PRYM VARIETIES OF TRIPLE COVERINGS

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Abstract. We show that the Prym variety associated to a triple covering \( f : Y \to X \) of curves is principally polarized of dimension \( \geq 2 \) if and only if \( f \) is non-cyclic, \( \acute{e}tale \) and \( X \) is of genus 2. We investigate some properties of these Prym varieties and their moduli.

1. Introduction

Let \( f : Y \to X \) be a covering of smooth projective curves. The Prym variety \( P(f) \) associated to this covering is, by definition, the connected component containing zero of the kernel of the norm map \( N_{JY} \) of the Jacobian \( JY \) onto the Jacobian \( JX \). We say that \( P(f) \) is a \( \text{principally polarized Prym variety} \) if the canonical principal polarization of \( JY \) restricts to a multiple of a principal polarization on \( P(f) \). In [BL, Proposition 12.3.3] it is claimed that \( P(f) \) is a principally polarized Prym variety of dimension at least 2 if and only if \( f \) is a double covering ramified at most at 2 points and \( X \) is of genus \( \geq 3 \). In this classification one case is missing, namely the Prym variety \( P(f) \) associated to a non-cyclic \( \acute{e}tale \) triple covering \( f \) of a curve \( X \) of genus 2, is principally polarized of dimension 2. The aim of this article is to fill this gap and carry out a study of these Prym varieties and their associated moduli spaces.

In the second section we recall some well known results about coverings with dihedral monodromy group. In Section 3 we prove that for a non-cyclic triple covering, \( P(f) \) is a principally polarized Prym variety of dimension \( \geq 2 \) if and only if \( f \) is \( \acute{e}tale \) and \( X \) is of genus 2. Let \( \mathcal{M}_2 \) be the moduli space of smooth curves of genus 2 and \( \mathcal{R}_{2,3}^{nc} \) the moduli space of non-cyclic \( \acute{e}tale \) triple coverings of curves of genus 2. In the fourth section we show that the forgetful map \( \mathcal{R}_{2,3}^{nc} \to \mathcal{M}_2 \) is \( \acute{e}tale \) (at the level of the corresponding Deligne-Mumford stacks), and of degree 60. If \( p : Z \to Y \) is the Galois closure of \( f : Y \to X \), then the composed map \( \delta \circ f \circ p : Z \to \mathbb{P}^1 \) is Galois with Galois group the dihedral group \( D_6 \) of order 12 (Corollary 4.4), where \( \delta : X \to \mathbb{P}^1 \) is the hyperelliptic covering. We use this to show that, surprisingly, the curve \( Y \) is hyperelliptic (Theorem 4.9). This fact allows us to describe the theta divisor \( \Xi \) of \( P(f) \). It turns out that \( P(f) \) is a Jacobian of a smooth curve \( \Xi \) (Theorem 4.17). The curve \( \Xi \) can be described explicitly in terms of the Weierstrass points of \( Y \) (see Proposition 4.18).

The map \( Pr : \mathcal{R}_{2,3}^{nc} \to \mathcal{A}_2 \) associating to any covering \( f \) the principally polarized abelian surface \( (P(f), \Xi) \) is called the Prym map. In Section 5 we show that \( Pr \) is of degree 10.

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onto its image and that $\mathcal{R}^{nc}_{\geq 2}$ is rational. Let $\alpha : Y \to JY$ denote the Abel map (depending on the choice of a point $y_0 \in Y$) and $\pi : JY \to P(f)$ the canonical projection. The composition $\alpha_f := \pi \circ \alpha : Y \to P(f)$ is called the Abel-Prym map of $P(f)$ (also dependent on $y_0$). In Theorem 6.3 we show that $\alpha_f$ is injective away from four of the Weierstrass points of $Y$, which have the same image.

Given a covering $f : Y \to X$ as above, the line bundle $\xi := \det f_* O_Y$ is a 2-torsion point of $JY$. Let $U_X(3, \xi)$ denote the moduli space of $S$-equivalence classes of semistable vector bundles of rank 3 and determinant $\xi$. Considering the elements of $P(f)$ as line bundles of degree zero, the direct image defines a morphism $f_* : P(f) \to U_X(3, \xi)$. In Proposition 7.3 we show that $f_*$ is injective.

To any non-cyclic étale triple covering $f$ as above, one can associate another abelian surface in a canonical way, namely the Prym variety $P(p, q)$ of the pair of maps $p : Z \to Y$ and $q : Z \to D$, where $p$ is as above and $q$ is the natural map onto the discriminant curve $D$ of $f$. By definition $P(p, q)$ is an abelian subvariety of the Prym variety $P(p)$ (see Section 8 for the definition). In the last section, we show that the restriction of the principal polarization of $P(p)$ to $P(p, q)$ is of type (1,1).

2. Coverings with monodromy the dihedral group

Let $f : Y \to X$ be a covering of degree $d$ of an irreducible smooth projective curve of genus $g_X$. Let $B$ be the finite subset of $X$ consisting of the branch locus of $f$ and fix a point $x_0 \in X \setminus B$. The covering induces a representation $\rho_f : \pi_1(X \setminus B, x_0) \to \mathcal{S}_{f^{-1}(x_0)}$ in the usual way, where $\mathcal{S}_{f^{-1}(x_0)}$ denotes the group of permutations of the fibre $f^{-1}(x_0)$. Choosing an identification $f^{-1}(x_0) = \{1, \ldots, d\}$, we get a representation of $\pi_1(X \setminus B, x_0)$ in the symmetric group $\mathcal{S}_d$ of degree $d$ which we also denote by $\rho_f$. Its image

$$\mathcal{M}(f) := \text{Im}(\rho_f)$$

does depend a priori on the base point $x_0$ and the identification $f^{-1}(x_0) = \{1, \ldots, d\}$. For a different choice of these the corresponding image is a conjugate subgroup. It is a transitive subgroup if and only if $Y$ is irreducible and called the monodromy group of the covering $f$.

Given $X$ and a finite subset $B \subset X$, it is well known that the map $f \mapsto \rho_f$ induces a bijection between the following sets:

(a) the set of isomorphism classes of irreducible coverings $f : Y \to X$ of degree $d$ whose branch points lie in $B$ and

(b) the set of representations $\rho : \pi_1(X \setminus B, x_0) \to \mathcal{S}_d$ with transitive image up to conjugacy in $\mathcal{S}_d$.

On the other hand, given an irreducible covering $f : Y \to X$, let $Z \to X$ denote the Galois closure of $f$; its Galois group

$$\mathcal{G}(f) := \text{Gal}(Z/X)$$

is called the Galois group of the covering $f$.

**Proposition 2.1.** An irreducible covering $f : Y \to X$ is Galois if and only if

$$\deg(f) = |\mathcal{M}(f)|.$$
Lemma 2.3. Let the notation be as in the previous lemma. Then
\[ B, x \] where
\[ \text{group} M \]
According to [H, Proposition p.689], for any irreducible covering \( f \) the monodromy group \( \mathcal{M}(f) \) coincides with the Galois group \( \mathcal{G}(f) \). Then Galois theory implies \( |\mathcal{M}(f)| = |\mathcal{G}(f)| = \deg(f) \) if and only if \( f \) is a Galois covering. \( \square \)

Consider a commutative diagram of smooth projective curves
(2.1)
\[
\begin{array}{ccc}
X & \xrightarrow{h} & D \\
\downarrow{f} & & \downarrow{g} \\
Y & \xrightarrow{p} & Z
\end{array}
\]
where \( Z \) is the normalization of the fibre product \( Y \times_X D \) of the maps \( f \) and \( g \). Later we assume \( f \) is an étale map. In that case the diagram is cartesian. The following lemma is an immediate consequence of the definitions.

Lemma 2.2. Suppose \( f \) and \( g \) in diagram (2.1) are given by representations \( \rho_f : \pi_1(X \setminus B, x_0) \to \mathcal{S}_{g^{-1}(x_0)} \) and \( \rho_g : \pi_1(X \setminus B, x_0) \to \mathcal{S}_{g^{-1}(x_0)} \) respectively. Then the composition \( h := p \circ f : Z \to X \) is given by
\[
\rho_h = \iota \circ (\rho_f, \rho_g) : \pi_1(X \setminus B, x_0) \to \mathcal{S}_{g^{-1}(x_0) \times g^{-1}(x_0)},
\]
where \( \iota : \mathcal{S}_{g^{-1}(x_0)} \times \mathcal{S}_{g^{-1}(x_0)} \to \mathcal{S}_{g^{-1}(x_0) \times g^{-1}(x_0)} \) is the obvious map. \( \square \)

Lemma 2.3. Let the notation be as in the previous lemma. Then \( \text{Im} \rho_h \simeq \text{Im} \rho_f \) if and only if \( \rho_g \) factors via \( \rho_f \).

Proof. Suppose \( \rho_g = \alpha \circ \rho_f \) with \( \alpha : \mathcal{S}_{g^{-1}(x_0)} \to \mathcal{S}_{g^{-1}(x_0)} \). Then \( \rho_h = \iota \circ (id, \alpha) \circ \rho_f \). Since \( \iota \circ (id, \alpha) \) is injective, it gives an isomorphism \( \text{Im} \rho_f \simeq \text{Im} \rho_h \).

