COMPACTIFICATIONS OF ADJOINT ORBITS AND THEIR HODGE DIAMONDS

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ABSTRACT. A recent theorem of [GGSM1] showed that adjoint orbits of semisimple Lie algebras have the structure of symplectic Lefschetz fibrations. We investigate the behaviour of their fibrewise compactifications. Expressing adjoint orbits and fibres as affine varieties in their Lie algebra, we compactify them to projective varieties via homogenisation of the defining ideals. We find that their Hodge diamonds vary wildly according to the choice of homogenisation, and that extensions of the potential to the compactification must acquire degenerate singularities.

CONTENTS

1. Hodge diamonds of Lefschetz fibrations 1
2. Lefschetz fibrations on adjoint orbits 2
3. Compactification of the orbit of $\mathfrak{sl}(2, \mathbb{C})$ 3
4. Smooth compactification of an $\mathfrak{sl}(3, \mathbb{C})$ orbit 4
5. Generalisations and computational corollaries 5
6. Singular compactifications of $\mathfrak{sl}(3, \mathbb{C})$ orbits 7
6.1. A fibration with 4 critical values 8
6.2. A fibration with 6 critical values 10
7. Open questions 11
References 11

1. HODGE DIAMONDS OF LEFSCHETZ FIBRATIONS

Given a symplectic manifold $X$, a symplectic Lefschetz fibration (SLF) on $X$ is a surjection $f : X \to \mathbb{C}$ with only Morse type singularities, giving $X$ the structure of a locally trivial fibration on the complement of the set of critical fibers, and whose regular fibres are symplectic submanifolds of $X$, see [Se]. A large family of new examples of noncompact SLFs was constructed in the recent paper [GGSM1] and we need to compactify these examples to obtain information provided by their Hodge diamonds (or simply the cohomological dimensions $h^p(X, \Omega^q)$ of the compactification in the singular case). Our motivation – coming from mathematical physics – is to eventually study categories of Lagrangian vanishing cycles. These play an essential role in the Homological Mirror Symmetry conjecture [Ko], where such a category appears as the Fukaya category of a Landau–Ginzburg (LG) model (that is, a Kähler manifold $X$ equipped with a holomorphic function $f : X \to \mathbb{C}$ called

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1
the superpotential). SLFs are nice examples of LG models where a precise definition of the Fukaya category of Lagrangian vanishing cycles is available, see [FOOO], [Se].

[GGSM1] showed the existence of the structure of SLFs on adjoint orbits of semisimple Lie algebras. These adjoint orbits are not compact. In fact, they are diffeomorphic to cotangent bundles of flag varieties [GGSM2]. We want to compare the behaviour of vanishing cycles on $X$ and on its compactifications. Expressing the adjoint orbit as an algebraic variety, we homogenise its ideal to obtain a projective variety, which serves as our compactification. To study such a compactification $\overline{X}$, we calculate its cohomological dimensions $h^p(\overline{X}, \Omega^q)$, as well as those of the compactified fibres of the SLF. Calculating such numbers is computationally heavy, so we used Macaulay2. Details of the computational algorithms we used appear in [CG]. In the smooth case, these dimensions give us the Hodge diamonds, from which we can read off topological data for the total space $X$ as well as for the fibres of the SLF.

Remark 1. Choosing a compactification is in general a delicate task: a different choice of generators for the defining ideal of the orbit can result in completely different cohomologies of the corresponding compactification. This happens because the homogenisation of an ideal $I$ can change drastically if we vary the choice of generators for $I$ (see Section 6.1).

In Section 2, we present the principal theorem that furnishes us with examples. In Section 3, we find all adjoint orbits of $\mathfrak{sl}(2, \mathbb{C})$ (up to isomorphism), and apply our compactification process to this simple case. In Section 4, we consider a more involved example of an adjoint orbit inside $\mathfrak{sl}(3, \mathbb{C})$, corresponding to the minimal flag variety, and show that any extension of the potential to the compactified orbit must acquire degenerate singularities, hence it would no longer remain a Lefschetz fibration. This is generalised in Section 5 to the minimal flag variety of $\mathfrak{sl}(n+1, \mathbb{C})$. We illustrate with an example in Section 6 just how delicate a task compactification can be.

