Constructing shells and their visualization in system "MathCad" on basis of vector equations of surfaces

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Abstract. One of the tasks of modern architecture and building is creation of new forms of civil and sport buildings. In world practice it is possible to see the examples of original forms of sport, exhibition, trade and other types of constructions. For creating new complex forms of spatial constructions it is necessary to have equations of middle surfaces of these constructive forms. Certainly, there are many sources of classic literature and articles on geometry of the surfaces including sources in Internet. "Encyclopedia of analytical surfaces" was published in 2012, Encyclopedia was published in English language in 2016. 35 classes of surfaces, more than 500 types of surfaces, are described in Encyclopedia. On the basis of the types of surfaces given at the encyclopedias it is possible to form various spatial forms. For the same time it is desirable to have a possibility to create other forms of surfaces. In 2016 the monograph was published. At the monographs vector equation of surfaces with the system of flat coordinate curves is given. Even knowing the law of surface formation, having its equation, it is not always possible to present the type of surface on all areas of change of coordinates at the certain values of parameters of surface. It is therefore important to use program facilities for constructing and visualizing compartments of surfaces. Presently there are many programmatic complexes, allowing reproducing the drafts of constructions, space graphic and surface. One of such complexes is a complex "MathCad". In this complex it is necessary to use equations of surfaces in the projections of the Cartesian system of coordinates. But the most comfortable form of equations of surfaces of complex forms are vector equations. Scalar self-reactance equations of complex surfaces usually are very bulky. It is difficult to set the method of formation of surfaces on scalar equations of surfaces of complex form. Vector equations are made with introduction of special additional vector-functions that shows the method of motion a formative curve. Methodology of development of program visualization of surfaces is shown in a complex "MathCad" with the use of vector equation.

1. Surfaces with family of flat co-ordinate lines
One of wide classes of surfaces are surfaces with family of flat co-ordinate lines. The most surfaces used in practice are the surfaces of this class: surfaces of rotation, surface of translation, ruled surfaces and others.

Surfaces with the system of flat co-ordinate lines are formed by motion of some flat curve (formative, genetrix) on any chosen law in space (Figure 1). Thus, at the moving a formative curve is also transformed on some law. It allows to create the most various spatial forms.

Vector equation of surfaces with the system of flat co-ordinate lines looks like:
\[ p(u,v) = r(u) + X_0(u,v)e_0(u) + Y_0(u,v)g_0(u). \] (1)

\( p(u,v) \) is radius-vector of surface; \( r(u) \) is a radius vector of directing curve; \( e_0(u) \), \( g_0(u) \) are unit vectors of rectangular system of coordinates in the secant plane of directing curve, where equations of formative curve \( X_0(u,v), Y_0(u,v) \) are set. The position of secant plane is determined in every point of directrix by the vector of normal \( n(u) \). Unit vectors \( e_0, g_0 \) contact with vectors of tangent, normal and bi-normal of directing curve or unit vectors of the Cartesian system of coordinates. Dependence of vector-functions \( e_0(u), g_0(u) \) and parameters \( X_0(u,v), Y_0(u,v) \) on a coordinate steam-meter \( u \) of directing curve allows transforming a formative curve in the process of her motion on a directing curve.

Vector equation (1) allows obtaining formulas of coefficients of quadratic forms of surface [6-8]. In a monograph [5] terms are got, when implementing that the system of flat co-ordinate lines is the system of lines of curvature of surface. These terms are described by the system of nonlinear equations based on that some classes of surfaces are got with the system of flat co-ordinate lines of curvature.

Character, the type of surface depends on the functional parameters of equation (1) set possibility to get different classes and subclasses of surfaces. If a formative curve moves in the normal plane of directing curve – \( n(u) = \tau \) (\( \tau \) it is a vector of tangent directing curve), then we will get the class of normal surfaces [5, 7] which are the surfaces of rotation (directing - straight line). Monge’s surfaces of double curve belong to the class of normal surfaces [6, 8] - a formative unchanging curve is hardly related to the directing curve. The system of co-ordinate lines of the Monge’s surfaces is the system of lines of curvature of surface. If the formative curve of normal surface is a circumference of variable radius we get a normal cyclic surface [9], in a case of permanent radius - tubular surface.

