On the stability of shear flows of Prandtl type for the steady Navier-Stokes equations

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Abstract In this paper, we study the stability of shear flows of Prandtl type as \((U(y/\sqrt{\nu}), 0)\) for the steady Navier-Stokes equations under a natural spectral assumption on the linearized NS operator. The key ingredient is to solve the Orr-Sommerfeld equation. For this, we develop a direct energy method combined with the compactness method, which may be of independent interest.

Keywords Navier-Stokes equations, shear flow, Prandtl expansion

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1 Introduction
1.1 The problem and the main result

In this paper, we study the steady Navier-Stokes (NS) equations with small viscosity \(\nu\) on the half-plane \(\Omega_\theta = T_\theta \times \mathbb{R}_+\), where \(T_\theta\) is a torus with the period \(2\pi\theta\):

\[
\begin{cases}
\nu' \cdot \nabla \nu' - \nu \Delta \nu' + \nabla q' = g', & (x, y) \in \Omega_\theta, \\
\nabla \cdot \nu' = 0, & (x, y) \in \Omega_\theta, \\
\nu' |_{y=0} = 0, & x \in T_\theta,
\end{cases}
\]

\((1.1)\)

where \(\nu' = (\nu_1', \nu_2')\) and \(q'\) are the unknown velocity and pressure of the fluid, respectively, and \(g'\) is a given external force.

We are concerned with the stability of shear flows of Prandtl type as \(U'(x, y) = (U(y/\sqrt{\nu}), 0)\), which is a solution of (1.1) with \(g' = -\nu \partial_{yy}^2 U'\) and \(q' = 0\). We make the following structure assumptions on \(U\):

\[
U(0) = 0, \quad U(Y) > 0 \quad \text{in} \quad Y > 0, \quad \lim_{Y \to +\infty} U(Y) = 1, \\
\partial Y U(0) > 0, \quad \sum_{k=1,2} \sup_{Y \geq 0} (1 + Y)^3 |\partial_{yy}^k U(Y)| < \infty.
\]

\((1.2)\)

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The assumption (1.2) implies that
\[ U(Y) \geq C^{-1} \min(1,Y) \sim Y/(1 + Y). \] (1.3)

In this paper, we take the external force \( g'' = -\nu \partial_y^2 U' + f'' \) so that \( u'' = v'' - U' \) satisfies
\[
\begin{cases}
U \left( \frac{y}{\sqrt{\nu}} \right) \partial_x u'' + \left( u_2^2 \partial_y U \left( \frac{y}{\sqrt{\nu}} \right), 0 \right) - \nu \Delta u'' + \nabla p'' = -u'' \cdot \nabla u'' + f'', \\
\nabla \cdot u'' = 0,
\end{cases}
\] (1.4)

We denote by \( P_n \ (n \in \mathbb{Z}) \) the projection on the \( n \)-th Fourier mode in the \( x \) variable:
\[ (P_n u)(x, y) = u_n(y)e^{inx}, \quad \tilde{n} \equiv \frac{n}{\theta}, \quad u_n(y) = \frac{1}{2\pi\theta} \int_0^{2\pi\theta} e^{-in\pi} u(x, y) dx. \]

We define
\[ Q_0 = I - P_0, \]
where \( I \) is the identity operator. It is easy to see that \( P_0 u'' = (u''_{0,1}, 0) \) due to \( \nabla \cdot u'' = 0 \).

In a recent work [9], Gerard-Varet and Maekawa proved the following stability result:
There exist small constants \( \nu_0 \) and \( \theta_0 \) so that if \( 0 < \nu \leq \nu_0 \), \( 0 < \theta \leq \theta_0 \) and \( f'' = Q_0 f'' \) with \( \|f''\|_{L^2} \leq \eta \nu^\frac{1}{2}\log \nu^{-1} \), then it holds that
\[
\|u''_{0,1}\|_{L^\infty} + \nu^{\frac{1}{2}} \|\partial_y u''_{0,1}\|_{L^2} + \sum_{n \neq 0} \|u''_n\|_{L^\infty} + \nu^{-\frac{1}{4}} \|Q_0 u''\|_{L^2} + \nu^{\frac{1}{4}} \|\nabla Q_0 u''\|_{L^2} \leq C \nu^{-\frac{1}{4}} \|f''\|_{L^2}. \]

The smallness assumption \( \theta \leq \theta_0 \) plays a crucial role in the proof of the stability. Gerard-Varet and Maekawa [9] used the Rayleigh-Airy iteration method introduced in [10, 13] to solve the Orr-Sommerfeld equation.

The aim of this paper is twofold. The first aim is to replace the smallness assumption \( \theta \leq \theta_0 \) by the following more natural spectral condition:
\[ \theta \in \Sigma(U, \nu), \]
where the set \( \Sigma(U, \nu) \) consists of the positive number \( \theta \) so that the following linearized Navier-Stokes equations have a unique solution \( u'' \in H^2(\Omega_\nu) \cap H_0^1(\Omega_\nu) \) for \( f'' \in L^2(\Omega_\nu) \):
\[
\begin{cases}
U \left( \frac{y}{\sqrt{\nu}} \right) \partial_x u'' + \left( u_2^2 \partial_y U \left( \frac{y}{\sqrt{\nu}} \right), 0 \right) - \nu \Delta u'' + \nabla p'' = f'', \\
\nabla \cdot u'' = 0,
\end{cases}
\]
\[ u''|_{y=0} = 0. \]

The set \( \Sigma(U, \nu) \) is not empty, since small \( \theta \) belongs to this set, which has been proved in [9]. In fact, our assumption is equivalent to the spectral condition: \( 0 \) is not an eigenvalue of the linearized operator \( \mathcal{L}_\nu \) defined by
\[ \mathcal{L}_\nu u = P'' \left( U \left( \frac{y}{\sqrt{\nu}} \right) \partial_x u + \left( u_2^2 \partial_y U \left( \frac{y}{\sqrt{\nu}} \right), 0 \right) \right) - \nu \partial_x u, \]
where \( P'' \) is the Helmholtz-Leray projection. It is very interesting to determine appropriate structure assumptions on \( U \) such that \( 0 \notin \sigma(\mathcal{L}_\nu) \).

The second aim is to solve the Orr-Sommerfeld equation by developing a direct method in the spirit of our paper [4], which should be of independent interest.

Now we state our main result.
**Theorem 1.1.** There exist positive constants \( \nu_0 \) and \( \eta \) such that the following result holds for \( 0 < \nu \leq \nu_0 \): if \( \theta \in \Sigma(U, \nu) \) and \( f' = Q_0 f'' \) with \( \|f''\|_{L^2} \leq \eta \nu^{\frac{3}{2}} \log \nu^{-1} \), then the unique solution \( u'' \) to (1.4) satisfies

\[
\|u''_{0,1}\|_{L^\infty} + \nu^{\frac{3}{2}} \|\partial_y u''_{0,1}\|_{L^2} + \sum_{n \neq 0} \|u''_n\|_{L^\infty} + \nu^{\frac{3}{2}} \|Q_0 u''\|_{L^2} + \nu^{\frac{3}{2}} \|\nabla Q_0 u''\|_{L^2} \leq C \nu^{-\frac{1}{2}} \|\log \nu\|^{\frac{1}{2}} \|f''\|_{L^2},
\]

where \( C \) is independent of \( \theta \) and \( \nu_0 \).

**Remark 1.2.** When \( \theta \) is small, the spectral condition holds and our result is the same as that in [9]. For \( \theta \) not small but in \( \Sigma(U, \nu) \), our result is new.

**Remark 1.3.** The works [4,10] consider the stability of shear flows of Prandtl type for the unsteady Navier-Stokes equations for the perturbations in the Gevrey class. In such a case, it is enough to establish the resolvent estimates for the linearized operator when the spectral parameter lies in an unstable part of the resolvent set, while in the steady case one needs to establish the resolvent estimates when the spectral parameter equals zero. However, it remains unknown whether \( 0 \) lies in the resolvent set of \( \sigma(L_v) \) for general period \( 2\pi \theta \) in \( x \). This is the essential reason why we impose the spectral condition \( 0 \notin \sigma(L_v) \).

**1.2 Some known results**

Most of earlier mathematical works are devoted to the study of the inviscid limit of the unsteady Navier-Stokes equations. In the absence of the boundary, the inviscid limit from the Navier-Stokes equations to the Euler equations has been justified in various functional settings [1,2,6,23,29,30,35]. In the presence of the boundary, under the Navier-slip boundary condition, the inviscid limit has also been established in [5,17,18,31,39,41].

For the non-slip boundary condition, due to the mismatch of the boundary condition between the Navier-Stokes equations and the Euler equations, Prandtl [33] introduced the following boundary layer expansion:

\[
\begin{aligned}
&u'_1(t,x,y) = u'_1(t,x,y) + u^{BL}(t,x,y) + O(\sqrt{\nu}), \\
&u'_2(t,x,y) = u'_2(t,x,y) + \sqrt{\nu} u^{BL}(t,x,y) + O(\sqrt{\nu}),
\end{aligned}
\]

(1.6)

where \( (u'_1, u'_2) \) is the solution of the 2-D Navier-Stokes equations on the half space with the non-slip boundary condition, \( (u_1', u_2') \) is the solution of the 2-D Euler equations, and \( (u^p, u^v) = (u_1'(t,x,0) + u^{BL}(t,x,Y), \partial_y u_2'(t,x,0) + \nu^{BL}(t,x,Y)) \) satisfies the Prandtl equation. To our knowledge, the Prandtl expansion (1.6) was justified only in some special cases: the analytic data [34,37], the initial vorticity vanishing near the boundary [7,28], as well as the domain and the data with a circular symmetry [27,32].

Initiated by Kato [24], there were many works devoted to the conditional convergence to the Euler equations [25,36,40]. These conditions can be confirmed for the data which is analytic near the boundary [26,38].

Grenier [12] studied the Prandtl expansion of shear type flows as

\[
u_1'' = (U^e(t,y),0) + (U^{BL}(t,y),0).
\]

(1.7)

When the shear flow \( U^{BL}(0,Y) \) is linearly unstable for the Euler equations, he proved the \( H^1 \) instability of this expansion, and also the \( L^\infty \) instability in [14]. In fact, even for the shear flows which are linearly stable for the Euler equations, Grenier et al. [13] proved that they could be linearly unstable for the Navier-Stokes equations when \( \nu \) is very small. When \( U^{BL}(t,Y) \) is a monotone and concave function, Gerard-Varet et al. [10] (see also [4]) proved the stability of the Prandtl expansion (1.7) for the perturbations in the Gevrey class (see [11] for general cases).
Guo and Nguyen [16] considered the Prandtl expansion of the steady Navier-Stokes equations over a moving plate (see [19–21] for more relevant works). For the steady Navier-Stokes equations with the non-slip boundary condition, Gerard-Varet and Maekawa [9] proved the stability of shear flows ($U(Y, 0)$) in the Sobolev space. Guo and Iyer [15] proved the local-in-time stability of the Blasius flow, and Iyer and Masmoudi [22] recently proved the global stability (see also [8] for a different method).

1.3 Sketch of the proof

The road map of the proof of Theorem 1.1 is as follows.

The key step is to study the following linearized NS system for each Fourier mode:

\[
\begin{aligned}
&i\tilde{n}U\left(\frac{y}{\sqrt{\nu}}\right)u_n + \left(u_{n,2}\partial_y U\left(\frac{y}{\sqrt{\nu}}\right), 0\right) - \nu(\partial_y^2 - \tilde{n}^2)u_n + (i\tilde{n}p_n, \partial_y p_n) = f_n, \quad y > 0, \\
i\tilde{n}u_{n,1} + \partial_y u_{n,2} = 0, \quad y > 0, \\
u_n|_{y=0} = 0,
\end{aligned}
\]

(1.8)

where $u_n = u_n(y)$ is the $n$-th Fourier mode of the velocity $u$. The divergence-free condition and the homogeneous Dirichlet condition imply $u_0 = (u_{0,1}, 0)$. The main ingredient of this paper is to establish the following uniform estimates (see Proposition 5.2 for details):

(1) If $0 < |\tilde{n}| < \delta_0\nu^{-\frac{1}{2}}$ and $\theta \in (0, \theta_0]$, then

\[
|\tilde{n}|^\frac{1}{2}\|u_n\|_{L^2} + |\tilde{n}|^\frac{1}{2}\|\partial_y u_n\|_{L^2} \leq C\|f_n\|_{L^2}.
\]

(2) If $|\tilde{n}| \geq \delta_0\nu^{-\frac{1}{2}}$ and $\theta \in (0, \theta_0]$, then

\[
|\tilde{n}|^2\|u_n\|_{L^2} + |\tilde{n}|\|\partial_y u_n\|_{L^2} \leq C\|f_n\|_{L^2}.
\]

(3) If $\theta > \theta_0$ and $\theta \in \Sigma(U, \nu)$, then

\[
|\tilde{n}|\|u_n\|_{L^2} + \nu^\frac{1}{2}|\tilde{n}|\|\partial_y u_n\|_{L^2} \leq C\|f_n\|_{L^2}.
\]

Compared with [9], the result in the case of $\theta > \theta_0$ and $\theta \in \Sigma(U, \nu)$ is completely new.

With the help of the above estimates for the linearized NS system, the nonlinear stability can be proved by using a fixed point argument in the following functional space:

\[
X_{\nu, \varepsilon} = \{u : \|u\|_{X_\nu} \leq \varepsilon\nu^\frac{1}{2}\log \nu^{-\frac{1}{2}}\},
\]

where

\[
\|u\|_{X_\nu} = \|u_{0,1}\|_{L^\infty} + \nu^\frac{1}{2}\|\partial_y u_{0,1}\|_{L^2} + \sum_{n \neq 0} \|u_n\|_{L^\infty} + \nu^{-\frac{1}{2}}\|Q_0u\|_{L^2} + \nu^\frac{1}{2}\|\nabla Q_0u\|_{L^2}
\]

(see Subsection 5.3 for details).

Next, we introduce main ideas of the proof for the estimates (1)–(3).

For the case of $\tilde{n} \geq C\nu^{-\frac{1}{2}}$, the diffusion part in (1.8) is dominant so that the desired estimate can be proved via the standard energy method for (1.8).

The case of $0 < \tilde{n} \leq C\nu^{-\frac{1}{2}}$ is much more complicated. As in [9], we reformulate the system in terms of the stream function $\phi_n$ and make a change of variables. Recall that

\[
u_{n,1} = \partial_y \phi_n, \quad u_{n,2} = -i\tilde{n}\phi_n.
\]

Hence, the stream function $\phi_n$ satisfies the following equation:

\[
\begin{aligned}
-i\tilde{n}U\left(\frac{y}{\sqrt{\nu}}\right)(\partial_y^2 - \tilde{n}^2)\phi_n + i\tilde{n}\partial_y^2 U\left(\frac{y}{\sqrt{\nu}}\right)\phi_n + \nu(\partial_y^2 - \tilde{n}^2)^2\phi_n = i\tilde{n}f_{n,2} - \partial_y f_{n,1}, \quad y > 0, \\
\phi_n|_{y=0} = \partial_y \phi_n|_{y=0} = 0.
\end{aligned}
\]
Now we rescale the variable $y$ to $Y = \frac{y}{\sqrt{\nu}}$ and introduce

$$
\phi(Y) = \nu^{-\frac{1}{2}}\phi_n(y), \quad f(Y) = \nu^{\frac{1}{2}}f_n(y).
$$

Then $\phi(Y)$ satisfies the following Orr-Sommerfeld equation:

$$
\begin{cases}
U(\partial_Y^2 - \alpha^2)\phi - U''\phi + i\varepsilon(\partial_Y^2 - \alpha^2)^2\phi = -f_2 - \frac{i}{\alpha}\partial_Y f_1, \quad Y > 0, \\
\phi|_{Y=0} = \partial_Y \phi|_{Y=0} = 0,
\end{cases}
$$

(1.9)

where $\alpha = \tilde{n}\sqrt{\nu}$ and $\varepsilon = \tilde{n}^{-1} = \frac{\beta}{\alpha}$.

To solve (1.9), we first make the following important decomposition for the solution $\phi = \phi_0 + \phi_1$ with $\phi_0$ and $\phi_1$ solving

$$
\begin{cases}
\partial_Y(U\partial_Y \phi_0) - \alpha^2U\phi_0 + i\varepsilon(\partial_Y^2 - \alpha^2)^2\phi_0 = -f_2 - \frac{i}{\alpha}\partial_Y f_1, \\
\phi_0|_{Y=0} = \partial_Y \phi_0|_{Y=0} = 0,
\end{cases}
$$

(1.10)

and

$$
\begin{cases}
U(\partial_Y^2 - \alpha^2)\phi_1 + i\varepsilon(\partial_Y^2 - \alpha^2)^2\phi_1 - U''\phi_1 = \partial_Y(U'\phi_0), \\
\phi_1|_{Y=0} = \partial_Y \phi_1|_{Y=0} = 0.
\end{cases}
$$

(1.11)

For the system (1.10), we can obtain our estimate by a direct energy method due to the good divergence structure of the main part (see Lemma 4.3), while for the system (1.11), the source term $\partial_Y(U'\phi_0)$ has a good decay in $Y$ and takes the divergence structure. To solve (1.11), we consider the following two cases.

In the case where $\varepsilon$ is small, we first solve the following system with artificial boundary conditions:

$$
\begin{cases}
U(\partial_Y^2 - \alpha^2)\varphi - U''\varphi + i\varepsilon(\partial_Y^2 - \alpha^2)^2\varphi = f, \\
w = (\partial_Y^2 - \alpha^2)\varphi, \quad \varphi|_{Y=0} = \partial_Y w|_{Y=0} = 0.
\end{cases}
$$

(1.12)

In this case, the diffusion term $i\varepsilon(\partial_Y^2 - \alpha^2)^2\varphi$ could be viewed as a perturbation in some sense. More precisely, we can close our estimates by solving the Airy equation

$$
Uw + i\varepsilon(\partial_Y^2 - \alpha^2)w = U''\varphi + f, \quad \partial_Y w|_{Y=0} = 0
$$

and the Rayleigh equation

$$
U(\partial_Y^2 - \alpha^2)\varphi - U''\varphi = f - i\varepsilon(\partial_Y^2 - \alpha^2)^2\varphi, \quad \varphi|_{Y=0} = 0.
$$

Thanks to the good boundary condition on $w$ and our structure assumption on $U$, we can obtain our desired estimates by a direct energy method and using Rayleigh’s trick (see Section 2 for details). In order to match the boundary condition, we need to construct the boundary layer corrector via solving the system

$$
\begin{cases}
i\varepsilon(\partial_Y^2 - \alpha^2)W_b + UW_b - U''\Phi_b = 0, \\
(\partial_Y^2 - \alpha^2)\Phi_b = W_b, \\
\Phi_b|_{Y=0} = 0, \quad \partial_Y \Phi_b|_{Y=0} = 1.
\end{cases}
$$

For $1 \leq |\alpha| < \infty$, the boundary corrector $W_b$ could be chosen as a perturbation of the Airy function. For $0 < |\alpha| \leq 1$, we need to use the slow mode and the fast mode to construct the boundary layer corrector as in [9].

In the case where $\varepsilon$ is not small, the diffusion term cannot be viewed as a perturbation. So the above argument or the Rayleigh-Airy iteration does not work. In this case, we develop the compactness method to solve the system

$$
\begin{cases}
U(\partial_Y^2 - \alpha^2)\varphi - U''\varphi + i\varepsilon(\partial_Y^2 - \alpha^2)^2\varphi = f, \\
w = (\partial_Y^2 - \alpha^2)\varphi, \quad \varphi|_{Y=0} = \partial_Y \varphi|_{Y=0} = 0.
\end{cases}
$$
More precisely, when \( \varepsilon \geq c \) and \( |\alpha| \leq C \), we can use the compactness method to establish the following estimate:

\[
||w||_{L^2} + ||(\partial_Y \varphi, \alpha \varphi)||_{L^2} \leq C||(1 + Y)^2 f||_{L^2}.
\]

To obtain strong convergence of the sequence, we also need to establish the weighted estimates such as

\[
\sum(\varepsilon^2 ||(1 + Y) (\partial_Y w, \alpha w)||_{L^2} + ||(1 + Y)^2 f||_{L^2}),
\]

Finally, to arrive at a contradiction, we need to assume that the homogeneous system

\[
\varepsilon \frac{d^2}{dY^2} \varphi - \alpha^2 \varphi = 0,
\]

\[
\varphi|_{Y=0} = 0
\]

has only a trivial solution \( \varphi = 0 \). This assumption exactly corresponds to our spectral condition \( 0 \not\in \Sigma(U, \nu) \) (see Subsection 4.2 for details).

2 The Orr-Sommerfeld equation with the artificial boundary condition

In this section, we study the Orr-Sommerfeld (OS) equation with the artificial boundary condition

\[
\begin{cases}
U(\partial_Y^2 - \alpha^2) \varphi - U'' \varphi + i\varepsilon (\partial_Y^2 - \alpha^2)^2 \varphi = f, & f(0) = 0, \\
w = (\partial_Y^2 - \alpha^2) \varphi, \quad \varphi|_{Y=0} = \partial_Y w|_{Y=0} = 0.
\end{cases}
\]

(2.1)

Compared with the non-slip condition, this kind of boundary condition allows us to pick enough information from the Airy type structure in the OS equation. Our main idea is as follows.

(1) We first treat the nonlocal term as a perturbation, i.e., we write (2.1) as the following Airy type equation:

\[
\text{Airy}[w] := Uw + i\varepsilon (\partial_Y^2 - \alpha^2)w = f := U'' \varphi + f, \quad \partial_Y w|_{Y=0} = 0.
\]

Then we establish the estimates for general \( f \) with \( (1 + Y)^2 f \in L^2 \).

(2) To close the estimates, we need to control \( ||\varphi||_{L^2} \). For this purpose, we treat \( i\varepsilon (\partial_Y^2 - \alpha^2)^2 \varphi \) as a perturbation, i.e., we write (2.1) as the following Rayleigh type equation:

\[
\text{Ray}[\varphi] := U(\partial_Y^2 - \alpha^2) \varphi - U'' \varphi = \tilde{f} := f - i\varepsilon (\partial_Y^2 - \alpha^2)^2 \varphi, \quad \varphi|_{Y=0} = 0,
\]

which will yield a useful estimate of \( ||\varphi||_{L^2} \).

Now we state our main result of this section.

**Proposition 2.1.** Assume that \( U \) satisfies (1.2). Then there exists a small positive number \( \varepsilon_1 > 0 \) such that if \( 0 < \varepsilon \leq \varepsilon_1 \) and \( \alpha \neq 0 \), then for any \( f \in H^1_0(\mathbb{R}_+) \) with \( (1 + Y)^2 f \in L^2(\mathbb{R}_+) \), there exists a unique solution \( \varphi \in H^4(\mathbb{R}_+) \cap H^1_0(\mathbb{R}_+) \) to (2.1) satisfying

\[
\varepsilon \frac{1}{2} ||(1 + Y)^2 f||_{L^2} + \varepsilon \frac{1}{2} ||(1 + Y)(\partial_Y w, \alpha w)\|_{L^2} + \varepsilon ||(1 + Y)(\partial_Y^2 - \alpha^2)w||_{L^2} + ||(\partial_Y \varphi, \alpha \varphi)||_{L^2}
\]

\[
\leq C \left( ||(1 + Y)^2 f||_{L^2} + \frac{1}{|\alpha|^2} \int_0^{+\infty} f dY \right).
\]

2.1 Estimates for the Airy equation

We consider the following Airy systems with different boundary conditions:

\[
\begin{cases}
U(\partial_Y^2 - \alpha^2) \varphi + i\varepsilon (\partial_Y^2 - \alpha^2)^2 \varphi = f, \\
w = (\partial_Y^2 - \alpha^2) \varphi, \quad \varphi|_{Y=0} = w|_{Y=0} = 0
\end{cases}
\]

(2.2)
Remark 2.3. Without the assumption $\varphi|_{Y=0} = 0$, we can also obtain the following estimate:

$$\|Uw\|_{L^2} + \varepsilon^\frac{1}{2} \|\sqrt{U}w\|_{L^2} + \varepsilon^\frac{1}{2} \|w\|_{L^2} \leq C \|f\|_{L^2}.$$  (2.4)
which implies
\[ \|w\|_{L^2} \leq C e^{-\frac{t}{2}} \|f\|_{L^2}. \]

Applying (2.4) again, we obtain
\[ \varepsilon \frac{t}{2} \|\sqrt{U}w\|_{L^2} + \varepsilon \frac{t}{2} \|w\|_{L^2} + \varepsilon \frac{t}{2} \|\langle \partial_Y w, \alpha w \rangle\|_{L^2} \leq C \|f\|_{L^2}. \]  

(2.7)

Multiplying f on both sides of the first equation in (2.2) or (2.3) and integrating by parts, we obtain
\[ \|Uw\|_{L^2}^2 + \varepsilon \frac{t}{2} \|\langle \partial_Y^2 - \alpha^2 \rangle w\|_{L^2}^2 + 2\varepsilon \text{Re}(i(Uw, \langle \partial_Y^2 - \alpha^2 \rangle w)) = \|f\|_{L^2}^2. \]  

(2.8)

Notice that
\[ |\text{Re}(i(Uw, \langle \partial_Y^2 - \alpha^2 \rangle w))| = |\text{Im}(Uw, \langle \partial_Y^2 - \alpha^2 \rangle w)| \leq \|Uw\|_{L^2} \|\partial_Y w\|_{L^2} \leq C \|w\|_{L^2} \|\partial_Y w\|_{L^2} \leq C \varepsilon^{-1} \|f\|_{L^2}^2, \]

where we used (2.7) in the last inequality. Then combining (2.8) and (2.7), we have that the above inequality implies
\[ \|Uw\|_{L^2} + \varepsilon \frac{t}{2} \|Uw\|_{L^2} + \varepsilon \frac{t}{2} \|\langle \partial_Y^2 - \alpha^2 \rangle w\|_{L^2} + \varepsilon \|\langle \partial_Y^2 - \alpha^2 \rangle w\|_{L^2} \leq C \|f\|_{L^2}, \]  

(2.9)

which is the first statement of this lemma.

