Discrete States of 2D String Theory in Polyakov’s Light-Cone Gauge

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Abstract

We find the discrete states of the $c = 1$ string in the light-cone gauge of Polyakov. When the state space of the gravitational sector of the theory is taken to be the irreducible representations of the $SL(2, R)$ current algebra, the cohomology of the theory is not the same as that in the conformal gauge. In particular, states with ghost numbers up to 4 appear. However, after taking the space of the theory to be the Fock space of the Wakimoto free-field representation of the $SL(2, R)$, the light-cone and conformal gauges are equivalent. This supports the contention that the discrete states of the theory are physical. We point out that the natural states in the theory do not satisfy the KPZ constraints.

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1 Introduction

Much effort has been devoted in recent years to the study of conformal matter systems coupled to two-dimensional quantum gravity. The barrier at $c = 1$, first discovered by Knizhnik, Polyakov and Zamolodchikov in the light-cone gauge [1] and confirmed later by David, Distler and Kawai in the conformal gauge [2,3], has restricted the study mainly to couplings to matter systems with central charge $c \leq 1$. Nevertheless, it turned out that these theories possess rich and quiet non-trivial structure and symmetries.

Among these models the $c = 1$ case is exceptional in that it has a two-dimensional space-time interpretation. In addition to the tachyon, the spectrum of this model consists of infinite set of discrete states, present for quantized values of momentum [4,5]. These special states appeared in the calculation of puncture operators correlation functions by Gross, Klebanov and Newman [6], and were found in the continuum by Polyakov, who interpreted them as remnants of transverse string excitations [7]. Witten showed that the spin zero ghost number zero discrete states generate a ground ring, and that one can combine antiholomorphic spin zero and holomorphic spin one states to yield $\mathcal{W}_\infty$ symmetry currents of the theory [8]. These currents have been used to facilitate the calculation of tachyon amplitudes [9]. A $\mathcal{W}_\infty$ symmetry structure has also been uncovered in the $c = 1$ matrix model [10–13].

Yet, it is not clear whether the discrete states are indeed physical objects. Although they appear as poles in tachyon amplitudes [6,7,14,15], one can interpret these poles simply as a renormalization of the external legs. Also, the correlation functions, in the continuum, of the discrete states themselves blow-up, and seem to vanish upon regularization [16].

Evidently, if discrete states are physical they should appear in any legitimate gauge fixing. So far, in the continuum, they have been found and analyzed in the conformal gauge, with the Liouville field treated as a free scalar with a background charge[4,5,8,17–19]. The aim of this paper is to study the spectrum of the theory in Polyakov’s light-cone gauge. Previous works in the literature aiming at the analysis of the spectrum of conformal matter coupled to gravity theories in the light-cone gauge exist [20], but do not reveal the structure of the discrete states in the spectrum. We will be interested in finding the discrete states and analyzing their structure in this gauge, and comparing the results to the conformal gauge. As we will discuss at the end of the paper, our results are applicable for all $c \leq 1$. However, we shall mainly deal with the $c = 1$ case.
The paper is organized as follows: In section 2 we state the BRST cohomology problem in the light-cone gauge and set the notations and conventions. In section 3 we start analyzing the BRST cohomology, taking the state space of the gravitational sector to be the irreducible Kac-Moody module of Polyakov’s residual $SL(2, R)$ light-cone gauge current algebra. We find the vacuum and tachyon states, and see that the states in the light-cone gauge fall into pairs of conjugate states. We also note that the KPZ constraints do not hold for the sector built on the vacuum. In section 4 we analyze the cohomology using a free field representation of the $SL(2, R)$, taking the state space to be the Fock space of the free fields. We again find a cohomology module and a module dual to it, and we prove that the cohomology is equivalent to that of the conformal gauge. Finally, we calculate the light-cone analogues of the generators of Witten’s ground ring, as well as the other operators in the cohomology. In section 5 we return to the cohomology on the Kac-Moody module. We examine the first level explicitly, and see that extra states appear in the current algebra cohomology that are not in the cohomology of the Fock space. We then develop the “Felder resolution”, that allows us to obtain the complete current algebra cohomology from that of the Fock space. We find that there are discrete states in the Kac-Moody case with ghost numbers ranging from $-2$ to $4$. This approach is therefore not equivalent to the conformal gauge. Section 6 is devoted to discussion and conclusions.

2 The BRST cohomology problem in the light-cone gauge.

An action describing the $c = 1$ conformal field theory coupled to two-dimensional gravity is given by:

$$ S = \int d^2 z \sqrt{g} g^{ab} \partial_a x \partial_b x , \quad (2.1) $$

where $g^{ab}$ is the two-dimensional metric and $x$ is a scalar field.

In the conformal gauge, the metric is fixed to the form $g_{ab} = e^{\phi} \delta_{ab}$, where $\phi$ is the Liouville field. The vanishing of the Weyl anomaly leads to the equation

$$ \nabla^2 \phi = 0 , \quad (2.2) $$

where we have set the cosmological constant to zero. The main problem with the conformal gauge is that, while $\phi$ appears like an ordinary massless scalar field, its quantization is not straightforward, since its measure in the functional integral is field dependent. Nevertheless, the quantization has been carried out with great success using the David, Disler and Kawai ansatz of considering $\phi$ to be a free scalar with a background charge fixed
at the quantum level [2,3]. In this gauge the fields of the theory, in addition to $\phi$, are the matter field $x$, and the standard $(b,c)$ and $(\bar{b},\bar{c})$ ghost systems associated with the secondary constraints $T_{zz} = T_{\bar{z}\bar{z}} = 0$, where $T_{ab}$ is the energy-momentum tensor.

In Polyakov’s light-cone gauge the metric is gauge-fixed to

$$ds^2 = dzd\bar{z} + h(z, \bar{z})dzd\bar{z} . \tag{2.3}$$

The nature of the metric implies that the measure of $h$ is not field dependent, so one does not encounter the difficulties of the Liouville field. In this case the gauge fixing conditions $g_{\bar{z}\bar{z}} = 0$ and $g_{zz} = \frac{1}{2}$ lead to the secondary constraints:

$$T_{zz} = 0 \quad \text{and} \quad T_{\bar{z}\bar{z}} = 0 . \tag{2.4}$$

We denote the ghost systems associated with these constraints by $(b,c)$ and $(\bar{b},\bar{c})$, respectively. They are both anticommuting, and have spins $(2,1)$ and $(0,1)$.

The vanishing of the gravitational anomaly leads to the equation

$$\partial_{\bar{z}}^2 h(z, \bar{z}) = 0 , \tag{2.5}$$

so $h$ is decomposed into three parts:

$$h(z, \bar{z}) = J^+(z) + 2\bar{z}J^0(z) + (\bar{z})^2J^-(z) . \tag{2.6}$$

As was shown by Polyakov, by analyzing the Ward identities of the theory, the $J$’s satisfy an $SL(2,R)$ Kac-Moody algebra [21], with OPE’s:

$$J^a(z)J^b(w) \sim -\kappa \eta^{ab} \left(\frac{1}{w-z}\right)^2 + \frac{f_{c}^{ab}}{w-z}J^c(w) . \tag{2.7}$$

Here $\kappa$ is the (renormalized) Kac-Moody central charge, the $f_{c}^{ab}$’s are the structure constants of $SL(2,R)$ algebra, and $\eta^{ab}$ its Cartan-Killing form*. The operator product expansion of the other fields of the theory are given by:

$$x(z)x(w) \sim -\log(z-w)$$

$$b(z)c(w) \sim \frac{1}{z-w}$$

$$\zeta(z)\eta(w) \sim \frac{1}{z-w} . \tag{2.8}$$

*In our notation, $f_{0}^{+-} = 2, f_{0}^{b+} = -1, f_{0}^{-} = 1$ and $\eta^{+-} = 2, \eta^{00} = -1$.
The “holomorphic” part of the energy-momentum tensor of the theory $T(z) \equiv T_{zz}(z)$ is given by

$$T(z) = T^{\text{grav}}(z) + T^{\text{matt}}(z) + T^{bc}(z) + T^{\zeta\eta}(z) \ ,$$  

where $T^{\text{grav}}(z), T^{\text{matt}}(z), T^{bc}(z)$ and $T^{\zeta\eta}(z)$ are the stress tensors of the gravity, matter and ghost sectors respectively. The stress tensors of the matter and ghost systems are, as usual, given by:

$$T^{\text{matt}}(z) = -\frac{1}{2} (\partial x)^2 :$$
$$T^{bc}(z) = -2\partial c - bc :$$
$$T^{\zeta\eta}(z) = :\partial \zeta \eta : .$$

Knizhnik, Polyakov and Zamolodchikov (KPZ) showed that the gravity stress tensor takes the form of a modified Sugawara construction [1]:

$$T^{\text{grav}}(z) = -\frac{1}{\kappa + 2} \eta_{ab} :J^a(z)J^b(z) : + \partial J^0(z) \ ,$$

where the second piece modifies the spins of the $SL(2,R)$ Kac-Moody currents such that they are compatible with the decomposition (2.6). Thus $\text{spin}(J^+) = 2$, $\text{spin}(J^-) = 0$ and $J^0$ remains of spin 1, but is no longer a primary field:

$$T(z)J^0(w) \sim -\kappa (z-w)^3 + J^0(w) (z-w)^2 + \partial J^0(w) .$$

The component of the gravity stress tensor $T_{z\bar{z}}$ is essentially the Kac-Moody current $J^- [1]$:  

$$T_{z\bar{z}} \sim \partial^2_{z\bar{z}} h \sim J^-(z) .$$

Thus the constraints algebra—the algebra of the residual symmetry of the light-cone gauge—takes the form:

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} ,$$

$$T(z)J^-(w) \sim \frac{\partial J^-(w)}{z-w} ,$$

$$J^-(z)J^-(w) \sim 0 .$$

The contribution to the central charge from the gravity sector, $c^{\text{grav}}$, can be found using eq. (2.11) to be $c^{\text{grav}} = 3\kappa/(-\kappa + 2) - 6\kappa$; $c^{\text{matt}} = 1$, $c^{bc} = -26$ and $c^{\zeta\eta} = -2$. The total central charge vanishes if

$$\kappa = -3 .$$

†Throughout the paper, we use conformal field theory normal ordering.
The system of constraints is then of first class type, and it is natural to impose them via the BRST formalism. The BRST operator corresponding to the algebra of constraints (2.14) is:

\[ Q_B = \int \frac{dz}{2\pi i} c(z) \left( T^{grav}(z) + T^{matt}(z) + \frac{1}{2} T^{bc}(z) + T^{\zeta\eta}(z) \right) + \eta(z) J^{-}(z) ; \]

The spectrum of the theory is then given by the cohomology of the BRST complex \((Q_B, H)\) where the quantum state space \(H\) is decomposed into \(H_{grav} \otimes H_{matt} \otimes H_{bc} \otimes H_{\zeta\eta}\). The matter and ghost state spaces are simply the Fock spaces built out of the oscillator modes of the fields. However, it is not \textit{a priori} apparent what should be chosen for the gravity sector state space.

3 \textbf{BRST cohomology on the SL}(2, R) \textbf{Kac-Moody module}

3.1 \textbf{Zero modes of the SL}(2, R).

