Online Incentive-Compatible Mechanisms for Traffic Intersection Auctions

David Rey\textsuperscript{a,b,*}, Michael W. Levin\textsuperscript{c}, Vinayak V. Dixit\textsuperscript{a}

\textsuperscript{a}School of Civil and Environmental Engineering, UNSW Sydney, NSW, 2052, Australia
\textsuperscript{b}Univ. Gustave Eiffel, Univ. Lyon, IFSTTAR, ENTPE, LICIT UMR_T9401, Lyon, F-69675, France
\textsuperscript{c}Department of Civil, Environmental, and Geo-Engineering, University of Minnesota, Minneapolis, MN 55455, United States

Abstract

We present novel online mechanisms for traffic intersection auctions in which users bid for priority service. We assume that users at the front of their lane are requested to declare their delay cost, i.e. value of time, and that users are serviced in decreasing order of declared delay cost. Since users are expected to arrive dynamically at traffic intersections, static pricing approaches may fail to estimate user expected waiting time accurately, and lead to non-strategyproof payments. To address this gap, we propose two Markov chain models to determine the expected waiting time of participants in the auction. Both models take into account the probability of future arrivals at the intersection. In a first model, we assume that the probability of future arrivals is uniform across lanes of the intersection. This queue-based model only tracks the number of lower-and higher-bidding users on access lanes, and the number of empty lanes. The uniformness assumption is relaxed in a second, lane-based model which accounts for lane-specific user arrival probabilities at the expense of an extended state space. We then design a mechanism to determine incentive-compatible payments in the dynamic sense. The resulting online mechanisms maximize social welfare in the long run. Numerical experiments on a four-lane traffic intersection are reported and compared to a static incentive-compatible mechanism. Our findings show that static incentive-compatible mechanisms may lead users to misreport their delay costs. In turn, the proposed online mechanisms are shown to be incentive-compatible in the dynamic sense and computationally efficient.

Keywords: Auctions/bidding; incentive-compatibility; online mechanism design; Markov chain; traffic intersection

1. Introduction

Traffic intersections are major bottlenecks of urban transport networks. Signalized traffic intersections are typically controlled so as to improve throughput, minimize vehicle delays or reduce emissions. While these design criteria have merits they are oblivious to users’ preferences. To address this limitation, auction-based mechanisms have emerged as promising alternatives to traditional traffic intersection control approaches (Schepperle and Böhm, 2007; Vasirani and Ossowski, 2012; Carlino et al., 2013). In auction-based mechanisms, users are assumed to be able to declare their preferences, e.g. value of time, to an intersection manager so as to obtain services commensurate to their need. The intersection can be viewed as a server which role is to process users’ service requests. In this context, the intersection manager acts as a controller which decides users’ service sequence and users’ payments.

In this paper, we propose novel online auction-based traffic intersection mechanisms that allows users in the front of their lanes to bid for intersection access. We show that the proposed online mechanisms are incentive-compatible, i.e. that users cannot achieve higher utility by bidding untruthfully. Vehicles bid for individual vehicle access to the intersection, as opposed to actuating traffic signal phases which serve multiple vehicles. Such individualized control is compatible with, for example, individual-vehicle signal lighting systems (such as ramp meters) or autonomous intersection management which controls individual vehicle trajectories (e.g. Dresner and Stone, 2004; Levin and Rey, 2017). The resulting online mechanisms

\*Corresponding author

Email address: d.rey@unsw.edu.au (David Rey)
are strategyproof and thus incentivize users to declare their true value of time, which is critical to promote auction-based mechanisms in urban transport networks.

Intersection auctions have previously been explored in the context of autonomous intersection management due to the natural relationship between auction-based intersection controls and autonomous intersection management, which both decide intersection access for individual connected vehicles. We first discuss the literature on autonomous intersection management before discussing pricing mechanisms in queueing systems and their relevance to traffic intersection auctions. We then position our paper with respect to the field and highlight our contributions.

1.1. Autonomous intersection management

Due to the real-time computing needs of auction-based traffic intersection mechanisms, such approaches are typically conceived in the context of autonomous intersection management (AIM), as proposed in the seminal work of Dresner and Stone (2004, 2008). In this paradigm, traffic control is assumed to be signal-free and users are assumed to be able reserve space-time trajectories through intersections. Several works have built on and extended the AIM protocol to richer configurations. Fajardo et al. (2011) and Li et al. (2013) proposed AIM protocols based on a First-Come-First-Served (FCFS) policies where vehicles are prioritized based on their arrival time at the intersection. De La Fortelle and Qian (2015) and Alché and De La Fortelle (2016) developed microscopic vehicle trajectory optimization formulations to coordinate vehicles through intersections. In these formulations, the intersection manager decides vehicles’ service time and speed while time is discretized. Zhang et al. (2016, 2017) and Malikopoulos et al. (2018) proposed decentralized approaches for traffic control based on FCFS conditions and considered energy consumption as well as vehicle separation and throughput in their formulations. Mirheli et al. (2019) developed a consensus-based approach for cooperative trajectory planning at signal-free intersections and shows that near-optimal solution can be achieved in competitive time. Wu et al. (2019) designed a multi-agent Markov decision process for cooperative trajectory planning in AIM. Levin and Rey (2017) proposed a conflict point model that obviates the need to discretize the intersection space using tiles. This conflict point model was then adapted by Rey and Levin (2019) to accommodate both legacy and automated vehicles with signalized and signal-free traffic phases; and more recently by Chen et al. (2020) to account for pedestrians movements.

1.2. Pricing in queueing systems

Traffic intersections can be viewed as queueing systems where users wait for service, i.e., intersection access, to pursue their journey. In this regard, auction mechanisms in traffic intersections are analogous to pricing mechanisms in queueing systems, which have been extensively studied over the past few decades. Naor (1969) was the first to study the management of queues via user payments: he considered a queueing system where arriving users faced the choice of joining a queue with a fixed payment or refrain from joining the queue. He showed that user payments could often lead to solutions maximizing social welfare. Dolan (1978) examined a queueing system where users are serviced in a sequence determined by their declared delay costs with priority given to higher delay costs. He proposed a dynamic mechanism to determine incentive-compatible user payments. Mendelson and Whang (1990) proposed mechanisms for optimal incentive-compatibility pricing for the M/M/1 queue. The authors considered several user classes that differ by their delay costs and identify optimality conditions for net value maximization under elastic demand. In a similar context, Bradford (1996) studied incentive-compatibility in the context of multiserver queues. Further details on equilibrium behavior in queueing systems can be found in a review by Hassin and Haviv (2003). Kittsteiner and Moldovanu (2005) studied a priority queueing system where users have varying service (processing) times and where queue lengths are unobservable. Users are allowed to bid for service and are privately informed of their processing time. In this incomplete information context, they show that the shortest processing time (SPT) discipline is approximately efficient if the curvature of the cost function is relatively low.

1.3. Traffic intersection auctions

Traffic intersection auctions have received a growing attention over the past few years, notably with the advent of connected and automated vehicle technology. Schepperle and Böhm (2007) proposed a subsidy-based mechanism for slot allocation which aim to balance vehicle waiting time. First-In First-Out (FIFO) constraints are accounted for but incentive-compatibility is not guaranteed. The “effect of starvation” and
its impact on fairness are discussed. Vasirani and Ossowski (2012) developed a policy based on combinatorial
auctions for the allocation of reservations at traffic intersections. The auction winner pays a price that is
exactly the bid that was submitted, which is not incentive-compatible. Carlino et al. (2013) proposed several
types of auctions where users can bid on phases or reservations (one vehicle at a time). All drivers can
participate in the auction but the candidates only include drivers at the front of their lane. Winners split
the cost of the second-highest bid with proportional payment, which yields a static incentive-compatible
mechanism and the strategyproofness of the dynamic case is not guaranteed. Sayin et al. (2018) proposed an
auction protocol and mechanism in which all vehicles within range of the intersection manager communicate
their value of time, and the intersection manager assigns reservations to those vehicles via a static auction.
The proposed mechanism is incentive-compatible in the static sense, i.e. assuming all participants in the
auction are known at the time payments are determined. Censi et al. (2019) introduced a credit-based auction
mechanism, i.e. a karma system in which agents pay other agents karma for priority. Agents with a low
priority today have an incentive to lose bids so they can achieve higher priority tomorrow by acquiring more
karma. The authors attempt to calculate a Nash equilibria of user bids, and they show that a centralized
strategy can be more unfair than some of the Nash equilibria identified. Recently, Lin and Jabari (2020)
proposed a mechanism for pricing intersection priority based on transferable utility games. In each game,
players have the possibility to trade time among themselves and “winners pay losers to gain priority”. The
authors provide empirical evidence that their approach is robust against adversarial user behavior, but no
formal proof of incentive-compatibility is presented.

