Precise Large Deviations of the First Passage Time

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Abstract. Let $S_n$ be partial sums of an i.i.d. sequence $\{X_i\}$. We assume that $EX_1 < 0$ and $P[ X_1 > 0 ] > 0$. In this paper we study the first passage time

$$\tau_u = \inf \{ n : S_n > u \}.$$ 

The classical Cramér’s estimate of the ruin probability says that

$$P[ \tau_u < \infty ] \sim C e^{-\alpha_0 u}$$

as $u \to \infty$, for some parameter $\alpha_0$. The aim of the paper is to describe precise large deviations of the first crossing by $S_n$ a linear boundary, more precisely for a fixed parameter $\rho$ we study asymptotic behavior of $P[ \tau_u = \lfloor u/\rho \rfloor ]$ as $u$ tends to infinity.

1. Introduction

Let $\{X_i\}$ be a sequence of independent and identically distributed (i.i.d.) real valued random variables. We denote by $S_n$ the partial sums of $X_i$, i.e. $S_0 = 0, S_n = X_1 + \cdots + X_n$. In this paper we are interested in the situation when $X_1$ has negative drift, but simultaneously $P[X_1 > 0] > 0$. Our primary objective is to describe the precise large deviations of the linearly normalized first passage time

$$\tau_u = \inf \{ n : S_n > u \},$$

as $u$ tends to infinity.

The stopping time $\tau_u$ arises in various contexts in probability, e.g. in risk theory, sequential statistical analysis, queueing theory. We refer to Siegmund [10] and Lalley [8] for a comprehensive bibliography. A celebrated result concerning $\tau_u$, playing a major role in the ruin theory, is due to Cramér, who revealed estimate of the ruin probability

(1.1) $P[ \tau_u < \infty ] \sim C e^{-\alpha_0 u},$ 

as $u \to \infty$, for some parameter $\alpha_0$ that will be described below (see Cramér [5] and Feller [7]).

Our aim is to describe the probability that at a given time partial sums $S_n$ first cross a linear boundary $\rho n$. This problem was studied e.g. by Siegmund [10] and continued by Lalley [8]. Up to our best knowledge all the known results concern probabilities of the form $P[ \tau_u < u/\rho ]$ or $P[u/\rho < \tau_u < \infty ]$, see Lalley [8] (see also Arfwedson [1] and Asmussen [2] for similar results related to compound Poisson risk model). In this paper we describe pointwise behavior of $\tau_u$, i.e. the asymptotic behavior of $P[ \tau_u = \lfloor u/\rho \rfloor ]$ as $u$ tends to infinity.

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2. Statement of the results

Our main result will be expressed in terms of the moment and cumulant generating functions of \(X_1\), i.e.
\[
\lambda(s) = \mathbb{E}[e^{sx_1}] \quad \text{and} \quad \Lambda(s) = \log \lambda(s),
\]
respectively. We assume that \(\lambda(s)\) exists for \(s\) in the interval \(D = [0, s_0)\) for some \(s_0 > 0\). It is well known that both \(\lambda\) and \(\Lambda\) are smooth and convex on \(D\). Throughout the paper we assume that there are \(\alpha \in D\) and \(\xi > 0\) such that
\[
\rho = \Lambda'(-\alpha) > 0
\]
and
\[
\lambda(\alpha + \xi) < \infty.
\]
Observe that (2.1) implies that \(\mathbb{P}[X_1 > 0] > 0\).

