Loop Amplitudes in Supergravity by Canonical Quantization

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ABSTRACT

Dirac’s approach to the canonical quantization of constrained systems is applied to $N=1$ supergravity, with or without gauged supermatter. Two alternative types of boundary condition applicable to quantum field theory or quantum gravity are contrasted. The first is the ‘coordinate’ boundary condition as used in quantum cosmology; the second type is scattering boundary conditions, as used in Feynman diagrams, applicable to asymptotically flat space-time. The first yields a differential-equation form of the theory, dual to the integral version appropriate to the second. Here, the first (Dirac) approach is found to be extremely streamlined for the calculation of loop amplitudes in these locally supersymmetric theories. By contrast, Feynman-diagram methods have led to calculations which are typically so large as to be unmanageable. Remarkably, the Riemannian quantum amplitude for coordinate boundary conditions in $N=1$ supergravity (without matter) is exactly semi-classical, being of the form $\exp(-I/\hbar)$, where $I$ is the classical action, allowing for the presence of fermions as well as gravity on the boundaries. Even when supermatter is included, typical one-loop amplitudes are often very simple, sometimes not even involving an infinite sum or integral. Specifically, the boundary conditions considered for a number of concrete one-loop examples are set on a pair of concentric 3-spheres in Euclidean 4-space. In the non-trivial cases the amplitudes appear to be exponentially convergent.
1. TWO ALTERNATIVE TYPES OF BOUNDARY CONDITION

One possibility is to use ‘coordinate’ or ‘quantum cosmology’ boundary conditions. For example, every undergraduate first learns quantum mechanics in terms of these variables. He or she is taught about the Schrödinger wave function $\psi(x,t)$. For the kind of boundary-value problem studied here, the analogue is to specify $x = x_1$ at $t = t_1$ and $x = x_2$ at $t = t_2$, and to ask for the amplitude to go between these data. A more advanced undergraduate or graduate student would learn that this can be computed as a Feynman path integral [1], or, alternatively, can be found in principle by solving the Schrödinger equation given the boundary conditions. Feynman showed that these two dual integral and differential formulations are equivalent, in that, for example, the path integral obeys the Schrödinger equation with the correct boundary conditions. In the celebrated book of Feynman and Hibbs [1] many examples of the calculational power of the Feynman path integral in ordinary quantum mechanics are given. At the same time, any reader will know that there are many other types of problem in quantum mechanics to which the Schrödinger approach may be much better suited. It should be clear, then, that the choice of boundary conditions and method can be purely a pragmatic one. Of course, Dirac has taught us that beautiful and elegant mathematics often leads to the best physics. We shall see below that this applies to the case of locally supersymmetric theories.

It should be pointed out that Dirac and Feynman themselves were certainly not insistent on the primacy of one approach over the other. Indeed, one could say that Dirac more or less invented the path integral, at least for infinitesimal time separations, as described in a paper published remarkably in 1933 in a Russian journal (but written in English) [2]. When Feynman learnt about this, he thought about it intensively. This led eventually to his celebrated Princeton PhD thesis during the war – the rest is history. Conversely, but entirely consistently (since Feynman was an ardent admirer of Dirac), the last major project on which Feynman worked concerned the ground state of Yang-Mills theory in 2 + 1 dimensions, treated by Dirac canonical methods [3]. Feynman attached great weight to this work, and spent at least three years on it. One can conclude that Feynman was not slavish about methods, as are some of his followers.

A further consideration is that our universe is an evolving cosmological model, and
not asymptotically flat at all. Hence, one cannot even set up an ‘infinity’ region in which to
describe the familiar scattering problems of particle physics. Another consideration is that
detectors in particle physics experiments are at distances of order meters or tens of meters
from the source, not at infinity. Therefore, it appears that we are forced, for the purposes
of comparison with experiment or observation, to use cosmological boundary conditions.
Indeed, the relation between ‘coordinate’ and scattering boundary conditions (the second
type of familiar boundary condition), allowing for gravity, is quite problematic. One might
attempt to construct scattering in- and out- states in regions of space-time which are not
quite at infinity, by taking outer products of single-particle harmonic-oscillator states of the
linearized theory with zero-particle states in all the asymptotic directions not occupied by
ingoing or outgoing particles. If one considers the one-particle wave function for gravitons,
one sees that, with a very small probability, one may have an arbitrarily large gravitational
wave excitation in that mode. If the amplitude of the wave is sufficiently large, then the
approximate classical infilling space-time will be very different from the nearly-flat space-
time that was originally assumed, and non-linearities will totally change the nature of
the quantum amplitude. Clearly, the process of taking the limit in which the asymptotic
regions are taken to infinity is a very awkward one, and needs much further detailed
investigation. This may very well account for the differences in the divergence structure
of quantum amplitudes for the same theory, when one adopts the two different types of
boundary condition above, which will be seen below. It is probable that such difficulties
are much less acute, say, in Yang-Mills theory, where such ambiguities have not so far been
detected.

