Non-local emergent hydrodynamics in a long-range quantum spin system

Alexander Schuckert,1,2,* Izabella Lovas,1,2,† and Michael Knap1,2,‡

1Department of Physics and Institute for Advanced Study, Technical University of Munich, 85748 Garching, Germany
2Munich Center for Quantum Science and Technology (MCQST), Schellingstr. 4, D-80799 München
(Dated: September 5, 2019)

Generic short-range interacting quantum systems with a conserved quantity exhibit universal diffusive transport at late times. We show how this universality is replaced by a more general transport process in the presence of long-range couplings that decay algebraically with distance as \( r^{-\alpha} \). While diffusion is recovered for large exponents \( \alpha > 1.5 \), longer-ranged couplings with \( 0.5 < \alpha \leq 1.5 \) give rise to effective classical Lévy flights; a random walk with step sizes following a distribution which falls off algebraically at large distances. We investigate this phenomenon in a long-range interacting XY spin chain, conserving the total magnetization, at infinite temperature by employing non-equilibrium quantum field theory and semi-classical phase-space simulations. We find that the space-time dependent spin density profiles are self-similar, with scaling functions given by the stable symmetric distributions. As a consequence, autocorrelations show hydrodynamic tails decaying in time as \( t^{-1/(2\alpha-1)} \) when \( 0.5 < \alpha \leq 1.5 \). We also extract the associated generalized diffusion constant, and demonstrate that it follows the prediction of classical Lévy flights; quantum many-body effects manifest themselves in an overall time scale depending only weakly on \( \alpha \). Our findings can be readily verified with current trapped ion experiments.

In quantum many-body systems, macroscopic inhomogeneities in a conserved quantity must be transported across the whole system to reach an equilibrium state. As this is in general a slow process compared to local dephasing, essentially classical hydrodynamics is expected to emerge at late times in the absence of long-lived quasiparticle excitations [1–8]. The universality of this effective classical description may be understood from the central limit theorem: in the regime of incoherent transport, short range interactions lead to an effective random walk with a finite variance of step sizes, leading to a Gaussian distribution at late times. This universality is only broken when quantum coherence is retained, such as in integrable models [9–13] or in the vicinity of a many-body localized phase, where rare region effects give rise to subdiffusive transport [14–20].

In this work, we show how this universal diffusive transport in short range interacting systems is replaced by a more general, non-local effective hydrodynamical description in systems with algebraically decaying long-range interactions. We use semi-analytical non-equilibrium quantum field theory calculations (referred to as spin-2PI below) and a discrete truncated Wigner approximation (dTWA) to show that in a long range interacting XY spin chain, spin transport at infinite temperature effectively obeys a classical master equation with long-range, algebraically decaying transition amplitudes. This effective description can be reformulated as a classical random walk with infinite variance of step sizes, giving rise to a generalized central limit theorem and to a late-time description in terms of classical Lévy flights [21], an example for superdiffusive anomalous transport. As a result, we demonstrate that the full spatio-temporal shape of the correlation function \( C(j, t) = \langle \hat{S}_j(t)\hat{S}_0(0) \rangle \), and in particular, the exponent of the hydrodynamic tail in the autocorrelation function \( C(j = 0, t) \), depends strongly on the long-range exponent.
\( \alpha \). While for \( \alpha > 1.5 \) we recover classical diffusion, the autocorrelation function shows hydrodynamic tails with an exponent \( 1/(2\alpha - 1) \) for \( 0.5 \leq \alpha \leq 1.5 \) as we show in Fig. 1. Furthermore, \( C(j,t) \) possesses a self-similar behavior, with the scaling function covering all stable symmetric distributions as a function of \( \alpha \), smoothly crossing over from a Gaussian at \( \alpha = 1.5 \) over a Lorentzian at \( \alpha = 1 \) to an even more sharply peaked function as \( \alpha \to 0.5 \). We also extract the generalized diffusion coefficient \( D_\alpha \) from the scaling functions, and explain its \( \alpha \) dependence by Lévy flights; quantum effects are incorporated in a many-body time scale depending only weakly on \( \alpha \). For \( \alpha \leq 0.5 \) no emergent hydrodynamic behavior is found as the system relaxes instantaneously in the thermodynamic limit [22].

This work not only shows how non-local transport phenomena emerge in long-range interacting systems, but also establishes both nonequilibrium quantum field theory and discrete truncated Wigner simulations as efficient tools to study transport phenomena in the thermalization dynamics of quantum many-body systems.

**Model.**—We study the long range interacting quantum XY chain with open boundary conditions, given by the Hamiltonian

\[
\hat{H} = -\frac{1}{2} \sum_{i,j=-L/2}^{L/2} \frac{J}{N_{L,\alpha}} |i-j|^{-\alpha} \left( \hat{S}_i^x \hat{S}_j^x + \hat{S}_i^y \hat{S}_j^y \right). \tag{1}
\]

