Self-Stabilizing Paxos

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Abstract

We present the first self-stabilizing consensus and replicated state machine for asynchronous message passing systems. The scheme does not require that all participants make a certain number of steps prior to reaching a practically infinite execution where the replicated state machine exhibits the desired behavior. In other words, the system reaches a configuration from which it operates according to the specified requirements of the replicated state-machine, for a long enough execution regarding all practical considerations.
1 Introduction

One of the most influential results in distributed computing is Paxos [12, 13], where repeated asynchronous consensus is used to replicate a state machine using several physical machines. The task is to use asynchronous consensus, where safety is guaranteed, and liveness is almost always achieved by using an unreliable failure detector to implement an abstraction of very reliable state machine on top of physical machines that can crash, though usually at least several machines stay alive at any particular moment. The extreme usefulness of such an approach is proven daily by the usage of this technique, by the very leading companies, to ensure their availability and functionality.

Unfortunately Paxos is not self-stabilizing and therefore a single transient fault may lead the system to stop functioning even when all the cluster machines operate. One example is the corruption of the time-stamp used to order operations in Paxos, where a single corruption of the value of this counter to the maximal value will cause the system to be blocked. In another scope, the occurrence of transient fault with the same nature caused the Internet to be blocked for a while [16]. Self-stabilization is a property that every on-going system should have, as self-stabilizing systems automatically recover from unanticipated states, i.e., states that have been reached due to insufficient error detection in messages, changes of bit values in memory [7], and in fact any temporary violation in the assumptions made for the system to operate correctly. The approach is comprehensive, rather than addressing specific fault scenarios (risking to miss a scenario that will later appear), the designer considers every possible configuration of the system, where configuration is a cartesian product of the possible values of the variables. Then the designer has to prove that from every such a configuration, the system converges to exhibit the desired behavior.

Self-stabilizing systems do not rely on the consistency of a predefined initial configuration and the application of correct steps thereafter. In contrast, self-stabilizing systems assume that the consistency can be broken along the execution and need to automatically recover thereafter. The designers assume an arbitrary configuration and prove convergence, not because they would like the system to be started in an arbitrary configuration, but because they are aware that the specified initial configuration and the defined steps consistency maybe temporarily broken, and would like the system to regain consistency. Of course, such an approach does not guarantee the convergence under a infinite stream of transient faults, which is clearly impossible, but guarantees the system recovery after the last transient fault. Therefore, although the system may lose safety properties, the safety is automatically regained, leading to a safer behavior than of non-stabilizing systems, namely, initially safe and eventually safe [3].

Self-stabilizing consensus and replicated state machine for shared memory system appeared in [8], the case of message passing being left to future investigation. One approach to gain a self-stabilizing consensus and replicated state machine in message passing is to implement the read-write registers used in [8], using message passing. A self-stabilizing implementation of such a single-writer multiple-reader register appeared in [2]. Unfortunately, the implementation had to assume that the writer is active forever. Thus, the implementation of self-stabilizing Paxos under the original assumptions was left open. In this paper we present the first self-stabilizing Paxos in message passing systems. One ingredient of the self-stabilizing Paxos algorithm is a recent construction of a self-stabilizing bounded time-stamp [2].

Note that the classical bounded time-stamp systems [4, 10] cannot be started with an arbitrary set of label values as the ordering is defined only for certain combination of labels (and missing labels). Such restricted combinations can be preserved by non-stabilizing algorithms as long as transient faults do not

\[1\] Note though that Paxos does not rely explicitly on a failure detector.

\[2\] A suggestion made by Eli Gafni.
occur. Another bounded weak time-stamp was designed for particular shared memory self-stabilizing systems, where participants have access to shared memory of others (while we deal with message passing where many label values can be in messages in transient). This bounded time-stamp allows a limited new-old version inversion, and therefore does not guarantee the eventual strict ordering of events, ordering that a replica state machine should (eventually) promise. Obtaining a self-stabilizing Paxos requires to cope with many aspects including a way to compose self-stabilizing bounded time stamps, such that each time stamp is governed by a distinct participant. In particular, the algorithm also needs to cope with an arbitrary set of messages that are stored in the system, as we demonstrate in the sequel. The paper starts with a background and description of techniques and correctness in a nutshell. Then we turn to a more formal and detailed description.

2 Self-Stabilizing Paxos Overview

In this section, we define the Repeated Consensus Problem, show how it can be used to implement a self-stabilizing replicated state machine and give an overview of the Paxos Algorithm. In addition, we give arguments for the need of a self-stabilizing algorithm that would solve the Repeated Consensus Problem. Doing so, we investigate a new kind of self-stabilizing behaviour, namely the practically self-stabilizing behaviour. Here and everywhere, the semantical synonym for practically self-stabilizing is essentially self-stabilizing.

Repeated Consensus. The processors have to perform successive instances of consensus on values proposed by some of them. Every processor is assumed to have an integer variable \( s \), namely the step variable, that denotes the current consensus instance it is involved in. In each consensus instance, processors decide on a value. For example, in the context of replicated state machines, the step variable denotes the current step of the state machine, and at each step, a processor may decide to apply a command to its copy of the state machine. Processors may have different views on what is the current step since some of them may have progressed faster than others. The Repeated Consensus Problem is defined by the following conditions: (Safety) for every step \( s \), if two processors decide on values in step \( s \), then the two decided values must be equal, (Integrity) for every \( s \), if a processor decides on some value, then this must have been proposed in step \( s \), (Liveness) every non-crashed processor decides in infinitely many steps.

Original Paxos. The original Paxos algorithm guarantees the safety and the integrity property in an asynchronous complete network of processors communicating by message-passing such that less than half of the processors are prone to crash failures. The algorithm uses unbounded integers and also assumes that the system starts in a consistent initial configuration. To guarantee the liveness property, additional assumptions must be made as discussed below. The Paxos algorithm defines three roles: proposer, acceptor and learner. The proposer basically tries to impose a consensus value for its current step. The acceptor accepts consensus values according to some specific rules. A value can be decided on for step \( s \) when a majority of acceptors have accepted it in step \( s \). Finally, the learner learns when some value has been accepted by a majority of acceptors for some step and decides accordingly. Here, we assume that every processor is a learner and an acceptor, while some processors can also be proposers. Every proposer has its own idea of what should the value be for step \( s \). For each step \( s \), a proposer executes one trial, or more, to impose some consensus value. Thus, each processor maintains a Paxos tag, namely a couple \( (s, t) \) where \( s \) denotes a step, i.e., a consensus instance, and \( t \) a trial within this step. The Paxos algorithm assumes that all the step variables and the trial variables are natural integers,

\[\text{Note that there might be more than one proposer in each step.}\]
hence unbounded, and initially set to zero. The Paxos tags are used to timestamp the proposals emitted by the proposers or accepted by the acceptors. To impose a proposal with tag \((s_t)\), a proposer must first (phase 1) reads the most recent accepted proposal from a majority of acceptors and try to impose its tag \((s_t)\) as the greatest tag on this majority of acceptors; knowing that an acceptor adopts the tag \((s_t)\) if it is strictly greater than its own. Secondly (phase 2), the proposer tries to make a majority of acceptors accept the previous read consensus value if it is not null, or its own value otherwise; knowing that an acceptor accepts the proposal if the tag \((s_t)\) is greater than or equal to its own. If the proposer succeeds in these two phases, it decides on the proposal and notifies the other processors.

The integrity property is guaranteed by the fact that a decided value always comes from a proposer in the system. The difficulty lies in proving that the safety property is ensured. Roughly speaking, the safety correctness is yielded by the claim that once a proposer has succeeded to complete the second phase, the consensus value is not changed afterwards for the corresponding step. Ordering of events in a common processor that answers two proposers yields the detailed argument, and the existence of such a common processor stems from the fact that any two majorities of acceptors always have non-empty intersection. The liveness property, however, is not guaranteed \[9\]. However, a close look at the behaviour of Paxos shows that only the liveness property cannot be guaranteed and why it is so. Indeed, since every proposer tries to produce a tag that is greater than the tags of a majority of acceptor, two such proposers may execute many trials for the same step without ever succeeding to complete a phase two. Intuitively though, it is clear that if, for any step, there is a single proposer in the system during a long enough period of time, then the processors eventually decide in that step.

**Self-Stabilizing Paxos.** As we pointed out in the previous section, the Paxos algorithm uses unbounded integers to tag data. In practice, however, every integer handled by the processors is bounded by some constant \(2^b\) where \(b\) is the integer memory size. Yet, if every integer variable is initialized to a very low value, the time needed for any such variable to reach the maximum value \(2^b\) is actually way larger than any reasonable system’s timescale. For instance, counting from 0 to \(2^{64}\) by incrementing every nanosecond takes roughly 500 years to complete. Such a long sequence is said to be practically infinite. This leads to the following important remark from which the current work stems.

**Remark 1 (Paxos and Bounded Integers).** Assuming that the integers are theoretically unbounded is reasonable only when it is ensured, in practice, that every step and trial variables are initially set to low values, compared to the maximum value. In particular, any initialized execution of the Paxos algorithm with bounded integers is valid as long as the counters are not exhausted.

In the context of self-stabilization, however, a transient fault may produce fake decision messages in the communication channels, or make an acceptor accepting a consensus value that was not proposed. Such transient faults only break the Repeated Consensus conditions punctually and nothing can be done except waiting. However, a transient fault may also corrupt the Paxos step and trial variables in the processors memory or in the communication channels, and set them to a value close to the maximum value \(2^b\). This leads to an infinite suffix of execution in which the Repeated Consensus conditions are never jointly satisfied. This issue is much more worrying than punctual breakings of the Repeated Consensus specifications. Intuitively though, if one can manage to get every integer variable (step and trials) to be reset to low values at some point in time, then there is consequently a finite execution (ending with step or trial variables reaching the maximum value \(2^b\)) during which the system behaves like an initialized original Paxos execution that satisfies the Repeated Consensus Problem conditions\[^1\]. Since we use bounded integers, we cannot prove the safe execution to be infinite, but we can prove that

\[^1\]Modulo the unavoidable punctual breakings due to, e.g., fake decision messages.
this safe execution is as long as counting from 0 to $2^b$, which is as long as the length of an initialized and safe execution assumed in the original Paxos prior to exhausting the counters (cd Remark$^1$). This is what we call a practically self-stabilizing behaviour.

Replicated state machines have to perform steps that are commonly decided. In the original Paxos, decisions on steps at a processor may be learned out of order, but eventually every decision arrives, and therefore the processor can also perform locally the agreed upon steps in a sequence. To avoid gaps in the sequence of agreed upon steps, it is possible to use the Generalized Paxos approach$^{[14]}$, where decisions are made on the entire known sequence of steps, together with the new proposed step. In the case of self-stabilization and when there is a need for (eventual) identical steps execution by each participant, rather than merely only a simulation of a global robust virtual state machine, the decision subject is histories rather than the last state and next step. We mainly focus on the repeated consensus version that can decide on the last state of the replicated state machine and the next step, and then detail in Appendix$^2$ the very few modifications needed to obtain the Generalized Self-Stabilizing Paxos.

The repeated consensus on both the current state and the step requires, on the one hand, more communication, but on the other hand, addresses a long standing technicality of memory garbage collection from the array used to accumulate decided steps, as the decided last current step encapsulates all step history prior to its execution. The proposers always propose a step that immediately follows the last decided state it knows, and does not propose a new step before deciding, or learning about a decision on this or a subsequent state and step.

3 System Settings

All the basic notions we use (state, configuration, execution, asynchrony, ...) can be found in, e.g.,[5,15]. Here, the model we work with is given by a system of $n$ asynchronous processors in a complete communication network. Each communication channel between two processors is a bidirectional asynchronous communication channel of finite capacity $C$. Every processor has a unique identifier and the set $\Pi$ of identifiers is totally ordered. If $\alpha$ and $\beta$ are two processor identifiers, the couple $(\alpha,\beta)$ denotes the communication channel between $\alpha$ and $\beta$. A configuration is the vector of states of every processor and communication channel. If $\gamma$ is a configuration of the system, we note $\gamma(\alpha)$ (resp. $\gamma(\alpha,\beta)$) for the state of the processor $\alpha$ (resp. the communication channel $(\alpha,\beta)$) in the configuration $\gamma$. We informally$^1$ define an event as the sending or reception of a message at a processor or as a local state transition at a processor. Given a configuration, an event induces a transition to a new configuration. An execution is denoted by a sequence of configurations $(\gamma_k)_{0\leq k< T}$, $T \in \mathbb{N} \cup \{+\infty\}$ related by such transitions$^2$. A local execution at processor $\lambda$ is the sequence of states obtained as the projection of an execution on $\lambda$. The initial configuration of every execution is arbitrary and at most $f$ processors are prone to crash failures. A quorum is any set of at least $n - f$ processors. For any execution $E$, we note $\text{Live}(E)$ the set of processors that do not crash during $E$, and we note $\text{Crashed}(E)$ the complement of $\text{Live}(E)$. We make the following resilience assumption.

Assumption 1 (Resilience). The maximum number of crash failures $f$ satisfies $n \geq 2 \cdot f + 1$. Thus, there always exists a responding majority quorum and any two quorums have a non-empty intersection.

We also use the “happened-before” strict partial order introduced by Lamport$^{[11]}$. In our case, we note $e \rightsquigarrow f$ and we say that $e$ happens before $f$, or $f$ happens after$^3 e$. In addition, every processor

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$^1$For a formal definition, refer to, e.g.,[5,15].
$^2$For sake of simplicity, the events and the transitions are omitted.
$^3$Note that the sentences “$f$ happens after $e$” and “$e$ does not happen before $f$” are not equivalent.
has access to a read-only boolean variable $\Theta_\alpha$, e.g., from an unreliable failure detector (Section 7) that satisfies the following condition.

**Assumption 2 (Module $\Theta$).** For every infinite execution $E_\omega = \langle \gamma_k \rangle_{k \in \mathbb{N}}$, there is a non-empty set $\mathcal{P}(E_\omega)$, namely the proposers in $E_\omega$, of processors in $\text{Live}(E_\omega)$, such that, for every processor $\lambda$ in $\mathcal{P}(E_\omega)$, the value of $\Theta_\lambda$ is always true, and for every live processor $\mu$ not in $\mathcal{P}(E_\omega)$, the value of $\Theta_\mu$ is eventually always false.

Note that this module is extremely weak in the sense that it simply guarantees that at least one proposer is active. This proposer is not required to be unique in order for our algorithm to stabilize. A unique proposer is required only for the liveness of Paxos.

### 4 Tag System Overview

This section presents the tag system used in our algorithm. For didactic reasons, we first describe a simpler tag system that works when there is a single proposer, before adapting it to the case of multiple proposers. Formal definitions of bounded integers, labels and tags are given in Appendix A.

