Space-squeezing optics in the microwave spectral region

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Optical systems often consist largely of empty space, as diffraction effects that occur through free-space propagation can be crucial to their function. Contracting these voids offers a path to the miniaturisation of a wide range of optical devices. Recently, a new optical element - coined a ‘spaceplate’ - has been proposed, that is capable of emulating the effects of diffraction over a specified propagation distance using a thinner non-local metamaterial [Nat. Commun. 12, 3512 (2021)]. The compression factor of such an element is given by the ratio of the length of free-space that is replaced to the thickness of the spaceplate itself. In this work we test a prototype spaceplate in the microwave spectral region (17-18 GHz) - the first such demonstration designed to operate in ambient air. Our device consists of a Fabry-Pérot cavity formed from two perforated conductive sheets, with a compression factor that can be directly tuned by varying the size of the perforations. Using a pair of directive horn antennas, we show evidence for a compression factor of up to ∼6.6. We also observe some distortion to the transmitted field, and we discuss future improvements to minimise aberrations. Finally, we investigate the fundamental trade-offs that exist between the compression factor, transmission efficiency, numerical aperture (NA) and bandwidth of this single resonator spaceplate design, and highlight that it can reach arbitrarily high compression factors by restricting its NA and bandwidth.

Introduction
Free-space optical devices implicitly rely on the redistribution of energy that occurs when light diffracts through empty space. For example, lenses, gratings and prisms typically modify an incident wavefront at an interface (or pair of closely spaced interfaces). Yet the desired effect of this modification only becomes apparent once the optical field has propagated some distance beyond the interface, e.g. by focusing a beam or separating it into distinct diffraction orders. This requirement for free-space propagation places limits on the minimum operational volume of a wide range of optical elements and devices, such as cameras, microscopes, telescopes and spectrometers. The issue of size becomes even more prominent at longer wavelengths in so-called quasi-optical systems common to the terahertz [1], millimetre-wave [2] and microwave domains [3]. In this regime, free-space diffraction is fundamental to the operation of antennas [4–7] and beam waveguides [8], and this can lead to very large optical systems [9].

Recently, Reshef et al. [10] and Guo et al. [11] introduced the intriguing new concept of a ‘spaceplate’ – an optical element capable of mimicking the effects of free-space propagation. Crucially, a spaceplate is thinner than the free-space distance it replaces, thus it can potentially be used to contract the volume of optical systems, as shown schematically in Fig. 1. The effect of free-space propagation over a distance $d_{\text{eff}}$ may be understood by decomposing an incident monochromatic optical field of wavelength $\lambda$ into its component plane-waves (i.e. spatial Fourier components). These plane-waves do not couple to one another during propagation, but each accumulates an angle-dependent phase shift of $\phi = kzd_{\text{eff}}$. Here, the wave-vector $k = [k_x, k_y, k_z]$ describes the direction each plane-wave is travelling in Cartesian coordinates, and $k_z = k \cos \theta$, where wavenumber $k = 2\pi/\lambda = |k|$, and $\theta$ is the polar angle of a plane-wave with respect to the optical axis. Therefore, in order to emulate free-space propagation, the action of a spaceplate must be ‘non-local’ [12] [13], i.e. it must independently act on the spatial Fourier components of the incident field, imparting an incident angle-dependent phase shift of

$$\phi_{\text{SP}}(\theta) = kd_{\text{eff}} \cos \theta.$$  

(1)

Designs fall into two main categories, which we term stochastic and deterministic spaceplates. Stochastic spaceplates, first introduced in ref. [10], are non-local...
metamaterials consisting of a multi-layer stack of homogeneous and isotropic layers distributed along the optical axis. Structuring in 1D in this way ensures there is no coupling between plane-waves incident at different angles, as required. The parameters of individual layers – the thicknesses and refractive indices – can be algorithmically optimised to approximate a spaceplate with a target set of performance characteristics, within certain constraints (see Supplementary Information (SI) §5).

Deterministic spaceplates are founded on the understanding that certain families of structure readily act as space compressing optics. For instance, ref. 10 demonstrated that this is the case for a slab of material of lower refractive index than the surrounding medium. Chen & Monticone, meanwhile, highlighted that the angular dispersion in a Fabry-Pérot cavity operating slightly off resonance imparts close to the necessary angle-dependent phase shifts to transmitted light 14.

However, understanding the fundamental limitations on spaceplate performance is an open problem. Of the different structures that have been theoretically shown to operate as spaceplates 10, 11, 13, 14, 15, all exhibit some degree of trade-off between the key parameters defining performance: the compression factor (C), given by the ratio of the emulated free-space propagation distance to the thickness of the spaceplate itself; the transmission efficiency as a function of incident angle; the numerical aperture (NA) and bandwidth (δω) over which the spaceplate operates; and the total space contraction length (L) – see Fig. 1. For example, in all devices proposed so far, prioritising a high compression factor tends to reduce the achievable NA and bandwidth 15. Furthermore, as the concept of space-compression optics is relatively new, the only experimental demonstration of a spaceplate to date relied on artificially increasing the refractive index of the ambient environment, and demonstrated a relatively modest compression factor of C ~ 1.2 over the visible spectrum 10. As such, it is not clear what maximum spatial compression is practically feasible in an air environment, where most envisioned applications lie.

In this work, we design and experimentally test a prototype deterministic spaceplate operating in ambient air in the microwave region (17-18 GHz). Our design consists of a two-layered resonator based on perforated conductive sheets which form a Fabry-Pérot cavity. We study the performance limits of single resonator-based spaceplates and show that arbitrarily high compression factors may be reached by tuning the reflectance of the cavity mirrors. Experimentally, we demonstrate a spaceplate with a peak compression factor of C = 6.6 and C > 4 over a bandwidth of 17.56–18 GHz (∼2.5 %). We discuss drawbacks associated with our proof-of-principle prototype, and describe future improvements, which merge the concepts of deterministic and stochastic spaceplate design.

Space-compression using a Fabry-Pérot cavity

Resonance features play a significant role in all spaceplate designs proposed so far. Therefore, following ref. 14 we examine the potential of a single Fabry-Pérot cavity, operating slightly off-resonance, to act as a spaceplate. We consider a cavity composed of two semi-transparent mirrors with equal reflectance (R1 = R2 = R), separated by a distance dSP, which corresponds to the thickness of the spaceplate (see Fig. 2a). The complex transmission coefficient, as a function of plane-wave incident angle θ and angular frequency ω, is the sum of the transmitted electric field after successive passes around the cavity, which converges to

\[ t(\theta, \omega) = \frac{t_1 t_2 \exp(i\beta)}{1 + r_1 r_2 \exp(2i\beta)}, \]

where

\[ \beta = \frac{\omega}{c} d_{SP} \cos \theta. \]

Here \( t_i \) and \( r_i \) are the fraction of the amplitude of the incident wave transmitted or reflected at mirror i respectively. For a single cavity in air, the Stokes relations connect transmission and reflection according to \( r_1 = -r_2, \ r_1^2 = r_2^2 = r^2 \) and \( t_1 t_2 = 1 - r^2 \), assuming absorption is negligible.

Within a limited angular range, the resonance frequency of a Fabry-Pérot cavity shifts approximately quadratically as a function of incidence angle – see SI §1. This results in a phase shift in the transmitted field that also depends quadratically on incident angle. Figure 2a shows the position of the resonance at three different incident angles. Figure 2b shows the behaviour of the transmitted intensity \( |t|^2 \) and phase \( \phi = \arg(t) \) as a function of incident angle, for three different frequencies located near to a resonance. We can see that, as observed in ref. 14, the phase change as a function of incidence angle mimics that of free-space over a finite numerical aperture, shaded in grey. The transmitted intensity is also >50% over this angular range, although evidently transmission does vary as a function of incident angle. The largest operating NA is found for \( \omega_3 \) - i.e. a frequency that is slightly higher than the resonance frequency at normal incidence.

Assuming high reflectance (so that \( 1 - R << 1 \), the compression factor \( C \) is related to the quality factor of the resonance (Q), and thus the reflectance \( R \) of the cavity mirrors:

\[ C = \frac{d_{eff}}{d_{SP}} = \frac{Q}{2\ell} \approx -\frac{\pi}{2 \ln R}, \]

where \( \ell \) is the order of the resonance (see SI §1.3 for derivation). Equation 4 demonstrates that arbitrarily high compression factors may be reached by tuning the reflectance \( R \) of the cavity mirrors alone, in a manner that is independent of \( \ell \). For example, a compression
Experimental demonstration of a spaceplate

An important step is to explore the extent to which space-compression is readily achievable under experimental conditions. To investigate this, we have built a prototype spaceplate based on a Fabry-Pérot cavity, designed to operate with a compression factor of up to \( C \approx 7 \) over an NA of 0.225 (i.e. maximum incident angle 13°) and a frequency range of 17.5-18 GHz. We note that although the optical properties of Fabry-Pérot cavities are well-understood, here we experimentally study them from the novel perspective of space-compression.

The partially reflecting cavity mirrors are implemented with a simple metamaterial: a conductive sheet perforated with sub-wavelength sized square holes, as depicted in Fig. 2d. Each layer behaves like a conductor with an effective plasma frequency considerably lower than the constituent metal [16]. The effective, frequency dependent permittivity of the layer is determined by the size and spacing of holes, allowing the creation of mirrors with well-controlled and near-arbitrary reflectivity. Here we choose the geometry of the perforations to yield a reflectance of \( R \approx 0.8 \) (see SI §2). The cavity consists of two layers of the conductive perforated sheet, each of area 0.5×0.5 m², separated by an air gap of ~15 mm. The 1st order (\( \ell = 1 \)) and 2nd order (\( \ell = 2 \)) Fabry Pérot resonances occur at 9 GHz, and 17.7 GHz respectively (see SI §3.2). We focus our study on the transmission properties around the 2nd order resonance between 17-19 GHz.

