The Riemann-Hilbert approach to double scaling limit of random matrix eigenvalues near the "birth of a cut" transition

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Abstract

In this paper we studied the double scaling limit of a random unitary matrix ensemble near a singular point where a new cut is emerging from the support of the equilibrium measure. We obtained the asymptotic of the correlation kernel by using the Riemann-Hilbert approach. We have shown that the kernel near the critical point is given by the correlation kernel of a random unitary matrix ensemble with weight \( e^{-x^2/\nu} \). This provides a rigorous proof of the previous results in [18].

1 Introduction

In this paper we studied a double scaling limit of the unitary random matrix model with the probability distribution

\[
Z_{n,N}^{-1} \exp(-N \text{tr}(V(M)))dM, \quad Z_{n,N} = \int_{\mathcal{H}_n} \exp(-N \text{tr}(V(M)))dM \tag{1.1}
\]
defined on the space \( \mathcal{H}_n \) of Hermitian \( n \times n \) matrices \( M \), where \( V \) is real analytic and satisfies

\[
\lim_{x \to \pm \infty} \frac{V(x)}{\log(x^2 + 1)} = +\infty.
\]

The eigenvalues \( x_1, \ldots, x_n \) of the matrices in this ensemble is distributed according to the probability distribution (See, e.g. [25], [12])

\[
\mathcal{P}^{(n,N)}(x_1, \ldots, x_n) d^n x = \hat{Z}_{n,N}^{-1} \exp(-N \sum_{i=1}^n V(x_i)) \prod_{j<k} (x_j - x_k)^2 dx_1 \ldots dx_n, \tag{1.2}
\]

where \( \hat{Z}_{n,N} \) is the normalization constant.

A particular important object is the \( m \)-point correlation function \( \mathcal{R}_m^{(n,N)}(x_1, \ldots, x_m) \)

\[
\mathcal{R}_m^{(n,N)}(x_1, \ldots, x_m) = \frac{n!}{(n-m)!} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \mathcal{P}^{(n,N)}(x_1, \ldots, x_n) dx_{m+1} \ldots dx_n. \tag{1.3}
\]

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The correlation function is a very useful quantity in the calculation of probabilities. In fact the 1-point correlation function \( R_1^{(n,N)}(x) \) gives the probability density of finding an eigenvalue at the point \( x \). (Note that, however, the \( m \)-point correlation function \( R_m^{(n,N)} \) is not a probability density in general.)

A well-known result concerning the \( m \)-point correlation function is that it admits a determinantal expression with a kernel constructed from orthogonal polynomials. (See e.g. [17], [25])

To be precise, let \( \pi_n(x) \) be the degree \( n \) monic orthogonal polynomials with weight \( e^{-N V(x)} \) on \( \mathbb{R} \). \[ \int_{\mathbb{R}} \pi_n(x) \pi_m(x) e^{-N V(x)} dx = h_n \delta_{nm}. \] (1.4)

Let us construct the correlation kernel by

\[
K_{n,N}(x, x') = e^{-\frac{1}{2} N (V(x) + V(x'))} \frac{\sum_{j=0}^{n-1} \pi_j(x) \pi_j(x')}{h_{n-1}(x - x')},
\]

By the Christoffel-Darboux formula, this kernel can be expressed in terms of the two orthogonal polynomials \( \pi_n(x) \) and \( \pi_{n-1}(x) \) instead of the whole sum:

\[
K_{n,N}(x, x') = e^{-\frac{1}{2} N (V(x) + V(x'))} \frac{\pi_n(x) \pi_{n-1}(x') - \pi_n(x') \pi_{n-1}(x)}{h_{n-1}(x - x')},
\]

Then the \( m \)-point correlation function can be written as the determinant of the kernel \([17, 25, 27]\)

\[
R_m^{(n,N)}(x_1, \ldots, x_m) = \det (K_{n,N}(x_j, x_k))_{1 \leq j, k \leq m}
\]

In the limit \( n, N \to \infty, \frac{n}{N} \sim 1 \), the 1-point correlation function \( R_1^{(n,N)}(x) \) of the ensemble \([14]\) is asymptotic to the equilibrium measure \( \rho(x) \) \([12, 22, 28]\):

\[
\lim_{n, N \to \infty, \frac{n}{N} \sim 1} R_1^{(n,N)}(x) = \rho(x),
\]

where the \( \rho(x) dx = d\mu_{\min}(x) \) is the density of the unique measure \( \mu_{\min}(x) \) that minimizes the energy

\[
I(\mu) = - \int_{\mathbb{R}} \int_{\mathbb{R}} \log |x - y| d\mu(x) d\mu(y) + \int_{\mathbb{R}} V(x) d\mu(x)
\]

among all Borel probability measures \( \mu \) on \( \mathbb{R} \). The fact that \( \mu_{\min}(x) \) admits a probability density follows from the assumption that \( V(x) \) is real and analytic \([14]\). Moreover, it was shown in \([14]\) that for real and analytic \( V(x) \), the equilibrium measure is supported on a finite union of intervals.

The following conditions are satisfied by the equilibrium density \( \rho(x) \) \([12, 28]\)

\[
2 \int_{\mathbb{R}} \log |x - s| \rho(s) ds - V(x) = l, \quad x \in \text{Supp}(\rho(x)),
\]

\[
2 \int_{\mathbb{R}} \log |x - s| \rho(s) ds - V(x) \leq l, \quad x \in \mathbb{R}/\text{Supp}(\rho(x)).
\]

(1.6)
For a generic potential \( V(x) \), the inequality in (1.6) is satisfied strictly

\[
2 \int_{\mathbb{R}} \log |x-s| \rho(s) ds - V(x) > l, \quad x \in \mathbb{R}/\text{Supp}(\rho(x)).
\]

However, for some special potential \( V(x) \), this inequality may not be strict and we may have

\[
2 \int_{\mathbb{R}} \log |x-s| \rho(s) ds - V(x) = l, \quad x = x^*
\]

at some point \( x^* \notin \text{Supp}(\rho(x)) \). In this case, if we change the potential slightly, a new interval may emerge from the support of the equilibrium measure. This is the ‘birth of new cut’ critical limit that we are going to consider in this paper.

According to \([15]\), the ‘birth of new cut’ critical limit is a singularity of type I for the random matrix model (1.1). Other singular cases include:

1. Type II singularity is the case where the equilibrium density vanishes at a point \( x^* \) inside the support.

2. Type III singularity is the case where the equilibrium density vanishes faster than a square-root at an edge point \( x^* \) of the support. (Generically it vanishes like a square-root at the edge)

The asymptotic behavior of a random matrix ensemble near singular points has been studied extensively \([2]\), \([4]\), \([5]\), \([7]\), \([8]\), \([9]\), \([10]\), \([11]\), \([13]\), \([20]\), \([30]\). In these studies, one considers a one or multi-parameter family of potential \( V_{t_j}(x) \) in which the singular point is achieved at \( t_j = t^c_j \). One then studies the asymptotic behavior of the random matrix model (1.1) when \( t_j \) is close to \( t^c_j \). The ‘double scaling limit’ is the study of the these asymptotic behavior when the differences between \( t_j \) and \( t^c_j \) are coupled with \( n \) and \( N \). A remarkable feature is that in the double scaling limit, a universality can be observed. Upon a suitable scaling of the variables \( x \) and \( x' \), the asymptotic behavior of the kernel (1.5) near the critical point \( x^* \) depends only on the type of singularity rather than the potential \( V(x) \) itself.

In many cases, the behavior of the kernel in a double scaling limit is described by integrable hierarchies such as the Painlevé equations. In the case of the type II singularity, \([3]\), \([8]\), \([10]\) and \([30]\) has shown that the kernel can be described by the Hastings-McLeod solution of the Painlevé II equation in the double scaling limit. While for the type III singularity, the kernel can be described by the Painlevé I transcendent \([11]\), \([13]\). In \([18]\), the double scaling limit of the ‘birth of new cut’ was studied and the kernel was described by the orthogonal polynomials with weight \( e^{-x^2} \) on the real axis. However, the formulae derived in \([18]\) have not been rigorously proven and it is the purpose of this paper to provide a rigorous proof of these results.
1.1 Statement of results

We should now introduce some notations and state the results in this paper.

In this paper, we should consider a one parameter family of potential  \( V_t(x) = \frac{V(x)}{t} \) parametrised by  \( t = \frac{1}{N} \). We should consider the double scaling limit of  \( t \to 1 \) and  \( N \to \infty \) such that

\[
\lim_{n, N \to \infty} \frac{\log n}{n} \left( \frac{n}{N} - 1 \right) = U_+ > 0, \quad n > N \tag{1.7a}
\]

\[
\lim_{n, N \to \infty} n^k \left( \frac{n}{N} - 1 \right) = U_- \leq 0, \quad n \leq N, \quad k \in \left[ 1 - \frac{1}{2\nu}, \infty \right) \tag{1.7b}
\]

exist. In particular, for  \( t \leq 1 \), we considered the regime where  \( t - 1 \) is of order  \( n^{-k} \) for any  \( k \) greater than or equal to  \( 1 - \frac{1}{2\nu} \), while the scaling for  \( t > 1 \) is fixed.

Let us now state the assumptions that are used in this study. Since the main point of this study is the treatment of the critical point  \( x^* \), we will assume the followings:

1. The support of the equilibrium density  \( \rho(x) \) consists of one interval only, that is, the first equation of (1.6) holds precisely on a single interval  \( (a, b) \). Without lost of generality, we will assume that  \( a = -2 \) and  \( b = 2 \).

2. The equilibrium measure does not vanish at any interior point of  \( (-2, 2) \).

3. The point  \( x^* \) is the only point outside  \( \text{Supp}(\rho(x)) \) where the inequality in (1.6) is not strict and we assume that  \( x^* > 2 \).

4. As pointed out in [16], the function  \( 2 \int_{\mathbb{R}} \log |x - s| \rho(x) ds - V(x) - l \) vanishes to an even order at  \( x^* \). We will assume that this order of vanishing is  \( 2\nu \).

Let the equilibrium measure of  \( V_t(x) \) be  \( \rho^t(x) \) such that

\[
2 \int_{\mathbb{R}} \log |x - s| \rho^t(s) ds - V_t(x) = l_t, \quad x \in \text{Supp}(\rho^t(x)),
\]

\[
2 \int_{\mathbb{R}} \log |x - s| \rho^t(s) ds - V_t(x) \leq l_t, \quad x \in \mathbb{R}/\text{Supp}(\rho^t(x)), \tag{1.8}
\]

and denote by  \( c_{x^*} \) the following

\[
c_{x^*} = n \left( \frac{V_t(x^*)}{2} + \frac{l_t}{2} - \int_{\mathbb{R}} \rho^t(s) \log |x^* - s| ds \right) \geq 0. \tag{1.9}
\]

It is known that both  \( t\rho^t(x) \) and the support of  \( \rho^t(x) \) are increasing with  \( t \) [24], [13], [28], [31]. In particular, for  \( t \leq 1 \), the equilibrium measure is supported on one interval while for  \( t \) slightly greater than 1, the equilibrium measure is supported on 2 intervals.
Let $\mathcal{S}_t$ be the support of $\rho^t(x)$. Then in [6], it was shown that the equilibrium measure $d\mu_t(x) = \rho^t(x)dx$ satisfies the Buyarov-Rakhmanov equation

$$
\mu_t = \frac{1}{t} \int_0^t \omega_{\mathcal{S}_\tau} d\tau,
$$

(1.10)

where $\omega_{\mathcal{S}_\tau}$ is the equilibrium measure of the set $\mathcal{S}_\tau$. Namely, it is the unique probability measure supported on $\mathcal{S}_\tau$ that minimizes the logarithmic potential

$$
I(\tilde{\mu}) = \int \int -\log |s-t|d\tilde{\mu}(s)d\tilde{\mu}(t)
$$

among all the Borel probability measures $\tilde{\mu}$ supported on $\mathcal{S}_\tau$.

If $\mathcal{S}_\tau$ consists of one interval only, then $\omega_{\mathcal{S}_\tau}(x)$ is given by

$$
\omega_{\mathcal{S}_\tau}(x) = \frac{1}{\pi \sqrt{(b_\tau - x)(x - a_\tau)}} dx, \quad x \in (a_\tau, b_\tau).
$$

In particular, we have, at $t = 1$

$$
\lim_{t \to 1} \frac{t\mu_t(x) - \mu(x)}{t - 1} = \frac{1}{\pi \sqrt{4-x^2}} dx = w(x)dx.
$$

(1.11)

The fact that $w(x)dx$ is the equilibrium measure on the interval $[-2, 2]$ means that

$$
\int_{-2}^{2} w(s) \log |x-s|ds = \frac{\zeta}{2}, \quad x \in [-2, 2],
$$

$$
\int_{-2}^{2} w(s) \log |x-s|ds = \log x + O(1), \quad x \to \infty
$$

(1.12)

for some constant $\zeta$.

Let us defined a function $\phi(x)$ that is closely related to $w(x)dx$.

$$
\phi(x) = \frac{\zeta}{2} + \int_{-2}^{2} w(s) \log(x^* - s)ds.
$$

(1.13)

In this paper, we will use an anzatz in [18] to construct an approximated equilibrium density $\tilde{\rho}^t(x)$ for $t > 1$ and use it to modify the Riemann-Hilbert problem of the orthogonal polynomials (1.4).

We shall denote the correlation kernel for the random matrix model

$$
Z_{m,\nu}^{-1} \exp(-\text{tr}(M^{2\nu}))dM, \quad Z_{m,\nu} = \int_{\mathcal{H}_m} \exp(-\text{tr}M^{2\nu})dM
$$

(1.14)

by $K_m^\nu(x, x')$. That is,

$$
K_m^\nu(x, x') = e^{-\frac{(x')^{2\nu}_+ x^{2\nu}}{4} \pi_m^\nu(x)\pi_{m-1}^\nu(x') - \pi_m^\nu(x')\pi_{m-1}^\nu(x)} \frac{h_m^\nu(x-x')}{h_{m-1}^\nu(x-x')},
$$

(1.15)
where $\pi_m^\nu(x)$ is the degree $m$ monic orthogonal polynomial on $\mathbb{R}$ with respect to the weight $e^{-x^{2\nu}}$ and $h_m^\nu$ is the corresponding normalization constant as in (1.4).

We can now state our main result.

**Theorem 1.1.** Let $V(x)$ be real and analytic on $\mathbb{R}$ such that $\lim_{x \to \pm \infty} \frac{V(x)}{\log(x^2+1)} = +\infty$. Let $\rho(x)$ be the density of the equilibrium measure of $V(x)$ supported on the interval $[-2, 2]$. Then

$$\rho(x) = \frac{\sqrt{4 - x^2} Q(x)(x - x^*)^{2\nu - 1}}{2\pi}, x \in [-2, 2],$$

where $x^* > 2$ and $Q(x)$ is real analytic on $\mathbb{R}$ with $Q(x^*) > 0$.

Let $n, N \to \infty$ such that (1.7a) and (1.7b) hold and let $u, \pi$ be the following

$$u = 2\nu \phi(x^*) U_+$$

$$\pi = \left[2\nu \phi(x^*) U_+ + \frac{1}{2}\right]$$

(1.16)

where $\phi(x^*)$ is defined in (1.13) and $\lfloor x \rfloor$ is the greatest integer that is smaller than or equal to $x$.