The perhaps most natural choice for the gravity sector state space \(H_{grav}\) is to take it to be the irreducible \(SL(2, R)\) Kac-Moody modules built out of the Kac-Moody currents. States in a Kac-Moody module can be described by acting creation modes of the currents, \(J_{-n}^a\), on a member of a representation of the \(SL(2, R)\) Lie algebra generated by the zero-modes \(J_0^a\). Because the Sugawara stress tensor of eq. (2.11) is twisted, the central term of the \(SL(2, R)\) Kac-Moody algebra is modified:

\[ [J_n^a, J_m^b] = f^{ab}_c J_{n+m}^c - \frac{\kappa}{2} \eta^{ab}(n + a) \delta_{n+m,0} , \]

where “n + a” denotes \(n \pm 1\) for \(a = \pm\), and \(n\) for \(a = 0\). The \(J_0^a\)’s thus do not quite generate \(SL(2, R)\). It is therefore convenient to define new generators

\[ j^\pm \equiv i J_0^\mp , \]

\[ j^z \equiv J_0^z + \frac{3}{2} , \]

with \(SU(2)\)-like commutation relations:

\[ [j^+, j^-] = 2j^z \]

\[ [j_z, j^\mp] = \pm j^\pm . \]
Then, defining the usual $SU(2)$ Casimir operator

$$C \equiv J(J+1) = (j^2)^2 + \frac{1}{2}\{j^+, j^-\},$$

(3.4)

one can represent the ground states of the gravity sector as $|J, M\rangle$ with $M$ being the eigenvalue of $j^z$.

The ground states in the other sectors of the theory describe the value of the $x$-momentum, $p$, and the usual $|\uparrow\rangle_{bc}$ of the zero-modes of the $(b, c)$ system. One also needs to define $|\downarrow\rangle_{\zeta\eta}$ for the $(\zeta, \eta)$ ghosts:

$$\zeta_0 \Downarrow_{\zeta\eta} = 0$$
$$\eta_0 \Uparrow_{\zeta\eta} = 0.$$  

(3.5)

3.2 The KPZ condition

Using the mode expansion of the fields, the BRST operator of eq. (2.16) takes the form

$$Q_B = \sum_m :c_{-m} (L_{m}^{\text{grav}} + L_{m}^{\text{c}} + \frac{1}{2}L_{m}^{\zeta\eta}) + \eta_{-m} J_- :,$$  

(3.6)

where the $L_m$'s are the Virasoro generators of the various sectors. $L_{n}^{\text{grav}}$ is given by

$$L_{n}^{\text{grav}} = -\frac{1}{K+2} \sum_m (\frac{1}{2}J^-_{m}J^-_{n-m} + \frac{1}{2}J^+_{m}J^+_{n-m} - J^0_{m}J^0_{n-m}) - (n+1) J^0_n.$$  

(3.7)

The other $L_n$'s are given explicitly in appendix A. As usual, the first step in finding the cohomology of $Q_B$ is to decompose it with respect to the $b, c$ zero modes:

$$Q_B = c_0 L_0 - b_0 \sum_{n \neq 0} n c_{-n} c_n + \hat{Q}.$$  

(3.8)

Since $L_0 = \{Q_B, b_0\}$, the cohomology of $Q_B$ is contained in the kernel of $L_0$ [22]. This means that one can first calculate the relative cohomology, i.e. the cohomology of $\hat{Q}$ restricted to the subspace of states $|\psi\rangle$ satisfying $b_0 |\psi\rangle = 0$, and then use this to derive the absolute cohomology.

$L_0$ can be found explicitly by carefully performing the normal ordering in the definitions of the $L$'s, and using the definition of the Casimir operator $C$ in eq. (3.4):

$$L_0 = \frac{1}{2} p^2 - C - \frac{1}{4} + N,$$  

(3.9)

*If $L_0 \Psi \neq 0$, and $\Psi$ is closed, then $\Psi$ is also exact: $\Psi = Q_B(L_0)^{-1} b_0 \Psi$. 

6
where $N$ is the level operator measured from the tachyon ground state. The appearance of the Casimir operator is not surprising, since $L_0$ commutes with the generators of the $SL(2,R)$. The requirement that $L_0 = 0$ can be rewritten as

$$\frac{1}{2} p^2 + N = (J + \frac{1}{2})^2 . \quad (3.10)$$

This can be compared to the KPZ relation [1]:

$$\Delta_0 = \Delta + \frac{\Delta(1 - \Delta)}{\kappa + 2} \quad \kappa = -3 \rightarrow \Delta_0 = \Delta^2 . \quad (3.11)$$

Here $\Delta_0$ is the dimension of the undressed physical operator, and $\Delta$ is its dimension dressed by the gravitational fluctuations. The KPZ formula is derived for states that satisfy the constraints (2.4) of the theory. Since $T_z \bar{T}_\bar{z} \sim J^{-}(z)$, this means that such states are annihilated by $J_{-} \sim j^{+}$, and are therefore highest weight states (HWS). Comparing eqs. (3.10) and (3.11), and noting that HWS satisfy $J = M = J_0^0 + \frac{3}{2}$, one sees that

$$\Delta = J_0^0 + 2 . \quad (3.12)$$

The reason for this shift, which is not apparent in the original paper of Polyakov [21], is that one needs to measure the dressed dimensions of operators with respect to that of the cosmological constant operator. A similar shift appears in the conformal gauge [3]. The (HWS) cosmological constant operator, a tachyon with $p = 0$, has $J = -\frac{1}{2}$ and therefore $J_0^0 = -2$.

### 3.3 The $SL(2,R)$ invariant vacuum

Before continuing with the general problem of finding the cohomology, it is instructive to consider the (conformal field theory, not Kac-Moody!) $SL(2,R)$ invariant vacuum of the theory, $|0\rangle$, defined by $A(0) |0\rangle \sim$ regular for all primary fields $A(z)$. As one would expect, the vacuum is indeed in the cohomology of $\hat{Q}$. This is shown in appendix A. Examining the spins of the fields in the theory, one sees that $|0\rangle$ is annihilated by $p$, $b_0$, $b_{-1}$, $\eta_0$, $J_0^0 = j^z - \frac{3}{2}$, $J_0^+ = j^-$ and $J_0^{-1}$, and can therefore be written as

$$|0\rangle = b_{-1} |p = 0, J = -\frac{3}{2}, M = \frac{3}{2} \rangle \otimes |\uparrow\rangle_{\xi\eta} \otimes |\downarrow\rangle_{bc} . \quad (3.13)$$

*Actually, as can be seen by eq. (2.12), $J^0_0$ is not a primary field. However $J^0_0$ still annihilates $|0\rangle$, since both $J^0_{+1}$ and $J^0_{-1}$ annihilate it, and $J^0_0$ is proportional to their commutator.
The vacuum therefore has several peculiar properties: First, it carries $M = 3/2$, so the theory has background charge 3. Second, since its $J$ is negative, it is not an element of a finite representation of $SL(2, R)$ but of a semi-infinite one. Third, since it is annihilated by $j^-$ it is a lowest weight state, with $J = -M$. The vacuum of the theory is not a KPZ state!

### 3.4 Zero’th-level states (tachyons)

At the lowest level of the theory states have no oscillators. The relative cohomology operator $\hat{Q}$ therefore reduces to

$$\hat{Q} \sim -i \eta_0 j^+ . \quad (3.14)$$

In the relative cohomology there are only two possible types of states: $|p, J, M\rangle \otimes |\downarrow\rangle_{\zeta \eta}$ and $|p, J, M\rangle \otimes |\uparrow\rangle_{\zeta \eta}$. (The $|\downarrow\rangle_{bc}$ state is not explicitly written.) Now

$$\hat{Q} \left( |p, J, M\rangle \otimes |\downarrow\rangle_{\zeta \eta} \right) = -i j^+ |p, J, M\rangle \otimes |\uparrow\rangle_{\zeta \eta}$$

$$\hat{Q} \left( |p, J, M\rangle \otimes |\uparrow\rangle_{\zeta \eta} \right) = 0 . \quad (3.15)$$

The first state is therefore closed if it is a highest-weight state (HWS), with $M = J$, while the second state is automatically closed. Also, while the first state can never be exact, the second state is exact unless it cannot be written as $j^+ |p, J, M\rangle$. This means that it is a lowest-weight state (LWS), with $M = -J$. The value of $J$ is determined from the $L_0 = 0$ condition (eq. (3.10)) with $N = 0$ to be either of the two solutions

$$J^\pm (p) = \pm \frac{|p|}{\sqrt{2}} - \frac{1}{2} , \quad (3.16)$$

and the relative cohomology is represented by the states

$$|p, J^\pm (p)\rangle, \quad J^\pm (p)\rangle \otimes |\downarrow\rangle_{\zeta \eta}$$

$$|p, J^\pm (p), - J^\pm (p)\rangle \otimes |\uparrow\rangle_{\zeta \eta} . \quad (3.17)$$

These states correspond to the tachyon, since they exist for all $p$.

This result can be compared to the tachyons of the conformal gauge, which have the form*

$$|p, p^\pm_\phi = -i(\sqrt{2} \pm |p|)\rangle . \quad (3.18)$$

*We use the radially-ordered notation that the vacuum has $p_\phi = 0$, rather than Witten’s symmetric notation that it carries $p_\phi = i\sqrt{2}$. 
We see that, because of the zero modes of the $(\zeta, \eta)$ ghost system, there appears to be a doubling of states in the light-cone gauge. However, it should be noted that, in general, the HWS are the largest members of semi-infinite dimensional representations with $M \leq J$, while the LWS are the smallest members of semi-infinite dimensional representations with $M \geq J$. These representations are conjugate, and cannot be obtained from each other. (The only exception to this is when $2J + 1 \in \mathbb{N}$, giving a finite dimensional representation with both a HWS and a LWS. This case corresponds to (half of) the “discrete tachyons” of the conformal gauge.) Since the vacuum is a LWS, only states built on LWS can be obtained from it, and their conjugate states should not be considered to give a duplication of the spectrum of the theory. Note that, as in the case of the vacuum state, the LWS representing the tachyons again do not satisfy the KPZ constraint $J_0^- = 0$.

**Absolute cohomology tachyons**

As is usual for tachyonic states, the absolute cohomology is obtained by simply taking the relative cohomology states and the states obtained from them by the raising operator $c_0$. The general level-zero cohomology is therefore given by the four types of states

\[
\begin{align*}
|p, J^{\pm}(p), J^{\pm}(p)\rangle \otimes |\downarrow\rangle_{\zeta\eta} \otimes |\downarrow\rangle_{bc} \\
|p, J^{\pm}(p), -J^{\pm}(p)\rangle \otimes |\uparrow\rangle_{\zeta\eta} \otimes |\downarrow\rangle_{bc}.
\end{align*}
\] (3.19)

It is not simple to solve the cohomology at higher levels in a straightforward way. We shall therefore turn to the use of free field representations of the current algebra in the next section, and return to the cohomology of the current algebra in section 5.

4 The BRST cohomology module in the Fock space

4.1 The Wakimoto free field representation

Free field representations of conformal field theories are very useful for the construction of operators and for the calculation of correlation functions [24, 25]. In this section we shall introduce the Wakimoto representation for the $SL(2, R)$ current algebra of the gravity sector, and find the cohomology of the BRST operator on the Fock space of the free fields. In the next section, we shall use this cohomology to find the cohomology on the irreducible Kac-Moody modules via the Felder resolution [26]. In this section we shall simply consider the Fock space to be the state space of the theory.