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2. Lefschetz fibrations on adjoint orbits

Let $H_0$ be an element in the Cartan subalgebra of a semisimple Lie algebra $\mathfrak{g}$, let $\mathfrak{g}(H_0)$ denote its adjoint orbit and $\langle \cdot, \cdot \rangle$ the Cartan-Killing form. It is proved in [GGSM1] that for each regular element $H \in \mathfrak{g}$, the function $f_H : \mathfrak{g}(H_0) \rightarrow \mathbb{C}$ given by $f_H(x) = \langle H, x \rangle$ gives the orbit the structure of a symplectic Lefschetz fibration. This includes the following properties for $f_H$:

1. The singularities are nondegenerate.
2. If $c_1, c_2 \in \mathbb{C}$ are regular values, then the level manifolds $f_H^{-1}(c_1)$ and $f_H^{-1}(c_2)$ are diffeomorphic.
There exists a symplectic form $\Omega$ in $\mathcal{O}(H_0)$ such that if $c \in \mathbb{C}$ is a regular value then the level manifold $f_H^{-1}(c)$ is symplectic; that is, the restriction of $\Omega$ to $f_H^{-1}(c)$ is a symplectic (nondegenerate) form.

If $c \in \mathbb{C}$ is a singular value, then $f_H^{-1}(c)$ is a union of affine subspaces (contained in $\mathcal{O}(H_0)$). These subspaces are symplectic with respect to the form $\Omega$ from the previous item.

We compactify the orbit by projectivisation; that is, we homogenise the polynomials with an extra variable $t$ to obtain a projective variety.

3. Compactification of the Orbit of $\mathfrak{sl}(2, \mathbb{C})$

Inside $\mathfrak{sl}(2, \mathbb{C})$, all adjoint orbits are of the same isomorphism type, which we now describe as an SLF with 2 critical values. In $\mathfrak{sl}(2, \mathbb{C})$, take

$$H = H_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which is regular since it has 2 distinct eigenvalues. The orbit $\mathcal{O}(H_0)$ is the set of matrices in $\mathfrak{sl}(2, \mathbb{C})$ with eigenvalues 1 and $-1$, which forms a submanifold of complex dimension 2 of $\mathfrak{sl}(2, \mathbb{C})$.

The Weyl group $W \cong S_2$ acts via conjugation by permutation matrices. The two singularities are thus $H$ and $-H$.

We can also express the orbit as an affine variety embedded in $\mathbb{C}^3$. Writing a general element $A \in \mathcal{O}(H_0)$ as

$$A = \begin{pmatrix} x & y \\ z & -x \end{pmatrix},$$

the characteristic polynomial of $A$ is

$$-(x - \lambda)(x + \lambda) - yz = \det(A - \lambda \text{id}) = \lambda^2 - 1,$$

the first equality being derived from explicit calculation and the second due to the fact that $\text{tr} A = 0$ and $\det A = -1$. This in turn implies that the orbit $\mathcal{O}(H_0) \subset \mathfrak{sl}(2, \mathbb{C}) \cong \mathbb{C}^3$ is an affine variety $X$ cut out by the equation

$$x^2 + yz - 1 = 0. \quad (1)$$

We can compactify this variety by homogenising eq. 1 and embedding $X$ into the corresponding projective variety. This gives the surface cut out by $x^2 + yz - t^2 = 0$ in $\mathbb{P}^3$. The Hodge diamond of this compactification is shown in figure 1.

