A HETEROCLINIC SOLUTION TO A VARIATIONAL PROBLEM
CORRESPONDING TO FITZHUGH-NAGUMO TYPE
REACTION-DIFFUSION SYSTEM WITH HETEROGENEITY

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(Communicated by Russell Johnson)

Abstract. Chen, Kung and Morita [5] studied a variational problem corre-
sponding to the FitzHugh-Nagumo type reaction-diffusion system (FHN type
RD system), and they proved the existence of a heteroclinic solution to the
system.

Motivated by [5], we consider a variational problem corresponding to FHN
type RD system which involves heterogeneity. We prove the existence of a hete-
roclinic solution to the problem under certain conditions on the heterogeneity.
Moreover, we give some information about the location of the transitions.

1. Introduction and main results. The FitzHugh-Nagumo type reaction-diffusion
system (FHN type RD systems) is introduced in the field of physiology, which essen-
tially describe neural excitability. This is also studied mathematically as a model
which generates complex patterns. A typical FHN type RD system is the following:
\[ \begin{align*}
    u_t(x,t) &= d \Delta u(x,t) + f(u(x,t)) - v(x,t), \quad x \in \Omega, \ t > 0, \\
    \tau v_t(x,t) &= D \Delta v(x,t) + u(x,t) - \gamma v(x,t), \quad x \in \Omega, \ t > 0,
\end{align*} \]
where \( \Omega \subset \mathbb{R}^N \) (\( N \geq 1 \)) is a domain, \( d, D, \tau, \gamma > 0 \) are constants and
\( f(t) = t(1-t) \beta (1-t) \) (\( 0 < \beta < 1/2 \)).

We present some previous works on the steady state problems of (1), that is
\( u_t = v_t = 0 \) in (1). There are a number of works on FHN type RD systems, and
then we pick up works only related to variational approaches. For simplicity, we
assume \( D = 1 \) throughout this paper.

In the case \( \Omega \) is a bounded domain with a smooth boundary, Oshita [10] proved
the existence of a solution to (1) under the Neumann boundary condition by consid-
ering a variational problem corresponding to (1). Oshita proved that the solution
to the variational problem oscillates rapidly. Under the Dirichlet boundary condi-
tion, Dancer and Yan [6] showed similar results. See also [8] for related results on
the three-component system.

In the case \( \Omega = \mathbb{R} \), Chen and Choi [3] showed the existence of a homoclinic
solution under the assumption that \( d, \gamma > 0 \) are small. In addition, Heijster et al.
[4] combined geometric singular perturbation techniques and variational techniques
to show various homoclinic solutions to the three component model. More detail, see

2000 Mathematics Subject Classification. Primary: 35J50, 35K57; Secondary: 35Q92.

Key words and phrases. Variational problem, FitzHugh-Nagumo type reaction diffusion sys-
tems, heteroclinic solution.
the existence of heteroclinic solutions in the case $\Omega = \mathbb{R}$ and references therein. On the other hand, Chen, Kung and Morita [5] showed

$$\psi \in C^\infty(\mathbb{R})$$

which is obtained by solving the following equation:

$$L\psi \in \mathbb{R}$$

by a variational approach. Now we give some notations. Let $a$ be a positive number and $\hat{\psi} \in C^\infty(\mathbb{R})$ be a function such that

$$\hat{\psi}(x) = \begin{cases} \frac{a}{\gamma}, & x \geq 1, \\ 0, & x \leq 0. \end{cases}$$

Then we define $\hat{u} = D\hat{\psi} = -\psi'' + \gamma \hat{\psi}$. We note that $\hat{u} \in C^\infty(\mathbb{R})$ satisfies the following:

$$\hat{u}(x) = \begin{cases} \frac{a}{\gamma}, & x \geq 1, \\ 0, & x \leq 0. \end{cases}$$

Moreover, we define the operator $L : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ as follows: for $\psi \in L^2(\Omega)$, $\phi = L\psi$ is the unique solution to

$$\begin{cases} -\phi''(x) + \gamma \phi(x) = \psi(x), & x \in \mathbb{R}, \\ \phi \in H^2(\mathbb{R}). \end{cases}$$

Then we define $J : H^1(\mathbb{R}) \to \mathbb{R}$ as follows:

$$J(\psi) = \int_{\mathbb{R}} \left[ \frac{\theta}{2} (u'(x))^2 + \frac{1}{2} \left( \frac{v'(x) - u'(x)}{\gamma} \right)^2 + \frac{\gamma}{2} \left( v(x) - \frac{u(x)}{\gamma} \right)^2 + W(u(x)) \right] dx$$

for $\psi \in H^1(\mathbb{R})$, where $u = \hat{u} + \psi$, $v = \hat{v} + L\psi$, $\theta = d - 1/\gamma^2$ and $W(t)$ is defined as follows:

$$W(t) = \frac{1}{4} t^2 (t - a)^2.$$ 

We suppose that $d > 1/\gamma^2$, that is, $\theta > 0$. One can see that the Euler-Lagrange equation corresponding to the minimizing problem

$$\sigma_0 = \inf \{ J(\psi) : \psi \in H^1(\mathbb{R}) \}$$

coincides with (1) under the assumption

$$(a, \gamma) = \left( \frac{2(\beta + 1)}{3}, \frac{9}{2\beta^2 - 5\beta + 2} \right),$$

which is obtained by solving the following equation:

$$\begin{cases} \int_0^a (f(t) - t/\gamma) \, dt = 0, \\ f(a) = a/\gamma. \end{cases}$$

With this strategy, Chen, Kung and Morita showed the existence of a heteroclinic solution of (1) connecting $(0, 0)$ to $(a, a/\gamma)$.

Motivated by [5], we consider the following minimizing problem (5) corresponding to the energy $\tilde{J} : H^1(\mathbb{R}) \to \mathbb{R}$ defined as (6), which involves the heterogeneity $\mu(x)$.

$$\sigma = \inf \{ \tilde{J}(\psi) : \psi \in H^1(\mathbb{R}) \},$$

$$\tilde{J}(\psi) = \int_{\mathbb{R}} \left[ \frac{\theta}{2} (u'(x))^2 + \frac{1}{2} \left( \frac{v'(x) - u'(x)}{\gamma} \right)^2 + \frac{\gamma}{2} \left( v(x) - \frac{u(x)}{\gamma} \right)^2 + \mu(x) W(u(x)) \right] dx,$$

for $\psi \in H^1(\mathbb{R})$, where $\theta$, $\gamma$ and $a$ are positive constants. Here $\mu \in L^\infty(\mathbb{R})$ is a function satisfying the following conditions:

$(\mu_1) \ 0 < \mu_1 \leq \mu(x) \leq 1$ for all $x \in \mathbb{R}$ and $\mu \neq 1$. 

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Let have the following result.

**Remark 1.** We can easily see $\sigma \leq \sigma_0$ from the definitions of $J(\psi), \tilde{J}(\psi)$ and $\mu(x)$. Moreover, we can prove $\sigma < \sigma_0$ (see Lemma 2.4).

For a related equation, there are some previous works on variational problems with a heterogeneity to find heteroclinic solutions. Bonheure and Sanchez [2] considered the Allen-Cahn type equation involving the heterogeneity $h(x)$ of the form

$$
\left\{ \begin{array}{ll}
u''(x) + h(x)\nu(x)(1 - \nu(x)^2) = 0, & x \in \mathbb{R}, \\
u(x) \to \pm 1, & x \to \pm \infty \\
u'(x) \to 0, & x \to \pm \infty.
\end{array} \right. 
$$

The energy corresponding (7) is the following:

$$
E(\nu) = \int_{\mathbb{R}} \left[ \frac{1}{2}(\nu')^2 + \frac{h(x)}{4}(\nu^2 - 1)^2 \right] dx.
$$

They obtained a heteroclinic solution connecting $-1$ to $1$ by a variational approach. We note that (6) is one of extended models of (7). We mention a related paper by Sourdis [11]. Sourdis proved the existence of a heteroclinic solution to the vector Allen-Cahn model with heterogeneity, which is another extended model of (7). In addition, we mention the works by Ikeda and Ei [7] or Nishiura, Teramoto and Yuan [9] as the works that dealt with the FHN type RD system with various heterogeneities. See also the references therein on their related works.

Now we state our main results. For the existence of the minimizer of (5), we have the following result.

**Theorem 1.1.** Let $a$ be a positive number and $\theta = 1 - 1/\gamma^2 > 0$. We assume that $\mu \in L^\infty(\mathbb{R})$ satisfies (μ1) and (μ2). Then the following statements hold.

(i) There exists a minimizer $\tilde{\psi} \in H^1(\mathbb{R})$ of the minimizing problem (5). We define $\tilde{\nu} = \nu + \tilde{\psi}, \tilde{\nu} = \nu + \tilde{\psi}$, and then $(\tilde{\nu}, \tilde{\nu})$ is a heteroclinic solution to the following equation:

$$
\left\{ \begin{array}{ll}
-\tilde{\nu}''(x) = \mu(x)(\tilde{\nu}(x) - a/2)(a - \tilde{\nu}(x)) - \tilde{\nu}(x) + \tilde{\nu}(x)/\gamma, & x \in \mathbb{R}, \\
-\tilde{\nu}''(x) = \tilde{\nu}(x) - \gamma \tilde{\nu}(x), & x \in \mathbb{R}, \\
(\tilde{\nu}(x), \tilde{\nu}(x)) \to (0, 0), & x \to -\infty, \\
(\tilde{\nu}(x), \tilde{\nu}(x)) \to (a, a/\gamma), & x \to \infty.
\end{array} \right.
$$

Moreover, if there exists a bounded interval $[x_1, x_2]$ such that supp $(1 - \mu) \subset [x_1, x_2]$, then $\tilde{y} \leq x_2$ and $\tilde{y}' \geq x_1$ hold. Here $\tilde{y}$ and $\tilde{y}'$ be defined as (9) and (10) for a small positive number $\rho$ such that $\rho < a/8$.

$$
\tilde{y} \equiv \min \left\{ x \in \mathbb{R} : |\tilde{\nu}(x)| = \frac{3\rho}{2} \right\},
$$

$$
\tilde{y}' \equiv \max \left\{ x \in \mathbb{R} : |\tilde{\nu}(x) - a| = \frac{3\rho}{2} \right\}.
$$

(ii) If $\theta \geq 1$, then there exists a positive constant $M$ independent on $d$ such that $|\tilde{y}' - \tilde{y}| < Md$.

**Remark 2.** If $(a, \gamma)$ satisfies (4) in Theorem 1.1, then we can readily see

$$
\left\{ \begin{array}{ll}
-\tilde{\nu}''(x) = \mu(x)(f(\tilde{\nu}) - \tilde{\nu}(x)/\gamma) - \tilde{\nu}(x) + \tilde{\nu}(x)/\gamma, & x \in \mathbb{R}, \\
-\tilde{\nu}''(x) = \tilde{\nu}(x) - \gamma \tilde{\nu}(x), & x \in \mathbb{R}, \\
(\tilde{\nu}(x), \tilde{\nu}(x)) \to (0, 0), & x \to -\infty, \\
(\tilde{\nu}(x), \tilde{\nu}(x)) \to (a, a/\gamma), & x \to \infty.
\end{array} \right.
$$
Theorem 1.1 (i) implies that there exists a minimizer of the minimizing problem (5) and that if \( \text{supp} (1 - \mu) \) is compact, then the transitions of \( \bar{u} = \hat{u} + \psi \) should have intersections with \( \text{supp} (1 - \mu) \). Here we use the term “transition of \( \bar{u} \)” to loosely mean the interval \([\bar{y}, \bar{y}']\) defined as (9) and (10), which is precisely defined in the next section. Moreover, Theorem 1.1 (ii) gives us the information about dependence of the width of \([\bar{y}, \bar{y}']\) on \( d \) in the case \( \theta \geq 1 \).

