Magnetically Stimulated Diffusion of Rydberg Gases

Yuri V. Dumin

Moscow State University, GAISh, Universitetski pr. 13, 119992, Moscow, Russia and
Space Research Institute (IKI) of Russian Academy of Sciences,
Profsoyuznaya str. 84/32, 117997, Moscow, Russia

(Dated: November 26, 2012)

The specific kind of diffusion stimulated (rather than suppressed) by the external magnetic field, which was predicted for the first time by Schmelcher and Cederbaum in 1992, is considered here for the case of high-angular-momentum (i.e., approximately “circular”) Rydberg atoms. The coefficient of such diffusion was calculated by a purely analytical approach and was found to be well relevant to the experiments on antihydrogen formation.

PACS numbers: 05.40.Fb, 05.40.Jc, 51.20.+d, 32.80.Ee

A well-known feature of the diffusion processes in various gaseous systems (ranging from the laboratory devices for plasma confinement to the large-scale astrophysical objects) is that it is strongly suppressed by imposition of the external magnetic fields. The aim of the present article is to discuss the case when Rydberg atoms, along with their other extraordinary properties (e.g., review [1]), may exhibit also a quite unexpected kind of diffusion, which is stimulated rather than suppressed by the magnetic field.

In fact, the possibility of such phenomenon was mentioned for the first time by Schmelcher and Cederbaum as early as 1992 [2,3]. These authors, using a classical approximation for the description of hydrogen-like atom in a magnetic field, studied dynamics of a “cyclic” canonical coordinate (i.e., the one not entering explicitly into the Hamiltonian), such as the center of mass of the atom. As a result, they revealed a possibility of chaos development in this coordinate and found that, in some circumstances, such chaotic motion follows the classical diffusion law.

Later, a profound mathematical treatment of the same problem was given in paper [4]. Its authors used the method of separation between the “slow” and “fast” degrees of freedom of a dynamical system. Then, the fast chaotic motions were treated as a stochastic force, resulting in the diffusion-like behavior of the slow degrees of freedom.

Experimental observation of such diffusion requires either an extremely strong magnetic field or very weak interatomic electric field (i.e., a low binding energy). Therefore, it was suggested to observe this phenomenon either in compact astrophysical objects with ultrastrong magnetic fields (e.g., pulsars) or in the experiments with highly-excited (Rydberg) atoms. Unfortunately, none of these opportunities was realized by now because, on the one hand, the diagnostic possibilities in astrophysical studies are too limited and, on the other hand, the standard atomic-beam experiments with Rydberg atoms are not well suited for studying the diffusion processes.

Fortunately, the situation changed in the recent decade: magneto-optical traps (MOT) became a new source of the Rydberg atoms, which are formed due to recombination of laser-produced ultracold plasmas (e.g., reviews [5,6]); and such devices are much better suited for tracing the long-term diffusional effects. Besides, quite similar apparatus is used now to create and study the antihydrogen by recombining antiprotons and positrons [7,8]. Such antihydrogen atoms are also born in the highly-excited states; and strong magnetic fields are applied to prevent antiparticles from the escape and annihilation.

However, it should be kept in mind that the above-cited theoretical studies were concentrated mostly on the case of atoms with zero or very low angular momenta. This was quite natural for the old atomic-beam experiments with Rydberg atoms produced by laser irradiation: such atoms can have a huge principal quantum number, but their orbital number is always limited to a few units, because any absorbed photon can change it only by unity. On the other hand, the Rydberg atoms formed by the recombination in MOT experiments, in general, should have large angular momenta (so that the respective orbital quantum numbers are on the order of the principal number). It is not clear in advance if such atoms will behave similarly in the external magnetic fields. So, a special treatment of the high-orbital-momentum case should be performed. Besides, to estimate importance of the magnetically-stimulated diffusion in various experimental setups, it would be desirable to have an analytical expression for the diffusion coefficient, with explicit dependences on all the relevant parameters. It is just the aim of the present work to provide such formula [9].