1.4. Our contributions

As highlighted in our review of the literature, existing payment mechanisms for traffic intersection auc-
tions are either only incentive-compatible for static auctions, or dynamic but not incentive-compatible. Static
auctions do not account for future arrivals when determining user payments. As we will show in our nu-
merical experiments, static payment mechanisms may incentivize users to misreport their preferences by
bidding lower than their true delay costs and obtaining a higher utility, thus potentially compromising the
acceptability of the corresponding mechanisms. Since users are expected to arrive dynamically at traffic
intersections, online incentive-compatible mechanisms are required to ensure truthful user behavior in the
long run.

We make the following contributions to the field. We propose two online payment mechanisms for traffic
intersection auctions. The two proposed mechanisms use Markov Chains (MC) to model the expected
waiting time of users in the system but differ in the way the expected waiting time of users is determined.
First, a queue-based model which assumes uniform arrival rates across the intersection is proposed and
shown to be computationally inexpensive. This approach is then extended to a lane-based model which is
able to capture non-uniform arrival rates across the lanes of the intersection. We show that both online
mechanisms are incentive-compatible in the dynamic sense. We conduct numerical experiments to explore
the behavior of the proposed online mechanisms and quantify the benefits of both queue-based and lane-
based mechanisms. Finally, our simulations highlight the limitations of static incentive-compatible payment
mechanisms compared to the proposed dynamic approaches.

We introduce the traffic intersection auction in Section 2 and present the proposed online mechanisms in
Section 3. Numerical experiments are reported in Section 4 and conclusions are summarized in Section 5.

2. Dynamic traffic intersection auction framework

We consider a traffic intersection with a finite number of access lanes. We consider discrete time and we
assume that at each time period, users arrive on access lanes with a known probability. Upon arriving at
the front of their lane, users are requested to declare their delay cost which will be used as their bid in a
combinatorial auction. We assume that users seek to minimize a generalized cost function which is a linear
combination of their expected waiting time and their payment to the intersection manager. We assume that
users are serviced sequentially by a single server, i.e. the traffic intersection and that service is nonpreemp-
tive, i.e. cannot be interrupted once started. This does not prevent several users to simultaneously traverse
the intersection. Instead, the proposed traffic intersection auction only aims to determine the sequence in
which users are to be serviced and the payments they should be charged based on their declared preferences.
The goal of the intersection manager is to maximize social welfare which is defined as the total generalized user cost. To achieve optimal social welfare, we seek to determine incentive-compatible user payments to ensure the truthful declaration of their delay costs. To ensure truthful user behavior, traffic intersection lanes are serviced in order of decreasing declared user delay costs. The expected waiting time of users is a function of the declared delay costs and the state of the system. In turn, the payment of users is to be determined by the intersection manager so as to maximize social welfare and ensure truthfulness.

To present the proposed online mechanism for determining incentive-compatible payments, we consider the case of a user \(i\) arriving at the front of its lane and declaring a delay cost of \(v_i\). Let \(Q\) be the number of access lanes of the intersection. We denote \(q_i\) (resp. \(q_i^\emptyset\)) the number of lanes which are occupied by users with greater (resp. lower) delay cost than \(i\). Further, we denote \(q_0\) the number of empty lanes. The following equation holds for any user \(i\) at any time period:

\[
Q = q_i + q_i^\emptyset + q_0 + 1. \tag{1}
\]

Accordingly, we can define the pricing queue of the proposed dynamic traffic intersection auction.

**Definition 1** (Pricing queue). In a traffic intersection with \(Q\) access lanes, the pricing queue is a dynamic set of at most \(Q\) users which represents the users at the front of their lane at a given time period.

The pricing queue consists of the set of users which have no users in front of them in their access lane to the intersection. In the proposed traffic intersection auction, only users in the pricing queue are participants. This implies that users who are behind other users are not eligible to participate until they reach the front of their lane. This auction design aims to obviate blocking effects induced by non-overtaking conditions in access lanes of the intersection. Under such typical, non-overtaking conditions users waiting behind other users in their lane cannot bypass users in front of them, thus preventing them to be serviced before reaching the front of their lane. In the proposed approach, only users in the pricing queue participate in the auction. While this dynamic auction framework restricts the number of participants to at most \(Q\) users, every user queuing in the intersection will eventually reach the front of its lane and participate in a single instance of the auction. This is in contrast to approaches in the literature where auction “losers” may participate in several auctions. We refer readers to Carlino et al. (2013) who presented pricing mechanisms for traffic intersection auctions in which blocked users are allowed to “vote” for their lane.

The objective of the proposed traffic intersection auction is to determine incentive-compatible user payments upon users joining the pricing queue. Let \(W_i(v_i)\) be the expected waiting time of user \(i\) declaring a delay cost of \(v_i\). Let \(v_i^\star\) be the true delay cost of user \(i\), the expected waiting time cost of user \(i\) declaring a delay cost of \(v_i\) is \(v_i^\star W_i(v_i)\). Let \(P_i(v_i)\) be the payment of user \(i\) declaring \(v_i\), which is to be determined by the intersection manager. We denote \(C_i(v_i)\) the generalized cost for user \(i\) and define the user objective function as:

**Definition 2** (User objective function). The user objective function of user \(i\) is:

\[
\min_{v_i} C_i(v_i) = \min_{v_i} (v_i^\star W_i(v_i) + P_i(v_i)). \tag{2}
\]

Following the approach of Dolan (1978), we will show that setting the payment \(P_i(v_i)\) equal to the expected marginal delay cost user \(i\) imposes to other users is incentive-compatible, i.e. the user objective is minimal for \(v_i = v_i^\star\).

We assume that the intersection manager has knowledge of the probability distribution of users’ valuation (delay costs), and of the arrival probability of users on access lanes of the intersection. These assumptions are reasonable since one can assume that the intersection manager can observe user behavior (arrival rate and valuations) and learn these probabilities over time. Let \(F(v_i)\) be the cumulative distribution function representing the probability that a user declares a delay cost \(v \leq v_i\). Further, let \(v^\ast\) and \(v_{\ast}\) be lower and upper bounds on users’ delay costs. For conciseness, we hereby use the term lower-bidding user (resp. higher-bidding user) to refer to a user with a declared delay cost lower (resp. higher) than that of user \(i\).

We next introduce the proposed online mechanisms for traffic intersection auctions.
3. Online incentive-compatible mechanisms

We propose two online mechanisms to determine the expected waiting time and payment of users. The proposed mechanisms differ in the way the expected waiting time of users is determined. We first consider a queue-based model which only tracks the number of users bidding higher or lower than the reference user (Section 3.1). We next extend this model to a lane-based approach which tracks users’ arrival lane (Section 3.2). A generic payment mechanism is proposed to determine incentive-compatible payments for both queue- and lane-based models which result in two online mechanisms (Section 3.3). The queue-based mechanism assumes that the arrival probability of users is uniform across lanes of the intersection and provides a relatively low-dimensional mathematical framework. This uniformness assumption is relaxed in the lane-based mechanism which can accommodate non-uniform lane arrival probabilities at the expense of an extended state space.

3.1. Queue-based model

We use a MC to determine the expected waiting time of user \( i \) under different system states. The state of the queuing system w.r.t. user \( i \) is represented by the number of lanes occupied by lower bidding vehicles \( q_i \), the number of empty lanes \( q_0 \). We denote \( q_i \) the vector \( q_i = (q_i, q_0) \) characterizing the state of the system w.r.t. user \( i \). User \( i \) is serviced when all lanes are either occupied by a lower-bidding user or empty. Observe that once a lane is occupied by a lower-bidding user w.r.t. user \( i \), it remains in this state until \( i \) is serviced. Hence, from the perspective of user \( i \), the state space of the MC can be characterized by the inequality \( q_i + q_0 \leq Q - 1 \). This leads to the following characterization of the state space.

**Definition 3** (State space in the queue-based model). In the queue-based model, the state space of the MC for user \( i \) declaring a delay cost of \( v_i \) in a pricing queue of size \( Q \) is:

\[
S^q_i = \left\{ q_i = (q_i, q_0) \in \mathbb{N}^2 : q_i + q_0 \leq Q - 1 \right\}.
\]

For each state \( q_i \in S^q_i \), we are interested in determining the cost-to-go from this state through the MC. If \( q_i + q_0 = Q - 1 \), then (1) yields \( \bar{q}_i = 0 \) which is equivalent to require that the pricing queue does not contain any user with declared delay costs higher than that of \( i \). Hence, if \( q_i + q_0 = Q - 1 \), then the MC converges and user \( i \) is serviced. Hence, to determine the state transition probabilities from state \( q_i \) to state \( q'_i \), we assume that \( q_i + q_0 < Q - 1 \).

