Existence of a tricritical point for the Blume-Capel model on $\mathbb{Z}^d$

Trishen S. Gunaratnam, Dmitrii Krachun, and Christoforos Panagiotis

October 25, 2022

Abstract

We prove the existence of a tricritical point for the Blume-Capel model on $\mathbb{Z}^d$ for every $d \geq 2$. The proof in $d \geq 3$ relies on a novel combinatorial mapping to an Ising model on a larger graph, the techniques of Aizenman, Duminil-Copin, and Sidoravicius (Comm. Math. Phys, 2015), and the celebrated infrared bound. In $d = 2$, the proof relies on a quantitative analysis of crossing probabilities of the dilute random cluster representation of the Blume-Capel. In particular, we develop a quadrichotomy result in the spirit of Duminil-Copin and Tassion (Moscow Math. J., 2020), which allows us to obtain a fine picture of the phase diagram in $d = 2$, including asymptotic behaviour of correlations in all regions. Finally, we show that the techniques used to establish subcritical sharpness for the dilute random cluster model extend to any $d \geq 2$.

1 Introduction

Let $L_d = (\mathbb{Z}^d, E^d)$ be the standard $d$-dimensional hypercubic lattice with vertex set $\mathbb{Z}^d$ and nearest-neighbour edges $E^d$. Given a finite subgraph $G = (V, E)$ of $L_d$, the Blume-Capel model on $G$ with inverse temperature $\beta > 0$, crystal field strength $\Delta \in \mathbb{R}$, and boundary condition $\eta \in \{-1, 0, 1\}^{\mathbb{Z}^d}$ is the probability measure $\mu_{G, \beta, \Delta}^\eta$ defined on spin configurations $\sigma \in \{-1, 0, 1\}^V$ by

$$\mu_{G, \beta, \Delta}^\eta(\sigma) = \frac{1}{Z_{G, \beta, \Delta}^\eta} e^{-H_{G, \beta, \Delta}^\eta(\sigma)},$$

where

$$H_{G, \beta, \Delta}^\eta(\sigma) = -\beta \sum_{xy \in E} \sigma_x \sigma_y - \Delta \sum_{x \in V} \sigma_x^2 - \beta \sum_{x \in V, y \in \mathbb{Z}^d \setminus V} \sigma_x \eta_y$$

is the Hamiltonian, and $Z_{G, \beta, \Delta}^\eta$ is the partition function. By convention, we write $\mu_{V, \beta, \Delta}^\eta$ to denote the Blume-Capel measure on the subgraph of $L_d$ spanned by $V$, and we denote expectations by $\langle \cdot \rangle_{V, \beta, \Delta}^\eta$. We denote by 0 (resp. +, −) the boundary conditions $\eta_x = 0$ (resp. $\eta_x = +1, \eta_x = -1$) for all $x \in \mathbb{Z}^d$.

The Blume-Capel model is closely related to one of the most famous models in statistical physics, the Ising model. The latter is defined analogously, with spins taking values in $\{-1, 1\}$ and the Hamiltonian only consisting of the terms involving $\beta$. In particular, one can view the Blume-Capel model as an Ising model on an annealed random environment, i.e. on the (random) set of vertices for which $\sigma_x \neq 0$. In the limit $\Delta \to \infty$, the underlying random environment becomes deterministically equal to $\mathbb{Z}^d$, and the Blume-Capel model converges to the classical Ising model on $\mathbb{Z}^d$. 

*Université de Genève, trishen.gunaratnam@unige.ch
†Université de Genève, dmitrii.krachun@unige.ch
‡Université de Genève, christoforos.panagiotis@unige.ch
The model was introduced independently by Blume [Blu66] and Capel [Cap66] in 1966 to study the magnetisation of uranium oxide and an Ising system consisting of triplet ions, respectively. Both papers were trying to explain first-order phase transitions that are driven by mechanisms other than external magnetic fields. Since then, it has been studied extensively by physicists due to its particularly rich phase diagram, i.e. as an archetypical example of a model that exhibits a multicritical point, see [BBS19] and references therein. Indeed, for each value of the parameter $\Delta \in \mathbb{R}$, the Blume-Capel model undergoes a phase transition at a critical parameter $\beta_c(\Delta)$, which is defined as

$$\beta_c(\Delta) = \inf \{ \beta > 0 \mid \langle \sigma_0 \rangle^{+}_{G,\beta,\Delta} > 0 \},$$

where $\langle \cdot \rangle^{+}_{G,\beta,\Delta}$ denotes the plus measure at infinite volume, which is defined as the limit of the finite volume measures $\langle \cdot \rangle_{G,\beta,\Delta}$ as $G$ tends to $\mathbb{L}^d$. It is expected that the critical behaviour of the model depends strongly on $\Delta$: as $\Delta$ increases, the phase transition changes from discontinuous, when $\langle \sigma_0 \rangle^{+}_{\beta_c(\Delta),\Delta} > 0$, to continuous, when $\langle \sigma_0 \rangle^{+}_{\beta_c(\Delta),\Delta} = 0$, and this transition happens as $\Delta$ crosses a single point $\Delta_{\text{tric}}$.

The model at the so-called tricritical point ($\beta_c(\Delta_{\text{tric}}), \Delta_{\text{tric}}$) is of particular interest. Indeed, in some dimensions it is expected to exhibit vastly different behaviour from the points on the critical line when $\Delta > \Delta_{\text{tric}}$, even though in both cases the phase transition is expected to be continuous. In particular, in $d = 2$, the scaling limit of the model at the critical points when $\Delta > \Delta_{\text{tric}}$ is expected to be in the Ising and $\phi^4$ universality class. On the other hand, at $\Delta_{\text{tric}}$, the scaling limit of the model is expected to be in a distinct universality class corresponding to the $\phi^6$ minimal conformal field theory with central charge $c = 7/10$ (whilst the Ising universality class is of central charge $c = 1/2$) – see Mussardo [Mus10]. A rigorous glimpse of this distinct universality class appears in the work of Shen and Weber [SW18], where near-critical scaling limits are considered. In $d = 3$, the scaling limit of the model at $\Delta > \Delta_{\text{tric}}$ is expected to be nontrivial, as conjecturally for Ising which is supported by e.g. conformal bootstrap methods, see Rychkov, Simmons-Duffin, and Zan [RSDZ17]. Whereas for $\Delta_{\text{tric}}, d = 3$ is predicted to be the upper-critical dimension for the model and one expects triviality of the scaling limit. This is supported by renormalisation group heuristics, which has recently been made rigorous for the $\phi^6$ model in the weak coupling regime by Bauerschmidt, Lohmann, and Slade [BLS20]. In dimensions $d \geq 5$, one expects that the model is trivial throughout the continuous phase transition regime – c.f. the case of Ising, where triviality was shown by Aizenman [Aiz82] and Fröhlich [Frö82]. We also refer to the review book [BBS19] for an account of renormalisation group approaches to this problem. In $d = 4$, there may be some distinction: for $\Delta > \Delta_{\text{tric}}$, one expects a marginal triviality result as in the case for Ising, which was recently shown by Aizenman and Duminil-Copin [ADC21], whereas at $\Delta_{\text{tric}}$ it is unclear whether logarithmic corrections are present.

Despite the interest of this model in physics and the interesting predictions about the tricritical point, there is a lack of rigorous understanding of the phase diagram of the Blume-Capel model. Indeed, many rigorous results about the model have been focused well within the discontinuous transition regime, where it is a good test case for Pirogov-Sinai theory – see [FV17, Chapter 7] and [BS89]. This is in contrast to the case of the Ising model, where much of the phase diagram is now well-understood [DC17]. Stochastic geometric methods have been at the heart of many recent developments, in particular probabilistic representations of spin correlations via the random cluster and random current representations. For the Blume-Capel model, an analogous random cluster representation, called the dilute random cluster representation, has been developed by Graham and Grimmett [GG06]. In this article, we show that the underlying philosophy of the recent techniques used to analyse the Ising model and its random cluster representation can be adapted to rigorously analyse the phase diagram of the Blume-Capel model in dimensions $d \geq 2$. The main result of this paper implies the existence of at least one separation point

\footnote{In the physics literature, $\beta_c$ is sometimes called a transition point or triple point when the phase transition is discontinuous, to distinguish from a critical point corresponding to continuous phase transition. In this article we adopt percolation terminology and call $\beta_c$ a critical point.}
between the points of continuous and discontinuous phase transitions, i.e. the existence of a tricritical point. Namely, we prove the following.

**Theorem 1.1.** Let \( d \geq 2 \). Then, there exist \( \Delta^{-}(d) \leq \Delta^{+}(d) \) such that

- for any \( \Delta < \Delta^{-}(d) \), \( \langle \sigma_{0} \rangle_{\beta_{c}(\Delta),\Delta} > 0 \)
- for any \( \Delta \geq \Delta^{+}(d) \), \( \langle \sigma_{0} \rangle_{\beta_{c}(\Delta),\Delta} = 0 \).

Moreover, one can take \( \Delta^{+}(d) := -\log 2 \) for \( d \geq 3 \), and \( \Delta^{+}(d) := -\log 4 \) for \( d = 2 \).

The proof of the existence of a discontinuous critical phase is a standard application of Pirogov-Sinai theory and is given in Section 3. The crux of the article is to establish the existence of a continuous critical phase, for which we use different techniques depending on the dimension. In the case \( d \geq 3 \), the proof relies on a new representation of the model as an Ising model on a larger (deterministic) graph, which is ferromagnetic only when \( \Delta \geq -\log 2 \) – see Section 2.2. This representation has the advantage that it naturally relates the correlation functions between the Blume-Capel and the Ising model, and allows us to use techniques developed for the latter to study the former. We use the celebrated infrared/Gaussian domination bound of Fröhlich, Simon, and Spencer [FSS76] to show that under the free boundary conditions, the two-point correlations decay, in an averaged sense, to 0 for every \( \beta \leq \beta_{c}(\Delta) \). This allows us to use the breakthrough result of Aizenmann, Duminil-Copin and Sidoravicius [ADCS15] on the continuity of the Ising model on \( \mathbb{Z}^{d} \) (see also [Rao20] for a relevant result for general amenable graphs) to conclude that the phase transition of the Blume-Capel model is continuous in \( d \geq 3 \) when the corresponding Ising model is ferromagnetic, namely, when \( \Delta \geq -\log 2 \).

In dimension 2, the infrared bound is no longer useful and the proof instead relies on quantitative estimates on crossing probabilities. In particular, we develop a Russo-Seymour-Welsh (RSW) theory for the dilute random cluster representation of the Blume-Capel model. Following the renormalisation strategy of [DCT20], we obtain a quadrichotomy result which describes the behaviour of crossing probabilities of macroscopic squares (and rectangles) under the effect of boundary conditions which are at macroscopic distance, and applies to both the critical and the off-critical phase. At criticality, when the phase transition is continuous, the crossing probabilities stay bounded away from 0 and 1 under both the wired (favourable boundary conditions) and the free measure (unfavourable boundary conditions). Whereas, when the phase transition is discontinuous, the crossing probabilities converge to 1 exponentially fast under the wired boundary conditions, and they converge to 0 exponentially fast under the free boundary conditions. As a consequence, in both cases, the two-point correlations decay to 0 under the free measure, and as in dimensions \( d \geq 3 \), the phase transition is continuous when \( \Delta \geq -\log 2 \).

The estimate on \( \Delta^{+} \) can be improved in dimension 2 to obtain continuity for \( -\log 4 \leq \Delta < -\log 2 \), where the corresponding Ising model is not ferromagnetic. Indeed, under the condition that the weak plus measure (defined as the measure with +\( \epsilon \) boundary condition) converges to the plus measure in the infinite volume limit, one can show that the Radon-Nikodym derivative between the plus and the free measure grows subexponentially fast with the boundary of the domain. This renders the behaviour of crossing probabilities when the phase transition is discontinuous impossible. An estimate on when this criterion holds, i.e. for \( \Delta \geq -\log 4 \), is obtained by showing a Lee-Yang type theorem on the complex zeros of the partition functions of the Blume-Capel model.

**Remark 1.2.** We note that \( \Delta = -\log 4 \) corresponds to the location of the tricritical point for the Blume-Capel model on the complete graph, see [EOT05, SW18]. We expect that the tricritical point on \( \mathbb{Z}^{d} \) converges to that of the complete graph as \( d \to \infty \), i.e. in the mean-field limit. One can imagine that the underlying philosophy of using the convergence of the weak plus measure to the plus measure is still a sufficient condition to yield continuity in higher dimensions. Thus, since our Lee-Yang type theorem holds for \( \Delta \geq -\log 4 \) on any graph, it seems that the underlying strategy becomes sharp as \( d \to \infty \).
Proving uniqueness of the tricritical point, which is the main omission of Theorem 1.1, amounts to showing that one can take $\Delta^{-}(d) = \Delta^{+}(d)$ in Theorem 1.1. Unfortunately, it is unclear whether there is monotonicity along the critical line and, in the absence of integrability, to the best of our knowledge all known techniques for showing discontinuity are intrinsically perturbative. Nevertheless, in dimension 2, the quantitative estimates on crossing probabilities, and other considerations that we describe shortly, allow us to obtain a fuller picture of the phase diagram, although we fall short of proving uniqueness of the tricritical point.

To state our next result, we first recall the definition of the truncated 2-point correlation:

$$(\sigma_0; \sigma_x)^+ = (\sigma_0 \sigma_x)^+ - \langle \sigma_0 \rangle^+ \langle \sigma_x \rangle^+.$$ 

In dimension $d = 2$ we obtain a fine picture of the phase diagram by showing that the truncated 2-point correlation decays exponentially everywhere except for the continuous critical regime, where they decay polynomially. We also give an alternative characterisation of the points of continuity in terms of the percolation of 0 and $-$ spins (equivalently, $*$-percolation properties of $+$ spins, where $*$-percolate refers to percolation on $\mathbb{Z}^2$ union the diagonals).

**Theorem 1.3.** Let $d = 2$. Then the following hold.

- **(OffCrit)** For all $\beta > 0$ and $\Delta \in \mathbb{R}$ such that $\beta \neq \beta_c(\Delta)$, there exists $c = c(\beta, \Delta) > 0$ such that
  
  $$(\sigma_0; \sigma_x)^+ \leq e^{-c\|x\|_{\infty}}, \quad \forall x \in \mathbb{Z}^2.$$ 

- **(DiscontCrit)** For all $\Delta \in \mathbb{R}$ such that $\langle \sigma_0 \rangle_{\beta_c(\Delta)}^+ > 0$, there exists $c = c(\Delta) > 0$ such that
  
  $$(\sigma_0; \sigma_x)^+ \leq e^{-c\|x\|_{\infty}}, \quad \forall x \in \mathbb{Z}^2.$$ 

- **(ContCrit)** For all $\Delta \in \mathbb{R}$ such that $\langle \sigma_0 \rangle_{\beta_c(\Delta)}^+ = 0$, there exist $c_1 = c_1(\Delta), c_2 = c_2(\Delta)$, such that for all $x \in \mathbb{Z}^2$ with $\|x\|_{\infty}$ large enough
  
  $$\frac{1}{\|x\|_{\infty}} \leq (\sigma_0 \sigma_x)^+_{\beta_c(\Delta), \Delta} \leq \frac{1}{\|x\|_{\infty}}.$$ 

- **(TriCrit)** The set of $\Delta \in \mathbb{R}$ such that $\langle \sigma_0 \rangle_{\beta_c(\Delta)}^+ > 0$ is open.

- **(Perc)** For all $\Delta \in \mathbb{R}$, $(\sigma_0)^+_{\beta_c(\Delta)} = 0$ if and only if $\{0, -\}$ spins do not percolate under $\langle \cdot \rangle^0_{\beta_c(\Delta)}$.

The proof of the behaviour at the critical points is based on the quadrichotomy for crossing probabilities for the dilute random cluster representation of the Blume-Capel model mentioned earlier. The fact that (TriCrit) holds has the implication that at any separation point (i.e. tricritical point) on the line of critical points, the phase transition is continuous and hence satisfies (ContCrit). The percolation characterisation (Perc) is an adaptation of an elegant argument for the continuity of nearest-neighbour Ising on $\mathbb{Z}^2$ by Werner [Wer09]. The proof of the subcritical behaviour relies on a generalisation of the OSSS inequality for monotonic measures by Duminil-Copin, Raoufi and Tassion [DCRT19] to the dilute random cluster model, and more generally to weakly monotonic measures, which we define in Section 8.1. The original OSSS inequality was obtained by O’Donell et. al. [OSSS05] for product measures. In fact, the technique for showing subcritical sharpness is robust enough to extend to all dimensions:

**Theorem 1.4.** Let $d \geq 2$ and $\Delta \in \mathbb{R}$. Then for every $\beta < \beta_c(\Delta)$ there exists $c = c(\beta, \Delta, d) > 0$ such that

$$\langle \sigma_0 \sigma_x \rangle_{\beta, \Delta}^+ \leq e^{-c\|x\|_{\infty}}$$ 

for every $x \in \mathbb{Z}^d$. 

4
We end on a natural follow up question on Theorem 1.3, which relates the percolative properties of the underlying random environment to the nature of the critical point.

**Question.** In $d = 2$, is it true that, for all $\Delta \in \mathbb{R}$, \((\sigma_0)_{\rho^2(\Delta), \Delta}^0 = 0\) if and only if \(\{0\}\) spins do not \(s\)-percolate under \((\cdot)_{\rho^2(\Delta), \Delta}^0\)?

### 1.1 Paper organisation

In Section 2 we define the dilute random cluster model and develop the necessary tools that we require for the rest of this article. We then establish basic facts about the phase transition for the Blume-Capel model in Section 3. In Section 4 we introduce a combinatorial mapping from the Blume-Capel model on $Z^d$ to an Ising model on an enlarged graph, and we use it to prove Theorem 1.1 for $d \geq 3$ in Section 5. In Section 6 we establish a quadrichotomy result for crossing probabilities under the dilute random cluster model in $d = 2$ and derive some basic consequences of it. We then apply this, together with Lee-Yang type arguments, to establish Theorem 1.1 for $d = 2$ in Section 7. In Section 8 we prove Theorem 1.4 via a generalisation of the OSSS argument to weakly monotonic measures. Finally, in Section 9 we finish off the proof of Theorem 1.3.

### Acknowledgements

Above all we thank Hugo Duminil-Copin for encouraging this collaboration. We thank Hendrik Weber for introducing us to this model. We thank Amol Aggarwal, Roman Kotecký, Vieri Mastropietro, Romain Panis, Sébastien Ott, Tom Spencer, Yvan Velenik for exciting discussions at various stages of this project.

TSG was supported by the Simons Foundation, Grant 898948, HD. DK and CP were supported by the Swiss National Science Foundation and the NCCR SwissMAP.

### 2 The dilute random cluster model

#### 2.1 Definitions

Let $\Psi = \{0, 1\}^{\mathbb{Z}^d}$ and $\Omega = \{0, 1\}^{\mathbb{E}^d}$. We say that $\psi = (\psi_x) \in \Psi$ and $\omega = (\omega_e) \in \Omega$ are compatible if $\omega_{xy} = 0$ whenever $\psi_x = 0$ or $\psi_y = 0$. Denote by $\Theta \subset \Psi \times \Omega$ the set of all possible compatible configurations, equipped with the $\sigma$-algebra generated by cylinder events, $\mathcal{F}$.

Let $((\psi, \omega)) \in \Theta$. A vertex $x \in V$ is called open if $\psi_x = 1$, and is called closed otherwise. An edge $e$ is called open if $\omega_e = 1$, and closed otherwise. Define $V_\psi = \{x \in \mathbb{Z}^d : \psi_x = 1\}$ to be the set of open vertices and $E_\psi = \{xy \in \mathbb{E}^d : \psi_x = \psi_y = 1\}$ to be the set of induced edges. In addition, define $o(\omega)$ to be the set of open edges. Observe that $((\psi, \omega)) \in \Theta$ if and only if $o(\omega) \subseteq E_\psi$.

For $\Lambda \subset \mathbb{Z}^d$ be finite, let $\partial \Lambda = \{x \in \mathbb{Z}^d \setminus \Lambda$ such that there exists $y \in \Lambda$ incident to $x \}$ and set $\Lambda := \Lambda \cup \partial \Lambda$. In addition, write $E_\Lambda$ for the set of edges with at least one endpoint in $\Lambda$, and $b(\Lambda) = \{xy \in \mathbb{E}^d : x, y \in \Lambda\}$ for the set of edges induced by $\Lambda$. We stress that the region $(\Lambda, E_\Lambda)$ is not a graph, but $(\Lambda, E_\Lambda)$ is a graph. Finally, given $\psi \in \Psi$, write $E_{\psi, \Lambda} = E_\psi \cap E_\Lambda$.

Let $\xi = (\kappa, \rho) \in \Theta$. Define $\Theta^\xi_\Lambda$ to be the set of $((\psi, \omega)) \in \Theta$ such that $\psi_x = \kappa_x$ for $x \in \mathbb{Z}^d \setminus \Lambda$ and $\omega_e = \rho_x$ for $e \in \mathbb{E}^d \setminus E_\Lambda$, i.e. configurations on $\Lambda$ with boundary condition $\xi$. Note that configurations in $\Omega^\xi_\Lambda$ are measurable with respect to the region $(\Lambda, E_\Lambda)$. We naturally identify the $\sigma$-algebra on $\Theta^\xi_\Lambda$ as a sub $\sigma$-algebra of $\mathcal{F}$. Finally, we write $k(\theta, \Lambda)$ for the number of connected components of $(V_\psi, o(\omega))$ that intersect $\Lambda$.

We now define the finite volume dilute random cluster model with parameters $a \in (0, 1)$ and $p \in (0, 1)$ and fixed boundary conditions.
Definition 2.1 (Dilute random cluster model). Let $p, a \in (0, 1)$ and write $r = \sqrt{1 - p}$. For $\Lambda \subset \mathbb{Z}^d$ finite and $\xi \in \Theta^\xi_{\Lambda}$, let $\varphi^\xi_{\Lambda, p, a}$ denote the probability measure on $\Theta^\xi_{\Lambda}$ defined, for $\theta \in \Theta^\xi_{\Lambda}$, by

$$
\varphi^\xi_{\Lambda, p, a}[\theta] = \frac{1}{Z^\xi_{\Lambda, p, a}} \prod_{x \in \Lambda} \left( \frac{a}{1 - a} \right)^{\psi_x} \prod_{e \in E_{\psi, \Lambda}} \left( \frac{p}{1 - p} \right)^{\omega_e} \mathcal{Z}^{[\psi, \Lambda]}_{\Lambda, \psi, a} [2^k(\theta, \Lambda)]
$$

(2.1)

where $Z^\xi_{\Lambda, p, a}$ is the normalisation constant.

When clear from context, we simply write $\varphi^\xi_{\Lambda, p, a} = \varphi^\xi_{\Lambda}$.

