Research Article

Hyers–Ulam Stability, Exponential Stability, and Relative Controllability of Non-Singular Delay Difference Equations

Sawitree Moonsuwan,1 Gul Rahmat ,2 Atta Ullah,2 Muhammad Yasin Khan,2 Kamran ,2 and Kamal Shah 3,4

1Mathematics Department, Faculty of Science and Technology, Suan Dusit University, Bangkok 10300, Thailand
2Department of Mathematics, Islamia College Peshawar, Khyber Pakhtunkhwa, Peshawar, Pakistan
3Department of Mathematics and Sciences, Prince Sultan University, P. O. Box 66833, Riyadh 11586, Saudi Arabia
4Department of Mathematics, University of Malakand, Chakdara Dir(L) 18000, Khyber Pakhtunkhwa, Pakistan

Correspondence should be addressed to Kamal Shah; kamalshah408@gmail.com

Received 8 August 2022; Accepted 22 September 2022; Published 18 October 2022

Academic Editor: Toshikazu Kuniya

Copyright © 2022 Sawitree Moonsuwan et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, we study the uniqueness and existence of the solutions of four types of non-singular delay difference equations by using the Banach contraction principles, fixed point theory, and Gronwall’s inequality. Furthermore, we discussed the Hyers–Ulam stability of all the given systems over bounded and unbounded discrete intervals. The exponential stability and controllability of some of the given systems are also characterized in terms of spectrum of a matrix concerning the system. The spectrum of a matrix can be easily obtained and can help us to characterize different types of stabilities of the given systems. At the end, few examples are provided to illustrate the theoretical results.

1. Introduction

In mathematics, we usually observed that many of the biological systems and models can be resolved by using differential equations. Differential equations have a lot of applications in various fields of natural sciences, economics, statistics, and engineering (see [1–4] and the references therein). Although differential equations are too useful, when we discuss a real-life problem, we need to take the sample in discrete form and show the model in a form of difference equations (for details, see [5, 6]). The applications of difference equation have appeared recently in many fields of sciences and technology, mathematical physics, and biological systems. The theory of difference equations will continue its role in mathematics as a whole because during the period of development of mathematics together with information revolution, there are many difference equations to describe the real problem such as the monographs and wind flow. Similarly, many models were described by fractional-order differential equation (FODE), in which the order of derivative is in fraction form rather than an integer form. These types of differential equations have a lot of applications in real life [7, 8]. In [7], the theoretical study of the Caputo–Fabrizio fractional modelling for hearing loss due to mumps virus with optimal control was discussed which is useful contribution in natural science. Also in [8] some novel mathematical analysis of fractal-fractional model of the AH1N1/09 virus and its generalized Caputo-type version was explained.

Any type of system has some properties (qualitative properties), in which the stability is more important. Every differential system has some qualitative properties, in which the stability plays a vital role. Using this, the system performance can be checked. A differential have various types of stabilities, but here we are interested in Hyers–Ulam stability, because nowadays many researchers wants to know about this stability. The idea of Hyers–Ulam stability started in 1940 [9]. Ulam in a seminar, in his presentation he
pointed out some problems associated with the stability of group homomorphism. After a year in [10], Hyers gave a positive answer to the Ulam’s question by considering Banach Space in place of that group. The general approach of this stability was given in 1978, by Rassias [11]. He also used this idea in the Cauchy difference system. Obloza [12] used this idea in differential equations, and later Jung [13] and Khan et al. [14] used it in the difference equations. This stability was also discussed in fractional differential equation by Gao et al. [15], and some results on Ulam-type stability of a first-order non-linear delay dynamic system were discussed by Shah et al. in [16]. Recently, the Hyers–Ulam stability of second order differential equations by using Mahgoub transform and generalized Hyers–Ulam stability of a coupled hybrid system of integro-differential equations involving $\phi$-caputo fractional operator was studied in [17,18]. The existence and Hyers–Ulam stability of solution for almost periodical fractional stochastic differential equation was discussed in [19]. Also in [20], the existence and Hyers–Ulam stability of random impulsive stochastic functional differential equations with finite delays was discussed, which showed that the Hyers–Ulam stability have a lot of contribution in fractional calculus.

Controllability is one of the fundamental concepts in modern mathematical control theory. Kalman’s result [21] on controllability assumes that controls are functions on time having values on some non-empty subset of $R^n$. This is a qualitative property of control systems and is of particular importance in control theory. Systematic study of controllability was started at the beginning of 1960s and theory of controllability is based on the mathematical description of the dynamical system. Many dynamical systems are such that the control does not affect the complete state of the dynamical system but only a part of it. On the other hand, very often in real industrial processes, it is possible to observe only a certain part of the complete state of the dynamical system. Therefore, it is very important to determine whether or not control of the complete state of the dynamical system is possible. Roughly speaking, controllability generally means that it is possible to steer dynamical system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. Controllability plays an essential role in the development of the modern mathematical control theory. There are important relationships between controllability, stability, and stabilizability of linear control systems [22, 23]. Controllability is also strongly connected with the theory of minimal realization of linear time-invariant control systems. Moreover, it should be pointed out that there exists a formal duality between the concepts of controllability and observability [24].

The delay difference system can be used in the characterization of automatic engine, control theory, and physiology system. Khusainov et al [25] solved the linear autonomous delay-time system with commutable matrices. Diblik and Khusainov [26] gave the description about the solutions of discrete delayed system using the idea [25]. Then, Wang et al. [27] studied relative controllability and exponential stability of non-singular systems. Recently, the generalized Hyers–Ulam–Rassias stability of impulsive difference equations was demonstrated by Almalki et al. [28]. Kuruklis [29] and Yu [30] studied the asymptotic behavior of the variable type delay difference equation. Kosmala and Teixeira [31] provided a good insight and discussed the behavior of solution of the difference equation of the type $U_{k+1} = (A + U_{k-1})/(BU_k + U_{k-1})$. Liu et al [32] designed the exponential behavior of switch discrete-time delay system. Marwen and Sakly [33] discussed the stability techniques about the switched non-linear time-delay difference equations. Yuanyuan [34] described the stability techniques of high-order difference systems. The stability of higher-order rational difference systems was studied by Khalil [35].

Our present study is focused on the Hyers–Ulam stability and exponential stability of non-singular delay difference system of the form

$$
\begin{align*}
EV_{n+1} &= AV_n + BV_{n-k}, n \geq 0, k \geq 0,
V_n &= \Phi m, -k \leq n \leq 0,
\end{align*}
$$

and

$$
\begin{align*}
EV_{n+1} &= AV_n + BV_{n-k} + f(n, V_n), n \geq 0, k \geq 0,
V_n &= \Psi m, -k \leq n \leq 0,
\end{align*}
$$

where the commutable constant matrices are $E, A, B \in R^{m \times n}$ and $E$ is non-singular. $\phi \in B(Z^*, X)$, the space of bounded sequences, and $F \in CS(Z^* \times X, X)$, the space of convergent sequences, where $J = [-k, -k + 1, \ldots, 0]$, $Z_* = \{0, 1, 2, \ldots\}$, and $X = R^n$. Also, our focus is on relative controllability of the system

$$
\begin{align*}
EV_{n+1} &= AV_n + BV_{n-k} + y(n, V_n) + CU_n, n \in I, k \geq 0,
V_n &= \Psi m, -k \leq n \leq 0,
\end{align*}
$$

where $I = \{0, 1, 2, \ldots, n\}, n > 0, C \in R^{m \times n}, y \in CS(Z^* \times X, X)$, and the control function $U(\cdot)$ takes values from $L^2(I, R^n)$. The continuous form of this work is given in [27]. The Hyers–Ulam stability of (3) was recently presented in [36].

2. Preliminaries

Here, we discuss some notations and definitions, which will be needed for our main work. By $R^n$ and $R^{m \times n}$, we will denote the $n$-dimensional Euclidean space with vector norm $\| \cdot \|$ and $n \times n$ matrices with real-valued entries. The vector infinite-norm is defined as $\|v\| = \max_{1 \leq i \leq n} |v_i|$ and the matrix infinite-norm is given as $\|A\| = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|$ where $v \in R^n$ and $A \in R^{m \times n}$, also, $v_i$ and $a_{ij}$ are the elements of the vector $v$ and the matrix $A$. $B(I, X)$ will be the space of all bounded sequences from $I$ to $X$ with norm $\|v\|_c = \sup_{t \in I} \|v_t\|$. We will use $R$, $Z$ and $Z_*$ for the set of real, integer, and non-native integer numbers, respectively. Also, we define $B(I, X) = \{v \in B(I, X) ; v' \in B'(I, X)\}$. 

