Affine geometric description of thermodynamics

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Abstract

Thermodynamics provides a unified perspective of thermodynamic properties of various substances. To formulate thermodynamics in the language of sophisticated mathematics, thermodynamics is described by a variety of differential geometries, including contact and symplectic geometries. Meanwhile affine geometry is a branch of differential geometry and is compatible with information geometry, where information geometry is known to be compatible with thermodynamics. By combining above, it is expected that thermodynamics is compatible with affine geometry, and is expected that several affine geometric tools can be introduced in the analysis of thermodynamic systems. In this paper affine geometric descriptions of equilibrium and nonequilibrium thermodynamics are proposed. For equilibrium systems, it is shown that several thermodynamic quantities can be identified with geometric objects in affine geometry, and that several geometric objects can be introduced in thermodynamics. Examples of these include: specific heat is identified with the affine fundamental form, a flat connection is introduced in thermodynamic phase space. For nonequilibrium systems, two classes of relaxation processes are shown to be described in the language of an extension of affine geometry. Finally this affine geometric description of thermodynamics for equilibrium and nonequilibrium systems is compared with a contact geometric description.

1 Introduction

Thermodynamics is a branch of physics, provides a unified perspective of thermodynamic properties of various substances, and has been applied to various branches of sciences and technologies [1]. Thus further developments in thermodynamics are beneficial in these branches. One way to develop thermodynamics further is to apply well-developed pure mathematics to thermodynamics [2], and differential geometry is one of such mathematics [3]. By introducing notions developed in differential geometry to thermodynamics, some new views and applications in thermodynamics were expected to be found. Such views and applications are described by contact geometry [4, 5], symplectic geometry [6], and so on [7, 8]. Here contact geometry is known as an odd-dimensional counterpart of symplectic geometry [9, 10, 11, 12], and is used in describing not only thermodynamics, but also singularities in hyper-surfaces [13]. Note that there have been other geometric formulations of thermodynamics [14, 15], and various developments are in progress.

Affine geometry is a branch of differential geometry, and studies invariant properties under affine transforms [16]. It is compatible with information geometry [17], where information geometry is a geometrization of mathematical statistics [18]. There are a variety of remarkable theorems relating affine geometry and information geometry. One of them is that divergence that plays a central role in information geometry has been extended in the framework of affine geometry [19]. Meanwhile information geometry is compatible with thermodynamics [20, 21, 22]. Note that affine geometry is deeply related to the so-called Hessian geometry [23].

By combining above, it is expected that thermodynamics is compatible with affine geometry, and is expected that several geometric tools can be introduced in the analysis of thermodynamic systems(see the
This paper is intended to discuss relations between thermodynamics and affine geometry, and to provide the first step towards the materialization of affine geometric descriptions of thermodynamics. To this end, an affine geometric description of thermodynamics and that of thermodynamic processes are proposed and discussed in this paper. More specifically the following are shown:

- The point of departure for this paper is to identify the set of equilibrium states in thermodynamics with the image of a graph immersion (see Interpretation 3.1 of this paper).

- For equilibrium systems it is shown that several thermodynamic quantities can be identified with geometric objects employed in affine geometry, and that several geometric objects can be introduced in thermodynamics (see Proposition 3.3 of this paper). Such examples include: specific heat is identified with affine fundamental form, a flat connection is introduced in thermodynamic phase space.

- For nonequilibrium systems two classes of relaxation processes are shown to be described in the language of an extension of affine geometry (see Theorems 3.9 and 3.14 of this paper). An analysis of a simple spin model with a unique set of equilibrium states shows how the description is justified.

- This affine geometric description of thermodynamics for equilibrium and nonequilibrium systems is compared with a contact geometric one, where the contact geometric thermodynamics is a representative existing theory. It is then shown that the present geometric formulation of relaxation processes are consistent with the existing theory (see Theorem 4.7 of this paper).

The rest of this paper is organized as follows. In Section 2, some of necessary background of geometries and thermodynamics are summarized. In Section 3 an affine geometric thermodynamics is proposed. In Section 4, the affine geometric description of thermodynamics is compared with the contact geometric thermodynamics. Finally Section 5 summarizes the present geometric formulation of equilibrium and nonequilibrium thermodynamics, and discusses future studies.

2 Preliminaries

This section is intended to provide a brief summary of the necessary background of geometries and thermodynamics, and is intended to fix notations here. Throughout this paper manifolds are connected, and every object on any manifold is smooth, unless otherwise stated. Given a manifold $\mathcal{M}$, its tangent and cotangent bundles are denoted by $T\mathcal{M}$ and $T^*\mathcal{M}$, respectively. Various formulæ and tools developed in differential geometry are known [2, 3]. For example, the Lie derivative of a $k$-form $\alpha \in \Gamma \Lambda^k \mathcal{M}$ on a manifold $\mathcal{M}$ along a vector field $X \in \Gamma T\mathcal{M}$ can be written as $L_X \alpha = d \iota_X \alpha + \iota_X d\alpha$, where $d$ is the exterior derivative and $\iota_X$ the interior product with $X$. This is known as the Cartan formula. This formula is valid even $k = 0$, where $\Gamma \Lambda^0 \mathcal{M}$ is identified with the space of functions on $\mathcal{M}$. If $\phi$ is a map from a manifold to another one, then $\phi_*$ denotes the push-forward of $\phi$, and $\phi^*$ denotes its pull-back. When a (affine) connection $\nabla$ is equipped on $\mathcal{M}$, the $(1,2)$-type tensor field $T^\nabla$ such that $T^\nabla(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$ for all $X,Y \in \Gamma T\mathcal{M}$, is called torsion tensor field, where $[X,Y] = XY - YX$ is the Lie bracket. If $T^\nabla(X,Y) \equiv 0$ for all $X,Y \in \Gamma T\mathcal{M}$, then $\nabla$ is said to be torsion-free. The $(1,3)$-type tensor field $R^\nabla$ such that $R^\nabla(X,Y,Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ for all vector fields $X,Y,Z$, is called the curvature tensor field. If $R^\nabla(X,Y,Z) \equiv 0$, then $\nabla$ is called curvature-free. If a connection $\nabla$ is torsion-free and curvature-free, then $\nabla$ is said to be flat. If a connection $\nabla$ is flat, then there exists a coordinate system such that all the connection coefficients vanish, $\Gamma_{ij}^k \equiv 0$. These coordinate systems are said to be affine coordinate systems.
2.1 Affine geometry

In this subsection necessary background in affine geometry are summarized (see Ref. [10] for more details. In Ref. [10], (co)tangent space at a point is identified with a point due to the property of affine space. Meanwhile in this paper, this identification is not adopted.). Let \( \mathcal{M} \) be an \( n \)-dimensional manifold \((n=1,2,\ldots)\), \( f \) an immersion of \( \mathcal{M} \) into \( \mathbb{R}^{n+1} \), and \( \xi \) a vector field along \( f \). If for arbitrary point \( p \in \mathcal{M} \) the condition
\[
T_{f(p)}\mathbb{R}^{n+1} = f_* (T_p\mathcal{M}) \oplus \text{span}\{\xi_p\},
\]
is satisfied, then the pair \((f,\xi)\) is referred to as an affine (hyper-surface) immersion, and \( \xi \) a transversal vector field.

There are various formulae associated with affine immersions. Let \((f,\xi)\) be an affine hyper-surface immersion, and \( D \) the standard flat affine connection on \( \mathbb{R}^{n+1} \). From (4), one calculates \( \nabla R \) immersion, and \( D \) the standard flat affine connection on \( \mathbb{R}^{n+1} \), and \( \xi \) a vector field along \( f \). If for arbitrary point \( p \in \mathcal{M} \) the condition \( T_{f(p)}\mathbb{R}^{n+1} = f_* (T_p\mathcal{M}) \oplus \text{span}\{\xi_p\} \)
is satisfied, then the pair \((f,\xi)\) is referred to as an affine (hyper-surface) immersion, and \( \xi \) a transversal vector field.

It follows from the flatness of \( D \) that \( D \) is a one-form being referred to as an affine shape operator, and \( \tau \) a one-form being referred to as a transversal connection form. When \( \tau = 0 \), the affine immersion \((f,\xi)\) is referred to as being equiaffine. If \( h \) is non-degenerate everywhere, then \( f \) is said to be non-degenerate.

The dual of \( \mathcal{M} \) provides various tools in differential geometry in general. In affine geometry, the dual of a vector space also provides useful geometric tools. Given an affine immersion \((f,\xi)\), introduce a map \( v : \mathcal{M} \to T^*_f(\mathbb{R}^{n+1}) \), and the pairing \( \langle -,\xi \rangle : T^*_f\mathbb{R}^{n+1} \times T_f\mathbb{R}^{n+1} \to \mathbb{R}, (p \in \mathcal{M}) \):
\[
\mathbb{R}^{n+1} \xrightarrow{f} \mathcal{M} \xrightarrow{v} T^*_f\mathbb{R}^{n+1} \leftarrow \text{dual} \xrightarrow{\xi} T_f\mathbb{R}^{n+1} \xrightarrow{f} \mathcal{M} \xrightarrow{v} T(T^*_f\mathbb{R}^{n+1}).
\]
If a map \( v \) satisfies the conditions
\[
\langle v(p),\xi_p \rangle = 1, \quad \text{and} \quad \langle v(p),f_*X_p \rangle = 0, \quad \forall X_p \in T_p\mathcal{M}, \quad \forall p \in \mathcal{M},
\]
then \( v \) is referred to as the conormal map.

The graph immersion is a class of affine immersions. This is explained below. Let \( \Omega \) be an \( n \)-dimensional region of \( \mathbb{R}^n \), and \( F \) a function defined on \( \Omega \), \( F : \Omega \to \mathbb{R} \). The coordinate system of \( \Omega \subset \mathbb{R}^n \) is denoted by \( x = (x^1,\ldots,x^n) \), and the coordinate of the other \( \mathbb{R} \) is denoted by \( z \). A graph immersion associated with \( F \) is a pair \((f,\xi)\) satisfying the conditions written in coordinates as
\[
x = (x^1,\ldots,x^n) \mapsto f(x) = (x,F(x)), \quad \text{and} \quad \xi = \frac{\partial}{\partial z}.
\]
It follows from the flatness of \( D \) that \( D_X\xi = 0 \) for all \( X \in T_p\mathcal{M} \). This and (2), yield \( S = 0 \) and \( \tau = 0 \). Since \( \tau = 0 \), any graph immersion is equiaffine. The affine fundamental form \( h \) is calculated as follows. First, one calculates
\[
f_* \frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^i} + \frac{\partial F}{\partial x^i} \frac{\partial}{\partial z}, \quad \cdots, \quad f_* \frac{\partial}{\partial x^n} = \frac{\partial}{\partial x^n} + \frac{\partial F}{\partial x^n} \frac{\partial}{\partial z},
\]
From (4), one calculates
\[
D_{\frac{\partial}{\partial x^j}} f_* \frac{\partial}{\partial x^i} = \frac{\partial^2 F}{\partial x^i \partial x^j} \frac{\partial}{\partial z}, \quad i,j = 1,\ldots,n.
\]
From (1) and (5), one has
\[
h \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = \frac{\partial^2 F}{\partial x^i \partial x^j}, \quad i,j = 1,\ldots,n,
\]

3
and
\[ \nabla \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} = 0, \quad i, j = 1, \ldots, n. \tag{7} \]

It follows from \( \[3 \] \) that if the matrix \( (\partial^2 F/\partial x^i \partial x^j) \) is strictly positive at \( x \), then \( h \) is non-degenerate. In addition, it follows from \( \[7 \] \) that the coordinate system \( x \) is affine with respect to \( \nabla \). If \( h \) is non-degenerate everywhere on \( M \), then the graph immersion is said to be non-degenerate. The conormal map at \( p \in \Omega \) is expressed as
\[ v(p) = dz - \sum_{a=1}^{n} \frac{\partial F}{\partial x^a} dx^a \in T^*_p \mathbb{R}^{n+1}. \tag{8} \]

### 2.2 Information geometry

Information geometry is a geometrization of mathematical statistics \([18]\), and a study of statistical manifolds \([17]\). The definition of statistical manifold is given from a viewpoint of differential geometry as follows. Let \((M, g)\) be an \( n \)-dimensional (pseudo-) Riemannian manifold, and \(\nabla\) a torsion-free affine connection. If \(\nabla\) satisfies the Codazzi equation \([16]\),
\[ \nabla X g(Y, -) = \nabla Y g(X, -), \quad \forall X, Y \in \Gamma TM \]
then the triplet \((M, \nabla, g)\) is referred to as a statistical manifold \([19]\).

