ON THE SINGULARITY OF QUILLEN METRICS

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Abstract. Let \(\pi : X \to S\) be a holomorphic map from a compact Kähler manifold \((X, g_X)\) to a compact Riemann surface \(S\). Let \(\Sigma_\pi\) be the critical locus of \(\pi\) and let \(\Delta = \pi(\Sigma_\pi)\) be the discriminant locus. Let \((\xi, h_{\xi})\) be a meromorphic Hermitian vector bundle on \(X\). We determine the singularity of the Quillen metric on \(\det R\pi_*\xi\) near \(\Delta\) with respect to \(g_X|_{TX/S}\) and \(h_{\xi}\).

1. Introduction

Let \(X\) be a compact Kähler manifold of dimension \(n+1\) with Kähler metric \(g_X\), and let \(S\) be a compact Riemann surface. Let \(\pi : X \to S\) be a surjective holomorphic map such that every connected component of \(X\) is mapped surjectively to \(S\). Let \(\Sigma_\pi := \{x \in X; d\pi(x) = 0\}\) be the critical locus of \(\pi\). For \(t \in S\), set \(X_t := \pi^{-1}(t)\). The relative tangent bundle of \(\pi\): \(X \to S\) is the subbundle of \(TX|_{X/\Sigma_\pi}\) defined as \(TX/S := \ker \pi_*|_{X/\Sigma_\pi}\). Set

\[
\Delta := \pi(\Sigma_\pi), \quad S^0 := S \setminus \Delta, \quad X^0 := X|_{S^0}, \quad \pi^0 := \pi|_{X^0}.
\]

Then \(\pi^0 : X^0 \to S^0\) is a holomorphic family of compact Kähler manifolds. Let \(g_{X/S} := g_X|_{TX/S}\) be the Hermitian metric on \(TX/S\) induced from \(g_X\).

Let \(\xi \to X\) be a holomorphic vector bundle on \(X\) equipped with a Hermitian metric \(h_\xi\). Let \(\lambda(\xi) = \det R\pi_*\xi\) be the determinant of the cohomologies of \(\xi\). By [5], [14], [15], \(\lambda(\xi)|_{S^0}\) is equipped with the Quillen metric \(||\cdot||_{\lambda(\xi), Q}^2\) with respect to the metrics \(g_{X/S}\) and \(h_\xi\).

Let \(0 \in \Delta\) be an arbitrary critical value of \(\pi\), and let \((U, t)\) be a coordinate neighborhood of \(S\) centered at \(0\) with \(U \cap \Delta = \{0\}\). Set \(U^0 := U \setminus \{0\}\).

Let \(\sigma\) be a nowhere vanishing holomorphic section of \(\lambda(\xi)|_U\). Then \(\log ||\sigma||_{\lambda(\xi), Q}^2\) is a \(C^\infty\) function on \(U^0\) by [5]. The purpose of this article is to study the behavior of \(\log ||\sigma(t)||_{\lambda(\xi), Q}^2\) as \(t \to 0\).

For a holomorphic vector bundle \(F\) over a complex manifold with zero-section \(Z\), define the projective-space bundle \(\mathbb{P}(F) = (F \setminus Z)/\mathbb{C}^*\). The dual projective-space bundle \(\mathbb{P}(F)^\vee\) is defined as \(\mathbb{P}(F)^\vee := \mathbb{P}(F^\vee)\), where \(F^\vee\) is the dual vector bundle of \(F\).

Following Bismut [3], we consider the Gauss map \(\mu : X \setminus \Sigma_\pi \to \mathbb{P}(TX)^\vee\) that assigns \(x \in X \setminus \Sigma_\pi\) the hyperplane \(\ker(\pi_*)|_x \subset \mathbb{P}(T_xX)^\vee\). Since \(\mu\) extends to a meromorphic map \(\mu : X \dashrightarrow \mathbb{P}(TX)^\vee\), there exists a resolution \(q : (\tilde{X}, E) \to (X, \Sigma_\pi)\) of the indeterminacy of \(\mu\) such that \(\tilde{\mu} := \mu \circ q\) extends to a holomorphic map from \(\tilde{X}\) to \(\mathbb{P}(TX)^\vee\) and such that \(E\) is a normal crossing divisor of \(\tilde{X}\). (For the scheme structure of \(E\), see Sect.3.) Let \(U\) be the universal hyperplane bundle of rank \(n = \dim X/S\) over \(\mathbb{P}(TX)^\vee\), and let \(H := \mathcal{O}(\mathbb{P}(TX)^\vee)(1)\).

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After Barlet [1], we define a subspace of $C^0(U)$ by
\[ B(U) := C^\infty(U) \oplus \bigoplus_{r \in \mathbb{Q} \cap (0,1]} \bigoplus_{k=0}^n |t|^{2r} (\log |t|)^k \cdot C^\infty(U). \]
A function $\varphi(t) \in B(U)$ has an asymptotic expansion at $0 \in \Delta$, i.e., there exist $r_1, \ldots, r_m \in \mathbb{Q} \cap (0,1]$ and $f_0, f_{1, k} \in C^\infty(U)$, $l = 1, \ldots, m$, $k = 0, \ldots, n$, such that
\[ \varphi(t) = f_0(t) + \sum_{l=1}^m \sum_{k=0}^n |t|^{2r_l} (\log |t|)^k f_{l, k}(t). \]
In what follows, if $f(t), g(t) \in C^\infty(U^p)$ satisfies $f(t) - g(t) \in B(U)$, we write $f \equiv_B g$.

For a complex vector bundle $F$ over a complex manifold, $c_i(F)$, $\text{Td}(F)$, and $\text{ch}(F)$ denote the $i$-th Chern class, the Todd genus, and the Chern character of $F$, respectively.

We can state the main result of this article, which generalizes [3, §5] and [16]:

**Theorem 1.1.** The following identity holds:
\[ \log \| \sigma \|^2_{Q, \lambda(\xi)} \equiv_B \left( \int_{E \cap \eta^{-1}(X_0)} \bar{\mu}^* \left\{ \text{Td}(U) \frac{\text{Td}(H) - 1}{c_1(H)} \right\} q^* \text{ch}(\xi) \right) \log |t|^2. \]

By Theorem 1.1, $\| \cdot \|^2_{Q, \lambda(\xi)}$ extends to a singular Hermitian metric on $\lambda(\xi)$. Let $\pi_\ast$ denote the integration along the fibers of $\pi$. As a consequence of Theorem 1.1 and the curvature formula for Quillen metrics [5], we get the following:

**Corollary 1.2.** The $(1,1)$-form $\pi_\ast(\text{Td}(TX/S, g_X/S) \text{ch}(\xi, h_\xi))^{(1,1)}$ lies in $L^p_{\text{loc}}(S)$ for some $p > 1$, and the curvature current of $(\lambda(\xi), \| \cdot \|_{Q, \lambda(\xi)})$ is given by the following formula on $U$:
\[ c_1(\lambda(\xi), \| \cdot \|_{Q, \lambda(\xi)}) = \pi_\ast(\text{Td}(TX/S, g_X/S) \text{ch}(\xi, h_\xi))^{(1,1)} \]
\[ - \left( \int_{E \cap \eta^{-1}(X_0)} \bar{\mu}^* \left\{ \text{Td}(U) \frac{\text{Td}(H) - 1}{c_1(H)} \right\} q^* \text{ch}(\xi) \right) \delta_0, \]
where $\delta_0$ denotes the Dirac $\delta$-current supported at 0.

The proof of Theorem 1.1 is quite similar to that of Bismut in [3, §5], and we just follow his argument. There are essentially no new ideas except a systematic use of the Gauss maps for the family $\pi: X \to S$; in fact, the Gauss maps were already used by Bismut in [3].

The existence of an asymptotic expansion of the Quillen norm $\log \| \sigma \|^2_{Q, \lambda(\xi)}$ was first shown by Bismut-Bost [4, Sect. 13.(b)] when $\pi: X \to S$ is a family of curves and by the author [16] when $\Sigma_\pi$ is isolated. In [9], Theorem 1.1 shall play an crucial role in the study of analytic torsion of Calabi-Yau threefolds.

Let $s_\Delta$ be a section of $O_S(\Delta)$ defining the reduced divisor $\Delta$. Let $\| \cdot \|$ be a $C^\infty$ Hermitian metric on $O_S(\Delta)$. By Theorem 1.1,
\[ \log \| \sigma(t) \|^2_{Q, \lambda(\xi)} - \left( \int_{E \cap \eta^{-1}(X_0)} \bar{\mu}^* \left\{ \text{Td}(U) \frac{\text{Td}(H) - 1}{c_1(H)} \right\} q^* \text{ch}(\xi) \right) \log \| s_\Delta(t) \|^2 \]
has a finite limit as $t \to 0$. In Section 6, we shall compute this limit in terms of various secondary objects, which extends some results in [3, §5].
This article is organized as follows. In Sections 2 and 3, we explain the Gauss maps associated to the family \( \pi: X \to S \) and their resolutions. In Sections 5 and 6, we prove the main theorem. In Sections 7 and 8, we verify the compatibility of Theorem 1.1 with the corresponding earlier results of Bismut [3] and the author [16]. In Sections 4 and 9, we prove some technical results. The problem treated in Section 9 seems to be related with the regularity problem of the star products of Green currents [8].

For a complex manifold, we set \( d^c = \frac{1}{2\pi i}(\partial - \bar{\partial}) \). Hence \( dd^c = \frac{1}{2\pi i}\bar{\partial}\partial \). We keep the notation in Sect. 1 throughout this article.

### 2. The Gauss maps

Let \( \Omega^1_X \) be the holomorphic cotangent bundle of \( X \). Let \( \Pi: \mathbb{P}(\Omega^1_X \otimes \pi^*TS) \to X \) be the projective-space bundle associated with \( \Omega^1_X \otimes \pi^*TS \). Since \( \dim S = 1 \), we have \( \mathbb{P}(\Omega^1_X \otimes \pi^*TS) = \mathbb{P}(\Omega^1_X) \). Let \( \Pi^\vee: \mathbb{P}(TX)^\vee \to X \) be the dual projective-space bundle of \( \mathbb{P}(TX) \), whose fiber \( \mathbb{P}(T_x X)^\vee \) is the set of hyperplanes of \( T_x X \) passing through the zero vector of \( T_x X \). We have the canonical isomorphisms

\[
\mathbb{P}(\Omega^1_X \otimes \pi^*TS) = \mathbb{P}(\Omega^1_X) \cong \mathbb{P}(TX)^\vee.
\]

Let \( x \in X \setminus \Sigma_x \). Let \( t \) be a holomorphic local coordinate of \( S \) near \( \pi(x) \in S \). We define the Gauss maps \( \nu: X \setminus \Sigma_x \to \mathbb{P}(\Omega^1_X \otimes \pi^*TS) \) and \( \mu: X \setminus \Sigma_x \to \mathbb{P}(TX)^\vee \) by

\[
\nu(x) := [d\pi_x] = \left[ \sum_{i=0}^n \frac{\partial(t \circ \pi)}{\partial z_i} (x) dz_i \otimes \frac{\partial}{\partial t}\right], \quad \mu(x) := [T_x X_{\pi(x)}].
\]

Under the canonical isomorphism \( \mathbb{P}(\Omega^1_X \otimes \pi^*TS) \cong \mathbb{P}(TX)^\vee \), one has

\[
\nu = \mu.
\]

Let

\[
L := \mathcal{O}_{\mathbb{P}(\Omega^1_X \otimes \pi^*TS)}(-1) \subset \Pi^*(\Omega^1_X \otimes \pi^*TS)
\]

be the tautological line bundle over \( \mathbb{P}(\Omega^1_X \otimes \pi^*TS) \), and set

\[
Q := \Pi^*(\Omega^1_X \otimes \pi^*TS)/L.
\]

