Thermo-electro-mechanical vibration of piezoelectric nanobeams resting on a viscoelastic foundation

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Abstract. Taking a piezoelectric nanobeam resting on a viscoelastic foundation as the research object, the free vibration of the nanobeam are studied by considering the thermo-electro-mechanical loadings. The vibration governing equations and boundary conditions are first derived using Hamilton’s principle and then the transfer function method are applied to calculate the natural frequencies under general boundary conditions. And then the influences of nonlocal parameter and material parameters of viscoelastic foundation on the vibration characteristics are also studied in detail. The results show that both nonlocal parameter and viscoelastic foundation have significant influence on the vibration of nanobeams. The conclusions of this study could provide important theoretical basis for the research and application of nanostructures in nanosensors, nanodrivers, nanoresonators and other intelligent devices.

1. Introduction

In recent years, nanotechnology has been recognized as one of the most important high-tech industries in the world and has become an important means to promote national economy and guarantee national security. Up still now, major powers in the world are working hard to develop nanotechnology to improve their overall national strength. Since the development of nanotechnology cannot be separated from the research on nanomaterials, the research on the mechanical behaviors of nanomaterials[1, 2] becomes an important part of nanomaterials, which plays an major role in the field of nanotechnology [3] and has become one of the research hotspots of today’s scholars.

Combining the modified strain theory with Timoshenko beam theory, both bending mechanical properties and vibration responses were investigated by Li et al. [4] for a functionally graded piezoelectric beam. The nonlocal Timoshenko beam model was applied by Ke et al. [5] to examine the nonlinear vibration characteristics of piezoelectric nanobeams with thermal and electrical loadings. Chen et al. [6] investigated the buckling and dynamic stability of piezoelectric nanobeams by combing Euler-Bernoulli beam theory with Galerkin method. Based on nonlocal strain gradient theory, the free vibration of strain gradient Timoshenko beams was studied by Li et al. [7], and the effects of nonlocal parameter and material parameters on the vibration characteristics were examined. Various refined beams theories were developed by Ebrahimi and Barati [8] to examine the coupling effect of moisture and temperature on vibration responses of functionally graded nanobeams embedded an elastic foundation. Furthermore, based on the nonlocal strain gradient theory, Ebrahimi and Barati [9] also investigated the hygrothermal effects on vibration responses of viscoelastic functionally graded nanobeams.

Up still now, the vibration of piezoelectric nanobeams resting on a viscoelastic foundation has not
been reported in the literature as thermo-electro-mechanical loadings are taken into consideration. In this contest, a constitutive model is established in this paper based on the nonlocal Euler-Bernoulli beam model. The transfer function method is then used to solve the governing equations to obtain the natural frequencies of the nano-beam. Also, the effects of nonlocal parameter and material parameters of viscoelastic foundation are carefully examined, which could provide a theoretical basis for the design and application of nanometer components.

2. Mathematical modelling
Considering nonlocal effects, the constitutive equation in a differential form for piezoelectric materials can be expressed as [2, 10]

\[
\left(1 - \left(\varepsilon_0 a\right)^2 \nabla^2\right)\sigma_{ij} = c_{ijkl}E_{kl} - e_{ijk}E_k - \lambda_{ij}\Delta T
\]

(1)

\[
\left(1 - \left(\varepsilon_0 a\right)^2 \nabla^2\right)D_i = e_{ijkl}E_{kl} + \kappa_{ik}E_k + p_i\Delta T
\]

(2)

\[
\sigma_{ij,j} = \rho u_i, D_{i,j} = 0, E_j = -\Phi_j
\]

(3)

where \(\varepsilon_0 a\) is the nonlocal parameter, \(\sigma_{ij}\) is the nonlocal stress tensor, \(c_{ijkl}\) is the elastic constants relating to materials, \(e_{ijk}\) is the classical strain tensor, \(D_i\) is the electric displacement tensor, \(E_k\) is the electric field tensor, \(e_{ijk}\) is the piezoelectric constants, \(\lambda_{ij}\) is the thermal moduli, \(\kappa_{ik}\) is the dielectric constants, \(p_i\) is the pyroelectric constants. Moreover, \(\Phi\) is the electric potential, \(\Delta T\) is the temperature change, and \(\nabla\) denotes the Hamilton arithmetic operator.

