An extended anyon Fock space and noncommutative Meixner-type orthogonal polynomials in infinite dimensions

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Abstract. Let $\nu$ be a finite measure on $\mathbb{R}$ whose Laplace transform is analytic in a neighbourhood of zero. An anyon Lévy white noise on $(\mathbb{R}^d, dx)$ is a certain family of noncommuting operators $\langle \omega, \varphi \rangle$ on the anyon Fock space over $L^2(\mathbb{R}^d \times \mathbb{R}, dx \otimes \nu)$, where $\varphi = \varphi(x)$ runs over a space of test functions on $\mathbb{R}^d$, while $\omega = \omega(x)$ is interpreted as an operator-valued distribution on $\mathbb{R}^d$. Let $L^2(\tau)$ be the noncommutative $L^2$-space generated by the algebra of polynomials in the variables $\langle \omega, \varphi \rangle$, where $\tau$ is the vacuum expectation state. Noncommutative orthogonal polynomials in $L^2(\tau)$ of the form $\langle P_n(\omega), f(n) \rangle$ are constructed, where $f(n)$ is a test function on $(\mathbb{R}^d)_n$, and are then used to derive a unitary isomorphism $U$ between $L^2(\tau)$ and an extended anyon Fock space $\mathcal{F}(L^2(\mathbb{R}^d, dx))$ over $L^2(\mathbb{R}^d, dx)$. The usual anyon Fock space $\mathcal{F}(L^2(\mathbb{R}^d, dx))$ over $L^2(\mathbb{R}^d, dx)$ is a subspace of $\mathcal{F}(L^2(\mathbb{R}^d, dx))$. Furthermore, the equality $\mathcal{F}(L^2(\mathbb{R}^d, dx)) = \mathcal{F}(L^2(\mathbb{R}^d, dx))$ holds if and only if the measure $\nu$ is concentrated at a single point, that is, in the Gaussian or Poisson case. With use of the unitary isomorphism $U$, the operators $\langle \omega, \varphi \rangle$ are realized as a Jacobi (that is, tridiagonal) field in $\mathcal{F}(L^2(\mathbb{R}^d, dx))$. A Meixner-type class of anyon Lévy white noise is derived for which the corresponding Jacobi field in $\mathcal{F}(L^2(\mathbb{R}^d, dx))$ has a relatively simple structure. Each anyon Lévy white noise of Meixner type is characterized by two parameters, $\lambda \in \mathbb{R}$ and $\eta > 0$. In conclusion, the representation $\omega(x) = \partial_x^3 + \lambda \partial_x^2 \partial_x + \eta \partial_x^2 \partial_x + \partial_x$ is obtained, where $\partial_x$ and $\partial_x^2$ are the annihilation and creation operators at the point $x$.

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1. Meixner polynomials in infinite dimensions

1.1. Meixner class of orthogonal polynomials

In 1934, Meixner [44] studied the following problem. Consider complex-valued functions $u(z)$ and $\Phi(z)$ which can be expanded in a power series with respect to $z \in \mathbb{C}$ in a neighbourhood of zero, and suppose that $u(0) = 1$, $\Phi(0) = 0$, and $\Phi'(0) = 1$. Then the function

$$G(x, z) = \exp[x\Phi(z)]u(z) = \sum_{n=0}^{\infty} \frac{P_n(x)}{n!}z^n$$

(1.1)

generates a system of monic polynomials $P_n(x)$. The problem arises of finding all such polynomials which are orthogonal with respect to a probability measure $\mu$ on $\mathbb{R}$. Such polynomials are sometimes called orthogonal polynomials with generating function of exponential type.

Meixner [44] proved that a system of polynomials $P_n(x)$ belongs to this class if and only if it satisfies the recurrence relation

$$xP_n(x) = P_{n+1}(x) + (l + n\lambda)P_n(x) + n(k + \eta(n-1))P_{n-1}(x), \quad n \in \mathbb{N}_0$$

(1.2)

where $l \in \mathbb{R}$, $k > 0$, $\lambda \in \mathbb{R}$, and $\eta \geq 0$. For each choice of the parameters, the corresponding measure of orthogonality $\mu$ is infinitely divisible. If $l = 0$, then $\mu$ becomes centered, whereas $l \neq 0$ corresponds to the shift of $\mu$ by $l$. For $l = 0$ and $k \neq 1$, the measure $\mu$ is the $k$th convolution power of the corresponding measure $\mu$ for $k = 1$. 

1.2. An infinite-dimensional extension

1.3. A noncommutative extension for anyons: introduction
One distinguishes five classes of polynomials satisfying (1.2) (see [23] and [44]):

(i) for $\lambda = \eta = 0$, $\mu$ is a Gaussian measure and $(P_n)_{n=0}^{\infty}$ is a system of Hermite polynomials;

(ii) for $\lambda \neq 0$ and $\eta = 0$, $\mu$ is analogous to a Poisson distribution ($\mu$ is a real Poisson distribution when $\lambda = 1$ and $l = 1$) and $(P_n)_{n=0}^{\infty}$ is a system of Charlier polynomials;

(iii) for $|\lambda| = 2\sqrt{\eta}$ and $\eta \neq 0$, $\mu$ is a gamma distribution and $(P_n)_{n=0}^{\infty}$ is a system of Laguerre polynomials;

(iv) for $|\lambda| > 2\sqrt{\eta}$ and $\eta \neq 0$, $\mu$ is a Pascal (negative binomial) distribution and $(P_n)_{n=0}^{\infty}$ is a system of Meixner polynomials of the first kind;

(v) for $|\lambda| < 2\sqrt{\eta}$ and $\eta \neq 0$, $\mu$ is a Meixner distribution and $(P_n)_{n=0}^{\infty}$ is a system of Meixner polynomials of the second kind (or Meixner–Polaczek polynomials).

We note that in each case

$$G(x, z) = \exp [x\Phi(z) - \mathcal{C}(\Phi(z))],$$  \hspace{0.5cm} (1.3)

for $z$ in a neighbourhood of zero in $\mathbb{C}$, where $\mathcal{C}(z) := \log (\int_{\mathbb{R}} e^{xz} \mu(dx))$ is the cumulant transform of $\mu$. We refer to [23] and [44] for explicit formulae for $\Phi(z)$ and $\mathcal{C}(z)$. If one introduces complex parameters $\alpha, \beta \in \mathbb{C}$ with $\alpha + \beta = -\lambda$ and $\alpha\beta = \eta$, then by using a Taylor series expansion one can write explicit formulae for $\Phi(z)$ and $\mathcal{C}(z)$ in a unique form for all parameters $\alpha$ and $\beta$ (see [47]).

The two observations below will be crucial for our considerations. First, setting $l = 0$ and $k = 1$, we can rewrite (1.2) as

$$x = \partial^\dagger + \lambda \partial \partial + \partial + \eta \partial \partial \partial,$$  \hspace{0.5cm} (1.4)

where (with abuse of notation) $x$ denotes the operator of multiplication by the variable $x$ in $L^2(\mathbb{R}, \mu)$, $\partial^\dagger$ is a creation (raising) operator $\partial^\dagger P_n(x) = P_{n+1}(x)$, and $\partial$ is an annihilation (lowering) operator $\partial P_n(x) = nP_{n-1}(x)$.

Second, the Kolmogorov representation of the Fourier transform of the infinitely divisible measure $\mu$ (with $l = 0$) has the form [48], [49] (see also [30])

$$\int_{\mathbb{R}} e^{iux} \mu(dx) = \exp \left[ k \int_{\mathbb{R}} (e^{ius} - 1 - ius)s^{-2} \nu(ds) \right], \quad u \in \mathbb{R},$$

where $\nu = \delta_\lambda$ (the Dirac measure with mass concentrated at $\lambda$) for $\eta = 0$ (the Gaussian and Poisson cases), whereas for $\eta \neq 0$ (cases (iii)–(v)) $\nu$ is the probability measure on $\mathbb{R}$ whose system $(p_n)_{n=0}^{\infty}$ of monic orthogonal polynomials satisfies the recurrence formula

$$sp_n(s) = p_{n+1}(s) + \lambda(n+1)s + \eta n(n+1)p_{n-1}(s).$$  \hspace{0.5cm} (1.5)

In particular, $(p_n)_{n=0}^{\infty}$ is again a system of orthogonal polynomials in the Meixner class.

### 1.2. An infinite-dimensional extension.

It appears that the Meixner class of orthogonal polynomials is fundamental for infinite-dimensional analysis, and in particular, for the theory of Lévy white noise (see, for example, [1], [41], [42], [48], [52] and the references therein). Let $X := \mathbb{R}^d$ and let

$$\mathcal{D}(X) \subset L^2(X, dx) \subset \mathcal{D}'(X)$$
be a standard triple of spaces in which $\mathcal{D}(X)$ is the nuclear space of smooth, compactly supported functions on $X$, and $\mathcal{D}'(X)$ is the dual space of $\mathcal{D}(X)$ with respect to the zero space $L^2(X, dx)$. For $\omega \in \mathcal{D}'(X)$ and $\varphi \in \mathcal{D}(X)$ we denote by $\langle \omega, \varphi \rangle$ the dual pairing between $\omega$ and $\varphi$. Let $\mu$ be a probability measure on $\mathcal{D}'(X)$, and assume that $\mu$ is a generalized stochastic process with independent values (in the sense of [27]), or in another terminology, a Lévy white noise measure [25]. We also assume that $\mu$ is centered and its Fourier transform has the Kolmogorov representation

$$\int_{\mathcal{D}'(X)} e^{i \langle \omega, \varphi \rangle} \mu(d\omega) = \exp \left[ \int_X \int_{\mathbb{R}} (e^{is\varphi(x)} - 1 - is\varphi(x)) s^{-2} \nu(ds) \, dx \right], \quad \varphi \in \mathcal{D}(X),$$

where $\nu$ is a probability measure on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} e^{\varepsilon |s|} \nu(ds) < \infty \quad \text{for some } \varepsilon > 0.$$  \hspace{1cm} (1.6)

The measure $s^{-2}\nu(ds)$ on $\mathbb{R} \setminus \{0\}$ is called the Lévy measure of $\mu$, while $\nu(\{0\})$ describes the Gaussian part of $\mu$ (for $s = 0$, the function under the integral sign in (1.6) is equal to $-(1/2)\varphi^2(x)$).

In the case $d = 1$, one can use approximation in $L^2(\mathcal{D}'(X), \mu)$ to define for each $t \geq 0$ the random variable $L_t(\omega) = \langle \omega, \chi_{[0,t]} \rangle$, where $\chi_{[0,t]}$ denotes the indicator function of $[0,t]$. Then $(L_t)_{t \geq 0}$ is a (version of a) Lévy process with the Kolmogorov measure $\nu$:

$$\int_{\mathcal{D}'(X)} e^{iuL_t(\omega)} \mu(d\omega) = \exp \left[ t \int_{\mathbb{R}} (e^{ius} - 1 - ius) s^{-2} \nu(ds) \right].$$

Thus, the measure $\mu$ is indeed a Lévy white noise.

Denote by $\mathcal{C} \mathcal{P}$ the set of all continuous polynomials on $\mathcal{D}'(X)$, that is, functions on $\mathcal{D}'(X)$ of the form

$$f^{(0)} + \sum_{i=1}^{n} \langle \omega^{\otimes i}, f^{(i)} \rangle, \quad \omega \in \mathcal{D}(X),$$

$$f^{(0)} \in \mathbb{R}, \quad f^{(i)} \in \mathcal{D}(X)^{\otimes i}, \quad i = 1, \ldots, n, \quad n \in \mathbb{N}.$$  \hspace{1cm} (1.8)

If $f^{(n)} \neq 0$, then one says that the polynomial in (1.8) has order $n$. The set $\mathcal{C} \mathcal{P}$ is dense in $L^2(\mathcal{D}'(X), \mu)$. So using the approach proposed by Skorohod [50], we may orthogonalize these polynomials. More precisely, we denote by $\mathcal{C} \mathcal{P}_n$ the linear space of all continuous polynomials on $\mathcal{D}'(X)$ of order $\leq n$. Let $\mathcal{M} \mathcal{P}_n$ denote the closure of $\mathcal{C} \mathcal{P}_n$ in $L^2(\mathcal{D}'(X), \mu)$ (the set of measurable polynomials of order $\leq n$). Let $\mathcal{O} \mathcal{P}_n := \mathcal{M} \mathcal{P}_n \ominus \mathcal{M} \mathcal{P}_{n-1}$ be the set of orthogonalized polynomials on $\mathcal{D}'(X)$ of order $n$. We clearly have

$$L^2(\mathcal{D}'(X), \mu) = \bigoplus_{n=0}^{\infty} \mathcal{O} \mathcal{P}_n.$$  \hspace{1cm} (1.9)

**Remark 1.1.** An alternative orthogonal decomposition of the $L^2$-space of a Lévy process was derived by Vershik and Tsilevich in [57].
For each \( f^{(n)} \in \mathcal{D}(X)^{\otimes n} \), we denote by \( \langle P_n(\omega), f^{(n)} \rangle \) the orthogonal projection of the continuous monomial \( \langle \omega^{\otimes n}, f^{(n)} \rangle \) on \( \mathcal{D}_n \). We denote by \( \mathcal{C} \mathcal{P} \) the linear space of orthogonalized continuous monomials, that is, the space of finite sums of functions of the form \( \langle P_n(\omega), f^{(n)} \rangle \) and the constants. It should be stressed that the function \( \langle P_n(\omega), f^{(n)} \rangle \) does not necessarily belong to \( \mathcal{C} \mathcal{P} \).

**Theorem 1.2.** Let \( \mu \) be a probability measure on \( \mathcal{D}'(X) \) which has Fourier transform (1.6) with \( \nu \) a probability measure on \( \mathbb{R} \) satisfying (1.7). Then

\[
\mathcal{C} \mathcal{P} = \mathcal{O} \mathcal{C} \mathcal{P}
\]

if and only if there exist \( \lambda \in \mathbb{R} \) and \( \eta > 0 \) such that if \( \eta = 0 \) then \( \nu = \delta_\lambda \), and if \( \eta > 0 \) then the system of monic polynomials \( (p_n)_{n=0}^{\infty} \) which are orthogonal with respect to the measure \( \nu \) satisfies the recurrence formula (1.5) with the indicated \( \lambda \) and \( \eta \).

This theorem can be derived from the main result of [9]. It will also be a corollary of Theorem 3.5 below.

We define the generating function of the family of orthogonal polynomials by

\[
G_\mu(\omega, \varphi) := \sum_{n=0}^{\infty} \frac{1}{n!} \langle P_n(\omega), \varphi^{\otimes n} \rangle
\]

and the cumulant transform of the measure \( \mu \) by

\[
\mathcal{C}_\mu(\varphi) := \log \left( \int_{\mathcal{D}'(X)} e^{\langle \omega, \varphi \rangle} \mu(d\omega) \right).
\]

The following theorem shows, in particular, that the formula (1.3) admits an extension to infinite dimensions (see [41] for a proof).

**Theorem 1.3.** Fix any \( \lambda \in \mathbb{R} \) and \( \eta \geq 0 \). Let \( \mu \) be the probability measure on \( \mathcal{D}'(X) \) which has Fourier transform (1.6) with \( \nu \) the probability measure on \( \mathbb{R} \) corresponding to the parameters \( \lambda \) and \( \eta \) as in Theorem 1.2. Let \( \mathcal{C}(\cdot) \) and \( \Phi(\cdot) \) be the functions in (1.3) for the parameters \( l = 0, k = 1, \lambda, \) and \( \eta \) as above. Then

\[
\mathcal{C}_\mu(\varphi) = \int_X \mathcal{C}(\varphi(x)) \, dx,
\]

\[
G_\mu(\omega, \varphi) = \exp \left[ \langle \omega, \Phi(\varphi) \rangle - \int_X \mathcal{C}(\Phi(\varphi(x))) \, dx \right],
\]

where the formulae hold for \( \varphi \) in (at least) a neighbourhood of zero in \( \mathcal{D}(X) \).

In the case \( \lambda = 0 \) and \( \eta = 0 \), \( \mu \) is a Gaussian white noise measure. (We refer the reader to [8], [25], and [33], for example, for a Gaussian white noise analysis.)

In the case \( \lambda \neq 0 \) and \( \eta = 0 \), \( \mu \) is a Poisson random measure (or point process) (see, for example, [36]). We refer to [55] for a discussion of representations of the group of diffeomorphisms in the Poisson space, to [35] for an analysis of Poisson white noise, and to [2] for a Poisson analysis on the configuration space.

For \( \eta \neq 0 \), the most important case of \( \mu \) is when \( \lambda = 2 \) and \( \eta = 1 \). Then \( \mu \) is the centered gamma measure. The gamma measure is concentrated on the cone.
of discrete Radon measures \( \sum_i s_i \delta_{x_i} \) on \( X \) such that the configuration of atoms \( \{x_i\} \) is a dense subset of \( X \). A very important property of the gamma measure is that it is quasi-invariant with respect to a natural group of transformations of the weights \( s_i \) (see [52] and the references there). Furthermore, as shown in [52], the gamma measure is the unique distribution law of a measure-valued Lévy process with an equivalent \( \sigma \)-finite measure which is projectively invariant with respect to the action of the natural group acting on the weights \( s_i \). In [52] this \( \sigma \)-finite measure is called the infinite-dimensional Lebesgue measure (see also [53]). We also note that in [26], [52], [54], and [56] the gamma measure was used in the representation theory of the group \( SL(2, F) \), where \( F \) is an algebra of functions on a manifold. The first analysis of white noise corresponding to the gamma measure was carried out in [37], and then further developed in [38]. Gibbs perturbations of the gamma measure were constructed in [32]. A Laplace operator associated with the gamma measure was constructed and studied in [31]. Finally, an infinite-dimensional analysis related to the case of a general \( \eta \neq 0 \) was carried out in [40] and [41].

It is well known that in the Gaussian and Poisson cases (\( \eta = 0 \)) the decomposition of \( L^2(\mathcal{D}'(X), \mu) \) into orthogonal polynomials yields the Wiener–Itô–Segal isomorphism between \( L^2(\mathcal{D}'(X), \mu) \) and the symmetric Fock space over \( L^2(X, dx) \). (An alternative derivation of this result is achieved by using multiple stochastic integrals; see, for example, [51] for the Poisson case.) This result admits the following extension (see [37], [38], and [41]).

**Theorem 1.4.** Let \( \lambda \in \mathbb{R} \) and \( \eta \geq 0 \), and let \( \mu \) be the corresponding probability measure on \( \mathcal{D}'(X) \) as in Theorem 1.3.