Unit vectors of the rectangular system of coordinates in the normal plane of directing curve, in that a formative curve is set:

\[ e_0(u) = v \cos \theta(u) + \beta \sin \theta(u); \quad g_0(u) = -v \sin \theta(u) + \beta \cos \theta(u), \] (2)

where \( v, \beta \) are normal and bi-normal of directing curve vectors; \( \theta(u) \) is a function of the law of rotation a formative curve in the normal plane of directing curve. If \( \theta = \theta_0 = \text{const} \), then a formative curve does not rotate in a secant plane.

In case of secant plane rotating about some axis we get the class of surfaces running around with the system of flat co-ordinate lines in the planes of pencil [7]. At the vertical axis of rotation of secant plane a normal vector of planes of pencil is determined by the vector-function of circumference of unit radius

\[ n(u) = -i \sin u + j \cos u, \] (3)

\( i, j, k \) – unit vectors of the Cartesian system of coordinates.

A formative curve is set in plane rotation with unit vectors:

\[ e_0(u) = h(u) \cos \theta(u) + k \sin \theta(u); \quad g_0(u) = -h(u) \sin \theta(u) + k \cos \theta(u); \quad h(u) = i \cos u + j \sin u. \] (4)

Surfaces of rotation, surface of Joachimsthal [10, 11] belong to the class of surfaces with the formative curves in the planes of pencil.
In case of unchanging normal \( n=\text{const} \) we receive class of surfaces with the plane of parallelism is a formative curve moving in parallel set of planes \([4, 5]\). In case of unchanging normal and unchanging formative curve, we receive a surface of transfer.

2. Using vector equation to visualize surfaces
Let us consider possibility of development of the program of surfaces visualization in the system MathCAD on the basis of vector equation of surface. For using vector equations of surfaces in the system MathCAD, it is necessary to enter unit vectors of the rectangular global system of coordinates of \( x, y, z \) and to bind them to vectors, used for the construction of vector equation of surface. We will enter some of the most often used vectors:

\[
\begin{align*}
ix &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} ; \quad jy = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} ; \quad kz = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \text{unit vectors of the rectangular system of coordinates.}
\end{align*}
\]

Further, we have to enter unit vectors in secant planes, binding them to unit vectors of the Cartesian system of coordinates. Consider this on the example of normal surface with a flat directing curve in the horizontal plane set by self-reactance equations of \( X_0(u), Y_0(u) \). For convenience, it is possible to create the matrix of the most often curves used \( a \) and the matrix of the first derivatives \( b \), required to determine vectors of tangent and normal of directing curve, is created also:

\[
\begin{align*}
a) \quad r(a, b, t) &= \begin{bmatrix} 0 & t & 0 \\ 1 & t & at^2 \\ 2 & a \cos(t) & b \sin(t) \\ 3 & a \cosh(t) & b \sinh(t) \end{bmatrix} \\
b) \quad r1(a, b, t) &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 2at \\ 2 & -a \sin(t) & b \cos(t) \\ 3 & a \sin(t) & b \cos(t) \end{bmatrix}
\end{align*}
\]

In the first column number of curve: 0 is a horizontal line, 1 is a parabola, 2 is an ellipse, 3 is a hyperbola etc.; in 2 - X (t); in 3 - Y (t). A matrix can be used both for directing and for formative curves. Further \( kn \) is number of directrix, \( ko \) is number of generatrix.

\[
sc(a, b, u, kn) := \sqrt{(r1(a, b, u)_{h, 1})^2 + (r1(a, b, u)_{h, 2})^2} \\
Sn(a, b, u, kn) := \frac{r1(a, b, u)_{h, 2}}{sc(a, b, u, kn)}
\]

\[
Cnc(a, b, u, kn) := \frac{r1(a, b, u)_{h, 1}}{sc(a, b, u, kn)} \\
vC(a, b, u, kn) := -ix \cdot Cnc(a, b, u, kn) + jy \cdot Cnc(a, b, u, kn)
\]

\( \nuC \) is a vector of normal of directing curve vector.