Fig. 1 A piezoelectric Euler-Bernoulli nanobeam resting on a viscoelastic foundation and subjected to thermo-electro-mechanical loadings.

As shown in Fig. 1, taking a piezoelectric nanobeam resting on a viscoelastic foundation as the research object in this study, whose length is denoted by \(L\) and thickness is denoted by \(h\). It assumed that the nanobeam is subjected to thermo-electro-mechanical loadings, where \(P_0\) is applied to denote the external applied biaxial force.

The basic Eqs. (1) and (2) can be rewritten as

\[
\left(1 - \left(\varepsilon_0 a\right)^2 \nabla^2\right)\sigma_{xx} = c_{11}E_{xx} - e_{31}E_z - \lambda_{11}\Delta T
\]

(4)

\[
\left(1 - \left(\varepsilon_0 a\right)^2 \nabla^2\right)D_x = \kappa_{11}E_x
\]

(5)

\[
\left(1 - \left(\varepsilon_0 a\right)^2 \nabla^2\right)D_z = e_{31}E_{xx} + \kappa_{33}E_z + p_i\Delta T
\]

(6)

The electric potential distribution can be assumed to satisfy the Maxwell equation for the present nonlocal piezoelectric Euler-Bernoulli beam model. Hence, \(\Phi(x, z, t)\) can be taken as follows [5, 13]

\[
\Phi(x, z, t) = -\cos\left(\frac{\pi z}{h}\right)\phi(x, t) + \frac{2zV_0}{h} e^{i\omega t}
\]

(7)
where $\phi(x,t)$ denotes the electric potential along the $x$-direction, $V_0$ denotes the external electric voltage, $\omega$ and $t$ is applied to denote natural frequency and time. From Eq. (7), the electric fields can be obtained as

$$E_x = -\frac{\partial \Phi}{\partial x} = \cos\left(\frac{\pi z}{h}\right) \frac{\partial \phi}{\partial x}$$

$$E_z = -\frac{\partial \Phi}{\partial z} = -\frac{\pi}{h} \sin\left(\frac{\pi z}{h}\right) \phi - \frac{2V_0}{h} e^{i\omega t}$$

(8)

(9)

To derive the governing equations of nanobeams, the Hamilton's principle is written as

$$\int_0^t (\delta \Pi_k + \delta \Pi_F - \delta \Pi_I) \, dt = 0$$

(10)

where $\Pi_I$ is the strain energy of the nanobeam and can be expressed as

$$\Pi_I = \frac{1}{2} \int_0^L \int_{-h/2}^{h/2} \left( \sigma_{rs} e_{ss} - D_x E_x - D_z E_z \right) \, dz \, dx$$

$$= -\frac{1}{2} \int_0^L M_x \frac{\partial^2 w}{\partial x^2} \, dx$$

$$- \frac{1}{2} \int_0^L \int_{-h/2}^{h/2} \left[ D_x \cos\left(\frac{\pi z}{h}\right) \frac{\partial \phi}{\partial x} - D_z \left(\frac{\pi}{h} \sin\left(\frac{\pi z}{h}\right) \phi + \frac{2V_0}{h} e^{i\omega t}\right) \right] \, dz \, dx$$

(11)

in which $M_x$ denotes the internal bending moment and is derived from

$$M_x = \int_{-h/2}^{h/2} \sigma_{rs} z \, dz$$

(12)

In addition, the kinetic energy $\Pi_k$ and the external work $\Pi_F$ of the nanobeam are defined by

$$\Pi_k = \frac{1}{2} \int_0^L \rho h \left( \frac{\partial w}{\partial t} \right)^2 \, dx$$

$$\Pi_F = \frac{1}{2} \int_0^L \rho h \left[ -N_Q w + \left( N_{P_x} + N_{E_x} + N_{E_x} \right) \left( \frac{\partial w}{\partial x} \right)^2 \right] \, dx$$

(13)

(14)

in which $N_Q$ is used to denote the reaction of the foundation[11, 12], $N_{P_x}$, $N_{E_x}$ and $N_{E_x}$ are the normal forces in the $x$-direction induced by $P_0$, $\Delta T$ and $V_0$, which can be calculated as follows

$$N_Q = k_w w - k_g \nabla^2 w + c_i \frac{\partial w}{\partial t}$$

$$N_{P_x} = P_0, \quad N_{E_x} = \lambda_1 \Delta T, \quad N_{E_x} = -2e_3 V_0$$

(15)

(16)

in which $k_w$ is Winkler's modulus parameter, $k_g$ is Pasternak's (shear) modulus parameter, and $c_i$ is damping parameter.

