AN ILLUSTRATED GUIDE TO D-BRANES IN SU\(_3\)

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Abstract. We give a systematic account of symmetric D-branes in the Lie group SU\(_3\). We determine both the classical and quantum moduli space of (twisted) conjugacy classes in terms of the (twisted) Stiefel diagram of the Lie group. We show that the allowed (twisted) conjugacy classes are in one-to-one correspondence with integrable highest weight representations of the (twisted) affine Lie algebra. In particular, we show how the charges of these D-branes fit in the twisted K-theory groups.

1. Introduction

Group manifolds provide an ideal laboratory for the study of D-branes in general backgrounds, as they are amenable to a variety of approaches, ranging from the algebraic techniques of BCFT to the lagrangian description based on the boundary WZW model. At present, the simplest and best understood class of D-brane configurations is obtained \([1, 2, 3]\) as solutions of the familiar gluing conditions on the chiral currents of the WZW background

\[
J(z) = R\bar{J}(\bar{z}),
\]

where \(R\) is a metric preserving Lie algebra automorphism. They describe symmetric D-branes, by which we mean D-branes wrapping (twisted) conjugacy classes in the group manifold \(G\). Both standard and twisted conjugacy classes are homogeneous spaces in the group manifold. In fact, it is often stated that ‘most’ D-brane configurations wrap conjugacy classes of ‘regular’ elements. This assertion is based on the fact that in a group manifold, regular points are dense while singular points form a set of codimension at least three. This seems enough to justify restricting ourselves to the above (twisted) conjugacy classes, as the corresponding D-brane configurations are indeed generic at the classical level. At the quantum level, however, one has a finite number of D-brane configurations which always include conjugacy classes of ‘singular’ elements. It therefore appears desirable to have a complete picture of all the possible D-brane configurations. The main aim of this paper is to give a such a complete picture of the consistent D-brane configurations on (twisted) conjugacy classes in the particular case of the SU\(_3\) group manifold, leaving the general analysis for a forthcoming paper \([4]\). D-branes in the SU\(_3\) manifold have been analysed
recently in [5, 6], using various approaches. Here we would like to offer a complementory description, from a geometric point of view, paying special attention to the twisted case.

The paper is organised as follows. Section 2 is devoted to the case $R = 1$. In this case, we provide a natural description of the space of D-branes in terms of the Stiefel diagram of $SU_3$. This is a figure in the Cartan subalgebra describing the singular points of the group in a maximal torus. The moduli space of classical D-branes is described by the fundamental domain of the extended Weyl group which is given by an equilateral triangle with the interior points describing ‘regular’ six-dimensional D-branes, whereas the boundary points describe lower-dimensional D-branes. The requirement of single-valuedness of the path integral of the boundary WZW model selects a finite number of consistent configurations at every given level, each of them being uniquely characterised by its intersections with certain cosets of the $SU_2^\alpha$ subgroups corresponding to the simple roots of $SU_3$.

In preparation for our study of the quantisation conditions for twisted conjugacy classes, in Section 3 we briefly describe the quantisation conditions in the case of the $SO_3$ WZW model, which will prove instrumental in our analysis of the twisted case for $SU_3$. We compare the space of D-submanifolds of the $SO_3$ and $SU_2$ theories, both at the classical and quantum levels.

In Section 4 we undertake the study of the D-brane configurations wrapping twisted conjugacy classes. Describing these conjugacy classes as orbits in a non-connected extension of $SU_3$ allows us to use the theory of non-connected Lie groups in order to describe their moduli space. To this end we introduce a generalisation of the Stiefel diagram and the notion of twisted regularity, which is more appropriate for the description of twisted conjugacy classes. We analyse the quantisation conditions imposed by the condition of single-valuedness of the path integral and we obtain a discrete spectrum of consistent D-brane configurations. We show that the admissible D-branes are in one-to-one correspondence with the integrable highest weight (IHW) representations of the twisted affine Lie algebra $\hat{su}(3)^{(2)}_k$. We describe the way in which the admissible twisted branes in $SU_3$ intersect the fixed point subgroup $SO_3$ in conjugacy classes which describe admissible D-branes of a restricted WZW model.

In Section 5 we discuss the charges of the D-branes in $SU_3$ and their relation to the twisted K-theory group $K^H(SU_3)$ which was analysed in [3]. For the untwisted branes, our results agree with the ones obtained previously in [5, 6]. For the twisted branes, we argue that the charge is given by the dimension of the inducing representation of the fixed point subalgebra. An explicit calculation shows that the charges of the configurations determined in Section 4 fit in the twisted K-theory group $K^H(SU_3)$. Our approach offers support for the idea that the charges carried by symmetric D-branes can be determined by entirely geometric and topological means, without any dynamical input.

In Section 6 we discuss our $SU_3$ results and we outline the general picture that emerges from this analysis. Finally, we include two appendices. In Appendix A we collect the basic facts about the twisted affine Lie algebra $\hat{su}(3)^{(2)}_k$ which we use in
our analysis of twisted branes. In Appendix B, written jointly with José Figueroa-
O’Farrill, we prove several results on the relative (co)homology groups of $SU_3$ modulo
a twisted conjugacy class. In particular we determine which twisted conjugacy classes
are quantum-mechanically consistent, a result which is used in Section 4.

2. Untwisted D-branes in $SU_3$

2.1. Classical analysis. We start with the simpler case where the gluing auto-
morphism $R$ is taken to be the identity. In this case the classical moduli space
$M_{cl} := M_{cl}(SU_3, 1)$ of D-branes is the space of conjugacy classes of $G = SU_3$. We
recall that a conjugacy class $C(g)$ of an element $g$ of $G$ is defined as the orbit of $g$
in $G$ under the adjoint action $\text{Ad}_k : g \mapsto kgk^{-1}$, for any $k$ in $G$. Since the stabiliser
of $g$ is given, in this case, by its centraliser $Z(g)$, the conjugacy class $C(g)$ can be
described as the homogeneous space

$$C(g) \cong G/Z(g).$$

(2)

We know that any point $g$ in the group manifold is conjugate, via the adjoint
action, to a point $h$ in a (fixed) maximal torus $T$, which is determined up to a Weyl
transformation in $W$. Furthermore every such element $h$ is the image, under the
exponential map, of an element $X$ in the Cartan subalgebra $t$ of $\text{su}(3)$, which is only
determined up to a translation in the integer lattice $\Lambda_I$. (Recall that the integer
lattice consists of the kernel of the exponential map.) Thus the space of conjugacy
classes in $SU_3$ is described by the quotient

$$M_{cl} = \frac{t}{W \ltimes \Lambda_I}.$$  

Moreover, since $SU_3$ is simply connected, its integral lattice $\Lambda_I$ agrees with its coroot
lattice $\Lambda_R^\vee$, and thus the semidirect product in the denominator is nothing but the
extended Weyl group $\tilde{W} = W \ltimes \Lambda_R^\vee$.

One can give a beautiful pictorial description of the space of conjugacy classes in $SU_3$
by using the Stiefel diagram (see, e.g., [4]), which is represented in Figure 1. This is a figure in the Cartan subalgebra $t$ describing the inverse image, under the
exponential map, of the singular points of a Lie group in its maximal torus. In our
case, this consists of 3 families of 1-dimensional hyperplanes: every positive root $\alpha_i$
in $\Phi_+ = \{\alpha_1, \alpha_2, \alpha_3 = \alpha_1 + \alpha_2\}$ gives rise to a family $\{L_{\alpha_i, n}\}_{n \in \mathbb{Z}}$ of affine lines in $t$, where each line $L_{\alpha_i, n}$ consists of the points $X$ in $t$ that satisfy

$$L_{\alpha_i, n} : \alpha_i(X) = n, \quad i = 1, 2, 3.$$

The set of regular points in $t$ decomposes in this way into convex connected compo-
nents, also known as the alcoves of the group. The group of isometries of $t$ generated
by the reflections on the lines $L_{\alpha_i, n}$ is nothing but the extended Weyl group of $SU_3$.
We therefore see that the space of conjugacy classes in $SU_3$ can be identified with the
fundamental domain of the extended Weyl group in $t$, which is given by the (solid)
equilateral triangle

$$M_{cl} = \{X \in t \mid 0 \leq \alpha_i(X) \leq 1, \quad i = 1, 2, 3\}. \quad (3)$$
Notice that the boundary of $\mathcal{M}_{cl}$ belongs to the singular lines $L_{\alpha_{i,n}}$; in particular, its vertices belong to the intersection of all three families of singular lines, and describe the central elements of SU$_3$.

The description of a conjugacy class as a homogeneous space given by (2) gives us a way to evaluate its dimension, as the codimension of $\mathcal{C}(h)$ is equal to the dimension of the corresponding centraliser $\mathcal{Z}(h)$. Clearly, if $h$ is regular—that is, if $h$ is contained in just one maximal torus—then the connected component of $\mathcal{Z}(h)$ agrees with the maximal torus and hence the dimension of $\mathcal{Z}(h)$ is equal to the rank of SU$_3$. On the other hand, if $h$ is singular that is, if $h$ is contained in more than one maximal torus, then the dimension of the centraliser will be strictly larger than the rank of the group. In general, we thus have

$$\dim \mathcal{C}(h) \leq \dim \text{SU}_3 - \text{rank SU}_3.$$ 

The interior points in $\mathcal{M}_{cl}$ are regular, and give rise to 6-dimensional conjugacy classes of the form SU$_3/\text{U}_2$. If we now consider an element $X$ in $\mathfrak{t}$ which belongs to one of the singular lines, say, $L_{\alpha_{i,n}}$, this describes a singular element $h = \exp(X)$ in $T$ whose centraliser includes the subgroup SU$_2^\alpha$. Thus the boundary points belonging to the three edges are singular, giving rise to 4-dimensional conjugacy classes of the form SU$_3/\text{S(U}_2 \times \text{U}_1)$. Finally, the three vertices corresponding to the three central elements of SU$_3$ describe point-like D-branes of the form SU$_3$/SU$_3$. We thus obtain the space of classical symmetric D-branes represented in Figure 2(a).

D-branes wrapping shifted conjugacy classes represent configurations that are indistinguishable, in their intrinsic properties, from their unshifted counterparts. A
brief inspection of the space $\mathcal{M}_{cl}$ reveals that one has a natural action of the centre of $SU_3$ on the space of conjugacy classes, such that for any element $z$ in the centre we can define a map

$$z : \mathcal{M}_{cl} \to \mathcal{M}_{cl}$$

$$\mathcal{C}(X) \mapsto \mathcal{C}(z \cdot X) = z\mathcal{C}(X),$$

which consists in shifting a given conjugacy class $\mathcal{C}(X)$ by the central element $z$. If we denote the centre by $\{e, \omega, \omega^2\}$, with $\omega^3 = e$, we see that the orbit $\mathcal{O}(X)$ of a point $X$ in $\mathcal{M}_{cl}$ under the above action of the centre consists (with one exception, the point in the centre of the equilateral triangle) of three points, $\mathcal{O}(X) = \{X, \omega \cdot X, \omega^2 \cdot X\}$, which determine an equilateral triangle inside $\mathcal{M}_{cl}$ and concentric to it. In particular, the three vertices of $\mathcal{M}_{cl}$ are one such orbit, $\mathcal{O}(0) = \{0, X_\omega, X_{\omega^2}\}$, where $\omega = \exp(X_\omega)$. One can think of this action, which shifts conjugacy classes, as defining a kind of equivalence relation on the space of D-branes; this will turn out to be instrumental in determining their charges (see Section 5).

