Semiclassical symmetry of the Gross-Pitaevskii equation with quadratic nonlocal Hamiltonian

Lisok A.L., Trifonov A.Yu., and Shapovalov A.V.

1 Mathematical Physics Department
Tomsk Polytechnic University,
Lenin ave., 30, Tomsk, Russia, 634034
2 Theoretical Physics Department
Tomsk State University,
Lenin ave., 36, Tomsk, Russia, 634050

Abstract
The Cauchy problem for the Gross–Pitaevsky equation with quadratic nonlocal nonlinearity is reduced to a similar problem for the correspondent linear equation. The relation between symmetry operators of the linear and nonlinear Gross–Pitaevskii equations is considered.

Introduction
Recent advances in formation of the Bose-Einstein condensates (BECs) of alkaline metal vapors [1–3] have stimulated study of the theoretical models describing behavior of nonlinear systems in external fields. In the BEC models the local and nonlocal Gross–Pitaevskii equations (GPEs) (which are also called the Hartree-type equations in the mathematical literature) are widely used. Besides the BEC theory, the nonlocal GPE serves as a basic equation in the models of quantum many-particle systems, nonlinear optics, collective excitations in molecular chains, etc.

Let us write down the nonlocal Gross–Pitaevskii equation as

\[ \begin{align*}
\{ -i\hbar \partial_t + \hat{\mathcal{H}}(\hat{z}, t) \} \Psi(\vec{x}, t) &= \{ -i\hbar \partial_t + \hat{\mathcal{H}}(t) + \hat{V}(t, \Psi(t)) \} \Psi(\vec{x}, t) = 0, \\
\Psi(\vec{x}, t) &\in L^2(\mathbb{R}_x^n), \\
\hat{V}(t, \Psi(t)) &= \int_{\mathbb{R}_y^n} \Psi^*(\vec{y}, t) \hat{V}(\hat{z}, \hat{w}, t) \Psi(\vec{y}, t) d\vec{y}.
\end{align*} \tag{1} \tag{2} \]

Here the linear operators \( \hat{\mathcal{H}}(t) = \mathcal{H}(\hat{z}, t) \) and \( V(\hat{z}, \hat{w}, t) \) are the Weyl-ordered functions [4] of time \( t \) and of noncommuting operators

\[ \hat{z} = (\hat{p}, \hat{x}) = (-i\hbar \partial/\partial \hat{x}, \hat{x}), \quad \hat{w} = (-i\hbar \partial/\partial \hat{y}, \hat{y}), \quad \hat{x}, \hat{y} \in \mathbb{R}_x^n, \]

with commutators

\[ [\hat{z}_k, \hat{z}_j]_- = [\hat{w}_k, \hat{w}_j]_- = i\hbar J_{kj}, \quad [\hat{z}_k, \hat{w}_j]_- = 0, \quad k, j = 1, 2n, \tag{3} \]

\( J = ||J_{kj}||_{2n \times 2n} \) is an identity symplectic matrix \( J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}_{2n \times 2n}, I = \begin{pmatrix} 1 \end{pmatrix}_{n \times n} \) is an identity \( n \times n \)-matrix.

In multidimensional space GPE (1) with variable coefficients of general form is non-integrable by known methods like, e.g., the Inverse Scattering Transform [5]. Therefore, analytical solutions of this equation can be constructed only approximately. An effective approach to construct such
solutions is provided by the method of semiclassical asymptotics. Thus for nonlinear self-consistent field operators, the theory of canonical operator with real phase was constructed for solution of the Cauchy problem in [6, 7]; for spectral problems including singular potentials in [8, 9] (see also [10–13]). Soliton-like solutions of the Hartree-type equation with some potentials of special form were constructed in [14]. A specific and attractive feature of the nonlocal GPE with nonlinearity presented in the equation only as a term under integral sign is that this equation can be referred to a class of nonlinear equations of mathematical physics which are close to linear ones in a sense [4]. Namely, among solutions there is a subset of solutions regularly depending on the nonlinearity parameter. Therefore, on the class of functions \( \mathcal{P}_1^+ \) called trajectory-concentrated functions [15], the problem of semiclassical asymptotics construction for the nonlinear equation is reduced to an auxiliary problem of construction of asymptotic solutions for associated linear Schrödinger equations. Finally, the WKB-Maslov complex germ method [16, 17] was generalized for Eq. (1).

In particular, formal solutions of the Cauchy problem asymptotic in formal small parameter \( \hbar \) (\( \hbar \to 0 \)) was constructed accurate to \( O(\hbar^{N/2}) \) where \( N \) is any natural number. The leading term of asymptotic solution of the spectral problem was found in [18].

