Effect of peristaltic flow of a third grade fluid in a tapered asymmetric channel

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Abstract: Peristaltic transport analysis of a third order fluid in a tapered asymmetric channel is made. The tapered asymmetric channel has been produced by choosing the peristaltic wave train on the narrow walls to have different amplitudes and phase. Long wavelength and low Reynolds number approach have been employed. The expressions for the horizontal velocity, stream function, pressure rise and frictional forces are obtained by a regular perturbation technique. Numerical computations have also been performed for the pressure rise, frictional force and amplitude of the shear stress and the effects of rheology parameter, amplitudes and phase difference of walls on the flow characteristics are further discussed in detail.

Keywords: Third grade fluid; Perturbation technique; Tapered asymmetric channel; Peristaltic transport;

1. INTRODUCTION

Peristaltic mechanism is one of the most important phenomenon which has been exploited the attention of many researchers due to its physiological and industrial applications. The peristaltic propulsive movement is observed in the esophagus, the gastrointestinal tract, bile ducts, the ureters and other glandular ducts throughout the body. The same principle has been adopted by engineers to pump corrosive material and fluids which are to be kept away from the pumping machinery. Peristalsis is also like industrial like sanitary fluid transport blood pumps in heart lung machines and transport of toxic liquid in nuclear industries [Hayat et al. (2007), Eldabe et al. (2007), Latham (1966), etc.].

A plenty of reports are now available in literature on peristalsis movement involving Newtonian and non-Newtonian flow. It is a well-known fact that non-Newtonian fluids are more appropriate than Newtonian fluids for industrial, medical and technological applications. Variety of consumer goods contains high concentration of glass or carbon fibres, paints and lubricants with polymer additives and biological fluids are non-Newtonian in nature. Several authors [Rajagopal (1982), (1984), Erdogan (1995), Radhakrishnamacharya (1982), Bandelli and Rajagopal (1995), Rajagopal, and Na (1983), 2003].
To find the solutions of non-Newtonian fluid flow, a perturbation approach plays an important role, due to the assumption of making the non-Newtonian parameter as small as the nonlinear part is perturbed and linear systems are made to solve the governing systems [Erdogan (1995), Asghar et al. (2004), (2003), Liao (1998)]. Elshehawey et al. (1998) deliberated the peristaltic motion of Carreau fluid in a non-uniform channel and they originated the solution in a perturbation series in powers of the Weissenberg number using long wavelength approximations. A fluid mechanics effect of peristaltic transport in a two-dimensional asymmetric channel under the assumptions of long wavelength and low Reynolds number in a waveframe of reference has been considered by Mishra and Rao (2003).

Physiological organs are generally monitored to be a non-uniform duct [Wiedeman (1963), Lee and Fung (1971)] and soperistaltic analysis of a Newtonian fluid in a uniform geometry cannot be utilized when elucidating the mechanism of transport of fluid in nearly bio-systems. The peristaltic transport of Newtonian and non-Newtonian fluid in non-uniform geometries has been studied by Srivastava et al. (1983), Haroun (2007) considered the effects of Deborah number and phase difference on the peristaltic transport of third order fluid in an asymmetric channel. Mekheimer (2004) investigated the effect of a uniform magnetic field on peristaltic transport of a blood in a non-uniform two-dimensional channel, when blood is acted by a couple-stress fluid. Also, Elshehawey et al. (2005) studied the asymmetric peristaltic motion of a viscous compressible liquid by a flexible pore of changing cross-section. Recently, physiologists studied that the intrauterine fluid flow suitable to myometrial contractions is peristaltic-type motion and the myometrial contractions may arise in mutually symmetric and asymmetric guidance (De Vries et al. 1990). Eytan and Elad (1999) have formulated a mathematical form of wall-attempt peristaltic fluid flow in a two-dimensional channel with wave trains having a phase difference moving independently of the upper and lower walls to reproduce intrauterine fluid motion in a sagittal cross-section of the uterus.

The purpose of this work is to study the peristaltic pumping of a third grade fluid in the tapered asymmetric channel under the assumptions of long wavelength approximation and the horizontal velocity and the pressure gradient could be elaborated in a perturbation technique in a small parameter ‘$\Gamma$’ that comprised the non-Newtonian coefficient suitable to shear thinning. To the best of authors knowledge, the present investigation may be the maiden attempt in the proposed generalized geometry. The resulting linear equations are solved up to the second order. Expressions for the stream function, velocity, pressure rise and frictional force per wavelength are presented. Numerical computations have also been performed for the pressure rise and frictional forces, amplitude of the shear stress and the variations of embedded flow parameters are discussed in detail.

