Integration of time-varying cocyclic one-forms against rough paths

Terry J. Lyons, Danyu Yang

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Abstract

We introduce the cocyclic one-form on a group, and recast the integration in the theory of rough paths as an example of the integration of a time-varying one-form against a group-valued path. The integration gives back, for example, the extension theorem and the integral developed by Lyons and Gubinelli. We define a family of Banach-space valued paths which can be represented as an integral of a one-form against a given group-valued path, and demonstrate that this family of paths is stable under certain operations.

1 Introduction

The integration in rough paths theory is an integration of a one-form against a group-valued path. When the metric on paths space gets weaker, it is problematic to define the integral of a path against itself, e.g. there exists a sequence of smooth paths \( \{ \gamma_n \}_n \) which converge to zero in the uniform norm while \( \{ \int \gamma_n d\gamma_n \}_n \) converge uniformly to a non-zero smooth path. Lyons [Lyo98] observed that, the map of a path to the integral of a regular one-form against the path becomes continuous (so closable), if one enhances the integral in Banach space to a consistent integral in a topological group. This enhancement is essentially nonlinear, due to the nonlinearity of the group. Gubinelli [Gub04], [Gub10] defined weakly controlled paths by those Banach-space valued paths whose local behavior is comparable to the increment of a given rough path. For a fixed reference rough path, the space of weakly controlled paths is linear, and there exists a canonical enhancement of a weakly controlled path to a group-valued path. The linearity of space and the existence of canonical enhancement are nice properties that general rough paths can not have, and give certain convenience e.g. when one solves a rough differential equation. Moreover, Gubinelli [Gub10] defined the branched rough paths, and established the relationship between the evolvement of a branched rough path and Connes-Kremier Hopf algebra [CK99] (see also Butcher group [But72]). More recently, Friz & Hairer [FH14] summarized the key theorems in the theory of rough paths by using Gubinelli’s approach, combined with a brief introduction to the recent breakthrough of the theory of regularity structures [Hai14].

The theory of rough paths has a wealth of literature, and there are many other formulations, e.g. [Dav07], [FV08], [FalP08], [HN09], etc. For more detailed expositions, see [LQ02], [Lej03], [Lej09], [LCL07], [FV10].

The original integration in the theory of rough paths tends to linearize the group-valued path and treat it as the composition of several Banach-space valued paths with certain product structure. An alternative approach is to develop direct integration for one-forms against group-valued paths. We allow the one-forms on the group to vary with time. As a consequence, the integral is not restricted to use the same one-form where the path intersects itself. By introducing the cocyclic one-form on a group, we define the integral of a time-varying one form against a group-valued path, and give back e.g. the extension theorem and the rough integral in [Lyo98] and [Gub04], [Gub10]. We used the graded algebraic structure to ensure the existence of integral and to define the set of dominated paths, so our setting is not far from the tensor algebra used in [Lyo98] and [Gub04] and the Connes-Kremier Hopf algebra used in [Gub10]. The theory of rough paths provides a natural framework to integrate a group-valued path, and is the incentive of this paper.

If we would like to define the integration of a time-varying cocyclic one-form against a group-valued path, then there are (at least) three basic questions to settle: 1, what is a cocyclic one-form; 2, how to compare two cocyclic one-forms; 3, how should the cocyclic one-form vary with time so that we can integrate it against a
given group-valued path. Here we try to illustrate the idea with the polynomial one-form used in [Lyo98], and the rigorous definitions will be given in section 2.

A cocyclic one-form is a one-form on a topological group which can be integrated against any continuous path taking value in the group, and the value of the integral only depends on the path through the starting point and the increment on a fixed time interval (e.g. constant one-forms on a Banach space). When the space is flat, the one-form is invariant as it translocates. While if the surface is bent (e.g. a Lie group), then one has to take into consideration how the translocation acts on the one form and the translocated one-form should only depend on the original one-form and the structure of the group (e.g. left invariant vector field on a Lie group). Suppose $\mathcal{A}$ and $\mathcal{B}$ are two Banach algebras and $G$ is a topological group in $\mathcal{A}$. We denote by $L(\mathcal{A},\mathcal{B})$ the set of continuous linear mappings from $\mathcal{A}$ to $\mathcal{B}$, and denote by $C(G,L(\mathcal{A},\mathcal{B}))$ the set of one-forms on the group $G$: for $\beta \in C(G,L(\mathcal{A},\mathcal{B}))$ and $a \in G$, $\beta(a,\cdot) \in L(\mathcal{A},\mathcal{B})$ is a one-form at $a$. Then we say $\beta \in C(G,L(\mathcal{A},\mathcal{B}))$ is cocyclic if there exists a group $\mathcal{H}$ in $\mathcal{B}$ such that $\beta(a,b) \in \mathcal{H}$, $\forall a,b \in G$, and

$$
\beta(a,b) \beta(ab,c) = \beta(a,b-c), \ \forall a,b,c \in G.
$$

Cocyclic one-forms are of specific form but abundant; they are fundamental in integration. One non-trivial example of cocyclic one-form is the polynomial one-form used in [Lyo98]. For Banach spaces $\mathcal{V}$ and $\mathcal{U}$, suppose $p \in C(\mathcal{V},L(\mathcal{V},\mathcal{U}))$ is a degree-$(m-1)$ polynomial one-form for some integer $m \geq 1$, defined by (with $\otimes$ denoting the tensor product)

$$
p(z)(v) = \sum_{l=0}^{m-1} (D^l p)(0) \frac{z^\otimes_l}{l!}(v) = \sum_{l=0}^{m-1} (D^l p)(y) \frac{(z-y)^\otimes_l}{l!}(v), \ \forall z,y,v \in \mathcal{V},
$$

where $(D^l p)(y) \in L(\mathcal{V}^{\otimes l},L(\mathcal{V},\mathcal{U}))$ denotes the value at $y$ of the $l$-th derivative of $p$. Suppose $x$ is a continuous bounded variation path on $[0,T]$ taking value in $\mathcal{V}$, and let $S_m(x)$ denote the step-$m$ Signature of $x$:

$$
S_m(x)_{s,t} := 1 + \sum_{k=1}^{m} x^k_{s,t} \text{ with } x^k_{s,t} := \int \cdots \int_{s u_1 < \cdots < u_k \leq t} dx_{u_1} \otimes \cdots \otimes dx_{u_k}, \ \forall 0 \leq s \leq t \leq T.
$$

Based on Chen [Che91], $S_m(x)$ takes value in the step-$m$ nilpotent Lie group $G^m(\mathcal{V})$, and is a group-valued path satisfying Chen’s identity:

$$
S_m(x)_{s,u} S_m(x)_{u,t} = S_m(x)_{s,t}, \ \forall 0 \leq s \leq u \leq t \leq T,
$$

where the multiplication on the l.h.s. is in the tensor algebra. For $0 \leq s \leq t \leq T$, since $(D^l p)(x_s) \in L(\mathcal{V}^{\otimes l},L(\mathcal{V},\mathcal{U}))$ is symmetric in $\mathcal{V}^{\otimes l}$ and the projection of $x^l_{s,t}$ to the space of symmetric tensors is $(l!)^{-1}(x_t - x_s)^\otimes_l$, we have

$$
(D^l p)(x_s) \frac{(x_t - x_s)^\otimes_l}{l!}(v) = (D^l p)(x_s)(x^l_{s,t})(v), \ \forall v \in \mathcal{V}, \ \forall 0 \leq s \leq t \leq T.
$$

Then, based on the expressions (2) and (3), and by using $x^{l+1}_{s,t} = \int_s^t x^l_{s,v} \otimes dx_v$, we have

$$
\int_s^t p(x_v) dx_v = \sum_{l=0}^{m-1} (D^l p)(x_s) \int_s^t \frac{(x_v - x_s)^\otimes_l}{l!} \otimes dx_v = \sum_{l=0}^{m-1} (D^l p)(x_s) \int_s^t x^l_{s,v} \otimes dx_v = \sum_{l=0}^{m-1} (D^l p)(x_s) x^{l+1}_{s,t}.
$$

Then for $0 \leq s \leq u \leq t \leq T$, by integrating the polynomial one-form $p$ at (2) against $x$ and by using the simple property of integration $\int_a^b + \int_b^c = \int_a^c$, we have

$$
\sum_{l=0}^{m-1} (D^l p)(x_s) x^{l+1}_{s,u} + \sum_{l=0}^{m-1} (D^l p)(x_u) x^{l+1}_{u,t} = \sum_{l=0}^{m-1} (D^l p)(x_s) x^{l+1}_{s,t}.
$$

As a result, if we define $y : [0,T] \to \mathcal{U}$ by

$$
y_t := \sum_{l=0}^{m-1} (D^l p)(x_0) x^{l+1}_{0,t},
$$

2
then
\[ y_t - y_s = \sum_{l=0}^{m-1} (D^l p) (x_s) x_{s,t}^{l+1}, \quad \forall 0 \leq s \leq t \leq T. \] (5)

Based on the definition of the Signature at \([4]\), combined with the representation of \( y \) at \([3]\), we have
\[ S_m (y)_{s,t} = 1 + \sum_{k=1}^{m} \sum_{l \in \{0, \ldots, m-1\}} \left( (D^l p) \otimes \cdots \otimes (D^k p) \right) (x_s) \int_{s}^{t} \cdots \int_{s<u_1<\cdots<u_k<t} dx_{s,u_1}^{l+1} \otimes \cdots \otimes dx_{s,u_k}^{l+1}. \] (6)

Based on the representation \([4]\), if we define \( P \) by (with \( g \) denoting the Signature of \( x \))
\[ P (g_s, g_{s,t}) := S_m (y)_{s,t}, \forall 0 \leq s \leq t \leq T, \] (7)
then, by using the ordered shuffle product \([1]\) p73-74), it can be checked that, \( P \) is a well-defined function, which depends on the polynomial one-form \( p \) but is independent of the selection of \( g \), and \( P \) is linear in the second component \( g_{s,t} \). Moreover, based on \([7]\), \( P \) takes value in the step-\( m \) nilpotent Lie group \( G^m (U) \) and satisfies Chen’s identity:
\[ P (g_s, g_{s,u}) P (g_u, g_{u,t}) = P (g_s, g_{s,t}), \forall 0 \leq s \leq u \leq t \leq T. \] (8)

Comparing \([8]\) with the cocyclic property defined at \([1]\), it can be checked that, \( P \) is a cocyclic one-form on one nilpotent Lie group taking value in another nilpotent Lie group. The polynomial one-form is the basic ingredient for the rough integral in \([1]\), and could be treated as an informative example.

Suppose we are given two cocyclic one-forms which are close at a point \( a \) in the group, and we would like to switch from one one-form to another one-form at \( a \) with a controllable error. Since they are cocyclic, these two one-forms will be close on the whole group pointwisely (by which we mean that if a sequence of cocyclic one-forms converge at a specific point in the group then they converge on the whole group pointwisely).

However, if we want fairly sharp regularity condition on the one-form to integrate against a given group-valued path, then it is reasonable to compare these two one-forms only around \( a \) (or say, on a bounded set including \( a \)), because their difference will propagate based on the structure of the group (considering the left invariant vector fields on a Lie group). Taking the polynomial one forms as an example, suppose \( p \) and \( q \) are two degree-(\( m - 1 \)) polynomial one-forms (as at \([2]\)) which are close at 0. Then for \( l = 0, 1, \ldots, m - 1, \)
\[ (D^l p) (z) - (D^l q) (z) = \sum_{k=0}^{m-1-l} \left( (D^{l+k} p) (0) - (D^{l+k} q) (0) \right) \frac{z^{\otimes k}}{k!}, \forall z \in \mathcal{V}. \] (9)

When \( \|z\| \) is large, one is likely to capture max\( \sum_{k=0}^{m-1-l} \|(D^{l+k} p) (0) - (D^{l+k} q) (0)\| \) instead of \( \|(D^l p) (0) - (D^l q) (0)\| \). While in the theory of rough paths we chose carefully the inhomogeneous regularity of the one-forms to compensate the regularity of the path which we integrate against. However, based on \([1]\) this inhomogeneous distance will not be preserved if we compare these two one-forms at a point which is far from our reference point. As a result, if we would like to switch from one cocyclic one-form to another at point \( a \), then we will compare these two one-forms around \( a \), and the way that we compare these two one-forms is to consider them as linear operators over a graded Banach-space, equipped with a inhomogeneous norm.

Suppose \( g \) is a continuous path on \([0, T]\) taking value in a group \( G \), and \( \beta \) is a time-varying cocyclic one-form (or say, a continuous path taking value in cocyclic one-forms on \( G \)). We define
\[ \int_{s}^{t} \beta_{u} (g_u) \, dg_u := \lim_{|D| \to 0, D = \{(t_k)_{k=0}^{\infty} \} \subset [s, t]} \beta_{s} (g_s, g_{s,t_1}) \beta_{t_1} (g_{t_1}, g_{t_1,t_2}) \cdots \beta_{t_{n-1}} (g_{t_{n-1}}, g_{t_{n-1},t_n}), \forall 0 \leq s \leq t \leq T. \]

For the integral to make sense, how \( t \mapsto \beta_{t} \) varies will be related to how \( t \mapsto g_t \) varies. In particular, if the cocyclic one-form does not vary with time, then it is possible to integrate the one-form against any continuous group-valued path. More specifically, if \( \beta_{u} \equiv \beta_{0} \) is a cocyclic one-form which does not vary with time, then based on the cocyclic property \([3]\), we have
\[ \int_{s}^{t} \beta_{u} (g_u) \, dg_u = \int_{s}^{t} \beta_{0} (g_u) \, dg_u = \beta_{0} (g_s, g_{s,t}), \forall 0 \leq s \leq t \leq T, \forall g \in C ([0, T], G). \]
More generally, if we equip the one-forms with a homogeneous norm, then the regularity of the one-form will be preserved after translocation (see (9)). In that case, we can integrate a given time-varying cocyclic one-form against a whole class of group-valued path with certain regularity (with Young integral as an example). For a given group-valued path, when the time-varying cocyclic one-form has the "dual" regularity which compensates the regularity of the path, it is possible to integrate the one-form against the group-valued path. For polynomial cocyclic one-forms (see P at (7)), we can vary it with time to incorporate Lipschitz one-forms as in [Ly09] and also incorporate time-varying Lipschitz one-forms.

We introduce the cocyclic one-form on a group, and provide a natural way to extend and compare two one-forms initially defined at two different points in the group. The cocyclic one-form bears some similarities to left invariant vector fields on Lie groups, but our group could be infinite dimensional topological group and our one-form could take value in another Banach algebra. The integral exists under certain compensating regularity condition between the time-varying cocyclic one-form and the driving group-valued path, and the integral path is another group-valued path. The integral we developed here is related to the integral of rough paths of inhomogeneous degree of smoothness as in [Gyn12, LV06] and some discussions about their relationship can be found in Section 3.2.

For a given group-valued path, we define the set of dominated paths as those Banach-space valued paths which can be represented as the integral of a time-varying cocyclic one-form against the group-valued path. Under some structural assumptions on the group, the set of dominated paths is both a linear space and an algebra, has a canonical enhancement to a group-valued path, and is stable under composition with regular functions. There are minor differences between dominated paths and weakly controlled paths (defined in [Gb04] and [Gb10]), and some discussions about their relationship can be found in Section 3.2.

2 Existence of integral

2.1 Cocyclic one-form

We denote by \( L(E,F) \) (\( C(E,F) \)) the set of continuous linear (continuous) mappings from \( E \) to \( F \).

Suppose \( A \) and \( B \) are two Banach algebras, and \( G \) is a topological group in \( A \). We denote by \( C(G, L(A,B)) \) the set of one-forms on group \( G \), i.e. for \( \beta \in C(G, L(A,B)) \) and \( a \in G \), \( \beta(a,\cdot) \in L(A,B) \) is a one-form at \( a \).

**Definition 1** We say \( \beta \in C(G, L(A,B)) \) is a cocyclic one-form, if there exists a group \( H \) in \( B \) such that \( \beta(a,b) \in H \), \( \forall a,b \in G \), and
\[
\beta(a,bc) = \beta(a,b) \beta(ab,c), \quad \forall a,b,c \in G.
\] (10)

Denote the set of cocyclic one-forms by \( B(G,H) \) (or \( B(G) \)).

Based on (10), we have \( \beta(a,1_G) = 1_H, \forall a \in G \), and \( \beta(a,b)^{-1} = \beta(ab,b^{-1}), \forall a,b \in G \).

The cocyclic one-form is a purely algebraic object, and topology will only come in when we want to vary it with time.