Here, \( \hat{S}_i^x = \frac{1}{2} \hat{\sigma}_i^x \) denotes spin-\( \frac{1}{2} \) operators given in terms of Pauli matrices, \( L \) is the (odd) length of the chain [23], and we set \( \hbar = 1 \). The interaction strength \( J \) is rescaled with the factor \( N_{L,\alpha} = \sqrt{\sum_{j \neq 0} |j|^{-2\alpha}} \) in order to remove the \( L \) and \( \alpha \) dependence of the time scale associated with the perturbative short time dynamics of the central spin at \( i = 0 \). The above model shows chaotic (Wigner-Dyson) level statistics for the whole range of \( \alpha \) considered here (\( 0.5 \leq \alpha \leq 2 \)) and is an effective description of the long range transverse field Ising model for large fields [24, 25]. In particular, it conserves the total \( S_z \) magnetization, with product states in the \( S_z \) basis evolving radically differently depending on the complexity of the corresponding magnetization sector. For just a few spin flips on top of the completely polarized state, the dynamics can be exactly solved and are described in terms of ballistically propagating spin waves, with a diverging group velocity at \( \alpha = 1 \) [24, 26, 27] related to the algebraic leakage of the Lieb-Robinson bound [28–31]. In contrast, here we show that the exponentially large Hilbert space sector for an extensive number of spin flips gives rise to rich transport phenomena, driven by the long-range nature of the interactions.

**Effective stochastic description of long-range transport.**—As the model in Eq. 1 is equivalent to long-range hopping hard core bosons, we conjecture the effective classical equation of motion for the transported local density \( f_j(t) \), in our case \( \left\langle \hat{S}_j^z(t) \right\rangle + \frac{1}{2} \), to be of the form [32]

\[
\partial_t f_j(t) = \sum_{i \neq j} (W_{i\to j} f_i - W_{j\to i} f_j). \tag{2}
\]

Here, the transition rate \( W_{i\to j} \) is determined by the microscopic transport processes present in the Hamiltonian, in our case the long-range hopping of spins. More specifically, from Fermi’s golden rule the transition rate for a flip flop process between spins \( i \) and \( j \) is proportional to \( |\langle \uparrow_i \downarrow_j | \hat{H} | \downarrow_i \uparrow_j \rangle|^2 \), and hence we phenomenologically set

\[
W_{i\to j} = W_{j\to i} = \frac{\lambda}{|i-j|^{2\alpha}}, \tag{3}
\]

where \( \lambda^{-1} \) is a characteristic time scale determined by the full many-body Hamiltonian.

Starting from an initial state with a single excitation in the center of the chain, the solution of this Master equation is given by [33]

\[
f_j(t) \approx \begin{cases} (D_\alpha t)^{-1/2} G \left( \frac{|j|}{(D_\alpha t)^{1/\alpha}} \right) & \text{for } \alpha > 1.5 \\ (D_\alpha t)^{-\beta_\alpha} F_\alpha \left( \frac{|j|}{(D_\alpha t)^{1/\alpha}} \right) & \text{for } 0.5 < \alpha \leq 1.5 \end{cases} \tag{4}
\]

in the limit of long times and large system sizes. Here, \( G(y) = \exp(-y^2/4)/8\sqrt{\pi} \) denotes the Gaussian distribution, indicating normal diffusion for \( \alpha > 1.5 \) with diffusion constant \( D_\alpha \propto \lambda \). For \( 0.5 < \alpha \leq 1.5 \), \( G(y) \) is replaced by the family of stable, symmetric distributions \( F_\alpha(y) \), given by

\[
F_\alpha(y) = \frac{1}{4} \int \frac{dk}{2\pi} \exp(-|k|^{1/\beta_\alpha}) \exp(iky), \tag{5}
\]

with the constant prefactor \( D_\alpha = \lambda c_\alpha \) constituting a generalized diffusion coefficient [34]. We find \( c_\alpha = -2\Gamma(1-2\alpha)\sin(\pi\alpha) \) from the classical Master equation, with \( \Gamma \) denoting the gamma function [33]. Furthermore, the exponent of the hydrodynamic tail \( \beta_\alpha \) is given by

\[
\beta_\alpha = \frac{1}{2\alpha - 1}. \tag{6}
\]

The Fourier transform in Eq. (5) only leads to elementary functions for \( \alpha = 3/2 \) and \( \alpha = 1 \), resulting in a Gaussian and a Lorentzian distribution, respectively [35]. The scaling functions \( F_\alpha(y) \) are the fixed point distributions in the generalized central limit theorem [36] of i.i.d. random variables with heavy tailed distributions. Importantly, \( F_\alpha(y) \) has diverging variance for \( \alpha < 1.5 \), undefined mean for \( \alpha \leq 1 \), and displays heavy tails \( \sim |y|^{-2\alpha} \). The classical Master equation hence predicts a cross-over from diffusive (\( \alpha \leq 1.5 \)) over ballistic (\( \alpha = 1 \)) to super-ballistic (\( 0.5 < \alpha < 1 \)) transport.

**Quantum dynamics from spin-2PI and dTWA.**—In the following, we demonstrate the emergence of these
effective classical dynamics in the quantum dynamics of
the Hamiltonian (1), by studying the unequal time corre-
lation function

$$C(j,t) := \text{Tr} \left[ \hat{S}_j^z(t)\hat{S}_0^z(0) \right]_{j=0,t=|t|}.$$  \hspace{1cm} (6)

Here, we perform the trace over product states in the $S^z$
basis, restricted to the Hilbert space sector with $\sum_S S^z =
\frac{1}{2} N$, such that $\langle S^z_i(t=0) \rangle = \frac{1}{2} \delta_{0,i}$ for all spins $i$. This
way, we probe the transport of a single spin excitation
moving in an infinite temperature bath with vanishing
total magnetization.

