FORWARD ATTRACTING SETS OF REACTION-DIFFUSION EQUATIONS ON VARIABLE DOMAINS

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Abstract. Reaction-diffusion equations on time-variable domains are intrinsically nonautonomous even if the coefficients in the equation do not depend explicitly on time. Thus the appropriate asymptotic concepts, such as attractors, are nonautonomous. Forward attracting sets based on omega-limit sets are considered in this paper. These are related to the Vishik uniform attractor but are not as restrictive since they depend only on the dynamics in the distant future. They are usually not invariant. Here it is shown that they are asymptotically positively invariant, in general, and, if the future dynamics is appropriately uniform, also asymptotically negatively invariant as well as upper semi continuous dependence in a parameter will be established. These results also apply to reaction-diffusion equations on a fixed domain.

1. Introduction. The nature of time in a nonautonomous dynamical system is very different from that in autonomous systems, which depend only on the time that has elapsed since starting rather on the actual time itself. Consequently, limiting objects may not exist in actual time as in autonomous systems. This is most apparent in the shortcomings in the definition of a forward attractor in a nonautonomous systems [7, 10]. The usual definition is an intuitive and natural counterpart of that of a pullback attractor: an invariant family of compact subsets that attracts bounded sets in the sense of forward rather than pullback convergence [1, 2, 11]. A forward attractor is a Lyapunov asymptotically stable family of sets and provides much information about the dynamics of the nonautonomous system in current as well as asymptotic future, but it does not exist in many systems. Moreover, the definition requires the subsets to exist for all time, also in the distant past, even though forward limiting behaviour is really only about what happens in the distant future.

Vishik [2, 15] proposed using the forward omega-limit set as the forward attractor, which he called the uniform attractor. This set indicates where the forward limit points are to be found. It is generally not invariant and provides little information about the dynamics in actual time on the approach to the limit set. In addition,
for nonautonomous dynamical systems defined in terms of processes, it requires the attraction to be uniform in the initial time, even those in the distant past.

In some recent developments by the authors [6, 7, 8, 13] it was shown that the forward omega-limit sets are, in general, asymptotically positively invariant and, if the future dynamics is appropriately uniform, also asymptotically negatively invariant. These are weaker than the usual invariance concepts, but nevertheless provide more information about the approach of the dynamics to the omega-limit set itself. As such they make Vishik’s definition of a uniform attractor and its weaker variations, which were called forward attracting sets in [13], much more useful.

Reaction-diffusion systems on time-variable domains are intrinsically nonautonomous even if the equation terms do not depend explicitly on time. Thus their attractors are nonautonomous. Pullback attractors for such systems were investigated by Kloeden et al [9, 12], while random attractors for stochastic versions were considered in Crauel et al [3, 4]. Here the forward limiting behaviour of deterministic reaction-diffusion systems on time-variable domains is investigated, in particular their asymptotical positively invariant and asymptotical negatively invariant as well as upper semi continuous dependence in a parameter will be established. The results also apply to systems on a fixed domain.

The set up with variable domains is formulated in the next section. They allow the dynamics to be transformed to a fixed domain \( O \) in Section 3. The asymptotic invariance results are established in Section 4 and those for upper semi continuous dependence in a parameter in Section 5. The asymptotic invariance and limiting results are formulated in terms of the time varying domains in Section 5.1. Finally, the existence of positively invariant absorbing sets in both \( L^2(O) \) and \( H^1_0(O) \), that is required in the previous proofs, is established in Section 6.

2. Variable domains. Let \( T^* \in \mathbb{R} \) and let \( O \) be a nonempty bounded open subset of \( \mathbb{R}^N \) with \( C^2 \) boundary \( \partial O \). Define \( \mathbb{R}_{T^*} = [T^*, \infty) \) and let \( r \in C^1(\mathbb{R}^N \times \mathbb{R}_{T^*}; \mathbb{R}^N) \) be such that

\[
  r(\cdot, t) : O \to O_t := r(O, t) \quad \text{is a } C^2\text{-diffeomorphism for each } t \in \mathbb{R}_{T^*}.
\]

For any \( \tau < T \) in \( \mathbb{R}_{T^*} \) define

\[
  Q_{\tau, T} := \bigcup_{t \in (\tau, T)} O_t \times \{t\}, \quad Q_{\tau} := \bigcup_{t \in (\tau, +\infty)} O_t \times \{t\},
\]

\[
  \Sigma_{\tau, T} := \bigcup_{t \in (\tau, T)} \partial O_t \times \{t\}, \quad \Sigma_{\tau} := \bigcup_{t \in (\tau, +\infty)} \partial O_t \times \{t\}.
\]

The set \( Q_{\tau, T} \) is an open subset of \( \mathbb{R}^{N+1} \) with boundary

\[
  \partial Q_{\tau, T} = \Sigma_{\tau, T} \cup (O_{\tau} \times \{\tau\}) \cup (O_T \times \{T\}).
\]

The following assumption is made on the domain transformation \( r \) and its spatial inverse \( \bar{r} \).

**Assumption 1.** The domain transformation \( r \) and \( \bar{r} \) are two times continuously differentiable with the derivatives uniformly bounded on \( Q_{T^*} \), i.e., \( r, \bar{r} \in C_b^2(Q_{T^*}; \mathbb{R}^N) \).
3. **A parabolic PDE on time-varying domains.** Consider the parabolic PDE of the reaction-diffusion type with a homogeneous Dirichlet boundary condition

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u - \lambda u - f(u) + g(t) \quad \text{in } Q_T, \\
\frac{\partial u}{\partial t} &= 0, \quad \text{on } \Sigma_T, \quad u(x, \tau) = u_\tau(x), \quad x \in \mathcal{O}_\tau, \quad \tau \geq T^*,
\end{aligned}
\]  

(3.1)

where \( u_\tau : \mathcal{O}_\tau \to \mathbb{R} \), \( f \in C^1(\mathbb{R}) \), \( g : \mathcal{Q}_T \to \mathbb{R} \) and \( \lambda > 0 \) are given.

