ALGEBRAIC TORSION VIA HEEGAARD FLOER HOMOLOGY

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Abstract. We use Hutchings’s prescription to define the notion of algebraic torsion for closed contact 3-manifolds via Heegaard Floer homology. Then we show that a closed 3-manifold equipped with an overtwisted contact structure has algebraic torsion of order zero.

1. Introduction

The goal of this note is to define a Heegaard Floer analog of Latschev and Wendl’s algebraic $k$-torsion [LW11]. The idea is to port Hutchings’s recipe (see appendix to [LW11]) to Heegaard Floer homology using the construction of an isomorphism between the latter and Seiberg–Witten Floer homology by Lee, Taubes, and the first named author (see [KLT10a], or [Kut13], for an outline). To be more explicit, given an open book decomposition and a basis of arcs, we define a relative filtration on the corresponding Heegaard Floer chain complex and a non-negative integer quantity associated to this data. The latter quantity is then used to define a contact invariant that is a refinement of the contact invariant in Heegaard Floer homology, the so-called Ozsváth–Szabó contact class [OS05].

We were recently informed that John Baldwin and David Shea Vela-Vick have independently come up with a closely related formulation of a refinement of the contact invariant in Heegaard Floer homology [BVV].

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2. Definition of Algebraic Torsion

To set the stage, let $M$ be a closed, connected, and oriented 3-manifold endowed with a co-oriented contact structure $\xi$. It is understood that the orientation on $M$ is induced by $\xi$. Fix an abstract open book decomposition $(S, \phi)$ of $M$ supporting $\xi$. Here, $S$ is a compact oriented surface of genus $g$ with $b$ boundary components, called the page, and $\phi$ is an orientation reversing diffeomorphism of $S$ which restricts to identity in a neighborhood
of the boundary, called the monodromy. The manifold $M$ is diffeomorphic to $S \times [0,1]/\sim$ where $(p,1) \sim (\phi(p),0)$ for any $p \in S$ and $(p,t) \sim (p,t')$ for any $p \in \partial S$ and $t,t' \in [0,1]$. Let $g = 2g + b - 1$, and fix a self-indexing Morse function on $S$ that has a single maximum and attains its minimum along $\partial S$. Then a suitable pseudo-gradient vector field for the latter defines a basis of arcs $a = \{a_1, \ldots, a_\ell\}$ on $S$, that is, a disjoint collection of properly embedded arcs cutting $S$ into a polygon. This basis together with the monodromy $\phi$ defines a Heegaard diagram $(\Sigma, \{\beta_1, \ldots, \beta_\ell\}, \{\alpha_1, \ldots, \alpha_\ell\})$ for $-M$ as in [HKM09, Section 3.1]. To be more explicit, let $b = \{b_1, \ldots, b_\delta\}$ be a collection of arcs on $S$ where $b_i$ is isotopic to $a_i$ via a small isotopy satisfying the following conditions:

- The endpoints of $b_i$ are obtained from the endpoints of $a_i$ by pushing along $\partial S$ in the direction of the boundary orientation,
- $a_i$ intersects $b_i$ transversally at one point in the interior of $S$,
- Having fixed an orientation of $a_i$, there is an induced orientation on $b_i$, and the sign of the oriented intersection $a_i \cap b_i$ is positive.