We employ two complementary, approximate meth-
ods to study the dynamics at long times and for large
system sizes, in a regime that is challenging to access
by numerically exact methods [37]. Schwinger boson
spin-2PI [38, 39], a non-equilibrium quantum field the-
ory method, employs an expansion in the inverse co-
ordination number $1/z$ to reduce the many-body prob-
lem to solving an integro-differential equation that scales
algebraically in system size. As the effective coordina-
tion number is large in a long-range interacting system,
we expect this approximation to be valid for small $\alpha$.
The discrete truncated Wigner approximation evolves
the classical equations of motion, while introducing quan-
turn fluctuations by sampling initial states from the
Wigner distribution [40–43] and was shown to be par-
cularly well suited for studying long-range interacting
systems [42, 44]. In both methods, we evaluate $C(j,t)$ by
starting from random initial product states in the $S^z$ ba-
sis and then averaging over sufficiently many such initial
states [45]. If not stated otherwise, all our results have
been converged with respect to system size, for which we
employed chains with $201 \times 601$ sites.

We study two distinct regimes in the dynamics. A
perturbative short time regime characterized by initial
dehasing is followed by the emergent effective classical
long-range transport described by the Master equation.

Perturbative short time dynamics.– At short
times, second-order perturbation theory yields

$$\text{Tr} \left[ \hat{S}_j^z(t)\hat{S}_0^z(0) \right] \approx \left\{ \begin{array}{ll}
\frac{1}{2}(1 - j^2 \alpha^2) & \text{for } j = 0 \\
\left( \frac{Jt}{4N_{L,\alpha}} \right)^{\frac{\alpha}{2}} & \text{for } j \neq 0
\end{array} \right.$$  \hspace{1cm} (7)

Physically, in this regime each spin is precessing in the
effective magnetic field created by all other spins. The
autocorrelation function is independent of $\alpha$ and $L$ due
to our choice of the normalization factor $N_{L,\alpha}$, ensuring
that the typical magnetic field at the center of the chain
remains of the order of $J$. The spatial correlation func-
tion at a fixed time inherits the algebraic behavior of the
interaction strength, falling off as $|j|^{-2\alpha}$ between spins
of distance $j$, which is reproduced by both dTWA (not
shown) and spin-2PI, see Fig. 2.

Hydrodynamic tails.– The scaling form from classi-
cal Lévy flights in Eq. 4 implies the presence of a hydro-
dynamic tail in the autocorrelation function $C(j = 0, t)$
with exponent $\beta_\alpha = 1/(2\alpha - 1)$, which replaces the
universal exponent $1/2$ for diffusion in 1D, see Fig. 1 for our
field theory results. For $\alpha \rightarrow 1.5$ we find slight devia-
tions from $\beta_\alpha$, these can however be explained by a subtle
finite-time effect also present in classical Lévy flights [33].
For $\alpha < 0.5$ we find no hydrodynamic tail for the numer-
ically accessible system sizes $L < 601$. This matches the
expectation that the system relaxes instantaneously in the
thermodynamic limit [22], which is also indicated by the
fact that the perturbative short time scale diverges,
$N_{L,\alpha,\infty} = \infty$, for $\alpha < 0.5$. Even on longer time scales,
the discretized Fourier transform underlying the deriva-
tion of Eq. 4 is dominated by the smallest wavenumber in
finite chains, and the hydrodynamic tail is replaced by an
exponential convergence towards the equilibrium value $0.25/L
with a rate $\sim (1/L)^{2\alpha-1}$.

Self-similar time evolution of correlations.– In
Fig. 3 we show the spreading of $t^{\beta_\alpha}C(j,t)$ for two val-
ues of $\alpha$. While for $\alpha = 2$ a diffusive cone is visible, the
spreading for $\alpha = 1$ is ballistic as expected from the Mas-
ter equation. The scaling collapse of these data shows
good agreement with classical Lévy flights, Eq. 4, at late
times. Interestingly, we find heavy tails even for $\alpha \geq 1.5$.
We explain these by sub-leading corrections to the scal-
ing ansatz Eq. (4) present in the Master equation [33].
They survive up to algebraically long times for $\alpha > 1.5$,
turning to a logarithmic correction at $\alpha = 1.5$ [46].

For $\alpha \geq 2$ we furthermore find signs of peaks propaga-
ting ballistically for intermediate times in the dTWA scal-
ing functions, which survive longer as $\alpha$ increases.
These peaks are remnants of the integrable point at $\alpha = \infty$.
Such behaviour is not present in the spin-2PI data as this
The correlation function $C(j, t)$ obtained from spin-2PI for chains of lengths $L = 201$ ($\alpha = 2$, subfigures (a-d)), $L = 301$ ($\alpha = 1$, subfigures (e-h)). (a,e) $C(j, t)$ multiplied by $t^{1/(2\alpha - 1)}$ to account for the overall decay expected from Lévy flights shows a diffusive cone for $\alpha = 2$, whereas for $\alpha = 1$ a ballistic light-cone emerges. The contour lines for $\alpha = 1, 2$ correspond to values $t^{1/(2\alpha - 1)}C(j, t) = 0.03, 10^{-4}$, respectively. (b,f) Rescaling of linearly spaced time slices for $23 \leq Jt \leq 84$ ($\alpha = 1$) and $42 \leq Jt \leq 226$ ($\alpha = 2$) (lines become darker as time increases) for the same data as in (a,e) agrees well with the scaling function expected from classical Lévy flights, Eq. 4. The only fitting parameter is the generalized diffusion coefficient. (d,h) Rescaled time slices ($2 \leq Jt \leq 28$) on a double-logarithmic scale reveal for $\alpha = 1$ the heavy tail $\sim y^{-2}$ expected from Lévy flights (Eq. 4), where the dashed-dotted line is the same fit as in (b). The tail $\sim y^{-4}$ (thick black line) for $\alpha = 2$ ($8 \leq Jt \leq 85$) is a finite time effect also present in classical Lévy flights. (c,g) Unscaled data.