Let us note that the semiclassical method, being approximate in essence, allows one to find exact solutions in some special cases. In [19, 20] the evolution operator for Eq. (1) with the quadratic nonlocal potential was found. In this case the linear operators \( \mathcal{H}(\hat{z}, t) \) and \( V(\hat{z}, \hat{\omega}, t) \) are quadratic in \( \hat{z}, \hat{\omega} \):

\[
\mathcal{H}(\hat{z}, t) = \frac{1}{2} \langle \hat{z}, \mathcal{H}_{zz}(t) \hat{z} \rangle + \langle \mathcal{H}_z(t), \hat{z} \rangle,
\]

\[
V(\hat{z}, \hat{\omega}, t) = \frac{1}{2} \langle \hat{z}, W_{zz}(t) \hat{z} \rangle + \langle \hat{z}, W_{zw}(t) \hat{\omega} \rangle + \frac{1}{2} \langle \hat{\omega}, W_{ww}(t) \hat{\omega} \rangle.
\]

Here, \( \mathcal{H}_{zz}(t), W_{zz}(t), W_{zw}(t), W_{ww}(t) \) are \( 2n \times 2n \)-matrices, \( \mathcal{H}_z(t) \) is a \( 2n \)-vector; \( \langle \cdot, \cdot \rangle \) is an Euclidean scalar product of vectors; \( \langle \vec{p}, \vec{x} \rangle = \sum_{j=1}^{n} p_j x_j ; \vec{p}, \vec{x} \in \mathbb{R}^n, \langle z, w \rangle = \sum_{j=1}^{2n} z_j w_j , z, w \in \mathbb{R}^{2n} \).

The Hartree-type equations (1), (4), (5) are integrated explicitly [19, 20] and possess fairly rich symmetries, the study of which gives varied information about solutions of the equation. It also shows how to reduce multidimensional equations to one-dimensional equations, to construct classes of exact and approximate solutions, to investigate asymptotic of special classes of solutions, etc. Moreover, as long as Eqs. (1), (4), (5) have nonlocal nonlinearity, its symmetry is of a special interest in the symmetry analysis. The matter is that standard methods of symmetry analysis [21–25], developed basically for partial differential equations (PDEs), face the problems when structure of equation under the study differs from the PDE because, for example, there are no regular rules to choose an appropriate structure of symmetries for a non-differential equation. Equation (1) enable one to avoid this problem as symmetry of this equation is closely connected with the symmetry of linear equation corresponding to the nonlinear one.

Here, we find symmetry operators of Eqs. (1), (4), (5) in explicit form which, by definition, leave invariant the solution set of the equation and allow to generate new solutions from the known ones, (see, e.g., [26, 27]).

1 Operators in terms of an operator Cauchy problem

Consider the Cauchy problem for Eq. (1)

\[
\Psi(\vec{x}, t, \hbar)|_{t=0} = \gamma(\vec{x}), \quad \gamma \in \mathbb{S}, \quad \|\gamma(\vec{x})\|^2 = 1
\]

where \( \mathbb{S} \) is a Schwartz space.

The second-order Hamilton–Ehrenfest related to the Cauchy problem (1) (6) has the form [20]

\[
\begin{align*}
\dot{z}_\Psi &= J\{\mathcal{H}_z(t) + [\mathcal{H}_{zz}(t) + \hat{\omega}(W_{zz}(t) + W_{zw}(t))]z_\Psi\}; \\
\Delta \Psi_2 &= J[\mathcal{H}_{zz}(t) + \hat{\omega}W_{zz}(t)]\Delta \Psi_2 - \Delta \Psi_2[\mathcal{H}_{zz}(t) + \hat{\omega}W_{zz}(t)]J.
\end{align*}
\]

Let us change the function \( \Psi(\vec{x}, t) \) by \( \Phi(\vec{x}, t) \) in Eq. (1) as

\[
\Psi(\vec{x}, t) = \exp \left\{ \frac{i}{\hbar} \int \left\{ S(t) + \langle \vec{\mathcal{U}}(t), \vec{x} - \vec{X}(t) \rangle \right\} \right\} \Phi(\vec{x} - \vec{X}(t), t),
\]

(8)
where \( S(t) \), \( \tilde{P}(t), \tilde{X}(t) \) are some differential functions to be determined.