Then $\Sigma = S \times \{\frac{1}{2}\} \cup_{\partial S} -S \times \{0\}$, $\alpha_i = a_i \times \{\frac{1}{2}\} \cup a_i \times \{0\}$, and $\beta_i = b_i \times \{\frac{1}{2}\} \cup \phi(b_i) \times \{0\}$. Alternatively, the pseudo-gradient vector field on $S$ and the monodromy $\phi$ can be used to define a self-indexing Morse function $f$ and a pseudo-gradient vector field $v$ on $-M$ which yield the desired Heegaard diagram. Note that there is a natural pairing of the index-1 and index-2 critical points since each index-1 critical point of the aforementioned Morse function on $S$ specifies a unique pair of index-1 and index-2 critical points of $f$. This fact manifests itself as the pairing of the $\alpha$ and $\beta$ curves. The definition of Heegaard Floer homology also requires the choice of a base point $z \in \Sigma \setminus \bigcup_{i \in \{1, \ldots, \ell\}} (\alpha_i \cup \beta_i)$. This is done according to the convention in [HKM09, Section 3.1]. By changing the monodromy in its isotopy class, one can make sure that the pointed Heegaard diagram $(\Sigma, \{\beta_1, \ldots, \beta_\ell\}, \{\alpha_1, \ldots, \alpha_\ell\}, z)$ is strongly admissible for the canonical $Spin^c$ structure $\xi$ defined by the contact structure $\xi$. Then, as in [KLT10b, Lemma 1.1], there exists an area form $w_{\Sigma}$ on $\Sigma$ with total area equal to 2, and the signed area of each periodic domain $P$ on this Heegaard diagram is equal to the pairing $\langle c_1(\xi), \mathcal{H}(P) \rangle$, where $\mathcal{H}(P) \in H_2(M; \mathbb{Z})$ is the homology class corresponding to $P$.

Next, as in [KLT10b, Section 1a], we construct a new manifold $Y$ out of $M$ by adding $g + 1$ 1-handles such that one, denoted $H_0$, is attached along the index-0 and index-3 critical points of $f$, whereas the remaining 1-handles are attached along pairs of index-1 and index-2 critical points of $f$ as prescribed by the pairing of the $\alpha$ and $\beta$ curves. We denote the latter kind of handles by $H_i$ for $i \in \{1, \ldots, g\}$. The resulting manifold $Y$ is diffeomorphic to $M \#_{g+1} S^1 \times S^2$. With the preceding understood, use $v$ and $w_{\Sigma}$, as in [KLT10b, Section 1a-1d], to construct a stable Hamiltonian structure on $Y$. The latter is a pair $(a, w)$ where $a$ is a smooth 1-form and $w$ is a smooth closed 2-form such that $da = hw$ for some smooth function $h : Y \to \mathbb{R}$, and $a \wedge w$ is nowhere zero. Associated to this stable Hamiltonian structure is a vector field $R$ satisfying $w(R, \cdot) = 0$ and $a(R) = 1$, which agrees with the pseudo-gradient vector field $v$ on $Y \setminus \bigcup_{i \in \{0,1, \ldots, g\}} H_i$. Of interest are periodic orbits of the vector field $R$. A complete description of periodic orbits of $R$ is provided in [KLT10b, Section 2]. In particular, there is a unique embedded periodic orbit passing through the base point $z$, denoted by $\gamma_z$. This orbit intersects every cross-sectional sphere of $H_0$ exactly once. Meanwhile, for each $i \in \{1, \ldots, g\}$,
there are exactly two distinct embedded periodic orbits inside $H_i$, denoted by $\gamma_i^+$ and $\gamma_i^-$, which are hyperbolic and homologically trivial.

