ON GEOMETRY OF SYMPLECTIC INVOLUTIONS

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Abstract. Let $V$ be a $2n$-dimensional vector space over a field $F$ and $\Omega$ be a non-degenerate symplectic form on $V$. Denote by $H_k(\Omega)$ the set of all $2k$-dimensional subspaces $U \subset V$ such that the restriction $\Omega|_U$ is non-degenerate. Our main result (Theorem 1) says that if $n \neq 2k$ and $\max(k, n-k) \geq 5$ then any bijective transformation of $H_k(\Omega)$ preserving the class of base subsets is induced by a semi-simplectic automorphism of $V$. For the case when $n \neq 2k$ this fails, but we have a weak version of this result (Theorem 2). If the characteristic of $F$ is not equal to 2 then there is a one-to-one correspondence between elements of $H_k(\Omega)$ and symplectic $(2k, 2n-2k)$-involutions and Theorem 1 can be formulated as follows: for the case when $n \neq 2k$ and $\max(k, n-k) \geq 5$ any commutativity preserving bijective transformation of the set of symplectic $(2k, 2n-2k)$-involutions can be extended to an automorphism of the symplectic group.

1. Introduction

Let $W$ be an $n$-dimensional vector space over a division ring $R$ and $n \geq 3$. We put $G_k(W)$ for the Grassmannian of $k$-dimensional subspaces of $W$. The projective space associated with $W$ will be denoted by $P(W)$.

Let us consider the set $G_k(W) \times G_{n-k}(W)$, where $S+U = W$. If $B$ is a base for $P(W)$ then the base subset of $G_k(W)$ associated with the base $B$ consists of all $(S, U)$ such that $S$ and $U$ are spanned by elements of $B$. If $n \neq 2k$ then any bijective transformation of $G_k(W)$ preserving the class of base subsets is induced by a semi-linear isomorphism of $W$ to itself or to the dual space $W^*$ (for $n = 2k$ this fails, but some weak version of this result holds true). Using Mackey’s ideas [7], J. Dieudonné [2] and C. E. Rickart [9] have proved this statement for $k = 1, n-1$. For the case when $1 < k < n-1$ it was established by author [8]. Note that adjacency preserving transformations of $G_k(W)$ were studied in [6].

Now suppose that the characteristic of $R$ is not equal to 2 and consider an involution $u \in \text{GL}(W)$. There exist two subspaces $S_+(u)$ and $S_-(u)$ such that

$$u(x) = x \text{ if } x \in S_+(u), \quad u(x) = -x \text{ if } x \in S_-(u)$$

and

$$W = S_+(u) + S_-(u).$$

We say that $u$ is a $(k, n-k)$-involution if the dimensions of $S_+(u)$ and $S_-(u)$ are equal to $k$ and $n-k$, respectively. The set of $(k, n-k)$-involutions will be denoted by $I_k(W)$. There is the natural one-to-one correspondence between elements of $I_k(W)$

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and $\mathfrak{S}_k(W)$ such that each base subsets of $\mathfrak{S}_k(W)$ corresponds to a maximal set of mutually permutable $(k, n - k)$-involutions. Thus any commutativity preserving transformation of $\mathfrak{S}_k(W)$ can be considered as a transformation of $\mathfrak{S}_k(W)$ preserving the class of base subsets, and our statement shows that if $n \neq 2k$ then any commutativity preserving bijective transformation of $\mathfrak{S}_k(W)$ can be extended to an automorphism of $\text{GL}(W)$.

In the present paper we give symplectic analogues of these results.

2. Results

2.1. Let $V$ be a $2n$-dimensional vector space over a field $F$ and $\Omega : V \times V \to F$ be a non-degenerate symplectic form. The form $\Omega$ defines on the set of subspaces of $V$ the orthogonal relation which will be denoted by $\perp$. For any subspace $S \subset V$ we put $S^\perp$ for the orthogonal complement to $S$. A subspace $S \subset V$ is said to be non-degenerate if the restriction $\Omega|_S$ is non-degenerate; for this case $S$ is even-dimensional and $S + S^\perp = V$. We put $\mathfrak{S}_k(\Omega)$ for the set of non-degenerate $2k$-dimensional subspaces. Any element of $\mathfrak{S}(\Omega)$ can be presented as the sum of $k$ mutually orthogonal elements of $\mathfrak{S}_1(\Omega)$.

Let us consider the projective space $\mathcal{P}(V)$ associated with $V$. The points of this space are 1-dimensional subspaces of $V$, and each line consists of all 1-dimensional subspaces contained in a certain 2-dimensional subspace. A line of $\mathcal{P}(V)$ is called hyperbolic if the corresponding 2-dimensional subspace belongs to $\mathfrak{S}_1(\Omega)$; otherwise, the line is said to be isotropic. Points of $\mathcal{P}(V)$ together with the family of isotropic lines form the well-known polar space. Some results related with the hyperbolic symplectic geometry (spanned by points of $\mathcal{P}(V)$ and hyperbolic lines) can be found in [1], [2], [3].

