Chiral Symmetry and Lattice Fermions

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# Contents

1 Chiral symmetry
   1.1 Introduction
   1.2 Spinor representations of the Lorentz group
   1.3 Chirality in even dimensions
   1.4 Chiral symmetry and fermion mass in four dimensions
   1.5 Weyl fermions
   1.6 Chiral symmetry and mass renormalization
   1.7 Chiral symmetry in QCD
   1.8 Fermion determinants in Euclidian space
   1.9 Parity and fermion mass in odd dimensions
   1.10 Fermion masses and regulators

2 Anomalies
   2.1 The $U(1)_A$ anomaly in 1+1 dimensions
   2.2 Anomalies in 3+1 dimensions
   2.3 Strongly coupled chiral gauge theories
   2.4 The non-decoupling of parity violation in odd dimensions

3 Domain Wall Fermions
   3.1 Chirality, anomalies and fermion doubling
   3.2 Domain wall fermions in the continuum
   3.3 Domain wall fermions and the Callan-Harvey mechanism
   3.4 Domain wall fermions on the lattice

4 Overlap fermions and the Ginsparg-Wilson equation
   4.1 Overlap fermions
   4.2 The Ginsparg-Wilson equation and its consequences
   4.3 Chiral gauge theories: the challenge

References
1

Chiral symmetry

1.1 Introduction

Chiral symmetries play an important role in the spectrum and phenomenology of both the standard model and various theories for physics beyond the standard model. In many cases chiral symmetry is associated with nonperturbative physics which can only be quantitatively explored in full on a lattice. It is therefore important to implement chiral symmetry on the lattice, which turns out to be less than straightforward. In these lectures I discuss what chiral symmetry is, why it is important, how it is broken, and ways to implement it on the lattice. There have been many hundreds of papers on the subject and this is not an exhaustive review; the limited choice of topics I cover reflects on the scope of my own understanding and not the value of the omitted work.

1.2 Spinor representations of the Lorentz group

To understand chiral symmetry one must understand Lorentz symmetry first. Since we will be discussing fermions in various dimensions of spacetime, consider the generalization of the usual Lorentz group to \( d \) dimensions. The Lorentz group is defined by the real matrices \( \Lambda \) which preserve the form of the \( d \)-dimensional metric

\[
\Lambda^T \eta \Lambda = \eta, \quad \eta = \text{diag} (1, -1, \ldots, -1).
\]  

(1.1)

With this definition, the inner product between two 4-vectors, \( v^\mu \eta_{\mu
u} w^\nu = v^T \eta w \), is preserved under the Lorentz transformations \( v \to \Lambda v \) and \( w \to \Lambda w \). This defines the group \( \text{SO}(d-1,1) \), which — like \( \text{SO}(d) \) — has \( d(d-1)/2 \) linearly independent generators, which may be written as \( M^{\mu\nu} = -M^{\nu\mu} \), where the indices \( \mu, \nu = 0, \ldots, (d-1) \) and

\[
\Lambda = e^{i \theta_{\mu\nu} M^{\mu\nu}},
\]  

(1.2)

with \( \theta_{\mu\nu} = -\theta_{\nu\mu} \) being \( d(d-1)/2 \) real parameters. Note that \( \mu, \nu \) label the \( d(d-1)/2 \) generators, while in a representation \( R \) each \( M \) is a \( d_R \times d_R \) matrix, where \( d_R \) is the dimension of \( R \). By expanding eqn. (1.1) to order \( \theta \) one sees that the generators \( M \) must satisfy

\[
(M^{\mu\nu})^T \eta + \eta M^{\mu\nu} = 0.
\]  

(1.3)

From this equation it is straightforward to write down a basis for the \( M^{\mu\nu} \) in the \( d \)-dimensional defining representation and determine the commutation relations for the algebra,
Chiral symmetry

\[ [M^{\alpha\beta}, M^{\gamma\delta}] = i (\eta^{\beta\gamma} M^{\alpha\delta} - \eta^{\alpha\gamma} M^{\beta\delta} - \eta^{\beta\delta} M^{\alpha\gamma} + \eta^{\alpha\delta} M^{\beta\gamma}) . \]  

(1.4)

A Dirac spinor representation can be constructed as

\[ M^{\alpha\beta} \equiv \Sigma^{\alpha\beta} = i \frac{4}{4} [\gamma^\alpha, \gamma^\beta] \]  

(1.5)

where the gamma matrices satisfy the Clifford algebra:

\[ \{\gamma^\alpha, \gamma^\beta\} = 2 \eta^{\alpha\beta} \]  

(1.6)

Solutions to the Clifford algebra are easy to find by making use of direct products of Pauli matrices. In a direct product space we can write a matrix as

\[ M = a \otimes A \]  

where \( a \) and \( A \) are matrices of dimension \( d_a \) and \( d_A \) respectively, acting in different spaces; the matrix \( M \) then has dimension \( (d_a \times d_A) \). Matrix multiplication is defined as \( (a \otimes A)(b \otimes B) = (ab) \otimes (AB) \). It is usually much easier to construct a representation when you need one rather than to look one up and try to keep the conventions straight! One finds that solutions for the \( \gamma \) matrices in \( d \)-dimensions obey the following properties:

1. For both \( d = 2k \) and \( d = 2k + 1 \), the \( \gamma \)-matrices are \( 2^k \) dimensional;
2. For even spacetime dimension \( d = 2k \) (such as our own with \( k = 2 \)) one can define a generalization of \( \gamma_5 \) to be

\[ \Gamma = i^{k-1} \prod_{\mu=0}^{2k-1} \gamma^\mu \]  

(1.7)

with the properties

\[ \{\Gamma, \gamma^\mu\} = 0 , \quad \Gamma = \Gamma^\dagger = \Gamma^{-1} , \quad \text{Tr} (\Gamma \gamma^{\alpha_1} \cdots \gamma^{\alpha_{2k}}) = 2^{k-1} i^{-k} \epsilon^{\alpha_1 \cdots \alpha_{2k}} , \]  

(1.8)

where \( \epsilon^{012 \ldots 2k-1} = +1 = -\epsilon^{012 \ldots 2k-1} \).

3. In \( d = 2k + 1 \) dimensions one needs one more \( \gamma \)-matrix than in \( d = 2k \), and one can take it to be \( \gamma^{2k} = i \Gamma \).

Sometimes it is useful to work in a specific basis for the \( \gamma \)-matrices; a particularly useful choice is a “chiral basis”, defined to be one where \( \Gamma \) is diagonal. For example, for \( d = 2 \) and \( d = 4 \) (Minkowski spacetime) one can choose

\[ d = 2 : \quad \gamma^0 = \sigma_1 , \quad \gamma^1 = -i \sigma_2 , \quad \Gamma = \sigma_3 \]  

(1.9)

\[ d = 4 : \quad \gamma^0 = -\sigma_1 \otimes 1 , \quad \gamma^i = i \sigma_2 \otimes \sigma_i , \quad \Gamma = \sigma_3 \otimes 1 . \]  

(1.10)

1.2.1 \( \gamma \)-matrices in Euclidian spacetime

In going to Euclidian spacetime with metric \( \eta^{\mu\nu} = \delta_{\mu\nu} \), one takes

\[ \partial_0^M \rightarrow i \partial_0^E , \quad \partial_i^M \rightarrow \partial_i^E \]  

(1.11)

and defines
\[
\gamma^0_M = \gamma^0_E, \quad \gamma^i_M = i\gamma^i_E,
\]
so that
\[
(\gamma^\mu_E)^\dagger = \gamma^\mu_E, \quad \{\gamma^\mu_E, \gamma^\nu_E\} = 2\delta_{\mu\nu}
\]
and \( D_M \to iD_E \), with
\[
D_E = -D_E^\dagger
\]
and the Euclidian Dirac operator is \( (D_E + m) \). The matrix \( \Gamma^{(2k)} \) in \( 2k \) dimensions is taken to equal \( \gamma^{2k}_E \) in \( (2k + 1) \) dimensions:
\[
\Gamma^{(2k)}_E = \gamma^{2k}_E = \Gamma^{(2k)}_M, \quad \text{Tr} (\Gamma_E \gamma^\alpha_1 \cdots \gamma^\alpha_{2k}) = -2^k k! \epsilon^{\alpha_1 \cdots \alpha_{2k}}
\]
where \( \epsilon_{012 \ldots 2k-1} = +1 = + \epsilon^{012 \ldots 2k-1} \).

### 1.3 Chirality in even dimensions

My lectures on chiral fermions follow from the properties of the matrix \( \Gamma \) in even dimensions. The existence of \( \Gamma \) means that Dirac spinors are reducible representations of the Lorentz group, which in turn means we can have symmetries ("chiral symmetries") which transform different parts of Dirac spinors in different ways. To see this, define the projection operators
\[
P_\pm = \frac{(1 \pm \Gamma)}{2},
\]
which have the properties
\[
P_+ + P_- = 1, \quad P_+^2 = P_+ \quad P_+ P_- = 0.
\]
Since in odd spatial dimensions \( \{\Gamma, \gamma^\mu\} = 0 \) for all \( \mu \), it immediately follows that \( \Gamma \) commutes with the Lorentz generators \( \Sigma^{\mu\nu} \) in eqn. [1.5]: \( [\Gamma, \Sigma^{\mu\nu}] = 0 \). Therefore we can write \( \Sigma^{\mu\nu} = \Sigma_+^{\mu\nu} + \Sigma_-^{\mu\nu} \) where
\[
\Sigma_\pm^{\mu\nu} = P_\pm \Sigma^{\mu\nu} P_\pm, \quad \Sigma_+^{\alpha\beta} \Sigma_-^{\mu\nu} = \Sigma_-^{\mu\nu} \Sigma_+^{\alpha\beta} = 0.
\]
Thus \( \Sigma^{\mu\nu} \) is reducible: spinors \( \psi_\pm \) which are eigenstates of \( \Gamma \) with eigenvalue \( \pm 1 \) respectively transform independently under Lorentz transformations.

The word "chiral" comes from the Greek word for hand, \( \chi\epsilon\iota\rho \). The projection operators \( P_+ \) and \( P_- \) are often called \( P_R \) and \( P_L \) respectively: what does handedness have to do with the matrix \( \Gamma \)? Consider the Lagrangian for a free massive Dirac fermion in \( 1 + 1 \) dimensions and use the definition of \( \Gamma = \gamma^0\gamma^1 \):
\[
\mathcal{L} = \bar{\psi} i \partial_0 \psi = \psi^\dagger \left( \partial_t + \Gamma \partial_x \right) \psi
\]
Chiral symmetry

\[ \Psi^\dagger_+ i (\partial_t + \partial_x) \Psi_+ + \Psi^\dagger_- i (\partial_t - \partial_x) \Psi_- \]  

(1.19)

where

\[ \Gamma \Psi_{\pm} = \pm \Psi_{\pm} . \]  

(1.20)

We see then that the solutions to the Dirac equation for \( m = 0 \) are

\[ \Psi_{\pm}(x, t) = \Psi_{\pm}(x \mp t) \]

so that \( \Psi_+ \) corresponds to right-moving solutions, and \( \Psi_- \) corresponds to left-moving solutions. This is possible since the massless particles move at the speed of light and the direction of motion is invariant under proper Lorentz transformations in \((1 + 1)\) dimensions.

In the chiral basis eqn. (1.9), the positive energy plane wave solutions to the Dirac equation are

\[ \Psi_+ = e^{-iE(t-x)} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \Psi_- = e^{-iE(t+x)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \]  

(1.21)

It is natural to call \( P_+ \) and \( P_- \) as \( P_R \) and \( P_L \) respectively.

Exercise 1.1 You should perform the same exercise in \(3 + 1\) dimensions and find that solutions \( \Psi_{\pm} \) to the massless Dirac equation satisfying \( \Gamma \Psi_{\pm} = \pm \Psi_{\pm} \) must also satisfy

\[ |\vec{p}| = E \]  

and (2\( \vec{p} \cdot \vec{S}/E \))\( \Psi_{\pm} = \pm \Psi_{\pm} \), where \( S_i = \frac{1}{2} \epsilon_{ijk} \Sigma^j \) are the generators of rotations. Thus \( \Psi_{\pm} \) correspond to states with positive or negative helicity respectively, and are called right- and left-handed particles.

1.4 Chiral symmetry and fermion mass in four dimensions

Consider the Lagrangian for a single flavor of Dirac fermion in \(3+1\) dimensions, coupled to a background gauge field

\[ \mathcal{L} = \bar{\Psi} (i \not{D} - m) \Psi = (\bar{\Psi}_L i \not{D} \Psi_L + \bar{\Psi}_R i \not{D} \Psi_R) - m (\bar{\Psi}_L \Psi_R + \bar{\Psi}_R \Psi_L) \]  

(1.22)

where I have defined

\[ \Psi_L = P_- \Psi , \quad \bar{\Psi}_L = P_+ \bar{\Psi} , \quad \Psi_R = P_+ \Psi , \quad \bar{\Psi}_R = P_- \bar{\Psi} . \]  

(1.23)

For now I am assuming that \( \Psi_{L,R} \) are in the same complex representation of the gauge group, where \( D \mu \) is the gauge covariant derivative appropriate for that representation. It is important to note that the property \( \{ \gamma_5, \gamma^\mu \} = 0 \) ensured that the kinetic terms in eqn. (1.22) do not couple left-handed and right-handed fermions; on the other hand, the mass terms \ref{eq:mass} do. The above Lagrangian has an exact \( U(1) \) symmetry, associated with fermion number, \( \Psi \to e^{i\alpha} \Psi \). Under this symmetry, left-handed and right-handed components of \( \Psi \) rotate with the same phase; this is often called a “vector symmetry”.

\(^1\)I will use the familiar \( \gamma_5 \) in \(3 + 1\) dimensions instead of \( \Gamma \) when there is no risk of ambiguity.
In the case where \( m = 0 \), it apparently has an additional symmetry where the left- and right-handed components rotate with the opposite phase, \( \Psi \rightarrow e^{i\alpha \gamma_5} \Psi \); this is called an “axial symmetry”, \( U(1)_A \).

Symmetries are associated with Noether currents, and symmetry violation appears as a nonzero divergence for the current. Recall the Noether formula for a field \( \phi \) and infinitesimal transformation \( \phi \rightarrow \phi + \epsilon \delta \phi \):

\[
J^\mu = -\frac{\partial L}{\partial (\partial_\mu \phi)} \delta \phi , \quad \partial_\mu J^\mu = -\delta L .
\] (1.24)

In the Dirac theory, the vector symmetry corresponds to \( \delta \Psi = i \Psi \), and the axial symmetry transformation is \( \delta \Psi = i\gamma_5 \Psi \), so that the Noether formula yields the vector and axial currents:

\[
\begin{align*}
U(1) : & \quad J^\mu = \bar{\Psi} \gamma^\mu \Psi , \quad \partial_\mu J^\mu = 0 \\
U(1)_A : & \quad J^\mu_A = \bar{\Psi} \gamma^\mu \gamma_5 \Psi , \quad \partial_\mu J^\mu_A = 2im \bar{\Psi} \gamma_5 \Psi .
\end{align*}
\] (1.25)

Some comments are in order:

- Eqn. (1.26) is not the whole story! We will soon talk about additional contributions to the divergence of the axial current from the regulator, called anomalies, which do not decouple as the regulator is removed.
- The fact that the fermion mass breaks chiral symmetry means that fermion masses get multiplicatively renormalized, which means that fermions can naturally be light (unlike scalars in most theories); more on this later.
- The variation of a general fermion bilinear \( \bar{\Psi} X \Psi \) under chiral symmetry is

\[
\delta \bar{\Psi} X \Psi = i\bar{\Psi} \{ \gamma_5, X \} \Psi .
\] (1.27)

This will vanish if \( X \) can be written as the product of an odd number of gamma matrices. In any even dimension the chirally invariant bilinears include currents, with \( X = \gamma^\mu \) or \( X = \gamma^\mu \Gamma \), while the bilinears which transform nontrivially under the chiral symmetry include mass terms, \( X = 1, \Gamma \). Thus gauge interactions can be invariant under chiral symmetry, while fermion masses are always chiral symmetry violating. In \( (3 + 1) \) dimensions, anomalous electromagnetic moment operators corresponding to \( X = \sigma_{\mu\nu}, \sigma_{\mu\nu} \gamma_5 \) are also chiral symmetry violating.
- A more general expression for the classical divergence of the axial current for a bilinear action \( \int \bar{\Psi} D\Psi \) in any even dimension is

\[
\partial_\mu J^\mu_A = i\bar{\Psi} \{ \Gamma, D \} \Psi .
\] (1.28)

On the lattice one encounters versions of the fermion operator \( D \) which violate chiral symmetry even for a massless fermion.

