Bounds for completely monotonic degree of a remainder for an asymptotic expansion of the trigamma function

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ABSTRACT
In the paper, the author presents bounds for completely monotonic degree of a remainder for an asymptotic expansion of the trigamma function. This result partially confirms one in a series of conjectures on completely monotonic degrees of remainders of asymptotic expansions for the logarithm of the gamma function and for polygamma functions.

1. Simple preliminaries
In the literature (Abramowitz & Stegun, 1972, Section 6.4), the functions
\[
\Gamma(w) = \int_0^\infty s^{w-1}e^{-s}ds \quad \text{and} \quad \psi(w) = [\ln \Gamma(w)]',
\]
for \( \Re(w) > 0 \) are known as the classical Euler gamma function and the digamma function. Moreover, the functions \( \psi^{(1)}(w), \psi^{(2)}(w), \psi^{(3)}(w), \) and \( \psi^{(4)}(w) \) are known as the tri-, tetra-, penta-, and hexa-gamma functions respectively. As a whole, all the derivatives \( \psi^{(k)}(w) \) for \( k \geq 0 \) are known as the polygamma functions (Berg & Henrik, 2012; Zhao & Chu, 2010).

Recall from the chapters Mitrinović et al. (1993, Chapter XIII), Schilling et al. (2012, Chapter 1), and Widder (1946, Chapter IV) that, if \( \phi(t) \) defines on an interval \( I \) and satisfies
\[
(-1)^k \phi^{(k)}(t) \geq 0 \quad \text{(1.1)}
\]
for all \( t \in I \) and \( k \geq 0 \), then we call it a completely monotonic function on \( I \). Theorem 12b on Widder (1946, p. 161) reads that \( \phi(t) \) is completely monotonic on \( (0, \infty) \) if and only if
\[
\phi(t) = \int_0^\infty e^{-st}ds
\]
converges for all \( t \in (0, \infty) \), where \( \sigma(s) \) is nondecreasing on \( (0, \infty) \). The integral representation (1.2) is equivalent to say that \( \phi(t) \) is completely monotonic on \( (0, \infty) \) if and only if it is a Laplace transform of \( \sigma(s) \) on \( (0, \infty) \). A result in Dubourdieu (1939–40, p. 98) and van Haeringen (1996, p. 395) asserts that, unless \( \phi(t) \) is a trivial completely monotonic function, that is, a nonnegative constant on \( (0, \infty) \), those inequalities in (1.1) are all strict on \( (0, \infty) \). Why do we investigate completely monotonic functions? One can find historic answers in two monographs (Schilling et al., 2012; Widder, 1946) and closely related references therein.

Let \( \phi(t) \) be defined on \( (0, \infty) \) and \( \phi(\infty) = \lim_{t \to \infty} \phi(t) \). If \( t^r[\phi(t) - \phi(\infty)] \) is completely monotonic on \( (0, \infty) \), but \( t^{r+c}[\phi(t) - \phi(\infty)] \) is not for any
number $\varepsilon > 0$, then $r$ is called the completely monotonic degree of $\phi(t)$ with respect to $t \in (0, \infty)$; if $t^r [\phi(t) - \phi(\infty)]$ is completely monotonic on $(0, \infty)$ for all $r \in \mathbb{R}$, then the completely monotonic degree of $\phi(t)$ with respect to $t \in (0, \infty)$ is said to be $\infty$.

The notation $\deg_{cm}^r[\phi(t)]$ was designed in Guo and Qi (2012) to denote the completely monotonic degree $r$ of $\phi(t)$ with respect to $t \in (0, \infty)$. Why do we investigate completely monotonic degrees? One can find significant answers in the second paragraph of Qi and Liu (2019, Remark 1) or in the papers Qi (2020a), Qi and Agarwal (2019), Qi and Li (2016) and closely related references therein.

