Physics of Finance

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We give a brief introduction to the Gauge Theory of Arbitrage \[1\]. Treating a calculation of net present values (NPV) and currencies exchanges as a parallel transport in some fibre bundle, we give geometrical interpretation of the interest rate, exchange rates and prices of securities as a proper connection components. This allows us to map the theory of capital market onto the theory of quantized gauge field interacted with a money flow field. The gauge transformations of the matter field correspond to a dilatation (redefinition) of security units which effect is eliminated by a proper tune of the connection. The curvature tensor for the connection consists of the excess returns to the risk-free interest rate for the local arbitrage operation. Free quantum gauge theory is equivalent to the assumption about the log-normal walks of assets prices. In general case the consideration maps the capital market onto QED, i.e. quantum system of particles with positive (securities) and negative (”debts”) charges which interact with each other through electromagnetic field (gauge field of the arbitrage). In the case of a local virtual arbitrage opportunity money flows in the region of configuration space (money poor in the profitable security) while ”debts” try to escape from the region. Entering positive charges and leaving negative ones screen up the profitable fluctuation and restore the equilibrium in the region where there is no arbitrage opportunity any more, i.e. speculators washed out the arbitrage opportunity.

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1 Informal sketch

There is a common belief that nature (and a human society, in particular) could be described in some unified terms and concepts of physics may find their use here. The present paper follows the last point and tries to draw parallels between the theory of financial markets and quantum gauge theory. Since it is difficult to be equally comprehensible for both physicists and economists we give the introduction which is basically intuitive and designed to explain main logic of the consideration. Further sections are more mathematical and use notions of quantum field theory.

1.1 Net present value as a parallel transport

First of all let us recall what the NPV method is. The NPV investment method works on the simple, but fundamental, principle that money have a time value. This time value has to be taken into account through so-called discounting process.

To elucidate the idea we use here an easy example from Ref. [2]: "Suppose you were made an offer: if you pay 500 pounds now, you will immediately receive 200 French Francs, 200 Japanese Yen and 200 German Marks. How would you go about deciding whether the offer was worthwhile? What you certainly would not do is to say: I have to give up 500 pieces of paper and, in return, I will get 600 pieces of paper. As I end up with 100 more pieces of paper, the deal is worthwhile! Instead, you would recognize that in order to evaluate the offer, you have to convert all the different currencies to a common currency, and then undertake the comparison" (please, do not consider this as a Euro propaganda!). In the same way that money in different currencies cannot be compared directly, but first has to be converted to a common currency, money in the same currency, but which occur at different points in time cannot be compared directly, but must first converted to a common point of time. This reflects the time value of money.

Intuitively it is clear that given the choice between £100 now or £100 in one year time, most people would take the £100 now since the money could be placed on deposit (risk free investment) at some interest rate. Then, in one year time the £100 turn itself into £(1 + r)100 instead of keeping £100 only. Therefore $r$ - the interest rate - represents the time value of money.

Skipping here some details (difference between simple and compound interest rate, continuous compounding, flat and effective interest rates [3]) we now are ready to formulate what the NPV is: if an amount of money $F$ is to be received in $T$ years’ time, the Present Value of that amount ($NPV(F)$) is the sum of money $P$ (principal) which, if invested today, would generate the compound amount $F$ in $T$ years’ time:

$$NPV(F) \equiv P = \frac{F}{(1+r)^T}.$$  

The interest rate involved in this calculation is known as the discount rate and the term $(1 + r)^{-T}$ is known as T-year discount factor $D_T$:

$$D_T = (1 + r)^{-T}.$$  \hspace{1cm} (1)
In a similar way, to calculate the present value of a stream of payments, the above formula is applied to each individual payment and the resulting individual present values are then summed. So, the NPV method states that if the NPV of an investment project has zero or positive value, the company should invest in the project; if it has negative value, it should not invest. There is also some use of the NPV for comparative analysis of several projects but we do not stop for the details (see [2]).

What is really important here for our goals is the following geometrical interpretation: discounting procedure plays a role of a "parallel transport" of a money amount through a time (though in fixed currency). The discounting factor (1) is then an element of a structural group of a fibre bundle (which still has to be defined) and the discount rate coincides with the time component of the connection vector field. Let us remind that in the differential geometry the connection field is responsible for pulling a fibre element from one point of a base space to another. Moreover, it is obvious that the "space" components of the connection has something to do with the exchange rates (see the example above). Indeed, the exchange rates or prices are responsible for converting money in different currencies or different securities (read points of discrete "space") to the same currency (point of the space) at a fixed moment of time. So, they can be interpreted as elements of the structural group which "transport" the money in "space" directions and are space analogues of the discount factor. Summing up, we see that there is some analogue between elements of a fibre bundle picture with some connection field and a capital market. We make the analogue precise below.

1.2 Arbitrage as a curvature of the connection

Next keyword is an arbitrage. What then the arbitrage is? Basically it means "to get something from nothing" and a free lunch after all. More strict definition states the arbitrage as an operational opportunity to make a risk-free profit [4] with a rate of return higher than the risk-free interest rate accrued on deposit (formalized version can be found in [5]).

