Geometrical Influences on the Vibration of Layered Plates

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This paper aims to study the influence of different geometric properties and support conditions on the vibration of layered plates of nonuniform thickness under shear deformation theory. The layered plates are supposed to have arbitrarily nonuniform thickness as linear, exponential, and sinusoidal. The spline approximation is used to approximate translational and angular displacement functions. Eigen frequency parameters are calculated by solving eigenvalue problem. The geometrical influences such as number of lay-ups, different ply orientations, each ply consisting of different material, side-to-thickness ratio, and aspect ratio are taken into consideration to examine the frequency variation of plates for two different support conditions.

1. Introduction

Plates with nonuniform thickness are extensively used as the aerospace and marine structural components (e.g., turbine disks and aircraft wings) therefore, attracting the attention of contemporary engineers. The plates with nonuniform thickness can help designer to modify resonant frequency, decrease the size and weight, increase the stiffness, and adjust flexural rigidity of the structure. Moreover, variation in weight and thickness effects the survivability and vulnerability issues of the components. Therefore, the frequency examination of plates having nonuniform thickness is significant subject of interest for investigators.

Plates with nonuniform thickness are generally difficult to analyze and get the analytical solution. Therefore, some researchers investigated such plates and obtained solution, as the investigations carried by Appl and Byers [1] on plates of linear thickness variation for simply supported plates. Moreover, Plunkett [2] examined linearly tapered rectangular plates. Subsequently, different types of thicknesses were studied such as stepped thickness [3], linearly varying thickness [4, 5], bilinear varying thickness [6, 7], and two-dimensional thickness variation [8].

A number of researchers studied composite plates such as Sahoo and Singh [9] who used zigzag model to examine the composite plates. Sharma et al. [10] examined the plates on elastic edge constraints. Guillaumie [11] analyzed symmetric Honeycomb Sandwich panels having composite faces. Le-Anh et al. [12] used the differential algorithm to investigate plate element to examine the static and frequency of folded layered plates. Sandwich panels vibration was investigated using piecewise SDT by Li et al. [13]. Mantari and Granados [14] studied the FG plates using novel FSDT. Moreover, Sandwich functionally graded plates were examined using refined FSDT for their static analysis by Mantari and Granados [15]. A new hyperbolic SDT method by Mahi et al. [16] was used to examine the Sandwich FG and laminated plates. Montemurrol et al. [17] considered the damping behavior of plates using two-level procedure for the global optimization. Dynamic Stiffness technique was used to examine the in-plane vibration of isotropic plates by Nefovska-Danilovic [18]. Curvilinearly stiffened plates were analyzed by Shi et al. [19] for their vibration and buckling analysis using FEM. Ullah et al. [20] characterized the mechanical behavior and damage of carbon fabric–reinforced polymer laminates under bending using finite element model. Arani et al. [21] examined vibration of shell made up of Boron Nitride nanotube reinforced composite. Plastic wrinkling formation for anisotropic laminated sheet was analyzed by Pourmoghadam et al. [22]. Composite plates were studied for assessment of variable separation for finite
element method for free-edge boundary conditions (Vidal et al., [23]). Zippo et al. [24] studied Sandwich plates for their active vibration under free-edge boundary condition using control method. Rayleigh–Ritz technique based on shear deformation theory was used to examine transverse vibration of triangular plates [25]. Thi et al. [26] studied the functionally graded composite plates. Zhang et al. [27] analyzed the vibration of orthotropic plates. Uzun et al. [28] examined functionally graded nanobeams using FEM. Bouderba et al. [29] investigated the thermal stability of FG-Sandwich plates using a simple shear deformation theory. Balubaid et al. [30] studied the free vibration of FG nanoscale plate using nonlocal two variables’ integral refined plate theory. Belbachir et al. [31] examined the thermal flexural analysis of antisymmetric crossply laminated plates. Bouderba et al. [32] studied the stability and dynamic analyses of SW-CNT reinforced concrete beam resting on elastic-foundation. A four-unknown refined plate theory for dynamic analysis of FG-sandwich plates under various boundary conditions were examined by Menasria et al. [33]. Allam et al. [34] studied the bending and free vibration of composite plates using refined higher-order shear deformation theory with four unknown variables. Moreover, Belbachir et al. [31] studied thermal flexural analysis of composite plates based on higher-order shear deformation theory. Static analysis of laminated plates was investigated using simple first-order shear deformation theory by Draiche et al. [35]. Thermomechanical analysis of antisymmetric plates was examined using trigonometric refined plate theory by Abualnour et al. [36]. Belbachir et al. [37] studied the bending analysis of laminated plates under non-linear thermal and mechanical loads. Sahla et al. [38] studied composite soft core Sandwich plates using principle of virtual work. Dastjerdi et al. [39] examined the bending analysis of spherical nanostructures made of functionally graded materials for nonhomogeneous nanostructures. Free vibration of carbon nanotubes were investigated using FSDT and discrete singular convolution method by Civalek, Ö. [40]. Viscoelastic porous functionally graded (FG) nanobeams embedded on visco-Pasternak medium subjected to axially oscillating loading as well as magnetic field were investigated using Kelvin–Voigt model by Jalaei et al. [41]. Civalek et al. [42] used the nonlocal finite element method to examine size-dependent transverse and longitudinal vibrations of embedded carbon and silica carbide nanotubes. Demir et al. [43] used nonlocal discrete models to study torsional and longitudinal frequency and wave response of microtubules. Ebrahimi et al. [44] used the Chebyshev–Ritz method for static stability and vibration analysis of nonlocal microstructure-dependent nanostructures. None of the above mentioned researchers used spline approximation method.

