ON SEQUENCES WITHOUT GEOMETRIC PROGRESSIONS

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Abstract. An improved upper bound is obtained for the density of sequences of positive integers that contain no \( k \)-term geometric progression.

1. A problem of Rankin

Let \( k \geq 3 \) be an integer. Let \( r \neq 0, \pm 1 \) be a real number. A geometric progression of length \( k \) with common ratio \( r \) is a sequence \((a_0, a_1, a_2, \ldots, a_{k-1})\) of nonzero real numbers such that

\[
r = \frac{a_i}{a_{i-1}}
\]

for \( 1, 2, \ldots, k - 1 \). For example, \((3/4, 3/2, 3, 6)\) and \((8, 12, 18, 27)\) are geometric progressions of length 4 with common ratios 2 and 3/2, respectively. A \( k \)-geometric progression is a geometric progression of length \( k \) with common ratio \( r \) for some \( r \).

If the sequence \((a_0, a_1, a_2, \ldots, a_{k-1})\) is a \( k \)-geometric progression, then \( a_i \neq a_j \) for \( 0 \leq i < j \leq k - 1 \).

A finite or infinite set of real numbers is \( k \)-geometric progression free if the set does not contain numbers \( a_0, a_1, \ldots, a_{k-1} \) such that the sequence \((a_0, a_1, \ldots, a_{k-1})\) is a \( k \)-geometric progression. Rankin \cite{3} introduced \( k \)-geometric progression free sets, and proved that there exist infinite \( k \)-geometric progression free sets with positive asymptotic density. For example, the set \( Q \) of square-free positive integers, with asymptotic density \( \frac{\pi^2}{6} \), contains no \( k \)-term geometric progression for \( k \geq 3 \).

Let \( A \) be a set of positive integers that contains no \( k \)-term geometric progression. Brown and Gordon \cite{2} proved that the upper asymptotic density of \( A \), denoted \( d_U(A) \), has the following upper bound:

\[
d_U(A) \leq 1 - \frac{1}{2^k} - \frac{2}{5} \left( \frac{1}{5^{k-1}} - \frac{1}{6^{k-1}} \right).
\]

Riddell \cite{4} and Beiglböck, Bergelson, Hindman, and Strauss \cite{1} proved that

\[
d_U(A) \leq 1 - \frac{1}{2^k}.
\]

The purpose of this note is to improve these results.

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\footnote{1}{If \( A(n) \) denotes the number of positive integers \( a \in A \) with \( a \leq n \), then the upper asymptotic density of \( A \) is \( d_U(A) = \limsup_{n \to \infty} A(n)/n \), and the asymptotic density of \( A \) is \( d(A) = \lim_{n \to \infty} A(n)/n \), if this limit exists.}

\footnote{2}{Brown and Gordon claimed a slightly stronger result, but their proof contains an (easily corrected) error.}
2. An upper bound for sets with no \( k \)-term geometric progression

**Theorem 1.** For integers \( k \geq 3 \) and \( n \geq 2^{k-1} \), let \( \text{GP}_k(n) \) denote the set of subsets of \( \{1, 2, \ldots, n\} \) that contain no \( k \)-term geometric progression. If \( A \in \text{GP}_k(n) \), then

\[
|A| \leq n - \left( \frac{1}{2k-1} + \frac{2}{5} \left( \frac{1}{5^{k-1}} - \frac{1}{6^{k-1}} \right) + \frac{4}{15} \left( \frac{1}{7^{k-1}} - \frac{1}{10^{k-1}} \right) \right) n + O\left( \frac{\log n}{k} \right).
\]

**Proof.** Let

\[
L = \left\lceil \frac{\log 2n}{k \log 2} \right\rceil.
\]

For \( 1 \leq \ell \leq L \) we have \( 2^{\ell k-1} \leq n \). Let \( a \) be an odd positive integer such that \( a \leq \frac{n}{2^{\ell k-1}} \).