A base $B = \{P_1, \ldots, P_{2n}\}$ of $\mathcal{P}(V)$ is called symplectic if for any $i \in \{1, \ldots, 2n\}$ there is unique $\sigma(i) \in \{1, \ldots, 2n\}$ such that $P_i \neq P_{\sigma(i)}$. Then the set $\mathfrak{S}_1$ consisting of all

$$S_i := P_i + P_{\sigma(i)}$$

is said to be the base subset of $\mathfrak{S}_1(\Omega)$ associated with the base $B$. For any $k \in \{2, \ldots, n - 1\}$ the set $\mathfrak{S}_k$ consisting of all $S_{i_1} + \cdots + S_{i_k}$ ($i_1, \ldots, i_k$ are different) will be called the base subset of $\mathfrak{S}_k(\Omega)$ associated with $\mathfrak{S}_1$ (or defined by $\mathfrak{S}_1$).

Now suppose that the characteristic of $F$ is not equal to 2. An involution $u \in \text{GL}(V)$ is symplectic (belongs to the group $\text{Sp}(\Omega)$) if and only if $S_+(u)$ and $S_-(u)$ are non-degenerate and $S_-(u) = (S_+(u))^\perp$. We denote by $\mathcal{I}_k(\Omega)$ the set of symplectic $(2k, 2n - 2k)$-involutions. There is the natural bijection

$$i_k : \mathcal{I}_k(\Omega) \to \mathfrak{S}_k(\Omega), \quad u \to S_+(u).$$

We say that $\mathcal{X} \subset \mathcal{I}_k(\Omega)$ is a $MC$-subset if any two elements of $\mathcal{X}$ commute and for any $u \in \mathcal{I}_k(\Omega) \setminus \mathcal{X}$ there exists $s \in \mathcal{X}$ such that $su \neq us$ (in other words, $\mathcal{X}$ is a maximal set of mutually permutable elements of $\mathcal{I}_k(\Omega)$).

**Fact 1.** [2], [3] $\mathcal{X}$ is a $MC$-subset of $\mathcal{I}_k(\Omega)$ if and only if $i_k(\mathcal{X})$ is a base subset of $\mathfrak{S}_k(\Omega)$. For any two commutative elements of $\mathcal{I}_k(\Omega)$ there is a $MC$-subset containing them.

Fact 1 shows that a bijective transformation $f$ of $\mathfrak{S}_k(\Omega)$ preserves the class of base subsets if and only if $f^{-1}i_k$ is commutativity preserving.
2.2. If \( l \) is an element of \( \Gamma \text{Sp}(\Omega) \) (the group of semi-linear automorphisms preserving \( \Omega \)) then for each number \( k \in \{1, \ldots, n-1\} \) we have the bijective transformation

\[
(l)_k : \mathfrak{H}_k(\Omega) \to \mathfrak{H}_k(\Omega), \quad U \to l(U)
\]

which preserves the class of base subsets. The bijection

\[
p_k : \mathfrak{H}_k(\Omega) \to \mathfrak{H}_{n-k}(\Omega), \quad U \to U^\perp
\]

sends base subsets to base subsets. We will need the following trivial fact.

Fact 2. Let \( f \) be a bijective transformation of \( \mathfrak{H}_k(\Omega) \) preserving the class of base subsets. Then the same holds for the transformation \( p_k f p_{n-k} \). Moreover, if \( f = (l)_k \) for certain \( l \in \Gamma \text{Sp}(\Omega) \) then \( p_k f p_{n-k} = (l)_{n-k} \).

Two distinct elements of \( \mathfrak{H}_1(\Omega) \) are orthogonal if and only if there exists a base subset containing them, thus for any bijective transformation \( f \) of \( \mathfrak{H}_1(\Omega) \) the following condition are equivalent:

— \( f \) preserves the relation \( \perp \),
— \( f \) preserves the class of base subsets.

It is not difficult to prove (see [2], p. 26-27 or [9], p. 711-712) that if one of these conditions holds then \( f \) is induced by an element of \( \Gamma \text{Sp}(\Omega) \). Fact 2 guarantees that the same is fulfilled for bijective transformations of \( \mathfrak{H}_{n-1}(\Omega) \) preserving the class of base subsets. This result was exploited by J. Dieudonné [2] and C. E. Rickart [9] to determining automorphisms of the group \( \text{Sp}(\Omega) \).

Theorem 1. If \( n \neq 2k \) and \( \max(k, n-k) \geq 5 \) then any bijective transformation of \( \mathfrak{H}_k(\Omega) \) preserving the class of base subsets is induced by an element of \( \Gamma \text{Sp}(\Omega) \).

Corollary 1. Suppose that the characteristic of \( F \) is not equal to 2. If \( n \neq 2k \) and \( \max(k, n-k) \geq 5 \) then any commutativity preserving bijective transformation \( f \) of \( \mathfrak{H}_k(\Omega) \) can be extended to an automorphism of \( \text{Sp}(\Omega) \).

Proof of Corollary. By Fact 1, \( i_k f i_k^{-1} \) preserves the class of base subsets. Theorem 1 implies that \( i_k f i_k^{-1} \) is induced by \( l \in \Gamma \text{Sp}(\Omega) \). The automorphism \( u \to lul^{-1} \) is as required. \( \square \)

2.3. For the case when \( n = 2k \) Theorem 1 fails.

Example 1. Suppose that \( n = 2k \) and \( X \) is a subset of \( \mathfrak{H}_k(\Omega) \) such that for any \( U \in X \) we have \( U^\perp \in X \). Consider the transformation of \( \mathfrak{H}_k(\Omega) \) which sends each \( U \in X \) to \( U^\perp \) and leaves fixed all other elements. This transformation preserves the class of base subsets (any base subset of \( \mathfrak{H}_k(\Omega) \) contains \( U \) together with \( U^\perp \)), but it is not induced by a semilinear automorphism if \( X \neq \emptyset \).

