MULTIPLIERS AND DUAL OPERATOR ALGEBRAS

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Abstract. In a previous paper we showed how the main theorems characterizing operator algebras and operator modules, fit neatly into the framework of the ‘noncommutative Shilov boundary’, and more particularly via the left multiplier operator algebra of an operator space. As well as giving new characterization theorems, the approach of that paper allowed many of the hypotheses of the earlier theorems to be eliminated. Recent progress of the author with Effros and Zarikian now enables weak*-versions of these characterization theorems. For example, we prove a result analogous to Sakai’s famous characterization of von Neumann algebras as the $C^*$-algebras with predual, namely, that the $\sigma$-weakly closed unital (not-necessarily-selfadjoint) subalgebras of $B(H)$ for a Hilbert space $H$, are exactly the unital operator algebras which possess an operator space predual. This removes one of the hypotheses from an earlier characterization due to Le Merdy. We also show that the multiplier operator algebras of dual operator spaces are dual operator algebras. Using this we refine several known results characterizing dual operator modules.
1. Introduction.

This paper is an application of non-commutative M-structure. Classically, M-structure is a theory which grew out of the seminal paper [1] (see [22] for more details). This theory has many facets, one of which is the theory and application of M-ideas (and, by duality, L-ideals). Recall that an arbitrary Banach space \( X \) can be embedded linearly isometrically in an ‘M-space’ \( C(K) \) (in many ways). There are also constructions that are ‘M in nature’ such as the \( \oplus^n \) direct sum. Part of classical ‘M-theory’ is concerned with what traces of the structure of the \( C(K) \) spaces, or the \( \oplus^n \) direct sum, are visible in a general \( X \). Via such considerations, a simple looking metric condition in \( X \) will often force a powerful conclusion of a structural nature. Another aspect of classical M-structure is the multiplier and centralizer algebras of a Banach space \( X \). These are, respectively, a function algebra \( \text{Mult}(X) \), and commutative \( C^* \)-algebra \( \text{Cent}(X) \), which may be associated with any Banach space \( X \), and which, in some sense, control the M-structure of \( X \). It is these algebras that we concentrate on here. In an earlier paper [8] we defined noncommutative generalizations of \( \text{Mult}(X) \) and \( \text{Cent}(X) \), which will be explained in more detail presently.

The main application of this paper is to theorems characterizing basic algebraic structures that are of interest to operator algebraists. A fundamental theorem in the field of operator algebras is the Gelfand and Naimark characterization of \( C^* \)-algebras (see [33] for example). For nonselfadjoint operator algebras the appropriate characterization is the BRS theorem [13] which may be stated as follows: the norm closed subalgebras of \( B(H) \) containing \( I_H \), are exactly the unital algebras \( A \) which are also operator spaces such that the identity in the algebra has norm 1, and also the condition:

\[
\|ab\|_n \leq \|a\|_n \|b\|_n
\]

holds, for matrices \( a, b \in M_n(A) \). Here the words “are exactly” mean “up to a homomorphism (i.e. an isomorphism) which is completely isometric”. We will give a short proof of BRS, and indeed of a new 1-sided version of BRS, at the end of this introduction.

With this theorem in mind, when we refer henceforth to an operator algebra in this paper, we mean either 1) a ‘concrete operator algebra’, i.e. a norm closed subalgebra of \( B(H) \) with an identity of norm 1, or 2) an operator space which is a unital algebra completely isometrically isomorphic to a concrete operator algebra. Note that any norm closed subalgebra \( A \) of \( B(H) \) with an identity of norm 1, may be assumed to have identity \( I_H \), by considering the completely isometric homomorphism on \( A \) which restricts \( A \) to act on the Hilbert space \( (AH) \). In this paper we will not consider operator algebras with contractive approximate identities, since they are automatically unital in the w*-situation (see the end of §2). It is worth remarking that all function algebras (i.e. unital norm closed subalgebras of \( C(K) \)) are operator algebras. Indeed these may be characterized as the operator algebras which are also \( MIN \) operator spaces [4].

There are also known characterization theorems similar to ‘BRS’ appropriate to the class of ‘operator modules’ (which are defined below) up to completely isometric module isomorphism (see [14] for the earliest such result).

In [8] we were able to fit these characterization theorems into a coherent context using the multiplier algebras just mentioned. This approach had three advantages of 1) being a
more unified treatment of these theorems, 2) allowing many of the hypotheses of the earlier theorems to be eliminated, and 3) giving new results. However it was not at all clear from [8] whether the new approach could be applied to the weak*-versions of such characterization theorems. For example, for $C^*$-algebras the appropriate weak*-version is Sakai's famous theorem, which characterizes von Neumann algebras as exactly the $C^*$-algebras with predual. Here the words “as exactly” mean “up to a *-homomorphism which is also a w*-w*-homeomorphism”. Sakai’s result is of course a noncommutative generalization of the classical characterization of $L^\infty$ spaces as exactly the $C(K)$ spaces with predual. One of the main results of the present paper is the following version of Sakai’s theorem for (not-necessarily-selfadjoint) operator algebras.

**Theorem 1.1.** Let $A$ be an operator algebra with an identity of norm 1, which is also a dual operator space (i.e. $A = X^*$ completely isometrically for an operator space $X$). Then $A$ is w*-w*-homeomorphically and completely isometrically isomorphic to a σ-weakly closed (i.e. w*-closed) subalgebra $B$ of $B(H)$ for some Hilbert space $H$. Conversely, any σ-weakly closed subalgebra (or indeed subspace) $B$ of $B(H)$ is a dual operator space.

Of course the isomorphism here is also multiplicative, linear, and w*-w*-homeomorphic. The converse direction of the theorem is an old and quite simple result (see 4.2.2 in [18]).

By virtue of the theorem, we shall henceforth refer to a σ-weakly closed subalgebra of $B(H)$ containing an identity of norm 1 as a ‘concrete dual operator algebra’, and an operator algebra which has an operator space predual as a ‘dual operator algebra’.

Some important remarks are in order. Firstly, one may assume as before that the identity of norm 1 in a concrete dual operator algebra is indeed $I_H$, by restricting $A$ to ($AH$). This restriction is clearly a completely isometric, weak*-continuous, homomorphism, and hence it is a w*-w*-homeomorphism (see Lemma 1.5 (3)).

The second important remark is that it turns out that the hypothesis that $A$ be a dual operator space, as opposed to a dual Banach space, is necessary for the conclusion of the theorem to hold. We will discuss an example in §2.

The third important remark is that Theorem 1.1 is a refinement of the following theorem of Christian Le Merdy, which appeared first in [28]. A very natural and elementary proof of it appears in [29] (Proposition 3.4 and Remark 3.5).

**Theorem 1.2.** (Le Merdy) Let $A$ be as in the statement of 1.1, but with the additional hypothesis that the multiplication on $A$ is separately weak*-continuous. Then $A$ is w*-w*-homeomorphically completely isometrically isomorphic to a concrete dual operator algebra.

Thus our proof of 1.1 will consist of showing that the multiplication on an operator algebra which is a dual operator space, is automatically separately weak*-continuous.

We shall prove Theorem 1.1 in §2. In §3 we will use this to deduce that the left multiplier algebra of any dual operator space is a dual operator algebra. We will also give a quick ‘M-structure’ proof of an operator space variant on important results of Zettl, and Effros, Ozawa and Ruan [46, 16]. In §4 we extend several results from [8] to the weak* situation. Before we describe some such results, we remark that hitherto this was quite problematic, because the ‘Hamana theory’ or ‘Shilov boundary’ approach did not seem to work well for dual spaces. In particular this is because the noncommutative Shilov boundary $T(X)$ and
injective envelope $I(X)$ of a dual operator space $X$ did not seem related to von Neumann algebras in any sense we were able to use. However the various left multiplier operator algebras of $X$ are dual algebras (as we show here in §3 and in [9]).

To describe the work in the last section we recall the following

**Definition 1.3.** An operator module is an operator space $X$, together with a left module action $A \times X \to X$ which satisfies:

(i) $ex = x$ for all $x \in X$,

(ii) $\|ax\|_n \leq \|a\|_n \|x\|_n$ for all matrices $a \in M_n(A)$, $x \in M_n(X)$ of any size ($n < \infty$).

Here $e$ is the identity of $A$. We have not said anything about what $A$ is, traditionally it is a unital $C^*$-algebra or operator algebra, but we will want to generalize this in §4. Note that (ii) may be rephrased as saying that the module action is completely contractive as a bilinear map (in the sense of [14, 32]).

As motivation for our work in §4 we recall the following

**Theorem 1.4.** (Effros-Ruan [17]) Let $M$ be a $W^*$-algebra, and $X$ a left operator $M$-module which is also a dual operator space, such that the module action $M \times X \to X$ is separately w*-continuous. Then there exist Hilbert spaces $H, K$, a unital normal *-representation $\pi : M \to B(K)$, and a weak*-homeomorphic complete isometry $\Phi : X \to B(H, K)$, such that $\Phi(bx) = \pi(b)\Phi(x)$ for all $x \in X, b \in M$.

A slightly more general result with a different proof may be found in [10]. Effros and Ruan call an $X$ satisfying the hypotheses of this result, a normal dual left operator module over $M$. A little thought will convince the reader that this result is an abstract characterization of w*-closed subspaces of $B(H, K)$ which are left invariant under multiplication by a von Neumann algebra acting on $K$.