One pair of connections explained below plays various roles in information geometry. Let \(g\) be a pseudo-Riemannian metric tensor field on \(M\), and \(\nabla\) an affine connection. If a connection \(\nabla^*\) satisfies the condition
\[ X[g(Y, Z)] = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \quad \forall X, Y, Z \in \Gamma TM \]
then \(\nabla^*\) is referred to as the dual connection of \(\nabla\) with respect to \(g\). It can be shown that if \(\nabla\) is flat then \(\nabla^*\) is flat. In information geometry, the tetrad \((M, g, \nabla, \nabla^*)\) is referred to as a dually flat space, and this class of manifolds has been well-studied \([17]\). On a dually flat space, let \(\theta = (\theta^1, \ldots, \theta^n)\) be an affine coordinate system. Then it can be shown that there exists an affine coordinate system \(\eta = (\eta_1, \ldots, \eta_n)\) for \(\nabla^*\) that satisfies
\[ g \left( \frac{\partial}{\partial \theta^a}, \frac{\partial}{\partial \theta^b} \right) = \delta^a_b, \quad a, b = 1, \ldots, n, \]
where \(\delta^a_b\) is the Kronecker delta giving unity when \(a = b\) and zero when \(a \neq b\). The coordinate system \(\eta\) is said to be the dual coordinate system of \(\theta\) with respect to \(g\). One basic proposition on dually flat spaces is as follows. Let \((M, g, \nabla, \nabla^*)\) be a dually flat space, \(\theta\) an affine coordinate system, \(\eta\) a dual affine coordinate system, where \(M\) is simply connected and has a global coordinate system. Then there exist functions \(\psi\) and \(\varphi\), and it follows that
\[ \frac{\partial \psi}{\partial \theta^a} = \eta_a, \quad \frac{\partial \varphi}{\partial \eta_a} = \theta^a, \quad \psi(p) + \varphi(p) - \sum_{a=1}^{n} \theta^a(p) \eta_a(p) = 0, \quad a, b = 1, \ldots, n, \quad \forall p \in M \tag{9} \]
\[ g_{ab} = \frac{\partial^2 \psi}{\partial \theta^a \partial \theta^b}, \quad g^{ab} = \frac{\partial^2 \varphi}{\partial \eta_a \partial \eta_b}, \quad a, b = 1, \ldots, n \]
where \(g_{ab} = g(\partial/\partial \theta^a, \partial/\partial \theta^b)\) and \((g^{ab})\) is the inverse matrix of \((g_{ab})\). In mathematical statistics and information geometry, divergence plays various roles \([18]\). The function \(M \times M \to \mathbb{R}\),
\[ D(p_1, p_2) = \psi(p_1) + \varphi(p_2) - \sum_{a=1}^{n} \theta^a(p_1) \eta_a(p_2), \]
is called the canonical divergence.

Several relations in affine geometry and information geometry are known in the literature, and one of them is as follows:
Proposition 2.1. ([17]). If \((f, \xi)\) is non-degenerate and equiaffine, then \((\mathcal{M}, \nabla, h)\) is a statistical manifold.

Another relation between affine and information geometries is found on the study of divergence. To define the geometric divergence, consider a graph immersion \((f, \xi)\) associated with \(\psi : \Omega \rightarrow \mathbb{R}\), written in coordinates as
\[ f : \theta \mapsto (\theta, \psi(\theta)), \quad \xi = \frac{\partial}{\partial z}. \]

If \(\psi\) is convex, then the explicit form of the conormal map acting on \(p \in \Omega\) is obtained in coordinates with (3) and (9) as
\[ v(p) = dz - \sum_{a=1}^{n} \eta_a(p) \, d\theta^a \in T_{f(p)}^* \mathbb{R}^{n+1}. \] (10)

Then, let \((f, \xi)\) be a non-degenerate equiafffine immersion, \(v\) its conormal map, and \(\Delta^f : \Omega \times \Omega \rightarrow T\mathbb{R}^{n+1}\) a map such that
\[ \Delta^f(p_1, p_2) = \sum_{a=1}^{n} (\theta^a(p_1) - \theta^a(p_2)) \frac{\partial}{\partial \theta^a} + (\psi(p_1) - \psi(p_2)) \frac{\partial}{\partial z} \in T_{f(p_2)}^* \mathbb{R}^{n+1}. \]

Then the function \(D^G : \Omega \times \Omega \rightarrow \mathbb{R}\), called geometric divergence,
\[ D^G(p_1, p_2) = \langle v(p_2), \Delta^f(p_1, p_2) \rangle, \] (11)
is introduced in the context of the study of affine geometry. A relation between \(D\) and \(D^G\) is obtained as shown explicitly in Ref. [17]. It is obtained from (10) and
\[ \varphi(p_2) = -\psi(p_2) + \sum_{a=1}^{n} \eta_a(p_2) \theta^a(p_2), \]
as
\[ D^G(p_1, p_2) = \langle v(p_2), \Delta^f(p_1, p_2) \rangle = \psi(p_1) - \psi(p_2) - \sum_{a=1}^{n} \eta_a(p_2)(\theta^a(p_1) - \theta^a(p_2)) = \psi(p_1) + \varphi(p_2) - \sum_{a=1}^{n} \eta_a(p_2) \theta^a(p_1) = D(p_1, p_2). \]

In more general case, a relation between the divergence and the geometric divergence has been known as follows:

Theorem 2.2. ([17]). Let \((\mathcal{M}, \nabla, g)\) be a simply connected dually flat space. Then the canonical divergence and the geometric divergence on \((\mathcal{M}, \nabla, g)\) coincides.

2.3 Thermodynamics and existing geometric formulations

In this subsection necessary background of thermodynamics for this study is briefly summarized. Symbols introduced here for thermodynamic quantities are duplicated with symbols for geometry introduced in Section 2.1. These duplicated symbols become consistent when affine geometric thermodynamics is constructed. In addition, thermodynamic processes in this paper are assumed to be quasi-static for simplicity.

To formulate equilibrium thermodynamics, we employ a subset of \(\mathbb{R}^n\) for thermodynamic variables. In addition we employ \(\mathbb{R}\) for a complete thermodynamic function, where complete thermodynamic functions are functions that derive equations of state at equilibrium and response functions [24]. Examples of complete
thermodynamic functions are the internal energy, entropy, Helmholtz and Gibbs free-energies with appropriate arguments. Thermodynamic variables and complete thermodynamic functions play fundamental roles, in the sense that these induce equations of state at equilibrium. More details about these variables and functions are explained below. Let \( x = (x^1, \ldots, x^n) \) be a set of thermodynamic variables in \( \mathbb{R}^n \) and \( F \) a complete thermodynamic function (free-energy or internal energy). If \( x \) is a set of arguments or equivalently variables of \( F \), then \( x^1, \ldots, x^n \) are called primal thermodynamic variables in this paper. Then the set of the thermodynamic conjugate variables \( y = (y_1, \ldots, y_n) \) is such that
\[
y_a = \frac{\partial F}{\partial x^a}, \quad a = 1, \ldots, n. \tag{12}
\]
In addition the value of a complete thermodynamic function \( F \) (a free-energy, entropy, or internal energy) at \( x \) should be the same as that of \( F(x) \),
\[
z = F(x). \tag{13}
\]
The thermodynamic phase space is where (12) and (13) are satisfied. Hence, the thermodynamic phase space is a subset of \( \mathbb{R}^{2n+1} \). The fundamental relation of thermodynamics can be written as
\[
dU - T dS + P dV = 0, \tag{14}
\]
where \( U \) is internal energy, \( T \) the absolute temperature, \( S \) entropy, \( P \) pressure, and \( V \) volume. In addition, the heat (one-form) \( Q \) and the work (one-form) \( W \) are defined as
\[
Q = T dS, \quad \text{and} \quad W = -P dV,
\]
respectively [2]. The first law of thermodynamics states that there exists the function \( U \) so that (14) holds with some processes, where processes are integral curves of vector fields. In other words, it states that the sum \( Q + W \) is an exact one-form \( dU \). Meanwhile, the second law of thermodynamics states that there exists the function \( S \) with some properties. To change variables, the Legendre transform is applied to functions, where the transformed functions are convex if the original functions are convex. Given a function \( F \), it follows from the theory of Legendre transform that there is a function \( F^* \) such that
\[
x^a = \frac{\partial F^*}{\partial y_a}, \quad a = 1, \ldots, n. \tag{17}
\]
After changing thermodynamic variables, (14) can be written in various forms, such as
\[
dS - \frac{1}{T} dU - \frac{P}{T} dV = 0, \tag{15}
\]
\[
dA + S dT + P dV = 0, \tag{16}
\]
and so on, where \( A \) denotes the Helmholtz free-energy. Response functions, such as heat capacity and specific heat, are obtained by differentiation of (12) as
\[
\chi_{ab} = \frac{\partial y_a}{\partial x^b} = \frac{\partial^2 F}{\partial x^a \partial x^b}, \quad a, b = 1, \ldots, n. \tag{17}
\]
Equilibrium states can be classified into at least three classes. They are unstable, metastable, and most stable equilibrium states. Since equilibrium states are fundamental objects in the study of thermodynamics, a classification of equilibrium states provides further understanding of thermodynamic properties of substances. The three classes above are explained below. The most stable equilibrium states are equilibrium states that are structurally stable within some time-length against some small external perturbation. Metastable equilibrium states are states that exist without very small external perturbation, but states deviate from the metastable states under some strength of external perturbation. Unstable equilibrium are states that are hard to be realized experimentally since the states will not return to the original unstable equilibrium states under very small perturbation.
Nonequilibrium thermodynamics is a developing branch of physics, and several theories have been proposed in the literature\cite{25, 26}. To provide a solid foundation for a nonequilibrium theory, we focus on a simple class for clarity. One simple class of nonequilibrium phenomena is that of relaxation processes. In this paper even in a nonequilibrium state, thermodynamic variables are assumed to be defined and described by extending the equilibrium thermodynamic variables. Let $y(t)$ and $F(t)$ be a set of thermodynamic variables and the value of a nonequilibrium free-energy at time $t \in I \subset \mathbb{R}$, respectively. In addition, $y$ and $F$ denote the corresponding variable set and the function defined at equilibrium, respectively. If a process (or a time-evolution) that satisfies
\[
\lim_{t \to \infty} y(t) = y, \quad \text{and} \quad \lim_{t \to \infty} F(t) = F,
\]
then the process is referred to as a relaxation process in this paper.

