On the Sombor characteristic polynomial and Sombor energy of a graph

Nima Ghanbari

Received: 22 September 2021 / Revised: 4 April 2022 / Accepted: 23 June 2022 / Published online: 12 July 2022
© The Author(s) 2022

Abstract
Let $G$ be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The Sombor matrix of $G$, denoted by $A_{SO}(G)$, is defined as the $n \times n$ matrix whose $(i, j)$-entry is $\sqrt{d_i^2 + d_j^2}$ if $v_i$ and $v_j$ are adjacent and 0 for another cases. Let the eigenvalues of the Sombor matrix $A_{SO}(G)$ be $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ which are the roots of the Sombor characteristic polynomial $\prod_{i=1}^{n}(\rho - \rho_i)$. The Sombor energy $E_{SO}$ of $G$ is the sum of absolute values of the eigenvalues of $A_{SO}(G)$. In this paper, we compute the Sombor characteristic polynomial and the Sombor energy for some graph classes, define Sombor energy unique and propose a conjecture on Sombor energy.

Keywords Sombor matrix · Sombor energy · Sombor characteristic polynomial · Regular graphs · Eigenvalues

Mathematics Subject Classification 05C12 · 05C50

1 Introduction

In this paper, we are concerned with simple finite graphs, without directed, multiple, or weighted edges, and without self-loops. Let $G = (V, E)$ be such a graph, with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. If two vertices $v_i$ and $v_j$ of $G$ are adjacent, then we use the notation $v_i \sim v_j$. For $v_i \in V(G)$, the degree of the vertex $v_i$, denoted by $d_i$, is the number of the vertices adjacent to $v_i$.

Let $A(G)$ be adjacency matrix of $G$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ its eigenvalues. These are said to be the eigenvalues of the graph $G$ and to form its spectrum (Cvetković et al. 1980). The

Communicated by Boris Furtula.

Nima Ghanbari
Nima.ghanbari@uib.no

1 Department of Informatics, University of Bergen, P.O. Box 7803, 5020 Bergen, Norway
energy $E(G)$ of the graph $G$ is defined as the sum of the absolute values of its eigenvalues

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$ 

Details and more information on graph energy can be found in Gutman (2001), Gutman (2005), Gutman et al. (2009), Majstorović et al. (2009). There are many kinds of graph energies, such as Randić energy (Alikhani and Ghanbari 2015; Bozkurt and Bozkurt 2013; Bozkurt et al. 2010; Das and Sorgun 2014; Gutman et al. 2014), distance energy (Stevanović et al. 2013), incidence energy (Bozkurt and Gutman 2013), matching energy (Chen and Shi 2015; Ji et al. 2013) and Laplacian energy (Das et al. 2013).

Sombor index is defined as $SO(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2}$ (see Gutman 2021a). More details on Sombor index can be found in Alikhani and Ghanbari (2021), Chen et al. (2022), Cruz et al. (2021), Das et al. (2021), Deng et al. (xx), Ghanbari and Alikhani (2021), Li et al. (2022), Redžepović (2021), Wang et al. (xx). Recently, in Gutman (2021b), Gutman introduced Sombor matrix of a graph $G$ as $ASO(G) = (r_{ij})_{n \times n}$, and

$$r_{ij} = \begin{cases} \sqrt{d_i^2 + d_j^2} & \text{if } v_i \sim v_j \\ 0 & \text{otherwise.} \end{cases}$$

The eigenvalues of $ASO(G)$ are denoted by $\rho_1 \geq \rho_2 \geq \ldots \geq \rho_n$, and are said to form the Sombor spectrum of the graph $G$. In Gutman (2021b), Gutman introduced Sombor characteristic polynomial $\phi_{SO}(G, \lambda)$ as

$$\phi_{SO}(G, \lambda) = \det(\lambda I - ASO(G)) = \prod_{i=1}^{n}(\lambda - \rho_i),$$

and Sombor energy $ESO(G)$ as

$$ESO(G) = \sum_{i=1}^{n} |\rho_i|.$$ 

We refer the reader to Gowtham and Swamy (2021); Gutman and Redžepović (2022); Jayanna and Gutman (2021) for more details on Sombor energy.