2. SUPERGRAVITY AND ITS DIVERGENCES

In this paper, we shall adopt the Dirac approach to the quantization of constrained
systems, which Dirac developed approximately between 1950 and 1965, particularly with
a view to the quantization of gravity [4]. Dirac’s approach was subsequently taken up by
Wheeler and DeWitt [5,6], around 1967-8. From this work there stemmed a ‘first era’ of
quantum cosmology, which lasted until around 1975 [7]. Subsequently, in 1983, Hartle and
Hawking made their famous proposal for the ground state of the universe, based on the
Riemannian Feynman path integral approach to quantum gravity [8].

When supergravity is treated by Dirac’s canonical quantization method, one studies physical quantum states such as $\Psi(e^{AA'}_i(x), \phi^A_i(x))$ [9]. Here, $e^{AA'}_i$ are the spatial components ($i = 1, 2, 3$) of the tetrad $e^{AA'}_\mu = \sigma^{AA'}_a e^a_\mu$, where $\mu = 0, 1, 2, 3$ is a world index and $a = 0, 1, 2, 3$ is a tetrad index. Further, $A = 0, 1$ and $A' = 0', 1'$ are two-component spinor indices, and $\sigma^{AA'}_a$ are the Infeld-van der Waerden symbols. The odd Grassmann quantities $\psi^A_\mu$ and $\tilde{\psi}^{A'}_\mu$ describe the four-dimensional gravitino field. With $\mu$ replaced by $i$, one has the fermionic Hamiltonian dynamical data. In $N = 1$ supergravity, the Dirac approach requires that a physical state should be annihilated by the generators $S^A(x), \tilde{S}^{A'}(x)$ of local supersymmetry and the generators $J^{AB}(x), J^{A'B'}(x)$ of local tetrad rotations. It is essentially trivial to satisfy the $J$ constraints, since they simply describe the invariance of the wave function under local rotations; equivalently, all physical wave functions must be made from spinor Lorentz invariants. Explicitly:

$$S_A = i\hbar^{3s}D_i \left[ \frac{\delta}{\delta \psi^A_i} \right] + \frac{1}{2} i\hbar^2 \kappa^2 \frac{\delta}{\delta e^{AA'}_i} \left[ D^{BA'}_{j i} \frac{\delta}{\delta \psi^B_j} \right]$$  \hspace{1cm} (2.1)

$$\tilde{S}^{A'} = \epsilon^{ijk} e^{AA'}_i \frac{3s}{3} D_j \psi^A_k + \frac{1}{2} i\hbar \kappa^2 \psi^A_i \frac{\delta}{\delta e^{AA'}}.$$  \hspace{1cm} (2.2)

Here, $\kappa^2 = 8\pi$, $3s D_i$ is the torsion-free three-dimensional covariant derivative on spinors [9]. $D^{AA'}_{j k} = -2i\hbar^{-1} \epsilon^{A'B'}_k e_{BB'} j n^{BA'}$, with $h = \det(h_{ij})$, the three-metric $h_{ij}$ being equal to the corresponding components $g_{ij}$ of the four-metric. Further, the unit outward normal $n^\mu$ corresponds to the spinor $n^{BA'} = e^{BA'}_{\mu} n^\mu$.

Since there are a number of younger members of the audience who did not live through the epoch of supergravity, I should now summarize what is known about the ultraviolet divergences of $N = 1$ supergravity, first without and then with gauged supermatter. Let us start with what is known in the case of scattering boundary conditions (i.e., by means of Feynman diagrams). We have to examine the historical record as far back as the rule of Rameses II; Egyptian mathematics lasted for three thousand years and has many achievements. The relevant record for the one-loop case is found on a very crumpled papyrus roll; of course, you have to be able to read hieratics in order to decipher it. But,
there is no doubt it reads that pure supergravity is finite at one loop. For two loops, we have to move forward into the Middle Ages, where in a monastery of the twelfth century there was found a frayed parchment document in Mediaeval Latin in which a mathematically-minded monk discovered that pure supergravity was also finite at this level. Beyond two loops, we look in vain at the historical record, whether in manuscript or stored in some computer. As far as anyone can tell, the question as to whether pure supergravity has a divergence at three loops has never been resolved. This limitation to our knowledge appears to be the result simply of human frailty. Perhaps if someone lived to the age of Methuselah, they might stand a chance of managing the enormous Feynman-diagram calculation. [For those of you who do not come from the Judaeo-Christian-Islamic tradition, Methuselah was the oldest man in the Bible; he lived to 969 years of age.] The notion that supergravity at three loops is divergent is no more than a myth.

For pure supergravity with quantum cosmology boundary conditions, one finds no ultraviolet divergences at any order. This can be seen from a general argument [9] and its workings can be examined in more detail in examples such as those at the end of this paper.