With the change of variables

\[
v(y, t) = u(r(y, t), t), \quad \text{for } y \in \mathcal{O}, \quad t \geq \tau,
\]

(3.2)

or, equivalently,

\[
u(x, t) = v(\tilde{r}(x, t), t), \quad \text{for } x \in \mathcal{O}_t, \quad t \geq \tau,
\]

(3.3)

the problem (3.1) on the time-varying domains is transformed into the problem

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \sum_{i,j=1}^N \frac{\partial}{\partial y_j} (a_{i,j}(y, \tau,t) \frac{\partial u}{\partial y_i}(y, \tau,t)) - b(y, \tau,t) \cdot \nabla_y v(y, \tau,t) \\
&\quad - \lambda v(y, \tau,t) - f(v(y, \tau,t)) + g(r(y, \tau,t)) \\
&\quad \text{in } \mathcal{O} \times (\tau, \infty),
\end{aligned}
\]

(3.4)

or, equivalently,

\[
\begin{aligned}
\frac{\partial v}{\partial t} &= \sum_{i,j=1}^N \frac{\partial}{\partial y_j} (\tilde{a}_{i,j}(\tilde{r}(y, \tau,t)) \frac{\partial v}{\partial y_i}(\tilde{r}(y, \tau,t))) \\
&\quad - \tilde{b}(\tilde{r}(y, \tau,t)) \cdot \nabla_y \tilde{r}(y, \tau,t) \\
&\quad - \tilde{\lambda} v(\tilde{r}(y, \tau,t)) - f(v(\tilde{r}(y, \tau,t))) + g(r(\tilde{r}(y, \tau,t))) \\
&\quad \text{in } \mathcal{O} \times (\tau, \infty),
\end{aligned}
\]

(3.5)

on the fixed spatial domain \( \mathcal{O} \). Here

\[
b(y, t) := (b_1(y, t), \ldots, b_N(y, t)) \in \mathbb{R}^N
\]

is defined componentwise by

\[
b_k(y, t) = \frac{\partial \tilde{r}_k}{\partial t} (\tilde{r}(y, t), t) - \Delta_x \tilde{r}_k(\tilde{r}(y, t), t) + \sum_{j=1}^N \frac{\partial a_{j,k}}{\partial y_j}(y, t)
\]

(3.6)

for \( k = 1, \ldots, N \).

Now consider the bilinear form \( a_\lambda(t, u, v) : \mathbb{R}_{T^*} \times H^1_0(\mathcal{O}) \times H^1_0(\mathcal{O}) \to \mathbb{R} \) for any \( T^* \) in \( \mathbb{R} \) defined by

\[
a_\lambda(t, u, v) := \int_{\mathcal{O}} \left( \sum_{i,j=1}^N a_{i,j}(y, t) \frac{\partial u}{\partial y_i}(y, t) \frac{\partial v}{\partial y_j}(y, t) \\
+ \sum_{i=1}^N b_i(y, t) \frac{\partial u}{\partial y_i} v(y, t) + \lambda u(y, t) v(y, t) \right) dy.
\]

(3.7)

The coefficients based on \( r \) and \( \tilde{r} \) defined above satisfy the following regularity properties. Assumption 1 is slightly stronger than required, but more easily described, since only the first derivative in time is in fact required.

**Lemma 3.1.** ([Lemma 4.2 in [12]]) If \( r \in C^2_b(\mathcal{O} \times \mathbb{R}_{T^*}; \mathbb{R}^N) \), then

\( a_{i,j} \in C^1_b(\mathcal{O} \times \mathbb{R}_{T^*}), \quad b_k \in C^0_b(\mathcal{O} \times \mathbb{R}_{T^*}). \)

In addition, there exists a \( \delta_0 = \delta_0(r) > 0 \) such that

\[
\sum_{i,j=1}^N \int_{\mathcal{O}} a_{i,j}(y, t) \frac{\partial u}{\partial y_i}(y, t) \frac{\partial u}{\partial y_j}(y, t) dy \geq \delta_0 \| u \|^2_{H^1_0(\mathcal{O})}
\]

(3.8)
for all \( t \in \mathbb{R}_{T^*} \) and \( u \in H^{1}_0(O) \). Finally, there exist constants \( \lambda_0 > 0, \delta_0 > 0 \) and \( \gamma > 0 \), which do not depend on \( u \) and \( v \), such that
\[
\begin{align*}
    a_\lambda(t, u, u) & \geq \delta_0 \|u\|_{H^1_0(O)}^2, \\
    |a_\lambda(t, u, v)| & \leq \gamma \|u\|_{H^1_0(O)} \|v\|_{H^1_0(O)},
\end{align*}
\]
for all \( \lambda \geq \lambda_0, t \in \mathbb{R}_{T^*} \) and \( u, v \in H^{1}_0(O) \).

**Lemma 3.2.** (Lemma 4.3 in [12]) If \( r \in C^2_b(O \times \mathbb{R}_{T^*}; \mathbb{R}^N) \), then there exist two positive constants \( c_0, c_1 \) which depend on \( r \) such that for any \( u \in H^2(O) \cap H^{1}_0(O) \), the following estimate holds
\[
c_0 \|\Delta u\|_{L^2}^2 - c_1 \|u\|_{L^2}^2 \leq \int_O \sum_{i,j=1}^{N} a_{i,j}(y,t) \frac{\partial^2 u}{\partial y_i \partial y_j} \Delta u \, dy.
\]

The following assumptions are also made on the nonlinear term \( f \) and forcing term \( g \).

**Assumption 2.** \( f : \mathbb{R} \to \mathbb{R} \) is continuously differentiable and there exist nonnegative constants \( \alpha_1, \alpha_2, \beta, C_1, C_2 \) as well as \( l \) and \( p \geq 2 \) such that
\[
\begin{align*}
    -\beta + \alpha_1 |s|^p & \leq f(s) s \leq \beta + \alpha_2 |s|^p, \\
    |f(s)| & \leq C_1 |s|^{p-1} + C_2, \\
    f'(s) & \geq -l
\end{align*}
\]
for all \( s \in \mathbb{R} \).