**Single Proposer.** We start by looking at Paxos tags $(s\ t)$ where the step $s$ and trial $t$ variables are integers bounded by a large constant $2^b$. Assume, for now, that there is a single proposer in the system, and let’s focus on its tag. The goal of this proposer is to succeed in imposing a consensus value for every step ranging from 0 to $2^b$, or at least from a low step value to a very high step value. The proposer can do a step increment, $(s\ t) \leftarrow (s + 1\ 0)$, or a trial increment within the same step, $(s\ t) \leftarrow (s + 1\ t)$. To impose some value in step $s$, it must reach a trial $t$ such that the tag $(s\ t)$ is lexicographically greater than every other processor tags in a majority of acceptors.

With an arbitrary initial configuration, some processors may have tags with step or trial value set to the maximum $2^b$, thus the proposer will not be able to produce a greater tag. We thus define a tag as a triple $(l\ s\ t)$ where $s$ and $t$ are the step and trial fields, and $l$ a label, which is not an integer but whose type is explicit below. We simply assume that it is possible to increment a label, and that two labels are comparable. The proposer can increment its trial variable, or increment its step variable and reset the trial variable, or increment the label and reset both the step and the trial variable. Now, if the proposer manages to produce a label that is greater than every label of the acceptors, then it will succeed in a practically infinite number of steps that mimicks the behaviour of the original Paxos tags. To do so, whenever the proposer notices an acceptor label which is not less than or equal to the proposer current label (such an acceptor label is said to cancel the proposer label), it records it in a history of canceling labels and produces a label greater than every label in its history.

Obviously, the label type cannot be an integer. Actually, it is sufficient to have some finite set of labels along with a comparison operator and a function that takes any finite (bounded by some constant) subset of labels and produces a label that is greater than every label in this subset. Such a device is called a finite labeling scheme. An implementation of such a finite labeling scheme was suggested in [2], and is formally presented in the Appendix B. Roughly saying, a label is a fixed length vector of integers from a bounded domain in which the first integer is called sting and the others are called antistings. A label $l_1$ is greater than a label $l_2$, noted $l_1 \prec l_2$, if the sting of $l_1$ does not appear in the antistings of $l_2$ but not vice versa. Given a finite set of labels $l_1, \ldots, l_r$, we can build a greater label $l$ by choosing a sting not present in the antistings of the $l_i$, and choosing the stings of the $l_i$ as antistings in $l$. It is important to note that the comparison relation between labels cannot be an order since transitivity does not hold.
Multiple Proposers. In the case of multiple proposers, the situation is a bit more complicated. Indeed, in the previous case, the single proposer is the only processor to produce labels, and thus it manages to produce a label greater than every acceptor label once it has collected enough information in its canceling label history. If multiple proposers were also producing labels, none of them would be ensured to produce a label that every other proposer will use. Indeed, the first proposer can produce a label \( l_1 \), and then a second proposer produces a label \( l_2 \) such that \( l_1 \prec l_2 \). The first proposer then sees that the label \( l_2 \) cancels its label and it produces a label \( l_3 \) such that \( l_2 \prec l_3 \), and so on.

To avoid such interferences between the proposers, we assume that the set of proposer identifiers is totally ordered and we define a tag to be a vector, say \( a \), whose entries are indexed by the proposer identifiers. Each entry \( a[\mu] \) of the tag \( a \) contains a tuple \( (l \; s \; t \; id \; cl) \) where \( l \) is a label, \( s \) and \( t \) are step and trial bounded integers, \( id \) is the identifier of the proposer that owns the tag, and \( cl \) is either a label that cancels \( l \) or the null value denoted by \( \bot \). The identifier of the proposer that owns the tag is included, so that two proposers never share the same content in any entry of their respective tags. The canceling field tells the proposer whether the corresponding label has been canceled by some label.

Therefore, a proposer, say \( \lambda \), has the possibility to use one of the entries of its tag, say \( a \), to specify the step and trial it is involved in. However, the entry used must be valid, i.e., the entry must contain a null canceling field value along with step and trial values strictly less than the maximum value \( 2^b \). The entry actually used by the proposer is determined by the lowest proposer identifier, noted \( \chi \). The entry corresponding to \( \chi(a) \) is valid. The entry \( a[\chi(a)] \) is referred to as the first valid entry in the tag. If the first valid entry is located at the left of the entry indexed by the proposer identifier, i.e., the identifier \( \chi(a) \) is less than the proposer identifier \( \lambda \), then the proposer can increment the step and trial values stored in the entry \( a[\chi(a)] \), but it cannot increment the label in the entry \( a[\chi(a)] \). The proposer can only increment the label, and thus reset the corresponding step and trial variables, stored in the entry indexed by its own identifier. In addition, whenever the entry indexed by the proposer identifier \( \lambda \) becomes invalid, the proposer \( \lambda \) produces a new label in the entry \( a[\lambda] \) and resets the integer variables to zero and the canceling field to the null value \( \bot \); this makes \( a[\lambda] \) a valid entry in the proposer tag. The important point is that, from a global point of view, the proposer identified by \( \lambda \) is the only proposer to introduce new labels in the entries indexed by \( \lambda \) in tags of the system. Besides, this also shows that any proposer \( \lambda \) has to record in its canceling label history only the canceling labels that are stored in the entry \( \lambda \) of tags.

A comparison relation is defined on tags so that every processor (proposer or acceptor) always try to use the valid entry with the lowest identifier. A tag \( b_1 \) is less than \( b_2 \), noted \( b_1 \prec b_2 \), when either the first valid entry of \( b_1 \) is located at the right of the first valid entry of \( b_2 \), or both first valid entries are indexed by the same identifier \( \mu \) and the tuple \( b_1[\mu].(l \; s \; t \; id) \) is lexicographically less than the tuple \( b_2[\mu].(l \; s \; t \; id) \). We note \( b_1 \simeq b_2 \) when both tags share the same first valid entry, and the corresponding contents are equal. We note \( b_1 \prec\sim b_2 \) when \( b_1 \prec b_2 \) or \( b_1 \simeq b_2 \). If there is no valid entry in both tags, or if the labels are not comparable, then the tags are not comparable.

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1 Which means that the label \( l \) is not canceled.
5 The Algorithm

In this section, we describe the self-stabilizing Paxos algorithm. We first present the variables before giving an overview of the algorithm. The details of the algorithm and the pseudo-code is given in Appendix C and F. In the sequel, we refer to the following datastructure.

Definition 1 (Fifo History). A fifo history \( H \) of size \( d \) on a set \( V \), is a vector of size \( d \) of elements of \( V \) along with an operator \( + \) defined as follows. Let \( H = (v_1, \ldots, v_d) \) and \( v \) an element in \( V \). If \( v \) does not appear in \( H \), then \( H + v = (v, v_1, \ldots, v_d) \), otherwise \( H + v = H \).

We define the tag storage limit \( K \) and the canceling label storage limit \( K^{cl} \) by \( K = n + Cn(n-1) \) and \( K^{cl} = (n+1)K \).

Variables. The state of a processor \( \alpha \) is defined by the following variables: the processor tag \( a_\alpha \), the processor proposal \( p_\alpha \) (a consensus value), the canceling label history \( H^{cl}_\alpha \) (fifo label history of size \( M = (K + 1)K^{cl} \)), the accepted proposal record \( r_\alpha \) and the label history \( H_\alpha \) described as follows. The accepted proposal record \( r_\alpha \) is a vector indexed by the processor identifiers. For each identifier \( \mu \), the field \( r_\alpha[\mu] \) contains either the null value \( \bot \) or a couple composed of a tag and consensus value. The variable \( H_\alpha \) is a vector indexed by the processor identifiers. For each identifier \( \mu \), the field \( H_\alpha[\mu] \) is a fifo label history of size \( K \). Note that all the label histories, and canceling label history are bounded by recent activity, since they accumulate only a polynomial number of the latest labels.

Tag Increment Functions. We define the step increment function, \( \nu^s \), and the trial increment function, \( \nu^t \). Both functions arguments are a processor identifier \( \lambda \), a tag \( x \), and the canceling label history \( H^{cl}_\lambda \), and they both return a tag (the incremented tag). First, a copy \( y \) of the tag \( x \) is created. The step increment function then increments the step in the first valid entry of \( y \) and resets the corresponding trial field to zero. The trial increment function only increments the trial field in the first valid entry of \( y \). Then, in both functions, it is checked whether the entry \( y[\lambda] \) is valid or not. If it is not, the label value \( \nu(\lambda)[x] \) is stored in the canceling label history, a new label is produced\(^1\) in \( x[\lambda] \) with the labels in the canceling label history, and the corresponding step and trial fields are reset to zero.

Protocol. Each processor can play two roles, namely, the acceptor role and the proposer role. A processor \( \alpha \) plays both the acceptor role and the proposer role as long as \( \Theta_\alpha \) is equal to \text{true}. When \( \Theta_\alpha \) is equal to \text{false}, the processor \( \alpha \) only plays the acceptor role. The current step and trial of a processor are determined by the step and trial values in the first valid entry of its tag. A proposer tries to impose some proposal for its current step. To do so, it executes the following two phases (cf. Algorithm \([3]\)).

(Phase 1). The proposer, say \( \lambda \), reads a new proposal and tries to recruit a quorum of acceptors by broadcasting a message (phase 1, message \( p1a \)) with its tag \( a_\lambda \) (Algorithm \([3]\), line \([7]\)). It waits for the replies from a majority of acceptors. When an acceptor \( \alpha \) receives this \( p1a \) message, it either adopts the proposer tag if the proposer tag is greater than its own tag \( a_\alpha \), or leaves its tag unchanged otherwise. The acceptor replies (phase 1, message \( p1b \)) to the proposer with its tag (updated or not) and the proposal, either null or a couple (tag, consensus value), stored in its accepted proposal variable \( r_\alpha(x(a_\alpha)) \).

Upon receiving the acceptor replies, the proposer \( \lambda \) knows if it has managed to recruit a majority of acceptors. In that case, the proposer \( \lambda \) can move to the second phase. Otherwise, \( \lambda \) has received at least one acceptor reply whose tag is not less than or equal to the proposer tag of \( \lambda \). At each reception

\(^1\)With the label increment function from the finite labeling scheme (cf. Definition \([3]\)).

\(^2\)One can think of having two threads on the same processor.
of such an acceptor tag, the proposer $\lambda$ modifies its tag in order for the proposer tag to be greater than the acceptor tag received. When messages are received from at least half of the processors, the proposer begins a new phase 1 with its updated tag.

(Phase 2). When the proposer $\lambda$ reaches this point, it has managed to recruit a quorum of acceptors and it knows all the latest proposals that they accepted for the entry $\chi(a_\lambda)$. Assume for instance that the proposer tag points to step $s$, i.e., the step value in the first valid entry $\mu$ of the proposer tag is equal to $s$. Then (Algorithm 4 line 11 to line 18) the proposer $\lambda$ first checks that the tags associated with the received proposals all share the same first valid entry and the same corresponding label as the tag of $\lambda$. If it is not the case, then $\lambda$ keeps its original proposal. Otherwise, it looks for non-null proposals for step $s$ and if there are some, it copies the proposal with the maximum tag (among those that point to step $s$) in its proposal variable. If there are more than two different proposals associated with this maximum tag, then $\lambda$ keeps its original proposal.

Next, the proposer $\lambda$ sends to all the acceptors a message (phase 2, message $p2a$) containing its tag along with the proposal it has computed (Algorithm 4 line 19) and waits for the replies of a majority of acceptors. When an acceptor $\alpha$ receives this $p2a$ message, if the proposer tag is greater than or equal to its own tag, then the acceptor adopts the proposer tag and stores the proposal in the variable $r_\alpha$. Otherwise, the acceptor leaves its tag and the accepted proposals record unchanged. Next, it replies (phase 2, message $p2b$) to the proposer with its tag (updated or not).

After having received the replies from a majority of acceptors, the proposer $\lambda$ knows if a majority have accepted its proposal. In that case, it broadcasts a decision message containing its proposer tag and the successful consensus value (Algorithm 4 line 21). At the reception of this message, any acceptor with a tag less than or equal to the proposer tag decides on the given proposal. The proposer $\lambda$ can then move to the next step. Otherwise, the proposer $\lambda$ has received tags that are not less than or equal to the proposer tag, and thus $\lambda$ updates its proposer tag accordingly, and starts a new phase 1.

Precisions. By “$\alpha$ adopts the tag $b$”, we mean that $\alpha$ copies the content of the first valid entry in $b$ to the same entry in $\alpha$’s acceptor tag, i.e., $a_\alpha[\chi(b)] \leftarrow b[\chi(b)]$. Furthermore, every time a processor $\alpha$ modifies its tag, it also does the following. If the label $l$, in some entry $\mu$ of the tag, is replaced by a new label, then the label $l$ is stored in the label history $H_\alpha[\mu]$ that corresponds to the identifier $\mu$ and a label that cancels the new label is looked for in the (bounded) label history $H_\alpha[\mu]$, updating the corresponding canceling field accordingly. If the label in the entry $\alpha$, i.e., the only entry in which the proposer $\alpha$ can create a label, gets canceled, then the associated canceling label is stored in the (bounded) canceling label history $H_{cl_\alpha}$. Any new label produced in the entry $\alpha$ of the tag at processor $\alpha$ is also stored in $H_{cl_\alpha}$. In addition, for every $\mu$, the accepted proposal $r_\alpha[\mu]$ is cleared, i.e., $r_\alpha[\mu] \leftarrow \bot$, whenever there is a label change in the entry $a_\alpha[\mu]$. A non-null field $r_\alpha[\mu] = (b,p)$ is also cleared whenever the label in the entry $b[\mu]$ is different than the label in the entry $a_\alpha[\mu]$, or the labels are equal but the entry $b[\mu]$ is lexicographically greater than the entry $a_\alpha[\mu]$. Finally, any processor $\alpha$ always checks that the entry $\alpha$ of its tag is valid. If it is not, the corresponding label is stored in the (bounded) canceling label history, a new label is produced instead and the step and trial fields are reset to zero.

6 Proof in a Nutshell

In this section, we present a summary of the main results of this work. Full details on the definitions, theorems and proofs are given in Appendix D. An epoch at processor $\lambda$ is a maximal local subexecution during which the first valid entry of its tag and the corresponding label remains constant (Appendix D.2).

\footnote{Note that only the entry $a_\alpha[\chi(b)]$ is modified. In fact, we have $a_\alpha \simeq b$ and not $a_\alpha = b$.}
Definition 10. Given a bounded integer \( h \), an \( h \)-safe epoch at processor \( \lambda \) is an epoch at \( \lambda \) that ends because the step or trial values in the first valid entry \( \mu \) of its tag have reached the maximum value \( 2^b \). In addition, in the configuration of the system that precedes this epoch, for any tag in the system, either the label in the entry \( \mu \) is different than the one used by \( \lambda \), or the corresponding step and trial values are less than \( h \). From the point of view of \( \lambda \), within an \( h \)-safe epoch, everything seems like an original Paxos execution initialized with integer values less than \( h \). For instance, we understand that for \( \lambda \) to jump, e.g., from step 10 to 15 within a 0-safe epoch, there must be a chain of events totally ordered by the happen-before relation that correspond to decision for steps 10 to 15. Thus, a \( h \)-safe epoch is actually as long as counting from \( h \) to \( 2^b \). The first main result (Appendix D.3, Theorem 1) states that there is some proposer \( \lambda \) at which there exists a 0-safe epoch. Note that this safe epoch is not necessarily unique, and it is not necessary to wait for it. Indeed, this results simply states that one has not to worry about having only very short epochs at processor \( \lambda \).

