**REMARKS ON CHERN-EINSTEIN HERMITIAN METRICS**

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**Abstract.** We study some basic properties and examples of Hermitian metrics on complex manifolds whose traces of the curvature of the Chern connection are proportional to the metric itself.

**Introduction**

In this note we aim to study certain “special” Hermitian metrics $\omega$ on compact complex (mostly non-Kähler) manifolds $X$, focusing on the geometry of the Chern connection. In a previous paper [ACS17] we started by looking at constant Chern-scalar curvature metrics in a conformal class of Hermitian metrics, giving partial results towards what we called Chern-Yamabe problem.

Here instead we collect some results on the Chern-Einstein problem(s). Namely, we look for Hermitian metrics whose Chern-Ricci forms are (non-necessarily constant) multiple of the metric itself. Note that, due to the lack of Bianchi symmetry, we have actually three different ways to contract the curvature.

A partial account on the literature includes [Gol56, Gau81, CM84, Bal85, Tos15, ST11, LY14, Pod18, AP18], see also the discussion at [MOF]. We now survey our main results.

**First-Chern-Einstein metrics.** The first-Chern-Ricci form $\text{Ric}^{(1)}(\omega)$ arises by tracing the endomorphism part of the Chern-curvature tensor. It yields a closed real $(1,1)$-form that represents the first Bott-Chern class $c_{BC}^1(X) \in H^{1,1}_{BC}(X)$. We claim that the corresponding Chern-Einstein problem, i.e., the search for Hermitian metrics $\omega$ satisfying

$$\text{Ric}^{(1)}(\omega) = \lambda \omega$$

for $\lambda \in C^\infty(X; \mathbb{R})$, is basically understood. If $c_{BC}^1(X) = 0$ there exists a Chern-Ricci-flat metric in any conformal class, and $X$ is called a (non-Kähler) Calabi-Yau manifold [Tos15], and non-Kähler examples actually do exist, e.g. Kodaira surfaces. If instead we are looking to not identically zero (non-necessarily constant, otherwise the statement is completely trivial) Einstein factor $\lambda$, it turns out that the manifold is actually Kähler of a very special type, and all first-Chern-Einstein metrics are easy to describe.

**Theorem A** (see Theorem 4). Let $(X^n, g)$ be a compact first-Chern-Einstein manifold, with non-identically-zero Einstein factor. Then $g$ is conformal to a Kähler metric in $\pm c_1(X)$, with conformal factor depending explicitly on the Ricci potential of the Kähler metric. Conversely, starting with a Kähler metric in $\pm c_1(X)$, one can always construct a Chern-Einstein metric, unique up to scaling, in the conformal class.

The above result is essentially telling that the first-Chern-Einstein problem is uninteresting for further investigations beside the fundamental Kähler-Einstein case or the non-Kähler...
Calabi-Yau situation [STW17]. So let us discuss the more promising second-Chern-Einstein problem.

**Second-Chern-Einstein metrics.** The second-Chern-Ricci form $Ric^{(2)}(\omega)$ is obtained by tracing the two other indices of the Chern-curvature tensor by means of the metric. Thus a second-Chern-Einstein metric is just a Hermitian metric on $TX$ that is Hermitian-Einstein by taking trace with itself. A straighforward computation for the conformal change of the second-Chern-Ricci form shows the second-Chern-Einstein problem depends only on the conformal class. In particular, we can apply conformal methods to study the Einstein factor. Here we distinguish between weak-second-Chern-Einstein and strong-second-Chern-Einstein metrics, according to the Einstein factor being a function, respectively, constant.

**Theorem B** (see Theorem 6, after [Gau81], [ACS17, Theorems 3.1-4.1]). Let $(X^n, g)$ be a compact weak-second-Chern-Einstein manifold. Then we can choose a representative in the conformal class of $g$ such that its Einstein factor has a sign, equal to the sign of the degree of the anti-canonical line bundle with respect to the conformal Gauduchon metric. Moreover, if this sign is non-positive, then there is representative in the conformal class which is actually strong-second-Chern-Einstein.

The assumption on the sign in the previous statement holds when the Kodaira dimension of $X$ is greater than or equal to zero, and it would be removed if one can prove the Chern-Yamabe conjecture [ACS17, Conjecture 2.1]. Thus, in this latter case, the weak-second-Chern-Einstein and strong-second-Chern-Einstein problems would became fully equivalent.

As a consequence, on compact manifolds, one gets a series of obstructions [Tel06, Gau81, Gau77a, Yan17]:

- the possible signs of the Einstein factor are determined according to $K_X$ and $K_X^{-1}$ being pseudo-effective or unitary flat (see Theorem 7);
- second-Chern-Ricci-flat metrics have Kod($X$) $\leq$ 0 (see Corollary 8);
- positive second-Chern-Einstein manifolds do not have non-trivial holomorphic $p$-forms; negative second-Chern-Einstein manifolds do not have non-trivial holomorphic $p$-vector-fields, for $p \geq 1$ (see Theorem 9).

We also noticed that second-Chern-Ricci metrics $g$ are weakly-$g$-Hermitian-Einstein [Kob80, LT95, Kob87], whence we get further obstructions as:

- the Bogomolov-Lübcke inequality (3.2) holds [Bog80, Lüb82];
- the Kobayashi-Hitchin correspondence assures that the holomorphic tangent bundle is $g$-semi-stable [Kob82, Lüb83].

We conclude by providing several examples. Clearly holomorphically-parallelizable manifolds are Chern-flat, hence strong-second-Chern-Einstein with zero Einstein factor. See [Boo58] for a characterization of compact Hermitian Chern-flat manifolds as quotients of complex Hermitian Lie groups. Furthermore, the classical Hopf manifold and the new examples by Fabio Podestà [Pod18] are strong-second-Chern-Einstein, both with positive Einstein factor. On the other end, in the literature we did not find any example of non-Kähler second-Chern-Einstein metric with negative Einstein factor, even local. The following result provide such examples on (non-compact) four-dimensional solvable Lie groups with invariant complex structures, as classified in [Sno86, Sno90, Ova00, ACHK15], see [Ova04]. Explicit computations are performed with the help of Sage [Sage].
There are four-dimensional solvable Lie groups endowed with invariant complex structures admitting strong-second-Chern-Einstein metrics with negative Einstein factors.

Compact quotients of four-dimensional solvable Lie groups [Has05] include also the non-Kähler examples of Inoue surfaces and Kodaira surfaces. As regards strong-second-Chern-Einstein metrics on these manifolds, we have the following:

**Theorem D** (see Theorem 11). Inoue surfaces and Kodaira surfaces, seen as quotients of solvable Lie groups endowed with invariant complex structures, do not admit any invariant second-Chern-Einstein metric. On the other hand, Kodaira surfaces admit first-Chern-Ricci-flat metrics.

