IMPROVED LOWER BOUND FOR THE RADIUS OF ANALYTICITY OF SOLUTIONS TO THE FIFTH ORDER KDV-BBM MODEL

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Abstract. We show that the uniform radius of spatial analyticity $\sigma(t)$ of solutions at time $t$ to the fifth order KdV-BBM equation cannot decay faster than $1/\sqrt{t}$ for large $t$, given initial data that is analytic with fixed radius $\sigma_0$. This improves a recent result by Belayneh, Tegene and the third author [1], where they obtained a $1/t$ decay of $\sigma(t)$ for large time $t$.

1. Introduction

In this paper we consider the Cauchy problem for fifth order KdV-BBM equation

\[
\begin{aligned}
\partial_t \eta + \partial_x \eta - \gamma_1 \partial_t \partial_x^2 \eta + \gamma_2 \partial_x^3 \eta + \delta_1 \partial_t \partial_x^4 \eta + \delta_2 \partial_x^5 \eta \\
= -\frac{3}{4} \partial_x (\eta^2) - \gamma \partial_x^3 (\eta^2) + \frac{7}{48} \partial_x (\eta_x^2) + \frac{1}{8} \partial_x (\eta^3),
\end{aligned}
\]

(1)

where $\eta : \mathbb{R}^{1+1} \to \mathbb{R}$ is the unknown function, and $\gamma, \gamma_1, \gamma_2, \delta_1, \delta_2$ are constants satisfying certain constraints; see [3,9] for more details. The fifth order KdV-BBM equation describes the unidirectional propagation of water waves, and was recently introduced by Bona et al. [3] using the second order approximation in the two way model, the so-called abcd-system derived in [3,4]. In the case $\gamma = \frac{7}{48}$, (1) satisfies the energy conservation

\[
E(t) := \frac{1}{2} \int_{\mathbb{R}} \eta^2 + \gamma_1 \eta_x^2 + \delta \eta_{xx}^2 \, dx = E(0) \quad (t > 0).
\]

The well-posedness theory for the Cauchy problem (1) was studied by Bona et al. in [2], where they established local well-posedness for the initial data $\eta_0 \in H^s(\mathbb{R})$ with $s \geq 1$. For $\gamma_1, \sigma_1 > 0$ and $\gamma = 7/48$, the authors [2] used the conservation of energy to prove global well-posedness of (1) for $\eta_0 \in H^s(\mathbb{R})$ with $s \geq 2$. Furthermore, they used the method of high-low frequency splitting to obtain global well-posedness for $\eta_0 \in H^s(\mathbb{R})$ with $3/2 \leq s < 2$. The global well-posedness result was further improved in [9] for $\eta_0 \in H^s(\mathbb{R})$ with $s \geq 1$.

The main concern of this paper is to study the property of spatial analyticity of the solution $\eta(x, t)$ to (1), given a real analytic initial data $\eta_0(x)$ with uniform radius of analyticity $\sigma_0$, so that there is a holomorphic extension to a complex strip

$$S_{\sigma_0} = \{x + iy \in \mathbb{C} : |y| < \sigma_0\}.$$

\[\text{2010 Mathematics Subject Classification.} \quad 35A01, 35Q53.\]
\[\text{Key words and phrases.} \quad \text{KdV-BBM model; Global well-posedness, Improved Lower bound; Radius of analyticity; Modified Gevrey spaces.}\]
Information about the domain of analyticity of a solution to a PDE can be used to gain a quantitative understanding of the structure of the equation, and to obtain insight into underlying physical processes. It is classical since the work of Kato and Masuda [17] that, for solutions of nonlinear dispersive PDEs with analytic initial data, the radius of analyticity, \( \sigma(t) \), of the solution might decrease with \( t \).

Bourgain [7] used a simple argument in the context of Kadomtsev Petviashvili equation to show that \( \sigma(t) \) decays exponentially in \( t \).

Rapid progress has been made lately in obtaining an algebraic decay rate of the radius, i.e., \( \sigma(t) \sim t^{-\alpha} \) for some \( \alpha \geq 1 \), to various nonlinear dispersive PDEs, see e.g., [1, 15, 22–27]. The method used in these papers was first introduced by Selberg and Tesfahun [24] in the context of the Dirac-Klein-Gordon equations, which is based on an approximate conservation laws and Bourgain’s Fourier restriction method. For earlier studies concerning properties of spatial analyticity of solutions for a large class of nonlinear partial differential equations, see e.g., [5–7, 11–14, 16, 17, 19–21].

By the Paley–Wiener Theorem, the radius of analyticity of a function can be related to decay properties of its Fourier transform. It is therefore natural to take initial data in Gevrey space \( G^{\sigma,s}(\mathbb{R}) \) defined by the norm

\[
\|f\|_{G^{\sigma,s}(\mathbb{R})} = \left\| \exp(\sigma|\xi|) \langle \xi \rangle^s \hat{f} \right\|_{L^2_x(\mathbb{R})} \quad (\sigma \geq 0),
\]

where \( \langle \xi \rangle = \sqrt{1 + \xi^2} \). For \( \sigma = 0 \), this space coincides with the Sobolev space \( H^s(\mathbb{R}) \), with norm

\[
\|f\|_{H^s(\mathbb{R})} = \left\| \langle \xi \rangle^s \hat{f} \right\|_{L^2_x(\mathbb{R})},
\]

while for \( \sigma > 0 \), any function in \( G^{\sigma,s}(\mathbb{R}) \) has a radius of analyticity of at least \( \sigma \) at each point \( x \in \mathbb{R} \). This fact is contained in the following theorem, whose proof can be found in [18] in the case \( s = 0 \); the general case follows from a simple modification.

