One-dimensional strongly interacting electrons with single impurity: conductance reemergence

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We show that conductance of 1D channel with one point-like impurity critically depends on asymptotic behavior of $e-e$ interaction at small momenta $k$ (about inverse length of a channel). Conductance reemerges (contrary to the case of point-like repulsive potential) if potential $V(k=0) \neq 0$. For example, this happens if the bare $e-e$ interaction is screened by a charges in the bulk. The relation of this phenomena to the long-range order present in the Luttinger model is discussed.

INTRODUCTION

Theory of one-dimensional interacting electrons is under investigation for a long time \[1-3\]. Its relativistic analog, two-dimensional QED, also attracted a lot of attention \[4\] in the past, since it is a simplest field theory with confinement. One-dimensional pure electronic systems (in particular, the Luttinger model \[2\]) are exactly solvable, their properties are very well understood. Situation changes if one introduces some impurities that are understood as short-range barriers with transition coefficient $K$ and reflection coefficient $R$. The simplest system of this kind (with only one impurity) was considered for the first time in Ref.\[5\] with surprising results. It turned out that properties of such system depend critically on the sign of the electron-electron ($e-e$) interaction. Conductance for attractive potentials is equal to the ballistic one and it is not affected by $e-e$ interaction (only the Fermi speed should be renormalized). Conductance for repulsive potentials vanishes. These results were obtained in \[5\] by the bosonization method.

Another approach with similar results was developed in \[6\]. The authors returned to the fermion language. Assuming that the interaction is short-range, $V = V_0 \delta(x)$ and small $V_0 \ll 1$, they summed up the leading infrared logs of frequency $\omega$ by the renorm-group method. Next-to-leading corrections to the conductivity were also found in \[5\] by the methods of current algebra.

The approaches of \[5\] and \[6\] have different (but overlapping) regions of applicability. The first approach employs perturbation theory in reflection (transition) coefficient for an arbitrary attractive (repulsive) potential, while the second one employs perturbation theory in potential for an arbitrary reflection or transition coefficients. A point-like $e-e$ interaction is assumed in both approaches.

We suggested in \[8\] an alternative approach to the problem based on the path integral formalism. Using the well-known trick \[9\] we see that Luttinger model can be interpreted as the system of non-interacting electrons in a random external field. Green functions of one-dimensional electrons in any external field can be found exactly. We used this fact to construct perturbatively a Green function of the system with impurity. At the end we integrate out fermions and arrive at a 0+1-dimensional field theory. This theory describes the evolution with time of the electron phase at the point where the impurity is located. It is completely equivalent to the original Luttinger model with one impurity.

Using this theory we were able to prove two theorems. First, conductance of the system is zero (for repulsion) or maximal (for attraction) for a wide class of potentials. The arguments in favor of this statement for a point-like potential were given earlier in both approaches of \[5\] and \[6\]. We will see below that a necessary condition for such behavior of conductance is that the Fourier transform of the potential $V(k)$ has a non-vanishing limit at $k \to 0$. The second theorem is a general exact property of the theory which one can call duality. It states that the effective reflection coefficient $|R_\omega|^2$ in a theory with an attractive potential is equivalent to the effective transition coefficient $|K_\omega|^2$ in a theory with repulsion if one exchanges $K \leftrightarrow R$ (for a precise formulation of duality transformation of potential, see below). The traces of this property were seen in the perturbation theory in \[5\] where duality transformation reduces to $v_c \to v_c^{-1}$ ($v_c$ is the renormalized Fermi speed). However, this statement is far more general. It means that it is enough to consider, say, only repulsive potentials.

For a repulsive potential conductance restores if potential vanishes at $k \to 0$. Such a situation takes place in the systems with a small density of carriers when the screening radius is large. In this case $e-e$ interaction is not point-like from one-dimensional point of view, and it is screened by the image charges on 3-dimensional gates, edges of the channel, etc. Renormalization of the ballistic conductance in such system is finite and will be calculated below.

