Complex hyperbolic and projective deformations of small Bianchi groups

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Abstract
The Bianchi groups \(\text{Bi}(d) = \text{PSL}(2, \mathcal{O}_d) < \text{PSL}(2, \mathbb{C})\) (where \(\mathcal{O}_d\) denotes the ring of integers of \(\mathbb{Q}(i\sqrt{d})\), with \(d \geq 1\) squarefree) can be viewed as subgroups of \(\text{SO}(3, 1)\) under the isomorphism \(\text{PSL}(2, \mathbb{C}) \cong \text{SO}_0(3, 1)\). We study the deformations of these groups into the larger Lie groups \(\text{SU}(3, 1)\) and \(\text{SL}(4, \mathbb{R})\) for small values of \(d\). In particular we show that \(\text{Bi}(3)\), which is rigid in \(\text{SO}(3, 1)\), admits a 1-dimensional deformation space into \(\text{SU}(3, 1)\) and \(\text{SL}(4, \mathbb{R})\), whereas any deformation of \(\text{Bi}(1)\) into \(\text{SU}(3, 1)\) or \(\text{SL}(4, \mathbb{R})\) is conjugate to one inside \(\text{SO}(3, 1)\). We also show that none of the deformations into \(\text{SU}(3, 1)\) are both discrete and faithful.

Keywords Lattices · Deformations · Bianchi groups

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1 Introduction

Let \(\Gamma\) be a discrete subgroup of a Lie group \(G\), and denote \(\iota : \Gamma \rightarrow G\) the inclusion map. A deformation of \(\Gamma\) in \(G\) is any continuous 1-parameter family of representations \(\rho_t : \Gamma \rightarrow G\) (for \(t\) in some interval \((-\varepsilon, \varepsilon)\)) satisfying \(\rho_0 = \iota\), and \(\rho_t\) not conjugate to \(\rho_0\) for any \(t \neq 0\). We say that \(\Gamma\) is locally rigid in \(G\) if it does not admit any deformations into \(G\).

Local rigidity and Mostow rigidity: when \(G\) is a semisimple real Lie group without compact factors local rigidity of lattices is known to hold in many cases. Weil proved in [32] that \(\Gamma\) is locally rigid in \(G\) if \(G/\Gamma\) is compact and \(G\) not locally isomorphic to \(\text{SL}(2, \mathbb{R})\). Garland and Raghunathan extended this result to the case where \(\Gamma\) is a non-cocompact lattice

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in a rank-1 semisimple group $G$ not locally isomorphic to $SL(2, \mathbb{R})$ or $SL(2, \mathbb{C})$ (Theorem 7.2 of Garland and Raghunathan [14]).

Mostow’s strong rigidity theorem [26] is both stronger and less general. It asserts that the inclusion map of any lattice $\Gamma$ in a rank-1 semisimple Lie group $G$, not locally isomorphic to $SL(2, \mathbb{R})$, is the unique faithful representation of $\Gamma$ into $G$ whose image is a lattice in $G$ (up to conjugation). In particular, $\Gamma$ has no deformations which are discrete and faithful with image a lattice in $G$.

Both local rigidity and Mostow rigidity fail dramatically in $SL(2, \mathbb{R})$. (From now on we consider $PSL(2, \mathbb{R})$ and $PSL(2, \mathbb{C})$ rather than $SL(2, \mathbb{R})$ and $SL(2, \mathbb{C})$ as these are the groups acting by isometries on hyperbolic space). Most lattices in $G = PSL(2, \mathbb{R})$ are well known to have large deformation spaces (of lattice embeddings) inside $PSL(2, \mathbb{R})$, which are known as the Teichmüller space of the corresponding lattice or quotient orbifold. For example the fundamental group of a compact surface of genus $g \geq 2$ enjoys a real $(6g-6)$-dimensional Teichmüller space. In the context of this paper it is also worth noting that some “small” lattices in $PSL(2, \mathbb{R})$ such as compact triangle groups are in fact locally rigid in $PSL(2, \mathbb{R})$.

Dehn surgery deformations: in the case of $G = PSL(2, \mathbb{C})$, all lattices satisfy Mostow rigidity and cocompact lattices satisfy local rigidity by Weil’s above result. Thurston showed however that many cusped lattices in $PSL(2, \mathbb{C})$ admit rich deformation spaces into $PSL(2, \mathbb{C})$, the so-called Dehn surgery spaces. More specifically (see Theorem 5.8.2 of Thurston [31]), if $M$ is a finite-volume complete orientable hyperbolic 3-manifold with $k \geq 1$ cusps, then $\pi_1(M)$ admits a (real) $2k$-dimensional deformation space into $PSL(2, \mathbb{C})$. (Here the original lattice embedding $\pi_1(M) \to PSL(2, \mathbb{C})$ is the holonomy representation of the unique complete, finite-volume hyperbolic structure on $M$). These deformations are not both discrete and faithful, but there exist infinitely many deformations with discrete and finite-covolume image in any neighborhood of the lattice embedding (corresponding to manifolds obtained by Dehn surgery on the cusps of $M$).

Dunbar and Meyerhoff showed in [12] that Thurston’s results carry through to the case of (complete, finite-volume, orientable) cusped hyperbolic 3-orbifolds, but only for those cusps whose cross-sections are a torus or a $2,2,2$-pillow, i.e. a sphere with four cone points of angle $\pi/2$. (Recall that each end of a complete, finite-volume hyperbolic $n$-manifold is diffeomorphic to $\mathbb{R}^n \times Q$ for some Euclidean $(n-1)$-orbifold $Q$ called the cusp cross-section of the cusp $\mathbb{R}^n \times Q$.) More specifically, let $Q$ be such an orbifold, denote $\Gamma = \pi_{orf}(Q)$ a lattice in $PSL(2, \mathbb{C})$ such that $Q = H_3/\Gamma$ and let $k$ be the number of cusps of $Q$ whose cross-sections are either tori or $2,2,2$-pillows. Theorem 5.3 of Dunbar and Meyerhoff [12] implies that the deformation space of $\Gamma$ into $PSL(2, \mathbb{C})$ contains a smooth (real) $2k$-dimensional family of deformations. We will abbreviate this as: $\dim_{hyp}(\chi(\Gamma, PSL(2, \mathbb{C}))) \geq 2k$, where $\chi(\Gamma, G)$ is the character variety of $\Gamma$ into $G = PSL(2, \mathbb{C})$, $\rho_{hyp}: \Gamma \to G$ is the lattice embedding (unique up to conjugation by Mostow), and $[\rho_{hyp}]$ its equivalence class in $\chi(\Gamma, G)$ - even if $\chi(\Gamma, G)$ is not necessarily smooth at $[\rho_{hyp}]$. (See Sect. 2 for a precise definition of $\chi(\Gamma, G)$ and more details on character varieties and smoothness).

Deformations of Bianchi groups: in this paper we focus on the Bianchi groups $\text{Bi}(d) = PSL(2, \mathcal{O}_d) < PSL(2, \mathbb{C})$, where $\mathcal{O}_d$ denotes the ring of integers of $\mathbb{Q}(i\sqrt{d})$, with $d \geq 1$ squarefree. These are well known to be (arithmetic) lattices in $PSL(2, \mathbb{C})$, with number of cusps equal to the class number $h_d$ of $\mathbb{Q}(i\sqrt{d})$ (see Sect. 9.1 of Maclachlan and Reid [24]). In particular, $\text{Bi}(d)$ has a single cusp exactly when $d = 1, 2, 3, 7, 11, 19, 43, 67, 163$ (by the Baker–Heegner–Stark theorem, see e.g. [29]; see also Table 1 in Section 5.1 for values of $h_d$ for small $d$). It is also known that the cusp cross-sections of $\text{Bi}(d)$ are all tori when $d \neq 1, 3$, and are a $2, 2, 2$-pillow when $d = 1$ and a $(3, 3, 3)$-turnover (sphere with three cone points of angle $\pi/3$) when $d = 3$. See [13, 24] for more details on Bianchi groups and
Sect. 2.4 of Baker and Reid [3] for a discussion of the Dunbar–Meyerhoff result in the case of the Bianchi groups. This gives the first part of the following:

**Theorem 1.1** With the above notation:

- \((\text{DM})\) \(\dim_{\rho_{\text{hyp}}} \chi(Bi(d), \text{PSL}(2, \mathbb{C})) \geq 2h_d\) for all \(d \neq 3\).
- \((\text{Se})\) \(\dim_{\rho_{\text{hyp}}} \chi(Bi(3), \text{PSL}(2, \mathbb{C})) = 0\).

