left( \frac{\zeta^{1+q}}{(1 + |\zeta|^2)^q} \right) (\eta_p) + 1 + \Delta_p.(\eta_p, ..., \eta_1) \left( \frac{\zeta^{1+q}}{(1 + |\zeta|^2)^q} \right) (\eta_{p+1}) \times \eta_{p+1}.$$ 

On the other hand,

$$\Delta_p.(\eta_p, ..., \eta_1) \left( \frac{\zeta^{1+q}}{(1 + |\zeta|^2)^q} \right) (\eta_{p+1}) =$$

...
Thus
\[ \Delta_p(\eta_p, \ldots, \eta_1) \left( \frac{\zeta}{1 + |\zeta|^2} \right)^q (\eta_{p+1}) = \]
\[ = \Delta_p(\eta_p, \ldots, \eta_1) \left( \frac{\zeta^q}{(1 + |\zeta|^2)^q} \right) (\eta_{p+1}) - \Delta_p(\eta_p, \ldots, \eta_1) \left( \frac{\zeta^q}{(1 + |\zeta|^2)^{q-1}} \right) (\eta_{p+1}). \]

Since \( s \leq q-1, s+1 \leq q \) and \((p-1)+q-(s+1) \leq p+(q-1)-s = p+q-(s+1) = m,\)
by induction and (3.9) it follows that
\[ \left| \Delta_p(\eta_p, \ldots, \eta_1) \left( \frac{\zeta}{1 + |\zeta|^2} \right)^q (\eta_{p+1}) \right| \leq \]
\[ \leq R_pQ^{q-1}S^{q-s} + R_p^{-1}Q^qS^{q-s-1} + \|\eta\|_\infty R_pQ^qS^{q-s-1} \]
\[ \leq R_p^{-1}Q^qS^{q-s-1}(RS + Q + \|\eta\|_\infty RQ) \]
\[ \leq R_p^{-1}Q^qS^{q-s-1}\left( RS\frac{Q}{3} + Q\frac{RS}{3} + S RQ \right) = R_pQ^qS^{q-s}, \]
and this proves (3.10).

Finally, if we set
\[ R'_\eta := \left[ \max(R, Q, S) \right]^2 = \left[ \max(3, 3\|\eta\|_\infty, R_\eta) \right]^2, \]
(3.10) yields to, for all \( p, q, s \geq 0 \) with \( s \leq q,\)
\[ \left| \Delta_p(\eta_p, \ldots, \eta_1) \left( \frac{\zeta}{1 + |\zeta|^2} \right)^q (\eta_{p+1}) \right| \leq R_pQ^qS^q \leq R'_\eta^{p+q}, \]
and the proof is achieved.

In the following the constant \( R_\eta \) will mean \( R'_\eta \) from Lemma (10). One can deduce as a consequence the following result.

**Lemma 11.** For all \( p, q \geq 0 \) and \( z \in \mathbb{C}^2,\)
\[ \left| \Delta_p(\eta_p, \ldots, \eta_1) \left[ \zeta \mapsto \left( \frac{z_2 + \zeta z_1}{1 + |\zeta|^2} \right)^q \right] (\eta_{p+1}) \right| \leq R_p^p(2R_\eta\|z\|)^q. \]
Proof. Indeed, Lemma 10 yields to
\[
\left| \Delta_{p,(\eta_1,\ldots,\eta_1)} \left( \frac{z_2 + \zeta z_1}{1 + |\zeta|^2} \right)^q \right| (\eta_{p+1}) = \nabla \\
= \left| \sum_{u=0}^{q} \frac{q!}{u!(q-u)!} z_2^{q-u} z_1^u \Delta_{p,(\eta_1,\ldots,\eta_1)} \left( \frac{\zeta^u}{(1 + |\zeta|^2)^q} \right) (\eta_{p+1}) \right| \\
\leq \sum_{u=0}^{q} \frac{q!}{u!(q-u)!} \|z\|^q \left| \Delta_{p,(\eta_1,\ldots,\eta_1)} \left( \frac{\zeta^u}{(1 + |\zeta|^2)^q} \right) (\eta_{p+1}) \right| \\
\leq \|z\|^q \sum_{u=0}^{q} \frac{q!}{u!(q-u)!} R_{\eta}^{p+q} = \|z\|^q 2^q R_{\eta}^{p+q} = R_{\eta}^{p}(2R_{\eta}\|z\|)^q.
\]

The next result will be usefull in order to prove the reciprocal sense of Theorem 1.

Lemma 12. For all \(n, p \geq 0\),
\[
A_n^p := \sum_{l_1=0}^{n} \sum_{l_2=0}^{l_1} \cdots \sum_{l_p=0}^{l_{p-1}} 1 = \text{card} \{(l_1, \ldots, l_p) \in \mathbb{N}^p, n \geq l_1 \geq l_2 \geq \cdots \geq l_p \geq 0\} \\
= \frac{(n + p)!}{n! p!}.
\]

Proof. First, we admit that if \(p = 0\) then \(l_0 = n\) and
\[
A_n^0 = \prod_{j=1}^{0} \sum_{l_j=0}^{l_{j-1}} 1 = 1.
\]

So one can assume that \(p \geq 1\) and we prove this result by induction on \(n + p \geq 1\).
If \(n + p = 1\) then \(p = 1\) and \(n = 0\) then
\[
A_0^1 = \text{card} \{s_1 \in \mathbb{N}, 0 \geq s_1 \geq 0\} = 1.
\]

Now we assume that it is true for all \(p, n\) such that \(1 \leq p + n \leq m\) with \(p \geq 1\) and \(n \geq 0\), and let be \(n \geq 0, p \geq 1\) such that \(n + p = m + 1\).
If \(n = 0\) then
\[
\text{card} \{0 \geq s_1 \geq \cdots \geq s_p \geq 0\} = \text{card} \{(0, \ldots, 0)\} = 1.
\]

If \(p = 1\) then
\[
\text{card} \{n \geq s_1 \geq 0\} = \frac{(n + 1)!}{n! 1!}.
\]

Otherwise \(n \geq 1\) and \(p \geq 2\). We claim that
\[
A_n^p = A_n^{p-1} + A_n^{p-1}.
\]
Indeed, for any element of \(\{(s_1, \ldots, s_p) \in \mathbb{N}^p, n \geq s_1 \geq \cdots \geq s_p \geq 0\}\), either \(s_1 = n\) or \(s_1 \leq n - 1\). Then
\[
\text{card} \{(s_1, \ldots, s_p) \in \mathbb{N}^p, n \geq s_1 \geq \cdots \geq s_p \geq 0\} =
\]
\[ A_{n} = A_{n+1} = n \cdot A_{n-1} + 1. \]

Since \( n + p - 1 = n - 1 + p = m \), one has by induction
\[ A_{n} = \frac{(n + p - 1)!}{n!(p-1)!} + \frac{(n - 1 + p)!}{(n-1)!p!} = \frac{(n + p - 1)!}{n!p!}(p + n) \]
and the lemma is proved.

\( \checkmark \)

Now Lemmas \[11\] and \[12\] yield to the following result.

**Lemma 13.** Let be \( f \in \mathcal{O}(\mathbb{C}^2) \). For all \( p \geq 0 \), \( N \geq 1 \), \( z \in \mathbb{C}^2 \) and \( R > 2\|\eta\|_{\infty} R_{\eta}^{p} \|z\| \), one has
\[
\left| \Delta_{n}(\eta_{\nu}, \ldots, \eta_{1}) \left[ \zeta \mapsto \sum_{m \geq N} \binom{\frac{z_2 + \bar{z}_1}{1 + |\zeta|^2}}{m-N+1} \sum_{k+l=m} a_{k,l}\zeta^{k} \right] (\eta_{p+1}) \right| \leq \frac{8||f||_{R} R_{\eta}^{p} ||z||}{||\eta||_{\infty}(1 - 2\|\eta\|_{\infty} R_{\eta}^{p} ||z||/R)} \left( \frac{R_{\eta}(1 + ||\eta||_{\infty})}{||\eta||_{\infty}} \right)^{p},
\]
with
\[
(3.13) \quad ||f||_{R} := \sup_{|z_1|, |z_2| \leq R} |f(z_1, z_2)|.
\]

**Proof.** First, for all \( m \geq N \), one has
\[
\left| \Delta_{n}(\eta_{\nu}, \ldots, \eta_{1}) \left[ \zeta \mapsto \sum_{m \geq N} \binom{\frac{z_2 + \bar{z}_1}{1 + |\zeta|^2}}{m-N+1} \sum_{k+l=m} a_{k,l}\zeta^{k} \right] (\eta_{p+1}) \right| = \sum_{v=0}^{p} \Delta_{v}(\eta_{\nu}, \ldots, \eta_{1}) \left[ \zeta \mapsto \sum_{m \geq N} \binom{\frac{z_2 + \bar{z}_1}{1 + |\zeta|^2}}{m-N+1} \sum_{k+l=m} a_{k,l}\zeta^{k} \right] (\eta_{v+1}) \Delta_{p-v}(\eta_{\nu}, \ldots, \eta_{1+v}) \left( \sum_{k+l=m} a_{k,l}\zeta^{k} \right)(\eta_{p+1}).
\]

Next, for all \( 0 \leq v \leq p \) and \( m \geq N \) (>, one has by Lemmas \[6\] and \[12\]
\[
\left| \Delta_{p-v}(\eta_{\nu}, \ldots, \eta_{1+v}) \left( \sum_{k+l=m} a_{k,m-k}\zeta^{k} \right)(\eta_{p+1}) \right| \leq \sum_{k=p-v}^{m} |a_{k,m-k}| \sum_{l_1=0}^{k-p+v} |\eta_{v+1}|^{k-p+v-l_1} \sum_{l_2=0}^{k-p+v-l_1} |\eta_{v+2}|^{l_2} \cdots \sum_{l_{p-v-1}=0}^{k-p+v-l_{p-v}} |\eta_{p+v}|^{l_{p-v}} |\eta_{p+1}|^{k!} \frac{1}{(p-v)!} \frac{1}{(k-p+v)!}.
\]