The Lagrangian for \( N_f \) flavors of massive Dirac fermions in odd \( d \), coupled to some background gauge field may be written as

\[
\mathcal{L} = (\bar{\Psi}_L^a iD \Psi_L^a + \bar{\Psi}_R^a iD \Psi_R^a) - \left( \bar{\Psi}_L^a M_{ab} \Psi_R^b + \bar{\Psi}_R^a M_{ab} M_{ab} \Psi_L^a \right) .
\] (1.29)

The index on \( \Psi \) denotes flavor, with \( a, b = 1, \ldots, N_f \), and \( M_{ab} \) is a general complex mass matrix (no distinction between upper and lower flavor indices). Again assuming the
fermions to be in a complex representation of the gauge group, this theory is invariant under independent chiral transformations if the mass matrix vanishes:

\[ \Psi_R^a \rightarrow U_{ab} \Psi_R^b , \quad \Psi_L^a \rightarrow V_{ab} \Psi_L^b , \quad U^\dagger U = V^\dagger V = 1 . \] (1.30)

where \( U \) and \( V \) are independent \( U(N_f) \) matrices. Since \( U(N_f) = SU(N_f) \times U(1) \), it is convenient to write

\[ U = e^{i(\alpha + \beta) R} , \quad V = e^{i(\alpha - \beta) L} , \quad R^\dagger R = L^\dagger L = 1 , \quad |R| = |L| = 1 , \] (1.31)

so that the symmetry group is \( SU(N_f)_L \times SU(N_f)_R \times U(1)_A \) with \( L \in SU(N_f)_L \), \( R \in SU(N_f)_R \).

If we turn on the mass matrix, the chiral symmetry is explicitly broken, since the mass matrix couples left- and right-handed fermions to each other. If \( M_{ab} = m \delta_{ab} \) then the “diagonal” or “vector” symmetry \( SU(N_f)_L \times U(1) \) remains unbroken, where \( SU(N_f) \subset SU(N_f)_L \times SU(N_f)_R \) corresponding to the transformation eqn. (1.30), eqn. (1.31) with \( L = R \). If \( M_{ab} \) is diagonal but with unequal eigenvalues, the symmetry may be broken down as far as \( U(1)^{N_f} \), corresponding to independent phase transformations of the individual flavors. With additional flavor-dependent interactions, these symmetries may be broken as well.

## 1.5 Weyl fermions

### 1.5.1 Lorentz group as \( SU(2) \times SU(2) \) and Weyl fermions

We have seen that Dirac fermions in even dimensions form a reducible representation of the Lorentz group. Dirac notation is convenient when both LH and RH parts of the Dirac spinor transform as the same complex representation under a gauge group, and when there is a conserved fermion number. This sounds restrictive, but applies to QED and QCD. For other applications — such as chiral gauge theories (where LH and RH fermions carry different gauge charges, as under \( SU(2) \times U(1) \)), or when fermion number is violated (as is the case for neutrinos with a Majorana mass), or when fermions transform as a real representation of gauge group — then it is much more convenient to use irreducible fermion representations, called Weyl fermions.

The six generators of the Lorentz group may be chosen to be the three Hermitian generators of rotations \( J_i \), and the three anti-Hermitian generators of boosts \( K_i \), so that an arbitrary Lorentz transformation takes the form

\[ \Lambda = e^{i(\theta_i J_i + \omega_i K_i)} . \] (1.32)

In terms of the \( M_{\mu\nu} \) generators in [12]

\[ J_i = \frac{1}{2} \epsilon_{0\mu\nu} M^{\mu\nu} , \quad K_i = M_{0i} . \] (1.33)

These generators have the commutation relations

\[ [J_i, J_j] = i\epsilon_{ijk} J_k , \quad [J_i, K_j] = i\epsilon_{ijk} K_k , \quad [K_i, K_j] = -i\epsilon_{ijk} J_k . \] (1.34)

It is convenient to define different linear combinations of generators
A \textit{Weyl fermions} 7

satisfying an algebra that looks like $SU(2) \times SU(2)$, except for the fact that the generators eqn. (1.35) are not Hermitian and therefore the group is noncompact:

\begin{align*}
[A_i, A_j] = i\epsilon_{ijk} A_k, & B_i = i\epsilon_{ijk} B_k, & \quad [A_i, B_j] = 0 .
\end{align*}

Thus Lorentz representations may be labelled with two $SU(2)$ spins, $j_A, j_B$ corresponding to the two $SU(2)$s: $(j_A, j_B)$, transforming as

\begin{align*}
\Lambda(\vec{\theta}, \vec{\omega}) = D^{j_A}(\vec{\theta} - i\vec{\omega}) \times D^{j_B}(\vec{\theta} + i\vec{\omega})
\end{align*}

where the $D^j$ is the usual $SU(2)$ rotation in the spin $j$ representation; boosts appear as imaginary parts to the rotation angle; the $D^{j_A}$ and $D^{j_B}$ matrices act in different spaces and therefore commute. For example, under a general Lorentz transformation, a LH Weyl fermion $\psi$ and a RH Weyl fermion $\chi$ transform as $\psi \rightarrow L \psi$, $\chi \rightarrow R \chi$, where

\begin{align*}
L = e^{i(\vec{\theta} - i\vec{\omega}) \cdot \vec{\sigma}/2} , & \quad R = e^{i(\vec{\theta} + i\vec{\omega}) \cdot \vec{\sigma}/2} .
\end{align*}

Evidently the two types of fermions transform the same way under rotations, but differently under boosts.

The dimension of the $(j_A, j_B)$ representation is $(2j_A + 1)(2j_B + 1)$. In this notation, the smaller irreducible Lorentz representations are labelled as:

- $(0, 0)$: scalar
- $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$: LH and RH Weyl fermions
- $(\frac{1}{2}, \frac{1}{2})$: four-vector
- $(1, 0)$, $(0, 1)$: self-dual and anti-self-dual antisymmetric tensors

A Dirac fermion is the reducible representation $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ consisting of a LH and a RH Weyl fermion.

Parity interchanges the two $SU(2)$s, transforming a $(j_1, j_2)$ representation into $(j_2, j_1)$. Similarly, charge conjugation effectively flips the sign of $K_i$ in eqn. (1.37) due to the factor of $i$ implying that if a field $\phi$ transforms as $(j_1, j_2)$, then $\phi^\dagger$ transforms as $(j_2, j_1)$. Therefore a theory of $N_L$ flavors of LH Weyl fermions $\psi_i$, and $N_R$ flavors of RH Weyl fermions $\chi_a$ may be recast as a theory of $(N_L + N_R)$ LH fermions by defining $\chi_a \equiv \omega_a^L$. The fermion content of the theory can be described entirely in terms of LH Weyl fermions then, $\{\psi_i, \omega_a\}$: this often simplifies the discussion of parity violating theories, such as the Standard Model or Grand Unified Theories. Note that if the RH $\chi_a$ transformed under a gauge group as representation $R$, the conjugate fermions $\omega_a$ transform under the conjugate representation $\overline{R}$.

\textsuperscript{2}For this reason, the combined symmetry CP does not alter the particle content of a chiral theory, so that CP violation must arise from complex coupling constants.
Chiral symmetry

For example, QCD written in terms of Dirac fermions has the Lagrangian:

$$\mathcal{L} = \sum_{i=u,d,s...} \Psi_n (i \partial^\mu - m_n) \Psi_n^\dagger,$$

(1.39)

where $D_\mu$ is the $SU(3)_c$ covariant derivative, and the $\Psi_n$ fields (both LH and RH components) transform as a 3 of $SU(3)_c$. However, we could just as well write the theory in terms of the LH quark fields $\psi_n$ and the LH anti-quark fields $\chi_n$. Using the $\gamma$-matrix basis in eqn. (1.10), we write the Dirac spinor $\Psi$ in terms of two-component LH spinors $\psi$ and $\chi$ as

$$\Psi = \begin{pmatrix} -\sigma_2 \chi^\dagger \\ \psi \end{pmatrix}. $$

(1.40)

Note that $\psi$ transforms as a 3 of $SU(3)_c$, while $\chi$ transforms as a 3. Then the kinetic operator becomes (up to a total derivative)

$$\overline{\psi} i D^\mu \sigma^\mu \psi + \overline{\chi} i D^\mu \sigma^\mu \chi, \quad \sigma^\mu \equiv \{1, -\vec{\sigma}\},$$

(1.41)

and the mass terms become

$$\overline{\Psi}_L \Psi_R = \chi \sigma_2 \psi = \psi \sigma_2 \chi \quad \overline{\Psi}_R \Psi_L = \psi^\dagger \sigma_2 \chi^\dagger = \chi^\dagger \sigma_2 \psi^\dagger,$$

(1.42)

where I used the fact that fermion fields anti-commute. Thus a Dirac mass in terms of Weyl fermions is just

$$m \overline{\psi} \psi = m(\psi \sigma_2 \chi + h.c),$$

(1.43)

and preserves a fermion number symmetry where $\psi$ has charge +1 and $\chi$ has charge −1. On the other hand, one can also write down a Lorentz invariant mass term of the form

$$m(\psi \sigma_2 \psi + h.c.),$$

(1.44)

which violates fermion number by two units; this is a Majorana mass, which is clumsy to write in Dirac notation. Experimentalists are trying to find out which form neutrino masses have — Dirac, or Majorana? If the latter, lepton number is violated by two units and could show up in neutrinoless double beta decay, where a nucleus decays by emitting two electrons and no anti-neutrinos.

The Standard Model is a relevant example of a chiral gauge theory. Written in terms of LH Weyl fermions, the quantum numbers of a single family under $SU(3) \times SU(2) \times U(1)$ are:

$$Q = (3, 2)_{+\frac{1}{6}} \quad L = (1, 2)_{-\frac{1}{2}}$$

$$U^c = (3, 1)_{-\frac{2}{3}} \quad E^c = (1, 1)_{+1}$$

$$D^c = (3, 1)_{+\frac{1}{3}}.$$

(1.45)

Evidently this is a complex representation and chiral. If neutrino masses are found to be Dirac in nature (i.e. lepton number preserving) then a partner for the neutrino
must be added to the theory, the “right handed neutrino”, which can be described by a LH Weyl fermion which is neutral under all Standard Model gauge interactions, \( N = (1, 1)_0 \).

If unfamiliar with two-component notation, you can find all the details in Appendix A of Wess and Bagger’s classic book on supersymmetry (Wess and Bagger, 1992); the notation used here differs slightly as I use the metric and \( \gamma \)-matrix conventions of Itzykson and Zuber (Itzykson and Zuber, 1980), and write out the \( \sigma_2 \) matrices explicitly.

**Exercise 1.2** Consider a theory of \( N_f \) flavors of Dirac fermions in a real or pseudo-real representation of some gauge group. (Real representations combine symmetrically to form an invariant, such as a triplet of \( SU(2) \); pseudo-real representations combine anti-symmetrically, such as a doublet of \( SU(2) \)). Show that if the fermions are massless the action exhibits a \( U(2N_f) = U(1) \times SU(2N_f) \) flavor symmetry at the classical level (the \( U(1) \) subgroup being anomalous in the quantum theory). If the fermions condense as in QCD, what is the symmetry breaking pattern? How do the resultant Goldstone bosons transform under the unbroken subgroup of \( SU(2N_f) \)?

**Exercise 1.3** To see how the \((\frac{1}{2}, \frac{1}{2})\) representation behaves like a four-vector, consider the \( 2 \times 2 \) matrix \( P = P_\mu \sigma^\mu \), where \( \sigma^\mu \) is given in eqn. (1.41). Show that the transformation \( P \rightarrow LPL^\dagger \) (with \( \text{det} \ L = 1 \)) preserves the Lorentz invariant inner product \( P_\mu P_\mu = (P_0^2 - P_i P_i) \).

**Exercise 1.4** Is it possible to write down an anomalous electric or magnetic moment operator in a theory of a single charge-neutral Weyl fermion?

### 1.6 Chiral symmetry and mass renormalization

Some operators in a Lagrangian suffer from additive renormalizations, such as the unit operator (cosmological constant) and scalar mass terms, such as the Higgs mass in the Standard Model, \( |H|^2 \). Therefore, the mass scales associated with such operators will naturally be somewhere near the UV cutoff of the theory, unless the bare couplings of the theory are fine-tuned to cancel radiative corrections. Such fine tuning problems have obsessed particle theorists since the work of Wilson and ’t Hooft on renormalization and naturalness in the 1970s. However, such intertemperate behavior will not occur for operators which violate a symmetry respected by the rest of the theory: if the bare couplings for such operators were set to zero, the symmetry would ensure they could not be generated radiatively in perturbation theory. Fermion mass operators generally fall into this benign category.

Consider the following toy model: QED with a charge-neutral complex scalar field coupled to the electron:

\[
\mathcal{L} = (i\overline{\Psi} (i\partial - m) \Psi + |\partial \phi|^2 - \mu^2 |\phi|^2 - g|\phi|^4 + y (\overline{\Psi}_R \phi \Psi_L + \overline{\Psi}_L \phi^* \Psi_R ) .
\] (1.46)

Note that in the limit \( m \rightarrow 0 \) this Lagrangian respects a chiral symmetry \( \Psi \rightarrow e^{i\gamma_5} \Psi \), \( \phi \rightarrow e^{-2i\alpha} \phi \). The symmetry ensures that if \( m = 0 \), a mass term for the fermion would not be generated radiatively in perturbation theory. With \( m \neq 0 \), this means that any renormalization of \( m \) must be proportional to \( m \) itself (i.e. \( m \) is “multiplicatively
Fig. 1.1 One-loop renormalization of the electron mass in QED due to photon exchange. A mass operator flips chirality, while gauge interactions do not. A contribution to the electron mass requires an odd number of chirality flips, and so there has to be at least one insertion of the electron mass in the diagram: the electron mass is multiplicatively renormalized. A scalar interaction flips chirality when the scalar is emitted, and flips it back when the scalar is absorbed, so replacing the photon with a scalar in the above graph again requires a fermion mass insertion to contribute to mass renormalization.

Fig. 1.2 One-loop additive renormalization of the scalar mass due to a quadratically divergent fermion loop.

Renormalized”). This is evident if one traces chirality through the Feynman diagrams; see Fig. 1.1. Multiplicative renormalization implies that the fermion mass can at most depend logarithmically on the cutoff (by dimensional analysis): \( \delta m \sim (\alpha/4\pi)m \ln m/\Lambda \).

In contrast, the scalar mass operator \(|\phi|^2\) does not violate any symmetry and therefore suffers from additive renormalizations, such as through the graph in Fig. 1.2. By dimensional analysis, the scalar mass operator can have a coefficient that scales quadratically with the cutoff: \( \delta \mu^2 \sim (y^2/16\pi^2)\Lambda^2 \). This is called an additive renormalization, since \( \delta \mu^2 \) is not proportional to \( \mu^2 \). It is only possible in general to have a scalar in the spectrum of this theory with mass much lighter than \( y\Lambda/4\pi \) if the bare couplings are finely tuned to cause large radiative corrections to cancel. When referring to the Higgs mass in the Standard Model, this is called the hierarchy problem.

If chiral symmetry is broken by operators other than the mass term, then the fermion mass will no longer in general be multiplicatively renormalized, and fine tuning may be necessary. This is particularly true if chiral symmetry is broken by “irrelevant” operators. Consider adding to QED a dimension five operator of the form \( W = \frac{r}{\Lambda} \bar{\Psi} D\mu D^\mu \Psi \), where \( r \) is a dimensionless coupling. This operator breaks chirality and therefore one can substitute it for the mass operator in Fig. 1.1. An estimate of the diagram then gives an additive renormalization of the fermion mass, \( \delta m \sim (\alpha/4\pi)r\Lambda \). Thus unless \( r \) is extremely small (e.g. \( r \lesssim m/\Lambda \)) the chiral symmetry breaking effects of this operator will be important and fine tuning will be necessary to ensure a light fermion in the spectrum. This example is relevant to Wilson’s method for putting fermions on the lattice, which does not respect chiral symmetry and entails adding
to the action a lattice version of $W$, where $a \approx 1/\Lambda$ is the lattice spacing and $r \approx 1$. Therefore Wilson fermions acquire an $O(1/a)$ correction to their mass which needs to be canceled by a bare contribution in order to describe a world with light fermions.

1.7 Chiral symmetry in QCD

1.7.1 Chiral symmetry breaking and Goldstone bosons

So far the discussion of chiral symmetry in terms of the effect on fermion masses has been appropriate for a weakly coupled theory. As was presented in M. Golterman’s lectures, the low energy spectrum of QCD is described by a chiral Lagrangian, encoding the interactions of the meson octet which are the approximate Nambu-Goldstone bosons of spontaneously broken $SU(3)_L \times SU(3)_R$ chiral symmetry. The Goldstone bosons would be massless in the limit of exact chiral symmetry, and so tuning away the leading finite lattice space correction for Wilson fermions can be accomplished by tuning the bare quark mass to eliminate the $1/a$ dependence of the square of the pion mass.