Although the proof of Theorem 1.1 is based on [5], we need to modify a part of the argument of [5]. The key to the proof of the existence of a minimizer of (5) is the uniform estimate of \( \psi_m \), where \( \{\psi_m\}_{m \in \mathbb{N}} \) is a minimizing sequence of (5). In the case \( \mu \equiv 1 \), Chen, Kung and Morita [5] called a transition from \( \rho \) to \( a - \rho \) by a “wavefront type oscillation” for a given small positive number \( \rho \), and they proved the uniform estimate of the width of the interval which contains all wavefront type oscillations of \( u_m = \hat{u} + \psi_m \). Noting that \( J(\psi) \) is invariant under translations of \( u = \hat{u} + \psi \),—more precisely, for given \( \psi \in H^1(\mathbb{R}) \) and \( L \in \mathbb{R} \), we define \( \hat{u}(x) = \hat{u}(x + L) + \psi(x + L) \) and \( \phi(x) = \hat{u}(x) - \hat{u}(x) \), and then we have \( J(\phi) = J(\psi) \)— this estimate implies that all wavefront type oscillations of \( u_m \) stay in a compact interval independent on \( m \in \mathbb{N} \). However in the case \( \mu \neq 1 \), \( J(\psi) \) is generally not invariant under translations, and thus it is necessary to show the location of the transitions of \( u_m \). To overcome this difficulty, we prove that there exists a positive constant \( M_1 \) such that \( y_m' - y_m < M_1 \) for all \( m \in \mathbb{N} \), where \( y_m \) and \( y_m' \) are defined as follows:

\[
y_m \equiv \min \left\{ x \in \mathbb{R} : |\hat{u}(x) + \psi_m(x)| = \frac{3}{2} \rho \right\},
\]

\[
y_m' \equiv \max \left\{ x \in \mathbb{R} : |\hat{u}(x) + \psi_m(x) - a| = \frac{3}{2} \rho \right\}.
\]

This uniform estimate of \( y_m' - y_m \) is a stronger estimate than the uniform estimate of the region of wavefront type oscillations due to Chen, Kung and Morita [5], and hence this estimate enables us to analyze more accurate behavior of \( u_m \). With this estimate and \( \sigma < \sigma_0 \) (see Lemma 2.4), we can show that the all intervals \([y_m', y_m] \) \( (m \in \mathbb{N}) \) are uniformly contained in a compact interval independent on \( m \in \mathbb{N} \).

We also consider the following equation:

\[
\begin{align*}
-\mu''(x) &= \mu(x)f(u) - \nu(x), & x \in \mathbb{R}, \\
-\nu''(x) &= u(x) - \gamma \nu(x), & x \in \mathbb{R}, \\
(u(x), \nu(x)) &\to (0, 0), & x \to -\infty, \\
(u(x), \nu(x)) &\to (a, a/\gamma), & x \to \infty,
\end{align*}
\]

where \( f(t) = t(t - \beta)(1 - t) \) \( (0 < \beta < 1/2) \). Moreover, we assume that \( d, \gamma, a \) are positive constants such that \( \theta \equiv d - 1/\gamma^2 > 0 \) and \( (a, \gamma) \) satisfies (4). We assume that \( \mu \in L^\infty(\mathbb{R}) \) is a function satisfying (\( \mu1 \)) and (\( \mu3 \)):

\( \mu3 \) 1 - \( \mu \) \( \in L^1(\mathbb{R}) \) holds, and there exists a positive constant \( L_0 \) such that

\[
F \left( \frac{\beta}{2} \right) \int_{L_0 - \beta^2 \theta / (8 \sigma_0)}^{L_0} (1 - \mu(x)) \, dx \geq |F(1)| \int_{L_0}^{\infty} (1 - \mu(x)) \, dx,
\]

where \( F(t) \equiv -\int_0^t f(s) \, ds \). We note that \( F(\beta/2) > 0 \) and \( F(1) < 0 \).
Remark 3. It is clear that if supp \((1 - \mu)\) is compact, then \(\mu\) satisfies \((\mu3)\).

Moreover, let \(\mu\) be a function such that
\[
1 - \mu(x) \sim e^{-\alpha x} \quad (x \to \infty)
\]
for sufficiently large \(\alpha > 0\). Then \(\mu\) also satisfies \((\mu3)\). In fact, let \(c_1, c_2\) and \(L_\ast\) be constants such that
\[
c_1 e^{-\alpha x} \leq 1 - \mu(x) \leq c_2 e^{-\alpha x} \quad \text{for all } x \geq L_\ast.
\]

Furthermore, we assume \(L_0 \geq L_\ast\). Then we obtain
\[
F\left(\frac{\beta}{2}\right) \int_{L_0-\beta^2\theta/(8\sigma_0)}^{L_0} (1 - \mu(x)) \, dx \geq F\left(\frac{\beta}{2}\right) c_1 e^{-\alpha L_0} \frac{\beta^2 \theta}{8\sigma_0}
\]
and
\[
|F(1)| \int_{L_0}^{\infty} (1 - \mu(x)) \, dx \leq |F(1)| \frac{c_2}{\alpha} e^{-\alpha L_0}.
\]

As a consequence, we can see that if \(\alpha \geq |F(1)| / |F(\beta/2)| \cdot c_2 / (c_1 \cdot 8\sigma / (\beta^2 \theta))\) and \(L_0 \geq L_\ast\), then \(\mu\) satisfies \((\mu3)\).

Similarly, we can prove that if \(\mu\) satisfies
\[
1 - \mu(x) \sim x^{-\alpha} \quad (x \to \infty)
\]
for sufficiently large \(\alpha > 0\), then \(\mu\) satisfies \((\mu3)\).

Now we define \(\tilde{J} : H^1(\mathbb{R}) \to \mathbb{R}\) as follows:
\[
\tilde{J}(\psi) = J(\psi) + \int_{\mathbb{R}} \frac{1 - \mu(x)}{2\gamma} u^2 \, dx
\]
\[
= \int_{\mathbb{R}} \left[ \frac{\theta (u')^2}{2} + \frac{1}{2} \left( v' - \frac{u'}{\gamma} \right)^2 + \frac{\gamma}{2} \left( v - \frac{u}{\gamma} \right)^2 + \mu(x) W(u) + \frac{1 - \mu(x)}{2\gamma} u^2 \right] \, dx.
\]

for \(\psi \in H^1(\mathbb{R})\), where \(\theta, \gamma\) and \(\alpha\) are positive constants. We can prove that the Euler-Lagrange equation corresponding to the minimizing problem
\[
\tilde{\sigma} = \inf \{ \tilde{J}(\psi) : \psi \in H^1(\mathbb{R}) \}
\]
coincides with \((13)\) under the assumptions \(\theta = d - 1/\gamma^2 > 0\) and \((4)\) (see Lemma 5.1).

We mention that \((15)\) is an extended model of \((2)\), which has an extra term \(\int_{\mathbb{R}} (1 - \mu(x)) u^2 / (2\gamma) \, dx\). For the existence of the minimizer of \((16)\), we have the following result.

**Theorem 1.2.** Let \(\theta = d - 1/\gamma^2 > 0\) and \((\alpha, \gamma)\) be constants satisfying \((4)\). We assume that \(\mu \in L^\infty(\mathbb{R})\) satisfies \((\mu1)\) and \((\mu3)\). Then \(\tilde{\sigma} < \sigma_0\) holds, and there exists a minimizer \(\psi \in H^1(\mathbb{R})\) of \((16)\). Moreover, \((\tilde{u}, \tilde{v}) = (\hat{u} + \psi, \hat{v} + \mathcal{L}\tilde{\psi})\) is a heteroclinic solution to \((13)\).

Theorem 1.2 implies that \((16)\) has a minimizer under the assumption that \(\mu\) satisfies more strict condition than Theorem 1.1. Basically we can prove it by repeating similar arguments as in the proof of Theorem 1.1, but we should modify a part of the arguments. In particular, the condition \((\mu3)\) is used for the proof of \(\sigma_0 > \tilde{\sigma}\).

This paper is arranged as follows. We first prepare several basic lemmas in Section 2: preliminaries in Section 2.1, and basic estimates in Section 2.2. In Section 2.2, we collect basic estimates in our setting, which are essentially same as
Lemma 2.1. Preliminaries and basic estimates.

2. Preliminaries. In this section, we present several lemmas. We begin with deriving the Euler-Lagrange equation.

**Lemma 2.1.** Assume that the minimizing problem (5) has a minimizer \( \psi \). Let \( u = \hat{u} + \psi \) and \( v = \hat{v} + L\psi \). Then \((u, v)\) satisfies (8).

**Proof.** Let \( \phi \in C_c^\infty(\mathbb{R}) \) and \( j[\phi]: \mathbb{R} \to \mathbb{R} \) be

\[
j[\phi](t) = \frac{1}{2} \int_\mathbb{R} \left[ \theta \{(u')^2 + 2t u \phi' + t^2 (\phi')^2\} + \left( v' - \frac{u'}{\gamma} \right)^2 + 2t \left( v' - \frac{u'}{\gamma} \right) \left( (L\phi)' - \frac{\phi'}{\gamma} \right) \right. \\
+ t^2 \left\{ (L\phi)' - \frac{\phi'}{\gamma} \right\}^2 + \gamma \left( v - \frac{u}{\gamma} \right) + 2t\gamma \left( v - \frac{u}{\gamma} \right) \left( L\phi - \frac{\phi}{\gamma} \right) \\
+ \gamma t^2 \left( L\phi - \frac{\phi}{\gamma} \right)^2 \right] dx + \int_\mathbb{R} \mu(x) W(u + t\phi) dx,
\]

Then it follows that

\[
(j[\phi])'(0) = \int_\mathbb{R} \left[ \theta u' \phi' + \left( v' - \frac{u'}{\gamma} \right) \left( (L\phi)' - \frac{\phi'}{\gamma} \right) + \gamma \left( v - \frac{u}{\gamma} \right) \left( L\phi - \frac{\phi}{\gamma} \right) + \mu(x) W'(u) \phi \right] dx = A_1 + A_2 + A_3 + A_4,
\]

where \( A_i \) \((i = 1, 2, 3, 4)\) are defined as follows:

\[
A_1 = \int_\mathbb{R} \theta u' \phi' dx = \int_\mathbb{R} \left( d - \frac{1}{\gamma^2} \right) u' \phi' dx = \int_\mathbb{R} \left( du' \phi' - u' \phi' \right) dx,
\]

\[
A_2 = \int_\mathbb{R} \left( v' - \frac{u'}{\gamma} \right) \left( (L\phi)' - \frac{\phi'}{\gamma} \right) dx = \int_\mathbb{R} \left( v'(L\phi)' - \frac{v' \phi'}{\gamma} - \frac{u'(L\phi)'}{\gamma} \right) dx,
\]

\[
A_3 = \int_\mathbb{R} \left( v - \frac{u}{\gamma} \right) \left( L\phi - \frac{\phi}{\gamma} \right) dx = \int_\mathbb{R} \left( v(L\phi) - \frac{v \phi}{\gamma} - u(L\phi) + \frac{u \phi}{\gamma} \right) dx,
\]

\[
A_4 = \int_\mathbb{R} \mu(x) W'(u) \phi dx = \int_\mathbb{R} \mu(x) u \left( u - \frac{a}{2} \right) (u - a) dx.
\]
Therefore we rewrite $A_2$ as follows.