Before proceeding to the detailed quantitative treatment, let us try to explain qualitatively why the magnetic field can stimulate rather than suppress a diffusion. In the case of high-angular-momentum (i.e., approximately “circular”) Rydberg atoms, this can be presented pictorially in Fig. 1. Indeed, if the magnetic field is absent, the electron orbit will be exactly closed, and average force experienced by the central ion after each revolution of the electron will equal exactly zero. On the other hand, when an external magnetic field is imposed on the sys-
tem, then the curvature radius of the electron orbit will either increase or decrease under the action of the additional Lorentz force. As a result, the orbit will no longer be closed; and a position of the electron after each revolution will be slightly shifted with respect to the initial point, as illustrated by the shaded circle in Fig. 1. Therefore, the average Coulomb force acting on the ion after such revolution becomes nonzero; and this ion will experience a kick in a quasi-random direction. A series of such kicks is equivalent to a random force, leading to the diffusion-like Brownian motion of the ion and, consequently, of the entire atom.

Quantitative description of the two-body system, coupled by Coulomb forces and imbedded in the external magnetic field, can be evidently given by the set of equations:

\[
\begin{align*}
  m_e \frac{d^2}{dt^2} \mathbf{r}_e &= -e^2 \frac{\mathbf{r}_e - \mathbf{r}_i}{|\mathbf{r}_e - \mathbf{r}_i|^3} - \frac{e}{c} \frac{d}{dt} \mathbf{v}_e \times \mathbf{B}, \\
  m_i \frac{d^2}{dt^2} \mathbf{r}_i &= e^2 \frac{\mathbf{r}_e - \mathbf{r}_i}{|\mathbf{r}_e - \mathbf{r}_i|^3} + \frac{e}{c} \frac{d}{dt} \mathbf{v}_i \times \mathbf{B},
\end{align*}
\]

(1)

where \(m_e\) and \(m_i\) are the electron and ion masses, \(\mathbf{r}_e\) and \(\mathbf{r}_i\) are their radius vectors, \(e\) is the absolute value of the electron charge, \(c\) is the speed of light, and \(\mathbf{B}\) is the magnetic field vector.

Next, let us introduce the standard definitions for the center of mass of the system

\[
\mathbf{R} = \frac{m_e \mathbf{r}_e + m_i \mathbf{r}_i}{m_e + m_i},
\]

(2)

the relative position of an electron with respect to ion

\[
\mathbf{r} = \mathbf{r}_e - \mathbf{r}_i,
\]

(3)

and the following mass parameters: the total mass of the system

\[
M = m_i + m_e \approx m_i,
\]

(4)

the reduced mass

\[
\mu = \frac{m_i m_e}{m_i + m_e} \approx m_e,
\]

(5)

and the relative mass difference

\[
\nu = \frac{m_i - m_e}{m_i + m_e} \approx 1.
\]

(6)

As a result, the initial set of equations (1) will take the form:

\[
\begin{align*}
  \frac{d^2}{dt^2} \mathbf{R} &= -\frac{e}{M c} \frac{d}{dt} \mathbf{v}_e \times \mathbf{B}, \\
  \frac{d^2}{dt^2} \mathbf{r} &= \frac{e^2}{\mu} \frac{\mathbf{r}}{r^3} - \frac{e}{\mu c} \frac{d}{dt} (\mathbf{R} + \nu \mathbf{r}) \times \mathbf{B}.
\end{align*}
\]

(7)

Next, it is convenient to normalize all spatial quantities to the initial radius \(a\) of the unperturbed orbit:

\[
\mathbf{r} = a \hat{\mathbf{r}}, \quad \mathbf{R} = a \hat{\mathbf{R}};
\]

(8)

and time, to the Keplerian period of revolution:

\[
t = \tau \hat{t}, \quad \text{where} \quad \tau = 2\pi \sqrt{\mu a^3/e^2}.
\]

(9)