Complexity
Lemma 1. The non-singular delay difference systems (1)–(4) have the solutions:

\[ V_n = A^n e^{-n} \Phi_0 + BA^{n-1} e^{-n} \sum_{i=0}^{k} A^{-i} E \Phi_{i-k} + BA^{n-1} e^{-n} \sum_{i=k+1}^{n} A^{-i} E V_{i-k}, \]

\[ V_n = A^n e^{-n} \psi_0 + A^{n-1} e^{-n} \sum_{i=0}^{k} A^{-i} E (B \Phi_{i-k} + f(i, V_i)) + A^{n-1} e^{-n} \sum_{i=k+1}^{n} A^{-i} E (BV_{i-k} + f(i, V_{i-k})), \]

\[ V_n = A^n e^{-n} \psi_0 + A^{n-1} e^{-n} \sum_{i=0}^{k} A^{-i} E (B(\psi_{i-k} + F(i, V_{i-k}))) + A^{n-1} e^{-n} \sum_{i=k+1}^{n} A^{-i} E (BV_{i-k} + F(i, V_{i-k})), \]

and

\[ V_n = A^n e^{-n} \psi_0 + A^{n-1} e^{-n} \sum_{i=0}^{k} A^{-i} E (B(\psi_{i-k} + y(i, V_i)) + CU_i) + A^{n-1} e^{-n} \sum_{i=k+1}^{n} A^{-i} E (BV_{i-k} + y(i, V_{i-k}) + CU_i), \]

respectively, where \( AE = EA, AB = BA, \) and \( EB = BE. \) The proofs can easily be obtained by successively putting the values of \( n \in \{-k, -k+1, \ldots\}. \)

Definition 1. The solution of system (1) is said to be exponentially stable if there exist positive real numbers \( \lambda_1 \) and \( \lambda_2, \) such that

\[ \| V_n \| \leq \lambda_1 e^{-\lambda_2 n}, \forall n \geq 0. \]

Definition 2. For a positive number \( \epsilon, \) the sequence \( \psi_n \) is said to be an \( \epsilon \)-approximate solution of (1)–(3) if the following holds:

\[ \| E\psi_{n+1} - A\psi_n - B\psi_{n-k} \| \leq \epsilon, n \geq 0, k \geq 0 \]

\[ \| \psi_n - \Phi_n \| \leq \epsilon, -k \leq n \leq 0 \]

\[ \| E\psi_{n+1} - A\psi_n - B\psi_{n-k} - f(n, \psi_n) \| \leq \epsilon, n \geq 0, k \geq 0 \]

\[ \| \psi_n - \Phi_n \| \leq \epsilon, -k \leq n \leq 0 \]

\[ \| E\psi_{n+1} - A\psi_n - B\psi_{n-k} - f(n, \psi_{n-k}) \| \leq \epsilon, n \geq 0, k \geq 0 \]

\[ \| \psi_n - \psi_n \| \leq \epsilon, -k \leq n \leq 0. \]

Definition 3. Systems (1)–(3) are said to be Hyers–Ulam stable if for every \( \epsilon \)-approximate solutions \( \psi_n \) of systems (1)–(3) there are exact solutions \( Y_n \) of (1)–(3) and a non-negative real number \( K \) such that

\[ \| Y_n - \psi_n \| \leq K \epsilon, n \in I. \]

Definition 4. System (4) is said to be relatively controllable, if for initial vector function \( \Psi \in \mathbb{B}(\mathbb{J}, X) \) and final state of the vector function \( v_1 \in X, \) there exists a control \( u \in \mathcal{L}^2(\mathbb{I}, X) \) such that (4) has a solution \( v \in \mathbb{B}([-v, \ldots, n_1]), X \) which satisfies the boundary condition \( v_{n_1} = v_1. \)

Remark 1. It is clear from (5) that \( Y \in \mathbb{B}(\mathbb{I}, X) \) satisfies (5) if and only if there exists \( f \in \mathbb{B}(\mathbb{I}, X) \) satisfying

\[ \| f_n \| \leq \epsilon, n \in I, \]

\[ E_2^n Y_{n+1} = A^2 Y_n + B^2 Y_{n-k} + f_n, n \in \mathbb{Z}_+, \]

\[ y_n = \Phi_n, -k \leq n \leq 0. \]

\[ \| f_n \| \leq \epsilon, n \in I, \]

\[ E_2^n Y_{n+1} = A^2 Y_n + B^2 Y_{n-k} + f(n, Y_{n-k}) + f_n, n \in \mathbb{Z}_+, \]

\[ y_n = \Phi_n, -k \leq n \leq 0. \]

\[ \| f_n \| \leq \epsilon, n \in I, \]

\[ E_2^n Y_{n+1} = A^2 Y_n + B^2 Y_{n-k} + f(n, Y_{n-k}) + f_n, n \in \mathbb{Z}_+, \]

\[ y_n = \Psi_n, -k \leq n \leq 0. \]

3. Existence and Uniqueness of Solutions

Here, we will discuss the existence and uniqueness of the solution of system (1). For this, we need the following assumptions:

\( \Lambda_1: \) the linear system \( AG_{n+1} = MG_n + NG_{n-k} \) is well modelled.

\( \Lambda_2: \) \( A^{n-1} \| E^{-n} \| L < 1. \)

Theorem 1. If assumptions \( \Lambda_1 \) and \( \Lambda_2 \) hold, then system (1) has a unique solution \( V \in \mathbb{B}(\mathbb{I}, X). \)

Proof. Define \( T: \mathbb{B}(\mathbb{I}, X) \rightarrow \mathbb{B}(\mathbb{I}, X) \) by

\[ (TV)_n = A^n e^{-n} \Phi_0 + BA^{n-1} e^{-n} \sum_{i=0}^{k} A^{-i} E \Phi_{i-k} + BA^{n-1} e^{-n} \sum_{i=k+1}^{n} A^{-i} E V_{i-k}. \]
Proof. System (1) is Hyers–Ulam stable over bounded interval.

Theorem 2. We will put the following assumptions:

\( \Lambda_1: \) the linear system \( EV_{n+1} = AV_n + BV_{n-k} \) is well posed.

\( \Lambda_2: \) there exists a constant \( \eta \) such that

\[
\sum_{r=1}^{n-k} \phi_r \leq \eta \text{ for each } n \in I. \tag{14}
\]

4. Hyers–Ulam Stability over Bounded Discrete Interval

In this part of the paper, we will discuss the Hyers–Ulam stability over bounded discrete interval. Before the result, we will put the following assumptions:

\( \Lambda_1: \) the linear system \( EV_{n+1} = AV_n + BV_{n-k} \) is well posed.

\( \Lambda_2: \) there exists a constant \( \eta \) such that

\[
\sum_{r=1}^{n-k} \phi_r \leq \eta \text{ for each } n \in I. \tag{14}
\]

Theorem 2. If \( \Lambda_1 \) and \( \Lambda_2 \) and Remark 1 are satisfied, then system (1) is Hyers–Ulam stable over bounded interval.

Proof. The solution of difference system (1) is

\[
V_n = A^n E^{-n} \Phi_0 + BA^{n-1} E^{-n} \sum_{i=0}^{k} A^{-i} E^i \Phi_{i-k} + BA^{n-1} E^{-n} \sum_{i=k+1}^{n} A^{-i} E^i V_{i-k}.
\]

From Remark 1, the solution of

\[
\begin{cases}
EU_{n+1} = AU_n + BU_{n-k} + f_n, n \geq 0, k \geq 0, \\
U_n = \Phi_{n-k}, -k \leq n \leq 0,
\end{cases}
\]

is

\[
(U_n)_{n} - (TV'_n) = \left\| A^n E^{-n} \Phi_0 + BA^{n-1} E^{-n} \sum_{i=0}^{k} A^{-i} E^i \Phi_{i-k} + BA^{n-1} E^{-n} \sum_{i=k+1}^{n} A^{-i} E^i V_{i-k} \right\|.
\]

(12)

This implies that

\[
\left\| (TV_n) - (TV'_n) \right\| \leq \left\| A^n \right\| \left\| E^{-n} \right\| \sum_{i=0}^{n} \left\| A^{-i} E^i \right\| \left\| V_{i-k} - V_{i-k} \right\|
\]

(13)

\[
\leq \left\| A^n \right\| \left\| E^{-n} \right\| \left\| L \right\| \left\| V - V' \right\|.
\]

Thus, \( T \) is contraction if \( \left\| A^n \right\| \left\| E^{-n} \right\| L < 1 \), so by BCP it has a unique fixed point and will be the solution of system (1). Similarly, we can show the existence and uniqueness of solutions of systems (2)–(4). For (3), we also refer to [36].

4 Complexity

Now, we have

\[
\left\| U_n \right\| = \left\| A^n E^{-n} \Phi_0 + BA^{n-1} E^{-n} \sum_{i=0}^{k} A^{-i} E^i \Phi_{i-k} + BA^{n-1} E^{-n} \sum_{i=k+1}^{n} A^{-i} E^i V_{i-k} \right\|
\]

(17)

\[
\left\| U_n - V_n \right\| = \left\| A^n E^{-n} \Phi_0 - BA^{n-1} E^{-n} \sum_{i=0}^{k} A^{-i} E^i \Phi_{i-k} + BA^{n-1} E^{-n} \sum_{i=k+1}^{n} A^{-i} E^i \right\|
\]

(18)

\[
\left\| U_n - V_n \right\| = \left\| A^{n-1} E^{-n} \sum_{i=0}^{k} A^{-i} E^i f_{i-k} \right\|
\]

where \( L = L^\eta. \) Hence, system (1) is Hyers–Ulam stable over bounded discrete interval.

Next, we will show that system (2) is Hyers–Ulam stable. Again, we need one more assumption:

\( \Lambda_3: \) the map \( F: I \times X \longrightarrow X \) satisfies the Carathéodory condition

\[
\left\| F(n, \bar{y}) - F(n, \bar{y}') \right\| \leq K \left\| \bar{y} - \bar{y}' \right\|,
\]

(19)

for some \( K \geq 0 \) and for all \( \bar{y}, \bar{y}' \in B(I, X). \)
Theorem 3. If $\Lambda_1 - \Lambda_3$ along with (2.6) and Remark 1 are satisfied, then system (2) is Hyers–Ulam stable over bounded interval.