If \( n = 2k \) then we denote by \( \overline{\mathfrak{H}}_k(\Omega) \) the set of all subsets \( \{U, U^\perp\} \subset \mathfrak{H}_k(\Omega) \).

Then every \( l \in \Gamma \text{Sp}(\Omega) \) induces the bijection

\[
(l)^_k : \overline{\mathfrak{H}}_k(\Omega) \to \overline{\mathfrak{H}}_k(\Omega), \quad \{U, U^\perp\} \to \{l(U), l(U^\perp) = l(U)^\perp\}.
\]

The transformation from Example 1 gives the identical transformation of \( \overline{\mathfrak{H}}_k(\Omega) \).

Theorem 2. Let \( n = 2k \geq 14 \) and \( f \) be a bijective transformation of \( \mathfrak{H}_k(\Omega) \) preserving the class of base subsets. Then \( f \) preserves the relation \( \perp \) and induces a bijective transformation of \( \overline{\mathfrak{H}}_k(\Omega) \). The latter mapping is induced by an element of \( \Gamma \text{Sp}(\Omega) \).
Corollary 2. Let \( n = 2k \geq 14 \) and \( f \) be a commutativity preserving bijective transformation of \( \mathcal{I}_k(\Omega) \). Suppose also that the characteristic of \( F \) is not equal to 2. Then there exists an automorphism \( g \) of the group \( \operatorname{Sp}(\Omega) \) such that \( f(u) = \pm g(u) \) for any \( u \in \mathcal{I}_k(\Omega) \).

3. Inexact subsets

In this section we suppose that \( n \geq 4 \) and \( 1 < k < n - 1 \).

3.1. Inexact subsets of \( \mathcal{G}_k(W) \). Let \( B = \{P_1, \ldots, P_n\} \) be a base of \( \mathcal{P}(W) \). For any \( m \in \{1, \ldots, n-1\} \) we denote by \( \mathcal{B}_m \) the base subset of \( \mathcal{G}_m(W) \) associated with \( B \) (the definition was given in section 1).

If \( \alpha = (M, N) \in \mathcal{B}_m \) then we put \( \mathcal{B}_k(\alpha) \) for the set of all \((S, U) \in \mathcal{B}_k \) where \( S \) is incident to \( M \) or \( N \) (then \( U \) is incident to \( N \) or \( M \), respectively), the set of all \((S, U) \in \mathcal{B}_k \) such that \( S \) is incident to \( M \) will be denoted by \( \mathcal{B}_k^+ (\alpha) \).

A subset \( \mathcal{X} \subset \mathcal{B}_k \) is called exact if there is only one base subset of \( \mathcal{G}_k(W) \) containing \( \mathcal{X} \); otherwise, \( \mathcal{X} \) is said to be inexact. If \( \alpha \in \mathcal{B}_2 \) then \( \mathcal{B}_k(\alpha) \) is a maximal inexact subset of \( \mathcal{B}_k \) (Example 1 in \([S]\)). Conversely, we have the following.

Lemma 1 (Lemma 2 of \([S]\)). If \( \mathcal{X} \) is a maximal inexact subset of \( \mathcal{B}_k \) then there exists \( \alpha \in \mathcal{B}_2 \) such that \( \mathcal{X} = \mathcal{B}_k(\alpha) \).

Lemma 2 (Lemmas 5 and 8 of \([S]\)). Let \( g \) be a bijective transformation of \( \mathcal{B}_k \) preserving the class of maximal inexact subsets. Then for any \( \alpha \in \mathcal{B}_{k-1} \) there exists \( \beta \in \mathcal{B}_{k-1} \) such that

\[
g(\mathcal{B}_k(\alpha)) = \mathcal{B}_k(\beta);
\]

moreover, we have

\[
g(\mathcal{B}_k^+(\alpha)) = \mathcal{B}_k^+(\beta)
\]

if \( n \neq 2k \).

3.2. Inexact subsets of \( \mathcal{H}_k(\Omega) \). Let \( \mathcal{S}_1 = \{S_1, \ldots, S_n\} \) be a base subsets of \( \mathcal{H}_1(\Omega) \).

For each number \( m \in \{2, \ldots, n-1\} \) we denote by \( \mathcal{S}_m \) the base subset of \( \mathcal{H}_m(\Omega) \) associated with \( \mathcal{S}_1 \).

Let \( M \in \mathcal{S}_m \). Then \( M^\perp \in \mathcal{S}_{n-m} \). We put \( \mathcal{S}_k(M) \) for the set of all elements of \( \mathcal{G}_k \) incident to \( M \) or \( M^\perp \). The set of all elements of \( \mathcal{G}_k \) incident to \( M \) will be denoted by \( \mathcal{G}_k^+(M) \).

