The Exact Solution to
the Schrödinger Equation with the Octic Potential

Shi-Hai Dong

Institute of High Energy Physics, P. O. Box 918(4), Beijing 100039, People's Republic of China

Zhong-Qi Ma

China Center for Advanced Science and Technology (World Laboratory), P. O. Box 8730, Beijing 100080

and Institute of High Energy Physics, P. O. Box 918(4), Beijing 100039, People's Republic of China

Abstract

The Schrödinger equation with the central potential is first studied in the arbitrary dimensional spaces and obtained an analogy of the two-dimensional Schrödinger equation for the radial wave function through a simple transformation. As an example, applying an ansatz to the eigenfunctions, we then arrive at an exact closed form solution to the modified two-dimensional Schrödinger equation with the octic potential, \( V(r) = ar^2 - br^4 + cr^6 - dr^4 + er^{10} \).

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*Electronic address: DONGSH@HPTC5. IHEP. AC. CN
1. Introduction

It is well known that the general framework of the nonrelativistic quantum mechanics is by now well understood[1, 2], however, whose predictions have been carefully proved against observations[3]. It is of importance to know whether some familiar problems are a particular case of a more general scheme. On behalf of this purpose, it is worthwhile to study the Schrödinger equation in the arbitrary dimensional spaces. This topic has attracted much more attention to many authors[4-8]. With respect to the arbitrary dimensional Schrödinger equation, it is readily to arrive at a simple analogy of the two-dimensional Schrödinger equation for the radial wave function through a simple transformation.

On the other hand, the exact solutions to the fundamental dynamical equations play an important role in physics. As we know, the exact solutions to the Schrödinger equation are possible only for several potentials and some approximation methods are frequently used to arrive at the solutions. Recently, the study of higher order anharmonic potentials have been much more desirable to physicists and mathematicians[10-12], who want to understand a few newly discovered phenomena (for instance, structural phase transitions[10], polaron formation in solids[11] and the concept of false vacuo in filed theory[12]) in the different fields of physics. Unfortunately, in these anharmonic potentials, not much work has been carried out on the octic potential except for some simpler study[13] by an ansatz to the eigenfunctions in the three-dimensional spaces. With the wide interest in the lower-dimensional field theory recently, however, it seems reasonable to study the two-dimensional Schrödinger equation with the octic potential. We has succeeded in dealing with the Schrödinger equation with some anharmonic potentials by this ansatz[15-17]. Consequently, we attempt to study the two-dimensional Schrödinger equation with the octic potential, to our knowledge, which is not appeared in the literature. The purpose of this paper is to demonstrate the modified Schrödinger equation in the arbitrary dimensional spaces and give a concrete application to the two-dimensional Schrödinger equation with the octic potential.

The paper is organized as follows. Section 2 studies the Schrödinger equation with
the central potential in the arbitrary dimensional spaces and obtains an analogy of the two-dimensional Schrödinger equation for the radial wave function through a simple transformation. In Sec. 3, as an example, applying an ansatz to the eigenfunctions, we obtain an exact closed form solution to the modified two-dimensional Schrödinger equation with the octic potential.

2. The modified Schrödinger equation

Throughout this paper the natural unit $\hbar = 1$ and $\mu = 1/2$ are employed. Following the Refs. [7, 8], in the $N$ dimensional Hilbert spaces, the radial wave function $\psi(r)$ for the Schrödinger equation for the stationary states can be written as

$$\left[ \frac{d^2}{dr^2} + \frac{(N-1)}{r} \frac{d}{dr} + (E - V(r)) - \frac{\ell(\ell + N - 2)}{r^2} \right] \psi(r) = 0, \quad (1)$$

where $\ell$ denotes the angular momentum quantum number. In order to make the coefficient of the first derivative vanish, we may furthermore define a new radial wave function $R(r)$ by means of the equation[9],

$$\psi(r) \equiv r^\rho R(r), \quad (2a)$$

where $\rho$ is an unknown parameter and will be given in the following. Substituting Eq. (2a) into Eq. (1), we will arrive at an algebraic equation containing the parameter $\rho$ as

$$2\rho + (N - 1) = 0.$$

Consequently, the Eq. (2a) will be read as

$$\psi \equiv r^{\frac{(N-1)}{2}} R(r), \quad (2b)$$

which will lead to the radial wave function $R(r)$ satisfying

$$\left\{ \frac{d^2}{dr^2} - \left[ \ell(\ell + N - 2) + \frac{1}{4}(N-1)(N-3) + (E - V(r)) \right] \frac{1}{r^2} \right\} R(r) = 0. \quad (3)$$

Through a simple deformation,

$$\ell(\ell + N - 2) + \frac{1}{4}(N-1)(N-3) = \left[ \ell + \frac{1}{2}(N-2) \right]^2 - \frac{1}{4},$$
we may introduce a parameter
\[ \eta \equiv \ell + \frac{1}{2}(N - 2), \] (4)
so that the Eq. (3) will be written as
\[
\left[ \frac{d^2}{dr^2} + (E - V(r)) - \frac{(\eta^2 - 1/4)}{r^2} \right] R(r) = 0,
\] (5)
which is our desired result. In other words, we have modified the Schrödinger equation in the arbitrary dimensional spaces into a simple analogy of the two-dimensional radial Schrödinger equation after introducing a parameter \( \eta \) given in Eq. (4), which relies on a linear combination between \( N \) and the angular momentum quantum number \( \ell \). As mentioned above, we want to solve this modified Schrödinger equation with the octic potential in two dimensions applying an ansatz to the eigenfunctions in the next section.