$$A_2 = \int_{\mathbb{R}} \left[ -v(L\phi)'' + \frac{v'' \phi}{\gamma} + \frac{u(L\phi)''}{\gamma} + \frac{u' \phi}{\gamma^2} \right] dx.$$  

Moreover, we note $(j[\phi])'(0) = 0$, and then we obtain

$$(j[\phi])'(0) = \int_{\mathbb{R}} \left[ du' \phi' + v \{ -(L\phi)'' + \gamma L\phi \} - \frac{1}{\gamma}(-v'' + \gamma v)\phi \right.$$

$$- \frac{u}{\gamma} \{ -(L\phi)'' + \gamma L\phi \} + \frac{\mu}{\gamma} + \mu(x)u \left( u - \frac{a}{2} \right) (u - a) \phi \left] dx \right.$$

$$= \int_{\mathbb{R}} \left[ du' \phi' + v \phi - \frac{u \phi}{\gamma} + \mu(x)u \left( u - \frac{a}{2} \right) (u - a) \phi \right] dx = 0,$$

that is,

$$\int_{\mathbb{R}} du' \phi' dx = \int_{\mathbb{R}} \left( -v + \frac{u}{\gamma} + \mu(x)u \left( u - \frac{a}{2} \right) (a - u) \right) \phi dx.$$  

For simplicity, we write the right hand side of the above equation as $\int_{\mathbb{R}} \eta \phi dx$, and we can easily see $\eta \in H^1(\mathbb{R})$. As a consequence, there exists $v'' \in H^1(\mathbb{R}) \subset C(\mathbb{R})$, and $u$ satisfies

$$-du''(x) = \mu(x)u(x)(u(x) - a/2)(a - u(x)) - v(x) + \frac{u(x)}{\gamma} \quad \text{for all } x \in \mathbb{R}.$$  

In addition, it is clear that $v$ satisfies

$$-v''(x) = u(x) - \gamma v(x) \quad \text{for all } x \in \mathbb{R}$$  

from the definition of $v$.  

Now we define the term “transition” as follows.

**Definition 2.2.** Let $u \in C(\mathbb{R})$ and $a_1, a_2 \in \mathbb{R}$. We say that $u$ has a transition on $[y_1, y_2]$ from $a_1$ to $a_2$ if $u$ satisfies as following conditions:

1. $u(y_1) = a_1$, $u(y_2) = a_2$.
2. $a_1 \leq u(x) \leq a_2$ for all $x \in (y_1, y_2)$.

Then we present some estimates, which are used to obtain the uniform estimate of $[y_m, y'_m]$ in the proof of Theorem 1.1, where $y_m$ and $y'_m$ are defined as (11) and (12).

The following lemma is well-known, but we present in the form to apply the lemmas shown later.

**Lemma 2.3.** Let $u = \dot{u} + \psi, \theta = d - (1/\gamma^2)$ and $\rho > 0$ be a small positive number satisfying $\rho \in (0, a/8)$. Moreover we assume that $u$ has a transition on $[y_1, y_2]$ from $\rho$ to $3\rho/2$ or from $a - \rho$ to $a - 3\rho/2$. Then the following inequality hold:

$$\int_{y_1}^{y_2} \theta \left( \frac{\theta}{2} u''^2 + \frac{u(x)}{4} u^2(u - a)^2 \right) dx \geq 4\epsilon_1,$$

where $\epsilon_1$ is defined as follows:
\[ \epsilon_1 = \frac{1}{8} \sqrt{\frac{\theta \mu_1}{2}} \rho^2 (a - \rho). \]

**Proof.** From the basic theorem of calculus and Hölder's inequality, we have
\[ |u(y_2) - u(y_1)| = \left| \int_{y_1}^{y_2} u'(x) \, dx \right| \leq |y_2 - y_1|^{\frac{1}{2}} \left( \int_{y_1}^{y_2} (u'(x))^2 \, dx \right)^{\frac{1}{2}}. \]

Thus we can see
\[ \int_{y_1}^{y_2} (u'(x))^2 \, dx \geq \frac{|u(y_2) - u(y_1)|^2}{|y_2 - y_1|^2}. \]

On the other hand, we can easily check
\[ \int_{y_1}^{y_2} \frac{1}{4} \mu(x) u^2(u - a)^2 \, dx \geq \frac{|y_2 - y_1| \mu_1 \rho^2 (a - \rho)^2}{4}. \]

Combining the above inequalities, we obtain
\[ \int_{y_1}^{y_2} \left\{ \frac{\theta}{2} (u')^2 + \frac{1}{4} \mu(x) u^2(u - a)^2 \right\} \, dx \geq \frac{\theta \rho^2}{8 |y_2 - y_1|} + \frac{|y_2 - y_1| \mu_1 \rho^2 (a - \rho)^2}{4} \]
\[ \geq \frac{1}{2} \sqrt{\frac{\theta \mu_1}{2}} \rho^2 (a - \rho) \]
and complete the proof. \( \square \)

**Remark 4.** It is easy to check that we can apply Lemma 2.3 to the case that \( u \) has a transition from \(-\rho\) to \(-3\rho/2\) or from \(a + \rho\) to \(a + 3\rho/2\).

The next lemma is almost obvious, but it is important for proving that the transitions of \( u_m = \hat{u} + \psi_m \) must stay in a bounded interval.

**Lemma 2.4.** \( \sigma_0 > \sigma \) holds, where \( \sigma_0 \) is defined as (3).

**Proof.** Let \( \psi_0 \in H^1(\mathbb{R}) \) be a minimizer of (3), \( u_0 = \hat{u} + \psi_0 \) and \( v_0 = \hat{v} + \mathcal{L}\psi_0 \). We may assume that \( \int_{\mathbb{R}} (1 - \mu(x))W(u_0) \, dx > 0 \) since \( J(\psi) \) is invariant under translations of \( u \). Thus we obtain
\[ \sigma_0 = J(\psi_0) = \tilde{J}(\psi_0) + \int_{\mathbb{R}} (1 - \mu(x))W(u_0) \, dx > \tilde{J}(\psi_0) \geq \sigma. \]

\( \square \)

The next lemma is used in the proof of Theorem 1.1 (ii).

**Lemma 2.5.** We write \( J(\psi) = J_d(\psi) \) to emphasize the parameter \( d \). Then we define \( \sigma_0^d \) as follows:
\[ \sigma_0^d = \inf \{ J_d(\psi) : \psi \in H^1(\mathbb{R}) \}. \]
Then we have \( \sigma_0^\alpha \leq \sigma_0^d \sqrt{d} \) for all \( d \geq 1 \), where \( d_\alpha = (1/\gamma)^2 + 1 \).

**Proof.** Let \( U, V \) be
\[ \{ U(X) = u(\sqrt{d}X), \quad V(X) = v(\sqrt{d}X). \]
By the change of variables \(x = \sqrt{d}X\) in (2), then we obtain

\[
\frac{J_d(\psi)}{\sqrt{d}} = \int_{\mathbb{R}} \left( \frac{1}{2} \left(1 - \frac{1}{d\gamma^2}\right)(U'(X))^2 + \frac{1}{2d} \left(V'(X) - \frac{U'(X)}{\gamma}\right)^2 \\
+ \frac{\gamma}{2} \left(V(X) - \frac{U(X)}{\gamma}\right)^2 + \frac{1}{4} U(X)^2(U(X) - a)^2 \right) dX
\]

\[
\leq \int_{\mathbb{R}} \left[ \frac{1}{2}(U')^2 + \frac{1}{2} \left(V' - \frac{U'}{\gamma}\right)^2 + \frac{\gamma}{2} \left(V - \frac{U}{\gamma}\right)^2 + \frac{1}{4} U^2(U - a)^2 \right] dX. \quad (17)
\]

Let \(\psi^*\) be a minimizer of the minimizing problem \(\sigma_d^0 = \inf \{J_d(\psi) : \psi \in H^1(\mathbb{R})\}\). Moreover we define \(U_0, V_0, u_0, v_0\) as follows:

\[
U_0 = \hat{u} + \psi^*, \quad V_0 = \hat{v} + \mathcal{L}\psi^*,
\]

\[
u_0(x) = U_0 \left( \frac{x}{\sqrt{d}} \right), \quad \psi_0(x) = u_0(x) - \hat{u}(x).
\]

We then substitute \(\psi_0\) into (17) and we have

\[
\frac{J_d(\psi_0)}{\sqrt{d}} \leq \int_{\mathbb{R}} \left[ \frac{1}{2}(U_0')^2 + \frac{1}{2} \left(V_0' - \frac{U_0'}{\gamma}\right)^2 + \frac{\gamma}{2} \left(V_0 - \frac{U_0}{\gamma}\right)^2 + \frac{1}{4} U_0^2(U_0 - a)^2 \right] dX
\]

\[
= J_d(\psi^*) = \sigma_d^0.
\]

This implies that \(\sigma_d^0 \leq \sigma_d^0 \sqrt{d}\). \(\square\)

2.2. Basic estimates. In this subsection, we present basic estimates of a minimizing sequence \(\{\psi_m\}_{m \in \mathbb{N}}\). Let \(\epsilon_0\) be a small positive number. Then we may assume that

\[
J(\hat{u} + \psi_m) \leq \sigma + \epsilon_0 < 2\sigma_0 \quad \text{for all} \quad m \in \mathbb{N}
\]

without loss of generality. Moreover, we note that \(\sigma > 0\) from Lemma 2.3.

Now we prove the uniformly boundedness of \(u_m = \hat{u} + \psi_m\).

**Lemma 2.6.** Let \(\{\psi_m\}_{m \in \mathbb{N}}\) be a minimizing sequence of (3), \(u_m = \hat{u} + \psi_m\) and \(v_m = \hat{v} + \mathcal{L}\psi_m\). Then there exists a positive constant \(M_1\) such that

\[
\|u_m\|_{L^\infty} \leq M_1 \quad \text{for all} \quad j \in \mathbb{N}.
\]

**Proof.** We prove by contradiction. Namely we suppose that there exists a subsequence \(\{m_j\}_{j \in \mathbb{N}}\) such that \(\|u_{m_j}\|_{L^\infty} \to \infty\) as \(j \to \infty\). For simplicity, we write \(j = m_j\). We may assume that there exists \(\xi_j \in \mathbb{R}\) such that \(u_j(\xi_j) = ja\) for all \(j \geq 3\). Then it follows that there exists \(\xi_j < \xi_j\) such that \(u_j(\xi) = 2a\) for all \(j \geq 3\).