In contrast to QCD, $N = 1$ super Yang Mills theory has a single Weyl fermion (the gaugino) transforming as an adjoint under the gauge group; the theory has a $U(1)_A$ symmetry at the classical level — phase rotations of the gaugino — but it is broken by anomalies to a discrete symmetry. This discrete symmetry is then spontaneously broken by a gluino condensate, but without any continuous symmetries, no Goldstone bosons are produced. What should the spectrum of this theory look like? Presumably a bunch of massive boson and fermion glueball-like states. They will form degenerate supersymmetric multiplets when the gluino mass is tuned to zero, but there is no particle that becomes massless in the chiral limit in this case, and therefore tuning the bare mass is difficult.

After tuning away the $O(1/a)$ mass correction, there remain for non-chiral lattice fermions the dimension-5 chiral symmetry violating operators in the Symanzik action which require $O(a)$ tuning, as discussed by Golterman. In contrast, chiral fermions receive finite lattice corrections only at $O(a^2)$, simply because one cannot write down a dimension-5 chiral symmetry preserving operator in QCD.

1.7.2 Operator mixing

One encounters additional factors of $1/a$ when computing weak processes. One of the most curious feature of the strong interactions is the $\Delta I = 1/2$ rule, which is the observation that $\Delta s = 1$ transitions in nature are greatly enhanced when they change isospin by $\Delta I = 1/2$, in comparison to $\Delta I = 3/2$. For example, one requires for the amplitudes for kaon decay $K \rightarrow \pi \pi$:

3With six favors of quarks in QCD, one might ask why an $SU(3)$ chiral symmetry instead of $SU(2)$ or $SU(6)$. The point is that the chiral symmetry is broken by quark masses, and whether the breaking is large or small depends on the ratio $m_q/\Lambda_{QCD}$, where here $\Lambda_{QCD}$ is some strong interaction scale in the 100s of MeV. The u and d quarks are much lighter than $\Lambda_{QCD}$, the strange quark is borderline, and the c, b, t quarks are much heavier. Therefore $SU(2) \times SU(2)$ is a very good symmetry of QCD; $SU(3) \times SU(3)$ is a pretty good symmetry of QCD, but assuming chiral symmetry for the heavier quarks is not justified. Radiative corrections in the baryon sector go as $\sqrt{m_q/\Lambda_{QCD}}$ and so there even $SU(3) \times SU(3)$ does not appear to be very reliable.
Chiral symmetry

\[ A(\Delta I = 1/2) \approx 20. \]

(1.47)

To compute this in the standard model, one starts with four-quark operators generated by W-exchange, which can be written as the linear combination of two operators

\[ \mathcal{L}_{\Delta S=1} = -V_{ud}V_{us}^* \frac{G_F}{\sqrt{2}} \left[ C_+(\mu, M_W) \mathcal{O}^+ + C_-(\mu, M_W) \mathcal{O}^- \right], \]

\[ \mathcal{O}^\pm = \left[ (\bar{s}d)_L(\bar{u}u)_L \pm (\bar{s}u)_L(\bar{p}d)_L \right] - [u \leftrightarrow c], \]

(1.48)

where \((\eta q')_L \equiv (\bar{q}\gamma^\mu P_L q').\) If one ignores the charm quark contribution, the \(\mathcal{O}^-\) transforms as an 8 under \(SU(3)_f,\) while \(\mathcal{O}^+\) transforms as a 27; therefore \(\mathcal{O}^\pm\) is not a good symmetry of the spectrum, it is only broken by quark masses which do not effect the log divergences of the theory. Thus the running of the operators respect \(SU(4)_f\) down to \(\mu = m_c,\) and there is no mixing between \(\mathcal{O}^\pm.\) At the weak scale \(\mu = M_W,\) one finds \(|C^+/C^-| = 1 + O(\alpha_s(M_W)),\) showing that there is no \(\Delta I = 1/2\) enhancement intrinsic to the weak interactions. One then scales these operators down to \(\mu \sim 2\) GeV in order to match onto the lattice theory; using the renormalization group to sum up leading \(\alpha_s \ln \mu/M_W\) corrections gives an enhancement \(|C^+/C^-| \approx 2 - \ldots\) which is in the right direction, but not enough to explain eqn. (1.47), which should then either be coming from QCD at long distances, or else new physics! This is a great problem for the lattice to resolve.

A wonderful feature about using dimensional regularization and \(\overline{MS}\) in the continuum is that an operator will never mix with another operator of lower dimension. This is because there is no UV mass scale in the scheme which can make up for the miss-match in operator dimension. This is not true on the lattice, where powers of the inverse lattice spacing \(1/a\) can appear. In particular, the the dimension-6 four fermion operators \(\mathcal{O}^\pm\) could in principle mix with dimension-3 two fermion operators. The only \(\Delta S = 1\) dimension-3 operator that could arise is \(\bar{s}\gamma_5 d,\) which is also \(\Delta I = 1/2.\) If the quarks were massless, the lattice theory would possess an exact discrete “SCP” symmetry under which on interchanges \(s \leftrightarrow d\) and performs a CP transformation to change LH quarks into LH anti-quarks; the operators \(\mathcal{O}_\pm\) are even under SCP while \(\bar{s}\gamma_5 d\) is odd, to the operator that could mix on the lattice is

\[ \mathcal{O}_p = (m_s - m_d) \bar{s}\gamma_5 d. \]

(1.49)

In a theory where the quark masses are the only source of chiral symmetry breaking, then \(\mathcal{O}_p = \partial_\mu A^{pd}_\mu,\) the divergence of the \(\Delta S = 1\) axial current. Therefore on-shell matrix elements of this operator vanish, since the derivative gives \((p_K - p_{2\pi}) = 0,\) i.e. no momentum is being injected by the weak interaction. We can ignore \(\mathcal{O}_p\) then when the \(K \rightarrow \pi\pi\) amplitude is measured with chiral lattice fermions with on-shell momenta.

\(^4\)The operator \(\bar{s}d\) is removed by rediagonalizing the quark mass matrix and does not give rise to \(K\pi\pi\).
For a lattice theory without chiral symmetry, \( \mathcal{O}_p = \partial_\mu A^\mu_n + O(a) \) and so has a nonvanishing \( O(a) \) matrix element. In this case operators \( \mathcal{O}_\pm \) from eqn. (1.48) in the continuum match onto the lattice operators

\[
\mathcal{O}_\pm(\mu) = Z_\pm(\mu a, g^2_0) \left[ \mathcal{O}_\pm(a) + \frac{C_\pm^2}{a^2} \mathcal{O}_p \right] + O(a) .
\]

In general then one would need to determine the coefficient \( C_\pm^2 \) to \( O(a) \) in order to determine the \( \Delta I = \frac{1}{2} am \) amplitude for \( K \to \pi \pi \) to leading order in an \( a \)-expansion, which is not really feasible. Other weak matrix elements such as \( B_K \) and \( \epsilon'/\epsilon \) similarly benefit from the use of lattice fermions with good chiral symmetry.

### 1.8 Fermion determinants in Euclidian space

Lattice computations employ Monte Carlo integration, which requires a positive integrand that can be interpreted as a probability distribution. While not sufficient for a lattice action to yield a positive measure, it is certainly necessary for the continuum theory one is approximating to have this property. Luckily, the fermion determinant for vector-like gauge theories (such as QCD), \( \det(\bar{D} + m) \), has this property in Euclidian space. Since \( \bar{D} = -D \) and \( \{ \Gamma, \bar{D} \} = 0 \), it follows that there exist eigenstates \( \psi_n \) of \( \bar{D} \) such that

\[
\bar{D}\psi_n = i\lambda_n \psi_n , \quad D\Gamma\psi_n = -i\lambda_n \Gamma\psi_n , \quad \lambda_n \text{ real} .
\]

For nonzero \( \lambda, \psi_n \) and \( \Gamma\psi_n \) are all mutually orthogonal and we see that the eigenvalue spectrum contains \( \pm i\lambda_n \) pairs. On the other hand, if \( \lambda_n = 0 \) then \( \psi_n \) can be an eigenstate of \( \Gamma \) as well, and \( \Gamma\psi_n \) is not an independent mode. Therefore

\[
\det(\bar{D} + m) = \prod_{\lambda_n>0} (\lambda_n^2 + m^2) \times \prod_{\lambda_n=0} m
\]

which is real and for positive \( m \) is positive for all gauge fields.

What about a chiral gauge theory? The fermion Lagrangian for a LH Weyl fermion in Euclidian space looks like \( \bar{\psi}D_L\psi \) with \( D_L = D_\mu \sigma_\mu \) and (in the chiral basis eqn. (1.10), continued to Euclidian space) \( \sigma_\mu = \{ 1, i\bar{\sigma} \} \). Note that \( D_L \) has no nice hermiticity properties, which means its determinant will be complex, its right eigenvectors and left eigenvectors will be different, and its eigenvectors will not be mutually orthogonal. Furthermore, \( D_L \) is an operator which maps vectors from the space \( \mathcal{L} \) of LH Weyl fermions to the space \( \mathcal{R} \) of RH Weyl fermions. In Euclidian space, these spaces are unrelated and transform independently under the \( SU(2) \times SU(2) \) Lorentz transformations. Suppose we have an orthonormal basis \( \{|n, \mathcal{R}\rangle \} \) for the RH Hilbert space and \( \{|n, \mathcal{L}\rangle \} \) for the LH Hilbert space; we can expand our fermion integration variables as

\[
\psi = \sum_n c_n |n, \mathcal{L}\rangle , \quad \bar{\psi} = \sum_n c_n \langle n, \mathcal{R}| \]

so that
The eigenvalue flow of the Dirac operator as a function of gauge fields, and two unsatisfactory ways to define the Weyl fermion determinant $\det D_L$ as a square root of $\det D$. The expression $\sqrt{\det D}$ corresponds to the picture on the left, where $\det D_L$ is defined as the product of positive eigenvalues of $D$; this definition is nonanalytic at $A_\ast$. The picture on the right corresponds to the product of half the eigenvalues, following those which were positive at some reference gauge field $A_0$. This definition is analytic, but not necessarily local. Both definitions are gauge invariant, which is incorrect for an anomalous fermion representation.

$$\int [d\psi][d\bar{\psi}] e^{-\int \bar{\psi} D_L \psi} = \det_{mn} \langle m, R| D_L |n, L \rangle .$$  \hspace{1cm} (1.54)

However, the answer we get will depend on the basis we choose. For example, we could have chosen a different orthonormal basis for the $L$ space $|n', L\rangle = U_{n' n} |n, L\rangle$ which differed from the first by a unitary transformation $U$; the resultant determinant would differ by a factor $\det U$, which is a phase. If this phase were a number, it would not be an issue — but it can in general be a functional of the background gauge field, so that different choices of phase for $\det D_L$ lead to completely different theories.

We do know that if $D_R$ is the fermion operator for RH Weyl fermions in the same gauge representation as $D_L$, then $\det D_R = \det D_L^*$ and that $\det D_R \det D_L = \det D$. Therefore the the norm of $|\det D_L|$ can be defined as

$$\det D_L = \sqrt{\det D} |e^{iW[A]}|$$ \hspace{1cm} (1.55)

where the phase $W[A]$ is a functional of the gauge fields. What do we know about $W[A]$?

1. Since $\det D$ is gauge invariant, $W[A]$ should be gauge invariant unless the fermion representation has a gauge anomaly, in which case it should correctly reproduce that anomaly;

2. It should be analytic in the gauge fields, so that the computation of gauge field correlators (or the gauge current) are well defined.

3. It should be a local functional of the gauge fields.

In Fig. 1.3 I show two possible ways to define $\det D_L$, neither of which satisfy the above criteria. The naive choice of just setting $W[A] = 0$ not only fails to reproduce the anomaly (if the fermion representation is anomalous) but is also nonanalytic and nonlocal. It corresponds to taking the product of all the positive eigenvalues $\lambda_n$ of $D$ (up to an uninteresting overall constant phase). This definition is seen to be nonanalytic where eigenvalues cross zero. Another definition might be to take the product
of positive eigenvalues at some reference gauge field \(A_0\), following those eigenvalues as they cross zero; this definition is analytic, but presumably not local, and is always gauge invariant.

There has been quite a few papers on how to proceed in defining this phase \(W[A]\) in the context of domain wall fermions, including a rather complicated explicit construction for \(U(1)\) chiral gauge theories on the lattice [Luscher, 1999] [Luscher, 2000a]; however, even if a satisfactory definition of \(W[A]\) is devised, it could be impossible to simulate using Monte Carlo algorithms due to the complexity of the fermion determinant.

1.9 Parity and fermion mass in odd dimensions

In these lectures I will be discussing fermions in \((2k + 1)\) dimensions with a spatially varying mass term which vanishes in some \(2k\)-dimensional region; in such cases we find chiral modes of a \(2k\)-dimensional effective theory bound to this mass defect. Such an example could arise dynamically when fermions have a Yukawa coupling to a real scalar \(\phi\) which spontaneously breaks a discrete symmetry, where the surface with \(\phi = 0\) forms a domain wall between two different phases; for this reason such fermions are called domain wall fermions, even though we will be putting the spatially dependent mass in by hand and not through spontaneous symmetry breaking.

To study domain wall fermions it is useful to say a few words about fermions in odd dimensions where there is no analogue of \(\Gamma\) and therefore there is no such thing as chiral symmetry. Nevertheless, fermion masses still break a symmetry: parity. In a theory with parity symmetry one has extended the Lorentz group to include improper rotations: spatial rotations \(R\) for which the determinant of \(R\) is negative. Parity can be defined as a transformation where an odd number of the spatial coordinates flip sign. In even dimensions parity can be the transformation \(x \rightarrow -x\) and

\[
\Psi(x, t) \rightarrow \gamma^0 \Psi(-x, t) \quad \text{(parity, } d \text{ even)}.
\]

(1.56)

Note that under this transformation \(\Psi_L\) and \(\Psi_R\) are exchanged and that a Dirac mass term is parity invariant.

However, in odd dimensions the transformation \(x \rightarrow -x\) is just a rotation; instead we can define parity as the transformation which just flips the sign of one coordinate \(x^1\), and

\[
\Psi(x, t) \rightarrow \gamma^1 \Psi(\bar{x}, t), \quad \bar{x} = (-x^1, x^2, \ldots, x^{2k})
\]

(1.57)

Remarkably, a Dirac mass term flips sign under parity in this case; and since there is no chiral symmetry in odd \(d\) to rotate the phase of the mass matrix, the sign of the quark mass is physical, and a parity invariant theory of massive quarks must have them come in pairs with masses \(\pm M\), with parity interchanging the two.

1.10 Fermion masses and regulators

We have seen that theories of fermions in any dimension can possess symmetries which forbid masses – chiral symmetry in even dimensions and parity in odd dimensions.
Chiral symmetry

This property obviously can have a dramatic impact on the spectrum of a theory. Supersymmetry ingeniously puts fermions and bosons in the same supermultiplet, which allows scalars to also enjoy the benefits of chiral symmetry, which is one reason theorists have been so interested in having supersymmetry explain why the Higgs boson of the standard model manages to be so much lighter than the Planck scale. However, precisely because mass terms violate these symmetries, it is difficult to maintain them in a regulated theory. After all, one regulates a theory by introducing a high mass scale in order to eliminate UV degrees of freedom in the theory, and this mass scale will typically violate chiral symmetry in even dimensions, or parity in odd. This gives rise to “anomalous” violation of the classical fermion symmetries, my next topic.
Anomalies

2.1 The $U(1)_A$ anomaly in 1+1 dimensions

One of the fascinating features of chiral symmetry is that sometimes it is not a symmetry of the quantum field theory even when it is a symmetry of the Lagrangian. In particular, Noether’s theorem can be modified in a theory with an infinite number of degrees of freedom; the modification is called “an anomaly”. Anomalies turn out to be very relevant both for phenomenology, and for the implementation of lattice field theory. The reason anomalies affect chiral symmetries is that regularization requires a cut-off on the infinite number of modes above some mass scale, while chiral symmetry is incompatible with fermion masses\(^1\).

Anomalies can be seen in many different ways. I think the most physical is to look at what happens to the ground state of a theory with a single flavor of massless Dirac fermion in (1 + 1) dimensions in the presence of an electric field. Suppose one adiabatically turns on a constant positive electric field $E(t)$, then later turns it off; the equation of motion for the fermion is $\frac{dp}{dt} = eE(t)$ and the total change in momentum is

$$\Delta p = e \int E(t) \, dt . \tag{2.1}$$

Thus the momenta of both left- and right-moving modes increase: if one starts in the ground state of the theory with filled Dirac sea, after the electric field has turned off, both the right-moving and left-moving sea has shifted to the right as in Fig. 2.1. The final state differs from the original by the creation of particle-antiparticle pairs: right moving particles and left moving antiparticles. Thus while there is a fermion current in the final state, fermion number has not changed. This is what one would expect from conservation of the $U(1)$ current:

$$\partial_\mu J^\mu = 0 , \tag{2.2}$$

However, recall that right-moving and left-moving particles have positive and negative chirality respectively; therefore the final state in Fig. 2.1 has net axial charge, even though the initial state did not. This is peculiar, since the coupling of the electromagnetic field in the Lagrangian does not violate chirality. We can quantify the effect: if

\(^1\)Dimensional regularization is not a loophole, since chiral symmetry cannot be analytically continued away from odd space dimensions.