Periodic orbits of $R$ and pseudo-holomorphic curves in $\mathbb{R} \times Y$ for a generic almost complex structure are used in [KLT10b, Appendix A] to define a variant of Hutchings’s embedded contact homology for certain homology classes $\Gamma \in H_1(Y; \mathbb{Z})$. First, we recall some basic definitions. An orbit set is a finite collection of the form $\{(\gamma, m)\}$ where $\gamma$ are distinct embedded periodic orbits of $R$ and $m$ is a positive integer. An orbit set $\{(\gamma, m)\}$ with the extra requirement that $m = 1$ if $\gamma$ is hyperbolic is called an admissible orbit set. The homology class of an orbit set $\Theta = \{(\gamma, m)\}$ is defined by $\sum m[\gamma]$. Of interest here are admissible orbit sets with homology class $\Gamma \in H_1(Y; \mathbb{Z})$ whose Poincaré dual has pairing 0 with $H_2(H_0; \mathbb{Z})$, and which has pairing 1 with the positive generator of $H_2(H_i; \mathbb{Z})$ for each $i \in \{0, \ldots, g\}$. Having fixed such $\Gamma \in H_1(Y; \mathbb{Z})$, it follows from [KLT10b, Section 2] that any $\gamma$ that belongs to an admissible orbit set with homology class $\Gamma$ is hyperbolic. Hence, we may drop the integer $m$ from the notation. With the preceding understood, the relationship between admissible orbit sets and Heegaard Floer generators is described in [KLT10b, Proposition 2.8]. To be more precise, the set of admissible orbit sets associated to a Heegaard Floer generator is in 1–1 correspondence with the set $\prod_{i \in \{1, \ldots, g\}} (\mathbb{Z} \times 0)$ where $\emptyset = \{0, +1, -1, \{+1, -1\}\}$.

Having fixed an almost complex structure $J$ on $\mathbb{R} \times Y$ satisfying the conditions in [KLT10b, Section 3a], the differential $\partial_{\text{ech}}$ on $\hat{\text{ecc}}(Y, \Gamma)$ is defined via a suitable count of $J$-holomorphic curves in $\mathbb{R} \times Y$ asymptotic to admissible orbit sets at $\pm \infty$. A detailed description of $J$-holomorphic curves in $\mathbb{R} \times Y$ is given in [KLT10b, Sections 3 and 4]. Given a pair of admissible orbit sets $\Theta_+$ and $\Theta_-$, the moduli space $M_1(\Theta_+, \Theta_-)$ of $J$-holomorphic curves asymptotic to $\Theta_+$ at $+\infty$ and to $\Theta_-$ at $-\infty$ is smooth, oriented, and 1-dimensional. The differential $\partial_{\text{ech}}$ is defined to be the endomorphism of $\hat{\text{ecc}}(Y, \Gamma)$ sending a generator $\Theta_+$ to

$$\sum \sigma(\Theta_+, \Theta_-)\Theta_-,$$

where $\sigma(\Theta_+, \Theta_-)$ is a signed count of elements of $M_1(\Theta_+, \Theta_-)$, modulo $\mathbb{R}$-translation, disjoint from $\mathbb{R} \times \gamma_\emptyset$. Note that $\mathbb{R} \times \gamma_\emptyset$ is a $J$-holomorphic cylinder. Hence, by positivity of intersections of pseudo-holomorphic curves, $\mathbb{R} \times \gamma_\emptyset$ introduces a filtration on the chain complex defined in [KLT10c, Section 1b], much like the filtration dictated by the base point in Heegaard Floer homology. Then it follows from [KLT10c] (cf. [HT07, HT09]) that $\partial_{\text{ech}} \circ \partial_{\text{ech}} = 0$, and

$$\hat{\text{ech}}(Y, \Gamma) := H_*(\hat{\text{ecc}}(Y, \Gamma), \hat{\partial}_{\text{ech}}) \cong \tilde{HF}(-M, s_\xi + PD(\Gamma|_M)) \otimes \mathbb{Z}^{2g}.$$

Given orbit sets $\Theta_+$ and $\Theta_-$ representing the same homology class, Hutchings defines variants of his ECH index for relative homology classes in $H_2(Y, \Theta_+, \Theta_-)$, denoted by $J_0$, $J_+$, and $J_-$ (see [Hut, Section 6]). If $C$ is an embedded $J$-holomorphic curve in $\mathbb{R} \times Y$ with positive ends at an admissible orbit set $\Theta_+$ and negative ends at an admissible orbit set $\Theta_-$, then $C$ has ends at distinct embedded periodic orbits of $R$, and hence

$$J_+(C) = -\chi(C) + |\Theta_+| - |\Theta_-|,$$
where $|\cdot|$ denotes the cardinality of an admissible orbit set. This is due to the fact that all relevant periodic orbits of $R$ are hyperbolic. We can rewrite the above formula as