Let $K_{n,N}$ be the correlation kernel (1.3), then for $u \notin \mathbb{N} + \frac{1}{2}$, the limit of the kernel is given by

$$\lim_{n,N \to \infty} \frac{1}{\varphi(x^*) n^{2\nu}} K_{n,N} \left(x^* + \frac{z}{\varphi(x^*) n^{\frac{1}{2}}}, x^* + \frac{z'}{\varphi(x^*) n^{\frac{1}{2}}} \right) = K_{\pi}^\nu(z, z'), \quad n > N,$$  

(1.17a)

$$\lim_{n,N \to \infty} e^{c x^*} K_{n,N} \left(x^* + \frac{z}{\varphi(x^*) n^{\frac{1}{2}}}, x^* + \frac{z'}{\varphi(x^*) n^{\frac{1}{2}}} \right)$$

$$= e^{-x^{2\nu}(z')^{2\nu}} \frac{1}{8\pi} \left(\frac{1}{x^* - \beta t} - \frac{1}{x^* - \alpha t}\right), \quad n \leq N,$$  

(1.17b)

where $K_{\pi}^\nu(z, z')$ is defined in (1.13) and $c_{x^*}$ is defined in (1.9) and $\varphi(x^*)$ is given by

$$\varphi(x^*) = \left(\frac{Q(x^*) \sqrt{(x^*)^2 - 4}}{2\nu}\right)^{\frac{1}{2\nu}}.$$  

The result shows that for $u \notin \mathbb{N} + \frac{1}{2}$, the correlation kernel near $x^*$ for $t > 1$ is given by the correlation kernel of a finite random matrix ensemble (1.14) with size $\lfloor u + \frac{1}{2}\rfloor$. This confirms the results in [18]. When $u$ goes pass a half integer, the size of the finite random matrix ensemble jumps by 1 and a non-trivial transition takes place. This is due to the non-uniform converges of (1.17a) in $u$ when $u$ is close to a half integer. When $u$ is close to a half integer, error terms that depends on $K_{\pi \pm 1}^\nu$ which are not seen in (1.17a) become significant and start taking over the $K_{\pi}^\nu$ terms, which results in a jump when $u$ goes pass a half integer.
Note that (1.17b) implies that the leading order term of the kernel at $x^*$ is $e^{c_{x^*}}$. This leading term tends to zero when $n, N \to \infty$ unless $t = 1$. This is not surprising as for $t < 1$, there is no eigenvalue near the point $x^*$ and the correlation kernel should be vanishing near $x^*$ in the limit.

**Remark 1.1.** Claeys [7] has simultaneously and independently used the Riemann-Hilbert method to study the birth of new cut double scaling limit. In Claeys [7], the case when $\nu = 1$ was studied and the Hermite polynomials was used to construct the asymptotic kernel. Despite the similarity of our work to [7], a very different treatment to the equilibrium measure was used in [7]. In [7], the equilibrium measure with total mass $1 - 2 \frac{t-1}{\log n} \phi(x^*)$ was used to construct the ‘g-function’ for the Deift-Zhou steepest decent method when $t > 1$. Whereas in this paper, we approximated the equilibrium measure by solving the Buyarov-Rakhmanov equation (1.10) up to a certain order in $t - 1$. We then use this approximated measure to construct the ‘g-function’ for the Deift-Zhou steepest decent method. Also worth remarking is that in [7], the behavior of the kernel when $u$ is close to a half integer was studied.

This paper is organized as follows. In section 2 we will use the ansatz obtained in [18] to construct an approximated equilibrium density for $t > 1$. We then show that conditions of the type (1.6) are satisfied for this approximated density outside some neighborhoods of the edge points and the critical point. We then study the error terms in these conditions.

In section 3 we will apply the Deift-Zhou steepest decent method to the Riemann-Hilbert problem of the orthogonal polynomials (1.4). We will use the approximated density to construct a ‘g-function’ and use it to modify the Riemann-Hilbert problem. We then approximate this modified Riemann-Hilbert problem by a Riemann-Hilbert problem that can be solved explicitly and construct parametrices to solve this approximated Riemann-Hilbert problem. These parametrices then give us the asymptotics of the orthogonal polynomials (1.4). These asymptotics will then be used to derive the asymptotics of the kernel (1.3) in section 4.

## 2 Equilibrium measure

We will now study the behavior of the equilibrium measure $\rho^t(x)$ (1.8) when $t$ is close to 1. Let $t$ be a real parameter and let us define

$$V_t(x) = \frac{1}{t} V(x), \quad t > 0.$$  

Then $V_1(x) = V(x)$. We shall consider the case when $t \leq 1$ and $t > 1$ separately. For $t > 1$, we will replace the eigenvalues on the newborn interval by a point charge. Let the support of the equilibrium measure $S_t$ be

$$S_t = [a_t, b_t], \quad t \leq 1$$  

$$S_t = [a_t, b_t] \cup [c_t, d_t], \quad t > 1.$$  

(2.1)
Let us define the function $h^t$ by

$$h^t(x) = \int_{\mathbb{R}} \log(x - s) d\mu_t(s)$$

where the principal branch of the logarithm is taken in the above,

$$\log(x - s) = \log|x - s| + i \arg(x - s)$$

$$0 < \arg(x - s) < \pi, \quad s \in \mathbb{R}, \quad \exists x > 0,$$

$$-\pi < \arg(x - s) < 0, \quad s \in \mathbb{R}, \quad \exists x, 0.$$

The boundary values of $h^t(x)$ on the real axis are then

$$h^t_\pm(x) = \int_{\mathbb{R}} \log|x - s| d\mu_t(s) \pm \pi i \int_{a}^{x} d\mu_t(s)$$

In particular, the function $h^t$ is analytic on $\mathbb{C}/[a_t, \infty)$ and it satisfies the following

$$h^t_+(x) + h^t_-(x) - V_t(x) + l_t = 0, \quad x \in [a_t, b_t] \cup [c_t, d_t]$$

$$h^t_+(x) + h^t_-(x) - V_t(x) + l_t < 0, \quad x \in \mathbb{R}/([a_t, b_t] \cup [c_t, d_t] \cup \{x^*\})$$

$$h^t_+(x) - h^t_-(x) = 2\pi i \int_{x}^{b_t} d\mu_t(s), \quad x \in \mathbb{R}$$

$$h^t(x) = \log x + O(x^{-1}) \quad x \to \infty$$

In [14], it was shown that for a real analytic potential $V(x)$ on $\mathbb{R}$, the equilibrium measure $d\mu_t(s)$ can be expressed in terms of the negative part of an analytic function $q_t(x)$.

**Theorem 2.1.** [14] Let $V(x)$ be real analytic in a neighborhood $\mathcal{V}$ of the real axis and let $q_t(x)$ be the following function

$$q_t(x) = \left(\frac{V'(x)}{2t}\right)^2 - \frac{1}{t} \int_{\mathbb{R}} \frac{V'(x) - V'(y)}{x - y} d\mu_t(y), \quad x \in \mathcal{V}. \quad (2.4)$$

Then the equilibrium measure has a density $\rho^t(x)$ which can be written as

$$\rho^t(x) = \frac{1}{\pi} \sqrt{-q^-_t(x)},$$

where $q^-_t(x)$ is the negative part of $q_t(x)$, that is,

$$q_t(x) = q^+_t(x) + q^-_t(x), \quad q^+_t(x) \geq 0, \quad q^-_t(x) \leq 0.$$

Moreover, we have the following

$$q_t(x) = \left(\int_{\mathbb{R}} \frac{\rho^t(y) dy}{y - x} + \frac{V'(x)}{2t}\right)^2, \quad x \in \mathcal{V}. \quad (2.5)$$
2.1 Approximated equilibrium measure for $t > 1$

For $t > 1$, a new cut in the support of the equilibrium measure is emerging at $x = x^*$. We would like to find an approximation to the equilibrium measure and study its properties.

The Buyarov-Rakhmanov equation (1.10) for the equilibrium measure is a nonlinear ODE which is difficult to solve. In [18], an ansatz was used to solve this differential equation up to some leading order terms in $t - 1$. As this ODE becomes singular at $t = 1$, it is difficult to prove rigorously that the solution in [18] does indeed give the equilibrium measure for $t$ slightly greater than 1.

Instead of showing that the solution obtained in [18] gives the correct equilibrium measure for $t > 1$, we would use the ansatz in [18] to construct an approximated density $\tilde{\rho}_t(x)$, together with a function $\tilde{h}_t(x)$ analogue to the function $h_t(x)$ defined in (2.2). We will then show that this approximated density satisfies conditions of the type (1.6) up to a certain order in $t - 1$.

First note that the function $q_t(x)$ defined in (2.4) has the following form at $t = 1$.

$$\sqrt{q_t(x)} = \frac{1}{2}Q(x)(x - x^*)^{2\nu - 1}\sqrt{x^2 - 4},$$

where $Q(x)$ is analytic in a neighborhood $V$ of the real axis.

We can now define a function $\tilde{q}_t(x)$ analogous to $q_t(x)$.

**Definition 2.1.** Let $\delta t = t - 1 > 0$. Then the function $\tilde{q}_t(x)$ is defined by

$$\sqrt{\tilde{q}_t(x)} = \frac{\sqrt{(x - \alpha_t)(x - \beta_t)}}{2}(Q(x)H_t(x)\sqrt{(x - x^*)^2 - 4y^2}\left(\frac{-\delta t}{\log \delta t}\right)^{\nu} + \eta(x)\delta t),$$

where $\alpha_t$ and $\beta_t$ are,

$$\alpha_t = -2 + \frac{\delta t}{(2 + x^*)^{2\nu - 1}Q(-2)}, \quad \beta_t = 2 - \frac{\delta t}{(x^* - 2)^{2\nu - 1}Q(2)}.$$

while $H_t(x)$ is a monic polynomial defined by

$$H_t(x) = (x - x^*)^{2\nu - 2}\sum_{k=0}^{\nu - 1} \frac{(2k)!}{k!k!}y^{2k}(x - x^*)^{-2k}\left(\frac{-\delta t}{\log \delta t}\right)^{\frac{k}{\nu}},$$

The function $\eta(x)$ is defined by

$$\eta(x) = \frac{Q(x)(x - x^*)^{2\nu - 1}}{2Q(2)(2 - x^*)^{2\nu - 1}(x - 2)} + \frac{Q(x)(x - x^*)^{2\nu - 1}}{2Q(-2)(2 + x^*)^{2\nu - 1}(x + 2)} - \frac{2}{x^2 - 4},$$

and the constant $y$ is defined by

$$y = \frac{4\nu^2\phi(x^*)(\nu - 1)!\nu!}{Q(x^*)\sqrt{(x^*)^2 - 4(2\nu)!}}$$

and $\phi(x^*)$ is defined in (1.13).
Remark 2.1. The function $\eta(x)$ is analytic in the neighborhood $\mathcal{V}$ of the real axis.

We will now show that the density defined by the function $\sqrt{q_t(x)}$ satisfies the Buyarov-Rakhmanov equation outside a fixed neighborhood of $x^*$.  

**Proposition 2.1.** Let $B^*_x$ be the set

$$B^*_x = \{x \mid |x - s| \leq \delta\}$$

and let $r_1 = -2$, $r_2 = 2$ and $r_3 = x^*$. Then for sufficiently small $\delta t$, there exist compact subset $\mathcal{K} \subset \mathcal{V}$ and $\delta > 0$ independent on $t$, such that the function $\bar{q}_t(x)$ satisfies

$$\sqrt{\bar{q}_t(x)} = - \frac{1}{\sqrt{x^2 - 4}} + O\left(\frac{\delta t}{\log \delta t}\right), \quad x \in \mathcal{K}/(\bigcup_{j=1}^{3} B^*_{r^j} \cup [-2, 2])$$

uniformly in $\mathcal{V}/(\bigcup_{j=1}^{3} B^*_{r^j} \cup [-2, 2])$, where $\delta t = t - 1$.

**Proof.** We will expand (2.7) in terms of $\delta t$ and $-\frac{\delta t}{\log \delta t}$. Let us first consider the product

$$H_t(x)\sqrt{(x - x^*)^2 - 4y^2 \left(-\frac{\delta t}{\log \delta t}\right)^{\frac{1}{\nu}}}.$$  

Let $\delta > 0$ be fixed. Then for small enough $\delta t$, the following Taylor series expansion is valid outside of $B^*_x$.

$$\left((x - x^*)^2 - 4y^2 \left(-\frac{\delta t}{\log \delta t}\right)^{\frac{1}{\nu}}\right)^{-\frac{1}{\nu}} = \sum_{j=0}^{\infty} \frac{(2j)!}{j! j! (1 - 2j)!} y^{2j}(x - x^*)^{-2j + 1}\left(-\frac{\delta t}{\log \delta t}\right)^{\frac{j}{\nu}}.  \quad (2.13)$$

Now from the Taylor series expansion of $\left((x - x^*)^2 - 4y^2 \left(-\frac{\delta t}{\log \delta t}\right)^{\frac{1}{\nu}}\right)^{-\frac{1}{\nu}}$, we see that (c.f. [18])

$$H_t(x) = \text{Pol} \left((x - x^*)^{2\nu - 1} \left((x - x^*)^2 - 4y^2 \left(-\frac{\delta t}{\log \delta t}\right)^{\frac{1}{\nu}}\right)^{-\frac{1}{\nu}}\right)$$

where $\text{Pol}(X)$ denotes the polynomial part of $X$.

