On the rigidity of nematic liquid crystal flow on $S^2$

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Abstract

In this paper we establish the uniformity property of a simplified Ericksen-Leslie system modelling the hydrodynamics of nematic liquid crystals on the two dimensional unit sphere $S^2$, namely the uniform convergence in $L^2$ to a steady state exponentially as $t$ tends to infinity. The main assumption, similar to Topping [15], concerns the equation of liquid crystal director $d$ and states that at infinity time, a weak limit $d_\infty$ and any bubble $\omega_i$ ($1 \leq i \leq l$) share a common orientation. As consequences, the uniformity property holds under various types of small initial data.

1 Introduction

The Ericksen-Leslie system modelling the hydrodynamics of nematic liquid crystals was proposed by Ericksen and Leslie during the period between 1958 and 1968 [2,4]. It is a macroscopic continuum description of the time evolution of the materials under the influence of both the flow velocity field $u(x,t)$ and the macroscopic description of the microscopic orientation configuration $d(x,t)$ of rod-like liquid crystals (i.e. $d(x,t)$ is a unit vector in $\mathbb{R}^3$). In order to effectively analyze it, Lin [5] proposed a simplified version of the Ericksen-Leslie system, which is called the nematic liquid crystal flow and is given by

$$
\begin{align*}
&u_t + u \cdot \nabla u + \nabla P = \Delta u - \nabla d \Delta d, \\
&\text{div } u = 0, \\
&d_t + u \cdot \nabla d = \Delta d + |\nabla d|^2 d.
\end{align*}
$$

Roughly speaking, the system (1.1) is a coupling between the non-homogeneous Naiver-Stokes equation and the transported heat flow of harmonic maps to $S^2$. The system (1.1) has generated great interests and activities among analysts, and there have been many research works on (1.1) recently (see, e.g. [7] [8] [9] [10] [3] [18] and [11]).

In this paper, we are mainly interested in the long time dynamics of the nematic liquid crystal flow (1.1) on $S^2 \times (0, +\infty)$, where $S^2 \hookrightarrow \mathbb{R}^3$ is the unit sphere that is equipped with the standard metric $g_0$. To begin with, we introduce some notations and explain terms in (1.1).

First, the fluid velocity field $u(x,t) \in T_xS^2 \times \{t\} \equiv T_xS^2$, the liquid crystal molecule director field $d(x,t) \in S^2$, and the pressure function $P(x,t) \in \mathbb{R}$ for $(x,t) \in S^2 \times (0, +\infty)$. In the system (1.1), $\nabla$ stands for the gradient operator on $S^2$, $\text{div}(= \text{div}_{g_0})$ and $\Delta(= \Delta_{g_0} = \text{div}_{g_0} \nabla)$

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represent the divergence operator and the Laplace-Beltrami operator on \((S^2, g_0)\). Hence the second equation \(1.1_2\) describes the incompressibility of the fluid. The convection term \(u \cdot \nabla u\) in the first equation \(1.1_1\) is the directed differentiation of \(u\) with respect to the direction \(u\) itself, which is interpreted as the covariant derivative \(D_u u\). Here \(D\) denotes the covariant derivative operator on \((S^2, g_0)\). Following the terminology in \(1.1\), we know that \(\Delta u\) is also accounted for by the Bochner Laplacian operator \(-D^* D u\), where \(D^*\) is the adjoint operator of \(D\) with respect to the \(L^2(S^2, TS^2)\) inner product. Write \(d = (d^1, d^2, d^3) \in S^2 \rightarrow \mathbb{R}^3\). Then in \(1.1_3\), \(\nabla d \Delta d = \sum_{i=1}^{3} \nabla d^i \Delta d^i\), the \(i\)-th component of \(u \cdot \nabla d\) equals to \(g_0(u, \nabla d^i)\), \(\Delta d = (\Delta g_0 d^1, \Delta g_0 d^2, \Delta g_0 d^3)\), and \(|\nabla d|^2 = g_0(\nabla d, \nabla d)\).

We will consider the nematic liquid crystal flow \((1.1)\) along with the initial condition:

\[
(u(x,0), d(x,0)) = (u_0(x), d_0(x)), \quad x \in S^2, \quad (1.2)
\]

where \(u_0 \in L^2(S^2, TS^2)\) is a divergence free tangential vector field on \((S^2, g_0)\) with zero average:

\[
\int_{S^2} u_0(x) \, dv_{g_0} = 0 \quad \text{and} \quad \text{div} \, u_0 = 0 \quad \text{on} \quad S^2, \quad (1.3)
\]

and \(d_0 \in H^1(S^2, S^2) := \left\{ d \in H^1(S^2, \mathbb{R}^3) : \, d(x) \in \mathbb{R}^3 \quad \text{a.e.} \, \, x \in S^2 \right\}\).