We define \(I_j = [\xi_j, \xi_j]\). From Hölder’s inequality, we have

\[
|u_j(\xi_j) - u_j(\xi_j)| = (j - 2)a \leq \int_{I_j} |u'_j(x)| \, dx \leq \left( \frac{4}{\sigma} \right)^{1/2} \cdot |I_j|^{1/2}
\]

for all \(j \geq 3\). Then we can see

\[
|I_j| \geq \frac{\theta(j - 2)^2a^2}{4\sigma}
\]

for all \(j \geq 3\), and then we obtain \(|I_j| \to \infty\) as \(j \to \infty\). On the other hand, we have

\[
2\sigma \geq \int_{I_j} \mu(x)W(u_j(x)) \, dx \geq \mu_1W(2a) |I_j| \to \infty.
\]
However, it clearly contradicts $\sigma < \infty$. \hfill \Box

The next lemma is essentially same as Lemma 2.4 of [5], and then we omit the proof.

Lemma 2.7. Let $\{\psi_m\}_{m \in \mathbb{N}}$ be a minimizing sequence corresponding to (5) and let $u_m = \hat{u} + \psi_m$, $v_m = \hat{v} + \mathcal{L}\psi_m$. For any positive number $k$, there exists $C = C(k)$ such that

$$
\sup_{m \geq 1} (\|u_m\|_{H^1(-k,k)} + \|v_m\|_{H^1(-k,k)}) < C.
$$

3. Local energy and key lemmas. In this section, we present key lemmas for the proof of the theorem. First, we consider the “local energy” $G_n (n \in \mathbb{Z})$ defined as follows:

$$
G_n(\psi) = \int_{n}^{n+1} \left[ \frac{\theta}{2} (u')^2 + \frac{1}{2} \left( v - \frac{u'}{\gamma} \right)^2 + \gamma \left( v - \frac{a}{\gamma} \right)^2 + \mu(x) W(u) \right] dx,
$$

where $u = \hat{u} + \psi$, $v = \hat{v} + \mathcal{L}\psi$. Then we obtain the next lemma. The lemma reveals the relationship between the local energy and the local behavior of $u$ and $v$. Although this lemma is almost same with Lemma 2.5 of [5], we present a proof in Appendix A to emphasize the dependence of the constant $M_2$.

Lemma 3.1. Let $\psi \in H^1(\mathbb{R})$. We assume that $\rho$ is a small positive number satisfying $\rho \in (0, a/8)$ and $\rho_1 > 0$ is a constant such that $\rho_1 < \epsilon_1 = \sqrt{\theta \mu_1/2 \rho^2 (a-\rho)/8}$. If there exists $n \in \mathbb{Z}$ such that

$$
G_n(\psi) < \rho_1,
$$

then there exists a constant $M_2 = M_2(\theta, \rho, \mu_1, \gamma)$ and either (18) or (19) holds.

$$
\|u - a\|_{H^1(I_n)} + \left\| v - \frac{a}{\gamma} \right\|_{H^2(I_n)} \leq M_2 \sqrt{\rho_1}, \tag{18}
$$

$$
\|u\|_{H^1(I_n)} + \|v\|_{H^2(I_n)} \leq M_2 \sqrt{\rho_1}, \tag{19}
$$

where $I_n = [n, n+1]$. Moreover, if $\theta \geq 1$, then we can take the constant $M_2 > 0$ which is independent on $d > \theta \geq 1$.

Next, we introduce some notations. Let $\chi \in C^\infty(\mathbb{R})$ be a cut-off function satisfying

$$
\chi(x) = \begin{cases} 
1, & x \leq 0, \\
0, & x \geq 1.
\end{cases}
$$

Let $\chi_\alpha(x) = \chi(x-\alpha)$, $\tilde{\chi}_\alpha(x) = 1 - \chi_\alpha(x)$ for $\alpha \in \mathbb{R}$. Moreover we define $S_\alpha$ and $\tilde{S}_\alpha$ as follows:

$$
S_\alpha \psi = \mathcal{D}(\chi_\alpha \mathcal{L}\psi),
$$

$$
\tilde{S}_\alpha \psi = \mathcal{D}(\tilde{\chi}_\alpha \mathcal{L}\psi),
$$

where $\mathcal{D} = -\partial^2/\partial x^2 + \gamma$. From the definition, it is easy to check

$$
\begin{cases}
S_\alpha \psi(x) = \psi(x), & x \leq \alpha, \\
S_\alpha \psi(x) = 0, & x \geq \alpha + 1.
\end{cases} \tag{20}
$$

$$
\begin{cases}
\tilde{S}_\alpha \psi(x) = 0, & x \leq \alpha, \\
\tilde{S}_\alpha \psi(x) = \psi(x), & x \geq \alpha + 1.
\end{cases} \tag{21}
$$

It is also easy to check

$$
\mathcal{L}(S_\alpha \psi) = \chi_\alpha (\mathcal{L}\psi), \tag{22}
$$
from the definition of $S_\alpha$ and $\bar{S}_\alpha$. Now we prove the following lemma, which play important roles to show Lemma 3.3 presented later.

**Lemma 3.2.** Let $\psi \in H^1(\mathbb{R})$, $u = \hat{u} + \psi$ and $v = \hat{v} + \mathcal{L}\psi$. We define $G_n^{(0)}$ as follows:

$$G_n^{(0)}(\psi) = \int_{I_n} \left[ \frac{\theta}{2} (\hat{u}')^2 + \frac{1}{2} \left( \hat{v}' - \frac{\hat{u}'}{\gamma} \right)^2 + \frac{\gamma}{2} \left( \hat{v} - \frac{\hat{u}}{\gamma} \right)^2 + \frac{1}{4} \hat{u}^2 (\hat{u} - a)^2 \right] dx.$$

Then we have the following statements.

(i) We assume that $\rho$ be a small number satisfying $\rho \in (0, a/8)$ and $\rho_1 > 0$ satisfies $\rho_1 < \min\{\epsilon_1, 1\}$. We also assume that there is $n \in \{n \in \mathbb{Z} : n \geq 1\}$ satisfying (18). Then there exists a constant $M_3 = M_3(\theta, \rho, \mu_1, \gamma, a) > 0$ such that

$$G_n^{(0)}(S_n\psi) < M_3\rho_1.$$

(ii) We assume that $\rho$ be a small number satisfying $\rho \in (0, a/8)$ and $\rho_1 > 0$ satisfies $\rho_1 < \min\{\epsilon_1, 1\}$. We also assume that there exists $n \in \{n \in \mathbb{Z} : n \leq -1\}$ satisfying (19). Then there exists a constant $M_3 = M_3(\theta, \rho, \mu_1, \gamma, a) > 0$ such that

$$G_n^{(0)}(\bar{S}_n\psi) < M_3\rho_1.$$

**Proof.** Since we can prove (ii) as in the case (i), it suffices to prove the case (i). Let $\bar{u}, \bar{v}$ be

$$\bar{u} = \hat{u} + S_n\psi$$
$$\bar{v} = \hat{v} + \mathcal{L}S_n\psi = \hat{v} + \chi_n\mathcal{L}\psi.$$

We note that

$$G_n^{(0)}(S_n\psi) = \int_{I_n} \left[ \frac{\theta}{2} (\hat{u}')^2 + \frac{1}{2} \left( \hat{v}' - \frac{\hat{u}'}{\gamma} \right)^2 + \frac{\gamma}{2} \left( \hat{v} - \frac{\hat{u}}{\gamma} \right)^2 + \frac{1}{4} \hat{u}^2 (\hat{u} - a)^2 \right] dx.$$

By direct calculation,

$$S_n\psi = -\left(\chi_n\mathcal{L}\psi\right)^n + \gamma (\chi_n\mathcal{L}\psi)$$

$$= -\left\{\chi''_n\mathcal{L}\psi + 2\chi'_n(\mathcal{L}\psi)' + \chi_n(\mathcal{L}\psi)''\right\} + \gamma \chi_n\mathcal{L}\psi.$$

Since $\hat{u}(x) = a, \hat{v}(x) = a/\gamma$ for all $x \in I_n$ with $n \geq 1$, (18) is equivalent to

$$\|\psi\|_{H^1(I_n)} + \|\mathcal{L}\psi\|_{H^1(I_n)} \leq M_2\sqrt{\rho_1}.$$
Hence we have the following estimates:
\[
\frac{\theta}{2} \int_{I_n} (\tilde{w}')^2 \, dx = \frac{\theta}{2} \int_{I_n} \{(S_n \psi)'\}^2 \leq dC_8(\gamma)M_2^2 \rho_1,
\]
\[
\frac{1}{2} \int_{I_n} \left( \frac{\tilde{v}' - \tilde{u}'}{\gamma} \right)^2 \leq \| (\chi_n L \psi)' \|_{L^2(I_n)}^2 + \frac{1}{\gamma^2} \| (S_n \psi)' \|_{L^2(I_n)}^2 \leq C_9(\gamma)M_2^2 \rho_1,
\]
\[
\frac{\gamma}{2} \int_{I_n} \left( \tilde{v} - \frac{\tilde{u}}{\gamma} \right)^2 \leq \gamma \| \chi_n L \psi \|_{L^2(I_n)}^2 + \frac{1}{\gamma} \| S_n \psi \|_{L^2(I_n)}^2 \leq C_{10}(\gamma)M_2^2 \rho_1,
\]
\[
\frac{1}{4} \int_{I_n} u^2 (u - a)^2 \, dx \leq \frac{1}{4} \| a + S_n \psi \|_{L^\infty(I_n)}^2 \| S_n \psi \|_{L^2(I_n)}^2 \leq \frac{1}{2} (a^2 + \| S_n \psi \|_{L^\infty(I_n)}) C_{11}(\gamma)M_2^2 \rho_1.
\]
By Sobolev inequality (see [1]), we have
\[
\| S_n \psi \|_{L^\infty(I_n)} \leq 8 \sqrt{2} \| S_n \psi \|_{H^1(I_n)},
\]
and thus we obtain
\[
\| S_n \psi \|_{L^\infty(I_n)}^2 \leq C_{11}(\gamma) \| L \psi \|_{H^3(I_n)}^2 \leq C_{12}(\gamma)M_2^2 \rho_1.
\]
As a result, we have
\[
\frac{1}{4} \int_{I_n} u^2 (u - a)^2 \, dx \leq \frac{1}{2} (a^2 + C_{12}M_2^2 \rho_1) C_{11}M_2^2 \rho_1 \leq \frac{1}{2} (a^2 + C_{12}M_2^2) C_{11}M_2^2 \rho_1
\]
with attention to the assumption \( \rho_1 < 1 \). Since \( M_2 \) depends on \( \theta, \rho, \mu_1 \) and \( \gamma \), there exists a constant \( M_3 = M_3(\theta, \rho, \mu_1, \gamma, a) > 0 \) such that
\[
G_n^{(0)}(S_n \psi) \leq M_3 \rho_1.
\]
This concludes the statement of Lemma 3.2. \( \square \)

**Remark 5.** If \( d \geq 1 \), then there exists a constant \( M' = M'(\rho, \mu_1, \gamma, a) \) such that \( M_3 = M'd \) because of Lemma 3.1.