Now we turn to prove the second statement. Let \((1 + Y)f \in L^2(\mathbb{R}_+).\) For this purpose, we introduce the following cut-off function. Let \(\chi \geq 0\) be a \(C^1(\mathbb{R}_+)\) cut-off function such that
\[ \chi(Y) = 1, \quad \text{if } Y \in [0, 1) \quad \text{and} \quad \chi(Y) = 0, \quad \text{if } Y \in [2, +\infty), \]

and let \(\chi_R(Y) = \chi(Y/R), \; R > 1.\) Then
\[ \chi(Y) = 1, \quad \text{if } Y \in [0, R); \quad \chi(Y) = 0, \quad \text{if } Y \in [2R, +\infty); \]
\[ R|\partial_Y \chi_R| + R|\partial_Y^2 \chi_R| \leq C, \]

which imply that
\[ |\partial_Y(Y\chi_R(Y))| + |\partial_Y^2(Y\chi_R(Y))| \leq C. \]  

(2.10)

Hence, \(Y\chi_Rw\) satisfies (2.2) with the source term \(Y\chi_Rf + 2i\varepsilon \partial_Y(Y\chi_R)\partial_Yw + i\varepsilon \partial_Y^2(Y\chi_R)w,\) i.e.,
\[ \begin{cases} 
 i\varepsilon (\partial_Y^2 - \alpha^2)(Y\chi_Rw) + U(Y\chi_Rw) = Y\chi_Rf + 2i\varepsilon \partial_Y(Y\chi_R)\partial_Yw + i\varepsilon \partial_Y^2(Y\chi_R)w, \\
 (Y\chi_Rw)|_{Y=0} = \lim_{Y \to +\infty} (Y\chi_Rw) = 0.
\end{cases} \]

Then by replacing \(w\) and \(f\) by \(Y\chi_Rw\) and \(Y\chi_Rf + 2i\varepsilon \partial_Y(Y\chi_R)\partial_Yw + i\varepsilon \partial_Y^2(Y\chi_R)w,\) respectively, in (2.9), we obtain
\[ \varepsilon \frac{t}{2} \|Y\chi_Rw\|_{L^2} + \varepsilon \frac{t}{2} \|\sqrt{U}Y\chi_Rw\|_{L^2} + \varepsilon \frac{t}{2} \|\partial_Y(Y\chi_Rw)\|_{L^2} + \varepsilon \|\langle \partial_Y(Y\chi_Rw), \alpha Y\chi_Rw \rangle\|_{L^2} \leq C \|Y\chi_Rf + 2i\varepsilon \partial_Y(Y\chi_R)\partial_Yw + i\varepsilon \partial_Y^2(Y\chi_R)w\|_{L^2} \]
\[ \leq C \|Y\chi_Rf\|_{L^2} + C\varepsilon \|\partial_Y(Y\chi_R)\|_{L^\infty} \|\partial_Y w\|_{L^2} + C\varepsilon \|\partial_Y^2(Y\chi_R)w\|_{L^\infty} \|w\|_{L^2} \leq C \|Yf\|_{L^2} + C\varepsilon \|\partial_Y w\|_{L^2} + C \varepsilon \|w\|_{L^2}. \]

Then by (2.9) again, we obtain
\[ \varepsilon \frac{t}{2} \|Y\chi_Rw\|_{L^2} + \varepsilon \frac{t}{2} \|\sqrt{U}Y\chi_Rw\|_{L^2} + \varepsilon \frac{t}{2} \|\partial_Y(Y\chi_Rw)\|_{L^2} + \varepsilon \|\langle \partial_Y(Y\chi_Rw), \alpha Y\chi_Rw \rangle\|_{L^2} \leq C \|Yf\|_{L^2} + C(\varepsilon^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}}) \|f\|_{L^2} \leq C \|(1 + Y)f\|_{L^2}. \]
On the other hand, by (2.9) and (2.10), we notice that
\[
\varepsilon \frac{2}{3} \| \partial_Y(Y \chi_R w) \|_{L^2} \geq \varepsilon \frac{2}{3} \| Y \chi_R \partial_Y w \|_{L^2} - C \varepsilon \frac{2}{3} \| w \|_{L^2} \geq \varepsilon \frac{2}{3} \| Y \chi_R \partial_Y w \|_{L^2} - C \varepsilon \frac{2}{3} \| f \|_{L^2}
\]
and
\[
\varepsilon \| \partial_Y^2 (Y \chi_R w) \|_{L^2} \geq \varepsilon \| Y \chi_R \partial_Y^2 w \|_{L^2} - C \varepsilon \| \partial_Y w \|_{L^2} - C \varepsilon \| w \|_{L^2} \geq \varepsilon \| Y \chi_R \partial_Y^2 w \|_{L^2} - C \varepsilon \| f \|_{L^2}.
\]
Therefore, we obtain
\[
\varepsilon \frac{2}{3} \| Y \chi_R w \|_{L^2} + \varepsilon \frac{2}{3} \| \sqrt{U} Y \chi_R w \|_{L^2} + \varepsilon \frac{2}{3} \| (Y \partial_Y w, \alpha Y \chi_R w) \|_{L^2} + \varepsilon \| Y \chi_R (\partial_Y^2 - \alpha^2) w \|_{L^2} \\
\leq C \| (1 + Y) f \|_{L^2}.
\]
Letting \( R \to +\infty \), we conclude
\[
\varepsilon \frac{2}{3} \| Y w \|_{L^2} + \varepsilon \frac{2}{3} \| \sqrt{U} Y w \|_{L^2} + \varepsilon \frac{2}{3} \| Y (\partial_Y w, \alpha w) \|_{L^2} + \varepsilon \| Y (\partial_Y^2 - \alpha^2) w \|_{L^2} \\
\leq C \| (1 + Y) f \|_{L^2}.
\]
Now we are left with the estimate of \( \| (\partial_Y \varphi, \alpha \varphi) \|_{L^2} \). Taking the inner product with \(-\varphi/(U + \varepsilon \frac{2}{3})\), we obtain
\[
\langle U w + i \varepsilon (\partial_Y^2 - \alpha^2) w, -\varphi/(U + \varepsilon \frac{2}{3}) \rangle = \langle f, -\varphi/(U + \varepsilon \frac{2}{3}) \rangle.
\]
We first notice that
\[
\langle U w, -\varphi/(U + \varepsilon \frac{2}{3}) \rangle = \langle w, -\varphi \rangle - \varepsilon \frac{2}{3} \langle w, -\varphi/(U + \varepsilon \frac{2}{3}) \rangle \\
\geq \| (\partial_Y \varphi, \alpha \varphi) \|^2_{L^2} - \varepsilon \frac{2}{3} \| w \varphi \|_{U^1},
\]
which implies that
\[
\| (\partial_Y \varphi, \alpha \varphi) \|^2_{L^2} \leq \varepsilon \frac{2}{3} \| w \varphi \|_{U^1} + \varepsilon \| \varphi/(U + \varepsilon \frac{2}{3}) \|_{U^1} + \varepsilon \| w \varphi \|_{U^1}.
\]
To deal with \( \| w \varphi \|_{U^1} \), we first notice that \( 1/U \leq C(1/Y + \sqrt{U}) \) from (2.5). Hence, we have
\[
\| w \varphi \|_{U^1} \leq C \| w \|_{L^2} \| \varphi \|_{Y^1} + C \| \sqrt{U} Y w \|_{L^2} \| \varphi \|_{Y^1} \\
\leq C \| w \|_{L^2} \| \partial_Y \varphi \|_{L^2} + C \| \sqrt{U} Y w \|_{L^2} \| \partial_Y \varphi \|_{L^2}.
\]
On the other hand, by the fact that \( |U|/Y \leq C \), we have
\[
\| \partial_Y (\varphi/(U + \varepsilon \frac{2}{3})) \|_{L^2} \leq \varepsilon \frac{2}{3} \| \partial_Y \varphi \|_{L^2} + \varepsilon \| \varphi \|_{L^2} |U|/Y \|_{L^\infty} \leq C \varepsilon \frac{2}{3} \| \partial_Y \varphi \|_{L^2}.
\]
Since \( \varphi/(U + \varepsilon \frac{2}{3}) \) \( |Y = 0 \), we obtain
\[
\| (i \varepsilon (\partial_Y^2 - \alpha^2) w, -\varphi/(U + \varepsilon \frac{2}{3})) \|_{L^2} \leq \| \partial_Y \varphi \|_{L^2} + \varepsilon \| \varphi \|_{L^2} |U|/Y \|_{L^\infty} \leq C \varepsilon \frac{2}{3} \| \partial_Y \varphi \|_{L^2}.
\]
By the fact that \( |Y/(1 + Y)U) | \leq C \) and \( (1 + Y) f \in L^2(\mathbb{R}_+), \) we have
\[
\| (f, -\varphi/(U + \varepsilon \frac{2}{3})) \|_{L^2} \leq \| (1 + Y) f \|_{L^2} |Y/(1 + Y)U | \| \varphi \|_{L^2} \leq C \| (1 + Y) f \|_{L^2} \| \partial_Y \varphi \|_{L^2}. 
\]
This completes the proof of this lemma.

2.2 Estimates for the Rayleigh equation

We consider the Rayleigh equation

\[
\begin{align*}
U(\partial_Y^2 - \alpha^2)\varphi - U''\varphi &= -\partial_Y f_1 + \alpha^2 f_2, \\
\varphi|_{Y=0} &= 0, \quad \partial_Y f_1|_{Y=0} = \alpha^2 f_2|_{Y=0}. 
\end{align*}
\]  

\tag{2.17} \label{2.17}

Lemma 2.5. Let \((\varphi, f_1, f_2)\) solve (2.17). Then we have

\[\|\varphi\|_{L^2} \leq C(1 + |Y|)f_1\|_{L^2} + C(1 + |\alpha|^2)|f_1(0)| + \|f_2\|_{H^1}.\]

Proof. We first construct two cut-off functions to correct \(\partial_Y f_1(0)\) and \(f_1(0)\). Let \(\rho_0\) be a cut-off function such that

\[\int_0^{+\infty} \rho_0 dY = 1, \quad \rho_0 \geq 0, \quad \rho_0(0) = 0, \quad \left\|\frac{\rho_0}{U}\right\|_{L^2} + \|(1 + Y)\sigma[\rho_0]\|_{L^2} \leq C,
\]

where \(\sigma[\rho_0] := \int_Y^{+\infty} \rho_0(Y) dY\). Let \(\rho_1\) be a cut-off function such that

\[\rho_1(0) = 1, \quad \int_0^{+\infty} \rho_1 dY = 0, \quad \|\rho_1\|_{H^1} + \|(1 + Y)\sigma[\rho_1]\|_{L^2} \leq C.
\]

Then we decompose \(-\partial_Y f_1\) as follows:

\[\text{\(-\partial_Y f_1 = F_{1,1} + F_{1,2},\)} \tag{2.18}
\]

where

\[
\begin{align*}
F_{1,1}(Y) &= -\partial_Y f_1 - f_1(0)\rho_0(Y) + \partial_Y f_1(0)\rho_1(Y), \\
F_{1,2} &= f_1(0)\rho_0(Y) - \partial_Y f_1(0)\rho_1(Y).
\end{align*}
\]
Clearly, \( \int_0^\infty F_{1,1} dY = 0 \) and \( F_{1,1} |_{Y=0} = 0 \). We decompose \( \varphi = \varphi_1 + \varphi_2 \) as
\[
\begin{align*}
U \partial_Y^2 \varphi_1 - U'' \varphi_1 &= F_{1,1}, \\
\varphi_1 |_{Y=0} &= 0,
\end{align*}
\]
and
\[
\begin{align*}
U(\partial_Y^2 - \alpha^2) \varphi_2 - U'' \varphi_2 &= \alpha^2 U \varphi_1 + F_{1,2} + \alpha^2 f_2, \\
\varphi_2 |_{Y=0} &= 0.
\end{align*}
\]

We first show the estimate of \( \varphi_1 \). We notice that the equation (2.19) can be written as
\[
\partial_Y \left[ U^2 \partial_Y \left( \frac{\varphi_1}{U} \right) \right] = F_{1,1}, \quad \varphi_1 |_{Y=0} = 0.
\]
Since \( F_{1,1} \) satisfies the compatible condition of (2.19), the solution \( \varphi_1 \) to (2.19) can be represented by the following formula:
\[
\varphi_1(Y) = U(Y) \int_Y^{+\infty} \frac{f_1(Y_{2})}{U^2(Y_{1})} dY_1 = L[\sigma[F_{1,1}]](Y),
\]
where the linear operator \( L[\cdot] \) is defined as
\[
L[f](Y) = U(Y) \int_Y^{\infty} \frac{f(Y)}{U^2(Y)} dY_1.
\]
According to Lemma A.4 and \( F_{1,1} = -\partial_Y f_1 - F_{1,2} \), we deduce that
\[
\| \varphi_1 \|_{L^2} \leq \| L[\sigma[F_{1,1}]] \|_{L^2} \leq C \| (1 + Y) \sigma[F_{1,1}] \|_{L^2}
\leq C \| (1 + Y) \sigma[\partial_Y f_1] \|_{L^2} + C \| (1 + Y) \sigma[F_{1,2}] \|_{L^2}
\leq C \| (1 + Y) f_1 \|_{L^2} + C \| (1 + Y) \sigma[F_{1,2}] \|_{L^2}.
\]
By the definition of \( F_{1,2} \) and the condition \( \partial_Y f_1(0) = \alpha^2 f_2(0) \), we have
\[
\| (1 + Y) \sigma[F_{1,2}] \|_{L^2} \leq \| f_1(0) \| \| (1 + Y) \sigma[\rho_0] \|_{L^2} + \alpha^2 \| f_2(0) \| \| (1 + Y) \sigma[\rho_1] \|_{L^2},
\]
which along with the fact \( \| (1 + Y) \sigma[\rho_0] \|_{L^2} + \| (1 + Y) \sigma[\rho_1] \|_{L^2} \leq C \) implies that
\[
\| (1 + Y) \sigma[F_{1,2}] \|_{L^2} \leq C \| f_1(0) \| + \alpha^2 \| f_2(0) \| \leq C \| f_1(0) \| + \alpha^2 \| f_2 \|_{H^1}.
\]
This shows that
\[
\| \varphi_1 \|_{L^2} \leq C \| (1 + Y) f_1 \|_{L^2} + \| f_1(0) \| + \| \alpha^2 \| f_2 \|_{H^1}). \tag{2.21}
\]
Now we turn to consider the estimate of \( \varphi_2 \). We also rewrite the equation as
\[
\begin{align*}
\partial_Y \left[ U^2 \partial_Y \left( \frac{\varphi_2}{U} \right) \right] - \alpha^2 U \varphi_2 &= \alpha^2 U \varphi_1 + F_{1,2} + \alpha^2 f_2, \\
\varphi_2 |_{Y=0} &= 0.
\end{align*}
\]
Multiplying both sides of the first equation by \( -\varphi_2/U \), we obtain by integration by parts that
\[
\left\| U \partial_Y \left( \frac{\varphi_2}{U} \right) \right\|_{L^2}^2 + \alpha^2 \| \varphi_2 \|^2_{L^2} = -\alpha^2 \langle \varphi_1, \varphi_2 \rangle - \langle F_{1,2} + \alpha^2 f_2, \varphi_2/U \rangle. \tag{2.22}
\]
For the second term on the right-hand side, we have
\[
\langle F_{1,2} + \alpha^2 f_2, \varphi_2/U \rangle = f_1(0) (\rho_0/U, \varphi_2) + \langle (-\partial_Y f_1(0) \rho_1 + \alpha^2 f_2)/U, \varphi_2 \rangle. \tag{2.23}
\]
From the definition of $\rho_0$, we infer that
\[ |f_1(0)(\rho_0/U, \varphi_2)| \leq |f_1(0)| \lVert \rho_0 / U \rVert_{L^2} \lVert \varphi_2 \rVert_{L^2} \leq C |f_1(0)| \lVert \varphi_2 \rVert_{L^2}. \tag{2.24} \]

According to the boundary condition $(-\partial_Y f_1(0)\rho_1 + \alpha^2 f_2) |_{Y=0} = 0$ and the fact that $U \geq C^{-1} Y/(1+Y)$, we obtain by Hardy’s inequality that
\[ |\langle (-\partial_Y f_1(0)\rho_1 + \alpha^2 f_2) / U, \varphi_2 \rangle| \leq C \|\langle (-\partial_Y f_1(0)\rho_1 + \alpha^2 f_2) (1+Y) / Y \|_{L^2} \|\varphi_2 \|_{L^2} \]
\[ \leq C \| - \partial_Y f_1(0)\rho_1 + \alpha^2 f_2 \|_{H^1} \|\varphi_2 \|_{L^2} \leq C (\|f_2(0)\| + \|f_2 \|_{H^1}) \alpha^2 \|\varphi_2 \|_{L^2} \]
\[ \leq C \alpha^2 \|f_2 \|_{H^1} \|\varphi_2 \|_{L^2}. \]

Summing up (2.22)–(2.24), we conclude that
\[ \lVert U \partial_Y \left( \frac{\varphi_2}{U} \right) \lVert_{L^2}^2 + \alpha^2 \|\varphi_2 \|_{L^2} \leq \alpha^2 \|\varphi_1 \|_{L^2} \|\varphi_2 \|_{L^2} + C |f_1(0)| \lVert \varphi_2 \rVert_{L^2} + C \alpha^2 \|f_2 \|_{H^1} \|\varphi_2 \|_{L^2}. \]

Therefore, we obtain
\[ \|\varphi_2 \|_{L^2} \leq C (\|\varphi_1 \|_{L^2} + |\alpha|^{-2} |f_1(0)|) + \|f_2 \|_{H^1}. \]

which along with (2.21) gives
\[ \|\varphi \|_{L^2} \leq \|\varphi_1 \|_{L^2} + \|\varphi_2 \|_{L^2} \leq C ((1 + Y) f_1 \|_{L^2} + C (1 + |\alpha|^2) (|\alpha|^{-2} |f_1(0)|) + \|f_2 \|_{H^1}). \]

This completes the proof of the lemma. \hfill \Box

### 2.3 Proof of Proposition 2.1

In this subsection, we prove Proposition 2.1. In fact, Proposition 2.1 is a consequence of the following two lemmas, which give the estimates of the solution $\varphi$ to (2.1) for small $|\alpha|$ and large $|\alpha|$, respectively.

**Lemma 2.6.** Let $(\varphi, w, f)$ solve (2.1). Then for any fixed $M \geq 0$, there exists a $c_0^{(0)} = c_0^{(0)}(M) > 0$ such that for any $|\alpha| \leq M$ and $0 < \varepsilon \leq c_0^{(0)}$, it holds that
\[ \varepsilon \frac{1}{2} \| (1 + Y) w \|_{L^2} + \varepsilon \frac{3}{2} \| (1 + Y) (\partial_Y w, \alpha w) \|_{L^2} + \varepsilon \| (1 + Y) (\partial_Y^2 - \alpha^2) w \|_{L^2} + \| (\partial_Y \varphi, \alpha \varphi) \|_{L^2} + \| \varphi \|_{L^2} \leq C \left( \| (1 + Y) f \|_{L^2} + \frac{1 + |\alpha|^2}{|\alpha|^2} \left| \int_0^\infty f dY \right| \right). \]

**Proof.** We first view the nonlocal term $U'' \varphi$ as a perturbation. Hence, we rewrite the equation as
\[ \begin{cases} U (\partial_Y^2 - \alpha^2) \varphi + i \varepsilon (\partial_Y - \alpha^2)^2 \varphi = f + U'' \varphi =: g, \\ w = (\partial_Y^2 - \alpha^2) \varphi, \quad \varphi |_{Y=0} = \partial_Y w |_{Y=0} = 0. \end{cases} \]

By the fact that $g(0) = 0$, and $|U''| \leq C (1 + Y)^{-3}$, we obtain
\[ \| (1 + Y) g \|_{L^2} \leq \| (1 + Y) f \|_{L^2} + C \| \varphi \|_{L^2}. \tag{2.25} \]

By Lemma 2.2 and (2.25), we obtain
\[ \varepsilon \frac{1}{2} \| (1 + Y) w \|_{L^2} + \varepsilon \frac{3}{2} \| (1 + Y) (\partial_Y w, \alpha w) \|_{L^2} + \varepsilon \| (1 + Y) (\partial_Y^2 - \alpha^2) w \|_{L^2} + \| (\partial_Y \varphi, \alpha \varphi) \|_{L^2} \leq C (\| (1 + Y) f \|_{L^2} + \| \varphi \|_{L^2}). \tag{2.26} \]

Next, we view $i \varepsilon (\partial_Y^2 - \alpha^2)^2 \varphi$ as a perturbation. So we rewrite the equation as
\[ \begin{cases} U (\partial_Y^2 - \alpha^2) \varphi - U'' \varphi = - \partial_Y (\sigma [f] - i \varepsilon \partial_Y w) + i \varepsilon \alpha^2 w, \\ w = (\partial_Y^2 - \alpha^2) \varphi, \quad \varphi |_{Y=0} = \partial_Y w |_{Y=0} = 0. \end{cases} \]
where $\sigma[f](Y) = \int_Y^{+\infty} f(Y_t) dY_t$. Then by Lemma 2.5 and $(\sigma[f] - i\varepsilon \partial_Y w)|_{Y=0} = \sigma[f](0)$, we obtain
\[
\|\varphi\|_{L^2} \leq C(\|1 + Y\)(\sigma[f] - i\varepsilon \partial_Y w))\|_{L^2} + (1 + |\alpha|^{-2})\|\sigma[f](0)\| + \varepsilon(1 + |\alpha|^2)\|w\|_{H^1}.
\]
By the definition of $\sigma[\cdot]$ and Lemma A.4, we have
\[
\|(1 + Y)\sigma[f]\|_{L^2} \leq C\|(1 + Y)^2 f\|_{L^2}, \quad |\sigma[f](0)| = \left| \int_0^{+\infty} f dY \right|.
\]
Hence, we conclude that
\[
\|\varphi\|_{L^2} \leq C\left(\|(1 + Y)^2 f\|_{L^2} + \frac{1 + |\alpha|^2}{|\alpha|^2} \left| \int_0^{+\infty} f dY \right| + \varepsilon(1 + M^2)\|1 + Y\|_{H^1}\right),
\]
which along with (2.26) implies that
\[
\varepsilon^{\frac{2}{3}}\|(1 + Y)w\|_{L^2} + \varepsilon^{\frac{2}{3}}\|(1 + Y)(\partial_Y w, \alpha w)\|_{L^2} + \varepsilon\|(1 + Y)(\partial_Y^2 f - \alpha^2)w\|_{L^2}
\]
\[
+ \|(\partial_Y \varphi, \alpha \varphi)\|_{L^2} + \|\varphi\|_{L^2}
\]
\[
\leq C\left(\|(1 + Y)^2 f\|_{L^2} + \frac{1 + |\alpha|^2}{|\alpha|^2} \left| \int_0^{+\infty} f dY \right| \right).
\]
Then we obtain
\[
\varepsilon^{\frac{2}{3}}\|(1 + Y)(\partial_Y^2 f - \alpha^2)w\|_{L^2} + \varepsilon\|(1 + Y)(\partial_Y \varphi, \alpha \varphi)\|_{L^2} + \|\varphi\|_{L^2}
\]
\[
\leq C\left(\|(1 + Y)^2 f\|_{L^2} + \frac{1 + |\alpha|^2}{|\alpha|^2} \left| \int_0^{+\infty} f dY \right| \right).
\]
Choosing $\varepsilon_0^{(1)}$ sufficiently small such that $(1 - C(1 + M^2)(\varepsilon_0^{(1)})^{\frac{2}{3}}) \geq 1/2$ and $(1 - C(1 + M^2)(\varepsilon_0^{(1)})^{\frac{2}{3}}) \geq 1/2$, we arrive at
\[
\varepsilon^{\frac{2}{3}}\|(1 + Y)w\|_{L^2} + \varepsilon^{\frac{2}{3}}\|(1 + Y)(\partial_Y w, \alpha w)\|_{L^2} + \varepsilon\|(1 + Y)(\partial_Y^2 f - \alpha^2)w\|_{L^2} + \|(\partial_Y \varphi, \alpha \varphi)\|_{L^2} + \|\varphi\|_{L^2}
\]
\[
\leq C\left(\|(1 + Y)^2 f\|_{L^2} + \frac{1 + |\alpha|^2}{|\alpha|^2} \left| \int_0^{+\infty} f dY \right| \right).
\]
The proof is completed. □

The following lemma gives the estimate for $|\alpha|$ sufficiently large, where we can regard the nonlocal term as a perturbation term.

