Geometric inequalities in Carnot groups

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Abstract

Let \( G \) be a sub-Riemannian \( k \)-step Carnot group of homogeneous dimension \( Q \). In this paper, we shall prove several geometric inequalities concerning smooth hypersurfaces (i.e. codimension one submanifolds) immersed in \( G \), endowed with the \( H \)-perimeter measure.

Key words and phrases: Carnot groups; Sub-Riemannian geometry; hypersurfaces; geometric inequalities.

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1. Introduction

During the last years there was an increasing interest in studying Analysis and Geometric Measure Theory in metric spaces (see [1], [3, 4], [17], [24], [31], [58] and bibliographic references therein, but this list is far from being exhaustive). In this regard, important examples of highly non-Euclidean geometries are represented by the so-called Carnot-Charathéodory (or sub-Riemannian) geometries; see [10], [33], [49], [50, 51, 52], [56], [59]. In this context, Carnot groups play the role of modeling the tangent space (in a suitable generalized sense, which is related with the Gromov-Hausdorff convergence) of a sub-Riemannian manifold; see [33], [49]. For this and many other reasons, Carnot groups are an intriguing field of research; see [5], [6], [7], [11], [13], [21], [23], [27], [28], [29], [30], [36], [40], [41], [43], [44], [54].

A \( k \)-step Carnot group \( (G, \cdot) \) is an \( n \)-dimensional, connected, simply connected, nilpotent, stratified Lie group (with respect to the group multiplication \( \cdot \)) whose Lie algebra \( \mathfrak{g} \cong \mathbb{R}^n \) satisfies:

\[
\mathfrak{g} = H_1 \oplus ... \oplus H_k, \quad [H_1, H_{i-1}] = H_i, \quad (i = 2, ..., k), \quad H_{k+1} = \{0\}.
\]

We assume that \( h_i = \text{dim}H_i (i = 1, ..., k) \) so that \( n = \sum_{i=1}^{k} h_i \). Any Carnot group \( G \) has a 1-parameter family of dilations, adapted to the stratification, that makes it a homogeneous group, in the sense of

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Stein’s definition; see [55]. We refer the reader to Section 1.1 for a more detailed introduction to Carnot groups.

In this paper, we shall prove some geometric inequalities concerning smooth hypersurfaces immersed in a sub-Riemannian $k$-step Carnot group $G$ of homogeneous dimension $Q := \sum_{i=1}^{k} i h_i$. We have to stress that hypersurfaces will be endowed with the so-called $H$-perimeter measure $\sigma_n^{n-1}$, which is a natural substitute of the intrinsic $(Q - 1)$-dimensional CC Hausdorff measure. In Section 1.2 we will discuss some preliminaries notions concerning homogeneous measures and the horizontal geometry of hypersurfaces. Then we will recall some tools which will be important in the sequel, such as a Coarea-type formula and the horizontal integration by parts theory; see Section 1.3.

In Section 2 we will extend to this setting some Isoperimetric-type Constants, introduced by Cheeger in the Seventies for compact Riemannian manifolds in [16] and later studied by Yau in [61].

In particular, we shall prove the validity of some global inequalities for smooth compact hypersurfaces with (or without) boundary, immersed into $G$. Here, we would like to remark that there is a strong relationship among these inequalities and some eigenvalue problems related to the 2nd order differential operator $L_{HS}$ (which is nothing but a horizontal version of the Laplace-Beltrami operator); see, more precisely, Definition 21 in Section 1.2.

Roughly speaking, after defining the isoperimetric constants (in purely geometric terms), we will show that they are equal to the infimum of some Rayleigh’s quotients. More precisely, let $S \subset G$ be a smooth hypersurface and assume $\partial S \neq \emptyset$. Furthermore, set

$$\text{Isop}(S) := \inf \frac{\sigma_n^{n-2}(N)}{\sigma_n^{n-1}(S_1)},$$

where $N \subset S$ is a smooth hypersurface of $S$ such that $N \cap \partial S = \emptyset$ and $S_1$ is the unique $(n-2)$-dimensional submanifold of $S$ such that $N = \partial S_1$. We have to stress that $\sigma_n^{n-1}$ and $\sigma_n^{n-2}$ denote homogeneous measures on $S_1$ and $N$, respectively. These measures can be thought of, respectively, as the $(Q-1)$-dimensional and the $(Q-2)$-dimensional CC Hausdorff measures on $S_1$ and $N$; see Section 1.2. Then, it will be shown that

$$\text{Isop}(S) = \inf \frac{\int_S |\text{grad}_{HS} \psi| \sigma_n^{n-1}}{\int_S |\psi| \sigma_n^{n-1}},$$

where the infimum is taken over suitably smooth functions on $S$ such that $\psi|_{\partial S} = 0$. As mentioned, this constant is related to the first non-zero eigenvalue $\lambda_1$ of the following Dirichlet-type problem:

$$\begin{cases} -L_{HS} \psi = \lambda \psi, \\ \psi|_{\partial S} = 0; \end{cases}$$

see Definition 21. Indeed, we shall see that

$$\lambda_1 \geq \left( \frac{\text{Isop}(S)}{4} \right)^2;$$

see Corollary 28. Some similar results concerning another isoperimetric constant will be proved; see Theorem 30 and Corollary 31. The proofs of these results follow the scheme of the Riemannian case, for which we refer the reader to Yau, [61]; see also [16] and [13, 14]. We also remark that the main technical tool in the original proofs is the Coarea formula.

In Section 3 we shall prove two geometric inequalities involving volume, $H$-perimeter and the 1st eigenvalue of the operator $L_{HS}$ on $S$. These results generalize an inequality of Chavel (see [12]) and an inequality of Reilly (see [53]), respectively.

In Section 4 we will extend to the Carnot setting some classical differential-geometric results (such as linear isoperimetric inequalities); see, for instance, [9] and references therein. The starting point is
an integral formula very similar to the Euclidean Minkowsky Formula; see Corollary \[20\] for a precise statement. In particular, we will show that

\[(h - 1)\sigma_h^n(S) \leq R \left( \int_S (|\mathcal{H}_h| + |C_h V_h|) \sigma_h^{n-1} + \sigma_h^{n-2}(\partial S) \right)\]

where \(S \subset \mathbb{G}\) is a compact hypersurface with boundary and \(R\) denotes the radius of a homogeneous \(\varrho\)-ball circumscribed about \(S\). From this linear (isoperimetric) inequality, it is possible to infer some geometric consequences and, among them, we prove a weak monotonicity inequality for the \(H\)-perimeter; see Section 4.1 Proposition \[38\] .

Section 5 contains a theorem about non-horizontal graphs in 2-step Carnot groups. This generalizes a classical result of Heinz [35]; see also Chern. [19].

Let us describe this result in the simpler case of the Heisenberg group \(\mathbb{H}^1\). So let \(S \subset \mathbb{H}^1\) be a \(T\)-graph associated with a function \(t = f(x,y)\) of class \(C^2\) over the \(xy\)-plane. If the horizontal mean curvature \(\mathcal{H}_h\) of \(S\) satisfies a bound \(|\mathcal{H}_h| \geq C > 0\), then

\[C^2_{Eu}(\mathcal{P}_{xy}(\mathcal{U})) \leq H_{Eu}^2(\mathcal{P}_{xy}(\partial \mathcal{U}))\]

for every \(C^1\)-smooth relatively compact open set \(\mathcal{U} \subset S\), where \(H_{Eu}^i(i = 1,2)\) is the usual \(i\)-dimensional Euclidean Hausdorff measure and \(\mathcal{P}_{xy}\) is the orthogonal projection onto the \(xy\)-plane. Hence, taking \(\mathcal{U} := S \cap C_r(T)\), where \(C_r(T)\) denotes a vertical cylinder of radius \(r\) around the \(T\)-axis of \(\mathbb{H}^1\), yields

\[r \leq \frac{2}{C}\]

for every \(r > 0\). It follows that any entire \(xy\)-graph of class \(C^2\), having either constant or only bounded horizontal mean curvature \(\mathcal{H}_h\), must be necessarily a \(H\)-minimal surface. An analogous result holds true in the framework of step 2 Carnot groups; see Theorem \[42\].

In Section 6 we shall study some (local) Poincaré-type inequalities, depending on the local geometry of the hypersurface \(S\) and, more precisely, on its characteristic set \(C_S\); see Theorem \[44\] Theorem \[45\].

For instance, let \(S \subset \mathbb{G}\) be a \(C^2\)-smooth hypersurface with bounded horizontal mean curvature \(\mathcal{H}_h\). Then, we shall prove that for every \(x \in S\) there exists \(R_0 \leq \text{dist}_g(x,\partial S)\) (which explicitly depends on \(C_S\)) such that:

\[\left( \int_{S_R} |\psi|^p \sigma_h^{n-1} \right)^{\frac{1}{p}} \leq C_{\rho} R \left( \int_{S_R} |\text{grad}_{\mathcal{H}_h} \psi|^p \sigma_h^{n-1} \right)^{\frac{1}{p}} \quad p \in [1, +\infty[\]

for all \(\psi \in C^1_0(S_R)\) and all \(R \leq R_0\), where \(S_R := S \cap B_\rho(x,R)\).

These results are obtained by means of elementary “linear” estimates starting from the horizontal integration by parts formula, together with a simple analysis of the role played by the characteristic set. Finally, in Section \[6.1\] we will prove the validity of a Caccioppoli-type inequality for weak solutions of the operator \(L_{\mathcal{H}_h}\).

1.1. Carnot groups. A \(k\)-step Carnot group \((\mathbb{G}, \bullet)\) is a finite-dimensional connected, simply connected, nilpotent and stratified Lie group with respect to a polynomial group law \(\bullet\). The Lie algebra \(\mathfrak{g} = \mathbb{R}^n\) fulfills the conditions: \(\mathfrak{g} = H_1 \oplus \ldots \oplus H_k\), \(H_i, H_{i+1} = H_i, \forall i = 2, \ldots, k+1\), \(H_{k+1} = \{0\}\), where \([\cdot, \cdot]\) denotes the Lie brackets and each \(H_i\) is a vector subspace of \(\mathfrak{g}\). In particular, we denote by 0 the identity of \(\mathbb{G}\) and assume that \(\mathfrak{g} \cong T_0 \mathbb{G}\). We also use the notation \(H := H_1\) and \(V := H_2 \oplus \ldots \oplus H_k\). The subspaces \(H\) and \(V\) are smooth subbundles of \(T \mathbb{G}\) called horizontal and vertical, respectively.

Notation 1. Throughout this paper, we denote by \(\mathcal{P}_H : T \mathbb{G} \rightarrow H_i\) the orthogonal projection map from \(T \mathbb{G}\) onto \(H_i\) for any \(i = 1, \ldots, k\). In particular, we set \(\mathcal{P}_H := \mathcal{P}_{H_1}\). Analogously, we set \(\mathcal{P}_V : T \mathbb{G} \rightarrow V\) to denote the orthogonal projection map from \(T \mathbb{G}\) onto \(V\).
Let $h_i := \text{dim} H_i$ for any $i = 1, \ldots, k$. Set $n_0 := 0$ and $n_i := \sum_{j=1}^i h_j$ for any $i = 1, \ldots, k$. Note that $n_0 = h_1, n_1 = h_1 + h_2, \ldots, n_k = n$.

**Notation 2.** Throughout this paper, we set $I_i := \{n_{i-1} + 1, \ldots, n_i\}$ for any $i = 1, \ldots, k$. We also set $I := \{h_1 + 1, \ldots, n\}$ and use Greek letters $\alpha, \beta, \gamma, \ldots$ for indices in $I$. For the sake of simplicity, we set $h := h_1$ and $I_h := I_i$.

The horizontal bundle $H$ is generated by a frame $X_h := \{X_1, \ldots, X_n\}$ of left-invariant vector fields. This frame can be completed to a global graded, left-invariant frame $X := \{X_1, \ldots, X_n\}$ for $T\mathbb{G}$. Note that the standard basis $\{e_i : i = 1, \ldots, n\}$ of $\mathbb{R}^n$ can be relabeled to be graded or adapted to the stratification. Any left-invariant vector field of the frame $X$ is given by $X_i(x) = L_{e_i}(x) = X (e_i)$ (i = 1, ..., n), where $L_{e_i}$ denotes the differential of the left-translation $L_x$, defined by $L_x y := x \cdot y \forall y \in \mathbb{G}$. We also fix a Euclidean metric on $\mathbb{G} := T_0\mathbb{G}$ such that $\{e_i : i = 1, \ldots, n\}$ is an orthonormal basis. This metric $g = \langle \cdot, \cdot \rangle$ extends to the whole tangent bundle by left-translations and makes $X$ an orthonormal left-invariant frame. Therefore $(\mathbb{G}, g)$ is a Riemannian manifold.

Let $\exp : \mathfrak{g} \to \mathbb{G}$ be the exponential map. Hereafter, we will use exponential coordinates of the 1st kind; see [60], Ch. 2, p. 88.

As for the case of nilpotent Lie groups, the multiplication $\cdot$ of $\mathbb{G}$ is uniquely determined by the “structure” of the Lie algebra $\mathfrak{g}$. This is the content of the Baker-Campbell-Hausdorff formula; see [20].

More precisely, one has
\[
\exp(X) \cdot \exp(Y) = \exp(X \ast Y) \quad \forall X, Y \in \mathfrak{g},
\]
where $\ast : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ denotes the so-called Baker-Campbell-Hausdorff product given by
\[
X \ast Y = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \text{brackets of length } \geq 3.
\]

Using exponential coordinates, the group multiplication $\cdot$ turns out to be polynomial and explicitly computable; see [20]. Moreover, $0 = \exp(0, \ldots, 0)$ and the inverse of $x \in \mathbb{G}$ ($x = \exp(x_1, \ldots, x_n)$) is $x^{-1} = \exp(-x_1, \ldots, -x_n)$.

A sub-Riemannian metric $g_H$ is a symmetric positive bilinear form on the horizontal bundle $H$. The $CC$-distance $d_{CC}(x,y)$ between $x, y \in \mathbb{G}$ is given by
\[
d_{CC}(x,y) := \inf \int \sqrt{g_H(\dot{\gamma}, \dot{\gamma})} \, dt,
\]
where the infimum is taken over all piecewise-smooth horizontal paths $\gamma$ joining $x$ to $y$. Later, we shall choose $g_H := g_H^2$.