For \( \Phi(\tilde{x}, t) \) we have the following equation:

\[
\left\{ -i\hbar \partial_t + \hat{S}(t) + \langle \hat{P}(t), \tilde{x} - \tilde{X}(t) \rangle - \langle \hat{P}(t), \tilde{X}(t) \rangle + \frac{1}{2} \langle \hat{z}_\Phi, \mathcal{H}_{zz}(t) \hat{z}_\Phi \rangle + \langle \mathcal{H}_z(t), \hat{z}_\Phi \rangle + + \mathcal{H}_z(t)) \right\} \Phi = 0, \tag{9}
\]

where we use the notations \( \hat{z}_\Phi = (-i\hbar \partial/\partial \tilde{x} \tilde{P}(t), \tilde{x}) \); \( \hat{w}_\Phi = (-i\hbar \partial/\partial \tilde{y} + \tilde{P}(t), \tilde{y}) \).

By changing variables \( \tilde{x} = \tilde{u} + \tilde{X}(t) \) and denoting \( \tilde{\Phi}(\tilde{u} + \tilde{X}(t), t) = \tilde{\Phi}(\tilde{u}, t) \), we obtain the following equation:

\[
\left\{ -i\hbar \partial_t + i\hbar \langle \hat{X}(t), \partial/\partial \tilde{u} \rangle + \hat{S}(t) + \langle \hat{P}(t), \tilde{u} \rangle - \langle \hat{P}(t), \tilde{X}(t) \rangle + \frac{1}{2} \langle \hat{z}_{\Phi_u}, \mathcal{H}_{zz}(t) \hat{z}_{\Phi_u} \rangle + + \langle \mathcal{H}_z(t), \hat{z}_{\Phi_u} \rangle + \kappa \int_{\mathbb{R}^n} d\tilde{y} \tilde{\Phi}^* (\tilde{y}, t) \left( \frac{1}{2} \langle \hat{z}_{\Phi_u}, W_{zz}(t) \hat{z}_{\Phi_u} \rangle + \langle \hat{z}_{\Phi_u}, W_{zu}(t) \hat{w}_{\Phi_u} \rangle + + \frac{1}{2} \langle \hat{w}_{\Phi_u}, W_{wu}(t) \hat{w}_{\Phi_u} \rangle \right) \tilde{\Phi}(\tilde{y}, t) \right\} \tilde{\Phi}(\tilde{u}, t) = 0, \tag{10}
\]

where \( \hat{z}_{\Phi_u} = \hat{z}_u + Z(t), \hat{w}_{\Phi_u} = \hat{w}_u + Z(t), \hat{z}_u = (-i\hbar \partial/\partial \tilde{u}, \tilde{u}), \hat{w}_u = (-i\hbar \partial/\partial \tilde{y}, \tilde{y}) \).

The functions \( \tilde{\Phi}(\tilde{u}, t) \) are centered by construction, i.e., satisfy the condition

\[
\int_{\mathbb{R}^n} \tilde{\Phi}^*(\tilde{u}, t) \tilde{\Phi}(\tilde{u}, t) d\tilde{u} = 0. \tag{11}
\]

Initial condition \( \Phi(\tilde{x}, 0) \) for Eq. \( 10 \) has the form

\[
\tilde{\Phi}(\tilde{u}, t)|_{t=0} = \exp \left\{ - \frac{i}{\hbar} [S(0) + \langle \tilde{P}(0), \tilde{u} \rangle] \right\} \gamma(\tilde{u} + \tilde{X}(0)). \tag{12}
\]

Let the vector \( z = Z(t) = (\tilde{P}(t), \tilde{X}(t)) \) satisfies the equation

\[
\dot{Z}(t) = J[\mathcal{H}_z(t) + \mathcal{H}_{zz}(t) + \mathcal{H}_{zw}(W_z(t) + W_{zu}(t))]Z(t) \tag{13}
\]

with the initial condition \( Z(0) = \langle \gamma(\tilde{x})|\tilde{z}|\gamma(\tilde{x}) \rangle \), and the function \( S(t) \) is determined by the relation

\[
S(t) = \int_0^t \left\{ \langle \hat{P}(t), \hat{X}(t) \rangle - S_j(t) \right\} dt, \tag{14}
\]

where

\[
S_j(t) = \frac{1}{2} \left[ \| \langle \gamma(\tilde{x})|\hat{z}_j|\gamma(\tilde{x}) \rangle \| \right] \Delta_2(t).
\]