In the recent paper \([6]\), Lin, Lin, and Wang have proved the global existence of weak solutions to the initial and boundary value problem of the nematic liquid crystal flow \((1.1)\) on any bounded domain \(\Omega \subset \mathbb{R}^2\), that are smooth away from at most finitely many singular times (see also \([3]\) for discussion in \(\mathbb{R}^2\)). The uniqueness of the global weak solution constructed by \([6]\) has been given by \([10]\) (see \([18]\) for a different proof). It is an open and challenging problem whether there exists a global weak solution to the problem \((1.1)-(1.2)\) in \(\mathbb{R}^3\). On the other hand, concerning the long time behavior of the global solution established in \([6]\), the authors have only shown the subsequential convergence at time infinity. Since the structure of the set of equilibrium to the problem \((1.1)\) is a continuum, whether the limiting point of \((1.1)\) at time infinity is unique is a very interesting problem.

The aim of this paper is to study this problem for \((1.1)-(1.2)\) under certain assumptions on the initial data. The key ingredient is to apply a Łojasiewicz-Simon type approach, which was originally studied by L. Simon in \([12]\) for a large class of nonlinear geometric evolution equations. Due to the nonlinear constraint \(|d| = 1\), we cannot establish the related Łojasiewicz-Simon type inequality for the Ginzburg-Landau approximation version of \((1.1)\) as in \([10]\)\([17]\). However, there is a counterpart for our problem if we consider the domain being the unit sphere, namely the heat flow of harmonic maps from \(S^2\) to \(S^2\) considered by Topping \([15]\).

We would like to point that by slightly modifying the proof of the regularity Theorem 1.2 in \([6]\), the existence Theorem 1.3 of \([6]\) and the uniqueness theorem of \([10]\) can be extended to \((1.1)-(1.2)\) on \(S^2\) without much difficulty. More precisely, we have

**Theorem 1.1.** Suppose \(u_0 \in L^2(S^2, TS^2)\) satisfies \((1.3)\) and \(d_0 \in H^1(S^2, S^2)\). Then there exists a unique, global weak solution \((u, d) \in (L^\infty_t L^2_x(S^2 \times [0, +\infty), TS^2)) \times (L^\infty_t H^1_x(S^2 \times [0, +\infty), S^2))\) of \((1.1)-(1.2)\) such that the following properties hold:
Unless explicitly specified, henceforth we will not distinguish the inner product between $T \mathbb{S}^2$ and $\mathbb{R}^3$, nor $d \in \mathbb{S}^2$ and its isometric embedding $(d^1, d^2, d^3)$ into $\mathbb{R}^3$. For the sake of simplicity, $\| \cdot \|_{L^2(\mathbb{S}^2)}$ will be shorthanded by $\| \cdot \|$, and $\| \cdot \|_{L^p(\mathbb{S}^2)}$ will be abbreviated by $\| \cdot \|_{L^p}$ for $p \neq 2$.

We will establish the $L^2$-convergence of the flow $(u, d)$ to (1.1)-(1.3) to a single steady state solution $(0, d_\infty)$, with $d_\infty \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ a harmonic map, as $t$ tends to infinity. We first recall some notations introduced by Topping [15].

### 2 Uniform limit in $L^2(\mathbb{S}^2)$

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Let us consider a map \( d \in H^1(S^2, S^2) \). We use \( z = x + iy \) as a complex coordinate on the domain \( S^2 \equiv \mathbb{C} \), via the stereographic projection. Set \( dz = dx + idy, d\bar{z} = dx - idy \), and write the metric \( g_0 \) on the domain \( S^2 \) as \( \sigma(z)^2 dz d\bar{z} \), where

\[
\sigma(z) = \frac{2}{1 + |z|^2}, \quad z \in \mathbb{C}
\]

Let \( u \) denote a complex coordinate on the target \( S^2 \), and write the metric \( g_0 \) on the target \( S^2 \) as \( \sigma(u)^2 du d\bar{u} \). and write

\[
d_z = \frac{1}{2}(d_x - id_y), \quad d_{\bar{z}} = \frac{1}{2}(d_x + id_y).
\]

The \( \partial \)-energy and \( \bar{\partial} \)-energy of \( d \) are given by

\[
E_{\partial}(d) = \frac{i}{2} \int_{C} \rho^2(d)|dz|^2 dz \wedge d\bar{z}, \quad E_{\bar{\partial}}(d) = \frac{i}{2} \int_{C} \rho^2(d)|d\bar{z}|^2 dz \wedge d\bar{z}.
\]

Recall the Dirichlet energy of \( d \) is defined by

\[
E(d) := \frac{1}{2} \int_{S^2} |\nabla d|^2 dv_{g_0}.
\]

It is easy to see that

\[
E(d) = E_{\partial}(d) + E_{\bar{\partial}}(d), \quad (2.1)
\]

\[
4\pi \deg(d) = E_{\partial}(d) - E_{\bar{\partial}}(d). \quad (2.2)
\]

Here \( \deg(d) \) denotes the topological degree of \( d : S^2 \rightarrow S^2 \), which is well-defined for maps \( d \in H^1(S^2, S^2) \) (see Brezis-Nirenberg [1]).