Carnot groups are homogeneous groups, that is, they admit a 1-parameter group of automorphisms $\delta_t : \mathbb{G} \to \mathbb{G}$ (t \geq 0) defined by $\delta_t x := \exp(\sum_{i,j} t_i x_i e_i)$, where $x = \exp(\sum_{i,j} x_i e_i) \in \mathbb{G}$. As already said, the homogeneous dimension of $\mathbb{G}$ is the integer $Q := \sum_{i=1}^k i h_i$ coinciding with the Hausdorff dimension of $(\mathbb{G}, d_{CC})$ as a metric space; see [49].

We recall that a continuous distance $\rho : \mathbb{G} \times \mathbb{G} \to \mathbb{R}_+ \cup \{0\}$ is a homogeneous distance if, and only if,
\[
\rho(x, y) = \rho(z \cdot x, z \cdot y) \quad \forall x, y, z \in \mathbb{G}; \quad \rho(\delta_t x, \delta_t y) = t \rho(x, y) \quad \forall t \geq 0.
\]

The structural constants of $\mathfrak{g}$ (see [13]) associated with the frame $X$ are defined by $C_i^j := \{[X_i, X_j], X_r\}$ for $i, j, r = 1, \ldots, n$. They are skew-symmetric and satisfy Jacobi’s identity. The stratification of the Lie algebra $\mathfrak{g}$ implies a fundamental “structural” property of Carnot groups, i.e. if $X_i \in H_l, X_j \in H_m$, then $[X_i, X_j] \in H_{l+m}$. It is worth remarking that, if $i \in l$, and $j \in k$, also, then
\[
C_{ij}^m \neq 0 \implies m \in l_{i+j}.
\]
Equivalently, if $C_{ij}^m \neq 0$, then $\text{ord}(i) + \text{ord}(j) = \text{ord}(r)$, where $\text{ord} : \{1, \ldots, n\} \to \{1, \ldots, k\}$ is the function defined as $\text{ord}(l) = i \iff l \in I_i$. 
Notation 3. Henceforth, we shall set
\[ C^n = \{ C_{ij} \}_{i,j=1,\ldots,n} \in M_{n\times n}(\mathbb{R}) \quad \forall \alpha \in I_{2h} = \{ h+1,\ldots, h+2 \}; \]
\[ C^\infty = \{ C_{ij} \}_{i,j=1,\ldots,n} \in M_{n\times n}(\mathbb{R}) \quad \forall \alpha \in I_n = \{ h+1,\ldots, n \}. \]

Remark 4. It is important to observe that (2) immediately implies that the matrices just defined are the only ones which can be non zero.

Let us define the left-invariant co-frame \( \omega := \{ \omega_i : i = 1,\ldots,n \} \) dual to \( X \), i.e. \( \omega_i = X_i^* \) for every \( i = 1,\ldots,n \). The left-invariant 1-forms \( \omega_i \) for \( i = 1,\ldots,n \) are uniquely determined by the condition \( \omega_i(X_j) = (X_i,X_j) = \delta^j_i \forall i, j = 1,\ldots,n \), where \( \delta^j_i \) denotes Kronecker delta.

Definition 5. We shall denote by \( \nabla \) the unique left-invariant Levi-Civita connection on \( \mathbb{G} \) associated with the left-invariant metric \( g = \langle \cdot, \cdot \rangle \). Moreover, if \( X, Y \in \mathcal{X}(H) := C^\infty(\mathbb{G}, H) \), we shall set
\[ \nabla^H_XY := P_n(\nabla_XY). \]
Let \( \mathbb{X} = \{ X_1,\ldots,X_n \} \) be the global left-invariant frame on \( T\mathbb{G} \). Then, it turns out that
\[ \nabla^H_XY = \sum_{i=1}^n \left( C^r_{ij} \right) X_j \quad \forall i = 1,\ldots,n; \]
see, for instance, Milnor’s paper [42], Section 5, pp. 310-311. Furthermore, we stress that \( \nabla^H \) is a partial connection, called horizontal \( H \)-connection; see [32] or [37]; see also [44] and references therein. Using Definition 5 together with (3) and (2), it is not difficult to show the following:

- \( \nabla^H \) is flat, i.e.
  \[ \nabla^H_XY = 0 \quad \forall i = 1,\ldots,n; \]

- \( \nabla^H \) is compatible with the sub-Riemannian metric \( g^H \), i.e.
  \[ X(Y,Z) = \langle \nabla^H_XY,Z \rangle + \langle Y,\nabla^H_XZ \rangle \quad \forall X,Y,Z \in \mathcal{X}(H) \]

- \( \nabla^H \) is torsion-free, i.e.
  \[ \nabla^H_XY - \nabla^H_YX - P_n[X,Y] = 0 \quad \forall X,Y \in \mathcal{X}(H). \]

Definition 6. If \( \psi \in C^\infty(\mathbb{G}) \) we define the horizontal gradient of \( \psi \) as the unique horizontal vector field \( \text{grad}^H \psi \) such that \( \langle \text{grad}^H \psi, X \rangle = d\phi(X) = X\psi \) for every \( X \in \mathcal{X}(H) \). The horizontal divergence of \( X \in \mathcal{X}(H) \), \( \text{div}^H X \), is defined, at each point \( x \in \mathbb{G} \), by
\[ \text{div}^H X(x) := \text{Trace} \left( Y \rightarrow \nabla^H_XY \right)(x) \quad (Y \in H_x). \]
For any \( Y = \sum_{j\in I_H} y_j X_j \in \mathcal{X}(H) \), we denote by \( \mathcal{J}_n Y \) the horizontal Jacobian matrix of \( Y \), i.e.
\[ \mathcal{J}_n Y := \left[ X_i(y_j) \right]_{j\in I_H}. \]

Example 7 (Heisenberg group \( \mathbb{H}^{2n}(n \geq 1) \)). The Lie algebra \( \mathfrak{h}_n \equiv \mathbb{R}^{2n+1} \) of the \( n \)-th Heisenberg group \( \mathbb{H}^{2n} \) can be described by means of a left-invariant frame \( \mathcal{Z} := \{ X_1, X_2,\ldots,X_n,Y_1,\ldots,Y_n,T \} \), where, at each \( p = \exp(x_1,y_1,x_2,y_2,\ldots,x_n,y_n) \in \mathbb{H}^{2n} \), we have set: \( X_i(p) := \frac{\partial}{\partial x_i} - \frac{y_i}{2} \frac{\partial}{\partial t} \), \( Y_i(p) := \frac{\partial}{\partial y_i} + \frac{x_i}{2} \frac{\partial}{\partial t} \) for every \( i = 1,\ldots,n \). One has \( \{ X_i, Y_i \} = T \) for every \( i = 1,\ldots,n \), and all other commutators vanish, so that \( T \) is the center of \( \mathfrak{h}_n \) and \( \mathfrak{h}_n \) turns out to be a nilpotent and stratified Lie algebra of step 2, i.e. \( \mathfrak{h}_n = H \oplus H_2 \). The structural constants of \( \mathfrak{h}_n \) are described by the skew-symmetric \( (2n \times 2n) \)-matrix
\[ C^{2n+1}_n := \begin{bmatrix} 0 & 1 & \cdots & 0 & \cdots & \cdots & 0 \\ -1 & 0 & \cdots & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & -1 & \cdots & \cdots & 0 \end{bmatrix}. \]
1.2. Hypersurfaces. The (Riemannian) left-invariant volume form of any Carnot group \( \mathbb{G} \) is defined as 
\[
\sigma^n_H := \bigwedge_{i=1}^n \omega_i \in \bigwedge^n (T^* \mathbb{G}).
\]
By integration of the \( n \)-form \( \sigma^n_H \), one obtains the Haar measure of \( \mathbb{G} \), which equals the push-forward of the \( n \)-dimensional Lebesgue measure \( \mathcal{L}^n \) on \( \mathfrak{g} \cong \mathbb{R}^s \). The symbols \( \mathcal{H}^{s}_{CC} \), \( \mathcal{H}^{s}_{Eu} \) will denote the intrinsic CC \( s \)-dimensional Hausdorff measure and the Euclidean \( s \)-dimensional Hausdorff measure, respectively. (Sometimes we will use the notation \( \sigma^n_H = \nu_{Vol}^H \).) Let \( S \subset \mathbb{G} \) be a hypersurface (i.e. a codimension 1 submanifold of \( \mathbb{G} \)) of class \( C^i \) \((i \geq 1)\). Let \( \nu \) denote the (Riemannian) unit normal vector along \( S \). Then \( x \in S \) is a characteristic point if and only if \( \dim H_x = \dim(T_x \cap S) \).
The characteristic set of \( S \) is given by \( C_S := \{ x \in S : \dim H_x = \dim(H_x \cap T_x S) \} \). In other words, a point \( x \in S \) is non-characteristic (hereafter abbreviated as NC) if and only if \( H \) is transversal to \( S \) at \( x \). Hence, one has \( C_S := \{ x \in S : [\mathcal{P}_H \nu(x)] = 0 \} \), where \( \mathcal{P}_H \) denotes orthogonal projection onto \( H \). It is of fundamental importance that the \((Q-1)\)-dimensional CC Hausdorff measure of the characteristic set \( C_S \) vanishes, i.e. \( \mathcal{H}^{Q-1}_{CC}(C_S) = 0 \); see, for instance, Theorem 6.6.2 in [40]. We also stress that if \( S \) is a hypersurface of class \( C^2 \), then precise estimates of the Riemannian Hausdorff dimension of \( C_S \) can be found in [8]; see also [6] for the case of the Heisenberg group \( \mathbb{H}^n \) \((n \geq 1)\).

The \((n-1)\)-dimensional Riemannian measure along \( S \) is defined by integration of the \((n-1)\)-differential form 
\[
\sigma^{n-1}_H \lrcorner S := (\nu \lrcorner \sigma^{n-1}_H)|_S, \text{ where } \nu \text{ denotes the "contraction" operator on differential forms; see } [25].
\]
We recall that \( \nu : \bigwedge^k (T^* \mathbb{G}) \rightarrow \bigwedge^{k-1} (T^* \mathbb{G}) \) is defined, for \( X \in T \mathbb{G} \) and \( \alpha \in \bigwedge^k (T^* \mathbb{G}) \), by setting \( (\nu \lrcorner \alpha)(Y_1, ..., Y_{k-1}) := \alpha(X, Y_1, ..., Y_{k-1}) \).

At each NC point \( x \in S \setminus C_S \) the unit \( H \)-normal is defined as \( \nu_H := \frac{\varphi_{\mathcal{P}_H \nu}}{d_{CC}} \). Similarly to the Riemannian case, we define an \((n-1)\)-differential form \( \sigma^{n-1}_H \lrcorner S \) by setting 
\[
\sigma^{n-1}_H \lrcorner S := (\nu_H \lrcorner \nu \lrcorner \sigma^{n-1}_H)|_S.
\]
By integration of \( \sigma^{n-1}_H \lrcorner S \), one gets a left-invariant and \((Q - 1)\)-homogeneous measure, which is called \( H \)-perimeter measure. This measure can be extended to the whole of \( S \) by setting \( \sigma^{n-1}_H \lrcorner C_S = 0 \). Note that \( \sigma^{n-1}_H \lrcorner S = [\mathcal{P}_H \nu] \lrcorner \sigma^{n-1}_H \lrcorner S \). Furthermore, denoting by \( S^{Q-1}_{CC} \) the \((Q-1)\)-dimensional spherical intrinsic CC Hausdorff measure (i.e. associated with the CC-distance \( d_{CC} \)), then 
\[
\sigma^{n-1}_H(S \cap B) = k(\nu_H) \cdot S^{Q-1}_{CC}(S \cap B) \quad \forall \ B \in \mathcal{B}(\mathbb{G}),
\]
where the density-function \( k(\nu_H) \), called metric factor, explicitly depends on \( \nu_H \) and \( d_{CC} \); see [40].

At each NC point \( x \in S \setminus C_S \), the horizontal tangent bundle \( HS := H \cap TS \subset TS \) and the horizontal normal bundle \( \nu_H S \subset H \) split the horizontal bundle \( H \) into an orthogonal direct sum, i.e. \( H = \nu_H \oplus HS \). The stratification of \( \mathfrak{g} \) induces a stratification of \( TS := \oplus_{i \geq 1} H_i S \), where we have set \( HS := H_1 S \); see [33]. Note that at any characteristic point \( x \in C_S \) one has \( H_x = H_x S \), so that 
\[
\dim(H_x S) = \begin{cases} h - 1 & \text{if } x \in S \setminus C_S \\ h & \text{if } x \in C_S \end{cases}.
\]

Notation 8. Throughout this paper, we denote by \( \mathcal{P}_{HS} : TS \rightarrow HS \) the orthogonal projection map from \( TS \) onto \( HS \).

Now let \( S \subset \mathbb{G} \) be a hypersurface of class \( C^2 \) and let \( \nabla^TS \) denote the induced connection on \( S \) from \( \nabla \). The tangential connection \( \nabla^TS \) induces a partial connection on \( HS \) defined by 
\[
\nabla^HS_X Y := \mathcal{P}_{HS} \left( \nabla^TS_X Y \right) \quad \forall \ X, Y \in \mathcal{X}^1(HS) := C^1(S, HS).
\]
It turns out that 
\[
\nabla^HS_X Y = \nabla^H_X Y - \langle \nabla^H_X Y, \nu_H \rangle \nu_H \quad \text{for every } X, Y \in \mathcal{X}^1(HS);
\]
see [44].
Definition 9 (see [44]). We call HS-gradient of \( \psi \in C^1(S) \) the unique horizontal tangent vector field \( \text{grad}_{HS} \psi \) such that

\[
\langle \text{grad}_{HS} \psi, X \rangle = d\psi(X) = X\psi \quad \forall \ X \in \mathfrak{x}^1(HS).
\]

We denote by \( \text{div}_{HS} \) the HS-divergence, i.e. if \( X \in \mathfrak{x}^1(HS) \) and \( x \in S \), then

\[
\text{div}_{HS} X(x) := \text{Trace}(Y \mapsto \nabla^H_Y X)(x) \quad (Y \in H_nS).
\]

The HS-Laplacian \( \Delta_{HS} \) is the 2nd order differential operator defined as

\[
\Delta_{HS} \psi := \text{div}_{HS}(\text{grad}_{HS} \psi) \quad \text{for every } \psi \in C^2(S).
\]

The horizontal 2nd fundamental form of \( S \setminus C_S \) is the map given by

\[
B_H(X,Y) := \langle \nabla^H_X Y, v_n \rangle \quad \forall \ X,Y \in \mathfrak{x}^1(HS).
\]

The horizontal mean curvature \( H_H \) is the trace of \( B_H \), i.e. \( H_H := \text{Tr} B_H = -\text{div}_{HS} v_n \).