2.2. Quantum analysis. The classical analysis of the possible D-brane configurations gives us a continuous family of conjugacy classes $[\mathfrak{g}]$, parametrised by the points of a solid polygon in $\mathfrak{t}$. In order to determine which of these configurations are consistent at the quantum level one has to analyse the quantisation conditions imposed by the requirement of single-valuedness of the path integral [8, 1, 9, 10]. This results in a number of quantisation conditions corresponding to evaluating the global worldsheet anomaly on the 3-cycles in $H_3(SU_3, \mathcal{C})$.

The data necessary for implementing this requirement is the relative 3-cocycle $(H, \omega)$ in $H^3(SU_3, \mathcal{C})$ and a basis of relative cycles in $H_3(SU_3, \mathcal{C})$. The relative form $(H, \omega)$ is well known: $H$ is the standard three-form field on the $SU_3$ and $\omega$ is the two-form field $[\mathfrak{g}^2] [\mathfrak{g}] [\mathfrak{g}]$ defined on the the D-brane which satisfies $d\omega = H|_{\mathfrak{e}}$. The other piece of data, the basis of relative 3-cycles, is provided by the $SU_2^\alpha$ subgroups of $SU_3$. In order to be more precise, let us fix a given conjugacy class $\mathcal{C}(h)$, with $h = \exp(X)$, for $X$ in $\mathfrak{t}$; then for every root $\alpha$ we can exhibit a relative cycle $(N_\alpha, \partial N_\alpha)$
in $H_3(SU_3, \mathcal{C})$, schematically represented in Figure 3. To this end we denote by $\{E_{\pm\alpha}, H_\alpha\}$ the standard basis of the corresponding $\mathfrak{su}(2)$. If we split $X = X^\perp + X_\alpha$, where $X_\alpha = \frac{1}{2}\alpha(X)H_\alpha$, we can write $h$ as a product, $h = h^\perp h_\alpha$, where $h_\alpha = \exp(X_\alpha)$ belongs to the $SU_2^\alpha$ subgroup, whereas $h^\perp = \exp(X^\perp)$ commutes with it. Using this, one can deduce that our conjugacy class $\mathcal{C}(h)$ intersects the coset $h^\perp SU_2^\alpha$ exactly in a conjugacy class of $SU_2^\alpha$, shifted in $SU_3$:

$$\mathcal{C}(h; SU_3) \cap h^\perp SU_2^\alpha = h^\perp \mathcal{C}(h_\alpha; SU_2), \quad (5)$$

where we have explicitly indicated to which group each of the two conjugacy classes belong.

This allows us to define the relative cycle $(N_\alpha, \partial N_\alpha)$ by taking $N_\alpha$ to be a $3$-submanifold of $h^\perp SU_2^\alpha$, whose boundary $\partial N_\alpha$ is given by the shifted conjugacy class above:

$$\partial N_\alpha = h^\perp \mathcal{C}(h_\alpha; SU_2). \quad (6)$$

Clearly, from (3) we have that $\partial N_\alpha \subset \mathcal{C}(X)$. Now it is easy to see that the global worldsheet anomaly evaluated for this particular relative 3-cycle of $SU_3$ is proportional to the one computed for the case of $SU_2$ and the above two-dimensional conjugacy class. Indeed, one obtains

$$\frac{1}{2\pi} \left( \int_{N_\alpha} H - \int_{\partial N_\alpha} \omega \right) = k\alpha(X), \quad (7)$$

since induced theory on $SU_2$ has level $k$ as well. Hence the requirement that the path integral be single-valued forces $k\alpha(X)$ to take integral values, for all roots of $SU_3$. Moreover, we have an additional quantisation condition which corresponds to the honest 3-cycle in $H_3(SU_3, \mathcal{C})$,

$$\frac{1}{2\pi} \int_{SU_2^\alpha} H = k,$$

which is nothing but the quantisation of the level, familiar from the standard WZW theory.
We have thus obtained the space \( \mathcal{M}_q := \mathcal{M}_q(SU_3, \mathbb{1}) \) of symmetric D-branes in \( SU_3 \) at level \( k \)

\[
\mathcal{M}_q = \{ X \in \mathfrak{h} \mid k\alpha_i(X) \in \mathbb{Z}, \ 0 \leq k\alpha_i(X) \leq k, \ i = 1, 2, 3 \}. \tag{8}
\]

Alternatively, we can use the isomorphism \( t \cong t^* \) to associate to every \( X \) in \( t \) an element \( \lambda \) in \( t^* \), in terms of which the space of D-brane configurations becomes \( \mathcal{M}_q = \{ \lambda \in \mathfrak{h}^* \mid \langle \alpha_i, \lambda \rangle \in \mathbb{Z}, \ 0 \leq \langle \alpha_i, \lambda \rangle \leq k, \ i = 1, 2, 3 \} \).

This shows that the set of consistent symmetric D-brane configurations in \( SU_3 \) at level \( k \) is in one-to-one correspondence with the set of IHW representations of the affine Lie algebra \( \hat{\mathfrak{su}}(3)^{(1)} \).

The space of symmetric D-brane configurations in \( SU_3 \) for the first few values of the level \( k \) is represented in Figure 4. At a given level \( k \) we have 3 point-like, \( 3(k - 1) \) 4-dimensional and \( \frac{1}{2}(k - 1)(k - 2) \) 6-dimensional symmetric D-branes. We also see that the 4-dimensional conjugacy classes are characterised by quantum numbers \( (\lambda_1, \lambda_2) \) with either one of the \( \lambda \)'s being equal to zero or \( \lambda_1 + \lambda_2 = k \); the point-like conjugacy classes are described by \( (0, 0), (0, k), (k, 0) \). In particular, the lower-dimensional conjugacy classes dominate the spectrum of D-branes for \( k \leq 9 \).

\[\text{Figure 4. Quantum moduli space for SU}_3 \text{ for lowest values of the level } k.\]

Finally, notice that the action of the centre on \( \mathcal{M}_d \) does induce a similar action on the discrete set of states \( \mathcal{M}_q \) which results in the various D-brane configurations in \( \mathcal{M}_q \) being organised into triplets, the three elements of a triplet differing from one another just by a translation by a central element in \( SU_3 \). From an algebraic point of view, this corresponds to the action of the extended Dynkin diagram on the IHW representations of the affine Lie algebra, as was recently pointed out in [6].

3. \( SO_3 \) versus \( SU_2 \)

We pause for a moment our analysis of D-branes in \( SU_3 \) in order to discuss the possible D-submanifolds in the non-simply connected group \( SO_3 \), which will prove instrumental in our discussion of twisted branes in the next section.

The groups \( SU_2 \) and its quotient \( SO_3 \) only admit standard conjugacy classes as they do not possess outer automorphisms. The moduli spaces of symmetric D-branes in \( SU_2 \) is well understood [7], and we briefly describe it here only for later comparison. The corresponding Stiefel diagram is represented in Figure 5. The classical moduli
space, represented in Figure 6, is given by the fundamental domain of the extended Weyl group in the Cartan subalgebra \( t \):

\[
\mathcal{M}_{cl}(SU_2) = \left\{ X \in t \mid 0 \leq \alpha(X) \leq 1 \right\},
\]

The two extremities correspond to the point-like D-branes, which are described as \( SU_2/SU_2 \), whereas the interior points describe regular elements in \( SU_2 \) which give rise to 2-dimensional D-branes of the form \( SU_2/U(1) \).

\[ \cdots L_{\alpha,-3} \cdots L_{\alpha,-2} \cdots L_{\alpha,-1} \cdots L_{\alpha,0} \cdots L_{\alpha,1} \cdots L_{\alpha,2} \cdots L_{\alpha,3} \cdots \]

**Figure 5.** Stiefel diagram of \( SU_2 \). Circles indicate the central lattice \( \Lambda_Z(SU_2) \) and filled circles indicate the integer lattice \( \Lambda_I(SU_2) \).

The requirement of single-valuedness of the path integral for the \( SU_2 \) WZW model imposes two quantisation conditions, corresponding to the two generating cycles of \( H_3(SU_2, \mathbb{C}) \), which are the \( SU_2 \) manifold itself and the relative 3-cycle \( (N_-, \partial N_-) \) with \( \partial N_- = \mathcal{C}_-(X) \) (see Figure 6). The first quantisation condition, corresponding to the honest 3-cycle, requires that the following

\[
\frac{1}{2\pi} \int_{SU_2} H_{SU_2} = k,
\]

take integer values, which essentially reiterates the quantisation of the level, \( k \in \mathbb{Z} \). The second quantisation condition, which imposes that

\[
\frac{1}{2\pi} \left( \int_{N_-} H_{SU_2} - \int_{\partial N_-} \omega_{SU_2} \right) = k\alpha(X),
\]

take integer values, selects a discrete set of conjugacy classes, which are labeled by the IHW representations of the affine Lie algebra \( \widehat{su}(2)_k \). Thus the space of consistent D-brane configurations is given by

\[
\mathcal{M}_{q}(SU_2) = \left\{ X \in t \mid k\alpha(X) \in \mathbb{Z} \; , \; 0 \leq k\alpha(X) \leq k \right\},
\]

\[ \begin{array}{ccc}
SU_2 & SU_2 & SU_2 \\
U_1 & SU_2 & SO_3 \\
SU_2 & SO_2 & O_2 \\
SO_3 & SO_2 & O_2 \\
\end{array} \]

**Figure 6.** Moduli space of conjugacy classes of \( SU_2 \) and \( SO_3 \).
The non-simply connected group $SO_3$ can be obtained from $SU_2$ by factoring out its centre: $SO_3 \cong SU_2/Z_2$. Let $Ad$ be the group homomorphism

$$Ad : SU_2 \to SO_3,$$

which projects the central subgroup $Z_2$ of $SU_2$ onto the identity element in $SO_3$. Using this projection, one can easily see that any conjugacy class $\mathcal{C}(h)$ in $SO_3$ is the image, under $Ad$, of a conjugacy class in the simply connected $SU_2$:

$$\mathcal{C}(h) = Ad(\mathcal{C}(Ad^{-1}(h))).$$

Notice that $Ad^{-1}(h)$ consists of two points in $SU_2$, which means that, generically, $Ad$ projects two conjugacy classes in $SU_2$ to one conjugacy class in $SO_3$. (The exception is provided by the ‘equatorial’ conjugacy class in $SU_2$.)

The groups $SU_2$ and $SO_3$ share the same root system, which implies that they also have the same Stiefel diagram and, in particular, the same fundamental domain of the extended Weyl group. Since the integral lattice of $SO_3$ is identical to its central lattice, in this case in the Stiefel diagram shown in Figure 5 all the circles are filled. Nevertheless, in the case of $SO_3$, the fundamental domain of the extended Weyl group can no longer be identified with the moduli space of symmetric D-branes; instead, the latter is obtained by a $Z_2$ quotient of the former

$$M_{cl}(SO_3) = \{ X \in t \mid 0 \leq \alpha(X) \leq \frac{1}{2} \}.$$ 

Here the point $\alpha(X) = 0$ describes a singular point (the identity in $SO_3$) and a point-like D-brane $SO_3/\mathbb{Z}_2$. The other extremity $\alpha(X) = \frac{1}{2}$ is a regular point with a non-connected centraliser, and the corresponding conjugacy class is a non-orientable submanifold $SO_3/O_2$. As before, the interior points are regular, and the corresponding symmetric D-branes wrap the homogeneous spaces $SO_3/\mathbb{Z}_2$. This is illustrated in Figure 6.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure7.png}
\caption{Relative 3-cycles in $SU_2$ and $SO_3$.}
\end{figure}

The quantisation conditions for $SO_3$ can be analysed using the $SU_2$ results. Let us start with the ‘generic’ case $\mathcal{C} \cong SO_3/\mathbb{Z}_2$. The requirement of single-valuedness of the path integral for the $SO_3$ WZW model imposes two quantisation conditions, corresponding to the two generating cycles of $H_3(SO_3, \mathcal{C}) \cong \mathbb{Z} \oplus \mathbb{Z}$. These relative cycles can be obtained from the $SU_2$ relative cycles, via the projection homomorphism.
Ad, as is schematically indicated in Figure 7. Indeed, the generic relative cycle in SU$_2$, given by $(N_-, \partial N_-)$, can be mapped into the relative cycle $(N, \partial N)$ of SO$_3$, where Ad$(N_-) = N$ and Ad$(\mathcal{C}_-) = \mathcal{C}$. By contrast, since SU$_2$ covers SO$_3$ twice, we obtain Ad$_*[^*]{SU}_2 = 2[SO_3]$ in homology.