Therefore we have

$$H_t(x)\sqrt{(x - x^*)^2 - 4y^2 \left(-\frac{\delta t}{\log \delta t}\right)^{\frac{1}{\nu}}} = (x - x^*)^{2\nu - 1}$$

$$+ \sum_{j=0}^{\infty} y^{2j+2\nu}(x - x^*)^{-2j - 1} \left(-\frac{\delta t}{\log \delta t}\right)^{1 + \frac{j}{\nu}} L_j,  \quad (2.14)$$

$$L_j = \sum_{p=0}^{\nu-1} \frac{(2p)!}{p! p!} \frac{(2j + \nu - p)!}{(j + \nu - p)! (j + \nu - p)! (1 - 2(j + \nu - p))}.$$
Then, for a small enough $\delta t$, we have, for $|x - x^*| > \delta$,

$$\sum_{j=0}^{\infty} y^{2j+2\nu}(x - x^*)^{-2j-1} \left( -\frac{\delta t}{\log \delta t} \right)^{1+\frac{j}{\nu}} L_j = O \left( \frac{\delta t}{\log \delta t} \right).$$

This means that, for $x \in \mathcal{K}/B_{x^*}^\delta$, we have

$$H_t(x) \sqrt{(x - x^*)^2 - 4y^2} \left( -\frac{\delta t}{\log \delta t} \right)^{\frac{1}{\nu}} = (x - x^*)^{2\nu - 1} + O \left( \frac{\delta t}{\log \delta t} \right), \quad x \notin B_{x^*}^\delta \quad (2.15)$$

Now let us look at the terms of order $\delta t$. Again, for small enough $\delta t$, the following Taylor series expansions are valid outside $B_{x^*}^2 \cup B_{x^*}^{-2}$.

$$\sqrt{x - \alpha_t} = \sqrt{x + 2} \sum_{j=0}^{\infty} \frac{(-1)^j (2j)!}{j! (1 - 2j)^4} \left( \frac{\Xi(-2)\delta t}{x + 2} \right)^j,$$

$$\sqrt{x - \beta_t} = \sqrt{x - 2} \sum_{j=0}^{\infty} \frac{(2j)!}{j! (1 - 2j)^4} \left( \frac{\Xi(2)\delta t}{x - 2} \right)^j, \quad (2.16)$$

where the function $\Xi(x)$ is defined by

$$\Xi(x) = \frac{1}{(x - x^*)^{2\nu - 1} Q(x)}.$$

The identity (2.16) implies that, for small enough $\delta t$, we have, for $x \in \mathcal{K}/ \left( B_{x^*}^{-2} \cup B_{x^*}^{2} \right)$,

$$\sqrt{(x - \alpha_t)(x - \beta_t)} = \sqrt{x^2 - 4 + \delta t \left( -\frac{\sqrt{x + 2} \Xi(2)}{2\sqrt{x - 2}} + \frac{\sqrt{x - 2} \Xi(-2)}{2\sqrt{x + 2}} \right)} + O((\delta t)^2). \quad (2.17)$$

Combining this with (2.15) and (2.10), we see that, outside of $B_{x^*}^\delta$, the limit (2.12) is given by

$$\sqrt{\tilde{q}(x) - \sqrt{q(x)}} = \left( \frac{\sqrt{x + 2} \Xi(2)}{4 \Xi(x) \sqrt{x - 2}} - \frac{\sqrt{x - 2} \Xi(-2)}{4 \Xi(x) \sqrt{x + 2}} \right) + O \left( \frac{\delta t}{\log \delta t} \right), \quad (2.18)$$

which is just

$$\sqrt{\tilde{q}(x) - \sqrt{q(x)}} = -\frac{1}{\sqrt{x^2 - 4}} + O \left( \frac{\delta t}{\log \delta t} \right), \quad x \in \mathcal{K}/ \left( \bigcup_{j=1}^{3} B_{x^*}^r \cup [-2, 2] \right).$$

This gives the assertion of the proposition.
Let us now define the approximated equilibrium density to be
\[ \tilde{\rho}_t(x) = \frac{1}{t\pi} \left( \sqrt{-\tilde{q}_t(x)} \right)_+, \quad x \in [\alpha_t, \beta_t] \]
\[ \tilde{\rho}_t(x) = \frac{\sqrt{(x - \alpha_t)(x - \beta_t)}}{2t\pi} Q(x) H_t(x) \sqrt{\sigma_t^2 - (x - x^*)^2}, \]
\[ x \in [x^* - \sigma_t, x^* + \sigma_t], \quad \sigma_t^\pm = 2y \left( -\frac{\delta t}{\log \delta t} \right)^{\frac{1}{2}}, \]
\[ \tilde{\rho}_t(x) = 0, \quad x \in \mathbb{R}/[\alpha_t, \beta_t] \cup [x^* - \sigma_t, x^* + \sigma_t], \]
and let \( \tilde{h}_t(x) \) be the following
\[ \tilde{h}_t(x) = \int_{\alpha_t}^{\beta_t} \tilde{\rho}_t(s) \log(x - s)ds + \int_{x^* - \sigma_t}^{x^* + \sigma_t} \tilde{\rho}_t(s) \log(x - s)ds. \]
Then we have the following analogue of (1.11) for \( \tilde{h}_t(x) \).

**Proposition 2.2.** For sufficiently small \( \delta t \), there exists \( \delta > 0 \) such that the following is satisfied for \( \tilde{h}_t(x) \)
\[ \tilde{h}_t(x) = \frac{h(x)}{t} + \frac{\delta t}{t} \left( \int_{-2}^{2} w(s) \log(x - s)ds \right) + O \left( \frac{\delta t \log(x + 2)}{\log \delta t} \right), \]
\[ x \in \mathbb{C}/ \bigcup_{j=1}^{3} B_{\delta}^{R_j} \cup \text{Supp} (\tilde{\rho}_t(x)), \]
where \( h(x) \) is the following
\[ h(x) = \int_{-2}^{2} \rho(s) \log(x - s)ds \]
and \( w(s) \) is the equilibrium measure of the interval \([-2, 2] \).

**Proof.** Let us first divide the real axis in to different parts
\[ \mathbb{R} = \bigcup_{j=1}^{6} \mathbb{R}_j \]
where the \( \mathbb{R}_j \) are the following intervals, that is,
\[ \mathbb{R}_1 = [\alpha_t, -2 - 2(2 + \alpha_t)], \quad \mathbb{R}_2 = [-2 - 2(2 - \alpha_t), -2 + \frac{\delta}{2}], \]
\[ \mathbb{R}_3 = \left[ -2 + \frac{\delta}{2}, 2 - \frac{\delta}{2} \right], \quad \mathbb{R}_4 = \left[ 2 - \frac{\delta}{2}, 2 - 2(\beta_t - 2) \right], \]
\[ \mathbb{R}_5 = [2 - 2(\beta_t - 2), \beta_t], \quad \mathbb{R}_6 = [x - \sigma_t, x + \sigma_t]. \]
Let us now define $\Gamma$ to be the line right above $\mathbb{R}_3$,
\[ \Gamma = \left\{ x \mid x = u + i\varepsilon, \ u \in \left[ -2 + \frac{\delta}{2}, 2 - \frac{\delta}{2} \right], \ \varepsilon \to 0^+ \right\}. \tag{2.23} \]

Then we have
\[ \int_{\mathbb{R}_3} \tilde{\rho}^t(s) \log(x-s)ds = \int_{\Gamma} \frac{\sqrt{-q^t(s)}}{t\pi} \log(x-s)ds, \]
\[ \int_{\mathbb{R}_3} \rho(s) \log(x-s)ds = \int_{\Gamma} \frac{\sqrt{-q(s)}}{\pi} \log(x-s)ds, \]
\[ \int_{\mathbb{R}_3} w(s) \log(x-s)ds = \int_{\Gamma} \frac{1}{\pi \sqrt{4 - s^2}} \log(x-s)ds. \tag{2.24} \]

Let $\delta > 0$ be such that the power series expansion of $Q(x)$ and $\eta(x)$ around $\pm 2$ are valid inside $B_{\pm 2}^\delta$.

First let us consider the integral on $\mathbb{R}_1$. On $\mathbb{R}_1$, the following power series expansions are valid.
\[ Q(s) = \sum_{j=0}^{\infty} Q_{(j,-2)}(s+2)^j, \quad \eta(s) = \sum_{j=0}^{\infty} \eta_j(s+2)^j \]
\[ \left( \sqrt{\beta_t - s} \right)_+ = \sqrt{\beta_t + 2} + \sum_{j=0}^{\infty} \lambda_j(s+2)^j, \tag{2.25} \]
\[ \log(x-s) = \log(x+2) - \sum_{j=1}^{\infty} \frac{1}{j} \left( \frac{s+2}{x+2} \right)^j, \]
where the branch of $\log(x+2)$ is chosen to be the principal branch.

It is not difficult to check that the coefficients in the above series remain finite as $\delta t \to 0$.

Moreover, from (2.15), we have
\[ H_t(x) \sqrt{(x - x^*)^2 - 4y^2 \left( -\frac{\delta t}{\log \delta t} \right)^{\frac{1}{2}}} = (x - x^*)^{2\nu - 1} + O \left( \frac{\delta t}{\log \delta t} \right), \quad x \in \mathbb{R}_1. \]

In particular, this means that on $\mathbb{R}_1$ the functions have the following estimates
\[ Q(s) = Q(-2) + O(\delta t), \quad \eta(s) = \eta(-2) + O(\delta t), \]
\[ \left( \sqrt{\beta_t - s} \right)_+ = \sqrt{\beta_t + 2} + O(\delta t), \quad \log(x-s) = \log(x+2) + O(\delta t). \]
\[ H_t(s) \sqrt{(s - x^*)^2 - 4y^2 \left( -\frac{\delta t}{\log \delta t} \right)^{\frac{1}{2}}} = -(2 + x^*)^{2\nu - 1} + O \left( \frac{\delta t}{\log \delta t} \right). \]
Therefore the integral on $\mathbb{R}_1$ can be evaluated as

$$
\int_{\mathbb{R}_1} \rho'(s) \log(x - s) ds = \frac{\sqrt{\beta} + 2 \log(x + 2)}{2\Xi(-2)\pi} \int_{\mathbb{R}_1} \sqrt{s - \alpha_t} ds (1 + O(\delta t)) = \frac{\sqrt{\beta} + 2 \log(x + 2) (\Xi(-2))^{\frac{1}{2}}}{3\pi} (3\delta t)^{\frac{3}{2}} (1 + O(\delta t)).
$$

Similarly, the following integrals for $\rho(x)$ and the equilibrium measure on $[-2, 2]$ are given by

$$
\int_{-2}^{-2-2(2+\alpha_t)} \frac{\rho(s)}{t} \log(x - s) ds = \frac{2^{\frac{3}{2}} \log(x + 2) (\Xi(-2))^{\frac{1}{2}}}{3t\pi} (1 + O(\delta t)),
$$

$$
\int_{-2}^{-2-2(2+\alpha_t)} \frac{w(s) \log(x - s)}{t} ds = \frac{\log(x + 2)}{t\pi} \sqrt{2\Xi(-2)} \delta t (1 + O(\delta t)).
$$

Therefore we have

$$
\int_{\mathbb{R}_1} \rho'(s) \log(x - s) ds - \int_{-2}^{-2-2(2+\alpha_t)} \left( \frac{\rho(s)}{t} - \frac{\delta t}{t} w(s) \right) \log(x - s) ds = O \left( (\delta t)^{\frac{3}{2}} \log(x + 2) \right).
$$

Next let us consider the integral on $\mathbb{R}_2$. Since $|x + 2| > \delta$, for $s \in \mathbb{R}_2$, we can find constants independent on $t$ and $s$ such that

$$
|Q(s)| < M_Q, \quad |\sqrt{s + 2}| < |\sqrt{s - \alpha_t}| < M_\alpha, \quad |\eta(s)| < M_\eta,
$$

$$
|s - x^*|^{2\nu - 1} < M_* \quad |\sqrt{2 - s}| < |\sqrt{\beta} - s| < M_\beta, \quad |\Xi(2)| < M_1,
$$

$$
\left| \sqrt{\beta} - s - \sqrt{2 - s} + \frac{\Xi(2)}{\sqrt{2 - s}} \delta t \right| < M_2(\delta t)^2,
$$

$$
|H_t(s) - (s - x^*)^{2\nu - 1}| < M_H \left( -\frac{\delta t}{\log \delta t} \right),
$$

$$
\log|x - s| < M_3 \log|x + 2|.
$$

Then, by using the the Taylor series expansion of $\sqrt{s - \alpha_t}$ in (2.16), we see that

$$
\left| t \rho'(s) - \rho(s) + \delta t \frac{1}{\pi \sqrt{4 - s^2}} \log|x - s| \right| \leq \left( E_1 \sum_{j=2}^{\infty} \frac{\sqrt{s + 2(2j)!}}{j! (2j - 1)4^j} \left( \frac{|\Xi(-2)|\delta t}{s + 2} \right)^j \right) \log|x + 2|,
$$

$$
+ E_2 \delta t \sum_{j=1}^{\infty} \frac{\sqrt{s + 2(2j)!}}{j! (2j - 1)4^j} \left( \frac{|\Xi(-2)|\delta t}{s + 2} \right)^j \log|x + 2|,
$$

$$
+ E_3 \sum_{j=0}^{\infty} \frac{\sqrt{s + 2(2j)!}}{j! (2j - 1)4^j} \left( \frac{|\Xi(-2)|\delta t}{s + 2} \right)^j \left( -\frac{\delta t}{\log \delta t} \right) \log|x + 2|.
$$

\(2.29\)
for some positive constants $E_1$, $E_2$ and $E_3$. One needs to be careful about the terms that contains negative power of $x + 2$ as they may become large in $\mathbb{R}_2$. If we integrate (2.29) and consider the leading order term in $\delta t$, we see that

$$\left| \int_{\mathbb{R}_2} (t\rho'(s) - \rho(s) + \delta tw(s)) \log |x - s| ds \right| \leq E \frac{\delta t}{\log \delta t} \log |x + 2|.$$ 

for some positive constant $E$.

This implies that

$$\int_{\mathbb{R}_2} (t\rho'(s) - \rho(s) + \delta tw(s)) \log |x - s| ds = O \left( \frac{\delta t \log(x + 2)}{\log \delta t} \right).$$

We then see that

$$\int_{\mathbb{R}_2} \left( \frac{\rho'(s)}{t} - \frac{\rho(s)}{t} + \frac{\delta t}{t} w(s) \right) \log |x - s| ds = O \left( \frac{\delta t \log(x + 2)}{\log \delta t} \right).$$

To compute the integral on $\mathbb{R}_3$, observe that for small enough $\delta t$, the relation (2.12) holds uniformly on $\Gamma$. Therefore by (2.24), the integral on $\mathbb{R}_3$ is given by

$$\int_{\mathbb{R}_3} \left( \frac{\rho'(s)}{t} - \frac{\rho(s)}{t} + \frac{\delta t}{t} w(s) \right) \log |x - s| ds = O \left( \frac{\delta t \log(x + 2)}{\log \delta t} \right).$$

By applying the argument used for $\mathbb{R}_1$ and $\mathbb{R}_2$ to $\mathbb{R}_4$ and $\mathbb{R}_5$, we obtain

$$\int_{\mathbb{R}_j} \left( \frac{\rho'(s)}{t} - \frac{\rho(s)}{t} + \frac{\delta t}{t} w(s) \right) \log |x - s| ds = O \left( \frac{\delta t \log(x + 2)}{\log \delta t} \right), \quad j = 4, 5.$$

Let us now consider the contribution from the interval $[x - \sigma_1, x + \sigma_1]$. From the power series expansions on $\mathbb{R}_6$, we have the following estimates,

$$Q(s) = Q(x^*) + O \left( \left( \frac{\delta t}{\log \delta t} \right)^\frac{1}{2} \right), \quad \eta(s) = \eta(x^*) + O \left( \left( \frac{\delta t}{\log \delta t} \right)^\frac{1}{2} \right)$$

$$\sqrt{(s - \alpha_t)(s - \beta_t)} = \sqrt{(x^*)^2 - 4} + O \left( \left( \frac{\delta t}{\log \delta t} \right)^\frac{1}{2} \right),$$

$$\log(x - s) = \log(x - x^*) + O \left( \left( \frac{\delta t}{\log \delta t} \right)^\frac{1}{2} \right).$$

Therefore, the integral on $\mathbb{R}_6$ satisfies the following estimate.

$$\int_{\mathbb{R}_6} \tilde{t}\rho'(s) \log(x - s) ds = \frac{Q(x^*)\sqrt{(x^*)^2 - 4} \log(x - x^*)}{2\pi} \times \int_{\mathbb{R}_6} H_t(s) \sqrt{\sigma_t^2 - (s - x^*)^2} ds \left( 1 + O \left( \left( \frac{\delta t}{\log \delta t} \right)^\frac{1}{2} \right) \right).$$