The second item (local rigidity of \(Bi(3)\) in \(\text{PSL}(2, \mathbb{C})\)) follows from the fact that \(Bi(3)\) has Serre’s Property FA, that is, whenever it acts on a tree it has a global fixed point (see Sect. 5 of Bridson et al. [7] and Serre [28]). In fact, more recently it was shown in [7] that \(Bi(3)\) has the much stronger Galois rigidity property, namely that the only two Zariski-dense representations of \(Bi(3)\) into \(\text{PSL}(2, \mathbb{C})\) are the lattice embedding and its complex conjugate.

**Projective and complex hyperbolic deformations:** under the isomorphism \(\text{PSL}(2, \mathbb{C}) \simeq \text{SO}^0(3, 1)\), one may also consider the Bianchi groups as subgroups of \(\text{SO}(3, 1)\) and ask whether the lattice embedding \(\rho_{\text{hyp}} : Bi(d) \longrightarrow \text{SO}(3, 1)\) admits any further deformations into other Lie groups \(G\) containing \(\text{SO}(3, 1)\). In this paper we consider complex hyperbolic and projective deformations of the Bianchi groups, that is deformations into the Lie groups \(G = \text{SU}(3, 1)\) and \(\text{SL}(4, \mathbb{R})\) respectively. Denote \(\iota\) the inclusion \(\text{SO}(3, 1) \longrightarrow \text{SU}(3, 1)\). By analogy with the 2-dimensional \(\mathbb{R}\)-Fuchsian terminology, we will call \(\iota \circ \rho_{\text{hyp}}\) the \(\mathbb{R}\)-Kleinian embedding of \(\Gamma\) into \(\text{SU}(3, 1)\). (Recall that a subgroup of \(\text{SU}(2, 1)\) is called \(\mathbb{R}\)-Fuchsian if it is contained in a conjugate of \(\text{SO}(2, 1)\)).

Projective deformations and complex hyperbolic deformations of real hyperbolic lattices/discrete groups are known to be equivalent in the following sense (Theorem 2.2 of Cooper et al. [11]):

**Theorem 1.2** [11] Let \(\Gamma\) be a finitely generated group, and let \(\rho : \Gamma \longrightarrow \text{SO}^0(n, 1)\) be a smooth point of the representation variety \(\text{Hom}(\Gamma, \text{SL}(n + 1, \mathbb{R}))\). Then \(\iota \circ \rho\) is also a smooth point of \(\text{Hom}(\Gamma, \text{SU}(n, 1))\), and near \(\rho\) the real dimensions of \(\text{Hom}(\Gamma, \text{SL}(n + 1, \mathbb{R}))\) and \(\text{Hom}(\Gamma, \text{SU}(n, 1))\) are equal.

Our first result is the following; in the next statement one can replace \(\text{SU}(3, 1)\) with \(\text{SL}(4, \mathbb{R})\) in view of the above result (denoting in that case \(\iota\) the inclusion \(\text{SO}(3, 1) \longrightarrow \text{SL}(4, \mathbb{R})\)).

**Theorem 1.3** For \(d = 3, 1\) the character variety \(\chi(Bi(d), \text{SU}(3, 1))\) is smooth at \([\iota \circ \rho_{\text{hyp}}]\), with:

- \(\dim_{[\iota \circ \rho_{\text{hyp}}]} \chi(Bi(3), \text{SU}(3, 1)) = 1\).
- \(\dim_{[\iota \circ \rho_{\text{hyp}}]} \chi(Bi(1), \text{SU}(3, 1)) = 2\).

The proof of Theorem 1.3 is in two steps. First we compute a linear approximation to the character variety \(\chi(\Gamma, G)\), usually called the Zariski tangent space, which provides an upper bound for the dimension of any smooth family of deformations (see Sect. 2.1). This has a somewhat sophisticated description as a cohomology group, denoted \(H^1(\Gamma, g)_\rho\), but is very simple to compute in practice, given a presentation of \(\Gamma\) and a parametrization of \(G\) (see Sect. 2.2).

The second step, in general much harder, is to produce a smooth family of representations realizing the upper bound given by the first step. In the present case, when \(d = 1\) the upper bound is 2 (see Proposition 5.1) and the Dunbar–Meyerhoff result (first item of Theorem 1.1) already provides a smooth 2-dimensional family of deformations of \(Bi(1)\) into \(\text{SO}(3, 1)\), giving the following:
Corollary 1.4 Any deformation of Bi(1) into SU(3, 1) or SL(4, \mathbb{R}) is conjugate to one inside SO(3, 1).

For the second step of proof of Theorem 1.3 when \( d = 3 \) we produce in Sect. 5.2 an explicit smooth family of deformations which realizes the upper bound, in this case 1. (We also produced an explicit 2-dimensional family of deformations when \( d = 1 \) but did not include it in the paper since it turns out to be conjugate into SO(3, 1)).

We also compute the dimension of the Zariski tangent spaces \( H^1(\text{Bi}(d), \mathfrak{sl}(4, \mathbb{R}))|_{\rho_{\text{hyp}}} \) when \( d = 2, 5, 6, 7, 11, 15, 19 \) in Proposition 5.1, but we do not know whether \( [\iota \circ \rho_{\text{hyp}}] \) is a smooth point in those cases, the computational aspects involved in the second step seeming currently out of reach for character varieties of dimension at least three. (The values in this list are those for which palatable presentations were computed by Swan, see Sect. 3 and Swan [30]).

We now discuss some related results about the existence of projective deformations of hyperbolic 3-manifold and orbifold groups, before turning to the questions of discreteness and faithfulness of deformations.

Projective deformations of cusped hyperbolic 3-manifolds: when \( \Gamma = \pi_1(M) \) with \( M \) a complete, finite-volume hyperbolic 3-manifold, the deformations of the holonomy representation \( \rho_{\text{hyp}} : \Gamma \longrightarrow \text{SO}(3, 1) \) into \( \text{SL}(4, \mathbb{R}) \) are in fact better understood when \( M \) is non-compact. More specifically, under the additional assumption that \( M \) is infinitesimally projectively rigid relative to its boundary (i.p.r.r.b.), Ballas–Danciger–Lee showed that \( [\iota \circ \rho_{\text{hyp}}] \) is a smooth point of \( \chi(\Gamma, \text{SL}(4, \mathbb{R})) \), its conjugacy class \( [\iota \circ \rho_{\text{hyp}}] \) is a smooth point of \( \chi(\Gamma, \text{SL}(4, \mathbb{R})) \), and \( \dim_{[\iota \circ \rho_{\text{hyp}}]} \chi(\Gamma, \text{SL}(4, \mathbb{R})) = 3k \) where \( k \) is the number of cusps of \( M \). (See Theorem 3.2 of Ballas et al. [4]; the dimension count appears in the proof of Lemma 3.6 therein). The i.p.r.r.b. condition, introduced by Heusener–Porti in [19], informally means that the relevant cohomology group of each cusp injects into the cohomology group of the manifold (see [19] or [4] for a precise definition). Heusener–Porti showed in particular that this condition is satisfied when \( \Gamma = \Gamma_8 \) or \( \Gamma_{WL} \), the fundamental groups of the figure-eight knot complement and Whitehead link complement respectively (as well as infinitely many Dehn fillings of each).

Combining these results, we see that the character varieties \( \chi(\Gamma_8, \text{SL}(4, \mathbb{R})) \) and \( \chi(\Gamma_{WL}, \text{SL}(4, \mathbb{R})) \) are smooth at \( [\iota \circ \rho_{\text{hyp}}] \), with dimension 3 and 6 respectively. As before, by Theorem 2.2 of Cooper et al. [11] (Theorem 1.2 above), one can replace \( \text{SL}(4, \mathbb{R}) \) with \( \text{SU}(3, 1) \) in these statements. These groups are closely related to the small Bianchi groups we consider. Indeed, Riley’s original parametrization showed that the fundamental group \( \Gamma_8 \) of the figure-eight knot complement is a subgroup of \( \text{Bi}(3) \) (with index 12), whereas \( \text{Bi}(1) \) contains the fundamental group \( \Gamma_{WL} \) of the Whitehead link complement (and \( \text{PGL}(2, \mathbb{C}_1) \) contains the Borromean rings group, see Sect. 9.2 of Maclachlan and Reid [24]).

Projective deformations of closed hyperbolic 3-manifolds: the situation is much more mysterious when \( \Gamma = \pi_1(M) \) with \( M \) a compact hyperbolic 3-manifold, as explored in [11]. There the authors tested the 4500 two-generator manifold groups in the Hodgson–Weeks census, and found that only 61 of these admit infinitesimal deformations into \( \text{SL}(4, \mathbb{R}) \) (that is, the Zariski-tangent space \( H^1(\Gamma, \mathfrak{sl}(4, \mathbb{R})) \) is non-zero, see Sect. 2.2). Of these, they proved rigorously that 25 admit deformations into \( \text{SL}(4, \mathbb{R}) \) whereas 3 do not. To quote the abstract of Cooper et al. [11]: “The set of closed hyperbolic manifolds for which one can do this seems mysterious”.