The second part of this work highlights the link between such a safe epoch at \( \lambda \) and the safety property on the global system. The idea is that \( \lambda \) is talking to quorums whose members cannot alter the first valid entry nor the corresponding label of \( \lambda \) during \( \sigma \), and must use the same first valid entry and corresponding label. Roughly saying, within a globally defined set of events related to the \( h \)-safe epoch at \( \lambda \), we show that for any two decision events for the same step \( s \geq h \) the two decided proposals are equal (Appendix D.5, Theorem 4).

These two results rely on a proper management of the labels. Indeed, the comparison relation on labels is not transitive; there might be cycles of labels. The algorithm uses histories of labels to detect such cycles. Precisely, the entry \( \lambda \) of the tag of a processor \( \alpha \) is associated with the label history \( H_{\alpha}[\lambda] \). Whenever, the corresponding label is replaced by a new label, the old label is stored in the label history, and a canceling label for the new label is looked for. This technique prevents the label field to follow a cycle whose length is less than the size of the label history. However, the size of the label history is chosen to equal the total label capacity of the system which implies that longer cycles are possible in the entry \( \lambda \) if and only if the processor \( \lambda \) produces at least one label meanwhile (Appendix D.3, Lemma 5). Thanks to this technique, it is possible to order events relatively to epochs occurring at \( \lambda \) since labels are produced by \( \lambda \) only at the end of some epochs occurring at \( \lambda \) (Appendix D.5, Lemma 8). If an epoch is practically infinite, it gives a way to discard events that happen after this epoch.

Besides, to guarantee liveness, the original Paxos algorithm requires a single proposer for each step; knowing that two steps may have different attributed proposers. In our model, an external module called \( \Theta \) is responsible for selecting the processors that act as proposers. For the tag system to stabilize, it is only needed that at least one processor acts infinitely often as a proposer. Nevertheless, this external module is generally implemented with a failure detector. For the sake of completeness we present a simple implementation of a self-stabilizing failure detector.

7 Self-Stabilizing Failure Detector

Liveness for some step \( s \) in Paxos is not guaranteed unless there is a unique proposer for this step \( s \). The original Paxos algorithm assumes that the choice of a distinguished proposer for a given step is done through an external module. In the sequel, we present an implementation of a self-stabilizing failure detector that works under a partial synchronism assumption. Note that this assumption is strong enough to implement a perfect failure detector, but a perfect failure detector is not mandatory for our algorithm to converge (i.e., the tag system). This brief section simply explain how a self-stabilizing implementation can be done; which is, although not difficult, not obvious either. Each processor \( \alpha \) has a vector \( L_{\alpha} \) indexed by the processor identifiers; each entry \( L_{\alpha}[\mu] \) is an integer whose value is comprised between 0 and some predefined maximum constant \( W \). Every processor \( \alpha \) keeps broadcasting a heartbeat message
containing its identifier (e.g., by using \[5, 6\]). When the processor \(\alpha\) receives a heartbeat from processor \(\beta\), it sets the entry \(L_\alpha[\beta]\) to zero, and increments the value of every entry \(L_\alpha[\rho]\), \(\rho \neq \beta\) that has value less than \(W\). The detector output at processor \(\alpha\) is the list \(F_\alpha\) of every identifier \(\mu\) such that \(L_\alpha[\mu] = W\). In other words, the processor \(\alpha\) assesses that the processor \(\beta\) has crashed if and only if \(L_\alpha[\beta] = W\).

(Interleaving of Heartbeats). For any two live processors \(\alpha\) and \(\beta\), between two receptions of heartbeat \(\langle \text{hb}, \beta \rangle\) at processor \(\alpha\), there are strictly less than \(W\) receptions of heartbeats from other processors. Under this condition, for every processor \(\alpha\), if the processor \(\beta\) is alive, then eventually the identifier \(\beta\) does not belong to the list \(F_\alpha\). The connection with the external module \(\Theta\) in Section 3 can be defined as follows: \(\Theta_\alpha = \text{true} \iff \alpha = \min(\mu; L_\alpha[\mu] < W)\). Under this hypothesis, we see that the module \(\Theta\) eventually satisfies the conditions in Assumption 2 Section 3.

8 Conclusion

The original Paxos algorithm provides a solution to the problem, for a distributed system, to reach successively several consensus on different requests to apply. A proper tagging system using natural integers is defined so that, although the liveness property, i.e., the fact that, in every consensus instance, every processor eventually decides, is not guaranteed, the safety property is ensured: no two processors decide on different values in the same consensus instance. The original formulation, however, does assume a consistent initial state and assumes that consistency is preserved forever by applying step transitions from a restricted predefined set of step transitions. This line of consistency preserving argument is fragile and error prone in any concrete system that should exhibit availability and functionality during very long executions. Hence, there is an urgent need for self-stabilizing on-going systems, and in particular for the very heart of asynchronous replicated state machine systems used by the leading companies to ensure robust services. One particular aspect of self-stabilizing systems is the need to re-examine the assumption concerning the use of (practically) unbounded time-stamps. While in practice it is reasonable for Paxos to assume that a bounded value, represented by 64 bits, is a natural (unbounded) number, for all practical considerations, in the scope of self-stabilization the 64 bits value may be corrupted by a transient fault to its maximal value at once, and still recovery following such a transient fault must be guaranteed. More generally, the designer of self-stabilizing systems, does not try to protect its system against specific “bad” scenarios. She assumes that some transient faults, whatever their origin is, corrupt (a part of) the system and ensures that the system recovers automatically after such fault occurrences.

Using a finite labeling scheme, we have defined a new kind of tag system that copes with such transient faults. The tag is defined as a vector indexed by the processor identifiers, such that each entry contains a label, a step and a trial value. Incrementing the label becomes a way to properly reset the step and trial values in a given entry of a tag. Each processor is responsible for producing labels only in the entry that corresponds to its identifier. Therefore, once it collects enough information about the labels present in its attributed entry, a processor is able to produce a label that no other processor can cancel. Hence, in a tag, there might be several entries with “winning” labels, and the owner of the tag uses the entry with the lowest identifier. Our algorithm ensures that at some point in time, almost all the processors uses the same entry, the same corresponding label and integer (step and trial) fields with low values. From this point, the system behaves like the original Paxos until the maximum value \(2^b\) is reached by some step or trial variables. This is what we named a “practically self-stabilizing” behaviour, since the length of the stable execution is not infinite as in classical self-stabilization but long enough for any concrete system’s timescale, just as assumed in the original Paxos algorithm.

\footnote{Recall that counting from 0 to \(2^{64}\) by incrementing every nanoseconds lasts about 500 years.}
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A Tag System - Formal Definitions

Definition 2 (Bounded Integer). Given a positive integer $b$, a $b$-bounded integer, or simply a bounded integer, is any non-negative integer less than or equal to $2^b$.

Definition 3 (Finite Labeling Scheme). A finite labeling scheme is a 4-tuple $\mathcal{L} = (\mathcal{L}, <, d, \nu)$ where $\mathcal{L}$ is a finite set whose elements are called labels, $<$ is a partial relation on $\mathcal{L}$ that is irreflexive ($l \neq l$) and antisymmetric ($\nexists (l, l') \mid l < l' \land l' < l$), $d$ is an integer, namely the dimension of the labeling scheme, and $\nu$ is the label increment function, i.e., a function that maps any finite set of at most $d$ labels to a label such that for every subset $A$ in $\mathcal{L}$ of at most $d$ labels, for every label $l$ in $A$, we have $l < \nu(A)$. We denote the reflexive closure of $<$ by $\simeq$.

Remark 2. The definition of a finite labeling scheme imposes that the relation $<$ is not transitive. Hence, it is not an order relation.

Definition 4 (Canceling Label). Given a label $l$, a canceling label for $l$ is a label $cl$ such that $cl \neq l$.

Definition 5 (Tag System). A tag system is given by a 4-tuple $(b, \Pi, \omega, \mathcal{L})$ where $b$ is positive integer, $\Pi$ is the totally ordered finite set of processor identifiers, $\omega$ is a special symbol such that $\omega \notin \Pi$ and $\mathcal{L}$ is a finite labeling scheme. In addition the order on $\Pi$ is extended as follows: for every $\mu \in \Pi$, $\mu < \omega$.

Definition 6 (Tag). Given a tag system $(b, \Pi, \omega, \mathcal{L})$, a tag is a vector $a[\mu] = (l \; t \; \text{id} \; cl)$ where $\mu$ and id are processor identifiers, $l$ is a label, $cl$ is either the null value noted $\bot$ or a canceling label for $l$, and $s$ and $t$ are $b$-bounded integers respectively called the step and trial fields. The entry indexed by $\mu$ in the tag $a$, or simply the entry $\mu$ in $a$, refers to the entry $a[\mu]$. The entry $\mu$ is said to be valid when the corresponding canceling field is null, $a[\mu].cl = \bot$, and both the corresponding step and trial values are strictly less than the maximum value, i.e., $a[\mu].s < 2^b$ and $a[\mu].t < 2^b$.

Definition 7 (First Valid Entry). Given a tag $a$, the first valid entry in the tag is defined by

$$\chi(a) = \min \{ \{ \mu \in \Pi \mid a[\mu] \text{ is valid} \} \cup \{ \omega \} \}$$

Definition 8 (Comparison of Tags). Given two tags $a$ and $a'$, we note $a < a'$ when either $\chi(a) > \chi(a')$ or $\chi(a) = \chi(a') = \mu < \omega$ and $\forall \mu \mid a[\mu].(l \; t \; \text{id}) < a'[\mu].(l \; t \; \text{id})$. We note $a \simeq a'$ when $\chi(a) = \chi(a')$ and $a[\chi(a)] = a'[\chi(a)]$. We note $a \leq a'$ when either $a < a'$ or $a \simeq a'$.

B Construction of a Finite Labeling Scheme

We show how to construct a finite labeling scheme $(\mathcal{L}, <, d, \nu)$. First, consider the set of integers $X = \{1, 2, \ldots, K\}$ with $K = d^2 + 1$. We define the set $\mathcal{L}'$ to be the set of every tuple $(z, A)$ where $z \in X$ is the sting, and $A \subset X$ with $|A| \leq d$ is called the antistings. The relation $<$ is defined as follows

$$l = (z, A) < l' = (z', A') \iff (z \in A') \land (z' \notin A) \quad (1)$$

The function $\nu$ is defined as follows. Given $r$ labels $(s_1, A_1), \ldots, (s_r, A_r)$ with $r \leq d$, the label $\nu(l_1, \ldots, l_r) = (s, A)$ is given by

$$s = \min \{ X - (A_1 \cup \cdots \cup A_r) \} \quad (2)$$

$$A = \{s_1, \ldots, s_r\} \quad (3)$$

The function is well-defined since $r \leq d$ and $|A_1 \cup \cdots \cup A_r| \leq d^2 < |X|$. In addition, for every $i$, we have $s \notin A_i$ and $s_i \in A_i$, thus $(s_i, A_i) < (s, A)$.

\(^1\)Lexicographical comparison using the corresponding relation on labels, integers and processor identifiers.
C Algorithm Details

We give more details about the algorithms. We consider a tag system \((b, \Pi, \omega, \mathcal{L}, \prec, d, v)\) such that \(\Pi\) is the set of processor identifiers and the labeling scheme dimension is equal to \((K + 1)K^c\).

C.1 Tag Procedures

Algorithm 1 defines a procedure clean that cleans the canceling fields of a given tag as follows. The procedure takes as input a processor identifier \(\lambda\) and a tag \(a\). After the completion of the procedure, for every entry \(\mu\) in the tag \(a\), if the canceling field \(a[\mu].cl\) is not null, then its value is a canceling label for the label in \(a[\mu].l\). In addition, every identifier value in \(a[\mu].id\) is equal to \(\lambda\). The second procedure fill_cl updates the canceling fields of two given tags \(x\) and \(y\) as follows. After the completion of the procedure, for any \(\mu \in \Pi\), if the label \(x[\mu].l\) or \(x[\mu].cl\) (not equal to \(\bot\)) cancels \(y[\mu].l\), then \(y[\mu].cl\) is not null. And if \(x[\mu].l = y[\mu].l\) with one of the integer fields in \(x[\mu]\) being equal to the maximum value \(2^b\), then both step and trial fields \(y[\mu].cl\) are equal to \(2^b\). The previous remarks also hold when exchanging \(x\) and \(y\).

Algorithm 2 defines the function check_entry whose arguments are a processor identifier \(\lambda\), a tag \(x\), and an history of labels \(L\). This function checks whether the entry \(x[\lambda]\) is valid or not. If this entry is invalid, it stores the label value \(x[\lambda].l\) in the history \(L\), produces\(^1\) a new label in \(x[\lambda]\) with the labels in the history \(L\) and resets the step and trial fields to zero. Algorithm 2 also defines the step increment function, \(\nu^s\), and the trial increment function, \(\nu^t\). Both functions arguments are a processor identifier \(\lambda\), a tag \(x\), and a fifo history of labels \(L\), and they both return a tag (the incremented tag). First, a copy \(y\) of the tag \(x\) is created. Then the tag \(y\) is cleaned with the procedure clean. The step increment function then increments the step in the first valid entry of \(y\) and resets the corresponding trial field to zero. The trial increment function only increments the trial field in the first valid entry of \(y\). Then, in both functions, it is checked whether the entry \(y[\lambda]\) is valid or not, and updated accordingly thanks to the function check_entry. Both functions return the tag \(y\).

C.2 Protocol

We focus on the reception of a proposer message by an acceptor (Algorithm 3). Say an acceptor \(\alpha\) receives a message \(\langle p1a, \lambda, b \rangle\) from proposer \(\lambda\). The acceptor \(\alpha\) first records in the canceling label history \(H^c_{\alpha}\) any label in the entry \(b[\alpha]\) that cancels the label \(a_{\alpha}[\alpha].l\) in the acceptor tag (line 4). Using the procedure fill_cl presented in Algorithm 1, the acceptor \(\alpha\) updates the canceling fields of both tags \(a_{\alpha}\) and \(b\). Then, it checks the validity of the entry \(a_{\alpha}[\alpha]\) with the procedure check_entry and updates it accordingly (line 5). If the updated tags satisfy \(a_{\alpha} \prec b\), then \(\alpha\) adopts the tag \(b\), i.e., it copies the content of the first valid entry \(b[\lambda(b)]\) to the entry \(a_{\alpha}[\lambda(b)]\) in \(a\) (line 7). If there has been a change of label in the entry \(a_{\alpha}[\lambda(b)]\), then the accepted proposal variable \(r_{\alpha}[\lambda(b)]\) is cleared, the old label is stored in the history \(H_{\alpha}[\lambda(b)]\), and \(\alpha\) looks in this history for labels that cancel the new label \(a_{\alpha}[\lambda(b)].l\), updating the corresponding canceling field accordingly (lines 9 to 11). Next, the acceptor checks for every identifier \(\mu\) if either the tag \(b\) in the accepted proposal \(r_{\alpha}[\mu]\) uses a label different than the label in the entry \(a_{\alpha}[\mu]\), or if the tuple \(a_{\alpha}[\mu].(l \ s \ i \ d)\) is less than the tuple \(b[\mu].(l \ s \ i \ d)\); in such a case, the entry \(r_{\alpha}[\mu]\) is cleared. In any case, whether it adopts the tag \(b\) or not, the acceptor \(\alpha\) replies to the proposer \(\lambda\) with a message \(\langle p1b, \alpha, a_{\alpha}, r_{\alpha}[\lambda(aa)] \rangle\) where \(aa\) is its updated (or not) acceptor tag and \(r_{\alpha}[\lambda(aa)]\) is the lastly accepted proposal for the entry \(\lambda(aa)\) (line 18).

