HOMOTOPY EQUIVALENCE FOR PROPER HOLOMORPHIC MAPPINGS

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Abstract. We introduce several homotopy equivalence relations for proper holomorphic mappings between balls. We provide examples showing that the degree of a rational proper mapping between balls (in positive codimension) is not a homotopy invariant. In domain dimension at least 2, we prove that the set of homotopy classes of rational proper mappings from a ball to a higher dimensional ball is finite. By contrast, when the target dimension is at least twice the domain dimension, it is well known that there are uncountably many spherical equivalence classes. We generalize this result by proving that an arbitrary homotopy of rational maps whose endpoints are spherically inequivalent must contain uncountably many spherically inequivalent maps. We introduce Whitney sequences, a precise analogue (in higher dimensions) of the notion of finite Blaschke product (in one dimension). We show that terms in a Whitney sequence are homotopic to monomial mappings, and we establish an additional result about the target dimensions of such homotopies.

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1. Introduction

This paper considers proper holomorphic mappings between balls in possibly different dimensional complex Euclidean spaces. Two notions of equivalence (spherical and norm) for such maps have been extensively used. See for example [Fa], [F2], [H], [HJ], [L], [D], [D3],[R]. The purpose of this paper is to introduce and investigate a natural but subtle third notion, homotopy equivalence, which is more useful for some purposes. Homotopy equivalence itself has several possible definitions, each of which is useful in different contexts.

The one-dimensional situation for homotopy equivalence is precise, beautiful, and easy to describe. See Proposition 2.1. It is natural to attempt to generalize that result to higher dimensions. Theorem 5.1 provides one precise analogue of Proposition 2.1. Several crucial differences arise, however, which we confront in this paper.

Let \( \mathbb{C}^n \) denote complex Euclidean space and let \( \mathbb{B}_n \) denote the unit ball in \( \mathbb{C}^n \). A holomorphic map \( f : \mathbb{B}_n \to \mathbb{B}_N \) is proper if and only if, for each compact subset \( K \) of the target ball, the inverse image \( f^{-1}(K) \) is compact in the domain ball.

The basic properties of homotopy developed in this paper do not depend on regularity assumptions of the mappings at the boundary. Definitions 2.1, 2.2, and 2.3 introduce the various notions of homotopy. One of the key issues, motivating Definition 2.3, allows a homotopy between maps whose target dimensions differ.

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Proposition 2.4 shows that any pair of proper maps from the same ball are *homotopy equivalent in target dimension* $M$ when $M$ is sufficiently large. Placing restrictions on $M$ then fits nicely into the general philosophy of complexity theory in CR Geometry. In particular, given proper maps $f$ and $g$ with the same domain ball, there is a minimal $M$ for which $f$ and $g$ are homotopy equivalent in target dimension $M$. Computing this dimension for explicit rational maps seems to be difficult.

When the domain dimension is at least 2, a proper mapping between balls that is smooth up to the boundary must be, by a well-known theorem of Forstneric [F1], a rational mapping. It is therefore important also to consider homotopies where all the maps in the family are rational (Definition 2.2).

Example 2.1 is striking; it shows that the degree of a family of rational proper mappings between balls is *not* a homotopy invariant. Theorem 5.1 provides large classes of homotopic proper rational maps (terms of Whitney sequences) for which the degree need not be a homotopy invariant and clarifies Example 2.1. The proof of this result illuminates a fundamental distinction between the one-dimensional case and the general case. A finite Blaschke product of degree $d$ is the $d$-th term of a Whitney sequence; in domain dimension at least two, however, the $d$-th term of a Whitney sequence can be of degree less than $d$. Furthermore, in the higher dimensional case, there exist rational proper mappings between balls that are not terms of Whitney sequences.

Theorem 3.1 gives a finiteness result: for $n \geq 2$ and $N$ fixed, the set of homotopy classes of rational proper maps from $B_n$ to $B_N$ is finite. Theorem 3.1 is useful, because, by contrast, the number of distinct spherical equivalence classes is infinite in general. Theorem 3.2 and Corollary 3.1 decisively illustrate the distinction between homotopy and spherical equivalence. Given two rational but spherically inequivalent maps, a homotopy between them must contain *uncountably many* spherically inequivalent maps. It follows that the four maps of Faran from $B_2$ to $B_3$ are not homotopic through rational maps in target dimension three, although they are homotopic in target dimension five. This particular result provides a new method for establishing that two rational proper maps are homotopically inequivalent through rational maps.

To further illuminate the situation for rational homotopies, in section 4 we connect our discussion to the so-called $X$-variety. The method for computing this variety from [D1] enables us to compute it simultaneously for all the maps in a rational homotopy.

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### 2. Definitions and Basic Properties of Homotopy Equivalence

We will denote the squared Euclidean norm on $\mathbb{C}^n$ by $||\cdot||^2$ without indicating the dimension. A holomorphic map $f : B_n \to B_N$ is *proper* if and only if $||f(z)||^2$ tends to 1 when $||z||^2$ tends to 1. It follows by standard complex analysis that $N \geq n$. It is also easy to see that the composition of proper mappings between balls is itself proper. Let $U(n)$ denote the group of unitary transformation of $\mathbb{C}^n$. 
Such transformations are of course the simplest examples of automorphisms of $B_n$. We note that $U(n)$, as a connected Lie group, is path connected.

Proper maps $f$ and $g$ between balls are **spherically equivalent** if there are automorphisms $\phi$ (of the domain ball) and $\chi$ (of the target ball) such that $g = \chi \circ f \circ \phi$.

Proper maps $f$ and $g$ are **norm equivalent** if $||f||^2 = ||g||^2$ as functions; this concept provides an equivalence relation for maps with possibly different target dimensions. When the target dimensions are the same, norm equivalence is a special case of spherical equivalence in which the domain automorphism is the identity and the target automorphism is unitary.

Let $f$ be a holomorphic mapping with values in $\mathbb{C}^N$. Its **embedding dimension** is the number of linearly independent components of $f$. An equivalent definition is the rank of the function $||f||^2$; in other words, the smallest possible number of terms in this squared norm.

At least three versions of **homotopy equivalence** between proper maps are sensible. In some situations we assume that $f$ and $g$ have the same target dimension, whereas in others we allow the target dimensions to differ. We also sometimes wish to demand that the maps in the family have an additional property, such as rationality.

By Proposition 2.4, given proper maps $f$ and $g$ with the same domain ball, there is a minimal $M$ for which $f$ and $g$ are homotopy equivalent in target dimension $M$.

Consider a continuous function $H : B_n \times [0,1] \to \mathbb{C}^N$ which is assumed to be holomorphic in the first variable. We write $H_t$ for the map $z \to H(z,t)$. Since $H_t : B_n \to \mathbb{C}^N$ is a holomorphic mapping, it can be written as a power series

$$H_t(z) = \sum c_\alpha(t)z^\alpha \quad (1)$$

in $B_n$. The series, which we have expressed in multi-index notation, converges uniformly on compact subsets of $B_n$. It follows (see Proposition 3.2) that each of these coefficients $c_\alpha$ depends continuously on $t$, and we say that $H_t$ is a **continuous family**. In our homotopy considerations, we assume, for each $t$, that $H_t$ is a proper mapping between balls.