**Proposition 2** \( \beta \in C(G, L(A,B)) \) is a cocyclic one-form iff there exist a group \( H \) in \( B \) and \( \alpha \in L(A,B) \) satisfying \( \alpha(G) \subseteq H \), such that
\[
\beta(a,b) = \alpha(b^{-1}) \alpha(ab), \quad \forall a,b \in G.
\]

**Proof.** \( \Leftarrow \) is clear. For \( \Rightarrow \), set \( \alpha(b) := \beta(1_G,b), \forall b \in G \). Then \( \beta(a,b) = \beta(1_G,a)^{-1} \beta(1_G,ab) = \alpha(a)^{-1} \alpha(ab), \forall a,b \in G \). □

Proposition 2 is simple, but is useful for constructing a cocyclic one-form.

**Definition 3** For \( g \in C([0,T],G) \) and \( \beta \in C([0,T],B(G,H)) \), the integral \( \int_0^t \beta_u(g_u) \, dg_u : [0,T] \to H \) is defined by \( g_{s,t} := g_s^{-1}g_t \)
\[
\int_0^t \beta_u(g_u) \, dg_u := \lim_{|D| \to 0, D = \{t_k\}_{k=0}^n \subset [0,t]} \beta_0(g_{0,0},g_{0,t_1}) \beta_{t_1}(g_{t_1,0},g_{t_1,t_2}) \cdots \beta_{t_{n-1}}(g_{t_{n-1},0},g_{t_{n-1},t}),
\] (11)

where \( 0 = t_0 < t_1 < \cdots < t_n = t \) and \( |D| := \max_{k=0}^{n-1}|t_{k+1} - t_k| \).
Moreover, we assume $T \triangleq \beta_\delta$ more specific, suppose there exists a set of linear mappings:

$$\beta(t) \text{ is induced by the comultiplication of the coalgebra on a finite family of graded projective mappings.}$$

To be specific, suppose there exists a consistent family of triples $\{\sigma_n\}_{n=1}^{\infty}$, where $\sigma_n : V \otimes V \rightarrow V$, $\forall n \in \mathbb{N}$.

For any $V \in \mathcal{P}$, we assume that $G$ is a consistent family of triples $\{\sigma_n\}_{n=1}^{\infty}$, where $\sigma_n : V \otimes V \rightarrow V$, $\forall n \in \mathbb{N}$.

We assume that $G$ is a closed topological group in $T^{(n)}(V)$ satisfying $\sigma_0(G_n) = 1$. ($G_n$ and $T^{(n)}(V)$ have consistent unit, multiplication and topology, but $G_n$ may be equipped with a different norm).

Assume that there exists a consistent family of triples $\{(T^{(n)}(V), G_n, P_n)\}_{n=1}^{\infty}$, i.e. for any $n \geq 1$, $P_n = \{\sigma|\sigma \in P_n, \sigma| \leq n\}$, and the mapping $1_n : T^{(n)}(V) \rightarrow T^{(n)}(V)$ defined by $1_n(a) = \sum_{\sigma \in P_n} \sigma(a)$, $\forall a \in T^{(n)}(V)$, is an algebra homomorphism, and is a group homomorphism from $G_n$ to $G_n$ satisfying $1_n(G_m) = G_n$. Moreover, we assume that there exist constants $N(n)$, $n \geq 0$, such that (with $N(\sigma)$ at (15))

$$(\#P_n) \lor \max_{\sigma \in P_n} N(\sigma) \leq N(n).$$

2.2 Algebraic formulation

Our algebraic structure is similar to the algebraic structure used in [Lyo98], [LCL07], [Gub04] and [Gub10].

Suppose $V$ is a Banach space. Following Def 1.25 [LCL07], we equip the tensor powers of $V$ with admissible norms: (with $Sym(k)$ denoting the symmetric group of degree $k$)

$$\|av\| = \|v\|, \forall a \in \mathcal{P}$$

$$\|u \otimes v\| \leq \|u\|\|v\|, \forall u \in V \otimes V.$$
2.3 Existence of integral

Definition 4 \( \omega : 0 \leq s \leq t \leq T \to \mathbb{R}^+ \) is a control if \( \omega \) is continuous, vanishes on the diagonal \( 0 \leq s = t \leq T \) and is super-additive: \( \omega (s, u) + \omega (u, t) \leq \omega (s, t), \forall 0 \leq s \leq u \leq t \leq T \).

Recall the triple \( (T^{(n)}(V), \mathcal{G}_n, \mathcal{P}_n) \) in Section 2.2 with the structural constant \( N(n) \) denoted at (10). Assume \( \mathcal{B} \) is a Banach algebra and \( \mathcal{H} \) is a topological group in \( \mathcal{B} \). (We do not assume that \( \mathcal{B} \) and \( \mathcal{H} \) satisfy the conditions in Section 2.2.)

Theorem 5 For \( g \in C([0, T], \mathcal{H}) \), suppose \( \beta \in C([0, T], B(\mathcal{H}, \mathcal{G}_n)) \) satisfy that, there exist \( M > 0 \), control \( \omega \) and \( \theta > 1 \) such that

\[
\max_{\sigma \in \mathcal{P}_n} \sup_{0 \leq s < t \leq T} \| \sigma (\beta_s(g_s, g_s)) \| \leq M, \tag{17}
\]

\[
\max_{\sigma \in \mathcal{P}_n} \| \sigma ((\beta_u - \beta_s)(g_u, g_u, t)) \| \leq \omega(s, t)^\theta, \forall 0 \leq s < u < t \leq T. \tag{18}
\]

Then the integral \( \int_0^T \beta_u(g_u) dg_u \in C([0, T], \mathcal{G}_n) \) exists, and there exists constant \( C_{n, \theta, M, \omega(0, T)} \), such that

\[
\max_{\sigma \in \mathcal{P}_n} \| \sigma \left( \int_0^t \beta_u(g_u) dg_u \right) - \sigma(\beta_s(g_s, g_s)) \| \leq C_{n, \theta, M, \omega(0, T)} \omega(s, t)^\theta, \forall 0 \leq s < t \leq T. \tag{19}
\]

Remark 6 Condition (18) represents the compensated regularity between \( \beta \) and \( g \).

Remark 7 The integral \( \int \beta(g) dg \) is continuous in the norm

\[
\max_{\sigma \in \mathcal{P}_n} \sup_{0 \leq s < t \leq T} \| \sigma (\beta_s(g_s, g_s)) \| + \max_{\sigma \in \mathcal{P}_n} \sup_{0 \leq s < u < t \leq T} \frac{\| \sigma ((\beta_u - \beta_s)(g_u, g_u, t)) \|}{\omega(s, t)^\theta}.
\]

Proof. Since \( \beta \in C([0, T], B(\mathcal{H}, \mathcal{G}_n)) \) and \( \sigma_0(\mathcal{G}_n) = 1 \), we have

\[
\sigma_0(\beta_s(a, b)) = 1, \forall a, b \in \mathcal{H}, \forall s \in [0, T]. \tag{20}
\]

For \([s, t] \subseteq [0, T]\) and finite partition \( D = \{t_j\}_{j=0}^l \subseteq [s, t], s = t_0 < t_1 < \cdots < t_l = t\), we denote

\[
\beta^{D}_{t_0, t_l} := \beta_{t_1}(g_{t_1}, g_{t_1, t_1 + 1}) \cdots \beta_{t_{l-1}}(g_{t_{l-1}, t_{l-1} + 1}).
\]

By using mathematical induction, we first prove that

\[
\max_{\sigma \in \mathcal{P}_n} \sup_{D, D \subseteq [s, t]} \| \sigma (\beta^{D}_{s, t}) - \sigma(\beta_s(g_s, g_s)) \| \leq C_{n, \theta, M, \omega(0, T)} \omega(s, t)^\theta, \forall 0 \leq s < t \leq T. \tag{21}
\]

which holds for \( \sigma_0 \), because based on (20), \( \sigma_0(\beta^{D}_{s, t}) = \sigma_0(\beta_s(g_s, g_s)) = 1 \). Suppose (21) holds for \( \{\sigma | |\sigma| \leq k, \sigma \in \mathcal{P}_n\} \) for some \( k = 0, 1, \ldots, n - 1 \). Combined with (17), we have

\[
M_k := \max_{|\sigma| \leq k, \sigma \in \mathcal{P}_n} \sup_{0 \leq s < u < t \leq T} \sup_{D, D \subseteq [s, t]} \| \sigma (\beta^{D}_{s, t}) \| \leq C_{n, \theta, M, \omega(0, T)}. \tag{22}
\]

Fix \( \sigma \in \mathcal{P}_n, |\sigma| = k + 1, [s, t] \subseteq [0, T] \) and a finite partition \( D = \{t_j\}_{j=0}^l \subseteq [s, t] \). By using that \( \beta_{t_{j-1}} \) is a cocyclic one-form, we have

\[
\beta^{D}_{s, t} - \beta^{D}_{s, t} = \beta^{D}_{s, t} \left( \beta_{t_j} - \beta_{t_{j-1}} \right) (g_{t_j}, g_{t_j, t_{j+1}}) \beta^{D}_{t_{j+1}, t}.
\]

Since the multiplication on \( \mathcal{G}_n \) is induced by the comultiplication on \( \mathcal{P}_n \), if

\[
((\triangle \otimes I_d) \circ \triangle) \sigma = \sum_{i} \sigma^{1,i} \otimes \sigma^{2,i} \otimes \sigma^{3,i}, \tag{23}
\]
then (by using the linearity of $\sigma \in \mathcal{P}_n$)

$$
\sigma \left( \beta_{s,t}^D \right) - \sigma \left( \beta_{s,t}^{D(t_j)} \right) = \sum_{i,|\sigma^2| \geq 1} \sigma^{1,i} \left( \beta_{s,t}^{D_{i,j}} \right) \sigma^{2,i} \left( (g_{t_j} - g_{t_{j+1}}) (g_{t_j}, g_{t_{j+1}}) \right) \sigma^{3,i} \left( \beta_{t_j,1,1}^D \right),
$$

(24)

where $|\sigma^{2,i}| \geq 1$ because $\sigma_0((\beta_{t_j} - g_{t_{j-1}})(g_{t_j}, g_{t_{j+1}})) = 0$. Since $|\sigma^{1,i}| + |\sigma^{2,i}| + |\sigma^{3,i}| = |\sigma|$, and $|\sigma^{2,i}| \geq 1$, we have $|\sigma^{1,i}| \vee |\sigma^{3,i}| \leq |\sigma| - 1 = k$. Combined with the definition of $N(n)$ at (16) and the inductive hypothesis at (22), we have, for $|\sigma| = k + 1$,

$$
\left\| \sigma \left( \beta_{s,t}^D \right) - \sigma \left( \beta_{s,t}^{D(t_j)} \right) \right\| \leq C_n M^2 \max_{|\sigma| \leq k+1, \sigma \in \mathcal{P}_n} \left\| \sigma' \left( \beta_{t_j} - \beta_{t_{j-1}} \right) (g_{t_j}, g_{t_{j+1}}) \right\| 
\leq C_n, \theta, M, \omega(0,T) \max_{|\sigma| \leq k+1, \sigma \in \mathcal{P}_n} \left\| \sigma' \left( \beta_{t_j} - \beta_{t_{j-1}} \right) (g_{t_j}, g_{t_{j+1}}) \right\|.
$$

Then, combined with (18), we have

$$
\left\| \sigma \left( \beta_{s,t}^D \right) - \sigma \left( \beta_{s,t}^{D(t_j)} \right) \right\| \leq C_n, \theta, M, \omega(0,T) \omega(t_{j-1}, t_{j+1})^\theta.
$$

For finite partition $D = \{t_j\}_{k=0}^l$ of $[s,t]$, as in Thm 1.16 [CL07] we select the interval $[t_{j-1}, t_{j+1}]$ such that

$$
\omega(t_{j-1}, t_{j+1}) \leq \frac{2}{l-1} \omega(s,t).
$$

(25)

By recursively removing partition point $t_j$ satisfying (26) (removing the middle point when $l = 2$), we have

$$
\left\| \sigma \left( \beta_{s,t}^D \right) - \sigma \left( \beta_{s,t}^{D(t_j)} \right) \right\| \leq C_n, \theta, M, \omega(0,T) \omega(s,t)^\theta.
$$

(26)

Then, (21) holds for $\sigma \in \mathcal{P}_n$, $|\sigma| = k + 1$, and we complete the mathematical induction. As a consequence, we have

$$
M_n := \max_{0 \leq s < t \leq T} \max_{D \subseteq [s,t]} \left\| \sigma \left( \beta_{s,t}^D \right) \right\| \leq C_n, \theta, M, \omega(0,T).
$$

(27)

Then we prove the existence of $\lim_{|D| \to 0} \beta_{s,t}^D$. If it exists, then based on (21), we have (19) holds. Suppose $D' = \{s_j\} \subseteq [s,t]$ is a refinement of $D = \{t_j\}_{j=0}^l \subseteq [s,t]$ i.e. for $j = 0, 1, \ldots, l$, there exists $n_j$ such that $s_{n_j} = t_j$. Similar as above, for $\sigma \in \mathcal{P}_n$, by using the comultiplication of $\sigma$ as at (23), we have

$$
\sigma \left( \beta_{s,t}^D \right) - \sigma \left( \beta_{s,t}^{D'} \right) = \sum_{j=0}^{l-1} \sigma \left( \beta_{s,t}^{D_{j+1},j} \right) \beta_{t_j,1,1}^{D_{j+1},j},
$$

$$
= \sum_{j=0}^{l-1} \sum_{i,|\sigma^2| \geq 1} \sigma^{1,i} \left( \beta_{s,t}^{D_{j+1},j} \right) \sigma^{2,i} \left( (g_{t_j} - g_{t_{j+1}}) (g_{t_j}, g_{t_{j+1}}) \right) \sigma^{3,i} \left( \beta_{t_j,1,1}^{D_{j+1},j} \right).
$$

Then, combined with the constant $N(n)$ defined at (16), $M_n$ defined at (27) and the estimate at (21), we have

$$
\left\| \sigma \left( \beta_{s,t}^D \right) - \sigma \left( \beta_{s,t}^{D'} \right) \right\| \leq C_n M^2 \sum_{j=0}^{l-1} \max_{\sigma \in \mathcal{P}_n} \left\| \sigma' \left( \beta_{t_j} - \beta_{t_{j+1}} \right) (g_{t_j}, g_{t_{j+1}}) \right\| 
\leq C_n, \theta, M, \omega(0,T) \sum_{j=0}^{l-1} \omega(t_j, t_{j+1})^\theta \leq C_n, \theta, M, \omega(0,T) \sup_{|u-v| \leq |D|} \omega(u,v)^{\theta-1}.
$$

For two general partitions $D$ and $\tilde{D}$ of $[s,t]$, we have

$$
\left\| \sigma \left( \beta_{s,t}^D \right) - \sigma \left( \beta_{s,t}^{\tilde{D}} \right) \right\| \leq \left\| \sigma \left( \beta_{s,t}^D \right) - \sigma \left( \beta_{s,t}^{D \cup \tilde{D}} \right) \right\| + \left\| \sigma \left( \beta_{s,t}^{D \cup \tilde{D}} \right) - \sigma \left( \beta_{s,t}^{\tilde{D}} \right) \right\| 
\leq C_n, \theta, M, \omega(0,T) \sup_{|u-v| \leq (|D| \vee |\tilde{D}|)} \omega(u,v)^{\theta-1}.
$$

Since $\theta > 1$, we have

$$
\int_s^t \beta_u (g_u) \, du := \lim_{|D| \to 0, D \subseteq [s,t]} \beta_{s,t}^D \text{ exists.}
$$
2.4 Extension theorem

As an application of the integral developed in Section 2.3 we prove the extension theorem (as in Thm 2.2.1 Lyon98, Prop 9 Gub11 and Thm 7.3 Gub10) and represent the extended group-valued path as the integral of a time-varying cocyclic one-form against the original group-value path.

Recall the Banach algebra $T^{(n)} (V)$ in Section 2.2 with multiplication induced by the comultiplication of $\mathcal{P}_n$ and $\sigma_0 \in \mathcal{P}_n$ denotes the projection of $T^{(n)} (V)$ to $\mathbb{R}$.

**Notation 8** For integer $n \geq 0$, we denote by $\mathcal{T}_n$ the closed topological group in $T^{(n)} (V)$ composed of all $a \in T^{(n)} (V)$ satisfying $\sigma_0 (a) = 1$.

For any $a \in \mathcal{T}_n$, $a^{-1}$ can be defined by $1 + \sum_{k=1}^n (a - 1)^k$. Since we assumed $\sigma_0 (G_n) = 1$, $G_n$ is a closed subgroup in $\mathcal{T}_n$.

