A HARMONIC MAP FLOW ASSOCIATED WITH
THE STANDARD SOLUTION OF RICCI FLOW

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Abstract. Let \((\mathbb{R}^n, g(t)), 0 \leq t \leq T, n \geq 3\), be a standard solution of the Ricci flow with radially symmetric initial data \(g_0\). We will extend a recent existence result of P. Lu and G. Tian and prove that for any \(t_0 \in [0,T]\) there exists a solution of the corresponding harmonic map flow \(\phi_t: (\mathbb{R}^n, g(t)) \rightarrow (\mathbb{R}^n, g(t_0))\) satisfying

\[
\frac{\partial \phi_t}{\partial t} = \Delta_{g(t)} g(\phi_t) \quad \text{of the form}
\]

\[
\phi_t(r, \theta) = (\rho(r, t), \theta) \quad \text{in polar coordinates in} \quad \mathbb{R}^n \times (t_0, T), \quad \phi_{t_0}(r, \theta) = (r, \theta),
\]

where \(r = r(t)\) is the radial co-ordinate with respect to \(g(t)\) and \(T_0 = \sup\{t_1 \in (t_0, T) : \|\tilde{\rho}(-, t)\|_{L^\infty(\mathbb{R}^+)} + \|\partial \tilde{\rho}/\partial r(-, t)\|_{L^\infty(\mathbb{R}^+)} < \infty \quad \forall t_0 < t \leq t_1\}\) with \(\tilde{\rho}(r, t) = \log(\rho(r, t)/r)\). We will also prove the uniqueness of solution of the harmonic map flow within the class of functions of the form \(\phi_t(r, \theta) = (\rho(r, t), \theta), \quad \rho(r, t) = re^{\tilde{\rho}(r, t)}, \) for some function \(\tilde{\rho}(r, t)\). We will also use the same technique to prove that the solution \(u\) of the heat equation in \((\Omega \setminus \{0\}) \times (0, T)\) has removable singularities at \(\{0\} \times (0, T), \quad \Omega \subset \mathbb{R}^m, m \geq 3,\) if and only if \(|u(x, t)| = O(|x|^{2-m})\) locally uniformly on every compact subset of \((0, T)\).

It is known that Ricci flow is a powerful method in studying the geometry of manifolds. A manifold \((M, g(t)), 0 \leq t \leq T,\) with an evolving metric \(g(t)\) is said to be a Ricci flow if it satisfies

\[
\frac{\partial}{\partial t} g_{ij} = -2R_{ij}
\]

in \(M \times (0,T).\) Short time existence of solution of Ricci flow on compact manifold was proved by R. Hamilton [H1] using the Nash-Moser Theorem. Short time existence of solutions of the Ricci flow on complete non-compact Riemannian manifold with bounded curvature was proved by W.X Shi [S1]. Global existence and uniqueness of solutions of the

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Ricci flow on non-compact manifold $\mathbb{R}^2$ was obtained by S.Y. Hsu in [Hs1]. We refer the readers to the lecture notes by B. Chow [C] and the book [CK] by B. Chow and D. Knopf on the basics of Ricci flow. Interested readers can also read the papers of R. Hamilton [H1-6], S.Y. Hsu [Hs1-7], B. Kleiner and J. Lott [KL], J. Morgan and G. Tian [MT], G. Perelman [P1], [P2], W.X. Shi [S1], [S2], L.F. Wu [W1], [W2], R. Ye [Ye] for some of the most recent results on Ricci flow.

Since the proof of existence of solution of the Ricci flow in [H1] is very hard and there is very few uniqueness results for Ricci flow, later D.M. DeTurck [D] devised another method to prove existence and uniqueness of solution of Ricci flow. For any $t_0 \in [0,T)$ he introduced an auxiliary harmonic map flow $\phi_t : (M, g(t)) \to (M, g(t_0))$ associated with the Ricci flow given by

$$\frac{\partial}{\partial t} \phi_t = \Delta_{g(t), g(t_0)} \phi_t$$

(0.1)

where

$$\Delta_{g(t), g(t_0)} \phi_t = \Delta_{g(t)} \phi_t + g^{ij}(x, t) \Gamma^\alpha_{\beta \gamma}(\phi_t(x)) \frac{\partial \phi_t(x)^\beta}{\partial x^i} \frac{\partial \phi_t(x)^\gamma}{\partial x^j}$$

in the local co-ordinates $x = (x^1, \ldots, x^n)$ of the domain manifold $(M, g(t))$ and the local co-ordinates $(y^\alpha)$ of the target manifold $(M, g(t_0))$ with $\Gamma^\alpha_{\beta \gamma}$ being the Christoffel symbols of $(M, g(t_0))$. This harmonic map flow then induces a push forward metric $\hat{g}(t) = (\phi_t)_*(g(t))$ on the target manifold $M$ which satisfies the Ricci-DeTurck flow [H5]

$$\frac{\partial}{\partial t} \hat{g}_{\alpha \beta} = (L_V \hat{g})_{\alpha \beta} - 2 \hat{R}_{\alpha \beta}$$

for some time varying vector field $V$ on the target manifold $M$ where $\hat{R}_{\alpha \beta}$ is the Ricci curvature associated with the metric $\hat{g}(t)$. The existence and uniqueness of solutions of Ricci flow on compact manifolds are then reduced to the study of existence and other properties of the harmonic map flow (0.1) and the Ricci-DeTurck flow [H5].

In [P1], [P2], G. Perelman proposed a scheme to study Ricci flow with singularities. Essential to this scheme is the construction of a standard solution of the Ricci flow which is used to replace the solution near the singularities during surgery. In [LT] P. Lu and G. Tian proved the short time existence of solution of the harmonic map flow associated with the standard solution $(\mathbb{R}^n, g(t))$, $0 \leq t \leq T$, $n \geq 3$, of Ricci flow with radially symmetric initial data $g_0$.

In this paper we will extend their result and prove that for any $t_0 \in [0,T)$ there exists a solution of the corresponding harmonic map flow $\phi_t : (\mathbb{R}^n, g(t)) \to (\mathbb{R}^n, g(t_0))$ satisfying

$$\begin{cases}
\frac{\partial \phi_t}{\partial t} = \Delta_{g(t), g(t_0)} \phi_t & \text{in } \mathbb{R}^n \times (t_0, T) \\
\phi_{t_0}(x) = x & \text{in } \mathbb{R}^n
\end{cases}$$

(0.2)

of the form

$$\phi_t(r, \theta) = (\rho(r, t), \theta), \rho(r, t) = re^{\tilde{\rho}(r, t)},$$

(0.3)
for some function \( \tilde{\rho}(r, t) \) in polar coordinates in \( \mathbb{R}^n \times (t_0, T) \), \( \phi_{t_0}(r, \theta) = (r, \theta) \), where \( r = r(t) \) is the radial co-ordinates with respect to the metric \( g(t) \) and

\[
T_0 = \sup\{ t_1 \in (t_0, T) : \| \tilde{\rho}(\cdot, t) \|_{L^\infty(\mathbb{R}^n)} + \| \partial \tilde{\rho} / \partial r(\cdot, t) \|_{L^\infty(\mathbb{R}^n)} < \infty \quad \forall t_0 < t \leq t_1 \}. \quad (0.4)
\]

Then

\[
\tilde{\rho}(r, t) = \log \left( \frac{\rho(r, t)}{r} \right).
\]

By (0.2),

\[
\rho(r, t_0) = r \quad \forall r > 0.
\]

We will also prove the uniqueness of solution of the harmonic map flow (0.2) within the class of functions of the form (0.3) for some function \( \tilde{\rho}(r, t) \).

The plan of the paper is as follows. In section one we will extend the existence result of [LT] and prove the existence of solution of (0.2) of the form (0.3) in \( \mathbb{R}^n \times (t_0, T) \) where \( T_0 \) is given by (0.4). In section two we will prove various estimates for the Green function of the heat equation in cylindrical and punctured cylindrical domains. In section three we will use the Green function estimates to prove that the transformed solution in \( (\mathbb{R}^{n+2} \setminus \{0\}) \times (t_0, T) \) has removable singularities on the line \( \{0\} \times (t_0, T) \). We will also prove the uniqueness of solution of (0.2) in section three. In section four we will prove that a solution \( u \) of the heat equation in \( (\Omega \setminus \{0\}) \times (0, T) \) has removable singularities at \( \{0\} \times (0, T) \), \( \Omega \subset \mathbb{R}^m, m \geq 3 \), if and only if there exists \( B_R(0) \subset \Omega \) such that

\[
|u(x, t)| = O(|x|^{2-m}) \quad \text{uniformly on} \ [t_1, t_2] \quad \forall 0 < |x| \leq R, 0 < t_1 < t_2 < T. \quad (0.7)
\]

We first start with a definition. Let \( n \geq 3 \). For any \( 0 \leq t < (n - 1)/2 \), let \( h(t) \) be the standard metric on \( S^{n-1} \) with constant scalar curvature

\[
\frac{1}{1 - \frac{2r}{n-1}}.
\]

Let \( g_0 \) be a fixed rotationally symmetric complete smooth metric with non-negative curvature operator on \( \mathbb{R}^n \) such that \( (\mathbb{R}^n \setminus \overline{B(0, 2)}, g_0) \) is isometric to the half infinite cylinder \( (S^{n-1} \times \mathbb{R}^+, h(1) \times ds^2) \) (cf. Section 1 of [LP] and definition 12.1 of [MT]). By the argument of section 1 of [LP] such \( g_0 \) exists. We say that a Ricci flow \( (\mathbb{R}^n, g(t)) \), \( 0 \leq t < T \), is a standard solution if \( g(0) = g_0 \) and the curvature \( Rm \) is locally bounded in time \( t \in [0, T) \).