Here the matrix \( \Delta_2 \) of the order \( 2n \times 2n \) satisfies the equation

\[
\dot{\Delta}_2 = J[\mathcal{H}_zz(t) + \mathcal{H}_{zz}(t)]\Delta_2 - \Delta_2[\mathcal{H}_zz(t) + \mathcal{H}_{zz}(t)]J \tag{15}
\]

and the initial condition

\[
\Delta_2(0) = \frac{1}{2} \| \langle \gamma(\tilde{x})|\hat{z}_j|\gamma(\tilde{x}) \rangle \|. \tag{16}
\]

Then the function \( \tilde{\Phi}(\tilde{u}, t) \) is a solution of the linear associated equation

\[
\left\{ -i\hbar \partial_t + \frac{1}{2} \langle \hat{z}_u, (\mathcal{H}_{zz}(t) + \mathcal{H}_{zz}(t))\hat{z}_u \rangle \right\} \tilde{\Phi}(\tilde{u}, t) = 0 \tag{17}
\]
with the initial condition \[12\].

Let \( \hat{A}(\bar{u}, t) = A(\bar{z}, t) \) be the operator whose Weyl symbol \( A(z, t) \) satisfies the relation

\[
\left[ -i\hbar \partial_t + \frac{1}{2} (\bar{z}_u, (\mathcal{H}_{zz}(t) + \varepsilon W_{zz})\bar{z}_u), \hat{A}(\bar{u}, t) \right] = 0
\]  

(18)

and the initial condition

\[
\hat{A}(\bar{u}, t)|_{t=0} = \hat{a}(\bar{u}),
\]

(19)

where \( \hat{a}(\bar{u}) : \mathbb{S} \to \mathbb{S} \) is an arbitrary operator.

Then, thereby \[13\] the operator \( \hat{A}(\bar{u}, t) \) is a symmetry operator of Eq. \[17\] that maps a solution \( \tilde{\phi}(\bar{u}, t) \) of Eq. \[17\] into another solution of this equation. Respectively, the function determined by the condition

\[
\Phi_A(\bar{u}, t) = \frac{1}{\alpha_A} \hat{A}(\bar{u}, t) \tilde{\phi}(\bar{u}, t),
\]

(20)

where \( \alpha_A = \|\hat{a}\tilde{\phi}(\bar{u}, 0)\| \) is also a solution of Eq. \[17\].

At \( t = 0 \) we have

\[
\begin{align*}
\Phi_A(\bar{u}, 0) &= \frac{1}{\alpha_A} \hat{a}(\bar{u})\phi(\bar{u}) = \phi_A(\bar{u}).
\end{align*}
\]

(21)

Here \( \|\phi_A(\bar{u})\| = 1 \), and we immediately obtain \( \|\Phi_A(\bar{u}, t)\| = 1 \). However, in general case the function \( \tilde{\Phi}_A(\bar{u}, t) \) can correspond no solution of the original nonlinear equation as it is centered not for all \( \tilde{A}(\bar{u}, t) \):

\[
\int_{\mathbb{R}^n} \Phi_A(\bar{u}, 0) \bar{z}_u \Phi_A(\bar{u}, 0) d\bar{u} \neq 0.
\]

(22)

To find solutions of the original nonlinear equation which would correspond to \( \tilde{\Phi}(\bar{u}, t) \), let us introduce the notations

\[
\lambda_0 = \int_{\mathbb{R}^n} \Phi_A(\bar{u}, 0) \bar{z}_u \Phi_A(\bar{u}, 0) d\bar{u} = \int_{\mathbb{R}^n} \phi_A(\bar{u}) \bar{z}_u \phi_A(\bar{u}) d\bar{u}
\]

(23)

and

\[
\lambda(t) = \int_{\mathbb{R}^n} \Phi_A(\bar{u}, t) \bar{z}_u \Phi_A(\bar{u}, t) d\bar{u}.
\]

(24)

We can verify immediately that if \( \lambda(t) \) is a solution of the Cauchy problem

\[
\begin{align*}
\dot{\lambda}(t) &= J(\mathcal{H}_{zz}(t) + \varepsilon W_{zz}(t))\lambda(t), \\
\lambda(t) &= \left( \begin{array}{c} \bar{X}_p(t) \\ \bar{X}_u(t) \end{array} \right), \\
\lambda(0) &= \lambda_0 = \left( \begin{array}{c} \bar{X}_{p_0} \\ \bar{X}_{u_0} \end{array} \right),
\end{align*}
\]