(i) For each \( n \in \mathbb{N} \) there exists a measure \( m_{\nu}^{(n)} \) on \( X^n \) which satisfies

\[
\int_{\mathcal{D}'(X)} \langle P_n(\omega), f^{(n)} \rangle^2 \mu(d\omega) = \int_{X^n} (\text{Sym}_n f^{(n)})^2 \, dm_{\nu}^{(n)}, \quad f^{(n)} \in \mathcal{D}(X)^{\otimes n},
\]

(1.10)

where \( \text{Sym}_n f^{(n)} \) denotes the usual symmetrization of a function \( f^{(n)} \), \( m_{\nu}^{(n)} = \frac{1}{n!} dx_1 \cdots dx_n \) for \( \eta = 0 \), and for \( \eta \neq 0 \) an explicit construction of the measure \( m_{\nu}^{(n)} \) is given below in §3.1.

(ii) Define the Hilbert space

\[
\mathbf{F}_{\text{sym}}(L^2(X, dx), \nu) := \mathbb{R} \oplus \bigoplus_{n=1}^{\infty} L^2_{\text{sym}}(X^n, m_{\nu}^{(n)}),
\]

(1.11)

where \( L^2_{\text{sym}}(X^n, m_{\nu}^{(n)}) \) is the subspace of \( L^2(X^n, m_{\nu}^{(n)}) \) consisting of all symmetric functions in this space. If \( \eta = 0 \), then \( \mathbf{F}_{\text{sym}}(L^2(X, dx), \nu) \) is the symmetric Fock space over \( L^2(X, dx) \). If \( \eta \neq 0 \), then \( \mathbf{F}_{\text{sym}}(L^2(X, dx), \nu) \) contains the symmetric Fock space as a proper subspace, and \( \mathbf{F}_{\text{sym}}(L^2(X, dx), \nu) \) is called an extended symmetric Fock space. The map

\[
f^{(0)} + \sum_{i=1}^{n} \langle P_i(\omega), f^{(i)} \rangle
\]

\[
\mapsto (f^{(0)}, \text{Sym}_1 f^{(1)}, \ldots, \text{Sym}_n f^{(n)}, 0, 0, \ldots) \in \mathbf{F}_{\text{sym}}(L^2(X, dx), \nu)
\]

(1.12)

extends by continuity to a unitary operator \( U : L^2(\mathcal{D}'(X), \mu) \to \mathbf{F}_{\text{sym}}(L^2(X, dx), \nu) \).
For each $\varphi \in \mathcal{D}(X)$ the notation $\langle \omega, \varphi \rangle$ is kept for the image of the operator of multiplication by the monomial $\langle \omega, \varphi \rangle$ in $L^2(\mathcal{D}'(X), \mu)$ under the action of the unitary operator $U$. Then in analogy with (1.4) the operator $\langle \omega, \varphi \rangle$ realized as acting in the (extended) symmetric Fock space $F_{\text{sym}}(L^2(X, dx), \nu)$ has the representation

$$
\langle \omega, \varphi \rangle = \int_X dx \varphi(x) (\partial_x^\dagger + \lambda \partial_x^\dagger \partial_x + \partial_x + \eta \partial_x^\dagger \partial_x \partial_x),
$$

where $\partial_x$ is the annihilation operator at the point $x$,

$$
(\partial_x f^{(n)})(x_1, \ldots, x_{n-1}) := nf^{(n)}(x, x_1, \ldots, x_{n-1}),
$$

and $\partial_x^\dagger$ is the creation operator at the point $x$, defined by

$$
\int_X dx \varphi(x) \partial_x^\dagger f^{(n)} := \text{Sym}_{n+1}(\varphi \otimes f^{(n)})
$$

(see [41] for further details).

Note that, in view of (1.13), we can heuristically write

$$
\omega(x) = \partial_x^\dagger + \lambda \partial_x^\dagger \partial_x + \partial_x + \eta \partial_x^\dagger \partial_x \partial_x.
$$

As follows from Theorem 1.4, (iii), the operators $\langle \omega, \varphi \rangle$ realized as acting in $F_{\text{sym}}(L^2(X, dx), \nu)$ form a Jacobi field, that is, they have a tridiagonal structure (compare, for example, with [7], [9], [20], [21], and [40]).

### 1.3. A noncommutative extension for anyons: introduction.

The results discussed above have noncommutative analogues in the framework of free probability [16], [17] (see also [4], [5], [10], [12], [13], [43], and the references there). See also [6] and [14] for further connections between the classical distributions in the Meixner class and free probability.

However, in this paper we will be interested in a noncommutative extension of Meixner polynomials for so-called anyon statistics [39], [28], [29] (see also [11]). This statistics, indexed by a complex number $q$ of modulus one, forms a continuous bridge between boson statistics ($q = 1$) and fermi statistics ($q = -1$). One of the main aims of the present paper is to show that, in the anyon setting, one naturally arrives at non-commutative Meixner-type polynomials having a representation of the type (1.13).

In fact, one might think it hopeless to expect an analogue of the formula (1.13) in the fermion setting. Indeed, if the operators $\partial_x$ and $\partial_y$ anticommute, that is, $\partial_x \partial_y = -\partial_y \partial_x$, then $\partial_x \partial_x = 0$, so that the term $\eta \partial_x^\dagger \partial_x \partial_x$ must be equal to zero. However, we do show that, even in the fermion setting, the integral $\int_X dx \varphi(x) \partial_x^\dagger \partial_x \partial_x$ leads to a well-defined, non-trivial operator acting in an extended antisymmetric Fock space $F_{\text{as}}(L^2(X, dx), \nu)$. The latter space contains the usual antisymmetric (fermion) Fock space $\mathcal{F}_{\text{as}}(L^2(X, dx))$ as a subspace. On the space $\mathcal{F}_{\text{as}}(L^2(X, dx))$ the operators $\partial_x$ and $\partial_y$ indeed anticommute. However, this anticommutation fails on the whole space $F_{\text{as}}(L^2(X, dx), \nu)$. As a result, use of the extended antisymmetric Fock space leads to a proper renormalization (more precisely, a non-trivial extension) of the operators $\partial_x$ and $\partial_x^\dagger$. 
Our discussion of this noncommutative extension is organized as follows. In §2 we follow [39], [18], [28] and briefly recall the construction of anyon Fock spaces, standard operators on them, the anyon commutation relations, and the construction of a Lévy white noise for anyon statistics as a family of non-commutative self-adjoint operators \( \langle \omega, \varphi \rangle \) acting in the anyon Fock space over \( L^2(X \times \mathbb{R}, dx \nu(ds)) \) (see [18] for details). We do not explain in §2 why the ‘increments’ of this process can be understood as being ‘anyon independent’, referring the reader instead to [18] for this. We only note that in the commutative, boson case \( q = 1 \), we can actually recover a classical Lévy white noise, realized as a family of commuting self-adjoint operators acting in the symmetric Fock space over \( L^2(X \times \mathbb{R}, dx \nu(ds)) \).

In §3 we formulate the main results of the paper. In particular, starting with the space \( \mathcal{C} \mathcal{P} \) of noncommutative continuous polynomials of an anyon white noise, we construct a space \( \mathcal{C} \mathcal{O} \mathcal{P} \) of orthogonalized continuous polynomials. By analogy with (1.10), we construct for each \( n \in \mathbb{N} \) a measure \( m^{(n)}_\nu \) on \( X^n \) and find the corresponding symmetrization operator \( \text{Sym}_n \). This symmetry extends the anyon symmetry (in particular, the fermion symmetry) in a non-trivial way. By analogy with (1.11) we define an extended anyon Fock space, and then by analogy with (1.12) we construct a unitary operator \( U \) between the noncommutative \( L^2 \)-space and the extended anyon Fock space. Under the unitary operator \( U \), each operator \( \langle \omega, \varphi \rangle \) takes a Jacobi form on the extended anyon Fock space. We show that this Jacobi field has the simplest form (in a certain sense) when \( \nu \) is the same measure as in Theorem 1.2, that is, \( \nu \) is the Kolmogorov measure of a white noise measure \( \mu \) in the Meixner class. Furthermore, analogues of the formulae (1.13)–(1.15) hold in this case.

Finally, §4 is devoted to proofs of the main results.

Among numerous open problems regarding the anyon Meixner-type white noise, we mention only two.

(i) In both the classical and the free cases the generating functions of the Meixner-type orthogonal polynomials are explicitly known and play an important role in the studies of these polynomials. In the anyon case, the form of the generating function is not yet known, even in the Gaussian case. The main difficulty lies in the fact that both the classical and free Meixner-type polynomials have corresponding systems of orthogonal polynomials on the real line. However, the anyon case is purely infinite-dimensional and does not have a corresponding one-dimensional theory.

(ii) As shown in [1], in the classical case the Lévy processes in the Meixner class with \( \eta > 0 \) are related to the renormalized squares of the boson white noise. Is it possible to interpret anyon Meixner-type white noises as white noises connected with renormalized squares of an anyon white noise?

2. Noncommutative Lévy white noise for anyon statistics

2.1. Anyon Fock space and anyon commutation relations. Let \( \mathcal{B}(X) \) denote the Borel \( \sigma \)-algebra on \( X \), and let \( \mathcal{B}_0(X) \) denote the family of all sets in \( \mathcal{B}(X) \) which have compact closure. Let \( m = m(dx) = dx \) denote the Lebesgue measure on \( (X, \mathcal{B}(X)) \).

For each \( n \geq 2 \) we define

\[
X^{(n)} := \{(x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j \forall 1 \leq i < j \leq n \}.
\]
Since the measure $m$ is non-atomic,
\begin{equation}
    m^\otimes n(X^n \setminus X^{(n)}) = 0. \tag{2.2}
\end{equation}

We introduce a strict total order on $X$ as follows: for any $x = (x^1, \ldots, x^d) \in X$ and $y = (y^1, \ldots, y^d) \in X$ with $x \neq y$, we set $x < y$ if $x^1 = y^1, \ldots, x^{j-1} = y^{j-1}$ and $x^j < y^j$ for some $j \in \{1, \ldots, d\}$.

We fix a number $q \in \mathbb{C}$ with $|q| = 1$ and define a function $Q: X^{(2)} \to \mathbb{C}$ by
\begin{align*}
    Q(x, y) &= \begin{cases} 
        q & \text{if } x < y, \\
        \bar{q} & \text{if } y < x. 
    \end{cases} 
\end{align*}

Note that the function $Q$ is Hermitian:
\begin{equation}
    Q(x, y) = \overline{Q(y, x)}, \quad (x, y) \in X^{(2)}.
\end{equation}

A function $f^{(n)}: X^{(n)} \to \mathbb{C}$ ($n \geq 2$) is called $Q$-symmetric if for each $j = 1, \ldots, n-1$
\begin{equation}
    f^{(n)}(x_1, \ldots, x_n) = Q(x_j, x_{j+1}) f^{(n)}(x_1, \ldots, x_{j-1}, x_{j+1}, x_j, x_{j+2}, \ldots, x_n). \tag{2.3}
\end{equation}

Let $\mathcal{H} := L^2(X, m)$ be the Hilbert space of all complex-valued square-integrable functions on $X$. Thus, $\mathcal{H}^\otimes n = L^2(X^n, m^\otimes n)$ for each $n \in \mathbb{N}$. In view of (2.2), we have $\mathcal{H}^\otimes n = L^2(X^{(n)}, m^\otimes n)$. We define the complex Hilbert space $\mathcal{H}^\otimes n$ to be the (closed) subspace of $\mathcal{H}^\otimes n$ consisting of all ($m^\otimes n$-versions of) $Q$-symmetric functions in $\mathcal{H}^\otimes n$. Let $\text{Sym}_n$ denote the orthogonal projection of $\mathcal{H}^\otimes n$ onto $\mathcal{H}^\otimes n$. This operator has the following explicit form: for each $f^{(n)} \in \mathcal{H}^\otimes n$,
\begin{equation}
    (\text{Sym}_n f^{(n)})(x_1, \ldots, x_n) = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} Q_{\pi}(x_1, \ldots, x_n) \\
    \times f^{(n)}(x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)}), \quad (x_1, \ldots, x_n) \in X^{(n)}, \tag{2.4}
\end{equation}

where $\mathfrak{S}_n$ denotes the group of all permutations of $1, \ldots, n$ and
\begin{equation}
    Q_{\pi}(x_1, \ldots, x_n) := \prod_{1 \leq i < j \leq n \atop \pi(i) > \pi(j)} Q(x_i, x_j), \quad (x_1, \ldots, x_n) \in X^{(n)}. \tag{2.5}
\end{equation}

We can now define the $Q$-symmetric tensor product $\otimes$: for any $m, n \in \mathbb{N}$ and any $f^{(m)} \in \mathcal{H}^\otimes m$ and $g^{(n)} \in \mathcal{H}^\otimes n$, we set
\begin{equation*}
    f^{(m)} \otimes g^{(n)} := \text{Sym}_{m+n}(f^{(m)} \otimes g^{(n)}).
\end{equation*}

Note that this tensor product is associative. Note also that $\otimes$ is the classical symmetric tensor product for $q = 1$, while it is the classical antisymmetric tensor product for $q = -1$.

We define an anyon Fock space by
\begin{equation*}
    \mathcal{F}^Q(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}^\otimes n n!.
\end{equation*}
Thus, $\mathcal{F}^Q(\mathcal{H})$ is the Hilbert space consisting of all sequences $F = (f(0), f(1), f(2), \ldots)$ with $f(n) \in \mathcal{H}^\otimes n$ ($\mathcal{H}^\otimes 0 := \mathbb{C}$) satisfying the condition

$$\|F\|^2_{\mathcal{F}^Q(\mathcal{H})} := \sum_{n=0}^{\infty} \|f(n)\|^2_{\mathcal{H}^\otimes n} n! < \infty.$$ 

(The inner product in $\mathcal{F}^Q(\mathcal{H})$ is induced by the norm in this space.) The vector $\Omega := (1, 0, 0, \ldots) \in \mathcal{F}^Q(\mathcal{H})$ is called the vacuum vector. We denote by $\mathcal{F}_{\text{fin}}^Q(\mathcal{H})$ the subspace of $\mathcal{F}^Q(\mathcal{H})$ consisting of all finite sequences

$$F = (f(0), f(1), \ldots, f(n), 0, 0, \ldots)$$

in which $f(i) \in \mathcal{H}^\otimes i$ for $i = 0, 1, \ldots, n$. This space can be endowed with the topology of the topological direct sum of the spaces $\mathcal{H}^\otimes n$. Thus, convergence in $\mathcal{F}_{\text{fin}}^Q(\mathcal{H})$ means uniform finiteness of the non-zero components and coordinate-wise convergence in $\mathcal{H}^\otimes n$.

For each $h \in \mathcal{H}$, we define a creation operator $a^+(h)$ and an annihilation operator $a^-(h)$ as the linear operators acting on $\mathcal{F}_{\text{fin}}^Q(\mathcal{H})$ given by

$$a^+(h)f(n) := h \otimes f(n), \quad f(n) \in \mathcal{H}^\otimes n, \quad a^-(h) := a^+(h)^* \upharpoonright \mathcal{F}_{\text{fin}}^Q(\mathcal{H}).$$

Both $a^+(h)$ and $a^-(h)$ act continuously on $\mathcal{F}_{\text{fin}}^Q(\mathcal{H})$. Indeed, for any $h \in \mathcal{H}$ and $f(n) \in \mathcal{H}^\otimes n$ we have

$$\begin{align*}
(a^+(h)f(n))(x_1, \ldots, x_{n+1}) &= \frac{1}{n+1} \left[ h(x_1)f(n)(x_2, \ldots, x_{n+1}) \\
&\quad + \sum_{k=2}^{n+1} Q(x_1, x_k)Q(x_2, x_k) \cdots Q(x_{k-1}, x_k) \\
&\quad \times h(x_k)f(n)(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1}) \right],
\end{align*}$$

$$\begin{align*}
(a^-(h)f(n))(x_1, \ldots, x_{n-1}) &= n \int_X \overline{h(y)}f(n)(y, x_1, \ldots, x_{n-1}) \, dy.
\end{align*}$$

(2.6)

The action of the annihilation operator can also be written in the following form: for any $h \in \mathcal{H}$ and $f(n) \in \mathcal{H}^\otimes n$,

$$\begin{align*}
(a^-(h)\text{Sym}_n f(n))(x_1, \ldots, x_{n-1}) &= \text{Sym}_{n-1} \left( \int_X \overline{h(y)} \left[ \sum_{k=1}^{n} Q(y, x_1)Q(y, x_2) \times \cdots \times Q(y, x_{k-1})f(n)(x_1, x_2, \ldots, x_{k-1}, y, x_k, \ldots, x_{n-1}) \right] \, dy \right).
\end{align*}$$

(2.7)

Let us now discuss the creation and annihilation operators at points of the space $X$. At least informally, we can consider the delta function $\delta_x$ at $x$ for each $x \in X$. Then we can heuristically define $\partial^+_x := a^+(\delta_x)$ and $\partial_x := a^-(\delta_x)$, so that

$$\begin{align*}
\partial^+_x f(n) &= \delta_x \otimes f(n), \\
\partial_x f(n) &= n f(n)(x, \cdot).
\end{align*}$$

(2.8)
Thus,
\[
a^+(h) := \int_X dx\, h(x) \partial_x^1, \quad a^-(h) = \int_X dx\, \overline{h(x)} \partial_x.
\] (2.9)

We note that the second formula in (2.8) is a rigorous definition of \(\partial_x\) (for \(m\)-a.e. \(x \in X\)), while the first formula in (2.9) is the rigorous definition of the integral \(\int_X dx\, h(x) \partial_x^1\).

Let \(B_0(X^n)\) denote the space of all complex-valued bounded measurable functions on \(X^n\) with compact support. For a fixed \(g^{(n)} \in B_0(X^n)\) and an arbitrary tuple \((z_1, \ldots, z_n)\) of length \(n \geq 2\) consisting of symbols + and −, it is easy to see that the expression
\[
\int_{X^n} dx_1 \cdots dx_n \, g^{(n)}(x_1, \ldots, x_n) \partial_{x_1} z_1 \cdots \partial_{x_n} z_n
\]
defines a continuous linear operator on \(\mathcal{F}_\text{fin}^Q(\mathcal{H})\), where we use the notation \(\partial_{x}^+ := \partial_x^1\) and \(\partial_{x}^- := \partial_x\).