**Paley-Wiener Theorem.** Let \( \sigma > 0 \) and \( s \in \mathbb{R} \), then the following are equivalent

(a) \( f \in G^{\sigma,s}(\mathbb{R}) \),

(b) \( f \) is the restriction to \( \mathbb{R} \) of a function \( F \) which is holomorphic in the strip

\[
S_{\sigma} = \{ x + iy \in \mathbb{C} : |y| < \sigma \}.
\]

Moreover, the function \( F \) satisfies the estimates

\[
\sup_{|y| < \sigma} \| F(\cdot + iy) \|_{H^s(\mathbb{R})} < \infty.
\]

Recently, Carvajal and Panthee [8] used the Gevrey space to obtain an exponential decay on the radius of spatial analyticity \( \sigma(t) \) for solution \( \eta(x,t) \) to (1), i.e., \( \sigma(t) \sim e^{-t} \) for large \( t \). This was improved, more recently, to a linear decay rate, \( \sigma(t) \sim 1/t \), by Belayneh, Teggen and the third author [1], using the method of almost conservation law. In the present paper, we improve the decay rate further to \( \sigma(t) \sim 1/\sqrt{t} \), by using a modified Gevrey space that was introduced recently in [10] and the method of almost conservation law.

The modified Gevrey space, denoted \( H^{\sigma,s}(\mathbb{R}) \), is obtained from the Gevrey space \( G^{\sigma,s}(\mathbb{R}) \) by replacing the exponential weight \( \exp(\sigma|\xi|) \) with the hyperbolic
weight $cosh(\sigma|\xi|)$, i.e.,
\[ \|f\|_{H^{\sigma,s}(\mathbb{R})} = \left\| \cosh(\sigma|\xi|)\hat{f} \right\|_{L^2(\mathbb{R})} \quad (\sigma \geq 0). \]

Observe that
\[ \frac{1}{2} \exp(\sigma|\xi|) \leq \cosh(\sigma|\xi|) \leq \exp(\sigma|\xi|), \]
and hence the $G^{\sigma,s}(\mathbb{R})$ and $H^{\sigma,s}(\mathbb{R})$-norms are equivalent, i.e.,
\[ \|f\|_{H^{\sigma,s}(\mathbb{R})} \sim \|f\|_{G^{\sigma,s}(\mathbb{R})} = \left\| \exp(\sigma|\xi|)\hat{f} \right\|_{L^2(\mathbb{R})}. \]

Therefore, the statement of Paley-Wiener Theorem still holds for functions in $H^{\sigma,s}(\mathbb{R})$.
Observe also that for $\sigma \geq 0$ the exponential weight $\exp(\sigma|\xi|)$ satisfies the estimate
\[ \frac{1 - \exp(-\sigma|\xi|)}{|\xi|} \leq \sigma \]
whereas the hyperbolic weight $\cosh(\sigma|\xi|)$ satisfies
\[ \frac{1 - |\cosh(\sigma|\xi|)|^{-1}}{|\xi|^2} \leq \sigma^2. \]

Consequently, the decay rate $\sigma(t) \sim 1/t$ that was obtained in [1] stems from the $\sigma$-factor on the r.h.s of (5) whereas the improved decay rate $\sigma(t) \sim 1/\sqrt{t}$ obtained in this paper stems from the $\sigma^2$-factor on the r.h.s of (6).

We state our main result as follows.

**Theorem 1** (Asymptotic lower bound for $\sigma$). Let $\gamma_1, \delta_1 > 0$, $\gamma = \frac{7}{48}$ and $\eta_0 \in H^{\sigma_0,2}(\mathbb{R})$ for $\sigma_0 > 0$. Then the global \footnote{As a consequence of the embedding $H^{\sigma_0,2}(\mathbb{R}) \hookrightarrow H^2(\mathbb{R})$ and the existing well-posedness theory in $H^2(\mathbb{R})$ (see [2]), the Cauchy problem (1) (with $\gamma_1, \delta_1 > 0$ and $\gamma = 7/48$) has a unique, smooth solution for all time, given initial data $\eta_0 \in H^{\sigma_0,2}$.} solution $\eta(t)$ of (1) satisfies
\[ \eta(t) \in H^{\sigma,2}(\mathbb{R}) \quad \text{for all} \quad t > 0, \]
with the radius of analyticity $\sigma$ satisfying the asymptotic lower bound
\[ \sigma := \sigma(t) \geq C/\sqrt{t} \quad \text{as} \quad t \to \infty, \]
where $C > 0$ is constant depending on the initial data norm $\|\eta_0\|_{H^{\sigma_0,2}(\mathbb{R})}$.

So it follows from Theorem 1 that the solution $\eta(t)$ at any time $t$ is analytic in the strip $S_{\sigma(t)}$ (due to (4) and the Paley-Wiener Theorem).

To prove Theorem 1 first we establish the following local well-posedness result, which states that for short time the radius of analyticity of solution remains constant.