**Assumption 3.** \( g \in L^2_{loc}(\mathbb{R}_{T^*}, L^2(O_1)) \) satisfies
\[
\int_{\mathbb{R}_{T^*}} e^{-k(t-s)} \|g(s)\|^2_{L^2(O_1)} \, ds \leq C_3 \quad \text{for all } t \in \mathbb{R}_{T^*}
\]
for a constant \( C_3 \) which is independent of \( t \geq T^* \), where \( k = \min\{\delta_0 \lambda_1, c_0 \lambda_1\} \), \( \lambda_1 \) is the first eigenvalue of \( -\Delta \) on \( H^2_0(O) \).

**Remark 3.3.** The results presented below also apply to a nonautonomous PDE of the form (3.1) on a fixed domain \( O \) simply be taking the domain transformation \( r(x,t) = x \), the identity mapping, so \( O_t \equiv O \), for each \( t \geq \mathbb{R}_{T^*} \).

4. **Existence of the nonautonomous forward attracting sets.** The evolution equation (3.4) generates a nonautonomous dynamical system formulated as a two-parameter semigroup or process \( \phi \) on \( L^2(O) \) [12].

It follows from the above assumptions that this system has positively invariant absorbing sets in both \( L^2(O) \) and \( H^1_0(O) \) (see section 6), which are uniformly absorbing, i.e., the time to be absorbed depends only on the elapsed time, and hence in both the pullback and forward directions. Since \( H^1_0(O) \) is compactly embedded in \( L^2(O) \), the absorbing set \( B \) in \( H^1_0(O) \) is a compact subset of \( L^2(O) \), \( \phi \)-positively invariant and absorbing in \( L^2(O) \).

**Theorem 4.1.** The process \( \phi \) on \( L^2(O) \) generated by the evolution equation (3.4) has compact positively invariant subset \( B \) of \( L^2(O) \), which is absorbing uniformly in the initial time and, hence, in both the pullback and forward senses.
This ensures the existence of a forward attracting set $\omega_B$ in $L^2(\mathcal{O})$, which is defined in terms of omega-limit sets (see [7, 13]) by

$$\omega_B := \bigcup_{t_0 \geq 0} \omega_{B,t_0}, \quad \omega_{B,t_0} := \bigcap_{t \geq t_0} \bigcup_{s \geq t} \phi(s, t_0, B).$$

Here $\omega_{B,t_0} \subset \omega_{B,t_0} \subset B$ for $t_0 \leq t_0'$, so $\omega_B \subset B$ is a compact subset of $L^2(\mathcal{O})$.

**Remark 4.2.** If $T^* = -\infty$ the above assumptions ensure the existence of a pullback attractor $\mathfrak{A} = \{A(t) : t \in \mathbb{R}\}$ with nonempty compact subsets $A(t) \in L^2(\mathcal{O})$ given by

$$A(t) = \bigcap_{t_0 \leq t} \phi(t, t_0, B), \quad t \in \mathbb{R}.$$ 

This need not be forward attracting, but it is if $\omega_A = \omega_B$, where

$$\omega_A := \bigcap_{s \geq 0} \bigcup_{t \geq s} A(t),$$

see Kloeden & Lorenz [7]. Note, in general, $\omega_A \subsetneq \omega_B$ and the inclusion may be strict. A pullback attractor uses information about the system in the distant past, whereas the forward attracting set $\omega_B$ only requires such information in the distant future.

### 4.1. Asymptotic invariance

Simple examples show that nonautonomous omega-limit sets need not be positively invariant or negatively invariant. They may, however, be asymptotically invariant. Since $B$ is compact, the proofs for ODEs in [6] also hold in this section for the process generated on $L^2(\mathcal{O})$ by the nonautonomous evolution equation (3.4).

**Definition 4.3.** A set $A$ is said to be **asymptotically positively invariant** if for any monotonic decreasing sequence $\varepsilon_p \to 0$ as $p \to \infty$ there exists a monotonic increasing sequence $T_p \to \infty$ as $p \to \infty$ such that

$$\phi(t, t_0, A) \subset B_{\varepsilon_p}(A), \quad t \geq t_0,$$

for each $t_0 \geq T_p$, where $B_{\varepsilon_p}(A) := \{u \in L^2(\mathcal{O}) : \text{dist}_{L^2(\mathcal{O})}(u, A) < \varepsilon_p\}$.

**Theorem 4.4.** Let the Assumptions 1, 2 and 3 hold. Then $\omega_B$ is asymptotically positively invariant.

**Definition 4.5.** A set $A$ is said to be **asymptotically negatively invariant** if for every $a \in A$, $\varepsilon > 0$ and $T > 0$, there exist $t_\varepsilon$ and $a_\varepsilon \in A$ such that

$$\|\phi(t, t_\varepsilon, T, a_\varepsilon) - a\| < \varepsilon.$$ 

To apply the results in [6] we first need to show that the process is Lipschitz continuous in initial conditions in the positively invariant compact set $B$ uniformly on intervals of finite length independently of the starting point of the interval.

**Lemma 4.6.** Let the Assumptions 1, 2 and 3 hold. Then for every $T > 0$ there exists a constant $K_T$ such that

$$\|\phi(t, t_0, u_0) - \phi(t, t_0, v_0)\| \leq K_T\|u_0 - v_0\|$$

in $L^2(\mathcal{O})$ for $t_0 \leq t \leq t_0 + T$ and all $t_0 \geq T^*$, $u_0, v_0 \in B$. 

Proof. We need a direct estimate on \( \|u(t) - v(t)\| \) for solutions in the compact set \( B \). For this we use a local Lipschitz condition on \( f \), which follows from its continuous differentiability.

