New Notions for Fuzzy Equivalence Using $\alpha$-cut Relation

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Abstract. This paper introduces a new notion to define equivalence to the non-reflexive fuzzy relation equation. The most important condition for the fuzzy equivalence relations is reflexive, but the condition of reflexive is unsuccessful in many cases of fuzzy relation in real life problems that creates problem to form partition tree. Therefore, this paper defines the equivalence for the cases of fuzzy relation that satisfies the symmetry, and transitive. That is, it proves the reflexive to non-reflexive fuzzy relations through alpha-cut relations. Further, this paper defines tolerance to the non-reflexive fuzzy relation equation through alpha-cut relations and it proves the entire upper left, lower right and centre sub matrices of every weakly-similarity relation matrix are weakly-similarity relation matrices.

Keywords: Fuzzy Equivalence, Weakly-Similarity, Constant $\varepsilon$-reflexivity, $\alpha$-cut Relation

1. Introduction

Fuzzy sets are the generalization of crisp sets with membership function. Fuzzy sets were first introduced by L. A. Zadeh[1965]. The concept of fuzzy relation has been done from that of fuzzy set in Zadeh’s very first paper [12]. Further developments on fuzzy relation have been done in his paper on 1971 paper [13]. Subsequently, the researchers [1,2,3,5,6,7,14] promoted the notions of a fuzzy equivalence relation and provided much more freedom to express the natural applications in modelling various problems like algebra, real analysis, topology, operations research, control theory, artificial intelligence, robotics, expert systems, decision theory, psychology, sociology and linguistics, etc. Fuzzy equivalence relations have been widely used to measure the degree of similarity between the highly complex objects of a given universe of discourse [4,6]. This provided thrust for research in the area of fuzzy relational equations. The fuzzy matrices are successfully used when fuzzy uncertainty occurs in different types of problem [8]. Fuzzy matrices arise in many applications, one of which is as adjacency matrices of fuzzy relations and fuzzy relational equations have important applications in pattern classification and in handing fuzziness in knowledge based systems [2,8,9].

The main motivation of this paper is to generate equivalence to the relational matrix that satisfies only symmetry and max-min transitivity. That is, this paper tries to construct an equivalence relations using $\alpha$-cut relation to the fuzzy relational matrices which are superior to proximity relation and inferior to similarity relation which is named as weakly-similarity relation. And, generation of tolerance relation by applying $\alpha$-cut relation to the fuzzy relational matrices which are superior to symmetric relation and inferior to proximity relation which is named as pseudo-proximity relation. Lastly, all the upper left, lower right, and centre square sub matrices of every weakly-similarity relation matrix are also weakly-similarity relation matrices. Generally, we can view that all the square sub matrix whose diagonal as consecutive diagonal elements of weakly-similarity relation matrix is also weakly-similarity relation matrix. Lastly, it is possible to count the minimum and maximum number of distinct entries are required to form weakly-similarity relation matrices.
similarity and weakly-proximity relation matrix for given order. And also, it is possible to count the number of distinct weakly-similarity and weakly-proximity relation submatrices of different orders for a weakly-similarity and weakly-proximity relation matrix respectively for a given order.

In the rest of the paper is organised as follows: Section 2 defines notation, terminology, basic definitions, and then it discusses some elementary results. Section 3 proposes the new notion of weakly-similarity, weakly-proximity relations and its related theorems. Finally, conclusions are made in Section 4.

2. Preliminary notion on Crisp Relations and Fuzzy Relations
This section gives brief account of some definitions about crisp relations and fuzzy relations which are necessary for the understanding of subsequent results.

Definition 1 (Fuzzy relation) [5,13,14]. The fuzzy relation $R$ from $X$ and $Y$ as a fuzzy subset of $X \times Y$ which is generalization of crisp relation with an extension to allow membership function $\mu_R(x, y)$ from $\{0,1\}$ to $[0,1]$.

That is, in mathematically, $\mu_R: X \times Y \rightarrow [0,1]$

$R = \{((x, y), \mu_R(x, y)) | \mu_R(x, y) \geq 0, x \in X, y \in Y\}$.