\(^2\)While in much of these lectures I will normalize gauge fields so that $D_\mu = \partial_\mu + iA_\mu$, in this section I need to put the gauge coupling back in. If you want to return to the nicer normalization, set the gauge coupling to unity, and put a $1/g^2$ factor in front of the gauge action.
we place the system in a box of size $L$ with periodic boundary conditions, momenta are quantized as $p_n = 2\pi n/L$. The change in axial charge is then

$$\Delta Q_A = 2 \frac{\Delta p}{2\pi} = \frac{e}{\pi} \int d^2 x \, E(t) = \frac{e}{2\pi} \int d^2 x \, \epsilon_{\mu\nu} F^{\mu\nu}, \quad (2.3)$$

where I expressed the electric field in terms of the field strength $F$, where $F^{01} = -F^{10} = E$. This can be converted into the local equation using $\Delta Q_A = \int d^2 x \, \partial_\mu J^\mu_A$, a modification of eqn. (1.26):

$$\partial_\mu J^\mu_A = 2im\Psi \Gamma \Psi + \frac{e}{2\pi} \epsilon_{\mu\nu} F^{\mu\nu}, \quad (2.4)$$

where in the above equation I have included the classical violation due to a mass term as well. The second term is the axial anomaly in $1+1$ dimensions; it would vanish for a nonabelian gauge field, due to the trace over the gauge generator.

So how did an electric field end up violating chiral charge? Note that this analysis relied on the Dirac sea being infinitely deep. If there had been a finite number of negative energy states, then they would have shifted to higher momentum, but there would have been no change in the axial charge. With an infinite number of degrees of freedom, though, one can have a “Hilbert Hotel”: the infinite hotel which can always accommodate another visitor, even when full, by moving each guest to the next room and thereby opening up a room for the newcomer. This should tell you that it will not be straightforward to represent chiral symmetry on the lattice: a lattice field theory approximates quantum field theory with a finite number of degrees of freedom — the lattice is a big hotel, but quite conventional. In such a hotel there can be no anomaly.

We can derive the anomaly in other ways, such as by computing the anomaly diagram Fig. 2.2 or by following Fujikawa (Fujikawa, 1979; Fujikawa, 1980) and carefully accounting for the Jacobian from the measure of the path integral when performing a chiral transformation. It is particularly instructive for our later discussion of lattice
fermions to compute the anomaly in perturbation theory using Pauli-Villars regulators of mass $M$. We replace our axial current by a regulated current

$$J_{\mu}^{\text{reg}} = \bar{\Psi} \gamma_{\mu} \Gamma \Psi + \bar{\Phi} \gamma_{\mu} \Gamma \Phi,$$

(2.5)

where $\Phi$ is our Pauli-Villars field; it follows then that

$$\partial_{\mu} J_{\mu}^{\text{reg}} = 2im\bar{\Psi} \Gamma \Psi + 2iM\bar{\Phi} \Gamma \Phi.$$

(2.6)

We are interested in matrix elements of $J_{\mu}^{\text{reg}}$ in a background gauge field between states without any Pauli-Villars particles, and so we need to evaluate $\langle 2iM\bar{\Phi} \Gamma \Phi \rangle$ in a background gauge field and take the limit $M \to \infty$ to see if $\partial_{\mu} J_{\mu}^{\text{reg}}$ picks up any anomalous contributions that do not decouple as we remove the cutoff.

To compute $\langle 2iM\bar{\Phi} \Gamma \Phi \rangle$ we need to consider all Feynman diagrams with a Pauli-Villars loop, and insertion of the $\bar{\Phi} \Gamma \Phi$ operator, and any number of external $U(1)$ gauge fields. By gauge invariance, a graph with $n$ external photon lines will contribute $n$ powers of the field strength tensor $F_{\mu\nu}$. For power counting, it is convenient that we normalize the gauge field so that the covariant derivative is $D_\mu = (\partial_\mu + ieA_\mu)$; then the gauge field has mass dimension 1, and $F_{\mu\nu}$ has dimension 2. In $(1 + 1)$ dimensions $\langle 2iM\bar{\Phi} \Gamma \Phi \rangle$ has dimension 2, and so simple dimensional analysis implies that the graph with $n$ photon lines must make a contribution proportional to $(F_{\mu\nu})^n / M^{2(n-1)}$. Therefore only the graph in Fig. 2.2 with one photon insertion can make a contribution that survives the $M \to \infty$ limit (the graph with zero photons vanishes). Calculation of this diagram yields the same result for the divergence of the regulated axial current as we found in eqn. (2.4).

**Exercise 2.1** Compute the diagram in Fig. 2.2 using the conventional normalization of the gauge field $D_\mu = (\partial_\mu + ieA_\mu)$ and verify that $2iM\langle \bar{\Phi} \Gamma \Phi \rangle = \frac{e^2}{2\pi} \epsilon_{\mu\nu} F^{\mu\nu}$ when $M \to \infty$. 

Note that in this description of the anomaly we (i) effectively rendered the number of degrees of freedom finite by introducing the regulator; (ii) the regulator explicitly broke the chiral symmetry; (iii) as the regulator was removed, the symmetry breaking effects of the regulator never decoupled, indicating that the anomaly arises when the two vertices in Fig. 2.2 sit at the same spacetime point. While we used a Pauli-Villars regulator here, the use of a lattice regulator will have qualitatively similar features, with the inverse lattice spacing playing the role of the Pauli-Villars mass,
and we can turn these observations around: A lattice theory will not correctly reproduce anomalous symmetry currents in the continuum limit, unless that symmetry is broken explicitly by the lattice regulator. This means we would be foolish to expect to construct a lattice theory with exact chiral symmetry. But can the lattice break chiral symmetry just enough to explain the anomaly, without losing the important consequences of chiral symmetry at long distances (such as protecting fermion masses from renormalization)?

2.2 Anomalies in 3+1 dimensions

2.2.1 The $U(1)_A$ anomaly

An analogous violation of the $U(1)_A$ current occurs in 3 + 1 dimensions as well.

One might guess that the analogue of $\epsilon_{\mu\nu}F^{\mu\nu} = 2E$ in the anomalous divergence eqn. (2.4) would be the quantity $\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} = 8E \cdot B$, which has the right dimensions and properties under parity and time reversal. So we should consider the behavior a massless Dirac fermion in (3 + 1) in parallel constant $E$ and $B$ fields. First turn on a $B$ field pointing in the $\hat{z}$ direction: this gives rise to Landau levels, with energy levels $E_n$ characterized by non-negative integers $n$ as well as spin in the $\hat{z}$ direction $S_z$ and momentum $p_z$, where

$$E_n^2 = p_z^2 + (2n + 1)eB - 2eBS_z. \quad (2.7)$$

The number density of modes per unit transverse area is defined to be $g_n$, which can be derived by computing the zero-point energy in Landau modes and requiring that it yields the free fermion result as $B \to 0$. We have $g_n \to p_\perp dp_\perp/(2\pi)$ with $[(2n + 1)eB - 2eBS_z] \to p_\perp^2$, implying that

$$g_n = eB/2\pi. \quad (2.8)$$

The dispersion relation looks like that of an infinite number of one-dimensional fermions of mass $m_{n,\pm}$, where

$$m_{n,\pm}^2 = (2n + 1)eB - 2eBS_z, \quad S_z = \pm \frac{1}{2}. \quad (2.9)$$

The state with $n = 0$ and $S_z = +\frac{1}{2}$ is distinguished by having $m_{n,+} = 0$; it behaves like a massless one-dimensional Dirac fermion (with transverse density of states $g_0$) moving along the $\hat{z}$ axis with dispersion relation $E = |p_z|$. If we now turn on an electric field also pointing along the $\hat{z}$ direction we know what to expect from our analysis in 1 + 1 dimensions: we find an anomalous divergence of the axial current equal to

$$g_0 eE/\pi = e^2 EB/2\pi^2 = \left(\frac{e^2}{16\pi^2}\right)\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}. \quad (2.10)$$

If we include an ordinary mass term in the 3 + 1 dimensional theory, then we get

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3Part of the content of this section comes directly from John Preskill’s class notes on the strong interactions, available at his web page: http://www.theory.caltech.edu/~preskill/notes.html.
Fig. 2.3 The $U(1)_A$ anomaly diagram in $3+1$ dimensions, with one Pauli-Villars loop and an insertion of $2iM \Phi \Gamma \Phi$.

\[
\partial_\mu J_\mu^A = 2im \Phi \Gamma \Psi + \left( \frac{e^2}{16\pi^2} \right) \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}. \tag{2.11}
\]

One can derive this result by computing $\langle M \Phi \Gamma \Psi \rangle$ for a Pauli-Villars regulator as in the $1+1$ dimensional example; now the relevant graph is the triangle diagram of Fig. 2.3.

If the external fields are nonabelian, the analogue of eqn. (2.11) is

\[
\partial_\mu J_\mu^A = 2im \Phi \Gamma \Psi + \left( \frac{g^2}{16\pi^2} \right) \epsilon_{\mu\nu\rho\sigma} F_a^{\mu\nu} F_b^{\rho\sigma} \text{Tr} T_a T_b. \tag{2.12}
\]

If the fermions transform in the defining representation of $SU(N)$, it is conventional to normalize the coupling $g$ so that $\text{Tr} T_a T_b = \frac{1}{2} \delta_{ab}$. This is still called an “Abelian anomaly”, since $J_\mu^A$ generates a $U(1)$ symmetry.

2.2.2 Anomalies in Euclidian spacetime

Continuing to Euclidian spacetime by means of eqns. (1.11)-(1.15) changes the anomaly equations simply by eliminating the factor of $i$ from in front of the fermion mass:

\[
2d: \quad \partial_\mu J_\mu^A = 2m \Phi \Gamma \Psi + \frac{e}{2\pi} \epsilon_{\mu\nu} F^{\mu\nu} \tag{2.13}
\]

\[
4d: \quad \partial_\mu J_\mu^A = 2m \Phi \Gamma \Psi + \left( \frac{g^2}{16\pi^2} \right) \epsilon_{\mu\nu\rho\sigma} F_a^{\mu\nu} F_b^{\rho\sigma} \text{Tr} T_a T_b. \tag{2.14}
\]

2.2.3 The index theorem in four dimensions

For nonabelian gauge theories the quantity on the far right of eqn. (2.14) is a topological charge density, with

\[
\nu = \frac{g^2}{64\pi^2} \int d^4x \epsilon_{\mu\nu\rho\sigma} F_a^{\mu\nu} F_b^{\rho\sigma} \tag{2.15}
\]

being the winding number associated with $\pi_3(G)$, the homotopy group of maps of $S_3$ (spacetime infinity) into the gauge group $G$.

Consider then continuing the anomaly equation eqn. (2.12) to Euclidian space and integrating over spacetime its vacuum expectation value in a background gauge field (assuming the fermions to be in the $N$-dimensional representation of $SU(N)$ so that...
22 Anomalies

\[ \text{Tr } T_a T_b = \frac{1}{2} \delta_{ab} \]. The integral of \( \partial_{\mu} \langle J^A_{\mu} \rangle \) vanishes because it is a pure divergence, so we get

\[ \int d^4x_E m(z) \langle \Psi \Gamma \Psi \rangle = -\nu . \] (2.16)

The matrix element above on the right equals

\[ \int [d\Psi] [d\bar{\Psi}] e^{-S_E (m \bar{\Psi} \Gamma \Psi)} / \int [d\Psi] [d\bar{\Psi}] e^{-S_E} . \] (2.17)

where \( S_E = \bar{\Psi} (\partial_E + m) \Psi \). We can expand \( \Psi \) and \( \bar{\Psi} \) in terms of eigenstates of the anti-hermitian operator \( \partial_E \), where

\[ \partial_E \psi_n = i \lambda_n \psi_n \ , \quad \int d^4x_E \psi^\dagger_n \psi_n = \delta_{mn} , \] (2.18)

with

\[ \Psi = \sum c_n \psi_n , \quad \bar{\Psi} = \sum \bar{c}_n \psi^\dagger_n . \] (2.19)

Then

\[ \int d^4x_E m \langle \Psi \Gamma \Psi \rangle = \left( \sum_n \int d^4x_E m \psi^\dagger_n \Gamma \psi_n \prod_{k \neq n} (i \lambda_k + m) \right) / \prod_k (i \lambda_k + m) \]

\[ = m \sum_n \int d^4x_E \psi^\dagger_n \Gamma \psi_n / (i \lambda_n + m) . \] (2.20)

Recall that \( \{ \Gamma, \partial_E \} = 0 \); thus

\[ \partial_E \psi_n = i \lambda_n \psi_n \] implies \( \partial_E (\Gamma \psi_n) = -i \lambda_n (\Gamma \psi_n) \). (2.21)

Thus for \( \lambda_n \neq 0 \), the eigenstates \( \psi_n \) and \( (\Gamma \psi_n) \) must be orthogonal to each other (they are both eigenstates of \( \partial_E \) with different eigenvalues), and so \( \psi^\dagger_n \Gamma \psi_n \) vanishes for \( \lambda_n \neq 0 \) and does not contribute to the sum in eqn. (2.20). In contrast, modes with \( \lambda_n = 0 \) can simultaneously be eigenstates of \( \partial_E \) and of \( \Gamma \); let \( n_+ , n_- \) be the number of RH and LH zeromodes respectively. The last integral in then just equals \( (n_+ - n_-) = (n_R - n_L) \), and combining with eqn. (2.16) we arrive at the index equation

\[ n_- - n_+ = \nu \] (2.22)

which states that the difference in the number of LH and RH zeromode solutions to the Euclidian Dirac equation in a background gauge field equals the winding number of the gauge field. With \( N_f \) flavors, the index equation is trivially modified to read

\[ n_- - n_+ = N_f \nu \] (2.23)

This link between eigenvalues of the Dirac operator and the topological winding number of the gauge field provides a precise definition for the topological winding number of a gauge field on the lattice (where there is no topology) — provided we have a definition of a lattice Dirac operator which exhibits exact zeromodes. We will see that the overlap operator is such an operator.
2.2.4 More general anomalies

Even more generally, one can consider the 3-point correlation function of three arbitrary currents as in Fig. 2.4

$$\langle J_a^\mu(k)J_b^\nu(p)J_c^\rho(q) \rangle,$$

and show that the divergence with respect to any of the indices is proportional to a particular group theory factor

$$k_\mu \langle J_a^\mu(k)J_b^\nu(p)J_c^\rho(q) \rangle \propto \left. \text{Tr } Q_a \{Q_b, Q_c\} \right|_{R-L} \epsilon^{\alpha\beta\rho\sigma} k_\alpha k_\sigma,$$

where the Qs are the generators associated with the three currents in the fermion representation, the symmetrized trace being computed as the difference between the contributions from RH and LH fermions in the theory. The anomaly $A$ for the fermion representation is defined by the group theory factor

$$\text{Tr } (Q_a \{Q_b, Q_c\}) \bigg|_{R-L} \equiv A d_{abc},$$

with $d_{abc}$ being the totally symmetric invariant tensor of the symmetry group. For a simple group $G$ (implying $G$ is not $U(1)$ and has no factor subgroups), $d_{abc}$ is only nonzero for $G = SU(N)$ with $N \geq 3$; even in the case of $SU(N)$, $d_{abc}$ will vanish for real irreducible representations (for which $Q_a = -Q_a^*$), or for judiciously chosen reducible complex representations, such as $\bar{5} \oplus 10$ in $SU(5)$. For a semi-simple group $G_1 \times G_2$ (where $G_1$ and $G_2$ are themselves simple) there are no mixed anomalies since the generators are all traceless, implying that if $Q \in G_1$ and $Q \in G_2$ then $\text{Tr } (Q_a \{Q_b, Q_c\}) \propto \text{Tr } Q_a = 0$. When considering groups with $U(1)$ factors there can be nonzero mixed anomalies of the form $U(1)G^2$ and $U(1)^3$ where $G$ is simple; the $U(1)^3$ anomalies can involve different $U(1)$ groups. With a little group theory it is not difficult to compute the contribution to the anomaly of any particular group representation.

If a current with an anomalous divergence is gauged, then the theory does not make sense. That is because the divergenceless of the current is required for the unphysical modes in the gauge field $A_\mu$ to decouple; if they do not decouple, their propagator has a piece that goes as $k_\mu k_\nu / k^2$ which does not fall off at large momentum, and the theory is not renormalizable.

Fig. 2.4 Anomalous three-point function of three currents.
Anomalies

When global $U(1)$ currents have anomalous divergences, that is interesting. We have seen that the $U(1)_A$ current is anomalous, which explains the $\eta'$ mass; the divergence of the axial isospin current explains the decay $\pi^0 \to \gamma\gamma$; the anomalous divergence of the baryon number current in background $SU(2)$ in the Standard Model predicts baryon violation in the early universe and the possibility of weak-scale baryogenesis.