We introduce the following definitions of the various homotopy equivalences.

**Definition 2.1.** Let $f,g : B_n \to B_N$ be proper holomorphic mappings. Then $f$ and $g$ are **homotopic** if, for each $t \in [0,1]$ there is a proper holomorphic mapping $H_t : B_n \to B_N$ such that

- $H_0 = f$ and $H_1 = g$.
- $H_t$ is a continuous family.

**Definition 2.2.** Let $f,g : B_n \to B_N$ be proper holomorphic mappings. Then $f$ and $g$ are **homotopic through rational maps** if, for each $t \in [0,1]$ there is a proper holomorphic mapping $H_t : B_n \to B_N$ such that

- $H_0 = f$ and $H_1 = g$.
- $H_t$ is a continuous family.
- Each $H_t$ is a rational mapping.

When $n \geq 2$ in Definition 2.2, we can replace the condition of rationality by demanding that each $H_t$ be smooth on the closed ball. See Section 4.

The third definition is required for a full understanding. It is sometimes important to identify a proper map $f$ between balls with the map $f \oplus 0 = (f,0)$. We write $f \sim (f \oplus 0)$. These maps are norm-equivalent, but they are not spherically
equivalent because the target dimensions differ. In this way we take advantage of the natural injection of a target ball into a ball in higher dimensions.

**Definition 2.3.** Let \( f : B_n \to B_{N_1} \) and \( g : B_n \to B_{N_2} \) be proper holomorphic mappings. Then \( f \) and \( g \) are homotopic in target dimension \( k \) if, for each \( t \in [0, 1] \), there is a proper holomorphic mapping \( H_t : B_n \to B_k \) such that

- \( H_0 \sim f \oplus 0 \) and \( H_1 \sim g \oplus 0 \).
- \( H_t \) is a continuous family.

It is evident that each of the notions of homotopy equivalence is an equivalence relation. We briefly motivate Definition 2.3. Given the target dimensions \( N \) and \( K \), and a larger integer \( M \), we have natural injections \( j_1 : C^N \to C^M \) and \( j_2 : C^K \to C^M \) each given by \( j(\zeta) = (\zeta, 0) \). The definition asks that the maps \( j_1 \circ f \) and \( j_2 \circ g \) be homotopy equivalent. Consider the simple example \( H_t : B_1 \to B_2 \) given by

\[
H_t(z) = (tz, \sqrt{1 - t^2}z^2).
\]

Then \( H_1(z) = (z, 0) \) and \( H_0(z) = (0, z^2) \). Note that \((z^2, 0) = U(0, z^2)\) for some unitary \( U \). We would like to say that \( z \) and \( z^2 \) are homotopic in target dimension 2, and hence we naturally identify \( z \) with \((z, 0)\) and \( z^2 \) with \((z^2, 0)\).

The following decisive result in one dimension holds:

**Proposition 2.1.** Suppose \( f : B_1 \to B_1 \) is proper holomorphic. Then there is a unique positive integer \( m \) such that \( f \) is homotopic in dimension 1 to the map \( z \mapsto z^m \).

**Proof.** It is well-known that each proper holomorphic self-map of the unit disk is a finite Blaschke product:

\[
f(z) = e^{i\theta} \prod_{j=1}^{m} \frac{z - a_j}{1 - a_jz}.
\]

(2)

For each \( j \), the point \( a_j \) in (2) satisfies \(|a_j| < 1\). These points need not be distinct. For \( t \in [0, 1] \), define \( H_t \) by replacing \( \theta \) in (2) with \((1 - t)\theta \) and each \( a_j \) in (2) with \((1 - t)a_j \). Then each \( H_t \) is proper, \( H_0 = f \) and \( H_1 = z^m \). The continuity in \( t \) is evident. Hence \( f \) is homotopic to \( z^m \), where \( m \) is the degree of the Blaschke product. Next we note the uniqueness. The maps \( z^m \) and \( z^d \) cannot be homotopic if \( m \neq d \) because we can recover the exponent \( m \) by a line integral:

\[
m = \frac{1}{2\pi i} \int_{|z|=1} \frac{f'(z)}{f(z)} \, dz.
\]

As usual, an integer-valued continuous function is locally constant. \( \square \)

The number \( m \) is the degree of the rational function \( f \); it is also the degree of the divisor defining the zero-set of \( f \).

Perhaps the most surprising result of this paper is that the degree of a proper rational mapping between balls is not invariant under homotopy. The following example illustrates this point and suggests ideas from the last section of the paper.

**Example 2.1.** We define proper polynomial maps \( f, g \) from \( B_2 \) to \( B_5 \). Both these maps have embedding dimension 5. These maps are of different degree but they are homotopic in target dimension 5.

\[
f(z, w) = (z, zw, zw^2, zw^3, w^4).
\]
\[
g(z, w) = (-w^2, zw, -zw^2, z^2w, zw^2, z^2w^2, z^2).
\]
Since each of \(f\) and \(g\) is a monomial map with five distinct monomials, the embedding dimension in each case is 5. We check that they are endpoints of a one-parameter family of proper maps.

First define a proper map \(h : B_2 \to B_3\) by \(h(z, w) = (z, zw, w)\). Next define a unitary matrix \(U\) on \(C^3\) by
\[
U = \begin{pmatrix}
cos(\theta) & 0 & -\sin(\theta) \\
0 & 1 & 0 \\
\sin(\theta) & 0 & \cos(\theta)
\end{pmatrix}.
\]
Finally let \(W : B_3 \to B_5\) be the Whitney map defined by \(\zeta \to (\zeta_1, \zeta_2, \zeta_1\zeta_3, \zeta_2\zeta_3, \zeta_3^2)\).

Put \(t = \cos(\theta)\). Define \(H_t\) by \(H_t = W \circ U \circ h\); then \(H_t : B_2 \to B_5\). Since the composition of proper maps is proper, each \(H_t : B_2 \to B_5\) is proper. Writing \(c\) for \(\cos(\theta)\) and \(s\) for \(\sin(\theta)\), we obtain
\[
H_t(z, w) = (c(z - s^2w^2), zw, (cz - sw^2)(sz + cw^2), zw(sz + cw^2), (sz + cw^2)^2).
\]
When \(t = 0\) in (3) we obtain \(f\) and when \(t = 1\) in (3) we obtain \(g\).

Other natural numerical invariants such as the maximum number of inverse images of a point also fail to be invariant under homotopy.

Remark 2.1. In Example 2.1, the maximum number of inverse images of a point is not a homotopy invariant, even though all maps in the family have the same embedding dimension. Note that \(f^{-1}(0, 0, 0, 0, a)\) has four points for \(a \neq 0\). No point in the image of \(g\) has more than two inverse images.

The next example does behave as in the one-dimensional case.