Recall the convex conjugate (or the Fenchel-Legendre transform) of \(\Lambda\) defined by
\[
\Lambda^*(x) = \sup_{s \in \mathbb{R}} \{sx - \Lambda(s)\}, \quad x \in \mathbb{R}.
\]
This rate function appears in studying large deviations problems for random walks. Its various properties can be found in Dembo, Zeitouni [6]. Given \(\alpha < s_0\) and \(\rho\) as in (2.1) we consider
\[
\alpha = \frac{1}{\rho} \Lambda^*(\rho).
\]
An easy calculation shows
\[
\alpha = \alpha - \frac{\Lambda(\alpha)}{\Lambda'(\alpha)}.
\]
The parameter \(\alpha\) arises in the classical large deviations theory for random walks. The Petrov’s theorem and the Bahadur-Rao theorem say that
\[
\mathbb{P}[S_n > n\rho] \sim C e^{-\alpha \rho \sqrt{n}} \quad \text{as} \quad n \to \infty,
\]
(see Petrov [9] and Dembo, Zeitouni [6]). As we will see below \(\alpha\) will play also the crucial role in our result. This parameter has a geometric interpretation: the tangent line to \(\Lambda\) at point \(\alpha\) intersects the \(x\)-axis at \(\alpha\). See the Figure 1 below.

\textbf{Figure 1.} \(\Lambda(s) = \log \mathbb{E}e^{sx_1}\)
We also introduce parameters $k_u$ and $\alpha_{\text{min}}$ defined by

$$\alpha_{\text{min}} = \arg\min \Lambda(s) \quad \text{and} \quad k_u = \frac{u}{\rho}.$$ 

Now we are ready to state our main result.

**Theorem 2.3.** Assume that $\{X_i\}$ is an i.i.d. sequence such that the law of $X_1$ is nonlattice, $\mathbb{E}X_1 < 0$ and $\rho = \Lambda'(\alpha) > 0$ for some $\alpha < s_0$. Then

$$\Pr[\tau_u = \lfloor k_u \rfloor] = C(\alpha)\lambda(\alpha)^{-\Theta(u)} \frac{e^{-u\pi}}{\sqrt{u}} (1 + o(1)) \quad \text{as} \quad u \to \infty$$

for some constant $C(\alpha) > 0$ and $\Theta(u) = k_u - \lfloor k_u \rfloor$.

Notice that the above formula gives the largest asymptotics when $\alpha = \alpha_0$ for $\alpha_0$ such that $\Lambda(\alpha_0) = 0$. Then $s_0 = \alpha_0$. For all the other parameters $\alpha$ we have $\alpha > \alpha_0$. The parameter $\alpha_0$ arises in the Cramér’s formula (1.1).

Similar results were obtained by Lalley, who proved that for $\alpha$ such that $\Lambda(\alpha) > 0$ we have

$$\Pr[\tau_u \leq k_u] = C_1(\alpha)\lambda(\alpha)^{-\Theta(u)} \frac{e^{-u\pi}}{\sqrt{u}} (1 + o(1)) \quad \text{as} \quad u \to \infty$$

and for $\alpha$ such that $\Lambda(\alpha) < 0$

$$\Pr[\tau_u > k_u] = C_2(\alpha)\lambda(\alpha)^{1-\Theta(u)} \frac{e^{-u\pi}}{\sqrt{u}} (1 + o(1)) \quad \text{as} \quad u \to \infty,$$

for some known, depending only on $\alpha$ constants $C_1(\alpha)$, $C_2(\alpha)$ (see Lalley [8], Theorem 5).

Notice that the function $\Theta(u)$ appears in all the formulas above only from purely technical reason. It reflects the fact that $\tau_u$ attains only integer values, whereas $k_u$ is continuous. Thus the function $\Theta$ is needed only to adjust both expressions for noninteger values of $k_u$. Below we will omit this point and without any saying we assume that $k_u$ is an integer.

### 3. Auxiliary results.

The proof of Theorem 2.3 bases on the Petrov’s theorem and the Bahadur-Rao theorem describing precise large deviations for random walks (2.2). We apply here techniques, which were recently used by Buraczewski et al. [3,4] to study the problem of the first passage time in a more general context of perpetuities. They obtained similar results as described above, but in our context the proof is essentially simpler and final results are stronger.