On any finite subset of $\mathbb{Z}^d$, there are two natural candidates for extremal dilute random cluster measures. These measures play a central role in this paper, so we formalise them in the following definition.

Definition 2.2. Let $p \in [0, 1)$ and $a \in [0, 1]$. For $\Lambda \subset \mathbb{Z}^d$ finite, we define

- the free measure $\varphi^0_{\Lambda} := \varphi^{(0,0)}_{\Lambda, p, a}$
- and, the wired measure $\varphi^1_{\Lambda} := \varphi^{(1,1)}_{\Lambda, p, a}$

where $(0,0)$ is the configuration consisting of only closed vertices and edges, and $(0,0)$ is the configuration consisting of only open vertices and edges.

Remark 2.3. It is also of interest to include the cases when $p, a \in \{0, 1\}$. By convention, for any $p \in [0, 1]$, we set: $\varphi^\xi_{\Lambda, p, 0} = \delta_{(0,0)}$; and, $\varphi^\xi_{\Lambda, p, 1}$ to be the usual random cluster measure of parameter $p$ and $q = 2$. On the other hand, for any $a \in (0,1]$, we set: $\varphi^\xi_{\Lambda, 0, a}$ to be the tensor product of Bernoulli site percolation of parameter $a$ and the Dirac measure centred at $0 \in \Omega$; and, $\varphi^\xi_{\Lambda, 1, a} = \delta_{(1,1)}$.

Finally, we define a probability measure on $\Psi$ that captures the law of the (dependent) site percolation induced by the dilute random cluster measure.

Definition 2.4. Let $p, a \in [0, 1]$. For $\Lambda \subset \mathbb{Z}^d$ and $\xi = (\kappa, \rho) \in \Theta$, let $\Psi^\xi_{\Lambda, p, a}$ denote the probability measure on $\Psi$ defined, for $\psi \in \Psi$, by

$$
\Psi^\xi_{\Lambda, p, a}[\psi] = \sum_{\omega \in \Omega} \varphi^\xi_{\Lambda, p, a}[\psi, \omega] 1\{\psi|_{\Lambda\setminus\Lambda} = \kappa\}.
$$

We call $\Psi^\xi_{\Lambda, p, a}$ the vertex marginal and write $\Psi^\xi_{\Lambda} = \Psi^\xi_{\Lambda, p, a}$ when clear from context.

2.2 Basic properties

We now list some important properties of the dilute random cluster model, the proofs of which follow either directly from the definition or by straightforward modifications of standard arguments, see e.g. [DCT20].

The first property tells us that, conditional on the random environment $\psi \in \Psi$, the dilute random cluster measure coincides with the usual random cluster measure on the random graph induced by $\psi$.

Proposition 2.5. Let $p, a \in [0, 1]$, $\Lambda \subset \mathbb{Z}^d$ finite, and $\xi = (\kappa, \rho) \in \Theta$. For every $\psi \in \Psi$ such that $\psi_x = \kappa_x$ for all $x \in \mathbb{Z}^d \setminus \Lambda$, we have

$$
\varphi^\xi_{\Lambda, p, a}[\omega | \psi] = \phi^{RC, \xi}_{\Lambda, \psi, a}[\omega].
$$

(2.2)
Above,
\[
\phi_{\Lambda,\psi,p}^{RC,\xi} \left[ \omega \right] = \frac{1}{Z_{\Lambda,\psi,p}^{RC,\xi}} 2^{k(\theta,\Lambda)} \prod_{e \in E_{\psi,\Lambda}} \left( \frac{p}{1 - p} \right)^{\omega_e}
\]
where \( Z_{\Lambda,\psi,p}^{RC,\xi} \) is the normalization constant. Note that \( \phi_{\Lambda,\psi,p}^{RC,\xi}[\omega] \) is the random cluster measure on \( V_\psi \cap \Lambda \) with boundary conditions \( \xi \), and parameters \( p \) and \( q = 2 \).

The second property concerns automorphism invariance of the measure.

**Proposition 2.6.** Let \( \tau \) be an automorphism of \( \mathbb{Z}^d \). Let \( p, a \in [0, 1] \), \( \Lambda \subset \mathbb{Z}^d \) finite, and \( \xi = (\kappa, \rho) \in \Theta \). Then, for every event \( A \) depending on the vertices and edges in \((\Lambda, E_\Lambda)\), we have
\[
\varphi^{\xi}_\Lambda[\tau A] = \varphi^{\xi}_\Lambda[A],
\]
where \( \tau A \) denotes the image of \( A \) under the action of \( \tau \).

Next, we formalise the following spatial Markov property.

**Proposition 2.7.** Let \( p, a \in [0, 1] \), \( \Lambda' \subset \Lambda \) finite subsets of \( \mathbb{Z}^d \), and \( \xi \in \Theta \). For every configuration \( \xi' = (\psi', \omega') \in \{0, 1\}^{\Lambda \setminus \Lambda'} \times \{0, 1\}^{E_\Lambda \setminus E_{\Lambda'}} \),
\[
\varphi^{\xi}_\Lambda[\cdot | \psi_x = \psi'_x \forall x \in \Lambda \setminus \Lambda', \omega_e = \omega'_e \forall e \in E_\Lambda \setminus E_{\Lambda'}] = \varphi^{\xi \cup \xi'}_{\Lambda'} \tag{SMP}
\]
where \( \xi \cup \xi' \in \Theta \) is the boundary condition which is equal to \( \xi' \) on \((\Lambda \setminus \Lambda') \times (E_\Lambda \setminus E_{\Lambda'})\), and otherwise coincides with \( \xi \).

Finally, we state the finite energy property of the measure.

**Proposition 2.8.** Let \( p, a \in [0, 1] \), \( \Lambda \subset \mathbb{Z}^d \) finite, and \( \xi \in \Theta \). For every \( x \in \Lambda \) and every configuration \( \xi' = (\psi', \omega') \in \{0, 1\}^{\Lambda \setminus \{x\}} \times \{0, 1\}^{E_\Lambda \setminus E_{\{x\}}} \),
\[
0 < \varphi^{\xi}_\Lambda[\psi_x = 1 \mid \psi_y = \psi'_y \forall y \in \Lambda \setminus \{x\}, \omega_e = \omega'_e \forall e \in E_\Lambda \setminus E_{\{x\}}] < 1.
\]
Moreover, for every edge \( xy \in b(\Lambda) \) and every configuration \( \xi' = (\psi', \omega') \in \{0, 1\}^{\Lambda \setminus \{x,y\}} \times \{0, 1\}^{E_\Lambda \setminus \{xy\}} \),
\[
0 < \varphi^{\xi}_\Lambda[\omega_{xy} = 1 \mid \psi_u = \psi'_u \forall u \in \Lambda \setminus \{x,y\}, \omega_e = \omega'_e \forall e \in E_\Lambda \setminus \{xy\}] < 1.
\]

### 2.3 Edwards-Sokal coupling with the Blume-Capel model

Let \( \Delta \in \mathbb{R} \) and \( \beta > 0 \). Set \( p = 1 - e^{-2\beta} \) and \( a = \frac{2\omega^\Delta}{1 + 2e^\Delta} \). For any \( \Lambda \subset \mathbb{Z}^d \) be finite, we define the measure \( \mathbf{P}_\Lambda \) on \( \Sigma_\psi \times \Psi \times \Omega \) by
\[
\mathbf{P}_\Lambda[(\sigma, \psi, \omega)] = \frac{1}{Z} \prod_{x \in \Lambda} \left( \frac{a}{1-a} \right)^{\psi_x} \prod_{e \in E_\psi,\Lambda} \left( \frac{p}{1-p} \right)^{\omega_e}
\]
where \( S \) is the set of triples \((\sigma, \psi, \omega) \in \Sigma_\Lambda \times \Psi \times \Omega \) such that the following hold:

1. \((\psi, \omega) \in \Theta_0^\Lambda\),
2. \( \sigma_x^2 = \psi_x \) for every \( x \in \Lambda \),
3. for every edge \( xy \), if \( \sigma_x \neq \sigma_y \) then \( \omega_{xy} = 0 \).

Note that the latter condition is equivalent to \( \sigma \) being constant on the clusters of \( \omega \).
Proposition 2.9. Let $\Delta \in \mathbb{R}$ and $\beta > 0$. For any $\Lambda \subset \mathbb{Z}^d$ finite, the marginal measure of $P_{\Lambda}$ on $\Sigma_{\Lambda}$ is $\mu_{\Lambda,\beta,\Delta}^1$, whilst the marginal on $\Psi \times \Omega$ is $\phi_{\Lambda,p,a}^0$.

Proof. See [GG06, Theorem 3.7].

Conditionally on $(\psi, \omega)$, the following statements hold for the law of $\sigma$:

(a) the spins are constants on the clusters of $\omega$, and the spin of each cluster is uniformly distributed on the set $\{\pm 1\}$,

(b) the spins on different clusters are independent random variables.

Conditionally on $\sigma$, the following statements hold for the law of $(\psi, \omega)$:

(i) $\psi_x = 0$ if and only if $\sigma_x = 0$,

(ii) the random variables $\omega_e$ are independent,

(iii) for an edge $xy$, $\omega_{xy} = 0$ if $\sigma_x \neq \sigma_y$, and $\omega_{xy} = 1$ with probability $p$ if $\sigma_x = \sigma_y$.

We now wish to obtain a coupling between $\mu_{\Lambda,\beta,\Delta}^1$ and $\phi_{\Lambda,p,a}^0$. To this end, let $\Lambda' \subset \mathbb{Z}^d$ be such that $\Lambda' \supset \Lambda$ and such that, if two vertices $x,y \in \partial \Lambda$ are in the same connected component of $\mathbb{Z}^d \setminus \Lambda$, then $x,y$ are in the same connected component of $\Lambda' \setminus \Lambda$. We define

$$P_{\Lambda}^0 = P_{\Lambda'}[\cdot | \sigma_x = \psi_x = 1 \forall x \in \Lambda' \setminus \Lambda, \omega_{uv} = 1 \forall uv \in b(\Lambda' \setminus \Lambda)].$$

As a corollary of the above and (SMP) we obtain the following.

Corollary 2.10. Let $\Delta \in \mathbb{R}$ and $\beta > 0$. For any $\Lambda \subset \mathbb{Z}^d$ finite, the marginal measure of $P_{\Lambda}^0$ on $\Sigma_{\Lambda}$ is $\mu_{\Lambda,\beta,\Delta}^1$, whilst the marginal on $\Psi \times \Omega$ is $\phi_{\Lambda,p,a}^1$.

Let $A \leftrightarrow B$ denote the event that $A$ is connected to $B$ by an open path in $\omega$. The following corollary is a standard application of the coupling, see e.g. [DCT20, Corollary 1.4].

Corollary 2.11. Let $\Delta \in \mathbb{R}$ and $\beta > 0$. For any $\Lambda \subset \mathbb{Z}^d$ finite, we have

$$\langle \sigma_x \sigma_y \rangle_{\Lambda,\beta,\Delta}^+ = \phi_{\Lambda,p,a}^1[x \leftrightarrow y], \quad \forall x,y \in \Lambda.$$  

2.4 Stochastic ordering and the FKG inequality

We introduce the following partial order on the set $\Theta$. For $\xi, \xi' \in \Theta$, write $\xi \leq \xi'$ to denote that $\psi_x \leq \psi_x'$ for every $x \in \mathbb{Z}^d$, and $\omega_e \leq \omega_e'$ for every $e \in \mathbb{E}^d$. A function $f : \Theta \mapsto \mathbb{R}$ is called increasing if $\xi \leq \xi'$ implies that $f(\xi) \leq f(\xi')$. An event $A$ is said to be increasing if $1_A$ is increasing.

We are going to show that the dilute random cluster measures all satisfy the FKG inequality. In the proof, we use that the vertex marginals satisfy an analogous inequality, as formalised below.

Lemma 2.12. Let $p,a \in [0,1]$, $\Lambda \subset \mathbb{Z}^d$ finite, and $\xi \in \Theta$. Then,

$$\Psi_{\Lambda}^\xi[A \cap B] \geq \Psi_{\Lambda}^\xi[A] \Psi_{\Lambda}^\xi[B]$$

for all increasing events $A$ and $B$ on $\Psi = \{0,1\}^{\mathbb{Z}^d}$ that are $\Lambda$-measurable.

Proof. This is a standard consequence of the positive association property established in [GG06, Theorem 5.3].
Proposition 2.13. Let $p, a \in [0, 1]$, $\Lambda \subset \mathbb{Z}^d$ finite, and $\xi \in \Theta$. For all increasing events $A, B \in \mathcal{F}$ that are $\Lambda$-measurable, we have

$$\varphi_{\Lambda, p, a}^\xi[A \cap B] \geq \varphi_{\Lambda, p, a}^\xi[A] \varphi_{\Lambda, p, a}^\xi[B]. \quad \text{(FKG)}$$

Proof. Let $A \in \mathcal{F}$ be an increasing event that is $\Lambda$-measurable. For each $\psi \in \Psi$, let $C_{\psi, A} := \{ \omega \in \Omega : (\psi, \omega) \in A \}$. The fact that $A$ is increasing has two straightforward consequences: first, $C_{\psi, A}$ is an increasing event on $\Omega$; second, $C_{\psi, A} \subset C_{\psi', A}$ whenever $\psi \leq \psi'$.

Thus, by the conditioning equality (2.2) and the usual FKG inequality for the random cluster measure, we have that

$$\varphi_{\Lambda}^\xi[A \cap B] = \Psi_{\Lambda}^\xi\left[\varphi_{\Lambda, \psi}^{RC}[C_{\psi, A \cap B}]\right] \geq \Psi_{\Lambda}^\xi\left[\varphi_{\Lambda, \psi}^{RC}[C_{\psi, A}] \varphi_{\Lambda, \psi}^{RC}[C_{\psi, B}]\right]. \quad (2.4)$$

Note that, for any $\tilde{A} \in \mathcal{F}$ increasing, the mapping $\psi \mapsto \varphi_{\Lambda, \psi}^{RC}[C_{\psi, \tilde{A}}]$ is increasing. Indeed, for any $\psi, \psi' \in \Psi$ that coincide with $\xi$ outside $\Lambda$ and such that $\psi' \geq \psi$, we have

$$\varphi_{\Lambda, \psi}^{RC}[C_{\psi, \tilde{A}}] \leq \varphi_{\Lambda, \psi'}^{RC}[C_{\psi', \tilde{A}}] \leq \varphi_{\Lambda, \psi'}^{RC}[C_{\psi', A}]$$

where the first inequality is by \cite{DC20} Proposition 5 and the second is due to inclusion of events. The desired result (FKG) then follows from applying the inequality (2.3) in (2.4). \qed

We now compare different dilute random cluster measures. For two measures $\mu_1, \mu_2$ on $(\Theta, \Sigma)$, we write $\mu_1 \preceq \mu_2$ if $\mu_1(f) \leq \mu_2(f)$ for every increasing function $f$. In this case, we say that $\mu_2$ stochastically dominates $\mu_1$. When comparing measures on sub $\sigma$-algebras of $\mathcal{F}$, we tacitly assume that the partial order for these events.

The following proposition states that the dilute random cluster measures are stochastically increasing in $a, p$, and the boundary condition.

Proposition 2.14. For every $a, a', p, p' \in [0, 1]$ and $\xi, \xi' \in \Theta$ such that $a \leq a', p \leq p'$ and $\xi \leq \xi'$,

$$\varphi_{\Lambda, a, p}^\xi \preceq \varphi_{\Lambda, a', p'}^{\xi'}.$$  

Proof. See \cite{GG05} Theorem 5.12]. \qed

We require a more refined stochastic domination that conveys that 0 boundary conditions are the most favourable boundary condition that induces no connections at the boundary. In the remainder of this subsection, we fix $a, p \in [0, 1]$. Let $V$ be a finite set. Given a configuration $\psi \in \{0, 1\}^V$ and $T \subset V$, we define the configurations $\psi_T$ and $\psi_T'$ by

$$\psi_T' = \begin{cases} \psi_y, & y \not\in T \\ 1, & y \in T, \end{cases} \quad (\psi_T)_y = \begin{cases} \psi_y, & y \not\in T \\ 0, & y \in T. \end{cases}$$

We first recall the following standard fact.

Lemma 2.15. Let $V$ be a finite set. Let $\mu_1, \mu_2$ be strictly positive probability measures on $\{0, 1\}^V$. If

$$\mu_2 \left[\psi_{\{x\}}\right] \mu_1 \left[\psi_{\{x\}}\right] \geq \mu_2 \left[\psi_{\{x\}}\right] \mu_1 \left[\psi_{\{x\}}\right] \quad (2.5)$$

and in addition, either $\mu_1$ or $\mu_2$ satisfies

$$\mu \left[\psi_{\{x, y\}}\right] \mu \left[\psi_{\{x, y\}}\right] \geq \mu \left[\psi_{\{x\}}\right] \mu \left[\psi_{\{y\}}\right] \quad (2.6)$$

then $\mu_1 \preceq \mu_2$. 

9
Proof. See [Gri04, Theorem 2.6]. □

For $S \subset \partial \Lambda$, let $E(S, \Lambda)$ be the set of edges with one endpoint in $S$ and the other in $\Lambda$. Given $\xi \in \Theta$, we let $\xi \cap 0_S \in \Theta$ be the configuration which is equal to 0 on $S \times E_S$, and otherwise coincides with $\xi$. In addition, we let $\xi \cap 0_{E(S, \Lambda)} \in \Theta$ be the configuration whose edge configuration is equal to 0 on $E(S, \Lambda)$, and otherwise coincides with $\xi$. Note that, in the latter, the vertices are not necessarily 0 on $S$.

We define

$$\varphi_\Lambda^{\xi \cap 0_{E(S, \Lambda)}} = \varphi_\Lambda^{| \cdot |} \omega_e = 0 \text{ for every } e \in E(S, \Lambda).$$

We write $\Psi_\Lambda^{\xi \cap 0_{E(S, \Lambda)}}$ and $\Psi_\Lambda^{\xi \cap 0_{E(S, \Lambda)}}$ for the vertex marginals of $\varphi_\Lambda^{\xi \cap 0_{S}}$ and $\varphi_\Lambda^{\xi \cap 0_{E(S, \Lambda)}}$, respectively.

Lemma 2.16. Let $\Lambda$ be a finite subset of $\mathbb{Z}^d$, and let $S \subset \partial \Lambda$. For every $\xi \in \Theta$,

$$\Psi_\Lambda^{\xi \cap 0_{E(S, \Lambda)}} \leq \Psi_\Lambda^{\xi \cap 0_{S}}.$$

Proof. We prove that (2.5) is satisfied for every $x \in \Lambda$. For $\psi \in \Psi$, write $N_x(\psi)$ and $N_{x,S}(\psi)$ to denote the number of neighbours of $x$ in the graph $(V_{\psi(x)}, E_{\psi(x)})$ and $(V_{\psi(x) \cap 0_S}, E_{\psi(x) \cap 0_S})$, respectively.

By direct calculation, we have

$$\frac{\Psi_\Lambda^{\xi \cap 0_{E(S, \Lambda)}}[\psi(x)]}{\Psi_\Lambda^{\xi \cap 0_{E(S, \Lambda)}}[\psi(x)]} = \frac{N_x(\psi)}{N_{x,S}(\psi)} = \frac{Z_{\Lambda,\psi(x)}^{\xi \cap 0_{E(S, \Lambda)}} a}{Z_{\Lambda,\psi(x)}^{\xi \cap 0_{E(S, \Lambda)}} 1 - a},$$

$$\frac{\Psi_\Lambda^{\xi \cap 0_{S}}[\psi(x)]}{\Psi_\Lambda^{\xi \cap 0_{S}}[\psi(x)]} = \frac{N_{x,S}(\psi)}{N_{x,S}(\psi)} = \frac{Z_{\Lambda,\psi(x)}^{\xi \cap 0_{S}} a}{Z_{\Lambda,\psi(x)}^{\xi \cap 0_{S}} 1 - a},$$

where $Z_{\Lambda,\psi(x)}^{\xi \cap 0_{E(S, \Lambda)}}$ denotes the partition function of the random cluster model on $(V_{\psi(x)}, E_{\psi(x)})$ with boundary conditions $\xi$, conditioned on $E(S, \Lambda)$ being closed, and $Z_{\Lambda,\psi(x)}^{\xi \cap 0_{E(S, \Lambda)}}$ is defined similarly.

Note that

$$\frac{Z_{\Lambda,\psi(x)}^{\xi \cap 0_{E(S, \Lambda)}}}{Z_{\Lambda,\psi(x)}^{\xi \cap 0_{E(S, \Lambda)}}} = \frac{Z_{\Lambda,\psi(x)}^{\xi \cap 0_{S}}}{Z_{\Lambda,\psi(x)}^{\xi \cap 0_{S}}}$$

because in both cases, the edges in $E(S, \Lambda)$ are closed. Since $r \leq 1$ and $N_{x,S} \leq N_x$, it follows that (2.5) is satisfied.

Applying [CG06, Proposition 5.5] for $\Phi_1 = \Phi_2 = \Psi_\Lambda^{\xi \cap 0_{S}}$, we obtain that (2.6) is satisfied by $\Psi_\Lambda^{\xi \cap 0_{S}}$. The desired result follows by Lemma 2.15. □

Proposition 2.17. Let $\Lambda$ be a finite subset of $\mathbb{Z}^d$, and let $S \subset \partial \Lambda$. For every $\xi \in \Theta$,

$$\varphi_\Lambda^{\xi \cap 0_{E(S, \Lambda)}} \leq \varphi_\Lambda^{\xi \cap 0_{S}}.$$

Proof of Proposition 2.17. Let $A \in \mathcal{F}$ be $\Lambda$-measurable and increasing. Recall the notation introduced in the proof of Proposition 2.13. By (2.2),

$$\varphi_\Lambda^{\xi \cap 0_{S}}[A] = \Psi_\Lambda^{\xi \cap 0_{S}} \left[ \phi_{\Lambda,\psi}^{RC,\xi \cap 0_{S}}[C_{\psi,A}] \right].$$

As argued above, the mapping $\psi \mapsto \phi_{\Lambda,\psi}^{RC,\xi \cap 0_{S}}[C_{\psi,A}]$ is increasing. Moreover, the measures $\phi_{\Lambda,\psi}^{RC,\xi \cap 0_{S}}$ and $\phi_{\Lambda,\psi}^{RC,\xi \cap 0_{E(S, \Lambda)}}$ coincide. The desired result then follows by applying Lemma 2.16 and undoing the conditioning. □
2.5 Monotonicity in the domain

We now develop stochastic comparison results for dilute random cluster measures on different domains. Let $\text{Cyl}(\mathbb{Z}^d \times \Theta) = \{(\Lambda, \xi) \in \mathbb{Z}^d \times \Theta : \Lambda \text{ is finite}\}$.