**Theorem 2.1.** There exist \( \epsilon_0 > 0 \) and \( T_0 \geq 1 \) such that if \( (u, d) : S^2 \times (0, +\infty) \rightarrow TS^2 \times S^2 \) is the global solution of the nematic liquid crystal flow (1.1)-(1.3) obtained by Theorem 1.1 satisfying

\[
\frac{1}{2}\|u(T_0)\|^2 + 2 \min \left\{ E_{\partial}(d(T_0)), E_{\bar{\partial}}(d(T_0)) \right\} \leq \epsilon_0. \quad (2.3)
\]

then there exist a smooth harmonic map \( d_\infty \in C^\infty(S^2, S^2) \), a nonnegative integer \( k \), and \( C_1, C_2 > 0 \) such that

(i) as \( t \rightarrow +\infty \), it holds that

\[
u(t) \rightarrow 0 \text{ strongly in } H^1(S^2), d(t) \rightarrow d_\infty \text{ weakly in } H^1(S^2) \text{ and strongly in } L^2(S^2).
\]

(ii)

\[
\|u(t)\| + \|d(t) - d_\infty\| \leq C_1 e^{-C_2 t}, \quad \forall t \geq T_0.
\]

(iii)

\[
|E(d(t)) - E(d_\infty) - 4\pi k| \leq C_1 e^{-C_2 t}, \quad \forall t \geq T_0.
\]

In order to prove Theorem 2.1 we need a key estimate, originally due to Topping [15], which provides a way to control the \( \partial \)-energy of \( d \) (or \( \bar{\partial} \)-energy of \( d \)) in terms of its tension field.
Lemma 2.1. There exist $\epsilon_0 > 0$ and $C_0 > 0$ such that if $d \in H^1(S^2, S^2)$ satisfies

$$\min \left\{ E_\theta(d), E_\theta(d) \right\} < \epsilon_0. \quad (2.6)$$

Then

$$\min \left\{ E_\theta(d), E_\theta(d) \right\} \leq C_0 \int_{S^2} |\Delta d + |\nabla d|d|^2| \, dv_{g_0}. \quad (2.7)$$

We need the energy inequality of the solution of (1.1)-(1.2) obtained by Theorem 1.1.

Lemma 2.2. Assume that $(u, d) : S^2 \times [0, +\infty) \to T S^2 \times S^2$ is the solution of (1.7)-(1.8) obtained by Theorem 1.1. Then for any $t \in (0, +\infty) \setminus \{ T_i \}_{i=1}^L$, the following basic energy law holds:

$$\frac{d}{dt} \left[ \frac{1}{2} \int_{S^2} |u|^2 \, dv_{g_0} + E(d) \right] = -\int_{S^2} \left( |\nabla u|^2 + |\Delta d + |\nabla d|^2 d|^2 \right) \, dv_{g_0} \quad (2.8)$$

Proof. (2.8) follows by multiplying both sides of the equation (1.1) by $u$ and both sides of the equation (1.3) by $\Delta d + |\nabla d|^2 d$ and integrating the resulting equations over $S^2$. We refer the interested readers to [6] Lemma 4.1 for the detail.

We also need the following simple fact on the average of $u$.

Lemma 2.3. Assume that $(u, d) : S^2 \times [0, +\infty) \to T S^2 \times S^2$ is the solution of (1.7)-(1.8) obtained by Theorem 1.1. If the condition (1.3) holds, then

$$\int_{S^2} u(x, t) \, dv_{g_0} = 0, \forall t \geq 0. \quad (2.9)$$

Proof. Under the assumption (1.1), (2.10) follows by integrating the equation (1.1) over $S^2$ and the fact that $\Delta d \nabla d = \text{div} (\nabla d \otimes \nabla d - \frac{1}{2} |\nabla d|^2)$. Here $(\nabla d \otimes \nabla d)_{ij} = \langle \frac{\partial d}{\partial x_i}, \frac{\partial d}{\partial x_j} \rangle$, for $1 \leq i, j \leq 3$.

Now we give the proof of Theorem 2.1.

Proof. Let $\epsilon_0 > 0$ be given by Lemma 2.1 and $T_0 \geq 1$ be sufficiently large so that $(u, d) \in C^\infty(S^2 \times [T_0, +\infty))$. Without loss of generality, we may assume that

$$E_\theta(d(T_0)) \leq E_\theta(d(T_0)).$$

It follows from (2.1) and (2.2) that

$$E_\theta(d) = \frac{1}{2} \left[ |E(d) + 4\pi \text{deg}(d)\right].$$

Since $\text{deg}(d(t))$ is constant for $t \geq T_0$, the basic energy law (2.8) implies that

$$\frac{d}{dt} \left[ \frac{1}{2} \|u\|^2 + 2E_\theta(d) \right] = \frac{d}{dt} \left[ \frac{1}{2} \|u\|^2 + E(d) \right] = -\left( \|\nabla u\|^2 + \|\Delta d + |\nabla d|^2 d|^2 \right), \forall t \geq T_0. \quad (2.10)$$

Therefore, we have that

$$\frac{1}{2} \|u(t)\|^2 + 2E_\theta(d(t)) \leq \frac{1}{2} \|u(T_0)\|^2 + 2E_\theta(d(T_0)) \leq \epsilon_0, \forall t \geq T_0. \quad (2.11)$$
Applying Lemma 2.1 to $d(t)$, and Lemma 2.3 and Poincaré’s inequality for $u(t)$, we have

$$\frac{1}{2} \left\| u(t) \right\|^2 + 2E_0(d(t)) \leq C\left[ \left\| \nabla u(t) \right\|^2 + \left\| \Delta d(t) + |\nabla d|^2 d(t) \right\|^2 \right], \forall t \geq T_0. \tag{2.12}$$

