MISO hierarchical inference engine with fuzzy implication satisfying $I(A(x, y), z) = I(x, I(y, z))$

Dechao Li* Qiannan Guo
School of Information and Engineering,
Zhejiang Ocean University, Zhoushan, 316000, China
Key Laboratory of Oceanographic Big Data Mining and Application of
Zhejiang Province, Zhoushan, 316022, China

Abstract

Fuzzy inference engine, as one of the most important components of fuzzy systems, can obtain some meaningful outputs from fuzzy sets on input space and fuzzy rule base using fuzzy logic inference methods. In order to enhance the computational efficiency of fuzzy inference engine in multi-input-single-output (MISO) fuzzy systems, this paper aims mainly to investigate three MISO fuzzy hierarchical inference engines based on fuzzy implications satisfying the law of importation with aggregation functions (LIA). We firstly find some aggregation functions for well-known fuzzy implications such that they satisfy (LIA). For a given aggregation function, the fuzzy implication which satisfies (LIA) with this aggregation function is then characterized. Finally, we construct three fuzzy hierarchical inference engines in MISO fuzzy systems applying aforementioned theoretical developments.

Key words: Fuzzy implication; Fuzzy inference engine; Aggregation function; Law of importation

1 Introduction

1.1 Motivation

As fuzzy systems can transform human knowledge into a nonlinear mapping, they have been successfully utilized in control, expert system, signal processing, decision making and so on. A fuzzy system mainly consists of fuzzyifier, fuzzy rule base, fuzzy inference engine and defuzzifier [39]. Where the rule base which constitutes a set of fuzzy IF-THEN rules is the heart of a fuzzy system. Usually, the fuzzy IF-THEN rules in a MISO (SISO) fuzzy system have the following form

(SISO) IF $x$ is $D_j$ THEN $y$ is $B_j (j = 1, 2, \cdots n)$,

(MISO) IF $x_1$ is $D_{1j}$ AND $x_2$ is $D_{2j}$ AND $\cdots$ AND $x_m$ is $D_{mj}$ THEN $y$ is $B_j (j = 1, 2, \cdots n)$.

Where $x = (x_1, x_2, \cdots, x_m) \in U = U_1 \times U_2 \times \cdots U_m$ and $y \in V$ are the input and output variables of the fuzzy system, $D_{ij}(D_j)$ and $B_j$ are respectively fuzzy sets on $U_i (i = 1, 2, \cdots m)$

*Email: dch1831@163.com
and $V$. From the fuzzy logical point of view, the fuzzy IF-THEN rules can be regarded as a series of fuzzy relations on $U \times V$. And they are often specified using the fuzzy implications. This brings about more fuzzy implications are studied in order to meet the various needs for fuzzy systems [2, 3, 9, 11, 16, 23, 31].

As another important component of the fuzzy system, the fuzzy inference engine transforms the fuzzy IF-THEN rules and fuzzy sets in $U$ into a fuzzy set on $V$ by some fuzzy logical principles [39]. Especially, the generalized modus ponens (GMP) are often utilized in case where the rule base consists of unique IF-THEN rule. The GMP introduced by Zadeh, as an extension of modus ponens (MP) in the classical logic, can be indicated straightforwardly as follows [42]:

Premise 1: IF $x$ is $D$ THEN $y$ is $B$
Premise 2: $x$ is $D'$

Conclusion: $y$ is $B'$,

where $D$ and $D'$, $B$ and $B'$ are fuzzy sets on $U$ and $V$, respectively.

In order to calculate $B'$, the compositional rule of inference (CRI) method is presented by Zadeh in 1973 [12]. After, the general CRI methods are discussed by many researchers. Unlike CRI method, Pedrycz proposed another inference method based on the Bandler-Kohout subproduct (BKS) composition denoted as $B' = D' \circ_{\text{BKS}} R$ [31]. In Pedrycz’s method, translated Premise 1 into a fuzzy relation $R$ using a fuzzy implication, the conclusion of GMP problem is computed as

$$B'_{\text{BKS}}(y) = \bigwedge_{x \in U} I(D'(x), (I(D(x), B(y))),$$

where $I$ is a fuzzy implication.

Notice that there are still some deficiencies in CRI method [5, 29, 38, 40]. To compensate these deficiencies, the similarity-based approximate reasoning (SBR) method and triple implication principle (TIP) are proposed [29, 32, 36, 38, 40]. Moreover, in order to judge the availability of these inference methods for the GMP problem, some commonly acknowledged axioms (also inferred as GMP rules) are provided by Magrez and Smets [19].

Similarly, some standards should be required in order to assess the goodness of fuzzy inference engine. Combined fuzzy inference engine with a great variety of practical applications, computational efficiency is all crucial to fuzzy inference engine. For this purpose, Jayaram represented a hierarchical CRI fuzzy inference engine [13]. Stepnicka and Jayaram suggested another hierarchical inferencing scheme based on Bandler-Kohout subproduct [37]. It is not difficult to see that the law of importation plays an important role in these hierarchical infer-
ence engines. This inspires people to investigate the fuzzy implications which satisfy the law of importation with t-norms and uninorms, respectively [4,20,25,27].

It is well known that the fuzzy negation, disjunction, conjunction, conditional and biconditional constitute the fuzzy logical connectives [2,14]. The t-norms (t-conorms) are usually employed to interpret the conjunction (disjunction) [2,14,39]. However, as de Soto et al. pointed out, a fuzzy mathematical model should be not symmetric always [7]. Indeed, in decision making and classification problems, the associativity or commutativity of conjunction and disjunction is not necessarily required [6,9]. Aggregation functions, as a better substitute for t-norms (t-conorms) (they indeed are some spacial cases of aggregation functions, See Definition 2.6) have been applied extensively in fuzzy logic, decision making and classification problems [6,8,9,12,17,20,21,30,33,34]. Thus, our motivation is to investigate the law of importation with aggregation functions to correspond with the actual needs. And then to develop three hierarchical inference engines based on the fuzzy implications satisfying the law of importation. Therefore, replacing the t-norms, the law of importation is firstly extended as follows: 

Definition 1.1 [34] Let $A$ be an aggregation function and $I$ a fuzzy implication. $I$ is said to satisfy the law of importation with an aggregation function $A$ (LIA) if for all $x, y, z \in [0,1],$

$$I(A(x, y), z) = I(x, I(y, z)).$$

(LIA)

1.2 Contribution of this paper

As the argument above, the aggregation functions and fuzzy implications satisfying the law of importation play a pivotal role in computational efficiency of a fuzzy inference engine. Moreover, the variety options of aggregation functions and fuzzy implications results in the flexibility of fuzzy inference engines. We therefore mainly develop three hierarchical inference engines utilized the fuzzy implications satisfying the law of importation with aggregation functions in this paper. We first investigate some properties of aggregation functions and fuzzy implications which satisfy the law of importation. And then we seek the aggregation functions for the well-known fuzzy implications such that they satisfy (LIA). Applied such aggregation functions and fuzzy implications, three hierarchical inference engines in MISO fuzzy system are developed. In a word, the contributions of this paper include:

(1) To study the properties of aggregation functions and fuzzy implications which satisfy (LIA).

(2) To seek the aggregation functions for the well-known fuzzy implications such that they satisfy (LIA).

(3) To characterize the fuzzy implications which satisfy (LIA) with a given aggregation function.
To construct three fuzzy hierarchical inference engines based on fuzzy implications satisfying (LIA).

This paper is organized as follows. Section 2 recalls some basic concepts utilized in this paper. In Section 3, we study the properties of aggregation functions and fuzzy implications when they satisfy (LIA). Section 4 shows necessary and sufficient conditions for $(A, N)$-implication and $R$-implications which satisfy (LIA) with some aggregation functions. In Section 5, some aggregation functions are constructed such that $f$-implication, $g$-implication, QL-implication, probabilistic implication, probabilistic $S$-implication and $T$-power implication satisfy (LIA) with them, respectively. Section 6 characterizes the fuzzy implication which satisfies (LIA) with a given aggregation function. In Section 7, three MISO hierarchical inference engines based on fuzzy implications satisfying (LIA) with aggregation functions are developed.

2 Preliminaries

This section will recall the definitions of fuzzy negation, aggregation function and fuzzy implication and their properties utilized in the remainder of this paper.

2.1 Fuzzy negation, aggregation function and fuzzy implication

Definition 2.1. A fuzzy negation $N$ is a mapping on $[0,1]$ satisfies

(N1) $N(0) = 1$, $N(1) = 0$,
(N2) $N(x) \geq N(y)$ if $x \leq y$, $\forall x, y \in [0,1]$.

A strict negation $N$ fulfills

(N3) $N$ is continuous,
(N4) $N(x) > N(y)$ if $x < y$.

A fuzzy negation $N$ is strong if

(N5) $N(N(x)) = x$, $\forall x \in [0,1]$.

Moreover, a fuzzy negation $N$ is said to be vanishing (non-vanishing) if $N(x) = 0$ for some $x \neq 1$ ($N(x) = 0 \iff x = 1$), and filling (non-filling) if $N(x) = 1$ for some $x \neq 0$ ($N(x) = 1 \iff x = 0$).

Examples 2.2.

