Superconductivity in a two-dimensional repulsive Rashba gas at low electron density

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Abstract

We study the superconducting instability and the resulting superconducting states in a two-dimensional repulsive Fermi gas with Rashba spin–orbit coupling at low electron density (namely the Fermi energy $E_F$ is lower than the energy $E_R$ of the Dirac point induced by Rashba coupling). We find that superconductivity is enhanced as the dimensionless Fermi energy $\epsilon_F (\epsilon_F \equiv E_F/E_R)$ decreases, due to two reasons. First, the density of states at $\epsilon_F$ increases as $1/\sqrt{\epsilon_F}$. Second, the particle–hole bubble becomes more anisotropic, resulting in an increasing effective attraction. The superconducting state is always in the total angular momentum $j_z = +2$ (or $j_z = -2$) channel with Chern number $C = 4$ (or $C = -4$), breaking time reversal symmetry spontaneously. Although a putative Leggett mode is expected due to the two-gap nature of the superconductivity, we find that it is always damped. More importantly, once a sufficiently large Zeeman coupling is applied to the superconducting state, the Chern number can be tuned to be $\pm 1$ and Majorana zero modes exist in the vortex cores.

Despite an effect originating from relativity, spin–orbit coupling (SOC) has found its way into nonrelativistic physics. In condensed matter physics, novel systems with SOC playing a significant role are found recently, such as topological insulators [1, 2], two-dimensional (2D) Rashba gases at interfaces of oxides [3, 4], Weyl semimetals [3] and SOC-induced Mott insulators [6] and other states in 5d series [7]; while in ultracold quantum gases, although atoms are neutral, synthetic SOC can be generated by atom-light interaction (see [8, 9] for review). Turning to superconductivity, non-centrosymmetric superconductors, where SOC mixes spin singlet and triplet pairings, have been extensively studied [10–14]; and in 2D, superconductivity related to SOC was observed at oxide interfaces [4, 15].

Here, we study a 2D repulsive gas with Rashba SOC at low density. The single-particle Hamiltonian is

$$H = \frac{k^2}{2m} + \alpha_R (\sigma \times \mathbf{k}) \cdot \hat{n},$$

where $m$ is the effective mass, $\alpha_R$ characterizes the strength of Rashba SOC, $\sigma$’s components are Pauli matrices, and $\hat{n}$ is the direction normal to the 2D system. By a unitary transformation to helicity basis, one finds the dispersion

$$E_{k\lambda} = \frac{(k - \lambda k_R)^2}{2m},$$

where $\lambda = \pm 1$ is the helicity and $k_R = m\alpha_R$ is the Rashba momentum. (We have shifted the energy by $k_R^2/(2m)$, which will be compensated by the shift of the Fermi energy.) The spin degeneracy is lifted, resulting in two bands touching at a Dirac point. In this system, the competition between the three energy scales—the Fermi energy $E_F$, Coulomb repulsion and the ‘Rashba energy’ $E_R = k_R^2/(2m)$—determines the system’s phases. We define the dimensionless Fermi energy by $\epsilon_F = E_F/E_R$. By ‘low density’, we refer to the regime $0 < \epsilon_F < 1$, as
We study the system with onsite repulsive interactions, the interacting Hamiltonian of which is
\[ H_{\text{int}} = \frac{u}{2} \sum_{k_1, k_2} \sum_{\sigma, \sigma'} \delta_{k_1 + k_2, k_1} \xi_{k_1 \sigma}^\dagger \xi_{k_2 \sigma'}^\dagger \xi_{k_1 \sigma'} \xi_{k_2 \sigma}, \]
where \( u \) is positive.
The interacting Hamiltonian in the helicity basis reads [17, 18]