On the other hand, since \( f \in \mathcal{O}(\mathbb{C}^2) \), one has for all \( R \geq 1 \)
\[
|a_{k,l}| = \frac{1}{(2\pi)^2} \int_{|\zeta_1|=|\zeta_2|=R} \frac{f(\zeta_1, \zeta_2) d\zeta_1 \wedge d\zeta_2}{\zeta_1^{k+1} \zeta_2^{l+1}} \leq \frac{||f||_{R} R^{k+l}}{R^{k+l}}.
\]
Thus
\[
\left| \Delta_{p,v}(\eta_0,\ldots,\eta_{p+1}) \left( \sum_{k=0}^{m} a_{k,m-k} \zeta_k \right) \right| \leq \\
\leq \frac{\|f\|_R}{R^m} \frac{k!}{(k-p-v)!} \|\eta\|_\infty^{k-p-v} = \frac{\|f\|_R}{R^m} \frac{1}{(p-v)!} \frac{\partial^{p-v}}{\partial \theta^{p-v}} |_{\theta = \|\eta\|_\infty} \left( \sum_{k=0}^{m} t^k \right) \\
= \frac{\|f\|_R}{R^m} \frac{1}{2\pi i} \int_{|t| = R} \sum_{k=0}^{m} t^k (t - \|\eta\|_\infty)^{p-v+1} dt \\
\leq \frac{\|f\|_R}{R^m} \frac{R_\eta \sum_{k=0}^{m} R^{k}_{\eta}}{(R_\eta - \|\eta\|_\infty)^{p-v+1}} = \frac{\|f\|_R}{R^m} \frac{R_\eta^{m+1} - 1}{(R_\eta - \|\eta\|_\infty)^{p-v+1}(1 - 1/R_\eta)} \\
\leq \frac{2\|f\|_R}{R^m} \frac{R_\eta^{m+1}}{\|\eta\|_\infty^{p-v+1}} = 2\|f\|_R \frac{R_\eta}{\|\eta\|_\infty} \left( \frac{R_\eta}{R} \right)^m \frac{1}{\|\eta\|_\infty^{p-v}},
\]
since by (11), $R_\eta \geq 2\|\eta\|_\infty^2$. It follows with Lemma[11] that
\[
\Delta_{p,(\eta_0,\ldots,\eta_1)} \left( \frac{z_0 + \zeta_1}{1 + |\zeta|^2} \right)^{m-N+1} \left( \sum_{k+i=m} a_{k,i} \zeta_k \right) (\eta_{p+1}) \leq \\
\leq \sum_{v=0}^{p} R_\eta^v (2R_\eta \|z\|)^{m-N+1} 2\|f\|_R R_\eta \left( \frac{R_\eta}{R} \right)^m \frac{1}{\|\eta\|_\infty} \\
= \frac{2\|f\|_R R_\eta \left( \frac{R_\eta}{R} \right)^m (2R_\eta \|z\|)^{m-N+1} \frac{1}{\|\eta\|_\infty} \sum_{v=0}^{p} (R_\eta \|\eta\|_\infty)^v}{\|\eta\|_\infty} \\
\leq \frac{2\|f\|_R R_\eta \left( \frac{2 \|\eta\|_\infty R_\eta^2 \|z\|}{R} \right)^m \frac{1}{\|\eta\|_\infty} \frac{(R_\eta (1 + \|\eta\|_\infty))^{p+1} - 1}{R_\eta (1 + \|\eta\|_\infty) - 1}}{\|\eta\|_\infty (2R_\eta \|z\|)^{N-1}} \\
\leq \frac{4\|f\|_R R_\eta \left( \frac{2 \|\eta\|_\infty R_\eta^2 \|z\|}{R} \right)^m \left( \frac{R_\eta (1 + \|\eta\|_\infty)}{\|\eta\|_\infty} \right)^p}{\|\eta\|_\infty (2R_\eta \|z\|)^{N-1}}.
\]
One can deduce that, for all $N \geq 1$, $0 \leq p \leq N - 1$, $z \in \mathbb{C}^2$ and $R > 2\|\eta\|_\infty R_\eta^2 \|z\|$,
\[
\left| \Delta_{p,(\eta_0,\ldots,\eta_1)} \left( \sum_{m \geq N} \left( \frac{z_0 + \zeta_1}{1 + |\zeta|^2} \right)^{m-N+1} \sum_{k+i=m} a_{k,i} \zeta_k \right) (\eta_{p+1}) \right| \\
\leq \sum_{m \geq N} \left| \Delta_{p,(\eta_0,\ldots,\eta_1)} \left( \frac{z_0 + \zeta_1}{1 + |\zeta|^2} \right)^{m-N+1} \sum_{k+i=m} a_{k,i} \zeta_k \right| (\eta_{p+1}) \\
\leq \frac{4\|f\|_R R_\eta \left( \frac{2 \|\eta\|_\infty R_\eta^2 \|z\|}{R} \right)^m \sum_{m \geq N} \left( \frac{2 \|\eta\|_\infty R_\eta^2 \|z\|}{R} \right)^m}{\|\eta\|_\infty (2R_\eta \|z\|)^{N-1}} \\
\leq \frac{4\|f\|_R R_\eta \left( \frac{R_\eta (1 + \|\eta\|_\infty)}{\|\eta\|_\infty} \right)^p \left( \frac{2 \|\eta\|_\infty R_\eta^2 \|z\|}{R} \right)^N}{\|\eta\|_\infty (2R_\eta \|z\|)^{N-1}} \\
= \frac{8\|f\|_R R_\eta^2 \|z\|}{\|\eta\|_\infty (1 - 2 \|\eta\|_\infty R_\eta^2 \|z\|/R)^N} \left( \frac{R_\eta (1 + \|\eta\|_\infty)}{\|\eta\|_\infty} \right)^p \left( \frac{2 \|\eta\|_\infty R_\eta^2 \|z\|}{R} \right)^N.\]
We can finally complete the proof of Theorem 1.

Proof. \( f \in \mathcal{O}(\mathbb{C}^2) \) and \( K \subset \mathbb{C}^2 \) compact subset being given, it follows from Lemma 13 that, for all \( N \geq 1, z \in K \) and \( R > 2 \| \eta \|_{\infty} R_{\eta}^2 \| z \| \),

\[
|R_N(f; \eta)(z)| \leq \sum_{p=0}^{N-1} \| z \|^{N-1-p} \prod_{j=1}^{p} |z_j - \eta_j z_j| \Delta_p(\eta_p; \ldots, \eta_1) \left| \sum_{k+l \geq N} a_k, \eta^k \left( \frac{z_j + \eta_j z_j}{1 + |\eta_j|^2} \right)^{k+l-N+1} \right| (\eta_p+1) \\
\leq \sum_{p=0}^{N-1} \| z \|^{N-1-p} \prod_{j=1}^{p} \sqrt{1 + |\eta_j|^2} \| R \|_{\infty} (1 - 2 |\eta| \| R \|_{\infty} \| z \| / R) \left( \frac{R \| \eta \|_{\infty}}{R} \right)^N \left( \frac{R \| (1 + |\eta| \|_{\infty})^2 \| z \|}{R} \right)^p \\
\leq \frac{16 \| f \|_{R \| R \|_{\infty}}}{(1 + |\eta| \|_{\infty})^2 (1 - 2 |\eta| \| R \|_{\infty} \| z \| / R)} \left( \frac{R \| (1 + |\eta| \|_{\infty})^2 \| z \|}{R} \right)^N.
\]

If we set

\[
R = R_{\eta, K} := 4 (1 + |\eta| \|_{\infty})^2 R_{\eta}^2 \sup_{z \in K} \| z \|,
\]

in particular \( 2 |\eta| \| R \|_{\infty} \| z \| / R_{\eta, K} \leq 1/2 < 1 \) and

\[
(3.14) \sup_{z \in K} |R_N(f; \eta)(z)| \leq \frac{32 R_{\eta} \| f \|_{R \| R \|_{\infty}} \sup_{z \in K} \| z \|}{(1 + |\eta| \|_{\infty})^2 4N} \rightarrow 0, \quad N \rightarrow \infty,
\]

and the proof of Theorem 1 is achieved.

Furthermore, (3.14) yields to a precision for the convergence of \( E_N(f; \eta) \).

Corollary 2. Assume that \( \{ \eta_j \}_{j \geq 1} \) satisfies (1.8). Let be \( K \subset \mathcal{O}(\mathbb{C}^2) \) (resp. \( K \subset \mathbb{C}^2 \)) a compact subset. Then there exists \( C_{K, K} \) such that, for all \( N \geq 1, \)

\[
\sup_{f \in K} \sup_{z \in K} |f(z) - E_N(f; \eta)(z)| \leq \frac{C_{K, K}}{4N}.
\]

Proof. It follows from above and the fact that \( f(z) = E_N(f; \eta)(z) + \sum_{k+l \geq N} a_k, z^k \eta^l, \) with \( |a_k, l| \leq \| f \|_{R} / R^{k+l}, \forall R \geq 1. \)

4. Proof of Theorems 2 and 3

4.1. Proof of Theorem 3 when \( \{ \eta_j \}_{j \geq 1} \) is bounded.
4.1.1. An equivalent condition for \( \{ \eta_j \} \) to be real-analytically interpolated. We begin with giving the following definition.

**Definition 2.** The (bounded) set \( \{ \eta_j \} \subset \mathbb{C} \) will be said of uniform exponential \( \Delta \) if there exist \( C, R \) such that, for all subsequence \( (j_k)_{k \geq 1} \) and for all \( p \geq 0 \),

\[
|\Delta_{p,(\eta_{j_1},\ldots,\eta_{j_0})}(\zeta \mapsto \zeta)(\eta_{j_{p+1}})| \leq C R^p.
\]

This condition looks like \((1.8)\) from Theorem \( \Pi \) with the difference that the constant \( R \) does not depend on the subsequence \( (\eta_{j_k})_{k \geq 1} \). This uniform condition for \( \{ \eta_j \} \) seems stronger, in particular as it is specified by the following result.

**Proposition 1.** The bounded set \( \{ \eta_j \} \subset \mathbb{C} \) is real-analytically interpolated if and only if it is of uniform exponential \( \Delta \).

We begin with the reciprocal sense of the equivalence.

**Lemma 14.** If \( \{ \eta_j \} \) is of uniform exponential \( \Delta \), then it is real-analytically interpolated.

**Proof.** We want to prove that, for all \( \zeta_0 \in \{ \eta_j \} \), there exist \( V \in \mathcal{V}(\zeta_0) \) and \( g \in \mathcal{O}(V) \) such that, \( \forall \eta_j \in V, \eta_j = g(\eta_j) \).

If \( \zeta_0 \) is isolated, then \( \zeta_0 = \eta_{j_0} \). Let be \( V \in \mathcal{V}(\eta_{j_0}) \) such that \( V \cap \{ \eta_j \}_{j \geq 1} = \{ \eta_{j_0} \} \).

One can choose the constant function \( g(\zeta) := \eta_{j_0} \).

Otherwise \( \zeta_0 \) is a limit point. Let be \( V = D(\zeta_0, 1/(4R)) \) and let be \( (\eta_{j_k})_{k \geq 1} \) a sequence that converges to \( \zeta_0 \). We can assume that \( \{ \eta_{j_k} \}_{k \geq 1} \subset V \) (by removing a finite number of points if necessary). Consider the Lagrange interpolation polynomial that is also by Lemma \( \Pi \)

\[
P_N(\zeta) = \sum_{p=0}^{N-1} \prod_{k=1}^{N} \frac{\zeta - \eta_{j_k}}{\eta_{j_p} - \eta_{j_k}} \Delta_{p,(\eta_{j_1},\ldots,\eta_{j_0})}(\zeta \mapsto \zeta)(\eta_{j_{p+1}}).
\]

For all \( \zeta \in V \) and \( k \geq 1 \), one has \( |\zeta - \eta_{j_k}| \leq (\zeta - \zeta_0) - (\eta_{j_k} - \zeta_0) \leq 1/(2R) \) then

\[
\sum_{p \geq 0} \sup_{\zeta \in V} \left| \prod_{k=1}^{p} (\zeta - \eta_{j_k}) \Delta_{p,(\eta_{j_1},\ldots,\eta_{j_0})}(\zeta \mapsto \zeta)(\eta_{j_{p+1}}) \right| \leq \sum_{p \geq 0} \left( \frac{1}{2R} \right)^p C R^p = 2 C R^p.
\]

The series \( \sum_{p \geq 0} \prod_{k=1}^{p} (\zeta - \eta_{j_k}) \Delta_{p,(\eta_{j_1},\ldots,\eta_{j_0})}(\zeta \mapsto \zeta)(\eta_{j_{p+1}}) \) is absolutely convergent on \( V \). The sequence \( (P_N)_{N \geq 1} \) uniformly converges on \( V \) to a function \( g_1 \in \mathcal{O}(V) \). Moreover

\[
\sup_{\zeta \in V} |g_1(\zeta)| \leq 2 C R
\]

and for all \( k \geq 1 \),

\[
g_1(\eta_{j_k}) = \lim_{N \to \infty} P_N(\eta_{j_k}) = \lim_{N \to \infty, N \geq k} P_N(\eta_{j_k}) = \eta_{j_k}.
\]
ie \( g_1 \) is a holomorphic and bounded function on \( V \) that interpolates the values \( \overline{\eta_j} \) on the points \( \eta_j \), \( k \geq 1 \).