Let \( \mathcal{X} \) be a subset of \( \mathcal{G}_k \). We say that \( \mathcal{X} \) is exact if it is contained only in one base subset of \( \mathcal{H}_k(\Omega) \); otherwise, \( \mathcal{X} \) will be called inexact. For any \( i \in \{1, \ldots, n\} \) we denote by \( \mathcal{X}_i \) the set of all elements of \( \mathcal{X} \) containing \( S_i \). If \( \mathcal{X}_i \) is not empty then we define

\[
U_i(\mathcal{X}) := \bigcap_{U \in \mathcal{X}_i} U,
\]

and \( U_i(\mathcal{X}) := \emptyset \) if \( \mathcal{X}_i \) is empty. It is trivial that our subset is exact if \( U_i(\mathcal{X}) = S_i \) for each \( i \).

Lemma 3. \( \mathcal{X} \) is exact if \( U_i(\mathcal{X}) \neq S_i \) only for one \( i \).

Proof. Let \( \mathcal{G}_1' \) be a base subset of \( \mathcal{H}_1(\Omega) \) which defines a base subset of \( \mathcal{H}_k(\Omega) \) containing \( \mathcal{X} \). If \( j \neq i \) then \( U_j(\mathcal{X}) = S_j \) implies that \( S_j \) belongs to \( \mathcal{G}_1' \). Let us take \( S' \in \mathcal{G}_1' \) which does not coincide with any \( S_j, j \neq i \). Since \( S' \) is orthogonal to all such \( S_j \), we have \( S' = S_i \) and \( \mathcal{G}_1' = \mathcal{G}_1 \). \( \square \)
Example 2. Let $M \in S_2$. Then $M = S_i + S_j$ for some $i, j$. We choose orthogonal $S'_i, S'_j \in S_1(\Omega)$ such that $S'_i + S'_j = M$ and $\{S_i, S_j\} \neq \{S'_i, S'_j\}$. Then

$$(S_1 \setminus \{S_i, S_j\}) \cup \{S'_i, S'_j\}$$

is a base subset of $S_1(\Omega)$ which defines another base subset of $S_k(\Omega)$ containing $S_k(M)$. Therefore, $S_k(M)$ is inexact. Any $U \in S_k \setminus S_k(M)$ intersects $M$ by $S_i$ or $S_j$ and

$$U_p(S_k(M) \cup \{U\}) = S_p$$

if $p = i$ or $j$; the same holds for all $p \neq i, j$. By Lemma 3, $S_k(M) \cup \{U\}$ is exact for any $U \in S_k \setminus S_k(M)$ Thus the inexact subset $S_k(M)$ is maximal.

Lemma 4. Let $X$ be a maximal inexact subset of $S_k$. Then $X = S_k(M)$ for certain $M \in S_2$.

Proof. By the definition, there exists another base subset of $S_1(\Omega)$ containing $X$; the associated base subset of $S_1(\Omega)$ will be denoted by $S'_1$. Since our inexact subset is maximal, we need to prove the existence of $M \in S_2$ such that $X \subset S_k(M)$.

Let us consider $i \in \{1, \ldots, n\}$ such that $U_i$ is not empty (from this moment we write $U_i$ in place of $U_i(\Omega)$). We say that the number $i$ is of first type if the inclusion $U_j \subset U_i$, $j \neq i$ implies that $U_j = \emptyset$ or $U_j = U_i$. If $i$ is not of first type and the inclusion $U_j \subset U_i$, $j \neq i$ holds only for the case when $U_j = \emptyset$ or $j$ is of first type then $i$ is said to be of second type. Similarly, other types of numbers can be defined.

Suppose that there exists a number $j$ of first type such that dim $U_j \geq 4$. Then $U_j$ contains certain $M \in S_2$. Since $j$ is of first type, for any $U \in X$ one of the following possibilities is realized:

- $U \in X_j$ then $M \subset U_j \subset U$,
- $U \in X \setminus X_j$ then $U \subset U_j^{\perp} \subset M^{\perp}$.

This means that $M$ is as required.

Now suppose that $U_j = S_j$ for all $j$ of first type, so $S_j \in S'_1$ if $j$ is of first type. Consider any number $i$ of second type. If $U_i \in S_m$ then $m \geq 2$ and there are exactly $m - 1$ distinct $j$ of first type such that $S_j = U_j$ is contained in $U_i$; since all such $S_j$ belong to $S'_1$ and $U_i$ is spanned by elements of $S'_1$, we have $S_i \in S'_1$. Step by step we establish the same for other types. Thus $S_i \in S'_1$ if $U_i$ is not empty. Since $X$ is inexact, Lemma 3 implies the the existence of two distinct numbers $i$ and $j$ such that $U_i = U_j = \emptyset$. We define $M := S_i + S_j$. Then any element of $X$ is contained in $M^{\perp}$ and we get the claim.

Let $S'_1$ be another base subset of $S_1(\Omega)$ and $S'_m$, $m \in \{2, \ldots, n - 1\}$ be the base subset of $S_m(\Omega)$ defined by $S'_1$.

Lemma 5. Let $h$ be a bijection of $S_k$ to $S'_k$ such that $h$ and $h^{-1}$ send maximal inexact subsets to maximal inexact subsets. Then for any $M \in S_k$ there exists $M' \in S'_k$ such that

$$h(S_k(M)) = S'_k(M')$$

moreover, we have

$$h(S_k^+(M)) = S'_k^+(M')$$

if $n \neq 2k$. 