Two graphs $G$ and $H$ are said to be Sombor energy equivalent, or simply $\mathcal{E}_{SO}$-equivalent, written $G \sim H$, if $ESO(G) = ESO(H)$. It is evident that the relation $\sim$ of being $\mathcal{E}_{SO}$-equivalence is an equivalence relation on the family $\mathcal{G}$ of graphs, and thus $\mathcal{G}$ is partitioned into equivalence classes, called the $\mathcal{E}_{SO}$-equivalent. Given $G \in \mathcal{G}$, let

$$[G] = \{H \in \mathcal{G} : H \sim G\}.$$ 

We call $[G]$ the equivalence class determined by $G$. A graph $G$ is said to be Sombor energy unique, or simply $\mathcal{E}_{SO}$-unique, if $[G] = \{G\}$.

A graph $G$ is called $k$-regular if all vertices have the same degree $k$. One of the famous graphs is the Petersen graph which is a symmetric non-planar 3-regular graph of order 10. There are exactly twenty one 3-regular graphs of order 10 Khosrovshahi and Maysoori (2001). In the study of Sombor energy, it is interesting to investigate the Sombor characteristic polynomial and Sombor energy of this graph and see if it can be recognised by its Sombor
energy and Sombor characteristic polynomial among other 3-regular graphs with the same order. We denote the Petersen graph by $P$.

In this paper, we consider the Sombor characteristic polynomial and Sombor energy of graphs. In Sect. 2, we bring some known results about Sombor characteristic polynomial and Sombor energy. In addition, the Sombor characteristic polynomial and Sombor energy of some special kind of graphs are computed. In Section 3, we consider to regular graphs, especially cubic graphs of order 10 and state a conjecture. In Section 4, we state some open problems for future direction of this research.

2 Sombor energy of specific graphs

In this section, we study the Sombor characteristic polynomial and the Sombor energy for certain graphs. The following result gives McClelland-type bound for the Sombor energy.

**Theorem 2.1** Gutman (2021b) If $G$ is a graph on $n$ vertices, and $F(G)$ is its forgotten topological index, then

$$E_{SO}(G) \leq \sqrt{2nF(G)}.$$

The following result gives Koolen–Moulton-type bound for the Sombor energy.

**Theorem 2.2** Gutman (2021b) Let $G$ be a graph on $n$ vertices, with Sombor and forgotten topological indices $SO(G)$ and $F(G)$, respectively. Then,

$$E_{SO}(G) \leq \frac{2SO(G)}{n} + \sqrt{(n-1) \left( \frac{2F(G)}{n} - \left( \frac{2SO(G)}{n} \right)^2 \right)}.$$

Here, we shall compute the Sombor characteristic polynomial of paths and cycles.

**Theorem 2.3** For every $n \geq 5$, the Sombor characteristic polynomial of the path graph $P_n$ satisfy

$$\phi_{SO}(P_n, \lambda) = \lambda^2 \Lambda_{n-2} - 10\lambda \Lambda_{n-3} + 25\Lambda_{n-4},$$

where for every $k \geq 3$, $\Lambda_k = \lambda \Lambda_{k-1} - 8\Lambda_{k-2}$ with $\Lambda_1 = \lambda$ and $\Lambda_2 = \lambda^2 - 8$. In addition, the characteristic polynomial of $P_2$, $P_3$ and $P_4$ are $\lambda^2 - 2$, $\lambda^3 - 10\lambda$ and $\lambda^4 - 18\lambda^2 + 25$, respectively.

**Proof** It is easy to see that the characteristic polynomial of $P_2$ is $\lambda^2 - 2$. In addition, for $P_3$ is $\lambda^3 - 10\lambda$ and for $P_4$ is $\lambda^4 - 18\lambda^2 + 25$. Now, for every $k \geq 3$ consider

$$M_k := \begin{pmatrix}
\lambda & -\sqrt{8} & 0 & \ldots & 0 & 0 \\
-\sqrt{8} & \lambda & -\sqrt{8} & \ldots & 0 & 0 \\
0 & -\sqrt{8} & \lambda & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & \lambda & -\sqrt{8} \\
0 & 0 & 0 & \ldots & -\sqrt{8} & \lambda
\end{pmatrix}_{k \times k}.$$
and let $\Lambda_k = \text{det}(M_k)$. One can easily check that $\Lambda_k = \lambda \Lambda_{k-1} - 8 \Lambda_{k-2}$. Now, consider the path graph $P_n$. Suppose that $\phi_{SO}(P_n, \lambda) = \text{det}(\lambda I - A_{SO}(P_n))$. We have

$$
\phi_{SO}(P_n, \lambda) = \text{det} \begin{vmatrix}
\lambda & -\sqrt{5} & 0 & 0 & \ldots & 0 & 0 \\
-\sqrt{5} & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & -\sqrt{5} \\
0 & 0 & 0 & 0 & \ldots & 0 & \lambda
\end{vmatrix}_{n \times n}.
$$

