Hodge gauge fixing in three dimensions

J. E. Hetrick

aPhysics Department, University of Arizona, Tucson, AZ 85721

A progress report on experiences with a gauge fixing method proposed in LATTICE 94 is presented. In this algorithm, an SU(N) operator is diagonalized at each site, followed by gauge fixing the diagonal (Cartan) part of the links to Coulomb gauge using the residual abelian freedom. The Cartan sector of the link field is separated into the physical gauge field \( \alpha^{(f)}_\mu \) responsible for producing \( f^{\text{Cartan}}_{\mu\nu} \), the pure gauge part, lattice artifacts, and zero modes. The gauge transformation to the physical gauge field \( \alpha^{(f)}_\mu \) is then constructed and performed. Compactness of the fields entails issues related to monopoles and zero modes which are addressed.

AZPH-TH/96-18

1. The Method

While gauge fixing is a central tool of lattice simulations, the effect of lattice artifact “Gribov copies” remains a delicate issue, particularly in chiral fermion models which often rely on gauge fixing essentially.

In [1], we proposed a method of gauge fixing to an 't Hooft like gauge which was called Gauge fixing by Hodge decomposition. We work here on a spatial torus in 3-dimensions, thus this presentation is for Coulomb gauge (the algorithm is done in parallel on each time slice). Landau gauge generalizes similarly. The method is as follows.

- Diagonalize some operator which transforms adjointly, \( O_x \equiv G_x^1 O_x G_x \), at each site. (The operator used here is the spatial sum of plaquette clovers and their adjoints at each site.)

- Define an Abelian \( \alpha_\mu \) field from the links.

- Decompose \( \alpha_\mu(x) = \alpha^{(f)}_\mu + \partial_\mu \phi + H_\mu + w_\mu \), and solve for the physical field \( \alpha^{(f)}_\mu \) which minimally produces the plaquettes and open (monopole) strings \( f^{\mu\nu} \).

- Add the continuum zero-mode: \( \alpha^{(f)}_\mu + w_\mu \).

- Construct and perform the \( U(1) \) gauge transformation which moves the links from \( \alpha_\mu \) to \( \alpha^{(f)}_\mu + w_\mu \).

1.1. Diagonalization of \( O \)

An iterative method can be used in which the operator \( O \) is hit with successive SU(2) subgroup gauge transformations that each minimize the modulus of resulting off-diagonal terms. If

\[
O' = G^d \left( \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right) G
\]

then \( \theta = \tan^{-1}(-2A/B)/4 \) minimizes \( |a'_{12}| + |a'_{21}| \). In the two cases:

**\( \sigma_x \) case:**

\[
G = \begin{pmatrix} \cos \theta & i \sin \theta \\ i \sin \theta & \cos \theta \end{pmatrix}
\]

\[
A = a_{11} r + a_{12} i - a_{12} r + a_{11} i
+ a_{21} r - a_{22} i + a_{22} r - a_{21} i
\]

\[
B = (a_{11} r)^2 + (a_{12} i)^2 + (a_{22} r)^2 + (a_{21} i)^2
- (a_{11} r)^2 - (a_{12} i)^2 - (a_{22} r)^2 - (a_{21} i)^2
+ 2a_{12} a_{21} - 2a_{11} a_{22} - 2a_{11} a_{22} - 2a_{11} a_{22}
\]

**\( \sigma_y \) case:**

\[
G = \begin{pmatrix} \cos \theta & -i \sin \theta \\ i \sin \theta & \cos \theta \end{pmatrix}
\]

\[
A = a_{11} r + a_{11} i + a_{12} r + a_{12} i
- a_{21} r - a_{21} i - a_{22} r - a_{22} i
\]

\[
B = (a_{11} r)^2 + (a_{12} i)^2 + (a_{22} r)^2 + (a_{21} i)^2
- (a_{11} r)^2 - (a_{12} i)^2 - (a_{22} r)^2 - (a_{21} i)^2
- 2a_{12} a_{21} + 2a_{11} a_{22} + 2a_{11} a_{22} - 2a_{11} a_{22}
\]
For $O \in SU(2)$ or $su(2)$, only one of each gauge transformation is required. For $SU(3)$ operators, about 10 iterations are required to get $\sum_{\text{off-diagonal}} |a_{ij}| < 10^{-7}$.