The physical reason for critical phenomena in the Luttinger model with an impurity is a long-range order which is present in a system of one-dimensional electrons. It is well-known that its analogue — the Schwinger model exhibits the anomalous breakdown of chiral symmetry. The strength of the interaction in the repulsive Lut-
tiner model is smaller: the system is in the Berezinskii-Kosterlitz-Thouless (BKT) phase\(^1\). Chiral condensate (consisting of pairs of \(R\) electron and \(L\) hole with finite density) arises only in the limit of an infinitely large interaction. In the case of an attractive potential there is a tendency to formation of a charged condensate of Cooper pairs. The Bose-Einstein principle implies that the chiral condensate increases the probability of reflection, i.e. the effective reflection coefficient \(|R_{\alpha}|^2\) at small frequencies, while the charged condensate increases the probability of transition. As a result, \(|R_{\alpha}|^2 = 1\) for repulsion and \(|K_{\omega}|^2 = 1\) for attraction. As we mentioned above, this does not happen if \(V(k \to 0) = 0\). We will see that in this case the long-range order in the Luttinger model also disappears. At last, conductance critical indices are simply related to the critical exponents in the BKT transition.

**EFFECTIVE TRANSITION/REFLECTION COEFFICIENTS AND CONDUCTANCE**

The Fermi surface in one dimension reduces to two isolated points \(\pm p_F\). The electrons with momenta close to the Fermi surface can be divided in right (R) and (L) movers,

\[
\Psi = e^{ip_Fx-i\varepsilon_F t}\psi_R + e^{-ip_Fx-i\varepsilon_F t}\psi_L,
\]

where \(\psi_{R,L}\) are slowly varying on the scale \(1/p_F\). The Schrödinger equation for the R,L electrons reduces to the Dirac equation in \(d = 1\) (in our units \(\hbar = v_F = 1\))

\[
[i\partial_t \pm i\partial_x - U]\psi_{R,L} = 0, \quad (1)
\]

where \(U\) is an external potential. The Luttinger liquid is a system which can be solved exactly. The ultimate reason for this is that a one-dimensional fermion Green function in the external field can be found exactly

\[
G_{R,L}(x,x') = G_{R,L}^{(0)}(x,x')e^{\gamma_{R,L}(x)-i\gamma_{R,L}(x')}
\]

\[
\gamma_{R,L}(x) = -\int d^2x'G^{(0)}(R,L)(x,x')U(x'), \quad (2)
\]

where \(G_{R,L}^{(0)}(x,x')\) is a free Green function.

A point-like impurity located at \(x = 0\) mixes left and right electrons. Impurity plays the role of a boundary condition, solutions of eq. (1) should be matched at \(x = 0\). Nevertheless, the general solution in the external field can be found. For example, for an electron moving from \(x = -\infty\) with energy \(\varepsilon = E - \varepsilon_F\)

\[
\psi_x = \begin{pmatrix} \psi_R \cr \psi_L \end{pmatrix} = \begin{pmatrix} |\theta(-x) + \theta(x)K|e^{-i\varepsilon(x-t)}e^{i\gamma_{R,L}(x,t)} \cr \theta(-x)Re^{-i\varepsilon(t+x)}e^{i\gamma_{R,L}(x,t)+i\alpha(t+x)} \end{pmatrix}, \quad (3)
\]

where \(K, R\) are the transition and reflection coefficients and \(\alpha(t) = \gamma_{R}(0,t) - \gamma_{L}(0,t)\) is the difference of the phases of the R and L electrons at the position of an impurity. Construction of the Green function with an impurity is impeded by the Feynman boundary conditions which lead to some integral equation. This equation can be solved perturbatively either in reflection or in transition coefficient (for details see \[8\]).

The Luttinger model has high symmetry: it is invariant both under gauge (vector) and chiral transformations (the latter symmetry is broken by the anomaly). The charge density \((\rho = \rho_R + \rho_L)\) and current \((j = \rho_R - \rho_L)\) can be completely determined from the conservation of the vector and axial currents,

\[
\partial_t \rho + \partial_x j = 0, \quad \partial_t j + \partial_x \rho = \frac{1}{\pi} \partial_x U + \mathcal{D}(t)\delta(x). \quad (4)
\]

Here the first term on the right-hand side is the Adler anomaly. The second term describes the influence of the impurity, and \(\mathcal{D}\) is the charge jump at \(x = 0\) which depends only on phase \(\alpha(t)\). It can be calculated if the Green function is known.