It is worth observing that the HS-connection admits, in general, a non-zero torsion because \( B_H \) is not symmetric; see [44].

Definition 10. Let \( U \subseteq S \) be an open set. We shall denote by \( C^i_{HS}(U) \), \( (i = 1, 2) \) the space of functions whose HS-derivatives up to i-th order are continuous on \( U \).

We stress that the previous definitions concerning the horizontal 2nd fundamental form \( B_H(\cdot, \cdot) \) and the HS-connection can also be reformulated by using the function space \( C^i_{HS}(U) \), \( (i = 1, 2) \) and, more precisely, by replacing \( \mathfrak{x}^1(HS) = C^1(S, HS) \) with \( \mathfrak{x}^i_{HS}(HS) := C^i_{HS}(S, HS) \).

Let \( S \subset \mathcal{G} \) be a hypersurface of class \( C^i \) \((i \geq 1)\) and let \( v \) be the outward-pointing unit normal vector field along \( S \). We need to define some important geometric objects. To this end, we first note that \( v = \mathcal{P}_H v + \mathcal{P}_V v \). By using the left-invariant frame \( \underline{X} = \{X_1, ..., X_n\} \), we see that \( \mathcal{P}_V v = \sum_{\alpha \in I_\nu} \gamma_\alpha X_\alpha \), where \( \gamma_\alpha := \langle \nu, X_\alpha \rangle \); see Notation 2.

Notation 11. Hereafter we shall set

- \( \sigma_\alpha := \frac{\nu_\alpha}{|\nu_\alpha|} \quad \forall \ \alpha \in I_\nu; \)
- \( \sigma := \sum_{\alpha \in I_\nu} \sigma_\alpha X_\alpha; \)
- \( C_H := \sum_{\alpha \in I_{H_2}} \sigma_\alpha C_\alpha^H; \)

see, for instance, Notation 3 and Remark 2.

1.3. Other tools. Let \( S \subset \mathcal{G} \) be a hypersurface of class \( C^i \) \((i \geq 1)\). Let \( \partial S \) be a \((n-2)\)-dimensional submanifold of \( S \) of class \( C^1 \), oriented by the outward pointing unit normal vector \( \eta \in TS \cap \text{Nor}(\partial S) \). We shall denote by \( \sigma_{\alpha}^{n-2} \) the Riemannian measure on \( \partial S \), i.e. \( \sigma_{\alpha}^{n-2} \langle \cdot, \cdot \rangle_{\partial S} = (\eta \otimes \sigma_{\alpha}^{n-1})_{|\partial S} \). In particular, note that \( (X \cdot \sigma_{\alpha}^{n-1})_{|\partial S} = (X, \eta) \langle \mathcal{P}_H v, \sigma_{\alpha}^{n-2} \rangle_{\partial S} \) for every \( X \in \mathfrak{x}^1(TS) := C^1(S, TS) \). The unit HS-normal along \( \partial S \) is given by \( \eta_{HS} := \frac{\mathcal{P}_H \eta}{|\mathcal{P}_H \eta|}. \)

In this way, we can define a homogeneous \((n-2)\)-dimensional measure \( \sigma_{\alpha}^{n-2} \in \wedge^{n-2}(T^* \partial S) \) by setting \( \sigma_{\alpha}^{n-2} \cdot \partial S = \langle \eta_{HS} \otimes \sigma_{\alpha}^{n-1} \rangle_{|\partial S} \). It follows that

\[
\sigma_{\alpha}^{n-2} \cdot \partial S = |\mathcal{P}_H v| |\mathcal{P}_H \eta| \sigma_{\alpha}^{n-2} \cdot \partial S
\]

and that \( (X \cdot \sigma_{\alpha}^{n-1})_{|\partial S} = (X, \eta_{HS}) \sigma_{\alpha}^{n-2} \cdot \partial S \) for every \( X \in \mathfrak{x}^1(HS) := C^1(S, HS) \).

Now let \( \nu \wedge \eta \in \Lambda^2(TS) \) be a unit 2-vector orienting \( \partial S \), where \( \nu \in \text{Nor}(S) \) and \( \eta \in TS \cap \text{Nor}(\partial S) \). Then, the characteristic set of \( \partial S \) is defined as \( \text{C}_{\partial S} := \{p \in \partial S : |\mathcal{P}_H (\nu \wedge \eta)| = 0\} \), where the orthogonal projection operator \( \mathcal{P}_H \) is extended to 2-vectors in the standard way.
Proposition 12. Let $S \subset \mathbb{S}^1$ be a compact hypersurface of class $C^1$ and let $\phi \in C^1_{\text{HS}}(S)$. Then

$$\int_S |\nabla_{\text{HS}} \phi(x)| \sigma_{H}^{n-1}(x) = \int_S \sigma_{H}^{n-2}(\phi^{-1}[s] \cap S) ds.$$  \hfill (4)

Proof. This formula follows from the Riemannian Coarea Formula; see [9], [15] or [46]. We have

$$\int_S \phi(x) |\nabla_{\text{TS}} \varphi(x)| \sigma_{H}^{n-1}(x) = \int_{\mathbb{R}} ds \int_{\varphi^{-1}[s] \cap S} \phi(y) \sigma_{H}^{n-2}(y)$$

for every $\phi \in L^1(S, \sigma_{H}^{n-1})$; see [9], [15]. Choosing $\phi = \frac{|\nabla_{\text{HS}} \varphi|}{|\nabla_{\text{TS}} \varphi|} |\mathcal{P}_H \nu|$, yields

$$\int_S \phi |\nabla_{\text{TS}} \varphi| \sigma_{H}^{n-1} = \int_S \frac{|\nabla_{\text{HS}} \varphi|}{|\nabla_{\text{TS}} \varphi|} |\mathcal{P}_H \nu| \sigma_{H}^{n-1} = \int_S |\nabla_{\text{HS}} \varphi| \sigma_{H}^{n-1}.$$  \hfill □

Below, we recall a basic integration by parts formula for horizontal vector fields; see [44].

Definition 13. Let $\mathcal{D}_{\text{HS}} : \mathfrak{X}^1_{\text{HS}}(HS) \longrightarrow C(S)$ be the 1st order differential operator given by

$$\mathcal{D}_{\text{HS}} X := \text{div}_{\text{HS}} X + \langle C_H \gamma_H X, X \rangle \quad \forall \ X \in \mathfrak{X}^1_{\text{HS}}(HS) \left( := C^1_{\text{HS}}(S, HS) \right).$$

Furthermore, let $\mathcal{L}_{\text{HS}} : C^2_{\text{HS}}(S) \longrightarrow C(S)$ be the 2nd order differential operator given by

$$\mathcal{L}_{\text{HS}} \varphi := \Delta_{\text{HS}} \varphi + \langle C_H \gamma_H, \nabla_{\text{HS}} \varphi \rangle \quad \forall \ \varphi \in C^2_{\text{HS}}(S);$$

see Definition [9] and Notation [17].

The horizontal matrix $C_H$ is a key object, related with the skew-symmetric part of the horizontal 2nd fundamental form $B_H$. Note that $\mathcal{D}_{\text{HS}} (\varphi X) = \varphi \mathcal{D}_{\text{HS}} X + \langle \nabla_{\text{HS}} \varphi, X \rangle$ for every $X \in \mathfrak{X}^1_{\text{HS}}(HS)$ and every $\varphi \in C^1_{\text{HS}}(S)$. Moreover, one has $\mathcal{L}_{\text{HS}} \varphi = \mathcal{D}_{\text{HS}} (\nabla_{\text{HS}} \varphi)$ for every $\varphi \in C^2_{\text{HS}}(S)$. These definitions are motivated by Theorem 3.17, Corollary 3.18 and Corollary 3.19 in [44].

Theorem 14 (see [44]). Let $S$ be a compact NC hypersurface of class $C^2$ with boundary $\partial S$ of class $C^1$. Then

$$\int_S \mathcal{D}_{\text{HS}} X \sigma_{H}^{n-1} = - \int_S \mathcal{H}_{\text{H}} \langle X, \gamma_H \rangle \sigma_{H}^{n-1} + \int_{\partial S} \langle X, \eta_{\text{HS}} \rangle \sigma_{H}^{n-2} \quad \forall \ X \in \mathfrak{X}^1(H).$$  \hfill (5)

Remark 15. We note that, in general, $\mathcal{H}_{\text{H}} \notin L^1_{\text{loc}}(S; \sigma_{H}^{n-1})$; see [22]. However, it is always true that $\mathcal{H}_{\text{H}} \in L^1_{\text{loc}}(S; \sigma_{H}^{n-1})$; see, for instance, [48].
Remark 16. Let \( S \subset \mathbb{G} \) be a hypersurface of class \( C^2 \) and \( v \) the outward-pointing unit normal vector along \( S \). Let for any \( X \in \mathcal{X}(\mathbb{G}) \) let us set \( X^\perp := \langle X, v \rangle v \) and \( X^\top := X - X^\perp \) to denote the Riemannian normal and tangential components of \( X \) at any point of \( S \). We would like to stress that formula (5) can be seen as a particular case of a general integral formula, the so-called 1st variation formula of the \( H \)-perimeter. More precisely, the 1st variation formula is given by

\[
I_1(X, \sigma_H^{n-1}) = \int_S \left( -\mathcal{H}_H(X^\perp, v) + \text{div}_T \left( X^\top \left[ \mathcal{P}_H v - \langle X^\perp, v \rangle v^\perp \right] \right) \right) \sigma_H^{n-1}
\]

where \( I_1(X, \sigma_H^{n-1}) \) denotes the 1st derivative of the \( H \)-perimeter under a smooth variation of \( S \) with initial velocity \( X \); see Theorem 4.6 in [48]. Formula (5) also holds if \( C_S \neq \emptyset \), but in this case we need to assume \( \mathcal{H}_H \in L^1_{\text{loc}}(S; \sigma_H^{n-1}) \). We observe that, in the case of the 1st Heisenberg group \( \mathbb{H}^1 \), this formula coincides with that of Ritoré and Rosales; see [54], Lemma 4.3, p. 14. Note that, if \( X = X_H \in \mathcal{X}(H) \), then

\[
X_H^\top \mathcal{P}_H v - \langle X_H^\perp, v \rangle v_H^\perp = \langle X_H - \mathcal{P}_H v(X_H, v_H) \rangle \mathcal{P}_H v - \langle \mathcal{P}_H v(X_H, v) \rangle (v_H - \mathcal{P}_H v | v) \\
= \langle X_H - (X_H, v) v_H \rangle \mathcal{P}_H v \\
= \mathcal{P}_{H_S}(X_H) \mathcal{P}_H v,
\]

where we have used the fact that \( v = |\mathcal{P}_H v| v_H + \sum_{a \in I_v} v_a X_a \) at each NC point. Finally, inserting this into (6), we obtain an equivalent form of (5). In particular, for any \( X \in \mathcal{X}(H) \) the function \( \mathcal{D}_{H_S} X \) turns out to be the Lie derivative of the differential \((n-1)\)-form \( \sigma_H^{n-1} \) over \( S \) with respect to the initial velocity \( X \) of a smooth variation of \( S \). Roughly speaking, this can be rephrased by saying that the differential \((n-1)\)-form \( \mathcal{D}_{H_S} X \sigma_H^{n-1} \in \Lambda^{n-1}(T^* S) \) is the “infinitesimal” 1st variation of \( S \).

Formula (5) holds true even if \( C_S \neq \emptyset \), at least under suitable assumptions.

Definition 17. Let \( X \in C^1(S \setminus C_S, H_S) \) and set \( \alpha_X := (X \cdot \sigma_H^{n-1})|S \). We say that \( X \) is admissible (for the horizontal divergence formula) if the differential forms \( \alpha_X \) and \( d\alpha_X \) are continuous on all of \( S \), or, more generally, if \( \alpha, d\alpha \in L^\infty(S) \) and \( \iota_S^* \alpha \in L^\infty(\partial S) \). We say that \( \phi \in C^2_{H_S}(S \setminus C_S) \) is admissible if \( \text{grad}_{H_S} \phi \) is admissible for the horizontal divergence formula.

We stress that, if the differential forms \( \alpha_X \) and \( d\alpha_X \) are continuous on all of \( S \) (or, more generally, if \( \alpha, d\alpha \in L^\infty(S) \) and \( \iota_S^* \alpha \in L^\infty(\partial S) \), where \( \iota_S : \partial M \rightarrow \overline{M} \) is the natural inclusion), then Stokes formula holds true; see, for instance, [57]. This fact motivates the following:

Corollary 18. Let \( S \subset \mathbb{G} \) be a compact hypersurface of class \( C^2 \) with boundary \( \partial S \) of class \( C^1 \). Then

(i) \( \int_S \mathcal{D}_{H_S} X \sigma_H^{n-1} = \int_{\partial S} \langle X, n_{H_S} \rangle \sigma_H^{n-2} \) for every admissible \( X \in C^1(S \setminus C_S, H_S) \);

(ii) \( \int_S \mathcal{L}_{H_S} \phi \sigma_H^{n-1} = \int_{\partial S} \langle \text{grad}_{H_S} \phi, n_{H_S} \rangle \sigma_H^{n-2} \) for every admissible \( \phi \in C^2_{H_S}(S \setminus C_S) \);

(iii) if \( \partial S = \emptyset \), then \( \int_S \phi \mathcal{L}_{H_S} \phi \sigma_H^{n-1} = \int_S |\text{grad}_{H_S} \phi|^2 \sigma_H^{n-1} \) for every \( \phi \in C^2_{H_S}(S \setminus C_S) \) such that \( \phi^2 \) is admissible.