\(^1\)With the label increment function from the finite labeling scheme (cf. Definition 3).
When an acceptor $\alpha$ receives a $2a$ message or a decision message containing a proposal $(b, p)$, the procedure is similar. It first updates the canceling label history $H^\lambda_a$, the canceling fields of $a_\lambda$ and $b$, and checks the validity of the entry $a_\alpha[\alpha]$ (lines 21 and 22). The difference with the previous case is that the condition to accept the proposal $(b, p)$ is $a_\alpha \preceq b$. In this case, the acceptor $\alpha$ adapts the tag $b$, updating the canceling field and the label history as in the case of a $1a$ message, stores the couple $(b, p)$ in its accepted proposal variable $r_\alpha[\chi(b)]$ and, in case of a decision message, decides on the couple $(b, p)$ (lines 24 and 25). In addition, if there has been a change of label in the entry $a_\alpha[\chi(b)]$, then the old label is stored in the history $H_b[\chi(b)]$, and $\alpha$ looks in this history for labels that cancel the new label $a_\alpha[\chi(b)].l$, updating the corresponding canceling field accordingly (lines 27 and 28). We say that the acceptor $\alpha$ has accepted the proposal $(b, p)$. Next, the acceptor checks for every identifier $\mu$ if either the tag $b$ in the accepted proposal $r_\alpha[\mu]$ uses a label different than the label in the entry $a_\alpha[\mu]$, or if the tuple $a_\alpha[\mu].(l \ s \ t \ id)$ is less than the tuple $b[\mu].(l \ s \ t \ id)$; in such a case, the entry $r_\alpha[\mu]$ is cleared. In case of a $2a$ message, whether it accepts the proposal or not, the acceptor $\alpha$ replies to the proposer $\lambda$ with a message $(2b, \alpha, a_\lambda)$ containing its updated (or not) acceptor tag (line 35). In case of a decision message, the acceptor does not reply.

At the end of any phase, a proposer executes a procedure named the preempting routine (Algorithm 5) that mainly consists in waiting for the replies from a majority of acceptors and suitably incrementing the proposer tag. The phase is considered successful if the routine returns $ok$ and failed otherwise. In this routine, the processor $\lambda$ waits for $n - f$ replies from the acceptors. Note that, although the pseudo-code suggests $\lambda$ receives only acceptor replies (Algorithm 5, line 5), the processor $\lambda$, as an acceptor, also processes messages ($1a$ or $2a$) from other proposers. The variable $a_{sent}$ stores the value of $a_\lambda$ that $\lambda$ has sent at the beginning of the phase. The variable $b$ is an auxiliary variable that helps filtering messages and is reset to $a_{sent}$ at the beginning of each new loop (line 4). For each message with tag $a_\alpha$ and proposal $r_\alpha$ received from a processor $\alpha$, the procedure updates the canceling fields of both $b$ and $a_\alpha$ (line 5).

If the current phase is a phase 1, then a reply is considered positive when the acceptor $\alpha$ has adopted the tag $\lambda$ sent, i.e., when $a_\alpha \simeq b$. If the current phase is a phase 2, the reply is considered positive when the acceptor $\alpha$ has adopted the tag $\lambda$ has sent and has accepted the corresponding proposal, i.e, $a_\alpha \simeq b$ and $p_\lambda = r_\alpha[\chi(b)].p$. The condition $C^+$ (line 7) summarizes these two cases. A reply is considered negative when received acceptor tag is not less than or equal to the tag the proposer $\lambda$ has sent, i.e., an acceptor tag $a_\alpha$ such that $a_\alpha \not\preceq b$ (condition $C^-$, line 8). The procedure discards any acceptor reply that does not satisfy the conditions $C^+$ nor $C^-$. The variable $M$ counts the number of positive replies. The routine returns $ok$ if all the replies are positive, i.e., $M = n - f$, and $nok$ otherwise (lines 36 and 37).

At each negative reply received, the routine updates the variable $a_\lambda$ so that $\lambda$ is always greater than the tag received. Precisely, it updates the canceling label history $H^\lambda_a$ (line 13), the canceling fields of $a_\alpha$ and $a_\lambda$ (line 14) and checks the validity of the entry $a_\lambda[\lambda]$ (line 15). Recall that this implies $\chi(a_\lambda) \leq \lambda$. Then, the routine checks if $a_\alpha$ is less than or equal to $a_\lambda$. If it is so, then the routine does not modify $a_\lambda$. Otherwise (lines 16 to 30), it checks if $a_\alpha$ has its first valid entry located at the left of $a_\lambda$’s first valid entry, i.e., $\chi(a_\alpha) < \chi(a_\lambda)$. In that case, the content of the entry $a_\alpha[\chi(a_\alpha)]$ is copied to the entry $a_\lambda[\chi(a_\alpha)]$ and the trial value is incremented. In addition, the previous label in $a_\lambda[\chi(a_\alpha)].l$ is stored in the label history $H^\lambda_a[\chi(a_\alpha)]$ and possible canceling labels for the new label in $a_\lambda[\chi(a_\alpha)].l$ are searched for in $H^\lambda_a[\chi(a_\alpha)]$ (lines 19 to 22). If the first valid entry $\chi(a_\alpha)$ in $a_\alpha$ is not located at the left of $a_\lambda$’s first valid entry, then necessarily $\chi(a_\alpha) = \chi(a_\lambda) = \mu$, since $a_\alpha \neq a_\lambda$. In that case, the routine compares the content of the entries indexed by $\mu$ in $a_\alpha$ and $a_\lambda$ (lines 24 to 30). Note that, since the routine has updated the
canceling fields, the corresponding labels are equal. If both entries $a_\alpha[\mu]$ and $a_\lambda[\mu]$ share the same step value, then $a_\lambda$ is updated with the time increment function $\nu^l$ (line 27). Otherwise, the step increment function is used (line 30).

D Proofs

D.1 Basics

**Lemma 1** (Pigeon-hole Principle). Consider a sequence $u = (u^i)_{1 \leq i \leq N}$ such that $\forall 1 \leq i \leq N, u^i \in \{0, 1\}$, and $N = (n + 1)m$ for some $n, m \in \mathbb{N} - \{0\}$. Assume that the cardinal of $\{i \mid u^i = 1\}$ is less than or equal to $n$. Then there exists $1 \leq i_0 \leq N$ such that for every $i_0 \leq i \leq i_0 + m - 1$, $u^i = 0$.

**Proof.** Divide the sequence $u$ in successive subsequences $\sigma^j$, $1 \leq j \leq n + 1$ such that each $\sigma^j$ length is $m$. If for every $1 \leq j \leq n + 1$, the sequence $\sigma^j$ contains at least one 1, then the number of 1 appearing in $u$ is at least $n + 1$, which leads to a contradiction. Hence, there is some $j_0$ such that the sequence $\sigma^{j_0}$ only contains 0.

**Lemma 2.** Any phase of the proposer algorithm eventually ends.

**Proof.** Let $\phi$ be a phase executed by some proposer $\lambda$. At the beginning of $\phi$, the proposer $\lambda$ has broadcast a message with its proposer tag $a_\lambda$, along with a consensus value $p$ in case of a $p2a$ message. Assumption 1 (Section 3) ensures that at least $n - f$ acceptors eventually reply. The only reason why $\phi$ would be endless is $\lambda$ discarding real replies from these acceptors in the preempting routine. For each such acceptor $\alpha$, when it receives the message sent by $\lambda$, it first updates the canceling fields in $a_\alpha$ and $a_\lambda$. Let $a, b$ respectively be the updated versions of $a_\alpha, a_\lambda$, and $\mu = \chi(a)$. According to the acceptor Algorithm 3 if the acceptor $\alpha$ adopts the tag $b$ then we have $a \simeq b$, and in case of a $p2a$ message, it also accepts the consensus value, i.e., $r_\alpha[\chi(a)] = (b, p)$; otherwise, we must have $a \not\simeq b$. These two cases correspond exactly to the conditions $C^+$ and $C^-$ in the Algorithm 5. In other words, real replies are not discarded by $\lambda$, and since there are at least $n - f$ such replies, phase $\phi$ eventually ends.

Given any configuration $\gamma$ of the system and any processor identifier $\mu$, let $S(\gamma)$ and $S^c(\mu, \gamma)$ be two sets as follows. The set $S(\gamma)$ is the set of every tag present either in a processor memory or in some message in a communication channel, in the configuration $\gamma$. The set $S^c(\mu, \gamma)$ denotes the collection of labels $l$ such that either $l$ is the value of the label field $x[\mu].l$ for some tag $x$ in $S(\gamma)$, or $l$ appears in the label history $H_\alpha[\mu]$ of some processor $\alpha$, in the configuration $\gamma$.

**Lemma 3** (Storage Limits). For every configuration $\gamma$ and every identifier $\mu$, we have $|S(\gamma)| \leq K$ and $|S^c(\mu, \gamma)| \leq K^{cl}$. In particular, the number of label values $x[\mu].l$ with $x \in S(\gamma)$ is less than or equal to $K$.

**Proof.** Consider a configuration $\gamma$. For each processor $\alpha$, there is one tag value (tag $a_\alpha$) in the processor state $\gamma(\alpha)$ of $\alpha$. For each communication channel $(\alpha, \beta)$, there are at most $C$ different messages in the channel state $\gamma(\alpha, \beta)$; all these messages have one tag each. Hence, the maximum number of tags present in the configuration $\gamma$ is $n$ plus $C$ times the number of communication channels. The network being complete, the number of communication channels is $C^{n(n-1)/2}$, thus we have $K \geq |S(\gamma)|$. For every $\alpha$, the maximum size of the history $H_\alpha[\mu]$ is $K$. Hence, the size of $S^c(\mu, \gamma)$ is bounded above by $K$ (labels $x[\mu].l$ for $x$ in $S(\gamma)$) plus $K$ times the number of processors (labels from $H_\alpha[\mu]$ for every processor $\alpha$), i.e., $(n + 1) \cdot K = K^{cl}$.

Otherwise, one would cancel the other and contradict the definition of the first valid counter.
D.2 Tag Stabilization - Definitions

Definition 9 (Interrupt). Let $\lambda$ be any processor and consider a local subexecution $\sigma = (\gamma_k(\lambda))_{k_0 \leq k \leq k_1}$ at $\lambda$. We note $a^k_\lambda$ for the value of $\lambda$’s tag in $\gamma_k(\lambda)$. We say that an interrupt has occurred at position $k$ in the local subexecution $\sigma$ when one of the following happens:

- $\mu < \lambda$, type $[\mu, \leftarrow]$ : the first valid entry moves to $\mu$ such that $\mu = \chi(a^{k+1}_\lambda) < \chi(a^k_\lambda)$, or the first valid entry does not change but the label does, i.e., $\mu = \chi(a^{k+1}_\lambda) = \chi(a^k_\lambda)$ and $a^k_\lambda[\mu].l \neq a^{k+1}_\lambda[\mu].l$.
- $\mu < \lambda$, type $[\mu, \rightarrow]$ : the first valid entry moves to $\mu$ such that $\mu = \chi(a^{k+1}_\lambda) > \chi(a^k_\lambda)$.
- type $[\lambda, \text{max}]$ : the first valid entry is the same but there is a change of label in the entry $\lambda$ due to the step or trial value having reached the maximum value $2^b$; we then have $\chi(a^{k+1}_\lambda) = \chi(a^k_\lambda) = \lambda$ and $a^k_\lambda[\lambda].l \neq a^{k+1}_\lambda[\lambda].l$.
- $[\lambda, \text{cl}]$ : the first valid entry is the same but there is a change of label in the entry $\lambda$ due to the canceling of the corresponding label; we then have $\chi(a^{k+1}_\lambda) = \chi(a^k_\lambda) = \lambda$ and $a^k_\lambda[\lambda].l \neq a^{k+1}_\lambda[\lambda].l$.

For each type $[\mu, \ast]$ ($\mu \leq \lambda$) of interrupt, we note $|[\mu, \ast]|$ the total number (possibly infinite) of interrupts of type $[\mu, \ast]$ that occur during the local subexecution $\sigma$.

Remark 3. If there is an interrupt like $[\mu, \leftarrow]$, $\mu < \lambda$, occurs at position $k$, then necessarily there is a change of label in the field $a^k_\lambda[\mu].l$. In addition, the new label $l'$ is greater than the previous label $l$, i.e., $l < l'$. Also note that, if $\chi(a^k_\lambda) = \lambda$, the proposer $\lambda$ never copies the content of the entry $\lambda$ of a received tag, say $a$, to the entry $\lambda$ of its proposer tag, even if $a^k_\lambda[\lambda].l < a[\lambda].l$. New labels in the entry $\lambda$ are only produced with the label increment function applied to the union of the current label and the canceling label history $H^\lambda_\ell$.

Definition 10 (Epoch). Let $\lambda$ be a processor. An epoch $\sigma$ at $\lambda$ is a maximal (for the inclusion of local subexecutions) local subexecution at $\lambda$ such that no interrupts occur at any position in $\sigma$ except for the last position. By the definition of an interrupt, every tag values within a given epoch $\sigma$ at $\lambda$ have the same first valid entry, say $\mu$, and the same corresponding label, i.e., for any two processor states that appear in $\sigma$, the corresponding tag values $a$ and $a'$ satisfies $\chi(a) = \chi(a') = \mu$. We note $\mu_\sigma$ and $l_\sigma$ for the first valid entry and associated label common to all the tag values in $\sigma$.

Definition 11 ($h$-Safe Epoch). Consider an execution $E$ and a processor $\lambda$. Let $\Sigma$ be a subexecution in $E$ such that the local subexecution $\sigma = \Sigma(\lambda)$ is an epoch at $\lambda$. Let $\gamma'$ be the configuration of the system right before the subexecution $\Sigma$, and $h$ be a bounded integer. The epoch $\sigma$ is said to be $h$-safe when the interrupt at the end of $\sigma$ is due to one of the integer fields in $a^k_\lambda[\mu_\sigma]$ having reached the maximum value $2^h$. In addition, for every processor $\alpha$ (resp. communication channel $(\alpha, \beta)$), for every tag $x$ in $\gamma'(\alpha)$ (resp. $\gamma'(\alpha, \beta)$), if $x[\mu_\sigma].l = l_\sigma$ then the step and trial values in $x[\mu_\sigma].l$ have values less than or equal to $h$.

