Chebyshev polynomial coefficient estimates for a class of analytic bi-univalent functions related to pseudo-starlike functions

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Abstract In this paper, we obtain initial coefficient bounds for functions belong to a subclass of analytic bi-univalent functions related to pseudo-starlike functions by using the Chebyshev polynomials and also we find Fekete-Szegö inequalities for this class.

Keywords Analytic functions · Bi-univalent functions · Coefficient bounds · Chebyshev polynomial · Fekete-Szegö problem · Subordination

Mathematics Subject Classification Primary 30C45

1 Introduction

Let $\mathbb{R} = (-\infty, \infty)$ be the set of real numbers, $\mathbb{C}$ be the set of complex numbers and

$\mathbb{N} := \{1, 2, 3, \ldots\} = \mathbb{N}_0 \setminus \{0\}$

be the set of positive integers.

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

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which are analytic in the open unit disk
\[ \Delta = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \} . \]

We also denote by \( \mathcal{S} \) the class of all functions in the normalized analytic function class \( \mathcal{A} \) which are univalent in \( \Delta \).

For two functions \( f \) and \( g \), analytic in \( \Delta \), we say that the function \( f \) is subordinate to \( g \) in \( \Delta \), and write
\[ f (z) \prec g (z) \quad (z \in \Delta), \]
if there exists a Schwarz function \( \omega \), which is analytic in \( \Delta \) with
\[ \omega (0) = 0 \quad \text{and} \quad |\omega (z)| < 1 \quad (z \in \Delta) \]
such that
\[ f (z) = g (\omega (z)) \quad (z \in \Delta). \]

Indeed, it is known that
\[ f (z) \prec g (z) \quad (z \in \Delta) \Rightarrow f (0) = g (0) \quad \text{and} \quad f (\Delta) \subset g (\Delta). \]

Furthermore, if the function \( g \) is univalent in \( \Delta \), then we have the following equivalence
\[ f (z) \prec g (z) \quad (z \in \Delta) \Leftrightarrow f (0) = g (0) \quad \text{and} \quad f (\Delta) \subset g (\Delta). \]

It is well known (e.g. see Duren [11]) that every function \( f \in \mathcal{S} \) has an inverse map \( f^{-1} \), defined by
\[ f^{-1} (f (z)) = z \quad (z \in \Delta) \]
and
\[ f (f^{-1} (w)) = w \quad \left( |w| < r_0 (f) : r_0 (f) \geq \frac{1}{4} \right). \]

In fact, the inverse function \( g = f^{-1} \) is given by
\[ g (w) = f^{-1} (w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \quad (1.2) \]

A function \( f \in \mathcal{A} \) is said to be bi-univalent in \( \Delta \) if both \( f \) and \( f^{-1} \) are univalent in \( \Delta \). We let \( \Sigma \) denote the class of bi-univalent functions in \( \Delta \) given by (1.1). For a history and examples of functions which are (or which are not) in the class \( \Sigma \), together with various other properties of subclasses of bi-univalent functions one can refer [1,4,6,13,15,17,19,21].

Recently, Babalola [3] defined the class \( \mathcal{L}_\lambda (\beta) \) of \( \lambda \)-pseudo-starlike functions of order \( \beta \) as follows:

Suppose \( 0 \leq \beta < 1 \) and \( \lambda \geq 1 \) is real. A function \( f \in \mathcal{A} \) given by (1.1) belongs to the class \( \mathcal{L}_\lambda (\beta) \) of \( \lambda \)-pseudo-starlike functions of order \( \beta \) in the unit disk \( \Delta \) if and only if
\[ \Re \left( \frac{z (f'(z))^\lambda}{f(z)} \right) > \beta \quad (z \in \Delta). \]

Remark 1 (see [3]) (i) If \( \lambda = 1 \), we have the class of starlike functions of order \( \beta \), that is, are 1-pseudo-starlike functions of order \( \beta \).

(ii) If \( \lambda = 2 \), we have the class \( \mathcal{L}_2 (\beta) \) consists of functions \( f \) satisfying
\[ \Re \left( \frac{f'(z)z f'(z)}{f(z)^2} \right) > \beta \]
which is a product combination of geometric expressions for bounded turning and starlike
functions.

(iii) The class \( L_{\infty} (\beta) \) is the singleton subclass of \( S \) containing the identity map only.