- The standard fuzzy negation $N_c(x) = 1 - x$.
- The smallest and the greatest fuzzy negations
  
  $N_{\bot}(x) = \begin{cases} 1 & x = 0 \\ 0 & \text{otherwise} \end{cases}$ and $N_{\top}(x) = \begin{cases} 0 & x = 1 \\ 1 & \text{otherwise} \end{cases}$.
- The natural negation of a fuzzy implication $I$ (see Definition 2.11) is defined by $N_I(x) = I(x, 0)$. 

Lemma 2.3 [2] Let the fuzzy negation $N$ be continuous. The mapping $	ilde{N}$ defined by

$$
\tilde{N}(x) = \begin{cases} 
N^{(-1)}(x), & x \in (0,1] \\
1 & x = 0
\end{cases}
$$

is a strict fuzzy negation, where $N^{(-1)}$ is the pseudo-inverse of $N$ given by $N^{(-1)}(x) = \sup \{ y \in [0,1] | N(y) > x \}$ for all $x \in [0,1]$. Moreover,

i. $\tilde{N}^{(-1)} = N$;

ii. $N \circ \tilde{N} = id$;

iii. $\tilde{N} \circ N|_{\text{Ran}(\tilde{N})} = id|_{\text{Ran}(\tilde{N})}$, where $\text{Ran}(\tilde{N})$ denotes the range of $\tilde{N}$.

Definition 2.4 [10] An aggregation function is a mapping $A : [0,1]^2 \to [0,1]$ which meets

(A1) Boundary conditions: $A(0,0) = 0$ and $A(1,1) = 1$,

(A2) Non-decreasing in two variables, respectively.

Suppose that $f$ is a binary function on $[0,1]$ and $\varphi$ an automorphism on $[0,1]$ (that is, an increasing bijection on $[0,1]$). Defining the function $f_{\varphi}(x, y) = \varphi^{-1}(f(\varphi(x), \varphi(y)))$, it is called as the $\varphi$-conjugate of $f$. Obviously, $\varphi$-conjugate of $A$, denoted by $A_{\varphi}$, is again an aggregation function. Especially, $A_N$ is known as the $N$-dual of $A$ chosen $\varphi$ as a strict negation $N$.

Definition 2.5 [10] Let $A_1$ and $A_2$ be two aggregation functions. We say $A_1 \leq A_2$ if $A_1(x, y) \leq A_2(x, y)$ holds for any $x, y \in [0,1]$.

Definition 2.6 [10] $e \in [0,1]$ is a left (right) neutral element of the binary aggregation function $A$ if $A(e, x) = x$ ($A(x, e) = x$) for any $x \in [0,1]$. Further, $e \in [0,1]$ is a neutral element of $A$ if $A(e, x) = A(x, e) = x$.

Definition 2.7 [34] Let $A$ be an aggregation function.

i. $A$ is a conjunctor if $A(1,0) = A(0,1) = 0$,

ii. $A$ is a disjunctor if $A(1,0) = A(0,1) = 1$,

iii. $A$ has zero divisors if there exist $x, y \in (0,1]$ such that $A(x, y) = 0$,

iv. $A$ has one divisors if there exist $x, y \in [0,1)$ such that $A(x, y) = 1$.

Definition 2.8 [33] Let $A$ be a binary aggregation function and $N$ a fuzzy negation. We say that $A$ satisfies the law of excluded middle principle (LEM) with respect to $N$ if $A(N(x), x) = 1$ holds for any $x \in [0,1]$. Obviously, $A$ is a disjunctor if it satisfies (LEM).

Definition 2.9 [10] Let $A$ denote an aggregation function. We say that $A$ is

i. associative if $A(x, A(y, z)) = A(A(x, y), z)$ for any $x, y, z \in [0,1]$,

ii. commutative if $A(x, y) = A(y, x)$ for any $x, y \in [0,1]$,

iii. a semi-copula if 1 is its neutral element,

iv. a t-norm if it is an associative and commutative semi-copula,

v. a t-conorm if it is $N$-dual of a t-norm,

vi. a uninorm if it is associative, commutative and $e \in (0,1)$ is its neutral element,
vii. a copula if it is a semi-copula which $A(x_1, y_1) - A(x_1, y_2) - A(x_2, y_1) + A(x_2, y_2) \geq 0$ holds for all $x_1 \leq x_2$ and $y_1 \leq y_2$.

**Example 2.10**[10,35] The following are some distinguished conjunctors:

- The smallest conjunctor, $C\perp(x, y) = \begin{cases} 1 & x = y = 1 \\ 0 & \text{otherwise} \end{cases}$
- The greatest averaging conjunctor, $(C_{avg})\top(x, y) = \begin{cases} 0 & x \lor y = 0 \\ x \lor y & \text{otherwise} \end{cases}$
- Representable aggregation functions, $A(x, y) = g^{-1}((g(x \land y) - g(N(x \lor y)) \lor 0))$, where $g : [0,1] \rightarrow [0, +\infty]$ is continuous strictly increasing with $g(0) = 0$ and $N$ is a strong negation;
- Weighted quasi-arithmetic mean (WQAM), $M_{\lambda,f}(x, y) = f^{-1}((1-\lambda)f(x) + \lambda f(y))$, where $f : [0,1] \rightarrow [-\infty, +\infty]$ is continuous and strictly monotone with $f(0) = \pm\infty$ and $\lambda \in (0,1)$;
- TS-functions, $TS_{\lambda,f}(x, y) = f^{-1}((1-\lambda)f(T(x, y)) + \lambda f(S(x, y)))$, where $T$ is a t-norm, $S$ is a t-conorm, $\lambda \in (0,1)$ and $f : [0,1] \rightarrow [-\infty, +\infty]$ is continuous and strictly monotone with $f(0) = \pm\infty$.

**Definition 2.11**[2] A fuzzy implication $I$ is a mapping $I : [0,1]^2 \rightarrow [0,1]$ satisfying

1. Non-increasing in the first variable, i.e., $I(x, z) \geq I(y, z)$ if $x \leq y$,
2. Non-decreasing in the second variable, i.e., $I(x, y) \leq I(x, z)$ if $y \leq z$,
3. $I(0, 0) = 1$,
4. $I(1, 1) = 1$,
5. $I(1, 0) = 0$.

According to Definition 2.11, the following facts for a fuzzy implication can be directly obtained

- **(LB)** Left boundary condition, $I(0, y) = 1, \forall y \in [0,1]$,
- **(RB)** Right boundary condition, $I(x, 1) = 1, \forall x \in [0,1]$.

**Definition 2.12**[2,25] We say that the fuzzy implication $I$ fulfills

1. **(NP)** Left neutrality property, $I(1, y) = y, \forall y \in [0,1]$,
2. **(IP)** Identity principle, $I(x, x) = 1, \forall x \in [0,1]$,
3. **(EP)** Exchange principle, $I(x, I(y, z)) = I(y, I(x, z)), \forall x, y, z \in [0,1]$,
4. **(CP(N))** Law of contraposition with a fuzzy negation $N$, $I(x, y) = I(N(y), N(x)), \forall x, y \in [0,1]$,
5. **(OP)** Ordering property, $I(x, y) = 1 \iff x \leq y, \forall x, y \in [0,1]$,
6. **(OP_{U})** Counterpart of ordering property for uninorms, $I(x, y) \geq e \iff x \leq y, \forall x, y \in [0,1]$ with $e \in (0,1)$. 

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Definition 2.13 [34] An \((A, N)\)-implication \(I_{A,N}\) is a mapping \(I_{A,N} : [0, 1]^2 \to [0, 1]\) defined by
\[
I_{A,N}(x, y) = A(N(x), y),
\]
where \(A\) is a disjunctor and \(N\) a fuzzy negation. Further, \(I_{A,N}\) is called an \(A\)-implication if \(N = \overline{N}\). Moreover, \(I_{S,N}\) is a strong implication or \(S\)-implication if it is generated by a t-conorm \(S\) and a strong negation \(N\).

Theorem 2.14 [34] \(I\) is a fuzzy implication if and only if \(I\) is an \(A\)-implication, i.e. there exists a disjunctor \(A\) such that \(I(x, y) = I_{A,N}(x, y) = A(1 - x, y)\).

Definition 2.15 [30] A function \(I_A : [0, 1]^2 \to [0, 1]\) is called an R-implication if
\[
I_A(x, y) = \sup\{t \in [0, 1] \mid A(x, t) \leq y\}
\]
is a fuzzy implication, where \(A\) is an aggregation function.

Definition 2.16 [34] A function \(I_{A_1,A_2} : [0, 1]^2 \to [0, 1]\) is called a QL-operation given by
\[
I_{A_1,A_2}(x, y) = A_1(N(x), A_2(x, y)),
\]
where \(A_1, A_2\) are two aggregation functions and \(N\) a fuzzy negation. Especially, a QL-operation \(I_{A_1,A_2}\) is called a QL-implication if it satisfies (I1) and (I3-I5).

Definition 2.17 [11] An \(f\)-implication \(I_f\) is a mapping \(I_f : [0, 1]^2 \to [0, 1]\) defined as \(I_f(x, y) = f^{-1}(xf(y))\) with the understanding \(0 \cdot \infty = 0\), where \(f : [0, 1] \to [0, +\infty]\) is a continuous and strict decreasing function with \(f(1) = 0\).