The arbitrage appears in the theory when we consider a curvature of the connection. In more details, a rate of excess return for an elementary arbitrage operation (a difference between rate of return for the operation and the risk-free interest rate) is an element of curvature tensor calculated from the connection. It can be understood keeping in mind that a curvature tensor elements are related to a difference between two results of infinitesimal parallel transports performed in different order. In financial terms it means that the curvature tensor elements measure a difference in gains accrued from two financial operations with the same initial and final points or, in other words, a gain from an arbitrage operation.

In a certain sense, the rate of excess return for an elementary arbitrage operation is an analogue of the electromagnetic field. In an absence of any uncertainty (or, in other words, in an absence of walks of prices, exchange and interest rates) the only state is realised is the state of zero arbitrage. However, if we place the uncertainty in the game, prices and the rates move and some virtual arbitrage possibilities to get more than less appear. Therefore we can say that the uncertainty play the same role in the developing theory as the quantization did for the quantum gauge theory.
Money flows as matter fields

Last principle ingredient to enters the theory is "matter" fields which interact through the connection. It is clear now that the "matter" fields are money flows fields which have to be gauged by the connection. Indeed, we started the introduction of the concept with the example of NPV method which shows how an amount of money units changes while the payment is "pulled" in time or one currency is converted to another.

Dilatations of money units (which do not change a real wealth) play a role of gauge transformation which eliminates the effect of the dilatation by a proper tune of the connection (interest rate, exchange rates, prices and so on) exactly as the Fisher formula does for the real interest rate in the case of an inflation. The symmetry of the real wealth to a local dilatation of money units (security splits and the like) is the gauge symmetry of the theory.

Following a formal analogy with $U(1)$ gauge theory (electrodynamics) case, we can say that an amount of a certain currency units at a particular time moment is analogues to a value of a phase cross section of the $U(1) \times M$ fibre bundle at some space-time point. If we want to compare values of the $U(1) \times M$ cross section at different points of the space-time we have to perform a parallel transport of the phase from one point of the base to another exactly as we have to convert one currency to another or discount money in the financial setting.

A theory may contain several types of the "matter" fields which may differ, for example, by a sign of the connection term as it is for positive and negative charges in the electrodynamics. In the financial stage it means different preferences of investors. Thus, in the paper we deal with cash flows and debts flows which behave differently in the same gauge field: cash tries to maximize it-self while debts try to minimize them-selves. This is equivalent to a behaviour of positive and negative charges in the same electric field where they move in different directions.

Investor’s strategy is not always optimal. It is due to partially incomplete information in hands, choice procedure, partially because of investor (or manager) internal objectives. It means that the money flows are not certain and fluctuates in the same manner as the prices and rates do. So this requires a statistical description of the money flows which, once again, returns us to an effective quantization of the theory.

1.3 All together is QED-like system

Collecting this all together, we are ready to map the capital market to a system of particles with positive (securities) and negative ("debts") charges which interact with each other through electromagnetic field (gauge field of the arbitrage). In the case of a local virtual arbitrage opportunity cash-debts flows in the region of configuration space (money poor in the profitable security) while "debts" try to escape from the region. Entering positive charges and leaving negative ones screen up the profitable fluctuation and restore the equilibrium so that there is no arbitrage opportunity any more. It means speculators wash up the arbitrage.

Taking into account the uncertainty mentioned above, we come to the quantum field theory with the gauge field and "matter" fields of opposite charges. At this point
standard machinery of quantum field theory may be applied to obtain distribution functions of the interest/exchange rates and cash-debts flows correlators (which are essential for the response of the system). It may answer questions about dynamical response of a financial market, dynamical portfolio theory and to be an approach to option pricing and other problems.

Last point to note in the subsection is a principal lattice nature of the capital market theory, since there always exists a natural minimal time interval (transaction time) and the "space" is a graph. This remove usual problems of the quantum field theory which has to deal with divergences due to continuous character of its space-time base.

### 1.4 Basic assumptions

In the previous subsections we discussed the main assumption (or postulate) which is a *local gauge invariance* with the dilatation gauge group. It can be shown that the postulate essentially dictates dynamical rules of the theory. However, there is a number of other assumptions which we want to list here. For the convinience we devide all assumptions into two parts. First of them deals with the prices and rates, i.e. contains assumptions concerning the connection field.

1. Exchange rates, prices of various securities and interest rate fluctuate and it bears local arbitrage opportunities.

2. The most probable configuration of the random connection is the configuration with a minimal absolute value of the excess rate of return.

3. Arbitrage opportunities, in absense of money flows, are uncorrelated in different space-time points, i.e. the events are statistically independant.

4. We suppose exponential distribution of the rate of return on a local arbitrage operation. Its characteristics, in general, depend on both "space" and time coordinates. However, for sake of simplicity, below we do not consider the details.