The sequence

\[
(2^{(\ell-1)k}a, 2^{(\ell-1)k+1}a, 2^{(\ell-1)k+2}a, \ldots, 2^{\ell k-1}a)
\]

is a geometric progression of length \( k \) with common ratio 2. If \( A \in \text{GP}_k(n) \), then \( A \) does not contain this geometric progression, and so at least one element in the set

\[
X_\ell(a) = \{2^{(\ell-1)k}a, 2^{(\ell-1)k+1}a, 2^{(\ell-1)k+2}a, \ldots, 2^{\ell k-1}a\}
\]

is not an element of \( A \). Because every nonzero integer has a unique representation as the product of an odd integer and a power of 2, it follows that, for integers \( \ell = 1, \ldots, L \) and odd positive integers \( a \leq 2^{1-\ell}n \), the sets \( X_\ell(a) \) are pairwise disjoint subsets of \( \{1, 2, \ldots, n\} \).

For every real number \( t \geq 1 \), the number of odd positive integers not exceeding \( t \) is strictly greater than \((t - 1)/2\). It follows that the cardinality of the set \( \{1, 2, \ldots, n\} \setminus A \) is strictly greater than

\[
\sum_{\ell=1}^{L} \frac{1}{2} \left( \frac{n}{2^{\ell k-1}} - 1 \right) = \frac{1}{2} \sum_{\ell=1}^{L} \left( \frac{n}{2^{\ell k}} - \frac{1}{2} \right) = n \sum_{\ell=1}^{L} \frac{1}{2^{\ell k}} + O\left( \frac{\log n}{k} \right) = \frac{n}{2^{k-1}} + O\left( \frac{\log n}{k} \right).
\]

Note that if \( r \) is an odd integer and \( r \in X_\ell(a) \), then \( \ell = 1 \) and \( r = a \).

Let \( b \) be an odd integer such that

\[
\frac{n}{6^{k-1}} < b \leq \frac{n}{5^{k-1}}
\]

and \( b \) is not divisible by 5, that is,

\[
b \equiv 1, 3, 7, \text{ or } 9 \pmod{10}.
\]

We consider the following geometric progression of length \( k \) with ratio 5/3:

\[
(3^{k-1}b, 3^{k-1}5b, \ldots, 3^{k-1-i}5^ib, \ldots, 5^{k-1}b).
\]

Every integer in this progression is odd, and

\[
\frac{n}{2^{k-1}} < 3^{k-1}b < \cdots < 5^{k-1}b \leq n.
\]
Let
\[ Y(b) = \{3^{k-1}b, 3^{k-2}5b, \ldots, 3^{k-1-i}5^ib, \ldots, 5^{k-1}b\}. \]
It follows that \( X_\ell(a) \cap Y(b) = \emptyset \) for all \( \ell, a, \) and \( b. \) If the integers \( b \) and \( b' \) satisfy (1) and (2) with \( b < b' \) and if \( Y(b) \cap Y(b') \neq \emptyset, \) then there exist integers \( i, j \in \{0, 1, 2, \ldots, k - 1\} \) such that
\[ 3^{k-1-i}5^ib = 3^{k-1-j}5^jb' \]
or, equivalently,
\[ 5^i b = 3^j b'. \]
The inequality \( b < b' \) implies that \( 0 \leq j < i \leq k - 1 \) and so \( b' \equiv 0 \pmod{5}, \) which contradicts (2). Therefore, the sets \( Y(b) \) are pairwise disjoint. The number of integers \( b \) satisfying inequality (1) and congruence (2) is
\[ \frac{2}{5} \left( \frac{1}{5^{k-1}} - \frac{1}{6^{k-1}} \right) n + O(1). \]

