ON KHINTCHINE TYPE INEQUALITIES FOR $k$-WISE INDEPENDENT RADEMACHER RANDOM VARIABLES

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Abstract. We consider Khintchine type inequalities on the $p$-th moments of vectors of $N$ $k$-wise independent Rademacher random variables. We show that an analogue of Khintchine’s inequality holds, with a constant $N^{1/2 - k/2p}$, when $k$ is even. We then show that this result is sharp for $k = 2$; in particular, a version of Khintchine’s inequality for sequences of pairwise Rademacher variables cannot hold with a constant independent of $N$. We also characterize the cases of equality and show that, although the vector achieving equality is not unique, it is unique (up to law) among the smaller class of exchangeable vectors of pairwise independent Rademacher random variables. As a fortunate consequence of our work, we obtain similar results for 3-wise independent vectors.

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1. Introduction

This short note concerns Khintchine’s inequality, a classical theorem in probability, with many important applications in both probability and analysis (see [1] [8] [9] [11] [12] among others). It states that the $L_p$ norm of the weighted sum of independent Rademacher random variables is controlled by its $L_2$ norm; a precise statement follows. We say that $\varepsilon_0$ is a Rademacher random variable if $\mathbb{P}(\varepsilon_0 = 1) = \mathbb{P}(\varepsilon_0 = -1) = \frac{1}{2}$. Let $\varepsilon_i$, $1 \leq i \leq N$, be independent copies of $\varepsilon_0$ and $a \in \mathbb{R}^N$. Khintchine’s inequality (see, for example, Theorem 2.b.3 in [9], Theorem 12.3.1 in [4] or the original work of Khintchine [7]) states that, for any $p > 0$

$$B(p) \left( \mathbb{E} \left| \sum_{i=1}^{N} a_i \varepsilon_i \right|^2 \right)^{\frac{1}{2}} = B(p) \|a\|_2 \leq \left( \mathbb{E} \left| \sum_{i=1}^{N} a_i \varepsilon_i \right|^p \right)^{\frac{1}{p}} \leq C(p) \|a\|_2 = C(p) \left( \mathbb{E} \left| \sum_{i=1}^{N} a_i \varepsilon_i \right|^2 \right)^{\frac{1}{2}}. \quad (1)$$

We will mostly be interested in the upper Khintchine inequality; that is, the second inequality in (1). Note here that the upper constant $C(p)$ depends only on $p$; in particular, it does not depend on $N$. In what follows, we take $C(p)$ to be the best possible constant in (1). This value is in fact known explicitly [9]:

$$C(p) = \begin{cases} 1 & 0 < p \leq 2 \\ \frac{\sqrt{2\pi (p/2-1)}}{\sqrt{\pi^{p/2}}} & p > 2. \end{cases}$$

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It is natural to ask whether the independence condition can be relaxed; indeed, random vectors with dependent coordinates arise in many problems in probability and analysis (see e.g. [5] and the references therein). In this short paper, we are interested in what can be said when the independence assumption on the coordinates is relaxed to pairwise (or, more generally, \( k \)-wise) independence.

**Definition 1.1.** We call an \( N \)-tuple \( \varepsilon = \{ \varepsilon_i \}_{i=1}^N \) of Rademacher random variables a Rademacher vector, or (finite) Rademacher sequence. For a fixed non-negative integer \( k \), a Rademacher vector is called \( k \)-wise independent if any subset \( \{ \varepsilon_{i_1}, \varepsilon_{i_2}, \ldots, \varepsilon_{i_k} \} \) of size \( k \) is mutually independent.

When \( k = 2 \) in the preceding definition, we will often use the terminology pairwise independent in place of 2-wise independent. For more on \( k \)-wise independent sequences and their construction, see, for example [2, 13, 14].

As it will be useful in what follows, we note that instead of random variables, it is equivalent to consider probability measures \( P \) on the set \( \{-1, 1\}^N \), where \( P = \text{law}(\varepsilon) \). The condition that \( \varepsilon \) is a Rademacher vector is then equivalent to the condition that the projections \( \text{law}(\varepsilon_{i_1}, \ldots, \varepsilon_{i_k}) \) of \( P \) onto each copy of \( \{-1, 1\} \) are all equal to \( P_1 := \frac{1}{2}[(\delta_{-1} + \delta_1)] \). The \( k \)-wise independence condition is equivalent to the condition that the projections \( \text{law}(\varepsilon_{i_1}, \ldots, \varepsilon_{i_k}) \) of \( P \) onto each \( k \)-fold product \( \{-1, 1\}^k \) is product measure \( \otimes^k P_1 \).

