HIGHER-DERIVATIVE SCALAR FIELD THEORIES
AS CONSTRAINED SECOND-ORDER THEORIES

by

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ABSTRACT

As an alternative to the covariant Ostrogradski method, we show that higher-derivative relativistic Lagrangian field theories can be reduced to second differential-order by writing them directly as covariant two-derivative theories involving Lagrange multipliers and new fields. Notwithstanding the intrinsic non-covariance of the Dirac’s procedure used to deal with the constraints, the Lorentz invariance is recovered at the end. We develop this new setting for a simple scalar model and then its applications to generalized electrodynamics and higher-derivative gravity are outlined. This method is better suited than Ostrogradski’s for a generalization to 2n-derivative theories.

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1. Introduction

Historically, field theories with higher order Lagrangians range from Higgs model regularizations [1] to generalized electrodynamics [2][3] and higher-derivative (HD) gravity [4]. A procedure was later devised to reduce them, by a Legendre transformation, to equivalent second-order theories [5] where a subsequent diagonalization explicitly displays the particle degrees of freedom [3][6][7].

The validity of the formal Lorentz covariant order-reducing method adopted there has been checked in an example of scalar HD theories by a rigorous study of the phase-space [7]. In this procedure, a generalization of the Ostrogradski formalism to continuous relativistic systems (2n-derivative because of Lorentz invariance) is carried out. In it, some of the field derivatives and the generalized conjugate momenta become, after a suitable diagonalization, new field coordinates describing the degrees of freedom (DOF) which were already identified in the particle propagators arising in the algebraic decomposition of the HD propagator.

By using Lagrange multipliers, a variant of the Ostrogradski method for mechanical discrete systems has been proposed which allows to show the quantum (Path Integral) equivalence between the modified action principle (first order Helmholtz Lagrangian) and the starting HD theory [8]. For relativistic field systems, a similarly inspired procedure can be followed in which the multipliers let to write the HD theory from the outset as a second order (constrained) covariant one which lends itself to a particle interpretation after diagonalization.

In this paper we implement this new setting by means of the use of Lagrange multipliers in a Lorentz invariant formulation of a relativistic scalar field theory. The Dirac method [9] prescribes the identification of the primary constraints arising in the definition of the momenta. These constraints are added to the starting Hamiltonian by means of new multipliers, and then they are required to be conserved by the time evolution driven by this enlarged ”total Hamiltonian” through the Poisson Bracket (PB). This may give rise to secondary constraints, the conservation of which can in turn generate more secondary constraints. The process stops when constraints are obtained that can be solved by fixing the values of the new multipliers. We then use the remaining constraints to eliminate the starting multipliers and the momenta, ending up with a two-derivative theory depending on just its true DOF. Since the latter appear mixed, a diagonalization works finally out the independently propagating DOF.

As long as time evolution is analyzed, the true mechanical Hamiltonian (i.e. the energy) of the system must be used. Then one cannot benefit of the compactness of the Lorentz invariant procedures introduced in [7], so we are forced to deal with non covariant objects and face the diagonalization of larger matrices. The relativistic invariance of the system is recovered at the end of the process.

From the methodological point of view, the new treatment of HD scalar theories that we present here provides a sharp departure from the more traditional Ostrogradski approach. Moreover, it is implementable and may prove advantageous in
generalized electrodynamics, HD Yang-Mills and linearized HD gravity as well. On the other hand, even if the explicit calculation already in the scalar case becomes practically intractable beyond six-derivative order, we have partial results that lend themselves to generalization to arbitrary $n$ better than the Ostrogradski method does.

In Section 2 we treat $n = 3$, the case $n = 2$ being too much trivial for the illustrative purposes we pursue here. Some results regarding the extension to arbitrary $n$ are presented in Section 3 which, with the help of a plausible ansatz, allows the explicit calculation for $n = 4$. Section 4 discusses some possible applications of the approach to more relevant vector and tensor field theories. The Conclusions are in Section 5. An Appendix is devoted to a general inductive proof of the pure algebraic character (i.e. absence of space derivatives) of the secondary constraints.
2. $n = 3$ theory.

We consider the six-derivative Lagrangian,

$$L^6 = -\frac{1}{2} \frac{\mu^2}{M} \phi[1][2][3] \phi - j \phi \ ,$$

(2.1)

where $\mu$ is an arbitrary mass parameter, $[i] \equiv (\square + m_i^2)$ are KG operators, $M \equiv \langle 12 \rangle \langle 13 \rangle \langle 23 \rangle$, $\langle ij \rangle \equiv m_i^2 - m_j^2 > 0$ for $i < j$, and mass dimensions $[\mu] = 1$, $[M] = 6$, $[\phi] = 1$, $[j] = 3$.

As discussed in [7], (2.1) displays the general form of the free part of a higher-derivative scalar theory with nondegenerate masses $m_1$, $m_2$, $m_3$, the source term embodying the remaining self interactions and the couplings to other fields.