2. Motivations

It is well-known in Abramowitz and Stegun (1972, p. 257, 6.1.40) and Abramowitz and Stegun (1972, p. 260, 6.4.11) that

$$
\ln \Gamma(w) \sim \left( w - \frac{1}{2} \right) \ln w - w + \frac{1}{2} \ln (2\pi) + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)w^{2k-1}}
$$

(2.1)

and

$$
\psi^{(n)}(w) \sim (-1)^{n-1} \left[ \frac{(n-1)!}{w^n} + \frac{n!}{2w^{n+1}} + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)} \frac{w^{2k}}{(2k)!w^{2k+n}} \right]
$$

(2.2)

as $w \to \infty$ in $|\arg w| < \pi$ for $n \geq 0$, where an empty sum is understood to be 0 and $B_{2k}$ for $k \geq 1$ are known as the Bernoulli numbers which can be generated (Qi, 2019; Shuang et al., 2021) by

$$
\frac{w}{e^w - 1} = 1 - \frac{w}{2} + \sum_{k=1}^{\infty} \frac{B_{2k} w^{2k}}{(2k)!} \quad |w| < 2\pi.
$$

Stimulated by the asymptotic expansions (2.1) and (2.2), many mathematicians considered and investigated the remainders

$$
R_n(t) = (-1)^n \left[ \ln \Gamma(t) - \left( t - \frac{1}{2} \right) \ln t + t - \frac{1}{2} \ln (2\pi) - \sum_{k=1}^{n} \frac{B_{2k}}{2k(2k-1)} \frac{1}{t^{2k-1}} \right]
$$

(2.3)

and its derivatives, where $n \geq 0$. For detailed information, please refer to the papers Allasia et al. (2002), Alzer (1997), Merkle (1996, Theorem 8), Guo et al. (2015, Section 1.4), Koumandos (2006, Theorem 2), Koumandos and Pedersen (2009, Theorem 2.1), Mortici (2010, Theorem 3.1), and closely related references therein.

In Qi and Liu (2019, Section 4) and Qi and Mahmoud (2019, 2021), the author posed, modified, and finalized the following conjectures:

1. the completely monotonic degrees

$$
\deg_{cm}^1 \left[ \ln t - \frac{1}{2t} - \psi(t) \right] = 1,
$$

(2.4)

$$
\deg_{cm}^1 \left[ \frac{1}{t} + \frac{1}{2t^2} + \frac{1}{6t^3} - \psi'(t) \right] = 3,
$$

(2.5)

$$
\deg_{cm}^1 \left[ \psi'(t) - \left( \frac{1}{t} + \frac{1}{2t^2} + \frac{1}{6t^3} - \frac{1}{30t^5} \right) \right] = 4
$$

(2.6)

with respect to $t \in (0, \infty)$ are valid;

2. when $m=0$, the completely monotonic degrees of $R_n(t)$ with respect to $t \in (0, \infty)$ satisfy

$$
\deg_{cm}^0[R_0(t)] = 0, \quad \deg_{cm}^0[R_1(t)] = 1
$$

(2.7)

and

$$
\deg_{cm}^0[R_n(t)] = 2(n-1), \quad n \geq 2;
$$

(2.8)

3. when $m=1$, the completely monotonic degrees of $-R_n(t)$ with respect to $t \in (0, \infty)$ satisfy

$$
\deg_{cm}^1[-R_0(t)] = 1, \quad \deg_{cm}^1[-R_1(t)] = 2
$$

(2.9)

and

$$
\deg_{cm}^1[-R_n(t)] = 2n - 1, \quad n \geq 2;
$$

(2.10)

4. when $m \geq 2$, the completely monotonic degrees of $(-1)^m R_n^{(m)}(t)$ with respect to $t \in (0, \infty)$ satisfy

$$
\deg_{cm}^m \left[ (-1)^m R_0^{(m)}(t) \right] = m,
$$

(2.11)

$$
\deg_{cm}^m \left[ (-1)^m R_1^{(m)}(t) \right] = m + 1,
$$

and

$$
\deg_{cm}^m \left[ (-1)^m R_n^{(m)}(t) \right] = m + 2(n-1), \quad n \geq 2.
$$

(2.12)

In Alzer (1997, Theorem 1), Guo and Qi (2010, Theorem 1), Qi and Mahmoud (2019, Theorem 2.1), Qi and Mahmoud (2021, Theorem 2.1), and Xu and Cen (2020, Theorem 3), the first conjecture in (2.7), which is equivalent to (2.4), was unconsciously or consciously verified again and again.