\[ H_{\text{int}} = \frac{\mu}{4\hbar^2} \sum_{k_{1} \cdots k_{4}} \sum_{\mu \nu} \delta_{k_{1}+k_{2}+k_{3}+k_{4}} \left( \mu e^{-i\theta_{k_{1}}} - \nu e^{-i\theta_{k_{2}}} \right) \times \left( \mu e^{i\theta_{k_{3}}} - \nu e^{i\theta_{k_{4}}} \right) a_{k_{1}}^\dagger a_{k_{2}}^\dagger a_{k_{3}} a_{k_{4}}, \]

where \( a_{k} \) is the annihilation operator in the helicity basis and \( \theta_{k} \) is the angle of \( k \). Following the weak-coupling RG approach developed in [17, 18] and integrating out high energy modes from the bandwidth \( A \) to a low-energy cutoff \( \Omega \), we derive the effective action for the low energy modes

\[ S'_{\text{int}} = \int_0^\beta d\tau \sum_{kk'j} e^{i\phi} \sum_{j} e^{i\phi_{j}} \times V^{(i)}_{\mu\nu}(\tau) a_{k\mu}^\dagger a_{k'\nu}(\tau) a_{-k'\nu}(\tau) a_{-k\nu}(\tau), \]

where \( \beta = 1/(k_{B}T) \) and \( \nu \)'s and \( \nu^* \)'s are Grassmann numbers. We have focused on the Cooper channel, the couplings of which are the only (marginally) relevant ones [36], and decomposed the couplings into angular momentum channels, where \( \phi = \theta_{k} - \theta_{k'} \).

Since in \( j_{z} = 0 \) channel, \( u \) dominates, one cannot get attractive interactions. Therefore, we go to higher orders and look for nonvanishing terms in higher angular momentum channels. At second order, we have particle–hole bubble and particle–particle bubble, the latter of which only has \( j_{z} = 0 \) component. The correction from the particle–hole bubble (shown in figure 6(a))

\[ \Pi(k, k') = \sum_{\alpha\beta} \int_0^{\beta} d\tau \frac{\mathcal{D}p}{(2\pi)^2} \frac{\lambda_{\alpha}(E_{p\alpha}) - \lambda_{\beta}(E_{p+k-k'}\beta)}{E_{p\alpha} - E_{p+k-k'\beta}} \times E_{\alpha\beta}(k, k', p), \]

where

\[ E_{\alpha\beta}(k, k', p) = (\alpha e^{-i\theta_{k'}} - \beta e^{-i\theta_{k}})(\alpha^{*} e^{i\theta_{k'}} - \beta^{*} e^{i\theta_{k}}) \times (\beta^{*} e^{-i\theta_{k'}-i\theta_{k}} - \alpha^{*} e^{-i\theta_{k}})(\beta e^{i\theta_{k'}-i\theta_{k}} - \alpha e^{i\theta_{k}}). \]

Since Cooper pairs are expected to form between electrons near the Fermi surfaces, \( k \) and \( k' \) are restricted to be at Fermi surface \( \mu \) and \( \lambda \), respectively. Straightforward calculations show that \( \Pi(k, k') \) can be written in the form

\[ \Pi(k, k') = e^{i\phi} 2m \Lambda_{\mu\lambda}(\epsilon_{F}, \cos \phi), \]

where \( \Lambda_{\mu\lambda}(\epsilon_{F}, \cos \phi) \) is a real function that depends on the dimensionless Fermi energy \( \epsilon_{F} \), but not on \( E_{0} \) and \( E_{k} \) independently. Then the renormalized coupling appearing in equation (5) reads

\[ V^{(i)}_{\mu\nu}(k) = \frac{u m^2}{2} V^{(i)}_{\mu\nu} + ..., \]