**Theorem 2.** (Local well-posedness). Let $\sigma_0 > 0$ and $\eta_0 \in H^{\sigma_0,2}(\mathbb{R})$. Then there exist a unique solution
\[ \eta \in C([0,T];H^{\sigma_0,2}(\mathbb{R})) \]
of the Cauchy problem (1), where the existence time is
\[ T \sim \left(1 + \|\eta_0\|_{H^{\sigma_0,2}(\mathbb{R})} \right)^{-2}. \]
Moreover, the data to solution map \( \eta_0 \mapsto \eta \) is continuous from \( H^{\sigma_0,2}(\mathbb{R}) \) to \( C([0,T];H^{\sigma_0,2}(\mathbb{R})) \).

Next, we derive an approximate energy conservation law for

\[ v_\sigma := \cosh(\sigma|D|)\eta, \]

where \( D = -i\partial_x \) and \( \eta \) is a solution to (1). To do this, we define a modified energy associated with \( v_\sigma \) by

\[ E_\sigma(t) = \frac{1}{2} \int_{\mathbb{R}} v_\sigma^2 + \gamma_1(\partial_x v_\sigma)^2 + \delta_1(\partial_x^2 v_\sigma)^2 \, dx. \]  

(8)

Note that since \( v_0 = \eta \), by (2) we have \( E_0(t) = E_0(0) \) for all \( t \).

**Theorem 3.** (Almost conservation law). Let \( \eta_0 \in H^{\sigma,2}(\mathbb{R}) \). Suppose that \( \eta \in C([0,T];H^{\sigma,2}(\mathbb{R})) \) is the local-in-time solution to the Cauchy problem (1) from Theorem 2. Then

\[ \sup_{0 \leq t \leq T} E_\sigma(t) = E_\sigma(0) + \sigma^2 T \cdot \mathcal{O} \left( \left[ 1 + (E_\sigma(0))^\frac{1}{2} \right] (E_\sigma(0))^\frac{3}{2} \right). \]

(9)

Observe that from (9), in the limit as \( \sigma \to 0 \), we recover the conservation \( E_0(t) = E_0(0) \) for \( 0 \leq t \leq T \). Applying the last two theorems repeatedly, and then by taking \( \sigma \) small enough we can cover any time interval \([0,T_\varepsilon]\) and obtain the lower bound in Theorem 1.

**Notation:** For any positive numbers \( p \) and \( q \), the notation \( p \lesssim q \) stands for \( p \leq cq \), where \( c \) is a positive constant that may vary from line to line. Moreover, we denote \( p \sim q \) when \( p \lesssim q \) and \( q \lesssim p \).

In the next sections we prove Theorems 2, 3 and 1.

## 2. Proof of Theorem 2

Taking the spatial Fourier transform of the first equation in (1) we obtain

\[
\begin{align*}
\partial_t \hat{\eta} + i\xi \hat{\eta} + \gamma_1 \xi^2 \partial_x \hat{\eta} - i\gamma_2 \xi^3 \hat{\eta} + \delta_1 \xi^4 \partial_x \hat{\eta} + i\delta_2 \xi^5 \hat{\eta} \\
= -\frac{3}{4} i\xi \hat{\eta}^2 + i\gamma \xi^3 \hat{\eta}^3 + \frac{7}{48} i\xi \hat{\eta}^2 + \frac{1}{8} i\xi \hat{\eta}^3.
\end{align*}
\]

Arranging the terms we have

\[
\left(1 + \gamma_1 \xi^2 + \delta_1 \xi^4\right) \partial_x \hat{\eta} + i\xi \left(1 - \gamma_2 \xi^2 + \delta_2 \xi^4\right) \hat{\eta} = \frac{1}{4} i\xi \left(-3 + 4\gamma \xi^2\right) \hat{\eta}^3 + \frac{7}{48} i\xi \hat{\eta}^2 + \frac{1}{8} i\xi \hat{\eta}^3.
\]

Dividing this equation by \( \varphi(\xi) := 1 + \gamma_1 \xi^2 + \delta_1 \xi^4 \) and multiplying by \( i \), we obtain

\[
i\partial_t \hat{\eta} - \varphi(\xi) \eta = \tau(\xi) \eta^2 - \frac{7}{48} \psi(\xi) \eta^2 - \frac{1}{8} \psi(\xi) \eta^3,
\]

(10)

where

\[
\begin{align*}
\varphi(\xi) &= \xi(1 - \gamma_2 \xi^2 + \delta_2 \xi^4) / \varphi(\xi), \\
\tau(\xi) &= \xi(3 - 4\gamma \xi^2) / 4\varphi(\xi), \\
\psi(\xi) &= \xi / \varphi(\xi).
\end{align*}
\]

In an operator form (10) can be rewritten as

\[
i\partial_t \eta - \varphi(D) \eta = \tau(D) \eta^2 - \frac{7}{48} \psi(D) \eta^2 - \frac{1}{8} \psi(D) \eta^3,
\]

(11)
where \( \phi(D), \psi(D) \) and \( \tau(D) \) are Fourier multiplier operators defined as

\[
\mathcal{F}[\phi(D)f](\xi) = \phi(\xi) \hat{f}(\xi), \quad \mathcal{F}[\psi(D)f](\xi) = \psi(\xi) \hat{f}(\xi), \quad \mathcal{F}[\tau(D)f](\xi) = \tau(\xi) \hat{f}(\xi).
\]

Now given initial data \( \eta(0) = \eta_0 \), the integral representation of (11) is

\[
\eta(t) = e^{-it\phi(D)} \eta_0 - i \int_0^t e^{-i(t-s)\phi(D)} F(\eta)(s) ds.
\]  (12)

By combining the estimates in [8, Lemma 2.2–2.4] and (4), we obtain the following a priori estimate for the \( H^{2}(\mathbb{R}) \)-norm of \( F(\eta) \).