Here, the value of grade of membership $\mu_R(x, y)$ represents the strength of the relation between $x$ and $y$. If $\mu_R(x, y) \geq \mu_R(x', y')$, then $(x, y)$ is more strongly related than $(x', y')$.

For the sake of simplicity, let us give the following example of Definition 1.

Example 1. Let us consider two (non-empty) finite universal sets $X$ and $Y$, assume that $X = \{15, 25, 50, 75\}$, and $Y = \{\text{Child(C), Young(Y), Adult(A), Senior(S)}\}$ are the age and different stages of human life period respectively. Let $R$ be a fuzzy relation between $X$ and $Y$ and the relational matrix $\mu_R$ is given by

\[
\begin{array}{cccc}
\text{C} & \text{S} & \text{A} & \text{Y} \\
15 & 0.8 & 0.2 & 0.1 & 0 \\
50 & 0 & 0.3 & 1 & 0.3 \\
75 & 0 & 0 & 1 & 1 \\
\end{array}
\]

$\mu_R = 25$

The fuzzy relation $R$ with $\alpha \leq \alpha'$, we have $R_\alpha \supseteq R_\alpha'$. Let us consider fuzzy relations from Example 1, one can have $\alpha$-cut relation matrices as following.

\[
\begin{array}{cccc}
\text{C} & \text{S} & \text{A} & \text{Y} \\
15 & 0.8 & 0.2 & 0.1 & 0 \\
50 & 0 & 0.3 & 1 & 0.3 \\
75 & 0 & 0 & 1 & 1 \\
\end{array}
\]

Definition 2 ($\alpha$-cut) [13,14]. $\alpha$-cut of a fuzzy relation $R$ is a crisp relation $R_\alpha$ containing all the those elements having membership values greater than or equal to $\alpha$. That is,

$R_\alpha = \{(x, y) | \mu_R(x, y) \geq \alpha, \alpha \in [0,1], x \in X, y \in Y\}$.

One can have the following observation of the Definition 2.

Remark 1. The fuzzy relation $R$ with $\alpha \leq \alpha'$, we have $R_\alpha \supseteq R_\alpha'$. Let us consider fuzzy relations from Example 1, one can have $\alpha$-cut relation matrices as following.
Then the $\alpha$-cut set is derived from fuzzy relation $R$ with $\alpha = 0.2$ and $\alpha = 0.8$, this means “the age and stages of human life with possibility not less than $\alpha$”.

If $\alpha = 0.2$, $R_{\alpha = 0.2} = \{(15, C), (15, Y), (25, Y), (25, A), (50, Y), (50, A), (50, S), (75, A), (75, S)\}$ and if $\alpha = 0.8$, $R_{\alpha = 0.8} = \{(15, C), (25, Y), (25, A), (50, A), (75, A), (75, S)\}$.

Since two $\alpha$-cut sets exist and $0.2 \leq 0.8$, then $R_{\alpha = 0.2} \supseteq R_{\alpha = 0.8}$.

**Definition 3 (Support) [13,14]**. The *support* of a fuzzy relation $R$ is a crisp relation containing all the those elements having membership values greater than $0$. That is, 

$$\text{support}(R) = \{(x, y) \in X \times Y | \mu_R(x, y) > 0\}.$$ 

From Example 1, 

$$\text{support}(R) = \{(15, C), (15, Y), (15, A), (25, C), (25, Y), (25, A), (50, Y), (50, A), (50, S), (75, A), (75, S)\}.$$ 

**Definition 4 (Level set) [13,14]**. The *level set* which is the values of the membership function, is in the range of $[0,1]$ and is obtained by the $\alpha$’s. That is, 

$$\Lambda_R = \{\alpha \mid \mu_R(x, y) =\alpha, \alpha \geq 0, (x, y) \in R \subseteq X \times Y\}.$$ 