(2.34)
To evaluate the integral on the right, let us note that $H_t(x)$ can be written in the following form [18],

$$H_t(z) = \left(-\frac{\delta t}{\log \delta t}\right)^{1-\frac{1}{\nu}} P \left( (z - x^*) \left(-\frac{\delta t}{\log \delta t}\right)^{-\frac{1}{\nu}} \right), \quad (2.35)$$

where $P(s)$ is the following polynomial of degree $2\nu - 2$,

$$P(s) = \sum_{j=0}^{\nu-1} \frac{(2j)!}{j!j!} s^{2(\nu-1-j)}. \quad (2.36)$$

Then by a change of variable

$$\xi = (s - x^*) \left(-\frac{\delta t}{\log \delta t}\right)^{-\frac{1}{\nu}}$$

in the integral on the right hand side of (2.34), we have

$$\int_{\mathbb{R}_6} \tilde{\rho}'(s) \log(x - s) ds = -\frac{\delta t}{\log \delta t} \frac{Q(x^*) \sqrt{(x^*)^2 - 4 \log(x - x^*)}}{2\pi} \quad (2.37)$$

$$\times \int_{-2y}^{2y} P(\xi) \left(\sqrt{4y^2 - \xi^2}\right) d\xi \left(1 + O \left( \left(\frac{\delta t}{\log \delta t}\right)^{\frac{1}{2\nu}} \right) \right).$$

To evaluate this integral, we will use the following differential equation for $P(\xi)$ in [18].

$$(2\nu - 2)P(\xi) - \xi P'(\xi) = \frac{4y^2}{\xi^2 - 4y^2} (P(\xi) - P(2y)).$$

Using this and integration by parts, we find that the integral in (2.37) is given by

$$\int_{-2y}^{2y} P(\xi) \sqrt{4y^2 - \xi^2} d\xi = \frac{2\pi y^2 P(2y)}{\nu}. \quad (2.38)$$

Hence the integral (2.37) is

$$\int_{\mathbb{R}_6} \tilde{\rho}'(s) \log(x - s) ds = -\frac{\delta t}{\log \delta t} \frac{y^2 P(2y)Q(x^*) \sqrt{(x^*)^2 - 4 \log(x - x^*)}}{\nu}$$

$$\times \left(1 + O \left( \left(\frac{\delta t}{\log \delta t}\right)^{\frac{1}{2\nu}} \right) \right).$$

Now by the use of induction, one can compute $P(2y)$ easily [18],

$$P(2y) = (2y)^{2\nu-2} \sum_{j=0}^{\nu-1} \frac{(2k)!}{k!k!} 4^{-k} = y^{2\nu-2} \frac{(2\nu)!}{2(\nu - 1)!\nu!}.$$
This, together with the expression \( (2.11) \) for \( y \) implies that
\[
\int_{B^*_δ} \tilde{\rho}'(s) \log(x-s)ds = -\frac{\delta t}{\log \delta t} 2\nu \phi(x^*) \log(x-x^*) \left( 1 + O \left( \frac{\delta t}{\log \delta t} \right) \right), \tag{2.39}
\]
which is of order \( \frac{\delta t}{\log \delta t} \). That is,
\[
\int_{B^*_δ} \tilde{\rho}'(s) \log(x-s)ds = O \left( \frac{\delta t \log(x+2)}{\log \delta t} \right). \tag{2.40}
\]
Now by adding (2.28), (2.30), (2.31), (2.32), (2.40), we arrive at (2.21).

Now from (1.12) and (2.21), we see that conditions of the type (2.3) are satisfied for \( \tilde{h}'(x) \).

**Corollary 2.1.** For sufficiently small \( \delta t \), there exist \( \delta > 0 \) such that \([x^*-\sigma_t, x^*+\sigma_t] \subset B^*_δ \) and that
\[
\tilde{h}'_+(x) + \tilde{h}'_-(x) - \frac{V(x)}{t} - \frac{\tilde{l}}{t} = v_t^h(x) \left( \frac{\delta t}{\log \delta t} \right), \quad x \in [\alpha_t, \beta_t] / (B^*_{δ^{-2}} \cup B^*_δ) \]
\[
\Re \left( \tilde{h}'_+(x) + \tilde{h}'_-(x) - \frac{V(x)}{t} - \frac{\tilde{l}}{t} \right) < 0, \quad x \in \mathbb{R}/ \left( \bigcup_{j=1}^{3} B^*_δ \cup [\alpha_t, \beta_t] \right), \tag{2.41}
\]
\[
\tilde{h}'(x) = \left( 1 + \kappa \left( \frac{\delta t}{\log \delta t} \right) \right) \log x + O(1), \quad x \to \infty.
\]
where \( \tilde{l} \) is the constant \( l + (\delta t)\zeta \). The function \( v_t^h(x) \) remains uniformly bounded in \([\alpha_t, \beta_t] \) as \( \delta t \to 0 \), while the constant \( \kappa \) remains finite in the limit. That is, if
\[
\lim_{\delta t \to 0} v_t^h(x) = v^h(x), \quad \lim_{\delta t \to 0} \kappa = \kappa,
\]
then \( v^h(x) \) is uniformly bounded in \([\alpha_t, \beta_t] \) and \( \kappa \) is finite.

Corollary 2.1 suggests that \( \tilde{\rho}'(x) \) is a good approximation to the actual equilibrium density \( \rho'(x) \).

The following corollary follows immediately from the proof of proposition 2.2 and the fact that \( h(x^*) - \frac{V(x^*)}{2} - \frac{l}{2} = 0 \).

**Corollary 2.2.** Inside \( B^*_δ \), the following is satisfied.
\[
\int_{\alpha_t}^{\beta_t} \tilde{\rho}'(s) \log(x-s)ds - \frac{h(x)}{t} = \delta t \int_{-2}^{2} w(s) \log(x-s)ds + O \left( \frac{\delta t}{\log \delta t} \right). \tag{2.42}
\]
In particular, by using \( h(x^*) = \frac{V(x^*)}{2} + \frac{l}{2} \), we see that the following is satisfied at \( x = x^* \).
\[
\int_{\alpha_t}^{\beta_t} \tilde{\rho}'(s) \log(x^*-s)ds - \frac{V(x^*)}{2t} - \frac{\tilde{l}}{2t} = \delta t \phi(x^*) + O \left( \frac{\delta t}{\log \delta t} \right), \tag{2.43}
\]
where \( \phi(x^*) \) is defined in (1.13).
This corollary is essential for the construction of the local parametrix inside the neighborhood $B^x_\delta$. (See section 3.3)

### 3 Riemann-Hilbert analysis

A result by Fokas, Its and Kitaev [19] shows that the orthogonal polynomials (1.4) can be expressed in terms of a Riemann-Hilbert problem. In this section we will apply the Deift-Zhou steepest decent method to approximate this Riemann-Hilbert problem by a Riemann-Hilbert problem that is solvable explicitly. We will achieve this by using the approximated equilibrium measure constructed in section 2. We will modify the measure $\tilde{\rho}(x)dx$ by replacing the charges on $[x^*-\sigma_t, x^*+\sigma_t]$ by a point charge. This then allows us to construct local parametrix near the critical point $x^*$ from orthogonal polynomials with weight $e^{-x^2\nu}$ on the real axis.

### 3.1 Riemann-Hilbert problem for the orthogonal polynomials

One important property of the orthogonal polynomials (1.4) is that they can be represented as a solution to a Riemann-Hilbert problem [19].

Consider the following Riemann-Hilbert problem for a matrix-valued function $Y(x) = Y_{n,N}(x)$.

1. $Y(x)$ is analytic on $\mathbb{C}/\mathbb{R}$
2. $Y_+(x) = Y_-(x) \begin{pmatrix} e^{-NV(x)} & 1 \\ 0 & 1 \end{pmatrix}$, $x \in \mathbb{R}$
3. $Y(x) = (I + O(x^{-1})) \begin{pmatrix} x^n & 0 \\ 0 & x^{-n} \end{pmatrix}$, $x \to \infty$ (3.1)

where $Y_+(x)$ and $Y_-(x)$ denotes the limiting values of $Y(x)$ as it approaches the left and right hand sides of the real axis. This Riemann-Hilbert problem has the following unique solution.

$$Y(x) = \begin{pmatrix} \pi_n(x) \\ \kappa_{n-1} \pi_{n-1}(x) \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\pi_n(s)e^{-NV(s)}}{s-x} ds \end{pmatrix}$$

where $\kappa_{n-1} = -2\pi i h_{n-1}^{-1}$ [12]. The correlation kernel (1.5) can be expressed in terms of the solution of the Riemann-Hilbert problem $Y(x)$ via [8]

$$K_{n,N}(x, y) = e^{-N Y(x) + Y(y)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Y_+^{-1}(y) Y_+(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$ (3.2)

We shall apply the Deift-Zhou steepest decent method to the Riemann-Hilbert problem (3.1) to obtain the asymptotics for the orthogonal polynomials and the correlation kernel.
3.2 Initial transformation of the Riemann-Hilbert problem

We shall perform a series of transformation to the Riemann-Hilbert problem (3.1) and approximate it with a Riemann-Hilbert problem that can be solved explicitly. We will then use the solution of the final model Riemann-Hilbert problem to compute the asymptotics of the orthogonal polynomials and the correlation kernel.

3.2.1 The \( g \)-function

To begin with, let us denote \( t \) by \( \frac{n}{N} \) and rewrite the jump matrix in (3.1) as

\[
\begin{pmatrix}
1 & e^{-NV(x)} \\
0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & e^{-nV_t(x)} \\
0 & 1
\end{pmatrix}
\]

where \( V_t(x) = \frac{1}{t} V(x) \). We will now define a function \( g^t(x) \) from the function \( \tilde{h}^t(x) \) constructed in section 2 to transform this Riemann-Hilbert problem.

Let \( u^t \) be the following

\[
u^t = n \int_{\mathbb{R}_6} \tilde{\rho}^t(s) ds
\]

where \( \mathbb{R}_6 \) is defined in \( \text{(2.22)} \).

For later convenience, let us denote by \( \bar{u}^t \) the non-negative integer closest to \( u^t \):

\[
\bar{u}^t = \left[ u^t + \frac{1}{2} \right], \quad u^t \geq 0
\]

\[
\bar{u}^t = 0, \quad u^t < 0.
\]

(3.3)

From (2.39), we see that if \( u^t > 0 \), then

\[
u^t = -n \frac{\delta t}{\log \delta t} 2\nu \phi(x^*) \left( 1 + O \left( \left( \frac{\delta t}{-\log \delta t} \right)^\frac{1}{\nu} \right) \right).
\]

(3.4)

By inserting the scaling (1.7a) into (3.4), we see that \( u^t \) is finite in this limit.

As mentioned before, we would like to replace the charges in the interval \([x^* - \sigma_t, x^* + \sigma_t] \) by a point charge when \( t > 1 \). We should therefore define the \( g \)-function to be the following.

\[
g^t(x) = \int_{\alpha_t}^{\beta_t} \left( \tilde{\rho}^t(s) - \frac{\delta t}{\log \delta t} \frac{8\pi \sqrt{(s - \alpha_t)(\beta_t - s)}}{(\alpha_t + \beta_t)^2} \right) \log(x - s) ds
\]

\[+ \frac{u^t}{n} \log(x - x^*), \quad t > 1,
\]

(3.5)

\[
g^t(x) = \int_{\alpha_t}^{\beta_t} \rho^t(s) \log(x - s) ds, \quad t \leq 1,
\]
where in the above equations, we have extended the definitions of the end points $\alpha_t$ and $\beta_t$ (2.8) to include the values of $t$ that are less than or equal to 1:

$$\begin{align*}
\alpha_t &= -2 + \frac{\delta t}{(2 + x^*)^{2\nu-1}Q(-2)}, \\
\beta_t &= 2 - \frac{\delta t}{(x^* - 2)^{2\nu-1}Q(2)}, \quad t > 1, \\
\alpha_t &= a_t, \quad \beta_t = b_t, \quad t \leq 1.
\end{align*}$$

(3.6)

Then, from (3.4) and (2.40), we see that, for $x \in \mathbb{C}/B^r_\delta$,

$$
\frac{u^t}{n} \log(x - x^*) = O\left(\frac{\delta t \log(x - x^*)}{\log \delta t}\right)
$$

and

$$
\int_{x^* - 1}^{x^* + 1} \bar{\rho}^t(s) \log(x - s) ds = O\left(\frac{\delta t \log(x - x^*)}{\log \delta t}\right).
$$

It then follows from corollary 2.1 and the properties (2.41) that the function $g^t(x)$ satisfies the following

**Proposition 3.1.** For sufficiently small $\delta t$, there exist $\delta > 0$ such that $[x^* - \sigma_t, x^* + \sigma_t] \subset B^r_\delta$ and that

$$
\begin{align*}
&g^t_+(x) + g^t_-(x) - \frac{V(x)}{t} - \bar{l} = v_t(x) \left(\frac{\delta t}{\log \delta t}\right), \quad x \in [\alpha_t, \beta_t]/(B^r_\delta \cup B^r_\delta) \\
&g^t_+(x) + g^t_-(x) - \frac{V(x)}{t} - \bar{l} < 0, \quad x \in \mathbb{R}/\left(\bigcup_{j=1}^3 B^r_\delta \cup [\alpha_t, \beta_t]\right), \\
&g^t(x) = \log x + O(1), \quad x \to \infty.
\end{align*}
$$

(3.7)

where $\bar{l}$ is the following constant

$$
\bar{l} = \begin{cases} 
\bar{l} - (\delta t)\varsigma, & t > 1; \\
\bar{l}t_t, & t \leq 1.
\end{cases}
$$

The function $v_t(x)$ remains uniformly bounded in $[\alpha_t, \beta_t]$ as $\delta t \to 0$. That is, if

$$
\lim_{\delta t \to 0} v_t(x) = v(x),
$$

then $v(x)$ is uniformly bounded in $[\alpha_t, \beta_t]$. In particular, $v_t(x)$ is zero when $t \leq 1$.

The function $g^t(x)$ is analytic on $\mathbb{C}/(-\infty, x^*)$ and has the following jump discontinuities on $(-\infty, \alpha_t)$ and $(\beta_t, x^*)$.

$$
\begin{align*}
&g^t_+(x) - g^t_-(x) = 2\pi i \int_x^{\beta_t} \left(\bar{\rho}^t(s) - \frac{\delta t}{\log \delta t} \frac{8\pi \sqrt{(s - \alpha_t)(\beta_t - s)}}{(\alpha_t + \beta_t)^2}\right) ds \\
&g^t_+(x) - g^t_-(x) = 2\pi i, \quad x \in (-\infty, \alpha_t) \\
&g^t_+(x) - g^t_-(x) = 2\pi i \frac{u^t}{n}, \quad x \in (\beta_t, x^*).
\end{align*}
$$

(3.8)
we can now transform the Riemann-Hilbert problem with the function \( g^t(x) \).

Let \( T(x) \) be the following function
\[
T(x) = e^{-\frac{n \delta t}{2t}} Y(x) e^{-n g^t(x) \sigma_3} e^{-\frac{n \delta t}{2t}},
\]
where \( \sigma_3 \) is the Pauli matrix
\[
\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Then \( T(x) \) is a solution to the following Riemann-Hilbert problem.

1. \( T(x) \) is analytic in \( \mathbb{C}/\mathbb{R} \);
2. \( T_+(x) = T_-(x) J_T(x), \quad x \in \mathbb{R} \);
3. \( T(x) = I + O(x^{-1}), \quad x \to \infty \).

where \( J_T(x) \) is the following matrix on \( \mathbb{R} \).
\[
J_T(x) = \begin{pmatrix} e^{-n(g^t_+(x)-g^t_-(x))} & e^n\left(g^t_+(x)+g^t_-(x)-\frac{t}{2}\right) \\ 0 & e^n(g^t_+(x)-g^t_-(x)) \end{pmatrix}, \quad x \in \mathbb{R}.
\]

3.2.2 Opening of the lens

We now perform a standard technique in the steepest decent method \([3], [15], [16]\). First note that, from (3.7), we see that \( J_T(x) \) becomes the following on the interval \([\alpha_t, \beta_t]\).
\[
J_T(x) = \begin{pmatrix} e^{-n(g^t_+(x)-g^t_-(x))} & e^{2D_n(x)} \\ 0 & e^n(g^t_+(x)-g^t_-(x)) \end{pmatrix}, \quad x \in [\alpha_t, \beta_t]
\]
where \( D_n(x) \) is the function
\[
D_n(x) = \nu_t(x) \frac{n \delta t}{\log \delta t}, \quad t > 1, \quad D_n(x) = 0, \quad t \leq 1
\]
which is bounded on \([\alpha_t, \beta_t]\) under the double scaling limit \((1.7a)\).