**Definition 2.18.** Let $\preceq_1$ be the partial order on $\text{Cyl}(\mathbb{Z}^d \times \Theta)$ defined, for $(\Lambda, \xi), (\Lambda', \xi') \in \text{Cyl}(\mathbb{Z}^d \times \Theta)$, by the relation

$$(\Lambda, \xi) \preceq_1 (\Lambda', \xi') \iff \Lambda' \subseteq \Lambda \text{ and } \xi' \supseteq \xi \cup 1_{\Lambda, \Lambda'}$$

where $\xi \cup 1_{\Lambda, \Lambda'} \in \Theta$ is the boundary condition which is equal to 1 on $(\Lambda \setminus \Lambda') \times (E_\Lambda \setminus E_{\Lambda'})$, and otherwise coincides with $\xi$.

**Definition 2.19.** Let $\preceq_0$ be the partial order on $\text{Cyl}(\mathbb{Z}^d \times \Theta)$ defined, for $(\Lambda, \xi), (\Lambda', \xi') \in \text{Cyl}(\mathbb{Z}^d \times \Theta)$, by the relation $\preceq_0$ if

$$(\Lambda, \xi) \preceq_0 (\Lambda', \xi') \iff \Lambda \subseteq \Lambda' \text{ and } \xi \subseteq \xi \cap 0_{\Lambda, \Lambda'}$$

where $\xi \cap 0_{\Lambda, \Lambda'} \in \Theta$ is the boundary condition which is equal to 0 on $(\Lambda \setminus \Lambda') \times (E_\Lambda \setminus E_{\Lambda'})$, and otherwise coincides with $\xi$.

**Proposition 2.20.** Let $(\Lambda, \xi), (\Lambda', \xi') \in \text{Cyl}(\mathbb{Z}^d \times \Theta)$ such that either $(\Lambda, \xi) \preceq_1 (\Lambda', \xi')$ or $(\Lambda, \xi) \preceq_0 (\Lambda', \xi')$. Then,

$$\varphi_\Lambda^\xi \preceq \varphi_{\Lambda'}^{\xi'}$$

in the sense that, for every increasing event $A \in \mathcal{F}$ that is measurable with respect to $(\Lambda \cap \Lambda', E_\Lambda \cap E_{\Lambda'})$, we have

$$\varphi_\Lambda^\xi[A] \leq \varphi_{\Lambda'}^{\xi'}[A]. \quad \text{(MON)}$$

**Proof.** The result follows from a standard application of [FKG], [SMP], and Proposition 2.14. See [DCT20, Proposition 5].

3 Basic facts about the phase transition

In this section, we use the dilute random cluster representation of the Blume-Capel model to prove various important properties about its phase diagram. First, we show that the critical point of the Blume-Capel model coincides with the critical percolation threshold of the dilute random cluster model and state some basic consequences of this. Then, we show that there exists a continuous line of critical points $\Delta \mapsto \beta_c(\Delta)$. Finally, we show that classical results using Pirogov-Sinai theory establish that the phase transition is discontinuous at the critical point for sufficiently negative values of $\Delta$.

We begin by collecting important facts about the natural infinite volume limits of the free and wired measures.

**Proposition 3.1.** Let $p, a \in [0, 1]$. Then, the weak limits of $\varphi_{\Lambda, p, a}^1$ and $\varphi_{\Lambda, p, a}^0$ as $\Lambda \uparrow \mathbb{Z}^d$, denoted by $\varphi_{p, a}^1$ and $\varphi_{p, a}^0$ respectively, exist and satisfy the following Gibbs property: let $\varphi = \varphi_{p, a}^1$ or $\varphi_{p, a}^0$. Then, for all $\Lambda \subset \mathbb{Z}^d$ finite and $\varphi$-a.s. $\lambda \in \Theta$,

$$\varphi[A \mid \mathcal{F}_\Lambda](\lambda) = \varphi_{\Lambda, p, a}^1[A], \quad A \in \mathcal{F}_\Lambda$$

where $\mathcal{F}_\Lambda$ (resp. $\mathcal{F}_{\Lambda'}$) consists of events in $\mathcal{F}$ that are $\Lambda$-measurable (resp. $\Lambda'$-measurable).

Furthermore, both $\varphi_{p, a}^1$ and $\varphi_{p, a}^0$ are invariant and mixing (hence, ergodic) under the group of automorphisms of $\mathbb{Z}^d$.

**Proof.** When $p, a \in (0, 1)$, the existence and Gibbs property follows from [GG06, Theorems 7.2 and 7.4]. The invariance and mixing under translations is then a standard consequence of the convergence, [FKG], and Proposition 2.14. See [DCT20, Lemma 1.10]. The cases when $a, p \in \{0, 1\}$ are classical. \hfill \square
3.1 Definition of the critical point

We begin by defining the critical point of the Blume-Capel model and the critical percolation threshold of the random cluster model.

**Definition 3.2.** Let $\Delta \in \mathbb{R}$. We define the critical temperature $\beta_c(\Delta)$ by

$$\beta_c(\Delta) := \inf \{ \beta > 0 : \langle \sigma_0 \rangle_{\beta,\Delta}^+ > 0 \}.$$

Furthermore, we set

$$\mathcal{L}_c := \{ (\beta, \Delta) : \beta = \beta_c(\Delta) \}$$

to be the graph of critical points.

A consequence of the GKS inequalities [GJ12, Theorem 4.1.3] is that for each $\Delta$, the magnetisation is monotone in $\beta$.

**Definition 3.3.** For $a \in (0, 1)$, define

$$p_c := \inf \{ p \in (0, 1) : \phi_{p,a}^{1}[0 \leftrightarrow \infty] > 0 \}.$$

Additionally, we set $p_c(0) = 1$ and $p_c(1) = p_{\text{Ising}}(\mathbb{Z}^d)$, where the latter is the critical point for the usual random cluster model (i.e. FK-Ising model) on $\mathbb{Z}^d$.

We now state the relation between the critical points of the dilute random cluster model and the Blume-Capel model.

**Proposition 3.4.** Let $\Delta \in \mathbb{R}$ and $a_\Delta = \frac{2e^{\Delta}}{1 + 2e^{\Delta}}$. Then,

$$p_c(a_\Delta) = 1 - e^{-2\beta_c(\Delta)}$$

where we recall $\beta_c(\Delta)$ is defined in Definition 3.2.

**Proof.** This is a direct consequence of the convergence in Proposition 3.1 and Corollary 2.11.

One important consequence of Proposition 3.4 is that it allows to prove rigorously that the critical point of the Blume-Capel model coincides with the point at which the model undergoes a long-range order transition.

**Corollary 3.5.** Let $\Delta \in \mathbb{R}$. Then, with $\beta_c(\Delta)$ as in Definition 3.2 we have that

$$\lim_{|x| \to \infty} \langle \sigma_0 \sigma_x \rangle^+ = \begin{cases} 0, & \beta < \beta_c \\ > 0, & \beta > \beta_c. \end{cases}$$

**Proof.** Follows from a straightforward modification of arguments in [DCT20, Corollary 1.13] by using Proposition 3.1.

**Remark 3.6.** The phase transition can also be defined with respect to a uniqueness to non-uniqueness transition for the set of Gibbs measures.
3.2 Non-triviality of critical points

We will now show that $p_c(a)$ is always non-trivial, i.e. $p_c(a) \in (0, 1)$, except when $a = 0$.

**Proposition 3.7.** For $a \in (0, 1)$, $p_c(a) \in (0, 1)$.

**Proof.** Let $a \in (0, 1)$ and set $p_c := p_c(a)$. Let $p < p_c^{\text{Ising}}(\mathbb{Z}^d)$. For $n \in \mathbb{N}$, let $B_n$ denote the ball of radius $n$ around 0. Fix $n$ and let $\Lambda \subset \mathbb{Z}^d$ finite such that $\Lambda \supset B_n$. Then,

$$\varphi_{\Lambda, p,a}[0 \leftrightarrow \partial B_n] = \Psi_{\Lambda, p,a}[\varphi_{\Lambda, p,a}^{\text{RC},1}[0 \leftrightarrow \partial B_n] \leq \varphi_{\Lambda, p,a}^{\text{RC},1}[0 \leftrightarrow \partial B_n]$$

where the first line is by (2.2) and the second line is by monotonicity in boundary conditions for the usual random cluster model, see [DCT20, Proposition 5]. Taking limits as $\Lambda \uparrow \mathbb{Z}^d$ and then $n \to \infty$, we obtain that $\varphi_{p,a}^{1}[0 \leftrightarrow \infty] = 0$. Thus, $p_c \geq p_c^{\text{Ising}}(\mathbb{Z}^d) > 0$.

In order to show $p_c < 1$, it is convenient to consider the edge marginal of $\varphi_{p,a}^{1}$, which is the probability measure $\Omega_{p,a}^{1}$ on $\Omega$ defined by

$$\Omega_{p,a}^{1}[\omega] = \sum_{\psi \in \Psi(\psi, \omega) \in \Theta} \varphi_{p,a}^{1}(\psi, \omega), \quad \omega \in \Omega.$$

We show that, for $p$ sufficiently close to 1,

$$\mathbf{P}_{2/3}^{\text{Ber}} \preceq \Omega_{p,a}^{1}$$

where $\mathbf{P}_{2/3}^{\text{Ber}}$ is the law of Bernoulli bond percolation on $\mathbb{Z}^d$ with parameter 2/3 (i.e., in the supercritical regime). Hence, $\varphi_{p,a}^{1}[0 \leftrightarrow \infty] > 0$ and so $p_c < 1$.

Indeed, it is classical [LSS97] that the stochastic domination follows once we show that for an edge $e := xy$, $\Omega_{\Lambda, p,a}^{1}[\omega_e = 1] \geq \rho$ for some $\rho$ close enough to 1, where $\xi$ is any boundary condition and $\Lambda = \{x, y\}$.

Note that, by Proposition 2.14 and a direct calculation,

$$\Omega_{\Lambda, p,a}^{1}[\omega_e = 1] \geq \Omega_{\Lambda, p,a}^{0}[\omega_e = 1] = \frac{\left(\frac{a}{1-a}\right)^2 \sqrt{1-p} \left(\frac{p}{1-p}\right)}{\left(\frac{a}{1-a}\right)^2 \sqrt{1-p} \left(\frac{p}{1-p}\right) (1 + f(p,a))}$$

where $f(p,a) = o(1)$ as $p \to 1$. Hence, $\Omega_{\Lambda, p,a}^{1}[\omega_e = 1]$ converges to 1 as $p$ tends to 1 uniformly in $\xi$, as desired. \qed

3.3 Continuity of the critical line

In order to prove continuity of the critical line, we need to show the following strengthening of Proposition 2.14 about stochastic domination.

**Proposition 3.8.** Let $d \geq 1$, $a \in (0, 1)$, and $p_1 > p_2$. Then, there exists $\varepsilon = \varepsilon(d, p_1, p_2, a) > 0$ such that, for any $a_1 > a - \varepsilon$ and any $a_2 < a + \varepsilon$,

$$\varphi_{p_2,a_2}^{1} \preceq \varphi_{p_1,a_1}^{1}.$$

**Proof.** We prove the stochastic domination for $\Lambda \subset \mathbb{Z}^d$ finite. The assertion then follows by taking the limit as $\Lambda$ tends to $\mathbb{Z}^d$.

We wish to find $\varepsilon > 0$ such that, for any edge $xy$ of $L^d$ and any boundary conditions $\xi_1 \geq \xi_2$,

$$\varphi_{\Lambda, p_2,a_2}^{\xi_2} \preceq \varphi_{\Lambda, p_1,a_1}^{\xi_1}$$

(3.1)
where $A' = \{x, y\}$, and $a_1$ and $a_2$ are as above. Then, by Strassen’s theorem [Str65], there exists an increasing coupling between $\varphi_{\Lambda',p_2,a_2}^\xi$ and $\varphi_{\Lambda',p_1,a_1}^\xi$. A standard Markov chain argument (see, for instance, [DC17] Lemma 1.5) then gives that

$$\varphi_{\Lambda',p_2,a_2}^\xi \preceq \varphi_{\Lambda',p_1,a_1}^\xi.$$  

In addition, since by Proposition 2.14 the measure $\varphi_{\Lambda',p_2,a_2}^\xi$ is stochastically increasing in $\xi$, it suffices to prove (3.1) for $\xi_1 = \xi_2 = \xi$.

First observe that, for any increasing and non-empty event $A \in \mathcal{F}$ that is measurable with respect to $(\Lambda', E_{\Lambda'})$, the function $p \mapsto \varphi_{\Lambda',p_2,a_2}^\xi[A]$ is analytic and non-constant. Thus, by Proposition 2.14 we have the strict inequality

$$\varphi_{\Lambda',p_1,a_1}^\xi[A] > \varphi_{\Lambda',p_2,a_2}^\xi[A].$$

Note that there are finitely many distinct maps $\varphi_{\Lambda',p_2,a_2}^\xi$ ranging over all choices of $\xi$ and over all possible $A$. Hence, by continuity and monotonicity (see Proposition 2.14) of the map $p \mapsto \varphi_{\Lambda',p_2,a_2}^\xi[A]$, we can choose $\varepsilon = \varepsilon(d, p_1, p_2, a) > 0$ such that

$$\varphi_{\Lambda',p_1,a_1}^\xi[A] > \varphi_{\Lambda',p_2,a_2}^\xi[A]$$

for all $a_1 > a - \varepsilon$ and $a_2 < a + \varepsilon$, and uniformly over all increasing, non-empty events $A$ and boundary conditions $\xi$ on $(\Lambda', E_{\Lambda'})$. The desired assertion follows.

**Proposition 3.9.** The function $a \mapsto p_\varepsilon(a)$ is decreasing and continuous on $[0, 1]$.

**Proof.** For the open interval $(0, 1)$, the assertion follows from Propositions 2.14 and 3.8. See the proof of [Gri04] Theorem 5.5.

For $a = 0$, we can argue as in the proof of Proposition 3.7 to see that for every $p \in (0, 1)$ there is $\varepsilon > 0$ such that for every $a \in (0, \varepsilon)$, $\Psi_{p,a}^1$ is dominated by subcritical Bernoulli site percolation on $\mathbb{Z}^d$. This implies that $\varphi_{p,a}^1$ is not supercritical, hence $p_\varepsilon(a) \geq p$. It follows that $p_\varepsilon(a)$ converges to $p_\varepsilon(0) = 1$ as $a$ tends to 0.

To handle $a = 1$, we wish to show that for every $p' > p$ in $(0, 1)$, there exists $a = a(p, p') \in (0, 1)$ such that $\Omega_{p,a}^{\text{RC},1}$ stochastically dominates $\varphi_{p',a}^{\text{RC},1}$, from which it follows that $p_\varepsilon(a)$ converges to $p_\varepsilon(1)$ as $a$ tends to 1. To this end, we prove the stochastic domination for $\Lambda \subset \mathbb{Z}^d$ finite.

We can argue as above to deduce that for any boundary conditions $\rho, \rho' \in \{0, 1\}^{\mathbb{Z}^d}$ with $\rho' > \rho$, and any increasing, non-empty event $A$ depending on $E_{\Lambda'}$, we have the strict inequality

$$\varphi_{\Lambda',p'}^{\text{RC},\rho}[A] > \varphi_{\Lambda',p}^{\text{RC},\rho}[A],$$

where $\Lambda' = \{x, y\}$ for some neighbours $x, y$. By continuity of the map $a \mapsto \Omega_{\Lambda',p,a}^{\xi}[A]$, where $\xi = (\kappa, \rho')$ with $\kappa = 1$, we have that

$$\Omega_{\Lambda',p,a}^{\xi}[A] > \varphi_{\Lambda',p}^{\text{RC},\rho}[A]$$

for every $a$ close enough to 1.

Arguing as in the proof of Proposition 3.7, we see that $\varphi_{\Lambda',p,a}^1[u] = 1 \forall u \in \partial \Lambda' \mid \rho'_e, e \in \mathbb{E}^d \setminus E_{\Lambda}$ converges to 1 uniformly in $\Lambda$ as $a$ tends 1. Hence,

$$\Omega_{\Lambda',p,a}^1[A \mid \rho'_e, e \in \mathbb{E}^d \setminus E_{\Lambda}] \geq \varphi_{\Lambda',p}^{\text{RC},\rho}[A]$$

for every $a$ sufficiently close to 1. The stochastic domination follows. \[\square\]

\[\text{Recall our measures depend only on the state of } \partial \Lambda' \text{ and which vertices of } \partial \Lambda' \text{ are connected to each other outside of } \Lambda'.\]
Remark 3.10. The arguments above can be adapted to show that the function $a \mapsto p_c(a)$ is strictly decreasing for $d \geq 2$.

Remark 3.11. Proposition [3.9] gives a rigorous proof that $\mathcal{L}_c$, as introduced in Definition [3.2], corresponds to a continuous, (strictly) decreasing curve of critical points whose left-limit is $+\infty$ and right-limit is $\beta_c(\text{Ising})$.

### 3.4 Discontinuity via Pirogov-Sinai theory

We now show that the first half of Theorem [1.1], namely the existence of $\Delta^- (d)$, follows from the classical result [BS89] in Pirogov-Sinai theory (see also [FV17] Chapter 7 for a textbook approach to Pirogov-Sinai theory applied to the Blume-Capel model). To state the latter result precisely, we introduce a slightly different parametrisation of the Blume-Capel model.

For this parametrisation, there are two parameters $\beta' > 0$ and $\lambda \in \mathbb{R}$. The Hamiltonian is defined by

$$H^n_{G,\lambda}(\sigma) = \sum_{xy \in E} (\sigma_x - \sigma_y)^2 - \lambda \sum_{x \in V} \sigma_x^2 + \sum_{x \in V, y \in \mathbb{Z}^d \setminus V} (\sigma_x - \eta_y)^2,$$

and the probability of a configuration is proportional to

$$e^{-\beta' H^n_{G,\lambda}(\sigma)}.$$

Expressed in our parametrisation, this corresponds to the change of variables $\beta = 2\beta'$ and $\Delta = \beta'(\lambda - 2d)$. In [BS89], it is proved that there exists $\beta'_0 > 0$ such that for every $\beta' > \beta'_0$ the following holds: there exists $\lambda_0(\beta') > 0$ such that

$$\langle \sigma_0 \rangle_{\beta',\lambda}^+ \begin{cases} = 0, & \lambda < \lambda_0(\beta') \\ > 0, & \lambda \geq \lambda_0(\beta'). \end{cases}$$

Let now $\beta = 2\beta'$ and $\Delta = \beta'(\lambda_0(\beta') - 2d)$. Note that either $\beta = \beta_c(\Delta)$ or $\beta > \beta_c(\Delta)$. If the latter holds, then $\langle \sigma_0 \rangle_{\beta',\lambda}^+ > 0$ for some $\lambda < \lambda_0(\beta')$ by Proposition [3.9] which is a contradiction.

### 4 Combinatorial mapping between the Blume-Capel model and the Ising model

Let $d \geq 2$, and let $G = (V, E)$ be a finite subgraph of $\mathbb{Z}^d$. We establish a correspondence between the Blume-Capel model on $G$ and an Ising model (not necessarily ferromagnetic) on a larger graph associated to $G$, described below. An immediate byproduct of the proof is that the coupling constants of the associated Ising model are ferromagnetic provided $\Delta \geq -\log 2$. We use this extensively later in the article.

First, we lift $G$ to the graph $\ell(G) = (\ell(V), \ell(E))$ with vertex set $\ell(V) = V \times \{0, 1\}$ and edge set $\ell(E) = E_1 \cup E_2$, where

$$E_1 = \bigcup_{xy \in E} \bigcup_{i,j=0}^1 \{(x, i), (y, j)\}, \quad E_2 = \bigcup_{x \in V} \{(x, 0), (x, 1)\}.$$
Note that $\ell(G)$ can be seen as a subgraph of the graph $\ell(\mathbb{Z}^d)$ with vertex set $\mathbb{Z}^d \times \{0, 1\}$ and edge set $E_1^d \cup E_2^d$, where
\[
E_1^d = \bigcup_{x,y \in \mathbb{Z}^d} \bigcup_{i,j = 0}^1 \{(x,i), (y,j)\}, \quad E_2^d = \bigcup_{x \in \mathbb{Z}^d} \{(x,0), (x,1)\}.
\]

**Lemma 4.1.** The graph $\ell(\mathbb{Z}^d)$ is amenable and transitive.

**Proof.** It is easy to see that $\ell(\mathbb{Z}^d)$ is a Cayley graph of the amenable group $\mathbb{Z}^d \times \{0, 1\}$. The desired result follows. \hfill \Box

Given some boundary conditions $\xi \in \{-1, 0, 1\}^{\mathbb{Z}^d}$, we define the boundary conditions $\ell(\xi) \in \{-1, 0, 1\}^{\mathbb{Z}^d \times \{0,1\}}$ by letting $\ell(\xi)(x,i) = \xi_x$ for every $x \in \mathbb{Z}^d$ and $i \in \{0,1\}$. Let also $T : \{\pm 1\}^V \times \{\pm 1\}^{V \rightarrow \{-1,0,1\}} \mapsto \sigma$, where $\sigma_x = \frac{1}{2}(x_0^1 + x_0^2)$. Identifying $\{\pm 1\}^V \times \{\pm 1\}^{V \rightarrow \{-1,0,1\}}$ with $\{\pm 1\}^{V \rightarrow \{-1,0,1\}}$, we also view $T$ as a map defined on $\{\pm 1\}^{V \rightarrow \{-1,0,1\}}$, i.e. a map defined over all possible Ising configurations on $\ell(G)$. We write $\tau$ for the spin variable of the Ising model to distinguish it from the spin variable $\sigma$ of the Blume-Capel model.