**Remark 1.** We haven’t investigated the third way to contract the Chern curvature yet, since we are very skeptical about the possible geometric meaning of such quantity.

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1. **Preliminaries and notation**

Let $X^n$ be a complex $n$-dimensional manifold endowed with a Hermitian structure $h = g - \sqrt{-1} \omega$, where $g$ is the associated Riemannian metric and $\omega = g(\cdot, J \cdot)$ is the associated $(1, 1)$-form; here, we consider $T^* X$ endowed with the dual complex structure $J^\alpha := \alpha(J^{-1})$. The holomorphic tangent bundle $T_X$ is then a holomorphic Hermitian bundle, and we consider its Chern connection $\nabla^{CH}$, namely, the unique Hermitian connection on $T_X$ extending the Cauchy-Riemann operator $\bar{\partial}$. We denote by $\Theta := (\nabla^{CH})^2 \in \wedge^2(X; \text{End} T_X) \cong \wedge^2(X; T_X \otimes T_X)$ the curvature tensor of the Chern connection. In local coordinates $z^j$, denoting with $h_{k\bar{l}}$ the Hermitian matrix representing the Hermitian structure $h$, the curvature $\Theta(\omega)$ has the following expression:

$$
\Theta(\omega) = \Theta_{ij\bar{k}\bar{l}} \sqrt{-1} \, dz^i \wedge d\bar{z}^j \otimes \sqrt{-1} \, dz^k \wedge d\bar{z}^\ell
$$

where

$$
\Theta_{ij\bar{k}\bar{l}} = - \frac{\partial^2 h_{k\bar{l}}}{\partial z^i \partial \bar{z}^j} + h^{pq} \frac{\partial h_{k\bar{q}}}{\partial z^i} \frac{\partial h_{p\ell}}{\partial \bar{z}^j}.
$$

1.1. **Chern-Ricci curvatures and Chern-Einstein metrics.** There are essentially three, in general different, ways to contract the above quantity $\Theta(\omega)$. We can define two real $(1, 1)$-forms as follows, [Gau84, Section I.4], see also [LY14, Section 3.1.2]. In local coordinates, the first Chern-Ricci form is

$$
\text{Ric}^{(1)}(\omega) := \text{tr} \, \Theta(\omega) = \text{tr} \, \Theta_{ij\bar{k}\bar{l}} \sqrt{-1} \, dz^i \wedge d\bar{z}^j
$$

$$
= h^{k\bar{l}} \Theta_{ij\bar{k}\bar{l}} \sqrt{-1} \, dz^i \wedge d\bar{z}^j
$$

$$
= h^{k\bar{l}} \frac{\partial^2 \log \det (h_{k\bar{l}})}{\partial z^i \partial \bar{z}^j} \, dz^i \wedge d\bar{z}^j,
$$
and the second Chern-Ricci form is
\[
\text{Ric}^{(2)}(\omega) = \text{tr}_g \Theta_{i\bar{k}\bar{l}} \sqrt{-1} dz^i \wedge dz^k = h^{i\bar{j}} \Theta_{i\bar{j}\bar{k}} \sqrt{-1} dz^i \wedge dz^k.
\]
Note that Ric\(^{(1)}(\omega)\) is a closed (1, 1)-form that represents the first Chern class \(c_1(X) := c_1(T_X) = c_1(K_X^{-1}) \in H^2(X; \mathbb{R})\). More precisely, it represents the first Bott-Chern class \(c_1^{BC}(X) \in H^{1,1}_{BC}(X)\). On the other hand, Ric\(^{(2)}(\omega)\) is in general not closed.

We also notice that there is a third way to contract the curvature \(\Theta\), namely, define the tensor
\[
\text{Ric}^{(3)}_{i\bar{j}\bar{k}}(\omega) = \text{tr}_g \Theta_{i\bar{j}\bar{k}} = h^{i\bar{k}} \Theta_{i\bar{j}k} = h^{i\bar{k}} \Theta_{j\bar{j}k}.
\]
On a Kähler manifold, (or more in general for Hermitian metrics being Kähler-like in the sense of [YZ16],) the three Chern-Ricci curvatures coincide.

**Definition 2.** For \(i \in \{1, 2\}\) a Hermitian manifold \((X^n, g)\) is called \((i)\)-Chern-Einstein if
\[
\text{Ric}^{(i)}(\omega) = \lambda \omega,
\]
for some real-valued function \(\lambda \in C^\infty(X; \mathbb{R})\), which is called the Einstein factor, where \(\text{Ric}^{(i)}(\omega)\) denotes the \((i)\)-contraction as above and \(\omega\) denotes the natural defined (1, 1)-form associated to the metric.

Sometimes we distinguish between weak-(\(i\))-Chern-Einstein and strong-(\(i\))-Chern-Einstein metrics, according to the Einstein factor being a function, respectively, constant.

The Chern-scalar curvature is defined as
\[
S^{Ch}(\omega) := \text{tr}_g \text{Ric}^{(1)}(\omega) = \text{tr}_g \text{Ric}^{(2)}(\omega) = h^{i\bar{j}} h^{k\bar{l}} \Theta_{i\bar{j}k\bar{l}},
\]
and the third Chern-scalar curvature is defined as
\[
S^{(3)}(\omega) := h^{i\bar{j}} h^{k\bar{l}} \Theta_{i\bar{j}k\bar{l}}.
\]
(See [LY14, Equation (4.18)] for a comparison between \(S^{Ch}\) and \(S^{(3)}\).) Clearly, if \(\omega\) is either \((1)\)-Chern-Einstein or \((2)\)-Chern-Einstein with Einstein factor \(\lambda\), then \(S^{Ch}(\omega) = n\lambda\).

**Remark 3.** Compare [Gol69, Dră05, DV17, LU17] in the almost-complex setting.

1.2. Formulas for the conformal changes. Let \(\omega_f := e^f \omega\) be a conformal change of \(\omega\). Then we compute for the curvature form \(\Theta_f := \Theta(e^f \omega)\) of the Chern connection for such metric:
\[
(\Theta_f)_{i\bar{j}k\bar{l}} = e^f (\Theta_{i\bar{j}k\bar{l}} - h^{k\bar{l}} \partial_{\bar{j}}^2 f).
\]
Thus:
\[
\begin{align*}
\text{Ric}^{(1)}(\omega_f) &= \text{Ric}^{(1)}(\omega) - n \sqrt{-1} \partial \bar{\partial} f, \\
\text{Ric}^{(2)}(\omega_f) &= \text{Ric}^{(2)}(\omega) - \omega \Delta^{Ch} f,
\end{align*}
\]
where \(\Delta^{Ch}\) denotes the Chern-Laplacian with respect to \(\omega\), namely, \(\Delta^{Ch} f := \frac{1}{2} (\omega, dd^c f)_\omega = \Delta_f + (df, \partial)^\omega = -2h^{j\bar{k}} \partial_{\bar{j}}^2 f\), where \(\partial\) is defined by \(d\omega^{n-1} =: \partial \wedge \omega^{n-1}\). See [LU17, Corollary 4.4] in the more general almost-Hermitian setting.