Then from (3.8), the jump matrix \( J_T(x) \) has the following factorization on \([\alpha_t, \beta_t]\).
\[
J_T(x) = \begin{pmatrix} 1 & 0 \\ e^n(V_t(x)-2g^t_+(x)+\frac{t}{2})-2D_n(x) & 1 \end{pmatrix} \begin{pmatrix} 0 & e^{2D_n(x)} \\ -e^{-2D_n(x)} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^n(V_t(x)-2g^t_+(x)+\frac{t}{2})-2D_n(x) & 1 \end{pmatrix}
\]

As in \([3], [15], [16]\), we can open a lens around the interval \([\alpha_t, \beta_t]\) as shown in Figure 1 and define the matrix \( S(x) \) to be the following
\[
S(x) = \begin{cases} T(x), & x \text{ outside the lens;} \\ T(x) \begin{pmatrix} 1 & 0 \\ -e^n(V_t(x)-2g^t(x)+\frac{t}{2})-2D_n(x) & 1 \end{pmatrix}, & x \in L_+; \\ T(x) \begin{pmatrix} 1 & 0 \\ e^n(V_t(x)-2g^t(x)+\frac{t}{2})-2D_n(x) & 1 \end{pmatrix}, & x \in L_-.
\end{cases}
\]
Figure 1: The opening of the lens and different regions in the lens.

Then the function $S(x)$ will satisfy the following Riemann-Hilbert problem.

1. $S(x)$ is analytic in $\mathbb{C}/\mathbb{R}$;
2. $S_+(x) = S_-(x)J_S(x)$, $x \in \mathbb{R}$;
3. $S(x) = I + O(x^{-1})$, $x \to \infty$.

where $J_S(x)$ is now defined by the following

$$J_S(x) = \begin{pmatrix} 1 & 0 \\ e^{n(V_t(x) - 2g_t(x) + \frac{i}{2}) - 2D_n(x)} & 0 \end{pmatrix}, \quad x \in C_1 \cup C_2$$

$$J_S(x) = \begin{pmatrix} 0 & e^{2D_n(x)} \\ -e^{-2D_n(x)} & 0 \end{pmatrix}, \quad x \in (\alpha_t, \beta_t)$$

$$J_S(x) = \begin{pmatrix} 1 & e^{n(2g_t(x) - V_t(x) - \frac{i}{2}) + 2D_n(x)} \\ 0 & 1 \end{pmatrix}, \quad x \in (-\infty, \alpha_t)$$

$$J_S(x) = \begin{pmatrix} e^{-2\pi i u^t} & e^{n(V_t(x) - 2g_t(x) - 2D_n(x))} \\ 0 & e^{2\pi i u^t} \end{pmatrix}, \quad x \in (\beta_t, x^*)$$

Then from (3.7), we see that for some large enough $n$ and $t$ close to 1 such that $\delta t \frac{n}{\log n} = o(1)$, we have

1. $D_n(x)$ is uniformly bounded on $[\alpha_t, \beta_t]$.

2. $e^{n(2g_t(x) - V_t(x) - \frac{i}{2})} \to 0$ on $\mathbb{R}/\left( \bigcup_{j=1}^{3} B_{\delta}^{r_j} \cup [\alpha_t, \beta_t] \right)$.

3. $e^{n(V_t(x) - 2g_t(x) + \frac{i}{2})} \to 0$ on $(C_1 \cup C_2) / (B_{\delta}^{-2} \cup B_{\delta}^2)$.

Therefore, in the double scaling limit, the jump matrix $J_S(x)$ behaves as

$$J_S(x) \to I, \quad (x \in \mathbb{R} \cup C_1 \cup C_2) / \left( \bigcup_{j=1}^{3} B_{\delta}^{r_j} \cup [\alpha_t, \beta_t] \right),$$

$$J_S(x) = \begin{pmatrix} 0 & e^{2D_n(x)} \\ -e^{-2D_n(x)} & 0 \end{pmatrix}, \quad x \in (\alpha_t, \beta_t)$$
3.3 Parametrix outside of the points $\alpha_t$, $\beta_t$ and $x^*$

We will now construct the parametrix outside of the singular points. We would like to find a solution to the following Riemann-Hilbert problem.

1. $S^\infty(x)$ is analytic in $\mathbb{C}/(\mathbb{R} \cup B_0^*)$;
2. $S^\infty(x) = S^\infty(x)J^\infty(x)$, $x \in \mathbb{R}$;
3. $S^\infty(x) = I + O(x^{-1})$, $x \to \infty$. \hfill (3.14)

where $J^\infty(x)$ is the following matrix-valued function.

\[
J^\infty(x) = \begin{pmatrix}
e^{-2\pi i u} & 0 \\
0 & e^{2\pi i u}
\end{pmatrix}, \quad x \in (\beta_t, x^*)
\]

\[
J^\infty(x) = \begin{pmatrix}0 & e^{2D_n(x)} \\
-e^{-2D_n(x)} & 0\end{pmatrix}, \quad x \in (\alpha_t, \beta_t). \hfill (3.15)
\]

We should construct several scalar functions and use them to ‘dress’ the 1-cut parametrix constructed in [15] so that it satisfies the Riemann-Hilbert problem (3.14) with the jumps (3.15).

3.3.1 The limiting Abelian differential

The discussions in sections 3.3.1 and 3.3.2 are only relevant when $t > 1$. Let us first construct a function $F(x)$ with the following jump discontinuities

\[
F^+ (x) = -F^-(x), \quad x \in [\alpha_t, \beta_t]
\]

\[
F^+ (x) = F^- (x) + 2\pi i, \quad x \in [\beta_t, x^*]
\]

We have the following

**Lemma 3.1.** The function $F(x)$ defined by

\[
F(x) = \int_{\alpha_t}^x \frac{\sqrt{(x^* - \alpha_t)(x^* - t)} ds}{\sqrt{(s - \alpha_t)(s - \beta_t)(s - x^*)}} = \log \frac{\sqrt{x^* - \alpha_t}}{\sqrt{x^* - \beta_t}} - \frac{\sqrt{x - \alpha_t}}{\sqrt{x - \beta_t}} + \frac{\sqrt{x - \alpha_t}}{\sqrt{x - \beta_t}}
\]

satisfies the following scalar Riemann-Hilbert problem.

1. $F(x)$ is analytic in $\mathbb{C}/[\alpha_t, x^*]$;
2. $F^+(x) = -F^-(x)$, $x \in [\alpha_t, \beta_t]$;
3. $F^+(x) = F^-(x) + 2\pi i$, $x \in [\beta_t, x^*]$;
4. $F(x) = \log(x - x^*) + O(1)$, $x \to x^*$;
5. $F(x) = F_0 + O(1)$, $x \to \infty$; \hfill (3.17)
6. $F(x)$ is bounded as $x$ approaches $\alpha_t$ or $\beta_t$. 

\[
F_0 = \log \frac{\sqrt{x^* - \alpha_t}}{\sqrt{x^* - \beta_t}} - 1, \quad \sqrt{\frac{x - \alpha_t}{x - \beta_t}} + 1
\]
where the square roots and logarithm are chosen such that the branch cut is on the negative real axis. Also, to avoid ambiguity, we set

\[ \sqrt{\frac{x - \alpha_t}{x - \beta_t}} = \frac{\sqrt{(x - \alpha_t)(x - \beta_t)}}{(x - \beta_t)^2} \]

Proof. First let us show that the only singularity of \( F(x) \) is at \( x^* \). The function \( F(x) \) can become singular when the argument becomes zero or infinity. The numerator and the denominator of the argument can become zero if

\[ \sqrt{x^* - \alpha_t} + \sqrt{x - \alpha_t} = 0, \]

which implies

\[ \frac{x^* - \alpha_t}{x^* - \beta_t} \times \frac{x - \alpha_t}{x - \beta_t} = \frac{x - \alpha_t}{x - \beta_t}, \]

\[ x = x^*. \]

Since the denominator is non-zero at \( x = x^* \), we see that the denominator does not vanish for all \( x \in \mathbb{C} \). Near \( x = x^* \), we can expand the numerator and the denominator in a power series of \( x - x^* \).

\[ \frac{\sqrt{x^* - \alpha_t}}{x^* - \beta_t} - \frac{\sqrt{x - \alpha_t}}{x - \beta_t} = c_0(x - x^*) + O((x - x^*)^2), \]

where \( c_0 \) is the following constant

\[ c_0 = -\frac{\frac{d}{dx} \left( \sqrt{\frac{x - \alpha_t}{x - \beta_t}} \right)_{x=x^*}}{2 \sqrt{x^* - \beta_t}} = \frac{\beta_t - \alpha_t}{4(x^* - \beta_t)(x^* - \alpha_t)} \neq 0. \]

Therefore near \( x = x^* \), \( F(x) \) behaves like

\[ F(x) = \log(x - x^*) + O(1), \quad x \to x^*, \]

this proves 4.

The other points where \( F(x) \) can be singular are the points \( x = \alpha_t, \beta_t \) or \( x = \infty \), where the argument may become infinite. However, near \( x = \alpha_t \), the function \( \sqrt{\frac{x - \alpha_t}{x - \beta_t}} \) remains finite and therefore \( F(x) \) does not have singularity near it. Let the argument inside the logarithm of (3.16) be \( \Phi(x) \).

\[ \Phi(x) = \frac{\sqrt{x^* - \alpha_t} - \sqrt{x - \alpha_t}}{\sqrt{x^* - \beta_t} + \sqrt{x - \beta_t}}. \quad (3.18) \]
Then near $x = \beta_t$, $\Phi(x)$ behaves like

$$
\Phi(x) = \frac{\sqrt{(x^* - \alpha_t)(x - \beta_t)}}{x^* - \beta_t} - \frac{\sqrt{x - \alpha_t}}{\sqrt{x^* - \beta_t}} + \frac{1 - \frac{x}{x^*}}{1 - \frac{\beta_t}{x^*}}
$$

which is bounded and non-zero as $x \to \beta_t$.

We now consider the point $x = \infty$. Near $x = \infty$, we can rewrite $\Phi(x)$ as

$$
\Phi(x) = \sqrt{\frac{x^* - \alpha_t}{x^* - \beta_t}} - \sqrt{\frac{1 - \frac{x}{x^*}}{1 - \frac{\beta_t}{x^*}}}
$$

Therefore the asymptotic behavior of $\Phi(x)$ near $x = \infty$ is given by

$$
\Phi(x) = \sqrt{\frac{x^* - \alpha_t}{x^* - \beta_t}} - 1 + O(x^{-1}).
$$

Hence $F(x)$ has a singularity at $x = x^*$ only and this proves 5 and 6.

We will now study the jump discontinuities of $F(x)$. First note that $F(x)$ can only have jump discontinuities outside $[\alpha_t, \beta_t]$ if

$$
\Phi(x) = \sqrt{\frac{x^* - \alpha_t}{x^* - \beta_t}} - \sqrt{\frac{\alpha_t}{\beta_t}} \in \mathbb{R}
$$

Simple calculations shows that $x \in \mathbb{R}$. Therefore $F(x)$ can only have jumps on the real axis.

We will first consider the jump on $[\alpha_t, \beta_t]$. On $[\alpha_t, \beta_t]$ the square function $\sqrt{\frac{x - \alpha_t}{x - \beta_t}}$ has the following jump discontinuity.

$$
\left(\sqrt{\frac{x - \alpha_t}{x - \beta_t}}\right)_+ - \left(\sqrt{\frac{x - \alpha_t}{x - \beta_t}}\right)_-
$$

Hence $F_-(x)$ is given by

$$
F_-(x) = \log \frac{\sqrt{\frac{x^* - \alpha_t}{x^* - \beta_t}} - \left(\sqrt{\frac{x - \alpha_t}{x - \beta_t}}\right)_-}{\sqrt{\frac{x^* - \alpha_t}{x^* - \beta_t}} + \left(\sqrt{\frac{x - \alpha_t}{x - \beta_t}}\right)_-} = \log \frac{\sqrt{\frac{x^* - \alpha_t}{x^* - \beta_t}} + \left(\sqrt{\frac{x - \alpha_t}{x - \beta_t}}\right)_-}{\sqrt{\frac{x^* - \alpha_t}{x^* - \beta_t}} - \left(\sqrt{\frac{x - \alpha_t}{x - \beta_t}}\right)_+} = -F_+(x).
$$

This proves property 2.
We now consider the jump on $[\beta_t, x^*]$. The function $\Phi(x)$ in (3.18) is real on $\mathbb{R}/[\alpha_t, \beta_t]$ and we need to show that it is negative on $(\beta_t, x^*)$ and positive elsewhere. Let us first consider a point $x \in (\beta_t, \infty)$. The denominator of $\Phi(x)$ is positive for $x \in (\beta_t, \infty)$. To study the signs of the numerator, let us consider its derivative
\[
\frac{d}{dx} \left( \frac{x^* - \alpha_t}{x^* - \beta_t} - \frac{x - \alpha_t}{x - \beta_t} \right) = \frac{1}{2} \frac{x - \beta_t}{x - \alpha_t} \frac{\beta_t - \alpha_t}{(x - \beta_t)^2} > 0, \quad x \in (\beta_t, \infty),
\]
we see that the numerator is a strictly increasing function on $(\beta_t, \infty)$. Since it vanishes at $x = x^*$, we see that it is negative on $(\beta_t, x^*)$ and positive on $(x^*, \infty)$.

Now let us look at the sign of $\Phi(x)$ on $(-\infty, \alpha_t)$. First note that, on $(\beta_t, \infty)$, the square root $\sqrt{\frac{x^* - \alpha_t}{x^* - \beta_t}}$ is positive and strictly decreasing from (3.19). Near $-\infty$, it approaches 1 while at $\alpha_t$, it becomes zero. Therefore on $(-\infty, \alpha_t)$, it takes values between 0 and 1.

Therefore we have
\[
\sqrt{x^* - \alpha_t} > 1.
\]
Now let $x \in (-\infty, \alpha_t)$. In this region, the square root $\sqrt{x^* - \alpha_t}$ is positive and strictly decreasing from (3.19). Near $-\infty$, it approaches 1 while at $\alpha_t$, it becomes zero. Therefore on $(-\infty, \alpha_t)$, it takes values between 0 and 1.

Hence the $\Phi(x)$ is positive on $(-\infty, \alpha_t)$. Summarizing, we have
\[
\begin{align*}
\Phi(x) &> 0, \quad x \in (-\infty, \alpha_t) \cup (x^*, \infty) \\
\Phi(x) &< 0, \quad x \in (\beta_t, x^*).
\end{align*}
\]
This proves 3.