Therefore,

$$
\phi_{SO}(P_n, \lambda) = \lambda \text{det} \begin{vmatrix}
M_{n-2} & 0 \\
0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & -\sqrt{5} & \lambda
\end{vmatrix} + \sqrt{5} \text{det} \begin{vmatrix}
-\sqrt{5} & -\sqrt{8} & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & -\sqrt{5} & \lambda
\end{vmatrix}.
$$

Therefore,

$$
\phi_{SO}(P_n, \lambda) = \lambda \left( \lambda \Lambda_{n-2} + \sqrt{5} \text{det} \begin{vmatrix}
M_{n-3} & 0 \\
0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & -\sqrt{5} & \lambda
\end{vmatrix} + \sqrt{8} \text{det} \begin{vmatrix}
M_{n-3} & 0 \\
0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & -\sqrt{5} & \lambda
\end{vmatrix} \right)
$$

Hence,

$$
\phi_{SO}(P_n, \lambda) = \lambda (\lambda \Lambda_{n-2} - 5 \Lambda_{n-3}).
$$
\[-5 \left( \lambda \Lambda_{n-3} + \sqrt{5} \det \begin{pmatrix} M_{n-4} & \vdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \right) \]

\[= \lambda (\lambda \Lambda_{n-2} - 5 \Lambda_{n-3}) - 5 (\lambda \Lambda_{n-3} - 5 \Lambda_{n-4}), \]

and, therefore, we have the result. \(\square\)

**Theorem 2.4** For every \(n \geq 3\), the Sombor characteristic polynomial of the cycle graph \(C_n\) satisfy

\[\phi_{SO}(C_n, \lambda) = \lambda \Lambda_{n-1} - 16 \Lambda_{n-2} - 2(\sqrt{8})^n,\]

where for every \(k \geq 3\), \(\Lambda_k = \lambda \Lambda_{k-1} - 8 \Lambda_{k-2}\) with \(\Lambda_1 = \lambda\) and \(\Lambda_2 = \lambda^2 - 8\).

**Proof** Similar to the proof of Theorem 2.3, for every \(k \geq 3\), we consider

\[M_k := \begin{pmatrix} \lambda & -\sqrt{8} & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\sqrt{8} & \lambda & -\sqrt{8} & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\sqrt{8} & \lambda & -\sqrt{8} & \cdots & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{8} & \lambda & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda & -\sqrt{8} & 0 \\ 0 & 0 & 0 & 0 & \cdots & -\sqrt{8} & \lambda & -\sqrt{8} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\sqrt{8} & \lambda \end{pmatrix}_{k \times k},\]

and let \(\Lambda_k = \det(M_k)\). We have \(\Lambda_k = \lambda \Lambda_{k-1} - 8 \Lambda_{k-2}\). Suppose that \(\phi_{SO}(C_n, \lambda) = \det(\lambda I - A_{SO}(C_n))\). We have

\[\phi_{SO}(C_n, \lambda) = \det \begin{pmatrix} \lambda & -\sqrt{8} & 0 & 0 & \cdots & 0 & 0 & -\sqrt{8} \\ -\sqrt{8} & \lambda & -\sqrt{8} & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\sqrt{8} & \lambda & -\sqrt{8} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda & -\sqrt{8} & 0 \\ 0 & 0 & 0 & 0 & \cdots & -\sqrt{8} & \lambda & -\sqrt{8} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\sqrt{8} & \lambda \end{pmatrix}_{n \times n}.