The performance of the prototype spaceplate is evaluated using a rotational test rig in a radio frequency (RF) anechoic chamber. An illuminating antenna (a linearly polarised horn antenna) approximates a point source, and is held in a fixed position with respect to the spaceplate. The source and spaceplate are then rotated in two dimensions with respect to the detection antenna, by further restricting the angular operating range of a Fabry-Pérot cavity to 0.5°, a compression factor of \( C \approx 3200 \) can be obtained. This behaviour can be encapsulated in a simple relation linking the NA to \( C \) and the desired operating bandwidth \( \delta \omega \), found by considering the Q-factor of the resonance (see SI §1.4):

\[
NA \sim \left[ 1 - \frac{1}{(2C\ell)} - \frac{\delta \omega / \omega_r}{2} \right]^{\frac{1}{2}},
\]

where \( \omega_r \) is the angular frequency of the resonance. In deriving Eqn. 4 we define the usable NA as the region over which the spaceplate transmits >50% of incident power, while the bandwidth must also satisfy \( \delta \omega \leq \omega_r / (2C\ell) \) to ensure the operational frequency range remains close to a resonance. Eqn. 4 demonstrates the key trade-offs inherent in all spaceplate designs: higher compression factors are obtained at the expense of a reduced NA and bandwidth. These trade-offs, which are discussed in more detail in SI §1.4, will profoundly impact the applicability of high-compression spaceplates.

FIG. 2: The operation of a Fabry-Pérot cavity based spaceplate. (a) The resonance line of a Fabry-Pérot cavity shifts approximately quadratically with incidence angle. (b) The transmittance (left column) and phase (right column) as a function of incident angle, for three different angular frequencies near resonance (\( \omega_1, \omega_2, \omega_3 \)). The maximum NA is achieved when the working angular frequency corresponds to \( \omega_3 = \omega_r + \Delta \omega / 2 \), where \( \omega_r \) and \( \Delta \omega \) are the resonance frequency and the half power line width. While this maximizes the NA, it also introduces a 50% reflective loss at normal incidence. Higher transmittance at normal incidence can be achieved if frequency is closer to the resonance (e.g. \( \omega_2 \)). The maximum transmittance and the lowest NA is achieved for \( \omega_1 = \omega_r \). (c) Schematic view of a general FP cavity formed by mirrors with equal reflectances \( R \). (d) Schematic view of the spaceplate proposed in this paper - the FP cavity is formed by two perforated conductive sheets acting as mirrors with tunable reflectance.
enabling both the transverse electric (TE) and transverse magnetic (TM) components of the field transmitted through the spaceplate to be mapped on a spherical surface, with polar and azimuthal spherical coordinates denoted by $\alpha_1$ and $\alpha_2$ respectively.

To interpret our measurements we first consider the spherical wavefronts emitted by an ideal point source, depicted in the absence of the spaceplate in Fig. 3a. If the radius of curvature of the spherical measurement surface/arc matches that of the wavefronts, and their centres are co-located, then the detector would sit on the same wavefront throughout the scan and we would measure no variation in phase across the measurement surface. However in our set-up, the source is offset behind the centre of rotation of the spherical measurement surface as shown in Fig. 3a. This means the arriving wavefront has a larger radius of curvature than the measurement surface, leading to a parabolic dependence of the theoretical phase $\phi_t$ as a function of rotation angles $\alpha_1$ and $\alpha_2$. This situation is depicted schematically in 1-dimension (along $\alpha_1$) in Fig. 3a where we see that a larger axial displacement of the source results in a steeper parabolic phase dependence along the measurement surface.

Shifting our attention to our experiment, there is an additional complication: the horn antenna source is not an ideal point source, and so the wavefronts are not perfectly spherical. Therefore, in order to calibrate the system, we first use a reference measurement without the spaceplate in place to determine the location of a point source which creates a theoretical field most closely matching the experimentally measured field emitted by the antenna. The location of this point source is referred
to as the phase-centre.

To calculate the location of the phase-centre we find the axial shift $\Delta_z$ of an ideal point source, from the centre of rotation, that generates a theoretical phase function $\phi_0(\alpha_1, \alpha_2, \Delta_z)$ across the measurement surface that is most highly correlated with our experimentally measured phase function $\phi_1(\alpha_1, \alpha_2)$ \cite{SI}. This is achieved by minimising $\sigma_{\text{phase}}$: the square root of the Euclidean norm of the residual phase difference between unwrapped functions $\phi_1$ and $\phi_0$. Detail of this phase-centre calculation, which is a single variable optimisation problem with one global minimum and no local minima, is given in SI §4.3.

Next, the spaceplate is mounted in the rotational test rig. The effect of the spaceplate is to make the source appear more distant, thus increasing the radius of curvature of the wavefronts arriving at the measurement surface, as shown schematically in Fig. 3. This further increases the magnitude of the 2nd order derivative of the parabolic phase as a function of angle across the measurement surface – a signature of space compression in our experiment, as shown in Fig. 3\textit{c}. The compression factor is calculated by finding the apparent axial displacement in the location of the phase-centre when the spaceplate is inserted into the set-up. This shift represents the amount of space contracted ($L$), and we can then infer the space compression factor using $C = (L + d_{\text{SP}}) / d_{\text{SP}}$.

Figure 3\textit{a} shows brute-force evaluations of the shift in phase-centre, at a frequency of 17.8 GHz, when the spaceplate is introduced into the set-up. Here we see that $\sigma_{\text{phase}}$ is minimised for axial translations of the phase-centre of 94 mm for TE, and 76 mm for TM polarisation. This corresponds to compression factors of $C = 6.6$ and $C = 5.5$ for TE and TM polarisations, respectively. The lower compression in the TM case is anticipated due to a slightly lower reflectance of the metamaterial layers forming the Fabry-Pérot cavity as a function of incident angle, as explained in SI §3. It is difficult to overcome this polarisation dependence of the compression factor in a single Fabry-Pérot cavity, as it is directly linked to the reflectivity of the cavity mirrors (see Eq. 4) which are inherently sensitive to both polarisation and incident angle. Nonetheless, in future there may be some degree of optimisation possible by introducing a rectangular symmetry to the shape and pitch of the perforations.

Optical components generally exhibit frequency dependent characteristics. For example, the focal length of a simple lens changes with frequency, which is a source of chromatic aberration in an imaging system. In a similar manner, the compression factor of a Fabry-Pérot spaceplate is also dependant on frequency: Fig. 3\textit{c} shows how the theoretically predicted and experimentally measured compression factors vary as a function of frequency for our spaceplate, within its design $\text{NA} = 0.225$. Here the theoretical compression factor is calculated using a finite element model (constructed in Ansys HFSS - see SI §3 for details) capturing the geometry of the perforated sheets. For frequencies below 18 GHz, we see reasonable agreement between the experimentally measured and simulated compression factors, with the measured compression factor slightly lower than predicted theoretically. However, at frequencies above 18 GHz we see larger discrepancies between measurement and modelling – namely ripples in the calculated compression factor as a function of frequency – which we discuss in more detail below.

Figure 4 shows the measured amplitude and phase of the reference field in the absence of the spaceplate (\textit{a,c}), and the field transmitted through the spaceplate (\textit{b,d}), at a frequency of 17.8 GHz. Here we plot only the co-polarised field component $E_\text{e}$ as a function of direction cosines $u, v$ \cite{SI}, which are related to the rotation angles $\alpha_1, \alpha_2$ as given in the figure. The white circle in all figures shows the approximate operational $\text{NA} = 0.225$.

FIG. 4: The normalised amplitude and phase proportional to the electric field as measured in the spherical setup without (\textit{a,c}) and with (\textit{b,d}) the spaceplate. The coordinates $u, v$ correspond to direction cosines and are tied to the rotation angles $\alpha_1, \alpha_2$ as given in the figure. The white circle in all figures shows the approximate operational $\text{NA} = 0.225$.\textit{c}}
ity (see Fig. 2b, top), which significantly reduces transmis-

sion for high incident angles which are further from

resonance.

In Fig. 4, we also observe some field distortion: the am-

plitude of the transmitted field is no longer smoothly vary-

ing, but has spatial variation within the NA of the

spaceplate (see Fig. 4b). We are able to identify and dis-

count two potential explanations for these aberrations.

The first possibility we consider is diffraction from the

edge of the finite sized spaceplate. To avoid this occur-

ring, the physical extent of both the incident and trans-

mitted fields must fall within the area of the spaceplate.

In our experiment, the divergence of the source means

that this condition is satisfied, and so we do not ex-

pect the transmitted field to interact with the edge of

the spaceplate (See SI §3 for experimental geometry).

A second possible source of aberrations emerges from

multiple reflections between different parts of the exper-

imental set-up. We do indeed observe multiple reflec-

tions between the source and the spaceplate, manifest-

ing as ringing in the frequency spectrum shown in SI

§3.

However, temporal filtering to remove these reflections

from our measurements does not significantly alter the

observed aberrations.

Having discounted other sources of aberrations from our

experiment, we attribute the observed distortion of the

transmitted field, along with the ripples in Fig. 3, to subtle

inhomogeneities in the layer spacing and the size and spacing of the perforations across the spaceplate. These will give rise to small spatial variations in the reflectivity and resonant frequency of the cavity, resulting in frequency dependent distortions of the transmitted field. These effects are also likely to cause inhomogeneous broadening of the resonance peak, resulting in the slightly lower than expected experimentally measured compression factors at the resonance peak (see Fig. 3). This indicates that improvement of spatial inhomogeneities is key to reducing the aberrations from our prototype spaceplate. Moreover, due to the intrinsic relationships between resonance Q-factor and compression ratio expressed in Eqns. 4 and 5 minimising spatial inhomogeneities will be even more crucial for the implementation of higher compression ratio spaceplates.