For any two solutions \( u(t) := \phi(t, t_0, u_0) \) and \( v(t) := \phi(t, t_0, v_0) \) in \( B \), the difference of the solutions \( w(y, t) := u(r(y, t), t) - v(r(y, t), t) \) satisfies the following equation

\[
\begin{align*}
\frac{\partial w}{\partial t} &= \sum_{i,j=1}^{N} \frac{\partial}{\partial y_i} (a_{i,j}(y,t) \frac{\partial w}{\partial y_j}(y,t)) - b(y,t) \cdot \nabla_y w(y,t) \\
- \lambda w(y,t) - (f(u(r(y, t), t)) - f(v(r(y, t), t))) & \quad \text{in } \mathcal{O} \times (t_0, \infty), \\
w &= 0 \quad \text{on } \partial \mathcal{O} \times (\tau, \infty),
\end{align*}
\]

where, from Lemma 3.1

\[
a_\lambda(t, w(t), w(t)) \geq \tilde{\delta}_0 \|w(t)\|_{H^1_\tau(\mathcal{O})}^2.
\]

In addition, from (3.14),

\[
- \int_\mathcal{O} (f(u(r(y, t), t)) - f(v(r(y, t), t))) w(t) dy \leq l \int_\mathcal{O} |w(t)|^2 dy.
\]

That is,

\[
\|u(r(y, t), t) - v(r(y, t), t)\|^2 \leq \|u_{t_0}(r(y, t_0)) - v_{t_0}(r(y, t_0))\|^2 \\
+ 2l \int_{t_0}^t |u(r(y, s), s) - v(r(y, s), s)|^2 ds,
\]

by Gronwall’s inequality, we have

\[
\|u(r(y, t), t) - v(r(y, t), t)\| \leq \|u_{t_0}(r(y, t_0)) - v_{t_0}(r(y, t_0))\| e^{2l(t-t_0)} \\
\leq \|u_{t_0}(r(y, t_0)) - v_{t_0}(r(y, t_0))\| e^{2lT}.
\]

Thus proofs in [6] hold and give the following result.

**Theorem 4.7.** Let the Assumptions 1, 2 and 3 hold. Then \( \omega_B \) is asymptotically negatively invariant.

5. **Upper semi continuity in a parameter.** Now consider a parameterised family of nonautonomous PDEs on \( \mathcal{O}_t \)

\[
\frac{\partial u}{\partial t} = \Delta u - \lambda u + f^\nu(u) + g(t), \quad t \geq T^*,
\]

for \( \nu \in [0, \nu^*] \), where each \( f^\nu : \mathbb{R}^1 \to \mathbb{R}^1 \) is (at least) continuously differentiable.

We assume that the \( f^\nu \) satisfy the same Assumption 2 as \( f \) uniformly in \( \nu \in [0, \nu^*] \) and that \( g \) satisfies Assumption 3. Then the processes \( \phi^\nu \) on \( L^2(\mathcal{O}) \) generated by the PDEs (5.1) have a common compact absorbing set \( B \) in \( L^2(\mathcal{O}) \), which is \( \phi^\nu \)-positively invariant for each \( \nu \in [0, \nu^*] \).
Lemma 5.1. Under the above assumptions there exists nonempty compact set $B$ which is \( \phi^\nu \)-positively invariant for each \( \nu \in [0, \nu^*] \) and for any bounded subset \( D \) of \( L^2(\mathcal{O}) \) and \( t_0 \geq T^* \) there exists a \( T_B \geq 0 \) (independent of \( \nu \)) such that

\[
\phi^\nu(t_0, x_0) \in B, \quad \forall t \geq t_0 + T_B, x_0 \in D, \nu \in [0, \nu^*].
\]

It will also be assumed that the nonlinear functions \( f^\nu \) in (5.1) converge to \( f^0 \) as the parameter \( \nu \to 0 \) uniformly on compact subsets of \( \mathbb{R}^1 \).

Assumption 4. For every \( \varepsilon > 0 \) and compact subset \( C \) of \( \mathbb{R}^1 \) there exists a \( \tilde{\delta}(\varepsilon) > 0 \) such that

\[
|f^\nu(x) - f^0(x)| < \varepsilon, \quad x \in C,
\]

for \( |\nu| < \tilde{\delta}(\varepsilon) \).

The continuous convergence of the solutions of the PDEs (5.1) as the parameter \( \nu \to 0 \) on the set \( B \) uniformly in bounded time sets independently of the initial time then follows

Lemma 5.2. Suppose that Assumptions 2 and 4 hold. For every \( \varepsilon > 0 \) and \( T > 0 \) there exists a \( \delta(\varepsilon, T) > 0 \) such that

\[
\|f^\nu(t, t_0, b) - f^0(t, t_0, b)\| < \varepsilon, \quad t_0 \leq t \leq t_0 + T, b \in B,
\]

for \( |\nu| < \delta(\varepsilon, T) \) and \( t_0 \geq T^* \).

Proof. For the two solutions \( u^\nu(t) := \phi^\nu(t, t_0, b) \) and \( v^0(t) := \phi^0(t, t_0, b) \), the difference of the solutions \( w^\nu(y, t) := u^\nu(r(y, t), t) - v^0(r(y, t), t) \) satisfies the equation

\[
\begin{cases}
\frac{\partial w^\nu}{\partial t} = \sum_{i,j=1}^N \frac{\partial}{\partial y_i}(a_{i,j}(y, t) \frac{\partial w^\nu}{\partial y_j}(y, t)) - b(y, t) \cdot \nabla_y w^\nu(y, t) \\
- \lambda w^\nu(y, t) - (f^\nu(u^\nu(r(y, t), t)) - f^0(v^0(r(y, t), t))) & \text{in } \mathcal{O} \times (t_0, \infty), \\
w = 0 & \text{on } \partial \mathcal{O} \times (\tau, \infty), \quad w(y, t_0) = 0, \quad y \in \mathcal{O}.
\end{cases}
\]