**Lemma 2.7.** Let $(\varphi, w, f)$ solve (2.1). There exists an $M_0 > 0$ such that if $|\alpha| \geq M_0$ and $0 \leq \varepsilon \leq 1$, then we have
\[
\varepsilon^{\frac{2}{3}}\|(1 + Y)w\|_{L^2} + \varepsilon^{\frac{2}{3}}\|(1 + Y)(\partial_Y w, \alpha w)\|_{L^2} + \varepsilon\|(1 + Y)(\partial_Y^2 f - \alpha^2)w\|_{L^2} + \|(\partial_Y \varphi, \alpha \varphi)\|_{L^2}
\]
\[
\leq C\|(1 + Y)f\|_{L^2}.
\]

**Proof.** We rewrite the equation as
\[
U(\partial_Y^2 - \alpha^2)\varphi + i\varepsilon(\partial_Y^2 - \alpha^2)^2\varphi = f + U''\varphi = g.
\]
It follows from Lemma 2.2 that
\[
\varepsilon^{\frac{2}{3}}\|(1 + Y)w\|_{L^2} + \varepsilon^{\frac{2}{3}}\|(1 + Y)(\partial_Y w, \alpha w)\|_{L^2} + \varepsilon\|(1 + Y)(\partial_Y^2 f - \alpha^2)w\|_{L^2} + \|(\partial_Y \varphi, \alpha \varphi)\|_{L^2}
\]
\[
\leq C\|(1 + Y)g\|_{L^2} \leq C\|(1 + Y)f\|_{L^2} + C\|(1 + Y)U''\varphi\|_{L^2}
\]
\[
\leq C\|(1 + Y)f\|_{L^2} + C|\alpha|^{-1}\|\varphi\|_{L^2} \leq C\|(1 + Y)f\|_{L^2} + CM_0^{-1}\|\varphi\|_{L^2},
\]
from which we can obtain our result by choosing $M_0$ sufficiently large so that $CM_0^{-1} \leq 1/2$. □
Now we prove Proposition 2.1.

Proof of Proposition 2.1. Let $M_0$ be the constant in Lemma 2.7. Then we take $M = M_0$ and $c_1 = c_0(0)(M)$ in Lemma 2.6. It follows from Lemmas 2.6 and 2.7 that

$$
\varepsilon^1 \|(1 + Y) w\|_{L^2} + \varepsilon^2 \|(1 + Y)(\partial_Y w, \alpha w)\|_{L^2} + \varepsilon\|(1 + Y)(\partial_Y^2 - \alpha^2)w\|_{L^2} + \|(\partial_Y \varphi, \alpha \varphi)\|_{L^2} \\
\leq C\left(\|(1 + Y)^2 f\|_{L^2} + \frac{1}{|\alpha|^2} \int_0^{+\infty} f dY\right).
$$

The uniqueness is a direct consequence of the above estimate.

Next, we prove the existence via the method of continuity. We consider the following system:

$$
\begin{cases}
U(\partial_Y^2 - \alpha^2)\varphi + \varepsilon(\partial_Y^2 - \alpha^2)\varphi + \lambda(\partial_Y^2 - \alpha^2)\varphi = f, & f(0) = 0, \\
w_\lambda = (\partial_Y^2 - \alpha^2)\varphi, & \varphi |_{Y=\lambda = 0} = \partial_Y w |_{Y=0} = 0,
\end{cases}
$$

where $\lambda \in [0, +\infty)$. It is obvious that $\varphi$ is the solution to (2.1) if $\lambda = 0$. It is also easy to check that for large enough $\lambda$, there exists a unique solution $\varphi_\lambda \in H^1 \cap H^1_0$ to the above system. Hence to prove the existence of (2.1), we only need to obtain a uniform estimate of $\varphi_\lambda$ with respect to $\lambda$. In fact, we can prove this by a similar argument to that in the proof of Lemma 2.6, which is guaranteed by the following facts:

(1) If we replace $\varepsilon(\partial_Y^2 - \alpha^2)^2$ by $\varepsilon(\partial_Y^2 - \alpha^2)^2 + \lambda(\partial_Y^2 - \alpha^2)$ with $\lambda \geq 0$, the estimates in Lemma 2.2 still hold true and are independent of $\lambda$.

(2) If we replace $U(\partial_Y^2 - \alpha^2)$ by $(U + \lambda)(\partial_Y^2 - \alpha^2)$, the estimates in Lemma 2.5 still hold true and are independent of $\lambda$. In fact, we just need to modify (2.19) and (2.20) to

$$
(U + \lambda)\partial_Y^2 \varphi_1 - U'' \varphi_1 = F_{1,1}, \quad \varphi_1 |_{Y=0} = 0
$$

and

$$
(U + \lambda)(\partial_Y^2 - \alpha^2)\varphi_2 - U'' \varphi_2 = \alpha^2 U \varphi_1 + F_{1,2} + \alpha^2 f_2, \quad \varphi_2 |_{Y=0} = 0.
$$

Combining the fact that $L_\lambda[\cdot]$ also satisfies the results in Lemma A.4 for any $\lambda > 0$, we obtain this conclusion. Here,

$$
L_\lambda[f](Y) := (U(Y) + \lambda) \int_{Y}^{\infty} \frac{f(Y_1)}{(U(Y_1) + \lambda)^2} dY_1.
$$

Therefore, we can obtain the existence of (2.1). 

\[\square\]

3 The boundary layer corrector

This section is devoted to constructing the boundary layer corrector, i.e., we need to construct the solution to the following homogeneous system:

$$
\begin{cases}
\varepsilon(\partial_Y^2 - \alpha^2)W_b + UW_b - U'' \Phi_b = 0, \\
(\partial_Y^2 - \alpha^2)\Phi_b = W_b, \\
\Phi_b |_{Y=0} = 0, \quad \partial_Y \Phi_b |_{Y=0} = 1.
\end{cases}
$$

We construct the solution $W_b$ by finding two special solutions (fast and slow modes) to the homogeneous Orr-Sommerfeld equation to match the boundary condition in (3.1). The fast mode is a solution to the homogeneous Orr-Sommerfeld equation built around the Airy function, and the slow mode is built around a solution to the Rayleigh equation.

We see that for $1 \leq |\alpha| < \infty$, the boundary corrector $W_b$ is a perturbation of the Airy function. In this case, the “stream function” $\Phi_a$ of the Airy function is equipped with the zero boundary condition, while for $0 < |\alpha| \leq 1$, we choose the fast decay part $\Phi_{a,f}$ of $\Phi_a$ as the “stream function” of the Airy function. Since $\Phi_{a,f}(0) \neq 0$, we need to use the slow mode to correct it.

We always assume $\varepsilon |\alpha|^3 \ll 1$ throughout this section.
3.1 Construction of the fast mode

We construct the fast mode $W$ around the Airy function. We first define the main part of $W$. Let $Ai(y)$ be the Airy function defined in Appendix B, which satisfies $Ai''(y) - ya_i(y) = 0$. Let $d = -i\varepsilon\alpha^2/(U'(0))$ and define

$$W_a(Y) = \frac{Ai(e^{i\frac{\pi}{6}}(U'(0)/\varepsilon)^\frac{1}{2}(Y + d))}{e^{i\frac{\pi}{6}}(U'(0)/\varepsilon)^\frac{1}{2}Ai'(0)}.$$  \hspace{1cm} (3.2)

Then we find that

$$i\varepsilon(\partial_Y^2 - \alpha^2)W_a + U'(0)YW_a = 0, \quad \partial_Y W_a(0) = \frac{Ai'(e^{i\frac{\pi}{6}}(U'(0)/\varepsilon)^\frac{1}{2}d)}{Ai'(0)}.$$  \hspace{1cm} (3.3)

We first notice that $|(U'(0)/\varepsilon)^\frac{1}{2}d| = U'(0)^{-\frac{2}{3}}\varepsilon^\frac{2}{3}\alpha^2$, which along with the facts $A_i(0) = 1/3$ and the smoothness of $Ai(y)$ implies that if $\varepsilon|\alpha|^3$ is small enough, we have

$$\frac{Ai(e^{i\frac{\pi}{6}}(U'(0)/\varepsilon)^\frac{1}{2}d)}{Ai'(0)} = 1 + O(\varepsilon^2 \alpha^2) \quad \text{and} \quad \frac{Ai'(e^{i\frac{\pi}{6}}(U'(0)/\varepsilon)^\frac{1}{2}d)}{Ai'(0)} = 1 + O(\varepsilon^2 \alpha^2).$$ \hspace{1cm} (3.4)

Let $\Phi_a$ solve $(\partial_Y^2 - \alpha^2)\Phi_a = W_a$ and $\Phi_a|_{Y=0} = 0$. By Lemma A.1, we know that

$$\Phi_a(Y) = -\alpha^{-1}e^{-\alpha Y} \int_0^{+\infty} W_a(Z) \sinh(\alpha Z) dZ + \alpha^{-1} \int_Y^{+\infty} W_a(Z) \sinh(\alpha(Z - Y)) dZ.$$  \hspace{1cm} (3.5)

We denote by $\Phi_{a,f}$ the fast decay part of $\Phi_a$, i.e.,

$$\Phi_{a,f} = \frac{1}{\alpha} \int_Y^{+\infty} W_a(Z) \sinh(\alpha(Z - Y)) dZ.$$  \hspace{1cm} (3.6)

We first establish estimates about $W_a$ and $\Phi_a$ before going further. Thanks to $\text{Im}(d) < 0$, we take

$$\kappa = (U'(0)/\varepsilon)^\frac{1}{2}, \quad \eta = d, \quad \tilde{A}(Y) = Ai(e^{i\frac{\pi}{6}}\kappa(Y + \eta))/Ai(e^{i\frac{\pi}{6}}\kappa\eta).$$

Then we have

$$W_a(Y) = \frac{Ai(e^{i\frac{\pi}{6}}\kappa\eta)}{e^{i\frac{\pi}{6}}(U'(0)/\varepsilon)^\frac{1}{2}Ai'(0)} \times \frac{Ai(e^{i\frac{\pi}{6}}(U'(0)/\varepsilon)^\frac{1}{2}(Y + d))}{Ai(e^{i\frac{\pi}{6}}\kappa\eta)} = \frac{Ai(e^{i\frac{\pi}{6}}\kappa\eta)\tilde{A}(Y)}{e^{i\frac{\pi}{6}}(U'(0)/\varepsilon)^\frac{1}{2}Ai'(0)}$$  \hspace{1cm} (3.7)

and

$$\Phi_{a,f}(Y) = \frac{Ai(e^{i\frac{\pi}{6}}\kappa\eta)\tilde{A}(Y)}{e^{i\frac{\pi}{6}}(U'(0)/\varepsilon)^\frac{1}{2}Ai'(0)}.$$  \hspace{1cm} (3.8)

Then we obtain by Lemma B.3 that

$$\Phi_{a,f}(0) = \frac{e^{i\frac{\pi}{6}}\kappa\eta Ai(e^{i\frac{\pi}{6}}\kappa\eta)}{(U'(0))^\frac{1}{2}Ai'(0)} + O(\varepsilon^2 \alpha^2), \quad \partial_Y \Phi_{a,f}(0) = -\frac{\varepsilon^2}{3(U'(0))^{\frac{3}{2}}Ai'(0)} + O(\varepsilon \alpha^2),$$

which along with (3.4) imply that

$$\Phi_{a,f}(0) = \frac{e^{i\frac{\pi}{6}}\kappa\eta Ai(0)}{(U'(0))^\frac{1}{2}Ai'(0)} + O(\varepsilon^2 \alpha^2), \quad \partial_Y \Phi_{a,f}(0) = -\frac{\varepsilon^2}{3(U'(0))^{\frac{3}{2}}Ai'(0)} + O(\varepsilon \alpha^2).$$ \hspace{1cm} (3.9)

We also notice that by (3.4),

$$|W_a(Y)| = \left| \frac{Ai(0)\tilde{A}(Y)}{e^{i\frac{\pi}{6}}(U'(0)/\varepsilon)^\frac{1}{2}Ai'(0)} \times \frac{Ai(e^{i\frac{\pi}{6}}\kappa\eta)}{Ai(0)} \right| \sim \varepsilon^{\frac{1}{3}}|\tilde{A}(Y)|.$$
According to the fact that $1 \leq 1 + |\alpha| \leq C(1 + \varepsilon|\alpha|^3)^{\frac{1}{2}}$ and applying Lemmas B.2 and B.3, we then have

\[
\varepsilon^{-\frac{1}{2}}\|W_a\|_{L^2} + \varepsilon^{-\frac{1}{2}}\|\partial_Y \Phi_a, \alpha \Phi_a\|_{L^2} + \varepsilon^{-\frac{1}{2}}\|Y W_a\|_{L^2} + \varepsilon^{-\frac{1}{2}}\|Y^2 W_a\|_{L^2} + \varepsilon^{-\frac{1}{2}}\|Y^3 W_a\|_{L^2} \leq C \varepsilon^\frac{1}{2}
\]

(3.7)

and

\[
|\partial_Y \Phi_a(0)| \geq C^{-1} \varepsilon^\frac{1}{2} (1 + \varepsilon|\alpha|^3)^{-\frac{1}{2}}.
\]

(3.8)

For the fast decay part, we also have

\[
\varepsilon^{-\frac{1}{2}}\|\partial_Y \Phi_a, \alpha \Phi_a\|_{L^2} + \varepsilon^{-\frac{1}{2}}\|\Phi_{a,f}\|_{L^2} + \varepsilon^{-\frac{1}{2}}\|Y \Phi_{a,f}\|_{L^2} \leq C \varepsilon^\frac{1}{2}
\]

(3.9)

and

\[
|\partial_Y \Phi_{a,f}(0)| \geq C^{-1} \varepsilon^\frac{1}{2} (1 + \varepsilon|\alpha|^3)^{-\frac{1}{2}}.
\]

(3.10)

As we mentioned before, we construct the fast mode $(W, \Phi)$ around $(W_a, \Phi_a)$ for the case of $1 \leq |\alpha| < \infty$ and $(W_f, \Phi_f)$ around $(W_{a,f}, \Phi_{a,f})$ for the case of $0 < |\alpha| \leq 1$. More precisely, we define $W = W_a + W_e$ with $W_e$ satisfying

\[
\begin{cases}
\varepsilon(\partial_Y^2 - \alpha^2) W_e + UW_e - U'' \Phi_e = -(U - U'(0)Y)W_a + U'' \Phi_a, \\
(\partial_Y^2 - \alpha^2) \Phi_e = W_e, \\
\Phi_e |_{Y=0} = \partial_Y W_e |_{Y=0} = 0
\end{cases}
\]

(3.11)

and $W_f = W_a + W_{e,f}$ with $W_{e,f}$ being the solution to

\[
\begin{cases}
\varepsilon(\partial_Y^2 - \alpha^2) W_{e,f} + UW_{e,f} - U'' \Phi_{e,f} = -(U - U'(0)Y)W_a + U'' \Phi_{a,f}, \\
(\partial_Y^2 - \alpha^2) \Phi_e = W_{e,f}, \\
\Phi_e |_{Y=0} = \partial_Y W_e |_{Y=0} = 0
\end{cases}
\]

(3.12)

Lemma 3.1. Let $c_1$ be the positive number in Proposition 2.1. Then for any $0 < \varepsilon \leq c_1$, the following statements hold:

- Let $(W_e, \Phi_e)$ be the solution to (3.11). Then for any $1 \leq |\alpha| < +\infty$, we have

\[
\varepsilon^\frac{1}{2}\|W_e\|_{L^2} + \varepsilon^\frac{1}{2}\|(1 + Y)W_e\|_{L^2} + \varepsilon^\frac{1}{2}\|(1 + Y)(\partial_Y W_e, \alpha W_e)\|_{L^2} \leq C \varepsilon.
\]

(3.13)

- Let $(W_{e,f}, \Phi_{e,f})$ be the solution to (3.12). Then for any $\alpha > 0$, we have

\[
\varepsilon^\frac{1}{2}\|W_{e,f}\|_{L^2} + \varepsilon^\frac{1}{2}\|(1 + Y)W_{e,f}\|_{L^2} + \varepsilon^\frac{1}{2}\|(1 + Y)(\partial_Y W_{e,f}, \alpha W_{e,f})\|_{L^2} \leq C \varepsilon^\frac{1}{2}.
\]

(3.14)

Remark 3.2. The existence of $(W_e, \Phi_e)$ and $(W_{e,f}, \Phi_{e,f})$ is also deduced from Proposition 2.1.

Proof of Lemma 3.1. We start our proof of (3.13). We first notice that the source term in (3.11) can be written as

\[
-(U - U'(0)Y)W_a + U'' \Phi_a
\]

\[
= -(\partial_Y [(U - U'(0)Y)] \partial_Y \Phi_a] + \partial_Y [(U' - U'(0)) \Phi_a]) + \alpha^2 (U - U'(0)Y) \Phi_a
\]

\[
=: \partial_Y R_1 + R_2,
\]

(3.15)

where

\[
R_1 = -(U - U'(0)Y) \partial_Y \Phi_a + (U' - U'(0)) \Phi_a,
\]

(3.16)

\[
R_2 = \alpha^2 (U - U'(0)Y) \Phi_a.
\]

(3.17)
Moreover, \( \lim_{Y \to +\infty} R_1(Y) = R_1|_{Y=0} = 0 \). Then
\[
\frac{1}{|\alpha|^2} \left| \int_0^{+\infty} (\partial_Y R_1 + R_2) dY \right| = \frac{1}{|\alpha|^2} \left| \int_0^{+\infty} R_2 dY \right| \leq C|\alpha|^{-2} \|(1 + Y)R_2\|_{L^2}.
\]
Then by Proposition 2.1, for any \( 0 < \varepsilon \leq c_1 \), we obtain
\[
\varepsilon^{\frac{1}{2}} \|(1 + Y)W_e\|_{L^2} + \varepsilon^{\frac{1}{2}} \|(1 + Y)(\partial_Y W_e, \alpha W_e)\|_{L^2} + \|(\partial_Y \Phi_e, \alpha \Phi_e)\|_{L^2} \\
\leq C \left( \|(1 + Y)^2(\partial_Y R_1 + R_2)\|_{L^2} + \frac{1}{|\alpha|^2} \left| \int_0^{+\infty} (\partial_Y R_1 + R_2) dY \right| \right) \\
\leq C \left( \|(1 + Y)^2(\partial_Y R_1 + R_2)\|_{L^2} + |\alpha|^{-2} \|(1 + Y)R_2\|_{L^2}. \right)
\]
On the other hand, thanks to \( |(U - U'(0)Y)W_a| \leq CY^2|W_a| \), we have
\[
\|(1 + Y)^2(\partial_Y R_1 + R_2)\|_{L^2} \leq C \left( \|(1 + Y)^2Y^2W_a\|_{L^2} + \|\Phi_a\|_{L^2} \right)
\]
and
\[
|\alpha|^{-2} \|(1 + Y)R_2\|_{L^2} \leq \|(1 + Y)^3\Phi_a\|_{L^2}.
\]
From (3.7), \( \varepsilon \leq 1 \) and \( |\alpha| \geq 1 \), we infer that
\[
\|(1 + Y)^2Y^2W_a\|_{L^2} + \|\Phi_a\|_{L^2} + \|(1 + Y)^3\Phi_a\|_{L^2} \leq C\varepsilon.
\]
Therefore, we obtain
\[
\|(1 + Y)^2(\partial_Y R_1 + R_2)\|_{L^2} + |\alpha|^{-2} \|(1 + Y)R_2\|_{L^2} \leq C\varepsilon.
\]
Thus, we conclude
\[
\varepsilon^{\frac{1}{2}} \|(1 + Y)W_e\|_{L^2} + \varepsilon^{\frac{1}{2}} \|(1 + Y)(\partial_Y W_e, \alpha W_e)\|_{L^2} + \|(\partial_Y \Phi_e, \alpha \Phi_e)\|_{L^2} \leq C\varepsilon.
\]
Now we turn to the proof of (3.14). Again by Proposition 2.1, for any \( 0 < \varepsilon \leq c_1 \) and a similar argument to that as above, we have
\[
\varepsilon^{\frac{1}{2}} \|(1 + Y)W_{e,f}\|_{L^2} + \varepsilon^{\frac{1}{2}} \|(1 + Y)(\partial_Y W_{e,f}, \alpha W_{e,f})\|_{L^2} + \|(\partial_Y \Phi_{e,f}, \alpha \Phi_{e,f})\|_{L^2} \\
\leq C \left( \|(1 + Y)^2Y^2W_{a,f}\|_{L^2} + \|\Phi_{a,f}\|_{L^2} + \|(1 + Y)^3\Phi_{a,f}\|_{L^2} \right),
\]
which along with (3.9) implies that
\[
\varepsilon^{\frac{1}{2}} \|(1 + Y)W_{e,f}\|_{L^2} + \varepsilon^{\frac{1}{2}} \|(1 + Y)(\partial_Y W_{e,f}, \alpha W_{e,f})\|_{L^2} + \|(\partial_Y \Phi_{e,f}, \alpha \Phi_{e,f})\|_{L^2} \leq C\varepsilon^{\frac{1}{2}}.
\]
This completes the proof. \( \square \)

**Lemma 3.3.** Let \( c_1 \) be the constant in Proposition 2.1. There exist \( c_2 \in (0, c_1] \) and \( \delta_* \in (0, 1] \) such that if \( |\alpha|^3 \leq \delta_* \) and \( 0 < \varepsilon \leq c_2 \), the following statements hold true:

1. For any \( 1 \leq |\alpha| < \infty \), there exists a unique solution \( \Phi \in H^4 \) to the homogeneous Orr-Sommerfeld equation satisfying
\[
\varepsilon^{-\frac{1}{2}} \|W\|_{L^2} + \varepsilon^{-\frac{1}{2}} \|\Phi - \Phi \|_{L^2} + \varepsilon^{-\frac{1}{2}} \|\partial_Y \Phi - \Phi \|_{L^\infty} \leq C\varepsilon^{\frac{1}{2}}.
\]

Moreover, we have
\[
|\partial_Y \Phi(0)| \geq C^{-1}\varepsilon^{\frac{3}{2}}, \quad \Phi(0) = 0.
\]

2. For any \( \alpha > 0 \), there exists a unique solution \( \Phi_f \in H^4 \) to the homogeneous Orr-Sommerfeld equation satisfying
\[
\varepsilon^{-\frac{1}{2}} \|W_f\|_{L^2} + \varepsilon^{-\frac{1}{2}} \|\partial_Y \Phi_f - \Phi_f \|_{L^2} + \varepsilon^{-\frac{1}{2}} \|\partial_Y \Phi_f - \Phi_f \|_{L^\infty} \leq C\varepsilon^{\frac{1}{2}}.
\]

Moreover, we have
\[
\partial_Y \Phi_f(0) = -\frac{\varepsilon^{\frac{3}{2}}}{3(U'(0))^2 A_{1r}(0)} + O(\varepsilon(1 + \alpha^2)), \quad \Phi_f(0) = \frac{\varepsilon^{\frac{3}{2}}}{(U'(0))^2 A_{1r}(0)} + O(\varepsilon^4 \alpha^2).
\]
Proof. We first show the first statement of this lemma. By (3.7) and (3.13), if $|\alpha|^3 \leq \delta_*\leq 1$, $0 < \varepsilon \leq c_2 \leq c_1$ and $|\alpha| \geq 1$, we have
\[
e^{-\frac{\varepsilon}{2}}\|W\|_{L^2} + e^{-\frac{\varepsilon}{2}}\|\partial_Y \Phi, \alpha \Phi\|_{L^2} \\
\leq C(e^{-\frac{\varepsilon}{2}}) + (1 + Y)W_0 \|\partial_Y \Phi, \alpha \Phi\|_{L^2} + e^{-\frac{\varepsilon}{2}}(1 + Y)W_c \|\partial_Y \Phi, \alpha \Phi\|_{L^2} \\
+ C\varepsilon(1 + \varepsilon^\frac{1}{2}) \leq C\varepsilon^\frac{1}{2}(1 + c_2^\frac{1}{2}) \leq C\varepsilon^\frac{1}{2}.
\] (3.18)

By the interpolation, we obtain
\[
\|\partial_Y \Phi, \alpha \Phi\|_{L^\infty} \leq \|W\|_{L^\frac{3}{2}} \|\partial_Y \Phi, \alpha \Phi\|_{L^2} \leq C\varepsilon^\frac{1}{2}.
\] (3.19)

Again by (3.13) and the interpolation, we have
\[
\|\partial_Y \Phi_c\|_{L^\infty} \leq \|W_c\|_{L^\frac{3}{2}} \|\partial_Y \Phi_c, \alpha \Phi_c\|_{L^2} \leq C\varepsilon^\frac{1}{2},
\]
which along with (3.8) implies that
\[
|\partial_Y \Phi(0)| \geq |\partial_Y \Phi_0(0)| - \|\partial_Y \Phi_c\|_{L^\infty} \geq C^{-1}\varepsilon^\frac{1}{2}(1 - C\varepsilon^\frac{1}{2}).
\]
Then taking $c_2$ sufficiently small so that $1 - Cc_2^\frac{1}{2} \geq 1/2$, we obtain
\[
|\partial_Y \Phi(0)| \geq C^{-1}\varepsilon^\frac{1}{2}.
\] (3.20)

Now we turn to the proof of the second statement. By an argument similar to (3.18), we have
\[
e^{-\frac{\varepsilon}{2}}\|W_f\|_{L^2} + e^{-\frac{\varepsilon}{2}}\|\partial_Y \Phi_f, \alpha \Phi_f\|_{L^2} + e^{-\frac{\varepsilon}{2}}\|\partial_Y \Phi_f, \alpha \Phi_f\|_{L^\infty} \leq C\varepsilon^\frac{1}{2}.
\]
We also notice that
\[
|\partial_Y \Phi_f(0) - \partial_Y \Phi_{a,f}(0)| = |\Phi_{c_f}(0)| \leq \|\Phi_{c_f}\|_{L^\infty} \leq \|W_{c,f}\|_{L^\frac{3}{2}} \|\partial_Y \Phi_{c_f}\|_{L^2} \leq C\varepsilon
\]
and $\Phi_f(0) = \Phi_{a,f}(0)$, which along with (3.6) imply that
\[
\partial_Y \Phi_f(0) = -\frac{\varepsilon^\frac{1}{2}}{3(U'(0))^2 A'(0)} + O(\varepsilon(1 + \alpha^2)), \quad \Phi_f(0) = \frac{e^{i\pi/2} \alpha Ai(0)}{(U'(0))^2 A'(0)} + O(\varepsilon^\frac{1}{2} \alpha^2).
\]
This completes the proof. \[\Box\]

3.2 Construction of the slow mode

In this subsection, we construct a solution to the homogeneous Orr-Sommerfeld equation around a solution $\varphi_{Ray}$ to the homogeneous Rayleigh equation when $0 < \alpha \leq 1$. Let $\varphi = \varphi_0 + \varphi_1 + \varphi_2$ be the solution to the homogeneous Rayleigh equation constructed in Proposition C.1. For $0 < \alpha \leq 1$, we define $\varphi_{Ray}$ as follows:
\[
\varphi_{Ray} = \frac{cE}{\alpha} \varphi, \quad c_E = \frac{\alpha}{\varphi_1(0)} = U'(0) + O(\alpha).
\] (3.21)

Then from Proposition C.1, we directly have the following lemma.