In the Wakimoto representation of $SL(2, R)$ Kac-Moody algebra the currents are given
by [27]

\[
J^+ = \beta \gamma^2 + \frac{2}{\alpha_+} \gamma \partial \varphi + \kappa \partial \gamma \\
J^0 = \beta \gamma + \frac{1}{\alpha_+} \partial \varphi \\
J^- = \beta ,
\]  

(4.1)

where \(2/\alpha_+ = -(\kappa+2) = 1\). Since the spins of the currents in our case have been modified, the bosonic fields \(\beta\) and \(\gamma\) should be taken to have spins 0 and 1 respectively, the reverse of the usual case. \(\varphi\) is a scalar field with background charge \(Q = -\frac{i}{2}(\alpha_+ + \frac{2}{\alpha_+}) = -\sqrt{2}i\).

Using the OPE’s:

\[
\gamma(z)\beta(w) = \frac{1}{z - w} \quad \text{and} \quad \varphi(z)\varphi(w) = -\log(z - w),
\]

(4.2)

one can show that the Wakimoto currents satisfy the \(SL(2, R)\) current algebra of eq. (2.7).

With this representation, the Sugawara stress tensor of eq. (2.11) takes the simple form:

\[
T^{grav} = :\partial \beta \gamma - \frac{1}{2}(\partial \varphi)^2 + iQ\partial^2 \varphi:,
\]

(4.3)

and the BRST operator of eq. (2.16) becomes

\[
Q_B = \int \frac{d^2z}{2\pi i} c(z) \left( -\frac{1}{2}(\partial x)^2 - \frac{1}{2}(\partial \varphi)^2 + iQ\partial^2 \varphi - b\partial c - \frac{1}{2}\partial bc + \partial \beta \gamma + \partial \zeta \eta \right) + \eta(z)\beta(z):,
\]

(4.4)

Note that the fields and stress tensor of the theory in the Wakimoto representation are in a one to one correspondence with those of the conformal gauge*, with \(\varphi\) the analogue of the Liouville field, except for the addition of the bosonic spin \(0,1\) fields \((\beta, \gamma)\) and the fermionic spin \(0,1\) fields \((\zeta, \eta)\). This leads one to expect that these extra fields will conspire to cancel the effects of each other, leaving behind the structure of the conformal gauge. This argument is supported by the fact that the \(\eta(z)\beta(z)\) piece of the BRST operator, coming from the constraint \(J^- (z) = 0\), can be viewed as a BRST operator generating the fermionic symmetry transformation

\[
\delta \gamma = -\varepsilon \eta \quad \delta \zeta = \varepsilon \beta ,
\]

(4.5)

in the topological field theory with action†

\[
S = \int (\partial \beta \gamma + \partial \zeta \eta).
\]

(4.6)

*At least with the holomorphic sector part of the conformal gauge. The structure of the antiholomorphic sector of the light-cone gauge is somewhat obscure.

†We would like to thank J. Sonnenschein for this point.
We shall see that this argument is essentially correct, but the situation is somewhat more complicated and the extra fields do appear in the operators representing the cohomology.

4.2 The BRST cohomology: Zero modes

As in the case of the current algebra, the first step in finding the cohomology is to remove the zero-modes of the fields. After achieving this, the procedure becomes relatively simple, and follows that of the conformal gauge. Once again, the zero modes of \( b \) and \( c \) are separated out by decomposing \( Q_B \) into:

\[
Q_B = c_0 L_0 - b_0 \sum_{n \neq 0} n c_{-n} c_n + \hat{Q} \ .
\] (4.7)

One can then obtain the absolute cohomology of \( Q_B \) from the relative cohomology of \( \hat{Q} \) acting on states annihilated by \( b_0 \) using the result [5, 18]:

**Theorem 1** States in the absolute cohomology of \( Q_B \) are of the form \( \Psi \) or \( a_0 \Psi \), where \( \Psi \) is in the relative cohomology of \( \hat{Q} \). As in the conformal gauge

\[
a = [Q, \varphi] = c \partial \varphi + \sqrt{2} \partial c \ ,
\] (4.8)

so \( a_0 \), the zero mode of \( a \), is essentially the BRST invariant inverse of \( b_0 \).

The proof of this result follows exactly the proof in the conformal gauge [18].

In the light cone, one still has to deal with the zero-modes of \( (\zeta, \eta) \) and \( (\beta, \gamma) \). The next step in doing this is to note that \( Q_B \) does not contain \( \zeta_0 \), and to separate out the \( \eta_0 \) part of \( \hat{Q} \):

\[
\hat{Q} \equiv \eta_0 \mathcal{X} + \hat{d} \ ,
\] (4.9)

with

\[
\mathcal{X} = \beta_0 + \sum_{n \neq 0} n c_n \zeta_{-n} \ .
\] (4.10)

We shall later denote \( \gamma_0 \) by \( \mathcal{P} \), to remind ourselves that \( \mathcal{X} \) and \( \mathcal{P} \) have commutation relations similar to those of a coordinate and a momentum: \([\mathcal{X}, \mathcal{P}] = 1\). The operator \( \hat{d} \) is still complicated, being given by

\[
\hat{d} = \sum_{n \neq 0} c_{-n} \left( L_n^{\text{mat}} + L_n^{\text{grav}} + L_n^{\zeta \eta} \right) + \sum_{n \neq 0} \eta_{-n} \beta_n - \frac{1}{2} \sum_{m,n \neq 0 \atop n + m \neq 0} (m - n) c_{-m} c_{-n} b_{n+m} \ ,
\] (4.11)
where the prime on $L'_n$ denotes the exclusion of the piece of $L^\xi_n$ containing $\eta_0$. $L^{grav}_n$ is given by
\[ L^{grav}_n = \sum_m \left( -m \beta_m \gamma_{n-m} + \frac{1}{2} \varphi_m \varphi_{n-m} \right) - Q(n + 1) \varphi_n ; \]
(4.12)
the other $L_n$'s are given in appendix A.

While the zero modes of $\eta$ and $\beta$ have been explicitly extracted, $\hat{d}$ still contains $\gamma_0$, via its dependence on $L^{grav}_n$. This dependence can be removed by performing a Bogolubov transformation from $(\beta_0, b_n, \eta_n)$ to $(\chi, \tilde{b}_n, \tilde{\eta}_n)$, with:
\[ \tilde{b}_n \equiv b_n + n \zeta_n \gamma_0 \]
\[ \tilde{\eta}_n \equiv \eta_n + n c_n \gamma_0 . \]
(4.13)
The new set of oscillators $(\tilde{b}_n, c_n, \zeta_n, \tilde{\eta}_n, \beta_n, \gamma_n)$ have the same commutation relations as the original oscillators and, in addition, all commute with $\chi$ and $\mathcal{P}$. The reason for performing the transformation is that:

**Lemma 1** $\hat{d}$ is independent of $\chi$ and $\mathcal{P}$, when written in terms of the “tilded” oscillators.

The lemma is proven by seeing that $\hat{d}$ commutes with $\chi$ and $\mathcal{P}$. The result for $\hat{d}$ is given in eqs. (4.25).

Because of the zero modes of the commuting $(\beta, \gamma)$ system, the Fock space of the theory splits into two conjugate infinitely degenerate Hilbert spaces which are not connected: the first contains states with an arbitrary number of $\gamma_0$'s acting on $|\beta_0 = 0\rangle$; the second contains states with $\beta_0$'s acting on $|\gamma_0 = 0\rangle$. After the Bogolubov transformation, states are built either on $|\chi = 0\rangle$ or on $|\mathcal{P} = 0\rangle$. Since $\mathcal{X} = \{\hat{Q}, \zeta_0\}$, one would expect that the cohomology of $\hat{Q}$ lies entirely within the kernel of $\chi$. In fact, the situation is slightly more complicated, since one should obtain both such states and their conjugates in the $|\mathcal{P} = 0\rangle$ sector. Using the result that $\hat{d}$ is independent of $\chi$ and $\mathcal{P}$, one can reduce the calculation of the cohomology of $\hat{Q}$ to that of the relative cohomology of $\hat{d}$, acting on the Fock space without the zero modes $\chi$, $\mathcal{P}$, $\zeta_0$ and $\eta_0$, by the following theorem:

**Theorem 2** States in the cohomology of $\hat{Q}$ are in one of the two conjugate forms:
\[ \Psi = |\psi\rangle \otimes |\chi = 0\rangle \otimes \downarrow \zeta_\eta \]
(4.14)
or
\[ \Psi = |\psi\rangle \otimes |\mathcal{P} = 0\rangle \otimes \uparrow \zeta_\eta , \]
(4.15)
where $|\psi\rangle$ is in the cohomology of $\hat{d}$, acting on the Fock space of the tilded oscillators without zero modes.
Proof: Consider first a $\Psi$ built on the state $|X = 0\rangle$:

$$\Psi \equiv \sum_{n=0}^{\infty} |\psi_n\rangle \otimes \mathcal{P}^n |X = 0\rangle,$$

(4.16)

where the $|\psi_n\rangle$'s are independent of $X$ and $\mathcal{P}$, but are not necessarily annihilated by $\zeta_0$ or $\eta_0$. Requiring that $\Psi$ be closed under $\hat{Q}$ yields the equations:

$$\hat{d}|\psi_n\rangle + (n + 1)\eta_0|\psi_{n+1}\rangle = 0 .$$

(4.17)

Using $X = \{\hat{Q}, \zeta_0\}$ and $X \sim \partial \mathcal{P}$, one sees that

$$\Psi = \zeta_0 \eta_0 |\psi_0\rangle \otimes |X = 0\rangle + \hat{Q} \left( \zeta_0 \sum_{n=0}^{\infty} \frac{1}{n+1} |\psi_n\rangle \otimes \mathcal{P}^{n+1} |X = 0\rangle \right).$$

(4.18)

$\Psi$ is therefore equivalent to a state $\Psi' = \zeta_0 \eta_0 |\psi_0\rangle \otimes |X = 0\rangle$, up to an exact state. Since $\zeta_0 \eta_0$ is simply the projection operator to the state $|\downarrow\rangle_{\zeta \eta}$, $\Psi'$ has the desired form of eq. (4.14). Now using eq. (4.17) on $\Psi'$, one sees that $|\psi\rangle$ is closed under $\hat{d}$, and it is clear that $\Psi'_1 \equiv \Psi'_2$ iff they differ by a state exact under $\hat{d}$. The theorem is therefore established for the states built on $|X = 0\rangle$. One can use a similar proof for the states built on $|\mathcal{P} = 0\rangle$, or simply argue that the conjugate of a state in the cohomology must also be in the cohomology.

As far as the spectrum of the theory is concerned, one should not consider the existence of cohomologically nontrivial states built on both $|X = 0\rangle$ and $|\mathcal{P} = 0\rangle$ as a duplication of the states of the theory. The fact that conjugate states can be in disjoint Hilbert spaces is well known, and occurs, for example, with the commuting ghosts of the superstring. In fact, as in the case of the superstring, these sectors are just two of an infinite number of sectors. In the superstring, one usually further “bosonizes” the ghosts [29], and is left with a system with no remaining bosonic fields. However, while such a bosonization of the $(\beta, \gamma)$ system may be useful for amplitude calculations, one can find the physical spectrum of the theory simply by restricting oneself to a particular sector. Since the vacuum of the theory $|0\rangle = b_{-1} |\mathcal{P} = 0\rangle \otimes |\uparrow\rangle_{\zeta \eta} |\downarrow\rangle_{bc}$ is in the $|\mathcal{P} = 0\rangle$ sector, we shall choose to work in this sector, which has the advantage that states in it can be represented by operators.