Turning to the more general case of $N = 1$ supergravity with gauged supermatter, as described clearly in the second edition of the book of Wess and Bagger [10], this is always divergent at all loops in the scattering formulation [11]. But an argument similar to that mentioned above shows that all amplitudes using quantum cosmology boundary conditions are finite, as in pure supergravity. This is partly based on the property that, when the boundary data including supermatter are purely bosonic, the amplitude is exactly semi-classical: $\Psi = \exp(-I/\hbar)$.

This connects with a very general pure-mathematical problem. To take the simplest example, consider the ‘Hartle-Hawking’ classical boundary-value problem for pure Einstein gravity. Suppose one takes a boundary manifold of topology $S^3$ with a given Riemannian three-metric $h_{ij}$ and an interior region with the usual topology. One then asks whether there is a unique (up to diffeomorphism) Riemannian four-metric $g_{\mu\nu}$ on the interior, agreeing with the boundary metric and obeying the vacuum Einstein equations $R_{\mu\nu} = 0$. It is easy to prove this in the case of small perturbations of the round sphere, using a
fixed-point method or equivalently the implicit function theorem [12], by first studying the problem linearized about flat Euclidean space. Some related results, found by working close to known manifolds, have recently been established [13,14]. Of course, the resulting metrics are the analogue of weak gravitational waves which are perturbations of Minkowski space-time. On the other hand, there has to date been no general study of this problem for large deformations of the sphere. One would think that this would be a wonderful arena for pure mathematicians! (G. Gibbons has pointed out that, if one replaces the usual interior topology by a suitable bundle topology, then for sufficiently deformed boundary data there may be two, not one, classical solutions inside, of Taub-Bolt type [15].) It is interesting to compare the historical situation with regard to the hyperbolic Cauchy evolution problem for the Einstein equations in the usual Lorentzian case, a question which Einstein himself might well have asked. By 1953, Yvonne Choquet-Bruhat had already begun to attack this problem [16]. Its resolution had reached an advanced state by 1970 [17]. By comparison, Riemannian quantum gravity began with the work of Hartle and Hawking on the positive-definite Schwarzschild solution in 1975 [18], and the boundary-value problem above, associated with the same authors, dates from 1983 [8].

3. SUITABLE BOUNDARY DATA

Starting with the Riemannian classical boundary-value problem for pure Einstein gravity, one might at first think that the simplest boundary conditions would be to specify the three-metric on two nearly-planar three-surfaces at different imaginary time coordinates, measured at spatial infinity, assuming asymptotic flatness at spatial infinity. Unfortunately, for weak perturbations of flat Euclidean four-space, the resulting four-metric $g_{\mu\nu}$ does not depend in a very smooth way on the boundary data $h_{ij}$ in a neighbourhood in which $h_{ij}$ is close to the flat metric $\delta_{ij}$, as pointed out by Stephen Hawking. To see this, suppose that the lower boundary is intrinsically flat, and that the upper boundary has a curved intrinsic three-metric, such that it can be embedded in flat Euclidean four-space with the other boundary as the lower boundary. Then note that one could have equally well turned the upper boundary surface upside down, and still had a flat solution of the same boundary-value problem. A more detailed investigation shows that the small devi-
ations in the classical four-metric go roughly as the square root of the deviations in the boundary three-metric, in this neighbourhood. Therefore, it is not really practicable to use Fourier analysis in studying this perturbation problem.

Instead, one takes the next-simplest possibility: two concentric three-spheres, such that, in the unperturbed configuration, the inner sphere has radius $\alpha$ and the outer sphere has radius $\beta$. It is then appropriate to decompose all perturbations in terms of harmonics on $S^3$. The original treatment, for bosonic perturbations, was given by Lifschitz in 1946. Greater detail is given by Lifschitz and Khalatnikov [19]. Workers in cosmology will be familiar with these: for spin $s = 0$, one has density perturbations, for $s = 1$, rotational perturbations, and for $s = 2$, one has cosmological gravitational waves. For example, one can write the scalar modes as

$$Q_{l|m}^n = \Pi_l^n(\chi)Y_{lm}(\theta,\phi)$$

$$Q_{i|i} = -(n^2 - 1)Q,$$

where $\chi, \theta, \phi$ are standard coordinates on the unit three-sphere as defined in Eq.(3.3), and $\Pi_l^n$ obeys a suitable radial equation.