We are able in §4 to improve on this useful characterization in four ways: 1) we show that the hypothesis that the action is separately w*-continuous it not necessary, one only needs that the action be w*-continuous in the first variable and then w*-continuity in the second variable is automatic; 2) we show that it can be arranged so that $H, K$ and $\Phi$ only depend on $X$, and not on $M$ or the particular action; 3) in the bimodule version of this theorem our general approach (§§5) shows that the left and right module actions on $X$ automatically commute with each other; and 4) we will generalize this result to allow $M$ and its action on $X$ to be replaced by any dual operator space which is a unital algebra such as perhaps a group algebra. Indeed $M$ need not even be an algebra, we will be able to characterize ‘oplications’ $m : Y \times X \to X$ which are w*-continuous in appropriate variables. An ‘oplication’ (see §§5) is a bilinear map $Y \times X \to X$, for operator spaces $X, Y$, which also satisfies (i) and (ii) of definition [13], where $e$ is a fixed element of $Ball(Y)$. We say that $e$ is the ‘unit’ of $Y$.

Thus we have extended the main results of §5 of [8] to dual modules and oplications.

We thank E. G. Effros, Christian Le Merdy, and N. Weaver for helpful comments.

We end this introduction with some notation and basic results. We first list a well known and basic functional analysis fact which we will use very frequently:

**Lemma 1.5.** (Krein-Smulian Theorem)
(1). Let $X$ be a dual Banach space, and $Y$ be a linear subspace of $X$. Then $Y$ is $w^*$-closed in $X$ if and only if $\text{Ball}(Y)$ is closed in the $w^*$-topology on $X$. In this case $Y$ is also a dual Banach space, with predual $X_*/Y_*$, and the inclusion of $Y$ in $X$ is $w^*$-continuous.

(2). A linear bounded map $T$ between dual Banach spaces is weak*-continuous if and only if whenever $x_i \to x$ is a bounded net converging weak*- in the domain space, then $T(x_i) \to T(x)$ weak*.

(3). Let $X$ and $Y$ be dual Banach spaces, and $T : X \to Y$ a $w^*$-continuous linear isometry. Then $T$ has $w^*$-closed range $V$ say, and $T$ is a $w^*$-$w^*$-homeomorphism onto $V$.

**Proof.** (1) and (2) may be found in any book on basic functional analysis. (2) is often stated for functionals $\phi$ but the result as stated follows from this by considering $\phi \circ T$. Item (3) is found in fewer books, the proof is quite obvious from (1). For by (1), $V$ is clearly $w^*$-closed in $Y$, and the restriction of $T$ to $\text{Ball}(X)$ then takes $w^*$-closed (and thus $w^*$-compact) sets to $w^*$-compact (and thus $w^*$-closed) sets in $V$. Thus the inverse of $T$ restricted to the ball is $w^*$-continuous, so $T^{-1}$ is $w^*$-continuous by (2). □

We will use the term $W^*$-algebra for a $C^*$-algebra with predual. In view of the aforementioned theorem of Sakai this is ‘the same as’ a von Neumann algebra.

We now turn to operator spaces and left multipliers. An operator space is a linear subspace of $B(H)$ for a Hilbert space $H$. In this paper all of our operator spaces are norm complete. Equivalently, there is an abstract characterization due to Ruan. We refer the reader to the books [18, 34] for more details on operator spaces. Any details needed which are omitted from these books may be found in [11, 6]. Details on completely bounded and completely positive maps may be found in [31]. We write $M_{nm}(X)$ for the operator space of $m \times n$ matrices with entries in $X$, and $C_n(X) = M_{n,1}(X)$ and $R_n(X) = M_{1,n}(X)$ as usual. For a map $T : X \to Y$ we write $T^{(n)}$ for the associated amplification $M_n(X) \to M_n(Y)$, that is $T \otimes I_n$. The ‘completely’ prefix to a property means that it passes to every $T^{(n)}$, thus for example $T$ is completely isometric means that each $T^{(n)}$ is isometric.

We now turn to duality; and here we have a notational problem, the use of the symbol $*$, which is used for three different things in this paper. Namely, we have the dual space $X^*$ of a space $X$; the adjoint or involution $S^*$ of an operator on a Hilbert space, and the adjoint operator $R^* : Y^* \to Z^*$ of an operator $R : Z \to Y$. We are forced by reasons of personal taste to leave it to the reader to determine which is meant in any given formula; although the third of these will only occur once or twice. We recall that if $X$ is an operator space then so is $X^*$; its matrix norms come from the identification $M_n(X^*) \cong CB(X, M_n)$. A dual operator space is one which is completely isometrically isomorphic to the operator space dual of another operator space. Any $\sigma$-weakly closed subspace of $B(H)$ is a dual operator space with predual determined by the predual of $B(H)$. Conversely, any dual operator space is linearly completely isometrically $w^*$-$w^*$-homeomorphic to a $\sigma$-weakly closed subspace of some $B(H)$. If $X$ is an operator space which is the dual Banach space of $Y$, then there is at most one operator space structure (i.e. matrix norms) on $Y$ with respect to which it is possible for $X = Y^*$ completely isometrically (since $Y \subseteq Y^{**} = X^*$). But one should be warned: there may in fact be no such operator space structure on $Y$ (see [20, 16] for
examples). However there is always one if; further, $X$ is a C*-algebra [1] (but not necessarily if $X$ is a nonselfadjoint operator algebra) or a MIN or MAX space.

It will be important for us to note that if $X$ is a dual operator space then so is $M_n(X)$, with predual equal to the operator space projective tensor product of the predual $T_n$ of $M_n$, and the predual of $X$. The duality pairing may be taken to be the obvious one. From this it follows that:

**Lemma 1.6.** If $X$ is a dual operator space, and $x_i$ is a net in $M_n(X)$, then $x_i \to x \in M_n(X)$ weak* in $M_n(X)$, if and only if each entry in $x_i$ converges weak* in $X$ to the corresponding entry in $x$.

We also recall [1], [8] that for a dual operator space $X = Y^*$, the space $CB(X)$ is canonically a dual operator space, with predual the operator space projective tensor product of $X$ and $Y$. A bounded net $T_i \in CB(X)$ thus converges in the w*-topology to $T \in CB(X)$ if and only if $T_i(x) \to T(x)$ w*- in $X$, for every $x \in X$. From this it is obvious that the multiplication on $CB(X)$, viewed as a bilinear map, is weak*-continuous in the first variable.

We will not use much about operator systems, but refer the reader to [18, 21] for details. For the purposes of this paper we will define a dual operator system to simply be an operator system which is a dual operator space. A unital operator space is a pair $(X,e)$, consisting of an operator space $X$ which may be completely isometrically linearly embedded in a unital C*-algebra $B$, with $e$ embedded as $1_B$. We call $e$ the ‘unit’ or ‘identity’ of $X$, and identify unital operator spaces up to ‘unital complete isometry’. Define a subset $\Delta(X) = \{ b \in X : b^* \in X \}$ of $B$. Then $\Delta(X)$ is an operator system which is independent (up to unital complete order isomorphism) of the actual $B$ containing $X$ unitally. This follows from Arveson’s result [3] that the space $\{x+y^* : x, y \in X\}$ is well defined up to complete order isomorphism. Therefore given a unital operator space $X$, the notation $\Delta(X)$ makes sense even if no particular $B$ containing $X$ is specified.

Notice further, that if $X$ in the last paragraph is an operator algebra $A$ with identity of norm 1, then we may suppose that $B$ above is a C*-algebra containing $A$ as a unital subalgebra. Hence the space $\Delta(A)$ above is a closed C*-subalgebra of $B$. This C*-algebra $\Delta(A)$ is independent up to *-isomorphism of the particular $B$. This is well-known fact.

We turn to the space $M_l(X)$ of left multipliers, and the space $A_l(X)$ of left adjointable maps, on an operator space $X$. These were defined first in [8], although earlier authors had considered variants valid for operator systems [12, 24]. They are natural generalizations of classical spaces, for example $M_l(X)$ generalizes the space of multipliers of a Banach space $Y$.

The reader not willing to refer to [8] for the original definitions, or to [8, 12, 24] for various equivalent formulations, may take the definitions from the following result from [8].

**Theorem 1.7.** Let $X$ be an operator space, and $T : X \to X$ a linear map. Then:

1. $T \in M_l(X)$ if and only if there exists a Hilbert space $H$, an $S \in B(H)$, and a completely isometric linear embedding $\sigma : X \to B(H)$ such that $\sigma(Tx) = S\sigma(x)$ for all $x \in X$.

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1There appears to be no way to define operator space left multipliers via left multipliers of an associated operator system.

2These may be defined to be the linear maps $T$ on the Banach space $X$ of the form $Tx = gx$ for $x \in X$, where $g \in C(K)$ for some $C(K)$ containing $X$ isometrically.
We have that $M_l(X)$ is an operator algebra, with algebra structure inherited from $B(X)$ and with norm the least value of $\|S\|$ possible in (1). This least value is achieved. This operator algebra has identity which is the identity operator on $X$. Also, $A_l(X) = \Delta(M_l(X))$, and this is a $C^*$-algebra.

In (1) we can replace $B(H)$ by $B(H, K)$ with no loss, or we can replace $B(H)$ with a $C^*$-algebra $A$, or a $C^*$-module.

**Corollary 1.8.** If $T$ is a left multiplier of an operator space $X$, then the ‘multiplier norm’ of $T$ is greater than, or equal to, its completely bounded norm. Also, if $Y$ is a closed subspace of $X$ with $T(Y) \subset Y$, then $T_{|Y}$ is a left multiplier of $Y$, with a smaller (or equal) ‘multiplier norm’ than that of $T$. If in addition, $T$ is adjointable on $X$ and $T^*(Y) \subset Y$, then $T_{|Y}$ is adjointable on $Y$.