Discussion

All of the spaceplate designs proposed thus far \cite{10, 11, 14, 15} exhibit trade-offs between the achievable compression factor, transmission efficiency, NA, bandwidth and total space contraction length. Here we have focussed on the simplest deterministic spaceplate design: a single Fabry-

Pérot cavity, and highlighted that it can achieve a compres-

sion factor of nearly an order-of-magnitude higher than the largest quoted compression factor recently found by stochastic optimisation, over an equivalent NA \cite{15}.

A single Fabry-Pérot cavity based spaceplate has two

main drawbacks: the dependence of the compression fac-

tor on the incident polarisation (see Fig. 3), and the modula-

tion of the transmitted intensity as a function of both frequency and angle (see Figs. 2b and 4b). How-

ever, coupling together a series of Fabry-Pérot cavities provides opportunities to overcome these issues. For example, Chen & Monticone theoretically showed that the transmission efficiency, along with the total space con-

traction length of a spaceplate, can be enhanced using coupled Fabry-Pérot cavities, whilst trading a modest re-

duction in compression factor \cite{14}.

Stochastic optimisation schemes also seem to generate spaceplate designs which are essentially coupled Fabry-

Pérot cavities. For example, in order to overcome the limitations of a single Fabry-Pérot cavity, we have ex-

plored the use of a genetic algorithm to optimise the layer-spacing of a spaceplate consisting of up to 15 el-

ements. SI §5 presents detail of our optimisation algo-

rithm and results. We give the optimiser freedom to merge, and thus reduce, the number of layers when seek-

ing an optimal solution. Using this method we find a locally optimal solution consisting of 3 Fabry-Pérot cavities separated by 2 optimised (\sim \lambda/6) coupling regions, which is very similar to the deterministic design presented by Chen & Monticone \cite{14}. These coupled cavity spaceplate designs are able to suppress the polarisation dependence exhibited by a single Fabry-Pérot cavity, and generate a roughly constant transmission as a function of incident angle over the operating NA. Nonetheless, no design has yet been found that can surpass the trade-offs captured in Eqn. 5, and so it seems likely that very high spatial compression always comes with the cost of very low NA and operational bandwidth.

Taken together, the apparent dependence of all space-

plate designs on Fabry-Pérot resonance effects suggests that the trade-offs inherent in a single Fabry-Pérot cav-

ity may be close to the fundamental limits on spaceplate performance. We speculate that a Fabry-Pérot cavity may be understood as a basic building block of a spaceplate, in the same way as a simple lens is a basic build-

ing block of a multi-element (e.g. objective) lens. We envisage that the majority of future spaceplate designs will feature coupled Fabry-Pérot cavities, which will be honed for specific applications – such as the optimisation of performance around three distinct colour channels for colour imaging – in a similar manner to the way com-

pound lenses are designed to suppress the chromatic and Seidel aberrations present in a single lens.

It is also worth noting that the ability of a Fabry-Pérot cavity to non-locally modify the angular spectrum of electromagnetic waves is well-known to the antenna and microwave communities. Fabry-Pérot resonator antennas, first proposed by Trentini et al. in ref. \cite{20}, increase the directivity of an antenna by coupling it to a Fabry-Pérot resonator with considerably larger lateral dimensions \cite{21}. It is also already understood that
wavefronts emanating from the structure have travelled a longer path length than the thickness of the antenna, as theoretically derived by Burghignoli in ref. 22, which is consistent with the theoretical behaviour of a spaceplate. Although, we emphasise that in these earlier studies, the applications in mind were very different from the concept of space-compression.

Conclusions
In summary, we have experimentally demonstrated a deterministically tunable space-squeezing optical element in the microwave spectral region. We observe a maximum space compression factor of ∼6.6 over an NA of 0.225 at 17.9 GHz. The compression factor is higher than 4 in the frequency band 17.56-18 GHz. We believe this is a significant step towards the introduction of the spaceplate concept into real-world quasi-optical systems. Our study hints that a Fabry-Pérot cavity may offer a close-to-optimal trade-off in capabilities, as encompassed by Eqn. 5, suggesting this type of simple spaceplate should be used as a benchmark for evaluating the performance of other designs.

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Contributions
MM devised the experiment and carried out the analytical and numerical modeling. JL built the spaceplate and conducted the measurements. LEB assisted with the simulations. DBP and EH supervised the work. DBP, EH and MM wrote the manuscript. All authors edited the manuscript and contributed to the interpretation of the experimental data.

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Supplementary Information

This supplementary document provides further, in-depth information on the theory, various trade-offs and design of the Fabry-Pérot cavity based spaceplate. The experimental setup and methods used to characterise the spaceplate are explained here. In the last section we include information on the design and performance of a stochastic spaceplate consisting of three coupled resonators.

§ 1 A Fabry-Pérot resonator as a spaceplate

It has been shown that the angular dispersion in a Fabry-Pérot cavity can be used to design a spaceplate [14]. Here, we analyze such a solution from a general point of view and find the trade-off between the compression ratio (C), numerical aperture (NA) and bandwidth (δω) of the spaceplate. In addition we show how the compression ratio can be obtained from the Q factor of the resonator or reflectances of its mirrors.

The transmission coefficient of a Fabry-Pérot cavity made of two semitransparent mirrors with equal reflectance (R₁ = R₂ = R) separated by a distance d_SP is

\[ t(\theta, \omega) = \frac{t₁t₂ \exp(i\beta)}{1 + r₁r₂ \exp(2i\beta)}, \]

with

\[ \beta = \frac{2\pi}{\lambda} d_{SP} \cos \theta = \frac{\omega}{c} d_{SP} \cos \theta, \]

where R = r² and r₁, r₂, t₁ ,t₂ are the reflection and transmission coefficients of the mirrors (Stokes relations r₁ = −r₂, t₁t₂ = 1 − r²), λ is the free space wavelength, θ is the incidence angle and ω is the angular frequency. We assume all the regions are free space (see Fig. 1).

![Fig. 1: Fabry-Pérot cavity formed by two mirrors with equal reflectances R.](image)

1.1 Even Fabry-Pérot resonances

The interesting behavior for a single FP resonator spaceplate occurs near the transmission peaks where the conditions for constructive interference are met. The relative phase shift between two successive reflections defining the constructive interference is \( 2\beta = 2\pi\ell \), where \( \ell \) is the order of the resonance (\( \ell = 1, 2, 3... \)). If we consider a cavity with thickness \( d_{SP} = \lambda_{r₁}/2 \), where \( \lambda_{r₁} = 2\pi c/\omega_{r₁} \) is the resonant wavelength of the first interference peak at normal incidence (i.e. \( \theta = 0^\circ \)), we can tie the resonance of the structure to the interference condition as follows:

\[ 2\beta = 2\pi\ell \rightarrow 2\frac{2\pi}{\lambda_r(\ell, \theta)} d_{SP} \cos \theta = 2\pi\ell \rightarrow \frac{\omega_r(\ell, \theta)}{\omega_{r₁}} \frac{2\pi c}{\omega_{r₁}} \cos(\theta) = 2\pi\ell. \]

The dispersion equation relating the resonant frequency \( \omega_r(\ell, \theta) \) corresponding to the angle \( \theta \) and resonance order \( \ell \) with respect to the normal incidence (and \( \ell = 1 \)) resonance \( \omega_{r₁} \):

\[ \omega_r(\ell, \theta) = \frac{\ell}{\cos \theta} \omega_{r₁}. \]

The resonant frequency of the FP cavity thus shifts with \( 1/\cos \theta \). However, for small angles in can be approximated by the first two terms of its Taylor expansion:
Equation (4) tells us that the higher order (even) resonant modes ($\ell > 1$) are more sensitive to the incidence angle by a factor $\ell$ (see Fig. 2). This suggests that they provide a limited NA compared to the first one ($\ell = 1$), however as we show later, the compression ratio $C$ remains the same. Effectively, this means that a resonator operating in its higher resonance can substitute a thicker slab of air but over a reduced angular range.

![Image](image_url)

**FIG. 2:** First three even resonances and their shift with incidence angle. The shift of $\ell^{th}$ order with angle $\theta$ is given by $\ell/\cos \theta$.

Now, we evaluate the phase of the transfer function of the spaceplate given by the dispersion according to eq. (4). To do this, we use a group delay (GD) concept described in the section 1.2. We linearize the phase of the spaceplate about the resonance and the slope of the linear portion of the phase is considered to be the approximate GD.

Figure 3 shows that the resonance of the spaceplate shifts with the incidence angle as $\ell/\cos \theta$. The phase $\phi_{SP}(\theta)$ is the phase difference (over the length of the SP) between $\theta = 0^\circ$ and $\theta > 0^\circ$ directions. This phase can be found from the triangle in Fig. 3 and is given as:

$$\phi_{SP}(\theta) = -\ell \frac{GD}{\cos \theta}$$

(6)

### 1.2 Compression factor of the spaceplate, $C$, and the properties of the FP resonator

The transmission curve of a resonator based spaceplate is fully given by its quality factor (Q-factor) and the order of the resonance $\ell$. The compression factor $C$ can be determined from the fundamental properties of the resonator (under certain assumptions) as follows.

If we consider an FP cavity formed by 2 mirrors with identical reflectance $R$ and assuming:

- $R$ does not depend on the angle of incidence $\theta$, and
- the phase of the transmission coefficient is linear about the resonance (see Fig. 4),

then the slope of the linearised phase determines the group delay of the structure

$$GD_{SP} = \frac{d\phi}{d\omega} \approx \frac{\Delta\phi}{\Delta \omega} = \frac{\pi/2}{2\pi \Delta f} = \frac{1}{4\Delta f} = \frac{Q}{4f_r} = \frac{\pi Q}{2\omega_r},$$

(7)

which must not change with the incidence angle and $\Delta f = \Delta \omega/2\pi$ is the line width of the resonator under normal incidence, $f_r = \omega_r/2\pi$ is the resonant frequency and the Q-factor is defined by $Q = f_r/\Delta f = \omega_r/\Delta \omega$.