The last formula holds true even if \( \partial S \neq \emptyset \), but for compactly supported functions. Moreover, it can be shown that \( \phi^2 \) is admissible if and only if \( \phi \in C^2_{H_S}(S \setminus C_S) \cap W^{1,2}_{H_S}(S, \sigma_H^{n-1}) \) where we have set \( W^{1,2}_{H_S}(S, \sigma_H^{n-1}) := \{ \phi \in L^2(S, \sigma_H^{n-1}) : |\text{grad}_{H_S} \phi| \in L^2(S, \sigma_H^{n-1}) \} \). We also remark that any vector field \( X \in C^1(S, H_S) \) turns out to be admissible. Analogously, any \( \phi \in C^2_{H_S}(S) \) is admissible.

Lemma 19. Let \( x_H := \sum_{i \in I_v} x_i I_i \) be the “horizontal position vector” and let \( g_H \) denote its component along the \( H \)-normal \( v_H \), i.e. \( g_H := \langle x_H, v_H \rangle \). In the sequel, the function \( g_H \) will be called “horizontal support function” of \( x_H \). Then, we have:

(i) \( \text{div}_H x_H = h \);

(ii) \( \mathcal{D}_{H_S} (x_H) = (h - 1) + g_H \mathcal{H}_H + \langle C_H v_H, x_{H_S} \rangle \) at each NC point \( x \in S \setminus C_S \), where \( x_{H_S} := x_H - g_H v_H \).
Proof. We have $\text{div}_{HS} x_H = \sum_{i=1}^{h} (\nabla_i x_H, X_i) = \sum_{i,j=1}^{h} (x_i(x_j) + \langle \nabla X_i X_j, X_i \rangle) = \sum_{i,j=1}^{h} \delta^j_i = h$, where $\delta^j_i$ denotes Kronecker’s delta. Note that we have used $\int_{H} (x_H) = \text{Id}_H$ and $\langle \nabla_X X_j, X_i \rangle = 0$ for all $i, j \in H$; see Definition 2.1 and formula 6. Furthermore, by definition, one has $\text{div}_{HS} x_H = \text{div}_H x_H - \langle \nabla_{y_H} x_H, y_H \rangle$. Hence $\text{div}_{HS} x_H = h - \langle y_H, y_H \rangle = h - 1$. Furthermore, by definition, we have

$$ \text{(7)} \quad \text{div}_{HS} x_H = \sum_{i=2}^{h} (\langle \tau_i, (x_H - g_H y_H), \tau_i \rangle), $$

where we have used an orthonormal horizontal frame $\tau_i := \{\tau_1, ..., \tau_h\}$ in an open neighborhood $U \subset \mathbb{G}$ of $S$ such that $\tau_i(x) = y_i(x)$ at every $x \in S \setminus C_S$; see, for instance, Definition 3.4 in [44]. Starting from (7), we compute

$$ \text{div}_{HS} x_H = \sum_{i=2}^{h} (\langle (\tau_i, \tau_i) - g_H \langle \nabla \tau_i y_H, \tau_i \rangle \rangle) = (h - 1) - g_H \text{div}_H y_H = (h - 1) + g_H H $$

for every $x \in S \setminus C_S$. The thesis easily follows from the definition of $D_{HS}$. □

A simple consequence of Corollary 18 and Lemma 19 is given by the following:

**Corollary 20** (Minkowsky-type formula). Let $S \subset \mathbb{G}$ be a compact hypersurface of class $C^2$. Let $x_H = \sum_{i \in H} x_i X_i$ be the horizontal position vector. Furthermore, set $g_H = \langle x_H, y_H \rangle$ and $x_{HS} = x - g_H y_H$ for every $x \in S \setminus C_S$. Then

$$ \int_S \left( (h - 1) + g_H H + \langle C_H y_H, x_{HS} \rangle \right) \sigma^{n-1}_H = 0. $$

**Proof.** It is enough to apply Corollary 18 to the horizontal tangent vector field $x_{HS} \in C^1(S \setminus C_S, HS)$. Using Remark 15 and Lemma 19 the thesis easily follows. □

**Definition 21** (Eigenvalue problems for $L_{HS}$). Let $S \subset \mathbb{G}$ be a compact hypersurface of class $C^2$ without boundary. Then we look for solutions of class $C^2_{HS}(S \setminus C_S) \cap W^{1,2}_{HS}(S, \sigma^{n-1}_H)$ to the problem:

$$ \text{(P1)} \quad \left\{ \begin{array}{l}
-L_{HS} \psi = \lambda \psi; \\
\int_S \psi \sigma^{n-1}_H = 0.
\end{array} \right. $$

If $\partial S \neq \emptyset$, we look for solutions of class $C^2_{HS}(S \setminus C_S) \cap W^{1,2}_{HS}(S, \sigma^{n-1}_H)$ to the problems:

$$ \text{(P2)} \quad \left\{ \begin{array}{l}
-L_{HS} \psi = \lambda \psi; \\
\psi_{\partial S} = 0;
\end{array} \right. \quad \text{and} \quad \text{(P3)} \quad \left\{ \begin{array}{l}
-L_{HS} \psi = \lambda \psi; \\
\frac{\partial \psi}{\partial N_{HS}} \mid_{\partial S} = 0.
\end{array} \right. $$

We explicitly remark that $\frac{\partial \psi}{\partial N_{HS}} = \langle \text{grad}_{HS} \psi, \eta_{HS} \rangle$.

The problems (P1), (P2) and (P3) generalize to our context the classical closed, Dirichlet and Neumann eigenvalue problems for the Laplace-Beltrami operator on Riemannian manifolds; see [13, 14].

Finally, we recall a recent general result about the size of horizontal tangencies to non-involutive distributions, which applies to our Carnot setting; see Theorem 4.5 in [8].

**Theorem 22** (Generalized Derridj’s Theorem). Let $\mathbb{G}$ be a k-step Carnot group.

(i) If $S \subset \mathbb{G}$ is a hypersurface of class $C^2$, the Euclidean-Hausdorff dimension of the characteristic set $C_S$ of $S$ satisfies $\dim_{\text{Eu-Hau}}(C_N) \leq n - 2$.

(ii) If $V = H_T \subset T\mathbb{G}$ satisfies $\dim V \geq 2$ and $N \subset \mathbb{G}$ is a $(n - 2)$-dimensional submanifold of class $C^2$, then the Euclidean-Hausdorff dimension of the characteristic set $C_N$ of $N$ satisfies $\dim_{\text{Eu-Hau}}(C_N) \leq n - 3$. 


Remark 23. Let $N \subset \mathbb{G}$ be a $(n-2)$-dimensional submanifold of class $C^2$. This smoothness condition is sharp, see [8]. Moreover, we stress that $\dim V = 1$ just for Heisenberg groups and 2-step Carnot groups having 1-dimensional center. For Heisenberg groups $\mathbb{H}^n$, $n > 1$, using Frobenius’ Theorem yields $\dim_{\text{Eu}}(C_N) \leq n$, where $n = \dim H$; see also [8]. On the contrary, 1-dimensional curves in $\mathbb{H}^1$, can be horizontal or transversal to $H$. For 2-step groups having 1-dimensional center (or, equivalently, horizontal bundle $H$ of codimension 1) a simple analysis shows that $\dim_{\text{Eu}}(C_N) = n - 2$ if, and only if, $\mathbb{G}$ reduces to the direct product of $\mathbb{H}^1$ and of a Euclidean space $\mathbb{R}^{n-2}$.

2. Isoperimetric constants and the 1st eigenvalue of $L_{\text{is}}$ on compact hypersurfaces

As a consequence of the Coarea Formula [4] we may generalize to the Carnot groups setting some results about isoperimetric constants and global Poincaré inequalities for which we refer the reader to [13, 14]; see also [16], [61].

Let $S \subset \mathbb{G}$ be a compact hypersurface of class $C^2$ with (or without) boundary. Similarly as in the Riemannian setting (see [16] and [61]), we may give the following:

Definition 24. The isoperimetric constant $\text{Isop}(S)$ of $S$ is defined as follows:

- if $\partial S = \emptyset$, we set
  $$\text{Isop}(S) := \inf \frac{\sigma_{n-2}^H(N)}{\min(\sigma_{n-1}^H(S_1), \sigma_{n-1}^H(S_2))},$$
  where the infimum is taken over all $C^2$-smooth $(n-2)$-dimensional submanifolds $N$ of $S$ which divide $S$ into two hypersurfaces $S_1, S_2$ with common boundary $N = \partial S_1 = \partial S_2$;
- if $\partial S \neq \emptyset$, we set
  $$\text{Isop}(S) := \inf \frac{\sigma_{n-2}^H(N)}{\sigma_{n-1}^H(S_1)},$$
  where $N \subset S$ is a smooth hypersurface of $S$ such that $N \cap \partial S = \emptyset$ and $S_1$ is the unique $C^2$-smooth $(n-2)$-dimensional submanifold of $S$ such that $N = \partial S_1$.

Here above $\partial S$, $S_1$, $S_2$ and $N = \partial S_i (i = 1, 2)$ are not assumed to be connected.

This definition requires some comments. As recalled in the introduction, in the Riemannian setting analogous isoperimetric constants were introduced by Cheeger in [16], in order to give a geometric lower bound for the smallest eigenvalue of the Laplace-Beltrami operator on smooth compact Riemannian manifolds. This definition was somewhat motivated by an example of Calabi, the so-called dumbbell manifold, homeomorphic to $S^2$. Actually, an analysis of this example shows that, in order to bound $\lambda$ from below, the diameter and the volume are not enough.

We also have to recall that these isoperimetric constants turn out to be strictly positive. Although, this claim turns out to be (more or less) elementary in dimension $n = 2$, it becomes a bit more difficult when $n > 2$; see [16]. Some years later after Cheeger result, Yau (see [61]) reconsidered the isoperimetric constants and demonstrated that $\lambda$ has a bound in terms of volume, diameter and (of a lower bound of the) Ricci curvature. See the survey [39] for a glimpse on this topic.

Below we shall generalize some of the results of [61]. Our results will follow the original scheme, which is based mainly on a suitable use of the Coarea formula for smooth functions. Note also that, instead of $C^\infty$-smooth hypersurfaces, here we are considering hypersurfaces of class $C^2$. We have to observe that all the results could also be stated for $C^1$ hypersurfaces. But the delicate matter here is that in our setting, new difficulties come from the presence of characteristic points and, in the $C^1$ case, it is not simple to prove that isoperimetric constants are strictly positive. Actually, the following further hypothesis seems to be unavoidable in order to have non-zero isoperimetric constants:

(H) every $C^2$-smooth $(n-2)$-dimensional submanifold $N \subset S$ satisfies $\dim C_N < n - 2$. 
This assumption can be overcome by using the generalized Derridj’s Theorem \cite{22}; see also Remark \cite{23}. As a consequence, the results of this section are “meaningful” (in the sense that the isoperimetric constants do not vanish) at least for any Carnot group \(\mathbb{G}\) such that \(\dim V \geq 2\) and for all Heisenberg groups \(\mathbb{H}^n\), with \(n > 1\).

**Theorem 25.** Let \(S \subset \mathbb{G}\) be a compact hypersurface of class \(C^2\).

(i) If \(\partial S = \emptyset\), then

\[
\text{Isop}(S) = \inf \int_S |\nabla_{\mathbb{H}} \psi| \sigma_n^{-1} \frac{dS}{\sigma_n^{-1}}.
\]

where the infimum is taken over all \(C^2\)-smooth functions on \(S\) such that \(\int_S \psi \sigma_n^{-1} = 0\).

(ii) If \(\partial S \neq \emptyset\), then

\[
\text{Isop}(S) = \inf \int_S |\nabla_{\mathbb{H}} \psi| \sigma_n^{-1} \frac{dS}{\sigma_n^{-1}}.
\]

where the infimum is taken over all \(C^2\)-smooth functions on \(S\) such that \(\psi|_{\partial S} = 0\).

**Warning 26.** The definition of \(\text{Isop}(S)\) can be weakened. For instance, (i) of Definition \cite{24} can be given by assuming \(S\) of class \(C^1\) and then by taking the infimum over all \((n-2)\)-dimensional submanifolds \(N\) of \(S\) of class \(C^1\) which divide \(S\) into two hypersurfaces \(S_1, S_2\) with common boundary \(N = \partial S_1 = \partial S_2\). In this case, (i) of Theorem \cite{25} holds, without modifications, by taking the infimum over \(C^1\)-smooth functions. If \(\partial S \neq \emptyset\) an analogous claim holds, for the other isoperimetric constant. Furthermore, equivalent remarks can be given for all the results of this section. Nevertheless, as already said, this weaker formulation seems to be less meaningful because of the presence of characteristic points.

**Warning 27.** Throughout this section, we shall fix a homogeneous distance \(\varrho\) on \(\mathbb{G}\) of class \(C^1\) outside the diagonal of \(\mathbb{G}\).