**FIG. 4. Generalized diffusion constant.** The $\alpha$ dependence of the diffusion constant obtained from fits with the scaling function of Lévy flights, Eq. 4. The qualitative behavior follows the Lévy flight prediction $D_\alpha \sim c_\alpha$ for $\alpha < 1.5$, constituting the quantum many-body time scale, depends only weakly on $\alpha$. As expected from their differing approximate treatment of the quantum fluctuations in the system, we find slight differences between the values of $\lambda$ determined by spin-2PI and dTWA, $\lambda_{2PI} \approx 0.25$ and $\lambda_{dTWA} \approx 0.15$. For $\alpha > 1.5$ we find considerable differences between the dTWA and spin-2PI results, because the emergent ballistic peaks, stemming from the nearby integrable point, accelerate the spreading in the dTWA simulations.

**Conclusions.** In this paper, we have shown that spin transport at high temperatures in long-range interacting XY-chains is well described by Lévy flights for long-range interaction exponents $0.5 < \alpha \leq 1.5$, effectively realizing a random walk with infinite variance of step sizes. In particular, we have shown that the scaling function of the unequal time spin correlation function covers the stable symmetric distributions, in accordance with the generalized central limit theorem. While the system relaxes instantly for $\alpha < 0.5$, standard diffusion was recovered for $\alpha > 1.5$, with heavy tails from finite time corrections surviving until extremely long times. We demonstrated the non-trivial dependence of the generalized diffusion coefficient $D_\alpha$ on $\alpha$, and found that it is captured by classical Lévy flights, with the quantum many body
time scale being approximately independent of $\alpha$. While we only studied one-dimensional systems, we expect this phenomenon to generalize straightforwardly to $d > 1$ dimensions. Assuming the effective classical Lévy flight picture persists, superdiffusive behaviour would be found for $d/2 < \alpha < 1 + d/2$ with the exponent of the hydrodynamic tails given by $d/(2\alpha - d)$ [33].

The long-range transport process found here can be experimentally studied in current trapped ion experiments [49], which can reach the required time scales [50–53]. The effective infinite temperature states can also be realized by sampling over random product states which are then evolved in time.

Acknowledgments.—We thank Rainer Blatt, Eleanor Crane, Iliya Esin, Philipp Hauke, Christine Maier, Asier Piñeiro Orioli, Tibor Rakovszky for insightful discussions and the Nanosystems Initiative Munich (NIM) funded by the German Excellence Initiative for access to their computational resources. We acknowledge support from the Max Planck Gesellschaft (MPG) through the International Max Planck Research School for Quantum Science and Technology (IMPRS-QST), the Technical University of Munich - Institute for Advanced Study, funded by the German Excellence Initiative, the European Union FP7 under grant agreement 291763, the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany’s Excellence Strategy–EXC-2111–390814868, the European Research Council (ERC) under starting grant ConsQuanDyn and the European Union’s Horizon 2020 research and innovation program grant agreement no. 77153, from DFG grant No. KN1254/1-1, and DFG TRR80 (Project F8).