The group delay in free space of effective thickness $d_{eff}$ which the spaceplate is to substitute is:

$$GD_{FS} = \frac{d\phi}{d\omega} = \frac{d}{d\omega} \left( \frac{\omega}{c} d_{eff} \right) = \frac{d_{eff}}{c}.$$
FIG. 3: Phase profile of the SP as a function of normalized angular frequency close to the \(1^{\text{st}}\) resonance. Blue curve represents normal incidence - the phase profile of the resonance is shifted by \(\ell/\cos \theta\) (where \(\ell = 1\)) for incidence angle \(\theta > 0^\circ\), plotted in red. The change in phase (\(\arg \{t\}\)) related to the change of incidence angle \(0 \rightarrow \theta\) at operating frequency \(\omega/\omega_{r1} = 1\) is \(\phi_{SP}(\theta)\).

Thus, by assuming \(GD_{SP} = GD_{FS}\) we obtain the equivalent thickness of free space, \(d_{\text{eff}}\), our resonator can substitute: \(d_{\text{eff}} = c \cdot \pi Q/(2\omega_r) = \lambda_r Q/4\).

If we operate the spaceplate of thickness \(d_{SP} = \ell \lambda_r/2\) at a frequency that matches the resonant frequency of the resonator \(f = f_r\) the transmission for the boresight direction is 1 and it slowly tapers off with incidence angle. The compression factor \(C\) in this case is given as:

\[
C = \frac{d_{\text{eff}}}{d_{SP}} = \frac{\lambda_r Q}{\ell 2Q} = \frac{Q}{2\ell}.
\]

(9)

Assuming a high reflectance \((R)\) of the mirrors that form the cavity (such as \(1 - R < < 1\)) we can tie the Q-factor to the reflectance as [23]

\[
Q = -\frac{\pi \ell}{\ln R},
\]

(10)

which combined with eq. 9 leads to the compression factor as a function of the reflectance of the mirrors:

\[
C = -\frac{\pi}{2 \ln R}.
\]

(11)

Under the assumptions given above, we can see that the compression factor \(C\) of the FP resonator spaceplate does not depend on the order of the resonance \(\ell\). However, we should keep in mind that the higher resonances will offer a reduced NA (see eq. [4]).

1.3 Numerical aperture and bandwidth of the FP resonator spaceplate

As shown below, there is a fundamental trade-off between the compression ratio, the numerical aperture of the space plate, and the frequency bandwidth it can operate over.

Starting with monochromatic waves, if we allow the transmittance of the spaceplate at normal incidence to be half of the maximum, the NA is maximized. The angles \(\theta_1\) and \(\theta_2\) describe the angles where the transmittance is increased to 1 (angle \(\theta_1\)) and where it drops to 0.5 again (angle \(\theta_2\)). In this case, the angle \(\theta_2\) defines the NA = \(\sin \theta_2\). The operating angular frequency is \(\omega_{\text{work}} = \omega_r + \Delta \omega/2\) (see Fig. 5).

If we wish to have the transmittance at normal incidence equal to 1, the numerical aperture is given by the \(\theta_1\) angle, NA = \(\sin \theta_1\) (operating frequency \(\omega_{\text{work}} = \omega_r\)).

The angles \(\theta_1\) and \(\theta_2\) are given as follows:

\[
\theta_1 = \cos^{-1} \frac{\omega_r}{\omega_r + \Delta \omega/2} = \cos^{-1} \frac{1}{1 + 1/2Q} = \cos^{-1} \frac{1}{1 + 1/4C\ell},
\]

(12)
The numerical aperture for the case with transmittance at normal incidence equal to 0.5 can be written using the trigonometric identity $\sin(\cos^{-1}(x)) = \sqrt{1 - x^2}$.

$$\theta_2 = \cos^{-1} \frac{\omega_r}{\omega_r + \Delta \omega} = \cos^{-1} \frac{1}{1 + 1/Q} = \cos^{-1} \frac{1}{1 + 1/2C\ell}.$$  \hspace{2cm} (13)

and it can be easily modified for the case with unity transmittance at $0^\circ$, $NA = \sin \theta_1$.

Now, we turn our attention from monochromatic waves and introduce a signal with a certain angular frequency bandwidth $\delta \omega = \omega_{\text{max}} - \omega_{\text{min}}$ and due to the nature of the spaceplate we must assume $\delta \omega < \Delta \omega$ (if we do not want to compromise on the performance even more than shown in Fig. 5). This effectively means that the maximum acceptable frequency shift of the resonance, which is $\Delta \omega$ for monochromatic waves will be reduced to $\Delta \omega - \delta \omega$. Using the dispersion equation (eq. (4)) the maximum angle $\theta_2$ is

$$\theta_2 = \cos^{-1} \frac{\omega_r}{\omega_r + \Delta \omega - \delta \omega} = \cos^{-1} \frac{1}{1 + 1/Q - \delta / \omega_r} = \cos^{-1} \frac{1}{1 + 1/2C\ell - \delta / \omega_r},$$ \hspace{2cm} (15)

and the NA of the spaceplate with given guaranteed frequency bandwidth $\delta \omega$ is

$$NA = \sin \theta_2 = \sqrt{1 - \left( \frac{1}{1 + 1/Q - \delta / \omega_r} \right)^2} = \sqrt{1 - \left( \frac{1}{1 + 1/2C\ell - \delta / \omega_r} \right)^2}. \hspace{2cm} (16)$$

The equations are very accurate for high Q cavities, but still work well for lower Q cases ($R \approx 0.5$).

1.4 Interesting trade-offs

Equations (9), (11), (14) and (16) are useful for studying the performance trade-offs and the limits of FP resonator spaceplates. To do so, we sweep the reflectance of the mirrors $R$ in the range 0.5 to 0.99 and plot the compression factor as a function of $R$ (see Fig. 6a) and as a function of the Q-factor of the cavity (see Fig. 6b). The insets show the compression factor $C$ for even higher reflectances ranging from 0.99 up to 0.9999. In limiting case, for $Q \to \infty$ (also $R \to \infty$), the compression ratio will tend to infinity (see eq. (9)) as the NA will be approaching 0 (eq. (14)):

$$\lim_{Q \to +\infty} C = \infty \lim_{Q \to +\infty} NA = 0$$ \hspace{2cm} (17)

An important trade-off for a resonant spaceplate is that between the numerical aperture (or maximum incidence angle $\theta_{\text{max}}$), the compression factor and the bandwidth ($\delta \omega$). In Fig. 6c, we can see the maximum acceptance angle (transmittance = 0.5 at normal incidence) as a function of the compression factor with the bandwidth as a parameter. The bandwidth is expressed as a fraction of the spectral linewidth of the resonator $\Delta \omega$. The blue curve ($\delta \omega = 0$) represents a monochromatic wave and thus sets the upper bound on achievable numerical aperture. As
FIG. 5: Operation of the spaceplate explained using how the amplitude (a) and phase (b) of the transfer function depend on the resonance shift of the FP cavity with the incidence angle. This case corresponds to the maximum NA, where the transmittance at normal incidence is reduced to 0.5. The operating angular frequency is $\omega_{\text{work}} = \omega_r + \Delta \omega/2$, where $\omega_r$ is the resonant frequency at normal incidence. The important angles $\theta_1$ and $\theta_2$ are shown here.

The bandwidth increases the NA goes down. The maximum theoretical bandwidth (i.e. $\delta \omega = \Delta \omega$) does not allow for any shift of the resonance and thus results in NA = 0. The inset in Fig. 5 shows that a compression factor of $C = 3282$ can be achieved with FP resonator spaceplate within NA corresponding to $\theta_{\text{max}} = 1^\circ$, and $C = 13130$ within $\theta_{\text{max}} = 0.5^\circ$.

It has been mentioned that the spaceplate trades off its numerical aperture for the transmittance at normal incidence (Fig. 5(a)). Figure 2 of the main paper shows that this depends on the selection of the operating frequency $\omega_{\text{work}}$. If we choose to operate at resonance frequency $\omega_{\text{work}} = \omega_r$ we achieve maximum transmittance $|t|^2 = 1$ at normal incidence, which tapers off to $|t|^2 = 0.5$ at incidence angle $\theta_1$. On the other hand, if we pick $\omega_{\text{work}} = \omega_r + \Delta \omega/2$, the transmittance at normal incidence is $|t|^2 = 0.5$, increases to $|t|^2 = 1$ (at $\theta_1$), and again drops to $|t|^2 = 0.5$ at incidence angle $\theta_2$. In Fig. 5(b) we can see the drop in normal incidence transmittance as a function of maximum incidence angle $\theta_{\text{max}}$. The red axis demonstrates a real world scenario of a resonant spaceplate based on mirrors with $R = 0.9$ and a compression factor $C = 14.9$ - here we can see that by allowing $|t|^2 = 0.5$ at normal incidence we can increase the NA angle from $\theta_{\text{max}} = 10.5^\circ$ to $\theta_{\text{max}} = 14.8^\circ$. Finally, the blue axis in Fig. 5 relates the range of maximum angles to the selection of operating frequency $\omega_{\text{work}}$. 
Even though the resonance order $\ell$ does not influence the maximum compression ratio of a spaceplate, it has a negative effect on its achievable numerical aperture (see eq. (4)). In Fig. 7 we examine this effect for $\ell = 1$ to $\ell = 5$. The main benefit of operating the spaceplate at a higher order resonance is the increase in effective thickness $d_{\text{eff}}$ since the length of the resonator is increased $d_{\text{SP}} = \ell \cdot \lambda / 2$ and the compression factor remains the same, $d_{\text{eff}} = C \cdot d_{\text{SP}}$. The second benefit is the mechanical simplicity with low weight and costs. The spaceplate consists of only three components - two dichroic mirrors separated by free space.