$L^6$ can be reexpressed directly as a second-order theory with constraints, namely

$$L^6 = \frac{1}{2} \frac{\mu^2}{M} \left[ -\bar{\psi}_3 [1] \bar{\psi}_1 + \lambda_1 (\bar{\psi}_1 - [2] \bar{\psi}_2) + \lambda_2 (\bar{\psi}_2 - [3] \bar{\psi}_3) \right] - j \bar{\psi}_3 \ ,$$

(2.2)

where $\bar{\psi}_3 = \phi$ and $\lambda_1$, $\lambda_2$ are Lagrange multipliers, so that (2.2) depends on five fields. Dropping total derivatives, in compact matrix notation, (2.2) reads

$$L^6 = \frac{1}{2} \dot{\Psi}^T K \dot{\Psi} + \frac{1}{2} \Psi^T M \Psi - J^T \Psi \ ,$$

(2.3)

where the vectors $\Psi$ and $J$, with components $\psi_i$, $J_i$, and the matrices $K$ and $M$ are

$$\Psi = \begin{pmatrix} \mu^{-4} \bar{\psi}_1 \\ \mu^{-2} \bar{\psi}_2 \\ \bar{\psi}_3 \\ \mu^{-2} \lambda_1 \\ \mu^{-4} \lambda_2 \end{pmatrix} \text{ so that } [\psi_i] = 1 \ i = 1, \ldots, 5 \ ; \ J_i = j \delta_{31} \ ;$$

(2.4)

$$K = \frac{\mu^6}{2M} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \ ; \ M = \frac{\mu^6}{2M} \begin{pmatrix} 0 & 0 & -M_1^2 & \mu^2 & 0 \\ 0 & 0 & 0 & -M_2^2 & \mu^2 \\ -M_1^2 & 0 & 0 & 0 & -M_4^2 \\ \mu^2 & -M_2^2 & 0 & 0 & 0 \\ 0 & \mu^2 & -M_3^2 & 0 & 0 \end{pmatrix} .$$

$M$ is an operator with space derivatives present in $M_i^2 \equiv m_i^2 - \Delta$. 


The canonical conjugate momenta are defined as
\[ \pi_i = \frac{\partial L^6}{\partial \dot{\psi}_i} . \] (2.5)
They are the components of a 5-vector \( \Pi \) for which one has
\[ \Pi = \mathcal{K} \dot{\Psi} . \] (2.6)

Since \( \mathcal{K} \) is not invertible, not all the velocities \( \dot{\psi}_i \) can be expressed in terms of the momenta and a primary constraint occurs, namely
\[ \Omega_1 \equiv \pi_5 - \pi_1 = 0 , \] (2.7)
as consequence of \( \pi_5 = \frac{\mu^6}{2M} \dot{\psi}_3 = \pi_1 \). There is only one such a constraint since the submatrix \( \mathcal{K}_{ab} \equiv \frac{\mu^6}{2M} \mathcal{K}_{ab}^\prime \) \( (a, b = 1, \ldots, 4) \) is regular. In the following, indices \( a, b, \ldots \) go from 1 to 4, while \( i, j, \ldots \) go from 1 to 5. The velocity \( \dot{\psi}_5 \) is not worked out, and from (2.6) we have
\[ \pi_a = \frac{\mu^6}{2M} \mathcal{K}_{ab}^\prime \dot{\psi}_b + \frac{\mu^6}{2M} \delta_{a3} \dot{\psi}_{a+2} \] (2.8)
\( \text{(do not sum over } a) \), and therefore
\[ \dot{\psi}_a = \frac{2M}{\mu^6} \mathcal{K}_{ab}^\prime \pi_b - \delta_{a1} \dot{\psi}_5 \] (2.9)

The Hamiltonian is
\[ \mathcal{H} = \pi_a \dot{\psi}_a + \pi_5 \dot{\psi}_5 - \frac{1}{2} \psi_a \mathcal{K}_{ab} \dot{\psi}_b - \frac{\mu^6}{2M} \dot{\psi}_3 \dot{\psi}_5 - \frac{1}{2} \psi_i \mathcal{M}_{ij} \psi_j + j \psi_3 \] (2.10)
where \( \dot{\psi}_a \) must be substituted according to (2.9). Then the dependence on \( \dot{\psi}_5 \) cancels out and we have
\[ \mathcal{H} = \frac{1}{2} \frac{2M}{\mu^6} \pi_a \mathcal{K}_{ab}^\prime \pi_b - \frac{1}{2} \psi_i \mathcal{M}_{ij} \psi_j + J_i \psi_i \] (2.11)

In (2.11) only four momenta appear, together with the five fields \( \psi_i \); not all of the five momenta \( \pi_i \) are independent because of the primary constraint (2.7). The "total Hamiltonian", with five independent momenta, accounting for this is
\[ \mathcal{H}_T = \mathcal{H} + \zeta \Omega_1 \] (2.12)
where \( \zeta \) is a Lagrange multiplier.
The stability of $\Omega_1$ requires

$$\dot{\Omega}_1 = \{\Omega_1, H_T\}_P \equiv \Omega_2 = \frac{\mu^6}{2M} \langle (13) \psi_3 - \mu^2 \psi_4 + \mu^2 \psi_2 \rangle = 0 \ .$$  \hspace{1cm} (2.13)$$

This secondary constraint yields

$$\psi_4 = \frac{\langle 13 \rangle}{\mu^2} \psi_3 + \psi_2 \ .$$  \hspace{1cm} (2.14)$$