2. First-Chern-Einstein metrics

Let \(X^n\) be a compact complex manifold endowed with a Hermitian metric \(\omega\). Consider the Chern connection \(\nabla^{Ch}\), and the \((1)\)-Chern-Ricci form. Denote by \(T(X, Y) := \nabla^{Ch}_X Y - \nabla^{Ch}_Y X - [X, Y]\) the torsion tensor of \(\nabla^{Ch}\), and by \(\tau := T_{jk}^k dz^j\) its trace-of-torsion form. In this
section, we investigate the condition weak-(1)-Chern-Einstein, namely, when \( \text{Ric}^{(1)}(\omega) = \lambda \omega \) for \( \lambda \in \mathcal{C}^\infty(X; \mathbb{R}) \).

First of all, we notice that the Einstein factor \( \lambda \) is constant non-zero if and only if the torsion form \( \tau \) vanishes identically, by [Gol56, Theorem 4]. In this case, the metric is actually Kähler, since \( d\text{Ric}^{(1)}(\omega) = 0 \). (The same conclusion by [Gol56] holds true for the strong-(3)-Chern-Ricci curvature, see [Bal85, Theorem 4.1]: namely, (3)-Chern-Einstein metrics with constant non-zero Einstein factor are actually Kähler.)

A way to construct weak-(1)-Chern-Einstein metrics is via the following completely elementary trick based on the \( \sqrt{-1}\partial\bar{\partial} \)-lemma. If \( X^n \) is compact and its first Chern class \( c_1(X) \) has a sign, i.e., there exists a Kähler metric \( \omega \in \pm 2\pi c_1(X) = \pm 2\pi c_1^{BC}(X) \in H^{1,1}_{BC}(X) \), then there exists a function \( f \in \mathcal{C}^\infty(X; \mathbb{R}) \) (the Ricci potential), unique up to a constant, such that \( \text{Ric}^{(1)}(\omega) = \pm \omega + \sqrt{-1}\partial\bar{\partial}f \). Consider the conformal metric \( \omega_f := e^f \omega \). Then, thanks to (1.1),

\[
\text{Ric}^{(1)}(\omega_f) = \text{Ric}^{(1)}(\omega) - \sqrt{-1}\partial\bar{\partial}f = \pm \omega = \pm e^{-\frac{f}{\lambda}} \omega_f.
\]

Note that if we instead assume \( c_1^{BC}(X) = 0 \), the same argument gives a weak-(1)-Chern-Ricci-flat metric, unique up to scaling in its conformal class.

Our first result shows that all weak-(1)-Chern-Einstein metrics with non-zero Einstein factor are constructed in the above way. Therefore, in non-Kähler geometry, the only interesting case is for (1)-Chern-Einstein-flat metrics. Note that our result has a flavor similar to the picture described in [LeB97, CLW08] in the case of complex surfaces.

**Theorem 4.** Let \((X^n, g)\) be a compact weak-(1)-Chern-Einstein manifold of dimension \( n \geq 2 \), and suppose that the Einstein factor \( \lambda \) is not identically equal to zero. Then:

- \( X \) is Fano or anti-Fano, i.e. the first Chern class \( c_1(X) \) has a sign;
- \( g \) is conformally equivalent to a Kähler metric \( \eta \in \pm 2\pi c_1(X) \);
- up to scaling, \( \lambda \) must be given by \( \pm e^{-\frac{f}{\lambda}} \), where \( f \) is a Ricci potential for \( \eta \).

In particular, such metrics are unique up to scaling in their conformal class.

**Proof.** Let \( \omega \) be a weak-(1)-Chern-Einstein metric with Einstein factor \( \lambda \) not identically equal to zero, and let \( \eta \) be the unique Gauduchon representative of volume one in its conformal class [Gau77b, Théorème 1]. Thus \( \omega = e^g \eta \) for some function \( g \). Define the function \( \tilde{\lambda} := \lambda e^g \) and consider the \((n-1,n)\)-form

\[
\tilde{\lambda}^{n-1}\sqrt{-1}\partial\bar{\lambda}^{n-1} \wedge \eta^{n-1}.
\]

Note that, by the Chern-Einstein condition,

\[
\partial(\tilde{\lambda}\eta) = \partial(\lambda \omega) = \partial(\text{Ric}^{(1)}(\omega)) = 0.
\]

Hence \( \tilde{\lambda} \partial \eta = -\partial \tilde{\lambda} \wedge \eta \). Now we compute

\[
\tilde{\lambda}^{n-1}\sqrt{-1}\partial\bar{\lambda}^{n-1} \wedge \eta^{n-1} = d\left(\tilde{\lambda}^{n-1}\sqrt{-1}\partial\bar{\lambda}^{n-1} \wedge \eta^{n-1}\right) - \partial\tilde{\lambda}^{n-1} \wedge \sqrt{-1}\partial\bar{\lambda}^{n-1} \wedge \eta^{n-1} + \tilde{\lambda}^{n-1}\sqrt{-1}\partial\bar{\lambda}^{n-1} \wedge \partial \eta^{n-1}
\]

\[
= d\left(\tilde{\lambda}^{n-1}\sqrt{-1}\partial\bar{\lambda}^{n-1} \wedge \eta^{n-1}\right).
\]
Recall that the Hodge-de Rham Laplacian $\Delta_{d,\eta} = [d, d^*\eta]$ and the Chern Laplacian $\Delta^{Ch}_{\eta} = 2\sqrt{-1} \text{tr}_{\eta} \partial \bar{\partial}$ on smooth functions are related by [Gau84, pages 502–503]:

$$\Delta^{Ch}_{\eta} f = \Delta_{d,\eta} f + (df, \theta_{\eta})_{\eta},$$

where $\theta_{\eta}$ denotes the co-closed torsion 1-form of the Gauduchon metric, that is, $d\eta^{n-1} = \theta_{\eta} \wedge \eta^{n-1}$, and $d^*\theta_{\eta} = 0$. Integrating on the compact manifold $X$, we get:

$$0 = \int_{X} \tilde{\lambda}^{n-1} \sqrt{-1} \partial \bar{\partial} \tilde{\lambda}^{n-1} \wedge \eta^{n-1} = \frac{1}{2n} \int_{X} \tilde{\lambda}^{n-1} \Delta^{Ch}_{\eta} \tilde{\lambda}^{n-1} \eta^{n}$$