Exercise 2.2 Verify that all the gauge currents are anomaly-free in the standard model with the representation in eqn. (1.45). The only possible $G^3$ anomalies are for $G = SU(3)$ or $G = U(1)$; for the $SU(3)^3$ anomaly use the fact that a LH Weyl fermion contributes $+1$ to $A$ if it transforms as a 3 of $SU(3)$, and contributes $-1$ to $A$ if it is a 3. There are two mixed anomalies to check as well: $U(1)SU(2)^2$ and $U(1)SU(3)^2$.

This apparently miraculous cancellation is suggestive that each family of fermions may be unified into a spinor of $SO(10)$, since the vanishing of anomalies which happens automatically in $SO(10)$ is of course maintained when the symmetry is broken to a smaller subgroup, such as the Standard Model.

Exercise 2.3 Show that the global $B$ (baryon number) and $L$ (lepton number) currents are anomalous in the Standard Model eqn. (1.45), but that $B - L$ is not.

2.3 Strongly coupled chiral gauge theories

Strongly coupled chiral gauge theories are particularly intriguing, since they can contain light composite fermions, which could possibly describe the quarks and leptons we see. A nice toy example of a strongly coupled chiral gauge theory is $SU(5)$ with LH fermions

$$\psi = 5, \quad \chi = 10.$$  (2.27)

It so happens that the $\psi$ and the $\chi$ contribute with opposite signs to the $(SU(5))^3$ anomaly $A$ in eqn. (2.26), so this seems to be a well defined gauge theory. Furthermore, the $SU(5)$ gauge interactions are asymptotically free, meaning that interactions become strong at long distances. One might therefore expect the theory to confine as QCD does. However, unlike QCD, there are no gauge invariant fermion bilinear condensates which could form, and which in QCD are responsible for baryon masses. That being the case, might there be any massless composite fermions in the spectrum of this theory? ’t Hooft came up with a nice general argument involving global anomalies that suggests there will be.

In principle there are two global $U(1)$ chiral symmetries in this theory corresponding to independent phase rotations for $\psi$ and $\chi$; however both of these rotations have global $\times SU(5)^2$ anomalies, similar to the global $\times SU(3)^2$ of the $U(1)_A$ current in QCD. This anomaly can only break one linear combination of the two $U(1)$ symmetries, and one can choose the orthogonal linear combination which is anomaly-free. With a little group theory you can show that the anomaly-free global $U(1)$ symmetry corresponds to assigning charges

$$\psi = 5_3, \quad \chi = 10_{-1}.$$  (2.28)

where the subscript gives the global $U(1)$ charge. This theory has a nontrivial global $U(1)^3$ anomaly, $A = 5 \times (3)^3 + 10 \times (-1)^3 = 125$. ’t Hooft’s argument is that
The non-decoupling of parity violation in odd dimensions

We have seen that chiral symmetry does not exist in odd space dimensions, but that a discrete parity symmetry can forbid a fermion mass. One would then expect a regulator — such as Pauli-Villars fields — to break parity. Indeed they do: on integrating the Pauli-Villars field out of the theory, one is left with a Chern Simons term in the Lagrangian with coefficient \( M/|M| \), which does not decouple as \( M \to \infty \). In \( 2k + 1 \) dimensions the Chern Simons form for an Abelian gauge field is proportional to

\[
\varepsilon^{a_1 \cdots a_{2k+1}} A_{a_1} F_{a_2 a_3} \cdots F_{a_{2k+1} a_{2k+2}}
\]

which violates parity; the Chern Simons form for nonabelian gauge fields is more complicated.

For domain wall fermions we will be interested in a closely related but slightly different problem: the generation of a Chern Simons operator on integrating out a

\[
\langle \chi \chi \chi \psi \rangle \text{ or } \langle \langle \psi \chi \rangle \langle \chi \psi \rangle \rangle
\]

You may wonder about whether fermion condensates form which break the global \( U(1) \) symmetry. Perhaps, but it seems unlikely. The lowest dimension gauge invariant fermion condensates involve four fermion fields — such as \( \langle \chi \chi \chi \psi \rangle \text{ or } \langle \langle \psi \chi \rangle \langle \chi \psi \rangle \rangle \) — which are all neutral under the \( U(1) \) symmetry. Furthermore, there are arguments that a Higgs phase would not be distinguishable from a confining phase for this theory.
Fig. 2.5 Integrating out a heavy fermion in three dimensions gives rise to the Chern Simons term in the effective action of eqn. (2.30).

heavy fermion of mass $m$. In 1+1 dimensions with an Abelian gauge field one computes the graph in Fig. 2.5 which gives rise to the Lagrangian

$$\mathcal{L}_{CS} = \frac{e^2}{8\pi} \frac{m}{|m|} \epsilon^{\alpha\beta\gamma} A_\alpha \partial_{\beta} A_\gamma .$$

What is interesting is that it implies a particle number current

$$J_\mu = \frac{\partial \mathcal{L}_{CS}}{\partial A_\mu} = \frac{e}{8\pi} \frac{m}{|m|} \epsilon^{\mu\alpha\beta} F_{\alpha\beta}$$

which we will see is related to the anomaly eqn. (2.4) in 1 + 1 dimensions.

**Exercise 2.4** Verify the coefficient in eqn. (2.30) by computing the diagram Fig. 2.5. By isolating the part that is proportional to $\epsilon_{\mu\nu\alpha} p^\alpha$ before performing the integral, one can make the diagram very easy to compute.
3

Domain Wall Fermions

3.1 Chirality, anomalies and fermion doubling

You have heard of the Nielsen-Ninomiya theorem: it states that a fermion action in $2k$ Euclidian spacetime dimensions

$$S = \int_{\pi/a}^{\pi/a} \frac{d^{2k}p}{(2\pi)^{2k}} \bar{\Psi}(p) D(p) \Psi(p)$$

cannot have the operator $\tilde{D}$ satisfy all four of the following conditions simultaneously:

1. $\tilde{D}(p)$ is a periodic, analytic function of $p_\mu$;
2. $D(p) \propto \gamma_\mu p_\mu$ for $a|p_\mu| \ll 1$;
3. $\tilde{D}(p)$ invertible everywhere except $p_\mu = 0$;
4. $\{\Gamma, \tilde{D}(p)\} = 0$.

The first condition is required for locality of the Fourier transform of $\tilde{D}(p)$ in coordinate space. The next two state that we want a single flavor of conventional Dirac fermion in the continuum limit. The last item is the statement of chiral symmetry. One can try keeping that and eliminating one or more of the other conditions; for example, the SLAC derivative took $\tilde{D}(p) = \gamma_\mu p_\mu$ within the Brillouin zone (BZ), which violates the first condition — if taken to be periodic, it is discontinuous at the edge of the BZ. This causes problems — for example, the QED Ward identity states that the photon vertex $\Gamma_\mu$ is proportional to $\partial \tilde{D}(p)/\partial p_\mu$, which is infinite at the BZ boundary. Naive fermions satisfy all the conditions except (3): there $\tilde{D}(p)$ vanishes at the $2^4$ corners of the BZ, and so we have $2^4$ flavors of Dirac fermions in the continuum. Staggered fermions are somewhat less redundant, producing four flavors in the continuum for each lattice field; Creutz fermions are the least redundant, giving rise to two copies for each lattice field. The discussion in any even spacetime dimension is analogous.

This roadblock in developing a lattice theory with chirality is obviously impossible to get around when you consider anomalies. Remember that anomalies do occur in the continuum but that in a UV cutoff on the number of degrees of freedom, there are no anomalies, and the exact symmetries of the regulated action are the exact symmetries of the quantum theory. The only way a symmetry current can have a nonzero divergence is if either the original action or the UV regulator explicitly violate that symmetry. The implication for lattice fermions is that any symmetry that is exact on the lattice will be exact in the continuum limit, while any symmetry anomalous in the continuum limit must be broken explicitly on the lattice.
A simple example to analyze is the case of a “naive” lattice action for a single RH fermion,

\[ S = \frac{1}{2a} \sum_{\mathbf{n}, \mu} \overline{\Psi}_+ (\mathbf{n}) \gamma_\mu [\Psi_+ (\mathbf{n} + \hat{\mu}) - \Psi_+ (\mathbf{n} - \hat{\mu})] \]

\[ = \frac{1}{2a} \sum_{\mathbf{p}, \mu} 2i \sin a p_\mu \overline{\Psi}_+ (-\mathbf{p}) \gamma_\mu \Psi_+ (\mathbf{p}) , \]

\[ \Gamma \Psi_+ = \Psi_+ , \quad \text{(3.2)} \]

so that \( \tilde{D}(\mathbf{p}) = i \gamma_\mu \sin ak_\mu/a \). This vanishes at every corner of the BZ; expanding about these points we write \( p_\mu = q_\mu + n_\mu \pi/a \) with \( n_\mu \in \{0, 1\} \) and \( aq_\mu \ll 1 \) and find

\[ \tilde{D}(\mathbf{p}) \simeq i \sum_\mu (-1)^{n_\mu} \gamma_\mu q_\mu , \quad \text{(3.3)} \]

These zeroes of \( \tilde{D}(\mathbf{p}) \) (at \( q_\mu = 0 \)) correspond to the \( 2^d \) doublers in \( d \)-dimensional Euclidean spacetime, violating condition (3) in the Nielsen-Ninomiya theorem. However, \( \tilde{D}(\mathbf{p}) \) does satisfy condition (4) and the action \( S \) is invariant under the symmetry

\[ \Psi_+(\mathbf{p}) \rightarrow e^{i\alpha} \Psi_+(\mathbf{p}) = e^{i\alpha \Gamma} \Psi_+(\mathbf{p}) , \quad \text{(3.4)} \]

which looks like a chiral symmetry — yet for a continuum theory with \( 2^d \) RH fermions, a phase symmetry would be anomalous, which we know cannot result from a symmetric lattice theory!

The resolution is that the continuum theory does not have \( 2^d \) RH fermions, but rather \( 2^{d-1} \) Dirac fermions, and the exact lattice \( U(1) \) symmetry corresponds to an exact fermion number symmetry in the continuum, which is not chiral and not anomalous. To show this, note that \( \tilde{D}(\mathbf{p}) \) in eqn. (3.3) has funny signs near the corners of the BZ. We can convert back to our standard gamma matrix basis using the similarity transformation \( P(n) \gamma_\mu P(n)^{-1} = (-1)^{n_\mu} \gamma_\mu \); but then \( P(n) \Gamma P(n)^{-1} = (-1)^{\sigma(n)} \Gamma \), where \( \sigma(n) = \sum_\mu n_\mu \) (since \( \Gamma \) is the product of all the \( \gamma^\mu \)). Therefore

\[ \Gamma[P(n)\Psi_+(\mathbf{q})] = (-1)^{\sigma(n)}[P(n)\Psi_+(\mathbf{q})] \quad \text{(3.5)} \]

and in the continuum we have \( 2^{d-1} \) RH fermions and \( 2^{d-1} \) LH fermions, and the exact and apparently chiral symmetry of the lattice corresponds to an exact and anomaly-free fermion number symmetry in the continuum. The redundancy of staggered and Creutz fermions serve the same purpose, ensuring that all exact lattice symmetries become anomaly-free vector symmetries in the continuum limit.

### 3.2 Domain wall fermions in the continuum

#### 3.2.1 Motivation

What we would like is a realization of chiral symmetry on the lattice which (i) is not exact, so we can correctly recover anomalies, but which (ii) retains all the good features of chiral symmetry in the continuum, such as protection from additive renormalization.
of fermion masses. There is a curious example in the continuum of such a system, which gave a clue on how to achieve this. The example has to do with fermions in odd dimension that interact with a domain wall. To be concrete, consider a system in three dimensions (coordinates \((x_0, x_1, x_2)\)), where the fermion has a mass which depends on \(x_2\) and switches sign at \(x_2 = 0\). For simplicity, I will take \(m(x_2) = mc(x_2) = mx_2/|x_2|\). Curiously enough, we will show that a massless fermion mode exists bound to this 2-dimensional surface, and that it is chiral: there exist a RH mode, and not a LH one. Thus the low energy limit of this theory looks like a 2-dimensional theory of a Weyl fermion with a chiral symmetry, even though we started with a 3-dimensional theory in which there can be no chirality.

Yet we know that the low energy effective theory is anomalous. Recall that for a massless Dirac fermion coupled to photons in two Euclidian dimensions, the vector current is conserved \((\partial_\mu J_\mu = 0)\), while the axial current is not in general \((\partial_\mu J_\mu^A = (e^2/\pi E))\). Thus the fermion current for a RH Weyl current satisfies

\[
\partial_\mu J_\mu^R = \frac{1}{2} \partial_\mu (J_\mu^R J_\mu^A) = \frac{e}{2\pi E} .
\]

If we turn on an electric field pointing in the \(x_1\) direction, then the charge on the mass defect must increase with time. Yet in the full 3-dimensional theory, there is only one fermion current \(J_\mu = \overline{\Psi} \gamma_\mu \Psi\), and it is conserved. So even though only massive states live off the mass defect, we see that they must somehow know about the anomaly and allow chiral symmetry to be violated as the anomaly requires.

You might be suspicious that there is some hidden fine tuning here to keep the chiral mode massless, but that cannot be: the low energy effective theory would need a LH mode as well in order for there to be a mass. Distortions of the domain wall mass function cannot change this result, unless the mass \(m(x_2)\) becomes small enough somewhere to change the spectrum of the low energy effective theory. But even in that case there is an index theorem that requires there to be a massless Weyl fermion as the sign of \(m(x_2)\) changes at an odd number of locations.

This looks like a useful trick to apply to the lattice: an anomalous chiral symmetry emerging at low energy from a full theory with no fundamental chiral symmetry, and without fine tuning. In this lecture I show how the continuum theory works, and then how it can be transcribed to the lattice. In the next lecture I will discuss how the effective theory can be described directly using the overlap formulation, without any reference to the higher dimensional parent theory.

3.2.2 The model

Even though fermions in even and odd dimensions look quite different, one finds an interesting connection between them when considering the Dirac equation with a space-dependent mass term. One can think of a space-dependent mass as arising from a Higgs mechanism, for example, where there is a topological defect trapped in the classical Higgs field, such as a domain wall or a vortex. A domain wall can naturally arise when the Higgs field breaks a discrete symmetry; a vortex when the Higgs field breaks a \(U(1)\) symmetry (see John Preskill’s lectures “Vortices and Monopoles” at the 1985 Les Houches Summer School (Preskill, 1985). Domain wall defects are pertinent to putting chiral fermions on the lattice, so I will consider that example.
Consider a fermion in Euclidian spacetime with dimension $d = 2k + 1$, where the coordinates are written as $\{x_0, x_1, \ldots, x_{2k-1}, s\} \equiv \{x_\mu, s\}$, where $\mu = 0, \ldots, 2k-1$ and $s$ is what I call the coordinate $x_{2k}$. The $(2k+1)$ $\gamma$ matrices are written as $\{\gamma_0, \ldots, \gamma_{2k}, \Gamma\}$. This fermion is assumed to have an $s$-dependent mass with the simple form

$$m(s) = m\epsilon(s) = \begin{cases} +m & s > 0, \\ -m & s < 0, \end{cases} \quad m > 0 \quad (3.7)$$

This mass function explicitly breaks the Poincaré symmetry of $2k+1$ dimensional spacetime, but preserves the Euclidian Poincaré symmetry of $2k$ dimensional spacetime. The fermion is also assumed to interact with $2k$-dimensional background gauge fields $A_\mu(x_\mu)$ which are independent of $s$. The Dirac equation may be written as:

$$[\not{D} + \Gamma \partial_s + m(s)] \Psi(x_\mu, s) = 0, \quad (3.8)$$

where $\not{D}$ is the lower dimension ($d = 2k$) covariant Dirac operator. The spinor $\Psi$ can be factorized as the product of a functions of $s$ times spinors $\psi_n(x_\mu)$,

$$\Psi(x_\mu, s) = \sum_n [b_n(s)P_+ + f_n(s)P_-] \psi_n(x_\mu), \quad P_\pm = \frac{1 \pm \Gamma}{2}, \quad (3.9)$$

satisfying the equations

$$[\partial_s + m(s)]b_n(s) = \mu_n f_n(s),$$

$$[-\partial_s + m(s)]f_n(s) = \mu_n b_n(s), \quad (3.10)$$

and

$$(\not{D} + \mu_n)\psi_n(x) = 0. \quad (3.11)$$

One might expect all the eigenvalues in eqn. (3.10) to satisfy $|\mu_n| \gtrsim O(m)$, since that is the only scale in the problem. However, there is also a solution to eqn. (3.10) with eigenvalue $\mu = 0$ given by

$$b_0 = Ne^{-\int_0^s m(s')ds'} = Ne^{-m|s|}. \quad (3.12)$$

This solution is localized near the defect at $s = 0$, falling off exponentially fast away from it. There is no analogous solution to eqn. (3.10) of the form

$$f_0 \sim e^{+ \int_0^s m(s')ds'},$$

since that would be exponentially growing in $|s|$ and not normalizable. Therefore as seen from eqn. (3.11) the spectrum consists of an infinite tower of fermions satisfying the $d = 2k$ Dirac equation: massive Dirac fermions with mass $O(m)$ and higher, plus a single massless right-handed chiral fermion. The massless fermion is localized at the defect at $s = 0$, whose profile in the transverse extra dimension is given by eqn. (3.12); the massive fermions are not localized. Because of the gap in the spectrum, at low energy the accessible part of the spectrum consists only of the massless RH chiral fermion.
Exercise 3.1 Construct a $d = 2k + 1$ theory whose low energy spectrum possesses a single light $d = 2k$ Dirac fermion with mass arbitrarily lighter than the domain wall scale $m$. There is more than one way to do this.