$$J_+(C) = \sum_{C_j \subset C} (2g_j - 2|\Theta_{j+}|),$$

(2.1)

where each $C_j$ denotes a connected component of $C$, $g_j$ denotes the genus of $C_j$, and each $\Theta_{j+} \subset \Theta_+$ is the collection of periodic orbits of $R$ that comprise the positive ends of $C_i$. Clearly, $2|J_+(C)|$. Meanwhile, no subcollection of periodic orbits from an admissible orbit set $\Theta$ can be homologically trivial unless $\Theta$ contains one or more of the homologically trivial $\gamma_i^+$, $\gamma_i^-$, and admissible orbit sets whose restriction to each $H_t$ correspond to the pair $(0,0) \in \mathbb{Z} \times \mathbb{O}$ generate a subcomplex, which we denote by $\hat{\text{c}c\text{c}}_0(Y,\Gamma)$. Furthermore, $\hat{\text{c}c\text{c}}_0(Y,\Gamma) \cong \hat{CF}(-M,\mathfrak{s}_\xi + PD(\Gamma|_M))$ as chain complexes via the aforementioned 1–1 correspondence between generators (see [KLT10c] for details). If $\Theta_+$ and $\Theta_-$ are generators of $\hat{\text{c}c\text{c}}_0(Y,\Gamma)$, then every connected component of an embedded $J$-holomorphic curve $C$ with positive ends at $\Theta_+$ and negative ends at $\Theta_-$ has non-empty positive and negative ends, and (2.1) implies that $J_+(C) \geq 0$.

The Heegaard Floer generator $x_\xi = \{x_1, \ldots, x_c\}$ that defines the Ozsváth–Szabó contact class in the Honda–Kazez–Matić description [HKM09] corresponds to a generator $\Theta_\xi$ of $\hat{\text{c}c\text{c}}_0(Y,\Gamma)$ which is a disjoint union of $g$ embedded periodic orbits of $R$ each representing a positive generator of $H_1(S^1 \times S^2; \mathbb{Z})$. Then, following almost verbatim an appendix to [LW11] by Hutchings, we decompose the differential $\partial_{\text{ech}}$ as

$$\partial_{\text{ech}} = \partial_0 + \partial_1 + \cdots + \partial_k + \cdots,$$

where $\partial_k$ counts admissible ECH index-1 $J$-holomorphic curves $C$ with $J_+(C) = 2k$ and $C \cap \mathbb{R} \times \gamma_z = \emptyset$. Since $J_+$ is additive under gluing of $J$-holomorphic curves (see [Hut, Proposition 6.5a]), the above decomposition induces a spectral sequence with pages

$$E^k(S,\phi,a) = H_*(E^{k-1}(S,\phi,a),\partial_{k-1}).$$

**Definition 2.1.** Let $AT(S,\phi,a)$ be the smallest non-negative integer $k$ such that the class $[\Theta_\xi]$ vanishes in $E^{k+1}(S,\phi,a)$. Note that if the Ozsváth–Szabó contact class vanishes, in other words,

$$(\partial_0 + \cdots + \partial_k)c = \Theta_\xi,$$

for some $c \in \hat{\text{c}c\text{c}}_0(Y,\Gamma)$, then $AT(S,\phi,a)$ is finite. But the converse is not true. We say that $(M,\xi)$ has algebraic $k$-torsion if the Ozsváth–Szabó contact class vanishes and $AT(S,\phi,a) = k$ for some choice of $(S,\phi,a)$.