There are several existing geometric formulations of thermodynamics in the literature. Such existing studies include Hesse geometry and contact geometry:

- (\cite{23}). A Hesse manifold is a manifold $M$ equipped with a structure $(D, g)$, where $D$ is a flat connection, and $g$ a pseudo Riemannian metric tensor field that can be written as $g = DdF$ with $F$ being a function.
- (\cite{11, 10}). A contact manifold is an odd-dimensional manifold $M$ equipped with a distribution $\ker \lambda$, where $\lambda$ is a one-form that satisfies the condition that the top-form $\lambda \wedge d\lambda \wedge \cdots \wedge d\lambda$ is a volume element on $M$.

In this section the calligraphic letter $\mathcal{F}$ is used to emphasize that the function is a complete thermodynamic function. In what follows this emphasis is not adopted, and thus the calligraphic letter is not used even in the case that a function is a complete thermodynamic function.

3 Affine geometric description of thermodynamics

In this section an affine geometric description of thermodynamics is proposed and discussed. This description consists of two cases, one is equilibrium case and the other nonequilibrium one. The description of the nonequilibrium case can be divided into two, one is the case of a unique set of equilibrium states, and the other is the case of two sets of equilibrium states:

- Equilibrium
- Nonequilibrium
  \[
  \begin{cases}
  \text{a unique set of equilibrium states} \\
  \text{two sets of equilibrium states}
  \end{cases}
  \]

3.1 Equilibrium

To describe thermodynamics in the language of affine geometry, the basic idea proposed in this paper is to employ graph immersions of a region $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^{n+1}$. In this paper the following interpretations are proposed.

Interpretation 3.1. (Equilibrium thermodynamics and affine geometry).

1. A point of $\mathbb{R}^{n+1}$ is identified as a set of primal thermodynamic variables and the value of a complete thermodynamic function (a free-energy, entropy or internal energy).

2. The coordinate system $x = (x^1, \ldots, x^n)$ for $\Omega \subset \mathbb{R}^n$ represents primal thermodynamic variables in thermodynamic systems. Then the coordinate $z$ for the additional space $\mathbb{R}$ represents the value of a complete thermodynamic function (a free-energy, entropy, or internal energy). This is written as $z = F(x)$ when a thermodynamic relation holds, and let $\xi = \partial / \partial z$. 
3. The image of a graph immersion \((f, \xi) \subset \mathbb{R}^{n+1}\) associated with \(F\) is identified with a thermodynamic phase space, where a thermodynamic relation holds. Accordingly, a point of this space is identified with an equilibrium state.

4. The conormal map is identified with one of the expressions of the fundamental relation of thermodynamics, \((14), (15),\) and \((16)\).

Note that as well as item 1 of Interpretation 3.1 where \((n + 1)\)-dimensional manifolds are involved, there are several geometric approaches with \((n + 1)\)-dimensional manifolds in the literature \([2, 27]\).

**Remark 3.2.** The above identifications induce the following.

1. The affine fundamental form \(h\), a \((0, 2)\)-type operator, is identified with a set of response functions at equilibrium (on the thermodynamic phase space) (see \((6)\) and \((17)\)).

2. The flat connection \(\nabla\) on \(\Omega\) is introduced at equilibrium (on the thermodynamic phase space). In addition, \(x\) is an affine coordinate system (see \((1)\) and \((7)\)).

3. The \(z\) component of the push-forward \(f_*(\partial/\partial x^a)\) is identified with the thermodynamic conjugate variable with respect to \(x^a\) (see \((4)\)). This is written as \(y_a = \partial F/\partial x^a\), where \(y_a\) is the thermodynamic conjugate variable with respect to a primal thermodynamic variable \(x^a\), and the collection is denoted by \(y = (y_1, \ldots, y_n)\). Hence thermodynamic primal and conjugate variables are expressed as a point of the tangent bundle \(T(f\Omega)\).

4. A succinct information about describing a set of equilibrium states is the image of a graph immersion \((f, \xi)\) into \(\mathbb{R}^{n+1}\), so that the definition of equilibrium state is given by item 3 of Interpretation 3.1.

When thermodynamic conjugate variables are needed, its tangent bundle is considered. For later purpose of describing nonequilibrium states, a relaxation (or, an extension) of the image of the graph immersion is discussed here. Since the values of a free-energy and thermodynamic variables in a nonequilibrium state are not written as \(F(x)\) and \((x, y(x))\) in coordinates, the manifold \(T(f\Omega)\) is relaxed to (or, extended to)

\[T\Omega \times \mathbb{R}.\]  

A point of \(T\Omega \times \mathbb{R}\) can be written in coordinates as \((x, y, z)\), where \((x, y)\) is the coordinates of \(T\Omega\) and \(z\) is the coordinate of \(\mathbb{R}\). Meanwhile the manifold \(T\Omega \times \mathbb{R}\) is redundant for the purpose of expressing equilibrium states. Since \(T(f\Omega)\) and \(T\Omega \times \mathbb{R}\) include redundant components for describing equilibrium states, we have defined equilibrium states as the image of a graph immersion \((f, \xi)\).

5. The conormal map, \((8)\) and \((3)\), provides the fundamental relation of thermodynamics, \((14), (15),\) and \((16)\).

6. In the case that \(F\) is convex for the graph immersion \((f, \xi)\), it follows from Proposition 2.1 that the equilibrium phase space is identified with a statistical manifold.

7. In the case that \(F\) is convex, the geometric divergence \((11)\) is introduced in the thermodynamic phase space.

Some of Remark 3.2 is summarized as follows. This is the main claim in this subsection:

**Proposition 3.3.** (equilibrium states in the language of affine geometry). On a set of equilibrium states that is a thermodynamic phase space in the sense of this paper, an affine fundamental form and a flat connection are induced. In addition, thermodynamic conjugate variables are described in the tangent bundle. The geometric divergence is introduced if the complete thermodynamic function is convex or concave.

Examples are given as follows. The first example below shows how to find an affine immersion and its geometric quantities from a given equation of state and a given complete thermodynamic function.
Example 3.4. (Ideal gas and its Helmholtz free energy). Consider the ideal gas, where the equation of state is

\[ PV = RT, \]

where \( R > 0 \) is constant, \( T > 0 \) temperature, \( P \) pressure, and \( V \) volume. This equation is written as

\[ P = -\frac{\partial A}{\partial V}, \quad A = -RT \ln V, \]

where \( A \) denotes the Helmholtz free energy. Let \( T \) be fixed, and identify \( \Omega = \mathbb{R}_{>0} \),

\[ x = V, \quad F(x) = -A(V) = RT \ln x, \]

and \( z \) the value of the free-energy. From the Helmholtz free energy as a complete thermodynamic function and the equation of state given above, the corresponding affine immersion is shown below. As a transversal vector field, take \( \xi = \partial / \partial z \). Then the image of the graph immersion \((f, \xi)\) associated with \( F \) is identified with the set of equilibrium states at temperature \( T \). The affine fundamental form \( h \) is such that

\[ h = \frac{\partial^2 F}{\partial x^2} dx \otimes dx, \quad \frac{\partial^2 F}{\partial x^2} = -\frac{RT}{x^2} \left( = \frac{\partial P}{\partial V} \right). \]

Thus \( h \) is non-degenerate on \( \Omega = \mathbb{R}_{>0} \), from which this graph immersion is non-degenerate. By applying Proposition 3.4 one can introduce the dual coordinate system \( x^* \) with respect to \( h \),

\[ F^*(x^*) = RT - RT \ln(RT) + RT \ln(x^*), \]

and the geometric divergence \( D^G \). The \( z \) component of \( f_*(\partial / \partial x) \), \( y \), is identified with \( P \).

The example below shows how to find an affine immersion and its geometric quantities from a given complete thermodynamic function and a given fundamental relation of thermodynamics.

Example 3.5. (Ideal gas and its entropy). Consider the ideal gas again (see Example 3.4). Introduce entropy \( S \) as a complete thermodynamic function

\[ S(U,V) = R \ln(U^c V), \]

where \( U \) is internal energy and \( c \) a positive constant. Since primal thermodynamic variables are the arguments of \( S \), the primal thermodynamic variables defined on \( \Omega = \mathbb{R}_{>0}^2 := \mathbb{R}_{>0} \times \mathbb{R}_{>0} \) for this model are

\[ x^1 = U, \quad \text{and} \quad x^2 = V, \]

so that \( S : \Omega \to \mathbb{R} \),

\[ S(x^1, x^2) = R \ln((x^1)^c x^2). \]

To specify an immersion, we let \( z \) be the value of \( S \) and let \( \xi = \partial / \partial z \) be a transversal vector field. The image of the graph immersion associated with \( S \) is identified with the thermodynamic phase space. The conormal map is expressed and specified as

\[ v = dz - \frac{\partial S}{\partial x^1} dx^1 - \frac{\partial S}{\partial x^2} dx^2 = dS - \frac{1}{T} dU - \frac{P}{T} dV. \]

The conjugate thermodynamic variables are then

\[ y_1 = \frac{\partial S}{\partial x^1} = \frac{cR}{x^1} = \frac{1}{T}, \quad \text{and} \quad y_2 = \frac{\partial S}{\partial x^2} = \frac{R}{V} = \frac{P}{T}. \]

These yield the expression of \( U \) and the equation of state,

\[ U = cRT, \quad \text{and} \quad PV = RT, \]
respectively. From the first equation above, the specific heat \( C = dU/dT \) is derived as \( C = cR \). The explicit form of the affine fundamental form \( h \) is shown as

\[
h = \sum_{a=1}^{2} \sum_{b=1}^{2} h_{ab} \, dx^a \otimes dx^b, \quad h_{ab} := \frac{\partial^2 S}{\partial x^a \partial x^b}, \quad \text{with} \quad h_{11} = -\frac{cR}{(x^1)^2}, \quad h_{12} = 0, \quad h_{22} = -\frac{R}{(x^2)^2}.
\]

Thus \( h \) is non-degenerate on \( \Omega = \mathbb{R}_0^2 \), from which this graph immersion is non-degenerate. By applying Proposition 2.1 one can introduce the function \( S^* \) so that

\[
x^1 = \frac{\partial S^*}{\partial y_1}, \quad \text{and} \quad x^2 = \frac{\partial S^*}{\partial y_2}.
\]

This \( S^*(y_1, y_2) \) is found to be

\[
S^*(y_1, y_2) = R \ln(y_1y_2) + \text{(constant)},
\]

and then the geometric divergence \( D^G \) can be introduced.

There are several complete thermodynamic functions for the ideal gas system, such as \( A \) in Example 3.4 and \( S \) in Example 3.5. Internal energy as a function of \( S \) and \( V \) is another choice as a complete thermodynamic function. In this choice, \( U(S, V) = c'V^{-1/c} \exp(S/(cR)) \) and (14) yield the equation of state \( PV = RT \) and the specific heat \( C = cR \), where \( c' \) is constant. In the examples above, the affine immersions are non-degenerate. Meanwhile in the following example, an affine immersion is shown to be degenerate.