Proof. Let $\mathcal{B}_m$, $m \in \{1, \ldots, n-1\}$ be as in subsection 3.1. For each $m$ there is the natural bijection $b_m : \mathcal{B}_m \rightarrow \mathfrak{S}_m$ sending $(S, U) \in \mathcal{B}_m$, $S = P_{i_1} + \cdots + P_{i_m}$ to $S_{i_1} + \cdots + S_{i_m}$. For any $M \in \mathfrak{S}_m$ we have

$$\mathfrak{S}_k(M) = b_k(\mathcal{B}_k(b_m^{-1}(M))) \quad \text{and} \quad \mathfrak{S}_k^+(M) = b_k(\mathcal{B}_k^+(b_m^{-1}(M))).$$

Let $b'_m$ be the similar bijection of $\mathcal{B}_m$ to $\mathfrak{S}'_m$. Then $(b'_k)^{-1}hb_k$ is a bijective transformation of $\mathcal{B}_k$ preserving the class of base subsets and our statement follows from Lemma 2.

4. Proof of Theorems 1 and 2

By Fact 2, we need to prove Theorem 1 only for $k < n - k$. Throughout the section we suppose that $1 < k \leq n - k$ and $n - k \geq 5$; for the case when $n = 2k$ we require that $n \geq 14$.

4.1. Let $f$ be a bijective transformation of $\mathcal{H}_k(\Omega)$ preserving the class of base subsets. The restriction of $f$ to any base subset satisfies the condition of Lemma 5.

For any subspace $T \subset V$ we denote by $\mathcal{H}_k(T)$ the set of all elements of $\mathcal{H}_k(\Omega)$ incident to $T$ or $T^\perp$, the set of all elements of $\mathcal{H}_k(\Omega)$ incident to $T$ will be denoted by $\mathcal{H}_k^+(T)$.

In this subsection we show that Theorems 1 and 2 are simple consequences of the following lemma.

Lemma 6. There exists a bijective transformation $g$ of $\mathcal{H}_{k-1}(\Omega)$ such that

$$g(\mathcal{H}_k^+(T)) = \mathcal{H}_k^+(g(T)) \quad \forall T \in \mathcal{H}_{k-1}(\Omega)$$

if $n \neq 2k$, and

$$g(\mathcal{H}_k(T)) = \mathcal{H}_k(g(T)) \quad \forall T \in \mathcal{H}_{k-1}(\Omega)$$

for the case when $n = 2k$.

Let $\mathfrak{S}_{k-1}$ be a base subset of $\mathcal{H}_{k-1}(\Omega)$ and $\mathfrak{S}_k$ be the associated base subset of $\mathcal{H}_k(\Omega)$ (these base subsets are defined by the same base subset of $\mathcal{H}_1(\Omega)$). By our hypothesis, $f(\mathfrak{S}_k)$ is a base subset of $\mathcal{H}_k(\Omega)$; we denote by $\mathfrak{S}'_k$ the associated base subset of $\mathcal{H}_k(\Omega)$. It is easy to see that $g(\mathfrak{S}_{k-1}) = \mathfrak{S}'_k$, so $g$ maps base subsets to base subsets. Since $f^{-1}$ preserves the class of base subset, the same holds for $g^{-1}$. Thus $g$ preserves the class of base subsets.

Now suppose that $g = (l)_{k-1}$ for certain $l \in \Gamma\text{Sp}(\Omega)$. Let $U$ be an element of $\mathcal{H}_k(\Omega)$. We take $M, N \in \mathcal{H}_{k-1}(\Omega)$ such that $U = M + N$. If $n \neq 2k$ then

$$\{U\} = \mathfrak{S}_k^-(M) \cap \mathfrak{S}_k^+(N) \quad \text{and} \quad \{f(U)\} = \mathfrak{S}_k^+(l(M)) \cap \mathfrak{S}_k^+(l(N)),$$

so $f(U) = l(M) + l(N) = l(U)$, and we get $f = (l)_{k}$. For the case when $n = 2k$ we have

$$\{U, U^\perp\} = \mathfrak{S}_k(M) \cap \mathfrak{S}_k(N) \quad \text{and} \quad \{f(U), f(U)^\perp\} = \mathfrak{S}_k(l(M)) \cap \mathfrak{S}_k(l(N));$$

since $l(M) + l(N) = l(U)$ and $l(M)^\perp \cap l(N)^\perp = l(U)^\perp = l(U)$,

$$\{f(U), f(U)^\perp\} = \{l(U), l(U)^\perp\};$$

the latter means that $f = (l)_{k}$. Therefore, Theorem 1 can be proved by induction and Theorem 2 follows from Theorem 1.

To prove Lemma 6 we use the following.

Lemma 7. Let $M \in \mathcal{H}_m(\Omega)$ and $N$ be a subspace contained in $M$. Then the following assertion are fulfilled:
(1) If $\dim N > m$ then $N$ contains an element of $\mathfrak{H}_1(\Omega)$.

(2) If $\dim N > m + 2$ then $N$ contains two orthogonal elements of $\mathfrak{H}_1(\Omega)$.