1.2. The Residual Abelian Fields

We must define residual $U(1)$ fields, which represent the part of $A_\mu^a$ in the Cartan subgroup in the continuum limit. For $SU(2)$ we use:

$$\alpha_\mu = \tan^{-1}(e_3/e_0)$$

where $U_\mu = e_0 + ie_k \sigma_k$.

For $SU(3)$ where a link has diagonal elements $\{ e^{i\beta^1}, e^{i\beta^2}, e^{i\beta^3} \}$, the angles

$$\alpha_\mu^1 = (2\beta^1 - \beta^2 - \beta^3)/3,$n$$

$$\alpha_\mu^2 = (2\beta^2 - \beta^1 - \beta^3)/3$$

are suitable Cartan fields.

2. Hodge Decomposition of $\alpha_\mu$

Any lattice vector field $\alpha_\mu(x)$ can be uniquely decomposed into:

$$\alpha_\mu(x) = \alpha_\mu^{(f)} + \partial_\mu \phi + H_\mu + w_\mu$$

where $\partial_\mu$ ($\partial_\mu^\dagger$) is the lattice forward (backward) derivative

- $\alpha_\mu^{(f)}$ is the physical part of the field and solely responsible for producing $f_{\mu\nu} = \partial_\mu \alpha_\nu^{(f)} - \partial_\nu \alpha_\mu^{(f)}$. Conveniently, $\alpha_\mu^{(f)}$ also naturally satisfies the Landau gauge condition: $\partial_\mu \alpha_\mu^{(f)} = 0$.

- $\partial_\mu \phi$ is the pure gauge part.

- $H_\mu$ is the lattice harmonic part responsible for Dirac string loops (and is the major source of Gribov copies).

- $w_\mu$ is the continuum harmonic part for a torus, i.e. a constant.

The decomposition is formal at this stage, but we remark that much of the work in solving for the lattice $\alpha_\mu^{(f)}$ is in identifying the necessary parts of $H_\mu$ which must be kept.

2.1. Solving for $\alpha_{\mu}^{(f)}$

$\alpha_\mu^{(f)}$ is determined (up to zero-modes) by

$$\alpha_\mu^{(f)} = \Delta^{-1} \partial_\mu f_{\nu\mu},$$

or

$$\hat{\alpha}_\mu^{(f)}(k) = \frac{ik_\mu}{-k^2} f_{\nu\mu}(k),$$

in Fourier space. $k_\mu$ is the lattice momentum; $2 \sin(\pi n_\mu/L_\mu)$. It is thus very easy to find $\alpha_\mu^{(f)}$ from $f_{\mu\nu}$ by FFT; however we must build the correct $f_{\mu\nu}$ in stages.

Since we want $\alpha_\mu^{(f)}$ to minimally reproduce the plaquette angles $f_{\mu\nu}^{\text{pla}}$:

- We first set $f_{\mu\nu} = f_{\mu\nu}^{\text{pla}}$.

3. Monopoles

For compact gauge fields, $\alpha_\mu^{(f)}$ must also the reproduce gauge invariant monopoles which are the ends of open lengths of Dirac string; thus we need to find all monopole sites on the dual lattice. Since the Dirac strings connecting these monopoles are gauge variant, we define a monopole-anti-monopole pairing which minimizes the string length between pairs. This can be done by simulated annealing for instance, but since this pairing is not unique, this step is a source of Gribov ambiguity. We could also connect pairs in order of finding them (violating rotational invariance), which is however fast and unique. Then,

- We next add to $f_{\mu\nu} = f_{\mu\nu}^{\text{pla}} + f_{\mu\nu}^{\text{monopole}}$

in otherwords, for each plaquette pierced by a Dirac string (as given by our minimal monopole pairing), we add $\pm 2\pi$ to $f_{\mu\nu}$.