2. Basic equations
In the absence of external forces, the equations of continuity and momentum for the flow of an incompressible fluid are given by

$$\text{div} \ V = 0, \quad (1)$$

$$\rho (V \cdot \nabla) = -\nabla p + \text{div} \ S. \quad (2)$$

In the above equation, $V$ is the velocity, $\rho$ is the fluid density, $p$ is the hydrostatic pressure, $S$ is the extra stress tensor for a third grade fluid.

For a third grade fluid, the extra stress can be written as

$$S = \mu A_i + \alpha_1 A_2 + \alpha_2 A_i^2 + \beta_1 A_i + \beta_2 (A_2 A_i + A_i A_2) + \beta_3 (\text{tr} A_i^2) A_i. \quad (3)$$
Here $\mu$ is the dynamic viscosity, $\alpha_i \ (i = 1, 2)$ and $\beta_i \ (i = 1 - 2)$ are the material constants corresponding to second and third order approximations respectively. The kinematical tensors $A_n$ are defined as [Rivlin, and Ericksen (1955)]

$$A_n = \nabla V + (\nabla V)^T,$$

$$A_n = \left(\frac{\partial}{\partial t} + (V \cdot V)A_{n-1} + A_{n-1}(\nabla V) + (\nabla V)^T A_{n-1}\right), \quad n = 2, 3, \ldots $$

Note that Eq. (3) is compatible with thermodynamics if [Fosdick and Rajagopal (1980)]

$$\mu \geq 0; \alpha_1 \geq 0; \mu \geq 0; |\alpha_1 + \alpha_2| \geq \sqrt{24 \mu \beta_3}; \beta_1 = \beta_2 = 0; \beta_3 \geq 0;$$

In which case Eq. (3) becomes

$$S = [\mu + \beta_3 (\text{tr} A^2)]A_1 + \alpha_1 A_2 + \alpha_2 A_3.$$  

3. Formulation of the problem

![Diagram of tapered asymmetric channel](image)

Fig. 1. Schematic diagram of tapered asymmetric channel

We consider the two dimensional flow of a third grade fluid in a non uniform asymmetric channel having width $2d$ at the inlet. Assume an infinite wave train travelling with velocity $c$ along the walls. We choose a rectangular coordinate system for the channel with $X$ along the direction of wave propagation and parallel to the center line and $Y$ transverse to it. The non uniform asymmetric channel is produced by choosing the peristaltic wave train on the walls to have different amplitudes and phase. The walls of the tapered asymmetric channel are given by the equations

$$H_2( X , t ) = d + m' X + a_2 \sin \left[ \frac{2\pi}{\lambda} (X - ct) \right]. \quad \text{... upper wall},$$

$$H_1( X , t ) = -d + m' X - a_1 \sin \left[ \frac{2\pi}{\lambda} (X - ct) + \phi \right]. \quad \text{... lower wall},$$

Where $a_1$ and $a_2$ are the amplitudes of the waves, $\lambda$ is the wave length, $c$ is the wave speed and $\phi$ is the phase difference which varies in the range $0 \leq \phi \leq \pi$. It should be noted that $\phi = 0$ corresponds to
symmetric channel with waves out of phase and $\phi = \pi$ the waves are in phase, and further $a_1, a_2, d$ and $\phi$ satisfies the condition

$$a_1^2 + a_2^2 + 2a_1a_2 \cos(\phi) \leq (2d)^2.$$  

(9)

The continuity and momentum equations for two-dimensional case are

$$\frac{\partial U}{\partial X} + \frac{\partial U}{\partial Y} = 0, \quad \rho \left( \frac{\partial U}{\partial t} + \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} \right) = \frac{\partial P}{\partial X} + \frac{\partial}{\partial X} S_{XX} + \frac{\partial}{\partial Y} (S_{XY}),$$

(10)

$$\rho \left( \frac{\partial V}{\partial t} + \frac{\partial V}{\partial X} + V \frac{\partial V}{\partial Y} \right) = -\frac{\partial P}{\partial Y} + \frac{\partial}{\partial X} S_{XY} + \frac{\partial}{\partial Y} (S_{YY}),$$

(11)

$$S_{XX} = 2U_X + \alpha_1 \left( 2U_X + 2U_{XX} + 2V_{XY} + 4U_X^2 + 2V_XU_Y + 2V_X^2 \right) + \alpha_2 \left( 4U_X^2 + U_Y^2 + V_X^2 + 2U_YV_X \right) + \beta_1 \left( 2U_{XX} + 2U_{XY} + 4U_{XX} + 2V_{XY} + 4V_{XX}, + 12U_X U_Y + 4V_X U_Y + 14U_X U_{XX} \right) + \beta_2 \left( 10V_X U_Y - 6V_X U_{XX} + 6V_Y U_{XX} + 6V_Y U_{XY} + 6V_Y U_{YY} + 4V_X U_{YY} + 8V_X^3 \right) + \beta_3 \left( 2U_X U_Y + 2V_X U_{XX} + 2U^2 U_{XX} + 2U^2 U_{XY} + 2U^2 U_{YY} \right) + S_{YY} + 8U_X U_Y + 8U_Y U_{XX} + 8U_X U_{XY} + 16U_X^3 + 8U_Y V_X U_Y + 4U_X U_Y^2 + 2U_Y U_Y + 2U_Y U_{XY} + 2U_Y U_{XX} + 2U_Y U_{YY} + 2V_X U_Y + 2V_X U_{XY} + 2V_Y U_Y + 2V_Y U_{XY} + 2V_Y U_{YY}$$