(3) If $\dim N > m + 4$ then $N$ contains three distinct mutually orthogonal elements of $\mathfrak{H}_1(\Omega)$.

Proof. The form $\Omega|_M$ is non-degenerate. If $\dim N > m$ then the restriction of $\Omega|_M$ to $N$ is non-zero. This implies the existence of $S \in \mathfrak{H}_1(\Omega)$ contained in $N$. We have

$$\dim N \cap S^\perp \geq \dim N - 2,$$

and for the case when $\dim N > m + 2$ there is an element of $\mathfrak{H}_1(\Omega)$ contained in $N \cap S^\perp$. Similarly, (3) follows from (2).

4.2. **Proof of Lemma 6 for** $k < n - k$. Let $T \in \mathfrak{H}_k-1(\Omega)$ and $\mathfrak{S}_1 = \{S_1, \ldots, S_n\}$ be a base subset of $\mathfrak{H}_1(\Omega)$ such that

$$T^\perp = S_1 + \cdots + S_{n-k+1} \quad \text{and} \quad T = S_{n-k+2} + \cdots + S_n.$$ We put $\mathfrak{S}_k$ for the base subset of $\mathfrak{H}_k(\Omega)$ associated with $\mathfrak{S}_1$. Then $\mathfrak{S}_k^+(T)$ consists of all

$$U_i := T + S_i,$$

where $i \in \{1, \ldots, n - k + 1\}$. By Lemma 5, there exists $T' \in \mathfrak{H}_{k-1}(\Omega)$ such that

$$f(\mathfrak{S}_k^+(T)) \subset \mathfrak{S}_k^+(T').$$ We need to show that $f(\mathfrak{S}_k^+(T))$ coincides with $\mathfrak{S}_k^+(T')$.

**Lemma 8.** Let $U \in \mathfrak{S}_k^+(T)$. Suppose that there exist two distinct $M, N \in \mathfrak{S}_k^+(T)$ such that $f(M), f(N)$ belong to $\mathfrak{S}_k^+(T')$ and there is a base subset of $\mathfrak{H}_k(\Omega)$ containing $M, N$ and $U$. Then $f(U)$ is an element of $\mathfrak{S}_k^+(T')$.

Proof. If there exists a base subset of $\mathfrak{H}_k(\Omega)$ containing $M, N$ and $U$ then $T$ belongs to the associated base subset of $\mathfrak{H}_{k-1}(\Omega)$ and Lemma 5 implies the existence of $T'' \in \mathfrak{H}_{k-1}(\Omega)$ such that $f(M), f(N)$ and $f(U)$ belong to $\mathfrak{S}_k^+(T'')$. On the other hand, $f(M)$ and $f(N)$ are different elements of $\mathfrak{S}_k^+(T')$ and $f(M) \cap f(N)$ coincides with $T'$. Hence $T' = T''$. □

For any $U \in \mathfrak{S}_k^+(T)$ we denote by $S(U)$ the intersection of $U$ and $T^\perp$, it is clear that $S(U)$ is an element of $\mathfrak{H}_1(\Omega)$.

If $S(U)$ is contained in $S_1 + \cdots + S_{n-k-1}$ then $S(U), S_{n-k}, S_{n-k+1}$ are mutually orthogonal and there exists a base subset of $\mathfrak{H}_k(\Omega)$ containing $U, U_{n-k}, U_{n-k+1}$. All $f(U_i)$ belong to $\mathfrak{S}_k^+(T')$ and Lemma 8 shows that $f(U) \in \mathfrak{S}_k^+(T')$.

Let $U$ be an element of $\mathfrak{S}_k^+(T)$ such that $S(U)$ is contained in $S_1 + \cdots + S_{n-k}$. We have

$$\dim(S_1 + \cdots + S_{n-k-1}) \cap S(U)^\perp \geq 2(n-k-2) > n-k-1$$

(the latter inequality follows from the condition $n-k \geq 5$) and Lemma 7 implies the existence of $S' \in \mathfrak{H}_1(\Omega)$ contained in

$$(S_1 + \cdots + S_{n-k-1}) \cap S(U)^\perp.$$ Then $S(U), S', S_{n-k+1}$ are mutually orthogonal and there exists a base subset of $\mathfrak{H}_k(\Omega)$ containing $U, T + S', U_{n-k+1}$. It was proved above that $f(T + S')$ belongs to $\mathfrak{S}_k^+(T')$. Since $f(U_i) \in \mathfrak{S}_k^+(T')$ for each $i$, Lemma 8 guarantees that $f(U)$ is an element of $\mathfrak{S}_k^+(T').$
Now suppose that \( S(U) \) is not contained in \( S_1 + \cdots + S_{n-k} \). Since \( n - k \geq 5 \),
\[
\dim(S_1 + \cdots + S_{n-k}) \cap S(U) \geq 2(n - k - 1) > n - k + 2.
\]
By Lemma 7, there exist two orthogonal \( S', S'' \in \mathcal{H}_1(\Omega) \) contained in
\[
(S_1 + \cdots + S_{n-k}) \cap S(U) \perp.
\]
Then \( S', S'', S(U) \) are mutually orthogonal and there exists a base subset of \( \mathcal{H}_k(\Omega) \)
containing \( S' + T, S'' + T \) and \( U \). We have shown above that \( f(S' + T), f(S'' + T) \)
belong to \( \mathcal{H}_k^+(T') \) and Lemma 8 shows that the same holds for \( f(U) \).
So \( f(\mathcal{H}_k^+(T)) \subset \mathcal{H}_k^+(T') \). Since \( f^{-1} \) preserves the class of base subsets, the
inverse inclusion holds true. We define \( g : \mathcal{H}_{k-1}(\Omega) \rightarrow \mathcal{H}_{k-1}(\Omega) \) by \( g(T) := T' \).
This transformation is bijective (otherwise, \( f \) is not bijective).

