On the thermodynamics of the black hole and hairy black hole transitions in the asymptotically flat spacetime with a box

Yan Peng\textsuperscript{1*}, Bin Wang\textsuperscript{2,3†}, Yunqi Liu\textsuperscript{4‡}

\textsuperscript{1} School of Mathematical Sciences, Qufu Normal University, Qufu, Shandong 273165, China
\textsuperscript{2} Center for Gravitation and Cosmology, College of Physical Science and Technology, Yangzhou University, Yangzhou 225009, China
\textsuperscript{3} Center of Astronomy and Astrophysics, Department of Physics and Astronomy, Shanghai Jiao Tong University, Shanghai 200240, China and
\textsuperscript{4} School of Physics, Huazhong University of Science and Technology, Wuhan, Hubei 430074, China

Abstract

We study the asymptotically flat quasi-local black hole/hairy black hole model with nonzero mass of the scalar field. We disclose effects of the scalar mass on transitions in a grand canonical ensemble with condensation behaviors of a parameter $\psi_2$, which is similar to approaches in holographic theories. We find that more negative scalar mass makes the phase transition easier to happen. We also obtain an analytical relation $\psi_2 \propto (T_c - T)^{1/2}$ around the critical phase transition points implying a second order phase transition. Besides the parameter $\psi_2$, we show that metric solutions can be used to disclose properties of transitions. In this work, we observe that phase transitions in a box are strikingly similar to holographic transitions in the AdS gravity and the similarity provides insights into holographic theories.

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* yanpengphy@163.com
† wang_b@sjtu.edu.cn
‡ liyunqi@hust.edu.cn
I. INTRODUCTION

As is well known, the asymptotically flat Schwarzschild black holes usually have negative specific heat and thus cannot be in equilibrium with the thermal radiation environment. In order to overcome this problem, York and other authors provided a simple way of enclosing the black hole in a box \[1, 2\]. It has been found that black holes could have positive specific heat in this quasi-local ensemble and are thermodynamically stable for certain range of parameters. On the other side, since the AdS boundary could play a role of the box boundary condition in a sense, the AdS black hole is usually stable \[3\]. According to the AdS/CFT correspondence \[4–6\], holographic superconductors constructed in the AdS spacetime have attracted a lot of attentions \[7–34\]. Partly with the interest of holography, there are literatures paying attention to the similarity between the asymptotically flat gravity with box boundary and the AdS gravity.

For anti-de Sitter space, the phase structure of charged black holes has served as a valuable test of the AdS/CFT correspondence. Compared with the AdS gravity, similar phase structures and same critical transition exponents have been found in the asymptotically flat gravity with a box boundary and it was argued that the holography first discovered in the AdS spacetimes may not be only limited to happen there \[35–37\]. The similarity of phase transitions observed in the quasi-local gravity may cast light on proposals for finite-volume holographic theories. It is more interesting to further disclose similarities in the AdS gravity and the asymptotically flat gravity with a box boundary.

The Einstein-Maxwell systems with box boundary conditions were studied in \[38–40\] and it was shown that overall phase structures of these gravity systems are similar to those of the AdS gravity. As a further step, it is interesting to generalize the system to a more completed transition model by including an additional scalar field. On the other side, it is also meaningful to study the similarity between transitions of this Einstein-Maxwell-scalar system in a box and those of the s-wave holographic superconductors constructed with scalar fields coupled to Maxwell fields in AdS background \[38–40\]. Recently, P. Basu, C. Krishnan and P.N.B. Subramanian initiated a thermodynamical study of such Einstein-Maxwell-scalar systems on asymptotically flat background with reflecting mirror-like boundary conditions for the scalar field \[41\]. For certain range of parameters, this model admits stable hairy black hole solutions, which provides a way to evade the flat space no-hair theorems. Another important conclusion is that the overall phase structure of this gravity system in a box is strikingly similar to that of holographic superconductor systems in the AdS gravity \[42, 43\]. By
choosing different values of the scalar charge, scalar mass and Stückelberg mechanism parameters, we found that effects of model parameters in this flat space/boson star system are qualitatively the same to those in the holographic insulator/superconductor model [43, 44]. Moreover, we also showed that operators on the box boundary can be used to detect properties of the bulk transitions and for the second order transitions, there exists a characteristic exponent in accordance with cases in AdS gravity systems. So it is interesting to extend the discussion in the horizonless spacetime in [44] to the background of black holes, which have been studied a lot in holographic theories. On the other hand, it is also meaningful to generalize the quasi-local black hole/hairy black hole transition model in [41] by considering nonzero mass of the scalar filed since the mass usually plays a crucial role in determining properties of transitions.