Definition 2.18 [11] Let \(g : [0, 1] \to [0, +\infty]\) be a continuous and strict increasing function with \(g(0) = 0\). A \(g\)-implication \(I_g\) generated by \(g\) is a mapping \(I_g : [0, 1]^2 \to [0, 1]\) defined as
\[
I_g(x, y) = g^{-1}\left(\frac{g(y)}{x}\right)
\]
with the understanding \(0 \cdot \infty = \infty\), where \(g^{-1}\) is pseudoinverse of \(g\) given by \(g^{-1}(x) = \begin{cases} g^{-1}(x) & x \leq g(1) \\ 1 & \text{otherwise} \end{cases}\).

Definition 2.19 [11] A probabilistic implication \(I_C\) generated by a copula \(C\) is defined by
\[
I_C(x, y) = \begin{cases} \frac{C(x, y)}{x} & x > 0 \\ 1 & \text{otherwise} \end{cases}
\]
if it satisfies (II).

Definition 2.20 [11] A probabilistic S-implication \(\tilde{I}_C\) is defined as \(\tilde{I}_C(x, y) = C(x, y) - x + 1\), where \(C\) be a copula.

Definition 2.21 [23] Let \(T\) be a t-norm. A \(T\)-power implication is a function \(I^T : [0, 1]^2 \to [0, 1]\) given by \(I^T(x, y) = \vee\{r \in [0, 1] \mid x^r y \geq x\}\) for all \(x, y \in [0, 1]\).

Lemma 2.22 [23] Let \(T\) be a continuous t-norm and \(I^T\) its power implication.
\begin{enumerate}
\item If \(T = T_M\) is the minimum t-norm, then \(I^{T_M}(x, y) = \begin{cases} 1 & x \leq y \\ 0 & x > y \end{cases}\);
\item If \(T\) is an Archimedean t-norm with additive generator \(t\), then \(I^T(x, y) = \begin{cases} \frac{1}{t(x)} & x \leq y \\ \frac{t(y)}{t(x)} & x > y \end{cases}\).
\end{enumerate}
2.2 Similarity based reasoning and triple implication method

Let \( F(U) \) be the set of fuzzy sets on \( U \). To solve the GMP problem, the algorithm for similarity based reasoning presented by Raha et al. as follows [36]:

Step 1. Combine premise 1 and calculate \( R(D, B) \) by some appropriate translating rules (such as a t-norm).

Step 2. Calculate \( S(D', D) \) combining \( D' \) and \( D \) using a similarity measure.

Step 3. Modify \( R(D, B) \) with \( S(D', D) \) in order to get \( R(D, B|D') \) utilized some schemes.

Step 4. Obtain \( B' \) as

\[
B'(y) = \bigvee_{x \in U} R(D, B|D')(x, y).
\]

To obtain \( R(D, B|D') \), they also proposed the following three axioms:

(AX1) \( R(D, B|D')(x, y) = R(D, B)(x, y) \) if \( S(D', D) = 1 \);

(AX2) \( R(D, B|D')(x, y) = 1 \) if \( S(D', D) = 0 \);

(AX3) \( R(D, B|D') \supseteq R(D, B) \) holds for any \( D' \in F(U) \).

Then \( R(D, B) \) is consider in the following ways:

Case 1. \( R(D, B)(x, y) = T(D(x), B(y)) \), where \( T \) is a t-norm.

Case 2. \( R(D, B)(x, y) = I(D(x), B(y)) \), where \( I \) is a fuzzy implication.

Finally, the conclusions \( B'_{SBR} \) and \( B''_{SBR} \) are obtained as

\[
B'_{SBR}(y) = \bigvee_{x \in U} I(S(D', D), T(D(x), B(y))),
\]

\[
B''_{SBR}(y) = \bigvee_{x \in U} I(S(D', D), I(D(x), B(y))).
\]

The following triple implication principle (TIP) for the GMP problem is proposed by Wang [40].

**Triple implication principle for GMP** Assume that the maximum of following formula

\[
M(x, y) = I(I(D(x), B(y)), I(D'(x), B'(y)))
\]  

exists for every \( x \in U \) and \( y \in V \), where \( I \) is a fuzzy implication on \([0,1]\). The solution \( B' \) of GMP problem should be the smallest fuzzy set on \( V \) such that Eq.(1) achieves its maximum.

**Lemma 2.23** [32] i. If \( I \) fulfills (I2), then

\[
\max_{x \in U, y \in V} M(x, y) = I(I(D(x), B(y)), I(D'(x), 1))
\]

ii. Moreover, if \( I \) is right-continuous with respect to the second variable, then the TIP solution of GMP problem is unique.
Theorem 2.24 Let $T$ be an R-implication generated by a left-continuous t-norm $T$. Then the TIP solution of GMP problem is given by

$$B'_\text{TIP}(y) = \bigvee_{x \in U} T(D'(x), I_T(D(x), B(y))).$$

3 Satisfaction of (LIA) with fuzzy implications and aggregation functions

This section will study some properties of fuzzy implications and aggregation functions when they satisfy (LIA).

Lemma 3.1 Let $I$ satisfy (LIA) with $A$. If $A$ is commutative, then $I$ satisfies (EP).

Proof. Straightforward.

Lemma 3.2 Let $I$ satisfy (LIA) with $A$. If $N_I(y_1) = N_I(y_2)$, then $N_I(A(x, y_1)) = N_I(A(x, y_2))$ holds for arbitrary and fixed $x \in [0, 1]$.

Proof. Let $N_I(y_1) = N_I(y_2)$. We then have $N_I(A(x, y_1)) = I(A(x, y_1), 0) = I(x, I(y_1, 0)) = I(x, N_I(y_1)) = I(x, N_I(y_2)) = I(x, I(y_2, 0)) = I(A(x, y_2), 0) = N_I(A(x, y_2))$.

Lemma 3.3 Let $I$ satisfy (EP) and $N_I$ be injective. If $I$ fulfills (LIA) with $A$, then $A$ is commutative.

Proof. It suffices to take $z = 0$ in (LIA).

Lemma 3.4 Let $I$ be a fuzzy implication such that $I(x, y) = 1$ iff $x = 0$ or $y = 1$. If $I$ satisfies (LIA) with $A$, then $A$ is conjunctor.

Proof. By (LIA), we have $I(A(0, 1), z) = I(0, I(1, z)) = 1$ for any $z \in [0, 1]$. This implies $A(0, 1) = 0$. We can similarly obtain $A(1, 0) = 0$. Thus, $A$ is a conjunctor.

Lemma 3.5 Let $I$ be a fuzzy implication such that $N_I$ is non-filling. If $I$ satisfies (LIA) with $A$, then $A$ is conjunctor.

Proof. By (LIA), we have $N_I(A(0, 1)) = I(A(0, 1), 0) = I(0, I(1, 0)) = 1$. This implies $A(0, 1) = 0$. We can similarly obtain $A(1, 0) = 0$. Thus, $A$ is a conjunctor.

Lemma 3.6 Let the mapping $h(z) = I(1, z)$ be continuous on $[0, 1]$. If $I$ satisfies (LIA) with $A$, then $I$ satisfies (NP).

Proof. For any $y \in [0, 1]$, there exist some $z \in [0, 1]$ such that $y = I(1, z)$ by the continuity of $h$. Therefore, $I(1, y) = I(1, I(1, z)) = I(A(1, 1), z) = I(1, z) = y$.

Definition 3.7 Let $I$ be a fuzzy implication and $A$ an aggregation function. The pair $(I, A)$ is called an adjoint pair if they satisfy the residuation property (RP), i.e.

$$A(x, y) \leq z \iff x \leq I(y, z), \forall x, y, z \in [0, 1].$$

Lemma 3.8 Let $I$ satisfy (OP) and (LIA) with $A$. We have
i. $A$ is conjunctor,

ii. $(I, A)$ is an adjoint pair.

**Proof.** i. By (LIA), we have $I(A(0, 1), 0) = I(0, I(1, 0)) = 1$. Since $I$ fulfills (OP), $A(0, 1) = 0$ holds. We can similarly obtain $A(1, 0) = 0$. Thus, $A$ is a conjunctor.

ii. Since $I$ satisfies (OP), $I(x, I(y, z)) = 1 \iff x \leq I(y, z)$ holds for any $x, y, z \in [0, 1]$. Similarly, $I(A(x, y), z) = 1 \iff A(x, y) \leq z$. By (LIA), we have $A(x, y) \leq z \iff x \leq I(y, z)$.

**Remark 1.** We can similarly obtain that $(I, A)$ forms an adjoint pair if $I$ satisfies (OP$_U$) and (LIA) with $A$.

**Lemma 3.9** Let $A$ be associative and commutative. If $I$ fulfills (RP) with $A$, then they satisfy (LIA).

**Proof.** By (RP), $A(I(x, y), x) = A(x, I(x, y)) \leq y$ holds for any $x, y \in [0, 1]$. We can then assert that $I(x, I(y, z)) \leq I(A(x, y), z)$. Indeed, $A(A(x, y), I(x, I(y, z))) = A(y, A(x, I(x, I(y, z)))) \leq z$. On the other hand, we have $A(A(x, y), I(A(x, y), z)) = A(y, I(y, z)) \leq z$. This means $A(x, I(A(x, y), z)) \leq I(y, z)$. And then $I(A(x, y), z) \leq I(x, I(y, z))$.

**Lemma 3.10** Let $A_\varphi$ and $I_\varphi$ be $\varphi$-conjugate of $A$ and $I$, respectively. If $I$ fulfills (LIA) with $A$, then $I_\varphi$ satisfies (LIA) with $A_\varphi$.