Here we need a reinforced version of (2.2), which is both uniform and allows to slightly perturb the parameters. As a direct consequence of Petrov’s theorem [9] the following results was proved in [3]:

**Lemma 3.1.** Assume that the law of $X_1$ is nonlattice and that $\rho$ satisfies $\mathbb{E}X_1 < \rho < A_0$. Choose $\alpha$ such that $\Lambda'(\alpha) = \rho$. If $\{\delta_n\}, \{j_n\}$ are two sequences satisfying

$$\max\{\sqrt{n} |\delta_n|, j_n/\sqrt{n}\} \leq \delta_n \to 0,$$

then

$$\Pr[S_{n-j_n} > n (\rho + \delta_n)] = C(\alpha) \frac{e^{-\pi\rho}}{\sqrt{n}} e^{-\alpha n\delta_n} \lambda(\alpha)^{-j_n} (1 + o(1)) \quad \text{as} \quad n \to \infty,$$
uniformly with respect to $\rho$ in the range
\[ \mathbb{E}X + \epsilon \leq \rho \leq A_0 - \epsilon, \]
and for all $\delta_n$, $j_n$ as in (3.2).

Let us define $M_n = \max_{1 \leq k \leq n} S_k$ and $S^n_i = S_n - S_{n-i} = X_{n-i+1} + \ldots + X_n$ for $0 \leq i \leq n$. The following Lemma will play a crucial role in the proof.

**Lemma 3.3.** Let $L$ and $M$ be two integers such that $L \geq 1$ and $-1 \leq M \leq L$. For any $\gamma \geq 0$, $\alpha_{\min} < \beta < \alpha$ and sufficiently large $u$, the following holds
\[
\Pr[M_k u - L > u, S_k u - M > u - \gamma] \leq C(\alpha, \beta)e^{\gamma\beta} \lambda(\alpha)^{-L} \lambda(\beta)^{L-M} e^{-u\alpha \lambda(\alpha)} e^{-ua \lambda(k u)},
\]
where $C(\alpha, \beta)$ is some constant depending on $\alpha$ and $\beta$.

**Proof.** We have
\[
\Pr[M_k u - L > u, S_k u - M > u - \gamma] \leq \sum_{i=0}^{k_u-1-L} \Pr[S_k u - M > u - \gamma, S_k u - i - L > u].
\]
Denote $\delta = \frac{\lambda(\beta)}{\lambda(\alpha)} < 1$. To estimate the above series, we divide the set of indices into two sets. 

**CASE 1.** First we consider $i$ satisfying $i > K \log k_u$ for some constant $K$ such that $\delta^{K \log k_u} < 1/u$. Notice that for any $u$ we have
\[
e^{-u\alpha \lambda(\alpha)} = e^{-ua \lambda(k u)}.
\]
Then, for any such $i$ we write
\[
\Pr[S_k u - M > u - \gamma, S_k u - i - L > u] \leq \sum_{m=0}^{\infty} \Pr[S_k u - M > u - \gamma, u + m < S_k u - i - L \leq u + m + 1]
\]
\[= \sum_{m=0}^{\infty} \Pr[S_k u - i - L + S_{L+i-M}^{k_u-M} > u - \gamma, u + m < S_k u - i - L \leq u + m + 1]
\]
\[\leq \sum_{m=0}^{\infty} \Pr[S_{L+i-M}^{k_u-M} > -\gamma - (m+1)] \Pr[S_k u - i - L > u + m]
\]
\[\leq \sum_{m=0}^{\infty} e^{\beta \gamma} e^{\beta (m+1)} \lambda(\beta)^{L+i-M} e^{-u\alpha \lambda(\alpha)} e^{-\alpha m \lambda(\alpha)^{k_u-i-L}}
\]
\[\leq C(\alpha, \beta)e^{\beta \gamma} \delta^i e^{-u\alpha \lambda(\alpha)^{-L}} \lambda(\beta)^{L-M},
\]
where in the third line we used Markov’s inequality with functions $e^{\beta x}$ and $e^{\alpha x}$. Summing over $i$ we obtain
The first term

\[
P \left[ S_{k_u-M} > u - \gamma, S_{k_u-i-L} > u \right] \leq C(\alpha, \beta) \sum_{i > K \log k_u} e^{\beta \gamma i} e^{-u\pi} \lambda(\alpha)^{-L} \lambda(\beta)^{L-M}
\]

\[
\leq C(\alpha, \beta) e^{\beta \gamma} \delta^i e^{-u\pi} \lambda(\alpha)^{-L} \lambda(\beta)^{L-M}
\]

\[
\leq C(\alpha, \beta) \frac{e^{-u\pi}}{u} \lambda(\alpha)^{-L} \lambda(\beta)^{L-M}.
\]