Function \( \theta(u), \) and also functions of coefficients of \( c(u), d(u) \) of the transformed directing curve is entered additionally (for the untransformed directing curve it is constants). For example \( \theta(u) := \cos(u) \)

\[
c(v) := 0.2 \cdot v \\
d(v) := 0.5 \cdot v
\]

Further get cycles organized on coordinates and calculation of coordinates of surface in the Cartesian system of coordinates. This part of the program is standard:

\[
\begin{align*}
M := 20 & \quad un := \ldots & \quad uk := \ldots & \quad du := \frac{uk - uu}{M} \\
u_n := un + du \cdot m & \quad N := 20 & \quad vn := \ldots & \quad vk := \ldots & \quad dv := \frac{vk - vn}{N} \\
v_n := vn + du \cdot n & \quad t_m := 0c(u_m) & \quad c_n := \cos(v_n) & \quad d_n := \operatorname{do}(v_n)
\end{align*}
\]
Further standard procedure of construction of surface in the system "MathCad".

In Figure 2 examples of formation of normal surfaces are made with directing and formative curves – ellipses. In an overhead row ellipsoid normal surfaces are with the unchanging parameters of formative ellipse; and is a not-involute normal surface - \( \theta = \text{const} \); in \( a \) - untwisted normal surfaces; in \( b, c - \theta = \sin pu \); in \( b - p = \pi/2 \) is an eventful turn on 180\(^\circ\) degrees (closed ellipsoid surface); in \( c - p = \pi/4 \) - at the round of directing ellipse a formative ellipse turns on 90\(^\circ\).

In a lower row, a formative ellipse moves on a linear law in the normal plane of directing ellipse. As a result an ellipsoid-helix-shaped surface appears: in \( a \) - with an unchanging formative ellipse; in \( b, c \) - with the linear changing axes of formative ellipse; it is an involute ellipsoid-spiral surface with the variable semiaxes of formative ellipse.

If in the program of visualization of surface one of coordinates is accepted as the constant \( (un=uk \) or \( vn=vk) \), then we will get the picture of the co-ordinate curve of the surface. In case that \( un=uk \) we receive the flat formative curves with permanent or variable parameters. In case of ellipsoid formative curve in the sections of \( vn=vk \) we get the oval closed curves (in general case by being not ellipses). In case of changeable parameters of formative curve and \( vn=vk \) we get spatial curves (Figure 3).

The center of formative ellipse of normal surfaces on Figure 2 moves on a directing ellipse (overhead and middle pictures) and on an elliptic cylinder on the pictures of lower row. The initial dimensions of formative ellipse are accepted \( c=1, d=0.75 \). The self-intersecting compartments of surface will turn out at the increase of parameters of formative ellipse (comparative with the parameters of directing ellipse). That it not took place, expedient to remove a formative curve from a directing curve in its plane at the direction of the normal of the directrix and formula (5) the will be as

\[
\rho(a,b,u,c,d,v,0,0,kn,ko) = m(a,b,u,ku) - \nu c(a,b,u,ku) \cdot f(u) + \ldots,
\]

further, according to a formula (5), \( f(u) \) is a function of moving of co-ordinate system a formative curve on the normal of directing curve.

In Figure 4, \( f(u) = c(u) \). For untwisted surfaces (Figure 4 (a) and (b)), a surface touches a directing ellipse: parameters of formative ellipse change: in \( a \) - sinusoid law; in \( b \) - linear. On Figure 4 (c) there is twisted surface with the unchanging parameters of formative ellipse. On a
Figure 4 (c) a directing ellipse is shown, a twisted surface touches a directing ellipse with an external formative ellipse at horizontal position (in plane of directing ellipse) of axis of formative ellipse.

In Figure 5, \( f(u) = c \cdot (2 + \cos ku) \). In Figure 6, ellipsoid-normal surfaces are presented with the simultaneous transferring of formative ellipse at the planes of directing ellipse and orthogonal to the plane. There is a linear law of moving of \( z = pu \) on a vertical line. On a figure 6 (a) in plane directing ellipse is a linear law of displacement: in \( a \) is a directing ellipse with permanent parameters, in \( b \) with linear variables by parameters. In the figure 6 (c), a horizontal displacement corresponds Figure 5 (k=3) (a kind from above is analogous to Figure 5 at \( k=3 \)).