The creation and annihilation operators satisfy the anyon commutation relations
\[
\partial_x \partial_y^i = \delta(x, y) + Q(x, y) \partial_y^i \partial_x, \quad (2.10)
\]
\[
\partial_x \partial_y = Q(y, x) \partial_y \partial_x, \quad (2.11)
\]
\[
\partial_x^i \partial_y^j = Q(y, x) \partial_y^j \partial_x^i, \quad (2.12)
\]
where \(\delta(x, y)\) is understood in the sense that
\[
\int_{X^2} dx\, dy\, g^{(2)}(x, y) \delta(x, y) := \int_X dx\, g^{(2)}(x, x).
\]
The formulae (2.10)–(2.12) make rigorous sense after smoothing with functions \(g^{(2)} \in B_0(X^2)\). Note that for \(q = 1\) the equations (2.10)–(2.12) become the canonical commutation relations, while for \(q = -1\) they become the canonical anticommutation relations.

**Remark 2.1.** Let \(D := \{(x, x) \mid x \in X\}\). For each \(g^{(2)} \in B_0(X^2)\) with support in \(D\) the operator \(\int_{X^2} dx\, dy\, g^{(2)}(x, y) \partial_y^i \partial_x\) is equal to zero. Hence, it does not influence (2.10) that we have not defined the function \(Q\) on \(D\).

For a bounded linear operator \(A\) on \(\mathcal{H}\) we define the differential second quantization \(d\Gamma(A)\) of \(A\) as a continuous linear operator on \(\mathcal{F}_n^Q(\mathcal{H})\) by the formulae
\[
d\Gamma(A)\Omega := 0 \quad \text{and} \quad \quad d\Gamma(A) | \mathcal{H}^{\otimes n} := \text{Sym}_n(A \otimes 1 \cdots \otimes 1 + 1 \otimes A \otimes 1 \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes A)
\]
for each \(n \in \mathbb{N}\). For each a.e.-bounded function \(h \in L^\infty(X, m)\) we define the neutral operator
\[
a^0(h) := \int_X dx\, h(x) \partial_x^1 \partial_x. \quad (2.13)
\]
According to the formulae (2.8) and (2.9), we have
\[
(a^0(h) f^{(n)})(x_1, \ldots, x_n) = \left(\int_X dx\, h(x) \partial_x^1 f^{(n)}(x, \cdot)\right)(x_1, \ldots, x_n)
\]
\[
= n \text{Sym}_n(h(x_1) f^{(n)}(x_1, x_2, \ldots, x_n)). \quad (2.14)
\]
From here one easily gets that
\[(a^0(h)f^{(n)})(x_1, \ldots, x_n) = (h(x_1) + \cdots + h(x_n))f^{(n)}(x_1, \ldots, x_n). \tag{2.15}\]
Hence, \(a^0(h) = d\Gamma(M_h)\), where \(M_h\) is the operator of multiplication by \(h\).

2.2. Anyon Lévy white noise and noncommutative orthogonal polynomials. We recall the construction of a Lévy white noise over \(X\) for anyon statistics (see \([18]\)). Let \(\nu\) be a probability measure on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\). (In fact, we can just assume that \(\nu\) is a finite measure, in which case the results below are modified in an obvious way.) Denote by \(\mathcal{P}(\mathbb{R})\) the linear space of polynomials on \(\mathbb{R}\). We assume that \(\mathcal{P}(\mathbb{R})\) is a dense subset of \(L^2(\mathbb{R}, \nu)\). This assumption is satisfied if, for example, \((1.7)\) holds.

We extend the function \(Q\) by setting
\[Q(x_1, s_1, x_2, s_2) := Q(x_1, x_2), \quad (x_1, x_2) \in X^{(2)}, \quad (s_1, s_2) \in \mathbb{R}^2.\]
Thus, the value of \(Q\) does not depend on \(s_1\) and \(s_2\). In analogy with \((2.3)\), we introduce the notion of a \(Q\)-symmetric function \(f^{(n)}\) defined on the set
\[\{(x_1, s_1, \ldots, x_n, s_n) \in (X \times \mathbb{R})^n \mid (x_1, \ldots, x_n) \in X^{(n)}\}.\]
For example, if \(n = 2\), then \(Q\)-symmetry means that
\[f^{(2)}(x_1, s_1, x_2, s_2) = Q(x_1, x_2)f^{(2)}(x_2, s_2, x_1, s_1).\]

Further, setting
\[\mathcal{G} := L^2(X \times \mathbb{R}, m \otimes \nu) = \mathcal{H} \otimes L^2(\mathbb{R}, \nu),\]
we consider the \(Q\)-symmetric Fock space \(\mathcal{F}^Q(\mathcal{G})\) constructed like \(\mathcal{F}^Q(\mathcal{H})\). Let \(\mathcal{F}^Q_{\text{fin}}(\mathcal{H} \otimes \mathcal{P}(\mathbb{R}))\) be the linear subspace of \(\mathcal{F}^Q(\mathcal{G})\) consisting of all finite sequences
\[F = (F^{(0)}, F^{(1)}, \ldots, F^{(n)}, 0, 0, \ldots), \quad n \in \mathbb{N}_0,\]
in which each element \(F^{(k)}\) with \(k \neq 0\) has the form
\[F^{(k)}(x_1, s_1, \ldots, x_k, s_k) = \text{Sym}_k\left[\sum_{(i_1, i_2, \ldots, i_k) \in \{0, 1, \ldots, N\}^k} f_{(i_1, i_2, \ldots, i_k)}(x_1, x_2, \ldots, x_k)s_1^{i_1}s_2^{i_2} \cdots s_k^{i_k}\right],\]
where \(f_{(i_1, i_2, \ldots, i_k)} \in \mathcal{H}^\otimes k\) and \(N \in \mathbb{N}\). Clearly, \(\mathcal{F}^Q_{\text{fin}}(\mathcal{H} \otimes \mathcal{P}(\mathbb{R}))\) is dense in \(\mathcal{F}^Q(\mathcal{G})\).

Let \(1(s) := 1\) and \(\text{id}(s) := s\) for \(s \in \mathbb{R}\). Thus, \(1, \text{id} \in \mathcal{P}(\mathbb{R})\). We denote by \(C_0(X \to \mathbb{R})\) the space of all real-valued continuous functions on \(X\) with compact support. For each \(f \in C_0(X \to \mathbb{R})\) we define an operator
\[\langle \omega, f \rangle := a^+(f \otimes 1) + a^0(f \otimes \text{id}) + a^{-}(f \otimes 1) \tag{2.16}\]
on \(\mathcal{F}^Q(\mathcal{G})\) with domain \(\mathcal{F}^Q_{\text{fin}}(\mathcal{H} \otimes \mathcal{P}(\mathbb{R}))\). Clearly, each operator \(\langle \omega, f \rangle\) maps \(\mathcal{F}^Q_{\text{fin}}(\mathcal{H} \otimes \mathcal{P}(\mathbb{R}))\) into itself. In fact, under the assumption \((1.7)\), each \(F \in \mathcal{F}^Q_{\text{fin}}(\mathcal{H} \otimes \mathcal{P}(\mathbb{R}))\) is an analytic vector for each operator \(\langle \omega, f \rangle\) with \(f \in C_0(X \to \mathbb{R})\), which implies that the operators \(\langle \omega, f \rangle\) are essentially self-adjoint on \(\mathcal{F}^Q_{\text{fin}}(\mathcal{H} \otimes \mathcal{P}(\mathbb{R}))\) (see, for example, \([46]\), § X.2).
Remark 2.2. We keep the notation $\langle \omega, f \rangle$ for the closure of this operator acting in $\mathcal{F}^Q(\mathcal{G})$. Then the operators $\langle \omega, f \rangle$ are self-adjoint. In the boson case when $q = 1$ these operators also commute in the sense that their resolutions of the identity commute. By using, for example, the projection spectral theorem [8], one shows [24] that there exists a unitary isomorphism between the symmetric Fock space $\mathcal{F}^Q(\mathcal{H})$ and the space $L^2(\mathcal{D}'(X), \mu)$, where $\mu$ is the Lévy white noise measure with Fourier transform (1.6). Under this unitary isomorphism, the vacuum vector $\Omega$ becomes the constant function 1, and each operator $\langle \omega, f \rangle$ becomes the operator of multiplication by the random variable $\langle \omega, f \rangle$ in $L^2(\mathcal{D}'(X), \mu)$. In other words, $\mu$ is the spectral measure of the family $((\omega, f))_{f \in C_0(X \to \mathbb{R})}$ of commuting self-adjoint operators. In particular, the operators $((\omega, f))_{f \in C_0(X \to \mathbb{R})}$ on the symmetric Fock space $\mathcal{F}^Q(\mathcal{G})$ can indeed be regarded as a Lévy white noise. We also remark that the unitary operator between $\mathcal{F}^Q(\mathcal{G})$ and $L^2(\mathcal{D}'(X), \mu)$ was originally derived by Itô [34] using multiple stochastic integrals.

Remark 2.3. Note that if the measure $\nu$ is concentrated at a single point $\lambda \in \mathbb{R}$, then $\mathcal{G} = \mathcal{H}$ and each operator $\langle \omega, f \rangle$ has in $\mathcal{F}^Q(\mathcal{H})$ the form
\begin{equation}
\langle \omega, f \rangle := a^+(f) + a^-(f) + \lambda a^0(f).
\end{equation}
The choice $\lambda = 0$ corresponds to an anyon Gaussian white noise, while $\lambda \neq 0$ corresponds to an anyon centered white noise. If we let
\begin{equation}
\omega(x) := \partial^\dagger_x + \lambda \partial_x \partial^\dagger_x + \partial_x, \quad x \in X,
\end{equation}
then by (2.9), (2.13), (2.17), and (2.18), we find that
\begin{equation}
\langle \omega, f \rangle = \int_X dx \omega(x)f(x), \quad f \in C_0(X \to \mathbb{R}),
\end{equation}
which justifies the notation $\langle \omega, f \rangle$. Thus, $(\omega(x))_{x \in X}$ is an anyon Gaussian or Poisson white noise. We also note that $\omega(x)$ can be informally regarded as an operator-valued distribution.

Further, let $C_0(X)$ be the space of all complex-valued continuous functions on $X$ with compact support. For $f \in C_0(X)$ we define $\langle \omega, f \rangle := \langle \omega, \text{Re} f \rangle + i \langle \omega, \text{Im} f \rangle$.

Let $\mathcal{P}$ denote the complex unital *-algebra generated by $((\omega, f))_{f \in C_0(X)}$, that is, the algebra of noncommutative polynomials in the variables $\langle \omega, f \rangle$. In particular, elements of $\mathcal{P}$ are linear operators acting on $\mathcal{F}^Q_\text{fin}(\mathcal{H} \otimes \mathcal{P}(\mathbb{R}))$, and for each $p \in \mathcal{P}$ the operator $p^*$ is the adjoint of $p$ on $\mathcal{F}^Q(\mathcal{G})$, restricted to $\mathcal{F}^Q_\text{fin}(\mathcal{H} \otimes \mathcal{P}(\mathbb{R}))$.

The vacuum state on $\mathcal{P}$ is defined by
\begin{equation}
\tau(p) := \langle p\Omega, \Omega \rangle_{\mathcal{F}^Q(\mathcal{G})}, \quad p \in \mathcal{P}.
\end{equation}
Next, we introduce an inner product on $\mathcal{P}$ by
\begin{equation}
(p_1, p_2)_{L^2(\tau)} := \tau(p_2^* p_1) = \langle p_1\Omega, p_2\Omega \rangle_{\mathcal{F}^Q(\mathcal{G})}, \quad p_1, p_2 \in \mathcal{P}.
\end{equation}
Let
\begin{equation}
\widetilde{\mathcal{P}} := \{ p \in \mathcal{P} \mid (p, p)_{L^2(\tau)} = 0 \}
\end{equation}
and define the noncommutative $L^2$-space $L^2(\tau)$ as the completion of the quotient space $\mathcal{P}/\tilde{\mathcal{P}}$ with respect to the norm generated by the inner product $(\cdot, \cdot)_{L^2(\tau)}$. Elements $p \in \mathcal{P}$ are regarded as representatives of the equivalence classes in $\mathcal{P}/\tilde{\mathcal{P}}$, and hence $\mathcal{P}$ becomes a dense subspace of $L^2(\tau)$. As shown in [18], the vacuum vector $\Omega$ is cyclic for the family of operators $(\langle \omega, f \rangle)_{f \in C_0(X \rightarrow \mathbb{R})}$. Consider the linear map $I: \mathcal{P} \rightarrow \mathcal{F}^Q(\mathcal{G})$ defined by

$$Ip := p\Omega \quad \text{for } p \in \mathcal{P}.\$$

Then $Ip_1 = Ip_2$ if $p_1, p_2 \in \mathcal{P}$ are such that $p_1 - p_2 \in \tilde{\mathcal{P}}$, and $I$ extends to a unitary operator $I: L^2(\tau) \rightarrow \mathcal{F}^Q(\mathcal{G})$.

Note that, for each $p \in \mathcal{P}$ and $f \in C_0(X)$,

$$I(\langle \omega, f \rangle p) = \langle \omega, f \rangle (Ip), \tag{2.20}$$

that is, under the unitary operator $I$ the operator of left multiplication by $\langle \omega, f \rangle$ in $L^2(\tau)$ becomes the operator $\langle \omega, f \rangle$ acting in $\mathcal{F}^Q(\mathcal{G})$.

Let us consider the topology on $C_0(X)$ which yields the following notion of convergence: $f_n \rightarrow f$ as $n \rightarrow \infty$ means that there exists a set $\Delta \in \mathcal{B}_0(X)$ such that $\text{supp}(f_n) \subset \Delta$ for all $n \in \mathbb{N}$ and

$$\sup_{x \in X} |f_n(x) - f(x)| \rightarrow 0, \quad n \rightarrow \infty. \tag{2.21}$$

By linearity and continuity the map

$$C_0(X)^n \ni (f_1, \ldots, f_n) \mapsto \langle \omega \otimes^n, f_1 \otimes \cdots \otimes f_n \rangle = \langle \omega, f_1 \rangle \cdots \langle \omega, f_n \rangle \in \mathcal{P}$$

extends to a map

$$C_0(X)^n \ni f^{(n)} \mapsto \langle \omega \otimes^n, f^{(n)} \rangle \in L^2(\tau),$$

and hence $\langle \omega \otimes^n, f^{(n)} \rangle$ can be thought of as a linear operator acting in $\mathcal{F}^Q(\mathcal{H} \otimes \mathcal{P}(\mathbb{R}))$. We will regard $\langle \omega \otimes^n, f^{(n)} \rangle$ as a continuous monomial of order $n$. Sums of such operators and the (complex) constants form the set $\mathcal{C}\mathcal{P}$ of continuous polynomials (in $\omega$). Obviously, $\mathcal{P} \subset \mathcal{C}\mathcal{P}$.

In complete analogy with (1.9) we obtain the orthogonal decomposition

$$L^2(\tau) = \bigoplus_{n=0}^{\infty} \mathcal{P}_n \tag{2.22}$$

(the notation is obvious). For any $f^{(n)} \in C_0(X^n)$ we denote by $\langle P_n(\omega), f^{(n)} \rangle$ the orthogonal projection of $\langle \omega \otimes^n, f^{(n)} \rangle$ on $\mathcal{P}_n$, and by $\mathcal{O}\mathcal{C}\mathcal{P}$ the set of finite linear sums of $\langle P_n(\omega), f^{(n)} \rangle$ and the (complex) constants (orthogonalized continuous polynomials).

Remark 2.4. We note that $\langle P_1(\omega), f \rangle = \langle \omega, f \rangle$.

Remark 2.5. In §1.2 we used functions $f^{(n)} \in \mathcal{D}(X)^{\otimes n}$ to define $\mathcal{C}\mathcal{P}$ and $\mathcal{O}\mathcal{C}\mathcal{P}$, while now we are using $f^{(n)} \in C_0(X^n)$ to define $\mathcal{C}\mathcal{P}$ and $\mathcal{O}\mathcal{C}\mathcal{P}$. The reason is that in the noncommutative case there is no need for $f^{(n)}$ to be smooth, while in the classical case $q = 1$, Theorem 1.2 still holds for the sets $\mathcal{C}\mathcal{P}$ and $\mathcal{O}\mathcal{C}\mathcal{P}$ as defined in this section.
3. Main results

3.1. The measures \( m_\nu^{(n)} \). Let \((p_k)_{k=0}^\infty\) denote the system of monic orthogonal polynomials in \( L^2(\mathbb{R}, \nu) \). (If the support of \( \nu \) is finite and consists of \( N \) points, then we set \( p_k := 0 \) for \( k \geq N \).) Hence, the system \((p_k)_{k=0}^\infty\) satisfies the recursion formula

\[
sp_k(s) = p_{k+1}(s) + b_k p_k(s) + a_k p_{k-1}(s), \quad k \in \mathbb{N}_0,
\]

where \( p_{-1}(s) := 0 \), \( a_k > 0 \), and \( b_k \in \mathbb{R} \). (If the support of \( \nu \) has \( N \) points, then \( a_k = 0 \) for \( k \geq N \).
We define

\[
c_k := a_0 a_1 \cdots a_{k-1}, \quad k \in \mathbb{N},
\]

where \( a_0 := 1 \) and the \( a_k \) for \( k \in \mathbb{N} \) are the coefficients in (3.1). Equivalently,

\[
c_k = \int_{\mathbb{R}} p_{k-1}(s)^2 \nu(ds), \quad k \in \mathbb{N},
\]

which is a well-known fact in the theory of orthogonal polynomials. Here \( c_1 = 1 \), and \( c_k = 0 \) for \( k \geq 2 \) if and only if the measure \( \nu \) is concentrated at a single point.

We denote by \( \Pi(n) \) the set of all (unordered) partitions of the set \( \{1, \ldots, n\} \). For a partition \( \theta = \{\theta_1, \ldots, \theta_l\} \in \Pi(n) \) we set \( |\theta| := l \). For each \( \theta \in \Pi(n) \) we denote by \( X^{(n)}_\theta \) the subset of \( X^n \) consisting of all \((x_1, \ldots, x_n) \in X^n \) such that for \( 1 \leq i < j \leq n \) the equality \( x_i = x_j \) holds if and only if \( i \) and \( j \) belong to the same element of the partition \( \theta \). Note that the sets \( X^{(n)}_\theta \) with \( \theta \in \Pi(n) \) form a partition of \( X^n \). Note also that, by (2.1), \( X^{(n)} = X^{(n)}_{(n)} \) for the minimal partition \( \theta = \{\{1\}, \{2\}, \ldots, \{n\}\} \).