Remark 4. If there is an epoch $\sigma$ at processor $\lambda$ such that $\mu_\sigma = \lambda$ and $\lambda$ has produced the label $l_\sigma$, then necessarily, at the beginning of $\sigma$, the step and trial value in $b_\lambda[\lambda]$ are equal to zero. However, other processors may already be using the label $l_\sigma$ with arbitrary corresponding step and trial values. The definition of a $h$-safe epoch ensures that the epoch is truly as long as counting from $h$ to $2^h$. 

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D.3 Tag Stabilization - Results

Lemma 4. Let \( \lambda \) be any processor. Then the first valid entry of its proposer tag is eventually always located at the left of the entry indexed by \( \lambda \), i.e., \( \chi(a_{\lambda}) \leq \lambda \).

Proof. This comes from the fact that whenever the entry \( a_{\lambda}[\lambda] \) is invalid, the processor \( \lambda \) produces a new label in \( a_{\lambda}[\lambda] \) and resets the step, trial and canceling field (cf. procedure check_entry, Algorithm 2). Once \( \chi(a_{\lambda}) \leq \lambda \), every consequent tag values is obtained either with the step or trial increment functions \((\nu^t, \nu^v)\), or by copying the content of a valid entry \( \mu < \lambda \) of some tag to the entry \( a_{\lambda}[\mu] \). Hence the first valid entry remains located before the entry \( \lambda \).

\[ \square \]

Remark 5. Thanks to this lemma, for every processor \( \lambda \), it is now assumed, unless stated explicitly, that the entry \( \chi(a_{\lambda}) \) is always located before the entry \( \lambda \), i.e., \( \chi(a_{\lambda}) \leq \lambda \).

Lemma 5 (Cycle of Labels). Consider a subexecution \( E \), a processor \( \lambda \) and an entry \( \mu < \lambda \) in the tag variable \( a_{\lambda} \). The label value in \( a_{\lambda}[\mu].l \) can change during the subexecution \( E \) and we note \((l^{i})_{1 \leq i \leq T+1}\) for the sequence of successive distinct label values that are taken by the label \( a_{\lambda}[\mu].l \) in the entry \( \mu \) during the subexecution \( E \). We assume that the first \( T \) labels \( l^{1}, \ldots, l^{T} \) are different from each other, i.e., for every \( 1 \leq i < j \leq T \), \( l^{i} \neq l^{j} \).

- If \( T > K \), then at least one of the label \( l^{i} \) has been produced by the processor \( \mu \) during \( E \).
- If \( T \leq K \) and \( l^{T+1} = l^{1} \), then when the processor \( \lambda \) adopts the label \( l^{T+1} \) in the entry \( \mu \) of its tag \( a_{\lambda} \), the entry \( \mu \) becomes invalid.

Proof. First note that a processor adopts a new label in the entry \( \mu \) of one of its tag, only when the old label is less than the new label. Hence, we have for every \( 1 \leq i \leq T \), \( l^{i} \prec l^{i+1} \) and, in particular, if \( l^{1} = l^{T+1} \), \( l^{2} \neq l^{T+1} \). Assume \( T > K \). Since in every configuration there is at most \( K \) tags in the system, and \( \mu \) is the only source of labels in the entry \( \mu \), the fact that \( \lambda \) has seen more than \( K \) different label values in the entry \( \mu \) is possible only if \( \mu \) has produced at least one label during \( E \). If \( T \leq K \) and \( l^{1} = l^{T+1} \), i.e., there is a cycle of length \( T \), then when \( \lambda \) adopts the label \( l^{T+1} = l^{1} \), the label history \( H_{\lambda}[\mu] \) contains the whole sequence \( l^{1}, \ldots, l^{T} \) since its size is \( K \). Hence, \( \lambda \) sees the label \( l^{2} \) that cancels the label \( l^{T+1} \), and the entry \( \mu \) becomes invalid.

\[ \square \]

Lemma 6 (Counting the Interrupts). Consider an infinite execution \( E_{\infty} \) and let \( \lambda \) be a processor identifier such that every processor \( \mu < \lambda \) produces labels finitely many times. Consider an identifier \( \mu < \lambda \) and any processor \( \rho \geq \lambda \). Then, the local execution \( E_{\infty}(\rho) \) at \( \rho \) induces a sequence of interrupts such that

\[ |[\mu, \leftarrow]| \leq R_{\mu} = (J_{\mu} + 1) \cdot (K + 1) - 1 \]  \hspace{1cm} (4)

where \( J_{\mu} \) is the number of times the processor \( \mu \) has produced a label since the beginning of the execution.

Proof. We note \((a_{\rho}^{k})_{k \in \mathbb{N}}\) the sequence of \( \rho \)'s tag values appearing in the local execution \( E_{\infty}(\rho) \). Assume on the contrary that \( |[\mu, \leftarrow]| \) is greater than \( R_{\mu} \). Note that after an interrupt like \( [\mu, \leftarrow] \), the first valid entry \( \chi(a_{\rho}) \) is equal to \( \mu \). In particular, the entry \( \mu \) is valid after such interrupts. Also, the label value in the entry \( a_{\lambda}[\mu].l \) does not change after an interrupt like \( [\mu, \rightarrow] \). We define an increasing sequence of integers \((f(i))_{1 \leq i \leq R_{\mu} + 1}\) such that the \( i \)-th interrupt like \( [\mu, \leftarrow] \) occurs at \( f(i) \) in the sequence \((a_{\rho}^{k})_{k \in \mathbb{N}}\). The sequence \( l^{i} = a_{\rho}^{f(i)+1}[\mu].l \) is the sequence of distinct labels successively taken by \( a_{\rho}[\mu].l \). We have \( l^{i} \prec l^{i+1} \) for every \( 1 \leq i \leq R_{\mu} \).

\[ ^{1} \text{Precisely, it has invoked the label increment function to update the entry } \mu \text{ of its tag } a_{\mu}. \]
Divide the sequence \((l^j)_{1 \leq i \leq R_\mu + 1}\) in successive segments \(u^j, 1 \leq j \leq J_\mu + 1,\) of size \(K + 1\) each. For any \(j,\) if all the \(K + 1\) labels in \(u^j\) are different, then, by Lemma 5, the processor \(\mu\) has produced at least one label. Since the processor \(\mu\) produces labels at most \(J_\mu\) many times, there is some sequence \(u^{h_\mu}\) within which some label appears twice. In other words, in \(u^{h_\mu}\) there is a cycle of length less than or equal to \(K.\) By Lemma 5, this implies that the entry \(\mu\) becomes invalid after an interrupt like \([\mu, \leftarrow]\); this is a contradiction. 

\[\text{Theorem 1 (Existence of a 0-Safe Epoch). Consider an infinite execution } E_\infty \text{ and let } \lambda \text{ be a processor such that every processor } \mu < \lambda \text{ produces labels finitely many times. We note } |\lambda| \text{ for the number of identifiers } \mu \leq \lambda, J_\mu \text{ for the number of times a proposer } \mu < \lambda \text{ produces a label and we define} \]

\[T_\lambda = \left( \sum_{\mu < \lambda} R_\mu \right) \cdot |\lambda| + 1 \cdot (K^{cl} + 1) \cdot (K + 1) \quad (5)\]

where \(R_\mu = (J_\mu + 1) \cdot (K + 1) - 1.\) Assume that there are more than \(T_\lambda\) interrupts at processor \(\lambda\) during \(E_\infty\) and consider the concatenation \(E_\mu(\lambda)\) of the first \(T_\lambda\) epochs, \(E_\mu(\lambda) = \sigma^1 \ldots \sigma^{T_\lambda}.\) Then \(E_\mu(\lambda)\) contains a 0-safe epoch.

**Proof.** By Lemma 6, we have \(\sum_{\mu < \lambda} |[\mu, \leftarrow]| \leq \sum_{\mu < \lambda} R_\mu\) in the local execution \(E_\mu(\lambda),\) a fortiori in the execution \(E_\mu(\lambda).\) By the pigeon-hole principle, there must be a local subexecution \(E_1(\lambda) = \sigma^1 \ldots \sigma^{i + X - 1}\) in \(E_\mu(\lambda),\) where \(X = (|\lambda| + 1) \cdot (K^{cl} + 1) \cdot (K + 1),\) that contains only interrupts like \([\mu, \rightarrow], [\lambda, \max]\) or \([\lambda, cl].\) Naturally, the number of interrupts like \([\mu, \rightarrow]\) in \(E_1(\lambda)\) is less than or equal to \(|\lambda|\). Hence, another application of the pigeon-hole principle gives a local subexecution \(E_2(\lambda) = \sigma^{j} \ldots \sigma^{j + Y - 1}\) in \(E_1(\lambda)\) where \(Y = (K^{cl} + 1) \cdot (K + 1)\) that contains only interrupts like \([\lambda, \max]\) or \([\lambda, cl].\)

Assume first that within \(E_2(\lambda),\) there is a subexecution \(E_3(\lambda) = \sigma^k \ldots \sigma^{k + Z - 1}\) where \(Z = K + 1\) in which there are only interrupts like \([\lambda, \max].\) Since \(K + 1 \leq M\) the size of the canceling label history, we have \(l_{\sigma^m}, \ldots, l_{\sigma^{k + h + 1}} \neq l_{\sigma^h},\) for every \(k < h < k + Z.\) In particular, all the labels \(l_{\sigma^m}, \ldots, l_{\sigma^{k + Z - 1}}\) are different. Since \(Z = K + 1\) and since there is at most \(K\) tags in a given configuration, there is necessarily some \(k \leq h < k + Z\) such that the label \(l_{\sigma^h}\) does not appear in the configuration \(\gamma^*\) that corresponds to the last position in \(\sigma^{h - 1}.\) Also, by construction, we have \(\mu_{\sigma^h} = \lambda\) and \(\sigma^h\) ends with an interrupt like \([\lambda, \max].\) Hence, \(\sigma^h\) is 0-safe.

Now, assume that there is no subexecution \(E_3\) in \(E_2\) as in the previous paragraph. This means that if we look at the successive interrupts that occur during \(E_2(\lambda),\) between any two successive interrupts like \([\lambda, cl],\) there is at most \(K\) interrupts like \([\lambda, \max].\) Since the length of \(E_2(\lambda)\) is \((K^{cl} + 1) \cdot (K + 1),\) there must be at least \(K^{cl} + 1\) interrupts like \([\lambda, cl].\) Let \(E_4(\lambda)\) be the local subexecution that starts with the epoch associated with the first interrupt like \([\lambda, cl]\) and ends with the epoch associated with the interrupt \([\lambda, cl]\) numbered \(K^{cl}.\) Let \(\sigma\) in \(E_2(\lambda)\) be the epoch right after \(E_4(\lambda).\) By construction, there is at most \(K^{cl} \cdot (K + 1)\) epochs in \(E_4(\lambda)\) which is the size \(M\) of the history \(H_{cl}^{\lambda}.\) Hence, at the beginning of \(\sigma,\) the history \(H_{cl}^{\lambda}\) contains all the labels the processor \(\lambda\) has produced during \(E_4\) as well as all the \(K^{cl}\) (exactly) labels it has received during \(E_4.\) Since there is at most \(K^{cl}\) candidates label for canceling in the system, necessarily, in the first configuration of \(\sigma,\) the history \(H_{cl}^{\lambda}\) contains every candidates label for canceling present in the whole system. And since \(l_{\sigma}\) is greater, by construction, than every label in the history \(H_{cl}^{\lambda}, l_{\sigma}\) was not present in the entry \(\lambda\) of some tag in the configuration that precedes \(\sigma\) and it cannot be canceled by any other label present in the the system. In addition, by construction, \(E_2\) only contains interrupts like \([\lambda, \max]\) or \([\lambda, cl].\) From what we said about \(l_{\sigma}\), the interrupt at the end of \(\sigma\) is necessarily \([\lambda, \max].\) In other words, the epoch \(\sigma\) is a 0-safe epoch. 

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1 Recall that the canceling label history also records the label produced in the entry \(\lambda.\)

2 Note that \(\lambda\) is the only processor to produce labels in entry \(\lambda,\) so during the subexecution that correspond to an epoch \(\sigma^h\) at \(\lambda,\) the set of labels in the entry \(\lambda\) of every tag in the system is non-increasing.
Remark 6. Note that the epoch found in the proof is not necessarily the unique 0-safe epoch in $E_c(\lambda)$. The idea is only to prove that there exists a practically infinite epoch. If the first epoch $\sigma$ at $\lambda$ ends because the corresponding label $l_{\sigma}$ in the entry $\alpha_{\sigma}$ gets canceled, but lasts a practically infinite long time, then this epoch can be considered, from an informal point of view, safe. One could worry about having only very “short” epochs at $\lambda$ due to some inconsistencies (canceling labels, or entries with high values in the step and trial fields) in the system. Theorem 7 shows that every time a “short” epoch ends, the system somehow loses one of its inconsistencies, and, eventually, the proposer $\lambda$ reaches a practically infinite epoch. Note also that a 0-safe epoch and a 1-safe or a 2-safe epoch are, in practice, as long as each other. Indeed, any $h$-safe epoch with $h$ very small compared to $2^b$ can be considered practically infinite. Whether $h$ can be considered very small depends on the concrete timescale of the system.

Remark 7. Besides, every processor $\alpha$ always checks that the entry $\alpha$ is valid, and, if not, it produces a new label in the entry $a_\alpha(\alpha)$ and resets the step, trial and canceling label field. Doing so, even if $\alpha$’s first valid entry $\mu$ is located before the entry $\alpha$, the processor $\alpha$ still works to find a “winning” label for its entry $\alpha$. In that case, if the entry $\mu$ becomes invalid, then the entry $\alpha$ is ready to be used, and a safe epoch can start without waiting any longer.

D.4 Safety - Definitions

To prove the safety property within a subexecution, we have to focus on the events that correspond to deciding a proposal, e.g., $(b, p)$ at processor $\alpha$. Such an event may be due to corrupted messages in the communication channels at any stage of the Paxos algorithm. Indeed, a proposer selects the proposal it will send in its phase 2 thanks to the replies it has received at the end of its phase 1. Hence, if one of these messages is corrupted, then the safety might be violated. However, there is a finite number of corrupted messages since the capacity of the communication channels is finite. Hence, violations of the safety do not happen very often. To formally deal with these issues, we define the notion of scenario that corresponds to specific chain of events involved in the Paxos algorithm.

Definition 12 (Scenario). Consider a subexecution $E = (\gamma_k)_{k_0 \leq k \leq k_1}$. A scenario in $E$ is a sequence $U = (U_i)_{0 \leq i \leq i_1}$ where each $U_i$ is a collection of events in $E$. In addition, every event in $U_i$ happens before every event in $U_{i+1}$. We use the following notations

- $\rho \xrightarrow{p_{1a}} (S, b)$: The proposer $\rho$ broadcasts a message $p_{1a}$ containing the tag $b$. Every acceptor in the quorum $S$ receives this message and adopts $\chi(b)$ the tag $b$.