It was shown in holographic superconductor theories that more negative mass corresponds to a larger holographic conductor/superconductor phase transition temperature or smaller mass makes the transition more easier to happen [45–47], so it is interesting to go on to compare effects of the scalar mass on transition in a box and those of holographic superconductor transitions constructed in the AdS gravity. On the other side, it was shown in [48] that hairy black holes in a box can be formed dynamically through the superradiant procedure and effects of scalar mass on dynamical stability of phases have been investigated in [49, 50]. So it is also interesting to further study thermodynamical properties of such quasi-local black hole/hairy black hole system with nonzero scalar mass and disclose how the scalar mass could affect the thermodynamical phase transitions.

This paper is organized as follows. In section II, we introduce the black hole/hairy black hole model in a box with nonzero mass of the scalar field away from the probe limit. In section III, we manage to use condensation behaviors of a parameter to disclose properties of phase transitions with various scalar mass in a grand canonical ensemble. The last section is devoted to conclusions.

II. EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

We study the formation of scalar hair on the background of four dimensional asymptotically flat spacetime in a box. In this paper, we choose a fixed radial coordinate $r = r_b$ as the time-like box boundary. And the corresponding Einstein-Maxwell-scalar Lagrange density reads [41]:

$$\mathcal{L} = R - F^{MN} F_{MN} - |\nabla \psi - iqA\psi|^2 - m^2 \psi^2,$$ (1)
where \( \psi(\mathbf{r}) \) is the scalar field with mass \( m^2 \) and \( A_M \) stands for the ordinary Maxwell field. \( q \) is the charge of the scalar field serving as a coupling parameter between the scalar field and the Maxwell field. Here, \( R \) is the Ricci scalar tensor and \( F = dA \).

For simplicity, we would like to consider the scalar field and the Maxwell field only depending on the radial coordinates as

\[
\psi = \psi(r), \quad A = \phi(r)dt.
\]

(2)

Since we are interested in including the matter fields’ backreaction on the background, we assume the ansatz of the geometry of the 4-dimensional hairy black hole solution in the form \[41\]

\[
ds^2 = -g(r)h(r)dt^2 + \frac{dr^2}{g(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2).
\]

(3)

where the Hawking temperature of the black hole reads

\[
T = \frac{g'(r_h)\sqrt{h(r_h)}}{4\pi}
\]

and \( r_h \) is the horizon of the black hole satisfying \( g(r_h) = 0 \).

We can obtain equations of motion for matter fields and metric solutions as

\[
\frac{1}{2} \psi''(r) + \frac{g'(r)}{rg(r)} + \frac{g^2 \psi(r)^2 \phi(r)^2}{2g(r)^2 h(r)} - \frac{1}{2r^2 g(r)} + \frac{1}{r^2} + \frac{m^2}{2g} \psi^2 = 0,
\]

(4)

\[
h'(r) - rh(r)\psi'(r)^2 - \frac{g^2 r \psi(r)^2 \phi(r)^2}{g(r)^2} = 0,
\]

(5)

\[
\phi'' + \frac{2\phi'(r)}{r} - \frac{h'(r)\phi'(r)}{2h(r)} - \frac{g^2 \psi(r)^2 \phi(r)}{2g(r)} = 0,
\]

(6)

\[
\psi''(r) + \frac{g'(r)\psi'(r)}{g(r)} + \frac{h'(r)\psi'(r)}{2h(r)} + \frac{2\psi'(r)}{r} + \frac{g^2 \psi(r) \phi(r)^2}{g(r)^2 h(r)} - \frac{m^2}{g} \psi = 0.
\]