**Example 3.6.** (van der Waals equation of state). The state equation for the van der Waals gas model in the dimension-less variables is written as

\[
\left( P + \frac{3}{V^2} \right)(3V - 1) = 8T,
\]

where \( T \) denotes temperature, \( V \) volume, and \( P \) pressure. This equation can be written as

\[
P = -\frac{\partial A}{\partial V}, \quad A = -\frac{3}{V} \frac{8T}{3} \ln(3V - 1),
\]

where \( A \) denotes the Helmholtz free-energy. Let \( T \) be fixed, and identify \( \Omega = \mathbb{R}_0^2 \),

\[
x = V, \quad F(x) = -A(V),
\]

and \( z \) the value of the free-energy. Choose \( \xi = \partial/\partial z \) as a transversal vector field. Then the image of the graph immersion \( (f, \xi) \) associated with \( F \) is identified with the set of equilibrium states at temperature \( T \). The affine fundamental form \( h \) is such that

\[
h = \frac{\partial^2 F}{\partial x^2} \, dx \otimes dx, \quad \frac{\partial^2 F}{\partial x^2} = \frac{6}{x^3} - \frac{24T}{(3x - 1)^2} \left( = \frac{\partial P}{\partial V} \right),
\]

from which this graph immersion is degenerate (not non-degenerate). The \( z \) component of \( f_*(\partial/\partial x) \), \( y \), is identified with \( P \). Note that there are several existing studies in the literature on this model in the language of Riemannian geometry with the Levi-Civita connection [15, 28].

### 3.2 Nonequilibrium

In this subsection, a nonequilibrium geometric theory is proposed. This is based on the equilibrium affine geometric theory developed in Section 3.1.

To construct a nonequilibrium geometric theory one needs to

1. introduce physical time,
2. extend or relax the equilibrium theory, where the equilibrium theory is based on a graph immersion \((f, \xi)\) into \(\mathbb{R}^{n+1}\),

3. verify that the nonequilibrium theory is consistent with the equilibrium theory in some limits. One of these limits is the time-asymptotic limit.

To develop a nonequilibrium theory, we

1. introduce time, and it is denoted by \(t \in I \subset \mathbb{R}\),

2. introduce the trivial fiber bundle \(\Omega \times \mathbb{R}\), where the restriction of this bundle to the set of equilibrium states is \(\Omega \times F(\Omega)\) being the image of a graph immersion \((f, \xi)\). Hence this trivial bundle is an extension of the graph immersion.

3. focus on the so-called relaxation process. Relaxation processes are time-dependent phenomena in which thermodynamic variables achieve an equilibrium state from a nonequilibrium state through a time evolution. Since there are a variety of classes of nonequilibrium systems and any confusion should be avoided, a simple formulation for a simple nonequilibrium phenomena is proposed in this paper. Then, time-asymptotic limits of the nonequilibrium theory is verified to be consistent with the equilibrium theory.

In what follows, the two cases are considered (see Fig. 1). They are relaxation processes for systems with • a unique set of equilibrium states, and • two sets of equilibrium states.

![Figure 1: Relaxation processes as integral curves of vector fields. (Left) System with a unique set of equilibrium states. The thick line represents the set of unique equilibrium states that is the graph of \(F\), and the arrows represent the relaxation generating vector field \(X_F\) (see Section 3.2.1). (Right) System with two sets of equilibrium states. The thick line represents the set of the most stable equilibrium state that is the graph of \(F_I\), the dashed line represents the set of metastable equilibrium states that is the graph of \(F_{II}\), and the arrows represent the relaxation generating vector field \(X_{II \rightarrow I}\) (see Section 3.2.2).](image)

### 3.2.1 Nonequilibrium, (i) case of unique set of equilibrium states

In the following, a nonequilibrium geometric theory is proposed for the case that there is a unique set of equilibrium states.

The following is the definition of relaxation process in the case that there is a unique set of equilibrium states.
Definition 3.7. (relaxation process). Consider a graph immersion \((f, \xi)\) associated with \(F\) from \(\Omega\) into \(\mathbb{R}^{n+1}\), \((\dim \Omega = n)\), and let \(x\) be a coordinate system of \(\Omega\), \(z \in \mathbb{R}\), \(\xi = \partial/\partial z\) and \(y_a = dz(f_a(\partial/\partial x^a))\), \((a = 1, \ldots, n)\). Note that \(y_a\) is the \(z\)-component of \(f_a(\partial/\partial x^a)\). Suppose that the image of \((f, \xi)\) is identified with a set of equilibrium states. Then let \(\phi: I \ni t \mapsto p(t) \in T\mathbb{R}^{n+1}\) be a curve, where the coordinates of \(p(t)\) are denoted by \((x(t), z(t), y(t), w(t))\), \((t \in I \subset \mathbb{R})\). For each \(x\) kept fixed in \(t\), if a curve satisfies the conditions

\[
\lim_{t \to \infty} y_a(t) = \frac{\partial F}{\partial x^a}(x), \quad \text{and} \quad \lim_{t \to \infty} z(t) = F(x), \quad a = 1, \ldots, n,
\]

then the image of the curve is said to be a relaxation process towards a point of the set of equilibrium states \((f, \xi)\).

To state an affine geometric description of this class of relaxation processes in terms of vector fields, a set of equilibrium states is placed as follows. Let \((f, \xi)\) be a graph immersion into \(\mathbb{R}^{n+1}\) written in coordinates as \(x \mapsto (x, F(x)) \in \mathbb{R}^{n+1}\) and \(\xi = \partial/\partial z\), that is, this immersion is associated with a function \(F: \Omega \to \mathbb{R}\). First, recognize that \(\Omega \times \mathbb{R} \subset \mathbb{R}^{n+1}\) is a trivial fiber bundle, \(\pi: \Omega \times \mathbb{R} \to \Omega, \pi(q, z) = q\). Hence \(z \in \mathbb{R}\) is a point of the fiber whose base point is \(q\). Then, consider the vector field on \(\mathbb{R}\) over the point \(q \in \Omega\) of the form written in coordinates as

\[
X_F = w_F(z; x) \frac{\partial}{\partial z}
\]

with \(w_F(-; x)\) being a function of \(z\) at a given \(x\) (see Fig. 1 (Left)). Integral curves of \((20)\) are obtained by solving the ordinary differential equation (ODE)

\[
\frac{dz}{dt} = w_F(z; x), \quad t \in I \subset \mathbb{R}.
\]

Equation \((21)\) is said to be the system of the ODE associated with \(X_F\) in this paper. If a solution \(z(t; x)\) of \((21)\) satisfies

\[
\lim_{t \to \infty} z(t; x) = F(x),
\]

then the system \((21)\) together with an appropriate initial condition is referred to as a relaxation generating system, and the corresponding vector field \(X_F\) a relaxation generating vector field in this paper.

The system in the following example is a relaxation generating system.

Example 3.8. Consider the dynamical system on \(\mathbb{R}\) associated with \(X_F\):

\[
\frac{dz}{dt} = w_F(z; x), \quad w_F(z; x) = F(x) - z, \quad \text{for each fixed } x, \quad I = \mathbb{R}.
\]

Then the explicit form of the solution \(z(t; x)\) is immediately obtained from \((22)\) as

\[
z(t; x) = (1 - e^{-t})F(x) + e^{-t}z(0; x) =: F_t(x).
\]

From \(\lim_{t \to \infty} z(t; x) = F(x)\) for each \(x\), it follows that the system \((22)\) is a relaxation generating system.

Roughly speaking a relaxation generating system is a seed for generating a relaxation process. This is refined as follows:

Theorem 3.9. (induced relaxation process 1). Consider the relaxation generating system \((21)\) with some initial condition. Then, introduce a family of graph immersions \(\{(f_t, \xi_t)\}_{t \in I}\) into \(\mathbb{R}^{n+1}\) written in coordinates as \(x \mapsto (x, F_t(x)) \in \mathbb{R}^{n+1}\), \((t \in I)\), that is, \(f_t\) is associated with a function \(F_t\). In addition,

1. choose the function \(F_t\) to be \(F_t(x) = z(t; x)\) with \(z(t; x)\) being a solution to \((21)\), and
2. let \(y_a(t; x)\) be such that

\[
y_a(t; x) = \frac{\partial F_t(x)}{\partial x^a}, \quad a = 1, \ldots, n.
\]
If the limit and differentiation are commute, then the image of the curve \( t \mapsto (x(z(t; x), y(t; x), w(t; x))) \) is a relaxation process in the sense of Definition 3.7:

\[
\lim_{t \to \infty} z(t; x) = F(x), \quad \text{and} \quad \lim_{t \to \infty} y_a(t) = \frac{\partial F}{\partial x^a}(x), \quad a = 1, \ldots, n.
\]

**Proof.** The statement on \( \lim z(t; x) \) is nothing but the definition of relaxation generating system. Then a proof for \( y_a(t) \) is given below. For all \( x \), it follows from

\[
\lim_{t \to \infty} F_t(x) = F(x),
\]

and commutability of the limit and differentiation that

\[
\lim_{t \to \infty} y_a(t) = \frac{\partial F}{\partial x^a}(x), \quad a = 1, \ldots, n.
\]

**Remark 3.10.** Note the following:

1. Given a relaxation generating system, the induced relaxation process is obtained by applying Theorem 3.9.
2. For each \( t \in \mathcal{I} \) of this family of graph immersions, \((f_t, \xi_t)\) can be interpreted as a set of equilibrium states.
3. For each \( t \in \mathcal{I} \) of this family of graph immersions, a flat connection is induced where the connection can be written as \( \nabla_t \).
4. For each \( t \in \mathcal{I} \) of this family of graph immersions, the geometric divergence is induced if \( F_t(x) \) is convex or concave with respect to \( x \), where this geometric divergence can be written as \( D^G_t \).

The following is a physical model explaining Theorem 3.9 in a physical language. The following example also shows how a physical system is translated into its affine geometric description of thermodynamics proposed in this paper.