To cover harmonics of all spins $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$, it is best to use two-component spinors. As a preliminary, we need to evaluate integrals of powers of $x^a$ over the unit three-sphere, where $a$ is a tetrad index. Suppose we are given $n = 2m(m = 0, 1, \ldots)$. Define

$$C_n = \int d\Omega u^n,$$

where $d\Omega$ is the measure on the unit three-sphere, and where

$$x = \sin \chi \sin \theta \cos \phi,$$

$$y = \sin \chi \sin \theta \sin \phi,$$

$$z = \sin \chi \cos \theta,$$

$$u = \cos \chi.$$  

One finds, using [20],

$$C_n = 4\pi \frac{1 \times 3 \ldots \times (n - 3) \times (n - 1) \times 1}{2 \times 4 \ldots \times (n - 2) \times n \times (n + 2)}.$$  

(3.4)
As \( n \to \), one finds from Stirling’s formula [20] that

\[
C_{n=2m} \sim \frac{2\pi^\frac{3}{2}}{(m + 1)^\frac{3}{2}}.
\]  

(3.5)

Now consider

\[
\int d\Omega x^{a_1} x^{a_2} ... x^{a_{2n}} = D_n \delta^{(a_1 a_2 \delta a_3 a_4 ... \delta a_{2n-1} a_{2n})},
\]  

(3.6)

with

\[
\int d\Omega u^{2n} = D_n = C_n.
\]  

(3.7)

We shall need the spinor version of these equations, giving

\[
\int d\Omega x^{A_1 A'_1} x^{A_2 A'_2} ... x^{A_{2m} A'_{2m}} = \frac{C_{2m}}{(2m)!} (\epsilon^{A_1 A_2} \epsilon^{A_3 A_4} ... \epsilon^{A_{2m-1} A_{2m}} ... + \text{all permutations on both primed and unprimed indices}).
\]  

(3.8)

The tensor and spinor harmonics on \( S^3 \) can now be described in a uniform way: (a) \( s = 0 \). Instead of the Lifschitz-Khalatnikov description above, one can write a normalized harmonic \( \phi^{npq} \) on the unit sphere as

\[
\phi^{npq} = T_{(A_1 ... A_n)(A'_1 ... A'_n)} x^{A_1 A'_1} ... x^{A_n A'_n},
\]  

(3.9)

where \( T_{...} \) is a constant array of the form

\[
T_{...} = E_n (T_{00 ... 01} ... 0^0 \; 0^0 \; ... 0^{1'} ... 1')
\]  

+ the remainder of the \( (n!)^2 \) permutations on both primed and unprimed indices).  

(3.10)

Here, there are \( p \) zeros and \( q \) primed zeros, and the quantity \( T_{...} \) on the righthand side of Eq.(3.10) is numerically equal to 1. The normalization constant \( E_n \) is fixed by the requirement

\[
\int d\Omega \phi^{npq} \bar{\phi}^{npq} = 1 = C_{2n} 2^n (n!)^2 |E_{np}|^2.
\]  

(3.11)

One can check from this definition that \( \phi^{npq} \) obeys the harmonic equation Eq.(3.1).
(b) \( s = \frac{1}{2} \). These harmonics are described in more detail in [21]. There are normalized positive frequency harmonics

\[ \rho_{A}^{npq} = (A \ T_{A_{1}...A_{n}}(A'_{1}...A'_{n})) x^{A_{1}'}...x^{A_{n}'} \]  

(3.12)

similar to the scalar harmonics in (a) above. One can check that they obey the eigenvalue equation

\[ e^{AA'}j \ 3sD_{j} \rho_{A} = -(n + \frac{3}{2}) e^{nAA'} \rho_{A}, \]  

(3.13)

where

\[ e^{nAA'} = -i n^{AA'} \]  

(3.14)

is the Euclidean normal. Similarly, there are positive frequency primed harmonics

\[ \sigma_{A'}^{npq} = A' \ T_{A_{1}...A_{n}A'_{1}...A'_{n}} x^{A_{1}'}...x^{A_{n}'} \]  

(3.15)

where \( A' \ T_{...} \) is again totally symmetric on primed and unprimed indices. The negative frequency modes are of the form

\[ \tau_{A} \propto e^{nAA'} \sigma_{A'}, \quad \mu_{A'} \propto e^{nAA'} \rho_{A}, \]  

(3.16)

which for example obey

\[ e^{AA'}j \ 3sD_{j} \tau_{A} = + (n + \frac{3}{2}) e^{nAA'} \tau_{A}. \]  

(3.17)

(c) \( s = 1 \). The harmonics are of the form

\[ v_{AA'}^{npq} = AA' \ T_{A_{1}...A_{n}A'_{1}...A'_{n}} x^{A_{1}'}...x^{A_{n}'} \]  

(3.18)

where, as always \( T \) is totally symmetric.

(d) \( s = \frac{3}{2} \). The true gravitino data are given by the harmonics

\[ \rho_{(ABC)}^{npq} = ABC \ T_{A_{1}...A_{n}A'_{1}...A'_{n}} x^{A_{1}'}...x^{A_{n}'} \]  

(3.19)

with the usual symmetry.
(e) $s = 2$. The true graviton data are analogously given by

$$e_{AA'B'B'}^{npq} = A_{A_1...A_n', A_1'...A'_n} x^{A_1} A_1'...x^{A_n} A'_n,$$  \hspace{1cm} (3.20)

again with total symmetry.