(iv) All pseudo-starlike functions are Bazilevič of type \( 1 - 1/\lambda \), order \( \beta^{1/\lambda} \) and univalent in \( \Delta \).

The significance of Chebyshev polynomials in numerical analysis is increased in both
theoretical and practical points of view. Out of four kinds of Chebyshev polynomials, many
researchers dealing with orthogonal polynomials of Chebyshev. For a brief history of Cheby-
shev polynomials of first kind \( T_n (t) \), the second kind \( U_n (t) \) and their applications one can
refer [2,10,12,16]. The Chebyshev polynomials of the first and second kinds are well known
and they are defined by

\[
T_n (t) = \cos n \theta \quad \text{and} \quad U_n (t) = \frac{\sin (n + 1) \theta}{\sin \theta} \quad (-1 < t < 1)
\]

where \( n \) denotes the polynomial degree and \( t = \cos \theta \).

**Definition 1** For \( \lambda \geq 1 \) and \( t \in (1/2, 1] \), a function \( f \in \Sigma \) given by (1.1) is said to be in
the class \( LB_{\Sigma} (\lambda, t) \) if the following conditions are satisfied:

\[
\frac{z (f'(z))^{\lambda}}{f(z)} < H(z, t) := \frac{1}{1 - 2tz + z^2} \quad (z \in \Delta)
\]

and

\[
\frac{w (g'(w))^{\lambda}}{g(w)} < H(w, t) := \frac{1}{1 - 2tw + w^2} \quad (w \in \Delta),
\]

where the function \( g = f^{-1} \) is defined by (1.2).

In addition to the identity map \( f(z) = z \), we give following nontrivial example of pseudo-
starlike functions for the special choice, \( \lambda = 2 \).

**Example 1** The function \( f(z) \) given by

\[
f(z) = \left( \arcsin \sqrt{1 - z} \right)^2 = z + \frac{1}{3} z^2 + \frac{8}{45} z^3 + \cdots
\]

belong to \( LB_{\Sigma} (2, t) \).

In particular, we set \( LB_{\Sigma} (1, t) = S^*_t \) (see [5]) for the class of functions \( f \in \Sigma \) given
by (1.1) and satisfying the following subordination conditions for all \( z, w \in \Delta \) :

\[
\frac{zf'(z)}{f(z)} < H(z, t) = \frac{1}{1 - 2tz + z^2}
\]

and

\[
\frac{wg'(w)}{g(w)} < H(w, t) = \frac{1}{1 - 2tw + w^2},
\]

where the function \( g = f^{-1} \) is defined by (1.2), (see also [2]).

We note that if \( t = \cos \alpha \), where \( \alpha \in (-\pi/3, \pi/3) \), then

\[
H(z, t) = \frac{1}{1 - 2 \cos \alpha z + z^2} = 1 + \sum_{n=1}^{\infty} \frac{\sin (n + 1) \alpha}{\sin \alpha} z^n \quad (z \in \Delta).
\]
Thus
\[ H(z, t) = 1 + 2 \cos \alpha z + (3 \cos^2 \alpha - \sin^2 \alpha)z^2 + \ldots \ (z \in \Delta). \]

From [20], we can write
\[ H(z, t) = 1 + U_1(t)z + U_2(t)z^2 + \ldots \ (z \in \Delta, \ t \in (-1, 1)) \]
where
\[ U_{n-1} = \frac{\sin(n \arccos t)}{\sqrt{1-t^2}} \ (n \in \mathbb{N}) \]
are the Chebyshev polynomials of the second kind and we have
\[ U_n(t) = 2t U_{n-1}(t) - U_{n-2}(t), \]
and \[ U_1(t) = 2t, \ U_2(t) = 4t^2 - 1, \ U_3(t) = 8t^3 - 4t, \ U_4(t) = 16t^4 - 12t^2 + 1, \ldots. \] (1.5)

The generating function of the first kind of Chebyshev polynomial \( T_n(t) \), \( t \in [-1, 1] \), is given by
\[ \sum_{n=0}^{\infty} T_n(t)z^n = \frac{1-tz}{1-2tz+z^2} \ (z \in \Delta). \]

The first kind of Chebyshev polynomial \( T_n(t) \) and second kind of Chebyshev polynomial \( U_n(t) \) are connected by:
\[ \frac{dT_n(t)}{dt} = nU_{n-1}(t); \quad T_n(t) = U_n(t) - tU_{n-1}(t); \quad 2T_n(t) = U_n(t) - U_{n-2}(t). \]

In this present paper, we use the Chebyshev polynomials expansions to provide the initial coefficients of bi-univalent functions in \( \mathcal{LB}_\Sigma(\lambda, t) \). We also solve Fekete-Szegö problem for functions in this class.