However, in the compact case the question of discreteness and faithfulness of deformations is well understood, by a general result of Guichard ([18], see also Theorem 1.2 of Cooper et al. [11] in the present case). Guichard’s theorem states that, given a convex-cocompact
subgroup $\Gamma$ of a rank 1 subgroup $H$ of a semisimple Lie group $G$ with finite center, the inclusion $\Gamma \hookrightarrow G$ has a neighborhood in $\text{Hom}(\Gamma, G)$ consisting entirely of discrete and faithful representations. Recall that $\Gamma$ is convex-cocompact in $H$ if there exists a (non-empty) $\Gamma$-invariant convex subset $C$ of the symmetric space $X$ associated to $H$ such that $\Gamma$ acts cocompactly on $C$. In particular if $\Gamma$ is a cocompact lattice in $H$ then it is convex-cocompact (taking $C = X$). Note that this fails in general when $\Gamma$ is a non-cocompact lattice in $H$, for example for the Dehn surgery deformations described above (and for the Bianchi groups with $d = 1, 3$, see Theorem 1.5 below).

Projective deformations of hyperbolic orbifolds: when $\Gamma$ is a lattice with torsion in $\text{SL}(2, \mathbb{R})$ or $\text{SL}(2, \mathbb{C})$ (so, $\Gamma = \pi^{\text{hyp}}(Q)$ with $Q$ a complete, finite-volume hyperbolic 2- or 3-orbifold), Porti studied in [27] deformations of $\Gamma$ into complex semi-simple Lie groups $G$ including $\text{SL}(n, \mathbb{C})$, where the original representation is obtained by post-composing the holonomy representation with the irreducible representation $\text{SL}(2, \mathbb{C}) \to \text{SL}(n, \mathbb{C})$ (so, analogous to the Fuchsian representation defining the Hitchin component when $G = \text{SL}(n, \mathbb{R})$, see e.g. [2, 23] for hyperbolic 2- and 3-orbifolds). When $\Gamma$ is a Bianchi group and $G = \text{SL}(4, \mathbb{C})$, Porti’s results apply to a point in a different component of the character variety than the one we consider here. In particular, he shows that $\chi(\text{Bi}(d), \text{SL}(4, \mathbb{C}))$ is smooth at that point, with dimension three times the number of cusps when all boundary components are tori, that is $3h_d$ when $d \neq 1, 3$.

Bending deformations: it is also well known that the modular surface $H^2_{\mathbb{R}}/\text{PSL}(2, \mathbb{Z})$ is embedded in the Bianchi orbifold $H^2_{\mathbb{R}}/\text{Bi}(d)$ for all $d \neq 1, 3$ (see e.g. Section 6.3.2 of Fine [13]), so that $\text{Bi}(d)$ admits a 1-parameter family of bending deformations into $\text{SL}(4, \mathbb{R})$, in the sense of Johnson–Millson [20]. Therefore there is at least one dimension worth of smooth deformations of $\text{Bi}(d)$ into $\text{SL}(4, \mathbb{R})$, transverse to the Dehn surgery deformations of Dunbar–Meyerhoff, for all $d \neq 1, 3$. In particular, when $d = 7$ this gives: $\dim_{\rho_{\text{hyp}}} \chi(\text{Bi}(7), \text{SL}(4, \mathbb{R})) \geq 3$ (assuming smoothness at $[\rho_{\text{hyp}}]$ for notational convenience).

Deformations of related Coxeter groups: the Bianchi groups $\text{Bi}(1)$ and $\text{Bi}(3)$ are also well known to be commensurable to several Coxeter groups (see e.g. Figures 13.2 and 13.3 of Maclachlan and Reid [24]); in particular $\text{Bi}(1)$ (resp. $\text{Bi}(3)$) is the index-2 orientation-preserving subgroup of a Coxeter group $\Gamma_1$ (resp. $\Gamma_3$), whose diagrams are pictured in Figs. 1 and 2. Projective deformations of tetrahedral Coxeter groups (that is, Coxeter groups in $\text{Isom}(H^3_{\mathbb{R}})$ admitting a tetrahedron as fundamental domain) were studied in [25], and it follows from the main result there that the corresponding Coxeter groups $\Gamma_1$ (resp. $\Gamma_3$) have no deformations (resp. a smooth 1-parameter family of deformations) into $\text{SL}(4, \mathbb{R})$. (According to Sect. 7.4 of Ballas et al. [4], the result for $\Gamma_3$ also follows from Benoist [5]). This shows in particular that $\dim_{\rho_{\text{hyp}}} \chi(\text{Bi}(3), \text{SU}(3, 1)) \geq 1$; our result shows that any deformation of $\text{Bi}(3)$ into $\text{SU}(3, 1)$ or $\text{SL}(4, \mathbb{R})$ is induced by a deformation of the Coxeter group $\Gamma_3$ (at least, in the irreducible component containing $[\rho_{\text{hyp}}]$).

Discrete and faithful deformations: from the point of view of geometric structures, discrete and faithful representations are of particular interest. Indeed they correspond to uniformizable (or complete) geometric structures, that is those structures whose associated developing map is a covering map, in the case where the model space carries an invariant Riemannian metric. More specifically, given a pair $(G, X)$ with $G$ a Lie group acting transitively by isometries on a Riemannian manifold $X$, for any manifold $M$ there is a bijection between the space of (marked) complete $(G, X)$-structures on $M$ and the discrete and faithful character variety $\chi_{DF}(\pi_1(M), G)$, which is the space of conjugacy classes of discrete and faithful representations of $\pi_1(M)$ into $G$. This bijection associates to a complete $(G, X)$-structure...
on $M$ its holonomy representation $\rho : \pi_1(M) \rightarrow G$ (see e.g. Section 8.1 of Goldman [15] for more details).

From this point of view the lattice embedding $\rho_{hyp} : \Gamma \rightarrow SO(3, 1)$ is the holonomy representation corresponding to the (unique) complete, finite-volume hyperbolic structure on the orbifold $Q = H^3_\mathbb{R} \Gamma$. Moreover it is well known that the $\mathbb{R}$-Kleinian embedding $t \circ \rho_{hyp} : \Gamma \rightarrow SU(3, 1)$ is the holonomy of a complete, complex hyperbolic structure on the tangent (orbi-)bundle of $Q$, in other words that $H^3_\mathbb{C} / t(\rho_{hyp}(\Gamma))$ is diffeomorphic to the tangent (orbi-)bundle of $Q$; see Sect. 3.3 for more details on orbifolds and tangent orbibundles.

**Theorem 1.5** For $d = 1, 3$, $[t \circ \rho_{hyp}]$ is the only point in its irreducible component of $\chi(Bi(d), SU(3, 1))$ comprising discrete and faithful representations. In particular, the complete complex hyperbolic structure on the tangent orbibundle to the corresponding Bianchi orbifold is locally rigid.

Theorem 1.5 with $d = 3$ will follow from a detailed analysis of the explicit deformations obtained in Sect. 5.2, see Proposition 5.2. When $d = 1$ it follows from Corollary 1.4, since the Dehn surgery deformations into SO(3, 1) are known to be non-discrete or non-faithful.

The paper is organized as follows. In Sect. 2 we review generalities about representation and character varieties and in Sect. 3 background on hyperbolic spaces and manifolds/orbifolds. In Sect. 4 we recall Swan’s presentations of small Bianchi groups, then in Sect. 5 specialize to deformations of Bianchi groups into SU(3, 1) and SL(4, $\mathbb{R}$), compute the corresponding Zariski-tangent spaces for $d = 2, 5, 6, 7, 11, 15, 19$ and explicit deformations of Bi(3), proving the results announced above.

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2 Representation varieties, character varieties and smoothness

Let $\Gamma$ be a finitely generated group and $G$ a linear algebraic group, more specifically a $\mathbb{Q}$-algebraic subgroup of $GL(n, \mathbb{R})$, or the set of real points in $GL(n, \mathbb{C})$ of a $\mathbb{Q}$-algebraic group (e.g. $G = SL(n, \mathbb{R})$, $SO(n, 1)$ or $SU(n, 1)$).

The representation variety $\text{Hom}(\Gamma, G)$ is the space of all representations of $\Gamma$ into $G$. Since $G$ is a linear algebraic group and $\Gamma$ is finitely generated, $\text{Hom}(\Gamma, G)$ is an affine algebraic variety.

The character variety $\chi(\Gamma, G)$ is the (maximal Hausdorff quotient of) the topological quotient $\text{Hom}(\Gamma, G)/G$ of the representation variety under $G$ (acting on representations by postcomposition by inner automorphisms). The latter is in fact also an algebraic variety, and can be alternatively described as the quotient in the sense of geometric invariant theory; see [1] for more details on character varieties in real forms of $SL(n, \mathbb{C})$.