Then

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathcal{O}} |w^\nu(t)|^2 dy + a_\lambda(t, w^\nu(t), w^\nu(t))
\]

\[
= - \int_{\mathcal{O}} (f^\nu(u^\nu(r(y, t), t)) - f^0(v^0(r(y, t), t)))w^\nu(t) dy
\]

\[
= - \int_{\mathcal{O}} (f^\nu(u^\nu(r(y, t), t)) - f^\nu(v^0(r(y, t), t)))w^\nu(t) dy
\]

\[
= - \int_{\mathcal{O}} (f^0(v^0(r(y, t), t)) - f^0(v^0(r(y, t), t)))w^\nu(t) dy
\]

where, from Lemma 3.1

\[
a_\lambda(t, w^\nu(t), w^\nu(t)) \geq \tilde{\delta}_0 \|w^\nu(t)\|_{L^2(\mathcal{O})}^2.
\]

In addition, from Assumption 2

\[
- \int_{\mathcal{O}} (f^\nu(u^\nu(r(y, t), t)) - f^\nu(v^0(r(y, t), t)))w^\nu(t) dy \leq \int_{\mathcal{O}} |w^\nu(t)|^2 dy
\]

and from Assumption 4

\[
| - \int_{\mathcal{O}} (f^0(v^0(r(y, t), t)) - f^0(v^0(r(y, t), t)))w^\nu(t) dy | \leq \frac{\varepsilon}{2\tilde{\delta}_0} + \frac{\tilde{\delta}_0 \varepsilon}{2} \int_{\mathcal{O}} |w^\nu(t)|^2 dy,
\]
by Gronwall’s inequality, we have, with \( \delta(\varepsilon, T) = \hat{\delta}(2\tilde{\delta}_0 \varepsilon e^{-dT}, T) \),
\[
\|w^\nu(r(y, t), t)\| \leq \frac{\varepsilon}{2\tilde{\delta}_0} e^{\nu T}.
\]

Then Theorem 4.1 in [6] holds and gives the upper semi continuous convergence of the forward attracting sets in a parameter.

**Theorem 5.3.** Suppose that Assumptions 1, 2 (uniformly in \( \nu \)), 3 and 4 hold. Then
\[
\text{dist}_{L^2(O)}(\omega^\nu_B, \omega^\nu_0) \to 0 \quad \text{as } \nu \to 0.
\]

5.1. **Observed dynamics.** The process \( \phi \) generated by the evolution equation (3.4) is defined on the state space \( L^2(O) \). The actual dynamics generated by the evolution equation (3.1) lives in the time dependent state spaces \( L^2(O_t) \). Processes on time dependent state spaces were considered in [9, 12]. Note that if \( v(\cdot) \in L^2(O) \), then \( v \circ r(\cdot, t) \in L^2(O_t) \). The process \( \phi \) can be transformed to one acting on the time dependent state spaces. Define
\[
\psi(t, t_0, v_0) : L^2(O_{t_0}) \to L^2(O_t), \quad t \geq t_0,
\]
by
\[
\psi(t, t_0, v_0)(\cdot) := \phi(t, t_0, u_0 \circ r(\cdot, t_0)) \circ r(\cdot, t), \quad t \geq t_0, u_0 \in L^2(O_0).
\]

Dynamical objects such as attractors need to be transformed similarly.

5.1.1. **Forward attracting sets.** The forward attracting set \( \omega_B \) of the process \( \phi \) takes values in the fixed state space \( L^2(O) \). To consider its behaviour under the transformed process \( \psi \) introduce
\[
\omega^\nu_B := \{ v \in L^2(O_t) : v \circ r(\cdot, t) \in \omega_B \}.
\]

Then one has forward convergence in the sense that
\[
\text{dist}_{L^2(O_t)}(\psi(t, t_0, v_0), \omega^\nu_B) \to 0 \quad \text{as } t \to \infty,
\]
i.e., the distance function and underlying norm are time dependent here.

The definitions of asymptotically positive and negative invariance need to be modified accordingly.

**Definition 5.4.**

i) The set \( \omega_B \) is **asymptotically positively invariant** if for any monotonic decreasing sequence \( \varepsilon_p \to 0 \) as \( p \to \infty \) there exists a monotonic increasing sequence \( T_p \to \infty \) as \( p \to \infty \) such that
\[
\psi(t, t_0, \omega^\nu_B) \subset B^t_{T_p}(\omega^\nu_B), \quad t \geq t_0,
\]
for each \( t_0 \geq T_p \), where \( B^t_{T_p}(\omega^\nu_B) := \{ v \in L^2(O_t) : \text{dist}_{L^2(O_t)}(v, \omega^\nu_B) < \varepsilon \} \).

ii) A set \( \omega_B \) is said to be **asymptotically negatively invariant** if for every \( v \in \omega_B \), \( \varepsilon > 0 \) and \( T > 0 \), there exist \( t_\varepsilon \) and \( v_\varepsilon \in \omega_B \) such that
\[
\|\psi(t_\varepsilon, t_\varepsilon - T, v_\varepsilon) - v \circ \tilde{r}(\cdot, t_\varepsilon - T)\|_{L^2(O_t)} < \varepsilon.
\]

The earlier results in subsection 4.1 for the process \( \phi \) show that the family \( \{ \omega^\nu_B : t \in \mathbb{R} \} \) is asymptotically positively invariant in general and also asymptotically negatively invariant if, in addition, the time dependence of the process is uniformly continuous in subintervals independently of their starting times after \( T^* \).
6. **Existence of absorbing sets.** The existence of positively invariant absorbing sets in both $L^2(O)$ and $H^1_0(O)$ are established in this section. The following a priori estimates will be deduced here by formally calculations. They can be justified by the Faedo-Galerkin approximation procedure.