Proof. The solution of delay difference system (2) is

$$V_n = A^nE^{-n}\phi_0 + A^{n-1}E^{-n} \sum_{i=0}^{k} A^{-i}E^i (B\phi_{i-k} + f(i, V_i))$$

+ $A^{n-1}E^{-n} \sum_{i=k+1}^{n} A^{-i}E^i (BV_{i-k} + f(i, V_i))$.

Also, from Remark 1, the solution of

$$\left\| U_n - V_n \right\| = \left\| A^nE^{-n}\phi_0 + A^{n-1}E^{-n} \sum_{i=0}^{k} A^{-i}E^i (B\phi_{i-k} + f(i, U_i)) \right\|$$

$$+ A^{n-1}E^{-n} \sum_{i=k+1}^{n} A^{-i}E^i (BU_{i-k} + f(i, U_i) + f_{i-k})$$

$$+ A^{n-1}E^{-n} \sum_{i=k+1}^{n} A^{-i}E^i (BV_{i-k} + f(i, V_i))$$

$$\leq \left\| A^n \right\| \left\| E^{-n} \right\| \sum_{i=k+1}^{n} \left\| A^{-i}E^i \right\| \left\| (BU_{i-k} - BV_{i-k}) \right\|$$

+ $\| f(i, U_i) - f(i, V_i) \| + \| f_{i-k} \|$.

$$\left\| U_n - V_n \right\| \leq \left\| A^n \right\| \left\| E^{-n} \right\| \sum_{i=k+1}^{n} \left\| A^{-i}E^i \right\| \left\| (BU_{i-k} - BV_{i-k}) \right\| + L \left\| U_i - V_i \right\| + \| f_{i-k} \|$$

$$= \left\| A^n \right\| \left\| E^{-n} \right\| \sum_{i=k+1}^{n} \left\| A^{-i}E^i \right\| \| f_{i-k} \|$$

$$\leq \left\| A^n \right\| \left\| E^{-n} \right\| \sum_{i=k+1}^{n} \left\| A^{-i}E^i \right\| \| \phi_{i-k} \|$$

$$= \epsilon \left\| A^n \right\| \left\| E^{-n} \right\| \sum_{r=1}^{n-k} \left\| A^{-k-r}E^{k+r} \right\| \| \phi_r \|$$

$$= L^4 \sum_{r=1}^{n-k} \phi_r$$

$$\leq L^4 \eta$$

$$= K\epsilon.$$
Thus, system (2) is Hyers–Ulam stable. The Hyers–Ulam stability of system (3) over bounded discrete interval is discussed in [36].

5. Hyers–Ulam Stability over an Unbounded Discrete Interval

Here, we discuss the Hyers–Ulam stability of systems (1)–(3) over an unbounded discrete interval; we have some assumptions:

$A_1$: the operator family $\|L^i\| \leq N e^{-\nu n}, n \geq 0, \nu \geq 0, N \geq 1$.

$A_2$: the linear system $AG_{n+1} = MG_n + NG_{n-k}$ is well posed.

$A_3$: also, assume that

$$\sum_{r=1}^{n-1} \Phi_r \leq \eta,$$  \hspace{1cm} (24)

for each $n \in \mathbb{Z}_+$, and for $\eta \geq 0$.

**Theorem 4.** If $A_1$–$A_3$ along with (2.6) and Remark 1 are satisfied, then system (1) is Hyers–Ulam stable over an unbounded interval.

**Proof.** The exact solution of non-autonomous difference system (1) is

$$V_n = A^nE^n\Phi_0 + BA^{n-1}E^{n-k} \sum_{i=0}^{k} A^{-i}E\Phi_{i-k} $$

$$+ BA^{n-1}E^{n-k} \sum_{i=k+1}^{n} A^{-i}E^iV_{i-k}.$$  \hspace{1cm} (25)

Let $Y$ be the approximate solution of system (1); then, clearly, for a sequence $f_n$, with $\|f_n\| \leq \epsilon$, we have

$$\begin{align*}
EY_{n+1} &= AY_n + BY_{n-k} + f_n, n \geq 0, k \geq 0, \\
Y_n &= \Phi_{n0}, -k \leq n \leq 0,
\end{align*}$$

and

$$\begin{align*}
Y_n &= A^nE^n\Phi_0 + A^{n-1}E^{n-k} \sum_{i=0}^{k} A^{-i}E^i\Phi_{i-k} $$

$$+ A^{n-1}E^{n-k} \sum_{i=k+1}^{n} A^{-i}E^i(BY_{i-k} + f_{i-k}).$$  \hspace{1cm} (27)

Now, we have

$$\|Y_n - U_n\| = \left\| A^nE^n\Phi_0 + BA^{n-1}E^{n-k} \sum_{i=0}^{k} A^{-i}E^i\Phi_{i-k} $$

$$+ A^{n-1}E^{n-k} \sum_{i=k+1}^{n} A^{-i}E^i(BY_{i-k} + f_{i-k}) \\
- A^nE^n\Phi_0 - BA^{n-1}E^{n-k} \sum_{i=0}^{k} A^{-i}E^i\Phi_{i-k} $$

$$- BA^{n-1}E^{n-k} \sum_{i=k+1}^{n} A^{-i}E^iU_{i-k} \\
+ BA^{n-1}E^{n-k} \sum_{i=k+1}^{n} A^{-k}E^i\Phi_{i-k} $$

$$\leq \|A^n\|\|E^n\| \sum_{i=k+1}^{n} A^{i-k}E^i\|\|BY_{i-k} + f_{i-k}\|$$

$$\leq \|A^n\|\|E^n\| \sum_{i=k+1}^{n} A^{i-k}E^i\|\|BY_{i-k} - BY_{i-k}\|$$

$$+ \|A^n\|\|E^n\| \sum_{i=k+1}^{n} A^{i-k}E^i\|\|f_{i-k}\|$$

$$\leq \|A^n\|\|E^n\| \sum_{i=k+1}^{n} A^{i-k}E^i\|\|\Phi_{i-k}\|$$

$$\leq \epsilon L^i \sum_{r=1}^{n} \Phi_r$$

$$\leq \epsilon L^i \eta$$

$$\leq Ne^{-\nu n} \eta K$$

$$= L.$$

(28)

where $L = Me^{-\nu n}$, thus, system (1) is Hyers–Ulam stable over an unbounded interval.

To prove the Hyers–Ulam stability of system (2), we have to add one more assumption:

$A_4$: the continuous function $H: \mathbb{Z}_+ \times X \rightarrow X$ satisfies the Carathéodory condition

$$\|H(n, \omega) - H(n, \omega')\| \leq K\|\omega - \omega'\|, K \geq 0, \hspace{1cm} (29)$$

for every $n \in \mathbb{Z}_+, \omega, \omega' \in X$.

**Theorem 5.** If $A_1$–$A_4$ along with (2.6) and Remark 1 are satisfied, then system (2) is Hyers–Ulam stable over an unbounded interval.
Proof. The solution of delay difference system (2) is

\[
V_n = A^n E^n \phi_0 + A^{n-1} E^{-n} \sum_{i=0}^{k} A^{-i} E^i (B\phi_{i-k} + f(i, V_i))
\]

\[
U_n = A^n E^n \phi_0 + A^{n-1} E^{-n} \sum_{i=0}^{k} A^{-i} E^i (B\phi_{i-k} + f(i, U_i))
\]

(30)

Also, from Remark 1, the solution of

\[
\begin{aligned}
\phi_{n+1} &= AU_n + BU_{n-k} + f(n, U_n) + \varepsilon n, n \geq 0, k \geq 0, \\
U_n &= \phi_n, -k \leq n \leq 0,
\end{aligned}
\]

(31)

Now, we have

\[
\|U_n - V_n\| = \|A^n E^n \phi_0 + A^{n-1} E^{-n} \sum_{i=0}^{k} A^{-i} E^i (B\phi_{i-k} + f(i, U_i)) +
\]

\[
A^{n-1} E^{-n} \sum_{i=0}^{k} A^{-i} E^i (BU_{i-k} + f(i, U_i) + f_{i-k}) -
\]

\[
A^n E^n \phi_0 - A^{n-1} E^{-n} \sum_{i=0}^{k} A^{-i} E^i (B\phi_{i-k} + f(i, V_i)) -
\]

\[
A^{n-1} E^{-n} \sum_{i=0}^{k} A^{-i} E^i (BV_{i-k} + f(i, V_i))
\]

\[
= \left| A^{n-1} E^{-n} \sum_{i=0}^{k} A^{-i} E^i (BU_{i-k} + f(i, U_i) + f_{i-k}) -
\]

\[
A^{n-1} E^{-n} \sum_{i=0}^{k} A^{-i} E^i (BV_{i-k} + f(i, V_i))
\]

\[
= \left| A^{n-1} E^{-n} \sum_{i=0}^{k} A^{-i} E^i (BU_{i-k} + f(i, U_i)) +
\]

\[
A^{n-1} E^{-n} \sum_{i=0}^{k} A^{-i} E^i f_{i-k} - A^{n-1} E^{-n} \sum_{i=0}^{k} A^{-i} E^i (BV_{i-k} + f(i, V_i))
\]

\[
\leq \|A^{n-1}\| \|E^{-n}\| \sum_{i=0}^{k} \|A^{-i} E^i\| \|BU_{i-k} - BV_{i-k}\| +
\]

\[
\|f(i, U_i) - f(i, V_i)\| + \|f_{i-k}\|
\]

\[
\leq \|A^{n-1}\| \|E^{-n}\| \sum_{i=0}^{k} \|A^{-i} E^i\| \|BU_{i-k} - BV_{i-k}\| + L \|U_i - V_i\| + \|f_{i-k}\|
\]

\[
= \|A^{n-1}\| \|E^{-n}\| \sum_{i=0}^{k} \|A^{-i} E^i\| \|f_{i-k}\|
\]

\[
\leq \|A^{n-1}\| \|E^{-n}\| \sum_{i=0}^{k} \|A^{-i} E^i\| \|\phi_{i-k}\|
\]

\[
= \|A^{n-1}\| \|E^{-n}\| \sum_{r=1}^{n-k} \|A^{-k-r} E^{k+r}\| \|\phi_r\|
\]

\[
= L \sum_{r=1}^{n-k} \|\phi_r\|
\]

\[
\leq N e^{-\eta n}
\]

\[
= K
\]
where $K = N e^{-m \eta}$. Thus, system (2) is Hyers–Ulam stable.