Therefore,

\[\phi_{SO}(P_n, \lambda) = \lambda \Lambda_{n-1} + \sqrt{8} \det \begin{pmatrix} -\sqrt{8} & -\sqrt{8} & 0 & \cdots & 0 \\ 0 & -\sqrt{8} & \cdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & -\sqrt{8} & \lambda \\ 0 & 0 & \cdots & 0 & -\sqrt{8} \end{pmatrix}_{n \times n}.

\[\square\]
(ii) The Sombor energy of $S_n$ is

\[
\text{Theorem 2.6} \quad \text{For } n \geq 2,
\]

(i) One can easily check that the Sombor matrix of $S_n$ is

\[
\text{Proof} \quad \text{(i) One can easily check that the Sombor matrix of } K_{1,n-1} \text{ is}
\]

\[
\phi_{SO}(P_n, \lambda) = \lambda \Lambda_{n-1} + \sqrt{8} \left( -\sqrt{8} \Lambda_{n-2} + (-1)^n (-\sqrt{8})^{n-1} \right)
\]

and, therefore, we have the result. \hfill \Box

Now, we consider to star graph $S_n$ and find its Sombor characteristic polynomial and Sombor energy. We need the following Lemma.

**Lemma 2.5** Cvetković et al. (1980) If $M$ is a nonsingular square matrix, then

\[
det \begin{pmatrix} M & N \\ P & Q \end{pmatrix} = det(M)det(Q - PM^{-1}N).
\]

**Theorem 2.6** For $n \geq 2$,

(i) The Sombor characteristic polynomial of the star graph $S_n = K_{1,n-1}$ is

\[
\phi_{SO}(S_n, \lambda)) = \lambda^{n-2} \left( \lambda^2 - (n - 1)(n^2 - 2n + 2) \right).
\]

(ii) The Sombor energy of $S_n$ is

\[
E_{SO}(S_n) = 2\sqrt{(n - 1)(n^2 - 2n + 2)}.
\]

**Proof** (i) One can easily check that the Sombor matrix of $K_{1,n-1}$ is

\[
A_{SO}(S_n) = \sqrt{n^2 - 2n + 2} \begin{pmatrix} 0_{1 \times 1} & J_{1 \times n-1} \\ J_{n-1 \times 1} & 0_{n-1 \times n-1} \end{pmatrix}.
\]

We have

\[
det(\lambda I - A_{SO}(S_n)) = det \left( \begin{array}{cc} \lambda & -\sqrt{n^2 - 2n + 2} J_{1 \times (n-1)} \\ -\sqrt{n^2 - 2n + 2} J_{(n-1) \times 1} & \lambda I_{n-1} \end{array} \right).
\]

Using Lemma 2.5,

\[
det(\lambda I - A_{SO}(S_n)) = \lambda det(\lambda I_{n-1} - \sqrt{n^2 - 2n + 2} J_{(n-1) \times 1} \cdot \frac{1}{\lambda} \sqrt{n^2 - 2n + 2} J_{1 \times (n-1)}).
\]

Since $J_{(n-1) \times 1} J_{1 \times (n-1)} = J_{n-1}$, therefore,

\[
det(\lambda I - A_{SO}(S_n)) = \lambda det(\lambda I_{n-1} - \frac{1}{\lambda} (n^2 - 2n + 2) J_{n-1})
\]

\[
= \lambda^{2-n} det(\lambda^2 I_{n-1} - (n^2 - 2n + 2) J_{n-1} - (n^2 - 2n + 2) J_{n-1}).
\]

On the other hand, the eigenvalues of $J_{n-1}$ are $n - 1$ (once) and 0 ($n - 2$ times), the eigenvalues of $(n^2 - 2n + 2) J_{n-1}$ are $(n - 1)(n^2 - 2n + 2)$ (once) and 0 ($n - 2$ times). Hence,

\[
\phi_{SO}(S_n, \lambda)) = \lambda^{n-2} \left( \lambda^2 - (n - 1)(n^2 - 2n + 2) \right).
\]
(ii) It follows from Part (i).

Here, we shall investigate the Sombor energy of complete graphs.

**Theorem 2.7** For \( n \geq 2 \),

(i) The Sombor characteristic polynomial of complete graph \( K_n \) is

\[
\phi_{SO}(K_n, \lambda) = (\lambda - (n - 1)^2\sqrt{2})(\lambda + (n - 1)^2\sqrt{2})^{n-1}.
\]

(ii) The Sombor energy of \( K_n \) is

\[
E_{SO}(K_n) = 2(n - 1)^2\sqrt{2}.
\]

**Proof** (i) The Sombor matrix of \( K_n \) is \((n - 1)\sqrt{2}(J - I)\). Therefore,

\[
\phi_{SO}(K_n, \lambda) = \det(\lambda I - (n - 1)\sqrt{2}J + (n - 1)\sqrt{2}I)
= \det((\lambda + (n - 1)\sqrt{2})I - (n - 1)\sqrt{2}J).
\]