### 4.3 The relative cohomology of $\hat{d}$

We now need to compute the relative cohomology of $\hat{d}$. This analysis will bear a close resemblance to the BRST analysis of the $c = 1$ theory in the conformal gauge.
by Bouwknegt, McCarthy and Pilch [5], and we shall follow their notations and proofs. Introduce the light-like combinations

\[ \alpha_n^\pm = \frac{1}{\sqrt{2}} (x_n \pm i \varphi_n) \quad n \neq 0 , \]  

(4.19)

where

\[ i \partial \varphi(z) \equiv \sum_n \varphi_n z^{-n-1} \]
\[ i \partial x(z) \equiv \sum_n x_n z^{-n-1} . \]  

(4.20)

The oscillators \( \alpha_n^\pm \) satisfy the commutation relations

\[ [\alpha_m^\pm, \alpha_n^+ \pm n] = m \delta_{n+m} . \]  

(4.21)

Define also

\[ p_n^\pm = \frac{1}{\sqrt{2}} (p \pm i (p \varphi + i \sqrt{2})) \]
\[ P^\pm(n) = p^\pm \mp n . \]  

(4.22)

Then \( L_0 \) becomes

\[ L_0 = p^+ p^- + \sum_{n \neq 0} \alpha_n^+ \alpha_n^- + n (c_{-n} \tilde{b}_n + \beta_{-n} \gamma_n + \zeta_{-n} \tilde{\eta}_n) : + 1 \]
\[ \equiv p^+ p^- + \hat{L}_0 , \]  

(4.23)

where \( \hat{L}_0 \) is the level operator for the oscillators with respect to the Fock-space vacuum.

The next step is to impose a grading of the state space, compatible with the algebra of
the theory, so that \( \hat{d} \) breaks into a finite sum \( \hat{d} = \sum_{i \geq 0} \hat{d}_i \), where \( \hat{d}_i \) has degree \( i \). One can then first compute the cohomology of \( \hat{d}_0 \) and then use a basic result from the cohomology theory of “filtered complexes” to derive the cohomology of \( \hat{d} \). In order to obtain a simple operator \( \hat{d}_0 \) we, extending the results of the conformal gauge, define the following grading for the tilded oscillators* with \( n \neq 0 \):

\[ \text{deg}(\alpha_n^+) = \text{deg}(c_n) = \text{deg}(\tilde{\eta}_n) = \text{deg}(\gamma_n) = 1 \]
\[ \text{deg}(\alpha_n^-) = \text{deg}(\tilde{b}_n) = \text{deg}(\zeta_n) = \text{deg}(\beta_n) = -1 . \]  

(4.24)

*This grading is consistent with the similar grading of the “untilded” oscillators, if one also chooses \( \text{deg}(\gamma_0) = \text{deg}(\beta_0) = 0 \).
With this grading, the operator \( \hat{d} \) decomposes into \( \hat{d} = \hat{d}_0 + \hat{d}_1 + \hat{d}_2 \), where the \( d_i \)'s are given by

\[
\begin{align*}
\hat{d}_0 &= \sum_{n \neq 0} P^+(n)c_{-n}\alpha_n + \tilde{n}_m\beta_{-n} \\
\hat{d}_1 &= \sum_{n,m \neq 0} c_{-n}\left(\alpha_{m}^{-} \alpha_{n+m}^{-} + \frac{1}{2}(m-n)c_{-m}\tilde{b}_{n+m} + mn\zeta_{m+1} + m\beta_{-m}\gamma_{n+m}\right) \\
\hat{d}_2 &= \sum_{n \neq 0} P^-(n)c_{-n}\alpha_n^+ .
\end{align*}
\] (4.25)

Following ref. [5] we distinguish two possible cases of momenta, and obtain the cohomology of \( \hat{d}_0 \) from the two following theorems:

**Theorem 3** If \( P^+(n) \neq 0 \) or \( P^-(n) \neq 0 \) \( \forall n \neq 0 \), the relative cohomology of \( \hat{d}_0 \) exists only at ghost number 0, and is one-dimensional. These states in the cohomology are the tachyons, and they satisfy the mass shell condition \( p^+ p^- = 0 \).

**Proof:** Consider the case \( P^+(n) \neq 0 \) \( \forall n \neq 0 \). (The proof for the other case is obtained by reversing the grading of eqs. (4.24).) Define the operator

\[
K = \sum_{n \neq 0} \left( \frac{1}{P^+(n)} \alpha_{-n}^+ b_n - n\gamma_{n}\zeta_{-n} \right)
\] (4.26)

Since

\[
\{ \hat{d}_0, K \} = \hat{L}_0 ,
\] (4.27)

any state of level \( \hat{L}_0 > 0 \) which is closed under \( \hat{d}_0 \) is also exact. At level zero, since \( L_0 = \hat{L}_0 = 0 \), the state satisfies \( p^+ p^- = 0 \). It is clearly in the cohomology of \( \hat{d}_0 \). Note that since all the terms of \( \hat{d}_1 \) and \( \hat{d}_2 \) contain oscillator modes, the tachyons are also in the cohomology of \( \hat{d} \).

**Theorem 4** If \( P^+(r) = P^-(s) = 0 \) for non-zero integers \( r \) and \( s \), then \( rs > 0 \) and the relative cohomology of \( \hat{d}_0 \) is represented by the following states:

(i) For \( r, s < 0 \) : \( (\alpha_{r}^-)^{-s} |p,p_{\phi}\rangle \) and \( \tilde{b}_{r}(\alpha_{r}^-)^{-s-1} |p,p_{\phi}\rangle \) ;

(ii) For \( r, s > 0 \) : \( (\alpha_{-r}^+)^{s} |p,p_{\phi}\rangle \) and \( c_{-r}(\alpha_{-r}^+)^{s-1} |p,p_{\phi}\rangle \).

\( \text{15} \)
Proof: If \( P^+(r) = P^-(s) = 0 \), eq. (4.22) implies that \( p^+p^- = -rs \). Thus, in order to have states in the kernel of \( L_0 = \hat{L}_0 + p^+p^- \), one needs \( rs > 0 \). Now, define the operator

\[
K_r = \sum_{n \neq 0,r} \frac{1}{P^+(n)} \alpha_{-n}^+ \tilde{b}_n - \sum_{n \neq 0} n\gamma_n \zeta_{-n} .
\]  

It satisfies:

\[
\{\hat{d}_0, K_r\} = \hat{L}_{0,r} ,
\]

giving \( \hat{L}_{0,r} \)—the level operator for all the oscillators except \( b_r, c_r \) and \( \alpha_r^-, \alpha_r^+ \). Since any state in the cohomology of the theory must now be in the kernel of \( \hat{L}_{0,r} \), it must be generated by these remaining oscillators. The only such states satisfying the condition \( \hat{L}_0 = rs \) are the ones written in (4.28) and (4.29). It is trivial to see that these states are indeed in the cohomology of \( \hat{d}_0 \).

4.4 The full cohomology of \( Q_B \)

We now have all the necessary ingredients to classify the cohomology of \( Q_B \). First, an elementary result of the cohomology theory of filtered complexes states that if the cohomology of \( \hat{d}_0 \) occurs at only one degree, the cohomology of \( \hat{d} \) is in a one to one correspondence with the cohomology of \( \hat{d}_0 \) [28]. Together with theorems 3 and 4, this immediately implies that:

**Theorem 5** The relative cohomology of \( \hat{d} \) is given by:

(i) The tachyons: A one-dimensional cohomology at ghost number 0, with mass-shell condition \( p^+p^- = 0 \).

(ii) A one-dimensional cohomology at ghost numbers \(-1\) and 0 for all \( p^+ = r, p^- = -s \), with \( r \) and \( s \) negative integers.

(iii) A one-dimensional cohomology at ghost numbers 0 and \(+1\) for all \( p^+ = r, p^- = -s \), with \( r \) and \( s \) positive integers.

We have already seen that the tachyonic states in the cohomology of \( \hat{d}_0 \) are in fact states in the full cohomology of \( \hat{d} \). It is less easy to find explicit representatives for the discrete states in the cohomology, but they are classified by this theorem.

The full cohomology of \( Q_B \) now follows from theorems 1 and 2: Denoting the states of theorem 5 by \( |\psi\rangle \), the absolute cohomology is generated by the states
\[ |\psi\rangle \otimes |P = 0\rangle \otimes |\uparrow\rangle_{\zeta\eta} \otimes |\downarrow\rangle_{bc} \]  
(4.32)

and

\[ a_0 \left( |\psi\rangle \otimes |P = 0\rangle \otimes |\uparrow\rangle_{\zeta\eta} \otimes |\downarrow\rangle_{bc} \right), \]
(4.33)

and their conjugates*:

\[ |\psi\rangle \otimes |\mathcal{X} = 0\rangle \otimes |\downarrow\rangle_{\zeta\eta} \otimes |\downarrow\rangle_{bc} \]
(4.34)

and

\[ a_0 \left( |\psi\rangle \otimes |\mathcal{X} = 0\rangle \otimes |\downarrow\rangle_{\zeta\eta} \otimes |\downarrow\rangle_{bc} \right). \]
(4.35)

4.5 Comparison to the conformal gauge

Recall that in studying the spectrum, we can restrict ourselves to the states built on \(|P = 0\rangle\). In order to compare the cohomology that we have found to that of the conformal gauge, it is convenient to change to a more conventional notation. First, the tachyon operators in the absolute cohomology can be written as:

\[ c e^{ipx+i\phi^+} \xrightarrow{a} ace^{ipx+i\phi^+} , \]
(4.36)

with

\[ p_\phi^\pm = -i(\sqrt{2} \pm |p|); \]
(4.37)

they agree with the tachyon states of the conformal gauge.

For the discrete states, associate the “\(b_r\)” states of theorem 4 with the operators \(O_{u,n}\), the “\(\alpha^-\)” states with the operators \(Y^+_{u+1,n}\), the “\(\alpha^+\)” states with the operators \(Y^-_{u+1,n}\) and the “\(c^-\)” states with the operators \(P_{u,n}\) states. \(u\) and \(n\) are related to \(r\) and \(s\) by:

\[ u = \frac{|r+s|}{2} - 1 \]

\[ n = \frac{r-s}{2}, \]
(4.38)

so \(u, n \in \mathbb{Z}/2\), and \(|n| \leq u\). Also note that the states with \(r, s\) negative have \(\phi\) “momenta” satisfying the Seiberg condition [23], while those with \(r, s\) positive are anti-Seiberg states. The discrete states now fall into the familiar “diamond” form of Witten and Zwiebach.

*Note that in this case, \(|\psi\rangle\) must be written in terms of the tilded oscillators.
and therefore agree with those of the conformal gauge [4, 5, 8]. We thus obtain the main result of this section:

*The cohomology of the light-cone theory in the Fock-space representation of the current algebra is equivalent to that of the Liouville theory.*

The ground ring in the light-cone gauge

Since the structure of the cohomology of the light-cone gauge (in the Fock space) is equivalent to that of the conformal gauge, the only problem remaining to us is to find explicit representatives of the cohomology. Start with the spin zero, ghost number zero operators \( O_{u,n} \). In the conformal gauge, they form a commutative “ground ring” [8], in that under the operator product expansion:

\[
O_s,n(z)O_{s',n'}(0) \sim O_{s+s',n+n'}(0),
\]

up to states exact under \( Q_B \). Denoting \( O_{1,1/2} \) by \( X \) and \( O_{1,-1/2} \) by \( Y \), one sees that the ring is a polynomial ring generated by \( X \) and \( Y \), with

\[
O_{s,n} = X^{s+n}Y^{s-n}. \tag{4.41}
\]

Note that \( O_{0,0} \) is the identity operator. In the conformal gauge \( X \) and \( Y \) are given by

\[
X = \left( cb + \frac{i}{\sqrt{2}}(\partial x - i\partial \phi) \right) e^{i(x+i\phi)/\sqrt{2}},
\]

\[
Y = \left( cb - \frac{i}{\sqrt{2}}(\partial x + i\partial \phi) \right) e^{-i(x-i\phi)/\sqrt{2}}. \tag{4.42}
\]

In the light-cone gauge this structure is reproduced, since the arguments leading to the existence of the ground ring are unchanged, and the two cohomologies are in a one-to-one correspondence. However, before we proceed, we need to check that the identity operator \( O_{0,0} = I \) or, rather, the vacuum state \(|0\rangle \) is in the cohomology. The vacuum state occurs
at the first level of the theory, with \( r = s = -1 \). In the cohomology of \( \hat{d}_0 \) at this level, theorem 4 gives us the state \( \tilde{b}_{-1} | p = 0, p_\varphi = 0 \rangle \), which is also in the cohomology of \( \hat{d} \). Using theorem 2, we obtain the relative-cohomology state \( b_{-1} | p = 0, p_\varphi = 0, \gamma_0 = 0 \rangle \otimes | \uparrow \rangle \zeta \eta \otimes | \downarrow \rangle \), which is indeed the vacuum.