By means of the Hubbard trick, the Luttinger model with an impurity can be reduced to a system of electrons in a random external field with a simple Gaussian weight. Integrating fermions out one can obtain an arbitrary Green function in the interacting theory as a product of the Green function in the external field and the determinant representing the sum of loop diagrams. The determinant consists of two parts: a well-known contribution of the Adler anomaly and a contribution of an impurity:

\[
\log \mathcal{D}_{\text{et imp}} = -\frac{i}{2} \int_0^1 d\lambda \frac{d\omega}{2\pi} \alpha(-\omega)\mathcal{D}(-\lambda\alpha)\omega, \quad (5)
\]

The effect of the impurity depends only on the charge jump and hence is a functional of \(\alpha\). The anomalous part of the determinant that depends on the full potential \(U(x,t)\) contributes to the effective Gaussian weight:

\[
\mathcal{S}_{\text{eff}}[U] = \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} \frac{U(k,\omega)U(-\omega,\omega)\omega^2 - k^2 v_F^2(k)}{\omega^2 - k^2 + i\delta} \quad (6)
\]

Now one can integrate out potential \(U(x,t)\) leaving the integration over the only dynamical variable — phase \(\alpha(t)\). This procedure reduces the 1+1-dimensional Luttinger model to a 0+1 field theory (non-local quantum mechanics of the phase).

Using eq. (2) and eq. (5) the conductivity can be written as \((\mathcal{E})\) is an infinitesimal electric field:

\[
j(k,\omega) = j_{\text{ballistic}} + \frac{d}{2\pi} \sigma(k,\omega)\mathcal{E}(q,\omega),
\]

\[
\sigma(k,q) = \frac{2|\mathcal{R}_{\omega}|^2}{\pi} \frac{\omega |\omega| v_F(\omega)}{(\omega^2 - v_F^2(k)k^2 + i\delta)(\omega^2 - v_F^2(q)q^2 + i\delta)} \quad (7)
\]
Here $v_r(k)$ is the renormalized speed of an electron ($v_c$ is the Fermi speed renormalized by interaction),
\[
v_r(k) = \sqrt{1 + \frac{V(k)}{\pi}}, \quad v_c = v_r(0),
\]
and $|\mathcal{R}_\omega|^2$ is the effective reflection coefficient. It follows from eq. (17) that conductance of the system is
\[
C(\omega) = \frac{|\mathcal{K}_\omega|^2}{2\pi v_r(\omega)},
\]
where we introduced an effective transition coefficient $|\mathcal{K}_\omega|^2 = 1 - |\mathcal{R}_\omega|^2$. It can be calculated in our 0+1 field theory as
\[
|\mathcal{R}_\omega|^2 2\pi \delta(\omega - \omega') = \frac{i\pi}{|\omega| W(\omega)v_r(\omega)} \langle \langle \alpha(-\omega') \mathcal{D}(\omega) \rangle \rangle.
\]
Here the average is understood as an integral with the effective action:
\[
\langle \langle \ldots \rangle \rangle = \frac{1}{Z} \mathcal{D} \ldots \mathcal{D} \text{det}_{\text{imp}} \exp \left[-\int d\omega \frac{\alpha(-\omega) \alpha(\omega)}{2\pi} W^2(\omega)\right],
\]
where the last factor arises from eq. (10), and $W(\omega)$ is
\[
W(\omega) = -\int \frac{dk}{2\pi i} \frac{4k^2 V(k)}{(\omega^2 - k^2 + i\delta)(\omega^2 - v_r^2(k)k^2 + i\delta)}.
\]
Another useful representation for the effective reflection coefficient has the form:
\[
|\mathcal{R}_\omega|^2 = \frac{i\pi}{|\omega| W^2(\omega)v_r(\omega)} [g(\omega) - g(0)],
\]
where $g(\tau - \tau') = \langle \langle \alpha(\tau) \alpha(\tau') \rangle \rangle$ is a Green function of the electron phase ($g_0$ is a Green function without impurity determinant). The representation (13) can be obtained from eq. (10) taking functional integral by parts, it is one of the Ward identities.