2.1 Infinitesimal deformations and cohomology

We start with an elementary computation to motivate the definition of the relevant cohomology groups due to Weil [32]; see [6] or Sect. 3.1 of Heusener and Porti [19] for more details. Given a finitely generated group $\Gamma$ and a linear Lie group $G$ as above, assume that $\rho_t : \Gamma \rightarrow G$ (with $t \in (\varepsilon, \varepsilon)$ for some $\varepsilon > 0$) is a $C^1$ family of representations, in the sense that for all $\gamma \in \Gamma$, the map $t \mapsto \rho_t(\gamma)$ is $C^1$. Then the tangent vectors to the various $\rho_t(\gamma)$ ($\gamma \in \Gamma$), appropriately (right-) translated into the Lie algebra $\mathfrak{g}$ will satisfy the following cocycle condition.

Cocycles: denote, for any $\gamma \in \Gamma$ and $t \in (\varepsilon, \varepsilon)$, $d_{\rho_0}(t, \gamma) = \rho_t(\gamma)\rho_0(\gamma)^{-1}$ and $h_{\rho_0}(\gamma) = \frac{d}{dt} |_{t=0} d_{\rho_0}(t, \gamma) \in \mathfrak{g}$. Then, for any $\gamma, \gamma' \in \Gamma$ we have:

\[ d_{\rho_0}(t, \gamma' \gamma') = \rho_t(\gamma' \gamma')\rho_0(\gamma' \gamma')^{-1} = \rho_t(\gamma)\rho_t(\gamma')\rho_0(\gamma')^{-1}\rho_0(\gamma)^{-1} = \left[\rho_t(\gamma)\rho_0(\gamma)^{-1}\right] \left[\rho_0(\gamma)\rho_t(\gamma')\rho_0(\gamma')^{-1}\rho_0(\gamma)^{-1}\right] = d_{\rho_0}(t, \gamma)\rho_0(\gamma) d_{\rho_0}(t, \gamma')\rho_0(\gamma')^{-1} \]

Therefore, taking derivatives and evaluating at $t = 0$ we get:

\[ h_{\rho_0}(\gamma' \gamma') = h_{\rho_0}(\gamma) + \rho_0(\gamma) h_{\rho_0}(\gamma')\rho_0(\gamma)^{-1} = h_{\rho_0}(\gamma) + \text{Ad} \rho_0(\gamma) \cdot h_{\rho_0}(\gamma') \]

where:

\[ \text{Ad} \rho_0 : \Gamma \rightarrow \text{GL}(\mathfrak{g}) \]

\[ \gamma \mapsto [v \mapsto \rho_0(\gamma)v\rho_0(\gamma)^{-1}] \]

Deformations by conjugation: as a special case, if $(g_t)$ ($t \in (\varepsilon, \varepsilon)$) is a $C^1$-path in $G$ with $g_0 = e$, we obtain a $C^1$-deformation $(c_t)$ of $\rho_0$ by defining: $c_t(\gamma) = g_t\rho_0(\gamma)g_t^{-1}$ for any $\gamma \in \Gamma$. Then:

\[ d_{c_0}(t, \gamma) = g_t\rho_0(\gamma)g_t^{-1}\rho_0(\gamma)^{-1}, \text{ and} \]

\[ h_{c_0}(\gamma) = X - \rho_0(\gamma)X\rho_0(\gamma)^{-1} = X - \text{Ad} \rho_0(\gamma) \cdot X, \]

where: $X = \frac{d}{dt} |_{t=0} g_t \in \mathfrak{g}$. This motivates the following definitions:

(Cocycles) $Z^1(\Gamma, \mathfrak{g})_\rho = \{u_1 : \Gamma \rightarrow \mathfrak{g} | (\forall \gamma, \delta \in \Gamma) u_1(\gamma \delta) = u_1(\gamma) + \text{Ad} \rho(\gamma) \cdot u_1(\delta)\}$.

(Coboundaries) $B^1(\Gamma, \mathfrak{g})_\rho = \{v_1 : \Gamma \rightarrow \mathfrak{g} | (\exists X \in \mathfrak{g}) (\forall \gamma \in \Gamma) v_1(\gamma) = X - \text{Ad} \rho(\gamma) \cdot X\}$.
(Cohomology groups) \( H^1(\Gamma, g)_\rho = Z^1(\Gamma, g)_\rho / B^1(\Gamma, g)_\rho. \)

Elements of the real vector space \( H^1(\Gamma, g)_\rho \) are called *infinitesimal deformations* of \( \rho \) into \( G \). The above computation shows that any \( C^1 \) family of deformations of \( \rho \) into \( G \), transverse to deformations by conjugation, gives rise to a non-zero element of \( H^1(\Gamma, g)_\rho \), but the converse is false in general, as there can be non-zero vectors in \( H^1(\Gamma, g)_\rho \) which are not tangent to a \( C^1 \) family of deformations, see for example [11]. More generally the dimension of the real vector space \( H^1(\Gamma, g)_\rho \) gives an upper bound for the dimension of any smooth family of pairwise non-conjugate deformations of \( \rho \) into \( G \).

### 2.2 Zariski tangent spaces and smoothness of character varieties

See Sect. 2 of Cooper et al. [11] and Sect. 3.1 of Ballas et al. [4] for related discussions. Assuming for simplicity that \( \Gamma \) is finitely presented and \( G = \text{SL}(n, \mathbb{R}) \), the affine variety \( \text{Hom}(\Gamma, G) \) has the following concrete description. Consider a finite presentation \( \langle g_1, ..., g_m \mid w_1(g_1, ..., g_m) = ... = w_k(g_1, ..., g_m) = 1 \rangle \) of \( \Gamma \), with \( w_1, ..., w_k \) words in the letters \( g_1, ..., g_m \) and their inverses. Then a representation \( \rho : \Gamma \to G \) is determined by an \( m \)-tuple \( (A_1, ..., A_m) \in G^m \) satisfying the equations \( w_1(A_1, ..., A_m) = ... = w_k(A_1, ..., A_m) = \text{Id} \). Moreover, since \( G = \text{SL}(n, \mathbb{R}) \supset M(n, \mathbb{R}) \), identifying \( M(n, \mathbb{R}) \) with \( \mathbb{R}^{n^2} \) will identify \( \text{Hom}(\Gamma, G) \) with \( V = f^{-1}(\{0\}) \), where:

\[
f : (\mathbb{R}^n)^m \to \mathbb{R}^m \times (\mathbb{R}^n)^k
(A_1, ..., A_m) \mapsto (\det A_1 - 1, ..., \det A_m - 1, w_1(A_1, ..., A_m) - \text{Id}, ..., w_k(A_1, ..., A_m) - \text{Id})
\]

From differential geometry we know that, if \( f \) is a submersion, \( V \) is smooth with tangent space \( T_p V = \ker df_p \) for any \( p \in V \). However this is often not the case, in particular it can never happen if \( k \geq m \), which will be the case for all the presentations we use in this paper.

The linear subspace \( T_p^2 V = \ker df_p \subset (\mathbb{R}^n)^m \) is usually called the Zariski tangent space to \( V \) at \( p \); by construction it contains the tangent vector at \( p \) to any smooth curve through \( p \) in \( V \), hence its dimension is an upper bound for the dimension of any smooth irreducible component of \( V \) containing \( p \).

It is well known that \( T_p^2 \text{Hom}(\Gamma, G) \cong Z^1(\Gamma, g)_\rho \) as real vector spaces (see e.g. p. 152 of Weil [32]). In particular, if the conjugation orbit \( C_\rho = \{g \rho g^{-1} \mid g \in G\} \subset \text{Hom}(\Gamma, G) \) is smooth at \( \rho \) then: \( H^1(\Gamma, g)_\rho \cong T_p^2 \text{Hom}(\Gamma, G) / TC_\rho \). This holds at any irreducible representation as follows.

**Lemma 2.1** Let \( \Gamma \) be a finitely generated group, \( G \) a Lie group with discrete center and \( \rho : \Gamma \to G \) an irreducible representation. Then the conjugation orbit \( C_\rho = \{g \rho g^{-1} \mid g \in G\} \subset \text{Hom}(\Gamma, G) \) is smooth at \( \rho \) with dimension \( \dim G \). In particular, \( \dim H^1(\Gamma, g)_\rho = \dim T_p^2 \text{Hom}(\Gamma, G) - \dim G \).

**Proof** Assume \( \rho(\Gamma) \) is generated by \( A_1, ..., A_m \in G \), and consider the map \( \Phi : G \to G^m \) defined by \( \Phi(g) = (g A_1 g^{-1}, ..., g A_m g^{-1}) \). We claim that \( \Phi \) is an immersion at the identity \( e \in G \), and the result follows.