In Koumandos and Pedersen (2009, Theorem 2.1), it was proved that

$$
\deg_{cm}^m[R_n(t)] \geq n, \quad n \geq 0.
$$

This result is weaker than those conjectures in (2.8). In Chen et al. (2010, Theorem 1), Qi and Liu (2019, Theorem 2), and Xu and Cen (2020, Theorem 4), the second conjecture in (2.9) was proved once again.

In Qi and Mahmoud (2019, Theorem 2.1) and Qi and Mahmoud (2021, Theorem 2.1), those five conjectures in (2.7), (2.9), and (2.10) were confirmed.
In Xu and Cen (2020, Theorems 1 and 2), it was
acquired that
deg_{cm}^t \left[ (-1)^2 R^t_0(t) \right] = 2 \quad \text{and} \quad deg_{cm}^t \left[ (-1)^2 R^t_1(t) \right] = 3.

These two results confirm conjectures in (2.11)
just for the case \( m = 2 \). The latter is equivalent to the conjecture (2.5).

The main aim of this paper is to partially confirm the conjecture (2.6), a special case \( m = n = 2 \) of the conjecture (2.12), by the double inequality
\[
4 \leq deg_{cm}^t \left[ (-1)^2 R^t_1(t) \right] \leq 5. \tag{2.13}
\]

3. Bounds and their proof
The conjecture (2.6), or say, a special case \( m = n = 2 \) of the conjecture in (2.12), can be partially confirmed by the following theorem.

**Theorem 3.1.** The completely monotonic degree of the function
\[
Q(t) = \psi'(t) - \frac{1}{t} - \frac{1}{2t^2} - \frac{1}{6t^3} + \frac{1}{30t^4}
\]
with respect to \( t \in (0, \infty) \) is not less than 4 and not greater than 5. In other words, the double inequality (2.13) is valid.

**Proof.** Making use of the integral representations
\[
\frac{1}{w^n} = \frac{1}{\Gamma(n)} \int_0^\infty s^{n-1} e^{-ws} ds, \quad \forall (w), \forall (r) > 0
\]
and
\[
\psi^{(k)}(w) = (-1)^{k+1} \int_0^\infty \frac{s^k}{1-e^{-s}} e^{-ws} ds, \quad \forall (w), \forall (k) > 0, k \geq 1
\]
in Abramowitz and Stegun (1972, p. 260, 6.4.1; p. 255, 6.1.1) and directly computing, we obtain
\[
Q(t) = \int_0^\infty \left( \frac{s}{1-e^{-s}} - 1 - \frac{s^2}{2} + \frac{s^3}{6} \right) e^{-ts} ds = H(s),
\]
\[
H(s) = s^3 + (s^3 - 30s + 90)e^{-s} - 2s^2 + 60e^{-s} - 30 - 90,
\]
\[
H'(s) = e^{s^2} (s^2 - 10) + 3(s^2 + 30s + 20)e^{-s} - 3(s^2 - 20s + 30)e^{s^2} - s^2 + 10,
\]
\[
H''(s) = e^{s^4} (90 - 34s^2 - 144s^4 - 4s^6) + 2(17s + 45)e^{s^2} + 1,
\]
\[
H'''(s) = e^{s^6} (5s + 6s^2 + 25s^3 + 10s + 33s^2 - 35s^3) - 10(33s + 35s^2 + 5s + 2s^2 - 1),
\]
\[
H^{(4)}(s) = e^{s^8} (30s^2 + 18s^4 + 27s^6 + 18s^4 + 5s^2 + 25s^4 - 1),
\]
\[
= \frac{1}{30(e^2 - 1)^2} \left[ \left( 5s^2 + \frac{(110k - 350k)3^3 + 5s(3k - 50k)2^{2k-1}}{5s^2 + 193s^3 + 85s^4 + 42s^5 + 5655s^6 + \ldots} \right) \right],
\]
\[
\lim_{n \to 0^+} H^{(k)}(t) = (1-k)(k-1)!
\]