**Lemma 1.** For \( \sigma \geq 0 \), we have nonlinear estimate

\[
\| F(\eta) \|_{H^{\sigma,2}(\mathbb{R})} \lesssim \| \eta \|_{H^{\sigma,2}(\mathbb{R})} \| \eta \|_{H^{\sigma,2}(\mathbb{R})}^2
\]

for all \( \eta \in H^{\sigma,2}(\mathbb{R}) \).

Next, we use the contraction mapping techniques and Lemma 1 to prove Theorem 2. To this end, define the mapping \( \eta \mapsto \Gamma(\eta) \) by

\[
\Gamma(\eta)(t) := e^{-it\phi(D)} \eta_0 - i \int_0^t e^{-i(t-s)\phi(D)} F(\eta)(s) ds
\]

and the space \( X_T \) by

\[
X_T = C([0,T]) \cap H^{\sigma,2}(\mathbb{R}) \quad \text{with norm} \quad \| u \|_{X_T} = \sup_{0 \leq t \leq T} \| u(t) \|_{H^{\sigma,2}(\mathbb{R})}.
\]

Then we look for a solution in the set

\[
S_T = \{ \eta \in X_T : \| \eta \|_{X_T} \leq r \},
\]

where \( 2r = \| \eta_0 \|_{H^{\sigma,2}(\mathbb{R})} \).

For \( \eta \in X_T \), we have by Lemma 1,

\[
\| \Gamma(\eta) \|_{X_T} \leq \| \eta_0 \|_{H^{\sigma,2}(\mathbb{R})} + cT \left[ 1 + \| \eta \|_{X_T} \right] \| \eta \|_{X_T}^2 \leq r/2 + cTr(1 + r)^2.
\]  (13)

Similarly, for \( \eta_1, \eta_2 \in X_T \), we obtain the difference estimate

\[
\| \Gamma(\eta_1) - \Gamma(\eta_2) \|_{X_T} \leq cT(1 + r)^2 \| \eta_1 - \eta_2 \|_{X_T}.
\]  (14)

By choosing

\[
T = \frac{1}{2c(1 + r)^2}
\]

in (13) and (14) we obtain

\[
\| \Gamma(\eta) \|_{X_T} \leq r \quad \text{and} \quad \| \Gamma(\eta_1) - \Gamma(\eta_2) \|_{X_T} \leq \frac{1}{2} \| \eta_1 - \eta_2 \|_{X_T}.
\]

Therefore, \( \Gamma \) is a contraction on \( S_T \) and therefore it has a unique fixed point \( \eta \in S_T \) solving the integral equation (12) on \( \mathbb{R} \times [0,T] \). Continuous dependence on the initial data can be shown in a similar way, using the difference estimate. This concludes the proof of Theorem 2.
3. Proof of Theorem 3

Fix $\gamma_1, \delta_2 > 0$ and $\gamma = \frac{7}{8}$. Recall that $v_\sigma := \cosh(\sigma|D|)\eta$, where $\eta$ is the solution to (1), and hence $\eta = \text{sech}(\sigma|D|)v_\sigma$.

Applying the operator $\cosh(\sigma|D|)$ to (1) we obtain

$$
\partial_t v_\sigma + \partial_x v_\sigma - \gamma_1 \partial_t \partial_x^2 v_\sigma + \gamma_2 \partial_x^3 v_\sigma + \delta_1 \partial_t \partial_x^4 v_\sigma + \delta_2 \partial_x^5 v_\sigma
$$

where

$$
N(v_\sigma) = \frac{3}{4} \gamma \partial_x^2 \partial_x(v_\sigma^2) + \frac{1}{8} \partial_x^3(v_\sigma^3) + N(v_\sigma),
$$

(15)

Differentiating the modified energy, $E_\sigma$, and using (15)–(17) we obtain

$$
\frac{d}{dt} E_\sigma(t) = \int_R v_\sigma \partial_t v_\sigma + \gamma_1 \partial_x v_\sigma \partial_t (\partial_x v_\sigma) + \delta_1 \partial_x v_\sigma \partial_t (\partial_x v_\sigma^3)
$$

$$
= \int_R v_\sigma [\partial_t v_\sigma - \gamma_1 \partial_t \partial_x^2 v_\sigma + \delta_1 \partial_t \partial_x^4 v_\sigma] dx
$$

$$
= - \int_R v_\sigma \left[ \partial_x v_\sigma + \gamma_2 \partial_x^3 v_\sigma + \delta_2 \partial_x^5 v_\sigma + \left( \frac{3}{4} + \gamma \partial_x^2 \right) \partial_x(v_\sigma^2) - \gamma \partial_x(\partial_x v_\sigma^2) - \frac{1}{8} \partial_x(v_\sigma^3) \right] dx
$$

$$
+ \int_R v_\sigma N(v_\sigma) dx.
$$

However, the integral on the third line is zero due to integration by parts (assuming sufficiently regular solution) and the following identities:

$$
u_\sigma \partial_x u = \frac{1}{2} \left( u^2 \right)_x, \quad \nu_\sigma \partial_x^2 u = (uu_{xx})_x - \frac{1}{2} \left( u^2 \right)_x,$$

$$
\nu_\sigma \partial_x^5 u = u \partial_x^4 u - \partial_x u \partial_x^3 u + \frac{1}{2} \left( u^2 \right)_{xx},
$$

and

$$
\nu_\sigma \left( u^2 \right)_x = \frac{2}{3} \left( u^3 \right)_x, \quad \nu_\sigma \left( u^3 \right)_x = \frac{3}{4} \left( u^4 \right)_x,$$

$$
\nu_\sigma \left( u^4 \right)_x = 2 \left( u^2 u_{xx} \right)_x + u \left( u^2 \right)_x.
$$