Substituting Eqs. (11), (13) and (14) into Hamilton's principle (10), the vibration governing equations can be obtained as follows

$$\delta w: \frac{\partial^2 M_x}{\partial x^2} - \left( k_w w - k_g \nabla^2 w + c_i \frac{\partial w}{\partial t} \right) - \left( N_{P_x} + N_{E_x} + N_{E_x} \right) \frac{\partial^2 w}{\partial x^2} = \rho h \frac{\partial^2 w}{\partial t^2}$$

$$\delta \phi: \int_{-h/2}^{h/2} \left[ \frac{\partial D_z}{\partial x} \cos\left(\frac{\pi z}{h}\right) + D_z \frac{\pi}{h} \sin\left(\frac{\pi z}{h}\right) \right] \, dz = 0$$

(17)

(18)

According to Eqs. (4), (6), (9) and (12), we have

$$\left(1 - \left(e_0 a \right)^2 \nabla^2 \right) M_x = -D_{11} \frac{\partial^2 w}{\partial x^2} + F_{31} \phi$$

(19)
\[
\int_{-h/2}^{h/2} \cos \left( \frac{\pi z}{h} \right) \left( 1-\left(e_0a\right)^2 \nabla^2 \right) D_z \, dz = X_{11} \frac{\partial \phi}{\partial x} 
\]
\[
\int_{-h/2}^{h/2} \frac{\pi}{h} \sin \left( \frac{\pi z}{h} \right) \left( 1-\left(e_0a\right)^2 \nabla^2 \right) D_z \, dz = -F_{31} \frac{\partial^2 \tilde{w}}{\partial x^2} - X_{33} \phi 
\]
where
\[
D_{11} = \int_{-h/2}^{h/2} c_{11} z^2 \, dz, \quad F_{31} = \int_{-h/2}^{h/2} c_{31} \frac{\pi}{h} z \sin \left( \frac{\pi z}{h} \right) \, dz,
\]
\[
X_{11} = \int_{-h/2}^{h/2} \kappa_{11} \cos^2 \left( \frac{\pi z}{h} \right) \, dz, \quad X_{33} = \int_{-h/2}^{h/2} \kappa_{33} \left( \frac{\pi}{h} \right)^2 \sin^2 \left( \frac{\pi z}{h} \right) \, dz.
\]
Substituting Eqs. (19)-(21) into the governing Eqs. (17) and (18), yields
\[
-d_{11} \frac{\partial^4 \tilde{w}}{\partial x^4} + \bar{F}_{31} \frac{\partial^2 \phi}{\partial x^2} - \left( 1-\alpha^2 \frac{\partial^2}{\partial x^2} \right) \left[ k_w \tilde{w} - k_G \frac{\partial^2 \tilde{w}}{\partial x^2} + \bar{c}_i \frac{\partial \tilde{w}}{\partial t} \right] + \left( \bar{N}_{p_x} + \bar{N}_{t_x} + \bar{N}_{e_x} \right) \frac{\partial^2 \tilde{w}}{\partial x^2} = \left( 1-\alpha^2 \frac{\partial^2}{\partial x^2} \right) \eta \frac{\partial^2 \tilde{w}}{\partial t^2} 
\]
\[
\bar{X}_{11} \frac{\partial^2 \phi}{\partial x^2} - \bar{F}_{31} \frac{\partial^2 \tilde{w}}{\partial x^2} - \bar{X}_{33} \tilde{\phi} = 0 
\]
where some dimensionless terms can be defined as below
\[
\bar{\tilde{w}} = \frac{\bar{w}}{L}, \quad \bar{\phi} = \frac{\phi}{L}, \quad \bar{\eta} = \frac{\eta}{L}, \quad \bar{\alpha} = \frac{\alpha}{L}, \quad \bar{c}_i = c_{11} \frac{\rho L}{c_{11}}, \quad \bar{T} = \frac{T}{L}, \quad \bar{\phi}_0 = \frac{\phi_0}{L}, \quad \bar{\phi}_{11} = \frac{\phi_{11}}{L}, \quad \bar{X}_{33} = \frac{X_{33}}{L}
\]
\[
k_w = \frac{k_w L^2}{c_{11}}, \quad \bar{k}_G = \frac{k_G}{c_{11}}, \quad \bar{c}_i = \frac{c_i L}{\rho L c_{11}}, \quad \bar{T} = \frac{T}{L}, \quad \bar{X}_{11} = \frac{X_{11}}{L}, \quad \bar{X}_{33} = \frac{X_{33}}{L}
\]
To solve the governing Eqs. (23) and (24), it is assumed that the solutions could be expressed as follows
\[
\bar{w}(\bar{x}) = \bar{W}(\bar{x}) \exp(i \Omega t), \quad \bar{\phi}(\bar{x}) = \bar{\Phi}(\bar{x}) \exp(i \Omega t)
\]
where \( \Omega \) and \( \bar{W}(\bar{x}) \) are the dimensionless angular frequency and corresponding mode shape of the piezoelectric nanobeam, respectively. In addition, the dimensionless angular frequency \( \Omega \) can be calculated from
\[
\Omega = \omega L \sqrt{\frac{\rho L}{c_{11}}} 
\]
Substituting Eq. (26) into the governing Eqs. (23) and (24) yields
\[
-d_{11} \frac{\partial^4 \bar{W}}{\partial x^4} + \bar{F}_{31} \frac{\partial^2 \bar{\phi}}{\partial x^2} - \left( 1-\alpha^2 \frac{\partial^2}{\partial x^2} \right) \left[ k_w \bar{W} - k_G \frac{\partial^2 \bar{W}}{\partial x^2} \right] + \eta \bar{W} \left( \bar{N}_{p_x} + \bar{N}_{t_x} + \bar{N}_{e_x} \right) \frac{\partial^2 \tilde{w}}{\partial x^2} = -\Omega^2 \left( 1-\alpha^2 \frac{\partial^2}{\partial x^2} \right) \eta \bar{W}
\]
\[ \ddot{X}_{11} \frac{\partial^2 \Phi}{\partial X^2} - F_{31} \frac{\partial^2 \bar{W}}{\partial X^2} - \ddot{X}_{33} \bar{\Phi} = 0 \]  