4. LOOP AMPLITUDES IN $N = 1$ SUPERGRAVITY

As remarked earlier, if we only had gravitational perturbations on the two spherical boundaries, then the full amplitude in quantum $N = 1$ supergravity would be exactly semi-classical, of the form $\exp(-I/\hbar)$, where $I$ is the classical gravitational action. For any hope of non-trivial quantum effects, one should put fermionic data on the boundaries. With our boundary data set on the concentric pair of spheres, the simplest fermionic weak-field case is to specify a harmonic, $\tilde{\psi}^{MPQ}_{A'B'C'}$, on the inner boundary, and the corresponding harmonic $\psi^{ABC}_{AB'C'}$ on the outer boundary. Of course, one could simulate scattering by taking a linear combination of two harmonics on each boundary, but the qualitative conclusions will not be greatly changed.

One proceeds by applying the supersymmetry constraint $S_C \Psi = 0$ to the amplitude

$$\Psi \sim (A + h A_1 + h^2 A_2 + ...) \exp(-I_{\text{class}}/\hbar).$$  \hspace{1cm} (4.1)

At the lowest order $h^0$, one obtains the classical constraint

$$S_C = 0,$$  \hspace{1cm} (4.2)

which is automatically satisfied by virtue of the classical field equations. At the next order, $h^1$ one finds

$$3s D_i \left[ \frac{\delta (log A)}{\delta \tilde{\psi}^C_i (x)} \right]$$

$$+ \frac{1}{2} h \kappa^2 \frac{\delta}{\delta e_{AA'}(x)} \left[ D_{j_1}^{BA'} \frac{\delta I}{\delta \tilde{\psi}^B_j (x)} \right] = 0.$$  \hspace{1cm} (4.3)

Here, $\kappa^2 = 8\pi$. The right-hand object in square brackets is in fact the classical $\tilde{\psi}^A_i (x)$, as one finds by consideration of the canonical fermionic momentum [9]. Note the double
functional derivative at the same point \( x \), but with respect to one fermionic and one bosonic argument. It turns out that this does not lead to the kind of infinities which are inevitably present in the canonical quantization of pure Einstein gravity, through terms of the kind

\[
\delta^2 \Psi \over \delta h_{ij}(x) \delta h_{kl}(x),
\]

as in the following example.

The classical solutions are derived from the Euclidean action of \( N = 1 \) supergravity;

\[
I = \int_{\text{VOL}} d^4x \left[ -\frac{1}{2\kappa^2} (\det e) R 
+ \frac{1}{2} e^{\mu
u\rho\sigma} (\tilde{\psi}_\mu^A e_{AA'\nu} D_\rho \psi_\sigma^A + \text{h.c.}) \right] 
+ \int_{\text{BDRY}} d^3x \left[ -\frac{1}{\kappa^2} h^{ij} (\text{tr} K) + \epsilon^{ijk} \psi_i^A e_{AA'j} \psi_{k'}^{A'} \right].
\] (4.4)

Here, \( K_{ij} \) is the second fundamental form [9] and \( \text{tr} K = h^{ij} K_{ij} \). Also, at a classical solution the volume integral vanishes, and the action \( I \) resides only in the boundary integral.

For the perturbation problem involving \( \psi^{MPQ}_{ABC} \) etc., there are several contributions to \( I \) of the type \( e \tilde{\psi} \), needed for the right hand side of Eq.\( (4.3) \). Let us take a typical one, of the largest possible size, arising from \( K_{ij} \), contributing to the covariant derivative \( D_\rho \) in the volume integral. One can calculate the change in \( \tilde{\psi}_{A'B'C'} \), due to the addition of (say) a small graviton perturbation \( \delta e_{AA'BB'}(x') \) by integrating the Rarita-Schwinger equation radially. The general graviton perturbation on the outer surface can be written as

\[
\delta e_{AA'BB'}(x') = \Sigma_{NRS} c_{NRS} e_{AA'BB'}^{NRS}(x').
\] (4.5)

Then, by orthogonality,

\[
c_{NRS} = \int d\Omega \delta e_{AA'BB'}(x') e_{CC'DD'}^{NRS}(x') n^{AC'} n^{BD'} n^{CA'} n^{DB'}.
\] (4.6)

Hence,

\[
\frac{\delta c_{NRS}}{\delta e_{AA'BB'}(x)} = e_{NRS}(x) nnnn.
\] (4.7)
where the indices on the right hand side are straightforward to calculate. The above-mentioned change in $\tilde{\psi}_{A'B'C'}$ depends on all the constants $c_{NRS}$. The total change of course involves a radial as well as an angular integral. But it turns out that the resulting dependence on the boundary radii $\alpha$ and $\beta$ is unimportant.

One finds that

$$\tilde{\psi}_i^{A'}(x) \sim \Sigma_{NRS} \int d\Omega' c_{NRS} \epsilon^{(NRS)}(x') \tilde{\psi}^{MPQ}(x')$$

$$\times \text{terms proportional to the normal } n,$$

the alternating symbol $\epsilon$ and the background spatial tetrad $e^i_{...}$.