**Lemma 4.2.** Let $\xi \in \{-1, 0, 1\}^{\mathbb{Z}^d}$. Let also $\mu_{\ell(G),\xi}^{\text{Ising},\ell(\xi)}$ denote the Ising model on $\ell(G)$ with boundary conditions $\ell(\xi)$ and coupling constants
\[
J_{x,y} = J(\beta, \Delta)_{x,y} = \frac{\beta}{4}1_{xy \in E_1} + \frac{(\Delta + \log 2)}{2}1_{xy \in E_2}.
\]

Then, for every $\eta \in \{-1, 0, 1\}^V$, we have
\[
\mu_{\ell(G),\xi}^{\text{Ising},\ell(\xi)}(\sigma = \eta) = \mu_{\ell(G),\xi}^{\text{Ising},\ell(\xi)}(\tau \in T^{-1}(\eta)).
\]

**Proof.** We start by showing that
\[
\sum_{\tau \in T^{-1}(\eta)} \prod_{u \in E_1} e^\frac{\beta}{4} \tau_u \prod_{v \in \partial(V)} e^\frac{\beta}{2} \tau_v \prod_{u \in E_2} e^\frac{(\Delta + \log 2)}{2} \tau_u \tau_v = \prod_{x \in V} e^\frac{(\Delta + \log 2)}{2} \prod_{x \in V} e^\Delta \eta_x = \prod_{x \in V} e^\frac{(\Delta + \log 2)}{2} \prod_{x \in V} e^\Delta \eta_x.
\]

Note that
\[
\sum_{\tau \in T^{-1}(\eta)} \prod_{u \in E_1} e^\frac{\beta}{4} \tau_u \prod_{v \in \partial(V)} e^\frac{\beta}{2} \tau_v \prod_{u \in E_2} e^\frac{(\Delta + \log 2)}{2} \tau_u \tau_v = \prod_{x \in V} e^\frac{(\Delta + \log 2)}{2} \prod_{x \in V} e^\Delta \eta_x
\]
\[
\sum_{\tau \in T^{-1}(\eta)} \prod_{u \in E_1} e^\frac{\beta}{4} \tau_u \prod_{v \in \partial(V)} e^\frac{\beta}{2} \tau_v \prod_{u \in E_2} e^\frac{(\Delta + \log 2)}{2} \tau_u \tau_v
\]
\[
\prod_{x \in V} e^\frac{(\Delta + \log 2)}{2} \prod_{x \in V} e^\Delta \eta_x = \prod_{x \in V} e^\frac{(\Delta + \log 2)}{2} \prod_{x \in V} e^\Delta \eta_x.
\]

\[
\sum_{\tau \in T^{-1}(\eta)} \prod_{u \in E_1} e^\frac{\beta}{4} \tau_u \prod_{v \in \partial(V)} e^\frac{\beta}{2} \tau_v \prod_{u \in E_2} e^\frac{(\Delta + \log 2)}{2} \tau_u \tau_v
\]
\[
\prod_{x \in V} e^\frac{(\Delta + \log 2)}{2} \prod_{x \in V} e^\Delta \eta_x = \prod_{x \in V} e^\frac{(\Delta + \log 2)}{2} \prod_{x \in V} e^\Delta \eta_x.
\]

\[
\sum_{\tau \in T^{-1}(\eta)} \prod_{u \in E_1} e^\frac{\beta}{4} \tau_u \prod_{v \in \partial(V)} e^\frac{\beta}{2} \tau_v \prod_{u \in E_2} e^\frac{(\Delta + \log 2)}{2} \tau_u \tau_v
\]
\[
\prod_{x \in V} e^\frac{(\Delta + \log 2)}{2} \prod_{x \in V} e^\Delta \eta_x = \prod_{x \in V} e^\frac{(\Delta + \log 2)}{2} \prod_{x \in V} e^\Delta \eta_x.
\]
where in the second equality, we used that
\[ \prod_{i,j=0}^{1} e^{\frac{\Delta}{2}\tau_{i,j}^y} = e^{\beta\eta_x\eta_y}, \quad \prod_{i,j=0}^{1} e^{\frac{\Delta}{2}\tau_{i,j}^y} = e^{\beta\eta_x\eta_y} \quad \text{and} \quad \tau_{x}^{0} + 1 = 2\eta_x^2 - 1. \]

It is easy to see that \( T \) is injective on the vertices \( x \in V \) such that \( \tau_{x}^{0} = \tau_{x}^{1} \) and is 2-to-1 on the vertices \( x \in V \) such that \( \tau_{x}^{0} \neq \tau_{x}^{1} \). Thus, \( T^{-1}(\eta) \) contains exactly \( 2^{\sum_{x \in V} (1-\eta_x^2)} \) elements.

Hence,
\[ (4.2) = e^{-\frac{(\Delta - \log 2)}{2}|V|} \prod_{x \in A} e^{\beta\eta_x\eta_y} \prod_{x \in V} e^{\beta\eta_x\eta_y} \prod_{x \in V} e^{\Delta\eta_x^2} \]
which establishes (4.1).

Similarly, we obtain
\[ Z_{G,\beta,\Delta}^{\text{Ising},J} = e^{-\frac{(\Delta - \log 2)}{2}|V|} Z_{G,\beta,\Delta}, \]
and the desired assertion follows readily.

\[ \square \]

Remark 4.3. \( J \) is ferromagnetic provided \( \Delta \geq - \log 2 \). We note that the edges between the vertices \( (x, 1) \) and \( (x, 2) \) are antiferromagnetic for \( \Delta < - \log 2 \) and their interaction becomes stronger as \( \Delta \to -\infty \), i.e. as 0 becomes more likely.

In what follows, we abuse the notation and write \( \tau_{x}^{1} \) instead of \( \tau_{(x,i)} \).

Corollary 4.4. Let \( G = (V, E) \) be a finite subgraph of \( \mathbb{Z}^d \), and let \( J \) be as in Lemma 4.2. For any non-empty set \( A \subset V \), any indices \( i_x \in \{0, 1\} \), \( x \in A \), and any boundary conditions \( \xi \in \{-1, 0, 1\}^V \),
\[ \langle \prod_{x \in A} \sigma_x \rangle_{G,\beta,\Delta}^{\text{Ising},J} = \langle \prod_{x \in A} \tau_{x}^{i_x} \rangle_{G,J}^{\text{Ising},\ell(G)} \]
Proof. Observe that, by symmetry,
\[ \langle \prod_{x \in A} \tau_{x}^{i_x} \rangle_{G,J}^{\text{Ising},\ell(G)} = \frac{1}{2^{|A|}} \sum_{j:A \to \{0,1\}} \langle \prod_{x \in A} \tau_{x}^{j_x} \rangle_{G,J}^{\text{Ising},\ell(G)} = \langle \prod_{x \in A} \tau_{x}^{0} + \tau_{x}^{1} \rangle_{G,J}^{\text{Ising},\ell(G),J}. \]
The latter is equal to \( \langle \prod_{x \in A} \sigma_x \rangle_{G,\beta,\Delta}^{\text{Ising},J} \) by Lemma 4.2.

\[ \square \]

5 Tricritical point in \( d \geq 3 \) via the infrared bound

In this section, we prove Theorem 1.1 in \( d \geq 3 \) for \( \Delta^+ = \log 2 \). The key idea is to use the mapping described in Section 4 to map the Blume-Capel model to a ferromagnetic Ising model on \( l(\mathbb{Z}^d) \). Since this graph is transitive and amenable, we can then apply the main result of [ADCS15] (see also [Rao20]), whose hypotheses is guaranteed in our setting by the infrared bound.

First, we claim that in \( d \geq 3 \) the two-point correlations \( \langle \tau_{x}^{1} \tau_{y}^{1} \rangle_{G,J,\Delta}^{\text{Ising},0} \) decay, in an averaged sense, to 0 for every \( \beta \leq \beta_e(\Delta) \).

Lemma 5.1. Let \( d \geq 3 \). Then, for any \( \Delta \in \mathbb{R} \) and \( \beta \leq \beta_e(\Delta) \),
\[ \inf_{B \subset \mathbb{Z}^d, |B| < \infty} \frac{1}{|B|^2} \sum_{x,y \in B} \langle \tau_{x}^{0} \tau_{y}^{1} \rangle_{J(\beta,\Delta)}^{\text{Ising},0} = 0 \]
where \( J(\beta, \Delta) \) is as in Lemma 4.2.\]
Proposition 1] that
\[ \langle \tau_0 \rangle_{\{J_{\beta,\Delta}\}, \beta > 0} = \langle \tau_0 \rangle_{J(\beta, \Delta)} \]
for every \( \beta > 0 \) and \( \Delta \geq -\log 2 \).

Arguing as in [DCT20, Proof of theorem 3] and using Lemma 5.1, we obtain
\[
\left( \langle \tau_0 \rangle_{\{J_{\beta,\Delta}\}, \beta > 0} \right)^2 \leq \inf_{B \subset \mathbb{Z}^d \times \{0,1\}, |B| < \infty} \frac{1}{|B|^2} \sum_{x,y \in B} \langle \tau_x \tau_y \rangle_{J(\beta, \Delta)} = 0.
\]

Hence, \( \langle \sigma_0 \rangle_{\{J_{\beta,\Delta}\}} = 0 \).

6 Quantitative analysis of crossing probabilities in \( d = 2 \)

6.1 A quadrichotomy on crossing probabilities via RSW and renormalisation inequalities

Given a rectangle \( R = [a, b] \times [c, d] \subset \mathbb{R}^2 \), we define the bottom, top, left, and right sides of \( R \) by
\[
\begin{align*}
B[R] &= [a, b] \times \{c\}, \\
T[R] &= [a, b] \times \{d\}, \\
L[R] &= \{a\} \times [c, d], \\
R[R] &= \{b\} \times [c, d].
\end{align*}
\]

We identify \( \mathbb{Z}^2 \) with its natural embedding in \( \mathbb{R}^2 \) and identify the subsets defined above with their corresponding subgraphs of \( \mathbb{Z}^2 \). We denote by \( H_R \) the event that \( \omega \cap R \) contains a path of open edges from \( L[R] \) to \( T[R] \), and we call such a path a horizontal crossing. Similarly, we denote by \( V_R \) the event that \( \omega \cap R \) contains a path of open edges from \( B[R] \) to \( R[R] \), and we call such a path a vertical crossing.

When \( R = \Lambda_n := [-n, n]^2 \) for some \( n \geq 1 \), we simply write \( H_n \) and \( V_n \), respectively.

The following quadrichotomy on crossing probabilities is at the heart of our approach in \( d = 2 \).

Proposition 6.1. Fix \( p, a \in (0, 1) \). Then, exactly one of the following properties is satisfied for some \( c > 0 \):

(SubCrit) \( \varphi_{A_{2n}} [H_n] \leq e^{-cn}, \quad \forall n \in \mathbb{N}; \)

(SupCrit) \( \varphi_{A_{2n}} [H_n] \geq 1 - e^{-cn}, \quad \forall n \in \mathbb{N}; \)

(ContCrit) \( c \leq \inf_{\xi} \varphi_{A_{2n}} [H_n] \leq \sup_{\xi} \varphi_{A_{2n}} [H_n] \leq 1 - e^{-cn}, \quad \forall n \in \mathbb{N}; \)

(DisContCrit) \( \varphi_{A_{2n}} [H_n] \leq e^{-cn} \) and \( \varphi_{A_{2n}} [H_n] \geq 1 - e^{-cn}, \quad \forall n \in \mathbb{N}. \)

The proof of Proposition 6.1 is based on the approach of [DCT20] and uses renormalisation inequalities.

We introduce the key ingredients of this approach and then sketch how they are used to prove Proposition 6.1 at the end of this subsection.

The first step is establishing an RSW estimate, which relates the horizontal crossings and vertical crossings in long rectangles. The RSW estimates apply to the wired and free measures in infinite volume, and also the following strip measures: let \( S_m = \mathbb{R} \times [-m, 2m] \) denote the infinite strip of width \( 3m \). We write \( \varphi_{S_m}^{1}, \varphi_{S_m}^{0}, \) and \( \varphi_{S_m}^{0/1} \) to denote the wired, free, and mixed boundary conditions (wired on bottom and free on top) random cluster measures on \( S_m \), respectively.
**Proposition 6.2 (RSW estimate).** Let \((p, a) \in (0, 1)\) and let \(\varphi\) denote either \(\varphi^0\) or \(\varphi^1\) or \(\varphi^\#\), where \# are as above. Then, for any \(\rho > 0\), there exists \(c_\rho > 0\) such that

\[
\varphi[H_{[0, pn]} \times [0, n]] \geq c_\rho \left(\varphi[V_{[0, pn]} \times [0, n]]\right)^{1/c_\rho}, \quad \forall n \geq 1/c_\rho.
\]

(RSW)

The proof of Proposition 6.2 is deferred to Subsection 6.2.

We now introduce strip densities.

**Definition 6.3.** Let \(n \in \mathbb{N}\). Define

\[
p_n := \limsup_{\alpha \to \infty} \left(\varphi^0_{[0, an] \times [-n, 2n]}[H_{[0, an] \times [0, n]}]\right)^{1/\alpha},
\]

\[
q_n := \limsup_{\alpha \to \infty} \left(\varphi^1_{[0, an] \times [-n, 2n]}[V^c_{[0, an] \times [0, n]}]\right)^{1/\alpha}.
\]

The complement of a vertical crossing can be interpreted using planar duality. Let \((\mathbb{L}^2)^* = \mathbb{L}^2 + (1/2, 1/2)\) be the dual lattice, which we identify with its natural embedding in \(\mathbb{R}^2\). We typically write \(\omega^*\) to denote set of edges in \((\mathbb{L}^2)^*\). Given \(\omega\) on \(\mathbb{Z}^2\), recall that there is an associated dual configuration \(\omega^*\) constructed by declaring that an edge in the dual graph is open if and only if it crosses a closed edge in the primal graph. Thus, if \(V^c_R\) occurs for some rectangle \(R\), then this implies a horizontal crossing in a suitable rectangle in the dual lattice. As such \(p_n\) is referred to as the crossing strip density and \(q_n\) is referred to as the dual crossing strip density.

The crossing and dual crossing densities are interrelated, albeit on different scales.

**Lemma 6.4 (Duality of strip densities).** Let \((a, p)\) be such that neither \((\text{SubCrit})\) nor \((\text{SupCrit})\) occur. Then, there exists \(C > 0\) such that, for all integers \(\lambda \geq 2\) and \(n \in 9\mathbb{N}\),

\[
p_{3n} \geq \frac{1}{\chi^3} q_{3n}^{3+3/\lambda}, \quad q_{3n} \geq \frac{1}{\chi^3} p_{3n}^{3+3/\lambda}.
\]

The proof of Lemma 6.4 is a relatively straightforward adaptation of the arguments in [DCT20, Section 5.1] to our setting. The adaptations required are already present in the proofs of Proposition 6.2 and the renormalisation inequalities of Lemma 6.5. Therefore, we omit it.

The following lemma contains the key renormalisation inequalities.

**Lemma 6.5 (Renormalisation of strip densities).** Let \((p, a)\) be such that neither \((\text{SubCrit})\) nor \((\text{SupCrit})\) occur. Then, there exists \(C > 0\) such that for all integers \(\lambda \geq 2\) and \(n \in 9\mathbb{N}\),

\[
p_{3n} \leq \lambda^{3+9/\lambda} p_{n}^{3+9/\lambda}, \quad q_{3n} \leq \lambda^{3+9/\lambda} q_{n}^{3+9/\lambda}.
\]

The proof of Lemma 6.5 is deferred to Subsection 6.3.

**Proof of Proposition 6.2 assuming Lemma 6.5.** First note that \((\text{SubCrit})\) and \((\text{SupCrit})\) are disjoint. Thus, it suffices to show that if \(\text{non(SubCrit)}\) and \(\text{non(SupCrit)}\) occur, then either \((\text{ContCrit})\) or \((\text{DiscontCrit})\) occur. Assume that \(\text{non(SubCrit)}\) and \(\text{non(SupCrit)}\) occur.

Note that by Lemmas 6.4 and 6.5 there exists \(c > 0\) such that either: (i) \(\inf_{n \in 9\mathbb{N}} p_n > 0\) and \(\inf_{n \in 9\mathbb{N}} q_n > 0\); or, (ii) there exists \(c > 0\) such that \(p_n \leq e^{-cn}\) and \(q_n \leq e^{-cn}\) for all \(n \in 9\mathbb{N}\). For claims (i) and (ii), one can take \(n \in \mathbb{N}\) because by \((\text{MON})\) and inclusion of events, we have \(p_m \geq p_{m/n}^{m/n}\) and \(q_m \geq q_{m/n}^{m/n}\) for every \(m \geq n\). For claim (ii) the fact that the decay is exponential and not just stretched exponential comes from a bootstrap argument: first, applying Lemma 6.5 for \(\lambda = 9\) we obtain stretch exponential decay along a geometric subsequence. Note that, by finite energy we have that \(p_n \geq e^{-c n}\) and so \(\sup_n p_n^{9/n} < \infty\). This implies that for \(\lambda = n\), we have that \(p_{3n} \leq C_1 n^{C_2} p_n^{3}\), for some constants.
$C_1$ and $C_2$. Thus, for some $C_3$ sufficiently large, the sequence $(n^{C_3}p_n)$ decays exponentially, and hence $p_n$ decays exponentially.

Arguing as in [DCT20, Section 5.4], we find that: if (i) holds, then (ContCrit) occurs; whereas, if (ii) holds, then (DiscontCrit) occurs. The adaptation of these arguments to our setting requires some care concerning the dual crossings. For the case of (ContCrit), see the proof of Lemma 5.5; for the case of (DiscontCrit), see the proof of Lemma 6.14.

### 6.2 RSW theory: Proof of Proposition 6.2

We reduce the proof of Proposition 6.2 to proving conditional probability estimates on the occurrence of tortuous paths, c.f. Lemma 6.8. An intermediate step is to reduce the proof to estimating the probability of certain bridge events.

Let $R = [a, b] \times [c, d]$ be a rectangle in $\mathbb{R}^2$. For $E, F \in \{B[R], T[R], L[R], R[R]\}$ and a rectangle $S \subset R$, we define the following connection events:

- we say $E \leftarrow S \rightarrow F$ if there exists a path in $\omega$ which lies in $S$, and intersects both $E$ and $F$;
- we say $E \leftarrow^* S \rightarrow F$ if there exists a path in $\omega^*$ which, except from its first and last edge, lies in $S$, and intersects both $E$ and $F$.

If $S = R$, then we drop $S$ from the notation. We write $H_R$ and $V_R$ to denote the events that $L[R] \leftarrow R[R]$ and $B[R] \leftarrow T[R]$, respectively. We write $H_R^*$ and $V_R^*$ to denote the events that $L[R] \leftarrow R[R]$ and $B[R] \leftarrow T[R]$, respectively.

Let $k = \lceil n/50 \rceil$. Define $R_0 = \{-17k, \ldots, 18k\} \times \{0, \ldots, n\}$ and $S_0 = \{0, \ldots, k\} \times \{0\}$. Let $R_j, S_j$ denote the translates of $R_0 + (jk, 0)$ and $S_0 + (jk, 0)$, respectively.

We consider the following bridge events:

$$A_j = \{S_j \leftarrow R_j \cup R_j + 4 \rightarrow S_j + 2 \cup S_j + 4\}$$
$$A_j^* = \{S_j^* \leftarrow^* R_j \cup R_j + 4 \rightarrow^* S_j + 2 \cup S_j + 4\}.$$

**Proposition 6.6.** There exists $c_1 > 0$ such that, for every $\lambda > 0$ and $n \in \mathbb{N}$,

$$\varphi[A_0] \geq \frac{c_1}{\lambda^3} \varphi[V_{[0, \lambda n]\times[0, n]}]^3 \text{ and } \varphi[A_0^*] \geq \frac{c_1}{\lambda^3} \varphi[V_{[0, \lambda n]\times[0, n]}^*]^3.$$  

**Proof of Proposition 6.2 assuming Proposition 6.6.** The proof of Proposition 6.2 follows from Proposition 6.6 by applying straightforward gluing arguments, see [DCT20, Lemma 9].

To prove Proposition 6.6, we follow the strategy of [DCT20, Section 3]. The main differences are the following:

1. conditioning on a set of edges being closed does not induce the free boundary conditions, but the conditional measure is dominated by the free measure (see Lemma 2.16);
2. the dual model is not a dilute random cluster measure, hence the estimate on dual crossings needs to be proved directly.

**Proof of Proposition 6.6** Without loss of generality, assume $\lambda \geq 1$. Note that $\bigcup_{j=0}^{C} R_j \supseteq [0, \lambda n] \times [0, n]$, where $C = \lfloor \lambda n/k \rfloor \in (0, 50\lambda]$. We bound the events $V_{[0, \lambda n]\times[0, n]}$ and $V_{[0, \lambda n]\times[0, n]}^*$ according to the crossings in the $R_j$ and $R_j^*$, respectively.
Define the events

\[ \mathcal{J}_j = \{ S_j \xleftarrow{R_j} T[R_j] \} \quad \mathcal{J}_j^* = \{ S_j \xrightarrow{R_j} T[R_j] \} \]
\[ \mathcal{L}_j^g = \{ S_j \xleftarrow{R_j, \ldots} L[R_j+4] \} \quad \mathcal{L}_j^{*,g} = \{ S_j \xrightarrow{R_j, \ldots} L[R_j+4] \} \]
\[ \mathcal{L}_j^b = \{ S_j \xleftarrow{R_j} L[R_j] \} \backslash \mathcal{L}_j^g \quad \mathcal{L}_j^{*,b} = \{ S_j \xrightarrow{R_j} L[R_j] \} \backslash \mathcal{L}_j^{*,g} \]
\[ \mathcal{B}_j^g = \{ S_j \xleftarrow{R_j, \ldots} R[R_j-4] \} \quad \mathcal{B}_j^{*,g} = \{ S_j \xrightarrow{R_j, \ldots} R[R_j-4] \} \]
\[ \mathcal{B}_j^b = \{ S_j \xleftarrow{R_j} R[R_j] \} \backslash \mathcal{B}_j^g \quad \mathcal{B}_j^{*,b} = \{ S_j \xrightarrow{R_j} R[R_j] \} \backslash \mathcal{B}_j^{*,g} . \]

**Remark 6.7.** The events \( \mathcal{J}_j, \mathcal{J}_j^* \) consist of up-down crossings and dual crossings in \( R_j \). The event \( \mathcal{L}_j^g \) (resp. \( \mathcal{L}_j^{*,g} \)) consists of a good left crossing in the sense that there exists a path (resp. dual path) from \( S_j \) to \( L[R_j] \) that does not explore to the right of the \( y \)-axis translated by the vector \((jk + 5k, 0)\). The event \( \mathcal{L}_j^b \) (resp. \( \mathcal{L}_j^{*,b} \)) consists of a bad left crossing in the sense that any path (resp. dual path) from \( S_j \) to \( L[R_j] \) must cross the \( y \)-axis translated by the vector \((jk + 5k, 0)\). There are similar observations for the events involving right crossings.