We will therefore consider $M_l(X)$ as a unital subalgebra of $CB(X)$, but possessing a possibly larger ‘norm’. On $A_l(X)$ the multiplier norm coincides with the ‘cb-norm’, as is shown in §4. We assign matrix norms to $M_l(X)$ via the relation $M_l(M_l(X)) \cong M_l(C\sigma(X)) \cong M_l(M_n(X))$, and similarly for $A_l$. There are many ways to see this last relation, for example using the proof of 4.8 in [8], or by using the ‘$J_{11}$’ definition of $M_l$ in [12].

One alternate characterization of $M_l(X)$ from [8] shows that for unital operator spaces $X$, we have

$$M_l(X) \cong \{a \in C^*_e(X) : aJ(X) \subset J(X)\}$$

where $C^*_e(X)$ is the $C^*$-envelope [20], and $J$ is the (unital) embedding of $X$ inside $C^*_e(X)$. The isomorphism here is the map taking $a$ in the right hand set to the map $x \mapsto J^{-1}(aJ(x))$ on $X$. This map clearly has inverse which is the map $T \mapsto J(T(1))$. Since $1 \in X$ it follows that $M_l(X) \subset X$ in this case. More specifically, the map $M_l(X) \to X$ given by $T \mapsto T(1)$ is a unital complete isometry. Since $\|T\|_{cb} \geq \|T(1)\|$, we see that for unital operator spaces the canonical completely contractive inclusion $M_l(X) \to CB(X)$ is completely isometric. In particular for a unital operator algebra $A$, we have $M_l(A) \cong A$ completely isometrically isomorphically. Of course viewed as a subset of $CB(A)$, these left multipliers are exactly the maps $T(b) = ab$ for fixed $a \in A$.

The following, our most recent characterization of left multipliers, is also the deepest and no doubt most useful. It is a generalization of a result of W. Werner characterizing multipliers on operator systems [43].

**Theorem 1.9.** [8] A linear map $T : X \to X$ on an operator space is a left multiplier of multiplier norm $\leq 1$, if and only if the following map $\tau_T$ is completely contractive on $C_2(X)$:

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} Tx \\ y \end{bmatrix}$$

Note this last theorem immediately gives a proof of the ‘BRS’ theorem mentioned in the second paragraph of our paper. More generally it gives a proof of the oplication theorem from [8] (from which BRS follows in one line as we pointed out in [8] 5.6).
Theorem 1.10. (The oplication theorem). Let $m : Y \times X \to X$ be an oplication. Then there exists a (necessarily unique) linear complete contraction $\theta : Y \to M_l(X)$ such that $\theta(e) = I$ and $\theta(y)(x) = m(y, x)$ for all $y \in Y, x \in X$. If in addition $(Y, e)$ is an operator system then $\theta$ maps into $A_l(X)$, whereas if $(Y, e)$ is a unital algebra then $\theta$ is a homomorphism if and only if $m$ is a module action.

Proof. If $m : Y \times X \to X$ is an oplication, consider the associated linear map $\tilde{m} : Y \to B(X)$. Note that

$$\tau_{\tilde{m}(y)} \left( \begin{bmatrix} x \\ x' \end{bmatrix} \right) = \begin{bmatrix} m(y, x) \\ x' \end{bmatrix},$$

for $x, x' \in X$. The last matrix may be regarded as the formal product, via $m$, of the diagonal matrix $y \oplus e$ and the column in $C_2(X)$ with entries $x$ and $x'$. By the hypothesis, one clearly sees that $\tau_{\tilde{m}(y)}$ is contractive, and an analogous argument with matrices shows that it is completely contractive. Thus by 1.9 we see that $\tilde{m}$ is a contractive map into $M_l(X)$. Given $[y_{ij}] \in \text{Ball}(M_n(Y))$, we consider $[\tilde{m}(y_{ij})]$ as a map $S : C_n(X) \to C_n(X)$, and argue as above using matrices to see that $S \in \text{Ball}(M_l(C_n(X)))$. Thus it follows that $\theta : y \mapsto \tilde{m}(y)$ is a complete contraction from $Y \to M_l(X)$. Alternatively the complete contractivity may be deduced from the contractive case, and the relation $M_l(M_n(X)) \cong M_n(M_l(X)) \cong M_l(C_n(X))$.

It is clear that $\theta(y)(x) = m(y, x)$ for all $y \in Y, x \in X$. The last few statements are immediate (see 5.2 in [8]).

We remark that Paulsen has recently found a much simpler proof of 1.9. Combining this with the last proof gives a short, and perhaps the best, route to the oplication and BRS theorems. Indeed this route gives a very interesting extension of BRS to operator algebras with a 1-sided identity:

Theorem 1.11. Let $A$ be an operator space which is an algebra with a right identity of norm 1 (or right c.a.i.). Then $A$ is completely isometrically isomorphic to a concrete operator algebra, if and only if we have

$$\| (x \oplus Id_m)y \|_{n+m} \leq \| x \|_n \| y \|_{n+m}$$

for all $n, m \in \mathbb{N}$ and $x \in M_n(X), y \in M_{n+m}(X)$.

To explain the notation of the theorem, we have written $Id$ for a formal identity, thus the expression $(x \oplus Id_m)y$ below means that the upper $n \times (m + n)$-submatrix of $y$ is left multiplied by $x$, and the lower submatrix is left alone.

Proof. Proceed exactly as in the proof above to obtain a complete contraction $\theta : A \to M_l(A)$ such that $\theta(a)(b) = ab$. One does need to permute the entries of the matrices involved in the calculation as to get them into the form of the hypothesis, but this is a simple matter. The result may then be completed as in our 1-line proof of BRS from [8] §5: for example $\| \theta(a) \| \geq \| a1 \| = \| a \|$.}

Clearly the same idea gives a version of ‘oplication’ theorem which requires no ‘identity element’ $e$, which needs a similar hypothesis to that of [1.11].
Finally, in connection with Theorem 1.9, we also point out again here that it gives a method to recover the product in a unital operator algebra from the linear operator space structure alone. What is more important, it appears to be a simple enough method to be useful in practice; for example it works well in conjunction with weak*-topologies, etc. Namely, suppose that $A$ is an operator algebra with an identity of norm 1, but that we have forgotten the product on $A$. Let us assume for a moment that we do remember the identity element $e$. Form $M_l(A)$ using 1.9 (or if $A$ is a $C^*$-algebra, using Lemma 4.5 in [9], (which is valid for $C^*$-algebras by going to the second dual)), and define $\theta : M_l(A) \to A$ by $\theta(T) = T(e)$. Then the product on $A$ is $ab = \theta(\theta^{-1}(a)\theta^{-1}(b))$.

If we have also forgotten the specific identity element $e$, then one may only retrieve the product on $A$ up to a unitary $u$ with $u, u^{-1} \in A$. Such unitaries form a group. Indeed such unitaries may be characterized by the Banach-Stone theorem for operator algebras (see e.g. the last page of [3], or [23] for the $C^*$-algebra case) as the elements $x_0$ with the property that the map $\pi : T \mapsto T(x_0)$ is a completely isometric surjection $M_l(A) \to A$ (Indeed if $A$ is a $C^*$-algebra one only needs this to be an isometry, by Kadison’s result [23]). We remark that by Lemma 4.5 in [9] the unitaries in a $C^*$-algebra correspond to linear $T : A \to A$ such that $\pi_T$ is a surjective isometry. However in this case there are other Banach space characterizations of unitaries - C. Akemann has shown me one such). Given such an $x_0$ and $\pi$, we may again recover the product as $ab = \pi(\pi^{-1}(a)\pi^{-1}(b))$. This is the operator algebra product on $A$ which has this unitary as the identity. This is all fairly easy to see from the Banach-Stone theorem and our notion of the left multiplier algebra. Nonetheless it is quite interesting that we can recover the product in this way.

2. A characterization of dual operator algebras

In this section $A$ is an operator algebra with identity of norm 1 possessing an operator space predual $A_*$ say. We need to show that:

**Theorem 2.1.** If $A$ is a unital operator algebra with operator space predual, then the multiplication on $A$ is separately w*-continuous.

We will consider the completely isometric unital homomorphism $L : A \to CB(A)$ given by $L(a)(b) = ab$ for $a, b \in A$. As we said in §1, the range of $L$ is $M_l(A)$, and quite clearly in this case, $M_l(A) \subseteq CB(A)$ completely isometrically. As we also said in §1, we know that $CB(A)$ is the dual of the projective tensor product of $A$ and $A_*$, completely isometrically. Thus a bounded net $T_i \to T$ weak*- in $CB(A)$ if and only of $T_i(b) \to T(b)$ w*- in $A$ for each $b \in A$. The multiplication on $CB(A)$ is weak*-continuous in the first variable, quite obviously, and we will show that this property descends to its subalgebra $A$.

**Lemma 2.2.** $M_l(A)$ is weak*-closed in $CB(A)$. Therefore $M_l(A)$ has a predual which gives $M_l(A)$ a weak*-topology coinciding with the relative w*-topology on $M_l(A)$ inherited from $CB(A)$.