6.1. **$L^2$-absorbing set.** Taking the inner product $(\cdot, \cdot)_{L^2(O)}$ of (3.4) with $v$ gives

$$\frac{1}{2} \frac{d}{dt} \int_O |v(t)|^2 \, dy + a_\lambda(t, v(t), v(t)) = \int_O (-f(v(y, t))v(y, t) + g(r(y, t), t) v(y, t)) \, dy.$$ 

From Lemma 3.1

$$a_\lambda(t, v(t), v(t)) \geq \delta_0 \|v(t)\|^2_{H^1_0(O)},$$

$$\int_O g(r(y, t), t) v(y, t) \, dy \leq \frac{1}{\delta_0} \int_O |g(r(y, t), t)|^2 \, dy + \frac{\delta_0}{4} \|v(t)\|^2_{L^2(O)}.$$ 

In addition, from (3.9),

$$\int_O -f(v(y, t))v(y, t) \, dy \leq -\alpha_1 \int_O |v(t)|^p \, dy + \beta |O|.$$ 

Combining the above estimates yields

$$\frac{d}{dt} \int_O |v(t)|^2 \, dy + \frac{3\delta_0}{2} \int_O |\nabla v(t)|^2 \, dy + 2\alpha_1 \int_O |v(t)|^p \, dy$$

$$\leq \frac{2}{\delta_0} \int_O |g(r(y, t), t)|^2 \, dy + 2\beta |O|.$$ 

Hence, by the Poincaré inequality

$$\frac{d}{dt} \int_O |v(t)|^2 \, dy + k \int_O |v(t)|^2 \, dy + \frac{\delta_0}{2} \int_O |\nabla v(t)|^2 \, dy + 2\alpha_1 \int_O |v(t)|^p \, dy$$

$$\leq \frac{2}{\delta_0} \int_O |g(r(y, t), t)|^2 \, dy + 2\beta |O|,$$ 

(6.1)

where $k = \min \{\delta_0 \lambda_1, c_0 \lambda_1\}$ and $\lambda_1$ is the first eigenvalue of $-\Delta$ on $H^1_0(O)$.

By Assumptions 1 and 2 and, in particular, Assumption 3 the positive number defined by

$$\rho_0^2 := 2 \sup_{t \geq T^*} \int_{T^*} e^{-k(t-s)} \left( \frac{1}{\delta_0} \|g(s)\|^2_{L^2(O)} + \beta |O| \right) \, ds$$

is finite.

Hence for any bounded subset $B$ of $L^2(O)$ there exists $T_B > 0$ for which

$$\|v(t, \tau; v_\tau)\|^2_{L^2(O)} \leq \rho_0^2$$

for all $t - \tau \geq T_B$,

for any $v_\tau \in B$, $\tau \geq T^*$.

Furthermore, integrating (6.1) over $[t, t + 1]$ gives

$$\int_t^{t+1} \|\nabla v(s)\|^2_{L^2(O)} \, ds \leq C$$

for all $t \geq \tau + T_B$

and

$$\int_t^{t+1} \|v(s)\|^p_{L^p(O)} \, ds \leq C$$

for all $t \geq \tau + T_B$,

which means that there exists a $T_1 > T_B$ such that

$$\|\nabla v(T_1 + \tau)\|^2_{L^2(O)} + \|v(T_1 + \tau)\|^p_{L^p(O)} \leq C.$$ 

(6.2)
Note that since the constant $C_3$ in the Assumption 3 is independent of $t$, the constants $C$ in the above inequalities are also independent of $t$.

6.2. $H^1_0$-absorbing set. Let $A(y, t) = [a_{i,j}(y, t)]_{N \times N}$, so $A(y, t) = \hat{T}(y, t)T(y, t)$, where $T(y, t) := G(r(y, t), t)$ with

$$G(x, t) = \begin{pmatrix} \frac{\partial \bar{r}_1(x, t)}{\partial x_1} & \frac{\partial \bar{r}_2(x, t)}{\partial x_1} & \cdots & \frac{\partial \bar{r}_N(x, t)}{\partial x_1} \\ \frac{\partial \bar{r}_1(x, t)}{\partial x_2} & \frac{\partial \bar{r}_2(x, t)}{\partial x_2} & \cdots & \frac{\partial \bar{r}_N(x, t)}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \bar{r}_1(x, t)}{\partial x_N} & \frac{\partial \bar{r}_2(x, t)}{\partial x_N} & \cdots & \frac{\partial \bar{r}_N(x, t)}{\partial x_N} \end{pmatrix}$$

and $\hat{T}(y, t)$ is the transpose of $T(y, t)$.

Then,

$$\int_{\mathcal{O}} \sum_{i,j=1}^{N} \frac{\partial}{\partial y_j} \left( a_{i,j}(y, t) \frac{\partial u}{\partial y_i} \right) \Delta u \, dy = \int_{\mathcal{O}} \sum_{i,j=1}^{N} \frac{\partial a_{i,j}(y, t)}{\partial y_j} \frac{\partial u}{\partial y_i} \Delta u \, dy + \int_{\mathcal{O}} \sum_{i,j=1}^{N} a_{i,j}(y, t) \frac{\partial^2 u}{\partial y_i \partial y_j} \Delta u \, dy \quad (6.3)$$

where, from Lemma 3.2,

$$c_0 \| \Delta u \|^2_{L^2} - c_1 \| u \|^2_{L^2} \leq \int_{\mathcal{O}} \sum_{i,j=1}^{N} a_{i,j}(y, t) \frac{\partial^2 u}{\partial y_i \partial y_j} \Delta u \, dy, \quad (6.4)$$

at the same time,

$$\left| \int_{\mathcal{O}} \sum_{i,j=1}^{N} \frac{\partial a_{i,j}(y, t)}{\partial y_j} \frac{\partial u}{\partial y_i} \Delta u \, dy \right| \leq \frac{c_0}{8} \int_{\mathcal{O}} \| \Delta u \|^2 \, dy + \frac{1}{2c_0} \| A \|_{C^2_0(\mathcal{O} \times \mathbb{R})} \| \nabla u \|^2_{L^2(\mathcal{O})}. \quad (6.5)$$