Further secondary constraints stem from the ensuing stability conditions

$$\dot{\Omega}_2 = \{\Omega_2, H_T\}_P \equiv \Omega_3 = \langle (13) \pi_1 - \mu^2 \pi_2 + \mu^2 \pi_4 \rangle = 0 \ ,$$  \hspace{1cm} (2.15)$$

so

$$\pi_4 = \pi_2 - \frac{\langle 13 \rangle}{\mu^2} \pi_1 \ ,$$  \hspace{1cm} (2.16)$$

and again

$$\dot{\Omega}_3 = \{\Omega_3, H_T\}_P \equiv \Omega_4 = \frac{\mu^6}{2M} \langle - (13) \langle 23 \rangle \psi_3 - \mu^2 \langle 13 \rangle \psi_2 - \mu^4 \psi_1 + \mu^4 \psi_5 \rangle = 0 \ ,$$  \hspace{1cm} (2.17)$$

(once (2.14) has been used), from which one gets

$$\psi_5 = \frac{\langle 13 \rangle \langle 23 \rangle}{\mu^4} \psi_3 + \frac{\langle 13 \rangle}{\mu^2} \frac{\psi_2 + \psi_1}{\psi_2 + \psi_1} \ .$$  \hspace{1cm} (2.18)$$

The next constraint, after using (2.16), gives

$$\dot{\Omega}_4 = \{\Omega_4, H_T\}_P \equiv \Omega_5 = \langle 13 \rangle \langle 12 \rangle \pi_1 - \mu^2 \langle 13 \rangle \pi_2 - \mu^4 \pi_3 + 2 \frac{\mu^6}{2M} \mu^4 \zeta = 0 \ ,$$  \hspace{1cm} (2.19)$$

and $\zeta$ can be obtained as a function of $\pi_1$, $\pi_2$ and $\pi_3$, thus bringing the generation of secondary constraints to an end.

$H_T$ being quadratic in $\psi$’s and $\pi$’s, guarantees an alternance of linear constraints involving the fields and the momenta. In spite of the occurrence of space derivatives in $\mathcal{M}$, they cancel out and the constraints are algebraic. From this set of constraints, the multipliers $\psi_4$ and $\psi_5$, together with their conjugate momenta $\pi_4$ and $\pi_5$, can be worked out.
The Lagrangian (2.3) can be expressed in terms of the independent variables $\psi_\alpha \ (\alpha = 1, 2, 3)$. Notice that implementing these constraints in $L^6$ is legitimate as long as this operation does not erase the dependence of $L^6$ on the other variables. One obtains

$$L^6 = \dot{\psi}_\alpha \bar{K}_{\alpha\beta} \dot{\psi}_\beta + \psi_\alpha \bar{M}_{\alpha\beta} \psi_\beta - j \psi_3, \quad (2.20)$$

where

$$\bar{K}_{\alpha\beta} \equiv \frac{1}{2} (K_{\alpha\beta} + K_{\alpha B} N_{B\beta} + N_{\alpha A} K_{A\beta}) = \frac{\mu^6}{2M} \begin{pmatrix} 0 & 0 & \frac{1}{2M} \\ 0 & 1 & \frac{\langle 13 \rangle}{\mu^2} \\ 1 & \frac{\langle 13 \rangle}{\mu^2} & \frac{\langle 13 \rangle \langle 23 \rangle}{\mu^4} \end{pmatrix}, \quad (2.21)$$

and

$$\bar{M}_{\alpha\beta} \equiv \frac{1}{2} (M_{\alpha\beta} + M_{\alpha B} N_{B\beta} + N_{\alpha A} M_{A\beta}) = \frac{\mu^6}{2M} \begin{pmatrix} 0 & \mu^2 & -M_3^2 \\ \mu^2 & \langle 13 \rangle - M_2^2 & -\frac{\langle 13 \rangle}{\mu^2} M_3^2 \\ -M_3^2 & \frac{\langle 13 \rangle}{\mu^2} - \frac{\langle 13 \rangle \langle 23 \rangle}{\mu^4} M_3^2 \\ \langle 13 \rangle & \frac{\langle 13 \rangle \langle 23 \rangle}{\mu^4} M_3^2 \end{pmatrix}, \quad (2.22)$$

with $\alpha, \beta, \ldots = 1, 2, 3$; $A, B, \ldots = 4, 5$; and

$$N_{A\beta} \equiv \begin{pmatrix} 0 & 1 \frac{\langle 13 \rangle}{\mu^2} \\ 1 & \frac{\langle 13 \rangle}{\mu^2} \frac{\langle 13 \rangle \langle 23 \rangle}{\mu^4} \end{pmatrix}, \quad (2.23)$$

that allows to embody (2.14) and (2.18) in the closed form

$$\psi_A = N_{A\beta} \psi_\beta. \quad (2.24)$$

The symmetric matrices $\bar{K}$ and $\bar{M}$ can be simultaneously diagonalized by the regular transformation

$$\psi_\alpha = R_{\alpha\beta} \phi_\beta, \quad (2.25)$$

where

$$R_{\alpha\beta} \equiv \begin{pmatrix} \frac{\langle 12 \rangle \langle 13 \rangle}{\mu^4} & 0 & 0 \\ \frac{\langle 13 \rangle}{\mu^2} & \frac{\langle 13 \rangle \langle 23 \rangle}{\mu^4} & 0 \\ 1 & 1 & 1 \end{pmatrix}. \quad (2.26)$$