To calculate the expected waiting time of user \( i \) in the queue-based model, we enumerate system states with varying number of lower-bidding and empty lanes w.r.t. user \( i \). We assume that users’ delay cost can be represented by a continuous random variable with a known distribution. Further, we assume that the arrival of users on intersection lanes can be represented by a boolean random variable. Let \( \xi \in \mathbb{R}^Q \times 2^Q \) represent the uncertain data in the system. Let \( f_q : S^q_i \times \mathbb{R}^Q \times 2^Q \to S^q_i \) be the transition function of the MC which maps the current state and the uncertain data to the next state. We denote \( q'_{i,q_i} \) the one-step-cost of the MC representing the service time of the system in state \( q_i \) and we assume that the one-step-cost is deterministic. This assumption is plausible since users’ behavior is assumed to be deterministic, i.e. users cannot change their declared delay cost over time.

Let \( W_q(v_i, q_i) \) be the expected waiting time of user \( i \) declaring \( v_i \) for state \( q_i \), which represents the cost-to-go of the MC representing the state evolution in the queue-based model. Let \( T^q_i = \{ q_i \in S^q_i : q_i + q_0 = Q - 1 \} \) be the set of terminal states in the queue-based mechanism. The expected waiting time of user \( i \) declaring \( v_i \) for state \( q \) can be determined as:

\[
W_q(v_i, q_i) = \begin{cases} 0 & \text{if } q_i \in T^q_i, \\ E \left[ W_q(v_i, f_q(q_i, \xi)) \right] + g_q(q_i) & \text{otherwise}. \end{cases}
\]

Note that the term \( E \left[ W_q(v_i, f_q(q_i, \xi)) \right] \) represents the expected value of the expected waiting time \( W_q(v_i, f_q(q_i, \xi)) \) after the transition function \( f_q \). If \( q_i \) is not a terminal state, i.e. \( q_i + q_0 < Q - 1 \), the expected waiting time of user \( i \) can be calculated using the probability \( \Pr \left[ q'_i | q_i \right] \) of transitioning from state \( q_i \) to state \( q'_i \). The expected waiting time of user \( i \) declaring \( v_i \) is:

\[
E \left[ W_q(v_i, f_q(q_i, \xi)) \right] = \sum_{q'_i \in S^q_i} \Pr \left[ q'_i | q_i \right] W_q(v_i, q'_i).
\]
Combining Eqs. (4) and (5) yields the following system of linear equations:

\[ W_q(v_i, q_i) = 0 \quad \forall q_i \in \mathcal{T}_i, \]
\[ W_q(v_i, q_i)(1 - \Pr[q_i|q_i]) - \sum_{q_i' \in S^i: q_i' \neq q_i} W_q(v_i, q_i') \Pr[q_i'|q_i] = g_q(q_i) \quad \forall q_i \in S^i_i \setminus \mathcal{T}_i. \]

Let \( p \) be the probability that a user arrives on a lane of the intersection at the next time period. Recall that \( F(v_i) \) is the cumulative distribution function of users bids. On each lane, three events may occur with the following probabilities: i) a lane can become or remain empty with probability \((1 - p)\); ii) a lower-bidding user can arrive with probability \( pF(v_i) \); and iii) a higher-bidding user can arrive with probability \( p(1 - F(v_i)) \). Observe that there are \((q_i + 1)\) ways to choose \( q_i \) empty lanes among \( q_0 + 1 \) lanes. There are \( q'_i - q_i \) lanes which become occupied by lower-bidding users and there are \((q_i + 1 - q'_i)q'_i \) ways to choose \( q'_i - q_i \) of the remaining \( q_0 + 1 - q'_i \) non-empty lanes to become occupied by a lower-bidding user. Accordingly, the transition probability from state \((q_i, q_0)\) to state \((q'_i, q'_0)\) is:

\[ \Pr[q_i'|q_i] = \left( \frac{q_0 + 1}{q'_0} \right) (1 - p)^{q_0} \left( \frac{q_i + 1 - q'_i}{q'_i - q_i} \right)^{q_0 + 1 - q'_i} \left( pF(v_i) \right)^{q'_i - q_i} \left( p(1 - F(v_i)) \right)^{q_0 + 1 - q'_i - q'_i}. \]

The expected waiting time defined in Eq. (4) is a recursive function which can be calculated by solving a series of systems of linear equations where each equation corresponds to a possible state \( q_i \in S^q_i \). Since all states such that \( q_i + q_0 = Q - 1 \) are terminal, their corresponding expected waiting time is null and these variables can be eliminated from the system of equations. Hence, we need only to consider the set of pairs of nonnegative integers \((q_i, q_0)\) such that \( q_i + q_0 \leq Q - 2 \). Observe that each variable \( q_i \) or \( q_0 \) can take \( Q - 2 + 1 \) values, thus the number of equations in the system is \( \sum_{k=0}^{Q-2} k = \binom{Q}{2} \).

Further, observe that \( q_i \) can never decrease while \( q_0 \) can increase or decrease. Hence, for any state \( q_i = (q_i', q_0) \), we can calculate \( W_q(v_i, q_i) \) in a dynamic programming fashion starting with the maximum number of lanes, i.e. \( k = Q - 1 \), and iterating downwards until \( k = q_i \). At each iteration \( k \), we solve the system of \( Q - k \) linear equations with \( Q - k \) variables corresponding to states \( q_i = (k, l) \) for \( 0 \leq l \leq Q - 1 - k \).

We illustrate the queue-based model in Example 1.

**Example 1.** Consider an intersection with \( Q = 4 \) access lanes. The state space \( S^q_i \) can be represented as in Table 1 where the bottom row corresponds to terminal states.

| (0,0) | (1,0) | (0,1) |
|-------|-------|-------|
| (2,0) | (1,1) | (0,2) |
| (3,0) | (2,1) | (1,2) |
|       |       | (0,3) |

Table 1: State space \( S^q_i \) for an intersection with \( Q = 4 \).

The reachable sets denoted \( S^q_i (q_i, q_0) \) of the six non-terminal states of this MC are:

\[
\begin{align*}
S^q_0(0,0) &= \{ (0,0), (0,1), (1,0) \} \\
S^q_0(0,1) &= \{ (0,0), (0,1), (0,2), (1,0), (1,1), (2,0) \} \\
S^q_0(0,2) &= \{ (0,0), (0,1), (0,2), (0,3), (1,0), (1,1), (1,2), (2,0), (2,1), (3,0) \} \\
S^q_0(1,0) &= \{ (1,0), (1,1), (2,0) \} \\
S^q_0(1,1) &= \{ (1,0), (1,1), (1,2), (2,0), (2,1), (3,0) \} \\
S^q_0(2,0) &= \{ (2,0), (2,1), (3,0) \}
\end{align*}
\]

We now illustrate the behavior of the queue-based model when determining the expected waiting time for the state \( q_i = (1,1) \).
1. At the first iteration, $k = Q - 1 = 3$ and there are $Q - k = 1$ equations corresponding to variable $W_q(v_i, (3, 0))$, which is trivial since $(3, 0)$ is a terminal state, i.e. $W_q(v_i, (3, 0)) = 0$.

2. At the second iteration, $k = Q - 2 = 2$ and there are $Q - k = 2$ equations corresponding to variables $W_q(v_i, (2, 0))$ and $W_q(v_i, (2, 1))$, i.e.:

   \[
   W_q(v_i, (2, 0))(1 - \Pr[(2, 0)|(2, 0)]) - W_q(v_i, (2, 1))\Pr[(2, 1)|(2, 0)] - W_q(v_i, (3, 0))\Pr[(3, 0)|(2, 0)] = g_q(v_i, (2, 0))
   \]

   \[
   W_q(v_i, (2, 1)) = 0.
   \]

   

   Since $W_q(v_i, (3, 0)) = 0$ was determined at the previous iteration, the above system collapses to:

   \[
   W_q(v_i, (2, 0)) = \frac{g_q(v_i, (2, 0))}{(1 - \Pr[(2, 0)|(2, 0)])}.
   \]

3. At the third iteration, $k = Q - 3 = 1$ and there are $Q - k = 3$ equations corresponding to variables $W_q(v_i, (1, 0))$, $W_q(v_i, (1, 1))$ and $W_q(v_i, (1, 2))$, i.e.:

   \[
   W_q(v_i, (1, 0))(1 - \Pr[(1, 0)|(1, 0)]) - W_q(v_i, (1, 1))\Pr[(1, 1)|(1, 0)] - W_q(v_i, (2, 0))\Pr[(2, 0)|(1, 0)] = g_q(v_i, (1, 0)),
   \]

   \[
   W_q(v_i, (1, 1))(1 - \Pr[(1, 1)|(1, 1)]) - W_q(v_i, (1, 0))\Pr[(1, 0)|(1, 1)] - W_q(v_i, (2, 1))\Pr[(2, 1)|(1, 1)] = g_q(v_i, (1, 1)),
   \]

   \[
   W_q(v_i, (1, 2)) = 0.
   \]

   Note that $W_q(v_i, (3, 0))$, $W_q(v_i, (2, 1))$ and $W_q(v_i, (2, 0))$ were determined at previous iterations. Since $W_q(v_i, (1, 2)) = 0$, the above system collapses to a system of two equations in the variables $W_q(v_i, (1, 0))$ and $W_q(v_i, (1, 1))$, the latter being the desired expected waiting time corresponding to state $q_i = (1, 1)$.