**Theorem 6.** System (3) is Hyers–Ulam stable over an unbounded interval.

For the proof, see [36].

### 6. Exponential Stability

In this part of the paper, we will present the exponential stability of system (1). First, we recall that a discrete system is said to be exponentially stable if there exist two positive constants $M$ and $\alpha$ such that $\|V_n\| \leq M e^{-\alpha n}$ for all $n \in \mathbb{Z}_+$. Before going to the result, we will consider the following assumptions:

1. Let $\sigma (AE^{-1}) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be the eigenvalues of $AE^{-1}$ with $|\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_n| \leq r, r \in (0, 1)$.

2. $\|A^nE^{-n}\| \leq Ne^{-\alpha n}$ for some positive number $\alpha$ and for all $n \in \mathbb{Z}_+$.

**Theorem 7.** Assume that (1)–(3) are satisfied. Then, system (1) is exponentially stable.

**Proof.** The solution of system (1) is

$$V_n = A^nE^{-n}\Phi_0 + BA^{n-1}E^{-n} \sum_{i=0}^{k} A^{-i}Ei\Phi_{i-k}$$

$$+ BA^{n-1}E^{-n} \sum_{i=k+1}^{n} A^{-i}EiV_{i-k}. \quad (35)$$

Now,

$$\|V_n\| = \left\| A^nE^{-n}\Phi_0 + BA^{n-1}E^{-n} \sum_{i=0}^{k} A^{-i}Ei\Phi_{i-k} + BA^{n-1}E^{-n} \sum_{i=k+1}^{n} A^{-i}EiV_{i-k} \right\|$$

$$\leq \| A^nE^{-n}\| \|\Phi_0\| + \|B\| \left\| A^{n-1}E^{-n} \sum_{i=0}^{k} A^{-i}Ei \right\| \|V_{i-k}\|$$

$$+ \|B\| \left\| A^{n-1}E^{-n} \sum_{i=k+1}^{n} A^{-i}Ei \right\| \|V_{i-k}\|$$

$$\leq Ne^{-\alpha n} \|\Phi_0\| + \|B\| \left( Ne^{-\alpha n} \sum_{i=0}^{k} \Phi_{i-k} + \|B\| \sum_{i=k+1}^{n} Ne^{-\alpha n} \right) \|V_{i-k}\|,$$  

$$\|V_n\| \leq Ne^{-\alpha n} \left( \|\Phi_0\| + \|B\| \sum_{i=0}^{k} Ne^{-\alpha n} \Phi_{i-k} + \|B\| \sum_{i=k+1}^{n} Ne^{-\alpha n} \|V_{i-k}\| \right),$$

$$e^{\alpha n} \|V_n\| \leq N \|\Phi_0\| + \|B\| \sum_{i=0}^{k} N^2 e^{-\alpha n} \|\Phi_{i-k}\| + \|B\| \sum_{i=k+1}^{n} N^2 e^{-\alpha n} \|V_{i-k}\|$$

$$= M (\phi, \phi) + \|B\| \sum_{i=k+1}^{n} a e^{-\alpha n} \|V_{i-k}\|,$$  

where $M (\phi, \phi) = N \|\Phi_0\| + \|B\| \sum_{i=0}^{k} N^2 e^{-\alpha n} \|\Phi_{i-k}\|; \quad \text{now,}$

using the Gronwall inequality, we have

$$e^{\alpha n} \|V_n\| \leq M (\phi, \phi) e^{N^2 \|B\| n}. \quad (37)$$

From this, we have

$$\|V_n\| \leq M (\phi, \phi) e^{(N^2 \|B\| - \alpha) n}. \quad (38)$$

Using definition of stability and assumption (3), the result follows.

**Theorem 8.** Assume that (1), (2), (4), and (6) are satisfied. Then, system (2) is exponentially stable.

**Proof.** The solution of (2) is in the form of

$$V_n = A^nE^{-n}\phi_0 + A^{n-1}E^{-n} \sum_{i=0}^{k} A^{-i}Ei (B\phi_{i-k} + f(i, V_i))$$

$$+ A^{n-1}E^{-n} \sum_{i=k+1}^{n} A^{-i}Ei (BV_{i-k} + f(i, V_i)). \quad (39)$$
Now,

\[ V_n = \left\| A^n E^{-n} \phi_0 + A^{n-1} E^{-n} \sum_{i=0}^{k} A^{-i} i \left( B \psi_{i-k} + f(i, V_i) \right) \right\|, \]

\[ V_n \leq \| A^n E^{-n} \| \| \phi_0 \| + \| A^{n-1} E^{-n} \| \sum_{i=0}^{k} \left\| A^{-i} i \right\| \| B \psi_{i-k} + f(i, V_i) \| \]

\[ \leq Ne^{-an} \| \phi_0 \| + Ne^{-an} \sum_{i=0}^{k} Ne^{-an} \| B \psi_{i-k} + f(i, V_i) \| + Ne^{-an} \sum_{i=0}^{k} Ne^{-an} \| f(i, V_i) \|. \] (40)

Using (4), we have

\[ e^{an} \| V_n \| \leq M(\phi, \phi_1) e^{(N1|B|^2-a)n}, \] (42)

where \( M(\phi, \phi_1) = N \| \phi_0 \| + \sum_{i=0}^{k} N2e^{-an} \| B \| \| \psi_{i-k} \| > 0. \) From this, we have

\[ \| V_n \| \leq M(\phi, \phi_1) e^{(N1|B|^2-a)n}. \] (43)

From (50), the desired result holds.

**Theorem 9.** Assume that (1), (2), (5), and (7) are satisfied. Then, system (3) is exponentially stable.

**Proof.** The solution of (3) is in the form of

\[ \Psi_n = A^n E^{-n} \psi_0 + A^{n-1} E^{-n} \sum_{i=0}^{k} A^{-i} i \left( B \psi_{i-k} + F(i, \psi_{i-k}) \right) \]

\[ + A^{n-1} E^{-n} \sum_{i=k+1}^{n} A^{-i} i \left( B \psi_{i-k} + F(i, \psi_{i-k}) \right). \] (44)

Now consider

\[ V_n = \left\| A^n E^{-n} \psi_0 + A^{n-1} E^{-n} \sum_{i=0}^{k} A^{-i} i \left( B \psi_{i-k} + F(i, \psi_{i-k}) \right) \right\|, \]

\[ + A^{n-1} E^{-n} \sum_{i=0}^{k} A^{-i} i \left( B \psi_{i-k} + F(i, \psi_{i-k}) \right) \]

\[ \leq Ne^{-an} \| \psi_0 \| + Ne^{-an} \sum_{i=0}^{k} Ne^{-an} \| B \psi_{i-k} + F(i, \psi_{i-k}) \| + Ne^{-an} \sum_{i=0}^{k} Ne^{-an} \| F(i, \psi_{i-k}) \|. \] (45)
\[ e^{an}\|\mathcal{V}_n\| \leq N\|\psi_0\| + \sum_{i=0}^{k} N^2 e^{-an} B\psi_{i-k} + \mathcal{F}(i, \psi_{i-k})] + \sum_{i=k+1}^{n} N^2 e^{-an} \|B\|\mathcal{V}_{i-k} \]
\[ + \sum_{i=k+1}^{n} N^2 e^{-an} \|\mathcal{V}_{i-k}\| \]
\[ = M(\psi, \psi_1) + \sum_{i=k+1}^{n} N^2 e^{-an} \|B\|\mathcal{V}_{i-k} + \sum_{i=k+1}^{n} N^2 e^{-an} \|\mathcal{F}(i, \psi_{i-k})\| \]

Using (5), we have
\[ e^{an}\|\mathcal{V}_n\| \leq M(\psi, \psi_1) + \sum_{i=k+1}^{n} N^2 e^{-an} \|B\|\mathcal{V}_{i-k} \tag{47} \]
where \(M(\psi, \psi_1) = N\|\psi_0\| + \sum_{i=0}^{k} N^2 e^{-an} B\psi_{i-k} + \mathcal{F}(i, \psi_{i-k})\|\).