An interesting general line of research in probability aims to understand which of the many known properties of mutually independent sequences carry over to the \( k \)-wise independent setting; how much independence is actually needed to assert various properties? Some results, including the second Borel-Cantelli lemma and the strong law of large numbers (see, for instance, [3] and [1]) hold true for pairwise independent sequences, whereas others, such as the central limit theorem, do not.

We found it surprising that little seems to be known about Khintchine’s inequality for \( k \)-wise independent sequences (except when \( k \geq p \), as we discuss briefly below).

It is therefore natural to ask whether Khintchine’s inequality holds for \( k \)-wise independent Rademacher random variables, and, if not, to understand how badly it fails. More precisely, we focus on the upper Khintchine inequality and define

\[
(2) \quad C(N, p, k) = \sup_{\varepsilon \text{ is a } k \text{-wise independent Rademacher vector}} \left( \mathbb{E} \left| \sum_{i=1}^N a_i \varepsilon_i \right|^p \right)^{1/p}.
\]

The questions we are interested in can then be formulated as:

1. Is \( C(N, p, k) \) bounded as \( N \to \infty \), for a fixed \( p > k \)?
2. If not, what is the growth rate of \( C(N, p, k) \)?

Note that the \( C(N, p, k) \) form a monotone decreasing sequence in \( k \), as the \( k \)-dependence constraint becomes increasingly stringent as \( k \) grows. Note that, as mutual independence implies \( k \)-wise independence for any \( k \), we have \( C(N, p, k) \geq C(p) \), where \( C(p) \) is the best constant in the classical Khintchine inequality [1].

Some properties of \( C(N, p, k) \) are easily discerned. For example, it is straightforward to see that \( C(N, 2, k) = 1 \). Let us also mention that, when \( p \) is an even integer, and \( k \geq p \), it is actually a straightforward calculation to show that \( C(N, p, k) = C(p) \) is independent of \( N \) (that is, Khintchine’s inequality for \( k \)-wise
independent random variables holds with the same constant as in the independence case).

For \( k < p \) and even, we first show that \( C(N, p, k) \leq C(k)^{k/p} N^{1/2 - k/2p} \), by combining a standard interpolation argument with the classical Khintchine inequality and the observation above. This provides some information on the second question above for general \( k \).

We then focus on the \( k = 2 \) case. We prove that for \( p \geq 2 \) and \( N \) even, \( C(N, p, 2) = N^{1/2 - 1/p} \), providing a negative answer to the first question above. We construct an explicit pairwise independent Rademacher sequence satisfying the equality. Finally, we characterize the cases of equality, and prove that although this equality may be achieved by multiple Rademacher vectors, the one we construct is the unique exchangeable equality case (up to law).

As a fortunate consequence of our work here, we obtain analogous results for \( k = 3 \). Understanding the \( k \geq 4 \) case remains an interesting open question.

2. A GENERAL ESTIMATE ON \( C(N, p, k) \)

We begin by establishing an upper bound on \( C(N, p, k) \) via a straightforward interpolation argument.

**Proposition 2.1.** For all \( p \geq k \geq 2 \) and \( k \) even, we have \( C(N, p, k) \leq C(k)^{k/p} N^{1/2 - k/2p} \).

**Proof.** Let \( \epsilon = (\epsilon_1, ..., \epsilon_N) \) be a \( k \)-wise independent Rademacher vector of length \( N \), and \( a = (a_1, ..., a_N) \in \mathbb{R}^N \). Set \( f = |\sum_{i=1}^{N} a_i \epsilon_i| \), so that \( f \) is a function on the underlying probability space. Writing \( f^p = f^k f^{p-k} \), we apply Holder’s inequality to get

\[
\|f\|_p \leq \|f\|_k^{k/p} \|f^{p-k}\|_\infty^{1/p}
\]

Now, note that the multinomial theorem implies

\[
\|f\|_k^k = E[|\sum_{j_1 + j_2 + ... + j_m = k} a_{j_1} \epsilon_{j_1} |^k] = \sum_{j_1 + j_2 + ... + j_m = k} \frac{k!}{j_1! j_2! ... j_m!} E\left( \Pi_{i=1}^{m} (a_i \epsilon_i)^{j_i} \right),
\]