Let \( c \) be an odd integer such that
\[ \frac{n}{10^{k-1}} < c < \frac{n}{7^{k-1}} \]
and \( c \) is not divisible by 3 or 5, that is,
\[ c \equiv 1, 7, 11, 13, 17, 19, 23, \) or \( 29 \pmod{30}. \]
We consider the following geometric progression of length \( k \) with ratio \( 7/5: \)
\[ (5^{k-1}c, 5^{k-2}7c, \ldots, 5^{k-1-i}7^ic, \ldots, 7^{k-1}c). \]
Every integer in this progression is odd, and
\[ \frac{n}{2^{k-1}} < 5^{k-1}c < \cdots < 7^{k-1}c \leq n. \]
Let
\[ Z(c) = \{5^{k-1}c, 5^{k-2}7c, \ldots, 5^{k-1-i}7^ic, \ldots, 7^{k-1}c\}. \]
It follows that \( X_\ell(a) \cap Z(c) = \emptyset \) for all \( \ell, a, \) and \( c. \) If \( c \) and \( c' \) satisfy (3) and (4) with \( c < c' \) and if \( Z(c) \cap Z(c') \neq \emptyset, \) then there exist integers \( i, j \in \{0, 1, 2, \ldots, k - 1\} \) such that
\[ 5^{k-1-i}7^ic = 5^{k-1-j}7^jc' \]
or, equivalently,
\[ 7^{i-j}c = 5^{i-j}c'. \]
The inequality \( c < c' \) implies that \( 0 \leq j < i \leq k - 1 \) and so \( c \equiv 0 \pmod{5}, \) which contradicts (4). Therefore, the sets \( Z(c) \) are pairwise disjoint.

If \( b \) and \( c \) satisfy inequalities (1) and (3), respectively, then \( c < b. \) If \( Y(b) \cap Z(c) \neq \emptyset, \) then there exist integers \( i, j \in \{0, 1, \ldots, k - 1\} \) such that
\[ 5^{k-1-i}7^ic = 5^{k-1-j}3^jb \]
or, equivalently,
\[ 5^i7^ic = 5^j3^jb. \]
Because \( bc \neq 0 \pmod{5}, \) it follows that \( i = j \) and so
\[ 7^ic = 3^jb. \]
Because $c < b$, we must have $i \geq 1$ and so $c \equiv 0 \pmod{3}$, which contradicts congruence (4). Therefore, $Y(b) \cap Z(c) = \emptyset$ and the sets $X_\ell(a)$, $Y(b)$, and $Z(c)$ are pairwise disjoint. The number of integers $c$ satisfying inequality (3) and congruence (4) is

$$\frac{4}{15} \left( \frac{1}{7^{k-1}} - \frac{1}{10^{k-1}} \right) n + O(1).$$

Because $A$ contains no $k$-term geometric progression, at least one element from each of the sets $X_\ell(a)$, $Y(b)$, and $Z(c)$ is not in $A$. This completes the proof. □

**Corollary 1.** If $A_k$ is a set of positive integers that contains no $k$-term geometric progression, then

$$d_U(A_k) \leq 1 - \frac{1}{2^k - 1} - \frac{2}{5} \left( \frac{1}{5^{k-1}} - \frac{1}{6^{k-1}} \right) - \frac{4}{15} \left( \frac{1}{7^{k-1}} - \frac{1}{10^{k-1}} \right).$$

Here is a table of upper bounds for $d_U(A)$ for various values of $k$:

| $k$ | $3$ | $4$ | $5$ | $6$ | $7$ | $10$ | $17$ |
|-----|-----|-----|-----|-----|-----|------|------|
| $d_U(A_k)$ | 0.84948 | 0.93147 | 0.96733 | 0.98404 | 0.99211 | 0.99902 | 0.99999 |

3. Open problems

For every integer $k \geq 3$, let $GP_{F_k}$ denote the set of sets of positive integers that contain no $k$-term geometric progression. It would interesting to determine precisely

$$\sup \left\{ d_U(A) : A \in GP_{F_k} \right\}$$

and

$$\sup \left\{ d(A) : A \text{ has asymptotic density and } A \in GP_{F_k} \right\}.$$

In the special case $k = 3$, Riddell [4, p. 145] claimed that if $A \in GP_{F_3}$, then $d_U(A) < 0.8339$, but wrote, "The details are too lengthy to be included here."

An infinite sequence $A = (a_i)_{i=1}^\infty$ of positive integers is *syndetic* if it is strictly increasing with bounded gaps. Equivalently, $A$ is syndetic if there is a number $c$ such that $1 \leq a_{i+1} - a_i \leq c$ for all positive integers $i$. Beiglböck, Bergelson, Hindman, and Strauss [1] asked if every syndetic sequence must contain arbitrarily long finite geometric progressions.

**References**

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