Conversely, suppose that \( \text{Im} \rho_h \simeq \text{Im} \rho_f \). Since \( \iota \) is injective, this gives an isomorphism \( \text{Im} (\rho_f, \rho_g) \simeq \text{Im} \rho_f \). If \( p \) and \( q \) denote the first and second projection of \( \mathcal{S}_{g^{-1}(x_0)} \times \mathcal{S}_{g^{-1}(x_0)} \), the equation \( p \circ (\rho_f, \rho_g) = \rho_f \) implies that \( p \) induces an isomorphism \( \text{Im} (\rho_f, \rho_g) \simeq \text{Im} \rho_f \). Let \( \beta \) denote the inverse of this isomorphism. Then \( \rho_g = q \circ (\rho_f, \rho_g) = q \circ \beta \circ \rho_f \) is the asserted factorization. \( \square \)

For a positive integer \( d \), consider the dihedral group
\[
D_d := \langle \sigma, \tau \mid \sigma^d = \tau^2 = 1, \tau \sigma \tau = \sigma^{-1} \rangle
\]
of order \( 2d \), and a covering
\[
f : Y \to X
\]
of degree \( d \) of irreducible curves with monodromy group \( \mathcal{M}(f) \simeq D_d \). Let \( \rho_f : \pi_1(X \setminus B, x_0) \to \mathcal{S}_d \) denote the monodromy representation. Consider the signature map \( \text{sign} : \mathcal{S}_d \to \mathcal{S}_2 \) and let
\[
g : D \to X
\]
be the covering corresponding to the composition \( \text{sign} \circ \rho_f : \pi_1(X \setminus B, x_0) \to \mathcal{S}_2 \). The curve \( D \) is sometimes called the discriminant curve of the covering \( f \), which explains the notation \( D \). With these maps we consider the commutative diagram (2.1) with \( \deg(f) = \deg(g) = d \) and \( \deg(g) = \deg(p) = 2 \).
Lemma 2.4. Under these assumptions the following conditions are equivalent:
(a) the curve $Z$ is irreducible;
(b) the curve $D$ is irreducible;
(c) the image of $\rho_f$ is not contained in the alternating group $A_d$.

Proof. By assumption $X$ and $Y$ are irreducible. This implies the equivalence of (a) and (b). The covering $g$ is of degree 2, hence $D$ is reducible if and only if $\rho_g$ is trivial. This is the case if and only if $\text{Im} \rho_f \subset A_d$.

Proposition 2.5. Suppose $f: Y \to X$ is a covering of degree $d$ with Galois group $D_d$ such that $\text{Im} \rho_f \not\subset A_d$. Then the covering $h = f \circ p: Z \to X$ is the Galois closure of $f$.

Proof. By Lemma 2.4 the curve $Z$ is irreducible. According to Proposition 2.1 it suffices to show that $\text{Im} \rho_h \simeq D_d$. This is clear by Lemma 2.3, since $\rho_g$ factors via $\rho_f$ by definition.

3. Non-cyclic coverings of degree 3

Let $f: Y \to X$ denote a covering of degree 3 of smooth irreducible curves. Since $S_3 = D_3$ and $A_3 = \mathbb{Z}_3$, we get as an immediate consequence of Proposition 2.5.

Proposition 3.1. The covering $f: Y \to X$ of degree 3 is non-cyclic if and only if the curve $Z$ of diagram (2.1) is Galois over $X$ with Galois group $S_3$.

As $f$ has degree 3, the ramification points of $f$ are either simple, i.e. of order 2, or total, i.e. of order 3. Let $s$ and $t$ denote the number of simple and total ramification points of $f$ respectively. Then the Hurwitz formula gives

$$g_Y = 3g_X - 2 + \frac{s}{2} + t.$$ 

Proposition 3.2. Let $f: Y \to X$ be a non-cyclic covering of degree 3 with $s$ simple and $t$ total ramification points.

(a) The covering $D \to X$ is ramified exactly over the $s$ simple branch points of $f$ and

$$g_D = 2g_X - 1 + \frac{s}{2};$$

(b) The covering $Z \to D$ is cyclic and ramified exactly over the $2t$ preimages in $D$ of the $t$ total branch points of $f$ and

$$g_Z = 6g_X - 5 + \frac{3}{2}s + 2t;$$

(c) The covering $Z \to Y$ is ramified exactly over the $s$ non-ramified points in the fibres over the simple branch points of $f$.

Proof. A simple ramification point corresponds to a transposition in the monodromy group of $f$. This and the Hurwitz formula imply (a). The statement (b) follows similarly, since a total ramification point corresponds to a cycle of length 3. Part (c) is a consequence of (a) and (b) using the commutativity of diagram (2.1).

The Prym variety $P(f)$ of the covering $f: Y \to X$ is by definition the complement of the abelian subvariety $f^*J_X$ in the canonically polarized Jacobian $J_Y$.
Proposition 3.3. The canonical polarization of $JY$ induces on $P(f)$ a polarization of type $(1, \cdots, 1, 3, \cdots, 3)$, where $1$ occurs $g_X - 2 + \frac{s}{2} + t$ times and $3$ occurs $g_X$ times.

Proof. The homomorphism $f^* : JX \rightarrow JY$ is injective, since $f$ is non-cyclic. Hence the canonical polarization of $JY$ induces a polarization of type $(3, \cdots, 3)$ on $f^*JX$. Since $\dim P(f) = 2g_X - 2 + \frac{s}{2} + t$, the assertion follows from [BL, Corollary 12.1.5].

We call $P(f)$ a principally polarized Prym variety if the induced polarization on $P(f)$ is a multiple of a principal polarization. In order to determine these Prym varieties, we may assume that $X \neq \mathbb{P}^1$, since in this case $P(f) = JY$.

Corollary 3.4. Let $f : Y \rightarrow X$ be a non-cyclic covering of degree 3 of smooth projective curves. Then $P(f)$ is a principally polarized Prym variety if and only if either

(i) $X$ has genus 2 and $f$ is étale, or
(ii) $X$ has genus 1 and $(s, t) = (2, 0)$ or $(0, 1)$.

Proof. According to Proposition 3.3 the abelian variety $P(f)$ is a principally polarized Prym variety if and only if $g_X - 2 + \frac{s}{2} + t = 0$. This shows the assertion.

Case (ii) is uninteresting, since here $P(f)$ is of dimension 1. In the next section we will study case (i) in more detail.

4. The Prym variety of a non-cyclic 3-fold covering of a genus-2 curve

4.1. The set up. Let $X$ be a curve of genus 2 and $f : Y \rightarrow X$ a connected non-cyclic étale covering of degree 3. Then we have diagram (2.1) where $D$ is the discriminant curve of $f$ and, according to Proposition 3.2, all maps in the diagram are étale. Hence (2.1) is a Cartesian diagram. According to Proposition 3.1 the curve $Z$ is the Galois closure of $f$ with Galois group $S_3$. For the genera of the curves we have that

$g_Y = 4, \quad g_D = 3, \quad g_Z = 7.$

It is easy to give an example of such a covering $f$. For this we exhibit a surjective homomorphism $\psi : \pi_1(X, x_0) \rightarrow S_3$. Since

\begin{equation}
\pi_1(X, x_0) = \left\langle a_1, a_2, b_1, b_2 \mid \prod_{i=1}^{2} a_i b_i a_i^{-1} b_i^{-1} = 1 \right\rangle,
\end{equation}

it suffices to give cycles $\sigma_i, \tau_i$, $i = 1, 2$ generating $S_3$ and satisfying $\prod_{i=1}^{2} \sigma_i \tau_i \sigma_i^{-1} \tau_i^{-1} = 1$. One can choose for example $\tau_1 = \tau_2 = (1, 2)$, $\sigma_1 = (1, 2, 3)$ and $\sigma_2 = (1, 3, 2)$.

In order to classify non-cyclic étale degree-3 coverings, we use the following proposition.

Proposition 4.1. There is a bijection between the following sets:

(1) connected non-cyclic étale coverings $f : Y \rightarrow X$ of degree 3 and
(2) étale Galois coverings $h : Z \rightarrow X$ with Galois group $S_3$,

up to isomorphisms.

In particular, there are only finitely many non-cyclic étale degree-3 coverings of $X$. 
Here we consider only connected Galois coverings. Note that two Galois coverings of the set (2) are isomorphic if they differ by an (inner) automorphism of $S_3$. Note also that any isomorphism of two coverings in (1) can be extended to an isomorphism of their Galois closures.

**Proof.** We already associated a Galois covering $h$ to every $f$. Conversely, let $h : Z \to X$ be a Galois covering as in (2). The group $S_3$ admits exactly 3 subgroups of index 3. Let $Y_i \to X$, $i = 1, 2, 3$ denote the sub-coverings corresponding to them. They are étale and non-cyclic, since the subgroups are not normal. The coverings $Y_i \to X$ are isomorphic to each other, since the subgroups are conjugate. The last assertion follows from the fact that there are only finitely many étale Galois coverings of $X$ with Galois group $S_3$ (in fact less than $2^4 \cdot 3^6$, which is the number of all possible double coverings of $X$ times the number of all possibilities triple cyclic coverings of $D$; see Remark 8.6). \qed

Let $M_2$ denote the moduli space of curves of genus 2 and $R_{2,3}^{nc}$ denote the moduli space of non-cyclic étale degree-3 coverings of curves of genus 2. It exists as a coarse moduli space, since the moduli space of étale Galois coverings of curves of genus 2 does.