![Graphs showing trade-offs and numerical aperture](image)

**FIG. 6:** Main trade-offs of a FP resonator spaceplate (with $\ell = 1$). $\delta \omega$ represents the available bandwidth and $\Delta \omega$ is the linewidth of the resonator.

**FIG. 7:** Numerical aperture as a function of compression ratio for the 5 lowest order resonances, assuming monochromatic operation. The drop in NA with increasing $\ell$ is obvious.
§2 Analytical model of periodically perforated metals

The electromagnetic interaction between light and the metal is driven by the free electrons of the metal, which give rise to a negative permittivity for frequencies below their plasma frequencies. It has been shown that one can mimic the negative permittivity of a metal near its plasma frequency by structuring the surface of a (near) perfect conductor by introducing periodic arrays of subwavelength holes. Such a structured layer of highly conducting material can support surface plasmon-like surface modes at frequencies well below the plasma frequency of the conductor [24–27], and also have the property of partial reflectance, similar to that thin of metal films [28].

In this paper, we use highly conducting sheets with two-dimensional arrays of subwavelength, open holes (dimensions as defined in Fig. 5) to mimic the partially reflecting mirrors of a Fabry Pérot cavity. Using a modified modal matching method, similar to that in refs [26, 27], we derive the effective permittivity and partial reflectance of these layers.

We start by introducing a rigorous modal matching model. Sub-wavelength hole arrays in metals at low frequencies (such as the THz range and below) have previously been modeled using modal-matching techniques [24–27]. We describe in brief how this technique works: the electromagnetic fields in the superstrate (assumed here to be air) and substrate (also assumed here to be air) are matched to the fields of the waveguide mode inside the subwavelength holes. By exploiting continuity of electric and magnetic fields at the boundaries, we can obtain explicit analytical expressions for transmission and reflection of a square array (period $d$) of square holes (side $a$) in a thin metallic sheet of thickness $h$.

We begin by defining expressions for the electric and magnetic fields in three regions of a hole array: in the incident vacuum region (superstrate), inside the holes, and in the substrate. For simplicity, note that in the following we omit the time ($t$) dependent component to the fields, $\sim \exp(i\omega t)$, where $\omega$ is the radial frequency. We consider a unit source field incident as shown in Fig. 3. We express the electric field on the incident side of the hole array (region 1) as a sum of our unit plane-wave with wavevector $(k_x, k_y, k_z)$ and a two dimensional Fourier-Floquet expansion of diffracted orders with wavevectors $(k_{x_1} m_1, k_{y_1} m_2, k_{z_1} m_2)$. Approximating the metal as perfectly conducting (a reasonable approximation at THz and microwave frequencies), the electric field inside the holes (region 2) is expressed by the fundamental mode of a square cavity of width $a$, while in the substrate (region 3) we have again a Fourier-Floquet expansion of diffracted orders. These definitions amount to $x$ components of the electric field of the form:

$$E_x^1 = \exp(i k_x x + i k_y y + i k_{z_1} z) + \sum_{m_1, m_2} r^{m_1, m_2} \psi^{m_1, m_2}_1 \exp(-i k_{z_2} m_2 z),$$

$$E_x^2 = \sum_{s_1, s_2} B^{s_1, s_2} \psi^{s_1, s_2}_2 \exp(i q_{s_1} s_2 z) - C^{s_1, s_2} \psi^{s_1, s_2}_2 \exp(-i q_{s_1} s_2 z),$$

$$E_x^3 = \sum_{n_1, n_2} t^{n_1, n_2} \psi^{n_1, n_2}_3 \exp(i k_{n_1} n_2 (z - h)),$$

where $\psi^{m_1, m_2}_1 = \exp \left( i (k_x + \frac{2m_1 \pi}{a}) x \right) \exp \left( i (k_y + \frac{2m_2 \pi}{a}) y \right)$ and $\psi^{s_1, s_2}_2 = \sin \left( \frac{2s_1 \pi}{a} y \right) \cos \left( \frac{2s_2 \pi}{a} x \right)$. Note that similar expressions for $y$ components of field can also be defined. The integer pairs $(m_1, m_2)$ and $(n_1, n_2)$ denote the diffracted orders, on the incident and substrate sides of the hole array, respectively, from the grating of pitch $d$. The factors $r^{m_1, m_2}$ and $t^{n_1, n_2}$ describe the complex field reflection and transmission coefficients. The $z$ component of the incident and transmitted wavevectors can be written as

$$k_{z_1}^{m_1, m_2} = \sqrt{k_0^2 - \left( k_x + \frac{2m_1 \pi}{d} \right)^2 - \left( k_y + \frac{2m_2 \pi}{d} \right)^2},$$

and

$$k_{z_1}^{s_1, s_2} = \sqrt{\varepsilon_{sub} k_0^2 - \left( k_x + \frac{2m_1 \pi}{d} \right)^2 - \left( k_y + \frac{2m_2 \pi}{d} \right)^2},$$

where $c$ is the speed of light, $k_0 = \omega / c$ is the wavenumber of the incident light and $\varepsilon_{sub}$ is the dielectric constant of the material inside the substrate (region 3). The factors $B^{s_1, s_2}$ and $C^{s_1, s_2}$ describe the electric field amplitudes of the decaying wave in the cavity and the reflected wave from the cavity bottom, respectively, while the integer pair $(s_1, s_2)$ define the waveguide mode within the cavity. For square holes, the propagation constant in the cavity is...
\[
q_{z_{1}^{s_{1}},s_{2}} = \sqrt{\epsilon_{0}k_{0}^{2} - \left( \frac{s_{1}\pi}{a} \right)^{2} - \left( \frac{s_{2}\pi}{a} \right)^{2}},
\]

(20)

where \(\epsilon_{h}\) is the dielectric constant of the material inside the cavity.

We can obtain the \(z\) components of the electric field in the three regions of space, and subsequently expressions for the magnetic field \(\mathbf{H}\), through the free space Maxwell’s relations \(\nabla \cdot \mathbf{E} = 0\) and \(\nabla \times \mathbf{E} = -\mu_{0}\partial\mathbf{H}/\partial t\). This gives the \(x\) and \(y\) components of the electric and magnetic fields in all regions in terms of the set of unknowns \(r, t, B,\) and \(C\). In order to eliminate some of these unknowns, we can use the fact that both the \(x\) and \(y\) components of the electric field must be continuous at the vacuum-sample interfaces (i.e. \(z = 0\) and \(z = h\), where \(h\) is the depth of the hole array) over the entire unit cell, while the magnetic field components are continuous only at the hole aperture. Matching the \(E\) fields in regions 1 and 2 at \(z = 0\), and in regions 2 and 3 at \(z = h\), (i.e. multiplying by \(\psi_{1}^{*}\) and integrating over \(x\) and \(y\) from 0 to \(d\)), and taking into account the orthogonality of the eigenmodes of the system, yields sets of continuity equations of the form

\[
(\delta^{m_{1},m_{2}} + r^{m_{1},m_{2}})d^{2} = \sum_{s_{1},s_{2}} (B^{s_{1},s_{2}} - C^{s_{1},s_{2}})Q_{1}^{m_{1},m_{2},s_{1},s_{2}},
\]

(21a)

\[
\ell^{n_{1},n_{2}}d^{2} = \sum_{s_{1},s_{2}} (B^{s_{1},s_{2}}e^{iq_{1}z} - C^{s_{1},s_{2}}e^{-iq_{1}z}h)Q_{1}^{n_{1},n_{2},s_{1},s_{2}},
\]

(21b)

where

\[
Q_{1}^{m_{1},m_{2},s_{1},s_{2}} = \int_{0}^{a} \sin \left( \frac{s_{1}\pi y}{a} \right) \cos \left( \frac{s_{2}\pi x}{a} \right) \times \exp \left[ -i \left( k_{x} + \frac{2m_{1}\pi}{d} \right) x \right] \exp \left[ -i \left( k_{y} + \frac{2m_{2}\pi}{d} \right) y \right] dxdy,
\]

(21c)

is the overlap integral between the diffracted order \((m_{1}, m_{2})\) and the waveguide mode \((s_{1}, s_{2})\), and \(\delta^{m_{1},m_{2}}\) represents the Kronecker delta function \(\delta (m_{1}) \delta (m_{2})\). We also obtain separate expressions for the continuity of the \(y\) components of electric field.

We can obtain a further set of equations by considering continuity of the \(H\) field over the holes at \(z = 0\) and \(z = h\) respectively, i.e. by multiplying \(H\) fields by \(\psi_{2}\), and integrating from 0 to \(a\) for \(x\) and \(y\). This gives a set of equations containing a second overlap

\[
Q_{2}^{m_{1},m_{2},s_{1},s_{2}} = \int_{0}^{a} \sin \left( \frac{s_{1}\pi y}{a} \right) \cos \left( \frac{s_{2}\pi x}{a} \right) \times \exp \left[ +i \left( k_{x} + \frac{2m_{1}\pi}{d} \right) x \right] \exp \left[ +i \left( k_{y} + \frac{2m_{2}\pi}{d} \right) y \right] dxdy.
\]

(22)

These equations, relating the \(x\) and \(y\) components of the electric and magnetic fields, define a complete set of equations describing the components of the fields in terms of the unknown sets \(r^{m_{1},m_{2}}, \ell^{n_{1},n_{2}}, B^{s_{1},s_{2}},\) and \(C^{s_{1},s_{2}}\). The number of equations present in the set depends on the number of diffracted orders and waveguide modes we include in the calculation, but the system of equations is always uniquely defined (i.e. the number of unknowns equals the number of equations). For a rigorous solution to the system of equations, we must include a high number of diffraction orders and waveguide modes. It is then straightforward, if rather laborious, to solve the continuity equations, eliminating the coefficients \(B^{s_{1},s_{2}}\) and \(C^{s_{1},s_{2}}\), to obtain the complex reflection and transmission coefficients, \(r^{m_{1},m_{2}}\) and \(\ell^{n_{1},n_{2}}\).