**Notation 9** We equip $\mathcal{T}_n$ (so $G_n$) with

$$|a| := \sum_{\sigma \in \mathcal{P}_n, |\sigma| \geq 1} \|\sigma (a)\|^{\frac{1}{|\sigma|}}, \forall a \in \mathcal{T}_n. \quad (28)$$

For $p \geq 1$ and $g \in C ([0, T], \mathcal{T}_n)$, we define the $p$-variation of $g$ by $(g_{t, s} := g^{-1}_t g_s)$

$$\|g\|_{p\text{-var}, [0, T]} := \sup_{D, D \subseteq [0, T]} \left( \sum_{k, t_k \in D} |g_{t_k, t_{k+1}}|^p \right)^{\frac{1}{p}}. \quad (29)$$

Denote the set of continuous paths of finite $p$-variation on $[0, T]$ taking value in the group $\mathcal{T}_n$ by $C^{p\text{-var}} ([0, T], \mathcal{T}_n)$ (similarly define the set of paths $C^{p\text{-var}} ([0, T], G_n)$).

For integers $m \geq n \geq 1$, recall the algebra homomorphism $1_n$ in Section 2.2 from $T^{(m)} (V)$ to $T^{(n)} (V)$ defined by $1_n (a) = \sum_{\sigma \in \mathcal{P}_n} \sigma (a), \forall a \in T^{(m)} (V)$, and $1_n$ is a group homomorphism from $G_m$ to $G_n$ satisfying $1_n (G_m) = G_n$.

Generally, the extended group-valued path takes value in $\mathcal{T}_n$. To guarantee that the extended path takes value in $G_n$, we further assume that $G_n$ is "large" enough to accommodate the extended path. More specifically, we assume that

**Condition 10** For any integer $n \geq 1$, there exists a constant $C_n > 0$ such that, for any $a \in G_n$ there exists $\tilde{a} \in G_{n+1}$ satisfying $1_n (\tilde{a}) = a$ and $|\tilde{a}| \leq C_n |a|$ (with $|$ defined at 28).

For the nilpotent Lie group, we could let $\tilde{a} := \exp_{n+1} (\log_n a)$ with log and exp defined by algebraic series and the lower index $n$ indicates the level of truncation. For Butcher group, we could let $\tilde{a} := a + \sum_{k=2}^{n+1} \sum_{\sigma \in \mathcal{P}_n, |\sigma|+\cdots+|\sigma_k|=n+1} \sigma_1 (a) \otimes \cdots \otimes \sigma_k (a)$. Then, it can be checked that, Condition 10 holds in both cases.

**Example 11 (Extension theorem)** For $g \in C^{p\text{-var}} ([0, T], \mathcal{T}_p)$ and integer $n \geq [p] + 1$, there exists a unique $g^n \in C^{p\text{-var}} ([0, T], \mathcal{T}_n)$ satisfying $g_0 = 1$ and $1_p (g^n_t) = g_{0, t}, \forall t \in [0, T]$. Moreover, there exists a $\beta \in C ([0, T], B (\mathcal{T}_p, \mathcal{T}_n))$ such that ($\beta, g$) satisfies the conditions of Theorem 5 and

$$g^n_t = \int_0^t \beta_g (g_s) dg_s, \forall t \in [0, T].$$

There exists a constant $C_{n, p}$ (which only depends on $n$ and $p$) such that

$$\|g^n\|_{p\text{-var}, [s, t]} \leq C_{n, p} \|g\|_{p\text{-var}, [s, t]}, \forall 0 \leq s \leq t \leq T. \quad (30)$$

If we further assume that $g$ takes value in $G_p$ and that Condition 10 holds, then $g^n$ takes value in $G_n$.

**Proof.** Define $\beta \in C ([0, T], B (\mathcal{T}_p, \mathcal{T}_n))$ by

$$\beta_{a,b} (a, b) := (1_p (g^{-1}_s a))^ {-1} 1_p (g^{-1}_s ab), \forall a, b \in \mathcal{T}_p, \forall s \in [0, T], \quad (31)$$
where we used the implicit identification of $T\{a\}$ as a subset of $T_n$, and the operations in $[31]$ are in $T_n$ except that the multiplication between $a$ and $b$ is in $T\{a\}$. Then for $s \leq t$, $\beta_s (g_s, g_{s,t}) = g_{s,t}$. For $s < u < t$,

$$
(\beta_u - \beta_s) (g_u, g_{u,t}) = 1\{p\} (g_{u,t}) - (1\{p\} (g_{u,u}))^{-1} 1\{p\} (g_{s,t})
$$

$$
= (1\{p\} (g_{u,u}))^{-1} (1\{p\} (g_{u,u}) 1\{p\} (g_{u,t}) - 1\{p\} (g_{s,t}))
$$

$$
= (1\{p\} (g_{u,u}))^{-1} (\sum_{[\sigma^1, \sigma^2, \sigma^3]} \sigma^1 (g_{u,u}) \sigma^2 (g_{u,t})) .
$$

Define control $\omega$ by

$$
\omega (s, t) := \|g\|^{p}_{p-\text{var},[s,t]}, \forall 0 \leq s \leq t \leq T .
$$

Hence, with the constant $N(n)$ defined at $[10]$, we have

$$
\max_{\sigma \in P_n} \|\sigma (\beta_u - \beta_s) (g_u, g_{u,t})\| \leq C_{n,\omega (0,T)} \|g\|^{[p]+1}_{p-\text{var},[s,t]}, \forall 0 \leq s \leq u \leq t \leq T .
$$

On the other hand, since $\beta_s (g_s, g_{s,t}) = g_{s,t}$, $\forall s < t$, we have

$$
\max_{\sigma \in P_n} \sup_{0 \leq s \leq t \leq T} \|\sigma (\beta_s (g_s, g_{s,t}))\| = \max_{\sigma \in P_n} \sup_{0 \leq s \leq t \leq T} \|\sigma (g_{s,t})\| \leq \omega (0,T)^\frac{1}{p} \vee \omega (0,T)^\frac{1}{p+1} < \infty .
$$

Then, based on Theorem $[5]$ we can integrate $\beta$ against $g$, and

$$
\max_{\sigma \in P_n} \left\| \sigma \left( \int_s^t \beta_u (g_u) \, dg_u \right) - \sigma (\beta_s (g_s, g_{s,t})) \right\| \leq C_{n,\omega (0,T)} \|g\|^{[p]+1}_{p-\text{var},[s,t]}, \forall 0 \leq s \leq t \leq T .
$$

Based on the definition of integral and that $\beta_s (g_s, g_{s,t}) = g_{s,t}$, $\forall s < t$, we have

$$
1\{p\} \left( \int_s^t \beta_u (g_u) \, dg_u \right) = \beta_s (g_s, g_{s,t}) = g_{s,t}, \forall 0 \leq s \leq t \leq T ,
$$

so $1\{p\} \left( \int_0^t \beta_u (g_u) \, dg_u \right) = g_{0,t}, \forall t \in [0, T]$. On the other hand, if $h \in C^{p-\text{var}} ([0,T], T_n)$ satisfies $1\{p\} (h_t) = g_{0,t}, \forall t \in [0, T]$, then $h_{0,t} = \int_0^t \beta_u (g_u) \, dg_u, \forall t \in [0, T]$. Indeed, based on $[32]$ and $[33]$, we have,

$$
\max_{\sigma \in P_n \setminus P\{p\}} \left\| \sigma \left( \int_s^t \beta_u (g_u) \, dg_u \right) \right\| \leq C_{n,\omega (0,T)} \|g\|^{[p]+1}_{p-\text{var},[s,t]}, \forall 0 \leq s \leq t \leq T .
$$

Since $1\{p\} (h_{s,t}) = g_{s,t} = 1\{p\} \left( \int_s^t \beta_u (g_u) \, dg_u \right), \forall s < t$, we have,

$$
\left\| \sigma \left( \int_s^t \beta_u (g_u) \, dg_u \right) - \sigma (h_{s,t}) \right\| = 0, \forall \sigma \in P\{p\}, \forall 0 \leq s \leq t \leq T .
$$

On the other hand, since $\int \beta_u (g_u) \, dg_u$ satisfies $[41]$ and $h$ is of finite $p$-variation, we have

$$
\left\| \sigma \left( \int_s^t \beta_u (g_u) \, dg_u \right) - \sigma (h_{s,t}) \right\| \leq C_{n,\omega (0,T)} \|g\|^{[p]+1}_{p-\text{var},[s,t]} + \|h\|^{[\sigma]}_{p-\text{var},[s,t]}, \forall \sigma \in P_n \setminus P\{p\}, \forall 0 \leq s \leq t \leq T .
$$

For $\sigma \in P_n \setminus P\{p\}$, suppose $((\Delta \otimes Id) \circ \Delta) \sigma = \sum_i \sigma^{1,i} \otimes \sigma^{2,i} \otimes \sigma^{3,i}$. Then, for any $0 \leq s \leq t \leq T$,

$$
\left\| \sigma \left( \int_s^t \beta_u (g_u) \, dg_u \right) - \sigma (h_{s,t}) \right\| \leq \lim_{|D| \to 0, D \subseteq [s,t]} \sum_{k,t_k \in D} \sum_{i} \gamma^{2,i} (h_{t_k,t_{k+1}}) \left\| \sigma^{3,i} (h_{t_k,t_{k+1}}) \right\| \leq C_{n,\omega (0,T)} \|h\|^{[p]+1}_{p-\text{var},[s,t]} + \lim_{|D| \to 0, D \subseteq [s,t]} \sum_{k,t_k \in D} \left( \|g\|^{[p]+1}_{p-\text{var},[t_k,t_{k+1}]} + \|h\|^{[p]+1}_{p-\text{var},[t_k,t_{k+1}]} \right) = 0 .
$$
Hence, if we can prove \( \| f_0^t \beta_u (g_u) \, dg_u \|_{p \text{-var}, [0, T]} < \infty \), then \( f_0^t \beta_u (g_u) \, dg_u \) is the unique path in \( C^{p \text{-var}} ([0, T], \mathcal{T}_n) \) which satisfies \( 1_{[p]} \left( f_0^t \beta_u (g_u) \, dg_u \right) = g_{0,t}, \forall t \in [0, T] \). For \( m = [p], \ldots, n \), we define \( g^m \in C ([0, T], \mathcal{T}_m) \) by

\[
\begin{align*}
g^m_t := 1_m \left( f_0^t \beta_u (g_u) \, dg_u \right), \quad \forall 0 \leq t \leq T, \\
\text{(so } g^m_t = g_{0,t}, \forall t\text{), define control } \omega^m \text{ by}
\end{align*}
\]

\[
\omega^m (s, t) := \| g^m \|_{p \text{-var}, [s, t]}^p, \quad \forall 0 \leq s \leq t \leq T,
\]

and, similar as at (31), define \( \beta^m \in C ([0, T], B(\mathcal{T}_m, \mathcal{T}_{m+1})) \) by

\[
\beta^m(a, b) := \left( 1_m \left( (g^m_s)^{-1} a \right) \right)^{-1} 1_m \left( (g^m_s)^{-1} ab \right), \quad \forall a, b \in \mathcal{T}_m, \forall s \in [0, T].
\]

Suppose \( \| g^m \|_{p \text{-var}, [0, T]} < \infty \), which holds when \( m = [p] \). Then by following similar arguments as above, the integral path \( f_0^t \beta^m \left( g^m_u \right) \, dg_u \in C ([0, T], \mathcal{T}_{m+1}) \) exists and satisfies

\[
\begin{align*}
1_m \left( f_0^t \beta^m \left( g^m_u \right) \, dg_u \right) = g^m_{0,t} = g^m_t = 1_m \left( f_0^t \beta_u (g_u) \, dg_u \right), \quad \forall 0 \leq t \leq T, \\
\max_{\sigma \in \mathcal{P}_{m+1} \setminus \mathcal{P}_m} \| \delta (f_0^t \beta^m \left( g^m_u \right) \, dg_u) \| & \leq C_{m+1, p, \omega^m (0, T)} \| g^m \|_{p \text{-var}, [s, t]}^{m+1}, \quad \forall 0 \leq s \leq t \leq T.
\end{align*}
\]

Then, (37) and (35) imply

\[
\| f_0^t \beta^m \left( g^m_u \right) \, dg_u \|_{p \text{-var}, [s, t]} \leq C_{m+1, p, \omega^m (0, T)} \| g^m \|_{p \text{-var}, [s, t]}, \quad \forall 0 \leq s \leq t \leq T.
\]

Combining (37), (38) with (34), and by using similar argument that used to prove (35), we have

\[
g^m_{t+1} = 1_{m+1} \left( f_0^t \beta^m \left( g^m_u \right) \, dg_u \right) = f_0^t \beta^m \left( g^m_u \right) \, dg_u, \quad \forall 0 \leq t \leq T.
\]

Then, based on (34), we have,

\[
\| g^m \|_{p \text{-var}, [s, t]} \leq C_{m+1, p, \omega^m (0, T)} \| g^m \|_{p \text{-var}, [s, t]}, \quad \forall 0 \leq s \leq t \leq T.
\]

Since (11) holds for \( m = [p], \ldots, n-1 \) and the constant \( C_{m+1, p, \omega^m (0, T)} \) can be chosen to be monotone in \( m + 1 \) and \( \omega^m (0, T) \), we have, for any \( 0 \leq s \leq t \leq T \),

\[
\begin{align*}
\| f_0^t \beta_u (g_u) \, dg_u \|_{p \text{-var}, [s, t]} &= \| g^m \|_{p \text{-var}, [s, t]} \\
&\leq C_{n, p, \omega^m (0, T)} \| g^p \|_{p \text{-var}, [s, t]} = C_{n, p, \omega^m (0, T)} \| g \|_{p \text{-var}, [s, t]} < \infty.
\end{align*}
\]

The constant \( C_{n, p, \omega^m (0, T)} \) in (12) can be chosen to be independent of \( \omega (0, T) = \| g \|_{p \text{-var}, [0, T]} \). Indeed, when when \( \| g \|_{p \text{-var}, [0, T]} = 0 \), we have \( g = 0 \) so (12) holds for any choice of the constant \( C_{n, p, \omega^m (0, T)} \), which surely could be chosen to be independent of \( \omega (0, T) \). When \( \| g \|_{p \text{-var}, [0, T]} > 0 \), denote \( c := \| g \|_{p \text{-var}, [0, T]} > 0 \) and define \( \delta_c g \in C^{p \text{-var}} ([0, T], \mathcal{T}_m) \) by \( \delta_c g := \sum_{\sigma \in \mathcal{P}_{[p]}} c^{\sigma} \sigma (g) \). Then \( \| \delta_c g \|_{p \text{-var}, [0, T]} = 1 \), and with \( \beta \in C ([0, T], B(\mathcal{T}_m, \mathcal{T}_{m+1})) \) defined at (31), for any \( 0 \leq s \leq t \leq T \), we have

\[
\begin{align*}
c \| f_0^t \beta_u (g_u) \, dg_u \|_{p \text{-var}, [s, t]} &= \| \delta_c \left( f_0^t \beta_u (g_u) \, dg_u \right) \|_{p \text{-var}, [s, t]} = \| \int_0^t \beta_u ((\delta_c g)_u) \, d(\delta_c g)_u \|_{p \text{-var}, [s, t]} \leq C_{p, n} \| \delta_c g \|_{p \text{-var}, [s, t]} \leq c C_{p, n} \| g \|_{p \text{-var}, [s, t]},
\end{align*}
\]

where \( \delta_c \left( f_0^t \beta_u (g_u) \, dg_u \right) := \sum_{\sigma \in \mathcal{P}_{[p]}} c^{\sigma} \sigma \left( f_0^t \beta_u (g_u) \, dg_u \right) \) and the equality

\[
\delta_c \left( f_0^t \beta_u (g_u) \, dg_u \right) = \int_0^t \beta_u ((\delta_c g)_u) \, d(\delta_c g)_u
\]
follows from the uniqueness of extension we just proved, because $1_{[p]}(\delta_c(f_0^β_u(g_u)\,dg_u)) = 1_{[p]}(\delta_c)g = 1_{[p]}(f_0^β_u((\delta_c g)_u)\,d(\delta_c g)_u)$, and both $\delta_c(f_0^β_u(g_u)\,dg_u)$ and $f_0^β_u((\delta_c g)_u)\,d(\delta_c g)_u$ are of finite $p$-variation based on (42).