Let us fix \( n \in \mathbb{N} \), a permutation \( \pi \in \mathfrak{S}_n \), and a partition \( \theta = \{\theta_1, \ldots, \theta_l\} \in \Pi(n) \) satisfying the condition

\[
\max \theta_1 < \max \theta_2 < \cdots < \max \theta_l.
\]

We define a measure \( m^{(n)}_{\nu, \theta} \) on \( X^{(n)} \) as the push-forward of the measure

\[
(c_{|\theta_1|} \cdots c_{|\theta_l|}) n!(|\theta_1|! \cdots |\theta_l|!)^{-1} m^{\otimes l}
\]

on \( X^{(l)} \) under the map

\[
X^{(l)} \ni y = (y_1, \ldots, y_l) \mapsto (R^{y_1}_{\theta_1} y, \ldots, R^n_{\theta_l} y) \in X^{(n)}_\theta,
\]

where \( R^y_{\theta_i} y = y_j \) for \( i \in \theta_j \). Here \( |\theta_i| \) denotes the number of elements of the set \( \theta_i \).

Recalling that the sets \( X^{(n)}_\theta \) with \( \theta \in \Pi(n) \) form a partition of \( X^n \), we define a measure \( m^{(n)}_\nu \) on \( X^n \) whose restriction to each \( X^{(n)}_\theta \) is equal to \( m^{(n)}_{\nu, \theta} \). Note that the restriction of \( m^{(n)}_\nu \) to \( X^{(n)} \) is equal to \( n! m^{\otimes n} \).

For example, for \( n = 2 \) we get that

\[
\int_{X^2} f^{(2)}(x_1, x_2) m^{(2)}_\nu(dx_1 \times dx_2)
\]

\[
= \int_{\{x_1 \neq x_2\}} f^{(2)}(x_1, x_2) dx_1 dx_2 + \int_X f^{(2)}(x, x) dx c_2
\]

\[
= \int_{X^2} f^{(2)}(x_1, x_2) dx_1 dx_2 2 + \int_X f^{(2)}(x, x) dx c_2.
\]
3.2. An extended anyon Fock space. In §2.1 (see, in particular, (2.3)) we defined the notion of a $Q$-symmetric function $f^{(n)} : X^n \to \mathbb{C}$. Our next aim is to extend this notion to a complex-valued function defined on the whole of $X^n$.

Let us fix a permutation $\pi \in \mathfrak{S}_n$ and a partition $\theta = \{\theta_1, \ldots, \theta_l\} \in \Pi(n)$ satisfying (3.4). The permutation $\pi$ maps the partition $\theta$ into a new partition
\[ \{\pi \theta_1, \ldots, \pi \theta_l\} \in \Pi(n), \]
which we call $\beta = \{\beta_1, \ldots, \beta_l\}$, with the elements of $\beta$ enumerated so that
\[ \max \beta_1 < \max \beta_2 < \cdots < \max \beta_l. \]

Thus, the permutation $\pi \in \mathfrak{S}_n$ determines a permutation $\hat{\pi} \in \mathfrak{S}_l$ (dependent on $\theta$) such that
\[ \pi \theta_i = \beta_{\hat{\pi}(i)}, \quad i = 1, \ldots, l. \]

Recall that the complex-valued function $Q_\pi(x_1, \ldots, x_n)$ on $X^{(n)}$ is defined in (2.5). We will now extend this function to the whole set $X^n$ as follows. Fix any $\theta = \{\theta_1, \ldots, \theta_l\} \in \Pi(n)$ satisfying (3.4) and any $(x_1, \ldots, x_n) \in X^{(n)}_\theta$. We denote by $x_{\theta_1}, x_{\theta_2}, \ldots, x_{\theta_l}$ the elements $x_{i_1}, x_{i_2}, \ldots, x_{i_l}$ with $i_1 \in \theta_1, i_2 \in \theta_2, \ldots, i_l \in \theta_l$, respectively. Let
\[ Q_\pi(x_1, \ldots, x_n) := \prod_{1 \leq i < j \leq l} Q(x_{\theta_i}, x_{\theta_j}), \quad \pi \theta_i = \beta_{\hat{\pi}(i)}, \quad i = 1, \ldots, l. \]

where the permutation $\hat{\pi} \in \mathfrak{S}_l$ is defined above. Note that, for the partition
\[ \theta = \{\{1\}, \{2\}, \ldots, \{n\}\} \]
the restriction of the function $Q_\pi$ to the set $X^{(n)}_\theta = X^{(n)}$ is indeed equal to the function $Q_\pi$ defined in (2.5).

We will say that a function $f^{(n)} : X^n \to \mathbb{C}$ is $Q$-symmetric if, for each permutation $\pi \in \mathfrak{S}_n$,
\[ f^{(n)}(x_1, \ldots, x_n) = Q_\pi(x_1, \ldots, x_n) f^{(n)}(x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)}), \quad (x_1, \ldots, x_n) \in X^n. \]

In particular, the restriction of such a function to $X^{(n)}$ is then $Q$-symmetric according to our definition in §2.1, that is, it satisfies (2.3).

Next, for a function $f^{(n)} : X^n \to \mathbb{C}$ we define
\[ (\text{Sym}_n f^{(n)})(x_1, \ldots, x_n) = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} Q_\pi(x_1, \ldots, x_n) \times f^{(n)}(x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)}), \quad (x_1, \ldots, x_n) \in X^n. \]

Clearly, the restriction of the function $\text{Sym}_n f^{(n)}$ to the set $X^{(n)}$ is still given by (2.4).

We denote by $\mathbf{F}^Q_n(\mathcal{H}, \nu)$ the subspace of the complex $L^2$-space $L^2(X^n, m^{(n)}_\nu)$ consisting of $(m^{(n)}_\nu)$-versions of $Q$-symmetric functions.
Corollary 3.3. There is a unitary isomorphism

$$\mathbf{F}^Q(\mathcal{H}, \nu) \cong (f(n))_{n=0}^\infty \mapsto f(0) + \sum_{n=1}^\infty \langle P_n(\omega), f(n) \rangle \in L^2(\tau).$$

(3.11)

We denote the inverse of the unitary operator in (3.11) by $U$. Then $U : L^2(\tau) \rightarrow \mathbf{F}^Q(\mathcal{H}, \nu)$ is a unitary operator (compare with Theorem 1.4, (ii) in the boson case $q = 1$).
3.3. Anyon Lévy white noise as a Jacobi field. In view of §2.2 and Corollary 3.3, we have the following chain of unitary operators:
\[ \mathbf{F}^Q(\mathcal{H}, \nu) \xleftarrow{U} L^2(\tau) \xrightarrow{I} \mathcal{F}^Q(\mathcal{G}). \]
We also consider the unitary operator
\[ U: \mathbf{F}^Q(\mathcal{H}, \nu) \rightarrow \mathcal{F}^Q(\mathcal{G}), \quad U := IU^{-1}. \]

Let \( h \in C_0(X) \). Recall the formula (2.20), which says that under \( I^{-1} \) the operator \( \langle \omega, h \rangle \) on \( \mathcal{F}^Q(\mathcal{G}) \) becomes the operator of left multiplication by \( \langle \omega, h \rangle \) in \( L^2(\tau) \).

Let \[ J(h) := U^{-1} \langle \omega, h \rangle U. \] (3.12)
Obviously, the operators \( J(h) \) form a Jacobi field in the extended anyon Fock space \( \mathbf{F}^Q(\mathcal{H}, \nu) \), that is, each operator \( J(h) \) has a representation
\[ J(h) = J^+(h) + J^0(f) + J^-(h), \] (3.13)
where \( J^+(h) \) is a creation operator, \( J^0(h) \) is a neutral operator, and \( J^-(h) \) is an annihilation operator. Equivalently,
\[ \langle \omega, h \rangle \langle P_n(\omega), f^{(n)} \rangle = \langle P_{n+1}(\omega), J^+(h) f^{(n)} \rangle + \langle P_n(\omega), J^0(h) f^{(n)} \rangle \]
\[ + \langle P_{n-1}(\omega), J^-(h) f^{(n)} \rangle. \]

Our next aim is to explicitly calculate the operators \( J^\sharp(h), \sharp = +, 0, - \).

Let \( \mathcal{F}_{\text{fin}}(B_0(X)) \) be the linear space of finite vectors \( (f^{(0)}, f^{(1)}, \ldots, f^{(n)}, 0, 0, \ldots) \) with \( f^{(0)} \in \mathbb{C} \) and \( f^{(i)} \in B_0(X^i) \) for \( i \geq 1 \). The vacuum vector \( \Omega \) obviously belongs to \( \mathcal{F}_{\text{fin}}(B_0(X)) \).

For each \( h \in C_0(X) \) we define a neutral operator \( \mathcal{J}^0(h) \) and an annihilation operator \( \mathcal{J}^-(h) \) acting on \( \mathcal{F}_{\text{fin}}(B_0(X)) \) as follows. We first set
\[ \mathcal{J}^0(h)\Omega = \mathcal{J}^-(h)\Omega := 0. \] (3.14)
Next, let
\[ (\mathcal{J}^0(h) f^{(n)})(x_1, \ldots, x_n) := \sum_{i=1}^n h(x_i) f^{(n)}(x_1, \ldots, x_n) R_i^{(n)}(x_1, \ldots, x_n). \] (3.15)
Here, for each \( \theta = \{\theta_1, \ldots, \theta_1\} \in \Pi(n) \), the restriction of the function \( R_i^{(n)}: X^n \rightarrow \mathbb{R} \) to the set \( X_\theta^{(n)} \) is given by
\[ R_i^{(n)} \upharpoonright X_\theta^{(n)} := \frac{b_{\gamma(i, \theta)-1}}{\gamma(i, \theta)}. \] (3.16)
In (3.16), \( \gamma(i, \theta) := |\theta_u| \) with \( \theta_u \in \theta \) chosen so that \( i \in \theta_u \), and the \( (b_k)_{k=0}^\infty \) are the coefficients in (3.1). Finally, let
\[ (\mathcal{J}^-(h) f^{(n)})(x_1, \ldots, x_{n-1}) := \sum_{1 \leq i < j \leq n} h(x_{j-1}) \]
\[ \times f^{(n)}(x_1, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_{j-1}, \ldots, x_{n-1}) \]
\[ \times S_j^{(n)}(x_1, \ldots, x_{n-1}), \] (3.17)
where for any \( \theta \in \Pi(n-1) \)
\[
S_{j-1}^{(n)} \upharpoonright X^{(n-1)}_{\theta} := \frac{2a_{(j-1, \theta)}}{\gamma(j-1, \theta) \gamma(j-1, \theta) + 1}.
\] (3.18)

Here the \( (a_k)_{k=1}^{\infty} \) are also the coefficients in (3.1).

We define
\[
\mathbf{F}_\text{fin}^Q(B_0(X)) := \text{Sym} \mathcal{F}_\text{fin}(B_0(X)),
\]
where Sym is the linear operator on \( \mathcal{F}_\text{fin}(B_0(X)) \) satisfying \( \text{Sym} f^{(n)} := \text{Sym}_n f^{(n)} \)
for \( f^{(n)} \in B_0(X^n) \). We also write \( B_0^Q(X^n) := \text{Sym}_n B_0(X^n) \).

On \( \mathbf{F}_\text{fin}^Q(B_0(X)) \) we define a \( Q \)-symmetric tensor product by setting, for any
\( f^{(m)} \in B_0^Q(X^m) \) and \( g^{(n)} \in B_0^Q(X^n) \),
\[
f^{(m)} \otimes g^{(n)} := \text{Sym}_{m+n}(f^{(m)} \otimes g^{(n)})
\] (3.19)
and then extending it by linearity. Here \( f^{(m)} \otimes g^{(n)} \in B_0(X^{m+n}) \) is given by
\[
(f^{(m)} \otimes g^{(n)})(x_1, \ldots, x_{m+n}) = f^{(m)}(x_1, \ldots, x_m)g^{(n)}(x_{m+1}, \ldots, x_{m+n}).
\]

We will prove below that the tensor product \( \otimes \) is associative. Furthermore, the restriction of \( f^{(m)} \otimes g^{(n)} \) to \( X^{(m+n)} \) obviously coincides with the product \( f^{(m)} \otimes g^{(n)} \) defined in § 2.1.

**Theorem 3.4.** For each \( h \in C_0(X) \) the operator \( \mathbf{J}(h) \) is linear on \( \mathbf{F}_\text{fin}^Q(B_0(X)) \) and has the representation (3.13). Moreover, for each \( F \in \mathbf{F}_\text{fin}^Q(B_0(X)) \)
\[
\mathbf{J}^+(h)F = h \otimes F, \quad \mathbf{J}^0(h)F = \text{Sym}(\mathcal{J}^0(h)F),
\]
and
\[
\mathbf{J}^-(h) = \mathbf{J}^-_1(h) + \mathbf{J}^-_2(h),
\] (3.20)
where
\[
\mathbf{J}^-_1(h)F = \text{Sym}(\mathcal{J}^-_1(h)F)
\]
and for each \( f^{(n)} \in B_0^Q(X^n) \)
\[
(\mathbf{J}^-_2(h)f^{(n)})(x_1, \ldots, x_{n-1}) = n \int_X dy h(y)f^{(n)}(y, x_1, \ldots, x_{n-1}).
\] (3.21)

### 3.4. A characterization of Meixner-type polynomials.

Recall that the operators \( \mathcal{J}^0(h) \) and \( \mathcal{J}^-(h) \) were defined using the coefficients of the recursion relation (3.1) (that is, in terms of the measure \( \nu \)), and these operators do not depend on the type of anyon statistics (that is, they are independent of \( Q \)). Also, recall the set \( \mathcal{OCP} \) of orthogonalized continuous polynomials defined in § 2.2.

Let us consider the following condition.

(C) For each \( h \in C_0(X \to \mathbb{R}) \), the linear operators \( \mathbf{J}^0(h) \) and \( \mathbf{J}^-_1(h) \) map the set \( \mathcal{OCP} \) into itself.
Theorem 3.5. Assume that either \(q \neq -1\) or \(q = -1\) and the support of the measure \(\nu\) does not consist of exactly two points. Then the condition (C) is satisfied if and only if there exist constants \(\lambda \in \mathbb{R}\) and \(\eta \geq 0\) such that the coefficients \(a_k\), \(b_k\) in the recursion formula (3.1) are given by

\[
a_k = \eta k(k + 1) \quad (k \in \mathbb{N}), \quad b_k = \lambda(k + 1) \quad (k \in \mathbb{N}_0).
\] (3.22)

In the latter case, for any \(h, f_1, \ldots, f_n \in C_0(X)\),

\[
J(h)f_1 \otimes \cdots \otimes f_n = h \otimes f_1 \otimes \cdots \otimes f_n
\]

\[
+ \lambda \sum_{i=1}^{n} f_1 \otimes \cdots \otimes f_{i-1} \otimes (hf_i) \otimes f_{i+1} \otimes \cdots \otimes f_n
\]

\[
+ 2\eta \sum_{1 \leq i < j \leq n} f_1 \otimes \cdots \otimes f_{i-1} \otimes f_{i+1} \otimes \cdots \otimes f_{j-1} \otimes (hf_i f_j) \otimes f_{j+1} \otimes \cdots \otimes f_n
\]

\[
+ n \int_X dy h(y)(f_1 \otimes \cdots \otimes f_n)(y, \cdot).
\] (3.23)

We see that in the classical case \(q = 1\) Theorem 3.5 gives exactly the Meixner class of infinite-dimensional polynomials discussed in §1.2. Note that the class of measures \(\nu\) obtained above is independent of \(q\). For such a choice of \(\nu\), we call \(\langle P_n(\omega), f^{(n)} \rangle\) a Meixner-type system of orthogonal (noncommutative) polynomials for the anyon statistics.

Remark 3.6. If in the fermion case \((q = -1)\) the support of the measure \(\nu\) consists of exactly two points, we have not been able to prove that the condition (C) always fails, but we conjecture this indeed to be so.

The following result is obvious.

Proposition 3.7. For each \(q \in \mathbb{C}\) with \(|q| = 1\) the equality \(\mathcal{CP} = \mathcal{OCP}\) holds in the anyon Gaussian or Poisson case, that is, when the formula (3.22) holds with \(\lambda \in \mathbb{R}\) and \(\eta = 0\).

However, due to the form (3.21) of the operator \(J^-_2(h)\), the equality \(\mathcal{CP} = \mathcal{OCP}\) fails if \(q \neq 1\) and the measure \(\nu\) is not concentrated at a single point. Still, in the classical case \((q = 1)\) Theorem 3.5 implies Theorem 1.2.

3.5. Anyon Meixner-type white noise. In this subsection (3.22) is assumed. We can, at least informally, define

\[
\omega(x) = \langle \omega, \delta_x \rangle, \quad x \in X,
\]

so that for \(h \in C_0(X)\)

\[
\langle \omega, h \rangle = \int_X dx \omega(x)h(x).
\] (3.24)

Hence, \((\omega(x))_{x \in X}\) can be regarded as an anyon Meixner-type white noise.