(12)

$$S_{XY} = U_Y + V_X + \alpha \left( U_Y + U_{XX} + V_Y + V_Y + V_{XX} + V_{XY} = 2U_X U_Y + 2U_Y U_{XY} \right) + \beta_1 \left( 4U_{XX} + U_{YY} + V_{XX} + 2V_{XX} + 2U_{XX} + 2V_{XX} + 4U_Y U_{XX} + 2U_X U_Y - 2U_X V_Y \right) + \beta_2 \left( 8U_X^3 + 8U_Y V_X + 2U_Y^3 + 8V_X U_X^2 + 6U_Y V_X^2 + 2V_X^3 \right) + \beta_3 \left( 8U_X^2 U_Y + 8U_X V_X + 2U_Y^3 \right) + 6U_Y V_X^2 + 6U_Y V_X + 2V_X^3$$

(13)
\[
S_{xy} = -2U_x + \alpha_1 \left( -2U_x - 2UU_{xx} - 2VU_{xy} + 4U_x^2 + 2V_xU_y + 2U_y^2 \right) + \alpha_2 \left( 4U_x^2 + U_y^2 + V_x^2 + 2U_xV_x \right) + \beta_1 \left( -2U_x - 2U_xU_{xx} - 4UU_{xx} - 2VU_{xy} - 4VU_{xy} + 12U_xU_{xx} + 2V_xU_y + 4V_yU_x + 6U_yU_{yy} \right) \\
+ \beta_2 \left( 2U_xU_y + 4U_yV_y + 2UU_yV_y - 2VU_xU_{xx} + 2UU_xU_{xy} + 2UU_yU_{yy} - 4U_x^2U_x \right) \\
+ \beta_3 \left( -8U_xV_xU_y + 2V_xU_{xy} + 2V_xV_x + 2UU_xV_{xx} - 2VU_xU_{xx} + 2VU_xV_{xy} + 2VV_xU_{yy} \right) \\
+ \beta_4 \left( 16U_x^2 - 4U_xU_y^2 - 4U_yV_y^2 - 8U_xU_yV_y \right),
\]

where \((U,V)\) and \(P\) are the velocity components and pressure and the subscripts denote the partial derivatives. If we employ these transformations in the governing equations of motion (10-15) and then introduce the following dimensionless variables:

\[
x = \frac{X}{\lambda}, \quad y = \frac{Y}{a}, \quad t = \frac{ct}{\lambda}, \quad u = \frac{V}{\rho c}, \quad \nu = \frac{d}{\lambda}, \quad \delta = \frac{d}{\lambda}, \quad h_1 = \frac{H_1}{a}, \quad h_2 = \frac{H_2}{a}, \quad p = \frac{d^2p}{c^2\mu},
\]

\[
S = \frac{d}{\mu} \left( x, R = \frac{\rho c d}{\mu}, \gamma_2 = \frac{\beta c^2}{\mu}, \gamma_5 = \frac{\beta c^2}{\mu} \right), \quad a = \frac{a_1}{a}, \quad b = \frac{a_2}{a}.
\]

we can introduce the stream function \(\psi(x,y)\), defined by

\[
u = \frac{\partial \psi}{\partial y}, \quad \nu = -\delta \frac{\partial \psi}{\partial x}.
\]

we obtain under the long wavelength and low Reynolds number approximation the following expressions [Asghar et al. (2003), Liao (1998), Elshehawey et al. (1998), Mishra, and Rao (2003), Wiedeman (1963)]

\[
-\frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \left( \frac{\partial^2 \psi}{\partial y^2} + 2I \left( \frac{\partial^2 \psi}{\partial y^2} \right)^3 \right) = 0,
\]

\[
-\frac{\partial p}{\partial y} = 0.
\]

in which \(I(= \gamma_2 + \gamma_5)\) is the Deborah number, \(\psi\) the stream function, \(R\) the Reynolds number, \(\rho\) the fluid density, \(\delta\) the wave number, \(\mu\) the constant viscosity and continuity equation is automatically satisfied. Further, Eq.(19) indicates \(p \neq p(y)\). From Eqs. (18-19), we have

\[
\frac{\partial^2}{\partial y^2} \left( S_{\psi} \right) = 0.
\]