where \( V^{(i)}_{\mu\nu}(k) \) is the \( i \)th Fourier component of \( \Lambda_{\mu\lambda}^{(S)}(\epsilon_{F}, \cos \phi) \equiv 1/2(\Lambda_{\mu\lambda}(\epsilon_{F}, \cos \phi) + \Lambda_{\mu\lambda}(\epsilon_{F}, -\cos \phi)) \). The functions \( \Lambda_{\mu\lambda}^{(S)}(\epsilon_{F}, \cos \phi) \) are plotted in figure 7. At \( \epsilon_{F} \rightarrow 1^{+} \), \( \Lambda_{\mu\lambda}^{(S)} \) and \( \Lambda_{\mu\lambda}^{(S)} \) connect with the same functions at \( \epsilon_{F} \rightarrow 1^{-} \) calculated in [17, 18], but \( \Lambda_{\mu\lambda}^{(S)} \) changes sign due to the change of the helicity of the inner Fermi surface. Clearly, the functions depend more strongly on \( \phi \) at smaller \( \epsilon_{F} \).

Up to the fourth order of \( u_{i} \), there is only one term that satisfies two conditions: (i) being finite in nonzero angular momentum channel; and (ii) having a logarithmic divergence \( \ln(A/\Omega) \), which may give rise to an instability. This term is shown in figure 6(b). Including it, the renormalized couplings become

\[ V^{(i)}_{\mu\nu}(k) = \frac{u m^2}{2} V^{(i)}_{\mu\nu} + \frac{u m^2}{2} \sum_{\alpha} N_{\alpha} V^{(i)}_{\mu\alpha} V^{(i)}_{\nu\alpha} \ln A \Omega + ... \]

Defining the dimensionless bare coupling \( g^{(i)}_{\mu\lambda} = \frac{u m}{16} \sqrt{N_{\mu} N_{\lambda}} V^{(i)}_{\mu\lambda} \) and the dimensionless renormalized coupling \( g^{(i)}_{\mu\lambda} = 2 \sqrt{N_{\mu} N_{\lambda}} V^{(i)}_{\mu\lambda} \), we find the RG flow equation.
\[ W = \ast \]

where ‘\( \ast \)’ is the matrix multiplication, and the bare couplings have been replaced by the renormalized couplings. The solution is

\[ W = W_{\pm} + W_{\mp} \]

with the initial condition

\[ g_{\pm}(\Omega) = \frac{1}{f_{\pm} + \ln(A/\Omega)} \]

where ‘\( f \)’ is the matrix multiplication, and the bare couplings have been replaced by the renormalized couplings. The solution is

\[ g_{\pm}(\Omega) = \frac{1}{f_{\pm} + \ln(A/\Omega)} \]

with the initial condition

\[ g_{\pm} = \frac{u^2 m}{16} \left( \frac{1}{2} (N_+ V^{(\pm)}_{++} + N_+ V^{(\pm)}_{+-}) \right) \pm \frac{1}{4} (N_+ V^{(\pm)}_{++} - N_- V^{(\pm)}_{--})^2 + N_+ N_- V^{(\pm)}_{+-} V^{(\pm)}_{-+} \]

The scale of the superconducting transition temperature is given by the largest energy at which the renormalized coupling diverges

\[ T_c \sim \Omega^{g_{\pm}(\Omega)} = Ae^{-\frac{1}{\epsilon_F(\Omega)}} \]

where \( j_{\text{loc}} \) is chosen in such a way that, at a given \( \epsilon_F \), \( g_{\pm}^{(\pm)\ast} \) is the most negative among all the \( g_{\pm}^{(\pm)} \)’s. We find that as long as \( \epsilon_F < 1 \), \( j_{\text{loc}} \equiv 2 \), although it increases in general with \( \epsilon_F \) when \( \epsilon_F > 1 \) \([17,18]\). The intraband and intraband couplings are shown in figure 2. The effective coupling \( g_{\pm}^{(\pm)\ast} \) is plotted in figures 3 and 4 in units of \( u^2 \nu_0^2 \) and \( u^2 N_0^2 \), respectively.

Figure 3. The effective coupling in units of \( u^2 \nu_0^2 \). At \( \epsilon_F \rightarrow 1^+ \), it goes to \(-0.0187\) that agrees with the value at \( \epsilon_F \rightarrow 1^+ \) derived in \([17,18]\).