**Proof.** $I_\varphi(A_\varphi(x, y), z) = \varphi^{-1}(I(\varphi(A_\varphi(x, y)), \varphi(z))) = \varphi^{-1}(I(A(\varphi(x), \varphi(y)), \varphi(z))) = \varphi^{-1}(I(\varphi(x), I(\varphi(y), \varphi(z)))) = I_\varphi(x, \varphi^{-1}(I(\varphi(y), \varphi(z)))) = I_\varphi(x, I_\varphi(y, z))$.

### 4 (LIA) with $(A, N)$- and R-implications

In this section, we shall seek some aggregation functions such that $(A, N)$- and R-implications satisfy (LIA) with them. We firstly consider the case when $(A, N)$-implications are generated by the small and greatest fuzzy negations, respectively.

**Lemma 4.1** Let $I_{A,N}$ be an $(A, N)$-implication generated by an associative disjunctior $A$ and the small fuzzy negation $N_\perp$. Then, $I_{A,N}$ satisfies (LIA) with any conjunctor $A'$ without zero divisors.

**Proof.** Let $A'$ be a conjunctor without zero divisors. We consider the following two cases.

i. $A'(x, y) = 0$. This case implies $x = 0$ or $y = 0$. We then obtain $I_{A,N}(A'(x, y), z) = I_{A,N}(0, z) = 1 = I_{A,N}(x, I_{A,N}(y, z))$.

ii. $A'(x, y) \neq 0$. In this case, we have $xy \neq 0$. This implies $I_{A,N}(A'(x, y), z) = A(0, z)$. On the other hand, $I_{A,N}(x, I_{A,N}(y, z)) = A(0, A(0, z)) = A(0, z)$. Therefore, $I_{A,N}(A'(x, y), z) = I_{A,N}(x, I_{A,N}(y, z))$.

**Lemma 4.2** Let $I_{A,N}$ be an $(A, N)$-implication generated by an associative disjunctior $A$ and the greatest fuzzy negation $N_\top$. Then, $I_{A,N}$ satisfies (LIA) with any conjunctor $A'$ without one divisors.
Proof. This proof is similar to that of Lemma 4.1.

However, it is not easy to seek some aggregation functions such that \((A, N)\)-implications obtained from other non-continuous fuzzy negations satisfy (LIA) with them. So, we next focus on the \((A, N)\)-implications generated by continuous fuzzy negations.

**Theorem 4.3** Let \(I_{A,N}\) be an \((A, N)\)-implication generated by an associative disjunctor \(A\) and a continuous fuzzy negation \(N\). Then \(I_{A,N}\) satisfies (LIA) with the aggregation function \(A'\) defined by \(A'(x, y) = \tilde{N}(A(N(x), N(y)))\).

**Proof.** Let \(I_{A,N}(A'(x, y), z) = A(N(A'(x, y)), z) = A(N(\tilde{N}(A(N(x), N(y))), z) = A(A(N(x), N(y)), z) = A(N(x), I_{A,N}(y, z)) = I_{A,N}(x, I_{A,N}(y, z)).\)

It is not difficult to see that other aggregation functions can be found such that the \(I_{A,N}\) satisfies (LIA) with them, because there exists other fuzzy negation \(N\) such that \(N \circ \tilde{N} = \text{id}\) holds. However, the following result shows that \(A_N\) is the only one for \(I_{A,N}\) satisfying (LIA) if \(N_{I_{A,N}}\) is strict.

**Theorem 4.4** Let \(I_{A,N}\) be an \((A, N)\)-implication generated by an associative disjunctor \(A\) and a strict negation \(N\). If \(N_{I_{A,N}}\) is an injective mapping, then \(I_{A,N}\) satisfies (LIA) with \(A'\) if and only if \(A'\) is the \(N\)-dual of \(A\).

**Proof.** \((\Longleftarrow)\) This proof is similar to that of Theorem 4.3.

\((\Longrightarrow)\) Assume that \(I_{A,N}\) satisfies (LIA) with \(A'\), that is, \(I_{A,N}(A'(x, y), z) = I_{A,N}(x, I_{A,N}(y, z))\) holds for any \(x, y, z \in [0, 1]\). Setting \(z = 0\), we have \(N_{I_{A,N}}(A'(x, y)) = A(N(x), A(N(y), 0)) = A(A(N(x), N(y)), 0) = N_{I_{A,N}}(N^{-1}(A(N(x), N(y))))\) for any \(x, y \in [0, 1]\). Since \(N_{I_{A,N}}\) is one-by-one, \(A'(x, y)) = N^{-1}(A(N(x), N(y)))\) holds for any \(x, y \in [0, 1]\). Therefore, \(A' = A_N\).

**Remark 3.** It is easy to see that \(N_{I_{A,N}} = N\) holds if 0 is a right neutral element of \(A\) if \(N\) is strict. In this case, \(I_{A,N}\) satisfies (LIA) with \(A'\) if \(A'\) is the \(N\)-dual of \(A\). Especially, an S-implication satisfies (LIA) with a t-norm \(T\) iff \(T\) is the \(N\)-dual of \(S\).

**Theorem 4.5** Let \(I_{A,N}\) be an \((A, N)\)-implication generated by a strict negation \(N\). Then \(I_{A,N}\) satisfies (LIA) with the \(N\)-dual of \(A\) if and only if \(A\) is associative.

**Proof.** It is sufficient to verify that \(A\) is associative. Since \(I_{A,N}\) satisfies (LIA) with the \(N\)-dual of \(A\), we have \(I_{A,N}(A_N(x, y), z) = A(A(N(x), N(y), z) = A(N(x), A(N(y), z)) = I_{A,N}(x, I_{A,N}(y, z))\). The continuity of \(N\) implies that \(A\) is associative.

In the rest of this section, we study the law of importation for R-implications generated by an associative and commutative aggregation functions.

**Theorem 4.6** Let \(I_A\) be an R-implication generated by an associative, commutative and left-continuous aggregation function \(A\). We have

i. \(I_A\) satisfies (LIA) with \(A\).

ii. If \(I_A\) fulfills (OP), then \(I_A\) satisfies (LIA) with \(A'\) if and only if \(A' = A\).
Proof. i. The proof comes from Lemma 3.1 in [17] and Lemma 3.9.

ii. Let $I_A$ satisfy (OP). Obviously, $I_A(A'(x, y), A'(x, y)) = 1$ holds for any $x, y \in [0, 1]$. This implies $I_A(x, I_A(y, A'(x, y))) = 1$ by (LIA). Again, we obtain $x \leq I_A(y, A'(x, y))$. Thus, $A(x, y) \leq A'(x, y)$.

On the other hand, $I_A(A'(x, y), A(x, y)) = I_A(x, I_A(y, A(x, y))) \geq I_A(x, x) = 1$ because $(I_A, A)$ is an adjoint pair. We then have $A'(x, y) \leq A(x, y)$.

Remark 4. i. Indeed, $A(x, 1) = A(1, x) = x$ holds for any $x \in [0, 1]$ iff $I_A$ satisfies (OP). This means that $I_A$ is an R-implication generated by the t-norm $T$. And then $I_T$ satisfies (LIA) with $A'$ iff $A' = T$. This result can be also found in Ref. [13].

ii. Similarly, we can obtain the fact that $A$ is a uninorm if $I_A$ satisfies (OPU). Thus, $I_A$ is an R-implication generated by the uninorm $U$. And then $I_U$ satisfies (LIA) with $A'$ iff $T' = U$. This result can be also found in Ref. [26].

iii. By Lemma 3.8, $(I_A, A')$ is an adjoint pair if $I_A$ satisfies (OP) (or (OPU)) and (LIA) with $A'$. Theorem 4.6 shows that $A$ is a unique aggregation function such that $I_A$ satisfies (RP) with it in this case.

iv. The R-implications generated by not left-continuous aggregation functions may not satisfy (LIA) with any aggregation function $A'$ or satisfy (LIA) with many aggregation functions as shown in the following examples.

Example 4.7 Consider the Weber implication $I_{WB}$ defined as $I_{WB}(x, y) = \begin{cases} 1 & x < 1 \\ y & x = 1 \end{cases}$.

Similar to the proof of Lemma 4.2, we can verify that $I_{WB}$ satisfies (LIA) with any conjunctor $A'$ without one divisors.

Example 4.8 Let $A(x, y) = \begin{cases} 0 & xy < 0.5 \\ \frac{x + y}{2} & \text{otherwise} \end{cases}$. It is obvious to see that $A$ is a not-left-continuous conjunctor. And then the R-implication $I_A$ generated by $A$ can be obtained as $I_A(x, y) = \begin{cases} 1 & x < 0.5 \\ \max(2y - x, \frac{1}{2}) & \text{otherwise} \end{cases}$.

For any aggregation function $A'$, a simple computation reveals $I_A(A'(1, 1), 0.8) = 0.6$ and $I_A(1, I_A(1, 0.8)) = 0.5$. This means that $I_A$ does not satisfy (LIA) with any aggregation function.