**Example 3.11.** (kinetic Ising model without spin-coupling [29, 30]). Consider a nonequilibrium system consisting of only one spin \( \sigma = \pm 1 \) in contact with a heat bath of fixed temperature \( T = 1/(k_B\beta) \) with \( k_B \) being the Boltzmann constant. The dynamics of the statistical average of this spin is governed by an externally applied static magnetic field \( \mathcal{H} \), where this \( \mathcal{H} \) yields a relaxation process towards a point of the unique set of equilibrium states. To describe this system, introduce a time-independent variable \( x := \beta \mu_B \mathcal{H} \in \mathbb{R} \). This \( x \) is a dimensionless variable for \( \mathcal{H} \) with \( \mu_B \) being the Bohr magneton. In the following after the equilibrium case is discussed, the nonequilibrium case is discussed. The equilibrium probability distribution function is assumed to be the canonical distribution,

\[
P_{eq}(\sigma, x) = \exp(x\sigma)/Z(x), \quad \text{where} \quad Z(x) = \sum_{\sigma = \pm 1} e^{x\sigma} = 2 \cosh x.
\]

At equilibrium, one can introduce the negative of a dimensionless free-energy \( F : \mathbb{R} \ni x \mapsto F(x) \in \mathbb{R} \) as

\[
F(x) := \ln Z(x) = \ln \cosh(x) + \ln 2,
\]

the magnetization at equilibrium is then written as

\[
\langle \sigma \rangle_{eq}(x) := \sum_{\sigma = \pm 1} P_{eq}(\sigma, x)\sigma = \tanh(x) = \frac{dF}{dx}.
\]
In the following a dynamical equation for this system is derived. Let $I = \mathbb{R}$. Introduce $P_t : \{\pm 1\} \times \mathbb{R} \to \mathbb{R}_{>0}$, $(t \in \mathcal{I})$, that is a probability distribution function of $\sigma$ at time $t$. Then the expectation variable at $t$ is denoted by $\langle \sigma \rangle (t; x) := \sum_{\sigma=\pm 1} P_t(\sigma; x) \sigma$, $(t \in \mathcal{I})$. This variable, $\langle \sigma \rangle (t; x)$, is identified with a nonequilibrium magnetization at $t$ for a fixed $x$. Imposing the simple assumptions for $P_t$,

- the equation for $P_t$ is the Pauli master equation, and
- the detailed balance condition holds for the master equation,

one derives the ODE:

$$\frac{d}{dt} \langle \sigma \rangle = \tanh(x) - \langle \sigma \rangle .$$

This system of the ODE is referred to as the kinetic Ising model without spin-coupling in Ref. [29]. Solving this ODE explicitly, one verifies that

$$\lim_{t \to \infty} \langle \sigma \rangle (t; x) = \langle \sigma \rangle_{eq} (x).$$

To write this system in affine geometry, identify $y(t; x) = \langle \sigma \rangle (t; x)$, and introduce the new variable $z(t; x) \in \mathbb{R}$. The physical meaning of $z$ is the value of a nonequilibrium extension of $F$, and $z(t; x)$ is denoted by $F_t(x)$. A relaxation generating system is introduced by letting $w_F(t; x) = F(x) - z$ as in [22],

$$\frac{dz}{dt} = \ln \cosh x + \ln 2 - z,$$

whose solution [23] is written as $z = F_t(x)$ with

$$F_t(x) = (1 - e^{-t})(\ln \cosh(x) + \ln 2) + e^{-t}z(0; x).$$

The set $\{(f_t, \xi_t)\}_{t \in \mathcal{I}}$ is a family of graph immersions, and the image of the curve $t \mapsto (x, y(t; x), z(t; x), w_F(t; x))$ is a relaxation process.

### 3.2.2 Nonequilibrium, (ii) case of two sets of equilibrium states

In the following, a nonequilibrium geometric theory is proposed for the case that there are two sets of equilibrium states. This is based on the geometric theories developed in Sections [3.1] and [3.2.1]. To avoid unnecessary confusion and to keep discussions in this paper simple, cases where more than three equilibrium states are not discussed in this paper.

The following is the definition of relaxation process from a point of the set of metastable equilibrium states to a point of the set of the most stable equilibrium states.

**Definition 3.12.** (relaxation process with metastable state). For $\Omega_0 \subset \Omega$, let $x$ be coordinates of a point $q_0 \in \Omega_0$, $z \in \mathbb{R}$, $\xi = \partial/\partial z$, and $y_a = dz(f_a(\partial/\partial x^a))$, $(a = 1, \ldots, n)$. In addition let $F_1 : \Omega_0 \to \mathbb{R}$ and $F_{II} : \Omega_0 \to \mathbb{R}$ be functions where the condition $F_1(q_0) < F_{II}(q_0)$ holds for any $q_0 \in \Omega_0$. Consider the two graph immersions given below:

- A graph immersion $(f_1, \xi)$ associated with $F_1$ into $\mathbb{R}^{n+1}$. Suppose that the image of $(f_1, \xi)$ is identified with the most stable equilibrium state set.
- A graph immersion $(f_{II}, \xi)$ associated with $F_{II}$ into $\mathbb{R}^{n+1}$. Suppose that the image of $(f_{II}, \xi)$ is identified with a metastable equilibrium state set.

In addition, let $\phi : \mathbb{R} \ni t \mapsto p(t) \in T\mathbb{R}^{n+1}$ be a curve, where the coordinates of $p(t)$ are denoted by $(x(t), z(t), y(t), w(t))$, $(t \in \mathbb{R})$. If a class of curves satisfies the conditions

$$z(0) < F_{II}(x), \quad x(t) = x(0), \quad \forall t \in \mathbb{R},$$

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1. \[ \lim_{t \to -\infty} z(t) = F_{II}(x), \quad \text{and} \quad \lim_{t \to -\infty} y_a(t) = \frac{\partial F_{II}}{\partial x^a}(x), \quad a = 1, \ldots, n, \]

2. \[ \lim_{t \to \infty} z(t) = F_1(x), \quad \text{and} \quad \lim_{t \to \infty} y_a(t) = \frac{\partial F_1}{\partial x^a}(x), \quad a = 1, \ldots, n, \]

then the image of the curve is said to be a relaxation process from a point of \((f_{II}, \xi)\) towards a point of \((f_1, \xi)\).

To state an affine geometric description of this class of relaxation processes, two sets of equilibrium states are placed as follows. Let \((f_1, \xi)\) and \((f_{II}, \xi)\) be graph immersions into \(\mathbb{R}^{n+1}\) written in coordinates as \(x \mapsto (x, F_1(x)) \in \mathbb{R}^{n+1}\) and \(x \mapsto (x, F_{II}(x)) \in \mathbb{R}^{n+1}\), respectively, where \(\xi = \partial / \partial z\). In addition, let \(\Omega_0 \subset \Omega\) be a region on which the condition \(F_{II}(x) > F_1(x)\) is satisfied. First, recognize that \(\Omega_0 \times \mathbb{R} \subset \mathbb{R}^{n+1}\) is a trivial fiber bundle, \(\pi : \Omega_0 \times \mathbb{R} \to \Omega_0, \pi(q_0, z) = q_0\). Hence \(z \in \mathbb{R}\) is a point of the fiber whose base point is \(q_0\). Then, consider the vector field on the fiber \(\mathbb{R}\) over a point \(q_0 \in \Omega_0 \subset \Omega\) of the form written in coordinates

\[ X_{II\rightarrow I} = w_{II\rightarrow I}(z; x) \frac{\partial}{\partial z} \quad (24) \]

with \(w_{II\rightarrow I}(\cdot; x)\) being a function of \(z\) at a given \(x\) (see Fig. 1 (Right)). Integral curves of \((24)\) are obtained by solving the ODE

\[ \frac{dz}{dt} = w_{II\rightarrow I}(z; x), \quad t \in \mathbb{R}. \quad (25) \]

If a solution \(z(t; x)\) whose initial condition is \(z(0) < F_{II}(x)\) satisfies

1. \[ \lim_{t \to -\infty} z(t; x) = F_{II}(x), \quad \text{and} \quad 2. \lim_{t \to \infty} z(t; x) = F_1(x), \]

then the system \((25)\) is referred to as a relaxation generating system with \(f_1\) and \(f_{II}\), and the corresponding vector field \(X_{II\rightarrow I}\) relaxation generating vector field with \(f_1\) and \(f_{II}\) in this paper.

**Example 3.13.** Consider the dynamical system on \(\mathbb{R}\) associated with \(X_{II\rightarrow I}\):

\[ \frac{dz}{dt} = w_{II\rightarrow I}(z; x), \quad w_{II\rightarrow I}(z; x) = -(z - F_1(x))(z - F_{II}(x))^2, \quad t \in \mathbb{R}, \quad \text{for a fixed} \; x, \quad (26) \]

where the graph of \(w_{II\rightarrow I}(z; x)\) is drawn in Fig. 2. In the case \(z(0) < F_{II}(x)\) in \(\mathbb{R}\) for a given \(x\), it follows that

\[ \lim_{t \to -\infty} z(t; x) = F_{II}(x), \quad \text{and} \quad \lim_{t \to \infty} z(t; x) = F_1(x). \quad (27) \]

Hence the system \((26)\) is a relaxation generating system.

There are various variants of \((26)\). One of them is

\[ w_{II\rightarrow I}(z; x) = (z - F_1(x))(z - F_{II}(x)). \]

Roughly speaking a relaxation generating system is a seed for generating a relaxation process. This is refined as follows:

**Theorem 3.14.** (induced relaxation process 2) Consider the relaxation generating system \((25)\). Then, introduce a family of graph immersions \(\{(f_t, \xi_t)\}_{t \in \mathbb{R}}\) into \(\mathbb{R}^{n+1}\) written in coordinates as \(x \mapsto (x, F_t(x)) \in \mathbb{R}^{n+1}, \; (t \in \mathbb{R})\), that is, \(f_t\) is associated with a function \(F_t\).

1. Choose \(F_t\) to be \(F_t(x) = z(t; x)\) with \(z(t; x)\) being a solution to \((25)\), and
2. let \(y_a(t; x)\) be

\[ y_a(t; x) = \frac{\partial F_t(x)}{\partial x^a}, \quad a = 1, \ldots, n. \]
If the limit and differential are commute, then the image of the curve \( t \mapsto (x, y(t; x), z(t; x), w_{II \rightarrow I}(t; x)) \) is a relaxation process in the sense of Definition 3.12:

1. 
\[
\lim_{t \to -\infty} z(t; x) = F_{II}(x), \quad \text{and} \quad \lim_{t \to -\infty} y_a(t; x) = \frac{\partial F_{II}}{\partial x^a}(t; x), \quad a = 1, \ldots, n.
\]

2. 
\[
\lim_{t \to \infty} z(t; x) = F_I(x), \quad \text{and} \quad \lim_{t \to \infty} y_a(t) = \frac{\partial F_I}{\partial x^a}(t; x), \quad a = 1, \ldots, n.
\]

**Proof.** The statement on \( \lim z(t; x) \) is nothing but the definition of relaxation generating system. Then a proof for \( y_a(t) \) is given below. For all \( x \), it follows from

\[
\lim_{t \to -\infty} F_t(x) = F_{II}(x), \quad \lim_{t \to \infty} F_t(x) = F_I(x),
\]

and commutability of the limit and differentiation that

\[
\lim_{t \to -\infty} y_a(t) = \frac{\partial F_{II}}{\partial x^a}(x), \quad \lim_{t \to \infty} y_a(t) = \frac{\partial F_I}{\partial x^a}(x), \quad a = 1, \ldots, n.
\]

**Remark 3.15.** Given a relaxation generating system, the induced relaxation process is obtained by applying Theorem 3.14. Note the following:

- For each \( t \in \mathbb{R} \), \((f_t, \xi_t)\) can be interpreted as a set of equilibrium states.
- For each \( t \in \mathbb{R} \), a flat connection is induced where the connection can be written as \( \nabla_t \).
- For each \( t \in I \), the geometric divergence is induced if \( F_t \) is convex or concave with respect to \( x \), where this geometric divergence can be written as \( D^G_t \).
4 Comparison with contact geometric thermodynamics

In this section the developed theory in Section 4 of this paper is compared with a representative existing theory. As a representative thermodynamic theory a contact geometric thermodynamics is summarized first in Section 4.1. Second a lift of the vector field \( X_{\ell} \) on \( \mathbb{R} \) is discussed in Section 4.2. The reason why lifted vector fields are considered is that the manifold on which \( X_{\ell} \) is defined is different to the manifold on which contact vector fields are defined. To compare these vector fields on the equal footing, one needs the same dimensional manifolds on which vector fields are defined. One way to realize such is to lift the vector fields on a lower dimensional manifold. For the same reason, a lift of \( X_{\Pi_{t=1}} \) is discussed. Then, in Section 4.3, the present affine geometric theory is compared with the existing contact geometric theory. For nonequilibrium systems, vector fields are compared by introducing appropriate identifications for different manifolds.