5. When a continuous limit will be taken, we assume all required smoothness properties of relevant objects.

Second part of main assumtions consists of ones about a behavior of cash-debts flows i.e. the "matter" fields.

1. We assume the perfect capital market enviroment, i.e. there are always a possibility to place money on a deposit and to borrow without any restrictions and at the same interest rate [5].

2. There are transaction costs. Their presense is not just unimportant complication. The transaction costs play a role of inertia for the cash-debts flows and stabilize the system.

3. Investors are (limited) rational and try to maximize their gain from the securities, minimizing debts at the same time.
4. Exactly as the rates and prices may fluctuate and are uncertain, flow trajectories fluctuate around the most profitable trajectory ("classical trajectory"). This reflects the fact that investors not always behave optimally.

Below we repeat the assumptions in more strict form which is useful for further formalisation of the theory.

2 Formal constructions

In this section we formalize the previous consideration. More precise, we give a description of the relevant fibre bundles, we construct the parallel transport rules using for this elements of the structural group and give an interpretation of the parallel transport operators. The corresponded curvature is also defined and it is shown that it is equal to the rate of excess return on the elementary plaquette arbitrage operation. This opens a way to a construction of the dynamics of the parallel transport factors what gives a lattice gauge theory formulation. The procedure of the dynamics construction is repeated then for the case of "matter" fields which represent cash-debts flows.

2.1 Fibre bundle construction

It is well-known that many important concepts in physics can be interpreted in terms of the geometry of fibre bundles. Maxwell’s theory of electromagnetism and Yang-Mills theories are essentially theories of the connections on principal bundles with a given gauge group $G$ as the fibre. Einstein’s theory of gravitation deals with the Levi-Civita connection on the frame bundle of the space-time manifold. In the section we show how the construction of fibre bundle can also be applied to describe a framework to develop the capital market theory.

**Construction of the base**

Now we have to construct a base of the fibre bundle. Let us order the complete set of assets (which we want to analyze) and label them by numbers from 0 to $N$. This set can be represented by $N$ (asset) points on a 2-dimensional plane (the dimension is a matter of convinience and can be choosen arbitrary). To add the time in the construction we attach a copy of $Z$-lattice (i.e. set of all integer number $\{...,−1,0,1,2,...\}$) to the each asset point. We use discretized time since anyway there is a natural time step and all real trades happen discretely. All together this gives the prebase set $L_0 = \{1,2,...,N\} \times Z$.

Next step in the construction is to define a connectivity of the prebase. To do this we start with an introduction of a matrix of links $\Gamma : L_0 \times L_0 \to \{0,\pm 1\}$ which is defined by the following rule for any $x \equiv (i,n) \in L_0$ and $y \equiv (k,m) \in L_0$: $\Gamma(x,y) = 0$ except for

1. $i = k$ and $n = m − 1$ accepting that the $i$-th security exists at the $n$-th moment and this moment is not an expiration date for the security;
2. \( n = m - 1 \) and at \( n \)-th moment of time the \( i \)-th asset can be exchanged on some quantity of \( k \)-th asset at some rate (we assume that the trasaction takes one unit of time).

In latter situations \( \Gamma(x, y) = 1 = -\Gamma(y, x) \).

Using the matrix \( \Gamma(., .) \) we define a curve \( \gamma(x, y) \) in \( L_0 \) which links two points \( x, y \in L_0 \). We call the set \( \gamma(x, y) \equiv \{x_j\}_j=1^p \) a curve in \( L_0 \) with ends at points \( x, y \in L_0 \) and \( p - 1 \) segments if \( x = x_1, x_p = y, \forall x_j \in L_0 \) and

\[
\Gamma(x_j, x_{j+1}) = \pm 1 \quad \text{for} \quad \forall j = 1, ..., p - 1 .
\]

The whole \( L_0 \) can be devided into a set of connected components. A connected component is a maximal set of elements of \( L_0 \) which can be linked by some curve for any pair of elements. The base \( L \) is defined now as the connected component contained US dollars at, say, 15.30 of 17-th of June 1997. This complete the construction of the base of fibre bundle.

**Structural group**

The structural group \( G \) to be used below is a group of dilatations. The corresponding irreducible representation is the following: the group \( G \) is a group of maps \( g \) of \( R_+ \equiv [0, +\infty) \) to \( R_+ \), which act as a multiplication of any \( x \in R_+ \) on some positive constant \( \lambda(g) \in R_+ \):

\[
g(x) = \lambda(g) \cdot x .
\]

Transition functions of a fibre bundle with the structure group correspond below to various swap rates, exchange rates, discount factors for assets.

**Fibres**

In the paper we use fibre bundles with the following fibres \( F \):

1. \( F = G \), i.e. the fibre coincides with the structure group. The corresponding fibre bundle is called Principle fibre bundle \( E_P \). A gauge theory in the fibre bundle in the next section corresponds to random walks of prices and rates.