$$= \frac{1}{2n} \int_{X} \tilde{\lambda}^{n-1} \Delta_{d,\eta} \tilde{\lambda}^{n-1} \eta^{n} + \frac{1}{2n} \int_{X} \tilde{\lambda}^{n-1} \langle d\tilde{\lambda}^{n-1}, \theta_{\eta} \rangle_{\eta} \eta^{n}$$

$$= \frac{1}{2n} \int_{X} |d\tilde{\lambda}^{n-1}|^2 \eta^{n} + \frac{1}{4n} \int_{X} \langle d\tilde{\lambda}^{2n-2}, \theta_{\eta} \rangle_{\eta} \eta^{n}$$

$$= \frac{1}{2n} \int_{X} |d\tilde{\lambda}^{n-1}|^2 \eta^{n}.$$ 

Thus our function $\tilde{\lambda}$ is a non-zero constant and $\eta$ is in fact Kähler.

Using again the Chern-Einstein condition we find

$$2\pi c_1(X) = 2\pi c_1^{BC}(X) \ni \text{Ric}^{(1)}(\eta) = \tilde{\lambda} \eta + \sqrt{-1} \partial \bar{\partial}(ng).$$

Hence the first Chern class $c_1(X) \leq 0$ depending on the value of $\tilde{\lambda}$, i.e., $X$ is Fano or anti-Fano. Thus our original weak-(1)-Chern-Einstein metric is conformally Kähler, and the Einstein factor $\lambda$ is given, up to scaling, by the Ricci potential of a Kähler metric in $\pm 2\pi c_1(X)$. Uniqueness up to scaling of weak-(1)-Chern-Einstein metrics in their conformal class follows immediately by the uniqueness up to constant of the Ricci potential. □

In particular, (1)-Chern-Einstein metrics on non-Kähler manifolds occur only for (1)-Chern-Ricci-flat metrics. In particular, $c_1^{BC}(X) = 0$, which also implies $c_1(X) = 0$, (but not the converse: compare e.g., the Hopf manifold in Example 3.3.1). In this case, the following result gives a complete description:

**Theorem 5** (see [Tos15, Proposition 1.1, Theorem 1.2], [TW10, Corollary 2]). On a compact Hermitian manifold $(X^n, g)$ with $c_1^{BC}(X) = 0$, there exists a strong-(1)-Chern-Einstein metric with Einstein factor $\lambda = 0$. It can be given either as a conformal transformation $e^\varphi g$ of $g$; or as associated to the form $\omega + \sqrt{-1} \partial \bar{\partial} \varphi$ where $\omega$ is the fundamental form of $g$. In both cases, $\varphi$ is unique up to additive constants.

In [STW17], see also [Pop15], by proving the Gauduchon’s generalization of the Calabi’s conjecture for compact complex manifolds satisfying $c_1^{BC}(X) = 0$, it is proved that it is also always possible to find Gauduchon (1)-Chern-Ricci-flat metrics, hence providing existence of non-Kähler special metrics satisfying both curvature and cohomological conditions.

3. Second-Chern-Einstein metrics

We consider now the second Chern-Ricci curvature, and the corresponding Chern-Einstein condition.

3.1. Second-Chern-Einstein metrics in the conformal class. The first fundamental remark is that, thanks to (1.2), the weak-(2)-Chern-Einstein property is a property of the
conformal Hermitian structure, and not just of the Hermitian structure. More precisely, under the conformal change \( \omega \mapsto \exp(-\lambda) \cdot \omega \), the Einstein factor changes as \( \lambda \mapsto \exp(\lambda) \cdot (\lambda + \Delta_{\omega}^Ch f) \). In particular, thanks to the Gauduchon conformal methods [Gau81], (see e.g., the preliminary step in the Proof of Theorem 4.1 in [ACS17],) we can assume that \( \lambda \) has a sign without loss of generality.

The sign is determined by an invariant of the conformal class as follows. In the conformal class \( \{\omega\} \), choose the unique Gauduchon metric \( \eta = \exp(\lambda) \cdot \omega \) with unitary volume, thanks to [Gau81, Théorème 1]. Define the Gauduchon degree of \( \{\omega\} \) as the degree of the anti-canonical line bundle \( K_X^{-1} \) with respect to the Gauduchon metric \( \eta \), namely,

\[
\Gamma_X(\{\omega\}) := \int_X c_1^{BC}(K_X^{-1}) \wedge \frac{1}{(n-1)!} \eta^{n-1} = \int_X S^Ch(\eta) \eta^n.
\]

Note indeed that, if \( \eta \) is weak-(2)-Chern-Einstein with Einstein factor \( \lambda_\eta \), so that \( \omega \) is weak-(2)-Chern-Einstein with Einstein factor \( \lambda_\omega = \exp(\lambda_\eta + \Delta_{\eta}^Ch f) \), then \( S^Ch(\eta) = n\lambda_\eta = ne^{-f}\lambda_\omega - \Delta_{\eta}^Ch f \), where \( \int_X \Delta_{\eta}^Ch f \eta^n = 0 \). In particular, if \( \text{Kod} X \geq 0 \), (respectively, \( \text{Kod} X > 0 \)) then by [Gau81], see also [Gau84, §1.17], we get that \( \Gamma_X(\{\eta\}) \leq 0 \), (respectively, \( \Gamma_X(\{\eta\}) < 0 \)) for any conformal class \( \{\eta\} \).

In the notation as above, the problem of finding a strong-(2)-Chern-Einstein metric in \( \{\omega\} \) with constant Einstein factor, reduces to solve the Liouville-type equation

\[
(\text{3.1}) \quad \Delta_{\eta}^Ch f + \frac{1}{n} S^Ch(\eta) = \lambda e^{-f}
\]

for \( (f, \lambda) \in C^\infty(X; \mathbb{R}) \times \mathbb{R} \). We introduced and investigated this equation (3.1) in [ACS17] under the name of Chern-Yamabe equation. In particular, we proved [ACS17, Theorems 3.1-4.1] that, if \( \Gamma_X(\{\omega\}) \) is non-positive, then the Chern-Yamabe equation admits a unique solution up to scaling. Summarizing:

**Theorem 6 ([Gau81], [ACS17, Theorems 3.1-4.1]).** Let \( (X^n, g) \) be a compact Hermitian manifold, and assume that the conformal class of \( g \) is weak-(2)-Chern-Einstein. Then we can choose a representative in the conformal class of \( g \) such that the Einstein factor has a sign, equal to the sign of the Gauduchon degree \( \Gamma_X(\{g\}) \).