Putting (2.12) together with (2.10), we obtain that for all $t \geq T_0$,

$$-\frac{d}{dt} \left[ \frac{1}{2} \left\| u \right\|^2 + 2E_0(d) \right] = -\frac{d}{dt} \left[ \frac{1}{2} \left\| u \right\|^2 + 2E_0(d) \right] \leq \frac{C}{\left[ \left\| \nabla u \right\|^2 + \left\| \Delta d + |\nabla d|^2 d \right\|^2 \right]^{\frac{1}{2}}} \geq C \left[ \left\| \nabla u \right\| + \left\| \Delta d + |\nabla d|^2 d \right\| \right] \geq C \left[ \frac{1}{2} \left\| u \right\|^2 + 2E_0(d) \right]^{\frac{1}{2}}. \tag{2.13}$$

Thus by Gronwall’s inequality we have

$$\left[ \left\| u(t) \right\|^2 + E_0(d(t)) \right] \leq \left[ \left\| u(T_0) \right\|^2 + E_0(d(T_0)) \right] e^{-C(t-T_0)} \leq C e^{-C(t-T_0)}, \forall t \geq T_0. \tag{2.14}$$

By integrating (2.13) over $[t, +\infty)$, we obtain that for any $t \geq 2T_0$,

$$\int_t^{\infty} \left( \left\| \nabla u \right\| + \left\| \Delta d + |\nabla d|^2 d \right\| \right) \, d\tau \leq C \left[ \left\| u(t) \right\|^2 + E_0(d(t)) \right]^{\frac{1}{2}} \leq C_1 e^{-C_2 t}. \tag{2.15}$$

Consequently, we infer from (1.13), (2.8), (2.13), (2.14) and (2.15) that for any $t_2 \geq t \geq 2T_0$,

$$\left\| d(t_2) - d(t) \right\|_{L^1} \leq \int_t^{t_2} \left\| d \right\|_{L^1} \, d\tau \\
\leq \int_t^{t_2} \left\| u \cdot \nabla d \right\|_{L^1} \, d\tau + \int_t^{t_2} \left\| \Delta d + |\nabla d|^2 d \right\|_{L^1} \, d\tau \\
\leq C \left[ \int_t^{t_2} \left\| u \right\|_{L^2} \left\| \nabla d \right\|_{L^2} \, d\tau + 2\sqrt{\pi} \int_t^{t_2} \left\| \Delta d + |\nabla d|^2 d \right\|_{L^2} \, d\tau \right] \\
\leq C \left[ \left\| u(t) \right\|^2 + E_0(d(t)) \right]^{\frac{1}{2}} \leq C_1 e^{-C_2 t}. \tag{2.16}$$

Thus

$$\left\| d(t_2) - d(t) \right\|_{L^2} \leq 2 \left\| d(t_2) - d(t) \right\|_{L^1} \leq C_1 e^{-C_2 t}, \tag{2.17}$$

which indicates that as $t \to +\infty$, $d(t)$ converges in $L^2(S^2)$. It follows from Theorem 1.1 (iii) that there exist a smooth harmonic map $d_\infty \in C^\infty(S^2, S^2)$, nontrivial harmonic maps $\{\omega_i\}_{i=1}^l$ for some nonnegative integer $l$, and a sequence $t_i \to +\infty$ such that

$$\begin{cases} d(t_i) \to d_\infty \text{ weakly in } H^1(S^2, S^2) \text{ and strongly in } L^2(S^2, S^2), \\
\int_{S^2} |\nabla d(t_i)|^2 \, dv_{g_0} \to \int_{S^2} |\nabla d_\infty|^2 \, dv_{g_0} + \sum_{i=1}^l \int_{S^2} |\nabla \omega_i|^2 \, dv_{g_0}. \end{cases} \tag{2.18}$$
Therefore, by choosing $t_2 = t_i$ in (2.17) and sending $i$ to $\infty$, we conclude that

$$\|d(t) - d_\infty\|_{L^2} \leq C_1 e^{-C_2 t}, \quad \forall t \geq 2T_0. \tag{2.19}$$

In particular, (2.19) implies that $d(t)$ converges to $d_\infty$ in $L^2(S^2)$ as $t \to +\infty$. To show that $d(t)$ converges to $d_\infty$ weakly in $H^1(S^2)$. Let $\{t_j\}$ be any sequence tending to $+\infty$. Since $\{d(t_j)\}$ is bounded in $H^1(S^2)$, there exists a subsequence $t_{j'} \to +\infty$ such that $d(t_{j'})$ weakly converges in $H^1(S^2)$ and strongly in $L^2(S^2)$ to a map $d_\ast \in H^1(S^2, S^2)$. Hence $d_\ast = d_\infty$. This shows that $d(t)$ converges to $d_\infty$ weakly in $H^1(S^2)$ as $t \to +\infty$. Hence (i) is proven.