Observe that, by translation invariance, we have \( \varphi[C_j] = \varphi[C_0] \), where \( C_j \) refers to any of the crossings or dual crossings defined above. Moreover, by reflection invariance along the \( y \)-axis translated by the vector \((jk + k, 0)\), we have that \( \varphi[C_j^L] = \varphi[C_j^R] \), where \( C_j^L \) and \( C_j^R \) refer to any of the same type of crossing/dual crossing event defined above. Therefore, by union bounds,

\[
\varphi[V_{0,\lambda n} \times [0,n]] \leq \sum_{j=0}^{C} \left( \varphi[\mathcal{J}_j] + \varphi[\mathcal{L}_j^g] + \varphi[\mathcal{L}_j^b] + \varphi[\mathcal{B}_j^g] + \varphi[\mathcal{B}_j^b] \right)
\]

\[
\varphi[V_{0,\lambda n} \times [0,n]] \leq \sum_{j=0}^{C} \left( \varphi[\mathcal{J}_j^*] + \varphi[\mathcal{L}_j^{*,g}] + \varphi[\mathcal{L}_j^{*,b}] + \varphi[\mathcal{B}_j^{*,g}] + \varphi[\mathcal{B}_j^{*,b}] \right)
\]

from which we deduce

\[
\max \{ \varphi[\mathcal{J}_0], \varphi[\mathcal{L}_0^g], \varphi[\mathcal{L}_0^b] \} \geq \frac{1}{5(C+1)^2} \varphi[V_{0,\lambda n} \times [0,n]] \quad \text{(6.1)}
\]

\[
\max \{ \varphi[\mathcal{J}_0^*], \varphi[\mathcal{L}_0^{*,g}], \varphi[\mathcal{L}_0^{*,b}] \} \geq \frac{1}{5(C+1)^2} \varphi[V_{0,\lambda n} \times [0,n]] . \quad \text{(6.2)}
\]

Note that by translation invariance, reflection invariance, and straightforward gluing arguments, it is sufficient to consider the case when the maximum is taken over the tortuous paths \( \mathcal{T}_0, \mathcal{T}_0^*, \mathcal{L}_0^b \), and \( \mathcal{L}_0^{*,b} \). Let \( C_0 \) be the maximiser amongst \( \{ \mathcal{T}_0, \mathcal{L}_0^b, \mathcal{T}_0^*, \mathcal{L}_0^{*,b} \} \). We show that the following conditional probability estimate, whose proof we postpone to afterwards, is sufficient to establish Proposition 6.6

**Lemma 6.8.** Let \( C_0 \) be as above. Then,

\[
\varphi[A_0 \mid C_0 \cap C_4] \geq \frac{1}{2} \varphi[C_2 \setminus (A_0 \cup A_2) \setminus C_0 \cap C_4] .
\]

By Lemma 6.8 and translation invariance, we have that

\[
q \varphi[A_0] \geq \varphi[(C_2 \setminus (A_0 \cup A_2)) \cap C_0 \cap C_4]
\]

and

\[
\varphi[A_0] \geq \frac{1}{2} \varphi[A_0 \cap (C_2 \cap C_0 \cap C_4)] + \frac{1}{2} \varphi[A_2 \cap (C_0 \cap C_2 \cap C_4)].
\]
Hence, by summing over disjoint events and using the FKG inequality, we obtain

$$(q + 2)\varphi[A_0] \geq \varphi[(C_1 \cap C_2) \cup (C_1 \cup C_2)] + \varphi[(C_1 \cap C_2) \cap (C_1 \cup C_2)]$$

$$= \varphi[C_1 \cap C_2 \cap C_4] \geq \varphi[C_0]$$

These estimates combined with (6.1) and (6.2) completes the proof.

**Proof of Lemma 6.8.** We only treat the case $C_0 = \mathcal{R}_0^*$ to illustrate the techniques; the other cases are similar, with the caveat that $\mathcal{L}_0^*$ and $\mathcal{L}_0^{*,\nu}$ require some additional geometric arguments that are purely deterministic (i.e. there is no change in adapting this to our setting). This difficulty was already treated in [DCT20, Section 3].

Let $\Gamma_L^*$ (resp. $\Gamma_R^*$) denote the left-most (resp. right-most) up-down dual crossing consisting of closed edges. We condition on $\Gamma_L^* = \gamma_L^*$ and $\Gamma_R^* = \gamma_R^*$, where $\gamma_L^*$ and $\gamma_R^*$ are dual paths in $R_0$ and $R_4$, respectively, such that all edges except the first and last lie in their respective domains. Identifying $\gamma_L$ and $\gamma_R$ with the corresponding set of edges in the primal, we let $v^-(\gamma_L)$ be the set of vertices incident to the left of $\gamma_L$ and $v^+(\gamma_R)$ be the set of vertices incident to the right of $\gamma_R$.

Let $\text{Sym} = [-17k, \ldots, 22k] \times [0, \ldots, 39k]$. Note that $L[\text{Sym}] = \{17k\} \times [0, \ldots, 39k]$ and $R[\text{Sym}] = \{22k\} \times [0, \ldots, 39k]$. Let $\Omega$ be set of vertices with boundary given by the union of $v^-(\gamma_L) \cup v^+(\gamma_R)$ and the appropriate top and bottom segments of $\text{Sym}$. Let $\text{mix}$ denote the following boundary conditions on $\Omega$:

- 0 vertices on $v^-(\gamma_L)$ and $v^+(\gamma_R)$
- wired on $T[\Omega]$
- wired on $B[\Omega]$.

Let $\xi$ denote a random boundary condition on $\Omega$ such that the left-most and right-most interior bonds are closed. Then, by Lemma 2.16 and monotonicity in the domain, we have that

$$\varphi[A_0^* | \Gamma_L^* = \gamma_L^*; \Gamma_R^* = \gamma_R^*] \geq \varphi[\varphi_{\text{mix}}^\xi[v^-(\gamma_L) \leftrightarrow v^+(\gamma_R)] | \Gamma_L^* = \gamma_L^*; \Gamma_R^* = \gamma_R^*]$$

$$\geq \varphi_{\text{mix}}^\xi[\varphi_{L[\text{Sym}]}[\gamma_L^*] \leftrightarrow \varphi_{R[\text{Sym}]}[\gamma_R^*]]$$

Now, we turn to the righthand side. We again partition $\mathcal{T}_0^* \cap \mathcal{T}_4^*$ according to $\Gamma_L^*$ and $\Gamma_R^*$. Fix $\Gamma_L^* = \gamma_L^*$ and $\Gamma_R^* = \gamma_R^*$. Conditionally on this, note that the event $\mathcal{T}_0^* \setminus (A_0 \cup A_2)$ implies the existence of random primal paths $\Pi_L$ (left-most) and $\Pi_R$ (right-most) separating $\mathcal{T}_2^*$ from $\gamma_L^*$ and $\gamma_R^*$. Conditional on $\Pi_R = \pi_R$ and $\Pi_L = \pi_L$, let $\Omega^*$ denote the set of vertices with boundary given by $\pi_R \cup \pi_L$ and the appropriate top and bottom segments of $\text{Sym}$, which we denote $T(\Omega^*)$ and $B(\Omega^*)$, respectively. Let $\xi$ denote a random boundary condition on $\Omega^*$ that is wired on $\pi_R \cup \pi_L$. Let also $\text{mix}^*$ denote the following boundary conditions on $\Omega^*$:

- 0 vertices on the appropriate top and bottom segments of $\Omega \cap \text{Sym}$
- wired on $\pi_L \cup \pi_R$.

By the spatial Markov property, monotonicity in boundary conditions, and inclusion of events,

$$\varphi[\mathcal{T}_2^* \setminus (A_0^* \cup A_1^*) | \gamma_L^*, \gamma_R^*, \pi_L, \pi_R] \leq \varphi[\varphi_{\text{mix}^*}[B[\Omega^*] \leftrightarrow T[\Omega^*]] | \gamma_L^*, \gamma_R^*, \pi_L, \pi_R]$$

$$\leq \varphi_{\text{mix}^*}[B[\Omega^*] \leftrightarrow T[\Omega^*]]$$

The desired conclusion follows from $\pi/2$ rotation invariance.
6.3 Renormalisation: Proof of Lemma 6.9

We require an intermediary lemma that gives estimates on crossing probabilities under balanced boundary conditions at macroscopic distance away from the rectangle. This allows us to push boundary conditions in the proof of the renormalisation inequality.

**Lemma 6.9.** There exists \( c > 0 \) such that, for all \( n \in 9\mathbb{N} \), one of the following must occur:

\[
\forall \alpha > 0, \quad \varphi_{R_0}^{\text{LPrim}}[H_R] \geq c^\alpha \quad \text{(PushPrim)}
\]

\[
\forall \alpha > 0, \quad \varphi_{R_0}^{\text{BPrim}}[V_R^c] \geq c^\alpha \quad \text{(PushDual)}
\]

where \( R = [0, \alpha_n] \times [0, n/9] \); \( R_0 = [0, \alpha_n] \times [0, 28n/9] \); LTR is the boundary condition that is wired on \( L[R] \cup T[R] \cup R[R] \) and 0 on \( B[R] \); and, B is the boundary condition that is 0 on \( L[R] \cup T[R] \cup R[R] \), and wired on \( B[R] \).

The proof of Lemma 6.9 follows the strategy of [DCT20, Sections 5.2]. The adaptation of the arguments involved to our setting is similar in spirit to the modifications presented in the proof of Proposition 6.2 and, in addition, in the proof of the renormalisation inequality below. Thus, we omit the proof.

**Proof of Lemma 6.5** We stress again that proof is an adaptation of the proof of [DCT20, Lemma 15], where the main differences are those discussed in the proof of Proposition 6.2. It is included for completeness. Without loss of generality, we prove the second inequality (renormalisation of the \( q_n \)'s). Moreover, without loss of generality we may assume that \( \text{(PushDual)} \) occurs.

Assume \( n \in 9\mathbb{N} \). Define the rectangles

\[
R = [0, \alpha_n] \times [0, 6\lambda n + 3n]
\]

\[
R_i = [0, \alpha_n] \times [6in + 3n, 6in + 6n], \quad 0 \leq i \leq \lambda - 1
\]

\[
R_i' = [0, \alpha_n] \times [6in, 6in + 3n], \quad 0 \leq i \leq \lambda.
\]

Note that the \( R_i \) and \( R_i' \) partition \( R \). We further subdivide each \( R_i \) horizontally into three thin rectangles. For \( 0 \leq i \leq \lambda - 1 \), define

\[
\tilde{R}_i^- := [0, \alpha_n] \times [6in + 3n + \frac{12}{9}n, 6in + 3n + \frac{13}{9}n]
\]

\[
\tilde{R}_i := [0, \alpha_n] \times [6in + 3n + \frac{13}{9}n, 6in + 3n + \frac{14}{9}n]
\]

\[
\tilde{R}_i^+ := [0, \alpha_n] \times [6in + 3n + \frac{14}{9}n, 6in + 3n + \frac{15}{9}n].
\]

Let \( \mathcal{E} \) denote the event that each rectangle \( R_i \) is crossed horizontally. Let \( \mathcal{F} \) denote the event that each rectangle \( R_i' \) is not crossed vertically, i.e. that there is a crossing of dual edges which cut closed edges in the primal. Let \( \tilde{\mathcal{E}} \) be the event that all of the \( \tilde{R}_i \) are crossed horizontally. Let \( \tilde{\mathcal{F}} \) be the event that none of the \( \tilde{R}_i^\pm \) are crossed vertically.

Let \( 1/0 \) denote the boundary condition that is wired on \( T[R] \cup B[R] \) and free on \( L[R] \cup R[R] \). Assume the following estimates hold:

\[
\varphi_{R}^{1/0}[\tilde{\mathcal{E}} \cap \mathcal{F}] \geq \frac{(2r)^{-12\lambda_1 n + 6n}}{\lambda_1^{2\lambda_1 n}} \varphi_{[0, \alpha_n] \times [-n, 2n]}^{1/0}[V_{[0, \alpha_n] \times [0, n]}]^{\lambda_1 + 1} \tag{6.3}
\]

\[
\varphi_{R}^{1/0}[\tilde{\mathcal{E}} \cap \tilde{\mathcal{F}}] \geq c^{2\lambda_1} \tag{6.4}
\]

\[
\varphi_{R}^{1/0}[\tilde{\mathcal{E}} \cap \tilde{\mathcal{F}}] \leq \varphi_{[0, \alpha_n] \times [-9, 2n/9]}^{0}[H_{[0, \alpha_n] \times [0, n/9]}]^{\lambda}. \tag{6.5}
\]

23
Then, by (6.3) and (6.4),
\[
\varphi_{R}^{1/0} [\tilde{E} \cap \tilde{F}] \geq \varphi_{R}^{1/0} [\tilde{F} | \tilde{E} \cap \tilde{F}] \varphi_{R}^{1/0} [\tilde{E} \cap \tilde{F}]
\]
where 1/0 denotes the balanced boundary condition that is wired on the top and bottom of \( R \), and free on the left and right of \( R \). Hence, by (6.5), we get that
\[
\varphi_{[0,\alpha n] \times [-n,2n]}^{1}[V_{\tilde{E} \cap \tilde{F}}^{c}] \leq \lambda^{C_{\alpha}} \varphi_{[0,\alpha n] \times [0,\alpha n]}^{0}[\tilde{F} | \tilde{E}] \varphi_{[0,\alpha n] \times [0,\alpha n]}^{\lambda} \alpha
\]
The renormalisation inequality then follows: i) raising to the power of \( 1/\lambda \); ii) taking \( \alpha \to \infty \) to express these events in terms of \( p_{m} \) and \( q_{m} \), where \( m = m(n) \); and, iii) applying the duality relation between \( p_{m} \) and \( q_{m} \) of Lemma 6.4. It remains to prove (6.3)-(6.5).

To prove (6.3), first note that
\[
\phi_{[0,\alpha n] \times [-n,2n]}^{1}[V_{\tilde{E} \cap \tilde{F}}^{c}] \leq \lambda^{C_{\alpha}} \varphi_{[0,\alpha n] \times [0,\alpha n]}^{0}[\tilde{F} | \tilde{E}] \varphi_{[0,\alpha n] \times [0,\alpha n]}^{\lambda} \alpha
\]
where above we abuse notation and write \( S_{[0,\alpha n] \times [-n,2n]}^{\lambda} = \mathbb{R} \times [0, 6\lambda n + 3n] \). On the other hand, note that we may partition the event \( \tilde{E} \) as follows. For each \( 1 \leq i \leq \lambda - 1 \), let \( \Gamma_{i} \) be the top-most path in \( \tilde{R}_{i} \), and set \( \Gamma_{0} = B[R] \) and \( \Gamma_{\lambda} = T[R] \). Let \( \Gamma_{i}(\Gamma) \) be the random domain with boundary given by \( \Gamma_{i+1}, \Gamma_{i} \), and the relevant left and right segments of the boundary of \( R \). Then, by conditioning on \( \Gamma_{i} \), the spatial Markov property, independence, monotonicity in boundary conditions, and translation invariance, we find
\[
\varphi_{[0,\alpha n] \times [-n,2n]}^{1}[V_{\tilde{E} \cap \tilde{F}}^{c}] \leq \varphi_{[0,\alpha n] \times [0,\alpha n]}^{1}[\tilde{F} \cap \tilde{F}] \geq \prod_{i=1}^{\lambda-1} \varphi_{[0,\alpha n] \times [0,\alpha n]}^{1}[\tilde{E} \cap \tilde{E}] \geq \lambda^{C_{\alpha}} \lambda
\]
Inserting (6.7) and (6.8) into (6.6) establishes (6.3).

A similar conditioning argument, this time on the dual paths occurring in the \( \tilde{R}_{\pm} \), together with the stochastic domination in Lemma 2.16 yields (6.5). Finally, in order to obtain (6.4), we note that by a similar conditioning (this time on the crossings in \( \tilde{E} \) and dual crossings of \( F \)) and pushing boundary conditions argument, together with translation invariance, we have the following estimate:
\[
\varphi_{[0,\alpha n] \times [0,2\alpha n]}^{1}[V_{\tilde{E} \cap \tilde{F}}^{c}] \geq \varphi_{[0,\alpha n] \times [0,\alpha n]}^{1}[V_{\tilde{E} \cap \tilde{F}}^{c}] \geq \lambda^{2} \lambda
\]
Applying the (PushDual) then completes the proof.

6.4 Applications of the quadrichotomy

We now explain some important consequences of Proposition 6.1 that allow us to get a better quantitative understanding of the phase diagram \( d = 2 \). We suppress notational dependence on \( p \) and \( \alpha \) when clear from context.
6.4.1 Off-critical behaviour

The following propositions characterise the off-critical behaviour. The proofs of the first two are standard and can be found in [DCT20 Section 5].

**Proposition 6.10.** Assume that (SubCrit) occurs. There exists $c > 0$ such that for every $n \geq 1$,

$$\varphi^1_{\Lambda_n}[0 \leftrightarrow \partial \Lambda_n] \leq e^{-cn}. \quad (6.9)$$

In particular, $\varphi^1[0 \leftrightarrow \infty] = 0$ and $\varphi^0 = \varphi^1$.

**Proposition 6.11.** Assume that (SupCrit) occurs. There exists $c > 0$ such that for every $n \geq 1$,

$$\varphi^0[\Lambda_n \leftrightarrow \infty] \leq e^{-cn}.$$ 

In particular, $\varphi^0[0 \leftrightarrow \infty] > 0$ and $\varphi^0 = \varphi^1$.

**Proposition 6.12.** Assume that (SupCrit) occurs. Then there exists $t > 0$ such that for every $n \geq 1$, we have $\varphi^0[C_n] \geq 1 - e^{-tn}$.

*Proof.* It suffices to show that $\varphi^0[H^*_n \times [-2n, 2n]]$ decays exponentially. This follows from our assumption that (SupCrit) occurs, (MON), and (RSW).

6.4.2 Discontinuous critical behaviour

The following proposition characterises the discontinuous critical behaviour.

**Proposition 6.13.** Assume that (DiscontCrit) occurs. There exists $c > 0$ such that for every $n \geq 1$,

$$\varphi^0[0 \leftrightarrow \partial \Lambda_n] \leq e^{-cn} \quad \text{and} \quad \varphi^1[\Lambda_n \leftrightarrow \infty] \leq e^{-cn}.$$ 

In particular, $\varphi^0[0 \leftrightarrow \infty] = 0$ and $\varphi^1[0 \leftrightarrow \infty] > 0$.

In order to prove Proposition 6.13, we require an intermediate lemma. Let $C_n$, respectively $C^*_n$, be the event that there is an open circuit in $\omega \cap (\Lambda_{2n} \setminus \Lambda_n)$, respectively $\omega^* \cap (\Lambda_{2n} \setminus \Lambda_n)$, surrounding the origin.

**Lemma 6.14.** Assume that (DiscontCrit) occurs. Then there exists $t > 0$ such that for every $n \geq 1$, we have $\varphi^0[C^*_n] \geq 1 - e^{-tn}$ and $\varphi^3[C_n] \geq 1 - e^{-tn}$.

*Proof.* We will show the assertion in the case of $\varphi^0$, with the case of $\varphi^1$ being similar. We start by showing that $\varphi^0_{\Lambda_{4n}}[C^*_n] \geq 1 - e^{-tn}$, and then proceed to show that the same holds at infinite volume.

Note that if the rectangles $[-2n, -n] \times [-2n, 2n]$ and $[n, 2n] \times [-2n, 2n]$ have dual vertical crossings, while $[-2n, 2n] \times [-2n, -n]$ and $[-2n, 2n] \times [n, 2n]$ have dual horizontal crossings, then $C^*_n$ occurs. By the $\pi/2$ rotational symmetry of the measure $\varphi^0_{\Lambda_{4n}}$, the FKG inequality, and duality, it suffices to show that $\varphi^0_{\Lambda_{4n}}[H_{[-2n, -n] \times [-2n, 2n]]] = 0$.

To this end, recall that $p_n$ decays exponentially by our assumption that (DiscontCrit) occurs. It follows from the union bound that there are vertices $x$ and $y$ on the left and right side of $[-2n, -n] \times [-2n, 2n]$, respectively, such that

$$\varphi^0_{\Lambda_{4n}}[x \leftrightarrow y \text{ in } [-2n, -n] \times [-2n, 2n]] \geq \frac{\varphi^0_{\Lambda_{4n}}[H_{[-2n, -n] \times [-2n, 2n]]]}{(4n + 1)^2}.$$ 

Moreover, for every $k \geq 1$, we can create a horizontal crossing of the rectangle $[0, 4kn] \times [0, 4n]$ by combining $4k$ translations and reflections of the event $\{x \leftrightarrow y \text{ in } [-2n, -n] \times [-2n, 2n]\}$, as
appropriate. Using the FKG inequality, (MON), and the reflection symmetry of the measure \( \varphi_{\Lambda_n}^0 \), we obtain
\[
\varphi_{\Lambda_n}^0 [H[0,4kn] \times [0,4n]] \geq \left( \frac{\varphi_{\Lambda_2n}^0 [H[-2n, -n] \times [-2n, 2n]]}{(4n + 1)^2} \right)^{4k}.
\]
By the finite energy property, there exists a constant \( c > 0 \) such that
\[
\varphi_{[0,4kn] \times [-4n,8n]}^0 [H[0,4kn] \times [0,4n]] \geq e^{-cn} \varphi_{\Lambda_n}^0 [H[0,4kn] \times [0,4n]],
\]
hence taking \( k \)th roots and sending \( k \) to infinity, we obtain
\[
\varphi_{\Lambda_n}^0 [H[-2n,-n] \times [-2n,2n]] \leq (4n + 1)^2 p_{4n}^{1/4}.
\]
This implies that \( \varphi_{\Lambda_2n}^0 [H[-2n,-n] \times [-2n,2n]] \) decays exponentially, as desired.

To prove the assertion at infinite volume, consider some \( k \geq 1 \), and note that
\[
\varphi_{\Lambda_{kn}}^0 [C_n^*] \geq \varphi_{\Lambda_{kn}}^0 \left[ \bigcap_{i=1}^k C_{2^k-i}^* \right] = \varphi_{\Lambda_{kn}}^0 \left[ \bigcap_{i=2}^k \varphi_{\Lambda_{2n}}^0 \left[ C_{2^{k-1}}^* \right] \right] = \prod_{i=2}^k \varphi_{\Lambda_{kn}}^0 \left[ C_{2^{k-1}}^* \right],
\]
By Lemma 2.17 and (SMP), (MON), we have
\[
\varphi_{\Lambda_{kn}}^0 \left[ C_{2^{k-1}}^* \right] \geq \varphi_{\Lambda_{kn}}^0 \left[ C_{2^{k-2}}^* \right],
\]
and \( \varphi_{\Lambda_{kn}}^0 \left[ C_{2^{k-1}}^* \right] \geq \varphi_{\Lambda_{kn}}^0 \left[ C_{2^{k-2}}^* \right] \), which implies that \( \varphi_{\Lambda_{kn}}^0 \left[ C_n^* \right] \geq 1 - e^{-t'n} \) for some constant \( t' > 0 \). Sending \( k \) to infinity we obtain the desired assertion. 

We are now ready to prove Proposition 6.13.

**Proof of Proposition 6.13** For the first inequality we use that \( \varphi_{[0] \leftarrow \partial \Lambda_n}^0 \leq 1 - \varphi_{[n/2]}^0 \) and Lemma 6.14. For the second inequality, we observe that when \( \Lambda_n \) is not connected to infinity, there is an open circuit in \( \omega^* \) surrounding \( \Lambda_n \). This implies that for some \( k \geq n \), the dual edge \( \{(k+1/2, 1/2), (k+1/2, 1/2)\} \) is connected to distance \( k \) in \( \omega^* \), hence the second inequality follows from Lemma 6.14.