2. \( F = R_+ \). This fibre bundle will be important to describe cash-debts flows. Indeed, a cross section (or simply a section) \( s \) (a rule which assigns a preferred point \( s(x) \) on each fibre to each point \( x \in L \) of the base) of the fibre bundle is a ”matter” field. In the context \( s(x \equiv (i, m)) \) gives a number of units of \( i \)-th asset at the moment of time \( m \).

Actions and the corresponding functional integrals will be written in terms of cross sections of the fibre bundles. The main property of the objects (actions and measures of the integrations) will be local gauge invariance, i.e. independance on a local action of the structural group.

The fibre bundle \( E \) we use below is trivial, i.e. \( E = L \times F \) and we do not stop for the definition of projections. Construction of the fibre bundles for the simple stock exchange, FX-market and financial derivatives can be found in Ref\[1\].
2.2 Parallel transport, curvature and arbitrage

A connection is a rule of the parallel transport of an element of a fibre from one point \((x)\) of a base to another point \((y)\). It means that an operator of the parallel transport along the curve \(\gamma\), \(U(\gamma) : F_x \rightarrow F_y\) is an element of the structural group of the fibre bundle \([7]\). Since we do not deal with continuous case and restrict our-selves on a lattice formulation, we do not need to introduce a vector-field of the connection but rather have to use elements of the structural group \(G\). By the definition, an operator of the parallel transport along a curve \(\gamma\), \(U(\gamma)\), defined as a product of operators of the parallel transport along the links which constitute the curve \(\gamma\):

\[
U(\gamma) = \prod_{i=1}^{p-1} U(x_i, x_{i+1}) , \quad \gamma \equiv \{x_i\}_{i=p-1}^{1}, \quad x_1 = x, \quad x_p = y .
\]

It means that we need to define only the parallel transport operators along elementary links. Since \(U(\gamma) = U^{-1}(\gamma^{-1})\), this restrict us on a definition of those along an elementary links with a positive connectivity. Summing up, the rules of the parallel transport in the fibre bundles are completely defined by a set of the parallel transport operators along elementary links with a positive connectivity. The definition of the set is equivalent to a definition of the parallel transport in the fibre bundle.

Since in subsection 2.1 the connectivity was defined by a possibility of assert movements in "space" and time, it allows us to give an interpretation of the parallel transport. In the subsection two principle kinds of links with positive connectivity were defined. First one connects two points \((i, n)\) and \((i, n+1)\) and represents a deposition the \(i\)-th assert for one unit of time. This deposition then results in a multiplication of the number of assert units by an interest factor (or internal rate of return factor) calculated as:

\[
U((i, n), (i, n + 1)) = e^{r_i \Delta} \in G ,
\]

where \(\Delta\) is a time unit and \(r_i\) is an appropriate rate of return for the \(i\) - th assert. In the continuous limit \(r_i\) becomes a time component of the corresponding connection vector-field at the point \((i, \Delta n)\).

By the same way the parallel transport operator is defined for the second kind of the elementary links, i.e. links between \((i, n)\) and \((k, n + 1)\) if there is a possibility to change at \(n\)-th moment a unit of \(i\)-th assert on \(S_{n,k}^{i}\) units of \(k\)-th assert:

\[
U((i, n), (k, n + 1)) = S_{n,k}^{i} \in G .
\]

In general, an operator of the parallel transport along a curve is a multiplier by which a number of assert units is multiplied as a result of an operation represented by the curve.

Results of parallel transports along two different curves with the same boundary points are not equal for a generic set of the parallel transport operators. A measure of the difference is a curvature tensor \(F\). Its elements are equal to resulting change in multiplier due to a parallel transport along a loop around an infinitesimal elementary plaquette with all nonzero links in the base \(L\):

\[
F_{\text{plaquette} \rightarrow 0} = \prod_{m} U_m - 1 .
\]
The index \( m \) runs over all plaquette links, \( \{ U_m \} \) are corresponding parallel transport operators and an agreement about an orientation is implied.

Now we show that the elements of the curvature tensor are, in fact, an excess returns on the operation corresponded to a plaquette. Since elements of the curvature tensor are local quantities, it is sufficient to consider an elementary plaquette on a "space"-time base graph. Let us, for example, consider two different assets (we will call them for the moment share and cash) which can be exchanged to each other with some exchange rate \( S_i \) (one share is exchanged on \( S_i \) units of cash) at some moment \( T_i \), and the reverse rate (cash to share) is \( S_i^{-1} \). We suppose that there exists a transaction time \( \Delta \) and this \( \Delta \) is taken as a time unit. So the exchange rates \( S_i \) are quoted on a set of the equidistant times: \( \{ T_i \} \) \((N, T_{i+1} - T_i = \Delta)\). Interest rate for cash is \( r_1 \) so that between two subsequent times \( T_i \) and \( T_{i+1} \) the volume of cash is increased by factor \( e^{r_1 \Delta} \). The shares are characterized by a rate \( r_2 \). As we will show latter, due to the gauge invariance we can fix \( r_1 \) to be risk free interest rate and \( r_2 \) related to the average rate of return of the share.