**Corollary 4.2.** The forgetful map $\phi : R_{2,3}^{nc} \to M_2$, $[f : Y \to X] \mapsto [X]$ is a finite, étale covering of Deligne-Mumford stacks of degree 60.

**Proof.** The finiteness of the map $\phi$ follows from Proposition 4.1. We have to show that for every curve $X$ of genus 2 there are exactly 60 classes of étale Galois coverings over $X$ with Galois group $S_3$. Using Riemann’s existence theorem and the fact that the group of (inner) automorphisms of $S_3$ is isomorphic to $S_3$ itself, it suffices to show that there are exactly 360 surjective homomorphisms

$$\psi : \pi_1(X, x_0) \to S_3.$$ Given a set of generators of $\pi_1(X, x_0)$ as in (11), we denote by $A_1, A_2, B_1, B_2$ their images in $S_3$ under a map $\psi : \pi_1(X, x_0) \to S_3$. Then it suffices to count the number of quadruples $(A_1, A_2, B_1, B_2) \in S_3^4$, whose entries generate $S_3$ and which verify

$$[A_1, B_1] = [B_2, A_2],$$

where $[A_i, B_i]$ is the commutator of $A_i, B_i$. Observe first that $\psi$ is surjective if the quadruple contains at least one element of order 3 and one transposition or 2 different transpositions. We shall consider 4 cases depending on the number of transpositions among the $A_i$’s and $B_i$’s.

Suppose first that there is only one transposition, say $A_1 = \tau$. Then $[B_2, A_2] = 1$ and relation (11) forces $B_1$ to be the identity and $A_2, B_2$ to be elements of order 3, not both trivial. By counting the choices for $\tau$ and the elements of order 3 we get $4 \cdot 3 \cdot 8 = 96$ possibilities.

Suppose now there are exactly 2 transpositions. In this case (11) imposes either $(A_1, B_1) = (\tau, \tau)$, $(A_2, B_2) = (\sigma^i, \sigma^j)$ with $\sigma^3 = 1$ and not both $A_2, B_2$ trivial, which gives $2 \cdot 3 \cdot 8 = 48$ possibilities, or $(A_1, B_1) = (\tau, \sigma^i)$, $(A_2, B_2) = (\tau', \sigma^j)$ with $4 \cdot 6$ choices if $\sigma^i = 1$ (and then $\tau \neq \tau'$) and $4 \cdot 9$ choices if $\sigma^j \neq 1$. So in this case we obtain $48 + 24 + 36 = 108$ possibilities.
Similarly, when we suppose exactly 3 transpositions in the quadruple, we have for example \((A_1, B_1) = (\tau, \tau), (A_2, B_2) = (1, \tau')\) with \(\tau \neq \tau'\), giving \(3 \cdot 2^3\) choices, or \((A_1, B_1) = (\tau, \tau'), (A_2, B_2) = (\tau'', \sigma')\) with \(\tau \neq \tau'\) and \(\sigma'\) determined by the other elements. In this way, we get \(3^2 \cdot 2^3\) choices. Hence we obtain \(24 + 72 = 96\) different possibilities in this case.

Finally, by assuming all non-trivial elements are transpositions and there are at least 2 of them, we add 60 more possibilities. Summing up the 4 cases we get the result. □

4.2. The extended diagram. Consider again diagram (2.1). We denote by \(\tau_Z\) and \(\tau_D\) the involutions corresponding to the double coverings \(p\) and \(g\) and by \(\sigma_Z\) the automorphism defining the cyclic covering \(q\) : \(Z \to D\).

As a curve of genus 2, \(X\) is hyperelliptic. According to [R, Example 1 p.64] the hyperelliptic involution \(\iota_X\) of \(X\) lifts to an involution on any cyclic covering. So \(\iota_X\) lifts to an involution \(\iota_D\) on \(D\) which moreover commutes with \(\tau_D\) (again by [R, Example 1 p.64]). Hence \(\iota_D\) and \(\tau_D\) generate the Klein group \(\mathbb{Z}_2 \oplus \mathbb{Z}_2\) acting on \(D\) and \(\kappa_D = \iota_D \tau_D = \tau_D \iota_D\) is also a lift of \(\iota_X\). According to [FK, Corollary of Proposition V.1.10, p.267], one of these liftings, say \(\iota_D\), is a hyperelliptic involution on \(D\).

In particular, \(D\) is hyperelliptic and \(\iota_D\) lifts to an involution \(\iota_Z\) on \(Z\). If \(r : Z \to B\) denotes the corresponding double covering, we have the following commutative diagram

\[
\begin{array}{ccc}
Y = Z/\tau_Z & \xrightarrow{\psi} & B = Z/\iota_Z \\
\downarrow f & & \downarrow g \\
D & \xrightarrow{\delta} & E = D/\kappa_D \\
\downarrow \delta & & \downarrow \delta \\
X = D/\tau_D & & \mathbb{P}^1 = D/\iota_D \\
\end{array}
\]

Observe that \(\iota_Z \tau_Z \iota_Z^{-1} \tau_Z^{-1}\) acts on the fibers of the map \(q : Z \to D\). So, \(\tau_Z \iota_Z\) and \(\iota_Z \tau_Z\) differ by a power of the automorphism \(\sigma_Z\), i.e. \(\tau_Z \iota_Z = \sigma_Z^i \tau_Z \iota_Z\) for some \(i, 0 \leq i \leq 2\). If \(i \geq 1\), multiplying this equation by \(\sigma^i\) we get

\[\sigma_Z^i \tau_Z \iota_Z = \sigma_Z^{-i} \tau_Z \iota_Z = \tau_Z \iota_Z^i \sigma_Z^i.\]

Replacing \(\iota_Z\) by \(\sigma_Z^i \iota_Z\) (note that with \(\iota_Z\) also \(\sigma_Z^i \iota_Z\) is an involution), we may assume that \(\iota_Z \tau_Z = \tau_Z \iota_Z\).

**Proposition 4.3.** The group generated by the automorphisms \(\iota_Z, \tau_Z\) and \(\sigma_Z\) is the dihedral group \(D_6\) of order 12.

**Proof.** For simplicity, we drop the indices of the automorphisms of \(Z\) in this proof. According to [R, Example 1 p.64] we have \(i \sigma = \sigma^i i\). It suffices to show that \(\psi := i \sigma \tau\) is of
order 6. Since \( \nu \tau = \tau \nu \) and \( \tau \sigma = \sigma^2 \tau \),
\[
\psi^2 = \nu \sigma \tau \sigma \tau = \sigma^2 \sigma^2 = \sigma,
\]
which implies the assertion. \( \square \)

**Corollary 4.4.** The covering \( \delta \circ f \circ p : Z \to \mathbb{P}^1 \) is Galois with Galois group \( D_6 \).

**Proof.** For the proof just note that the covering \( \delta \circ f \circ p : Z \to \mathbb{P}^1 \) is Galois if and only if \( |Aut(Z/\mathbb{P}^1)| = \text{deg}(\delta \circ f \circ p) \). \( \square \)

**Corollary 4.5.** The curve \( B \) is of genus 2.

**Proof.** According to [R, Theorem 1] the Prym variety \( P(q) \) of the covering \( q : Z \to D \) is isomorphic to \( JB \times JB \), which gives the assertion, since \( \dim P(q) = 4 \). For another proof of this fact see [O, Proposition 2.3]. \( \square \)

**Remark 4.6.** Let \( \phi_{P(q)} \) denote the polarization of \( P(q) \) induced by the canonical polarization of \( JZ \). If \( JB \) is identified with its dual via the canonical polarization, then the polarization on \( JB \times JB \) induced by \( \phi_{P(q)} \) via the isomorphism \( P(q) \simeq JB \times JB \) of [R, Theorem 1] is given by the matrix \( \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \).

**Proposition 4.7.** The map \( p^* : JY \to JZ \) is injective on \( P(f) \). Moreover, \( p^*(P(f)) \) is contained in \( P(q) \).

