Note on Quantum Newtonian Cosmology

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Abstract

It is well known that, for pressureless matter, Newtonian and relativistic cosmologies are equivalent. We show that this equivalence breaks down in the quantum level. In addition, we find some cases for which quantum Newtonian cosmology can be related to quantum cosmology in (2+1) dimensions. Two exact solutions for the wave function of the Newtonian universe are also obtained.

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1 Introduction

An interesting outcome from Newtonian cosmology is that Friedmann equation for pressureless matter can be obtained from it. That is, for this kind of matter the dynamics obtained from both Newtonian and relativistic cosmologies is equivalent. The first work on this topic dates back to 1934 when E A Milne [1] showed that by combining fluid equations and the cosmological principle one gets to Friedmann equation for pressureless matter. Extensions of this work have been carried out [2] and interestingly it has been shown [3] that the same Friedmann equation can also be obtained from classical mechanics. A recent discussion on this approach can be found in Ref. [4].

Because of the equivalence between Newtonian and relativistic cosmologies just mentioned, it is natural to expect them also to be equivalent in the quantum level. That is, for pressureless matter, one would expect the wave equation of the Newtonian universe to coincide with that of its relativistic counterpart. In this work we show that such a thing does not happen. In fact, despite both cosmologies yield the same equations of motion for pressureless matter, they have different phase spaces. Therefore, after implementing quantization rules, they yield different quantum systems. By considering a matterless space, but with cosmological constant, we also find that Newtonian quantum cosmology is close to quantum cosmology in a (2+1) dimensional space. To end up, we present two cases where the wave function of the Newtonian universe can be calculated exactly.

The work in this paper is organized as follows: In Section 2 we quickly review the Newtonian cosmology and write down the wave equation of the Newtonian universe. In Section 3 we find the differences between Newtonian quantum cosmology and quantum cosmology in (3+1) dimensions. Some cases for which Newtonian quantum cosmology can be mapped to quantum cosmology in (2+1) dimensions are found in Section 4. Section 5 presents a couple of exact solutions for the wave equation of the Newtonian universe and finally in Section 6 conclusions are drawn.

2 Newtonian Cosmology

We concentrate first in obtaining Friedmann equation from classical mechanics. For this, let us assume we have a system of particles interacting gravi-
tationally. Its energy is therefore
\[ E = \frac{1}{2} \sum_{i}^{n} m_{i} r_{i}^{2} - G \sum_{i>j}^{n} \frac{m_{i} m_{j}}{|r_{i} - r_{j}|} - \frac{\Lambda}{6} \sum_{i}^{n} r_{i}^{2}, \tag{1} \]

Now, by considering the cosmological principle we find that \( r_{i}(t) = S(t)r_{i}(t_{0}) \), and thus the energy can be rewritten as
\[ E = \frac{1}{2} A \ddot{S}^{2} - G \frac{B}{S} - \frac{\Lambda}{6} AS^{2}, \tag{2} \]

with \( A = \sum_{i}^{n} m_{i} r_{i}^{2}(t_{0}) \) and \( B = \sum_{i>j}^{n} \frac{m_{i} m_{j}}{|r_{i}(t_{0}) - r_{j}(t_{0})|} \). It is not difficult to see that Eq. (2) is an alternative form of Friedmann equation. By introducing the rescaling \( a = \mu S \), with \( \mu = \text{const} \), this equation becomes
\[ H^{2} = \left( \frac{\dot{a}}{a} \right)^{2} = -\frac{k}{a^{2}} + \frac{\Lambda}{3} + \frac{8\pi G}{3} \rho, \tag{3} \]

with
\[ k = -\frac{2E\mu^{2}}{A}, \quad \rho = \frac{\rho_{0}}{a^{3}}, \quad \text{and} \quad \rho_{0} = \frac{3B\mu^{3}}{4A\pi}. \tag{4} \]

Notice that if \( E = 0 \), then \( k = 0 \). However, if \( E \neq 0 \) one may take \( \mu^{2} = A/2|E| \); which implies either \( k = +1 \) or \( k = -1 \). We see then that \( \mu \) can be chosen in such a way that \( k \) only takes values 1, -1, 0; and therefore Eq. (3) is Friedmann equation. One should realize, however, that only the case \( \rho \propto a^{-3} \), for pressureless matter, is possible.