Consequently, the function \( t^4 Q(t) \) is completely monotonic on \( (0, \infty) \). This means that
\[
deg_{cm}^t [Q(t)] \geq 4. \tag{3.2}
\]

For \( \forall (w), \forall (k) \geq 1 \), we have the recurrent relation
\[
\psi^{(k-1)}(w + 1) = \psi^{(k-1)}(w) + (-1)^k (k - 1)! \frac{(k - 1)!}{w^k}.
\]

See Abramowitz and Stegun (1972, p. 260, 6.4.6). From this, it follows that
\[
\lim_{t \to 0^+} \left[ t^4 \psi^{(k-1)}(t) \right] = \lim_{t \to 0^+} \left[ t^4 \left( \psi^{(k-1)}(t + 1) + (-1)^k (k - 1)! \right) \right] = (1-k)(k-1)! \tag{3.3}
\]

for \( k \geq 1 \). If \( t^{4+k} Q(t) \) for \( \forall t \in \mathbb{R} \) were completely monotonic on \( (0, \infty) \), by virtue of the result mentioned on page 1 and proved in Dubourdieu (1939–40, p. 98) and van Haeringen (1996, p. 395), we see that the first derivative of \( t^{4+k} Q(t) \) should be negative on \( (0, \infty) \), that is,
\[\left( t^{\alpha}Q(t)\right)' = (4 + \varepsilon)t^{\alpha+1}Q(t) + t^{\alpha+1}Q'(t) = Q(t)t^{\alpha+1}\left(4 + \varepsilon + \frac{Q'(t)}{Q(t)}\right)\]

should be negative on \((0, \infty)\). Therefore, we acquire

\[
\varepsilon < -4 - \frac{Q'(t)}{Q(t)}
\]

\[
= -4 - t\psi''(t) - \frac{1}{5t^2} + \frac{1}{5t^2} + \frac{1}{t^2} + \frac{1}{t^2} \rightarrow 1, \quad t \to 0^+
\]
on \((0, \infty)\), where we used the limit (3.3) in the last step. Consequently, we arrive at

\[
\deg_{\text{com}}^1 Q(t) \leq 5. \quad (3.4)
\]

Combining (3.2) with (3.4) leads to the inequality (2.13). The proof of Theorem 3.1 is complete. \(\square\)

**Remark 3.1.** This paper is a revised version of the preprints (Qi, 2020b, 2021).

### 4. Conclusions

The main result in this paper is Theorem 3.1 which reads that the completely monotonic degree \(\deg_{\text{com}}^1 Q(t)\) of the function \(Q(x) = R^c_2(t)\) defined by (2.3) and (3.1) is bounded between 4 and 5.

We emphasize that we conjecture \(\deg_{\text{com}}^1 Q(t) = 4\), as stated in (2.6).

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The author states that there is no conflict of interest.

### Availability of data and material

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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### References

Abramowitz, M., & Stegun I. A. (Eds.). (1972). *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. National Bureau of Standards, Applied Mathematics Series 55, 10th Printing, Dover Publications.