3.2. Lane-based model

We now consider an alternative approach where lane arrival probabilities can be non-uniform across lanes of the intersection. We abuse notation and denote $p_j$ the probability of a user arriving on lane $j$ at the next time period. Let $L_i = \{\sigma_{\emptyset}, \sigma_z, \sigma_{\pi}\}$ be the set of possible lane-based states with regards to user $i$ bidding $v_i$; where $\sigma_{\emptyset}$ represents an empty lane, $\sigma_z$ the arrival of a user bidding lower than $v_i$ and $\sigma_{\pi}$ the arrival of a user bidding higher than $v_i$. Let $z_{ij} \in L_i$ be the state of lane $j$ w.r.t. user $i$. We denote $z_i = (z_{i,1}, \ldots, z_{i,Q-1})$ the state w.r.t. to user $i$ in the lane-based model.

**Definition 4** (State space for lane-based model). In the lane-based model, the state space of the MC for user $i$ declaring a delay cost of $v_i$ in a pricing queue of size $Q$ is

\[
S^i = \{z_i = (z_{i,j}, \ldots, z_{i,jQ-1}) \in L_i^{Q-1}\},
\]

In the lane-based model, the number of possible states w.r.t. to user $i$ in an intersection with $Q$ lanes is $|L_i|^{Q-1}$, hence a four-lane intersection has $3^3 = 27$ possible states.

Let $f_z : S^i \times \mathbb{R} \times 2^2 \rightarrow S^i$ be the transition function of the MC and let $g_z(z_i)$ be the one-step cost in the lane-based model. Let $T^i \subset S^i$ be the set of terminal states, i.e. $T^i = \{z_i \in S^i : z_{ij} = \sigma_{\emptyset} \lor z_{ij} = \sigma_{\pi}, j = 1, \ldots, Q-1\}$. The expected waiting time $W_z(v_i, z_i)$ of user $i$ declaring $v_i$ for state $z_i$ in the lane-based model is thus:

\[
W_z(v_i, z_i) = \begin{cases} 
0 & \text{if } z_i \in T^i, \\
E[W_z(v_i, f_z(z_i, \xi))] + g_z(z_i) & \text{otherwise},
\end{cases}
\]

with:

\[
E[W_z(v_i, f_z(z_i, \xi))] = \sum_{z_i' \in S^i} \Pr[z_i'|z_i] W_z(v_i, z_i').
\]
This leads to the following system of linear equations:

\[ W_z(v_i, z_i) = 0 \quad \forall z_i \in T_i^z, \]  
\[ W_z(v_i, z_i)(1 - \Pr[z_i|z_i]) - \sum_{z'_i \in S_i^z \setminus z_i} W_z(v_i, z'_i)\Pr[z'_i|z_i] = g_z(z_i) \quad \forall z_i \in S_i^z \setminus T_i^z. \]  

Unlike the systems of equations (6), the system (11) does not admit an intuitive decomposition which could be used to solve the system of equations via a recursive algorithm. Thus, in our numerical experiments, (11) is solved in a single step using traditional linear algebra codes.

To determine transition probabilities in the lane-based mechanism, it is necessary to track which user will be serviced at the next time period. For any non-terminal state, there is at least one higher-bidder in the pricing queue and the highest-bidder in the queue will be serviced next. Let \( V_{>1} \) be the set of higher-bidders, i.e. \( V_{>1} = \{ j_1, j_2, \ldots, j_{\sigma_i} \} \) with \( v_{j_1} < v_{j_2} < \cdots < v_{j_{\sigma_i}} \). Since the bids of future arrivals are unknown, the probability distribution that a higher-bidder is the highest-bidder is uniform.

Let \( z_{ij}(k) \) be the state of lane \( j \) with regards to user \( i \) assuming lane \( k \) is the moving lane, i.e. the lane occupied with the highest-bidder in the pricing queue. Let \( \Pr[z'_{ij}|z_{ij}(k)] \) be the transition probability of lane \( j \) with regards to user \( i \) from state \( z_{ij}(k) \) to state \( z'_{ij} \). Since the arrival process of a lane is assumed to be independent of that of other lanes, the transition probability from state \( z_i \) to \( z'_i \) denoted \( \Pr[z'_i|z_i] \) is

\[ \Pr[z'_i|z_i] = \Pr[(z'_{i1}, \ldots, z'_{iQ-1})|(z_{i1}, \ldots, z_{iQ-1})] = \frac{1}{q_i} \sum_{k \in V_{>1}} \prod_{j=1}^{Q-1} \Pr[z'_ij|z_{ij}(k)]. \]  

In the lane-based mechanism, transition probabilities are function of current lane state. If lane \( j \) is empty or if lane \( j \) is occupied by the highest bidder in the pricing queue, then we say that this lane is open. Otherwise, lane \( j \) is either occupied by a lower-bidding user or a higher-bidding user who will not be serviced next and we say that his lane is closed. Let \( \mathcal{V}_{-i} \) be the set of users in the pricing queue other than user \( i \). Let \( \mathcal{O}_{i,j} \subseteq \mathcal{L}_i \) and \( \mathcal{C}_{i,j} \subseteq \mathcal{L}_i \) be the set of open and closed states for lane \( j \) with regards to user \( i \), respectively:

\[ \mathcal{O}_{i,j} = \begin{cases} \{ \sigma_{\emptyset}, \sigma_{\pi_i} \} & \text{if } j \in \arg \max \{ v_k : k \in \mathcal{V}_{-i} \}, \\ \{ \sigma_{\emptyset} \} & \text{otherwise}, \end{cases} \]  
\[ \mathcal{C}_{i,j} = \begin{cases} \{ \sigma_{\emptyset} \} & \text{if } j \in \arg \max \{ v_k : k \in \mathcal{V}_{-i} \}, \\ \{ \sigma_{\emptyset}, \sigma_{\pi_i} \} & \text{otherwise}. \end{cases} \]

For any lane \( j \) in state \( \sigma \in \mathcal{O}_{i,j} \), we have the following transition probabilities:

\[ \Pr[\sigma_{\emptyset}|\sigma] = 1 - p_j \quad \forall j \in \{1, \ldots, Q - 1\}, \forall \sigma \in \mathcal{O}_{i,j}, \]  
\[ \Pr[\sigma_{\emptyset}, \sigma_{\pi_i}|\sigma] = p_j F(v_i) \quad \forall j \in \{1, \ldots, Q - 1\}, \forall \sigma \in \mathcal{O}_{i,j}, \]  
\[ \Pr[\sigma_{\pi_i}|\sigma] = p_j (1 - F(v_i)) \quad \forall j \in \{1, \ldots, Q - 1\}, \forall \sigma \in \mathcal{O}_{i,j}. \]

Otherwise if lane \( j \) is in state \( \sigma \in \mathcal{C}_{i,j} \), this lane remains in its current state with probability one, since the corresponding user cannot be serviced at the next time period.

\[ \Pr[\sigma'|\sigma] = \begin{cases} 1 & \text{if } \sigma' = \sigma, \\ 0 & \text{otherwise}, \end{cases} \quad \forall j \in \{1, \ldots, Q - 1\}, \forall \sigma \in \mathcal{C}_{i,j}. \]  

Using these lane-based transition probabilities, we can determine intersection-based transition probabilities via Eq. (12).

The queue-based model can be viewed as a special case of the lane-based model. Let \( 1_{z_{ij} = \sigma} \) be the indicator function taking value 1 if \( z_{ij} = \sigma \) and 0 otherwise. We first define the mapping between the state spaces of both queue- and lane-based models.

**Definition 5.** Let \( h_i : S_i^Q \to S_i^Q \) be a function mapping the state space of the lane-based model to the state space of the queue-based model: \( h_i : z_i \mapsto q_i = h_i(z_i) \) with \( q_i = \sum_{j=1}^{Q-1} 1_{z_{ij} = \sigma_{\emptyset}} \) and \( q_0 = \sum_{j=1}^{Q-1} 1_{z_{ij} = \sigma_{\emptyset}}. \)
The proposition below highlights the correspondence between queue- and lane-based models.