To prove the claim, recall that for a fixed \( h \in G \), the map \( \varphi : G \to G \) defined by \( \varphi(g) = ghg^{-1} \) has derivative at \( e \) \( d_e \varphi : g \to g \) given by \( X \mapsto Xh - hX \) (assuming for concreteness that \( G \) is linear; otherwise replace \( Xh - hX \) with \( \text{Ad}_h(X) \)). Therefore \( \ker d_e \varphi = \mathfrak{z}(h) \), the Lie algebra of the centralizer \( Z(h) \).

Hence the derivative of \( \Phi \) at \( e \), \( d_e \Phi : g \to g^m \) is given by \( d_e \Phi(X) = (X A_1 - A_1 X, ..., X A_m - A_m X) \) and its kernel is: \( \ker d_e \Phi = \mathfrak{z}(A_1) \cap \cdots \cap \mathfrak{z}(A_m) = \mathfrak{z}(\rho) \). Since \( \rho \)
is irreducible, $\mathfrak{z}(\rho) \subset \mathfrak{z}(G)$ and the latter is $\{0\}$ since $G$ has discrete center. This proves the claim. \hfill $\square$

We will give concrete examples of these computations in Sect. 5.1. Summarizing the results of this discussion, we use the following criterion to show smoothness of the character variety:

**Proposition 2.1** Let $\Gamma$ be a finitely generated group, $G$ a Lie group with discrete center and $\rho: \Gamma \longrightarrow G$ an irreducible representation. Denote $k = \dim H^1(\Gamma, g)_\rho$ and assume that there exist a neighborhood $U$ of $0$ in $\mathbb{R}^{k}$ and a smooth family of representations $\rho_t: \Gamma \longrightarrow G$ ($t \in U$) such that $\rho_0 = \rho$ and $\rho_t$ not conjugate to $\rho_{t'}$ for any $t \neq t' \in U$. Then $\chi(\Gamma, G)$ is smooth at $\rho$ with dimension $k$.

## 3 Hyperbolic spaces, manifolds and orbifolds

### 3.1 Hyperbolic spaces

We now give a brief summary of basic definitions and facts about real and complex hyperbolic geometry, and refer the reader to Goldman [16] or Chen and Greenberg [9] for more details. In this section we take $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ (the same description holds with $\mathbb{K} = \mathbb{H}$, the Hamiltonian quaternions, with the usual caveats of linear algebra over a non-commutative division algebra, see [9]).

**Projective models of $H^n_\mathbb{K}$:** denote $\mathbb{K}^{n,1}$ the vector space $\mathbb{K}^{n+1}$ endowed with a Hermitian form $\langle \cdot, \cdot \rangle$ of signature $(n, 1)$. Define $V^- = \{ Z \in \mathbb{K}^{n,1} \mid \langle Z, Z \rangle < 0 \}$ and $V^0 = \{ Z \in \mathbb{K}^{n,1} \mid \langle Z, Z \rangle = 0 \}$. Let $\pi: \mathbb{K}^{n+1} - \{ 0 \} \longrightarrow \mathbb{K}P^n$ denote projectivization. One may then define (real or complex) hyperbolic $n$-space $H^n_\mathbb{K}$ as $\pi(V^-) \subset \mathbb{K}P^n$, with the distance $d$ (corresponding to the Bergman metric when $\mathbb{K} = \mathbb{C}$) given by:

$$\cosh^2 \frac{1}{2} d(\pi(X), \pi(Y)) = \frac{\lVert \langle X, Y \rangle \rVert^2}{\langle X, X \rangle \langle Y, Y \rangle}$$

(3.1)

The boundary at infinity $\partial_{\infty}H^n_\mathbb{K}$ is then naturally identified with $\pi(V_0)$.

**Isometries:** it is clear from (3.1) that $PU(n, 1)$ acts by isometries on $H^n_\mathbb{K}$, denoting $U(n, 1)$ the subgroup of $GL(n + 1, \mathbb{K})$ preserving the Hermitian form, and $PU(n, 1)$ its image in $PGL(n + 1, \mathbb{K})$. ($PU(n, 1)$ is usually denoted $PO(n, 1)$ when $\mathbb{K} = \mathbb{R}$).

It turns out that $PO(n, 1)$ is the full group of isometries of $H^n_\mathbb{R}$. In dimensions $n = 2$ and $3$ the orientation-preserving isometries enjoy an alternative description as $\text{Isom}^+(H^n_\mathbb{R}) \simeq PSL(2, \mathbb{R})$ and $\text{Isom}^+(H^3_\mathbb{C}) \simeq PSL(2, \mathbb{C})$.

When $\mathbb{K} = \mathbb{C}$, $PU(n, 1)$ is the group of holomorphic isometries of $H^n_\mathbb{C}$, and the full group of isometries is $PU(n, 1) \ltimes \mathbb{Z}/2$, where the $\mathbb{Z}/2$ factor corresponds to a real reflection (see below). Any $g \in PU(n, 1) \setminus \{ \text{Id} \}$ is of one of the three following types:

- **elliptic** if $g$ has a fixed point in $H^n_\mathbb{R}$,
- **parabolic** if $g$ has (no fixed point in $H^n_\mathbb{R}$ and) exactly one fixed point on $\partial_{\infty}H^n_\mathbb{R}$,
- **loxodromic:** if $g$ has (no fixed point in $H^n_\mathbb{R}$ and) exactly two fixed points in $\partial_{\infty}H^n_\mathbb{R}$.

These three types can be distinguished by the following properties of their matrix lifts in $U(n, 1)$ (Theorem 3.4.1 of Chen and Greenberg [9]). Let $g \in PU(n, 1) \setminus \{ \text{Id} \}$ and $A \in U(n, 1)$ a lift of $g$. Then:

- $g$ is elliptic $\iff$ $A$ is diagonalizable with all eigenvalues of unit modulus,
\begin{itemize}
  \item $g$ is parabolic $\iff$ $A$ is not diagonalizable, and
  \item $g$ is loxodromic $\iff$ $A$ is diagonalizable with all but two eigenvalues of unit modulus.
\end{itemize}

**Totally geodesic subspaces:** a complex $k$-plane is a projective $k$-dimensional subspace of $\mathbb{C}P^n$ intersecting $\pi(V^-)$ non-trivially (so, it is an isometrically embedded copy of $H^k_{\mathbb{C}} \subset H^k_{\mathbb{C}}$).

A real $k$-plane is the projective image of a totally real $(k + 1)$-subspace $W$ of $\mathbb{C}^{n,1}$, i.e., a $(k + 1)$-dimensional real linear subspace such that $\langle v, w \rangle \in \mathbb{R}$ for all $v, w \in W$ (so, it is an isometrically embedded copy of $H^k_{\mathbb{R}} \subset H^k_{\mathbb{R}}$). Every real $n$-plane in $H^k_{\mathbb{R}}$ is the fixed-point set of an (antiholomorphic) isometry of order $2$ called a real reflection or $\mathbb{R}$-reflection. The prototype of such an isometry is the map given in affine coordinates by $(z_1, \ldots, z_n) \mapsto (\overline{z_1}, \ldots, \overline{z_n})$; this is an isometry provided that the Hermitian form has real coefficients.

It is a basic feature of complex hyperbolic space that any totally geodesic subspace of $H^k_{\mathbb{C}}$ is a real or complex $k$-plane for some $0 \leq k \leq n$ (Proposition 2.5.1 of Chen and Greenberg [9]).

### 3.2 Hyperbolic manifolds and orbifolds

From the point of view of geometric structures, a (real or complex) hyperbolic $n$-manifold is a quotient $M = H^n_{\mathbb{K}} / \Gamma$ with $(\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and) $\Gamma$ a discrete, torsion-free subgroup of $\text{Isom}(H^n_{\mathbb{K}})$. In particular, the quotient map $H^n_{\mathbb{K}} \to M$ is a covering map, hence $M$ is a smooth manifold and $\pi_1(M) \simeq \Gamma$. When $\mathbb{K} = \mathbb{R}$ this is equivalent to $M$ being a complete Riemannian $n$-manifold with constant sectional curvature $-1$. The discrete subgroup $\Gamma < \text{Isom}(H^n_{\mathbb{K}})$ is called the holonomy of the hyperbolic structure.

Without the torsion-free assumption on $\Gamma$ the quotient $H^n_{\mathbb{K}} / \Gamma$ is called a (real or complex) hyperbolic $n$-orbifold. It is not smooth, but is an orbifold in the sense that certain points (the orbifold points) are allowed to have neighborhoods diffeomorphic to a quotient of $\mathbb{R}^n$ by a finite isotropy group (rather than just $\mathbb{R}^n$). Likewise, the tangent orbibundle of an orbifold is defined analogously to the tangent bundle of a manifold, but at the orbifold points the tangent space $\mathbb{R}^n$ is replaced by its quotient under the isotropy group. See Sect. 3.1 of Caramello [8] for more details on orbifolds and tangent orbibundles.