$$
\begin{array}{ccc}
1 & & \\
0 & 2 & 0 \\
0 & & 0 \\
1 & & \\
\end{array}
$$

**Figure 1.** The Hodge diamond of the projectivisation of $\mathcal{O}(\text{Diag}(1, -1))$.

The height function is

$$f_H(A) = \text{tr} HA = \text{tr} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x & y \\ z & -x \end{pmatrix} = 2x.$$
Note that the two critical points belong to distinct fibres. We can also express the regular fibre (over zero) as the affine variety in \((y, z) \in \mathbb{C}^2\) cut out by the equation 

\[ yz - 1 = 0 \]

since it must satisfy eq. 1 and \(x = 0\). As with the orbit, we homogenise this equation and embed the fibre into the corresponding projective variety cut out by the equations \(x = 0\) and \(yz - t^2 = 0\) in \(\mathbb{P}^3\). This yields the Hodge diamond shown in fig. 2. Note that these compactified fibres have no middle homology.

\[
\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \\
\end{array}
\]

**Figure 2.** The Hodge diamond of the projectivisation of the regular fibre over zero, where \(H = H_0 = \text{Diag}(1, -1)\).

### 4. Smooth compactification of an \(\mathfrak{sl}(3, \mathbb{C})\) orbit

The adjoint orbits of \(\mathfrak{sl}(3, \mathbb{C})\) fall into one of three isomorphisms types. Here we present an SLF with 3 critical values. In \(\mathfrak{sl}(3, \mathbb{C})\), consider the orbit \(\mathcal{O}(H_0)\) of

\[
H_0 = \begin{pmatrix}
2 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1 \\
\end{pmatrix}
\]

under the adjoint action. We fix the element

\[
H = \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}
\]

to define the potential \(f_H\). A general element \(A \in \mathfrak{sl}(3, \mathbb{C})\) has the form

\[
A = \begin{pmatrix}
x_1 & y_1 & y_2 \\
z_1 & x_2 & y_3 \\
z_2 & z_3 & -x_1 - x_2 \\
\end{pmatrix}.
\]

In this example, the adjoint orbit \(\mathcal{O}(H_0)\) consists of all the matrices with the minimal polynomial

\[
(A + \text{id})(A - 2\text{id}).
\]

So, the orbit is the affine variety cut out by the ideal \(I\) generated by the polynomial entries of \((A + \text{id})(A - 2\text{id})\). To obtain a projectivisation of \(X\), we first homogenise its ideal \(I\) with respect to a new variable \(t\), then take the corresponding projective variety. In this case, the projective variety \(\overline{X}\) is a smooth
compactification of $X$. We used Macaulay2 \([M2]\) to calculate the Hodge diamonds of a compactification of the adjoint orbit $O(H_0)$, obtaining:

\[
\begin{array}{ccccccc}
1 & & & & & & \\
0 & 2 & 0 & & & & \\
0 & 0 & 0 & 0 & & & \\
0 & 0 & 3 & 0 & 0 & & \\
0 & 0 & 0 & 0 & & & \\
0 & 2 & 0 & & & & \\
0 & 0 & & & & & \\
1 & & & & & &
\end{array}
\]

We now calculate the Hodge diamond of a compactified regular fibre. The potential corresponding to our choice of $H$ is $f_H = x_1 - x_2$. The critical values of this potential are $\pm 3$ and 0. Since all regular fibres of an SLF are isomorphic, it suffices to choose the regular value 1. We then define the regular fibre $X_1$ as the variety in $\mathfrak{sl}(3, \mathbb{C}) \cong \mathbb{C}^8$ corresponding to the ideal $J$ obtained by summing $I$ with the ideal generated by $f_H - 1$. We then homogenise $J$ to obtain a projectivisation $\overline{X}_1$ of the regular fibre $X_1$. The Hodge diamond of $\overline{X}_1$ is:

\[
\begin{array}{ccccccc}
1 & & & & & & \\
0 & 2 & 0 & & & & \\
0 & 0 & 0 & 0 & & & \\
0 & 2 & 0 & & & & \\
0 & 0 & & & & & \\
1 & & & & & &
\end{array}
\]

**Remark 2.** We used the same method to calculate the Hodge diamonds for the singular fibre over 0 and obtained the same Hodge diamond as for the regular fibres.