**Proof of Theorem 25.** The proof repeats almost verbatim the arguments of Theorem 1 in \cite{61}. We just prove the theorem for \(\partial S = \emptyset\) since the other case is analogous. First, let us prove the inequality

\[
\text{Isop}(S) \leq \inf \int_S |\nabla_{\mathbb{H}} \psi| \sigma_n^{-1} \frac{dS}{\sigma_n^{-1}}
\]

where \(\psi \in C^2(S)\) and \(\int_S \psi \sigma_n^{-1} = 0\). To prove this inequality let us consider the auxiliary functions \(\psi^+ = \max\{0, \psi\}, \psi^- = \max\{0, -\psi\}\). By applying the Coarea Formula \eqref{4} and the definition of \(\text{Isop}(S)\) we get that

\[
\int_S |\nabla_{\mathbb{H}} \psi^\pm| \sigma_n^{-1} \int_0^{\sigma_n^{-2}} \sigma_n^{-2} \{x \in S : \psi^\pm = t\} dt \geq \text{Isop}(S) \int_S |\psi^\pm| \sigma_n^{-1}.
\]

Now we shall prove the reversed inequality. So let us assume that \(\sigma_n^{-1}(S_1) \leq \sigma_n^{-1}(S_2)\) and let \(\varepsilon > 0\). By making use of the fixed homogeneous distance \(\varrho\) on \(\mathbb{G}\), we now define a function \(\psi_\varepsilon : S \rightarrow \mathbb{R}\) by setting

\[
\psi_\varepsilon(x)|_{S_1} := \begin{cases} \frac{\varrho(x,N)}{\varepsilon} & \text{if} \ \varrho(x,N) \leq \varepsilon \\ 1 & \text{if} \ \varrho(x,N) > \varepsilon \end{cases}, \quad \psi_\varepsilon(x)|_{S_2} := \begin{cases} -\frac{\varrho(x,N)}{\varepsilon} & \text{if} \ \varrho(x,N) \leq \varepsilon \\ -1 & \text{if} \ \varrho(x,N) > \varepsilon \end{cases},
\]

where the constant \(\alpha\) depends on \(\varepsilon\) and is chosen in a way that \(\int_S \psi_\varepsilon \sigma_n^{-1} = 0\). Obviously

\[
\lim_{\varepsilon \to 0} \alpha = \frac{\sigma_n^{-1}(S_1)}{\sigma_n^{-1}(S_2)}.
\]
Since
\[ \int_S |\nabla_{\mathcal{H}S} \psi| \sigma_n^{n-1} = \frac{1 + \alpha}{\epsilon} \int_{S_{\epsilon}} |\nabla_{\mathcal{H}S} \varphi(x, N)| \sigma_n^{n-1} \]
\[ = \frac{1 + \alpha}{\epsilon} \int_0^\epsilon \sigma_n^{n-2} \{ x \in N_\epsilon : \varphi(x, N) = t \} \, dt, \]
one gets
\[ \lim_{\epsilon \to 0} \int_S |\nabla_{\mathcal{H}S} \psi| \sigma_n^{n-1} = (1 + \alpha) \sigma_n^{n-2}(N). \]
Moreover \( \lim_{\epsilon \to 0} \int_S |\psi| \sigma_n^{n-1} = \sigma_n^{n-1}(S_1) + \alpha \sigma_n^{n-1}(S_2) \). Putting all together we get
\[ \inf_{\psi} \frac{\int_S |\nabla_{\mathcal{H}S} \psi| \sigma_n^{n-1}}{\int_S |\psi| \sigma_n^{n-1}} \leq \lim_{\epsilon \to 0} \frac{\int_S |\nabla_{\mathcal{H}S} \psi| \sigma_n^{n-1}}{\int_S |\psi| \sigma_n^{n-1}} \leq \frac{\sigma_n^{n-1}(N)}{\sigma_n^{n-2}(S_1)}. \]
If we take the infimum over \( N \) and \( S_1 \), the inequality follows. \( \square \)

**Corollary 28.** Let \( \lambda_1 \) be the first non-zero eigenvalue of either the closed eigenvalue problem or the Dirichlet eigenvalue problem; see Definition 27. Then \( \lambda_1 \geq \frac{(\text{Isop}(S))^2}{4} \).

**Proof.** We just prove the first claim, as the second claim is similar. Let \( \psi \) be an eigenfunction of \( \mathcal{L}_{\mathcal{H}S} \) corresponding to \( \lambda_1 \). Then
\[ \lambda_1 = -\frac{\int_S \psi \mathcal{L}_{\mathcal{H}S} \psi \sigma_n^{n-1}}{\int_S |\psi| \sigma_n^{n-1}} = \frac{\int_S |\nabla_{\mathcal{H}S} \psi|^2 \sigma_n^{n-1}}{\int_S |\psi|^2 \sigma_n^{n-1}} \]
\[ = \frac{\int_S |\nabla_{\mathcal{H}S} \psi|^2 \sigma_n^{n-1}}{\left( \int_S |\psi|^2 \sigma_n^{n-1} \right)^2} \int_S |\psi|^2 \sigma_n^{n-1} \]
\[ \geq \frac{\left( \int_S |\nabla_{\mathcal{H}S} \psi|^2 \varphi \sigma_n^{n-1} \right)^2}{\left( \int_S |\psi|^2 \sigma_n^{n-1} \right)^2} \]
\[ = \frac{1}{4} \left( \frac{\int_S |\nabla_{\mathcal{H}S} \psi|^2 \sigma_n^{n-1}}{\int_S |\psi|^2 \sigma_n^{n-1}} \right)^2 \geq \frac{(\text{Isop}(S))^2}{4}, \]
where we have used Theorem 25 together with Cauchy-Schwarz inequality. \( \square \)

We now extend, to Carnot groups, another isoperimetric constant and some related facts which, in the Riemannian case, were studied in [61].

**Definition 29.** The isoperimetric constant \( \text{Isop}_0(S) \) of any \( C^2 \)-smooth compact hypersurface \( S \subset \mathbb{G} \) with boundary \( \partial S \) is given by
\[ \text{Isop}_0(S) := \inf \left\{ \frac{\sigma_n^{n-2}(\partial S_1 \cap \partial S_2)}{\min(\sigma_n^{n-1}(S_1), \sigma_n^{n-1}(S_2))} \right\}, \]
where the infimum is taken over all decompositions \( S = S_1 \cup S_2 \) such that \( \sigma_n^{n-1}(S_1 \cap S_2) = 0 \).

**Theorem 30.** Let \( S \subset \mathbb{G} \) be a compact hypersurface of class \( C^2 \) with boundary. Then
\[ \text{Isop}_0(S) = \inf \left\{ \frac{\int_S |\nabla_{\mathcal{H}S} \psi| \sigma_n^{n-1}}{\inf_{\beta \in \mathbb{R}} \int_S |\psi - \beta| \sigma_n^{n-1}} \right\}, \]
where the inf is taken over all \( C^2 \)-functions defined on \( S \).
Proof. The proof is analogous to that of Theorem 6 in [61]. First, let us prove the inequality

$$\text{Isop}(S) \leq \inf \frac{\int_S |\text{grad}_{HS} \psi| \sigma^n_{H}}{\int_S |\psi| \sigma^n_{H}}.$$ 

To this purpose, let us define the functions $\psi^+ := \max(0, \psi - k), \psi^- := -\min(0, \psi - k)$, where $k \in \mathbb{R}$ is any constant such that:

$$\sigma^n_{H}(x) = \begin{cases} \sigma^n_{H}(S), & x \in S : \psi^+ > 0 \\ \frac{1}{2}\sigma^n_{H}(S), & x \in S : \psi^- > 0 \end{cases}$$

By using again the Coarea Formula (4) together with the definition of $\text{Isop}_0(S)$ we get that

$$\int_S |\text{grad}_{HS} \psi^\pm | \sigma^n_{H} = \int_0^{+\infty} \sigma^n_{H}(x) \in S : \psi^\pm = t \, dt \geq \text{Isop}(S) \int_S |\psi^\pm| \sigma^n_{H}.$$ 

We prove the other inequality. Assuming $\sigma^n_{H}(S_1) \leq \sigma^n_{H}(S_2)$ and $\epsilon > 0$, we define the function

$$(9) \quad \psi_{\epsilon}(x)|_{S_1} := 1, \quad \psi_{\epsilon}(x)|_{S_2} := \begin{cases} 1 - \frac{\epsilon(x, \partial S_1 \cap \partial S_2)}{\epsilon} & \text{if } \varrho(x, \partial S_1 \cap \partial S_2) \leq \epsilon \\ 0 & \text{if } \varrho(x, \partial S_1 \cap \partial S_2) > \epsilon. \end{cases}$$

Furthermore, one can find a constant $k(\epsilon)$ satisfying

$$\int_S |\psi_{\epsilon} - k(\epsilon)| \sigma^n_{H} = \inf_{\beta \in \mathbb{R}} \int_S |\psi_{\epsilon} - \beta| \sigma^n_{H}$$

and such that $k(\epsilon) \to 0$ for $\epsilon \to 0^+$. Hence

$$\lim_{\epsilon \to 0^-} \left\{ \frac{\int_S |\text{grad}_{HS} \psi_{\epsilon}| \sigma^n_{H}}{\inf_{\beta \in \mathbb{R}} \int_S |\psi_{\epsilon} - \beta| \sigma^n_{H}} \right\} \leq \frac{\sigma^n_{H}(\partial S_1 \cap \partial S_2)}{\min\{\sigma^n_{H}(S_1), \sigma^n_{H}(S_2)\}}.$$ 

□

Corollary 31. Let $S \subset \mathbb{G}$ be a compact hypersurface of class $C^2$. Then

$$(10) \quad \int_S |\psi - k|^2 \sigma^n_{H} \leq \frac{4}{(\text{Isop}_0(S))^2} \int_S |\text{grad}_{HS} \psi|^2 \sigma^n_{H}$$

for every $\psi \in C^2(S)$ and every $k \in \mathbb{R}$ such that

$$\sigma^n_{H}(x) = \begin{cases} \sigma^n_{H}(S), & x \in S : \psi \geq k \\ \frac{1}{2}\sigma^n_{H}(S), & x \in S : \psi \leq k \end{cases}$$

Furthermore, if $\psi \in C^2(S)$ and $\int_S \psi \sigma^n_{H} = 0$, then

$$(11) \quad \int_S |\psi|^2 \sigma^n_{H} \leq \frac{4}{(\text{Isop}_0(S))^2} \int_S |\text{grad}_{HS} \psi|^2 \sigma^n_{H}.$$

Proof. One has $\int (\psi^+ \cdot \psi^-) \sigma_H^{n-1} = 0$, where the functions $\psi^\pm$ are defined as in the proof of Theorem\hspace{1cm}30. Moreover, by using once more Coarea Formula, we get
\[
\int_S |\psi - k|^2 \sigma_H^{n-1} = \int_S |\psi^+ + \psi^-|^2 \sigma_H^{n-1}
\leq \int_S |\psi^+|^2 \sigma_H^{n-1} + \int_S |\psi^-|^2 \sigma_H^{n-1}
\leq \frac{1}{|\text{Iso}(S)|} \left( \int_S |\text{grad}_{\text{H}} (\psi)^2 | \sigma_H^{n-1} + \int_S |\text{grad}_{\text{H}} (\psi^-)^2 | \sigma_H^{n-1} \right)
\leq \frac{2}{|\text{Iso}(S)|} \int_S (\psi^+ + \psi^-)|\text{grad}_{\text{H}} \psi| \sigma_H^{n-1}
\leq \frac{2}{|\text{Iso}(S)|} \|\psi^+ + \psi^-\|_{L^2(S, \sigma_H^{n-1})} \|\text{grad}_{\text{H}} \psi\|_{L^2(S, \sigma_H^{n-1})}.
\]
This proves (10). In order to prove (11) we note that the hypothesis $\int_S \psi \sigma_H^{n-1} = 0$ actually implies that
\[
\int_S \psi^2 \sigma_H^{n-1} = \inf_{k \in \mathbb{R}} \int_S (\psi - k)^2 \sigma_H^{n-1},
\]
which together with (10), implies the thesis of the theorem. \qed

3. Two upper bounds on $\lambda_1$

Below we shall extend two (nowadays classical) inequalities obtained, respectively, by Chavel and Reilly in the Euclidean/Riemannian setting. An important feature of these results is in that they give explicit upper bounds for the first non-trivial eigenvalue (of the Laplacian) of a compact submanifold of $\mathbb{R}^n$. For further details we refer to [12 and 53]; see also [34]. To begin with, let $\Omega \subset \mathbb{R}^n$ be a bounded domain and assume that $S := \partial \Omega$ is a connected hypersurface of class $C^2$, with orientation given by the outward normal vector $\nu$. Moreover, let $x_0$ be the horizontal position vector field and let us apply the usual divergence formula. We also set $\sigma_H = \text{Vol}^H$. We have
\[
h \text{Vol}^H(\Omega) = \int \text{div}_H x_0 \sigma_H^n = \int_{\partial \Omega} \langle x_0, \nu \rangle \sigma_H^{n-1} = \int_S \langle x_0, \nu \rangle \sigma_H^{n-1},
\]
where we have used identity (i) of Lemma\hspace{1cm}19. Furthermore, we may further assume that the “center of mass”of $\partial \Omega$ (with respect to the $H$-perimeter) is placed at the identity $0 \in \mathbb{G}$. In other words, let us assume that $\int_S x_i \sigma_H^{n-1} = 0$ for every $i \in I_H = \{1, \ldots, h\}$, where $x_0 \equiv (x_1, \ldots, x_i, \ldots, x_h)$ is the horizontal position vector; see Lemma\hspace{1cm}19.

The last assumption is justified by the following:

Lemma 32. Let $S \subset \mathbb{G}$ be a compact hypersurface of class $C^i$ ($i \geq 1$). We can always choose a system of exponential coordinates $x = \exp(x_1, \ldots, x_h)$ on $\mathbb{G}$ such that $\int_S x_i \sigma_H^{n-1}(x) = 0$ for every $i \in I_H = \{1, \ldots, h\}$.