- $(S, b) \xrightarrow{p_{1b}} \rho$: Every processor $\alpha$ in the quorum $S$ sends to the proposer $\rho$ a message containing the tag $b$, and containing the last proposal $r_a(\chi(a))$ they accepted. These messages are received by $\rho$.

- $\rho \xrightarrow{p_{2a}} (Q, b, p)$: The proposer $\rho$ broadcasts a message containing a proposal $(b, p)$. Every acceptor in the quorum $Q$ accepts the proposal $(b, p)$.

- $(Q, b, p) \xrightarrow{p_{2b}} \rho$: Every acceptor $\alpha$ in the quorum $Q$ sends to the proposer $\rho$ a message containing that it has accepted the proposal $(b, p)$. The proposer $\rho$ receives these messages.

- $\rho \xrightarrow{\text{dec}} (\alpha, b, p)$: the proposer $\rho$ sends a decision message containing the proposal $(b, p)$. The processor $\alpha$ receives this message, accepts and decides on the proposal $(b, p)$.

---

1Recall that this means it copies the entry $b[\chi(b)]$ in the entry $a_{\beta}[\chi(b)]$. 

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**Definition 13** (Simple Acceptation Scenario). Given a quorum of acceptors, b a tag, p a consensus value, ρ a proposer and α an acceptor, a simple acceptation scenario U of the first kind is defined as follows.

(U₀) A proposer ρ broadcasts a p₁a message with tag b.

(U₁) Every processor β from a quorum S receives this p₁a message, adopts the tag b and replies to ρ a p₁b message containing its tag aβ ≃ b and the lastly accepted proposal rβ[χ(αβ)].

(U₂) The proposer ρ receives these messages at the end of its Paxos phase 1, moves to the second phase of Paxos, and sends a p₂a message to a processor α telling it to accept the proposal (b, p).

(U₃) The processor α receives the p₂a message and accepts the proposal (b, p).

Given quorums S and Q, b a tag, p a consensus value, ρ a proposer and α an acceptor, a simple acceptation scenario V of the second kind is defined as follows.

(V₀) A proposer ρ broadcasts a p₁a message with tag b.

(V₁) Every processor β from a quorum S receives this p₁a message, adopts the tag b and replies to ρ a p₁b message containing its tag aβ ≃ b and the lastly accepted proposal rβ[χ(αβ)].

(V₂) The proposer ρ receives these messages at the end of its Paxos phase 1, moves to the second phase of Paxos, and sends a p₂a message to every processor in Q telling it to accept the proposal (b, p).

(V₃) Every processor in Q receives the p₂a message, accepts the proposal and replies to the proposer ρ with a p₂b message.

(V₄) The proposer ρ receives the replies from the acceptors in Q, and sends to the acceptor α a decision message containing a proposal (b, p).

(V₅) The acceptor α receives the decision message, accepts and decides on the proposal (b, p).

With the notations introduced, we have

\[
\begin{align*}
\text{(1-st kind)} & \quad \rho \xrightarrow{p₁a} (S,b) \xrightarrow{p₁b} \rho \xrightarrow{p₂a} (\alpha,b,p) \\
\text{(2-nd kind)} & \quad \rho \xrightarrow{p₁a} (S,b) \xrightarrow{p₁b} \rho \xrightarrow{p₂a} (Q,b,p) \xrightarrow{p₂b} \rho \xrightarrow{\text{dec}} (\alpha,b,p)
\end{align*}
\]

If the kind of scenario is not relevant, we note S ⇝ (α, b, p).

**Remark 8.** A simple acceptation scenario is simply a basic execution of the Paxos algorithm that leads a processor to either accept a proposal, or decide on a proposal (accepting it by the way).

**Definition 14** (Fake Message). Given a subexecution E = (γₖ)₀≤k≤₁ of a fake message relatively to the subexecution E, or simply a fake message, is a message that is in the communication channels in the first configuration γ₀ of the subexecution E.

**Remark 9.** This definition of fake messages comprises the messages at the beginning of E that were not sent by any processor, but also messages produced in the prefix of execution that precedes E.
**Definition 15** (Simple Fake Acceptation Scenario). Given a subexecution \( E \), we note \( \circ \rightarrow X \) if there exists an event \( e \) in \( X \) that corresponds to the reception of a fake message relatively to \( E \). With the previous notation, a simple fake acceptation scenario relatively to \( E \) is one of the following scenario.

\[
\begin{align*}
\circ & \xrightarrow{p_{2a}} (\alpha, b, p) \\
\circ & \xrightarrow{p_{1b}} \rho \xrightarrow{p_{2a}} (\alpha, b, p) \\
\circ & \xrightarrow{\text{dec}} (\alpha, b, p) \\
\circ & \xrightarrow{p_{2b}} \rho \xrightarrow{\text{dec}} (\alpha, b, p) \\
\circ & \xrightarrow{p_{2a}} (Q, b, p) \xrightarrow{p_{2b}} \rho \xrightarrow{\text{dec}} (\alpha, b, p) \\
\circ & \xrightarrow{p_{1b}} \rho \xrightarrow{p_{2a}} (Q, b, p) \xrightarrow{p_{2b}} \rho \xrightarrow{\text{dec}} (\alpha, b, p)
\end{align*}
\]

If the exact type is not relevant, we note \( \circ \rightsquigarrow (\alpha, b, p) \).

**Remark 10.** A simple fake acceptation scenario is somehow similar to a simple acceptation scenario except the fact that at least one fake message (relatively to the given subexecution) is involved during the scenario.

**Definition 16** (Composition). Consider two simple scenarios \( U = X \rightsquigarrow (\alpha_1, b_1, p_1) \), where \( X = \circ \) or \( X = (S_1, b_1) \), and \( V = S_2 \rightsquigarrow (\alpha_2, b_2, p_2) \) such that the following conditions are satisfied.

- The processor \( \alpha_1 \) belongs to \( S_2 \).
- Let \( e_2 \) be the event that corresponds to \( \alpha_1 \) sending a \( p_{1b} \) message in scenario \( V \). Then the event “\( \alpha_1 \) accepts the proposal \( (b_1, p_1) \)” is the last acceptation event before \( e_2 \). In addition, the proposer involved in the scenario \( V \) selects the proposal \( (b_1, p_1) \) as the highest-numbered proposal at the beginning of the Paxos phase 2. In particular, \( p_1 = p_2 \).
- All the tags involved share the same first valid entry, the same corresponding label and step value.

Then the composition of the two simple scenarios is the concatenation the scenarios \( U \) and \( V \). This scenario is noted

\[
X \rightsquigarrow (\alpha_1, b_1, p_1) \rightarrow S_2 \rightsquigarrow (\alpha_2, b_2, p_2)
\]

Note that the trial value is strictly increasing along the simple scenarios.

**Definition 17** (Acceptation Scenario). Given a subexecution \( E \), an acceptation scenario is the composition \( U \) of simple acceptation scenarios \( U_1, \ldots, U_r \), where \( U_1 \) is either a simple acceptation scenario or a simple fake acceptation scenario relatively to \( E \). We note

\[
X \rightsquigarrow (\alpha_1, b_1, p) \rightarrow S_2 \rightsquigarrow (\alpha_2, b_2, p) \ldots S_r \rightsquigarrow (\alpha_r, b_r, p)
\]

An acceptation scenario whose first simple scenario is not fake relatively to \( E \) is called real acceptation scenario relatively to \( E \). An acceptation scenario whose first simple scenario is fake relatively to \( E \) is called fake acceptation scenario relatively to \( E \). Given an event \( e \) that corresponds to some processor accepting a proposal, we note \( Sc(e) \) the set of acceptation scenarios that ends with the event \( e \).
Remark 11. Given an acceptation event or a decision event, there is always at least one way to trace back the scenario that has lead to this event. If one of these scenarios involve a fake message, then we cannot control the safety property for the corresponding step. Besides, note that all the tags involved share the same first valid entry $\mu$, the same corresponding label $l$, step value $s$ and consensus value $p$. Also, the trial value is increasing along the acceptation scenario.

Definition 18 (Scenario Characteristic). The characteristic of an acceptation scenario $U$ in which all tags have first valid entry $\mu$, corresponding label $l$, step value $s$ and consensus value $p$, is the tuple $\text{char}(U) = (\mu, l, s, p)$.

Definition 19 (Fake Characteristics). Consider a subexecution $E = (\gamma_k)_{k_0 \leq k \leq k_1}$. Given a scenario characteristic $(\mu, l, s, p)$, we note $\delta(E, \mu, l, s, p)$ the set of events in $E$ that correspond to accepting a proposal $(b, p)$ with $\chi(b) = \mu$ and $b[\mu], (l, s) = (l, s)$. A characteristic $(\mu, l, s, p)$ is said to be fake relatively to $E$ if there exists an event $e$ in $\delta(E, \mu, l, s, p)$ such that the set $\text{Sc}(e)$ contains a fake acceptation scenario relatively to $E$. We note $\mathcal{FC}(E)$ the set of fake characteristics relatively to $E$.

Definition 20 (Unsafe Steps). If we fix the identifier $\mu$ and the label $l$, we define the set of unsafe step values $\mathcal{US}(E, \mu, l)$ as the set of values $s$ such that there exists a consensus value $p$ with $(\mu, l, s, p) \in \mathcal{FC}(E)$.

Remark 12. Given an identifier $\mu$ and the label $l$, an unsafe step $s$ is a step in which an accepted proposal might be induced by fake messages, and thus, we cannot control the safety for this step.

Definition 21 (Observed Zone). Consider an execution $E$. Let $\lambda$ be a proposer and let $\Sigma$ be a subexecution such that the local execution $\sigma = \Sigma(\lambda)$ at $\lambda$ is a $h$-safe epoch. We note $F$ the suffix of the execution that starts with $\Sigma$. Assume that $\lambda$ executes at least two trials during its epoch $\sigma$. Let $Q^0$, $Q^f$ be the first and last quorums respectively whose messages are processed by the proposer $\lambda$ during $\sigma$. For each processor $\alpha$ in $Q^0$ (resp. $Q^f$), we note $e^0(\alpha)$ (resp. $e^f(\alpha)$) the event that corresponds to $\alpha$ sending to $\lambda$ a message received in the trial that corresponds to $Q^0$ (resp. $Q^f$).

The zone observed by $\lambda$ during the epoch $\sigma$, noted $Z(F, \lambda, \sigma)$, is the set of real acceptation scenarios relatively to $F$ described as follows. A real acceptation scenario relatively to $F$ belongs to $Z(F, \lambda, \sigma)$ if and only if it ends with an acceptation event that does not happen after the end of $\sigma$ and its first simple acceptation scenario $U = (S, b) \leadsto (\beta, b, p)$ is such that there exists an acceptor $\alpha$ in $S \cap Q^0 \cap Q^f$ at which the event $e^0(\alpha)$ happens before the event $e$ that corresponds to sending a $p\lambda b$ message in $U$, and the event $e$ happens before the event $e^f(\alpha)$ (cf. Figure 3).
Remark 13. The observed zone models a globally defined time period during which we will prove, under specific assumptions, the safety property (cf. Theorem 4).

D.5 Safety - Results

Lemma 7 (Fake Acceptation Scenarios). Consider a fake message \( m \), and two acceptation scenarios of characteristics \( (\mu, l, s, p) \) and \( (\mu', l', s', p') \) that begins with the reception of \( m \). Then both scenarios share the same characteristics, i.e., \( (\mu, l, s, p) = (\mu', l', s', p') \).

Proof. We have two scenarios that begins with the reception of \( m \). Focus on the first simple scenario of each acceptation scenario. Assume, for instance, that the message \( m \) is a \( p_1 b \) message and both simple fake acceptation scenarios are as follows

\[
\begin{align*}
\textcircled{1} & \overset{p_1 b}{\rightarrow} \rho \overset{p_2 a}{\rightarrow} (\alpha, b, p) \\
\textcircled{2} & \overset{p_1 b}{\rightarrow} \rho' \overset{p_2 a}{\rightarrow} (\alpha', b', p')
\end{align*}
\]

Since once a message is received, it is not in the communication channels anymore, the event “reception of \( m \) at \( \rho \)” and “reception of \( m \) at \( \rho' \)” must be the same. In particular \( \rho = \rho' \). Thanks to the messages it has received, the processor \( \rho \) computes a proposal \( (b, p) \) and broadcasts it. Hence, the processors \( \alpha \) and \( \alpha' \) receives (and accepts) the same proposal \( (b, p) \). By definition, \( \chi(b) = \mu, b[\mu].(l s) = (l s) \) and \( \chi(b') = \mu', b[\mu'].(l s) = (l' s') \). Therefore, \( (\mu, l, s, p) = (\mu', l', s', p') \). The other cases are analogous.

Theorem 2 (Fake Characteristics). Consider an execution \( E \). Let \( \lambda \) be a proposer, and let \( \Sigma \) be a subexecution such that the local execution \( \sigma = \Sigma(\lambda) \) at \( \lambda \) is an \( h \)-safe epoch, with first valid entry \( \mu_{\sigma} \) and label \( l_{\sigma} \). We note \( F \) the suffix of execution that starts with \( \Sigma \). Then, for every fake characteristic \( (\mu_{\sigma}, l_{\sigma}, s, p) \in \mathcal{F}_C(F) \), we have \( s < h \). In other words, every step \( s \in \mathcal{U}_F(F, \mu_{\sigma}, l_{\sigma}) \) satisfies \( s < h \).

Proof. Let \( \gamma' \) denote the configuration right before \( \Sigma \). Consider any fake scenario of characteristic \( (\mu_{\sigma}, l_{\sigma}, s, p) \) relatively to \( F \). The scenario begins by the reception of one or more fake messages. But, each of these fake messages carry tags with first valid entry \( \mu_{\sigma} \) and label \( l_{\sigma} \) that were present in \( \gamma' \). Hence, the corresponding step fields must have values less than \( h \).

Lemma 8 (Epoch and Cycle of Labels). Consider an execution \( E \). Let \( \lambda \) be a processor and consider a subexecution \( \Sigma \) such that the local execution \( \sigma = \Sigma(\lambda) \) is an epoch at \( \lambda \). We note \( F \) the suffix of the
execution $E$ that starts with $\Sigma$. Consider a processor $\rho$ and a finite subexecution $G$ in $F$ as follows: $G$ starts in $\Sigma$ and induces a local execution $G(\rho)$ at $\rho$ such that it starts and ends with the first valid entry of the tag $a_\rho$ being equal to $\mu_\sigma$ and containing the label $l_\sigma$, and the label field in the entry $a_\rho[\mu_\sigma]$ undergoes a cycle of labels during $G(\rho)$. Assume that, if $\mu_\sigma < \lambda$, then the processor $\mu_\sigma$ does not produce any label during $G$. Then $\mu_\sigma = \lambda$ and the last event of $\sigma$ happens before the last event of $G(\rho)$.