Figure 4. The effective coupling in units of \( u^2 N_0^2 \).
The effective coupling in the justiﬁcation approach, the result is not justiﬁed parameter regime. However, \( T_c \) can increase by many orders of magnitude as \( \epsilon_F \) decreases. If \( u \) is ﬁxed, a superconductor-insulator transition is expected as the density of the gas decreases since Wigner crystal state should exist at strong coupling [16], but we are not concerned with this case.

**Topological phase diagram with Zeeman ﬁeld**

As discussed in [17, 18], the ground state breaks TRS spontaneously and the system goes to either \( j_z = 2 \) or \( j_z = -2 \) superconducting state, both of which have the same energy. Due to SOC, the pairing term in the Hamiltonian has both triplet and singlet part, which reads

\[
\Delta_\ell e^{i\theta_k} \epsilon_{\ell-k_{\parallel}} + \Delta_\ell e^{i\theta_{-k_{\parallel}}} \epsilon_{\ell-k_{\parallel}} - \epsilon_{\ell+k_{\parallel}} + h.c.,
\]

where \( \Delta_\ell \) and \( \Delta_\ell \) are the triplet and singlet pairing strength, respectively, and \( \theta_k \) is the angle between \( k \) and \( k_{\parallel} \)-axis. To search for Majorana fermions, instead of solving the Bogoliubov-de Gennes equations in the presence of vortices, we apply an index theorem proved in [28] that superconductors with an odd Chern number can support Majorana zero modes. In [17, 18], the Chern number \( C \) of this state has been shown to be 2, if \( E_F < E_R \), and 0 if \( E_F < E_R \). To have an odd \( C \), we apply a Zeeman ﬁeld \( h_z \) [26, 29, 30] which couples to the system as

\[
h_z (c_{k_{\parallel}}^+ c_{k_{\parallel}} - c_{k_{\parallel}}^+ c_{k_{\parallel}}).
\]

Since now the Hamiltonian breaks TRS, the two states with \( j_z = \pm 2 \) have different energies. As pointed in [29], if \( h_z < 0 \), the system favors \( j_z = 2 \) state; and if \( h_z > 0 \), it favors \( j_z = -2 \) state. These two cases are related by time reversal operation, so we just consider the former case. We calculate the Chern number and ﬁnd three topological phases, depending on the position of Fermi level and \( h_z \), as shown in Figure 5. The topological phase diagram can be explained as follows. When \( h_z < 0 \), the Dirac point is gapped, and Chern number is well deﬁned for each band in the superconducting state. Due to the winding of spin around \( k_{\parallel} = 0 \), the band with helicity \(-1 \) (\(+1 \)) carries Chern number \( +1 \) (\(-1 \)). In addition, the phase winding of the order parameter superimposes \( j_z \) to the Chern number of each band. Therefore, the inner and outer band carry Chern number \( j_z + 1 \) and \( j_z - 1 \), respectively. When \( E_F - E_R > |h_z| \), the Fermi level crosses both bands, and the Chern number is the sum of the two, \( C = 2j_z \). When \( E_F - E_R < |h_z| \), the Fermi level either crosses the outer band twice, in which case the electron pocket and hole pocket contribute opposite Chern number, thus \( C = 0 \), or does not cross any band and hence \( C = 0 \). Between these two parameter regimes, i.e. when \( |h_z| > |E_F - E_R| \), the Fermi level only crosses the
outer band, and then $C = \frac{1}{j_z} - 1$. In this case, according to the index theorem mentioned above, Majorana zero modes exist at the edge and in the vortex cores. At low density, only the left part of the phase diagram in figure 5(b) is available.