Proof. Let
\[
a_i := \int_S x_i \sigma_H^{n-1}(x) \sigma_H^{-1}(S) \quad \forall \ i \in I_H = \{1, \ldots, h\}
\]
and $a_0 \equiv (a_1, \ldots, a_i, \ldots, a_h)$. Set $a := \exp(a_H, 0_H)$, where the symbol $0_H$ denotes the origin of $V \subset g$. Consider the change of variables $y := \Phi(x) = a^{-1} \cdot x$ ($x \in \mathbb{G}$). Equivalently, we have $\Phi(x) = L_{a^{-1}}(x)$, where $L_{a^{-1}}$ is the left-translation by $a^{-1} = -a$; see Section\hspace{1cm}1.1. The usual Change of Variables formula together with standard properties of the pull-back imply the following chain of equalities:
\[
(12) \quad \int_{\Phi(S)} f(y) \sigma_H^{n-1}(y) = \int_S f(\Phi(x)) \text{Jac}(\Phi)(x) \sigma_H^{n-1}(x) = \int_S \Phi^* (f \sigma_H^{n-1}) = \int_S (f \circ \Phi)(\Phi^* \sigma_H^{n-1})
\]
for every smooth function $f : S \to \mathbb{R}$; see, for instance, Lee’s book [38] Lemma 9.11, p. 214. Using the left-invariance of the $H$-perimeter yields $J_{ac}(\Phi) = 1$, or equivalently, $\Phi^* \sigma^{n-1}_H = \sigma^{n-1}_H$. Now, let us assume that $f(y) := y_i$ for any $i \in I_H$. Equivalently, let $f$ be the $i$-th exponential coordinate of the variable $y \in \mathbb{G}$. Note also that $(f \circ \Phi)(x) = \Phi_i(x) = -a_i + x_i$ for any $i \in I_H$. Actually, this follows from the fact that the group law $\bullet$ acts linearly on the horizontal layer; see (1). Then, using (12) yields

$$\int_{\Phi(S)} y_i \sigma^{n-1}_H(y) = \int_S (-a_i + x_i) \sigma^{n-1}_H(x) = 0 \quad \forall \ i \in I_H,$$

which achieves the proof. \qed

We therefore get that

$$h^*\text{Vol}^n(\Omega) = \int_S (x_0, y_0) \sigma^{n-1}_H$$

$$\leq \int_S |x_0| \sigma^{n-1}_H$$

$$\leq \sqrt{\sigma^{n-1}_H(S)} \sqrt{\int_S |x_0|^2 \sigma^{n-1}_H}$$

$$= \sqrt{\sigma^{n-1}_H(S)} \sqrt{\int_S \sum_{i=1}^n x_i^2 \sigma^{n-1}_H}$$

$$\leq \sqrt{\sigma^{n-1}_H(S)} \frac{\lambda_1}{\lambda_1} \sqrt{\int_S \sum_{i=1}^n |\text{grad}_H x_i|^2 \sigma^{n-1}_H},$$

where the last identity follows from Lord Rayleigh’s characterization of the 1st non-trivial eigenvalue $\lambda_1$ of the operator $L_\text{hs}$ on $S$. Now a direct computation gives the pointwise identity $\sum_{i \in I_H} |\text{grad}_H x_i|^2 = h - 1$. Hence, putting all together, we have shown the following:

**Theorem 33.** Let $\Omega \subseteq \mathbb{G}$ be a bounded domain with $C^2$ boundary $S = \partial D$. Moreover, let $\lambda_1$ be the 1st (non-trivial) eigenvalue of the operator $L_\text{hs}$ on $S$. Then

$$\sqrt{\lambda_1} \frac{\text{Vol}^n(\Omega)}{\sigma^{n-1}_H(S)} \leq \frac{\sqrt{h - 1}}{h}.$$

We now discuss another geometric inequality, which looks very similar to the last one. More precisely, let $S$ be a $C^2$-smooth compact hypersurface without boundary. So let us make use of Rayleigh’s principle

$$\int_S \varphi^2 \sigma^{n-1}_H \leq \int_S |\text{grad}_H \varphi|^2 \sigma^{n-1}_H$$

for any function $\varphi \in C^2(S \setminus C_S) \cap W^{1,2}_{\text{hs}}(S, \sigma^{n-1}_H)$ satisfying $\int_S \varphi \sigma^{n-1}_H = 0$. Again, we assume that the center of mass of $S = \partial \Omega$ is placed at $0 \in \mathbb{G}$ so that $\int_S x_i \sigma^{n-1}_H = 0$ for every $i \in I_H$. Hence, similarly as above, we get that

$$\lambda_1 \int_S |x_0|^2 \sigma^{n-1}_H \leq \lambda_1 \sum_{i \in I_H} \int_S x_i^2 \sigma^{n-1}_H \leq \lambda_1 \sum_{i \in I_H} \int_S |\text{grad}_H x_i|^2 \sigma^{n-1}_H = (h - 1) \sigma^{n-1}_H(S).$$

At this point, we reformulate Corollary 20 as follows:

$$\int_S ((h - 1) + \langle H_\text{hs} y_H + C_\text{hs} y_H, x_H \rangle) \sigma^{n-1}_H = 0.$$
From this identity and Cauchy-Schwartz inequality, we easily get that
\[
(h - 1) \sigma_H^{n-1}(S) \leq \sqrt{\int_S |x_H|^2 \sigma_H^{n-1}} \sqrt{\int_S \left| \mathcal{H}_H v_H + C_H v_H \right|^2 \sigma_H^{n-1}}
\]
and hence
\[
\int_S \left( \mathcal{H}_H^2 + |C_H v_H|^2 \right) \sigma_H^{n-1} \leq \int_S |x_H|^2 \sigma_H^{n-1}.
\]
Therefore
\[
\frac{(h - 1) \sigma_H^{n-1}(S)}{\int_S \left( \mathcal{H}_H^2 + |C_H v_H|^2 \right) \sigma_H^{n-1}} \leq (h - 1) \sigma_H^{n-1}(S),
\]
and hence
\[
\lambda_1 \frac{(h - 1) \sigma_H^{n-1}(S)}{\int_S \left( \mathcal{H}_H^2 + |C_H v_H|^2 \right) \sigma_H^{n-1}} \leq (h - 1) \sigma_H^{n-1}(S),
\]
which proves the following:

**Theorem 34.** Let \( \Omega \subseteq G \) be a bounded domain with \( C^2 \) boundary \( S = \partial D \) and \( \nu \) the outward-pointing unit normal vector along \( S \). Moreover, let \( \lambda_1 \) be the 1st eigenvalue of the operator \( \mathcal{L}_{\text{HS}} \) on \( S \). Then, the following upper bound for \( \lambda_1 \) holds
\[
\lambda_1 \leq \frac{\int_S \left( \mathcal{H}_H^2 + |C_H v_H|^2 \right) \sigma_H^{n-1}}{(h - 1) \sigma_H^{n-1}(S)} = \frac{\int_S \left( \mathcal{H}_H^2 + |C_H v_H|^2 \right) \sigma_H^{n-1}}{h - 1}.
\]

4. **Horizontal Linear Isoperimetric Inequalities**

Let \( S \subseteq G \) be a compact hypersurface of class \( C^2 \) with (or without) boundary. Let \( x_H \) be the horizontal position vector of \( S \) and set \( x_{HS} := x_H - g_H v_H \) where \( g_H = (x_H, v_H) \) is the horizontal support function of \( S \); see Lemma [19]. We recall that
\[
\int_S ((h - 1) + g_H \mathcal{H}_H + \langle C_H v_H, x_{HS} \rangle) \sigma_H^{n-1} = \int_{\partial S} \langle x_H, \eta_{HS} \rangle \sigma_H^{n-2},
\]
see Corollary [20]. Note that, if \( \partial S = \emptyset \), then the boundary integral vanishes. From this we easily get that
\[
(h - 1) \sigma_H^{n-1}(S) \leq \int_S \left( |g_H| \mathcal{H}_H | + |(C_H v_H, x_{HS}) \right) \sigma_H^{n-1} + \int_{\partial S} \langle x_H, \eta_{HS} \rangle \sigma_H^{n-2}.
\]

**Remark 35** (Assumptions on \( \varrho \)). Let \( \varrho(x) = \varrho(0, x) = ||x||_0 \) be a homogeneous norm on \( G \) and let \( \varrho(x, y) = ||x||_0 \) be the associated (homogeneous) distance on \( G \). In this section we assume the following:

(i) \( \varrho \) is piecewise \( C^1 \) outside the diagonal of \( G \);
(ii) \( |\text{grad}_o \varrho| \leq 1 \) at each regular point of \( \varrho \);
(iii) \( |x_o| \leq \varrho(x, 0) \quad \forall \; x \in G \).

**Example 36.** On the Heisenberg group \( \mathbb{H}^n \), the CC-distance \( d_{CC} \) satisfies these assumptions. Another example is the distance associated with the Korany norm defined as \( ||x||_0 := \sqrt{|x_H|^4 + 16t^2} \) for \( x = \exp(x_H, t) \in \mathbb{H}^n \). This norm is homogeneous and \( C^\infty \)-smooth out of \( 0 \in \mathbb{H}^n \) and satisfies conditions (ii) and (iii). This example can easily be generalized to any Carnot group having step 2 and satisfying \( C_H^2 \mathcal{H}_H = -1_H \delta_\alpha \delta_\beta \), \( (\alpha, \beta) \in I_2 \). Actually, in this case, one can show that the homogeneous norm \( || \cdot ||_{0} \), defined by \( ||x||_0 := \sqrt{|x_H|^4 + 16|x_o|^2} \) \( \forall \; x = \exp(x_H, x_o) \), satisfies all the conditions in Remark [35]
Let \( R \) be the radius of the \( \mathcal{O} \)-ball \( B_{\mathcal{O}}(0,R) \), centered at the identity \( 0 \) of the group \( \mathcal{G} \) and circumscribed about \( S \). It is important to remark that, because of the left-invariance of the \( H \)-perimeter, we may replace \( 0 \) with any \( x \in \mathcal{G} \). Below, we shall estimate (by Cauchy-Schwarz inequality) the right-hand side of (14). To this aim, note that \( g_n \leq |x| \leq \|x\|_0 \). So we have

\[
(h - 1) \sigma_n^{n-1}(S) \leq R \left( \int_S (|\mathcal{H}_n| + |C_n \nu_n|) \sigma_n^{n-1} + \sigma_n^{n-2}(\partial S) \right),
\]

which is a linear inequality. Obviously, if \( S \) is \( H \)-minimal, i.e. \( \mathcal{H}_n = 0 \), it follows that

\[
(h - 1) \sigma_n^{n-1}(S) \leq R \left( \int_S |C_n \nu_n| \sigma_n^{n-1} + \sigma_n^{n-2}(\partial S) \right).
\]

Furthermore, if \( \mathcal{H}_n^0 := \max\{\mathcal{H}_n(x)|x \in S\} \), one gets

\[
\sigma_n^{n-1}(S) \left( h - 1 - R \mathcal{H}_n^0 \right) \leq R \left( \int_S |C_n \nu_n| \sigma_n^{n-1} + \sigma_n^{n-2}(\partial S) \right).
\]

Equivalently, we have

\[
R \geq \frac{(h - 1) \sigma_n^{n-1}(S)}{\mathcal{H}_n^0 \sigma_n^{n-1}(S) + \left( \int_S |C_n \nu_n| \sigma_n^{n-1} + \sigma_n^{n-2}(\partial S) \right)},
\]

and, by assuming \( R \mathcal{H}_n^0 < h - 1 \), we also get that

\[
\sigma_n^{n-1}(S) \leq \frac{R \left( \int_S |C_n \nu_n| \sigma_n^{n-1} + \sigma_n^{n-2}(\partial S) \right)}{(h - 1) - R \mathcal{H}_n^0}.
\]

Here, we just remark that there are no closed compact \( H \)-minimal hypersurfaces immersed in Carnot groups. This fact can be proved by using the 1st variation formula of the \( H \)-perimeter; see [48]. The previous formulae have been proved for hypersurfaces with boundary, but they hold even if \( \partial S = \emptyset \). More precisely we have:

**Proposition 37.** Let \( S \subset \mathcal{G} \) be a compact hypersurface of class \( C^2 \) without boundary. Let \( R \) be the radius of the \( \mathcal{O} \)-ball \( B_{\mathcal{O}}(0,R) \), centered at the identity \( 0 \) of the group \( \mathcal{G} \) and circumscribed about \( S \). Then:

\[
\begin{align*}
(h - 1) \sigma_n^{n-1}(S) & \leq R \int_{\mathcal{U}} (|\mathcal{H}_n| + |C_n \nu_n|) \sigma_n^{n-1}; \\
R & \geq \frac{(h - 1) \sigma_n^{n-1}(S)}{\mathcal{H}_n^0 \sigma_n^{n-1}(S) + \int_S |C_n \nu_n| \sigma_n^{n-1}}; \\
\sigma_n^{n-1}(S) & \leq \frac{R \left( \int_S |C_n \nu_n| \sigma_n^{n-1} \right)}{(h - 1) - R \mathcal{H}_n^0}.
\end{align*}
\]

4.1. **Application: a weak monotonicity formula.** In the sequel, we shall set \( S_t = S \cap B_{\mathcal{O}}(x,t) \). The “natural” monotonicity formula which can be deduced from the inequality (15) is contained in the next:

**Proposition 38.** The following inequality holds

\[
- \frac{d}{dt} \frac{\sigma_n^{n-1}(S_t)}{t^{n-1}} \leq \frac{1}{t^{n-1}} \left( \int_{S_t} (|\mathcal{H}_n| + |C_n \nu_n|) \sigma_n^{n-1} + \sigma_n^{n-2}(\partial S \cap B_{\mathcal{O}}(x,t)) \right)
\]

for \( L^1 \)-a.e. \( t > 0 \).
From the last inequality we infer that least piecewise- satisfying

\[ (24) \quad (h-1) \sigma_{h}^{n-1}(S_{t}) \leq t \left( \int_{S_{t}} (|\mathcal{H}_{u}| + |C_{u} v_{u}|) \sigma_{h}^{n-1} + \sigma_{h}^{n-2}(\partial S_{t}) \right), \]

where \( t \) is the radius of a \( q \)-ball centered at \( x \) and intersecting \( S \). Since we begin by describing our result in the simpler setting of the first Heisenberg group \( \mathbb{H}^{1} \); see also [45]. So we may apply the Coarea Formula to this function. Since \( |\text{grad}_{\mathbb{H}^{1}} \psi| \leq |\text{grad}_{\mathbb{H}^{1}} \psi| \), we easily get that

\[ (h-1) \sigma_{h}^{n-1}(S_{t}) \leq t \left( \int_{S_{t}} (|\mathcal{H}_{u}| + |C_{u} v_{u}|) \sigma_{h}^{n-1} + \sigma_{h}^{n-2}(\partial S_{t}) \right). \]

Now let us consider the function \( \psi(y) := |y - x|_{0} \forall y \in S \). By hypothesis, \( \psi \) is a \( C^{1} \)-smooth function -at least piecewise- satisfying \( |\text{grad}_{\mathbb{H}^{1}} \psi| \leq 1 \); see Remark [35]. We have

\[ \sigma_{h}^{n-1}(S_{t}) = \int_{S_{t}} |\text{grad}_{\mathbb{H}^{1}} \psi| \sigma_{h}^{n-1} \]

\[ = \int_{t}^{t_{1}} \sigma_{h}^{n-2}(\partial S_{t}) \psi^{-1}(s) \, ds \]

\[ = \int_{t}^{t_{1}} \sigma_{h}^{n-2}(\partial B_{q}(x, s) \cap S) \, ds. \]

From the last inequality we infer that

\[ \frac{d}{dt} \sigma_{h}^{n-1}(S_{t}) \geq \sigma_{h}^{n-2}(\partial B_{q}(x, t) \cap S) \]

for \( L^{1} \)-a.e. \( t > 0 \). Hence, from this inequality and (24), we obtain

\[ (h-1) \sigma_{h}^{n-1}(S_{t}) \leq t \left( \mathcal{A}(t) + \mathcal{B}(t) + \frac{d}{dt} \sigma_{h}^{n-1}(S_{t}) \right), \]

which is an equivalent form of (23).