For \(x \in X\) we define an annihilation operator \(\partial_x\) as the linear operator acting on \(F^Q_{\text{fin}}(B_0(X))\) by the formula

\[
(\partial_x f^{(n)})(x_1, \ldots, x_{n-1}) := nf^{(n)}(x, x_1, \ldots, x_{n-1}), \quad (x_1, \ldots, x_{n-1}) \in X^{n-1},
\] (3.25)
where \( f^{(n)} \in \mathcal{B}_0^Q(X^n) \). Then by (3.21), for \( h \in C_0(X) \) we can interpret the operator \( J^{-}_2(h) \) as the integral

\[
J^{-}_2(h) = \int_X dx \, h(x) \partial_x.
\]  

(3.26)

Next, we introduce an ‘operator-valued distribution’ \( X \ni x \mapsto \partial^\dagger_x \) as follows:

\[
\int_X dx \, h(x) \partial^\dagger_x f^{(n)} := h \circledast f^{(n)} ,
\]  

(3.27)

where \( h \in C_0(X) \) and \( f^{(n)} \in \mathcal{B}_0^Q(X^n) \). In other words, the action of this distribution can be understood as \( \partial^\dagger_x f^{(n)} = \delta_x \circledast f^{(n)} \). Thus,

\[
J^+(h) = \int_X dx \, h(x) \partial^\dagger_x.
\]  

(3.28)

For \( h \in C_0(X) \), we will now need the operators

\[
\int_X dx \, h(x) \partial^\dagger_x \partial_x, \quad \int_X dx \, h(x) \partial^\dagger_x \partial_x \partial_x,
\]

acting on \( \mathcal{F}^Q_{fin}(B_0(X)) \). In view of (3.25) and (3.27), for each \( f^{(n)} \in \mathcal{B}_0^Q(X^n) \) we get that

\[
\left( \int_X dx \, h(x) \partial^\dagger_x \partial_x f^{(n)} \right)(x_1, \ldots, x_n) = n \left( \int_X dx \, h(x) \partial^\dagger_x f^{(n)}(x, \cdot) \right)(x_1, \ldots, x_n) = n \text{Sym}_n(h(x_1)f^{(n)}(x_1, x_2, \ldots, x_n))
\]

(compare with (2.14)), and

\[
\left( \int_X dx \, h(x) \partial^\dagger_x \partial_x \partial_x f^{(n)} \right)(x_1, \ldots, x_{n-1}) = n(n-1) \left( \int_X dx \, h(x) \partial^\dagger_x f^{(n)}(x, x, \cdot) \right)(x_1, \ldots, x_{n-1}) = n(n-1) \text{Sym}_{n-1}(h(x_1)f^{(n)}(x_1, x_1, x_2, x_3, \ldots, x_{n-1})).
\]  

(3.29)

**Theorem 3.8.** Assume that (3.22) holds. Then for \( h \in C_0(X) \)

\[
J^0(h) = \int_X dx \, h(x) \lambda \partial^\dagger_x \partial_x,
\]  

(3.30)

\[
J^1_1(h) = \int_X dx \, h(x) \eta \partial^\dagger_x \partial_x \partial_x.
\]  

(3.31)

Thus,

\[
J(h) = \int_X dx \, h(x)(\partial^\dagger_x + \lambda \partial^\dagger_x \partial_x + \eta \partial^\dagger_x \partial_x \partial_x + \partial_x).
\]  

(3.32)
In view of (3.12), the operator \( J(h) \) is a realization of the operator \( \langle \omega, h \rangle \) acting in the extended anyon Fock space \( \mathbf{F}^Q(\mathcal{H}, \nu) \). So with abuse of notation we can denote \( J(h) \) by \( \langle \omega, h \rangle \). Then by (3.24) and (3.32), we get the following representation of the anyon Meixner-type white noise (realized in the extended anyon Fock space \( \mathbf{F}^Q(\mathcal{H}, \nu) \)):

\[
\omega(x) = \partial_1^\lambda + \lambda \partial_1^\lambda \partial_x + \eta \partial_1^\lambda \partial_x \partial_x + \partial_x.
\]

**Remark 3.9.** We note that, for \( q \)-commutation relations with \( q \) real and either in \((-1,0)\) or in \((0,1)\) (see [3], [15], [19]), there is no analogue of a \( q \)-Lévy process which would have a representation like (3.32). Nevertheless, as shown in [22], there exist classical Markov processes whose transition probabilities are measures of orthogonality for \( q \)-Meixner (orthogonal) polynomials on the real line.

### 4. Proofs

**4.1. Proof of Proposition 3.1.** Since \( \text{Sym}_1 = 1 \), it suffices to prove the assertion for \( n \geq 2 \). Following [18], we first briefly recall how to show that the operator \( \text{Sym}_n \) given on the space \( L^2(X^n, m^{\otimes n}) \) by (2.4) is an orthogonal projection. For each \( \pi \in \mathfrak{S}_n \) we define

\[
(\Psi_\pi f^{(n)})(x_1, \ldots, x_n) = Q_\pi(x_1, \ldots, x_n) f^{(n)}(x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)})
\]

for \((x_1, \ldots, x_n) \in X^{(n)}\). Thus, \( \text{Sym}_n = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \Psi_\pi \). We then have \( \Psi_\pi^* = \Psi_{\pi^{-1}} \), which implies that \( \text{Sym}_n^* = \text{Sym}_n \). Furthermore, for each permutation \( \kappa \in \mathfrak{S}_n \), we have

\[
\Psi_\pi \Psi_\kappa = \Psi_{\pi \kappa}.
\]

Therefore, on \( X^{(n)} \)

\[
\text{Sym}_n^2 = \frac{1}{(n!)^2} \sum_{\pi, \kappa \in \mathfrak{S}_n} \Psi_\pi \Psi_\kappa = \frac{1}{(n!)^2} \sum_{\pi \in \mathfrak{S}_n} \sum_{\kappa \in \mathfrak{S}_n} \Psi_{\pi \kappa} = \frac{1}{n!} \sum_{\kappa \in \mathfrak{S}_n} \Psi_\kappa = \text{Sym}_n.
\]

Thus, \( \text{Sym}_n \) is an orthogonal projection. Note that (4.2) implies that

\[
\left( \prod_{1 \leq i < j \leq n \atop \pi(i) > \pi(j)} Q(x_i, x_j) \right) \left( \prod_{1 \leq k < l \leq n \atop \kappa(k) > \kappa(l)} Q(x_{\pi^{-1}(k)}, x_{\pi^{-1}(l)}) \right) = \prod_{1 \leq i < j \leq n \atop (\kappa \pi)(i) > (\kappa \pi)(j)} Q(x_i, x_j), \quad (x_1, \ldots, x_n) \in X^{(n)}
\]

for \( \kappa, \pi \in \mathfrak{S}_n \).

Now let us consider the bounded linear operator \( \text{Sym}_n \) on \( L^2(X^n, m_n^{(n)}) \). We represent it as

\[
\text{Sym}_n = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \Psi_\pi,
\]

where \( \Psi_\pi f^{(n)} \) is defined on the whole of \( X^n \) by (4.1) with the function \( Q_\pi(x_1, \ldots, x_n) \) on the right-hand side defined on \( X^n \) in § 3.2. We fix a permutation \( \pi \in \mathfrak{S}_n \) and
a partition \( \theta = \{\theta_1, \ldots, \theta_l\} \in \Pi(n) \) satisfying (3.4), and let (3.5) and (3.6) hold. Further, let \( \kappa \in \mathfrak{S}_n \) and let \( \zeta = \{\zeta_1, \ldots, \zeta_l\} \in \Pi(n) \) be such that
\[
\max \zeta_1 < \max \zeta_2 < \cdots < \max \zeta_l
\]
(4.6) and
\[
\kappa \beta_i = \zeta_{\hat{\pi}(i)}, \quad i = 1, \ldots, l,
\]
where \( \hat{\pi} \in \mathfrak{S}_l \).

Then for any function \( f^{(n)}: X^n \to \mathbb{C} \) and any \( (x_1, \ldots, x_n) \in X^n \) we have
\[
(\Psi_\pi \Psi_\kappa f^{(n)})(x_1, \ldots, x_n) = \left( \prod_{1 \leq i < j \leq l} Q(x_{\theta_i}, x_{\theta_j}) \right)(\Psi_\kappa f^{(n)})(x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)}).
\]
(4.7)

For \( i = 1, \ldots, n \) let \( y_i = x_{\pi^{-1}(i)} \), or equivalently, \( y_{\pi(i)} = x_i \). Thus, \( y_{\pi(i)} = y_{\pi(j)} \) if and only if \( i \) and \( j \) belong to the same element of the partition \( \theta \). Equivalently, \( y_i = y_j \) if and only if \( i \) and \( j \) belong to the same element of the partition \( \beta \). Therefore,
\[
(\Psi_\kappa f^{(n)})(y_1, \ldots, y_n) = \left( \prod_{1 \leq u < v \leq l} Q(y_{\beta_u}, y_{\beta_v}) \right)f^{(n)}(y_{\pi^{-1}(1)}, \ldots, y_{\pi^{-1}(n)}).
\]

Hence,
\[
(\Psi_\kappa f^{(n)})(x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)})
\]
\[
= \left( \prod_{1 \leq u < v \leq l} Q(x_{\pi^{-1} \beta_u}, x_{\pi^{-1} \beta_v}) \right)f^{(n)}(x_{\pi^{-1} \beta^{-1}(1)}, \ldots, x_{\pi^{-1} \beta^{-1}(n)})
\]
\[
= \left( \prod_{1 \leq u < v \leq l} Q(x_{\theta_{\hat{\pi}(u)}}, x_{\theta_{\hat{\pi}(v)}}) \right)f^{(n)}(x_{(\theta \pi^{-1})(1)}, \ldots, x_{(\theta \pi^{-1})(n)}),
\]
(4.8)

where we used the fact that for each \( u = 1, \ldots, l \)
\[
\pi^{-1} \beta_u = \theta_{\hat{\pi}^{-1}(u)}.
\]

By (4.4),
\[
\left( \prod_{1 \leq i < j \leq l} Q(x_{\theta_i}, x_{\theta_j}) \right) \left( \prod_{1 \leq u < v \leq l} Q(x_{\theta_{\hat{\pi}(u)}}, x_{\theta_{\hat{\pi}(v)}}) \right) = \left( \prod_{1 \leq i < j \leq l} Q(x_{\theta_i}, x_{\theta_j}) \right),
\]
(4.9)

where \( \hat{\pi} \) is the permutation in \( \mathfrak{S}_l \) induced by the permutation \( \kappa \pi \in \mathfrak{S}_n \) and the partition \( \theta \). In deducing (4.9) we used the fact that \( \hat{\pi} \) = \( \hat{\pi} \). Substituting (4.8) in (4.7) and using (4.9), we now conclude that
\[
\Psi_\pi \Psi_\kappa = \Psi_{\kappa \pi},
\]
(4.10)
and hence in analogy with (4.3) we get that \( \text{Sym}_n^2 = \text{Sym}_n \).

Next, we note that the measure \( m_\nu^{(n)} \) remains invariant under the transformation

\[
X^n \ni (x_1, \ldots, x_n) \mapsto (x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)}) \in X^n.
\]

Furthermore, it is easily seen that

\[
\overline{Q_{\pi^{-1}}(x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)})} = Q_{\pi}(x_1, \ldots, x_n)
\]

for each \( (x_1, \ldots, x_n) \in X^n \). Hence, \( \Psi^*_\pi = \Psi_{\pi^{-1}} \) for each \( \pi \in \mathcal{S}_n \), which implies that \( \text{Sym}_n^* = \text{Sym}_n \).

Thus, \( \text{Sym}_n \) is an orthogonal projection defined on \( L^2(X^n, m_\nu^{(n)}) \). In analogy with Proposition 2.5 in [18], we easily conclude that the image of \( \text{Sym}_n \) is indeed \( \mathbf{F}_\nu^n(\mathcal{H}, \nu) \), and hence Proposition 3.1 is proved.

Recall that the tensor product \( \otimes \) defined on \( \mathbf{F}_\text{fin}^Q(B_0(X)) \) is given by (3.19). Using (4.5) and (4.10), it is easy to show that

\[
(\text{Sym}_m f^{(m)}) \otimes (\text{Sym}_n g^{(n)}) = \text{Sym}_{m+n}(\text{Sym}_m f^{(m)} \otimes (\text{Sym}_n g^{(n)}))
\]

for any \( f^{(m)} \in B_0(X^m) \) and \( g^{(n)} \in B_0(X^n) \). Therefore, the tensor product \( \otimes \) is associative on \( \mathbf{F}_\text{fin}^Q(B_0(X)) \).

### 4.2. Proof of Theorem 3.2.

In connection with the unitary operator \( I: L^2(\tau) \rightarrow \mathcal{F}^Q(\mathcal{G}) \) defined in 2.2, our next goal is to obtain an explicit form for the subspace \( I(\mathcal{F}^Q_\mathcal{P}_n) \) of \( \mathcal{F}^Q(\mathcal{G}) \).

Denote by \( \mathbb{N}_{0,\text{fin}}^\infty \) the set of all infinite sequences \( \alpha = (\alpha_0, \alpha_1, \alpha_2, \ldots) \in \mathbb{N}_{0,\text{fin}}^\infty \) such that only finitely many elements \( \alpha_j \) are not zero. Let \( |\alpha| := \alpha_0 + \alpha_1 + \alpha_2 + \cdots \). For each \( \alpha \in \mathbb{N}_{0,\text{fin}}^\infty \) with \( |\alpha| \geq 1 \) we denote by \( \mathcal{F}_\alpha \) the subspace of the Fock space \( \mathcal{F}^Q(\mathcal{G}) \) consisting of all elements of the form

\[
\text{Sym}_{|\alpha|}(f^{(|\alpha|)}(x_1, \ldots, x_{|\alpha|})p_0(s_1) \cdots p_0(s_{\alpha_0}) \times p_1(s_{\alpha_0+1}) \cdots p_1(s_{\alpha_0+\alpha_1})p_2(s_{\alpha_0+\alpha_1+1}) \cdots),
\]

where \( f^{(|\alpha|)} \in \mathcal{H}^{|\alpha|} \). For \( \alpha \in \mathbb{N}_{0,\text{fin}}^\infty \) with \( |\alpha| = 0 \) we set \( \mathcal{F}_\alpha := \{c\Omega \mid c \in \mathbb{C}\} \). The following proposition is proved in [18], §7, and is an analogue of the Nualart–Schoutens decomposition of the \( L^2 \)-space of a classical Lévy process [45] (see also [48]).

**Proposition 4.1.**

\[
\mathcal{F}^Q(\mathcal{G}) = \bigoplus_{\alpha \in \mathbb{N}_{0,\text{fin}}^\infty} \mathcal{F}_\alpha. \tag{4.11}
\]

For each \( n \in \mathbb{N}_0 \), we define

\[
\mathcal{F}_n := \bigoplus_{\alpha \in \mathbb{N}_{0,\text{fin}}^\infty, \alpha_0+2\alpha_1+3\alpha_2+\cdots=n} \mathcal{F}_\alpha.
\]
Note that, by (4.11),
\[ F_Q(G) = \bigoplus_{n=0}^{\infty} F_n. \]

**Proposition 4.2.** For each \( n \in \mathbb{Z}_+ \),
\[ I \mathcal{O} \mathcal{P}_n = F_n. \]

**Proof.** It suffices to prove that, for each \( n \in \mathbb{N} \),
\[ I \mathcal{M} \mathcal{P}_n = \bigoplus_{\alpha \in \mathbb{N}_0^{\infty}} \mathcal{F}_\alpha =: \mathbb{M}_n. \]  \( \text{(4.12)} \)

**Lemma 4.3.** The space \( \mathbb{M}_n \) consists of all finite sums of elements of the form
\[ \text{Sym}_k \left( f^{(k)}(x_1, \ldots, x_k) s_1^{i_1} s_2^{i_2} \cdots s_k^{i_k} \right), \]  \( \text{(4.13)} \)
where \( f^{(k)} \in \mathcal{H} \otimes^k \) and \( i_1 + i_2 + \cdots + i_k + k \leq n \).

**Proof.** For each \( \pi \in \mathfrak{S}_k \) we define a unitary operator \( \Psi_\pi \) on \( (\mathcal{H} \otimes L^2(\mathbb{R}, \nu))^\otimes k \) by
\[ (\Psi_\pi g^{(k)})(x_1, s_1, \ldots, x_k, s_k) = Q_\pi(x_1, \ldots, x_k) \times g^{(k)}(x_{\pi^{-1}(1)}, s_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(k)}, s_{\pi^{-1}(k)}), \]
where the function \( Q_\pi \) is defined by (2.5). Then by [18], the operators \( \Psi_\pi \) form a unitary representation of the symmetric group \( \mathfrak{S}_k \), and for each \( \pi \in \mathfrak{S}_k \) we have \( \text{Sym}_k = \text{Sym}_k \Psi_\pi \). Hence,
\[ \text{Sym}_k \left( f^{(k)}(x_1, \ldots, x_k) r^{(k)}(s_1, \ldots, s_k) \right) = \text{Sym}_k \left( u^{(k)}(x_1, \ldots, x_k) r^{(k)}(s_{\pi^{-1}(1)}, \ldots, s_{\pi^{-1}(k)}) \right) \]
for any permutation \( \pi \in \mathfrak{S}_k \), any \( f^{(k)} \in \mathcal{H} \otimes^k \), and any polynomial \( r^{(k)}(s_1, \ldots, s_k) \) in the variables \( s_1, \ldots, s_k \), where
\[ u^{(k)}(x_1, \ldots, x_k) = Q_\pi(x_1, \ldots, x_k) f^{(k)}(x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(k)}) \]
in particular, \( u^{(k)} \in \mathcal{H} \otimes^k \).