Next we give a description of the function $J_+$, and hence, a description of algebraic torsion, in terms of the data on the Heegaard diagram associated to $(S,\phi,a)$. To do so, we use the isomorphism between Heegaard Floer homology and $\text{ech}$ [$\text{KLT10b, KLT10c}]$. The construction of this isomorphism exploits the cylindrical reformulation of Heegaard Floer homology due to Lipshitz [Lip06]. In the above context, where we restrict ourselves to the subcomplex $\hat{\text{c}c\text{c}}_0(Y,\Gamma)$, there is a 1–1 correspondence between $J$-holomorphic curves with ends at generators of this subcomplex having ECH index 1 and pseudo-holomorphic curves that appear in Lipshitz’s reformulation with Maslov index 1. Having said that, denote by $C_L$ the
Maslov index-1 pseudo-holomorphic curve in Lipshitz’s reformulation that corresponds to an ECH index-1 $J$-holomorphic curve $C \subset \mathbb{R} \times Y$ with positive ends at an admissible orbit set $\Theta_+$ and negative ends at an admissible orbit set $\Theta_-$, both of which are generators of $\hat{cc}_{0}(Y, \Gamma)$. Then, $C_L \subset \mathbb{R} \times [0,1] \times \Sigma$ is an embedded surface with boundary and $2g$ punctures on its boundary half of which are at $+\infty$ and the other half are at $-\infty$. Topologically, $C$ is obtained from $C_L$ by adding 2-dimensional 1-handles such that for each $i = 1, \ldots, g$ a 1-handle is attached along a point in $\alpha_i$ and a point in $\beta_i$. As a result,

$$J_+(C) = -\chi(C) + |\Theta_+| - |\Theta_-|,$$

$$= -\chi(C_L) + g + |\Theta_+| - |\Theta_-|.$$  

Lipshitz gives a formula for $\chi(C_L)$ in terms of the data in the Heegaard diagram (see [Lip06, Proposition 4.2]); more specifically,

$$\chi(C_L) = g - n_{x_+}(D) - n_{x_-}(D) + e(D).$$

Here, $D$ denotes the holomorphic domain in the Heegaard diagram that represents $C_L$, $x_+$ denotes the Heegaard Floer generator that corresponds to $\Theta_+$, $x_-$ denotes the Heegaard Floer generator that corresponds to $\Theta_-$, $n_p(D)$ denotes the point measure, namely, the average of the coefficients of $D$ for the four regions with corners at $p \in \alpha_i \cap \beta_j$, and $e(D)$ is the Euler measure of $D$. On the other hand, with the labeling of the $\alpha$ and $\beta$ curves in mind, $|\Theta_\pm|$ is equal to the number of cycles in the element of the symmetric group $S_g$ which represents $x_+$ or $x_-$, respectively. We will denote the latter quantity also by $|\cdot|$. Hence, we obtain the following formula for $J_+(C)$:

$$J_+(C) = n_{x_+}(D) + n_{x_-}(D) - e(D) + |x_+| - |x_-|,$$

or by [Lip06, Corollary 4.10],

$$J_+(C) = 2(n_{x_+}(D) + n_{x_-}(D)) - 1 + |x_+| - |x_-|,$$

(2.2) since $D$ has Maslov index 1. The right hand side of (2.2) depends only on the domain $D$. Hence, from now on, we denote it by $J_+(D)$.

With the preceding understood, note that there is one important caveat in the definition of algebraic $k$-torsion. That is to say, every contact 3-manifold has algebraic 0-torsion. This is due to works of Sarkar–Wang [SW10] and Plamenevskaya [Pla07] which show that the Heegaard diagram resulting from an arbitrary choice of $(S, \phi, a)$ can be made nice by choosing $\phi$ appropriately in its isotopy class. On a nice Heegaard diagram, every Maslov index-1 holomorphic domain is represented by an empty embedded bigon or an empty embedded square [SW10, Theorem 3.3]. It is easy to see from (2.2) that such domains have vanishing $J_+$. Therefore, it is unlikely to extract any non-trivial information about the contact structure considering all supporting open book decompositions and all bases of arcs. Instead, we restrict attention to $(S, \phi, a)$ where $(\phi, a)$ is minimally intersecting, that is, $a$ intersects $\phi(b)$ minimally in $S$.