Some comments are in order:

- It is not a problem that the low energy theory of a single right-handed chiral fermion violates parity in $d = 2k$ since the mass for $\Psi$ breaks parity in $d = 2k + 1$;
- Furthermore, nothing is special about right-handed fermions, and a left handed mode would have resulted if we had chosen the opposite sign for the mass in eqn. (3.7). This makes sense because choosing the opposite sign for the mass can be attained by flipping the sign of all the space coordinates: a rotation in the $(2k + 1)$ dimensional theory, but a parity transformation from the point of view of the $2k$-dimensional fermion zeromode.
- The fact that a chiral mode appeared at all is a consequence of the normalizability of $\exp(-\int_0^s m(s')ds')$, which in turn follows from the two limits $m(\pm\infty)$ being nonzero with opposite signs. Any function $m(s)$ with that boundary condition will support a single chiral mode, although in general there may also be a number of very light fermions localized in regions wherever $|m(s)|$ is small — possibly extremely light if $m(s)$ crosses zero a number of times, so that there are widely separated defects and anti-defects.
- Gauge boson loops will generate contributions to the fermion mass function which are even in $s$. If the coupling is sufficiently weak, it cannot effect the masslessness of the chiral mode. However if the gauge coupling is strong, or if the mass $m$ is much below the cutoff of the theory, the radiative corrections could cause the fermion mass function to never change sign, and the chiral mode would not exist. Or it could still change sign, but become small in magnitude in places, causing the chiral mode to significantly delocalize. An effect like this can cause trouble with lattice simulations at finite volume and lattice spacing; more later.

3.3 Domain wall fermions and the Callan-Harvey mechanism

Now turn on the gauge fields and see how the anomaly works, following (Callan and Harvey, 1985). To do this, I integrate out the heavy modes in the presence of a background gauge field. Although I will be interested in having purely $2k$-dimensional gauge fields in the theory, I will for now let them be arbitrary $2k + 1$ dimensional fields. And since it is hard to integrate out the heavy modes exactly, I will assume perform the calculation as if their mass was constant, and then substitute $m(s)$; this is not valid where $m(s)$ is changing rapidly (near the domain wall) but should be adequate farther away. Also — in departure from the work of (Callan and Harvey, 1985), I will include a Pauli-Villars field with constant mass $M < 0$, independent of $s$; this is necessary to regulate fermion loops in the wave function renormalization for the gauge fields, for example.

When one integrates out the heavy fields, one generates a Chern Simons operator in the effective Lagrangian, as discussed in §2.4
32 \textit{Domain Wall Fermions}

\[ \mathcal{L}_{CS} = \left( \frac{m(s)}{|m(s)|} + \frac{M}{|M|} \right) \mathcal{O}_{CS} = (\epsilon(s) - 1) \mathcal{O}_{CS} \]  

(3.13)

Note that with \( M < 0 \), the coefficient of the operator equals \(-2\) on the side where \( m(s) \) is negative, and equals zero on the side where it is positive. For a background \( U(1) \) gauge field one finds in Euclidian spacetime:

\[ d = 3 : \quad \mathcal{O}_{CS} = -\frac{e^2}{8\pi} \epsilon_{abc}(A_a \partial_a A_c) , \]  

(3.14)

\[ d = 5 : \quad \mathcal{O}_{CS} = -\frac{e^3}{48\pi^2} \epsilon_{abcde}(A_a \partial_b A_c \partial_d A_e) . \]  

(3.15)

Differentiating \( \mathcal{L}_{CS} \) by \( A_\mu \) and dividing by \( e \) gives the particle number current:

\[ J^{(CS)}_a = (\epsilon(s) - 1) \begin{cases} -\frac{e}{8\pi} \epsilon_{abc}(F_{bc}) & d = 3 \\ -\frac{e^2}{64\pi^2} \epsilon_{abcde}(F_{bc}F_{de}) & d = 5 \end{cases} \]  

(3.16)

where I use Latin letters to denote the coordinates in \( 2k + 1 \) dimensions, while Greek letters will refer to indices on the \( 2k \)-dimensional defect. So when we turn on background \( 2k \) dimensional gauge fields, particle current flows either onto or off of the domain wall along the transverse \( s \) direction on the left side (where \( m(s) = -m \)). If we had regulated with a positive mass Pauli Villars field, the current would flow on the right side. But in either case, this bizarre current exactly accounts for the anomaly. Consider the case of a \( 2 \)-dimensional domain wall embedded in \( 3 \)-dimensions. If we turn on an \( E \) field we know that from the point of view of a \( 2 \)-d creature, RH Weyl particles are created, where from eqn. (2.14),

\[ \partial_\mu J^{(CS)}_{\mu R} = \frac{1}{2} \partial_\mu J_{\mu A} = \frac{e}{4\pi} \epsilon_{\mu \nu} F_{\mu \nu} . \]  

(3.17)

We see from eqn. (3.16) this current is exactly compensated for by the Chern Simons current \( J^{(CS)}_2 = \frac{e}{4\pi} \epsilon_{2\mu\nu} F_{\mu\nu} \) which flows onto the domain wall from the \( -s = -x_2 \) side. The total particle current is divergenceless.

This is encouraging: (i) we managed to obtain a fermion whose mass is zero due to topology and not fine tuning; (ii) the low energy theory therefore has a chiral symmetry even though the full \( 3 \)-d theory does not; (iii) the only remnant of the explicit chiral symmetry breaking of the full theory is the anomalous divergence of the chiral symmetry in the presence of gauge fields. One drawback though is the infinite dimension in the \( s \) direction, since we will eventually want to simulate this on a finite lattice; besides, it is always disturbing to see currents streaming in from \( s = -\infty \)!

One solution is to work in finite \((2k + 1)\) dimensions, in which case we end up with a massless RH mode stuck to the boundary on one side and a LH mode on the other (which is great for a vector like theory of massless Dirac fermions, but not for chiral gauge theories). This is what one does when simulating domain wall fermions. The other solution is more devious, leads to the “overlap operator”, and is the subject of another day’s lecture.
3.3.1 Domain wall fermions on a slab

To get a better understanding for how the theory works, it is useful to consider a compact extra dimension. In particular, consider the case of periodic boundary conditions \( \Psi(x_\mu, s + 2s_0) = \Psi(x_\mu, s) \); we define the theory on the interval \(-s_0 \leq s \leq s_0\) with boundary conditions \( \Psi(x_\mu, -s_0) = \Psi(x_\mu, s_0) \) and mass \( m(s) = m_{s_0}^s \). Note the the mass function \( m(s) \) now has a domain wall kink at \( s = 0 \) and an anti-kink at \( s = \pm s_0 \). There are now two exact zeromode solutions to the Dirac equation,

\[
\begin{align*}
    b_0(s) &= Ne^{-\int_{-s_0}^{s_0} m(s') ds'}, \\
    f_0(s) &= Ne^{+\int_{-s_0}^{s_0} m(s') ds'}.
\end{align*}
\]

Both solutions are normalizable since the transverse direction is finite; \( b_0 \) corresponds to a right-handed chiral fermion located at \( s = 0 \), and \( f_0 \) corresponds to a left-handed chiral fermion located at \( s = \pm s_0 \). However, in this case the existence of exactly massless modes is a result of the fact that \( \int_{-s_0}^{s_0} m(s) ds = 0 \) which is not a topological condition and not robust. For example, turning on weakly coupled gauge interactions will cause a shift the mass by \( \delta m(s) \propto \alpha m \) (assuming \( m \) is the cutoff) which ruins this property. However: remember that to get a mass in the \( 2k \)-dimensional defect theory, the RH and LH chiral modes have to couple to each other. The induced mass will be

\[
\delta \mu_0 \sim \delta m \int ds b_0(s) f_0(s) = \delta m N^2 \sim \alpha m \times \frac{2m_{s_0}}{\cosh[ms_0]} \equiv m_{\text{res}}
\]

which vanishes exponential fast as \( (Ms_0) \rightarrow \infty \). Nevertheless, at finite \( s_0 \) there will always be some chiral symmetry breaking, in the form of a residual mass, called \( m_{\text{res}} \). If, however, one wants to work on a finite line segment in the extra dimension instead of a circle, we can take an asymmetric mass function,

\[
m(s) = \begin{cases} 
    -m & -s_0 \leq s \leq 0 \\
    +\infty & 0 < s < s_0
\end{cases}
\]

This has the effect of excluding half the space, so that the extra dimension has boundaries at \( s = -s_0 \) and \( s = 0 \). Now even without extra interactions, one finds

\[
m_{\text{res}} \sim 2me^{-2ms_0}
\]

Any matrix element of a chiral symmetry violating operator will be proportional to the overlap of the LH and RH zeromode wave functions, which is proportional to \( m_{\text{res}} \). On the lattice the story of \( m_{\text{res}} \) is more complicated — as discussed in §3.4 — both because of the discretization of the fermion action, and because of the presence of rough gauge fields. Lattice computations with domain wall fermions need to balance the cost of simulating a large extra dimension versus the need to make \( m_{\text{res}} \) small enough to attain chiral symmetry.

3.3.2 The (almost) chiral propagator

Before moving to the lattice, I want to mention an illuminating calculation by Lüscher (Lüscher, 2000a) who considered noninteracting domain wall fermions with a semi-infinite fifth dimension, negative fermion mass, and and LH Weyl fermion zeromode
bound to the boundary at $s = 0$. He computed the Green function for propagation of the zeromode from $(x, s = 0)$ to $(y, s = 0)$ and examined the chiral properties of this propagator. The differential operator to invert should be familiar now:

$$D_5 = \partial_4 + \gamma_5 \partial_s - m , \quad s \geq 0 .$$  \hfill (3.22)

We wish to look at the Green function $G$ which satisfies

$$D_5 G(x, s; y, t) = \delta^{(4)}(x - y) \delta(s - t) , \quad P_+ G(x, 0; y, t) = 0 .$$  \hfill (3.23)

The solution Lüscher found for propagation along the boundary was

$$G(x, s; y, t) \bigg|_{s=t=0} = 2 P_- D^{-1} P_+ ,$$  \hfill (3.24)

where $D$ is the peculiar looking operator

$$D = [1 + \gamma_5 \epsilon(H)] , \quad H = \gamma_5(\partial_4 - m) = H^\dagger , \quad \epsilon(\mathcal{O}) = \frac{\mathcal{O}}{\sqrt{\mathcal{O}^\dagger \mathcal{O}}} .$$  \hfill (3.25)

This looks pretty bizarre! Since $H$ is hermitian, in a basis where $H$ is diagonal, $\epsilon(H) = \pm 1$! But don’t conclude that in this basis the operator is simply $D = (1 \pm \gamma_5)$ — you must remember, that in the basis where $H$ is diagonal, $\epsilon(H) \gamma_5$ is not (by which I mean $\langle m | \epsilon(H) \gamma_5 | n \rangle$ is in general nonzero for $m \neq n$ in the $H$ eigenstate basis). In fact, eqn. (3.25) looks very much like the overlap operator discovered some years earlier and which we will be discussing soon.

A normal Weyl fermion in four dimensions would have a propagator $P_- (\partial_4)^{-1} P_+$; here we see that the domain wall fermion propagator looks like the analogous object arising from the fermion action $\bar{\Psi} D \Psi$, with $D$ playing the role of the four-dimensional Dirac operator $\partial_4$. So what are the properties of $D$?

- For long wavelength modes (e.g. $k \ll m$) we can expand $D$ in powers of $\partial_4$ and find

$$D = \frac{1}{m} \left( \partial_4 - \frac{\partial_4^2}{2m} + \ldots \right) ,$$  \hfill (3.26)

which is reassuring: we knew that at long wavelengths we had a garden variety Weyl fermion living on the boundary of the extra dimension (the factor of $1/m$ is an unimportant normalization).

- A massless Dirac action is chirally invariant because $\{\gamma_5, \partial_4\} = 0$. However, the operator $D$ does not satisfy this relationship, but rather:

$$\{\gamma_5, D\} = D \gamma_5 D ,$$  \hfill (3.27)

or equivalently,

$$\{\gamma_5, D^{-1}\} = \gamma_5 .$$  \hfill (3.28)

This is the famous Ginsparg-Wilson equation, first introduced in context of the lattice (but not solved) many years ago \[Ginsparg and Wilson, 1982\]. Note the
right hand side of the above equations encodes the violation of chiral symmetry that our Weyl fermion experience; the fact that the right side of eqn. (3.28) is local in spacetime implies that violations of chiral symmetry will be seen in Green functions only when operators are sitting at the same spacetime point. We know from our previous discussion, the only chiral symmetry violation that survives to low energy in the domain wall model is the anomaly, and so it must be that the chiral symmetry violation in eqns. (3.27)-(3.28) encode the anomaly and nothing else, at low energy.

3.4 Domain wall fermions on the lattice

The next step is to transcribe this theory onto the lattice. If you replace continuum derivatives with the usual lattice operator $D \to \frac{1}{2}(\nabla^* + \nabla)$ (where $\nabla$ and $\nabla^*$ are the forward and backward lattice difference operators respectively) then one discovers... doublers! Not only are the chiral modes doubled in the $2k$ dimensions along the domain wall, but there are two solutions for the transverse wave function of the zero mode, $b_0(s)$, one of which alternates sign with every step in the $s$ direction and which is a LH mode. So this ends up giving us a theory of naive fermions on the lattice, only in a much more complicated and expensive way!

However, when we add Wilson terms $\frac{r}{2}\nabla^*\nabla$ for each of the dimensions, things get interesting. You can think of these as mass terms which are independent of $s$ but which are dependent on the wave number $k$ of the mode, vanishing for long wavelength. What happens of we add a $k$-dependent spatially constant mass $\Delta m(k)$ to the step function mass $m(s) = m\epsilon(s)$? The solution for $b_0(s)$ in eqn. (3.12) for an infinite extra dimension becomes

$$b_0 = Ne^{-\int_0^s[m(s') + \Delta m(k)]ds'},$$

(3.29)

which is a normalizable zeromode solution — albeit, distorted in shape — so long as $|\Delta m(k)| < m$. However, for $|\Delta m(k)| > m$, the chiral mode vanishes. What happens to it? It becomes more and more extended in the extra dimension until it ceases to be normalizable. What is going on is easier to grasp for a finite extra dimension: as $|\Delta m(k)|$ increases with increasing $k$, eventually the $b_0$ zeromode solution extends to the opposing boundary of the extra dimension, when $|\Delta m(k)| \sim (m - 1/s_0)$. At that point it can pair up with the LH mode and become heavy.

So the idea is: add a Wilson term, with strength such that the doublers at the corners of the Brillouin zone have $|\Delta m(k)|$ too large to support a zeromode solution. Under separation of variables, one looks for zeromode solutions with $\Psi(x, s) = e^{ipx}\phi_\pm(s)\psi_\pm$ with $\Gamma\psi_\pm = \pm\psi$. One then finds (for $r = 1$)

$$\Psi_4\psi_\pm = 0, \quad -\phi_\pm(s \mp 1) + (m_{\text{eff}}(s) + 1)\phi_\pm(s) = 0,$$

(3.30)

1A lattice solution to eqn. (3.27) (the only solution in existence) is the overlap operator discovered by Neuberger [Neuberger, 1998a, Neuberger, 1998b]; it was a key reformulation of earlier work [Narayanan and Neuberger, 1993, Narayanan and Neuberger, 1995] on how to represent domain wall fermions with an infinite extra dimension (and therefore exact chiral symmetry) in terms of entirely lower dimensional variables. We will discuss overlap fermions and the Ginsparg-Wilson equation further in the next lecture.
Fig. 3.1 Domain wall fermions in $d = 2$ on the lattice: dispersion relation plotted in the Brillouin zone. Chiral modes exist in white regions only. For $0 < |m/r| < 2$ there exists a single RH mode centered at $(k_1, k_2) = (0, 0)$. For $2 < |m/r| < 4$ there exist two LH modes centered at $(k_1, k_2) = (\pi, 0)$ and $(k_1, k_2) = (0, \pi)$; for $4 < |m/r| < 6$ there exists a single RH mode centered at $(k_1, k_2) = (\pi, \pi)$. For $|m/r| > 6$ there are no chiral mode solutions.

where

$$m_{\text{eff}}(s) = m\epsilon(s) + \sum_{\mu} (1 - \cos p_{\mu}) \equiv m\epsilon(s) + F(p).$$ \hspace{1cm} (3.31)$$

Solutions of the form $\phi_{\pm}(s) = z_{\pm}^s$ are found with

$$z_{\pm} = (1 + m_{\text{eff}}(s))^{\mp 1} = (1 + m\epsilon(s) + F(p))^{\mp 1};$$ \hspace{1cm} (3.32)

they are normalizable if $|z'^{(s)}_{\epsilon} < 1$. Solutions are found for $\psi_+$ only, and then provided that $m \in$ is the range $F(p) < m < F(p) + 2$. (For $r \neq 1$, this region is found by replacing $m \rightarrow m/r$.) However, even though the solution is only found for $\psi_+$, the chirality of the solutions will alternate with corners of the Brillouin zone, just as we found for naive fermions, eqn. (3.5). The picture for the spectrum in 2d is shown in Fig. 3.1

It was first shown in (Kaplan, 1992) that doublers could be eliminated for domain wall fermions on the lattice; the rich spectrum in Fig. 3.1 was worked out in (Jansen and Schmaltz, 1992), where for 4d they found the number of zeromode solutions to be the Pascal numbers $(1, 4, 6, 4, 1)$ with alternating chirality, the critical values for $|m/r|$ being $0, 2, \ldots, 10$. One implication of their work is that the Chern Simons currents must also change discontinuously on the lattice at these critical values of $|m/r|$; indeed that is the case, and the lattice version of the Callan-Harvey mechanism was verified analytically in (Golterman, Jansen and Kaplan, 1993).