The first quantisation condition, corresponding to the honest 3-cycle, requires that the following be integer valued:

$$\frac{1}{2\pi} \int_{SO_3} H_{SO_3} = \frac{1}{4\pi} \int_{SU_2} H_{SU_2} = \frac{k}{2},$$

where we used Ad$^* H_{SO_3} = H_{SU_2}$. This recovers the well known fact that the level in the SO$_3$ theory has to be even, so that $k \in 2\mathbb{Z}$. On the other hand, a similar argument, employing Ad$(N_-, \partial N_-) = (N, \partial N)$ and Ad$^*(H_{SO_3}, \omega_{SO_3}) = (H_{SU_2}, \omega_{SU_2})$, shows that the second quantisation condition for the SO$_3$ theory is identical to the corresponding one in the case of SU$_2$

$$\frac{1}{2\pi} \left( \int_N H_{SO_3} - \int_{\partial N} \omega_{SO_3} \right) = k\alpha(X).$$

This selects a discrete set of consistent D-brane configurations which can also be labeled by the IHW representations of the affine Lie algebra $\hat{su}(2)_k^{(1)}$.

For the two conjugacy classes at the boundary of the moduli space that is, the point-like SO$_3$/SO$_3$ and the non-orientable SO$_3$/O$_2$, we have $H_3(SO_3, \mathcal{C}) \cong \mathbb{Z}$, thus the only quantisation condition is the one corresponding to the honest 3-cycle, which we know to be satisfied.

Therefore, by contrast with SU$_2$, in this case we only get $\frac{k}{2} + 1$ admissible D-submanifolds at a given level $k$, these being described by

$$\mathcal{M}_q(SO_3) = \left\{ X \in t \mid k\alpha(X) \in \mathbb{Z}, 0 \leq k\alpha(X) \leq \frac{k}{2} \right\},$$

which is exemplified for the first few levels in Figure 8.

![Figure 8. Quantum moduli spaces for SO$_3$ (black circles) and SU$_2$ for lowest values of the level $k$.](image)

4. Twisted D-branes in SU$_3$

The group of (metric-preserving) outer automorphisms can be defined as the factor group

$$\text{Out}_o(G) = \text{Aut}_o(G)/\text{Inn}_o(G),$$
where Aut$_o(G)$ denotes the group of metric-preserving automorphisms of $G$ and Inn$_o(G) \subset$ Aut$_o(G)$ denotes the invariant subgroup corresponding to inner automorphisms.$^1$ In the particular case of SU$_3$ this group contains only one non-trivial element. One particular representative for this outer automorphism is given by complex conjugation. For the purpose of our analysis we will however take the outer automorphism to be the Dynkin diagram automorphism $\tau$ illustrated in Figure 9, which acts on the simple roots of SU$_3$ by interchanging $\alpha_1$ and $\alpha_2$. This type of automorphisms have been studied in some detail in the mathematics literature, since they are central to the construction of twisted affine Lie algebras (see, for instance, [13]). Notice that $\tau$ has order two, that is, $\tau^2 = 1$.

![Figure 9. Dynkin diagram automorphism of SU$_3$](image)

The diagram automorphism $\tau$ gives rise to a ‘folded’ root system, denoted by $\Phi^\tau$, which is generated by the linear combinations of the roots of $\mathfrak{su}(3)$ that are invariant under $\tau$:

$$\bar{\alpha} = \begin{cases} 
\alpha, & \text{for } \tau(\alpha) = \alpha \\
\frac{1}{2}(\alpha + \tau(\alpha)), & \text{for } \tau(\alpha) \neq \alpha
\end{cases} \quad (9)$$

We thus have that the set of positive roots is given by $\Phi^\tau_+ = \{(\alpha_1 + \alpha_2)/2, \alpha_1 + \alpha_2\}$. Notice that the folded root system $\Phi^\tau$ is not a subset of the original root system $\Phi$. In fact, in this case, $\Phi^\tau = BC_1$ is not even the root system of a Lie algebra.

This automorphism, defined at the level of $\Delta$, can be extended by linearity to $\mathfrak{t}^*$, and further, by duality, to the Cartan subalgebra, in such a way that for any $X$ in $\mathfrak{t}$

$$\tau(\alpha)(\tau(X)) = \alpha(X).$$

If we then define

$$\tau(E_\alpha) = E_{\tau(\alpha)}, \quad \alpha \in \Delta,$$

one extends $\tau$ to a Lie algebra automorphism. Furthermore, since SU$_3$ is compact, connected, and simply connected, we can lift $\tau$ to a Lie group automorphism which, by a small abuse of notation, we will also denote by $\tau$. It is known that there exists a maximal torus $T$ which is left invariant by $\tau$. If we denote by SU$_3^\tau$ the fixed point subgroup of SU$_3$ under $\tau$, and by $T^\tau$ the fixed point set of $T$, we clearly have that $T^\tau$ is the maximal torus of SU$_3^\tau$ and $\tau|_{SU_3^\tau} = 1$. Let $\mathfrak{su}(3)^\tau$ and $\mathfrak{t}^\tau$ be the Lie algebras

$^1$It was suggested in [12] that this group plays a role analogous to that of the T-duality group in toroidal compactifications.
of $\text{SU}_3^\tau$ and $\mathfrak{t}^\tau$ respectively; these are nothing but the fixed point sets of $\tau$ in $\mathfrak{su}(3)$ and $\mathfrak{t}$, respectively. The root system of $\text{SU}_3^\tau$ has one generator $\bar{\alpha}$ which is given by

$$\bar{\alpha} = \frac{\alpha_1 + \alpha_2}{2}. \quad (10)$$

One can determine the fixed point subgroup $\text{SU}_3^\tau$ in a variety of ways (see, for instance, [14]) and obtain

$$\text{SU}_3^\tau \cong \text{SO}_3,$$

with the simple root $\bar{\alpha}$ given by (11). The coroot lattice $\Lambda_R^\vee$ of $\text{SU}_3^\tau$ acts on $\mathfrak{t}^\tau$ by translations, and is generated by

$$\bar{\alpha}^* = 2(\alpha_1^* + \alpha_2^*). \quad (11)$$

4.1. Twisted D-branes. The twisted conjugacy class $\mathcal{C}_\tau(g)$ of a group element $g$ can be defined as the orbit of $g$ under the twisted conjugation $\text{Ad}_{\tau,k} : g \mapsto \tau(k)gk^{-1}$, for any $k$ in the group manifold. Similarly to the untwisted case, this takes the form of a homogeneous space

$$\mathcal{C}_\tau(g) \cong G/\mathcal{Z}_\tau(g),$$

where $\mathcal{Z}_\tau(g)$ denotes the twisted centraliser of $g$ defined as the subgroup

$$\mathcal{Z}_\tau(g) = \{k \in G \mid \tau(k)g = gk\}.$$

Let $\Gamma = \{1, \tau\}$ be the finite group generated by the diagram automorphism and let

$$\mathcal{G} = \Gamma \ltimes \text{SU}_3$$

be the principal extension of $\text{SU}_3$ by $\tau$. It can be characterised as the smallest Lie group containing $\text{SU}_3$ for which $\tau$ is an inner automorphism. It is the (disjoint) union of two connected components, which we write symbolically as $\mathcal{G} = \text{SU}_3 + \tau\text{SU}_3$, with $\text{SU}_3$ being the connected component of the identity. $\text{SU}_3$ acts on $\mathcal{G}$ by conjugation, and since $\text{SU}_3$ is connected, this action stabilises each of the connected components of $\mathcal{G}$. On $\text{SU}_3$ it is just the standard conjugation of $\text{SU}_3$ on itself; but on $\tau\text{SU}_3$ it is the twisted conjugation defined above (provided that $\tau^2 = 1$). It then follows that the conjugacy class $O(\tau g)$ of an element $\tau g$ under $\text{SU}_3$ is nothing but the twisted conjugacy class shifted by $\tau$; that is,

$$O(\tau g) = \tau \mathcal{C}_\tau(g). \quad (12)$$

This relation is clearly valid for any Lie group $G$, provided $\tau^2 = 1$; the connection between adjoint orbits in $G\tau$ and twisted adjoint orbits in $G$ allows us to use the theory of nonconnected Lie groups to to give a detailed description of the space of twisted conjugacy classes.

One of the key notions which we will need to use in the next paragraph is that of $\tau$-regularity, which can be thought of as a translation, at the level of $G$, of the notion

\footnote{Alternatively, one can choose to work with a different representative $\rho$ of the outer automorphism of $\text{SU}_3$ which is given by complex conjugation, $\rho(g) = \bar{g}$. The resulting fixed point subgroup $\text{SU}_3^\rho = \text{SO}_4$ will then be conjugate and hence isomorphic to $\text{SU}_3^\tau$.}
of regularity in $\tau G$. An element $\tau g$ is said to be regular in $\tau G$ if it belongs to only one Cartan subgroup $S$ of $G$. Alternatively, we can say that $\tau g$ is regular in $\tau G$ if the connected component of the identity of its centraliser $Z(\tau g)$ is abelian. Using the relation between $Z(\tau g)$ and $Z_\tau(g)$, we say that an element $g$ in $G$ is $\tau$-regular if the connected component of its twisted centraliser is abelian. Otherwise, $g$ is said to be $\tau$-singular in $G$.

4.2. Classical analysis. Using the theory on non-connected Lie groups, one can show \cite{13,14,4} that twisted conjugacy classes in a group $G$ are parametrised by the quotient

$$M_{cl}(G, \tau) := T(G^\tau)/W_\tau(G),$$

where the ‘twisted’ Weyl group $W_\tau(G)$ is given \cite{14} by the semidirect product

$$W_\tau(G) = W(G^\tau) \ltimes \Lambda_T,$$

of the Weyl group of $G^\tau$ with a certain discrete group, $\Lambda_T = (T/T^\tau)^\tau$, which acts on $t^\tau$ by translations. The generators $\alpha_T$ of this translation group stem from those coroots $\alpha^*$ of $G$ for which $\tau(\alpha^*) \neq \alpha^*$, and are given by $\tau$-invariant linear combinations of these:

$$\alpha^*_T = \frac{\alpha^* + \tau(\alpha^*)}{2}.$$

The space of twisted conjugacy classes in $SU_3$ is therefore described by the quotient

$$M_{cl}(SU_3, \tau) = \frac{t^\tau}{W(SO_3) \times (\Lambda_I(SO_3) \times \Lambda_T)}.$$

Notice that, in the absence of $\Lambda_T$, the above quotient would be nothing but the space of standard conjugacy classes in the fixed point subgroup $SU_3^\tau \cong SO_3$. In order to fully understand the structure of this space and thus the spectrum of possible twisted D-branes in $SU_3$, we need to understand better the twisted Weyl group $W_\tau(SU_3)$. This is generated by the Weyl reflection

$$s_\alpha : X \mapsto -X,$$

and by the translation in $\Lambda_T$

$$\gamma_{\alpha_T} : X \mapsto X + \alpha^*_T, \quad \alpha^*_T = \frac{1}{4} \tilde{\alpha}^*.$$

To these we have to add of course the translations corresponding to the integral lattice of $SO_3$.