**Definition 2.2.** Let $(M, \xi)$ be a closed contact 3-manifold. Suppose that $(M, \xi)$ has vanishing Oszváth–Szabó contact class. Then define the order of algebraic torsion via Heegaard Floer homology as

$$\text{AT}(M, \xi) := \min \{ AT(S, \phi, a) \mid (\phi, a) \text{ is minimally intersecting} \}.$$
Otherwise, define \( \mathcal{AT}(M, \xi) \) to be \( \infty \).

A celebrated theorem of Giroux states that there is a 1–1 correspondence between contact structures up to isotopy and open book decompositions up to positive stabilization [Gir02]. It follows as a consequence that the above definition yields an invariant of the contact structure. Moreover,

**Theorem 2.3.** Let \( \xi_{\text{ot}} \) be an overtwisted contact structure on a closed 3-manifold \( M \). Then \( \mathcal{AT}(M, \xi_{\text{ot}}) = 0 \).

**Proof.** To see this, note that an overtwisted contact structure is supported by an open book decomposition \((S, \phi)\) where the monodromy \( \phi \) is not right-veering [HKM07, Theorem 1.1]. One can find a basis of arcs \( a \) on \( S \) so that in the corresponding Heegaard diagram \( \widehat{\partial y} y = x_{\xi_{\text{ot}}} \) where \( y = \{y_1, x_2, \ldots, x_g\} \) and there is exactly one Maslov index-1 holomorphic domain \( D \), a bigon, that contributes to the differential [HKM09, Lemma 3.2]. Therefore, \( n_y(D) = \frac{1}{4} \), \( n_{x_{\xi_{\text{ot}}}}(D) = \frac{1}{4} \), \( |y| = g \), and \( |x_{\xi_{\text{ot}}}| = g \). Applying (2.2), we find \( J_+(D) = 0 \). Moreover, after an isotopy of \( \phi \) relative to \( \partial S \), we may assume that \( (\phi, a) \) is minimally intersecting. As a result, \( \mathcal{AT}(M, \xi_{\text{ot}}) = 0 \). \( \square \)

Unfortunately, the above definition cannot be used to argue that a contact 3-manifold has non-zero order of algebraic torsion. Such an argument would require finding a non-zero lower bound for \( \mathcal{AT}(S, \phi, a) \) over all choices of \( (S, \phi, a) \) where \( (\phi, a) \) is minimally intersecting. Ideally, one would rather show that \( \mathcal{AT}(S, \phi, a) \) does not depend on \( (S, \phi, a) \) as long as \( (\phi, a) \) is minimally intersecting; this is the content of work in progress. Assuming that one can show this, we provide evidence that the order of algebraic torsion is a non-trivial contact invariant. The contact 3-manifold we provide as evidence is motivated by [CKK14]. The open book decomposition that describes this contact 3-manifold is depicted in Figure 1a. We claim that this contact 3-manifold has non-zero order of algebraic torsion. In order to see this, note by (2.1) that \( J_+(C) = 0 \) if and only if \( C \) is a disjoint union of genus-0 curves with connected positive ends. As a result, a Maslov index-1 holomorphic domain \( D \) with \( J_+(D) = 0 \) has to be an immersed 2n-gon where \( 1 \leq n \leq g \). It is not hard to see from Figure 1b that any such domain has positive multiplicity at the base point \( z \). In fact, this is true more generally if we increase the number of left-handed Dehn twists. It follows from an application of [Bal13, Corollary 1.3] that all of the resulting contact 3-manifolds have vanishing Ozsváth–Szabó contact class. Theorem 2.3 then implies that the contact structures supported by these open book decompositions are tight. The latter result also follows from [IK14, Corollary 1.2] (cf. [Wan12, Theorem 3.10]).