Since $F(x)$ cannot have any jump discontinuities and singularities other than the ones that are considered here, property 1. is true. \hfill $\square$

**Remark 3.1.** The function $F(x)$ can be thought of as the limit of an Abelian integral on an elliptic curve. Let us consider the following elliptic curve
\[
z^2 = (x - \alpha_t)(x - \beta_t)((x - x^*)^2 - \sigma_t^2)
\]
and define the $a$ and $b$-cycles of this curve as in Figure 2. Then the normalized holomorphic Abelian differential on this curve is given by
\[
\Omega(x) = \frac{Cdx}{\sqrt{(x - \alpha_t)(x - \beta_t)((x - x^*)^2 - \sigma_t^2)}}
\]
for some constant $C$ such that
\[ \oint_a \Omega(x) = 1 \]

In the limit $\sigma_t \to 0$, the curve becomes degenerate and the Abelian integral $\int_x \Omega(s)$ degenerates into the function $F(x)$.

### 3.3.2 Scalar function with jump $D_n(x)$

We will now seek a scalar function $K_n(x)$ that is bounded at infinity and has jump $2D_n(x)$ on $[\alpha_t, \beta_t]$. (cf. the Szego function used in [23], [32])

We shall construct a function $K(x)$ that satisfies the following Riemann-Hilbert problem.

1. $K(x)$ is analytic on $\mathbb{C}/[\alpha_t, \beta_t]$;
2. $K_+(x) = -K_-(x) + 2D_n(x), \quad x \in [\alpha_t, \beta_t]$;
3. $K(x) = K_0 + O(x^{-1}), \quad x \to \infty$,

\begin{equation}
K_0 = -\frac{1}{2\pi i} \int_{\alpha_t}^{\beta_t} \frac{2D_n(s)ds}{\left(\sqrt{(s-\alpha_t)(s-\beta_t)}\right)_+}.
\end{equation}

This function can be constructed by the use of Cauchy transform easily [26].

\begin{equation}
K(x) = \frac{\sqrt{(x-\alpha_t)(x-\beta_t)}}{2\pi i} \int_{\alpha_t}^{\beta_t} \frac{2D_n(s)ds}{\left(\sqrt{(s-\alpha_t)(s-\beta_t)}\right)_+ (s-x)}
\end{equation}

**Lemma 3.2.** The function $K(x)$ defined in (3.23) satisfies the Riemann-Hilbert problem (3.22).

**Proof.** Let $C(f)$ be the Cauchy transform

\[ C(f)(x) = \frac{1}{2\pi i} \int_{\alpha_t}^{\beta_t} \frac{f(s)ds}{s-x} \]
Then from the Plemelj formula (See, e.g. [26]), we have
\[ C_{\pm}(f)(x) = \pm \frac{1}{2} f + \frac{1}{2\pi i} \int_{\alpha_t}^{\beta_t} \frac{f(s)ds}{s-x}, \quad x \in [\alpha_t, \beta_t], \]
where the principal value is taken in the integral on the right hand side. Taking into account the change of sign of \( \sqrt{(x-\alpha_t)(x-\beta_t)} \) across \([\alpha_t, \beta_t]\), we have
\[ K_+(x) + K_-(x) = \left( \sqrt{(x-\alpha_t)(x-\beta_t)} \right)_+ + \frac{2D_n(x)}{\sqrt{(x-\alpha_t)(x-\beta_t)}} \]
\[ = 2D_n(x). \]
To see that \( K(x) \) has the desired property at \( x = \infty \), let us write the factor \( \frac{1}{s-x} \) in a power series near \( x = \infty \).
\[ \frac{1}{s-x} = -\frac{1}{x} \left( 1 + \frac{s}{x} + O(x^{-2}) \right). \]
Therefore the function \( K(x) \) behaves as follows as \( x \to \infty \).
\[ K(x) = -\frac{\sqrt{(x-\alpha_t)(x-\beta_t)}}{2\pi i x} \left( \int_{\alpha_t}^{\beta_t} \frac{2D_n(s)ds}{\sqrt{(s-\alpha_t)(s-\beta_t)}} + O(x^{-1}) \right) \]
\[ = -\frac{1}{2\pi i} \int_{\alpha_t}^{\beta_t} \frac{2D_n(s)ds}{\sqrt{(s-\alpha_t)(s-\beta_t)}} + O(x^{-1}) \]
which is property 3. in (3.22).

### 3.3.3 Parametrix outside of special points

We are now in a position to construct the parametrix outside of the special points. First let us consider the following matrix.
\[ \Pi(x) = \begin{pmatrix} \gamma(x) + \gamma(x)^{-1} & 0 & 2 \gamma(x)^{-1} \\ \frac{\gamma(x)^2 - 2\gamma(x)^{-1}}{2i} & \gamma(x) & \frac{2i}{2i} \\ \frac{2i}{2i} & \frac{2i}{2i} \end{pmatrix} \]
where \( \gamma(x) = \gamma_t(x) \) is defined by
\[ \gamma(x) = \left( \frac{x-\beta_t}{x-\alpha_t} \right)^{\frac{1}{4}} \]
Recall that this matrix satisfies the following Riemann-Hilbert problem [15], [12], [3].
1. \( \Pi(x) \) is analytic on \( \mathbb{C}/[\alpha_t, \beta_t] \);
2. \( \Pi_+(x) = \Pi_-(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad x \in [\alpha_t, \beta_t] \);
3. \( \Pi(x) = I + O(x^{-1}), \quad x \to \infty \).
We can now combine \( \Pi(x) \), \( K(x) \) and \( F(x) \) to form our parametrix outside the special points. The main result is the following.

**Proposition 3.2.** The matrix \( S^\infty(x) \) defined by

\[
S^\infty(x) = e^{(K_0 + (u^t - \nu^t)F_0)} \Pi(x) e^{-(K(x) + (u^t - \nu^t)F(x))} \sigma_3
\]

(3.25)
satisfies the Riemann-Hilbert problem (3.14) and (3.15). In particular, when \( t \leq 1 \), both \( u^t - \nu^t \) and \( K(x) \) vanishes and the above equation reduces to

\[
S^\infty(x) = \Pi(x), \quad t \leq 1.
\]

(3.26)

**Proof.** First let us consider the asymptotic behavior near \( x = \infty \). We have

\[
S^\infty(x) = e^{(K_0 + (u^t - \nu^t)F_0)} \Pi(x) e^{-(K(x) + (u^t - \nu^t)F(x))} \sigma_3 = (I + O(x^{-1})
\]

This proves 3. in (3.14). Next we consider the jump discontinuities. It is given by

\[
S^\infty_+(x) = S^\infty_-(x) \begin{pmatrix} e^{A_1(x)} & 0 \\ 0 & e^{-A_1(x)} \end{pmatrix}, \quad x \in (-\infty, x^*)/[\alpha_t, \beta_t]
\]

(3.27)

where \( A_i(x) \) is given by

\[
A_1(x) = (u^t - \nu^t)(-F_+(x) + F_-(x)), \quad x \in (\beta_t, x^*),
\]

\[
A_2(x) = (\nu^t - u^t)(F_+(x) + F_-(x)) + K_+(x) + K_-(x), \quad x \in (\alpha_t, \beta_t).
\]

From (3.17) and (3.22), we see that \( A_1(x) \) and \( A_2(x) \) are in fact the following

\[
A_1(x) = -2\pi i (u^t - \nu^t), \quad x \in (\beta_t, x^*)
\]

\[
A_2(x) = 2D_n(x), \quad x \in (\alpha_t, \beta_t).
\]

Since \( \nu^t \) is an integer, we see that

\[
e^{A_1(x)} = e^{-2\pi i u^t}, \quad x \in (\beta_t, x^*).
\]

Substituting these back into (3.27), we see that the matrix \( S^\infty(x) \) does indeed satisfy the jump conditions (3.15).

**Remark 3.2.** The appearance of the degenerate Abelian integral \( F(x) \) (3.16) in the parametrix (3.25) comes from the appearance of the elliptic theta function in the 1-cut parametrix. Let the 1-cut parametrix outside of the special points be \( M^\infty(x) \). Then \( M^\infty(x) \) is given by (3.23):

\[
M^\infty(x) = H \begin{pmatrix} \frac{\gamma + 1}{2} \frac{\theta(W(x) - (u^t - \nu^t) + d)}{\theta(W(x) - (u^t - \nu^t) - d)} & \frac{\gamma - 1}{2} \frac{\theta(-W(x) - (u^t - \nu^t) + d)}{\theta(-W(x) - (u^t - \nu^t) - d)} \\ \frac{\gamma + 1}{2} \frac{\theta(W(x) + d)}{\theta(W(x) - d)} & \frac{\gamma - 1}{2} \frac{\theta(-W(x) + d)}{\theta(-W(x) - d)} \end{pmatrix}, \quad \Im(z) > 0
\]

\[
M^\infty(x) = H \begin{pmatrix} \frac{\gamma + 1}{2} \frac{\theta(W(x) - (u^t - \nu^t) + d)}{\theta(W(x) - (u^t - \nu^t) - d)} & \frac{\gamma - 1}{2} \frac{\theta(-W(x) - (u^t - \nu^t) + d)}{\theta(-W(x) - (u^t - \nu^t) - d)} \\ \frac{\gamma + 1}{2} \frac{\theta(W(x) + d)}{\theta(W(x) - d)} & \frac{\gamma - 1}{2} \frac{\theta(-W(x) + d)}{\theta(-W(x) - d)} \end{pmatrix}, \quad \Im(z) < 0.
\]
for some scalar constant $d$ and constant diagonal matrix $H$, where $W(x)$ is the Abelian integral $\int^{x} \Omega(s)$ and $\theta(z)$ is the elliptic theta function. One can think of the parametrix $S^{\infty}(x)$ as a degenerate version of $M^{\infty}(x)$ as the branch cut $[x^* - \sigma_t, x^* + \sigma_t]$ in $(3.20)$ is closing up and the curve degenerates into a genus zero curve. In this case, the Abelian integral degenerates into the function $F(x)$ and while the theta function degenerates into an exponential function. In the multi-cut case, one could apply the analysis similar to those in [1], [21] to obtain degenerate hyper-elliptic theta functions and use them to construct the suitable parametrix.

### 3.4 Parametrix near the edge points $\alpha_t$ and $\beta_t$

At the edge points $\alpha_t$ and $\beta_t$ the approximated density $\tilde{\rho}(x)$ vanishes like a square root and the local parametrices $S^{\pm 2}(x)$ near these points can be constructed by the use of Airy-functions. Such construction has been done many times in the literature and we should not repeat the details here. An interested reader can consult [15], [16], [3] for example.

### 3.5 Local parametrix near the critical point $x^*$ for $t > 1$

We will now consider the parametrix near the critical point $x^*$. As in [18], the parametrix will be constructed out of the monic orthogonal polynomial $\pi_{\nu}^{\mu}(\zeta)$ of degree $\nu$ and weight $e^{-\zeta^2 \nu}$, where $2\nu$ is the order of vanishing of $2h(x) - V(x) + l$ at $x^*$.

We would like to construct a parametrix $S^{x^*}(x)$ in $B_\delta^{x^*}$ such that

1. $S^{x^*}(x)$ is analytic in $B^{x^*}_\delta / \left(B^{x^*}_\delta \cap \mathbb{R}\right)$;
2. $S^{x^*}_+(x) = S^{x^*}_-(x) J_S(x), \quad x \in B^{x^*}_\delta \cap \mathbb{R}$; (3.28)
3. $S^{x^*}(x) = (I + o(1)) S^{\infty}(x)$ as $n \to \infty, t \to 1$, uniformly in $\partial B^{x^*}_\delta$.

#### 3.5.1 Conformal map in $B^{x^*}_\delta$

Let us define a conformal map $\zeta = f(x)$ that maps the neighborhood $B^{x^*}_\delta$ into the complex plane, such that, as $n \to \infty$, the boundary of $B^{x^*}_\delta$ is mapped into infinity. We will define $\zeta$ as follows.

$$\zeta = f(x) = (-n(2h(x) - V(x) - l))^{\frac{1}{2\nu}},$$

(3.29)

where the $\frac{1}{2\nu}$th-root is chosen such that the intervals $[x^* - \delta, x^*]$ and $[x^*, x^* + \delta]$ are mapped onto the negative and positive real axis respectively. This is possible because $h(x) - \frac{V(x)}{2} - \frac{l}{2}$ vanishes to order $2\nu$ at $x^*$ and that it is real and negative on the interval $[x^* - \delta, x^* + \delta]$ due to (2.41).

Since $h(x) - \frac{V(x)}{2} - \frac{l}{2}$ vanishes to order $2\nu$ at $x^*$, the function $\zeta$ is of the form

$$\zeta = n^{\frac{1}{2\nu}}(x - x^*) \varphi(x),$$

(3.30)
such that \( \varphi(x) \) is independent on \( n \) and \( \varphi(x^*) \neq 0 \). By choosing \( \delta t \) and \( \delta \) smaller if necessary, we can assume that \( \varphi(x) \) and hence \( \zeta \) is conformal inside the neighborhood \( B^x_\delta \).

Then \( \zeta \) maps the neighborhood \( B^x_\delta \) into the complex \( \zeta \)-plane such that the boundary of \( B^x_\delta \) is mapped into infinity.

Let us now define the constant \( Z_t \) and function \( \tau_t(x) \) by

\[
Z_t = n \left( g_1^t(x^*) - \frac{V(x^*)}{2t} - \frac{i}{2t} \right) - u^t \left( \log \varphi(x^*) + \frac{1}{2\nu} \log n \right),
\]

\[
\tau_t(x) = \frac{n \left( 2g_1^t(x) - \frac{V(x)}{t} - \frac{i}{t} \right) - 2Z_t + \zeta^{2\nu} - 2u^t \log \left( n^{\frac{1}{2\nu}} \varphi(x) \right)}{\zeta},
\]

\[
g_1^t(x) = \int_{\alpha t}^{\beta t} \left[ \rho^t(s) - \frac{\delta t}{\log \delta t} \frac{8\pi \sqrt{(s - \alpha t)(\beta t - s)}}{(\alpha t + \beta t)^2} \right] \log(x - s) ds.
\]

Note that \( \tau_t(x) \) does not have a pole at \( x = x^* \) and that by taking \( \delta t \) and \( \delta \) smaller if necessary, we can assume that \( \tau_t(x) \) is analytic inside \( B^x_\delta \).

Then it is easy to see that \( \zeta, \tau_t(x) \) and \( Z_t \) satisfy

\[
n \left( g^t(x) - \frac{V(x)}{2t} - \frac{i}{2t} \right) = -\frac{\zeta^{2\nu}}{2} + \frac{\tau_t(x)\zeta}{2} + u^t \log \zeta + Z_t.
\]

Moreover, we have the following

**Proposition 3.3.** As \( n \to \infty \) under the scaling (1.7a), the constant \( Z_t \) and function \( \tau_t(x) \) are of order

\[
Z_t = -\frac{u^t}{2\nu} \log \log n + O(1),
\]

\[
\tau_t(x) = O \left( \frac{\log n}{n^{2\nu}} \right),
\]

uniformly in \( B^x_\delta \).

**Proof.** The first part of the proposition follows from (2.43). By (2.43) and (3.31), we have

\[
Z_t = n \delta t \varphi(x^*) - u^t \left( \log \varphi(x^*) + \frac{1}{2\nu} \log n \right) + O \left( \frac{\delta t}{\log \delta t} \right).
\]

Then by using (3.34) to eliminate \( n \delta t \), we obtain the following,

\[
Z_t = -\frac{u^t}{2\nu} (\log n + \log \delta t) - u^t \log \varphi(x^*) + O \left( \frac{\delta t}{\log \delta t} \right).
\]

Now by taking the logarithm of (1.7a), we see that,

\[
\log n + \log \delta t = \log \log n + O(1).
\]
Hence we obtain
\[ Z_t = -\frac{u^t}{2\nu} \log \log n + O(1). \]

This proves the 1st equation in (3.33).

