An aggregating strategy for shifting experts in
discrete sequence prediction

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Abstract

We study how we can adapt a predictor to a non-stationary environment with advices from multiple experts. We study the problem under complete feedback when the best expert changes over time from a decision theoretic point of view. Proposed algorithm is based on popular exponential weighing method with exponential discounting. We provide theoretical results bounding regret under the exponential discounting setting. Upper bound on regret is derived for finite time horizon problem. Numerical verification of different real life datasets are provided to show the utility of proposed algorithm.

I. INTRODUCTION

Online prediction is a widely studied topic with applications including user activity prediction, webpage requests prediction, location prediction in wireless mobile networks etc. It deals with the problem of predicting future symbols by observing an online stream of symbols in the sequence. Multiple approaches are proposed from diverse fields such as information theory (See [17], [31], [13]), machine learning (See [10], [5], [14], [2]) etc. One of the widely successful approach for online prediction is to assume Markov property while modeling the sequence. By assuming that a finite history of past observations will be enough for predicting the future symbols, Markovian models make predictions based on the observed history. This finite history of symbols is referred to as context and the length of context is the order of Markov model.

Creating a Markovian model based predictor comprises of two steps - building a frequency table for each context based on past observations and devising a method to make predictions from the table. A tree structure is one of the popular methods for storing the frequency counts. These frequency trees can be stored as special data structures known as tries (See [18]). In a trie, each node can represent a context from the frequency table and prediction is made based on the symbol distribution at each node. One of the main challenges while constructing the frequency tree is the loss of information at cross phrase boundaries. Rate of convergence to optimal predictability is another challenge during frequency tree construction. By updating the count at each node (context) based on the observed symbol, a trie structure can converge to the Markovian model of underlying sequence generator with enough data.
Once such a tree is built, prediction task merely becomes selecting the most probable symbol based on the context. Challenge lies in how to predict while the tree is still being built. A few notable works in this area include Prediction by Partial Matching (See [8], [7]), Context Tree Weighting (See [31]), Probabilistic Suffix Trees (See [28]), Compact Prediction Trees (See [22], [21]) etc.

[8] proposed Prediction by Partial Matching (PPM) initially for data compression as a method for encoding symbols in a sequence. Since predictability and compressibility are related (See [17]), PPM can also be used in predicting future symbols from discrete sequences. By combining predictions from multiple sub-contexts from the tree, PPM attempts to model the symbol probability as combination of multiple sub-contexts within the given context. [11] showed a successful application of PPM on frequency trie based on LeZi update for the task of inhabitant action prediction in a smart home environment. Since sequences are broken into sub-sequences and update of trie is performed based on these *phrases*, information about the relationship between the symbols at both ends of sub-sequences are lost. This loss at *cross-phrase boundary*, along with the complexity of sequence and number of symbols in the alphabet contribute together to the slow convergence of frequency tree. [20] proposed Active LeZi as a method to optimally predict the symbol without losing the information at cross-phrase boundaries. To address the problem of slow convergence, they used a sliding window over the phrases with variable length. For the prediction phase, Active LeZi employs PPM. One of the useful measures in analyzing convergence of a predictor is *FS Predictability*. Finite State (FS) Predictability of an infinite sequence is defined as the minimum fraction of prediction errors that can be made by any finite state predictor (See [17]). Active LeZi is able to build predictors which are able to converge to FS Predictability at the rate $O(\sqrt{n})$.

Another challenge in building frequency trees is to decide on the maximum depth of the tree. This is an important issue from a practical point of view, as it is not possible to enjoy an infinite memory requirement. [23] proposed a finite bounded length context tree approach and used distribution entropy for order estimation focusing on data compression. Even though the proposed method did not perform better than other PPM improvements, it showed that by properly bounding the context length, it is possible to obtain a performance comparable to other methods.

The success of Markovian predictors depends on how strong the assumption of stationarity in underlying generating process is. If the sequence is non-stationary, then the predictor needs to adapt. [27] proposed an adaptive model combining algorithm which is loosely based on Active LeZi and has empirically shown to achieve better results than conventional PPM methods. Following the work of [23], they employed a fixed length sliding window for constructing the trie, rather than using a variable length sliding window over cross phrase boundaries. PPM algorithm with multiple context lengths are ran on this trie and the algorithm weighs each model based on the past performance. These weights are later used to combine the predictions from each model. But no theoretical guarantees on the performance of predictor is provided.

Prediction by combining multiple experts is a long studied problem. Assuming that the decision maker (predictor) has complete knowledge about all the past decisions and performance of experts, the goal is to perform as good as the best expert in the pool. The method of using advices from multiple experts is introduced by [29] and is generalized as a strategies aggregating framework by [24]. [19] gave a decision theoretic generalization to the problem. By adopting a multiplicative update of weight parameter, they were able to produce algorithms performing
almost as well as the best expert in the pool, within a loss of \( O(\sqrt{N \ln K}) \), where \( N \) denotes time-steps and \( K \) is the number of experts in the pool.

Given a training set of sequences, \([16]\) describes a procedure for learning a set of experts that will work on online prediction. On this pool of experts, they run traditional setting of statistical learning to produce a model whose expected loss is as small as possible over an online sequence. They achieve this by first minimizing training loss on the experts and then minimizing hindsight loss in online prediction for the learning model. \([9]\) introduces a series of learning algorithms for designing accurate ensembles of structured prediction task. Their goal is, given a set of labeled training examples, exploit sub structures present in the problem domain to design experts and combine these experts to form an accurate ensemble. Here the experts are trained on a set of labeled samples and the ensemble algorithm is trained on a distinct set of samples. This ensemble is then used to predict labels for a given sequence of labels.

In both the cases above (\([16] \) and \([9]\) ), the models are first trained on a dataset that is considered to be uniformly sampled from the problem domain and then prediction is performed online. We are in search of methods which does not require pre-training as the aforementioned methods, but are able to combine multiple experts on an online manner to produce better results. Our goal considerably varies from the above methods as we want neither our experts nor the combining forecaster to be pre-trained. Our goal is to train the experts and the final forecaster online while they are expected to make predictions. This pose challenges of experiencing a greater loss during the initial stages of prediction.