5 Other implications satisfying (LIA) with aggregation function

In this section, we investigate if the QL-, $f$-, $g$-, probabilistic, probabilistic S- and $T$-power implications satisfy (LIA) with some aggregation functions. Let $I_{A_1, A_2}$ be a QL-operation. It is easy to see that $I_{A_1, A_2}$ satisfies (I3) and (I5) when $A_1$ is a disjunctor and $A_2$ is a conjunctor. We therefore only consider the case where $I_{A_1, A_2}$ is obtained from a disjunctor, a conjunctor
and a fuzzy negation in the rest of this section. Further, if $A_2$ has a right neutral element 1, then $I_{A_1,A_2}$ being a QL-implication implies that $A_1$ satisfies (LEM). And then the following statements hold.

**Lemma 5.1** Let $I_{A_1,A_2}$ be a QL-implication generated by an associative disjunctive without one divisor $A_1$, a semi-copula $A_2$ and a fuzzy negation $N$. Then $I_{A_1,A_2}$ satisfies (LIA) with any aggregation function $A$ without zero divisor.

**Proof.** Since $A_1$ has not one divisor, it is not difficult to verify that $A_1$ satisfies (LEM) with respect to $N$ if and only $N = N_\top$. This implies that $I_{A_1,A_2}$ becomes an $(A,N)$-implication generated by an associative disjunctive $A_1$ and the greatest fuzzy negation $N_\top$. By Lemma 4.2, $I_{A_1,A_2}$ satisfies (LIA) with any aggregation function $A$ without zero divisor.

**Theorem 5.2** Let $I_{A_1,A_2}$ be a QL-implication generated by a disjunctive $A_1$ such that the mapping $h(x) = A_1(x,0)$ is continuous on $[0,1]$, a conjunctive $A_2$ and a continuous fuzzy negation $N$. If the aggregation function $A$ is commutative, then $I_{A_1,A_2}$ satisfies (LIA) with $A$ if and only if $A(x,y) = \tilde{N}_{I_{A_1,A_2}}(I_{A_1,A_2}(\tilde{N}_{I_{A_1,A_2}}(x),\tilde{N}_{I_{A_1,A_2}}(y)))$, where 

$$
\begin{cases}
\tilde{N}_{I_{A_1,A_2}}^{-1}(x), & x \in (0,1] \\
1, & x = 0
\end{cases}
$$

**Proof.** Since $A$ is commutative, $I_{A_1,A_2}$ satisfies (EP) according to Lemma 3.1. The continuity of $h(x)$ implies that $I_{A_1,A_2}(x) = A_1(N(x),0)$ is continuous. Therefore, $I_{A_1,A_2}$ can be rewritten as an $(S,N)$-implication generated by a $t$-conorm $S$ and the natural negation $N_{I_{A_1,A_2}}$ according to Theorem 2.4.10 in [2], where $S(x,y) = I_{A_1,A_2}(\tilde{N}_{I_{A_1,A_2}}(x),y)$. By Theorem 4.3, $I_{A_1,A_2}$ satisfies (LIA) with $A$ if and only if $A(x,y) = \tilde{N}_{I_{A_1,A_2}}(S(N_{I_{A_1,A_2}}(x),N_{I_{A_1,A_2}}(y))) = \tilde{N}_{I_{A_1,A_2}}(I_{A_1,A_2}(\tilde{N}_{I_{A_1,A_2}}(N_{I_{A_1,A_2}}(x)),N_{I_{A_1,A_2}}(y))).$ 

**Theorem 5.3** Let $I_f$ be an $f$-implication. $I_f$ satisfies (LIA) with an aggregation function $A$ if and only if $A(x,y) = xy$. 

**Proof.** ($\Rightarrow$) This can be verified directly. 

($\Rightarrow$) By Theorem 2.14, $I_f$ can be rewritten as an $A$-implication $I_{A,N}$ generated by the disjunctive $A(x,y) = f^{-1}((1-x)f(y))$. Suppose that $I_f$ satisfies (LIA) with an aggregation function $A'$. Then, $A$ is associative according to Corollary 4.5. This means that $f(1-(1-x)(1-y)) = (1-x)f(y)$ holds for any $x,y \in [0,1]$. We therefore obtain $A'(x,y) = A_N(x,y) = 1-A(1-x,1-y) = xy$. That is, $I_f$ satisfies (LIA) with an aggregation function $A'$ if and only if $A'(x,y) = xy$. 

**Lemma 5.4** Let $I_g$ be a $g$-implication. If $I_g$ satisfies (LIA) with an aggregation function $A$, then $A$ has not zero divisor.

**Proof.** On the contrary, we assume that there exist $x,y \in (0,1]$ such that $A(x,y) = 0$. Since $I_g$ satisfies (LIA) with $A$, we have $I_g(A(x,y),0) = 1$. However, $I_g(x,I_g(y,0)) = I_g(x,0) = 0$ by Proposition 3.2.7 in [2]. This is a contradiction.
Theorem 5.5 Let $I_g$ be a $g$-implication. $I_g$ satisfies (LIA) with an aggregation function $A'$ if and only if $A'(x, y) = xy$.

Proof. ($\iff$) This can be verified directly.

($\implies$) By Theorem 2.14, $I_g$ can be rewritten as an $A$-implication $I_{A,N}$ generated by the disjunctor $A(x, y) = g^{-1}(\frac{2}{1+x} \land g(1))$. Assume that $I_g$ satisfies (LIA) with an aggregation function $A'$. Then, $A$ is associative and has not zero-divisor by Corollary 4.3 and Lemma 5.5. This implies that $A(x, A(y, z)) = 1$ if and only if $A(x, A(y, z)) = 1$. And then $\frac{g(z)}{1-x} = \frac{g(z)}{1-x}$ holds for any $x, y, z \in [0,1]$. Thus, $A'(x, y) = A_N(x, y) = 1 - A(1 - x, 1 - y) = xy$. That is, $I_g$ satisfies (LIA) with an aggregation function $A'$ if and only if $A'(x, y) = xy$.

Remark 5. Theorems 5.3 and 5.5 also appeared in [2][13], respectively. However, our proofs can help to understand them from another perspective.

Theorem 5.6 Let $I_C$ be a probabilistic implication. If the equation $x^2C\left(1 - \frac{C(x, y)}{x}, z\right) = xC\left(x, \frac{C(1-y, z)}{1-y}\right) - C(x, y)C\left(x, \frac{C(1-y, z)}{1-y}\right)$ holds for any $x, y, z \in [0,1]$ with understanding $\frac{C(x, y)}{x} = 1$, then $I_C$ satisfies (LIA) with an aggregation function $A'$ if and only if $A'(x, y) = \begin{cases} 1 - \frac{C(x, y)}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$.

Proof. By Theorem 2.14, $I_C$ can be rewritten as an $A$-implication $I_{A,N}$ generated by the disjunctor $A(x, y) = \begin{cases} \frac{C(1-y, z)}{1-y} & x \neq 1 \\ 1 & x = 1 \end{cases}$. The equation $x^2C\left(1 - \frac{C(x, y)}{x}, z\right) = xC\left(x, \frac{C(1-y, z)}{1-y}\right) - C(x, y)C\left(x, \frac{C(1-y, z)}{1-y}\right)$ can ensure that the disjunctor $A$ is associative. According to Theorem 4.4, $I_C$ satisfies (LIA) with an aggregation function $A'$ if and only if $A'$ is $N$-dual of $A$. That is, $A'(x, y) = \begin{cases} 1 - \frac{C(x, y)}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$.

Remark 6. Notice that there exist some probabilistic implications which satisfy (LIA) with an aggregation function $A'$ without the condition of Theorem 5.6 (See Example 5.2 in [12]).

Theorem 5.7 Let $I_C$ be a probabilistic S-implication. If the equation $C(x, C(1-y, z)+y) = C(x, y) + C(x - C(x, y), z)$ holds for any $x, y, z \in [0,1]$, then $I_C$ satisfies (LIA) with an aggregation function $A'$ if and only if $A'(x, y) = x - C(x, 1 - y)$.

Proof. By Theorem 2.14, $I_C$ can be rewritten as an $A$-implication $I_{A,N}$ generated by the disjunctor $A(x, y) = x + C(1 - x, y)$. The equation $C(x, C(1-y, z)+y) = C(x, y) + C(x - C(x, y), z)$ implies that the disjunctor $A$ is associative. According to Theorem 4.4, $I_C$ satisfies (LIA) with an aggregation function $A'$ if and only if $A'$ is $N$-dual of $A$. That is, $A'(x, y) = x - C(x, 1 - y)$.

Theorem 5.8 Let $T$ be a nilpotent t-norm with additive generator $t$. Then, its power implication $I_T$ does not satisfy (LIA) with any aggregation function.

Proof. Suppose that $I_T$ satisfies (LIA) with an aggregation function $A$, that is, $I_T(A(x, y), z) = I_T(x, I_T(y, z))$. Taking $z = 0$, we have $\frac{t(A(x, y))}{t(0)} = \frac{t(x)}{t(\frac{1}{t(0)})}$ and $1$. This means that $A$ must be formed as $A(x, y) = t^{-1}\left(\frac{t(0) t(x)}{t(\frac{1}{t(0)})} \land t(0)\right)$. This case implies that $A$ has a right neutral element 1. And
then \( I^T(A(x, 1), z) = \frac{t(x)}{t(z)} \) holds if \( 1 > x > z \). However, \( I^T(x, I^T(1, z)) = I^T(x, 0) = \frac{t(x)}{t(0)} \). Thus, \( I^T \) does not satisfy (LIA) with any aggregation function.

**Theorem 5.9** Let \( T \) be the minimum t-norm and a strict t-norm, respectively. Then, their power implications does not satisfy (LIA) with any aggregation function having zero divisors or being commutative.