We have the exact sequence of holomorphic vector bundles on \( \mathbb{P}(\Omega^1_X \otimes \pi^*TS) \):

\[
\mathcal{S}: 0 \to L \to \Pi^*(\Omega^1_X \otimes \pi^*TS) \to Q \to 0.
\]

Let \( H = \mathcal{O}_{\mathbb{P}(TX)^\vee}(1) \), and let \( U \) be the universal hyperplane bundle of \( (\Pi^\vee)^*TX \). Then the dual of \( \mathcal{S} \) is given by

\[
\mathcal{S}^\vee: 0 \to U \to (\Pi^\vee)^*TX \to H \to 0.
\]

Since \( T_x X_{\pi(x)} = \{ v \in T_x X; d\pi_x(v) = 0 \} \), we have on \( X \setminus \Sigma_x \)

\[
T X/S = \mu^* U.
\]

Let \( g_U \) be the Hermitian metric on \( U \) induced from \( (\Pi^\vee)^* g_X \), and let \( g_H \) be the Hermitian metric on \( H \) induced from \( (\Pi^\vee)^* g_X \) by the \( C^\infty \)-isomorphism \( H \cong U^\perp \). On \( X \setminus \Sigma_x \), we have

\[
(TX/S, g_{X/S}) = \mu^*(U, g_U).
\]

Let \( g_S \) be a Hermitian metric on \( S \). Let \( g_{\Omega^1_X} \) be the Hermitian metric on \( \Omega^1_X \) induced from \( g_X \). Let \( g_L \) be the Hermitian metric on \( L \) induced from the metric \( \Pi^*(g_{\Omega^1_X} \otimes \pi^*g_S) \) by the inclusion \( L \subset \Pi^*(\Omega^1_X \otimes \pi^*TS) \). Let \( g_q \) be the Hermitian metric on \( Q \) induced from \( \Pi^*(g_{\Omega^1_X} \otimes \pi^*g_S) \) by the \( C^\infty \)-isomorphism \( Q \cong L^\perp \).

\[
\mathcal{S}^\vee: 0 \to U \to (\Pi^\vee)^*TX \to H \to 0.
\]
Let \( c_1(L, g_L) \) be the Chern form of \((L, g_L)\). Since \( d\pi \) is a nowhere vanishing holomorphic section of \( \nu^*L|_{X\setminus \Sigma_\pi} \), we get the following equation on \( X\setminus \Sigma_\pi \)
\[-dd^c \log \|d\pi\|^2 = \nu^*c_1(L, g_L).\]

3. Resolution of the Gauss maps

Since \( \Sigma_\pi \) is a proper analytic subset of \( X \), the maps \( \nu : X\setminus \Sigma_\pi \to \mathbb{P}(\Omega_X^1 \otimes \pi^* T_S) \) and \( \mu : X\setminus \Sigma_\pi \to \mathbb{P}(T_X)^\vee \) extend to meromorphic maps \( \nu : X \to \mathbb{P}(\Omega_X^1 \otimes \pi^* T_S) \) and \( \mu : X \to \mathbb{P}(T_X)^\vee \) by [13, Th. 4.5.3]. By Hironaka, there exists a compact Kähler manifold \( \tilde{X} \), a normal crossing divisor \( E \subset \tilde{X} \), a birational holomorphic map \( q : \tilde{X} \to X \), and holomorphic maps \( \tilde{\nu} : \tilde{X} \to \mathbb{P}(\Omega_X^1 \otimes \pi^* T_S) \) and \( \tilde{\mu} : \tilde{X} \to \mathbb{P}(T_X)^\vee \) satisfying the following conditions:

(i) \( q_{\tilde{X}\setminus q^{-1}(\Sigma_\pi)} : \tilde{X} \setminus q^{-1}(\Sigma_\pi) \to X\setminus \Sigma_\pi \) is an isomorphism;
(ii) \( q^{-1}(\Sigma_\pi) = E; \)
(iii) \( (\pi \circ q)^{-1}(b) \) is a normal crossing divisor of \( \tilde{X} \) for all \( b \in \Delta; \)
(iv) \( \tilde{\nu} = \nu \circ q \) and \( \tilde{\mu} = \mu \circ q \) on \( \tilde{X} \setminus E. \)

Then \( \tilde{\nu} = \tilde{\mu} \) under the canonical isomorphism \( \mathbb{P}(\Omega_X^1 \otimes \pi^* T_S) \cong \mathbb{P}(T_X)^\vee \). We set
\[ \tilde{\pi} := \pi \circ q \]
and \( \tilde{X}_s := \tilde{\pi}^{-1}(s) \) for \( s \in S \). Similarly, we set \( E_b := E \cap \tilde{X}_b \) for \( b \in \Delta \). Since \( E = q^{-1}(\Sigma_\pi) \subset \tilde{\pi}^{-1}(\Delta) \), we have \( E = \bigcup_{b \in \Delta} E_b. \)

Let \( I_{\Sigma_\pi} \) be the ideal sheaf of \( \Sigma_\pi \). For every \( p \in \Sigma_\pi \), the sheaf \( I_{\Sigma_\pi} \) has the following expression on a neighborhood of \( p: \)
\[ I_{\Sigma_\pi} = \mathcal{O}_X \left( \frac{\partial (t \circ \pi)}{\partial z_0}(z), \ldots, \frac{\partial (t \circ \pi)}{\partial z_n}(z) \right). \]

Define the ideal sheaf \( I_E \) of \( E \) as
\[ I_E = q^{-1}I_{\Sigma_\pi}. \]

Denote by \( \delta_E \) the \((1,1)\)-current on \( \tilde{X} \) defined as the integration over \( E \), i.e.,
\[ \delta_E(\psi) := \int_E \psi \lvert_E \] for all \( C^\infty(n,n) \)-form \( \psi \) on \( \tilde{X} \). Since \( \tilde{\nu}^*L = q^*\nu^*L \), \( q^*d\pi \) extends to a holomorphic section of \( \tilde{\nu}^*L \) with zero divisor \( E \) by the definition of the ideal sheaf \( I_E \). By the Poincaré-Leung formula, the following identity of currents on \( \tilde{X} \) holds
\[ -dd^c (q^*\log \|d\pi\|^2) = \tilde{\nu}^*c_1(L, g_L) - \delta_E. \]

4. Regularity of the direct image of differential forms

Recall that \( (U, t) \) is a coordinate neighborhood of \( S \) centered at the critical value \( 0 \in \Delta \). Set \( D := \{ (s, t) \in S \times U; s = t \} \). Then \( D \) is a divisor of \( S \times U \). Let \( [D] \) be the line bundle on \( S \times U \) defined by the divisor \( D \). Let \( s_D \) be a section of \( [D] \) with zero divisor \( D \). Let \( B \subset S \) be a finite subset with \( 0 \in B \). By shrinking \( U \) if necessary, we may assume that \( U \cap B = \{ 0 \} \). Let \( \| \cdot \|_D \) be a \( C^\infty \) Hermitian metric on \([D]\) such that
\[ \|s_D(b, t)\|_D = 1, \quad \forall (b, t) \in (B \setminus \{0\}) \times U. \]

We set \( s_t := s_D|_{S \times \{t\}} \) and \( \| \cdot \|_t := \| \cdot \|_D|_{S \times \{t\}} \) for \( t \in U \). Then \( \text{div}(s_t) = \{ t \} \) and \( \|s_t\|^2 \in C^\infty(S \times U). \)

Let \( V \) be a compact connected complex manifold with \( \dim V = n + 1 \). Let \( f : V \to S \) be a proper surjective holomorphic map. We set \( V_t := f^{-1}(t) \) for \( t \in S \).
Let $\mathcal{F} := (F, \| \cdot \|_F)$ be a holomorphic Hermitian line bundle on $V$, and let $\alpha$ be a holomorphic section of $F$ with

$$\text{div}(\alpha) \subset \sum_{b \in B} V_b.$$ 

Denote by $f_*$ the integration along the fibers of $f$. In Section 4, we assume that $\varphi$ is a $\partial$-closed and $\bar{\partial}$-closed $C^\infty(n, n)$-form on $V$.

**Lemma 4.1.** There exists a Hölder continuous function $\eta$ on $\mathcal{U}$ such that

$$f_* \{ (\log \| \alpha \|_F^2) \varphi \}^{(0, 0)} - \left( \int_{\text{div}(\alpha) \cap V_0} \varphi \right) \log \| s_0 \|_0^2 = \eta.$$ 

**Proof.** Since $\log \| \alpha \|_F^2 \varphi$ is a locally integrable differential form on $V$, we have $f_* \{ (\log \| \alpha \|_F^2) \varphi \}^{(0, 0)} \in L^1_\log(S) \cap C^\infty(S^*)$. Since $dd^c$ commutes with $f_*$ and since $\varphi$ is $d$ and $d^c$-closed, we get the following equation of currents on $\mathcal{U}$:

$$dd^c f_* \{ (\log \| \alpha \|_F^2) \varphi \}^{(0, 0)} = [f_* \{ dd^c((\log \| \alpha \|_F^2) \wedge \varphi) \}]^{(1, 1)} = -[f_* \{ (c_1(\mathcal{F}) - \delta_{\text{div}(\alpha)}) \wedge \varphi \}]^{(1, 1)} = \left( \int_{\text{div}(\alpha) \cap V_0} \varphi \right) \delta_0 - [f_* \{ c_1(\mathcal{F}) \wedge \varphi \}]^{(1, 1)}.$$

By Lemma 9.2 below, there exists $\psi \in \mathcal{B}(\mathcal{U})$ such that

$$[f_* \{ c_1(\mathcal{F}) \wedge \varphi \}]^{(1, 1)}(t) = \psi(t) \frac{dt \wedge d\bar{t}}{|t|^2}, \quad \psi(0) = 0.$$ 

Since $\psi(0) = 0$, there exists $\nu \in \mathbb{Q} \cap (0, 1]$ such that $\psi(t) \in \sum_{k \leq n} |t|^{2 \nu} (\log |t|)^k \mathcal{B}(\mathcal{U})$. Hence $|t|^{-2} \psi(t) \in L^1_{\text{loc}}(\mathcal{U})$ for some $p > 1$. By the ellipticity of the Laplacian and the Sobolev embedding theorem, there exists a Hölder continuous function $\chi$ on $\mathcal{U}$ satisfying the following equation of currents on $\mathcal{U}$

$$[f_* \{ c_1(\mathcal{F}) \wedge \varphi \}]^{(1, 1)} = dd^c \chi.$$ 

This, together with (4.2) and the equation of currents $dd^c \log |t|^2 = \delta_0$ on $\mathcal{U}$, implies the assertion, because $\log \| s_0 \|_0^2 - \log |t|^2 \in C^\infty(\mathcal{U})$.

**Lemma 4.2.** The following identity holds for all $t \in \mathcal{U}^c$:

$$\int_{V_t} (\log \| \alpha \|_F^2) \varphi = \left( \int_{\text{div}(\alpha) \cap V_0} \varphi \right) \log \| s_t(0) \|_t^2 - \int_V (f^* \log \| s_t \|_t^2) c_1(\mathcal{F}) \wedge \varphi + \int_V (\log \| \alpha \|_F^2) f^* c_1([t], \| \cdot \|_t) \wedge \varphi.$$ 

**Proof.** Since $V_t \cap \text{div}(\alpha) = \emptyset$ for $t \in \mathcal{U}^c$, $V_t$ meets $\text{div}(\alpha)$ properly. Since $\varphi$ is $\partial$ and $\bar{\partial}$-closed, we deduce from [11, Th. 2.2.2] the following identity by setting $X = W = V, Y = V_t, Z = \text{div}(\alpha)$, and $g_Y = -f^* \log \| s_t \|_t^2, g_Z = -\log \| \alpha \|_F^2$ in [11,
The following identity of functions on $\mathcal{U}$ holds:

\[
\int_{V_t} \left( \log ||\alpha||^2_{F} \right) \varphi = \sum_{b \in B} \left( \int_{\text{div}(\alpha) \cap V_b} \varphi \right) \log ||s_1(b)||^2_t - \int_{V_t} (f^* \log ||s_1||^2_t) c_1(F) \wedge \varphi \\
+ \int_{V_t} (\log ||\alpha||^2_{F}) f^* c_1([t], \| \cdot \|) \wedge \varphi,
\]

where we used the assumption $\text{div}(\alpha) \subset \sum_{b \in B} V_b$. (See also [15, p.59, 1.3-1.7].) Since $||s_1(b)||_t = 1$ for $(b, t) \in (B \setminus \{0\}) \times \mathcal{U}$ by (4.1), the result follows from (4.3).