**Proof.** The kernel of \( p^* \) is generated by a 2-division point \( \lambda \in JY \). The fact that \( p \) is a lift of the covering \( g \) means that \( \lambda \in f^*(JX) \). Now \( f^*(JX) \cap P(f) \) consists of 3-division points, so does not contain \( \lambda \). This implies the first assertion. For the last assertion, note that \( P(f) \) is the kernel of the norm map \( Nm_f \). Hence \( p^*(P(f)) \subset \text{Ker}(Nm_q) \) (in fact, \( Nm_q \circ p^* = g^* \circ Nm_f \) holds). Since \( p^*(P(f)) \) is connected, it is contained even in \( (\text{Ker} Nm_q)^0 = P(q) \); for details, see [LR, Proposition 2.2]. \( \square \)

4.3. **The full extended diagram.** According to Corollary [R, Theorem 1] the covering \( Z \to \mathbb{P}^1 \) is Galois with group \( D_6 \), for which we will use for simplicity the following notation

\[
D_6 = \langle \psi, \tau \mid \psi^6 = \tau^2 = 1, \ \tau \psi \tau = \psi^{-1} \rangle,
\]

with \( \tau = \tau_Z \) and \( \psi = \nu_Z \sigma_Z \tau_Z \) as in the previous section (and thus \( \nu_Z = \psi^3 \tau \)). The subgroup diagram of \( D_6 \), which contains one representative of every conjugacy class of subgroups together with their natural inclusions, gives us the following commutative diagram of
sub-coverings of the curve $Z$. We keep the notation of the previous diagram.

\begin{equation}
Y = Z/\langle \tau \rangle \quad Z \quad B = Z/\langle \psi \tau \rangle \quad A = Z/\langle \psi^3 \rangle
\end{equation}

\begin{equation}
X = Z/\langle \psi^2, \tau \rangle \quad D = Z/\langle \psi^2 \rangle 
\end{equation}

\begin{equation}
C = Z/\langle \psi^3, \tau \rangle \quad \mathbb{P}^1 = Z/\langle \psi^3 \rangle 
\end{equation}

The maps of $Z$ to the curves in the second (respectively third, fourth and fifth) row of the diagram are of degree 2 (respectively 3, 4 and 6).

**Proposition 4.8.** (a) The curves $A$ and $E$ are of genus 1.
(b) The curve $C$ is of genus 0.

**Proof.** For both parts of the proposition we use the following Theorem of Accola [A, Theorem 5.9]:

Let $Z$ be a compact Riemann surface of genus $p$. Suppose that $Z$ admits a finite group of automorphisms $G_0$ and that this group admits a partition (i.e. $G_0 = \bigcup_{i=1}^{s} G_i$, where $\{G_i\}_{i=1}^{s}$ is a collection of subgroups of $G_0$ with $G_i \cap G_j = \{e\}$, $1 \leq i < j \leq s$). Let $n_i = |G_i|$, $X_i = X/G_i$ and $p_i$ be the genus of $X_i$ for $i = 0, \ldots, s$. Then

\[(s - 1)p + n_0p_0 = \sum_{i=1}^{s} n_ip_i.\]

To show (a) we apply the Theorem of Accola to the partition $G_0 = D_6 = \bigcup_{i=1}^{7} G_i$ with

\[G_1 = \langle \psi \rangle, \ G_2 = \langle \tau \rangle, \ G_3 = \psi G_2 \psi^{-1}, \ G_4 = \psi^2 G_2 \psi^{-2}, \]

\[G_5 = \langle \psi^3 \rangle, \ G_6 = \psi G_5 \psi^{-1}, \ G_7 = \psi^2 G_5 \psi^{-2}, \]

to give

\[42 = 6 \cdot 7 + 0 = 6g(Z/\langle \psi \rangle) + 3(2g(Z/\langle \tau \rangle)) + 3(2g(Z/\langle \psi^3 \rangle)) = 6g(E) + 6g(Y) + 6g(B).\]

Since $g(Y) = 4$ and $g(B) = 2$ according to Corollary 15 this implies $g(E) = 1$. Consider the upper right commutative square, i.e. $\mu \circ \alpha = \beta \circ q$. Since $f$ is étale, so ist $q$. From the facts that $\deg \mu$ and $\deg \beta$ are coprime and $q$ is cyclic étale, we conclude that $\mu$ is étale. This implies $g(A) = 1$. 

\[\]
For the assertion (b), we apply the Theorem of Accola to the group \( G_0 = \langle \psi^3, \tau \rangle \) of automorphisms of the curve \( Z \) with partition \( G_0 = \bigcup_{i=1}^{3} G_i \) with 

\[
G_1 = \langle \psi^3 \rangle, \quad G_2 = \langle \psi^3 \tau \rangle, \quad G_3 = \langle \tau \rangle,
\]
to obtain the equality 

\[
14 + 4g(C) = 2g(A) + 2g(B) + 2g(Y) = 2 + 4 + 8,
\]
which implies the assertion.

Since \( \deg \gamma = \deg r = 2 \), we get as an immediate consequence,

**Theorem 4.9.** Any non-cyclic étale 3-fold covering \( Y \) of a curve of genus 2 is hyperelliptic.

### 4.4. The theta divisor of the Prym variety \( P \)

According to Theorem 4.9 the curve \( Y \) is hyperelliptic. Let \( q_1, \ldots, q_{10} \) denote its Weierstrass points and let \( p_1, \ldots, p_6 \) be the Weierstrass points of \( X \). From the commutativity of the diagram

\[
\begin{array}{ccc}
Y & \overset{\gamma}{\longrightarrow} & \mathbb{P}^1 = C \\
\downarrow f & & \downarrow \text{id} \\
X & \overset{\delta}{\longrightarrow} & \mathbb{P}^1 = Z/D_6
\end{array}
\]

we conclude that the image of any Weierstrass point under the map \( f \) is a Weierstrass point of \( X \) and that there are 2 points \( p_i \) such that \( f^{-1}(p_i) \) consists of 3 Weierstrass points and 4 points \( p_i \) such that \( f^{-1}(p_i) \) contains only one Weierstrass point. Without loss of generality we may assume that 

\[
\begin{align*}
f(q_1) &= f(q_2) = f(q_3) = p_1, \\
f(q_4) &= f(q_5) = f(q_6) = p_2
\end{align*}
\]
and

\[
f(q_{6+i}) = p_{2+i} \quad \text{for} \quad i = 1, \ldots, 4.
\]
Observe that, if \( h_X \) and \( h_Y \) denote hyperelliptic divisors of \( X \) and \( Y \), we have 

\[
f^{-1}(h_X) \sim 3h_Y.
\]
Finally, denote by \( \Theta \) the canonical theta divisor in \( \text{Pic}^3(Y) \), given by the image of the map

\[
Y^{(3)} \to \text{Pic}^3(Y), \quad y_1 + y_2 + y_3 \mapsto \mathcal{O}_Y(y_1 + y_2 + y_3).
\]

In order to describe the restriction of the theta divisor of \( Y \) to the Prym variety \( P \), we will work in \( \text{Pic}^2(Y) \). Let \( q \) be a Weierstrass point of \( Y \) and \( p = f(q) \). We consider the following translate of \( \Theta \):

\[
\Theta_q := \Theta - q \subset \text{Pic}^2(Y).
\]
Let \( \text{Nm} \) denote the norm map \( \text{Pic}^2(Y) \to \text{Pic}^2(X) \) (and also the norm map from \( \text{Div}^2(Y) \) onto \( \text{Div}^2(X) \)). In the following we will consider \( h_X \) (respectively \( h_Y \)) also as an element of \( \text{Pic}^2(X) \) (respectively of \( \text{Pic}^2(Y) \)).

Define \( \tilde{P} \) as the translate of the Prym variety \( P \),

\[
\tilde{P} := \text{Nm}^{-1}(h_X) \subset \text{Pic}^2(Y).
\]
We can consider \( \widetilde{P} \) also as an abelian surface, since it contains a distinguished point, namely \( h_Y \).

**Lemma 4.10.** For any \( y_1 + y_2 + y_3 \in Y^{(3)} \) the following conditions are equivalent

(a) \( \mathcal{O}_Y(y_1 + y_2 + y_3 - q) \in \widetilde{P} \cap \Theta_q \);
(b) after possibly renumbering the \( y_i \) we have \( f(y_3) = p \) and \( f(y_2) = f(\iota_Y y_1) \).

**Proof.** (a) is equivalent to \( \Nm(y_1 + y_2 + y_3 - q) \sim h_X \) and this occurs if and only if

\[
\begin{align*}
\text{(4.6)} & \quad f(y_1) + f(y_2) + f(y_3) \sim h_X + p.
\end{align*}
\]

The linear system \( |h_X + p| \) has \( p \) as a base point which implies that we have \( f(y_3) = p \) after possibly renumbering the \( y_i \). Then (4.6) reads

\[ f(y_1) + f(y_2) \sim h_X, \]

which implies

\[ f(y_2) = \iota_X f(y_1) = f(\iota_Y y_1). \]

The converse implication is obvious. \( \square \)

Now we choose \( q = q_1 \), so that \( f(q_1) = p_1 \) and, with the notation of the previous lemma, \( y_3 \in f^{-1}(p_1) = \{ q_1, q_2, q_3 \} \). Hence, if we define for \( i = 1, 2, 3 \),

\[ \Xi_i := \{ \mathcal{O}_Y(y_1 + y_2 + q_i - q_1) \in \Pic^2(Y) \mid y_1, y_2 \in Y \text{ with } f(y_2) = f(\iota_Y y_1) \} \]

with reduced subscheme structure, we get as an immediate consequence the following set-theoretical equality:

\[ \Xi = \Xi_1 \cup \Xi_2 \cup \Xi_3. \]

**Lemma 4.11.** For \( i = 1, 2, 3 \) the scheme \( \Xi_i \) is the disjoint union of a complete curve \( \Xi_i \) and a point. More precisely,

\[ \Xi_i = \Xi_i \cup \mathcal{O}_Y(h_Y + q_i - q_1). \]