4.1 Contact geometric thermodynamics

One of developing geometric theories of thermodynamics employs contact geometry, where contact geometry is known as an odd-dimensional counterpart of symplectic geometry. In the contact geometric thermodynamics, the manifold \( T^*Q \times \mathbb{R} \) is often considered as an ambient manifold \([31, 32]\), where \( Q \) is an \( n \)-dimensional manifold. Let \( x \) be a coordinate system of \( Q \), \( y \) that of \( T^*_q Q \), and \( z \) that of \( \mathbb{R} \). In addition, equip the one-form \( \lambda = dz - \sum_{a=1}^n y_a \, dx^a \) on \( T^*Q \times \mathbb{R} \), where the top-form \( \lambda \wedge \, d\lambda \wedge \cdots \wedge d\lambda \) does not vanish anywhere. An odd-dimensional manifold with ker \( \lambda \) is called a contact manifold where ker \( \lambda \) denotes the kernel of \( \lambda \). Then the pair \((T^*Q \times \mathbb{R}, \ker \lambda)\) is a contact manifold.

A Legendrian submanifold of a contact manifold can be interpreted as a thermodynamic phase space\([5]\), where the equation of state at equilibrium is described. Here a Legendrian submanifold is an \( n \)-dimensional submanifold of a contact manifold satisfying the condition that the pull-back of \( \lambda \) vanishes. There are some useful projections \([13]\). In terms of the coordinate system \((x, y, z)\), the projection of a Legendrian submanifold onto the \((x, z)\)-plane is called a Legendre map, its image is called a wave front. In addition, the projection of a Legendrian submanifold onto the \((x, y)\)-plane is called a Lagrange map.

A diffeomorphism that preserves the contact structure ker \( \lambda \) is called a contact transform, and its vector field is called a contact vector field. This vector field is sometimes employed as a tool for expressing nonequilibrium time-evolution of thermodynamic systems \([29, 32, 33, 34]\) (see Ref. \[35\] for another thermodynamic interpretation of a contact vector field). A way to provide a contact vector field is to provide a function, called a contact Hamiltonian, and the derived contact vector field is called a contact Hamiltonian vector field. A contact Hamiltonian vector field \( Y_h \) associated with contact Hamiltonian \( h \) is determined by

\[ \iota_{Y_h} \lambda = h, \quad \text{and} \quad \iota_{Y_h} \, d\lambda = -dh + (R \lambda) \lambda, \]

where \( R \) is called the Reeb vector field defined such that

\[ \iota_R \lambda = 1, \quad \text{and} \quad \iota_R \, d\lambda = 0. \]

In the case where the contact form is expressed as above, the contact Hamiltonian vector field \( Y_h \) on \( T^*Q \times \mathbb{R} \) is written in coordinates as

\[ Y_h = \sum_{a=1}^n \left( x^a \frac{\partial}{\partial x^a} + y_a \frac{\partial}{\partial y_a} \right) + \dot{z} \frac{\partial}{\partial z}. \]

By identifying \( \dot{\cdot} = d/dt \), one derives

\[ \frac{d}{dt} x^a = -\frac{\partial h}{\partial y_a}, \quad \frac{d}{dt} y_a = y_a \frac{\partial h}{\partial x^a} + \frac{\partial h}{\partial y}, \quad \frac{d}{dt} z = h - \sum_{b=1}^n y_b \frac{\partial h}{\partial y_b}, \quad a = 1, \ldots, n. \]

When \( h \) does not depend on \( y \), \( h = h(x, z) \), one immediately has that

\[ \frac{d}{dt} x^a = 0, \quad \frac{d}{dt} y_a = y_a \frac{\partial h}{\partial z} + \frac{\partial h}{\partial x^a}, \quad \frac{d}{dt} z = h, \quad a = 1, \ldots, n. \]

The following two contact Hamiltonian systems will be focused in Section 4.3.
Choose $h$ to be $h_F$, where

$$h_F(x, z) = F(x) - z,$$

with $F$ being some function. Substituting this $h_F$ into (28), one has

$$\frac{d x^a}{d t} = 0, \quad \frac{d y_a}{d t} = -y_a + \frac{\partial F}{\partial x^a}, \quad \frac{d z}{d t} = h_F, \quad a = 1, \ldots, n. \quad (29)$$

The contact Hamiltonian system (29) has been studied in Refs. [29, 36], so that a class of relaxation processes is described on a contact manifold.

Choose $h$ to be $h_{II-1}$, where

$$h_{II-1}(x, z) = -(z - F_I(x))(z - F_{II}(x))^2,$$

with $F_I$ and $F_{II}$ being functions of $x$. Substituting this $h_{II-1}$ into (28), one has

$$\frac{d x^a}{d t} = 0, \quad \frac{d y_a}{d t} = -(z - F_{II})^2 \left( y_a - \frac{\partial F}{\partial x^a} \right) - 2(z - F_I)(z - F_{II}) \left( y_a - \frac{\partial F_{II}}{\partial x^a} \right), \quad \frac{d z}{d t} = h_{II-1}, \quad (30)$$

where $a = 1, \ldots, n$. The contact Hamiltonian system (30) has been studied in Ref. [32], so that a class of relaxation processes with sets of multiple equilibrium states is described on a contact manifold. In Ref. [32], the dimension of the contact manifold is three, and suffix $a$ in (30) has been omitted.

### 4.2 Lift of relaxation generating vector field

In this subsection vector fields on a higher dimensional manifold are defined based on the developed theories in Section 3.2, so that the vector fields on the higher dimensional manifold will be compared with the existing vector fields defined on a contact manifold. In particular, a vector field lifted from vector fields defined on a contact manifold. In Section 3.2, so that the vector fields on the higher dimensional manifold will be compared with the existing vector fields defined on a contact manifold. In particular, a vector field lifted from $X_F$ in (20) and a vector field lifted from $X_{II-1}$ in (24) are discussed here, where the lifted vector fields are denoted by $\tilde{X}_F$ and $\tilde{X}_{II-1}$, respectively. Recall that the vector fields $X_F$ and $X_{II}$ generate relaxation processes for the unique equilibrium state systems, and that vector fields $X_{II-1}$ and $\tilde{X}_{II-1}$ generate relaxation processes for non-unique equilibrium state systems.

Before discussing lifted vector fields for the both cases, how to discuss stability of vector fields on a manifold $M$ of dimension $n$ is summarized here. Let $\mu$ be a volume-element of $M$, and $X$ a vector field on $M$. Then the phase space compressibility of $X$ with respect to $\mu$, denoted $\text{div}_\mu X \in \Gamma \Lambda^0 \Lambda M$, is defined such that

$$\mathcal{L}_X \mu = (\text{div}_\mu X) \mu,$$

where $\mathcal{L}_X$ is the Lie derivative along $X$. This function is sometimes considered when stability of vector fields on manifolds is discussed [33]. If the system associated with $X$ has the property that $(\text{div}_\mu X)(p) < 0$ at $p \in M$, then this dynamical system is said to be contracting at $p \in M$ with respect to $\mu$ in this paper. A coordinate expression of $\text{div}_\mu X$ is obtained as follows. Let $\xi = (\xi^1, \ldots, \xi^n)$ be coordinates, and $\mu$ be such that

$$\mu = d \xi^1 \wedge \cdots \wedge d \xi^n.$$

Note that the coordinates $\xi$ are nothing to do with any transversal vector field of an affine immersion. Consider the vector field written in coordinates $\xi$ of the form

$$X = \sum_{a=1}^n \dot{\xi}^a(\xi) \frac{\partial}{\partial \xi^a},$$

where $\dot{\xi} = (\dot{\xi}^1, \ldots, \dot{\xi}^n)$ is a set of functions of $\xi$. Then the Lie derivative of $\mu$ along $X$ is calculated to be

$$\mathcal{L}_X \mu = (d \dot{\xi}^1 \wedge d \xi^2 \wedge \cdots \wedge d \xi^n) + \cdots + (d \dot{\xi}^1 \wedge \cdots \wedge d \xi^{n-1} \wedge d \xi^n) = \sum_{a=1}^n \frac{\partial \dot{\xi}^a}{\partial \xi^a} \mu.$$
from which one can write $\text{div}_\mu X$ as a function of $\xi$,

$$(\text{div}_\mu X)(\xi) = \sum_{a=1}^{n} \frac{\partial \xi^a}{\partial x^a}.$$  

To see the role of this function more clearly, consider the one-dimensional case, $X = \dot{z}\partial/\partial z$ on $\mathcal{M} = \mathbb{R}$ with $\mu = dz$ and $\dot{z} = -z$. Integral curves of $X$ are obtained by solving the ODE $\dot{z} = -z$, where $\dot{z} = dz/dt$. It follows from the explicit solution for $z$ that the absolute value $|z(t)|$ is decreasing (contracting) as time develops from $t = 0$ to any $t_0 > 0$. Meanwhile, for this system the phase space compressibility as a function of $z$ is obtained by the calculation $\partial(-z)/\partial z = -1$ as $(\text{div}_\mu X)(z) = -1$ on $\mathbb{R}$, and one concludes that this system is contracting. This example justifies the terminology “contracting.”

Below, the case of the unique equilibrium set and the case of non-unique equilibrium state sets are discussed separately.

### 4.2.1 Unique set of equilibrium states

In Theorem 3.9, the variables $y$ and $z$ evolve in time. Meanwhile the corresponding vector field $X_F$ and the dynamical system are written only in terms of $z$ for a fixed $q$ and $x$. Hence it is natural to consider a dynamical system involving $y$ and $z$, and is natural to consider its corresponding vector field denoted by $\tilde{X}_F$. This vector field is studied here. Since there are various classes of ODEs, one focuses on systems where

1. a fixed point $(y_{*a}, z_*)$ is $(\partial F/\partial x^a, F(x))$, $(a = 1, \ldots, n)$ with $x$ being a set of coordinate values of a point $q \in \Omega$, and
2. the corresponding vector field is a lift of $X_F$, in the sense that $\pi_* \tilde{X}_F = X_F$:

   $\begin{align*}
   T\mathbb{R} &\xleftarrow{\pi_*} T(T_q \Omega \times \mathbb{R}) , \\
   T\Omega \times \mathbb{R} &\xrightarrow{\pi} T\mathbb{R} , \\
   (z, \dot{z}) &\xleftarrow{\pi_*} (y, \dot{y}, z, \dot{z}) .
   \end{align*}$

   Note that the manifold $T\Omega \times \mathbb{R}$ has been discussed in [18] as a redundant manifold for describing equilibrium states.

This $\tilde{X}_F$ is written of the form:

$$\tilde{X}_F = \sum_{a=1}^{n} u_{Fa}(z, y; x) \frac{\partial}{\partial y_a} + w_F(z; x) \frac{\partial}{\partial z},$$  

with some functions $u = (u_{F1}, \ldots, u_{Fn})$. The reason why item 1 is needed is as follows. If $(\partial F/\partial x^a, F)$ is a fixed point and attractive in some sense, then integral curves of $\tilde{X}_F$ are relaxation processes, which we seek. Here relaxation process in $T(T_q \Omega \times \mathbb{R})$ is defined as follows.