Pictures over of surfaces are higher brought with a directing ellipse with unchanging parameters and formative ellipse with unchanging and variable parameters and different laws of moving of directing ellipse at motion in the normal plane of directing ellipse. Wide possibilities of constructing spatial forms can be seen from these examples.

\[
\begin{align*}
\mathbf{r}(u,v) &= R(u)h(u) + z(u)k + X_0(u,v)e_0(u) + Y_0(u,v)g_0(u) + z(u),
\end{align*}
\]

where \( R(u) \) is a radius of flat directing curve in the polar system of coordinates; \( z(u) \) it is displacement of the co-ordinate system a formative curve in the secant plane in vertical direction; \( h(u) \) determines position of secant plane (4); \( e_0(u) \), \( g_0(u) \) determine rotation of a formative curve in the secant plane (4). So we enter functions in the system "MathCad":

\[
\begin{align*}
&h(u) := ix\cdot \cos(u) + jy\cdot \sin(u) \quad e_0(u, \theta) := h(u) \cdot \cos(\theta) + kz\cdot \sin(\theta) \\
g_0(u, \theta) := -h(u) \cdot \sin(\theta) + kz\cdot \cos(\theta)
\end{align*}
\]

and radius vector of surface \( \rho(R, z, u, c, d, v, \theta, kn, ko) := R \cdot h(u) + z \cdot kz \). Further like a formula (5). The process of calculation of surface coordinates and its visualization is analogous to the previous examples.
In Figure 8, $a$-$f$ is an axis of rotation in a center surfaces with a directing external ellipse $a=1,5$, $b=0.75$ and formative ellipses, $g$, $h$, $i$ - in the pole of directing ellipse. In the first row, formative ellipse does not rotate. In the second row, external formative ellipses rotate $\theta = \sin(0.25\pi u)$ with changing sizes of semiaxes. In a lower row, $\theta = \sin(0.25\pi u)$, the axis of formative ellipse is changed on the sinewave law: $c = c_0 + c_1 \sin pu$; $d = d_0 + d_1 \sin pu$.

If the co-ordinate system of directing curve is hardly related to the directing curve, and the vertical parameter of curve changes on sinusoid law $Y_o = Y_0(v)\sin(pu)$, we get a wavy surface (Figure 9). At the integer values of parameter of $p$ we get the wavy closed surface. Directing curve is a circle of radius $b$. In a case of the integer values of parameter $p$ we get a wavy monolayer surface, at a fractional - multi-layered. In a 2-nd (3,g) row generatrix sinusoid $Z(u) = b\sin(\mu u)$. In a lower row, generatrix is cosine $Z(u) = b\cos(\mu u)$.

![Figure 8. Ellipsoid surfaces with formative in the planes of pencil.](image8)

![Figure 9. Wavy surfaces with formative in the planes of pencil.](image9)

![Figure 10. Spiral and spiral-shaped cyclic surfaces.](image10)

We get spiral surfaces set by the function of the vertical moving of formative curve; in a case of simultaneous horizontal and vertical moving are spiral-shaped surfaces with formative in the planes of pencil (figure 10).

The publication is made with the support of the program RUDN “5-100”

**References**

[1] Shulikovskiy V I 1963 *Classical Differential GEOMETRY* (Moscow: GIFMAL) p 540
[2] Catalan E 1843 Memoire sur les Surfaces Gauches a Plan Directeur *J. Ecole Polyt.* XVII, cah. 29
[3] Farin G 2002 *Curves and Surfaces for CAGD A Practical Guide* Morgan Kaufmann Series in Computer Graphics Book
[4] Krivoshapko S N and Ivanov V N 2015 *Encyclopedia of Analytical Surfaces* (Switzerland: Springer) p 752
[5] Ivanov V N and Romanova V A 2016 *Constructive forms of space constructions. Visualization*
of the surfaces at the systems MathCAD and AUTOCAD (Moscow: ASV Press) p 412
[6] Ivanov V N 2010 Structural Mechanics of Engineering Constructions and Buildings 3 6–15
[7] Ivanov V N 2011 Structural Mechanics of Engineering Constructions and Buildings 4 6–14
[8] Monge G 1787 Memoire sur L’integration de Quelques Equations aux Derives Partielles (Mem. C SCI) p 309
[9] Ivanov V N and Shmeleva A A 2016 Structural Mechanics of Engineering Constructions and Buildings 6 3–8
[10] Joachimsthal F J 1846 Math Dd. 30 347–50
[11] Ivanov V N and Nasr 1997 Unes Abbushy Investigation of geometry of the Joachimsthal canal surfaces Problems of the theory and practice of the engineering investigation (Moscow: RUDN) pp 115–8