**Proof.** Note first that \( \Xi_i = \Xi_1 + q_i - q_1 \) for \( i = 2 \) and \( 3 \). So it suffices to prove the assertion for \( i = 1 \). Certainly the scheme \( \Xi_1 \) is of dimension 1, since \( \widetilde{P} \cap \Theta_{q_1} \) is a divisor. To see this, it suffices to show that \( \widetilde{P} \not\subseteq \Theta_{q_1} \). For instance, note that \( \mathcal{O}_Y(q_2 + q_3 + q_4 + q_5 - h_Y) \in \widetilde{P} \), as \( \Nm(q_2 + q_3 + q_4 + q_5 - h_Y) = 2p_1 + 2p_2 - h_X \sim h_X \). We claim that the line bundle \( \mathcal{O}_Y(q_1 + q_2 + q_3 + q_4 + q_5 - h_Y) \) has no sections, hence \( \mathcal{O}_Y(q_2 + q_3 + q_4 + q_5 - h_Y) \not\subseteq \Theta_{q_1} \). Since \( h_Y \sim 2q_5 \), \( \mathcal{O}_Y(q_1 + q_2 + q_3 + q_4 + q_5 - h_Y) \) has a non zero section if and only if \( h^0(\mathcal{O}_Y(q_1 + \cdots + q_4)) \geq 2 \). This occurs if and only if the Serre dual \( h^0_Y - q_1 - \cdots - q_4 - q_5 - q_1 + q_2 + q_3 - q_4 \) is effective, that is there exist \( p_1, p_2 \in Y \) such that \( q_1 + q_2 + q_3 \sim q_4 + p_1 + p_2 \), i.e., \( h^0(q_1 + q_2 + q_3) \geq 2 \); but any linear series \( g^1_3 \) on an hyperelliptic curve has a fixed point (see (6.1)), this leads to the contradiction \( q_4 = q_i \) for some \( i \neq 4 \). This proves the claim.

Clearly the point \( h_Y \) is contained in \( \Xi_1 \). Define

\[ \Xi_1 := \Xi_1 \setminus h_Y. \]

We claim that \( h_Y \) is not in the closure of \( \Xi_1 \). In order to prove this, note that any \( L \in \Xi_1 \) is of the form \( L = \mathcal{O}_Y(y_1 + y_2) \) with \( y_1 \) and \( \iota_Y y_2 \) in the same fibre of the map \( f \). The fact
that $L \neq h_Y$ means that $y_1$ and $\nu_Y y_2$ are different points of the fibre. So, since $f$ is étale, $h_Y$ cannot be a limit of a sequence of line bundles in $\Xi_1$.

It remains to prove that $\Xi_1$ is purely one-dimensional. This will be shown in Corollary 4.13 below.

**Corollary 4.12.** The curves $\Xi_i$ are divisors in the Prym variety $\tilde{P}$ and we have
\[ \tilde{P} \cap \Theta_{q_1} = \Xi_1 \cup \Xi_2 \cup \Xi_3. \]

**Proof.** This is an immediate consequence of Lemma 4.11 and (4.7) just noting that $h_Y = O_Y(q_1 + q_2 + q_3 - q_1) = O_Y(q_1 + \nu_Y q_2 + q_3 - q_1) \in \Xi_2$ and similarly for the isolated points of $\Xi_2$ and $\Xi_3$. □

**Theorem 4.13.** The principal polarization $\Xi$ of the Prym variety $\tilde{P}$ is given by each of the algebraically equivalent divisors
\[ \Xi_1 \equiv \Xi_2 \equiv \Xi_3. \]

**Proof.** Note first that $\Xi_i = \Xi_1 + q_i - q_1$ for $i = 2$ and 3. So for $i = 2, 3$, $\Xi_i$ is the translate of $\Xi_1$ by the 2-division point $q_i - q_1$. It remains to show that the curve $\Xi_1$ defines the principal polarization $\Xi$. For this note that $\tilde{P} \cap \Theta_{q_1}$ defines the polarization $3\Xi$ and, if $n_i$ is the multiplicity of the curve $\Xi_i$ in $\tilde{P} \cap \Theta_{q_1}$, the Corollary 4.12 implies
\[ 3\Xi \equiv n_1 \Xi_1 + n_2 \Xi_2 + n_3 \Xi_3 \equiv (n_1 + n_2 + n_3)\Xi_1. \]

Since $\Xi$ is a principal polarization and $n_i \geq 1$ for all $i$, this is only possible if $n_1 = n_2 = n_3 = 1$ and $\Xi_1$ is also a principal polarization. This implies the assertion. □

**4.5. The curve** $\Xi := \Xi_1$. Recall that the scheme $\Xi := \Xi_1$ is defined by
\[ \Xi = \{ O_Y(y + z) \in \text{Pic}^2(Y) \mid f(\nu_Y z) = f(y), \nu_Y z \neq y \} \]
with reduced subscheme structure. In particular $\Xi$ is contained in the Brill-Noether locus
\[ W_2 = \{ L \in \text{Pic}^2(Y) \mid h^0(L) \geq 1 \}. \]

Recall that the symmetric product $Y^{(2)}$ is the blow-up of $W_2$ at the hyperelliptic line bundle $h_Y$:
\[ \rho : Y^{(2)} \to W_2. \]

Consider the canonical double covering $Y \times Y \to Y^{(2)}$, $(y, z) \mapsto y + z$. By assumption, the map $f : Y \to X$ is étale, so for every point $y \in Y$ the fibre $f^{-1}(y)$ consists of exactly 3 points. Let us denote these points by $f^{-1}(y) = \{ y, y', y'' \}$; we define
\[ D := \{(y, \nu_Y y'), (y, \nu_Y y'') \in Y \times Y \mid y \in Y \} \]
with reduced subscheme structure.

**Lemma 4.14.** The scheme $D$ is an étale double covering of $Y$. In particular it is a complete curve in $Y \times Y$.

**Proof.** Let $p_1 : D \to Y$ denote the projection onto the first factor. By definition, every point of $Y$ has exactly 2 preimages under the map $p_1$. This implies the assertion. □
According to the definition, the scheme $\Xi$ does not contain the point $h_Y$. This implies that the map $\rho : \rho^{-1}(\Xi) \rightarrow \Xi$ is an isomorphism. In the sequel we identify $\rho^{-1}(\Xi)$ with $\Xi$. A direct consequence of the previous lemma is

**Corollary 4.15.** The scheme $\Xi$ is a complete curve in $Y^{(2)}$ and thus also in $\tilde{P}$.

For the Weierstrass points $q_i, \ i = 7, \ldots, 10$ we have

(4.9) \[ f^{-1}f(q_i) = \{q_i, z_i, \iota_Y z_i\} \]

with some non-Weierstrass points $z_i$. With this notation we get

**Lemma 4.16.** The double covering $j : D \rightarrow \Xi$ induced by the double covering $Y \times Y \rightarrow Y^{(2)}$ is ramified exactly at the 8 points $(z_i, z_i), (\iota_Y z_i, \iota_Y z_i) \in D, \ i = 7, \ldots, 10$.

**Proof.** Note first that $(z_i, z_i) = (z_i, \iota_Y(z_i)) \in D$ and similarly $(\iota_Y z_i, \iota_Y z_i) \in D$. Hence it suffices to show that $D$ intersects the diagonal of $Y \times Y$ exactly in these 8 points. For this observe that only on the fibers over the Weierstrass points $q_i, \ i = 7, \ldots, 10$ two different points of the fiber are exchanged under the hyperelliptic involution, namely $z_i, \iota_Y z_i$ for $i = 7, \ldots, 10$. \[ \square \]

**Theorem 4.17.** The curve $\Xi$ is smooth and irreducible of genus 2 and the principally polarized abelian surface $(\tilde{P}, \Xi)$ is the Jacobian of $\Xi$.

**Proof.** Recall that the curve $\Xi$ defines a principal polarization of the abelian surface $\tilde{P}$. So either it is smooth and irreducible of genus 2 or the union of 2 elliptic curves intersecting transversally at one point. Suppose $\Xi$ is the union of 2 elliptic curves $\Xi = \Xi^1 \cup \Xi^2$. Then, by Lemma 4.14 also the curve $D$ is reducible: $D = D_1 \cup D_2$ with $D_i \simeq Y$ and thus of genus 4 for $i = 1, 2$. Then without loss of generality and according to Lemma 4.16 we may assume that $D_1 \rightarrow \Xi^1$ is a double covering ramified in $\leq 4$ points of $D_1$. This contradicts the Hurwitz formula. \[ \square \]

Recall that the curve $\Xi$ is given by

$\Xi = \{[y + \iota_Y z] \in Y^{(2)} \mid f(y) = f(z), \ y \neq z\}$

with reduced subscheme structure. The involution $\iota_Y$ induces the hyperelliptic involution on $\Xi$ by

$\iota_{\Xi} : \Xi \rightarrow \Xi, \ [y + \iota_Y z] \mapsto \iota_Y([y + \iota_Y z]) = [z + \iota_Y y].$

So the following proposition is immediate.

**Proposition 4.18.** If $\{q_1, \ldots, q_{10}\}$ denote the Weierstrass points of $Y$ as above, i.e. with $f^{-1}(p_1) = \{q_1, q_2, q_3\}$ and $f^{-1}(p_2) = \{q_4, q_5, q_6\}$, then

$\{[q_1 + q_2], [q_1 + q_3], [q_2 + q_3], [q_4 + q_5], [q_4 + q_6], [q_5 + q_6]\}$

are the Weierstrass points of $\Xi$. 