By the results of [LP], there exists a standard solution \( (\mathbb{R}^n, g(t)) \) of the Ricci flow on \( (0, T) \) for some \( T \in (0, \frac{n-1}{2}) \) with \( g(0) = g_0 \) which has non-negative curvature operator \( Rm(t) \) for each \( t \in [0, T) \).

We will now let \( (\mathbb{R}^n, g(t)), 0 \leq t < T, \) be the standard solution of Ricci flow for the rest of the paper. By the result of [P2] and [LT] for each \( 0 \leq t < T, g(t) \) is a rotationally symmetric metric of \( \mathbb{R}^n \). Let \( \tilde{r} \) be the standard radial co-ordinate in \( \mathbb{R}^n \). As observed by P. Lu and G. Tian [LT] if \( d\sigma = h(1) \) is the standard metric on \( S^{n-1} \) with constant scalar curvature 1, then there exists a function \( f(r, t) \geq 0 \) such that

\[
g(t) = d\tilde{r}^2 + f(r, t)^2 d\sigma
\]
where $r = r(\tilde{r}, t)$ is the radial co-ordinate on $\mathbb{R}^n$ with respect to the metric $g(t)$. We fix a $t_0 \in [0, T)$ and consider the harmonic map flow $\phi_t : (\mathbb{R}^n, g(t)) \to (\mathbb{R}^n, g(t_0))$ (0.2) of the form (0.3). Let $\tilde{\rho}(r, t)$ be given by (0.5) and let $f_0(r) = f(r, t_0)$. Then by (0.5) and (0.6),

$$\tilde{\rho}(r, t_0) = 0.$$ 

By the computation in [LT] and [MT],

$$f(r, t) = re^{\tilde{f}(r^2, t)}, f_0(\rho) = \rho e^{\tilde{f}_0(\rho^2)}$$

for some smooth functions $\tilde{f}(w, t)$ and $\tilde{f}_0(w)$ of $w \geq 0$ and $t$ with $\tilde{f}(w, t_0) = \tilde{f}_0(w)$. Moreover $\tilde{\rho}(r, t)$ satisfies

$$\frac{\partial \tilde{\rho}}{\partial t} = \frac{\partial^2 \tilde{\rho}}{\partial r^2} + \frac{n + 1}{r} \frac{\partial \tilde{\rho}}{\partial r} + \left[ (n - 1) \frac{\partial \tilde{f}}{\partial r} - r \xi \right] (r^2, t) \frac{\partial \tilde{\rho}}{\partial r} + \left( \frac{\partial \tilde{\rho}}{\partial r} \right)^2 + G(\tilde{\rho}, r^2, t) = 0 \quad (0.8)$$

where

$$G(\tilde{\rho}, w, t) = \frac{n - 1}{w} \left[ 1 - e^{2\tilde{f}_0(\rho^2) - 2\tilde{f}(w, t)} \right] + 2(n - 1) \frac{\partial \tilde{f}}{\partial w}(w, t)$$

$$- 2(n - 1)e^{2\tilde{f}_0(\rho^2) + 2\tilde{f}(w, t)} \frac{\partial \tilde{f}_0}{\partial w}(\rho^2) - 2\xi(w, t)$$

and $\xi(w, t)$ is a smooth function satisfying

$$\xi(r^2, t) = \frac{1}{r} \frac{\partial r}{\partial t}.$$ 

Let $x = (x^1, \ldots, x^{n+2}) \in \mathbb{R}^{n+2}$, $|x| = \left( \sum_{i=1}^{n+2} (x^i)^2 \right)^{1/2}$, and

$$\tilde{\rho}(x, t) = \tilde{\rho}(|x|, t), \quad \tilde{f}(x, t) = \tilde{f}(|x|^2, t), \quad G_1(s, x, t) = G(s, |x|^2, t). \quad (0.9)$$

If $\tilde{\rho} \in C^{2,1}([0, \infty) \times (t_0, T'))$ is a solution of (0.8) in $\mathbb{R}^+ \times (t_0, T')$ for some $T' \in (t_0, T)$ [LT], then $\tilde{\rho}(x, t)$ is a radially symmetric solution of

$$\frac{\partial \tilde{\rho}}{\partial t} = \Delta \tilde{\rho} + \nabla [((n - 1)\tilde{f} - \underline{f}) \cdot \nabla \rho + |\nabla \rho|^2 + G_1(\rho, x, t)] \quad (0.10)$$

in $(\mathbb{R}^{n+2} \setminus \{0\}) \times (t_0, T')$ where

$$B(w, t) = \frac{1}{2} \int_0^w \xi(u, t) du$$

is a smooth even function of $w$ and $\underline{B}(x, t) = B(|x|^2, t)$ is a smooth function of $x \in \mathbb{R}^{n+2}$.
Conversely if \( \overline{\rho}(x,t) \) is a radially symmetric solution of (0.10) in \( \mathbb{R}^{n+2} \times (t_0, T') \), then \( \tilde{\rho} \in C^{2,1}([0, \infty) \times (t_0, T')) \) is a solution of (0.8) in \( \mathbb{R}^{+} \times (t_0, T') \).

As in [LT] we rewrite (0.10) as

\[
\frac{\partial \overline{\rho}}{\partial t} = \Delta \overline{\rho} + F(x, \overline{\rho}, \nabla \overline{\rho}, t) \tag{0.11}
\]

where

\[
F(x, \overline{\rho}, \nabla \overline{\rho}, t) = \nabla[(n-1)\overline{f} - \overline{B}](x, t) \cdot \nabla \overline{\rho} + |\nabla \overline{\rho}|^2 + G_1(\overline{\rho}, x, t).
\]

We will fix \( t_0 \in (0, T) \) for the rest of the paper.

For any set \( A \) we let \( \chi_A \) be the characteristic function of the set \( A \). For any \( R > 0 \), \( m \geq 2 \), let \( B_R = \{ x : |x| < R \} \subset \mathbb{R}^m \) and let \( G_R = G_R(x, t, y, s) \) be the Green function of the heat equation in \( B_R \times (-\infty, \infty) \). Then ([LSU] P.408)

\[
G_R(x, t, y, s) = \Gamma(x, t, y, s) - g_R(x, t, y, s) \quad \forall x, y \in B_R, t > s,
\]

where

\[
\Gamma(x, t, y, s) = \frac{1}{(4\pi(t-s))^\frac{n+2}{2}} e^{-\frac{|x-y|^2}{4(t-s)}}
\]

and \( \forall y \in B_R, s \in \mathbb{R} \),

\[
\begin{cases}
\partial_t g_R = \Delta_x g_R & \forall x \in B_R, t > s \\
g_R(x, t, y, s) = \Gamma(x, t, y, s) & \forall x \in \partial B_R, t > s \\
g_R(x, s, y, s) = 0 & \forall x \in \partial B_R.
\end{cases} \tag{0.12}
\]

Note that by the maximum principle (cf. [A],[F]),

\[
0 \leq G_R(x, t, y, s) \leq G_{R'}(x, t, y, s) \leq \Gamma(x, t, y, s) \quad \forall x, y \in B_R, 0 < R < R', t > s. \tag{0.13}
\]

Section 1

In this section we will extend the existence result of P. Lu and G. Tian [LT] and prove the existence of solution of (0.2) of the form (0.3) in \( \mathbb{R}^n \times (t_0, T_0) \) for some constant \( T_0 \in (t_0, T] \) given by (0.4). We first recall a result of [LT].

**Theorem 1.1.** (Section 2.2.3 of [LT]) There exists \( T' \in (t_0, T] \) such that (0.11) has a radially symmetric solution \( \overline{\rho} \) in \( \mathbb{R}^{n+2} \times (t_0, T') \) satisfying

\[
\overline{\rho}(x, t_0) \equiv 0 \quad \text{in } \mathbb{R}^{n+2}, \tag{1.1}
\]

\[
\overline{\rho}(x, t) = \int_{t_0}^{t} \int_{\mathbb{R}^{n+2}} \frac{1}{(4\pi(t-s))^{\frac{n+2}{2}}} e^{-\frac{|x-y|^2}{4(t-s)}} F(y, \overline{\rho}, \nabla \overline{\rho}, s) \, dy \, ds \tag{1.2}
\]

for all \( x \in \mathbb{R}^{n+2}, t_0 \leq t < T' \), and

\[
\sup_{t_0 \leq t < T'} \left( \|\overline{\rho}(\cdot, t)\|_{L^\infty(\mathbb{R}^{n+2})} + \|\nabla \overline{\rho}(\cdot, t)\|_{L^\infty(\mathbb{R}^{n+2})} \right) < \infty. \tag{1.3}
\]
Theorem 1.2. There exists $T_0 \in (t_0, T]$ such that (0.11) has a radially symmetric solution $\bar{\rho}$ in $\mathbb{R}^{n+2} \times (t_0, T_0)$ satisfying (1.1), (1.2), in $\mathbb{R}^{n+2} \times (t_0, T_0)$ where $T_0$ is given by (0.4) with $\bar{\rho}$ and $\bar{\rho}$ being related by (0.9).