Let us consider an elementary (arbitrage) operation between two subsequent times \( T_i \) and \( T_{i+2} \). There are two possibilities for an investor, who posseses a cash unit at the moment \( T_i \), to get shares by the moment \( T_{i+2} \). The first one is to put cash on a bank deposit with the interest rate \( r_1 \) at the moment \( T_i \), withdraw money back at the moment \( T_{i+2} \) and buy shares for price \( S_{i+1} \) each. In this way investor gets \( e^{r_1 \Delta} S_{i+1}^{-1} \) shares at the moment \( T_{i+2} \) for each unit of cash he had at the moment \( T_i \). The second way is to buy the shares for price \( S_i \) each at the moment \( T_i \). Then, at the moment \( T_{i+2} \) investor will have \( S_i^{-1} e^{r_2 \Delta} \) shares for each unit of cash at the moment \( T_i \). If these two numbers \( (e^{r_1 \Delta} S_{i+1}^{-1} \text{ and } S_i^{-1} e^{r_2 \Delta}) \) are not equal then there is a possibility for an arbitrage. Indeed, suppose that \( e^{r_1 \Delta} S_{i+1}^{-1} < S_i^{-1} e^{r_2 \Delta} \), then at the moment \( T_i \) an arbitrager can borrow one unit of cash, buy \( S_i^{-1} \) shares and get \( S_i^{-1} e^{r_2 \Delta} S_{i+1} \) units of cash from selling shares at the moment \( T_{i+1} \). The value of this cash discounted to the moment \( T_i \) is \( S_i^{-1} e^{r_2 \Delta} S_{i+1} e^{-r_1 \Delta} > 1 \). It means that \( S_i^{-1} e^{r_2 \Delta} S_{i+1} e^{-r_1 \Delta} - 1 \) is an arbitrage excess return on the operation. In the other hand, as we have shown above, this represent lattice regularization of an element of the curvature tensor along the plaquette. If \( e^{r_1 \Delta} S_{i+1}^{-1} > S_i^{-1} e^{r_2 \Delta} \) then arbitrager can borrow one share at the moment \( T_i \), sell it for \( S_i \) units of cash, put cash in the bank and buy \( S_i e^{r_2 \Delta} S_{i+1} \) shares at the moment \( T_{i+2} \). We have an arbitrage situation again.

We consider the following quantity

\[
(S_i^{-1} e^{r_2 \Delta} S_{i+1} e^{-r_1 \Delta} + S_i e^{r_1 \Delta} S_{i+1}^{-1} e^{-r_2 \Delta} - 2) / 2 \Delta .
\]  

(2)

It is a sum of excess returns on the plaquette arbitrage operations. In continuous limit this quantity converges as usual to a square of the curvature tensor element. The absense of the arbitrage is equivalent to the equality

\[
S_i^{-1} e^{r_2 \Delta} S_{i+1} e^{-r_1 \Delta} = S_i e^{r_1 \Delta} S_{i+1}^{-1} e^{-r_2 \Delta} = 1 ,
\]

and we can use quantity (2) to measure the arbitrage (excess rate of return). In more formal way the expression (2) may be written as

\[
R = (U_1 U_2 U_3^{-1} U_4^{-1} + U_2 U_3 U_2^{-1} U_1^{-1} - 2) / 2 \Delta .
\]
In this form it can be generalized for other plaquettes such as, for example, the "space"-"space" plaquettes.

As we saw above, excess return is an element of the lattice curvature tensor calculated from the connection. In this sense, the rate of excess return for an elementary arbitrage operation is an analogue of the electromagnetic field. In the absence of uncertainty (or, in other words, in the absence of walks of prices, exchange and interest rates) the only state is realized — the state of zero arbitrage. However, if we introduce the uncertainty in the game, prices and the rates move and some virtual arbitrage possibilities appear. Therefore, we can say that the uncertainty plays the same role in the developing theory as the quantization does for the quantum gauge theory.

Last point to add in the section is a notion of the gauge transformation. The gauge transformation means a local change of a scale in fibres:

\[ f_x \rightarrow g(x)f_x \equiv f'_x, \quad f_x \in F_x, \quad g(x) \in G, \quad x \in E \]

together with the following transformation of the parallel transport operators:

\[ U(y, x) \rightarrow g(y)U(y, x)g^{-1}(x) \equiv U'(y, x) \in G. \]

It is easy to see that the parallel transport operation commutes with a gauge transformation:

\[ g(y)(U(y, x)f_x) = U'(y, x)f'_x \quad \text{(3)} \]

and the curvature tensor is invariant under the transformation:

\[ U_1U_2U_3^{-1}U_4^{-1} = U'_1U'_2(U'_3)^{-1}(U'_4)^{-1}. \quad \text{(4)} \]

### 2.3 Gauge field dynamics

In the previous subsections we showed that the exchange rates and interest rate (or, more general, internal rate of return) discount factors are elements of the structural group of the fibre bundle. Moreover, they are responsible for the parallel transport in "space" and time directions correspondingly. In the present subsection we address a question about a dynamics for the exchange/discount factors.