Then, equations (7a), (7b) can be rewritten in terms of the dimensionless variables (marked by tildes) as

\[
\begin{align*}
  \frac{d^2}{d\tau^2} \hat{\mathbf{R}} &= -2\pi \Omega_M \frac{d}{d\tau} \hat{\mathbf{v}} \times \hat{\mathbf{b}}, \\
  \frac{d^2}{d\tau^2} \hat{\mathbf{r}} &= -(2\pi)^2 \frac{\hat{\mathbf{r}}}{\tau^3} - 2\pi \Omega_e \frac{d}{d\tau} (\mathbf{R} + \nu \mathbf{r}) \times \mathbf{b},
\end{align*}
\]

(10)

where

\[
\Omega_M = \frac{eB}{Mc} \approx \Omega_i = \frac{eB}{m_i c}
\]

(11)

is the gyrofrequency of the center of mass,

\[
\Omega_e = \frac{eB}{\mu c} \approx \Omega_e = \frac{eB}{m_e c}
\]

(12)

is the gyrofrequency of the relative motion,

\[
\omega_e = 2\pi/\tau = \sqrt{e^2/\mu a^3} \approx \sqrt{e^2/m_e a^3}
\]

(13)

is the Keplerian frequency, and \(\mathbf{b} = \mathbf{B}/B\) is the unit vector in the direction of the magnetic field.

Since \(m_e \ll m_i\), it can be reasonably assumed that the center-of-mass motion is much slower than the relative motion of the electron and ion [10].

\[
|d\mathbf{R}/d\tau| \ll |d\mathbf{r}/d\tau|.
\]

(14)

Consequently, the term \(d\mathbf{R}/d\tau\) in the right-hand side of Eq. (10b) can be neglected, and this equation becomes completely independent of Eq. (10a).

\[
\frac{d^2}{d\tau^2} \hat{\mathbf{r}} = -(2\pi)^2 \frac{\hat{\mathbf{r}}}{\tau^3} - 2\pi \Omega_e \frac{d}{d\tau} \hat{\mathbf{r}} \times \mathbf{b}.
\]

(15)
Therefore, we can solve this equation alone and then substitute the obtained solution to Eq. (10a).

Rewriting Eq. (15) in the polar coordinate system \((r, \varphi)\) leads to the following set of equations:

\[
\begin{align*}
\ddot{r} - \dot{r}^2 + (2\pi)^2 \frac{\Omega_e}{\omega_e} \dot{r} \dot{\varphi} + (2\pi)^2 \frac{1}{\tilde{t}^2} & = 0, \\
\ddot{\varphi} + 2\dot{\varphi} \dot{\varphi} - (2\pi)^2 \frac{\Omega_e}{\omega_e} \dot{r} & = 0,
\end{align*}
\]

where \(\dot{\cdot}\) denotes a derivative with respect to the dimensional time \(\tilde{t}\).

Let us seek for the solution of equations (16) as perturbation of the purely circular motion:

\[
\begin{align*}
\tilde{r}(\tilde{t}) & = \tilde{r}_0 + \delta \tilde{r}(\tilde{t}), \\
\varphi(\tilde{t}) & = 2\pi \tilde{t} + \varphi_0 + \delta \varphi(\tilde{t}),
\end{align*}
\]

where \(\tilde{r}_0 = \text{const}, \varphi_0 = \text{const}\).

Then, it can be easily shown that, due to the influence of a magnetic field, the mean orbital radius changes from 1 (in dimensionless units) to

\[
\tilde{r}_0 = \left[1 - \frac{\Omega_e}{\omega_e}\right]^{-1/3},
\]

while perturbations \(\delta \tilde{r}(\tilde{t})\) and \(\delta \varphi(\tilde{t})\) experience small harmonic oscillations with the frequency

\[
\omega = (2\pi) \left[ \left(1 + \frac{1}{\tilde{r}_0^2}\right)^2 - \frac{3}{\tilde{r}_0^3} \right]^{1/2} \approx (2\pi) \left[ 1 - \frac{1}{2} \frac{\Omega_e}{\omega_e} \right], \quad \text{at} \; \Omega_e / \omega_e \ll 1.
\]