As above mentioned Example 1, the level $\Lambda_R$ set is given by 

$$\Lambda_R = \{0, 0.1, 0.2, 0.3, 0.8, 0.9, 1.0\}.$$ 

**Proposition 1 (Decomposition of fuzzy relation) [10, 13]**. For every fuzzy relation matrix $R$ on $X \times Y$, one has the resolution of the form 

$$R = \bigcup_{\alpha \in \Lambda_R} \alpha \cdot R_{\alpha}, 0 < \alpha \leq 1,$$

where $\alpha$ is a value in level set, $\alpha R_{\alpha}$ is a fuzzy relation on $X \times Y$ and is defined by 

$$\mu_{\alpha R_{\alpha}}(x, y) = \alpha \cdot \mu_{R_{\alpha}}(x, y), (x, y) \in R \subseteq X \times Y.$$ 

or, more explicitly 

$$\mu_{\alpha R_{\alpha}}(x, y) = \begin{cases} 
\alpha, & \text{for } (x, y) \in R_{\alpha} \\
0, & \text{otherwise}.
\end{cases}$$ 

As one can see from Example 1 and Remark 1, the fuzzy relation matrix $\mu_R$ of a fuzzy relation $R$ has the resolution form and can be decomposed as following.

$$\mu_R = 0.1 \cdot \mu_{R_{0.1}} \cup 0.2 \cdot \mu_{R_{0.2}} \cup 0.3 \cdot \mu_{R_{0.3}} \cup 0.8 \times \mu_{R_{0.8}} \cup 0.9 \cdot \mu_{R_{0.9}} \cup 1.0 \cdot \mu_{R_{1.0}}.$$ 

### 3. Notion of Weakly-Similarity and Weakly-Proximity Relation
For the purpose of this paper, let us introduce a motivation for employment of weakly-similarity and weakly-proximity relation, and its relevant theorems. To introduce these notions, we have presented some formal definitions and theorems as follows.

### 3.1. Weakly-Similarity Relation

**Definition 5** *(Similarity relation or Fuzzy Equivalence relation)* [6,13]. The relation \( R \) on \( X \) is called a similarity relation if

1. **Reflexivity** i.e., \( \mu_R(x, x) = 1 \), \( \forall x \in X \),
2. **Symmetry** i.e., \( \mu_R(x, y) = \mu_R(y, x) \), \( \forall x, y \in X \),
3. **Transitivity** i.e., \( \mu_R(x, z) \geq \max_{y \in X} \min \{ \mu_R(x, y), \mu_R(y, z) \} \), \( \forall x, z \in X \).

Let us present an example to visualize the previous definition 5.

**Example 2.** Let the similarity relation matrix \( \mu_R \) of a similarity relation \( R \) on \( X = \{ 1, 2, 3, 4, 5, 6 \} \) be given by

\[
\begin{array}{cccccc}
1 & 0.3 & 0.6 & 0.3 & 0.6 \\
0.3 & 1 & 0.3 & 0.9 & 0.3 \\
1 & 0.3 & 1 & 0.3 & 0.6 \\
0.6 & 0.3 & 0.6 & 1 & 0.3 & 0.9 \\
0.3 & 0.9 & 0.3 & 0.3 & 1 & 0.3 \\
0.6 & 0.3 & 0.6 & 0.9 & 0.3 & 1 \\
\end{array}
\]

The corresponding two dimensional fuzzy graph is given below:

![Graphical representation of similarity relation.](image)

The conditions of a similarity relation as defined in Definition 5, rather restrictive and not quite in accordance in fuzzy set thinking: Reflexivity condition in similarity relation could be considered as being too restrictive and weakened by substituting the conditions by constant-\( \varepsilon \)-reflexive and weakly reflexive [11,14].