Therefore,

$$
\frac{d}{dt} E_\sigma(t) = \int_R v_\sigma(x, t) N(v_\sigma(x, t)) dx.
$$

Consequently, integrating with respect to time we get

$$
E_\sigma(t) = E_\sigma(0) + \int_0^t \int_R v_\sigma(x, s) N(v_\sigma(x, s)) dx ds.
$$

(18)

Combining (18) with the following key lemma, which will be be proved in the last section, we obtain (9).
Lemma 2. For $N(v_\sigma)$ as in (16)–(17) we have

$$\int_0^T v_\sigma N(v_\sigma) \, dx \leq c\sigma^2 \left[ 1 + \|v_\sigma\|_{H^2(\mathbb{R})} \right] \|v_\sigma\|_{L^2(\mathbb{R})}^2$$

(19)

for all $v_\sigma \in H^2(\mathbb{R})$.

Indeed, applying (19) to (18) we obtain

$$\sup_{0 \leq t \leq T} E_\sigma(t) = E_\sigma(0) + \sigma^2 T \cdot \left( 1 + \|v_\sigma\|_{L^2(\mathbb{R})}^3 \right)$$

(20)

where $L^2(\mathbb{R}) = L^2([0, T] \times \mathbb{R})$ with $T$ is as in Theorem 2.

As a consequence of Theorem 2 we have the bound

$$\|v_\sigma\|_{L^2(\mathbb{R})}^2 = \|\eta\|_{L^2(\mathbb{R})}^2 \leq c \|\eta_0\|_{L^2(\mathbb{R})} = c \|v_\sigma(\cdot, 0)\|_{H^2(\mathbb{R})}.$$  

(21)

On the other hand,

$$E_\sigma(0) = \frac{1}{2} \int_\mathbb{R} |v_\sigma(x, 0)|^2 + \gamma_1 (\partial_x v_\sigma(x, 0))^2 + \delta_1 \left( \partial_{xx}^2 v_\sigma(x, 0) \right)^2 \, dx$$

$$\sim \|v_\sigma(\cdot, 0)\|_{H^2(\mathbb{R})}.$$  

(22)

From (21) and (22) we get

$$\|v_\sigma\|_{L^2(\mathbb{R})} \sim (E_\sigma(0))^\frac{1}{2},$$

which can combined with (20) to obtain the desired estimate (9).

4. Proof of Theorem 1

Suppose that $\eta(\cdot, 0) = \eta_0 \in H^{\sigma_0}(\mathbb{R})$ for some $\sigma_0 > 0$. This implies $v_\sigma(\cdot, 0) = \cosh(\sigma_0 |D|) \eta_0 \in H^2$, and hence

$$E_{\sigma_0}(0) \sim \|v_\sigma(\cdot, 0)\|_{H^2(\mathbb{R})}^2 < \infty.$$  

Now following the argument in [24] (see also [22]) we can construct a solution on $[0, T_\ast]$ for arbitrarily large time $T_\ast$. This is achieved by applying the approximate conservation (9), so as to repeat the local result in Theorem 3 on successive short time intervals of size $T$ to reach $T_\ast$, by adjusting the strip width parameter $\sigma \in (0, \sigma_0]$ of the solution according to the size of $T_\ast$.

In what follows we prove that

$$\sup_{0 \leq t \leq T_\ast} E_\sigma(t) \leq 2E_{\sigma_0}(0) \quad \text{for} \quad \sigma \geq C/\sqrt{T_\ast}$$

(23)

for arbitrarily large $T_\ast$ and $C > 0$ depending on $E_{\sigma_0}(0)$. This would in turn imply

$$\sup_{0 \leq t \leq T_\ast} \|\eta(t)\|_{H^{\sigma_2}(\mathbb{R})} < \infty \quad \text{for} \quad \sigma \geq C/\sqrt{T_\ast}$$

which proves Theorem 1.

It remains to prove (23). To do this, first observe that for $\sigma \in (0, \sigma_0]$ and $\tau \in (0, T)$, we have by Theorems 2 and 3,

$$\sup_{0 \leq t \leq \tau} E_\sigma(t) \leq E_\sigma(0) + c\sigma^2 \tau \left[ 1 + (E_\sigma(0))^{1/2} \right] (E_\sigma(0))^{3/2}$$

$$\leq E_{\sigma_0}(0) + c\sigma^2 \tau \left[ 1 + (E_{\sigma_0}(0))^{1/2} \right] (E_{\sigma_0}(0))^{3/2}.$$
To get the second line we used the fact the \( \mathcal{E}_{\sigma}(0) \leq \mathcal{E}_{\sigma_0}(0) \) which holds for \( \sigma \leq \sigma_0 \) as \( \cosh r \) is increasing for \( r \geq 0 \). Thus,

\[
\sup_{0 \leq t \leq \tau} E_{\sigma}(t) \leq 2E_{\sigma_0}(0) \tag{24}
\]

provided that

\[
c\cosh \left[ 1 + (E_{\sigma_0}(0))^{1/2} \right] (E_{\sigma_0}(0))^{3/2} \leq E_{\sigma_0}(0). \tag{25}
\]