Lemma 3.4. For any $0 < \alpha \leq 1$, there exists a solution $\varphi_{Ray} \in H^1(\mathbb{R}_+)$ to the homogeneous Rayleigh equation satisfying the following properties:
\[
\varphi_{Ray} = \varphi_{Ray,0} + \varphi_{Ray,1} + \varphi_{Ray,2}
\]
According to the last statement in Lemma 2.2 and (3.22), we have

\[ \varphi_{Ray,0} = \frac{cE}{\alpha} U e^{-\alpha Y}, \quad \varphi_{Ray,1}(0) = 1, \]
\[ \| \partial_Y \varphi_{Ray,1} \|_{L^2} + \| \varphi_{Ray,1} \|_{L^2} \leq C, \]
\[ \| \partial_Y \varphi_{Ray,2} \|_{L^2} + \alpha \| \varphi_{Ray,2} \|_{L^2} \leq C \alpha^{\frac{1}{2}}. \]  

(3.22)

In particular, we have

\[ \varphi_{Ray}(0) = 1. \]  

(3.23)

If \( \frac{U''}{U} \in L^2(\mathbb{R}^+) \) in addition, then \( \varphi_{Ray,1}, \varphi_{Ray,2} \in H^2(\mathbb{R}^+) \).

We define \( OS = i\varepsilon (\partial_Y^2 - \alpha^2)^2 + U(\partial_Y^2 - \alpha^2) - U'' \). We observe that

\[ OS[\varphi_{Ray}] = i\varepsilon (\partial_Y^2 - \alpha^2)^2 \varphi_{Ray}, \]

whose source term contains too much singularity. Hence, we introduce \( \psi \) being the solution to the following system:

\[
\begin{align*}
\begin{cases}
&i\varepsilon (\partial_Y^2 - \alpha^2)\psi + U' \psi = i\varepsilon (\partial_Y^2 - \alpha^2)\varphi_{Ray}, \quad Y > 0, \\
&\psi(0) = 0.
\end{cases}
\end{align*}
\]

(3.24)

**Lemma 3.5.** Let \( \alpha \in (0,1] \) and \( \psi \) be the solution of (3.24). Then \( \psi = \psi_0 + \psi_1 \) with

\[ \varepsilon^{\frac{1}{2}} \| \partial_Y \psi_0 \|_{L^2} + \| \psi_0 \|_{L^2} \leq C \alpha^{-1} \varepsilon^{\frac{1}{2}}, \]
\[ \varepsilon^{\frac{1}{2}} \| \partial_Y \psi_1 \|_{L^2} + \alpha \| \psi_1 \|_{L^2} \leq C \varepsilon^{\frac{1}{2}}. \]

Proof. We first notice that

\[ (\partial_Y^2 - \alpha^2)\varphi_{Ray} = \frac{U''}{U} \varphi_{Ray} = \frac{U''}{U} \varphi_{Ray,0} + \frac{U''}{U} (\varphi_{Ray,1} + \varphi_{Ray,2}) \]
\[ = \frac{cE}{\alpha} U'' e^{-\alpha Y} + \frac{U''}{U} (\varphi_{Ray,1} + \varphi_{Ray,2}). \]

Then we decompose \( \psi \) as \( \psi = \psi_0 + \psi_1 \), where \( \psi_0 \) is the solution to

\[ i\varepsilon (\partial_Y^2 - \alpha^2)\psi_0 + U' \psi_0 = i\varepsilon \frac{cE}{\alpha} U'' e^{-\alpha Y}, \quad \psi_0(0) = 0, \]  

(3.25)

and \( \psi_1 \) is the solution to

\[ i\varepsilon (\partial_Y^2 - \alpha^2)\psi_1 + U' \psi_1 = i\varepsilon \frac{U''}{U} (\varphi_{Ray,1} + \varphi_{Ray,2}), \quad \psi_1(0) = 0. \]  

(3.26)

By the first statement in Lemma 2.2, we have

\[ \| \partial_Y \psi_0 \|_{L^2} \leq C \varepsilon^{\frac{1}{2}} \alpha^{-1} \| U'' e^{-\alpha Y} \|_{L^2} \leq C \varepsilon^{\frac{1}{2}} \alpha^{-1}, \]
\[ \| \psi_0 \|_{L^2} \leq C \varepsilon^{\frac{1}{2}} \alpha^{-1} \| U'' e^{-\alpha Y} \|_{L^2} \leq C \varepsilon^{\frac{1}{2}} \alpha^{-1}. \]

According to the last statement in Lemma 2.2 and (3.22), we have

\[ \| \partial_Y \psi_1 \|_{L^2} \leq C \left\| \frac{Y U''}{U} (\varphi_{Ray,1} + \varphi_{Ray,2}) \right\|_{L^2} \leq C, \]
\[ \alpha \| \psi_1 \|_{L^2} \leq C \alpha \varepsilon^{\frac{1}{2}} \left\| \frac{Y U''}{U} (\varphi_{Ray,1} + \varphi_{Ray,2}) \right\|_{L^2} \leq C \alpha \varepsilon^{\frac{1}{2}}. \]

This completes the proof.
Now we are ready to construct the slow mode. We define $\phi_s = \varphi_{\text{Ray}} + \psi + \tilde{\phi}$, where $\tilde{\phi}$ is the solution to
\[
\begin{cases}
  i\varepsilon(\partial^2_Y - \alpha^2)\tilde{w} + U\tilde{w} - U''\tilde{\phi} = -2\partial_Y(U'\psi), \\
  (\partial^2_Y - \alpha^2)\tilde{\phi} = \tilde{w}, \\
  \tilde{\phi}|_{Y=0} = \partial_Y\tilde{w}|_{Y=0} = 0.
\end{cases}
\] (3.27)

**Lemma 3.6.** Let $\alpha \in (0, 1]$ and $0 < \varepsilon \ll c_1$, where $c_1$ is the small positive constant in Proposition 2.1. Suppose that $\tilde{\phi}$ is the solution to (3.27). Then we have
\[
\varepsilon^{\frac{1}{2}}\|(1 + Y)\tilde{w}\|_{L^2} + \varepsilon^{\frac{1}{2}}\|(1 + Y)(\partial_Y\tilde{w}, \alpha\tilde{w})\|_{L^2} + \varepsilon\|(1 + Y)(\partial^2_Y - \alpha^2)\tilde{w}\|_{L^2} + \|(\partial_Y\tilde{\phi}, \alpha\tilde{\phi})\|_{L^2} 
\leq C(\alpha^{-1}\varepsilon^{\frac{1}{4}} + 1).
\]

**Proof.** From Proposition 2.1 and the fact that $U'\psi|_{Y=0} = 0$, we have
\[
\varepsilon^{\frac{1}{2}}\|(1 + Y)\tilde{w}\|_{L^2} + \varepsilon^{\frac{1}{2}}\|(1 + Y)(\partial_Y\tilde{w}, \alpha\tilde{w})\|_{L^2} + \varepsilon\|(1 + Y)(\partial^2_Y - \alpha^2)\tilde{w}\|_{L^2} + \|(\partial_Y\tilde{\phi}, \alpha\tilde{\phi})\|_{L^2} 
\leq C||(1 + Y)^2\partial_Y(U'\psi)||_{L^2}.
\]
On the other hand, we obtain by Lemma 3.5 that
\[
||(1 + Y)^2\partial_Y(U'\psi)||_{L^2} \leq ||(1 + Y)^2U''\psi||_{L^2} + ||(1 + Y)^2U'\partial_Y\psi||_{L^2} 
\leq ||(1 + Y)^2U''\psi||_{L^2} + ||(1 + Y)^2U'\partial_Y\psi||_{L^2} 
\leq C\|\partial_Y\psi\|_{L^2} \leq C(\alpha^{-1}\varepsilon^{\frac{1}{4}} + 1).
\]
Therefore, we obtain
\[
\varepsilon^{\frac{1}{2}}\|(1 + Y)\tilde{w}\|_{L^2} + \varepsilon^{\frac{1}{2}}\|(1 + Y)(\partial_Y\tilde{w}, \alpha\tilde{w})\|_{L^2} + \varepsilon\|(1 + Y)(\partial^2_Y - \alpha^2)\tilde{w}\|_{L^2} + \|(\partial_Y\tilde{\phi}, \alpha\tilde{\phi})\|_{L^2} 
\leq C(\alpha^{-1}\varepsilon^{\frac{1}{4}} + 1).
\]
This completes the proof. \(\square\)

**Proposition 3.7.** Let $\alpha \in (0, 1]$ and $0 < \varepsilon \ll c_1$, where $c_1$ is the small positive constant in Proposition 2.1. Then there exists a solution $\phi_s \in H^4$ to the homogeneous Orr-Sommerfeld equation such that
\[
\phi_s = \frac{c_E}{\alpha} U e^{-\alpha Y} + \phi_{s, \text{re}}, \quad \phi_{s, \text{re}}(0) = 1,
\]
where
\[
|||\partial_Y\phi_{s, \text{re}}, \alpha\phi_{s, \text{re}}|||_{L^2} \leq C(1 + \alpha^{-1}\varepsilon^{\frac{1}{4}}),
\]
\[
||\partial_Y\phi_{s, \text{re}}||_{L^\infty} \leq C(\varepsilon^{-\frac{1}{2}} + \alpha^{-1}\varepsilon^{\frac{1}{4}}),
\]
\[
|||\partial^2_Y - \alpha^2|||_{L^2} \leq C(\varepsilon^{-\frac{1}{4}} + \alpha^{-1}\varepsilon^{\frac{1}{2}}).
\]
In particular,
\[
\partial_Y\phi_s(0) = \frac{c_E U'(0)}{\alpha} + O(\varepsilon^{-\frac{1}{4}} + \alpha^{-1}\varepsilon^{\frac{1}{2}}).
\]

**Proof.** Recall that $\phi_s = \varphi_{\text{Ray}} + \psi + \tilde{\phi}$. First of all, we show $\phi_s \in H^2$. For this moment, we assume that $U''/U \in L^2(\mathbb{R}_+)$ and later we recover the $H^2$ regularity of $\phi_s$ without the assumption $U''/U \in L^2(\mathbb{R}_+)$. By Lemmas 3.4–3.6, we obtain $\phi_s \in H^2$. Hence, $\phi_s$ satisfies the homogeneous Orr-Sommerfeld equation in the following weak sense:
\[
(U(\partial^2_Y - \alpha^2)\phi_s - U''\phi_s, h) + i\varepsilon((\partial^2_Y - \alpha^2)\phi_s, (\partial^2_Y - \alpha^2)h) = 0, \quad h \in H^2, \quad \partial_Y h(0) = 0,
\]
which imply that \( w_s = (\partial_x^2 - \alpha^2)\phi_s \) is a weak solution to the Poisson equation \( i\varepsilon(\partial_x^2 - \alpha^2)w_s = -Uw_s + U''\phi_s \) with the Neumann boundary condition \( \partial_Y w_s|_{Y=0} = 0 \). Therefore, we have \( w_s \in H^2 \) and

\[
\langle Uw_s - U''\phi_s, w_s \rangle - i\varepsilon(\|\partial_Y w_s\|^2_{L^2} + \alpha^2\|w_s\|^2_{L^2}) = 0.
\] (3.28)

According to Lemma 2.2 and Remark 2.4, we have

\[
\|w_s\|_{L^2} \leq C\varepsilon^{-\frac{1}{2}}\|U''\phi_s\|_{L^2}.
\]

Hence, the \( H^2 \) regularity of \( \phi_s \) is controlled by \( \|U''\phi_s\|_{L^2} \), which means that the assumption \( U''/U \in L^2 \) is not required. Now we complete the proof of \( \phi_s \in H^4 \).

By Lemmas 3.4–3.6, we directly have

\[
\|(\partial_Y \phi_{s, re}, \alpha \phi_{s, re})\|_{L^2} \leq C(1 + \alpha^{-1}\varepsilon^{\frac{1}{2}}).
\]

Now we show the estimate about \( \|(\partial_x^2 - \alpha^2)\phi_{s, re}\|_{L^2} \). Recall that

\[
\phi_{s, re} = \phi_s - \frac{cE}{\alpha}U e^{-\alpha Y}, \quad w_{s, re} = (\partial_x^2 - \alpha^2)\phi_{s, re},
\]

which along with (3.28) imply

\[
\langle -2cEUU'e^{-\alpha Y} + Uw_{s, re} - U''\phi_{s, re}, cE\alpha^{-1}(U''e^{-\alpha Y} - 2\alpha U'e^{-\alpha Y}) + w_{s, re}\rangle
\]

\[
- i\varepsilon(\|\partial_Y w_{s, re}\|^2_{L^2} + \alpha^2\|w_{s, re}\|^2_{L^2}) = 0.
\] (3.29)

By taking the real part of the above equality, we obtain

\[
\|\sqrt{U}w_{s, re}\|^2_{L^2} = \text{Re}[2cEUU'e^{-\alpha Y} + U'\phi_{s, re}w_{s, re}] - \text{Re}\left(Uw_{s, re}, \frac{cE}{\alpha}(U''e^{-\alpha Y} - 2\alpha U'e^{-\alpha Y})\right)
\]

\[
+ \text{Re}\left(2cEUU'e^{-\alpha Y} + U''\phi_{s, re}, \frac{cE}{\alpha}(U''e^{-\alpha Y} - 2\alpha U'e^{-\alpha Y})\right),
\]

which shows that

\[
\|\sqrt{U}w_{s, re}\|^2_{L^2} \leq C\|U\partial_x U'e^{-\alpha Y}\|^2_{L^2} + \|U''\phi_{s, re}\|_{L^2}\|w_{s, re}\|_{L^2}
\]

\[
+ C\alpha^{-2}\|U\partial_x U'e^{-\alpha Y} - 2\alpha U'e^{-\alpha Y}\|^2_{L^2} + C\alpha^{-1}(1 + \|U''\phi_{s, re}\|_{L^2})
\]

\[
\leq C\alpha^{-2} + \|U''\phi_{s, re}\|_{L^2}\|w_{s, re}\|_{L^2} + C\alpha^{-1}(1 + \|U''\phi_{s, re}\|_{L^2}).
\]

On the other hand, notice that \( \phi_{s, re} = \varphi_{Ray, 1} + \varphi_{Ray, 2} + \psi + \varepsilon \), and then

\[
\|U''\phi_{s, re}\|_{L^2} \leq \|U''\varphi_{Ray, 1}\|_{L^2} + \|U''(\phi_{s, re} - \varphi_{Ray, 1})\|_{L^2},
\]

which along with the fact \( (\phi_{s, re} - \varphi_{Ray, 1})|_{Y=0} = 0 \) implies that

\[
\|U''\phi_{s, re}\|_{L^2} \leq C\|\varphi_{Ray, 1}\|_{L^2} + C\left|\frac{\phi_{s, re} - \varphi_{Ray, 1}}{Y}\right|_{L^2}
\]

\[
\leq C\|\varphi_{Ray, 1}\|_{L^2} + C(\|\partial_Y \phi_{s, re}\|_{L^2} + \|\partial_Y \varphi_{Ray, 1}\|_{L^2}) \leq C\left(1 + \frac{\varepsilon^\frac{1}{2}}{\alpha}\right).
\]

Hence, we have

\[
\|\sqrt{U}w_{s, re}\|^2_{L^2} \leq C\alpha^{-2} + C\left(1 + \frac{\varepsilon^\frac{1}{2}}{\alpha}\right)\|w_{s, re}\|_{L^2}.
\] (3.30)

From the imaginary part of (3.29), we have

\[
\varepsilon(\|\partial_Y w_{s, re}\|^2_{L^2} + \alpha^2\|w_{s, re}\|^2_{L^2}) \leq C\alpha^{-1}\|\sqrt{U}w_{s, re}\|_{L^2} + C\alpha^{-1}\|U''\phi_{s, re}\|_{L^2} + C\|w_{s, re}\|_{L^2}
\]
+ C\|U''\phi_{s,rc}\|_{L^2}\|w_{s,rc}\|_{L^2} \\
\leq C\alpha^{-2} + C\left(1 + \frac{\epsilon^2}{\alpha}\right)\|w_{s,rc}\|_{L^2}.

(3.31)

Thanks to \(w_{s,rc} = w_s - cE\alpha^{-1}U''e^{-\alpha Y} + 2cE'U' e^{-\alpha Y}\), we have
\[
\|\partial_Y w_{s,rc}\|_{L^2}^2 + \alpha^2\|w_{s,rc}\|_{L^2}^2 \leq \|\partial_Y w_s\|_{L^2}^2 + \alpha^2\|w_s\|_{L^2} + C\alpha^{-1}.
\]

Thus, we have
\[
\epsilon(\|\partial_Y w_{s,rc}\|_{L^2}^2 + \alpha^2\|w_{s,rc}\|_{L^2}^2) \leq C\alpha^{-2} + C\left(1 + \frac{\epsilon^2}{\alpha}\right)\|w_{s,rc}\|_{L^2}.

(3.32)

By a similar interpolation inequality to (2.6), we obtain
\[
\|w_{s,rc}\|_{L^2} \leq C\epsilon^{\frac{1}{2}}\|\partial_Y w_{s,rc}\|_{L^2}^{\frac{1}{2}}\|w_{s,rc}\|_{L^2}^{\frac{1}{2}} + C\epsilon^{-\frac{1}{2}}\|U w_{s,rc}\|_{L^2},
\]

which along with (3.30) and (3.32) implies that
\[
\|w_{s,rc}\|_{L^2} \leq C\epsilon^{\frac{1}{2}}(\alpha^{-\frac{1}{2}}\epsilon^{-\frac{1}{2}} + (1 + \epsilon^{-\frac{1}{2}}\alpha^{-\frac{1}{2}}))\|w_{s,rc}\|_{L^2}^{\frac{1}{2}}\|w_{s,rc}\|_{L^2}^{\frac{1}{2}} + C\epsilon^{-\frac{1}{2}}(1 + (1 + \epsilon^{-\frac{1}{2}}\alpha^{-\frac{1}{2}}))\|w_{s,rc}\|_{L^2}^{\frac{3}{2}}
\]
\[
\leq C(\epsilon^{-\frac{1}{2}}\alpha^{-1} + \epsilon^{-\frac{1}{2}}).
\]

Then we finally have
\[
\|\partial_Y^2 - \alpha^2\|\phi_{s,rc}\|_{L^2} = \|w_{s,rc}\|_{L^2} \leq C(\epsilon^{-\frac{1}{2}}\alpha^{-1} + \epsilon^{-\frac{1}{2}}),
\]

and by the interpolation,
\[
\|\partial_Y \phi_{s,rc}\|_{L^\infty} \leq C(\epsilon^{-\frac{1}{2}} + \alpha^{-1}\epsilon^{-\frac{1}{2}})(1 + \alpha^{-1}\epsilon^{-\frac{1}{2}}) = C(\epsilon^{-\frac{1}{2}} + \alpha^{-1}\epsilon^{-\frac{1}{2}}).
\]

This completes the proof of the proposition. \(\square\)

### 3.3 The boundary corrector

Here, we construct the boundary corrector \(\Phi_b\). Since \(|\partial_Y \Phi(0)| \geq C^{-1}\epsilon^\frac{1}{2}\), we know that the solution \(W_b\) to (3.1) can be given by \(W_b = W/\partial_Y \Phi(0)\) for the case of \(1 \leq |\alpha| < \infty\). For the case of \(0 < |\alpha| \leq 1\), we need to use the slow mode \(\phi_s\) to modify the boundary value of \(\Phi_f\).