Example 2.2. Each automorphism \(\phi\) of \(B_n\) is homotopic to the identity map. The proof is easy; each automorphism is a composition of a unitary transformation \(U\) and a linear fractional automorphism of the form
\[
z \to \frac{L_\alpha(z) - \alpha}{1 - \langle z, \alpha \rangle}.
\]
Here \(L_\alpha\) is a linear map depending continuously on \(\alpha\), and \(\alpha\) is a point in the unit ball. By multiplying \(\alpha\) by \(1 - t\), and deforming \(U\) into the identity, we obtain a family \(H_t\) where \(H_0(z) = \phi(z)\) and \(H_1(z) = z\).

The following result relates the various equivalence relations.

Proposition 2.2. Let \(f, g : B_n \to B_N\) be proper holomorphic maps.

- If \(f\) and \(g\) are norm equivalent, then they are spherically equivalent. The converse fails.
- If \(f\) and \(g\) are spherically equivalent, then they are homotopy equivalent. The converse fails.

Proof. By [D], the equality \(||f||^2 = ||g||^2\) implies that there is a unitary map \(U\) such that \(g = Uf\) and hence \(f\) and \(g\) are spherically equivalent. The converse fails: if \(f\) is the identity and \(g\) is an automorphism which moves the origin, then \(f\) and \(g\) are spherically equivalent but not norm equivalent. Consider the second statement. Suppose there are automorphisms \(\chi\) and \(\phi\) for which \(g = \chi \circ f \circ \phi\). We may then
deform each automorphism as in Example 2.2 to obtain a homotopy between $f$ and $g$. The converse fails: consider a Blaschke product $f$ with three distinct factors. By Proposition 2.1, $f$ is homotopic to $z^3$; it is easy to see that $f$ is not spherically equivalent to $z^3$. The same idea works in higher dimensions upon replacing product by tensor product.

Homotopy in the equi-dimensional case is easy. Proposition 2.1 handled the one-dimensional case. For $n \geq 2$ we have:

Proposition 2.3. For $n \geq 2$, let $f : B_n \to B_n$ be a proper holomorphic map. Then $f$ is homotopic to the identity.

Proof. By a well-known result of Pincuk, $f$ must be an automorphism. By Example 2.2, $f$ is homotopic to the identity. □

We note that if $f$ and $g$ are homotopy equivalent in target dimension $M_0$, then they are homotopy equivalent in target dimension $M$ if $M \geq M_0$.

We have the following simple result, noted years ago (in different language) by the first author in [D]. The map $H_t$ in the proof of this proposition is called the juxtaposition of $f$ and $g$.

Proposition 2.4. Let $f : B_n \to B_N$ and $g : B_n \to B_K$ be proper holomorphic maps. Then $f$ and $g$ are homotopic in target dimension $M$ if $M \geq n + \max(N, K)$. Furthermore, if $f$ and $g$ are rational, then the same conclusion holds with homotopy replaced by homotopy through rational maps.

Proof. First define $H_t$ by $H_t = \sqrt{1-t^2} f \oplus t g$. Then

$$||H_t(z)||^2 = (1-t^2)||f(z)||^2 + t^2||g(z)||^2. \tag{4}$$

When $||z||$ tends to one, $||H_t(z)||$ also does. It follows that $H_t$ is a proper mapping from $B_n$ to $B_{N+K}$. The continuity in $t$ is obvious. Formula (4) makes evident the norm equivalence at the endpoints 0 and 1. Thus $f$ and $g$ are homotopy equivalent in target dimension $N + K$. We can lower this dimension to $n + \max(N, K)$. To do so, let $I$ denote the identity mapping. By the same reasoning, $f$ is homotopic to $I \oplus 0$ in dimension $n + N$ and $g$ is homotopic to $I \oplus 0$ in dimension $n + K$. Since homotopy is an equivalence relation, $f$ and $g$ are homotopic in target dimension $n + \max(N, K)$, and hence also for any larger target dimension. When $f$ and $g$ are rational, the same argument provides a rational homotopy between them. □

3. Homotopy Equivalence Through Rational Maps (General Results)

In Example 3.1 below, the number of spherical equivalence classes is finite. In general this number is infinite. For example, as soon as $N \geq 2n$, there are one-dimensional families of spherically inequivalent proper mappings. These mappings can be taken to be quadratic polynomials. See pages 168-169 of [D] and also [L]. Quadratic polynomial proper maps also appear in [JZ]. By contrast, when the domain dimension is at least 2, we will show in Theorem 3.1 that the number of homotopy classes for rational proper mappings, in each target dimension, is finite.

Proposition 3.1. Fix integers $d$ and $n$. There exists a constant $B = B(n, d)$ such that the following holds: If $h : B_n \to B_N$ is a proper rational map of degree at most $d$, written as $h = \frac{p}{q}$ with $p, q$ of degree at most $d$, where $q(0) = 1$, and $q$ is not zero on $B_n$, then the coefficients of (the polynomials) $p$ and $q$ are bounded by $B$.
Proof. Let $p$ and $q$ be as in the statement. Let us first show that coefficients of $q$ are bounded. It is enough to assume that $q(0) = 1$ and $q$ is nonzero on the unit ball. In one dimension, the claim follows by simply factorizing $q$ as

$$q(z) = \prod_j (1 - z a_j).$$

Then the $a_j$ are all of modulus at most 1 and the coefficients of $q$ are in fact bounded by the largest binomial coefficient of the form $\binom{d}{k}$.

Next, in $n$ dimensions, decompose $q$ into homogeneous parts as $q(z) = 1 + q_1(z) + q_2(z) + \cdots + q_d(z)$. For each $z \in \mathbb{B}_n$ we have

$$q(\xi z) = 1 + \xi q_1(z) + \xi^2 q_2(z) + \cdots + \xi^d q_d(z).$$

This polynomial in $\xi \in \mathbb{C}$ has no zeros on the unit disk. Therefore all its coefficients $q_j(z)$ are bounded by $B(1, d)$ by the argument above. Hence $|q(z)| \leq (d + 1)B(n, d)$ for $z \in \mathbb{B}_n$. Via the Cauchy formula all the coefficients of $q$ are bounded by some constant depending only on $n$ and $d$.

Next, since $||h|| = 1$ on the unit sphere, we have

$$||p(z)|| = |q(z)| \leq dB(n, d)$$

for $z$ in the unit sphere. Again via the Cauchy formula, the coefficients of $p$ are bounded by a constant only depending on $n$ and $d$. \hfill \Box

The proposition says that we can always find a representative $\frac{p}{q}$ with bounded coefficients, while at the same time normalizing with $q(0) = 1$ and not requiring $p$ and $q$ to be in lowest terms. The following set will be useful.

**Definition 3.1.** Fix a positive integer $d$. Let $\mathcal{R}$ denote the set of polynomial maps $(p, q) : \mathbb{C}^n \to \mathbb{C}^N \times \mathbb{C}$ with the following properties:

- The degree of $q$ and of each component polynomial of $p$ is at most $d$.
- $q(0) = 1$, and $q$ is not zero on $\mathbb{B}_n$.
- $\frac{p}{q} : \mathbb{B}_n \to \mathbb{B}_N$ is proper.