As the simplest solution, we can limit ourselves to considering only the first order waveguide mode \((s_{1} = 1, s_{2} = 0)\) in the cavity and specular reflection/transmission \((m_{1} = m_{2} = n_{1} = n_{2} = 0)\). This is the approximation used in Refs. 24, 26 to derive analytical dispersion relations for spoof surface plasmons on dimpled conducting surfaces, and shown to be an excellent approximation in refs 27, 28 for arrays similar to those use here. The approximation is valid only in the limit \(a < d << \lambda_{0}\), where \(\lambda_{0}\) is the vacuum wavelength. Then, the summations in equations 21a, 21b, and
those in the continuity equations for magnetic fields, are removed, and one can easily solve the continuity equations, eliminating the coefficients $B_1^0,0$ and $C_1^0,0$. The far field transmission coefficient found is

$$t_{0,0}^0 = -\frac{2k_0^2}{k_z^2} \frac{Q_1^{0,0,1,0} Q_2^{0,0,1,0}}{d^2} \exp(i q_z h) \left(F - \frac{a^2 q_z^2}{2}\right)^2 - \exp(-i q_z h) \left(F + \frac{a^2 q_z^2}{2}\right)^2,$$

(23a)

where

$$F = \frac{Q_1^{0,0,1,0} Q_2^{0,0,1,0}}{d^2} \left(\frac{k_0^2}{k_0^2}\right).$$

(23b)

From this solution, one can also infer an effective permittivity and permeability for the layer: this describes the parameters required if you were to replace the structured layer by a homogeneous one, while retaining the amplitude and phase of a transmitted plane-wave.

Effective parameters of the hole array used here were obtained by numerically fitting Fresnel predictions to the reflection and transmission coefficients found using our modal matching model. The effective permittivity, along with the calculated reflectance, for one of our layers is plotted in Fig. 9. The effective parameters reveal a large negative permittivity at low frequencies which follows a Drude-like dispersion with an imaginary permittivity equal to zero - i.e. the hole array behaves as a homogeneous metal with a plasma frequency higher than the frequencies used here.

FIG. 8: Two dimensional, periodic array of square holes in a perfect conductor. The holes considered here are of depth $h$, width $a$, separated by distance $d$ and filled by dielectric material with $\varepsilon_h$, while the substrate (region 3) is defined by material with dielectric constant $\varepsilon_{sub}$.

§3 Design of a resonant spaceplate

A Fabry-Pérot resonator spaceplate can be built similarly as shown in section §5 of SI as an air (free-space) cavity surrounded by high index dielectric sheets. Indeed, this approach was proposed in [14], where the authors suggested using thin sheets of material with $\varepsilon_r = 15$ operating near the quarter wavelength resonance to maximize the reflectance of the sheets.

Here, we use metallic hole arrays (described in the previous section) made of thin aluminium sheets to design the mirrors of the FP cavity. This approach has several practical advantages such as its low cost, reduced thickness and, most importantly, the tunability. By adjusting the filling ratio one can design a dichroic mirror with arbitrary reflectance. This in turn gives direct and continuous control over the Q-factor of the cavity, thus defining the compression ratio of the spaceplate.

The analytical model of the hole arrays that we use to determine their reflectance was described in the previous section. The model allows us to include the angle dependent reflectance of the mirrors in eq. (1). We verify the analytical design of the spaceplate with numerical simulations in a commercial finite element software, Ansys HFSS.

3.1 Numerical analysis of dichroic mirrors

We numerically analyze the hole array as a periodic structure with a unit cell shown in Fig. 11a) using Ansys HFSS. First, we sweep the frequency in the range 10-20 GHz (limited by the source antenna) and observe the reflectance (see Fig. 11c) of one sheet. Using these values, we can predict the compression ratio of a single FP resonator spaceplate made of two sheets, as a function of frequency. It is worth remembering that the compression ratio is solely a function of the reflectance of the mirrors (see eq. 11).
At operating frequency 17.7 GHz and normal incidence, the reflectance of the mirrors is about 0.79 which gives a theoretical spaceplate with compression factor of $C = 6.7$. Unfortunately, the reflectance of the mirrors changes with the incidence angle in a way that differs for the two polarization states (TE and TM). We plot this in Fig. 12 where we sweep the incidence angle in the range of $\theta = 0$ to 25 deg. It is obvious that the discrepancy in reflectance between the two polarizations increases with increasing the incidence angle - as we show later this directly influences the performance of the SP operating with the two polarizations by increasing the theoretical compression factor for TE waves while reducing it for the TM waves (see Fig. 4 of the main paper).
3.2 Analysis and design of the experimental spaceplate

Since the measurement distance between the source and the detector antennas is relatively large, we decided to use the second \( (d_{SP} = 1\lambda, \ell = 2) \) resonance of the FP spaceplate to double the effective distance \( d_{eff} \) the SP adds without resorting to the coupled FP cavities [14].

To simplify the fabrication of the spaceplate we decided to base it on commercially available perforated aluminium sheets. We used sheets from RS components [29] with lateral size 0.5x0.5 m², thickness 1.2 mm, square holes of 6x6 mm² and a period 7.6 mm, corresponding to the effective open area percentage 62% (see Fig. 10). The unit cell of the periodic structure is given in Fig. 11a.

We used 3D printed spacers with length of about 14.9 mm to keep the two sheets apart. However, as the sheets arrived substantially warped and bent to a right angle from the vendor, we did not manage to perfectly flatten them and a certain residual warping remained present. This in turn caused a rippled spectral line of the resonant spaceplate (see Fig. 13).
The simulated transmittance and the phase of the transmission coefficient are shown in Fig. 14b,c at four frequency points 17.7, 17.8, 17.9, and 18 GHz with $f_r = 17.7$ GHz. Both polarisations are shown in the figures together with a free-space fit. As a result of the unequal reflectances of the mirrors for the two polarisations, the phase curves of TE polarisation are below the free-space fit whereas phase for the TM polarisation is above it. This corresponds to a slightly higher compression ratio and a smaller NA for the TE polarisation compared to the TM case.

\[ \theta \text{ (deg)} \]

\[ R (-) \]

17.7 GHz

\[ 0 \quad 5 \quad 10 \quad 15 \quad 20 \quad 25 \]

\[ 0.75 \quad 0.76 \quad 0.77 \quad 0.78 \quad 0.79 \quad 0.8 \quad 0.81 \quad 0.82 \]

**FIG. 12:** Reflectance of the mirrors as a function of incidence angle for the TE and TM polarisation at 17.7 GHz.

**FIG. 13:** Three lowest simulated and measured resonance lines of the FP resonator spaceplate. We work with the $\ell = 2$ resonance at 17.7 GHz.

§4 Experimental Setup

The spaceplate is experimentally characterised in a spherical near field range according to Fig. 15. As is usual for such a range it is located inside an RF anechoic chamber.

The setup consists of two antennas - source and detector separated by a certain distance. The source antenna is placed on a roll over azimuth positioner which allows it to rotate about the centre of rotation. In this way the fields radiated by the source can be sampled on a spherical surface in a standard $\theta - \phi$ spherical coordinate system with $\hat{\theta} - \hat{\phi}$ polarization basis. To avoid confusion we denote these spherical angles $\alpha_1$ and $\alpha_2$ (and the polarisation basis $\hat{\alpha}_1 - \hat{\alpha}_2$) to distinguish them from the symbol for the plane-wave angle $\theta$ and the for phase $\phi$.

The spherical angle $\alpha_2$ corresponds to the roll and $\alpha_1$ to the azimuth rotation angle. The detector antenna is also mounted on a roll stage which is used to control the polarisation.

Wideband (2-20 GHz), linearly polarized, double ridge horn antennas are used as the source and the detector, and are connected to a vector network analyser (VNA). This enables us to capture both amplitude and phase of the signal as a function of the angle, frequency and polarisation.

4.1 Phase centre of an antenna and its determination

The phase centre (PC) of an antenna is the key concept in our experimental demonstration of a spaceplate and is thus described in more detail here. We explain how the phase centre can be calculated from measured phase patterns and we describe the way it can be used to characterise a spaceplate. We present raw measured data as well as
FIG. 14: a) Unit cell of the FP resonator spaceplate. b) Simulated transmission phase as a function of incidence angle at the resonance frequency 17.7 GHz and 3 other frequency points above the resonance. Compression factor of almost 7 is achieved. c) Transmittance of the SP at 4 frequency points. Dashed lines represent characteristics of free space with effective thickness $d_{\text{eff}}$. The behaviour of the SP depends on the polarisation state as the change of the reflectance of the mirrors with incidence angle is sensitive to the polarisation - see Fig. 12.

Extracted positions of phase centres in a setup with and without the spaceplate - the difference is directly related to the compression factor.

4.2 Phase of a real antenna vs. a point source
A phase centre is an imaginary point associated with an antenna that defines the origin of emanating spherical wavefronts as measured in antenna’s far field. For antenna measurement purposes the far field starts at approximately $2D^2/\lambda$, where $D$ is the largest dimension of the antenna and this corresponds to a phase error of 11° measured over the antenna’s aperture when illuminated by a spherical wavefront. Determining a phase centre of an antenna is one of the fundamental measurements in antenna metrology.