**Lemma 4.3.** The following identity holds

\[
\lim_{t \to 0^+} \left\{ \int_{V_t} \left( \log ||\alpha||^2_{F} \right) \varphi - \left( \int_{\text{div}(\alpha) \cap V_o} \varphi \right) \log ||s_0(t)||^2_o \right\} = \\
\int_{V_t} \left( \log ||\alpha||^2_{F} \right) f^* c_1([0], \| \cdot \|) \wedge \varphi - \int_{V_t} (f^* \log ||s_0||^2_o) c_1(F) \wedge \varphi.
\]

**Proof.** By Lemma 4.2, we have

\[
\int_{V_t} \left( \log ||\alpha||^2_{F} \right) \varphi = \left( \int_{\text{div}(\alpha) \cap V_o} \varphi \right) \log ||s_0(t)||^2_o - \int_{V_t} (f^* \log ||s_1||^2_t) c_1(F) \wedge \varphi \\
+ \int_{V_t} (\log ||\alpha||^2_{F}) f^* c_1([t], \| \cdot \|) \wedge \varphi.
\]

Since $\lim_{s \to 0}(\log(||s_1(0)||^2_t/||s_0(t)||^2_o) = 0$, the assertion follows from (4.4).

**Lemma 4.4.** The following identity of functions on $\mathcal{U}^\circ$ hold:

\[
f_*(\left( \log ||\alpha||^2_{F} \right) \varphi)^{(0,0)} \equiv_B \left( \int_{\text{div}(\alpha) \cap V_o} \varphi \right) \log ||s_0||^2_o.
\]

**Proof.** For $t \in \mathcal{U}^\circ$, set

\[
I_1(t) := \int_{V_t} (f^* \log ||s_1||^2_t) c_1(F) \varphi, \quad I_2(t) := \int_{V_t} (\log ||\alpha||^2_{F}) f^* c_1([t], \| \cdot \|) \varphi.
\]

By (4.4), it suffices to prove that $I_1 \in \mathcal{B}(\mathcal{U})$ and $I_2 \in \mathcal{B}(\mathcal{U})$.

Let $\{(W_\lambda, z_\lambda)\}_{\lambda \in A}$ be a system of local coordinates on $V$. Since $V$ is compact, we may assume $\#A < +\infty$. For every $\lambda \in A$, there exist $F_\lambda \in \mathcal{O}(W_\lambda)$, $G_\lambda \in \mathcal{O}(W_\lambda)$, $A_\lambda \in C^\infty(W_\lambda)$, and $B_\lambda \in C^\infty(W_\lambda \times \mathcal{U})$ such that

\[
\bar{\pi}^* \log ||s_1||^2_{W_\lambda}(z_\lambda) = \log |F_\lambda(z_\lambda)| - t^2 + B_\lambda(z_\lambda, t),
\]

\[
\log ||\alpha||^2_{W_\lambda}(z_\lambda) = \log |G_\lambda(z_\lambda)|^2 + A_\lambda(z_\lambda).
\]

Let $\{\varrho_\lambda\}_{\lambda \in A}$ be a partition of unity of $V$ subject to the covering $\{W_\lambda\}_{\lambda \in A}$. We set $\chi_\lambda := \varrho_\lambda c_1(F) \varphi$. Then

\[
I_1(t) = \sum_{\lambda \in A} \int_{W_\lambda} \log |F_\lambda(z_\lambda)| - t^2 \cdot \chi_\lambda(z_\lambda) + \sum_{\lambda \in A} \int_{W_\lambda} B_\lambda(z_\lambda, t) \chi_\lambda(z_\lambda).
\]

Since the first term of the right hand side of (4.5) lies in $\mathcal{B}(\mathcal{U})$ by Theorem 9.1 below, we get $I_1 \in \mathcal{B}(\mathcal{U})$. 
We set \( \theta_\lambda := \varrho_\lambda \bar{\pi}^* c_1([t], \| \cdot \|_t) \varphi \). Then \( \theta_\lambda(z_\lambda, t) \) is a \( C^\infty (n + 1, n + 1) \)-form on \( W_\lambda \times \mathcal{U} \). Since

\[
I_2(t) = \sum_{\lambda \in \Lambda} \int_{W_\lambda} \log |G_\lambda(z_\lambda)|^2 \cdot \theta_\lambda(z_\lambda, t) + \sum_{\lambda \in \Lambda} \int_{W_\lambda} A_\lambda(z_\lambda) \theta_\lambda(z_\lambda, t),
\]

we get \( I_2 \in C^\infty(\mathcal{U}) \). This completes the proof. \( \Box \)

**Corollary 4.5.** The following identity holds

\[
\lim_{t \to 0} \left\{ \int_{X_1} q^* (\log \|d\pi\|^2) \varphi - \left( \int_{E_0} \varphi \right) \log \|s_0(t)\|_0^2 \right\} = \int_X (q^* \log \|d\pi\|^2) \bar{\pi}^* c_1([0], \| \cdot \|_0) \wedge \varphi - \int_X \left( \bar{\pi}^* \log \|s_0\|_0^2 \right) \nu^* c_1(L, g_L) \wedge \varphi.
\]

**Proof.** Setting \( V = \tilde{X}, f = \tilde{\pi}, \mathcal{F} = \nu^*(L, g_L) \) and \( \alpha = q^*(d\pi) \) in Lemma 4.3, we get the result. \( \Box \)

**Corollary 4.6.** The following identity of functions on \( U^n \) hold:

\[
\bar{\pi}_*(q^*(\log \|d\pi\|^2) \varphi)^{(0, 0)} = \left( \int_{E_0} \varphi \right) \log \|s_0\|_0^2.
\]

**Proof.** Setting \( V = \tilde{X}, f = \tilde{\pi}, \mathcal{F} = \nu^*(L, g_L) \) and \( \alpha = q^*(d\pi) \) in Lemma 4.4, we get the result. \( \Box \)

5. **Behavior of the Quillen norm of the Knudsen-Mumford section**

Let \( \Gamma \subset X \times S \) be the graph of \( \pi \), which is a smooth divisor on \( X \times S \). Let \( [\Gamma] \) be the holomorphic line bundle on \( X \times S \) associated to \( \Gamma \). Let \( s_\Gamma \in H^0(X \times S, [\Gamma]) \) be the canonical section of \( [\Gamma] \), so that \( \text{div}(s_\Gamma) = \Gamma \). We identify \( X \) with \( \Gamma \).

Let \( i: \Gamma \hookrightarrow X \times S \) be the inclusion. Let \( p_1: X \times S \to X \) and \( p_2: X \times S \to S \) be the projections. On \( X \times S \), we have the exact sequence of coherent sheaves,

\[
0 \to \mathcal{O}_{X \times S}([\Gamma]^{-1}) \otimes p_1^* \xi \xrightarrow{s_\Gamma} \mathcal{O}_{X \times S}(p_1^* \xi) \to i_\ast \mathcal{O}_\Gamma(p_1^* \xi) \to 0.
\]

Let \( \lambda(p_1^* \xi), \lambda([\Gamma]^{-1}) \otimes p_1^* \xi), \lambda(\xi) \) be the determinants of the direct images \( R(p_1)^\ast p_1^! \xi, R(p_2)^\ast ([\Gamma]^{-1}) \otimes p_1^! \xi, R \pi \ast \xi \), respectively. By definition [5], [12], [15],

\[
\lambda(\xi) = \bigotimes_{q \geq 0} (\det R^q \pi_\ast \xi)^{(-1)^q}.
\]

Under the isomorphism \( p_1^! \xi|_V \cong \xi \) induced from the identification \( p_1 : \Gamma \to X \), the holomorphic line bundle on \( S \)

\[
\lambda := \lambda([\Gamma]^{-1}) \otimes p_1^! \xi \otimes \lambda(p_1^! \xi)^{-1} \otimes \lambda(\xi)
\]

carries the canonical nowhere vanishing holomorphic section \( \sigma_{KM} \) by [7], [12].

Let \( \mathcal{V} \subset \mathcal{U} \) be a relatively compact neighborhood of \( 0 \in \Delta \), and set \( \mathcal{V}^0 := \mathcal{V} \setminus \{0\} \). On \( \pi^{-1}(\mathcal{U}) \), we identify \( \pi \) (resp. \( d\pi \)) with \( т \pi \) (resp. \( d(t \pi \pi) \)). Hence \( \pi \in \mathcal{O}(\pi^{-1}(\mathcal{U})) \) and \( d\pi \in H^0(\pi^{-1}(\mathcal{U}), \Omega^1_X) \) in what follows.

Let \( h_{[\Gamma]} \) be a \( C^\infty \) Hermitian metric on \( [\Gamma] \) with

\[
h_{[\Gamma]}(s_\Gamma, s_\Gamma)(w, t) = \begin{cases} |\pi(w) - t|^2 & \text{if } (w, t) \in \pi^{-1}(\mathcal{V}) \times \mathcal{V}, \\ 1 & \text{if } (w, t) \in (X \setminus \pi^{-1}(\mathcal{U})) \times \mathcal{V}. \end{cases}
\]
Let \( h_{[\Gamma]}^{-1} \) be the metric on \([\Gamma]^{-1}\) induced from \( h_{[\Gamma]}\).

Let \( \| \cdot \|_{Q,L(\xi)} \) be the Quillen metric on \( \lambda(\xi) \) with respect to \( g_{X/S}, h_{\xi} \). Let \( \| \cdot \|_{Q,L([\Gamma]^{-1} \otimes \xi^1)} \) (resp. \( \| \cdot \|_{Q,L(p_1^*\xi)} \)) be the Quillen metric on \( \lambda([\Gamma]^{-1} \otimes p_1^*\xi) \) (resp. \( \lambda(p_1^*\xi) \)) with respect to \( g_X, h_{[\Gamma]}^{-1} \otimes h_{\xi} \) (resp. \( g_X, h_{\xi} \)). Let \( \| \cdot \|_{Q,L} \) be the Quillen metric on \( \lambda \) defined as the tensor product of those on \( \lambda([\Gamma]^{-1} \otimes p_1^*\xi), \lambda(p_1^*\xi)^{-1}, \lambda(\xi) \).

For a complex manifold \( Y \), \( A^{p,q}(Y) \) denotes the vector space of \( C^\infty(p,q) \)-forms on \( Y \). We set \( A(Y) := \bigoplus_{p \geq 0} A^{p,p}(Y)/\text{Im} \partial + \text{Im} \bar{\partial} \).

For a Hermitian vector bundle \((F,H)\) over \( Y \), \( c_i(F,H), Td(F,H), ch(F,H) \in \bigoplus_{p \geq 0} A^{p,p}(Y) \) denote the \( i \)-th Chern form, the Todd form, and the Chern character form of \((F,H)\) with respect to the holomorphic Hermitian connection, respectively. Let \( R(F) \) denote the R-genus of Gillet-Soule [7, (0.4)], [15, p. 160].