The 3rd. row of the non-orthogonal matrix $R$ in (2.26), has been chosen so as to yield the correct source term in (2.29) (see below). The remaining six elements are uniquely determined by requiring $R$ to diagonalize $\bar{K}$ and $\bar{M}$.
The diagonalized matrices are

\[ R^T \tilde{K} R = \frac{\mu^6}{2M} \text{diag} \left( \frac{\langle 12 \rangle \langle 13 \rangle}{\mu^4}, -\frac{\langle 12 \rangle \langle 23 \rangle}{\mu^4}, -\frac{\langle 13 \rangle \langle 23 \rangle}{\mu^4} \right) , \]  

(2.27)

\[ R^T \tilde{M} R = \frac{\mu^6}{2M} \text{diag} \left( -M_1^2 \frac{\langle 12 \rangle \langle 13 \rangle}{\mu^4}, M_2^2 \frac{\langle 12 \rangle \langle 23 \rangle}{\mu^4}, -M_3^2 \frac{\langle 13 \rangle \langle 23 \rangle}{\mu^4} \right) , \]  

(2.28)

so that (2.20) finally writes

\[ \mathcal{L}^6 = -\frac{1}{2} \frac{\mu^2}{\langle 23 \rangle} \phi_1 [1] \phi_1 + \frac{1}{2} \frac{\mu^2}{\langle 13 \rangle} \phi_2 [2] \phi_2 - \frac{1}{2} \frac{\mu^2}{\langle 12 \rangle} \phi_3 [3] \phi_3 - j \phi_1 + \phi_2 + \phi_3 \] .  

(2.29)

This shows again the Ostrogradski-based result [7] of the equivalence between the second-order theory (2.1) and the LD version (2.29) that reproduces the propagator structure.
3. Theories with arbitrary \( n \).

The general Lagrangian

\[
L^{2n} = -\frac{1}{2 M} \phi [1][2] \ldots [n] \phi - j \phi ,
\]

where \( M \equiv \prod \langle ij \rangle \), and \( \beta = n(n-3) + 2 \) for dimensional convenience, can be dealt with along similar lines. The 2-derivative constrained recasting of (3.1) is

\[
L^{2n} = \frac{1}{2 M} \left[ -\bar{\psi}_n [1] \psi_1 + \lambda_1 (\bar{\psi}_1 - [2] \bar{\psi}_2) + \ldots + \lambda_{n-1} (\bar{\psi}_{n-1} - [n] \bar{\psi}_n) \right] - j \bar{\psi}_n ,
\]

with \( \bar{\psi}_n \equiv \phi \), and \( \lambda_1, \ldots, \lambda_{n-1} \) being Lagrange’s multipliers. In order to have a more compact notation we define

\[
\psi_\alpha = \mu^{-2(n-\alpha)} \bar{\psi}_\alpha \quad \alpha = 1, \ldots, n
\]

\[
\psi_A = \mu^{-2\alpha} \lambda_\alpha \quad A = n + \alpha ; \quad \alpha = 1, \ldots, n - 1
\]

so that \( [\psi_i] = 1 \) \((i = 1, \ldots, 2n-1)\). Then

\[
L^6 = \frac{1}{2} \dot{\Psi}^T K \dot{\Psi} + \frac{1}{2} \Psi^T M \Psi - J^T \Psi
\]

with \( J_i = j \delta_{in} \), and the \((2n-1) \times (2n-1)\) matrices \( K \) and \( M \) are given by

\[
K_{ij} \equiv \sigma (\delta_{i,j-n+1} + \delta_{j,i-n+1})
\]

\[
M_{ij} \equiv \sigma \left[ -(M_1^2 \delta_{i,j-n+1} + M_2^2 \delta_{j,i-n+1}) + \mu^2 (\delta_{i,j-n} + \delta_{j,i-n}) \right]
\]

with \( \sigma \equiv \frac{\mu^{n(n-1)}}{2 M} \). This ”mass” matrix contains again space derivatives. Here, as before in (2.8) and in the following, an underlined index means that Einstein summation convention does not apply. The canonical conjugate momenta are now

\[
\pi_i = \frac{\partial L^{2n}}{\partial \dot{\psi}_i} ,
\]

i.e., in closed notation,

\[
\Pi = K \dot{\Psi} .
\]

Defining the matrix \( K' \)

\[
K'_{ab} = \frac{1}{\sigma} K_{ab} \quad (a, b = 1, \ldots, 2n-2)
\]

one sees that \( \det K' \neq 0 \), while \( \det K = 0 \). This means that we only have one primary constraint, namely

\[
\Omega_1 \equiv \pi_{2n-1} - \pi_1 = 0 .
\]
Then $\dot{\psi}_{2n-1}$ is not worked out, while $\dot{\psi}_a \ (a = 1, \ldots, 2n - 2)$ can be expressed in terms of $\pi_a$ and $\dot{\psi}_{2n-1}$. The first $2n - 2$ components of eq.(3.7), namely

$$\pi_a = \sigma K_{ab} \dot{\psi}_b + \sigma \delta_{an} \dot{\psi}_{a+n-1} \quad (3.10)$$

give

$$\dot{\psi}_a = \frac{1}{\sigma} K_{ab} \pi_b - \delta_{a1} \dot{\psi}_{2n-1} \quad (3.11)$$

After checking that the terms in $\dot{\psi}_{2n-1}$ cancel out, the Hamiltonian has the simple expression

$$\mathcal{H} = \frac{1}{2} \sigma \pi_a K_{ab} \pi_b - \frac{1}{2} \psi_i M_{ij} \psi_j + j \psi_n \quad (3.12)$$

In $\mathcal{H}$ only $2n - 2$ momenta $\pi_a$ occur against $2n - 1$ fields $\psi_i$, because of the primary constraint (3.9). One may restore the dependence on $2n - 1$ momenta by introducing the "total Hamiltonian"

$$\mathcal{H}_T = \mathcal{H} + \zeta \Omega_1 \quad , \quad (3.13)$$

where $\zeta$ is a Lagrange multiplier.