Moreover, if \( \Gamma_X(\{g\}) \leq 0 \), (for example, if \( \text{Kod} X \geq 0 \),) then we can choose a representative in the conformal class having non-positive constant Einstein factor. If one can prove the Chern-Yamabe conjecture [ACS17, Conjecture 2.1], then the same holds true without any assumption on the sign of the Gauduchon degree.

Note in particular that the existence of a constant positive Einstein factor does not force its uniqueness in the conformal class, compare [ACS17, Section 5.5].

The possible values for the sign of weak-(2)-Chern-Einstein metrics are summarized in the following:

**Theorem 7 ([Tel06], [Yan17, Theorem 1.1, Theorem 3.4]).** Let \( X^n \) be a compact complex manifold. We look at the image of the application \( \Gamma_X \) that associates to each Hermitian conformal class \( \{\eta\} \in \text{HermConf}(X) \), its Gauduchon degree \( \Gamma_X(\{\eta\}) \in \mathbb{R} \):

- \( \Gamma_X(\text{HermConf}(X)) = \mathbb{R} \) if and only if neither \( K_X \) nor \( K_X^{-1} \) is pseudo-effective;
- \( \Gamma_X(\text{HermConf}(X)) = \mathbb{R}^{>0} \) if and only if \( K_X^{-1} \) is pseudo-effective and non-unitary-flat;
- \( \Gamma_X(\text{HermConf}(X)) = \mathbb{R}^{<0} \) if and only if \( K_X \) is pseudo-effective and non-unitary-flat;
• $\Gamma_X(\text{HermConf}(X)) = \{0\}$ if and only if $K_X$ is unitary-flat.

We recall that a holomorphic line bundle over $(X^n, g)$ compact Hermitian manifold is called pseudo-effective if it admits a (possibly singular) Hermitian metric with non-negative curvature (in the sense of currents); it is called unitary-flat if it admits a smooth Hermitian metric with zero curvature.

In particular, one gets an obstruction for the existence of weak-(2)-Chern-Einstein-flat metrics:

**Corollary 8 ([Gau81], [Yan17, Theorem 1.4]).** Let $X^n$ be a compact complex manifold. It admits a weak-(2)-Chern-Einstein metric with Einstein factor $\lambda = 0$ only if:
- either: Kod $X = -\infty$ and neither $K_X$ nor $K_X^{-1}$ is pseudo-effective;
- or: Kod $X = -\infty$ and $K_X$ is unitary-flat;
- or: Kod $X = 0$ and $K_X$ is holomorphically-torsion (namely, there exists $m \in \mathbb{N}$ such that $K_X^{\otimes m}$ is trivial; in particular, $K_X$ is unitary-flat).

Obstructions à la Bochner follow by [Gau77a], see also [KW70, Theorem at page 1] and, more in general, [LY12, Theorem 1.1], give further obstructions:

**Theorem 9 ([Gau77a, Corollaire 2 at page 124], [LY12, Corollary 1.2]).** Let $X^n$ be a compact complex manifold. If it admits a weak-(2)-Chern-Einstein metric with Einstein factor $\lambda \neq 0$, then:
- in case $\lambda \geq 0$, then $H^0_p(X; \mathcal{O}_X) = 0$ for any $p \geq 1$. In particular, the arithmetic genus is $\chi(X; \mathcal{O}_X) = 1$, by the Hirzebruch-Riemann-Roch theorem, and $X$ does not admit finite covers;
- in case $\lambda \leq 0$, then $H^0(X; \mathcal{T}^*_X) = 0$ for any $p \geq 1$. In particular, there are no non-trivial holomorphic vector fields.

As for strong-(2)-Chern-Einstein metric with Einstein factor $\lambda = 0$, then any holomorphic $p$-form and any holomorphic $p$-vector field is parallel with respect to the Chen connection.

By Theorem 6, we then know that the hypothesis of the above Theorem 9 hold depending on the sign of the Gauduchon degree of a (2)-Chern-Einstein conformal class.

### 3.2. Second-Chern-Einstein metrics as Hermitian-Einstein metrics on the tangent bundle.

Another observation is that weak-(2)-Chern-Einstein metrics $g$ yield that that the holomorphic tangent bundle $(T_X, g)$ is weakly-$g$-Hermitian-Einstein, see e.g. [Kob80, LT95, Kob87]. In particular, one has the obstruction given by the Bogomolov-Lübke inequality [Bog80, Lüb82], see e.g. [LT95, Theorem 2.2.3]:

\[(n-1)c_1(X)^2 - 2nc_2(X) \wedge \omega^{n-1} \leq 0,\]

and the Kobayashi-Hitchin correspondence [Kob80, Kob82, Lüb83], see e.g. [LT95, Theorem 2.3.2], namely, the holomorphic tangent bundle is $g$-semi-stable.

In analogy to the Kähler-Einstein/K-stable case, it seems interesting to understand if there is a more refined version of stability of the complex manifold itself (plus some extra data, such as the Aeppli class of the $(n-1)$th power of the Gauduchon representative of a conformal class) that characterizes existence of (2)-Chern-Einstein metrics.

**Remark 10.** Projectively flat metrics on the holomorphic tangent bundle of a compact complex manifold are a special instance of second-Chern-Einstein metrics. In [Cal17] is given a classification of them and is proven that each of them is strong-second-Chern-Einstein;
3.3. Examples of second-Chern-Einstein metrics. In this section we concentrate on examples of Chern-Einstein metrics. We begin with a first, classical and illustrative example:

3.3.1. Hopf manifold. The Hopf surface admits a strong-(2)-Chern-Einstein Hermitian metric with constant positive Einstein factor, see e.g. [Yan17, Example 5.1], [LY12, Section 6], [LY14, Theorem 1.5].

Let $X^2 = (\mathbb{C}^2 \setminus 0)/\Gamma$ be a Hopf surface, namely, a compact complex surface with universal cover $\mathbb{C}^2 \setminus 0$ and fundamental group isomorphic to $\mathbb{Z}$. We have $\text{Kod} X = -\infty$. The anti-canonical bundle $K_X^{-1}$ is pseudo-effective, while the canonical bundle $K_X$ is not pseudo-effective. In particular, the Gauduchon degree is positive, $\Gamma_X(\{\eta\}) > 0$, for any conformal class $\{\eta\}$. The first Bott-Chern class is not zero, $c_{BC}^1(X) \neq 0$, while its image in de Rham cohomology is zero, $c_1(X) = 0$. We have that $h^{p,0}(X) = 0$ and $H^0(X; \wedge^p T_X) = 0$ for any $p \geq 1$, and that the holomorphic tangent bundle $T_X$ is semi-stable.