**6.4.3 Continuous critical behaviour**

The following proposition, which gives polynomial bounds on one-arm events, is a standard consequence of Proposition 6.1. See [DC17, Section 5].

**Proposition 6.15.** Assume that (ContCrit) occurs. There exists \( c > 0 \) such that for every \( n \geq 1 \),
\[
\frac{c}{n} \leq \varphi_{[0] \leftarrow \partial \Lambda_n} \leq \frac{1}{n^c}.
\]
In particular, \( \varphi_{[0] \leftarrow \infty} = 0 \) and \( \varphi_{[0] \leftarrow \infty} = \varphi^0. \)
6.4.4 Exact correspondence of critical behaviour

The following proposition makes rigorous the intuitively clear statement that (ContCrit) or (DiscontCrit) can only occur at $p_c$.

**Proposition 6.16.** Let $a \in (0, 1)$. If $p = p_c(a)$, then either (ContCrit) or (DiscontCrit) occurs. Furthermore, the set of $a \in (0, 1)$ for which (DiscontCrit) occurs at $p_c(a)$ is open.

**Proof.** The fact that (ContCrit) or (DiscontCrit) must occur at $p_c(a)$ follows from a finite size criterion. Indeed, if (SupCrit) holds at $p_c(a)$, then arguing as in [DCT20] Lemma 10, this would imply that there is no infinite cluster under $\phi^1_{p_c(a)+\epsilon}$, which is a contradiction. Similar considerations hold with (SupCrit).

In addition, a standard consequence of the renormalisation inequality in Lemma 6.5 is that the set of points at which (DiscontCrit) occurs is open. See [DCT20] Section 5.4.

Finally, we establish that (ContCrit) and (DiscontCrit) correspond exactly to continuous and discontinuous critical points for the Blume-Capel model, respectively.

**Proposition 6.17.** Let $\Delta \in \mathbb{R}$ and $a = \frac{2e^\Delta}{1 + 2e^\Delta}$. If (ContCrit) occurs at $p = p_c(a)$, then $\langle \sigma_0 \rangle^{+}_{\beta_c(\Delta), \Delta} = 0$. If (DiscontCrit) occurs at $p = p_c(a)$, then $\langle \sigma_0 \rangle^{+}_{\beta_c(\Delta), \Delta} > 0$.

**Proof.** Recall that $\langle \sigma_0 \rangle^{+} = \varphi^1[0 \leftrightarrow \infty]$ from the Edwards-Sokal coupling. If (ContCrit) occurs, then $\varphi^1[0 \leftrightarrow \infty] = 0$ by Proposition 6.15. If (DiscontCrit) occurs, then $\varphi^1[0 \leftrightarrow \infty] > 0$ by Proposition 6.13. This proves the desired result.

7 Tricritical point in $d = 2$ via crossing probabilities

7.1 A direct proof from the quadrichotomy

The importance of Proposition 6.16 is that it allows to deduce that the two point function decays to 0 at $p_c(a)$ for any $a \in (0, 1)$. This, together with the mapping described in Section 4, implies Theorem 1.1 for $\Delta^+ = -\log 2$.

**Proof of Theorem 1.1 in $d = 2$ for $\Delta^+ = -\log 2$.** Let $\Delta \geq -\log 2$ and set $a = \frac{2e^\Delta}{1 + 2e^\Delta}$. By Proposition 6.16, we know that either (ContCrit) or (DiscontCrit) occurs at $p_c(a)$. Using Propositions 6.13 and 6.15, we obtain that in any case, $\lim_{|x| \to \infty} \phi^{0}_{p_c(a),d}[0 \leftrightarrow x] = 0$. As a consequence of Corollary 4.4, we have that

$$\lim_{|x| \to \infty} \langle \tau_0 \tau_x \rangle^{\text{Ising,0}}_{f(\beta_c(\Delta), \Delta)} = 0.$$

Hence, by [Rao20] Proposition 1, translation invariance, and mixing,

$$\left( \langle \tau_0 \tau_x \rangle^{\text{Ising,0}}_{f(\beta_c(\Delta), \Delta)} \right)^2 = \lim_{|x| \to \infty} \langle \tau_0 \tau_x \rangle^{\text{Ising,0}}_{f(\beta_c(\Delta), \Delta)} = \lim_{|x| \to \infty} \langle \tau_0 \tau_x \rangle^{\text{Ising,0}}_{f(\beta_c(\Delta), \Delta)} = 0.$$

Thus $\langle \sigma_0 \rangle^{+}_{\beta_c(\Delta), \Delta} = 0$.

7.2 A sufficient criterion for continuity

We give a sufficient criterion to prove continuity in dimension 2 that avoids the use of the coupling with Ising explained in Section 4 thus having the potential to extend beyond $\Delta^+ = -\log 2$. In the next subsection, we show that this is indeed the case.
Let $G = (V, E)$ be a finite subgraph of $\mathbb{Z}^2$, and let $\varepsilon > 0$. We denote by $\mu^{+,\varepsilon}_{G,\beta,\Delta}$ the probability measure defined on spin configurations $\sigma \in \{-1, 0, 1\}^V$ by

$$
\mu^{+,\varepsilon}_{G,\beta,\Delta}(\sigma) = \frac{1}{Z^{+,\varepsilon}_{G,\beta,\Delta}} e^{-H^{+,\varepsilon}_{G,\beta,\Delta}(\sigma)}
$$

where

$$
H^{+,\varepsilon}_{G,\beta,\Delta}(\sigma) = -\beta \sum_{xy \in E} \sigma_x \sigma_y - \Delta \sum_{x \in V} \sigma_x^2 - \varepsilon \sum_{xy \in E_0} \sigma_x.
$$

and $Z^{+,\varepsilon}_{G,\beta,\Delta}$ is the partition function. This corresponds to the Blume-Capel model with $\varepsilon$-boundary conditions.

When the phase transition is continuous, $\mu^{+,\varepsilon}_{G,\beta,\Delta}$ converges to $\mu^{+}_{\beta,\Delta}$ for every $\varepsilon > 0$. On the other hand, when the phase transition is discontinuous, it is unclear whether this property holds as one expects that $\mu^0$ is extremal. Therefore, the following criterion is natural.

**Definition 7.1.** For $\beta > 0$ and $\Delta \in \mathbb{R}$, we say (H) is satisfied if

$$
\lim_{n \to \infty} \mu^{+,\varepsilon}_{\Lambda_n,\beta,\Delta} [\sigma_0 | \sigma_0 \neq 0] = \mu^{+}_{\beta,\Delta} [\sigma_0 | \sigma_0 \neq 0], \quad \forall \varepsilon > 0.
$$

Given $\delta \in (0, 1)$, we now define a dilute random cluster measure $\varphi^{1,\delta}_{\Lambda_n,p,a}$, which corresponds to $\mu^{+,\varepsilon}_{\Lambda_n,\beta,\Delta}$ for $\varepsilon = -\frac{1}{2} \log (1 - \delta)$ via the Edwards-Sokal coupling. The measure $\varphi^{1,\delta}_{\Lambda_n,p,a}$ is defined by

$$
\varphi^{1,\delta}_{\Lambda_n,p,a}[\theta] = \frac{1}{Z^{1,\delta}_{\Lambda_n,p,a}} 2^{k(\theta,\Lambda)} \prod_{x \in V} \left( \frac{a}{1-a} \right)^{\psi_x} \prod_{e \in E_0} r_e \left( \frac{p_e}{1-p_e} \right)^{\omega_e},
$$

where $Z^{1,\delta}_{\Lambda_n,p,a}$ is the normalisation constant. Here $r_e = \sqrt{1-p}$ if both endpoints of $e$ belong to $\Lambda$, and $r_e = \sqrt{1-\delta}$ otherwise, while $p_e = p$ if both endpoints of $e$ belong to $\Lambda$, and $p_e = \delta$ otherwise. Note that samples from $\varphi^{1,\delta}_{\Lambda_n,p,a}$ are supported on $\Theta^1_{\Lambda_n}$, although we stress that not all vertices in $\Lambda_n$ are required to be open.

In the next lemma, we obtain a comparison between $\varphi^{1,\delta}_{\Lambda_n,p,a}$ and $\varphi^{0}_{\Lambda_n,p,a}$ which states that, up to an exponential rewiring cost that is proportional to $n$ and small when $\delta$ is small, we may bound probabilities under $\varphi^{0}_{\Lambda_n,p,a}$ from below by probabilities under $\varphi^{1,\delta}_{\Lambda_n,p,a}$. We recall that $b(\Lambda_n)$ is the set of edges induced by $\Lambda_n$. We write $\partial_{E} \Lambda = \{xy \in \mathbb{Z}^2 : x \in \Lambda, y \in \mathbb{Z}^2 \setminus \Lambda \}$ to denote the edge boundary of $\Lambda$.

**Lemma 7.2.** For every $\delta \in (0, 1)$, $n \in \mathbb{N}$, and event $A$ depending on $(\Lambda_n, b(\Lambda_n))$, we have

$$
\varphi^1_{\Lambda_n,p,a}[A] \leq \left( \frac{1}{1-\delta} \right)^{12n+6} \varphi^0_{\Lambda_n,p,a}[A].
$$

**Proof.** Let $\theta \in \{0,1\}^{\Lambda_n} \times \{0,1\}^{b(\Lambda_n)}$ be a configuration on $\Lambda_n$. Given $\xi \in \Theta^\delta_{\Lambda_n}$, we write $\xi \sim_{\Lambda_n} \theta$ if $\xi|_{\Lambda_n \setminus b(\Lambda_n)} = \theta$, i.e. if the two configurations coincide exactly in $(\Lambda_n, b(\Lambda_n))$.

Let $Z^{1,\delta}_{\Lambda_n,p,a}[\theta]$ and $Z^{0}_{\Lambda_n,p,a}[\theta]$ be defined such that $\varphi^{1,\delta}_{\Lambda_n,p,a}[\theta] = \frac{Z^{1,\delta}_{\Lambda_n,p,a}[\theta]}{Z^{1,\delta}_{\Lambda_n,p,a}}$ and $\varphi^{0}_{\Lambda_n,p,a}[\theta] = \frac{Z^{0}_{\Lambda_n,p,a}[\theta]}{Z^{1,\delta}_{\Lambda_n,p,a}}$, respectively. Note that

$$
\varphi^{1,\delta}_{\Lambda_n,p,a}[\theta] = \frac{1}{Z^{1,\delta}_{\Lambda_n,p,a}} \sum_{\theta' \succeq_{\Lambda_n} \theta} Z^{1,\delta}_{\Lambda_n,p,a}[\theta'], \quad \text{and} \quad \varphi^{0}_{\Lambda_n,p,a}[\theta] = \frac{1}{Z^{0}_{\Lambda_n,p,a}} \sum_{\theta' \succeq_{\Lambda_n} \theta} Z^{0}_{\Lambda_n,p,a}[\theta'],
$$

where
Thus, the proof is immediate if we can prove:

\[
Z_{\Lambda_n,p,a}^{1,\delta} [\theta] \leq \left( \frac{1}{1 - \delta} \right)^{8n+4} Z_{\Lambda_n,p,a}^0 [\theta], \quad \text{and} \quad Z_{\Lambda_n,p,a}^{1,\delta} [\theta] \geq (1 - \delta)^{4n+2} Z_{\Lambda_n,p,a}^0 [\theta]. \tag{7.1}
\]

The left-hand side of (7.1) follows from three observations. First, note that for every \( \theta^1 \in \Theta_{\Lambda_n}^1 \) and \( \theta^0 \in \Theta_{\Lambda_n}^0 \) that coincide with \( \theta \) on \( \Lambda_n \), we have \( k(\theta^1, \Lambda_n) \leq k(\theta^0, \Lambda_n) \). Second, note that \( r_e \leq 1 \). Finally, note that

\[
\sum_{\omega \in \{0,1\}^\partial E_{\Lambda_n}} \prod_{e \in \partial E_{\Lambda_n}} \left( \frac{\delta}{1 - \delta} \right)^{\omega_e} \leq \left( \frac{1}{1 - \delta} \right)^{8n+4}
\]

where \( \theta_{b(\Lambda_n)} \) refers to the edge configuration (i.e. the projection on the second coordinate) of \( \theta \) inside \( b(\Lambda_n) \).

For the right-hand side of (7.1), we truncate the sum in \( Z_{\Lambda_n,p,a}^{1,\delta} [\theta] \) over all configurations \( \theta^1 \) whose edge projection is equal to 0 for every edge in \( \partial E_{\Lambda_n} \). Thus, for any such \( \theta^1 \) and for any \( \theta^0 \in \Theta_{\Lambda_n}^0 \), we have \( k(\theta^1, \Lambda_n) = k(\theta^0, \Lambda_n) \). The estimate follows from this consideration together with the bound

\[
\prod_{x \in \Lambda_n \cup \partial E_{\Lambda_n}} r_{xy} \geq (1 - \delta)^{8n+4} = (1 - \delta)^{4n+2}
\]

where \( \psi^1 \) is the projection of \( \theta^1 \) onto its first coordinate. \( \square \)

We now show that \( (H) \) is a sufficient condition for continuity.

**Proposition 7.3.** Let \( \Delta \in \mathbb{R} \). Assume that \( (H) \) is satisfied at \((\beta_c(\Delta), \Delta)\). Then

\[
\langle \sigma_0 \rangle^{+}_{\beta_c(\Delta), \Delta} = 0.
\]

**Proof.** Let \( (\beta_c(\Delta), \Delta) \) be such that \( (H) \) is satisfied and assume that we have \( \langle \sigma_0 \rangle^{+}_{\beta_c(\Delta), \Delta} > 0 \). By Proposition 6.17 (DiscontCrit) occurs at \( p = p_c(a) \) for \( a = \frac{\Delta}{1+2\kappa_0} \). Let \( t \) be the constant of Lemma 6.14. Then, by Lemma 7.2 there exists \( \delta > 0 \) such that

\[
\varphi_{\Lambda_{2n}}^{1,\delta} [\mathcal{C}_n \mid \psi_0 = 1] \leq e^{t_n/2} \varphi_{\Lambda_{2n}}^0 [\mathcal{C}_n \mid \psi_0 = 1]
\]

for every \( n \geq 1 \). Note that

\[
\varphi_{\Lambda_{2n}}^0 [\mathcal{C}_n \mid \psi_0 = 1] \leq \varphi_{\Lambda_{2n}}^0 [\mathcal{C}_n] \leq C \varphi_{\Lambda_{2n}}^0 [\mathcal{C}_n] \leq Ce^{-t_n}
\]

for some constant \( C > 0 \) independent of \( n \), which implies that

\[
\varphi_{\Lambda_{2n}}^{1,\delta} [\mathcal{C}_n \mid \psi_0 = 1] \leq Ce^{-t_n/2}.
\]

By the Edwards-Sokal coupling,

\[
\mu_{\Lambda_{2n}}^{+,e} [\sigma_0 \mid \sigma_0 \neq 0] = \varphi_{\Lambda_{2n}}^{1,\delta} [0 \leftrightarrow \Lambda_{2n} \mid \psi_0 = 1] \leq \varphi_{\Lambda_{2n}}^{1,\delta} [\mathcal{C}_n \mid \psi_0 = 1] \leq Ce^{-t_n/2},
\]

where \( \varepsilon = -\frac{1}{2} \log(1 - \delta) \). Letting \( n \) go to infinity and using Lemma 7.2, we obtain \( \mu^+ [\sigma_0 \mid \sigma_0 \neq 0] = 0 \), hence \( \mu^+ [\sigma_0] = 0 \), which concludes the proof by contradiction. \( \square \)
7.3 $\Delta^+ = - \log 4$ via Lee-Yang theory

In order to prove Theorem 1.1 in two dimensions with $\Delta^+ = - \log 4$, we need the following general statement of Lee-Yang type on the complex zeros of partition functions, which works for any finite graph.

Let us first define the Blume-Capel model with a magnetic field. Given a finite graph $G = (V, E)$ and a real magnetic field $h : V \to \mathbb{R}$, we denote by $\mu_{G, \beta, h}^\Delta$ the probability measure defined on spin configurations $\sigma \in \{-1, 0, 1\}^V$ by

$$\mu_{G, \beta, h}^\Delta(\sigma) = \frac{1}{Z_{G, \beta, h}^\Delta} e^{-h \sigma \cdot \sigma}$$

where

$$Z_{G, \beta, h}^\Delta(\sigma) = -\beta \sum_{x \in V} \sigma_x \sigma_y - \Delta \sum_{x \in V} \sigma_x^2 - \sum_{x \in V} h_x \sigma_x.$$ 

and $Z_{G, \beta, h}^\Delta$ is the partition function.

One can extend the definition of $\mu_{G, \beta, h}^\Delta$ to a complex magnetic field $h : V \to \mathbb{C}$, which is in general a complex measure, as long as the partition function $Z_{G, \beta, h}^\Delta$ does not vanish. Our aim is to study the complex zeros of $Z_{G, \beta, h}^\Delta$ and, more generally, the complex zeros of partition functions of events. Given an event $\mathcal{E}$, we define $Z_{G, \beta, h}[\mathcal{E}]$ as

$$Z_{G, \beta, h}[\mathcal{E}] := \sum_{\sigma \in \mathcal{E}} \exp \left\{ \sum_{x \in V} \beta \sigma_x \sigma_y + \Delta \sum_{x \in V} \sigma_x^2 + \sum_{x \in V} h_x \sigma_x \right\}.$$

In the following lemma we study the Lee-Yang zeros of the partition function $Z_{G, h}[\sigma_A \neq 0]$ with complex magnetic field $h$. The proof is based on an adaptation of [Dun77].

**Lemma 7.4.** Let $\beta > 0$ and $\Delta \geq - \log 4$. Consider a finite graph $G = (V, E)$, and let $A \subset V$. Let $h : V \to \mathbb{C}$ be a magnetic field such that for each $x \in V$, we have $\Re(h_x) \geq |\Im(h_x)|$. Then $Z_{G, \beta, h}[\sigma_A \neq 0] \neq 0$.

**Proof.** Let $\mathcal{E} = \{\sigma_A \neq 0\}$. We first express $|Z_{G, \beta, h}[\mathcal{E}]|^2$ as a sum over pairs of configurations:

$$|Z_{G, \beta, h}[\mathcal{E}]|^2 = \sum_{\sigma, \sigma' \in \mathcal{E}} \exp \left\{ \beta \sum_{x \in E} (\sigma_x \sigma_y + \sigma_x' \sigma_y') + \Delta \sum_{x \in V} (\sigma_x^2 + \sigma_x'^2) + \sum_{x \in V} (h_x \sigma_x + h_x' \sigma_x') \right\}.$$

We now express our duplicated system as follows. For each vertex $x$ we define $\nu_x := e^{-i\pi/4}, (\sigma_x + i\sigma_x') \in \mathbb{C}$. We then have

$$\sigma_x \sigma_y + \sigma_x' \sigma_y' = \frac{(\sigma_x + i\sigma_x')(\sigma_y - i\sigma_y') + (\sigma_x - i\sigma_x')(\sigma_y + i\sigma_y')}{2} = \frac{\nu_x \overline{\nu_y} + \overline{\nu_x} \nu_y}{2},$$

$$\sigma_x^2 + \sigma_x'^2 = |\nu_x|^2,$$

and

$$h_x \sigma_x + \overline{h_x} \sigma_x' = \Re(h_x)(\sigma_x + \sigma_x') + i \Im(h_x)(\sigma_x - \sigma_x') = \Re(h_x) \frac{\nu_x + \overline{\nu_x}}{\sqrt{2}} + \Im(h_x) \frac{\overline{\nu_x} - \nu_x}{\sqrt{2}},$$

$$= \nu_x \frac{\Re(h_x) - \Im(h_x)}{\sqrt{2}} + \overline{\nu_x} \frac{\Re(h_x) + \Im(h_x)}{\sqrt{2}}.$$
Putting this together we obtain

\[
\left| Z_{G, \beta, \Delta}^h [\mathcal{E}] \right|^2 = \sum_{\nu \in \mathcal{S}_{BC}^A \times \mathcal{S}_{\text{Ising}}^A} \left( \prod_{xy \in E} \exp \left\{ \beta \nu_x \overline{\nu_y} / 2 \cdot \exp \left\{ \beta \nu_x \nu_y / 2 \right\} \right\} \cdot \left( \prod_{x \in V} e^{\Delta \nu_x / 2} \right) \cdot \left( \prod_{x \in V} \nu_x \cdot \left\{ \text{Re} (h_x) - \text{Im} (h_x) / \sqrt{2} \right\} \cdot \exp \left\{ \nu_x \cdot \left( \frac{\text{Re}(h_x) + \text{Im}(h_x)}{\sqrt{2}} \right) \right\} \right) ,
\]

where the sets \( \mathcal{S}_{\text{Ising}} \) and \( \mathcal{S}_{BC} \) are given by

\[
\mathcal{S}_{\text{Ising}} := \sqrt{2} \cdot \{ 1, i, -1, -i \}, \quad \mathcal{S}_{BC} := \frac{1}{\sqrt{2}} \cdot \{ 0, 2i, -2i, -2, 1 + i, -1 + i, -1 - i, 1 - i \}.
\]

Now, since both \( \text{Re} (h_x) \pm \text{Im} (h_x) \) are non-negative, when we expand all exponentials, except for \( e^{\Delta |\nu_x|^2} \), as \( e^x = \sum s^k / k! \), and exchange summations, we get

\[
\left| Z_{G, \beta, \Delta}^h [\mathcal{E}] \right|^2 = \sum_{n, n' : V \to \mathbb{Z}_{\geq 0}} f(n, n') \cdot \prod_{x \in V} \left( \sum_{\nu_x} \nu_x^n \nu_x^{n'_x} e^{\Delta |\nu_x|^2} \right) ,
\]

where \( \nu_x \) ranges over \( \mathcal{S}_{\text{Ising}} \) for \( x \in A \) and over \( \mathcal{S}_{BC} \) otherwise, and all factors \( f(n, n') \) are non-negative

\[
\sum_{\nu_x} \nu_x^n \nu_x^{n'_x} e^{\Delta |\nu_x|^2} = 4 \cdot \sqrt{2^{n_x + n'_x}} \cdot e^{2\Delta} 1 \{ |n_x - n'_x| \leq 3 \} ,
\]

wheras for \( x \in V \setminus A \) we have

\[
\sum_{\nu_x} \nu_x^n \nu_x^{n'_x} e^{\Delta |\nu_x|^2} = 1 \{ n_x = 0, n'_x = 0 \} + 1 \{ |n_x - n'_x| \leq 3 \} \cdot \left[ 4 \cdot \sqrt{2^{n_x + n'_x}} \cdot e^{2\Delta} + 4 e^{\Delta} \cdot (-1)^{n_x - n'_x} \right] .
\]

Since \( n_x - n'_x = 4 \mod 8 \) implies \( n_x + n'_x \geq 4 \), all these sums are positive if \( \Delta \geq - \log 4 \), except when \( n_x + n'_x = 4 \), in which case the sum is equal to \( 0 \). This means that all summands on the right hand side of (7.2) are non-negative. Since \( f(0, 0) = 1 \), the contribution from \( \{ n, n' \} = \{ 0, 0 \} \) is strictly positive, hence \( Z_{G, \beta, \Delta}^h [\mathcal{E}] \neq 0 \).