As each term \( \Pi_{i=1}^{m} (a_i \epsilon_i)^{j_i} \) contains only at most \( k \) distinct \( \epsilon_i \), and the distribution is \( k \)-wise independent, the expected value is identical to what we would obtain from a mutually independent series, \( \bar{\epsilon} \), and so we have, by the classical Khintchine inequality,

\[
\|f\|_k^k = E[|\sum_{j_1} a_{j_1} \epsilon_{j_1} |^k] = E\| \sum_{j_1} a_{j_1} \epsilon_{j_1} |^k \| \leq C(k)^k \|a\|_2^k.
\]

On the other hand,

\[
\|f^{p-k}\|_\infty \leq \|a_{j_1}^{1-k/p} \leq \|\|a\|_2^{1/2} N^{1/2 - k/2p} \|a\|_2^{1-k/p} = C(k)^{k/p} N^{1/2 - k/2p} \|a\|_2^{1-k/p}
\]

where the last inequality is by Cauchy-Schwartz. Combining these, we have

\[
\|f\|_p \leq C(k)^{k/p} \|a\|_2^{k/p} N^{1/2 - k/2p} \|a\|_2^{1-k/p} = C(k)^{k/p} N^{1/2 - k/2p} \|a\|_2
\]

which implies the desired result. \( \Box \)
3. Pairwise Independence: the precise value of $C(N, p, 2)$.

The following theorem provides a negative answer to the first question in the introduction when $k = 2$, and provides the precise answer to the second question in the same case. In particular, it shows that the bound $C(N, p, 2) \leq N^{1/2-1/p}$ from the previous proposition is sharp for even $N$.

**Theorem 3.1.** Let $N = 2n$ be even. There exists a pairwise independent Rademacher vector $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_N)$ such that, for $a = (1, 1, \ldots, 1) \in \mathbb{R}^N$ and all $p \geq 2$,

$$\left( \mathbb{E} \left| \sum_{i=1}^{N} a_i \varepsilon_i \right|^p \right)^{1/p} = N^{1/2-1/p} \|a\|_2$$

Consequently, $C(N, p, 2) = N^{1/2-1/p}$.

**Proof.** We will construct the probability measure $P = \text{law}(\varepsilon)$.

We define $P = \frac{1}{N} P_a + \frac{N-1}{N} P_b$ where $P_a = \frac{1}{2} \{\delta_{1,1, \ldots, 1} + \delta_{-1,-1, \ldots, -1}\}$ is uniform measure on the two points $(1, 1, \ldots, 1), (-1, -1, \ldots, -1) \in \{-1, 1\}^N$ and $P_b$ is uniform measure on the set of all points with an equal number of $1$’s and $-1$’s; that is, points which are permutations of $\{1, 1, \ldots, 1 -1, -1, \ldots, -1\}$. We first verify that this is a pairwise independent probability measure; that is, that it’s twofold marginals are $\frac{1}{2}(\delta_{1,1} + \delta_{-1,-1})$. By symmetry between the coordinates, it suffices to verify this fact for the projection $P_2$ on the first two copies of $\{-1, 1\}$. We have

$$P_2(1, 1) = \frac{1}{N} P_a(1, 1, \ldots, 1) + \frac{N-1}{N} P_b\{\varepsilon : (\varepsilon_1, \varepsilon_2) = (1, 1)\}.$$

Now, $P_a(1, 1, \ldots, 1) = \frac{1}{2}$, and it is easy to see that $P_b\{\varepsilon : (\varepsilon_1, \varepsilon_2) = (1, 1)\} = \frac{N}{2(N-1)}$, implying

$$P_2(1, 1) = \frac{1}{2N} + \frac{(N-1)(N/2 - 1)}{2N(N-1)} = \frac{1}{4}.$$  

Similar calculations imply $P_2(1, -1) = P_2(-1, 1) = P_2(-1, -1) = \frac{1}{4}$, and so $P$ is pairwise independent.