**Proof.** We will use Krein–Smulian. Let $T_i \in Ball(M_l(A))$ be a net converging weak* in $CB(A)$ to $T \in Ball(CB(A))$, say. Using Theorem 1.9 we will now check that $T$ is in $Ball(M_l(A))$. For suppose that $v \in Ball(C_2(A))$. Write $v$ as a column $[x \ y]^t$. Then $w_i = [T_i(x) \ y]^t \in Ball(C_2(A))$ by 1.3. Let $w = [T(x) \ y]^t$. Consider the operator space
preidual $Z$ of $C_2(A)$; any $G \in \text{Ball}(Z)$ is given by a pair $[\phi \; \psi]$ of functionals in $A_*$. The duality pairing is:

$$\langle G, v \rangle = \langle \phi, x \rangle + \langle \psi, y \rangle.$$ 

Since

$$|\langle G, w \rangle| = |\langle \phi, T(x) \rangle + \langle \psi, y \rangle| \leq 1$$

in the limit we have that

$$|\langle G, w \rangle| = |\langle \phi, T(x) \rangle + \langle \psi, y \rangle| \leq 1.$$ 

Thus $\tau_T$ is contractive. A similar argument, picking $G$ in the unit ball of the predual of $M_{2n,n}(A)$, and using Lemma 1.6, shows that $\tau_T$ is completely contractive. Thus $T \in \text{Ball}(M_l(X))$, so that $\text{Ball}(M_l(X))$ is $w^*$-closed.

\textbf{Proof.} (Of Theorem 2.1:) By symmetry it is enough to show that the multiplication is weak*-continuous in the first variable. One can see this symmetry by considering the ‘opposite algebra’ $A^{op}$ (c.f. [5] Proposition 1) which has operator space predual which is $A_*$ with the transposed matrix norm structure.

From the lemma it is evident that the map $L^{-1} : M_l(A) \to A$ is weak*-continuous with respect to the weak*-topology of the Lemma. For, if $L(a_\lambda) \to L(a)$ in $M_l(A)$, then $a_\lambda = L(a_\lambda)(1) \to L(a)(1)$ in $A$. By Lemma 1.3 (3), $L$ is weak*-continuous. But this says exactly that the multiplication on $A$ is weak*-continuous in the first variable. \hfill \Box

We next show that the hypothesis that $A$ is a dual operator space is necessary to get the full conclusion of Theorem 1.1 in general. Note however that if $A$ is a function algebra, i.e. a MIN space (see [5]), then if $A$ has a Banach space predual then it has an operator space predual [1]. In fact it is quite tricky to come up with examples of Banach space preduals of an operator space, which are not operator space preduals.

Another important remark is that, in contrast to Sakai’s theorem, the preduals of non-selfadjoint operator algebras are not necessarily unique. See [39] for more detailed information. As pointed out by Derek Westwood, this nonuniqueness may be seen quickly from the fact that Banach space preduals are not unique, together with the well known Arveson $2 \times 2$ matrix trick (played also in the proof below). Nonetheless, the automatic separate $w^*$-continuity of the multiplication makes some kind of uniqueness statement, which perhaps may be more fully exploited.

\textbf{Proposition 2.3.} (cf. [28]) There exists an operator algebra $A$ with identity of norm 1, i.e. a closed subalgebra of some $B(H)$ containing $I_H$, which is the dual of a Banach space $X$, and for which the multiplication on $A$ is separately $w^*$-continuous, but for which there is no weak*-homeomorphic completely isometric isomorphism of $A$ onto a dual operator algebra. Indeed there exists an operator algebra $A$ with identity of norm 1, with no operator space predual at all, but which is a dual Banach space.

\textbf{Proof.} Take any operator space $Y \subset B(H)$ which is the dual of a Banach space $X$, such that $Y$ with the associated $w^*$-topology derived from $X$ is not completely isometrically isomorphic via a $w^*,w^*$-homeomorphic linear map to a dual operator space (see [28] for such an example, and [16] for an example which has a unique Banach space predual). Build the
‘canonical $2 \times 2$ unital operator algebra’ $A \subset B(H \oplus H)$ which has $Y$ contained completely isometrically as the 1-2-corner and scalars on the diagonal. This is an adaption of an example from [28] where Le Merdy puts zeroes on the diagonal. Suppose that $i : Y \to \ell^\infty(I)$ is any isometric $w^*$-homeomorphic linear embedding (such embeddings exist since any Banach space is a quotient of a $\ell^1(I)$). Then $A$ may be identified isometrically with a subalgebra $B$ of $M_2(\ell^\infty(I))$ via the obvious map

$$\begin{bmatrix} \lambda I_H & yH \\ 0 & \mu I_H \end{bmatrix} \mapsto \begin{bmatrix} \lambda 1_I & i(y) \\ 0 & \mu 1_I \end{bmatrix}.$$ 

Here $1_I$ is the identity of $\ell^\infty(I)$. That this is an isometry follows for example from [19] (Chapter IV §2). It is clear that this subalgebra $B$ is $w^*$-closed, so that $A$ is a dual Banach space. Moreover the inclusion of $Y$ into $A$ as the corner, is a $w^*$-homeomorphism. That is, $Y$ is completely isometrically isomorphic to a $w^*$-closed subspace of $A$. Therefore if $A$ with its given weak*-topology was $w^*$-homeomorphic and completely isometric to a $\sigma$-weakly closed subspace of some $B(H)$, then so would $Y$ be, which is false.

Finally, suppose that $Y$ is as in the example at the end of [16], an operator space which has a unique Banach space predual, but no operator space predual. The $A$ constructed above is a dual Banach space. If $A$ were a dual operator space, there would exist by Theorem 1.1, a completely isometric unital homomorphism $\pi$ of $A$ onto a concrete dual operator algebra $B$. However it is very easy to check that the diagonal idempotents in $A$ force the Hilbert space on which $B$ acts, to split as a direct sum $K \oplus N$, such that $B$ may be written as $2 \times 2$ matrices exactly like our original $A$. Moreover, since there are projections onto $K$ and $N$, the 1-2-corner of $B$ is $w^*$-closed in $B(N, K)$. Also, $\pi$ restricted to the copy of $Y$, maps $Y$ completely isometrically onto this $1-2$ corner of $B$. But this forces $Y$ to be a dual operator space.

Our theorem [1.1] is thus best possible in the operator space category. An obvious question which remains open is the same question in the Banach category: if $A$ is a unital operator algebra which is a dual Banach space, then is $A$ weak*-homeomorphic via an isometric unital homomorphism to a $\sigma$-weakly closed subalgebra of some $B(H)$?

Note that our characterization gives as a quick deduction, the fact due to Le Merdy [28], and independently Arias and Popescu [2], that if $A$ is a concrete dual operator algebra (unita or otherwise), and if $I$ is a weak*-closed two-sided ideal in $A$, then $A/I$ is a dual operator algebra. For if $A$ is a weak*-closed subalgebra of $B(H)$, then by basic functional analysis so is $A + \mathbb{C}I_H$, and $(A + \mathbb{C}I_H)/I$ satisfies the hypotheses of our theorem. This gives the result. However, this proof is not a good proof for this result, since as Le Merdy has pointed out to me privately, this $A/I$ result already follows from the much simpler [1.2]. Indeed one need only note that the product on $A/I$ is automatically separately weak* continuous. Le Merdy sees this by defining for each $b \in A$, a map $u : A \to A/I$ by $u(a) = [ab]$. Clearly $u$ is weak* -continuous. Thus its kernel is weak*-closed, and we get an induced weak*-continuous map $A/(Ker u) \to A/I$. (Indeed the last sentence works for weak*-continuous continuous maps between any dual Banach spaces.) The desired separate weak*-continuity follows immediately. It is worth pointing out that this $A/I$ result whose proof we have just sketched is the major step in Le Merdy’s earlier characterization of general dual operator
algebras (assuming separately weak*-continuous product) in [28]. Thus Le Merdy’s later characterization simplifies this step.

It is interesting that the idea in the proof of our theorem [11] gives another proof of the following classical result. Our argument has the advantage too of being applicable to dual function algebras (see comments after the proof).

**Theorem 2.4.** (Classical) Suppose that \( A \) is a \( C(K) \) space, or a commutative \( C^* \)-algebra, with a predual. Then \( A \) is \( w^* \)-homeomorphically isometrically \(*\)-isomorphic to a commutative von Neumann algebra.

**Proof.** We may assume that \( A = C(K) \) for compact \( K \), by the usual argument (if \( A = C_0(K) \) for locally compact \( K \) then Krein-Milman implies the existence of extreme points. Any such extreme point, by a simple application of Urysohn’s lemma has constant absolute value 1, so that \( K \) is compact). Replace the argument above for Theorem 2.1 by an almost identical argument in \( B(A) \) instead of \( CB(A) \), and using the very easy [22] Proposition I.3.9 instead of our theorem [1.9] to obtain that the multiplication on \( C(K) \) is separately \( w^* \)-continuous. Then we may use Le Merdy’s proof of [1.2] from [29] to finish. Alternatively, there is a commutative proof using measure theory which avoids Le Merdy’s argument. We will omit this since this result is quite well known.

The argument above may be adapted, using the M-bounded characterization of multipliers [22], to any unital function algebra (i.e. uniform algebra) with a predual, to show that any such algebra is \( w^* \)- and isometrically isomorphic to a commutative dual operator algebra. Of course this fact also follows almost immediately from our theorem [1.1]. This leads us to repeat from [10] the following very interesting question which seems to be open: is a unital function algebra with Banach space predual, \( w^* \)- and isometrically isomorphic to a \( w^* \)-closed subalgebra of an \( L^\infty \) space? We know, by the above, that it is a commutative dual operator algebra. If the answer to this question is negative then this would show that one must leave the category of function algebras to study ‘dual function algebras’!