Thanks to Theorem 4, the manifold $X$ does not admit any (1)-Chern-Einstein metric, because $c_{BC}^1(X) \neq 0$ and it is not Kähler. On the other hand, consider Hopf surfaces with $\Gamma$ generated by $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ where $0 < |\alpha| = |\beta| < 1$. Consider the standard locally conformally Kähler Hermitian metric induced by a conformal change of the flat metric on $\mathbb{C}^2$ as below with $r = 1$: it is straightforward to see that it is strong-(2)-Chern-Einstein with constant positive Einstein factor $\lambda = 2$ (e.g. [LY12, Section 6]). More concretely, looking at the diffeomorphism type $X \simeq S^1 \times SU(2)$, the manifold $X$ is described by a coframe of global $(1,0)$-forms $\{\varphi^1, \varphi^2\}$ with structure equations

$$d\varphi^1 = \sqrt{-1} \varphi^1 \wedge \varphi^2 + \sqrt{-1} \varphi^1 \wedge \bar{\varphi}^2, \quad d\varphi^2 = -\sqrt{-1} \varphi^1 \wedge \bar{\varphi}^1.$$

For the invariant metric with fundamental form

$$\omega = \frac{\sqrt{-1}}{2} r^2 \varphi^1 \wedge \bar{\varphi}^1 + \frac{\sqrt{-1}}{2} r^2 \varphi^2 \wedge \bar{\varphi}^2,$$

where $r \in \mathbb{R} \setminus \{0\}$, we compute the Chern curvature: the only non-zero components are

$$\Theta_{1111} = \frac{1}{2} r^2, \quad \Theta_{1122} = -\frac{1}{2} r^2,$$

and the ones corresponding to the symmetries. In particular, the first-Chern-Ricci form is

$$\text{Ric}^{(1)} = 2 \sqrt{-1} \varphi^1 \wedge \bar{\varphi}^1$$

and the second-Chern-Ricci form is

$$\text{Ric}^{(2)} = \sqrt{-1} \varphi^1 \wedge \bar{\varphi}^1 + \sqrt{-1} \varphi^2 \wedge \bar{\varphi}^2 = \frac{2}{r^2} \omega.$$

namely, the standard metric $\omega$ is strong-(2)-Chern-Einstein with constant Einstein factor $\lambda = \frac{2}{r^2}$. The Chern-scalar curvature is then $S^{\text{Ch}} = \frac{2}{r^2}$.

We also notice that the third Chern-Ricci tensor has the only non-zero component $\text{Ric}^{(3)}_{111} = 1$, and it gives the same third Chern-scalar curvature $S^{(3)} = S^{\text{Ch}}$.

3.3.2. Podestà examples. More examples of strong-(2)-Chern-Einstein metrics with constant positive Einstein factor are provided by F. Podestà in [Pod18] among $\mathcal{C}$-manifolds. They are
compact simply-connected complex homogeneous spaces, given by the product of two $G_j$-homogeneous spaces of the form $G_j/L_j$ where $L_j$ is a connected subgroup of $G_j$ that coincides with the semisimple part of the centralizer of a torus in $G_j$, for $j \in \{1, 2\}$. Manifolds in such class are $\mathbb{T}^2$-bundle over the product of two compact irreducible Hermitian symmetric spaces, and can be endowed with a two-parameter family of inequivalent invariant complex structures, which do not admit either Kähler or balanced metrics. Examples are given by the Calabi-Eckmann manifolds. In [Pod18, Theorem 3], it is shown that, for any such invariant complex structure, there exists an invariant Hermitian metric being strong-(2)-Chern-Einstein with Einstein factor equal to 1.

### 3.3.3. Compact complex surfaces diffeomorphic to solvmanifolds

Other than the Kähler-Einstein case, compact complex surfaces being first-Chern-Einstein are completely understood thanks to the condition $c_1^{BC}(X) = 0$: they include Kodaira surfaces beside Calabi-Yau surfaces (namely, complex 2-dimensional tori, Enriques surfaces, bi-elliptic surfaces, K3 surfaces). We consider now compact complex (non-Kähler) surfaces diffeomorphic to solvmanifolds, namely, compact quotients of solvable Lie groups, according to [Has05, Theorem 1]. Besides the Kähler case (complex torus, hyperelliptic surface), we have Inoue surfaces and Kodaira surfaces, endowed with invariant complex structures. We consider invariant Hermitian metrics: in the notation above (we recall that $g = \omega(J_{-1})$, and $J_{\alpha} = \alpha(J^{-1}_{-1})$ on the dual), with respect to a chosen coframe $\{\varphi^1, \varphi^2\}$ of invariant $(1, 0)$-forms, invariant Hermitian metrics are given by

$$\omega = \frac{\sqrt{-1}}{2} r^2 \varphi^1 \wedge \varphi^1 + \frac{\sqrt{-1}}{2} s^2 \varphi^2 \wedge \varphi^2 + \frac{1}{2} u \varphi^1 \wedge \varphi^2 - \frac{1}{2} \bar{u} \varphi^2 \wedge \varphi^1$$

where $r, s \in \mathbb{R} \setminus \{0\}$ and $u \in \mathbb{C}$ such that $r^2 s^2 - |u|^2 > 0$.

We give here their curvature tensors, showing by explicit computations that:

**Theorem 11.** On Inoue and Kodaira surfaces, seen as quotients of solvable Lie groups endowed with invariant complex structures, there is no invariant Hermitian metric being strong-(2)-Chern-Einstein.

**Proof.** The proof is based on a case-by-case check.

**Inoue $S_M$:** The structure equations can be given as $d \varphi^1 = \sqrt{-1} r^2 \varphi^1 \wedge \varphi^2 - \sqrt{-1} s^2 \varphi^1 \wedge \varphi^2$, $d \varphi^2 = \sqrt{-1} s^2 \varphi^2 \wedge \varphi^2$. The first-Chern-Ricci form is $\text{Ric}^{(1)} = -\sqrt{-1} s^2 \varphi^2 \wedge \varphi^2$ and the Chern-scalar curvature is $S^{Ch} = -\frac{2}{8(r^2 s^2 - |u|^2)^2} < 0$. We also report the third-Chern-scalar curvature: $S^{(3)} = -\frac{r^2 (8 r^2 s^2 + |u|^2)^2}{8(r^2 s^2 - |u|^2)^2}$. Looking at the second-Chern-Einstein tensor $\text{Ric}^{(2)}(\omega) - \lambda \omega$ (whose vanishing forces $\lambda = \frac{1}{2} S^{Ch} < 0$) as for the coefficient in $\varphi^1 \wedge \varphi^1$, we get $8\lambda r^2 r^2 s^2 - |u|^2) < 0$, so the second-Chern-Einstein equation does not have any solution.