Using again the Gronwall inequality, we have
\[ e^{an}\|\mathcal{V}_n\| \leq M(\psi, \psi_1)e^{(N^2M[\beta])n}. \tag{48} \]
That is,
\[ \|\mathcal{V}_n\| \leq M(\psi, \psi_1)e^{(N^2M[\beta]-\alpha)n}. \tag{49} \]

From (51), the desired result holds. \[ \square \]

### 7. Controllability

In this portion, we will discuss the controllability of system (4). First, we will discuss the linear problem and then the non-linear problem.

**Linear Problem.** We assume that \(y = 0\); then, (4) reduces to the linear system

\[ y_{n+1} = A^nE^{-n}\psi_0 + A^{n-1}E^{-n} \sum_{i=0}^{k} A^{-i}E^i (BY_{i-k} + CU) \]
\[ + A^{n-1}E^{-n} \sum_{i=k+1}^{n} A^{-i}E^iCU_i \]
\[ = A^nE^{-n}\psi_0 + A^{n-1}E^{-n} \sum_{i=0}^{k} A^{-i}E^i (BY_{i-k} + CU) \]
\[ + A^{n-1}E^{-n} \sum_{i=k+1}^{n} (A^{-i}E^i)CC^T(A^{n-1}E^{-n})^T((A^{-1}E)^T)^TW_e^{-1}[0,n_1] \eta \]
\[ = y_1. \tag{54} \]

Clearly, from Definition 4, we have that (6) is relatively controllable.
Necessity. We will prove by contradiction; assume that \( W_{c}[0,n_1] \) is singular, i.e., there exists at least one non-zero state \( \hat{v} \in X \) such that

\[
\hat{v} W_{c}[0,n_1] \hat{v} = 0.
\]  

(55)

So, we obtained

\[
0 = \hat{v} W_{c}[0,n_1]\nu
\]

\[
= \sum_{i=0}^{k} \hat{v} (A^{-1}E)(A^{n_1}E^{-n})CC^{T}(A^{n_1}E^{-n})^{T}(A^{-1}E)^{T} \nu
\]

\[
= \sum_{i=0}^{k} \hat{v} (A^{-1}E)(A^{n_1}E^{-n})C \|C\|^2,
\]

which implies that

\[
\hat{v}^{T}(A^{-1}E)(A^{n_1}E^{-n})C = (0,\ldots,0) = 0 \forall n \in I.
\]  

(57)

Since (6) is relatively controllable, from Definition 4, there exists \( U_1(n) \) that drives the initial state to zero at \( n_1 \), that is,

\[
v_{n_1} = A^{n_1}E^{-n} \nu_0 + A^{n_1}E^{-n} \sum_{i=0}^{k} A^{-1}E^{i} (B\Psi_{i-k} + C U_i)
\]

\[
+ A^{n_1}E^{-n} \sum_{i=k+1}^{n} A^{-1}E^{i} C U'_1(n) = 0.
\]  

(58)

Similarly, there also exists a control \( U_2(n) \) that drives the initial state to the state \( \mathfrak{m} \) at \( n_1 \):

\[
v_{n_1} = A^{n_1}E^{-n} \nu_0 + A^{n_1}E^{-n} \sum_{i=0}^{k} A^{-1}E^{i} (B\Psi_{i-k} + C U_i)
\]

\[
+ A^{n_1}E^{-n} \sum_{i=k+1}^{n} A^{-1}E^{i} C U'_2(n) = \hat{v}.
\]  

(59)

From the above, we have

\[
v = A^{n_1}E^{-n} \sum_{i=k+1}^{n} A^{-1}E^{i} C [U_2(n) - U_2(n)].
\]  

(60)

Multiplying both sides of (60) by \( v \) and via (8), we have

\[
v^{T}v = A^{n_1}E^{-n} \sum_{i=k+1}^{n} v^{T}A^{-i}E^{i} C U_2(n) - U_2(n) = 0.
\]  

(61)

This implies that \( \hat{v} = 0 \), which contradicts the fact that \( \hat{v} \) is non-zero. So, the delay Gramian matrix \( W_{c}[0,n_1] \) is non-singular, which completes the proof.

Non-Linear Problem. To discuss the controllability of a non-linear system (4), consider the following conditions:

(1) The operator \( W: L^2(I,X) \longrightarrow X \) defined by

\[
W_u = \sum_{i=0}^{n} (A^{-1}E)^{i}(A^{n-1}E^{n})CU_n
\]  

has inverse operator \( W^{-1} \), which takes values from \( L^2(I,X)/\ker W \) and the set \( M = \|W^{-1}\|_{L^2(I,X)/\ker W} \). For the next result, we put another assumption.

(2) The map \( u: I \times X \longrightarrow X \) is continuous and there exist a constant \( p > 1 \) and \( L_{\gamma}(\cdot) \in L^p(I,X) \) such that

\[
\|u(n,b) - u(n,b)\| \leq L_{\gamma}(n) \|b - a\|, b,a \in X.
\]  

(63)

\[\square\]

**Theorem 11.** Let us suppose that (1)–(3), (8), and (9) are satisfied. Then, system (4) is relatively controllable if

\[
b\left[1 + N M_{1} \|C\| A^{n_1}E^{-n} \|1 - e^{-an_1}\| \right] < 1,
\]  

(64)

where \( b = \|A^{n_1}E^{-n}\| N \|1 - e^{-an_1}/1 - e^{-n}\|^{1/q} \|L_{\gamma}\|_{L^p(I,X)} \)

and \( 1/q + 1/p = 1, q, p > 1 \).

**Proof.** Using (8), for an arbitrary \( u(c) \in \mathfrak{B}(I,X) \), we define a control function \( u_{n} \) by

\[
u_{n} = W^{-1} \left[ v_{1} - A^{n_1}E^{-n} \nu_0 - A^{n_1}E^{-n} \sum_{i=0}^{k} A^{-i}E^{i} (B\Psi_{i-k} + y(i,V_i)) \right]
\]

\[
A^{n_1}E^{-n} \sum_{i=k+1}^{n} A^{-i}E^{i} (BV_{i-k} + y(i,V_i)) \right], n \in I.
\]  

(65)

We show that the operator \( P: \mathfrak{B}(I,X) \longrightarrow \mathfrak{B}(I,X) \) defined by
(P_v)_n = A^n E^{-n} \Psi_0 + A^{n-1} E^{-n} \sum_{i=0}^{n} A^{-i} E^i (B \Psi_{i-k} + y(i, V_i))
+ A^{n-1} E^{-n} \sum_{i=k+1}^{n} A^{-i} E^i (B V_{i-k} + y(i, V_i))
+ A^{n-1} E^{-n} \sum_{i=0}^{n} A^{-i} E^i C U_i
\tag{66}

```markdown
has a fixed point, which is the solution of (4), by using the above control function.
```