5. The Prym map

Let $\mathcal{A}_2$ denote the moduli space of principally polarized abelian surfaces. The Prym map for non-cyclic triple coverings is the map

$$Pr : \mathcal{R}_{2,3}^{nc} \to \mathcal{A}_2, \quad [f : Y \to X] \mapsto P(f).$$

According to Theorem 4.17 the image of the Prym map is contained in the open subset of $\mathcal{A}_2$ consisting of Jacobians of smooth curves of genus 2.

**Theorem 5.1.** The Prym map $Pr : \mathcal{R}_{2,3}^{nc} \to \mathcal{A}_2$ is finite of degree 10 onto its image.

**Proof.** Let $[f : Y \to X] \in \mathcal{R}_{2,3}^{nc}$ and suppose $P(f) = J\Xi$. We have the following commutative diagram

$$\begin{array}{ccc}
Y & \xrightarrow{\gamma} & \mathbb{P}^1 \\
\downarrow f & & \downarrow \phi \\
X & \xrightarrow{\delta} & \mathbb{P}^1
\end{array}$$

where the left hand square is diagram (4.1), $\phi$ is the hyperelliptic covering and $\psi$ the composed map. Let the notation of the Weierstrass points of $X$ and $Y$ be as above. Then the Weierstrass points of $\Xi$ are given by Proposition 4.18. The map $\psi$ is given by a pencil $g_6^1 \subset |3K_\Xi|$ whose ramification divisor consists of the 6 Weierstrass points of $\Xi$ and the 8 preimages of the 4 ramification points over $\delta(p_3), \ldots, \delta(p_6)$.

Conversely, let $\Xi$ be a smooth curve of genus 2 with Weierstrass points $\{w_1^\prime, \ldots, w_6^\prime\}$. Consider the pencil $g_6^1 \subset |3K_\Xi|$ generated by 2 divisors of the form $2w_i + 2w_j + 2w_k$ where the union of the $w_i$'s in both divisors equals $\{w_1^\prime, \ldots, w_6^\prime\}$. Hence the corresponding 6:1 covering $\psi : \Xi \to \mathbb{P}^1$ factors via the hyperelliptic covering, because the generating divisors are sums of divisors linear equivalent to $K_\Xi$. So we have the commutative triangle of diagram (5.1).

Let $x_1, x_2$ be the branch points of $\psi$ in $\mathbb{P}^1$ corresponding to the two generating divisors of the $g_6^1$. Thus $J$ is étale over $x_1$ and $x_2$. Assume for the moment that $J$ is simply ramified, say over the points $x_3, \ldots, x_6 \in \mathbb{P}^1$. Let $\delta : X \to \mathbb{P}^1$ be the double covering ramified in $x_1, \ldots, x_6$ and let $f : Y \to X$ be the normalization of the pull-back of $J$ by $\delta$. Clearly $f$ is étale. Moreover, we claim that $f$ is non-cyclic. If $f$ was cyclic the corresponding automorphism of order 3 would permute the Weierstrass points of $Y$; but since there are 10 of them, this automorphism would have fixed points, contradicting the fact that $f$ is étale. So we are in the situation of diagram (5.1). In particular, we have $P(f) = J\Xi$. Clearly every covering $f$ with $Pr(f) = J\Xi$ arises in this way. Moreover, $f$ is uniquely determined by the choice of a partition of $\{w_1^\prime, \ldots, w_6^\prime\}$ into 2 subsets of three elements. Hence, under the above assumption, we conclude that there are exactly $\frac{1}{2}\binom{6}{3} = 10$ elements in the fibre $Pr^{-1}(J\Xi)$.

Observe that if $J$ is not simply ramified, then either it has 2 triple ramification points, or it has 1 triple ramification point and 2 simple ones. In the former case $\psi$ is ramified over 4 branch points, in the last case $\psi$ ramifies over 5 points. We claim that a general $\Xi$ does not admit such coverings over $\mathbb{P}^1$. This will show that for a general $\Xi$ all 10 pencils satisfy the assumption above. To prove the claim, we estimate the dimension of
the following locus in $\mathcal{M}_2$

$$\mathcal{N}_b := \{ [C] \in \mathcal{M}_2 \mid C \text{ has a base point free } g_6^1 \text{ branched over } b \text{ points } \},$$

for $b = 4, 5$. Let $\mathcal{H}_{6,b}$ denote the Hurwitz scheme of coverings $\psi : C \to \mathbb{P}^1$ of degree 6 branched over $b$ points. It is known that the Hurwitz schemes have expected dimension $b$, the number of branch points. Hence the dimension of $\mathcal{N}_b$, for $b = 4, 5$, is at most $\dim \mathcal{H}_{6,b} - \dim \text{Aut}(\mathbb{P}^1) \leq 2$, which is less than the dimension of $\mathcal{M}_2$. This completes the proof.

\[ \square \]

**Proposition 5.2.** The moduli space $R_{2,3}^{nc}$ is rational.

**Proof.** Consider the diagram (5.1). The moduli space $\mathbb{V}$ of triple coverings $\bar{f} : \mathbb{P}^1 \to \mathbb{P}^1$ is a rational curve according to [EEHS, pp. 96-98]. Let $V$ denote its open subset consisting of triple coverings branched over 4 distinct points. Let $B(\bar{f}) \subset \mathbb{P}^1$ be the branch locus of $\bar{f}$. A pair $(p_1 + p_2, \bar{f})$ in a Zariski open subset of $(\mathbb{P}^1)^{(2)} \times V$ determines uniquely a genus 2 curve $X$ whose hyperelliptic covering $\delta$ has branched locus $B(\bar{f}) \cup \{p_1, p_2\}$ and a triple covering $\bar{f} : Y \to X$ defined as the normalization of the pullback of $\bar{f}$ by $\delta$. Conversely, it is clear from what we have said above (see diagram (5.1)), that every element $[\bar{f} : Y \to X] \in R_{2,3}^{nc}$ defines a point in $(\mathbb{P}^1)^{(2)} \times V$. Thus we conclude that $R_{2,3}^{nc}$ is birational to $\mathbb{P}^3$. \[ \square \]

6. The Abel-Prym map

Let $\alpha_{y_0} : Y \to JY$ denote the Abel map with respect to a point $y_0 \in Y$ and $\pi : JY \to P$ the canonical projection. The composition

$$\alpha_f : Y \xrightarrow{\alpha_{y_0}} JY \xrightarrow{\pi} P$$

is called the **Abel-Prym map** of $P = P(\bar{f})$.

Recall that every endomorphism of the Jacobian $JY$ is given by a correspondence on $Y$. In order to study the Abel-Prym map $\alpha_f$, we need a correspondence inducing $\pi : JY \to P$. For this we use the following notation: if $y$ is a point of $Y$, let $y'$ and $y''$ denote the other 2 points of the fibre $f^{-1}(f(y))$.

**Lemma 6.1.** The projection $\pi : JY \to P$ is induced by the following correspondence on $Y$:

$$y \mapsto 2y - y' - y''.$$ 

**Proof.** $P$ is the complement of the abelian subvariety $f^*JX$ in the principally polarized abelian variety $JY$. The projection $JY \to f^*JX$ is the image of the trace correspondence $T : y \mapsto y + y' + y''$. According to [BL, Section 5.3], if an abelian subvariety of exponent $e$ is induced by a correspondence $D$, the complementary abelian subvariety is the image of $e \cdot id_{JY} - D$. This implies the assertion, since $f^*JX$ is of exponent 3 in $JY$. \[ \square \]

**Lemma 6.2.** (1) For any 2 points $y_1, y_2 \in Y$,

$$\alpha_f(y_1) = \alpha_f(y_2) \iff 2y_1 + y'_2 + y''_2 \sim 2y_2 + y'_1 + y''_1.$$ 

(2) For 2 points $y, y'$ in a fibre of $f$,

$$\alpha_f(y) = \alpha_f(y') \iff 3y \sim 3y'.$$
Proof. Lemma 6.1 implies that \( \alpha_f(y_1) = \alpha_f(y_2) \) if and only if \( 2y_1 - y''_1 - y''_2 \sim 2y_2 - y''_2 \), which gives (1). Inserting \( y_1 = y \) and \( y_2 = y' \), (1) gives \( 2y + y'' \sim 2y' + y' + y'' \) which implies (2).

Recall the notation of the Weierstrass points of \( Y \) and \( X \) from subsection 4.3. In particular, according to (4.9), \( f^{-1}(q_i) = \{ q_i, z_i, \nu_Y z_i \} \) for the Weierstrass points \( q_i, i = 7, \ldots, 10 \) with non-Weierstrass points \( z_i \). Then we have

**Theorem 6.3.** The Abel-Prym map \( \alpha_f : Y \to P \) is injective away from the Weierstrass points \( q_7, \ldots, q_{10} \), which have the same image.