**Definition 4.1.** (relaxation process in the lifted space 1). Let $\phi : I \ni t \mapsto p(t) \in T(T_q \Omega \times \mathbb{R})$ be a curve, where the coordinates of $p(t)$ are denoted by $(y(t), z(t), \dot{y}(t), \dot{z}(t))$, $(t \in I \subset \mathbb{R})$. For each $x$ kept fixed in $t$, if a curve satisfies the conditions

$$\lim_{t \to \infty} y_a(t) = \frac{\partial F}{\partial x^a}, \quad \text{and} \quad \lim_{t \to \infty} z(t) = F(x), \quad a = 1, \ldots, n,$$  

then the image of the curve is said to be a relaxation process towards a point of the set of equilibrium states $(f, \xi)$.
Simple calculations yield
\[ L_{X_F}dz = (\text{div}_{d_z} X_F)dz. \]

To study properties of \( \overline{X}_F \), similar to the case of \( X_F \), one defines the function \( \text{div}_\mu \overline{X}_F \in \Gamma \Lambda^0(T_q \Omega \times \mathbb{R}) \) such that
\[ L_{\overline{X}_F} \mu = (\text{div}_\mu \overline{X}_F) \mu, \quad \text{where} \quad \mu = dy_1 \wedge \cdots \wedge dy_n \wedge dz. \]

The following Lemma is about the system of the ODEs associated with \( X_F \) and that with \( \overline{X}_F \). When \( (y_*, z_*) = (\partial F/\partial x^a, F(x)) \) is a fixed point for the system of the ODEs associated with \( X_F \) and conditions for Theorem 3.9 are satisfied, this fixed point is an attractor for the dynamical system.

**Lemma 4.2.** Consider a dynamical system for \( (y, z) \) associated with \( \overline{X}_F \),
\[
\frac{dz}{dt} = w_F(z; x), \quad \frac{dy_a}{dt} = u_{Fa}(y, z; x), \quad a = 1, \ldots, n,
\]
where \( u_{Fa}(-, -; x) \) is a function of \( z \) and \( y \) for a given \( x \). Choose
\[
u = u_{Fa}(y, z; x) = y_a \frac{\partial w_F}{\partial z} + \frac{\partial w_F}{\partial x^a}, \quad a = 1, \ldots, n.
\]
Let \( z_* \) be a point that satisfies \( w_F(z_*; x) = 0 \), i.e., the point \( z_* \) is a fixed point for the ODE \( \dot{z} = w_F \) in (33). Then the following hold.

1. The contracting property is preserving under the lift in the sense that
\[
(\text{div}_{d_z} X_F)(z; x) < 0 \quad \text{on} \quad Z \quad \Rightarrow \quad (\text{div}_\mu \overline{X}_F)(z, y; x) < 0, \quad \text{on} \quad T_q \Omega \times Z
\]
with some \( Z \subset \mathbb{R} \).

2. In a subset \( Z \subset \mathbb{R} \) containing \( z_* \), if \( w_F(z; x) \geq 0, \partial w_F/\partial z \leq 0 \), and \( \partial w_F/\partial z|_{z_*} = 0 \), then
\[
\lim_{t \to \infty} (y, z) = (y_*, z_*).
\]

3. If
\[
\frac{\partial w_F}{\partial z}(z_*; x) = -1,
\]
then \( y_* \) of a fixed point \( (y_*, z_*; x) \) is given by
\[
y_{*a} = \frac{\partial w_F}{\partial x^a}(z_*; x), \quad a = 1, \ldots, n.
\]

**Proof.** (Proof for 1.) The explicit form of \( \text{div}_\mu \overline{X}_F \) is obtained as
\[
(\text{div}_\mu \overline{X}_F)(z, y; x) = (n + 1) \frac{\partial w_F}{\partial z}(z; x).
\]
From this and the condition \( (\text{div}_{d_z} X_F)(z; x) = \partial w_F/\partial z < 0 \), item 1 holds.
(Proof for 2.) This follows from the Theorem of Lyapunov [39].
(Proof for 3.) Substituting (36) into (34), one has
\[
u = \left. u_{Fa}(z_*, y_*; x) = y_{*a} \frac{\partial w_F}{\partial z} \bigg|_* + \frac{\partial w_F}{\partial x^a} \bigg|_* \right. - y_{*a} + \left. \frac{\partial w_F}{\partial x^a} \bigg|_* \right., \quad a = 1, \ldots, n.
\]
To find an explicit form of a fixed point for \( \dot{y}_a = u_{Fa} \), letting \( \dot{y}_{*a} = u_{Fa}(z_*, y_*; x) = 0 \) for each \( x \), one has (37).
In Lemma 4.2 note the following.

- In Example 3.8, the case $w_F(z; x) = F(x) - z$ has been considered. In this example it follows that $(\text{div}d_z L_z)(z; x) = 0$. Hence the condition for item 1 and that for 3 are satisfied. In addition, the subset $Z$ for item 2 is found as $Z = \{z | F(x) - z \geq 0\}$. These calculations are summarized as Table 1.

| $z$ | $F(x)$ |
|-----|--------|
| $w_F(z; x)$ | + | 0 | - |
| $(\text{div}d_z L_z)(z; x)$ | -1 | -1 | -1 |

- The system consisting of (33), (34), and $\dot{x}^a = 0$ ($a = 1, \ldots, n$), is formally the same as (29), where (29) is the contact Hamiltonian system with the contact Hamiltonian $h(x, z) = F(x) - z$. Hence, it is expected that there is a relation between the lifted relaxation generating systems and contact Hamiltonian systems. In particular, the function $w_F(z; x)$ is expected to play a role of $h(x, z)$. This role will be discussed in Section 4.3.

The following Proposition shows how to describe a relaxation process in terms of $\tilde{X}_F$.

**Proposition 4.3.** Consider the dynamical system (33) with (34). In the case where $w_F(z; x) = F(x) - z$, with an initial condition $F(x) > z(0)$, integral curves of $\tilde{X}_F$ connect points on $T_q \Omega \times \mathbb{R}$ and the fixed point $(\partial F/\partial x^a, F(x))$.

**Proof.** This follows from item 2 of Lemma 4.2.

Lemmas 4.2 and 4.3 are associated with Theorem 3.9 that is the case of a unique set of equilibrium states. Meanwhile the following are associated with Theorem 3.14.

### 4.2.2 Two sets of equilibrium states

In Theorem 3.14, the variables $y$ and $z$ evolve in time. Meanwhile the corresponding vector field $X_{II \rightarrow I}$ and the dynamical system are written only in terms of $z$ for a fixed $q_0$ and $x$. Hence it is natural to consider a dynamical system involving $y$ and $z$, and its corresponding vector field denoted by $\tilde{X}_{II \rightarrow I}$. This vector field is studied here. Since there are various classes of ODEs, one focuses on systems where

1. a fixed point $(y_x, z_x)$ is $(\partial F_1/\partial x^a, F_1(x))$, ($a = 1, \ldots, n$), with $x$ being the coordinate system for $\Omega_0$,

2. the corresponding vector field is a lift of $X_{II \rightarrow I}$, in the sense that $\pi_* \tilde{X}_{II \rightarrow I} = X_{II \rightarrow I}$:

$$
\begin{align*}
\begin{array}{c}
\pi_* \tilde{X}_{II \rightarrow I} = X_{II \rightarrow I} \\
\pi \arrow{<} T_{q_0} \Omega_0 \times \mathbb{R} \\
\pi \arrow{<} T \Omega \times \mathbb{R}
\end{array}
\end{align*}
$$

Note that the manifold $T \Omega \times \mathbb{R}$ has been discussed in 18 as a redundant manifold for describing equilibrium states.
This $\tilde{X}_{\Gamma \rightarrow \Omega}$ is written of the form:

$$\tilde{X}_{\Gamma \rightarrow \Omega} = \sum_{a=1}^{n} u_{\Gamma \rightarrow \Omega}(z, y; x) \frac{\partial}{\partial y_a} + \omega_{\Gamma \rightarrow \Omega}(z; x) \frac{\partial}{\partial z}, \quad (38)$$

with some functions $u_{\Gamma \rightarrow \Omega} = (u_{\Gamma \rightarrow \Omega,1}, \ldots, u_{\Gamma \rightarrow \Omega,n})$. The reason why item 1 is needed is as follows. If $(\partial F_i/\partial x^a, F_i)$ is a fixed point and attractive in some sense, then integral curves of $\tilde{X}_{\Gamma \rightarrow \Omega}$ are relaxation processes, which we seek. Here relaxation process in $T(Q_0, \Omega_0 \times \mathbb{R})$ is defined as follows.

**Definition 4.4.** (Relaxation process in the lifted space 2). Let $\phi : \mathbb{R} \ni t \rightarrow p(t) \in T(Q_0, \Omega_0 \times \mathbb{R})$ be a curve, where the coordinates of $p(t)$ are denoted by $(y(t), z(t), \dot{y}(t), \dot{z}(t))$, $(t \in \mathbb{R})$. For each $x$ kept fixed in $t$, if a curve satisfies the conditions

1. $$\lim_{t \to -\infty} y_a(t) = \frac{\partial F_{\Gamma}}{\partial x^a}; \quad \text{and} \quad \lim_{t \to -\infty} z(t) = F_{\Gamma}(x), \quad a = 1, \ldots, n, \quad (39)$$

2. $$\lim_{t \to \infty} y_a(t) = \frac{\partial F_{I}}{\partial x^a}; \quad \text{and} \quad \lim_{t \to \infty} z(t) = F_{I}(x), \quad a = 1, \ldots, n, \quad (40)$$

then the image of the curve is said to be a relaxation process towards a point of the set of equilibrium states $(f, \xi)$.

To compare $X_{\Gamma \rightarrow \Omega}$ with $\tilde{X}_{\Gamma \rightarrow \Omega}$, a property of $X_{\Gamma \rightarrow \Omega}$ is studied first. In particular, a stability of the dynamical system of the ODE (25) associated with $X_{\Gamma \rightarrow \Omega}$ on $\mathbb{R}$, $\dot{z} = w_{\Gamma \rightarrow \Omega}(z; x)$, is discussed here. Similar to the case of $\text{div}_dz X_F \in \Gamma \Lambda^0 \mathbb{R}$, let $\text{div}_dz X_{\Gamma \rightarrow \Omega} \in \Gamma \Lambda^0 \mathbb{R}$ be the phase space compressibility with respect to $dz$ on $\mathbb{R}$, that is,

$$\mathcal{L}_{X_{\Gamma \rightarrow \Omega}} dz = (\text{div}_dz X_{\Gamma \rightarrow \Omega})dz.$$ 

Simple calculations yield

$$(\text{div}_dz X_{\Gamma \rightarrow \Omega})(z, y; x) = \frac{\partial w_{\Gamma \rightarrow \Omega}}{\partial z}(z; x).$$

If $\partial w_{\Gamma \rightarrow \Omega}/\partial z < 0$ in some $Z \subset \mathbb{R}$, then the system is contracting in $Z$.