The next lemma is the key to show boundedness of the interval \([y_m, y'_m]\), where \( y_m \) and \( y'_m \) are defined as (11) and (12).

**Lemma 3.3.** Let \( \rho \) be a small positive number satisfying \( \rho \in (0, \alpha/8) \) and \( \rho_1 \) be a small positive number satisfying \( \rho_1 < \min\{\epsilon_1, 1/3M_3\} \). We assume \( \psi \in H^1(\mathbb{R}) \) and define \( u = \tilde{u} + \psi \) and \( v = \tilde{v} + L \psi \). Then the following statements hold.

(i) We suppose that \( u \) has a transition on \([y_1, y_2]\) from \( a - \rho \) to \( a - 3\rho/2 \) or from \( a + \rho \) to \( a + 3\rho/2 \). Moreover we suppose that there exists \( j \in \mathbb{Z} \) such that \( j + 1 < y_1 \) and \( \psi \) satisfies (18) with \( n = j \). Then there exists \( \phi \in H^1(\mathbb{R}) \) satisfying
\[
\tilde{J}(\phi) < \tilde{J}(\psi) - 3\epsilon_1.
\]
In particular, if \( j \geq 1 \), then we may take \( \phi = S_j \psi \).

(ii) We suppose that \( u \) has a transition on \([y_1, y_2]\) from \( \rho \) to \( 3\rho/2 \) or from \( -\rho \) to \( -3\rho/2 \). Moreover we suppose that there exists \( j \in \mathbb{Z} \) such that \( j > y_2 \) and \( \psi \) satisfies (19) with \( n = j \). Then there exists \( \phi \in H^1(\mathbb{R}) \) satisfying
\[
\tilde{J}(\phi) < \tilde{J}(\psi) - 3\epsilon_1.
\]
In particular, if \( j \leq -1 \), then we may take \( \phi = \hat{S}_j \psi \).

**Proof.** Since we can prove \((ii)\) as in the case \((i)\), it suffices to prove the case \((i)\). In addition, we may assume that \( u \) has a transition from \( a - \rho \) to \( a - 3\rho/2 \) because of Remark 4.

First, we prove the statement in the case \( j \geq 1 \). We set \( \hat{u} = u + S_j \psi, \hat{v} = \hat{v} + \mathcal{L}(S_j \psi) \). Then it is easy to see \( \hat{v} = \hat{v} + \chi_J(\mathcal{L}\psi) \) from (22). Moreover we can see from (20). Hence we obtain the following estimate of \( \tilde{J}(S_j \psi) \).

\[
\tilde{J}(S_j \psi) = \sum_{n=-\infty}^{-1} G_n(S_j \psi) + G_j(S_j \psi) + \sum_{n=j+1}^{\infty} G_n(S_j \psi)
\]

\[
= \sum_{n=-\infty}^{-1} G_n(\psi) + G_j(S_j \psi) + 0
\]

\[
= \left\{ \tilde{J}(\psi) - G_j(\psi) - \sum_{n=j+1}^{\infty} G_n(\psi) \right\} + G_j(S_j \psi)
\]

\[
\leq \tilde{J}(\psi) - \sum_{n=j+1}^{\infty} G_n(\psi) + G_j^{(0)}(S_j \psi). \tag{24}
\]

From the assumption, there is at least one transition from \( a - \rho \) to \( a - 3\rho/2 \) in the interval \([j+1, \infty)\), and thus we obtain by (2.3)

\[
\sum_{n=j+1}^{\infty} G_n(\psi) > 4\epsilon_1.
\]

Moreover, since \( u, v \) satisfy (18) with \( n = j \), we have \( G_j^{(0)}(S_j \psi) < \epsilon_1 \) by Lemma 3.2

\((i)\). Substituting these estimates into (24), we obtain

\[
\tilde{J}(S_j \psi) < \tilde{J}(\psi) - 3\epsilon_1.
\]

Next, we prove in the case \( j \in \mathbb{Z} \). We set \( U, \varphi \) and \( V \) as follows:

\[
U(x) = u(x + j - 1),
\]

\[
\varphi(x) = U(x) - \hat{u}(x),
\]

\[
V(x) = \hat{v}(x) + \mathcal{L}\varphi(x).
\]

We readily see \( \varphi \in H^1(\mathbb{R}) \). Moreover, we can see that \( V \) is the unique solution to

\[
\begin{cases}
- V'' + \gamma V = U & \text{on } \mathbb{R}, \\
V'(\pm \infty) = 0.
\end{cases}
\]

On the other hand, \( v(x + j - 1) \) is also the solution to the above equation, and thus we have \( V(x) = v(x + j - 1) \). As a consequence, \( U, V \) satisfy

\[
\|U - a\|_{H^1(I_j)} + \left\| \frac{V - a}{\gamma} \right\|_{H^3(I_j)} \leq \|u - a\|_{H^1(I_j)} + \left\| \frac{v - a}{\gamma} \right\|_{H^3(I_j)} < M_2\rho_1.
\]
This means that $\varphi$ satisfies (18) with $n = 1$. Subsequently, we define $\bar{U}, \bar{V}$ as follows:

$$
\bar{U}(x) = \tilde{u}(x) + S_1\varphi(x),
\bar{V}(x) = \tilde{v}(x) + \mathcal{L}(S_1\varphi(x)) = \tilde{v}(x) + \chi_1\mathcal{L}\varphi(x).
$$

We note that there are the following relationships between $U, V, \tilde{u}, \tilde{v}, \psi, \varphi$

$$
\begin{align*}
\text{Moreover noting that } & \varphi \text{ satisfies (18) with } n = 1, \text{ we see } G^{(0)}_j(S_1\varphi) \leq M_3\rho_1. \\
& \text{Since } \rho_1 < \epsilon_1/M_3, \text{ we obtain}
\end{align*}
$$

$$
\bar{J}(\varphi) < \bar{J}(\psi) - 4\epsilon_1 + G^{(0)}_j(S_1\varphi).
$$

In view of $\bar{u}, \bar{v}, \bar{U}, \bar{V}, \tilde{u}, \tilde{v}$, we have

$$
G^{(0)}_j(\varphi) = \int_1^{(j+1)} \left[ \frac{\theta}{2}(\tilde{u}'')^2 + \frac{1}{2} \left( \bar{v}' - \tilde{u}' \right)^2 + \frac{\gamma}{2} \left( \bar{v} - \tilde{u} \right)^2 + \frac{1}{4} \bar{u}^2(\bar{u} - a)^2 \right] dx
$$

$$
= \int_1^{(j+1)} \left[ \frac{\theta}{2} \left( \bar{U}' - \tilde{u}' \right)^2 + \frac{1}{2} \left( \bar{V}' - \tilde{v}' \right)^2 + \frac{\gamma}{2} \left( \bar{V} - \tilde{v} \right)^2 + \frac{1}{4} \bar{V}^2(\bar{V} - a)^2 \right] dx
$$

Moreover noting that $\varphi$ satisfies (18) with $n = 1$, we can see $G^{(0)}_j(S_1\varphi) \leq M_3\rho_1$. Since $\rho_1 < \epsilon_1/M_3$, we obtain

$$
\bar{J}(\varphi) < \bar{J}(\psi) - 4\epsilon_1 + M_3 \cdot \frac{\epsilon_1}{M_3} = \bar{J}(\psi) - 3\epsilon_1
$$

from (25) and (26). As a consequence, we complete the proof of (i). Similarly we prove (ii) with Lemma 3.2 (ii).

4. **Proof of Theorem 1.1.** In this section, we present the proof of Theorem 1.1.

**Proof of Theorem 1.1.** We fix a positive constant $\rho_0$ satisfying

$$
W\left( 3\rho_0 \right) \int_{\mathbb{R}} (1 - \mu(x)) \, dx < \frac{\epsilon_2}{4} \quad \text{and} \quad \rho_0 < \frac{a}{8},
$$

where $\epsilon_2 = \min\{\epsilon_1, \sigma_0 - \sigma\}/2$. We note that $\epsilon_2 > 0$ from Lemma 2.4. Let $\rho \in (0, \rho_0)$ and $\{\psi_m\}_{m \in \mathbb{N}}$ be a minimizing sequence of (5). We may assume that

$$
\bar{J}(\psi_m) \leq \sigma + \epsilon_2 \quad \text{for all } m \in \mathbb{N}.
$$
Let \( u_m = \hat{u} + \psi_m, v_m = \hat{v} + \mathcal{L}\psi_m \) and \( \rho_1 \) be a positive constant defined as follows:

\[
\rho_1 = \frac{1}{2} \min \left\{ 1, \epsilon_1, \frac{\epsilon_1}{M_3}, \frac{\rho^2}{(8\sqrt{2}M_2)^2} \right\}.
\]

(27)

Then we have the following statements by Lemma 3.1 and the Sobolev inequality.

If \( x \in I_n \) and \( u_n(x) \in [\rho, a - \rho] \), then \( G_n(\psi_m) \geq \rho_1 \).

(28)

If \( u_m, v_m \) satisfy (18), then \( |a - u_m| < \rho \) on \( I_n \).

(29)

If \( u_m, v_m \) satisfy (19), then \( |u_m| < \rho \) on \( I_n \).

(30)

Moreover, we recall \( y_m, y'_m \) defined as (11) and (12):

\[
y_m \equiv \min \left\{ x \in \mathbb{R} : |u_m(x)| = \frac{3}{2}\rho \right\},
\]

\[
y'_m \equiv \max \left\{ x \in \mathbb{R} : |u_m(x) - a| = \frac{3}{2}\rho \right\}.
\]

(28)

We can easily see \(-\infty < y_m < y'_m < \infty \) for all \( m \in \mathbb{N} \) since \( \psi_m \in H^1(\mathbb{R}) \). Then let \( z_m \) be an integer such that \( y_m \in [z_m, z_m + 1] \). We note that \( G_{z_m}(\psi_m) \geq \rho_1 \) from (28).

Theorem 1.1 is proved by the following three claims.

**Claim 1.** There exists a positive constant \( M_4 \) independent of \( m \in \mathbb{N} \) such that

\[
|y'_m - y_m| < M_4 \text{ for all } m \in \mathbb{N}.
\]

(31)

Moreover, if \( \theta \geq 1 \), then there exists a constant \( M \) independent on \( d \) such that \( M_4 = M d \).

**Claim 2.** There exists a positive constant \( M_5 \) such that

\[
[y_m, y'_m] \subset [-M_5, M_5] \text{ for all } m \in \mathbb{N}.
\]

Furthermore, if there exists a bounded interval \([x_1, x_2]\) such that \( \text{supp} \ (1 - \mu) \subset [x_1, x_2] \), then \( y_m \geq x_2 \) and \( y_m \leq x_1 \) for all \( m \in \mathbb{N} \).

**Claim 3.** There exist a minimizer \( \tilde{\psi} \in H^1(\mathbb{R}) \) of the minimizing problem (5). Thus the statement of the theorem is true.