Not going into details of the perturbed relative motion, let us mention only one property, which will be very important further: As follows from Eq. (19), the electron after each revolution will be shifted by some distance approximately in the same direction in the course of

\[
n \approx \frac{\pi}{2(2\pi - \omega)} \approx \frac{1}{2} \frac{\omega_e}{\Omega_e},
\]

revolutions; and then the direction of these shifts will gradually change, as shown by the dotted arrow within the shaded circle in Fig. 1. Therefore, it can be estimated as difference between the average radius of the electron orbit in presence of the magnetic field, as given by Eq. (15), and the radius without the field (which is equal just to unity). As a result, we get

\[
\Delta \tilde{r} \approx n \approx \frac{1}{2} \frac{\Omega_e}{\omega_e}.
\]

On the other hand, the above-mentioned radial increment \(\Delta \tilde{r}\) (involving \(n\) revolutions of the electron) is just the typical size of the shaded circle in Fig 1. Therefore, it can be estimated as difference between the average radius of the electron orbit in presence of the magnetic field, as given by Eq. (15), and the radius without the field (which is equal just to unity). As a result, we get

\[
\Delta \tilde{r} \approx \tilde{r}_0 - 1 \approx \frac{1}{3} \frac{\Omega_e}{\omega_e}.
\]

At last, the total number of random-walk steps, appearing in Eq. (21), can be evidently obtained as ratio of the total time interval to the duration of one step:

\[
N_{\text{tot}} = \tilde{t} / n \approx \frac{2\Omega_e}{\omega_e} \tilde{t}.
\]

Finally, substituting formulas (22)–(25) into (21), we arrive at

\[
\sqrt{\langle \tilde{R}^2(\tilde{t}) \rangle} \approx \Delta \tilde{R} \sqrt{N_{\text{tot}}},
\]

which is just the diffusion law, where the coefficient of diffusion (in dimensionless units) is

\[
\tilde{D} \approx \left(\frac{\Omega_e}{\omega_e}\right)^2 \left(\frac{\Omega_e}{\omega_e}\right)^{1/2} \tilde{t}^{1/2},
\]

Here, \(\Delta \tilde{R}\) is the typical length of one uncorrelated step, comprising \(n\) correlated shifts given by Eq. (20); and \(N_{\text{tot}}\) is the total number of uncorrelated steps during a given time.

Next, estimating the differential equation (10a) at the scale of one uncorrelated step, we can easily get the following relation:

\[
\Delta \tilde{R} \approx (2\pi) \frac{\Omega_e}{\omega_e} \Delta \tilde{r} \Delta \tilde{t},
\]

where \(\Delta \tilde{r}\) and \(\Delta \tilde{t}\) are the radial and temporal increments corresponding to such an uncorrelated step.

The last-mentioned time interval (in dimensionless units) is just the typical number of revolutions in which the correlations still survive, which is given by Eq. (20); so that

\[
\Delta \tilde{t} \approx n \approx \frac{1}{2} \frac{\Omega_e}{\omega_e}.
\]
This seems to be the first analytical formula derived for the coefficient of magnetically-stimulated diffusion of the high-angular-momentum (“circular”) Rydberg atoms.

Now, let us present some numerical estimates of the effect under consideration. Rewriting expression (25) in terms of the “elementary” physical quantities and using relation $a = a_0 n_q^2$, we get

$$D \approx \frac{\epsilon a_0^5 B^3}{m_i^2 c^3} n_q^{10}, \quad (29)$$

where $a_0$ is Bohr radius, and $n_q$ is the principal quantum number of the Rydberg atom.

Taking $B = 3\, T = 3 \times 10^4\, G$ (which can be achieved in the installations for antihydrogen production), and identifying $m_i$ with the mass of (anti-)proton, we arrive at

$$D \approx 7 \times 10^{-22} n_q^{10} \, \text{cm}^2/\text{s}. \quad (30)$$

Since antihydrogen atoms are typically formed by the three-body recombination in the states with $n_q \approx 100$ or somewhat greater, Eq. (30) gives $D \sim 1 \, \text{cm}^2/\text{s}^{-1}$. Such value of the diffusion coefficient should be very important for the experiments, because they involve trapping of the antihydrogen atoms in a chamber of typical size about a centimeter during the time intervals about a second. A significant practical conclusion following from our consideration is that it might be unreasonable to increase the magnetic field strength above some critical value: otherwise, the magnetically-stimulated diffusion of the created atoms will override trapping the charged particles by the magnetic field.