Now, letting $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_N)$ be a random variable with $\text{law}(\varepsilon) = P$, and noting that $|\sum_{i=1}^{N} \varepsilon_i|^p$ is 0 for points in the support of $P_0$ and $N$ for points in the support of $P_a$, we have

$$\mathbb{E} \left| \sum_{i=1}^{N} \varepsilon_i \right|^p = \frac{1}{N} N^p = N^{p-1}$$

Noting that $\|a\|_2 = \sqrt{N}$, it follows that

$$\left( \mathbb{E} \left| \sum_{i=1}^{N} \varepsilon_i \right|^p \right)^{1/p} = N^{1-1/p} = \sqrt{N} N^{1/2-1/p} = \|a\|_2 N^{1/2-1/p}$$

$\square$
We record next a partial uniqueness result on the $\alpha$ and $\varepsilon$ giving equality in (3). The result establishes that the $|\alpha_i|$ must all be equal, and that the support of $\text{law}(\varepsilon)$ must be as in the preceding proposition, up to permutation of the signs of the $\alpha_i$. If, in addition, $\varepsilon$ is exchangeable, its law is uniquely determined.

**Proposition 3.2.** Suppose $\alpha' = (\alpha'_1, \alpha'_2, \ldots, \alpha'_N) \in \mathbb{R}^N$ and $\varepsilon' = (\varepsilon'_1, \ldots, \varepsilon'_N)$ satisfy

$$\left(\mathbb{E} \left| \sum_{i=1}^{N} \alpha'_i \varepsilon'_i \right|^p \right)^{1/p} = N^{1/2-1/p} \|\alpha'\|_2,$$

where $N$ is even, $\varepsilon'$ is pairwise independent and $p > 2$. Then we must have $|\alpha'_i| = c$ for some constant $c$, for all $i$. Moreover, almost surely, either:

(a) $\varepsilon'_i = \text{sgn}(\alpha'_i)$ for all $i$,

(b) $\varepsilon'_i = -\text{sgn}(\alpha'_i)$ for all $i$,

or

(c) $\varepsilon'_i = \text{sgn}(\alpha'_i)$ for exactly half of the $i$ (so that $\sum_{i=1}^{N} \alpha'_i \varepsilon'_i = 0$).

Furthermore, we have

$$P'\{\varepsilon'_i = \text{sgn}(\alpha'_i) \text{ for all } i\} = P'\{\varepsilon'_i = -\text{sgn}(\alpha'_i) \text{ for all } i\} = \frac{1}{2N},$$

where $P' = \text{law}(\varepsilon')$.

Finally, if in addition $\varepsilon'$ is exchangeable and $N \geq 6$, we have $P' = P$, where $P = \text{law}(\varepsilon)$ from the preceding theorem.

**Proof.** One can only have equality in (3) if each $|\alpha_i| = c$ (to have equality in Cauchy-Schwartz), and if either $\varepsilon'_i = \text{sgn}(\alpha'_i)$ for all $i$ or $\varepsilon'_i = -\text{sgn}(\alpha'_i)$ for all $i$ with a positive probability (to have equality in the first inequality in (4)). One can only have equality in (3) if the function $f$ takes on only two values (one of them being zero); taken together, then, we can only have equality in both (3) and (4) if, almost surely, one of the conditions (a) - (c) holds.

Now, the fact that the $\varepsilon'_i$ are Rademacher variables implies that $\mathbb{E}(\sum_{i=1}^{N} \alpha'_i \varepsilon'_i) = 0$; as $\sum_{i=1}^{N} \alpha'_i \varepsilon'_i$ is either $0$, $Nc$ or $-Nc$ almost surely, we must have $\sum_{i=1}^{N} \alpha'_i \varepsilon'_i = \pm Nc$ with equal probability. The constraint $\left(\mathbb{E} \left| \sum_{i=1}^{N} \alpha_i \varepsilon_i \right|^p \right)^{1/p} = N^{1/2-1/p} \|\alpha\|_2$ then implies that each of these probabilities must be $\frac{1}{2N}$.

Turning to the final assertion, if $\varepsilon'$ is exchangeable, we claim that each of the $\alpha'_i$ must then share the same sign. To see this, note that if not, we can assume without loss of generality that $\alpha'_1 > 0$ and $\alpha'_2 < 0$. As we have $P\{\varepsilon'_i = \text{sgn}(\alpha'_i) \forall i\} = \frac{1}{2N}$, symmetry implies that

$$P\{\varepsilon'_1 = -\text{sgn}(\alpha'_1), \varepsilon'_2 = -\text{sgn}(\alpha'_2) \text{ and } \varepsilon'_i = \text{sgn}(\alpha'_i) \forall i \geq 3\} = \frac{1}{2N} > 0$$

as well. But these values of $\varepsilon'$ do not satisfy any of the conditions (a) - (c) for $N \geq 6$. This establishes the claim.