Proof. By Lemma 5, since the entry $a_\rho[\lambda]$ remains valid after the readoption of the label $l$ at the end of $G(\rho)$, the proposer $\mu_\sigma$ must have produced some label $l'$ during $G$ (hence $\mu_\sigma = \lambda$) that was received by $\rho$ during $G$. Necessarily, the production of $l'$ happens after the last event of $\sigma$ at $\lambda$, thus the last event of $G(\rho)$ at $\rho$ also happens after the last event of $\sigma$ at $\lambda$.

Theorem 3 (Weak Safety). Consider an execution $E$. Let $\lambda$ be a processor and let $\Sigma$ be a subexecution such that the local execution $\sigma = \Sigma(\lambda)$ at $\lambda$ is an $h$-safe. We note $F$ the suffix of the execution that starts with $\Sigma$. Consider a step value $s$ and the two following simple scenarios

$U_1 = \rho_1 \xrightarrow{p_1 a} (S_1, b_1) \xrightarrow{p_1 b} (Q_1, b_1, p_1) \xrightarrow{p_2 a} (Q_1, b_1, p_1) \xrightarrow{p_2 b} \rho_1 \xrightarrow{d e c} (\alpha_1, b_1, p_1)$ \hspace{0.5cm} (18)

$U_2 = (S_2, b_2) \xrightarrow{} (\alpha_2, b_2, p_2)$ \hspace{0.5cm} (19)

with characteristics $(\mu_\sigma, l_\sigma, s, p_1)$ and $(\mu_\sigma, l_\sigma, s, p_2)$ respectively. In addition, we assume that $b_1[\mu_\sigma].t > h$ and $\tau_1 \leq \tau_2$ where $\tau_2 = b_1[\lambda].t(id)$. We note $e_1$ for the acceptance event $(\alpha_1, b_1, p_1)$. Assume that both events $e_1$ and $e_2$ occur in $F$ and $s \notin \mathcal{U}.\mathcal{G}(F, \mu_\sigma, l_\sigma)$. In addition, assume that, if $\mu_\sigma < \lambda$, then the processor $\mu_\sigma$ does not produce any label during $F$. Then either $p_1 = p_2$ or the last event of $\sigma$ happens before one of the event $e_1$ or $e_2$.

Proof. We assume that both events $e_1$ and $e_2$ do not happen after the last event of $\sigma$ and we prove that $p_1 = p_2$. Since $s$ is not in $\mathcal{U}.\mathcal{G}(F, \mu_\sigma, l_\sigma)$, every scenario in $\mathcal{S}(e_1)$ or $\mathcal{S}(e_2)$ are real acceptance scenarios relatively to $F$. We note $\gamma^*$ the configuration right before the subexecution $\Sigma$. We prove the result by induction on the value of $\tau_2$.

(Bootstrapping) We first assume that $\tau_2 = \tau_1$. In particular, $\rho_1 = \tau_1.id = \tau_2.id = \rho_2$. If $p_1 \neq p_2$, this means that $\rho_1$ has sent two $p2a$ messages with different proposals and the same tag. Note $e$ and $f$ the events that correspond to these two sendings. None of the events $e$ and $f$ occurs in the execution prefix $A$. Otherwise, since $e_1$ and $e_2$ occur in $F$, the configuration $\gamma^*$ would contain a tag $x$ with $x[l_\sigma].t = l_\sigma$ and $x[l_\sigma].t > h$; this is a contradiction since $\sigma$ is $h$-safe. Hence, $e$ and $f$ occur in $F$. Then, there must be a cycle of labels in the entry $a_\rho[\mu_\sigma]$ between the $e$ and $f$. By Lemma 8, this implies that the last event of $\sigma$ happens before the event $e_1$ or $e_2$; this is a contradiction. Hence, $p_1 = p_2$.

(Induction) Now, $\tau_2$ is any value such that $\tau_2 < \tau_2$ and we assume the result holds for every value $\tau$ such that $\tau_1 < \tau < \tau_2$. Pick some acceptor $\beta$ in $Q_1 \cap S_2$. From its point of view, there are two events $f_1$ and $f_2$ at $\beta$ that respectively correspond to the acceptance of the proposal $(b_1, p_1)$ in the scenario $U_1$ (reception of a $p2a$ message), and the adoption of the tag $b_2$ in the scenario $U_2$ (reception of a $p1a$ message). First, the events $f_1$ and $f_2$ do not occur in the execution prefix $A$. Otherwise there would exist a tag value $x$ in $\gamma^*$ such that $x[l_\sigma] = l_\sigma$ and $x[l_\sigma].t > h$; this is a contradiction, since $\sigma$ is $h$-safe. Hence, $f_1$ and $f_2$ occur in the suffix $F$.

We claim that $f_1$ happens before $f_2$. Otherwise, since $\tau_2 > \tau_1$, there must be a cycle of labels in the field $a_\beta[\mu_\sigma].l$. By Lemma 8, this implies that the last event of $\sigma$ happens before the event $f_1$, and thus

\[\text{Modulo } \equiv.\]
before the event $e_1$; contradiction. Hence, $f_1$ happens before $f_2$. We claim that the $p1b$ message the acceptor $\beta$ has sent contains a non-null lastly accepted proposal $r_\beta[\mu_\sigma] = (b, p)$ such that $\chi(b) = \mu_\sigma$, $b[\mu_\sigma], (l s) = (l_\sigma s)$ and $\tau_1 \leq b[\mu_\sigma].(t id) < \tau_2$. Otherwise, there must be a cycle of labels between $f_1$ and $f_2$, which implies that $f_2$, and thus $e_2$, happens after the end of $\sigma$.

Now, the proposer $\rho_2$ receives a set of proposals from the acceptors of the quorum $S_2$, including at least one non-null proposal from $\beta$. It first checks that every tag received uses the entry $\mu_\sigma$ and the label $l_\sigma$ and that there is no different proposals with two tags that share the same content in entry $\mu_\sigma$ before continuing to the second phase of Paxos, and if it is not the case, it updates its proposer tag and executes another phase 1 of Paxos. Hence, since $p_2$ has moved to the second phase of Paxos, it means that no such issue has happened. Then, it selects among the proposals whose tags point to the step $s$ the proposal $(b_c, p_c)$ with the highest tag. In particular, $\chi(b_c) = \mu_\sigma$, $b_c[\mu_\sigma].(l s) = (l_\sigma s)$. Since $\rho_2$ has received the proposal $(b, p)$ from $\beta$, we have $\tau_1 \leq \tau_c < \tau_2$, where $\tau_c = \beta_c[\mu_\sigma].(t id)$. Let $\beta_c$ be the proposer in $S_2$ which has sent to $\rho_2$ the proposal $(b_c, p_c)$ in the $p1b$ message. There is an event $f_c$ in $F$ that corresponds to $\beta_c$ accepting the proposal $(b_c, p_c)$. Otherwise there would exist a tag value $x$ in $\gamma$ such that $x[\mu_\sigma].l = l_\sigma$ and $x[\mu_\sigma].t > \lambda$; this is a contradiction, since $\sigma$ is $h$-safe. Next, since $s \notin \mathcal{U}(F, \mu_\sigma, l_\sigma)$, $\chi(b_c) = \mu_\sigma$, and $b_c[\mu_\sigma].(l s) = (l_\sigma s)$, the set $Sc(\epsilon_2)$ does not contain any fake acceptation scenario relatively to $F$, thus neither the set $Sc(f_c)$. We can pick a real scenario in $Sc(f_c)$ and apply the induction hypothesis, which shows that $p_2 = p_1$. Hence, $p_1 = p_2$, since $p_c$ is the consensus value the proposer $\rho_2$ sends during the corresponding Paxos phase 2.

**Corollary 1 (Weak Safety).** Consider an execution $E$. Let $\lambda$ be a processor and let $\Sigma$ be a subexecution such that the local execution $\sigma = \Sigma(\lambda)$ at $\lambda$ is an $h$-safe epoch. We note $F$ the suffix of the execution that starts with $\Sigma$. Consider a step value $s$ and two decision events $e_1 = (\alpha_i, b_i, p_i)$, $i = 1, 2$, such that $\chi(b_i) = \mu_\sigma$, $b_i[\mu_\sigma].(l s) = (l_\sigma s)$ and $b_i[\mu_\sigma].t > h$. Assume that both events $e_1$ and $e_2$ occur in $F$ and $s \geq h$. In addition, assume that, if $\mu_\sigma < \lambda$, then the processor $\mu_\sigma$ does not produce any label during $F$. Then either $p_1 = p_2$ or the last event of $\sigma$ happens before one of the event $e_1$ or $e_2$.

**Proof.** Since $e_1$ and $e_2$ are decision events, and since $s$ is not in $\mathcal{U}(F, \mu_\sigma, l_\sigma)$ ($s \geq h$, cf Theorem 3), there are two real acceptation scenarios in $Sc(e_1)$ and $Sc(e_2)$ relatively to $F$ respectively that contains simple acceptation scenarios of the second kind as follows:

$U_1 = \rho_1 \xrightarrow{p1a} (S_1, c_1) \xrightarrow{p1b} \rho_1 \xrightarrow{p2a} (Q_1, c_1, p_1) \xrightarrow{p2b} \rho_1 \xrightarrow{dec} (\beta_1, c_1, p_1)$

$U_2 = \rho_2 \xrightarrow{p1a} (S_2, c_2) \xrightarrow{p1b} \rho_2 \xrightarrow{p2a} (Q_2, c_2, p_2) \xrightarrow{p2b} \rho_2 \xrightarrow{dec} (\beta_2, c_2, p_2)$

with characteristics $(\mu_\sigma, l_\sigma, s, p_1)$ and $(\mu_\sigma, l_\sigma, s, p_2)$ respectively and trial values $c_i[\mu_\sigma].t$ greater than $h$. We note $\tau_i = c_i[\mu_\sigma].(t id)$. Whether $\tau_1 \leq \tau_2$ or $\tau_2 \leq \tau_1$, Theorem 1 yields the result.

**Theorem 4 (Safety).** Consider an execution $E$, a proposer $\lambda$ proposer and a subexecution $\Sigma$ such that the local execution $\sigma = \Sigma(\lambda)$ at $\lambda$ is a $h$-safe epoch for some bounded integer $h$. We note $F$ the suffix of execution that starts with $\Sigma$. Assume that the observed zone $Z(F, \lambda, \sigma)$ is defined and that, if $\mu_\sigma < \lambda$, then the processor $\mu_\sigma$ does not produce any label during $F$. Consider two scenarios $U_1, U_2$ in $Z(F, \lambda, \sigma)$ with characteristics $(\mu_1, l_1, s_1, p_1)$ and $(\mu_2, l_2, s_2, p_2)$ such that $\mu_\sigma \leq \min(\mu_1, \mu_2)$ and both scenarios contain simple acceptation scenarios with tags whose associated trial values are greater than $h$. Then $(\mu_1, l_1) = (\mu_2, l_2) = (\mu_\sigma, l_\sigma)$, and if $s_1 = s_2 \geq h$ then $p_1 = p_2$.

**Proof.** Assume that the scenario $U_1$ is such that $\mu_1 > \mu_\sigma$. Let $V = (S, b) \sim (\beta, b, p)$ be its first simple acceptation scenario. By definition of the observed zone $Z(F, \lambda, \sigma)$, there exists an acceptor $\alpha$ in $S \cap Q^0 \cap Q^f$ such that we have the happen-before relations $e^0(\alpha) \sim e \sim e^f(\alpha)$, where $e$ is the event that...
corresponds to $\alpha$ sending a $p1b$ message in the scenario $V$. At $e^0(\alpha)$ and $e^f(\alpha)$, messages are sent to $\lambda$ and are processed during $\sigma$. Hence, the corresponding tag values of the variable $a_\alpha$ must use the entry $\mu_\sigma$ and the label $l_\sigma$. Otherwise, the message either is not processed or causes an interrupt at processor $\lambda$. Now, at event $e$, the first valid entry of the variable $a_\alpha$ is $\mu_1 > \mu_\sigma$ which implies that the entry $\mu_\sigma$ is invalid. Hence, between $e^0(\alpha)$ and $e^f(\alpha)$, the entry $a_\alpha[\mu_\sigma]$ becomes invalid and valid again. There must be a cycle of labels in the label field $a_\alpha[\lambda].l$. Lemma \[8\] implies that the last event of $\sigma$ happens before $e^f(\alpha)$; by the definition of $e^f(\alpha)$, this is a contradiction. Therefore $\mu_1 = \mu_\sigma$. If $l_1 \neq l_\sigma$, then there must also be a cycle of labels in the entry $a_\alpha[\mu_\sigma]$ between $e^0(\alpha)$ and $e^f(\alpha)$, which leads to a contradiction again, thanks to the same argument. Therefore, $l_1 = l_\sigma$. Of course, the previous demonstration also shows that $(\mu_2,l_2) = (\mu_\sigma,l_\sigma)$. If $s_1 = s_2 \geq h$, then Corollary \[1\] the fact that the trial values associated to the scenarios $U_1$ and $U_2$ are greater than $h$ and the fact that the two acceptance events in scenarios $U_1$ and $U_2$ do not happen after the end of $\sigma$ imply that $p_1 = p_2$.

Remark 14. In the case $\mu_\sigma < \lambda$, assuming that $\mu_\sigma$ does not produce any label during $F$ means that the proposer $\lambda$ should be the live processor with the lowest identifier. To deal with this issue, one can use a failure detector.

E Generalized Self-Stabilizing Paxos

A command history is a sequence $c_1c_2\ldots c_r$ of state-machine commands of length at most $r < 2^b$. Given sequences $p_1$ and $p_2$, we note $p_1 \sqsubseteq p_2$ when $p_1$ is a prefix of $p_2$. Each learner $\alpha$, has a command history variable $\text{learned}_\alpha$ that represents the sequence of commands decided so far. We denote the empty sequence by $\bot$, the concatenation of two command histories by $p_1 \circ p_2$ and the length of a command history $p$ by $|p|$. Following [14], the Generalized Consensus specifications are:

- (Non-triviality) For any learner $\alpha$, $\text{learned}_\alpha$ is always a sequence of proposed commands.

- (Stability) For any learner $\alpha$, the value of $\text{learned}_\alpha$ at any time is a prefix of its value at any later time.

- (Consistency) For any learners $\alpha$ and $\beta$, it is always the case that one of the sequences $\text{learned}_\alpha$ and $\text{learned}_\beta$ is a prefix of the other.

- (Liveness) If command $cmd$ is proposed and $\alpha$ is a learner, then eventually $\text{learned}_\alpha$ will contain the command $cmd$.

E.1 Algorithm

Our generalized self-stabilizing Paxos, assigns empty history of state-machine commands whenever an epoch change takes place. Thus, when the epoch is not changed for practically infinite execution the accumulated history of state-machine commands is extended by practically infinitely many new globally decided upon state-machine commands, and therefore act as a virtual single state machine. Such an execution allows the replicated state machine to stabilize in the interaction with the (possibly interactive) algorithms that use the replicated state machine as their virtual machine.