2 Coefficient bounds for the function class \( \mathcal{LB}_\Sigma(\lambda, t) \)

**Theorem 1** For \( \lambda \geq 1 \) and \( t \in (1/2, 1] \), let the function \( f \in \Sigma \) given by (1.1) be in the class \( \mathcal{LB}_\Sigma(\lambda, t) \). Then
\[ |a_2| \leq \min \left\{ \frac{2t}{2\lambda - 1}, \frac{2t\sqrt{2t}}{(2\lambda - 1)\sqrt{4(1-\lambda)t^2 + 2\lambda - 1}} \right\}, \] (2.1)
\[ |a_3| \leq \frac{4t^2}{(2\lambda - 1)^2} + \frac{2t}{3\lambda - 1}, \] (2.2)
and for some \( \mu \in \mathbb{R} \),
\[ |a_3 - \mu a_2^2| \leq \left\{ \begin{array}{ll}
\frac{2t}{3\lambda - 1}, & |\mu - 1| \leq \frac{(2\lambda - 1)(4(1-\lambda)t^2 + 2\lambda - 1)}{4(3\lambda - 1)t^2} \\
\frac{8|\mu - 1|^3}{(2\lambda - 1)(4(1-\lambda)t^2 + 2\lambda - 1)}, & |\mu - 1| \geq \frac{(2\lambda - 1)(4(1-\lambda)t^2 + 2\lambda - 1)}{4(3\lambda - 1)t^2}.
\end{array} \right. \] (2.3)
Proof Let the function \( f \in \Sigma \) given by (1.1) be in the class \( \mathcal{L}B_{\Sigma} (\lambda, t) \). From (1.3) and (1.4), we have
\[
\frac{z \left( f'(z) \right)^\lambda}{f(z)} = 1 + U_1(t) p(z) + U_2(t) p^2(z) + \cdots \quad (2.4)
\]
and
\[
\frac{w \left( g'(w) \right)^\lambda}{g(w)} = 1 + U_1(t) q(w) + U_2(t) q^2(w) + \cdots \quad (2.5)
\]
for some analytic functions
\[
p(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots \quad (z \in \Delta),
\]
and
\[
q(w) = d_1 w + d_2 w^2 + d_3 w^3 + \cdots \quad (w \in \Delta),
\]
such that \( p(0) = q(0) = 0 \), \( |p(z)| < 1 \) \((z \in \Delta)\) and \( |q(w)| < 1 \) \((w \in \Delta)\). It is well-known that if \( |p(z)| < 1 \) and \( |q(w)| < 1 \), then
\[
|c_j| \leq 1 \quad \text{and} \quad |d_j| \leq 1 \quad \text{for all} \quad j \in \mathbb{N}. \quad (2.8)
\]
From (2.4), (2.5), (2.6) and (2.7), we have
\[
\frac{z \left( f'(z) \right)^\lambda}{f(z)} = 1 + U_1(t) c_1 z + \left[ U_1(t) c_2 + U_2(t) c_1^2 \right] z^2 + \cdots \quad (2.9)
\]
and
\[
\frac{w \left( g'(w) \right)^\lambda}{g(w)} = 1 + U_1(t) d_1 w + \left[ U_1(t) d_2 + U_2(t) d_1^2 \right] w^2 + \cdots . \quad (2.10)
\]
Equating the coefficients in (2.9) and (2.10), we get
\[
(2\lambda - 1) a_2 = U_1(t) c_1 \quad (2.11)
\]
\[
(2\lambda - 4\lambda + 1) a_2^2 + (3\lambda - 1) a_3 = U_1(t) c_2 + U_2(t) c_1^2 \quad (2.12)
\]
\[
- (2\lambda - 1) a_2 = U_1(t) d_1 \quad (2.13)
\]
and
\[
(2\lambda^2 + 2\lambda - 1) a_2^2 - (3\lambda - 1) a_3 = U_1(t) d_2 + U_2(t) d_1^2 . \quad (2.14)
\]
From (2.11) and (2.13) we obtain
\[
c_1 = -d_1 \quad (2.15)
\]
and
\[
2 (2\lambda - 1)^2 a_2^2 = U_1^2(t) \left( c_1^2 + d_1^2 \right) . \quad (2.16)
\]
Also, by using (2.12) and (2.14), we obtain
\[
2\lambda (2\lambda - 1) a_2^2 = U_1(t) \left( c_2 + d_2 \right) + U_2(t) \left( c_1^2 + d_1^2 \right) . \quad (2.17)
\]
By using (2.16) in (2.17), we get
\[
\left[ 2\lambda (2\lambda - 1) - \frac{2U_2(t)}{U_1^2(t)} (2\lambda - 1)^2 \right] a_2^2 = U_1(t) \left( c_2 + d_2 \right) . \quad (2.18)
\]
From (1.5), (2.8) and (2.18), we have the desired inequality (2.1) comparing with (2.11). Next, by subtracting (2.14) from (2.12), we have
\[
2 (3\lambda - 1) a_3 - 2 (3\lambda - 1) a_2^2 = U_1(t) \left( c_2 - d_2 \right) + U_2(t) \left( c_1^2 - d_1^2 \right) . \quad (2.19)
\]
Further, in view of (2.15), we obtain
\[
a_3 = a_2^2 + \frac{U_1(t)}{2 (3\lambda - 1)} (c_2 - d_2) . \quad (2.20)
\]
Hence using (2.16) and applying (1.5), we get desired inequality (2.2).