* alexander.schuckert@tum.de
† izabella.lovas@tum.de
‡ michael.knap@ph.tum.de

[1] S. Mukerjee, V. Oganesyan, and D. Huse, Phys. Rev. B 73, 035113 (2006).
[2] J. Lux, J. Müller, A. Mitra, and A. Rosch, Physical Review A 89, 053608 (2014).
[3] A. Bohrdt, C. B. Mendel, M. Endres, and M. Knap, New Journal of Physics 19, 063001 (2017).
[4] M. Medenjak, K. Klobas, and T. Prosen, Physical Review Letters 119, 110603 (2017).
[5] E. Leviatan, F. Pollmann, J. H. Bardarson, D. A. Huse, and E. Altman, arXiv:1702.08894 (2017).
[6] V. Khemani, A. Vishwanath, and D. A. Huse, Phys. Rev. X 8, 031057 (2018).
[7] T. Rakovszky, F. Pollmann, and C. W. von Keyserlingk, Phys. Rev. X 8, 031058 (2018).
[8] D. E. Parker, X. Cao, A. Avdoshkin, T. Scaffidi, and E. Altman, arXiv:1812.08657 (2018).
[9] S. Gopalakrishnan and R. Vasseur, Physical Review Letters 122, 127202 (2019).
[10] M. Ljubotina, L. Zadnik, and T. Prosen, Phys. Rev. Lett. 122, 150605 (2019).
[11] M. Ljubotina, M. Žnidarič, and T. Prosen, Phys. Rev. Lett. 122, 210602 (2019).
[12] P. Prelovšek, J. Bonča, and M. Mierzejewski, Phys. Rev. B 98, 125119 (2018).
[13] M. Šroda, P. Prelovšek, and M. Mierzejewski, Phys. Rev. B 99, 121110 (2019).
[14] K. Agarwal, S. Gopalakrishnan, M. Knap, M. Müller, and E. Demler, Physical Review Letters 114, 160401 (2015).
[15] Y. Bar Lev, G. Cohen, and D. R. Reichman, Phys. Rev. Lett. 114, 100601 (2015).
[16] R. Vosk, D. A. Huse, and E. Altman, Phys. Rev. X 5, 031032 (2015).
[17] A. C. Potter, R. Vasseur, and S. A. Parameswaran, Phys. Rev. X 5, 031033 (2015).
[18] P. Bordia, H. Lüschen, S. Scherg, S. Gopalakrishnan, M. Knap, U. Schneider, and I. Bloch, Physical Review X 7, 041047 (2017).
[19] S. Gopalakrishnan, K. R. Islam, and M. Knap, Physical Review Letters 119, 046601 (2017).
[20] K. Agarwal, E. Altman, S. Gopalakrishnan, D. A. Huse, and M. Knap, Annalen der Physik 529, 1600326 (2017).
[21] V. Zaburdaev, S. Denisov, and J. Klafter, Reviews of Modern Physics 87, 483 (2015).
[22] R. Bachelard and M. Kastner, Phys. Rev. Lett. 110, 170603 (2013).
[23] We always assume integer divisions when we write $\frac{x}{y}$.
[24] P. Jurečević, B. P. Lanyon, P. Hauke, C. Hempel, P. Zöller, R. Blatt, and C. F. Roos, Nature 511, 202 (2014).
[25] J. Smith, A. Lee, P. Richerme, B. Neyenhuis, P. W. Hess, P. Hauke, M. Heyl, D. A. Huse, and C. Monroe, Nature Physics 12, 907 (2016).
[26] P. Hauke and L. Tagliacozzo, Physical Review Letters 111, 20720 (2013).
[27] P. Richerme, Z.-X. Gong, A. Lee, C. Senko, J. Smith, M. Foss-Feig, S. Michalakis, A. V. Gorshkov, and C. Monroe, Nature 511, 198 (2014).
[28] E. H. Lieb and D. W. Robinson, Communications in Mathematical Physics 28, 251 (1972).
[29] M. B. Hastings and T. Koma, Communications in Mathematical Physics 265, 781 (2006).
[30] J. Eisert, M. Van Den Worm, S. R. Manmana, and M. Kastner, Physical Review Letters 111, 260401 (2013).
[31] M. C. Tran, A. Y. Guo, Y. Su, J. R. Garrison, Z. El-dredge, M. Foss-Feig, A. M. Childs, and A. V. Gorshkov, Phys. Rev. X 9, 031006 (2019).
[32] A Master equation in terms of a local probability density may be obtained by normalizing the local density. See supplementary material.
[33] The prefactor $1/4$ accounts for the normalization of the correlation function, $C(j,t = 0) = \delta_0,j/4$. Furthermore, reinstating a lattice spacing $a$, the units of $D_\alpha$ depend on $\alpha$, in particular it is a velocity for $\alpha = 1$.
[34] Other closed form solutions exist [54], for example for $\alpha = 1.25$ in terms of hypergeometric functions.
[35] B. Gnedenco and A. N. Kolmogorov, Limit Distributions for Sums of Independent Random Variables (Addison-Wesley, 1954).
[36] B. Kloss and Y. Bar Lev, Phys. Rev. A 99, 032114 (2019).
[37] M. Babadi, E. Demler, and M. Knap, Phys. Rev. X 5, 041005 (2015).
Here, we derive the scaling collapse given by Eqs. (4) and (5) from the master equation Eq. (2),

\[ \partial_t f_j = \sum_{i \neq j} W_{ij} (f_i - f_j) , \tag{8} \]

with transition rates

\[ W_{ij} = \frac{\lambda}{|i - j|^{2\alpha}} . \tag{9} \]

By taking the Fourier transform of this equation, we arrive at

\[ \partial_t f(k) = [W(k) - W(k = 0)] f(k) , \tag{10} \]

where

\[ f(k) = \sum_{j=-L/2}^{L/2} e^{-ikj} f_j , \tag{11} \]

and

\[ W(k) - W(k = 0) = \lambda \left[ \sum_{j=-L/2}^{L/2} \sum_{j=-L/2}^{L/2} (e^{-ikj} - 1) / |j|^{2\alpha} \right] \approx 2 \lambda \int_{-L/2}^{L/2} dx \frac{\cos kx - 1}{x^{2\alpha}} , \tag{12} \]

with \( k = 2\pi n/L, n = -L/2, \ldots, L/2 \).

The long time behavior is dominated by large wavelengths \( k \ll 1 \). For \( 0.5 < \alpha < 1.5 \) the integral remains convergent.

SUPPLEMENTARY MATERIAL

Scaling functions from the classical master equation
when we remove both the upper and lower cutoffs, leading to the following approximation in the regime $k \ll 1$, 

$$W(k) - W(k = 0) \approx 2\lambda \int_0^{L/2} dx (\cos kx - 1) / x^{2\alpha} - 2\lambda \int_0^1 dx (\cos kx - 1) / x^{2\alpha}$$ 

$$\approx 2\lambda \int_0^{\infty} dx (\cos kx - 1) / x^{2\alpha} + \lambda k^2 \int_0^1 dx x^{2-2\alpha}$$ 

$$= \left( -c_\alpha |k|^{2\alpha - 1} + \frac{k^2}{3 - 2\alpha} \right) \lambda. \quad (13)$$

Here 

$$c_\alpha = -2 \int_0^{\infty} dz (\cos z - 1) / z^{2\alpha} = -2\Gamma(1 - 2\alpha) \sin(\alpha\pi),$$

with $\Gamma$ denoting the gamma function. Note that $c_\alpha > 0$ for $0.5 < \alpha < 1.5$.