An expression for the electron determinant was constructed in Ref. [8] as a series:
\[
\log \text{det}_{\text{imp}}[\alpha] = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left( \frac{|R|}{|K|} \right)^{2n} B_{2n-1}[\alpha],
\]
where
\[
B_n = \int \frac{d\tau_1 \ldots d\tau_n}{(2\pi i)^n} \frac{1 - \cos[\alpha(\tau_0) - \alpha(\tau_1) + \ldots + \alpha(\tau_n)\]}{(\tau_0 - \tau_1 - i\delta) \ldots (\tau_n - \tau_0 - i\delta)}.
\]
It was proved in [8] that eq. (14) can be presented also in a dual form as a series in the inverse parameter
\[
\log \text{det}_{\text{imp}}[\alpha] = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left( \frac{|K|}{|R|} \right)^{2n} B_{2n-1}[\tilde{\alpha}] + \int \frac{d\omega}{2\pi} \frac{|\omega|}{4\pi} \tilde{\alpha}(-\omega)\tilde{\alpha}(\omega),
\]
where $\tilde{\alpha}(\omega) = \text{sign}(\omega)\alpha(\omega)$. It is natural to call the first term a dual determinant and combine the second one with the quadratic form in eq. (12). Introducing $\tilde{\alpha}$ as a new variable we obtain a dual theory with the quadratic form
\[
\tilde{W}^{-1}(\omega) = -W^{-1}(\omega) - \frac{|\omega|}{2\pi}.
\]
We see that the coefficients $|\mathcal{R}_\omega|^2$ and $|\mathcal{K}_\omega|^2$ can be written either as functional integrals eq. (11) in the original theory or as the functional integrals in the dual theory after the transformation:
\[
|R|^2 \leftrightarrow |K|^2, \quad |\mathcal{R}_\omega|^2 \leftrightarrow |\mathcal{K}_\omega|^2, \quad W \leftrightarrow \tilde{W}.
\]
A naive perturbation theory works if the quadratic form in the exponent of a functional integral is positive. For attractive potentials $V(k) < 0$ one has to use the original $W(\omega)$ and calculate $|\mathcal{R}_\omega|^2$ as a series in the reflection coefficient $|R|^2$. For repulsive potentials $\tilde{W}(\omega)$ is positive and we should use the dual theory to obtain $|\mathcal{K}_\omega|^2$ as an expansion in $|K|^2$.

Let us consider the first term of the expansion in $|R|^2$. Expanding $\text{det}_{\text{imp}}$, we obtain the functional integral of a product an exponential of the quadratic form and of a cosine (see eq. (13)) that can be written as a sum of exponentials of $\alpha$. This functional integral is Gaussian and therefore can be easily calculated:
\[
|\mathcal{R}_\omega|^2 = \frac{|R|^2}{4\pi v_r(\omega)} \int d\tau \frac{1 - \cos \omega \tau}{\tau^2} \Xi(\tau),
\]
where
\[
\log \Xi(\tau) = \int \frac{d\omega'}{2\pi} W(\omega')(1 - \cos \omega' \tau).
\]
This expression is valid for an arbitrary electron-electron interaction provided that $|R|^2 \ll 1$. For a point-like potential $V(k) = V_0$ the quadratic form in eq. (12) is singular at small $\omega$:
\[
W(\omega) = \frac{2\pi \nu}{|\omega|}, \quad \nu = \frac{1}{v_c} - 1,
\]
and we obtain
\[
|\mathcal{R}_\omega|^2 = \Gamma(-1 - 2\nu) \frac{2\sin \pi \nu}{\pi v_c} |R|^2 \left( \frac{\omega}{M} \right)^{2\nu},
\]
where $M$ is an ultraviolet cutoff which we need for point-like potential. For a non-point-like potential of finite size the effective cutoff $M$ can be calculated from eq. (12).