Proof. By Theorem 1.1 there exists $T' \in (t_0, T]$ such that (0.11) has a radially symmetric solution $\bar{\rho}$ in $\mathbb{R}^{n+2} \times (t_0, T')$ satisfying (1.1) and (1.2). Let $T_1 \in (t_0, T]$ be the maximal existence time of a radially symmetric solution $\bar{\rho}$ of (0.11) in $\mathbb{R}^{n+2} \times (t_0, T)$ that satisfies (1.1), (1.2), and (1.3) for any $t_0 < T' < T_1$. Then $T_1 \leq T_0$. We claim that $T_1 = T_0$. Suppose $T_1 < T_0$. Then

$$
C_1 = \sup_{t_0 \leq t < T_1} \left( \|\bar{\rho}(\cdot, t)\|_{L^\infty(\mathbb{R}^{n+2})} + \|\nabla \bar{\rho}(\cdot, t)\|_{L^\infty(\mathbb{R}^{n+2})} \right) < \infty.
$$

Let $T_2 \in (t_0, T_1)$ be a constant to be determined later. We will now use a modification of the argument of [LiT] and [LT] to construct a solution of (0.11) with initial data $\bar{\rho}(x, T_2)$. For any $T_2 \leq t < T$, $x \in \mathbb{R}^{n+2}$, let $\bar{\rho}_1(x, t) = \bar{\rho}(x, T_2)$, and

$$
\bar{\rho}_1(x, t) = v(x, t) + \int_{T_2}^t \int_{\mathbb{R}^{n+2}} \frac{1}{(4\pi(t-s))^{n+2/2}} e^{-\frac{|x-y|^2}{4(t-s)}} F(y, \bar{\rho}_{i-1}, \nabla \bar{\rho}_{i-1}, s) \, dy \, ds \quad \forall i \geq 2
$$

with

$$
v(x, t) = \begin{cases} 
\frac{1}{(4\pi(t-T_2))^{n+2/2}} \int_{\mathbb{R}^{n+2}} e^{-\frac{|x-y|^2}{4(t-T_2)}} \bar{\rho}(y, T_2) \, dy & \text{for } t > T_2 \\
\bar{\rho}(x, T_2) & \text{for } t = T_2.
\end{cases}
$$

Then

$$
\|v(\cdot, t)\|_{L^\infty(\mathbb{R}^{n+2})} + \|\nabla v(\cdot, t)\|_{L^\infty(\mathbb{R}^{n+2})} \leq \|\bar{\rho}_1(\cdot, t)\|_{L^\infty(\mathbb{R}^{n+2})} + \|\nabla \bar{\rho}_1(\cdot, t)\|_{L^\infty(\mathbb{R}^{n+2})} \\
\leq C_1 \quad \forall t \geq T_2.
$$

We claim that there exists $\delta_1 \in (0, T - T_1)$ independent of $T_2$ such that

$$
\|v(\cdot, t)\|_{L^\infty(\mathbb{R}^{n+2})} + \|\nabla v(\cdot, t)\|_{L^\infty(\mathbb{R}^{n+2})} \leq 2C_1 \quad \forall T_2 \leq t \leq T_2 + \delta_1, i \in \mathbb{Z}^+.
$$

We will prove this claim by induction. By (1.5), (1.6) holds for $i = 1$. Suppose there exists $\delta_1 \in (0, T - T_1)$ such that (1.6) holds for $i = k - 1$ for some $k \geq 2$. As in [LT] there exist a constant $C_2 > 0$ depending on $C_1$ and a constant $C_3 > 0$ such that $\forall(x, t) \in \mathbb{R}^{n+2} \times [T_2, T_2 + \delta_1),$ 

$$
\|G_1(\bar{\rho}_{k-1}, x, t)\| \leq C_2 \\
|F(x, \bar{\rho}_{k-1}, \nabla \bar{\rho}_{k-1}, t)| \leq C_3(2C_1) + (2C_1)^2 + C_2 = C_4 \text{ (say)}.
$$

Then by (1.4),

$$
|\bar{\rho}_k(x, t)| \leq |v(x, t)| + C_4(t - T_2) \quad \forall(x, t) \in \mathbb{R}^{n+2} \times [T_2, T_2 + \delta_1]
$$

(1.8)
and
\[ |\nabla \rho_k(x, t)| \leq |\nabla v(x, t)| + \int_{T_2}^t \int_{\mathbb{R}^{n+2}} \frac{1}{(4\pi(t-s))^{n+2 \over 2}} |\nabla e^{-|x-y|^2 \over 4(t-s)}||F(y, \bar{\rho}_{k-1}, \nabla \bar{\rho}_{k-1}, s)| \, dy \, ds \]
\[ \leq |\nabla v(x, t)| + C_1 \int_{T_2}^t \int_{\mathbb{R}^{n+2}} \frac{|x-y|}{(4\pi(t-s))^{n+2 \over 2}} e^{-|x-y|^2 \over 4(t-s)} \, dy \, ds \]
\[ \leq |\nabla v(x, t)| + C_4 C_5 \int_{T_2}^t \int_{\mathbb{R}^{n+2}} \frac{1}{(t-s)^{n+2 \over 2}} e^{-|x-y|^2 \over 4(t-s)} \, dy \, ds \]
\[ \leq |\nabla v(x, t)| + C_4 C_5' \sqrt{t-T_2} \quad \forall (x, t) \in \mathbb{R}^{n+2} \times [T_2, T_2 + \delta_1] \]  
(1.9)
for some constants $C_5 > 0$, $C_5' > 0$, independent of $T_2$ and $\delta_1$. Let
\[ \delta_1 = \min\left(1, \frac{T-T_1}{2}, \left(\frac{C_1}{C_4 + C_4 C_5'}\right)^2\right). \]

Then by (1.5), (1.8) and (1.9), $\forall T_2 \leq t \leq T_2 + \delta_1$,
\[ \|\bar{\rho}_k(\cdot, t)\|_{L^\infty(\mathbb{R}^{n+2})} + \|\nabla \bar{\rho}_k(\cdot, t)\|_{L^\infty(\mathbb{R}^{n+2})} \leq C_1 + C_4 \delta_1 + C_4 C_5' \sqrt{\delta_1} \leq 2C_1. \]

Hence by induction (1.6) holds. Since
\[ \bar{\rho}_i(x, t) - \bar{\rho}_{i-1}(x, t) \]
\[ = \int_{T_2}^t \int_{\mathbb{R}^{n+2}} \frac{1}{(4\pi(t-s))^{n+2 \over 2}} e^{-|x-y|^2 \over 4(t-s)} \left(F(y, \bar{\rho}_{i-1}, \nabla \bar{\rho}_{i-1}, s) - F(y, \bar{\rho}_{i-2}, \nabla \bar{\rho}_{i-2}, s)\right) \, dy \, ds \]
for any $i \geq 3$, by (1.6) and an argument similar to the proof on P.10–11 of [LT] there exists a constant $\delta_2 \in (0, \delta_1)$ independent of $T_2$ and depending only on $\delta_1$, $C_1$, $T_1$, and $T$ such that $\{\bar{\rho}_i\}_{i=1}^\infty$ is a Cauchy sequence in $C^1(\mathbb{R}^{n+2} \times (T_2, T_2 + \delta_2))$ with norm given by
\[ \|\psi\| = \|\psi\|_{L^\infty(\mathbb{R}^{n+2} \times (T_2, T_2 + \delta_2))} + \|\nabla \psi\|_{L^\infty(\mathbb{R}^{n+2} \times (T_2, T_2 + \delta_2))} \quad \forall \psi \in C^1(\mathbb{R}^{n+2} \times (T_2, T_2 + \delta_2)). \]

Let $T_2 = T_1 - \delta_2/2$. Then $T_2 + \delta_2 > T_1$ and there exists $\bar{\rho}_\infty \in C^1(\mathbb{R}^{n+2} \times (T_2, T_2 + \delta_2))$ such that $\bar{\rho}_i$ converges uniformly to $\bar{\rho}_\infty$ in $C^1(\mathbb{R}^{n+2} \times (T_2, T_2 + \delta_2))$ as $i \to \infty$. We now extend $\bar{\rho}$ beyond the time $T_2$ by setting $\bar{\rho}(x, t) = \bar{\rho}_\infty(x, t)$ for any $x \in \mathbb{R}^{n+2}, T_2 < t \leq T_2 + \delta_2$. Letting $i \to \infty$ in (1.4),
\[ \bar{\rho}(x, t) = v(x, t) + \int_{T_2}^t \int_{\mathbb{R}^{n+2}} \frac{1}{(4\pi(t-s))^{n+2 \over 2}} e^{-|x-y|^2 \over 4(t-s)} F(y, \bar{\rho}, \nabla \bar{\rho}, s) \, dy \, ds \]  
(1.10)
for any $x \in \mathbb{R}^{n+2}, T_2 < t \leq T_2 + \delta_2$. By Theorem 1.1 and the semi-group property of the heat equation,
\[ v(x, t) = \frac{1}{(4\pi(t-T_2))^{n+2 \over 2}} \int_{\mathbb{R}^{n+2}} e^{-|x-y|^2 \over 4(t-T_2)} \int_{T_2}^T \int_{\mathbb{R}^{n+2}} \frac{1}{(4\pi(T_2-s))^{n+2 \over 2}} e^{-|x-y|^2 \over 4(T_2-s)} F(y, \bar{\rho}, \nabla \bar{\rho}, s) \, dy \, ds \, dz \]
\[ = \int_{T_0}^T \int_{\mathbb{R}^{n+2}} \frac{1}{(4\pi(t-s))^{n+2 \over 2}} e^{-|x-y|^2 \over 4(t-s)} F(y, \bar{\rho}, \nabla \bar{\rho}, s) \, dy \, ds. \]  
(1.11)
By (1.10) and (1.11), $\overline{\rho}$ satisfies (1.2) in $\mathbb{R}^{n+2} \times (t_0, T_2 + \delta_2)$. By (1.2) and the same argument as [LT] $\overline{\rho}^T$ is a classical solution of (0.11). Since $T_2 + \delta_2 > T_1$ and (1.6) holds, there is a contradiction to the definition of $T_1$. Hence $T_1 = T_0$ and the theorem follows.