**Proof.** We only consider the case where \( T \) is minimum t-norm. Another case can be similarly proved. Suppose that the aggregation function \( A \) has zero divisors and \( A(x, y) = 0 \). We then have \( I^{T_m}(A(x, y), 0) = 1 \). However, \( I^{T_m}(x, I^{T_m}(y, 0)) = 0 \).

Moreover, we assume that \( I^{T_m} \) satisfies (LIA) with a commutative aggregation function. By Lemma 3.1, \( I^{T_m} \) satisfies (EP). However, \( I^{T_m} \) does not satisfy (EP) (See Proposition 13 in [23]).

**Remark 7.** We can similarly verify that these \( T \)-power implications do not satisfy (LIA) with any aggregation function having a neutral element \( e \), too. However, we can not ensure whether they do not satisfy (LIA) with any aggregation function.

### 6 (LIA) with a given aggregation function

For a fixed aggregation function \( A \), this section aims to seek fuzzy implications such that they satisfy (LIA) with this aggregation function \( A \). We firstly extend Definition 6 in [27] as follows.

**Definition 6.1** Let \( A \) be an aggregation function and \( N \) a fuzzy negation. We say that \( N \) is \( A \)-compatible if \( N(y_1) = N(y_2) \) implies \( N(A(x, y_1)) = N(A(x, y_2)) \) for any \( x \in [0, 1] \).

**Lemma 6.2** Let \( I \) be a fuzzy implication and \( N_I \) be continuous. If \( I \) satisfies (LIA) with a given conjunctor \( A \), then \( I \) has the form of \( I(x, y) = N_I(A(x, \tilde{N}_I(y))) \).

**Proof.** Since \( I \) satisfies (LIA) with a given conjunctor \( A \), \( I(A(x, y), z) = I(x, I(y, z)) \) holds for any \( x, y, z \in [0, 1] \). Taking \( z = 0 \), we have \( N_I(A(x, y)) = I(x, N_I(y)) \). Let us consider the following two options.

i. \( y \in \text{Ran}(\tilde{N}_I) \). In this case, we have \( \tilde{N}_I(N_I(y)) = y \). This implies that \( I(x, N_I(y)) = N_I(A(x, \tilde{N}_I(N_I(y)))) \). Since \( N_I \) is continuous, we have \( I(x, y) = N_I(A(x, \tilde{N}_I(y))) \).

ii. \( y \notin \text{Ran}(\tilde{N}_I) \). This case implies that there exists \( y' \in \text{Ran}(\tilde{N}_I) \) such that \( N_I(y) = N_I(y') \). We then have \( I(x, N_I(y)) = I(x, N_I(y')) = N_I(A(x, \tilde{N}_I(N_I(y')))) = N_I(A(x, \tilde{N}_I(N_I(y)))) \). Since \( N_I \) is continuous, \( I(x, y) = N_I(A(x, \tilde{N}_I(y))) \) holds.

Further, considering \( A \) is a conjunctor, it can be verified that \( I(x, y) = N_I(A(x, \tilde{N}_I(y))) \) is a fuzzy implication.

**Lemma 6.3** Let \( A \) be an associative conjunctor and \( N \) an \( A \)-compatible continuous fuzzy negation. Then \( I(x, y) = N(A(x, \tilde{N}(y))) \) satisfies (LIA) with \( A \).

**Proof.** \( I(A(x, y), z) = N(A(A(x, y), \tilde{N}(z))) = N(A(x, A(y, \tilde{N}(z)))) \). Let us consider the follow-
compatible. Without loss of generality, we suppose that $N_0$ is non-filling fuzzy negation. Otherwise, there exists $I$ the greatest fuzzy implication, that is, $(I(x, y), z)) = I(x, y') = I(y') = I(A(x, y'), z)$. 

**Theorem 6.4** Let $I$ be a fuzzy implication and $A$ a conjunctor. If $A$ is associative and $N_I$ is continuous, then $I$ satisfies (LIA) with $A$ if and only if $N_I$ is $A$-compatible and $I(x, y) = N_I(A(x, N_I(y)))$.

**Proof.** $(\Rightarrow)$ The proof comes from Lemma 6.3.

$(\Rightarrow)$ Assume that $I$ satisfies (LIA) with $A$. By Lemma 3.2, $N_I$ is $A$-compatible. And then $I(x, y) = N_I(A(x, N_I(y)))$ according to Lemma 6.2.

In the rest of this section, we will characterize fuzzy implications for some distinguished conjunctors such that they satisfy (LIA).

**Lemma 6.5** Let $I$ be a fuzzy implication. Then $I$ satisfies (LIA) with $C_\perp$ if and only if $I$ is the greatest fuzzy implication, that is, $I(x, y) = \begin{cases} 0 & x = 1, y = 0 \\ 1 & \text{otherwise} \end{cases}$.

**Proof.** $(\Rightarrow)$ Assume that $I$ satisfies (LIA) with $C_\perp$. Then, $I(C_\perp(x, y), z) = I(x, y, z)$ holds for any $x, y, z \in [0, 1)$. For any $x \neq 1$, we have $I(x, 0) = I(x, y, z)) = I(C_\perp(x, y), 0) = I(1, 1) = 1$. This implies that $I$ can be written as $I(x, y) = \begin{cases} 0 & x = 1, y = 0 \\ 1 & \text{otherwise} \end{cases}$.

$(\Rightarrow)$ It is easy to verify that $I$ satisfies (LIA) with $C_\perp$.

**Lemma 6.6** Let $I$ be a fuzzy implication and $N_I$ a continuous fuzzy negation. Then $I$ satisfies (LIA) with $(C_{avg})^\top$ if and only if $N_I$ is non-filling and $I(x, y) = \begin{cases} 1 & x = 0 \text{ or } y = 1 \\ N_I(x) & y = 0 \\ N_I(x) \land y & \text{otherwise} \end{cases}$.

**Proof.** $(\Rightarrow)$ Suppose that $I$ satisfies (LIA) with $(C_{avg})^\top$. It can firstly be asserted that $N_I$ is a non-filling fuzzy negation. Otherwise, there exists $y_0 \neq 0$ such that $N_I(y_0) = 1$. Notice that $N_I(0) = N_I(y_0) = 1$. This implies that $N_I((C_{avg})^\top(1, 0)) = N_I(0) = 1 > N_I((C_{avg})^\top(1, y_0)) = N_I(1) = 0$.

According to Lemma 6.2, we have $I(x, y) = \begin{cases} 1 & x = 0 \text{ or } y = 1 \\ N_I(x) & y = 0 \\ N_I(x) \land y & \text{otherwise} \end{cases}$.

$(\Rightarrow)$ Obviously, $(C_{avg})^\top$ is associative. Further, we can ensure that $N_I$ is $(C_{avg})^\top$-compatible. Without loss of generality, we suppose that $N_I(y_1) = N_I(y_2) < 1$ holds for $0 < y_1 < y_2$. In order to obtain $N_I(C_{avg})^\top(x, y_1)) = N_I(C_{avg})^\top(x, y_2))$ for any $x \in [0, 1]$, let us consider the following three options:

i. $x \leq y_1 < y_2$. In this case, we have $N_I(C_{avg})^\top(x, y_1)) = N_I(y_1) = N_I(C_{avg})^\top(x, y_2)) = 0$. 

ii. $y_1 < x \leq y_2$. In this case, we have $N_I(C_{avg})^\top(x, y_1)) = N_I(y_1) = N_I(C_{avg})^\top(x, y_2)) = 0$. 

iii. $y_2 < x$. In this case, we have $N_I(C_{avg})^\top(x, y_1)) = N_I(y_1) = N_I(C_{avg})^\top(x, y_2)) = 0$. 


\(N_I(y_2)\).

ii. \(y_1 < x \leq y_2\). This case implies \(N_I(y_1) = N_I(x) = N_I(y_2)\). We therefore obtain
\[N_I((C_{\text{avg}})^\top(x, y_1)) = N_I(x) = N_I((C_{\text{avg}})^\top(x, y_2)) = N_I(y_2)\].

iii. \(y_1 < y_2 < x\). In this case, we have \(N_I((C_{\text{avg}})^\top(x, y_1)) = N_I(x) = N_I((C_{\text{avg}})^\top(x, y_2))\).

Based on the argument above, \(I\) satisfies (LIA) with \((C_{\text{avg}})^\top\) according to Lemma 6.3.

According to Theorem 3.7 in [13], \(N\) is a fuzzy negation iff there exists a continuous strictly increasing function \(g : [0, 1] \to [0, +\infty]\) with \(g(0) = 0\) such that \(N(x) = g^{-1}(g(1) - g(x))\) for any \(x \in [0, 1]\). In this case, the representable aggregation functions can be rewritten as
\[A(x, y) = g^{-1}((g(x) + g(y) - g(1)) \vee 0)\]. We then obtain the following result.

**Lemma 6.7** Let \(I\) be a fuzzy implication and \(N_I\) a strict fuzzy negation. \(I\) satisfies (LIA) with the representable aggregation functions defined as \(A(x, y) = g^{-1}((g(x) + g(y) - g(1)) \vee 0)\) if and only if \(I(x, y) = \begin{cases} 1 & f^{-1}(f(N_I(x)) + f(y)) \\ f(N_I(x)) + f(y) & \text{otherwise} \end{cases}\), where \(f = g \circ N_I^{-1}\).