What is the effect of the ‘special’ twisted Weyl transformations in $W_\tau(G)$ which are generated by the translations in $\Lambda_T$? It turns out that these translations result in some of the regular points in $T^\tau$ becoming singular with respect to the twisted adjoint action. In other words, the effect of the translations in $\Lambda_T$ amounts to having an additional number of singular lines in $t^\tau$. These new singular lines can be thought of as being generated by the element in $(t^\tau)^* \text{dual to } \alpha^*_T$:

$$\alpha_T = 2(\alpha + \tau(\alpha)).$$
We can thus construct a figure that could be called the ‘twisted Stiefel diagram’ of $\text{SU}_3$, which is a diagram in $\mathfrak{t}^\tau$ defined as the inverse image, under the exponential map, of the $\tau$-singular points in $T^\tau$. This consists of two types of 0-dimensional singular hyperplanes:

\[ L_{\vec{a},n} : \, \vec{a}(X) = n , \]
\[ S_{\alpha_T,n} : \, \alpha_T(X) = n , \quad \text{with} \quad \alpha_T = 4\vec{a} . \]

The ‘ordinary’ singular hyperplanes $\{L_{\vec{a},n}\}_{n \in \mathbb{Z}}$ do not require much explanation. They are generated by the $\text{SO}_3$ root $\vec{a}$, and describe the singular points in $T^\tau$, which give rise to lower-dimensional standard conjugacy classes in $\text{SO}_3$. By contrast with the standard case, we have an additional family $\{S_{\alpha_T,n}\}_{n \in \mathbb{Z}}$ of singular points, which describe the elements in $T^\tau$ which are left invariant by a twisted Weyl transformation. Each of these singular points $X$ in $\mathfrak{t}^\tau$, although might be regular in $\text{SO}_3$, turn out to be $\tau$-singular in $\text{SU}_3$, giving rise to lower-dimensional twisted conjugacy classes.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure10}
\caption{‘Twisted’ Stiefel diagram of $\text{SU}_3$. Circles indicate $\tau$-singular points and filled circles indicate singular points in the maximal torus of $\text{SU}_3^\tau$.}
\end{figure}

If the inverse image of $h$, under the exponential map, does not belong to one of the special singular hyperplanes, then we have

\[ T_h \mathcal{Z}_\tau(h, \text{SU}_3) = T_h \mathcal{Z}(h, \text{SO}_3) , \]

from which we immediately deduce that

\[ \text{codim}_{\text{SU}_3}\mathcal{C}_\tau(h) = \text{codim}_{\text{SO}_3}\mathcal{C}(h) . \quad (14) \]

Let us now show that the points belonging to $S_{\alpha_T,n}$ are indeed $\tau$-singular. Notice first of all that $S_{\alpha_T,4n} = L_{\vec{a},n}$, and the corresponding points are singular already in $\text{SO}_3$. Furthermore, every point on $S_{\alpha_T,4n+2}$ or $S_{\alpha_T,2n+1}$ is related, via a twisted Weyl transformation, to a point on $S_{\alpha_T,2}$ or $S_{\alpha_T,1}$, respectively. Hence it is sufficient to discuss these latter two cases.

Let us start with the point $X$ on $S_{\alpha_T,2}$. The corresponding group element $h$ is regular in $\text{SO}_3$, since $X$ lies inside an alcove of $\text{SO}_3$; however, since $X$ is conjugate, via a twisted Weyl transformation (a $\Lambda_T$ translation, to be more precise), to $X = 0$, which is the point on $S_{\alpha_T,0} = L_{\vec{a},0}$, we deduce that $h$ is $\tau$-singular in $\text{SU}_3$, as its twisted centraliser is $\text{SO}_3$.

Finally, let us consider the case where $X$ belongs to $S_{\alpha_T,1}$; this means that $X$ satisfies $2(\alpha_1 + \alpha_2)(X) = 1$; it also means that $X$ is regular in $\text{SO}_3$. In this case we claim that the corresponding twisted centraliser is nothing but $\text{SU}_2^{\alpha_1+\alpha_2}$. If we denote by $\{H_{\alpha_1+\alpha_2}, E_{\pm(\alpha_1+\alpha_2)}\}$ the standard basis of $\mathfrak{su}(2)_{\alpha_1+\alpha_2}$, the condition that
$E_{\alpha_1+\alpha_2}$ belong to $T_\epsilon Z_\tau(h;SU_3)$, in other words, that it describe a vector normal to the D-brane is given by

$$\text{Ad}_h E_{\alpha_1+\alpha_2} = \tau(E_{\alpha_1+\alpha_2}) .$$

This can be verified using the fact that $\tau(E_{\alpha_1+\alpha_2}) = -E_{\alpha_1+\alpha_2}$. A similar argument can be carried out for the other basis elements of $\text{su}(2)_{\alpha_1+\alpha_2}$.

We thus obtain that the classical moduli space of twisted D-branes in $SU_3$ is given by roughly a 'one half fraction' of the moduli space of symmetric D-branes in $SO_3$. More precisely,

$$\mathcal{M}_{cl}(SU_3, \tau) = \left\{ X \in \mathfrak{t} \mid 0 \leq \bar{\alpha}(X) \leq \frac{1}{4} \right\} .$$

The point $\bar{\alpha}(X) = 0$ is singular in $SO_3$ and $\tau$-singular in $SU_3$, giving rise to a 5-dimensional twisted D-brane of the form $SU_3/SO_3$. The other endpoint $4\bar{\alpha}(X) = 1$ is regular in $SO_3$, but $\tau$-singular in $SU_3$. The corresponding twisted class is also 5-dimensional, but has the form $SU_3/SU_2$. Finally, the interior points are regular and give rise to 7-dimensional twisted conjugacy classes of the form $SU_3/SO_2$. The resulting space of classical twisted D-branes in $SU_3$ is described in Figure 11. Notice that in this case we have that the dimension of these twisted conjugacy classes is always odd, due to the fact that the difference between the ranks of $SU_3$ and $SO_3$ is odd.

Notice also that in this case, by contrast with the case of standard conjugacy classes described in Section 2, the action of the centre of $SU_3$ on the space of twisted conjugacy classes is trivial since we have

$$\mathcal{C}_\tau(z \cdot X) = \mathcal{C}_\tau(X) .$$

![Figure 11. Moduli space of twisted conjugacy classes of SU_3](image)

Let us also mention that in the case of $SU_3$ one can determine the twisted conjugacy classes by direct calculation. If we can think of $SU_3$ as the group of special unitary $3 \times 3$ matrices, it is convenient to use a different representative for the outer automorphism, which is given by complex conjugation, that is, $\rho(g) = \bar{g}$, for any $g$ in $SU_3$. Clearly, $\tau$ and $\rho$ are related by an inner automorphism, which is equivalent, at the level of the group, to a change of coordinates. The fixed point subgroup in this case is the $SO_3$ subgroup of $SU_3$ given by the real valued matrices $g$. Its maximal
\[
C_\rho(h_\theta) = C_\rho(h_{\pi/2-\theta}) = C_\rho(h_{\theta+\pi/2}) = C_\rho(h_{-\theta})
\]
which is a manifestation of the action of the twisted Weyl group. This implies that the space of twisted conjugacy classes is parametrised by \( \theta \in [0, \pi/4] \), with \( C_\rho(h_\theta) \cong SU_3/SO_3 \), \( C_\rho(h_\pi) \cong SU_3/SO_2 \) for \( 0 < \theta < \pi/4 \), and \( C_\rho(h_{\pi/2}) \cong SU_3/SU_2 \). Also, one can explicitly check in this case that the elements \( h_\theta \), for \( \theta = \pi/4 \), and \( \theta = \pi/2 \), despite being regular in \( SO_3 \), possess nontrivial twisted centralisers, and give rise to lower-dimensional twisted D-branes.

**Table 1.** Comparison between conjugacy classes in \( SU_2 \), \( SO_3 \) and twisted conjugacy classes in \( SU_3 \). (Here, \( SO_3 = SU_3^{\tau} \) and \( SU_2 \) is its double cover.)

| \( \alpha(X) \) | \( SU_2 \) | \( SO_3 \) | \( SU_3, \tau \) |
|------------------|-----------|-----------|------------------|
| \( X \) | \( C(X) \) | \( X \) | \( C(X) \) | \( X \) | \( C_{\tau}(X) \) |
| 0 | sing , \( SU_2^{\frac{\tau}{2}} \) | sing , \( SO_3^{\frac{\tau}{2}} \) | \( \tau \)-sing , \( SU_3^{\frac{\tau}{2}} \) |
| | reg , \( SU_2^{\frac{\tau}{2}} \) | reg , \( SO_3^{\frac{\tau}{2}} \) | \( \tau \)-reg , \( SU_3^{\frac{\tau}{2}} \) |
| \( \frac{1}{4} \) | reg , \( SU_2^{\frac{\tau}{2}} \) | reg , \( SO_3^{\frac{\tau}{2}} \) | \( \tau \)-sing , \( SU_3^{\frac{\tau}{2}} \) |
| | reg , \( SU_2^{\frac{\tau}{2}} \) | reg , \( SO_3^{\frac{\tau}{2}} \) | \( \tau \)-reg , \( SU_3^{\frac{\tau}{2}} \) |
| \( \frac{1}{2} \) | reg , \( SU_2^{\frac{\tau}{2}} \) | reg , \( SO_3^{\frac{\tau}{2}} \) | \( C_{\tau}(X) = C_{\tau}(\frac{\alpha^*}{4} - X) \) |
| | reg , \( SU_2^{\frac{\tau}{2}} \) | \( C(X) = C(\frac{\alpha^*}{2} - X) \) | \( C_{\tau}(X) = C_{\tau}(X + \frac{\alpha^*}{4}) \) |
| \( \frac{3}{4} \) | reg , \( SU_2^{\frac{\tau}{2}} \) | \( C(X) = C(\frac{\alpha^*}{2} - X) \) | \( C_{\tau}(X) = C_{\tau}(X + \frac{\alpha^*}{4}) \) |
| | reg , \( SU_2^{\frac{\tau}{2}} \) | \( C(X) = C(\alpha^* - X) \) | \( C_{\tau}(X) = C_{\tau}(X + \frac{\alpha^*}{4}) \) |
| 1 | sing , \( SU_2^{\frac{\tau}{2}} \) | \( C(X) = C(\alpha^* - X) \) | \( C_{\tau}(X) = C_{\tau}(X + \frac{\alpha^*}{4}) \) |

Torus can be parametrised as follows:

\[
T(SO_3) = \left\{ h_\theta = \begin{pmatrix} \cos 2\theta & \sin 2\theta & 0 \\ -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \quad \theta \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]
\]
We find it instructive to compare, in Table 1, the moduli spaces of conjugacy classes in SU_2 and SO_3 with the moduli space of twisted conjugacy classes in SU_3, as all three of them can be described in terms of the same parameter. This example illustrates the fact that the notion of regularity in $G^\tau$ is not appropriate for describing the space of twisted conjugacy classes; instead, the relevant notion is that of $\tau$-regularity. It is also important to remark that, although twisted conjugacy classes are, roughly speaking, parametrised by $T_0^\tau$, regular elements in $T_0^\tau$ do not necessarily give rise to top-dimensional twisted classes of the form $G/T_0^\tau$, as previously stated in the literature [2].

4.3. Quantum analysis. The classical analysis of the previous paragraph gave us the space (15) of twisted conjugacy classes in SU_3, which forms a continuous family parametrised by the points of an interval in $\tau^\tau$. In order to determine the twisted D-brane configurations that are consistent at the quantum level, we must analyse the quantisation conditions imposed by the requirement of single-valuedness of the path integral. In this case, the global worldsheet anomaly is measured by the periods of $(H, \omega)/2\pi$ over the relative cycles in $H_3(SU_3, \mathcal{C}_\tau)$, where $H$ is the same three-form field of the SU_3 WZW model, whereas the two-form field $\omega$ defined on the D-brane is the one determined in [11] for a general automorphism.