Section 2

In this section we will prove various estimates for the Green function of the heat equation in cylindrical and punctured cylindrical domains. We first choose a monotone increasing function $\eta \in C^\infty(\mathbb{R})$, $0 \leq \eta \leq 1$, such that $\eta(\tau) = 0$ for any $\tau \leq 1/2$ and $\eta(\tau) = 1$ for any $\tau \geq 1$. For any $0 < \delta \leq 1$, $s \in \mathbb{R}$, let $\eta_\delta(\tau) = \eta(\tau/\delta)$ and let $g_{R,\epsilon,\delta}$ be the solution of

$$
\begin{aligned}
\partial_t g_{R,\epsilon,\delta} &= \Delta_x g_{R,\epsilon,\delta} & \forall x \in B_R \setminus B_\epsilon, t > s \\
g_{R,\epsilon,\delta}(x, t, y, s) &= \Gamma(x, t, y, s)\eta((t-s)/\delta) & \forall (x, t) \in (\partial B_\epsilon \cup \partial B_R) \times (s, \infty) \\
g_{R,\epsilon,\delta}(x, s, y, s) &= 0 & \forall x \in B_R
\end{aligned}
$$

where $B_R \subset \mathbb{R}^m$ for some $m \in \mathbb{Z}^+$. Then by the maximum principle,

$$
g_{R,\epsilon,\delta}(x, t, y, s) \geq g_{R,\epsilon,\delta'}(x, t, y, s) \geq 0 \quad \forall \epsilon \leq |x|, |y| \leq R, 0 < \delta < \delta' \leq 1, t > s. \quad (2.1)
$$

**Lemma 2.1.** Let $s \in \mathbb{R}$ and $\epsilon \leq |y| \leq R$. Then there exists a sequence $\{\delta_i\}_{i=1}^\infty$, $\delta_i \rightarrow 0$ as $i \rightarrow \infty$, such that $g_{R,\epsilon,\delta_i}(\cdot, t, y, s)$ converges uniformly in $C^{2,1}((\overline{B}_R \setminus B_\epsilon) \times [t_1, t_2])$ to the solution $g_{R,\epsilon}(\cdot, t, y, s)$ of the problem

$$
\begin{aligned}
\partial_t g_{R,\epsilon} &= \Delta_x g_{R,\epsilon} & \forall x \in B_R \setminus B_\epsilon, t > s \\
g_{R,\epsilon}(x, t, y, s) &= \Gamma(x, t, y, s) & \forall x \in \partial B_\epsilon \cup \partial B_R, t > s \\
g_{R,\epsilon}(x, s, y, s) &= 0 & \forall x \in B_R
\end{aligned}
$$

as $i \rightarrow \infty$ for any $t_2 > t_1 > s$. Moreover

$$
g_{R,\epsilon,\delta}(x, t, y, s) \leq \Gamma(x, t, y, s) \quad \forall \epsilon \leq |x|, |y| \leq R, 0 < \delta \leq 1, t > s \quad (2.3)
$$

and

$$
0 \leq g_{R,\epsilon}(x, t, y, s) \leq \Gamma(x, t, y, s) \quad \forall \epsilon \leq |x|, |y| \leq R, t > s. \quad (2.4)
$$

**Proof.** Note that the result (2.4) is well-known (cf. [A]). For the sake of completeness we will give a short proof of (2.4) here. We will use a modification of the technique of [DK] to prove (2.3). Let $h \in C^\infty_0(B_R \setminus B_\epsilon)$ be such that $0 \leq h \leq 1$. For any $t > s$, let $\phi(x, \tau)$ be the solution of

$$
\begin{aligned}
\partial_\tau \phi + \Delta \phi &= 0 & \text{in } (B_R \setminus B_\epsilon) \times (s, t) \\
\phi(x, \tau) &= 0 & \text{on } (\partial B_R \cup \partial B_\epsilon) \times (s, t) \\
\phi(x, t) &= h(x) & \text{in } B_R \setminus B_\epsilon.
\end{aligned}
$$

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By the maximum principle $0 \leq \phi \leq 1$ on $(B_R \setminus B_\varepsilon) \times (s,t)$. Hence $\partial \phi / \partial n \geq 0$ on $(\partial B_R \cup \partial B_\varepsilon) \times (s,t)$ where $\partial / \partial n$ is the derivative at the boundary in the direction of the inward normal of the domain $B_R \setminus B_\varepsilon$. Then for any $t > t_1 > s, \varepsilon \leq |y| \leq R$,

$$
\int_{B_R \setminus B_\varepsilon} (g_{R,\varepsilon,\delta}(x,t,y,s) - \Gamma(x,t,y,s)) h(x) \, dx \\
\quad - \int_{B_R \setminus B_\varepsilon} (g_{R,\varepsilon,\delta}(x,t_1,y,s) - \Gamma(x,t_1,y,s)) \phi(x,t_1) \, dx \\
= \int_{t_1}^{t} \frac{\partial}{\partial \tau} \left( \int_{B_R \setminus B_\varepsilon} (g_{R,\varepsilon,\delta}(x,\tau,y,s) - \Gamma(x,\tau,y,s)) \phi(x) \, dx \right) \, d\tau \\
= \int_{t_1}^{t} \int_{B_R \setminus B_\varepsilon} \phi \frac{\partial}{\partial \tau} (g_{R,\varepsilon,\delta} - \Gamma) \, dx \, d\tau + \int_{t_1}^{t} \int_{B_R \setminus B_\varepsilon} (g_{R,\varepsilon,\delta} - \Gamma) \phi_{\tau} \, dx \, d\tau \\
= \int_{t_1}^{t} \int_{B_R \setminus B_\varepsilon} \phi \Delta (g_{R,\varepsilon,\delta} - \Gamma) \, dx \, d\tau + \int_{t_1}^{t} \int_{B_R \setminus B_\varepsilon} (g_{R,\varepsilon,\delta} - \Gamma) \phi_{\tau} \, dx \, d\tau \\
= \int_{t_1}^{t} \int_{B_R \setminus B_\varepsilon} (g_{R,\varepsilon,\delta} - \Gamma)(\phi_{\tau} + \Delta \phi) \, dx \, d\tau + \int_{t_1}^{t} \int_{\partial(B_R \setminus B_\varepsilon)} (g_{R,\varepsilon,\delta} - \Gamma) \frac{\partial \phi}{\partial n} \, d\sigma \, d\tau.
$$

Hence

$$
\int_{B_R \setminus B_\varepsilon} (g_{R,\varepsilon,\delta}(x,t,y,s) - \Gamma(x,t,y,s)) h(x) \, dx \\
\leq \int_{B_R \setminus B_\varepsilon} (g_{R,\varepsilon,\delta}(x,t_1,y,s) - \Gamma(x,t_1,y,s)) \phi(x,t_1) \, dx \quad \forall \, t > t_1 > s \\
\Rightarrow \int_{B_R \setminus B_\varepsilon} (g_{R,\varepsilon,\delta}(x,t,y,s) - \Gamma(x,t,y,s)) h(x) \, dx \leq 0 \quad \forall \, t > s, \varepsilon \leq |y| \leq R \quad \text{as} \; t_1 \searrow s. \quad (2.5)
$$

We now choose a sequence of functions $\{h_i\}_{i=1}^{\infty} \subset C_0^\infty(B_R \setminus B_\varepsilon), \; 0 \leq h_i \leq 1$ for any $i \in \mathbb{Z}^+$, such that $h_i(x)$ converges to $\chi_A(x)$ a.e. as $i \to \infty$ where $A = \{x \in B_R \setminus B_\varepsilon : g_{R,\varepsilon,\delta}(x,t,y,s) > \Gamma(x,t,y,s)\}$. Putting $h = h_i$ in (2.5) and letting $i \to \infty$,

$$
\int_{B_R \setminus B_\varepsilon} (g_{R,\varepsilon,\delta}(x,t,y,s) - \Gamma(x,t,y,s))_+ \, dx \leq 0 \quad \forall \, t > s, \varepsilon \leq |y| \leq R
$$

and (2.3) follows. By (2.1), (2.3), and the Schauder estimates [LSU] for any $t_2 > t_1 > s$, $\varepsilon \leq |y| \leq R$, the sequence $\{g_{R,\varepsilon,\delta}(\cdot,\cdot,y,s)\}_{0<\delta \leq 1}$ is equi-Holder continuous on $C^{2,1}((\overline{B_R \setminus B_\varepsilon}) \times [t_1,t_2])$. Hence by the Ascoli theorem and a diagonalization argument there exists a sequence $\{\delta_i\}_{i=1}^{\infty}, \delta_i \to 0$ as $i \to \infty$, such that $g_{R,\varepsilon,\delta_i}(\cdot,\cdot,y,s)$ converges uniformly in $C^{2,1}((\overline{B_R \setminus B_\varepsilon}) \times [t_1,t_2])$ to the solution $g_{R,\varepsilon}(\cdot,\cdot,y,s)$ of (2.2). Putting $\delta = \delta_i$ in (2.1), (2.3), and letting $i \to \infty$ we get (2.4) and the lemma follows.
For any $0 < \delta \leq 1$, $s \in \mathbb{R}$, let $g^\delta_R$ be the solution of

\[
\begin{align*}
\partial_t g^\delta_R &= \Delta_x g^\delta_R & \forall x \in B_R, t > s \\
g^\delta_R(x, t, y, s) &= \Gamma(x, t, y, s)\eta((t-s)/\delta) & \forall x \in \partial B_R, t > s \\
g^\delta_R(x, s, y, s) &= 0 & \forall x \in B_R.
\end{align*}
\]

By an argument similar to the proof of (2.1) and Lemma 2.1 we have

**Lemma 2.2.** Let $s \in \mathbb{R}$, $0 < |y| \leq R$, and let $\{\delta_i\}_{i=1}^\infty$ be the sequence given by Lemma 2.1. Then there exists a subsequence $\{\delta_{i'}\}_{i'=1}^\infty$ of $\{\delta_i\}_{i=1}^\infty$ such that $g^\delta_R(\cdot, \cdot, y, s)$ converges uniformly in $C^{2,1}(\overline{B_R \times [t_1, t_2]})$ to the solution $g_R(\cdot, \cdot, y, s)$ of (0.12) as $i \to \infty$ for any $t_2 > t_1 > s$. Moreover

\[
0 \leq g^\delta_R(x, t, y, s) \leq g^\delta_{R}(x, t, y, s) \leq \Gamma(x, t, y, s) \quad \forall 0 < |x|, |y| \leq R, 0 \leq \delta \leq \delta' \leq 1, t > s.
\]

**Lemma 2.3.** Let $m \geq 3$, $s \in \mathbb{R}$ and $0 < |y| \leq R$. Then there exists a sequence $\{\varepsilon_i\}_{i=1}^\infty$, $0 < \varepsilon_i < R/3 \quad \forall i \in \mathbb{Z}^+$ and $\varepsilon_i \to 0$ as $i \to \infty$, such that the sequence $\{g_R(\cdot, \cdot, y, s)\}$ converges uniformly to $g_R(\cdot, \cdot, y, s)$ in $C^{2,1}(K)$ for any compact set $K \subset (\overline{B_R \setminus \{0\}} \times (s, \infty)$ as $i \to \infty$.