**Proposition 1.** If lane arrival probabilities are uniform, i.e. \( p = p_j \) for all \( j = 1, \ldots, Q - 1 \), and if the one-step-costs of the queue- and lane-based models are such that \( g_z(z_i) = g_q(q_i) \) for all states \( z_i \in S_i^z \) and \( q_i \in S_i^q \) such that \( q_i = h_i(z_i) \), then \( W_z(v_i, z_i) = W_q(v_i, q_i) \) for all such states.

**Proof.** The probability of state \( q_i \) occurring is:

\[
\Pr[q_i] = (1 - p)^{q_i} p^{\sum q_i} F(v_i)^{q_i} (1 - F(v_i))^{p - q_i}.
\]

The probabilities of occurrence for lane-specific states \( z_{ij} \) are:

\[
\Pr[z_{ij}] = \begin{cases} 
1 - p_j & \text{if } z_{ij} = \sigma_0, \\
p_j F(v_i) & \text{if } z_{ij} = \sigma_{w_i}, \\
p_j (1 - F(v_i)) & \text{if } z_{ij} = \sigma_{s_i}.
\end{cases}
\]

Hence, if lane arrival probabilities are uniform, i.e. \( p = p_j \), then

\[
\Pr[z_i] = \prod_{j=1}^{Q-1} \Pr[z_{ij}] = \Pr[q_i] \text{ with } q_i = h_i(z_i).
\]

Accordingly, if lane arrival probabilities are uniform, transition probabilities given by (7) and (12) are equal for any states \( q_i, q_i' \in S_i^q \) and \( z_i, z_i' \in S_i^z \) such that \( q_i = h_i(z_i) \) and \( q_i' = h_i(z_i') \), i.e. \( \Pr[z_i'|z_i] = \Pr[h_i(z_i')|h_i(z_i)] \). Hence, if the one-step-costs are such that \( g_z(z_i) = g_q(h_i(z_i)) \), the solutions of the systems of equations (6) and (11) verify \( W_z(v_i, z_i) = W_q(v_i, q_i) \) for any pair of states \( z_i \in S_i^z \) and \( q_i \in S_i^q \) such that \( q_i = h_i(z_i) \). \( \square \)

Proposition 1 establishes a correspondence between the proposed queue- and lane-based models for determining the expected waiting times of users and shows that under uniform lane arrival probabilities and one-step-costs, both models are equivalent.

We next illustrate the lane-based model and the mapping \( q_i = h_i(z_i) \).

**Example 2.** We consider an intersection with \( Q = 4 \) lanes. To determine the expected waiting of a state \( z_i \in S_i^z \) requires solving the systems of equations (11) with \( |T_i^z| = 2^4 - 1 = 8 \) terminal states and \( |S_i^z| - |T_i^z| = 3^Q - 1 - 8 = 19 \) non-terminal states.

As in Example 1, we examine the queue-based state \( q_i = (1, 1) \). The set of lane-based states \( z_i \in S_i^z \) which are antecedents of \( q_i \) via the mapping \( q_i = h_i(z_i) \) is, denoted \( S_i^z(h_i^{-1}((1, 1))) \) is:

\[
S_i^z(h_i^{-1}((1, 1))) = \{(\sigma_0, \sigma_{w_1}, \sigma_{w_2}, \sigma_{w_3}, \sigma_{w_4}, \sigma_{\tau_1}, \sigma_{\tau_2}, \sigma_{\tau_3}, \sigma_{\tau_4}, \sigma_{\tau}, \sigma_{w}), \sigma_{\tau}, \sigma_{w}, \sigma_0, \sigma_0, \sigma_{w_1}, \sigma_{w_2}, \sigma_{w_3}, \sigma_{w_4}, \sigma_{\tau_1}, \sigma_{\tau_2}, \sigma_{\tau_3}, \sigma_{\tau_4}, \sigma_{\tau}, \sigma_{w})\}
\]

According to Proposition 1, if \( p_j = p \) for all lanes \( j \) of the intersection and if \( g_z(z_i) = g_q(h_i(z_i)) \) for all states \( z_i \), then \( W_q(v_i, (1, 1)) = W_z(v_i, z_i) \) for all \( z_i \in S_i^z((1, 1)) \).

We next present a payment mechanism that can be used with both of these models to determine incentive-compatible payments.

### 3.3. Payment mechanism

The proposed payment mechanism can be used with both queue- and lane-based models for determining the expected waiting time of users. Hence, in this section, we simply denote \( W_i(v_i) \) the expected waiting time of user \( i \) bidding \( v_i \). This expected waiting time can be replaced by \( W_q(v_i, q_i) \) for the queue-based model, or by \( W_z(v_i, z_i) \) for the lane-based model. To determine incentive-compatible user payments, we calculate the expected marginal cost user \( i \) imposes on other users. For this, we first examine the period of time over which an extra user in the lane of user \( i \) is expected to have an impact on other users.

The busy period w.r.t. to user \( i \) is the expected time required to clear the queues if \( i \) declares a minimal delay cost, i.e. \( v_i = \bar{v} \). Observe that if \( v_i = \bar{v} \), then \( q_i = 0 \), since no user can bid lower than \( \bar{v} \), i.e. \( F(\bar{v}) = 0 \). We measure the impact of user \( i \) onto other users throughout the remaining busy period, i.e. the duration
between the start of service of user \( i \) if \( i \) declares \( v_i \), and the duration between the start of service of \( i \) if \( i \) had declared a minimal delay cost. Hence, the remaining busy period w.r.t. user \( i \) bidding \( v_i \) denoted \( \Delta W_i(v_i) \) can be defined as:

\[
\Delta W_i(v_i) = W_i(u) - W_i(v_i).
\]  

(20)

We can split the remaining busy period w.r.t. user \( i \) into two components: the expected portion of \( \Delta W_i(v_i) \) which affects users who arrived before user \( i \); and the expected portion of \( \Delta W_i(v_i) \) which affects users who are expected to arrive after user \( i \). We denote \( B_i(v_i) \) and \( A_i(v_i) \) these quantities, respectively, and impose:

\[
\Delta W_i(v_i) = B_i(v_i) + A_i(v_i).
\]  

(21)

To determine the marginal delay caused by user \( i \) onto users already in the pricing queue, we need only to consider those \( q \) users which bid lower than \( i \) since others are not delayed by user \( i \). Let \( V_{<i} \) be the set of all users bidding lower than \( i \) ordered by increasing declared delay costs, i.e. \( V_{<i} = \{ j_1, j_2, \ldots, j_{q_i} \} \) with \( v_{j_1} < v_{j_2} < \ldots < v_{j_{q_i}} < v_i \).

To determine the expected portion of the remaining busy period w.r.t. user \( i \) which affects user \( j \) in the lane of user \( i \). The expected period of time during which this extra user affects user \( j \) is the difference between the waiting time of the extra user if the extra user is bypassed by user \( j \) and that if the extra user bypasses user \( j \) in the pricing queue. We abuse notation and denote \( W_i(v_j : z_{ij} = \sigma_{\pi}) \) and \( W_i(v_j : z_{ij} = \sigma_{\omega}) \) these waiting times, respectively. We define \( B_i(v_i) \) as:

\[
B_i(v_i) \equiv \sum_{k=1}^{q_i} W_i(v_{j_k} : z_{ij_k} = \sigma_{\pi}) - W_i(v_{j_k} : z_{ij_k} = \sigma_{\omega}).
\]  

(22)

Accordingly, the expected marginal delay cost that user \( i \) imposes on users who arrived before her is:

\[
MB_i(v_i) = \sum_{k=1}^{q_i} (W_i(v_{j_k} : z_{ij_k} = \sigma_{\pi}) - W_i(v_{j_k} : z_{ij_k} = \sigma_{\omega}))v_{j_k}.
\]  

(23)

We next show that the period of time \( B_i(v_i) \) as defined via Eq. (22) cannot exceed the remaining busy period \( \Delta W_i(v_i) \) as given by Eq. (20).