**Theorem 5.1.** The following identity of functions on \( U^\alpha \) holds

\[
\log \|\sigma_{KM}\|^2_{Q,L} \equiv_{B} \left( \int_{E_0} \bar{\mu} \left\{ \frac{Td(U)}{c_1(U)} - 1 \right\} q^*ch(\xi) \right) \log |t|^2.
\]

**Proof.** We follow Bismut [3, Sect.5]. (See also [17, Th.6.3].)

**Step 1** Let \([X_t]\) be the holomorphic line bundle on \( X \) associated to the divisor \( X_t \). Then \([X_t] = [\Gamma]|_{X_t}\). We define the canonical section \( s_t \) of \([X_t]\) by \( s_t := s_t|_{X_t \times \{t\}} \in H^0(\{t\}, [X_t])\). Then \( \text{div}(s_t) = X_t \). Let \( i_t: X_t \hookrightarrow X \) be the embedding, and set \( \xi_t := \xi|_{X_t}\). By (5.1), we get the exact sequence of coherent sheaves on \( X \),

\[
0 \rightarrow \mathcal{O}_X([X_t]^{-1} \otimes \xi) \xrightarrow{s_{t\xi}} \mathcal{O}_X(\xi) \rightarrow (i_t)_*\mathcal{O}_{X_t}(\xi) \rightarrow 0.
\]

Let \( \lambda([X_t]^{-1} \otimes \xi) \) and \( \lambda(\xi_t) \) be the determinants of the cohomology groups of \([X_t]^{-1} \otimes \xi \) and \( \xi_t \), respectively. Then \( \lambda_t = \lambda([X_t]^{-1} \otimes \xi) \otimes \lambda(\xi_t) \).

Set \( h_{[X_t]} = h_{[\Gamma]|_{X_t \times \{t\}}} \) for \( t \in V \). Then \( h_{[X_t]} \) is a Hermitian metric on \([X_t]\). Let \( h_{[X_t]}^{-1} \) be the Hermitian metric on \([X_t]^{-1}\) induced from \( h_{[X_t]}\).

Let \( N_t = N_{X_t/X} \) (resp. \( N_t^* = N_{X_t/X}^* \)) be the normal (resp. conormal) bundle of \( X_t \) in \( X \). Then \( d\pi|_{X_t} \in H^0(X_t, N_t^*) \) generates \( N_t^* \) for \( t \in U^\alpha \). Let \( h_{N_t^*} \) be the Hermitian metric on \( N_t^* \) defined by

\[
h_{N_t^*}(d\pi|_{X_t}, d\pi|_{X_t}) = 1.
\]

Let \( N_t \) be the Hermitian metric on \( N_t \) induced from \( h_{N_t^*} \). Then we have the identity \( c_1(N_t, h_{N_t}) = 0 \) for \( t \in V^\alpha \).

For \((w,t) \in \pi^{-1}(U) \times U\), set

\[
\bar{s}_t(w,t) = \frac{st(w,t)}{\pi(w)} - t.
\]

Since \( \bar{s}_t(w) - t \) is a holomorphic function on \( \pi^{-1}(U) \times U \) with divisor \( \Gamma \), \( \bar{s}_t \) is a nowhere vanishing holomorphic section of \( [\Gamma]|_{\pi^{-1}(U) \times U} \). Set \( \bar{s}_t|_{X_t} = \bar{s}_t|_{X_t \times \{t\}} \in H^0(X_t, [X_t]|_{X_t}) \) and

\[
ds_t|_{X_t} := d\pi \otimes \bar{s}_t|_{X_t} \in H^0(X_t, N_t^* \otimes [X_t]|_{X_t}).
\]

By (5.2), (5.4), the isomorphism

\[
\otimes ds_t|_{X_t}: [X_t]^{-1} \otimes \xi|_{X_t} \ni v \mapsto ds_t|_{X_t}(v) \in N_t^* \otimes \xi_t
\]

gives an isometry of holomorphic Hermitian vector bundles

\[
([X_t]^{-1} \otimes \xi, h_{[X_t]}^{-1} \otimes h_{\xi})|_{X_t} \cong (N_t^* \otimes \xi_t, h_{N_t^*} \otimes h_{\xi}|_{X_t})
\]

Let \( h_{[\Gamma]}^{-1} \) be the metric on \([\Gamma]^{-1}\) induced from \( h_{[\Gamma]}\).
for all $t \in \mathcal{V}_\infty$. Hence the metrics $h_{[X_t]^{-1}} \otimes h_\xi$ and $h_\xi$ verify assumption (A) of Bismut [2, Def.1.5] with respect to $h_{N_t}$ and $h_\xi|_{X_t}$.

**Step 2** Associated to the exact sequence of holomorphic vector bundles on $X_t$,

$$\mathcal{E}_t: 0 \longrightarrow T X_t \longrightarrow T X|_{X_t} \longrightarrow N_t \longrightarrow 0,$$

one can define the Bott-Chern class $\widetilde{\text{Td}}(\mathcal{E}_t; g_{X_t}, g_X, h_{N_t}) \in \widetilde{A}(X_t)$ by [5, I, f]), [10, I, Sect. 1], [15, Chap. IV, Sect. 3] such that

$$dd^c \widetilde{\text{Td}}(\mathcal{E}_t; g_{X_t}, g_X, h_{N_t}) = Td(TX_t, g_{X_t}) Td(N_t, h_{N_t}) - Td(TX, g_X)|_{X_t}.$$ Notice that our $\widetilde{\text{Td}}(\mathcal{E}_t; g_{X_t}, g_X, h_{N_t})$ and Bismut-Lebeau's $\widetilde{\text{Td}}(T X_t, T X|_{X_t}, h_{N_t})$ are related as follows:

$$\widetilde{\text{Td}}(\mathcal{E}_t; g_{X_t}, g_X, h_{N_t}) = -\widetilde{\text{Td}}(T X_t, T X|_{X_t}, h_{N_t}).$$

Let $Z$ be a general fiber of $\pi: X \rightarrow S$. By applying the embedding formula of Bismut-Lebeau [7, Th. 0.1] (see also [3, Th. 5.6]) to the embedding $i_t: X_t \hookrightarrow X$ and to the exact sequence (5.3), we get for all $t \in \mathcal{V}_\infty$:

$$\log \|\sigma_{KM}(t)\|^2_{Q, \lambda} = \int_{X_t \times \{t\}} \frac{\text{Td}(TX_t, g_{X_t}) \log h_\xi(s \tau)}{\text{ch}(\{s \tau\})} \big| X_t \times \{t\} \big| \chi_N \log h_{[\tau]}(s \tau)$$

$$- \int_{X_t} \frac{\widetilde{\text{Td}}(\mathcal{E}_t; g_{X_t}, g_X, h_{N_t}) \log h_\xi(\xi)}{\text{ch}(\chi_N)}$$

$$- \int_X Td(TX_t) R(TX_t) \log h_\xi(\chi_N) + \int_Z Td(TX_t) R(TX_t) \log h_\xi(\chi_N).$$

Here we used the explicit formula for the Bott-Chern current [6, Rem. 3.5, especially (3.23), Th. 3.15, Th. 3.17] to get the first term of the right hand side of (5.5). Notice that the dual of our $A(\xi)$ was defined as $A(\xi)$ in [7].

By Theorem 9.1 below, the first term of the right hand side of (5.5) lies in $B(\mathcal{U})$. Substituting $c_1(N_t, h_{N_t}) = 0$ into (5.5), we get

$$\log \|\sigma_{KM}(t)\|^2_{Q, \lambda} \equiv B \int_{X_t} -\widetilde{\text{Td}}(\mathcal{E}_t; g_{X_t}, g_X, h_{N_t}) \log h_\xi(\chi_N).$$

**Step 3** Let $g_{N_t}$ be the Hermitian metric on $N_t$ induced from $g_X$ by the $C^\infty$ isomorphism $N_t \cong (TX_t)^\perp$. Let $\widetilde{\text{Td}}(N_t; h_{N_t}, g_{N_t}) \in \widetilde{A}(X_t)$ be the Bott-Chern class [5, I, e]), [10, Sect. 1.2.4], [15, Chap. IV, Sect. 3] such that

$$dd^c \widetilde{\text{Td}}(N_t; h_{N_t}, g_{N_t}) = Td(N_t, h_{N_t}) - Td(N_t, g_{N_t}).$$

By [10, I, Prop. 1.3.2 and Prop. 1.3.4] (see also Lemma 5.3 below),

$$\widetilde{\text{Td}}(\mathcal{E}_t; g_{X_t}, g_X, h_{N_t}) = \widetilde{\text{Td}}(\mathcal{E}_t; g_{X_t}, g_X, g_{N_t}) + Td(TX_t, g_{X_t}) \widetilde{\text{Td}}(N_t; h_{N_t}, g_{N_t}).$$

Since $c_1(N_t, h_{N_t}) = 0$ and $g_{N_t} = \|d\pi\|^2 \cdot h_{N_t}$, we deduce from [10, I, Prop. 1.3.1 and (1.2.5.1)] the identity

$$\widetilde{\text{Td}}(N_t; h_{N_t}, g_{N_t}) = 1 - Td(d^d c \log \|d\pi\|^2) \log \|d\pi\|^2$$

$$= \nu^* \left\{ \frac{1 - Td(-c_1(L, g_L))}{-c_1(L, g_L)} \right\} \log \|d\pi\|^2 \big|_{X_t}. $$
the proof of Theorem 5.1.

Substituting (5.8) and $(TX_i, g_{X_i}) = \mu^*(U, g_U)|_{X_i}$ into (5.7), we get
\begin{equation}
\tilde{Td}(E_i; g_{X_i}, g_X, h_{N_i}) = \tilde{Td}(E_i; g_{X_i}, g_X, h_N) + \mu^* Td(U, g_U) \nu^* \left\{ \frac{1 - Td(-c_1(L, g_L))}{-c_1(L, g_L)} \right\} \log \|d\pi\|^2|_{X_i}.
\end{equation}

Since
\[ E_i = \mu^* S^\vee|_{X_i}, \quad g_{X_i} = \mu^* g_U|_{X_i}, \quad g_X = \mu^* (\Pi^\vee)^* g_X|_{X_i}, \quad g_N = \mu^* g_H|_{X_i}, \]
we deduce from [10, I, Th. 1.2.2 (ii)] that
\begin{equation}
\tilde{Td}(E_i; g_{X_i}, g_X, h_{N_i}) = \mu^* \tilde{Td}(S^\vee; g_U, (\Pi^\vee)^* g_X, g_H)|_{X_i}.
\end{equation}
Comparing (5.9) and (5.10), we get
\begin{equation}
\tilde{Td}(E_i; g_{X_i}, g_X, h_{N_i}) = \mu^* \tilde{Td}(S^\vee; g_U, (\Pi^\vee)^* g_X, g_H)|_{X_i},
\end{equation}
\begin{equation}
+ \mu^* Td(U, g_U) \nu^* \left\{ \frac{1 - Td(-c_1(L, g_L))}{-c_1(L, g_L)} \right\} \log \|d\pi\|^2|_{X_i}.
\end{equation}

Substituting (5.11) into (5.6), we get
\begin{equation}
\log \|\sigma_{K_M}\|_{Q, \lambda}^2 = B - \pi_* \left[ \mu^* \tilde{Td}(S^\vee; g_U, (\Pi^\vee)^* g_X, g_H) \chi(\xi, h_\xi) \right]^{(0,0)}
- \pi_* \left[ \mu^* Td(U, g_U) \nu^* \left\{ \frac{1 - Td(-c_1(L, g_L))}{-c_1(L, g_L)} \right\} \chi(\xi, h_\xi) \log \|d\pi\|^2 \right]^{(0,0)}
\equiv_B - \bar{\pi}_* \left[ \tilde{\mu}^* \tilde{Td}(S^\vee; g_U, (\Pi^\vee)^* g_X, g_H) \tilde{\nu}^* \chi(\xi, h_\xi) \right]^{(0,0)}
+ \bar{\pi}_* \left[ \tilde{\mu}^* Td(U, g_U) \tilde{\nu}^* \left\{ \frac{Td(-c_1(L, g_L)) - 1}{-c_1(L, g_L)} \right\} \tilde{\nu}^* \chi(\xi, h_\xi) \log \|d\pi\|^2 \right]^{(0,0)}.
\end{equation}