Finally, we remark that Le Merdy’s theorem [1.2] is true even without the presence of an identity (see [28]); but we have no idea if the ‘separate \( w^* \)-continuity’ hypothesis may be removed in the nonunital case, except in the case that \( A \) has a contractive approximate identity. Notice that if \( A \) is an operator algebra with a contractive approximate identity \( \{ e_\alpha \} \), and if \( A \) has a predual and separately \( w^* \)-continuous product, then if \( e_\alpha \to e \) weak* in \( A \), then \( e \) is clearly an identity in \( A \) of norm 1. However it seems to be much more difficult to remove the ‘separately \( w^* \)-continuity’ hypothesis in the last line. We will do this by using an additional result from [9]. In fact we have a more general result (which does not assume ‘separate \( w^* \)-continuity’ of the product):

**Theorem 2.5.** Let \( A \) be an operator algebra with a right contractive approximate identity, and suppose that \( A \) is a dual Banach space. Then \( A \) has a right identity of norm 1.

**Proof.** We know that \( M_l(A) \) is a unital operator algebra, and it is easy to check that the natural map \( L : A \to M_l(A) \) is a completely isometric homomorphism. Let \( B = \{ T \in M_l(A) : TL(a) = L(T(a)) \text{ for all } a \in A \} \). It is easy to see that \( L(A) \subset B \), and that \( B \) is a unital subalgebra of \( M_l(A) \). Clearly \( B \) contains \( L(A) \) as a left ideal. Therefore by §6 of
L(A) is a complete left M-ideal in B. By Theorem 3.10 (4) in [4], L(A) is a complete left M-summand of B. Thus L(A) = Be for a projection e ∈ B. Clearly e ∈ L(A), so that L(A) = L(A)e, from which it follows that e is a right identity for L(A).

3. Multiplier algebras of dual operator spaces

We begin with a general functional analytic result and its operator space variant. We will not need anything above for its proof, except for Lemma 3.1 and 3.2.

Lemma 3.1. (1) Let X and Y be Banach spaces, with Y a dual Banach space, and let T : X → Y be a 1-1 linear map. Then the following are equivalent:

(i) X is a dual Banach space and T is w*-continuous, and
(ii) T(Ball(X)) is w*-compact.

(2) Let X and Y be operator spaces, with Y a dual operator space, and let T : X → Y a 1-1 linear map such that T(\langle Ball(M_n(X)) \rangle) is w*-compact for every positive integer n. Then the predual of X given in the proof of (1), is an operator space predual of X, and T is w*-continuous.

Proof. (1) is undoubtedly well known, we give a proof since we have no reference for it. Thanks go to C. Le Merdy for helping to simplify my original argument. That (i) implies (ii) is clear (as indeed is the converse of statement (2), by the way). Given (ii), we see by the Principle of Uniform Boundedness that T(Ball(X)) and therefore T is bounded. We may assume wlog that T is a contraction. Suppose that Z is the predual of Y, and let W = T*(Z), a linear subspace of X*. The canonical map j : X → W* is 1-1 and contractive and

\langle j(x), T*(z) \rangle = T*(z)(x) = z(T(x)) = T(x)(z).

On the other hand, given g ∈ Ball(W*) let ˜g(z) = g(T*(z)) for z ∈ Z, then ˜g ∈ Ball(Z*) = Ball(Y). If ˜g = T(x) for an x ∈ Ball(X) then we’d be done, for in this case its clear that g = j(x). So suppose, by way of contradiction that ˜g /∈ T(Ball(X)). By assumption T(Ball(X)) is w*-closed, so by the Hahn-Banach theorem, there exists z ∈ Z such that ˜g(z) > 1 and |⟨T(x), z⟩| ≤ 1 for all x ∈ Ball(X). The latter condition implies that \|T*(z)\| ≤ 1, whereas the former condition implies the contradictory assertion that g(T*(z)) > 1.

That T is w*-continuous with respect to this predual of X is now clear.

Now we prove (2). We will use the fact (see [20]) that if an operator space X is the dual of a Banach space W, and if W is equipped with its natural matrix norms as a subspace of X* via the natural inclusion, then X is the dual operator space of W if and only if the unit ball of M_n(X) is σ(X,W)-closed. To check that the latter condition holds, let x_λ = [x_{ij}^λ] be a net in Ball(M_n(X)), converging to x = [x_{ij}] in M_n(X), the convergence being the σ(X,W)-convergence. That is, \langle x_{ij}^λ, w \rangle → \langle x_{ij}, w \rangle for all i, j and w ∈ W. Equivalently, \langle T(x_{ij}^λ), z \rangle → \langle T(x_{ij}), z \rangle for all z ∈ Z. By Lemma 3.2 the matrices [T(x_{ij}^λ)] converge w*- to [T(x_{ij})] in M_n(Y). By hypothesis, [T(x_{ij})] ∈ T(\langle Ball(M_n(X)) \rangle), so that x ∈ Ball(M_n(X)), and we are done.

\[\square\]
In our situations below, we will know in advance that the operator $T$ is bounded, and so we need only check $w^*$-closedness instead of $w^*$-compactness in the Lemma.

We will now show that the canonical inclusion $M_l(X) \to CB(X)$ satisfies the hypothesis of (2) of the previous result.

**Corollary 3.2.** If $X$ is a dual operator space then

1. $M_l(X)$ is a dual operator space.
2. A bounded net $\{a_i\}$ in $M_l(X)$ converges weak* to $a$ in $M_l(X)$ iff $a_ix \to ax$ weak* in $X$ for all $x \in X$.
3. The product on $M_l(X)$ is separately weak*-continuous, and $M_l(X)$ is a dual operator algebra.
4. If $M_l^{w^*}(X)$ is the set of $T \in M_l(X)$ which are weak*-continuous as maps on $X$, then $M_l^{w^*}(X)$ is a unital norm closed subalgebra of $M_l(X)$, and the product on $M_l^{w^*}(X)$ is separately weak*-continuous.

**Proof.** (1): Using [3.4], we need to show that if $T_\lambda = [T_{i,j}^\lambda]$ is a net in $Ball(M_n(M_l(X)))$ converging $w^*$- in $M_n(CB(X))$ to $T = [T_{i,j}]$, then $T \in Ball(M_n(M_l(X)))$. We will use the fact from [8] §4 that $M_n(M_l(X)) \cong M_l(C_n(X))$, and we will test for $T \in Ball(M_l(C_n(X)))$ using [1.9]. So we begin with two matrices $x = [x_{pq}(k)], y = [y_{pq}(k)] \in M_m(C_n(X))$ (rows indexed by $p$, columns indexed by $q, k$), and we need to check that

$$\left\| \begin{bmatrix} \mu(T)^{(m)}(x) \\ y \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|$$

where $\mu : M_n(M_l(X)) \to M_l(C_n(X))$ is the canonical completely isometric identification. However we do know that

$$\left\| \begin{bmatrix} \mu(T_\lambda)^{(m)}(x) \\ y \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\| .$$

And of course $T_{i,j}^\lambda(x) \to T_{i,j}(x)$ weak* in $X$ for all $i, j$ and $x \in X$, by hypothesis. The first matrix in the last displayed equation is an element of $M_{2mn,m}(C_n(X)) = M_{2mn,m}(X)$. Now pick a norm 1 functional $G$ in the predual of the $M_{2mn,m}(X)$, and apply it to this first matrix in displayed equation. Using Lemma [1.3] as in the proof of [2.2] shows that indeed $T \in Ball(M_l(C_n(X)))$. Thus we have now proved (1).

Item (2) follows from (1) and the definition of the weak*-topologies concerned (see proof of [3.4]). Item (3) follows from Theorem [1.4].

Finally, the assertions of (4) are now fairly obvious. If $T_n \in M_l^{w^*}(X)$, with $T_n \to T$ in multiplier norm, then $T_n \to T$ in cb-norm. Since the canonical image of $CB(X_*)$ in $CB(X)$ is norm closed, we see that $M_l^{w^*}(X)$ is norm closed.

Notice that the proof above provides, if $X$ is a dual operator space such that the natural map $M_l(X) \to CB(X)$ is a complete isometry, a canonical predual for $M_l(X)$ which is a natural quotient of the operator space projective tensor product of $X$ and $X_*$.

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aIn fact a uniform version of this is true, namely that this map is what is known in the Banach space literature as a ‘semi-embedding’, however we see no use yet for this fact.
The main part we wish to stress here is that for any dual operator space \( X \), we have that \( M_l(X) \) is a dual operator algebra, that is it may be \( \sigma \)-weakly-homeomorphically and completely isometrically identified with a \( \sigma \)-weakly closed subalgebra \( B \) of some \( B(K) \).

From this fact we can deduce another proof of the following result from [4]:

**Corollary 3.3.** If \( X \) is a dual operator space, then \( A_l(X) \) is a \( W^* \)-algebra.

**Proof.** From the above we now know that we can represent \( M_l(X) \) as a \( \sigma \)-weakly closed subalgebra \( B \) of some \( B(K) \), with \( I_K \in B \). Then the set of adjoints of operators in \( B \) is a \( \sigma \)-weakly closed subalgebra of \( B(K) \), and \( A_l(X) = \Delta(B) \subset B(K) \). So \( A_l(X) \) is \( \sigma \)-weakly closed in \( B(K) \), and consequently is a dual space.

The last result was crucial for much of the theory in our project with Effros and Zarikian.

