Sorted Range Selection and Range Minima Queries

Waseem Akram¹ and Sanjeev Saxena¹

Dept. of Computer Science and Engineering, Indian Institute of Technology, Kanpur, INDIA-208 016
{akram,ssax}@iitk.ac.in

Abstract. Given an array \( A[1 : n] \) of \( n \) elements drawn from an ordered set, the sorted range selection problem is to build a data structure that can be used to answer the following type of queries efficiently: Given a pair of indices \( i, j \) (\( 1 \leq i \leq j \leq n \)), and a positive integer \( k \), report the \( k \) smallest elements from the sub-array \( A[i : j] \) in order. Brodal et al. [Brodal, G. S., Fagerberg, R., Greve, M., and López-Ortiz, A., Online sorted range reporting. Algorithms and Computation (2009) pp. 173–182] introduced the problem and gave an optimal solution. After \( O(n \log n) \) time for preprocessing, the query time is \( O(k) \). The space used is \( O(n) \).

In this paper, we propose the only other possible optimal trade-off for the problem. We present a linear space solution to the problem that takes \( O(k \log k) \) time to answer a range selection query. The preprocessing time is \( O(n) \). Moreover, the proposed algorithm reports the output elements one by one in non-decreasing order. Our solution is simple and practical.

We also describe an extremely simple method for range minima queries (most of whose parts are known) which takes almost (but not exactly) linear time. We believe that this method may be, in practice, faster and easier to implement in most cases.

Keywords: Range Minimum Query · Range Reporting · Algorithms · Data Structures

1 Introduction

The range minimum query (RMQ) problem is a well-studied problem [1,3,4,6,11]. Given an array, the RMQ problem is to find the position (i.e., index) of the smallest element (in an index range). In this paper, we study the sorted range selection problem [2], a generalization of the range minimum query problem. Given an input array \( A[1 : n] \) of \( n \) elements drawn from an ordered set, the problem is to preprocess the array \( A[1 : n] \) so that the queries of the following type can be answered efficiently [2]:

Given a pair of indices \( i, j \) with \( 1 \leq i \leq j \leq n \) and a positive integer \( k \), report the \( k \) smallest elements in the index range \([i, j]\) in sorted order.
Akram and Saxena

Brodal, Fagerberg, Greve, and López-Ortiz \[2\] introduced the problem and gave a linear space data structure with \(O(k)\) query time. The preprocessing time to build the structure is \(O(n \log n)\). By reporting all \(n\) elements \((k = n)\), one can sort the elements in \(O(n)\) time; thus, \(\Omega(n \log n)\) preprocessing time is required, and their algorithm is optimal. The solution uses a fairly complicated result due to Frederickson and Johnson \[10\].

We propose the only other possible optimal trade-off for the sorted range selection problem. The preprocessing time is \(O(n)\) with linear space, and the query time is \(O(k \log k)\). Our solution is also optimal as by reporting all elements in the array \((k = n)\), we can sort the elements of the input array in \(O(n \log n)\) time.

Note that our solution offers the only other possible optimal trade-off for the problem. If the preprocessing time is \(p(n)\) and the query algorithm takes \(q(n)\) time to report any element in the query range in the worst case, then we can sort \(n\) elements in \(p(n) + \sum_{i=1}^{n} q(n)\) time. Thus, if \(p(n) = o(n \log n)\), then \(q(n) = \Omega(\log n)\). Our algorithm is extremely simple and only uses range minimum queries in addition to the usual binary heap. Moreover, the algorithm reports the output elements one by one in non-decreasing order.

A related and more general problem that has been studied in the past is the range selection problem \[2,7\], where the output elements are not required to be reported in sorted order. Brodal et al. \[2\] suggested that an array can be preprocessed using linear space and time to answer a range selection query in \(O(k)\) time in the RAM model. Brodal et al. \[2\] also suggested a method to solve the problem in the pointer machine model by using the priority search tree \[12\] and Frederickson’s \(O(k)\)-time algorithm \[9\] for finding the \(k\) smallest elements in a binary heap. The resultant structure takes linear space and can report \(k\) smallest elements in \(O(\log n + k)\) time. Skala \[14\] presented a survey of array range query problems.

2 Proposed Solution

We use the following notations. Let \(A[1 : n]\) be the input array of \(n\) elements drawn from a totally ordered set. For any \(i, j \in \{1, 2, \ldots, n\}\) with \(i \leq j\), \(A[i : j]\) denotes the sub-array starting at index \(i\) and ending at index \(j\), and \(A[i]\) denotes the array element at index \(i\). For any two parameters \(i\) and \(j\), \(1 \leq i \leq j \leq n\), the range minimum query, denoted by \(RMQ(i, j)\), is to find the index of a minimum element among \(A[i], A[i + 1], \ldots, A[j]\).