Next, we apply Theorem 2 with initial time \( t = \tau \) and time-step size \( T \) as in (7) to extend the solution from \( [0, \tau] \) to \( [\tau, \tau + T] \). By Theorem 3 and (24) we obtain

\[
\sup_{\tau \leq t \leq \tau + T} E_{\sigma}(t) \leq E_{\sigma}(\tau) + c \cosh \left[ 1 + (2E_{\sigma_0}(0))^{1/2} \right] (2E_{\sigma_0}(0))^{3/2} \tag{26}
\]

In this way we cover all time intervals \([0, T], [T, 2T], \ldots\), and obtain

\[
E_{\sigma}(T) \leq E_{\sigma}(0) + c \cosh \left[ 1 + (2E_{\sigma_0}(0))^{1/2} \right] (2E_{\sigma_0}(0))^{3/2}
\]

\[
E_{\sigma}(2T) \leq E_{\sigma}(T) + c \cosh \left[ 1 + (2E_{\sigma_0}(0))^{1/2} \right] (2E_{\sigma_0}(0))^{3/2}
\]

\[
\vdots
\]

\[
E_{\sigma}(nT) \leq E_{\sigma}(0) + n \cosh \left[ 1 + (2E_{\sigma_0}(0))^{1/2} \right] (2E_{\sigma_0}(0))^{3/2}.
\]

This argument can be continued as long as

\[
n \cosh \left[ 1 + (2E_{\sigma_0}(0))^{1/2} \right] (2E_{\sigma_0}(0))^{3/2} \leq E_{\sigma_0}(0) \tag{27}
\]

as this would imply \( E_{\sigma}(nT) \leq 2E_{\sigma_0}(0) \).

Thus, the induction stops at the first integer \( n \) for which

\[
n \cosh \left[ 1 + (2E_{\sigma_0}(0))^{1/2} \right] (2E_{\sigma_0}(0))^{3/2} > E_{\sigma_0}(0)
\]

and then we have reached the finite time \( T_\sigma = nT \) when

\[
c \cosh \left[ 1 + (2E_{\sigma_0}(0))^{1/2} \right] (2E_{\sigma_0}(0))^{1/2} > 1.
\]

This proves \( \sigma \geq C/\sqrt{T_\sigma} \) for some \( C > 0 \) depending on \( E_{\sigma_0}(0) \).

5. Proof of Lemma 2

To prove (19) we need the following estimate from [10, Lemma 3] in the special cases of \( p = 2 \) and \( p = 3 \).

**Lemma 3.** Let \( \xi = \sum_{j=1}^{p} \xi_j \) for \( \xi_j \in \mathbb{R} \), where \( p \geq 1 \) is an integer. Then

\[
\left| 1 - \cosh |\xi| \prod_{j=1}^{p} \text{sech} |\xi_j| \right| \leq 2^p \sum_{j \neq k=1}^{p} |\xi_j||\xi_k|.
\]

**Proof.** For the readers convenience we include the proof in the case \( p = 2 \). Note that

\[
\cosh |\xi_1| \cosh |\xi_2| = \frac{1}{2} [\cosh(|\xi_1| - |\xi_2|) + \cosh(|\xi_1| + |\xi_2|)].
\]

(29)
On the other hand, we have (see [10, Lemma 2]),

$$|\cosh b - \cosh a| \leq \frac{1}{2} |b^2 - a^2| (\cosh b + \cosh a).$$  \hspace{1cm} (30)

for $a, b \in \mathbb{R}$.

Then by (29) and (30),

$$|\cosh |\xi_1| \cosh |\xi_2| - \cosh |\xi|| = \left| \frac{1}{2} \left( \sum_{\pm} \cosh (|\xi_1| \pm |\xi_2|) - \cosh |\xi| \right) \right|$$

$$\leq \frac{1}{2} \left| \left( |\xi_1| \pm |\xi_2| \right)^2 - |\xi|^2 \right| (\cosh (|\xi_1| \pm |\xi_2|) + \cosh |\xi|)$$

$$\leq \frac{1}{2} \cdot 4|\xi_1| |\xi_2| \cdot 4 \cosh(|\xi_1|) \cosh(|\xi_2|)$$

$$= 8|\xi_1| |\xi_2| \cosh(|\xi_1|) \cosh(|\xi_2|).$$

Dividing by $\cosh(|\xi_1|) \cosh(|\xi_2|)$ yields the desired estimate (28) in the case $p = 2$.  \hspace{1cm} $\Box$

Next we prove (19). For $N(v_\sigma)$ as in (16)–(17), we use Plancherel theorem to write

$$\int_\mathbb{R} v_\sigma N(v_\sigma) dx = \int_\mathbb{R} v_\sigma \left( \frac{3}{4} + \gamma \partial_x^2 \right) \partial_x N_1(v_\sigma) - \gamma v_\sigma \partial_x N_2(v_\sigma) - \frac{1}{8} \gamma v_\sigma \partial_x N_3(v_\sigma) dx$$

$$= \int_\mathbb{R} \left( \frac{3}{4} + \gamma \partial_x^2 \right) v_\sigma \partial_x N_1(v_\sigma) dx + \gamma \int_\mathbb{R} \partial_x v_\sigma \cdot N_2(v_\sigma) dx + \frac{1}{8} \gamma \int_\mathbb{R} \partial_x v_\sigma \cdot N_3(v_\sigma) dx.$$