We end this section with another application of ‘noncommutative \( M \)-structure’. We give our main result:

**Theorem 3.5.** Suppose that \( X \) is a TRO, ternary system, or right \( C^* \)-module, which is also a dual operator space. Then

1. \( X \) is a corner of a von Neumann algebra. That is, \( X \cong e R(1 - e) \) completely isometrically and a weak*-homeomorphically, for a von Neumann algebra \( R \) and orthogonal projection \( e \in R \).
2. If in addition \( X \) is injective, then \( R \) may be taken to be injective too.
3. Suppose that \( X \) is full on the right over a \( C^* \)-algebra \( A \). Let \( M \) be the multiplier algebra of \( A \). Then \( M \) and the algebra of adjointable operators on \( X \), are von Neumann algebras, and \( X \) is a self-dual \( C^* \)-module over \( M \) (and therefore also over \( A \)).

**Proof.** We prove (3) first. We will use basic Hilbert \( C^* \)-module facts [25, 36]. We know from [3] Appendix A.4 that for a right \( C^* \)-module \( X \) we have that \( A_l(X) \) is the algebra of adjointable (in the usual \( C^* \)-module sense) right module maps on \( X \). From [3], \( A_l(X) \) is a von Neumann algebra. By Morita symmetry, and using Kasparov’s result asserting that the algebra of adjointable maps is the multiplier algebra of the algebra of ‘compact’ adjointable maps, we have that the multiplier algebra \( M \) of \( A \) is isomorphic to the algebra \( A_r(X) \) of adjointable operators acting on the right of \( X \). By symmetry this is also a von Neumann algebra. By [1,2] in the next section, the product map \( A_l(X) \times X 
\times M \to X \) is separately weak*-continuous. We will now check that the inner product \( X 
\times X \to M \) is separately weak*-continuous. To this end suppose that we have a bounded net \( y_\lambda \to y \) weak* in \( X \). Then letting \( T \) be the rank 1 adjointable operator \( w \otimes x \) for \( w, x \in X \), we have by the above that \( Ty_\lambda \to Ty \) weak*, so that \( w \langle x, y_\lambda \rangle \to w \langle x, y \rangle \) weak*. The net \( \{ \langle x, y_\lambda \rangle \} \) is bounded, and if it has a weak*-convergent subnet \( \{ \langle x, y_{\lambda_0} \rangle \} \), converging to \( b \in M \) say, then by the first part \( w \langle x, y_{\lambda_0} \rangle \to wb \) weak*. Hence \( wb = w \langle x, y \rangle \). Since this is true for all \( w \in X \), it follows that \( b = \langle x, y \rangle \). Hence the net \( \langle x, y_\lambda \rangle \to \langle x, y \rangle \) weak*.

We now can see that \( X \) is a self-dual right \( M \)-module by the following trick. Suppose that \( f : X \to M \) is a bounded \( M \)-module map. It is well known that one may choose a contractive
approximate identity \( \{ T_n \} \) for the ‘imprimitivity \( C^* \)-algebra’ \( K_A(X) \) with terms of the form \( \sum_{k=1}^n x_k \otimes x_k \) for \( x_k \in X \). Note that \( f(T_n(x)) = \sum_{k=1}^n f(x_k) \langle x_k, x \rangle = \langle \sum_{k=1}^n x_k f(x_k)^*, x \rangle \). The element \( \sum_{k=1}^n x_k f(x_k)^* \) depends on \( \alpha \), let us name it \( w_\alpha \). Then \( \{ w_\alpha \} \) is a bounded net in \( X \), so has a subnet converging \( w^* \)-to \( w \in X \). By replacing the net with the subnet, and using the first part, \( \langle w_\alpha, x \rangle \to \langle w, x \rangle \). Since \( f(T_n(x)) \to f(x) \) in norm, we obtain \( f(x) = \langle w, x \rangle \) for all \( x \in X \). So \( X \) is self-dual over \( M \).

(1) follows immediately from (3) as is well known \([31, 33]\). For example this is easily seen by considering the linking von Neumann algebra, which may be taken to be \( B_M(X \oplus_c M) \), where \( X \oplus_c M \) is is the \( C^* \)-module direct sum. Indeed the \( C^* \)-module direct sum \( X \oplus_c M \) is clearly selfdual over \( M \), and hence by \([30]\), \( B_P(X \oplus_c M) \) is a von Neumann algebra.

(2): If \( X \) is injective in addition, then we know from \([12]\) (see the introduction and Corollary 1.8 there) that (using the notations of that paper) \( I_{22} \) is the multiplier algebra of \( X^*X = A \), so that \( I_{22} = M \). Similarly, the linking von Neumann of the last paragraph, is what we called \( I(S(X)) \) in \([12]\) and is therefore injective. So \( X \) is a corner of an injective von Neumann algebra.

There are some similar arguments to the above in the thesis of J. Schweizer \([10]\). We also observe that another proof of the last corollary proceeds on the following lines. If \( X \) is a dual operator space then the natural projection \( P : X^{**} \to X \) is completely contractive. One then sees from results in \([13]\) that \( P \) is a ‘conditional expectation’. This allows one to circumvent a large part of the argument in \([16]\) (which is exactly what one would expect, given our stronger hypothesis).

4. CHARACTERIZATIONS OF DUAL ACTIONS ON AN OPERATOR SPACE

We will use the following simple result from \([9]\). For completeness we give a short proof.

**Proposition 4.1.** If \( X \) is a dual operator space then any \( T \in A_1(X) \) is weak*-continuous.

**Proof.** We may assume that \( 0 \leq T \leq 1 \) in the \( C^* \)-algebra \( A_1(X) \). By basic operator theory, any such \( T \) is the 1-1-corner of an orthogonal projection \( P \in M_2(A_1(X)) \). However \( M_2(A_1(X)) \cong A_\ell(C_2(X)) \). Moreover the left adjointable projections are exactly the complete left \( M \)-projections, which are shown in \([9]\) to be weak*-continuous (the argument for this is quite simple and very similar to the classical one). Thus \( P \) is weak*-continuous. We have \( T = \pi_1 P j_1 \), where \( \pi_1 : C_2(X) \to X \) and \( j_1 : X \to C_2(X) \) are the canonical maps. Since these latter maps are clearly weak*-continuous, so is \( T \).

**Corollary 4.2.** Any dual operator space \( X \) is a normal dual \( A_1(X) - A_1(X) \)-bimodule. Also the left module action of \( M_1(X) \) or \( A_1(X) \) on \( X \) is weak*-continuous in the first variable.

The proof of the last corollary is elementary from \([1]\) and the definition of the \( w^* \)-topology on \( A_1(X) \) from §3 (i.e. Corollary 3.2 (2)).

Unfortunately, at this point we do not know whether the action of \( M_1(X) \) on \( X \) is weak*-continuous in second variable, or equivalently whether left multipliers on \( X \) are automatically weak*-continuous on \( X \). This makes it necessary to make a definition: we say that a dual operator space is **left normal** if every \( T \in M_1(X) \) is weak*-continuous on \( X \). Unital dual operator algebras, dual TRO’s and reflexive spaces are all left normal, for example.
Theorem 4.3. Let \( m : M \times X \to X \) be an operator module action of a \( W^* \)-algebra on a dual operator space, or more generally an oplication of a dual operator system on a dual operator space. If \( m \) is \( w^* \)-continuous in the first variable, then it is separately \( w^* \)-continuous. Moreover, in this case there exist Hilbert spaces \( H, K \), a unital \( w^* \)-continuous completely positive map \( \pi : M \to B(K) \), and a \( w^* \)-continuous complete isometry \( \Phi : X \to B(H,K) \), such that \( \Phi(m(b,x)) = \pi(b)\Phi(x) \) for all \( x \in X, b \in M \). Moreover \( H, K, \Phi \) can be chosen to only depend on \( X \), and not on \( M \) or the particular action. If \( M \) is a \( W^* \)-algebra, then \( \pi \) is an \( \ast \)-homomorphism if and only if \( m \) is a module action.

Proof. First apply Theorem 1.4 to the \( A_l(X) \) action on \( X \) (using Corollary 1.2). This provides the Hilbert spaces \( H, K \) and the map \( \Phi \). We also obtain a normal \( \ast \)-homomorphism \( \rho : A_l(X) \to B(K) \), with \( \rho(T)\Phi(x) = \Phi(Tx) \) for all \( x \in X, T \in A_l(X) \). Next apply the oplication theorem, to get a unique unital completely positive \( \psi : M \to A_l(X) \) such that \( \psi(b)x = m(b,x) \) for all \( x \in X, b \in M \). The last assertion in the statement of our theorem follows also from the oplication theorem. It is elementary to check from the definition of the \( w^* \)-topology on \( A_l(X) \) from \( \S 3 \), the fact that \( m \) is \( w^* \)-continuous in the first variable, and (2) of the Krein-Smulian theorem, that \( \psi \) is \( w^* \)-continuous. Let \( \pi = \rho \circ \psi \). Finally, if \( b \in M, \) and \( x_i \to x \) \( w^* \)- in \( X \), then \( \pi(b)\Phi(x_i) \to \pi(b)\Phi(x) \) \( w^* \)- in \( B(H,K) \). That is, \( m(b,x_i) \to m(b,x) \) (since \( \Phi \) is a \( w^* \)-\( w^* \)-homeomorphism). Thus \( m \) is \( w^* \)-continuous in the second variable. \( \square \)

We should also remark on the necessity of the hypothesis that \( m \) be \( w^* \)-continuous in the first variable, in the above theorem. To see this consider the natural action of \( B(H) \) on \( B(H)'' \). This is an operator module action which is clearly not \( w^* \)-continuous in the first variable.