Our solution is based on the following observation.

Consider any two fixed indices \(i\) and \(j\) with \(i \leq j\). If \(A[r]\) is the smallest element in the sub-array \(A[i : j]\), then the next smallest element in \(A[i : j]\) will either be in the sub-array \(A[i : r - 1]\) or in the sub-array \(A[r + 1 : j]\).

We first preprocess the given array \(A[1 : n]\) for the range minimum queries \((RMQ)\) \[11,13\]. The preprocessing takes \(O(n)\) time and space. For each subsequent range minimum query, \(RMQ(i, j)\) with \(i \leq j\), the RMQ data structure
returns the index \( r \) of the minimum element \( A[r] \) in the subarray \( A[i : j] \) in \( O(1) \) worst-case time.

In Section 3, we describe, for completeness, an extremely simple method for range minima queries which takes \( O(n \log^k n) \) preprocessing time, for any \( \log(n) \), and can answer queries in \( O(1) \) time, as \( \log(5) n \leq 2 \) for \( n \leq 2^{65536} \approx 10^{19728} \), this method takes almost (but not exactly) linear time. We believe that the method will be, in practice, faster and easier to implement in most cases.

Assume we are to report the \( k \) smallest elements from the sub-array \( A[i : j] \). An RMQ will give the index \( r \) of the smallest element in the subarray \( A[i : j] \) in \( O(1) \) time. We report \( A[r] \) as the smallest element in \( A[i : j] \). We then split the interval into two pieces \([i, r - 1]\) and \([r + 1, j]\) and use RMQs to find the minimum element in each instance. We insert these elements into (an initially empty) binary heap (see, e.g., Chapter 6 in [8]). We keep performing the following step until \( k \) elements are reported or the min heap becomes empty:

Remove the minimum element from the heap and report it as the next smallest element in the subarray \( A[i : j] \). Split the interval of the minimum element into two pieces and insert their minimum elements into the heap.

Algorithm 1 is the pseudo-code of the sorted range selection query procedure. We illustrate Algorithm 1 with an example in Subsection 2.1.

**Algorithm 1 Sorted Range Selection**

**Input** \( i, j, k \) with \( 1 \leq i, j \leq n \) and \( 1 \leq k \leq j - i + 1 \)

**Output** \( k \) smallest elements in the subarray \( A[i : j] \)

1: \( Q \leftarrow \emptyset \) \Comment{Initializing min-heap}
2: \( x_l, x_r \leftarrow -1 \) \Comment{index variables}
3: \( r \leftarrow \text{RMQ}(i, j) \)
4: insert the element \( A[r] \) into the heap \( Q \). \Comment{[i, j] is stored as satellite data}
5: repeat
6: delete the minimum element \( A[x] \) from heap \( Q \)
7: report \( A[x] \) as \( t \)th smallest element if the current iteration is \( t \)th iteration
8: let \( [p, q] \) be the satellite information associated with \( A[x] \)
9: if \( p < x \), then \( x_l \leftarrow \text{RMQ}(p, x - 1) \)
10: if \( x < q \), then \( x_r \leftarrow \text{RMQ}(x + 1, q) \)
11: insert \( A[x_l] \) (resp. \( A[x_r] \)) into the heap \( Q \) if \( x_l \neq -1 \) (resp. \( x_r \neq -1 \)).
12: \( x_l, x_r \leftarrow -1 \) \Comment{initializing for next iteration}
13: until \( k \) elements have been reported or the heap \( Q \) gets exhausted

**Remark 1.** As the next smallest element of \( A[p : q] \) will be either in \( A[p : x - 1] \) or \( A[x + 1 : q] \), the next smallest element (to be reported) will always be in a heap.

\( \log(1) n = \log \log(n) \)
Remark 2. Our algorithm reports or outputs the required elements one by one in non-decreasing order.

As in each iteration, we are deleting one element from the heap and inserting at most two more; after \( i \) iterations, we will have at most \( i + 1 \) elements in the heap. Insertion or deletion in a binary heap takes \( O(\log s) \) time, where \( s \) is the size of the heap before the operation. Therefore, each insertion or deletion of an element in \( i \) iteration will take \( O(\log i) \) time, or the total time will be \( O(\sum_{i=1}^{k} \log i) = O(k \log k) \). We have the following theorem.