Our strategy for analysing the quantisation conditions in this case will be to try and make use of the knowledge we have acquired about the consistent D-branes in SO_3. More precisely, we will show how the quantisation conditions satisfied by a D-brane in SU_3 wrapping a twisted conjugacy class $\mathcal{C}_\tau(h)$, where $h = \exp(X)$, for some $X$ in $\mathcal{C}_\tau$, can be evaluated in terms of the quantisation conditions for D-branes wrapping standard conjugacy classes in the fixed point subgroup SO_3. To this end, let us determine the intersection of the twisted conjugacy class $\mathcal{C}_\tau(h)$ with this SO_3.

We first of all notice that the standard conjugacy class $\mathcal{C}(h)$ in SO_3, corresponding to the same element $h$ in the maximal torus of SO_3, is a submanifold of $\mathcal{C}_\tau(h)$, since $\tau$ restricts to the identity on SO_3:

$$\{\tau(g)hg^{-1} \mid g \in SO_3\} = \mathcal{C}(h; SO_3).$$

Contrary to appearance, this does not fully account for the intersection we are trying to determine. In order to see this, we recall that $X$ and $\alpha^\tau - X$ are related by a twisted Weyl transformation, which implies that

$$\mathcal{C}_\tau(X) = \mathcal{C}_\tau(\alpha^\tau - X).$$

Since, on the other hand, $X$ and $\alpha^\tau - X$ are not related by a Weyl transformation, this indicates that our twisted conjugacy class $\mathcal{C}_\tau$ intersects SO_3 in two generically distinct conjugacy classes of SO_3:

$$\mathcal{C}_\tau(X; SU_3) \cap SO_3 = \mathcal{C}(X; SO_3) \cup \mathcal{C}(\alpha^\tau - X; SO_3).$$

Depending on $X$, we have three cases, which are schematically drawn in Figure 12. For $X$ such that $\bar{\alpha}(X) = 0$, the intersection of the 5-dimensional $\mathcal{C}_\tau$ with SO_3 consists in the point-like conjugacy class of the identity $\mathcal{C}(e)$ and the non-orientable 2-dimensional conjugacy class $\mathcal{C} \cong SO_3/O_2$. For generic $X$, the intersection consists
of two spherical conjugacy classes in $\text{SO}_3$, denoted for simplicity by $C$ and $C'$ corresponding to $X$ and $\alpha_\tau^* - X$, respectively. Finally, for $\bar{\alpha}(X) = 1/4$, $C_\tau$ intersects $\text{SO}_3$ in only one conjugacy class $C \cong \text{SO}_3/\text{SO}_2$.

The fact that $C_\tau$ intersects the fixed point subgroup $\text{SO}_3$ in what turn out to be conjugacy classes of $\text{SO}_3$ prompts us to investigate the possibility of describing the relative cycles of $H_3(\text{SU}_3, C_\tau)$ in terms of those of $H_3(\text{SO}_3, C)$, which we already know. Indeed, in the previous section we have seen that for the generic $C \cong \text{SO}_3/\text{SO}_2$, $H_3(\text{SO}_3, C) \cong \mathbb{Z} \oplus \mathbb{Z}$ and the generating 3-cycles are $\text{SO}_3$ itself and the relative cycle denoted by $(N, \partial N)$, where $N$ is given by a 3-submanifold in $\text{SO}_3$ such that $\partial N = C$. We claim that these two 3-cycles can be promoted to relative cycles in $H_3(\text{SU}_3, C_\tau) \cong \mathbb{Z} \oplus \mathbb{Z}$. In order to show this, we notice that $\partial N \subset C_\tau$; furthermore one has to check that $H$ and $(H, \omega)$ do not vanish when integrated on $i(\text{SO}_3)$ and $(i(N), \partial i(N))$, respectively, where $i : \text{SO}_3 \rightarrow \text{SU}_3$ denotes the embedding of $\text{SO}_3$ in $\text{SU}_3$. This is indeed true, as we will see in a moment.

The fact that every relative cycle in $H_3(\text{SO}_3, C)$ gives rise to one in $H_3(\text{SU}_3, C_\tau)$ implies that a necessary condition for a twisted brane to wrap $C_\tau$ is that the global worldsheet anomaly evaluated on the ‘induced’ relative cycle should take integer values. In other words, the following quantisation conditions must be be satisfied:

$$\frac{1}{2\pi} \int_{i(\text{SO}_3)} H \in \mathbb{Z},$$

for the honest 3-cycle, and

$$\frac{1}{2\pi} \left( \int_{i(N)} H - \int_{\partial i(N)} \omega \right) \in \mathbb{Z},$$

for the second relative cycle. A brief inspection of these conditions shows that they are nothing but the quantisation conditions for the admissible brane configurations.

Figure 12. Twisted conjugacy classes in $\text{SU}_3$ and their intersections with the fixed point subgroup $\text{SO}_3$: (a) $C_\tau \cong \text{SU}_3/\text{SO}_3$, (b) $C_\tau \cong \text{SU}_3/\text{SO}_2$, (c) $C_\tau \cong \text{SU}_3/\text{SU}_2$. 

The fact that $C_\tau$ intersects the fixed point subgroup $\text{SO}_3$ in what turn out to be conjugacy classes of $\text{SO}_3$ prompts us to investigate the possibility of describing the relative cycles of $H_3(\text{SU}_3, C_\tau)$ in terms of those of $H_3(\text{SO}_3, C)$, which we already know. Indeed, in the previous section we have seen that for the generic $C \cong \text{SO}_3/\text{SO}_2$, $H_3(\text{SO}_3, C) \cong \mathbb{Z} \oplus \mathbb{Z}$ and the generating 3-cycles are $\text{SO}_3$ itself and the relative cycle denoted by $(N, \partial N)$, where $N$ is given by a 3-submanifold in $\text{SO}_3$ such that $\partial N = C$. We claim that these two 3-cycles can be promoted to relative cycles in $H_3(\text{SU}_3, C_\tau) \cong \mathbb{Z} \oplus \mathbb{Z}$. In order to show this, we notice that $\partial N \subset C_\tau$; furthermore one has to check that $H$ and $(H, \omega)$ do not vanish when integrated on $i(\text{SO}_3)$ and $(i(N), \partial i(N))$, respectively, where $i : \text{SO}_3 \rightarrow \text{SU}_3$ denotes the embedding of $\text{SO}_3$ in $\text{SU}_3$. This is indeed true, as we will see in a moment.

The fact that every relative cycle in $H_3(\text{SO}_3, C)$ gives rise to one in $H_3(\text{SU}_3, C_\tau)$ implies that a necessary condition for a twisted brane to wrap $C_\tau$ is that the global worldsheet anomaly evaluated on the ‘induced’ relative cycle should take integer values. In other words, the following quantisation conditions must be be satisfied:

$$\frac{1}{2\pi} \int_{i(\text{SO}_3)} H \in \mathbb{Z},$$

for the honest 3-cycle, and

$$\frac{1}{2\pi} \left( \int_{i(N)} H - \int_{\partial i(N)} \omega \right) \in \mathbb{Z},$$

for the second relative cycle. A brief inspection of these conditions shows that they are nothing but the quantisation conditions for the admissible brane configurations.
in the fixed point subgroup WZW theory, that is, in $SO_3$, whose fields are given by $H_{SO_3} = i^* H$ and $\omega_{SO_3} = i^* \omega$.

In order to evaluate the first quantisation condition we use the fact that the ‘induced’ $SO_3$ WZW model on the fixed point group has a level $k_{SO_3}$ given by

$$k_{SO_3} = k \frac{\langle \alpha_3, \alpha_3 \rangle}{\langle \bar{\alpha}, \bar{\alpha} \rangle} = 4k,$$

where, we recall, $\alpha_3 = \alpha_1 + \alpha_2$ is the maximal root of $SU_3$. We thus deduce that

$$\frac{1}{2\pi} \int_{i(SO_3)} H = \frac{1}{2\pi} \int_{SO_3} H_{SO_3} = 2k,$$

where $H_{SO_3} = i^* H$ is the three-from field of the ‘induced’ WZW model on $SO_3$. In light of this result, the quantisation condition corresponding to the honest 3-cycle reiterates the condition that the level be an integer, $k \in \mathbb{Z}$. The second quantisation condition can be evaluated using a similar line of argument and the calculation of the global worldsheet anomaly for the $SO_3$ theory performed in the previous section, thus obtaining

$$\frac{1}{2\pi} \int_{i(N)} H - \int_{\partial i(N)} \omega = \frac{1}{2\pi} \left( \int_{N} H_{SO_3} - \int_{\partial N} \omega_{SO_3} \right) = 4k\bar{\alpha}(X).$$

Hence the consistent D-branes wrapping twisted conjugacy classes in $SU_3$ are characterised by integral values of $4k\bar{\alpha}(X)$.

Knowing that the relative cycles of the fixed point subgroup can be promoted to relative cycles in $SU_3$ is however not enough if we want to accurately describe the quantisation conditions for the twisted branes. What we need is to be able to exhibit a basis of generators for the relative homology group $H_3(SU_3, \mathbb{C}_\tau)$, for a fixed $\mathbb{C}_\tau$, and evaluate the global worldsheet anomaly on these generators. It turns out that the most convenient way of solving this problem is to resort to a different representative for the outer automorphism of $SU_3$, namely the one provided by complex conjugation $\rho$. This is done in Appendix B, where the generators of $H_3(SU_3, \mathbb{C}_\rho)$ are determined and the corresponding quantisation conditions are derived. Using the fact that $\tau$ and $\rho$ are related by an inner automorphism, we deduce that the correct quantisation condition for $\mathbb{C}_\tau(X)$ reads

$$2k\bar{\alpha}(X) = 2m - k, \quad m \in \mathbb{Z}.$$  \hspace{1cm} (20)

Let us now consider separately the quantisation conditions corresponding to the two ‘singular’ twisted D-branes. Let us begin with the twisted class of the identity, $\mathbb{C}_\tau \cong SU_3/SO_3$. The relevant relative homology group in this case is $H_3(SU_3, \mathbb{C}_\tau) \cong \mathbb{Z}$, where the $SO_3$ subgroup is four times the generator. We thus obtain the following quantisation condition

$$\frac{1}{2\pi} \int_{i(SO_3)} H \in 4\mathbb{Z},$$

\[3\text{Alternatively, as done in Appendix } \Box \text{, one can show that } SO_3 \text{ is twice the generator of } H_3(SU_3), \text{ which leads us to a similar conclusion regarding the level of the } SO_3 \text{ theory.} \]
which imposes that $k \in 2\mathbb{Z}$. In other words, this particular state only appears as a consistent configuration for even values of the level. Finally, for the other 5-dimensional twisted brane wrapping $\mathcal{C}_T \cong SU_3/SU_2$, we have $H_3(SU_3, \mathcal{C}_T) \cong \mathbb{Z}$, with the $SO_3$ subgroup being twice the generator, so that the quantisation condition reads

\[
\frac{1}{2\pi} \int_{i(SO_3)} H \in 2\mathbb{Z},
\]

imposing that the level $k$ be an integer.

We thus obtain that the space of twisted D-branes in $SU_3$ is given by

\[
\mathcal{M}_q(SU_3, \tau) = \begin{cases} 
\{ X \in t^\tau \mid 4k\alpha(X) = 1, 3, \ldots, k \}, & \text{for } k \text{ odd,} \\
\{ X \in t^\tau \mid 4k\alpha(X) = 0, 2, \ldots, k \}, & \text{for } k \text{ even,}
\end{cases}
\]

and the states corresponding to the first few values of the level are represented in Figure 13. At a given odd level $k$ we have $\frac{1}{2}(k-1)$ 7-dimensional and one 5-dimensional branes, whereas for an even level $k$ we have $(\frac{1}{2}k - 1)$ 7-dimensional and two 5-dimensional branes.