(29)

3. Transfer function method

According to the TFM, one can define the state vector \( \eta(\bar{x}, \Omega) \) as

\[ \eta(\bar{x}, \Omega) = \left[ \bar{W}, \frac{d\bar{W}}{d\bar{x}}, \frac{d^2\bar{W}}{d\bar{x}^2}, \frac{d^3\bar{W}}{d\bar{x}^3}, \bar{\Phi}, \frac{\partial \bar{\Phi}}{\partial \bar{X}} \right]^T \]  

(30)

Then both governing Eqs. (28)-(29) and boundary conditions can be expressed in a matrix form as

\[ \frac{d\eta(\bar{x}, \Omega)}{d\bar{x}} = F(\Omega)\eta(\bar{x}, \Omega) \]  

(31)

where

\[ F(\Omega) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ f_1 & 0 & f_2 & 0 & f_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{F_{31}}{\bar{X}_{11}} & 0 & \frac{\bar{X}_{33}}{\bar{X}_{11}} & 0 \end{bmatrix} \]  

(32)

and

\[ f_1 = \frac{\Omega^2 \eta - \bar{k}_w - i\Omega \bar{c}_t}{d_{11} + \alpha^2 \bar{k}_G - \alpha^2 (\bar{N}_{\pi} + \bar{\bar{N}}_{\tau} + \bar{\bar{N}}_{\pi\tau})}, \]

\[ f_2 = \frac{\bar{k}_G + \alpha^2 \bar{k}_w + \alpha^2 i\Omega \bar{c}_t - \alpha^2 \Omega^2 \eta - (\bar{N}_{\pi} + \bar{\bar{N}}_{\tau} + \bar{\bar{N}}_{\pi\tau}) + \frac{F_{33}^2}{\bar{X}_{11}}}{d_{11} + \alpha^2 \bar{k}_G - \alpha^2 (\bar{N}_{\pi} + \bar{\bar{N}}_{\tau} + \bar{\bar{N}}_{\pi\tau})}, \]