(25)

then the function

\[
\tilde{\Phi}_A(\bar{u}, t) = \exp \left\{ \frac{i}{\hbar} \left[ S_{\lambda}(t) + \langle \bar{X}_p(t), \bar{u} + \bar{X}_u(t) \rangle \right] \right\} \Phi_A(\bar{u} + \bar{X}_u(t), t),
\]

(26)

where

\[
S_{\lambda}(t) = \int_0^t \left\{ \langle \bar{X}_p(t), \dot{\lambda}_u(t) \rangle - \mathcal{S}_{\lambda}(t) \right\} dt
\]

(27)

and

\[
\mathcal{S}_{\lambda}(t) = \frac{1}{2} \langle \dot{\lambda}(t), [\mathcal{H}_{zz}(t) + \varepsilon (W_{zz}(t) + 2W_{zw}(t) + W_{ww}(t))]\lambda(t) \rangle + \frac{1}{2} \varepsilon \text{Sp}(W_{ww}(t)\Delta_2)
\]

is a solution of Eq. \[17\] and satisfies the condition

\[
\int_{\mathbb{R}^n} \Phi_A^*(\bar{u}, t) \bar{z}_u \Phi_A(\bar{u}, t) d\bar{u} = 0.
\]

(28)
Let us correlate according to (8) the functions \( \tilde{\Phi}_A(\vec{u}, t) = \Phi(\vec{x} + \vec{X}_A(t), t) \) and \( \hat{\Phi}(\vec{u}, t) = \Phi(\vec{x} + \vec{X}(t), t) \) with the functions \( \Psi_A(\vec{x}, t) \) and \( \Psi(\vec{x}, t) \), respectively. Then we have the relation

\[
\exp \left\{ -\frac{i}{\hbar} [S_A(t) + \langle \vec{P}_A(t), \vec{x} - \vec{X}_A(t) \rangle] \right\} \Psi_A(\vec{x}, t) = \\
= \frac{1}{\alpha_A} \exp \left\{ \frac{i}{\hbar} [S_A(t) + \langle \vec{P}_A(t), \vec{x} - \vec{X}_A(t) \rangle] \right\} A(\vec{x} - \vec{X}_A(t) + \lambda_u(t), t) \times \\
\times \exp \left\{ -\frac{i}{\hbar} [S(t) + \langle \vec{F}(t), \vec{x} - \vec{X}(t) \rangle] \right\} \Psi(\vec{x} + \vec{X}(t) - \vec{X}_A(t) + \lambda_u(t), t)
\]

from which it follows

\[
\Psi_A(\vec{x}, t) = \exp \left\{ \frac{i}{\hbar} [S_A(t) + \langle \vec{P}_A(t), \vec{x} - \vec{X}_A(t) \rangle] \right\} \times \\
\times \frac{1}{\alpha_A} \exp \left\{ \frac{i}{\hbar} [S_A(t) + \langle \vec{P}_A(t), \vec{x} - \vec{X}_A(t) \rangle] \right\} A(\vec{x} - \vec{X}_A(t) + \lambda_u(t), t) \times \\
\times \exp \left\{ -\frac{i}{\hbar} [S(t) + \langle \vec{F}(t), \vec{x} - \vec{X}(t) \rangle] \right\} \Psi(\vec{x} + \vec{X}(t) - \vec{X}_A(t) + \lambda_u(t), t).
\]

The operator

\[
\Psi_A(\vec{x}, t) = \tilde{A}_m \Psi(\vec{x}, t)
\]

determined by (30) is a symmetry operator of the original nonlinear equation.

## 2 Symmetry operators obtained via auxiliary equation

Let \( \tilde{a} : S \to S \) be an operator and \( \gamma(\vec{x}) \in S \) is a function. Let us set for Eq. (1) the following Cauchy problems: \( \Psi(\vec{x}, t, \hbar)|_{t=0} = \gamma(\vec{x}) \), \( \gamma(\vec{x}) \in P^0_{\hbar}, ||\gamma(\vec{x})||^2 = 1 \), and \( \Psi_A(\vec{x}, t, \hbar)|_{t=0} = \tilde{a} \gamma(\vec{x}) = \gamma_A(\vec{x}), \gamma_A(\vec{x}) \in P_{\hbar}^0, ||\gamma_A(\vec{x})||^2 = 1 \).

