Numerical study of turbulent surface waves on cryogenic fluids

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Abstract. We studied turbulent surface waves by performing direct numerical simulations of hydrodynamic equations for cryogenic fluids. In the statistically stationary state of the turbulence, the frequency-spectrum of the surface elevation is found to be broadened over a wide range of frequencies, and shows the energy cascade from large to small scales. This spectrum obeys the power law $\omega^{-4.46}$ which is different from weak turbulence theory.

1. Introduction

Nonlinear surface waves on liquids give a typical example of wave turbulence. Wave turbulence is comprised of nonlinear dispersive waves extending from small to large wave-numbers. These waves transfer energies between different modes to make Kolmogorov-like power-law spectrum.

The phenomena are usually understood by the wave turbulence theory which is based on two chief assumptions [1]. The first is the random phase approximation to wave fields. The second is that nonlinear interaction between different modes is reduced to three- or four-body collisions. This sort of theory predicts some statistics observed experimentally [2,3]. The first assumption may be, however, questionable from numerical and experimental works; the probability density function of the surface elevation deviates weakly from the Gaussian statistics [4,5].

Turbulence of capillary waves on the surface of liquid $^4$He and hydrogen have been reported recently [2,3]. Because the kinematic viscosities of these liquids, in particular of $^4$He, are very small, the wide inertial range over the frequencies can be observed. The low densities of these liquids make it easy to excite surface waves. Hence it is easier to address wave turbulence in a cryogenic fluid than a conventional fluid.

In this paper, we made the turbulent gravity-capillary surface waves by performing numerical simulations of the hydrodynamic equations, and studied the statistical behavior.

2. Numerical simulation

2.1. Hydrodynamic equations

We consider the surface waves on the irrotational and incompressible liquid of infinite depth characterized by a velocity potential $\phi(r,t)$ and a surface elevation $\eta(r,t)$ with $r = (x,y)$.

The equation and the boundary conditions governing the motion of the surface are [6]

$$\frac{\partial \eta}{\partial t} + \nabla \cdot \nabla \eta = \frac{\partial \phi}{\partial z} \quad (z = \eta), \quad (1)$$
\[ \frac{\partial \phi}{\partial t} + \frac{1}{2} \left| \nabla \phi \right|^2 + g \eta - \sigma \text{div} \left( \frac{\nabla \eta}{\sqrt{1 + \left| \nabla \eta \right|^2}} \right) = 0 \quad (z = \eta), \tag{2} \]

\[ \Delta \phi = 0 \quad (-\infty < z < \eta), \tag{3} \]

\[ \nabla \phi = 0 \quad (z = -\infty). \tag{4} \]

where \( g \) is the gravity acceleration, \( \sigma \) is the surface tension divided by the liquid density, \( \nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \) and \( \nabla_\perp = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) \). Furthermore, introducing a velocity potential on the surface such as \( \psi(x, y, t) = \phi(t, x, y, z = \eta(t, x, y)) \), these equations are reduced to

\[ \frac{\partial \eta}{\partial t} = -\nabla_\perp \psi \cdot \nabla_\perp \eta + W \{1 + \left| \nabla \eta \right|^2\}, \tag{5} \]

\[ \frac{\partial \psi}{\partial t} = -g \eta - \frac{1}{2} \left| \nabla \psi \right|^2 - \frac{1}{2} W^2 \{1 + \left| \nabla \eta \right|^2\} + \sigma \text{div} \left( \frac{\nabla \eta}{\sqrt{1 + \left| \nabla \eta \right|^2}} \right), \tag{6} \]

where \( W = \frac{\partial \phi}{\partial z} |_{z=\eta} \). Then in terms of the functions \( \eta \) and \( \psi \), the system is given by a canonical form

\[ \frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \psi}, \quad \frac{\partial \psi}{\partial t} = -\frac{\delta H}{\delta \eta}. \]

Here \( H \) is the total energy of the fluid consisting of the kinetic and potential components

\[ H = \frac{1}{2} g \int \eta^2 \, d\mathbf{r} + \sigma \int (\sqrt{1 + \left| \nabla \eta \right|^2} - 1) \, d\mathbf{r} + \frac{1}{2} \int d\mathbf{r} \int_{-\infty}^{\eta} dz |\nabla \phi|^2. \]

In the right-hand side of the equation, the first term is the gravity energy, the second term is the capillary energy and the third term is the kinetic energy.