Let us now express Eq. (3) in terms of the canonical variables. The Lagrangian of the system is
\[ L = \frac{1}{2} \sum_{i}^{n} m_{i} r_{i}^{2} + G \sum_{i>j}^{n} \frac{m_{i} m_{j}}{|r_{i} - r_{j}|} + \frac{\Lambda}{6} \sum_{i}^{n} r_{i}^{2}. \tag{5} \]

By assuming the cosmological principle this reads
\[ L = \frac{1}{2} AS^{2} + G \frac{B}{S} + \frac{\Lambda}{6} AS^{2}, \]
\[ = \frac{A}{2\mu^{2}} a^{2} + G \frac{\mu B}{a} + \frac{\Lambda A}{6\mu^{2}} a^{2}. \tag{6} \]

From this we can construct two phase spaces: one defined as \((S, P_{S})\) with \( P_{S} = AS \) and the other as \((a, P_{a})\) with
\[ P_{a} = \frac{A}{\mu^{2}} \dot{a}, \tag{7} \]
which are equivalent to each other. In terms of the \((S,P_S)\) variables, the Hamiltonian takes the form

\[
H = E = \frac{P_S^2}{2A} - \frac{GB}{S} - \frac{\Lambda A S^2}{6};
\]

whereas Friedmann equation (3) in terms of the canonical variables \((a,P_a)\) reads

\[
\frac{P_a^2}{a^2} + \left(\frac{A}{\mu^2}\right)^2 \left(\frac{k}{a^2} - \frac{\Lambda}{3} - \frac{8\pi G}{3} \rho\right) = 0.
\]

In order to quantize this system we must write down the wave equation

\[
\hat{H}\psi(S) = \left(\frac{\hat{P}_S^2}{2A} - \frac{GB}{S} - \frac{\Lambda A S^2}{6}\right)\psi(S) = E\psi(S).
\]

This is equivalent to request physical states, \(\psi(a)\), which vanish within Friedmann equation (9), i.e.

\[
\left[\frac{1}{a^2} \hat{P}_a^2 + \left(\frac{A}{\mu^2}\right)^2 \left(\frac{k}{a^2} - \frac{\Lambda}{3} - \frac{8\pi G}{3} \rho\right)\right]\psi(a) = 0,
\]

with \(\hat{P}_a = -i\hbar \partial/\partial a\). Solutions to this equation are the quantum states of the Newtonian universe. Another proposal of quantum Newtonian cosmology can be found in Ref. [5].

### 3 Quantum Cosmology in \((3+1)\) Dimensions

We describe now the differences in the quantum level between Newtonian and relativistic cosmologies. It is a known fact that Friedmann equation (3) can also be obtained from Einstein equations for a space with cosmological constant and the Robertson-Walker metric [6]. For this metric, the kinetic term in the Lagrangian from general relativity is

\[
L = -\frac{\alpha}{2} a \dot{a}^2.
\]

Therefore, for this case the canonical momentum is given by

\[
\frac{\partial L}{\partial \dot{a}} = \Pi_a = -\alpha a \dot{a}.
\]
Note that this differs from the canonical momentum from classical mechanics in Eq. (7). Now, by using Eq. (13) Friedmann equation (3) becomes

\[ H = \frac{\Pi^2}{a^4} + \left( \frac{1}{\alpha} \right)^2 \left( \frac{k}{a^2} - \frac{\Lambda}{3} - \frac{8\pi G}{3} \rho \right) = 0. \]  

(14)

In the classical level, both Eqs. (14) and (9) describe the same dynamical system as just different auxiliary variables were used to rewrite them. Though this is irrelevant in the classical regime, it yields notable differences quantumly.

Within the canonical formalism Eq. (14) represents a first-class constraint [7]. According to that formalism, the quantity \( H \) is substituted by an operator and the physical states, \( \psi(a) \), are found as the null vectors of this operator. Therefore, the equation determining the physical states is

\[ \left[ \frac{\hat{\Pi}^2}{a^4} + \left( \frac{1}{\alpha} \right)^2 \left( \frac{k}{a^2} - \frac{\Lambda}{3} - \frac{8\pi G}{3} \rho \right) \right] \psi(a) = 0, \]  

(15)

with \( \hat{\Pi}_a = -i\hbar \partial/\partial a \). According to general relativity, this equation determines the quantum state of the universe. This is the so-called Wheeler-De Witt equation [8].

As it can be seen, Eq. (11) differs from (15). This is because the classical phase space of both systems is different. Thus, in the classical level, Newtonian and relativistic in (3+1) dimensions cosmologies are equivalent, but quantumly they are different things. As an example, by considering the \( \rho = 0 \) and \( \Lambda \neq 0 \) case we can see that: for the Newtonian cosmology the system behaves as an oscillator both in the classical and quantum levels. However, for the relativistic cosmology the system in the classical level behaves as an oscillator, but in the quantum level it is no longer an oscillator. It could be thought that the inequivalence arises from the canonical momentum definition and that, by using another quantization formalism, the equivalence could be preserved. However, due to the difference in actions, after carrying out the path integrals one would obtain that difference again.
4 Cosmology in (2+1) Dimensions

Now we concentrate in the connection between Newtonian cosmology and cosmology in a (2+1) dimensional space. Let us then consider the metric
\[ ds^2 = dt^2 - a^2 \left( \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 \right). \]  
(16)

For pressureless matter, and this metric, Einstein equations are
\[ \left( \frac{\dot{a}}{a} \right)^2 = -\frac{k}{a^2} + \Lambda + 8\pi G \rho, \]  
(17)
\[ \frac{\ddot{a}}{a} = \Lambda. \]  
(18)