### 3.3 $\mathbb{R}$-Kleinian embeddings and tangent bundles

The following is well known, see for example Sect. 2.5 of Goldman et al. [17] in dimension 2. It is interesting to note that the result holds in all dimensions, as opposed to the analogous interpretation of $\mathbb{C}$-Fuchsian representations which is only valid in dimension 2, by a coincidence of dimensions. We include a short proof, essentially the same as in [17], for the reader’s convenience.

**Theorem 3.1** Let $n \geq 2$, $M$ a complete hyperbolic $n$-manifold, $\rho_{\text{hyp}} : \Gamma = \pi_1(M) \to \text{SO}(n,1)$ the corresponding holonomy representation and $\iota$ the inclusion $\text{SO}(n,1) \to \text{SU}(n,1)$. Then $H^n_{\mathbb{C}}/\iota(\rho_{\text{hyp}}(\Gamma))$ is diffeomorphic to the tangent bundle $TM$.

**Proof** The totally real $n$-plane $H^n_{\mathbb{R}}$ is a Lagrangian subspace of $X = H^n_{\mathbb{C}}$, so for any $p \in L$, $J \cdot T_pL$ is orthogonal to $T_pL$, where $J$ denotes the complex structure on $X$. In particular, $J \cdot T_pL \subset T_pX$ is the normal subspace to $L$ at $p$, hence $J$ induces an $\text{SO}(n,1)$-equivariant diffeomorphism $TL \simeq NL$ (the latter denoting the normal bundle to $L$ in $X$). On the other hand, orthogonal projection onto $L$ induces a diffeomorphism $X \simeq NL$, also $\text{SO}(n,1)$-equivariant. Therefore: $TM \simeq TL/\rho_{\text{hyp}}(\Gamma) \simeq NL/\iota(\rho_{\text{hyp}}(\Gamma)) \simeq X/\iota(\rho_{\text{hyp}}(\Gamma))$. \hfill $\square$
The same result holds, with the same proof, replacing $M$ by a hyperbolic $n$-orbifold and its tangent bundle by its tangent orbibundle.

4 Bianchi groups

For any squarefree $d \geq 1$ we consider the Bianchi group $\text{Bi}(d) = \text{PSL}(2, \mathcal{O}_d) < \text{PSL}(2, \mathbb{C})$, where $\mathcal{O}_d$ denotes as usual the ring of integers of $\mathbb{Q}(i\sqrt{d})$. Recall that $\mathcal{O}_d = \mathbb{Z}[\tau]$, where $\tau = i\sqrt{d}$ if $d \equiv 1, 2 \mod 4$ and $\tau = \frac{1+i\sqrt{d}}{2}$ if $d \equiv 3 \mod 4$. Swan found explicit presentations for most Bianchi groups $\text{Bi}(d)$ with $d \leq 19$ in [30]. We now list Swan’s generators and presentations, for the 1-cusped groups with $d = 1, 2, 3, 7, 11, 19$ and the 2-cusped groups with $d = 5, 6, 15$ (when $d = 3$ we use $\tau = \frac{-1+i\sqrt{3}}{2}$ to unify notation). The generators include for all $d$:

$$T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & \tau & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

as well as, when $d = 1, 3$:

$$L = \begin{pmatrix} \tau^{-1} & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

when $d = 19$:

$$B = \begin{pmatrix} 1 - \tau & 2 & \tau \\ 2 & \tau & \tau \end{pmatrix},$$

when $d = 15$:

$$C = \begin{pmatrix} 4 & 1 - 2\tau & 2\tau \\ 2\tau - 1 & 4 & \tau \end{pmatrix},$$

when $d = 5$:

$$B = \begin{pmatrix} -\tau & 2 & \tau \\ 2 & \tau & \tau \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} -\tau - 4 & 2\tau & \tau - 4 \\ 2\tau & \tau & -4 \end{pmatrix},$$

and when $d = 6$:

$$B = \begin{pmatrix} -1 - \tau & -\tau & \tau \\ 2 & \tau & \tau \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 5 & -2\tau & \tau \\ 2\tau & \tau & 5 \end{pmatrix}.$$

The presentations given by Swan are the following:

\begin{align*}
\text{Bi}(1) &= \langle T, U, L, A \mid [T, U] = L^2 = (TL)^2 = (UL)^2 = (AL)^2 = A^2 = (TA)^3 = (UAL)^3 = 1 \rangle \\
\text{Bi}(3) &= \langle T, U, L, A \mid [T, U] = L^2 = (AL)^2 = A^2 = (TA)^3 = (UAL)^3 = 1, L^{-1}UL = T, L^{-1}TL = T^{-1}U^{-1} \rangle \\
\text{Bi}(2) &= \langle T, U, A \mid [T, U] = A^2 = (TA)^3 = (AU^{-1}AU)^2 = 1 \rangle \\
\text{Bi}(7) &= \langle T, U, A \mid [T, U] = A^2 = (TA)^3 = (ATU^{-1}AU)^2 = 1 \rangle \\
\text{Bi}(11) &= \langle T, U, A \mid [T, U] = A^2 = (TA)^3 = (ATU^{-1}AU)^3 = 1 \rangle \\
\text{Bi}(19) &= \langle T, U, A, B \mid [T, U] = A^2 = (TA)^3 = B^3 = (BT^{-1})^3 = (AB)^2 = (AT - 1UBU^{-1})^2 = 1 \rangle 
\end{align*}
\[ \begin{align*}
\text{Bi}(15) &= \langle T, U, A, C \mid [T, U] = [A, C] = A^2 = (TA)^3 = 1, UCUAT = TAU CU \rangle \\
\text{Bi}(5) &= \langle T, U, A, B, C \mid [T, U] = A^2 = (TA)^3 = B^2 = (AB)^2 \\
&= (ATUBU^{-1})^2 = 1, ACA = TCT^{-1} = UBU^{-1}CB \rangle \\
\text{Bi}(6) &= \langle T, U, A, B, C \mid [T, U] = [A, C] = A^2 = (TA)^3 = B^2 \\
&= (ATB)^3 = (ATUBU^{-1})^3 = 1, CTUB = TBCU \rangle
\end{align*} \]

5 Deformations of Bianchi groups into SU(3, 1)

5.1 Infinitesimal deformations

We first compute the spaces of infinitesimal deformations \( H^1(\text{Bi}(d), \mathfrak{sl}(4, \mathbb{R}))_{\rho_0} \). The principle of the computation is very simple, but requires computing the rank of rather large matrices (from 48 by 67 to 80 by 133 in the cases considered here) so are best handled with formal computation software such as Mathematica or Maple. We write the details of the computation when \( d = 7 \); the others are similar but may have more generators and/or relations. Recall Swan’s presentation:

\[ \text{Bi}(7) = \langle T, U, A \mid [T, U] = A^2 = (TA)^3 = (ATU^{-1}AU)^2 = 1 \rangle \]

To decrease degrees, since we are considering linear representations, we may rewrite the relations as: Rel1 : \( TU - UT \), Rel2 : \( A^2 - \text{Id} \), Rel3 : \( TAT - A^{-1}T^{-1}A^{-1} \), Rel4 : \( ATU^{-1}AU - U^{-1}A^{-1}UT^{-1}A^{-1} \). The following elements of \( \text{SO}(3, 1) \) generate the image of the representation \( \rho_0 \) (they are obtained by composing the generators in \( \text{SL}(2, \mathbb{C}) \) listed above with an isomorphism \( \text{SL}(2, \mathbb{C}) \rightarrow \text{SO}^0(3, 1) \)):

\[
T_0 = \begin{pmatrix}
3/2 & -1/2 & 1 & 0 \\
1/2 & 1/2 & 0 & 1 \\
1 & -1 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad U_0 = \begin{pmatrix}
2 & -1 & 1/2 & \sqrt{7}/2 \\
1 & 0 & 1/2 & \sqrt{7}/2 \\
1/2 & -1/2 & 1 & 0 \\
\sqrt{7}/2 & -\sqrt{7}/2 & 0 & 1
\end{pmatrix},
\]

\[
A_0 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

Consider the polynomial map:

\[
\text{Eqs} : (\mathbb{R}^{16})^3 \rightarrow \mathbb{R}^3 \times (\mathbb{R}^{16})^4
\]

\[
(T, U, A) \mapsto (\det T - 1, \det U - 1, \det A - 1, \text{Rel1}, ..., \text{Rel4})
\]

(Taking the inverse of a matrix is only a rational map, but since we are requiring the determinants to be 1 it is in fact polynomial in this case).