Now if \( V \cap \{ \eta_j \}_{j \geq 1} = \{ \eta_j \}_{j \geq 1} \), the function \( g_1 \) satisfies the required conditions. Otherwise we set \( S_1 := (\eta_{j_k})_{k \geq 1} \), choose an element \( \eta_{j_0} \in V \setminus S_1 \) and set \( S_2 := (\eta_{j_0}, S_1) := (\eta_{j_0}, \eta_{j_1}, \ldots, \eta_{j_k}, \ldots) \). Then \( S_2 \subset V \) is another subsequence of \( \{ \eta_j \}_{j \geq 1} \) that converges to \( \zeta_0 \). One can construct the same sequence of Lagrange polynomials that converges to a function \( g_2 \in \mathcal{O}(V) \) (since \( \{ \eta_j \}_{j \geq 1} \) is of uniform exponential \( \Delta \)). With the same argument \( g_2 \) is bounded (by \( 2C_n \)) and interpolates the values \( \overline{\eta_j} \) on the points \( \eta_{j_0}, \eta_{j_1}, \ldots, \eta_{j_k}, \ldots \).

We can follow this process as long as there is \( \eta_j \in V \) that is not reached. If there is \( r \geq 1 \) such that \( S_r = V \cap \{ \eta_j \}_{j \geq 1} \), the associate function \( g_r \) will satisfy the required conditions. Otherwise we can construct a sequence \( \{ g_r \}_{r \geq 1} \) with \( S_1 = (\eta_{j_0}, S_{r-1}) \) and \( g_r \in \mathcal{O}(V) \), bounded that interpolates the values \( \overline{\eta_j} \) on \( S_r \). Since \( \{ \eta_j \}_{j \geq 1} \) is countable, for all \( \eta_j \in V \), there exists \( r \geq 1 \) such that \( \eta_j \in S_r \) and \( \forall s \geq r, g_s(\eta_j) = g_r(\eta_j) = \overline{\eta_j} \).

On the other hand the sequence \( \{ g_r \}_{r \geq 1} \) is uniformly bounded on \( V \) (by \( 2C_n \)). By the Stiltjes-Vitali-Montel Theorem, there is a subsequence \( \{ g_r \}_{r \geq 1} \) that uniformly converges on \( V \) to a function \( g_\infty \in \mathcal{O}(V) \). So \( \forall \eta_j \in V, \exists r \geq 1, \eta_j \in S_r \),

\[
g_\infty(\eta_j) = \lim_{l \to \infty} g_r(\eta_j) = \lim_{l \to \infty} g_r(\eta_j) = \lim_{l \to \infty} \overline{\eta_j} = \overline{\eta_j},
\]

ie the function \( g_\infty \) interpolates the values \( \overline{\eta_j} \) on all the points \( \eta_j \in V \).

Now we prove the first sense of the equivalence.

**Lemma 15.** If \( \{ \eta_j \}_{j \geq 1} \) is real-analytically interpolated, then it is of uniform exponential \( \Delta \).

**Proof.** For all \( \zeta \in \overline{\{ \eta_j \}_{j \geq 1}} \), reducing \( V_\zeta \) if necessary, we can assume that \( V_\zeta = D(\zeta, 3\varepsilon_\zeta) \). Since

\[
\overline{\{ \eta_j \}_{j \geq 1}} \subset \bigcup_{\zeta \in \overline{\{ \eta_j \}_{j \geq 1}}} D(\zeta, \varepsilon_\zeta)
\]

and \( \overline{\{ \eta_j \}_{j \geq 1}} \) is a compact subset, there exists a finite number \( \zeta_1, \ldots, \zeta_L \) such that

\[
\overline{\{ \eta_j \}_{j \geq 1}} \subset \bigcup_{l=1}^L D(\zeta_l, \varepsilon_{\zeta_l}).
\]

There also exists \( \varepsilon_0 \) such that, for all \( \zeta \in \overline{\{ \eta_j \}_{j \geq 1}}, \exists l, 1 \leq l \leq L, D(\zeta, \varepsilon_0) \subset D(\zeta_l, \varepsilon_{\zeta_l}).
\]

Now we begin with giving the proof in the following special case.

**Lemma 16.** Let be \( p \geq 1 \) and \( \eta_{j_1}, \ldots, \eta_{j_{p+1}} \), such that, for all \( 1 \leq k < l \leq p + 1 \), \( |\eta_{j_k} - \eta_{j_l}| < \varepsilon_0 \). Then \( \exists C_n, \varepsilon_\eta \) (that do not depend on \( p \)),

\[
\left| \Delta_p(\eta_{j_p}, \ldots, \eta_{j_1}) (\zeta) (\eta_{j_{p+1}}) \right| \leq \frac{C_n}{\varepsilon_\eta}.
\]

**Proof.** In particular, \( \eta_{j_2}, \ldots, \eta_{j_{p+1}} \in D(\eta_{j_1}, \varepsilon_0) \). On the other hand, \( \exists l, 1 \leq l \leq L, D(\eta_{j_1}, \varepsilon_0) \subset D(\zeta_l, \varepsilon_{\zeta_l}) \). Since the function \( g_{\zeta_l} \in \mathcal{O}(D(\zeta_l, 3\varepsilon_{\zeta_l})) \) interpolates the values \( \overline{\eta_j} \) on the points \( \eta_j \in D(\zeta_l, 3\varepsilon_{\zeta_l}) \), one has

\[
\Delta_p(\eta_{j_p}, \ldots, \eta_{j_{1}}) (\zeta \mapsto \overline{\eta_j}) (\eta_{j_{p+1}}) = \Delta_p(\eta_{j_p}, \ldots, \eta_{j_{1}}) (\zeta \mapsto g_{\zeta_l}(\zeta)) (\eta_{j_{p+1}}).
\]
Let consider for all $|\zeta - \zeta_i| < 3\varepsilon_\zeta$,

$$g_\zeta(\zeta) = \sum_{n \geq 0} a_n(\zeta)(\zeta - \zeta_i)^n$$

the Taylor expansion of $g_\zeta$ on $\zeta_i$. Since $\eta_j_1, \ldots, \eta_j_p, \eta_{j+p+1} \in D(\zeta, \varepsilon_\zeta)$, it follows by Lemmas 6 and 12 that

$$\left| \Delta_p(\eta_{j_1}, \ldots, \eta_{j_p}) (\zeta \mapsto g_\zeta(\zeta)) (\eta_{j+p+1}) \right| \leq$$

$$\leq \sum_{n \geq 0} |a_n(\zeta)| \sum_{l_1=0}^{n-p} |\eta_{j_1} - \zeta|^{n-p-l_1} \sum_{l_p=0}^{l_p-1} |\eta_{j_p} - \zeta|^{l_p-1} \cdot |\eta_{j_{p+1}} - \zeta|^n$$

$$\leq \sum_{n \geq 0} |a_n(\zeta)| \sum_{l_1=0}^{n-p} \sum_{l_p=0}^{l_p-1} \varepsilon_\zeta^{n-p-l_1} \sum_{l_p=0}^{l_p} \varepsilon_\zeta^{l_p-1} \cdot \sum_{n \geq 0} |a_n(\zeta)| \varepsilon_\zeta^{-p} \frac{n!}{p! (n-p)!}$$

$$= \frac{1}{p!} \frac{\partial^p}{\partial \zeta^p} \varepsilon_\zeta (\sum_{n \geq 0} |a_n(\zeta)| \varepsilon_\zeta^n) = \frac{1}{2\pi i} \int_{|t|=2\varepsilon_\zeta} \frac{\sum_{n \geq 0} |a_n(\zeta)| \varepsilon_\zeta^n dt}{(t - \zeta_\zeta)^p+1}$$

$$\leq 2\varepsilon_\zeta \sup \sum_{n \geq 0} |a_n(\zeta)| \varepsilon_\zeta^n = \frac{2}{\varepsilon_\zeta} \sum_{n \geq 0} |a_n(\zeta)| (2\varepsilon_\zeta)^n.$$ 

For all $l = 1, \ldots, L$, we set

$$M_\zeta := 2 \sum_{n \geq 0} |a_n(\zeta)| (2\varepsilon_\zeta)^n < +\infty,$$

$$C_\eta := \max_{1 \leq l \leq L} M_\zeta < +\infty$$

and

$$\varepsilon_\eta := \min \left\{ \varepsilon_0/2, \min_{1 \leq l \leq L} \varepsilon_\zeta \right\} > 0.$$

Thus

$$\left| \Delta_p(\eta_{j_1}, \ldots, \eta_{j_1}) (\zeta \mapsto g_\zeta(\zeta)) (\eta_{j+p+1}) \right| \leq \frac{C_\eta}{\varepsilon_\eta}.$$ 

Now we can give the proof in the general case by induction on $p \geq 0$ with the above choice of $C_\eta$, $\varepsilon_\eta$.

Let be $p = 0$ and $j_1 \geq 1$. Then $\exists l, 1 \leq l \leq L, \eta_{j_1} \in D(\zeta, \varepsilon_\zeta)$, thus

$$|\Delta_0(\zeta)(\eta_{j_1})| = |\eta_{j_1}| = |g_\zeta(\eta_{j_1})| \leq \sum_{n \geq 0} |a_n(\zeta)| |\eta_{j_1} - \zeta|^n$$

$$\leq \sum_{n \geq 0} |a_n(\zeta)| \varepsilon_\zeta^n \leq M_\zeta \leq C_\eta.$$
Now if it is true for \( p - 1 \geq 0 \), let be \( p \geq 1 \) and \( \eta_{j_1}, \ldots, \eta_{j_{p+1}} \in \{ \eta_j \}_{j \geq 1} \). If for all \( 1 \leq k < l \leq L \), \( |\eta_{j_k} - \eta_{j_l}| < \varepsilon_0 \), then it is still true by Lemma 16. Otherwise, \( \exists k, l \) with \( 1 \leq k < l \leq L \) such that \( |\eta_{j_k} - \eta_{j_l}| \geq \varepsilon_0 \), then by Lemma 4
\[
\left| \Delta_{p,(\eta_{j_1}, \ldots, \eta_{j_{p+1}})}(\zeta)(\eta_{j_{p+1}}) \right| = \left| \Delta_{p,(\eta_{j_k}, \ldots, \eta_{j_{p+1}})}(\zeta)(\eta_{j_l}) \right|
\leq \left| \Delta_{p-1,(\eta_{j_1}, \ldots, \eta_{j_{p+1}})}(\zeta)(\eta_{j_l}) - \Delta_{p-1,(\eta_{j_k}, \ldots, \eta_{j_{p+1}})}(\zeta)(\eta_{j_k}) \right|
\leq \frac{2C_\eta/\varepsilon_0^{-1}}{\varepsilon_0} \leq \frac{C_\eta}{\varepsilon_0^p}.
\]

4.1.2. Proof of Theorem 3 when \( \|\eta\|_\infty < +\infty \). Now we will give the proof of Theorem 3 in the special case when \( \{\eta_j\}_{j \geq 1} \) is bounded.