**Remark 3.** More details of this example appear in \([C]\).

### 5. Generalisations and Computational Corollaries

We generalise our example of $\mathfrak{sl}(3, \mathbb{C})$ to $\mathfrak{sl}(n+1, \mathbb{C})$. To obtain the case where the adjoint orbit is diffeomorphic to the cotangent bundle of the minimal flag, we set $H_0 = \text{Diag}(n, -1, \ldots, -1)$ and $H = \text{Diag}(1, -1, 0, \ldots, 0)$. Then the diffeomorphism type of the adjoint orbit is given by $O(H_0) = T^*\mathbb{P}^n$ (see \([GGSM2, \text{sec. 2.2}]\)), and $H$ gives the potential $x_1 - x_2$ as before. If we compactify this orbit to $\mathbb{P}^n \times \mathbb{P}^n$ (this may be done holomorphically by \([GGSM2, \text{Sec. 4.2}]\)), then the Hodge classes of the compactification are given by

\[
h^{p,p} = n + 1 - |n - p|
\]

and the remaining Hodge numbers are 0. An application of the Lefschetz hyperplane theorem determines all but the Hodge numbers of the middle row of the compactification of the regular fibre, and computations shows the latter are zero.

**Remark 4.** We observe that there are various ways to look at the isomorphism type of the adjoint orbit $O(H_0)$ depending on the point of view best suited to a given problem.
Firstly, the adjoint orbit $\text{Ad}(G)H_0$ can be identified with the homogeneous space $G/Z_{H_0}$ where $Z_{H_0}$ is the centralizer of $H_0$. The compact subgroup $K$ of $G$ cuts out the subadjoint orbit $\text{Ad}(K)H_0$, which can be identified with the flag manifold $\bar{F}_{H_0} = G/P_{H_0}$ where $P_{H_0}$ is the parabolic subgroup associated to $H_0$. In [GGSM1] the symplectic structure on $\text{Ad}(G)H_0$ is chosen as the imaginary part of the Hermitian form inherited from $\mathfrak{g}$. With this choice, the flag $\bar{F}_{H_0}$ is the Lagrangian in $\mathcal{O}(H_0) \cong T^*\bar{F}_{H_0}$ corresponding to the zero section of the cotangent bundle. From a Riemannian point of view this is also diffeomorphic to $T\bar{F}_{H_0}$.

Secondly, $\mathcal{O}(H_0)$ can be identified with the open orbit of the diagonal action of $G$ on the product $F_{H_0} \times \bar{F}_{H_0}^*$ [GGSM2, sec. 4.2]. A vector bundle structure on $\mathcal{O}(H_0)$ is obtained by observing that

$$\mathcal{O}(H_0) = \text{Ad}(G)H_0 = \text{Ad}(K)(H_0 + n^+),$$

where $n^+$ is the sum of the eigenspaces of $\text{ad}(H_0)$ associated to its positive eigenvalues. The process of projectivisation then transforms the affine space $H_0 + n^+$ into a projective space of the same dimension as the flag.

**Example 1.** Let $\Sigma = \{\alpha_{12}, \alpha_{23}\}$ be the usual choice of simple roots for $\mathfrak{sl}(3, \mathbb{C})$. In the case $H_0 = \text{Diag}(2, -1, -1)$, the corresponding positive nilpotent part is $n^+ = \mathfrak{g}_{\alpha_{12}} \oplus \mathfrak{g}_{\alpha_{13}}$, which consists of matrices of the form:

$$\begin{pmatrix}
0 & y_1 & y_2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.$$