Recall that for a $C^\infty$ differential form $\varphi$ on $\tilde{X}$, one has $\bar{\pi}_*(\varphi)^{(0,0)} \in B(\mathcal{U})$ by Barlet [1, Th. 4bis]. Since $q^* \chi(\xi, h_\xi)$ and
\[ \tilde{\mu}^* \tilde{Td}(S^\vee; g_U, (\Pi^\vee)^* g_X, g_H), \quad \tilde{\mu}^* Td(U, g_U), \quad \tilde{\nu}^* \left\{ \frac{Td(-c_1(L, g_L)) - 1}{-c_1(L, g_L)} \right\} \]
are $C^\infty$ differential forms on $\tilde{X}$, we deduce from (5.12), [1, Th. 4bis], and Corollary 4.6 that
\begin{equation}
\log \|\sigma_{K_M}\|_{Q, \lambda}^2 \equiv_B \left( \int_{E_0} \tilde{\mu}^* \left\{ \frac{Td(H) - 1}{c_1(H)} \right\} q^* \chi(\xi) \right) \log |t|^2.
\end{equation}
Here we used the identity $c_1(H) = -c_1(L) + (\Pi^\vee)^* \pi^* c_1(S)$ in $H^2(\mathbb{P}(TX)^\vee, \mathbb{Z})$ and the triviality of the line bundle $\tilde{\mu}^* (\Pi^\vee)^* \pi^* (TS)|_{\tilde{\pi}^{-1}(\mathcal{U})}$ to get (5.13). This completes the proof of Theorem 5.1. \qedsymbol
For simplicity, we set \( \mathcal{L} := (L, g_L) \), \( \mathcal{U} := (U, g_U) \), \( \xi := (\xi, h_\xi) \) in what follows.

Let \( \widetilde{Td}(S^\nu) \colon g_U, (\Pi^\nu)^*g_X, g_H \) be the Bott-Chern secondary class associated with the Todd genus and the exact sequence of holomorphic vector bundles

\[ S^\nu : 0 \to U \to (\Pi^\nu)^*TX \to H \to 0 \]
equipped with the Hermitian metrics \( g_U, (\Pi^\nu)^*g_X, g_H \), such that

\[
\dd^c \widetilde{Td}(S^\nu) \colon g_U, (\Pi^\nu)^*g_X, g_H = Td(U, g_U) Td(H, g_H) - (\Pi^\nu)^* Td(TX, g_X).
\]

Recall that \( Z \) is a general fiber of \( \pi \colon X \to S \).

**Theorem 5.2.** The following identity holds

\[
\lim_{t \to 0} \left[ \log \| \sigma_{KM}(t) \|_{Q, \lambda}^2 - \left( \int_{E_0} \overline{\mu}^* \left\{ \frac{Td(U) (H) - 1}{c_1(H)} \right\} q^* \chi(\xi) \right) \log \| s_\Gamma \|^2_{X \times \{t\}} \right] =
\]

\[
- \int_{X \times \{t\}} \frac{\overline{\mu}^* Td(TX, g_X) \chi(\xi)}{Td([\Gamma], h_{[\Gamma]})} \log \| s_\Gamma \|^2_{X \times \{t\}}
\]

\[
- \int_{X_0} \overline{\mu}^* \widetilde{Td}(S^\nu) ; g_U, (\Pi^\nu)^* g_X, g_H q^* \chi(\xi)
\]

\[
\int_X (q^* \log \| d\pi \|^2) \overline{\pi}^* c_1([0], \| \cdot \|_0) \left[ \overline{\mu}^* Td(\overline{U}) \overline{\nu}^* \left\{ \frac{Td(-c_1(\overline{L})) - 1}{-c_1(\overline{L})} \right\} q^* \chi(\overline{\xi}) \right]
\]

\[
- \int_X (\overline{\pi}^* \log \| s_0 \|^2) \overline{\nu}^* c_1(\overline{L}) \left[ \overline{\mu}^* Td(\overline{U}) \overline{\nu}^* \left\{ \frac{Td(-c_1(\overline{L})) - 1}{-c_1(\overline{L})} \right\} q^* \chi(\overline{\xi}) \right]
\]

\[
- \int_X Td(TX) R(TX) \chi(\xi) + \int_Z Td(TZ) R(TZ) \chi(\xi|_Z).
\]

**Proof.** Define topological constants \( C_0 \) and \( C_1 \) by

\[
C_0 := \int_{E_0} \overline{\mu}^* \left\{ \frac{Td(U) (H) - 1}{c_1(H)} \right\} q^* \chi(\xi),
\]

\[
C_1 := - \int_X Td(TX) R(TX) \chi(\xi) + \int_Z Td(TZ) R(TZ) \chi(\xi|_Z).
\]

Substituting (5.11) and \( c_1(N_t, h_{N_t}) = 0 \) into (5.5), we get for \( t \in \mathcal{U}^o \)

(5.14)

\[
\log \| \sigma_{KM}(t) \|_{Q, \lambda}^2 = - \int_{X \times \{t\}} \overline{\mu}^* Td(TX, g_X) \chi(\xi) \log \| s_\Gamma \|^2_{X \times \{t\}}
\]

\[
- \int_X \overline{\mu}^* Td(\overline{U}) \overline{\nu}^* \left\{ \frac{1 - Td(-c_1(\overline{L}))}{-c_1(\overline{L})} \right\} \chi(\overline{\xi}) \log \| d\pi \|^2 + C_1
\]

\[
= - \int_{X \times \{t\}} \frac{Td(TX, g_X) \chi(\xi)}{Td([\Gamma], h_{[\Gamma]})} \log \| s_\Gamma \|^2_{X \times \{t\}}
\]

\[
- \int_X \overline{\mu}^* Td(\overline{U}) \overline{\nu}^* \left\{ \frac{Td(-c_1(\overline{L})) - 1}{-c_1(\overline{L})} \right\} q^* \chi(\overline{\xi}) q^* (\log \| d\pi \|^2) + C_1,
\]

where

\[
\overline{\mu}^* Td(\overline{U}) \overline{\nu}^* \left\{ \frac{Td(-c_1(\overline{L})) - 1}{-c_1(\overline{L})} \right\} q^* \chi(\overline{\xi}) q^* (\log \| d\pi \|^2) + C_1.
\]
which yields that

\begin{equation}
(5.15)
\log ||\sigma_{KM}(t)||_{Q, t}^2 - C_0 \log ||s_0(t)||_0^2 = \\
- \int_{X \times \{t\}} \frac{Td(TX, g_X) \ch(\xi)}{Td([F], [h_F])} \log ||s_t||_2^2 - \int_{\tilde{X}_t} \mu^* \overline{Td}(S^\vee; g_U, (\Pi^\vee)^* g_X, g_H) q^* \ch(\xi) \\
+ \int_{\tilde{X}_t} \mu^* Td(\overline{U}) \tilde{\nu}^* \left\{ \frac{Td(-c_1(\tilde{L})) - 1}{-c_1(\tilde{L})} \right\} q^* \ch(\xi) q^* (\log ||d\pi||^2) - C_0 \log ||s_0(t)||_0^2 \\
+ C_1.
\end{equation}

By Corollary 4.5,

\begin{equation}
(5.16)
\int_{\tilde{X}_t} \left[ \mu^* Td(\overline{U}) \tilde{\nu}^* \left\{ \frac{Td(-c_1(\tilde{L})) - 1}{-c_1(\tilde{L})} \right\} q^* \ch(\xi) \right] q^* (\log ||d\pi||^2) - C_0 \log ||s_0(t)||_0^2 \\
= \int_{\tilde{X}} (q^* \log ||d\pi||^2) \bar{\pi}^* c_1([0], \| \cdot \|_0) \left[ \mu^* Td(\overline{U}) \tilde{\nu}^* \left\{ \frac{Td(-c_1(\tilde{L})) - 1}{-c_1(\tilde{L})} \right\} q^* \ch(\xi) \right] \\
- \int_{\tilde{X}} (\bar{\pi}^* \log ||s_0||_0^2) \tilde{\nu}^* c_1(\tilde{L}) \left[ \mu^* Td(\overline{U}) \tilde{\nu}^* \left\{ \frac{Td(-c_1(\tilde{L})) - 1}{-c_1(\tilde{L})} \right\} q^* \ch(\xi) \right] + o(1).
\end{equation}

From (5.15) and (5.16), we get

\begin{equation}
(5.17)
\lim_{t \to 0} [\log ||\sigma_{KM}(t)||_{Q, t}^2 - C_0 \log ||s_0(t)||_0^2] = \\
- \int_{X \times \{0\}} \frac{Td(TX, g_X) \ch(\xi)}{Td([F], [h_F])} \log ||s_t||_2^2\big|_{X \times \{0\}} \\
- \int_{\tilde{X}_0} \mu^* \overline{Td}(S^\vee; g_U, (\Pi^\vee)^* g_X, g_H) q^* \ch(\xi) \\
+ \int_{\tilde{X}} (q^* \log ||d\pi||^2) \bar{\pi}^* c_1([0], \| \cdot \|_0) \left[ \mu^* Td(\overline{U}) \tilde{\nu}^* \left\{ \frac{Td(-c_1(\tilde{L})) - 1}{-c_1(\tilde{L})} \right\} q^* \ch(\xi) \right] \\
- \int_{\tilde{X}} (\bar{\pi}^* \log ||s_0||_0^2) \tilde{\nu}^* c_1(\tilde{L}) \left[ \mu^* Td(\overline{U}) \tilde{\nu}^* \left\{ \frac{Td(-c_1(\tilde{L})) - 1}{-c_1(\tilde{L})} \right\} q^* \ch(\xi) \right] + C_1.
\end{equation}

This completes the proof of Theorem 5.2. \hfill \Box

**Lemma 5.3.** Let $\mathcal{E}: 0 \to E' \to E \to E'' \to 0$ be an exact sequence of holomorphic vector bundles over a complex manifold $Y$. Let $h'$ and $h$ be Hermitian metrics on $E'$ and $E$, respectively. Let $h''$ and $g''$ be Hermitian metrics on $E''$. Then

\[ Td(\mathcal{E}; h', h, h'') - \overline{Td}(\mathcal{E}; h', h, g'') = Td(E', h') \overline{Td}(E''; h'', g''). \]

**Proof.** Setting $\mathcal{L}_1 = (\mathcal{E}, h', h, h'')$, $\mathcal{L}_2 = (\mathcal{E}, h', h, g'')$, $\mathcal{L}_3 = 0$ in [10, I, Prop. 1.3.4], we get

\[ Td(\mathcal{E}; h', h, h'') - \overline{Td}(\mathcal{E}; h', h, g'') = \overline{Td}(E' \oplus E''; h' \oplus h'', h' \oplus g''). \]

Since $\overline{Td}(E' \oplus E''; h' \oplus h'', h' \oplus g'') = Td(E', h') \overline{Td}(E''; h'', g'')$ by [10, I, Prop. 1.3.2], we get the result. \hfill \Box
6. The divergent term and the constant term

Let \( \alpha \) be a nowhere vanishing holomorphic section of \( \lambda([\Gamma]^{-1} \otimes p_1^* \xi)^{-1} \otimes \lambda(p_1^* \xi) \) defined on \( \mathcal{U} \).