\textbf{Proof.} One has by (3.4) from the proof of Lemma 9
\[
R_N(f; \eta)(\zeta) = \sum_{p=0}^{N-1} \frac{\varepsilon_0^{-1-p}}{p!} \prod_{j=1}^{p} (z_1 - \eta_j z_2) \Delta_{p,(\eta_{j_1}, \ldots, \eta_{j_1})}(\zeta \mapsto r_N(\zeta, z))(\eta_{p+1}).
\]

We know that \( \forall \zeta_0 \in \{ \eta_j \}_{j \geq 1}, \exists V_{\zeta_0} \in \mathcal{V}(\zeta_0), g_{\zeta_0} \in O(V_{\zeta_0}), \text{ such that } \forall \eta_j \in V_{\zeta_0}, \eta_j = g_{\zeta_0}(\eta_j). \) In particular, \( \zeta_0 = g_{\zeta_0}(\zeta_0). \) Indeed, if \( \zeta_0 \) is isolated, \( \zeta_0 = \eta_j. \) Reducing \( V_{\zeta_0} \) if necessary, one has \( \{ \eta_j \}_{j \geq 1} \cap V_{\zeta_0} = \{ \eta_j \} \) then \( \zeta_0 = \eta_j = g_{\zeta_0}(\eta_j) = g_{\zeta_0}(\zeta_0). \) Otherwise \( \zeta_0 \) is a limit point so there is a subsequence \( (\eta_{j_k})_{k \geq 1} \) that converges to \( \zeta_0, \) then \( \zeta_0 = \lim_{k \to \infty} \eta_{j_k} = \lim_{k \to \infty} g_{\zeta_0}(\eta_{j_k}) = g_{\zeta_0}(\zeta_0). \)

In particular, \( \zeta_0 g_{\zeta_0}(\zeta_0) = |\zeta_0|^2 \geq 0 \) then reducing \( V_{\zeta_0} \) if necessary, \( \forall \zeta \in V_{\zeta_0}, \Re(\zeta g_{\zeta_0}(\zeta)) > -1/2. \) Finally, reducing \( V_{\zeta_0} \) again if necessary, one can choose \( V_{\zeta_0} = D(\zeta_0, 4\varepsilon_{\zeta_0}) \) such that
\[
\begin{align*}
\forall \eta_j \in V_{\zeta_0}, & \quad \eta_j = g_{\zeta_0}(\eta_j), \\
\forall \zeta \in V_{\zeta_0}, & \quad \Re(\zeta g_{\zeta_0}(\zeta)) > -1/2, \\
\|g_{\zeta_0}\|_{V_{\zeta_0}} := & \sup_{\zeta \in V_{\zeta_0}} |g_{\zeta_0}(\zeta)| < +\infty.
\end{align*}
\]

Now let consider for all \( \zeta \in V_{\zeta_0} \) and \( z \in \mathbb{C}^2 \),
\[
(4.1) \quad \tilde{r}_{N,\zeta_0}(\zeta, z) := \sum_{m \geq N} \left( \frac{z_2 + z_1 g_{\zeta_0}(\zeta)}{1 + \zeta g_{\zeta_0}(\zeta)} \right)^{m-N+1} \sum_{k+l=m} a_{k,l} \zeta^k.
\]

This function is well-defined for all \( N \geq 1, \zeta \in V_{\zeta_0} \) and \( z \in \mathbb{C}^2 \) since
\[
\sum_{m \geq N} \left| \left( \frac{z_2 + z_1 g_{\zeta_0}(\zeta)}{1 + \zeta g_{\zeta_0}(\zeta)} \right)^{m-N+1} \sum_{k+l=m} a_{k,l} \zeta^k \right| \leq
\]
\begin{align*}
\leq & \sum_{m \geq N} \left( \frac{\|z\|}{\|1 + \zeta g_\zeta(\zeta)\|} \right)^{m-N+1} \sum_{k+l=m} \|a_{k,l}\| |\zeta|^k \\
\leq & \sum_{m \geq N} \left( \frac{\|z\|}{\|R(1 + \zeta g_\zeta(\zeta))\|} \right)^{m-N+1} \sum_{k+l=m} \|f\|_R \|\eta\|_\infty^k \\
\leq & \|f\|_R \sum_{m \geq N} \frac{(2\|z\||(1 + \|g_\zeta\|_{V_{\zeta_0}}))}{(2\|z||\|1 + \|g_\zeta\|_{V_{\zeta_0}}))^{N-1}} \sum_{m \geq N} \left( \frac{2\|z\|(2 + \|\eta\|_\infty)(1 + \|g_\zeta\|_{V_{\zeta_0}})}{R} \right)^m \\
= & \frac{2\|f\|_R}{(2\|z||\|1 + \|g_\zeta\|_{V_{\zeta_0}}))^{N-1}} \frac{(2(2 + \|\eta\|_\infty))^{\|z||\|1 + \|g_\zeta\|_{V_{\zeta_0}})/R)}{1 - 2(2 + \|\eta\|_\infty)||z||\|1 + \|g_\zeta\|_{V_{\zeta_0}})/R} \left( \frac{2 + \|\eta\|_\infty}{R} \right)^N \\
\text{for all } R > 2(2 + \|\eta\|_\infty)||z||(1 + \|g_\zeta\|_{V_{\zeta_0}}). & \text{ Moreover } \tilde{r}_{N,\zeta_0} \in \mathcal{O}(V_{\zeta_0} \times \mathbb{C}^2) \text{ and for any compact subset } K \in \mathbb{C}^2 \text{ and all } R \geq 4(2 + \|\eta\|_\infty)||z||K(1 + \|g_\zeta\|_{V_{\zeta_0}}), \\
\sup_{(\zeta, z) \in V_{\zeta_0} \times K} |\tilde{r}_{N,\zeta_0}(\zeta, z)| & \leq \frac{4\|f\|_R||z||K(1 + \|g_\zeta\|_{V_{\zeta_0}})}{1 - 2(2 + \|\eta\|_\infty)||z||K(1 + \|g_\zeta\|_{V_{\zeta_0}})/R} \left( \frac{2 + \|\eta\|_\infty}{R} \right)^N \\
(4.3) & \leq 8\|f\|_R||z||K(1 + \|g_\zeta\|_{V_{\zeta_0}}) \left( \frac{2 + \|\eta\|_\infty}{R} \right)^N. \\
\text{Moreover, } \forall \eta_j \in V_{\zeta_0}, & \text{ (4.4)} \\
\tilde{r}_N(\eta_j, z) = \tilde{r}_{N,\zeta_0}(\eta_j, z). \\
\text{Now let consider the Taylor expansion of } \tilde{r}_{N,\zeta_0}(\cdot, z) \text{ on } V_{\zeta_0}, \\
\tilde{r}_{N,\zeta_0}(\zeta, z) = & \sum_{n \geq 0} a_n(N, \zeta_0, z)(\zeta - \zeta_0)^n. \\
\text{One has, for all } n \geq 0, \\
|a_n(N, \zeta_0, z)| = & \left| \frac{1}{2i\pi} \int_{|\zeta - \zeta_0|=3\varepsilon_{\zeta_0}} \tilde{r}_{N,\zeta_0}(\zeta, z) d\zeta \right| \leq \frac{\|\tilde{r}_{N,\zeta_0}(\cdot, z)||V_{\zeta_0}}{(3\varepsilon_{\zeta_0})^n}, \\
\text{then} \\
\sup_{z \in K} |a_n(N, \zeta_0, z)| & \leq \frac{\|\tilde{r}_{N,\zeta_0}(\cdot, z)||V_{\zeta_0} \times K}{(3\varepsilon_{\zeta_0})^n}. \\
\text{It follows that} \\
\sup_{z \in K} \sum_{n \geq 0} |a(N, \zeta_0, z)|(2\varepsilon_{\zeta_0})^n & \leq \sum_{n \geq 0} \|\tilde{r}_{N,\zeta_0}(\cdot, z)||V_{\zeta_0} \times K \left( \frac{2}{3} \right)^n \\
(4.5) & \leq 3\|\tilde{r}_{N,\zeta_0}(\cdot, z)||V_{\zeta_0} \times K =: M_{N,\zeta_0,K}. \\
\text{Since it is true for any } \zeta_0 \in \{\eta_j\}_{j \geq 1} \text{ and } \{\eta_j\}_{j \geq 1} \text{ is compact, there are } \zeta_1, \ldots, \zeta_L \text{ such that} \\
\{\eta_j\}_{j \geq 1} \subset \bigcup_{l=1}^L D(\zeta_l, \varepsilon_{\zeta_l}).
\end{align*}
Moreover, there is \( \varepsilon_0 > 0 \) such that, for all \( \zeta \in \{ \eta_j \}_{j \geq 1}, \exists \ell, 1 \leq \ell \leq L, D(\zeta, \varepsilon_0) \subset D(\zeta_L, \varepsilon_0). \) Set

\[
M_{N,K} := 2 \max_{1 \leq l \leq L} M_{N,\zeta_l,K}
\]

and

\[
\varepsilon := \min \left\{ \frac{1}{2}, \varepsilon_0/2, \min_{1 \leq l \leq L} \varepsilon_{\zeta_l} \right\}.
\]

For all \( R \geq 4(2 + \| \eta \|_\infty) \| z \|_K \left( 1 + \max_{1 \leq l \leq L} \| g_{\zeta_l} \|_{V_{\zeta_l}} \right), \)

one has by (4.3), (4.5) and (4.6),

\[
(4.8)
\]

\[
(4.9)
\]

\[
(4.10)
\]

Now we want to prove by induction on \( p \geq 0 \) that

\[
\sup_{z \in K} |\Delta_{p,(\eta_{p+1}, \ldots, \eta_1)}(\zeta \mapsto r_N(\zeta, z))(\eta_{p+1})| \leq \frac{M_{N,K}}{\varepsilon^p}. \]

If \( p = 0 \), let be \( \zeta_L \) such that \( \eta_1 \in D(\zeta_L, \varepsilon_{\zeta_L}). \) One has

\[
\sup_{z \in K} |\Delta_0(\zeta \mapsto r_N(\zeta, z))(\eta_1)| = \sup_{z \in K} |r_N(\eta_1, z)| = \sup_{z \in K} |\overline{r}_{N,\zeta_L}(\eta_1, z)| \leq \| \overline{r}_{N,\zeta_L} \|_{V_{\zeta_L} \times K} \leq M_{N,\zeta_L,K} \leq M_{N,K}.
\]

Now if it is true for \( p \geq 0 \), let be \( \eta_1, \ldots, \eta_{p+1}, \eta_{p+2} \). If there exist \( 1 \leq k < \ell \leq p+2 \) such that \( |\eta_k - \eta_\ell| \geq \varepsilon_0 \), then by permuting if necessary the \( \eta_j \), \( 1 \leq j \leq p+2 \), (that does not change \( \Delta_{p+1} \) by Lemma [7], we can assume that \( |\eta_{p+2} - \eta_{p+1}| \geq \varepsilon_0 \). Then

\[
\sup_{z \in K} |\Delta_{p+1,(\eta_{p+1}, \ldots, \eta_1)}(\zeta \mapsto r_N(\zeta, z))(\eta_{p+2})| \leq \frac{M_{N,K}}{\varepsilon^p} + \sup_{z \in K} |\Delta_{p,(\eta_{p+1}, \ldots, \eta_1)}(r_N(\zeta, z))(\eta_{p+2})| + \sup_{z \in K} |\Delta_{p,(\eta_{p+1}, \ldots, \eta_1)}(r_N(\zeta, z))(\eta_{p+1})|
\]

\[
\leq \frac{M_{N,K}/\varepsilon^p + M_{N,K}/\varepsilon^p}{\varepsilon_0} = \frac{M_{N,K}/\varepsilon^p}{\varepsilon_0/2} \leq \frac{M_{N,K}}{\varepsilon^p+1}.
\]