CASE 2. Now consider \( i \leq K \log k_u \). Let \( N \) be a constant such that \(-\alpha N + 1 < 0, \) for \( \Lambda(\alpha) \geq 0 \) and \(-\alpha N + 1 - \Lambda(\alpha)K < 0 \) for \( \Lambda(\alpha) < 0 \). We have

\[
P \left[ S_{k_u-M} > u - \gamma, S_{k_u-i-L} > u \right] \leq P \left[ S_{k_u-i-L} \geq u + N \log k_u \right]
\]

\[
+ P \left[ S_{k_u-M} > u - \gamma, u < S_{k_u-i-L} < u + N \log k_u \right]
\]

\[
= P_1 + P_2
\]

The first term \( P_1 \) we estimate using Markov’s inequality with function \( e^{\alpha x} \) and we obtain

\[
P_1 \leq e^{-u\alpha k_u^{-\alpha N} \lambda(\alpha)^{k_u-i-L}} = e^{-u\pi} k_u^{-\alpha N} \lambda(\alpha)^{-i-L} \leq C(\alpha) e^{-u\pi} \frac{1}{u} k_u^{-\alpha N+1} e^{-i(\alpha) \lambda(\alpha)^{-L}}
\]

\[
\leq C(\alpha) e^{-u\pi} \frac{1}{u} \lambda(\alpha)^{-L}.
\]

To estimate \( P_2 \) we apply Lemma \([3.1]\) and again Markov’s inequality with function \( e^{\beta x} \).

\[
P_2 = P \left[ S_{k_u-i-L} + S_{L+1-i-M}^{k_u-M} > u - \gamma, u < S_{k_u-i-L} < u + N \log k_u \right]
\]

\[
\leq \sum_{m=0}^{[N \log k_u-1]} P \left[ S_{k_u-i-L} + S_{L+1-i-M}^{k_u-M} > u - \gamma, u + m < S_{k_u-i-L} < u + m + 1 \right]
\]

\[
\leq \sum_{m=0}^{[N \log k_u-1]} P \left[ S_{k_u-i-L} > u + m \right] P \left[ S_{L+1-i-M}^{k_u-M} > - \gamma - (m + 1) \right]
\]

\[
\leq \sum_{m=0}^{[N \log k_u-1]} C(\alpha) \frac{e^{-u\pi \gamma}}{\sqrt{k_u}} \lambda(\alpha)^{-i-L} e^{-\alpha m} e^{\beta \gamma} e^{\beta (m+1)} \lambda(\beta)^{i+L-M}
\]

\[
\leq \sum_{m=0}^{[N \log k_u-1]} C(\alpha, \beta) \frac{e^{-u\pi}}{\sqrt{k_u}} \lambda(\beta)^{-L} \lambda(\beta)^{L-M}
\]

\[
\leq C(\alpha, \beta) \frac{e^{-u\pi}}{u} \lambda(\alpha)^{-L} \lambda(\beta)^{L-M}.
\]

Now we sum over \( i \)
\[
\sum_{i \leq K} \mathbb{P}[S_{k_u - M} > u - \gamma, S_{k_u - i - L} > u] \\
\leq \sum_{i \leq K} (P_1 + P_2) \\
\leq \sum_{i \leq K} \left( C(\alpha) \frac{e^{-u\alpha}}{u} \lambda(\alpha)^{-L} + C(\alpha, \beta) \frac{e^{-u\alpha}}{\sqrt{u}} \delta^i e^{-\beta\gamma \lambda(\alpha)^{-L} \lambda(\beta)^{L - M}} \right) \\
\leq C(\alpha) e^{-u\alpha} \log \frac{k_u}{u} \lambda(\alpha)^{-L} + C(\alpha, \beta) \frac{e^{-u\alpha}}{\sqrt{u}} e^{\beta\gamma \lambda(\alpha)^{-L} \lambda(\beta)^{L - M}}.
\]