Hence,

$$\frac{\delta}{\delta e_i^{AA'}}(x) [\tilde{\psi}_i^{A'}(x)]$$

$$\sim \Sigma_{NRS} \int d\Omega' e^{NRS}(x') e^{NRS}(x) \tilde{\psi}^{MPQ}(x')$$

$$\times \text{other terms, as above.}$$

But, by completeness, the sum over $NRS$ reduces to a product of delta functions with respect to the indices. Since on the left hand side there is only one free index $A$, the right hand side must consist of a multiple of $\tilde{\psi}_{ABC}(x)$ with the two indices $BC$ contracted, namely zero. Hence the corresponding log $A$ is also zero. A similar result would have been obtained if we had allowed for a linear combination of two harmonics on the boundaries.

A more general argument, leading to a similar conclusion, can be given (say) when the boundary data on both spheres consist only of a weak-field mixture of spin-$3/2$ harmonics. This depends on examining the left hand term of Eq. (4.3), instead of the right hand term. This arises because one cannot make log $A$ out of some quantity contracted with $\psi^{MPQ}_{ABC}$, etc., because as one can check,

$$3s D_i \psi^{MPQ}_{A} = 0,$$  \hspace{1cm} (4.10)

where

$$\psi^{MPQi}_{A} = e^{BB'}e_C^{B'}\psi^{MPQ}_{ABC}.$$  \hspace{1cm} (4.11)
This is analogous to the property [19]

\[ S^i |i = 0, \]  

(4.12)

where \( S^i \) represents a generic spin-1 harmonic in the language of Lifschitz and Khalatnikov [19]. Hence one must have \( \log A = 0 \) in this case. Of course, one could instead have checked that the right hand term in Eq. (4.3) was also zero.

One could only make a non-zero \( \log A \) if the boundary data contained a spin-1/2 part \( \psi_A^{MPQ} \), which is forbidden in our simple example by the \( \tilde{S}_{A'} = 0 \) classical constraint:

\[ \tilde{S}_{A'} = \epsilon^{ijk} e^{3s}_{AA'i} D_j \psi_k^A + \frac{1}{2} i \kappa^2 \psi_i^A p_{AA'}^i = 0, \]  

(4.13)

where \( p_{AA'}^i \) is the momentum conjugate to \( e_i^{AA'} \) [9]. The next simplest generalization of these boundary data would be to include in addition a weak-field graviton mode \( e_{AA'B'B'}^{RST} \) on the outer boundary (say). The classical constraint (4.13) will enforce an extra non-zero spin-1/2 term on the outer boundary, which is at least quadratic in the gravitino and graviton perturbations (starting with a cross term). Again, this cannot match the right hand term in Eq. (4.3). The same holds for more general gravitino and graviton perturbations. We conclude that

\[ \Psi = \exp(-I_{\text{class}}/\hbar) \]  

(4.14)

for pure \( N = 1 \) supergravity.

This might, at first sight, seem shocking. It says that there are no quantum corrections for \( N = 1 \) supergravity with these boundary conditions. All the dynamics resides in the classical motion. However, it was previously known [9] that Eq. (4.14) held for purely bosonic boundary data, and so it does not seem unreasonable that the same should be true when one includes fermionic data, given the local supersymmetry of the theory. Further, all our experimental knowledge of loop effects comes, of course, from particle physics at ‘low’ energies, which only involves non-gravitational interactions.

5. **\( N = 1 \) SUPERGRAVITY WITH GAUGED SUPERMATTER**

Due to the work of many authors, not listed here, the general locally supersymmetric \( N = 1 \) model of gravity interacting with a gauge theory has been found [10]. Because of
the huge amount of local symmetry – local supersymmetry, local coordinate invariance, local tetrad rotation invariance, and local gauge invariance – these models are of a very restricted type. The Lagrangian $\mathcal{L}$ can be split as

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{pot}},$$

(5.1)

where $\mathcal{L}_{\text{kin}}$ is determined once the symmetry group such as SU(2), SU(3), etc., and the gauge coupling constant $g$ are specified. The remaining part $\mathcal{L}_{\text{pot}}$ depends on a potential $P$ which is a function of the scalar fields. For simplicity, in this section we shall set $P = 0$.

In the simplest non-trivial case, with SU(2) gauge group [10], there is one complex scalar $(a, a^*)$ with Kähler potential

$$K = \log(1 + aa^*)$$

(5.2)

and Kähler metric

$$g_{11} = \frac{\partial^2 K}{\partial a \partial a^*} = \frac{1}{(1 + aa^*)^2}.$$  

(5.3)

Now,

$$ds^2 = \frac{dada^*}{(1 + aa^*)^2}$$

(5.4)

is the metric on a unit two-sphere (really, $CP^1$). The point with $a = a^* = 0$ may be regarded as the North pole, and the point $a = a^* = \infty$ is then the South pole; there is nothing preferred about these points – for example, the scalar field may move freely through $a = \infty$. The connection between the Kähler scalar part of the theory and the gauge theory is that the isometry group $SU(2)$ for the scalars is, then, the gauge group of the full theory.