4.3. Proof of Lemma 6 for \( n = 2k \). We start with the following.

**Lemma 9.** If \( n = 2k \) then \( f(U^\perp) = f(U) \) for any \( U \in \mathcal{H}_k(\Omega) \).

**Proof.** We take a base subset \( \mathcal{G}_k \) containing \( U \). Then \( U^\perp \in \mathcal{G}_k \). Denote by \( \mathcal{G}_{k-1} \)
the base subset of \( \mathcal{H}_{k-1}(\Omega) \) associated with \( \mathcal{G}_k \). Let \( \mathcal{G}_k' \)
be the base subset of \( \mathcal{H}_{k-1}(\Omega) \) associated with \( \mathcal{G}_k' := f(\mathcal{G}_k) \). We choose \( M, N \in \mathcal{G}_{k-1} \)
such that \( U = M + N \). Then
\[
\{U, U^\perp\} = \mathcal{G}_k(M) \cap \mathcal{G}_k(N)
\]
and Lemma 5 guarantees that
\[
\{f(U), f(U^\perp)\} = \mathcal{G}_k'(M') \cap \mathcal{G}_k'(N')
\]
for some \( M', N' \in \mathcal{G}_k' \). The set \( \mathcal{G}_k'(M') \cap \mathcal{G}_k'(N') \) is not empty if one of the
following possibilities is realized:

- \( M' + N' \) and \( M'^\perp \cap N'^\perp \) are elements of \( \mathcal{H}_{k-1}(\Omega) \) and \( \mathcal{G}_k'(M') \cap \mathcal{G}_k'(N') \)
  consists of these two elements.
- \( M' \subset N'^\perp \) and \( N' \subset M'^\perp \), then \( \mathcal{G}_k'(M') \cap \mathcal{G}_k'(N') \) consists of 4 elements.

Thus
\[
\{f(U), f(U^\perp)\} = \{M' + N', M'^\perp \cap N'^\perp\}.
\]
Since \( M' + N' \) and \( M'^\perp \cap N'^\perp \) are orthogonal, we get the claim. \( \square \)

Let \( T \in \mathcal{H}_{k-1}(\Omega) \). As in the previous subsection we consider a base subset
\( \mathcal{G}_1 = \{S_1, \ldots, S_n\} \) of \( \mathcal{H}_1(\Omega) \) such that
\[
T^\perp = S_1 + \cdots + S_{n-k+1} \text{ and } T = S_{n-k+2} + \cdots + S_n.
\]
We denote by \( \mathcal{G}_k \) the base subset of \( \mathcal{H}_k(\Omega) \) associated with \( \mathcal{G}_1 \). Then \( \mathcal{G}_k(T) \)
consists of
\[
U_i := T + S_i, \quad i \in \{1, \ldots, n - k + 1\}
\]
and their orthogonal complements. Lemma 5 implies the existence of \( T' \in \mathcal{H}_{k-1}(\Omega) \)
such that
\[
f(\mathcal{G}_k(T)) \subset \mathcal{H}_k(T').
\]
We show that \( f(U) \) belongs to \( \mathcal{H}_k(T') \) for any \( U \in \mathcal{H}_k(T) \).

We need to establish this fact only for the case when \( U \) is an element of \( \mathcal{H}_k^+(T) \). Indeed, if \( U \in \mathcal{H}_k^+(T^\perp) \) then \( U^\perp \) is an element of \( \mathcal{H}_k^+(T) \) and \( f(U^\perp) \in \mathcal{H}_k(T') \)
implies that \( f(U) = f(U^\perp)^\perp \) belongs to \( \mathcal{H}_k(T') \).
Lemma 10. Let $U \in \mathcal{F}_k^+(T)$. Suppose that there exist distinct $M_i \in \mathcal{F}_k^+(T)$, $i = 1, 2, 3$ such that each $f(M_i)$ belongs to $\mathcal{F}_k(T')$ and there is a base subset of $\mathcal{F}_k(\Omega)$ containing $M_1, M_2, M_3$ and $U$. Then $f(U) \in \mathcal{F}_k(T')$.

Proof. By Lemma 5, there exists $T'' \in \mathcal{F}_{k-1}(\Omega)$ such that $f(U)$, all $f(M_i)$, and their orthogonal complements belong to $\mathcal{F}_k(T''')$. For any $i = 1, 2, 3$ one of the subspaces $f(M_i)$ or $f(M_i)^\perp$ is an element of $\mathcal{F}_k^+(T''')$; we denote this subspace by $M_i'$. Then

$$T'' = \bigcap_{i=1}^3 M_i'$$

and $T'' = M_i'^\perp + M_j'^\perp$, $i \neq j$;

note also that the intersection of any $M_i'$ and $M_j'^\perp$ does not belong to $\mathcal{F}_{k-1}(\Omega)$. Since all $M_i'$ and $M_j'^\perp$ belong to $\mathcal{F}_k(T')$, we have $T'' = T''$.