Proposition 3.1 implies that the set $\mathcal{R}$ is relatively compact. Let $\mathcal{R}^*$ denote the closure of $\mathcal{R}$. The first property is obviously preserved under closure. The second property is preserved because of the Hurwitz theorem; since $q(0) = 1$, the limit function is not identically 0, and hence never 0. The condition that $\frac{p}{q}$ must be a proper map need not be preserved, but we can easily identify the limits. By continuity, if $(p, q) \in \mathcal{R}^*$, we must have $||p||^2 = ||q||^2$ on the unit sphere. Then $\frac{p}{q}$ is a proper map of balls if it is nonconstant. Hence, $\mathcal{R}^*$ is a compact set that includes $\mathcal{R}$ and also the polynomials $(p, q)$, where $\frac{p}{q}$ is constant.

Given a sequence $\{\omega_j\}$ of positive weights, we let $\ell^1(\omega)$ denote the space of sequences $\{x_j\}$ for which

$$\sum \omega_j |x_j| < \infty.$$  

We say that $\{x_j\}$ is in weighted $\ell^1$ with weights $\omega_j$. We will use this space when the sequence is indexed by multi-indices.

Let $H_t : \mathbb{B}_n \to \mathbb{B}_N$ be a family of proper mappings between balls. We can expand each $H_t$ as a power series converging uniformly on compact subsets of $\mathbb{B}_n$. If we assume that the function $t \mapsto H_t(z)$ is continuous, it follows that each Taylor...
coefficient depends continuously on \( t \). It also follows that the Hermitian matrix of Taylor coefficients of \( \|H_t\|^2 \) depends continuously on \( t \).

Here we do not require the homotopies to have any regularity on the boundary. Let \( H_t \) be a family of holomorphic mappings defined on the unit ball. We write

\[
H_t(z) = \sum c_\alpha(t) z^\alpha
\]

(5)

\[
\|H_t(z)\|^2 = \sum_{\alpha,\beta} a_{\alpha\beta}(t) z^\alpha \overline{z}^\beta.
\]

(6)

Thus \( a_{\alpha\beta}(t) = \langle c_\alpha(t), c_\beta(t) \rangle \).

**Proposition 3.2.** Let \( H_t(z) : B_n \to B_N \) be a homotopy of proper maps. The Taylor coefficients \( a_{\alpha\beta}(t) \) of \( \|H_t\|^2 \) and \( c_\alpha(t) \) of \( H_t \) are continuous functions of \( t \). In fact, there exist fixed weights \( \{\omega_{\alpha\beta}\} \) such that the map \( t \mapsto [a_{\alpha\beta}(t)]_{\alpha\beta} \) is a continuous map from \([0,1]\) to \( \ell^1(\omega) \).

**Proof.** For each \( \alpha \), the coefficient \( c_\alpha(t) \) is given by a Cauchy integral over the distinguished boundary \( T \) of a polydisc \( P \) centered at 0 whose closure lies in \( B_n \):

\[
c_\alpha(t) = \frac{1}{(2\pi i)^n} \int_T \frac{H_t(\zeta)}{\zeta^{\alpha+1}} d\zeta.
\]

(9)

Since \( H_t \) is continuous in \( t \), also each \( c_\alpha \) is continuous in \( t \). Furthermore, for a constant \( C_\alpha \), (9) yields the estimates

\[
\|c_\alpha(t) - c_\alpha(s)\| \leq C_\alpha \sup_{z \in T} \|H_t(z) - H_s(z)\|
\]

(10)

\[
\|c_\alpha(t)\| \leq C_\alpha.
\]

(11)

Note that (11) holds for all \( t \). Next consider the coefficients \( a_{\alpha\beta} \) in the squared norm. There are constants \( C_{\alpha\beta} \) such that

\[
\|a_{\alpha\beta}(t)\| = |\langle c_\alpha, \overline{c_\beta} \rangle| \leq C_\alpha C_\beta.
\]

For an appropriate choice of weights \( \omega_{\alpha\beta} \) (independent of \( H \)) we obtain that \( [a_{\alpha\beta}] \) is in \( \ell^1(\omega) \).

Estimate

\[
|a_{\alpha\beta}(t) - a_{\alpha\beta}(s)| \leq |c_\alpha(t) - c_\alpha(s)||c_\beta(t)|| + |c_\beta(t) - c_\beta(s)||c_\alpha(s)||
\]

\[
\leq C_{\alpha\beta} \sup_{z \in T} \|H_t(z) - H_s(z)\|.
\]

As \( s \) tends to \( t \), \( \sup_{z \in T} \|H_t(z) - H_s(z)\| \) tends to zero. Therefore, for the same weights, the map \( t \mapsto [a_{\alpha\beta}] \) is continuous from \([0,1]\) to \( \ell^1(\omega) \). \( \square \)

The following lemma is used in verifying the finiteness of the homotopy classes. Given a rational family \( H_t \), for each \( t \) we can write \( H_t = \frac{p_t}{q_t} \) for a polynomial \( q_t \) and a polynomial map \( p_t \). Since this choice is not unique, we need to establish continuity in \( t \).

**Lemma 3.1.** Let \( H_t \) be a homotopy of rational functions of degree at most \( d \) in target dimension \( N \). Then there exists a choice of degree \( d \) polynomial maps \( p_t : C^n \to C^N \) and \( q_t : C^n \to C \) whose coefficients are continuous functions of \( t \) such that

\[
z \mapsto \frac{p_t(z)}{q_t(z)}
\]
is a rational proper mapping of $B_n$ to $B_N$. Furthermore

$$H_0 = \frac{p_0}{q_0}, \quad \text{and} \quad H_1 = \frac{p_1}{q_1}.$$ 

Note that the homotopy obtained in the proof is not necessarily the same as $H_t$.

Proof. Let $\mathcal{R}$ be as in Definition 3.1 and let $\mathcal{R}^*$ be its closure as above.

Let $\Phi$ be the map from $\mathcal{R}^*$ to the weighted $\ell^3$ defined by letting $\Phi(p, q)$ be the sequence of Taylor coefficients at 0 of $||\frac{p}{q}||^2$. The map $\Phi$ is continuous, by similar estimates as in the proof of Proposition 3.2. If $\frac{p}{q}$ is a proper map of balls, then by [CS] $\frac{p}{q}$ extends holomorphically past the sphere. Thus $\frac{p}{q}$ has only removable singularities and stays bounded on the sphere and hence on the torus $T$ (as defined above).

Let $(p_1, q_1)$ and $(p_2, q_2)$ be two elements in $\mathcal{R}^*$. These two elements are close together in the standard topology on $\mathcal{R}^*$ if their coefficients are close together. Write $q_j(z) = 1 - Q_j(z)$, where $Q_j(0) = 0$. Suppose $Q_1$ is close enough to $Q_2$, such that on $T$

$$|q_1(z)q_2(z)| = |1 - Q_1(z) - Q_2(z) + Q_1(z)Q_2(z)| > \frac{1}{2}.$$ 

Then suppose that $(p_1, q_1)$ and $(p_2, q_2)$ are close enough so that on $T$

$$\|p_1(z)q_2(z) - p_2(z)q_1(z)\| < \epsilon.$$

Then

$$\left\| \frac{p_1(z)}{q_1(z)} - \frac{p_2(z)}{q_2(z)} \right\| < 2\epsilon.$$ 

In other words, the sup norm

$$\sup_{z \in T} \left\| \frac{p_1(z)}{q_1(z)} - \frac{p_2(z)}{q_2(z)} \right\|$$

can be bounded in terms of the difference of the coefficients of $(p_1, q_1)$ and $(p_2, q_2)$.