Given the first equation in (3.33), the second equation in (3.33) now follows easily from the definition of \( \zeta \) (3.29), \( \tau_t(x) \) (3.31) and the relation (2.42) from corollary 2.2.

### 3.5.2 Construction of the parametrix

We should now construct the parametrix by using orthogonal polynomials with weight \( \exp (-\zeta^{2\nu} + \tau_t(x) \zeta) \).

Let \( \pi_k^\nu(\zeta, \tau) \) be the monic orthogonal polynomial of degree \( k \) with respect to the weight \( -\zeta^{2\nu} + \tau \zeta \),
\[
\int_{\mathbb{R}} \pi_k^\nu(\zeta) \pi_j^\nu(\zeta) \exp \left( -\zeta^{2\nu} + \tau \zeta \right) \, d\zeta = h_k^\nu(\tau) \delta_{kj},
\]
where \( h_k^\nu(\tau) \) is the normalization constant as a function of \( \tau \). Let us denote by \( \Psi^\nu(\zeta, s) \) the following matrix constructed from the orthogonal polynomial \( \pi_k^\nu(\zeta, \tau) \).
\[
\Psi^\nu(\zeta, \tau) = \begin{pmatrix}
\pi_k^\nu(\zeta, \tau) & \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\pi_k^\nu(s, \tau) \exp \left( -\zeta^{2\nu} + \tau \zeta \right)}{s-x} \, ds \\
\kappa_{k-1}^\nu(\tau) \pi_{k-1}^\nu(\zeta, \tau) & \frac{1}{2\pi i} \int_{\mathbb{R}} \pi_{k-1}^\nu(s, \tau) \exp \left( -\zeta^{2\nu} + \tau \zeta \right) \, ds
\end{pmatrix},
\]
where \( \kappa_{k-1}^\nu(\tau) = -\frac{2\pi i}{h_k^\nu(\tau)} \).

Then the matrix \( \Psi^\nu(\zeta, \tau) \) satisfies the following Riemann-Hilbert problem.

1. \( \Psi^\nu(\zeta, \tau) \) is analytic on \( \mathbb{C}/\mathbb{R} \);
2. \( \Psi^\nu_+(\zeta, \tau) = \Psi^\nu_-(\zeta, \tau) \begin{pmatrix} 1 & \exp \left( -\zeta^{2\nu} + \tau \zeta \right) \\ 0 & 1 \end{pmatrix}, \ z \in \mathbb{R}; \)\n3. \( \Psi^\nu(\zeta, \tau) = (I + O(\zeta^{-1})) \begin{pmatrix} \zeta^\nu & 0 \\ 0 & \zeta^{-\nu} \end{pmatrix}, \ z \to \infty. \)

Let \( E(x) \) be the following matrix-valued function,
\[
E(x) = S^\infty(x) \zeta^{(u^t - \pi^t)}_{\sigma_3} e^{Z_t\sigma_3}
\]
(3.37)

Then from (3.25) we see that
\[
E(x) = e^{(K_0 + (u^t - \pi^t)F_0)\sigma_3} \Pi(x) e^{-K(x)\sigma_3} \left( e^{-(u^t - \pi^t)F(x)\sigma_3} \zeta^{(u^t - \pi^t)\sigma_3} \right) e^{Z_t\sigma_3}.
\]
(3.38)

From property 4. of (3.17), we see that the factor
\[
e^{-(u^t - \pi^t)F(x)\sigma_3} \zeta^{(u^t - \pi^t)\sigma_3}
\]
is analytic inside \( B^{x^*}_\delta \). Then, since both \( K(x) \) and \( \Pi(x) \) are analytic inside \( B^{x^*}_\delta \), the function \( E(x) \) is analytic inside \( B^{x^*}_\delta \). Hence we have
Proposition 3.4. Let the matrix \( S^x(x) \) be
\[
S^x(x) = E(x)\Psi^x(\zeta, \tau_t(x))\zeta^{-u^i\sigma_3}e^{-Z_t\sigma_3}, \quad x \in B^*_x,
\] (3.39)

Then, under the double scaling limit (1.7a), \( S^x(x) \) satisfies the conditions

1. \( S^x(x) \) is analytic in \( B^*_x / (B^*_x \cap \mathbb{R}) \);
2. \( S^x(x) = S^x(x)J_S(x), \quad x \in B^*_x \cap \mathbb{R} \);
3. \( S^x(x) = \left( I + O\left( \frac{1}{n}\right) \right) S(x) + \frac{1}{2} S(x) \), \( x \to \infty \), uniformly in \( \partial B^*_x \).

\[
S^x(x) = S^\infty(x) + \frac{1}{2} S(x), \quad x \to \infty.
\]

Proof. The properties 1. and 2. follows immediately from (3.32) and property 2. of (3.36).

We should now prove property 3.

At the boundary of \( B^*_x \), we have \( \zeta \to \infty \) and hence the function \( S^x(x) \) behaves as
\[
S^x(x) = S^\infty(x)\zeta^{(u^i-m)^\sigma_3}e^{Z_t\sigma_3} \left( I + O(\zeta^{-1}) \right) e^{-Z_t\sigma_3(\zeta-m)^\sigma_3}, \quad \zeta \to \infty.
\]

From (3.29), we see that \( \zeta^{-1} = O(n^{-\frac{1}{2\nu}}) \) at the boundary of \( B^*_x \), hence the above equation becomes
\[
S^x(x) = S^\infty(x) \left( I + O\left( \frac{1}{n}\right) \right) S(x),
\]
where the second equality follows from the fact that \( S^\infty(x) \) is bounded in \( B^*_x \) as \( n \to \infty \). This proves the proposition.

3.6 Local parametrix near the critical point \( x^* \) for \( t \leq 1 \)

We will now construct the parametrix in \( B^*_{\delta} \) when \( t \leq 1 \). In this case, the parametrix can be constructed from the Cauchy transform [16].

3.6.1 Conformal map in \( B^*_{\delta} \)

We will use the same conformal map (3.29) defined in section 3.5.1. However, the function \( \tau_t(x) \) and the constant \( Z_t \) are now defined to be
\[
Z_t = n \left( g^t(x^*) - \frac{V_t(x^*)}{2} - \frac{l_t}{2} \right)
\]
\[
\tau_t(x) = \frac{n(2g^t(x) - V_t(x) - l_t) - 2Z_t + \zeta^{2\nu}}{\zeta}.
\] (3.41)
Then by \((1.7b)\) and the Buyarov-Rakhmanov equation \((1.11)\), we see that \(Z_t\) is of order
\[
Z_t = O(n\delta t) = O(n^{1-k}) \quad (3.42)
\]
As we see in \((1.17b)\), the limiting kernel will be of order \(e^{Z_t}\). However, one should bear in mind that \(Z_t\) is negative and therefore the term \(e^{Z_t}\) is bounded as \(n \to \infty\).

We can now deduce the order of \(\tau_t(x)\) from \((1.11)\), \((1.7b)\) and \((3.42)\).
\[
\tau_t(x) = 2n^{1-k-\frac{1}{2\nu}} \frac{U_- (\phi(x) - \phi(x^*))}{(x - x^*)\varphi(x)} + o(1) \quad (3.43)
\]
From the definition \((3.41)\), we see that \(Z_t\) and \(\tau_t(x)\) together satisfies the following
\[
l_n \left( g'(x) - \frac{V_t(x)}{2} - \frac{l_t}{2} \right) = -\frac{\zeta^{2\nu}}{2} + \frac{\tau_t(x)\zeta}{2} + Z_t \quad (3.44)
\]

3.6.2 Construction of the parametrix

Let us now construct the local parametrix by using Cauchy transform (cf. \[16\]). Let
\[
\Psi(\zeta, \tau_t(x)) \text{ be the following matrix.}
\]
\[
\Psi(\zeta, \tau_t(x)) = \left( \begin{array}{cc} 1 & \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\exp(-\zeta^{2\nu} + \tau_t(x)\zeta)}{s-x} ds \\ 0 & 1 \end{array} \right) \quad (3.45)
\]
Then \(\Psi(\zeta, \tau_t(x))\) is the unique solution to the following Riemann-Hilbert problem.
\[
1. \quad \Psi(\zeta, \tau_t(x)) \text{ is analytic on } \mathbb{C}/\mathbb{R};
2. \quad \Psi_+ (\zeta, \tau_t(x)) = \Psi_- (\zeta, \tau_t(x)) \left( \begin{array}{cc} 1 & \exp(-\zeta^{2\nu} + \tau_t(x)\zeta) \\ 0 & 1 \end{array} \right), \quad \zeta \in \mathbb{R}; \\
3. \quad \Psi(\zeta, \tau_t(x)) = \left( I + O(\zeta^{-1}) \right), \quad \zeta \to \infty. 
\]
We can use \(\Psi(\zeta, \tau_t(x))\) to construct the local parametrix \(S^{x^*}(x)\). Let us define the matrix \(E(x)\) to be
\[
E(x) = S^{\infty}(x)e^{Z_t\sigma_3} \quad (3.47)
\]
then \(E(x)\) is analytic and invertible inside of \(B^{x^*}_0\). From \((3.44)\) and \((3.46)\), we have the following.

**Proposition 3.5.** Let the matrix \(S^{x^*}(x)\) be
\[
S^{x^*}(x) = E(x)\Psi(\zeta, \tau_t(x))e^{-Z_t\sigma_3}, \quad x \in B^{x^*}_0, \quad (3.48)
\]
Then, under the double scaling limit \((1.7b)\), \(S^{x^*}(x)\) satisfies the conditions
\[
1. \quad S^{x^*}(x) \text{ is analytic in } B^{x^*}_0 / (B^{x^*}_0 \cap \mathbb{R});
2. \quad S^{x^*}_+(x) = S^{x^*}_-(x)J_S(x), \quad x \in B^{x^*}_0 \cap \mathbb{R};
3. \quad S^{x^*}(x) = \left( I + O \left( n^{-\frac{1}{2}} \right) \right) S^{\infty}(x) \text{ as } n \to \infty, \text{ uniformly in } \partial B^{x^*}_0. 
\]
Proof. As in the proof of proposition 3.4, properties 1. and 2. are clear from (3.44) and (3.46). Let us take a look at the condition at the boundary of $B_{\delta}^{x^*}$. We need to be careful as $Z_t$ may contain powers of $n$ in it. At the boundary of $B_{\delta}^{x^*}$, we have $\zeta \to \infty$, and

$$S^{x^*}(x) = S^{\infty}(x) \begin{pmatrix} 1 & O(e^{2zt} \zeta^{-1}) \\ 0 & 1 \end{pmatrix}$$

but from the expression of $Z_t$ (3.41) and the property of the equilibrium measure (2.3), we see that $Z_t < 0$ and hence $e^{2zt}$ is bounded. Hence we have

$$S^{x^*}(x) = S^{\infty}(x) \left( I + O \left( n^{-\frac{1}{2}} \right) \right) = \left( I + O \left( n^{-\frac{1}{2}} \right) \right) S^{\infty}(x)$$

This completes the proof of the proposition. \qed

3.7 Last transformation of the Riemann-Hilbert problem

Let us now define $R(x)$ to be the following function.

$$R(x) = \begin{cases} S(x) \left( S^{x}(x) \right)^{-1}, & x \text{ inside } B_{\delta}^{r_1}; \\ S(x) \left( S^{\infty}(x) \right)^{-1}, & x \text{ outside of } B_{\delta}^{r_1}. \end{cases}$$

(3.50)

where $r_1 = -2$, $r_2 = 2$, $r_3 = x^*$ and $S^{\pm 2}(x)$ are the local parametrices near the edge points $\alpha_t$ and $\beta_t$. Then the function $R(x)$ has jump discontinuities on the contour $\Sigma$ shown in Figure 3.

In particular, $R(x)$ satisfies the Riemann-Hilbert problem

1. $R(x)$ is analytic on $\mathbb{C}/\Sigma$
2. $R_+(x) = R_-(x) J_R(x)$
3. $R(x) = I + O(x^{-1})$, $x \to \infty$
4. $R(x)$ is bounded.

From the definition of $R(x)$ (3.50), it is easy to see that the jumps $J_R(x)$ has the following order of magnitude.

$$J_R(x) = \begin{cases} I + O(n^{-1}), & x \in \partial B_{\delta}^{-2} \cup B_{\delta}^{2}; \\ I + O \left( n^{\frac{2i\pi - \pi - 1}{2x}} \log n \right), & x \in \partial B_{\delta}^{x}, \ t > 1; \\ I + O \left( n^{-\frac{1}{2}} \right), & x \in \partial B_{\delta}^{x}, \ t \leq 1; \\ I + O(e^{-n\gamma}), & \text{for some fixed } \gamma > 0 \text{ on the rest of } \Sigma. \end{cases}$$

(3.52)
Since $|u^t - \overline{u^t}| < \frac{1}{2}$, for sufficiently large $n$, $n^{\frac{2|u^t - \overline{u^t}| - 1}{2\nu}}(\log n)^{\frac{t}{2\nu}}$ and $n^{-\frac{1}{2\nu}}$ are small. Then by the standard theory, [12], [15], [16], we have

$$R(x) = I + O\left( n^{\frac{2|u^t - \overline{u^t}| - 1}{2\nu}}(\log n)^{\frac{t}{2\nu}} \right), \quad t > 1$$

(3.53)



$$R(x) = I + O\left( n^{-\frac{1}{2\nu}} \right), \quad t \leq 1.$$ 

uniformly in $\mathbb{C}$.

In particular, the solution $S(x)$ of the Riemann-Hilbert problem (3.12) can be approximated by $S^\infty(x)$ and $S^\nu(x)$ as

$$S(x) = \begin{cases} 
R(x)S^\nu(x), & x \text{ inside } B_\delta^\nu; \\
R(x)S^\infty(x), & x \text{ outside of } B_\delta^\nu.
\end{cases}$$

(3.54)

When $t > 1$, this approximation becomes poor as $|u^t - \overline{u^t}|$ gets close to $\frac{1}{2}$. However, if we restrict our attention to a small neighborhood of $x^*$ such that

$$z = (x - x^*) n^{\frac{1}{2\nu}} \varphi(x^*),$$

(3.55)

is finite, then we can still use this approximation to obtain the asymptotic kernel (1.17a).

4 Asymptotics of the correlation kernel

We should now compute the kernel using the the asymptotics obtained in section 3. Recall that the kernel and the solution $Y(x)$ of the Riemann-Hilbert problem (3.1) are related by (3.2).