The obvious representations
\[ p_l(s) = \sum_{i=0}^{l} \alpha_{il} s^i, \quad s^l = \sum_{i=0}^{l} \beta_{il} p_l(s) \]
easily give us the lemma. □

We now finish the proof of (4.12). Let \( \mathcal{F}_{\text{fin}}(\mathcal{H} \otimes \mathcal{P}(\mathbb{R})) \) be the linear subspace of the full Fock space over \( \mathcal{H} \otimes L^2(\mathbb{R}, \nu) \) consisting of finite sums of the elements \( c \Omega \) \( (c \in \mathbb{C}) \) and elements of the form
\[ f^{(k)}(x_1, \ldots, x_k) s_1^{i_1} s_2^{i_2} \cdots s_k^{i_k}, \]  \( \text{(4.14)} \)
where \( f^{(k)} \in \mathcal{H}^\otimes k \) and \( i_1, i_2, \ldots, i_k \in \mathbb{Z}_+ \) for \( k \in \mathbb{N} \). We set
\[
\text{Sym} := 1 \oplus \text{Sym}_1 \oplus \text{Sym}_2 \oplus \text{Sym}_3 \oplus \cdots.
\] (4.15)

This operator projects \( \mathcal{F}_{\text{fin}}(\mathcal{H} \otimes \mathcal{P}(\mathbb{R})) \) onto \( \mathcal{F}_{\text{fin}}^Q(\mathcal{H} \otimes \mathcal{P}(\mathbb{R})) \). For each \( h \in C_0(X) \) and \( F \in \mathcal{F}_{\text{fin}}(\mathcal{H} \otimes \mathcal{P}(\mathbb{R})) \) we have
\[
a^+(h \otimes 1) \text{Sym} F = \text{Sym}(J^+(h \otimes 1)F),
\]
\[
a^-(h \otimes 1) \text{Sym} F = \text{Sym}(J^-(h \otimes 1)F),
\] (4.16)
and
\[
a^0(h \otimes \text{id}) \text{Sym} F = \text{Sym}(J^0(h \otimes \text{id})F),
\] (4.17)
where, for each \( F \) of the form (4.14),
\[
(J^+(h \otimes 1)F)(x_1, s_1, \ldots, x_{k+1}, s_{k+1}) = h(x_1) 1(s_1)f^{(k)}(x_2, \ldots, x_{k+1})s_2^{i_1} s_3^{i_2} \cdots s_{k+1}^{i_k},
\]
\[
(J^0(h \otimes \text{id})F)(x_1, s_1, \ldots, x_k, s_k) = (h(x_1)s_1 + \cdots + h(x_k)s_k)
\]
\[
\times f^{(k)}(x_1, \ldots, x_k)s_1^{i_1} s_2^{i_2} \cdots s_k^{i_k},
\]
\[
(J^-(h \otimes 1)F)(x_1, s_1, \ldots, x_{k-1}, s_{k-1})
\]
\[
= \sum_{j=1}^k \int_X dy \int_{\mathbb{R}} \nu(dt) h(y)Q(y, x_1) \cdots Q(y, x_{j-1})
\]
\[
\times f^{(k)}(x_1, \ldots, x_{j-1}, y, x_j, \ldots, x_{k-1})s_1^{i_1} \cdots s_{j-1}^{i_{j-1}} t^{i_j} s_j s_{j+1} \cdots s_k^{i_k}.
\] (4.18)

Hence, it follows by induction from Lemma 4.3 and (4.16)–(4.18) that
\[
\langle \omega, h_1 \rangle \cdots \langle \omega, h_n \rangle \Omega \subset \mathbb{M}_n
\]
for any \( h_1, \ldots, h_n \in C_0(X) \), \( n \in \mathbb{N} \). Since \( \mathbb{M}_n \) is a closed subspace of \( \mathcal{F}^Q(\mathcal{G}) \), we have \( I.M. \mathcal{P}_n \subset \mathbb{M}_n \). On the other hand, it follows directly from the proof of Proposition 6.7 in [18] that each element of \( \mathbb{M}_n \) with the form (4.13) belongs to \( I.M. \mathcal{P}_n \). Hence, we get the inverse inclusion \( \mathbb{M}_n \subset I.M. \mathcal{P}_n \). \( \square \)

Note that for each \( h \in C_0(X) \)
\[
a^0(h \otimes \text{id}) = d\Gamma(M_h \otimes \text{id}) = d\Gamma(M_h \otimes M_{\text{id}}),
\] (4.19)
where \( M_h \) is the operator of multiplication by the function \( h(x) \) in \( \mathcal{H} \) and \( M_{\text{id}} \) is the (restricted to \( \mathcal{P}(\mathbb{R}) \)) operator of multiplication by the monomial \( \text{id}(s) = s \) in \( L^2(\mathbb{R}, \nu) \). It should be noted that the operator \( M_{\text{id}} \) is unbounded in \( L^2(\mathbb{R}, \nu) \) if the support of the measure \( \nu \) is unbounded, and the second quantization operator has domain \( \mathcal{F}_{\text{fin}}^Q(\mathcal{H} \otimes \mathcal{P}(\mathbb{R})) \). In view of the recursion formula (3.1), we get the representation
\[
M_{\text{id}} = A^+ + A^0 + A^-,
\]
where \( A^+ \), \( A^0 \), and \( A^- \) are the linear operators on \( \mathcal{P}(\mathbb{R}) \) given by
\[
A^+ p_k := p_{k+1}, \quad A^0 p_k := b_k p_k, \quad A^- p_k := a_k p_{k-1}.
\] (4.20)
Lemma 4.5. Let $\mathcal{A}^0(h \otimes \text{id}) = d\Gamma(M_h \otimes A^+) + d\Gamma(M_h \otimes A^0) + d\Gamma(M_h \otimes A^-).$ (4.21)

Furthermore, by (2.16) and (4.21),

$$\langle \omega, h \rangle = \mathcal{A}^+(h) + \mathcal{A}^0(h) + \mathcal{A}^-(h)$$ (4.22)

for all $h \in C_0(X)$, where

$$\mathcal{A}^+(h) := a^+(h \otimes 1) + d\Gamma(M_h \otimes A^+),$$

$$\mathcal{A}^0(h) := d\Gamma(M_h \otimes A^0),$$

$$\mathcal{A}^-(h) := a^-(h \otimes 1) + d\Gamma(M_h \otimes A^-).$$ (4.23)

**Proposition 4.4.** For each $h \in C_0(X)$,

$$\mathcal{A}^+(h) : \mathbb{F}_n \to \mathbb{F}_{n+1}, \quad \mathcal{A}^0(h) : \mathbb{F}_n \to \mathbb{F}_n, \quad \mathcal{A}^-(h) : \mathbb{F}_n \to \mathbb{F}_{n-1}.$$

**Proof.** Let $\sharp = +, 0, -. For each $h \in C_0(X)$ we define an operator $N(M_h \otimes A^\sharp)$ on $\mathcal{F}_{\text{fin}}(\mathcal{H} \otimes \mathcal{P}(\mathbb{R}))$ as follows: $N(M_h \otimes A^\sharp)\Omega := 0$, and for each $n \in \mathbb{N}$

$$N(M_h \otimes A^\sharp) \upharpoonright \left( \mathcal{F}_{\text{fin}}(\mathcal{H} \otimes \mathcal{P}(\mathbb{R})) \cap \mathcal{G}^\otimes_n \right) := (M_h \otimes A^\sharp) \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes (M_h \otimes A^\sharp) \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes (M_h \otimes A^\sharp).$$

**Lemma 4.5.** Let $\sharp = +, 0, -. For any $h \in C_0(X \to \mathbb{R})$ and $F \in \mathcal{F}_{\text{fin}}(\mathcal{H} \otimes \mathcal{P}(\mathbb{R}))$

$$d\Gamma(M_h \otimes A^\sharp) \text{Sym } F = \text{Sym}(N(M_h \otimes A^\sharp)F).$$

**Proof.** Fix any $F \in \mathcal{F}_{\text{fin}}(\mathcal{H} \otimes \mathcal{P}(\mathbb{R}))$ of the form

$$F(x_1, s_1, \ldots, x_n, s_n) = f^{(n)}(x_1, \ldots, x_n)p_{i_1}(s_1) \cdots p_{i_n}(s_n).$$

By (2.4),

$$(\text{Sym}_n F)(x_1, s_1, \ldots, x_k, s_k) = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} Q_\pi(x_1, \ldots, x_n) \times f^{(n)}(x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)})p_{i_1}(s_{\pi(1)}) \cdots p_{i_n}(s_{\pi(n)}).$$ (4.24)

Note that

$$d\Gamma(M_h \otimes A^+) = \text{Sym}(N(M_h \otimes A^+)).$$ (4.25)

By (4.24),

$$(N(M_h \otimes A^+) \text{Sym}_n F)(x_1, s_1, \ldots, x_n, s_n)$$

$$= \frac{1}{n!} \sum_{j=1}^n \sum_{\pi \in \mathfrak{S}_n} Q_\pi(x_1, \ldots, x_n) h(x_{\pi^{-1}(j)}) f^{(n)}(x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)})$$

$$\times p_{i_1}(s_{\pi^{-1}(1)}) \cdots p_{i_j+1}(s_{\pi^{-1}(j)}) \cdots p_{i_n}(s_{\pi^{-1}(n)})$$

$$= \frac{1}{n!} \sum_{j=1}^n \sum_{\pi \in \mathfrak{S}_n} Q_\pi(x_1, \ldots, x_n) g_j^{(n)}(x_{\pi^{-1}(1)}, s_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)}, s_{\pi^{-1}(n)}),$$ (4.26)
where, for \( j = 1, \ldots, n \),
\[
g_j^{(n)}(x_1, s_1, \ldots, x_n, s_n) := h(x_j) f^{(n)}(x_1, \ldots, x_n)p_{i_1}(s_1) \cdots p_{i_j+1}(s_j) \cdots p_{i_n}(s_n).
\]

Then by (4.25) and (4.26),
\[
(d\Gamma(M_h \otimes A^+) \Sigma_n F)(x_1, s_1, \ldots, x_n, s_n)
= \frac{1}{(n!)^2} \sum_{j=1}^{n} \sum_{\sigma \in S_n} \sum_{\pi \in S_n} Q_{\sigma}(x_1, \ldots, x_n) Q_{\pi}(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})
\times g_j^{(n)}(x_{\sigma^{-1}(1)}, s_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}, s_{\sigma^{-1}(n)}).
\]

Hence,
\[
d\Gamma(M_h \otimes A^+) \Sigma_n F = \sum_{j=1}^{n} \Sigma_n^2 g_j^{(n)} = \sum_{j=1}^{n} \Sigma_n g_j^{(n)}
= \Sigma_n \left( \sum_{j=1}^{n} g_j^{(n)} \right) = \Sigma_n \left( N(M_h \otimes A^+) F \right).
\]

The arguments for \( A^0 \) and \( A^- \) are analogous. \( \square \)

Proposition 4.4 now follows directly from the definition of the spaces \( \mathbb{F}_n \), the formula (4.16), and Lemma 4.5. \( \square \)

**Proposition 4.6.** For any \( h_1, \ldots, h_n \in C_0(X) \),
\[
I(P_n(\omega), h_1 \otimes \cdots \otimes h_n) = \mathcal{A}^+(h_1) \cdots \mathcal{A}^+(h_n) \Omega.
\]

**Proof.** Recall that \( \langle P_n(\omega), h_1 \otimes \cdots \otimes h_n \rangle \) is the orthogonal projection of the monomial
\[
\langle h_1, \omega \rangle \cdots \langle h_n, \omega \rangle = \langle h_1 \otimes \cdots \otimes h_n, \omega^{\otimes n} \rangle
\]
on \( \mathcal{O}_n \). The assertion follows from Propositions 4.2 and 4.4 if we note that
\[
I(P_n(\omega), h_1 \otimes \cdots \otimes h_n)
\]
is equal to the orthogonal projection of
\[
\langle \omega, h_1 \rangle \cdots \langle \omega, h_n \rangle \Omega
= (\mathcal{A}^+(h_1) + \mathcal{A}^0(h_1) + \mathcal{A}^-(h_1)) \cdots (\mathcal{A}^+(h_n) + \mathcal{A}^0(h_n) + \mathcal{A}^-(h_n)) \Omega
\]
on \( \mathbb{F}_n \). \( \square \)

We will now explicitly calculate the vector \( I(P_n(\omega), h_1 \otimes \cdots \otimes h_n) \). We introduce a topology on \( B_0(X^n) \) which yields the following notion of convergence: \( f_n \to f \) as \( n \to \infty \) means that there exists a set \( \Delta \in \mathcal{B}_0(X) \) such that \( \text{supp}(f_n) \subset \Delta \) for all \( n \in \mathbb{N} \) and (2.21) holds. Note that \( C_0(X^n) \) is a topological subspace of \( B_0(X^n) \).

For each \( \theta = \{\theta_1, \ldots, \theta_l\} \in \Pi(n) \) with \( \theta_1, \ldots, \theta_l \) satisfying the condition (3.4),
we define
\[
(\mathcal{E}_\theta f^{(n)})(x_1, s_1, \ldots, x_1, s_1) := f^{(n)}(x_1, \ldots, x_l)p_{|\theta_1|-1}(s_1) p_{|\theta_2|-1}(s_2) \cdots p_{|\theta_l|-1}(s_l),
\]
(4.27)
where \( f^{(n)} \in B_0(X^n) \), \((x_1, \ldots, x_l) \in X^{(l)}\), \((s_1, \ldots, s_l) \in \mathbb{R}^l\), and \( f^{(n)}_\theta(x_1, \ldots, x_l) \) is obtained from \( f^{(n)}(y_1, \ldots, y_n) \) by replacing \( y_{i_1} \) by \( x_1 \) for all \( i_1 \in \theta_1 \), \( y_{i_2} \) by \( x_2 \) for all \( i_2 \in \theta_2 \), and so on. Note that the function \( f^{(n)}_\theta : X^{(l)} \to \mathbb{C} \) is completely determined by the restriction of the function \( f^{(n)} : X^n \to \mathbb{C} \) to the set \( \mathcal{X}_\theta^{(n)} \).

For example, let \( n = 6 \) and let \( \theta = \{\theta_1, \theta_2, \theta_3\} \in \Pi(6) \) be
\[
\theta_1 = \{1, 3\}, \quad \theta_2 = \{2, 4, 6\}, \quad \theta_3 = \{5\}.
\]
Then for each \( (x_1, x_2, x_3) \in X^{(3)} \) and \((s_1, s_2, s_3) \in \mathbb{R}^3\),
\[
(\mathcal{E}_\theta f^{(6)})(x_1, x_2, x_3, s_1, s_2, s_3) = f^{(6)}(x_1, x_2, x_3, x_2)p_1(s_1)p_2(s_2)p_0(s_3).
\]

**Proposition 4.7.** For any \( n \in \mathbb{N} \) the map
\[
(C_0(X))^n \ni (h_1, \ldots, h_n) \mapsto \langle P_n(\omega), h_1 \otimes \cdots \otimes h_n \rangle \in L^2(\tau)
\]
can be extended by linearity and continuity to a map
\[
B_0(X^n) \ni f^{(n)} \mapsto \langle P_n(\omega), f^{(n)} \rangle \in L^2(\tau).
\]
Furthermore, for any \( f^{(n)} \in B_0(X^n)\),
\[
I\langle P_n(\omega), f^{(n)} \rangle = \text{Sym}\left( \sum_{\theta \in \Pi(n)} \mathcal{E}_\theta f^{(n)} \right). \tag{4.28}
\]

**Proof.** Fix any \( h_1, \ldots, h_n \in C_0(X) \) and let \( f^{(n)}(x_1, \ldots, x_n) = h_1(x_1) \cdots h_n(x_n) \). Then by Proposition 4.6 the formula (4.28) is equivalent to
\[
(a^+(h_1 \otimes 1) + d\Gamma(M_{h_1} \otimes A^+)) \cdots (a^+(h_n \otimes 1) + d\Gamma(M_{h_n} \otimes A^+)) \Omega = \text{Sym}\left( \sum_{\theta \in \Pi(n)} \mathcal{E}_\theta f^{(n)} \right), \tag{4.29}
\]
and to prove the latter it suffices by (4.16) and Lemma 4.5 to show that
\[
(J^+(h_1 \otimes 1) + N(M_{h_1} \otimes A^+)) \cdots (J^+(h_n \otimes 1) + N(M_{h_n} \otimes A^+)) \Omega = \sum_{\theta \in \Pi(n)} \mathcal{E}_\theta f^{(n)}. \tag{4.30}
\]

Let \( \beta = \{\beta_1, \ldots, \beta_k\} \) be an (unordered) partition of the set \( \{i + 1, i + 2, \ldots, n\} \). Then
\[
J^+(h_i \otimes 1)\mathcal{E}_\beta(h_{i+1} \otimes h_{i+2} \otimes \cdots \otimes h_n) = \mathcal{E}_\beta^+(h_i \otimes h_{i+1} \otimes \cdots \otimes h_n), \tag{4.31}
\]
where \( \beta^+ := \{\{i\}, \beta_1, \ldots, \beta_k\} \) is a partition of the set \( \{i, i + 1, \ldots, n\} \). Further,
\[
N(M_{h_i} \otimes A^+)\mathcal{E}_\beta(h_{i+1} \otimes h_{i+2} \otimes \cdots \otimes h_n) = \sum_{j=1}^k \mathcal{E}_{\beta_j}(h_i \otimes h_{i+1} \otimes \cdots \otimes h_n), \tag{4.32}
\]
where \( \beta_j^0 \) is the partition of \( \{i, i + 1, \ldots, n\} \) obtained from \( \beta \) by adding \( i \) to the set \( \beta_j \), that is,
\[
\beta_j^0 := \{\beta_1, \ldots, \beta_j \cup \{i\}, \ldots, \beta_k\}.
\]
By (4.31) and (4.32), the formula (4.30) follows by induction.

Finally, the extension of the formula (4.28) to the case of a general \( f^{(n)} \in B_0(X^n) \) follows by linearity and approximation arguments. □

We will now prove Theorem 3.2. In fact, a bit more generally, we will prove that (3.10) holds for any \( f^{(n)}, g^{(n)} \in B_0(X^n) \).

We first note that it suffices to prove (3.10) in the case when \( f^{(n)} = g^{(n)} = h_1 \otimes \cdots \otimes h_n \) with \( h_1, \ldots, h_n \in B_0(X) \). By Proposition 4.7,

\[
\left( \langle P_n(\omega), f^{(n)} \rangle, \langle P_n(\omega), f^{(n)} \rangle \right)_{L^2(\tau)} = \sum_{\theta \in \Pi(n)} \text{Sym}_{|\theta|}(\delta_\theta f^{(n)}), \sum_{\zeta \in \Pi(n)} \text{Sym}_{|\zeta|}(\delta_\zeta f^{(n)}) \right)_{\mathcal{F}^Q(\mathcal{G})}
\]

\[
= \sum_{l=1}^n \sum_{\theta, \zeta \in \Pi(n), |\theta| = l} \left( \text{Sym}_l(\delta_\theta f^{(n)}), \delta_\zeta f^{(n)} \right)_{L^2((X \times \mathbb{R})^l, (m \otimes \nu) \otimes i)^{(l)}}.
\]

(4.33)

Note that, by Proposition 3.1,

\[
\left( \text{Sym}_n f^{(n)}, \text{Sym}_n f^{(n)} \right)_{\mathcal{F}^Q_n(\mathcal{H}, \nu)} = \int_{X^n} (\text{Sym}_n f^{(n)})^2 dm^{(n)}_{\nu} = \sum_{\zeta \in \Pi(n)} \int_{X^n} (\text{Sym}_n f^{(n)})^2 dm^{(n)}_{\nu, \zeta}.
\]

(4.34)

By (4.33) and (4.34), (3.10) will follow if we show that

\[
\sum_{\theta \in \Pi(n), |\theta| = l} \left( \text{Sym}_l(\delta_\theta f^{(n)}), \delta_\zeta f^{(n)} \right)_{L^2((X \times \mathbb{R})^l, (m \otimes \nu) \otimes i)^{(l)}}
\]

\[
= \int_{X^n} (\text{Sym}_n f^{(n)})^2 dm^{(n)}_{\nu, \zeta}
\]

(4.35)

for a fixed \( \zeta \in \Pi(n) \) with \( |\zeta| = l \).

