Canard solutions near a degenerate turning point

T Forget
Laboratoire de Mathématiques et Applications Pôle, Sciences et Technologies - Université de
La Rochelle, Avenue Michel Crépeau, 17042 La Rochelle, France
E-mail: thomas.forget@univ-lr.fr

Abstract. We are interested in the study of canard solutions in a singularly perturbed real
first order ODE of the form
\[ \varepsilon u' = \Psi(x, u, a, \varepsilon), \]
where \( \varepsilon > 0 \) is a small parameter, and \( a \in \mathbb{R} \) is a control parameter. An existence result for such solutions is given, and the method used
in the demonstration allow us to conjecture the existence of a generalized \( \varepsilon^{1/(p+1)} \)-asymptotic
expansion for those solutions.

1. Introduction
In the theory of first order singularly perturbed real
first order ODEs, equations of the form
\[ \varepsilon u' = \Psi(x, u, a, \varepsilon) \]  \hspace{1cm} (1)
are considered, where \( \Psi \) is \( C^\infty \), \( x \in [-x_0, x_0] \) and \( u \) is real, \( a \) is a control parameter, and \( \varepsilon > 0 \)
is a small real parameter.
As we are concerned with canard solutions, we have to make assumptions to be in the appropriate
region for the appearance of canard solutions. For this reason, we have to suppose that the
following two assumptions are satisfied by (1):
(H1) (1) has a slow curve.
This means that there exists \((a_0, u_0)\) with \( a_0 \) real, and \( u_0 \) a real \( C^\infty \) function, such that
\[ \forall x \in [x_0, -x_0], \Psi(x, u_0(x), a_0, 0) = 0. \]
Without loss of generality, we may suppose that \( a_0 = 0 \) and \( u_0(x) = 0 \) for all \( x \in [-x_0, x_0] \).
(H2) \( \frac{\partial}{\partial u} \Psi(x, 0, 0, 0) \) is \[ \begin{cases} < 0 \text{ if } x < 0 \\ > 0 \text{ if } x > 0. \end{cases} \]
This assumption allows us to suppose that the slow curve has a turning point at \( x = 0 \), which
means that the slow curve is attractive to the left, and repulsive to the right, which are the
conditions for the appearance of a canard.
Moreover \( \frac{\partial^2}{\partial u^2} \Psi(x, 0, 0, 0) \) has a zero at \( x = 0 \) of order \( p \), where \( p \) is odd.
Solutions of interest for (1) are the so-called canard solutions [4][7][3] which are families of pairs
\((a_\varepsilon, u_\varepsilon)\) which tends to the slow curve \((0, 0)\) as \( \varepsilon \to 0. \)
A similar study has been done in the case of complex variables, where solutions \((a_\varepsilon, u_\varepsilon)\) which
tends to the slow curve \((0, 0)\) as \( \varepsilon \to 0 \) for all \( x \) in a complex neighborhood of 0 are the so-called
overstable solutions [6][2].

© 2006 IOP Publishing Ltd
In our study, the difference is that we are using a 1 dimensional real parameter $a$, where the complex study required $p$ complex parameters, this difference being due to the real characteristic of the neighbourhood studied.

In the continuation of our study, we will restrict ourselves to the equations

$$
\varepsilon u' = x^p u + ax^L + \sum_{i} a^{k_i} x^{l_i} + \varepsilon P(x, u, a, \varepsilon)
$$

(2)

where $L < p$ is even and, for all $i, l_i > L, k_i \geq 1$.

By assuming some strong assumptions, it is possible to transform equation (1) to such a general equation.

From those equations, we will define an operator $\Xi : (b, v) \mapsto (a, u)$. It will be proved that it is a contraction, with its Lipschitz constant equal to $K\varepsilon^{1/(p+1)}$.

Then, a fixed point theorem will prove the existence of a canard solution by giving an iterative sequence $(a_n, \varepsilon, u_n, \varepsilon)$ which has the expected canard solution $(a^*, \varepsilon, u^*)$ in the limit as $n \to +\infty$.

It has to be noted that this result has been proved by Peter de Maesschalck and Freddy Dumortier who have used a blowing up method [5], but the demonstration we propose allows us to assume that the canard solution has an asymptotic expansion, and this will be the subject of the second part of this paper:

By writing

$$
u^*_\varepsilon = \sum_{n \geq 1} (u_{n, \varepsilon} - u_{n-1, \varepsilon})
$$

we see that for all $n$, $u_{n, \varepsilon} - u_{n-1, \varepsilon} = O(\varepsilon^{n/(p+1)})$. Thus, the study of the existence and the uniqueness of an $\varepsilon^{1/(p+1)}$-asymptotic expansion for the canard solution is a subject of interest.