By combining Mixing Past Posteriors (MPP) (see \([6]\) ) and AdaHedge (see \([12]\) ), \([30]\) proposed an online aggregation algorithm for the problem of shifting experts. By using the adaptive learning rate of AdaHedge, they modified MPP and obtained regret bounds of signed unbounded losses under adversarial setting. Empirical results provided show that the modified algorithm outperform AdaHedge in both synthetic and real data, even when the losses of experts are volatile.

Motivated from Decision Theoretic Online Learning view of combining multiple experts and Information Theoretic techniques for discrete sequence prediction, this paper propose a discrete sequence predictor that trains online and adaptively adjusts to the changes in model. By applying exponential filtering over the past performance of experts, we present a modified version of HEDGE algorithm. We also prove the convergence of the model to FS Predictability (under stationary assumptions) and obtain an upper bound on the regret of the algorithm.

Section II formally introduces the problem along with the mathematical notations used in this paper. In Section III, we discuss the method for constructing the pool of experts online and prove the optimal convergence rate. Section IV introduces proposed algorithm and in Section V we derive the rate of convergence to the best predictor. Experiments conducted to validate the proposed method are included in Section VI.

II. Problem Formulation

Let \( \mathcal{S} \) denotes the symbol space alphabet. We assume there exists a source which emits a symbol \( s[n] \in \mathcal{S} \) at discrete time instant \( n \). We want to create a predictor, who observes all the symbols emitted from the source till time instant \( n \), \( s_{1:n} = (s[1], s[2], \ldots, s[n]) \), and predicts the next symbol \( \hat{s}[n + 1] \). We also assume there exists a
rewarding mechanism which, after observing the actual symbol $s[n + 1]$ at time instant $n + 1$, will appropriately reward the predictor. In generalized online method for prediction, the predictor cannot be assumed to have the knowledge about the sequences and the symbols in the sequences. Hence, the predictor will only have information about the symbols it has seen so far. This subset of symbols from the alphabet constitutes the decision space for the predictor. Let $D_n$ denotes the decision space of the predictor at time instant $n$ with $D_n \subseteq S$. By construction, $D_n \triangleq \{s_{1:n}\}$ where $\{\cdot\}$ is the set operator which returns the unique members in the input.

Let $K$ denotes the set of experts available to the predictor with $|K| = K < \infty$, where $|\cdot|$ stands for cardinality of the set. The pool of predictors we consider in this paper are Markovian models of order $k$ with $k \in \{0, \ldots, K - 1\}$. Let $\hat{s}^{(k)}[n] \in D^{(k)}_n$ be the prediction from $k^{th}$ predictor for time instant $n$. Instantaneous loss of $k^{th}$ predictor is defined as $l^{(k)}[n] = \mathcal{L}(s[n], \hat{s}^{(k)}[n])$, $\mathcal{L}(\cdot)$ is the loss function. Define the cumulative loss incurred by $k^{th}$ predictor after $N$ time steps as

$$L_{k,N}(\gamma) = \sum_{n=1}^{N} \gamma^{N-n} l^{(k)}[n],$$

where $0 < \gamma \leq 1$ is the discounting factor.

The predictor maintains a probability distribution $p[n] \in [0, 1]^K$ over the pool of experts. Defining the instantaneous loss of predictor as $l[n] = \sum_{k=1}^{K} p^{(k)}[n] \cdot l^{(k)}[n]$, the discounted cumulative loss of predictor till time instant $N$ can be written as

$$L_N(\gamma) = \sum_{n=1}^{N} \gamma^{N-n} l[n] = \sum_{n=1}^{N} \sum_{k=1}^{K} \gamma^{N-n} p^{(k)}[n] \cdot l^{(k)}[n] = \sum_{n=1}^{N} \gamma^{N-n} p[n] \cdot l[n]$$

(2)

where $l[n]$ is the vector of instantaneous loss functions of all the experts at instant $n$.

Now, objective of the prediction algorithm to perform as good as the best expert in the pool can be represented as

$$\min_p \left\{ L_N(\gamma) - \min_k L_{k,N}(\gamma) \right\}$$

(3)

In this paper, we deal with a pool of predictors that share a common decision space which is based on only the observed symbols from a finite cardinality symbol space. Hence, $D^{(k)}_n = D_n \forall k \in K$. The experts we consider for this problem setup are finite context length PPM predictors which are Markovian predictors with different depth levels.

### III. Model Construction

This section introduces creating the pool of experts. These experts will be used in the second stage by the adaptive predictor to make final predictions. It is empirically shown by [23] that predictor will not incur a remarkable loss by bounding the depth. Following this observation, we create experts who are fixed context $K^{th}$-order Markov Models and then apply PPM approach to make predictions. Our trie building procedure is detailed in Algorithm 1. PPM is used to calculate the probability of each symbol from the model and the prediction is made as,

$$d^{(k)}[n] = \arg \max_d P(d|s[n - 1], s[n - 2], \ldots, s[n - k])$$

(4)

where $d$ is a symbol in the alphabet captured in trie.
**Algorithm 1 KOM(K)**

1. **Parameters:** Context Length, \( k \in I_0^+ \)
2. **Initialization:** Current Window, \( W \leftarrow \phi \)
3. for \( n = 1, 2, \ldots \) do
4. Predict next symbol \( d^{(k)}[n] \) based on \( (13) \)
5. Observe actual symbol \( s[n] \) and incur loss \( l^{(k)}[n] \)
6. \( W \leftarrow [W \ s[n]] \) (Append \( s[n] \) to \( W \))
7. if \( \text{length}(W) > k \) then
8. \( W \leftarrow W - w[1] \) (Remove first entry from \( W \))
9. end if
10. for \( i = 1, \ldots, K \) do
11. if context \( W[i : k] \) is available in trie then
12. Increment context count by 1
13. else
14. Insert new context to trie and set value to 1
15. end if
16. end for
17. end for