![Figure 13. Quantum moduli space for $\tau$-twisted D-branes in $SU_3$ at level $k$ (black circles) compared with that of $SU_3 \cong SO_3$ (white circles) at level $4k$.](image)

If we now compare our quantisation condition for the 7-dimensional branes (21) and the additional conditions obtained for the 5-dimensional ones with the spectrum (25) of IHW representations of the twisted affine Lie algebra $\tilde{su}(3)_k^{(2)}$ we can conclude that the admissible twisted D-brane configurations in $SU_3$ are in one-to-one correspondence with the IHW representations of the corresponding twisted affine Lie algebra $\tilde{su}(3)_k^{(2)}$.

In particular, this implies that for even $k$ the twisted D-branes intersect the fixed point subgroup in $SO_3$ D-brane configurations corresponding to even highest weights, whereas for $k$ odd the corresponding $SO_3$ branes correspond to odd highest weights.
5. D-brane charges

In the previous sections we have analysed in a systematic fashion which are the possible D-brane configurations in $SU_3$ described by the gluing conditions (1). The next important question to be addressed is: How can one classify these D-brane configurations?

One possible classification is provided by the very result of our previous analysis. Namely, we have seen that the consistent (twisted) D-brane configurations are in one-to-one correspondence with the integrable representations of the corresponding (twisted) affine Lie algebra. This can be rephrased by saying that every such brane configuration is characterised by a set of $\text{rank}(G^r)$ quantum numbers, where $r$ stands for an arbitrary group automorphism. These quantum numbers are obtained as a result of imposing that the path integral be well defined; they are the possible values taken by the quantity

$$\frac{1}{2\pi} \left( \int_N H - \int_{\partial N} \omega \right),$$

where $(N, \partial N)$ are taken to be the generators of the appropriate relative homology group. Thus (22) can be thought of as defining a topological charge whose values measure the quantum numbers characterising the state of the system. In the framework of BCFT, they are the quantum numbers which describe the possible boundary conditions. Notice also that (22) can be understood both algebraically and geometrically. From an algebraic point of view, we have seen that this describes the highest weights of the integrable representations associated to a given quantum configuration. On the other hand, from a geometric point of view, the vanishing of the global worldsheet anomaly translates into a set of quantisation conditions imposed on the boundary values of the fields normal to the D-brane. This is a typical feature of topological charges.

In Section 2 we have seen that the space of allowed D-brane configurations falls into triplets, where the states belonging to one such triplet describe D-branes that wrap shifted conjugacy classes. Since shifted conjugacy classes are indistinguishable from one another, in their intrinsic properties, it is natural to demand that the charges carried by them be equal. This suggests that, in order to obtain a suitable notion of charge, a further reduction prescription is necessary.

In the case of untwisted D-branes, where the admissible configurations are in one-to-one correspondence with the IHW representations of the affine Lie algebra, it has been argued from various points of view (see, for instance, [5, 6]) that it is natural to define a charge which is given by the dimension of the IHW representation of the horizontal subalgebra. Thus in the case of $SU_3$, a D-brane configuration described by a $\widehat{su}(3)_k^{(1)}$ IHW representation is labelled by the $su(3)$ IHW representation $(\lambda_1, \lambda_2)$ whose dimension is given by

$$\dim(\lambda_1, \lambda_2) = \frac{1}{2}(\lambda_1 + 1)(\lambda_2 + 1)(\lambda_1 + \lambda_2 + 2).$$

If one now takes into account the equivalence relation (4) described in Section 2, which translates into the physical requirement that branes corresponding to shifted
conjugacy classes should carry the same charge, one obtains \[ Q_{(\lambda_1, \lambda_2)} = \dim(\lambda_1, \lambda_2) \mod n(k), \]
where
\[ n(k) = \begin{cases} k + 3, & \text{for } k \text{ even}, \\ \frac{1}{2}(k + 3), & \text{for } k \text{ odd}. \end{cases} \] (23)
This completely determines the charges of the SU\(_3\) branes wrapping conjugacy classes, which turn out to fit in the twisted K-theory group \[ K^0_H(SU_3) = \begin{cases} \mathbb{Z}/(k + 3)\mathbb{Z}, & \text{for } k \text{ even}, \\ \mathbb{Z}/\frac{1}{2}(k + 3)\mathbb{Z}, & \text{for } k \text{ odd}. \end{cases} \]
Furthermore, one can easily see that the admissible branes at a given level \(k\) fall into \(k\) multiplets characterised by equal values of the charge. Generically, these multiplets consist of 6 different states, although there exist multiplets of 1 or 3 states as well. The distribution of the various charges for the first few values of the level is represented in Figure 14.

Let us now analyse the case twisted case. In the previous section we have shown that a twisted brane configuration can be uniquely characterised by an IHW representation \(\hat{\mu}\) of the twisted affine Lie algebra \(\hat{su}(3)^{(2)}_k\) which, in turn, is labelled by a IHW representation \(\mu\) of the fixed point subalgebra \(\mathfrak{so}(3)\). Moreover, since the action of the centre of SU\(_3\) on the space of twisted conjugacy classes is trivial, in this case we do not have multiplets of shifted twisted conjugacy classes. This prompts us to define the charge of such a D-brane to be given by the dimension of the corresponding representation of \(\mathfrak{so}(3)\)
\[ Q_{\mu} = \dim(\mu). \]
Notice that for these configurations no further reduction is necessary. It is easy to explicitly compute the charges of the allowed D-branes at level \(k\) and obtain:
\[ Q_{\mu} = \begin{cases} 1, 3, ..., k + 1, & \text{for } k \text{ even}, \\ 2, 4, ..., k + 1, & \text{for } k \text{ odd}. \end{cases} \] (24)
We now see that these charges fit into the charge group predicted by twisted K-theory. 

\[ K^1_H(SU_3) = \begin{cases} \mathbb{Z}/(k+3)\mathbb{Z} & \text{for } k \text{ even}, \\ 2\mathbb{Z}/(k+3)\mathbb{Z} & \text{for } k \text{ odd}. \end{cases} \]

Let us end this section with a few remarks regarding the D-brane configurations wrapping the (twisted) conjugacy class of the identity. It is interesting to notice a certain similarity between these configurations. First of all, notice that, both in the twisted and in the untwisted case, the brane wrapping the conjugacy class of the identity is the configuration carrying the unit of charge. However, in the twisted case, this configuration only exists for even values of the level. Moreover, one can show that the two-form field \( \omega \), given by

\[ \omega = -\frac{1}{2} \langle g^{-1}dg, \frac{1 + \text{Ad}_g^{-1} \tau}{1 - \text{Ad}_g^{-1} \tau} g^{-1}dg \rangle, \]

vanishes on the twisted conjugacy class of the identity. In order to see this is suffices to parametrise the point \( g \) on \( C_\tau(e) \) by \( g = \tau(h)h^{-1} \), in which case \( \omega \) can be written as

\[ \omega = -\langle h^{-1}dh, \tau(h^{-1}dh) \rangle, \]

from which one deduces that \( \omega \) vanishes on \( C_\tau(e) \) as it does, trivially, on \( C(e) \). From this it follows, in particular, that also the three-form field \( H \) vanishes, when pulled back to \( C_\tau(e) \).

6. Conclusions and outlook

In this paper we have constructed a detailed and systematic picture of the consistent D-branes in SU_3 described by the gluing conditions \([1]\). The aim of this paper is twofold. On the one hand, we provide a detailed description of the possible branes in SU_3, which is an important case in its own right, as it can be used as a building block for type II string backgrounds. On the other hand, we introduce a systematic framework for the study of the general case, which makes the subject of a companion paper \([4]\). The discussion of the SU_3 case introduces all the ingredients necessary for the discussion of D-branes in general group manifolds; moreover, many of the arguments presented here are valid for any compact, connected, simply connected Lie group. In fact, it is worth noting that, in many ways, the SU_3 case is the most intricate among the simply connected groups, essentially due to the subtleties related to the outer automorphisms of the \( A_2 \) groups.

At the classical level, the space of (twisted) D-branes is given by the fundamental domain of the (twisted) Weyl group. For standard conjugacy classes this has a natural description in terms of the Stiefel diagram. We have shown that the space of twisted conjugacy classes can be described in a similar way by constructing a natural generalisation of the Stiefel diagram which we call ‘twisted Stiefel diagram’. The key notion that allows us to distinguish between the various twisted conjugacy classes is that of ‘twisted regularity’ — \( \tau \)-regularity, for short — which, as the name suggests,
is a natural generalisation of the familiar notion of regularity obtained when the adjoint action is replaced with the twisted adjoint action. Thus the twisted Stiefel diagram is a figure in the Cartan subalgebra $\mathfrak{t}$ of the fixed point subalgebra $\mathfrak{g}^\tau$ which is constructed as the inverse image, under the exponential map, of the $\tau$-singular points in the fixed point subset $T^\tau$ of the maximal torus.

The quantum analysis of the consistent D-brane configurations centers around the requirement that the path integral of the boundary WZW model be well defined or, in other words, that we have a vanishing global worldsheet anomaly. As we know, this amounts to the condition that $\frac{1}{2\pi}[(H, \omega)]$ defines a class in $H^3(G, \mathcal{E}_{(\tau)}; \mathbb{Z})$. For untwisted D-branes it was known that the quantisation conditions select a discrete subset of configurations labelled by the IHW representations of $\hat{\mathfrak{g}}^{(1)}_k$. One of the main results here is that twisted D-branes are uniquely characterised by IHW representations of the corresponding twisted affine Lie algebra, in our case, $\hat{\mathfrak{su}}(3)_k^{(2)}$, which offers an a posteriori justification for the notion of twisted conjugacy class introduced in [3].

One of the most important consequences of this result is a definition of the charge of the twisted D-branes, which is given by the dimension of the representation of the fixed point subalgebra. An explicit evaluation of the charges for the set of consistent configurations determined in Section 4 shows that they fit in the twisted K-theory group $K^1_H(\text{SU}_3)$.

The approach presented here is ideally suited for determining the admissible D-submanifolds; that is, the submanifolds on which D-branes can wrap. One of the most important challenges for the future is to construct a suitable generalisation able to discern configurations of more than one D-brane wrapping a certain (twisted) conjugacy class. This limitation notwithstanding, our approach shows that the correct charges for configurations consisting of one D-brane on a (twisted) conjugacy class can be obtained solely on geometrical (and topological) grounds, without resorting to any dynamical information.

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APPENDIX A. BASIC FACTS ABOUT $\hat{\mathfrak{su}}(3)_k^{(2)}$  

We collect here a few known facts about the twisted affine Lie algebra $\hat{\mathfrak{su}}(3)_k^{(2)}$. For more details see [13, 17]. Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{su}_3$ and $\tau$ the Dynkin diagram.
automorphism of \( g \), such that \( \tau^2 = 1 \). The finite dimensional Lie algebra \( g \) can be split into eigenspaces \( g(m) \) of \( \tau \)

\[
\mathfrak{g} = \mathfrak{g}(0) \oplus \mathfrak{g}(1)
\]

with \( \tau |_{\mathfrak{g}(0)} = 1, \tau |_{\mathfrak{g}(1)} = -1 \). Moreover, we have \([\mathfrak{g}(m), \mathfrak{g}(n)] = \mathfrak{g}(m+n)\). We know from Section 3 that the fixed point subalgebra \( \mathfrak{g}(0) \cong \mathfrak{so}_3 \), and its simple root is given by \( \bar{\alpha} = \frac{1}{2}(\alpha_1 + \alpha_2) \).