By using Eq. (16) the kinetic term in the Lagrangian from general relativity becomes
\[ L = -\frac{\beta}{2} \dot{a}^2; \]  
(19)
so that the canonical momentum is
\[ P_a = -\beta \dot{a}. \]  
(20)

From this, Friedmann equation becomes
\[ \frac{\dot{a}^2}{a^2} + \left( \frac{1}{\beta} \right)^2 \left( \frac{k}{a^2} - \Lambda - 8\pi G \rho \right) = 0. \]  
(21)

By quantizing this system we find physical states, \( \psi(a) \), such that
\[ \left[ \frac{\dot{P}_a^2}{a^2} + \left( \frac{1}{\beta} \right)^2 \left( \frac{k}{a^2} - \Lambda - 8\pi G \rho \right) \right] \psi(a) = 0. \]  
(22)

Clearly, up to ordering terms, Eq. (9) is analogous to (21). Nevertheless, in two dimensions, for pressureless matter one has \( \rho \propto a^{-2} \), which is different to the three-dimensional case; but if we take \( \rho \approx 0 \) and perform the changes
\[ \beta^{-1} \rightarrow \frac{A}{\mu^2}, \quad \Lambda \rightarrow \frac{\Lambda}{3}, \]  
(23)
then Eq. (21) becomes equal to (9) and Eq. (22) analogous to (11). That is, for the matterless case, both in the classical and quantum levels Newtonian cosmology can be mapped into the relativistic one in (2+1) dimensions.
5  Wave Function of the Newtonian Universe

Let us now look at the wave function of the Newtonian universe. Because $S$ and $a$ are positive, the wave equation is defined only in the positive axis. This leads one to require appropriate boundary conditions. By imposing $\psi(\infty) = 0$, one must also impose $\psi(0) = 0$ [9].

We first consider the case of matter dominated by a negative cosmological constant; i.e. $\Lambda = -|\Lambda|$ and $\rho \approx 0$. For this, Eq. (10) reads

$$\hat{H}\psi(S) = \left[ -\frac{\hbar^2}{2A} \frac{\partial^2}{\partial S^2} - \frac{A\Lambda S^2}{6} \right] \psi(S) = E\psi(S). \quad (24)$$

By introducing the variable $z = \left( \frac{A}{\hbar} \sqrt{\frac{A\Lambda}{3}} \right)^{1/2} S$, Eq. (24) can be rewritten as

$$\left[ \frac{\partial^2}{\partial z^2} + \left( \frac{2E}{h\sqrt{3|\Lambda|}} - z^2 \right) \right] \psi(z) = 0; \quad (25)$$

which has solutions

$$\psi_n(z) = H_{2n+1}(z)e^{-z^2/2}, \quad n = 0, 1, \ldots \quad (26)$$

where $H_N(z)$ is the Hermite polynomial of order $N$; and energy spectrum

$$E_n = \hbar \sqrt{\frac{|\Lambda|}{3}} \left( 2n + \frac{3}{2} \right). \quad (27)$$

Note that this does not depend on $A$.

Now we turn to the dust case, i.e. $\rho = \rho_0/a^3$ and $\Lambda = 0$. For this, the wave equation reads

$$\left[ \frac{\partial^2}{\partial S^2} + \left( \frac{2EA}{\hbar^2} + \frac{2ABG}{\hbar^2} \right) \right] \psi(S) = 0. \quad (28)$$

By using $z = \left( \sqrt{-\frac{8EA}{h^2}} \right) S$, one gets to the equation

$$\left[ \frac{z^2}{\partial z^2} + \left( \gamma - \frac{z}{4} \right) \right] \psi(z) = 0, \quad (29)$$

$$7$$
where $\gamma = \frac{2ABG}{\hbar^2} \sqrt{\frac{h^2}{8EA}}$. In this case the complete set of solutions vanishing at the origin and infinity is

$$\psi_n(z) = e^{-z/2}z L_n^1(z);$$  \hspace{1cm} (30)

where $L_n^1(z)$ is the associate Laguerre polynomial of order $n$; whereas the energy spectrum is given by

$$E_n(z) = -\frac{AB^2G^2}{2h^2(n + 1)^2}. \hspace{1cm} (31)$$

We can see then that there are two cases where exact solutions to the wave equation of the Newtonian cosmology can be obtained. For the same two cases in the relativistic cosmology in (3+1) dimensions no exact solutions are known.

It is remarkable that, for a cosmology with cosmological constant and pressureless matter, there are theories which in the classical regime yield the same dynamics, but in the quantum level are inequivalent. This phenomenon has been also found in other physical systems [10].

6 Conclusions

It was shown that despite, for pressureless matter, Newtonian cosmology is equivalent to a relativistic one in (3+1) dimensions, in the quantum level they are inequivalent. It was also shown, however, that especial cases exist where the quantum Newtonian cosmology can be mapped to a quantum cosmology in (2+1) dimensions. At the end, two cases were presented where the exact wave function of the Newtonian universe could be found.

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