---

**Fig. 1:** Example tree constructed based on Algorithm 1 for sequence \( s = a, b, c, c, d, b, c, d, c, b, c, b, c \) with context length 2.
Example:

Assume a sequence \( s = a, b, c, c, d, b, c, d, c, b, c, b, c \). We consider a tree with depth 3; i.e., it can consider a context length of up to 2. Tree constructed based on Algorithm 1 is given in Fig 1. At each node, the letter inside the node denotes the symbol stored at that node and the numeric denotes the frequency of occurrence of that context. At root node, i.e., the node with zero context length, the frequency will be the sequence length - 13 in this example. At the end of this sequence, we have a context \( \{b, c\} \). Applying PPM, we can get the symbol prediction probabilities, \( P(\cdot|\cdot) \), as 
\[
P(a|bc) = \frac{1}{312}, \quad P(b|bc) = \frac{108}{312}, \quad P(c|bc) = \frac{97}{312} \text{ and } P(d|bc) = \frac{106}{312}.
\]
Using (4), we get
\[
\hat{d}^{(2)}[14] = b.
\]

A pool of experts is created as explained above in Algorithm 1 with values of \( k = 0, \ldots, K - 1 \) forming the set \( K \). Rather than maintaining different tries for each expert, all experts can co-exist in the largest depth trie - the trie with context length equal to \( K - 1 \). This helps to keep the memory requirement low and also satisfies our assumption of having a common decision space for all the experts.

**Theorem III.1.** Algorithm 1 attains Finite State (FS) convergence at the rate of \( O\left(\sqrt{\frac{1}{n}}\right) \).

**Proof.** A predictor can be defined by the pair \((f, g)\), where \( f \) is the next symbol prediction function and \( g(\cdot) \) is the next-state function. Let \( \pi(g; s^n_1) \) be the minimum fraction of prediction errors, \( s^n_1 \) be sequence. Also let \( S \) be the states in the predictor. Then finite state predictability is defined as [17], 
\[
\pi(x) = \lim_{\chi \to \infty} \lim_{n \to \infty} \sup_g \min \pi(g; s^n_1)
\]
This is the minimum fraction of error a predictor makes over the set of available next-state functions \( G_\chi \), when both the number of states \( \chi \) and the sequence length \( n \) tend to infinity. Let \( \hat{\pi}(g; s^n_1) \) be the expected fraction of errors over the randomization in \( f(\cdot) \). Then, by Theorem 1 of [17],
\[
\hat{\pi}(g; s^n_1) \leq \pi(g; s^n_1) + \frac{\chi}{n} \sqrt{\frac{n}{\chi}} + 1 + \frac{1}{2n}
\]
Thus, \( \hat{\pi}(g; s^n_1) \) approaches \( \pi(g; x^n_1) \) at least as fast as \( O\left(\frac{\chi}{n} \sqrt{\frac{n}{\chi}} + 1 + \frac{1}{2n}\right) \). 

In the case of Algorithm 1, the number of states is evolving and an upper bound on the number of states is \( |S^{k+1}| \) as \( n \to \infty \). Substituting this upper bound, we can get \( O\left(\sqrt{\frac{5}{n}}\right) = O\left(\sqrt{\frac{|S^{k+1}|}{n}}\right) = O\left(\sqrt{\frac{1}{n}}\right) \). Thus, we can conclude that Algorithm 1 converges to FS predictability at a rate of \( O\left(\frac{1}{\sqrt{n}}\right) \) under a stationary environment. This result assumes that the optimal order Markov Model exists in the frequency trie.

This result is consistent with the result derived in [10], where the authors show the results of Bayes predictors for which the expected proportion of errors of a Bayes predictor differ from the observed \( k^{th} \) order Markov structure by \( O(n^{-1/2}) \).

**IV. Adaptive Prediction**

In this section, we describe the algorithm for combining predictions from multiple experts. As there is no one single best expert for the whole time of prediction, the final predictor is required to adaptively combine the expert
predictions based on observed performance. We propose a modified version of HEDGE algorithm (See [19]) with a forgetting (discounting) factor to deal with this problem of shifting experts.

By maintaining a set of weights over the experts and doing multiplicative weight updation based on the observed loss, HEDGE is able to give more weight to best performing experts and almost zero weight to non-performing experts. The weight updation requires a factor $\beta \in (0, 1]$ to be set prior hand. By setting this parameter appropriately, HEDGE can achieve a upper bound of $O(\sqrt{N \ln K})$ for regret. These weights can be normalized to get the required weight distribution over the pool of experts.

To introduce an adaptive behaviour for combining the outputs from multiple predictors, Algorithm 2 includes a discounting factor $\gamma$, and hence give more importance to the experts which have been performing well in the recent past. We modify weight update to include this forgetting effect to the final predictor. The proposed algorithm is given in Algorithm 2 Discounted HEDGE with PPM requests prediction from each expert at every time instant and combine those based on the maintained distribution over experts. After actual symbol is observed, individual losses are calculated and weights are updated based on the performance of each expert.