We recall that an asymptotic expansion is a series $\sum_{k} u_k \varepsilon^{k/(p+1)}$ such that

$$
\forall K, \left\| \sum_{k=0}^{K-1} u_k \varepsilon^{k/(p+1)} \right\| = O(\varepsilon^{K/(p+1)})
$$

where the properties of the coefficients $u_k$ have to be defined.

We will show that intermediary functions have to be defined in order to seek coefficients $u_k$ as $C^\infty$ functions for the variable $x$ and those intermediary functions.

Implementing such formulae in the case $p = 0$, to study limit layers in the case of an attractive slow curve, allows us to retrieve the so-called Combined Asymptotic Expansion [1].

When $p \geq 1$, implementing it implies the existence of the expansion, but the question of uniqueness is not solved, as interactions between those intermediary functions need to be better controlled.

In what follows, we will denote for convenience $\eta := \varepsilon^{1/(p+1)}$.

2. Existence of canard solutions in (2)

Let $\Omega$ be an open subset of $\mathbb{R}^4$ containing 0, and let $x_0 > 0$, $A > 0$, $B > 0$, $\tilde{\eta} > 0$, such that $[-x_0, x_0] \times [-A, A] \times [-B, B] \times [0, \tilde{\eta}] \subset \Omega$.

We will denote by $E$ the set of real continuous functions over $[-x_0, x_0]$, and note that the spaces $(E, ||.||_\infty)$ and $(\mathbb{R} \times E, ||.||_\infty)$ are two Banach spaces.

For all $M > 0$, $B_M$ will be the set $\{v \in E; ||v||_\infty \leq M\}$.

For convenience, the polynomial $\sum_{i} a^{k_i} x^{l_i}$ will be denoted by $s(x, a)$. 

Theorem 1 We suppose \((b, v)\) and \((a, u)\) are in \([-B, B] \times \mathcal{B}_A\).
For all \(\eta \in [0, b]\) and \((b, v)\) bounded, the equation
\[
\begin{align*}
\eta^{p+1}u'(x) &= x^p u(x) + ax^L + s(x, a) + \eta^{p+1}P(x, v(x), b, \eta) \\
u(x_0) &= 0 = u(-x_0)
\end{align*}
\]
has an unique bounded solution \((a, u)\), that will be denoted by \(\Xi_\eta(b, v)\).
Moreover, as \(\delta\)
\[
\text{Finally, for all } \delta_1 \in [0, B] \text{ and } \delta_2 \in [0, A], \text{ there exists } \eta_0 > 0 \text{ such that for all } \eta \in [0, \eta_0], \text{ } \Xi_\eta \text{ is an operator mapping from } [-\delta_1, \delta_1] \times \mathcal{B}_{\delta_2} \text{ to itself.}
\]
For \((b, v)\) bounded, the linear equation can be solved, and the matching conditions in 0 imply the uniqueness for a bounded parameter \(a\), and so the uniqueness of a bounded solution \(u\) follows.
By writing \(\Xi_\eta\) as a composition of two operators \(\Xi_\eta = \Theta \circ \varphi\) defined as follows:
\[
\varphi(b, v)(x) := P(x, v(x), b, \eta^{p+1})
\]
\(\Theta\): linear operator such that \(\Theta(w)\) is the solution of
\[
\begin{align*}
\eta^{p+1}u'(x) &= x^p u(x) + ax^L + s(x, a) + \eta^{p+1}w \\
u(x_0) &= 0 = u(-x_0),
\end{align*}
\]
we prove, by estimating integrals like \(e^{(x/\eta)^{p+1}} \int_0^x e^{-(\xi/\eta)^{p+1}} d\xi\), that the operator \(\Theta\) satisfies a Lipschitz condition, with Lipschitz constant equal to \(C\eta\), with \(C\) independent of \(\eta\).
As the operator \(\varphi\) is a \(L\)-Lipschitz function, we conclude that \(\Xi_\eta\) satisfies a Lipschitz condition, with Lipschitz constant equal to \(K\eta\), where \(K\) is independent of \(\eta\).
This result implies the last part of the theorem.

As \([-\delta_1, \delta_1] \times \mathcal{B}_{\delta_2}\) is a closed set of the Banach space \((\mathbb{R} \times E, ||.||_\infty)\), the fixed point theorem implies that \(\Xi_\eta\) has a fixed point \((a^*_\eta, u^*_\eta) \in [-\delta_1, \delta_1] \times \mathcal{B}_{\delta_2}\) which satisfies the equation
\[
\begin{align*}
\eta^{p+1}u^*_\eta'(x) &= x^p u^*_\eta(x) + ax^L + s(x, a^*_\eta) + \eta^{p+1}P(x, u^*_\eta(x), a^*_\eta, \eta) \\
u^*_\eta(x_0) &= 0 = u^*_\eta(-x_0)
\end{align*}
\]
(3)

As a direct consequence, the existence of a canard solution for (2) follows:

Corollary 1 For all \(\delta_1 \in [0, B] \text{ and } \delta_2 \in [0, A], \text{ there exists } \eta_0 > 0 \text{ such that for all } \eta \in [0, \eta_0], \text{ the equation } (3) \text{ has an unique solution } (a_\eta, u_\eta) \in [-\delta_1, \delta_1] \times \mathcal{B}_{\delta_2}.
Moreover, as \(\delta := \min\{\delta_1, \delta_2\}\) is fixed, the family \((a_\eta, u_\eta)\) is uniformly convergent to 0, over \([-x_0, x_0]\), as \(\eta \to 0\).