At a first sight the dynamics is difficult to specify since it is not restricted and any attempt to formulate the dynamics seems to be voluntary and not enough motivated. However, as we show below the dynamics can be derived from a few quite general and natural assumptions. The main postulate of the present analysis is an assumption about local gauge invariance with the dilatation group as a gauge group.

**Postulate 1: Gauge invariant dynamics**

We assume that all observable properties of the financial environment (in particular, rules of dynamical processes) do not depend on a choice of units of the assets. It means that all effects of, say, change of currency units or shares splittings may be eliminated by a corresponding change of interest rates, exchange rates and prices [8]. This is a very natural assumption which allow us, however, to make a step to a specification of the dynamics. Indeed, due to the gauge invariance an action which governs the dynamics has to be constructed from gauge invariant quantities.
Postulate 2: Locality
Furthermore, we assume local dynamics of the exchange/interest rate factors. This locality means that in the framework the dynamics of an asset is influenced by connected (in the sense of Γ connectivity on the base graph L) asserts only.

These two postulates allow us to make the following conclusion: the action $s_{\text{gauge}}$ has to be a sum over plaquettes in the base graph of some function of the (gauge invariant) curvature of the connection. It means that the action is a sum over plaquettes of a function of the excess return on the plaquette arbitrage operation as it was shown in the previously.

Postulate 3: Free field theory – correspondance principle
We postulate that the action is linear on the plaquette curvatures on the base graph since this is a simplest choice. It will be shown later that the postulate is equivalent to the free field theory description in an absense of matter fields and produce quasi-Brownian walks for the exchange/interest rates in the continuous time limit. This means that the approach generalizes standard constructions of the mathematical finance. This fact serves as a correspondance principle.

Postulate 4: Extremal action principle
In fully rational and certain economic enviroment it should be no possibility to have "something from nothing", i.e. to have higher return than the riskless rate of return. In more general form, the excess rate of return (rate of return above the riskless rate) on any kind of operations takes the smallest possible value which is allowed by external economic enviroment. Together with the locality of the action this give the extremality principle for the action.

Postulate 5: Limited rationality and uncertainty
The real enviroment is not certain and fully rational and there exist nonzero probabilities to get other excess rates of return (exchange rates, prices and interest rates fluctuate and its bears local arbitrage opportunities). We assume that the possibilities to have the excess return $R(x,T)$ at point of ”space” $x$ and time moment $T = 0$ are statistically independant for different $x$, $T$ and are distributed with an exponential probability weight $e^{-\beta R(x,T)}$ with some effective measure of rationality $\beta$. If $\beta \rightarrow \infty$, we return to a fully rational and certain economic enviroment.

Formally, we state that the probability $P(\{U_{i,k}\})$ to find a set of the exchange rates/interest rates $\{U_{i,k}\}$ is given by the expression:

$$P(\{U_{i,k}\}) \sim e^{-\beta \sum_{(x,T)} R(x,T)} \sim e^{-\beta s_{\text{gauge}}}.$$ 

Now we are ready to writing down the general action for the exchange/interest rate factors. However, before writing we would like to consider in more details a very simple example which gives some insight of the general framework. Let us, once again, consider two-assert system (cash-shares).

The first moment to mention is a gauge fixing. Since the action is gauge invariant it is possible to perform a gauge transformation which will not change the dynamics but will simplify further calculation. In lattice gauge theory there are several standard choices of the gauge fixing and the axial gauge fixing is one of them. In the axial gauge an element of the structural group are taken constant on links in time direction (we
keep them $e^{r_{1,2}\Delta}$) and one of exchange rates (an element along "space" direction at some particular chosen time) is also fixed. Below we fix the price of the shares at moment $T = 0$ taking $S_0 = S(0)$. It means that in the situation of the ladder base the only dynamical variable is the exchange rate (price) as a function of time and the corresponding measure of integration is the invariant measure $dS / S_i$.

From above derivation, the definition of the distribution function for the exchange rate (price) $S = S(T)$ at the moment $T = N\Delta$ under condition that at the moment $T = 0$ the exchange rate was $S_0 = S(0)$ is given by:

$$P(0, S_0; T, S) = \int_0^\infty \ldots \int_0^\infty \prod_{i=1}^{N-1} \frac{dS_i}{S_i} \exp \left[ -\frac{\beta}{2\Delta} \sum_{i=0}^{N-1} \left( S_i^{-1} e^{r_{2,2}\Delta} S_{i+1} e^{-r_{1,1}\Delta} + S_i e^{r_{1,2}\Delta} S_{i+1}^{-1} e^{-r_{2,2}\Delta} - 2 \right) \right].$$

(5)

It is not difficult to see that in the limit $\Delta \to 0$ the expression in brackets converges to the integral

$$-\beta \int_0^T d\tau \left( \frac{\partial S(\tau)}{\partial \tau} / S(\tau) + (r_2 - r_1) \right)^2$$

which corresponds to the geometrical random walk. Evaluating the integral and taking into account the normalization condition we come to the following expression for the distribution function of the price $S(T)$:

$$P(S(T)) = \frac{1}{\sigma S \sqrt{2\pi T}} e^{-\left( \ln(S(T)/S(0)) - (\mu - \frac{1}{2}\sigma^2)T \right)^2/(2\sigma^2T)}$$

(6)

Here we introduced so-called volatility $\sigma$ as $\beta = \sigma^{-2}$ and the average rate of share return $\mu$ as $\mu = r_1 - r_2$.