Let $\hat{P}_{W_{-j}}$ be the escape proabability calculated by PPM for the symbols in context window from $i$ to $j$. Prediction from Algorithm 1 and from Algorithm 2 can be deduced to matrix operations as shown below.

$$
\begin{align*}
&\hat{p}(s_1) = \begin{pmatrix} \hat{P}_{s_1} & \hat{P}_{s_1} & \cdots & \hat{P}_{s_1} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{P}_{s_1} & \hat{P}_{s_1} & \cdots & \hat{P}_{s_1} \end{pmatrix} \\
&\hat{P}(s_2) = \begin{pmatrix} P_{s_2} & P_{s_2} & \cdots & P_{s_2} \\ \vdots & \vdots & \ddots & \vdots \\ P_{s_2} & P_{s_2} & \cdots & P_{s_2} \end{pmatrix} \\
&\hat{p}(s_3) = \begin{pmatrix} \hat{P}_{s_3} & \hat{P}_{s_3} & \cdots & \hat{P}_{s_3} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{P}_{s_3} & \hat{P}_{s_3} & \cdots & \hat{P}_{s_3} \end{pmatrix}
\end{align*}
$$

$$
\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} \hat{P}_{W_{-1},K-1} & \cdots & \hat{P}_{W_{-1},K-2} & \cdots & \hat{P}_{W_{-1},K-3} \cdots & \hat{P}_{W_{-1},K-1} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \hat{P}_{W_{-1},K-1} & \cdots & \hat{P}_{W_{-1},K-2} & \cdots & \hat{P}_{W_{-1},K-3} \cdots & \hat{P}_{W_{-1},K-1} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \hat{P}_{W_{-1},K-1} & \cdots & \hat{P}_{W_{-1},K-2} & \cdots & \hat{P}_{W_{-1},K-3} \cdots & \hat{P}_{W_{-1},K-1} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \end{pmatrix} \begin{pmatrix} \hat{p}(K) \\ \hat{p}(K-1) \\ \vdots \end{pmatrix}
$$

Algorithm 2 Discounted HEDGE with PPM

1: **Parameters:** $\beta \in (0, 1], \gamma \in (0, 1] K \in I^+$
2: **Initialization:** Set $w_k(1) = W > 0 \forall k$.
3: for $n = 1, 2, \ldots$ do
4: for $k = 1, \ldots, K$ do
5: $p^{(k)}[n] \leftarrow \frac{w_k[n]^{\gamma}}{\sum_{j=1}^{K} w_j[n]^{\gamma}}$
6: end for
7: Get symbol distribution from each expert $p_k[n]$
8: Calculate $\hat{P} = p[n] \cdot [p_1[n] \ldots p_K[n]]$
9: Predicted symbol $\hat{s}[n] = \arg\max_s \hat{P}$
10: Observe actual symbol $s[n]$
11: Calculate loss $l^k[n] \forall k \in \{1, \ldots, K\}$
12: Update weights as $w_k[n + 1] \leftarrow w_k[n]^{\gamma} \cdot \beta l^k[n]$
13: end for
V. REGRET UPPER BOUND

Next we derive regret upper bound for Algorithm 2. Our algorithm analysis follows the same method as in \[19\] but with two key observations from majorization theory.

First step is to relate the probability distributions with two different discounting exponents through majorization, as given below.

**Lemma V.1.** Let \( p^{(k)}(\gamma) = \frac{w_i^\gamma}{\sum_{j=1}^{N} w_j^\gamma} \) where \( \gamma \in (0,1] \), we have \( p^{(k)}(\gamma^M) \prec p^{(k)}(\gamma) \) \( \forall M \geq 1 \). That is \( p(\gamma^M) \) is majorized by \( p(\gamma) \).

**Proof.** Observing that \( \gamma \in (0,1] \) and \( \gamma^M \leq \gamma \) for all \( M \geq 1 \), this is a direct consequence of 5.B.2.b in \[25\].

**Lemma V.2.** Let the experts are ordered as \( p^{(1)}(\gamma) \leq p^{(2)}(\gamma) \leq \ldots \leq p^{(K)}(\gamma) \). If the instantaneous losses of the experts in same ordering obeys \( l^{(1)}(k) \geq l^{(2)}(k) \geq \ldots \geq l^{(K)}(k) \), then \( \sum_{k=1}^{K} p^{(k)}(\gamma^M) \cdot l^{(k)}(k) \geq \sum_{k=1}^{K} p^{(k)}(\gamma) \cdot l^{(k)}(k) \)

**Proof.** When \( p^{(1)}(\gamma) \leq p^{(2)}(\gamma) \leq \ldots \leq p^{(K)}(\gamma) \), we can directly observe that \( p^{(1)}(\gamma^M) \leq p^{(2)}(\gamma^M) \leq \ldots \leq p^{(K)}(\gamma^M) \). By Lemma V.1, \( p(\gamma^M) \prec p(\gamma) \). Hence we have,

\[
\sum_{i=1}^{K} p^{(i)}(\gamma^M) \geq \sum_{i=1}^{K} p^{(i)}(\gamma), \quad \forall k = 1, \ldots, K - 1 \quad \text{and} \quad \sum_{i=1}^{K} p^{(i)}(\gamma^M) = \sum_{i=1}^{K} p^{(i)}(\gamma).
\]

Let \( m_j \) be some non-negative numbers. Then consider the sequence on inequalities,

\[
m_1 \cdot p^{(1)}(\gamma^M) \geq m_1 \cdot p^{(1)}(\gamma)
\]

\[
m_2 \cdot \left( p^{(1)}(\gamma^M) + p^{(2)}(\gamma^M) \right) \geq m_2 \cdot \left( p^{(1)}(\gamma) + p^{(2)}(\gamma) \right)
\]

\[
\vdots
\]

\[
m_{K-1} \left( \sum_{k=1}^{K-1} p^{(k)}(\gamma^M) \right) \geq m_{K-1} \cdot \left( \sum_{k=1}^{K-1} p^{(k)}(\gamma) \right)
\]

\[
m_K \left( \sum_{k=1}^{K} p^{(k)}(\gamma^M) \right) = m_K \cdot \left( \sum_{k=1}^{K} p^{(k)}(\gamma) \right)
\]

Summing over all of them,

\[
\sum_{k=1}^{K} \left( \sum_{j=k}^{K} m_j \right) \cdot p^{(k)}(\gamma^M) \geq \sum_{k=1}^{K} \left( \sum_{j=k}^{K} m_j \right) \cdot p^{(k)}(\gamma)
\]

Taking \( \sum_{j=k}^{K} m_j = l^{(k)} \) and noting that \( \sum_{j=k}^{K} m_j \geq \sum_{j=k+1}^{K} m_j \), we have \( \sum_{k=1}^{K} l^{(k)} \cdot p^{(k)}(\gamma^M) \geq \sum_{k=1}^{K} l^{(k)} \cdot p^{(k)}(\gamma) \). This completes the proof.