**Theorem 6.1.** Let \( \sigma \) be a nowhere vanishing holomorphic section of \( \lambda(\xi) \) defined on \( \mathcal{U} \). Then

\[
\log \|\sigma\|^2_{Q,\lambda(\xi)} \equiv_B \left( \int_{E_0} \mu^* \left\{ \frac{Td(U) Td(H) - 1}{c_1(H)} \right\} q^* \sigma(\xi) \right) \log |t|^2.
\]

**Proof.** There exists a nowhere vanishing holomorphic function \( f(t) \) on \( \mathcal{U} \) such that

\[
\sigma(t) = f(t) \sigma_{KM}(t) \otimes \alpha(t).
\]

Since \( \log |f(t)|^2 \) and \( \log \|\alpha\|^2_{Q,\lambda(\xi)^{-1} \otimes p_1^* \xi \otimes \lambda(p_1^* \xi)} \) are \( C^\infty \) functions on \( \mathcal{U} \), we deduce from Theorem 5.1 that

\[
\log \|\sigma(t)\|^2_{Q,\lambda(\xi)} = \log |f(t)|^2 + \log \|\sigma_{KM}(t)\|^2_{Q,\lambda} + \log \|\alpha(t)\|^2_{Q,\lambda([\Gamma]^{-1} \otimes p_1^* \xi)^{-1} \otimes \lambda(p_1^* \xi)}
\]

\[
\equiv_B \left( \int_{E_0} \mu^* \left\{ \frac{Td(U) Td(H) - 1}{c_1(H)} \right\} q^* \sigma(\xi) \right) \log |t|^2.
\]

This completes the proof of Theorem 6.1. \( \square \)

**Theorem 6.2.** The following identity holds:

\[
\lim_{t \to 0} \left[ \log \|\sigma_{KM} \otimes \alpha\|^2_{Q,\lambda(\xi)}(t) - \left( \int_{E_0} \mu^* \left\{ \frac{Td(U) Td(H) - 1}{c_1(H)} \right\} q^* \sigma(\xi) \right) \log \|s_0(t)\|^2_{0} \right]
\]

\[
= \log \|\alpha(0)\|^2_{Q} - \int_{\mathcal{X} \times \{0\}} \frac{Td(TX,g_X) \sigma(\xi)}{Td([\Gamma],h_\Gamma)} \log \|s_0\|^2_{\mathcal{X} \times \{0\}}
\]

\[
- \int_{\mathcal{X}} (q^* \log \|\sigma(\xi)\|^2_{\mathcal{X}}) \pi^* c_1([0],\| \cdot \|_0) \left[ \mu^* Td(U) \sigma^* \left\{ \frac{Td(-c_1(L)) - 1}{-c_1(L)} \right\} q^* \sigma(\xi) \right]
\]

\[
+ \int_{\mathcal{X}} (\pi^* \log \|s_0(\xi)\|^2_{\mathcal{X}}) c_1(L) \left[ \mu^* Td(U) \sigma^* \left\{ \frac{Td(-c_1(L)) - 1}{-c_1(L)} \right\} q^* \sigma(\xi) \right]
\]

\[
- \int_{\mathcal{X}} \left( \mu^* Td(U) \sigma^* \left\{ \frac{Td(-c_1(L)) - 1}{-c_1(L)} \right\} q^* \sigma(\xi) \right)
\]

\[
- \int_{\mathcal{X}} Td(TX) R(TX) \sigma(\xi) + \int_{\mathcal{X}} Td(TZ) R(TZ) \sigma(\xi) |z).
\]

**Proof.** Since

\[
\log \|\sigma_{KM} \otimes \alpha\|^2_{Q,\lambda(\xi)} = \log \|\sigma_{KM}\|^2_{Q,\lambda} + \log \|\alpha\|^2_{Q,\lambda([\Gamma]^{-1} \otimes p_1^* \xi)^{-1} \otimes \lambda(p_1^* \xi)},
\]

the result follows from Theorem 5.2. \( \square \)

7. Critical points defined by a quadric polynomial of rank 2

In this section, we assume that for every \( x \in \Sigma_\pi \cap X_0 \), there exists a system of coordinates \( (z_0, \ldots, z_n) \) centered at \( x \) such that

\[
\pi(z) = z_0 z_1.
\]

Hence \( \Sigma_\pi \subset X \) is a complex submanifold of codimension 2 defined locally by the equation \( z_0 = z_1 = 0 \). Let \( N_{\Sigma_\pi}/X \) be the normal bundle of \( \Sigma_\pi \) in \( X \). In [3, Def. 5.1,
Prop. 5.2], Bismut introduced the additive genus $E(\cdot)$ associated with the generating function
\[
E(x) := \frac{Td(x) Td(-x)}{2x} \left( \frac{Td^{-1}(x) - 1}{x} - \frac{Td^{-1}(-x) - 1}{-x} \right),
\]
where $Td^{-1}(x) := (1 - e^{-x})/x$.

The following result was proved by Bismut [3, Th. 5.9].

**Theorem 7.1.** The following equation of functions on $U^\circ$ holds:
\[
\log \|\sigma(t)\|^2_{\lambda(\xi), Q} \equiv B \frac{1}{2} \left( \int_{\Sigma_\pi \cap X_0} -Td(T\Sigma_\pi) E(N_{\Sigma_\pi / X}) \ch(\xi) \right) \log |t|^2.
\]

**Remark 7.2.** As mentioned before, the dual of our $\lambda(\xi)$ was defined as $\lambda(\xi)$ in [3, Th. 5.9], which explains the difference of the sign of the coefficient of $\log |t|^2$ in Theorem 7.1 with that of [3, Th. 5.9].

**Proof.** Let $q: \tilde{X} \rightarrow X$ be the blowing-up along $\Sigma_\pi$ with exceptional divisor
\[
E = \mathbb{P}(N_{\Sigma_\pi / X}).
\]

Then $\tilde{\nu} = \nu \circ q$ extends to a holomorphic map from $\tilde{X}$ to $\mathbb{P}(\Omega^1_X)$.

Since the Hessian of $\pi$ is a non-degenerate symmetric bilinear form on $N_{\Sigma_\pi / X}$, we have $N_{\Sigma_\pi / X} \cong N^*_{\Sigma_\pi / X}$. Under the identification $\mathbb{P}(N_{\Sigma_\pi / X}) = \mathbb{P}(N^*_{\Sigma_\pi / X})$ induced from the Hessian of $\pi$, $\tilde{\nu}$ is identified with the natural inclusion $\mathbb{P}(N^*_{\Sigma_\pi / X}) \hookrightarrow \mathbb{P}(\Omega^1_X|_{\Sigma_\pi})$, which yields that
\[
(7.1) \quad \tilde{\nu}^* L|_E = \mathcal{O}_{\mathbb{P}(N^*_{\Sigma_\pi / X})}(-1), \quad \tilde{\mu}^* H|_E = \mathcal{O}_{\mathbb{P}(N_{\Sigma_\pi / X})}(1).
\]

Set $F := \mathcal{O}_{\mathbb{P}(N_{\Sigma_\pi / X})}(1)$.

By the exact sequence $S^\nu$, we get
\[
(7.2) \quad Td(U) = \frac{Td((H)^*TX)}{Td(H)}.
\]

Since $H^\nu \circ \tilde{\mu} = q$, we deduce from the exact sequence of vector bundles on $\Sigma_\pi$
\[
0 \rightarrow T\Sigma_\pi \rightarrow TX|_{\Sigma_\pi} \rightarrow N_{\Sigma_\pi / X} \rightarrow 0
\]
the identity
\[
(7.3) \quad \tilde{\mu}^* Td((H)^*TX)|_E = q^* \left\{ Td(T\Sigma_\pi) Td(N_{\Sigma_\pi / X}) \right\}.
\]

Substituting (7.3) into (7.2), we get
\[
(7.4) \quad \tilde{\mu}^* Td(U)|_E = \frac{q^* \left\{ Td(T\Sigma_\pi) Td(N_{\Sigma_\pi / X}) \right\}}{\tilde{\mu}^* Td(H)|_E} = \frac{q^* \left\{ Td(T\Sigma_\pi) Td(N_{\Sigma_\pi / X}) \right\}}{Td(F)},
\]
where we used (7.1) to get the second equality.
Let \( p_* \) be the integration along the fibers of the projection \( p : \mathbb{P}(N_{\Sigma_*/X}) \to \Sigma_* \). Since \( q|E = p \), we deduce from (7.1), (7.4) and the projection formula that

\[
\int_{E \cap \Sigma \setminus X_0} \mu^* \left\{ \frac{\text{Td}(U) \cdot \text{Td}(H) - 1}{c_1(H)} \right\} q^* \text{ch}(\xi) = \int_{\Sigma \cap X_0} \text{Td}(T\Sigma \cap F) \text{Td}(N_{\Sigma_*/X}) \text{ch}(\xi) p_* \left\{ \frac{1}{\text{Td}(F)} \cdot \frac{\text{Td}(F) - 1}{c_1(F)} \right\}.
\]

Since \( N_{\Sigma_*/X} \cong N_{\Sigma_*/X}^* \), we have

\[
c_1(N_{\Sigma_*/X}) = 0,
\]

which, together with \( \text{rk}(N_{\Sigma_*/X}) = 2 \), yields that

\[
0 = c_1(F)^2 - c_1(N_{\Sigma_*/X}^*) c_2(N_{\Sigma_*/X}) + p^* c_2(N_{\Sigma_*/X}) = c_1(F)^2 + p^* c_2(N_{\Sigma_*/X}).
\]

Since \( p_* c_1(F) = 1 \), this implies that for \( m \geq 0 \)

\[
p_* c_1(F)^m = \begin{cases} 
(-1)^k c_2(N_{\Sigma_*/X})^k & (m = 2k + 1) \\
0 & (m = 2k).
\end{cases}
\]

For a formal power series \( f(x) = \sum_{j=0}^{\infty} a_j x^j \in \mathbb{C}[[x]] \), set

\[
f_-(x) := \frac{f(x) - f(-x)}{2x} \in \mathbb{C}[[x]].
\]

By (7.6), we get

\[
p_* f(1)(F) = \sum_{k} a_{2k+1} c_2(N_{\Sigma_*/X})^k = \sum_{k} (-1)^k a_{2k+1} c_2(N_{\Sigma_*/X})^k.
\]

Let \( f_-(N_{\Sigma_*/X}) \) be the additive genus associated with \( f_-(x) \in \mathbb{C}[[x]] \). Let \( x_1, x_2 \) be the Chern roots of \( N_{\Sigma_*/X} \). Since \( c_1(N_{\Sigma_*/X}) = x_1 + x_2 = 0 \), we get

\[
f_-(N_{\Sigma_*/X}) = \frac{f(x_1) - f(-x_1)}{2x_1} + \frac{f(x_2) - f(-x_2)}{2x_2}
\]

\[
= \sum_{k=0}^{\infty} a_{2k+1} (x_1^{2k} + x_2^{2k})
\]

\[
= \sum_{k=0}^{\infty} a_{2k+1} (-x_1 x_2)^k
\]

\[
= 2 \sum_{k=0}^{\infty} (-1)^k a_{2k+1} c_2(N_{\Sigma_*/X})^k = 2 p_* f(1)(F).
\]
Since $\Sigma$ is discrete, we may identify $\Sigma_x / X$ with the trivial projective-space bundle on a neighborhood of $\Sigma_x$. Then we have the following expression on a neighborhood of each $p \in \Sigma_x$:

$$
\mu(z) = \nu(z) = \left( \frac{\partial \pi}{\partial z_0}(z) : \cdots : \frac{\partial \pi}{\partial z_n}(z) \right).
$$

For a formal power series $f(x) \in \mathbb{C}[[x]]$, let $f(x)|_{x^m}$ denote the coefficient of $x^m$. Let $\mu(\pi, p) \in \mathbb{N}$ be the Milnor number of the isolated critical point $p$ of $\pi$. The following result was proved by the author [16, Main Th.].