**Theorem 1.** An array \( A[1:n] \) of \( n \) elements drawn from a totally ordered set can be preprocessed so that given a pair of indices \( i, j \) with \( 1 \leq i \leq j \leq n \) and a parameter \( k \), we can report the \( k \) smallest elements in the subarray \( A[i:j] \) in \( O(k \log k) \) time. The preprocessing takes \( O(n) \) space and time.

Remark 3. If elements in the input array \( A[1:n] \) are integers from a set \([1..U]\), then we can use van Emde Boas structure (see Chapter 20 in [8]) for implementing the pool. As a result, the query time will become \( O(k \log \log U) \).

## 3 Range Minimum Query

In range minima query we are given an array \( A[1:n] \) which we can preprocess. We have to answer queries of kind:

\[
\text{RMQ}(i, j): \text{Find the index of the smallest element in } A[i], A[i+1], \ldots, A[j-1], A[j].
\]

Query time should be \( O(1) \).

Assume that after preprocessing, for each position \( i \) in the array we know the (index of) minimum element in each of the following cases

\[
A[i : i + 1], A[i : i + 2], A[i : i + 2^2], \ldots, A[i : i + 2^j]
\]

for \( i + 2^j \leq n \).

Then query \( \text{RMQ}(i, j) \) to find the minimum element in \( A[i : j] \) can be answered as follows

1. Let \( r \) be the largest integer s.t., \( i + 2^r \leq j \), or equivalently, \( 2^r \leq j - i \)

   **Remark 4.** \( r \) is the index of the most significant bit in binary representation of \( j - i \).

2. Using precomputations, we can find the (index of) minimum element in \( A[i : i + 2^r] \).

3. If \( j' = j - 2^r \), then again using precomputations, we can find the (index of) minimum element in \( A[j' : j'] \) or \( A[j' : j] \).

4. The (index of) the required minimum element is (the index of) smaller of these two values, and hence can be found in \( O(1) \) time.
As \( r \) is the largest integer s.t., \( i + 2^r \leq j \), or \( i + 2^{r+1} > j \), or (subtracting \( 2^r \)), \( i + 2^r > j - 2^r = j' \). Thus, the union of intervals \([i : i + 2^r]\) and \([j' : j]\) is \([i : j]\) (portion \([j' : i + 2^r]\) is common to both intervals). Thus, we are computing minimum of elements exactly in the range \( i . . j \) (some elements are however considered twice).

Let us now look at the precomputation. Assume for each \( 1 \leq i \leq n \) (and for some \( j \)) we have computed the (index of) minimum element in \( A[i : \min\{i + 2^j, n\}] \). Then we can compute the (index of) minimum element in \( A[i : \min\{i + 2^j + 1, n\}] \), for each \( 1 \leq i \leq n \) as follows:

1. If \( i + 2^j \geq n \), then \( i + 2^{j+1} \geq n \). Or (index of) minimum element in \( A[i : \min\{i + 2^j, n\}] \) is also the (index of) minimum element in \( A[i : \min\{i + 2^{j+1}, n\}] \).
2. Else \( (i + 2^j < n) \) let \( i' = i + 2^j \). By hypothesis (precomputation) we know the (index of) minimum element in \( A[i' : \min\{i' + 2^j, n\}] \) (or \( A[i + 2^j : \min\{i + 2^{j+1}, n\}] \)).
3. The (index of) minimum element in \( A[i : \min\{i + 2^{j+1}, n\}] \) is the (index of) the smaller of the two numbers:
   minimum element in \( A[i : i + 2^j] \) and the minimum element in \( A[i + 2^j : \min\{i + 2^{j+1}, n\}] \).

Thus, for each \( 1 \leq i \leq n \), we can find \( A[i : \min\{i + 2^{j+1}, n\}] \) in \( O(1) \) time, or for all \( 1 \leq i \leq n \) in \( O(n) \) time. As \( 0 \leq j \leq \log n \), entire precomputation takes \( O(n \log n) \) time.

The complete algorithm is:

**Algorithm 2**

1: for \( i \leftarrow 1 \) to \( n \) do
2: \( B_1[i] \leftarrow i; \)
3: \( \text{if } i + 1 \leq n \text{ and } (A[i + 1] < A[i]) \text{ then} \)
4: \( B_1[i] \leftarrow i + 1; \)
5: for \( k \leftarrow 2 \) to \( \log n \) do
6: \( \text{for } i \leftarrow 1 \) to \( n \) do
7: \( B_k[i] \leftarrow B_{k-1}[i]; \)
8: \( \text{if } i + 2^k < n \text{ then} \)
9: \( r \leftarrow B_{k-1}[i] \text{ and } s \leftarrow B_{k-1}[i + 2^k]; \)
10: \( \text{if } (A[s] < A[r]) \text{ then} \)
11: \( B_k[i] \leftarrow s; \)
Thus we have Lemma 1.