**Proof.** Let $\eta$ be as before and let $\tilde{\eta}_\delta(x) = \eta(|x|/\delta)$ for any $0 < \delta \leq 2R/3$, $x \in \mathbb{R}^m$. Then $|\nabla \tilde{\eta}_\delta(x)| \leq C/\delta$ and $|\Delta \tilde{\eta}_\delta(x)| \leq C/\delta^2$ on $B_R$ for some constant $C > 0$ independent of $\delta$ and $\tilde{\eta}_\delta(x) \equiv 0$ for any $|x| \leq \delta/2$. Let $h \in C_0^\infty(B_R)$ be such that $0 \leq h \leq 1$. For any $t > s$, let $\phi(x, \tau)$ be the solution of

\[
\begin{align*}
\begin{cases}
\partial_{\tau} \phi + \Delta \phi = 0 & \text{in } B_R \times (s, t) \\
\phi(x, \tau) = h(x) & \text{in } B_R.
\end{cases}
\]

(2.7)
Now by (2.1), (2.3), and (2.6), for any $0 < \varepsilon \leq R/3$, $0 < \delta \leq 1$, $t > s$,
\[
\int_{B_R} (g_R(x, t, y, s) - g_{R, \varepsilon, \delta}(x, t, y, s) h(x) \tilde{\eta}_{2\varepsilon}(x) \, dx
\]
\[
= \int_{s}^{t} \frac{\partial}{\partial \tau} \left( \int_{B_R \setminus B_{\varepsilon}} (g_R(x, \tau, y, s) - g_{R, \varepsilon, \delta}(x, \tau, y, s)) \phi \tilde{\eta}_{2\varepsilon} \, dx \right) \, d\tau
\]
\[
= \int_{s}^{t} \int_{B_R \setminus B_{\varepsilon}} \phi \tilde{\eta}_{2\varepsilon} \Delta (g_R^\delta - g_{R, \varepsilon, \delta}) \, dx \, d\tau + \int_{s}^{t} \int_{B_R \setminus B_{\varepsilon}} (g_R^\delta - g_{R, \varepsilon, \delta}) \phi \tau \tilde{\eta}_{2\varepsilon} \, dx \, d\tau
\]
\[
= \int_{s}^{t} \int_{B_R \setminus B_{\varepsilon}} (g_R^\delta - g_{R, \varepsilon, \delta}) (\phi \tau + \Delta \phi) \tilde{\eta}_{2\varepsilon} \, dx \, d\tau
\]
\[
+ \int_{s}^{t} \int_{|x| \leq 2\varepsilon} (g_R^\delta - g_{R, \varepsilon, \delta})(2\nabla \phi \cdot \nabla \tilde{\eta}_{2\varepsilon} + \phi \Delta \tilde{\eta}_{2\varepsilon}) \, dx \, d\tau
\]
\[
\leq \frac{C}{\varepsilon^2} \int_{s}^{t} \int_{|x| \leq 2\varepsilon} \Gamma(x, \tau, y, s) \, dx \, d\tau + \frac{C}{\varepsilon} \int_{s}^{t} \int_{|x| \leq 2\varepsilon} \Gamma(x, \tau, y, s) |\nabla \phi| \, dx \, d\tau
\]
\[
= I_1 + I_2 \quad (2.8)
\]
for some constant $C > 0$ independent of $0 < \varepsilon \leq R/3$ and $0 < \delta \leq 1$. Let $0 < \varepsilon \leq \min(|y|/10, R/3)$. Then for any $\varepsilon \leq |x| \leq 2\varepsilon$,
\[
|x| \leq \frac{|y|}{5} \quad \Rightarrow \quad |x - y| \geq |y| - |x| \geq \frac{4}{5} |y|.
\]
Hence
\[
\Gamma(x, \tau, y, s) \leq \frac{1}{(4\pi (\tau - s))^m} e^{-\frac{4|y|^2}{2m(\tau - s)}} \left( \frac{|y|}{(4\pi (\tau - s))^{\frac{1}{2}}} \right)^m e^{-\frac{2|y|^2}{2m(\tau - s)}} \cdot e^{-\frac{2|y|^2}{2m(\tau - s)}} |y|^m
\]
\[
\leq \frac{C}{|y|^m} e^{-\frac{2|y|^2}{2m(\tau - s)}} \quad \forall \varepsilon \leq |x| \leq 2\varepsilon, \tau > s. \quad (2.9)
\]
Thus
\[
I_1 \leq C \frac{\varepsilon^{m-2}}{|y|^m} e^{-\frac{2|y|^2}{2m(\tau - s)}} (t - s). \quad (2.10)
\]
Now by (2.7),
\[
\frac{\partial}{\partial \tau} \int_{B_R} |\nabla \phi|^2 \, dx = 2 \int_{B_R} \nabla \phi \cdot \nabla \phi \, dx = -2 \int_{B_R} \Delta \phi \cdot \phi \, dx = 2 \int_{B_R} (\Delta \phi)^2 \, dx
\]
\[
\Rightarrow \quad \sup_{s \leq \tau \leq t} \int_{B_R} |\nabla \phi|^2 \, dx \leq \int_{B_R} |\nabla h|^2 \, dx. \quad (2.11)
\]
By (2.9) and (2.11),
\[
I_2 \leq C \varepsilon \cdot e^{-\frac{2|y|^2}{25(t-s)}} \int_s^t \int_{B_R} |\nabla \phi| \, dx \, d\tau
\leq C \varepsilon \frac{m-1}{m} \cdot e^{-\frac{2|y|^2}{25(t-s)}} \int_s^t \left( \int_{B_R} |\nabla h|^2 \, dx \right)^{\frac{1}{2}} \, d\tau
\leq C \varepsilon \frac{m-1}{m} \cdot e^{-\frac{2|y|^2}{25(t-s)}} \left( \int_{B_R} |\nabla h|^2 \, dx \right)^{\frac{1}{2}} (t-s).
\] (2.12)

By (2.8), (2.10), and (2.12),
\[
\int_{B_R \setminus B_\varepsilon} (g_R^i(x, t, y, s) - g_{R, \varepsilon}(x, t, y, s)h(x)\eta_{2\varepsilon}(x)) \, dx
\leq C \varepsilon^{m-2} \frac{m-1}{m} e^{-\frac{2|y|^2}{25(t-s)}} (t-s) + C \varepsilon \frac{m-1}{m} \left( \int_{B_R} |\nabla h|^2 \, dx \right)^{\frac{1}{2}} (t-s) \quad \forall t > s.
\] (2.13)

Let \( \{\delta'_i\}_{i=1}^\infty \) be the sequence given by Lemma 2.2. Putting \( \delta = \delta'_i \) in (2.13) and letting \( i \to \infty \) by Lemma 2.1, Lemma 2.2, and the Lebesgue dominated convergence theorem,
\[
\int_{B_R \setminus B_\varepsilon} (g_R(x, t, y, s) - g_{R, \varepsilon}(x, t, y, s)h(x)\eta_{2\varepsilon}(x)) \, dx
\leq C \varepsilon^{m-2} \frac{m-1}{m} e^{-\frac{2|y|^2}{25(t-s)}} (t-s) + C \varepsilon \frac{m-1}{m} \left( \int_{B_R} |\nabla h|^2 \, dx \right)^{\frac{1}{2}} (t-s) \quad \forall t > s.
\] (2.14)

By (2.4) and the Schauder estimates [LSU] the sequence \( \{g_{R, \varepsilon}(x, t, y, s)\}_{0<\varepsilon \leq R/3} \) is equi-Hölder continuous in \( C^{2,1}(K) \) for any compact subset \( K \subset (\overline{B_R \setminus \{0\}}) \times [t_1, t_2] \) for any \( t_2 > t_1 > s \). By the Ascoli theorem and a diagonalization argument there exists a sequence \( \{\varepsilon_i\}_{i=1}^\infty \), \( 0 < \varepsilon_i < R/3 \) \( \forall i \in \mathbb{Z}^+ \) and \( \varepsilon_i \to 0 \) as \( i \to \infty \), such that the sequence \( \{g_{R, \varepsilon_i}(\cdot, \cdot, y, s)\} \) converges uniformly to some function \( \tilde{g}_R(\cdot, \cdot, y, s) \) on every compact subset of \( (\overline{B_R \setminus \{0\}}) \times (s, \infty) \) as \( i \to \infty \).