Then we check that, if $g \in C^{p-var}([0,T],G_{[p]})$ and Condition (10) holds, then the integral path $\int_0^β_u(g_u)\,dg_u$ takes value in $G_m$. For $m = [p], \ldots, n - 1$, with $g^m$ defined at (33), we suppose that $g^m$ takes value in $G_m$, which holds when $m = [p]$. Based on Condition (10) there exists a constant $C_m > 0$ such that for any $0 \leq s < t \leq T$ there exists $a^{s,t}_{m+1} \in G_{m+1}$ such that

$$1_m(a^{s,t}_{m+1}) = g^m_{s,t} \text{ and } \sum_{\sigma \in P_{m+1}\setminus P_m} \|\sigma(a^{s,t}_{m+1})\| \leq C_m \|g^m\|_{p-var,[s,t]}^{m+1}.$$  (43)

For $t \in [0,T]$ and finite partition $D = \{t_j\}_{j=0}^l$ of $[0,t]$, $0 = t_0 < \cdots < t_l = t$, we denote $g^{D,m+1}_t \in \mathcal{T}_{m+1}$ and $a^{D,m+1}_t \in G_{m+1}$ by (with multiplications in $\mathcal{T}_{m+1}$)

$$g^{D,m+1}_t := 1_m(g^{m}_{t_0,t_1}) \cdots 1_m(g^{m}_{t_{l-1},t_l}) \text{ and } a^{D,m+1}_t := a^{t_0,t_1}a^{t_1,t_2} \cdots a^{t_{l-1},t_l}.$$  

Then since $1_m(g^{m}_{t_j,t_{j+1}}) = 1_m(a^{t_j,t_{j+1}})$, $j = 0, 1, \ldots, l - 1$, it can be checked that,

$$g^{D,m+1}_t - a^{D,m+1}_t = - \sum_{j=0}^{l-1} \sum_{\sigma \in P_{m+1}\setminus P_m} \sigma(a^{t_j,t_{j+1}}).$$

Combined with (43), we have $(m + 1 \geq [p] + 1 > p)$

$$\|g^{D,m+1}_t - a^{D,m+1}_t\| \leq \sum_{j=0}^{l-1} \sum_{\sigma \in P_{m+1}\setminus P_m} \|\sigma(a^{t_j,t_{j+1}})\| \leq C_m \sum_{j=0}^{l-1} \|g^m\|_{p-var,[t_j,t_{j+1}]}^{m+1} \leq \|g^m\|_{p-var,[0,T]}^{m+1} \text{ for all } u \in [0,T].$$

which tends to zero as $|D| \to 0$, so if $g^{D,m+1}_t$ converge to a limit as $|D| \to 0$, then $a^{D,m+1}_t$ converge to the same limit as $|D| \to 0$. Hence, by using (40), we have

$$g^{m+1}_t = \int_0^t \beta_u^m(g_u^m)\,dg_u = \lim_{|D| \to 0, D = \{t_j\}_{j=0}^l \subset [0,t]} 1_m(g^{m}_{t_0,t_1}) \cdots 1_m(g^{m}_{t_{l-1},t_l}) = \lim_{|D| \to 0, D \subset [0,t]} g^{D,m+1}_t = a^{D,m+1}_t, \forall t \in [0,T].$$

Since $a^{D,m}_t \in G_m$ and $G_{m+1}$ is a closed topological group, we have that $t \mapsto g^{m+1}_t = 1_m(\int_0^t \beta_u(g_u)\,dg_u)$ takes value in group $G_{m+1}$.

3 Dominated path

For Banach space $\mathcal{V}$, suppose $\{(T^{(n)}(\mathcal{V}),G_n,P_n)\}_{n=0}^{\infty}$ is a family of triples as in Section 2.2

3.1 Structural assumptions on the group

The set of dominated paths are Banach-space valued paths which can be represented as the integral of a time-varying cocyclic one-form against a given group-valued path. We would like the set of dominated paths to be stable under certain operations, which imposes some structural assumptions on the group.

Condition 12 $T^{(n)}(\mathcal{V})$ is the smallest Banach space which includes $G_n$, in the sense that, for Banach space $\mathcal{U}$ and $\alpha \in L(T^{(n)}(\mathcal{V}),\mathcal{U})$, if $\alpha(g) = 0, \forall g \in G_n$, then $\alpha(v) = 0, \forall v \in \mathcal{V}^{\otimes k}$, $k = 0, \ldots, n$.

Condition 13 For $\{\sigma^i\}_{i=1}^k \subset \mathcal{P}_n$, $k \geq 1$, there exists $\sigma^1 \ast \sigma^2 \ast \cdots \ast \sigma^k \in L(\mathcal{V}^{\otimes |\sigma|},\mathcal{V}^{\otimes |\sigma|} \otimes \cdots \otimes \mathcal{V}^{\otimes |\sigma|})$ (where $|\sigma| := \sum_{i=1}^k |\sigma^i|$) satisfying

$$(\sigma^1 \ast \sigma^2 \ast \cdots \ast \sigma^k)(a) = \sigma^1(a) \otimes \sigma^2(a) \otimes \cdots \otimes \sigma^k(a), \forall a \in G_n, \forall n \geq |\sigma|. \quad (44)$$
The equality (14) holds when \( a \in G_n \), but (14) may not hold when \( a \in T^{(n)}(V) \).

It is always possible to extend the algebra (and group) by adding in monomials of projective mappings so that Condition (13) holds.

**Condition 14** There exists a continuous linear mapping \( \mathcal{I} \) from \( T^{(2n)}(V) \) to \( T^{(n)}(V)^{\otimes 2} \) satisfying

\[
\mathcal{I}(V^{\otimes k}) \subseteq \sum_{j_1+j_2=k,j_1=1,\ldots,n} \delta g_{s,\sigma_1} \otimes \delta g_{s,\sigma_2}, \quad k = 0, 1, \ldots, 2n,
\]

and (with \( 1_n \) denoting the projection to \( T^{(n)}(V) \))

\[
\mathcal{I}(ab) = \mathcal{I}(a) + (1_n(a) \otimes 1_n(a)) \mathcal{I}(b) + (1_n(a) - 1) \otimes (1_n(a)(1_n(b) - 1)), \quad \forall a, b \in G_{2n},
\]

where \((a_1 \otimes b_1)(a_2 \otimes b_2) := (a_1a_2) \otimes (b_1b_2), \forall a_1, b_1 \in T^{(n)}(V)\).

For a possible choice of \( \mathcal{I} \), if for any \( g \in C([0, T], G_n) \), the "formal" integral \( \int \int_{0 < u_1 < u_2 < T} \delta g_{s,u_1} \otimes \delta g_{s,u_2} \) is well-defined and can be represented as a universal continuous linear function of extended \( g_{0,T} \), then we can define

\[
\mathcal{I}(a) := \int \int_{0 < u_1 < u_2 < T} \delta g_{0,u_1} \otimes \delta g_{0,u_2}, \quad \forall g \in C([0, T], G_n), \quad g_{0,T} = 1_n(a), \quad \forall a \in G_{2n},
\]

which is defined on \( G_{2n} \) but extends linearly to \( T^{(2n)}(V) \) based on Condition (12). In the definition of \( \mathcal{I} \) at (17), we assumed that \( \mathcal{I}(a) = \mathcal{I}(b) \) if \( 1_n(a) = 1_n(b) \) for \( a, b \in G_{2n} \), and we assumed that \( \int \int_{0 < u_1 < u_2 < T} \delta g_{0,u_1} \otimes \delta g_{0,u_2} \) is a linear function of extended \( g_{0,T} \), which is independent of the selection of \( g \) and independent of the fine structure of \( g \) as a path from \( g_0 \) to \( g_T \). Then (16) follows from \( \int \int_{s < u_1 < u_2 < t} = \int \int_{s < u_1 < u_2 < u} + \int \int_{u_1 < u_2 < t} + \int \int_{s < u_1 < u < t} \int \int_{u < u_2} \) for \( s < u < t \), and (15) holds if \( \delta (1) = 0 \).

For the space of (weak) geometric rough paths, \( G_n \) is the step-\( n \) nilpotent Lie group \( G^{(n)}(V) \); \( \mathcal{P}_n \) is the set of projective mappings \( \{\pi_k\}_{k=0}^n \) with \( \pi_k \) denoting the projection to \( V^{\otimes k} \) and \( \Delta \pi_k = \sum_{j=0}^{k} (\pi_j \otimes \pi_{k-j}) \), \( k = 0, 1, \ldots, n \). Condition (12) holds, because, if \( \alpha \in L(T^{(n)}(V), U) \) satisfies \( \alpha(g) = 0, \forall g \in G_n \), then by using finite dimensional argument we have \( \alpha(v_1 \otimes \cdots \otimes v_k) = 0, \forall \{v_i\}_{i=1}^k \subseteq V, k = 0, 1, \ldots, n \), which implies \( \alpha(v) = 0, \forall v \in \mathcal{V}^{\otimes k} \). Condition (13) is satisfied by using the shuffle product (p36 [LCL07]). Condition (14) is satisfied because any \( g \in C([0, T], G_n) \) satisfies the formal differential equation \( \delta g_t = g_t \delta x_t \) with \( x \) denoting the first level of \( g \). Then,

\[
\int \int_{0 < u_1 < u_2 < T} \delta g_{0,u_1} \otimes \delta g_{0,u_2} = \int T_0(g_0, u - 1) \otimes g_0 \delta x_u, \quad \forall g \in C([0, T], G_n).
\]

Then combined with Condition (13) the mapping \( \mathcal{I} \) can be defined as at (17). (Equivalently, \( \mathcal{I} \) can be defined by using the ordered shuffle product, see Def 4.2 [LCL07].)

For the space of branched rough paths, \( G_n \) is the Butcher group (i.e. \( G_n \) is a group in \( T^{(n)}(V) \) whose elements are indexed by forests and whose multiplication is induced by the comultiplication in the Connes-Kreimer Hopf algebra); \( \mathcal{P}_n = \{\sigma| \sigma \leq n\} \) is the set of forests of degree less or equal to \( n \) and \( \Delta \sigma \) denotes the comultiplication in the Connes-Kreimer Hopf algebra. Condition (12) holds for similar reasons as for the nilpotent Lie group. Condition (13) holds, because \( \sigma^1(a) \otimes \cdots \otimes \sigma^k(a) \) is part of \( a \) (see e.g. [CK99]). Moreover, (based on Thm 1 [Gub04] and Thm 8.5 [Gub010]), any \( g \in C([0, T], G_n) \) satisfies the formal differential equations that, for trees \( \{\sigma^1\}_{i=1}^k \subseteq \mathcal{P}_n, k \geq 1 \), with \( \sigma^1 \cdots \sigma^k \) denoting the tree obtained by attaching \( \{\sigma^i\}_{i=1}^k \) to a new root and \( x \) denoting the first level of \( g \),

\[
\delta \left[ \sigma^1 \cdots \sigma^k \right](g_t) = \sigma^1(g_t) \cdots \otimes \sigma^k(g_t) \otimes \delta x_t
\]

\[
\delta \left( \sigma^1 \cdots \sigma^k \right)(g_t) = \sum_{i=1}^k \sigma^1(g_t) \otimes \cdots \otimes \delta \left( \sigma^i(g_t) \right) \otimes \cdots \otimes \sigma^k(g_t).
\]

Then, based on these differential equations and Condition (13) the mapping \( \mathcal{I} \) can be defined as at (17).

### 3.2 Definition of dominated path

**Notation 15** For Banach space \( U \) and \( \alpha \in L(T^{(n)}(V), U) \), we denote

\[
\|\alpha(\cdot)\| := \sup_{v \in T^{(n)}(V), \|v\| = 1} \|\alpha(v)\|,
\]

\[
\|\alpha(\cdot)\|_k := \sup_{v \in V^{\otimes k}, \|v\| = 1} \|\alpha(v)\|, \quad k = 0, 1, \ldots, n.
\]
Condition 16 Suppose \( \{(T^{(n)}(\mathcal{V}), \mathcal{G}_n, \mathcal{P}_n)\}_{n=0}^{\infty} \) satisfies Conditions 12 and 13. For \( g \in C^{p,\text{var}}([0,T], \mathcal{G}_p) \) and Banach space \( \mathcal{U} \), assume \( \beta \in C([0,T], B(\mathcal{G}_p, \mathcal{U})) \) satisfies that there exist \( M > 0 \), control \( \omega \) and \( \theta > 1 \), such that
\[
\| \beta_t(g_t, \cdot) \| \leq M, \forall t,
\]
\[
\| (\beta_t - \beta_s) (g_t, \cdot) \|_k \leq \omega(s,t)^{1-k} \| p \|, \forall 0 \leq s < t \leq T, \quad k = 1, 2, \ldots, [p].
\] (48)

If \( \beta \) satisfies Condition 16, then \( \beta \) satisfies the conditions of Theorem 5, so \( \beta \) is integrable against \( g \) and the integral \( \int \beta_u(g_u) \, dg_u \) exists. Indeed, for \( s < u < t \), we have,
\[
\| (\beta_u - \beta_s) (g_u, g_{u,t}) \| \leq \sum_{\sigma \in \mathbb{P}_p} \| (\beta_u - \beta_s) (g_u, \sigma (g_{u,t})) \| \leq \sum_{\sigma \in \mathbb{P}_p} \| (\beta_u - \beta_s) (g_u, \cdot) \|_\sigma \| \sigma (g_{u,t}) \| \leq \sum_{\sigma \in \mathbb{P}_p} \omega(s,u) \| p \| \| p \|_{\text{var},[s,t]} \leq C_p \omega(s,t) + \| p \| \| p \|_{\text{var},[s,t]}^\theta.
\]

Definition 17 (Dominated path) Suppose \( g \in C^{p,\text{var}}([0,T], \mathcal{G}_p) \) and \( \mathcal{U} \) is a Banach space. We say \( h \in C([0,T], \mathcal{U}) \) is dominated by \( g \), if there exists \( \beta \in C([0,T], B(\mathcal{G}_p, \mathcal{U})) \) satisfying Condition 16, such that
\[
h_t = \int_0^t \beta_u(g_u) \, dg_u, \forall t \in [0,T].
\] (49)

The dominated path defined here is similar to the weakly controlled path in [Gub04] and [Gub10]. Suppose \( \mathcal{G} \) is the Butcher group. Based on [Gub04] and [Gub10], \( \gamma \in C([0,T], \mathcal{U}) \) is weakly controlled by \( g \), if there exists a family of paths \( \gamma^\sigma \in C([0,T], L(\mathcal{G}^\sigma \mathcal{G}, \mathcal{U})), \sigma \in \mathbb{P}_p-1, |\sigma| \geq 1 \), and constants \( C > 0, \theta > 1 \), such that, \( \gamma \) satisfies
\[
\| \gamma_t - \gamma_s - \sum_{\sigma \in \mathbb{P}_p-1, |\sigma| \geq 1} \gamma^\sigma (g_{s,t}) \| \leq C(\| g \|_{\text{var},[s,t]})^{\theta - \frac{1}{p}}, \forall 0 \leq s < t \leq T,
\] (50)
and \( \gamma^\sigma, \sigma \in \mathbb{P}_p-1, |\sigma| \geq 1 \), satisfies
\[
\| \gamma_t^\sigma - \gamma_s^\sigma - \sum_{\sigma \in \mathbb{P}_p-1, |\sigma| \geq 1} c(\sigma^1, \sigma^2, \sigma) \gamma^\sigma (g_{s,t}) \| \leq C(\| g \|_{\text{var},[s,t]})^{\theta - \frac{1+|\sigma|}{p}}, \forall 0 \leq s < t \leq T,
\] (51)
where \( c(\sigma^1, \sigma^2, \sigma) \) counts the number of \( \sigma^2 \otimes \sigma \) in the reduced co-multiplication \( \Delta^1 \sigma^1 = \Delta \sigma^1 - \sigma_0 \otimes \sigma_1 - \sigma_1 \otimes \sigma_0 \) (with \( \sigma_0 \) denoting the projection to \( \mathbb{R} \)).