**Proposition 3.8.** Let \(c_1\) be the constant in Proposition 2.1. There exist \(c_2 \in [0, c_1]\) and \(\delta_s \in (0, 1]\) such that if \(\epsilon|\alpha|^3 \leq \delta_s\) and \(0 < \epsilon \leq c_2\), there exists a unique solution \(\Phi_b \in H^3 \cap H_0^1\) to the homogeneous Orr-Sommerfeld equation (3.1) satisfying the following properties:

- If \(1 \leq |\alpha| < \infty\), then we have
  \[
  \epsilon^{\frac{1}{2}}\|W_b\|_{L^2} + \epsilon^{-\frac{1}{2}}\|\partial_Y \Phi_b(\alpha \Phi_b)\|_{L^2} + \|\partial_Y \Phi_b(\alpha \Phi_b)\|_{L^\infty} \leq C.
  \]

- If \(0 < |\alpha| \leq 1\), then we have
  \[
  \epsilon^{\frac{1}{2}}\|W_b\|_{L^2} + \frac{\alpha + \epsilon^{\frac{1}{2}}}{\alpha^2 \epsilon^{\frac{1}{2}} + \epsilon^{\frac{1}{2}}}\|\partial_Y \Phi_b(\alpha \Phi_b)\|_{L^2} + \|\partial_Y \Phi_b(\alpha \Phi_b)\|_{L^\infty} \leq C.
  \]

**Proof.** We first consider the case of \(1 \leq |\alpha| < \infty\). We define \(\Phi_b(Y) = \Phi(Y)/\partial_Y \Phi(0)\). By Lemma 3.3, we have
\[
\epsilon^{\frac{1}{2}}\|W_b\|_{L^2} + \epsilon^{-\frac{1}{2}}\|\partial_Y \Phi_b(\alpha \Phi_b)\|_{L^2} + \|\partial_Y \Phi_b(\alpha \Phi_b)\|_{L^\infty} \leq C.
\]
Now we turn to the case of $0 < |\alpha| \leq 1$. We may assume $0 < \alpha \leq 1$. Recall that

$$\Phi_f(0) = \Phi_{a,f}(0) = \alpha^{-1} \int_0^{+\infty} W_a(Z) \sinh(\alpha Z) dZ.$$ 

We define

$$\Phi_b = A\Phi_f + B\phi_s,$$

where

$$A = \frac{1}{\partial_Y \Phi_f(0) - \Phi_f(0) \partial_Y \phi_s(0)}, \quad B = -\frac{\Phi_f(0)}{\partial_Y \Phi_f(0) - \Phi_f(0) \partial_Y \phi_s(0)}.$$ 

It is easy to check that $\Phi_b(0) = 0$ and $\partial_Y \Phi_b(0) = 1$. From Lemma 3.3 and Proposition 3.7, we infer that

$$\partial_Y \Phi_f(0) - \Phi_f(0) \partial_Y \phi_s(0) = -\frac{\varepsilon^\frac{2}{3}}{3(U'(0))^\frac{2}{3} Ai'(0)} + \frac{\varepsilon^2 \varepsilon^\frac{2}{3} Ai(0) (U'(0))^\frac{2}{3}}{Ai'(0) \alpha} + O(\varepsilon + \varepsilon^\frac{12}{17} \alpha^{-1}).$$

On the other hand, notice that

$$\left| -\frac{\varepsilon^\frac{2}{3}}{3(U'(0))^\frac{2}{3} Ai'(0)} + \frac{\varepsilon^2 \varepsilon^\frac{2}{3} Ai(0) (U'(0))^\frac{2}{3}}{Ai'(0) \alpha} \right| \geq C \left( \varepsilon^\frac{2}{3} + \frac{\varepsilon^\frac{2}{3}}{\alpha} \right).$$

Therefore, we obtain

$$|\partial_Y \Phi_f(0) - \Phi_f(0) \partial_Y \phi_s(0)| \geq C \left( \varepsilon^\frac{2}{3} + \frac{\varepsilon^\frac{2}{3}}{\alpha} \right).$$

Thus,

$$|A| \leq C \frac{\alpha}{\varepsilon^\frac{2}{3} (\alpha + \varepsilon^\frac{2}{3})}, \quad |B| \leq C \frac{\alpha \varepsilon^\frac{2}{3}}{\alpha + \varepsilon^\frac{2}{3}},$$

which along with Lemma 3.3 and Proposition 3.7 imply that

$$\|\partial_Y \Phi_b, \alpha \Phi_b\|_{L^2} \leq C \frac{\varepsilon^\frac{2}{3} + \alpha \varepsilon^\frac{2}{3}}{\varepsilon^\frac{2}{3} + \alpha}, \quad \|\partial_Y^2 - \alpha^2\Phi_b\|_{L^2} \leq C \varepsilon^{-\frac{1}{3}}, \quad \|\partial_Y \Phi_b, \alpha \Phi_b\|_{L^\infty} \leq C.$$

This finishes the proof of the proposition. \(\square\)

4 The Orr-Sommerfeld equation with the non-slip boundary condition

This section is devoted to solving the Orr-Sommerfeld equation (1.9). Since the source term of (1.9) belongs to $H^{-1}$, we decompose the solution $\phi$ as $\phi = \phi_0 + \phi_1$, where $\phi_0$ and $\phi_1$ solve the following systems, respectively:

$$\begin{cases}
\partial_Y (U \partial_Y \phi_0) - \alpha^2 U \phi_0 + i\varepsilon (\partial_Y^2 - \alpha^2) \phi_0 = -f_2 - \frac{i}{\alpha} \partial_Y f_1, \\
\phi_0 |_{Y=0} = \partial_Y \phi_0 |_{Y=0} = 0
\end{cases} \tag{4.1}$$

and

$$\begin{cases}
U (\partial_Y^2 - \alpha^2) \phi_1 + i\varepsilon (\partial_Y^2 - \alpha^2) \phi_1 - U'' \phi_1 = \partial_Y (U' \phi_0), \\
\phi_1 |_{Y=0} = \partial_Y \phi_1 |_{Y=0} = 0.
\end{cases} \tag{4.2}$$

Remark 4.1. The main reasons that we use this decomposition are as follows:
Lemma 4.2. Let \( \varepsilon > 0 \) be the constants in Proposition 3.8. Then if \( \varepsilon |\alpha|^3 \leq \delta_* \) and \( 0 < \varepsilon \leq c_2 \), then for any \( f \in H^1 \) with \( f(0) = 0 \), there exists a unique solution \( \varphi \in H^4 \cap H_0^2 \) to the system (4.3) satisfying

\[
\varepsilon^\frac{1}{2} ||w||_{L^2} + \varepsilon^\frac{3}{2} ||(\partial_Y \varphi, \alpha \varphi)||_{L^\infty} + ||(\partial_Y \varphi, \alpha \varphi)||_{L^2} \leq C \left( \| (1 + Y)^2 f \|_{L^2} + \frac{1}{|\alpha|^2} \left| \int_0^\infty \int dY \right| \right).
\]

Proof. Let \((w_{ar}, \varphi_{ar})\) solve the following OS equation with the Neumann boundary condition:

\[
\begin{align*}
U(\partial_Y^2 - \alpha^2)\varphi - U''\varphi + i\varepsilon(\partial_Y^2 - \alpha^2)^2\varphi &= f, \quad \varphi(0) = 0, \\
w_{ar} = (\partial_Y^2 - \alpha^2)\varphi_{ar}, \quad \varphi_{ar}|_{Y=0} = \partial_Y w_{ar}|_{Y=0} &= 0.
\end{align*}
\]

We take \((W_b, \Phi_b)\) to be the boundary layer corrector satisfying (3.1). By comparing the boundary conditions, we can decompose \( w \) as

\[
w = w_{ar} - \partial_Y \varphi_{ar}(0)W_b, \quad \varphi = \varphi_{ar} - \partial_Y \varphi_{ar}(0)\Phi_b.
\]

We point out that the existence of \((w_{ar}, \varphi_{ar})\) and \((W_b, \Phi_b)\) is guaranteed by Proposition 2.1 and Proposition 3.8, respectively. By Proposition 2.1, we have

\[
\varepsilon^\frac{1}{2} ||w_{ar}||_{L^2} + ||(\partial_Y \varphi_{ar}, \alpha \varphi_{ar})||_{L^2} \leq C \left( \| (1 + Y)^2 f \|_{L^2} + \frac{1}{|\alpha|^2} \left| \int_0^\infty \int dY \right| \right).
\]

By the interpolation, we obtain

\[
|\partial_Y \varphi_{ar}(0)| \leq ||(\partial_Y \varphi_{ar}, \alpha \varphi_{ar})||_{L^\infty} \leq C \|w_{ar}\|_{L^2} \left( ||(\partial_Y \varphi_{ar}, \alpha \varphi_{ar})||_{L^2} \right)^\frac{3}{2}
\]

\[
\leq C \varepsilon^{-\frac{1}{2}} \left( \| (1 + Y)^2 f \|_{L^2} + \frac{1}{|\alpha|^2} \left| \int_0^\infty \int dY \right| \right).
\]

For \((W_b, \Phi_b)\), we obtain by Proposition 3.8 that if \( 1 \leq |\alpha| < +\infty \),

\[
\varepsilon^\frac{1}{2} ||W_b||_{L^2} + \varepsilon^\frac{1}{2} ||(\partial_Y \Phi_b, \alpha \Phi_b)||_{L^2} + \| (\partial_Y \Phi_b, \alpha \Phi_b)\|_{L^\infty} \leq C,
\]

and if \( 0 < |\alpha| \leq 1 \),

\[
\varepsilon^\frac{1}{2} ||W_b||_{L^2} + \frac{\alpha + \varepsilon^\frac{1}{2}}{\alpha \varepsilon^\frac{1}{2}} ||(\partial_Y \Phi_b, \alpha \Phi_b)||_{L^2} + \| (\partial_Y \Phi_b, \alpha \Phi_b)\|_{L^\infty} \leq C.
\]

Summing up (4.4)-(4.8), we conclude that for \( 1 \leq |\alpha| < \infty \),

\[
\varepsilon^\frac{1}{2} ||w||_{L^2} + \varepsilon^\frac{3}{2} ||(\partial_Y \varphi, \alpha \varphi)||_{L^\infty} + ||(\partial_Y \varphi, \alpha \varphi)||_{L^2} \leq C \left( \| (1 + Y)^2 f \|_{L^2} + \frac{1}{|\alpha|^2} \left| \int_0^\infty \int dY \right| \right).
\]
and for $0 < |\alpha| \leq 1$, 
\[
\varepsilon^{\frac{1}{2}} \|w\|_{L^2} + \frac{\alpha + \varepsilon^{\frac{1}{2}}}{\alpha + \varepsilon^{\frac{1}{2}}} \|\nabla Y \cdot \varphi, \alpha \varphi\|_{L^2} + \varepsilon^{\frac{1}{2}} \|\nabla Y \cdot \varphi, \alpha \varphi\|_{L^\infty} 
\leq C \left( \|(1 + Y)^2 f\|_{L^2} + \frac{1}{|\alpha|^2} \int_0^{+\infty} f dY \right).
\]

In particular, we have 
\[
\|\nabla Y \cdot \varphi, \alpha \varphi\|_{L^2} \leq C \alpha^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}} \left( \|(1 + Y)^2 f\|_{L^2} + \frac{1}{|\alpha|^2} \int_0^{+\infty} f dY \right) 
\leq C \alpha^{\frac{1}{2}} \varepsilon \left( \|(1 + Y)^2 f\|_{L^2} + \frac{1}{|\alpha|^2} \int_0^{+\infty} f dY \right). 
\]

This finishes the proof of this lemma.

**Lemma 4.3.** Let $0 < \varepsilon \leq 1$. Then for any $f \in L^2$, there exists a unique solution to (4.1) satisfying 
\[
\varepsilon^{\frac{1}{2}} \|\nabla Y \cdot \varphi, \alpha \varphi\|_{L^2} + \varepsilon^{\frac{1}{2}} \|\nabla^2 Y \cdot \varphi, \alpha \varphi\|_{L^2} + \varepsilon^{\frac{1}{2}} \|\nabla Y \cdot \varphi, \alpha \varphi\|_{L^\infty} \leq C |\alpha|^{-1} \|(f_1, f_2)\|_{L^2}. 
\]

**Proof.** This elliptic problem is uniquely solvable. Hence, it suffices to provide the a priori estimate. Taking the inner product with $\varphi_0$, we obtain 
\[
\int_0^{+\infty} -U(\|\nabla Y \cdot \varphi_0\|^2 + \alpha^2 |\varphi_0|^2) dY + i\varepsilon \|\nabla^2 Y \cdot \varphi_0\|^2_{L^2} = -\int_0^{+\infty} \left( f_2 + \frac{i}{\alpha} \partial_Y f_1 \right) \varphi_0 dY. 
\]

Taking the real part and the imaginary part of the above equation, we obtain 
\[
\|\nabla Y \cdot \varphi_0, \alpha \varphi_0\|_{L^2} + \varepsilon \|\nabla^2 Y \cdot \varphi_0, \alpha \varphi_0\|_{L^2} \leq \left| \int_0^{+\infty} \left( f_2 + \frac{i}{\alpha} \partial_Y f_1 \right) \varphi_0 dY \right| 
\leq C |\alpha|^{-1} \|(f_1, f_2)\|_{L^2} \|(\nabla Y \cdot \varphi_0, \alpha \varphi_0)\|_{L^2}. 
\]

On the other hand, using the structure assumptions on $U$, we can actually obtain the control of 
\[
\|\nabla Y \cdot \varphi_0, \alpha \varphi_0\|_{L^2} \leq C \|(U/Y)^{\frac{1}{2}} + C \sqrt{U(Y)}\) \partial_Y \varphi_0, \alpha \varphi_0\|_{L^2} \text{ and } \partial_Y \varphi_0 |_{Y=0} = \varphi_0 |_{Y=0}, \text{ which imply that} 
\]
\[
\|\nabla Y \cdot \varphi_0, \alpha \varphi_0\|_{L^2} \leq C \left( \int_0^{+\infty} \frac{\|\nabla Y \cdot \varphi_0\|_{L^2}}{Y} \left( U^{\frac{1}{2}} |\partial_Y \varphi_0|^2 dY \right)^{\frac{1}{2}} + C \left( \int_0^{+\infty} \frac{|\partial_Y \varphi_0|^2}{Y} \left( U^{\frac{1}{2}} |\varphi_0|^2 dY \right)^{\frac{1}{2}} 
\right. 
\leq C \frac{\varepsilon}{\alpha} \|\nabla Y \cdot \varphi_0, \alpha \varphi_0\|_{L^2} \left. \leq C \|\nabla Y \cdot \varphi_0, \alpha \varphi_0\|_{L^2} \right) 
\leq C \frac{\varepsilon}{\alpha} \|\nabla Y \cdot \varphi_0, \alpha \varphi_0\|_{L^2} \leq C \frac{\varepsilon}{\alpha} \|\nabla Y \cdot \varphi_0, \alpha \varphi_0\|_{L^2}, 
\]

which along with (4.9) gives 
\[
\|\nabla Y \cdot \varphi_0, \alpha \varphi_0\|_{L^2} \leq C (\varepsilon^{-\frac{1}{2}} + 1) |\alpha|^{-\frac{1}{2}} \|(f_1, f_2)\|_{L^2} \|(\nabla Y \cdot \varphi_0, \alpha \varphi_0)\|_{L^2}. 
\]

Applying (4.9) again and using $\varepsilon \leq 1$, we conclude 
\[
\varepsilon^{\frac{1}{2}} \|\nabla Y \cdot \varphi_0, \alpha \varphi_0\|_{L^2} + \varepsilon^{\frac{1}{2}} \|\nabla^2 Y \cdot \varphi_0, \alpha \varphi_0\|_{L^2} + \varepsilon^{\frac{1}{2}} \|\nabla Y \cdot \varphi_0, \alpha \varphi_0\|_{L^\infty} \leq C |\alpha|^{-1} \|(f_1, f_2)\|_{L^2}. 
\]

This completes the proof.
Proposition 4.4. Let $f = (f_1, f_2) \in L^2(\mathbb{R}_+^2)$. Then there exist small positive numbers $\delta_0$ and $c_*$ such that for any $\varepsilon|a|^3 \leq \delta_0$, and $0 < \varepsilon \leq c_*$, there exists a unique solution $\phi \in H^4 \cap H^2_0(\mathbb{R}_+)$ to (1.9) satisfying
\[
\varepsilon^3 |a| \|\partial_y^2\phi - a^2\phi\|_{L^2} + \varepsilon^3 |a| \|\partial_y \phi, \alpha \phi\|_{L^\infty} + \varepsilon^3 |a| \|\partial_y \phi, \alpha \phi\|_{L^2} \leq C \|f_1, f_2\|_{L^2}.
\]

Proof. We decompose $\phi$ as $\phi = \phi_0 + \phi_1$, where $\phi_0$ and $\phi_1$ solve (4.1) and (4.2), respectively. By Lemma 4.2, we obtain
\[
\varepsilon^3 \|\partial_y^2 \phi_0\|_{L^2} + \varepsilon^3 \|\partial_y \phi_1, \alpha \phi_1\|_{L^\infty} + \varepsilon^3 \|\partial_y \phi_1, \alpha \phi_1\|_{L^2} \leq C \|f_1, f_2\|_{L^2}.
\]

Summing up the above two inequalities, we finish the proof.

4.2 Estimates when $c_2 \leq \varepsilon \leq 1$

Definition 4.5. We say that $\varphi$ is a weak solution to (4.3) if $\varphi \in H^3_0(\mathbb{R}_+)$ and $\partial_y \varphi \in H^1_0(\mathbb{R}_+)$, and for any $\xi \in C^\infty(\mathbb{R}_+)$, it holds that
\[
\langle \varepsilon w, \partial_y^2 \xi \rangle + \langle Uw, \xi \rangle - \langle U'' \varphi, \xi \rangle = \langle f, \xi \rangle, \quad w = \partial_y^2 \varphi.
\]

In the rest of this subsection, we always assume the following solvability.

Solvability Assumption (S-A). For $c_2 \leq \varepsilon \leq 1$ and $\alpha \neq 0$, there is no nontrivial weak solution to the homogeneous equation
\[
\begin{cases}
\varepsilon (\partial_y^2 \varphi - a^2)w + Uw - U'' \varphi = 0, \\
(\partial_y^2 \varphi - a^2) \varphi = w, \quad \partial_y \varphi |_{Y=0} = \varphi |_{Y=0} = 0.
\end{cases}
\] (4.10)

Remark 4.6. The solvability assumption is ensured by our spectral condition. For small $0 < \varepsilon \leq c_2$, Lemma 4.2 can ensure that there only exists the zero solution to (4.10). When $\alpha = 0$, it is easy to find by the energy method that for any $0 < \varepsilon \leq 1$, there only exists the zero solution to (4.10).

Lemma 4.7. Let $0 < \varepsilon \leq 1$, $\alpha \neq 0$, and $(1 + Y)^2f \in L^2(\mathbb{R}_+)$ and $f(0) = 0$ be given. Suppose that $\varphi$ is the solution to (4.3) with the source term $f$ and $\varphi \in H^4(\mathbb{R}_+)$.

Then we have
\[
\varepsilon \|\sqrt{Y}(\partial_y \varphi, \alpha \varphi)\|_{L^2} \leq C \|\partial_y \varphi, \alpha \varphi\|_{L^2} \|f\|_{L^2} + \|f\|_{L^2},
\]

\[
\varepsilon \|\sqrt{Y}(\partial_y w, \alpha w)\|_{L^2} + \varepsilon \|\sqrt{Y}w\|_{L^2} \leq C \|\partial_y \varphi\|_{L^2} + \|Yf\|_{L^2} + \varepsilon \|\partial_y w\|_{L^2}.
\]

Proof. First of all, we construct a $C^2(\mathbb{R}_+)$ cut-off function $\chi \geq 0$ such that
\[
\chi(Y) = 1, \quad \text{if } Y \in [0, 1) \quad \text{and} \quad \chi(Y) = 0, \quad \text{if } Y \in [2, +\infty).
\]

Let $\chi_R(Y) = \chi(Y/R)$ and $R > 1$. Then
\[
\sum_{k=1,2} |R^k \partial^k_y \chi_R| \leq C,
\]
which imply that

$$\sum_{k=1,2} |(1 + Y)^k \partial_k \chi_R| \leq C.$$ 

Taking the inner product with $-Y \chi_R \varphi / U \in L^2$, we obtain

$$i \varepsilon \langle \partial_Y^2 - \alpha^2 \rangle w, -Y \chi_R \varphi / U \rangle + \langle w, -Y \chi_R \varphi \rangle + \left\| \sqrt{U^2 Y \chi_R / U} \varphi \right\|_{L^2}^2 = \langle f, -Y \chi_R \varphi / U \rangle. \quad (4.11)$$

Thanks to $|\partial_Y^2 (Y \chi_R)| \leq CR^{-1}$, we obtain

$$\text{Re} \langle w, -Y \chi_R \varphi \rangle = \text{Re} \langle (\partial_Y \varphi, \partial_Y (Y \chi_R) \varphi) + (\partial_Y \varphi, Y \chi_R \partial_Y \varphi) + \alpha^2 \langle \varphi, Y \chi_R \varphi \rangle \rangle = \langle \partial_Y |\varphi|^2 / 2, \partial_Y (Y \chi_R) \rangle + \| \sqrt{Y \chi_R} (\partial_Y \varphi, \alpha \varphi) \|_{L^2}^2$$

$$= -\frac{1}{2} \int_0^{+\infty} \partial_Y^2 (Y \chi_R) |\varphi|^2 dY + \| \sqrt{Y \chi_R} (\partial_Y \varphi, \alpha \varphi) \|_{L^2}^2 \geq -CR^{-1} \|\varphi\|_{L^2}^2 + \| \sqrt{Y \chi_R} (\partial_Y \varphi, \alpha \varphi) \|_{L^2}^2.$$ 

Hence, we obtain

$$\| \sqrt{Y \chi_R} (\partial_Y \varphi, \alpha \varphi) \|_{L^2}^2 \leq \text{Re} \langle w, -Y \chi_R \varphi \rangle + CR^{-1} |\alpha|^{-2} \|\varphi\|_{L^2}^2$$

$$\leq \varepsilon \|\langle \partial_Y^2 - \alpha^2 \rangle w, -Y \chi_R \varphi / U \rangle + \| f, -Y \chi_R \varphi / U \rangle$$

$$- \left\| \sqrt{U^2 Y \chi_R / U} \varphi \right\|_{L^2}^2 + CR^{-1} |\alpha|^{-2} \|\varphi\|_{L^2}^2 \leq \varepsilon \|\langle \partial_Y^2 - \alpha^2 \rangle w, -Y \chi_R \varphi / U \rangle + \| f, -Y \chi_R \varphi / U \rangle$$

$$+ CR^{-1} |\alpha|^{-2} \|\varphi\|_{L^2}^2. \quad (4.12)$$

We obtain by integration by parts that

$$\|\langle \partial_Y^2 - \alpha^2 \rangle w, -Y \chi_R \varphi / U \rangle$$

$$= \|\partial_Y w, \partial_Y (Y \chi_R \varphi / U) \rangle + \alpha^2 \|w, Y \chi_R \varphi / U \rangle ||$$

$$\leq \| (1 + Y) \partial_Y w, (Y \chi_R / (1 + Y)) \partial_Y \varphi \rangle + \| (1 + Y) w, (Y \chi_R / (1 + Y)) \alpha \varphi \|$$

$$+ \| \langle \partial_Y w, \partial_Y (Y \chi_R / U) \rangle \|$$

$$\langle \partial_Y \varphi \rangle_{L^2} \| (1 + Y) \partial_Y w \|_{L^2} \| \partial_Y \varphi \|_{L^2} + \| (1 + Y) \alpha \varphi \|_{L^2} \| \alpha \varphi \|_{L^2}$$

$$+ \| \partial_Y \varphi \|_{L^2} \| (1 + Y) \partial_Y w \|_{L^2} \| \varphi / U \|_{L^2},$$

which along with the facts that $U \geq C^{-1} Y / (1 + Y)$ and $|\partial_Y (Y \chi_R / U) \rangle \leq C(1 + Y) / Y$ implies

$$\|\langle \partial_Y^2 - \alpha^2 \rangle w, -Y \chi_R \varphi / U \rangle \leq C \| (1 + Y) \partial_Y \varphi \|_{L^2} \| \partial_Y \varphi \|_{L^2} \| \partial_Y \varphi, \alpha \varphi \|_{L^2}. \quad (4.13)$$

Using the fact that $U \geq C^{-1} Y / (1 + Y)$ again, we have

$$\| f, -Y \chi_R \varphi / U \rangle \leq \| Y \chi_R (1 + Y) / U \|_{L^\infty} \| (1 + Y) Y f \|_{L^2} \| \varphi / Y \|_{L^2}$$

$$\leq C \| (1 + Y)^2 f \|_{L^2} \| \partial_Y \varphi \|_{L^2}. \quad (4.14)$$

Summing up (4.12)–(4.14), we arrive at

$$\| \sqrt{Y \chi_R} (\partial_Y \varphi, \alpha \varphi) \|_{L^2}^2 \leq C \| (\partial_Y \varphi, \alpha \varphi) \|_{L^2} \left( R^{-1} |\alpha|^{-2} \|\varphi\|_{L^2} + \varepsilon \| (1 + Y) \partial_Y w, \alpha \varphi \|_{L^2} + \| (1 + Y)^2 f \|_{L^2}. $$
from which we take \( R \to +\infty \) and obtain that
\[
\| \sqrt{Y} (\partial_Y \varphi, \alpha \varphi) \|_{L^2}^2 \leq C \| (\partial_Y \varphi, \alpha \varphi) \|_{L^2} (1 + Y)(\partial_Y w, \alpha w) \|_{L^2} + (1 + Y)^2 f \|_{L^2}.
\]
This shows the first inequality of this lemma.

On the other hand, \( Y \chi_R w \) satisfies
\[
\varepsilon (\partial_Y^2 - \alpha^2) (Y \chi_R w) + U (Y \chi_R w) = Y \chi_R U'' \varphi + Y \chi_R f + 2i \varepsilon\partial_Y (Y \chi_R) \partial_Y w + i \varepsilon \partial_Y^2 (Y \chi_R) w.
\]
For the source term on the right-hand side, we have
\[
\| Y \chi_R U'' \varphi + Y \chi_R f + 2i \varepsilon\partial_Y (Y \chi_R) \partial_Y w + i \varepsilon \partial_Y^2 (Y \chi_R) w \|_{L^2} \leq C (\| \partial_Y \varphi \|_{L^2} + \| f \|_{L^2} + \varepsilon \| \partial_Y w \|_{L^2} + R^{-1} \| w \|_{L^2}).
\]
Taking \( R \to +\infty \), we obtain
\[
\varepsilon^\frac{3}{2} (\| \partial_Y, \alpha \| (Y \chi_R w) \|_{L^2} + \varepsilon^\frac{1}{2} |Y \chi_R w| \|_{L^2} \leq C (\| \partial_Y \varphi \|_{L^2} + \| f \|_{L^2} + \varepsilon \| \partial_Y w \|_{L^2} + R^{-1} \| w \|_{L^2}).
\]
This completes the proof.