**Theorem 8.1.** The following identity of functions on $U^\circ$ holds:

$$
\log \|\sigma\|^2_{\lambda(\xi), Q} \equiv_B \left( \frac{(-1)^n}{(n + 2)!} \operatorname{rk}(\xi) \sum_{\mu(\pi, p) \in \mathbb{N}} \mu(\pi, p) \log |t|^2 \right).
$$

**Proof.** In Theorem 6.1, we can identify $U$ (resp. $L$) with the universal hyperplane bundle (resp. tautological line bundle) on $\mathbb{P}^n$. Then $H = L^{-1}$. Set $x := c_1(H)$. Hence $\int_{x^n} x^n = 1$. From the exact sequence $0 \to U \to \mathbb{C}^{n+1} \to H \to 0$, we get

$$
\operatorname{Td}(U) = \operatorname{Td}^{-1}(x) = \frac{1 - e^{-x}}{x}.
$$

By substituting this and the equation $q^*\operatorname{ch}(\xi)|_{E \cap X_0} = \operatorname{rk}(\xi)$ into the formula of Theorem 6.1, we get

$$
\begin{aligned}
\int_{E_0} \overline{\mu}^* \operatorname{Td}(U) \overline{\nu}^* \left\{ \frac{\operatorname{Td}(c_1(H)) - 1}{c_1(H)} \right\} q^* \operatorname{ch}(\xi) \\
= \frac{1}{\operatorname{Td}(x)} \cdot \frac{\operatorname{Td}(x) - 1}{x} \cdot \operatorname{rk}(\xi) \int_{E_0} \overline{\mu}^* c_1(H)^n \\
= \left\{ 1 - \frac{1 - e^{-x}}{x^2} \right\} \cdot \operatorname{rk}(\xi) \int_{E_0} \overline{\mu}^* c_1(H)^n \\
= \frac{(-1)^n}{(n + 2)!} \operatorname{rk}(\xi) \int_{E_0} \overline{\mu}^* c_1(H)^n.
\end{aligned}
$$

8. Isolated critical points

In this section, we assume that $\operatorname{Sing}(X_0) = \Sigma_x \cap X_0$ consists of isolated points. Setting $f(x) = (\operatorname{Td}^{-1}(x) - 1)/x$, we get

$$
E(N_{\Sigma_x / X}) = \operatorname{Td}(x_1) \operatorname{Td}(x_2) \left\{ \frac{f(x_1) - f(-x_1)}{2x_1} + \frac{f(x_2) - f(-x_2)}{2x_2} \right\}
$$

$$
= 2 \operatorname{Td}(N_{\Sigma_x / X}) p_* f(c_1(F))
$$

$$
= -2 \operatorname{Td}(N_{\Sigma_x / X}) p_* \left( \frac{1 - \operatorname{Td}^{-1}(F)}{c_1(F)} \right).
$$

By comparing (7.5) and (7.7), the desired formula follows from Theorem 6.1. □
Since
\[ \pi_* \{ \mu^*( -c_1(L, g_L) )^n q^*( \log \| d\pi \|^2 ) \} = \pi_* \{ q^*( \log \| d\pi \|^2 ) ( dd^c \log \| d\pi \|^2 )^n \} = \sum_{p \in \text{Sing}(X_0)} \mu(\pi, p) \log |t|^2 + O(1) \]
by [16, Th. 4.1], we get
\[ (8.2) \quad \int_{E_0} \mu^* c_1(H)^n = \sum_{p \in \text{Sing}(X_0)} \mu(\pi, p) \]
by Corollary 4.6. The result follows from Theorem 6.1 and (8.1), (8.2). \hfill \Box

9. Some results on asymptotic expansion

Let \( \mathcal{A}_C \) (resp. \( \mathcal{C}_C \)) be the sheaf of germs of \( C^\infty \) (resp. \( C^0 \)) functions on \( \mathbb{C} \). The stalk of \( \mathcal{A}_C \) (resp. \( \mathcal{C}_C \)) at the origin is denoted by \( \mathcal{A}_0 \) (resp. \( \mathcal{C}_0 \)). We define
\[ \mathcal{B}_0 := \mathcal{A}_0 \oplus \bigoplus_{r \in \mathbb{Q} \cap (0,1)} \bigoplus_{k=0}^n |t|^{2r} (\log |t|)^k \cdot \mathcal{A}_0 \subset \mathcal{C}_0. \]
In this section, we prove the following

**Theorem 9.1.** Let \( \Omega \subset \mathbb{C}^n \) be a relatively compact domain. Let \( F(z) \) be a holomorphic function on \( \Omega \) with critical locus \( \Sigma_F := \{ z \in \Omega; dF(z) = 0 \} \). Let \( \chi(z) \) be a \( C^\infty \) \((n,n)\)-form with compact support in \( \Omega \). Define a germ \( \psi \in \mathcal{C}_0 \) by
\[ \psi(t) := \int_{\Omega} \log |F(z) - t|^2 \chi(z). \]
If \( \Sigma_F \subset F^{-1}(0) \), then \( \psi(t) \in \mathcal{B}_0 \).

The continuity of similar integrals was studied by Bost-Gillet-Soulé [8, Sect. 1.5] in relation with the regularity of the star products of Green currents.

For the proof of Theorem 9.1, we prove some intermediary results.

**Lemma 9.2.** Let \( \Phi \) be a \( C^\infty \) \((n,n)\)-form with compact support in \( \Omega \). Let \( F_*(\Phi) \) be the locally integrable \((1,1)\)-form on \( \mathbb{C} \) defined as the integration of \( \Phi \) along the fibers of \( F: \Omega \to \mathbb{C} \). If \( \Sigma_F \subset F^{-1}(0) \), then there exists a germ \( A(t) \in \mathcal{B}_0 \) such that
\[ F_*(\Phi)(t) = A(t) \frac{dt \wedge d\bar{t}}{|t|^2}, \quad A(0) = 0 \]
near \( 0 \in \mathbb{C} \).

**Proof.** By Hironaka, there exists a proper holomorphic modification \( \varpi: \tilde{\Omega} \to \Omega \) such that
(i) \( \varpi: \tilde{\Omega} \setminus \varpi^{-1}(\Sigma_F) \to \Omega \setminus \Sigma_F \) is an isomorphism;
(ii) \( (F \circ \varpi)^{-1}(\Sigma_F) \) is a normal crossing divisor of \( \Omega \).

Set \( \tilde{F} := F \circ \varpi \). For any \( z \in F^{-1}(0) \), there exist a system of coordinates \((U, (w_1, \ldots, w_n))\) and integers \( k_1, \ldots, k_l \geq 1, l \leq n \), such that \( \tilde{F}(w) = w_1^{k_1} \cdots w_l^{k_l} \).

Define a holomorphic \((n-1)\)-form on \( U \) by
\[ \tau := \frac{1}{l} \sum_{i=1}^l \frac{1}{k_i} (-1)^{l-1} w_i \, dw_1 \wedge \cdots \wedge dw_{i-1} \wedge dw_{i+1} \wedge \cdots \wedge dw_n. \]
Let \( g_U \) be a \( C^\infty \) function with compact supported in \( U \). Since \( \varpi^* \Phi \) is a \( C^\infty (n, n) \)-form on \( \Omega \), there exists \( h(w) \in C_0^\infty (U) \) such that
\[
g_U \varpi^* \Phi = h(w) \, dw_1 \wedge \cdots \wedge dw_n \wedge d\bar{w}_1 \wedge \cdots \wedge d\bar{w}_n.
\]
We define a germ \( B(t) \in \mathcal{C}_0 \) by
\[
B(t) := \int_{\tilde{F}^{-1}(t) \cap U} h(w) \, \tau \wedge \bar{\tau}.
\]
Then \( B(t) \in \mathcal{B}_0 \) by [1, p.166, Th. 4bis]. Since
\[
\tilde{F}^*(\frac{dt}{t}) \wedge \tau = dw_1 \wedge \cdots \wedge dw_n,
\]
we get by the projection formula (9.1)
\[
\tilde{F}^*(g_U \varpi^* \Phi)(t) = \tilde{F}^*(h(w) \, dw_1 \wedge \cdots \wedge dw_n \wedge d\bar{w}_1 \wedge \cdots \wedge d\bar{w}_n)(t)
\]
\[
= \frac{dt \wedge d\bar{t}}{|t|^2} \tilde{F}^*(h(w) \, \tau \wedge \bar{\tau}) = B(t) \frac{dt \wedge d\bar{t}}{|t|^2}.
\]
For an \( \epsilon > 0 \) small enough, set \( \Delta(\epsilon) := \{ t \in \mathbb{C}; |t| < \epsilon \} \). Since
\[
\left| \int_{\Delta(\epsilon)} \tilde{F}^*(g_U \varpi^* \Phi) \right| = \left| \int_{\tilde{F}^{-1}(\Delta(\epsilon))} g_U \varpi^* \Phi \right| < \infty,
\]
the \((1, 1)\)-form \( B(t) \, dt \wedge d\bar{t}/|t|^2 \) is locally integrable near the origin. Hence \( B(0) = 0 \).

Let \( \{ U_\beta \}_{\beta \in B} \) be a locally finite open covering of \( \Omega \) and let \( \{ \varrho_\beta \}_{\beta \in B} \) be a partition of unity subject to \( \{ U_\beta \}_{\beta \in B} \). By (9.1), there exists \( B_\beta(t) \in \mathcal{B}_0 \) for each \( \beta \in B \) such that
\[
\tilde{F}^*(\varrho_\beta \varpi^* \Phi) = B_\beta(t) \frac{dt \wedge d\bar{t}}{|t|^2}, \quad B_\beta(0) = 0.
\]
There exist finitely many \( \beta \in B \) with \( B_\beta(t) \neq 0 \) by the compactness of the support of \( \varpi^* \Phi \). Since
\[
F^*_\tau(\Phi) = \sum_{\beta \in B} \tilde{F}^*(\varrho_\beta \varpi^* \Phi) = \left( \sum_{\beta \in B} B_\beta(t) \right) \frac{dt \wedge d\bar{t}}{|t|^2},
\]
we get \( A(t) = \sum_{\beta \in B} B_\beta(t) \in \mathcal{B}_0 \) and \( A(0) = 0 \). \( \square \)

We regard \( \Omega \) as a domain in \((\mathbb{P}^1)^n\). Hence \( \chi \) is a \( C^\infty (n, n) \)-form on \((\mathbb{P}^1)^n\). Let \( z = (z_1, \ldots, z_n) \) be the inhomogeneous coordinates of \((\mathbb{P}^1)^n\). For \( 1 \leq i \leq n \), set
\[
\omega_i := \sqrt{-1} \frac{dz_i \wedge d\bar{z}_i}{2\pi(1 + |z_i|^2)^2}.
\]

**Lemma 9.3.** Assume that \( F(z) = z_1^{\nu_1} \cdots z_n^{\nu_n}, \nu_1, \ldots, \nu_n \geq 0 \) and set
\[
\alpha := \int_{(\mathbb{P}^1)^n} \chi(z).
\]
Then there exists \( \eta(t) \in \mathcal{B}_0 \) such that
\[
\psi(t) = \alpha \int_{(\mathbb{P}^1)^n} \log |z_1^{\nu_1} \cdots z_n^{\nu_n} - t|^2 \omega_1 \wedge \cdots \wedge \omega_n + \eta(t).
\]
Proof. Let $((\zeta_1 : \xi_1), \ldots, (\zeta_n : \xi_n))$ be the homogeneous coordinates of $(\mathbb{P}^1)^n$ such that $z_i = \zeta_i / \xi_i$. For $t \in \mathbb{C}$, set

$$Y_t := \{((\zeta_1 : \xi_1), \ldots, (\zeta_n : \xi_n)) \in (\mathbb{P}^1)^n; \frac{\zeta_1^\nu \cdots \zeta_n^\nu - t \xi_1^\nu \cdots \xi_n^\nu}{\zeta_1^\nu \cdots \xi_n^\nu} = 0\},$$

$$D := \{((\zeta_1 : \xi_1), \ldots, (\zeta_n : \xi_n)) \in (\mathbb{P}^1)^n; \zeta_1^\nu \cdots \xi_n^\nu = 0\}.$$