We now explain how to adapt the self-stabilizing repeated consensus algorithm to obtain a Generalized Self-Stabilizing Paxos. We choose the type of consensus value to be a command history. We keep the same variables as before, and we simply add command history $\text{learned}_\alpha$ that is modified only on decisions (Algorithm \[6\]). The acceptor algorithm and the preempting routine are not modified.
We only add minor modifications to the proposer algorithm (Algorithm 7) as follows. At the beginning of the loop (line 3), the proposer, say $\lambda$, reads a command $cmd$, and initializes its variable $p_\lambda$ to $p^* \circ cmd$ (i.e. the command history with a single command) where $p^*$ is the value of $p_\lambda$ at the beginning of the loop. The phase 1 remains the same. At the beginning of phase 2, the proposer $\lambda$ has collected replies (set $\Gamma$, line 16) from a majority of acceptors. If the tags in $\Gamma$ satisfies a coherence condition, the proposer selects the proposals $(a, p)$ such that the tag $a$ is maximal, and discards those that do not satisfy $a[\chi(a_\lambda), s = |p|]$. Noting $\Gamma_0$ the filtered proposals, $\lambda$ selects the command sequence $p_{\text{max}}$ in $\Gamma_0$ that is maximal according to some lexicographical order (to break ties), and $\lambda$ sets its command history $p_\lambda$ to the concatenation $p_{\text{max}} \circ cmd$. Note that, if there are only null collected replies, or if all of the replies were incoherent, then $p_\lambda$ keeps its value $p^* \circ cmd$. Next, the proposer $\lambda$ executes the second phase as in the previous algorithm.

In addition, any time the proposer undergoes a change of first valid entry, or a change of label, the proposer either cuts its command sequence $p_\lambda$ (via $p^*$ in the pseudo-code) or fill it with $nop$ operations in order to have a length equal to the step field in the first valid entry of $a_\lambda$ (command $\text{truncate}$).

E.2 Proofs

From the four requirements of Generalized Consensus, we only outline the proofs for stability and consistency requirements. Indeed, the non-triviality condition follows from the fact that histories are extended only by processors, namely by a concatenation of a new command to existing histories (Algorithm 7, lines 7 and 22). And the liveness condition relies on a failure detector, or more precisely, on the possibility for a proposer to complete the Paxos phases; which is common to the Paxos algorithm.

Theorem 1, that ensures the existence of a safe epoch, is still valid in this framework since the tag system is not related to the type of consensus values and to the way they are processed. Theorem 4 can be reformulated as follows.

**Theorem 5** (Generalized Paxos Stability and Consistency). Consider an execution $E$, a proposer $\lambda$, proposer and a subexecution $\Sigma$ such that the local execution $\sigma = \Sigma(\lambda)$ at $\lambda$ is a $h$-safe epoch for some bounded integer $h$. We note $F$ the suffix of execution that starts with $\Sigma$. Assume that the observed zone $Z(F, \lambda, \sigma)$ is defined and that, if $\mu_\sigma < \lambda$, then the processor $\mu_\sigma$ does not produce any label during $F$. Consider two scenarios $U_1$, $U_2$ in $Z(F, \lambda, \sigma)$ with characteristics $(\mu_1, l_1, s_1, p_1)$ and $(\mu_2, l_2, s_2, p_2)$ such that $\mu_\sigma \leq \min(\mu_1, \mu_2)$ and both scenarios contain simple acceptance scenarios with tags whose associated trial values are greater than $h$. Then $(\mu_1, l_1) = (\mu_2, l_2) = (\mu_\sigma, l_\sigma)$, and if $h \leq s_1 \leq s_2$ then $p_1 \subseteq p_2$.

This theorem ensures the stability and consistency condition of the Generalized Consensus problem.
F Algorithm Pseudo-Code

Algorithm 1: Tags - Procedures
1 function clean(λ : processor identifier, a : tag)
2    foreach μ ∈ Π do
3       if a[μ].cl ≤ a[μ].l then a[μ] ← ⊥
4       a[μ].id ← λ
5    end
6 end
7 function fill_cl(x, y : tags)
8    x_c ← x, y_c ← y
9    foreach μ ∈ Π do
10       if y_c[μ].(l or cl) ≠ x[μ].l then
11          x[μ].cl ← y_c[μ].(l or cl)
12       if y_c[μ].l = x[μ].l and y_c[μ].(s or t) = 2^b then
13          x[μ].(s or t) ← (2^b + 2^b)
14       idem by exchanging (x, x_c) and (y, y_c)
15    end
16 end

Algorithm 2: Tags - Increment functions
1 function check_entry(λ : identifier, x : tag, L : history of labels)
2    if x[λ] is invalid then
3       L ← L + x[λ].l
4       x[λ].(l s t i d) ← (ν(L) 0 0 λ)
5       x[λ].cl ← ⊥
6    end
7 function ν^(λ : identifier, x : tag, L : label history)
8    y ← x
9    clean(λ, y)
10   if x(y) ≤ λ then
11      (case ν^y) y[x(y)].(s t) ← (1 + y[x(y)].s 0)
12      (case ν^y) y[x(y)].t ← 1 + y[x(y)].t
13    check_entry(λ, y, L)
14    return y
15 end

Algorithm 3: Acceptors: Acceptors
1 switch receive() do
2   case (p1a, λ, b)
3      a_α α ← a_α
4      if b[α].(l or cl) ≠ a_α[α].l then
5         H^l_α ← H^l_α + b[α].(l or cl)
6      fill_cl(a_α, b), check_entry(a_α, a_α.H^l_α)
7      if a_α < b then
8         a_α[x(b)] ← b[x(b)]
9         if a_α[x(b)].l ≠ a_α[α].l then
10        r_α[x(b)] ← ⊥
11        H_α[x(b)] ←
12        H_α[x(b)] + a_α[x(b)].l
13        if ∃l ∈ H_α[x(b)], l ≠ a_α[x(b)].l
14        then a_α[x(b)].cl ← l
15    end
16    foreach μ ∈ Π do
17      c ← r_α[μ].b
18      if c[μ].l ≠ a_α[μ].l or a_α[μ].(l s t i d) < c[μ].(l s t i d) then
19        r_α[μ] ← ⊥
20    end
21    send(λ, (p1b, α, a_α, r_α[x(a_α)]))
22   case (p2a, λ, b, p) or (decision, λ, b, p)
23      a_α α ← a_α
24      if b[α].(l or cl) ≠ a_α[α].l then
25         H^l_α ← H^l_α + b[α].(l or cl)
26      fill_cl(a_α, b), check_entry(a_α, a_α.H^l_α)
27      if a_α < b then
28         a_α[x(b)] ← b[x(b)], r_α[x(b)] ← [b, p]
29      if it is a decision message then
30         decide(b, p)
31      if a_α[x(b)].l ≠ a_α[α].l then
32         H_α[x(b)] ←
33         H_α[x(b)] + a_α[x(b)].l
34         if ∃l ∈ H_α[x(b)], l ≠ a_α[x(b)].l
35         then a_α[x(b)].cl ← l
36    end
37    foreach μ ∈ Π do
38      c ← r_α[μ].b
39      if c[μ].l ≠ a_α[μ].l or a_α[μ].(l s t i d) < c[μ].(l s t i d) then
40        r_α[μ] ← ⊥
41    end
42    if it is p2a message then
43       send(λ, (p2b, α, a_α, r_α))
44 endsw
Algorithm 4: Proposer $\lambda$ - Main loop

1. loop As long as $\Theta_\lambda = \text{true}$
2. \quad $p^* \leftarrow \text{input}()$
3. \quad $a_\lambda \leftarrow v^*(\lambda, a_\lambda, H^{cl}_\lambda)$
4. [Ph. 1]
5. \quad $\forall \alpha \in \Pi$, send($\alpha, (p1\alpha, \lambda, a_\lambda)$)
6. if PR(1) returns nok then go to [Ph. 1]
7. [Ph. 2]
8. \quad let $\mu = \chi(a_\lambda)$, and $\Gamma$ be the set of non-null proposals $r_\alpha[\mu]$ received at the end of [Ph. 1] in
9. if $\Gamma \neq \emptyset$ then
10. \qquad if $\forall x, y \in \Gamma, \chi(x.a) = \chi(y.a) = \mu \land x.a[\mu].l = y.a[\mu].l = a_\lambda[\mu].l$ then
11. \qquad \quad $\Gamma_0 \leftarrow \{(a, p) \in \Gamma | a = \max(b|\exists q, (b, q) \in \Gamma, b[\mu].s = a_\lambda[\mu].s)\}$
12. \qquad if $\Gamma_0 = \{(a, p)\}$ then $p_\lambda \leftarrow p$
13. \quad else $p_\lambda \leftarrow p^*$
14. else $p_\lambda \leftarrow p^*$
15. \quad $\forall \alpha \in \Pi$, send($\alpha, (p2\alpha, \lambda, a_\lambda, p_\lambda)$)
16. if PR(2) returns nok then go to [Ph. 1]
17. $\forall \alpha \in \Pi$, send($\alpha, (\text{decision}, \lambda, a_\lambda, p_\lambda)$)
18. end loop

Algorithm 5: Proposer $\lambda$ - Preempting Routine

function PR($\phi : \text{phase 1 or phase 2}$)

1. \quad $N \leftarrow \emptyset$, $M \leftarrow 0$, $a_{\text{sent}} \leftarrow a_\lambda$
2. while $|N| < n - f$ do
3. \qquad $b \leftarrow a_{\text{sent}}$
4. \qquad $(\lambda b, \alpha, a_\alpha, q_\alpha) \leftarrow \text{receive}((\lambda b, *, *, *))$
5. \quad $\text{fill_cl}(a_\lambda, b)$
6. \quad $C^+ = (a_\lambda \ni b) \land (\phi = 2 \Rightarrow p_\lambda = q_\alpha.p)$
7. \quad $C^- = (a_\lambda \ni b)$
8. if $a_\lambda \ni N$ then
9. \qquad if $C^+ \lor C^-$ then $N \leftarrow N \cup \{\alpha\}$
10. \qquad if $C^+$ then $M \leftarrow M + 1$
11. else
12. \qquad if $a_\alpha[\lambda].(l \lor cl) \neq a_\lambda[\lambda].l$ then
13. \qquad \quad $H^{cl}_\lambda \leftarrow H^{cl}_\lambda + a_\alpha[\lambda].(l \lor cl)$
14. \qquad \quad $\text{fill_cl}(a_\alpha, a_\lambda)$
15. \qquad \quad $\text{check_entry}(\lambda, a_\lambda, H^{cl}_\lambda)$
16. \qquad \quad let $\mu = \chi(a_\alpha)$ in
17. \qquad if $a_\alpha \neq a_\lambda$ then
18. \qquad \quad if $\mu < \chi(a_\lambda)$ then
19. \qquad \qquad $H^{cl}_\lambda[\mu] \leftarrow H^{cl}_\lambda[\mu] + a_\lambda[\mu].l$
20. \qquad \qquad $a_\lambda[\mu].l \leftarrow a_\alpha[\mu].l$
21. \qquad \qquad if $\exists l \in H^{cl}_\lambda[\mu], l \neq a_\lambda[\mu].l$ then $a_\lambda[\mu].cl \leftarrow l$
22. \qquad \quad \quad $a_\lambda \leftarrow v^*(\lambda, a_\lambda, H^{cl}_\lambda)$
23. \qquad \quad else
24. \qquad \quad \quad (we have $\chi(a_\lambda) = \mu$ and $a_\lambda[\mu].l = a_\alpha[\mu].l$)
25. \qquad \quad \quad if $a_\alpha[\mu].s = a_\lambda[\mu].s$ then
26. \qquad \qquad \quad $a_\lambda[\mu].l \leftarrow a_\alpha[\mu].l$
27. \qquad \qquad \quad $a_\lambda \leftarrow v^*(\lambda, a_\lambda, H^{cl}_\lambda)$
28. \qquad \quad \quad else
29. \qquad \qquad \quad $a_\lambda[\mu].s \leftarrow a_\alpha[\mu].s$
30. \qquad \qquad \quad $a_\lambda \leftarrow v^*(\lambda, a_\lambda, H^{cl}_\lambda)$
31. \quad \quad end
32. \quad \quad end
33. \quad \quad end
34. \quad \quad end
35. \quad \quad if $M = n - f$ then return ok
36. \quad else return nok
37. \quad end
38. end
Algorithm 6: Generalized Paxos - Procedure decide and truncate, acceptor $\alpha$

```plaintext
1 function decide(b : tag, p : command sequence)
2     learned$\alpha$ ← p
3 end
4 function truncate(a : tag, p : command sequence)
5     if $|p| > a[a(a)].s - 1$ then $p ←$ the suffix of $p$ of length $a[a(a)].s - 1$
6     if $|p| < a[a(a)].s - 1$ then append $nop$ to $p$ until $|p| = a[a(a)].s - 1$
7 end
```

Algorithm 7: Generalized Paxos - Main loop, proposer $\lambda$

```plaintext
1 loop As long as $\Theta\lambda = true$
2     $p^* ← p_\lambda$
3     cmd ← input()
4     $a_\lambda ← V'(\lambda, a_\lambda, H^\lambda_0)$
5     truncate($a_\lambda, p^*$)
6     [Ph. 1]
7     $p_\lambda ← p^* \circ cmd$
8     $\forall \alpha \in \Pi, send(\alpha, \langle p1a, \lambda, a_\lambda \rangle)$
9     if PR(1) returns nok then
10        if $a_\lambda[a(a)].l$ has changed then
11           truncate($a_\lambda, p^*$)
12           go to [Ph. 1]
13 [Ph. 2]
14        let $\mu = a(a).l$, and $\Gamma$ be the set of non-null proposals $r_\alpha[\mu]$ received at the end of [Ph. 1] in
15        if $\Gamma \neq \emptyset$ then
16           if $\forall x, y \in \Gamma, \chi(x.a) = \chi(y.a) = \mu \wedge x.a[\mu].l = y.a[\mu].l = a_\lambda[\mu].l$ then
17              $\Gamma_0 ← \{(a, p) \in \Gamma | a = \max(b) \exists q, (b, q) \in \Gamma, b[\mu].s = a_\lambda[\mu].s = |q|\}$
18              if $\Gamma_0$ is not empty then
19                 let $p_{max}$ be the maximum (lexicographically) command history in $\Gamma_0$
20                 $p_\lambda ← p_{max} \circ cmd$
21                 $\forall \alpha \in \Pi, send(\alpha, \langle decision, \lambda, a_\lambda, p_\lambda \rangle)$
22                 end loop
23        else $p_\lambda ← p^* \circ cmd$
24        else $p_\lambda ← p^* \circ cmd$
25    if PR(1) returns nok then
26        if $a_\lambda[a(a)].l$ has changed then
27           truncate($a_\lambda, p^*$)
28           go to [Ph. 1]
29    $\forall \alpha \in \Pi, send(\alpha, \langle decision, \lambda, a_\lambda, p_\lambda \rangle)$
30 end loop
```