Allasia, G., Giordano, C., & Pečarić, J. (2002). Inequalities for the gamma function relating to asymptotic expansions. *Mathematical Inequalities & Applications*, 5(3), 543–555. doi:10.7153/mia-05-54

Alzer, H. (1997). On some inequalities for the gamma and psi functions. *Mathematics of Computation*, 66(217), 373–389. doi:10.1090/S0025-5718-97-00807-7

Berg, C., & Henrik, L. (2012). A completely monotonic function used in an inequality of Alzer. *Computational Methods and Function Theory*, 12(1), 329–341. doi:10.1007/BF03321830

Chen, C.-P., Qi, F., & Srivastava, H. M. (2010). Some properties of functions related to the gamma and psi functions. *Integral Transforms and Special Functions*, 21(2), 153–164. doi:10.1080/1065265090364216

Dubourdieu, J. (1939–40). Sur un théorème de M. S. Bernstein relatif à la transformation de Laplace-Stieltjes. *Compositio Mathematica*, 7, 96–111. [http://www.numdam.org/item?id=CM_1940__7__96_0](http://www.numdam.org/item?id=CM_1940__7__96_0) (French).

Guo, B.-N., & Qi, F. (2010). Two new proofs of the complete monotonicity of a function involving the psi function. *Bulletin of the Korean Mathematical Society*, 47(1), 103–111. doi:10.4134/BKMS.2010.47.1.103

Guo, B.-N., & Qi, F. (2012). A completely monotonic function involving the tri-gamma function and with degree one. *Journal of Applied Mathematics and Computing*, 218(19), 9890–9897. doi:10.1007/jamc.2012.03.075

Guo, B.-N., Qi, F., Zhao, J.-L., & Luo, Q.-M. (2015). Sharp inequalities for polygamma functions. *Mathematica Slovaca*, 65(1), 103–120. doi:10.1515/ms-2015-0010

Koumandos, S. (2006). Remarks on some completely monotonic functions. *Journal of Mathematical Analysis and Applications*, 324(2), 1458–1461. doi:10.1016/j.jmaa.2005.12.017

Koumandos, S., & Pedersen, H. L. (2009). Completely monotonic functions of positive order and asymptotic expansions of the logarithm of Barnes double gamma function and Euler’s gamma function. *Journal of Mathematical Analysis and Applications*, 355(1), 33–40. doi:10.1016/j.jamaa.2009.01.042

Merkle, M. (1996). Logarithmic convexity and inequalities for the gamma function. *Journal of Mathematical Analysis and Applications*, 203(2), 369–380. doi:10.1006/jmaa.1996.0385

Mitrović, D. S., Pečarić, J. E., & Fink, A. M. (1993). *Classical and new inequalities in analysis*. Kluwer Academic Publishers. doi:10.1007/978-94-017-1043-5

Mortici, C. (2010). Very accurate estimates of the poly-gamma functions. *Asymptotic Analysis*, 68(3), 125–134. doi:10.3233/ASY-2010-0983

Qi, F. (2019). A double inequality for the ratio of two non-zero neighbouring Bernoulli numbers. *The Journal of Computational and Applied Mathematics*, 351, 1–5. doi:10.1016/j.cam.2018.10.049

Qi, F. (2020a). Completely monotonic degree of a function involving trigamma and tetragamma functions. *AIMS Mathematics*, 5(4), 3391–3407. doi:10.3934/math.20200219

Qi, F. (2020b). Completely monotonic degree of remainder of asymptotic expansion of trigamma function, arXiv. [https://arxiv.org/abs/2003.05300v1](https://arxiv.org/abs/2003.05300v1)

Qi, F. (2021). A double inequality for completely monotonic degree of an asymptotic expansion of the trigamma function. arXiv. [https://arxiv.org/abs/2003.05300v2](https://arxiv.org/abs/2003.05300v2)

Qi, F., & Agarwal, R. P. (2019). On complete monotonicity for several classes of functions related to ratios of gamma functions. *Journal of Inequalities and Applications*, 2019 (36), 42 pages. doi:10.1186/s13660-019-1976-z
Qi, F., & Li, W.-H. (2016). Integral representations and properties of some functions involving the logarithmic function. *Filomat*, 30(7), 1659–1674. doi:10.2298/FIL1607659Q

Qi, F., & Liu, A.-Q. (2019). Completely monotonic degrees for a difference between the logarithmic and psi functions. *The Journal of Computational and Applied Mathematics*, 361, 366–371. doi:10.1016/j.cam.2019.05.001

Qi, F., & Mahmoud, M. (2019). Completely monotonic degrees of remainders of asymptotic expansions of the digamma function. HAL preprint. https://hal.archives-ouvertes.fr/hal-02415224v1.