From the stability condition on $\Omega_1$, a cascade of secondary constraints follows, eventually ending with an equation that determines the value of $\zeta$. We outline here the steps closely following the lines of section 2.

$$\hat{\Omega}_1 = \{ \Omega_1, \mathcal{H}_T \} \equiv \Omega_2 = 0 \quad \Rightarrow \quad \psi_{n+1} = \psi_{n-1} + \frac{(1n)}{\mu^2} \psi_n \quad . \quad (3.14)$$

Then

$$\hat{\Omega}_2 = \{ \Omega_2, \mathcal{H}_T \} \equiv \Omega_3 = \mu^2 \pi_{2n-2} + (1n) \pi_1 - \mu^2 \pi_2 (1 - \delta_{n2}) - 2\sigma \zeta \delta_{n2} = 0 \quad . \quad (3.15)$$

If $n = 2$, eq.(3.15) gives $\zeta$ in terms of $\pi_1$ and $\pi_2$, and the cascade stops here. If $n > 2$, eq.(3.15) yields

$$\pi_{2n-2} = - \frac{(1n)}{\mu^2} \pi_1 + \pi_2 \quad . \quad (3.16)$$

The next step is $\hat{\Omega}_3 = \{ \Omega_3, \mathcal{H}_T \} \equiv \Omega_4 = 0$, which together with (3.14) gives

$$\psi_{n+2} = \psi_{n-2} + \frac{1}{\mu^2} ( (1n) + (2, n - 1) ) \psi_{n-1} + \frac{1}{\mu^4} (1n) \langle 2n \rangle \psi_n \quad , \quad (3.17)$$
and, proceeding further, we obtain for the momenta

\[ \dot{\Omega}_4 = \{ \Omega_4, H_T \} \equiv \Omega_5 = \mu^4 \pi_{2n-3} - \langle 1n \rangle \langle 1, n-1 \rangle \pi_1 + \]
\[ + \mu^2 (\langle 1n \rangle + \langle 2, n-1 \rangle) \pi_2 - \mu^4 \pi_3 (1 - \delta_{n3}) - 2\sigma \mu^4 \zeta \delta_{n3} = 0 \]  

(3.18)

where (3.16) has been taken into account. Again, if \( n = 3 \), the process stops here and we have reproduced the results of section 2. If \( n > 3 \), eq.(3.18) yields

\[ \pi_{2n-3} = \frac{1}{\mu^4} \langle 1n \rangle \langle 1, n-1 \rangle \pi_1 - \frac{1}{\mu^2} (\langle 1n \rangle \langle 2, n-1 \rangle) \pi_2 + \pi_3 \]  

and the process goes on.

For illustrative purposes, we complete here the steps that cover the case \( n = 4 \).

\[ \dot{\Omega}_5 = \{ \Omega_5, H_T \} \equiv \Omega_6 = 0 \]  

yields

\[ \psi_{n+3} = \psi_{n-3} + \frac{1}{\mu^2} (\langle 3, n-2 \rangle + \langle 2, n-1 \rangle + \langle 1n \rangle) \psi_{n-2} + \]
\[ + \frac{1}{\mu^4} (\langle 2, n-1 \rangle \langle 3, n-1 \rangle + \langle 1n \rangle (\langle 2, n-1 \rangle + \langle 3n \rangle)) \psi_{n-1} + \]
\[ + \frac{1}{\mu^6} \langle 1n \rangle \langle 2n \rangle \langle 3n \rangle \psi_n \]  

and \( \dot{\Omega}_6 = \{ \Omega_6, H_T \} \equiv \Omega_7 = 0 \) gives \( \zeta \) in terms of \( \pi_1, \pi_2, \pi_3 \) and \( \pi_4 \).

In general, for a fixed \( n \), the quadratic dependence of \( H \) on \( \pi_i \) and \( \psi_i \), together with the primary constraint \( \Omega_1 \), leads to a set of secondary constraints \( \Omega_k \) that splits in two classes according to \( k \) being even or odd. A constraint \( \Omega_{2j} \) \((j = 1, \ldots, n-1)\) is linear in \( \psi_i \) and gives \( \psi_{n+j} \) in terms of \( \psi_n, \ldots, \psi_{n-j} \). A constraint \( \Omega_{2j-1} \) \((j = 2, \ldots, n-1)\) is a linear combination of \( \pi_i \) and gives \( \pi_{2n-j} \) in terms of \( \pi_1, \ldots, \pi_j \). Finally, \( \Omega_{2n-1} \) fixes the value of \( \zeta \) and stops the process.

One can prove that the constraints \( \Omega_{2n-1} \) do not contain space derivatives, even though the elements of \( M \) involved in their calculation contain the Laplacian operator. This will be shown in the Appendix.