Collective modes

Collective modes in superconductors were first studied by Bogoliubov [31] and Anderson [32, 33]. They found a Goldstone mode accompanying the spontaneous breaking of $U(1)$ gauge symmetry in a neutral system, which corresponds to the phase oscillations of the superconducting order parameter. In a charged system, this so called BAG mode is pushed up to the plasma frequency. In two-band superconductors, Leggett predicted another collective mode corresponding to the oscillations of the relative phase of the two superconducting condensates [34]. In the superconducting state derived above, due to the two-gap nature, in addition to BAG mode, Leggett mode is also expected to exist. The detailed calculations are carried out in the supplemental material. Effectively, in $j_z = 2$ channel, we have a two-band $p + ip$ superconductor. Actually, if the bare interaction were attractive, superconductivity would occur in $j_z = 0$ channel, and the superconducting state would also have a two-band $p + ip$ nature. Thus our approach also applies to that case. We find both BAG and Leggett modes, while the former is pushed to plasma energy, the latter has a dispersion

$$\omega^2 = \omega_0^2 + v^2 K^2,$$

where

$$\omega_0^2 = \frac{N_1 + N_2}{2N_1N_2} \frac{8|g_1|\Delta_1\Delta_2}{g_{11}g_{22} - g_{12}^2},$$

$$v^2 = \frac{(N_1 + N_2)c_2^2}{N_1c_1^2 + N_2c_2^2}.$$
In the above, \( N_{\lambda} \)'s are the DOS, \( g_{\mu \lambda} \sim V_{\mu \lambda} \), and

\[
\zeta_i^2 = \frac{m_i}{mN_{\lambda}} + \frac{2}{k_F^2} \lambda \left( \frac{1}{a} - \frac{1}{2 \Delta_{\lambda}} \right) \tag{21}
\]

where \( n_\lambda \) is the total particle number in band \( \lambda \) and \( \lambda = 1, 2 \) labels the outer and inner band. Note that the second term in equation (21) does not appear in a two-band \( s \)-wave superconductor. In order for Leggett mode to exist, it is necessary for \( \omega_0^2 \) to be positive, hence \( \frac{\Delta_{\lambda}}{k_F^2} - \frac{1}{a} > 0 \). However, the whole parameter regime shown in figure 2 does not satisfy this condition, thus this mode must be damped in our model. But in other SOC superconductors, Leggett mode could exist [35].

**Discussion and conclusion**

Looking for various ways to increase the critical temperature of superconductors is an important direction in condensed matter physics. We showed that increasing Rashba SOC or decreasing the Fermi energy can strongly enhance superconductivity in a 2D repulsive Fermi gas in the low density regime. As shown in figure 4, if we assume \( n N_{\text{tot}} \) is fixed to be \( \lesssim 1 \), the effective attraction can be increased by ten times by decreasing \( \epsilon_F \), resulting in a much higher \( T_c \) than that in the high density regime. Although such superconductivity is still too weak to observe in our weak-coupling approach, we wonder if SOC helps in strongly correlated systems, which is worthy of further study.

In real materials, the underlying lattice will affect the model in several aspects. The DOS is suppressed: although increasing with lowering \( \epsilon_F \), it will not diverge. The particle–hole bubble will also be affected. Due to these two consequences, the behavior of \( T_c \) will be modified, but the trend that \( T_c \) increases with lowering \( \epsilon_F \) will probably remain the same. Moreover, the spontaneous TRS breaking state may not be stable unless the two pairing components of the \( j_x = 2 \) state, \( d_{xy} \) and \( d_{xy} \) order parameter, belong to the same irreducible representation of the point group, which can be satisfied if the lattice has a threefold rotational symmetry.

In conclusion, we have investigated the superconducting instability of a 2D Rashba gas with repulsive interaction at low density, in the weak-coupling limit. As the density decreases, the superconducting transition temperature increases significantly. The superconducting state is always in \( j_x = 2 \) channel. When a Zeeman field is applied, the state can have an odd Chern number, and hence Majorana zero modes are supported. Although Leggett mode does not exist due to the specific parameters, we expect it to appear in other spin-orbit coupled superconductors because of the two-gap nature.

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