Then the Taylor coefficient

$$\frac{1}{(2\pi i)^n} \int_T \frac{p(\zeta)/q(\zeta)}{\zeta^{n+1}} d\zeta$$

can be bounded in terms of the sup norm of $\frac{p}{q}$ over $T$. Hence, the map $\Phi$ is continuous.

Fix $(P, Q) \in \mathcal{R}^*$. Next we consider the fibre $\Phi^{-1}(\Phi(P, Q))$. That is, the set of all choices $(p, q) \in \mathcal{R}^*$, such that $||\frac{p}{q}|| = ||\frac{P}{Q}||$.

It is easy to check that the set of all $(p, q) \in \mathcal{R}^*$ with $\frac{p}{q} = \frac{P}{Q}$ is convex. If $||h_1|| = ||h_2||$ for two maps of spheres $h_1$ and $h_2$, then there is a unitary matrix $U$ such that $h_1 =Uh_2$ (see [D]), and the set of unitary matrices is connected. Therefore, the fibre $\Phi^{-1}(\Phi(P, Q))$ is connected.

Our homotopy $H_t$ is represented by a path $\gamma: [0, 1] \to \ell^3(\omega)$ given by the coefficients of $||H_t(z)||^2$. The set $\Phi^{-1}(\gamma)$ is closed in $\mathcal{R}^*$ and therefore compact. It is also connected: if it were disconnected it would be a union of two disjoint relatively open sets $X_1$ and $X_2$. For every $t$, as the fibre is connected, $\Phi^{-1}(\gamma(t))$ is a subset of $X_1$ or $X_2$ but not both. We could therefore obtain $[0, 1]$ as a union of two disjoint compact sets $\Phi(X_1) \cup \Phi(X_2)$, a contradiction.

The set $\Phi^{-1}(\gamma)$ is therefore connected. Furthermore $\Phi^{-1}(\gamma) \subset \mathcal{R}$. We next claim that every connected topological component of $\mathcal{R}$ is path connected. This
fact follows by showing that $R$ is a semialgebraic set (a set defined by polynomial equations and inequalities). The set $R^*$ is an intersection of a closed real algebraic subvariety of the space of polynomials $(p,q)$ of degree at most $d$ with a polydisc of fixed radius $B$, and thus $R^*$ is semialgebraic. The set of polynomials $(p,q)$ such that $\frac{p}{q}$ is a constant (the set $R^* \setminus R$) is also a subvariety of the space of polynomials, and hence $R$ is semialgebraic.

Then there must exist a path in $R$ from $(p_0,q_0)$ to $(p_1,q_1)$ where $H_0 = \frac{p_0}{q_0}$ and $H_1 = \frac{p_1}{q_1}$. This path yields the desired $p_t$ and $q_t$. □

Theorem 3.1. Let $S$ denote the set of homotopy classes (of rational mappings and in target dimension $N$) of proper rational mappings $f : B_n \to B_N$. Assume that $n \geq 2$. Then $S$ is a finite set. (For $n = 1$, $S$ is countable by Proposition 2.1.)

Proof. We first need to know that a degree bound holds. That is, for $n \geq 2$ and $f$ as in the statement of the theorem, the degree of $f$ is bounded by some expression $c(n,N)$. The sharp bound is not known, but any bound will do here. For example, by [DL2]

$$d \leq \frac{N(N-1)}{2(2n-3)}.$$ 

Let $f = \frac{p}{q} : B_n \to B_N$ be a rational proper map, reduced to lowest terms, and of degree $d$. We may write

\begin{align*}
p(z) &= \sum_{|\alpha| = 0}^{d} C_{\alpha} z^{\alpha}, \quad (12.1) \\
q(z) &= \sum_{|\alpha| = 0}^{d} D_{\alpha} z^{\alpha}, \quad (12.2)
\end{align*}

where $C_{\alpha} \in \mathbb{C}^N$ and $D_{\alpha} \in \mathbb{C}$. Consider the Hermitian polynomial $R$ defined by $R = ||p||^2 - ||q||^2$. It is of degree at most $d$ in $z$ and of total degree at most $2d$, and it is divisible by $||z||^2 - 1$. A proper holomorphic mapping $f$ of degree at most $d$ thus determines a Hermitian form

$$R(z,\overline{z}) = \sum_{|\alpha|,|\beta| \leq d} c_{\alpha\beta} z^{\alpha} \overline{z}^{\beta}$$ 

on the vector space of polynomials of degree at most $d$. Note that

$$c_{\alpha\beta} = \langle C_{\alpha}, C_{\beta} \rangle - D_{\alpha} \overline{D_{\beta}}. \quad (14)$$

We next find the explicit condition for the expression in (13) to vanish on the sphere. Put $z_j = r_j e^{i\theta_j}$. Thus $r = (r_1,\ldots,r_n) = (|z_1|,\ldots,|z_n|)$ and $\theta = (\theta_1,\ldots,\theta_n)$. Assume that (13) vanishes on the sphere. Equating Fourier coefficients shows the following: on the set $S$ defined by $\sum r_j^2 = 1$, and for each multi-index $\nu$, we have

$$\sum_{\beta} \left( (C_{\beta+\nu}, C_{\beta}) - D_{\beta+\nu} \overline{D_{\beta}} \right) r^{2\beta} = 0. \quad (15)$$

By putting $x_j = r_j^2$ we can regard these conditions as equalities on the hyperplane $\sum x_j = 1$.

Thus a proper holomorphic rational mapping $f$ of degree at most $d$ determines a Hermitian form

$$\sum_{|\alpha|,|\beta| \leq d} c_{\alpha\beta} z^{\alpha} \overline{z}^{\beta}.$$
The space of forms is a finite-dimensional real vector space $V$. The conditions in (10) are linear in the coefficients $c_{\alpha\beta}$, and hence determine a subspace of $V$. The forms are restricted further because each such form must have at most $N$ positive and exactly one negative eigenvalue. This restriction is determined by finitely many polynomial inequalities on the $c_{\alpha\beta}$. Since dividing all the coefficients by the same constant does not change the proper map $f$, we may assume that the squared norm of the coefficients equals one. Hence (the norm equivalence class of) a proper rational map corresponds to the intersection of the unit sphere in a finite-dimensional real vector space with a set described by finitely many polynomial inequalities. Such a set can have at most a finite number of components. By the lemma, each component corresponds to a collection of homotopic rational proper mappings with target dimension at most $N$. □

**Theorem 3.2.** Assume $n \geq 2$. Let $H_t : B_n \to B_N$ be a homotopy of rational proper maps. Fix $t_0 \in [0, 1]$. The set

$$\{ t \in [0, 1] : H_t \text{ is spherically equivalent to } H_{t_0} \}$$

is closed in $[0, 1]$.