In the above equations \(\alpha\) is the viscosity parameter, \(\beta\) is the non-dimensional velocity slip parameter and

\[
S_{\psi} = \frac{\partial^2 \psi}{\partial y^2} + 2I \left( \frac{\partial^2 \psi}{\partial y^2} \right)^3.
\]

The subjected boundary condition in dimensionless form become
\[ \psi = \frac{F}{2} \frac{\partial \psi}{\partial y} = 0, \text{ at } y = h_2 = 1 + mx + b \sin(2\pi(x-t)), \quad (22) \]
\[ \psi = -\frac{F}{2} \frac{\partial \psi}{\partial y} = 0, \text{ at } y = h_1 = -1 - mx - a \sin(2\pi(x-t) + \phi). \quad (23) \]

**Rate of volume flow and boundary conditions**

If we introduce the wave frame having coordinates \( (X,Y) \) which travel in the \( X \) direction with the same wave velocity \( (c) \), then the unsteady flow in the laboratory frame \( (x,y) \) can be treated as steady. The coordinates and velocities in the two frames are related by
\[ X = x - ct, \quad Y = y, \quad U = u - c, \quad V = v. \quad (24) \]
where \( u, v \) are the velocity components in the wave frame \( (x,y) \).

In laboratory frame, the dimensional volume flow rate is
\[ Q(X,t) = \int_{H_1(x,t)}^{H_2(x,t)} U(X,Y,t) \, dY. \quad (25) \]
in which \( H_1 \) and \( H_2 \) are function of \( X \) and \( t \).

Substituting Eqs.(25 - 26), we obtain,
\[ Q(X,t) = q + cH_1 - cH_2. \quad (27) \]

The time-averaged flow over a period \( (T = \lambda/c) \) at a fixed position \( \bar{x} \) is defined as
\[ \bar{Q} = \frac{1}{T} \int_0^T Q \, dt. \quad (28) \]

If we substitute Eq.(26) into Eq.(28) and integrating, we get [Srivastava and Srivastava (1983), Vajravelu et al. (2012), Kothandapani et al. 2015, Kothandapani and Prakash, 2016]
\[ \bar{Q} = q + a_c \sin \left( \frac{2\pi}{\lambda} (X - ct) \right) + a_c \sin \left( \frac{2\pi}{\lambda} (X - ct) + \phi \right). \quad (29) \]

One finds that Eq. (27) becomes, after using Eq. (16)
\[ F(x,t) = \Theta + a \sin 2\pi(x-t) + b \sin \left[ 2\pi(x-t) + \phi \right]. \quad (30) \]
where
\[ F = \frac{\bar{Q}}{cd}, \quad \Theta = \frac{q}{cd}, \quad F = \int_{h_1}^{h_2} u \, dy = \psi(h_2) - \psi(h_1) \text{ and } \Theta \text{ in the time average of flow one period of the wave.} \]
We note that \( h_1 \) and \( h_2 \) represent the dimensionless form of the surfaces of the peristaltic walls
\[ h_2 = 1 + mx + b \sin(2\pi(x-t)) \text{ and } h_1 = -1 - mx - a \sin(2\pi(x-t) + \phi). \quad (31) \]

Where \( a = \frac{a_1}{d} \) and \( b = \frac{a_2}{d} \) are the occlusions or the amplitude ratios. If we select the value of steam function at the lower wall as \( \psi(h_1) = -\frac{F}{2} \), then the value of \( \psi \) at the upper wall is gives \( \psi(h_2) = \frac{F}{2} \).

The appropriate dimensionless boundary conditions can be put into the following forms:
\[
\psi = \frac{F}{2}, \frac{\partial \psi}{\partial y} = 0, \text{ at } y = h_2 = 1 + mx + b \sin(2\pi(x-t)), \tag{32}
\]

\[
\psi = -\frac{F}{2}, \frac{\partial \psi}{\partial y} = 0, \text{ at } y = h_1 = -1 - mx - a \sin(2\pi(x-t) + \phi). \tag{33}
\]

which satisfy, at the inlet of channel,

\[
a^2 + b^2 + 2ab\cos(\phi) \leq (2d)^2. \tag{34}
\]

4. **Method of solution**

Since the governing differential equations is extremely non-linear. Here the resulting system consists of non-linear differential equation and boundary conditions. Exact analytical solution of the governing system is not possible and hence the attention is focused to the perturbation solution for small Deborah number \( \Gamma \). We can be expand \( \psi, p \) and \( F \) as:

\[
\psi = \psi_0 + \Gamma \psi_1 + \Gamma^2 \psi_2 + ... \\
p = p_0 + \Gamma p_1 + \Gamma^2 p_2 + ... \\
F = F_0 + \Gamma F_1 + \Gamma^2 F_2 + ... \tag{35}
\]