\[ f_3 = \frac{1}{d_{11} + \alpha^2 \bar{k}_G - \alpha^2 (\bar{N}_{\pi} + \bar{\bar{N}}_{\tau} + \bar{\bar{N}}_{\pi\tau})} \frac{F_{31}}{\bar{X}_{11}}. \]

\[ M(\Omega)\eta(0, \Omega) + N(\Omega)\eta(1, \Omega) = 0 \]  

(34)

\( M(\Omega) \) and \( N(\Omega) \) in Eq. (34) are respectively applied to denote the boundary condition set matrices at the left and the right ends of the beam.

Obviously, the solution of Eq. (31) can be expressed as

\[ \eta(\bar{x}, \Omega) = e^{F(\Omega)\bar{x}} \eta(0, \Omega) \]  

(35)

Substituting Eq. (35) into Eq. (34) leads to

\[ \left[ M(\Omega) + N(\Omega)e^{F(\Omega)\bar{x}} \right] \eta(0, \Omega) = 0 \]  

(36)

Based on the TFM, \( \Omega \) can be obtained by solving the following equation

\[ \det \left[ M(\Omega) + N(\Omega)e^{F(\Omega)\bar{x}} \right] = 0 \]  

(37)

According to Eq. (27), the natural frequency \( \omega \) of the nanobeam can be calculated from
\[ \omega = \frac{\Omega}{L} \sqrt{\frac{c_{11}}{\rho L}} \]  

(38)

4. Numerical results and discussion

To verify the accuracy of the model and solution method established above, the piezoelectric nanobeam is assumed to be PZT-4 with material properties listed in Table 1 [5]. Unless otherwise stated, we take the length of the nanobeam \( L = 12 \text{nm} \) and thickness \( h = 2 \text{nm} \).

| Material properties of PZT-4. |  |  |  |  |
|-------------------------------|---|---|---|---|
| \( c_{11}/\text{GPa} \)       | \( e_{33}/(\text{C/m}^2) \) | \( \kappa_{11}/(\text{C/Vm}) \) | \( \kappa_{33}/(\text{C/Vm}) \) | \( \lambda_{1}/(\text{N/m}^2 \text{K}) \) | \( p_{1}/(\text{C/m}^2 \text{K}) \) | \( \rho/(\text{kg/m}^3) \) |
| 132                           | -4.1 | 5.841 \times 10^{-9} | 7.124 \times 10^{-9} | 4.738 \times 10^{-9} | 2.5 \times 10^{-5} | 7500 |

The first three natural frequencies for the nanobeam without foundations (i.e., \( k_w = k_G = c_t = 0 \)) and with foundations (\( k_w = 0.1 \text{GPa/nm}, k_G = 0.25 \text{GPa nm}, c_t = 10^{-4} \text{GPa ns/nm} \)) under various boundary conditions and nonlocal parameter \( \alpha \) are listed in Table 2 (here we take \( V_0 = \Delta T = P_0 = 0 \)). From the table we can see that the imaginary part (i.e. damping ratios) of the natural frequency is nonzero when the beam is supported by viscoelastic medium. This is because that as the viscoelastic foundation is taken into consideration, the damping effect introduced into the system.