**Lemma V.3.** Loss incurred by Discounted HEDGE algorithm can be upper bounded by the loss of best expert in pool as

\[
L_N(\gamma) \leq \frac{\ln(1/\beta)}{1 - \beta} L_{k^*,N}(\gamma) + \frac{\ln K}{1 - \beta}.
\]
Proof. Our proof is partly based on the analysis of HEDGE(\(\beta\)) (See [19]). But the method of discounting we have introduced to the HEDGE algorithm leads to certain technical difficulties in the proof which are addressed using Lemma V.1 and Lemma V.2. Due to space constraints, only the key steps of the proof is provided below. For complete version of the proof, refer supplementary material.

From Equation 1 and Step 12 of Algorithm 2 we get \(w_k[N + 1] = w_k[1] \gamma^N \cdot \beta^{L_k,N(\gamma)}\). Consider the sum of weights of all experts at time instant \(n + 1\).

\[
\sum_{k=1}^{K} w_k[n + 1] = \sum_{k=1}^{K} (w_k[n])^\gamma \cdot \beta^{l(k)[n]}.
\]  

(8)

Applying Bernoulli’s Inequality, we get

\[
\sum_{k=1}^{K} w_k[n + 1] \leq \sum_{k=1}^{K} (w_k[n])^\gamma - (1 - \beta) \sum_{k=1}^{K} (w_k[n])^\gamma \cdot l(k)[n]
\]

Noting that \(p^{(k)[n]} = \frac{w_k[n]}{\sum_{j=1}^{K} w_j[n]}\) and applying Lemma V.2 we can write

\[
\sum_{k=1}^{K} w_k[n + 1] \leq \left( \sum_{k=1}^{K} (w_k[n])^\gamma \right) \cdot (1 - (1 - \beta)p[n] \cdot l[n])
\]

Applying \(1 + x \leq \exp(x)\) and expanding the terms recursively,

\[
\sum_{k=1}^{K} w_k[N + 1] \leq \sum_{k=1}^{K} (w_k[1])^\gamma \prod_{n=1}^{N-1} \left( 1 - (1 - \beta) \cdot \left( \sum_{k=1}^{K} (w_k[n])^\gamma \frac{\gamma^{N-n+1}}{\sum_{j=1}^{K} (w_j[n])^\gamma} \cdot l(k)[n] \right) \right) \cdot \exp((-1 - \beta)p[N] \cdot l[N])
\]  

(9)

Because of the discounting that has been introduced to the HEDGE algorithm, we get terms of the form \(\frac{(w_k[n])^\gamma}{\sum_{j=1}^{K} (w_j[n])^\gamma} \gamma^{N-n+1}\), which cannot be readily used to calculate the expected loss of final predictor, as done in [19]. Without loss of generality, when the experts are arranged in ascending order of their weights, if their instantaneous losses follow a descending pattern then by Lemma V.2 we can write

\[
\sum_{k=1}^{K} w_k[N + 1] \leq \left( \sum_{k=1}^{K} w_k[N - 1] \right)^\gamma \exp(-\gamma(1 - \beta)p[N - 1] \cdot l[N - 1]) \exp((-1 - \beta)p[N] \cdot l[N])
\]

Combining the product terms and simplifying, we get

\[
\sum_{k=1}^{K} w_k[N + 1] = \left( \sum_{k=1}^{K} w_k[1] \right)^\gamma \exp((-1 - \beta) \cdot L_N(\gamma)) \quad (: \text{Eqn} 2)
\]

Taking logarithm on both sides and rearranging,

\[
L_N(\gamma) \leq - \frac{\ln \left( \sum_{k=1}^{K} w_k[N + 1] \right) - \ln \left( \sum_{k=1}^{K} w_k[1] \right) \gamma^N}{1 - \beta}
\]  

(10)

Let \(k^* = \arg \min_{k \in \mathcal{K}} L_{k,N}(\gamma)\) be the set index for the best expert in collection \(\mathcal{K}\). Then we have \(\sum_{k=1}^{K} w_k[N + 1] \geq L_{k^*,N}(\gamma) \ln(\beta) + \ln \left( w_{k^*}[1] \right) \gamma^N\). Applying this to (10) and by setting \(w_k[1] = W \forall k \in \mathcal{K}\) with \(W > 0\), we get

\[
L_N(\gamma) \leq \frac{\ln(1/\beta)}{1 - \beta} L_{k^*,N}(\gamma) + \frac{\ln K}{1 - \beta}
\]  

(11)
This completes the proof.

Theorem V.4. By optimally setting value of $\beta$, Net Loss of PPM-HEDGE algorithm can be bounded by $O(\sqrt{N \ln K})$.

Proof. From Lemma 4 in [19], Suppose $0 \leq L \leq \tilde{L}$ and $0 \leq R \leq \tilde{R}$ and $\beta = g(\tilde{L}/\tilde{R})$ where $g(z) = 1/(1 + \sqrt{2}/z)$.

\[
- \frac{L \ln(\beta) + R}{1 - \beta} \leq L + \sqrt{2 \tilde{L} \tilde{R}} + R
\]

Taking $L \leq L^*, N(\gamma)$ and $R = \ln(K)$, (11) can be rewritten as,

\[
L_N(\gamma) = L_{k^*, N}(\gamma) + \sqrt{2 \tilde{L} + \ln(K)}
\]