Then we rewrite (50) and (51) in term of time-varying cocyclic one-forms. Define \( \beta \in C([0,T], B(\mathcal{G}_p, \mathcal{U})) \) by
\[
\beta_s(a,b) := \sum_{\sigma \in \mathbb{P}_p-1, |\sigma| \geq 1} \gamma^\sigma (g_{s-1}(a - \sigma_0(b))), \forall a, b \in \mathcal{G}_p, \forall 0 \leq s \leq T,
\]
where \( \beta_s \in B(\mathcal{G}_p, \mathcal{U}) \) for any \( s \in [0,T] \) because \( \sigma_0(b) = 1 \), \( \forall b \in \mathcal{G}_p \). Then (50) can be rewritten as
\[
\| \gamma_t - \gamma_s - \beta_s (g_{s,t}) \| \leq C(\| g \|_{\text{var},[s,t]})^{\theta - \frac{1}{p}}, \forall 0 \leq s < t \leq T,
\] (52)
and (51) implies
\[
\| (\beta_t - \beta_s) (g_t, \cdot) \|_k \leq C(\| g \|_{\text{var},[s,t]})^{\theta - \frac{1+|\sigma|}{p}}, \forall 0 \leq s < t \leq T, \quad k = 1, \ldots, [p] - 1.
\] (53)

Indeed, to get (53) from (51), for any \( a \in \mathcal{G}_p \), we have
\[
(\beta_t - \beta_s) (g_t, a) = \sum_{\sigma \in \mathbb{P}_p-1, |\sigma| \geq 1} \gamma^\sigma (a - \sigma_0(a)) - \sum_{\sigma \in \mathbb{P}_p-1, |\sigma| \geq 1} \gamma^\sigma (g_{s,t} (a - \sigma_0(a)))
\]
\[
\sum_{\sigma \in \mathbb{P}_p-1, |\sigma| \geq 1} \left( \gamma_t^\sigma - \gamma_s^\sigma - \sum_{\sigma' \in \mathbb{P}_p-1, |\sigma'| \geq 1} c(\sigma^1, \sigma^2, \sigma) \gamma^\sigma (g_{s,t}) \right) \sigma(a),
\] (54)
where the constant \( c(\sigma^1, \sigma^2, \sigma) \) is defined as in (51). Since both ends of (54) are linear in \( a \), based on Condition 12 (which holds for the Butcher group as we checked), (54) holds for any \( v \in \mathcal{V}^\otimes k, k = 1, \ldots, [p] - 1 \). Hence,
\[
\| (\beta_t - \beta_s) (g_t, \cdot) \|_k \leq \sum_{\sigma \in \mathbb{P}_p-1, |\sigma| \geq 1} \left\| \left( \gamma_t^\sigma - \gamma_s^\sigma - \sum_{\sigma' \in \mathbb{P}_p-1, |\sigma'| \geq 1} c(\sigma^1, \sigma^2, \sigma) \gamma^\sigma (g_{s,t}) \right) \sigma(\cdot) \right\|_k
\]
\[
\leq C_p(\| g \|_{\text{var},[s,t]})^{\theta - \frac{1+|\sigma|}{p}}, \forall 0 \leq s < t \leq T, \quad k = 1, \ldots, [p] - 1.
\]
Then, if we define the set of paths weakly controlled by \( g \) as those Banach-space valued paths satisfying (52) and (53), then combined with the definition of dominated path, we have that, if \( \gamma \) is a path dominated by \( g \), then \( \gamma \) is weakly controlled by \( g \) (possibly with a different control). The set of weakly controlled paths is a linear space and is preserved under composition with regular functions. Moreover, when \( 2 \leq p < 3 \), for paths \( \gamma^1 \) and \( \gamma^2 \) weakly controlled by \( g \in CP^{-\varpi}([0,T], \mathcal{G}_{[p]}) \), the integral path \( \int_0^t \gamma_u^1 \otimes d\gamma_u^2 \) is canonically defined and is again a path weakly controlled by \( g \) (see Thm 1 [Gub04]). (When \( p \geq 3 \), the integral path \( \int_0^t \gamma_u^1 \otimes dx_u \) is canonically defined with \( x \) denoting the first level of \( g \), see Thm 8.5 [Gub10].)

In the definition of dominated path e.g. \( h = \int_0^t \beta_u(g_u) \, dg_u \) as at (49), we assumed that \( (\beta, g) \) satisfies the conditions in Theorem 5 so \( \beta \) is integrable against \( g \) and \( h \) is determined by \( (\beta, g) \). Indeed, the dominated path is all about integrable one-forms, and the path is defined from the integral. On the other hand, based on (52) and (53), for weakly controlled path \( \gamma \), \( (\beta, g) \) does not necessarily satisfy the integrable conditions in Theorem 5 and \( \gamma \) is not uniquely determined by \( (\beta, g) \). (Indeed, for \( (\beta, g) \) satisfying (52), there does not necessarily exist a path \( \gamma \) that satisfies (52), and if there exists a \( \gamma \) which satisfies (52) then \( \gamma + \eta \) also satisfies (52) for any \( \eta : [0,T] \to \mathcal{U} \) satisfying \( ||\eta - \eta_s|| \leq C ||g||_{p^{-\varpi}, [s,t]} \) for \( s < t \).) That the time-varying one-form is not sufficiently integrable and that the path is not uniquely determined by the one-form will always be there for a weakly controlled path, which makes the existence of canonical enhancement of a weakly controlled path an interesting result.

Actually, the existence of canonical enhancement is not solely about one-forms, it is the result of the interplay between one-forms and the integration developed in Section 3.3. In Example 24 we represent the path \( \int_0^t \gamma_u^1 \otimes d\gamma_u^2 \) as the integral of a time-varying cocyclic one-form against the group-valued path \( \gamma^2 \otimes g \in C([0,T], \mathcal{U}^2 \otimes \mathcal{G}_{[p]}) \). As a consequence, for \( g \in CP^{\varpi}([0,T], \mathcal{G}_{[p]}) \), \( p \geq 2 \), and \( \gamma \) weakly controlled by \( g \), there exists a canonical enhancement of \( \gamma \) to a path taking value in the Butcher group, which we call the signature of \( \gamma \): (with \( \mathcal{P}_n = \{1\} \) denoting the set of forests of degree less or equal to \( n \))

\[
\Gamma^n = 1 + \sum_{\sigma \in \mathcal{P}_n} x_t^\sigma \text{ with } x_t^\sigma := \gamma_t - \gamma_0, \quad x_t^{\sigma_1 \sigma_2 \cdots \sigma_k} := x_t^{\sigma_1} \otimes \cdots \otimes x_t^{\sigma_k}, \quad x_t^{[\sigma]} := \int_0^t x_t^\sigma \otimes d\gamma_u, \forall t \in [0,T].
\]

As a result, for a weakly controlled path \( \gamma \), one could enhance \( \gamma \) to \( \Gamma \) via integration. The set of paths dominated by \( \Gamma \) clearly includes \( \gamma \). When \( \gamma \) is dominated by \( g \), the set of paths which are dominated by \( \Gamma \) is a subset of the paths dominated by \( g \). Intuitively, one could split the space of weakly controlled paths to subspaces of dominated paths (dominated by a slightly perturbed group-valued path), and each subspace is a linear space and an algebra, stable under iterated integration and composition with regular functions (as in Section 3.3). It is also possible to union finitely many of these subspaces, which will be dominated by the joint signature of these weakly controlled paths.

In the construction of \( \Gamma^n \), it is possible to relax the regularity requirement on the time-varying one-form (i.e., we could let the one-form vary more quickly with time), provided that we know more about the signature of \( \gamma \). More specifically, based on (52), when \( x^\sigma = \gamma - \gamma_0 \) is given with approximating accuracy \( \theta - p^{-1} \), we can construct \( \Gamma^n \). Then if we assume that \( \{x^{\sigma}\}_{|\sigma| \leq k, \sigma \in \mathcal{P}_n} \) are given, then it is likely that one could construct \( \Gamma^n \) if the approximating accuracy of \( x^{\sigma} \) is \( \theta - (k - |\sigma| + 1)p^{-1} \), \( \sigma \in \mathcal{P}_n, |\sigma| \leq k \), (for \( x^\sigma \) it is \( \theta - kp^{-1} \)).

### 3.3 Stableness of dominated paths

The set of dominated paths admits a canonical enhancement to a group-valued path, is an algebra and is stable under composition with regular functions.

For Banach space \( \mathcal{U} \) and \( \alpha \in L \left( T^{[p]}(\mathcal{V}), \mathcal{U} \right) \), if we define \( \beta \in C \left( \mathcal{G}_{[p]}, L \left( T^{[p]}(\mathcal{V}), \mathcal{U} \right) \right) \) by

\[
\beta(a, b) := \alpha(a (b - 1)), \forall a, b \in \mathcal{G}_{[p]},
\]

then \( \beta \in B \left( \mathcal{G}_{[p]}, \mathcal{U} \right) \). Indeed, for \( a \in \mathcal{G}_{[p]} \), \( \beta(a, \cdot) \) is a linear mapping because \( \sigma_0 \left( \mathcal{G}_{[p]} \right) = 1 \) so \( \beta(a, b) = \alpha(a (b - \sigma_0(b))) \), \( \forall b \in T^{[p]}(\mathcal{V}) \). As a result,

\[
\beta(a, bv) = \beta(ab, v), \forall a, b \in \mathcal{G}_{[p]}, \forall v \in \mathcal{V}^\otimes k, k = 1, \ldots, [p]. \tag{55}
\]

**Proposition 18 (Enhancement)** For \( g \in CP^{\varpi}([0,T], \mathcal{G}_{[p]}) \) and Banach spaces \( \mathcal{U}^i, i = 1, 2 \), suppose \( \int_0^t \beta_u^i (g_u) \, dg_u : [0,T] \to \mathcal{U}^i, i = 1, 2 \), are dominated paths. With the mapping \( \mathcal{I} \in L(T^{[2][p]}(\mathcal{V}), T^{[p]}(\mathcal{V})^\otimes 2) \)
Then we get (61) based on (62) and (63).

\[ \beta_s^{1,2}(a, b) := (\beta_s^1(a, \cdot) \otimes \beta_s^2(a, \cdot)) (b) \), \forall a \in \mathcal{G}_{[p]}, \forall b \in T(2[p])(\mathcal{V}) , \] (56)

where \( \beta_s^1(a, \cdot) \otimes \beta_s^2(a, \cdot) \) denotes the unique continuous linear operator from \( T([p])(\mathcal{V}) \otimes \mathcal{U} \) to \( \mathcal{U} \otimes \mathcal{U} \) satisfying \( (\beta_s^1(a, \cdot) \otimes \beta_s^2(a, \cdot))(b_1 \otimes b_2) = \beta_s^1(a, b_1) \otimes \beta_s^2(a, b_2), \forall b_1, b_2 \in T([p])(\mathcal{V}). \)

We define \( \beta \in C([0, T], (\mathcal{G}_{[p]}, \mathcal{U}^1 \otimes \mathcal{U}^2)) \) by (with \( 1_{[p]} \) denoting the projection to \( T([p])(\mathcal{V}) \))

\[ \beta_s(a, b) := \int_0^s \beta_s^1(u)(g_{d\alpha})(dg_s) \otimes \beta_s^2(gs_g^{-1}(a(b-1))) + \beta_s^{1,2}(gs_g^{-1}(a(b-1))), \forall a, b \in \mathcal{G}_{[p]}, \forall s \in [0, T] . \]

Then \( \int_0^t \beta_s(g_u)(dg_u) : [0, T] \rightarrow \mathcal{U}^1 \otimes \mathcal{U}^2 \) is a dominated path and satisfies, for some control \( \omega \) and \( \theta > 1 \),

\[ \left\| \int_0^s \beta_u(g_u)(dg_u - \int_0^s \beta_s^1(g_u)(dg_u) \otimes \beta_s^2(g_u, s_s) - \beta_s^{1,2}(g_u, s_s) \right\| \leq \omega(s, t)\theta, \forall 0 \leq s < t \leq T . \] (57)

**Proof.** We check that \( \beta \) satisfies Condition [14]. Then the estimate [57] follows from Theorem 6.

Recall the properties of the mapping \( \mathcal{I} \) in Condition [14] that

\[ \mathcal{I}(\mathcal{V}^\otimes k) \subseteq \sum_{j_1 + j_2 = k, j_1 = 1, \ldots, [p]} \mathcal{V} \otimes \mathcal{V}^\otimes j_2, \ k = 0, 1, \ldots, [p] , \] (58)

\[ \mathcal{I}(ab) = \mathcal{I}(a) + (1_{[p]}(a) \otimes 1_{[p]}(a)) \mathcal{I}(b) + (1_{[p]}(a) - 1) \otimes (1_{[p]}(a) - 1) , \forall a, b \in \mathcal{G}_{[p]} . \]

Fix \( s < t \). Based on [58], for \( k = 1, \mathcal{I}(\mathcal{V}) = 0 \), so we have

\[ \left\| (\beta_s^{1,2} - \beta_s^{1,2})(g_t, \cdot) \right\|_1 = \left\| (\beta_s^1(a, \cdot) \otimes \beta_s^2(a, \cdot)) (g_t) \right\|_1 = 0 . \]

For \( k = 2, \ldots, [p] \), since \( \beta \) satisfies Condition [14] we have (with \( C \) independent of \( s \) and \( t \))

\[ \left\| (\beta_s^{1,2} - \beta_s^{1,2})(g_t, \cdot) \right\|_k \leq C \max_{i=1,2} \max_{j=1,\ldots,k} \left\| (\beta_s^i - \beta_s^i)(g_t, \cdot) \right\|_j \]

\[ \leq C \max_{j=1,\ldots,k} \omega(s, t)^{\theta - \frac{1}{\theta}} \leq C \omega(s, t)^{\theta - \frac{1}{\theta}} \leq C \omega(s, t)^{\theta - \frac{1}{\theta}}, \forall s < t . \] (60)

On the other hand, it can be checked that,

\[ \beta_s^{1,2}(a, bc) = \beta_s^1(a, b) + \beta_s^1(ab, c) + \beta_s^1(a, b) \otimes \beta_s^2(ab, c), \forall a, b, c \in \mathcal{G}_{[p]}, \forall s \in [0, T] , \]

where \( \beta_s^2(\cdot, \cdot) := \beta_s^2(1_{[p]}(\cdot), \cdot) \) and \( \beta_s^i(\cdot, \cdot) := \beta_s^i(1_{[p]}(\cdot), 1_{[p]}(\cdot)), i = 1, 2 \). Indeed, based on [59], we have

\[ \beta_s^{1,2}(a, bc) = (\beta_s^1(a, \cdot) \otimes \beta_s^2(a, \cdot))(bc) \]

\[ = (\beta_s^1(a, \cdot) \otimes \beta_s^2(a, \cdot))(b)(c) + (1_{[p]}(b) \otimes 1_{[p]}(c))(bc) + (1_{[p]}(b) - 1) \otimes (1_{[p]}(c) - 1)(bc) \]

\[ = \beta_s^2(a, b) + \beta_s^1(a, b \cdot) \otimes \beta_s^2(a, b \cdot)(bc) + \beta_s^1(a, b) \otimes \beta_s^2(ab, c) , \]

where we used \( \beta_s^2(a, b - 1) = \beta_s^2(a, b) \) and \( \beta_s^2(a, b(c - 1)) = \beta_s^2(ab, c) \). Suppose \( \mathcal{I}(c) = \sum_{j_1} v^{1, j_1} \otimes v^{2, j_2} \) with \( v^{i, j_1} \in \mathcal{V}^\otimes j_1 \), then based on [58], we have \( j_i \geq 1 \). Then by using [55], we have

\[ \beta_s^1(a, b \cdot) \otimes \beta_s^2(a, b \cdot)(c) \]

\[ = \sum_{j_1} \beta_s^1(ab, v^{1, j_1}) \otimes \beta_s^2(ab, v^{2, j_2}) = \sum_{j_1} \beta_s^1(ab, v^{1, j_1}) \otimes \beta_s^2(ab, v^{2, j_2}) \]

\[ = \beta_s^1(ab, \cdot) \otimes \beta_s^2(ab, \cdot)(c) = \beta_s^{1,2}(ab, c) . \] (63)

Then we get [61] based on [62] and [63].
Let $g^{2[p]} \in C^{p\text{-var}}([0, T], T_{2[p]})$ denote the unique extension of $g$ as in Example 11. Then, by using (61) and that $\beta^*_s \in B \left( G, \mathcal{U}^2 \right)$, we have, for $s < t$ and $a \in G_{2[p]}$, 

\begin{align}
(\beta^*_s - \beta^*_a) (g_s, a) &= \int_0^1 \beta^*_s (g_u) \, du \otimes \beta^*_t (g_t, a) + \beta^*_s (g_t, 1_{[p]} (a)) \\
&\quad - \int_0^s \beta^*_u (g_u) \, du \otimes \beta^*_t (g_s, g_s, t) (a - 1) - \beta^*_t (g_t, 1_{[p]} (a)) \\
&= \int_0^1 \beta^*_s (g_u) \, du \otimes (\beta^*_t - \beta^*_u) (g_u, a) \\
&\quad + \left( \int_0^1 \beta^*_u (g_u) \, du - \beta^*_t (g_u, g_s, t) \right) \otimes \beta^*_s (g_t, a) + \left( \beta^*_t - \beta^*_s \right) (g_t, 1_{[p]} (a)) \\
&\quad - \sum_{\sigma \in \mathcal{P}_{2[p]} \setminus \mathcal{P}_p} \beta^*_s (g_t, \sigma (a)) + \sum_{\sigma \in \mathcal{P}_{2[p]} \setminus \mathcal{P}_p} \beta^*_s (g_t, \sigma (g_{2[p]} (a - 1))) .
\end{align}

(64)

Since $(\beta^*_s - \beta^*_a) (g_s, a) = (\beta^*_s - \beta^*_a) (g_t, 1_{[p]} (a))$, the three equivalent expressions in (64) only depend on $1_{[p]} (a)$. Indeed, one can check that, the terms in (65) containing $\sigma (a)$, $\sigma \in \mathcal{P}_{2[p]} \setminus \mathcal{P}_p$, cancel with each other. The expressions in (64) are linear in $a$ (because $\sigma_0 (a) = 1$), so based on Condition 12 of [4] holds for any $\nu, \forall \nu \in C^{\otimes k}$, $\forall k = 1, 2, \ldots, [p]$. Since we assumed that $\beta^i, i = 1, 2$, satisfies Condition 16 combined with (60) and that $\|g^{2[p]} \| \leq C_p \| g \|_{p\text{-var}, [s, t]}$, $\forall s \leq t \leq T$, (see 39), it can be checked that $\beta$ satisfies Condition 16.