We have to notice however that, in order to prove an “intrinsic” isoperimetric inequality, the number \( (h-1) \) in the previous differential inequality is not the correct one, which is \( (Q-1) \). This fact motivates a further study, made by the author in [46, 47].

5. A theorem about non-horizontal graphs in 2-step Carnot groups

We begin by describing our result in the simpler setting of the first Heisenberg group \( \mathbb{H}^{1} \); see also [45]. For the notation, see Example [7].

**Theorem 39** (Heinz’s estimate for \( T \)-graphs). Let \( S = \{ p = \exp(x, y, t) \in \mathbb{H}^{1} : t = f(x, y) \forall (x, y) \in \mathbb{R}^{2} \} \) be a \( T \)-graph of class \( C^{2} \) over the xy-plane. If \( |\mathcal{H}_{u}| \geq C > 0 \), then

\[ C \mathcal{H}_{Eu}^{2}(\mathcal{P}_{xy}(U)) \leq \mathcal{H}_{Eu}^{1}(\mathcal{P}_{xy}(\partial U)). \]
for every $C^1$-smooth relatively compact open set $U \subset S$. Hence, taking $U := S \cap C_r(T)$, where $C_r(T)$ denotes a vertical cylinder of radius $r$ around the $T$-axis $T := \{ p = \exp(0, 0, t) \in \mathbb{H}^1, t \in \mathbb{R} \}$, yields

$$r \leq \frac{2}{C}$$

for every $r > 0$.

It follows that any entire $xy$-graph of class $C^2$ having constant (or just bounded) horizontal mean curvature $H_a$ must be necessarily a $H$-minimal surface. To see this fact, it is enough to send $r \to +\infty$.

The proof of the previous theorem is elementary. More precisely, one uses the following identity:

$$- \int_{U} H_a \sigma_r^2 = \int_{\partial U} \nu \cdot d\theta,$$

where $\theta = T^* = dt + \frac{y dy - x dx}{2}$ denotes the dual 1-form to the vertical direction $T$. We also have to remark that $\sigma_r^2 = -d\theta = dx \wedge dy$. The previous theorem is a generalization to our context of a classical result obtained by Heinz in [35]. This was generalized by Chern in [19] and then by other authors in a number of different directions.

Below, we shall restrict ourselves to consider only 2-step Carnot groups.

**Definition 40** (Non-horizontal graphs in 2-step Carnot groups). Let $\mathbb{G}$ be a 2-step Carnot group and let $Z = \sum_{\alpha \in I_v} z_{\alpha} X_{\alpha} \in V$ be a constant vertical vector. In this case, for the sake of simplicity, we reorder the variables in $g$ as $x \equiv (x_Z, x_{\perp})$, where $x_Z := (x, Z) \in \mathbb{R}$ and $x_{\perp} := x - x_{\perp} Z \in \mathbb{R}^\perp$. Then, we say that $S \subset \mathbb{G}$ is a $Z$-graph (over the hyperplane $\mathbb{R}^\perp$) if there exists a function $\psi : \mathbb{R}^\perp \to \mathbb{R}$ such that $S = \{ p = \exp(x_Z, \psi(x_{\perp})) \in \mathbb{G}, x_{\perp} \in \mathbb{R}^\perp \}$.

So let us fix a constant vertical vector $Z \in V$ and let $S = \{ p = \exp(x_Z, \psi(x_{\perp})) \in \mathbb{G}, x_{\perp} \in \mathbb{R}^\perp \}$ be a $Z$-graph of class $C^2$ over the $\mathbb{R}^\perp$-hyperplane. For the sake of simplicity and without loss of generality, we may take $Z = X_{\alpha}$ for a fixed index $\alpha \in I_v = \{ h + 1, ..., n \}$.

Now let us define a differential $(n - 2)$-form on $S \subset \mathbb{G}$ by setting

$$\xi^\alpha := (\nu \cdot \bigwedge_{\alpha} X_{\alpha} \bigwedge_{\sigma_r^2})|_{S \setminus C_S} \in \Lambda^2(T^*S).$$

This differential $(n - 2)$-form $\xi^\alpha$ is well-defined out of $C_S$ and we have to compute its exterior derivative. Below we will briefly sketch a proof, which can also be found in [44], see Claim 3.22.

**Lemma 41.** We have $d\xi^\alpha|_{S \setminus C_S} = -H_a \sigma^2 \nu|_{S \setminus C_S}$, at each NC point.

**Proof.** Let us set $\xi_j := (X_j \bigwedge X_j \bigwedge \sigma_r^2)|_S$ for any $\alpha \in I_v$ and $j \in I_h$ and compute $d\xi_j := d(X_j \bigwedge X_j \bigwedge \sigma_r^2)|_S$. Let $\mathbb{G}$ be a 2-step Carnot group. We claim that

$$d\xi_j|_{S \setminus C_S} = \sum_{k=a+1}^n C_{\alpha_j k}^k (X_k \bigwedge \sigma_r^2)|_{C_S} = \sum_{k=a+1}^n C_{\alpha_j k}^k \nu_k \sigma_r^{n-1}|_{C_S}.$$

The proof of this claim is just a long, but elementary, calculation. Since we are assuming that $\mathbb{G}$ has step 2, using the properties of the Carnot structural constants yields $C_{\alpha_j}^k = 0$ whenever $j, k \in I_h$ and $\alpha \in I_v$. Hence $d\xi_j = 0$ for every $j \in I_h$. By linearity $\xi^\alpha = -\sum_{j \in I_h} \nu_j \xi_j$, where $\nu_j = \langle \nu_j, X_j \rangle$ for any $j \in I_h$. It follows easily that $d\xi^\alpha = -H_a \sigma^2 \nu|_{C_S}$, as wished. \qed

**Theorem 42** (Heinz’s estimate for non-horizontal graphs in 2-step Carnot groups). Let $\mathbb{G}$ be a 2-step Carnot group and let $Z \in V$ be a constant vertical vector. Furthermore, let $S$ be a $Z$-graph of class $C^2$ over the $\mathbb{R}^\perp$-hyperplane. If $|H_a| \geq C > 0$, then

$$C \mathcal{H}_E^{n-1}(P_Z^{-1}(U)) \leq \mathcal{H}_E^{n-2}(P_Z^{-1}(\partial U)).$$
for every $C^1$-smooth relatively compact open set $\mathcal{U} \subset S$. Hence, taking $\mathcal{U} := S \cap C_\rho(Z)$, where $C_\rho(Z)$
denotes a Euclidean cylinder of radius $r$ around the $Z$-axis given by $Z := \{ p = \exp(0Z^\perp, t) \in \mathbb{G}, t \in \mathbb{R} \}$,\n
(27) $r \leq \frac{n-1}{C}$

for every $r > 0$.

Proof. Without loss of generality, we may assume $-\mathcal{H}_h \geq C > 0$ and take $Z = X_\alpha$ for some fixed index $\alpha \in I_\nu$. In this case, one has\n
$$\sup_{x \in \mathcal{U}} |\omega_\alpha \sigma_{n-1}^\alpha|_S = \nu_\alpha \sigma_{n-1}^\alpha|_S = (X_\alpha \mathcal{L} \sigma_{n}^\alpha)|_S = d\mathcal{H}_{Eu}^{n-1}\mathcal{L} X_\alpha^\perp,$$

where the last identity follows from our assumption that $S$ is a $X_\alpha$-graph. By using Lemma 41 and Stokes’ formula, we obtain the integral identity\n
$$- \int_{\mathcal{U}} \mathcal{H}_h \omega_\alpha \sigma_{n-1}^\alpha = \int_{\partial \mathcal{U}} \nu_\alpha \mathcal{L} X_\alpha \mathcal{L} \sigma_{n}^\alpha.$$

Furthermore, we have\n
$$- \int_{\mathcal{U}} \mathcal{H}_h \omega_\alpha \sigma_{n-1}^\alpha = - \int_{\mathcal{P}_{X_\alpha^\perp}(\partial \mathcal{U})} \mathcal{H}_h d\mathcal{H}_{Eu}^{n-1}$$

and\n
$$\int_{\mathcal{P}_{X_\alpha^\perp}(\partial \mathcal{U})} (\nu_\alpha, \eta) d\mathcal{H}_{Eu}^{n-2} \mathcal{L} \mathcal{P}_{X_\alpha^\perp}(\partial \mathcal{U}),$$

Putting all together, we get that\n
$$C \mathcal{H}_{Eu}^{n-1}(\mathcal{P}_{X_\alpha^\perp}(\mathcal{U})) \leq \mathcal{H}_{Eu}^{n-2}(\mathcal{P}_{X_\alpha^\perp}(\partial \mathcal{U})),$$

which proves (26) when $Z = X_\alpha$. Clearly, the thesis follows by linearity. Finally, (27) follows from (26) and the elementary calculation $\frac{\mathcal{H}_{Eu}^{n-2}(\mathcal{P}_{B_{Eu}^{n-1}})}{\mathcal{H}_{Eu}^{n-1}(\mathcal{P}_{B_{Eu}^{n-1}})} = n-1$, where $B_{Eu}^{n-1}$ denotes a Euclidean unit ball in $Z^\perp \cong \mathbb{R}^{n-1}$.

It follows that an entire $Z$-graph of class $C^2$ over the $Z^\perp$-hyperplane having constant (or bounded) horizontal mean curvature $\mathcal{H}_h$ must be necessarily a $H$-minimal hypersurface.

6. Local Poincaré-type inequality

By using an elementary technique, somehow analogous to the one used in Section 4, we will state a local Poincaré-type inequality for smooth compactly supported functions on NC domains. First we need the following:

Definition 43. Let $S \subset \mathbb{G}$ be a hypersurface of class $C^2$ and let $\mathcal{U} \subseteq S$ be an open domain. We say that $\mathcal{U}$ is uniformly non-characteristic (abbreviated UNC) if

$$\sup_{x \in \mathcal{U}} |\pi(x)| = \sup_{x \in \mathcal{U}} \frac{|\mathcal{P}_h \nu(x)|}{|\mathcal{P}_h \nu(x)|} < +\infty.$$

We stress that

(28) $|C_\nu \nu_\alpha| = \sum_{\alpha \in I_\nu} \omega_\alpha C_\nu^\alpha \nu_\alpha \leq \sum_{\alpha \in I_\nu} |\omega_\alpha||C_\nu^\alpha||\nu_\alpha| \leq \frac{C}{|\mathcal{P}_h \nu|},$
where $C := \sum_{\alpha \in I_{\nu}} \|C^\nu_{\alpha}\|_{\nu}$ only depends on the structural constants of $g$. Let us set

$$R_{U} := \frac{1}{2 \left[ \|H_{\nu}\|_{L^\infty(U)} + C \|\sigma\|_{L^\infty(U)} \right]}.$$ 

From (28) we have $|C_{H} v_{\mu}| \leq C \max_{\alpha \in I_{\nu}} |\sigma_{\alpha}|$. Moreover $\int_{B} |\sigma_{\alpha}|^{n-1} = \int_{B} |\sigma|^{n-1} \leq \sigma_{H}^{n-1}(B)$ for every Borel set $B \subseteq S$.

**Theorem 44.** Let $S \subset \mathbb{G}$ be a hypersurface of class $C^{2}$. Let $U \subset S$ be a uniformly NC open domain. Then, for all $x \in U$ and for all $R \leq \min(\text{dist}_{g}(x, \partial U), R_{U})$, the following holds

$$\int_{U} |\psi|^{p} \sigma_{\nu}^{n-1} \leq C_{p} \left( \int_{U} |\text{grad}_{H_{\nu}} \psi|^{p} \sigma_{\nu}^{n-1} \right)^{\frac{1}{p}} \quad p \in [1, +\infty[$$

for every $\psi \in C_{H_{\nu}}^{1}(U) \cap C_{0}(U)$. More generally, let $\overline{U} \subset U$ be a bounded open subset of $U$ with smooth boundary and such that $\text{diam}_{g}(\overline{U}) \leq 2 \min(\text{dist}_{g}(x, \partial U), R_{U})$. Then

$$\int_{\overline{U}} |\psi|^{p} \sigma_{\nu}^{n-1} \leq C_{p} \text{diam}_{g}(\overline{U}) \left( \int_{\overline{U}} |\text{grad}_{H_{\nu}} \psi|^{p} \sigma_{\nu}^{n-1} \right)^{\frac{1}{p}} \quad p \in [1, +\infty[$$

for every $\psi \in C_{H_{\nu}}^{1}(\overline{U}) \cap C_{0}(\overline{U})$.

In the above theorem one can take $C_{p} := \frac{2p}{2h-3}$.