**Proof of Claim 1.** For each \( m \in \mathbb{N} \), we define \( K_m \equiv \{ n \in \mathbb{Z} : G_n(\psi_m) \geq \rho_1 \}, l_m = |K_m| \). Then we easily see that

\[
l_m \leq 2\sigma/\rho_1 \text{ for all } m \in \mathbb{N}.
\]

(32)

Thus it suffices to show the next equation:

\[
[y_m, y'_m] \subset [z_m, z_m + l_m] \text{ for all } m \in \mathbb{N}.
\]

Since \( G_{z_m}(\psi_m) \geq \rho_1 \), there exists \( n \in \mathbb{N} \) satisfying

\[
G_n(\psi_m) < \rho_1 \text{ and } n \in \{ z_m + 1, z_m + 2, \ldots, z_m + l_m \}.
\]

(33)

By Lemma 3.1, \( u_m, v_m \) should satisfy either (18) or (19), but the case (19) never happen. In fact, \( u_m \) should have a transition from \( \rho \) to \( 3\rho/2 \) or from \( -\rho \) to \( -3\rho/2 \) in the interval \( (-\infty, y_m] \) by the definition of \( y_m \). If the case (19) happens, there exists \( \phi_m \in H^1(\mathbb{R}) \) such that

\[
\tilde{J}(\phi_m) < \tilde{J}(\psi_m) - 3\epsilon_1
\]

by Lemma 3.3 (ii). However this implies \( \tilde{J}(\phi_m) < \sigma - 2\epsilon_1 \) and it is contradiction. Therefore \( \psi_m \) should satisfy (18) for \( n \in \mathbb{Z} \) defined in (33). \( u_m \) should also satisfy
|a - u_m| < ρ on I_n by (29). We can also prove that u_m has transitions neither from a - ρ to a - 3ρ/2 nor from a + ρ to a + 3ρ/2 in the interval [n, ∞) by Lemma 3.3 (i). This implies

\[ |u_m(x) - a| < \frac{3}{2}\rho \quad \text{for all } x \in [n, \infty). \]

Combining with \( n \leq z_m + l_m, z_m \leq y_m \) and (32), we obtain

\[ y'_m \leq y_m + \frac{2\sigma}{\rho_1} \quad \text{for all } m \in \mathbb{N}. \]

Moreover, from (27), Lemma 3.1 and Remark 5, if \( \theta \geq 1 \), then we can write \( \rho_1 \geq M_6/\sqrt{d} \), where \( M_6 \) is a positive constant independent on \( d \). Combining with Lemma 2.5, we obtain

\[ y'_m - y_m \leq \frac{2\sigma\rho}{M_6}d \quad \text{for all } m \in \mathbb{N}. \]

Thus we conclude the claim.

**Proof of Claim 2.**

We prove by contradiction. Namely we assume that there exists a subsequence \( \{m_j\}_{j \in \mathbb{N}} \) such that \( y'_j \rightarrow \infty \) as \( j \rightarrow \infty \). For simplicity, we write \( j = m_j \). From Claim 1, \( y_j \rightarrow \infty \) as \( j \rightarrow \infty \). By the definition of \( J(\psi_j) \) and \( \tilde{J}(\psi_j) \), we have

\[ \tilde{J}(\psi_j) = J(\psi_j) - \int_{\mathbb{R}} (1 - \mu(x))W(u_j) \, dx. \]

Since \( \tilde{J}(\psi_j) - J(\psi_j) \leq \sigma + \epsilon_2 - \sigma_0 \leq -\epsilon_2 \), we have

\[ -\epsilon_2 \geq -\int_{-\infty}^{y_j} (1 - \mu(x))W(u_j) \, dx - \int_{y_j}^{\infty} (1 - \mu(x))W(u_j) \, dx. \quad (34) \]

\( \|u_j\|_{L^\infty} \) is uniformly bounded from Lemma 2.6, and \( (1 - \mu) \in L^1(\mathbb{R}) \). Thus we have

\[ \int_{y_j}^{\infty} (1 - \mu(x))W(u_j) \, dx < \frac{\epsilon_2}{4} \quad \text{for sufficiently large } j \in \mathbb{N}. \]

Moreover, we can see

\[ \int_{-\infty}^{y_j} (1 - \mu(x))W(u_j) \, dx \leq W \left( \frac{3}{2}\rho \right) \int_{\mathbb{R}} (1 - \mu(x)) \, dx < \frac{\epsilon_2}{4} \]

by \( \rho \in (0, \rho_0) \). As a result, we have \( -\epsilon_2 > -\epsilon_2/2 \), but it is clearly contradiction. Thus it never happens that \( y'_j \rightarrow \infty \) as \( j \rightarrow \infty \). We can also prove that it never happens that \( y_j \rightarrow -\infty \) as \( j \rightarrow \infty \) in the same way.

In the case \( \text{supp} \ (1 - \mu) \subset [x_1, x_2] \), we suppose that \( x_2 < y_m \). Then we can see

\[ -\epsilon_2 \geq -\int_{-\infty}^{y_m} (1 - \mu(x))W(u_m) \, dx > -\frac{\epsilon_2}{4} \]

in the same way as (34). Clearly it is contradiction. Thus we arrive at the claim.

**Proof of Claim 3.** Letting \( k_1 = -M_5, k_2 = M_5 \) and \( K = [k_1, k_2] \), we can see that there exists a constant \( C = C(K) \) such that

\[ ||\psi_m||_{H^1(K)} + \|L\psi_m\|_{H^1(K)} < C \]

from Lemma 2.7. By taking \( k_1 \) smaller and \( k_2 \) larger, if necessary, we may assume \( k_1 \leq 0 \) and \( k_2 \geq 1 \). Thus we have

\[ |u_m(x) - a| < \frac{3}{2}\rho \quad \text{for all } x \in (k_2, \infty), \quad |u_m(x)| < \frac{3}{2}\rho \quad \text{for all } x \in (-\infty, k_1). \]
In addition, since \( \rho < a/8 \), we can easily see
\[
|u_m(x)| > \frac{a}{2} \quad \text{for all } x \in (k_2, \infty), \quad |u_m(x) - a| > \frac{a}{2} \quad \text{for all } x \in (-\infty, k_1).
\]
Hence we obtain
\[
\int_{k_2}^{\infty} \psi_m^2 \, dx = \int_{k_2}^{\infty} (u_m - a)^2 \, dx = \int_{k_2}^{\infty} \frac{4}{a^2} u_m^2 (u_m - a)^2 \, dx
\]
\[
< \frac{16}{a^2} \int_{k_2}^{\infty} u_m^2 (u_m - a)^2 \, dx < \frac{16(\sigma + \epsilon_0)}{a^2}
\]
for all \( m \in \mathbb{N} \). Similarly, we also have
\[
\int_{-\infty}^{k_1} \psi_m^2 \, dx < \frac{16(\sigma + \epsilon_0)}{a^2}
\]
for all \( m \in \mathbb{N} \). Moreover it is clear that
\[
\|\psi_m'\|_{L^2(\mathbb{R})} \leq \sqrt{\frac{2}{\beta}} (\sigma + \epsilon_0) + \|\tilde{u}'\|_{L^2(\mathbb{R})}
\]
for all \( m \in \mathbb{N} \). As a consequence, it follows that \( \{\psi_m\}_{m \in \mathbb{N}} \) is bounded in \( H^1(\mathbb{R}) \) and there exists \( \tilde{\psi} \in H^1(\mathbb{R}) \) such that
\[
\psi_m \to \tilde{\psi} \quad \text{weakly in } H^1(\mathbb{R}) \quad \text{and strongly in } C_{\text{loc}}(\mathbb{R}) \quad \text{as } m \to \infty.
\]
In addition, it easy to check that
\[
\mathcal{L}\psi_m \to \mathcal{L}\tilde{\psi} \quad \text{weakly in } H^1(\mathbb{R}) \quad \text{and strongly in } C_{\text{loc}}(\mathbb{R}) \quad \text{as } m \to \infty.
\]
Now we prove \( \hat{J}(\tilde{\psi}) \leq \sigma \). From (35), we can easily see
\[
\liminf_{m \to \infty} \int_{\mathbb{R}} (u_m')^2 \, dx \geq \int_{\mathbb{R}} (\tilde{u}')^2 \, dx
\]
by lower semicontinuity and we can also see
\[
\liminf_{m \to \infty} \int_{\mathbb{R}} u_m^2 (u_m - a)^2 \, dx \geq \int_{\mathbb{R}} \tilde{u}^2 (\tilde{u} - a)^2 \, dx
\]
by Fatou’s lemma. Moreover, we have
\[
v_m - \frac{u_m}{\gamma} \to \tilde{v} - \frac{\tilde{u}}{\gamma} \quad \text{weakly in } H^1(\mathbb{R})
\]
in view of (35) and (36) and thus we obtain the following inequalities:
\[
\liminf_{m \to \infty} \int_{\mathbb{R}} \left( v_m - \frac{u_m}{\gamma} \right)^2 \, dx \geq \int_{\mathbb{R}} \left( \tilde{v} - \frac{\tilde{u}}{\gamma} \right)^2 \, dx,
\]
\[
\liminf_{m \to \infty} \int_{\mathbb{R}} \left( v_m' - \frac{u_m'}{\gamma} \right)^2 \, dx \geq \int_{\mathbb{R}} \left( \tilde{v}' - \frac{\tilde{u}'}{\gamma} \right)^2 \, dx.
\]
As a consequence, we conclude
\[
\hat{J}(\tilde{\psi}) \leq \liminf_{m \to \infty} \hat{J}(\psi_m) = \sigma.
\]
Let \( \tilde{u} = \tilde{\psi} + \tilde{v}, \tilde{v} = \tilde{\psi} + \mathcal{L}\tilde{\psi}, \) and we can see that \( (\tilde{u}, \tilde{v}) \) satisfy (8) by Lemma 2.1. we can also see that \( \tilde{u}(x) \to 0 \) as \( x \to -\infty \) and \( \tilde{u}(x) \to a \) as \( x \to \infty \) since \( \tilde{\psi} \in H^1(\mathbb{R}) \), moreover, we obtain \( |\tilde{y}' - \hat{y}'| \leq M_4 \) from (31), where \( \tilde{y}, \tilde{y}' \) is defined by (9) and (10). From Claim 1, it is clear that \( |\tilde{y}' - \hat{y}'| < Md \). We complete the proof. \( \square \)
5. Proof of Theorem 1.2. We give a proof of Theorem 1.2 in this section. Basically we can prove it as in the proof of Theorem 1.1, but we slightly modify the arguments. Throughout this section, we always assume (4), that is, \( a = 2(\beta + 1)/3 \) and \( \gamma = 9/(2\beta^2 - 5\beta + 4) \).

First, we derive the Euler-Lagrange equation.

Lemma 5.1. If (16) has a minimizer \( \psi \), then \((u, v) = (\hat{u} + \psi, \hat{v} + \mathcal{L}\psi)\) satisfies (13).