For $0.5 < \alpha < 1.5$, the first term in Eq. (13), $\sim |k|^{2\alpha - 1}$, will dominate the long time behavior, leading to 

$$\partial_t f(k) \approx -\lambda c_\alpha |k|^{2\alpha - 1} \Rightarrow f(k, t) = f(k, 0) e^{-\lambda c_\alpha |k|^{2\alpha - 1} t}. \quad (14)$$

In particular, taking an initial state with a single localized excitation, $f_j(t = 0) = \delta_j,0/4$ and hence $f(k, 0) \equiv 1/4$ with the factor 1/4 stemming from $(\hat{S}^z)^2 = 1/4$, we arrive at the scaling ansatz 

$$f_j(t) \approx \frac{1}{4} \int \frac{dk}{2\pi} \exp \left( i k j - \lambda t c_\alpha |k|^{2\alpha - 1} \right) = (\lambda c_\alpha t)^{-1/(2\alpha - 1)} F_\alpha \left( \frac{|j|}{(\lambda c_\alpha t)^{1/(2\alpha - 1)}} \right), \quad (15)$$

with $F_\alpha(y)$ given by Eq. (5).

**Diffusion for $\alpha > 1.5$.** While we used $\alpha < 1.5$ in the derivation of Eq. 13, the expression is in fact valid for all $\alpha > 0.5$. This can be shown by evaluating the following integral exactly,

$$W(k) - W(k = 0) \approx 2\lambda |k|^{2\alpha - 1} \int_0^{\infty} dz (\cos z - 1) / z^{2\alpha},$$

and expanding the resulting expression around $k = 0$. Noting that for $\alpha > 1.5$ the $|k|^{2\alpha - 1}$ term is subdominant, we arrive at 

$$f_j^{\alpha>1.5}(t) \approx \frac{1}{4} \int \frac{dk}{2\pi} \exp \left( i k j - \frac{\lambda}{2\alpha - 3} k^2 t \right) = (D_\alpha t)^{-1/2} G \left( \frac{|j|}{(D_\alpha t)^{1/2}} \right), \quad (16)$$

reproducing diffusive behaviour for $\alpha > 1.5$ with diffusion coefficient $D_\alpha = \lambda/(2\alpha - 3)$ and a Gaussian $G(y) = \exp(-y^2/4)/8\sqrt{\pi}$. We note that in contrast to the regime $\alpha < 1.5$, here the dTWA results for the quantum many-body time scale $\lambda$ show a strong $\alpha$ dependence, with $D_\alpha$ increasing as a function of $\alpha$ due to the approach to the integrable point at $\alpha \rightarrow \infty$.

**Exponential late time decay of the autocorrelation function.** For finite system sizes $L$, approximating the discrete Fourier sums by integrals eventually breaks down at very long times. In this regime the time evolution will be dominated by the two smallest non-zero wave-numbers, $k = \pm 2\pi/L$, leading to an exponential decay 

$$f_j(t) \approx \frac{1}{L} \left[ f(k = 0, t) + \sum_{k_0=\pm2\pi/L} e^{ik_0j} f(k_0, t) \right] = \frac{1}{4L} \left[ 1 + 2 \cos(2\pi j/L) e^{-\lambda c_\alpha (2\pi/L)^{2\alpha - 1}} \right] \quad (17)$$

for the case of $\alpha \leq 1.5$. The exponent of this exponential decay is hence expected to scale with the system size as $\sim (1/L)^{2\alpha - 1}$. This prediction is in agreement with our spin-2PI numerical results. Moreover, this result can be used to extract the diffusion coefficient $D_\alpha = \lambda c_\alpha$ from finite size data.

**Corrections to scaling**

For finite times, the two terms in Eq. 13 compete, leading to corrections to the leading-order scaling ansatz, Eq. 4. As we show below, these corrections survive until algebraically long times for $\alpha > 1.5$, and they add a logarithmic
correction to the scaling ansatz at the threshold value $\alpha = 1.5$, explaining all major deviations from scaling (4) we observe in our numerical data.

**Diffusive regime, $\alpha > 1.5$.** Taking into account both terms in Eq. 13, then rescaling as $k \to k\sqrt{D_\alpha t}$ and $j \to y = j/\sqrt{D_\alpha t}$ where $D_\alpha = \lambda/(2\alpha - 3)$, and finally expanding the remaining time dependent term for $t \to \infty$, we arrive at

$$\sqrt{D_\alpha t} f_j(t) \approx \frac{1}{4} \int \frac{dk}{2\pi} \exp\left(iky - k^2\right) \left(1 - (D_\alpha t)^{-\alpha + 3/2} \zeta(2\alpha - 3)|k|^{2\alpha - 1}\right)$$

$$= G(y) - (D_\alpha t)^{-\alpha + 3/2} \zeta(2\alpha - 3) \Gamma(\alpha)\frac{16}{\pi} {}_1 F_1 \left[\alpha, \frac{1}{2}, -\frac{y^2}{4}\right],$$

with ${}_1 F_1 [\cdot, \cdot; \cdot]$ denoting the Kummer confluent hypergeometric function. Most importantly, the latter exhibits heavy tails $\sim y^{-2\alpha}$ for $y \to \infty$, reproducing the finite time data for $\alpha = 2$ in Fig. 3 of the main text.