The other fields in the kinetic theory may be summarised as follows. There is a spinor field $(\chi_A, \tilde{\chi}_{A'})$, which has no Yang-Mills index, and which is the partner of $(a, a^*)$. The Yang-Mills field $v^{(a)}_\mu$, with $(a) = 1, 2, 3$ has fermionic partners $(\lambda^{(a)}_A, \tilde{\lambda}^{(a)}_{A'})$. As usual, one also has the tetrad $e_{AA'\mu}$ and the gravitino $(\psi_{A\mu}, \bar{\psi}_{A'\mu})$. The relevant Lagrangian may be found in Wess and Bagger [10].
If, say, one wanted to extend this model to $SU(3)$, one could use the corresponding Kähler metric given in [22]. For $SU(n)$, one can similarly use [15].

As in the case above of pure $N = 1$ supergravity, one proceeds to find loop terms iteratively using the quantum supersymmetry constraint $S_A \Psi = 0$. In the present $SU(2)$ case, the operator $S_A$ has the general structure

\begin{align}
\text{const.} & 3s D_i \left( \frac{\delta}{\delta \psi^C_i(x)} \right) \\
& + \text{const.} \frac{\delta}{\delta e_i^{AA'}(x)} \left[ D_{ji}^{BA'} \frac{\delta}{\delta \psi_j^B(x)} \right] \\
& + \text{const.} \epsilon^{ijk} e_{AB'} k_{n}^{BB'} F_{ij}^{(a)} \left( \frac{\delta}{\delta \lambda^{(a)B}} \right) \\
& + \text{const.} n_{BB'} e_{AB'} \frac{\delta}{\delta \psi^{(a)}_i} \left( \frac{\delta}{\delta \lambda^{(a)B}} \right) \\
& + \text{const.} g_{11'} \left( \tilde{D}_i a^{*} \right) e_{A'B'} \frac{\delta}{\delta \tilde{\chi}^{B'}} \\
& + \text{const.} n_{A'A'} \frac{\delta}{\delta a} \left( \frac{\delta}{\delta \tilde{\chi}^{A'}} \right) \\
& + \text{higher – order terms}\.
\end{align}

Here, the fermionic coordinates are being regarded as $\psi^A_i, \lambda^{(a)B}, \tilde{\chi}^{A'}$, while $F_{ij}^{(a)}$ are the spatial components of the Yang-Mills field strength. The covariant derivative $\tilde{D}_i a^{*}$ is defined in [10].

Once a loop term has been found (iteratively, if necessary), one must further check that the conjugate quantum constraint $\bar{S}_{A'} \Psi = 0$ is also satisfied. This occurs in the examples below.

The Dirac approach to the computation of loop terms in such a locally supersymmetric theory is extremely streamlined by comparison with the corresponding path-integral calculation. As can be seen from Eq.(5.5), in the Dirac approach one only needs to concentrate on related fermionic and bosonic partners to find the dependence of the amplitude on those variables. This removes many complications. In contrast, a path-integral treatment would inevitably involve integration over the relatively large number of fields, and one would
always have to be verifying cancellation effects between bosons and fermions.

Now consider some examples of loop calculations, with the usual pair of spherical boundaries, in the simplest $SU(2)$ model. This work has been carried out jointly with M.M. Akbar. Note, from [10], that there is a negative cosmological constant, of order $g^2$ in the theory. Strictly, this implies that the background four-geometry is a Riemannian space of constant negative curvature. In this case, the angular harmonics used are the same, but the radial dependence of the corresponding classical solutions for a given principal quantum number $n$ changes from a power law to an exponential. This makes no qualitative difference to the outcomes of the calculations below. Alternatively, the reader may wish to imagine that $g$ is exceedingly small.

Example (1) Find the one-loop correction in the case that the data on the inner sphere, of radius $\alpha$, are a harmonic of the first kind of spinor field above: $\chi_A = \chi_{A}^{PMN}$, while the data on the outer sphere are the corresponding $\tilde{\chi}_{A'} = \tilde{\chi}_{A'}^{PMN}$. From Eq.(5.5), we see that we need to make a variation $\delta a^*(x)$, which can be expanded in scalar harmonics as

$$\delta a^*(x) = \Sigma_{QST} c_{QST} \phi_{QST}.$$  (5.6)

Ignoring the details caused by the radial dependence (which lead to a possible dependence on $\alpha$ and $\beta$, one finds schematically from Eq.(5.5) that

$$\phi^{BWX}_R(x)e^{i}_{BA} n^{BB'}(\frac{\delta \log A}{\delta \tilde{\chi}_{B'}(x)})$$

$$= \Sigma_{YZL} n_{CA'} \phi_{YZL} B(x)e^{i}_{BB'}(\partial_i \phi^{BWX}_R)(\tilde{\phi}_{YZL} B(x)\phi_{C(PMN)}(x)).$$  (5.7)

Now note that, when one fixes $Y$ in the summation, but sums over all $ZL$ consistent with this, the term $\phi^{B(YZL)}(x)e^{i}_{BB'}(\partial_i \phi^{BWX}_R)(\tilde{\phi}_{YZL} B(x)\phi_{C(PMN)}(x))$ cancels out. Hence, in this case, $\log A = 0$.