Let us change the functions \( \Psi(\vec{x}, t) \) and \( \Psi_A(\vec{x}, t) \) in Eq. (1) by the functions \( \tilde{\Phi}(\vec{u}, t) \) and \( \hat{\Phi}(\vec{u}, t) \), respectively, according to formulas (8), (13), (14), and (15). Then the function \( \Phi(\vec{u}, t) \) satisfies the equation

\[
\left\{ -i\hbar \partial_t + \frac{1}{2} \langle \vec{z}_u, (\mathcal{H}_{zz}(t) + \gamma W_{zz}) \vec{z}_u \rangle \right\} \Phi(\vec{u}, t) = 0
\]

and the initial condition \( \tilde{\Phi}(\vec{u}(t))|_{t=0} = \exp \left\{ -\frac{i}{\hbar} [S(0) + \langle \vec{F}(0), \vec{u} \rangle] \right\} \gamma(\vec{u} + \vec{X}(0)) \), where the vector \( Z(t) \) is a solution of Eq. (13) with the initial condition \( Z(0) = \langle \gamma(\vec{x})|\vec{z} \gamma(\vec{x}) \rangle \). Similarly, \( \hat{\Phi}(\vec{u}, t) \) is a solution of the equation

\[
\left\{ -i\hbar \partial_t + \frac{1}{2} \langle \vec{z}_u, (\mathcal{H}_{zz}(t) + \gamma W_{zz}) \vec{z}_u \rangle \right\} \hat{\Phi}(\vec{u}, t) = 0
\]

with the initial condition \( \hat{\Phi}(\vec{u}(t))|_{t=0} = \exp \left\{ -\frac{i}{\hbar} [S_A(0) + \langle \vec{P}_A(0), \vec{u} \rangle] \right\} \gamma_A(\vec{u} + \vec{X}_A(0)) \).

Vector \( Z_A(t) \) satisfies Eq. (13) with the initial condition \( Z_A(0) = \langle \gamma_A(\vec{x})|\vec{z} \gamma_A(\vec{x}) \rangle \), and

\[
S_A(t) = \int_0^t \left\{ \langle \vec{P}_A(t), \dot{\vec{X}}_A(t) \rangle - \delta_A(t) \right\} dt,
\]

where

\[
\delta_A(t) = \frac{1}{2} \langle Z_A(t), [\mathcal{H}_{zz}(t) + \gamma W_{zz}(t) + 2W_{zw}(t) + W_{ww}(t)]Z_A(t) + \rangle \\
+ \langle \mathcal{H}_z(t), Z_A(t) \rangle + \frac{1}{2} \gamma \text{Sp}(W_{ww}(t) \Delta_{2A}).
\]

Matrix \( \Delta_{2A}(t) \) is a solution of Eq. (13) with the initial condition

\[
\Delta_{2A}(0) = \frac{1}{2} \| \langle \gamma_A(\vec{x}, t) | [\Delta \hat{z}_j \Delta \hat{z}_k + \Delta \hat{z}_k \Delta \hat{z}_j] | \gamma_A(\vec{x}, t) \rangle \|.
\]
The relation $\gamma_A(\vec{x}) = \hat{a}\gamma(\vec{x})$ determines operator $\hat{a}(\vec{u})$ such that $\hat{\Phi}_A(\vec{u}, 0) = \hat{a}(\vec{u})\hat{\Phi}(\vec{u}, 0)$.

Consider the Cauchy problem for an operator $\hat{A}(\vec{u}, t) = \hat{A}(\vec{z}_u, t)$:

$$\left[ -i\hbar\partial_t + \frac{1}{2}\left( \hat{z}_u, (\hat{H}_{zz}(t) + \hat{\kappa}W_{zz})\hat{z}_u, \hat{A}(\vec{u}, t) \right) \right] = 0$$  \hspace{1cm} (36)

with the initial condition

$$\hat{A}(\vec{u}, t)|_{t=0} = \hat{a}(\vec{u}).$$  \hspace{1cm} (37)

In accordance with (36) $\hat{A}(\vec{u}, t)$ is a symmetry operator of Eq. (31) and it maps solution $\hat{\Phi}(\vec{u}, t)$ of Eq. (31) into its solution $\hat{\Phi}_A(\vec{u}, t)$.