The construction of the generators of the ring \( X \) and \( Y \) is a little more complicated. They occur at the second level, with \( rs = 2 \). The relevant states of the \( \hat{d}_0 \) cohomology are given by theorem 4:

\[
\begin{align*}
    &r = -1, s = -2 : & |X_0\rangle &= \tilde{b}_{-1} \alpha \zeta^{-1} \left| \frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}} \right\rangle \\
    &r = -2, s = -1 : & |Y_0\rangle &= \tilde{b}_{-2} \left| \frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}} \right\rangle .
\end{align*}
\]

In order to obtain the corresponding states in the cohomology of \( \hat{d} \), we use an inductive procedure of ref. [5], which we briefly illustrate in our case: Starting with a state \( \psi_0 \) in the cohomology of \( \hat{d}_0 \), construct the state \( \psi_1 \) as

\[
\psi_1 = -\hat{L}^{-1}_{0,r} K_r \hat{d}_1 \psi_0 ,
\]

where the operators \( \hat{L}_{0,r}, K_r \) are given in eqs. (4.30) and (4.31). \( \psi_1 \) is of degree one greater than \( \psi_0 \). Continue inductively, defining

\[
\psi_{k+1} = -\hat{L}^{-1}_{0,r} K_r (\hat{d}_1 \psi_k + \hat{d}_2 \psi_{k-1}) .
\]

Then \( \psi = \psi_0 + \psi_1 + \psi_2 + \cdots \) is in the cohomology of \( \hat{d} \). In our case the procedure stops after two steps, giving the states

\[
\begin{align*}
    &|X\rangle = \left( \tilde{b}_{-2} + \tilde{b}_{-1} \alpha \zeta^{-1} - \zeta^{-1} \gamma^{-1} \right) \left| \frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}} \right\rangle \\
    &|Y\rangle = \left( \tilde{b}_{-2} + \alpha^{+} \tilde{b}_{-1} + \zeta^{-1} \gamma^{-1} \right) \left| -\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}} \right\rangle .
\end{align*}
\]

Using theorem 2 to turn these states into relative-cohomology states, we see that the ground ring in the light-cone gauge is generated by the operators:

\[
\begin{align*}
    X^{lc} &= \left( cb + \partial \alpha^+ + c(\partial \zeta) \gamma \right) e^{\frac{i(x+i \varphi)}{\sqrt{2}}} \\
    Y^{lc} &= \left( cb - \partial \alpha^+ + c(\partial \zeta) \gamma \right) e^{-\frac{i(x-i \varphi)}{\sqrt{2}}} .
\end{align*}
\]

These operators have extra pieces depending on \( \zeta \) and \( \gamma \) compared to \( X \) and \( Y \) of the conformal gauge. Since the \((\beta, \gamma)\) and \((\zeta, \eta)\) fields appear here, the light-cone theory cannot be regarded simply as the Liouville theory plus a topological theory.

*Note that on all states with \( P = \gamma_0 = 0 \), one can drop the “tildes” on the oscillators, since the difference between the tilded and untilded oscillators is proportional to \( \gamma_0 \) (see eqs. (4.13)).*
Higher ghost number states, and the chiral $w_\infty$ algebra

We can continue in the same manner to study the higher ghost number states. Since, by theorem 1, the operator $a$ needed to obtain the absolute cohomology has the same form in the light-cone as in the conformal gauge, we can restrict ourselves to studying the relative cohomology operators $Y^{\pm}_{j,m}$ and $P_{u,n}$. The usual way of constructing the $Y^{\pm}_{j,m}$ operators in the conformal gauge is to start with the extra primary states of the $c = 1$ conformal field theory of a scalar compactified on a circle with self-dual radius $R = \sqrt{2}$, $V_{j,m}$ [30], and to dress them with the Liouville field to produce the operators $W^{\pm}_{j,m}$:

$$W^{\pm}_{j,m} = V_{j,m}e^{(\sqrt{2}(1\mp j)\phi)}.$$  

(4.48)

As was shown in [17], the charges $Q^{\pm}_{j,m} = \frac{1}{2\pi i} \oint W^{\pm}_{j,m}$ satisfy a chiral $w_\infty$ algebra. One can then form the desired BRST invariant operators $Y^{\pm}_{j,m}$:

$$Y^{\pm}_{j,m} = cW^{\pm}_{j,m}.$$  

(4.49)

However, one can also calculate these operators directly from the $\hat{d}_0$ cohomology states, using the method illustrated in the previous section. Note that the states of the light-cone $\hat{d}_0$ cohomology have the same form as those of the conformal gauge in ref. [5]. Also, except for the ground-ring states, the new light-cone oscillators $\beta_n, \gamma_n, \zeta_n, \eta_n$ do not appear when $\hat{d}_1$ and $\hat{d}_2$ act on the states. The procedure therefore gives exactly the same result in the light-cone as in the conformal gauge, and the operators $Y^{\pm}_{j,m}$ that we had previously are representatives of the cohomology in the light-cone gauge. The ghost number two operators $P_{u,n}$ are also the same as those of the conformal gauge.

5 The Kac-Moody BRST cohomology from Felder’s resolution

5.1 The Kac-Moody and Fock-space cohomologies at the first level

We can now return to the problem of obtaining the full cohomology on the irreducible Kac-Moody modules, from the cohomology on the Fock space. First, in order to illustrate the issues involved in this procedure, we shall examine both of these cohomologies at the first level, restricting ourselves to the relative cohomology of $\hat{Q}$. The states in the Fock space are easily derived using the results of the previous section, and are summarized in table 1*. As was shown in the previous section, the light-cone Fock-space states are equivalent to those of the conformal gauge; in fact, at this level they are identical. The

*From this section on, we measure ghost numbers from the vacuum state $|0\rangle$.  

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Table 1: First-level states in the Fock space.

| ghost number | state \( \langle p_\varphi \rangle \otimes |\gamma_0 = 0 \rangle \otimes |\uparrow\rangle \zeta \eta \rangle \) | operator |
|--------------|-------------------------------------------------|----------|
| 0            | \(-b_1 |0\rangle\)                                 | \(I\)    |
| 1            | \(-x_1 |0\rangle\)                                 | \(c \partial x\) |
|              | \(-x_1 | -2i\sqrt{2}\rangle\)                       | \(c \partial x e^{2\sqrt{2}\varphi}\) |
| 2            | \(-c_1 | -2i\sqrt{2}\rangle\)                       | \(c \partial c e^{2\sqrt{2}\varphi}\) |

states of the relative cohomology of the current algebra at the first level are derived in appendix A, and the states built on lowest weight states (LWS) are given in table 2. (All the states in the cohomology at level 1 have \(p = 0\), so we shall suppress \(p\) in this section, and write only \(p_\varphi\) or \(J\) and \(M\). We also suppress the \(|\downarrow\rangle_{bc}\).) Finally, we show the conjugate states of the Fock-space theory in table 3, and the HWS of the current algebra in table 4.

In order to compare the current algebra to the Fock space, one needs the relation between the \(|J, M\rangle\) states and the “Liouville” states \(|p_\varphi\rangle\). For generic\(^1\) HWS and LWS, this is determined from the definitions of the currents in eqs. (4.1) to be:

\[\langle J, J \rangle \leftrightarrow |p_\varphi = \sqrt{2i(J - \frac{1}{2})} \rangle \otimes |\beta_0 = 0\rangle\]

\(^1\)If \(2J + 1 \in \mathbb{N}\), the HWS and LWS are both in the same finite \(SU(2)\) representation so this identification is not unique. We shall see the importance of this later.
Table 3: First-level conjugate states in the Fock space.

| ghost number | state \( (|p_\varphi\rangle \otimes |\beta_0 = 0\rangle \otimes |\downarrow\rangle_{\zeta\eta}) \) |
|--------------|--------------------------------------------------|
| -1           | \( (b_{-1} - \zeta_{-1}\gamma_0) |0\rangle \) |
| 0            | \( x_{-1} |0\rangle \) |
|              | \( x_{-1} \left|{-2i\sqrt{2}}\right\rangle \) |
| 1            | \( c_{-1} \left|{-2i\sqrt{2}}\right\rangle \) |

Table 4: First-level conjugate states in the current algebra.

| ghost number | state \( (|J, M\rangle \otimes |\downarrow\rangle_{\zeta\eta}) \) |
|--------------|--------------------------------------------------|
| -1           | \( \zeta_{-1} \left|{-\frac{3}{2}, -\frac{3}{2}}\right\rangle \) |
|              | \( (b_{-1} - \zeta_{-1}J_0^+) \left|{\frac{1}{2}, \frac{1}{2}}\right\rangle \) |
| 0            | \( x_{-1} \left|{\frac{1}{2}, \frac{1}{2}}\right\rangle \) |
|              | \( x_{-1} \left|{-\frac{3}{2}, -\frac{3}{2}}\right\rangle \) |
| 1            | \( c_{-1} \left|{-\frac{3}{2}, -\frac{3}{2}}\right\rangle \) |
Using these relations, one sees from the tables that the states in the current algebra are in a one-to-one correspondence with those of the Fock space (or equivalently with those of the conformal gauge), except for the extra pair of conjugate states \( \eta_{-1} \mid -\frac{3}{2}, \frac{3}{2} \rangle \otimes \uparrow \rangle_\zeta \eta \) and \( \zeta_{-1} \mid -\frac{3}{2}, -\frac{3}{2} \rangle \otimes \downarrow \rangle_\zeta \eta \) in the current algebra. The existence of these extra states at the first level indicates that the BRST cohomology on the irreducible \( SL(2, R) \) Kac-Moody modules is inequivalent to that of the conformal gauge. However, since there are an infinite number of discrete states in both cohomologies, we can not yet draw a definite conclusion. This will be possible after we obtain the full Kac-Moody cohomology in section 5.3.

**From free fields to current algebra at the first level**

In order to understand why extra states appeared in the current algebra formulation, it is useful to recall some facts about free-field representations of theories: A free field description of a conformal field theory provides a realization of the chiral algebra—in our case the \( SL(2, R) \) Kac-Moody algebra—by free fields. In addition, one needs to have a projection from the free-field Fock space to the irreducible representations of the chiral algebra. This projection is essentially a null states decoupling, and needs to take care of two problems that are dual to each other:

(i) Singular states: Null states in the Kac-Moody module do not necessarily vanish in the free-field Fock space.