To show (iii), integrating (2.10) over $[t, t_2]$ for $t \geq 2T_0$ and applying (2.13) yields

$$\frac{1}{2}\|u(t)\|^2 + E(d(t)) \leq \frac{1}{2}\|u(t_2)\|^2 + E(d(t_2)) \tag{2.20}$$

This implies

$$|E(d(t) - E(d(t_2))| \leq (\|u(t)\|^2 + \|u(t_2)\|^2) + C_1 e^{-C_2 t}, \quad \forall t \geq 2T_0. \tag{2.21}$$

Let $t_2 \to +\infty$ be the sequence such that (2.13) holds. Since each harmonic map $\omega_i$, $1 \leq i \leq l$, is nontrivial and has its energy

$$\int_{S^2} |\nabla \omega_i|^2 dv_{g_0} = 8\pi m_i,$$

for some positive integer $m_i$, there exists a nonnegative integer $k$ such that

$$\lim_{t_2 \to +\infty} E(d(t_2)) = E(d_\infty) + 8\pi k.$$

Sending $t_2$ to infinity in (2.21), this implies

$$|E(d(t) - E(d_\infty) - 8\pi k| \leq C_1 e^{-C_2 t}, \quad \forall t \geq 2T_0. \tag{2.22}$$

The proof is now complete. \(\square\)

It is well-known that any harmonic map from $S^2$ to $S^2$ is either holomorphic or anti-holomorphic. Inspired by [15] Theorem 2, we have

**Corollary 2.1.** Suppose $(u, d) : S^2 \times (0, +\infty) \to TS^2 \times S^2$ is the global solution of the nematic liquid crystal flow (1.1)-(1.3) obtained by Theorem 1.1.

(i) Suppose that the weak limit $d_\infty$ and the bubbles $\omega_i$ ($1 \leq i \leq l$), associated with a sequence $t_i \uparrow +\infty$, are all holomorphic or all anti-holomorphic. Then there exist a nonnegative integer $k$, and $C_1, C_2 > 0$ such that

$$u(t) \to 0 \text{ strongly in } H^1(S^2), d(t) \to d_\infty \text{ weakly in } H^1(S^2) \text{ and strongly in } H^1(S^2), \text{ as } t \to +\infty,$$

and

$$\|u(t)\| + \|d(t) - d_\infty\| + |E(d(t)) - E(d_\infty) - 4\pi k| \leq C_1 e^{-C_2 t}, \quad \forall t \geq T_0. \tag{2.22}$$

(ii) The same conclusions of (i) hold if the initial data $(u_0, d_0)$ satisfies

$$\frac{1}{2}\|u_0\|^2 + 2\min \left\{E_\partial(d_0), E_\partial(d_0)\right\} < 8\pi. \tag{2.23}$$
Proof. For the part (i), it suffices to verify that the condition (2.3) holds. For simplicity, assume that \( d_\infty \) and all \( \omega_i \)'s are anti-holomorphic. Thus we have
\[
E_\partial(d_\infty) = E_\partial(\omega_1) = \cdots = E_\partial(\omega_l) = 0.
\]
From (1.5), we know that
\[
\lim_{t_i \uparrow +\infty} \frac{1}{2} \|u(t_i)\|^2 + E_\partial(d(t_i)) = E_\partial(d_\infty) + \sum_{i=1}^l E_\partial(\omega_i) = 0.
\]
This clearly implies that there exists a sufficiently large \( i_0 \) such that
\[
\left[ \frac{1}{2} \|u(t_{i_0})\|^2 + 2E_\partial(d(t_{i_0})) \right] \leq \epsilon_0,
\]
which implies (2.3).

For the part (ii), we will show that (2.23) implies the condition in the part (i). For simplicity, assume that \( E_\partial(d_0) \leq E_\partial(d_0) \). Let \( 0 < T_1 < +\infty \) be the first singular time of the flow (1.1)-(1.3). Since \( (u(t), d(t)) \in C^\infty(S^2 \times (0, T_1)) \) and
\[
\lim_{t \downarrow 0} \left[ \|u(t) - u_0\| + \|\nabla(d(t) - d_0)\| \right] = 0,
\]
it is not hard to see
\[
\deg(d(t)) = \deg(d_0), \forall 0 \leq t < T_1.
\]
Thus the basic energy law (2.8) implies that
\[
\frac{d}{dt} \left( \frac{1}{2} \|u\|^2 + 2E_\partial(d) \right) = \frac{d}{dt} \left[ \frac{1}{2} \|u\|^2 + E(d) \right] \leq 0. \tag{2.24}
\]
Integrating (2.24) from 0 to 0 < \( t \leq T_1 \) yields
\[
\frac{1}{2} \|u(t)\|^2 + 2E_\partial(d(t)) \leq \frac{1}{2} \|u_0\|^2 + 2E_\partial(d_0), \quad 0 \leq t \leq T_1. \tag{2.25}
\]
Let \( T_2 \in (T_1, +\infty) \) be the second singular time. Then the same argument yields
\[
\frac{1}{2} \|u(t)\|^2 + 2E_\partial(d(t)) \leq \frac{1}{2} \|u(T_2)\|^2 + 2E_\partial(d(T_2)), \quad T_1 \leq t \leq T_2. \tag{2.26}
\]
Since there are at most finitely many finite singular times \( \{T_i\}_{i=1}^L \) for the flow (1.1)-(1.3), by repeating the argument we would reach that for any \( t \geq 0 \), it holds that
\[
E(t) := \frac{1}{2} \|u(t)\|^2 + 2E_\partial(d(t)) \leq \frac{1}{2} \|u_0\|^2 + 2E_\partial(d_0)
= \frac{1}{2} \|u_0\|^2 + 2\min \{ E_\partial(d_0), E_\partial(d_0) \}
< \frac{1}{2} \|u_0\|^2 + 2\min \{ E_\partial(d_0), E_\partial(d_0) \} \tag{2.27}
\]
By the lower semicontinuity, we have that
\[
2E_\partial(d_\infty) \leq \lim_{t \to \infty} E(t) < 8\pi,
\]
and
\[
2E_\partial(\omega_i) \leq \lim_{t \to \infty} E(t) < 8\pi, \quad 1 \leq i \leq l.
\]
This implies that \( \omega_1, \cdots, \omega_l \) are all nontrivial anti-holomorphic maps. If \( d_\infty \) is not a constant, then \( d_\infty \) has to be anti-holomorphic. Therefore \( d_\infty \) and all \( \omega_i \)'s are anti-holomorphic.