Otherwise, for all \( 1 \leq \ell < k \leq p+2 \), \( |\eta_k - \eta_\ell| < \varepsilon_0 \). In particular, \( \eta_2, \ldots, \eta_{p+1} \in D(\eta_1, \varepsilon_0). \) On the other hand, \( \exists \zeta_L, D(\eta_1, \varepsilon_0) \subset D(\zeta_L, \varepsilon_{\zeta_L}). \) Then \( \forall z \in K \), one has by (4.4), Lemmas [6] and [12]

\[
|\Delta_{p+1,(\eta_{p+1}, \ldots, \eta_1)}(r_N(\zeta, z))(\eta_{p+2})| = \]
Then one has by (4.7) and (4.9), for all $N \geq 1$,

$$\sup_{z \in K} |R_N(f;z)(z)| \leq \frac{2}{\varepsilon} \sup_{z \in K} \sum_{n \geq 0} |a_n(N, \xi(z), z)| (2\varepsilon)^n \leq \frac{2M_{N,K}}{\varepsilon+1} \leq \frac{M_{N,K}}{\varepsilon+1},$$

and this proves (4.10). Then one has by (4.7) and (4.9), for all $N \geq 1$,

$$\sup_{z \in K} |R_N(f;z)(z)| \leq \sup_{z \in K} \sum_{n=0}^{N-1} \left| z_l \right|^{N-1-p} \prod_{j=1}^{p} \left| z_1 - \eta_j z_2 \right| \left| \Delta_{p,(\eta_1, \ldots, \eta_n)}(\xi(z)) (\eta_{p+1}) \right|$$

$$\leq \sum_{p=0}^{N-1} \| z \|_{K}^{N-1-p} \prod_{j=1}^{p} \left( \| z \|_{K} \sqrt{1+|\eta_j|^2} \right) \left| \Delta_{p} \right|_{K} \left| r_{N}(\xi,z) \right| (\eta_{p+1})$$

$$\leq \| z \|_{K}^{N-1} \sum_{p=0}^{N-1} (1+\| \eta \|_{\infty})^{p} \frac{M_{N,K}}{\varepsilon^{p}} = M_{N,K} \| z \|_{K}^{N-1} \frac{(1+\| \eta \|_{\infty}/\varepsilon)^{N} - 1}{(1+\| \eta \|_{\infty}/\varepsilon - 1)}$$

$$\leq M_{N,K} \| z \|_{K}^{N-1} (1+\| \eta \|_{\infty}/\varepsilon)^{N}$$

$$\leq 48 \| f \|_{K} \left( 1 + \max_{1 \leq l \leq L} \| g_{l} \|_{V_{\xi(l)}} \right) \left( \| z \|_{K} (2+\| \eta \|_{\infty}) (1+\| \eta \|_{\infty}/\varepsilon) \right)^{N}. $$

It follows by (4.8) that if we fix

$$R = R_K := 4 \| z \|_{K} (2+\| \eta \|_{\infty})^{2} \left( 1 + \max_{1 \leq l \leq L} \| g_{l} \|_{V_{\xi(l)}} \right) / \varepsilon,$$

we get

$$\sup_{z \in K} |R_N(f;z)(z)| \leq 48 \| f \|_{R_K} \left( 1 + \max_{1 \leq l \leq L} \| g_{l} \|_{V_{\xi(l)}} \right) \frac{1}{4N} \xrightarrow{N \to \infty} 0,$$

and the proof is achieved.

$\sqrt{}$
4.2. Proof of Theorems 2 and 3. In this part we do not assume any more that \( \{ \eta_j \}_{j \geq 1} \) is bounded, else it is not dense in \( \mathbb{C} \). Then there exists \( \eta_{\infty} \in \mathbb{C} \), \( \exists V \in \mathcal{V}(\eta_{\infty}) \), \( \forall j \geq 1, \eta_j \notin V \). We can assume that \( \eta_{\infty} \neq \infty \) because otherwise it means that \( \{ \eta_j \}_{j \geq 1} \) is bounded and the results we want to prove have already been proved in Section 3 and Subsection 4.1. Then

\[
\exists \varepsilon_{\infty} > 0, \forall j \geq 1, |\eta_j - \eta_{\infty}| \geq \varepsilon_{\infty}.
\]

4.2.1. Proof of Theorem 2. Let consider

\[
U_{\eta_{\infty}} := \frac{1}{\sqrt{1 + |\eta_{\infty}|^2}} \begin{pmatrix} \eta_{\infty} & 0 \\ 1 & -\eta_{\infty} \end{pmatrix}.
\]

Then

\[
U_{\eta_{\infty}} \in \mathcal{U}(2, \mathbb{C}), \quad \forall j \geq 1, \quad \frac{1}{\sqrt{1 + |\eta_{\infty}|^2}} \begin{pmatrix} \eta_{\infty} & 0 \\ 1 & -\eta_{\infty} \end{pmatrix}.
\]

and

\[
U_{\eta_{\infty}} \left( \{ z_1 - \eta_{\infty} z_2 = 0 \} \right) = \{ z_2 = 0 \}, \quad \forall j \geq 1, \quad \frac{1}{\sqrt{1 + |\eta_{\infty}|^2}} \begin{pmatrix} \eta_{\infty} & 0 \\ 1 & -\eta_{\infty} \end{pmatrix}.
\]

We remind the definition of \( \theta_j \) associate to \( \eta_{\infty} \) (Introduction, (1.9)),

\[
\forall j \geq 1, \quad \theta_j = \frac{1 + \eta_{\infty} \eta_j}{\eta_j - \eta_{\infty}},
\]

and we begin with this preliminar result.

**Lemma 17.** The following assertions are equivalent:

1. the formula \( R_N(f; \eta) \) converges for every function \( f \in \mathcal{O}(\mathbb{C}^2) \)
2. \( \exists \eta_{\infty} \notin \{ \eta_j \}_{j \geq 1} \cup \{ \infty \} \) such that the formula \( R_N(f; \theta) \) (constructed with the associate \( \theta_j \)) converges for every \( f \in \mathcal{O}(\mathbb{C}^2) \)
3. \( \forall \eta_{\infty} \notin \{ \eta_j \}_{j \geq 1} \cup \{ \infty \} \), the formula \( R_N(f; \theta) \) converges for every \( f \in \mathcal{O}(\mathbb{C}^2) \).

**Proof.** We begin with reminding this equality (Introduction, Corollary 1): \( \forall f \in \mathcal{O}(\mathbb{C}^2), \forall N \geq 1 \) and \( \forall z \in \mathbb{B}_2, \)

\[
\prod_{j=1}^{N} (z_1 - \eta_j z_2) \lim_{\varepsilon \to 0} \frac{1}{(2\pi)^2} \int_{\Omega_{\varepsilon}} \frac{f(\zeta) \omega(\zeta)}{\prod_{j=1}^{N} (\zeta_1 - \eta_j \zeta_2)} (1 - \zeta_1 z_2)^2 = R_N(f; \eta)(z) - \sum_{k+l \geq N} a_{k,l} z_1^k z_2^l,
\]

with

\[
\Omega_{\varepsilon} = \left\{ \zeta \in \mathbb{B}_2, \left| \prod_{j=1}^{N} (\zeta_1 - \eta_j \zeta_2) \right| > \varepsilon \right\}.
\]
On the other hand,

\[
\prod_{j=1}^{N} (z_1 - \eta_j z_2) \lim_{\varepsilon \to 0} \frac{1}{(2\pi)^2} \int_{\Omega} \frac{f(\zeta) \omega'(\Omega) \wedge \omega(\zeta)}{\prod_{j=1}^{N} (\zeta_1 - \eta_j \zeta_2) (1 - <\zeta, z>)^2} = 
\]

\[
= \prod_{j=1}^{N} (z_1 - \eta_j z_2) \times 
\frac{1}{(2\pi)^2} \int_{\Omega} \frac{f(U^*_\eta \zeta) \omega(U^*_\eta U^*_\eta \zeta \wedge \omega(U^*_\eta U^*_\eta \zeta))}{\prod_{j=1}^{N} (1 - <U^*_\eta \zeta, z>)^2} 
\]

\[
= \prod_{j=1}^{N} (z_1 - \eta_j z_2) \times 
\frac{1}{(2\pi)^2} \int_{U^*_\eta \zeta} \frac{f(U^*_\eta \zeta) \omega(U^*_\eta \zeta) \wedge \omega(U^*_\eta \zeta)}{\prod_{j=1}^{N} (1 - <U^*_\eta \zeta, z>)^2}.
\]

Since

\[
\begin{cases}
(U^*_\eta \zeta)_1 = \frac{\eta_\infty z_1 + \zeta_2}{\sqrt{1 + |\eta_\infty|^2}} \\
(U^*_\eta \zeta)_2 = \frac{\zeta_1 - \eta_\infty \zeta_2}{\sqrt{1 + |\eta_\infty|^2}}
\end{cases}
\]

then

\[
\omega' (U^*_\eta \zeta) = \frac{1}{1 + |\eta_\infty|^2} \left[ (\eta_\infty \zeta_1 + \zeta_2)(d\zeta_1 - \eta_\infty d\zeta_2) - (\zeta_1 - \eta_\infty \zeta_2)(\eta_\infty d\zeta_1 + d\zeta_2) \right]
\]

\[
= \frac{1}{1 + |\eta_\infty|^2} \left[ (1 + |\eta_\infty|^2) \zeta_2 d\zeta_1 + (-1 - |\eta_\infty|^2) \zeta_1 d\zeta_2 \right]
\]

\[
= -\omega' (\zeta),
\]

and

\[
\omega (U^*_\eta \zeta) = \frac{1}{2} d(\omega' (U^*_\eta \zeta)) = -\frac{1}{2} d\omega' (\zeta) = -\omega (\zeta) = -\omega(\zeta).
\]

One has also, \(\forall j = 1, \ldots, N,\)

\[
(U^*_\eta \zeta)_1 - \eta_j (U^*_\eta \zeta)_2 = \frac{1}{\sqrt{1 + |\eta_\infty|^2}} ((\eta_\infty - \eta_j) \zeta_1 + (1 + \eta_\infty \eta_j) \zeta_2)
\]

\[
= \frac{\eta_\infty - \eta_j}{\sqrt{1 + |\eta_\infty|^2}} (\zeta_1 - \theta_j \zeta_2).
\]

Moreover, since \(U^*_\eta \in \mathcal{U}(2, \mathbb{C}),\)

\[
< U^*_\eta \zeta, z > = < U^*_\eta \zeta, U^*_\eta z >
\]

\[
= < \zeta, U^*_\eta z >
\]

and \(\forall j = 1, \ldots, N,\)

\[
z_1 - \eta_j z_2 = (U^*_\eta z)_1 - (U^*_\eta z)_2
\]

\[
= \frac{\eta_\infty - \eta_j}{\sqrt{1 + |\eta_\infty|^2}} ((U^*_\eta z)_1 - \theta_j (U^*_\eta z)_2).
\]
Finally, since \( U_{\eta_{\infty}}^{-1}(S_2) = S_2 \), one has

\[
U_{\eta_{\infty}}(\Omega_z) = (U_{\eta_{\infty}}^{-1})(\Omega_z)
\]

\[
= \left\{ \zeta \in S_2, \prod_{j=1}^{N} \left| (U_{\eta_{\infty}}^{-1})(U_{\eta_{\infty}} \zeta) \right| > \varepsilon \right\}
\]

\[
= \left\{ \prod_{j=1}^{N} \frac{|\eta_{\infty} - \eta_j|}{\sqrt{1 + |\eta_{\infty}|^2}} \prod_{j=1}^{N} (\zeta_1 - \theta_j \zeta_2) > \varepsilon \right\}
\]

\[
= \left\{ \zeta \in S_2, \prod_{j=1}^{N} (\zeta_1 - \theta_j \zeta_2) > \varepsilon/C_N \right\}
\]

\[
= \Omega'_{\varepsilon/C_N}.
\]