**Proof.** \((\Rightarrow)\) Suppose that \(I\) satisfies (LIA) with \(A\). By Lemma 6.2, we have \(I(x, y) = N_I(A(x, N_I^{-1}(y))) = N_I(g^{-1}((g(x) + g(N_I^{-1}(y)) - g(1)) \vee 0)) = f^{-1}(f(N_I(x)) + f(y) - f(0))\).

\((\Leftarrow)\) This proof comes from Lemma 6.3.

However, we cannot use Lemma 6.2 to characterize fuzzy implications for \(M_{\lambda,f}\) and \(TS_{\lambda,f}\) by the aforementioned method because they are not associative.

## 7 Fuzzy hierarchical inference engine with fuzzy implications satisfying (LIA)

In this section, we will present three fuzzy hierarchical inference engines in MISO fuzzy systems based on the fuzzy implications satisfying (LIA). Here, we therefore assume that the fuzzy implication \(I\) satisfies (LIA) with an aggregation function \(A\). Firstly, let us study the solution of GMP problem in Pedrycz’s, Raha’s and TIP methods, respectively.

### 7.1 Three fuzzy hierarchical inference engines with fuzzy implication satisfying (LIA)

**Lemma 7.1** The conclusion of GMP problem in Pedrycz’s method can be rewritten as

\[B'_{\text{BKS}}(y) = I\left(\bigvee_{x \in U} A(D'(x), D(x)), B(y))\right).\]

**Proof.** \(B'_{\text{BKS}}(y) = \bigwedge_{x \in U} I(D'(x), B(y))) = \bigwedge_{x \in U} I(A(D'(x), D(x)), B(y)) = I\left(\bigvee_{x \in U} A(D'(x), D(x)), B(y)\right).\)
**Lemma 7.2** The conclusion $B''_{\text{SBR}}$ of GMP problem in Raha’s method is

$$B''_{\text{SBR}}(y) = \bigvee_{x \in U} I(A(S(D', D), D(x)), B(y)).$$

**Proof.** Obvious.

**Lemma 7.3** Let $I$ be a fuzzy implication which is right continuous with respect to the second variable and satisfies (OP). Then the TIP solution of GMP problem is

$$B'_{\text{TIP}}(y) = \bigvee_{x \in U} A(I(D(x), B(y)), D'(x)).$$

**Proof.** Since $I$ is right-continuous with respect to the second variable, the TIP solution of GMP problem is unique and Eq.(1) takes its maximum 1 by Lemma 2.23. It is not difficult to verify that $I(I(D(x), B(y)), I(D'(x), \bigvee_{x \in U} A(I(D(x), B(y)), D'(x)))) \equiv 1$ holds for any $x \in V$ and $y \in U$ according to Lemma 3.8.

On the other hand, assume that $C$ is an arbitrary fuzzy set on $V$ such that $I(I(D(x), B(y)), I(D'(x), C(y))) \equiv 1$ holds for any $x \in V$ and $y \in U$. Since $I$ satisfies (LIA) with the aggregation function $A$, we have $I(I(D(x), B(y)), I(D'(x), C(y))) = I(A(I(D(x), B(y)), D'(x)), C(y)) \equiv 1$ for any $x \in V$ and $y \in U$. The ordering property of $I$ implies that $C(y) \geq \bigvee_{x \in U} A(I(D(x), B(y)), D'(x))$. Therefore, $B'_{\text{TIP}}(y) = \bigvee_{x \in U} A(I(D(x), B(y)), D'(x))$.

In order to construct the fuzzy hierarchical inference engine in MISO fuzzy system, we combine the input and IF-THEN rules into the output by the above three methods to solve the GMP problem. For convenience to show three fuzzy hierarchical inference engines, we only consider this case when $m = 2$ and $n = 1$ (that is, two-input-one-output fuzzy system and the fuzzy rule base including only one rule). Assume that the fuzzifier is the singleton fuzzifier [39] and that the aggregation function $A$ is used to combine the antecedent of IF-THEN rule in fuzzy inference engine. For an arbitrary input $x_0 = (x_{01}, x_{02}) \in U_1 \times U_2$, we have the following results.

**Theorem 7.4** Let $I$ satisfy (NP) and $A$ be a conjunctor having a left neutral element 1. If the conjunctor $A$ which is employed to combine the antecedent of IF-THEN rule and $I$ satisfy (LIA), then the BKS inference engine is $B'_{\text{BKS}} = (D'_1, D'_2) \circ_{\text{BKS}} I((D_1, D_2), B) = D'_1 \circ_{\text{BKS}} I(D_1, D'_2 \circ_{\text{BKS}} I(D_2, B)).$

**Proof.** By Lemma 7.1, $B'_{\text{BKS}}(y) = I(\bigvee_{(x_1, x_2) \in U_1 \times U_2} A(D'_1(x_1), D'_2(x_2)), A(D_1(x_1), D_2(x_2))), \quad B(y) = I(A(D_1(x_{01}), D_2(x_{02})), B(y)) = I(D_1(x_{01}), I(D_2(x_{02}), B(y))) = I(\bigvee_{x_1 \in U_1} A(D'_1(x_1), D_1(x_1)), A(D_1(x_{01}), D_2(x_{02})), B(y))).$ This can be shortened as $B'_{\text{BKS}} = (D'_1, D'_2) \circ_{\text{BKS}} I((D_1, D_2), B) = D'_1 \circ_{\text{BKS}} I(D_1, D'_2 \circ_{\text{BKS}} I(D_2, B)).$

For convenience, we shorten the conclusions $B''_{\text{SBR}}$ and $B'_{\text{TIP}}$ in Lemmas 7.2 and 7.3 as $B''_{\text{SBR}} = D' \circ_{\text{SBR}} I(D, B)$ and $B'_{\text{TIP}} = D' \circ_{\text{TIP}} I(D, B)$, respectively. Similar to Theorem 7.4,
we obtain the following results.

**Theorem 7.5** Let $A$ be an associative and commutative conjunctor and $S(A(D_1', D_2'), A(D_1, D_2)) = A(S(D_1', D_1), S(D_2', D_2))$. If the conjunctor $A$ which is employed to combine the antecedent of IF-THEN rule and $I$ satisfy (LIA), then the TIP inference engine is $B''_{\text{SBR}} = (D_1', D_2') \circ_{\text{SBR}} I(D_1, D_2, B) = D_1' \circ_{\text{SBR}} I(D_1, D_2 \circ_{\text{SBR}} I(D_2, B))$.

**Proof.** By Lemma 7.2, $B''_{\text{SBR}}(y) = \bigvee_{(x_1, x_2) \in U_1 \times U_2} I(A(S(A(D_1', D_2'), A(D_1, D_2)), A(D_1(x_1), D_2(x_2))), B(y)) = I(A(A(S(D_1', D_1), S(D_2', D_2)), A(D_1(x_01), D_2(x_02))), B(y)) = I(A(A(S(D_1', D_1), D_1(x_01)), A(S(D_2', D_2), D_2(x_02))), B(y)) = I(A(S(D_1', D_1), D_1(x_1)), \bigvee_{x_2 \in U_2} I(A(S(D_2', D_2), D_2(x_2))), B(y))).$ This can be shortened as $B'_{\text{BKS}} = (D_1', D_2') \circ_{\text{BKS}} I(D_1, D_2, B) = D_1' \circ_{\text{BKS}} I(D_1, D_2 \circ_{\text{BKS}} I(D_2, B))$.

**Remark 8.** Since $D_1'$ and $D_2'$ are singleton fuzzy sets, the condition $S(A(D_1', D_2'), A(D_1, D_2)) = A(S(D_1', D_1), S(D_2', D_2))$ can be meet by some measures of similarity. For example, the several measures of similarity mentioned in Ref. [32] satisfy this condition for any conjunctor.

**Theorem 7.6** Let the conjunctor $A$ which is employed to combine the antecedent of IF-THEN rule and $I$ satisfy (LIA) with $A$. If $I$ satisfies (OP) and is right-continuous with respect to the second variable, then the TIP inference engine is $B'_{\text{TIP}} = (D_1', D_2') \circ_{\text{TIP}} I((D_1, D_2), B) = D_1' \circ_{\text{TIP}} I(D_1, D_2 \circ_{\text{TIP}} I(D_2, B))$.

**Proof.** According to Lemma 7.3, $B'_{\text{TIP}}(y) = \bigvee_{(x_1, x_2) \in U_1 \times U_2} A(I(A(D_1(x_1), D_2(x_2)), B(y)), A(D_1(x_1), D_2(x_2))) = I(A(D_1(x_01), D_2(x_02), B(y)) = I(D_1(x_01), I(D_2(x_02), B(y))) = \bigvee_{x_1 \in U_1} A(I(D_1(x_1), B(y)), I(D_2(x_2), D_1(x_1))).$ This means $B'_{\text{TIP}} = (D_1', D_2') \circ_{\text{TIP}} I((D_1, D_2), B) = D_1' \circ_{\text{TIP}} I(D_1, D_2 \circ_{\text{TIP}} I(D_2, B))$.

### 7.2 Discussion

It is not difficult to see that we can extend these three hierarchical inference engines to any MISO fuzzy system. This implies that the MISO fuzzy system employing these three inference engines can be converted into an SISO hierarchical fuzzy system employing these three inference engines. Similar to that mentioned in [13], it is sufficient to calculate the two-dimensional matrices at each stage and to store the antecedent of fuzzy rules in the SISO hierarchical fuzzy system. In brief, these three inference engines have the advantages in storing and computing.