Radio antennas are not perfect point sources. Nevertheless, the wavefronts of the radiation in the angular range corresponding to the main beam tend to be almost perfect spheres and this assumption is valid provided we are measuring in the far field. We show this in the reference measurement of our double ridge waveguide horn (DRH) which we measured in configuration according to Fig. 15a. We numerically compensate for the offset of the antenna from the centre of rotation (see the following section). The results comparing the phase of our antenna with a perfect point source are given in Fig. 17. The reference measurement is then used in our normalization routine to determine the properties of the spaceplate.

4.3 Methodology of phase centre determination
Here, we briefly review how the phase centre of an antenna can be determined from the measured phase profile [17]. Since in the experiment we only work with a shift in the z-direction (along the optical axis), we simplify the explanation for our geometry. Following Fig. 15a, the setup includes 2 antennas – detector and source. The detector can revolve about the centre of rotation (in fact, in the actual experiment it is the source that is being rotated and the detector is kept stationary) and sample the fields radiated by the source. Using coherent detection with a vector network analyzer (VNA) we detect the amplitude and phase of the signal along the arc as a function of $\alpha$ on two orthogonal planes. The source antenna is misplaced from the centre of rotation by a vector $\vec{R}'$, the radius of curvature of the measurement arc is $R$, and the vector connecting the centre of rotation with the detector is $\vec{R}$. The difference
FIG. 15: Schematic view of the experimental setup. Fig. a) represents the reference measurement, where the field of the source antenna is measured on a spherical surface by the detector. Instead of revolving the detector about the source, we rotate the source about the centre of rotation by the spherical angles $\alpha_1$ and $\alpha_2$. In b) the spaceplate is introduced by mounting it onto the source antenna. The radiation of the source passing through the spaceplate as they both rotate is measured. All the dimensions are in millimeters and also relative to the free space wavelength at 17.8 GHz ($\lambda = 16.9 \text{ mm}$).

FIG. 16: Spaceplate fixed to the source antenna by four nylon rods keeping the distance between the spaceplate and the aperture of the horn at 240 mm.

The vector $|\vec{R} - \vec{R}'|$ as a function of the angle $\alpha_1$ defines the free-space propagation factor that is added to the phase of the source antenna as it revolves about the centre of rotation. Provided that the actual phase pattern of the antenna is $F(\alpha_1)$ we measure a different phase profile $F_M$ given as

$$F_M(\alpha_1) = -k_0 |\vec{R} - \vec{R}'| + F(\alpha_1),$$  \hspace{1cm} (24)
FIG. 17: Measured phase of the DRH antenna with respect to its phase centre (blue). And comparison to the phase that would be measured if the antenna were a perfect point source radiating perfect spherical wavefronts (red). The oscillation of ±5.5° amounts to a phase centre location ambiguity of ±255 µm at 18 GHz (λ = 16.7 mm)

where

\[ | \vec{R} - \vec{R}' | = R \left[ 1 - 2 \frac{\Delta_z \cos \alpha_1}{R} + \frac{(-\Delta_z)^2}{R^2} \right]^{1/2}, \]  

and \( \Delta_z \) is the distance between the antenna and the centre of rotation along the z-axis.

To determine the phase centre position of the source antenna means to find the \( R' \) (or \( \Delta_z \)) with respect to the origin (centre of rotation) from the measured \( F_M(\alpha_1) \).  

Now, if we wish to determine the phase centre with measured phase pattern \( F_M \), we optimize the position \( R' \) in eq. 24 in order to match the measured pattern \( F_M \) to the theoretical one \( F \). Since for point sources, the theoretical phase \( F'(\alpha_1) \) is a constant with \( \alpha_1 \), the process includes adding an angularly dependent phase shift \( k_0|\vec{R} - \vec{R}'| \) to the measured phase \( F_M(\alpha_1) \) to flatten the measured phase curve. The position \( R' \) which provides the maximum flatness is then considered to be the PC position (see the optimisation described in the following paragraph).

Provided the source antenna radiates almost spherical waves (point-like source) at least within the main lobe (in our case within the NA of the spaceplate ±15 deg which is much smaller than the half-power beamwidth of the source antenna), we can assume the phase \( F \) to be constant with angle (the radius of curvature of our measurement arc would be aligned with the radius of the wavefront). Thus, we can find the location of the phase centre through optimization with the following objective function:

\[ OBJ = \min \{ \sigma_{\text{Phase}} \}, \]  

with

\[ \sigma_{\text{Phase}} = \sqrt{ \frac{1}{N_{\alpha_1}} \sum_{\alpha_1,i} [ (F_M(\alpha_{1,i}) + k_0|\vec{R} - \vec{R}'|) - \text{mean}(F_M(\alpha_{1,i}) + k_0|\vec{R} - \vec{R}'|) ]^2 } \],

where \( N_{\alpha_1} \) is the number of angular positions along the arc at which we measure the phase. Objective function corresponds to a minimisation of the standard deviation of a difference between the measured phase \( F_M(\alpha_{1,i}) \) and a theoretical phase of a point source misaligned by \( \vec{R}' \) from the centre of the measurement arc.

If we restrict ourselves to 1D phase centre misalignment (e.g. in the z-direction) we can easily evaluate the objective function with \( \vec{R}' \) being the optimization variable (\( \Delta_z \) to be more specific, see eq. 25). In such a case, the value \( \Delta_z \) where the minimum of the objective function occurs determines the phase centre position of the source antenna. The value of the objective function \( \sigma_{\text{Phase}} \) at its minimum is then related to the quality of the phase centre (e.g. quality of the focus) and could be understood/assessed through the theory of optical aberrations. Zero objective value \( \sigma_{\text{Phase}} = 0 \)° at some arbitrary \( \Delta_z \) corresponds to a point source antenna being misplaced by \( \Delta_z \) with respect to the centre of the measurement arc.
4.4 Measured phase and the curvature of wavefronts

During the measurement (see Fig. 15), we measured the phase \( \text{arg } S_{21} \) of the DRH antenna as a function of angle in the angular range \( \pm 25 \text{ deg} \) in a setup without (reference measurement) and with the spaceplate. It is essential to notice, the source antenna whose pattern we are measuring, is not placed in the centre of rotation. This also means that our measurement surface (arc in 1D case) has a different radius of curvature than the wavefronts and the centre of the measurement arc (centre of rotation) and the antenna’s position do not coincide.

To relate the measured phase to the apparent position of the source (its PC), we show how the measured phase \( F_M(\alpha_1) \) is supposed to behave as a function of its misplacement from the centre of rotation. To do this we consider our source to radiate perfect spherical waves with origin at position \( \vec{R}' \) (for axial shift only \( \vec{R}' = \vec{\Delta}_z \)). The frequency considered is 17.7 GHz (to match the experiment) and the wavelength is 16.9 mm. We start with \( \Delta_z = 0 \) mm where the measured phase is constant (measuring on a wavefront). As the point source is shifted away from the centre of rotation \( \Delta_z \) is increasing, the curvature of the measured phase is being increased according to eq. 25. This shows that if the source is moved away from the centre of rotation the effect is a steeper measured phase pattern with the angle \( \alpha_1 \). This is shown in Fig. 3b of the main paper and is repeated here in Fig.18.

The apparent shift of the phase center of the source, i.e. the apparent movement of the source place behind the spaceplate farther away from it is a direct consequence of its function and we use it for the experimental demonstration.

![FIG. 18: a) Geometry to show the theoretical behaviour of the phase \( \phi \) as a function of its misplacement \( \vec{\Delta}_z \). In the case of axial misplacement along the z-direction only, it follows \( \vec{R}' = \vec{\Delta}_z \). b) As the shift \( \Delta_z \) increases, the phase gets steeper.](image)

4.5 Measurement results - raw phase data, extracted phase centre and the compression factor

We present here the measured transmission phase \( \phi_e \) which corresponds to the phase measured at points on the measurement arc (see Fig. 18). Left hand columns in Fig. 19 and 20 show the unwrapped phase as a function of angle \( \alpha_1 \) for the measurement without the spaceplate (blue curves) and with the spaceplate (red curves). The measurements shown in Fig. 19 and 20 correspond to two orthogonal planes where the polarisation of the wave is either TM or TE with respect to the spaceplate, respectively. As a result of the apparent shift of the phase centre away from the centre of measurement arc, we measure a steeper phase profile when the SP is introduced.

The right hand column in Fig. 19 and 20 shows the locations of phase centres determined by optimisation of the phase profiles given in the left hand column. The quantity on the vertical axes is the standard deviation \( \sigma_{\phi_e} \) (given by eq. 27) between the measured phase and the phase of a point source with on offset \( \Delta_z \) with respect to the centre of rotation. The position \( \Delta_z \), where \( \sigma_{\phi_e} \) is minimised (eq. 26) is the location of the phase centre.