Now we consider the case where \( |\alpha| \) has an upper bound and \( \varepsilon \) has a lower bound.

**Lemma 4.8.** Let \( M_1 > 0, \ 0 < c_3 \leq 1 \) and \( 0 \neq |\alpha| \leq M_1 \). Assume that the solvability assumption (S-A) holds. Let \( \varphi \) be the solution to (4.3) with \( (1 + Y)^2 f \in L^2(\mathbb{R}_+) \) and \( f \in H^1_0(\mathbb{R}_+) \). Then it holds that for any \( 1 \geq \varepsilon \geq c_3 \),
\[
\| w \|_{L^2} + \| (\partial_Y \varphi, \alpha \varphi) \|_{L^2} \leq C (1 + Y)^2 f \|_{L^2},
\]
where the constant \( C \) depends on \( c_3 \) and \( M_1 \).

**Proof.** Without loss of generality, we may assume \( \alpha \geq 0 \). We prove this lemma by a contradiction argument. Assume that this lemma is not true. Then there exists a sequence \( \{ \varepsilon_n, \alpha_n, \varphi(n), w(n), f(n) \} \) satisfying
\[
\begin{cases}
\c_3 \leq \varepsilon_n \leq 1, & 0 \neq |\alpha_n| \leq M_1, \\
\varepsilon_n (\partial_Y^2 - \alpha_n^2) w(n) + U w(n) - U'' \varphi(n) = f(n), \\
(\partial_Y^2 - \alpha_n \varphi(n) = w(n), \quad \varphi(n) |_{Y=0} = \partial_Y \varphi(n), \quad \varphi(n) |_{Y=0} = 0, \\
\| w(n) \|_{L^2} + \| (\partial_Y \varphi(n), \alpha_n \varphi(n)) \|_{L^2} = 1, \quad \| (1 + Y)^2 f(n) \|_{L^2} \to 0
\end{cases}
\]
\] such that we can take a subsequence \( \{ \varepsilon_n, \alpha_n, \varphi(n), w(n), f(n) \} \) (denoted by the same index) satisfying
\[
\varepsilon_n \to \varepsilon \geq c_3, \quad \alpha_n \to \alpha \in [0, M_1], \quad \text{as } n \to \infty.
\]
Then we obtain the following weak convergence results when \( n \to \infty \):

- If \( \alpha = 0 \), we have
\[
\| \partial_Y \varphi(n) \|_{H^1} + \| \alpha_n \varphi(n) \|_{L^2} \leq C (\| w(n) \|_{L^2} + \| (\partial_Y \varphi(n), \alpha_n \varphi(n)) \|_{L^2}) \leq C.
\]

Then
\[
\partial_Y \varphi(n) \to u_1 \quad \text{in } H^1_0(\mathbb{R}_+) \text{ with } u_1 \in H^1_0(\mathbb{R}_+),
\]
\[
\alpha_n \varphi(n) \to i u_2 \quad \text{in } H^1_0(\mathbb{R}_+) \text{ with } i u_2 \in H^1_0(\mathbb{R}_+),
\]
and hence,
\[
(w(n), \partial_Y \varphi(n), \alpha_n \varphi(n)) \to (\partial_Y u_1, u_1, i u_2) \quad \text{in } L^2(\mathbb{R}_+) \text{ with } (u_1, u_2) \in H^1_0(\mathbb{R}_+).
\]
If $\alpha \neq 0$, we may assume that $\alpha_n > 0$, and thanks to $\alpha_n \geq \alpha/2$ for large $n$, we can deduce that
\[
\|\varphi^{(n)}\|_{H^2} \leq (|\alpha_n|^{-1} + 1)(\|w^{(n)}\|_{L^2} + \|\partial_Y \varphi^{(n)} + \alpha_n \varphi^{(n)}\|_{L^2}) \leq (2\alpha^{-1} + 1).
\]
Then $\|\varphi^{(n)}\|_{H^2}$ is uniformly bounded, and hence there exists a $\varphi \in H^2(\mathbb{R}_+)$ such that $\varphi^{(n)} \rightharpoonup \varphi$ in $H^2(\mathbb{R}_+)$. We obtain
\[
(w^{(n)}, \partial_Y \varphi^{(n)}, \alpha_n \varphi^{(n)}) \rightharpoonup ((\partial_Y^2 - \alpha^2)\varphi, \partial_Y \varphi, \alpha \varphi) =: (w, u_1, iu_2) \text{ in } L^2(\mathbb{R}_+) \text{ with } \varphi \in H^2(\mathbb{R}_+) \text{ and } (w, u_1, iu_2) \in L^2(\mathbb{R}_+). \quad (4.18)
\]

Step 1. Strong convergence.

Since $w$, $\varphi$ and $f$ all belong to $L^2(\mathbb{R}_+)$, we know that $(\partial_Y^2 - \alpha^2)w$ is also in $L^2(\mathbb{R}_+)$. Moreover, the following bound holds:
\[
\varepsilon_n \|((\partial_Y^2 - \alpha_n^2)w^{(n)})\|_{L^2} \leq \|Uw^{(n)}\|_{L^2} + \|U''\varphi^{(n)}\|_{L^2} + f^{(n)}\|_{L^2} \leq C\|w^{(n)}\|_{L^2} + \|\partial_Y \varphi^{(n)}\|_{L^2} + \|f^{(n)}\|_{L^2}.
\]
Thanks to $\varepsilon_n \geq c_1 > 0$, we deduce that $\|(\partial_Y^2 - \alpha_n^2)w^{(n)}\|_{L^2}$ are uniformly bounded with respect to $n$. We obtain by Lemma A.2 that
\[
\|\partial_Y w^{(n)}\|_{L^2} \leq C\|((\partial_Y^2 - \alpha_n^2)w^{(n)})\|_{L^2}^{\frac{1}{2}} w^{(n)}\|_{L^2}^{\frac{1}{2}}.
\]
Then we have
\[
\|\partial_Y w^{(n)}\|_{L^2} \text{ are uniformly bounded.} \quad (4.19)
\]
Thanks to $\|\partial_Y (\partial_Y \varphi^{(n)} + \alpha_n \varphi^{(n)})\|_{L^2} \leq C\|w^{(n)}\|_{L^2} \leq C$, we obtain
\[
\|\partial_Y (\partial_Y \varphi^{(n)} + \alpha_n \varphi^{(n)})\|_{L^2} \text{ are uniformly bounded.} \quad (4.20)
\]
Thanks to Lemma 4.7, we obtain
\[
\|\sqrt{Y}(\partial_Y \varphi^{(n)} + \alpha_n \varphi^{(n)})\|_{L^2} \leq C\|((\partial_Y^2 - \alpha_n^2)w^{(n)})\|_{L^2} + \varepsilon_n\|Yw^{(n)}\|_{L^2} \leq C\|\partial_Y \varphi^{(n)}\|_{L^2} + \|f^{(n)}\|_{L^2} + \varepsilon_n\|\partial_Y w^{(n)}\|_{L^2}.
\]
Summing up, we arrive at
\[
\|((1 + Y)(\partial_Y^2 - \alpha_n^2)w^{(n)} + \sqrt{Y}(\partial_Y \varphi^{(n)} + \alpha_n \varphi^{(n)}))\|_{L^2} + \|Yw^{(n)}\|_{L^2} \leq C\|\partial_Y \varphi^{(n)}\|_{L^2} + \|f^{(n)}\|_{L^2} + \varepsilon_n\|\partial_Y w^{(n)}\|_{L^2}.
\]
are uniformly bounded. Then
\[
\lim_{R \to +\infty} \sup_n \|(w^{(n)}, \partial_Y \varphi^{(n)} + \alpha_n \varphi^{(n)})\|_{L^2([R, +\infty))} = 0. \quad (4.22)
\]
Combining (4.19), (4.20) and (4.22), we deduce that as $n \to +\infty$,
\[
(w^{(n)}, \partial_Y \varphi^{(n)} + \alpha_n \varphi^{(n)}) \rightharpoonup (\partial_Y u_1, u_1) \text{ strongly in } L^2(\mathbb{R}_+). \quad (4.23)
\]
Moreover if $\alpha \neq 0$, we have
\[
\varphi^{(n)} \rightharpoonup \varphi \text{ strongly in } H^2(\mathbb{R}_+). \quad (4.24)
\]

Step 2. The limit equation.

Let $\xi$ be any $C_\infty^\infty(\mathbb{R}_+)$ test function. Then we have
\[
i\varepsilon_n \langle w^{(n)}, (\partial_Y^2 - \alpha_n^2)\xi \rangle + \langle Uw^{(n)}, \xi \rangle - \langle U''\varphi^{(n)}, \xi \rangle = \langle f^{(n)}, \xi \rangle, \quad w^{(n)} = (\partial_Y^2 - \alpha_n^2)\varphi^{(n)}.
\]
• If \( \alpha \neq 0 \), by (4.24) and passing to the limit for \( n \to \infty \), we obtain
\[
\iota \langle w, (\partial_{Y}^{2} - \alpha_{2}^{2}) \xi \rangle + \langle U w, \xi \rangle - \langle U'' \varphi, \xi \rangle = 0, \quad w = (\partial_{Y}^{2} - \alpha_{2}^{2}) \varphi.
\]

For the boundary condition, thanks to \( \varphi^{(n)} \in H^{2}_{0}(\mathbb{R}_{+}), \partial_{Y} \varphi^{(n)} \in H^{1}_{0}(\mathbb{R}_{+}) \) and (4.24), we deduce that \( \varphi \in H^{2}_{0}(\mathbb{R}_{+}) \) and \( \partial_{Y} \varphi \in H^{1}_{0}(\mathbb{R}_{+}) \). This means that \( \varphi \) is a weak solution to (4.10). By the solvability assumption (S-A), we obtain \( \varphi = 0 \).

• If \( \alpha = 0 \), we define \( \varphi(Y) = \int_{0}^{Y} u_{1}(Z) dZ \). By (4.23), we obtain
\[
\iota \langle w^{(n)}, (\partial_{Y}^{2} - \alpha_{2}^{2}) \xi \rangle + \langle U w^{(n)}, \xi \rangle \to \iota \langle \partial_{Y} u_{1}, \partial_{Y}^{2} \xi \rangle + \langle U \partial_{Y} u_{1}, \xi \rangle.
\]

On the other hand, we notice that
\[
\left\| U'' \varphi^{(n)} - U'' \int_{0}^{Y} u_{1}(Z) dZ \right\|_{L^{2}} \leq C \left\| \varphi^{(n)} - \int_{0}^{Y} u_{1}(Z) dZ \right\|_{L^{2}} \leq C \| \partial_{Y} \varphi^{(n)} - u_{1} \|_{L^{2}} \to 0,
\]
which implies that
\[
\langle U'' \varphi^{n}, \xi \rangle \to \left\langle U'' \int_{0}^{Y} u_{1}(Z) dZ, \xi \right\rangle, \quad \langle f^{n}, \xi \rangle \to 0.
\]

Thus, we have \( w = \partial_{Y}^{2} \varphi, \varphi \in H^{2}_{0}(\mathbb{R}_{+}), \partial_{Y} \varphi \in H^{1}_{0}(\mathbb{R}_{+}) \) and
\[
\iota \langle w, \partial_{Y}^{2} \xi \rangle + \langle U w, \xi \rangle - \langle U'' \varphi, \xi \rangle = 0.
\]

Then \( \varphi \) is a weak solution to (4.10) with \( \alpha = 0 \), and \( \varphi = 0 \) by (S-A).

**Step 3.** Contraction.

• If \( \alpha \neq 0 \), we have
\[
(1 + \alpha_{n}^{2}) \| \varphi^{(n)} \|_{H^{2}} \geq C^{-1} (\| w^{(n)} \|_{L^{2}} + \| (\partial_{Y} \varphi^{(n)}, \alpha_{n} \varphi^{(n)}) \|_{L^{2}}) \geq C^{-1},
\]
which along with (4.24) and \( \varphi = 0 \) implies
\[
0 = (1 + \alpha^{2}) \| \varphi \|_{H^{2}} = \lim_{n} (1 + \alpha_{n}^{2}) \| \varphi^{(n)} \|_{H^{2}} \geq C^{-1} > 0,
\]
which leads to a contradiction.

• If \( \alpha = 0 \), by (4.23) and \( \varphi = 0 \),
\[
0 = \| (\partial_{Y}^{2} \varphi, \partial_{Y} \varphi) \|_{L^{2}} = \lim_{n \to +\infty} \| (w^{(n)}, \partial_{Y} \varphi^{(n)}) \|_{L^{2}} \geq C^{-1} > 0.
\]

This is a contradiction.

This finishes the proof of the lemma. \( \square \)

Next, we consider the case where \( |\alpha| \) is large enough and \( \varepsilon \) has a lower bound.

**Lemma 4.9.** Let \( 0 < c_{3} \leq 1, \, c_{3} \leq \varepsilon \leq 1, \, (1 + Y)^{2} f \in L^{2}(\mathbb{R}_{+}) \) and \( f \in H^{1}_{0}(\mathbb{R}_{+}) \). Assume that \( \varphi \) is the solution to (4.3). Then there exists an \( M_{2} > 0 \) sufficiently large such that for any \( |\alpha| \geq M_{2} \), it holds that
\[
\| w \|_{L^{2}} + \| (\partial_{Y} \varphi, \alpha \varphi) \|_{L^{2}} \leq C \| f \|_{L^{2}},
\]
where the constant \( C \) depends on \( c_{3} \).

**Proof.** Notice that
\[
\varepsilon \| (\partial_{Y}^{2} - \alpha^{2}) w \|_{L^{2}} \leq \| U w \|_{L^{2}} + \| U'' \varphi \|_{L^{2}} + \| f \|_{L^{2}},
\]
Choosing then we have

\[ c_3 \alpha^2 \|w\|_{L^2} \leq C \varepsilon \| (\partial_Y^2 - \alpha^2) w \|_{L^2} \leq C (\|w\|_{L^2} + \alpha^{-2} \|\varphi\|_{L^2} + \|f\|_{L^2}). \]

Then we have

\[ c_3 \alpha^2 \|w\|_{L^2} \leq C (1 + M_2^{-2}) \|w\|_{L^2} + C \|f\|_{L^2}. \]

Choosing \( M_2 \) sufficiently large so that \( c_3 M_2^2 \geq 2C (1 + M_2^{-2}) \), we obtain

\[ c_3 \alpha^2 \|w\|_{L^2} \leq C \|f\|_{L^2}. \]

This along with \( \| (\partial_Y \varphi, \alpha \varphi) \|_{L^2} \leq |\alpha|^{-1} \|w\|_{L^2} \) gives our result.

**Proposition 4.10.** Let \( \phi \) solve \((1.9)\) with \((f_1, f_2) \in L^2(\mathbb{R}_+^2)\). For any fixed \( 0 < c_3 \leq 1 \), if \( c_3 \leq \varepsilon \leq 1 \) and \( \alpha \neq 0 \), then we have

\[ |\alpha| \| (\partial_Y^2 - \alpha^2) \phi \|_{L^2} + |\alpha| \| (\partial_Y \varphi, \alpha \varphi) \|_{L^2} + |\alpha| \| (\partial_Y \phi, \alpha \phi) \|_{L^2} \leq C \| (f_1, f_2) \|_{L^2}, \]

where the constant \( C \) depends on \( c_3 \).

**Proof.** We decompose \( \phi \) as \( \phi = \phi_0 + \phi_1 \), where \( \phi_0 \) and \( \phi_1 \) solve \((4.1)\) and \((4.2)\), respectively. By Lemmas 4.8 and 4.9, we have

\[ \| (\partial_Y^2 - \alpha^2) \phi_0 \|_{L^2} + \| (\partial_Y \phi_0, \alpha \phi_0) \|_{L^2} + \| (\partial_Y \phi_1, \alpha \phi_1) \|_{L^2} \leq C \| (1 + Y)^2 \partial_Y (U' \phi_0) \|_{L^2}. \]

By Lemma 4.3 and the interpolation, we have

\[ \| (\partial_Y^2 - \alpha^2) \phi_0 \|_{L^2} + \| (\partial_Y \phi_0, \alpha \phi_0) \|_{L^2} + \| (\partial_Y \phi_1, \alpha \phi_1) \|_{L^2} \leq C (\varepsilon^2 \| (\partial_Y^2 - \alpha^2) \phi_0 \|_{L^2} + \varepsilon \| (\partial_Y^2 - \alpha^2) \phi_0 \|_{L^2}). \]

Summing up the above two inequalities, we finish the proof.

5 Nonlinear stability

This section is devoted to the proof of Theorem 1.1.

5.1 The estimate for the zero mode

In this case, the linearized system \((1.8)\) can be written as

\[ \begin{cases} 
\nu \partial_y^2 u_{0,1} = f_{0,1}, & u_{2,0} = 0, \quad \partial_y p_0 = f_{0,2}, \\
u u_{0,1} \big|_{y=0} = 0.
\end{cases} \]

**Proposition 5.1.** Let \( f_0 \in L^1(\mathbb{R}_+) \) and \( f_{0,1} = \partial_y F_{0,1} \) with \( F_{0,1} \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+) \). Then there exists a unique solution \( u_0 = (u_{0,1}, 0) \) to \((1.8)\) with \( \bar{n} = 0 \) such that

\[ \| u_{0,1} \| \leq \nu^{-1} \| F_{0,1} \|_{L^1}, \quad \| \partial_y u_{0,1} \| \leq \nu^{-1} \| F_{0,1} \|_{L^2}. \]

Moreover, we have

\[ \lim_{y \to +\infty} u_{0,1}(y) = \nu^{-1} \int_0^{y} F_{0,1}(y') dy', \]

\[ \lim_{y \to +\infty} \partial_y u_{0,1}(y) = \nu^{-1} \int_0^{y} F_{0,1}(y') dy', \]

which along with \( F_{0,1} \in L^1 \) implies that \( \| u_{0,1} \|_{L^\infty} \leq \nu^{-1} \| F_{0,1} \|_{L^1} \) and \( \lim_{y \to +\infty} u_{0,1} = \nu^{-1} \int_0^{y} F_{0,1}(y') dy' \). Moreover, since \( \partial_y u_{0,1}(y) = \nu^{-1} F_{0,1}(y) \) and \( F_{0,1} \in L^2 \), we have \( \| \partial_y u_{0,1} \| \leq \nu^{-1} \| F_{0,1} \|_{L^2}. \)
5.2 Estimates for non-zero modes

Proposition 5.2. There exist positive numbers $\theta_0$, $\nu_0$ and $\delta_0$ so that the following statements hold. For any $0 < \nu \leq \nu_0$, $(0, \theta_0) \subset \Sigma(U, \nu)$. Moreover, for any $f_n \in L^2(\mathbb{R}_+)$, it holds that

(i) if $0 < |\tilde{n}| \leq \delta_0 \nu^{-\frac{1}{2}}$ and $\theta \in (0, \theta_0)$, then

$$|\tilde{n}|^{\frac{2}{3}} \| u_n \|_{L^2} + |\tilde{n}|^{\frac{1}{2}} \| \partial_y u_n \|_{L^2} \leq C \| f_n \|_{L^2};$$

(ii) if $|\tilde{n}| \geq \delta_0 \nu^{-\frac{1}{2}}$ and $\theta \in (0, \theta_0)$, then

$$|\tilde{n}|^{\frac{1}{2}} \nu \| u_n \|_{L^2} + |\tilde{n}| \| \partial_y u_n \|_{L^2} \leq C \| f_n \|_{L^2};$$

(iii) if $\theta > \theta_0$ and $\theta \in \Sigma(U, \nu)$, then

$$|\tilde{n}| \| u_n \|_{L^2} + \nu^{-\frac{1}{2}} |\tilde{n}| \| \partial_y u_n \|_{L^2} \leq C \| f_n \|_{L^2}.$$ 

Proof. Let $\delta_0 \leq \delta_\ast$ and we take $\theta_0 = c_\ast$ and $\nu_1 = \delta_\ast^2$, where $c_\ast$ and $\delta_\ast$ are small positive numbers in Proposition 4.4. For any fixed $0 < \nu \leq \nu_1$, the estimates for the case of $\theta \in (0, \theta_0)$, $0 < |\tilde{n}| \leq \delta_\ast \nu^{-\frac{1}{2}}$ or $\theta > \theta_0$, $\theta \in \Sigma(U)$ are deduced from the estimates for the Orr-Sommerfeld equation in Section 4. Indeed, for each given $\tilde{n} \neq 0$, we have

$$u_{n,1} = \partial_Y \phi \left( \frac{y}{\sqrt{\nu}} \right), \quad u_{n,2} = -i\alpha \phi \left( \frac{y}{\sqrt{\nu}} \right) \quad \text{and} \quad f_n(y) = \nu^{-\frac{1}{2}} f \left( \frac{y}{\sqrt{\nu}} \right), \quad (5.1)$$

where $(\phi, f)$ satisfies (1.9).