Combining both cases we end up with
\[
\mathbb{P}[M_{k_u - L} > u, S_{k_u - M} > u - \gamma] \leq C(\alpha, \beta) e^{-u\alpha} \log \frac{k_u}{u} \lambda(\alpha)^{-L} + C(\alpha) e^{-u\alpha} \log \frac{k_u}{u} \lambda(\alpha)^{-L} \\
+ C(\alpha, \beta) \frac{e^{-u\alpha}}{\sqrt{u}} e^{\beta\gamma \lambda(\alpha)^{-L} \lambda(\beta)^{L - M}} \\
\leq C(\alpha, \beta) \frac{e^{-u\alpha}}{\sqrt{u}} e^{\beta\gamma \lambda(\alpha)^{-L} \lambda(\beta)^{L - M}}.
\]

\[
\]

4. Lower and upper estimates

The goal of this section is to prove the following

**Proposition 4.1.** There is a constant \( C > 0 \) such that for large \( u \)

\[
\frac{1}{C} \frac{e^{-u\alpha}}{\sqrt{u}} \leq \mathbb{P}[\tau_u = k_u + 1] \leq C \frac{e^{-u\alpha}}{\sqrt{u}}.
\]

**Proof.** First, observe that the upper estimate is an immediate consequence of Petrov’s theorem (Lemma 3.1) used with \( \gamma_n = 0 \). Indeed, we have

\[
\mathbb{P}[\tau_u = k_u + 1] = \mathbb{P}[M_{k_u} \leq u, S_{k_u + 1} > u] \leq \mathbb{P}[S_{k_u + 1} > u] \leq C(\alpha) \frac{e^{-u\alpha}}{\sqrt{u}}.
\]

For the lower estimate we write for any positive \( \gamma \) and any positive integer \( L \)

\[
\mathbb{P}[\tau_u = k_u + 1] = \mathbb{P}[M_{k_u} \leq u, S_{k_u + 1} > u] \geq \mathbb{P}[M_{k_u} \leq u, S_{k_u + 1} > u, S_{k_u + 1}^L > \gamma].
\]

For any \( 0 < r < \gamma \) one has

\[
\mathbb{P}[M_{k_u} \leq u, S_{k_u + 1} > u, S_{k_u + 1}^L > \gamma] \geq \mathbb{P}[M_{k_u} \leq u, u - \gamma < S_{k_u - L} < r + u - \gamma, S_{k_u + 1}^L > \gamma].
\]
Let $M^n_t = \max(0, S_1^{n-1}, S_2^{n-2}, S_3^{n-3}, \ldots, S_{L-1}^{n-1}, S_L^n)$. Note that $M_k = \max(M_{k_u-L}, S_{k_u-L+M_{k_u}^b})$.

Hence we have

$$\mathbb{P}[M_{k_u} \leq u, u - \gamma < S_{k_u-L} < r + u - \gamma, S_{L+1}^{k_u+1} > \gamma]$$

$$= \mathbb{P}[M_{k_u-L} \leq u, u - \gamma < S_{k_u-L} < r + u - \gamma, S_{L+1}^{k_u+1} > \gamma]$$

$$\geq \mathbb{P}[M_{k_u-L} \leq u, M_{k_u}^b \leq -r + \gamma, u - \gamma < S_{k_u-L} < r + u - \gamma, S_{L+1}^{k_u+1} > \gamma].$$