**Proposition 2.** For any user \( i \) bidding \( v_i \), \( B_i(v_i) \leq \Delta W_i(v_i) \).

**Proof.**

\[
B_i(v_i) = \sum_{k=1}^{q_i} W_i(v_{j_k} : z_{ij_k} = \sigma_{\pi}) - W_i(v_{j_k} : z_{ij_k} = \sigma_{\omega})
\]

\[
= W_i(v_{j_1} : z_{ij_1} = \sigma_{\pi}) - W_i(v_{j_1} : z_{ij_1} = \sigma_{\omega}) + W_i(v_{j_2} : z_{ij_2} = \sigma_{\pi}) - W_i(v_{j_2} : z_{ij_2} = \sigma_{\omega})
\]

\[
+ \ldots + W_i(v_{j_{q_i}} : z_{ij_{q_i}} = \sigma_{\pi}) - W_i(v_{j_{q_i}} : z_{ij_{q_i}} = \sigma_{\omega}).
\]  

(24)

Recall that \( v_{j_k} < v_{j_{k+1}} < v_i \) for any \( k = 1 \ldots q_i - 1 \). Observe that the state corresponding to the extra user bidding \( v_{j_k} \) and bypassing user \( j_k \) is equivalent to the state of the extra user bidding \( v_{j_{k+1}} \) and being bypassed by user \( j_{k+1} \), i.e. these states have identical number and location of lower/higher-bidders and empty lanes w.r.t. the perspective of the extra user. Since \( v_{j_k} < v_{j_{k+1}} \), the waiting time of the extra user bidding \( v_{j_{k+1}} \) and bypassing user \( v_{j_{k+1}} \) is greater than the waiting time of the extra user bidding \( v_{j_k} \) and being bypassed by user \( v_{j_k} \), i.e.

\[
W_i(v_{j_{k+1}} : z_{ij_{k+1}} = \sigma_{\pi}) - W_i(v_{j_k} : z_{ij_k} = \sigma_{\omega}) \leq 0 \quad \forall k = 1 \ldots q_i - 1.
\]

Omitting these negative terms from Eq. (24) yields:

\[
B_i(v_i) \leq W_i(v_{j_1} : z_{ij_1} = \sigma_{\pi}) - W_i(v_{j_{q_i}} : z_{ij_{q_i}} = \sigma_{\omega}).
\]  

(25)

The state corresponding to the extra user bidding \( v_{j_{q_i}} \) and bypassing user \( j_{q_i} \) is equivalent to the state from the perspective of user \( i \). Thus, since \( v_{j_{q_i}} < v_i \), \( W_i(v_{j_{q_i}} : z_{ij_{q_i}} = \sigma_{\omega}) > W_i(v_i) \). Finally, \( W_i(u) \) is
an upper bound on the waiting time for any user on the lane of user $i$, thus $W_i(v_{k,i} : z_{ik} = \sigma_i) < W_i(v)$. Substituting in Eq. (25) gives

$$B_i(v_i) \leq W_i(v) - W_i(v_i) = \Delta W_i(v_i).$$

Proposition 2 together with Eq. (21) ensure that the expected portion of the remaining busy period corresponding to future arrivals is nonnegative, i.e. $\Delta W_i(v_i) - B_i(v_i) = A_i(v_i) \geq 0$.

To determine the expected marginal delay cost that user $i$ imposes on future arrivals, denoted $MA_i(v_i)$, we use the result of Dolan (1978) who observed that this marginal delay cost can be calculated by integrating $\frac{dA_i(v)}{dv}$ over the bid range $(v_i, v_i)$. Further, he observed that the state is constant for any bid comprised between two consecutive bids of users in the pricing queue. Thus, for any segment corresponding to a pair of consecutive bids in the sequence $\{v_i, v_{j_1}, \ldots, v_{j_{k-1}}, v_{j_k}, v_i\}$, the expected marginal delay cost imposed by user $i$ onto users bidding in this segment can be calculated by integrating $\frac{dA_i(v)}{dv}$ over the corresponding domain. Specifically, let $v_{j_0} = v_i$ and let $v_{j_{k+1}} = v_i$. Let $D(k) = (v_{j_k}, v_{j_{k+1}})$ be the domain corresponding to the bid segment $(v_{j_m}, v_{j_n})$ for $m < n$ such that $j_m, j_n \in V_{\leq i}$. Accordingly, the expected marginal delay cost imposed by user $i$ on future arrivals can be determined as:

$$MA_i(v_i) = \sum_{k=0}^{d} \int_{D(k)} \frac{dA_i(v)}{dv} dv = \sum_{k=0}^{d} \int_{D(k)} -\frac{dW_i(v)}{dv} dv. \quad (26)$$

The expected marginal delay cost imposed by user $i$ declaring $v_i$ on other users is thus:

$$MC_i(v_i) = MB_i(v_i) + MA_i(v_i). \quad (27)$$

We next give the main result.

**Theorem 1.** If the user objective function is $\min v_i^* W_i(v_i) + P_i(v_i)$ and users are serviced in order of decreasing declared delay costs, the payment $P_i(v_i) = MC_i(v_i)$ is incentive-compatible.

Proof. The proof follows the same logic as that of Dolan (1978). We next recall its main steps and detail elements which are specific to our mechanism. Consider a pricing queue of size $Q$ and a user with true delay cost $v_i^*$ declaring $v_i^* + \delta$ with $\delta > 0$. Increasing the declared delay cost may reduce waiting time in two ways: i) bypassing a user in the queue, and, ii) having fewer future arrivals bypass the bidder.

For i) the reduction in expected waiting time is $B_i(v_i^* + \delta) - B_i(v_i^*) = W_i(v_{ik} : z_{ik} = \sigma_i) - W_i(v_{ik}) = (z_{ik} = \sigma_i, v_{ik}^* + \delta)$ such that $v_{ik} \in (v_i^*, v_i^* + \delta)$. This reduction is valued $v_i^*(B(v_i^* + \delta) - B(v_i^*))$ but the added cost is $v_{ik}(B(v_i^* + \delta) - B(v_i^*))$ with $v_{ik} \geq v_i^*$.

For ii) the reduction in expected waiting time is $A_i(v_i^* + \delta) - A_i(v_i^*)$ which is valued $v_i^*(A_i(v_i^* + \delta) - A_i(v_i^*))$. However, the induced expected marginal cost is $MA_i(v_i^* + \delta) - MA_i(v_i^*) = \int_{v_i^*}^{v_i^* + \delta} \frac{dA_i(v)}{dv} dv = v_i^*(A_i(v_i^* + \delta) - A_i(v_i^*))$. A similar reasoning can be applied for users declaring a delay cost lower than their true delay cost.

Theorem 1 proves that the expected marginal delay cost as defined by Eq. (27) represents an incentive-compatible user payment in the dynamic sense, i.e. by taking into account the expected impact of users on future arrivals. This provides a basis to implement the proposed queue- and lane-based mechanisms in an online fashion. Upon reaching the front of their lane, users are asked to declare their delay cost $v_i$ and receive a corresponding payment $P_i(v_i)$ which ensures strategyproof user behavior in the long run.

The proposed queue- and lane-based traffic intersection mechanisms are computationally efficient. In terms of computational resources, both mechanisms require the solution of Eqs. (23) and (26) to determine user payments. Eq. (23) requires the solution of multiple systems of linear equations ((6) or (11) depending on the mechanism selected) which can be accomplished using standard linear algebra codes. To compute Eq. (26), standard numerical integration and differentiation techniques can be used in combination with codes for systems of linear equations. As will be shown in our numerical experiments, both mechanisms have low average computation time per user and can thus be expected to be executed in real-time to determine incentive-compatible payments.
4. Numerical experiments

In this section, we conduct numerical simulations of the proposed online traffic intersection mechanisms to explore their behavior.

We implement the proposed online mechanisms by simulating the arrival and departure of users at an intersection with $Q = 4$ access lanes using a discrete time process. At every time period, random trials are conducted for each empty lane to simulate the stochastic arrival process of users. For all our experiments, we use a unit one-step cost, i.e. $g(q_i) = g(z_i) = 1$ for all states $q_i \in S_i^q$ and $z_i \in S_i^z$.

Users’ delay costs are randomly drawn from a uniform distribution $\mathcal{U}(5, 10)$, i.e. $v_i = 5 \$/hour and $\tau = 10 \$/hour. These values are chosen based on a study on value of time for automated vehicles (Wong et al., 2018). In each experiment, we simulate the service of one million users and report the average user behavior by segmenting users’ delay cost. Specifically, we segment the delay cost range [5$, 10$] into 30 uniform bins and group all serviced users throughout the simulation into these bins. We then report average waiting times, payments and generalized user costs based on the average quantities obtained for each of these 30 bins.

The simulations are implemented in Python and the linear systems of equations (6) and (11) are solved using Numpy’s linear algebra solver on a Windows 10 machine with 8 Gb of RAM and a CPU of 2.7 Ghz. Under this configuration, the average computation time per user for the queue-based mechanism is 0.007 s and that of the lane-based mechanism is 0.126 s, which highlights the computational efficiency of the proposed online mechanisms.

4.1. Comparison of dynamic and static mechanisms

To quantify the value of using a dynamic auction compared to a static auction, we implement the static mechanism outlined by Dolan (1978) for priority queueing systems. This static mechanism is a Vickrey-Clarke-Groves auction which is known to be incentive-compatible in the static sense, i.e. all participating users are assumed to be known when determining payments (Vickrey, 1961; Clarke, 1971; Groves and Loeb, 1975).