**Proof.** Let $H_{t_0} = p_{0, q_0}$. We must show that the set of $t$ for which $H_t$ is spherically equivalent to $H_{t_0}$ is closed. To do so, we must determine all rational maps spherically equivalent to a given one.

The degree bounds imply that the degrees of $H_t$ for $t \in [0, 1]$ are uniformly bounded by some $d = d(n, N)$. Let $R$ be as in Definition 3.1. As before, let $R^*$ denote the closure of $R$ in the space of all polynomial mappings. The set $R^*$ is compact. Define $\Phi$ as in Lemma 3.1. Thus $\Phi : R^* \to \ell^1(\omega)$, and we have shown that $\Phi$ is continuous. Since $R^*$ is compact, $\Phi$ is a closed map.

Next we want to see the effect of composition on both sides by automorphisms. Each domain automorphism has the form

$$U \frac{a - L_a(z)}{1 - \langle z, a \rangle}$$

for $U \in U(n)$ and $a \in B_n$. Here

$$L_a(z) = \frac{\langle z, a \rangle a}{s + 1} + sz$$

and $s^2 = 1 - ||a||^2$. Thus $Aut(B_n)$ can be identified with $U(n) \times B_n$. We compactify by allowing $a$ to lie in the unit sphere. When $||a|| = 1$, we see that $s = 0$ and

$$\frac{a - L_a(z)}{1 - \langle z, a \rangle} = \frac{a - \langle z, a \rangle a}{1 - \langle z, a \rangle} = a.$$

Therefore the only new maps in $\tilde{Aut}(B_n)$ after compactification are constants. We may identify an automorphism with the linear map on $\mathbb{C}^{n+1}$ given by

$$(z, w) \mapsto (wa - L_za, w - \langle z, a \rangle),$$

and the constant map with the linear map

$$(z, w) \mapsto (a(w - \langle z, a \rangle), w - \langle z, a \rangle).$$

We do the same construction in the target $\mathbb{C}^N$. Composition on both sides with automorphisms (and with the degenerate maps in the closure) now becomes
a continuous map 
\[ T : \mathcal{R}^* \times \overline{\text{Aut}(B_n)} \times \overline{\text{Aut}(B_N)} \to \mathcal{R}^*. \]

This map \( T \) is continuous from a compact set to a compact set, and therefore maps closed sets to closed sets. Since the maps with \( ||a|| = 1 \) correspond to constant maps, the image of the automorphisms lies in \( \mathcal{R} \).

As before, the homotopy \( H_t \) defines a curve \( \gamma : [0, 1] \to \ell^1(\omega) \) given by the coefficients of \( ||H_t(z)||^2 \). Let \( X \) denote the image \( T((p_0, q_0) \times \overline{\text{Aut}(B_n)} \times \overline{\text{Aut}(B_N)}) \).

Then \( X \) is closed, and \( X \cap \mathcal{R} \) is the image under \( T \) of all projectivized maps spherically equivalent to \( H_t \). The set \( \Phi^{-1}(\gamma) \cap X \) is compact because, as in the proof of Lemma 3.1, \( \Phi^{-1}(\gamma) \subset \mathcal{R} \). But \( \Phi^{-1}(\gamma) \cap X \) is the set of projectivized maps spherically equivalent to \( H_t \). Since \( \Phi \) is continuous and \( X \) is closed, the set of maps in \( H_t \) that are spherically equivalent to \( H_t \) is a closed subset of \([0, 1]\). \( \square \)

A homotopy between two spherically inequivalent maps must contain uncountably many spherically inequivalent maps.

**Corollary 3.1.** Suppose \( H_t \) is a homotopy of proper rational maps between balls in dimension \( N \). If \( H_0 \) and \( H_1 \) are not spherically equivalent, then \( H_t \) contains uncountably many spherically inequivalent maps.

In particular, the juxtaposition \( J_t(f, g) \) of any two spherically inequivalent rational maps always contains uncountably many inequivalent maps.

**Proof.** Each spherical equivalence class intersects the path in a closed set, as we have shown above. The interval \([0, 1]\) cannot be written as a union of countably many disjoint closed sets, by Sierpinski’s Theorem. Hence, if there are at least two inequivalent maps in the homotopy, then there must be uncountably many. \( \square \)

**Corollary 3.2.** All four Faran maps from the Example 3.1 below are homotopically inequivalent in target dimension 3 through rational maps.

**Proof.** By Faran’s theorem in [Fa1], there are only 4 spherical equivalence classes of rational maps from \( B_2 \) to \( B_3 \). \( \square \)

**Example 3.1.** Here are representatives of the four spherical equivalence classes:

\[
\begin{align*}
f(z, w) &= (z, w, 0) \\
g(z, w) &= (z^2, zw, w) \\
h(z, w) &= (z^2, \sqrt{2}zw, w^2) \\
\phi(z, w) &= (z^3, \sqrt{3}zw, w^3).
\end{align*}
\]

By Corollary 3.2, none of these maps are homotopy equivalent (through rational maps) in target dimension 3. By Proposition 2.4, all are homotopy equivalent in dimension 5. We analyze what happens in dimensions 4 and 5.

The maps \( f \) and \( g \) are homotopic in dimension 4:

\[ H_t(z, w) = (\sqrt{1 - t^2}z, tz^2, tw, w). \]

The maps \( g \) and \( h \) are homotopic in dimension 4:
\[ H_t(z, w) = (z^2, \sqrt{2-t^2}zw, tw, \sqrt{1-t^2}w^2). \] (17)

It follows that \( f, g, h \) are members of the same equivalence class in dimension 4. We suspect that \( \phi \) is not, a question we posed at AIM in June 2014. Note that \( h \) and \( \phi \) are homotopy equivalent in dimension 5; here is an explicit homotopy:

\[ H_t(z, w) = (tz^2, tw^2, \sqrt{1-t^2}z^3, \sqrt{1-t^2}w^3, \sqrt{1-t^2}zw). \]

Thus, in target dimension 5 these four maps lie in the same homotopy class, whereas in target dimension 3 they lie in 4 distinct homotopy classes.

Remark 3.1. By Corollary 3.1, the family connecting \( f \) and \( g \) in Example 3.1 consists of spherically inequivalent maps. This result also follows from an old result in [D2]. If polynomial proper maps preserve the origin and are spherically equivalent, then they must be unitarily equivalent. It is easy to check that unitary equivalence fails for the maps in Example 3.1.

4. Rational proper mappings between balls

Example 2.1 reveals that the degree of a rational proper mapping between balls is not a homotopy invariant. It is natural to further investigate the rational case, because of the following theorem of Forstneric [F1]. One of the key tools in the proof is a variety called the \( X \)-variety of \( f \). The method of proof involves the Schwarz reflection principle. The variety \( X_f \) contains the graph of \( f \), and Forstneric showed that this variety is rational. Assuming the map is rational, the first author developed in [D1] an efficient method for computing this variety, which we will recall and use in this section. The method allows us to understand how this variety depends on \( t \) when \( H_t \) is a family of rational proper mappings.