It is easy to give an interpretation of the last relation. The system at whole is not conservative and both the interest rate $r_1$ and the rate $r_2$ come outside of the system (from banks and the corresponding company production). Let us imagine that the world is certain and that because of the company production the capitalization of the firm increased. For this amount new shares with the same price $S_1$ have been issued (no dividends have been payed). The number of new shares for each old share is equal $e^{r_{2,2}\Delta}$. It means that the cumulative (old) share will have a price $S_1 e^{r_{2,2}\Delta}$ while the original price (at zero moment) was $S_0$. Taking into account discounting and certainty, we end with the following expression:

$$S_1 = e^{(r_1 - r_2)\Delta} S_0,$$

which tells that the rate of return on the share is equal $r_1 - r_2$. After introducing an uncertainty last expression turns into an average rate of return on the share.

Eqn (6) returns us to a justification of Postulate 3 which, together with other Postulates, are equivalent to a log-normal model for a price walks in an absence of matter fields, which we consider in details in the next section.

Now we give the general expression for probability distribution of a given exchange rates and internal rates of return profile:

$$P(\{S_{i,k}\}, \{r_{i,k}\}) \sim \exp(-\frac{\beta}{\Delta} \sum_{\text{plaquettes}} \left( \prod_m U_m + \prod_m U_m^{-1} - 2 \right)),$$

(7)
where the sum is calculated over all plaquettes in the base graph with all links with nonzero connectivity $\Gamma$, the index $m$ runs over all links of a plaquette, $\{U_m\}$ are corresponding elements of the structural group, which perform the parallel transport along the links.

To complete the gauge field consideration we want to return once again to the gauge invariance principle. Next section is devoted to the consideration of the “matter” field which interact through the connection. It is clear now that the field is a field which represents a number of assets (as a phase in the electrodynamics) and has to be gauged by the connection. Dilatations of money units (which do not change rules of investors behaviour) play a role of gauge transformation which eliminates the effect of the dilatation by a proper tune of the connection (interest rate, exchange rates, prices and so on) exactly as it is in the Fisher formula for the real interest rate in the case of inflation [2]. The symmetry to a local dilatation of money units (security splits and the like) is the gauge symmetry of the theory.

2.4 Effective theory of cash-debts flows: matter fields

Now let us turn our attention to ”matter” fields. These fields represent cash-debts flows on the market. Importance of the cash-debts flows for our consideration is caused by their role in a stabilization of market prices. Indeed, if, say, some bond prices eventually go down and create a possibility to get bigger return than from other assets, then an effective cash flows appear, directed to these more valuable bonds. This causes an upward shift of the prices because of the demand-supply mechanism. All together these effects smoothen the price movements. The same picture is valid for debts flows if there is a possibility for debts restructurisation. As we will see all this features finds their place in the constructed framework.

To formulate an effective theory for the flows we will assume that:

1. Any particular investor tries to maximize his return on cash and minimize his debts.

2. Investor’s behaviour is limited rational i.e. there are deviations from pure rational strategy because of, for example, lack of complete information, specific financial manager’s objectives [2] and so on.

We start with a construction of an effective theory for the cash flows and the generalize it for the case of debts presence. The first assumption tells us that an investor tries to maximize the following expression for the multiplier of a value of his investment (in the case of cash, shares and securities):

$$s(C) = \ln(U_1 U_2 \ldots U_N)/\Delta$$

by a certain choice of the strategy which is resulted in the corresponding trajectory in ”space”-time for assets. Here $\{U_i\}_{i=1}^N$ are exchange (price) or interest factors which came from a choice of the investor behaviour at $i$-th step on the trajectory $C$ and boundary points (at times $T = 0$ and $T = N$) are fixed. We assume that there is a
transaction time which is smallest time in the systems equal to $\Delta$. In other words, a rational investor will choose the trajectory $C_0$ such that

$$s(C_0) = \max_{C} s(C).$$

Choosing the best strategy, fully rational investor maximizes his return $s$. However, as we assumed limited rationality, in analogy with the corresponding consideration of the connection field probability weights, we define the following probability weight for a certain trajectory $C$ with $N$ steps:

$$P(C) \sim e^{\beta s(C)/\Delta} \equiv e^{\beta s(C)},$$

with a some "effective temperature" $1/\beta$ which represents a measure of average unrationality of investors in a unit time.