In what follows, we condition on \( \sigma_0 \neq 0 \) and show that this convergence is true for every \( \Delta \geq - \log 4 \), using the above lemma. For that we use the monotonicity of the conditional measure in \( \varepsilon \) and \( G \), which follows the GKS inequality and the FKG inequality.

**Proposition 7.5.** Let \( \beta > 0 \) and \( \Delta \geq - \log 4 \). Then (H) is satisfied, i.e.

\[
\lim_{n \to \infty} \mu_{\Lambda_n, \beta, \Delta}^{+, \varepsilon} [\sigma_0 \mid \sigma_0 \neq 0] = \mu_{\beta, \Delta}^+ [\sigma_0 \mid \sigma_0 \neq 0] , \quad \forall \varepsilon > 0 .
\]

**Proof.** Note that for every \( \varepsilon \geq \beta \), \( \mu_{\Lambda_n}^{+, \varepsilon} [\cdot \mid \sigma_0 \neq 0] \) stochastically dominates \( \mu_{\Lambda_n}^+ [\cdot \mid \sigma_0 \neq 0] \). On the other hand, the restriction of \( \mu_{\Lambda_n}^{+, \varepsilon} [\cdot \mid \sigma_0 \neq 0] \) on \( \Lambda_{n-1} \) is stochastically dominated by \( \mu_{\Lambda_{n-1}}^+ [\cdot \mid \sigma_0 \neq 0] \). This implies that

\[
\lim_{n \to \infty} \mu_{\Lambda_n, \beta, \Delta}^{+, \varepsilon} [\sigma_0 \mid \sigma_0 \neq 0] = \mu_{\beta, \Delta}^+ [\sigma_0 \mid \sigma_0 \neq 0]
\]

for every \( \varepsilon \geq \beta \). Hence it suffices to show that \( \mu_{\Lambda_n, \beta, \Delta}^{+, \varepsilon} [\sigma_0 \mid \sigma_0 \neq 0] \) converges to an analytic function of \( \varepsilon > 0 \).
Note that
\[
\mu_{\Lambda_n,\beta,\Delta}^{+,\varepsilon}[\sigma_0 \mid \sigma_0 \neq 0] = \frac{Z_{\Lambda_n,\beta,\Delta}^{+,\varepsilon}[\sigma_0 = 1] - Z_{\Lambda_n,\beta,\Delta}^{+,\varepsilon}[\sigma = -1]}{Z_{\Lambda_n,\beta,\Delta}^{+,\varepsilon}[\sigma_0 = 1] + Z_{\Lambda_n,\beta,\Delta}^{+,\varepsilon}[\sigma = -1]} = \frac{1 - W_{\Lambda_n,\beta,\Delta}^{\varepsilon}}{1 + W_{\Lambda_n,\beta,\Delta}^{\varepsilon}},
\]
where
\[
W_{\Lambda_n,\beta,\Delta}^{\varepsilon} = \frac{Z_{\Lambda_n,\beta,\Delta}^{+,\varepsilon}[\sigma_0 = -1]}{Z_{\Lambda_n,\beta,\Delta}^{+,\varepsilon}[\sigma_0 = 1]}.
\]
For \(\varepsilon \geq \beta\), \(W_{\Lambda_n,\beta,\Delta}^{\varepsilon}\) converges to a constant, i.e.
\[
\lim_{n \to \infty} W_{\Lambda_n,\beta,\Delta}^{\varepsilon} = \frac{1 - \mu_{\beta,\Delta}^{\varepsilon}[\sigma_0 \mid \sigma_0 \neq 0]}{1 + \mu_{\beta,\Delta}^{\varepsilon}[\sigma_0 \mid \sigma_0 \neq 0]}.
\]

We wish to show that \(W_{\Lambda_n,\beta,\Delta}^{\varepsilon}\) remains bounded for \(\varepsilon\) in a complex neighbourhood of \((0, \infty)\). For \(h, \eta\) with \(\text{Re}(h) > |\text{Im}(h)|\) and \(\text{Re}(\eta) > |\text{Im}(\eta)|\), let \(h : \Lambda_n \to \mathbb{C}\) be the function which is defined by
\[
h_x = \eta 1_{x=0} + \sum_{y \in \mathbb{Z}^d \setminus \Lambda_n} h 1_{y \neq x}.
\]
Then we can write
\[
Z_{\Lambda_n,\beta,\Delta}^{h}[\sigma_0 \neq 0] = e^{\eta} Z_{\Lambda_n,\beta,\Delta}^{+,h}[\sigma_0 = 1] + e^{-\eta} Z_{\Lambda_n,\beta,\Delta}^{+,h}[\sigma_0 = -1]
\]
Since \(Z_{\Lambda_n,\beta,\Delta}^{h}[\sigma_0 \neq 0] \neq 0\), it follows that \(W_{\Lambda_n,\beta,\Delta}^{h} \neq -e^{2\eta}\). Note that for any \(z \in \mathbb{C}\) with \(|z| > e^{2\pi}\) there is some \(\eta\) with \(\text{Re}(\eta) > |\text{Im}(\eta)|\) such that \(z = -e^{2\eta}\), hence \(|W_{\Lambda_n,\beta,\Delta}^{h}| \leq e^{2\pi}\). It follows from (7.3) and Vitali’s theorem that
\[
\lim_{n \to \infty} W_{\Lambda_n,\beta,\Delta}^{h} = \frac{1 - \mu_{\beta,\Delta}^{+}[\sigma_0 \mid \sigma_0 \neq 0]}{1 + \mu_{\beta,\Delta}^{+}[\sigma_0 \mid \sigma_0 \neq 0]}
\]
for every \(h \in \mathbb{C}\) with \(\text{Re}(h) > |\text{Im}(h)|\). The desired assertion follows readily from (7.3) by taking the limit as \(n\) tends to infinity. \(\square\)

**Proof of Theorem** 1.1 for \(\Delta^+ = -\log 4\). This is a direct corollary of Propositions 7.3 and 7.5. \(\square\)

**Remark 7.6.** We expect the overall strategy above to extend to higher dimensions. Given that the tricritical point of the mean-field Blume-Capel model is \(\Delta = -\log 4\), see [Liu05], we expect that for any \(\Delta < -\log 4\) there is a sequence of graphs whose Lee-Yang zeroes have an accumulation point on the positive real line.

## 8 Subcritical sharpness for dilute random cluster on \(\mathbb{Z}^d\)

### 8.1 Weakly monotonic measures and the OSSS inequality

In general, the dilute random cluster measure is not monotonic, as was already observed in [GG06, p.12]. In this section, we show that the dilute random cluster measure satisfies a weaker notion of monotonicity, once we restrict to certain types of boundary conditions. This allows us to prove an OSSS inequality for the dilute random cluster measure.
Let $\Lambda \subset \mathbb{Z}^d$ be a finite set of vertices, and let $n = |\Lambda| + |E_\Lambda|$. We define $D = \Lambda \cup E_\Lambda$, and we call it a domain. We also define $S_\Lambda$ to be the set of sequences $(d_1, d_2, \ldots, d_k), k \leq n$, such that each element of $D$ appears at most once in $(d_1, d_2, \ldots, d_k)$, and for every edge $e$ appearing in $(d_1, d_2, \ldots, d_k)$, all the endpoints of $e$ lying in $\Lambda$ appear in $(d_1, d_2, \ldots, d_k)$, and they all precede $e$ (if $e \in \partial E_\Lambda$, then only one endpoint lies in $\Lambda$). We write $U_\Lambda$ for the corresponding set of unordered objects, i.e. $\{d_1, \ldots, d_k\}$ belongs to $U_\Lambda$ whenever each element of $D$ appears at most once, and for every edge $e \in \{d_1, \ldots, d_k\}$, all the endpoints of $e$ lying in $\Lambda$ belong to $\{d_1, \ldots, d_k\}$.

A decision tree is a pair $T = ((d_1, \phi_1), \ldots, (d_k, \phi_k), \eta(d_1, \ldots, d_k))$, where $d_i \in D$, and for each $t > 1$, the function $\phi_t$ takes a pair $((d_1, \ldots, d_{t-1}), \eta(d_1, \ldots, d_{t-1}))$ as an input, where $d_i \in D$ and $\eta \in \{0, 1\}^D$, and returns an element $d_t \in D \setminus \{d_1, \ldots, d_{t-1}\}$. For a comprehensive introduction to decision trees, we refer the reader to [O'D]. We call a decision tree admissible if $d_1 \in \Lambda$ and for every $t > 1$, if $(d_1, \ldots, d_{t-1}) \in S_\Lambda$, then $(d_1, \ldots, d_t) \in S_\Lambda$. In other words, an admissible decision tree starts always from a vertex, and queries the state of an edge $e$ only if its endpoints lying in $\Lambda$ have been queried at previous steps.

Our result applies beyond the dilute random cluster measure to measures satisfying the following property. We call a measure $\mu$ on $\{0, 1\}^D$ weakly monotonic if for every $\{d_1, \ldots, d_k\} \in U_\Lambda$ and every $\eta^1, \eta^2 \in \{0, 1\}^{\{d_1, \ldots, d_k\}}$ such that $\eta^1 \leq \eta^2$ pointwise, $\mu[\eta_0 = 1 | \eta^1] \leq \mu[\eta_0 = 1 | \eta^2]$ for every $\eta_0 \in D$.

For an $n$-tuple $d[n] = (d_1, \ldots, d_n)$ and $t \leq n$, write $d[t] = (d_1, \ldots, d_t)$ and $\eta[d[t]] = (\eta_1, \ldots, \eta_d)$. Let $T$ be an admissible decision tree and let $f : \{0, 1\}^D \rightarrow \mathbb{R}$. Given a pair $(d, \eta)$ produced by $T$, we define $\tau_f(\eta) = \tau_f,\eta(\eta) := \min\{t \geq 1 : \forall \eta^t \in \{0, 1\}^D, \eta[d[t]] = \eta[d[t]] \implies f(\eta) = f(\eta^t)\}$.

We can now state the main technical result of this section.

**Theorem 8.1** (OSSS inequality for weakly monotonic measures). Let $\Lambda \subset \mathbb{Z}^d$ be a finite set of vertices, and let $f : \{0, 1\}^D \rightarrow [0, 1]$ be an increasing function. For any weakly monotonic measure $\mu$ on $\Lambda$ and any admissible decision tree $T$, $\begin{align*} \text{Var}_\mu(f) \leq \sum_{d_0 \in D} \delta_{d_0}(f, T) \text{Cov}_\mu(f, \eta_{d_0}), \end{align*}$

where $\delta_{d_0}(f, T) := \mu[\exists t \leq t_f(\eta) : d_t = d_0]$ is the revealment (of $f$) for the decision tree $T$. 

**Proof.** The proof is the same as that of Theorem 1.1 in [DCRT].

The following lemma, which is analogous to [DCRT] Lemma 3.2, can be viewed as a sharp threshold result for the event $\{0 \leftarrow \partial \Lambda_n\}$.

**Lemma 8.2.** Let $\Lambda \subset \mathbb{Z}^d$ be a finite set containing 0. For every weakly monotonic measure $\mu$ on $\Lambda$ and every $n \geq 1$, we have $\begin{align*} \sum_{d \in D} \text{Cov}_\mu(1_{0 \leftarrow \partial \Lambda_n}, \eta_d) \geq \frac{n}{4dQ_n} \mu[0 \leftarrow \partial \Lambda_n] (1 - \mu[0 \leftarrow \partial \Lambda_n]), \end{align*}$

where $Q_n := \max_{x \in \Lambda_n} \sum_{k=0}^{n-1} \mu[x \leftarrow \partial \Lambda_k(x)]$. 

**Proof.** For any $k \in \{1, 2, \ldots, n\}$, we wish to construct an admissible decision tree $T = T(k)$ determining $1_{0 \leftarrow \partial \Lambda_n}$ such that for every $u \in \Lambda_n$, $\begin{align*} \delta_u(T) \leq 1_{u \in \partial \Lambda_k} + 1_{u \not\in \partial \Lambda_k} \sum_{x \in \Lambda_n, x \sim u} \mu[x \leftarrow \partial \Lambda_k] \quad (8.1) \end{align*}$
and for each edge $e = uv$, \[
    \delta_e(T) \leq \mu[u \leftrightarrow \partial \Lambda_k] + \mu[v \leftrightarrow \partial \Lambda_k].
\] (8.2)

Note that for $u \in \Lambda_n$, \[
    \sum_{k=1}^{n} \mu[u \leftrightarrow \partial \Lambda_k] \leq \sum_{k=1}^{n} \mu[u \leftrightarrow \partial \Lambda_{|u|}](u) \leq 2Q_n
\]
from which the desired assertion follows by applying Theorem $8.1$ for each $T(k)$ with $f = 1_{0 \leftrightarrow \partial \Lambda_n}$ and then summing over $k \in \{1, 2, \ldots, n\}$.

The decision tree $T$ is defined at follows. We first fix an ordering of the vertices and the edges. Then we explore the state of each vertex of $\partial \Lambda_k$, one at a time, according to the ordering, and once we have explored the state of all vertices in $\partial \Lambda_k$, we explore the state of the edges with both endpoints being open and in $\partial \Lambda_k$. Let $F_0$ be the set of vertices explored thus far, namely $\partial \Lambda_k$, and $V_0$ the set of open vertices in $F_0$. Next, we explore the state of all the vertices $u \in \Lambda_n \setminus F_0$ adjacent to $F_0$, one at a time, according to the ordering, and if $u$ is open, then we explore the state of the edges connecting $u$ to $V_0$, according to the ordering. We let $F_1$ be the union of $F_0$ with the set of vertices $u \in \Lambda_n \setminus F_0$ adjacent to $V_0$, and $E_1$ be the set of open edges explored thus far.

We then proceed inductively. Assuming we have defined $F_t$ and $E_t$ for some $t \geq 1$, we then consider the first vertex $u \in \Lambda_n \setminus F_t$ incident to $E_t$, and if $u$ is open, then we explore the state of the edges connecting $u$ to $E_t$, according to the ordering. We let $F_{t+1} = F_t \cup \{u\}$, and $E_{t+1}$ be the union of $E_t$ with the open edges between $u$ and $E_t$. As long as there is an open vertex incident to $E_{t+1}$, we keep exploring the cluster of $\partial \Lambda_k$. Otherwise, the exploration process stops, in which case the event $\{0 \leftrightarrow \partial \Lambda_n\}$ has been determined, and we can define our decision tree after that point arbitrarily.

It is clear from the construction that $T$ is an admissible decision tree and that (8.2) and (8.1) are satisfied. This concludes the proof.

\section{A generalisation of the dilute random cluster}

In this section, we introduce the generalized dilute random cluster measure, which depends on three parameters $p, a, r$ and is defined as the standard dilute random cluster measure, except that we are not requiring $r = \sqrt{1-p}$. We show that for certain values of $r$, the generalized dilute random cluster measure is weakly monotonic, which together with Theorem $8.1$ will be used in the proof of subcritical sharpness for the two parameter dilute random cluster measure.

Given $a \in (0, 1)$, $p \in (0, 1)$, and $r > 0$, we let $\psi^\xi_{\Lambda,p,a,r}$ denote the measure on $\Lambda$ with boundary condition $\xi$, that is, \[
    \psi^\xi_{\Lambda,p,a,r} [\theta] = \frac{1_{\theta \in \Theta^\xi_{\Lambda,p,a,r}}[E_{\psi,\Lambda}] 2^{k(\theta, \Lambda)}}{Z^\xi_{\Lambda,p,a,r}} \prod_{x \in V} \left( \frac{a}{1-a} \right) \psi_\xi \prod_{e \in E_{\psi,\Lambda}} \left( \frac{p}{1-p} \right)^{\omega_e},
\]
where $Z^\xi_{\Lambda,p,a,r}$ is the normalisation constant.

**Proposition 8.3.** Let $a, p \in (0, 1)$ and $r \geq \frac{2(1-p)}{2-p}$. Then the measure $\psi^\xi_{\Lambda,p,a,r} \rho \Xi$ is weakly monotonic.

**Proof.** Let $\{d_1, \ldots, d_k\} \in U_\Lambda$ and let $\eta^1, \eta^2 \in \{0, 1\}^{d_1, \ldots, d_k}$ be such that $\eta^1 \leq \eta^2$. We wish to show that the vertex marginal $\Psi_2 := \Psi^\xi_{\Lambda,p,a,r}[\cdot | \eta^2]$ stochastically dominates the vertex marginal $\Psi_1 := \Psi^\xi_{\Lambda,p,a,r}[\cdot | \eta^1]$. Then the desired assertion follows as in the proof of Proposition $2.17$.  

34
It suffices to verify (2.5) and (2.6). To this end, let $\Lambda' = \Lambda \setminus \{d_1, d_2, \ldots, d_k\}$, and let $x \in \Lambda'$. Then, for every configuration $\psi \in \{0, 1\}^\Lambda$,

$$\frac{\Psi_2[\psi(x)]}{\Psi_2[\psi(x)]} = r N_x^2 \frac{a_z}{1 - a_z} Z_{\Lambda, \psi(x)}^{\{A, \xi\}_2}$$

and

$$\frac{\Psi_1[\psi(x)]}{\Psi_1[\psi(x)]} = r N_x^2 \frac{1}{1 - a_z} Z_{\Lambda, \psi(x)}^{\{A, \xi\}_1},$$

where $N_x^2$ is the number of neighbours of $x$ in $V_\psi \cup (V_{\xi \cup \psi} \setminus \Lambda')$. Define $A(x)$ to be the event that all edges incident to $x$ are closed, and note that

$$Z_{\Lambda, \psi(x)}^{\{A, \xi\}_2} \equiv 1 \quad \text{and} \quad Z_{\Lambda, \psi(x)}^{\{A, \xi\}_1} \equiv 1.$$

It remains to show that

$$\phi_{\Lambda, \psi}^{A, \xi \cup \psi} [A(x)] \leq r N_x^2 \phi_{\Lambda, \psi}^{A, \xi \cup \psi} [A(x)].$$

For this, recall that by the finite energy property for the random cluster model [Gri04 Theorem 3.1],

$$\phi_{\Lambda, \psi}^{A, \xi \cup \psi} [A(x)] \leq e_p N_x^2 \phi_{\Lambda, \psi}^{A, \xi \cup \psi} [A(x)],$$

where $e_p = \max \left\{ 1 - p, 1 - \frac{p}{p+2(1-p)} \right\} = \frac{2(1-p)}{2-p}$. Since $r \geq \frac{2(1-p)}{2-p}$, it follows that (8.3) holds. Thus we have verified (2.6).

We wish to verify (2.6) for $\Psi_1$. It is not hard to see that

$$\frac{\Psi_1[\psi(x, y)]}{\Psi_1[\psi(x, y)]} = \phi_{\Lambda, \psi}^{A, \xi \cup \psi} [A(x)] \quad \text{and} \quad \frac{\Psi_1[\psi(y, x)]}{\Psi_1[\psi(y, x)]} = \phi_{\Lambda, \psi}^{A, \xi \cup \psi} [A(y)].$$

As above, we have

$$\frac{Z_{\Lambda, \psi(x, y)}^{\{A, \xi\}_1}}{Z_{\Lambda, \psi(x, y)}^{\{A, \xi\}_1}} = \frac{\phi_{\Lambda, \psi}^{A, \xi \cup \psi} [A(x)]}{\phi_{\Lambda, \psi}^{A, \xi \cup \psi} [A(y)]}.$$

Let $B(x, y)$ be the event that all edges $xu \in \{x\}$ are closed. By the FKG inequality

$$\phi_{\Lambda, \psi}^{A, \xi \cup \psi} [A(y)] \leq \phi_{\Lambda, \psi}^{A, \xi \cup \psi} [A(y) | B(x, y)].$$

Using the finite energy property we see that by closing $xy$ we get

$$\phi_{\Lambda, \psi}^{A, \xi \cup \psi} [A(y) | B(x, y)] \leq \left( \frac{2(1-p)}{2-p} \right) \phi_{\Lambda, \psi}^{A, \xi \cup \psi} [A(y)],$$

which completes the proof.

The following result is a standard calculation similar to the case of the classical random cluster model [Gri04 Theorem 3.12].

**Proposition 8.4.** Let $\alpha, \rho \in (0, 1)$ and $r > 0$. Let also $\Lambda \subset \mathbb{Z}^d$ finite. For every event $A$ that is $\Lambda$-measurable, we have

$$\frac{d\phi^\xi_{\Lambda, \rho, \rho, \alpha} [A]}{d\rho} = \frac{1}{p(1-p)} \sum_{e \in E_\Lambda} \text{Cov}(A, \eta_e)$$

and

$$\frac{d\phi^\xi_{\Lambda, \rho, \rho, \alpha} [A]}{da} = \frac{1}{a(1-a)} \sum_{x \in \Lambda} \text{Cov}(A, \eta_x).$$
8.3 Proof of subcritical sharpness

The main result of this section is the following.

**Theorem 8.5.** Let $a \in (0, 1)$. For every $p < p_c(a)$ there exists $c = c(p, a) > 0$ such that for every $n \geq 0$,

$$\varphi_{\Lambda_n, p, a}^1[0 \leftrightarrow \partial\Lambda_n] \leq e^{-cn}.$$

Before proving this result, we use it to prove Theorem 1.4.

**Proof of Theorem 1.4 assuming Theorem 8.5.** For $x \in \mathbb{Z}^d$ with $\|x\|_{\infty} = n$, we have

$$\langle \sigma_0 \sigma_x \rangle_{p, \Delta}^+ \leq \varphi_{p, a}^1[0 \leftrightarrow \partial\Lambda_n] \leq \varphi_{\Lambda_n, p, a}^1[0 \leftrightarrow \partial\Lambda_n],$$

by the Edwards-Sokal coupling and (MON). The assertion follows from Theorem 8.5. \qed

We are now ready to prove Theorem 8.5.

**Proof of Theorem 8.5.** Let $a_0 \in (0, 1)$ and consider $p_0 < p_c(a_0)$. We prove the assertion for the pair $(p_0, a_0)$.