**Proof.** According to Theorem (1.9) the curve \( Y \) is hyperelliptic. It is a consequence of the geometric version of the Riemann-Roch theorem that any complete linear system \( g^r_1 \) is of the form (see [ACGH], p. 13)

\[
(6.1) \quad rh_Y + a_1 + \cdots + a_{d-2r}.
\]

Suppose now \( y_1, y_2 \in Y \), with \( \alpha_f(y_1) = \alpha_f(y_2) \). If \( f(y_1) = f(y_2) \) then, according to Lemma 6.2, \( 3y_1 \sim 3y_2 \), which gives \( y_1 = y_2 \) by equation (6.1).

If \( f(y_1) \neq f(y_2) \), Lemma 6.2 implies \( 2y_1 + y'_2 + y''_2 \sim 2y_2 + y'_1 + y''_1 \). If this linear system is a complete \( g^1_1 \), then, by (6.1), it has two fixed points, which implies \( y_1 = y_2 \). So suppose it is a subsystem of a two-dimensional linear system \( g^2_1 \). Then it follows from (6.1) that \( y_1 \) and \( y_2 \) are Weierstrass points and \( y''_1 = \nu_Y y'_1 \) as well as \( y''_2 = \nu_Y y'_2 \). This implies

\[
y_1, y_2 \in \{ q_7, \ldots, q_{10} \}.
\]

Conversely, if \( y_1, y_2 \in \{ q_7, \ldots, q_{10} \} \), then, according to (1.9), we have \( y'_i = \nu_Y y'_i \) for \( i = 1, 2 \). Now Lemma 6.2 implies that \( \alpha_f(y_1) = \alpha_f(y_2) \). \( \square \)

7. **DIRECT IMAGES OF LINE BUNDLES ON \( Y \)**

According to [M], Proposition 4.7

\[
f_*O_Y = O_X \oplus E,
\]

with \( E \) a vector bundle of rank 2 such that \( (\det E)^2 = O_X \). In the case of a cyclic triple covering \( f \) given by a line bundle \( \eta \in JX[3] \setminus \{ 0 \} \) it is well known that \( E \cong \eta \oplus \eta^2 \). In the non-cyclic case we have

**Lemma 7.1.** The vector bundle \( E \) is stable.

**Proof.** Consider the commutative diagram (2.1) and let \( h : Z \to X \) denote the composition \( h = f \circ p = g \circ q \). Let \( \eta \in JD[3] \setminus \{ 0 \} \) and \( \lambda \in JX[2] \setminus \{ 0 \} \) be the line bundles associated to the coverings \( q \) and \( g \). So \( \text{Ker } q^* = \langle \eta \rangle \), \( \text{Ker } g^* = \langle \lambda \rangle \). Since \( f^* : JX \to JY \) is injective, \( | \text{Ker } h^* | \leq | \text{Ker } p^* | = 2 \). If \( \eta \) was in the image of \( g^* \) then the inverse images of \( \eta \) and \( \eta^2 \) would be in the kernel of \( h^* \), but \( \text{Ker } h^* \) contains at most one non-trivial element. So, \( \eta \notin \text{Im } g^* \). On the other hand, by flat base change

\[
O_D \oplus \eta \oplus \eta^2 = q_*O_Z = q_*p^*O_Y = g^*f_*O_Y = O_D \oplus g^*E.
\]
Thus \( g^*E \simeq \eta \oplus \eta^2 \). The semistability of \( E \) follows from the fact that \( f_*\mathcal{O}_Y \) is semistable (see [B, Proof of Proposition 4.1]) and \( E \) is of degree zero. Suppose that \( E \) has a line subbundle of degree 0, \( L_1 \hookrightarrow E \). Without lost of generality, we might assume that the composition map

\[
g^*L_1 \hookrightarrow g^*E \simeq \eta \oplus \eta^2 \to \eta
\]

is non zero, where the last arrow is the projection. Since \( \eta \) is also of degree 0, this map gives an isomorphism \( g^*L_1 \simeq \eta \), which contradicts the fact \( \eta \notin g^*(JX) \). Therefore, \( E \) is stable. \( \square \)

As a consequence of the proof, we can say more about the determinant of \( E \):

**Corollary 7.2.** Let \( \lambda \) be the 2-division point defining the covering \( g : D \to X \). Then \( \det E \in \{ \mathcal{O}_X, \lambda \} \).

**Proof.** In the proof of the previous lemma, we saw that \( g^*E = \eta \oplus \eta^2 \), where \( \eta \) is the 3-division point of \( JD \) defining the covering \( q : Z \to D \). This implies that

\[
g^* \det E = \det g^*E = \mathcal{O}_D
\]

which implies the assertion. \( \square \)

Let \( \xi \) denote the 2-division point \( \det E \) of \( \text{Pic}^0(X) \) and let \( U_X(3, \xi) \) denote the moduli space of S-equivalence classes of semistable rank 3 vector bundles over \( X \) with determinant \( \xi \). The direct image \( f_*L \) of a line bundle \( L \) on \( Y \) is vector bundle of rank 3 on \( X \) and by [B, Proof of Proposition 4.1] it is semi-stable. Moreover, for any line bundle \( L \) on \( Y \) the determinant of \( f_*L \) is given by the following formula

\[
(7.1) \quad \det f_*L = \text{Nm} L \otimes \det f_*\mathcal{O}_Y
\]

(observe that the formula is trivial for \( L = \mathcal{O}_Y \), then one can apply induction on the degree of \( L \) using the fact that any line bundle \( L' \) fits in an exact sequence \( 0 \to L \to L' \to \mathcal{O}_p \to 0 \), where \( \deg L = \deg L' - 1 \) and \( \mathcal{O}_p \) is the skyscraper sheaf supported at the point \( p \in Y \)). Thus \( f \) induces a morphism \( f_* : \text{Pic}(f) \to U_X(3, \xi) \).

**Proposition 7.3.** The direct image morphism

\[
f_* : \text{Pic}(f) \to U_X(3, \xi)
\]

is injective.

**Proof.** Recall the diagram (2.1). According to Proposition 3.1, the curve \( Z \) is Galois over \( X \) with Galois group \( D_3 = \langle \sigma, \tau \mid \sigma^3 = \tau^2 = 1, \tau \sigma \tau = \sigma^2 \rangle \). For \( i = 0, 1, 2 \) denote \( Z_i = Z \) and define \( p_0 = p : Z_0 \to Y \), \( p_1 = p \circ \sigma : Z_1 \to Y \) and \( p_2 = p \circ \sigma^2 : Z_2 \to Y \). Then the following diagram is cartesian

\[
\begin{array}{ccc}
\bigcup_{i=0}^2 Z_i & \xrightarrow{f'} & Z \\
\downarrow h' & & \downarrow h \\
Y & \xrightarrow{f} & X
\end{array}
\]

where \( f' \) is the identity on each \( Z_i \) and \( h'|_{Z_i} = p_i \) for \( i = 0, 1, 2 \).
Suppose now that $L, L' \in P(f)$ with $f_sL \simeq f_sL'$. Applying flat base change to diagram (7.2), we get
\[
\bigoplus_{i=0}^{2} p_i^*L = f'_s h^* L \simeq h^* f_sL = h^* f_sL' \simeq f'_s h^* L' = \bigoplus_{i=0}^{2} p_i^*L'.
\]

This implies
\[
p^*L = p_0^*L \simeq p_i^*L' = (\sigma^i)^* p^* L'
\]
for some $i \in \{0, 1, 2\}$. Hence, since $p^*|_{P(f)}$ is injective according to Proposition 4.7 it suffices to show that $p^*L \simeq (\sigma^i)^* p^* L'$ implies $p^*L \simeq p^* L''$ for $i = 1$ and 2. Moreover, replacing $\sigma$ by $\sigma^2$, we may assume $i = 1$.

So assume that $p^*L = \sigma^* p^* L'$ for some $L, L' \in P(f)$. It suffices to show that this implies
\[
(7.3) \quad p^*L \in \text{Fix}(\sigma^*),
\]
since then $p^*L' \simeq (\sigma^2)^* p^* L = p^*L$ and thus $L \simeq L'$.

Now the assumption means that the pair $(L, L'^{-1})$ is in the kernel of the morphism
\[
r : P(f) \times P(f) \to JZ, \quad (L, L') \mapsto p^*L \otimes \sigma^* p^* L'.
\]

It is well known that $r$ is a $D_3$-equivariant isogeny onto its image (for the first proof of this see [RR Theorem 5.4]). Here the action on the left hand side is the standard action of $D_3$ given by
\[
\tau \mapsto \tilde{\tau} = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \quad \sigma \mapsto \tilde{\sigma} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}
\]
and the action on the right hand side is the action induced on $JZ$ from $Z$. In particular the kernel of the homomorphism $r$ is a $D_3$-invariant subgroup of $P(f) \times P(f)$.

By hypothesis $(L, L'^{-1}) \in \text{Ker}(r)$. So also $\tilde{\sigma}(L, L'^{-1}) = (L', L \otimes L') \in \text{Ker}(r)$ and $\tilde{\tau}(L, L'^{-1}) = (L \otimes L', L') \in \text{Ker}(r)$. Together with the assumption $p^*L = \sigma^* p^* L'$ (or equivalently $(\sigma^2)^* p^* L = p^* L'$) this gives the equalities
\[
(\sigma^2)^* p^* L \otimes \sigma^* p^* L \otimes p^* L = 0, \quad (\sigma^2)^* p^* L \otimes (p^* L)^2 = 0.
\]

These equations imply
\[
\sigma^* p^* L \simeq (\sigma^2)^* p^* L^{-1} \otimes p^* L^{-1} \simeq p^* L
\]
which was to be shown.