We now turn away from the case where \( M \) is a \( W^* \)-algebra or dual operator system, and we will replace it by a dual operator algebra, or general dual operator space. Thus we consider oplications \( A \times X \to X \), where \( A, X \) are dual operator spaces. One immediately has:

Theorem 4.4. Suppose that \( m : A \times X \to X \) is an oplication of a dual operator space \( A \) on a dual operator space \( X \), which is \( w^* \)-continuous in the first variable. Then there exists a unique \( w^* \)-continuous unital completely contractive \( \pi : A \to M_l(X) \) such that \( \pi(a)x = m(a,x) \) for all \( a \in A, x \in X \). Also:

1. If \( A \) is an algebra then \( \pi \) is a homomorphism iff \( m \) is a module action.
2. If \( m \) is separately \( w^* \)-continuous then the range of \( \pi \) is contained in \( M_l^w(X) \).

Proof. This follows immediately from the oplication theorem which provides us with a unital \( \pi : A \to M_l(X) \) such that \( \pi(a)x = m(a,x) \) for all \( a \in A, x \in X \). As in 1.3 \( \pi \) is \( w^* \)-continuous. A similar argument proves (2.). \( \square \)

The following characterizes dual oplications as corresponding to appropriate concrete \( w^* \)-closed spaces of operators on Hilbert space. It is a generalization of Theorem 3.1 in [10] (which gave a similar characterization for dual operator modules over dual operator algebras):

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### Theorem 4.3

Let \( m : M \times X \to X \) be an operator module action of a \( W^* \)-algebra on a dual operator space, or more generally an oplication of a dual operator system on a dual operator space. If \( m \) is \( w^* \)-continuous in the first variable, then it is separately \( w^* \)-continuous. Moreover, in this case there exist Hilbert spaces \( H, K \), a unital \( w^* \)-continuous completely positive map \( \pi : M \to B(K) \), and a \( w^* \)-continuous complete isometry \( \Phi : X \to B(H,K) \), such that \( \Phi(m(b,x)) = \pi(b)\Phi(x) \) for all \( x \in X, b \in M \). Moreover \( H, K, \Phi \) can be chosen to only depend on \( X \), and not on \( M \) or the particular action. If \( M \) is a \( W^* \)-algebra, then \( \pi \) is an \( \ast \)-homomorphism if and only if \( m \) is a module action.

Proof. First apply Theorem 1.4 to the \( A_l(X) \) action on \( X \) (using Corollary 1.2). This provides the Hilbert spaces \( H, K \) and the map \( \Phi \). We also obtain a normal \( \ast \)-homomorphism \( \rho : A_l(X) \to B(K) \), with \( \rho(T)\Phi(x) = \Phi(Tx) \) for all \( x \in X, T \in A_l(X) \). Next apply the oplication theorem, to get a unique unital completely positive \( \psi : M \to A_l(X) \) such that \( \psi(b)x = m(b,x) \) for all \( x \in X, b \in M \). The last assertion in the statement of our theorem follows also from the oplication theorem. It is elementary to check from the definition of the \( w^* \)-topology on \( A_l(X) \) from \( \S 3 \), the fact that \( m \) is \( w^* \)-continuous in the first variable, and (2) of the Krein-Smulian theorem, that \( \psi \) is \( w^* \)-continuous. Let \( \pi = \rho \circ \psi \). Finally, if \( b \in M, \) and \( x_i \to x \) \( w^* \)- in \( X \), then \( \pi(b)\Phi(x_i) \to \pi(b)\Phi(x) \) \( w^* \)- in \( B(H,K) \). That is, \( m(b,x_i) \to m(b,x) \) (since \( \Phi \) is a \( w^* \)-\( w^* \)-homeomorphism). Thus \( m \) is \( w^* \)-continuous in the second variable. \( \square \)

We should also remark on the necessity of the hypothesis that \( m \) be \( w^* \)-continuous in the first variable, in the above theorem. To see this consider the natural action of \( B(H) \) on \( B(H)'' \). This is an operator module action which is clearly not \( w^* \)-continuous in the first variable.

We now turn away from the case where \( M \) is a \( W^* \)-algebra or dual operator system, and we will replace it by a dual operator algebra, or general dual operator space. Thus we consider oplications \( A \times X \to X \), where \( A, X \) are dual operator spaces. One immediately has:

Theorem 4.4. Suppose that \( m : A \times X \to X \) is an oplication of a dual operator space \( A \) on a dual operator space \( X \), which is \( w^* \)-continuous in the first variable. Then there exists a unique \( w^* \)-continuous unital completely contractive \( \pi : A \to M_l(X) \) such that \( \pi(a)x = m(a,x) \) for all \( a \in A, x \in X \). Also:

1. If \( A \) is an algebra then \( \pi \) is a homomorphism iff \( m \) is a module action.
2. If \( m \) is separately \( w^* \)-continuous then the range of \( \pi \) is contained in \( M_l^w(X) \).

Proof. This follows immediately from the oplication theorem which provides us with a unital \( \pi : A \to M_l(X) \) such that \( \pi(a)x = m(a,x) \) for all \( a \in A, x \in X \). As in 1.3 \( \pi \) is \( w^* \)-continuous. A similar argument proves (2.). \( \square \)

The following characterizes dual oplications as corresponding to appropriate concrete \( w^* \)-closed spaces of operators on Hilbert space. It is a generalization of Theorem 3.1 in [10] (which gave a similar characterization for dual operator modules over dual operator algebras):
**Theorem 4.5.** Suppose that \( m : A \times X \to X \) is an oplication of a dual operator space \( A \) on a dual operator space \( X \), with \( e \) the ‘identity’ of \( A \). Suppose that either \( m \) is separately weak*-continuous, or that \( X \) is left normal and \( m \) is weak*-continuous in the first variable. Then there exist Hilbert spaces \( H \) and \( K \) a weak*-homeomorphic completely isometric embedding \( \Phi : X \to B(H,K) \), and a weak*-continuous completely contractive linear map \( \theta : A \to B(K) \), such that \( \theta(e) = I_K \), and \( \Phi(m(a,x)) = \theta(a)\Phi(x) \) for all \( a \in A, x \in X \). If \( A \) is also an algebra, and \( m \) is a module action then \( \theta \) is a homomorphism. Finally, the map \( \Phi \) and spaces \( H \) and \( K \) may be chosen to only depend on \( X \), i.e. they are independent of \( A \) and \( m \).

**Proof.** By 4.4 there is a unique w*-continuous unital complete contraction \( \rho : A \to M_l^\mu(X) \) such that \( m(a,x) = \rho(a)x \). If \( X \) is left normal then the proof proceeds exactly as in 4.3, but using Theorem 3.1 in [10] instead of 1.4, to obtain the result. So we assume henceforth that \( m \) is separately weak*-continuous.

Let \( B = M_l^\mu(X) \), a closed unital subalgebra of the dual operator algebra \( M_l(X) \). On \( B \) we consider the relative weak*-topology inherited from \( M_l(X) \). Consider the natural operator module action \( n : B \times X \to X \). Then \( n \) is separately weak*-continuous. If one follows through the proof of Theorem 3.1 in [10] carefully one finds that, even without assuming that \( B \) is a dual space, we have a weak*-homeomorphic completely isometric embedding \( \Phi : X \to B(H,K) \), and a completely contractive unital homomorphism \( \pi : B \to B(K) \), such that \( \Phi(n(b,x)) = \pi(b)\Phi(x) \) for all \( b \in B, x \in X \). Moreover for any bounded net \( b_\lambda \to b \) weak*-in \( B \), we have \( \pi(b_\lambda) \to \pi(b) \) weak*-in \( B(K) \).

Let \( \theta = \pi \circ \rho \). Then \( \theta \) restricted to \( Ball(A) \) is w*-continuous, so \( \theta \) is w*-continuous. The rest is clear.

There is a bimodule version of the last theorem which we leave to the reader. A quite interesting feature of these theorems, and 4.3, is that it provides for any dual operator space \( X \), a completely isometric w*-w*-homeomorphic representation of \( X \) inside \( B(H,K) \) for some Hilbert spaces, and dual operator algebras inside \( B(K) \) and \( B(H) \), which are universal in that every ‘normal dual bimodule’ action on \( X \) has to ‘factor through’ this one.

Theorem 4.5 is not quite a complete analogue of Theorem 4.3, in that we were not able to answer the question: does w*-continuity in the first variable imply w*-continuity in the second? But it turns out that this is exactly the same as the question of the existence of a non-weak*-continuous left multiplier on \( X \):

**Corollary 4.6.** Let \( X \) be a dual operator space. There exists a left operator module action of a dual operator algebra on \( X \) which is weak*-continuous in the first variable but not the second, if and only if there exist left multipliers which are not weak*-continuous on \( X \). We are not sure if this ever happens.

**Proof.** If \( X \) is not left normal, then the action of \( M_l(X) \) on \( X \) is only weak*-continuous in the left variable. Conversely, if \( X \) is left normal, and we are given a left module action of a dual operator algebra on \( X \) which is weak*-continuous in the left variable, then it is weak*-continuous in the second variable exactly as in 4.3, but using [10] Theorem 3.1 instead of 4.4.
Next we make a remark on unital operator spaces which are also dual operator spaces. In this case, as explained in the introduction, the map $M(I(X)) \to X$ given by $T \mapsto T(1)$ is a unital complete isometry. However it is clearly weak*-continuous with the topology on $M(I(X))$ from §3, so that by [13] it is a weak*-homeomorphism onto its w*-closed range $W$. Thus $X$ contains a w*-closed unital subspace which is also a dual operator algebra. It is clear that any $T \in M(I(X))$ has the property that $T(W) \subset W$, so that $T$ restricted to $W$ is a multiplier of $W$ by Corollary [13]. Consequently $T$ is weak*-continuous on $W$, since $W$ is a dual operator algebra.