The above presentation then identifies \( \text{Hom}(\text{Bi}(7), \text{SL}(4, \mathbb{R})) \) with \( \text{Eqs}^{-1}((0, ..., 0)) \), and the Zariski tangent space to \( \text{Hom}(\text{Bi}(7), \text{SL}(4, \mathbb{R})) \) at the point \((T_0, U_0, A_0)\) is identified with the kernel of the differential \( d_{(T_0, U_0, A_0)}\)Eqs. Coding the map Eqs and computing the rank of \( d_{(T_0, U_0, A_0)}\)Eqs (a real 48-by-67 matrix) is routine in any formal software system; Maple and Mathematica tell us that in the present case the rank is 30, hence the dimension of the kernel is 48 – 30 = 18. Now we conclude by Lemma 2.1 that the dimension of \( H^1(\text{Bi}(7), \text{SL}(4, \mathbb{R}))_{\rho_0} \) is 18 – \text{dim} \( \text{SL}(4, \mathbb{R}) \) = 18 – 15 = 3. Similar computations give the following:
Table 1  Class number $h_d$ of $\mathbb{Q}(i\sqrt{d})$ for small values of $d$

| $d$ | 1 | 2 | 3 | 5 | 6 | 7 | 10 | 11 | 13 | 14 | 15 | 17 | 19 | 21 |
|-----|---|---|---|---|---|---|----|----|----|----|----|----|----|----|
| $h_d$ | 1 | 1 | 1 | 1 | 2 | 2 | 1 | 2 | 1 | 2 | 4 | 2 | 4 | 1 | 4 |

Proposition 5.1  The space of infinitesimal deformations $H^1(\mathrm{Bi}(d), \mathfrak{sl}(4, \mathbb{R})))_{\rho_0}$ has dimension:

- 1 when $d = 3$,
- 2 when $d = 1$,
- 3 when $d = 2$ or 7,
- 4 when $d = 11$,
- 5 when $d = 19$,
- 7 when $d = 5$ or 15,
- 9 when $d = 6$.

The values of $d$ appearing in this list are those for which palatable presentations were computed by Swan. It is interesting to compare these values with $3h_d$ which, as noted in the introduction, is the actual dimension of the character variety $\chi(\Gamma, \mathrm{SL}(4, \mathbb{R}))$ at $\rho_{\text{hyp}}$ when $\Gamma = \pi_1(M)$ with $M$ a cusped hyperbolic 3-manifold which is infinitesimally projectively rigid relative to its boundary [4]. See Table 1 for values of $h_d$ with small $d$; we include more values of $d$ than covered in the Proposition to give the reader some context.

5.2 Computing the deformation spaces

In [10] a method was introduced for exact computation of character varieties of fundamental groups of low-dimensional manifolds and orbifolds. The method is not algorithmic, in that we do not have a proof that it will always yield a result, but when it does yield a result it is rigorous and has to date been effective for varieties that have dimension at most 2, this being the case for many interesting examples. There is no reason to suppose that it would not be effective for varieties of dimension greater than 2, but we have not attempted this on account of the large volume of data that would have to be processed.

Here, to begin with, is a short summary of the steps for implementing the method. Although the overall plan is not complicated in its conception, any attempt to implement it will encounter some mild technical issues, for which the reader is referred to Cooper et al. [10].

1. Noting that the group relations correspond to a system of polynomial equations in matrix entries of group generators, we can obtain by means of Newton’s iterative method a numerical representation of the group in question to high accuracy. By adding extra constraints we can steer the iteration so that it converges (numerically) to representations where certain traces of elements, chosen as parameters of the variety, have specified values. In this way a sequence of numerical representations is generated, with parameter values evenly spaced rational numbers, with a view to performing polynomial interpolation in step 4 below. For varieties with a single parameter, 20 representations usually suffices, whereas for two parameters we generate a $20 \times 20$ “grid” of representations, providing 20 values independently for each parameter.

2. The representations of the previous step are conjugated to some normal form, so as to guarantee that all matrix entries are verified by the LLL algorithm [21] to be close approximations of algebraic numbers of small degree. In [10, 22] various ways of accomplishing this are described. A good choice of basis depends on the situation at hand, for example...
one can take basis vectors that are eigenvectors of generators, or one can start by putting a generator into rational canonical form.

3. A basis is obtained for the number field generated by the matrix entries, and LLL is used to express each matrix entry of the group generators as a rational linear combination of basis elements. This number field and its basis are functions of the parameters.

4. The rational coefficients obtained in the previous step are subjected to polynomial interpolation, so as to express the coefficients for each matrix entry of the generators as a function of the parameters. In this way we obtain generating matrices whose entries are exact algebraic expressions in parameters and basis elements.

5. Verify by formal manipulation that the generating matrices of the previous step satisfy the group relations. These matrices generate what is known as a tautological representation for the variety: any point of the variety can be evaluated by substituting values for the parameters in the generating matrices. We verify that appropriate values of the parameters give the original holonomy representation.

For this project we computed the full deformation space for $\text{Bi}(3)$. We also computed a 1-dimensional subvariety of the 2-dimensional deformation space for $\text{Bi}(1)$ and a 2-dimensional subvariety of the 3-dimensional deformation space for $\text{Bi}(7)$. We do not use the latter results here so have not included them in the paper.

There now follows a more detailed description of the determination of the 1-dimensional deformation space for $\text{Bi}(3)$; it is hoped that this amount of detail is sufficient for the interested reader to perform similar computations.

The starting point is the holonomy representation $\rho_{\text{hyp}}$ of $\text{Bi}(3)$ into the group of isometries of 3-dimensional hyperbolic space $H^3_{\mathbb{R}}$. We take this target group to be the Lorentz group $O(3, 1)$, as our aim is to deform into the larger group $\text{SL}(4, \mathbb{R})$ (see Fig. 3).

Before attempting actual deformations, a good plan is to calculate the Zariski tangent dimension for this starting representation. As explained in Sect. 5.1, this is purely an exercise in linear algebra; in down-to-earth terms, for each generating matrix in $O(3, 1)$ as above, say $(a_{ij})$, we add a matrix of indeterminates $(h_{ij})$, and then solve the linear system in the $h_{ij}$ given by the requirement that the adjusted generators $(a_{ij} + h_{ij})$ satisfy the group relations to first order, in the spirit of elementary differential calculus (we neglect all terms in the $h_{ij}$ of degree greater than 1). In the case at hand, namely $\text{Bi}(3)$, it was found that the space of solutions had dimension 16, so after factoring out the 15-dimensional subspace of (inessential) solutions given by conjugations in $\text{SL}(4, \mathbb{R})$, we are left with a 1-dimensional
quotient space of essential “linearized deformations” that could lead to actual deformations of \( \rho_{\text{hyp}} \). It is important to realize that in general these linearized deformations are not necessarily integrable to actual deformations; examples of non-integrability are given in [11]. However we now know that the deformation space has dimension at most 1; the actual dimension will be ascertained shortly.

We apply random small perturbations to the generating matrices \( T_0, U_0, A_0, L_0 \), thus destroying the group relations. We can however recover a representation by instructing Newton’s method to converge to matrices which do obey the group relations. Since the Newton iterative process is unlikely to have awareness of the special nature of the original holonomy representation, and since we already hope from the Zariski tangent computation that the variety has dimension 1, we are not surprised when the process converges to a representation \( \rho' \) close to, but not conjugate to \( \rho_{\text{hyp}} \), and indeed from the existence of this \( \rho' \) we infer that the dimension of the deformation space for \( \mathrm{Bi}(3) \) is very likely 1.

Let the images of \( T, U, A, L \) under the new representation \( \rho' \) be \( T'_0, U'_0, A'_0, L'_0 \). One quickly observes various features of the characteristic polynomials of these matrices. In particular, the characteristic polynomials of \( A'_0, L'_0 \) are equal to those of \( A_0, L_0 \), respectively, and furthermore the characteristic polynomials of \( T'_0, U'_0 \) are equal, of form

\[
1 - (2 + t)x + (2 + 2t)x^2 - (2 + t)x^3 + x^4.
\]

It therefore seems reasonable to take \( t \) as a parameter for the 1-dimensional variety. We choose a sequence \( t_1, t_2, \ldots, t_n \) of rational values of \( t \) (typically \( n = 20 \) suffices), and run the Newton process again \( n \) times, forcing it to converge to a sequence of representations \( \rho_1, \rho_2, \ldots, \rho_n \), where \( \rho_i(T) \) has characteristic polynomial

\[
1 - (2 + t_i)x + (2 + 2t_i)x^2 - (2 + t_i)x^3 + x^4.
\]

In order to “recognize” algebraic numbers in the next stage we often require accuracy to a large number \( m \) of significant figures, typically \( m = 500 \) or sometimes even \( m = 1000 \).