(ii) Cosingular states: There can exist states in the Fock space that do not have analogous states in the irreducible Kac-Moody module.

The extra states that we have found in the first level relative cohomology illustrate both cases: The state \( \zeta_{-1} \mid -\frac{3}{2}, -\frac{3}{2} \rangle \otimes \downarrow \rangle_\zeta \eta \) is closed, since

\[
\hat{Q} \left( \zeta_{-1} \mid -\frac{3}{2}, -\frac{3}{2} \rangle \otimes \downarrow \rangle_\zeta \eta \right) = J_{-1} \mid -\frac{3}{2}, -\frac{3}{2} \rangle \otimes \downarrow \rangle_\zeta \eta ,
\]

and \( J_{-1} \mid -\frac{3}{2}, -\frac{3}{2} \rangle \) is a null state, being annihilated by all \( J_{n>0} \)'s. However, acting \( \hat{Q} \) on the equivalent state in the Fock-space realization, one obtains

\[
\hat{Q} \left( \zeta_{-1} \mid p_\varphi = -2\sqrt{2}i, \beta_0 = 0 \rangle \otimes \downarrow \rangle_\zeta \eta \right) = \beta_{-1} \mid p_\varphi = -2\sqrt{2}i, \beta_0 = 0 \rangle \otimes \downarrow \rangle_\zeta \eta ,
\]

which is a nonvanishing singular vector. Thus the state is not closed in the Fock-space analysis, and one has an extra state in the current algebra.

In the conjugate case, the analogue of the extra state \( \eta_{-1} \mid -\frac{3}{2}, \frac{3}{2} \rangle \) in the Fock space is exact:

\[
\eta_{-1} \mid p_\varphi = 0, \gamma_0 = 0 \rangle \otimes \uparrow \rangle_\zeta \eta = \hat{Q} \left( \gamma_{-1} \mid p_\varphi = 0, \gamma_0 = 0 \rangle \otimes \uparrow \rangle_\zeta \eta \right) .
\]
However, one cannot construct the analogue of the cosingular state \( \gamma_{-1} |p_\varphi = 0, \gamma_0 = 0 \rangle \) in the Kac-Moody module, since

\[
\gamma_{-1} |p_\varphi, \gamma_0 = 0 \rangle = \frac{i}{\sqrt{2p_\varphi}} J_{-1}^+ |p_\varphi, \gamma_0 = 0 \rangle
\]  

(5.5)

becomes ill defined as \( p_\varphi \to 0 \). Thus, again, an extra state appears in the analysis on the Kac-Moody module.

### 5.2 The Felder resolution

We have seen at the first level that the BRST cohomology on the irreducible \( SL(2, R) \) Kac-Moody module is different than the BRST cohomology on the free field Fock space. The general mechanism for obtaining cohomologies on irreducible modules of chiral algebras from those of free field Fock spaces is called a Felder resolution [31]. It is needed whenever the Fock-space module \( F_\Lambda \) carries a reducible representation of the chiral algebra. One then needs a projection from the free-field Fock space to the irreducible representations of the chiral algebra. This is done by a BRST like procedure: First one needs operators \( Q^{(i)} \) that commute with the chiral algebra. These operators are built out of screening operators. They change the momentum of the Fock space, giving a map between different modules \( F^{(i)}_\Lambda \), with \((i)\) denoting the Felder index. One then has a complex, which is generically of the form:

\[
\cdots \rightarrow F^{(-1)}_\Lambda \xrightarrow{Q^{(-1)}} F^{(0)}_\Lambda \xrightarrow{Q^{(0)}} F^{(1)}_\Lambda \rightarrow \cdots
\]  

(5.6)

Since the \( Q^{(i)} \)'s commute with the chiral algebra, the cohomology groups of the complex are chiral algebra modules. The cohomology is nontrivial only at one Felder index, chosen to be 0, where it is isomorphic to the irreducible representation \( L_\Lambda \) of the chiral algebra.

In our case the chiral algebra is an \( A_1^{(1)} \sim SL(2, R) \) Kac-Moody algebra of level \( \kappa = -3 \). Representations are characterized by their vacuum, either a HWS or a LWS with spin \( J \). When \( \kappa = -3 \), one can see from the Kac-Kazhdan formula [32] that representations can be reducible only when \( J \in \mathbb{Z}/2 \). The basic inclusion diagrams, illustrating which representations are contained in each other, are given in figure 1. We shall denote the Fock spaces modules built on \( |\gamma=0\rangle \) by \( F_J \), and their conjugate modules built on \( |\beta=0\rangle \) by \( F_J^* \), where \( J \) is determined from \( p_\varphi \) by eqs. (5.1). Recall that the vacuum state of \( F_J \) is a LWS, while that of \( F_J^* \) is a HWS. The BRST operators are built from the basic screening operator which, in our case, is given by:

\[
V \equiv \beta e^{\sqrt{2}p_\varphi}.
\]  

(5.7)
The operator $Q_n$ is made by taking appropriate contour integrals of the product of $n$ V’s [31]. Thus, as can be seen from eq. (5.1), it raises the $J$’s of the Fock spaces, $Q_n : F_J \rightarrow F_{J+n}$, and lowers the $J$’s of the conjugate Fock spaces, $Q_n : F^*_J \rightarrow F^*_{J-n}$.

In order to obtain the resolution of the algebra, we now need to build the Felder complex. The Felder resolution for $A_1^{(1)}$ has been carried out for the cases where $\kappa + 2$ is a “generic” complex number, giving an inclusion diagram with just two representations, and when $\kappa + 2$ is a positive rational number, giving a double-branch inclusion diagram [26]. It has not been carried out for our degenerate single-branch case. However, the Felder resolutions of the Virasoro algebra and of the $A_1^{(1)}$ Kac-Moody algebra share many properties, and the resolution of the single-branch Virasoro case has been carried out* [33]. In that case it was found that the complex reduces to short complexes of the type $0 \rightarrow A \rightarrow B \rightarrow 0$. We shall therefore assume that such a structure appears in our case.

Examining the form of the inclusion diagrams of figure 1, this leads us to the following conjecture for the Felder complex:

**Claim 6** For $J \in \mathbb{Z}/2$, $J \neq -\frac{1}{2}, -1$, the Felder complex of the $\kappa = -3$ $A_1^{(1)}$ Kac-Moody

*We would like to thank Profs. P. Bernard and G. Felder for e-mail concerning this issue.
algebra is:

\[
J \geq 0 : \quad 0 \rightarrow F_{-J-1}^{(1)} \xrightarrow{Q_{2J+1}} F^{(0)}_{J} \rightarrow 0
\]

\[
J \leq -\frac{3}{2} : \quad 0 \rightarrow F_{J}^{(0)} \xrightarrow{Q_{-2J-3}} F_{-J-2}^{(1)} \rightarrow 0 ,
\]

with the dual complex:

\[
J \geq 0 : \quad 0 \rightarrow F_{-J-1}^{*}^{(0)} \xrightarrow{Q_{2J+1}} F^{*}_{-J-1}^{(-1)} \rightarrow 0
\]

\[
J \leq -\frac{3}{2} : \quad 0 \rightarrow F_{-J-2}^{*}^{(-1)} \xrightarrow{Q_{-2J-2}} F^{*}_{J}^{(0)} \rightarrow 0 .
\]

(5.8)

(5.9)

For all other J’s the complex is trivial. The only nontrivial cohomology of the complex occurs at index \( i = 0 \), and is isomorphic to the irreducible Kac-Moody module \( \Lambda_J \).

We have not proven this conjecture, but have checked that it gives the correct results on Fock spaces with small J’s. In particular the conjectured Felder cohomology takes care of the leading singular vector \( (\beta_0)^{2J+1} |J, -J\rangle \) for \( J \geq 0 \), and the leading cosingular vector \( (\gamma_{-1})^{-2J-2} |J, -J\rangle \) for \( J \leq -\frac{3}{2} \).

5.3 The Kac-Moody cohomology

The relative cohomology of \( \hat{Q} \) acting on the current algebra at ghost number \( n \) is given by \( H_{rel}^{n}(\Lambda_J) = H_{rel}^{n}(H_F^{0}) \). This cohomology can be evaluated using the theorem:

**Theorem 7** If the Felder complex exists only at degrees 0 and either +1 or -1, then \( H_{rel}^{0}(H_F^{0}) = H_{F}^{0}(H_{rel}) + H_{F}^{\pm 1}(H_{rel}^{n+\pm 1}) \).

The proof is given in appendix B.

In order to see the implications of this theorem, we need to find \( H_{F}^{0}(H_{rel}) \) and \( H_{F}^{\pm 1}(H_{rel}^{n+\pm 1}) \). \( H_{rel}^{n} \) consists of the states of ghost number \( n \) that we have found previously in the cohomology of the Fock space. Denote such states by \( \Psi_{J}^{n} \). One can simplify the analysis by noting that, except for the “ground-ring” states, these \( \Psi \)'s are all in the vacuum state of the Felder Fock space, i.e. they contain no oscillator modes of the Wakimoto fields. Also, the ground-ring states do not have to be considered separately, since their conjugates are Felder vacuum states.

Now, consider first a Felder complex of the form \( 0 \rightarrow F_{J}^{(0)} \xrightarrow{Q_{F}} F_{J'}^{(1)} \rightarrow 0 \). In this case, \( H_{F}^{0}(H_{rel}) \) consists of the \( \Psi_{J}^{n} \)'s that are annihilated by \( Q_{F} \). Since the \( \Psi \)'s are vacuum vectors, and are therefore not cosingular, the cohomology consists simply of the states in the current algebra corresponding directly to the \( \Psi_{J}^{n} \)'s. In order to find states of
one needs to pull back (the vacuum state) \( \Psi_J^{n-1} \) to a cosingular vector \( \chi_J^{n-1} \), satisfying \( Q_F \chi_J^{n-1} = \Psi_J^{n-1} \). This can always be done since \( Q_F \) maps \( F_J \) onto \( F_{J'} \). The desired state in the current-algebra cohomology is then the state corresponding to \( \hat{Q} \) acting on the cosingular vector \( \chi_J^{n-1} \). Schematically,

\[
\Psi_J^n \sim \hat{Q} Q_F^{-1} \Psi_{J'}^{n-1} \tag{5.10}
\]

Note that this procedure has increased the ghost number of the state, as required by theorem 7. As an example of such a state, start with the LWS cosmological constant state

\[
\Psi_{1/2}^{1} = |\gamma_0 = 0, p_\varphi = 0, p = 0\rangle \otimes |\uparrow\rangle_{\zeta\eta} \tag{5.11}
\]

One can then find the cosingular state

\[
\chi_{-3/2}^1 = \gamma -1 |\gamma_0 = 0, p_\varphi = 0, p = 0\rangle \otimes |\uparrow\rangle_{\zeta\eta} \tag{5.12}
\]

satisfying \( Q_1 \chi_{-3/2}^1 = \Psi_{-1/2}^1 \). As seen in eq. (5.4),

\[
\Psi_{-3/2}^2 = \hat{Q} \chi_{-3/2}^1 \tag{5.13}
\]

then corresponds to the extra first-level state \( \eta_{-1} \left| J = -\frac{3}{2}, M = \frac{3}{2}, p = 0 \right\rangle \otimes |\uparrow\rangle_{\zeta\eta} \) found in section 5.1.