For numerical simulation, we have used the high-order spectral method developed by West et al.[7] This method enables the evaluation of \( W \) from \( \eta \) and \( \psi \) at a time, and once \( W \) is obtained, \( \eta \) and \( \psi \) at the next time are calculated from eqs.(5) and (6) by a fourth-order Runge-Kutta method.

### 2.2. Dissipation and excitation

The dissipation and the excitation terms are introduced into eq.(6) to realize turbulent surface waves. Here we suppose that the Fourier-component of the dissipation term is \( 2\nu |k|^2 \psi_k \) where \( \psi_k \) is the Fourier-component of \( \psi \) and \( \nu \) is a kinematic viscosity; the excitation term creates waves with a wave-number \( k \) so that the amplitude may be random at each time-step. Note that these terms are not introduced into eq.(5). This means that fluid elements on the surface remain there, because turbulent surface waves should be caused through the energy excitation to the fluid rather than the elevation of the surface.

### 3. Results and discussions

For the simulation, the number of grid points in physical space was \( 128 \times 128 \) and the system size was \( 2 \text{ cm} \times 2 \text{ cm} \). The parameters were set for \( ^4\text{He} \) at 4 K except for the kinematic viscosity. The value of kinematic viscosity in our simulation is supposed to be a thousand times as large as that of the original one, because such a large effective dissipation is essential to realize statistically stationary turbulence. In statistically stationary turbulence, the excitation (pumping) energy at large scales and dissipation energy at small scales should be comparable. The original kinematic viscosity is, however, too small for our simulation to cause the effective dissipation at small scales. By increasing the kinematic viscosity, we can make the dissipation effective at the wider scales, which enables us to obtain the statistically stationary state. Although the value of kinematic
Figure 1. Evolution of the energy components. At $t = 2$ s, the components are total, kinetic, gravity and capillary energy in descending order.

Figure 2. The surface elevation $\eta(t)$ at a point.

Figure 3. Spectrum of the surface elevation $|\eta_\omega|^2$ at $t = 3$ s. The straight line shows the power of $\omega^{-4.46}$.

Figure 4. Probability density function of the normalized surface elevation $\eta(t)/\sigma_\eta$ and Gaussian (dotted line). The standard deviation $\sigma_\eta = 0.047$ mm, the skewness $= 0.059$ and kurtosis $= 2.7$.

viscosity in our simulations is unreal, we suppose that this situation do not affect the statistical structure of turbulence, such as power-law behaviors and probability density functions.

In the experiments for surface waves on the liquid $^4$He, the waves were excited at 18.5 Hz and the inertial range was observed to 1 kHz [2]. We excited our system at 38 Hz to compare the spectra effectively. In our simulation, because of numerical parameters, the effective wave-number range is limited. This wave-number range corresponds to the frequency range from 9 Hz to 414 Hz.

Figure 1 shows the evolution of the total energy and the components, by which we can find
the surface waves become statistically stationary. A typical example of the evolution of the surface elevation is shown in Fig. 2.

In the stationary state, the frequency-spectrum of the surface elevation $|\eta_\omega|^2$ is found to be broadened over a wide range of frequencies (Fig. 3). From the formation of this spectrum, the power-law of $\omega^{-4.46}$ is confirmed. Therefore, the energy is transferred to small scales self-similarly by the cascading nonlinear interactions of dispersive waves. We can conclude that turbulent surface waves have been realized. The power law for spectra of capillary turbulence is predicted as $-17/6$ by the theory [8]. The observed power is different from the predicted one. We do not know the reason of these difference. Figure 4 shows that the probability density function of the surface elevation deviates very weakly from the Gaussian statistics; the deviation is too small to conclude that the system is intermittent.

4. Conclusions
In this paper, we performed direct numerical simulations of surface waves on cryogenic fluids. In the statistically stationary state, the surface wave turbulence clearly shows that the energy injected in large scales is self-similarly transferred to small scales in a wide inertial range sustained by the nonlinear interaction between different modes of surface waves. The power-law obtained in our simulation was different from that predicted by the weak turbulence theory. We could not confirm the intermittent surface wave turbulence from the probability density function.

Considering the difference between the experimental results and our numerical results, more improved numerical simulation is required. A larger number of grid points would improve our simulation. First, the dissipation range will be more realistic. Secondly, we can observe the fine structure of surface waves. Then, the surface waves may be intermittent, which would make the probability density function deviate from the Gaussian.

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