We denote by $1 \otimes (\mathcal{U}^1, \mathcal{U}^2) \oplus (\mathcal{U}^1 \otimes \mathcal{U}^2)$ the group with multiplication

$$(1, (u^1, v^1), x^1) (1, (u^2, v^2), x^2) = (1, (u^1 + u^2, v^1 + v^2), x^1 + x^2 + u^1 \otimes v^2),$$

$$\forall u^i \in \mathcal{U}^i, \forall v^i \in \mathcal{U}^i, \forall x^i \in \mathcal{U}^1 \otimes \mathcal{U}^2.$$ Based on (61), for any $s \in [0, T], \beta_s := (1, (\beta^1_s, \beta^2_s), (\beta^3_s, \beta^4_s))$ is a cocyclic one-form taking value in group $1 \otimes (\mathcal{U}^1, \mathcal{U}^2) \oplus (\mathcal{U}^1 \otimes \mathcal{U}^2)$, and $\beta \in C \left( [0, T], B \left( G, 1 \otimes (\mathcal{U}^1, \mathcal{U}^2) \oplus (\mathcal{U}^1 \otimes \mathcal{U}^2) \right) \right)$ is integrable against $g$ satisfying some control $\omega$ and $\theta > 1$

$$\left\| \int_0^1 \beta (g_u) \, du - (1, (\beta^1_s (g_s, g_s, t), \beta^2_s (g_s, g_s, t), \beta^3_s (g_s, g_s, t)) \right\| \leq \omega (s, t) \theta, \forall s < t.$$

For path $\gamma$ dominated by $g$, the integral $\int_0^1 \gamma_u \otimes du$ (where $g$ is treated as a Banach-space valued path) is again a dominated path and is a special case of Proposition 11. Indeed, if we let $\beta^*_s = \beta^*_a$ with $\beta^*_a (a, b) = (a - 1), \forall a, b \in G_{[p]}$, then $\int_0^1 \beta^*_s (g_u) \, du = \beta^*_a (g_u, g_s, t) = g_t - g_s$. Similarly, for $\sigma \in \mathcal{P}_{[p]}$ and $x := \sigma (g)$, if we let $\beta^*_a (a, b) = (a (b - 1)), \forall a, b \in G_{[p]}, \forall s$, then we have $\int_0^1 \beta^*_s (g_u) \, du = \sigma (g_t) - \sigma (g_s)$.

**Proposition 19 (Algebra) For** $g \in C^{p\text{-var}}([0, T], G_{[p]}), \suppose \text{that } \int_0^T \beta^*_s (g_u) \, du : [0, T] \rightarrow \mathcal{U}^i, i = 1, 2,\text{ are dominated paths. Then } \int_0^T \beta^*_s (g_u) \, du \oplus \int_0^T \beta^*_a (g_u) \, du : [0, T] \rightarrow \mathcal{U}^1 \otimes \mathcal{U}^2 \text{ is a dominated path.}$

**Proof.** Based on the definition of dominated path, we check that, there exists $\beta \in C \left( [0, T], B \left( G_{[p]}, \mathcal{U}^1 \otimes \mathcal{U}^2 \right) \right)$ satisfying Condition 16 such that

$$\int_0^T \beta^*_s (g_u) \, du \oplus \int_0^T \beta^*_a (g_u) \, du = \int_0^T \beta (g_u) \, du, \forall t \in [0, T].$$

(66)

Since $\beta^*_i, i = 1, 2$, satisfy Condition 16 based on Theorem 5 there exist a control $\omega$ and $\theta > 1$, s.t. $\forall s < t$,

$$\left\| \int_0^T \beta^*_s (g_u) \, du \oplus \int_0^T \beta^*_a (g_u) \, du - \int_0^T \beta^*_s (g_u) \, du \oplus \int_0^T \beta^*_a (g_u) \, du \right\| = \beta^*_s (g_s, g_s, t) \otimes \int_0^T \beta^*_a (g_u) \, du + \int_0^T \beta^*_s (g_u) \, du \otimes \beta^*_s (g_s, g_s, t) + \beta^*_a (g_s, g_s, t) \otimes \beta^*_s (g_s, g_s, t) + O \left( \omega (s, t) \theta \right).$$

(67)

We define $\beta \in C \left( [0, T], B \left( G_{[p]}, \mathcal{U}^1 \otimes \mathcal{U}^2 \right) \right)$ by

$$\beta (a, b) := \sum_{i=1}^3 A^*_i (a (b - 1)), \forall a, b \in G_{[p]},$$

16
where $\alpha^i_s \in L(T^{(p)}(V), U^1 \otimes U^2)$, $i = 1, 2, 3$, are defined by, for any $a \in T^{(p)}(V)$,
\[
\alpha^1_s(a) := \int^s_0 \beta^1_s(g_s, g_s^{-1} a) \, dg_s, \quad \alpha^2_s(a) := \int^s_0 \beta^2_s(g_u) \, dg_u, \quad \alpha^3_s(a) := \sum_{\sigma^i \in P_{[p], |\sigma^i|+|\sigma^i| \leq [p]}} \beta^1_s(g_{\sigma^i}, \cdot) \otimes \beta^2_s(g_{\sigma^i \cdot}) (\sigma^1 \ast \sigma^2) (g^{-1}_s a),
\]
where $\sigma^1 \ast \sigma^2 \in L(V \otimes \{\sigma^1\} \otimes V \otimes \{\sigma^2\})$ is defined in Condition [13] and $\beta^1_s(g_{\sigma^i}, \cdot) \otimes \beta^2_s(g_{\sigma^i \cdot})$ denotes the unique continuous linear mapping from $T^{(p)}(V) \otimes U^1 \otimes U^2$ satisfying $\beta^1_s(g_{\sigma^i}, \cdot) \otimes \beta^2_s(g_{\sigma^i \cdot}) (b_1 \otimes b_2) = \beta^1_s(g_{b_1} b_2) \otimes \beta^2_s(g_{b_1} b_2), \forall b_i \in T^{(p)}(V)$. Denote by $g^{2[p]} \in C^{0-\var}$ $\left([0, T], T_{2[p]}(V)\right)$ the extension of $g$ as in Example [11]. Then, based on Condition [13] we have, for any $s < t$,
\[
\begin{align*}
(a^1 \ast a^2) \left( g^{2[p]}_{s,t} \right) &= \left( a^1 \ast a^2 \right) (g^{2[p]}_{s,t}), \quad \forall \sigma^i \in P_{[p]}, \quad |\sigma^1| + |\sigma^2| \leq [p], \\
(a^1 \ast a^2) \left( g^{2[p]}_{s,t} \right) &= a^1(g_{s,t}) \ast a^2(g_{s,t}), \quad \forall \sigma^i \in P_{[p]}.
\end{align*}
\]
Hence, for $s < t$, we have
\[
\begin{align*}
\sum_{\sigma^i \in P_{[p], |\sigma^i|+|\sigma^i| \leq [p]}} \beta^1_s(g_{\sigma^i}, \cdot) \otimes \beta^2_s(g_{\sigma^i \cdot}) (\sigma^1 \ast \sigma^2) (g^{2[p]}_{s,t}) &= \beta^1_s(g_{s,t} \ast g^{2[p]}_{s,t}) \left( g^{2[p]}_{s,t} \right) - \sum_{\sigma^i \in P_{[p], |\sigma^i|+|\sigma^i| \geq [p]}} \beta^1_s(g_{\sigma^i}, \cdot) \otimes \beta^2_s(g_{\sigma^i \cdot}) (\sigma^1 \ast \sigma^2) (g^{2[p]}_{s,t}) \\
&= \sum_{\sigma^i \in P_{[p], |\sigma^i|+|\sigma^i| \leq [p]}} \beta^1_s(g_{s,t}, g^{2[p]}_{s,t}) \otimes \beta^2_s(g^{2[p]}_{s,t}) - \sum_{\sigma^i \in P_{[p], |\sigma^i|+|\sigma^i| \geq [p]}} \beta^1_s(g_{\sigma^i}, \cdot) \otimes \beta^2_s(g_{\sigma^i \cdot}) (\sigma^1 \ast \sigma^2) (g^{2[p]}_{s,t}) \\
&= \beta^1_s(g_{s,t} \ast g^{2[p]}_{s,t}) \left( g^{2[p]}_{s,t} \right) - \sum_{\sigma^i \in P_{[p], |\sigma^i|+|\sigma^i| \geq [p]}} \beta^1_s(g_{\sigma^i}, \cdot) \otimes \beta^2_s(g_{\sigma^i \cdot}) (\sigma^1 \ast \sigma^2) (g^{2[p]}_{s,t}).
\end{align*}
\]
Since $\sigma^1 \ast \sigma^2 \in L(V \otimes \{\sigma^1\} \otimes V \otimes \{\sigma^2\})$ is a continuous linear mapping, so is bounded. Hence, combined with $\|g^{2[p]}_{p-\var} \|_{p-\var} \leq C_p \|g\|_{p-\var} \leq C$, we have, for $s < t$, (as at [33] in Example [11], we have, for $i \in P_{[p]}$, $|\sigma^i| + |\sigma^i| \geq [p] + 1$, (with the constant $C$ independent of $s$ and $t$)
\[
\| (a^1 \ast a^2) (g^{2[p]}_{s,t}) \| \leq C \sum_{\sigma^i \in P_{[p], |\sigma^i|+|\sigma^i| \leq [p]}} \| (a^1 \ast a^2) (g^{2[p]}_{s,t}) \| \leq C \|g^{2[p]}_{p-\var} \|_{p-\var} \leq C \|g\|_{p-\var} + 1.
\]
As a result, we have, for $s < t$,
\[
\begin{align*}
\beta_s(g_{s,t}) &= \beta_s^1(g_{s,t} \ast g^{2[p]}_{s,t}) \left( g^{2[p]}_{s,t} \right) - \sum_{\sigma^i \in P_{[p], |\sigma^i|+|\sigma^i| \geq [p]}} \beta^1_s(g_{\sigma^i}, \cdot) \otimes \beta^2_s(g_{\sigma^i \cdot}) (\sigma^1 \ast \sigma^2) (g^{2[p]}_{s,t}) \\
&= \beta_s^1(g_{s,t} \ast g^{2[p]}_{s,t}) \left( g^{2[p]}_{s,t} \right) - \sum_{\sigma^i \in P_{[p], |\sigma^i|+|\sigma^i| \geq [p]}} \beta^1_s(g_{\sigma^i}, \cdot) \otimes \beta^2_s(g_{\sigma^i \cdot}) (\sigma^1 \ast \sigma^2) (g^{2[p]}_{s,t}) \\
&= \beta^1_s(g_{s,t} \ast g^{2[p]}_{s,t}) \left( g^{2[p]}_{s,t} \right) + \beta^1_s(g_{s,t} \ast g^{2[p]}_{s,t}) \left( g^{2[p]}_{s,t} \right) \otimes \left( |g| \right)^{p-1} \left( g^{2[p]}_{p-\var} \right) \left( g^{2[p]}_{p-\var} \right).
\end{align*}
\]
If $(\beta, g)$ satisfies Condition [16] then based on [67] and [83], we have [66] holds. Indeed, suppose $\varphi^i$, $i = 1, 2$, are two paths taking value in a Banach space, and there exist a control $\omega$ and $\theta > 1$ s.t.
\[
\|\varphi^1_t - \varphi^2_t - (\varphi^1_t - \varphi^2_t)\| \leq \|\omega(s, t)^\theta\|, \forall s < t.
\]
Then, we check that $(\beta, g)$ satisfies Condition [16]. For $s < t$, by using that $\beta_s^1 \in B(V, U^1 \otimes U^2)$, we have, for any $a \in U^1_{2[p]}$, (with $\beta_s^1(a) := (\beta_s - \beta_s)(g_{1[p]}(a))$)
\[
\begin{align*}
\beta_s^1(g_{s,t}) &= \beta_s^1(g_{s,t} \ast g^{2[p]}_{s,t}) \left( g^{2[p]}_{s,t} \right) - \sum_{\sigma^i \in P_{[p], |\sigma^i|+|\sigma^i| \geq [p]}} \beta^1_s(g_{\sigma^i}, \cdot) \otimes \beta^2_s(g_{\sigma^i \cdot}) (\sigma^1 \ast \sigma^2) (g^{2[p]}_{s,t}) \\
&= \left( \beta^1_s - \beta^1_s \right) \left( g_{s,t} \ast g^{2[p]}_{s,t} \right) \left( g^{2[p]}_{s,t} \right) + \left( \beta^1_s - \beta^1_s \right) \left( g_{s,t} \ast g^{2[p]}_{s,t} \right) \left( g^{2[p]}_{s,t} \right) \otimes \left( |g| \right)^{p-1} \left( g^{2[p]}_{p-\var} \right) \left( g^{2[p]}_{p-\var} \right).
\end{align*}
\]
Both sides of (69) are linear in \(a \in G_{[p]}\) and actually only depends on 1\([p]\)(a). Based on Condition 12 (69) holds for any \(v \in V^{\otimes k}, k = 1, 2, \ldots, [p]\).

We assumed that \(\beta^i, i = 1, 2,\) satisfies Condition 16 so there exist a control \(\omega\) and \(\theta > 1\), such that

\[
\| (\beta^i_s - \beta^j_s) (g_t, \cdot) \| \leq \omega (s, t)^{\theta - \frac{1}{2}}, \quad k = 1, 2, \ldots, [p], \quad \text{and} \quad \left\| \int_s^t \beta^i_s (g_v) \, dv - \beta^i_s (g_s, g_s, t) \right\| \leq \omega (s, t)^{\theta}.
\]

On the other hand, for \(\sigma^1, \sigma^2 \in P_{[p]}, \) since \(\sigma^1 \ast \sigma^2 \in L(V^{\otimes (|\sigma^1| + |\sigma^2|)}, V^{\otimes 1} \otimes V^{\otimes 2})\), if \(\sigma^1 \ast \sigma^2 \) (\(v\) \(\neq 0\) for some \(v \in V^{\otimes k}\), then \(|\sigma^1| + |\sigma^2| = k\) and \(|\sigma| \leq k\). Then by using \(\beta^i_s (a, 1) = 0\), we have, for \(k = 1, 2, \ldots, [p]\),

\[
\left\| (\beta^1_t (g_t, \cdot) \otimes \beta^2_t (g_t, \cdot) - \beta^1_s (g_t, \cdot) \otimes \beta^2_s (g_t, \cdot)) (\sigma^1 \ast \sigma^2) (\cdot) \right\|_k \leq C \max_{i = 1, 2, \ldots, k} \left\| (\beta^i_t - \beta^i_s) (g_t, \cdot) \right\|_j \leq C \omega (s, t)^{\theta - \frac{k - 1}{2}} \leq C \omega (s, t)^{\theta - \frac{k}{2}}, \quad \forall s < t.
\]

Based on (30) in Example 11 \(\| g^{2;p} \|_{p-\text{var}, [s, t]} \leq C_p \| g \|_{p-\text{var}, [s, t]}, \quad \forall s < t\). Then for \(k = 1, \ldots, [p]\) and \(\sigma^1 \in P_{[p]}, i = 1, 2, |\sigma^1| + |\sigma^2| = n \geq [p] + 1,\)

\[
\left\| \beta^i_s (g_s, \cdot) \otimes \beta^2_s (g_s, \cdot) (\sigma^1 \ast \sigma^2) (\cdot) \right\|_k \leq C \| g \|_{p-\text{var}, [s, t]}^{n-k} \leq C \| g \|_{p-\text{var}, [s, t]}^{n-1-k}, \quad \forall s < t.
\]

As a result, it can be checked that, \(\beta\) satisfies Condition 16.