\[ \sum_{i=0}^{n} e^{-ai} \| y(i, 0) \| \leq F \sum_{i=0}^{n} e^{-ai} = F (1 - e^{-an}/1 - e^{-a}). \]

```

We need to check that \((P_v)_n = v_1\) and \((P_v)_0 = v_0\), which means that \(u_i\) steers system (4) from \(v_0\) to \(v_1\) in finite \(n_1\) and this implies that system (4) is relatively controllable on \(I\).

For every real number \(r\), let \(B_r = \{ v \in \mathbb{B}(I, X): \| v \| \leq r \}\).
Set \( F = \sup_{n \geq 1} \| y(n, 0) \|\). We will prove this theorem in following three steps.

**Step 1.** We claim that there exists a positive real number \(r\) such that \(P(B_r) \subseteq B_r\).
Note that

\[ \sum_{i=0}^{n} e^{-ai} \| y(i, 0) \| \leq F \sum_{i=0}^{n} e^{-ai} = F (1 - e^{-an}/1 - e^{-a}). \]

Now using (1), (2), (8), and (60), we have

\[ \| u_n \| = \| W^{-1} \left[ v_1 - A^n E^{-n_1} \Psi_0 - A^{n-1} E^{-n_1} \sum_{i=0}^{k} A^{-i} E^i (B \Psi_{i-k} + y(i, V_i)) - A^{n-1} E^{-n_1} \sum_{i=k+1}^{n} A^{-i} E^i (B V_{i-k} + y(i, V_i)) \right] \]
\[ \leq \| W^{-1} \| \sum_{i=0}^{n} \| A^{-i} E^i \| \| B \Psi_{i-k} + y(i, V_i) \| \]
\[ + \| A^{n-1} E^{-n_1} \| \sum_{i=0}^{n} \| A^{-i} E^i \| \| B V_{i-k} + y(i, V_i) \| \]
\[ \leq M_1 \left( \| v_1 \| + \sum_{i=0}^{n} \| A^{-i} E^i \| \| B \Psi_{i-k} + y(i, V_i) \| + \sum_{i=0}^{n} \| A^{-i} E^i \| \| B V_{i-k} + y(i, V_i) \| \right) \]
\[ \leq M_1 \left( \| v_1 \| + \sum_{i=0}^{n} \| A^{-i} E^i \| \| B \Psi_{i-k} \| + \sum_{i=0}^{n} \| A^{-i} E^i \| \| y(i, V_i) \| \right) \]
\[ = M_1 \left( \| v_1 \| + \sum_{i=0}^{n} \| A^{-i} E^i \| \| B \Psi_{i-k} \| + \sum_{i=0}^{n} \| A^{-i} E^i \| \| y(i, V_i) - y(i, 0) + y(i, 0) \| \right) \]
\[ \leq M_1 \left( \| v_1 \| + \sum_{i=0}^{n} \| A^{-i} E^i \| \| B \Psi_{i-k} \| + \sum_{i=0}^{n} \| A^{-i} E^i \| \| y(i, V_i) \| \right) \]
\[ + \sum_{i=0}^{n} \| A^{-i} E^i \| \| B V_{i-k} \| \]
\[ \leq M_1 \left( \| v_1 \| + \sum_{i=0}^{n} \| A^{-i} E^i \| \| B \Psi_{i-k} \| + \sum_{i=0}^{n} \| A^{-i} E^i \| \| y(i, V_i) \| \right) \]
\[ + \sum_{i=0}^{n} \| A^{-i} E^i \| \| B V_{i-k} \| \]

So, we obtain $P(B) \subseteq B$. 

Step 2. Now, we define a map $P_1$ on $B$, and we will show that it is a contraction mapping.

$$(P_1v)_n = A^nE^{-n}\Psi_0 + A^{n-1}E^{-n} \sum_{i=0}^{k} A^{-i}E^i (BV_{i-k} + y(i, V_i))$$

(73)

Let $r, q \in B$. Using (57) and (60) for each $n \in I$, we have
which implies that it is a compact and continuous operator. Here we define a map $\rho_n$ which satisfies the inequality $\|\rho_n\| \leq A^{-n} E^{-n} \|\rho\|$ for all $n \in \mathbb{N}$. From (10), $M < 1$, so $\rho_n$ is contraction.

Step 3. Here we define a map $P_2: B_r \rightarrow B(I, X)$ and will show that it is a compact and continuous operator.

$$
\| (P_2 \psi)_n - (P_2 \psi) \| = \left\| \frac{A^{-n} E^{-n}}{1 - e^{-\alpha n}} \sum_{i=0}^{n} A^{-i} E^i \psi \right\| 
\leq M \|\psi\| \left\| \sum_{i=0}^{n} e^{-ai} \psi \right\| 
\leq M \|\psi\| \|\psi\| 
\rightarrow 0 \text{ as } n \rightarrow \infty,
$$

which implies that $P_2$ is continuous on $B_r$. For $\| P_2 \| = M$, where $M = \frac{A^{-n} E^{-n}}{1 - e^{-\alpha n}}$, for $n \in \mathbb{N}$, we have $P_2 \rightarrow \psi$ in $B(I, X)$ as $n \rightarrow \infty$. Using (74), we have $y_n \rightarrow y$ in $B(I, X)$ as $n \rightarrow \infty$, and thus $P_2 \psi_n \rightarrow P_2 \psi$ in $B(I, X)$ as $n \rightarrow \infty$.
To show that $P_2$ is compact on $B_r$, we have to prove that $P_2(B_r)$ is equicontinuous and bounded. For any $v \in B_r$, $n_1 \geq n + h \geq 0$, note that

\[
(P_2 v)_{n+h} - (P_2 v)_n = A^{n+h-1} E^{-n+h} \sum_{i=0}^{n+h} A^{-i} E^i (y(i,v_i)) - A^{n-1} E^{-n} \sum_{i=0}^{n} A^{-i} E^i (y(i,v_i)),
\]

\[
\| (P_2 v)_{n+h} - (P_2 v)_n \| = \| A^{n+h-1} E^{-n+h} \sum_{i=0}^{n+h} A^{-i} E^i (y(i,v_i)) - A^{n-1} E^{-n} \sum_{i=0}^{n} A^{-i} E^i (y(i,v_i)) \|
\]

\[
\leq \| A^{n+h-1} E^{-n+h} \| \left( \sum_{i=0}^{n+h} e^{-ai} L_y(i) \right) \| y(i,v) \| + \| y(i,0) \|
\]

\[
\leq \| A^{n+h-1} E^{-n+h} \| \left( \sum_{i=0}^{n+h} e^{-ai} \right)^{1/q} \| y \|_{L^p(I,X)}
\]

\[
= \| A^{n+h-1} E^{-n+h} \| NF \left( \frac{1 - e^{-a(n+h)}}{1 - e^{-a}} \right)
\]

\[
\rightarrow \delta_2 \text{ as } h \rightarrow 0,
\]

which implies that

\[
\| (P_2 v)_{n+h} - (P_2 v)_n \| \rightarrow 0 \text{ as } h \rightarrow 0.
\]

Therefore, $P_2(B_r)$ is equicontinuous.

Next, we show that $P_2(B_r)$ is bounded, and we have

\[
\| (P_2 v) \| = \| A^{n-1} E^{-n} \sum_{i=0}^{n} A^{-i} E^i (y(i,v_i)) \|
\]

\[
\leq \| A^{n-1} E^{-n} \| \sum_{i=0}^{n} e^{-ai} L_y(i) \| y(i,0) \|
\]

\[
\leq \| A^{n-1} E^{-n} \| \sum_{i=0}^{n} \frac{1 - e^{-aq}}{1 - e^{-a}}
\]

Hence, $P_2(B_r)$ is bounded. From the Arzelà–Ascoli theorem, $P_2(B_r)$ is compact in $B(I,X)$. Thus, $P_2(B_r)$ is a compact and continuous operator.

Now, Krasnoselskii’s fixed point theorem guarantees that $P$ has a fixed point $v$ in $B_r$. Clearly, $v$ is a solution of (4) satisfying $v_n = v_1$, and the boundary condition $v_n = \Psi_n, -k \leq n \leq 0$ holds from the solution of system (4), which completes the proof.

8. Numerical Examples

In this section, we give some examples on Hyers–Ulam stability and controllability for the theoretical results.

Example 1. Consider the following non-singular delay difference equation:

\[
EV_{n+1} = AV_n + BV_{n-3}, V_0 = 1, n \in \{0, 1, 2, 3\},
\]

\[
V_n = \Phi_n, -3 \leq n \leq 0,
\]

with inequality
\[
\begin{align*}
\|EV_{n+1} - AV_n - BV_{n-3}\| & \leq 0.8, V_0 = 1, n \in \{0, 1, 2, 3\}, \\
\|V_n - \Phi_{n}\| & \leq 1, -3 \leq n \leq 0,
\end{align*}
\]

(86)

where \( k = 3 \).

If we fixed
\[
A = \begin{pmatrix} -5 & -2 \\ -4 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}, \quad E = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},
\]

and
\[
\phi_n = [\cos(n + \pi/2)\cos(n + \pi/2)], \quad \text{obviously,} \quad \phi_n = [00]^T,
\]

when \( n = 0 \). Then, we get that
\[
AB = \begin{pmatrix} -24 & -11 \\ -22 & -13 \end{pmatrix} = BA, \quad AE = \begin{pmatrix} -10 & -4 \\ -8 & -6 \end{pmatrix} = EA,
\]

\[
BE = \begin{pmatrix} 8 & 2 \\ 4 & 6 \end{pmatrix} = EB, \quad E^{-1} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}, \quad AE^{-1} = \begin{pmatrix} 2 & 0.5 \\ 0 & 1.5 \end{pmatrix} = E^{-1}B.
\]

Moreover, if \( V \) satisfied (86), then there exists \( f_n \) such that \( \|f_n\| \leq 0.8 \), and
\[
\begin{align*}
\left\{ \begin{array}{l}
EV_{n+1} = AV_n + BV_{n-3} + f_n, V_0 = 1, n \in \{0, 1, 2, 3\} \\
V_n = \Phi_n, -3 \leq n \leq 0.
\end{array} \right.
\end{align*}
\]

(87)

Also, the solution of (85) is
\[
V_n = A^nE^{-n}\Phi_0 + BA^{-n-1}E^{-n} \sum_{i=0}^{k} A^{-i}E\Phi_{i-k}
\]

\[+ BA^{-n-1}E^{-n} \sum_{i=k+1}^{n} A^{-i}E\Phi_{i-k}. \]

(88)

Let \( \epsilon = 0.8 \), and \( f: Z_+ \rightarrow X \) be as given below.
\[
f_n = \left[ 0.6 \cos(n + \pi/2) \ 0.6 \sin(n + \pi/2) \right]^T.
\]

Then, clearly
\[
\|f_n\| = \sqrt{\left(0.6 \cos(n + \pi/2)^2 + (0.6 \sin(n + \pi/2))^2 \right)}
\]

\[= \sqrt{(0.6)^2 \cos^2\left(n + \frac{\pi}{2}\right) + (0.6)^2 \sin^2\left(n + \frac{\pi}{2}\right)} \]

\[= \sqrt{0.36} = 0.6 \leq 0.8.
\]

(89)

Now, the perturbed delay difference systems (11)–(13) have the solution
\[
H_n = A^nE^{-n}\Phi_0 + BA^{-n-1}E^{-n} \sum_{i=0}^{k} A^{-i}E\Phi_{i-k}
\]

\[+ BA^{-n-1}E^{-n} \sum_{i=k+1}^{n} A^{-i}E\Phi_{i-k} + f_{n-k}. \]

(90)