To study properties of $\tilde{X}_{\Gamma \rightarrow \Omega}$, similar to the case of $X_{\Gamma \rightarrow \Omega}$, one defines the function $\text{div}_{\mu} \tilde{X}_{\Gamma \rightarrow \Omega} \in \Gamma \Lambda^0 (T_q \Omega_0 \times \mathbb{R})$ such that

$$\mathcal{L}_{\tilde{X}_{\Gamma \rightarrow \Omega}} \mu = (\text{div}_{\mu} \tilde{X}_{\Gamma \rightarrow \Omega})\mu, \quad \text{where} \quad \mu = dy_1 \wedge \cdots \wedge dy_n \wedge dz.$$

The following Lemma is about the system of the ODEs associated with $X_{\Gamma \rightarrow \Omega}$ and that with $\tilde{X}_{\Gamma \rightarrow \Omega}$.

**Lemma 4.5.** Consider a dynamical system for $(y, z)$ associated with $\tilde{X}_{\Gamma \rightarrow \Omega}$,

$$\frac{dz}{dt} = w_{\Gamma \rightarrow \Omega}(z; x), \quad \frac{dy_a}{dt} = u_{\Gamma \rightarrow \Omega, a}(z, y; x), \quad a = 1, \ldots, n, \quad (41)$$

where $u_{\Gamma \rightarrow \Omega, a}(-, -; x)$ is a function of $z$ and $y$ for a given $x$. Choose

$$u_{\Gamma \rightarrow \Omega, a}(z, y; x) = y_a \frac{\partial w_{\Gamma \rightarrow \Omega}}{\partial z} + \frac{\partial w_{\Gamma \rightarrow \Omega}}{\partial x^a}, \quad a = 1, \ldots, n. \quad (42)$$

Then the contracting property is preserving under the lift in the sense that

$$(\text{div}_dz X_{\Gamma \rightarrow \Omega})(z; x) < 0 \quad \text{on} \quad Z \quad \Rightarrow \quad (\text{div}_{\mu} \tilde{X}_{\Gamma \rightarrow \Omega})(z, y; x) < 0, \quad \text{on} \quad T_q \Omega_0 \times Z, \quad (43)$$

with some $Z \subset \mathbb{R}$.
Proof. A way to prove this is analogous to the proof of item 1 of Lemma 4.2.

In Lemma 4.5 note the following.

- In Example 3.13 the case \( w_{\Pi \rightarrow 1}(z; x) = -(z - F_1(x))(z - F_{II}(x))^2 \) has been considered. In this example it follows that \( \left< \text{div}_{d_z} X_{\Pi \rightarrow 1}(z; x) \right> < 0 \) in some \( \mathcal{Z} \subset \mathbb{R} \). Hence there is some non-empty \( \mathcal{Z} \) such that the condition for Lemma 4.3 is satisfied. More precisely, one has

\[
(\text{div}_{d_z} X_{\Pi \rightarrow 1})(z; x) = \begin{cases} 
\text{positive} & z_0(x) < z < F_{II}(x) \\
0 & z = z_0(x), \quad z = F_{II}(x) \\
\text{negative} & z_0(x) := \frac{2F_1(x) + F_{II}(x)}{3}.
\end{cases}
\]

Note that \( F_1(x) < z_0 < F_{II}(x) \) due to \( F_1(x) - z_0(x) = (F_1(x) - F_{II}(x))/3 < 0 \) and \( F_{II}(x) - z_0(x) = (2/3)(F_{II}(x) - F_1(x)) > 0 \). These calculations are summarized in Tab. 2. From this summary, it follows that \( (\text{div}_{d_z} X_{\Pi \rightarrow 1})(z; x) < 0 \) and \( (\text{div}_{w} \hat{X}_{\Pi \rightarrow 1})(z; y; x) < 0 \) around \( z = F_1(x) \).

| \( \frac{z}{-F_{II}(x)}(z_0) \) | \( F_{II}(x) \) |
|-----------------|---------|
| \( w_{\Pi \rightarrow 1}(z; x) \) | + | 0 | - | - | - | 0 | - |
| \( (\text{div}_{d_z} X_{\Pi \rightarrow 1})(z; x) \) | - | - | - | 0 | + | 0 | - |

Moreover, in this Example the point \((y_{*, a}, z_*)\) in the system 41 with 42,

\[
(y_{*, a}(x), z_*(x)) = \left( \frac{\partial F_1}{\partial x^a}(x), F_1(x) \right), \quad a = 1, \ldots, n,
\]

is verified to be a fixed point.

- The system consisting of 41, 42, and \( \dot{x}^a = 0 \) (\( a = 1, \ldots, n \)), is formally the same as 30, where 30 is the contact Hamiltonian system with the contact Hamiltonian \( h(x, z) = -(z - F_1(x))(z - F_{II}(x))^2 \). Hence, it is expected that there is some relation between the lifted relaxation generating systems and contact Hamiltonian systems. In particular, the function \( w_F(z; x) \) is expected to play a role of \( h(x, z) \). This role will be discussed in Section 4.3.

The following Proposition shows how to describe a relaxation process in terms of \( \hat{X}_{\Pi \rightarrow 1} \).

**Proposition 4.6.** Consider the dynamical system 41 with 42. In the case of Example 3.13

\[
w_{\Pi \rightarrow 1}(z; x) = -(z - F_1(x))(z - F_{II}(x))^2,
\]

with an initial condition \( z(0) < F_{II}(x) \), integral curves of \( \hat{X}_{\Pi \rightarrow 1} \) connect points on \( T_{q_0} \Omega_0 \times \mathbb{R} \) and the fixed point \( (\partial F_1/\partial x^a, F_1(x)) \).

**Proof.** The strategy for proving this Lemma is to find Lyapunov functions, ans this proof is similar to the proof of Theorem 3.1 in Ref. 32.

Let \( V_I(-; x) \) and \( V_{II}(-; x) \) be the functions for a given \( x \),

\[
V_I(z; x) = \frac{1}{2}(z - F_1(x))^2, \quad \text{on} \quad \mathcal{Z}_I(x) = \{ z | z < F_{II}(x) \}, \quad \text{and} \\
V_{II}(z; x) = z - F_{II}(x), \quad \text{on} \quad \mathcal{Z}_{II}(x) = \{ z | F_{II}(x) \leq z \}.
\]

It follows that \( V_I(z; x) \geq 0 \) on \( \mathcal{Z}_I(x) \), and that

\[
\frac{dV_I}{dt} = \dot{z}(z - F_1(x)) = -(z - F_1(x))^2(z - F_{II}(x))^2 \leq 0, \quad \text{on} \quad \mathcal{Z}_I(x).
\]
The equality holds when \( z = F_1(x) \). Hence \( V_1(\cdot; x) \) is a Lyapunov function on \( Z_1(x) \). Next, it follows that \( V_1(z; x) \geq 0 \) on \( Z_1(x) \), and that
\[
\frac{dV_1}{dt} = \dot{z} = -(z - F_1(z - F_1(x)))^2 \leq 0, \quad \text{on} \ Z_1(x).
\]
The equality holds when \( z = F_1(x) \). Hence \( V_1(\cdot; x) \) is a Lyapunov function on \( Z_1(x) \).

Applying the Theorem of Lyapunov, one completes the proof.

4.3 Comparisons

In the following, the present affine geometric thermodynamics is compared with a contact geometric thermodynamics for equilibrium and nonequilibrium systems. Note that contact geometric thermodynamics is an developing branch of mathematical physics, and these identifications may differ among theories.

4.3.1 Equilibrium

Table 3 shows identifications of notions used in equilibrium thermodynamics in the languages of contact geometry and affine geometric thermodynamics.

| Thermodynamics         | Contact geometric theory | Affine geometric theory |
|------------------------|--------------------------|-------------------------|
| Primal variables \( x \) | Coordinates of \( Q \)   | Coordinates of \( \Omega \) |
| Conjugate variables \( y \) | Coordinates of \( T_q^*Q \) | Some of coordinates of \( T_{f(q)}\mathbb{R}^{n+1} \) |
| Complete function \( F \) | A function on \( Q \)     | \( F \) for a graph immersion |
| Fundamental relation   | Contact form              | Conormal map             |
| State equation         | Legendrian submfd         | Graph immersion          |
| Response function      | Hessian of \( F \)        | Affine fundamental form \( h \) |
| Graph \( (x, F) \)     | Image of Legendre map     | Image of graph immersion |
| Graph \( (x, y) \)     | Image of Lagrange map     | (Unnamed projection)     |

4.3.2 Nonequilibrium

Notice that the system (28) is formally the same as (33) with (34), and is also formally the same as (41) with (42). In addition, since \( x \) is a coordinate of a point \( q \) or \( q_0 \), one recognizes that the equations for \( y \) and \( z \) in (28) are coordinate expressions for the restricted vector field \( Y_{h(q)} \in \Gamma(T^*_qQ \times \mathbb{R}) \) at \( q \in Q \).

One then immediately arrives at the following.

Theorem 4.7. (relation between vector fields in affine and contact geometries). A class of contact Hamiltonian vector fields \( Y_{h|q} \in \Gamma(T^*_qQ \times \mathbb{R}) \) is formally the same as the lifted relaxation generating vector field \( \widetilde{X}_F \in \Gamma(T_q\Omega \times \mathbb{R}) \) in (31) with (33) and (34), and is formally the same as \( \widetilde{X}_{II_{\omega}} \in \Gamma(T^*_qQ \times \mathbb{R}) \) in (38) with (41) and (42). How to identify \( Y_{h|q} \) with \( \widetilde{X}_F \) and \( \widetilde{X}_{II_{\omega}} \) is to put \( Q = \Omega \) or \( Q = \Omega_0 \) and identify \( T^*Q \cong T\Omega \) or \( T^*Q \cong T\Omega_0 \).

From Theorem 4.7, it follows that the present affine geometric formalism is consistent with a contact geometric formalism [29, 32, 33]. In addition, this Theorem indicates how a contact vector field is constructed from a given relaxation generating vector field on \( \mathbb{R} \). Conversely, a relaxation generating vector field on \( \mathbb{R} \) is easily obtained from a contact Hamiltonian vector field on a contact manifold by defining an appropriate projection.
5 Concluding remarks

This paper offers an affine geometric formulation of thermodynamic systems. This formulation covers both equilibrium and simple nonequilibrium systems. The main claims of this paper are as follows:

- A set of equilibrium states is identified with the image of a graph immersion into $\mathbb{R}^{n+1}$ (see Interpretation 3.1).
- Several affine geometric objects can be introduced in thermodynamics (see Proposition 3.3).
- A class of relaxation processes in thermodynamic systems are described (see Theorems 3.9 and 3.14).
- The present geometric formulation of relaxation processes are consistent with the existing theory (see Theorem 4.7).

The significance of this study includes, as indicated the diagram below,

- shedding light on a potential link between affine geometry and thermodynamics, so that several methodologies in affine geometry will be introduced to the study of thermodynamics.
- showing how to compare vector fields on an extended affine immersion with those on a contact manifold, which provides relations between the study of affine geometry and that of contact geometry.

There remain unsolved problems that have not been addressed in this paper. They include

- showing applications of the present approach to various thermodynamic systems and electric circuits [51, 58],
- showing applications of the affine immersion theory of codimension two to thermodynamic systems [17],
- exploring relations between affine geometry and statistical mechanics [37, 52, 56],
- exploring relations among affine geometry, symplectic and Liouville geometries, since symplectic and Liouville geometries are employed to describe thermodynamics [7, 53].

By addressing these, it is expected that a relevant and sophisticated geometric methodology will be established for dealing with various thermodynamic systems and related systems. In addition, related mathematics are expected to be developed.

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Conflict of interest
The author has no conflicts to disclose.

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Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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