SYMMETRIC GROUP REPRESENTATIONS AND \( \mathbb{Z} \)

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Let \( S_n \) be the symmetric group of permutations of \( \{1, 2, \ldots, n\} \). A representation is a homomorphism \( \rho : S_n \to \text{GL}(V) \) where \( V \) is a vector space over \( \mathbb{C} \). Equivalently, \( V \) is an \( S_n \)-module under the action defined by \( \sigma \cdot v = \rho(\sigma)v \), for \( \sigma \in S_n \) and \( v \in V \). Then \( \rho \) is irreducible if there is no proper \( S_n \)-submodule of \( V \). Conjugacy classes and hence irreducible representations of \( S_n \) biject with \( \text{Par}(n) \), the partitions of size \( n \).

Consider three families of numbers from the theory:

(I) The character of \( \rho \) is \( \chi^\rho : S_n \to \mathbb{C} ; \ \sigma \mapsto \text{tr}(\rho(\sigma)) \).

Textbooks focus on the case \( V = V_\lambda \) is irreducible (because of Maschke’s theorem). Since characters are constant on each conjugacy class \( \mu \), one needs only \( \chi^\lambda(\mu) \).

These are computed by the Murnaghan-Nakayama rule (see below). More recent results include bounds on (normalized) character evaluations [Ro96, FePi11].

(II) If \( V_\lambda \) and \( V_\mu \) are irreducible \( S_m \) and \( S_n \)-modules, respectively, then \( V_\lambda \otimes V_\mu \) is an irreducible \( S_m \times S_n \)-module. If \( V_\nu \) is an irreducible \( S_{m+n} \)-representation, it restricts to a \( S_m \times S_n \)-representation \( V_\nu \downarrow_{S_m \times S_n} \). The Littlewood-Richardson coefficient is

\[
c_{\lambda,\mu}^\nu = \text{multiplicity of } V_\lambda \otimes V_\mu \text{ in } V_\nu \downarrow_{S_m \times S_n}.
\]

Many Littlewood-Richardson rules are available to count \( c_{\lambda,\mu}^\nu \) [St99].

(III) If \( V_\lambda, V_\mu \) are \( S_n \)-modules then so is \( V_\lambda \otimes V_\mu \). Hence we may write

\[
V_\lambda \otimes V_\mu \cong \bigoplus_{\nu \in \text{Par}(n)} V_\nu ^{\otimes \mu_{\lambda,\nu}}.
\]

Here, \( g_{\lambda,\mu,\nu} \) is the Kronecker coefficient. One has an \( S_3 \)-symmetric but cancellative formula \( g_{\lambda,\mu,\nu} = \frac{1}{n!} \sum_{\sigma \in S_n} \chi^{\lambda}(\sigma) \chi^{\mu}(\sigma) \chi^{\nu}(\sigma) \); it is an old open problem to give a manifestly nonnegative combinatorial rule. The study of Kronecker coefficients has been given new impetus from \textit{Geometric Complexity Theory}, an approach to the \textit{P} vs \textit{NP} problem; see [BlMuSo15] and the references therein.

This note visits a rudimentary point. While for finite groups, character evaluations are algebraic integers, for \( S_n \), in fact \( \chi^\lambda(\mu) \in \mathbb{Z} \). Moreover, by definition, \( c_{\lambda,\mu}^\nu, g_{\lambda,\mu,\nu} \in \mathbb{Z}_{\geq 0} \). We remark the three converses hold.\(^1\) The proof uses standard facts, but we are unaware of any specific reference in the textbooks [Ja78, FuHa99, St99, Sa01], or elsewhere.

**Theorem.** Every integer is infinitely often an irreducible \( S_n \)-character evaluation. Every non-negative integer is infinitely often a Littlewood-Richardson coefficient, and a Kronecker coefficient.

**Corollary A.** There exists a value-preserving multiset bijection between the Littlewood-Richardson and Kronecker coefficients.

\(^{1}\) Inspired by P. Polo [Po99]: every \( f \in 1 + q\mathbb{Z}_{\geq 0}[q] \) is a Kazhdan-Lusztig polynomial for some \( S_n \).
Proof. Clearly, the Theorem implies that for each $k \in \mathbb{Z}_{\geq 0}$, the sets
\[
\text{LR}_k = \{(\lambda, \mu, \nu) : c_{\lambda, \mu}^{\nu} = k\} \quad \text{and} \quad \text{Kron}_k = \{(\lambda, \mu, \nu) : g_{\lambda, \mu, \nu} = k\}
\]
are countably infinite and thus in bijection. □

Desirable would be a construction of an injection $\text{Kron}_k \hookrightarrow \text{LR}_k$ for each $k \in \mathbb{Z}_{\geq 0}$ (avoiding the countable axiom of choice). That should solve the Kronecker problem in (III), by reduction to (II). This we cannot do. However, there has been success in this vein \cite{KnMiSh04} on another counting problem. See the Remark at the end of this paper.