With the description of $\mathcal{O}(H_0)$ as a vector bundle above, the closure of the fibre $H_0 + n^+$ inside the compactification of section 4 consists of matrices of the form:

$$A = \begin{bmatrix}
2t & y_1 & y_2 \\
0 & -t & 0 \\
0 & 0 & -t
\end{bmatrix}.$$

Two matrices of this form are equivalent if one is a scalar multiple of the other and a priori one might expect there to be further relations between the matrices. However, it can be verified by inspecting the generators of the defining ideal that there are no further relations. Therefore, we can embed $H_0 + n^+$ into $\mathbb{P}^2$ by mapping $A$ to $[t, y_1, y_2]$. The case of $\mathfrak{sl}(n, \mathbb{C})$ is similar, with a map into $\mathbb{P}^n$ given by $[t, y_1, \ldots, y_n]$.

**Proposition 2.** Let $H_0 = \text{Diag}(n, -1, \ldots, -1)$. Then the adjoint orbit of $H_0$ in $\mathfrak{sl}(n+1, \mathbb{C})$ compactifies holomorphically to a trivial product.

**Proof.** For the case $H_0 = \text{Diag}(n, -1, \ldots, -1)$, [GGSM2, Thm. 5.11] showed that $\mathcal{O}(H_0)$ can be embedded holomorphically into $\bar{X} = \mathbb{P}^n \times \mathbb{P}^{n*}$ as the open orbit of the diagonal action of $G$ on $\bar{F}_{H_0} \times \bar{F}_{H_0}^* = \mathbb{P}^n \times \mathbb{P}^{n*}$. We claim that the complement of the image of $\mathcal{O}(H_0)$ in the compactification $\bar{X} = \mathbb{P}^n \times \mathbb{P}^{n*}$ can be identified with the adjoint orbit of the nilpotent matrix

$$N = \begin{pmatrix}
0 & 1 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & 0
\end{pmatrix}.$$
Since $G$ acts by conjugation, it is clear that if $N$ belongs to $\overline{X}$ so does its entire adjoint orbit $O(N)$. Notice that $O(H_0)$ contains all matrices of the form

$$N = \begin{pmatrix} n & t & \ldots & 0 \\ 0 & -1 & 0 & \vdots & \ddots \\ 0 & \ldots & -1 \end{pmatrix},$$

this can be verified by direct calculation. Now, dividing by $t$ and taking limit when $t \to \infty$ shows that $N$ belongs to $\overline{X}$ (in fact, this argument shows that any manifold that serves as a compactification of $O(X_0)$ contains a copy of $N$). It remains to prove that $\mathbb{P}^n \times \mathbb{P}^n \setminus O(X_0)$ isomorphic to $\overline{O}(H_0)$ and the closed one isomorphic to $\overline{O}(N)$. □

Remark 5. As mentioned in Remark 4, under the real diffeomorphism, the flag $F_{H_0}$ corresponds to the zero section of the vector bundle $\overline{O}(H_0)$ and consequently is Lagrangian in the orbit. This flag remains Lagrangian when embedded into the product $\mathbb{P} \times \mathbb{P}$ as the anti-diagonal. Therefore, by Weinstein’s theorem, it has a neighbourhood which is symplectomorphic to the cotangent bundle $T^* \mathbb{P}$. However, via the equivariant real diffeomorphism $\mathbb{P}^n \times \mathbb{P}^n \setminus \overline{O}(X_0)$ exhibited in [GGSM2, Thm. 2.1] the canonical symplectic form on the cotangent bundle pulls back to the Kostant–Kirillov–Souriau (KKS) form on the adjoint orbit $\overline{O}(H_0)$, thus the real diffeomorphism can not be made holomorphic.

The following corollary follows immediately from observing the Hodge diamonds we obtained.

**Corollary 3.** An extension of the potential $f_H$ to the compactification $\mathbb{P}^n \times \mathbb{P}^n$ cannot be of Morse type; that is, it must have degenerate singularities.