Proof. Let \( \phi \in C_c^\infty(\mathbb{R}) \) and \( j[\phi]: \mathbb{R} \to \mathbb{R} \) be
\[
\tilde{j}[\phi](t) = J(\psi + t\phi).
\]
Then we calculate \((\tilde{j}[\phi])'(0)\) as in Lemma 2.1, and we obtain
\[
(\tilde{j}[\phi])'(0) = \int_\mathbb{R} du' \phi' + v\phi - \frac{u\phi}{\gamma} + \mu(x)u \left( u - \frac{a}{2} \right) (u - a)\phi + \frac{1 - \mu(x)}{\gamma} u\phi \, dx.
\]
Under the assumption (4), we can easily see
\[
u \left( u - \frac{a}{2} \right) (u - a) = -f(u) + \frac{u}{\gamma}.
\]
Thus we have
\[
0 = (\tilde{j}[\phi])'(0) = \int_\mathbb{R} du' \phi' + v\phi - \mu(x)f(u)\phi \, dx.
\]
Repeating almost the same argument as in Lemma 2.1, we can prove that \((u, v)\) is a solution to (13). \(\square\)

Next we reveal the relationship between \( \sigma_0 \) and \( \hat{\sigma} \).

Lemma 5.2. \( \sigma_0 > \hat{\sigma} \) holds, where \( \sigma_0, \hat{\sigma} \) are defined as (3) and (16).

Proof. Let \( \psi_0 \in H^1(\mathbb{R}) \) be a minimizer of (3), \( u_0 = \hat{u} + \psi_0 \) and \( v_0 = \hat{v} + \mathcal{L}\psi_0 \). Then we have
\[
\sigma_0 - \hat{\sigma} \geq J(\psi_0) - \tilde{J}(\psi_0) = \int_\mathbb{R} (1 - \mu(x)) \left( W(u_0) - \frac{u_0^2}{2\gamma} \right) \, dx.
\]
Under the assumption \( a = 2(\beta + 1)/3 \) and \( \gamma = 9/(2\beta^2 - 5\beta + 4) \), we can see
\[
W(t) - \frac{t^2}{2\gamma} = -\int_0^t f(s) \, ds = F(t).
\]
Let \( L_0 \) be a positive constant satisfying (14). Then we may assume that \( u_0(L_0) = \beta \) and \( u_0(x) \leq \beta \) for all \( x \in (-\infty, L_0] \) since \( J(\psi) \) is invariant under translation of \( u \). Moreover, we define \( I_1 \equiv \{ x \in \mathbb{R} : x \leq x_2 \) and \( |u_0(x)| \geq \beta/2 \} \), and then we can easily see that \( |I_1| \geq \beta^2/8\sigma_0 \) with the Hölder’s inequality. From (\( \mu \beta \)), we obtain
\[
\sigma_0 - \sigma \geq \int_{-\infty}^{L_0} (1 - \mu(x)) F(u_0) \, dx + \int_{L_0}^{\infty} (1 - \mu(x)) F(u_0) \geq F \left( \frac{\beta}{2} \right) \int_{I_1} (1 - \mu(x)) \, dx - |F(1)| \int_{L_0}^{\infty} (1 - \mu(x)) \, dx > 0.
\]
Hence we conclude the lemma. \(\square\)
Lemma 5.4. Let \( \rho > 0 \) be a small constant. Then there exists a constant \( L_1 \geq 1 \) such that
\[
\int_{L_1}^{\infty} (1 - \mu(x)) \, dx < \delta.
\]
Now we set \( q = 2L_1 \) and define \( \tilde{G}_n \) as follows:
\[
\tilde{G}_n(\psi) = \int_{-\infty}^{\infty} \left[ \frac{\theta}{2} (u')^2 + \frac{1}{2} \left( v' - \frac{u'}{\gamma} \right)^2 + \frac{\gamma}{2} \left( v - \frac{u}{\gamma} \right)^2 + \mu(x)W(u) + \frac{1 - \mu(x)}{2\gamma} u^2 \right] \, dx,
\]
where \( u = \hat{u} + \psi \), \( v = \hat{v} + L\psi \). Then we obtain the next lemma. Since we can see it by repeating almost the same arguments, we omit the proof.

**Lemma 5.3.** Let \( \rho \) be a small positive number satisfying \( \rho \in (0, a/8) \) and let \( \rho_1 > 0 \) be a constant such that \( \rho_1 < \epsilon_1 = \sqrt{\theta \mu_1} / 2 \rho^2 (a - \rho) / 8 \). Moreover let \( \psi \in H^1(\mathbb{R}) \). If there exists \( n \in \mathbb{Z} \) such that
\[
\tilde{G}_n(\psi) < \rho_1,
\]
then there exists a constant \( \tilde{M}_2 = \tilde{M}_2(\theta, \rho, \mu_1, \gamma, q) \) and one of the following estimates holds:
\[
||u - a||_{H^1(I_n')} + ||v - \frac{a}{\gamma}||_{H^1(I_n')} \leq \tilde{M}_2 \sqrt{\rho_1},
\]
\[
||u||_{H^1(I_n')} + ||v||_{H^1(I_n')} \leq \tilde{M}_2 \sqrt{\rho_1},
\]
where \( I_n' = [nq - L_1, nq + L_1] \).

It is also easy to check the next lemma, and we omit the proof.

**Lemma 5.4.** Let \( \psi \in H^1(\mathbb{R}) \), \( u = \hat{u} + \psi \) and \( v = \hat{v} + L\psi \). We define \( \tilde{G}_n^{(0)} \) as follows:
\[
\tilde{G}_n^{(0)}(\psi) = \int_{-\infty}^{\infty} \left[ \frac{\theta}{2} (u')^2 + \frac{1}{2} \left( v' - \frac{u'}{\gamma} \right)^2 + \frac{\gamma}{2} \left( v - \frac{u}{\gamma} \right)^2 + W(u) \right] \, dx.
\]
Then we have the following statements.

(i) We assume that \( \rho \) be a small number satisfying \( \rho \in (0, a/8) \) and \( \rho_1 > 0 \) satisfies \( \rho_1 < \min\{\epsilon_1, 1\} \). We also assume that there is \( n \in \mathbb{N} \) satisfying (5.3). Then there exists a constant \( \tilde{M}_3 = \tilde{M}_3(\theta, \rho, \mu_1, \gamma, a, q) > 0 \) such that
\[
\tilde{G}_n^{(0)}(\tilde{S}_{nq-L_1}, \psi) < \tilde{M}_3 \rho_1. \tag{37}
\]

(ii) We assume that \( \rho \) be a small number satisfying \( \rho \in (0, a/8) \) and \( \rho_1 > 0 \) satisfies \( \rho_1 < \min\{\epsilon_1, 1\} \). We also assume that there exists \( n \in \mathbb{N} \) satisfying (38). Then there exists a constant \( \tilde{M}_3 = \tilde{M}_3(\theta, \rho, \mu_1, \gamma, a, q) > 0 \) such that
\[
\tilde{G}_n^{(0)}(\tilde{S}_{nq-L_1}, \psi) < \tilde{M}_3 \rho_1. \tag{38}
\]

The next lemma corresponds to Lemma 3.3. Although the statement is almost same, we need to modify a part of the proof because of the extra term \( \int_{\mathbb{R}} (1 - \mu)u^2 / (2\gamma) \, dx \).
Lemma 5.5. Let \( \rho \) be a small positive number satisfying \( \rho \in (0, a/8) \) and \( \rho_1 \) be a small positive number satisfying \( \rho_1 < \min\{\epsilon_1, 1, \epsilon_1/M_3\} \) and
\[
32\sqrt{2a\bar{M}_2}\sqrt{\rho_1} \int_{\mathbb{R}} \frac{1-\mu(x)}{2\gamma} \, dx < \epsilon_1.
\]
We assume \( \psi \in H^1(\mathbb{R}) \) and define \( u = \hat{u} + \psi \) and \( v = \hat{v} + L\psi \). Then the following statements hold.

(i) We suppose that \( u \) has a transition on \( [y_1, y_2] \) from \( a - \rho \) to \( a - 3\rho/2 \) or from \( a + \rho \) to \( a + 3\rho/2 \). Moreover we suppose that there exists \( j \in \mathbb{Z} \) such that \( j + 1 < y_1 \) and \( \psi \) satisfies (37) with \( n = j \). Then there exists \( \phi \in H^1(\mathbb{R}) \) satisfying
\[
\tilde{J}(\phi) < \tilde{J}(\psi) - \epsilon_1.
\]
In particular, if \( j \geq 1 \), then we may take \( \phi = S_{jq-L_1}\psi \).

(ii) We suppose that \( u \) has a transition on \( [y_1, y_2] \) from \( \rho \) to \( 3\rho/2 \) or from \( -\rho \) to \( -3\rho/2 \). Moreover we suppose that there exists \( j \in \mathbb{Z} \) such that \( j > y_2 \) and \( \psi \) satisfies (38) with \( n = j \). Then there exists \( \phi \in H^1(\mathbb{R}) \) satisfying
\[
\tilde{J}(\phi) < \tilde{J}(\psi) - \epsilon_1.
\]
In particular, if \( j \leq -1 \), then we may take \( \phi = \bar{S}_{jq-L_1}\psi \).

Proof. In the case \( j \geq 1 \), with paying attention to
\[
\sum_{n=j+1}^{\infty} \bar{G}_n(S_{jq-L_1}\psi) = \frac{a^2}{2\gamma} \int_{(j+1)q-L_1}^{\infty} (1-\mu(x)) \, dx < \epsilon_1,
\]
we can easily check
\[
\tilde{J}(S_{jq-L_1}\psi) < \tilde{J}(\psi) - 2\epsilon_1
\]
by repeating almost the same arguments as in Lemma 3.3.

Next, we consider the case \( j = 0 \). As in the proof if Lemma 3.3, we construct a function \( \phi \in H^1(\mathbb{R}) \) satisfying the following conditions:
\[
G_0^{(0)}(\phi) < \bar{M}_3\rho_1 < \epsilon_1, \tag{39}
\]
\[
\begin{aligned}
\bar{u}(x) &= \hat{u}(x) + \phi(x) = u(x), & x \leq x_1, \\
\bar{u}(x) &= a, & x \geq x_2,
\end{aligned}
\]
\[
\begin{aligned}
\bar{v}(x) &= \hat{u}(x) + \phi(x) = v(x), & x \leq x_1, \\
\bar{v}(x) &= a/\gamma & x \geq x_2.
\end{aligned}
\]
Then we obtain
\[
\tilde{J}(\phi) = \sum_{n=-\infty}^{-1} \bar{G}_n(\phi) + \bar{G}_0(\phi) + \sum_{n=1}^{\infty} \bar{G}_n(\phi).
\]
As in the proof of Lemma 3.3, we can easily see
\[
\sum_{n=-\infty}^{-1} \bar{G}_n(\phi) < \tilde{J}(\psi) - \bar{G}_0(\psi) - 4\epsilon_1.
\]
From the definition of \( \tilde{G}_n \) and \( \phi \), we obtain the following inequalities:

\[
- \tilde{G}_0(\psi) \leq - \int_{-L_1}^{L_1} (1 - \mu(x)) \frac{u^2}{2\gamma} \, dx,
\]

\[
\tilde{G}_0(\phi) \leq \tilde{G}_0(\phi) + \int_{-L_1}^{L_1} (1 - \mu(x)) \frac{\bar{u}^2}{2\gamma} \, dx < \epsilon_1 + \int_{-L_1}^{L_1} (1 - \mu(x)) \frac{\bar{u}^2}{2\gamma} \, dx,
\]

\[
\sum_{n=1}^{\infty} \tilde{G}_n(\phi) = \int_{L_1}^{\infty} \frac{1 - \mu(x)}{2\gamma} a^2 \, dx < \epsilon_1.
\]

Thus we have

\[
\tilde{J}(\phi) < \tilde{J}(\psi) - 2\epsilon_1 + \int_{-L_1}^{L_1} \frac{1 - \mu(x)}{2\gamma} |\bar{u}^2 - u^2| \, dx.
\]

Since \( u \) satisfies \( \|u - a\|_{H^1((I')^c)} < M_2\sqrt{\rho_1} \), we obtain

\[
\|u - a\|_{L^\infty((I')^c)} \leq 4\sqrt{2}(1 + 1/q)M_2\sqrt{\rho_1} \leq 8\sqrt{2}M_2\sqrt{\rho_1}
\]

by Sobolev inequality. Similarly from (39) and Lemma 5.3, we can see

\[
\|\bar{u} - a\|_{L^\infty((I')^c)} \leq 8\sqrt{2}M_2\sqrt{\rho_1}.
\]

Thus we have

\[
\int_{-L_1}^{L_1} \frac{1 - \mu(x)}{2\gamma} |\bar{u}^2 - u^2| \, dx < 32\sqrt{2aM_2\sqrt{\rho_1}} \int_{-L_1}^{L_1} \frac{1 - \mu(x)}{2\gamma} \, dx < \epsilon_1.
\]

As a consequence, it follows \( \tilde{J}(\phi) < \tilde{J}(\psi) - \epsilon_1 \).