If we correlate the functions $\hat{\Phi}_A(\vec{u}, t) = \Phi(\vec{x} + \vec{X}(t), t)$ and $\hat{\Phi}(\vec{u}, t) = \Phi(\vec{x} + \vec{X}(t), t)$ with the functions $\Psi_A(\vec{x}, t)$ and $\Psi(\vec{x}, t)$, respectively, according to formula (3), we obtain

$$\exp\left\{ -\frac{i}{\hbar}[S_A(t) + \langle \vec{P}_A(t), \vec{x} - \vec{X}_A(t) \rangle] \right\} \Psi_A(\vec{x}, t) = \hat{A}(\vec{x} - \vec{X}_A(t), t) \times$$

$$\times \exp\left\{ -\frac{i}{\hbar}[S(t) + \langle \vec{P}(t), \vec{x} - \vec{X}(t) \rangle] \right\} \Psi(\vec{x} + \vec{X}(t) - \vec{X}_A(t), t)$$  \hspace{1cm} (38)

or

$$\Psi_A(\vec{x}, t) = \exp\left\{ \frac{i}{\hbar}[S_A(t) + \langle \vec{P}_A(t), \vec{x} - \vec{X}_A(t) \rangle] \right\} \hat{A}(\vec{x} - \vec{X}_A(t), t) \times$$

$$\times \exp\left\{ -\frac{i}{\hbar}[S(t) + \langle \vec{P}(t), \vec{x} - \vec{X}(t) \rangle] \right\} \Psi(\vec{x} + \vec{X}(t) - \vec{X}_A(t), t).$$  \hspace{1cm} (39)

An operator

$$\Psi_A(\vec{x}, t) = \hat{A}_{nl}\Psi(\vec{x}, t)$$

determined by relation (39) is also a symmetry operator of the initial nonlinear equation.

3 Conclusion

It should be noted that a direct calculation of symmetry operators for a nonlinear equation is, as a rule, difficult because of the complexity of the determining equations [28]. In construction of symmetry operators, we have used the fact that the original nonlinear equation can be connected with an associated linear equation, and for quadratic operators of the form (1) and (3) these associated equations coincide. In general, this is not true and therefore, analytical solutions of the GPE can be constructed only approximately.

Many approximate methods are based on appropriate anzatses representing the general element of a class of functions in which approximate solution is constructed. To such methods it is possible to relate, e.g., the method of collective variables (see, e.g., [29] and references therein), the Lagrangian method [30, 31], etc.

It appears that the most effective method in various multidimensional problems of mathematical physics is the method of semiclassical approximations. Peculiarity that differs this method from the usual method of expanding in power series in a small asymptotic parameter is that the asymptotic small parameter is included into the solution both regularly and singularly. This fact allows us, in particular, to construct space-localized solutions having an important physical meaning. This possibility is of a special importance in nonlinear problems where stable localized excitations (patterns), such as, for example, solitons are the object of research. The advantage of the semiclassical asymptotic method is that it allows to estimate accuracy of the constructed solution with a given power of the asymptotic parameter.

Symmetry analysis provides another approach to construct analytical solutions of the nonlinear equations [21–25]. Taking into account invariant properties of the equation under consideration, one can find classes of special solutions which can serve as a prototype of anzatses for classes of exact solutions.

However, variable coefficients, as a rule, reduce the equation symmetry and even can exclude it completely that reduces the symmetry analysis in such cases. Comparison of these two approaches results in a problem of consideration of approximate symmetry for the GPE in the formalism of semiclassical approximation method (see, e.g., [32]).
Acknowledgements

The work was supported in part by President of the Russian Federation, Grant No SS-5103.2006.2, MK-4629.2006.2, DFG.

References

[1] Cornell E A and Wieman C E 2002 Nobel lecture: Bose–Einstein condensation in a dilute gas, the first 70 years some recent experiments Rev. Mod. Phys. 74 875–893; Ketterle W Nobel lecture: When atoms behave as waves: Bose–Einstein condensation and the atom laser Ibid. 1131–1151

[2] Pitaevskii LP 1961 Vortex lines in an imperfect Bose gas Zh. Eksper. Teor. Fiz. 40 646–651 (in Russian)

[3] Gross EP 1961 Structure of a quantized vortex in boson systems Nuovo Cimento 20:3 454–477.

[4] Karasev MV and Maslov VP 1993 Nonlinear Poisson Brackets: Geometry and Quantization (Providence, RI: Amer. Math. Soc.) Ser. Traslations of Mathematical Monographs, V. 119

[5] Zakharov VE, Manakov SV, Novikov SP and Pitaevsky LP 1984 Theory of Solitons: The Inverse Scattering Method (New York: Plenum)

[6] Maslov VP 1976 Complex Markov Chains and The Continual Feynmann Integral (Moscow: Nauka)

[7] Maslov VP 1978 Self-consistent field equations Modern Mathematical Problems 11 153–234

[8] Karasev MV and Maslov VP 1979 Algebras with common commutative relationships and their applications Ibid. 13 145–267