**Proof.** Our potential has singularities at $wH_0, w \in W$. Now observe that the Hodge diamond of our compactified regular fibres have only zeroes in the middle row, hence any extension of the fibration to the compactification will have no vanishing cycles. However, the existence of a Lefschetz fibration with singularities and without vanishing cycles is precluded by the fundamental theorem of Picard–Lefschetz theory. □

**6. Singular compactifications of $\mathfrak{sl}(3,\mathbb{C})$ orbits**

We show that the compactified regular fibre for $f_H$ can change drastically according to the choice of homogenisation of the ideal cutting out the orbit as an affine variety. The compactifications obtained in this section turn out to be singular. Nevertheless, we wish to depict diamonds with their sheaf cohomological information. It is well known, see e.g. [St] that every complex algebraic variety has a mixed Hodge structure. We do not attempt to describe
mixed Hodge structures, instead we calculate the numbers $h^p(X, \Omega^q)$, where $\Omega$ is the cotangent sheaf. Although we do not explore here how the diamond containing such numbers might be related to the topology of $X$, such diamonds do provide us with enough information to show that 2 natural choices of compactification differ.

6.1. A fibration with 4 critical values. In $\mathfrak{sl}(3, \mathbb{C})$ we take

$$H = H_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which is regular since it has 3 distinct eigenvalues. Then $X = \mathcal{O}(H_0)$ is the set of matrices in $\mathfrak{sl}(3, \mathbb{C})$ with eigenvalues $1, 0, -1$. This set forms a submanifold of real dimension 6 (a complex threefold).

In this case $W = S_3$, the permutation group in 3 elements, and acts via conjugation by permutation matrices. Therefore, the potential $f_H = x_1 - x_2$ has 6 singularities; namely, the 6 diagonal matrices with diagonal entries $1, 0, -1$. The four singular values of $f_H$ are $\pm 1, \pm 2$. Thus, 0 is a regular value for $f_H$. Let $A \in \mathfrak{sl}(3, \mathbb{C})$ be a general element written as in (2), and let $p = \det(A)$, $q = \det(A - \text{id})$. The ideals $\langle p, q \rangle$ and $\langle p - q, q \rangle$ are clearly identical and either of them defines the orbit though $H_0$ as an affine variety in $\mathfrak{sl}(3, \mathbb{C})$.

Now

$$I = \langle p, q, f_H \rangle \quad J = \langle p, p - q, f_H \rangle$$

are two identical ideals cutting out the regular fibre $X_0$ over 0. Let $I_{\text{hom}}$ and $J_{\text{hom}}$ be the respective saturated homogenisations and notice that $I_{\text{hom}} \neq J_{\text{hom}}$, so that they define distinct projective varieties, and thus two distinct compactifications

$$\overline{X}_0^I = \text{Proj}(\mathbb{C}[x_1, x_2, y_1, y_2, y_3, z_1, z_2, z_3, t]/I_{\text{hom}}) \quad \text{and} \quad \overline{X}_0^J = \text{Proj}(\mathbb{C}[x_1, x_2, y_1, y_2, y_3, z_1, z_2, z_3, t]/J_{\text{hom}})$$

of $X_0$. Their diamonds are given in figure 3. Remark 7 explains the computational issues.

Remark 6. The variety $\overline{X}_0^I$ is an irreducible component of $\overline{X}_0^J$. Indeed, we find that $I \subset J$ and that $J$ is a prime ideal (whereas $I$ is not), thus the variety $\overline{X}_0^I$ is irreducible and contained in $\overline{X}_0^J$.