Accordingly, let us fix a partition \( \zeta = \{ \zeta_1, \ldots, \zeta_l \} \in \Pi(n) \) and assume that (4.6) holds. Let \( k_i := |\zeta_i|, i = 1, \ldots, l \). By the definition of \( \delta_\zeta f^{(n)} \),

\[
\delta_\zeta f^{(n)} = \left( \prod_{i_1 \in \zeta_1} h_{i_1} \right) \otimes p_{k_1 - 1} \otimes \cdots \otimes \left( \prod_{i_l \in \zeta_l} h_{i_l} \right) \otimes p_{k_l - 1}.
\]

(4.36)

Let \( \theta = \{ \theta_1, \ldots, \theta_l \} \in \Pi(n) \) and assume that (3.4) holds. Let \( r_i := |\theta_i| \) for \( i = 1, \ldots, l \). We can assume that there exists a permutation \( \tilde{\pi} \in \mathcal{S}_l \) such that

\[
\tilde{\pi} = k_{\sigma(i)} = 1, \ldots, l
\]

(4.37)

(indeed, otherwise the corresponding term in the sum on the left-hand side of (4.35) vanishes). In analogy with (4.36), we have

\[
\begin{align*}
&\mathcal{L} \text{Sym}_l(\delta_\theta f^{(n)})(y_1, s_1, \ldots, y_l, s_l) = \sum_{x \in S_l} Q_x(y_1, \ldots, y_l) \\
&\times \left( \prod_{j_i \in \theta_{\sigma(i)}} h_{j_i} \right) \otimes p_{r_{\sigma(i)} - 1} \otimes \cdots \otimes \left( \prod_{j_l \in \theta_{\sigma(i)}} h_{j_l} \right) \otimes p_{r_{\sigma(i)} - 1}(y_1, s_1, \ldots, y_l, s_l).
\end{align*}
\]
Hence, by (3.3),
\[
\begin{aligned}
& \left( \text{Sym}_l \left( \varepsilon_{\theta} f^{(n)} \right), \varepsilon_{\zeta} f^{(n)} \right)_{L^2((X \times \mathbb{R})^l, (m \otimes \nu) \otimes l)} ! \\
= & \sum_{\pi} \int_{X^l} Q_{\pi} (y_1, \ldots, y_l) \left( \prod_{j_1 \in \theta_{\pi}(1)} h_{j_1}(y_1) \right) \left( \prod_{i_1 \in \zeta_1} h_{i_1}(y_1) \right) \times \cdots \\
& \times \left( \prod_{j_l \in \theta_{\pi}(l)} h_{j_l}(y_l) \right) \left( \prod_{i_l \in \zeta_l} h_{i_l}(y_l) \right) dy_1 \cdots dy_l c_{k_1} \cdots c_{k_l},
\end{aligned}
\]
where the summation is over all permutations \( \pi \in S_l \) satisfying (4.37). Let us fix such a permutation \( \pi \). Then there exist exactly
\[
r_1! \cdots r_l! = k_1! \cdots k_l! \]
permutations \( \pi \in S_n \) such that
\[
\pi \zeta_i = \theta_{\pi(i)}, \quad i = 1, \ldots, l. \tag{4.39}
\]
We note that
\[
f^{(n)} (x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)}) = (h_1 \otimes \cdots \otimes h_n) (x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)})
\]
\[
= (h_{\pi(1)} \otimes \cdots \otimes h_{\pi(n)}) (x_1, \ldots, x_n)
\]
\[
= \left( \prod_{j_1 \in \pi \zeta_1} h_{j_1}(y_1) \right) \cdots \left( \prod_{j_l \in \pi \zeta_l} h_{j_l}(y_l) \right)
\]
\[
= \left( \prod_{j_1 \in \theta_{\pi}(1)} h_{j_1}(y_1) \right) \cdots \left( \prod_{j_l \in \theta_{\pi}(l)} h_{j_l}(y_l) \right) \tag{4.40}
\]
for any permutation \( \pi \) satisfying (4.39) and any \( (x_1, \ldots, x_n) \in X^{(n)} \zeta \), where
\[
y_1 = x_{i_1} \quad \text{for } i_1 \in \zeta_1, \quad \ldots, \quad y_l = x_{i_l} \quad \text{for } i_l \in \zeta_l.
\]

Let \( \zeta, \theta \in \Pi(n) \) be such that (4.37) holds for some permutation \( \widehat{\pi} \in S_l \). That is, the corresponding sequences \((k_1, \ldots, k_l)\) and \((r_1, \ldots, r_l)\) coincide up to a permutation. Denote by \( S_n[\zeta, \theta] \) the set of all permutations \( \pi \in S_n \) satisfying (4.39) for some permutation \( \widehat{\pi} \in S_l \). Note that the permutation \( \widehat{\pi} \) is then completely determined by \( \pi, \zeta, \) and \( \theta \) and automatically satisfies (4.39). Clearly, if \( \theta \) and \( \theta' \) are in \( \Pi(n) \) with \( \theta \neq \theta' \) and \( |\theta| = |\theta'| = l \) and if both satisfy (4.39), then
\[
S_n[\zeta, \theta] \cap S_n[\zeta, \theta'] = \emptyset. \tag{4.41}
\]
Furthermore,
\[
\bigcup_{\theta \in \Pi(n), \quad |\theta| = l \quad \theta \text{ satisfying } (4.39)} S_n[\zeta, \theta] = S_n. \tag{4.42}
\]
Therefore, by the definition of the measure $m_{\nu,\zeta}^{(n)}$ and the formulae (3.7), (4.38), (4.40)–(4.42),

$$\left(\text{Sym}_l(\mathcal{E}_\theta f^{(n)}), \mathcal{E}_\zeta f^{(n)}\right)_{L^2((X\times\mathbb{R})^l,(\mathfrak{m}\otimes\nu)^\otimes l)!}$$

$$= \frac{1}{n!} \sum_{\pi \in S_n[\zeta,\theta]} \int_{X^{(n)}_\zeta} Q_\pi(x_1, \ldots, x_n) f^{(n)}(x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)})$$

$$\times f^{(n)}(x_1, \ldots, x_n) m_{\nu,\zeta}^{(n)}(dx_1 \times \cdots \times dx_n).$$

Hence

$$\sum_{\theta \in \Pi(n), |\theta|=l} \left(\text{Sym}_l(\mathcal{E}_\theta f^{(n)}), \mathcal{E}_\zeta f^{(n)}\right)_{L^2((X\times\mathbb{R})^l,(\mathfrak{m}\otimes\nu)^\otimes l)!}$$

$$= \frac{1}{n!} \sum_{\theta \in \Pi(n), |\theta|=l} \sum_{\pi \in S_n[\zeta,\theta]} \int_{X^{(n)}_\zeta} Q_\pi(x_1, \ldots, x_n)$$

$$\times f^{(n)}(x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)}) f^{(n)}(x_1, \ldots, x_n) m_{\nu,\zeta}^{(n)}(dx_1 \times \cdots \times dx_n)$$

$$= \frac{1}{n!} \sum_{\pi \in S_n} \int_{X^{(n)}_\zeta} Q_\pi(x_1, \ldots, x_n) f^{(n)}(x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)})$$

$$\times f^{(n)}(x_1, \ldots, x_n) m_{\nu,\zeta}^{(n)}(dx_1 \times \cdots \times dx_n)$$

$$= \int_{X^{(n)}_\zeta} (\text{Sym}_n f^{(n)}) f^{(n)} dm_{\nu,\zeta}^{(n)},$$

which proves Theorem 3.2.

4.3. Proof of Theorem 3.4. Let us first prove the following result.

**Lemma 4.8.** Let $h \in C_0(X)$ and $f^n \in B_0(X^n)$, $n \in \mathbb{N}$. Then (3.13) and (3.20) hold with

$$J^+(h) \text{Sym}_n f^{(n)} = \text{Sym}_{n+1}(h \otimes f^{(n)}),$$

$$J^0(h) \text{Sym}_n f^{(n)} = \text{Sym}_n(\mathcal{J}^0(h) f^{(n)}),$$

$$J^-_{1}(h) \text{Sym}_n f^{(n)} = \text{Sym}_{n-1}(\mathcal{J}^{-1}_1(h) f^{(n)}),$$

$$J^-_{2}(h) \text{Sym}_n f^{(n)} = \text{Sym}_{n-1}(\mathcal{J}^{-2}_2(h) f^{(n)}).$$

Here

$$(\mathcal{J}^-_{2}(h) f^{(n)})(x_1, \ldots, x_{n-1}) := \sum_{i=1}^{n} \int_{X} dy h(y)$$

$$\times f^{(n)}(x_1, \ldots, x_{i-1}, y, x_i, \ldots, x_{n-1}) T_i(y, x_1, \ldots, x_{n-1}), \quad (4.43)$$
where for any \( \theta \in \Pi(n-1) \)
\[
T_i^{(n)} \upharpoonright X \times X_\theta^{(n-1)} := \prod_{\theta_u \in \theta: \max \theta_u \leq i-1} Q(y, x_{\theta_u}). \tag{4.44}
\]

**Proof.** By (4.22) and (4.23), we have
\[
\langle \omega, h \rangle = \mathcal{A}^+(h) + \mathcal{A}^0(h) + d\Gamma(M_h \otimes A^-) + a^- (h \otimes 1). \tag{4.45}
\]

(i) (\( J^+(h) \) part). From the arguments in the proof of Proposition 4.7 it follows that
\[
U^{-1} \mathcal{A}^+(h) U \text{ Sym}_n f^{(n)} = \text{ Sym}_{n+1}(h \otimes f^{(n)}) = J^+(h) \text{ Sym}_n f^{(n)}. \tag{4.46}
\]

(ii) (\( J^0(h) \) part). By Lemma 4.5, Proposition 4.7, and the formulae (4.15), (4.23), (3.15), and (3.16),
\[
U^{-1} \mathcal{A}^0(h) U \text{ Sym}_n f^{(n)} = U^{-1} \mathcal{A}^0(h) \text{ Sym} \left( \sum_{\theta \in \Pi(n)} \mathcal{E}_\theta f^{(n)} \right)
= U^{-1} \text{ Sym} \left( N(M_h \otimes A^0) \sum_{\theta \in \Pi(n)} \mathcal{E}_\theta f^{(n)} \right)
= U^{-1} \text{ Sym} \left( \sum_{\theta \in \Pi(n)} \sum_{i=1}^{n} \mathcal{E}_\theta (h \times_i f^{(n)}) b_{\gamma(i, \theta)-1} \gamma(i, \theta)^{-1} \right)
= \text{ Sym}_n (\mathcal{J}^0(h) f^{(n)})
= J^0(h) \text{ Sym}_n f^{(n)}, \tag{4.47}
\]

where
\[
(h \times_i f^{(n)})(x_1, \ldots, x_n) := h(x_i) f^{(n)}(x_1, \ldots, x_n).
\]

(iii) (\( J^-_1(h) \) part). As above,
\[
U^{-1} d\Gamma(M_h \otimes A^-) U \text{ Sym}_n f^{(n)} = U^{-1} \text{ Sym} \left( N(M_h \otimes A^-) \sum_{\theta \in \Pi(n)} \mathcal{E}_\theta f^{(n)} \right)
= U^{-1} \text{ Sym} \left( \sum_{l=1}^{n} \sum_{\theta \in \Pi(n)} \sum_{|\theta|=l} 1^{\otimes(k-1)} \otimes (M_h \otimes A^-) \otimes 1^{\otimes(l-k)} \mathcal{E}_\theta f^{(n)} \right)
= U^{-1} \text{ Sym} \left( \sum_{l=1}^{n} \sum_{\theta = \{\theta_1, \ldots, \theta_l\} \in \Pi(n)} \sum_{1 \leq k \leq l \atop |\theta_k| \geq 2} 1^{\otimes(k-1)} \otimes (M_h \otimes A^-) \otimes 1^{\otimes(l-k)} \mathcal{E}_\theta f^{(n)} \right), \tag{4.48}
\]

where (3.4) is assumed to hold. For \( \theta = \{\theta_1, \ldots, \theta_l\} \in \Pi(n) \) satisfying (3.4) and \( k \in \{1, \ldots, l\} \) such that \( |\theta_k| \geq 2 \), we have
\[
(1^{\otimes(k-1)} \otimes (M_h \otimes A^-) \otimes 1^{\otimes(l-k)} \mathcal{E}_\theta f^{(n)})(x_1, s_1, \ldots, x_l, \xi_s)
= a_{|\theta_k|-1} h(x_k) f^{(n)}_\theta(x_1, \ldots, x_k, \ldots, x_l) p_{|\theta_1|-1}(s_1) \cdots p_{|\theta_{k-1}|-1}(s_{k-1})
\times p_{|\theta_k|-2}(s_k) p_{|\theta_{k+1}|-1}(s_{k+1}) \cdots p_{|\theta_l|-1}(s_l). \tag{4.49}
\]
Let us fix any $i, j \in \{1, \ldots, n\}$ with $i < j$ and consider the set
\[ L_i := \{1, 2, \ldots, i-1, i+1, \ldots, n\} \]
of $n - 1$ elements. Then any partition $\zeta = \{\zeta_1, \ldots, \zeta_l\} \in \Pi(n-1)$ determines a partition $\tilde{\zeta} = \{\tilde{\zeta}_1, \ldots, \tilde{\zeta}_l\}$ of the set $L_i$ with $\tilde{\zeta}_u := K_i^u\zeta_u$ for $u = 1, \ldots, l$, where
\[ K_i^u : = \begin{cases} v & \text{if } v \leq i - 1, \\ v + 1 & \text{if } v \geq i. \end{cases} \]
Let $\tilde{\zeta}_k$ be the element of $\tilde{\zeta}$ containing $j$, and let
\[ \theta_u := \begin{cases} \tilde{\zeta}_u & \text{if } u \neq k, \\ \tilde{\zeta}_k \cup \{i\} & \text{if } u = k. \end{cases} \]
Thus, we have constructed a partition $\theta = \{\theta_1, \ldots, \theta_l\} \in \Pi(n)$ with $l \leq n - 1$. Next, consider an arbitrary partition $\theta = \{\theta_1, \ldots, \theta_l\} \in \Pi(n)$ with $l \leq n - 1$. Choose any $k \in \{1, \ldots, l\}$ such that $|\theta_k| \geq 2$. In how many ways can we obtain $\theta$ from $i, j$ and $\zeta \in \Pi(n-1)$ as above? This number is obviously equal to the number of ways of choosing $i, j \in \{1, \ldots, n\}$ with $i < j$ and $i, j \in \theta_k$, that is,
\[ \frac{1}{2} |\theta_k|(|\theta_k| - 1) = \frac{1}{2} (|\tilde{\zeta}_k| + 1)|\tilde{\zeta}_k| = \frac{1}{2} (|\zeta_k| + 1)|\zeta_k|, \]
where $j \in \tilde{\zeta}_k$ (or equivalently, $j - 1 \in \zeta_k$). Hence, by (3.4) (3.17), (3.18), (4.48), and (4.49), we get that
\[ \mathbf{U}^{-1}d\Gamma(M_h \otimes A^-)U \text{Sym}_n f^{(n)} = \text{Sym}_{n-1}(\mathcal{J}_1^- (h) f^{(n)}) = \mathbf{J}_1^- (h) \text{Sym}_n f^{(n)}. \]  
(iv) ($\mathbf{J}_2^- (h)$ part). For each $\theta = \{\theta_1, \ldots, \theta_l\} \in \Pi(n)$ satisfying (3.4), we have
\[ (a^- (h \otimes 1) \text{Sym}_l (\mathcal{S}_\theta f^{(n)}))) (x_1, s_1, \ldots, s_l-1) \]
\[ = \text{Sym}_{l-1} \left( \int_X dy \sum_{1 \leq i \leq l, |\theta_i| = 1} h(y)Q(y, x_1)Q(y, x_2)\cdots Q(y, x_{i-1}) \right. \]
\[ \times f^{(n)}_\theta(x_1, \ldots, x_{i-1}, y, x_i, \ldots, x_{l-1}) \]
\[ \left. \times p_{|\theta_1|-1}(s_1)\cdots p_{|\theta_{i-1}|-1}(s_{i-1})p_{|\theta_{i+1}|-1}(s_i)\cdots p_{|\theta_l|-1}(s_{l-1}) \right), \]  
where we used (2.7) and (4.27). Hence, by (4.43), (4.44), and (4.51),
\[ \mathbf{U}^{-1} a^- (h \otimes 1) \mathbf{U} \text{Sym}_n f^{(n)} = \text{Sym}_{n-1}(\mathcal{J}_2^- (h) f^{(n)}) = \mathbf{J}_2^- (h) \text{Sym}_n f^{(n)}, \]  
concluding the proof. $\square$

**Lemma 4.9.** For any $h \in C_0(X)$ and $f^{(n)} \in \mathcal{B}_0^Q(X^n)$,
\[ (\mathbf{J}_2^- (h) f^{(n)})(x_1, \ldots, x_{n-1}) = (\mathcal{J}_2^- (h) f^{(n)})(x_1, \ldots, x_{n-1}) \]
\[ = n \int_X dy \, h(y) f^{(n)}(y, x_1, \ldots, x_{n-1}). \]  

(4.53)
Proof. Fix any \( n \geq 2 \) and \( i \in \{2, \ldots, n\} \). Let a permutation \( \pi \in \mathfrak{S}_n \) be given by \( \pi(1) = i, \pi(j) = j - 1 \) for \( j = 2, \ldots, i \), and \( \pi(j) = j \) for \( j = i + 1, \ldots, n \). By (3.7) and (4.44) we have, for each \((x_1, \ldots, x_n) \in X^n\) such that \( x_1 \neq x_j \) for \( j \in \{2, \ldots, n\} \),

\[
(\Psi_\pi f^{(n)})(x_1, \ldots, x_n) = f^{(n)}(x_2, x_3, \ldots, x_i, x_1, x_{i+1}, \ldots, x_n)T_i(x_1, x_2, \ldots, x_n),
\]

where \( \Psi_\pi \) was defined in (4.1).