Proof of the Theorem: The Murnaghan-Nakayama rule states $\chi^\lambda(\mu) = \sum_T (-1)^{\text{ht}(T)}$, where $T$ is a tableaux of shape $\lambda$ with $\mu_i$ many labels $i$, the entries are weakly increasing along rows and columns, and the labels $i$ form a connected skew shape $T_i$ with no $2 \times 2$ subsquare; $\text{ht}(T)$ is the sum of the heights of each $T_i$, i.e., one less than the number of rows of $T_i$.

We sharpen the assertion about $\chi^\lambda(\mu)$. In particular, for a given $n$, we consider the intervals of consecutive integers achievable as character evaluations for $S_n$. From the rule, the character of the defining representation satisfies $\chi^{(n-1,1)}(\mu) = \#(1^\prime s \text{ in } \mu) - 1$ (see also \cite{Ja78} Lemma 6.9). Hence, $\chi^{(n-1,1)}$ takes the values $[0,n-2]$. Similarly, $\chi^{(2,1^{n-2})}$ achieves an interval of negative integers: Take $k \in [1,n-5] \cup \{n-3\}$. If $k \not\equiv n \mod 2$, let $\mu = (n-k-1,1^{k+1})$. Otherwise, if $k \equiv n \mod 2$, let $\mu = (n-k-4,3,1^{k+1})$. Note that if $k = n-6$, let $\mu$ be these parts in decreasing order. In either case, the rule shows $\chi^{(2,1^{n-2})}(\mu) = -k$. Thus, for $n \geq 5$, $[-(n-5),n-2] \subseteq \{\chi^\lambda(\mu) : \lambda, \mu \in \text{Par}(n)\}$. Taking $n \to \infty$ implies the statement regarding character evaluations.

The Kostka coefficient $K_{\lambda, \mu}$ is the number of semistandard Young tableaux of shape $\lambda$ with content $\mu$, i.e., fillings of $\lambda$ with $\mu_i$ many $i$’s such that rows are weakly increasing and columns are strictly increasing.

**Lemma.** Every nonnegative integer is infinitely often a Kostka coefficient.

Proof. Clearly, $K_{(1^j,1^{k-1}),j,1^k} = k$ for $j \geq 1$. The lemma then follows. □

The Littlewood-Richardson coefficient claim holds since it is long known that Kostka coefficients are a special case. To be specific, $K_{\lambda, \mu} = c_{\lambda, \mu}^{\nu}$ where
\[
\tau_i = \mu_i + \mu_{i+1} + \cdots, \quad i = 1, 2, \ldots, \ell(\mu), \quad \text{and} \quad \sigma_i = \mu_i + 1 + \mu_{i+1} + \cdots, \quad i = 1, 2, \ldots, \ell(\mu) - 1.\]

For $\lambda = (\lambda_1, \lambda_2, \ldots)$, let $\lambda[N] := (N - |\lambda|, \lambda_1, \lambda_2, \ldots)$. F. D. Murnaghan \cite{Mu38} proved that for an integer $N > 0$, $\chi^{\lambda[N]} \otimes \chi^{\nu[N]} = \sum_{\nu} g_{\lambda, \mu, \nu} \chi^{\nu[N]}$. The $g_{\lambda, \mu, \nu}$ are called stable Kronecker coefficients and are evidently a special case of Kronecker coefficients. When $|\lambda| + |\mu| = |\nu|$ one has $g_{\lambda, \mu, \nu} = c_{\lambda, \mu}^{\nu}$. Hence one infers the Kronecker coefficient assertion. □

\[\text{There is debate about the idiomatic meaning of counting rule or manifestly nonnegative combinatorial rule}
\[\text{etc. Consider the (adjusted) Fibonacci numbers (1,1,2,3,5,8,13,\ldots). A counting rule is that $F_n$ counts the number of (1,2)-lists whose sum is $n$. The recursive (and computationally efficient) description is}
\[F_n = F_{n-1} + F_{n-2} \quad (n \geq 2), \quad F_0 = F_1 = 1.
\[\text{Construct a binary tree $T_n$ with root labelled $F_n$; each node of label $F_i$ has}
\[\text{a left child $F_i-1$ and right child $F_i-2$. Leaves of $T_n$ are labelled $F_1$ or $F_0$. $F_n$ counts the number of leaves of $T_n$. The latter description restates the recurrence and is not, per se, a counting rule.}
\[\text{This reduction is used by H. Narayanan \cite{Na06} to show computing}
\[c_{\lambda, \mu}^{\nu} \text{ is a \#P problem.}
When, e.g., \( n = 25 \), all of \([-853, 949]\) appear as some \( \chi^\lambda(\mu) \), but the proof merely guarantees \([-20, 23] \). Let \( \ell_n \) be the maximum size of an interval of consecutive character evaluations for \( \mathfrak{S}_n \). Trivially, the results of [Ro96, FePi11] imply upper bounds for \( \ell_n \). Can one prove better upper or lower bounds for \( \ell_n \)?