Thus the conclusions in (i) and (ii) follow from Theorem (2.1). The proof is complete. \( \square \)
3 Uniform limit in $H^k(S^2)$ for $k \geq 1$

This subsection is to consider the convergence issues of the nematic liquid crystal flow (1.1)-(1.3) in higher order Sobolev spaces at $t = +\infty$.

**Theorem 3.1.** Suppose $(u, d) : S^2 \times [0, +\infty) \to T S^2 \times S^2$ is the global solution of (1.1)-(1.3) obtained by Theorem 1.1. Suppose that there exist a sequence $t_i \uparrow +\infty$ and a smooth harmonic map $d_\infty \in C^\infty(S^2, S^2)$ such that

$$\lim_{t_i \uparrow +\infty} \left[ \|u(t_i)\| + \|d(t_i) - d_\infty\| + \|\nabla (d(t_i) - d_\infty)\| \right] = 0. \quad (3.2)$$

Then for any $k \geq 1$ there exist $C_1, C_2 > 0$ depending only on $k$ such that

$$\left\| u(t) \right\|_{H^k(S^2)} + \left\| d(t) - d_\infty \right\|_{H^k(S^2)} \leq C_1 e^{-C_2t}. \quad (3.3)$$

In particular, for any $k \geq 1$, $d(t) \to d_\infty$ in $H^k(S^2)$ as $t \to +\infty$.

**Proof.** For simplicity, assume that $d_\infty$ is anti-holomorphic, i.e., $\partial_z d_\infty \equiv 0$. Thus we have

$$E_\partial(d(t_i)) = E_\partial(d(t_i) - d_\infty) \leq E(d(t_i) - d_\infty).$$

This, combined with (3.2), implies

$$\lim_{t_i \uparrow +\infty} \left[ \|u(t_i)\| + E_\partial(d(t_i)) \right] = 0. \quad (3.4)$$

Hence we can apply Theorem 2.1 to conclude that

$u(t) \to 0$ strongly in $H^1(S^2)$, $d(t) \to d_\infty$ weakly in $H^1(S^2)$ and strongly in $L^2(S^2)$, as $t \to \infty$, and

$$\|u(t)\| + \|d(t) - d_\infty\| \leq C_1 e^{-C_2t}. \quad (3.5)$$

Since it follows from the basic energy law (2.5) and

$$\lim_{t_i \uparrow +\infty} \left( \frac{1}{2} \|u(t_i)\|^2 + E(d(t_i)) \right) = E(d_\infty)$$

that

$$\lim_{t \to +\infty} E(d(t)) = E(d_\infty),$$

we can conclude that

$$\lim_{t \to +\infty} \left\| d(t) - d_\infty \right\|_{H^1(S^2)} = 0. \quad (3.6)$$

For $\epsilon_1 > 0$, let $r_0 = r_0(\epsilon_1) > 0$ be such that

$$\max_{x \in S^2} \int_{B_{r_0}(x) \cap S^2} |\nabla d_\infty|^2 \, dv_{g_0} \leq \frac{\epsilon_1}{2}.$$

By (3.6), there exists $T_0 > 0$ such that

$$\sup_{t \geq T} \max_{x \in S^2} \int_{B_{r_0}(x) \cap S^2} |\nabla d(t)|^2 \, dv_{g_0} \leq \epsilon_1. \quad (3.7)$$
As in [10], (3.7) then implies the following inequality:
\[
\int_{T_0}^\infty \int_{S^2} (|\nabla u|^2 + |\nabla^2 d|^2) \leq C \left( \|u(T_0)\|^2 + E(d(T_0)) \right). \tag{3.8}
\]

With the estimates (3.7) and (3.8), we can apply the regularity Theorem 1.2 of [6] to get that for any \( k \geq 0 \), there exists \( C_k > 0 \) such that
\[
\sup_{t \geq T_0} \left( \|u(t)\|_{C^{k}(S^2)} + \|d(t)\|_{C^{k+1}(S^2)} \right) \leq C_k. \tag{3.9}
\]

By standard interpolation inequalities, (3.7) and (3.8) imply that (3.3) holds. The proof is now complete.