3. Asymptotic expansion
The fixed point theorem allows us to define an iterative sequence \(((a_n, u_n, \varepsilon))_n\) whose limit is the canard solution \((a^*_\eta, u^*_\eta)\), and so we have that
\[
a^*_\eta = \sum_{n \geq 1} (a_n, u_n, \varepsilon)
\]
where, as the operator \(\Xi_\eta\) is a \((K\eta)\)-Lipschitz function, we have for all \(n\),
\[
a_n, u_n - a_{n-1}, u_{n-1} = O(\eta^n)
\]
This implies that we have an $\eta$-asymptotic expansion for $a^*_\eta$: 

$$a^*_\eta \simeq \sum_k \tilde{a}_k \eta^k,$$

i.e.:

$$\forall K, ||a^*_\eta - \sum_{k=0}^{K-1} \tilde{a}_k \eta^k|| = \mathcal{O}(\eta^K)$$

In what follows, we will try to propose a similar result for $u^*_\eta$ but, as it is a function of $x$ and $\eta$, some problems will appear:

3.1. Implementation of a general structure for $u$

Similarly to the result for $a$, we write

$$u^*_\eta = \sum_{n \geq 1} (u_{n,\eta} - u_{n-1,\eta}).$$

where, as the operator $\Xi_\eta$ is a $(K\eta)$-Lipschitz function, for all $n$,

$$u_{n,\eta} - u_{n-1,\eta} = \mathcal{O}(\eta^n)$$

This brings us to the question of the existence and the uniqueness of an $\eta$-asymptotic expansion for the function $u^*_\eta$, $u^*_\eta \simeq \sum_k u_k \eta^k$, i.e.:

$$\forall K, ||u^*_\eta - \sum_{k=0}^{K-1} u_k \eta^k|| = \mathcal{O}(\eta^K).$$

Unfortunately, the natural $\eta$-asymptotic expansion $\sum_k u_k \eta^k$ where all coefficients $u_k$ are analytic in the variable $x$ is not sufficient:

A formal substitution shows that an $\eta$-asymptotic expansion $\sum_k u_k \eta^k$, where the coefficients $u_k$ are independent of $\eta$, implies that the coefficients $u_k$ may have a pole at $x = 0$.

So, we will seek coefficients $u_k$ to be analytic in $x$ and in some intermediary functions $\varphi$ which have to be well defined.

Some possible first choices for such functions are $\varphi(x, \eta) = e^{-(x/\eta)^p + 1}$ or $\varphi(x, \eta) = (e^{(x/\eta)^p + 1} \int_x^{+\infty} \xi e^{(-\xi/\eta)^p + 1} d\xi, \ldots, e^{(x/\eta)^p + 1} \int_x^{+\infty} \xi^{p-1} e^{-\xi/\eta)^p + 1} d\xi)$.

In the second case, the integrals considered are ones which appeared when we tried to make a decomposition of $\Theta$. As they do not have an $\eta$-asymptotic expansion, it is natural to consider them as the expected intermediary functions, but as they are not bounded in $\mathbb{R}$, we have to use a modified version of those functions:

Seeing that those functions are the most practical ones we are willing to use them in our study, but the lack of control at 0 may cause problems and thus, by pointing out that $e^{(x/\eta)^p + 1} \int_x^{+\infty} v(\xi)e^{-\xi/\eta)^p + 1} d\xi$ is a solution of the differential equation

$$\begin{cases}
\eta^{p+1}u'(x) = x^p u(x) + \eta^{p+1}v \\
 0 = 0
\end{cases}$$

we will introduce a parameter $a$ to give us a control at 0, and so, the function denoted by $I(v)$ will be a solution of the differential equation

$$\begin{cases}
\eta^{p+1}u'(x) = x^p u(x) + a + \eta^{p+1}v \\
 0 = 0
\end{cases}$$
where the parameter $a$ is calculated such that $I(v)$ is continuous at 0.