(7)

In order to describe transitions in detail, we apply the shooting method to integrate these coupled nonlinear differential equations from \( r = r_h \) to box boundary \( r = r_b \) to search for the numerical solutions with box boundary conditions. At the horizon of the black hole \( r = r_h \), we impose Taylor expansions of solutions as \[41\]

\[
\psi(r) = aa + bb(r - r_h) + cc(r - r_h)^2 + \cdots, \\
\phi(r) = aaa(r - r_h) + bbb(r - r_h)^2 + \cdots, \\
g(r) = AAA(r - r_h) + BBB(r - r_h)^2 + \cdots, \\
h(r) = 1 + AAA(r - r_h) + \cdots,
\]

(8)
where $aa, bb, \cdots, AAA$ are parameters and the dots denote higher order terms. Putting these expansions into equations of motion, we could use three independent parameters $r_h, aa$ and $aaa$ to describe the solutions. The scaling symmetry $r \to \alpha r$ can be used to set $r_h = 1$. Around the box boundary ($r_h = 1$), we assume asymptotic behaviors of the scalar field and the Maxwell field as

$$\psi \to \psi_1 + \psi_2(1 - r) + \cdots,$$  \hspace{1cm} (9)

$$\phi \to \phi_1 + \phi_2(1 - r) + \cdots,$$  \hspace{1cm} (10)

where $\mu = \phi(1) = \phi_1$ is interpreted as the chemical potential. In this paper, we will fix the chemical potential and work in a grand canonical ensemble. With the symmetry $h \to \beta^2 h$, $\phi \to \phi$, $t \to \frac{t}{\beta}$, we make a transformation to set $g_{tt}(1) = -1$. Since there are reflecting mirror-like boundary conditions for the scalar field as $\psi(r_b) = 0$, we fix $\psi_1 = 0$ instead and try to use another parameter $\psi_2$ to describe the phase transition, which is similar to approaches in holographic superconductor theories. Here, we point out that this box boundary condition $\psi_1 = 0$ is independent of the scalar mass, which is different from cases in holographic superconductor theories where asymptotic behaviors of scalar fields at the infinity boundary usually depend on the scalar mass. And we will show in the following section that $\psi_2$ is a good probe to the critical temperature and also the order of transitions in a box.

### III. PROPERTIES OF PHASE TRANSITIONS IN A BOX

In this part, we firstly plot the numerical solutions as a function of the radial coordinate with $q = 100$, $m^2 = 0$, $\mu = 0.15$ and $\psi(r_h) = 0.1$ in Fig. 1. In this paper, we take $q = 100$ as an example for reasons that hairy black holes in a box are usually only global stable for large charge of the scalar field [41]. When choosing $m^2 = 0$, our results are related to the right panel of Fig. 11 in [41]. It can be easily seen from the left panel of Fig. 1 that this gravity system admits scalar hairy black hole solutions and at the boundary there is reflecting condition for the scalar field or $\psi(r_b) = 0$. We also represent behaviors of the metric solutions $h(r)$ in the right panel. Since we have $h(r) = 1$ for cases in the probe limit, behaviors of curves in the right panel show that the metric is deformed when considering the matter fields’ backreaction on the background.

According to the no-hair theorem, the scalar field cannot attach to the black hole in the usual situation [51–57]. The hairy black holes constructed in AdS spacetimes provide a challenge to the black hole no-hair theorem. In the AdS gravity, there is an infinite potential wall at the AdS boundary to confine the scalar field [58]. The dynamical formation of scalar hairy black hole due to the AdS boundary was discussed in [59]. In
the previous discussion, we have obtained scalar hairy black holes in asymptotically flat gravity with a box boundary. Comparing with the AdS gravity, the reflecting box boundary plays the similar role to confine the scalar field, thus it is natural for the scalar field to condense around a black hole mimicking the condensation we observed in the AdS black hole.