It is an interesting question to find sufficient conditions that guarantee the global solution \((u, d): S^2 \times [0, +\infty) \to TS^2 \times S^2\) of the flow (1.1)-(1.3) by Theorem 1.1 has a sequence \( t_i \uparrow +\infty \) such that \((u(t_i), d(t_i)) \to (0, d_\infty)\) strongly in \( L^2(S^2) \times H^1(S^2) \).

In this context, we have the following result.

**Corollary 3.1.** Suppose \((u, d): S^2 \times [0, +\infty) \to TS^2 \times S^2\) is the global solution of (1.1)-(1.3) by Theorem 1.1. Then there exists a smooth harmonic map \( d_\infty \in C^\infty(S^2, S^2) \) such that for any \( k \geq 1 \),
\[
\|u(t)\|_{H^k(S^2)} + \|d(t) - d_\infty\|_{H^k(S^2)} \leq C_k e^{-C_k t}, \tag{3.10}
\]
provided that one of the following conditions holds:

i) \( \frac{1}{2} \|u_0\|^2 + E(d_0) \leq 4\pi \).

ii) \( d_0(S^2) \) is contained in the hemisphere (e.g. \( d_0^3 \geq 0 \)).

iii) there exists \( C_L > 0 \) such that \((u_0, d_0)\) satisfies
\[
\exp \left( 108 C_L^8 \left( \|u_0\|^2 + \frac{1}{8C_L^4} \right)^2 \right) \|\nabla d_0\|^2 \leq \frac{1}{8C_L^4}. \tag{3.11}
\]

**Proof.** We will establish that under any one of the three conditions, there exists a sequence \( t_i \uparrow +\infty \) such that \( u(t_i) \to 0 \) in \( H^1(S^2) \) and \( d(t_i) \) is strongly convergent in \( H^1(S^2) \).

Let us first consider the condition (i). It has been proved by Theorem 1.1 (iv) that \((u, d)\) has neither finite time singularity nor energy concentration at \( t = \infty \). In particular, \((u, d) \in C^\infty(S^2 \times (0, +\infty))\), and there exists \( t_i \uparrow +\infty \) and a harmonic map \( d_\infty \in C^\infty(S^2, S^2) \) such that
\[
\|u(t_i)\| + \|d(t_i) - d_\infty\|_{H^1(S^2)} \to 0.
\]

Moreover, it has been shown in [10] that \( d_\infty \) is constant, unless \( \left( \frac{1}{2} \|u_0\|^2 + E(d_0) \right) = 4\pi \) which would imply that \( u \equiv u_0 \equiv 0 \), and \( d \equiv d_0 \equiv d_\infty \) is a harmonic map of degree one.

Now let us consider the condition (ii). Since \( d_0^3 \geq 0 \), it follows from the maximum principle on the equation (1.1), that \( d^3(t) \geq 0 \) for all \( t \geq 0 \). Since there doesn’t exist non-constant harmonic maps from \( S^2 \) to the hemisphere \( S^2_+ \), we can apply [9] Theorem 1.3 to conclude that there is neither finite time singularity nor any energy concentration at \( t = \infty \) for \((u, d)\) (see also [11] for a different proof). In particular, there exist a point \( p \in S^2_+ \) and \( t_i \uparrow +\infty \) such that
\[
\|u(t_i)\| + \|d(t_i) - p\|_{H^1(S^2)} \to 0.
\]
Finally let us consider the condition (iii). Recall that under the condition (iii), it has been proven by Xu-Zhang [18] that \((u, d)\) is smooth when the domain is \(\mathbb{R}^2\). Here we indicate how to extend the argument by [18] to \(S^2\).

Multiplying both sides of the equation \((1.1)_3\) by \(-\langle \Delta d + |\nabla d|^2 d \rangle\) and integrating the resulting equation over \(S^2\) yields

\[
\frac{1}{2} \frac{d}{dt} \|\nabla d\|^2 + \int_{S^2} |\Delta d + |d|^2 d|^2 \, dv_g = \int_{S^2} u \cdot \nabla d \Delta d \, dv_g. \tag{3.12}
\]

To estimate the right side of \((3.12)\), we apply the Ricci identity on \((S^2, g_0)\) and the Poincaré inequality for \(u\) and \(\nabla d\) to obtain

\[
\int_{S^2} |\Delta d|^2 \, dv_g = \int_{S^2} (|\nabla d|^2 + |d|^2) \, dv_g \geq \int_{S^2} |\nabla d|^2 \, dv_g, \tag{3.13}
\]

\[
\|u\| \leq C_p \|\nabla u\| \quad \text{and} \quad \|\nabla d\| \leq C_p \|\nabla d\| \leq C_p \|\Delta d\|, \tag{3.14}
\]

where \(C_p > 0\) is the constant in the Poincaré inequality. Also recall the Ladyzhenskaya inequality on \(S^2\):

\[
\|f\|_{L^4} \leq C_1 \|f\|_1^\frac{1}{2} \|\nabla f\|_1^\frac{1}{2} + C_2 \|f\|. \tag{3.15}
\]