We suspect that the contact 3-manifold resulting from the open book decomposition depicted in Figure 1 has algebraic torsion of order at most 2. A holomorphic domain \( D \) with \( J_+(D) = 4 \) is highlighted in Figure 2. The corresponding pseudo-holomorphic curve in Lipshitz’s reformulation would be the disjoint union of a genus-1 curve with two boundary components and four punctures on its boundary, and a disk with two punctures on its boundary. On the ech side, this pseudo-holomorphic curve corresponds to a \( J \)-holomorphic curve that is the disjoint union of an embedded genus-1 curve asymptotic to two distinct embedded Reeb orbits at both ends, and a trivial cylinder.
Figure 1. An open book decomposition for a tight contact 3-manifold that has non-zero order of algebraic torsion. On the left hand side, black circles are the boundary components of a page $S$ of the planar open book decomposition, and the grey circles are the curves along which we apply the indicated Dehn twists, whose product gives the monodromy $\phi$ of the open book decomposition. On the right hand side, a basis of arcs is indicated by the dotted red, green, and blue line segments, while the effect of the monodromy on this basis is indicated by the solid arcs in the same colors.

Figure 2. A holomorphic domain $\mathcal{D}$ with $J_+(\mathcal{D}) = 4$. The lighter shaded regions have multiplicity 1; the darker shaded regions have multiplicity 2.

Remark. Baldwin and Vela-Vick are working on a proof that their version of a refinement of the contact invariant in Heegaard Floer homology does not depend on the choice of minimally intersecting open book decomposition and basis of arcs.
3. Conclusion

We end our discussion with a few questions that we are currently pursuing. First, we would like to know if the order of algebraic torsion detects overtwistedness. In this regard, we ask the following question:

Question 3.1. Is there a tight contact 3-manifold with algebraic torsion of order 0?

Even though this question might have an affirmative answer, it is conceivable that the order of algebraic torsion detects overtwistedness on contact 3-manifolds supported by planar open book decompositions. Regardless of the answer to Question 3.1, it would also be interesting to compare algebraic torsion to Wendl’s planar torsion \[\text{Wen13}\]. As is stated in \[\text{LW11, Theorem 6}\], planar torsion provides an upper bound to Latschev and Wendl’s algebraic torsion. Moreover, planar torsion detects overtwistedness. We expect a similar relationship between the algebraic torsion defined in this note and Wendl’s planar torsion.

Question 3.2. Suppose that the closed contact 3-manifold \((M, \xi)\) has planar \(k\)-torsion. Does it have algebraic torsion of order \(k\)?

Next, we would like to understand how algebraic torsion behaves under exact symplectic cobordisms:

Question 3.3. Suppose that there exists an exact symplectic cobordism from \((M_+, \xi_+)\) to \((M_-, \xi_-)\), and that \((M_+, \xi_+)\) has algebraic torsion of order \(k\). Does \((M_-, \xi_-)\) have algebraic torsion of order \(k\) as well?

Here \((M_+, \xi_+)\) and \((M_-, \xi_-)\) are respectively the convex and the concave boundary of the exact symplectic cobordism in question. Since our algebraic torsion is defined using open book decompositions, and there is currently no topological description of exact symplectic cobordisms in general, it is hard to answer this question. However, the following restricted version might be more accessible:

Question 3.4. Suppose that \((M_+, \xi_+)\) is obtained from \((M_-, \xi_-)\) via contact \((-1)\)-surgery on a null-homologous Legendrian knot \(K \subset M_-\), and that \((M_+, \xi_+)\) has algebraic torsion of order 0. Does \((M_-, \xi_-)\) have algebraic torsion of order 0 as well?

To put things in perspective, an effective version of algebraic torsion together with an affirmative answer to Question 3.4 would provide an alternative proof of the following theorem, which has recently been proved by the last named author in \[\text{Wan14}\]:

Theorem 3.5. Let \(\xi\) be a tight contact structure on \(M\), and \(K \subset M\) be a null-homologous Legendrian knot. Then, contact \((-1)\)-surgery on \(K\) produces a 3-manifold with a tight contact structure.
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