From (1), we get $L_{k, N} = \sum_{n=1}^{N} \gamma^{N-n} l(k)[n] \leq \sum_{n=1}^{N} \gamma^{N-n} = \sum_{n=0}^{N-1} \gamma^n = \frac{1-\gamma^N}{1-\gamma}$. This is the maximum value that the loss can take and hence, $\tilde{L} = \frac{1-\gamma^N}{1-\gamma}$. As a limiting case, when $\gamma = 1.00$, $\tilde{L} = \lim_{\gamma \to 1} \frac{1-\gamma^N}{1-\gamma} = N$. For a particular prediction task, $R$ is fixed and hence, $\tilde{R} = \ln(K)$. This will give the optimal value of $\beta$ as

\[
\beta = \begin{cases} 
\frac{1}{1 + \sqrt{2 \ln(K)} \cdot \frac{1-\gamma^N}{1-\gamma}} & ; \gamma \in [0, 1) \\
\frac{1}{1 + \sqrt{2 \ln(K)} / N} & ; \gamma = 1
\end{cases}
\]

Applying above results in (11), we get

\[
L_N(\gamma) \leq \begin{cases} 
L_{k^*, N}(\gamma) + \sqrt{2 \cdot \ln(K) \cdot \frac{1-\gamma^N}{1-\gamma}} + \ln(K) & ; \gamma \in [0, 1) \\
L_{k^*, N}(\gamma) + \sqrt{2 \cdot \ln(K) \cdot N} + \ln(K) & ; \gamma = 1
\end{cases}
\]

This completes the proof.

Corollary V.4.1. By setting $\gamma = 1$, Discounted HEDGE algorithm becomes HEDGE with no forgetting factor.

VI. EXPERIMENTAL RESULTS

To evaluate usefulness of the proposed method, this section provides comparison of the proposed algorithm with six other widely used and state-of-the-art algorithms - LZ78 (See [32]), Dependency Graphs (See [26]), LeZi Update (See [4]), Active LeZi (See [20]), Error Weighted PPM (See [27]) (referred as ewPPM in results) and Adaptive MPP (See [30]). We show the results on four different real life datasets - Reality Mining Dataset (RM) (See [15]), Building Activity (BA) (See [1]), Cognitive Assessment (CA) (See [1]) and a proprietary dataset of Modulation Scheme prediction (MCS) from a LTE cellular network comprising of 19 cells, 3 sector layout containing MCS values for 570 users corresponding to the rate feedback from the cellphones. In the results, dHedgePPM refers to the proposed method. For ewPPM, Adaptive MPP, and dHedge, we used a model built according to Algorithm 1 with $K = 4$ and hence, have 5 experts to predict with.
A. Loss Model for Experiments

In the derivation of the bound, the loss function is defined as $L(x, y) \in [0, 1]$, where $x$ is the observed symbol and $y$ is the predicted symbol. This enables us to use any loss function satisfying the above criteria and derive different bounds based on the applications. For the purpose of validating the claims presented above, and in order to prove the utility of the algorithm, we are using the following discrete loss function for the experiments mentioned below.

$$L(x, y) = \mathbb{I}_x \neq y$$

(15)

where $\mathbb{I}$ is the indicator function. Hence $L(x, y) \in \{0, 1\}$.

Even though we made some strict assumptions about the ordering of the weights of the experts and their corresponding instantaneous losses, our experiments show that the results hold even when these conditions are not always met. This shows the possibility of having a wider set of scenarios where the proposed analysis can hold.

B. Performance Comparison

In order to compare the accuracy evolution with time, we considered a prediction task on first 5000 symbols of each of the sequences in all the datasets except for MCS dataset. In MCS dataset, there are 210 sequences of 992 symbols each. For RM, BA, and CA datasets, we have 95, 9 and 350 sequences respectively. For the algorithms which require a context length to work with, we set it to 4 after making a few empirical observations. We provide the result of average accuracy over all the sequences in the dataset with respect to time. Figure 2 shows the results.

In each dataset, although different algorithms are performing better, the proposed method is able to match the performance of the best algorithm in all the datasets. Even though Active-LeZi is able to perform satisfactorily in RM, BA and CA, it suffers a severe hit in MCS dataset. In MCS dataset, DG eventually performs better. But in all cases, the proposed method can be seen performing nearly as good as the best performer. Another interesting observation is the performance of ewPPM; even though its performance is comparable to that of proposed method, it lacks the theoretical guarantees enjoyed by the proposed method.

Another interesting observation in the case of BA and MCS prediction dataset is the decline in prediction accuracy after some time. This is due to the increase in entropy of the underlying model as the time progresses. Particularly, the performance of Active-LeZi in MCS dataset is interesting - the accuracy reduces sharply as the time progresses. Even though we know that the entropy in the system increases, it will be worthwhile to examine this behavior to get better insights on how Active-LeZi works.

C. Comparison of execution time

All the algorithms are implemented on the programming language Julia (See [3]) and is ran on a Linux machine with Intel Core i5@2.90Ghz CPU and 16GB RAM. The time taken for the running each of the algorithms over the entire dataset is listed in Table I.

We can see that DG and LZ-Update are fast algorithms, but their performances are not consistent across different datasets. A notable observation is the runtime of LZ78; even though it is supposed to be low, the additional check during the trie building process contributes to a high observed runtime of LZ78.
### TABLE I: Execution Time for algorithms (in seconds)

| Algorithm            | RM  | CA  | BA  | MCS |
|----------------------|-----|-----|-----|-----|
| Active-LeZi          | 158.1 | 170.3 | 8.1  | 12.6 |
| Dependency Graph     | 8.9 | 21.4 | 1.1  | 1.6  |
| LZ78                 | 1422.6 | 4950.4 | 193.6 | 123.1 |
| LZ Update            | 43.6 | 68.9 | 3.4  | 5.3  |
| ewPPM                | 157.1 | 254.7 | 16.4 | 10.2 |
| Adaptive MPP         | 154.1 | 558.8 | 15.5 | 8.7  |
| Proposed Method      | 156.1 | 256.0 | 16.7 | 8.5  |

D. Comparison of memory requirements

For comparing the memory requirements, we provide the number of symbol nodes constructed by each of the algorithms. Each symbol node will hold space for the actual symbol and a number representing its support over its child nodes. The comparison is provided in Table I. As mentioned earlier, ewPPM, Adaptive MPP, and the proposed method use the same trie building strategy and hence, have the same memory requirements. We can observe that the proposed algorithm is not as efficient as LZ78 and DG in terms of memory requirement. But, the advantage is
TABLE II: Number of symbol nodes created

clearly visible in the prediction accuracy.