Qi, F., & Mahmoud, M. (2021). Partial solutions to several conjectures on completely monotonic degrees for remainders in asymptotic expansions of the digamma function. arXiv. https://arxiv.org/abs/1912.07989v3.

Schilling, R. L., Song, R., & Vondraček, Z. (2012). *Bernstein Functions* (Vol. 37, de Gruyter Studies in Mathematics, 2nd ed.). Walter de Gruyter. doi:10.1515/9783110269338

Shuang, Y., Guo, B.-N., & Qi, F. (2021). Logarithmic convexity and increasing property of the Bernoulli numbers and their ratios. *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas*, 115(3), 135. doi:10.1007/s13398-021-01071-x

van Haeringen, H. (1996). Completely monotonic and related functions. *Journal of Mathematical Analysis and Applications*, 204(2), 389–408. doi:10.1006/jmaa.1996.0443

Xu, A.-M., & Cen, Z.-D. (2020). Qi’s conjectures on completely monotonic degrees of remainders of asymptotic formulas of di- and tri-gamma functions. *Journal of Inequalities and Applications*, (83), 10. doi:10.1186/s13660-020-02345-5

Widder, D. V. (1946). *The Laplace transform*. Princeton University Press.

Zhao, T.-H., & Chu, Y.-M. (2010). A class of logarithmically completely monotonic functions associated with a gamma function. *Journal of Inequalities and Applications*, 2010(1), 392431. doi:10.1155/2010/392431

Qi, F., & Li, W.-H. (2016). Integral representations and properties of some functions involving the logarithmic function. *Filomat*, 30(7), 1659–1674. doi:10.2298/FIL1607659Q

Qi, F., & Liu, A.-Q. (2019). Completely monotonic degrees for a difference between the logarithmic and psi functions. *The Journal of Computational and Applied Mathematics*, 361, 366–371. doi:10.1016/j.cam.2019.05.001

Qi, F., & Mahmoud, M. (2019). Completely monotonic degrees of remainders of asymptotic expansions of the digamma function. HAL preprint. https://hal.archives-ouvertes.fr/hal-02415224v1.

Qi, F., & Mahmoud, M. (2021). Partial solutions to several conjectures on completely monotonic degrees for remainders in asymptotic expansions of the digamma function. arXiv. https://arxiv.org/abs/1912.07989v3.

Schilling, R. L., Song, R., & Vondraček, Z. (2012). *Bernstein Functions* (Vol. 37, de Gruyter Studies in Mathematics, 2nd ed.). Walter de Gruyter. doi:10.1515/9783110269338

Shuang, Y., Guo, B.-N., & Qi, F. (2021). Logarithmic convexity and increasing property of the Bernoulli numbers and their ratios. *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas*, 115(3), 135. doi:10.1007/s13398-021-01071-x

van Haeringen, H. (1996). Completely monotonic and related functions. *Journal of Mathematical Analysis and Applications*, 204(2), 389–408. doi:10.1006/jmaa.1996.0443

Xu, A.-M., & Cen, Z.-D. (2020). Qi’s conjectures on completely monotonic degrees of remainders of asymptotic formulas of di- and tri-gamma functions. *Journal of Inequalities and Applications*, (83), 10. doi:10.1186/s13660-020-02345-5

Widder, D. V. (1946). *The Laplace transform*. Princeton University Press.

Zhao, T.-H., & Chu, Y.-M. (2010). A class of logarithmically completely monotonic functions associated with a gamma function. *Journal of Inequalities and Applications*, 2010(1), 392431. doi:10.1155/2010/392431