The next step relies in an essential way on the LLL lattice reduction algorithm [21], which is implemented in most computer algebra systems. It is the means by which one can journey from the world of floating point numbers to that of exact algebraic numbers. By choosing a suitable basis for \( \mathbb{R}^4 \) for each \( \rho_i \), one can conjugate by the corresponding transition matrix to obtain a representation where LLL detects that all matrix entries are close approximations of algebraic numbers. With the information provided by LLL we identify the number field, dependent on \( \rho_i \) and hopefully of small degree, generated by these algebraic numbers. We also choose a basis for this number field over the rationals, and LLL will express each matrix entry as a rational linear combination of the elements of the basis. There is no “magic formulà" for this step: one has to use specific features of the current situation to find a suitable basis of \( \mathbb{R}^4 \) for the conjugation. In this instance one can use suitable eigenvectors of generators. Several examples are given for this step in [10, 22].

By means of polynomial interpolation over the sequence of 20 representations (here the dimension of the variety is 1), we express each matrix entry as a linear combination of basis elements where both coefficients and basis elements are now rational functions of the parameters. Denoting the parameter of the variety by \( u \), our matrix entries are now in a finite extension of the transcendental extension \( \mathbb{Q}(u) \), which in general can be a nontrivial extension; a welcome surprise for the case \( \mathrm{Bi}(3) \) was that all matrix entries were in the field \( \mathbb{Q}(u) \) itself, see Fig. 4.

The result up to this point was obtained by judicious guesswork, but we establish that the matrices of Fig. 4 do indeed determine the desired deformation space by checking with formal algebraic manipulation that the resulting generating matrices over \( \mathbb{Q}(u) \) satisfy the
Using formal computation software, one can readily check that when \( |u| = 1 \), writing \( s = \text{Re} u \) and \( t = \text{Im} u \), the four generators for \( \rho_u(\text{Bi}(3)) \) preserve the Hermitian form given by the following matrix, where we denote \( f(s, t) = -1 - s + 4s^2 + i(-1 + 4s)t \):

\[
H_u = \begin{pmatrix}
1 & 0 & 0 & 0 \\
\frac{-2 - u + u^2}{u^2} & \frac{1}{u} + u & -1 & \frac{1}{u} + u \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
\rho_u(T) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
\frac{-2 - u + u^2}{u^2} & \frac{1}{u} + u & -1 & \frac{1}{u} + u \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
\rho_u(A) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\frac{-2 - u + u^2}{u^2} & \frac{1}{u} + u & -1 & \frac{1}{u} + u \\
0 & 1 & 0 & 0
\end{pmatrix}
\]

\[
\rho_u(L) = \begin{pmatrix}
1 + u & 0 & 0 & -u \\
\frac{-2}{u + u^2} & \frac{1}{1 + u} & \frac{1}{1 + u} & \frac{u}{1 + u} \\
\frac{-2 - u + u^2}{u^2 + u^2} & \frac{1}{u + u^2} & \frac{1}{u + u^2} & \frac{1}{u + u^2} \\
1 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\rho_u(U) = \begin{pmatrix}
1 + u & 0 & 0 & -u \\
\frac{-2}{u + u^2} & \frac{1}{1 + u} & \frac{1}{1 + u} & \frac{u}{1 + u} \\
\frac{-2 - u + u^2}{u^2 + u^2} & \frac{1}{u + u^2} & \frac{1}{u + u^2} & \frac{1}{u + u^2} \\
1 & 0 & 0 & 0
\end{pmatrix}
\]

Fig. 4 The tautological representation for Bi(3)

...group relations, and that the appropriate value 1 of the parameter \( u \) gives a representation conjugate to the “base” holonomy representation.

We can think of \( \rho_u \) as a family of representations parametrized by a complex number \( u \in \mathbb{C} \setminus \{0, -1\} \), where, as just mentioned, the original holonomy representation \( \rho_{\text{hyp}} \) is recovered up to conjugacy by setting \( u = 1 \).

**Proposition 5.2** 1. For any \( u \in \mathbb{C} \setminus \{0, -1\} \), \( \rho_u : \text{Bi}(3) \rightarrow \text{SL}(4, \mathbb{C}) \) is a representation, i.e. the matrices \( \rho_u(T) \), \( \rho_u(U) \), \( \rho_u(A) \) and \( \rho_u(L) \) satisfy the relations in the Swan presentation of Bi(3).

2. When \( u = 1 \), \( \rho_1 = \rho_{\text{hyp}} \); for any \( u, u' \in \mathbb{C} \setminus \{0, -1\} \) with \( u' \neq \{u, 1/u\} \), \( \rho_u \) is not conjugate to \( \rho_{u'} \).

3. For any \( u \in \text{U}(1) \setminus \{-1\} \), \( \rho_u \) preserves a Hermitian form \( H_u \), with signature \((3, 1)\) when \( \text{Re} u > 1/4 \) and \((4, 0)\) when \( \text{Re} u < 1/4 \). In other words, for \( u \in \text{U}(1) \setminus \{-1\} \) the image of \( \rho_u \) is contained in \( \text{SU}(3, 1) \) when \( \text{Re} u > 1/4 \) and \( \text{SU}(4) \) when \( \text{Re} u < 1/4 \).

4. For any \( u \in \text{U}(1) \setminus \{-1\} \), \( \rho_u(T) \) and \( \rho_u(U) \) are elliptic; in particular \( \rho_u \) is not discrete or not faithful.

5. For any \( u \in \text{U}(1) \) with \( 1/4 < \text{Re} u < 1 \), \( \rho_u(\text{Bi}(3)) \) is Zariski-dense in \( \text{SU}(3, 1) \).

**Proof** 1. This follows from explicit computations best handled by formal computation software such as Mathematica or Maple.

2. By construction, \( \rho_1 = \rho_{\text{hyp}} \). By a straightforward computation, the eigenvalues of \( \rho_u(T) \) (as well as \( \rho_u(U) \)) are \((1, 1, u, 1/u)\), so the claim about non-conjugate \( \rho_u, \rho_{u'} \) follows.

3. Using again formal computation software one can readily check that when \( |u| = 1 \), writing \( s = \text{Re} u \) and \( t = \text{Im} u \), the four generators for \( \rho_u(\text{Bi}(3)) \) preserve the Hermitian form given by the following matrix, where we denote \( f(s, t) = -1 - s + 4s^2 + i(-1 + 4s)t \):

\[
H_u = \begin{pmatrix}
-2 + 4s & f(s, t) & f(s, t) & f(s, t) \\
f(s, t) & -2 + 4s & -1 & -1 \\
f(s, t) & -1 & -2 + 4s & -1 \\
f(s, t) & -1 & -1 & -2 + 4s
\end{pmatrix}
\]

The determinant of \( H_u \) vanishes exactly when \( s = 1/4 \); since the form has signature \((3, 1)\) when \( u = s = 1 \), \( H_u \) has signature \((3, 1)\) for all \( u \in \text{U}(1) \) with \( \text{Re} u > 1/4 \). Likewise, by testing a value in the other interval, we see that the signature is \((4, 0)\) when \( \text{Re} u < 1/4 \).
4. We compute the dimension of the eigenspace of $\rho_u(T)$ for the eigenvalue 1. Note that:

$$\rho_u(T) - \text{Id} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
-\frac{2-u+u^2}{u^2} & \frac{1}{u} + u - 1 & -\frac{1}{u} + u & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

has rank 2, as its two non-zero lines are linearly independent. Therefore $\rho_u(T)$ is diagonalizable when $u \neq \pm 1$, so by the classification of isometries of complex hyperbolic space $H^3_\mathbb{C}$ (see Sect. 3.1), $\rho_u(T)$ is elliptic. The computation for $\rho_u(U)$ is similar. Recall that in the original representation, $\rho_{\text{hyp}}(T)$ and $\rho_{\text{hyp}}(U)$ are parabolic, and in particular have infinite order. Now for $u \neq \pm 1$, either the elliptic isometry $\rho_u(T)$ has finite order, in which case $\rho_u$ is not faithful, or it has infinite order, in which case $\rho_u$ has non-discrete image.

5. Let $G$ denote the Zariski-closure of $\rho_u(\text{Bi}(3))$ in $\text{SU}(3, 1)$. In particular, $G$ is a closed subgroup of $\text{SU}(3, 1)$ with its classical topology. By Theorem 4.4.2 of Chen and Greenberg [9], either $G = \text{SU}(3, 1)$ or $G$ has a global fixed point in $H^3_\mathbb{C}$, or preserves a totally geodesic submanifold of $H^3_\mathbb{C}$. By the classification of totally geodesic subspaces of $H^n_\mathbb{C}$ (see Sect. 3.1), in all these cases $G$ would be reducible, or contained in (a copy of) $\text{SO}(3, 1)$. It is easy to see that $\rho_u$ is irreducible for all $u$; now since $\rho_u$ is not conjugate to $\rho_1$ for any $u \neq 1$, $\rho_u(\text{Bi}(3))$ cannot be contained in a copy of $\text{SO}(3, 1)$ without violating rigidity of $\text{Bi}(3)$ inside $\text{SO}(3, 1)$ (second item of Theorem 1.1).

\[\square\]

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