Let \( A_n \) be the alternating group of even permutations in \( \mathfrak{S}_n \). Sources about the representation theory of \( A_n \) include [JaKe09, Section 2.5] and [FuHa99, Section 5.1]. Character evaluations of \( A_n \) are not always integral, however:

**Corollary B.** Every integer appears infinitely often as an \( A_n \)-irreducible character evaluation.

**Proof.** Let \( \psi^\lambda = \chi^\lambda |_{A_n} \) be the character of the restriction of the \( \mathfrak{S}_n \)-irreducible \( V_\lambda \). If \( \mu \) is not a partition with distinct odd parts then the conjugacy class in \( \mathfrak{S}_n \) of cycle type \( \mu \) is also a conjugacy class of \( A_n \). If \( \lambda \) is not a self-conjugate partition, the restriction is an \( A_n \)-irreducible and also \( \psi^\lambda(\mu) = \chi^\lambda(\mu) \). Repeat the Theorem’s character argument, since for \( n \geq 4 \) neither \( \lambda \) used is self-conjugate, and since for \( k \geq 1 \), \( \mu \) has equal parts. \( \Box \)

**Definition.** For a countable indexing set \( A \), a family of nonnegative integers \((a_n)_{n \in A}\) is entire if every \( k \in \mathbb{Z}_{\geq 0} \) appears infinitely often.

Many of the nonnegative integers arising in algebraic combinatorics are entire. For example, this is true for the theory of Schubert polynomials (we refer to [Ma01] for references). If \( w_0 \in \mathfrak{S}_n \) is the longest permutation then \( S_{w_0}(x_1, \ldots, x_n) = x_1^{n-1} x_2^{n-2} \cdots x_{n-1} \). If \( w \neq w_0 \), \( w(i) < w(i + 1) \) for some \( i \). Then \( S_w(x_1, \ldots, x_n) = \partial_i S_{w_0}(x_1, \ldots, x_n) \) where \( \partial_i = \frac{f^{-s_i}}{x_{i-1} x_{i+1}} \) and \( s_i \) is the simple transposition interchanging \( i, i + 1 \). Nontrivially, each \( S_w \in \mathbb{Z}_{\geq 0}[x_1, x_2, \ldots] \). Moreover, \( S_w = S_{w \times 1} \) where \( w \times 1 \in \mathfrak{S}_{n+1} \) is the usual image of \( w \in \mathfrak{S}_n \). Thus we can discuss \( S_w \) for \( w \in \mathfrak{S}_\infty \); these form a \( \mathbb{Z} \)-linear basis of \( \mathbb{Z}[x_1, x_2, \ldots] \). The Schubert structure constants \( C_{w,w}^w := \langle [w] S_u S_v \rangle \in \mathbb{Z}_{\geq 0} \) for geometric reasons. The Stanley symmetric function is defined by \( F_w = \lim_{m \to \infty} S_{w \times 1}^m \in \mathbb{Z}[x_1, x_2, \ldots] \); here \( 1^m \times w \in \mathfrak{S}_{m+n} \) sets \( 1^m \times w(i) \) equal to \( i \) if \( 1 \leq i \leq m \) and equal to \( w(i - m + 1) + m \) otherwise. \( F_w \) is Schur-nonnegative.

**Corollary C.** These families of nonnegative integers are entire:

(a) The coefficients of monomials in Schubert polynomials.

(b) The Schubert structure constants.

(c) The coefficients of Schur functions in Stanley symmetric functions.

**Proof.** (a) is true by the Lemma since when \( w \) is Grassmannian (has at most one descent), \( S_w(x_1, \ldots, x_n) \) is a Schur polynomial \( s_\lambda \). When \( u, v \) and \( w \) are Grassmannian with descent position \( d \), then \( C_{u,v}^w \) is a Littlewood-Richardson coefficient so the Theorem implies (b). Finally, when \( w \) is 321-avoiding (i.e., there does not exist indices \( i < j < k \) such that \( w(i) > w(j) > w(k) \)), \( F_w = s_{w/\lambda} = \sum \mu C_{\lambda,\mu} \) is a skew Schur function. Hence, here the coefficient (c) is \( c_{\lambda,\mu} \) and we apply the Theorem. \( \Box \)

Abstractly, all entire families are mutually in value-preserving bijection. However, for Corollary C one can say more: (a) and (c) are a special cases of (b) (see [BeSo98] and [BuSoYo05]). Can one construct a “wrong way map” (as in \( \mathbb{Q} \leftrightarrow \mathbb{N} \)) for either (b)\( \to\) (a) or (b)\( \to\) (c) (thereby finding a rule for \( C_{u,w}^w \))? A special case indicating the difficulty is:

**Problem.** Construct an explicit value-preserving injection between Littlewood-Richardson and Kostka coefficients.
Remark. Finding a wrong way map has solved a significant counting rule problem concerning A. Buch-W. Fulton’s quiver coefficients. These arise in the study of degeneracy loci of vector bundles over a smooth projective algebraic variety. It was conjectured by those two authors that these integers are nonnegative, with a conjectural counting rule. Also, A. Buch showed that special cases of the quiver coefficients are the numbers from (c) above. The resolution of this problem, due to A. Knutson-E. Miller-M. Shimozono, came by establishing the opposite: quiver coefficients are special cases of the well-understood numbers (c). We refer to the solution [KnMiSh04] for background and references.

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