As regards ordinary ultracold plasmas, the diffusion with coefficient $D \sim 1 \, \text{cm}^2/\text{s}^{-1}$ will be less relevant to the current experiments, because they are usually performed at the much less time scales ($\lesssim 10^{-4} \, \text{s}$). Besides, the diffusion coefficient should be further suppressed for heavy ions by the term $m_i^2$ in the denominator of formula (29). Let us emphasize also that, since the ordinary ultracold gas density is usually not so small ($\sim 10^9 \, \text{cm}^{-3}$), the characteristic interparticle separation will not be much greater than the typical size of the Rydberg atoms. As a result, the approximation of non-interacting atoms may no longer be sufficiently adequate. (However, this approximation should work very well for antihydrogen plasmas, whose density is extremely low, about $1 \, \text{cm}^{-3}$.)

At last, let us mention that, as distinct from the case of “linear” atoms [2, 3], our formula (28) does not show any sharp (threshold-like) onset of the diffusion as function of the magnetic field. This is not surprising because a distinctive feature of the linear atoms is singularity of the electron trajectories near the Coulomb center. Such trajectories are specifically perturbed by the external magnetic field, resulting in the formation of kink-like peculiarities (cf. Fig. 2 in Ref. [2]), the intermittent domains of various dynamics, etc. On the other hand, the electron trajectories of “circular” Rydberg atoms are always far away from the Coulomb center and, therefore, everything changes smoothly.

In summary, we presented a pictorial treatment and derived an explicit analytical formula for the coefficient of magnetically-stimulated diffusion of the high-angular-momentum (“circular”) Rydberg atoms (in contract to the previous studies, which were based on the numerical methods and dealt with the “linear” atoms). As follows from the resulting formulas, the case of diffusion considered in our work should be especially relevant to the Rydberg atoms formed in the experiments on the production and trapping of antihydrogen.

A considerable part of the present work was carried out during my visit to the Max Planck Institute for the Physics of Complex Systems (Dresden, Germany). I am grateful to Prof. H. Kantz for valuable discussions and comments.

\[\text{dumin@yahoo.com}\]

1 On leave from: Theoretical Department, IZMIRAN, Russian Academy of Sciences, Troitsk, Moscow reg., 142190 Russia

1 [1] T. Gallagher, \textit{Rydberg Atoms} (Cambridge Univ. Press, Cambridge, UK, 1994).

2 [2] P. Schmelcher and L. Cederbaum, Phys. Lett. \textbf{A164}, 305 (1992).

3 [3] P. Schmelcher and L. Cederbaum, Z. Phys. D \textbf{24}, 311 (1992).

4 [4] N. Baba, W. Just, H. Kantz, and A. Riegert, Phys. Rev. E \textbf{73}, 066228 (2006).

5 [5] P. Gould and E. Eyler, Phys. World \textbf{14(3)}, 19 (2001).

6 [6] S. Bergeson and T. Killian, Phys. World \textbf{16(2)}, 37 (2003).

7 [7] E. Reich, Nature \textbf{468}, 355 (2010).

8 [8] G. Andresen, M. Ashkezari, M. Baquero-Ruiz, W. Bertsche, P. Bowe, E. Butler, C. Cesar, S. Chapman, M. Charlton, A. Deller, et al., Nature \textbf{468}, 673 (2010).

9 [9] Let us mention also one more subtle point of the earlier theoretical studies: it is not clear if the classical equations of motion have a real physical sense for the Rydberg atoms with very low angular momenta because, generally speaking, all quantum numbers should be large for the classical mechanics to work well. Fortunately, the case of high-angular-momentum Rydberg atoms considered in our paper satisfies this requirement.

10 [10] This is actually the same assumption as in the time-scale separation method [2].

11 [11] To avoid unnecessary details, Fig. 1 represents, in fact, the case $n \sim 1$. 