Finally, we combine above, use independence of $(M_{k_u}^b, S_{L+1}^{k_u+1})$ and $(M_{k_u-L}, S_{k_u-L})$ and the identity $\mathbb{P}[A \cap B] = \mathbb{P}[A] - \mathbb{P}[A \cap B^c]$ to obtain

$$\mathbb{P}[\tau_u = k_u + 1] \geq \mathbb{P}[M_{k_u-L} \leq u, M_{k_u}^b \leq -r + \gamma, u - \gamma < S_{k_u-L} < r + u - \gamma, S_{L+1}^{k_u+1} > \gamma]$$

$$\geq \mathbb{P}[M_{k_u-L} \leq u, u - \gamma < S_{k_u-L} < r + u - \gamma] \mathbb{P}[M_{k_u}^b \leq -r + \gamma, S_{L+1}^{k_u+1} > \gamma]$$

$$\times (\mathbb{P}[u - \gamma < S_{k_u-L} < r + u - \gamma] - \mathbb{P}[M_{k_u-L} > u, u - \gamma < S_{k_u-L} < r + u - \gamma]).$$

Lemma 3.1 gives an asymptotics

$$\mathbb{P}[u - \gamma < S_{k_u-L} < r + u - \gamma] \sim C(\alpha, r)e^{\alpha \gamma} \lambda(\alpha)^{-L} \frac{c^{-u \alpha}}{\sqrt{u}} \quad \text{as } u \to \infty.$$

Using Lemma 3.3 with $M = L$ we obtain

$$\mathbb{P}[M_{k_u-L} > u, u - \gamma < S_{k_u-L} < r + u - \gamma] \leq \mathbb{P}[M_{k_u-L} > u, S_{k_u-L} > u - \gamma]$$

$$\leq C(\alpha, \beta) \frac{e^{-u \alpha}}{\sqrt{u}} e^{\beta \gamma} \lambda(\alpha)^{-L},$$

where $\beta < \alpha$. From (4.3), (4.4) and (4.5) we have

$$\mathbb{P}[\tau_u = k_u + 1] \geq \mathbb{P}[S_{L+1}^{k_u+1} > \gamma, M_{L}^b \leq -r + \gamma]$$

$$\times (\mathbb{P}[u - \gamma < S_{k_u-L} < r + u - \gamma] - \mathbb{P}[M_{k_u-L} > u, u - \gamma < S_{k_u-L} < r + u - \gamma])$$

$$\geq \mathbb{P}[S_{L+1}^{k_u+1} > \gamma, M_{L}^b \leq -r + \gamma] \left(C(\alpha, r)\lambda(\alpha)^{-L} \frac{e^{-u \alpha}}{\sqrt{u}} e^{\alpha \gamma} - C(\alpha, \beta) \frac{e^{-u \alpha}}{\sqrt{u}} e^{\beta \gamma} \lambda(\alpha)^{-L}\right)$$

$$= \mathbb{P}[S_{L+1}^{k_u+1} > \gamma, M_{L}^b \leq -r + \gamma] \lambda(\alpha)^{-L} \frac{e^{-u \alpha}}{\sqrt{u}} \left(C(\alpha, r)e^{\alpha \gamma} - C(\alpha, \beta)e^{\beta \gamma}\right).$$

Notice that $(M_{i}^{a}, S_{i+1}^{a+1}) \overset{d}{=} (M_i, S_{i+1})$. To make constants in the last term strictly positive firstly pick $r > 0$ such that $\mathbb{P}[X_1 > 2r] > 0$. Next, take $\gamma > 0$ big enough to ensure that $C(\alpha, r)e^{\alpha \gamma} - C(\alpha, \beta)e^{\beta \gamma} > 0$ and $\gamma - 2r > 0$. Now we choose large $L$ to have $\mathbb{P}[LX_1 > -2r + \gamma] > 0$. Since $\gamma$ is continuous parameter, if necessary, we can increase it to
get $\mathbb{P}[-2r + \gamma < L X_1 < -r + \gamma] > 0$. For such constants we have

$$0 < \mathbb{P} \left[ X_{L+1} > 2r \right] \prod_{i=1}^{L} \mathbb{P} \left[ -2r + \gamma < L X_i < -r + \gamma \right]$$

$$\leq \mathbb{P} \left[ X_{L+1} > 2r, S_L \geq S_{L-1} \geq \ldots \geq S_1, -2r + \gamma < S_L < -r + \gamma \right]$$