Substitution of above equations into Eqs. (18), (19) and (21) and collection of terms with respect to like powers of \( \Gamma \), yields the following systems:

4.1. **For the system of order zero,**

\[
\frac{\partial^4 \psi_0}{\partial y^4} = 0, \tag{36}
\]

\[
\frac{\partial p_0}{\partial x} = \frac{\partial^3 \psi_0}{\partial y^3}, \tag{37}
\]

\[
\psi_0 = \frac{F_0}{2}, \frac{\partial \psi_0}{\partial y} = 0, \text{ at } y = h_2, \tag{38}
\]

\[
\psi_0 = -\frac{F_0}{2}, \frac{\partial \psi_0}{\partial y} = 0, \text{ at } y = h_1, \tag{39}
\]

4.2. **For the system of order one,**

\[
\frac{\partial^4 \psi_1}{\partial y^4} = -2 \frac{\partial^2}{\partial y^2} \left[ \left( \frac{\partial^3 \psi_0}{\partial y^3} \right) \right], \tag{40}
\]

\[
\frac{\partial p_1}{\partial x} = \frac{\partial^2 \psi_1}{\partial y^2} + 2 \left( \frac{\partial^2 \psi_0}{\partial y^2} \right)^3, \tag{41}
\]

\[
\psi_1 = \frac{F_1}{2}, \frac{\partial \psi_1}{\partial y} = 0, \text{ at } y = h_2, \tag{42}
\]

\[
\psi_1 = -\frac{F_1}{2}, \frac{\partial \psi_1}{\partial y} = 0, \text{ at } y = h_1, \tag{43}
\]

4.3. **For the system of order second,**

\[
\frac{\partial^4 \psi_2}{\partial y^4} = -6 \frac{\partial^2}{\partial y^2} \left[ \left( \frac{\partial^2 \psi_0}{\partial y^2} \right)^2 \left( \frac{\partial^2 \psi_1}{\partial y^2} \right) \right]. \tag{44}
\]
\[
\frac{\partial P_2}{\partial x} = \frac{\partial^3 \psi_2}{\partial y^3} + 6 \frac{\partial}{\partial y} \left[ \frac{\partial^2 \psi_2}{\partial y^2} \right] \left[ \frac{\partial \psi_1}{\partial y^2} \right] \]  
(43)

\[
\psi_2 = \frac{F_2}{2} \frac{\partial \psi_2}{\partial y} = 0, \text{ at } y = h_2,
\]

\[
\psi_2 = -\frac{F_2}{2} \frac{\partial \psi_2}{\partial y} = 0, \text{ at } y = h_1,
\]  
(44)

4.4. Zeroth-Order Solution

The axial velocity and axial pressure gradient at this order are, respectively, given by

\[
\psi_0 = A_1 + A_2 y + A_3 y^2 + A_4 y^3,
\]  
(45)

\[
\frac{\partial P_0}{\partial x} = 6A_4.
\]  
(46)

where

\[
A_1 = \frac{-2F_0}{2(h_3^2 - h_3^3) - 6h_1^2(h_2 - h_1) - 3(h_1^3 - h_2^3)(h_1 - h_2)},
\]

\[
A_2 = \frac{-3A_1}{2(h_1 - h_2)},
\]

\[
A_3 = -2A_1 h_1 - 3A_1 h_2,
\]

\[
A_4 = \frac{F_0}{2} - A_2 h_2 - A_3 h_2^2 - A_4 h_2^3.
\]

4.5. First-Order Solution

Substituting Eq. 45 into Eq. 39 and solving the resulting equation subject to the boundary conditions 41 we obtain

\[
\psi_1 = B_1 + B_2 y + B_3 y^2 + B_4 y^3 - 36A_3 A_4^2 y^3 - \frac{2592}{120} A_4^3 y^5,
\]  
(47)

\[
\frac{\partial P_1}{\partial x} = 6B_4 + 144A_3^2 A_4.
\]  
(48)

where

\[
B_1 = \frac{2F_1 - A_1 A_2^2}{3(h_1^3 - h_2^3)(h_1 - h_2) - 2\left[3h_2^3(h_2 - h_1) - (h_1^3 - h_1^3) \right]},
\]

\[
A_1 A_2^2 \left[ 144h_2^3(h_2 - h_1) - 36(h_2^3 - h_1^3) \right] + A_1 \left[ 108h_2^3(h_2 - h_1) - \frac{2592}{120} (h_2^3 - h_1^3) \right] + \frac{144A_3 A_4^2 h_2^3(h_2 - h_1)}{F_0},
\]

\[
B_3 = \frac{-B_4 \left[ 9h_2^5 - 2(h_3^2 - h_3^3) \right] + F}{(h_1 - h_2)^2},
\]

\[
B_2 = 144A_3 A_4^2 h_2^3 + 108A_3^2 h_2^3 - 2B_3 h_2^2 - 3B_4 h_2^2,
\]

\[
B_1 = \frac{F_1}{2} - B_2 h_2^2 - B_3 h_2^2 - B_4 h_2^2 + 36A_3 A_4^2 h_2^3 + \frac{2592}{120} A_4^3 h_2^3.
\]