The static mechanism utilizes the same priority queueing model than that of the proposed online mechanism, i.e. users in the pricing queue are sorted by decreasing declared delay costs, but differ in the determination of waiting time and user payments. In the static mechanism, the waiting time of user $i$ is only function of the position of user $i$ in the pricing queue, e.g. if user $i$ is third in the pricing queue, her waiting time is the sum of the two one-step costs for transitioning from the current state to the state where user $i$ served. The payment of user $i$ is sum of user valuations over lower-bidders in the pricing queue. This payment is incentive-compatible in the static sense (Dolan, 1978).
The static mechanism is compared to the queue-based model. We first set the expected arrival rate of users at the intersection to 1 so as to simulate a stable regime. Since the arrival rate is uniform across all \( Q = 4 \) lanes, this corresponds to an arrival probability of \( p = 0.25 \). The results are reported in Figure 1 which depicts the relative difference of \( C_i(v_i) \) to \( C_i(v^*_i) \) in percentage, which represents the difference between the generalized cost of user \( i \) declaring a delay cost \( v_i \) and the generalized cost of user declaring its true delay cost \( v^*_i \). Strictly positive values indicate that declaring \( v_i \) results in a higher cost than declaring the user true delay cost \( v^*_i \), whereas negative values indicate that misreporting delay cost \( v_i \) achieves a lower cost than truthful reporting. For clarity, relative user generalized cost values are bounded within \( \pm 10\% \), i.e. if \( (C_i(v_i) - C_i(v^*_i))/C_i(v^*_i) \) is greater (resp. lower) than 10\% (resp. \(-10\%\)), this value is shown as 10\% (resp. \(-10\%\)).

The relative user generalized cost obtained via the online queue-based mechanism is shown to be, on average, incentive-compatible almost everywhere, as highlighted by positive relative generalized user costs. In turn, using the static mechanism, a user misreporting her delay cost by declaring a higher delay cost can achieve a lower generalized cost than truthfully reporting her delay cost, as highlighted by the top-right negative relative user generalized costs. This highlights the benefits of online mechanisms over static approaches for traffic intersection auctions.

We next examine the impact of varying the arrival probability around the stable regime. We run simulations from \( p = 0.15 \), which corresponds to an expected arrival rate of 0.6, to \( p = 0.35 \), which corresponds to an expected arrival rate of 1.4, in steps of 0.05. The results of these simulations are reported in Fig-
Figure 4: Results of simulations with 1 million users with a varying arrival probability of $p$. Figure 4a shows user payments obtained using the queue-based online mechanism and Figure 4b shows user payments obtained using the static mechanism.

To quantify the accuracy of the proposed MC models for estimating expected waiting times, we report the difference of experienced minus expected waiting times for the queue-based and the static mechanisms in Figure 3. For clarity y-axis scales are different in Figure 3a and Figure 3b representing the queue-based and static mechanisms, respectively. Figure 3a shows that the average expected waiting times obtained via the MC queue-based model are very close to the average experienced waiting times, with maximum deviations approximately 0.05 time units reported for low-bidding users. In contrast, Figure 3b shows that the static mechanism almost systematically underestimates the average experienced waiting times. Further, the loss of quality increases with the arrival probability at the intersection. These trends are consequences of the static mechanism not accounting for the probability of future arrivals. Figure 4 shows the average payments of users under both queue-based and static mechanisms. User payments increase with delay cost and are comparatively larger in the queue-based mechanism (Figure 4a) than in the static mechanism (Figure 4b). This emphasizes the potential impact of future arrivals which are neglected in static mechanisms. The queue-based mechanism also exhibits more concave-shaped payments with regards to user delay cost compared to the static mechanism which is more linear. We also find that the curvature of the payment curves increase with the arrival probability, highlighting the impact of the demand regime onto the payment mechanisms.

4.2. Comparison of queue- and lane-based mechanisms

To quantify the trade-offs between queue- and lane-based mechanisms, we implement both mechanisms for a fully asymmetric case. We use cardinal directions, i.e. East (E), South (S), West (W) and North (N) to refer to all $Q = 4$ lanes of the intersection and assign lane arrival probabilities of 0.50, 0.25, 0.15 and 0.10, respectively. For the queue-based mechanism, we use the average lane arrival probability, i.e. $p = \sum_{j=1}^{Q} p_j = 0.25$. We report lane-based heatmaps of the relative difference in terms of relative user generalized cost $(C_i(v_i) - C_i(v^*_i))/C_i(v^*_i)$ for both queue- and lane-based mechanisms in Figure 5. In Figure 5a, the outcome of these simulations highlight that using the average lane arrival probability in the queue-based mechanism fails to yields incentive-compatible outcome for the East, West and North lanes which have a lane arrival probability different that $p = 0.25$. Only users on the South lane with $p_S = 0.25 = p$ receive incentive-compatible payments. In turn, Figure 5b illustrates that the lane-based mechanism is incentive-compatible for each lane of the intersection.
Figure 5: Relative user generalized cost \( (C_i(v_i) - C_i(v^*_i))/C_i(v^*_i) \). Results of a simulation with 1 million users with lane-based arrival probabilities of 0.50, 0.25, 0.15 and 0.10. Figure 5a shows relative user generalized costs obtained using the queue-based online mechanism and Figure 5b shows relative user generalized costs obtained using the lane-based mechanism.
4.3. Non-uniform lane arrival probabilities

We next further examine the behavior of the lane-based mechanism in the context of non-uniform lane arrival probabilities. We compare three configurations which correspond to a stable regime. The first configuration consists of uniform lane arrival probabilities of $p_j$, $j \in \{E,S,W,N\}$. The second configuration consists of an arrival probability of $p_j = 0.35$ for East and West lanes, and $p_j = 0.15$ for South and North lanes. The third configuration consists of fully asymmetric lane arrival probabilities identical to those used in Section 4.2. These three configurations are denoted Ewww-Sxxx-Wyyy-Nzzz where the three digits next to the lane direction represent the associated probability with two decimal values, e.g. E025 correspond to $p_E = 0.25$. For each of the three configurations tested, we simulate the arrival of 1 million users at an intersection.

Figures 6 and 7 depict the average experienced waiting time and average user payment, respectively, in the three lane arrival probabilities configurations. In both figures, the top left sub-figure (6a and 7a) shows the average behavior over all four lanes of the intersection, whereas each of the other three subfigures focus on one of the three configuration tested and shows lane-specific behavior. We find that the uniform configuration (E025-S025-W025-N025) leads to higher average waiting times and user payments compared the other two configurations tested. As expected, the uniform configuration yields a uniform behavior all lanes. In contrast, non-uniform configurations lead to lane-specific waiting times and payments. The outcome of the partially asymmetric configuration (E035-S015-W035-N015) reveals that payments for the lower arrival probability lanes (South and North) are only marginally higher than that of the uniform configuration (E025-S025-W025-N025), whereas payments for the higher arrival probability lanes (East and West) are substantially lower. Results for the fully asymmetrical configuration (E050-S025-W015-N010) show that waiting times are higher on low arrival probability lanes and decrease with the arrival probability. Conversely, user payments are lower on high arrival probability lanes and increase for on less demanded lanes. This can be explained by observing that users on low arrival probability lanes have a higher chance of delaying more users since those are expected to arrive more frequently on higher arrival probability lanes.
5. Conclusions

We next summarize the findings of this study before discussing its limitations and outlining potential extensions and future research directions.

5.1. Summary of findings

In this paper, we have presented novel online mechanisms for traffic intersection auctions. We assume that users can declare their delay cost privately to the intersection manager and focus on determining incentive-compatible payments for users at the front of their lane. We also assume that the intersection manager has knowledge of the distribution of users’ delay cost. The proposed mechanisms are designed to minimize user’s generalized cost which is defined as a linear combination of expected waiting time and user payment. We introduced two Markov chain models to determine users’ expected waiting time, and presented a payment mechanism that can be implemented with both models. We showed that the proposed online traffic intersection mechanisms are incentive-compatible in the dynamic sense and thus maximize social welfare. We conducted numerical experiments on a four-lane intersection to explore the behavior of the proposed online mechanisms and compared their performance to that of a static incentive-compatible mechanism. Our findings highlight that static mechanisms, i.e. which do not account for future arrivals, may not yield incentive-compatible in dynamic sense. That is, static incentive-compatible mechanisms may fail to ensure truthful user behavior by incentivizing users to misreport their delay cost in the long run. We also provide empirical evidence that the proposed online mechanisms are computationally efficient and could be used to manage traffic operations in real-time. Finally, we quantified the trade-offs between the queue- and lane-based mechanisms and find that the lane-based mechanism can accommodate lane-specific arrival probabilities in competitive time.
5.2. Limitations and future research directions

We next discuss the assumptions and mechanism design choices made in this research.