Remark 7 (Computational matters). Macaulay2 greatly facilitates cohomological calculations that are unfeasible by hand. The Macaulay2 algorithm that computes $h^p(X, \Omega^q)$ is written for a smooth variety $X$. However, the algorithm proceeds by resolving the cotangent sheaf and calculating its exterior powers to compute sheaf cohomology, all of which works out reasonably well for our singular examples. The only drawback is that the memory requirements rise steeply with the dimension of the variety – especially for the classes $h^{p,p}$. In fact, the unknown entries in our diamonds (marked with a ?) exhausted the 48GB of RAM of the computers of our collaborators at IACS without producing an answer.
6.1.1. Expected Euler characteristic. To reassure ourselves about the much larger values occurring for the diamond of $X_0^/ \text{left}$ in comparison to $X_0^/ \text{right}$, we perform the rather amusing calculation of the expected Euler characteristic of both varieties, which give out quite surprising numbers.

Remark 8. Let $Y = Y_1 \cap \cdots \cap Y_r$ be a complete intersection. If $Y$ is smooth, then the Euler characteristic of $Y$ is uniquely determined by its cohomology class. However, for a singular variety this is no longer true, and the cohomological classes $Y_i$ do not determine the topological Euler characteristic. They determine only what is called the expected Euler characteristic of $Y$ (equal to the Fulton–Johnson class), see [Cy].

To calculate the expected Euler characteristic we use the following basic formulae from intersection theory. Let $X := V(f_1, \ldots, f_k) \subset \mathbb{P}^{n+k}$ be a complete intersection with inclusion $i : X \rightarrow \mathbb{P}^{n+k}$. Define $\alpha := i^*(c_1(\mathcal{O}_{\mathbb{P}^{n+k}}(1))) \in H^2(X)$. Then

$$\int_X \alpha^{n} = d,$$

where $d = \prod_{i} d_i$ and $d_i = \deg f_i$. Moreover,

$$c(X) = \frac{(1 + \alpha)^{n+k+1}}{\prod_{i}(1 + d_i \alpha)} = 1 + c_1(X) + \cdots + c_n(X),$$

and the Euler characteristic is given by

$$\chi(X) = \int_X c_n(X),$$

where $c_i(X) \in H^{2i}(X)$ is the $i$-th Chern class.

Example 4. We first illustrate the formula with two elementary cases.

For a conic $C$ in $\mathbb{P}^2$, expression 6 produces $(1 + a)^3/(1 + 2a)$, whose expansion at zero is $1 + a + a^2 + o(a^3)$. Here, $\int a = 2$ and we get $\chi(C) = 2$, which was to be expected since the conic is topologically isomorphic to $\mathbb{P}^1$.

For the quartic $Q$ in $\mathbb{P}^3$, expression 6 gives $(1 + a)^4/(1 + 4a)$, whose expansion at zero is $1 + 6a^2 + o(a^3)$. Here, $\int a^2 = 4$ and so $\chi(Q) = 6 \times 4 = 24$, which was to

**Figure 3.** The diamonds of two projectivisations $X_0^/ \text{left}$ and $X_0^/ \text{right}$ of the regular fibre corresponding to $H = H_0 = \text{Diag}(1, -1, 0)$. 
be expected since the quartic is a $K3$ surface, whose Hodge diamond is well known to be
\[
\begin{array}{cccc}
1 & 0 & 0 \\
1 & 20 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{array}
\]

Now let us return to our two projectivisations $\overline{X}_0^I$ and $\overline{X}_0^J$. For the ideal $I_{\text{hom}}$, we have degrees $d_1 = d_2 = 3$ and $d_3 = 1$. The orbit was embedded in $\mathbb{P}^8$. So expression 6 gives
\[
c(\overline{X}_0^I) = \frac{(1 + \alpha)^9}{(1 + 3\alpha)(1 + 3\alpha)(1 + \alpha)} = \frac{(1 + \alpha)^6}{(1 + 3\alpha)^2}.
\]
The Taylor series expansion around zero is given by $1 + 2\alpha + 7\alpha^2 - 4\alpha^3 + 31\alpha^4 - 94\alpha^5 + o(\alpha^5)$. Here $\int \alpha^5 = 9$ and we get the expected Euler characteristic to be
\[
\chi(\overline{X}_0^I) = -94 \times 9 = -846.
\]