Finally, we consider the case \( j \leq -1 \). Let \( \rho_2 \) be a small constant such that

\[
\max \left\{ W(\rho_2), \frac{\rho_2^2}{2\gamma} \right\} \int_{\mathbb{R}} (1 - \mu(x)) \, dx < \frac{\epsilon_1}{4}.
\]

Moreover, let \( L_2 > 0 \) be a large constant such that

\[
\frac{\|u\|_{L^\infty}^2}{2\gamma} \int_{L_2}^{\infty} (1 - \mu(x)) \, dx < \frac{\epsilon_1}{4}.
\]

We fix a sufficient large \( N \in \mathbb{N} \), and then we define \( U(x) \equiv u(x - Nq) \), \( \varphi(x) \equiv U(x) - \bar{u}(x) \) and \( V(x) \equiv \bar{v}(x) + L\varphi(x) \). We may assume the following conditions:

\[
N + j \geq 1,
\]

\[
U(x) < \rho_2 \quad \text{for all } x \in \max\{jq + L_1, L_2\},
\]

\[
\|W(u)\|_{L^\infty} \int_{j+1}^{\infty} (1 - \mu(x)) \, dx < \frac{\epsilon_1}{4}.
\]

By an easy calculation, we have

\[
\tilde{J}(\varphi) < \tilde{J}(\psi) + \int_{\mathbb{R}} \mu(x)(W(U) - W(u)) \, dx + \int_{\mathbb{R}} \frac{1 - \mu(x)}{2\gamma} U(x)^2 \, dx.
\]

From (40) and (42), we can see

\[
\int_{\mathbb{R}} \frac{1 - \mu(x)}{2\gamma} U^2 \, dx < \frac{\rho_2^2}{2\gamma} \int_{-\infty}^{L_2} (1 - \mu(x)) \, dx + \frac{\|u\|_{L^\infty}^2}{2\gamma} \int_{L_2}^{\infty} (1 - \mu(x)) \, dx < \frac{\epsilon_1}{2}.
\]

On the other hand, it is easy to check that

\[
\int_{\mathbb{R}} \mu(x)(W(U) - W(u)) \, dx = \int_{\mathbb{R}} W(U)(\mu(x) - \mu(x - Nq)) \, dx.
\]
by changing variables. Thus we obtain the following estimates:

\[
\int_{-\infty}^{\infty} W(U)(\mu(x) - \mu(x - Nq)) \, dx < W(\rho_2) \int_{\mathbb{R}} (1 - \mu(x - Nq)) \, dx < \frac{\epsilon_1}{4},
\]
\[
\int_{J q + L_1} W(U)(\mu(x) - \mu(x - Nq)) \, dx < \|W(u)\|_{L^\infty} \int_{(j+N)q+L_1}^{\infty} (1 - \mu(x)) \, dx < \frac{\epsilon_1}{4}.
\]

As a consequence, we have
\[
\bar{J}(\varphi) < \bar{J}(\psi) + \epsilon_1.
\]

Since \( \varphi \) satisfies (37) with \( n = N + j \geq 1 \), there exists \( \phi \in H^1(\mathbb{R}) \) such that \( \bar{J}(\phi) < \bar{J}(\varphi) - 2\epsilon_1 \). Hence we obtain \( \bar{J}(\phi) < \bar{J}(\psi) - \epsilon_1 \). \( \square \)

We have proved all lemmas to show Theorem 1.2. With these lemmas, we can prove the existence of a minimizer as in Theorem 1.1: First, the uniform boundedness of \( |y'_m - y_m| \), where \( y_m, y'_m \) are defined as (11) and (12); Second, the existence of a bounded interval \([-M_5, M_5]\) such that \( [y_m, y'_m] \subset [-M_5, M_5] \) for all \( m \in \mathbb{N} \); Finally, the existence of a minimizer. We omit the detail. Moreover, we have already proved \( \bar{\sigma} < \sigma_0 \) in Lemma 5.2. Thus we complete the proof of Theorem 1.2.

**Appendix A. Proof of Lemma 3.1.**

**Proof of Lemma 3.1.** From the assumption (3.1) and \( \rho_1 < \epsilon_1 \), it is easy to see

\[
\int_{I_n} \left\{ \frac{\theta}{2}(u')^2 + \frac{\mu(x)}{4}u^2(u - a)^2 \right\} \, dx < \epsilon_1.
\]

By Lemma 2.1, \( u \) has transition neither from \( \rho \) to \( 3\rho/2 \) nor from \( a - \rho \) to \( a - 3\rho/2 \) in \( I_n \). This implies that either \( u(x) > \rho \) for all \( x \in I_n \) or \( a - u(x) > \rho \) for all \( x \in I_n \) holds. Here we assume \( u(x) > \rho \) for all \( x \in I_n \). Thus we have

\[
\frac{\rho^2}{4} \int_{I_n} (u - a)^2 \, dx < \frac{1}{4} \int_{I_n} \mu(x)u^2(u - a)^2 \, dx < \rho_1
\]

from the assumption and we have

\[
\|u - a\|_{L^2(I_n)} < \frac{2}{\rho} \sqrt{\frac{\rho_1}{\mu_1}}.
\]

In addition, we easily see \( \|(u - a)'\|_{L^2(I_n)} < \sqrt{2}\rho_1/\theta \), and then we obtain

\[
\|u - a\|_{H^1(I_n)} < C_5\sqrt{\rho_1},
\]

where \( C_5 \) is defined by

\[
C_5 = C_5(\theta, \rho, \mu_1) = \sqrt{\frac{2}{\theta} + \frac{2}{\rho\sqrt{\mu_1}}}.
\]

Moreover, we have

\[
\int_{I_n} \left( v - \frac{a}{\gamma} \right)^2 \, dx \leq 2 \int_{I_n} \left( v - \frac{u}{\gamma} \right)^2 \, dx + \frac{2}{\gamma^2} \int_{I_n} (u - a)^2 \, dx \leq \frac{4}{\gamma} \rho_1 + \frac{2}{\gamma^2} C_5^2 \rho_1 \leq \left( \frac{2}{\sqrt{\gamma}} + \frac{\sqrt{2}}{\gamma} C_5 \right)^2 \rho_1.
\]


and we similarly obtain
\[
\int_{I_n} \left\{ \left( v - \frac{a}{\gamma} \right)' \right\}^2 \, dx \leq 2 \int_{I_n} \left\{ \left( v - \frac{u}{\gamma} \right)' \right\}^2 \, dx + \frac{2}{\gamma^2} \int_{I_n} \{(u-a)'\}^2 \, dx
\]
\[
\leq 4\rho_1 + \frac{2}{\gamma^2} C_5^2 \rho_1 \leq \left( 2 + \frac{\sqrt{2}}{\gamma} C_5 \right)^2 \rho_1.
\]
Thus letting
\[
C_6 = C_6(\theta, \rho, \mu_1, \gamma) = \frac{2}{\sqrt{\gamma}} + \frac{\sqrt{2}}{\gamma} C_5 + 2 + \frac{\sqrt{2}}{\gamma} C_5,
\]
we can see
\[
\| v - \frac{a}{\gamma} \|_{H^1(I_n)} \leq C_6 \sqrt{\rho_1}.
\]
Thus letting
\[
C_7 = C_7(\theta, \rho, \mu_1, \gamma) = C_5 + C_6,
\]
we estimate
\[
\| u - a \|_{H^1(I_n)} + \| v - \frac{a}{\gamma} \|_{H^2(I_n)} \leq C_7 \sqrt{\rho_1}
\]
under the assumption \( u(x) > \rho \) for all \( x \in I_n \). Moreover, from the definition of \( u, v \), we have
\[
-v' = -\frac{a}{\gamma} + (u - a).
\]
Multiplying the above equation by \( (v - a/\gamma)'' \) and integrating over \( I_n \), we have
\[
\int_{I_n} \left\{ (v - a/\gamma)'' \right\}^2 \, dx \leq \gamma \int_{I_n} \left\| v - \frac{a}{\gamma} \right\|_{L^2(I_n)}^2 + \| u - a \|_{L^2(I_n)} \left\| (v - a/\gamma)'' \right\|_{L^2(I_n)}
\]
by Hölder’s inequality. Hence we have
\[
\left\| (v - a/\gamma)'' \right\|_{L^2(I_n)} \leq (\gamma + 1) C_7 \sqrt{\rho_1}.
\]
In addition, we can see
\[
-v'' = -\frac{a}{\gamma} + (u - a)'
\]
and thus we similarly obtain
\[
\left\| (v - a/\gamma)''' \right\|_{L^2(I_n)} \leq (\gamma + 1) C_7 \sqrt{\rho_1}.
\]
As a consequence, letting \( M_2 = M_2(\theta, \rho, \mu, \gamma) = (3 + 2\gamma) C_7 \), we can estimate
\[
\| u - a \|_{H^1(I_n)} + \| v - \frac{a}{\gamma} \|_{H^2(I_n)} \leq M_2 \sqrt{\rho_1}
\]
under the assumption \( u(x) > \rho \) for all \( x \in I_n \). Similarly we also obtain
\[
\| u \|_{H^1(I_n)} + \| v \|_{H^2(I_n)} \leq M_2 \sqrt{\rho_1}
\]
under the assumption \( a - u(x) > \rho \) for all \( x \in I_n \).

Moreover, in view of the above proof, we can easily see that if \( \theta \geq 1 \), then we can take the constant \( M_2 > 0 \) which is independent on \( d > \theta \geq 1 \).
Acknowledgments. We would like to thank the referee for carefully reading our manuscript and for giving useful comments.

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Received January 2017; revised July 2017.

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