Like in (2.24), we take

\[ \psi_A = N_{A\beta} \psi_\beta \]  

(3.21)

with indices \((\alpha, \beta, \ldots = 1, \ldots, n)\) and \((A, B, \ldots = n+1, \ldots, 2n-1)\).
Now $\mathcal{N}$ is a $(n-1) \times n$ numerical matrix whose three first rows can be read from (3.14), (3.17) and (3.20). One sees that the elements of the $j$-th row became more and more complicated for bigger $j$; we have always 0 in the $n-j-1$ first places of this row and 1 in the $n-j$ position (see Appendix), although we lack a general expression in closed form except for some elements.

Then, the Lagrangian again is

$$\mathcal{L}^{2n} = \dot{\psi}_\alpha \tilde{K}_{\alpha\beta} \dot{\psi}_\beta + \psi_\alpha \tilde{M}_{\alpha\beta} \psi_\beta - j \psi_n .$$  \hspace{1cm} (3.22)

The $n \times n$ matrices $\tilde{K}$ and $\tilde{M}$ have the same structure in terms of $K$, $M$ and $\mathcal{N}$ given in (2.21) and (2.22). However, the difficulty of explicitly finding the elements of $\mathcal{N}$, makes $\tilde{K}$ and $\tilde{M}$ hard to calculate.

The diagonalization of (3.22) will be accomplished, as in (2.25), by a $n \times n$ real matrix $\mathcal{R}$. We again impose $\mathcal{R}_{\alpha\beta} = 1$ ($\beta = 1, \ldots, n$) to ensure the correct form of the final source term. The requirement of simultaneously diagonalizing $\tilde{K}$ and $\tilde{M}$, yield $n(n-1)$ quadratic equations that determine the $n(n-1)$ remaining elements of $\mathcal{R}$. The existence of such a regular $\mathcal{R}$ with real elements is by no means guaranteed a priori, but the results of [7], showing the equivalence of the HD and the LD Lagrangians, spurs us to look for it. Although obtaining $\mathcal{R}$ is almost impossible already for $n = 4$, we guess his general form, namely:

$$\mathcal{R}_{\alpha\beta} = 1 \quad ; \quad (\alpha = n)$$

$$\mathcal{R}_{\alpha\beta} = (-1)^{n-\alpha} \mu^{-2(n-\alpha)} \langle \beta, \alpha + 1 \rangle \langle \beta, \alpha + 2 \rangle \cdots \langle \beta, n \rangle \quad ; \quad (\beta \leq \alpha < n)$$

$$\mathcal{R}_{\alpha\beta} = 0 \quad ; \quad (\alpha < \beta) .$$  \hspace{1cm} (3.23)

Of course, this $\mathcal{R}$ is just (2.26) for $n = 3$. For $n = 4$ in fact, from (3.14), (3.17) and (3.20) the matrix $\mathcal{N}$ is known, and assuming (3.23) we obtain the LD Lagrangian

$$\mathcal{L}^8 = \frac{1}{2} \frac{1}{\langle 1 \rangle} \phi_1[1] \phi_1 - \frac{1}{2} \frac{1}{\langle 2 \rangle} \phi_2[2] \phi_2 + \frac{1}{2} \frac{1}{\langle 3 \rangle} \phi_3[3] \phi_3 - \frac{1}{2} \frac{1}{\langle 4 \rangle} \phi_4[4] \phi_4$$

$$- j(\phi_1 + \phi_2 + \phi_3 + \phi_4) .$$  \hspace{1cm} (3.24)

where $\langle i \rangle \equiv \frac{1}{\mu^2} M \prod_{j \neq i} \frac{1}{\langle ij \rangle}$, and $M$ is given in (3.1). This is the result expected from the covariant Ostrogradski method shown in [7]. This success strongly backs the ansatz (3.23).

Finally, we want to remark that the case $n = 2$ is trivially contained in the general $n$ case considered in this section.
4. Applications to other theories.

The constraint method we have developed for scalar theories can be implemented for HD vector and tensor theories as well.

In the case of HD vector theories, a most general example is the gauge-fixed generalized QED, given by

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4m^2} F_{\mu\nu} \square F^{\mu\nu} - \frac{1}{2} \zeta^2 (\partial_{\mu} A_{\mu})^2 - \frac{\zeta^2}{2M^2} (\partial_{\mu} A_{\mu}) \square (\partial_{\mu} A_{\mu}) - j_{\mu} A_{\mu}.
\]

The structure of its constraints has been studied in [3] by a canonical forcefully non-covariant analysis carried out on the Ostrogradski-based order-reduction procedure.

A recasting of a higher-derivative gauge-invariant Yang-Mills theory as a two-derivative one by means of constraints has been done in a non-covariant 3+1 way [10], while we are interested in keeping the explicit Lorentz covariance at this stage.