It follows from (4.15) that

\[
R_N(f; \eta)(z) - \sum_{k+l \geq 0} \frac{1}{k!l!} \frac{\partial^{k+l}f}{\partial z_1^k \partial z_2^l} (0) z_1^k z_2^l
\]

\[
= \prod_{j=1}^{N} (\zeta_1 - \eta_j \zeta_2) \lim_{\varepsilon \to 0} \frac{1}{C_N} \int_{\Omega'_{\varepsilon/C_N}} \frac{f(U_{\eta_{\infty}} z) - \theta_j (U_{\eta_{\infty}} z)_{2}}{\prod_{j=1}^{N} (\zeta_1 - \theta_j \zeta_2) (1 - \zeta, \eta_{\infty}, z > 2)}
\]

\[
\times \lim_{\varepsilon \to 0} \frac{1}{C_N} \int_{\Omega'_{\varepsilon/C_N}} \frac{f(U_{\eta_{\infty}} z)(\zeta - \omega(\zeta)) \omega(\zeta)}{\prod_{j=1}^{N} (\zeta_1 - \theta_j \zeta_2) (1 - \zeta, \eta_{\infty}, z > 2)}
\]

\[
= \prod_{j=1}^{N} (\zeta_1 - \eta_j \zeta_2) \lim_{\varepsilon \to 0} \frac{1}{C_N} \int_{\Omega'_{\varepsilon/C_N}} \frac{(f \circ U_{\eta_{\infty}})(\zeta) \omega(\zeta)}{\prod_{j=1}^{N} (\zeta_1 - \theta_j \zeta_2) (1 - \zeta, \eta_{\infty}, z > 2)}
\]

\[
= R_N(f \circ U_{\eta_{\infty}} : \theta) (U_{\eta_{\infty}} z) - \sum_{k+l \geq 0} \frac{1}{k!l!} \frac{\partial^{k+l} f \circ U_{\eta_{\infty}}}{\partial z_1^k \partial z_2^l} (0) (U_{\eta_{\infty}} z)_1^k (U_{\eta_{\infty}} z)_2^l.
\]

Since \( f \circ U_{\eta_{\infty}} \in \mathcal{O}(\mathbb{C}^2) \) and for any compact subset \( K \subset \mathbb{C}^2 \), \( U_{\eta_{\infty}}(K) \) is still compact, one has

\[
\sup_{z \in K} \left\| \sum_{k+l \geq 0} \frac{1}{k!l!} \frac{\partial^{k+l} f}{\partial z_1^k \partial z_2^l} (0) z_1^k z_2^l \right\| \to 0,
\]

and

\[
\sup_{z \in K} \left\| \sum_{k+l \geq 0} \frac{1}{k!l!} \frac{\partial^{k+l} (f \circ U_{\eta_{\infty}})}{\partial z_1^k \partial z_2^l} (0) (U_{\eta_{\infty}} z)_1^k (U_{\eta_{\infty}} z)_2^l \right\| \to 0.
\]

Therefore \( R_N(f; \eta) \) converges for every function \( f \in \mathcal{O}(\mathbb{C}^2) \) if and only if \( R_N(f \circ U_{\eta_{\infty}} : \theta)(U_{\eta_{\infty}} z) \) converges for every \( f \in \mathcal{O}(\mathbb{C}^2) \), then if and only if \( R_N(f; \theta) \) converges for every
function \( f \in \mathcal{O}(\mathbb{C}^2) \). Moreover, this equivalence is true for all \( \eta_\infty \notin \{ \eta_j \}_{j \geq 1} \cup \{ \infty \} \).

This yields to (2) \( \Rightarrow \) (1) \( \Rightarrow \) (3) (and (3) \( \Rightarrow \) (2) since \( \mathbb{C} \setminus \{ \eta_j \}_{j \geq 1} \neq \emptyset \)).

Now we see that if \( |\eta_j| \leq 2|\eta_\infty| \), then by (4.11)

\[
|\theta_j| = \frac{1 + \eta_\infty \eta_j}{|\eta_j - \eta_\infty|} \leq \frac{1 + 2|\eta_\infty|^2}{\varepsilon_\infty} < +\infty.
\]

Otherwise \( |\eta_j| > 2|\eta_\infty| \) then

\[
|\theta_j| = \frac{|\eta_\infty| + 1/|\eta_j|}{|1 - \eta_\infty/\eta_j|} \leq \frac{|\eta_\infty| + 1/(2|\eta_\infty|)}{1 - 1/2} = 2 \left( |\eta_\infty| + 1/(2|\eta_\infty|) \right) < +\infty,
\]

and as long as \( \eta_\infty \neq 0 \); if it is not the case, one has \( |\eta_j| \geq \varepsilon_\infty \), \( \forall j \geq 1 \), then

\[
|\theta_j| = 1/|\eta_j| \leq 1/\varepsilon_\infty < +\infty.
\]

It follows that

\[
(4.17) \quad ||\theta||_\infty = \sup_{j \geq 1} |\theta_j| < +\infty.
\]

On the other hand, if we admit that for \( \eta_\infty = 0 \) (i.e. \( ||\eta||_\infty < +\infty \)) one has

\[
\theta_j := \lim_{x \to \infty, x > 0} \frac{1 + i\varepsilon \eta_j}{\eta_j - i\varepsilon},
\]

then \( \theta_j = \eta_j \). Moreover, if we more generally choose

\[
\theta_j := \lim_{x \to \infty, x > 0} \frac{1 + e^{i\varphi} \eta_j}{\eta_j - e^{i\varphi} x},
\]

then \( \theta_j = -e^{-2i\varphi} \eta_j \) and one still has

\[
\left| \Delta_{p_i}(\theta_{p_1},\ldots,\theta_1) \left[ \left( \frac{\zeta}{1 + |\zeta|^2} \right)^q \right] (\theta_{p+1}) \right| = \left| \Delta_{p_i}(-e^{-2i\varphi} \eta_{p_1},\ldots,-e^{-2i\varphi} \eta_1) \left[ \left( \frac{\zeta}{1 + |\zeta|^2} \right)^q \right] (-e^{-2i\varphi} \eta_{p+1}) \right|
\]

(4.18)

This allows us to give the proof of Theorem 2.

**Proof.** Let be \( \eta_\infty \notin \{ \eta_j \}_{j \geq 1} \cup \{ \infty \} \) and the associate \( \{ \theta_j \}_{j \geq 1} \). We know by Lemma [17] that \( R_N(f; \eta) \) converges for every \( f \in \mathcal{O}(\mathbb{C}^2) \) if and only if so does \( R_N(f; \theta), \forall f \in \mathcal{O}(\mathbb{C}^2) \). On the other hand, \( \{ \theta_j \}_{j \geq 1} \) being bounded by (4.17), it follows by Theorem [1] that it is true if and only if \( \exists R_\theta, \forall p, q \geq 0, \)

\[
\left| \Delta_{p_i}(\theta_{p_1},\ldots,\theta_1) \left[ \left( \frac{\zeta}{1 + |\zeta|^2} \right)^q \right] (\theta_{p+1}) \right| \leq R_\theta^{p+q}.
\]

\[
\checkmark
\]

**Remark 4.1.** It follows from (4.18) that Theorem 2 can be extended in the case when \( \eta_\infty = \infty \), so it is an extension of Theorem 1.
4.2.2. Proof of Theorem 3

We begin with specifying a point about a real-analytically interpolated set \( \{\eta_j\}_{j \geq 1} \).

Remark 4.2. If \( \{\eta_j\}_{j \geq 1} \) is not bounded, then \( \zeta_0 = \infty \) is a limit point. Let be \( (\eta_k)_{k \geq 1} \) a subsequence that converges to \( \infty \), then from Definition 1 \( g_\infty(\infty) = \lim_{k \to \infty} g_\infty(\eta_k) = \lim_{k \to \infty} \eta_k = \infty \). It follows that the associate function \( g_\infty \) is holomorphic from a neighborhood of \( \infty \) to a neighborhood of \( \infty \), ie the function

\[
g_\infty : V_0 \to V'_0
\]

\[
\zeta \mapsto \begin{cases} 
1 & \text{if } \zeta \neq 0, \\
0 & \text{if } \zeta = 0
\end{cases}
\]

(with \( V_0, V'_0 \in \mathcal{V}(0) \)), is holomorphic.

In Definition 1 we do not need to assume that \( \{\eta_j\}_{j \geq 1} \) is not dense. The following result specifies that it cannot be the case.

Lemma 18. Let \( \{\eta_j\}_{j \geq 1} \subset \mathbb{C} \) be any subset. If it is real-analytically interpolated, then it is not dense.

Proof. On the contrary, assume that \( \{\eta_j\}_{j \geq 1} \) is dense. Then in particular \( 0 \) is limit point. Let be the associate \( V \in \mathcal{V}(0) \) and \( g \in \mathcal{O}(V) \). Then \( V \subset \{\eta_j\}_{j \geq 1} \) and for all \( \eta_j \) close to \( 0 \) one has

\[
\frac{\eta_j - 0}{\eta_j} = \frac{\eta_j}{\eta_j} = \frac{g(\eta_j) - g(0)}{\eta_j - 0} \xrightarrow[\eta_j \to 0]{} \frac{\partial g}{\partial \zeta}(0).
\]

In particular \( |\frac{\partial g}{\partial \zeta}(0)| = 1 \) then one has \( \frac{\partial g}{\partial \zeta}(0) = e^{i\theta} \). We set

\[
w_p = \frac{1}{p} \ e^{-i\theta/2}
\]

with \( p \) large enough such that \( w_p \in V \). Let \( (\eta_{jp})_{p \geq p_0} \) be a subsequence of \( \{\eta_j\}_{j \geq 1} \) such that, for all \( p \geq p_0 \), \( w_p \in V \) and

\[
|\eta_{jp} - w_p| \leq \frac{1}{2p^2}.
\]

Then (since \( (\eta_{jp})_{p \geq p_0} \) converges to \( 0 \))

\[
e^{i\theta} = \frac{\partial g}{\partial \zeta}(0) = \lim_{p \to \infty} \frac{\eta_{jp}}{w_p} = \lim_{p \to \infty} \frac{w_p + O(1/p^2)}{w_p + O(1/p^2)} = \lim_{p \to \infty} \frac{-i e^{i\theta/2} + O(1/p)}{i e^{-i\theta/2} + O(1/p)} = -e^{i\theta},
\]

and that is impossible.

\( \square \)

The following result will be usefull for the proof of Theorem 3.

Lemma 19. Assume that \( \{\eta_j\}_{j \geq 1} \) is not bounded and real-analytically interpolated. Then for all \( \eta_\infty \notin \{\eta_j\}_{j \geq 1} (\neq \mathbb{C}) \), the associate bounded subset \( \{\theta_j\}_{j \geq 1} \) is real-analytically interpolated.
Proof. Let be any $\eta_\infty \notin \{\eta_j\}_{j \geq 1}$ (such an element exists by Lemma 18). Necessarily
$\eta_\infty \neq \infty$. By definition, $\theta_j = h(\eta_j)$, where
\[(4.19) \quad h : \overline{\mathbb{C}} \to \overline{\mathbb{C}} \quad \quad z \mapsto \frac{\eta_\infty z + 1}{z - \eta_\infty},\]
is homographic. In particular, $h$ is bicontinuous from $\overline{\mathbb{C}}$ to $\overline{\mathbb{C}}$ then there is a correspondence
between the limit (resp. isolated) points of $\{\eta_j\}_{j \geq 1}$ and the ones of $\{\theta_j\}_{j \geq 1}$.