Putting \( \varepsilon = \varepsilon_i \) in (2.14) and letting \( i \to \infty \),
\[
\int_{B_R \setminus \{0\}} (g_R(x, t, y, s) - \tilde{g}_R(x, t, y, s))h(x) \, dx \leq 0 \quad \forall t > s, 0 < |y| \leq R.
\] (2.15)

We now choose a sequence of functions \( \{h_i\}_{i=1}^\infty \subset C_0^\infty(B_R), 0 \leq h_i \leq 1 \) for any \( i \in \mathbb{Z}^+ \), such that \( h_i \) converges to \( \chi_A \) a.e. as \( i \to \infty \) where \( A = \{x \in B_R : g_R(x, t, y, s) > \tilde{g}_R(x, t, y, s)\} \). Putting \( h = h_i \) in (2.15) and letting \( i \to \infty \),
\[
\int_{B_R \setminus \{0\}} (g_R(x, t, y, s) - \tilde{g}_R(x, t, y, s))_+ \, dx \leq 0 \quad \forall t > s, 0 < |y| \leq R
\Rightarrow g_R(x, t, y, s) \leq \tilde{g}_R(x, t, y, s) \quad \forall 0 < |x|, |y| \leq R, t > s.
\] (2.16)
Interchanging the role of \( g_R(x, t, y, s) \) and \( \tilde{g}_R(x, t, y, s) \) and repeating the above argument we get

\[
g_R(x, t, y, s) \geq \tilde{g}_R(x, t, y, s) \quad \forall 0 < |x|, |y| \leq R, t > s. \tag{2.17}
\]

By (2.16) and (2.17),

\[
\tilde{g}_R(x, t, y, s) = g_R(x, t, y, s) \quad \forall 0 < |x|, |y| \leq R, t > s \tag{2.18}
\]

and the lemma follows.

For any \( 0 < \varepsilon < R \), let \( G_{R,\varepsilon} = G_{R,\varepsilon}(x, t, y, s) \) be the Green function for the heat equation in \( (B_R \setminus B_{\varepsilon}) \times (-\infty, \infty) \). Then

\[
G_{R,\varepsilon}(x, t, y, s) = \Gamma(x, t, y, s) - g_{R,\varepsilon}(x, t, y, s) \quad \forall \varepsilon \leq |x|, |y| \leq R, t > s. \tag{2.19}
\]

Then by Lemma 2.3 and the uniqueness of the Green function for the heat equation we have the following corollary.

**Corollary 2.4.** Let \( m \geq 3 \), \( s \in \mathbb{R} \) and \( 0 < |y| \leq R \). Then \( G_{R,\varepsilon}^{*}(\cdot, \cdot, y, s) \) will converge uniformly to \( G_R^{*}(\cdot, \cdot, y, s) \) in \( C^{2,1}(K) \) for any compact set \( K \subset (\overline{B}_R \setminus \{0\}) \times (s, \infty) \) as \( \varepsilon \to 0 \).

We now let \( G_{R,\varepsilon}^{*} = G_{R,\varepsilon}^{*}(y, s, x, t) \) and \( G_R^{*} = G_R^{*}(y, s, x, t), s < t, \) be the Green function for the adjoint heat equation \( \partial_s u + \Delta u = 0 \) in \( (B_R \setminus B_{\varepsilon}) \times (-\infty, \infty) \) and \( B_R \times (-\infty, \infty) \) respectively. Then by a similar argument as the proof of Lemma 2.3 and Corollary 2.4 we have the following result.

**Corollary 2.5.** Let \( m \geq 3 \). Then for any \( t \in \mathbb{R} \), \( 0 < |x| \leq R \), \( G_{R,\varepsilon}^{*}(\cdot, \cdot, x, t) \) will converge uniformly to \( G_R^{*}(\cdot, \cdot, x, t) \) in \( C^{2,1}(K) \) for any compact set \( K \subset (\overline{B}_R \setminus \{0\}) \times (-\infty, t) \) as \( \varepsilon \to 0 \).

Now by [F],

\[
G_{R,\varepsilon}(x, t, y, s) = G_{R,\varepsilon}^{*}(y, s, x, t) \quad \forall \varepsilon \leq |x|, |y| \leq R, t > s \tag{2.20}
\]

and

\[
G_R(x, t, y, s) = G_R^{*}(y, s, x, t) \quad \forall |x|, |y| \leq R, t > s. \tag{2.21}
\]

Hence by (2.20), (2.21), and Corollary 2.5 we have the following result.

**Corollary 2.6.** Let \( m \geq 3 \). Then for any \( t \in \mathbb{R} \), \( 0 < |x| \leq R \), \( G_{R,\varepsilon}(x, t, \cdot, \cdot) \) will converge uniformly to \( G_R(x, t, \cdot, \cdot) \) in \( C^{2,1}(K) \) for any compact set \( K \subset (\overline{B}_R \setminus \{0\}) \times (-\infty, t) \) as \( \varepsilon \to 0 \).

Let \( G_{\infty,1}(x, t, y, s) \) be the Green function for the heat equation in the domain \((\mathbb{R}^{m-1} \setminus B_1) \times (-\infty, \infty)\). By an argument similar to the proof of (ii) of Lemma 1.3 of [Hu] we have
Corollary 2.8. For any \( \delta \) holds where \( \partial/\partial n \) holds where \( \delta_1(y) = \text{dist}(y, \partial B_1) \) and \( \delta_R(y) = \text{dist}(y, \partial B_R) \) respectively.

**Lemma 2.7.** For any \( T > 0 \), there exist constants \( C > 0 \) and \( c > 0 \) such that

\[
0 \leq G_{\infty,1}(x,t,y,s) \leq C \frac{\delta_1(y)}{(t-s)^{m+1/2}} e^{-c|x-y|^2/(t-s)} \quad \forall |x|, |y| > 1, 0 \leq s < t \leq T
\]

and

\[
0 \leq G_R(x,t,y,s) \leq C \frac{\delta_R(y)}{(t-s)^{m+1/2}} e^{-c|x-y|^2/(t-s)} \quad \forall |x|, |y| < R, 0 \leq s < t \leq T, R > 0
\]

holds where \( \delta_1(y) = \text{dist}(y, \partial B_1) \) and \( \delta_R(y) = \text{dist}(y, \partial B_R) \) respectively.

**Corollary 2.8.** For any \( T > 0 \) there exist constants \( C > 0 \) and \( c > 0 \) such that

\[
0 \leq \frac{\partial G_{\infty,1}}{\partial n_y}(x,t,y,s) \leq C \frac{1}{(t-s)^{m+1/2}} e^{-c|x-y|^2/(t-s)} \quad \forall |x| > 1, |y| = 1, 0 \leq s < t \leq T
\]

holds where \( \partial/\partial n_y \) is the derivative in the direction of the inward normal \( n_y = y/|y| \) at the point \( y \in \partial B_1(0) \) with respect to the domain \( \mathbb{R}^m \setminus \overline{B}_1 \).

**Corollary 2.9.** For any \( T > 0 \) there exist constants \( C > 0 \) and \( c > 0 \) such that

\[
0 \leq \frac{\partial G_R}{\partial n_y}(x,t,y,s) \leq C \frac{1}{(t-s)^{m+1/2}} e^{-c|x-y|^2/(t-s)} \quad \forall |x| < R, |y| = R, 0 \leq s < t \leq T, R > 0
\]

holds where \( \partial/\partial n_y \) is the derivative in the direction of the inward normal \( n_y = y/|y| \) at the point \( y \in \partial B_R \) with respect to the domain \( B_R \).

**Corollary 2.10.** For any \( T > 0 \) there exist constants \( C > 0 \) and \( c > 0 \) such that

\[
0 \leq \frac{\partial G_{R,\varepsilon}}{\partial n_y}(x,t,y,s) \leq C \frac{1}{(t-s)^{m+1/2}} e^{-c|x-y|^2/(t-s)} \quad \forall |y| = \varepsilon, \varepsilon < |x| < R, 0 \leq s < t \leq T
\]

holds where \( \partial/\partial n_y \) is the derivative in the direction of the inward normal \( n_y = y/|y| \) at the point \( y \in \partial B_\varepsilon \) with respect to the domain \( B_R \setminus \overline{B}_\varepsilon \).

**Proof.** By scaling,

\[
G_{R,\varepsilon}(x,t,y,s) = \varepsilon^{-m} G_{R/\varepsilon,1}(x/\varepsilon, t/\varepsilon^2, y/\varepsilon, s/\varepsilon^2).
\]

Hence

\[
\frac{\partial G_{R,\varepsilon}}{\partial n_y}(x,t,y,s) = \varepsilon^{-m-1} \frac{\partial G_{R/\varepsilon,1}}{\partial n_{y'}}(x/\varepsilon, t/\varepsilon^2, y/\varepsilon, s/\varepsilon^2), y' = y/\varepsilon.
\]

(2.23)

By the maximum principle (cf. [F]), \( \forall R_2 > R_1 > 1, \)

\[
0 \leq G_{R_1,1}(x,t,y,s) \leq G_{R_2,1}(x,t,y,s) \leq G_{\infty,1}(x,t,y,s) \quad \forall 1 < |x|, |y| \leq R_1, t > s
\]

\[
\Rightarrow 0 \leq \frac{\partial G_{R_1,1}}{\partial n_y}(x,t,y,s) \leq \frac{\partial G_{\infty,1}}{\partial n_y}(x,t,y,s) \quad \forall 1 < |x| < R_1, |y| = 1, t > s, R_1 > 1.
\]

(2.24)

By (2.23), (2.24), and Corollary 2.8 we get (2.22) and the lemma follows.

By an argument similar to the proof of Corollary 2.10 but with \( G_R \) and Corollary 2.9 replacing \( G_{\infty,1} \) and Corollary 2.8 in the proof we have
Corollary 2.11. For any $T > 0$ there exist constants $C > 0$ and $c > 0$ such that

$$0 \leq \frac{\partial G_{R,\varepsilon}(x, t, y, s)}{\partial n_y} \leq \frac{C}{(t-s)^{n+1}} e^{-c|x-y|^2/(t-s)} \quad \forall |y| = R, \varepsilon < |x| < R, 0 \leq s < t \leq T$$

holds where $\partial/\partial n_y$ is the derivative in the direction of the inward normal $n_y = -y/|y|$ at the point $y \in \partial B_R$ with respect to the domain $B_R \setminus \overline{B}_\varepsilon$.