$$\leq \mathbb{P} \left[ X_{L+1} > 2r, S_L = M_L, -2r + \gamma < S_L < -r + \gamma \right]$$

$$\leq \mathbb{P} \left[ M_L < -r + \gamma, S_{L+1} > \gamma \right],$$

and (4.2) follows. \qed

5. Asymptotics

Proof of Theorem 2.3 We will show that the limit

$$\lim_{u \to \infty} e^{u\alpha} \sqrt{u} \mathbb{P}[\tau_u = k_u + 1]$$

exists, which combined with Proposition 4.1 gives us Theorem 2.3

Fix an arbitrary $L$. Since $M_{k_u} = \max(M_{k_u-L}, S_{k_u-L} + M_{k_u})$ we have

$$\mathbb{P} \left[ S_{k_u-L} + M_{k_u} < u, S_{k_u+1} > u \right] = \mathbb{P} \left[ S_{k_u-L} + M_{k_u} < u, S_{k_u+1} > u, M_{k_u-L} > u \right]$$

$$+ \mathbb{P} \left[ S_{k_u-L} + M_{k_u} < u, S_{k_u+1} > u, M_{k_u-L} \leq u \right]$$

$$= \mathbb{P} \left[ S_{k_u-L} + M_{k_u} < u, S_{k_u+1} > u, M_{k_u-L} > u \right]$$

$$+ \mathbb{P} \left[ M_{k_u} < u, S_{k_u+1} > u \right]$$

$$= \mathbb{P} \left[ S_{k_u-L} + M_{k_u} < u, S_{k_u+1} > u, M_{k_u-L} > u \right]$$

$$+ \mathbb{P} \left[ \tau_u = k_u + 1 \right].$$

From Lemma 3.3 with $M = -1$ and $\gamma = 0$ we obtain

$$\mathbb{P} \left[ M_{k_u-L} > u, S_{k_u+1} > u \right] \leq C(\alpha, \beta) \lambda(\alpha)^{-L} \lambda(\beta)^{L+1} \frac{e^{-u\alpha}}{\sqrt{u}} = C(\alpha, \beta) \delta^L \frac{e^{-u\alpha}}{\sqrt{u}},$$

where $\delta = \frac{\lambda(\beta)}{\lambda(\alpha)} < 1$ provided $\beta < \alpha$. Thus to get (5.1) it is sufficient to show that for some large fixed $L$

$$\lim_{u \to \infty} e^{u\alpha} \sqrt{u} \mathbb{P} \left[ S_{k_u-L} + M_{k_u} \leq u, S_{k_u+1} > u \right]$$

exists. Indeed, multiply both sides of (5.2) by $e^{u\alpha} \sqrt{u}$, let first $u \to \infty$ and then $L \to \infty$. We write

$$\mathbb{P} \left[ S_{k_u-L} + M_{k_u} \leq u, S_{k_u+1} > u \right] = \mathbb{P} \left[ u - u^4 < S_{k_u-L} < u, S_{k_u-L} + M_{k_u} \leq u, S_{k_u+1} > u \right]$$

$$+ \mathbb{P} \left[ u - u^4 > S_{k_u-L}, S_{k_u-L} + M_{k_u} < u, S_{k_u+1} > u \right].$$
To estimate the second summand fix $\beta > \alpha$ and observe that by Markov’s inequality with functions $e^{\alpha x}$ and $e^{\beta x}$ we have

$$\mathbb{P}[S_{k_u-L} \leq u - u^\frac{1}{2}, S_{k_u+1} > u]$$

\[ \leq \sum_{m \geq 0} \mathbb{P} \left[ u - u^\frac{1}{2} - (m + 1) < S_{k_u-L} \leq u - u^\frac{1}{2} - m, S_{k_u-L} + S_{L+1}^k > u \right] \]

\[ \leq \sum_{m \geq 0} \mathbb{P} \left[ S_{k_u-L} > u - u^\frac{1}{2} - (m + 1) \right] \mathbb{P} \left[ S_{L+1} > u^\frac{1}{2} + m \right] \]

\[ \leq \sum_{m \geq 0} \lambda(\alpha)^{k_u-L} e^{-\alpha \sum_{k=1}^{m} \frac{1}{2} \lambda(\beta)^{L+1} e^{-\beta (u^\frac{1}{2} + m)} \sum_{m \geq 0} \lambda(\beta)^{L+1} e^{-\beta m} = o \left( \frac{e^{-\nu \pi}}{\sqrt{u}} \right). \]