**Definition 6** *(\( \varepsilon \)-reflective)* [11]. \( R \) is called \( \varepsilon \)-reflective if

\[
\mu_R(x, x) \geq \varepsilon \text{ for } \varepsilon \in (0,1], \forall x \in X.
\]

In particularly,
\[
\mu_R(x, x) = \varepsilon \text{ for } \varepsilon \in (0,1], \forall x \in X,
\]

one has named as constant-\( \varepsilon \)-reflexive.
Definition 7 (weakly reflexive) [11]. A relation $\mu_R$ is called weakly reflexive if
\[ \mu_R(x, y) \leq \mu_R(x, x) \text{ and } \mu_R(y, x) \leq \mu_R(x, x), \forall x, y \in X. \]

Now, one can define weakly-similarity relation as following.

Definition 8 (Weakly-Similarity Relation). The relation $\mu_R \subseteq X \times X$ is called an weakly-similarity relation if the following conditions are satisfied:

1. Constant $\varepsilon$-reflexivity i.e., $\mu_R(x, x) = \varepsilon$ for $\varepsilon \in (0, 1]$, $\forall x \in X$,
2. Weak reflexivity i.e., $\mu_R(x, y) \leq \mu_R(x, x)$ and $\mu_R(y, x) \leq \mu_R(x, x)$, $\forall x, y \in X$,
3. Symmetry i.e., $\mu_R(x, y) = \mu_R(y, x)$, $\forall x, y \in X$,
4. Transitivity i.e., $\mu_R(x, z) \geq \max_{y \in X} \{ \min \{ \mu_R(x, y), \mu_R(y, z) \} \}$, $\forall x, z \in X$.

Remarks 2. If we consider a special case, $\varepsilon = 1$ in Definition 8 then we will obtain the similarity relation.

For the sake of understanding, let us provide the following example of Definition 8.

Example 3. Let $R$ be a weakly-similarity relation on $X = \{1, 2, 3, 4, 5, 6\}$ that is given by the following matrix.

\[
\begin{array}{cccccc}
0.9 & 0.3 & 0.9 & 0.6 & 0.3 & 0.6 \\
0.3 & 0.9 & 0.3 & 0.8 & 0.3 & \\
0.9 & 0.3 & 0.9 & 0.6 & 0.3 & 0.6 \\
0.6 & 0.3 & 0.6 & 0.9 & 0.3 & 0.8 \\
0.3 & 0.8 & 0.3 & 0.9 & 0.3 & \\
0.6 & 0.3 & 0.6 & 0.8 & 0.3 & 0.9 \\
\end{array}
\]

One may think that, whether all the submatrices of weakly-similarity matrix is also weakly-similarity matrix. As will be displayed, the answer is partially true, which have explained in the following propositions and theorem.

Proposition 2. For a weakly-similarity relation matrix, each upper left square submatrix is also weakly-similarity relation matrix. (Given in Figure 2)

![Figure 2](image-url) Graphical illustration of the upper left square submatrices.

![Figure 3](image-url) Graphical illustration of the lower right square submatrices.
Proof. Let $\mu_R$ be a weakly-similarity relation matrix of order $n$ of a weakly-similarity relation $R$ on $X = \{1, 2, ..., n\}$. We have to prove that, each $m \times m$ upper left square submatrix $\mu_S$ of $\mu_R$ on $Y = \{1, 2, ..., m\} \subseteq X$ is also weakly-similarity relation matrix, where $m \leq n, \forall m, n \in \mathbb{N}$.

1. Constant $\varepsilon$-reflexivity: Since $\mu_R(x, x) = \varepsilon$ for $\varepsilon \in (0, 1]$, $\forall x \in X = \{1, 2, ..., n\}$ i.e. all the diagonal elements of $\mu_R$ are $\varepsilon$. So, all diagonal elements of upper left square submatrix $\mu_S$ is also $\varepsilon$, i.e. $\mu_S(x, x) = \varepsilon, \forall x \in Y = \{1, 2, ..., m\}$.

2. Weak reflexivity: Since $\mu_R(x, y) \leq \mu_R(x, x)$ and $\mu_R(y, x) \leq \mu_R(x, x), \forall x, y \in X$. So, $\mu_S(x, y) \leq \mu_S(x, x)$ and $\mu_S(y, x) \leq \mu_S(x, x), \forall x, y \in Y \subseteq X$.