**Lemma 1.** An array $A[1:n]$ can be preprocessed in $O(n \log n)$ time and space such that queries of kind:

$$RMQ(i, j): \text{Find the index of the smallest element in } A[i], A[i+1], \ldots, A[j-1], A[j]$$

can be answered in $O(1)$ time.

### 3.1 Linear space solution

The space can be reduced to $O(n)$ as follows [15].

1. The array $A[1:n]$ is conceptually split into $n/\log n$ blocks of size $\log n$.
2. The minimum of each block of $\log n$ elements is computed, in $O(\log n)$ time.
   As there are $n/\log n$ blocks, total time is $O(n)$ overall. We also compute the prefix minimum (smallest element from start of block) and suffix minimum (smallest element till end of the block). This can also be done in same time bounds.
3. These minima are stored in another array of length $n/\log n$, say $S[1:n/\log n]$.
4. The array $S$ is preprocessed as per Lemma 1.
   As $S$ has $n/\log n$ elements, it will take $O((n/\log n) \log (n/\log n)) = O(n)$ time and space.

Thus, preprocessing time and space is $O(n)$.

A query $RMQ(l, r)$ when two elements are not in the same block can be answered in $O(\log n)$ time as follows:

1. Find $i = \lfloor l/\log n \rfloor$ and $j = \lfloor r/\log n \rfloor$, the the block(s) containing the two indices.
2. If $i < j$, then find $k = RMQ(i + 1, j - 1)$.
   Basically, the minima of all blocks contained completely inside the range is computed using a query to the data structure built over array $S$ in $O(1)$ time.
3. If $i \neq j$, then as we know the suffix minima at location $l$ in block $i$ and prefix minima at location $r$ in block $j$. Comparing these two elements with the element computed in previous step, we get the overall minimum.

We are left with the case, when both $l$ and $r$ are in the same group, i.e., when $i = j$. This is the usual range minima query, restricted to a block.

If we preprocess each block (independently and separately) for range minima query using algorithm of Lemma 1, the time for each block is $O((\log n) \log(\log n)) = O(n \log \log n)$. As there are $n/\log n$ blocks, total time is $O((n/\log n)(\log n / \log \log n)) = O(n \log \log n)$.

Thus we have:

**Corollary 1.** An array $A[1:n]$ of $n$ elements drawn from a totally ordered set can be preprocessed in $O(n \log \log n)$ time and $O(n)$ space such that range minima queries can be answered in $O(1)$ time.
If we use the method of Cor. 1 for preprocessing each block for range minima queries, the preprocessing time using $k$-level structure, the preprocessing time can be easily reduced to $O(n \log^k n)$. We next describe an almost linear time solution. We have to only consider the case when the two elements are in same block. The complete preprocessing algorithm is:

### 3.2 Almost Linear Time Method

1. The array $A[1 : n]$ is conceptually split into $n / \log n$ blocks of size $\log n$ (as before).
2. The minimum of each block of $\log n$ elements is computed, in $O(\log n)$ time. As there are $n / \log n$ blocks, total time is $O(n)$ overall.
   (a) We also compute the prefix minimum (smallest element from start of block) and suffix minimum (smallest element till end of the block). This can also be done in same time bounds.
   (b) If number of elements in a block is less than 32 (a constant), we preprocess each block for range minima queries using the method.
3. These minima are stored in another array of length $n / \log n$, say $S \left[1 : \frac{n}{\log n}\right]$.
4. The array $S$ is preprocessed as per Lemma 1.
   As $S$ has $n / \log n$ elements, it will take $O((n / \log n) \log (n / \log n)) = O(n)$ time and space.

Thus, preprocessing time and space is $O(n)$.

A query $RMQ(l, r)$ when two elements are not in the same block can be answered in $O(\log n)$ time as follows:

1. Find $i = \lfloor l / \log n \rfloor$ and $j = \lfloor r / \log n \rfloor$, the the block(s) containing the two indices.
2. If $i < j$, then find $k = RMQ(i + 1, j - 1)$.
   Basically, the minima of all blocks contained completely inside the range is computed using a query to the data structure built over array $S$ in $O(1)$ time.
3. If $i \neq j$, then as we know the suffix minima at location $l$ in block $i$ and prefix minima at location $r$ in block $j$. Comparing these two elements with the element computed in previous step, we get the overall minimum.

It is further possible to reduce the preprocessing time to $O(n)$, but for that a popular way of doing uses Cartesian tree, Euler tour traversal, and table look-up.