Now let be $w_0 \in \{\theta_j\}_{j \geq 1}$. $\exists! \zeta_0 \in \{\eta_j\}_{j \geq 1}$, $w_0 = h(\zeta_0)$. One has $\zeta_0 = \infty$ if and
only if $w_0 = \eta_\infty$. First, assume that $\zeta_0 \neq \infty$ and let be the associate $V_{\zeta_0} \in \mathcal{V}(\zeta_0)$,
$g_{\zeta_0} \in \mathcal{O}(V_{\zeta_0})$. In particular, $g_{\zeta_0}(\zeta_0) = \eta_\infty$. One has, for all $\eta_j \in V_{\zeta_0}$,
\[\overline{\theta_j} = \frac{1 + \eta_\infty \eta_j}{\eta_j - \eta_\infty} = \frac{1 + \eta_\infty g_{\zeta_0}(\eta_j)}{g_{\zeta_0}(\eta_j) - \eta_\infty}.\]
On the other hand, one has $\eta_j = h^{-1}(\theta_j)$, with
\[h^{-1} : \mathbb{C} \to \mathbb{C} \quad \quad w \mapsto \frac{\eta_\infty w + 1}{w - \eta_\infty}.\]
Notice that $h^{-1}(w) = \overline{h(w)}$, then
\[\overline{\theta_j} = h^{-1}(g_{\zeta_0}(\eta_j)).\]
Finally, for all $\theta_j \in (h^{-1})^{-1}(V_{\zeta_0}) = h(V_{\zeta_0})$,
\[(4.20) \quad \overline{\theta_j} = h^{-1} \left[ g_{\zeta_0} \left( h^{-1}(\theta_j) \right) \right].\]
Since $\text{dist} \left( \{\eta_j\}_{j \geq 1}, \eta_\infty \right) \geq \varepsilon_\infty$ and $w_0 \neq \eta_\infty$, then $\exists V_{w_0} \in \mathcal{V}(w_0)$, $\forall w \in V_{w_0}$, $w \neq \eta_\infty$. One the other hand, $g_{\zeta_0}(\zeta_0) = \eta_\infty \neq \eta_\infty$, then by reducing $V_{\zeta_0}$ if necessary, one can assume that, $\forall \zeta \in V_{\zeta_0}$, $g_{\zeta_0}(\zeta) \neq \eta_\infty$. Finally, $\exists W_{w_0}$, $\forall w \in W_{w_0}$, $h^{-1}(w) \in V_{\zeta_0}$.

This allows us to define
\[g_{w_0} : W_{w_0} \to \mathbb{C} \quad \quad w \mapsto h^{-1} \left[ g_{\zeta_0} \left( h^{-1}(w) \right) \right],\]
the composed function of
\[\left( \begin{array}{ccc}
W_{w_0} & \rightarrow & V_{\zeta_0} \\
\zeta & \mapsto & g_{\zeta_0}(\zeta)
\end{array} \right), \left( \begin{array}{ccc}
V_{\zeta_0} & \rightarrow & \mathbb{C} \setminus \{\eta_\infty\} \\
\eta_j & \mapsto & g_{\zeta_0}(\eta_j)
\end{array} \right) \quad \text{and} \quad \left( \begin{array}{ccc}
\mathbb{C} \setminus \{\eta_\infty\} & \rightarrow & \mathbb{C} \\
u & \mapsto & h^{-1}(u)
\end{array} \right).\]
It follows that $g_{w_0}$ is holomorphic and satisfies by (4.20): $\forall \theta_j \in W_{w_0}$, $g_{w_0}(\theta_j) = \overline{\theta_j}$.

Now assume that $\zeta_0 = \infty$ (then $w_0 = \eta_\infty$) and let be the associate $V_\infty \in \mathcal{V}(\infty)$,
$g_\infty : V_\infty \to \overline{\mathbb{C}}$, holomorphic that satisfies: $\forall \eta_j \in V_\infty$, $\overline{\eta_j} = g_\infty(\eta_j)$. On the other
hand, let be $W_0, V_0 \in \mathcal{V}(0)$ such that
\[(4.21) \quad \widetilde{g}_\infty : W_0 \to V_0 \quad \quad \zeta \mapsto \begin{cases}
\frac{1}{g_\infty(1/\zeta)} & \text{if } \zeta \neq 0, \\
0 & \text{if } \zeta = 0,
\end{cases}\]
is holomorphic. By reducing $V_0$ and $W_0$ if necessary, one can assume that, $\forall \zeta \in W_0 \setminus \{0\}$, $1/\zeta \in V_{\infty}$ and $\forall w \in V_0$, $1 - \frac{1}{\eta_{\infty}}w \neq 0$.

Now let be $W_\infty \in \mathcal{V}(\infty)$ such that $W_\infty \subset V_\infty$, $0 \notin W_\infty$ and $\forall \zeta \in W_\infty$, $1/\zeta \in W_\infty$.

Since we still have, $\forall \eta_j \in W_\infty \subset V_\infty$, $\eta_j = h^{-1}(\theta_j) = \frac{\eta_{\infty} \theta_j + 1}{\theta_j - \eta_{\infty}}$, we get, for all $\theta_j \in h(V_\infty)$,

$$\theta_j = h_\infty \left( \frac{1}{h^{-1}(\theta_j)} \right),$$

with

$$h_\infty : \mathbb{C} \to \mathbb{C}, \quad w \mapsto \frac{\eta_{\infty} + w}{1 - \eta_{\infty} w}.$$

In particular, the restriction $h_\infty : V_0 \to \mathbb{C}$, is still holomorphic. If we choose $V_\infty \in \mathcal{V}(\eta_{\infty})$ such that $V_\infty \subset h(V_\infty)$ and, $\forall w \in V_\infty$, one has $1/h^{-1}(w) = \frac{w - \eta_{\infty}}{\eta_{\infty} w + 1} \in W_0$ and $h(w) = \frac{\eta_{\infty} w + 1}{w - \eta_{\infty}} \in W_\infty$, the function

$$g_\infty : V_\infty \to \mathbb{C}, \quad w \mapsto \begin{cases} h_\infty \left( \frac{1}{h^{-1}(w)} \right) & \text{if } w \neq \eta_{\infty}, \\ h_\infty(\eta_{\infty}) = \eta_{\infty} & \text{if } w = \eta_{\infty}, \end{cases}$$

is holomorphic and satisfies by (4.22): $\forall \theta_j \in V_\infty$, $g_\infty(\theta_j) = \overline{\theta_j}$. \hfill \checkmark

**Remark 4.3.** We could also prove that the assertion is reciprocal but it will not be useful for the result that we want to prove.

Now we can give the proof of Theorem 3.

*Proof.* Let $\{\eta_j\}_{j \geq 1}$ be a real-analytically interpolated subset. If it is bounded, then Theorem 3 follows by Section 4, Subsection 4.1. Otherwise, we know by Lemma 18 that $\{\eta_j\}_{j \geq 1}$ cannot be dense, then there is $\eta_{\infty} \notin \{\eta_j\}_{j \geq 1} \cup \{\infty\}$. Let consider the associate bounded subset $\{\theta_j\}_{j \geq 1}$. Thus by Lemma 19, $\{\theta_j\}_{j \geq 1}$ is bounded and real-analytically interpolated. It follows by Subsection 4.1 that $R_N(f; \theta)$ converges for every function $f \in \mathcal{O}(\mathbb{C}^2)$. Finally, by Lemma 17, $R_N(f; \eta)$ (then $E_N(f; \eta)$) converges for every $f \in \mathcal{O}(\mathbb{C}^2)$, and the theorem is proved. \hfill \checkmark
5. A COUNTEREXAMPLE

Let consider the subset \( \mathbb{R} \cup i\mathbb{R} \). It is a union of real-analytical manifolds of \( \mathbb{C} \) (\( \mathbb{R} \) and \( i\mathbb{R} \)), but is not a manifold (problem on \( 0 \)). Here we will deal with which can happen when the sequence \( (\eta_j)^2 \geq 1 \) converges to \( 0 \) without staying in one or the other line (see Introduction) and show the following result.

**Proposition 2.** There exists a (bounded) subset \( \{\eta_j\}_{j \geq 1} \subset \mathbb{R} \cup i\mathbb{R} \) that does not satisfy the condition (1.8). It follows by Theorem 7 that there is (at least) a function \( f \in \mathcal{O}(\mathbb{C}) \) such that the formula \( E_N(f;\eta) \) cannot converge.

We begin with the following result that gives the construction in a more general case.

**Lemma 20.** Let \( f \) be a function of class \( C^2 \) in a neighborhood \( V \) of \( 0 \) and that is not \( \mathbb{C} \)-differentiable on \( 0 \), ie
\[
\frac{\partial f}{\partial \zeta}(0) \neq 0.
\]
Then there exists a bounded sequence \( (\eta_j)^2 \geq 1 \subset \mathbb{R} \cup i\mathbb{R} \) such that, for all \( p \geq 1 \),
\[
|\Delta_{3p-1,(\eta_{3p-1},\ldots,\eta_1)}(\zeta) \mapsto f(\zeta))(\eta_{3p})| \geq p^p.
\]

**Proof.** For all different and nonzero \( \eta_1, \ldots, \eta_p \in \mathbb{R} \cup i\mathbb{R} \), the function \( \zeta \mapsto \Delta_{p,(\eta_p,\ldots,\eta_1)}(f)(\zeta) \) is still of class \( C^2 \) on \( V \setminus \{\eta_1, \ldots, \eta_p\} \). Moreover,
\[
\frac{\partial}{\partial \zeta} \Delta_p(f)(\zeta) = \frac{\partial}{\partial \zeta} \left[ \frac{\Delta_{p-1}(f)(\zeta) - \Delta_{p-1}(f)(\eta_p)}{\zeta - \eta_p} \right] = \frac{1}{\zeta - \eta_p} \frac{\partial}{\partial \zeta} \Delta_{p-1}(f)(\zeta) = \cdots = \frac{\partial}{\partial \zeta} \Delta_0(f)(\zeta) = \frac{\partial f}{\partial \zeta}(\zeta).
\]
Now let be \( \eta_1, \ldots, \eta_{3p-1}, \eta_{3p} \) all different and nonzero, and let be \( \eta_{3p+1} \in (\mathbb{R} \cup i\mathbb{R}) \cap (V \setminus \{0, \eta_1, \ldots, \eta_{3p}\}) \). Since \( \frac{\partial f}{\partial \zeta}(0) \neq 0 \), one has
\[
\frac{\partial}{\partial \zeta} \Delta_{3p+1}(f)(0) = \frac{\partial f}{\partial \zeta}(0) = \frac{(-\eta_{3p+1})(-\eta_{3p}) \cdots (-\eta_1)}{\eta_1 \cdots \eta_{3p} \eta_{3p+1}} \frac{\partial f}{\partial \zeta}(0) \xrightarrow{\eta_{3p+1} \to \infty} \infty.
\]
Let fix \( \eta_{3p+1} \in \mathbb{R} \cup i\mathbb{R} \) with \( 0 < |\eta_{3p+1}| < \min_{1 \leq i \leq 3p} |\eta_i| \) (then \( \eta_{3p+1} \in V \setminus \{0, \eta_1, \ldots, \eta_{3p}\} \)) and such that
\[
(5.1) \quad \left| \frac{\partial}{\partial \zeta} \Delta_{3p+1}(f)(0) \right| \geq (p+1)^{p+1} + 1.
\]
Now for all $\eta_{3p+2}, \eta_{3p+3} \in \mathbb{R} \cup i\mathbb{R}$, different and such that $0 < |\eta_{3p+2}|, |\eta_{3p+2}| < \min_{1 \leq i \leq 3p+1} |\eta_i|$, one has

$$\Delta_{3p+2}(\eta_{3p+2}; \eta_{3p+3}, \ldots, \eta_1)(f)(\eta_{3p+3}) =$$

$$= \frac{\Delta_{3p+1}(\eta_{3p+1}, \ldots, \eta_1)(f)(\eta_{3p+3}) - \Delta_{3p+1}(\eta_{3p+1}, \ldots, \eta_1)(f)(\eta_{3p+2}) + \Delta_{3p+1}(\eta_{3p+1}, \ldots, \eta_1)(f)(\eta_{3p+3})}{\eta_{3p+3} - \eta_{3p+2}}$$

$$- \frac{1}{\eta_{3p+3} - \eta_{3p+2}} \left[ \eta_{3p+3} \frac{\partial}{\partial \zeta} \Delta_{3p+1}(f)(0) + \frac{\partial}{\partial \zeta} \Delta_{3p+1}(f)(0) + O(\eta_{3p+3}) \right]$$

$$- \frac{1}{\eta_{3p+3} - \eta_{3p+2}} \left[ \eta_{3p+2} \frac{\partial}{\partial \zeta} \Delta_{3p+1}(f)(0) + \frac{\partial}{\partial \zeta} \Delta_{3p+1}(f)(0) + O(\eta_{3p+2}) \right]$$

$$= \frac{\partial}{\partial \zeta} \Delta_{3p+1}(f)(0) + \frac{\eta_{3p+3} - \eta_{3p+2}}{\eta_{3p+3} - \eta_{3p+2}} \frac{\partial}{\partial \zeta} \Delta_{3p+1}(f)(0) + O(\eta_{3p+3}) + O(\eta_{3p+2}).$$

On the other hand, since $\Delta_{3p+1}(f)(\zeta)$ is of class $C^2$ on $V \setminus \{\eta_1, \ldots, \eta_{3p+1}\}$, there is $\epsilon_{p+1} > 0$ small enough with $0 < \epsilon_{p+1} < \min_{1 \leq i \leq 3p+1} |\eta_i|$, $D(0, \epsilon_{p+1}) \subset V$, such that

$$M_{p+1} := \sup_{0 < |\zeta| \leq \epsilon_{p+1}} \left| \frac{O(\zeta^2)}{\zeta^2} \right| < +\infty$$

$(\epsilon_{p+1}$ and $M_{p+1}$ will only depend on $f$ and $\eta_1, \ldots, \eta_{3p+1}$).