Therefore, we must have $P(\varepsilon'_i = 1 \forall i) = P(\varepsilon'_i = -1 \forall i) = \frac{1}{2N}$, and then symmetry implies that each $\varepsilon'$ such that $\sum_{i=1}^{N} \varepsilon'_i = 0$ must have the same probability, which means that $\text{law}(\varepsilon') = P$. \hfill $\square$

**Remark 3.3.** As was pointed out to us by an anonymous referee on an earlier version of this manuscript, when $N = 2^n$ is a power of 2, an alternate construction can yield $C(N, 2, p) = N^{1/2-1/p}$; let $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n)$ be a family of $n+1$ mutually independent Rademacher variables. It is straightforward to see that the family $\{\varepsilon_S := \varepsilon_0 \Pi_{i \in S} \varepsilon_i\}$, where $S$ runs over the set of subsets of $\{1, 2, \ldots, n\}$, is a pairwise
independent family of $2^n$ Rademacher random variables (here we have taken the convention that $\Pi_{i\in S} \epsilon_i = 1$ when $S = \emptyset$ is empty). It is straightforward to check that almost surely either each $\epsilon_S$ is 1, each is $-1$ or exactly half are 1, and the probability of each of the first two events is $\frac{1}{2^{n+1}} = \frac{1}{2^n}$. The claim follows.

This construction also shows that we do not have uniqueness up to law(ε) without the additional exchangeability condition, for $n \geq 3$. To see this, note that the subset $\{\epsilon_0, \epsilon_0 \epsilon_1, \epsilon_0 \epsilon_1 \epsilon_2, \epsilon_0 \epsilon_1 \epsilon_2 \epsilon_3\}$ is not mutually independent (as the product of the first three elements is exactly the fourth). On the other hand, the subset, $\{\epsilon_0, \epsilon_0 \epsilon_1, \epsilon_0 \epsilon_2, \epsilon_0 \epsilon_3\}$ is mutually independent and so $\{\epsilon_S\}_{S \subseteq \{1, 2, \ldots, n\}}$ is clearly not exchangeable. Therefore, law{ε} $\neq$ law(ε).

**Remark 3.4.** It is straightforward to verify that the measure $P = \text{law}(\varepsilon)$ derived in Theorem is in fact 3-wise independent, and thus we immediately obtain analogues of the preceding results for $k = 3$. In particular,

$$C(N, p, 3) = N^{1/2 - 1/p}.$$  

It is clear that for $p > k \geq 4$, the estimate in Proposition 2.1 cannot be sharp; as the argument involved applying Holder’s inequality to the function $f$, it can only be sharp if $f$ takes on at most two values (one of them being 0). However, a direct calculation verifies that if $f$ takes on less than three values, then $\varepsilon$ cannot be 4-wise independent. Whether or not the growth rate in $N$ of $C(N, p, k)$ is proportional to $N^{1/2 - k/2p}$ (that is, whether $C(N, p, k) = K(p, k) N^{1/2 - k/2p}$ for some constant $K(p, k) > 1$) when $k \geq 4$ is an interesting open question.

**Remark 3.5.** Generally speaking, one can identify exchangeable, $k$-wise independent random variables $X_1, \ldots, X_N$ on $\mathbb{R}$ having equal fixed marginals $P_1 = \text{law}(X_i)$ with permutation symmetric probability measures $P = \text{law}(X_1, \ldots, X_N)$ on $\mathbb{R}^N$ whose $k$-fold marginals are $\otimes^k P_1$. The set of measures satisfying these constraints is a convex set, and identifying the set of extremal points, or vertices, of this set is an interesting and nontrivial question.

Proposition 3.2 tells us that the measure law(ε) we construct is the unique maximizer of the linear functional $\text{law}(\varepsilon) \mapsto \mathbb{E}(\sum_{i=1}^N \epsilon_i)$ on the convex set of exchangeable, pairwise independent Rademacher probability measures. As a consequence, we have therefore identified an extremal point of this set.

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