That is,

$$\frac{7c_0}{8} \| \Delta u \|^2_{L^2} - c_1 \| u \|^2_{L^2} - \frac{1}{2c_0} \| A \|_{C^2_0(\mathcal{O} \times \mathbb{R})} \| \nabla u \|^2_{L^2(\mathcal{O})} \leq \int_{\mathcal{O}} \sum_{i,j=1}^{N} \frac{\partial}{\partial y_j} \left( a_{i,j}(y, t) \frac{\partial u}{\partial y_i} \right) \Delta u \, dy. \quad (6.6)$$

Taking the inner product $(\cdot, \cdot)_{L^2(\mathcal{O})}$ of (3.4) with $-\Delta v$ gives

$$\frac{1}{2} \frac{d}{dt} \int_{\mathcal{O}} |\nabla v|^2 \, dy + \int_{\mathcal{O}} \sum_{i,j=1}^{N} \frac{\partial}{\partial y_j} \left( a_{i,j}(y, t) \frac{\partial v}{\partial y_i} \right) \Delta v \, dy - \int_{\mathcal{O}} b(y, t) \cdot \nabla v \Delta v \, dy$$

$$+ \lambda \int_{\mathcal{O}} |\nabla v|^2 \, dy = \int_{\mathcal{O}} f(v) \Delta v \, dy - \int_{\mathcal{O}} g(t) \Delta v \, dy.$$
Consider the terms one by one. According to (6.6),
\[ \frac{7c_0}{8} \| \Delta v \|^2_{L^2} - c_1 \| v \|^2_{L^2} - \frac{1}{2c_0} \| A \|^2_{C^1([0,\infty) \times \mathbb{R})} \| \nabla v \|^2_{L^2(\mathcal{O})} \]
and
\[ \int_{\mathcal{O}} \sum_{i,j=1}^N \frac{\partial}{\partial y_j} \left( a_{i,j}(y, t) \frac{\partial v}{\partial y_i} \right) \Delta v \, dy \]
\[ \left| \int_{\mathcal{O}} b(t, y) \cdot \nabla v \Delta v \, dy \right| \leq \frac{c_0}{8} \int_{\mathcal{O}} \| \Delta v \|^2 \, dy + \frac{1}{2c_0} \| b \|^2_{C^2([0,\infty) \times \mathbb{R})} \int_{\mathcal{O}} \| \nabla v \|^2 \, dy. \]
In addition,
\[ \int_{\mathcal{O}} f(v) \Delta v \, dy = \int_{\mathcal{O}} -f'(v) \| \nabla v \|^2 \, dy \leq t \int_{\mathcal{O}} \| \nabla v \|^2 \, dy \]
and
\[ \left| -\int_{\mathcal{O}} g(t) \Delta v \, dy \right| \leq \frac{c_0}{8} \int_{\mathcal{O}} \| \Delta v \|^2 \, dy + \frac{2}{c_0} \int_{\mathcal{O}} |g(t)|^2 \, dy. \]
Combining above estimates gives
\[ \frac{d}{dt} \int_{\mathcal{O}} \| \nabla v(y, t) \|^2 \, dy + \frac{5c_0}{4} \int_{\mathcal{O}} \| \Delta v(y, t) \|^2 \, dy + 2\lambda \int_{\mathcal{O}} \| \nabla v(y, t) \|^2 \, dy \]
\[ \leq \frac{1}{c_0} \left( \| A \|^2_{C^1([0,\infty) \times \mathbb{R})} + \| b \|^2_{C^2([0,\infty) \times \mathbb{R})} + 2c_0 \right) \int_{\mathcal{O}} \| \nabla v(y, t) \|^2 \, dy \]
\[ + \frac{4}{c_0} \int_{\mathcal{O}} |g(t)|^2 \, dy + 2c_1 \| v \|^2_{L^2}. \]
Set
\[ k_3 := \max \left\{ \frac{1}{c_0} \left( \| A \|^2_{C^1([0,\infty) \times \mathbb{R})} + \| b \|^2_{C^2([0,\infty) \times \mathbb{R})} + 2c_0 \right), \frac{4}{c_0}, 2c_1 \right\}. \]
Using the Poincaré inequality implies that
\[ \frac{d}{dt} \int_{\mathcal{O}} \| \nabla v(t) \|^2 \, dy + k_2 \int_{\mathcal{O}} \| \nabla v \|^2 \, dy \]
\[ \leq k_3 \left( \int_{\mathcal{O}} \| \nabla v \|^2 \, dy + \int_{\mathcal{O}} |g(t)|^2 \, dy + \| v \|^2_{L^2} \right), \]
with \( k_2 = c_0 \lambda_1 \). Let \( k = \min \{ c_0 \lambda_1, \delta_0 \lambda_1 \} \), then the Gronwall Lemma yields
\[ \int_{\mathcal{O}} \| \nabla v(y, t) \|^2 \, dy \leq e^{-k(t-T_1-\tau)} \int_{\mathcal{O}} \| \nabla v(\tau + T_1) \|^2 \, dy \]
\[ + k_3 \int_{T+T_1}^{t} e^{-k(t-s)} \left( \int_{\mathcal{O}} \| \nabla v(s) \|^2 \, dy + \int_{\mathcal{O}} |g(s)|^2 \, dy + \| v \|^2_{L^2} \right) \, ds. \quad (6.7) \]
Now (6.1) implies that
\[ \frac{d}{ds} \left( e^{-k(t-s)} \int_{\mathcal{O}} |v(s)|^2 \, dy \right) + e^{-k(t-s)} \frac{\delta_0}{2} \int_{\mathcal{O}} |\nabla v(s)|^2 \, dy \]
\[ \leq e^{-k(t-s)} \left( \frac{2}{\delta_0} \int_{\mathcal{O}} |g(s)|^2 \, dy + 2\beta |\mathcal{O}| \right). \]
Integrating this inequality from $\tau + T_1$ to $t$ w.r.t $s$ gives
\[
\int_{\tau + T_1}^{t} e^{-k(t-s)} \int_{\mathcal{O}} |\nabla v(s)|^2 ds dy
\leq e^{-k(t-\tau - T_1)} \int_{\mathcal{O}} |v(\tau + T_1)|^2 dy + \int_{\tau + T_1}^{t} e^{-k(t-s)} \left( \frac{2}{\delta_0} \int_{\mathcal{O}} |g(s)|^2 dy + 2\beta |\mathcal{O}| \right) ds,
\]
that is,
\[
\int_{\tau + T_1}^{t} e^{-k(t-s)} \int_{\mathcal{O}} |\nabla v(s)|^2 ds dy
\leq \frac{2}{\delta_0} e^{-k(t-\tau - T_1)} \int_{\mathcal{O}} |v(\tau + T_1)|^2 dy + \frac{4}{\delta_0^3} \int_{\tau + T_1}^{t} e^{-k(t-s)} \int_{\mathcal{O}} |g(s)|^2 dy ds + \frac{4\beta}{\delta_0 k} |\mathcal{O}|.
\]
Combining this with (6.7) then leads to
\[
\int_{\tau + T_1}^{t} e^{-k(t-s)} \int_{\mathcal{O}} |\nabla v(y,t)|^2 dy \leq e^{-k(t-T_1-\tau)} \int_{\mathcal{O}} |\nabla v(\tau + T_1)|^2 dy
\]
\[
+ \frac{2k_3}{\delta_0} e^{-k(t-\tau - T_1)} \int_{\mathcal{O}} |v(\tau + T_1)|^2 dy
\]
\[
+ k_3 \int_{\tau + T_1}^{t} e^{-k(t-s)} \int_{\mathcal{O}} |v(s)|^2 dy ds
\]
\[
+ k_3 \left( \frac{4}{\delta_0^3} + 1 \right) \int_{\tau + T_1}^{t} e^{-k(t-s)} \int_{\mathcal{O}} |g(s)|^2 dy ds + \frac{4\beta k_3}{\delta_0 k} |\mathcal{O}|
\]
\[
\leq e^{-k(t-T_1-\tau)} \int_{\mathcal{O}} |\nabla v(\tau + T_1)|^2 dy
\]
\[
+ \frac{2k_3}{\delta_0} e^{-k(t-\tau - T_1)} \int_{\mathcal{O}} |v(\tau + T_1)|^2 dy
\]
\[
+ k_3 \left( \frac{4}{\delta_0^3} + 1 \right) \int_{\tau + T_1}^{t} e^{-k(t-s)} \int_{\mathcal{O}} |g(s)|^2 dy ds
\]
\[
+ k_3 \rho_2^2 + \frac{4\beta k_3}{\delta_0 k} |\mathcal{O}|.
\]
Hence, by (6.2) and Assumption 3, there is a $\rho_2 > 0$ such that for any bounded subset $B$ of $L^2(\mathcal{O})$ there exists a $T_B > 0$ such that for any $v_\tau \in B$, $\tau \geq T^*$,
\[
\|v(t,\tau;v_\tau)\|_{H^1(\mathcal{O})} \leq \rho_2^2 \quad \text{for all } t - \tau \geq T_B.
\]