An attractive potential ($V_0 < 0$) is implied in eq. (22). We can restore the result for a repulsive potential and small $|K|^2$ by duality
\[
|\mathcal{K}_\omega|^2 = \Gamma(-1 - 2\tilde{\nu}) \frac{2\nu \sin \pi \tilde{\nu}}{\pi} |K|^2 \left( \frac{\omega}{M} \right)^{2\tilde{\nu}}, \quad \tilde{\nu} = v_c - 1.
\]
Let us notice that eq. (22) is valid only for $2\nu < 1$ while eq. (23) is valid only for $2\nu < 1$. For $2\nu > 1$ the main contribution to the integral in eq. (19) comes from $\tau \leq M^{-1}$ and $|K_\omega|^2 \sim \omega$ at small $\omega$ for any such $\nu$. The same is true for $|K_\omega|^2$ when $2\nu > 1$.

The expressions in eq. (22) and eq. (23) coincide with the formulae obtained in Ref. [5]. One can also verify that the results of [6] are reproduced by our theory in the corresponding limit (to be published elsewhere). Our approach is valid for an arbitrary $e - e$ potential, not only for a point-like one as in [6].

**CONDUCTANCE REEMERGENCE**

The coefficient $|R_\omega|^2 \rightarrow 0$ for an attractive potential and $|K_\omega|^2 \rightarrow 0$ for a repulsive one when $\omega \rightarrow 0$ (see eq. (22) and eq. (23)). This happens only if $V(k=0) \neq 0$ due to an infrared divergency. An example of a different behavior (for a repulsive potential with $V(k=0) = 0$) is given by a system of one-dimensional electrons with small concentration. We will see that conductance remains finite in this case.

Indeed, if the concentration is small then the screening radius can be much larger than the width $d$ of the channel (which is considered to be zero in our one-dimensional theory). The bare $e - e$ interaction is screened by the "image charges" that arise on the split gates. The screened interaction is Coulombic at distances smaller than the distance $l$ to the gate and it is dipole-dipole in the opposite case [12]:

$$V(k) = -\frac{2}{\zeta} \left\{ \log |k|d, \frac{kl}{(kl)^2 \log |k|d} \right\},$$

Here $\zeta \equiv a_{BPF} \ll 1$, $a_B$ is the Bohr radius. We assume that $l \ll L$ where $L$ is the length of the one-dimensional channel.

The quadratic form $\tilde{W}(\omega)$ in eq. (17) is nonsingular for this repulsive potential at $\omega \rightarrow 0$. Therefore $\Xi(\tau)$ has a finite limit $\Xi_{\infty}$ at $\tau \rightarrow \infty$ which determines the conductance at small frequencies (see eq. (21)),

$$C(\omega=0) = \frac{|K|^2}{2\pi} \Xi_{\infty}.$$  

The renormalized Fermi speed is equal one, $v_c = 1$, for the potential in eq. (24). The expression in eq. (25) is valid at $|K|^2 \ll 1$.

It is easy to estimate $\Xi_{\infty}$

$$\sigma_{\infty} \equiv -\log \Xi_{\infty} = \int \frac{d\omega}{2\pi} \tilde{W}(\omega) \sim \frac{4}{\zeta \log \frac{1}{\zeta d}}.$$ 

The main contribution to the integral at $1/(\zeta d) \gg 1$ comes from $kl \gg 1$, where the potential is not screened. Let us notice that in this case conductance in eq. (25) is small.