Indeed, it owes to the law of importation that the MISO fuzzy system with these three inference engines converted into a SISO hierarchical fuzzy system. In the MISO fuzzy system, chosen the fuzzy implications (such as R-implication, $(A, N)$-implication, QL-implication and
so on) to interpret the fuzzy rules in rule base, we can construct the aggregation functions such that they satisfy the law of importation by which obtained in Section 4. To enhance the storage and computational efficiency, people ought to accordingly employ these aggregation functions to combine the antecedent of fuzzy rules in rule base.

By the results in Section 5, if a given aggregation function is employed to translate the antecedent of fuzzy rules in rule base, we can also construct a fuzzy implication such that they satisfy the law of importation. Similarly, people should utilize the fuzzy implication to translate the fuzzy rules in rule base in order to advance the computational and storage efficiency.

8 Conclusions

We firstly have studied the fuzzy implications which satisfy the law of importation with aggregation functions. And then three hierarchical inference engines based on fuzzy implications satisfying (LIA) have been investigated. Specifically, we have

(1) Analyzed the properties of aggregation functions and fuzzy implications when they satisfy (LIA);

(2) Given the necessary and sufficient conditions for \( (A, N) \)-implications and R-implications which satisfy (LIA) with some aggregation functions;

(3) Found some aggregation functions for \( f \)-implication, \( g \)-implication, QL-implication, probabilistic implication, probabilistic S-implication and \( T \)-power implications such that they satisfy (LIA);

(4) Characterized the fuzzy implications which satisfy (LIA) with a given aggregation function;

(5) Constructed three fuzzy hierarchical inference engines in MISO fuzzy systems based on the aggregation functions and fuzzy implications satisfying (LIA).

Our results can help to improve the effectiveness of fuzzy inference engine in MISO fuzzy systems. In the future, we wish to study the capability of fuzzy system using these hierarchical inference engines. We also will apply them in real-life control problems and decision making.

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References

[1] M. Baczyński, P. Grzegorzewski, P. Helbin, W. Niemyska, Properties of the probabilistic implications and S-implications, Information Sciences 331(2016)2-14.
[2] M. Baczyński, B. Jayaram, Fuzzy Implications, Springer, Berlin, 2008.

[3] M. Baczyński, B. Jayaram, R. Mesiar, On special fuzzy implications, Fuzzy Sets and Systems 160(2009)2063-2085.

[4] M. Baczyński, B. Jayaram, R. Mesiar, Fuzzy implications: alpha migrativity and generalised laws of importation, Information Sciences 531(2020)87-96.

[5] J. Baldwin, A new approach to approximate reasoning using a fuzzy logic, Fuzzy Sets and Systems 2(1979)309-325.

[6] H. Bustince, M. Pagola, R. Mesiar, E. Hüllermeier, F. Herrera, Grouping, overlap, and generalized bientropic functions for fuzzy modeling of pairwise comparisons, IEEE Transactions on Fuzzy Systems 20(2012)405-415.

[7] A.R. de Soto, A. Sobrino, E. Trillas, C. Alsina, Reflections on an old problem: That of preserving the logical forms and symmetry, Fuzzy Sets and Systems 401(2020)150-162.

[8] G.P. Dimuro, B. Bedregal, On residual implications derived from overlap functions, Information Sciences 312(2015)78-88.

[9] J.C. Fodor, T. Keresztfalvi, Nonstandard conjunctions and implications in fuzzy logic, International Journal of Approximate Reasoning 12(2)(1995)69-84.

[10] M. Grabisch, J.L. Marichal, R. Mesiar, E. Pap, Aggregation Functions, Cambridge University Press, New York, 2009.

[11] P. Grzegorzewski, Probabilistic implications, Fuzzy Sets and Systems 226(2013)53-66.

[12] P. Helbin, M. Baczyński, P. Grzegorzewski, W. Niemyska, Some properties of fuzzy implications based on copulas, Information Sciences 502(2019)1-17.

[13] B. Jayaram, On the law of importation \((x \land y) \rightarrow z \equiv (x \rightarrow (y \rightarrow z))\) in fuzzy logic, IEEE Transactions on Fuzzy Systems 16(2008)130-144.

[14] E.P. Klement, R. Mesiar, E. Pap, Triangular Norms, Kluwer Academic Publishers, Dordrecht, Boston, London, 2000.

[15] G.J. Klir, B. Yuan, Fuzzy Sets and Fuzzy Logic: Theory and Applications. Prentice Hall, Upper Saddle River, 1995.

[16] A.A. Lima, B. Bedregal, L. Mezzomo, Ordinal sums of the main classes of fuzzy negations and the natural negations of t-norms, t-conorms and fuzzy implications, International Journal of Approximate Reasoning 116(2020)19-32.

[17] D.C. Li, Q.X. Zeng, Approximate reasoning with aggregation functions satisfying GMP rules, Artificial Intelligence Review, 2022, DOI: 10.1007/s10462-022-10136-1.

[18] R. Lowen, On fuzzy complements, Information Sciences 14(2)(1978)107-113.
[19] P. Magrez, P. Smets, Fuzzy modus ponens: A new model suitable for applications in knowledge-based systems, International Journal of Intelligent Systems 4(1989)181-200.

[20] M. Mas, M. Monserrat, J. Torrens, A characterization of (U,N), RU, QL and D-implications derived from uninorms satisfying the law of importation, Fuzzy Sets and Systems, 161(2010)1369-1387.

[21] M. Mas, M. Monserrat, J. Torrens, D. Ruiz-Aguilera, RU and (U, N)-implications satisfying Modus Ponens, International Journal of Approximate Reasoning 73(2016)123-137.

[22] S. Massanet, A. Pradera, D. Ruiz-Aguilera, J. Torrens, Equivalence and characterization of probabilistic and survival implications, Fuzzy Sets and Systems 359(2019)63-79.

[23] S. Massanet, J. Recasens, J. Torrens, Fuzzy implication functions based on powers of continuous t-norms, International Journal of Approximate Reasoning 83(2017)265-279.

[24] S. Massanet, J. Recasens, J. Torrens, Some characterizations of T-power based implications, Fuzzy Sets and Systems 359(2019)42-62.

[25] S. Massanet, D. Ruiz-Aguilera, Joan Torrens, Characterization of a class of fuzzy implication functions satisfying the law of importation with respect to a fixed uninorm (Part I), IEEE Transactions on Fuzzy Systems 26(2018)1983-1994.

[26] S. Massanet, J. Torrens, The law of importation versus the exchange principle on fuzzy implications, Fuzzy Sets and Systems 168(2011)47-69.

[27] S. Massanet, J. Torrens, Characterization of fuzzy implication functions with a continuous natural negation satisfying the law of importation with a fixed t-norm, IEEE Transactions on Fuzzy Systems 25(2017)100-113.

[28] K. Miś, M. Baczyński, A note on “On special fuzzy implications”, Fuzzy Sets and Systems 359(2019)90-94.

[29] M. Mizumoto, Fuzzy reasoning under new compositional rules of inference, Kybernetes 12(1985)107-117.

[30] Y. Ouyang, On fuzzy implications determined by aggregation operators, Information Sciences 193(2012)153-162.

[31] W. Pedrycz, Applications of fuzzy relational equations for methods of reasoning in presence of fuzzy data, Fuzzy Sets and Systems 16(2)(1985)163-175.

[32] D.W. Pei, Unified full implication algorithms of fuzzy reasoning, Information Sciences 178(2)(2008)520-530.

[33] A. Pradera, G. Beliakov, H. Bustince, Aggregation functions and contradictory information, Fuzzy Sets and Systems 191(2012)41-61.
[34] A. Pradera, G. Beliakov, H. Bustince, B.D. Baets, A review of the relationships between implication, negation and aggregation functions from the point of view of material implication, Information Sciences 329(2016)357-380.

[35] A. Pradera, S. Massanet, D. Ruiz-Aguilera, J. Torrens, The non-contradiction principle related to natural negations of fuzzy implication functions, Fuzzy Sets and Systems 359(2019)3-21.

[36] S. Raha, N.R. Pal, K.S. Ray, Similarity-based approximate reasoning: methodology and application, IEEE Trans. Syst. Man Cybern., Part A, Syst. Hum. 32(4)(2002)541-547.

[37] M. Stepnicka, B. Jayaram, On the suitability of the Bandler-Kohout subproduct as an inference mechanism, IEEE Transaction on Fuzzy Systems 18(2)(2010)285-298.

[38] I.B. Turksen, Z. Zhong, An approximate analogical reasoning approach based on similarity measures, IEEE Trans. Syst. Man Cybern. 18(1988)1049-1056.

[39] L.X. Wang, A Course in Fuzzy Systems and Control, Prentice Hall PTR, Upper Saddle River, 1997.

[40] G.J. Wang, On the logic foundation of fuzzy reasoning, Information Sciences 117(1999)47-88.

[41] R. Yager, On some new classes of implication operators and their role in approximate reasoning, Information Sciences 167(2004)193-216.

[42] L.A. Zadeh, Outline of a new approach to the analysis of complex systems and decision processes, IEEE Trans. Syst. Man Cybernet. 3(1)(1973):28-44.