Combining these inequalities with Young’s inequality and \((2.8)\), we have

\[
\left| \int_{S^2} u \cdot \nabla d \Delta d \, dv_g \right| \leq \|\Delta d\| \|\nabla d\|_{L^4} \|u\|_{L^4}
\leq \||\Delta d||C_1 \|\nabla d\|_2^\frac{1}{2} \|\Delta d\|_2^\frac{1}{2} \|C_2 \|\nabla d\|_2^\frac{1}{2} + C_2 \|u\|_2 + C_2 \|u\|_2 \|
\leq \|\Delta d\|^\frac{1}{2} \|\nabla d\|_2^\frac{1}{2} \|u\|_2 \|\nabla u\|_2 + C_2 \|\Delta d\|_2^\frac{1}{2} \|\nabla u\|_2 \|\nabla d\|_2
\leq \frac{1}{8} \|\Delta d\|^2 + 54 C_8^2 \|u\|_2^2 \|\nabla u\|_2^2 \|\nabla d\|_2^2 \tag{3.16}
\]

Here

\[
C_L = (C_1 + C_2 C_8^2). \tag{3.16}
\]

On the other hand, since

\[
|\Delta d + |\nabla d|^2 d|^2 = |\Delta d|^2 - |\nabla d|^4,
\]

we have

\[
\int_{S^2} \left| |\nabla d|^2 + \Delta d\right|^2 \, dv_g = \|\Delta d\|^2 - \|\nabla d\|^4 \geq \|\Delta d\|^2 - C_L^2 \|\nabla d\|^2 \|\Delta d\|^2. \tag{3.17}
\]

If \(d_0\) satisfies \(\|\nabla d_0\|^2 < \frac{1}{8 C_L^2}\), then there exists \(T_1 > 0\) such that for any \(t \in [0, T_1]\),

\[
\|\nabla d(t)\|^2 \leq \frac{1}{8 C_L^2}. \tag{3.18}
\]

Let \(T_1^*\) denote the maximal time such that \((3.18)\) holds on \([0, T_1]\). Then, by \((3.12)-(3.17)\) we have that for any \(t \in [0, T_1^*]\),

\[
\frac{d}{dt} \|\nabla d\|^2 + \frac{1}{4} \|\Delta d\|^2 \leq 108 C_L^2 \left(\|u_0\|^2 + \frac{1}{8 C_L^4}\right) \|\nabla u\|^2 \|\nabla d\|_2^2. \tag{3.19}
\]
Using Gronwall’s inequality and (2.8), we deduce from (3.19) that for any \(0 \leq t \leq T^*_1\),

\[
\|\nabla d(t)\|^2 + \frac{1}{4} \int_0^t \|\Delta d(\tau)\|^2 d\tau \leq \exp \left(108C_L^8 (\|u_0\|^2 + \frac{1}{8C_L^2}) \int_0^{T^*_1} \|\nabla u(\tau)\|^2 d\tau \right) \|\nabla d_0\|^2
\]

\[
\leq \exp \left(108C_L^8 (\|u_0\|^2 + \frac{1}{8C_L^2})^2 \|\nabla d_0\|^2 \right) \|\nabla d_0\|^2,
\]

(3.20)

which implies that \(T^*_1 = T\) and

\[
\|\nabla d(t)\|^2 + \frac{1}{4} \int_0^t \|\Delta d(\tau)\|^2 d\tau \leq \frac{1}{8C_L^2},
\]

(3.21)

holds for all \(0 \leq t \leq T\), provided that \((u_0, d_0)\) satisfies

\[
\exp \left(108C_L^8 (\|u_0\|^2 + \frac{1}{8C_L^2})^2 \|\nabla d_0\|^2 \right) \leq \frac{1}{8C_L^2}.
\]

(3.22)

Let \(T^*\) be the maximal existence time for the solution \((u, d)\). Then (2.8) and (3.21) ensure that \(T^* = +\infty\) by the continuity argument. Moreover, \((u, d) \in C^\infty(S^2 \times (0, +\infty))\) by Theorem 1.2 in [9]. Note that (3.21) and (3.13) imply that

\[
\int_0^\infty \|\nabla^2 d(\tau)\|^2 d\tau \leq \frac{1}{2C_L^2}.
\]

(3.23)

It follows from (2.8), (3.22), and (3.13) that there is a sequence \(t_i \uparrow +\infty\) such that

\[
\lim_{t_i \uparrow +\infty} \left(\|u(t_i)\| + \|\nabla d(t_i)\| + \|\nabla^2 d(t_i)\|\right) = 0.
\]

In particular, there exists \(p \in S^2\) such that \(d(t_i) \to p\) strongly in \(H^1(S^2)\) as \(t_i \uparrow +\infty\).

We have verified the condition of Theorem 3.1 holds under all the three conditions. Thus the conclusion follows from Theorem 3.1.

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