On the other hand, for the ideal $J_{\text{hom}}$, we have degrees $d_1 = 2$, $d_2 = 3$, and $d_3 = 1$. Expression 6 gives
\[
c(\overline{X}_0^J) = \frac{(1 + \alpha)^9}{(1 + 2\alpha)(1 + 3\alpha)(1 + \alpha)} = \frac{(1 + \alpha)^8}{(1 + 3\alpha)(1 + 2\alpha)}.
\]
The Taylor series expansion around zero is $1 + 3\alpha + 7\alpha^2 + 3\alpha^3 + 13\alpha^4 - 27\alpha^5 + o(\alpha^5)$. In this case, $\int \alpha^5 = 6$ and we obtain
\[
\chi(\overline{X}_0^J) = -27 \times 6 = -162.
\]
The difference between $\chi(\overline{X}_0^J)$ and $\chi(\overline{X}_0^I)$ is a concrete topological difference between our two compactifications.

6.2. A fibration with 6 critical values. In $\mathfrak{sl}(3, \mathbb{C})$ we now take
\[
H_0 = \begin{pmatrix}
3 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -2
\end{pmatrix},
\]
which is regular since it has 3 distinct eigenvalues. Then $\mathfrak{g}(H_0)$ is the set of matrices in $\mathfrak{sl}(3, \mathbb{C})$ with eigenvalues 3, $-1$, $-2$. We choose
\[
H = \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]
giving the potential $f_H(A) = x_1 - x_2$, with critical values $\pm 1, \pm 4, \pm 5$. This fibration is only mildly different from the previous one by the fact that 2 singular fibres contain 2 singularities each. The orbit is diffeomorphic to the one of subsection 6.1. The regular fibres are pairwise diffeomorphic.

As in 6.1, let $A \in \mathfrak{sl}(3, \mathbb{C})$, and $p = \det(A + \text{id})$, $q = \det(A + 2\text{id})$. Once again, the ideals $\langle p, q \rangle$ and $\langle p - q, q \rangle$ are clearly equal and either of them defines the orbit though $H_0$ as an affine variety in $\mathfrak{sl}(3, \mathbb{C})$. The matrix $A$ belongs to the regular fibre $X_0$ if in addition it satisfies $f_H = x_1 - x_2 = 0$. Now, let
\[
I = \langle p, q, f_H \rangle \quad J = \langle p, p - q, f_H \rangle
\]
be two equal ideals cutting out the regular fibre $X_0$ through 0 and let $I_{\text{hom}}$ and $J_{\text{hom}}$ be the respective homogenisations. However, $I_{\text{hom}} \neq J_{\text{hom}}$, so they define distinct projective varieties. Performing the necessary computations, we obtain the same cohomological diamonds, and the same Euler characteristics as for the corresponding varieties of 6.1.

We then went further to check for the appearances of 16’s and 1’s in the diamonds of the singular fibres at 1 and indeed, they reappeared.

\begin{center}
\begin{tabular}{cccc}
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & ? & 0 \\
0 & 0 & 16 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{tabular}
\end{center}

\textbf{Figure 4}. The diamonds of two projectivisations of the singular fibre over 1 corresponding to $H_0 = \text{Diag}(3, -2, -1)$, $H = \text{Diag}(1, -1, 0)$.

\textit{Remark} 9. While we were making the amendments to an earlier version of this work. Katzarkov, Kontsevich, and Pantev posted [KKP], which gives 3 definitions of Hodge numbers for Landau–Ginzburg models. Understanding the relation between the diamonds we gave here and those Hodge numbers now provides and entirely new perspective for our work.

7. \textbf{Open Questions}

We finish by posing the following open questions. How many compactifications can be obtained via homogenisation? Is there a preferred choice in the sense that it maintains the topology closest to the original variety? Given two compactifications with distinct numerical invariants, do there exist compactifications realising the intermediate values of the invariants?

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