The same argument proves

$$\mathbb{P}[S_{k_u-L} > u - u^\frac{1}{2}, S_{L+1}^{k_u+1} > u^\frac{1}{2}] = o \left( \frac{e^{-\nu \pi}}{\sqrt{u}} \right).$$

Now we see that

$$\mathbb{P}[S_{k_u-L} + M_{L}^{k_u} \leq u, S_{k_u+1} > u]$$

\[ = \mathbb{P} \left[ u - u^\frac{1}{2} < S_{k_u-L} < u, S_{k_u-L} + M_{L}^{k_u} \leq u, S_{k_u+1} > u \right] + o \left( \frac{e^{-\nu \pi}}{\sqrt{u}} \right) \]

\[ = \mathbb{P} \left[ u - u^\frac{1}{2} < S_{k_u-L} < u, S_{k_u-L} + M_{L}^{k_u} \leq u, S_{k_u+1} > u, S_{L+1}^{k_u+1} < u^\frac{1}{2} \right] + o \left( \frac{e^{-\nu \pi}}{\sqrt{u}} \right) \]

and hence we reduced our problem to finding

$$\lim_{u \to \infty} e^{\nu \pi} \sqrt{u} \mathbb{P} \left[ u - u^\frac{1}{2} < S_{k_u-L} < u, S_{k_u-L} + M_{L}^{k_u} \leq u, S_{k_u+1} > u, S_{L+1}^{k_u+1} < u^\frac{1}{2} \right].$$

For this purpose we write

$$\mathbb{P} \left[ u - u^\frac{1}{2} < S_{k_u-L} < u, S_{k_u-L} + M_{L}^{k_u} \leq u, S_{k_u-L} + S_{L+1}^{k_u+1} > u, S_{L+1}^{k_u+1} < u^\frac{1}{2} \right]$$

(5.3)

$$= \int_{0 \leq y < u^\frac{1}{2}} \mathbb{P} \left[ u - x < S_{k_u-L} < u - y \right] \mathbb{P} \left[ M_{L}^{k_u} \leq dy, S_{L+1}^{k_u+1} \leq dx \right].$$

Now we apply Lemma 3.1 with $n = k_u$, $j_n = L$, $\delta_n = C n^{-\frac{1}{2}}$ and $\delta_n = -\frac{n}{n}$. We have

$$\mathbb{P} \left[ S_{k_u-L} \geq u - y \right] = C(\alpha) \frac{e^{-\nu \pi}}{\sqrt{u}} e^{\nu \alpha (1 + o(1))},$$

provided $\max \left\{ \frac{\nu \pi y}{u}, L/\sqrt{u} \right\} \leq C u^{-\frac{1}{2}}$. But since $y < u^\frac{1}{2}$ all the assumptions of the Lemma are satisfied. Analogously

$$\mathbb{P} \left[ S_{k_u-L} \geq u - x \right] = C(\alpha) \frac{e^{-\nu \pi}}{\sqrt{u}} e^{\nu \alpha (1 + o(1))}. $$
Back to (5.3) we end up with
\[
\mathbb{P} \left[ u - u^\frac{1}{2} < S_{k_u - L} < u, S_{k_u - L} + M_{L}^{k_u} \leq u, S_{L+1}^{k_u+1} > u \right]
\]
\[
= C(\alpha) \frac{e^{-u\alpha}}{\sqrt{u}} e^{-L\Lambda(\alpha)} \mathbb{E} \left[ (e^{\alpha S_{L+1} - e^{\alpha M_L}})_+ \right] (1 + o(1)) \quad \text{as } u \to \infty.
\]
Note that by the moment assumptions the expectation above is finite, hence we conclude (5.1).

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