4.6. Second-Order Solution

The axial velocity and axial pressure gradient at this order are, respectively, given by
\[
\psi_2 = C_1 + C_2 y + C_3 y^2 + C_4 y^3 + \frac{y^4}{24} \left( 20736A_1^2 A_4^2 - 864A_1^2 B_2 - 1728B_4 A_4 \right) + \frac{y^5}{120} \left( 435456A_1^2 A_4^2 - 7776A_1^2 B_4 \right) + 5184A_1 A_4^2 y^6 + \frac{1866240}{840} A_4^3 y^7,
\]
(49)

\[
\frac{\partial \psi}{\partial x} = 6C_4.
\]
(50)

where

\[
C_4 = \frac{2K_1 - 2K_2 - K_4(h_2-h_1) + K_4(h_2-h_1) - 2F_2}{6h^5(h_2-h_1) - 2(h_2^3-h_1^3) + 3(h_2^2-h_1^2)(h_2-h_1)},
\]

\[
C_3 = \frac{K_3 - K_4 - 3C_4(h_2^2-h_1^2)}{2(h_2-h_1)},
\]

\[
C_2 = -2C_3h_2 - 3C_4h_2^2 - K_4,
\]

\[
C_1 = \frac{F_1}{2} - C_3h_2 - C_4h_2^2 - C_4h_2^4 - K_4,
\]

\[
K_i = \frac{h^4}{24} \left[ 20736A_1^2 A_4^2 - 864A_1^2 B_2 - 1728B_4 A_4 \right] + \frac{h^5}{120} \left( 435456A_1^2 A_4^2 - 7776A_1^2 B_4 \right) + 5184A_1 A_4^2 h^6 + \frac{1866240}{840} A_4^3 h^7,
\]

The expressions of stream function and axial pressure gradient up to the first order may be written as

\[
\psi = A_1 + A_2 y + A_3 y^2 + A_4 y^3 + \Gamma \left( B_1 + B_2 y + B_3 y^2 + B_4 y^3 - 36A_1 A_4^2 y^4 - \frac{2592}{120} A_4^3 y^5 \right)
\]

\[
+ \frac{y^5}{120} \left( 435456A_1^2 A_4^2 - 7776A_1^2 B_4 \right) + 5184A_1 A_4^2 y^6 + \frac{1866240}{840} A_4^3 y^7,
\]

\[
\frac{\partial \psi}{\partial x} = 6A_4 + \Gamma \left( 6B_4 + 144A_4^2 A_4 \right) + \Gamma^2 6C_4.
\]
(52)
The perturbation series solutions up to second order for \( \psi \), \( dp/dx \) and \( \Delta p \) may be summarized as

\[
\psi = \psi_0 + \Gamma \psi_1 + \Gamma^2 \psi_2, \quad \frac{dp}{dx} = \frac{dp_0}{dx} + \Gamma \frac{dp_1}{dx} + \Gamma^2 \frac{dp_2}{dx}, \quad \Delta p = \Delta p_0 + \Gamma \Delta p_1 + \Gamma^2 \Delta p_2, \quad (53)
\]

Defining

\[
F = F_0 + \Gamma F_1 + \Gamma^2 F_2,
\]

Using \( F_0 = F - \Gamma F_1 - \Gamma^2 F_2 \), and then neglecting the terms greater than \( \mathcal{O}(\Gamma^2) \) the results given by Eq. (53) can be expressed up to second order.

The non-dimensional expressions for the pressure rise \( \Delta p_\lambda(t) \), frictional forces on the upper and lower walls \( F_{\lambda,1}(t) \) and \( F_{\lambda,2}(t) \) per wavelengths and amplitude of the shear stress distribution \( S_{xy} \) on the lower wall are given respectively as follows:

\[
\Delta p_\lambda(t) = \int_0^1 \frac{\partial \psi}{\partial x} \, dx, \quad (54)
\]

\[
F_{\lambda,1}(t) = \int_0^1 \left( h_1^2 \left( - \frac{\partial \psi}{\partial x} \right) \right) \, dx, \quad (55)
\]

\[
F_{\lambda,2}(t) = \int_0^1 \left( h_2^2 \left( - \frac{\partial \psi}{\partial x} \right) \right) \, dx, \quad (56)
\]

\[
S_{xy} = \frac{\partial^2 \psi}{\partial y^2} + 2\Gamma \left( \frac{\partial^2 \psi}{\partial y^2} \right)^3. \quad (57)
\]