Another problem may appear since $x$ and $x\varphi$ may not have the same estimation. For example,

\[ ||e^{-(x/\eta)p+1}||_{x\in[-1,1]} = 1 \quad \text{and} \quad ||xe^{-(x/\eta)p+1}||_{x\in[-1,1]} = \frac{e^{-1/(p+1)}}{(p+1)^{p+1}}. \]

which implies that those two elements are not associated with the same coefficients $u_k$ in the $\eta$-asymptotic expansion of the solution. Thus we have to order the monomials $x^i\varphi^j\eta^l$ with respect to their estimate in $\eta$.

For this purpose, we define an order

\[ \bigtriangleup(x^i\varphi^j\eta^l) := \lim_{\eta \to 0} \ln \frac{||x^i\varphi(x, \eta)^j\eta^l||_{x\in[-1,1]}}{\ln \eta}. \]

Note that the set

\[ A_k = \text{Vect}\{x^i\varphi^j\eta^l; \bigtriangleup(x^i\varphi^j\eta^l) \leq k\} \]

will be the set of the principal terms with order $k$ for the $\eta$-asymptotic expansion of $u^*_\eta$ and, by using the transformation $\pi_k : A \rightarrow A_k$, where $A := \bigcup_k A_k$, we are able to construct an iterative sequence defined as

\[ \begin{cases} u_0 := 0 \\ u_k := \pi_k(\Theta \circ \varphi(u_{k-1})), \end{cases} \]

We are able to prove that this sequence is well defined and convergent, by making a decomposition of $\Theta$ as a linear combination of monomials $x^i\varphi^j\eta^l$.

The limit will be the expected $\eta$-asymptotic expansion, which will prove the existence of the asymptotic expansion.

The current problem we are trying to solve is the uniqueness of such asymptotic expansions, because intermediary functions have some interactions.

For example, it can be proved that

\[ I(x^kI(u)) = \frac{1}{k+1} \left( x^{k+1}I(u) - I(x^{k+1}u) - c_{u}I(x^{k+1}) \right). \]

We can find an answer to this question in the case $p = 1$ which is the non degenerate case, and also in the case $p = 0$, even if it is not related to a study of canard solutions:

\subsection{Application to the case $p = 0$}

We will apply our general structure for the study of an asymptotic expansion in the case of a limit layer for a singularly perturbed real ODE with an attractive slow curve.

Thus, we are studying the equation

\[ \begin{cases} \varepsilon u' = -u + \varepsilon P(x, u, \varepsilon) \\ u(0) = u_0 \neq 0. \end{cases} \tag{4} \]

We note that the slow curve of (4) is $u = 0$.

For this study, we will choose for an intermediary function $\varphi(x, \varepsilon) = e^{-x/\varepsilon}$.

The application of this structure gives the existence and uniqueness of an $\varepsilon$-asymptotic expansion for (4) of the form

\[ u^*_\varepsilon(x) \approx \sum_{i,l} a_{i,0,l}x^i\varepsilon^l + \sum_{i,j,l \geq 1} a_{i,j,l} \left( \frac{x}{\varepsilon} \right)^i \varphi(x, \varepsilon)^j\varepsilon^{l+i}. \]
This is a particular form of the so-called Combined Asymptotic Expansion, which has the following form

\[ u^*_\varepsilon(x) \approx \sum_n \left( f_n(x) + g_n \left( \frac{x}{\varepsilon} \right) \right) \varepsilon^n, \]

where for all \( n \), \( f_n \) is analytic, and \( g_n \) is an exponentially decreasing function. Functions \( f_n \) are related to the asymptotic expansion of the slow curve, whereas the functions \( g_n \) are related to the solution in the fast layer.

3.3. Application to the case \( p = 1 \)

In this case, the turning point is not degenerate, and no intermediary functions are needed to give the asymptotic expansion of the canard solution. Indeed, in this case, we prove that \( \Theta(x^k) \) can be written as a linear combination of monomials \( x^l \varepsilon^l \), which directly implies the uniqueness of the asymptotic expansion.

Acknowledgments

I am grateful to Eric Benoît for the scientific framing of this PhD research, and to Guy Wallet for helpful discussions as co-advisor. This work was supported by the Région Poitou-Charentes under grant no 03/RPC-R-148.

References

[1] Benoît E El Hamidi A and Fruchard A 2002 *Electron. J. Differential Equations* 51
[2] Benoît E Fruchard A Schäfke R and Wallet G 1998 *C. R. Acad. Sci. Paris Sér. I Math.* 326 7 873-8
[3] Cartier P. 1982 *Bourbaki Seminar* 92 21-44
[4] Diener F and Diener M 1981 *Collect. Math.* 32 1 37
[5] de Maesschalck P and Dumortier F 2006 *Trans. Amer. Math. Soc.* 358 5 2291
[6] Wallet G 1994 *Bull. Soc. Math. France* 122 2 185
[7] Zvonkin A K and Shubin M A 1984 *Uspekhi Mat. Nauk* 39 2(236) 77