It is possible to generalize the approach to a case of many investors operating with cash and debts. The corresponding functional integral representation for the transition probability (up to a normalization constant) has the form [1]:

$$\int D\psi_1 D\psi_2 D\chi_1 D\chi_2 e^{\beta(s_1+s_1')} \equiv e^{\beta s(1)},$$

with the actions for cash and debts flows:

$$s_1 = \frac{1}{\beta} \sum_i \left( \psi_{1,i+1} e^{\beta r_1} \psi_{1,i} - \psi_{1,i+1} \psi_{1,i} + \psi_{2,i+1} e^{\beta r_2} \psi_{2,i} - \psi_{2,i} \psi_{2,i} \right)$$

$$+ (1-t)^{\beta} S_i^{\beta} \psi_{1,i+1} \psi_{2,i} + (1-t)^{\beta} S_i^{\beta} \psi_{2,i+1} \psi_{1,i},$$

$$s_1' = \frac{1}{\beta} \sum_i \left( \chi_{1,i+1} e^{-\beta r_1} \chi_{1,i} - \chi_{1,i+1} \chi_{1,i} + \chi_{2,i+1} e^{-\beta r_2} \chi_{2,i} - \chi_{2,i} \chi_{2,i} \right)$$

$$- \chi_{2,i} \chi_{2,i} + (1+t)^{-\beta} S_i^{\beta} \chi_{1,i+1} \chi_{2,i} + (1+t)^{-\beta} S_i^{\beta} \chi_{2,i+1} \chi_{1,i}.$$
represents a particular asset (a point in ”space”) and the index \( k \) labels time intervals. By the definition of the base \( L \) any two points of the base can be linked by a curve \( \gamma \), each formed by elementary segments with a nonzero connectivity \( \Gamma \). In its own, each of the segments is provided with an element of the structural group \( U \) which performs a parallel transport along this (directed) link. These allow us to give a most general form of the action \( s_1 \) \(\left((i_{12},k_{12}) \equiv (i_1,k_1),(i_2,k_2) \in L : \Gamma((i_1,k_1),(i_2,k_2)) = 1)\right) \):

\[
s_1 = \frac{1}{\beta} \sum_{(i_{12},k_{12})} (\psi_{ij_1,k_1}^+ U_{ij_2,k_2}^\beta (1-t\delta_{i_1-1,i_2})^\beta \psi_{ij_2,k_2} - \delta_{ij_1,k_1}\psi_{ij_2,k_2}^+ \psi_{ij_1,k_1}) . \quad (12)
\]

In the same way the action for the debts flows can be written:

\[
s_1' = \frac{1}{\beta} \sum_{(i_{12},k_{12})} (\chi_{ij_1,k_1}^+ U_{ij_2,k_2}^{-\beta} (1+t\delta_{i_1-1,i_2})^{-\beta} \chi_{ij_2,k_2} - \delta_{ij_1,k_1}\chi_{ij_2,k_2}^+ \chi_{ij_1,k_1}) . \quad (13)
\]

### 3 Conclusion

In conclusion, we proposed a mapping of the capital market theory in the lattice quantum gauge theory where the gauge field represents the interest rate and prices and ”matter” fields are cash-debts flows. Basing on the mapping we derived action functionals for both the gauge field and ”matter” fields assuming several postulates. The main assumption is a gauge invariance of the dynamics which means that the dynamics does not depend on particular values of units of assets and a change of the values of the units may be compensated by a proper change of the gauge field. The developed formalism has been applied to some issues of the capital market theory. Thus, it was shown in Ref [10] that a deviation of the distribution function from the log-normal distribution may be explained by an active trading behavior of arbitragers. In this framework such effects as changes in shape of the distribution function, “screening” of its wings for large values of the price and non-Markovian character (memory) of price random walks turned out to be consequences of damping of the arbitrage and directed price movements caused by speculators. In Ref [14] consequences of the bid-ask spread and the corresponding gauge invariance breaking were examined. In particular, it turned out that the distribution function is also influenced by the bid-ask spread and the change of its form may be explained by this factor as well. So, the complete analysis of the statistical characteristics of prices have to account both these factors. Besides, it is possible to show [12] that the Black-Scholes equation for the financial derivatives can be obtained in the present formalism in an absence of speculators (i.e. absence of the arbitrage game).

Let us now make two final remarks.

1. In all cited above references a very simple model of a stock exchange was considered, where the only one kind of security is traded. The consideration can be generalized on more realistic situation with a set of traded securities. Following this line, dynamical portfolio theory can be constructed and, in static (equilibrium) limit, will coincide with standard portfolio theory [13]. In the dynamical theory time-dependent correlation functions will play a role of response functions...
of the market to an external perturbation such as a new information or a change in macroeconomic environment. Account of virtual arbitrage fluctuations will lead to a time-dependent modification of CAPM [3].

2. Since the influence of the speculators leads to a non-Markoffian character of a price walks there is no possibility to eliminate a risk using arbitrage arguments to derive an equation for a price of a derivative. Then the virtual arbitrage and corresponding asset flows have to be considered. It will lead to a correction of Black-Scholes equation.

We will return to these points in our forthcoming papers.

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