\section{The Prym variety of the pair $(p, q)$}

Let $f : Y \to X$ be a non-cyclic étale degree 3 covering of a curve $X$ of genus 2 and consider the associated diagram (2.1). According to [LR] the Prym variety $P(p, q)$ of the pair of coverings $(p, q)$ is defined to be the complement of the abelian subvariety $q^* P(g)$ in the Prym variety $P(p)$ with respect to the restriction of the canonical polarization of $J(Z)$.

**Proposition 8.1.** $P(p, q)$ is an abelian surface.

**Proof.** $\dim P(p, q) = \dim JZ - \dim JY - \dim P(g) = 7 - 4 - 1 = 2$. \hfill $\Box$
Hence we have associated to the abelian surface $P(f)$ another abelian surface $P(p,q)$. In this section we denote $Ξ$ the principal polarization of the Prym variety $P(p)$. Recall the curves $A$ and $B$ of diagram (4.4) and let $Θ_A$ and $Θ_B$ denote the canonical principal polarizations of their Jacobians $JA$ and $JB$. Moreover, we identify the elliptic curve $A$ with its Jacobian.

**Theorem 8.2.** (a) There is a canonical isomorphism of principally polarized abelian varieties

$$(P(p), Ξ) \simeq (A, Θ_A) \times (JB, Θ_B).$$

(b) If we consider $JB$ as an abelian subvariety of $JZ$, we have an equality of polarized abelian varieties

$$(P(p,q), Ξ|_{P(p,q)}) = (JB, Θ_B).$$

**Proof.** (a): The Theorem of Mumford on Prym varieties of étale double covering of hyperelliptic curves (see [Mu, Page 346]) applied to the following subdiagram of the full extended diagram

\[
\begin{array}{c}
\begin{array}{ccc}
Z & \overset{p}{\rightarrow} & Y \\
\downarrow{r} & & \downarrow{γ} \\
B & \overset{α}{\leftarrow} & A \\
\end{array}
\end{array}
\]

\[C = \mathbb{P}^1\]

gives an isomorphism of principally polarized abelian varieties

\[(8.2) \quad (P(p), Ξ) \simeq (JA \times JB, Θ_A \times JB + JA \times Θ_B) = (A, Θ_A) \times (JB, Θ_B).\]

(b): Let $Ξ_g$ denote the principal polarization of the Prym variety $P(g)$. The theorem of Mumford applied to the following subdiagram of the full extended diagram

\[
\begin{array}{c}
\begin{array}{ccc}
D & \overset{g}{\rightarrow} & X \\
\downarrow{β} & & \downarrow{δ} \\
\mathbb{P}^1 & \overset{μ}{\leftarrow} & \mathbb{P}^1 \\
\end{array}
\end{array}
\]

\[\mathbb{P}^1 = Z/D_6\]

yields

\[(8.4) \quad (P(g), Ξ_g) \simeq (JE \times J\mathbb{P}^1, Θ_E \times J\mathbb{P}^1 + Θ_{\mathbb{P}^1} \times JE) = (E, Θ_E).\]

where we use $J\mathbb{P}^1 = 0$ and identify $E = JE$.

The full extended diagram consists of diagrams (8.1) and (8.3) together with the maps $f, q, ν, μ$ and $f$ from (8.1) to (8.3) Since the Theorem of Mumford is certainly compatible with pull-backs with respect to $f, q, ν, μ$ and $f$, we get that $q^*$ respects the decompositions (8.2) and (8.4). In particular we get

$$q^*(P(g)) = μ^*(E) × ν^*(J\mathbb{P}^1) = A × \{0\} = A.$$
Since \( \alpha \) and \( r \) are ramified, we can consider \( A \) and \( JB \) as abelian subvarieties in \( JZ \) and thus of the Prym variety \( P(p) \). Then (8.2) implies that the complement of the abelian subvariety \( A \) in \( P(p) \) with respect to the polarization \( \Xi \) is \( JB \).

Now let \( \Theta \) denote the canonical principal polarization of the Jacobian \( JZ \). Since \( \Xi \) is a principal polarization and \( \Theta \) is the canonical polarization of \( JZ \), we have the following lemma.

Lemma 8.3. Let \( A \subset JD \) be an abelian subvariety with \( \Theta_D|_A \) of type \( (d_1,\ldots,d_r) \). Then the polarization

\[
\Theta|_{P(p)} = 2\Xi,
\]

the abelian variety \( JB \) is also the complement of \( A \) with respect to the polarization \( \Theta|_{P(p)} \).

On the other hand, the Prym variety \( P(p,q) \) of the pair of maps \( (p,q) \) is defined to be the complement of the abelian subvariety \( q^*P(g) = A \) in \( P(p) \) with respect to the polarization \( \Theta|_{P(p)} \). Since the complement of an abelian subvariety with respect to a polarization is uniquely determined, we get \( P(p,q) = JB \). The equality of the polarizations is a consequence of (8.2).

We want to determine the 3-division point \( \eta \) of the Jacobian \( JD \) inducing the cyclic covering \( q : Z \to D \) of the diagram (2.1). More generally, consider \( f : C \to D \) a cyclic étale covering of prime degree \( p \) and denote by \( \Theta_C \) and \( \Theta_D \) the canonical polarizations. The associated pull-back map of line bundles \( f^* : JD \to JC \) is an isogeny onto its image with kernel a cyclic subgroup of order \( p \) generated by an element \( \eta \in JD \). With this setting we have the following lemma.

Lemma 8.4. The 3-division point \( \eta \) of \( JD \) corresponding to the étale covering \( q : Z \to D \) is contained in the elliptic curve \( P(g) = E \subset JD \).

Proof. According to [BL, Proposition 12.3.1], \( (f^*)^*\Theta_C \equiv p\Theta_D \). This implies that \( (f^*)^*\Theta_C|_A \equiv p\Theta_D|_A \) is of type \( (pd_1,\ldots,rd) \). Since \( f^*|_A : A \to f^*(A) \) is an isomorphism if \( \eta \notin A \) (here we use that \( p \) is a prime) and an isogeny of degree \( p \) if \( \eta \in A \), this implies the assertion.

Proposition 8.4. The 3-division point \( \eta \) of \( JD \) corresponding to the étale covering \( q : Z \to D \) is contained in the elliptic curve \( P(g) = E \subset JD \).

Proof. Recall that \( \Theta \) and \( \Xi \) are the canonical principal polarizations of \( JZ \) and \( P(p) \). Since \( g : D \to X \) is an étale double covering, \( \Theta_{|P(g)} \) is of type (2). Hence by Lemma 8.3, the polarization

\[
\Theta|_{q^*(P(g))} \quad \text{is of type} \quad \left\{ \begin{array}{ll}
(6) & \text{if } \eta \notin P(g) \\
(2) & \eta \in P(g).
\end{array} \right.
\]

Now \( q^*(P(g)) \subset P(p) \) and \( \Theta|_{P(p)} = 2\Xi \). This yields

\[
\Xi|_{q^*(P(g))} \quad \text{is of type} \quad \left\{ \begin{array}{ll}
(3) & \text{if } \eta \notin P(g) \\
(1) & \eta \in P(g).
\end{array} \right.
\]

Since \( \Xi \) is a principal polarization and \( P(p,q) \) is the complement of \( q^*(P(g)) \) with respect to \( \Xi_{P(p)} \), it follows that

\[
\Xi|_{P(p,q)} \quad \text{is of type} \quad \left\{ \begin{array}{ll}
(1,3) & \text{if } \eta \notin P(g) \\
(1,1) & \eta \in P(g).
\end{array} \right.
\]
But we know from Theorem 8.2 that $\Xi_{P(p,q)}$ is of type $(1,1)$. Hence $\eta \in P(g)$. The equality $P(g) = E$ is [8.4].

**Remark 8.5.** According to [LR, Proposition 2.4] the Prym variety $P(p,q)$ can also be defined as the complement of $p^*P(f)$ in the Prym variety $P(q)$. So we have an isogeny

$$P(q) \sim P(f) \times P(p,q).$$

Recall that $P(f)$ is a principally polarized Prym variety and according to Theorem 8.2 (b) $P(p,q)$ is also principally polarized. On the other hand, the restriction of the canonical principal polarization of $JZ$ to $P(q)$ is of type $(1,1,3,3)$ and thus the above isogeny is not an isomorphism. Moreover, since $P(q) \simeq JB \times JB$ and $P(p,q) = JB$ by Theorem 8.2 (b), one deduces that $P(f)$ is isogenous to $JB$.

**Remark 8.6.** Observe that the number of étale (connected) double coverings $g : D \to X$ is $2^4 - 1 = 15$. On the other hand, according to Proposition 8.4 the étale triple coverings $q : Z \to D$ corresponding to the Galois closure of $f$ are determined by a subgroup of order $3$ of $E[3]$ and there are $(3^2 - 1)/2 = 4$ of them. Therefore, for every genus 2 curve $X$ there are $15 \cdot 4 = 60$ choices for the Galois covering $Z \to X$, which agrees with Corollary 4.2.

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