We can further see the similarity from the effective potential behavior. Considering a scalar perturbation $\psi$ on the normal charged black hole, $g(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$ with charge $Q$ and mass $M$, the scalar field perturbation can be described in the Schrödinger-like equation \[ \frac{d^2 R(r)}{d r^*} = V(r) R(r), \tag{11} \]

The effective potential $V(r) = (1 - \frac{2M}{r} + \frac{Q^2}{r^2})(m^2 + \frac{2M}{r} - \frac{2Q^2}{r^3}) - \frac{q^2 Q^2}{r^2}$, is a function of the radius coordinate $r$. It can be seen from Fig. 2 that there is no potential well outside the black hole for the scalar field to accumulate. However, if we impose a box reflecting boundary (or infinity repulsive potential) at $r_b = 1$, we see that a potential well emerges, so that the scalar field can be confined and condensed in the well to lead to the formation of the scalar hair. The formation of hairy black holes with low frequency scalar perturbation if an additional box was placed far enough from the black hole horizon was observed in [48, 49]. The appearance of the potential well in the asymptotically flat gravity with a box boundary is similar to that observed in the AdS case and plays the similar role for the scalar field to condense.

It is well known that the free energy is powerful in disclosing properties of phase transitions. The authors in [41] have proposed a way to calculate the free energy of the system by doing a subtraction of the flat space background. In this way, the free energy of the Minkowski box is set to be zero. We show the free energy of the gravity system as a function of the temperature in cases of $q = 100$, $m^2 = 0$ and $\mu = 0.15$ in the left panel of Fig. 3. Since the physical procedure is with the lowest free energy, we can choose only one phase.
FIG. 2: (Color online) We plot $V(r)$ as a function of the radial coordinate $r$ with $q = 100$, $m^2 = 0$, $Q = 0.001$ and $M = 2$. The vertical dashed blue line $r_b = 1$ corresponds to the box boundary and the horizon is around $r_h \approx 0.06667$. Here, we have used the scaling symmetry $r \rightarrow \alpha r$ to set $r_b = 1$ with $\alpha = \frac{1}{7}$. For every fixed value of the temperature. It can be easily seen from the left panel that the solid blue line is physical and there is a critical temperature $T_c = 0.4910$, below which the normal black hole phase changes into the hairy black hole phases. As the free energy is smooth with respect to the temperature around phase transition points, this black hole/hairy black hole transition is of the second order.

Inspired by approaches in the holographic superconductor theory, we also want to disclose properties of transitions from condensation diagram directly related to asymptotical behaviors of the scalar field on the boundary. In this work, we use $\psi_2$ as a parameter to describe properties of phase transitions. In the right panel of Fig. 3, we plot $\psi_2$ with respect to $T$ in cases of $q = 100$, $m^2 = 0$ and $\mu = 0.15$. It can be seen from the right panel that $\psi_2$ becomes larger as we choose a smaller temperature, which is qualitatively the same with properties in holographic conductor/superconductor transitions [12, 43]. We also obtain a critical transition temperature $T = 0.4910$ in the right panel, below which the parameter $\psi_2$ becomes nonzero. We mention that this critical transition temperature $T = 0.4910$ is equal to $T_c = 0.4910$ from behaviors of the free energy in the left panel. As a summary, we conclude that the parameter $\psi_2$ can be used to determine the
critical temperature of the black hole/hairy black hole transition in a box.

![Graph](image)

**FIG. 4:** (Color online) We plot hairy black hole phases with solid blue points. And the solid red line corresponds to the function of 
\[ \psi_2 = 148(T - T_c)^{1/2} \] with \( T_c = 0.4910 \).

By fitting the numerical data around the phase transition points, we arrive at an analytical relation \( \psi_2 \propto (T_c - T)^\beta \) with \( \beta = \frac{1}{2} \), which also holds for the condensed scalar operator in the holographic conductor/superconductor system in accordance with mean field theories signaling a second order phase transition [7, 62–64]. We have plotted the fitting formula \( \psi_2 \approx 148(T_c - T)^{1/2} \) with \( T_c = 0.4910 \) in Fig. 4 with red solid line. It can be seen from the picture that the solid blue points representing hairy black hole phases almost lie on the solid red line around the critical phase transition temperature. Comparing with the results in Fig. 3, we make a conclusion that this relation between condensation and temperature is a general property of second order phase transitions for both AdS gravity and asymptotically flat gravity in a box. Here, the parameter \( \psi_2 \) plays a role strikingly similar to the condensed scalar operator in holographic conductor/superconductor theories.