3.1. Zero-modes of $f_{\mu\nu}$: globally wrapping string

Due to compactness again, $f_{\mu\nu}$ may have a zero-mode. This zero-mode can be viewed as a length of non-contractable Dirac string, i.e. one that stretches across the torus.

$$\frac{1}{2\pi L^3} \int dx^\rho dx^\nu (f_{\mu\nu}^{\text{pla}} + f_{\mu\nu}^{\text{monopole}}) \equiv e^\rho \neq 0$$

$(\mu, \nu \perp \rho)$ indicates the occurrence of $e^\rho$ global non-contractable Dirac strings in the $\rho$-direction.
in the initial configuration which must be added to \( f_{\mu\nu} \).

We are at liberty to place these strings wherever we like, either randomly for pseudo translational invariance, or along some particular axis so that we know where they are. Along these strings we add \( \pm 2\pi \) to the \( f_{\mu\nu} \) perpendicular to the path so that \( f_{\mu\nu} = f_{\mu\nu}^{\text{plq}} + f_{\mu\nu}^{\text{mnple}} + f_{\mu\nu}^{\text{global}} \).

We are now ready to solve eq. 6, which will give the minimal \( \alpha_\mu \) that reproduces the plaquette angles, monopole, and global strings, and which is in Landau gauge:

\[
\alpha_\mu^{(f)} = \Delta^{-1} \partial_\nu (f_{\mu\nu}^{\text{plq}} + f_{\mu\nu}^{\text{mnple}} + f_{\mu\nu}^{\text{global}}).
\]

3.2. One more zero-mode \( w_\mu \)

There is one more zero mode that is undetermined by eq. 6 which is responsible for producing the correct Wilson loops. After solving for \( \alpha_\mu^{(f)} \) we must add to it the correct constant \( w_\mu \)

\[
w_\mu = \frac{1}{L^3} \int \int dx^\rho dx^\nu \int dx^\mu \left( \alpha_\mu - \alpha_\mu^{(f)} \right)
\]

in order that Wilson loops are preserved mod(2\( \pi \)).

This is the zero-mode of the original \( \alpha_\mu \) field, and does not contribute to \( f_{\mu\nu} \).

4. The Gauge Transformation

Because of the zero-modes, the only way to find the gauge transformation taking us from \( \alpha_\mu \rightarrow \alpha_\mu^{(f)} + w_\mu \) is by constructing a tree, or in other words integrating the equation

\[
\partial_\mu g = (\alpha_\mu^{(f)} + w_\mu) - \alpha_\mu
\]

5. Tests

Figure 2 shows three extremization gauge fixed copies derived from the initial configuration in figure 1. In figure 3, the only two copies obtained by the Hodge method are displayed. The \( SU(2) \) starting configuration is relatively smooth though, generated at \( \beta = 44 \), followed by a random gauge transformations.

6. Conclusions

- The method seems to work reasonably well; it returns a uniquely gauge fixed configuration, up to the connectivity of monopole pairs. However, the Cartan sector of \( SU(N) \) fields at typical \( \beta \) values is fairly rough, and thus the monopole density is also relatively high.

- Closed (contractable) loops of Dirac string are removed, which are a primary source of Gribov copies in extremization methods.

- For high monopole densities, simulated annealing seems to give poorer connectivity for the large number of monopole pairs than extremization, i.e. extremization uses less string to connect monopoles.

- For rough fields, \( |\alpha^{(f)}_\mu| \) can occasionally be larger than \( \pi \). This means that the physical (non-compact) field \( \alpha^{(f)}_\mu \) is “clipped” by compactness, resulting in a site with \( \partial_\mu \alpha_\mu = \pm 2\pi \).
REFERENCES

1. Ph. de Forcrand and J.E. Hetrick, Nucl. Phys. B (Proc. Suppl.) 42 (1995) 861

Figure 2. Three of the many Gribov copies from gauge fixing by extremization.
Figure 3. Hodge method: only 2 final configurations occur.