Since

$$z_1^\nu \cdots z_n^\nu - t = \frac{\zeta_1^\nu \cdots \zeta_n^\nu - t \xi_1^\nu \cdots \xi_n^\nu}{\zeta_1^\nu \cdots \xi_n^\nu},$$

we get the following equation of currents on $(\mathbb{P}^1)^n$ by the Poincaré-Lelong formula:

$$dd^c \log |z_1^\nu \cdots z_n^\nu - t|^2 = \delta_{Y_t} - \delta_D.$$  

Since $\chi(z)$ is cohomologous to $\alpha \omega_1 \wedge \cdots \wedge \omega_n$, there exists a $C^\infty (n-1, n-1)$-form $\gamma$ on $(\mathbb{P}^1)^n$ by the $dd^c$-Poincaré lemma, such that

$$\chi(z) - \alpha \omega_1 \wedge \cdots \wedge \omega_n = dd^c \gamma.$$ 

Hence we get by (9.3)

$$\psi(t) = \alpha \int_{(\mathbb{P}^1)^n} \log |z_1^\nu \cdots z_n^\nu - t|^2 \omega_1 \wedge \cdots \wedge \omega_n + \int_{(\mathbb{P}^1)^n} \log |z_1^\nu \cdots z_n^\nu - t|^2 dd^c \gamma$$

$$= \alpha \int_{(\mathbb{P}^1)^n} \log |z_1^\nu \cdots z_n^\nu - t|^2 \omega_1 \wedge \cdots \wedge \omega_n + \int_{(\mathbb{P}^1)^n} dd^c (\log |z_1^\nu \cdots z_n^\nu - t|^2) \wedge \gamma$$

$$= \alpha \int_{(\mathbb{P}^1)^n} \log |z_1^\nu \cdots z_n^\nu - t|^2 \omega_1 \wedge \cdots \wedge \omega_n + \int_{Y_t} \gamma - \int_{D} \gamma.$$ 

For $t \in \mathbb{C}$, set

$$\eta(t) := \int_{Y_t} \gamma - \int_{D} \gamma.$$ 

Define a divisor of $(\mathbb{P}^1)^n \times \mathbb{C}$ by

$$Y := \{((\zeta_1 : \xi_1), \ldots, (\zeta_n : \xi_n), t) \in (\mathbb{P}^1)^n \times \mathbb{C}; \frac{\zeta_1^\nu \cdots \zeta_n^\nu - t \xi_1^\nu \cdots \xi_n^\nu}{\zeta_1^\nu \cdots \xi_n^\nu} = 0\}.$$ 

Let $pr_1 : (\mathbb{P}^1)^n \times \mathbb{C} \to (\mathbb{P}^1)^n$ and $pr_2 : (\mathbb{P}^1)^n \times \mathbb{C} \to \mathbb{C}$ be the projections. Then $Y_t = Y \cap pr_2^{-1}(t)$. Let $P : \tilde{Y} \to Y$ be the resolution of the singularities of $Y$. Then $pr_2|_Y \circ P$ is a proper holomorphic function on the complex manifold $\tilde{Y}$. Since $P^*(pr_1)^* \gamma$ is a $C^\infty (n-1, n-1)$-form on $\tilde{Y}$, we get

$$\eta(t) = \int_{(pr_2|_Y \circ P)^{-1}(t)} P^*(pr_1)^* \gamma - \int_{D} \gamma \in B_0$$

by [1, Th. 4bis]. The result follows from (9.4), (9.5), (9.6). 

Define a germ $f \in C_0$ by

$$f(t) := \int_{(\mathbb{P}^1)^n} \log |z_1^\nu \cdots z_n^\nu - t|^2 \omega_1 \wedge \cdots \wedge \omega_n.$$ 

Lemma 9.4. There exists a germ $g(t) \in B_0$ such that

$$dd^c f(t) = \frac{1}{4\pi} g(t) \frac{dt \wedge dt}{|t|^2}, \quad g(0) = 0.$$
The assertion follows from (9.9). □

Lemma 9.5. The germ \( f(t) \) is \( S^1 \)-invariant, i.e., \( f(t) = f(|t|) \).

Proof. Without loss of generality, we may assume that \( \nu_n > 0 \). Since
\[
\int_{P^1} |Az_n^{\nu_n} + B|^2 \omega_n = \log(|A|^{2/\nu_n} + |B|^{2/\nu_n})
\]
when \( (A, B) \neq (0, 0) \), we get by Fubini’s theorem
\[
(9.9) \quad f(t) = \int_{(P^1)^n} \log |z_1^{\nu_1} \cdots z_n^{\nu_n} - t|^2 \omega_1 \wedge \cdots \wedge \omega_n
\]
\[
= \int_{(P^1)^{n-1}} \left( \int_{P^1} \log |z_1^{\nu_1} \cdots z_n^{\nu_n} - t|^2 \omega_n \right) \omega_1 \wedge \cdots \wedge \omega_{n-1}
\]
\[
= \int_{(P^1)^{n-1}} \log \left( |z_1^{\nu_1} \cdots z_{n-1}^{\nu_{n-1}}|^{2/\nu_n} + |t|^{2/\nu_n} \right) \omega_1 \wedge \cdots \wedge \omega_{n-1}.
\]
The assertion follows from (9.9). □

Let \((r, \theta) \) be the polar coordinates of \( C \). Hence \( t = re^{i\theta} \).

Lemma 9.6. Let \( \lambda(t) \in C^\infty(\Delta^*) \). Assume that \( \lambda(t) \) is \( S^1 \)-invariant, i.e., \( \lambda(t) = \lambda(r) \). If \( r \partial_r \lambda(t) \in \mathcal{B}_0 \), then \( \lambda(t) \in \mathcal{B}_0 \).

Proof. By the definition of \( \mathcal{B}_0 \), there exist a finite set \( A \subset \mathbb{Q} \cap (0,1] \) and germs \( \mu_{\alpha,k}(t) \in \mathcal{A}_0, \alpha \in A, 0 \leq k \leq n \) such that
\[
(9.10) \quad r \partial_r \lambda(r) = \sum_{\alpha \in A} \sum_{k=0}^n r^{2\alpha} (\log r)^k \mu_{\alpha,k}(t).
\]
We may assume that \(\mu_{\alpha,k}(t) \in C^\infty(\Delta(2\epsilon))\) for some \(\epsilon > 0\). Since the left hand side of (9.10) is \(S^1\)-invariant, we may assume that \(\mu_{\alpha,k}(t) = \mu_{\alpha,k}(r)\) for all \(\alpha\) and \(k\) after replacing \(\mu_{\alpha,k}(t)\) by \(\int_0^{2\pi} \mu_{\alpha,k}(e^{i\theta}) \, d\theta / 2\pi\). By (9.10), we get

\[
\lambda(\epsilon) - \lambda(r) = \sum_{\alpha \in A} \sum_{k=0}^n \int_0^\epsilon u^{2\alpha - 1} (\log u)^k \mu_{\alpha,k}(u) \, du.
\]

(9.11)

By (9.11), we see that \(\lambda(t) \in \mathcal{C}_0\) by setting

\[
\lambda(0) := \lambda(\epsilon) - \sum_{\alpha \in A} \sum_{k=0}^n \int_0^\epsilon u^{2\alpha - 1} (\log u)^k \mu_{\alpha,k}(u) \, du.
\]

Since \(\lambda(t) \in \mathcal{C}_0\), we get by (9.11)

\[
\lambda(r) = \lambda(0) + \sum_{\alpha \in A} \sum_{k=0}^n \int_0^r u^{2\alpha - 1} (\log u)^k \mu_{\alpha,k}(u) \, du
\]

\[
= \lambda(0) + \sum_{\alpha \in A} \sum_{k=0}^n r^{2\alpha} \int_0^1 u^{2\alpha - 1} (\log r + \log u)^k \mu_{\alpha,k}(u) \, du
\]

\[
= \lambda(0) + \sum_{\alpha \in A} \sum_{k=0}^n \sum_{l=0}^k \left(\frac{k}{l}\right) r^{2\alpha} (\log r)^l \int_0^1 u^{2\alpha - 1} (\log u)^{k-l} \mu_{\alpha,k}(u) \, du,
\]

which implies that \(\lambda(t) \in \mathcal{B}_0\).

\[\square\]

**Lemma 9.7.** If \(F(z) = z^v_1 \cdots z^v_n, \nu_1, \ldots, \nu_n \geq 0\), then \(\psi(t) \in \mathcal{B}_0\).

**Proof.** By Lemma 9.3, it suffices to prove that \(f \in \mathcal{B}_0\). Since \(f(t) = f(r)\) by Lemma 9.5, we deduce from Lemma 9.4 the equation

\[
\frac{1}{2\pi} \partial_t \partial_r f(t) = \frac{1}{4\pi} \{ f''(r) + r^{-1} f'(r) \} = \frac{g(t)}{4\pi r^2}.
\]

Hence \(g(t)\) is invariant under the rotation, i.e., \(g(t) = g(r)\), and the following equation holds

\[
(r \partial_r)^2 f(r) = g(r).
\]

(9.12)

Since \(g(t) \in \mathcal{B}_0\), we deduce from Lemma 9.6 and (9.12) that \(r \partial_r f(r) \in \mathcal{B}_0\). By Lemma 9.6 again, we get \(f(t) \in \mathcal{B}_0\).

\[\square\]

**Proof of Theorem 9.1**

We keep the notation in the proof of Lemma 9.2. There exists a system of coordinate neighborhoods \(\{(U_\beta, w_\beta = (w_{1,\beta}, \ldots, w_{n,\beta}))\}_{\beta \in B}\) of \(\tilde{\Omega}\) and integers \(k_1, \beta, \ldots, k_{n, \beta} \geq 0\) for each \(\beta \in B\) such that \(\tilde{F}|_{U_\beta}(w_\beta) = w_{1,\beta}^{k_{1,\beta}} \cdots w_{n,\beta}^{k_{n,\beta}}\). Without loss of generality, we may assume that the covering \(\{U_\beta\}_{\beta \in B}\) of \(\tilde{\Omega}\) is locally finite. Let \(\{\varrho_\beta\}_{\beta \in B}\) be a partition of unity subject to the covering \(\{U_\beta\}_{\beta \in B}\). Then \(\chi_\beta := \varrho_\beta \varpi^* \chi\) is a \(C^\infty\) \((n, n)\)-form with compact support in \(U_\beta\). Since \(\varpi^* \chi\) has a compact support in \(\Omega\), \(\chi_\beta = 0\) except finitely many \(\beta \in B\). By Lemma 9.7,

\[
\psi_\beta(t) := \int_{U_\beta} \log |w_{1,\beta}^{k_{1,\beta}} \cdots w_{n,\beta}^{k_{n,\beta}} - t|^2 \chi_\beta(w_\beta) \in \mathcal{B}_0.
\]

(9.13)
Since
\[ \psi(t) = \int_{\Omega} \varpi^* \log |F - t|^2 \varpi^* \chi = \sum_{\beta \in B} \int_{U_{\beta}} \log \left| \tilde{F}_{\beta}(w_{\beta}) - t \right|^2 \varpi_{\beta} \varpi^* \chi = \sum_{\beta \in B} \psi_{\beta}(t), \]
we get \( \psi(t) \in \mathcal{B}_0 \) by (9.13). This completes the proof of Theorem 9.1. \( \Box \)

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