We show this behaviour more explicitly in Fig. 5, where we compare our numerical results to a scaling function involving a single fit parameter $D_2$,

$$\sqrt{D_2 t} f_j(t) \approx G(y) - \frac{1}{96\sqrt{D_2 t}} {}_1 F_1 \left[\alpha, \frac{1}{2}, -\frac{y^2}{4}\right],$$

following from Eq. 18 using $\lim_{\alpha \to \infty} \sin(\alpha \pi)\Gamma(1 - 2\alpha) = \pi/12$. Furthermore, according to Eq. 18 the approach to Gaussian scaling is algebraically slow with exponent $1.5 - \alpha \to 0$ for $\alpha \to 1.5$. As we show in the following, at this special value $\alpha = 1.5$ this algebraic convergence to scaling is replaced by a persistent logarithmic correction to the scaling ansatz.

**Crossover point $\alpha = 1.5$.** In the limit $\alpha \to 1.5$, both prefactors in Eq. (13), $\zeta(\lambda)$ and $1/(3 - 2\alpha)$, diverge with their difference remaining finite, $-\zeta(\lambda) + 1/(3 - 2\alpha) \to \gamma - 3/2 \approx -0.92$, with $\gamma$ denoting the Euler-Mascheroni constant, resulting in a Gaussian leading order term, $\partial_t f(k) \approx -0.92 \lambda k^2$. However, an additional logarithmic correction from $\lim_{\alpha \to 1.5} \zeta(\lambda)|k|^{2\alpha - 1} - k^2) = 0.5k^2 \log k^2$ also contributes. Following the derivation in Ref. [46], we rescale $k$ as $k \to k\sqrt{\lambda \Omega(\lambda t)/2}$, with $\Omega(\lambda t)$ a function to be determined, and get

$$\sqrt{\lambda \Omega(\lambda t)/2} f_j(t) \approx \frac{1}{4} \int \frac{dk}{2\pi} \exp(ik\tilde{y}) \exp\left(\frac{k^2}{\Omega(\lambda t)} \ln\left(\frac{2 \exp(2\gamma - 3)}{\lambda t \Omega(\lambda t)}\right) + \frac{k^2}{\Omega(\lambda t)} \ln(k^2)\right)$$

with scaling variable $\tilde{y} = j/\sqrt{\lambda \Omega(\lambda t)/2}$. The function $\Omega(\lambda t)$ is chosen in such a way that the first term in the exponent is equal to $-k^2$, reproducing the leading order Gaussian behavior [46]. This leads to

$$\Omega(\lambda t) = \left| W_{-1} \left( \frac{2 \exp(2\gamma - 3)}{\lambda t} \right) \right|,$$

**FIG. 5. Corrections to scaling.** The heavy tails found in the scaling functions for $\alpha \geq 1.5$ are completely captured by the finite time corrections to scaling in the classical Master equation, with fit functions given in Eqs. 19 and 21. Note that there is only a single fit parameter (given by the quantum many body time scale $1/\lambda$), with its numerical value being approximately equal in the fit to the (scaling) functions obtained from the LO and LO+NLO in a simultaneous $k \to 0, t \to \infty$ expansion.
with \( W_{-1} \) the secondary branch of the Lambert \( W \)-function. As discussed in [46], this gives \( \Omega(\lambda t) \approx \ln(\lambda t) \sim \ln t \), for \( t \to \infty \), yielding a logarithmic correction to the scaling ansatz. Finally, we expand the resulting expression for large time \( \Omega(\lambda t) \sim \ln(\lambda t) \gg 1 \) and arrive at

\[
\sqrt{M} \Omega(\lambda t)/2 f_j(t) \approx \frac{1}{4} \int \frac{dk}{2\pi} \exp(ik\tilde{y}) \exp(-k^2) \left( 1 + \frac{k^2}{\Omega(\lambda t)} \ln(k^2) \right)
= G(\tilde{y}) \left( 1 + \frac{1}{4\Omega(\lambda t)} \left( -2 + \tilde{y}^2 \right) \left( -2 + \gamma + \ln(4) \right) + 2 \exp(\tilde{y}^2/4) J_1^{(1,0,0)} \left[ \frac{3}{2}, \frac{1}{2}, -\tilde{y}^2/4 \right] \right),
\]

with the superscript \((1,0,0)\) denoting the derivative with respect to the first argument. This expression shows the logarithmically slow convergence towards the Gaussian scaling function for \( \alpha = 1.5 \), as well as a persistent logarithmic correction to the scaling form, with scaling variable \( \tilde{y} = j/\sqrt{M} \Omega(M) \). Furthermore, for finite \( t \), the above function exhibits a heavy tail \( \sqrt{M} \Omega(\lambda t) f_{j,\alpha=1.5}(t) \sim y^{-3} \) and matches our 2PI results as shown in Fig. 5, using the single fitting parameter \( \alpha \). As times were not large enough in the simulations to be in the regime where \( \Omega(\lambda t) \approx \ln(t) \), we used the full expression in Eq. 20 for \( \Omega(\lambda t) \) to fit the unrescaled \( f_j(t) \) at a fixed time \( t \).