4.1 Asymptotics of the kernel when $t > 1$

First let us recover the asymptotics of $Y(x)$ from that of $S(x)$. By reversing the series of transformations (3.11) and (3.9), we find that, for $x \in B_\delta^\nu$, the matrix $S(x)$ and $Y(x)$ are related by

$$Y(x) = e^{n\frac{i}{2\nu} \sigma_3} S(x) e^{n\left(g'(x) - \frac{i}{2}\right) \sigma_3}, \quad x \in B_\delta^\nu.$$ 

We now use the estimate (3.54) and the expression of (3.39) to obtain

$$Y(x) = e^{n\frac{i}{2\nu} \sigma_3} R(x) E(x) \Psi^\nu(x, \tau_1(x)) e^{n\left(g'(x) - \frac{i}{2\nu} - u^t \log \zeta - Z_t\right) \sigma_3}, \quad x \in B_\delta^\nu.$$ 

Now from (3.32), we see that the above is equal to

$$Y(x) = e^{n\frac{i}{2\nu} \sigma_3} R(x) E(x) \Psi^\nu(x, \tau_1(x)) e^{n\left(\frac{x_{\nu}(x)}{2\nu} + \frac{\phi(x)}{2\nu}\right) \sigma_3}, \quad x \in B_\delta^\nu.$$ 

(4.1)
Let us now study the behavior of $E(x)$ and $R(x)$ in the vicinity of $x^*$ when $z$ defined by (3.55) is finite. First let us consider $E(x)$. From (3.35), we see that $E(x)$ is analytic inside the neighborhood $B_{\delta}^{x^*}$. Then from the power series expansion of $E(x)$ inside $B_{\delta}^{x^*}$, we obtain

$$E(x) = \left( E^0 + E^1 zn^{-\frac{1}{2\nu}} + O \left( n^{-\frac{1}{2\nu}} \right) \right) n^{\frac{u_0'}{2\nu}\sigma_3} (\log n)^{-\frac{u_1'}{2\nu}\sigma_3},$$

(4.2)

for some constants $E^0$ and $E^1$ that are bounded as $n \to \infty$.

Now consider $R(x)$. Let $m$ be the biggest integer such that

$$m \left( 1 - 2|u^t - \overline{u^t}| \right) < 1.$$

Then from the Riemann-Hilbert problem (3.50), we see that $R(x)$ is analytic inside $B_{\delta}^{x^*}$, hence in terms of $z$, we have the following estimate

$$R(x) = I + \sum_{j=1}^{m} \Lambda_j \left( n^{\frac{2|u^t - \overline{u^t}| - 1}{2\nu}} (\log n)^{\frac{u_1'}{2\nu}} \right)^j + O \left( zn^{-\frac{1}{2\nu}} (\log n)^{\frac{u_1'}{2\nu}} \right) + O \left( n^{-\frac{1}{2\nu}} \right),$$

(4.3)

where $O \left( zn^{-\frac{1}{2\nu}} (\log n)^{\frac{u_1'}{2\nu}} \right)$ denotes $z$ dependent terms that are of order $n^{-\frac{1}{2\nu}} (\log n)^{\frac{u_1'}{2\nu}}$. The constants $\Lambda_j$ are finite in the limit $n \to \infty$.

In particular, from (4.2) we see that $E^{-1}(x')E(x)$ satisfies the following

$$E^{-1}(x')E(x) = n^{\frac{u_1'}{2\nu}\sigma_3} (\log n)^{\frac{u_1'}{2\nu}\sigma_3} \left( I + O \left( \frac{z' - z}{n^{\frac{1}{2\nu}}} \right) \right) n^{\frac{u_1'}{2\nu}\sigma_3} (\log n)^{-\frac{u_1'}{2\nu}\sigma_3},$$

(4.4)

while $R^{-1}(x')R(x)$ satisfies the following estimate

$$R^{-1}(x')R(x) = I + O \left( (z' - z)n^{-\frac{1}{2\nu}} (\log n)^{\frac{u_1'}{2\nu}} \right)$$

(4.5)

Hence the product $E^{-1}(x')R^{-1}(x')R(x)E(x)$ satisfies the following estimate

$$E^{-1}(x')R^{-1}(x')R(x)E(x) = I + O \left( (z' - z)n^{\frac{2|u^t - \overline{u^t}| - 1}{2\nu}} (\log n)^{\frac{u_1'}{2\nu}} \right)$$

(4.6)

If we now substitute (4.1) and (4.5) back to (3.2), we obtain the following estimate for the kernel

$$K_{n,N}(x, x') = \frac{\phi(x^*) n^{\frac{1}{2\nu}}}{2\pi i(z - z')} \begin{pmatrix} 0 & 1 \\ (\Psi_+^{\nu}(\zeta', \tau_1(x')))^{-1} \Psi_+^{\nu}(\zeta, \tau_1(x)) & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\times \exp \left( \frac{\tilde{V}(\zeta)}{2} + \frac{\tilde{V}(\zeta')}{2} \right) \left( 1 + O \left( n^{\frac{2|u^t - \overline{u^t}| - 1}{2\nu}} (\log n)^{\frac{u_1'}{2\nu}} \right) \right)$$

(4.7)
where $\tilde{V}(\zeta) = -\zeta^{2\nu} + \tau_i(x)\zeta$.

Now recall that from (3.33) and (3.30), we have

$$\lim_{n \to \infty} \zeta = z, \quad \lim_{n \to \infty} \tau_i(x) = 0$$

If we now take these into account and substitute (3.35) into (4.7), then we obtain the limit of the kernel as

$$\lim_{n,N \to \infty} \frac{1}{\varphi(x^*)n^{\frac{1}{2\nu}}} K_{n,N}(x,x') = \kappa_{\Pi-1}^{\nu}e^{-\frac{(x')^{2\nu} + z^{2\nu}}{2} \frac{\pi^{\nu}(z') \pi^{\nu}_{m-1}(z)}{2\pi i(z - z')} - \frac{\pi^{\nu}(z) \pi^{\nu}_{m-1}(z')}{2\pi i(z - z')}}$$

(4.8)

where $u = \lim_{n \to \infty} u^t$ and $\kappa_\Pi^{\nu} = \kappa_{\Pi}^{\nu}(0)$.

To complete the proof of the first part of theorem 1.1, we need to show that $\varphi(x^*) = \left(\frac{Q(x^*)\sqrt{(x^*)^{2\nu} - 4}}{2\nu}\right)^{\frac{1}{2\nu}}$. This can be seen from expression (2.5). From (2.5), we have

$$\sqrt{q(x)} = \int_R \frac{\rho(y)}{y-x} dy + \frac{V'(x)}{2} = -h'(x) + \frac{V'(x)}{2},$$

then from the fact that $2h(x^*) - V(x^*) - l = 0$ and the expressions of $q(x)$ (2.6) and $\zeta$ (3.29), we see that, upon integration, we have

$$n(x - x^*)^{2\nu} \varphi^{2\nu}(x^*) = n(x - x^*)^{2\nu} \frac{Q(x^*)\sqrt{(x^*)^{2\nu} - 4}}{2\nu}$$

this completes the proof of theorem 1.1 for the case $t > 1$.

4.2 Asymptotics of the kernel when $t \leq 1$

We will now use the local parametrix $S_{x^*}(x)$ (3.48) constructed for $t \leq 1$ to compute the kernel. In this case, the solution $Y(x)$ to (3.1) is given by (4.1) with $\Psi^{\nu}(x, \tau_i(x))$ replaced by $\Psi(x, \tau_i(x))$ and $\tau_i(x)$ defined by (3.41).

Let $z$ be the variable defined by (3.35) and assume that $z$ is finite. Then by using the power series expansion of $E(x)$ and $R(x)$ inside $B_{\xi_t}^*$, we obtain

$$E(x) = \left(\Pi(x^*) + \Pi'(x^*)\frac{z}{\varphi(x^*)n^{\frac{1}{2\nu}}} + O\left(zn^{-\frac{1}{2\nu}}\right)\right) e^{Z_t(x^*)},$$

(4.9)

$$R(x) = I + R^0 n^{-\frac{1}{2\nu}} + O\left(n^{-\frac{1}{2\nu}}\right) + O\left(zn^{-\frac{1}{2\nu}}\right)$$

where we have used (3.26) to replace $S_{x^*}(x)$ by $\Pi(x)$ and $O\left(zn^{-\frac{1}{2\nu}}\right)$ denotes $z$ dependent terms with order $n^{-\frac{1}{2\nu}}$. Therefore the product $E^{-1}(x')R(x')R(x)E(x)$ is of order

$$E^{-1}(x')R(x')R(x)E(x) = e^{-Z_t(x)} \left(\Pi^{-1}(x^*)\Pi(x^*)z - z'\right) + O\left(\frac{z - z'}{n^{\frac{1}{2\nu}}}\right) e^{Z_t(x^*)}.$$  

(4.10)
From (3.45), one can easily check that

\[
\begin{pmatrix}
0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
-\frac{N V(x')}{2} - \frac{\tilde{V}(\zeta)}{2} \\
\end{pmatrix} + \Psi_+(\zeta, \tau_t(x)) e^{-Z_t \sigma_3} = e^{\left(Z_t + N \frac{V(x')}{2} + \frac{\tilde{V}(\zeta)}{2}\right)} \begin{pmatrix}
0 & 1 \\
\end{pmatrix}.
\]

(4.11)

where \( \tilde{V}(\zeta) = -\zeta^{2\nu} + \tau_t(x)\zeta \). We then substitute (4.11) and (3.45) into (3.2) and arrive at

\[
K_{n,N}(x, x') = e^{\frac{(2Z_t + \tilde{V}(\zeta) + \tilde{V}(\zeta')}{2}} \left(0 \ 1 \right) \Pi^{-1}(x^*) \Pi(x) \begin{pmatrix}
1 \\
0 \\
\end{pmatrix} \left(1 + O\left(n^{-1}\right)\right).
\]

This gives the double scaling limit of the kernel

\[
\lim_{n,N \to \infty} e^{-2Z_t} K_{n,N}(x, x') = e^{-\frac{x^2 + (x')^2}{2}} \frac{1}{8\pi} \left(\frac{1}{x^* - \beta_t} - \frac{1}{x^* - \alpha_t}\right).
\]

This completes the proof of theorem 1.1.

References

[1] E. D. Belokolos, A. I. Bobenko, V. Z. Enolskii, A. R. Its and V. B. Matveev. Algebro-geometric approach to nonlinear integrable equations. Springer series in nonlinear dynamics, Springer-Verlag. (1995)

[2] P. Bleher and B. Eynard. Double scaling limit in random matrix models and a nonlinear hierarchy of differential equations. J. Phys. A, 36 (2003), no. 12, 3085–3105.

[3] P. Bleher and A. Its. Semiclassical asymptotics of orthogonal polynomials, Riemann-Hilbert problem, and universality in the matrix model. Ann. of Maths. (2), 150 (1999), no. 1, 185–266.

[4] P. Bleher and A. Its. Double scaling limit in the random matrix model: the Riemann-Hilbert approach. Comm. Pure Appl. Math., 56 (2003), no. 4, 433–516.

[5] P. Bleher and A. B. J. Kuijlaars. Large \( n \) limit of Gaussian random matrices with external source, Part III: Double scaling limit. Comm. Math. Phys., 270 (2007), no. 2, 481–517.

[6] V. S. Buyarov and E. A. Rakhmanov. On families of measures that are balanced in the external field on the real axis. Mat. Sb., 190 (1999), no. 6, 11–22.

[7] T. Claeyss. The birth of a cut in unitary random matrix ensembles. arXiv:0711.26.09.
[8] T. Claeys and A. B. J. Kuijlaars. Universality of the double scaling limit in random matrix models. *Comm. Pure Appl. Math.*, 59 (2006), no. 11, 1573–1603.

[9] T. Claeys and A. B. J. Kuijlaars. Universality in unitary random matrix ensembles when the soft edge meets the hard edge. [arXiv:math-ph/0701003](https://arxiv.org/abs/math-ph/0701003).

[10] T. Claeys, A. B. J. Kuijlaars and M. Vanlessen. Multi-critical unitary random matrix ensembles and the general Painlevé II equation. [arXiv:math-ph/0508062](https://arxiv.org/abs/math-ph/0508062).

[11] T. Claeys and M. Vanlessen. Universality of a double scaling limit near singular edge points in random matrix models. *Comm. Math. Phys.*, 273 (2007), no. 2, 499–532.

[12] P. Deift. *Orthogonal polynomials and random matrices: A Riemann-Hilbert approach*. Courant lecture notes 3. New York University. (1999).

[13] M. Duits, A. B. J. Kuijlaars. Painlevé I asymptotics for orthogonal polynomials with respect to a varying quartic weight. *Nonlinearity*, 19 (2006), no. 10, 2211–2245.

[14] P. Deift, T. Kriecherbauer and K. T. R. McLaughlin. New results on the equilibrium measure for logarithmic potentials in the presence of an external field. *J. Approx. Theory*, 95 (1998), no. 3, 388–475.

[15] P. Deift, T. Kriecherbauer, K. T. R. McLaughlin and S. Venakides. Strong asymptotics of orthogonal polynomials with respect to exponential weights. *Comm. Pure Appl. Math.*, 52 (1999), no. 12, 1491–1552.

[16] P. Deift, T. Kriecherbauer, K. T. R. McLaughlin and S. Venakides. Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory. *Comm. Pure Appl. Math.*, 52 (1999), no. 11, 1335–1425.

[17] F. J. Dyson. Correlation between the eigenvalues of a random matrix. *Comm. Math. Phys.*, 19 (1970), 235–250.

[18] B. Eynard. Universal distribution of random matrix eigenvalues near the ”birth of a cut” transition. *J. Stat. Mech.*, 7 (2006), P07005.

[19] A. S. Fokas, A. R. Its and A. V. Kitaev. The isomonodromy approach to matrix models in 2D quantum gravity. *Comm. Math. Phys.*, 147 (1992), no. 2, 395–430.

[20] A. R. Its, A. B. J. Kuijlaars and J. Ostensson. Critical edge behavior in unitary random matrix ensembles and the thirty fourth Painlevé transcendent. [arXiv:0704.1972](https://arxiv.org/abs/0704.1972).

[21] A. R. Its, F. Mezzadri and M. Y. Mo. Entanglement entropy in quantum spin chains with finite range interaction. [arXiv:0708.0161](https://arxiv.org/abs/0708.0161).
[22] K. Johansson. On fluctuations of eigenvalues of random Hermitian matrices. Duke Math. J., 91 (1998), no. 1, 151–204.

[23] A. B. J. Kuijlaars and M. Vanlessen. Universality for eigenvalue correlations at the origin of the spectrum. Comm. Math. Phys., 243 (2003), no. 1, 163–191.

[24] A. B. J. Kuijlaars and K. T. R. McLaughlin. Generic behavior of the density of states in random matrix theory and equilibrium problems in the presence of real analytic external fields. Comm. Pure Appl. Math., 53 (2000), no. 6, 736–785.

[25] M. L. Mehta. Random matrices. Elsevier, Amsterdam. (2004).

[26] N. I. Muskhelishvili. Singular integral equations. Boundary problems of function theory and their application to mathematical physics. Translation by J. R. M. Radok. Noordhoff, Groningen (1953).

[27] C. E. Porter. ed. Statistical theories of spectra: Fluctuations, a collection of reprints, original papers, with an introductory review. Academic press, New York. (1965).

[28] E. B. Saff and V. Totik. Logarithmic potentials with external fields., Grundlehren der Mathematischen Wissenschaften 316, Springer-Verlag, Berlin. (1997).

[29] G. Szego. Orthogonal polynomials., American Mathematical Society, Colloquium Publications, v. 23. American Mathematical Society. (1939).

[30] M. Shcherbina. Double scaling limit for matrix models with non analytic potentials. arXiv:math-ph/0508062

[31] V. Totik. Weighted approximation with varying weight. Lecture notes in mathematics 1569, Springer-Verlag, Berlin. (1994).

[32] M. Vanlessen. Strong asymptotics of the recurrence coefficients of orthogonal polynomials associated to the generalized Jacobi weight. J. Approx. Theory, 125 (2003), no. 2, 198–237.

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