| The first three natural frequencies (GHz) of piezoelectric nanobeams with different boundary conditions and nonlocal parameter \( \alpha \). |
|-------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| BCs                           | In the absence of foundation     | In the presence of viscoelastic foundation |
|                               | \( \alpha = 0.0 \) | \( \alpha = 0.1 \) | \( \alpha = 0.2 \) | \( \alpha = 0.0 \) | \( \alpha = 0.1 \) | \( \alpha = 0.2 \) |
| C-F                           | 9.4700                          | 9.5214                          | 9.6508                          | 15.9999+0.5305i | 16.0457+0.5305i | 16.1908+0.5305i |
| 59.4129                       | 55.7609                         | 47.3966                         | 61.1338+0.5305i | 57.6663+0.5305i | 49.7679+0.5305i |
| 166.3032                      | 137.6441                        | 99.2294                         | 167.2122+0.5305i | 138.8653+0.5305i | 101.1094+0.5305i |
| 26.6505                       | 25.4253                         | 22.5658                         | 30.1269+0.5305i | 29.0489+0.5305i | 26.5827+0.5305i |
| S-S                           | 106.5347                        | 90.2065                         | 66.3367                         | 107.8609+0.5305i | 91.7689+0.5305i | 68.4463+0.5305i |
| 239.5231                      | 174.3075                        | 112.2525                        | 240.4170+0.5305i | 175.5338+0.5305i | 114.1474+0.5305i |
| 60.2480                       | 56.8477                         | 43.8721                         | 62.3012+0.5305i | 58.6987+0.5305i | 46.4644+0.5305i |
| C-C                           | 166.0604                        | 137.2979                        | 89.3509                         | 167.4149+0.5305i | 138.5232+0.5305i | 91.2851+0.5305i |
| 325.4342                      | 230.7514                        | 134.8691                        | 326.6846+0.5305i | 231.8967+0.5305i | 136.8211+0.5305i |

From Table 2 it also can be found that the damping ratios of the nanobeam remain 0.5305 no matter how nonlocal parameter \( \alpha \) and boundary conditions change. Because of this reasons, only the effect of nonlocal parameter \( \alpha \) on the real parts are discussed in Fig. 2 and Fig. 3. Fig. 2 shows that the real parts (i.e. damped frequencies) of the first natural frequencies decrease significantly with increasing \( \alpha \) for both C-C and S-S beams but increase slightly for C-F beams. A similar phenomenon is also discussed by Zhang et al. [15] and Lu et al. [16]. On the other hand, we can observe from Fig. 3 that the damped frequencies decrease significantly with an increase in \( \alpha \) no matter which boundary conditions are considered.
To examine the effect of viscoelastic foundation on the vibration responses of piezoelectric nanobeams, the variations of the first natural frequencies versus damping parameter $c_t$ are shown in Fig. 4 with various Winkler’s modulus parameter $k_w$ and nonlocal parameter $\alpha$. As can be seen from the Fig. 4(a), when the damping coefficient $c_t$ is greater than the critical damping coefficient $(c_t)_{crit}$, the real parts of the natural frequencies remain zero, which indicating that the nanobeam will not have reciprocating vibration. Accordingly, the imaginary parts have a sharp change at the critical damping coefficient $(c_t)_{crit}$. On the other hand, as $c_t < (c_t)_{crit}$, we can see that the real parts of the first natural frequencies decrease nonlinearly with an increase in $c_t$ but increase significantly with Winkler’s modulus parameter $k_w$. In addition, the damping ratios increase almost linearly as damping parameter $c_t$ increases and Winkler’s modulus parameter $k_w$ has almost no effect.

(a) The real parts of natural frequencies.  
(b) The imaginary parts of natural frequencies.

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(a) The real parts of natural frequencies.  
(b) The imaginary parts of natural frequencies.

5. Conclusions
In this paper, the nonlocal elasticity theory combing with Euler-Bernoulli beam model is developed for vibration analysis of piezoelectric nanobeams resting on a viscoelastic foundation. The vibration governing equations and boundary conditions are first derived by utilizing Hamilton’s principle and the transfer function method is then applied to obtain the natural frequencies. The results show that the natural frequencies under different boundary conditions can be obtained by using the model and
solution method established above. On this basis, the influence laws of nonlocal parameter and material parameters of viscoelastic medium on the vibration characteristics of nanobeams are systematically studied. The obtained laws are as follows:

(1) The damping ratios of the system are only affected by the damping of the viscoelastic foundation, which are independent with the nonlocal parameter \( \alpha \) and boundary conditions.

(2) The effect of nonlocal parameter \( \alpha \) on the natural frequencies becomes less pronounced when viscoelastic foundation and softer constrains are considered.

(3) The damped frequency of the beam remains zero as \( c_\ell \) of the foundation is greater than \( (c_\ell)_{\text{crit}} \), which decreases significantly with a decrease in either Winkler’s modulus parameter \( k_w \) and mode numbers.

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