5. Results and discussion

The particular interest of the problem is to study the effects of \( \Gamma, m, \phi \) and \( a \) on \( \Delta P_\lambda \). Fig.2 (a) shows the plots of \( \Delta P_\lambda \) against dimensionless flow rate \( \Theta \) for various values of Deborah number \( \Gamma \). It indicates that pressure rise for \( \Gamma = 0 \) is linear whereas it is non-linear to the non-zero values of \( \Gamma \). Fig.2 (b) shows that the pressure rise decreases with an increase in the non-uniform parameter \( (m) \). Fig.2(c) shows the effect of phase difference \( (\phi) \) on \( \Delta p \). It is clear that \( \Delta p \) decreases for large values of \( \phi \) both in pumping and co-pumping regions. The effects of wave amplitudes \( a \) on \( \Delta P_\lambda \) is illustrated in Fig.2 (d). Fig.2 (d) elucidate the for adverse pressure gradient \( (\Delta P_\lambda > 0) \) in pumping region and to the free pumping \( (\Delta P_\lambda = 0) \) the pumping increases with \( a \). It has also been noticed from that in co-pumping, for the appropriately chosen values \( (\Delta P < 0) \), \( \Theta \) increases with an increase in \( a \). In Fig.3(a-d) plots of frictional force \( F_{\lambda,1}(t) \) at the lower wall \( (y = h(y, x, t)) \) of the channel with dimensionless time mean flow \( \Theta \) have been provident. In these figures, it is observed that there exists a critical value of \( \Theta \) below which \( F_{\lambda,1}(t) \) resists the flow along the channel wall. Further, the frictional force on the upper wall behaves complementarily with pressure rise.
Fig. 2 (a) Pressure rise distribution for $a = 0.3, b = 0.4, m = 0.01, \phi = \pi / 4,$

Fig. 2 (b) Pressure rise distribution for $a = 0.4, b = 0.5, \phi = \pi / 2, \Gamma = 0.02,$

Fig. 2 (c) Pressure rise distribution for $a = 0.2, b = 0.3, m = 0.25, \Gamma = 0.05,$

Fig. 2 (d) Pressure rise distribution for $b = 0.3, m = 0.3, \phi = \pi / 3, \Gamma = 0.04,$
The distribution of axial velocity ($u$) is plotted against $y$ in Figs. 4(a–e). It is noticed from Fig. 4(a) that the velocity profile is parabolic in nature and it increases with an increase in dimensionless amplitude of lower wall $a$. In Fig. 4(b), influence of flow rate $\Theta$ on $u$ is captured. It has been detected that with an increase in $\Theta$ the axial velocity increases. Fig. 4(c) shows that with an increase in Deborah number $\Gamma$, the axial velocity increases. The axial velocity for the non-uniform parameter $m$ is shown in Fig. 4(d). It was observed that an increase in $m$ leads to the increase in the axial velocity $u$ at the boundaries and at the center of the channel $u$ decreases for higher values of $m$. In Fig. 4(e) the variation of $\Delta p$ versus $\Theta$ is shown for different values of the phase difference $\phi$ by fixing the other parameters as constants. The distribution of the axial velocity $u$ at $\phi = 0$ is depicted for symmetric channel and for the asymmetric channel, it is $\pi / 3$ in Fig. 4(e). It was observed that $\Delta p$ decreases as $\phi$ increases because more occlusion will be produced in the gap between the walls and this in turn decreases $\Delta p$ values by increasing $\phi$ in narrow part of the channel and also in the wider parts.
An attractive phenomenon in peristaltic motion is trapping. In a reference wave frame, streamlines under certain conditions split to trap a bolus of fluid which is pushed ahead along with the peristaltic wave with the speed of the wave. The effects of upper wall amplitude parameter $b$ on trapping could be seen from Fig. 5. It has been observed further that the trapped bolus increases with an increase in $b$. Fig. 6 depicts the effect of Deborah number $\Gamma$ on the streamlines for fixed values of other parameters. It was also noticed that the size of the channel decreases with increasing $\Gamma$. In Fig. 7 shown that the size of the trapping bolus decreases by increasing $m$ (in the upper and lower parts of the channel). Fig. 8 provides the variation of $\phi$ on trapping and it was found that the trapped bolus is symmetric about the center line for $\phi = 0$, but it decreases in size and move towards left as $\phi$ increases.