Since \( f \in \mathcal{B}_0^Q (X^n) \), by (4.5) and (4.10)

\[
\Psi_\pi f^{(n)} = \Psi_\pi \text{Sym}_n f^{(n)} = \text{Sym}_n f^{(n)} = f^{(n)}.
\]

By (4.54) and (4.55), for \((x_1, \ldots, x_{n-1}) \in X^{n-1}\)

\[
\int_X dy \, h(y) f^{(n)}(x_1, \ldots, x_{i-1}, y, x_i, \ldots, x_{n-1})T_i(y, x_1, \ldots, x_{n-1})
= \int_{X \setminus \{x_1, \ldots, x_{n-1}\}} dy \, h(y) f^{(n)}(x_1, \ldots, x_{i-1}, y, x_i, \ldots, x_{n-1})T_i(y, x_1, \ldots, x_{n-1})
= \int_X dy \, h(y) f^{(n)}(y, x_1, \ldots, x_{n-1}).
\]

Hence, by (4.43),

\[
(\mathcal{F}^-_2(h) f^{(n)})(x_1, \ldots, x_{n-1}) = n \int_X dy \, h(y) f^{(n)}(y, x_1, \ldots, x_{n-1})
= g^{(n-1)}(x_1, \ldots, x_{n-1}).
\]

Since \( f^{(n)} \in \mathcal{B}_0^Q (X^n) \), the formula (3.8) holds for each \( \pi \in \mathfrak{S}_n \). Hence, for each \( \pi \in \mathfrak{S}_{n-1} \)

\[
g^{(n-1)}(x_1, \ldots, x_{n-1}) = Q_\pi(x_1, \ldots, x_{n-1})g^{(n)}(x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)})
\]

(see (3.7)). Therefore,

\[
\text{Sym} \, g^{(n-1)} = g^{(n-1)}.
\]

Finally, the lemma follows from (4.56) and (4.57). \( \square \)

Theorem 3.4 now follows from Lemmas 4.8 and 4.9.

4.4. Proof of Theorem 3.5. Assume that (3.22) holds. Then by (3.16) and (3.18) we get that \( R_i^{(n)} \equiv \lambda \) and \( S_j^{(n)} \equiv 2\eta \). Hence by (3.15) and (3.17), the operators \( \mathcal{F}^0(h) \) and \( \mathcal{F}^-(h) \) map \( \mathcal{F}_\text{fin}(C_0(X)) \) into itself for any \( h \in C_0(X) \), so that the condition (C) is satisfied. Furthermore, the equality (3.23) follows from Theorem 3.4.

To show that (3.22) is necessary for the condition (C) to hold, we proceed as follows. First assume that \( \nu = \delta_\lambda \) for some \( \lambda \in \mathbb{R} \) (Gaussian/Poisson). Then \( a_k = 0 \) for all \( k \in \mathbb{N} \), \( b_0 = \lambda \), and the values of \( b_k \) for \( k \in \mathbb{N} \) can be chosen arbitrarily. Thus, (3.22) holds in this case with \( \eta = 0 \).

We next assume that the support of the measure \( \nu \) contains infinitely many points. Then \( a_k > 0 \) for all \( k \in \mathbb{N} \).
Lemma 4.10. Suppose that \( q \neq -1 \) and \( a_k > 0 \) for all \( k \in \mathbb{N} \). Further, let \( n \geq 2 \) and assume that the functions \( f^{(n)} \in C_0(X^n) \) are such that \( \text{Sym}_n f^{(n)} = 0 \) \( m^\nu \)-a.e. on the set \( X_\theta^{(n)} \), where \( \theta = \{ \theta_1, \theta_2 \} \in \Pi(n) \) with \( \theta_1 = \{1\} \) and \( \theta_2 = \{2, \ldots, n\} \). Then \( f^{(n)}(x_1, \ldots, x) = 0 \) for all \( x \in X \).

In the fermion case \( q = -1 \) the above result remains true for \( n \geq 3 \).

Proof. Let \( x_1, x_2 \in X \) be such that \( x_1^1 < x_2^1 \). (Recall that \( x^i \) denotes the \( i \)-th coordinate of \( x = (x^1, \ldots, x^d) \in X \).) In particular, \( x_1 < x_2 \). Then

\[
\begin{align*}
(\text{Sym}_n f^{(n)})(x_1, x_2, x_2, \ldots, x_2) &= \frac{1}{n} (f^{(n)}(x_1, x_2, x_2, \ldots, x_2) \\
&\quad + f^{(n)}(x_2, x_1, x_2, \ldots, x_2) + \cdots \\
&\quad + f^{(n)}(x_2, \ldots, x_2, x_1, x_2) + q f^{(n)}(x_2, \ldots, x_2, x_1)) = 0. \quad (4.58)
\end{align*}
\]

Since the function \( f^{(n)} \) is continuous, \( (4.58) \) holds pointwise on the open set

\[
\{(x_1, x_2) \in X^2 \mid x_1^1 < x_2^1\}.
\]

Therefore, \((n - 1 + q)/n) f^{(n)}(x, \ldots, x) = 0 \) for \( x \in X \). Thus, \( f^{(n)}(x, \ldots, x) = 0 \) if either \( q \neq -1 \) and \( n \geq 2 \), or \( q = -1 \) and \( n \geq 3 \). \( \square \)

We now set \( \lambda := b_0 \). Let us show that if \( (C) \) holds, then \( b_k = \lambda(k + 1) \) for all \( k \in \mathbb{Z}_+ \). The proof below works for any anyon statistics, but when \( q \neq -1 \) this proof can be significantly simplified.

Let \( \varepsilon \in \mathbb{R} \) be such that \( b_1 = 2\lambda + \varepsilon \). We will now show by induction that

\[
b_k = \lambda(k + 1) + \varepsilon, \quad k \geq 1. \quad (4.59)
\]

Assume that the equality in \((4.59)\) holds for \( k = 1, \ldots, n \). We fix an arbitrary \( h \in C_0(X) \) and an \( f^{(n+2)} \in C_0(X^{n+2}) \) and define a function \( g^{(n+2)} \in C_0(X^{n+2}) \) by

\[
g^{(n+2)}(x_1, \ldots, x_{n+2}) := f^{(n+2)}(x_1, \ldots, x_{n+2}) \left( \lambda h(x_1) + h(x_2)(\lambda(n+1)+\varepsilon) \right). \quad (4.60)
\]

Let \( \theta = \{ \theta_1, \theta_2 \} \in \Pi(n+2) \) with \( \theta_1 = \{1\} \) and \( \theta_2 = \{2, \ldots, n+2\} \). By \((3.15)\) and \((3.16)\), we have

\[
(\mathcal{F}^0(h) f^{(n+2)})(x_1, \ldots, x_{n+2}) \]

\[
= f^{(n+2)}(x_1, \ldots, x_{n+2}) \left( \lambda h(x_1) + (n+1) \frac{h(x_2)(\lambda(n+1)+\varepsilon)}{n+1} \right) \\
= g^{(n+2)}(x_1, \ldots, x_{n+2})
\]
m\((n+2)\)-a.e. on \( X_\theta^{(n+2)} \). Since \( (C) \) holds, there exists a function \( u^{(n+2)} \in C_0(X^{n+2}) \) such that

\[
\text{Sym}_{n+2} (\mathcal{F}^0(h) f^{(n+2)}) = \text{Sym}_{n+2} u^{(n+2)} \quad (4.61)
\]
m\((n+2)\)-a.e. on \( X^{n+2} \). Hence,

\[
\text{Sym}_{n+2}(g^{(n+2)} - u^{(n+2)})(x_1, \ldots, x_{n+2}) = 0
\]
for \( m^{(n+2)} \)-a.e. \((x_1, \ldots, x_{n+2}) \in X_\theta^{(n+2)}\). Noting that \( g^{(n+2)} - u^{(n+2)} \in C_0(X^{n+2})\), we conclude from Lemma 4.10 that
\[
u^{(n+2)}(x, \ldots, x) = g^{(n+2)}(x, \ldots, x), \quad x \in X. \tag{4.62}
\]
By (4.60)–(4.62),
\[
(J_0^\nu(f^{(n+2)}))(x, \ldots, x) = (\lambda(n + 2) + \varepsilon)h(x)f^{(n+2)}(x, \ldots, x) \tag{4.63}
\]
for all \( x \in X \). By (3.15), (3.16), and (4.63), we therefore get that \( b_{n+1} = \lambda(n+2)+\varepsilon \). Thus, (4.59) is proved.

Our next aim is to show that \( \varepsilon = 0 \). We first derive the following analogue of Lemma 4.10.

Lemma 4.11. Let \( a_k > 0 \) for all \( k \in \mathbb{N} \), and let \( f^{(5)} \in C_0(X^5) \) be such that \( \text{Sym}_5 f^{(5)} = 0 \) \( m^{(5)} \)-a.e. on the set \( X^{(5)}_\theta \), where \( \theta = \{\theta_1, \theta_2\} \in \Pi(5) \) with \( \theta_1 = \{1, 2\} \) and \( \theta_2 = \{3, 4, 5\} \). Then \( f^{(5)}(x, \ldots, x) = 0 \) for all \( x \in X \).

Proof. The proof is similar to that of Lemma 4.10. In fact, from the condition of Lemma 4.11, we get that
\[
6 + 4q f^{(5)}(x, \ldots, x) = 0,
\]
which implies the assertion. \( \square \)

By (3.15), (3.16), and (4.59), we have
\[
(J_0^\nu(f^{(5)}))(x_1, \ldots, x_5) = f^{(5)}(x_1, \ldots, x_5)(h(x_1)(2\lambda + \varepsilon) + h(x_3)(3\lambda + \varepsilon)) \tag{4.64}
\]
for \( m^{(5)} \)-a.e. \((x_1, \ldots, x_5) \in X^{(5)}_\theta \), where \( \theta \in \Pi(5) \) was defined in Lemma 4.11.

In analogy with the derivation of (4.63), we conclude from the condition (C), Lemma 4.11, and (4.64) that for all \( x \in X \)
\[
(J_0^\nu(f^{(5)}))(x, \ldots, x) = f^{(5)}(x, \ldots, x)h(x)(5\lambda + 2\varepsilon). \tag{4.65}
\]
On the other hand, by (3.15), (3.16), and (4.59), we have
\[
(J_0^\nu(f^{(5)}))(x, \ldots, x) = f^{(5)}(x, \ldots, x)h(x)(5\lambda + \varepsilon) \tag{4.66}
\]
for all \( x \in X \). Comparing (4.65) and (4.66), we see that \( \varepsilon \) must be equal to zero.

The proof of the equality \( a_k = \eta k(k+1) \) for \( k \in \mathbb{N} \) is similar, so we only outline it. Let \( \eta := a_1/2 \). Using Lemma 4.10 and the formulae (3.17), (3.18), we get the recursion formula
\[
a_{n+1} = 2\eta + ((n+1)(n+2) - 2)\frac{a_n}{n(n+1)} \tag{4.67}
\]
for \( n \geq 2 \). Choose \( \varepsilon \in \mathbb{R} \) so that \( a_2 = 6\eta + \varepsilon \). Then by (4.67),
\[
a_3 = 12\eta + \frac{10}{6}\varepsilon, \quad a_4 = 20\eta + \frac{5}{2}\varepsilon, \quad a_5 = 30\eta + \frac{7}{2}\varepsilon. \tag{4.68}
\]
On the other hand, by Lemma 4.11,
\[
a_5 = a_2 + 2a_3. \tag{4.69}
\]
From (4.68) and (4.69) we get that \( \varepsilon = 0 \). Hence the recursion formula (4.67) holds for all \( n \geq 1 \), and the desired equality follows.

We consider finally the case when the support of the measure \( \nu \) consists of \( l \) points with \( l \geq 2 \) finite. In the case when \( q = -1 \) we will additionally assume that \( l \geq 3 \). Then \( a_1 > 0, a_2 > 0, \ldots, a_{l-1} > 0 \), and \( a_i = 0 \) for \( i \geq l \). Furthermore, it follows from (3.2) that \( c_1 > 0, c_2 > 0, \ldots, c_l > 0 \), and \( c_i = 0 \) for \( i \geq l+1 \). Assume that the condition (C) is satisfied. Then in view of the analogous construction of the measure \( m^{(n)}_\nu \), we conclude as above that (4.67) holds for \( n = 1, 2, \ldots, l-1 \). In particular, we get that

\[
a_l = a_1 + (l(l+1) - 2) \frac{a_{l-1}}{(l-1)l}.
\]

Since \( a_1 > 0 \) and \( a_{l-1} > 0 \), we therefore get that \( a_l > 0 \), which contradicts the equality \( a_l = 0 \). Thus, the condition (C) cannot be satisfied, and Theorem 3.5 is proved.

We leave the easy proof of Proposition 3.7 to the reader. But we do indicate how Theorem 1.2 can now easily be derived.

Let \( q = 1 \) and assume that \( \mathcal{CP} = \mathcal{OP} \). Then for any \( h \in C_0(X) \) and \( f^{(n)} \in C_0(X^n) \) we have

\[
\langle \omega, h \rangle \langle P_n(\omega), f^{(n)} \rangle \in \mathcal{OP}
\]

(we used the fact that a product of polynomials in \( \mathcal{CP} \) is again in \( \mathcal{CP} \)). Since

\[
\mathcal{J}_2^-(h) \langle f^{(n)}, P_n(\omega) \rangle = \langle \mathcal{J}_2^-(h) f^{(n)}, P_{n-1}(\omega) \rangle \in \mathcal{OP},
\]

we conclude from Theorem 3.4 and (4.70) that (C) holds. Hence, (3.22) holds by Theorem 3.5.

Let us now assume that (3.22) holds. Then as follows from the proof of Theorem 3.5, \( h \in C_0(X) \) and the operators \( \mathcal{J}^0(h) \) and \( \mathcal{J}_1^-(h) \) map \( \mathcal{F}_{\text{fin}}(C_0(X)) \) into itself. Then (4.70) holds for any \( f^{(n)} \in C_0(X^n) \), and the equality \( \mathcal{CP} = \mathcal{OP} \) can now be deduced as in the proof of Theorem 4.1 in [16].

**4.5. Proof of Theorem 3.8.** We will only prove the equality (3.31), since the proof of the equality (3.30) is similar and simpler. We note also that the formula (3.32) will follow from (3.26)–(3.31).

It suffices to prove that for any \( h \in C_0(X) \)

\[
\mathcal{J}_1^-(h) g^{(n)} = \int_X dx h(x) \eta \partial_x^4 \partial_x g^{(n)}(x),
\]

where \( g^{(n)} \in B_0^Q(X^n) \) is of the form \( g^{(n)} = f_1 \otimes \cdots \otimes f_n \), with \( f_1, \ldots, f_n \in B_0(X) \). We have

\[
g^{(n)}(x_1, \ldots, x_n) := \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} Q_{\pi}(x_1, \ldots, x_n) f_{\pi(1)}(x_1) \cdots f_{\pi(n)}(x_n).
\]
It now follows from (3.29) that
\[
\left( \int_X dx \ h(x) \partial_x^1 \partial_x^2 \partial_x^3 \partial_x^4 \cdots \partial_x^n \right) (x_1, \ldots, x_{n-1})
\]
\[
= \text{Sym}_{n-1} \left( \frac{1}{(n-2)!} \sum_{\pi \in S_n} Q_{\pi}(x_1, x_1, x_2, \ldots, x_{n-1}) \right)
\]
\[
\times (hf_{\pi(1)}f_{\pi(2)})(x_1)f_{\pi(3)}(x_2) \cdots f_{\pi(n)}(x_{n-1})\right)
\]
\[
= \sum_{1 \leq i < j \leq n} \frac{1}{(n-2)!} \sum_{\pi \in S_n, \{1, 2\} = \{i, j\}} \text{Sym}_{n-1}(Q_{\pi}(x_1, x_1, x_2, \ldots, x_{n-1})
\]
\[
\times (hf_{i}f_{j})(x_1)f_{\pi(3)}(x_2) \cdots f_{\pi(n)}(x_{n-1})). \tag{4.71}
\]

By (3.7),
\[
Q_{\pi}(x_1, x_1, x_2, \ldots, x_{n-1}) = Q_{\sigma_{ij}(\pi)}(x_1, x_2, \ldots, x_{n-1}) \tag{4.72}
\]
for any \(\pi \in S_n\) satisfying \(\{1, 2\} = \{i, j\}\) with \(i < j\) and any \((x_1, x_2, \ldots, x_{n-1}) \in X^{n-1}\), where the permutation \(\sigma_{ij}(\pi) \in S_{n-1}\) is defined as follows:
\[
\sigma_{ij}(\pi)(1) := j
\]
and for \(k = 2, \ldots, n - 1\)
\[
\sigma_{ij}(\pi)(k) := \begin{cases} 
\pi(k + 1) & \text{if } \pi(k + 1) < i, \\
\pi(k + 1) - 1 & \text{if } \pi(k + 1) > i.
\end{cases}
\]

By (4.72), for any \(\pi \in S_n\) satisfying \(\{1, 2\} = \{i, j\}\) with \(i < j\),
\[
Q_{\pi}(x_1, x_1, x_2, \ldots, x_{n-1})(hf_{i}f_{j})(x_1)f_{\pi(3)}(x_2) \cdots f_{\pi(n)}(x_{n-1})
\]
\[
= Q_{\sigma_{ij}(\pi)}(x_1, x_2, \ldots, x_{n-1})(f_1 \otimes \cdots \otimes f_{i-1} \otimes f_{i+1} \otimes \cdots \otimes f_j)
\]
\[
\otimes (hf_{i}f_{j}) \otimes f_{j+1} \otimes \cdots \otimes f_n)(x_{\sigma_{ij}(\pi)^{-1}(1)}, \ldots, x_{\sigma_{ij}(\pi)^{-1}(n-1)})
\]
\[
= \Psi_{\sigma_{ij}(\pi)}(f_1 \otimes \cdots \otimes f_{i-1} \otimes f_{i+1} \otimes \cdots \otimes f_j)
\]
\[
\otimes (hf_{i}f_{j}) \otimes f_{j+1} \otimes \cdots \otimes f_n)(x_1, \ldots, x_{n-1}).
\]

Hence, by (4.5) and (4.10),
\[
\text{Sym}(Q_{\pi}(x_1, x_1, x_2, \ldots, x_{n-1})(hf_{i}f_{j})(x_1)f_{\pi(3)}(x_2) \cdots f_{\pi(n)}(x_{n-1}))
\]
\[
= (f_1 \otimes \cdots \otimes f_{i-1} \otimes f_{i+1} \otimes \cdots \otimes f_j)
\]
\[
\otimes (hf_{i}f_{j}) \otimes f_{j+1} \otimes \cdots \otimes f_n)(x_1, \ldots, x_{n-1}). \tag{4.73}
\]
By (4.71) and (4.73), we get finally that

\[
\int_X d\mathbf{x} \frac{h}{\partial x} \frac{1}{\partial x} \frac{\partial g}{\partial x}^{(n)} = 2 \sum_{1 \leq i < j \leq n} f_1 \otimes \cdots \otimes f_{i-1} \otimes f_{i+1} \otimes \cdots \otimes f_{j-1} \otimes (h f_i f_j) \otimes f_{j+1} \otimes \cdots \otimes f_n,
\]

and (3.31) follows from this.

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