Since the eigenvalues of \( J_n \) are \( n \) (once) and \( 0 \) \((n - 1) \) times), the eigenvalues of \((n - 1)\sqrt{2}J_n\) are \( n(n - 1)^2\sqrt{2} \) (once) and \( 0 \) \((n - 1) \) times. Therefore,

\[
\phi_{SO}(K_n, \lambda) = (\lambda - (n - 1)^2\sqrt{2})(\lambda + (n - 1)^2\sqrt{2})^{n-1}.
\]

(ii) It follows from Part (i).

We end this section by finding Sombor characteristic polynomial of complete bipartite graphs and their Sombor energy.

**Theorem 2.8** For natural number \( m, n \neq 1 \),

(i) The Sombor characteristic polynomial of complete bipartite graph \( K_{m,n} \) is

\[
\phi_{SO}(K_{m,n}, \lambda) = \lambda^{m+n-2}(\lambda^2 - mn(m^2 + n^2)).
\]

(ii) The Sombor energy of \( K_{m,n} \) is \( 2\sqrt{mn(m^2 + n^2)} \).

**Proof** (i) It is easy to see that the Sombor matrix of \( K_{m,n} \) is \( \sqrt{m^2 + n^2}\begin{pmatrix} 0_{m \times m} & J_{m \times n} \\ J_{n \times m} & 0_{n \times n} \end{pmatrix} \).

Using Lemma 2.5, we have

\[
\det(\lambda I - \text{SOM}_{K_{m,n}}) = \det\begin{pmatrix} \lambda I_m & -\sqrt{m^2 + n^2}J_{m \times n} \\ -\sqrt{m^2 + n^2}J_{m \times n} & \lambda I_n \end{pmatrix}.
\]

Therefore,

\[
\det(\lambda I - \text{SOM}_{K_{m,n}}) = \det(\lambda I_m)\det(\lambda I_n - \sqrt{m^2 + n^2}J_{m \times n} + \frac{1}{\lambda}I_m\sqrt{m^2 + n^2}J_{m \times n}).
\]

We know that \( J_{n \times m}J_{m \times n} = mJ_n \). Therefore,

\[
\det(\lambda I - \text{SOM}_{K_{m,n}}) = \lambda^m\det(\lambda I_n - \frac{1}{\lambda}m(m^2 + n^2)J_n)
= \lambda^{m-n}\det(\lambda^2 I_n - m(m^2 + n^2)J_n).
\]

The eigenvalues of \( J_n \) are \( n \) (once) and \( 0 \) \((n - 1) \) times. Therefore, the eigenvalues of \( m(m^2 + n^2)J_n \) are \( mn(m^2 + n^2) \) (once) and \( 0 \) \((n - 1) \) times. Hence,

\[
\phi_{SO}(K_{m,n}, \lambda) = \lambda^{m+n-2}(\lambda^2 - mn(m^2 + n^2)).
\]

(ii) It follows from Part (i).
3 Sombor energy of 2-regular and 3-regular graphs

In this section, we consider 2-regular and 3-regular graphs. As a beginning of this section, we have the following easy lemma:

**Lemma 3.1** Let $G = G_1 \cup G_2 \cup G_3 \cup \ldots \cup G_n$. Then

(i) $\phi_{SO}(G) = \prod_{i=1}^{n} \phi_{SO}(G_i)$.
(ii) $E_{SO}(G) = \sum_{i=1}^{n} E_{SO}(G_i)$.

As an immediate result of Lemma 3.1, we have the following results:

**Proposition 3.2** (i) If $e = v_r v_{r+1} \in E(P_n)$, then $E_{SO}(P_n - e) = E_{SO}(P_r) + E_{SO}(P_s)$, where $r + s = n$.
(ii) If $e \in E(C_n)$, $(n \geq 3)$, then $E_{SO}(C_n - e) = E_{SO}(P_n)$.
(iii) Let $S_n$ be the star on $n$ vertices and $e \in E(S_n)$. Then, for any $n \geq 3$,

$$E_{SO}(S_n - e) = E_{SO}(S_{n-1}).$$

Now, consider to the 2-regulars. Every 2-regular graph is a disjoint union of cycles. By Theorem 2.4, we can find all the eigenvalues of Sombor matrix of cycle graphs. Therefore, by Lemma 3.1, we can find Sombor characteristic polynomial and Sombor energy of 2-regular graphs. Now, we consider to the characteristic polynomial of 3-regular graphs of order 10. In addition, we shall compute Sombor energy of this class of graphs. There are exactly 21 cubic graphs of order 10 given in Fig. 1 (see Khosrovshahi and Maysoori (2001)).