**Inoue $S^\pm$:** The structure equations can be given as $d \varphi^1 = \frac{1}{2\sqrt{-1}} r^2 \varphi^1 \wedge \varphi^2 + \frac{1}{2\sqrt{-1}} s^2 \varphi^2 \wedge \varphi^1$, $d \varphi^2 = \frac{1}{2\sqrt{-1}} s^2 \varphi^2 \wedge \varphi^2$. The first-Chern-Ricci form is $\text{Ric}^{(1)} = -\frac{1}{2\sqrt{-1}} s^2 \varphi^1 \wedge \varphi^2$ and the Chern-scalar curvature is $S^{Ch} = -\frac{1}{8(r^2 s^2 - |u|^2)^2} < 0$. In fact, recall that, by [Tel06, Remark 4.2], it holds $\Gamma_X(\{\omega\}) < 0$ for any conformal class $\{\omega\}$. Looking at the second-Chern-Einstein tensor $\text{Ric}^{(2)}(\omega) - \lambda \omega$ (whose vanishing forces $\lambda = \frac{1}{2} S^{Ch} < 0$) as for the coefficient in $\varphi^1 \wedge \varphi^1$, we get $2\lambda r^2 (r^2 s^2 - |u|^2)^2 - r^4 (r^2 + r^2 s^2 + |u|^2 + 2 Re u^2) < 0$ since $r^2 s^2 + |u|^2 + 2 Re u^2 \geq r^2 s^2 + |u|^2 - 2 |u|^2 = r^2 s^2 - |u|^2 > 0$, so, there is no
second-Chern-Einstein metric on $S^+$. In fact, $S^+$ admits non-trivial holomorphic vector fields by [Ino74, Proposition 3].

**primary Kodaira:** The structure equations can be given as $d\varphi^1 = 0$, $d\varphi^2 = \sqrt{-1} \varphi^1 \wedge \bar{\varphi}^1$. It is known that the first-Chern-Ricci form is zero: $\text{Ric}^{(1)} = 0$, for any invariant metric. The Chern-scalar curvature is then zero, too; the third-Chern-scalar curvature is $S^{(3)} = \frac{-\ell^2 s^4 u}{4 (r^2 s^2 - |u|^2)^2}$. Looking at the second-Chern-Einstein equation

\[ \text{Ric}^{(2)}(\omega) - \lambda \omega = 0 \]

(whence $\lambda = \frac{1}{2} S^{Ch} = 0$) as for the coefficient in $\varphi^1 \wedge \varphi^2$, we get $u (2\lambda (r^2 s^2 - |u|^2)^2 - s^6) = 0$, whence $u = u_0$; then the coefficient in $\varphi^1 \wedge \bar{\varphi}^1$, that is $\sqrt{-1} s^4 (r^2 s^2 - 2|u|^2)$, becomes $\sqrt{-1} r s^6$, which is never zero.

**secondary Kodaira:** The structure equations can be given as $d\varphi^1 = -\frac{1}{2} \varphi^1 \wedge \varphi^2 + \frac{1}{2} \varphi^1 \wedge \bar{\varphi}^2$, $d\varphi^2 = \sqrt{-1} \varphi^1 \wedge \bar{\varphi}^1$. Clearly we known that the invariant metrics are first-Chern-Ricci-flat. In particular, $S^{Ch} = 0$. Looking at the second-Chern-Einstein equation $\text{Ric}^{(2)}(\omega) - \lambda \omega = 0$, where in fact $\lambda = 0$, as for the coefficient in $\varphi^1 \wedge \varphi^2$, we get $-us^2 (\sqrt{-1} (r^2 s^2 - |u|^2)^2 + (r^4 + s^4)) = 0$, whence $u = u_0$; then the coefficient in $\varphi^1 \wedge \bar{\varphi}^1$ becomes $\sqrt{-1} s^2 \ell^2 u$, which is never zero. $\square$

### 3.3.4. Snow manifold $S^5$. Here we give an example of a non-compact strong-(2)-Chern-Einstein manifold with constant zero Einstein factor. It is given by an invariant Hermitian metric on a four-dimensional simply-connected solvable real Lie group endowed with an invariant complex structure. Complex structures on four-dimensional Lie algebras are classified by [Sno86, Sno90, Ova00, ACHK15], see [Ova04].

More precisely, we consider the Lie group $S^5$ with structure equations $[X,Y] = Y$, $[X,W] = \ell W$ with $\ell \neq 0$ a real parameter, the other brackets being zero, with respect to a basis $\{X,Y,W,Z\}$. The invariant complex structure is given by the coframe of $(1,0)$-forms $\{\varphi^1, \varphi^2\}$ such that

\[ dp^1 = 0, \quad dp^2 = \frac{\ell}{2} \varphi^1 \wedge \varphi^2 - \frac{\ell}{2} \varphi^2 \wedge \bar{\varphi}^1. \]

It is not holomorphically-parallelizable. Consider the generic Hermitian metric

\[ \omega = \sqrt{-1} r s \varphi^1 \wedge \bar{\varphi}^1 + \sqrt{-1} s^2 \varphi^2 \wedge \bar{\varphi}^2 + u \varphi^1 \wedge \bar{\varphi}^2 - \bar{u} \varphi^2 \wedge \bar{\varphi}^1 \]

where $r \neq 0$, $s \neq 0$, $r^2 s^2 - |u|^2 > 0$. One sees that it is not Kähler. We notice that such a metric is locally conformally Kähler if and only if $u = u_0$, and in this case the Lee form is $\theta = \frac{1}{2} \ell \varphi^1 + \frac{1}{2} \ell \bar{\varphi}^1$ (we recall that this means that $d\omega = \theta \wedge \omega$ with $d\theta = 0$). Explicit computations show that

\[ \text{Ric}^{(1)} = 0, \]

and that the non-zero component of $\text{Ric}^{(2)}$ are

\[ \text{Ric}^{(2)}_{11} = \frac{\ell^2 s^4 |u|^2}{4 (r^2 s^2 - |u|^2)^2}, \quad \text{Ric}^{(2)}_{12} = -\frac{\sqrt{-1} \ell^2 r s^4 u}{4 (r^2 s^2 - |u|^2)^2}, \quad \text{Ric}^{(2)}_{22} = \frac{\ell^2 s^4 |u|^2}{4 (r^2 s^2 - |u|^2)^2}, \]

and the symmetric ones. Moreover, the Chern-scalar curvatures is clearly $S^{Ch} = 0$. Summarizing, $\omega$ is always strong-(1)-Chern-Einstein with zero Einstein factor, for any value of the parameters $r$, $s$, $u$. Moreover, metrics with $u = 0$ are also strong-(2)-Chern-Einstein with zero Einstein factor (in fact, they are Chern-flat).