The proposed online mechanisms focus on determining incentive-compatible payments for users at the front of their lane queues. Since all users queueing on a lane of a traffic intersection are eventually going to reach the front of their lanes, all users will eventually join the pricing queue and be required to declare their delay cost so as to obtain their corresponding payment. However, the proposed mechanisms do not provide the option for “blocked” users, i.e., users queueing behind another user on their lane, to “vote” for the user at the front of their lane queues. Although this is a limitation of the proposed online mechanisms, since blocked users are assumed to be unaware of other users’ delay cost, it appears challenging to ensure that they could reap benefits from bidding for their lane leader.

The geometry of traffic intersections is only implicitly accounted for within the proposed online mechanisms. Specifically, users’ service times are captured within the one-step-costs of the proposed Markov chain models used to determine the expected waiting time of users. Yet, in our numerical experiments, we have assumed that all service times are deterministic and uniform. The uniformness assumption could be relaxed by substituting the unit one-step costs with movement-specific service times. However, relaxing this assumption would also require the introduction of a detailed model of users’ travel time within the intersection which is function of other users’ route choice and service time. This is expected to require an extension of the state space in the proposed lane-based Markov chain model to a movement-based model. Further, the simulation of realistic traffic movements within the intersection is expected to require traffic control model that provides the flexibility of servicing in signal-free context. Such models have been widely explored in the connected and autonomous vehicles’ literature and we leave the extension of the proposed online mechanisms to realistic traffic models for future works. Finally, the deterministic service time assumption could be relaxed at the expense of introducing more complex modeling of users’ trajectory interactions within the intersection which is beyond the scope of this research.

The proposed research may also be extended by considering batches or coalitions of users. This may lead to improved social benefits if, for instance, platoons of vehicles are priced to traverse the intersection in an effective manner. Such market-driven approaches could be combined with recent efforts on platoon coordination at traffic intersections (Jiang et al., 2006; Lioris et al., 2017; Monteil et al., 2018; Niroumand et al., 2020). The proposed online mechanisms could be revisited under the light of fairness. As observed by Schepperle and Böhm (2007) “starvation” effects may occur if intersection access is blocked by low delay cost users or dominated by high delay cost users on specific lanes. Future research is thus needed to advance the development of market-driven mechanisms that are able to combine dynamic incentive-compatibility with equity factors. Finally, methods on load balancing in parallel queueing systems as proposed by Down and Lewis (2006) could be used to develop lane assignment policies to assign users to small queue-length lanes so as to improve the social welfare of the system.

References

Altché F., De La Fortelle A. Analysis of optimal solutions to robot coordination problems to improve autonomous intersection management policies. In: Intelligent Vehicles Symposium (IV), 2016 IEEE. IEEE; 2016. p. 86–91.

Bradford R.M. Pricing, routing, and incentive compatibility in multiserver queues. European Journal of Operational Research 1996;89(2):226–36.

Carlino D., Boyles S.D., Stone P. Auction-based autonomous intersection management. In: 16th International IEEE Conference on Intelligent Transportation Systems (ITSC 2013). IEEE; 2013. p. 529–34.

Censi A., Bolognani S., Zilly J.G., Mousavi S.S., Frazzoli E. Today me, tomorrow thee: Efficient resource allocation in competitive settings using karma games. In: 2019 IEEE Intelligent Transportation Systems Conference (ITSC). IEEE; 2019. p. 866–93.

Chen R., Hu J., Levin M.W., Rey D. Stability-based analysis of autonomous intersection management with pedestrians. Transportation Research Part C: Emerging Technologies 2020;114:463–83.

Clarke E.H. Multipart pricing of public goods. Public choice 1971; 17:17–33.
De La Fortelle A., Qian X. Autonomous driving at intersections: combining theoretical analysis with practical considerations. In: ITS World Congress. 2015.

Dolan R.J. Incentive mechanisms for priority queuing problems. The Bell Journal of Economics1978;421–36.

Down D.G., Lewis M.E. Dynamic load balancing in parallel queueing systems: Stability and optimal control. European Journal of Operational Research2006;168(2):509–19.

Dresner K., Stone P. Multiagent traffic management: A reservation-based intersection control mechanism. In: Proceedings of the Third International Joint Conference on Autonomous Agents and Multiagent Systems-Volume 2. IEEE Computer Society; 2004. p. 530–7.

Dresner K., Stone P. A multiagent approach to autonomous intersection management. Journal of Artificial Intelligence Research2008;31:591–656.

Fajardo D., Au T.C., Waller S., Stone P., Yang D. Automated intersection control: Performance of future innovation versus current traffic signal control. Transportation Research Record: Journal of the Transportation Research Board2011;(2259):223–32.

Groves T., Loeb M. Incentives and public inputs. Journal of Public economics1975;4(3):211–26.

Hassin R., Haviv M. To queue or not to queue: Equilibrium behavior in queueing systems. volume 59. Springer Science & Business Media, 2003.

Jiang Y., Li S., Shamo D.E. A platoon-based traffic signal timing algorithm for major–minor intersection types. Transportation Research Part B: Methodological2006;40(7):543–62.

Kittsteiner T., Moldovanu B. Priority auctions and queue disciplines that depend on processing time. Management Science2005;51(2):236–48.

Levin M.W., Rey D. Conflict-point formulation of intersection control for autonomous vehicles. Transportation Research Part C: Emerging Technologies2017;85:528–47.

Li Z., Chitturi M., Zheng D., Bill A., Noyce D. Modeling reservation-based autonomous intersection control in Vissim. Transportation Research Record: Journal of the Transportation Research Board2013;(2381):81–90.

Lin D., Jabari S.E. Pay for intersection priority: A free market mechanism for connected vehicles. arXiv preprint arXiv:2001018132020;.

Lioris J., Pedarsani R., Tascikaraoglu F.Y., Varaiya P. Platoons of connected vehicles can double throughput in urban roads. Transportation Research Part C: Emerging Technologies2017;77:292–305.

Malikopoulos A.A., Cassandras C.G., Zhang Y.J. A decentralized energy-optimal control framework for connected automated vehicles at signal-free intersections. Automatica2018;93:244–56.

Mendelson H., Whang S. Optimal incentive-compatible priority pricing for the m/m/1 queue. Operations research1990;38(5):870–83.

Mirheli A., Tajallil M., Hajibaba L., Hajibabaie A. A consensus-based distributed trajectory control in a signal-free intersection. Transportation research part C: emerging technologies2019;100:161–76.

Monteil J., Bouroche M., Leith D.J. $L_2$ and $L_\infty$ stability analysis of heterogeneous traffic with application to parameter optimization for the control of automated vehicles. IEEE Transactions on Control Systems Technology2018;27(3):934–49.

Naor P. The regulation of queue size by levying tolls. Econometrica: journal of the Econometric Society1969;15:15–24.
Niroumand R., Tajalli M., Hajibabai L., Hajibabaie A. Joint optimization of vehicle-group trajectory and signal timing: Introducing the white phase for mixed-autonomy traffic stream. Transportation Research Part C: Emerging Technologies 2020;116.

Rey D., Levin M.W. Blue phase: Optimal network traffic control for legacy and autonomous vehicles. Transportation Research Part B: Methodological 2019;130:105–29.

Sayin M.O., Lin C.W., Shiraishi S., Shen J., Başar T. Information-driven autonomous intersection control via incentive compatible mechanisms. IEEE Transactions on Intelligent Transportation Systems 2018;20(3):912–24.

Schepperle H., Böhm K. Agent-based traffic control using auctions. In: International Workshop on Cooperative Information Agents. Springer; 2007. p. 119–33.

Vasirani M., Ossowski S. A market-inspired approach for intersection management in urban road traffic networks. Journal of Artificial Intelligence Research 2012;43:621–59.

Vickrey W. Counterspeculation, auctions, and competitive sealed tenders. The Journal of finance 1961;16(1):8–37.

Wong T.W., Saxena N., Dixit V.V. A study of route choice behavior of drivers in autonomous vehicles. In: Transportation Research Board 97th Annual Meeting. Transportation Research Board; 2018.

Wu Y., Chen H., Zhu F. Dcl-aim: Decentralized coordination learning of autonomous intersection management for connected and automated vehicles. Transportation Research Part C: Emerging Technologies 2019;103:246–60.

Zhang Y., Malikopoulos A.A., Cassandras C.G. Decentralized optimal control for connected automated vehicles at intersections including left and right turns. In: Decision and Control (CDC), 2017 IEEE 56th Annual Conference on. IEEE; 2017. p. 4428–33.

Zhang Y.J., Malikopoulos A.A., Cassandras C.G. Optimal control and coordination of connected and automated vehicles at urban traffic intersections. In: American Control Conference (ACC), 2016. IEEE; 2016. p. 6227–32.