Define $c_0 = \frac{p_0(1-a_0)}{(1-p_0)p_0}$. For $n \geq 1$, $p \in (0, 1)$, $a(p) = \frac{p}{p+r_0(1-p)}$, and $r_0 = \sqrt{1 - p_0}$, define

$$\mu_n := \varphi_{\Lambda_2n, p, a(p), r_0}^1, \quad \theta_k(p) := \mu_k[0 \leftrightarrow \partial\Lambda_k], \quad \text{and} \quad S_n := \sum_{k=0}^{n-1} \theta_k.$$  

Note that there exists $\varepsilon > 0$ such that $r_0 \geq \frac{2(1-p)}{2-p}$ for every $p \geq p_0 - \varepsilon$, hence $\mu_n$ is weakly monotonic for every $p \geq p_0 - \varepsilon$ by Proposition 8.3. Note also that $a(p_0) = a_0$ and

$$a'(p) = \frac{a(p)(1-a(p))}{p(1-p)},$$

hence

$$\theta_k'(p) = \frac{1}{p(1-p)} \sum_{x \in \partial\Lambda_k} \text{Cov}(1_{0 \leftrightarrow \partial\Lambda_k}, \omega_x) + \frac{a'(p)}{a(p)(1-a(p))} \sum_{x \in \partial\Lambda} \text{Cov}(1_{0 \leftrightarrow \partial\Lambda_k}, \psi_x).$$

Now, the comparison between boundary conditions together with the fact that, for $k \leq n/2$, $\Lambda_{2k}(x) \subset \Lambda_{2n}$, imply that for $x \in \Lambda_n$,

$$\sum_{k=1}^{n-1} \mu_n[x \leftrightarrow \partial\Lambda_k(x)] \leq 2 \sum_{k \leq n/2} \mu_n[x \leftrightarrow \partial\Lambda_k(x)] \leq 2 \sum_{k \leq n/2} \mu_k[0 \leftrightarrow \partial\Lambda_k] \leq 2S_n.$$  

By Lemma 8.2 there exists a constant $t > 0$ such that

$$\theta'_n(p) \geq t \frac{n}{S_n} \theta_n(p)$$

for $p \leq 1 - \varepsilon$, where we used that $1 - \theta_n(p) \geq 1 - \theta_1(p)$, and the latter is bounded away from 0. Using Lemma 3.1 in [DCRT19] for $f_n = \theta_n/t$, we obtain the existence of $p_1 > 0$ such that the following holds: for every $p_0 - \varepsilon \leq p < p_1$ there exists $c > 0$ such that

$$\theta_n(p) \leq e^{-cn} \text{ for every } n \geq 0,$$
and for every \( p_1 < p \leq p_0 - \varepsilon \),
\[
\varphi^1_{p,a(p),r_0} [0 \leftrightarrow \infty] \geq t(p - p_1).
\]

Note that \( \mu_n[0 \leftrightarrow \partial \Lambda_n] = \theta_n(p) \), and \( \varphi^1_{\Lambda_n, p,a(p), r_0} [0 \leftrightarrow \partial \Lambda_n] \) is decreasing in \( n \), hence to conclude it suffices to show that in the above one could take \( p_1 > p_0 \). To this end, consider some \( p_0 < p' < p_c(a_0) \). We wish to show that \( \varphi^1_{p', a} \) stochastically dominates \( \varphi^1_{p''', a(p''), r_0} \) for some \( p'' > p_0 \). This can be easily done by checking conditional probabilities of the form \( \varphi^1_{\Lambda', p,a,r} [A] \) as in the proof of Proposition [3.8] noting that \( \varphi^1_{\Lambda', p,a,r} [A] \) is strictly increasing in \( p \) for \( r = \sqrt{1 - p} \) and fixed \( a \), and continuous in \( a, p \) for \( r = r_0 \). It follows that \( \varphi^1_{p'', a(p''), r_0} [0 \leftrightarrow \infty] = 0 \), hence \( p_1 \geq p'' > p_0 \), as desired.

\[\square\]

9 Further analysis of critical behaviour in dimension 2

In this section, we prove Theorem [1.3] The statements (OffCrit), (DiscontCrit), (ContCrit), and (TriCrit) are almost direct consequences of the quadrichotomy in Proposition [6.1]. We therefore first develop the tools to establish (Perc) before writing the proof. Let us introduce the following notions. We say that the \( \{0, -\} \) spins percolate if with positive probability there is an infinite connected component \( \mathcal{C} \) such that \( \sigma_x \in \{0, -\} \) for every \( x \in \mathcal{C} \). Define
\[
\mathcal{L}_c = \{ (\Delta, \beta_c(\Delta)) \in \mathcal{L} : \{0, -\} \text{ spins do not percolate under } \mu^0 \}.
\]

9.1 Werner’s argument for continuity

We start by showing the first direction of (Perc). Let us first recall some standard facts. Consider a finite subset \( A \) of \( \mathbb{Z}^2 \), which induces a connected graph. Let \( C \) be the set of vertices \( x \in A \) for which there exists an adjacent vertex \( y \in \mathbb{Z}^d \setminus A \) such that \( y \) can be connected to infinity in \( \mathbb{Z}^d \setminus A \). In general, \( C \) is not connected in \( \mathbb{Z}^d \), but it is a \( * \)-connected circuit. The latter means that \( C \) can be represented as a walk \( w = (w(0), w(1), \ldots, w(n)) \) such that \( \|w(i) - w(i + 1)\|_\infty = 1 \), \( w(0) = w(n) \), and otherwise \( w(i) \neq w(j) \) for \( i \neq j \). Given a \( * \)-connected circuit \( C \) surrounding 0, we write \( C^{\text{int}} \) for the union of \( C \) with the connected component of \( \mathbb{Z}^d \setminus C \) that contains 0.

We now prove the following fact, which is a modification of Werner’s argument for the planar Ising model [Wer09].

**Proposition 9.1.** For every \( (\Delta, \beta_c(\Delta)) \in \mathcal{L}_c \), \( \langle \sigma_0 \rangle_{\beta_c(\Delta), \Delta}^+ = 0 \).

**Proof.** We fix some \( (\Delta, \beta_c(\Delta)) \in \mathcal{L}_c \) and drop it from the notation. Let \( A_n \) be the event that there is no path \( II \) connecting 0 to \( \partial \Lambda_n \) such that \( \sigma_x \in \{0, -\} \) for every vertex \( x \in II \). On this event, there is a \( * \)-connected circuit of \( \{+\} \) spins surrounding 0 which is contained in \( \Lambda_n \). Let \( C_n \) denote the outermost such circuit in \( \Lambda_n \). Then, for every \( n \geq 1 \)
\[
\langle \sigma_0 \rangle^0 = \langle \sigma_0 \rangle_{\sigma \not\in A_n}^0 + \langle \sigma_0 \rangle_{\sigma \in A_n}^0 = \langle \sigma_0 \rangle_{\sigma \not\in A_n}^0 + \sum_C \langle \sigma_0 \rangle_{C_{\text{int}} = C}^0
\]
\[
= \langle \sigma_0 \rangle_{\sigma \not\in A_n}^0 + \sum_C \langle \sigma_0 \rangle_{C_{\text{int}}}^+ \mu^0[C = C] \geq \langle \sigma_0 \rangle_{\sigma \not\in A_n}^0 + \langle \sigma_0 \rangle^+ \mu^0[A_n].
\]

In the second line we have partitioned the event \( A_n \) over all possibilities for \( C_n \). In the third line we have used the spatial Markov property and the fact that the event \( \{C_n = C\} \) is measurable with respect to the configuration on \( C \cup (\mathbb{Z}^d \setminus C_{\text{int}}) \). In the final line we have used the monotonicity of the \( + \) boundary conditions in the domain.

Taking limits as \( n \to \infty \) and using that \( \mu^0[A_n] \) converges to 1, we obtain that \( 0 = \langle \sigma_0 \rangle^0 \geq \langle \sigma_0 \rangle^+ \).

Since \( \langle \sigma_0 \rangle^+ \geq 0 \), the desired assertion follows. \[\square\]
9.2 Infinite cluster and crossing probabilities

Our aim now is to relate the existence of an infinite \( \{0, -\} \) cluster with crossing probabilities, from which the desired result will follow. In order to do so, we first need to show that \( \mu^0 \) is mixing at the critical line. We remark that \( \mu^0 \) is not mixing when \( \varphi^0[0 \leftarrow \infty] > 0 \), since there is probability 1/2 that there is an infinite cluster of \( \{+\} \) spins, and probability 1/2 that there is an infinite cluster of \( \{-\} \) spins. For this reason, our proof utilises that \( \varphi^0[0 \leftarrow \infty] = 0 \) at the critical line.

Let us introduce the following definition. For every \( x \in \mathbb{Z}^2 \), we define \( \mathcal{C}_x = \{x\} \cup \{y \in \mathbb{Z}^2 : \text{\( x \) is connected to \( y \) by an open path in \( \omega \)} \} \) to be the cluster of \( x \) in a configuration \((\psi, \omega)\). Note that \( x \in \mathcal{C}_x \) even if \( \psi_x = 0 \).

**Lemma 9.2.** Let \( \beta > 0, \Delta \in \mathbb{R} \) and let \( p = 1 - e^{-2\beta}, \ a = \frac{2e^\Delta}{1+2e^\Delta} \). Assume that \( \varphi^0_{p,\alpha[0 \leftarrow \infty] = 0} \). Then \( \mu^0_{\beta,\Delta} \) is mixing.

**Proof.** We fix \( \beta, \Delta, p \) and \( a \) and drop them from the notation. It suffices to show that
\[
\lim_{x \to \infty} \mu^0[A, \tau_x B] = \mu^0[A] \mu^0[B]
\]
for all events \( A, B \) depending on the spins \( \sigma_x \), \( x \in \Lambda_k \) for some \( k \geq 1 \).

To this end, let us define \( \mathcal{C}_x = \bigcup_{x \in \Lambda_k} \mathcal{C}_x \). Since \( \varphi^0(o \leftarrow \infty) = 0 \), \( \mathcal{C}_x \) is finite almost surely. Note that \( \mathcal{C}_x \) spans a connected subgraph of \( \mathbb{Z}^2 \), hence almost surely there is a dual circuit surrounding \( \Lambda_k \).

Fix \( \varepsilon > 0 \) and let \( n > k \) be such that \( \varphi^0[C_{k,n}] \geq 1 - \varepsilon \), where \( C_{k,n} \) denotes the event that there is a dual circuit surrounding \( \Lambda_k \) which is contained in \( \Lambda_n \). For every \( x \in \mathbb{Z}^2 \), we have
\[
P[A, C_{k,n}, \tau_x B, \tau_x C_{k,n}] \leq \mu^0[A, \tau_x B] \leq P[A, C_{k,n}, \tau_x B, \tau_x C_{k,n}] + 2\varepsilon.
\]
Consider some \( \|x\|_{\infty} \geq 2n + 1 \), and a pair of configurations \( \theta = (\psi, \omega), \theta' = (\psi', \omega') \) living on \( \Lambda_n \) such that \( \theta, \theta' \in C_{n,n} \). Then
\[
P[A, \tau_x B, \tau_x \theta'] = \varphi^0[\theta, \tau_x \theta'] P[A | \theta] P[B | \theta']
\]
because conditioned on \( \theta \) and \( \tau_x \theta' \), the spins on the vertices of \( \Lambda_k \) and \( \tau_x \Lambda_k \) are assigned independently, and the assignment is measurable with respect to \( \theta \) in the first case and with respect to \( \theta' \) in the second case, due to the existence of the dual circuits. By the ergodicity of \( \varphi^0, \varphi^0[\theta, \tau_x \theta'] \) converges to \( \varphi^0[\theta] \varphi^0[\theta'] \) as \( \|x\|_{\infty} \) tends to infinity, hence summing over all possible \( \theta \) and \( \theta' \) we obtain
\[
\lim_{x \to \infty} P[A, C_{k+1,n}, \tau_x B, \tau_x C_{k+1,n}] = P[A, C_{k+1,n}] P[B, C_{k+1,n}].
\]
Sending \( n \) to infinity and \( \varepsilon \) to 0, we obtain that \( P[A, C_{k+1,n}] \) converges to \( \mu^0[A] \), and \( P[B, C_{k+1,n}] \) converges to \( \mu^0[B] \). The desired assertion follows readily. \( \square \)

As a corollary of the above lemma and Propositions 6.13 and 6.15 we obtain the following.

**Corollary 9.3.** Let \( \Delta \in \mathbb{R} \). Then \( \mu^0_{\beta,\Delta} \) is mixing.

Let \( N_{0,-} \) be the number of infinite clusters of \( \{0, -\} \) spins. Since \( \mu^0 \) satisfies the finite energy property and the FKG inequality, a Burton-Keane argument implies the following.

**Lemma 9.4.** Let \( \Delta \in \mathbb{R} \) and \( \beta > 0 \). Assume that \( \mu^0_{\beta,\Delta} \) is ergodic. Then either \( \mu^0_{\beta,\Delta}[N_{0,-} = 0] = 1 \) or \( \mu^0_{\beta,\Delta}[N_{0,-} = 1] = 1 \).

Let \( H_{n,-}^0 \) be the event that there is a horizontal crossing \( \gamma \) in \( \Lambda_n \) such that \( \sigma_x \in \{0, -\} \) for every vertex \( x \in \gamma \). Define \( V_{n,-}^0 \), \( H_{n}^+ \) and \( V_{n}^+ \) analogously. The following result tells us that when \( \mu^0_{\beta,\Delta} \) is ergodic and there is a unique infinite cluster of \( \{0, -\} \) spins, crossing probabilities go to 1. It is an adaptation of Zhang’s argument and its proof is given in [DCT17] Proposition 2.1.

**Lemma 9.5.** Let \( \Delta \in \mathbb{R} \) and \( \beta > 0 \). Assume that \( \mu^0_{\beta,\Delta}[N_{0,-} = 1] = 1 \). Then
\[
\lim_{n \to \infty} \mu^0_{\beta,\Delta}[H_{n,-}^0] = 1.
\]
9.3 Proof of Theorem 1.3

Proof of Theorem 1.3 Since $\beta, \Delta$ are fixed, we drop them from the notation. Note that (TriCrit) is a standard consequence of the renormalisation inequalities of Lemma 6.5 We first establish (OffCrit) and (DiscontCrit). For $\beta < \beta_c(\Delta)$, we have $\langle \sigma_0 ; \sigma_x \rangle^+ = \langle \sigma_0 \sigma_x \rangle^+$. In this case, the assertion follows from Theorem 1.4.

Let $\beta \geq \beta_c(\Delta)$ and $\Delta \in \mathbb{R}$ be such that $\langle \sigma_0 \rangle > 0$. Recall that $C_n$ is the event that there is an open circuit in $\omega \cap (\Lambda_{2n} \setminus \Lambda_n)$. Also recall that Propositions 6.11 6.13 and Propositions 6.14 6.12 state that there exists $c > 0$ such that for every $n \geq 1$,

$$\varphi^1[\Lambda_n \leftrightarrow \infty] \leq e^{-cn} \quad \text{and} \quad \varphi^1[C_n] \geq 1 - e^{-cn}. \quad (9.1)$$

On the one hand, for every $\|x\|_\infty > 4n$ we have

$$\langle \sigma_0 \sigma_x \rangle^+ \leq (\varphi^1_{\Lambda_{2n}}[0 \leftrightarrow \partial \Lambda_{2n}])^2$$

by the Edwards-Sokal coupling and (MON). On the other hand, when each of $C_n$, $\{0 \leftrightarrow S\}$, and $\{\Lambda_n \leftrightarrow \infty\}$ happens, where $S$ denotes the exterior most circuit in $\omega \cap (\Lambda_{2n} \setminus \Lambda_n)$, 0 is connected to infinity. Hence,

$$\langle \sigma_0 \rangle = \varphi^1[0 \leftrightarrow \infty] \geq \varphi^1[C_n, 0 \leftrightarrow S, \Lambda_n \leftrightarrow \infty] \geq (1 - e^{-cn})\varphi^1[0 \leftrightarrow S, \Lambda_n] - e^{-cn} \geq (1 - e^{-cn})\varphi^1_{\Lambda_{2n}}[0 \leftrightarrow \partial \Lambda_{2n}] - e^{-cn}$$

by (9.1), (SMP) and (MON). The desired assertion for $\beta \geq \beta_c(\Delta)$ such that $\langle \sigma_0 \rangle > 0$ follows.

We now establish (ContCrit). Let $\beta = \beta_c(\Delta)$ and $\langle \sigma_0 \rangle = 0$. For the upper bound, we use Proposition 6.10. For the lower bound, if $\|x\|_\infty = 2n$, then one way for 0 to be connected to $x$ is if both $C_n$ and $x + C_n$ happen, 0 is connected to $\partial \Lambda_{2n}$, and $x$ is connected to $x + \partial \Lambda_{2n}$. Then the assertion follows from Propositions 6.1 and 6.10 and the FKG inequality.

Finally, we prove (Perc). The reverse implication follows from Proposition 9.1. For the forward implication, let us assume that $\langle \sigma_0 \rangle > 0$. Then (ContCrit) for the dilute random cluster model (i.e. in the sense of Proposition 6.1) occurs at $p = p_c(\alpha)$ by Proposition 6.17. It follows from the monotonicity in the domain and the $\pi/2$ rotational symmetry that for every $n \geq 1$, $\varphi^0[\nu^+_{n}] \geq c$. Since we have probability $1/2$ to colour a vertical crossing with $\{+\}$ spins, it follows that for every $n \geq 1$, $\varphi^0[\nu^+_{n}] \geq c/2$. Note that when $\nu^+_{n}$ happens, the event $H^n_{0, -}$ does not happen, hence

$$\varphi^0[H^n_{0, -}] \leq 1 - \varphi^0[\nu^+_{n}] \leq 1 - c/2.$$ 

It follows from Corollary 9.3 and Lemmas 9.4 and 9.5 that the $\{0, -\}$ spins do not percolate under $\mu^0$, as desired.

References

[ADC21] M. Aizenman and H. Duminil-Copin. Marginal triviality of the scaling limits of critical 4D Ising and $\lambda \phi^4_3$ models. Annals of Mathematics, 194(1), 2021.

[ADCS15] M. Aizenman, H. Duminil-Copin, and V. Sidoravicius. Random currents and continuity of Ising model’s spontaneous magnetization. Communications in Mathematical Physics, 334(2):719–742, 2015.

[Aiz82] M. Aizenman. Geometric analysis of $\varphi^4$ fields and Ising models. Parts I and II. Communications in Mathematical Physics, 86(1):1–48, 1982.
[BBS19] R. Bauerschmidt, D.C. Brydges, and G. Slade. Introduction to a renormalisation group method, volume 2242. Springer Nature, 2019.

[BLH95] H.W.J. Blote, E. Luijten, and J.R. Heringa. Ising universality in three dimensions: a Monte Carlo study. Journal of Physics A: Mathematical and General, 28(22), 1995.

[BLS20] R. Bauerschmidt, M. Lohmann, and G. Slade. Three-dimensional tricritical spins and polymers. Journal of Mathematical Physics, 61(3), 2020.

[Blu66] M. Blume. Theory of the First-Order Magnetic Phase Change in UO2. Journal of Applied Physics, 37(3), 1966.

[BP18] P. Butera and M. Pernici. The Blume–Capel model for spins S=1 and 3/2 in dimensions d=2
and 3. Physica A: Statistical Mechanics and its Applications, 507(C), 2018.

[BS89] J. Bricmont and J. Slawny. Phase transitions in systems with a finite number of dominant ground states. Journal of Statistical Physics, 54(1):89–161, 1989.

[Cap66] H.W. Capel. On the possibility of first-order phase transitions in Ising systems of triplet ions with zero-field splitting. Physica, 32(5), 1966.

[DC17] H. Duminil-Copin. Lectures on the Ising and Potts models on the hypercubic lattice. In PIMS-CRM Summer School in Probability, pages 35–161. Springer, 2017.

[DCRT19] H. Duminil-Copin, A. Raoufi, and V. Tassion. Sharp phase transition for the random-cluster and Potts models via decision trees. Annals of Mathematics, 2019.

[DCT20] H. Duminil-Copin and V. Tassion. Renormalization of crossing probabilities in the planar random-cluster model. Moscow Mathematical Journal, 20(4):711–740, 2020.

[Dun77] F. Dunlop. Zeros of partition functions via correlation inequalities. Journal of Statistical Physics, 17(4):215–228, 1977.

[EOT05] R.S. Ellis, P.T. Otto, and H. Touchette. Analysis of phase transitions in the mean-field Blume–Emery–Griffiths model. Annals of Applied Probability, 15(3), 2005.

[Frö82] J. Fröhlich. On the triviality of $\lambda\phi^4_d$ theories and the approach to the critical point in $d_{(-)}>4$ dimensions. Nuclear Physics B, 200(2):281–296, 1982.

[FSS76] J. Fröhlich, B. Simon, and T. Spencer. Infrared bounds, phase transitions and continuous symmetry breaking. Communications in Mathematical Physics, 50(1):79–95, 1976.

[FV17] S. Friedli and Y. Velenik. Statistical Mechanics of Lattice Systems: A Concrete Mathematical Introduction. Cambridge University Press, 2017.

[GG06] B.T. Graham and G.R. Grimmett. Random-cluster representation of the Blume–Capel model. Journal of Statistical Physics, 125(2):283–316, 2006.

[GJ12] J. Glimm and A. Jaffe. Quantum physics: A Functional Integral Point of View. Springer Science & Business Media, 2012.

[Gri04] G. Grimmett. The random-cluster model. In Probability on discrete structures, pages 73–123. Springer, 2004.

[LSS97] T.M. Liggett, R.H. Schonmann, and A.M. Stacey. Domination by product measures. The Annals of Probability, 25(1):71–95, 1997.
[Mus10] G. Mussardo. *Statistical Field Theory: An Introduction to Exactly Solved Models in Statistical Physics*. Oxford University Press, 2010.

[O’D14] R. O’Donnell. *Analysis of Boolean Functions*. Cambridge University Press, 2014.

[OSSS05] R. O’Donnell, M. Saks, O. Schramm, and R. A. Servedio. Every decision tree has an influential variable. In *FOCS*, pages 31–39, 2005.

[Rao20] A. Raoufi. Translation-invariant Gibbs states of the Ising model: general setting. *The Annals of Probability*, 48(2):760–777, 2020.

[RSDZ17] S. Rychkov, D. Simmons-Duffin, and B. Zan. Non-gaussianity of the critical 3d Ising model. *SciPost Physics*, 2:001, 2017.

[Str65] V. Strassen. The existence of probability measures with given marginals. *The Annals of Mathematical Statistics*, 36(2):423–439, 1965.

[SW18] H. Shen and H. Weber. Glauber dynamics of 2D Kac–Blume–Capel model and their stochastic PDE limits. *Journal of Functional Analysis*, 275(6), 2018.

[Wer09] W. Werner. *Percolation et modèle d’Ising*. Société mathématique de France Paris, 2009.