In the cases when the Felder complex is of the form \( 0 \to F_{J'}^{(1)} \xrightarrow{Q_F} F_{J'}^{(0)} \to 0, H_F^{(n)}(H_{rel}) \) again consists of the states in the current algebra corresponding to the \( \Psi_{J'}^{n} \)'s, since none of the \( \Psi \)'s are singular vectors. For \( H_F^{(n)}(H_{rel}) \), one maps the state \( \Psi_J^n \) to the singular vector \( \chi_J^n = Q_F \Psi_{J'}^n \). Since this singular vector is not in the Fock-space cohomology, it can be written as \( \chi_J^n = \hat{Q} \Psi_J^{n-1} \), and \( \Psi_J^{n-1} \) corresponds to the desired state. Thus

\[
\Psi_J^{n-1} \sim \hat{Q}^{-1} Q_F \Psi_{J'}^n \ , \tag{5.14}
\]

and the ghost number of the state has been reduced by 1, as required by theorem 7. As an example of this type, start with the HWS cosmological constant state

\[
\Psi_{-1/2}^0 = |\beta_0 = 0, p_\varphi = -\sqrt{2}i, p = 0\rangle \otimes |\downarrow\rangle_{\zeta\eta} \tag{5.15}
\]

Acting on it by \( Q_1 \), one finds the singular vector

\[
\chi_{-3/2}^0 = \beta_{-1} |\beta_0 = 0, p_\varphi = -2\sqrt{2}i, p = 0\rangle \otimes |\downarrow\rangle_{\zeta\eta} \tag{5.16}
\]

As seen in eq. (5.3), solving \( \hat{Q} \Psi_{-3/2}^1 = \chi_{-3/2}^0 \) gives one the analogue of the extra first-level state \( \zeta_{-1} \left| J = -\frac{3}{2}, M = -\frac{3}{2}, p = 0 \right\rangle \otimes |\downarrow\rangle_{\zeta\eta} \) of section 5.1.
Figure 2: The relative-cohomology states in the current algebra built on (a) the HWS and (b) the LWS of the Fock space. Only the $p \geq 0$ sector is shown. Discrete states are labeled by the ghost number of the operators, discrete tachyons by “blobs”. The starred states are the states obtained nontrivially by the Felder resolution. Reflections through the dashed lines at $J = -\frac{1}{2}$ and $J = -1$ give the $J$’s related by the Felder operation for positive and negative $J$’s, respectively.
The complete relative cohomology on the current algebra

In order to keep track of this procedure for arbitrary \( J \), it is convenient to plot the starting Fock-space cohomology with respect to \( p \) and \( J(p_\varphi) \). This is done in figure 2, to which the reader is referred. There, the unstared states and the tachyons refer to the states of the relative Fock-space cohomology. Graph 2a contains the HWS states in the sector \( |\beta_0\rangle \); graph 2b the LWS states in the sector \( |\gamma_0 = 0\rangle \). We now use the Felder procedure to obtain the states of the current algebra. As was argued above, all the Fock-space states have direct analogues in the current algebra, but there are other states in addition.

Consider first the case \( J \geq 0 \). The alert reader may have noticed that we found no new states of this form in the cohomology of the current algebra at the first level. That this persists to all orders can be seen most easily by considering the HWS states in graph 2a. We remind the reader that these states are of the form \( |\downarrow\rangle_{\zeta\eta} \). In this case, the \( J \geq 0 \) Felder complex of theorem 6 is of the form \( 0 \to F_J^{(0)} \xrightarrow{Q} F_{J-1}^{(1)} \to 0 \). Thus one can find new states \( \Psi_J^n \) from the states \( \Psi_{J-1}^{n-1} \), which are found by reflecting \( \Psi_J^n \) through the line \( J = -\frac{1}{2} \) in the graph. The cosingular states \( \chi_J^{n-1} \) satisfying

\[
Q_{2J+1} \chi_J^{n-1} = \Psi_{J-1}^{n-1}
\]  

(5.17)

can be taken to be simply

\[
\chi_J^{n-1} = \gamma_0 2^{J+1} e^{-\sqrt{2}(2J+1)\varphi} \Psi_{J-1}^{n-1}.
\]

(5.18)

Since \( [\hat{Q}, \gamma_0] = -\eta_0 \), the desired state is given by:

\[
\Psi_J^n = \hat{Q} \chi_J^{n-1} = \left( \hat{Q} + \gamma_0 2^{J+1} e^{-\sqrt{2}(2J+1)\varphi} \right) \Psi_{J-1}^{n-1} \\
\sim \eta_0 \gamma_0 2^{J} e^{-\sqrt{2}(2J+1)\varphi} \Psi_{J-1}^{n-1} \\
\sim \eta_0 \left( j^{-} \right)^{2J} e^{-\sqrt{2}(2J+1)\varphi} \Psi_{J-1}^{n-1}.
\]

(5.19)

Note that this new state is at the same level as the original state, and its ghost number is one larger because the \( |\downarrow\rangle_{\zeta\eta} \) is raised by \( \eta_0 \). Also, the new state is a LWS of spin \( J \), since the HWS has been lowered \( 2J \) times. Comparing graphs 2a and 2b, one can see that these new states obtained by the Felder procedure are simply the LWS that we already know. The reason for this is that the HWS and LWS are conjugate states in the same representation of the current algebra, for \( 2J + 1 \in \mathbb{N} \), but are not in the same representation in the Fock space. They can therefore be found from both sectors of the Fock space, after performing the Felder resolution.
On the other hand, when \( J \leq -\frac{3}{2} \) one gets genuine new states. We have already seen this at the first level. Particularly interesting states of this type are the states of ghost number 3, with conjugates of ghost number \(-2\), since they can not have any analogues in the conformal gauge. As can be seen from graph 2b, the first such state occurs at \( J = -\frac{5}{2} \), and comes from the \( \Psi_{1/2}^2 \) state \( c_{-1} |\gamma_0 = 0, p_\varphi = -2i\sqrt{2}, p = 0 \rangle \otimes |\uparrow\rangle_{\varsigma\eta} \) of table 5.1. In this case, the cosingular vector \( \chi_{-5/2}^2 \) satisfying \( Q_3 \chi_{-5/2}^2 = \Psi_{1/2}^2 \) can be taken to be:

\[
\chi_{-5/2}^2 = c_{-1}^{-1} |\gamma_0 = 0, p_\varphi = i\sqrt{2}, p = 0 \rangle \otimes |\uparrow\rangle_{\varsigma\eta} .
\]  

Then the desired new ghost number 3 state is

\[
\Psi_{-5/2}^3 = \hat{Q} \chi_{-5/2}^2 \\
= \eta_{-1} c_{-1} \gamma_1^2 |\gamma_0 = 0, p_\varphi = i\sqrt{2}, p = 0 \rangle \otimes |\uparrow\rangle_{\varsigma\eta} \\
\rightarrow \eta_{-1} c_{-1} (J_+^2) |J = -\frac{5}{2}, M = \frac{5}{2}, p = 0 \rangle \otimes |\uparrow\rangle_{\varsigma\eta} .
\]  

(5.21)

Using the procedures given above, one can also obtain its conjugate state with ghost number \(-2\):

\[
\Psi_{-5/2}^{-2} = \zeta_{-1} \left(b_{-1} J_{-1} - 2\zeta_{-2}\right) J_{-1} |J = -\frac{5}{2}, M = -\frac{5}{2}, p = 0 \rangle \otimes |\downarrow\rangle_{\varsigma\eta} .
\]  

(5.22)

One can check that this state is not exact, and that it closes up to a singular vector.

**The absolute cohomology on the current algebra**

The only way to avoid the conclusion that the current algebra cohomology is different than that of the Fock space, would be for these extra states with funny ghost numbers to not appear in the absolute cohomology. The absolute cohomology can be found from the relative cohomology using a long exact sequence, with the operators \( i \) (inclusion), \( b_0 \) and \( M = \sum_{n \neq 0} n c_{-n} c_n \) [34]. If the relative cohomology exists at only two consecutive ghost numbers the operator \( M \) must be trivial, so one ends up with a short exact sequence. This means that the absolute cohomology is simply a doubling of the relative cohomology. In our case the cohomology exists at several ghost numbers, so the argument does not go through. However, for the highest ghost number states, the exact sequence reduces to

\[
0 = H^4_{rel} \xrightarrow{i} H^4_{abs} \xrightarrow{b_0} H^3_{rel} \xrightarrow{M} H^5_{rel} = 0 ,
\]  

(5.23)

so \( H^4_{abs} \simeq H^3_{rel} \). At \( J = -\frac{5}{2} \), the ghost number 4 state of the absolute cohomology is given simply by raising the relative-cohomology state of eq. (5.21) with \( c_0 \), giving:

\[
\eta_{-1} c_{-1} (J_+^2) |J = -\frac{5}{2}, M = \frac{5}{2}, p = 0 \rangle \otimes |\uparrow\rangle_{\varsigma\eta} \otimes |\uparrow\rangle_{bc} .
\]  

(5.24)
Since there are no states with ghost number larger than 2 in the absolute cohomology of the conformal gauge, the existence of this state proves that

The BRST cohomology in the light-cone gauge, with the gravity state space being the irreducible $SL(2, R)$ Kac-Moody modules, is inequivalent to that of the conformal gauge.

We conclude that one should not work on irreducible $SL(2, R)$ Kac-Moody representations in the gravitational sector. It is interesting that a similar conclusion also holds in the conformal gauge, where the Liouville field is not taken to give an irreducible representation of the Virasoro algebra. It is also the case in topological $G/G$ theories; this is implicit in the works of refs. [35, 36], and can be seen in ref. [37].

6 Conclusions

In order to study the BRST cohomology of a theory coupled to gravity, one needs to define the state space of the gravity sector. In the conformal gauge in two dimensions the gravity sector is represented by the Liouville field, which is usually quantized as a free scalar with a background charge. In Polyakov’s light-cone gauge the gravity sector is represented by an $SL(2, R)$ Kac Moody algebra. In this case there are two reasonable choices for the gravity state space: one can work on irreducible $SL(2, R)$ modules, or on the Fock space of the (modified) Wakimoto free-field representation of the $SL(2, R)$. These two choices lead to different cohomologies. We have analyzed the cohomology of the $c = 1$ theory on both these spaces, and have shown how the cohomology on the current algebra can be derived from that on the Fock space using a Felder resolution.

In the Wakimoto representation, the theory contains a scalar field $\varphi$ playing the role of the Liouville field. In addition, there are extra fermionic ghost fields $(\eta, \zeta)$ and extra bosonic fields $(\beta, \gamma)$. Unlike the conformal gauge, the quantization of all these fields is straightforward, since the measure in the gravity sector is field independent. Since the theory in the Wakimoto representation has more fields and a more complicated structure than the Liouville theory, it is probably impractical for amplitude calculations. It is possible, however, to find the BRST cohomology in this gauge, and we see that the spectrum of the light-cone theory on the Fock space is equivalent to that of the Liouville theory. The operators generating the cohomology are identical to those of the Liouville theory, except for the generators of the ground ring. The light-cone ground ring operators depend on the extra fields, but their algebra is identical to that of the conformal gauge.

The BRST cohomology on irreducible $SL(2, R)$ Kac-Moody modules leads to a coho-
ology structure and spectrum different than that of the conformal gauge. In particular, the cohomology contains operators with ghost numbers up to 3 in the relative cohomology, and up to 4 in the absolute cohomology. These operators have no analogues in the conformal gauge. Since the Fock space cohomology agrees with that of the conformal gauge, and one would like to obtain gauge-independent results, we conclude that one should not work with irreducible representations in the gravitational sector.