Now we assume that $\frac{\partial}{\partial \zeta} \Delta_{3p+1}(f)(0) \neq 0$. Then

$$\frac{\partial}{\partial \zeta} \Delta_{3p+1}(f)(0) = \frac{\partial}{\partial \zeta} \Delta_{3p+1}(f)(0) e^{i\varphi_1} = \frac{\partial}{\partial \zeta} \Delta_{3p+1}(f)(0) e^{i\varphi_2}.$$

Three cases can happen.

1. If $e^{i\theta} = 1$, we choose $\eta_{3p+2}, \eta_{3p+3} \in \mathbb{R}$ with $\eta_{3p+3} = -\eta_{3p+2}$ and $0 < |\eta_{3p+2}| = |\eta_{3p+3}| \leq \min(\epsilon_{p+1}, 1/M_{p+1})$. One has

$$\frac{O(\eta_{3p+3}^2) + O(\eta_{3p+2}^2)}{\eta_{3p+3} - \eta_{3p+2}} \leq \frac{2M_{p+1} \eta_{3p+2}^2}{2|\eta_{3p+2}|} = M_{p+1} |\eta_{3p+2}| \leq 1,$$
then by (5.1)

$$\left| \Delta_{3p+2}(f)(\eta_{3p+3}) \right| \geq \left| \frac{\partial}{\partial \zeta} \Delta_{3p+1}(f)(0) + \frac{\eta_{3p+3} - \eta_{3p+2}}{\eta_{3p+3} - \eta_{3p+2}} \frac{\partial}{\partial \zeta} \Delta_{3p+1}(f)(0) \right| - \frac{O(\eta_{3p+3}) + O(\eta_{3p+2})}{\eta_{3p+3} - \eta_{3p+2}} - 1$$

$$\geq \left| \frac{\partial}{\partial \zeta} \Delta_{3p+1}(f)(0) + \frac{\partial}{\partial \zeta} \Delta_{3p+1}(f)(0) \right| - 1$$

$$= \left| \frac{\partial}{\partial \zeta} \Delta_{3p+1}(f)(0) - \frac{\partial}{\partial \zeta} \Delta_{3p+1}(f)(0) \right| - 1$$

$$\geq \left| \frac{\partial}{\partial \zeta} \Delta_{3p+1}(f)(0) \right| - 1 \geq (p+1)^{p+1}.$$ (2)

If $e^{i\theta} = -1$, we choose $\eta_{3p+2}, \eta_{3p+3} \in i\mathbb{R}$ with $\eta_{3p+3} = -\eta_{3p+2}$, $0 < |\eta_{3p+2}| = |\eta_{3p+3}| \leq \min(\epsilon_{p+1}, 1/M_{p+1})$, such that

$$\left| \frac{O(\eta_{3p+3}) + O(\eta_{3p+2})}{\eta_{3p+3} - \eta_{3p+2}} \right| \leq M_{p+1} |\eta_{3p+2}| \leq 1,$$

then by (5.1)

$$\left| \Delta_{3p+2}(f)(\eta_{3p+3}) \right| \geq \left| \frac{\partial}{\partial \zeta} \Delta_{3p+1}(f)(0) + \frac{\eta_{3p+3} - \eta_{3p+2}}{\eta_{3p+3} - \eta_{3p+2}} \frac{\partial}{\partial \zeta} \Delta_{3p+1}(f)(0) \right| - 1$$

$$= \left| \frac{\partial}{\partial \zeta} \Delta_{3p+1}(f)(0) - \frac{\partial}{\partial \zeta} \Delta_{3p+1}(f)(0) \right| - 1$$

$$\geq \left| \frac{\partial}{\partial \zeta} \Delta_{3p+1}(f)(0) \right| - 1 \geq (p+1)^{p+1}.$$ (3)

Otherwise $e^{i\theta} \neq \pm 1$, i.e $e^{i\theta} = \cos \theta + i \sin \theta$ with $\sin \theta \neq 0$. We choose

$$\begin{cases} 
\eta_{3p+2} := r_{p+1} \cos(\theta/2) \\
\eta_{3p+3} := ir_{p+1} \sin(\theta/2),
\end{cases}$$

with $0 < r_{p+1} \leq \min(\epsilon_{p+1}, 1/M_{p+1})$. Since $\theta/2 \neq 0 \pmod{\pi/2}$ then $\eta_{3p+2}, \eta_{3p+3}$ are nonzero and different. One has

$$\left| \frac{O(\eta_{3p+3}) + O(\eta_{3p+2})}{\eta_{3p+3} - \eta_{3p+2}} \right| \leq \frac{M_{p+1}(r_{p+1} \cos(\theta/2))^2 + M_{p+1}(r_{p+1} \sin(\theta/2))^2}{r_{p+1} \cos(\theta/2) - ir_{p+1} \sin(\theta/2)}$$

$$= \frac{M_{p+1}r_{p+1}^2}{r_{p+1}} \leq M_{p+1}r_{p+1} \leq M_{p+1}r_{p+1} \leq 1.$$
On the other hand,
\[
\frac{\eta_{3p+3} - \eta_{3p+2}}{\eta_{3p+3} - \eta_{3p+2}} = \frac{-ir_{p+1} \sin(\theta/2) - r_{p+1} \cos(\theta/2)}{r_{p+1} \sin(\theta/2) - r_{p+1} \cos(\theta/2)} = \frac{\cos(\theta/2) + i \sin(\theta/2)}{\cos(\theta/2) - i \sin(\theta/2)} = e^{i\theta}.
\]

It follows by (3.1) that
\[
|\Delta_{3p+2}(f)(\eta_{3p+3})| \geq \left| \frac{\partial}{\partial \zeta} \Delta_{3p+1}(f)(0) + \frac{\eta_{3p+3} - \eta_{3p+2}}{\eta_{3p+3} - \eta_{3p+2}} \frac{\partial}{\partial \zeta} \Delta_{3p+1}(f)(0) \right| - 1
\]
\[
= \left| \frac{\partial}{\partial \zeta} \Delta_{3p+1}(f)(0) + \frac{\partial}{\partial \zeta} \Delta_{3p+1}(f)(0) + e^{i\theta} \frac{\partial}{\partial \zeta} \Delta_{3p+1}(f)(0) \right| - 1
\]
\[
\geq \left| \frac{\partial}{\partial \zeta} \Delta_{3p+1}(f)(0) \right| - 1 \geq (p + 1)^{p+1}.
\]

Now if \( \frac{\partial}{\partial \zeta} \Delta_{3p+1}(f)(0) = 0 \), one still has for example
\[
\frac{\partial}{\partial \zeta} \Delta_{3p+1}(f)(0) = \frac{\partial}{\partial \zeta} \Delta_{3p+1}(f)(0) = 0,
\]
and we go back to the first case that is already done.

Finally, there are \( \eta_{3p+1}, \eta_{3p+2}, \eta_{3p+3} \in \mathbb{R} \cup i \mathbb{R} \), different with \( 0 < |\eta_{3p+1}|, |\eta_{3p+2}|, |\eta_{3p+3}| < \min_{1 < i < 3p} |\eta_i| \) and such that
\[
|\Delta_{3p+2,(...,\eta_{p+2},...,\eta_1)}(f)(\eta_{3p+3})| \geq (p + 1)^{p+1}.
\]

This allows us to construct the sequence \( (\eta_j)_{j \geq 1} = (\eta_{3p+1}, \eta_{3p+2}, \eta_{3p+3})_{p \geq 0} \) by induction on \( p \geq 0 \). We begin with \( \eta_1, \eta_2, \eta_3 \in (\mathbb{R} \cup i \mathbb{R}) \cap V \), different and such that
\[
|\Delta_{2,(...,\eta_2,\eta_1)}(f)(\eta_3)| \geq 1.
\]

Now if assume having constructed \( \eta_1, \ldots, \eta_{3p-1}, \eta_{3p} \) such that, \( \forall j = 1, \ldots, p, \)
\[
|\Delta_{3j-1,(...,\eta_{j-1},...,\eta_1)}(f)(\eta_j)| \geq j^j,
\]
we can construct \( \eta_{3p+1}, \eta_{3p+2} \) and \( \eta_{3p+3} \), different, distinct from \( \eta_1, \ldots, \eta_{3p} \) and that satisfy by (5.2)
\[
|\Delta_{3p+2,(...,\eta_{p+2},...,\eta_1)}(f)(\eta_{3p+3})| \geq (p + 1)^{p+1},
\]
and this proves the induction. On the other hand, the sequence \( (\eta_j)_{j \geq 1} \) is bounded, thus the proof of the lemma is achieved.

\(\sqrt{\text{Remark } 5.1.\text{ We can also construct } (\eta_j)_{j \geq 1} \text{ such that it converges to 0. Indeed, in the proof, we can also assume that } 0 < |\eta_{3p+1}| < \min (\min_{1 < i < 3p} |\eta_i|, 1/(3p + 1)). \text{ Since by construction } 0 < |\eta_{3p+2}|, |\eta_{3p+3}| < |\eta_{3p+1}|, \text{ then } \lim_{j \to \infty} \eta_j = 0.\}
\]

Now we can give the proof of Proposition 2.
Proof. We consider
\[
    f(\zeta) = \frac{\zeta}{1 + |\zeta|^2} = \frac{\zeta}{1 + \zeta^2}.
\]

Then \( f \in C^\infty(\mathbb{C}) \) and \( \forall \zeta \in \mathbb{C}, \)
\[
    \frac{\partial f}{\partial \zeta}(\zeta) = \frac{1 + |\zeta|^2 - \zeta \zeta}{(1 + |\zeta|^2)^2} = \frac{1}{(1 + |\zeta|^2)^2}.
\]

In particular, \( \frac{\partial f}{\partial \zeta}(0) \neq 0 \), then \( f \) satisfies the conditions of Lemma 20. It follows that there is a (bounded) sequence \((\eta_j)_{j \geq 1} \subset \mathbb{R} \cup i\mathbb{R}\) that satisfies: \( \forall p \geq 1, \)
\[
    \left| \Delta_{p-1}(\eta_{3p-1}, \ldots, \eta_1) \left( \zeta \mapsto \frac{\zeta}{1 + |\zeta|^2} \right) (\eta_{3p}) \right| \geq p^p.
\]

Therefore the condition (1.8) of Theorem 1 is not satisfied with \( \{\eta_j\}_{j \geq 1} \) since there cannot exist \( R \geq 1 \) such that, for all \( p \geq 1 \), one has
\[
    p^p \leq R^{3p-1+1} = (R^3)^p.
\]

\[ \checkmark \]

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