Using Maple, we computed the Sombor characteristic polynomials of 3-regular graphs of order 10 in Table 1. By finding the roots of Sombor characteristic polynomial of cubic graphs of order 10, we can have the Sombor energy of these graphs. We compute them to three decimal places. Therefore, we have them in Table 2.

**Proposition 3.3** Six cubic graphs of order 10 are not $E_{SO}$-unique.

**Proof** By observing Table 2, we see that $[G_1] = [G_1, G_8], [G_{12}] = [G_{12}, G_{17}]$ and $[G_{16}] = [G_{16}, G_{20}]$. Therefore, we have 15 cubic graphs of order 10 which are $E_{SO}$-unique.

As an immediate result of Proposition 3.3, we have

**Corollary 3.4** In general, two $k$-regular graphs of the same order may not have same Sombor energy.

**Theorem 3.5** Let $\mathcal{G}$ be the family of 3-regular graphs of order 10. For the Petersen graph $P$ (Fig. 2), we have the following properties:

(i) $P$ is not $E_{SO}$-unique in $\mathcal{G}$.
(ii) $P$ has the maximum Sombor energy in $\mathcal{G}$.
Fig. 1  Cubic graphs of order 10
Proof (i) The Sombor matrix of $P$ is

$$A_{SO}(P) = \begin{pmatrix}
0 & 3\sqrt{2} & 0 & 0 & 3\sqrt{2} & 0 & 0 & 0 \\
3\sqrt{2} & 0 & 3\sqrt{2} & 0 & 0 & 0 & 3\sqrt{2} & 0 \\
3\sqrt{2} & 0 & 3\sqrt{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 3\sqrt{2} & 0 & 3\sqrt{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 3\sqrt{2} & 0 & 3\sqrt{2} & 0 & 0 & 0 \\
3\sqrt{2} & 0 & 0 & 3\sqrt{2} & 0 & 0 & 0 & 0 \\
3\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 3\sqrt{2} & 3\sqrt{2} \\
0 & 3\sqrt{2} & 0 & 0 & 0 & 0 & 3\sqrt{2} & 3\sqrt{2} \\
0 & 0 & 0 & 3\sqrt{2} & 0 & 3\sqrt{2} & 3\sqrt{2} & 0 \\
0 & 0 & 0 & 3\sqrt{2} & 0 & 3\sqrt{2} & 3\sqrt{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 3\sqrt{2} & 3\sqrt{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 3\sqrt{2} & 3\sqrt{2} \\
\end{pmatrix}.$$ 

Therefore,

$$\phi_{SO}(P, \lambda) = \det(\lambda I - A_{SO}(P)) = (\lambda - 9\sqrt{2})(\lambda + 6\sqrt{2})^4(\lambda - 3\sqrt{2})^5.$$ 

Therefore, we have

$$\lambda_1 = 9\sqrt{2}, \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = -6\sqrt{2}, \lambda_6 = \lambda_7 = \lambda_8 = \lambda_9 = \lambda_{10} = 3\sqrt{2},$$

and so we have $E_{SO}(P) = 48\sqrt{2}$. By Table 2, we have $P \in \{G_{12}, G_{17}\}$. Hence, $P$ is not $E_{SO}$-unique in $G$.

(ii) It follows from Part (i) and Table 2.

Now, we check that is there any relationship between Sombor energy and permanent of adjacency matrix of two connected $k$-regular graphs of the same order?

Observation 3.6 If two connected $k$-regular graphs have the same Sombor energy, then their adjacency matrices may have or have not the same permanent.