3. An ansatz to the eigenfunctions

Consider the two-dimensional Schrödinger equation with a potential \( V(r) \) that depends only on the distance \( r \) from the origin
\[
H \psi(r, \varphi) = \left( \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \right) \psi(r, \varphi) + V(r) \psi(r, \varphi) = E \psi(r, \varphi),
\] (6)
where here and hereafter the potential
\[
V(r) = ar^2 - br^4 + cr^6 - dr^8 + er^{10}, \quad d < 0.
\] (7)
Due to the symmetry of the potential, let
\[
\psi(r, \varphi) = r^{-1/2} R_m(r) e^{\pm im\varphi}, \quad m = 0, 1, 2, \ldots.
\] (8)
It is easy to find from Eq. (5) that the radial wave function \( R_m(r) \) satisfies the following radial equation
\[
\frac{d^2 R_m(r)}{dr^2} + \left[ E - V(r) - \frac{m^2 - 1/4}{r^2} \right] R_m(r) = 0,
\] (9)
where the parameter $\lambda = m = \ell, N = 2$; $m$ and $E$ denote the angular momentum number and energy, respectively. For the solution of Eq. (9), we make an ansatz for the ground state

$$R_{m0}(r) = \exp[p_{m0}(r)],$$

where

$$p_{m0}(r) = \frac{1}{2} \alpha r^2 - \frac{1}{4} \beta r^4 + \frac{1}{6} \tau r^6 + \kappa \ln r.$$  

After calculating, we arrive at the following equation

$$\frac{d^2 R_{m0}(r)}{dr^2} - \left[ \frac{d^2 p_{m0}(r)}{dr^2} + \left( \frac{dp_{m0}(r)}{dr} \right)^2 \right] R_{m0}(r) = 0.$$  

Compare Eq. (12) with Eq. (9) as before and obtain the following set of equations

$$\kappa(\kappa - 1) = (m + 1/2)(m - 1/2), \quad \tau^2 = e,$$
$$\beta^2 + 2\alpha \tau = c, \quad 2\beta \tau = d,$$
$$\alpha^2 - 2\beta \kappa - 3\beta = a,$$
$$5\tau + 2\tau \kappa - 2\alpha \beta = -b,$$
$$E = -\alpha(1 + 2\kappa).$$

It is not difficult to obtain the values of parameters $\tau$ and $\kappa$ from the Eq. (13a) written as

$$\tau = \pm \sqrt{e}, \quad \kappa = -m + 1/2 \quad \text{or} \quad m + 1/2.$$  

In order to retain the well-behaved solution at the origin and at infinity, we choose positive sign in $\tau$ and $\kappa$ as $m + 1/2$. According to these choices, the Eq. (13b) will give the other parameter values as

$$\beta = -\frac{d}{2\sqrt{e}}, \quad \alpha = \frac{d^2 - 4ce}{8e\sqrt{e}}.$$  

Besides, it is readily to obtain from the Eqs. (13c) and (13d) that

$$a = \frac{d^4 - 8ced^2 + 16e^2c^2 + 64de^2\sqrt{e}(\kappa + 3/2)}{64e^3}.$$
which are the constraints on the parameters of the octic potential.

The eigenvalue $E$, however, will be given by Eq. (13e) as

$$E = -\frac{(1 + 2\kappa)(d^2 - 4ce)}{8e\sqrt{e}}. \quad (17)$$

The corresponding eigenfunctions Eq. (10) can now be read as

$$R_{m0} = N_0 r^\kappa \exp \left[ \frac{1}{2} \alpha^2 - \frac{1}{4} \beta r^4 + \frac{1}{6} \tau r^6 \right], \quad (18)$$

where $N_0$ is the normalized constant and here and hereafter the parameters $\alpha, \beta$ and $\kappa$ are the same as the values given above. As a matter of fact, the normalized constant $N_0$ can be calculated in principle from the normalized relation

$$\int_0^\infty |R_{m0}|^2 dr = 1, \quad (19)$$

which implies that

$$N_0 = \left[ \frac{1}{\omega} \right]^{1/2}, \quad (20)$$

where

$$\omega \equiv \int_0^\infty r^{2\kappa} \exp \left[ \alpha r^2 - \frac{1}{2} \beta r^4 + \frac{1}{3} \tau r^6 \right] dr. \quad (21)$$

In this case, however, the normalization of the eigenfunctions becomes a very difficult task. Considering the values of the parameters of the potential, we fix them as follows. The value of parameter $c, d, e$ are first fixed, for example $c = 1.0, d = -2$ and $e = 4.0$, the value of the parameters $a$ and $b$ are given by Eq. (11) for $m = 0$. By this way, the parameters turn out to $a = -1.96, b = 11.8, c = 4.0, d = -2.0, e = 4.0$ and $\alpha = -0.188, \beta = 0.5, \tau = -2$. The ground state energy corresponding to these values is obtained as $E = 0.375$. Actually, when we study the property of the ground state, as we know, the unnormalized radial wave function will not affect the main features of the wave function. We have plotted the unnormalized radial wave function $R_{00}(r)$ in fig. 1 for the ground state.

To summarize, we first deal with the Schrödinger equation with the central potential in the arbitrary dimensional spaces and obtain an analogy of the two-dimensional
Schrödinger equation for the radial wave function through a simple transformation. As an example, we obtain an exact closed solution to the Schrödinger equation with the octic potential using a simpler ansatz and simultaneously two constrains on the parameters of the potential are arrived at from the compared equation. The other studies to the Schrödinger equation with the related anharmonic potential in two dimensions are in progress.

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Figure 1: The ground state wave functions $R_{00}(r)$ as a function of $r$ for the potential (2) with the values $a = -1.96$, $b = 11.8$, $c = 1.0$, $d = -2.0$, $e = 4.0$. The $x$-axis denotes the variable $r$ and the $y$-axis denotes the values of wave functions.