The plots of exact and perturbed solutions obtained using Mathematica are shown in Figure 1.

Figure 1: The plots of exact and perturbed solutions.

**Example 2.** Consider the following non-singular delay difference equation:
\[
\begin{cases}
EV_{n+1} = AV_n + BV_{n-3} + f(n, V), \\
V_0 = 1, n \in \{0, 1, 2, 3\}, \\
V_n = \phi_n, -3 \leq n \leq 0,
\end{cases}
\]

(91)

with inequality
\[
\left\{ \begin{array}{l}
\|EV_{n+1} - AV_n - BV_{n-3} - f(n, V)\|
\leq 0.8, V_0 = 1, n \in \{0, 1, 2, 3\}, \\
\|V_n - \Phi_n\| \leq 1, -3 \leq n \leq 0,
\end{array} \right.
\]

(92)

where \( k = 3 \).

If again we fixed
\[
A = \begin{pmatrix} 4 & -3 \\ 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -3 \\ 2 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \quad \phi_n = [\cos(n + \pi/2)\cos(n + \pi/2)]^T,
\]

obviously, \( \phi_n = [00]^T \), when \( n = 0 \). Then, we get
\[
AB = \begin{pmatrix} 2 & -15 \\ 10 & -3 \end{pmatrix} = BA, \quad AE = \begin{pmatrix} 12 & -9 \\ 6 & 9 \end{pmatrix} = EA, \quad BE = \begin{pmatrix} 6 & -9 \\ 3 & 3 \end{pmatrix} = EB, \quad E^{-1} = \begin{pmatrix} 0.333 & 0 \\ 0 & 0.333 \end{pmatrix}, \quad \text{AE}^{-1} = \begin{pmatrix} 1.332 & -0.999 \\ 0.666 & 0.999 \end{pmatrix} = E^{-1}A, \quad \text{and} \ BE^{-1} = \begin{pmatrix} 0.666 & -0.999 \\ 0.666 & 0.333 \end{pmatrix} = E^{-1}B.
\]

Moreover, if \( V \) satisfied (15), then there exists \( f_n \) such that \( \|f_n\| \leq 0.8 \), and
\[
\begin{cases}
EV_{n+1} = AV_n + BV_{n-3} + f(n, V) + f_n, \\
V_0 = 1, n \in \{0, 1, 2, 3\}, \\
V_n = \phi_n, -3 \leq n \leq 0.
\end{cases}
\]

(93)

Also, the solution of (91) is
\[ V_n = A^n E^{-n} \phi_0 + A^{n-1} E^{-n} \sum_{i=0}^{k} A^{-i} E^i (B\phi_{i-k} + f(i, V_i)) \]
+ \[ A^{n-1} E^{-n} \sum_{i=k+1}^{n} A^{-i} E^i (B\phi_{i-k} + f(i, V_i)). \]

Let \( \epsilon = 0.8 \), and \( f: \mathcal{X} \rightarrow \mathcal{X} \) be as given below. \( f_n = \left(0.7 \cos (n + \pi/2) 0.7 \sin (n + \pi/2)\right)^T \). Then, clearly
\[
\|f_n\|^2 = \left(0.7 \cos \left(n + \frac{\pi}{2}\right)\right)^2 + \left(0.7 \sin \left(n + \frac{\pi}{2}\right)\right)^2 = 0.7 \leq 0.8.
\]

Now, the perturbed delay difference systems (91)–(93) have the solution
\[
H_n = A^n E^{-n} \phi_0 + A^{n-1} E^{-n} \sum_{i=0}^{k} A^{-i} E^i (B\phi_{i-k} + f(i, V_i)) \]
+ \[ A^{n-1} E^{-n} \sum_{i=k+1}^{n} A^{-i} E^i (B\phi_{i-k} + f(i, V_i) + f_{i-k}). \]

The plots of exact and perturbed solutions obtained using Mathematica are shown in Figure 2.

**Example 3.** Set \( n_1 = 3 \). Consider the following delay difference controlled system:
\[
\begin{align*}
EV_{n+1} &= AV_n + BV_{n-1} + y(n, V_n) \\
&+ CU_n, n \in \mathbb{N} = \{0, 1, 2, 3\}, \\
V_n &= \Psi_{n-3} \leq n \leq 0,
\end{align*}
\]
which has the solution
\[
V_n = A^n E^{-n} \psi_0 + A^{n-1} E^{-n} \sum_{i=0}^{k} A^{-i} E^i (B\phi_{i-3} + y(i, V_i) + CU_i)
\]
+ \[ A^{n-1} E^{-n} \sum_{i=k+1}^{n} A^{-i} E^i (B\phi_{i-3} + y(i, V_i) + CU_i), \]
\begin{align*}
\text{where} & \ k = 3, \\
A &= \begin{pmatrix} 7 & 3 \\ 3 & 7 \end{pmatrix}, & B &= \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}, & E &= \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, \\
C &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & y(n, V_n) &= \begin{pmatrix} 0.2nV_2 \\ 0.1nV_1 \end{pmatrix}.
\end{align*}

Note that \[ AB = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix} = BA, \ AE = \begin{pmatrix} 28 & 12 \\ 24 & 16 \end{pmatrix} = EA, \ BE = \begin{pmatrix} 4 & -4 \\ -8 & 8 \end{pmatrix} = EB, \ E^{-1} = \begin{pmatrix} 0.25 & 0 \\ 0 & 0.25 \end{pmatrix} \text{ AE}^{-1} = \begin{pmatrix} 1.75 & 0.75 \\ 1.5 & 1 \end{pmatrix} = E^{-1} A, \text{ and } BE^{-1} = \begin{pmatrix} 0.25 & -0.25 \\ -0.5 & 0.5 \end{pmatrix} = E^{-1} B. \]

Now \( \|C\| = 1. \) Also, \( \|A^n E^{-n}\| \leq Ne^{-\alpha n} \) with \( N = 1, \alpha = 2 \), and \( n \in \{0, 1, 2, 3\}. \)

Now consider
\[
W_c[0, n_1] = \sum_{i=0}^{n_1} (A^{-i} E^i)(A^{n_1-i} E^{-n_1}) \]
\[
= \sum_{i=0}^{n_1} \left( A^{-i} E^i \right) \left( A^{n_1-i} E^{-n_1} \right) = \sum_{i=0}^{n_1} \left( A^{-i} E^i \right)^T \left( A^{n_1-i} E^{-n_1} \right)^T
\]
\[
= \left( \left( A^{-1} E \right)^T \right)^T
\]
\[
= \left( -0.00997 2.37257 \right)^T \text{ and } M_1 = \sqrt{\|W_c[0, n_1]\|} = 0.399932. \]

Further for any \( \nu, \mu \in \mathcal{X} \), we have
\[
\|y(n, \nu) - y(n, \mu)\| = \max \{0.2n\|v_1 - \mu_1\|, 0.1n\|v_2 - \mu_2\|\}
\]
\[
\leq 0.2n\max \{\|v_1 - \mu_1\|, \|v_2 - \mu_2\|\}
\]
\[
= 0.2n\|v - \mu\|.
\]

Now we set \( L_y(n) = 0.2n \in L^2(1, \mathcal{X}) \) with \( p = q = 2 \), so
\[
L_y \|L^2(1, \mathcal{X}) = \left(\sum_{n=0}^{\infty} L_y(n)\right)^{1/2} = \left(\sum_{n=0}^{\infty} 0.2n\right)^{1/2} = 1.09545.
\]
\[
b = \|A^{n_1-i} E^{-n_1}N\left(\frac{1 - e^{-\alpha n_1}}{1 - e^{-\alpha}}\right)^{1/2}
\]
\[
= \|A^2 E^{-2}\left(\frac{1 - e^{-12}}{1 - e^{-4}}\right)^{1/2}
\]
\[
= 0.3546,
\]
and
\[
b\|A^{n_1-i} E^{-n_1}\| \left\|M_1N(1 - e^{-\alpha n_1})\right\| \left(\frac{1 - e^{-\alpha}}{1 - e^{-\alpha n_1}}\right)^{1/2}
\]
\[
= (0.3546)\left[\|A^2 E^{-2}\| \left(M_1\right)(1 - e^{-6})\right]
\]
\[
= 0.8573
\]
\[ \leq 1. \]
Example 4. Consider the following non-singular delay difference equation:

$$\begin{align*}
EV_{n+1} &= AV_n + BV_{n-0.3}, n \in \{0, 1, 2, 3, \ldots\}, \\
V_n &= (0.3, 0.2)^T, -0.3 \leq n \leq 0,
\end{align*}$$

(103)

where \( k = 0.3 \) and we set \( A = \begin{pmatrix} -5 & -2 \\ -4 & -3 \end{pmatrix}, B = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}, \)

\[ E = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad \text{and} \quad \phi_n = [\cos(n + n/2)\cos(n + n/2)]^T; \]

obviously, \( \phi_n = [00]^T \), when \( n = 0 \). Then, we get that \( AB = \begin{pmatrix} -24 & -11 \\ -22 & -13 \end{pmatrix} = BA, \quad AE = \begin{pmatrix} -10 & -4 \\ -8 & -6 \end{pmatrix} = EA, \quad BE = \begin{pmatrix} 8 & 2 \\ 4 & 6 \end{pmatrix} \)

\[ EB, E^{-1} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}, \quad \text{and} \quad AE^{-1} = \begin{pmatrix} -2.5 & -1 \\ -2 & -1.5 \end{pmatrix} = E^{-1}A, \text{and } BE^{-1} = \begin{pmatrix} 2 & 0.5 \\ 1 & 1.5 \end{pmatrix} = E^{-1}B. \]

Now,

\[ \|B\| = \left\| \begin{pmatrix} 0.5 & 1 \\ 0.6 & 0.34 \end{pmatrix} \right\| = 0.47 < Ne^{-0.3\alpha} = 0.656101, \]

choosing \( \alpha = 1.4048, N = 1 \).

Also, \( \|\phi\| = 0.3; \text{ now,} \)

\[ M(\phi, \phi) = N\|\phi\| + \|B\| \sum_{i=0}^{k} N^2 e^{-\alpha n}\|\phi_{i-k}\| = 0.4324, \]

\[ (N^2\|B\| - \alpha) \leq -0.3725 < 0, \]

\[ \|V_n\| \leq M(\phi, \phi)e^{(N^2\|B\| - \alpha)n} \longrightarrow 0, \text{as} n \longrightarrow \infty. \]

(104)

Hence, system (1) is exponentially stable.

9. Conclusion

In recent years, the qualitative behavior of delay difference equations has been considered as one of the important topics of the literature, in which different types of conditions have been used in the form of inequalities and mostly results have been obtained through discrete Gronwall inequality. In this paper, we have investigated the existence and uniqueness of the solution through Banach contraction principle, Hyers–Ulam stability over bounded and unbounded discrete interval, exponential stability, and controllability of the delay difference system with the help of Gronwall inequality and Carathéodory condition.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