As the lateral size of the experimental spaceplate was 500x500 mm², we did not perform the measurement in the far-field of the SP (starts at about 28 m). However, for the definition of the far field distance, it is important to consider the non-local nature of the spaceplate in our measurement. A perfect spaceplate does not introduce any local effects, such as edge diffraction and scattering. Therefore the far field distance is not determined by the size of the SP but rather by the size of the source illuminating the SP provided the spaceplate has sufficient lateral size to avoid diffraction. The assumption is also valid if the amount of power scattered from the local (parasitic) sources is relatively small and we dominantly measure the properties of the source as appearing through the spaceplate. This is a common practice in free space material characterization, where a focused beam is used to measure properties of a dielectric slab of unknown refractive index [31].
| Frequency (GHz) | TM polarisation |
|----------------|----------------|
|                | Measured Phase - arg (S_{21}) | Phase centre shift |
| 17.7 GHz       | ![Graph](image1) | ![Graph](image2) |
| 17.8 GHz       | ![Graph](image3) | ![Graph](image4) |
| 17.9 GHz       | ![Graph](image5) | ![Graph](image6) |
| 18.0 GHz       | ![Graph](image7) | ![Graph](image8) |
| 18.1 GHz       | ![Graph](image9) | ![Graph](image10) |

FIG. 19: Measured phase $\phi_e$ of the reference DRH antenna (blue) and of the DRH antenna behind the spaceplate (red) as a function of the rotation angle $\alpha_1$ for TM polarisation at 5 frequency points. In the right hand column, the position $\Delta_z$ of the minimum $\sigma_{Phase}$ determines the phase centre location. The path length apparently added by the spaceplate is shown in each figure of the right hand column.
FIG. 20: Measured phase $\phi_e$ of the reference DRH antenna (blue) and of the DRH antenna behind the spaceplate (red) as a function of the rotation angle $\alpha_1$ for TM polarisation at 5 frequency points. In the right hand column, the position $\Delta z$ of the minimum $\sigma_{\text{Phase}}$ determines the phase centre location. The path length apparently added by the spaceplate is shown in each figure of the right hand column.
4.6 Electric field measurement results on a 2D surface

The near-field measurement setup (see section §4 of SI and 15) allowed us to sample the normalised electric field on a 2D spherical surface in spherical coordinates, where the quantity being directly measured was the scattering coefficient $S_{21}$. The range of spherical coordinates corresponded to $\alpha_1$ (0-25°) and $\alpha_2$ (0-360°) and we measured two orthogonal polarisations. We use the $\alpha_1$ and $\alpha_2$ notation for the spherical angles instead of the conventional $\theta$, $\phi$ to distinguish them from the angle of plane-wave propagation and the phase, where we already use greek symbols for $\theta$ and $\phi$.

The raw data from such a measurement are not very intuitive to interpret because of the rotating nature of the polarisation basis vectors in the angular measurement [19] - i.e. the direction of unit vectors $\hat{\alpha}_1$, $\hat{\alpha}_2$ at certain point $(\alpha_1, \alpha_2)$ depends on the location of this point.

It is convenient to use a different polarisation basis for plotting 2D fields, such as Ludwig III [18] where we define the two components as co-polar and cross-polar. The co-polar field component preserves the polarisation matching between the source and the detector antennas for all the points on the spherical surface.

The transformation from the spherical (i.e. components $E_{\alpha_1}, E_{\alpha_2}$) to the Ludwig III polarisation basis ($E_{\text{co}}, E_{\text{cross}}$) is defined as:

$$
\begin{bmatrix}
E_{\text{co}} \\
E_{\text{cross}}
\end{bmatrix} =
\begin{bmatrix}
\cos \alpha_2 & -\sin \alpha_2 \\
\sin \alpha_2 & \cos \alpha_2
\end{bmatrix}
\begin{bmatrix}
E_{\alpha_1} \\
E_{\alpha_2}
\end{bmatrix}
$$

Furthermore, it is customary to plot the measured 2D field as a function of the so called direction cosines $u$, $v$ which are related to the coordinates of the spherical coordinate system $\alpha_1$, $\alpha_2$ as follows:

$$
u = \sin \alpha_1 \cos \alpha_2$$

$$
v = \sin \alpha_1 \sin \alpha_2$$

During a far-field measurement, the direction cosines would be related to the transverse components of the wavevector as $u = k_x/|\mathbf{k}|$, $v = k_y/|\mathbf{k}|$.

The Fig. 5 of the main paper presents the amplitude and phase of the measured co-polarised E-field on a 2D surface as a function of the above described direction cosines. The increase in the curvature of the measured phase is obvious. As explained earlier, this corresponds to an increasing radius of curvature of the wavefronts compared to the radius of the measurement sphere.

§5 Evolutionary optimization of a stochastic spaceplate

A non-local metamaterial spaceplate can also be realised as a multilayer stack of homogeneous and isotropic layers distributed along the optical axis [10]. The parameters of individual layers – the thicknesses and refractive indices – are then optimised by a stochastic optimisation algorithm. Such a design strategy does not inherently rely on any knowledge about the behavior of the elements. Theoretically (if the search space allows it), it can take advantage of e.g. higher order modes within layers, creating multiple resonant cavities coupled by arbitrary coefficients (unlike the design in [3]) etc. However, the results of this approach presented in [10] tell us that finding a feasible solution can be quite challenging (for example in [10], with 21 layers and $R = 4.9$, a spaceplate with an angular range of about 12-15 deg showed transmittance below $\sim 1.5\%$ for normal incidence). The other disadvantage is the non-scalability of the solution. Designing the spaceplate for other effective thicknesses requires, in general, starting the optimization from a scratch. A suitable optimisation strategy could be the combination of the global search with a local gradient-based optimization method.

The problem can be formulated as either a single- or multi-objective. In our experience, the single-objective formulation with three discernable sub-objectives (amplitude, phase, compression factor $C$) led to a feasible solution more quickly. Generally, the objective is defined as the minimisation of the error between the desired transmission coefficient of the free space $t_0$ and a transmission coefficient of a spaceplate $t_{SP}$, while the compression factor $C$ should be maximized. If we break the transmission coefficient into the amplitude and the phase derivative criterion and we sum through a finite number of incidence angles we can write for both polarisations (superscripts TE, TM)
\[
OBJ = \frac{1}{N} \sum_{\theta_i}^N c_1 \cdot (|t_{SP}^{TM}(\theta_i)| - |t_0(\theta_i)|)^2 + c_2 \left( \frac{\partial \arg(t_{SP}^{TM}(\theta_i))}{\partial \theta} - \frac{\partial \arg(t_0(\theta_i))}{\partial \theta} \right)^2 +
\]
\[
\frac{1}{N} \sum_{\theta_i}^N c_3 \cdot (|t_{SP}^{TE}(\theta_i)| - |t_0(\theta_i)|)^2 + c_4 \left( \frac{\partial \arg(t_{SP}^{TE}(\theta_i))}{\partial \theta} - \frac{\partial \arg(t_0(\theta_i))}{\partial \theta} \right)^2 +
\]
\[c_5 \cdot d_{SP}/d_{\text{eff}}\]

Here, the derivatives are used to remove any constraints on the global phase of the transmitted field, which is irrelevant for most of the spaceplate applications. The five weighing coefficients $c_1$ - $c_5$ allow us to tune the importance of the amplitude ($c_1, c_3$) and phase ($c_2, c_4$) errors and the maximum achieved compression factor ($1/c_5$). The inputs of the algorithm are the maximum number of the layers and the effective thickness $d_{\text{eff}}$ which we are trying to squeeze. During the optimisation, the transmission coefficient of the free-space is pre-calculated (as it does not change during the optimisation) and the transmission coefficient of the SP is evaluated by a transfer (characteristic) matrix method [32], which gives a full-wave field solution for a multilayer stack.

In our design, we tried to reduce the search space as much as possible while still producing a solution with relatively high $C$ (in this case we set $C > 3$). With the experimental demonstration in mind, we opted for a two material combination with alternating high/low refractive index medium – as the low $n$ medium, we selected air and for the high $n$ medium a commercially available microwave substrate from Rogers ($\varepsilon_r = 10.2$). We fixed the thicknesses of the high $n$ dielectric layers to 1.52 mm (thickness of the substrate). Thus, the optimization parameters are only the thicknesses of the air gaps and the total number of the layers. By allowing the thicknesses of the layers to go to zero (skipping the layer), the optimiser can effectively double the maximum thickness limitations defined during the initialisation.

In the optimisation results below, we set the frequency of operation to $f = 15$ GHz, maximum number of layers ($NL_{\text{max}}$) to 15, the effective thickness to be substituted by SP to $d_{\text{eff}} = 330$ mm. An evolutionary optimization strategy based on a genetic algorithm in Matlab was used to search for feasible solutions. With maximum of 15 layers (8 layers of microwave substrate of fixed thickness, separated by 7 layers of air), we are optimizing only 7 parameters allowing the thicknesses of the air gaps and the total number of the layers. By allowing the thicknesses of the layers to go to zero (skipping the layer), the optimiser can effectively double the maximum thickness limitations defined during the initialisation.

One of the optimised structures is shown in Fig. 21. We can see that the first, third and fifth gaps have lengths very close to the even FP resonances whereas the second and the fourth are relatively small. The resemblance of this stochastically optimised structure to the empirical multi-cavity design proposed in [14] is clear.

Figure 22 gives the transmittance $|t|^2$ and the phase of the transmission coefficient for the two polarisations and compares them to the free-space fit. In Fig. 23 and 24 we can see the transmittance and the transmission phase as functions of the frequency and the incidence angle.

![Diagram](image_url)

FIG. 21: Spaceplate designed by a genetic algorithm with total thickness of $d_{SP} = 67.44$ mm substitutes a slab of air with equivalent thickness $d_{\text{eff}} = 330$ mm corresponding to a compression factor $C \approx 4.9$. 

\[
d_1 = 1.52 \text{ mm}
\]
\[
d_2 = 20.52 \text{ mm} = 1.026 \lambda_0
\]
\[
d_3 = 3.27 \text{ mm} = 0.1635 \lambda_0
\]
\[
d_4 = 20.85 \text{ mm} = 1.042 \lambda_0
\]
\[
d_5 = 3.12 \text{ mm} = 0.156 \lambda_0
\]
\[
d_6 = 10.56 \text{ mm} = 0.528 \lambda_0
\]
FIG. 22: Transmittance and the transmission phase of the spaceplate as a function of incidence angle at 15 GHz. The free space fit corresponds to a distance $d_{eff} = 330$ mm.

FIG. 23: Transmittance and the phase of the spaceplate as a function of incidence angle and frequency for TE polarisation.

FIG. 24: Transmittance and the phase of the spaceplate as a function of incidence angle and frequency for TM polarisation.