Section 3

In this section we will use the Green function estimates obtained in section two to prove that under an uniform boundedness condition on a solution of (0.11) in $(\mathbb{R}^{n+2} \setminus \{0\}) \times (t_0, T_0)$ the solution has removable singularities on the line $\{0\} \times (t_0, T_0)$. We will also prove the uniqueness of solution of (0.8).

Lemma 3.1. Let $T' \in (t_0, T]$ and let $\overline{\rho}$ be a solution of (0.11) in $(\mathbb{R}^{n+2} \setminus \{0\}) \times (t_0, T')$ which satisfies (1.1) and

$$\sup_{t_0 \leq t < T'} (\|\overline{\rho}(\cdot, t)\|_{L^\infty(\mathbb{R}^{n+2} \setminus \{0\})} + \|\nabla \overline{\rho}(\cdot, t)\|_{L^\infty(\mathbb{R}^{n+2} \setminus \{0\})}) < \infty. \quad (3.1)$$

Then $\overline{\rho}$ can be extended to a solution of (0.11) in $\mathbb{R}^{n+2} \times (t_0, T')$ and $\overline{\rho}$ satisfies (1.2) and (1.3) in $\mathbb{R}^{n+2} \times (t_0, T')$.

Proof. Let $|x| > 0$. By standard parabolic theory (cf. [A] and [LSU]), for any $t_0 < t < T'$ and $R > \varepsilon > 0$ such that $\varepsilon \leq |x| \leq R$,

$$\overline{\rho}(x,t) = \int_{t_0}^{t} \int_{B_R \setminus B_\varepsilon} G_{R,\varepsilon}(x, t, y, s) F(y, \overline{\rho}, \nabla \overline{\rho}, s) \, dy \, ds + \int_{t_0}^{t} \int_{\partial B_\varepsilon} \frac{\partial G_{R,\varepsilon}(x, t, y, s)}{\partial n_y} \overline{\rho}(y, s) \, d\sigma(y) \, ds$$

$$+ \int_{t_0}^{t} \int_{\partial B_R \setminus \partial B_\varepsilon} \frac{\partial G_{R,\varepsilon}(x, t, y, s)}{\partial n_y} \overline{\rho}(y, s) \, d\sigma(y) \, ds$$

$$= I_{1,R}^\varepsilon + I_{2,R}^\varepsilon + I_{3,R}^\varepsilon \quad (3.2)$$

where $\partial/\partial n_y$ is the derivative in the direction of the inward normal of the domain $B_R \setminus B_\varepsilon$ at $y \in \partial B_R \cup \partial B_\varepsilon$. Since $\overline{\rho}$ satisfies (3.1), by an argument similar to the proof on P.10 of [LT] there exists a constant $C > 0$ such that

$$|F(x, \overline{\rho}, \nabla \overline{\rho}, t)| \leq C \quad \forall x \in \mathbb{R}^{n+2} \setminus \{0\}, t_0 \leq t < T'. \quad (3.3)$$

By (2.4), (2.19), (3.3), Corollary 2.6, and Lebesgue dominated convergence theorem,

$$\lim_{\varepsilon \to 0} I_{1,R}^\varepsilon = \int_{t_0}^{t} \int_{B_R \setminus \{0\}} G_R(x, t, y, s) F(y, \overline{\rho}, \nabla \overline{\rho}, s) \, dy \, ds. \quad (3.4)$$
By Corollary 2.10 (2.22) holds with \( m = n + 2 \). Hence

\[
|I_{2,R}^ε| \leq C \int_{t_0}^t \int_{\partial B_R} \frac{1}{(t-s)^{n+3}} e^{-c|x-y|^2/(t-s)} \, dσ(y) \, ds \leq \frac{Cε^{n+1}}{(|x| - ε)^{n+3}}
\]

⇒ \[ \lim_{ε \to 0} I_{2,R}^ε = 0. \] (3.5)

Now

\[
I_{3,R}^ε = \int_{t_0}^{t-δ} \int_{\partial B_R} \frac{∂G_Rε(x, t, y, s)}{∂n_y} \, dσ(y) \, ds + \int_{t_0}^t \int_{\partial B_R} \frac{∂G_Rε(x, t, y, s)}{∂n_y} \, dσ(y) \, ds
\]

\[= J_{1,δ}^ε + J_{2,δ}^ε \quad \forall 0 < δ < t - t_0. \] (3.6)

By (3.1) and Corollary 2.6,

\[
\lim_{ε \to 0} J_{1,δ}^ε = \int_{t_0}^{t-δ} \int_{\partial B_R} \frac{∂G_R(x, t, y, s)}{∂n_y} \, dσ(y) \, ds \quad \forall 0 < δ < t - t_0. \] (3.7)

By Corollary 2.11 (2.25) holds with \( m = n + 2 \). Then

\[
|J_{2,δ}^ε| \leq C \int_{t_0}^t \int_{\partial B_R} \frac{1}{(t-s)^{n+3}} e^{-c|x-y|^2/(t-s)} \, dσ(y) \, ds \leq \frac{Cδ}{(R-|x|)^{n+3}}. \] (3.8)

Similarly by Corollary 2.9,

\[
\left| \int_{t_0}^t \int_{\partial B_R} \frac{∂G_R(x, t, y, s)}{∂n_y} \, dσ(y) \, ds \right| \leq \frac{Cδ}{(R-|x|)^{n+3}}. \] (3.9)

Now

\[
|I_{3,R}^ε - \int_{t_0}^t \int_{\partial B_R} \frac{∂G_R(x, t, y, s)}{∂n_y} \, dσ(y) \, ds| \leq |J_{1,δ}^ε - \int_{t_0}^{t-δ} \int_{\partial B_R} \frac{∂G_R(x, t, y, s)}{∂n_y} \, dσ(y) \, ds| + |J_{2,δ}^ε|
\]

\[+ \int_{t_0}^t \int_{\partial B_R} \frac{∂G_R(x, t, y, s)}{∂n_y} \, dσ(y) \, ds|. \] (3.10)

Letting \( ε \to 0 \) in (3.10), by (3.7), (3.8), and (3.9),

\[
\lim_{ε \to 0} \sup |I_{3,R}^ε - \int_{t_0}^t \int_{\partial B_R} \frac{∂G_R(x, t, y, s)}{∂n_y} \, dσ(y) \, ds| \leq \frac{Cδ}{(R-|x|)^{n+3}} \quad \forall 0 < δ < t - t_0
\]

⇒ \[ \lim_{ε \to 0} |I_{3,R}^ε - \int_{t_0}^t \int_{\partial B_R} \frac{∂G_R(x, t, y, s)}{∂n_y} \, dσ(y) \, ds| = 0 \quad \text{as} \ δ \to 0. \] (3.11)
By Corollary 2.9,
\[
\bar{p}(x, t) = \int_{t_0}^{t} \int_{B_R \setminus \{0\}} G_R(x, t, y, s) F(y, \bar{p}, \nabla \bar{p}, s) \, dy \, ds + \int_{t_0}^{t} \int_{\partial B_R} \frac{\partial G_R}{\partial n_y}(x, t, y, s) \bar{p}(y, s) \, d\sigma(y) \, ds.
\] (3.12)

Since \( G_R(x, t, y, s) \) increases to \( \Gamma(x; t; y; s) \) as \( R \to \infty \), by (0.13), (3.3), and the Lebesgue dominated convergence theorem,
\[
\lim_{R \to \infty} \int_{t_0}^{t} \int_{B_R \setminus \{0\}} G_R(x, t, y, s) F(y, \bar{p}, \nabla \bar{p}, s) \, dy \, ds = \int_{t_0}^{t} \int_{\mathbb{R}^{n+2} \setminus \{0\}} \Gamma(x, t, y, s) F(y, \bar{p}, \nabla \bar{p}, s) \, dy \, ds.
\] (3.13)

By Corollary 2.9,
\[
\left| \int_{t_0}^{t} \int_{\partial B_R} \frac{\partial G_R}{\partial n_y}(x, t, y, s) \bar{p}(y, s) \, d\sigma(y) \, ds \right| \leq C \int_{t_0}^{t} \int_{\partial B_R} \frac{1}{(t-s)^{n+3}} e^{-\frac{|x-y|^2}{4(t-s)}} \, d\sigma(y) \, ds \leq \frac{CR^{n+1}}{(R - |x|)^{n+3}}.
\] (3.14)

Letting \( R \to \infty \) in (3.12), by (3.13) and (3.14) we get that \( \bar{p} \) satisfies
\[
\bar{p}(x, t) = \int_{t_0}^{t} \int_{\mathbb{R}^{n+2} \setminus \{0\}} \frac{1}{(4\pi(t-s))^{\frac{n+2}{2}}} e^{-\frac{|x-y|^2}{4(t-s)}} F(y, \bar{p}, \nabla \bar{p}, s) \, dy \, ds
\] (3.15)

\( \forall (x, t) \in (\mathbb{R}^{n+2} \setminus \{0\}) \times [t_0, T') \). Since the right hand side of (3.15) is a continuous function on \( \mathbb{R}^{n+2} \times (t_0, T') \), we can extend \( \bar{p}(x, t) \) to a function on \( \mathbb{R}^{n+2} \times (t_0, T') \) by letting \( \bar{p}(x, t) \) equal to the right hand side of (3.15) for \( x = 0, t_0 < t < T' \). Then the extended function \( \bar{p}(x, t) \) is a continuous function on \( \mathbb{R}^{n+2} \times (t_0, T') \) and has continuous first derivatives in \( x \) on \( \mathbb{R}^{n+2} \times (t_0, T') \). Hence \( \bar{p}(x, t) \) satisfies (1.2) on \( \mathbb{R}^{n+2} \times (t_0, T') \). By the same argument as that on P.11–12 of [LT] \( \bar{p} \) is a classical solution of (0.11) in \( \mathbb{R}^{n+2} \times (t_0, T') \). By (3.1) \( \bar{p} \) satisfies (1.3) and the lemma follows.