This example may look too simple, in that we have only taken one harmonic for both surfaces. However, the same conclusion arises when one considers a ‘scattering’ problem with two harmonics added together on each surface.

Example (2) The intention here is to illustrate a gravitational effect on a one-loop term $A$. The data chosen are as in Example (1), except that one adds in a weak field scalar harmonic $a(x) = \phi^{RWX}(x)$ on the outer surface. Clearly, the presence of $a$ will
induce a non-trivial gravitational field at quadratic and higher orders in $a$, which will then contribute to $I, A, ...$ Since the calculation is a little complicated, we take here the simplest non-trivial case with the lowest spinor harmonic $P = 0$, giving $\chi_A = \text{constant}$ if we were in flat Euclidean four-space.

One finds, without detailed attention to the radial dependence,

\[
\frac{\delta (\log A)}{\delta \tilde{\chi}_{B'}(x)} = \sum_{QNY} \phi^{QNY}(x) \phi^{(Q+R,N+W,Y+X)}(x) \chi^B(x) n_{BB'}
\]

\[
\times \left[ \frac{QR}{(Q + R)(Q + R + 1)} \right] \frac{(Q + R)!Q!R!}{C_Q R C_Q C_R C_{2(Q+R)} 2^{Q+R}(2Q + 2R)!}.
\]

(5.8)

When one does include the radial dependence, it only makes a difference of $O(1)$, except that, as usual in quantum gravity, a single power of $\hbar$ is associated with two negative powers of radius. This means that the present one-loop term is smaller than the kind of one-loop term which might be found by studying the interactions between particles of spins $0, \frac{1}{2}, 1$ by a factor of order $(\text{Planck length}/\beta)^2$. If, say, $\beta$ were $1 \text{ cm.}$, our factor would be down by $10^{-66}$ on a typical one-loop factor.

To understand the rate of convergence of the sum in Eq.(5.8), one uses Stirling’s formula [20], which shows that, for large $Q$, the sum has the form

\[
\sum_Q (\text{slow}) 2^{-\text{const}.Q}.
\]

(5.9)

Here, the ‘slow’ terms are typically polynomial, and one sees that the convergence is exponential; this is of course much faster than in any Feynman diagram. One might similarly ask about the corresponding two- and higher-loop terms. Because of the way in which the spinor indices combine in the spinor harmonics above, the dominant structure is always the same: the sum of the terms inside the factorial signs on the top line is always the same as the sum of the corresponding terms on the bottom line. But the terms on the bottom line are always combined in larger fragments. Because of the way in which Stirling’s theorem works, this means that one always will find negative exponentials for large values of the principal quantum numbers, which will overwhelm any ‘slow’ polynomial
terms arising from ‘gravitational vertices’. It is of course the dreaded polynomials in the momentum which lead to the non-renormalizability of Einstein quantum gravity.

Had we, in Example (1), say, taken data which give a non-trivial sum on the right hand side, for log\(A\) or for higher loops, but which do not perhaps involve gravitational interactions in the classical action, we would still have found the same dominant structure in the sum on the right hand side, leading again to an exponential convergence.

Example (3) Here we choose the quark-like fermionic data, given by a harmonic \(\tilde{\lambda}_{(P MN)}^{(a)B'}\) on the inner sphere, and \(\lambda_{(P MN)}^{(a)B}\) on the outer sphere. Recall that the bosonic partner of \(\lambda_{(P MN)}^{(a)B}\) is \(v_{m}^{(b)}\). The constraint, as given by Eq.(5.5), yields

\[
\epsilon^{ijk} e_{BA'k} F_{ij}^{(a)} \left( \frac{\delta \log A}{\delta \lambda_{(a)B}(x)} \right) m^{B'B} \\
\sim \Sigma_{QNP} \cdots \epsilon^{abc} v_{l}^{(a)(QNP)} v_{l}^{(b)(QNP)} \cdots
\]

\[
= 0.
\]  

Hence, \(\log A = 0\) in this case also.

Of course, there are many interactions between particles of spins 0, \(\frac{1}{2}\), 1, which one would expect to lead to various loop effects. However, the ‘gravitational’ example (2) should be sufficient to illustrate what happens in such cases.

Since the loop behaviour of this model appears reasonable, it seems worthwhile to investigate the model further with regard to its physical consequences, and to try to predict effects which are observable at accelerator energies.

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