**Remark 20** The statement that the set of dominated paths is an algebra does not necessarily follow from the statement that the iterated integral of two dominated paths is canonically defined. It will depend on the definition of the formal integral \(\int_{0}^{T} \delta g_{[0, u_1]} \otimes \delta g_{0, u_2}\), and the integration by parts formula may not hold:

\[
\int_{0}^{T} \delta g_{[0, u_1]} \otimes \delta g_{0, u_2} \int_{0}^{T} \delta g_{[0, u_2]} \otimes \delta g_{0, u_2} \leq g_{0,T} \otimes g_{0,T}, \quad \forall g \in C([0, T], G).
\] (70)

When \(G\) is the nilpotent Lie group, (70) holds; when \(G\) is the Butcher group, generally (71) does not hold.

For \(\gamma > 0\), let \([\gamma]\) denote the largest integer which is strictly less than \(\gamma\). For Banach spaces \(U\) and \(W\), we denote \(f \in C^\gamma (U, W)\), if \(f : U \to W\) is \([\gamma]\)-times Fréchet differentiable and

\[
\sup_{x \neq \gamma, y \in U, \|x\| \leq R, \|y\| \leq R} \frac{\| (D^{[\gamma]} f)(x) - (D^{[\gamma]} f)(y) \|}{\| x - y \|^{\gamma-1}} \leq C_R, \quad \forall R > 0.
\] (71)

**Proposition 21 (Composition)** For \(g \in C^{p-\text{var}} ([0, T], G_{[p]}),\) suppose \(\int_{0}^{T} \beta_u (g_u) \, du : [0, T] \to U\) is a dominated path. For \(X_0 \in U\), define \(X_t := X_0 + \int_{0}^{t} \beta_u (g_u) \, du, \quad \forall t \in [0, T].\) Then for Banach space \(W\) and \(f \in C^\gamma (U, W), \gamma > p,\) \(f (X_t) - f (X_0) : [0, T] \to W\) is a dominated path.

**Proof.** We check that, there exists \(\bar{\beta} \in C ([0, T], B (G_{[p]}, W))\) satisfying Condition 16, such that

\[
f (X_t) - f (X_0) = \int_{0}^{t} \bar{\beta}_u (g_u) \, du, \quad \forall t \in [0, T].
\]

For \(s \in [0, T],\) define \(\alpha_s \in L(T([p])(V), W)\) by

\[
\alpha_s (a) := \sum_{i = 1}^{[p]} \frac{1}{i} D^i f (X_T) \sum_{\sigma \in P_{[p]}, |\sigma^1| + \ldots + |\sigma^i| \leq [p]} \beta_s (g_{s, \cdot}) \otimes \sigma^1 \ast \ldots \ast \sigma^i (g^{-1} a),
\]

where \(\sigma^1 \ast \ldots \ast \sigma^i \in L(V^{\otimes (|\sigma^1| + \ldots + |\sigma^i|)} \otimes \sigma^1 \otimes \ldots \otimes V^{\otimes |\sigma^i|})\) is defined in Condition 13 and \(\beta_s (g_{s, \cdot}) \otimes \sigma^1 \otimes \ldots \otimes V^{\otimes |\sigma^i|})\) denotes the unique continuous linear mapping from \(T([p])(V)\) to \(U^{\otimes [p]}\) satisfying \(\beta_s (g_{s, \cdot}) \otimes \sigma^1 \otimes \ldots \otimes V^{\otimes |\sigma^i|})\) \(g_{s, b_1} \otimes \ldots \otimes g_{s, b_l}, \quad \forall b \in T([p])(V)\). Define \(\bar{\beta} \in C ([0, T], B (G_{[p]}, W))\) by

\[
\bar{\beta}_s (a, b) := \alpha_s (a (b - 1)), \quad \forall a, b \in G_{[p]}.
\]

18
Let \( g^{[p]} \in C_{p\text{-var}}([0,T], \mathbb{T}^{[p]}_{[T]}) \) denote the extension of \( g \) as in Example 11 with the estimate at \( 10 \):
\[
\|g^{[p]}\|_{p\text{-var},[s,t]} \leq C_p \|g\|_{p\text{-var},[s,t]}, \quad \forall s < t.
\]
Then, for \( s < t \), by using \( \beta_s(g_s,1) = 0 \), we have
\[
\tilde{\beta}_s(g_s,g_{s,t}) = \sum_{l=1}^{[p]} \frac{1}{l!} D^l f(X_s) \sum_{\sigma \in P_{[p],|\sigma^1| + \ldots + |\sigma^l| \leq [p]} \beta_s(g_s,\sigma^1 \ldots \sigma^l) (\sigma^1 \ldots \sigma^l) (g_{s,t})
\]
\[
= \sum_{l=1}^{[p]} \frac{1}{l!} D^l f(X_s) \left( \beta_s(g_s,g_{s,t}) \otimes (g_{s,t}) - \beta_s(g_s,1) \otimes \sum_{\sigma \in P_{[p],|\sigma^1| + \ldots + |\sigma^l| \geq [p]+1} (\sigma^1 \ldots \sigma^l) (g_{s,t}^2) \right)
\]
\[
= \sum_{l=1}^{[p]} \frac{1}{l!} D^l f(X_s) \beta_s(g_s,g_{s,t}) \otimes l + O \left( \|g\|_{p\text{-var},[s,t]} \right).
\]
On the other hand, since \( X_s - X_0 = \int_0^s \beta_a(g_a) \, dg_a \) is a dominated path (with control \( \omega \) and \( \theta \in (1, \frac{1}{n}) \)), it can be checked that
\[
f(X_s) - f(X_0) = \sum_{l=1}^{[p]} \frac{1}{l!} D^l f(X_s) (\beta_s(g_s,g_{s,t}) \otimes l + O \left( (\omega(s,t) + \|g\|_{p\text{-var},[s,t]}^p) \right)).
\]
Hence, if \( \tilde{\beta} \) satisfies Condition 16, then comparing (73) with (72), we have
\[
f(X_t) - f(X_0) = \lim_{|D| \to 0, D \subset [0,t]} \sum_{j,t \in D} \sum_{l=1}^{[p]} \frac{1}{l!} D^l f(X_t) \beta_s(g_t,gt,t_{j,t+1}) \otimes l
\]
\[
= \lim_{|D| \to 0, D \subset [0,t]} \sum_{j,t \in D} \tilde{\beta}(gt,gt,t_{j,t+1}) = \int_0^t \tilde{\beta}(g_s) \, dg_s.
\]
Then we check that \( \tilde{\beta} \) satisfies Condition 16. For \( s < t \) and \( l = 1, 2 \), define \( R^l \in L(T([p]^2), \mathcal{V}, \mathcal{W}) \) by
\[
R^1(v) = \sum_{l=1}^{[p]} \frac{1}{l!} D^l f(X_s) (\beta_s(g_t,a) \otimes l - \beta_s(g_s,a) \otimes l) + R^2(a)
\]
\[
= \sum_{l=1}^{[p]} \frac{1}{l!} \left( D^l f(X_s) - \sum_{j=0}^{[p]-l} \frac{1}{j!} D^{l+j} f(X_s) \right) (\beta_s(g_t,a) \otimes l)
\]
\[
+ \sum_{l=1}^{[p]} \frac{1}{l!} \left( \sum_{j=0}^{[p]-l} \frac{1}{j!} D^{l+j} f(X_s) \right) (\beta_s(g_t,a) \otimes l) - R^1(a) + R^2(a).
\]
Then by using \( \beta_s \in B(G_{[p]},U) \), we have, for \( a \in G_{[p]^{2}}, \) (with \( (\tilde{\beta}_s - \bar{\beta}_s)(g_t,a) := (\tilde{\beta}_s - \bar{\beta}_s)(g_t,1_{[p]}(a)) \))
\[
(\tilde{\beta}_s - \bar{\beta}_s)(g_t,a)
\]
\[
= \sum_{l=1}^{[p]} \frac{1}{l!} D^l f(X_s) \left( \beta_s(g_t,a) \otimes l - \beta_s(g_s,a) \otimes l \right) + R^1(a) + R^2(a).
\]
For \( s < t, i = 1, 2, \ldots, [p] \), we define \( L^i \in L(T([p]^2), \mathcal{V}, \mathcal{U}^{\otimes l}) \) by
\[
L^1(v) = \sum_{\sigma \in P_{[p]}} \beta_t(g_t,\sigma^1 \ldots \sigma^l) (\sigma^1 \ldots \sigma^l) (v),
\]
\[
L^2(v) = \sum_{\epsilon_1, 1 \sigma \in P_{[p]}} (x^{\epsilon_1 \otimes \ldots \otimes x^{\epsilon_1}} (\sigma^1 \ldots \sigma^{\epsilon_1 + \ldots + \epsilon_k}) (v), x^0 = \beta_s(g_s,g_{s,t}), x^1 = \beta_t(g_t,\cdot)
\]
\[
L^3(v) = \sum_{\epsilon_1, 1 \sigma \in P_{[p]}} (y^{\epsilon_1 \otimes \ldots \otimes y^{\epsilon_1}} (\sigma^1 \ldots \sigma^{\epsilon_1 + \ldots + \epsilon_k}) (v), y^0 = \beta_s(g_s,g_{s,t}), y^1 = \beta_s(g_t,\cdot)
\]
Then for \( l = 1, \ldots, [p] \) and any \( a \in G_{[p]^{2}}, \) we have (with \( \beta_t(g_s,\cdot) := \beta_t(g_t,1_{[p]}(\cdot)) \)),
\[
L^1(a) = (\beta_t(g_t,a))^\otimes l,
\]
\[
L^2(a) = (\beta_s(g_s,g_{s,t}) + \beta_t(g_t,a))^{\otimes l},
\]
\[
L^3(a) = (\beta_s(g_s,g_{s,t}) + \beta_t(g_t,a))^{\otimes l}.
\]
and for any $a \in \mathcal{G}_{[p]}$, 
\[
\left(\overline{\beta}_t - \overline{\beta}_s\right)(g_t, a) = \sum_{i=1}^{[p]} \frac{1}{i!} \left(D^i f(X_t) - \sum_{j=0}^{[p]-i} \frac{1}{j!} D^{i+j} f(X_s) (X_t - X_s)^{\otimes j}\right) L^{1,i}(a) \\
+ \sum_{i=1}^{[p]} \sum_{j=0}^{[p]-i} \frac{1}{i!} \frac{1}{j!} D^{i+j} f(X_s) \left((X_t - X_s)^{\otimes j} - (\beta_s(g_s, g_s,t))^{\otimes j}\right) L^{1,i}(a) \\
+ \sum_{i=1}^{[p]} \frac{1}{i!} D^i f(X_s) \left(L^{2,i}(a) - L^{3,i}(a)\right) \\
- R^1(a) + R^2(a).
\] (76)

Both sides of (76) are linear in $a \in \mathcal{G}_{[p]}$ and actually only depends on $1_{[p]}(a)$ (because $(\overline{\beta}_t - \overline{\beta}_s)(g_t, a) := (\overline{\beta}_t - \overline{\beta}_s)(g_t, 1_{[p]}(a))$). Hence, based on Condition 12, (76) holds for any $v \in \mathcal{V}^{\otimes k}$, $k = 1, 2, \ldots, [p]$.

For $\{\sigma^i\}_{i=1}^{[p]} \in \mathcal{P}_{[p]}$, if there exists $\sigma_i$ satisfying $|\sigma_i| = 0$, then we have $\beta_i(g_t, c)^{\otimes i} (\sigma_1 \ast \cdots \ast \sigma^i)(v) = 0$, $\forall v \in T^{([p]^2)}(\mathcal{V})$, because $\beta_i(g_t, c) = 0, \forall c \in \mathbb{R}$. Then, if $\beta_i(g_t, c)^{\otimes i} (\sigma_1 \ast \cdots \ast \sigma^i)(v) \neq 0$ for some $v \in T^{(\mathcal{V}^2)}(\mathcal{V})$ then $|\sigma_i| \geq 1$. If we further assume $v \in \mathcal{V}^{\otimes k}$ and $\beta_i(g_t, c)^{\otimes i} (\sigma_1 \ast \cdots \ast \sigma^i)(v) \neq 0$, then since $\sigma_1 \ast \cdots \ast \sigma^i \in \mathcal{L}(\mathcal{V}^{\otimes (|\sigma_1| + \cdots + |\sigma^i|)}, \mathcal{V}^{\otimes |\sigma_1| \ast \cdots \otimes \mathcal{V}^{\otimes |\sigma^i|}})$, we have $k = |\sigma_1| + \cdots + |\sigma^i| \leq l$ (since $|\sigma_i| \geq 1$, $\forall i$). Hence, for $k = 1, \ldots, [p]$ and $v \in \mathcal{V}^{\otimes k}$, (the summation of $l$ is from 1 to $k$)

\[
\left(\overline{\beta}_t - \overline{\beta}_s\right)(g_t, v) = \sum_{i=1}^{[p]} \frac{1}{i!} \left(D^i f(X_t) - \sum_{j=0}^{[p]-i} \frac{1}{j!} D^{i+j} f(X_s) (X_t - X_s)^{\otimes j}\right) L^{1,i}(v) \\
+ \sum_{i=1}^{[p]} \sum_{j=0}^{[p]-i} \frac{1}{i!} \frac{1}{j!} D^{i+j} f(X_s) \left((X_t - X_s)^{\otimes j} - (\beta_s(g_s, g_s,t))^{\otimes j}\right) L^{1,i}(v) \\
+ \sum_{i=1}^{[p]} \frac{1}{i!} D^i f(X_s) \left(L^{2,i}(v) - L^{3,i}(v)\right) \\
- R^1(v) + R^2(v) \\
= : I(v) + II(v) + III(v) + IV(v).
\] (77)

Without loss of generality, we assume $\frac{2}{p} \in (1, \theta]$. Then, (with the constant $C$ independent of $s$ and $t$)

\[
||\beta_s(g_s, g_s,t)|| = ||\beta_s(g_s, g_s,t) - 1|| \leq C ||g_{s,t} - 1|| \leq C ||g_{s,t}|| \leq C ||g||_{p-var,s,t}.
\] Since $\beta$ satisfies Condition 5 and $X_t = X_0 + \int_0^t \beta_s(g_s) \, dg_s$, based on Theorem 5 there exist control $\omega$ and $\theta > 1$, such that

\[
||X_t - X_s - \beta_s(g_s, g_s,t)|| \leq \omega(s, t)^\theta,
\] (78)

and

\[
||X_t - X_s|| \leq ||X_t - X_s - \beta_s(g_s, g_s,t)|| + ||\beta_s(g_s, g_s,t)|| \leq C \left(\omega(s, t)^\frac{\theta}{2} + ||g||_{p-var,s,t}\right).
\] (79)

For the estimation of $I$, by using Taylor’s theorem, $||L^{1,i}(-)|| < \infty$ and (29), we have,

\[
||I(-)||_k \leq \sum_{i=1}^{[p]} \left||D^i f(X_t) - \sum_{j=0}^{[p]-i} \frac{1}{j!} D^{i+j} f(X_s) (X_t - X_s)^{\otimes j}\right|| \left||L^{1,i}(-)\right|| \\
\leq C \sum_{i=1}^{[p]} ||X_t - X_s||^{\gamma-i} \leq C \sum_{i=1}^{[p]} \left(\omega(s, t)^\frac{\theta}{2} + ||g||_{p-var,s,t}\right) \leq C \left(\omega(s, t) + ||g||_{p-var,s,t}\right)^{\frac{\theta}{2} - \frac{\theta}{2}}.
\] (80)

For the estimation of $II$, by using that $f \in C^\gamma(\mathcal{V}, \mathcal{U})$ so $\{D^i f\}_{i=0}^{[p]}$ are bounded on bounded set, that $||L^{1,i}(-)|| < \infty$ and (28), we have $(x^{\otimes 0} = 1)$

\[
||II(-)||_k \leq \sum_{i=1}^{[p]} \sum_{j=0}^{[p]-i} \frac{1}{i!} \frac{1}{j!} \left||D^{i+j} f(X_s)\right|| \left||X_t - X_s\right||^{\otimes j} - (\beta_s(g_s, g_s,t))^{\otimes j} \left||L^{1,i}(-)\right|| \\
\leq C \max_{j=0,1,\ldots,[p]} \left||X_t - X_s\right||^{\otimes j} - (\beta_s(g_s, g_s,t))^{\otimes j} \leq C \omega(s, t)^\theta.
\] (81)
For the estimation of $III$, for $v \in \mathcal{V}^\otimes k$ and $\{\sigma^i\}_{i=1}^j \subset \mathcal{P}[p]$, since $\sigma^1 \cdots \sigma^j \in L(\mathcal{V}^\otimes |\sigma^1| + \cdots + |\sigma^j|), \mathcal{V}^\otimes |\sigma^1| \otimes \cdots \otimes \mathcal{V}^\otimes |\sigma^j|$, we have
\[
(\sigma^1 \cdots \sigma^j)(v) \neq 0 \Rightarrow |\sigma^1| + \cdots + |\sigma^j| = k = |\sigma^i| \leq k.
\]
Based on the definition of $L^2, l$ and $L^3, l$ at (74) and (75), and by using that $(\beta, g)$ satisfies Condition 10 we have
\[
\|L^{2, l}(\cdot) - L^{3, l}(\cdot)\|_k \leq C \max_{i \leq k} \|((\beta - \beta_s)(g, \cdot))\|_i \leq C \omega(s, t)^{\theta - \frac{k}{p}}, \ l = 1, \ldots, [p].
\]
As a result, by using that $f \in C^\gamma(\mathcal{V}, \mathcal{U})$ so $\{D^k f\}_{k=0}^l$ are bounded on bounded set, we have
\[
\|III(\cdot)\|_k \leq \sum_{p=1}^{[p]} \frac{1}{p!} \|D^p f(X_s)\| \|L^{2, l}(\cdot) - L^{3, l}(\cdot)\|_k \leq C \omega(s, t)^{\theta - \frac{k}{p}}. \tag{82}
\]
For the estimation of $IV$, since $\|R^i(\cdot)\|_k = 0, \ k = 1, \ldots, [p]$, by using $\|g|^{p, t}\|_{p-\text{var},[s,t]} \leq C_p \|g\|_{p-\text{var},[s,t]}, \ \forall s < t$, as at (30), we have, for $k = 1, \ldots, [p],$
\[
\|IV(\cdot)\|_k \leq \|R^1(\cdot)\|_k + \|R^2(\cdot)\|_k \leq C \|g\|_{p-\text{var},[s,t]}^{[p] - k}. \tag{83}
\]
As a result, by combining (77), (80), (81), (82) and (83), we have $(\frac{p}{p+1} \in (1, \theta \wedge \frac{[p] + 1}{p})$\]
\[
\left\| \left( \beta - \beta_s \right)(g, \cdot) \right\|_k \leq C \left( \omega(s, t) + \|g\|_{p-\text{var},[s,t]} \right)^{\frac{\theta - k}{p}}, \ \forall s < t, \ k = 1, 2, \ldots, [p].
\]
Combined with $\sup_{t \in [0,T]} \left\| \beta - \beta_s (g, \cdot) \right\| < \infty$, we have $\beta$ satisfies Condition 10.