In order to describe $c < 1$ matter theories coupled to 2d gravity, the free scalar describing the matter of the $c = 1$ theory should be replaced by a coloumb gas description. The gravity sector is still built from the Wakimoto fields representing the current algebra. As in the conformal gauge, the Fock-space cohomology consists only of tachyon states [5]. After employing the Virasoro-algebra Felder resolution [31] for the matter sector, one will see that the cohomology in the Fock space is the same as that of the conformal gauge [38, 5]. In fact, one has the stronger result that the same operators can represent the cohomology in both gauges. As in the $c = 1$ case, one could now find the cohomology on the irreducible $SL(2, R)$ modules using the Felder resolution for the gravity sector, and the resulting cohomology would be different from that of the conformal gauge.

In conclusion, we have seen that the cohomology structure in the light-cone gauge, with the appropriate choice of state space, is identical to that of the conformal gauge. This supports the claim that discrete states are physical objects.

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Appendix A  First level BRST cohomology on the $SL(2, R)$ Kac-Moody module

In this appendix we calculate explicitly the first level relative BRST cohomology of $\hat{Q}$ on the $SL(2, R)$ Kac-Moody modules. The expressions for the modes of the various stress tensors needed for the calculation are:

\begin{align*}
L_{m}^{\text{matt}} & = \frac{1}{2} \sum_{m} :x_{m}x_{n-m}: \\
L_{n}^{bc} & = \sum_{m} (2n-m):b_{m}c_{n-m}: \\
L_{n}^{\zeta \eta} & = - \sum_{m} m: \zeta_{m} \eta_{n-m}: \\
L_{n}^{\text{grav}} & = \sum_{m} \left( \frac{1}{2} J_{m}^{-} J_{n-m}^{-} + \frac{1}{2} J_{m}^{+} J_{n-m}^{+} - J_{n-m}^{0} J_{m}^{0} : - (n+1) J_{n}^{0} \right) .
\end{align*}

(A.1)

It is also useful to have the commutation relations of the Virasoro and Kac-Moody operators (recalling, from eq. (2.12) that $J^{0}$ is not a primary field):

\begin{align*}
[L_{n}, J_{m}^{a}] = (na - m) J_{n+m}^{a} - \frac{\kappa}{2} n(n+1) \delta_{n+m,0} \delta_{a,0} .
\end{align*}

(A.2)

At the first level, the relative BRST operator $\hat{Q}$ reduces to:

\begin{align*}
\hat{Q} & \sim \eta_{0} (J_{0}^{-} + c_{1} \zeta_{-1} - c_{-1} \zeta_{1}) + \eta_{-1} J_{-1}^{-} + \eta_{1} J_{1}^{-} + c_{-1} (L_{-1}^{\text{grav}} + p x_{1}) + c_{1} (L_{1}^{\text{grav}} + p x_{-1}) .
\end{align*}

(A.3)

Since the operators are conformally normal ordered, $L_{-1}^{\text{grav}}$ is given by

\begin{align*}
L_{-1}^{\text{grav}} & \sim J_{0}^{-} J_{-1}^{+} + J_{-1}^{-} J_{0}^{+} - 2 J_{1}^{0} J_{-1}^{0}
\end{align*}

(A.4)

Note that the order of the $J$’s is not the naive one. Also, at the first level, the $L_{0} = 0$ condition of eq. (3.10) means that

\begin{align*}
p = 0 \iff J = \frac{1}{2} \text{ or } J = \frac{3}{2} .
\end{align*}

(A.5)

It will turn out that $p$ vanishes for all the states in the cohomology at this level, and it will be suppressed in writing the states.
Ghost number $-1$

\begin{align*}
\hat{Q}(\zeta_{-1} \downarrow) &= J_{-1} \downarrow - J_0 \zeta_{-1} \uparrow \\
\hat{Q}(b_{-1} \downarrow) &= (p x_{-1} + L^{grav}_{-1}) \downarrow + (\zeta_{-1} - J_0 b_{-1}) \uparrow
\end{align*} \tag{A.6}

For the state $\zeta_{-1} \downarrow$ to be closed, it must be a highest weight state (HWS) and must be annihilated by $J_{-1}$. Upon acting with $J^+_0$, the latter condition yields $M = -\frac{3}{2}$, using the commutation relations of eq. (3.1) and the definition of $M$ from eqs. (3.2). Since $J = -\frac{3}{2}$, $p = 0$ by eq. (A.5). In addition, using eq. (A.4), one can verify that the state $(b_{-1} - \zeta_{-1} J^+_0) \frac{1}{2}, \frac{1}{2} \rangle$ is closed. Since there are no states of ghost number lower than $-1$, the closed states are not exact and thus belong to the relative cohomology.

Ghost number 0

\begin{align*}
\hat{Q}(b_{-1} \uparrow) &= (p x_{-1} + L^{grav}_{-1}) \uparrow \\
\hat{Q}(x_{-1} \downarrow) &= J_0^+ x_{-1} \uparrow + p c_{-1} \downarrow
\end{align*} \tag{A.7}

For the state $b_{-1} \uparrow$ to be closed $p$ must vanish, and $L^{grav}_{-1}$ on the state must give 0. Using eq. (A.4), we see that the state must be a lowest weight state (LWS) with $J = -M = -\frac{3}{2}$. This state corresponds to the identity operator. The state $x_{-1} \downarrow$ is closed provided it is a HWS with $p = 0$. Thus, again, $J = -\frac{3}{2}$ or $J = \frac{1}{2}$. One can verify that all these states are not exact.

Ghost number 1

\begin{align*}
\hat{Q}(x_{-1} \uparrow) &= p c_{-1} \uparrow \\
\hat{Q}(c_{-1} \downarrow) &= -J_0^- c_{-1} \uparrow
\end{align*} \tag{A.8}

The first state is closed for $p = 0$, and so has $J = -\frac{3}{2}$ or $J = \frac{1}{2}$. Unless it is a LWS, it is exact, being $\hat{Q}(x_{-1} \downarrow)$. The second state is closed if it is a HWS. It is then exact, $\hat{Q}(x_{-1} \downarrow)$, unless $p = 0$. It must have $J = -\frac{3}{2}$, since the $J = \frac{1}{2}$ state is exact, being $\hat{Q}(J_0^+ \downarrow)$.

Ghost number 2

\begin{align*}
\hat{Q}(c_{-1} \uparrow) &= 0 \\
\hat{Q}(\eta_{-1} \uparrow) &= 0
\end{align*} \tag{A.9}

These states are trivially closed. $c_{-1} \uparrow$ is exact, $\hat{Q}(c_{-1} \downarrow)$, unless it is a LWS. It is also exact, $\hat{Q}(x_{-1} \uparrow)$, unless $p = 0$. Using:

\begin{align*}
\hat{Q}(\eta_{-1} \downarrow) &= c_{-1} \uparrow - J_0^- \eta_{-1} \uparrow
\end{align*}
\[ \hat{Q}(J^+_1 |\uparrow\rangle) = -2(M - \frac{3}{2})\eta_{-1} |\uparrow\rangle + 2c_{-1}J^+_0 |\uparrow\rangle \]
\[ \hat{Q}(J^+_0 |\uparrow\rangle) = (-J^-_0 \eta_{-1} |\uparrow\rangle) + (M + \frac{3}{2})c_{-1} |\uparrow\rangle , \quad (A.10) \]

one sees that the \( J = -\frac{3}{2} \) state is exact, being \( \hat{Q}(\eta_{-1} |\downarrow\rangle - J^+_0 |\uparrow\rangle) \). Thus, \( J = \frac{1}{2} \). Using eqs. (A.10), one can also see that the state \( \eta_{-1} |\uparrow\rangle \) is exact unless it is a LWS with \( J = -\frac{3}{2} \).

There are no other nontrivial cohomology states at this level.

**Appendix B  A proof of theorem 7 via spectral sequences**

In this appendix we briefly introduce the concept of spectral sequences in association with double complexes [39] and use it to prove theorem 7.

A spectral sequence is a sequence of two-dimensional arrays of abelian groups \( E_r = \{ E^{p,q}_r; p, q \in \mathbb{Z} \} \) \( r = 1, 2, \ldots \), with group homomorphisms \( d_r \) that map the array to itself:

\[ d_r : E^{p,q}_r \to E^{p+r,q-r+1}_r . \quad (B.1) \]

A property of the spectral sequence is that \( E^{p,q}_{r+1} \) is the cohomology of \( E_r \) with respect to the map \( d_r \). Moreover, it has a well defined limit \( E^{pq}_{\infty} = \lim_{r \to \infty} E^{p,q}_r \).

A double complex is a complex with two anticommuting operators \( d \) and \( \delta \) that raise two gradings \( p \) and \( q \), respectively. One can also define the operator \( D = d + \delta \), satisfying \( D^2 = 0 \), that raises the total grading \( n = p + q \). To any double complex one can associate a spectral sequence

\[ E^{p,q}_1 \equiv H^{p,q}_d \]
\[ E^{p,q}_2 \equiv H^{p,q}_\delta (H^{p,q}_d) \]
\[ \vdots \quad , \quad (B.2) \]

which converges to the cohomology of \( D \):

\[ H^0_D = \oplus_{p+q=n} E^{p,q}_\infty . \quad (B.3) \]

One can relate \( H_\delta(H_d) \) to \( H_d(H_\delta) \) by using the spectral sequence first to calculate the cohomology of the double complex directly, and then after interchanging the roles of \( d \) and \( \delta \). Thus,

**Theorem:** If the Felder complex exists only at degrees 0 and either +1 or −1, then 
\[ H^n_{rel}(H^0_F) = H^0_F(H^n_{rel}) + H^{±1}_F(H^{n±1}_{rel}) . \]
Proof: In the definition above, let \( d \) be the Felder operator and \( \delta \) the relative BRST operator: \( d = Q_F, \delta = \hat{Q} \). Then \( E^{p,q}_{1} = H^{p,q}_F \) vanishes for all \( q \neq 0 \), since the Felder cohomology exists only at Felder index 0. This means that \( E^{p,q}_{2} = H^{p}_{rel}(H^{q}_F) \) also is nontrivial only for \( q = 0 \). Now, \( E^{p,q}_{3} \) is defined as the the cohomology of \( E^{pq}_{2} \) with respect to the operator \( d_2 : E^{p,q}_{2} \rightarrow E^{p+2,q-1}_{2} \). Since this changes the value of \( q \), the operator must vanish and \( E^{pq}_{3} = E^{pq}_{2} \). Therefore the sequence has converged:

\[
E^{pq}_{r} = E^{pq}_{2} = H^{p}_{rel}(H^{0}_F) \delta_{q,0} \quad \forall r \geq 2.
\]

(B.4)

Using eqs. (B.3) and (B.4), the cohomology of the double complex is given by:

\[
H^n_D = H^n_{rel}(H^0_F).
\]

(B.5)

Now let \( d = \hat{Q}, \delta = Q_F \). In this case,

\[
E^{pq}_{2} = H^{p}_{F}(H^{q}_{rel})
\]

vanishes for \( p \neq 0, \pm 1 \), since the Felder complex exists only at \( p = 0 \) and either \( p = +1 \) or \( p = -1 \). Since \( d_2 : E^{p,q}_{2} \rightarrow E^{p+2,q-1}_{2} \) shifts the Felder index by 2, it again vanishes, so the sequence has converged. Eqs. (B.3) and (B.6) now show that

\[
H^n_D = H^0_F\langle H^n_{rel} \rangle + H^{\pm 1}_F(H^{n+1}_{rel})\ .
\]

(B.7)

The theorem is proved by equating eqs. (B.5) and (B.7).
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