Fig. 3.1 suggests that chiral fermions will exist in two spacetime dimensions so long as $0 < |m/r| < 6$, with critical points at $|m/r| = 0, 2, \ldots, 6$ where the numbers of massless flavors and their chiralities change discontinuously. In four spacetime dimensions a similar calculation leads to chiral fermions for $0 < |m/r| < 10$ with critical points at $|m/r| = 0, 2, \ldots, 10$. However, this reasoning ignores the gauge fields. In perturbation theory one would expect the bulk fermions to obtain a radiative mass correction of size $\delta m \sim O(\alpha)$ in lattice units, independent of the extra dimension $s$. Extrapolating shamelessly to strong coupling, one then expects the domain wall form

\[ \text{...} \]
of the mass to be ruined when $\alpha \sim 1$ for $|m/r| \sim (2n + 1)$, $n = 0, \ldots, 4$ causing a loss of chiral symmetry; near the critical points in $|m/r|$ the critical gauge coupling which destroys chiral symmetry will be smaller.

While qualitatively correct, this argument ignores the discrete nature of the lattice. On the lattice, the exponential suppression $m_{\text{res}} \sim \exp(-2ms_0)$ found in eqn. (3.19) is replaced by $\hat{T}^{L_s} = \exp(-L_s\hat{h})$, where $\hat{T}$ is a transfer matrix in the fifth dimension which is represented by $L_s$ lattice sites. Good chiral symmetry is attained when $\hat{h}$ exhibits a “mass gap”, i.e. when all its eigenvalues are positive and bounded away from zero. However one finds that at strong coupling, rough gauge fields can appear which give rise to near zero-modes of $\hat{h}$, destroying chiral symmetry, with $m_{\text{res}} \propto 1/L_s$ (Christ, 2006; Antonio et al., 2008). To avoid this problem, one needs to work at weaker coupling and with an improved gauge action which suppresses the appearance of rough gauge fields.

At finite lattice spacing the phase diagram is expected to look something like in Fig. 3.2 where I have plotted $m$ versus $g^2$, the strong coupling constant. On this diagram, $g^2 \to 0$ is the continuum limit. Domain wall fermions do not require fine tuning so long as the mass is in one of the regions marked by an “X”, which yield $\{1, 4, 6, 4, 1\}$ chiral flavors from left to right. The shaded region is a phase called the Aoki phase (Aoki, 1984); it is presently unclear whether the phase extends to the continuum limit (left side of Fig. 3.2) or not (right side) (Golterman, Sharpe and Singleton, 2005). In either case, the black arrow indicates how for Wilson fermions one tunes the mass from the right to the boundary of the Aoki phase to obtain massless pions and chiral symmetry; if the Aoki phase extends down to $g^2 = 0$ than the Wilson program will work in the continuum limit, but not if the RH side of Fig. 3.2 pertains. See (Golterman and Shamir, 2000) (Golterman and Shamir, 2003) for a sophisticated discussion of the physics behind this diagram.

Of course, in the real world we do not see exact chiral symmetry, since quarks and leptons do have mass. A mass for the domain wall fermion can be included as a
coupling between the LH mode at $s = 1$, and the RH mode at $s = N_s$:

$$m_q \left[ \overline{\psi}(x, 1) P_+ \psi(x, N_s) + \overline{\psi}(x, N_s) P_- \psi(x, 1) \right]$$  \hspace{1cm} (3.33)

and correlation functions are measured by sewing together propagators from one boundary to itself for chiral symmetry preserving operators, or from one boundary to the other for operators involving a chiral flip. The latter will require insertions of the mass operator above to be nonzero (assuming a negligible $m_{\text{res}}$) — just like it should be in the continuum.

### 3.4.1 Shamir’s formulation

Domain wall fermions are used by a number of lattice collaborations these days, using the formulation of Shamir (Shamir, 1993; Furman and Shamir, 1995), which is equivalent to the continuum version of domain wall fermions on a slab described above. The lattice action is given by:

$$\sum_{b=1}^{5} \sum_{x} \sum_{s=1}^{N_s} \left[ \frac{1}{2} \overline{\psi} (\partial^b \overline{\psi} + \partial^b \overline{\psi}) - m \overline{\psi} \psi - \frac{r}{2} \overline{\psi} \partial^b \partial^b \psi \right]$$ \hspace{1cm} (3.34)

where the lattice coordinate on the 5d lattice is $n = \{x, s\}$, $x$ and $s$ being the 4d and fifth dimension lattice coordinates respectively. The difference operators are

$$\partial^b \psi(n) = \psi(n + \hat{\mu}_b) - \psi(n) , \quad \partial^b \psi(n) = \psi(n) - \psi(n - \hat{\mu}_b)$$ \hspace{1cm} (3.35)

where $\hat{\mu}_b$ is a unit vector in the $x_b$ direction. In practice of course, these derivatives are gauged in the usual way by inserting gauge link variables. The boundary conditions are defined by setting fields to zero on sites with $s = 0$ and $s = N_s + 1$. I have reversed the sign of $m$ and $r$ from Shamir’s original paper, since the above sign for $r$ appears to be relatively standard now. For domain wall fermions, $m$ has the opposite sign from standard Wilson fermions, which is physics, not convention. The above action gives rise to a RH chiral mode bound to the $s = 1$ boundary of the lattice, and a LH chiral mode bound at the $s = N_s$ boundary.

### 3.4.2 The utility of domain wall fermions

Theoretically, chiral symmetry can be as good a symmetry as one desires if one is close enough to the continuum limit (to avoid delocalization of the zeromode due to large gauge field fluctuations) and large extra dimension. In practical simulations, the question is whether the residual mass term can be small enough to warrant the simulation cost. This was reviewed dispassionately and at length in (Sharpe, 2007), and I refer you to that article if you are interested in finding out the details. Currently, domain wall fermions are being extensively applied to QCD; for some diverse examples from the past year see (Yamazaki, 2009; Chiu, Hsieh and Tseng, 2009b; Gaiav and Sharma, 2009; Torok et al., 2009; Ohta, RBC and Collaborations, 2009; Cheng et al., 2009; Chiu et al., 2009a), and a recent overview (Jansen, 2008). Another recent application has been to $N = 1$ supersymmetric Yang-Mills theory (Giedt, Brower, Catterall, Fleming and Vranas, 2008; Giedt, Brower, Catterall, Fleming and Vranas, 2009b).
2008; Endres, 2009a; Endres, 2009b) based on a domain wall formulation for Majorana fermions (Kaplan and Schmaltz, 2000) and earlier numerical work (Fleming, Kogut and Vranas, 2001).
Overlap fermions and the Ginsparg-Wilson equation

4.1 Overlap fermions

We have seen that the low energy limit of a domain wall fermion in the limit of large extra dimension is a single massless Dirac fermion, enjoying the full extent of the chiral symmetry belonging to massless fermions in the continuum. In this low energy limit, the effective theory is four-dimensional if the original domain wall fermion lived in five dimensions. One might wonder whether one could dispense with the whole machinery of the extra dimension and simply write down the low energy four-dimensional theory to start with. Furthermore, one would like a four dimensional formulation with exact chiral symmetry, which could only occur for domain wall fermions with infinite extent in the time direction, which is not very practical numerically!

Neuberger and Narayanan found an extremely clever way to do this, leading to the four-dimensional “overlap operator” which describes lattice fermions with perfect chiral symmetry. The starting point is to consider a five dimensional fermion in the continuum with a single domain wall, and to consider the fifth dimension to be time (after all, it makes no difference in Euclidian space). Then $\gamma_5(D_4 + m(s))$ looks like the Hamiltonian, where $s$ is the new time coordinate, and $m(-\infty) = -m_1$, $m(\infty) = +m_2$, where $m_1,2 > 0$. The path integral projects onto ground states, and so the partition function for this system is $Z = \langle \Omega, -m_1 | \Omega, +m_2 \rangle$, where the state $|0, m\rangle$ is the ground state of $H_4(m) = \gamma_5(D_4 + m)$. We know that this should describe a massless Weyl fermion. Note that the partition function is in general complex with an ill-defined phase (we can redefine the phase of $|\Omega, -m_1\rangle$ and $|\Omega, +m_2\rangle$ separately and arbitrarily).

If we now instead imagine that the fermion mass function $m(s)$ exhibits a wall-antiwall pair, with the two defects separated infinitely far apart, we recognize a system that will have a massless Dirac fermion in the spectrum, and $Z = |\langle \Omega, -m_1 | \Omega, +m_2 \rangle|^2$, which is real, positive, and independent on how we chose the phase for the groundstates.

We can immediately transcribe this to the lattice, where we replace $D_4$ with the four dimensional Wilson operator,

$$H(m) = \gamma_5(D_w + m) = \gamma_5 \left( D_\mu \gamma_\mu - \frac{r}{2} D_\mu^2 + m \right)$$

(4.1)

with $D_\mu$ being the symmetric covariant derivative on the lattice, and $D_\mu^2$ being the covariant lattice Laplacian. Note that $H(m)$ is Hermitian, and so its eigenvalues are real. Furthermore, one can show that it has equal numbers of positive and negative eigenvalues.
We can account for the \( \gamma \)-matrix structure of \( \mathcal{H}(m) \) explicitly in a chiral basis where \( \gamma_5 = \sigma_3 \otimes 1 \):

\[
\mathcal{H}(m) = \begin{pmatrix} B + m & C \\ C^\dagger & -B - m \end{pmatrix} \tag{4.2}
\]

where \( B = -\frac{\tau}{2} \nabla^2 \) is the Wilson operator and \( C = D \sigma_\mu \) where \( \sigma_\mu = \{ i, \vec{\sigma} \} \). For simplicity for \( \langle \Omega, m_2 | \Omega, -m_2 \rangle \) will represent a massless Dirac fermion on the lattice, so long as \( 0 < m_1 < 2r \), with \( m_2 \) arbitrary. The groundstates of interest may be written as Slater determinants of all the one-particle wave functions with negative energy. Let us designate the one-particle energy eigenstates of \( \mathcal{H}(-m_1) \) and \( \mathcal{H}(m_2) \) to be \( |n, -m_1\rangle \) and \( |n, m_2\rangle \) respectively, with

\[
\langle n, m_2 | n', -m_1 \rangle \equiv U_{nn'} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}_{nn'}, \quad U^\dagger U = 1 , \tag{4.3}
\]

where the block structure of \( U \) is in the same \( \gamma \)-matrix space that we introduced in writing \( \mathcal{H} \) in block form, eqn. (4.2). Now, we want to only fill negative energy eigenstates, so it is convenient to introduce the sign function

\[
\varepsilon(\lambda) \equiv \frac{\lambda}{\sqrt{\lambda^\dagger \lambda}} . \tag{4.4}
\]

With \( m_2 \to \infty \) we have

\[
\varepsilon(\mathcal{H}(m_2)) \xrightarrow{m_2 \to \infty} \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{4.5}
\]

Assuming \( \mathcal{H}(-m_1) \) has no exact zeromodes then, it follows that all eigenvalues come in \( \pm \) pairs (just like the operator \( \gamma_5 \)) and we can choose our basis \( |n, -m_1\rangle \) so that

\[
\varepsilon(\mathcal{H}(m_2)) = U \gamma_5 U^\dagger = U \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} U^\dagger . \tag{4.6}
\]

Therefore the Slater determinant we want is

\[
Z = |\langle \Omega, m_2 | \Omega, -m_1 \rangle|^2 = |\det U_{22}|^2 = |\det \delta| \det \delta = \det \left( 1 + \gamma_5 \varepsilon(\mathcal{H}(-m_1)) \right) . \tag{4.7}
\]

Some steps have been omitted from this derivation (Narayanan, 2001); see exercise 4.2.

**Exercise 4.1** Prove the assertion that if \( \mathcal{H}(-m_1) \) has no zeromodes, it has equal numbers of positive and negative eigenvalues.
42 Overlap fermions and the Ginsparg-Wilson equation

Exercise 4.2 You should prove the last step in eqn. (4.7), breaking it down to the following steps:

(a) Show that \( \det \delta^\dagger = \det \alpha \det U^\dagger \);

(b) ...so that \( \det \delta^\dagger \det \delta = \det \delta \det \alpha \det U^\dagger = \det \left[ \frac{1}{2} (U + \gamma_5 U \gamma_5) U^\dagger \right] \);

(c) ...which combines with eqn. (4.6) to yield eqn. (4.7).

On the other hand, \( Z \propto \det D \), where \( D \) is the fermion operator. So we arrive at the overlap operator (dropping the subscript from \( m_1 \)):

\[
D = 1 + \gamma_5 \varepsilon (H(-m)) \\
= 1 + \gamma_5 \frac{H(-m)}{\sqrt{H(-m)^2}} \\
= 1 + \frac{D_w - m}{\sqrt{(D_w - m)(D_w - m)}} .
\]  (4.8)

a remarkable result. It was subsequently shown explicitly that this fermion operator can be derived directly from lattice domain wall fermions at infinite wall separation [Neuberger, 1998c; Kikukawa and Noguchi, 1999]. Recall from our discussion of domain wall fermions that at least for weak gauge fields, we need \( 0 < m < 2r \) in order to obtain one flavor of massless Dirac fermion (where I have set the lattice spacing \( a = 1 \)).

4.1.1 Eigenvalues of the overlap operator

Recall that the eigenvalues of the Dirac operator in the continuum are \( \pm i \lambda_n \) for real nonzero \( \lambda_n \), plus \( n_+ \) RH and \( n_- \) LH zero modes, where the difference is constrained by the index theorem to equal the topological winding number of the gauge field. Thus the spectrum looks like a line on the imaginary axis. What does the spectrum of the overlap operator look like? Consider

\[
(D - 1) \cdot (D - 1) = \epsilon (H)^2 = 1 .
\]  (4.9)

Thus \( (D - 1) \) is a unitary matrix and the eigenvalues of \( D \) are constrained to lie on a circle of unit radius in the complex plane, with the center of the circle at \( z = 1 \). If you put the lattice spacing back into the problem, \( D \rightarrow aD \) in the above expression to get the dimensions right, and so the eigenvalues sit on a circle of radius \( 1/a \) centered at \( 1/a \). Thus, as \( a \rightarrow 0 \) the circle gets bigger, and the eigenvalues with small magnitude almost lie on the imaginary axis, like the continuum eigenvalues. See the problem below, where you are to show that the eigenfunctions of \( D \) with real eigenvalue are chiral.

4.1.2 Locality of the overlap operator

If just presented with the overlap operator eqn. (4.7) without knowing how it was derived, one might worry that its unusual structure could entail momentum space
singualarities corresponding to unacceptable nonlocal behavior in coordinate space. (From its derivation from domain wall fermions this would be very surprising for sufficiently weakly coupled gauge fields, since the domain wall theory looks well defined and local with a mass gap.) The locality of the overlap operator (i.e. that it falls off exponentially in coordinate space) was proven analytically in [Hernandez, Jansen and Luscher, 1999], under the assumption of sufficiently smooth gauge link variables, namely that $|1 - U| < 1/30$. They also claimed numerical evidence for locality that was less restrictive.