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REFERENCES

[1] A. N. Carvalho, J. A. Langa and J. C. Robinson, *Attractors of Infinite Dimensional Nonautonomous Dynamical Systems*, Applied Mathematical Sciences, 182, Springer, New York, 2013.

[2] V. V. Chepyzhov and M. I. Vishik, *Attractors for equations of mathematical physics*, Amer. Math. Soc., Providence, Rhode Island, 2002.

[3] H. Crauel, P. E. Kloeden and J. Real, *Stochastic partial differential equations on time-varying domains*, Boletín de la Sociedad Española de Matemática Aplicada., 51 (2010), 41–48.
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[4] H. Crauel, P. E. Kloeden and M. Yang, Random attractors of stochastic reaction-diffusion equations on variable domains, Stochastics & Dynamics 11 (2011), 301–314.

[5] J. K. Hale, Asymptotic Behavior of Dissipative Systems, American Mathematical Society, Providence, 1988.

[6] P. E. Kloeden, Asymptotic invariance and the approximation of nonautonomous forward attracting sets, J. Comput. Dynamics, 3 (2016), 179–189.

[7] P. E. Kloeden and T. Lorenz, Construction of nonautonomous forward attractors, Proc. Amer. Mat. Soc., 144 (2016), 259–268.

[8] P. E. Kloeden, T. Lorenz and M. Yang, Forward attractors in discrete time nonautonomous dynamical systems, in Differential and Difference Equations with Applications, Springer Proceedings in Mathematics & Statistics, 164, Editors: O. Dosly, P.E. Kloeden, S. Pinelas; Springer, Heidelberg, (2016), 313–322.

[9] P. E. Kloeden, P. Marín-Rubio and J. Real, Pullback attractors for a semilinear heat equation in a non-cylindrical domain, J. Differential Eqns., 244 (2008), 2062–2090.

[10] P. E. Kloeden, C. Pötzsche and M. Rasmussen, Limitations of pullback attractors of processes, J. Difference Eqns. Applns., 18 (2012), 693–701.

[11] P. E. Kloeden and M. Rasmussen, Nonautonomous Dynamical Systems, Amer. Math. Soc., Providence, 2011.

[12] P. E. Kloeden, J. Real and C. Y. Sun, Pullback attractors for a semilinear heat equation on time-varying domains, J. Differential Eqns., 246 (2009), 4702–4730.

[13] P. E. Kloeden and M. Yang, Forward attraction in nonautonomous difference equations, J. Difference Eqns. Applns., 22 (2016), 513–525.

[14] J. P. Lasalle, The Stability of Dynamical Systems, SIAM-CBMS, Philadelphia, 1976.

[15] M. I. Vishik, Asymptotic Behaviour of Solutions of Evolutionary Equations, Cambridge University Press, Cambridge, 1992.

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