Notice that $\epsilon |\alpha|^2 \leq \delta_\ast$ and $\epsilon \leq c_\ast$ when $\nu \leq \nu_1$, $\theta \leq \theta_0$ and $0 < |\tilde{n}| \leq \delta_\ast \nu^{-\frac{1}{2}}$. Hence, by Proposition 4.4 and (5.1), we deduce that for any $0 < \nu \leq \nu_1$, $\theta \in (0, \theta_0)$ and $0 < |\tilde{n}| \leq \delta_\ast \nu^{-\frac{1}{2}}$, there exists a unique solution $u_n \in H^2 \cap H^1_0$. Moreover, it holds that

$$\| u_n \|_{L^2} \leq \nu^{\frac{1}{2}} \| (\partial_Y \phi, \alpha \phi) \|_{L^2} \leq C \nu^{\frac{1}{2}} |\tilde{n}|^{-\frac{1}{2}} \| f_n(\sqrt{\nu}) \|_{L^2} \leq C |\tilde{n}|^{-\frac{1}{2}} \| f_n \|_{L^2}$$

and

$$\| \partial_y u_n \|_{L^2} + |\tilde{n}| \| u_n \|_{L^2} \leq \nu^{-\frac{1}{2}} \| (\partial_Y^2 - \alpha^2) \phi \|_{L^2} \leq C \nu^{-\frac{1}{2}} |\tilde{n}|^{-\frac{1}{2}} \| f_n(\sqrt{\nu}) \|_{L^2} \leq \frac{C \| f_n \|_{L^2}}{|\tilde{n}|^{\frac{1}{2}} \nu^{\frac{1}{2}}}.$$ 

For the case of $\theta > \theta_0$ and $\theta \in \Sigma(U, \nu)$, by Proposition 4.10 and (5.1), we have

$$\| u_n \|_{L^2} \leq \nu^{\frac{1}{2}} \| (\partial_Y \phi, \alpha \phi) \|_{L^2} \leq C \nu^{\frac{1}{2}} |\tilde{n}|^{-1} \| f_n(\sqrt{\nu}) \|_{L^2} \leq C |\tilde{n}|^{-1} \| f_n \|_{L^2}$$

and

$$\| \partial_y u_n \|_{L^2} \leq \nu^{-\frac{1}{2}} \| (\partial_Y^2 - \alpha^2) \phi \|_{L^2} \leq C \nu^{-\frac{1}{2}} |\tilde{n}|^{-1} \| f_n(\sqrt{\nu}) \|_{L^2} \leq C \nu^{-\frac{1}{2}} |\tilde{n}|^{-1} \| f_n \|_{L^2}.$$ 

It remains to prove the second statement. Instead of considering the Orr-Sommerfeld equation, we are back to the original system (1.8). Taking the inner product with $u_n$, we obtain

$$\nu (\| \partial_y u_n \|_{L^2}^2 + |\tilde{n}|^2 \| u_n \|_{L^2}^2) \leq \text{Re} \left( u_{n,2} \partial_y U \left( \frac{y}{\sqrt{\nu}} \right), u_{n,1} \right) + \| f_n \|_{L^2} \| u_n \|_{L^2} \quad (5.2)$$

and

$$\tilde{n} \left( U \left( \frac{y}{\sqrt{\nu}} \right) u_n, u_n \right) - \text{Re} \left( \left( \partial_y U \left( \frac{y}{\sqrt{\nu}} \right) + \partial_y \phi_n \phi_n \right) \phi_n, \partial_y \phi_n \phi_n \right) = \text{Im} (f_n, u_n). \quad (5.3)$$

We consider two cases: (i) $|\tilde{n}| \geq \delta_0^{-1} \nu^{-\frac{1}{2}}$ and (ii) $\delta_0 \nu^{-\frac{1}{2}} \leq |\tilde{n}| \leq \delta_0^{-1} \nu^{-\frac{1}{2}}$, where we take $\delta_0 = \min(\delta_\ast, \| \partial_Y U \|_{L^2}^{-\frac{1}{2}})$.
For the case (i), we notice that \( \delta_0 = \min(\delta_* , \| \partial_Y U \|_{L^\infty}^{-\frac{3}{2}} / 2) \). Then for any \( |\tilde{n}| \geq \delta_0^{-1} \nu^{-\frac{3}{2}} \), we have
\[
\left| \Re \left( u_{n,2} \partial_y U \left( \frac{y}{\sqrt{n}} \right), u_{n,1} \right) \right| \leq \nu^{-\frac{3}{2}} \| \partial_Y U \|_{L^\infty} \| u_{n,2} \|_{L^2}^2 \leq \nu \tilde{n}^{-1} \delta_0 \| \partial_Y U \|_{L^\infty} \| u_{n,2} \|_{L^2}^2 \leq \frac{\nu \tilde{n}^2}{4} \| u_{n} \|_{L^2}^2,
\]
which along with (5.2) gives
\[
\nu(\| \partial_y u_{n,2} \|_{L^2}^2 + \tilde{n}^2 \| u_{n} \|_{L^2}^2) \leq C \| f_n \|_{L^2} \| u_{n} \|_{L^2}.
\]
This implies that
\[
|\tilde{n}|^2 \nu \| u_{n} \|_{L^2} + |\tilde{n}| \nu \| \partial_y u_{n} \|_{L^2} \leq C \| f_n \|_{L^2}.
\]

For the case (ii), by (5.3), we obtain
\[
\int_0^{+\infty} U \left( \frac{y}{\sqrt{n}} \right) (|\partial_y \phi_n|^2 + |\tilde{n}|^2 |\phi_n|^2) dy + \frac{1}{2} \int_0^{+\infty} \left( \partial_y^2 U \left( \frac{y}{\sqrt{n}} \right) \right) |\phi_n|^2 dy = \frac{1}{n} \Im(f_n, u_n).
\]
Let \( \chi \) be a cut-off function such that \( \chi(Y) = 1 \) for \( 0 \leq Y < 1 \) and \( \chi(Y) = 0 \) for \( Y \geq 2 \). Let \( Y_3 > 0 \) such that \( \partial_Y U > 0 \) for \( Y \in [0, 4Y_3] \). Then
\[
\left| \partial_y^2 U \left( \frac{y}{\sqrt{n}} \right) \right| \leq C \nu^{-\frac{3}{2}} \partial_y U \left( \frac{y}{\sqrt{n}} \right)
\]
for \( 0 \leq Y \leq 2Y_3 \nu^\frac{3}{4} \), which gives
\[
\left| \int_0^{+\infty} \left( \partial_y^2 U \left( \frac{y}{\sqrt{n}} \right) \right) |\phi_n|^2 dy \right| \leq C \nu^{-\frac{3}{2}} \int_0^{+\infty} (\partial_y U \left( \frac{y}{\sqrt{n}} \right)) \chi \left( \frac{y}{2 \sqrt{n} Y_3} \right) |\phi_n|^2 dy + C \nu^{-1} \int_{Y_3 \nu^\frac{3}{4}}^{+\infty} |\phi_n|^2 dy
\]
\[
= -C \nu^{-\frac{3}{2}} \int_0^{+\infty} U \left( \frac{y}{\sqrt{n}} \right) \partial_y \left( \chi \left( \frac{y}{2 \sqrt{n} Y_3} \right) |\phi_n|^2 \right) dy + C \nu^{-1} \int_{Y_3 \nu^\frac{3}{4}}^{+\infty} |\phi_n|^2 dy
\]
\[
\leq C \nu^{-\frac{3}{2}} \int_0^{+\infty} U \left( \frac{y}{\sqrt{n}} \right) \partial_y \phi_n |\phi_n| dy + C \nu^{-1} \int_{Y_3 \nu^\frac{3}{4}}^{+\infty} |\phi_n|^2 dy
\]
\[
\leq C \nu^{-\frac{3}{2}} \frac{\| U \left( \frac{y}{\sqrt{n}} \right) \tilde{n} \phi_n \|_{L^2}}{\sqrt{n}} \| U \left( \frac{y}{\sqrt{n}} \right) \partial_y \phi_n \|_{L^2} + C \nu^{-1} \frac{\| U \left( \frac{y}{\sqrt{n}} \right) \tilde{n} \phi_n \|_{L^2}}{\sqrt{n}}
\]
\[
\leq C \nu^{-\frac{3}{4}} \frac{\| U \left( \frac{y}{\sqrt{n}} \right) \tilde{n} \phi_n \|_{L^2}}{\delta_0} \| U \left( \frac{y}{\sqrt{n}} \right) \partial_y \phi_n \|_{L^2} + C \nu^{-1} \frac{\| U \left( \frac{y}{\sqrt{n}} \right) \tilde{n} \phi_n \|_{L^2}}{\delta_0}
\]
Take \( \nu_0 = \min(\nu_1, (C^{-1} \delta_0^4)\nu) \). If \( 0 < \nu \leq \nu_0 \), we obtain by (5.4) that
\[
\| U \left( \frac{y}{\sqrt{n}} \right) u_n \|_{L^2}^2 \leq \| U \left( \frac{y}{\sqrt{n}} \right) \partial_y \phi_n \|_{L^2}^2 + |\tilde{n}|^2 \| U \left( \frac{y}{\sqrt{n}} \right) \phi_n \|_{L^2}^2 \leq C |\tilde{n}|^{-1} \| f_n \|_{L^2} \| u_{n,2} \|_{L^2}^2.
\]
Using the fact that \( \tilde{n} u_{n,1} + \partial_y u_{n,2} = 0 \) and Hardy’s inequality, we can deduce from (5.2) that
\[
\nu(\| \partial_y u_{n,2} \|_{L^2}^2 + \tilde{n}^2 \| u_{n} \|_{L^2}^2) \leq C \nu^{-\frac{3}{2}} |\tilde{n}| \| u_{n,2} \|_{L^2}^2 + \| f_n \|_{L^2} \| u_{n} \|_{L^2}.
\]
As in the proof of (2.6), we have
\[
\| u_{n} \|_{L^2} \leq C \nu^{-\frac{3}{2}} |\tilde{n}|^{-\frac{3}{2}} \| \partial_y u_{n,2} \|_{L^2}^2 \| u_{n,2} \|_{L^2}^2 + C |\tilde{n}|^\frac{1}{2} \| U \left( \frac{y}{\sqrt{n}} \right) u_n \|_{L^2}^2.
\]
Plugging (5.6) into (5.7), we obtain
\[
\|u_n\|_{L^2} \leq C|\tilde{n}|^{-\frac{1}{4}} \|f_n\|_{L^2}^\frac{1}{4} \|u_n\|_{L^2}^{\frac{3}{4}} + C|\tilde{n}|^{\frac{1}{4}} \left\| U \left( \frac{y}{\sqrt{\nu}} \right) u_n \right\|_{L^2} + \nu^{-\frac{3}{4}} |\tilde{n}|^{-\frac{1}{4}} \left\| U \left( \frac{y}{\sqrt{\nu}} \right) u_n \right\|_{L^2}^{\frac{1}{4}} \|u_n\|_{L^2}^{\frac{3}{4}},
\]
which gives
\[
\|u_n\|_{L^2} \leq C|\tilde{n}|^{-\frac{1}{4}} \|f_n\|_{L^2} + C(|\tilde{n}|^{\frac{1}{4}} + \nu^{-\frac{1}{4}} |\tilde{n}|^{-\frac{1}{4}}) \left\| U \left( \frac{y}{\sqrt{\nu}} \right) u_n \right\|_{L^2}.
\]
This along with (5.5) shows
\[
\|u_n\|_{L^2} \leq C(|\tilde{n}|^{-\frac{1}{4}} + \nu^{-\frac{1}{4}} |\tilde{n}|^{-\frac{1}{4}})|f_n|_{L^2} \leq C|\tilde{n}|^{-\frac{3}{4}} \nu^{-\frac{1}{2}} |f_n|_{L^2},
\]
where in the last inequality we used $|\tilde{n}| \sim \nu^{-\frac{1}{2}}$ in the case (ii). Putting this inequality into (5.5) and (5.6), and using the fact that $|\tilde{n}| \sim \nu^{-\frac{1}{2}}$, we conclude that $|\tilde{n}|^2 \nu \|u_n\|_{L^2} + |\tilde{n}| \nu |\partial_y u_n|_{L^2} \leq C |f_n|_{L^2}$.

The existence of the solution can be proved by using the method of continuity. After replacing $-\nu(\partial_y^2 - \tilde{n}^2)$ by $-\nu(\partial_y^2 - \tilde{n}^2) + l$ with $l > 0$, we can easily show that there exists a unique solution for any $f_n \in L^2$ and $l$ large enough, and the above a priori estimates still hold true for any $l > 0$.

### 5.3 Proof of Theorem 1.1

With the estimates for the linearized system, the proof of nonlinear stability is similar to that in [9]. For the completeness, we present a sketch.

We first introduce the functional space
\[
X_{\nu, \varepsilon} := \{ u : \|u\|_{X_{\nu}} \leq \varepsilon \nu^{\frac{1}{2}} \|\log \nu\|^{-\frac{1}{2}} \},
\]
where
\[
\|u\|_{X_{\nu}} = \|u_{0,1}\|_{L^\infty} + \nu^{\frac{1}{2}} \|\partial_y u_{0,1}\|_{L^2} + \sum_{n \neq 0} \|u_n\|_{L^\infty} + \nu^{-\frac{1}{2}} \|Q_0 u\|_{L^4} + \nu^{\frac{1}{4}} \|\nabla Q_0 u\|_{L^2}.
\]

For $v \in X_{\nu, \varepsilon}$, we define the map $\Psi[v] = u$ as the solution to the system
\[
\begin{cases}
U \left( \frac{y}{\sqrt{\nu}} \right) \partial_x u + \left( u_2 \partial_y U \left( \frac{y}{\sqrt{\nu}} \right), 0 \right) - \nu \Delta u + \nabla p = -v \cdot \nabla v + f^\nu, \\
\nabla \cdot u = 0, \\
u |_{y=0} = 0.
\end{cases}
\]

Notice that
\[
-v \cdot \nabla v = -v_{0,1} \partial_y Q_0 v - (Q_0 v_2 \partial_y v_{0,1}, 0) - Q_0 v \cdot \nabla Q_0 v,
\]
which implies
\[
P_0(-v \cdot \nabla v + f^\nu) = \partial_y (-P_0 (Q_0 v \cdot \nabla Q_0 v)).
\]
Then we obtain by Proposition 5.1 that
\[
\|u_{0,1}\|_{L^\infty} \leq \nu^{-1} \|Q_0 v\|_{L^4} \quad \text{and} \quad \|\partial_y u_{0,1}\|_{L^2} \leq \nu^{-1} \|Q_0 v\|_{L^2} \|Q_0 v\|_{L^2}.
\]

For non-zero modes, we have
\[
Q_0(-v \cdot \nabla v + f^\nu) = -v_{0,1} \partial_y Q_0 v - (Q_0 v_2 \partial_y v_{0,1}, 0) - Q_0 (Q_0 v \cdot \nabla Q_0 v) + f^\nu.
\]
Proposition 5.2 implies that
\[ \|Q_0v\|_{L^2} \leq C(\|v_{0,1}\|_{L^\infty} \|\nabla Q_0 v\|_{L^2} + \|Q_0 v\|_{L^\infty} \|\partial_y v_{0,1}\|_{L^2} + \|Q_0 v\|_{L^\infty} \|\nabla Q_0 v\|_{L^2} + \|f'\|_{L^2}) \]  
(5.9)
and
\[ \|\nabla Q_0 u\|_{L^2} \leq C \nu^{-\frac{1}{2}} (\|v_{0,1}\|_{L^\infty} \|\nabla Q_0 v\|_{L^2} + \|Q_0 v\|_{L^\infty} \|\partial_y v_{0,1}\|_{L^2} + \|Q_0 v\|_{L^\infty} \|\nabla Q_0 v\|_{L^2} + \|f'\|_{L^2}). \]  
(5.10)

For the case of \( n \neq 0 \), notice that
\[ P_n(-v \cdot \nabla v + f'') = -v_{0,1}\partial_x P_n v - (P_n v_2 \partial_y v_{0,1}, 0) - P_n(Q_0 v \cdot \nabla Q_0 v) + P_n f''. \]

Then it follows from Proposition 5.2 and the interpolation that for \( 0 < |\tilde{n}| = n/\theta \leq \delta_0 \nu^{-\frac{3}{4}} \),
\[ \|P_n u\|_{L^\infty} \leq C \nu^{-\frac{1}{4}} |\tilde{n}|^{-\frac{1}{2}} (\|v_{0,1}\|_{L^\infty} \|\nabla P_n v\|_{L^2} + \|P_n v\|_{L^\infty} \|\partial_y v_{0,1}\|_{L^2} + \|P_n(Q_0 v \cdot \nabla Q_0 v)\|_{L^2} + \|P_n f''\|_{L^2}), \]
and for \( |\tilde{n}| \geq \delta_0 \nu^{-\frac{3}{4}} \),
\[ \|P_n u\|_{L^\infty} \leq C \nu^{-\frac{1}{4}} |\tilde{n}|^{-\frac{1}{2}} (\|v_{0,1}\|_{L^\infty} \|\nabla P_n v\|_{L^2} + \|P_n v\|_{L^\infty} \|\partial_y v_{0,1}\|_{L^2} + \|P_n(Q_0 v \cdot \nabla Q_0 v)\|_{L^2} + \|P_n f''\|_{L^2}), \]
which imply by the Parseval’s equality that
\[ \sum_{n \neq 0} \|P_n u\|_{L^\infty} \leq C \nu^{-\frac{1}{4}} \left( |\log \nu|^{\frac{1}{2}} \|v_{0,1}\|_{L^\infty} \|\nabla Q_0 v\|_{L^2} + \sum_{n \neq 0} \|P_n v\|_{L^\infty} \|\partial_y v_{0,1}\|_{L^2} + \|P_n f''\|_{L^2} \right). \]  
(5.11)

By collecting (5.9)–(5.11), we arrive at
\[ \|\Psi[v]\|_{X_{\nu}} \leq C \nu^{-\frac{1}{2}} |\log \nu|^{\frac{1}{2}} \|v\|_{X_{\nu}} + C \nu^{-\frac{1}{4}} |\log \nu|^{\frac{1}{2}} \|f''\|_{L^2} \]
\[ \|\Psi[v] - \Psi[v']\|_{X_{\nu}} \leq C \nu^{-\frac{1}{2}} |\log \nu|^{\frac{1}{2}} (\|v\|_{X_{\nu}} + \|v'\|_{X_{\nu}}) \|v - v'\|_{X_{\nu}}. \]

Therefore, \( \Psi \) is a contraction from \( X_{\nu, \varepsilon} \) to itself if \( \varepsilon \) and \( \|f''\|_{L^2} \) are small enough. Hence, by the fixed point theorem, for any \( f'' \in L^2 \) with \( \|f''\|_{L^2} \leq \varepsilon |\log \nu|^{-1} \nu^{\frac{3}{4}} \), there exists a unique solution \( v'' \) to (1.4) in \( X_{\nu, \varepsilon} \). Using the elliptic regularity of the Stokes equation, we obtain \( \nabla^2 v'' \in L^2(\Omega_0) \). The proof of Theorem 1.1 is completed.

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then it holds that

\[ \partial_Y \phi(Y) = e^{-\alpha Y} \int_0^Y w(Z) \sinh(\alpha Z) dZ - \cosh(\alpha Y) \int_Y^{+\infty} w(Z) e^{-\alpha Z} dZ, \]

\[ \alpha \phi(Y) = -\sinh(\alpha Y) \int_Y^{+\infty} w(Z) e^{-\alpha Z} dZ - e^{-\alpha Y} \int_0^Y w(Z) \sinh(\alpha Z) dZ \]

\[ = -e^{-\alpha Y} \int_Y^{+\infty} w(Z) \sinh(\alpha Z) dZ + \int_Y^{+\infty} w(Z) \sinh(\alpha (Z - Y)) dZ. \]

Specially, we have

\[ \partial_Y \phi(0) = -\int_0^{+\infty} w(Y) e^{-\alpha Y} dY. \]

Proof. Integration by parts gives

\[ \int_Y^{+\infty} w(Z) e^{-\alpha Z} dZ = \int_Y^{+\infty} ((\partial_Z^2 - \alpha^2) \phi(Z)) e^{-\alpha Z} dZ \]

\[ = \int_Y^{+\infty} \phi((\partial_Z^2 - \alpha^2)e^{-\alpha Z}) dZ + ((\partial_Z \phi)e^{-\alpha Z})|_Y^{+\infty} - (\partial_Z \phi_Z e^{-\alpha Z})|_Y^{+\infty} \]

\[ = -\partial_Y \phi(Y) e^{-\alpha Y} - \alpha \phi(Y) e^{-\alpha Y}, \]

i.e.,

\[ \partial_Y \phi(Y) e^{-\alpha Y} + \alpha \phi(Y) e^{-\alpha Y} = -\int_Y^{+\infty} w(Z) e^{-\alpha Z} dZ. \quad (A.1) \]

Specially, we have

\[ \partial_Y \phi(0) = -\int_0^{+\infty} w(Y) e^{-\alpha Y} dY. \quad (A.2) \]

Integration by parts again gives

\[ \int_0^Y w(Z) e^{\alpha Z} dZ = \int_0^Y ((\partial_Z^2 - \alpha^2) \phi(Z)) e^{\alpha Z} dZ \]

\[ = (\partial_Z \phi(Z) e^{\alpha Z})|_0^Y - (\phi(Z) \partial_Z (e^{\alpha Z}))|_0^Y = \partial_Y \phi(Y) e^{\alpha Y} - \alpha \phi(Y) e^{\alpha Y} - \partial_Y \phi(0). \]

Combining (A.1) and (A.2), we obtain

\[ \partial_Y \phi(Y) = \frac{1}{2} e^{-\alpha Y} \int_0^Y w(Z) e^{\alpha Z} dZ - \frac{1}{2} e^{\alpha Y} \int_Y^{+\infty} w(Z) e^{-\alpha Z} dZ + \frac{1}{2} e^{-\alpha Y} \partial_Y \phi(0) \]

\[ = e^{-\alpha Y} \int_0^Y w(Z) \sinh(\alpha Z) dZ - \cosh(\alpha Y) \int_Y^{+\infty} w(Z) e^{-\alpha Z} dZ. \]

Similarly, we have

\[ \alpha \phi(Y) = -\frac{1}{2} e^{-\alpha Y} \int_0^Y w(Z) e^{\alpha Z} dZ - \frac{1}{2} e^{\alpha Y} \int_Y^{+\infty} w(Z) e^{-\alpha Z} dZ - \frac{1}{2} e^{-\alpha Y} \partial_Y \phi(0). \]
This completes the proof. □

**Lemma A.2.** Let \( \alpha \neq 0 \). Suppose that \((\partial_Y^2 - \alpha^2)w \in L^2(\mathbb{R}_+)\) with \((\partial_Y^2 - \alpha^2)\varphi = w\) and \(\partial_Y\varphi|_{Y=0} = \varphi|_{Y=0} = 0\). Then it holds that

\[
|\alpha|^2 \|\partial_Y\varphi, \alpha\varphi\|_{L^2} + \alpha^2 \|w\|_{L^2} + |\alpha| \|\partial_Y w\|_{L^2} + \|\partial_Y^2 w\|_{L^2} \leq C\|\partial_Y^2 - \alpha^2\|w\|_{L^2}
\]

and

\[
\|\partial_Y w, \alpha w\|_{L^2} \leq C\|\partial_Y^2 - \alpha^2\|w\|_{L^2}^\frac{1}{2} w\|_{L^2}^\frac{1}{2}.
\]

**Proof.** Without loss of generality, we may assume \(\alpha > 0\). Clearly, \(e^{-\alpha Y} \in H^2(\mathbb{R}_+)\) and \((\partial_Y^2 - \alpha^2)e^{-\alpha Y} = 0\). Let \((w_d, \varphi_d)\) be the solution to the following elliptic equations:

\[
\begin{aligned}
\left\{ \begin{array}{l}
(\partial_Y^2 - \alpha^2)w_d = (\partial_Y^2 - \alpha^2)w, \\
(\partial_Y^2 - \alpha^2)\varphi_d = w_d, \quad w_d|_{Y=0} = \varphi_d|_{Y=0} = 0.
\end{array} \right.
\end{aligned}
\]

Using the energy method and the interpolation, we can easily show that

\[
|\alpha|^2 \|\partial_Y\varphi_d\|_{L^\infty} + |\alpha|^2 \|\partial_Y w_d\|_{L^2} + \alpha^2 \|w_d\|_{L^2} + |\alpha| \|\partial_Y w_d\|_{L^2} + \|\partial_Y^2 w_d\|_{L^2} \leq C\|\partial_Y^2 - \alpha^2\|w\|_{L^2}.
\]

(A.3)

On the other hand, by Lemma A.1, we know that

\[
\partial_Y(\partial_Y^2 - \alpha^2)^{-1}(e^{-\alpha Y})|_{Y=0} = -1/(2\alpha).
\]

Then \(w = w_d + 2\alpha \partial_Y \varphi_d(0)e^{-\alpha Y}\), which along with (A.3) implies that

\[
|\alpha|^2 \|\partial_Y\varphi, \alpha\varphi\|_{L^2} + \alpha^2 \|w\|_{L^2} + |\alpha| \|\partial_Y w\|_{L^2} + \|\partial_Y^2 w\|_{L^2} \leq C\|\partial_Y^2 - \alpha^2\|w\|_{L^2}.
\]

This gives the first inequality of this lemma.

For the second inequality, we first notice that

\[
\langle (\partial_Y^2 - \alpha^2)w, -w \rangle = \|\partial_Y w, \alpha w\|_{L^2}^2 + \partial_Y w(0)w(0),
\]

\[
\|\partial_Y w, \alpha w\|_{L^2}^2 \leq \|\partial_Y^2 w\|_{L^2} \|w\|_{L^\infty}^2.
\]

Thanks to \(\partial_Y w\) (\(w \in L^2\)), we have \(\lim_{Y \to \pm \infty} w(Y) = \lim_{Y \to \pm \infty} \partial_Y w(Y) = 0\), and hence,

\[
\|\partial_Y w\|_{L^\infty}^2 \leq 2\|\partial_Y^2 w\|_{L^2} \|\partial_Y w\|_{L^2}, \quad \|w\|_{L^\infty}^2 \leq 2\|\partial_Y w\|_{L^2} \|w\|_{L^2}.
\]

Summing up, we obtain

\[
\|\partial_Y w, \alpha w\|_{L^2}^2 \leq \|\partial_Y^2 w\|_{L^2} \|\partial_Y w\|_{L^2}^2 + 2\|\partial_Y^2 w\|_{L^2} \|\partial_Y w\|_{L^2} \|w\|_{L^2}^2
\]

\[
\leq \|\partial_Y^2 - \alpha^2\|w\|_{L^2} \|w\|_{L^2} + \|\partial_Y^2 w\|_{L^2} \|\partial_Y w\|_{L^2} \|w\|_{L^2}^2,
\]

which implies

\[
\|\partial_Y w, \alpha w\|_{L^2}^2 \leq C\|\partial_Y^2 - \alpha^2\|w\|_{L^2} \|w\|_{L^2}.
\]

This completes the proof. □
Lemma A.3. There exists a positive constant $C > 0$ such that for any $z \in \mathbb{C}$ and $t > 0$, it holds that
\[
\int_0^t |z - s|^2 \, ds \geq C^{-1} |z|^2 t.
\]

Proof. Let $z_r = \text{Re}(z)$ and $z_i = \text{Im}(z)$. Let us first claim that
\[
\int_0^t |z_r - s|^2 \, ds \geq C^{-1} |z_r|^2 t.
\]
Once (A.4) holds, we have
\[
\int_0^t |z - s|^2 \, ds \geq C^{-1} |z| t.
\]

It remains to prove (A.4).