[9] Karasev MV and Pereskokov AV. 1989 The quantization rule for self-consistent field equations with a local rapidly descending nonlinearity Teor. Mat. Fiz. 79: 2 198–208 [English Transl. 1989 Teor. Math. Phys. 79]

[10] Babich VM, Buldyrev VS and Molotkov IA. 1985 The Space-Time Beam Method. Linear and Nonlinear Waves (Leningrad: Leningrad State University Publishers)

[11] Molotkov IA and Vakulenko SA 1988 Concentrated Nonlinear Waves (Leningrad: Leningrad State University Publishers)

[12] Vakulenko SA, Maslov VP, Molotkov IA and Shafarevich AI 1995 Asymptotical solutions of the Hartree equation concentrated at \( t \to 0 \) in a small neighborhood of a curve Dokl. RAN 345 743–745 [English Transl. Rus. Math. Doct.]

[13] Simenog IV 1977 The asymptotic behavior of the solution of a stationary nonlinear Hartree equation. Teoret. Mat. Fiz.30:3 408–14 [English Transl. 1977 Theor. and Math. Phys. 30:3 263–68]

[14] Vo Khagn Fuk and Chetverikov VM 1978 Generalized solitons of the Srödinger equation with unitary nonlinearity Teor. Mat. Fiz. 36 345–351 [English Transl. 1978 Teor. Math. Phys. 36].

[15] Belov VV, Trifonov AYu and Shapovalov AV 2002 The trajectory-coherent approximation and the system of moments for the Hartree type equation, Int. J. Math. and Math. Sci. 32:6 325–370

[16] Maslov VP 1994 The Complex WKB Method for Nonlinear Equations. I. Linear Theory (Basel, Boston, Berlin: Birkhauser Verlag)

[17] Belov VV and Dobrokhotov SYu 1992 Semiclassical Maslov asymptotics with complex phases. I. General approach Teor. Mat. Fiz. 130:2 215–254 [English transl.: 1992 Theor. Math. Phys. 92:2].
[18] Belov V V, Litvinets F N and Trifonov A Yu 2007 The semiclassical spectral series for a Hartree-type operator corresponding to a rest point of the Hamilton–Ehrenfest system Theor. Math. Phys. - 150:1 20–32

[19] Lisok A L, Trifonov A Yu and Shapovalov A V 2004 The evolution operator of the Hartree-type equation with a quadratic potential J. Phys. A.: Math. Gen. 37 4535–4556

[20] Lisok A L, Trifonov A Yu and Shapovalov A V 2005 Exact solutions and symmetry operators for the nonlocal Gross–Pitaevskii equation with quadratic potential Symmetry, Integrability and Geometry: Methods and Applications 1 1–14

[21] Ovsjannikov L V 1982 Group Analysis of Differential Equations (New-York: Academic Press)

[22] Anderson R L and Ibragimov N H 1979 Lie–Backlund Transformations in Applications (Philadelphia: SIAM)

[23] Olver P J 1986 Application of Lie Groups to Differential Equations (New-York: Springer)

[24] Fushchych W I, Shtelen W M and Serov N I 1993 Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics (Dordrecht: Kluwer)

[25] Fushchich W I and Nikitin A G 1994 Symmetries of Equations of Quantum Mechanics (New-York: Allerton Press Inc.)

[26] Miller W, Jr. 1977 Symmetry and Separation of Variables (London, Amsterdam, Ontario, Tokio, New-York: Addison-Wesley)

[27] Malkin M A and Manko V I 1979 Dynamic Symmetries and Coherent States of Quantum Systems (Moscow: Nauka) (In Russian)

[28] Meirmanov A.M., Pukhnachov V V., and Shmarev S I., Evolution equations and lagrangian coordinates, Berlin, New York, Walter de Gruyter, 1994.

[29] Sanchez A. and Bishop A R 1998 Collective coordinates and length–scale competition in spatially inhomogeneous soliton–bearing equations SIAM Rev. 40:3 579–615

[30] Malomed B A 1992 Perturbative analysis of the interaction of a $\phi^4$ kink with inhomogeneities J. Phys. A: Math. Gen. 25 755–764

[31] Kivshar Y S, Fei Z and Vazquez L 1991 Resonant soliton-impurity interactions Phys. Rev. Lett. 67 1177–1180; 1992 Resonant kink-impurity interactions in the sine-Gordon model Phys. Rev. A 45 6019–6030; 1992 Resonant kink-impurity interactions in the $\phi^4$ model Ibid. 5214–5220

[32] Shvedov O Yu 2002 Semiclasical symmetries Ann. Phys. (NY) 296 51–89