**Proof.** Let us set $\psi_{x} := \sqrt{\epsilon^{2} + \psi^{2}}$ ($\epsilon \geq 0$). By applying Theorem [14] with $X = \psi_{x} x_{H}$ we get

$$\int_{U} \{ \psi_{x} ((h-1) + g_{H} H_{\nu} + \langle C_{H} v_{\mu}, x_{H} \rangle) + \langle \text{grad}_{H_{\nu}} \psi_{x}, x_{H} \rangle \} \sigma_{\nu}^{n-1} = \int_{\partial U} \psi_{x} (x_{H}, \eta_{H}) \sigma_{\nu}^{n-2},$$

and so

$$(h-1) \int_{U_{\nu}} \psi_{x} \sigma_{\nu}^{n-1} \leq R \left( \int_{U_{\nu}} \psi_{x} (\|H_{\nu}\| + \|C_{H} v_{\mu}\|) + \|\text{grad}_{H_{\nu}} \psi_{x}\| \sigma_{\nu}^{n-1} + \int_{\partial U_{\nu}} \psi_{x} \sigma_{\nu}^{n-2} \right)$$

$$\leq R \left( \|H_{\nu}\|_{L^\infty(U_{\nu})} + C \|\sigma\|_{L^\infty(U_{\nu})} \right) \int_{U_{\nu}} \psi_{x} \sigma_{\nu}^{n-1}$$

$$+ R \left( \int_{U_{\nu}} \|\text{grad}_{H_{\nu}} \psi_{x}\| \sigma_{\nu}^{n-1} + \int_{\partial U_{\nu}} \psi_{x} \sigma_{\nu}^{n-2} \right).$$

By using Fatou’s Lemma and the estimate $R \leq R_{U}$ we get that

$$(h-1) \int_{U_{\nu}} \psi \sigma_{\nu}^{n-1} \leq (h-1) \liminf_{\epsilon \to 0^{+}} \int_{U_{\nu}} \psi_{x} \sigma_{\nu}^{n-1}$$

$$\leq \frac{1}{2} \lim_{\epsilon \to 0^{+}} \int_{U_{\nu}} \psi_{x} \sigma_{\nu}^{n-1} + R \lim_{\epsilon \to 0^{+}} \left( \int_{U_{\nu}} \|\text{grad}_{H_{\nu}} \psi_{x}\| \sigma_{\nu}^{n-1} + \int_{\partial U_{\nu}} \psi_{x} \sigma_{\nu}^{n-2} \right).$$

Obviously, $\psi_{x} \to \psi$ and $\|\text{grad}_{H_{\nu}} \psi_{x}\| \to \|\text{grad}_{H_{\nu}} \psi\|$ as long as $\epsilon \to 0$; moreover $|\psi_{x}| = 0$ along $\partial U_{\nu}$. Now since, as it is well-known, $\|\text{grad}_{H_{\nu}} \psi\| \leq \|\text{grad}_{H_{\nu}} \psi\|$, we easily get the claim by Lebesgue’s Dominate Convergence Theorem. So we have shown that

$$\int_{U_{\nu}} \psi \sigma_{\nu}^{n-1} \leq \frac{2R}{2h-3} \int_{U_{\nu}} \|\text{grad}_{H_{\nu}} \psi\| \sigma_{\nu}^{n-1}.$$
for every $\psi \in C^1_{H^S}(U_R) \cap C_0(U_R)$. Finally, the general case follows by Hölder’s inequality. More precisely, let us use the last inequality with $|\psi|$ replaced by $|\psi|^p$. This implies

\[
\int_{U_R} |\psi|^p \sigma_H^{n-1} \leq \frac{2R}{(2h-3)} \int_{U_R} p |\psi|^{p-1} |\text{grad}_{H^S} \psi| \sigma_H^{n-1}
\]

\[
\leq \frac{2pR}{(2h-3)} \left( \int_{U_R} |\psi|^{(p-1)d} \sigma_H^{n-1} \right)^{\frac{1}{p}} \left( \int_{U_R} |\text{grad}_{H^S} \psi| \sigma_H^{n-1} \right)^{\frac{1}{q}},
\]

where $\frac{1}{p} + \frac{1}{q} = 1$. This achieves the proof of (29). Finally, (30) can be proved by repeating the same arguments as above, just by replacing $R$ with $\text{diam}(\mathcal{U})$.

\[\square\]

With some extra hypotheses one can show that (29) still holds up to the characteristic set.

**Theorem 45.** Let $S \subset \mathbb{G}$ be a hypersurface of class $C^2$ with (or without) boundary $\partial S$. We assume that $S$ has bounded horizontal mean curvature $H^S$ and that $\dim C_S < n-2$. Furthermore, let $\mathcal{U}_e (e > 0)$ be a family of open subsets of $S$ with $C^1$ boundaries, such that:

(i) $C_S \subset \mathcal{U}_e$ for every $e > 0$;

(ii) $\sigma^{-1}_H(\mathcal{U}_e) \rightarrow 0$ for $e \rightarrow 0^+$;

(iii) $\int_{\mathcal{U}_e} |\mathcal{H}^n_{\mathcal{U}}| \sigma^{-2}_H \rightarrow 0$ for $e \rightarrow 0^+$.

Then, for every $x \in S$ and every (small enough) $e > 0$ there exists $R_0 := R_0(x, e) \leq \text{dist}_e(x, \partial S)$ such that

\[\left( \int_{S_R} |\psi|^p \sigma_H^{n-1} \right)^{\frac{1}{p}} \leq C_R \left( \int_{S_R} |\text{grad}_{H^S} \psi|^p \sigma_H^{n-1} \right)^{\frac{1}{p}} \quad p \in [1, +\infty[\]

holds for every $\psi \in C^1_{H^S}(S_R) \cap C_0(S_R)$ and every $R \leq R_0$, where

\[R_0 := \min \left\{ \text{dist}_e(x, \partial S), \frac{1}{2} \left[ C \left( 1 + \|\mathcal{H}^n\|_{L^\infty(S_R \backslash \mathcal{U}_e)} \right) + \|\mathcal{H}^n\|_{L^\infty(S_R)} \right] \right\}.\]

**Proof.** Set $\psi_e := \sqrt{e^2 + \psi^2}$ ($0 \leq e < 1$). We shall prove the theorem for $p = 1$. The general case will follow by using Hölder’s inequality. Let $\mathcal{U}_e (e > 0)$ be as above. Fix $e_0 > 0$. For every $e \leq e_0$ one has

\[
\int_{\mathcal{U}_e} \psi_e |\mathcal{H}^n_{\mathcal{U}}| \sigma_H^{n-1} \leq 2C \|\psi\|_{L^\infty(S_R \backslash \mathcal{U}_e)} \sigma_H^{n-1}(\mathcal{U}_e),
\]

where we have put $C := \sum_{\alpha \in \mathcal{I}_\nu} \|c\|_{L^\infty}$. Furthermore (ii) implies that for every $\delta > 0$ there exists $\epsilon_\delta > 0$ such that $\sigma_H^{n-1}(\mathcal{U}_e) < \delta$ whenever $e < \epsilon_\delta$. Taking $\delta \leq \left( \frac{\int_{S_R} \psi_e \sigma_H^{n-1}}{2 \|\psi\|_{L^\infty(S_R \backslash \mathcal{U}_e)}} \right)$, one gets

\[
\int_{\mathcal{U}_e} \psi_e |\mathcal{H}^n_{\mathcal{U}}| \sigma_H^{n-1} \leq C \int_{S_R} \psi_e \sigma_H^{n-1}
\]

for every $e \leq \min\{\epsilon_\delta, e_0\}$. Moreover, for any $e \in [0, \min\{\epsilon_\delta, e_0\}]$, one has

\[
\int_{S_R \backslash \mathcal{U}_e} \psi_e |\mathcal{H}^n_{\mathcal{U}}| \sigma_H^{n-1} \leq C \|\mathcal{H}^n\|_{L^\infty(S_R \backslash \mathcal{U}_e)} \int_{S_R} \psi_e \sigma_H^{n-1}.
\]

It follows that

\[
\int_{S_R} \psi_e |\mathcal{H}^n_{\mathcal{U}}| \sigma_H^{n-1} \leq C (1 + \|\mathcal{H}^n\|_{L^\infty(S_R \backslash \mathcal{U}_e)}) \int_{S_R} \psi_e \sigma_H^{n-1}.
\]

Since, by hypothesis, the horizontal mean curvature is bounded, we clearly have

\[
\int_{S_R} \psi_e |\mathcal{H}^n_{\mathcal{U}}| \sigma_H^{n-1} \leq \|\mathcal{H}^n\|_{L^\infty(S_R)} \int_{S_R} \psi_e \sigma_H^{n-1}.
\]
Applying Theorem \[4\] with \(X = \psi_e x_N\) (and arguing as in the proof of Theorem \[4\]) yields
\[
(h - 1) \int_{S_R} \psi_e \sigma_H^{n-1} \leq R \left( \int_{S_R} \psi_e (|\mathcal{H}_0| + |C_H \nu_R|) + |\text{grad}_H \psi_e| \sigma_H^{n-1} + \int_{\partial S_R} \psi_e \sigma_H^{n-2} \right) 
\]
\[
\leq R \left[ C|\sigma| L^\infty(S_R) + \mathcal{H}_0 \right] \int_{S_R} \psi_e \sigma_H^{n-1} + R \left( \int_{S_R} |\text{grad}_H \psi_e| \sigma_H^{n-1} + \int_{\partial S_R} \psi_e \sigma_H^{n-2} \right). 
\]
So if \(R \leq R_0\), one gets
\[
\int_{S_R} \psi_e \sigma_H^{n-1} \leq \frac{2R}{2h - 3} \left( \int_{S_R} |\text{grad}_H \psi_e| \sigma_H^{n-1} + \int_{\partial S_R} \psi_e \sigma_H^{n-2} \right). 
\]
We have \(\psi_e \rightarrow |\psi|\) and \(|\text{grad}_H \psi_e| \rightarrow |\text{grad}_H \psi|\) as long as \(e \rightarrow 0\) and \(|\psi| = 0\) along \(\partial S_R\). Since \(|\text{grad}_H \psi| \leq |\text{grad}_H \psi|\), the thesis follows from Fatou’s lemma and Lebesgue’s Dominated Convergence Theorem. \(\square\)

6.1. A Caccioppoli-type inequality. Our final result is a generalization of the classical Caccioppoli inequality (see, for instance, \[2\]) for the operator \(L_{HS}\) on smooth hypersurfaces.

Let \(S \subset \mathbb{G}\) be a hypersurface of class \(C^2\) and set \(S_R := S \cap B_R(x, R)\) for any \(x \in \mathbb{G}\). We are going to consider the functions satisfying, in the distributional sense, the following problem:
\[
(32) \quad -L_{HS} \phi = \psi \quad \text{on} \quad S_R, 
\]
whenever \(\psi \in L^2(S_R, \sigma_H^{n-1})\).

So let us take a function \(\zeta \in C^1_{\text{loc}}(S_R) \cap C_0(S_R)\) such that \(0 \leq \zeta \leq 1\), \(\zeta = 1\) on \(S_{R/2} = S \cap B_R(0, R/2)\) and \(|\text{grad}_H \zeta| \leq C_0/R\). Inserting into the above equation the function \(\varphi = \zeta^2(\phi - \phi_0)\), where \(\phi_0 \in \mathbb{R}\) is a fixed constant, and then integrating over \(S_R\), yields
\[
\int_{S_R} \zeta^2 |\text{grad}_H \phi|^2 \sigma_H^{n-1} + 2 \int_{S_R} \zeta (\phi - \phi_0) (\text{grad}_H \zeta, \text{grad}_H \phi) \sigma_H^{n-1} = \int_{S_R} \psi \zeta^2 (\phi - \phi_0) \sigma_H^{n-1}. 
\]
We have
\[
I_2 \leq \frac{1}{2} \int_{S_R} |\zeta|^2 |\text{grad}_H \phi|^2 \sigma_H^{n-1} + 2 \int_{S_R} |\phi - \phi_0|^2 |\text{grad}_H \phi|^2 \sigma_H^{n-1}. 
\]
Moreover \(I_4 \leq 2C_0^2/R^2|\phi - \phi_0|L^2(S_R)\). Now let us estimate the third integral \(I_3\). We have
\[
\int_{S_R} \psi \zeta^2 (\phi - \phi_0) \sigma_H^{n-1} = \int_{S_R} 2 \left( 2R \psi \zeta^2 (\phi - \phi_0) \right) \sigma_H^{n-1} 
\]
\[
\leq 4R^2 \int_{S_R} \psi^2 \sigma_H^{n-1} + \frac{1}{16R^2} \int_{S_R} \zeta^4 |\phi - \phi_0|^2 \sigma_H^{n-1} 
\]
\[
\leq 4R^2 \int_{S_R} 2\psi^2 \sigma_H^{n-1} + \frac{1}{R^2} \int_{S_R} |\phi - \phi_0|^2 \sigma_H^{n-1}. 
\]
Since \(\zeta = 1\) on \(S_{R/2}\), using the previous estimates yields
\[
\int_{S_{R/2}} |\text{grad}_H \phi|^2 \sigma_H^{n-1} \leq \frac{2C_0^2 + 1}{R^2} \int_{S_R} |\phi - \phi_0|^2 \sigma_H^{n-1} + 4R^2 \int_{S_R} \psi^2 \sigma_H^{n-1}. 
\]
We summarize these calculations, as follows:
Theorem 46. Let $S \subset \mathbb{G}$ be a hypersurface of class $C^2$; let $\phi_0 \in \mathbb{R}$ and let $\phi$ be a distributional solution to the equation $-\mathcal{L}_S \phi = \psi$ on $S_R$, where $\psi \in L^2(S_R, \sigma_n^{-1})$. Then, there exists a positive constant $C > 0$ such that the following “Caccioppoli-type” inequality holds:

$$\int_{S_{R/2}} |\text{grad}_{HS} \phi|^2 \sigma_n^{-1} \leq C \left( \frac{1}{R^2} \int_{S_R} |\phi - \phi_0|^2 \sigma_n^{-1} + R^2 \int_{S_R} \psi^2 \sigma_n^{-1} \right)$$

for every (small enough) $R > 0$, where $S_R := S \cap B_R(x, R)$, for any $x \in S$.

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