3.4 Rough integral and weakly controlled path

Example 22 (Rough integral) For Banach space $\mathcal{V}$, suppose $\mathcal{G}[p]$ is the nilpotent Lie group $\mathcal{G}[p](\mathcal{V})$. For $g \in C^{p, \text{var}}([0, T], \mathcal{G}[p]), \gamma > p - 1$ and Banach space $\mathcal{U}$, suppose $f \in C^\gamma(\mathcal{V}, \mathcal{L}(\mathcal{V}, \mathcal{U}))$ (as defined at (71)). We define $\beta \in ([0, T], B(\mathcal{G}[p], \mathcal{U}))$ by (with $x_s := \pi_1(g_s)$ and $\pi_k$ denoting the projection to $\mathcal{V}^\otimes k$)
\[
\beta_s(a, b) := \sum_{l=0}^{[p] - 1} \sum_{j=0}^{[p] - 1 - l} (D^{l+j} f) (x_s) \left( \frac{(-x_s)^{\otimes j}}{j!} \right) \otimes \pi_{l+1}(a (b - 1)), \ \forall a, b \in \mathcal{G}[p], \ \forall s. \tag{84}
\]
Then $\beta$ satisfies Condition 10 so $y_t := \int_t^s \beta_u (g_u) \, dy_u, \ \forall t \in [0, T]$, is a dominated path. Moreover, the enhancement of $y$ into a group-valued path
\[
y_t = 1 + \sum_{k=1}^{[p]} y^k_t \text{ with } y^1_t := y_t \text{ and } y^{k+1}_t := \int_t^s y^k_u \otimes d y_u, \ \forall t \in [0, T],
\]
coincides with the rough integral in [Lyo98] (where $\int y^k \otimes dy$ is defined as in Proposition 15).

Proof. Since $D^j f \in C^\gamma(\mathcal{V}, \mathcal{L}(\mathcal{V}^\otimes j, \mathcal{L}(\mathcal{V}, \mathcal{U})))$ is symmetric in $\mathcal{V}^\otimes j$ and the projection of $\pi_i(a), a \in \mathcal{G}[p], \text{ to the space of symmetric tensors is } (i!)^{-1} (\pi_1(a))^{\otimes i}$, it can be computed that, for $a, b \in \mathcal{G}[p],$
\[
\beta_s(a, b) = \sum_{l=0}^{[p] - 1} \sum_{j=0}^{[p] - 1 - l} (D^{l+j} f) (x_s) \left( \frac{(-x_s)^{\otimes j}}{j!} \right) \otimes \pi_{l+1}(a (b - 1)), \text{ (since } \pi_0(b) = 1) = \sum_{l=0}^{[p] - 1} \sum_{j=0}^{[p] - 1 - l} \sum_{i=0}^{l} (D^{l+j} f) (x_s) \left( \frac{(-x_s)^{\otimes j}}{j!} \right) \otimes \left( \pi_1(a) \right)^{\otimes i} \otimes \pi_{l+1-i}(b) = \sum_{L=0}^{[p] - 1} \sum_{j=0}^{[p] - 1 - L - L-i} (D^{L+i+j} f) (x_s) \left( \frac{(-x_s)^{\otimes j}}{j!} \right) \otimes \left( \pi_1(a) \right)^{\otimes i} \otimes \pi_{L+1}(b) = \sum_{L=0}^{[p] - 1} \sum_{j=0}^{[p] - 1 - L} (D^{L+j} f) (x_s) \left( \pi_1(a - x_s)^{\otimes j} \right) \otimes \pi_{L+1}(b), \tag{86}
\]
where in the last step we used the fact that (as in Lemma 4.7 \([\text{LCL07}]\)), for \(x, y \in \mathcal{V}\), the projection of \(\sum_{i=0}^{n} n^{-i} (it)^{-1} x^{(i)} \otimes y^{(j)}\) to symmetric tensors is \((nt)^{-1} (x + y)^{(n)}\).

Then we check that \(\beta\) satisfies Condition \([16]\). Since \([50]\) is linear in \(b\), based on Condition \([12]\) \([80]\) holds for any \(v \in \mathcal{V}^{\otimes k}\), \(k = 1, \ldots, [p]\). Then, for \(s < t\) and \(k = 1, 2, \ldots, [p]\), we have

\[
(\beta_t - \beta_s)(g_t, v) = \left( (D^{k-1}f)(x_t) - \sum_{j=0}^{[p]-k} (D^{k+j-1}f)(x_s) \frac{(x_t - x_s)^{\otimes j}}{j!} \right) \otimes v, \quad \forall v \in \mathcal{V}^{\otimes k}.
\]

Since \(f \in C^\gamma(\mathcal{V}, L(\mathcal{V}, \mathcal{U}))\) for \(\gamma > p - 1\), by using Taylor’s theorem and \([87]\), we have

\[
\| (\beta_t - \beta_s)(g_t, \cdot) \|_k \leq C \| x_t - x_s \|^\gamma \leq C \| g \|^p_{p-\text{var}, [s,t]} \frac{t-s}{p}.
\]

Since \(f \in C^\gamma(\mathcal{V}, L(\mathcal{V}, \mathcal{U}))\), \(\{D^k f\}_{k=0}^{[p]-1}\) are bounded on bounded set and \(\sup_{s \in [0, T]} \| \beta_s(g_s, \cdot) \| < \infty\). Hence, \(\beta\) satisfies Condition \([16]\).

Since \(\beta\) satisfies Condition \([16]\), \(t \mapsto y_t := \int_0^t \beta_u(g_u) \, dg_u\) is a path dominated by \(g\). Based on the enhancement defined in Proposition \([18]\) and \([83]\), the \(k\)th iterated integral of \(y\) on \([s, t]\) (denoted by \(y^{\otimes k}_k\)) satisfies

\[
\left\| y^{\otimes k}_k - \beta_s(g_s, \cdot) \right\| \leq C \| D^k f \|_{p-\text{var}, [s,t]} \theta^{k-1} \theta^\gamma \left( \frac{t-s}{p} \right),
\]

where the formal integral \(\int \cdots \int_{s < u_1 < \cdots < u_k < t} \delta g_{u_1} \otimes \cdots \otimes \delta g_{u_k}\) can be defined by

\[
\sum_{j_1, \ldots, j_k} \sum_{\rho \in \text{OS}(j_1, \ldots, j_k)} \rho^{-1} (\pi_{j_1 + \cdots + j_k}(g_t))
\]

with \(\text{OS}(j_1, \ldots, j_k)\) denoting the ordered shuffles (as on p73 \([\text{LCL07}]\)). Then we check that, the step-[\(p\)] enhancement \(Y\) (defined at \([84]\)) coincides with the rough integral \(\int f(x) \, dg\) in \([\text{Lyo98}]\). Indeed, based on \([89]\), \(X\) defined by

\[
X_{s,t} := 1 + \sum_{k=1}^{[p]} \beta_s(g_s, \cdot) \otimes k \int \cdots \int_{s < u_1 < \cdots < u_k < t} \delta g_{u_1} \otimes \cdots \otimes \delta g_{u_k}, \quad \forall s < t,
\]

is a almost multiplicative functional (Def 4.2 \([\text{LCL07}]\)), and \(Y\) is a \(p\)-rough path associated with \(X\) (i.e. there exist control \(\omega\) and \(\theta > 1\) such that \(Y_{s,t}\) and \(X_{s,t}\) are close up to an error of \(\omega(s,t)^\theta\) for any \(s < t\)). On the other hand, based on Def 3.2.2 and Thm 3.2.1 \([\text{Lyo98}]\), the rough integral \(\int f(x) \, dg\) is another \(p\)-rough path associated with \(X\). Since the \(p\)-rough path associated with \(X\) is unique (Thm 3.3.1 \([\text{Lyo98}]\), the enhancement of \(y\) coincides with the rough integral in \([\text{Lyo98}]\).}

**Remark 23** In the framework of Example \([22]\), suppose \(f: [0, T] \to C^\gamma(\mathcal{V}, L(\mathcal{V}, \mathcal{U}))\) is a time-varying Lipschitz one-form satisfying that, there exist a control \(\omega\) and \(\theta > 1\), such that

\[
\| (D^{k-1}f_t - D^{k-1}f_s)(x_t) \| \leq \omega(s,t)^{\theta - \frac{k}{p}}, \quad \forall s < t, \quad k = 1, 2, \ldots, [p].
\]

Then if we modify \(\beta \in ([0, T], B(\mathcal{G}, \mathcal{U}))\) as at \([84]\) by replacing \(f\) with \(f_s\), then similar to Example \([22]\), it can be checked that \(y := \int_0^t \beta_u(g_u) \, dg_u\) is a dominated path.

**Example 24 (Weakly controlled path)** Suppose \(\{(T^{(n)}(\mathcal{V}), \mathcal{G}_n, \mathcal{P}_n)\}_{n=0}^{\infty}\) satisfies \([12]\) \([13]\) and \([14]\) and \(g \in C^{p-\text{var}}([0, T], \mathcal{G}_{[p]})\) for \(p \geq 2\). For \(i = 1, 2\) and Banach spaces \(\mathcal{U}',\) suppose \(\gamma^i \in C([0, T], \mathcal{U}')\) and \(\beta^i \in C([0, T], B(\mathcal{G}, \mathcal{U}'))\) satisfy that there exist control \(\omega\) and \(\theta > 1\), s.t.

\[
\left\| \gamma^i_t - \gamma^i_s - \beta^i_t(g_s, g_t) \right\| \leq \omega(s,t)^{\theta - \frac{i}{p}}, \quad \forall s < t,
\]

\[
\left\| (\beta^i_t - \beta^i_s)(g_t, \cdot) \right\|_k \leq \omega(s,t)^{\theta - \frac{k+1}{p}}, \quad \forall s < t, \quad \forall k = 1, \ldots, [p] - 1.
\]

Define \(h \in C([0, T], \mathcal{U}^2 \otimes \mathcal{G}_{[p]})\) by \(h := \gamma^2 \otimes g\) with \(h_{s,t} = (\gamma^2_t - \gamma^2_s) \otimes g_{s,t}, \quad \forall s < t\). Then there exists \(\beta \in C([0, T], B(\mathcal{U}^2 \otimes \mathcal{G}_{[p]}, \mathcal{U}^1 \otimes \mathcal{U}^2))\) such that \((\beta, h)\) satisfies the conditions of Theorem \([15]\) and (with \(I\) in Condition \([14]\))

\[
\| \int_s^t \beta_u(h_u) \, dh_u - \gamma^2_t - \beta^2_t(g_s, \cdot) \beta^2_s(g_s, \cdot) I(g_{s,t}) \| \leq C \left( \| g \|^p_{p-\text{var}, [s,t]} + \omega(s,t)^{\theta - \frac{p+1}{p}} \right), \quad \forall s < t.
\]
Proof. With the mapping $\mathcal{I} \in L(T^{(2[p])}(\mathcal{V}), T^{(p)}(\mathcal{V})) \otimes \mathcal{U}$ in Condition [14] for $s \in [0, T]$, we define $\beta^{1,2}_s \in C(\mathcal{G}_{[p]}, L(T^{(2[p])}(\mathcal{V}), \mathcal{U} \otimes \mathcal{U}))$ by

$$\beta^{1,2}_s(a, b) := \beta^{1}_s(a, \cdot) \otimes \beta^{2}_s(\cdot, b) \mathcal{I}(b), \forall a \in \mathcal{G}_{[p]}, \forall b \in T^{(2[p])}(\mathcal{V}),$$

and define

$$\beta_s(u \oplus v \oplus b) := \gamma^1_s \otimes v + \beta^{1,2}_s(g_s, 1[p] (g^{-1}_s a (b - 1))), \forall u, v \in \mathcal{U}^2, \forall a, b \in \mathcal{G}_{[p]}, \forall s.$$

Then we check that $(\beta, h)$ satisfies the conditions of Theorem [5]. Similar to the proof in Proposition [18] for $s < u < t$, we have

$$\left\| (\beta^{1,2}_s - \beta^{1,2}_u) (g_{u,t}) \right\| \leq C \left\| (\beta^{1,2}_s - \beta^{1,2}_u) (g_{u,t}) \right\|_{\mathcal{P}[p]} \left\| \sigma (g_{u,t}) \right\| \leq C \left( \omega(s, t) + \|g\|_{p-var,s,t} \right)^\theta.$$

(92)

With $g^{2[p]} \in C^{p-var}([0, T], T_{2[p]})$ denoting the enhancement of $g$ as in Example [11] for $s < u < t$,

$$(\beta_s - \beta_u) (h_{u,t})$$

(93)

Since $p \geq 2$, $\theta - p^{-1} \geq p^{-1}$, based on [10], we have (with the constant $C$ independent of $s, u$ and $t$)

$$\left\| \beta^{1}_s(g_s, g_{s,u}) \right\| \leq C \left\| g_{s,u} - 1 \right\| \leq C \left\| g_{s,u} \right\| \leq C \left\| g \right\|_{p-var,s,u},$$

$$\left\| \gamma^2 - \gamma^2_u \right\| \leq \left\| \beta^2_s(g_{s,u}, g_{u,t}) \right\| + \left\| \gamma^2 - \gamma^2_u - \beta^2_s(g_{u,t}) \right\| \leq C \left\| g \right\|_{p-var,u,t} + \omega(s, t)^{\theta - \frac{1}{p}} \leq C \left\| g \right\|_{p-var,s,t} + \omega(s, t)^{\theta}.$$

Then, combined with (90) and (91), we have

$$\left\| (\gamma^1_s - \gamma^1_u) \otimes (\gamma^2 - \gamma^2_u) - \beta^1_s(g_s, g_{s,u}) \otimes \beta^2_s(g_{u,t}) \right\|$$

$$\leq \left\| (\gamma^1_s - \gamma^1_u) \otimes (\gamma^2 - \gamma^2_u) \right\| + \left\| \beta^1_s(g_s, g_{s,u}) \otimes (\gamma^2 - \gamma^2_u - \beta^2_s(g_{u,t}) \right\|$$

$$\leq C \left\| g \right\|_{p-var,s,t} + \omega(s, t)^{\theta}.$$

On the other hand, by using that $\|g^{2[p]}\|_{p-var,s,t} \leq C_p \|g\|_{p-var,s,t}$, at (30) in Example [11] on p8, the last two terms in (93) are bounded by $C \|g\|_{p-var,s,t}$. Then combined with (93) and (92), $\beta$ and $h$ satisfy the conditions of Theorem [5].

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