The simple roots of the twisted affine Lie algebra \( \hat{\mathfrak{su}}(3)_k^{(2)} \) are given by

\[
a(0) = ( -\phi , 0 , \frac{1}{2} ) , \\
a(1) = ( \bar{\alpha} , 0 , 0 ) ,
\]

where \( \phi = \alpha_1 + \alpha_2 \) is the highest root of \( g \) and \( \bar{\alpha} = \frac{1}{2}(\alpha_1 + \alpha_2) \) is the simple root of \( \mathfrak{g}(0) \). An IHW representation \( \hat{\mu} \) of \( \hat{\mathfrak{su}}(3)_k^{(2)} \) is labelled by an IHW representation \( \mu \) of \( \mathfrak{g}(0) \cong \mathfrak{so}_3 \) together with a value of the level \( k \), which we write as \( \hat{\mu} = (\mu, k; 0) \). The conditions for having an IHW representation for the twisted affine algebra require that the quantity

\[
\frac{2(a,\mu)}{(a,a)}
\]

take positive integral values for any root \( a \), and also that

\[
0 \leq 2(\phi, \mu) \leq k .
\]

One thus obtains that the highest weights of \( \hat{\mathfrak{su}}(3)_k^{(2)} \) are of the form

\[
n_0 l(0) + n_1 l(1) ,
\]

where \( l(0) \) and \( l(1) \) are the fundamental weights of \( \hat{\mathfrak{su}}(3)_k^{(2)} \), which read

\[
l(0) = ( 0 , 2 , 0 ) , \\
l(1) = ( \lambda , 1 , 0 ) ,
\]

with \( \bar{\lambda} = \frac{\phi}{2} \) the fundamental weight of \( \mathfrak{so}_3 \). The positive integers \( n_0 \) and \( n_1 \) are determined as solutions of the equation

\[
\frac{2k}{(\bar{\alpha}, \bar{\alpha})} = M(2n_0 + n_1) ,
\]

where \( M = \frac{(\phi,\phi)}{(\bar{\alpha}, \bar{\alpha})} = 4 \). This equation can be rewritten in the simpler form

\[
k = 2n_0 + n_1 . \tag{25}
\]

Thus the IHW representations of \( \hat{\mathfrak{su}}(3)_k^{(2)} \) are labelled by the \( \mathfrak{so}_3 \) highest weights \( \mu = n_1 \bar{\lambda} \), where the integer \( n_1 \) takes the following values:

\[
n_1 = \begin{cases} 
1, 3, ..., k , & \text{for } k \text{ odd} , \\
0, 2, ..., k , & \text{for } k \text{ even} .
\end{cases} \tag{26}
\]
Appendix B. On the relative (co)homology of SU\(_3\)  
(with José Figueroa-O’Farrill)

In this appendix we collect some topological results which are used in the main body of the paper concerning the (co)homology of SU\(_3\) relative its twisted conjugacy classes. Let \( G = SU_3 \) throughout this appendix, identified with the group of \( 3 \times 3 \) unitary matrices with unit determinant. Complex conjugation defines an automorphism \( \rho \) which is not inner and whose fixed point set \( G^{\rho} \) is the subgroup consisting of \( 3 \times 3 \) real orthogonal matrices with unit determinant; that is, SO\(_3\). This is a closed submanifold of \( G \) and hence a (geometric) 3-cycle defining a homology class in \( H_3(G) \). In the first part of this appendix we will show (by wholly elementary means) that this is class is twice the generator of \( H_3(G) \cong \mathbb{Z} \). In the second part of this appendix we exhibit generators for \( H_3(G, C_{\rho}) \), where \( C_{\rho} \) is a twisted conjugacy class and we also prove that \( H_2(G, C_{\rho}) \) vanishes. In the third and final part we determine which twisted conjugacy classes are admissible in the sense of [10].

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B.1. The homology class of SO\(_3\) in SU\(_3\). Since \( G \) is compact and simple, \( H_3(G) \cong \mathbb{Z} \) and the isomorphism is realised by integrating a suitably normalised 3-form \( \Omega \) on the cycle: \([Z] \mapsto \int_Z \Omega\). Up to normalisation, there is a unique harmonic 3-form on \( G \): \( \Omega = \lambda \text{Tr} (g^{-1}dg)^3 \), for some \( \lambda \). Our strategy is the following:

- we fix \( \lambda \) by demanding that \( \int_Z \Omega = 1 \), where \( Z \) is the SU\(_2\) subgroup associated to any one of the roots, since any of these subgroups is a generator of \( H_3(G) \); and
- we integrate the suitably normalised \( \Omega \) on the SO\(_3\) subgroup.

In summary we will show that the homology class of SO\(_3\) in \( G \) is twice that of the generator of \( H_3(G) \).

Consider the SU\(_2\) subgroup of \( G \) consisting of matrices of the form
\[
\begin{pmatrix}
a & b & 0 \\
-\bar{b} & \bar{a} & 0 \\
0 & 0 & 1
\end{pmatrix}
\]  
(27)

where \(|a|^2 + |b|^2 = 1\). It is a geometric 3-cycle which generates \( H_3(G) \). We will parametrise this submanifold of \( G \) with coordinates \((\theta, \phi, \psi)\) as follows
\[
a = \cos \theta e^{i\phi} \quad \text{and} \quad b = \sin \theta e^{i\psi},
\]  
(28)

where \( \phi \) and \( \psi \) are angles (ranging from 0 to 2\( \pi \)) and \( \theta \) ranges from 0 to \( \pi/2 \), to render \( \sin \theta \) and \( \cos \theta \) non-negative, as they play the role of radial coordinates.
Restricting the left-invariant (for definiteness) Maurer–Cartan 1-forms to this subgroup, we find
\[ g^{-1}dg = \sum_{i=1}^{3} \theta^i \tau_i \]
where \( \tau_i \) are a basis for the generators of \( \mathfrak{su}_2 \) inside \( \mathfrak{su}_3 \). Explicitly, we have
\[
\tau_1 = \begin{pmatrix} 0 & -i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} -i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]
and a calculation reveals that
\[
\begin{align*}
\theta^1 &= \sin(\phi - \psi)d\theta - \frac{1}{2} \sin(2\theta) \cos(\phi - \psi)(d\phi + d\psi) \\
\theta^2 &= -\cos(\phi - \psi)d\theta - \frac{1}{2} \sin(2\theta) \sin(\phi - \psi)(d\phi + d\psi) \\
\theta^3 &= -(\cos \theta)^2 d\phi + (\sin \theta)^2 d\psi,
\end{align*}
\]
from where it follows that
\[
\Omega = \lambda \text{Tr} \left( g^{-1}dg \right)^3 = 6\lambda \sin(2\theta)d\theta \wedge d\phi \wedge d\psi.
\]
Integrating this over the range of our coordinates, we find
\[
\int_{SU_2} \Omega = \lambda 24\pi^2,
\]
whence
\[
\Omega = \frac{1}{24\pi^2} \text{Tr} \left( g^{-1}dg \right)^3
\]
is a de Rham representative for the generator of \( H^3(G) \).

We will now parametrise the \( SO_3 \) subgroup consisting of real matrices in \( G \). It is convenient to realise \( SO_3 \) as the adjoint group of \( SU_2 \) acting on the Lie algebra \( \mathfrak{su}_2 \). In other words, let \( \text{Ad} : SU_2 \to SO_3 \) be the adjoint action. Relative to the basis \( \tau_i \) for \( \mathfrak{su}_2 \), we can write the entries of the adjoint matrix \( \text{Ad}(g) \) of an element \( g \) in \( SU_2 \) as
\[
\text{Ad}(g)_{ij} = -\frac{1}{2} \text{Tr} \left( g \tau_j g^{-1} \tau_i \right).
\]
For
\[
g = \begin{pmatrix} \cos \theta e^{i\phi} & \sin \theta e^{i\psi} \\ -\sin \theta e^{-i\psi} & \cos \theta e^{-i\phi} \end{pmatrix}
\]
the adjoint matrix \( \text{Ad}(g) \) is
\[
\begin{pmatrix}
(\cos \theta)^2 \cos(2\phi) - (\sin \theta)^2 \cos(2\psi) & (\cos \theta)^2 \sin(2\phi) + (\sin \theta)^2 \sin(2\psi) - \sin(2\theta) \cos(\phi + \psi) \\
(\sin \theta)^2 \sin(2\psi) - (\cos \theta)^2 \sin(2\phi) & (\cos \theta)^2 \cos(2\phi) + (\sin \theta)^2 \cos(2\psi) \\
\cos(\phi - \psi) \sin(2\theta) & \sin(\phi - \psi) \sin(2\theta) \end{pmatrix}.
\]
Because \( \text{Ad} \) is a two-to-one map, this parametrisation covers \( SO_3 \) twice.
Pulling back the Maurer–Cartan form to $SU_2$, we have
\[
\text{Ad}(g)^{-1}d\text{Ad}(g) = \sum_{i=1}^{3} \text{Ad}^* \theta^i \text{Ad}_* \tau_i ,
\]
where
\[
\text{Ad}^* \theta^1 = \sin(\psi - \phi)d\theta + \frac{1}{2} \sin(2\theta) \cos(\psi - \phi)(d\phi + d\psi)
\]
\[
\text{Ad}^* \theta^2 = \cos(\psi - \phi)d\theta + \frac{1}{2} \sin(2\theta) \sin(\psi - \phi)(d\phi + d\psi)
\]
\[
\text{Ad}^* \theta^3 = (\cos \theta)^2 d\phi - (\sin \theta)^2 d\psi ,
\]
and
\[
\text{Ad}_* \tau_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{pmatrix} \quad \text{Ad}_* \tau_2 = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \quad \text{Ad}_* \tau_3 = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .
\]

Therefore the pull-back of the normalised 3-form $\Omega$ is
\[
\text{Ad}^* \Omega = \frac{1}{\pi^2} \sin(2\theta) d\theta \wedge d\phi \wedge d\psi .
\]
Integrating this over $SU_2$ we find
\[
\int_{SU_2} \text{Ad}^* \Omega = 4 .
\]
On the other hand,
\[
\int_{SU_2} \text{Ad}^* \Omega = \int_{\text{Ad}(SU_2)} \Omega = 2 \int_{SO_3} \Omega ,
\]
whence we conclude that
\[
\int_{SO_3} \Omega = 2 .
\]

B.2. On $H_2(SU_3, \mathcal{C}_\rho)$ and $H_3(SU_3, \mathcal{C}_\rho)$. In this section we exhibit generators for the relative homology group $H_3(G, \mathcal{C}_\rho)$, where $i : \mathcal{C}_\rho \hookrightarrow G$ is a twisted conjugacy class. We also show that $H_2(G, \mathcal{C}_\rho)$ vanishes. As shown in the body of the paper, twisted conjugacy classes are parametrised by (a quotient of) the maximal torus of the fixed point subgroup $G^\rho = SO_3$. More concretely, let $\mathcal{C}_\rho(\vartheta)$, for $0 \leq \vartheta \leq \pi/4$, denote the twisted conjugacy class of the element
\[
h_\vartheta := \begin{pmatrix} \cos 2\vartheta & \sin 2\vartheta & 0 \\ -\sin 2\vartheta & \cos 2\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix} .
\]
Then these exhaust all the twisted conjugacy classes of $G$. The two extreme conjugacy classes $\mathcal{C}_\rho(0)$ and $\mathcal{C}_\rho(\pi/4)$ are 5-dimensional, whereas for $0 < \vartheta < \pi/4$, $\mathcal{C}_\rho(\vartheta)$ is 7-dimensional. It is not hard to show that $\mathcal{C}_\rho(\pi/4)$ is a homology sphere, whereas the
integral homology of the other twisted conjugacy classes is given by

$$\begin{align*}
H_p(\mathcal{C}_\rho(0)) &\cong \begin{cases} 
\mathbb{Z} & p = 0, 5, \\
\mathbb{Z}_2 & p = 2, \\
0 & p = 1, 3, 4
\end{cases}
\quad \text{and} \quad H_p(\mathcal{C}_\rho(\vartheta)) &\cong \begin{cases} 
\mathbb{Z} & p = 0, 2, 5, 7, \\
0 & p = 1, 3, 4, 6
\end{cases}
\end{align*}$$

for $0 < \vartheta < \pi/4$.