So (19) follows from

$$|I_1| \lesssim \sigma^2 \|v_\sigma\|^3_{H^2(\mathbb{R})}, \quad (j = 1, 2)$$  \hspace{1cm} (31)

$$|I_3| \lesssim \sigma^2 \|v_\sigma\|^4_{H^2(\mathbb{R})}. \quad \hspace{1cm} (32)

5.1. Proof of (31) when $j = 1$. By Cauchy-Schwarz inequality,

$$|I_1| \leq \left\| \left( \frac{3}{4} + \gamma \partial_x^2 \right) v_\sigma \right\|_{L^2_x(\mathbb{R})} \|\partial_x N_1(v_\sigma)\|_{L^2_x(\mathbb{R})}$$

$$\lesssim \|v_\sigma\|_{H^2(\mathbb{R})} \|\partial_x N_1(v_\sigma)\|_{L^2_x(\mathbb{R})}.$$

So the proof reduces to

$$\|\partial_x N_1(v_\sigma)\|_{L^2_x(\mathbb{R})} \lesssim \sigma^2 \|v_\sigma\|^2_{H^2(\mathbb{R})},$$  \hspace{1cm} (33)

where

$$N_1(v_\sigma) = v_\sigma^2 - \cosh(\sigma|D|) [\text{sech}(\sigma|D|) v_\sigma]^2.$$
Now taking the Fourier Transform of $\partial_x N_1(v_\sigma)$ and applying (28) with $p = 2$, we obtain
\[
\mathcal{F} [\partial_x N_1(v_\sigma)](\xi) = \left| \int_{\xi = \xi_1 + \xi_2} i\xi \left( 1 - \cosh(\sigma|\xi|) \prod_{j=1}^{2} \text{sech}(\sigma|\xi_j|) \right) \hat{\nu}_\sigma(\xi_1)\hat{\nu}_\sigma(\xi_2) d\xi_1 d\xi_2 \right|
\leq 4\sigma^2 \int_{\xi = \xi_1 + \xi_2} |\xi| \left( \sum_{j \neq k=1}^{2} |\xi_j||\xi_k| \right) |\hat{\nu}_\sigma(\xi_1)||\hat{\nu}_\sigma(\xi_2)| d\xi_1 d\xi_2.
\]
By symmetry, we may assume $|\xi_1| \leq |\xi_2|$. Then
\[
\mathcal{F} [\partial_x N_1(v_\sigma)](\xi) \leq 16\sigma^2 \int_{\xi = \xi_1 + \xi_2} |\xi_1||\hat{\nu}_\sigma(\xi_1)||\hat{\nu}_\sigma(\xi_2)| d\xi_1 d\xi_2
= 16\sigma^2 \mathcal{F}_\chi^{-1}(\|D|w_\sigma|D^2 w_\sigma\|)(\xi),
\]
where $w_\sigma = \mathcal{F}_\chi^{-1}(\|\hat{\nu}_\sigma\|)$. Finally, by Plancherel, Hölder inequality and Sobolev embedding,
\[
\|\partial_x N_1(v_\sigma)\|_{L^2_x(R)} \leq 16\sigma^2 \left\| |D|w_\sigma|D^2 w_\sigma\| \right\|_{L^2_x(R)}
\leq \sigma^2 \left\| |D|w_\sigma\|_{L^\infty_x(R)} \right\| \left\| |D^2 w_\sigma\| \right\|_{L^2_x(R)}
\leq \sigma^2 \left\| w_\sigma\|_{H^2_x(R)} \right\| \leq \sigma^2 \left\| v_\sigma\|_{H^2_x(R)} \right\|^2.
\]
which proves (33).

5.2. **Proof of (31) when $j = 2$.** By Plancherel and Cauchy-Schwarz inequality,
\[
|I_2| = \left| \int_R \partial_x v_\sigma. N_2(v_\sigma) dx \right| = \left| \int_R \langle D\partial_x v_\sigma, (D)^{-1} N_2(v_\sigma) \rangle dx \right|
\leq \| \langle D\partial_x v_\sigma \|_{L^2_x(R)} \| (D)^{-1} N_2(v_\sigma) \|_{L^2_x(R)}
\leq \| v_\sigma\|_{H^2_x(R)} \| N_2(v_\sigma)\|_{H^{-1}_x(R)}.
\]
So the proof reduces to
\[
\| N_2(v_\sigma)\|_{H^{-1}_x(R)} \lesssim \sigma^2 \left\| v_\sigma\|_{H^2_x(R)} \right\|^2,
\]
which
\[
N_2(v_\sigma) = (\partial_x v_\sigma)^2 - \cosh(\sigma|D|) \text{sech}(\sigma|D|) \partial_x v_\sigma)^2.
\]
Taking the spatial Fourier Transform of \( N_2(v_\sigma) \) and using (28) with \( p = 2 \), we obtain
\[
\mathcal{F}[N_2(v_\sigma)](\xi) = \left| \int_{\xi_1 + \xi_2} \left( 1 - \cosh(\sigma|\xi|) \prod_{j=1}^{2} \text{sech}(\sigma|\xi_j|) \right) i\xi_1 \hat{v}_\sigma(\xi_1) i\xi_2 \hat{v}_\sigma(\xi_2) d\xi_1 d\xi_2 \right|
\]
\[
\leq 8\sigma^2 \int_{\xi_1 + \xi_2} |\xi_1|^2 |\xi_2|^2 |\hat{v}_\sigma(\xi_1)||\hat{v}_\sigma(\xi_2)| d\xi_1 d\xi_2
\]
\[
\leq 8\sigma^2 \mathcal{F}_x \left[ |D^2 v_\sigma| |D^2 v_\sigma| \right](\xi),
\]
where \( w_\sigma = \mathcal{F}_x^{-1}(\hat{v}_\sigma) \).