### 4 Conclusion

In this paper, we studied the range selection problem and gave a linear space solution with $O(k \log k)$ query time and $O(n)$ preprocessing time. The output elements are reported individually in non-decreasing order. The proposed solution
offers the only possible trade-off other than the one given by Brodal et al. [2].
Our solution is simple and easy to implement. The data structure of the solution
consists of an RMQ structure and a usual binary min heap.

To the best of our knowledge, the sorted range selection problem has not
been studied in a dynamic setting. One can consider the problem in the dynamic
setting where an update operation can change the element value stored at an
index without changing the element values stored at any other indices. We leave
this as an open problem.

References

1. Fischer, J., and Heun, V. Space-Efficient Preprocessing Schemes for Range
Minimum Queries on Static Arrays. SIAM Journal on Computing 40, 2 (2011),
465–492.
2. Brodal, G. S., Fagerberg, R., Greve, M., and López-Ortiz, A. Online
sorted range reporting. In Algorithms and Computation: 20th International Sym-
posium, ISAAC (2009), pp. 173–182.
3. Fischer, J., and Mäkinen, V., and Navarro, G. Faster entropy-bounded
compressed suffix trees in Theoretical Computer Science (2009), pp.5354-5364.
4. Amir, A. and Landau, G. M. and Vishkin, U. Efficient pattern matching with
scaling Journal of Algorithms 13(1), 1992, 2–32.
5. Kunihiko, S. Succinct data structures for flexible text retrieval systems In Journal
of Discrete Algorithms 5(1) (2007) 12-22
6. Saxena, S. Dominance Made Simple In Information Processing Letters, 109(9),
2009, 419-421
7. Afshani, P., Brodal, G. S., and Zeh, N. Ordered and unordered top-k range
reporting in large data sets. SODA 2011, 390-400.
8. Cormen, T. H., Leiserson, C. E., Rivest, R. L., and Stein, C. Introduction
to Algorithms, Third Edition, 3rd ed. The MIT Press, 2009.
9. Frederickson, G. An optimal algorithm for selection in a min-heap. Information
and Computation 104, 2 (1993), 197–214.
10. Frederickson, G. N., and Johnson, D. B. The complexity of selection and
ranking in x + y and matrices with sorted columns. Journal of Computer and
System Sciences 24, 2 (1982), 197–208.
11. Gabow, H. N. and Bentley, J. L. and Tarjan, R. E. Scaling and Related
Techniques for Geometry Problems ACM STOC ’84 135-143
12. McCreight, E. M. Priority search trees. SIAM Journal on Computing 14, 2
(1985), 257–276.
13. Schieber, B., and Vishkin, U. On finding lowest common ancestors: Simplifi-
cation and parallelization. SIAM Journal on Computing 17, 6 (1988),
1253–1262.
14. Skala, M. Array Range Queries In Space-Efficient Data Structures, Streams, and
Algorithms, LNCS, vol 8066. Springer (2013), 333–350.
15. Range Minimum Query. https://en.wikipedia.org/wiki/Range_minimum_query
16. Fischer, J., and Heun, V. Theoretical and Practical Improvements on the
RMQ-Problem, with Applications to LCA and LCE In Combinatorial Pattern
Matching, Springer, 2006, 36–48.
17. Bender, M. A., and Farach-Colton, M. The LCA Problem Revisited InLATIN 2000: Theoretical Informatics, 2000, 88-94.
18. Bender, M. A., Farach-Colton, M., Pemmasani, G., Skiena, S., and Sumazin, P. Lowest common ancestors in trees and directed acyclic graphs *Journal of Algorithms*, 57(2), 2005, 75-94.

19. Schieber, B., and Vishkin, U. On finding lowest common ancestors: Simplification and parallelization *SIAM Journal on Computing*, 17(6), 1988, 1253-1262.

20. Berkman, O., Schieber, B., and Vishkin, U. Optimal doubly logarithmic parallel algorithms based on finding all nearest smaller values *Journal of Algorithms*, 14(3), 1993, 344-370.

21. Berkman, O., and Vishkin, U. Recursive star-tree parallel data structure *SIAM Journal on Computing*, 22(2), 1993, 221-242.

22. Berkman, O., and Matias, Y. Fast parallel algorithms for minimum and related problems with small integer inputs *Proceedings of 9th International Parallel Processing Symposium*, 1995, 203-207.

23. Berkman, O., Matias, Y., and Ragde, P. Triply-logarithmic parallel upper and lower bounds for minimum and range minima over small domains *Journal of Algorithms*, 28(2), 1998, 197-215.