Case 1. $z_r \leq 0$. In this case, we have
\[
\int_0^t |z_r - s|^2 \, ds \geq \int_0^t |z_r|^2 \, ds = |z_r|^2 t.
\]

Case 2. $0 \leq z_r \leq t/2$. In this case, we have
\[
\int_0^t |z_r - s|^2 \, ds \geq \int_{t/2}^t |z_r - s|^2 \, ds = \int_{t/2}^t (s - z_r)^2 \, ds \geq \int_{t/2}^t (s - t/2)^2 \, ds
\]
\[
= \frac{2(t-t/2)^2}{3} = \frac{2(t/2)^2}{3} \geq \frac{z_r^2 t}{3} \geq C|z_r|^2 t.
\]

Case 3. $z_r \geq t/2$. In this case, we have
\[
\int_0^t |z_r - s|^2 \, ds \geq \int_0^{t/4} |z_r - s|^2 \, ds \geq \int_0^{t/4} (z_r - t/4)^2 \, ds
\]
\[
= \frac{(z_r - t/4)^2 t}{4} \geq \frac{|z_r/2|^2 t}{4},
\]
where we used $z_r - t/4 \geq z_r/2 = |z_r|/2$. Combining the three cases, we conclude our proof. \qed

The following Hardy’s type inequalities come from [9].

Lemma A.4. (1) Let $\sigma[\cdot]$ be a linear operator defined by
\[
\sigma[f](Y) = \int_Y^{+\infty} f(Y_1) \, dY_1, \quad f \in C_0^\infty(\mathbb{R}_+).
\]
Then for $1 \leq p \leq +\infty$ and $k = 0, 1, \ldots$, we have
\[
\|Y^k \sigma[f]\|_{L_p^Y} \leq C_p \|Y^{k+1} f\|_{L_p^Y}.
\]

(2) Let $\mathcal{L}[\cdot]$ be a linear operator defined by
\[
\mathcal{L}[f](Y) = U(Y) \int_Y^{+\infty} \frac{f(Y_1)}{U^2} \, dY_1, \quad f \in C_0^\infty(\mathbb{R}_+).
\]
Then for $1 < p < +\infty$ and $k = 0, 1, \ldots$, we have
\[
\|Y^k \mathcal{L}[f]\|_{L_p^Y} \leq C \|Y^k (1 + Y) f\|_{L_p^Y},
\]
\[
\|\partial_Y \mathcal{L}[f]\|_{L_p^Y} \leq C (\|f\|_{L_p^Y} + \|f\|_{L_p^Y} + \|\partial_Y f\|_{L_p^Y}).
\]
Appendix B  Some estimates of the Airy function

Let $Ai(y)$ be the Airy function, which is a nontrivial solution of $f'' - yf = 0$. We define

$$A_0(z) = \int_0^{+\infty} Ai(t) e^{i\pi/6} t dt = e^{i\pi/6} \int_0^{+\infty} Ai(e^{i\pi/6} t) dt.$$ 

The following lemma comes from [3].

**Lemma B.1.** There exist $c > 0$ and $\delta_0 > 0$ so that for $\text{Im}(z) \leq \delta_0$,

$$\left| \frac{A_0'(z)}{A_0(z)} \right| \leq 1 + \left| z \right|^2, \quad \text{Re} \frac{A_0'(z)}{A_0(z)} \leq \min(-1/3, -c(1 + \left| z \right|^4)). \quad (B.1)$$

Moreover, for $\text{Im} z \leq \delta_0$, we have

$$\left| \frac{A_0'(z)}{A_0(z)} \right| \leq C(1 + \left| z \right|).$$

We define

$$\tilde{A}(Y) = Ai(e^{i\frac{\pi}{6}} \kappa(Y + \eta))/Ai(e^{i\frac{\pi}{6}} \kappa \eta),$$

where $\kappa > 0$ and $\text{Im} \eta < 0$. We define $\hat{\Phi}(Y)$ as the solution of

$$(\partial_Y^2 - \alpha^2)\hat{\Phi} = \tilde{A}, \quad \hat{\Phi}(0) = 0.$$

Under the assumption $\tilde{A} \in L^2(\mathbb{R}_+) \cap L^1(\mathbb{R}_+; e^{\alpha Y})$, by Lemma A.1, we know that

$$\hat{\Phi}(Y) = -\frac{e^{-\alpha Y}}{\alpha} \int_0^{+\infty} \tilde{A}(Z) \sinh(\alpha Z) dZ + \frac{1}{\alpha} \int_Y^{+\infty} \tilde{A}(Z) \sinh(\alpha(Z - Y)) dZ.$$

We define the fast decay part of $\hat{\Phi}(Y)$ as

$$\hat{\Phi}_f(Y) = \frac{1}{\alpha} \int_Y^{+\infty} \tilde{A}(Z) \sinh(\alpha(Z - Y)) dZ.$$

Then $(\partial_Y^2 - \alpha^2)\hat{\Phi}_f = \tilde{A}$.

**Lemma B.2.** Let $\kappa > 0$ and $\text{Im} \eta < 0$. Then there exists a $c > 0$ such that

$$\left| \tilde{A}(Y) \right| \leq Ce^{-c \kappa Y(1 + \left| \kappa \eta \right|^{\frac{1}{2}})},$$

$$\| Y^\beta \tilde{A} \|_{L^2} \leq C \kappa^{-\beta - \frac{1}{2} - \frac{1}{2}} (1 + \left| \kappa \eta \right|)^{-\frac{\beta}{2} - \frac{1}{2}}, \quad \beta \geq 0,$$

$$\left\| (\partial_Y \hat{\Phi}, \alpha \hat{\Phi}) \right\|_{L^2} \leq C \kappa^{-\frac{1}{2}} (1 + \left| \kappa \eta \right|)^{-\frac{1}{4}}.$$

Moreover, if $c\kappa \geq 2\alpha > 0$, we have

$$\left| \hat{\Phi}_f(Y) \right| \leq C \kappa^{-2} (1 + \left| \kappa \eta \right|)^{-1} e^{-c \kappa Y(1 + \left| \kappa \eta \right|^{\frac{1}{2}})^{1/2}},$$

$$\left| \partial_Y \hat{\Phi}_f(Y) \right| \leq C \kappa^{-1} (1 + \left| \kappa \eta \right|)^{-\frac{1}{2}} e^{-c \kappa Y(1 + \left| \kappa \eta \right|^{\frac{1}{2}})^{1/2}},$$

$$\| Y^\beta \hat{\Phi}_f \|_{L^2} \leq C \kappa^{-\frac{2\beta + 5}{2}} (1 + \left| \kappa \eta \right|)^{-\frac{2\beta + 5}{2}}, \quad \beta \geq 0,$$

$$\left\| (\partial_Y \hat{\Phi}_f, \alpha \hat{\Phi}_f) \right\|_{L^2} \leq C \kappa^{-\frac{1}{2}} (1 + \left| \kappa \eta \right|)^{-\frac{1}{4}}$$

and

$$\left| \hat{\Phi}(Y) \right| \leq C \kappa^{-2} (1 + \left| \kappa \eta \right|)^{-1} e^{-\alpha Y},$$

$$\| Y^\beta \hat{\Phi} \|_{L^2} \leq C \kappa^{-2} (1 + \left| \kappa \eta \right|)^{-1} \alpha^{-\frac{2\beta + 1}{2}}, \quad \beta \geq 0.$$
Proof. By Lemmas B.1 and A.3, we have
\[
\frac{A_0(t + B)}{A_0(B)} = \left| \exp \left( \ln \left( \frac{A_0(t + B)}{A_0(B)} \right) \right) \right| = \left| \exp \left( \int_0^t \frac{A_0(s + B)}{A_0(s + B)} \, ds \right) \right|
\leq \exp \left( \int_0^t \mathbf{Re} \frac{A_0(s + B)}{A_0(s + B)} \, ds \right) \leq \exp \left( - \int_0^t \max(1/3, c(1 + |s + B|^\frac{1}{2})) \, ds \right),
\]
which along with Lemma A.3 implies
\[
\left| \frac{A_0(t + B)}{A_0(B)} \right| \leq \exp(- \max(t/3, c(1 + |B|^\frac{1}{2})t)). \tag{B.2}
\]
Thanks to \( \mathbf{Re} \frac{A_0(z)}{A_0(\zeta)} \leq \min(-1/3, -c(1 + |z|^\frac{1}{2})) < 0 \), we deduce that \( \mathbf{Re} \frac{A_0(z)}{A_0(\zeta)} \geq c(1 + |z|^\frac{1}{2}) \) and
\[
\left| \frac{A_0(z)}{A_0(\zeta)} \right| = \left| \frac{A_0(z)}{A_0(\zeta)} \right|^{-1} \leq \left| \mathbf{Re} \frac{A_0(z)}{A_0(\zeta)} \right|^{-1} \leq c^{-1}(1 + |z|^\frac{1}{2})^{-1}. \tag{B.3}
\]
Now we are ready to show the estimates of \( \tilde{A}(Y) \). Lemma B.1 gives
\[
\left| \tilde{A}(Y) \right| = \left| \frac{A_0(\kappa(Y + \eta))}{A_0(\kappa\eta)} \right| = \left| A_0(\kappa\eta) \frac{A_0(\kappa(Y + \eta))}{A_0(\kappa\eta)} \frac{A_0(\kappa(Y + \eta))}{A_0(\kappa\eta)} \right| \leq C(1 + |\kappa\eta|)^{-\frac{1}{2}}(1 + |\kappa\eta| + \kappa Y)^{\frac{1}{2}} e^{-\frac{\kappa Y}{1 + |\kappa|} Y} \leq C e^{-\frac{\kappa Y}{1 + |\kappa|} Y}.
\]
Then it is easy to find that for \( \beta \geq 0 \),
\[
\left\| Y^\beta \tilde{A} \right\|_{L^2} \leq C \left\| e^{-\frac{\kappa Y}{1 + |\kappa|} Y} \right\|_{L^2} \leq C(1 + |\kappa Y|)^{\frac{1}{2}} \left( 1 + |\kappa Y| \right)^{-\frac{1}{2}}. \tag{4.1}
\]
Now we turn to deal with \( \tilde{\Phi}(Y) \). By Hardy’s inequality, we have
\[
\left\| (\partial_Y \tilde{\Phi}, \alpha \tilde{\Phi}) \right\|_{L^2}^2 = \left\| (\tilde{A}, \tilde{\Phi}) \right\| \leq \left\| Y \tilde{A} \right\|_{L^2} \left\| \tilde{\Phi}/Y \right\|_{L^2} \leq 2 \left\| Y \tilde{A} \right\|_{L^2} \left\| \partial_Y \tilde{\Phi} \right\|_{L^2},
\]
which gives
\[
\left\| (\partial_Y \tilde{\Phi}, \alpha \tilde{\Phi}) \right\|_{L^2} \leq 2 \left\| Y \tilde{A} \right\|_{L^2} \leq C \left( 1 + |\kappa Y| \right)^{-\frac{1}{2}}. \tag{4.2}
\]
Thanks to the definition of \( \tilde{\Phi}_f \) and \( \left| \tilde{A}(Y) \right| \leq C e^{-\frac{\kappa Y}{1 + |\kappa|} Y}, \) we have
\[
\left| \tilde{\Phi}_f(Y) \right| = \alpha^{-1} \int_Y^{+\infty} \tilde{A}(Z) \sinh(\alpha(Z - Y)) \, dZ
\leq C \alpha^{-1} \int_Y^{+\infty} e^{-\frac{\kappa Z}{1 + |\kappa|} Y} \sinh(\alpha(Z - Y)) \, dZ
= C \kappa^{-1} \left( 1 + |\kappa\eta|^\frac{1}{2} \right)^{-1} \int_Y^{+\infty} e^{-\frac{\kappa Z}{1 + |\kappa|} Y} \cosh(\alpha(Z - Y)) \, dZ
\leq C \kappa^{-1} \left( 1 + |\kappa\eta|^\frac{1}{2} \right)^{-\frac{1}{2}} \int_Y^{+\infty} e^{-\frac{\kappa Z}{1 + |\kappa|} Y} \exp(\alpha(Z - Y)) \, dZ
\leq C \kappa^{-1} \left( 1 + |\kappa\eta|^\frac{1}{2} \right)^{-\frac{1}{2}} (c\kappa(1 + |\kappa|^\frac{1}{2}) - \alpha)^{-1} \exp(-c\kappa(1 + |\kappa|^\frac{1}{2}) - Y) Y.
\]
Therefore, for \( \kappa \geq 2 \alpha > 0 \), we have \( c\kappa(1 + |\kappa|^\frac{1}{2}) - \alpha \geq c\kappa(1 + |\kappa|^\frac{1}{2})/2 \) and
\[
\left| \tilde{\Phi}_f(Y) \right| \leq C \kappa^{-2} \left( 1 + |\kappa\eta|^\frac{1}{2} \right)^{-1} e^{-c\kappa(1 + |\kappa|^\frac{1}{2}) Y/2}
\]
which implies that for \( \beta \geq 0 \),
\[
\left\| Y^\beta \tilde{\Phi}_f \right\|_{L^2} \leq C \kappa^{-\frac{\beta + 1}{2} - \frac{1}{2}} \left( 1 + |\kappa\eta|^\frac{1}{2} \right)^{-\frac{\beta + 1}{2}}.
\]
Notice that
\[
\partial_Y \tilde{\Phi}_f(Y) = -\int_Y^{+\infty} \tilde{A}(Z) \cosh(\alpha(Z - Y))dZ.
\]
Similarly, we can obtain
\[
|\partial_Y \tilde{\Phi}_f(Y)| \leq C_{\kappa}^{-1}(1 + |\kappa\eta|)^{-\frac{1}{2}}e^{-c_2\kappa Y(1 + |\kappa\eta|)^{\frac{1}{2}}},
\]
\[
\|Y^{\frac{1}{2}} \tilde{\Phi}_f\|_{L^2} \leq C_{\kappa}^{-\frac{1}{2}}(1 + |\kappa\eta|)^{-\frac{2\alpha+1}{4}}.
\]
Thence we obtain that |\tilde{A}(Y)| \leq Ce^{-c_2(1 + |\kappa\eta|)^{\frac{1}{2}}}Y , we obtain
\[
\alpha^{-1} \int_0^{+\infty} \tilde{A}(Z) \sinh(\alpha Z) dZ \leq C\alpha^{-1} \int_0^{+\infty} e^{-c_2(1 + |\kappa\eta|)^{\frac{1}{2}}}Y \sinh(\alpha Z) dZ
\]
\[
= C_{\kappa}^{-1}(1 + |\kappa\eta|)^{-\frac{1}{2}} \int_0^{+\infty} e^{-c_2(1 + |\kappa\eta|)^{\frac{1}{2}}}Y \cosh(\alpha Z) dZ
\]
\[
\leq C_{\kappa}^{-1}(1 + |\kappa\eta|)^{-\frac{1}{2}} \int_0^{+\infty} e^{-c_2(1 + |\kappa\eta|)^{\frac{1}{2}}}Y 2 \exp(\alpha Z) dZ
\]
\[
\leq C_{\kappa}^{-1}(1 + |\kappa\eta|)^{-\frac{1}{2}}(c_2(1 + |\kappa\eta|)^{\frac{1}{2}} - \alpha)^{-1}.
\]
Then for \( c\kappa \geq 2\alpha > 0 \), we have
\[
\alpha^{-1} \int_0^{+\infty} \tilde{A}(Z) \sinh(\alpha Z) dZ \leq C_{\kappa}^{-2}(1 + |\kappa\eta|)^{-1}.
\]
Recall that
\[
\tilde{\Phi}(Y) = -\frac{e^{-\alpha Y}}{\alpha} \int_0^{+\infty} \tilde{A}(Z) \sinh(\alpha Z) dZ + \tilde{\Phi}_f(Y).
\]
Then we obtain
\[
|\tilde{\Phi}(Y)| \leq C_{\kappa}^{-2}(1 + |\kappa\eta|)^{-1}(e^{-\alpha Y} + e^{-c_2(1 + |\kappa\eta|)^{\frac{1}{2}}}Y^{1/2}) \leq C_{\kappa}^{-2}(1 + |\kappa\eta|)^{-1}e^{-\alpha Y},
\]
which yields that \( \|Y^{\frac{1}{2}} \tilde{\Phi}\|_{L^2} \leq C_{\kappa}^{-2}(1 + |\kappa\eta|)^{-1}e^{-\frac{2\alpha+1}{2}}. \)

**Lemma B.3.** Let \( c \) be the constant in Lemma B.2, \( c\kappa \geq 2\alpha > 0 \) and \( \text{Im}\eta < 0 \). Then it holds that
\[
|\partial_Y \tilde{\Phi}(0)| \geq C^{-1}(1 + |\kappa\eta|)^{-\frac{1}{2}}(\kappa + 3\alpha)^{-1},
\]
\[
\partial_Y \tilde{\Phi}_f(0) = -\frac{e^{i\frac{\pi}{2}}}{3\kappa \text{Ai}(e^{i\pi} \kappa\eta)} + \mathcal{O} \left( \frac{\eta}{\text{Ai}(e^{i\pi} \kappa\eta)} \right) + \mathcal{O} \left( \frac{\alpha^2}{(c_2(1 + |\kappa\eta|)^{1/2} - \alpha)^3} \right),
\]
\[
\tilde{\Phi}_f(0) = i\kappa^{-3} + \mathcal{O}(\kappa^{-1} Y) + \mathcal{O} \left( \frac{\alpha^2}{(c_2(1 + |\kappa\eta|)^{1/2} - \alpha)^3} \right).
\]

**Proof.** By Lemma A.1, we have
\[
-\partial_Y \tilde{\Phi}(0) = \int_0^{+\infty} \tilde{A}(Y) e^{-\alpha Y} dY = \int_0^{+\infty} \frac{\text{Ai}(e^{i\pi} \kappa(Y + \eta))}{\text{Ai}(e^{i\pi} \kappa\eta)} e^{-\alpha Y} dY
\]
\[
= \int_0^{+\infty} \frac{\text{Ai}(e^{i\pi} \kappa Y)}{\kappa \text{Ai}(e^{i\pi} \kappa\eta)} e^{-\alpha Y} dY
\]
\[
= \frac{\text{Ai}(0) e^{i\pi}}{\kappa \text{Ai}(e^{i\pi} \kappa\eta)} - \int_0^{+\infty} \frac{\text{Ai}(e^{i\pi} \kappa Y)}{\kappa \text{Ai}'(e^{i\pi} \kappa\eta)} \partial_Y (e^{-\alpha Y}) dY,
\]
which along with Lemma B.2 gives
\[ \kappa |A_0'(\kappa\eta)||\partial_Y \tilde{\Phi}(0)| \geq |A_0(\kappa\eta)| - \int_0^{+\infty} |A_0(\kappa(Y + \eta))||\partial_Y (e^{-\alpha Y})|dY \]
\[ \geq |A_0(\kappa\eta)| - \int_0^{+\infty} e^{-\kappa Y/3} |A_0(\kappa\eta)||\partial_Y (e^{-\alpha Y})|dY \]
\[ = |A_0(\kappa\eta)| - |A_0(\kappa\eta)| \int_0^{+\infty} e^{-\kappa Y/3} \partial_Y (e^{-\alpha Y})dY \]
\[ = \frac{\kappa}{3} |A_0(\kappa\eta)| \int_0^{+\infty} e^{-\kappa Y/3 - \alpha Y} dY \]
\[ = |A_0(\kappa\eta)| \frac{\kappa}{\kappa + 3\alpha}. \]

This along with Lemma B.1 shows that
\[ |\partial_Y \tilde{\Phi}(0)| \geq \frac{|A_0(\kappa\eta)|}{(\kappa + 3\alpha)|A_0'(\kappa\eta)|} \geq C^{-1}(1 + |\kappa\eta|)^{-\frac{3}{2}}(\kappa + 3\alpha)^{-1}. \]

Now we estimate \( \tilde{\Phi}_f(0) \). We first have
\[ \partial_Y \tilde{\Phi}_f(0) = -\int_0^{+\infty} \tilde{A}(Y) \cosh(\alpha Y)dY =: I + II, \]
where
\[ I = -\int_0^{+\infty} \tilde{A}(Y)dY, \quad II = -\int_0^{+\infty} \tilde{A}(Y)(\cosh(\alpha Y) - 1)dY. \]

For \( I \), we notice that by the definition of \( \tilde{A}(Y) \),
\[ I = -\frac{1}{\kappa A_1(e^{\frac{1}{3}} \kappa\eta)} \int_0^{+\infty} \frac{1}{\kappa A_1(e^{\frac{1}{3}} \kappa\eta)} \int_0^{+\infty} A_1(e^{\frac{1}{3}} Z)dZ \]
\[ = -\frac{1}{\kappa A_1(e^{\frac{1}{3}} \kappa\eta)} \int_0^{+\infty} Ai(e^{\frac{1}{3}} Y)dY + \frac{1}{\kappa A_1(e^{\frac{1}{3}} \kappa\eta)} \int_0^{+\infty} Ai(e^{\frac{1}{3}} Z)dZ. \]

On the other hand, we know that \( \int_0^{+\infty} Ai(Y)dY = 1/3 \) and observe that
\[ \left| \int_0^{\kappa\eta} Ai(e^{\frac{1}{3}} Z)dZ \right| \lesssim \kappa. \]

Then we obtain
\[ I = -\frac{e^{\frac{1}{3}}}{3\kappa A_1(e^{\frac{1}{3}} \kappa\eta)} + O\left(\frac{\eta}{A_1(e^{\frac{1}{3}} \kappa\eta)}\right). \]  \hspace{1cm} (B.4)

For \( II \), we obtain by Lemma B.2 that
\[ |II| \leq C \int_0^{+\infty} \alpha^2 Y^2 e^{-\kappa Y(1 + |\kappa\eta|^\frac{1}{2})} e^{\alpha Y} dY \leq C \frac{\alpha^2}{(c\kappa(1 + |\kappa\eta|^\frac{1}{2}) - \alpha)^3}. \]

which along with (B.4) implies
\[ \partial_Y \tilde{\Phi}_f(0) = -\frac{e^{\frac{1}{3}}}{3\kappa A_1(e^{\frac{1}{3}} \kappa\eta)} + O\left(\frac{\eta}{A_1(e^{\frac{1}{3}} \kappa\eta)}\right) + O\left(\frac{\alpha^2}{(c\kappa(1 + |\kappa\eta|^\frac{1}{2}) - \alpha)^3}\right). \]
Now we turn to the estimate of $\tilde{\Phi}_f(0)$. Notice that

$$\tilde{\Phi}_f(0) = \alpha^{-1} \int_0^{+\infty} \tilde{A}(Y) \sinh(\alpha Y) dY =: I_0 + I_{I_0},$$

where

$$I_0 = \alpha^{-1} \int_0^{+\infty} \tilde{A}(Y) \alpha Y dY, \quad I_{I_0} = \alpha^{-1} \int_0^{+\infty} \tilde{A}(Y)(\sinh(\alpha Y) - \alpha Y) dY.$$

Thanks to the definition of $\tilde{A}(Y)$, we have

$$Y \tilde{A}(Y) = -\frac{i}{\kappa^3} \partial^2_Y \tilde{A}(Y) - \eta \tilde{A}(Y).$$

Then we obtain

$$I_0 = -\frac{i}{\kappa^3} \int_0^{+\infty} \partial_Y \tilde{A}(Y) dY - \eta \int_0^{+\infty} \tilde{A}(Y) dY$$

$$= i\kappa^{-3} \partial_Y \tilde{A}(0) + O(\kappa^{-1} \eta) = i\kappa^{-3} + O(\kappa^{-1} \eta)$$

and

$$|I_{I_0}| \leq C \alpha^2 \int_0^{+\infty} Y^3 e^{-\alpha Y(1+|\kappa|^{1/2})} e^{\alpha Y} dY \leq C \frac{\alpha^2}{(\kappa(1+|\kappa|^{1/2})^{1/2} - \alpha^2)}.$$

Thus, we conclude that

$$\tilde{\Phi}_f(0) = i\kappa^{-3} + O(\kappa^{-1} \eta) + O\left(\frac{\alpha^2}{(\kappa(1+|\kappa|^{1/2})^{1/2} - \alpha^2)}\right).$$

This completes the proof. 

**Appendix C  The homogeneous Rayleigh equation**

Here, we recall a result about the homogeneous Rayleigh equation from [9].

**Proposition C.1.** For any $0 < \alpha < 1$, there exists a function $\varphi \in H^1(\mathbb{R}_+)$ such that

$$U(\partial_Y^2 - \alpha^2) \varphi - U'' \varphi = 0, \quad Y > 0,$$

and the following properties hold:

$$\varphi = \varphi_0 + \varphi_1 + \varphi_2$$

with

$$\varphi_0 = U e^{-\alpha Y}, \quad \varphi_1 |_{Y=0} = \frac{\alpha}{U'(0)} + O(\alpha^2),$$

$$\|\partial_Y \varphi_1\|_{L^2} + \|\varphi_1\|_{L^2} \leq C \alpha,$$

$$\|\partial_Y \varphi_2\|_{L^2} + \alpha \|\varphi_2\|_{L^2} \leq C \alpha^{3/2}.$$

Here, $C$ is independent of $\alpha$. If $\frac{U''}{U} \in L^2(\mathbb{R}_+)$ in addition, then $\varphi_1$ and $\varphi_2$ belong to $H^2(\mathbb{R}_+)$. 