![Fig. 4 (a) velocity distribution for fixed value $(x=0.4, t=0.2)$ and $b=0.3, Q=1.4,$ $m=0.2, \phi = \pi / 4,$](image1)

![Fig. 4 (b) velocity distribution for fixed value $(x=0.4, t=0.2)$ and $a=0.3, b=0.2,$ $m=0.1, \phi = \pi / 3,$](image2)

![Fig. 4 (c) velocity distribution for fixed value $(x=0.4, t=0.2)$ and $a=0.6, b=0.4,$ $m=0.1, \phi = \pi / 8, Q=1.2.$](image3)

![Fig. 4 (d) velocity distribution for fixed value $(x=0.4, t=0.2)$ and $a=0.5, b=0.3,$ $\phi = \pi / 4, Q=1.5.$](image4)
Amplitude of the shear stress distribution \((S_{\alpha y})\) on the lower wall of the tapered asymmetric channel for different values of \(m\) has been presented in Fig. 9(a). It was noticed that the shear stress distribution decreases with increase of \(m\) values. The amplitude of the shear stress found to be maximum at \(m = 0\). The effects of Deborah number \(\Gamma\) with respect to shear stress distribution is shown in Fig.9 (b). It has been observed from the figure that the amplitude of the shear stress distribution increases with the increase in \(\Gamma\) values. Figs. 10(c) and 10(d) show that the impression of \(a\) and \(\phi\) on the shear stress. It has been studied that the shear stress is exhibiting a cyclic wave of oscillatory behaviour, probity probity due to peristaltic motion.
$m = 0.1$, $\phi = \pi/4$, $t = 0.4$, $Q = 1.3$, $a = 0.2$,

Fig.5. Streamlines for (a) $b = 0.1$ (b) $b = 0.2$ (c) $b = 0.3$ (d) $b = 0.4$. 
\( m = 0.1, \phi = \pi/4, \quad t = 0.2, \quad Q = 1.4, \quad a = 0.3, \quad b = 0.4, \)

Fig. 6. Streamlines for (a) \( \Gamma = 0 \), (b) \( \Gamma = 0.05 \), (c) \( \Gamma = 0.1 \), (d) \( \Gamma = 0.15 \),
\[ \Gamma = 0.02, \phi = \pi / 3, \ t = 0.2, \ Q = 1.2, \ a = 0.2, \ b = 0.3, \]

Fig.7. Streamlines for (a) \( m = 0 \), (b) \( m = 0.1 \), (c) \( m = 0.2 \), (d) \( m = 0.3 \),
\( \Gamma = 0.06, \ m = 0.15, \ Q = 1.6, \ t = 0.3, \ a = 0.3, \ b = 0.1, \)

Fig. 8. Streamlines for (a) \( \phi = 0 \), (b) \( \phi = \pi / 6 \), (c) \( \phi = \pi / 3 \), (d) \( \phi = \pi / 2 \),

\[
\Gamma = 0, \ m = 0.03, \ m = 0.06, \ m = 0.09
\]

Fig. 9 (a) Amplitude of the shear stress distribution at the lower wall \( (t = 0.2) \) for \( a = 0.4, \ b = 0.3, \ Q = 1.4, \ \phi = \pi / 2, \ \Gamma = 0.05 \),

\[
\Gamma = 0, \ \Gamma = 0.02, \ \Gamma = 0.04, \ \Gamma = 0.06
\]

Fig. 9 (b) Amplitude of the shear stress distribution at the lower wall \( (t = 0.2) \) \( a = 0.3, \ b = 0.2, \ m = 0.1, \ \phi = \pi / 3, \ \Theta = 1.2, \)
Fig. 9 (c) Amplitude of the shear stress distribution at the lower wall ($t = 0.2$) and $a = 0.2$, $b = 0.3$, $m = 0.1$, $\Theta = 1.4$, $\Gamma = 0.01$.

Fig. 9 (d) Amplitude of the shear stress distribution at the lower wall ($t = 0.2$) and $a = 0.3$, $m = 0.05$, $\phi = \pi / 3$, $Q = 1.5$, $\Gamma = 0.05$.

6. Conclusion
In this study the effects of the peristaltic flow of a third order fluid in the tapered asymmetric channel are analyzed under longwave length and low Reynolds number situations. The effects of amplitude ($a$), non-uniform parameter ($m$), Deborah number ($\Gamma$) and phase difference ($\phi$) on pumping characteristics, frictional forces, axial velocity and trapping have been studied in detail. The following observations have been made from the above study:

- The pressure rise increases with an increase in $\phi$.
- The pressure rise decreases with an increase in the non-uniform parameter ($m$).
- The axial velocity increases at the boundaries by increasing $m$. However, it decreases with increasing $m$ at the centre of the channel.
- The size of a trapped bolus increases with an increase in $a$ and $\phi$ while the size of the channel decreases with increase in $\phi$ and $m$ values.
- The shear stress shows cyclic wave oscillatory behaviour in $\phi$ and $b$, which may be attributed to peristalsis.

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