3. Symmetry: Since $\mu_R(x, y) = \mu_R(y, x), \forall x, y \in X$ implies $\mu_S(x, y) = \mu_S(y, x), \forall x, y \in Y \subseteq X$.

4. Transitivity: Since $\mu_R(x, y) \geq \max\{\min(\mu_R(x, x_1), \mu_R(x_1, y)), ...., \min(\mu_R(x, x_n), \mu_R(x_n, y))\}, \forall x, y, z \in X$ implies $\mu_S(x, y) \geq \max\{\min(\mu_S(x, x_1), \mu_S(x_1, y)), ...., \min(\mu_S(x, x_m), \mu_S(x_m, y))\}, \forall x, y, z \in Y$.

This completes the proof. ■

Proposition 3. For a weakly-similarity relation matrix, each lower right square submatrix is also weakly-similarity relation matrix.(Given in Figure 3)

Proof. It can be proved analogously to Proposition 2. ■

Proposition 4. For a weakly-similarity relation matrix, each centre square submatrix is also weakly-similarity relation matrix.(Given in Figure 4-5)

![Figure 4](image1.png)  
Figure 4. Graphical illustration of the centre square submatrices for an odd order matrix.  

![Figure 5](image2.png)  
Figure 5. Graphical illustration of the centre square submatrices for an even order matrix.

Proof. It can also be proved analogously to Proposition 2 for the cases of $n = odd$ and $n = even$.

One may inquire at this moment that, whether the above cases in Propositions 2-4 are the only square submatrices. As will be revealed, the answer is negative. The following Theorem 1 discusses all the possibilities of weakly-similarity relation submatrix.
Theorem 1. Each square submatrix whose diagonal asconsecutive diagonal elements of weakly-similarity relation matrix is also weakly-similarity relation matrix. (Given in Figure 4)

![Figure 6: Graphical illustration of the square submatrices.](image)

Proof. It can be proved analogously to Proposition 3 for each order $1, 2, 3, \ldots, n$. □

Remark 3. The propositions 2-4 are the special cases of Theorem 1.

The weakly-similarity relation is generalisation of equivalence relation that is defined as follows.

Definition 9. (Equivalence relation). The relation $\mathcal{R} \subseteq X \times X$ is called an equivalence relation if

1. Reflexivity i.e., $x \in X \rightarrow (x, x) \in \mathcal{R}$,
2. Symmetry i.e., $(x, y) \in \mathcal{R} \rightarrow (y, x) \in \mathcal{R}$,
3. Transitivity i.e., $(x, y) \in \mathcal{R}, (y, z) \in \mathcal{R} \rightarrow (x, z) \in \mathcal{R}$.

Now it is the time to observe the relation between similarity relation and equivalence relation.

Proposition 5 [10, 13]. If $\mathcal{R}$ is a max-min similarity relation; then for each $0 < \alpha \leq 1$, $\mathcal{R}_\alpha$ possesses an equivalence relation.

One may ask if a fuzzy relation doesn’t satisfy the all the conditions of similarity relation and whether it is still possible to produce equivalence relation based on $\alpha$-cut relation. It is seen that weakly-similarity relation is able to generate equivalence relation.

The following proposition shows a fuzzy relation is not necessary to satisfy all conditions of similarity relation to produce equivalence relation.

Proposition 6. If $\mathcal{R}$ is weakly-similarity relation; then for each $0 < \alpha \leq 1$, $\mathcal{R}_\alpha$ possesses either a null relation or an equivalence relation.

Proof. The proof is based on the Definition 8 and Proposition 5.

For each $0 < \alpha, \varepsilon \leq 1$

Case 1: For $\varepsilon \geq \alpha$

1. Constant $\varepsilon$-reflexivity: $\mu_{\mathcal{R}}(x, x) = \varepsilon \geq \alpha$ for $\varepsilon \in (0, 1], \forall x \in X$. Then $(x, x) \in \mathcal{R}_\alpha, \forall x \in X$, i.e., $\mu_{\mathcal{R}_\alpha}(x, x) = 1, \forall x \in X$.