Then by Plancherel, the Sobolev embedding
\[
H^1_x(\mathbb{R}) \hookrightarrow L^\infty_x(\mathbb{R}) \quad \leftrightarrow \quad L^1_x(\mathbb{R}) \hookrightarrow H^{-1}_x(\mathbb{R})
\]
and Cauchy-Schwarz, we obtain
\[
\|N_2(v_\sigma)\|_{H^1_x(\mathbb{R})} \leq 8\sigma^2 \left\| |D^2 v_\sigma| |D^2 v_\sigma| \right\|_{H^{-1}_x(\mathbb{R})}
\]
\[
\leq \sigma^2 \left\| |D^2 v_\sigma| |D^2 v_\sigma| \right\|_{L^1_x(\mathbb{R})}
\]
\[
\leq \sigma^2 \left\| |D^2 v_\sigma| \right\|_{L^2_x(\mathbb{R})} \left\| |D^2 v_\sigma| \right\|_{L^2_x(\mathbb{R})}
\]
\[
\leq \sigma^2 \|v_\sigma\|^2_{H^1_x(\mathbb{R})},
\]
which proves (34).

5.3. **Proof of (32).** By Cauchy-Schwarz inequality,
\[
|I_3| = \frac{1}{8} \left| \int_{\mathbb{R}} \partial_x v_\sigma N_3(v_\sigma) dx \right| \leq \| \partial_x v_\sigma \|_{L^2_x(\mathbb{R})} \| N_3(v_\sigma) \|_{L^2_x(\mathbb{R})}
\]
\[
\leq \| v_\sigma \|_{H^1(\mathbb{R})} \| N_3(v_\sigma) \|_{L^2_x(\mathbb{R})}.
\]
So it remains to prove
\[
\| N_3(v_\sigma) \|_{L^2_x(\mathbb{R})} \lesssim \sigma^2 \| v_\sigma \|^3_{H^1(\mathbb{R})},
\]  

(35)

where
\[
N_3(v_\sigma) = v_\sigma^3 - \cosh(\sigma|D|) |\text{sech}(\sigma|D|)|v_\sigma|^3.
\]

Taking the Fourier Transform of \( N_3(v_\sigma) \) and applying (28) with \( p = 3 \), we obtain
\[
\mathcal{F}_x [N_3(v_\sigma)](\xi)
\]
\[
= \left| \int_{\xi_1 + \xi_2 + \xi_3} \left( 1 - \cosh(\sigma|\xi|) \prod_{j=1}^{3} \text{sech}(\sigma|\xi_j|) \right) \hat{v}_\sigma(\xi_1) \hat{v}_\sigma(\xi_2) \hat{v}_\sigma(\xi_3) d\xi_1 d\xi_2 d\xi_3 \right|
\]
\[
\leq 8\sigma^2 \int_{\xi_1 + \xi_2 + \xi_3} \left( \sum_{j \neq k=1}^{3} |\xi_j||\xi_k| \right) |\hat{v}_\sigma(\xi_1)||\hat{v}_\sigma(\xi_2)||\hat{v}_\sigma(\xi_3)| d\xi_1 d\xi_2 d\xi_3
\]
By symmetry, we may assume $|\xi_1| \leq |\xi_2| \leq |\xi_3|$, which implies
\[
|\mathcal{F}_\chi [N_3(\cdot \sigma)](\xi)| \leq 48c^2 \int_{\xi = \xi_1 + \xi_2 + \xi_3} |\tilde{\nu}(\xi_1)| |\nu(\xi_2)| |\xi_3|^2 |\tilde{\nu}(\xi_3)| d\xi_1 d\xi_2 d\xi_3
\]
\[
= 48c^2 |\mathcal{F}_\chi (w_\sigma, \cdot |D|^2 w_\sigma)(\xi)|,
\]
where $w_\sigma = \mathcal{F}_\chi^{-1}(|\nu|_\sigma)$.

Then by Plancherel and Hölder inequality we get
\[
\|\mathcal{F}_\chi [N_3(\cdot \sigma)](\xi)\|_{L^2(\mathbb{R})} \lesssim \sigma^2 \|\nu\|_{L^2(\mathbb{R})} \|D^2 w_\sigma\|_{L^2(\mathbb{R})} 
\]
\[
\lesssim \sigma^2 \|w_\sigma\|^2_{H^2(\mathbb{R})} \|\nu\|_{H^2(\mathbb{R})} 
\]
\[
\lesssim \sigma^2 \|w_\sigma\|^3_{H^2(\mathbb{R})} 
\]
\[
\lesssim \sigma^2 \|\nu\|^3_{H^2(\mathbb{R})}
\]
which proves (35).

Acknowledgments A. Tesfahun acknowledges support from the Social Policy Research Grant (SPG), Nazarbayev University.

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