Consider now also the opposite limit $\sigma_{\infty} \ll 1$. It can be implemented at intermediate concentrations if a 3-dimensional screening radius is of the order of the channel thickness. In this case we are dealing with a weak interaction and the conductance is determined by an expansion in powers of $\alpha$

$$\mathcal{B}_{2n-1} = \frac{n}{2^2} \int \frac{d\omega}{(2\pi)^2} |\omega| \alpha(\omega) \alpha(-\omega) - \frac{n^2}{4!} \int \frac{d\omega_1 \ldots d\omega_k}{(2\pi)^4} \times$$

$$\times \delta \left( \sum_{i=1}^{\infty} \omega_i \right) \Gamma_4(\omega_i) \alpha(\omega_1) \ldots \alpha(\omega_k) + \ldots, \quad (27)$$

where the vertex $\Gamma_4$ is

$$\Gamma_4(\omega_i) = \sum_i |\omega_i| - \frac{1}{2} \sum_{i<j} |\omega_i + \omega_j|.$$  

This vertex is equal zero if any frequency $\omega_i$ vanishes. Substituting this expression into eq. (14) and taking an integral over $\alpha$ with the quadratic form $\tilde{W}$ we obtain the effective transition coefficient

$$|K_\omega|^2 = |K|^2 - |R|^2 |K|^2 \sigma_{\infty} + O(\sigma_{\infty}^2),$$

which again means that conductance is non-zero according to eq. (9). Notice that this expression obeys duality which requires that the term linear in interaction should be symmetrical under $K \leftrightarrow R$ exchange. At small $|K|^2$ we return here to the expression in eq. (25) expanded at small $\sigma_{\infty}$.

The physical reason for restoration of the conductance is, in fact, disappearance of long-range order in the system. Let us see that the Luttinger model is in the BKT phase for $V(k=0) \neq 0$. Indeed, consider, first, a repulsive point-like potential, not only

$$G(T) = \langle \psi_R(0,T)\psi_R^+(0,T)\psi_R^+(0)\psi_R(0) \rangle.$$  

Using expressions for the Green function in the external field in eq. (4), eq. (5) for the effective action and taking the Gaussian integral over $U$ we obtain (see also Appendix in Ref. [13])

$$G(T) = \exp \left[ -\frac{1}{4} \int \frac{d\omega}{4\pi^2} \tilde{W}(\omega) \left( 1 - e^{i\omega T} \right)^2 \right] \frac{\Xi(T)}{4\pi^2 T^2},$$

where $\Xi(\tau)$ is a function introduced in eq. (20). In other words, the behavior of $G(T)$ and of the conductance (see eq. (19)) for an arbitrary $e - e$ potential is controlled by one and the same function $\Xi(T)$.

The function $G(T)$ for a repulsive point-like potential at large $T$ behaves as

$$G(T) \sim \left( \frac{1}{T} \right)^{2/v_c}.$$  

In the limit $v_c \rightarrow \infty$ (an infinitely strong interaction) $G(T)$ goes to some constant at large $T$. This means that
a chiral condensate $\langle \psi_R \psi_L^\dagger \rangle \neq 0$ is present in the system. This phenomenon is known to happen also in the Schwinger model (see, e.g., [14]). For a finite interaction there is no condensate but the correlator $G(T)$ decays slower than for free electrons, long-range order still exists. The number of correlated $RL$ pairs is macroscopically large $\sim L^{1-1/v_c}$ and the system is in the BKT phase. We have constructed an exact wave function of the ground state of the Luttinger model and investigated the nature of this phase in Ref. [15].

If $V(k) = 0$ at $k = 0$ the long-range order disappears. In this case $v_c = 1$ and $G(T) \sim 1/T^2$ as for free electrons. Simultaneously all critical phenomena in electron transport disappear as well and the system has finite conductance (see above).

Similar situation takes place for an attractive interaction but for the charged condensate (as for superconductivity). To reveal this condensate one considers the correlator:

$$\tilde{G}(T) = \langle \psi_R^+(0,T)\psi_L^+(0,0)\psi_R(0)\psi_L(0) \rangle \sim \left( \frac{1}{T} \right)^{2v_c}$$

for a point-like potential. Superconducting condensate arises for $v_c = 0$, while for $0 < v_c < 1$ the system is in the BKT phase. The BKT phase disappears for $v_c = 1$ when $V(k \to 0) = 0$.

Thus in all cases disappearance or restoration of the conductance is related to the long-range order in the Luttinger model.

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