Finally we consider the ‘central 2-sided’ version of the theory of this section. We will assume familiarity with some basic ideas from [8]. Recalling that $A_l(X)$ corresponds to left operator module actions of a $C^*$-algebra on $X$, and that $A_r(X)$ corresponds to right actions, one is led to consider the subset $A_l(X) \cap A_r(X)$ inside $CB(X)$. This space is a commutative $C^*$-algebra which we have thoroughly studied elsewhere, and for which we have quite a large number of characterizations. We will call it the operator space centralizer algebra $Z(X)$, not to be confused with the Banach space centralizer algebra $Cent(X)$ (see [22]). Nonetheless, $Z(X)$ may be developed entirely analogously to the classical centralizer theory [1, 22]. The ensuing theory might be called ‘central complete M-structure’, as opposed to ‘complete right M-structure’. ‘Central complete M-structure’ is much less interesting in some ways than the ‘1-sided’ theory precisely because it is so close to the classical, commutative theory surveyed in [22]. We will present these details elsewhere. Here we will simply discuss the ‘central versions’ of the characterization theorems of the type above.

We first note that in the development of $A_r(X)$ as opposed to $A_l(X)$ there is a slight twist, the subset $A_r(X)$ of right adjointable maps on $X$ should be given the opposite of the usual multiplication of $CB(X)$ if we want it to correspond *-isomorphically to a subalgebra of $\mathcal{F}(X)$ in the language of [8] (or $I_{22}(X)$ in the language of [12]). This will not really be an issue for us though. It follows easily by looking at these latter algebras, that $ST = TS$ if $S \in A_l(X), T \in A_r(X)$. Thus the subalgebra $Z(X) = A_l(X) \cap A_r(X)$ of $A_l(X)$ is a $C^*$-algebra which is also commutative. It also follows from [3, 3] now that if $X$ is a dual operator space then $Z(X)$ is a commutative $W^*$-algebra.

We will say that a left operator module $X$ over a $C^*$-algebra $A$ is a central operator module if the map $X \otimes_h A \to X$ given by $(x, a) \mapsto ax$, is also completely contractive. Or, in other words, this latter map is a right oplication. If $A$ is commutative, this is just saying that $X$ is an operator $A - A$-bimodule with respect to this action $ax = xa$. It should be quite clear now that any operator space $X$ is a central operator $Z(X)$-module. Conversely, it is easy to see from the oplication theorem that for any central operator $A$-module $X$ there exists a *-homomorphism $\pi : A \to Z(X)$ such that $\pi(a)x = ax$ for all $a \in A, x \in X$. If $A$ is unital then so is $\pi$.

There is a similar definition for ‘central oplications’, and an analogous argument showing that every central oplication can be written as in the last equation, for a completely positive $\pi$.

It is clear that one way to produce central operator modules is to consider subalgebras of the commutant $X'$ of a concrete operator space $X$. More particularly, if $X \subset B(H)$, and if $B \subset X' \subset B(H)$ is such that $BX \subset X$, then $X$ has a central $B$-action. It is fairly obvious how to show that these are essentially the only ones; namely by first applying Theorem 2.1 of
Suppose that Corollary 3.6 in [8] that $\mathcal{X}$ spaces with the MIN structure. In this case we are in the setting of [10] and [8] $\mathcal{X}$.

That is, in this case $\mathcal{X}$ is actually a commutative unital $C^*$-algebra.

We leave the omitted details to the reader.

\begin{proof}
The idea is the same as for the proof of [13]. We first observe that the $Z(X)$ action on $X$ (which clearly makes $X$ a normal dual $Z(X)$-bimodule by [4,2]) has such a representation. To see this use [17] Theorem 4.1 that there exist such $H, \Phi, \pi$ as above, $\pi : Z(X) \to B(H)$, such that $\pi(z_1)\Phi(x)\pi(z_2) = \Phi(z_1xz_2)$ for all $z_1, z_2 \in Z(X), x \in X$. Then, since $\pi(z)\Phi(x) = \Phi(x)\pi(z)$ we have that $\pi$ maps into $\Phi(X)'$. Now proceed as in [13].

We leave the omitted details to the reader.
\end{proof}

We now classify the singly generated central modules:

\begin{theorem}
Let $X$ be a central operator $A$-module over a $C^*$-algebra $A$. Suppose that there exists an $x_0 \in X$ such that the closure of $AX_0$ is norm dense in $X$ (or weak*-dense in $X$, if $X$ is a dual operator space). Then $X$ is a MIN space, so that $X$ is a function $A$-module in the sense of [10]. That is, there is a compact space $K$, and a completely isometric linear $\Phi : X \to C(K)$, and a *-homomorphism $\pi : A \to C(K)$, such that $\Phi(ax) = \pi(a)\Phi(x)$ for all $a \in A, x \in X$. Conversely, any function $A$-module is a central operator module. If $X$ is an algebraically singly generated (i.e. $AX_0 = X$) central operator module over a $C^*$-algebra then in fact $X$ is a $C(K)$ space (linearly completely isometrically).
\end{theorem}

\begin{proof}
Suppose that $AX_0$ is norm dense in $X$. By the result three paragraphs above, [17], it follows that $Z(X)x_0$ is norm dense in $X$. It is easy to see that $Z(X)$ commutes with $J(x_0)J(x_0)^*$ inside the multiplier algebra of the $C^*$-algebra we called $E(X)$ in [8]. We refer to that paper, particular §6 there, for background on what follows. Since $Z(X)J(x_0)$ is norm dense in $J(X)$, it follows from the definition of $E(X)$, that the latter will be a commutative $C^*$-algebra. Note also, that since $X$ is also singly generated as a right $Z(X)$ module, if $E = (J(x_0)J(x_0)^*)^{\frac{1}{2}}$, then by [8] Lemma 6.2, $E$ is strictly positive in $E(X)$. Hence the set $E E(X)$ is dense in $E(X)$, by Stone-Weierstrass. By [25] Lemma 4.4, we may write $J(x_0) = E^{\frac{1}{2}}w$, for some $w \in T(X)$. Since $E(X)J(x)$ is dense in $T(X)$ (see the beginning of §4 in [8]), the map $Ez \to z E^{\frac{1}{2}}w$ from $EE(X) \to T(X)$, is an isometric $E(X)$-module map onto a dense subset of $T(X)$. Hence it extends to a completely isometric surjective $E(X)$-module map $T$ say, from $E(X) \to T(X)$. Thus $T(X)$ and therefore also $X$, are Banach spaces with the MIN structure. In this case we are in the setting of [10] and [8] §2 and 3. That is, $X$ is a topologically singly generated function module over $Z(X)$. We see from Corollary 3.6 in [8] that $X$ is algebraically singly generated iff $X$ is e.n.v. as a Banach space, and in this case $X$ is actually a commutative unital $C^*$-algebra.

\begin{theorem}
Let $B$ be a $W^*$-algebra (resp. dual operator system), and let $X$ be a dual operator space, and let $m : B \times X \to X$ be a central operator module action (resp. central oplication) which is $w^*$-continuous in the $B$-variable. Then there exists a Hilbert space $H$, a $w^*$-homeomorphic complete isometry $\Phi : X \to B(H)$, and a unital weak*-continuous $*$-homomorphism (resp. completely positive map) $\pi : B \to \Phi(X)' \subset B(H)$ such that $\pi(b)\Phi(x) = \Phi(m(b,x))$ for all $b \in B, x \in X$. As above, $H, \Phi$ may be chosen independently of $B, m$.
\end{theorem}

\begin{proof}
The idea is the same as for the proof of [13]. We first observe that the $Z(X)$ action on $X$ (which clearly makes $X$ a normal dual $Z(X)$-bimodule by [4,2]) has such a representation. To see this use [17] Theorem 4.1 that there exist such $H, \Phi, \pi$ as above, $\pi : Z(X) \to B(H)$, such that $\pi(z_1)\Phi(x)\pi(z_2) = \Phi(z_1xz_2)$ for all $z_1, z_2 \in Z(X), x \in X$. Then, since $\pi(z)\Phi(x) = \Phi(x)\pi(z)$ we have that $\pi$ maps into $\Phi(X)'$. Now proceed as in [13].

We leave the omitted details to the reader.
\end{proof}
Finally, if $X$ is a dual space and $Ax_0$ is w*-dense, then the above shows that the norm closure of $Z(X)x_0$ is a MIN space. However the weak*-closure of a MIN space is also a MIN space (we leave this fact as an exercise), so that $X$ is a MIN space.

Thus the topologically singly generated central operator modules over a $C^*$-algebra $A$ are exactly the topologically singly generated function modules over $A$.

Notice that every element $x_0$ in an operator space $X$, is contained in a subspace of $X$ which is a topologically singly generated central operator $Z(X)$-module, namely the closure of $Z(X)x_0$. We know this is a MIN space by the last result, but no doubt it will in most cases be 1-dimensional.

We also remark that the result above that an algebraically singly generated central operator module (or equivalently, function module) over a $C^*$-algebra is linearly isometric to a commutative $C^*$-algebra, is false with ‘algebraically’ replaced by ‘topologically singly generated’. Simple examples abound (see remark after 3.6 in [8]).

One final remark: there are a host of well known characterization theorems which are appropriate for the ‘completely isomorphic’ case of the theory, as opposed to the ‘completely isometric’ type results above (see [11], [10], [28] for example). Of course our techniques here have no hope of working in the ‘completely isomorphic’ case.

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