Theorem 4.1 (Forstneric 1989). Assume \( n \geq 2 \). Suppose \( f : B_n \rightarrow B_N \) is proper, holomorphic, and smooth up to the boundary. Then \( f \) is a rational mapping.

By a follow-up result of Cima-Suffridge ([CS]), \( f \) extends holomorphically past the sphere. Hence, for \( n \geq 2 \), we can identify rational proper mappings between balls with holomorphic proper mappings between balls that extend smoothly to the unit sphere.

We now develop the properties of \( X_f \). Assume \( R(z, \overline{z}) \) and \( r(z, \overline{z}) \) are real-analytic, and \( \{ r = 0 \} \) is a hypersurface. Suppose \( R \) vanishes on \( \{ r = 0 \} \). Then \( R(z, \overline{w}) = 0 \) on the set defined by \( r(z, \overline{w}) = 0 \). This result is known as polarization.

Let \( f : B_1 \rightarrow B_1 \) be proper (hence a finite Blaschke product). By polarization, if we know the value of \( f(z) \) for some \( z \) inside the circle, then we automatically know the value of \( f \) at the reflected point \( \frac{1}{\overline{z}} = \frac{1}{|z|^2} \) from the formula

\[ f\left( \frac{z}{|z|^2} \right) = \frac{1}{f(z)} \]

One of the difficulties in homotopy considerations is that reflection in higher dimensions is much more subtle. Suppose \( n \geq 2 \), and that \( f : B_n \rightarrow B_N \) is proper and smooth up to the sphere. Then \( f \) is rational and holomorphic past the sphere. What do we get from polarization and reflection?

\[ \langle z, w \rangle = 1 \implies \langle f(z), f(w) \rangle = 1. \]
Given $z$ we know
\[ (f(z), f\left(\frac{z}{\|z\|^2}\right)) = 1, \]
but this equation does not determine the value of $f$ at the reflected point.

**Definition 4.1.** (Forstneric) Suppose $f : \mathbb{B}_n \to \mathbb{B}_N$ is proper.
\[ X_f = \{(w, \zeta) \in \mathbb{C}^n \times \mathbb{C}^N : \langle z, w \rangle = 1 \implies \langle f(z), \zeta \rangle = 1.\} \quad (X) \]

It is convenient to decree that $(0, f(0)) \in X_f$. In $(X)$ we could insist that $w$ lie in the domain of $f$, or we could allow $\infty$. For us $X_f$ will be the union of the set defined by $(X)$ with the point $(0, f(0))$, assuming that $f(w)$ is defined.

Note that $(w, f(w)) \in X_f$ by polarization. In general $X_f$ is a proper superset of the graph. If $N$ is minimal for $f$, then the fibre over a generic $w$ will be $f(w)$, but in most cases exceptional fibers exist. If $N$ is larger than the embedding dimension of $f$, then the fibres are all positive-dimensional. Saying that $X_f$ equals the graph of $f$ amounts to saying $f(w)$ is the unique solution to the polarized equation.

The method in [D1] uses homogenization techniques to create a matrix $\mathcal{C}(w)$ of holomorphic polynomial functions with the following properties. Suppose $f : \mathbb{B}_n \to \mathbb{B}_N$ is rational of degree $d$. The matrix $C(w)$ has $N$ columns. It has $K(n,d)$ rows, where $K(n,d)$ is the number of homogeneous monomials of degree $d$ in $n$ variables. Thus we can think of $\mathcal{C}(w)$ as a linear map from $\mathbb{C}^N$ to $H^n(\mathbb{P}_n, O(d))$.

**Theorem 4.2.** Let $f = \frac{g}{h} : \mathbb{B}_n \to \mathbb{B}_N$ be a proper rational holomorphic mapping. Let $C(\pi)$ denote the linear map from above. For each nonzero $w$ in the domain of $f$, we have $(w, \zeta) \in X_f$ if and only if $\zeta - f(w)$ lies in the null space of $\mathcal{C}(w)$. Thus $X_f$ equals the graph of $f$ if and only if, for each nonzero $w$ in the domain of $f$, the null space of $\mathcal{C}(w)$ is trivial. Furthermore, if $H_t$ is a homotopy of rational mappings, then the corresponding linear maps $C_t(\pi)$ depends continuously on $t$.

**Corollary 4.1.** Let $f$ be a rational proper holomorphic mapping between balls. For each nonzero $w$ in the domain of $f$, the fibre $X_f(w)$ over $w$ is the affine space $f(w) + \text{null space}(\mathcal{C}(w))$. In particular, the null space of $C(w)$ is trivial if and only if the fibre over $w$ is zero-dimensional, when it is the single point $f(w)$.

**Corollary 4.2.** Let $H_t$ be a homotopy of rational proper maps. If the $X$-variety of $H_{t_0}$ is the graph of $H_{t_0}$, then the same holds for $t$ near $t_0$.

**Example 4.1.** Let $f : \mathbb{B}_2 \to \mathbb{B}_4$ be the group-invariant map
\[ f(z_1, z_2) = (z_5^5, \sqrt{5}z_1^2z_2, \sqrt{5}z_1^2z_2^2, z_2^5). \]
We compute $X_f$ as follows. Homogenize:
\[ (z_1^5, \sqrt{5}z_1^2z_2(z_1\overline{w}_1 + z_2\overline{w}_2), \sqrt{5}z_1^2z_2^2(z_1\overline{w}_1 + z_2\overline{w}_2)^2, z_2^5) \quad (19) \]
From $(19)$ we obtain the matrix $C(\overline{w})$.
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \sqrt{5}\overline{w}_1 & 0 & 0 \\
0 & \sqrt{5}\overline{w}_2 & \sqrt{5}\overline{w}_1^2 & 0 \\
0 & 0 & 2\sqrt{5}\overline{w}_1\overline{w}_2 & 0 \\
0 & 0 & \sqrt{5}\overline{w}_2^2 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \quad (20)
\]
The degree of a rational proper map between balls equals the degree of its numerator. The map $C(w)$ is independent of $w$ if and only if $f$ is homogeneous.

Suppose $f_t$ is a family, all of degree at most $d$, and some $f_t$ is of degree $d$. If all $f_t$ have embedding dimension $N$, then $C$ is of size $K$ by $N$.

We can recover $f$ from $C$. Let $E_1,\ldots,E_N$ be the usual basis for $C^N$. Then $C(\pi_k(E_k))$ is the $k$-th component of the numerator of $f$, homogenized by writing $1 = (z,w)$. Then we can dehomogenize.

**Example 4.2.** We recall Example 2.1. Put $t = \cos(\theta) = c$ and $s = \sin(\theta)$.

$$H_t(z_1, z_2) = (cz_1 - sz_2^2, z_1z_2, (cz_1 - sz_2^2)(sz_1 + cz_2^2), z_1z_2(sz_1 + cz_2^2), (sz_1 + cz_2^2)^2). \quad (21)$$

For each $t$, the map $H_t$ has embedding dimension 5. When $t = 1$, the degree drops, and hence the degree is not a homotopy-invariant.