We have used operators on the box boundary to study the bulk gravity and it works well in revealing the critical phase transition temperature and also the order of transitions. We also found a critical exponent \( \beta = \frac{1}{2} \) as a characteristic of second order transitions, which usually holds in holographic transitions in asymptotically AdS gravities. These properties imply that the operator on the hard cut-off box boundary covers some information of the bulk transition. Our discussion here provides additional signatures that boundary/bulk correspondence may exist in asymptotically flat spacetimes.

Besides the condensed scalar operator, it has been found that metric solutions also can be used to disclose properties of holographic conductor/superconductor transitions and the jump of metric solutions with respect to the temperature corresponds to a second order phase transition [64]. As a further step, we plan to examine whether the metric solutions can be used to study transitions in this quasi-local black hole/hairy black hole transition model. We show behaviors of \( h(1) \) as a function of the temperature in Fig. 5 with \( q = 100, m^2 = 0 \).
FIG. 5: (Color online) We show behaviors of $h(1)$ as a function of the temperature $T$ with $q = 100$, $m^2 = 0$ and $\mu = 0.15$. and $\mu = 0.15$. It can be seen from the picture that $h(1)$ has a jump of the slope with respect to the temperature at the critical phase transition points $T_c = 0.4910$ implying second order phase transitions in accordance with results in Fig. 3. We conclude that metric solutions can be used to disclose the threshold phase transition temperature and the order of transitions in asymptotically flat black hole/hairy black hole systems in a box, which is similar to properties of holographic conductor/superconductor transitions in AdS gravity [64].

FIG. 6: (Color online) In the left panel, we plot the parameter $\Psi_2$ with respect to the temperature with $q = 100$, $\mu = 0.15$ and various scalar mass $m^2$ from left to right as: $m^2 = -1$ (red), $m^2 = 0$ (blue) and $m^2 = 1$ (green). In the right panel, we show the critical temperature $T_c$ as a function of the scalar mass $m^2$ with $q = 100$ and $\mu = 0.15$.

For every set of parameters, we obtain a critical temperature $T_c$, below which normal black hole phases transform into hairy black hole phases. By choosing $q = 100$, $\mu = 0.15$ and various scalar mass $m^2$, we disclose how the scalar mass could affect the critical temperature $T_c$ in the left panel of Fig. 6. We can easily see from curves in the left panel that $T_c$ decreases as we choose a larger $m^2$. With more detailed calculations, we go on to plot $T_c$ as a function of the scalar mass in the right panel of Fig. 6. We again arrive at a conclusion that $T_c$ becomes smaller as we choose a less negative scalar mass or larger mass makes the phase transitions more difficult to happen, which is qualitatively similar to cases in holographic transitions in AdS gravity.
IV. CONCLUSIONS

We studied a general four dimensional black hole/hairy black hole transition model in a box with nonzero mass of the scalar field in a grand canonical ensemble. Similar to approaches in holographic superconductor theories, we disclosed properties of transitions through condensation behaviors of a parameter $\psi_2$. With various scalar mass, we examined how the scalar mass can affect the critical temperature mainly from behaviors of $\psi_2$. We found the more negative scalar mass corresponds to a larger critical temperature and makes the black hole/hairy black hole transition easier to happen. In particular, we obtained an analytical relation $\psi_2 \propto (T_c - T)^{1/2}$ implying a second order phase transition. Besides the parameter $\psi_2$, we also showed that the metric solutions can be used to detect the critical phase transition temperature and the order of transitions.

We mention that condensation behaviors of the parameter $\psi_2$ is strikingly similar to those of the scalar operator in holographic transition system and properties of transitions in this general quasi-local asymptotically flat gravity are qualitatively the same to those of holographic conductor/superconductor theories in AdS gravity.

In summary, we obtained properties of bulk transitions once also observed in holography in AdS gravity. Our results provide additional signature that holographic theories may also exist in quasi-local asymptotically flat gravity.

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