VII. CONCLUDING REMARKS

We introduced an algorithm for adaptively combining multiple experts in a non-stationary environment and applied it to the task of sequence prediction. We also derived an upper bound on the regret of its loss. Numerical verification is performed with six other widely used sequence prediction/aggregating strategies to prove the utility of the proposed algorithm. The main advantage is that the proposed method can be used as a drop in replacement for conventional PPM methods without changing other parts of the system, and yet offer an improvement in performance.

Even though we applied our adaptive expert combining method over PPM for the task of discrete sequence prediction, the method proposed in this work can be applied to a wide variety of problems that require combining opinions from multiple experts when the best expert keeps on changing. One of the potential applications could be to consider different algorithms like Context Tree Weighting, Probability Suffix Tree along with PPM to create a pool of experts and then use the proposed method to predict based on the knowledge acquired by all experts.

An interesting direction for further studies will be the derivation of tighter bounds as the experiments show that the bound proposed in this work can be improved. Prior knowledge about the accuracy of the experts in the pool might be the key to a tighter bound.

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A. Proof of Lemma 4

Proof. Our proof is partly based on the analysis of HEDGE(β) (See [19]). But the method of discounting we have introduced to the HEDGE algorithm leads to certain technical difficulties in the proof which are addressed using majorization theory. From Algorithm 2, we have

\[ w_k[N+1] = w_k[N]^{\gamma} \cdot \beta^{l(k)[N]} \]
\[ = (w_k[N-1] \cdot \beta^{l(k)[N-1]})^{\gamma} \cdot \beta^{l(k)[N]} \]
\[ = w_k[1]^{\gamma^N} \cdot \beta^{L_k,N(\gamma)} \] (From (Eqn.1)). (16)

Now consider the sum of weights of all experts at time instant \( n + 1 \).

\[ \sum_{k=1}^{K} w_k[n+1] = \sum_{k=1}^{K} (w_k[n])^{\gamma} \cdot \beta^{l(k)[n]} . \] (17)

By Bernoulli’s Inequality, for \( \beta \geq -1 \) and \( l(k)[n] \in [0, 1] \),

\[ \beta^{l(k)[n]} \leq 1 - (1 - \beta)l(k)[n]. \]

For Discounted HEDGE, we have \( \beta \in (0, 1] \). Applying this in (8), we have

\[ \sum_{k=1}^{K} w_k[n+1] \leq \sum_{k=1}^{K} (w_k[n])^{\gamma} \cdot \left(1 - (1 - \beta)l(k)[n]\right) \]
\[ = \sum_{k=1}^{K} (w_k[n])^{\gamma} - (1 - \beta) \sum_{k=1}^{K} (w_k[n])^{\gamma} \cdot l(k)[n] \]

From Algorithm 2, noting that \( p(k)[n] = \frac{(w_k[n])^{\gamma}}{\sum_{j=1}^{K} (w_j[n])^{\gamma}} \), we can write

\[ \sum_{k=1}^{K} w_k[n+1] \leq \sum_{k=1}^{K} (w_k[n])^{\gamma} - (1 - \beta) \left( \frac{K}{\sum_{j=1}^{K} (w_j[n])^{\gamma}} \right) \left( \sum_{k=1}^{K} p(k)[n] \cdot l(k)[n] \right) \]
\[ = \left( \sum_{k=1}^{K} (w_k[n])^{\gamma} \right) \left(1 - (1 - \beta) \left( \sum_{k=1}^{K} p(k)[n] \cdot l(k)[n] \right) \right) \]
\[ = \left( \sum_{k=1}^{K} (w_k[n])^{\gamma} \right) \cdot (1 - (1 - \beta)p[n] \cdot l[n]) \]
Applying $1 + x \leq \exp(x)$, we can write
\[
\sum_{k=1}^{K} w_k[N + 1] \leq \left( \sum_{k=1}^{K} (w_k[N])^{\gamma} \right) \exp \left( -(1 - \beta)p[N] \cdot I[N] \right)
\]
\[
= \left( \sum_{k=1}^{K} (w_k[N - 1])^{\gamma} \cdot \beta^{\gamma(k)[N-1]} \right) \exp \left( -(1 - \beta)p[N] \cdot I[N] \right)
\]
\[
\leq \sum_{k=1}^{K} (w_k[N - 1])^{\gamma} \left( 1 - (1 - \beta) \cdot \sum_{k=1}^{K} \frac{(w_k[N - 1])^{\gamma}}{\sum_{j=1}^{K} (w_j[N - 1])^{\gamma}} \cdot l(k)[N - 1] \right)
\]
\[
\cdot \exp \left( -(1 - \beta)p[N] \cdot I[N] \right)
\]
\[
\leq \sum_{k=1}^{K} (w_k[1])^{\gamma} \prod_{n=1}^{N-1} \left( 1 - (1 - \beta) \cdot \sum_{k=1}^{K} \frac{(w_k[n])^{\gamma}}{\sum_{j=1}^{K} (w_j[n])^{\gamma}} \cdot l(k)[n] \right)
\]
\[
\cdot \exp \left( -(1 - \beta)p[N] \cdot I[N] \right)
\]