As in the previous subsection for any $U \in \mathcal{F}_k^+(T)$ we denote by $S(U)$ the intersection of $U$ and $T^\perp$, it is an element of $\mathcal{F}_1(\Omega)$.

If $S(U)$ is contained in $S_1 + \cdots + S_{n-k-2}$ then $S(U), S_{n-k-1}, S_{n-k}, S_{n-k+1}$ are mutually orthogonal and there exists a base subset of $\mathcal{F}_k(\Omega)$ containing $U, U_{n-k-1}, U_{n-k}, U_{n-k+1}$. Since $f(U_i) \in \mathcal{F}_k(T')$ for each $i$, Lemma 10 shows that $f(U)$ belongs to $\mathcal{F}_k(T')$.

Suppose that $S(U)$ is contained in $S_1 + \cdots + S_{n-k-1}$. We have

$$\dim(S_1 + \cdots + S_{n-k-2}) \cap S(U)^\perp \geq 2(n-k-3) > n-k-2$$

(since $k = n-k \geq 7$) and Lemma 7 implies the existence of $S' \in \mathcal{F}_1(\Omega)$ contained in

$$(S_1 + \cdots + S_{n-k-2}) \cap S(U)^\perp.$$

Then $S(U), S', S_{n-k}, S_{n-k+1}$ are mutually orthogonal, so $U, T + S', U_{n-k}, U_{n-k+1}$ are contained in a certain base subsets of $\mathcal{F}_k(\Omega)$. It was shown above that $f(T+S')$ is an element of $\mathcal{F}_k(T')$ and Lemma 10 guarantees that $f(U) \in \mathcal{F}_k(T')$ (recall that all $f(U_i)$ belong to $\mathcal{F}_k(T')$).

Consider the case when $S(U)$ is contained in $S_1 + \cdots + S_{n-k}$. We have

$$\dim(S_1 + \cdots + S_{n-k-1}) \cap S(U)^\perp \geq 2(n-k-2) > (n-k-1) + 2$$

(recall that $k = n-k \geq 7$) and there exist two orthogonal $S', S'' \in \mathcal{F}_1(\Omega)$ contained in

$$(S_1 + \cdots + S_{n-k-1}) \cap S(U)^\perp.$$

(Lemma 7). Then $S(U), S', S'', S_{n-k+1}$ are mutually orthogonal and there exists a base subsets of $\mathcal{F}_k(\Omega)$ containing $U, T+S', T+S'', U_{n-k+1}$. It follows from Lemma 10 that $f(U) \in \mathcal{F}_k(T')$ (since $f(T+S')$, $f(T+S'')$ and any $f(U_i)$ belong to $\mathcal{F}_k(T')$).

Let $U$ be an element of $\mathcal{F}_k(T')$ such that $S(U)$ is not contained in $S_1 + \cdots + S_{n-k}$. Since $n = 2k \geq 14$,

$$\dim(S_1 + \cdots + S_{n-k}) \cap S(U)^\perp \geq 2(n-k-1) > n-k+4.$$ 

By Lemma 7, there exist mutually orthogonal $S', S'', S''' \in \mathcal{F}_1(\Omega)$ contained in

$$(S_1 + \cdots + S_{n-k}) \cap S(U)^\perp.$$

A base subsets of $\mathcal{F}_k(\Omega)$ containing $U, T+S', T+S'', T+S'''$ exists. It was shown above that $f(T+S')$, $f(T+S'')$ and $f(T+S''')$ belong to $\mathcal{F}_k(T')$ and Lemma 10 implies that the same holds for $f(U)$. 

Thus $f(\mathfrak{H}_k(T)) \subset \mathfrak{H}_k(T')$. As in the previous subsection we have the inverse inclusion and define $g : \mathfrak{H}_{k-1}(\Omega) \rightarrow \mathfrak{H}_{k-1}(\Omega)$ by $g(T) := T'$. 

References

[1] Cuypers H., Symplectic geometries, transvection groups and modules, J. Combin. Theory A 65(1994), 39-59.
[2] Dieudonné J., On the automorphisms of the classical groups, Memoirs Amer. Math. Soc. 2(1951) 1-95.
[3] Dieudonné J., La Géométrie des Groupes Classiques, Springer-Verlag, Berlin, 1971.
[4] Gramlich R., On the hyperbolic symplectic geometry, J. Combin. Theory A 105(2004) 97-110.
[5] Hall J. I., The hyperbolic lines of finite symplectic spaces, J. Combin. Theory A 47(1988), 284-298.
[6] H. Havlicek and M. Pankov, Transformations on the product of Grassmann spaces. Demonstratio Math. XXXVIII (2005), to appear.
[7] Mackey G. W., Isomorphisms of normed linear spaces, Ann. of Math. 43 (1942) 244-260.
[8] Pankov M., On geometry of linear involution, Adv. Geom. (to appear).
[9] Rickart C. E., Isomorphic groups of linear transformations I, II, Amer. J. Math. 72(1950) 451-464, 73(1951), 697-716.

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