**Theorem 3.2.** Let \( T' \in (t_0, T] \). Suppose \( \bar{p}_1 \) and \( \bar{p}_2 \) are solutions of (0.11) and (1.1) in \( (\mathbb{R}^{n+2} \setminus \{0\}) \times (t_0, T') \) which satisfies (3.1). Then \( \bar{p}_1 \equiv \bar{p}_2 \) in \( (\mathbb{R}^{n+2} \setminus \{0\}) \times (t_0, T') \).

**Proof.** Let \( T_1 \geq t_0 \) be the maximal time such that \( \bar{p}_1 \equiv \bar{p}_2 \) in \( (\mathbb{R}^{n+2} \setminus \{0\}) \times (t_0, T_1) \). Suppose \( T_1 < T' \). By Lemma 3.1 both \( \bar{p}_1 \) and \( \bar{p}_2 \) can be extended to solutions of (0.11) in \( \mathbb{R}^{n+2} \times (t_0, T') \) and both satisfy (1.2) and (1.3) in \( \mathbb{R}^{n+2} \times (t_0, T') \). Then by (1.2)
\[
\bar{p}_1(x, t) - \bar{p}_2(x, t) = \int_{T_1}^{t} \int_{\mathbb{R}^{n+2}} \frac{1}{(4\pi(t-s))^{\frac{n+2}{2}}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[ F(y, \bar{p}_1, \nabla \bar{p}_1, s) - F(y, \bar{p}_2, \nabla \bar{p}_2, s) \right] \, dy \, ds
\] (3.16)
By (3.16), (3.17) and (3.18), there exist constants $C > 0$ such that

$$|\nabla \overline{p}_1(x, t) - \nabla \overline{p}_2(x, t)|$$

$$\leq \int_{T_1}^{t} \int_{\mathbb{R}^{n+2}} \frac{|x - y|}{2(t - s) (4\pi(t - s))^{n+2}} e^{-\frac{|x-y|^2}{4(t-s)}} |F(y, \overline{p}_1, \nabla \overline{p}_1, s) - F(y, \overline{p}_2, \nabla \overline{p}_2, s)| \, dy \, ds$$

$$\leq \int_{T_1}^{t} \int_{\mathbb{R}^{n+2}} \frac{C}{(t - s)^{n+2}} e^{-\frac{|x-y|^2}{4(t-s)}} |F(y, \overline{p}_1, \nabla \overline{p}_1, s) - F(y, \overline{p}_2, \nabla \overline{p}_2, s)| \, dy \, ds$$

(3.17)

By (1.3) and the argument on P.10–11 of [LT] there exists a constant $C > 0$ such that

$$|F(y, \overline{p}_1, \nabla \overline{p}_1, s) - F(y, \overline{p}_2, \nabla \overline{p}_2, s)| \leq C(|\overline{p}_1 - \overline{p}_2| + |\nabla \overline{p}_1 - \nabla \overline{p}_2|) \quad \forall y \in \mathbb{R}^{n+2}, s \in [t_0, T']$$

(3.18)

Let

$$E(t) = \sup_{T_1 \leq s < t} (||\overline{p}_1(\cdot, s) - \overline{p}_2(\cdot, s)||_{L^\infty(\mathbb{R}^{n+2})} + ||\nabla \overline{p}_1(\cdot, s) - \nabla \overline{p}_2(\cdot, s)||_{L^\infty(\mathbb{R}^{n+2})}).$$

By (3.16), (3.17) and (3.18), there exist constants $C_5 > 0$ and $C_6 > 0$ such that

$$E(t) \leq C_5(t - T_1)E(t) + C_6 \sqrt{t - T_1}E(t) \quad \forall T_1 \leq t < T'.$$

(3.19)

Let

$$\delta = \min(1, 1/(4(C_5 + C_6)^2)).$$

Then by (3.19) for any $T_1 \leq t \leq T_1 + \delta$,

$$E(t) \leq \frac{E(t)}{2} \quad \Rightarrow \quad E(t) = 0 \quad \forall T_1 \leq t \leq T_1 + \delta$$

$$\Rightarrow \quad \overline{p}_1(x, t) = \overline{p}_2(x, t) \quad \forall x \in \mathbb{R}^{n+2}, T_1 \leq t \leq T_1 + \delta.$$ 

This contradicts the choice of $T_1$. Hence $T_1 = T'$ and the theorem follows.

By (0.9) and Theorem 3.2 we have the following uniqueness result.

**Theorem 3.3.** Let $0 < t_0 < T_0 < T$. Suppose $\overline{p}_1, \overline{p}_2$, are solutions of (0.8) in $\mathbb{R}^+ \times (t_0, T_0)$ satisfying

$$\tilde{p}_i(r, t_0) = 0 \quad \forall r \geq 0, i = 1, 2$$

and

$$\sup_{t_0 \leq t < T'} (||\tilde{p}_i(\cdot, t)||_{L^\infty(\mathbb{R}^+)} + ||\partial_\rho \tilde{p}_i(\cdot, t)||_{L^\infty(\mathbb{R}^+)}) < \infty \quad \forall i = 1, 2$$

for any $T' \in (t_0, T_0)$. Then

$$\tilde{p}_1(r, t) = \tilde{p}_2(r, t) \quad \forall r \geq 0, t_0 \leq t < T_0.$$

Section 4

In this section we will prove the removable singularities property of the solution of the heat equation.

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Theorem 4.1. Let $m \geq 3$ and let $\Omega \subset \mathbb{R}^m$ be a domain. Suppose $u$ is a solution of the heat equation in $(\Omega \setminus \{0\}) \times (0, T)$. Then $u$ has removable singularities at $\{0\} \times (0, T)$ if and only if there exists $\overline{B_\delta} \subset \Omega$ such that (0.7) holds.

Proof. If $u$ has removable singularities at $\{0\} \times (0, T)$, then there exists a solution $v$ of the heat equation in $\Omega \times (0, T)$ such that $u = v$ on $(\Omega \setminus \{0\}) \times (0, T)$. Choose $\delta > 0$ such that $\overline{B_\delta} \subset \Omega$. Then (0.7) holds for $v$. Hence (0.7) holds for $u$.

Now suppose that there exists $\overline{B_R} \subset \Omega$ such that (0.7) holds for $0 < t_1 < t_2 < T$. By standard parabolic theory, $\forall 0 < \varepsilon < |x| < R$, $t_1 < t \leq t_2$,

$$u(x, t) = \int_{B_R \setminus B_\varepsilon} G_{R, \varepsilon}(x, t, y, t_1) u(y, t_1) \, dy + \int_{t_1}^t \int_{\partial B_\varepsilon} \frac{\partial G_{R, \varepsilon}}{\partial n_y}(x, t, y, s) u(y, s) \, d\sigma(y) \, ds$$

$$= I_{1, \varepsilon} + I_{2, \varepsilon} + I_{3, \varepsilon} \quad (4.1)$$

where $\partial / \partial n_y$ is the derivative in the direction of the inward normal of the domain $B_R \setminus B_\varepsilon$ at $y \in \partial B_R \cup \partial B_\varepsilon$. By an argument similar to the proof of Lemma 3.1 we get

$$\lim_{\varepsilon \to 0} I_{1, \varepsilon} = \int_{B_R \setminus \{0\}} G_R(x, t, y, t_1) u(y, t_1) \, dy \quad \forall 0 < |x| < R, t_1 < t \leq t_2 \quad (4.2)$$

and

$$\lim_{\varepsilon \to 0} I_{3, \varepsilon} = \int_{t_1}^t \int_{\partial B_R} \frac{\partial G_R}{\partial n_y}(x, t, y, s) u(y, s) \, d\sigma(y) \, ds \quad \forall 0 < |x| < R, t_1 < t \leq t_2. \quad (4.3)$$

By (0.7) and Corollary 2.10, $\forall 0 < \varepsilon < |x| < R$,

$$|I_{2, \varepsilon}| \leq C \int_{t_1}^t \int_{\partial B_\varepsilon} \frac{e^{-c|x-y|^2/(t-s)}}{(t-s)^{m+1/2}} |y|^{2-m} \, d\sigma(y) \, ds \leq \frac{C\varepsilon}{(|x| - \varepsilon)^{m+1}}. \quad (4.4)$$

Letting $\varepsilon \to 0$ in (4.1), by (4.2), (4.3), and (4.4), $\forall 0 < \varepsilon < |x| < R$, $t_1 < t \leq t_2$,

$$u(x, t) = \int_{B_R \setminus \{0\}} G_R(x, t, y, t_1) u(y, t_1) \, dy + \int_{t_1}^t \int_{\partial B_R} \frac{\partial G_R}{\partial n_y}(x, t, y, s) u(y, s) \, d\sigma(y) \, ds. \quad (4.5)$$

Since the right hand side of (4.5) is a $C^\infty$ function of $(x, t) \in B_R \times (t_1, t_2)$, we can extend $u(x, t)$ to a continuous function on $B_R \times (t_1, t_2)$ by defining $u(0, t)$ to be equal to the right hand side of (4.5). Then $u \in C^\infty(B_R \times (t_1, t_2))$ satisfies

$$u(x, t) = \int_{B_R} G_R(x, t, y, t_1) u(y, t_1) \, dy + \int_{t_1}^t \int_{\partial B_R} \frac{\partial G_R}{\partial n_y}(x, t, y, s) u(y, s) \, d\sigma(y) \, ds. \quad (4.6)$$

Since $G_R(x, t, y, s)$ satisfies the heat equation, by (4.6) the extended $u$ satisfies the heat equation in $B_R \times (t_1, t_2)$. Since $0 < t_1 < t_2 < T$ is arbitrary, $u$ has removable singularities at $\{0\} \times (0, T)$ and the theorem follows.
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