2. Weak reflexivity: Since $\mu_{\mathcal{R}}(x, y) \leq \mu_{\mathcal{R}}(x, x)$ and $\mu_{\mathcal{R}}(y, x) \leq \mu_{\mathcal{R}}(x, x), \forall x, y \in X$. Suppose if $\mu_{\mathcal{R}}(x, y) \geq \alpha$, then $(x, y) \in \mathcal{R}_\alpha$ and if $\mu_{\mathcal{R}}(y, x) \geq \alpha$, then $(y, x) \in \mathcal{R}_\alpha$. 
Symmetry: $\mu_R(x,y) = \mu_R(y,x), \forall x, y \in X$. Suppose $(x,y) \in R_\alpha$, i.e. $\mu_R(x,y) \geq \alpha$.
Then $\mu_R(y,x) \geq \alpha$ i.e. $(y,x) \in R_\alpha$.
(4) Transitivity: Suppose $(x,y) \in R_\alpha$ and $(y,z) \in R_\alpha$, i.e. $\mu_R(x,y) \geq \alpha$ and $\mu_R(y,z) \geq \alpha$.
Then by definition of transitivity of $R$, $\mu_R(x,z) \geq \alpha$. That is, $(x,z) \in R_\alpha$.
Therefore, $R_\alpha$ is an equivalence relation.
Case 2: For $\alpha > \epsilon$, clearly $R_\epsilon$ is an null relation.
This completes the proof. 

3.2. Weakly-proximity Relation

**Definition 10 (Proximity relation)** [10]. The relation $R$ on $X$ is called a proximity relation if
(1) Reflexivity i.e., $\mu_R(x,x) = 1$, $\forall x \in X$,
(2) Symmetry i.e., $\mu_R(x,y) = \mu_R(y,x), \forall x, y \in X$.

**Definition 11 (Tolerance relation)** [10]. The crisp relation $R \subseteq X \times X$ is called a tolerance relation if
(1) Reflexivity i.e., $x \in X \rightarrow (x,x) \in R$,
(2) Symmetry i.e., $(x,y) \in R \rightarrow (y,x) \in R$.

**Proposition 7** [10]. If $R$ is a proximity relation; then for each $0 < \alpha \leq 1$, $R_\alpha$ possesses a tolerance relation.

The following definition is a general case of the Definition 10.

**Definition 12 (Weakly-Proximity relation)**. The relation $R$ on $X$ is called a proximity relation if
(1) Constant $\epsilon$-reflexivity i.e., $\mu_R(x,x) = \epsilon$ for $\epsilon \in (0,1]$, $\forall x \in X$,
(2) Weak reflexivity i.e., $\mu_R(x,y) \leq \mu_R(x,x)$ and $\mu_R(y,x) \leq \mu_R(x,x), \forall x, y \in X$,
(3) Symmetry i.e., $\mu_R(x,y) = \mu_R(y,x), \forall x, y \in X$.

**Proposition 8**. If $R$ is a weakly-proximity; then for each $0 < \alpha \leq 1$, $R_\alpha$ possesses either a null relation or a tolerance relation.

**Proof.** This is a straightforward consequence of Proposition 6. ■

**Theorem 2**. Each square submatrix whose diagonal as consecutive diagonal elements of weakly-proximity relation matrix is also weakly-proximity relation matrix.

**Proof.** The proof is a straightforward consequence of Theorem 1. ■

4. Conclusion
This paper has investigated the equivalence of fuzzy relation equation which fails to satisfy the most important condition of reflexive. The main work of weakly-similarity and weakly-proximity relations have been given and discussed with numerous examples. By introducing the alpha cut relations on fuzzy relations with symmetry and transitive, this paper has obtained the weakly-similarity relation which enables the fuzzy relations with symmetry and transitive into equivalence relations. The new notion and the presented results on this paper are new to the research topic of fuzzy relation.

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