Kamal Shah would like to thank Prince Sultan University for the support through the TAS research lab, and Sawitree Moonsuwan would like to thank Suan Dusit University. This research was funded by National Science, Research and Innovation Fund (NSRF) and Suan Dusit University.

References

[1] S. Frassu and G. Viglialoro, “Boundedness for a fully parabolic Keller–Segel model with sublinear segregation and superlinear aggregation,” Acta Applicandae Mathematicae, vol. 171, no. 1, pp. 19–20, 2021.

[2] T. Li, N. Pintus, and G. Viglialoro, “Properties of solutions to porous medium problems with different sources and boundary conditions,” Zeitschrift für Angewandte Mathematik und Physik, vol. 70, no. 3, pp. 86–18, 2019.

[3] T. Li and G. Viglialoro, “Boundedness for a non local reaction chemotaxis model even in the attraction-dominated regime,” Differential and Integral Equations, vol. 34, no. 5/6, pp. 315–336, 2021.
[4] T. Li and G. Viglialoro, "Analysis and explicit solvability of degenerate torsional problems," *Boundary Value Problems*, vol. 2018, no. 1, 13 pages, Article ID 1366, 2018.

[5] S. Brianzoni, C. Mammana, E. Michetti, and F. Zirilli, "A stochastic cobweb dynamical model," *Discrete Dynamics in Nature and Society*, vol. 2008, Article ID 219653, 18 pages, 2008.

[6] J. Diblik, I. Dzhalladova, and M. Ružičková, "Stabilization of company’s income modeled by a system of discrete stochastic equations," *Advances in Difference Equations*, vol. 2014, no. 1, 2014.

[7] H. Mohammadi, S. Kumar, S. Rezapour, and S. Etemad, "A theoretical study of the Caputo-Fabrizio fractional modeling for hearing loss due to Mumps virus with optimal control," *Chaos, Solitons & Fractals*, vol. 144, Article ID 110668, 2021.

[8] S. Etemad, I. Avci, P. Kumar, D. Baleanu, and S. Rezapour, "Some novel mathematical analysis on the fractal-fractional model of the AH1N1/09 virus and its generalized Caputo-type version," *Chaos, Solitons & Fractals*, vol. 162, Article ID 112511, 2022.

[9] S. M. Ulam, *A Collection of Mathematical Problems*, Inter science Publishers, no. 8, New York, USA, 1960.

[10] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences*, vol. 27, no. 4, pp. 222–224, 1941.

[11] T. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.

[12] M. Obloza, *Connections between Hyers and Lyapunov Stability of the Ordinary Differential Equations*, 1997.

[13] S. M. Jung, "Hyers-Ulam stability of the first order matrix difference equations," *Advances in Difference Equations*, vol. 2015, no. 1, Article ID 614745, 2015.

[14] A. Khan, G. Rahmat, and A. Zada, "On uniform exponential stability and exact admissibility of discrete semigroups," *International Journal of Differential Equations*, vol. 2013, Article ID 268309, 4 pages, 2013.

[15] Z. Gao, X. Yu, and J. Wang, "Exp-type Ulam-Hyers stability of fractional differential equations with positive constant coefficient," *Advances in Difference Equations*, vol. 2015, no. 1, Article ID 13662, 2015.

[16] S. O. Shah, A. Zada, M. Muzamil, M. Taryab, and R. Rizwan, "On the bielecki ulam type stability results of first order non-linear impulsive delay dynamic systems on time scales," *Qualitative Theory of Dynamical Systems*, vol. 19, no. 3, pp. 98–118, 2020.

[17] A. R. Aruldass, D. Pachaiyappan, and C. Park, "Hyers-Ulam stability of second-order differential equations using Mahgoub transform," *Advances in Difference Equations*, vol. 2021, no. 1, pp. 1–10, 2021.

[18] A. Boutiara, S. Etemad, A. Hussain, and S. Rezapour, "The generalized Ulam-Hyers and Ulam-Hyers stability and existence analysis of a coupled hybrid system of integro-differential IVPs involving O-Caputo fractional operators," *Advances in Difference Equations*, vol. 2021, no. 1, pp. 1–21, 2021.

[19] Y. Guo, M. Chen, X. B. Shu, and F. Xu, "The existence and Hyers-Ulam stability of solution for almost periodical fractional stochastic differential equation with fBm," *Stochastic Analysis and Applications*, vol. 39, no. 4, pp. 643–666, 2021.

[20] S. Li, L. Shu, X. B. Shu, and F. Xu, "Existence and Hyers-Ulam stability of random impulsive stochastic functional differential equations with finite delays," *Stochastics*, vol. 91, no. 6, pp. 857–872, 2019.

[21] S. Avdonin and L. de Teresa, "The Kalman condition for the boundary controllability of coupled 1-d wave equations," *Evolution Equations and Control Theory*, vol. 9, no. 1, pp. 255–273, 2019.

[22] C. T. Chen, *Introduction to Linear System Theory*, Holt Rinehart & Winston, New York, 1970.

[23] T. Kaczorek, *Linear Control Systems*, Research Studies Press and John Wiley, New York, 1993.

[24] J. Klama, *Controllability of Dynamical Systems*, Kluwer Academic Publishers, Dordrecht, 1991.

[25] Khushainov, D. Ya, and G. V. Shuklin, "Linear autonomous time-delay system with permutation matrices solving," *Stud. Univ. Zilina.*, vol. 17, no. 1, Article ID 101108, 2003.

[26] J. Diblik and D. Y. Khushainov, "Representation of solutions of discrete delayed system $x(k+1)=Ax(k)+Bx(k-m)+f(k)$ with commutative matrices," *Journal of Mathematical Analysis and Applications*, vol. 318, no. 1, pp. 63–76, 2006.

[27] J. Wang, Z. You, and D. O’Regan, "Exponential stability and relative controllability of non singular delay systems," *Bulletin of the Brazilian Mathematical Society, New Series*, vol. 50, no. 2, pp. 457–479, 2019.

[28] Y. Almalki, G. Rahmat, A. Ullah, F. Shehryar, M. Numan, and M. U. Ali, "Generalized Hyers-Ulam-Rassias stability of impulsive difference equations," *Computational Intelligence and Neuroscience*, vol. 2022, Article ID 9462424, 2022.

[29] S. A. Kuruklis, "The Asymptotic Stability of $x(n+1) = an + bxn−k = 0$," *Journal of Mathematical Analysis and Applications*, vol. 188, no. 3, pp. 719–731, 1994.

[30] J. S. Yu, "Asymptotic stability for a linear difference equation with variable delay," *Computers & Mathematics with Applications*, vol. 36, no. 10–12, pp. 203–210, 1998.

[31] W. Kosmala and C. Teixeira, "More on the Difference Equation $y(n+1) = p + qn + r(n−1)$," *Applicable Analysis*, vol. 81, no. 1, pp. 143–151, 2002.

[32] Z. F. Liu, C. X. Cai, and Y. Zou, "Switching signal design for exponential stability of uncertain discrete-time switched time-delay systems," *Journal of Applied Mathematics*, pp. 1–11, 2013.

[33] K. Marwen and A. Sakly, "On stability analysis of discrete-time uncertain switched nonlinear time-delay systems," *Advances in Difference Equations*, vol. 2014, no. 1, pp. 1–22, 2014.

[34] L. Yuanjuan and F. Meng, "Stability analysis of a class of higher order difference equations and its applications," *Abstract and Applied Analysis*, vol. 2014, Article ID 436261, 7 pages, 2014.

[35] A. Khalqi, H. S. Alayachi, M. S. M. Noorani, and A. Q. Khan, "On stability analysis of higher-order rational difference equations," *Discrete Dynamics in Nature and Society*, vol. 2020, Article ID 30994, 10 pages, 2020.

[36] G. Rahmat, A. Ullah, A. U. Rahman, M. Sarwar, T. Abdeljawad, and A. Mukheimer, "Hyers-Ulam stability of non-autonomous and non-singular delay difference equations," *Advances in Difference Equations*, vol. 2021, no. 1, Article ID 133662, 2019.

[37] S. Avdonin and L. de Teresa, "The Kalman condition for the boundary controllability of coupled 1-d wave equations," *Evolution Equations and Control Theory*, vol. 9, no. 1, pp. 255–273, 2019.