Now, by using (2.18) and (2.20) for some $\mu \in \mathbb{R}$, we get

$$a_3 - \mu a_2^2 = (1 - \mu) \left[ \frac{U_1^3(t)(c_2 + d_2)}{2(2\lambda - 1)} \left( \lambda U_1^2(t) - (2\lambda - 1) U_2(t) \right) \right] + \frac{U_1(t) (c_2 - d_2)}{2(3\lambda - 1)}$$

$$= \frac{U_1(t)}{2} \left[ \left( h(\mu) + \frac{1}{3\lambda - 1} \right) c_2 + \left( h(\mu) - \frac{1}{3\lambda - 1} \right) d_2 \right],$$

where $h(\mu) = \frac{U_1^2(t)(1 - \mu)}{(2\lambda - 1) \left[ \lambda U_1^2(t) - (2\lambda - 1) U_2(t) \right]}$.

So, we conclude that

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2t}{3\lambda - 1}, & 0 \leq |h(\mu)| \leq \frac{1}{3\lambda - 1} \\ 2 |h(\mu)| t, & |h(\mu)| \geq \frac{1}{3\lambda - 1} \end{cases}.$$ 

This completes the proof of Theorem 1.

Taking $\mu = 1$ in Theorem 1, we get the following consequence.

**Corollary 1** For $\lambda \geq 1$ and $t \in (1/2, 1]$, let the function $f \in \Sigma$ given by (1.1) be in the class $LB_\Sigma(\lambda, t)$. Then

$$|a_3 - a_2^2| \leq \frac{2t}{3\lambda - 1}.$$ 

Taking $\lambda = 1$ in Theorem 1, we get the following consequence.

**Corollary 2** For $t \in (1/2, 1]$, let the function $f \in \Sigma$ given by (1.1) be in the class $S^*_\Sigma(t)$. Then

$$|a_2| \leq 2t,$$

$$|a_3| \leq 4t^2 + t,$$

and for some $\mu \in \mathbb{R}$,

$$|a_3 - \mu a_2^2| \leq \begin{cases} t, & |\mu - 1| \leq \frac{1}{8t^2} \\ 8 |\mu - 1| t^3, & |\mu - 1| \geq \frac{1}{8t^2} \end{cases}.$$ 

Taking $\mu = 1$ in Corollary 2, we get the following consequence.

**Corollary 3** For $t \in (1/2, 1]$, let the function $f \in \Sigma$ given by (1.1) be in the class $S^*_\Sigma(t)$. Then

$$|a_3 - a_2^2| \leq t.$$ 

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