**Superdiffusive regime, \( \alpha < 1.5 \).** While there are no qualitative corrections to the scaling function in this regime, the term \( \sim k^2 \) in Eq. 13 leads to a correction to the exponent of the hydrodynamic tail as \( \alpha \nearrow 1.5 \). When evaluating the Fourier transform numerically with the full expression in Eq. (13) for \( \alpha \lesssim 1.5 \), we still find an approximate hydrodynamic tail with a modified exponent reproducing the finite-time dTWA and spin-2PI results more closely than the 'bare' expression \( \beta_0 \) and hence accounting for the slight deviations between the numerical results and \( \beta_0 \) mentioned in the main text. For example, we get \( \beta_{\alpha=1.5} \approx 0.57 \) from this procedure, in agreement with 2PI \( (\beta_{\alpha=1.5} \approx 0.58 \pm 0.02) \) and dTWA \( (\beta_{\alpha=1.5} \approx 0.59 \pm 0.02) \).

**Classical master equation in dimension \( d > 1 \)**

In the following, we extend the results of the classical Master equation to spins at locations \( r_i \) in \( d \) dimensions. The Master equation (8) then reads

\[
\partial_t f_{r_j} = \sum_{i \neq j} W_{|r_i - r_j|}(f_{r_i} - f_{r_j}), \quad \text{with} \quad W_{|r_i - r_j|} = \frac{\lambda}{|r_i - r_j|^{2\alpha}}.
\]

Fourier transforming again diagonalizes the differential equation, yielding

\[
f(|k|, t) = f(|k|, 0) \exp \{ (W(|k|) - W(|k| = 0)t) \}.
\]

**Two spatial dimensions \( d=2 \).** Denoting \( k \equiv |k| \) in the following, we evaluate the Fourier transform of the transition amplitudes in continuous space with both an IR (system size \( L \)) and UV (lattice spacing \( a = 1 \)) cutoff, yielding

\[
W(k) - W(k = 0) = \lambda L \int_0^L \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \left( e^{-ikr \cos(\theta)} - 1 \right) \frac{1}{r^{2\alpha}}
= \lambda 2\pi \int_1^L \mathrm{d}r r^{1-2\alpha} (J_0(kr) - 1),
\]

with \( J_0(kr) \) denoting the zeroth order Bessel function of the first kind. For \( \alpha < 1 \) we get a divergence in the thermodynamic limit \( L \to \infty \), hence we expect the dynamics to be described by the infinite ranged mean field model in that regime. Concentrating on \( \alpha \geq 1 \), we can remove the IR cutoff and arrive at

\[
W(k) - W(k = 0) \approx 2\pi \lambda \int_1^\infty \mathrm{d}r r^{1-2\alpha} \left[ J_0(kr) - 1 \right]
\approx 2\pi \lambda \left[ k^{2\alpha-2} \int_0^\infty \mathrm{d}x x^{1-2\alpha} (J_0(x) - 1) + \frac{k^2}{4} \int_0^1 r^{3-2\alpha} \mathrm{d}r \right],
\]

\[
W(k) - W(k = 0) \approx \lambda \left[ -c_0 k^{2\alpha-2} + \frac{k^2 \pi}{2(4-2\alpha)} \right],
\]
with
\[ c_\alpha = 2\pi \int_0^\infty dx x^{1-2\alpha} (1 - \mathcal{J}_0(x)) = -\frac{2^{2-2\alpha} \pi \Gamma(1-\alpha)}{\Gamma(\alpha)} \quad \text{for} \quad 1 < \alpha < 2. \] (29)

We see that a superdiffusive solution is obtained for \( 1 < \alpha \leq 2 \), where the first term \( \sim k^{2\alpha-2} \) dominates. Neglecting all other terms, we hence arrive at the scaling ansatz for a localized excitation
\[ f_r(t) = (\lambda c_\alpha t)^{-\frac{2}{\alpha-2}} F^{2D}_\alpha \left( \frac{|r|}{(\lambda c_\alpha t)^{1/2}} \right), \] (30)
with
\[ F^{2D}_\alpha(y) = \frac{1}{8\pi} \int_0^\infty dk k J_0(ky) e^{-k^{2\alpha-2}}. \] (31)

**Three spatial dimensions \( d=3 \).** We similarly get
\[ W(k) - W(k = 0) = 2\pi \lambda \int_{1}^{L} dr r^2 \int_0^\pi d\theta \sin(\theta) \left( e^{-ikr\cos(\theta)} - 1 \right) \frac{1}{y^{2\alpha}} \] (32)
\[ = 4\pi \lambda \int_{1}^{L} dr r^{2-2\alpha} \left( \sin(kr) - 1 \right), \] (33)
where we get an IR divergence and hence expect mean-field behavior for \( \alpha < 3/2 \). Considering only \( \alpha \geq 3/2 \), we set \( L \to \infty \) and get for small \( k \)
\[ W(k) - W(k = 0) \approx \lambda \left[ -c_\alpha k^{2\alpha-3} + \frac{k^2 2\pi}{3(5-2\alpha)} \right], \] (34)
with
\[ c_\alpha = -4\pi \sin(\pi\alpha)\Gamma(2-2\alpha) \quad \text{for} \quad 1.5 < \alpha < 2.5. \] (35)

We see immediately, that now superdiffusive behavior is seen for \( 1.5 < \alpha < 2.5 \) with a scaling ansatz in real space for a localized excitation
\[ f_r(t) = (\lambda c_\alpha t)^{-\frac{3}{2\alpha-3}} F^{3D}_\alpha \left( \frac{|r|}{(\lambda c_\alpha t)^{1/2}} \right), \] (36)
with
\[ F^{3D}_\alpha(y) = \frac{1}{8\pi^2 y} \int_0^\infty dk k \sin(ky) e^{-k^{2\alpha-3}}. \] (37)