Proof We consider to the cubic graphs of order 10. By Table 2, $E_{SO}(G_1) = E_{SO}(G_8)$ and $E_{SO}(G_{16}) = E_{SO}(G_{20})$. Now, we find $\text{per}(A(G_1))$, $\text{per}(A(G_8))$, $\text{per}(A(G_{16}))$ and $\text{per}(A(G_{20}))$. For graph $G_1$, we have

$$A(G_1) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
\end{pmatrix}.$$ 

By Ryser’s method, we have $\text{per}(A(G_1)) = 72$. Similarly, we have

$$\text{per}(A(G_8)) = 72, \quad \text{per}(A(G_{16})) = 144, \quad \text{per}(A(G_{20})) = 180.$$ 

Therefore, we have the result.
By Observation 3.6, we know that if two connected \( k \)-regular graphs have the same Sombor energy, we can say nothing about the permanent of their adjacency matrices. Now by the following Remark, we show that if two graphs have the same permanent, then we cannot conclude that they have same Sombor energy. Therefore, in general, there is no relation between Sombor energy and permanent of adjacency matrices of \( k \)-regular graphs with the same order.
Remark 3.7 In the class of cubic graphs of order 10, we have \( \text{per}(A(G_7)) = \text{per}(A(G_{11})) = 85 \), but as we see in Table 2, \( E_{SO}(G_7) \neq E_{SO}(G_{11}) \).

As an observation, we see that every graph does not have integer-valued Sombor energy. We end this section with the following conjecture:

Conjecture 3.8 There is no graph with integer-valued Sombor energy.

4 Conclusions

In this paper, we obtained the Sombor characteristic polynomial and Sombor energy of specific graphs such as paths, cycles, stars, complete bipartite graphs and complete graphs. In addition, we studied Sombor energy of 2-regular and 3-regular graphs.

Future topics of interest for future research include the following suggestions:

- Proving Conjecture 3.8 or giving a graph with integer-valued Sombor energy.

Table 2 Sombor energy of cubic graphs of order 10

| \( G_i \) | \( E_{SO}(G_i) \) | \( G_i \) | \( E_{SO}(G_i) \) | \( G_i \) | \( E_{SO}(G_i) \) |
|---|---|---|---|---|---|
| \( G_1 \) | 64.161 | \( G_8 \) | 64.161 | \( G_{15} \) | 62.767 |
| \( G_2 \) | 63.043 | \( G_9 \) | 64.981 | \( G_{16} \) | 59.396 |
| \( G_3 \) | 62.880 | \( G_{10} \) | 61.399 | \( G_{17} \) | 67.882 |
| \( G_4 \) | 57.336 | \( G_{11} \) | 62.375 | \( G_{18} \) | 57.517 |
| \( G_5 \) | 60.638 | \( G_{12} \) | 67.882 | \( G_{19} \) | 66.096 |
| \( G_6 \) | 63.403 | \( G_{13} \) | 61.000 | \( G_{20} \) | 59.396 |
| \( G_7 \) | 63.969 | \( G_{14} \) | 65.835 | \( G_{21} \) | 50.911 |

Fig. 2 Petersen graph
• What is the relationship between $ESO(G)$ and $ESO(G - e)$ where $e \in E(G)$?
• What can we say about $ESO(G)$ and $ESO(G - v)$ where $v \in V(G)$?
• What is the Sombor energy of $G \ast H$ where $\ast$ is some kind of operation on two graph?
• If two graphs of the same order be $\mathcal{E}_{SO}$-equivalent, do they have any properties in common?

Acknowledgements
The author would like to express his gratitude to the referee for her/his careful reading and helpful comments. In addition, he would like to thank the Research Council of Norway and Department of Informatics, University of Bergen for their support.

Funding
Open access funding provided by University of Bergen (incl Haukeland University Hospital)

Open Access
This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder.