Because of the discounting that has been introduced to the HEDGE algorithm, we get terms of the form
\[
\frac{(w_k[n])^{\gamma} N - n + 1}{\sum_{j=1}^{K} (w_j[n])^{\gamma} N - n + 1},
\]
which cannot be readily used to calculate the expected loss of final predictor, as done in [19]. However, if

\[
\sum_{k=1}^{K} \frac{(w_k[n])^{\gamma} N - n}{\sum_{j=1}^{K} (w_j[n])^{\gamma} N - n} \cdot l(k)[n] \geq \sum_{k=1}^{K} \frac{(w_k[n])^{\gamma}}{\sum_{j=1}^{K} (w_j[n])^{\gamma}} \cdot l(k)[n]
\]

we can upper bound the LHS in (9) with expression involving instantaneous probability terms. Hence if (19) holds, we will be able proceed in a manner similar to the analysis in [19]. We show that such an inequality does hold provided certain conditions are met. Without loss of generality, when the experts are arranged in ascending order of their weights, if their instantaneous losses follow a descending pattern then the above inequality holds. Refer to Appendix 2 for the proof. Now using (19), we can rewrite (9) as

\[
\sum_{k=1}^{K} w_k[N + 1] \leq \left( \sum_{k=1}^{K} w_k[N - 1]^{\gamma} \right) \exp \left( -\gamma(1 - \beta)p[N - 1] \cdot I[N - 1] \right) \exp \left( -(1 - \beta)p[N] \cdot I[N] \right)
\]

Combining the product terms and simplifying, we get

\[
\sum_{k=1}^{K} w_k[N + 1] \leq \left( \sum_{k=1}^{K} w_k[N - 1]^{\gamma} \right) \prod_{n=N-1}^{N} \exp \left( -(1 - \beta) \cdot \gamma^{N-n} p[n][I[n]] \right)
\]
\[
= \left( \sum_{k=1}^{K} w_k[N - 1]^{\gamma} \right) \left( (1 - \beta) \sum_{n=N-1}^{N} \gamma^{N-n} p[n][I[n]] \right)
\]
\[
\leq \left( \sum_{k=1}^{K} w_k[1]^{\gamma} \right) \exp \left( -(1 - \beta) \sum_{n=1}^{N} \gamma^{N-n} p[n][I[n]] \right)
\]
\[
= \left( \sum_{k=1}^{K} w_k[1]^{\gamma} \right) \exp \left( -(1 - \beta) \cdot L_N(\gamma) \right) \quad (\because \text{Eqn.1 from main paper})
\]
Taking logarithm on both sides,
\[
\ln \left( \sum_{k=1}^{K} w_k [N+1] \right) \leq -(1 - \beta) \cdot L_N(\gamma) + \ln \left( \sum_{k=1}^{K} w_k [1] \gamma^N \right)
\]

Rearranging,
\[
L_N(\gamma) \leq -\frac{\ln \left( \sum_{k=1}^{K} w_k [N+1] \right) - \ln \left( \sum_{k=1}^{K} w_k [1] \gamma^N \right)}{1 - \beta}
\] (20)

Let \( k^* = \arg\min_{k \in \mathcal{K}} L_{k,N}(\gamma) \) be the set index for the best expert in collection \( \mathcal{K} \). Then we have
\[
\sum_{k=1}^{K} w_k [N+1] = \sum_{k=1}^{K} w_k [1] \gamma^N \cdot \beta L_{k,N}(\gamma) \quad (\because \text{From (16)})
\]
\[
\geq \beta L_{k^*,N}(\gamma) \cdot w_{k^*}[1] \gamma^N
\]

Taking logarithm,
\[
\ln \left( \sum_{k=1}^{N} w_k [N+1] \right) \geq L_{k^*,N}(\gamma) \ln(\beta) + \ln \left( w_{k^*}[1] \gamma^N \right)
\]

Applying this to (10), we have
\[
L_N(\gamma) \leq -\frac{L_{k^*,N}(\gamma) \ln(\beta)}{1 - \beta} - \frac{\ln \left( w_{k^*}[1] \gamma^N \right) - \ln \left( \sum_{k=1}^{K} w_k [1] \gamma^N \right)}{1 - \beta}
\]
\[
= \frac{\ln(1/\beta)}{1 - \beta} L_{k^*,N}(\gamma) - \frac{1}{1 - \beta} \ln \left( \frac{w_{k^*}[1] \gamma^N}{\sum_{k=1}^{K} w_k [1] \gamma^N} \right)
\]

By setting \( w_k[1] = W \forall k \in \mathcal{K} \) with \( W > 0 \), we can observe that
\[
L_N(\gamma) \leq \frac{\ln(1/\beta)}{1 - \beta} L_{k^*,N}(\gamma) + \frac{\ln K}{1 - \beta}
\] (21)

\[\square\]

**B. Validity of Proposed Bound**

To verify the validity of the bound derived in Theorem 5, experiments are done in a synthetic dataset and the results are reported after averaging over multiple runs. To generate test sequences for this experiment a Markov model is created as follows. Sequence of User#0 from the Cognitive Assessment dataset of CASAS project (See [1]) is taken and first 5000 symbols are trained into a trie with context length 6. Then this trie is used to generate sequences which are in turn fed into Discounted HEDGE with PPM for the prediction task. This is repeated over 100 independent runs and the average results are reported.

Experiments are conducted in three different scenarios for four different values of \( \gamma \). In each experiment, we varied the context lengths available in the pool of experts. Since the synthetic data is coming from a model having context length 6, we tried with context length of 2, 6 and 10. This in turn results in 3, 7 and 11 experts in the pool respectively. In all the experiments, optimal value of \( \beta \) is calculated as mentioned in Eqn. 13 in main paper.
Results are shown in Fig. 3 - Fig. 6. Loss is normalized by dividing by $\hat{L}$ to facilitate a direct comparison results with different $\gamma$ values.

From the above results, we can observe that the proposed bound holds for in all the experimental scenarios. Loss are normalized to $\hat{L}$, i.e., the fraction of observed loss to the maximum attainable loss is plotted.
(a) No. of Experts = 3  
(b) No. of Experts = 7  
(c) No. of Experts = 11

Fig. 5: Loss of Predictor with $\gamma = 0.99$

Fig. 6: Loss of Predictor with $\gamma = 1.00$. This is equivalent to HEGDE.