Dropping total derivatives, (4.1) may be written as

\[
\mathcal{L} = \frac{1}{2} A^{\mu} (\theta_{\mu} + \zeta^2 \omega_{\mu}) \square \left( \Theta_{\mu} + \frac{\omega_{\mu\nu}}{m^2} + \eta_{\mu\nu} \right) A^{\nu} - j_{\mu} A_{\mu}
\]

where \( \theta_{\mu\nu} = \eta_{\mu\nu} - \frac{\partial_{\mu} \partial_{\nu}}{\square} \) and \( \omega_{\mu\nu} = \frac{\partial_{\mu} \partial_{\nu}}{\square} \), with Minkowski metric \( \eta_{\mu\nu} = diag(1, -1, -1, -1) \). Then, omitting indices, the covariant two-derivative constrained version may be readily written as

\[
\mathcal{L} = \frac{1}{2} \left\{ A \square (\theta + \zeta^2 \omega) B + \Lambda \left( B - \left( \Theta - \frac{\omega}{M^2} + \eta \right) A \right) \right\} - j A
\]

where the four-vector \( \Lambda \) is the multiplier and \( B \) is a new vector field. The Lagrangian (4.3) is local and is regular in the time derivatives of the fields involved. Therefore it is adequate for defining conjugate momenta \( \pi^A_{\mu}, \pi^B_{\mu} \) and \( \pi^A_{\mu} \) upon which the Dirac method and subsequent diagonalization can be implemented.

The covariant Ostrogradski order-reduction of the four-derivative gravity leads to a two-derivative equivalent in which the particle DOF can be fully diagonalized [6].

The constraint technique for the order-reduction of a pure four-derivative conformally invariant gravitational Lagrangian has been already used in a 3+1 non-covariant form [11], where further first class constraints from Diff-invariance occur. In a covariant treatment and for the general case including also two-derivative terms [12], a seemingly similar method is adopted where in place of the Lagrange multiplier a less trivial auxiliary field featuring a squared (mass)term is used. A little work shows
however that this method is identical to the covariant Ostrogradski’s [7], as shall be discussed elsewhere. We illustrate this here on the grounds of the scalar model.

Consider the four-derivative Lagrangian,

\[ \mathcal{L}^4 = -\frac{1}{2} \langle 12 \rangle \phi [1][2] \phi - j \phi \ , \quad (4.4) \]

which also reads

\[ \mathcal{L}^4 = -\frac{1}{2} \langle 12 \rangle [p \phi^2 + s \phi(\Box \phi) + (\Box \phi)^2] - j \phi \ , \quad (4.5) \]

where \( p = m_1^2 m_2^2 \) and \( s = m_1^2 + m_2^2 \). The covariant Ostrogradski method, in a slightly less refined version that the one presented in [7], would define a conjugate generalized momentum \( \pi = \frac{\partial \mathcal{L}}{\partial (\phi')} \). The Legendre transformation performed on it leads to a Hamiltonian-like density from which the following two-derivative Helmholtz Lagrangian is derived

\[ \mathcal{L}_H = \pi \Box \phi + \frac{1}{2} \langle 12 \rangle \pi^2 + \frac{1}{2} \langle 12 \rangle \phi^2 + \frac{1}{2} s \pi \phi . \quad (4.6) \]

On the other hand, by using the auxiliary field technique of [12] the higher-derivative term is brought to second order by writing (4.5) as

\[ \mathcal{L}^4 = -\frac{1}{2} \langle 12 \rangle [p \phi^2 + s \phi(\Box \phi) + \Lambda(\Box \phi) - \frac{1}{4} \Lambda^2] - j \phi \ , \quad (4.7) \]

where the equation of motion for \( \Lambda \) recovers (4.5) when substituted back in (4.7). Now, in spite of their quite different look, (4.6) and (4.7) are related by the simple field redefinition \( \pi = -\frac{1}{2} \langle 12 \rangle (\Lambda + s \phi) \).

The covariant constraint method introduced in this paper provides a new approach. The most immediate application in higher-derivative gravity regards the linearized theory, usually considered when analyzing the DOF. Take for example the four-derivative Lagrangian

\[ \mathcal{L} = \sqrt{-g} [a R + b R^2 + c R_{\mu \nu} R^{\mu \nu}] . \quad (4.8) \]

The linearization around the flat Minkowski metric, namely \( g_{\mu \nu} = \eta_{\mu \nu} + h_{\mu \nu} \), simplifies it to

\[ \mathcal{L} = \frac{1}{2} h_{\mu \nu} [(a G + 2b \eta \Box P + 2c \eta \Box P) \Box P]^{\mu \nu, \rho \sigma} h_{\rho \sigma} , \quad (4.9) \]
where  
\[ \bar{\eta}^{\mu\nu,\rho\sigma} \equiv \frac{1}{2}(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}) , \quad \bar{\bar{\eta}}^{\mu\nu,\rho\sigma} \equiv \eta^{\mu\nu}\eta^{\rho\sigma} , \quad G = \frac{1}{2} \bar{\bar{\eta}} - \bar{\eta} \quad \text{and} \quad P_{\mu\nu,\rho\sigma} = \frac{1}{2}(\omega_{\mu\rho}\eta_{\nu\sigma} + \omega_{\nu\rho}\eta_{\mu\sigma} - \omega_{\mu\nu}\eta_{\rho\sigma} - \bar{\bar{\eta}}_{\mu\nu,\rho\sigma}) . \]

Omitting indices, the order-reduction of the theory by means of a Lagrange multiplier yields the two-derivative local Lagrangian

\[ \mathcal{L} = \frac{1}{2} [ h(aG + 2b\bar{\eta} \Box P + 2c\bar{\eta} \Box P)f + \Lambda(f - \bar{\eta} P)] , \quad (4.10) \]

where  \( f_{\mu\nu} \)  is a new field and  \( A_{\mu\nu} \)  is the multiplier. Of course, because of the Diff-invariance, first class constraints will remain when the Dirac procedure is carried out.