The basic tool for computing relative homology groups is the long exact sequence

$$\cdots \to H_p(\mathcal{C}_\rho) \xrightarrow{i_*} H_p(G) \to H_p(G, \mathcal{C}_\rho) \to H_{p-1}(\mathcal{C}_\rho) \xrightarrow{i_*} H_{p-1}(G) \to \cdots$$

relating the relative homology of $(G, \mathcal{C}_\rho)$ to the homologies of $G$ and $\mathcal{C}_\rho$. Since $H_1(G) = H_2(G) = 0$, it follows that $H_2(G, \mathcal{C}_\rho) \cong H_1(\mathcal{C}_\rho)$. Since $H_1(\mathcal{C}_\rho)$ vanishes for all $\mathcal{C}_\rho = \mathcal{C}_\rho(\vartheta)$, we conclude that $H_2(G, \mathcal{C}_\rho(\vartheta)) = 0$ for all $\vartheta$.

Now let us study $H_3(G, \mathcal{C}_\rho)$ for each of the twisted conjugacy classes in turn.

B.2.1. $\mathcal{C}_\rho = \mathcal{C}_\rho(0)$. Since $H_3(\mathcal{C}_\rho) = 0$ and $H_2(G) = 0$, the long exact sequence truncates to a short exact sequence

$$0 \to H_3(G) \to H_3(G, \mathcal{C}_\rho) \to H_2(\mathcal{C}_\rho) \to 0.$$ 

Since $H_3(G) \cong \mathbb{Z}$ and $H_2(\mathcal{C}_\rho) \cong \mathbb{Z}_2$, we see that $H_3(G, \mathcal{C}_\rho) \cong \mathbb{Z}$ and moreover that the map $H_3(G) \to H_3(G, \mathcal{C}_\rho)$ is multiplication by 2.

B.2.2. $\mathcal{C}_\rho = \mathcal{C}_\rho(\pi/4)$. Since $H_3(\mathcal{C}_\rho) = H_2(\mathcal{C}_\rho) = 0$, we have that $H_3(G, \mathcal{C}_\rho) \cong H_3(G) \cong \mathbb{Z}$. Therefore $H_3(G, \mathcal{C}_\rho)$ is generated by the “honest” 3-cycle generating $H_3(G)$.

B.2.3. $\mathcal{C}_\rho = \mathcal{C}_\rho(\vartheta)$, $0 < \vartheta < \pi/4$. Since $H_3(\mathcal{C}_\rho) = 0$ and $H_2(G) = 0$, we have again the short exact sequence

$$0 \to H_3(G) \to H_3(G, \mathcal{C}_\rho) \to H_2(\mathcal{C}_\rho) \to 0.$$ 

Since $H_3(G) \cong \mathbb{Z}$ and $H_2(\mathcal{C}_\rho) \cong \mathbb{Z}$, we see that $H_3(G, \mathcal{C}_\rho) \cong \mathbb{Z} \oplus \mathbb{Z}$ with generators given by the generating cycle of $H_3(G)$ and the generating cycle $\Sigma$ of $H_2(\mathcal{C}_\rho)$. Since $H_2(G) = 0$, $\Sigma$ bounds in $G$, whence there exists a 3-chain $N$ in $G$ such that $\partial N = \Sigma$. The relative cycle $(N, \Sigma)$ is then one of the generators for $H_3(G, \mathcal{C}_\rho)$. Let us determine the 2-cycle $\Sigma$ in $\mathcal{C}_\rho$.

Consider again the SU$_2$ subgroup of SU$_3$ with elements of the form $[\rho]$, and let $\mathcal{C}_\rho$ be the twisted conjugacy class of the element $h_\vartheta \in$ SU$_2$. Let $\pi : G \to \mathcal{C}_\rho$ denote the twisted adjoint action of $G$ on $h_\vartheta$

$$\pi(g) = \rho(g) h_\vartheta g^{-1} = \tilde{g} h_\vartheta g^{-1}.$$ 

Let $\Sigma \subset \mathcal{C}_\rho$ denote the submanifold of $\mathcal{C}_\rho$ obtained by restricting the action to SU$_2$. We claim that $\Sigma$ is the generator of $H_2(\mathcal{C}_\rho)$. To establish this claim we will prove that $\int_\Sigma F = 1$, where $F$ is the generator of $H^2(\mathcal{C}_\rho)$. We first need to determine $F$. We do this as follows.
The map $\pi : G \to \mathcal{C}_\rho$ is a fibration with fibre a circle. Recall (see, for example, [18]) that the Leray spectral sequence for such a fibration degenerates to a long exact sequence (the Gysin sequence)

$$\cdots \to H^3(\mathcal{C}_\rho) \xrightarrow{\pi^*} H^3(G) \xrightarrow{\pi_*} H^2(\mathcal{C}_\rho) \xrightarrow{e \cup} H^4(\mathcal{C}_\rho) \to \cdots$$

where $\pi^*$ is the pull-back, $\pi_*$ is “integration along the fibre” and $e \cup$ is the cup product with the Euler class $e$ of the circle bundle. Since $H^3(\mathcal{C}_\rho) = 0 = H^4(\mathcal{C}_\rho)$, we see that integration along the fibre defines an isomorphism $H^3(G) \cong H^2(\mathcal{C}_\rho)$. This means that the generator of $H^2(\mathcal{C}_\rho)$ is given by $F = \pi_\ast \Omega$, where $\Omega$ is the generator of $H^3(G)$ defined in (31). Equivalently, there exists a one-form $A$ on $G$ such that $\pi^* F = dA$ and such that

$$\int_{\text{fibre}} A = 1 \quad \text{and} \quad \int_{\text{SU}_2} A \wedge dA = 1.$$  \hspace{1cm} (33)

This means that $\Omega = A \wedge dA + \Omega_h$, where $\Omega_h$ is horizontal, so that

$$\pi_* \Omega = \pi_*(A \wedge dA) = \pi_*(A \wedge \pi^* F) = \left( \int_{\text{fibre}} A \right) F = F.$$

It is easy to find an explicit expression for the one-form $A$ and to check that it satisfies the properties (33) above. Indeed, let $K \subset G$ be the circle subgroup which leaves invariant the point $h = h_0$, namely

$$K = \left\{ \begin{pmatrix} \cos 2\pi t & \sin 2\pi t & 0 \\ -\sin 2\pi t & \cos 2\pi t & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| t \in [0, 1] \right\}.$$  

Equivalently, $K = \pi^{-1}(h)$ is the fibre at $h$. The fibre at the point $\pi(g) = \bar{g}hg^{-1}$ is the left translate $gK$ of $K$ by $g$. To ensure that the one-form $A$ integrates to 1 on all the fibres we choose it to be left-invariant, namely

$$A(g) = \text{Tr} Xg^{-1}dg$$

for some constant $X \in \mathfrak{su}_3$. A choice of $A$ which satisfies the properties (33) is

$$A(g) = \frac{1}{4\pi} \text{Tr} \tau_2 g^{-1}dg,$$

where $\tau_2$ is defined in (29). Indeed, $A$ restricts to the global angular form $dt$ on the fibres, and on $\text{SU}_2$ to $-(1/2\pi)\theta^2$, where $\theta^2$ is defined in (31). This implies that $A \wedge dA$ agrees with the restriction of $\Omega$ to $\text{SU}_2$. Therefore we could have restricted the discussion to the fibration $\pi : \text{SU}_2 \to \Sigma$ obtained by restricting the map $\pi$ to $\text{SU}_2$ and we would not alter any of the conclusions. In particular, we find that the 2-form $F = \pi_\ast \Omega$, where $\pi$ is now restricted to $\text{SU}_2$, is the generator of $H^2(\Sigma)$, whence $\int_\Sigma F = 1$. 

B.3. Quantisation conditions. We are now ready to apply the above results to select those twisted conjugacy classes for which the global worlds heet anomaly vanishes. Let us call these classes admissible. Recall [10] that a submanifold $i : C_ρ ↪ G$ is admissible if the following three conditions hold:

1. $H_2(G, C_ρ) = 0$,
2. the 3-form $H$ in the Wess–Zumino term satisfies $i^*H = dω$ for some 2-form $ω$ on $C_ρ$; and
3. the relative de Rham cocycle $\frac{1}{2\pi}(H, ω)$ is integral.

We have proven above that first condition is satisfied; whereas the second condition follows from an explicit calculation [1, 9, 11]. The third condition means that evaluating the class $\frac{1}{2\pi}[(H, ω)] ∈ H^3(G, C_ρ)$ on any relative 3-cycle gives an integer. As we have seen above, for any twisted conjugacy class $C_ρ$, the homology group $H_3(G, C_ρ)$ is freely generated; hence it is sufficient (and necessary) to check that

$$\frac{1}{2\pi} \int_N H - \frac{1}{2\pi} \int_{\partial N} ω ∈ \mathbb{Z},$$

(34)

where $(N, ∂N) ⊂ (G, C_ρ)$ is a generator of $H_3(G, C_ρ)$. In the previous section we exhibited these generators in terms of the SU$_2$ subgroup given by [27]. Our strategy will consist in pulling back the class $\frac{1}{2\pi}[(H, ω)]$ to $H^3(SU_2, Σ)$ and use the known results for SU$_2$.

B.3.1. $C_ρ = C_ρ(0)$. The fundamental 3-cycle SU$_2$ is twice the generator of $H_3(G, C_ρ)$; hence evaluating $\frac{1}{2\pi}[(H, ω)]$ on SU$_2$ should give an even integer. On the other hand, $\frac{1}{2\pi} \int_{SU_2} H = k$, whence this class is admissible only when the level $k$ is even.

B.3.2. $C_ρ = C_ρ(π/4)$. The fundamental 3-cycle SU$_2$ is the generator, hence $C_ρ$ is admissible for all integer levels.

B.3.3. $C_ρ = C_ρ(θ), 0 < θ < π/4$. There are two generators of $H_3(G, C_ρ)$. The first generator is the fundamental cycle SU$_2$ and the quantisation condition simply says that the level $k$ is an integer. For the second generator we can take any relative cycle $(N, ∂N)$ where $∂N = Σ$. Therefore $C_ρ$ is admissible if and only if $Σ$ is admissible for SU$_2$ at level $k$. We observe that because complex conjugation is an inner automorphism for SU$_2$, Σ is a shifted conjugacy class in SU$_2$. Indeed, notice that

$$Σ = \{gh_θg^{-1}|g ∈ SU_2\} = \{jgj^{-1}h_θg^{-1}|g ∈ SU_2\} = jC_ρ(h_θ+π/4),$$

where $j$ is defined by

$$j := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and where $C_ρ(h_θ+π/4)$ is the conjugacy class of $h_θ+π/4$ in SU$_2$. Since $j ∈ SU_2$ and SU$_2$ is connected, the homotopy invariance of relative (co)homology says that $jC_ρ$ is
admissible if and only if so is \( C_\rho \). At level \( k \), \( C_\rho(h_{\theta+\pi/4}) \) is admissible if and only if
\[
2\vartheta + \frac{\pi}{2} = \frac{n\pi}{k}
\]
for some integer \( n \); equivalently,
\[
\vartheta = \frac{(2n - k)\pi}{2k}
\]
for some integer \( n \), which is equivalent to equation (21).

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