4.1.3 The value of $m$ and the number of fermions

For domain wall fermions we found the interesting phase structure as a function of $m/r$, where in the intervals between the critical values $m/r = \{0, -2, -4, \ldots, -2(2k + 1)\}$ there were $\{1, 2k, \ldots, 1\}$ copies of chiral fermions with alternating chirality, as shown in Fig. 3.1. One would expect something analogous then for the overlap operator, since it is equivalent to a domain wall fermion on a $2k$-dimensional lattice with infinite continuous extra dimension. The equation of motion for the domain wall modes is slightly different than found in eqn. (3.30) due to the continuous dimension:

$$\gamma_{2k} \psi_\pm = 0, \quad \pm \phi'_\pm(s) + (m_{\text{eff}}(s) + 1)\phi_\pm(s) = 0,$$

where

$$m_{\text{eff}}(s) = m\epsilon(s) + r \sum_\mu (1 - \cos p_\mu) \equiv m\epsilon(s) + rF(p).$$

Solutions are of the form $\phi_\pm(s) = e^{\mp \int m_{\text{eff}}(t) dt}$. With $rF(p) > 0$, $\phi_-$ is never normalizable, while for $\phi_+$, normalizability requires $m/r > F(p)$. Thus the overlap operator eqn. (4.8) represents $\{1, 1 + 2k, \ldots, 2^2k\}$ massless Dirac fermions for $m/r$ in the intervals $(0, 2), (2, 4), \ldots, (4k, \infty)$. In $2k = 4$ dimensions, these flavor numbers are $\{1, 5, 11, 15, 16\}$.

4.1.4 Simulating the overlap operator

The overlap operator has exact chiral symmetry, in the sense that it is an exact solution to the Ginsparg Wilson relation, which cannot be said for domain wall fermions at finite $N_s$; furthermore, it is a four-dimensional operator, which would seem to be easier to simulate than a 5d theory. However, the inverse square root of an operator is expensive to compute, and requires some approximations. The algorithms for computing it are described in detail in an excellent review by A. Kennedy [Kennedy, 2006]. Amusingly, he explains that the method for computing the overlap operator can be viewed as simulating a five-dimensional theory, albeit one with more general structure than the domain wall theory. For a recent review comparing the computational costs of different lattice fermions, see the recent review [Jansen, 2008].

**Exercise 4.3** Show that the overlap operator in eqn. (4.8) has the following properties:

(a) At zero gauge field and acting on long wavelength fermion modes, $D \simeq \frac{\partial}{\partial s}$, the ordinary Dirac operator for a massless fermion.
(b) It satisfies the Ginsparg-Wilson equation, eqn. \(3.27\):
\[
\{\gamma_5, D\} = D\gamma_5 D .
\]  
(4.12)

Exercise 4.4
(a) Show that one can write \(D = 1 + V\) where \(V^\dagger V = 1\), and that therefore \(D\) can be diagonalized by a unitary transformation, with its eigenvalues lying on the circle \(z = 1 + e^{i\phi}\).

(b) Show that, despite \(D\) being non-hermitian, normalized eigenstates satisfying \(D|z\rangle = z|z\rangle\) with different eigenvalues are orthogonal, satisfying \(\langle z'|z\rangle = \delta_{z'z}\)

(c) Show that if \(D|z\rangle = z|z\rangle\) then \(D^\dagger|z\rangle = z^*|z\rangle\)

(d) Assuming that \(\gamma_5 D\gamma_5 = D^\dagger\), show that \(\langle z|\gamma_5|z\rangle = 0\) unless \(z = 0\) or \(z = 2\), in which case \(\langle z|\gamma_5|z\rangle = \pm 1\)

4.2 The Ginsparg-Wilson equation and its consequences

In 1982 Paul Ginsparg and Kenneth Wilson wrote a paper about chiral lattice fermions which was immediately almost completely forgotten, accruing 10 citations in the first ten years and none in the subsequent five; today it is marching toward 700 citations. The reason for this peculiar history is that they wrote down an equation they speculated should be obeyed by a fermion operator in the fixed point action of a theory tuned to the chiral point — but they did not solve it. After domain wall and overlap fermions were discovered in the early 1990s, it was realized that they provided a solution to this equation (the domain wall solution only being exact in the limit of infinite extra dimension). Shortly afterward, M. Lüscher elaborated on how the salient features of chirality flowed from the Ginsparg-Wilson equation — in particular, how anomalies and multiplicative mass renormalization were consequences of the equation, which provided a completely explicit four-dimensional explanation for the success of the overlap and domain wall fermions.

4.2.1 Motivation

A free Wilson fermion with its mass tuned to the critical value describes a chiral fermion in the continuum. As we have seen, chiral symmetry does not exist on the lattice, but its violation is not evident at low energy, except through correctly reproducing the anomaly. However, imagine studying this low energy effective theory by repeatedly performing block spin averages. One would eventually have a lattice theory with all the properties one would desire: chiral fermions and chiral anomalies. What is the fermion operator in this low energy theory, and how does it realize chiral symmetry? Motivated by this question, Ginsparg and Wilson performed a somewhat simpler calculation: they took a continuum theory with chiral symmetry and anomalies, and performed a average of spacetime cells to create a lattice theory, and asked how the chiral symmetry in the original theory was expressed in the resulting lattice theory.
The starting point is the continuum theory

\[ Z = \int [d\psi] [d\bar{\psi}] e^{-S(\psi, \bar{\psi})} \]  

(4.13)

I assume there are \( N_f \) identical flavors of fermions, and that \( S \) is invariant under the full \( U(N_f) \times U(N_f) \) chiral symmetry. We define \( \psi_n \) to be localized averages of \( \psi \),

\[ \psi_n = \int d^4x \psi(x) f(x - an) \]  

(4.14)

where \( f(x) \) is some function with support in the region of \(|x| \lesssim a\). Then up to an irrelevant normalization, we can rewrite

\[ Z = \int [d\psi] [d\bar{\psi}] \prod_n d\chi_n d\bar{\chi}_n e^{-\sum_n \alpha(\chi_n - \bar{\psi}_n)(\chi_n - \psi_n) - S(\psi, \bar{\psi})} \]

\[ = \int \prod_n d\chi_n d\bar{\chi}_n e^{-S_{\text{lat}}(\chi_n, \bar{\chi}_n)} \equiv e^{-\chi D \chi}, \]  

(4.15)

where \( \alpha \) is a dimensionful parameter, where \( D \) is the resulting lattice fermion operator. Since there are \( N_f \) copies of all the fields, the operator \( D \) is invariant under the vector \( U(N_f) \) symmetry, so that if \( T \) is a \( U(N_f) \) generator, \([T, D] = 0\). The lattice action is therefore defined as

\[ e^{-\chi D \chi} = \int [d\psi] [d\bar{\psi}] e^{-\sum_n \alpha(\chi_n - \bar{\psi}_n)(\chi_n - \psi_n) - S(\psi, \bar{\psi})}, \]  

(4.16)

Note that explicit chiral symmetry breaking has crept into our definition of \( S_{\text{lat}} \) through the fermion bilinear we have introduced in the Gaussian in order to change variables.

Now consider a chiral transformation on the lattice variables, \( \chi_n \rightarrow e^{i\gamma_5 T} \chi_n \), \( \bar{\chi}_n \rightarrow \bar{\chi}_n e^{i\gamma_5 T} \), where \( T \) is a generator for a \( U(N_f) \) flavor transformation. This is accompanied by a corresponding change of integration variables \( \psi, \bar{\psi} \):

\[ e^{-\chi e^{i\gamma_5 T} D e^{i\gamma_5 T} \chi} = \int [d\psi] [d\bar{\psi}] e^{i \epsilon A \text{Tr} F_{\alpha\beta} F_{\gamma\delta}} e^{2i\gamma_5 T(\chi_n - \bar{\psi}_n) - S(\psi, \bar{\psi})}, \]  

(4.17)

where \( A \) is the anomaly due to the non-invariance of the measure \([d\psi] [d\bar{\psi}]\) as computed by Fujikawa [Fujikawa, 1979]:

\[ A = \frac{1}{16\pi^2} \epsilon_{\beta\gamma\delta} \text{Tr} F_{\alpha\beta} F_{\gamma\delta} \]  

(4.18)

with

\[ \int A = 2\nu, \]  

(4.19)

\( \nu \) being the topological charge of the gauge field.
Overlap fermions and the Ginsparg-Wilson equation

Expanding to linear order in $\epsilon$ gives

$$
-\bar{\chi}\{\gamma_5, D\} T \chi e^{-\tau D} = \int [d\psi] [d\bar{\psi}] \left( 2 \nu \operatorname{Tr} T + \sum_n [(\bar{\chi}_n - \bar{\psi}_n)2\alpha \gamma_5 T(\chi_n - \psi_n)] \right)
$$

$$
\times \exp \left[ -\sum_m (\bar{\chi}_m - \bar{\psi}_m)(\chi_m - \psi_m) - S(\psi, \bar{\psi}) \right]
$$

$$
= \sum_n \left( 2 \nu \operatorname{Tr} T - \frac{2}{\alpha} \frac{\delta}{\delta \chi_n} \gamma_5 T \frac{\delta}{\delta \chi_n} \right) e^{-\tau D} \chi
$$

$$
= \left( \operatorname{Tr} \gamma_5 DT + 2\nu \operatorname{Tr} T - \frac{2}{\alpha} \chi_n D \gamma_5 DT \chi_n \right) e^{-\tau D} \chi
$$

(4.20)

Defining $\alpha \equiv 2/a$ this yields the operator identity

$$
(\gamma_5, D) = a D \gamma_5 D.
$$

(4.21)

If $T$ is taken to be a traceless generator of $U(N_f)$, multiplying both sides by $T$ and taking the trace yields the Ginsparg-Wilson equation:

$$
\{\gamma_5, D\} = a D \gamma_5 D.
$$

(4.22)

If on the other hand we take $T$ to be the unit matrix and use eqn. (4.22) we find

$$
\operatorname{Tr} \gamma_5 D = 2 N_f \nu,
$$

(4.23)

This latter equation was not derived in the original Ginsparg-Wilson paper; from our discussion of the index theorem eqn. (2.23), it follows from eqn. (4.23) that $\operatorname{Tr} \gamma_5 D = (n_- - n_+)$, where $n_{\pm}$ are the number of $\pm$ chirality zeromodes. We will see that that is indeed the case.

Note that the GW relation eqn. (4.22) is the same equation satisfied by the overlap operator (Neuberger, 1998b) — and therefore by the domain wall propagator at infinite wall separation on the lattice, being equivalent as shown in (Neuberger, 1998a; Neuberger, 1998c) — as well as by the infinitely separated domain wall propagator in the continuum (Luscher, 2000a). In fact, the general overlap operator derived by Neuberger

$$
D = 1 + \gamma_5 \epsilon(\mathcal{H})
$$

(4.24)

is the only explicit solution to the GW equations that is known.

4.2.2 Exact lattice chiral symmetry

Missing from the discussion so far is how the overlap operator is able to ensure multiplicative renormalization of fermion masses (and similarly, multiplicative renormalization of pion masses). In the continuum, both phenomena follow from the fact that fermion masses are the only operators breaking an otherwise good symmetry. The GW relation states exactly how chiral symmetry is broken on the lattice, but does
not specify a symmetry that is exact on the lattice and capable of protecting fermion masses from additive renormalization.

Lüscher was able to solve this problem by discovering the GW relation implied the existence of an exact symmetry of the lattice action: \( \int \bar{\psi} D\psi \) is invariant under the transformation

\[
\delta \psi = \gamma_5 \left( 1 + \frac{a}{2} D \right) \psi, \quad \delta \bar{\psi} = \bar{\psi} \left( 1 - \frac{a}{2} D \right) \gamma_5.
\]

(4.25)

Note that this becomes ordinary chiral symmetry in the \( a \to 0 \) limit, and that it is broken explicitly by a mass term for the fermions.

### 4.2.3 Anomaly

If this symmetry were an exact symmetry of the path integral, we would run afoul of all the arguments we have made so far: it becomes the anomalous \( U(1)_A \) symmetry in the continuum, so it cannot be an exact symmetry on the lattice! The answer is that this lattice chiral transformation is not a symmetry of the measure of the lattice path integral:

\[
\delta [d\psi][d\bar{\psi}] = [d\psi][d\bar{\psi}] \left( \text{Tr} \left[ \gamma_5 \left( 1 + \frac{a}{2} D \right) \right] + \text{Tr} \left[ \left( 1 - \frac{a}{2} D \right) \gamma_5 \right] \right)
\]

\[
= [d\psi][d\bar{\psi}] \times a \text{Tr} \gamma_5 D,
\]

(4.26)

where I used the relation \( d \det M/dx = \det[M] \text{Tr} M^{-1} dM/dx \). Unlike the tricky non-invariance of the fermion measure in the continuum under a \( U(1)_A \) transformation — which only appears when the measure is properly regulated — here we have a perfectly ordinary integration measure and a transformation that gives rise to a Jacobean with a nontrivial phase (unless, of course, \( \text{Tr} \gamma_5 D = 0 \)). To make sense, \( \text{Tr} \gamma_5 D \) must map into the continuum anomaly...and we have already seen that it does, from eqn. (4.23).

What remains is to prove the index theorem [Hasenfratz, Laliena and Niedermayer, 1998; Luscher, 1998], the lattice equivalent of eqn. (2.22). From exercise 4.4 it follows that for states \( |z\rangle \) satisfying \( D|z\rangle = z|z\rangle \)

\[
\text{Tr} \gamma_5 D = \sum_z \langle z | \gamma_5 D | z \rangle = 2N_f (n_+^{(2)} - n_-^{(2)}),
\]

(4.27)

where \( n_+^{(2)} \) are the number of positive and negative chirality states with eigenvalue \( z = 2 \). We also know that

\[
0 = \text{Tr} \gamma_5 = \sum_z \langle z | \gamma_5 | z \rangle = (n_+ - n_-) + (n_+^{(2)} - n_-^{(2)}),
\]

(4.28)

where \( n_\pm \) are the number of \( \pm \) chirality zeromodes at \( z = 0 \). Therefore we can write

\[
\text{Tr} \gamma_5 D = 2(n_- - n_+).
\]

(4.29)

Substituting into eqn. (4.23), we arrive at the lattice index theorem,

\[
(n_- - n_+) = \nu N_f
\]

(4.30)

which is equivalent to the continuum result eqn. (2.23), and provides an interesting definition for the topological charge of a lattice gauge field. A desirable feature of
the overlap operator is the existence of exact zeromode solutions in the presence of
topology; it is also a curse for realistic simulations, since the zeromodes make it difficult
to sample different global gauge topologies. And while it cannot matter what the global
topology of the Universe is, fixing the topology in a lattice QCD simulation gives rise
to spurious effects which only vanish with a power of the volume (Edwards, 2002).

4.3 Chiral gauge theories: the challenge

Chiral fermions on the lattice make an interesting story whose, final chapter on chiral
gauge theories has barely been begun. It is a story that is both theoretically amusing
and of practical importance, given the big role chiral symmetry plays in the standard
model. I have tried to stress that the understanding of anomalies has been the key
to both understanding the puzzling doubling problem and its resolution. In terms of
practical application they are more expensive than other fermion formulations, but
have advantages when studying physics where chirality plays an important role. For a
recent review comparing different fermion formulations, see (Jansen, 2008).

While domain wall and overlap fermions provide a way to represent any global
chiral symmetry without fine tuning, it may be possible to attain these symmetries
by fine tuning in theories with either staggered or Wilson fermions. In contrast, there
is currently no practical way to regulate general nonabelian chiral gauge theories on
the lattice. (There has been a lot of papers in this area, however, in the context
of domain wall - overlap - Ginsparg-Wilson fermions; for a necessarily incomplete
list of references that gives you a flavor of the work in this direction, see (Kaplan,
1992; Kaplan, 1993; Narayanan and Neuberger, 1993; Narayanan and Neuberger, 1995;
Narayanan and Neuberger, 1996; Kaplan and Schmaltz, 1996; Luscher, 1999; Aoyama
and Kikukawa, 1999; Luscher, 2000; Kikukawa and Nakayama, 2001; Kikukawa, 2002;
Kadoh and Kikukawa, 2008; Hasenfratz and von Allmen, 2008).) Thus we lack of
a nonperturbative regulator for the Standard Model — but then again, we think
perturbation theory suffices for understanding the Standard Model in the real world.
If a solution to putting chiral gauge theories on the lattice proves to be a complicated
and not especially enlightening enterprise, then it probably is not worth the effort
(unless the LHC finds evidence for a strongly coupled chiral gauge theory!). However,
if there is a compelling and physical route to such theories, that would undoubtedly
be very interesting.

Even if eventually a lattice formulation of the Standard Model is achieved, we must
be ready to address the sign problem associated with the phase of the fermion deter-
minant in such theories. A sign problem has for years plagued attempts to compute
properties of QCD at finite baryon chemical potential; the same physics is responsible
for poor signal/noise ratio experienced when measuring correlators in multi-baryon
states. To date there have not been any solutions which solve this problem. We can at
least take solace in the fact that the sign problems encountered in chiral gauge theo-
ries and in QCD at finite baryon density are not independent! After all, the standard
model at fixed nontrivial $SU(2)$ topology with a large winding number can describe a
transition from the QCD vacuum to a world full of iron atoms and neutrinos!
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