5. Conclusions

We have shown how to deal with 2\( n \)-derivative relativistic scalar theories by writing them directly as second-order constrained Lagrangians with more fields and suitable Lagrange multipliers. The corresponding canonical conjugate momenta are subject to primary constraints, whose conservation in time gives rise to a finite chain of secondary constraints according to the Dirac’s procedure. Though expected, a non-trivial result is that these constraints, later used to extract the final DOF, are purely algebraic relations that do not involve the space derivatives.

Once the constraints have been implemented, we are left with a second-order Lagrangian for the DOF of the system. We have performed explicitly the diagonalization for \( n=3 \) and used an ansatz to work out the case \( n=4 \), confirming the result obtained in [7]. Though not proven, this ansatz is given a plausible expression for arbitrary \( n \), namely (3.23). This step towards the explicit generalization to higher \( n \), gives this method an advantage over Ostrogradski’s.

The applications to more interesting theories like HD generalized electrodynamics and HD Diff-invariant gravity illustrate also the fact that the order-reducing methods used in the literature fall in two categories: the one based in the covariant Ostrogradski and the one based in the contraints by Lagrange multipliers. The methods based on auxiliary fields with a quadratic term, which may look like a variant of the multipliers, actually belong to the first category and have no obvious extension beyond the four-derivative order.

In vector and tensor field theories where gauge symmetries occur, the corresponding first class constraints live together with the second class ones worked out in this paper and survive the order-reducing procedure as long as gauge fixings are not considered. The method may then prove useful for a detailed analysis of the constraints from gauge (or Diff-)invariance in these HD theories, chiefly of the fate of the scalar and vector constraints of Hamiltonian gravity.

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Appendix

We prove, by induction, that the constraints $\Omega_{2j}$ ($j = 1, \ldots, n-1$) involving the fields, do not contain space derivatives because the Laplacian operators cancel out.

One first sees, by inspection, that this statement is true for $\Omega_2$: the calculation leading to (3.14) is

$$\Omega_2 \equiv \sigma \left[ \mu^2 \psi_{n-1} + (M_1^2 - M_n^2) \psi_n - \mu^2 \psi_{n+1} \right] = 0 \ ,$$

(A.1)

where the cancelation of the Laplacian operator is apparent, i.e.

$$M_1^2 - M_n^2 = m_1^2 - m_n^2 \equiv \langle 1n \rangle \ ,$$

(A.2)

and obviously no summation is understood for repeated indices. Then, let us suppose that, after taking into account the preceding constraints, one has that in the constraint

$$\Omega_{\alpha} = \sigma \left[ \mu^{2\alpha} \psi_{n-\alpha} + a_1 \psi_{n-\alpha + 1} + a_2 \psi_{n-\alpha + 2} + \ldots + a_{\alpha-1} \psi_{n-1} + a_\alpha \psi_n - \mu^{2\alpha} \psi_{n+\alpha} \right] = 0 \ ,$$

(A.3)

for $\alpha = 1, \ldots, n-2$, the coefficients $a_1, \ldots, a_\alpha$ are real numbers, as are those found in (3.14), (3.17) and (3.20). We now prove that this is also true for $\Omega_{\alpha+1}$. In fact

$$\Omega_{\alpha+1} = \mu^{2\alpha} \pi_{2\alpha+1} = a_1 \pi_{n-\alpha} + a_2 \pi_{n-\alpha + 1} + \ldots + a_{\alpha-1} \pi_{n-\alpha + (\alpha-1)} + a_\alpha \pi_{n-\alpha + \alpha} -$$

$$- \mu^2 \pi_{\alpha+1} (1 - \delta_{n,\alpha+1}) - 2\sigma \mu^2 \zeta \delta_{n,\alpha+1} \ ,$$

(A.4)

from which

$$\Omega_{2(\alpha+1)} \equiv \Omega_{\alpha+1} = \left\{ \Omega_{\alpha+1}, \mathcal{H}_T \right\}_{PB} =$$

$$= \sigma \left[ -\mu^{2\alpha} M_{n-\alpha}^2 \psi_{n-\alpha} - a_1 M_{n-\alpha+1}^2 \psi_{n-\alpha+1} - \ldots - a_{\alpha-1} M_{n-1}^2 \psi_{n-1} -$$

$$- a_\alpha M_n^2 \psi_n + \mu^{2\alpha} M_{n+\alpha}^2 \psi_{n+\alpha} + \mu^{2(\alpha+1)} \psi_{n-\alpha+1} + \mu^2 a_1 \psi_n + \ldots +$$

$$+ \mu^2 a_{\alpha-1} \psi_{n-2} + \mu^2 a_\alpha \psi_{n+1} - \mu^{2(\alpha+1)} \psi_{n+\alpha+1} \right] = 0 \ .$$

(A.5)
The crucial point now is that, when working $\psi_{n+\alpha}$ out of (A.3) and substituting it in (A.5), only differences of squared masses $M_i^2$ occur as in (A.2), thus cancelling out the operators $\Delta$. Then, by substituting also $\psi_{n+1}$ from (A.1), one gets $\psi_{n+\alpha+1}$ as a sum of linear terms in $\psi_n, \psi_{n-1}, \ldots, \psi_{n-\alpha-1}$, the coefficient for the last one being the unity. This ends the inductive proof.

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