To clarify this example, we compute the $X$-variety for the maps $H_t$. Here is the matrix $C(w)$:

$$
\begin{pmatrix}
   cw_1^3 & 0 & csw_1^2 & 0 & s^2w_1^2 \\
   3cw_1^2w_2 & w_1^2 & 2csw_1w_2 & sw_1 & 2s^2w_1w_2 \\
   3cw_1w_2^2 - sw_1^2 & 2w_1w_2 & csw_1^2 + (c^2 - s^2)w_1 & sw_2 & 2scw_1 + s^2w_2^2 \\
   cw_2^3 - 2csw_1w_2 & w_2^2 & (c^2 - s^2)w_2 & c & 2cw_2 \\
   -sw_2^2 & 0 & 0 & -sc & 0 & c^2
\end{pmatrix}
$$

The determinant of this matrix is $c^2w_1^6$. Thus, unless $c = 0$, it is generically invertible. When $c \neq 0$, there is an exceptional fibre when $w_1 = 0$. When $c = 0$ we have a map of degree 3; hence the rank cannot exceed 4.

This example illustrates a general method for constructing homotopies, which we elaborate in the last section.

### 5. Whitney sequences

Let $f : B_n \to B_N$ be a proper rational mapping. Let $A$ be a subspace of $C^N$, and let $\pi_A$ denote orthogonal projection onto $A$. Following the first author’s approach from [D], we may form the new proper mapping $E_A(f)$, defined by

$$E_A(f) = (\pi_A f \otimes z) \oplus (1 - \pi_A)(f).$$

Suppose that $B$ is another subspace of $C^N$ of the same dimension $d$ as $A$, and $A \cap B = \{0\}$. Then there is a unitary mapping $U \in U(N)$ such that $U(A) = B$. Since the unitary group is path connected, we can find a one-parameter family of unitary mappings connecting $U$ to the identity. It follows that the maps $E_A(f)$ and $E_B(f)$ are homotopic in dimension $K$, where $K = N + d(n - 1)$. Example 2.1 is obtained via this construction.

**Definition 5.1.** A Whitney sequence is a collection $F_0, F_1, \ldots$ of rational proper maps from $B_n$ to $B_{N_k}$ defined as follows. Put $F_0(z) = \phi_0$, where $\phi_0$ is an automorphism of $B_n$. Given $F_k : B_n \to B_{N_k}$, let $A_k$ be a non-zero subspace of $C^{n_k}$, and let $\pi_k$ denote orthogonal projection onto $A_k$. Choose an automorphism $\phi_k$ of $B_n$. Choose a linear, norm-preserving injection $j_k$ to whatever target dimension we wish. Define $F_{k+1}(z)$ by

$$F_{k+1} = j_k \circ ((\pi_k F_k \otimes \phi_k) \oplus (1 - \pi_k)F_k). \quad (22)$$
The degree of the rational function $F_k$ is at most $k + 1$, but it can be smaller. The following result provides an analogue of the one-dimensional situation.

**Theorem 5.1.** Let $\{F_k\}$ denote a Whitney sequence of proper mappings. Each $F_k$ is homotopic to a monomial proper mapping of degree $k + 1$.

**Proof.** The idea of the proof comes from Proposition 2.1. We proceed by induction on the number of factors. When $k = 0$, the function $F_0$ is an automorphism, and hence homotopic to the identity map (a monomial map of degree 1) by Example 2.2. Suppose for some $k$ that $F_k$ is homotopic to a monomial mapping $G_k$ of degree $k + 1$. Find a homotopy connecting $\phi_k$ to the identity. Then $F_{k+1}$ is homotopic to the mapping

$$G_{k+1} = (\pi_k G_k \otimes z) \oplus (1 - \pi_k) G_k. \quad (23)$$

Note that $G_{k+1}$ if of degree $k + 2$ if $\pi_k G_k$ is of degree $k + 1$. By the induction hypothesis $G_k$ is of degree $k + 1$. Since $A_k$ is not the trivial subspace, there is a unitary map such that $\pi_k U G_k$ is also of degree $k + 1$. Also $(1 - \pi_k) G_k$ has degree at most $k + 1$. Since the unitary group is path-connected, $U G_k$ is homotopic to $G_k$. Hence $G_{k+1}$ is homotopic to a monomial mapping of degree $k + 2$. $\square$

The mappings $H_t$ in Example 2.1 are each part of a Whitney sequence. The degree is not a homotopy invariant because the tensor products are taken on different subspaces, and hence the tensor product need not increase the degree.

Not every proper rational mapping is a term of a Whitney sequence. For example, even the monomial map $(z, w) \rightarrow (z^3, \sqrt{3} zw, w^3)$ cannot be obtained in this fashion. One must allow also the inverse operation of replacing $F_{k+1}$ in (22) with $F_k$.

The following result indicates the significance of the target dimension in the definition of homotopy.

**Theorem 5.2.** Let $F_k : B_n \rightarrow B_N$ be a term in a Whitney sequence. Then, $F_k \oplus 0$ is homotopic in target dimension $N + 1$ to the injection $z \rightarrow z \oplus 0$.

**Proof.** We have already shown that $F_k$ is homotopy equivalent to a monomial mapping in target dimension $N$. Furthermore this monomial mapping is in the image of the tensor product construction. We claim that such monomial maps are always homotopically equivalent to the identity in target dimension $N + 1$. This conclusion is trivial for maps of degree 1. For $d \geq 2$, consider an arbitrary monomial mapping $f$ of degree $d$, of embedding dimension $N$, and in the image of the tensor product operation. We will show that it is homotopic in target dimension $N + 1$ to a monomial mapping of degree at most $d - 1$, of embedding dimension at most $N$, and also in the image of the tensor product mapping. To do so, we order the monomials of degree $d$ lexicographically. Choose the last monomial $m$ that occurs. Then there is a monomial $q$ of degree $d - 1$ such that the $n$ monomials $z_1 q, \ldots, z_n q$ include $m$. After renumbering we may assume that $m = z_n q$. For some polynomial map $g$ of degree at most $d$ we can write

$$f = g \oplus z_1 q \oplus z_2 q \oplus \cdots \oplus z_n q. \quad (24)$$

Now we replace the last $n$ of these components with $\lambda$ times each, and we add an $(N + 1)$-st component $\sqrt{1 - \lambda^2} q$. For each $\lambda \in [0, 1]$ the result is a proper map to the $N + 1$ ball. When $\lambda = 0$ we obtain a map whose last component is $q$ and for which $m$ no longer appears. We continue in this fashion one monomial (of degree
at a time, obtaining homotopies in target dimension \(N + 1\) that (in finitely many steps) eliminate all terms of degree \(d\). Since homotopy is an equivalence relation, the composition defines a homotopy \(H_t\) in target dimension \(N + 1\). Since \(f\) is a term of a Whitney sequence, we are in the same situation as before, with \(d\) lowered. Eventually we reach a linear map and the result follows.

More information on the tensor product operation appears, for example, in [D] and [D3].

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