For the sake of completeness, we also list the non-zero component of the third Chern-Ricci tensor: $\text{Ric}^{(3)}_{12} = -\frac{\sqrt{-1} \ell^2 s^2 u}{4 (r^2 s^2 - |u|^2)^2}$, and the third Chern-scalar curvature $S^{(3)} = -\frac{\ell^2 s^2 |u|^2}{2 (r^2 s^2 - |u|^2)^2}$. 

We finally describe examples of complete negative Chern-Einstein metrics on Ovando manifolds:

3.3.5. Ovando Kähler manifold $\mathfrak{t}_2\mathfrak{t}_2$. We give an example of a non-compact complete Kähler-Einstein manifold with negative Einstein factor. It is given on the four-dimensional Lie group $\mathfrak{t}_2\mathfrak{t}_2 = (0, -12, 0, -34)$ in Salamon notation, with invariant complex structure characterized by the coframe $(\varphi^1, \varphi^2)$ of invariant $(1, 0)$-forms with structure equations

$$d\varphi^1 = -\frac{1}{2} \varphi^1 \wedge \varphi^1, \quad d\varphi^2 = -\frac{1}{2} \varphi^2 \wedge \varphi^2.$$  

We consider an invariant metric $\omega$ of the form (3.3). We compute the Chern-Ricci forms:

$$\text{Ric}^{(1)}(\omega) = -\sqrt{-1} \frac{1}{2} \varphi^1 \wedge \bar{\varphi}^1 - \sqrt{-1} \frac{1}{2} \varphi^2 \wedge \bar{\varphi}^2,$$

$$\text{Ric}^{(2)}(\omega) = -\sqrt{-1} \left( 2 r^4 s^4 - \sqrt{-1} r^2 u^2 \bar{u} - \left( -\sqrt{-1} r^2 u^2 + (r^4 + 3 r^2 s^2) u \bar{u} \right) \right) \varphi^1 \wedge \bar{\varphi}^1$$

$$- \sqrt{-1} \left( r^2 s^2 u^2 - (r^2 s^2 u - (\sqrt{-1} r^2 + \sqrt{-1} s^2) u \bar{u}) \right) \varphi^1 \wedge \bar{\varphi}^2$$

$$+ \sqrt{-1} \left( r^2 s^2 u^2 - (r^2 s^2 - (\sqrt{-1} r^2 + \sqrt{-1} s^2) u \bar{u}) \right) \varphi^2 \wedge \bar{\varphi}^1$$

$$- \sqrt{-1} \left( 2 r^4 s^4 - \sqrt{-1} r^2 u^2 \bar{u} - \left( -\sqrt{-1} s^2 u^2 + (3 r^2 s^2 + s^4) u \bar{u} \right) \right) \varphi^2 \wedge \bar{\varphi}^2.$$  

For the Kähler diagonal metric

$$\omega_{\text{diag}} = \sqrt{-1} \frac{1}{2} \varphi^1 \wedge \bar{\varphi}^1 + \sqrt{-1} \frac{1}{2} \varphi^2 \wedge \bar{\varphi}^2$$

corresponding to the parameters $r = 1$, $s = 1$, $u = 0$, we get

$$\text{Ric}^{(2)}(\omega_{\text{diag}}) = \text{Ric}^{(1)}(\omega_{\text{diag}}) = -\sqrt{-1} \frac{1}{2} \varphi^1 \wedge \bar{\varphi}^1 - \sqrt{-1} \frac{1}{2} \varphi^2 \wedge \bar{\varphi}^2 = -\omega_{\text{diag}},$$

that is, the diagonal metric is complete Kähler-Einstein with negative Einstein factor.

3.3.6. Ovando non-Kähler manifold $\mathfrak{t}_{4,-1,-1}$. We give an example of a non-compact non-Kähler strong-(2)-Chern-Einstein manifold with negative Einstein factor. It is given on the four-dimensional Lie group $\mathfrak{t}_{4, \alpha, \beta} = (14, \alpha 24, \beta 34, 0)$ with $\alpha = \beta = -1$, in Salamon notation, with invariant complex structure characterized by the coframe $(\varphi^1, \varphi^2)$ of invariant $(1, 0)$-forms with structure equations

$$d\varphi^1 = -\sqrt{-1} \frac{1}{2} \varphi^1 \wedge \bar{\varphi}^1, \quad d\varphi^2 = -\sqrt{-1} \frac{1}{2} \varphi^1 \wedge \bar{\varphi}^2 - \sqrt{-1} \frac{1}{2} \varphi^2 \wedge \bar{\varphi}^1.$$  

We consider an invariant metric $\omega$ of the form (3.3). By computing $d\omega = -\sqrt{-1} \bar{u} \varphi^{121} - \frac{1}{2} s^2 \varphi^{122} + \sqrt{-1} u \varphi^{112} - \frac{1}{2} s^2 \varphi^{212}$, we notice that $\omega$ is never Kähler.

We compute the Chern-Ricci forms:

$$\text{Ric}^{(1)}(\omega) = -\sqrt{-1} \frac{1}{2} \varphi^1 \wedge \bar{\varphi}^1,$$

$$\text{Ric}^{(2)}(\omega) = -\sqrt{-1} \frac{1}{2} \varphi^1 \wedge \bar{\varphi}^1 - \frac{s^2 u}{2 (r^2 s^2 - |u|^2)} \varphi^1 \wedge \bar{\varphi}^1.$$
\[
+ \frac{s^2 \bar{u}}{2 (r^2 s^2 - |u|^2)} \varphi^2 \wedge \bar{\varphi}^1 - \frac{\sqrt{-1} s^4}{2 (r^2 s^2 - |u|^2)} \varphi^2 \wedge \bar{\varphi}^2 \\
= - \frac{s^2}{r^2 s^2 - |u|^2} \left( \frac{\sqrt{-1}}{2} r^2 \varphi^1 \wedge \bar{\varphi}^1 + \frac{1}{2} u \varphi^1 \wedge \bar{\varphi}^2 - \frac{1}{2} \bar{u} \varphi^2 \wedge \bar{\varphi}^1 + \frac{\sqrt{-1}}{2} s^2 \varphi^2 \wedge \bar{\varphi}^2 \right) \\
= \frac{1}{2} S_{Ch}^\omega(\omega).
\]

where the Chern-scalar curvature is
\[
S_{Ch}^\omega(\omega) = - \frac{2 s^2}{r^2 s^2 - |u|^2} < 0.
\]

Clearly, \( \omega \) is never \((1)\)-Chern-Einstein, and is always strong-\((2)\)-Chern-Einstein with negative Einstein factor.

**Remark 12.** It would be very interesting to find (if any!) examples of negative compact non-Kähler \((2)\)-Chern-Einstein metrics.

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