To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

References
Alikhani S, Ghanbari N (2015) Randić energy of specific graphs. Appl Math Comput 269:722–730
Alikhani S, Ghanbari N (2021) Sombor index of polymers. MATCH Commun Math Comput Chem 86:715–728
Bozkurt SB, Bozkurt D (2013) Sharp upper bounds for energy and Randić Energy. MATCH Commun Math Comput Chem 70:669–680
Bozkurt SB, Gutman I (2013) Estimating the incidence energy. MATCH Commun Math Comput Chem 70:143–156
Bozkurt SB, Güngör AD, Gutman I, Ćevik AS (2010) Randić matrix and Randić energy. MATCH Commun Math Comput Chem 64:239–250
Chen H, Li W, Wang J (2022) Extremal values on the Sombor index of trees. MATCH Commun Math Comput Chem 87:23–49. https://doi.org/10.46793/match.87-1.023
Chen L, Shi Y (2015) Maximal matching energy of tricyclic graphs. MATCH Commun Math Comput Chem 73:105–119
Das R, Gutman I, Rada J (2021) Sombor index of chemical graphs. Appl Math Comp 399:126018
Cvetković D, Doob M, Sachs H (1980) Spectra of graphs - theory and application. Academic Press, New York
Das KC, Ćevik AS, Çangul IN, Shang Y (2021) On Sombor index. Symmetry 13:140
Das KC, Gutman I, Ćevik AS, Zhou B (2013) On Laplacian energy. MATCH Commun Math Comput Chem 70:689–696
Das KC, Sorgun S (2014) On Randić energy of graphs. MATCH Commun Math Comput Chem 72:227–238
Deng H, Tang Z, Wu R, Molecular trees with extremal values of Sombor indices, Int. J. Quantum Chem. https://doi.org/10.1002/qua.26622.
Ghanbari N, Alikhani S (2021) Sombor index of certain graphs. Iranian J Math Chem 12(1):27–37
Gowtham KJ, Swamy NN (2021) On Sombor energy of graphs. Nanosyst Phys Chem Math 12(4):411–417
Gutman I (2021) Geometric approach to degree based topological indices. MATCH Commun Math Comput Chem 86(1):11–16
Das KC, Sorgun S (2014) On Randić energy of graphs. MATCH Commun Math Comput Chem 72:227–238
Deng H, Tang Z, Wu R, Molecular trees with extremal values of Sombor indices, Int. J. Quantum Chem. https://doi.org/10.1002/qua.26622.
Ghanbari N, Alikhani S (2021) Sombor index of certain graphs. Iranian J Math Chem 12(1):27–37
Gowtham KJ, Swamy NN (2021) On Sombor energy of graphs. Nanosyst Phys Chem Math 12(4):411–417
Gutman I (2021) Geometric approach to degree based topological indices. MATCH Commun Math Comput Chem 86(1):11–16
Gutman I (2021) Spectrum and energy of the Sombor matrix. Vojnotehnicki Glasnik 69(3):551–561
Gutman I (2001) The energy of a graph: old and new results. In: Betten A, Kohnert A, Laue R, Wassermann A (eds) Algebraic combinatorics and applications. Springer, Berlin, pp 196–211
Gutman I (2005) Topology and stability of conjugated hydrocarbons. The dependence of total $\pi$-electron energy on molecular topology. J Serb Chem Soc 70:441–456
Gutman I, Furtula B, Bozkurt SB (2014) On Randić energy. Linear Algebra Appl 442:50–57
Gutman I, Li X, Zhang J (2009) Graph energy. In: Dehmer M, Emmert-Streib F (eds) Analysis of complex. From biology to linguistics. Wiley, Weinheim, pp 145–174
Gutman I, Redžepović I (2022) Sombor energy and Hückel rule. Discrete Math Lett 9:67–71. https://doi.org/10.47443/dml.2021.s211
Jayanna GK, Gutman I (2021) On characteristic polynomial and energy of Sombor matrix. Open J Discrete Appl Math 4(3):29–35. https://doi.org/10.30538/psrp-odam2021.0062
Ji S, Li X, Shi Y (2013) Extremal matching energy of bicyclic graphs. MATCH Commun Math Comput Chem 70:697–706
Khosrovshahi GB, Maysoori Ch (2001) Tayfeh-Rezaie, A note on 3-factorizations of $K_{10}$. J Combin Des 9:379–383
Li S, Wang Z, Zhang M (2022) On the extremal Sombor index of trees with a given diameter. Appl Math Comput 416:126731
Majstorović S, Kloučar A, Gutman I (2009) Selected topics from the theory of graph energy: hypoenergetic graphs. In: Cvetković D, Gutman I (eds) Applications of graph spectra. Math. Inst, Belgrade, pp 65–105
Redžepović I (2021) Chemical applicability of Sombor indices. J Serb Chem Soc 86:445–457
Stevanović D, Milošević M, Hic P, Pokorný M (2013) Proof of a conjecture on distance energy of complete multipartite graphs. MATCH Commun Math Comput Chem 70:157–162
Wang Z, Mao Y, Li Y, Furtula B (2021) On relations between Sombor and other degree-based indices. J Appl Math Comput. https://doi.org/10.1007/s12190-021-01516-x

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.