The main contribution of the present research is that plates are of nonuniform thickness, and each layer consists of different material and is oriented at different angle. Moreover, spline approximation method is used which was not used by any of abovementioned researcher except author’s work [45–47]. Spline is used to approximate the equations in terms of translational and angular displacement functions. A spline technique is used due to more accuracy than a global high-order approximation (Bickley [48]). Eigenfrequency parameter is calculated by solving eigenvalue problem. The frequency variation of plates is examined for different geometrical influences such as number of layers, different ply orientations, material used, side-to-thickness ratio, and aspect ratio for two different support conditions. Graphs and tables ascertain the results of the present study.

2. Mathematical Design

Plate of nonuniform thickness is shown in Figures 1(a) and 1(b). It can be seen from the figure that a plate in Cartesian coordinate system x, y, and z taken transverse direction to the plate. a and b are the length and width of the plate, respectively. The plate is assumed to have a constant thickness h, and h_k is the thickness of the k-th layer.

2.1. Displacement Field. According to YNS theory [49], the displacement components are considered as follows:

\[ u = u_0(x, y, t) + z\psi_x(x, y, t), \]
\[ v = v_0(x, y, t) + z\psi_y(x, y, t), \]
\[ w = w(x, y, t), \]

where \( u, v, \) and \( w \) are the displacement components in the \( x, y, \) and \( z \) directions, respectively, \( u_0 \) and \( v_0 \) are the out-of-plane displacements of the middle plate, and \( \psi_x \) and \( \psi_y \) are the shear rotations of any point on the middle surface of the plate. Meanwhile, some authors such as Bourada et al. [32] used an integral first-order shear deformation beam theory, integral Timoshenko beam theory by Matouk et al. [50], and novel integral first-order shear deformation theory by Bousaha et al. [51] in their analysis.

2.2. Kinematics Relations. The kinematic relations are as follows:

\[ \varepsilon_x = \frac{\partial u_0}{\partial x} + z \frac{\partial \psi_x}{\partial x}, \]
\[ \varepsilon_y = \frac{\partial v_0}{\partial y} + z \frac{\partial \psi_y}{\partial y}, \]
\[ \gamma_{xy} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + z \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right), \]
\[ \gamma_{xz} = \psi_x + \frac{\partial w}{\partial x}, \]
\[ \gamma_{yz} = \psi_y + \frac{\partial w}{\partial y}. \]
2.3. Constitutive Relations. These are as follows:

\[
\begin{pmatrix}
\sigma_x^{(k)} \\
\sigma_y^{(k)} \\
\tau_{xy}^{(k)} \\
\tau_{yz}^{(k)} \\
\tau_{zx}^{(k)}
\end{pmatrix}
= \begin{pmatrix}
C_{11}^{(k)} & C_{12}^{(k)} & C_{16}^{(k)} & 0 & 0 \\
C_{12}^{(k)} & C_{22}^{(k)} & C_{26}^{(k)} & 0 & 0 \\
C_{16}^{(k)} & C_{26}^{(k)} & C_{66}^{(k)} & 0 & 0 \\
0 & 0 & 0 & C_{44}^{(k)} & C_{45}^{(k)} \\
0 & 0 & 0 & C_{45}^{(k)} & C_{55}^{(k)}
\end{pmatrix}
\begin{pmatrix}
x_x^{(k)} \\
x_y^{(k)} \\
y_{xy}^{(k)} \\
y_{yz}^{(k)} \\
y_{zx}^{(k)}
\end{pmatrix},
\]

(3)

where \(Q_{ij}^{(k)}\) are the transformed constitutive relations of the materials oriented at angle \(\theta\) with the \(x\)-axis.

2.4. Equations of Motion. The equations of motion are

\[
\begin{align*}
N_{x,x} + N_{x,y,y} &= P \frac{\partial^2 u}{\partial t^2}, \\
N_{x,y,x} + N_{y,y} &= P \frac{\partial^2 v}{\partial t^2}, \\
Q_{x,x} + Q_{y,y} &= P \frac{\partial^2 w}{\partial t^2}, \\
M_{x,x} + M_{x,y,y} - Q_x &= I \frac{\partial^2 \psi_x}{\partial t^2}, \\
M_{x,y,x} + M_{y,y} - Q_y &= I \frac{\partial^2 \psi_y}{\partial t^2}.
\end{align*}
\]

(5)

where \(P\) and \(I\) are the normal and rotary inertia coefficients defined by

\[
(P, I) = \int \rho^{(k)} (1, z^2) \, dz.
\]

Using (2) and (4) into (7), one can obtain the equations of stress-resultants and displacements in the form

\[
\begin{pmatrix}
N_x \\
N_y \\
N_{x,y} \\
M_x \\
M_y \\
M_{x,y}
\end{pmatrix}
= \begin{pmatrix}
A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\
A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\
A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\
B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\
B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\
B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial u_0}{\partial x} \\
\frac{\partial v_0}{\partial y} \\
\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \\
\frac{\partial \psi_x}{\partial x} \\
\frac{\partial \psi_y}{\partial y} \\
\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}
\end{pmatrix},
\]

(7)

in which \(A_{ij}, B_{ij}, D_{ij}\) are the laminate stiffnesses, \(K\) (Pai and Schultz, [53]) is the shear correction coefficient, and \(z_k\) and \(z_{k-1}\) are the boundaries of the \(k\)-th layer.

The thickness of the \(k\)-th layer is considered as

\[
h_k (x) = h_{0k} g(x),
\]

(9)

where \(h_{0k}\) is a constant thickness.
In general, the nonuniform thickness of each layer is assumed in the form
\[ h(x) = h_0 g(x), \]
\[ g(x) = 1 + C_1 x + C_2 \exp\left(\frac{x}{\ell}\right) + C_3 \sin\left(\frac{n x}{\ell}\right). \]  \tag{10}

2.5. Thickness Variation
Case (i):
If \( C_1 = C_2 = 0 \), then the thickness variation becomes linear. In this case it can easily be shown that \( C_\ell = (1/\eta) - 1 \), where \( \eta \) is the taper ratio \( h_0(0)/h_0(1) \).
Case (ii):
If \( C_1 = C_2 = 0 \), then the excess thickness over uniform thickness varies exponentially.
Case (iii):
If \( C_1 = C_2 = 0 \), then the excess thickness varies exponentially.

It may be noted that the thickness of any layer at the end \( X = 0 h_0 \), for cases (i) and (iii) but is \( h_0(1 + C_\ell) \) for case (ii).

Therefore, with the elastic coefficients \( A_{ij}, B_{ij}, \) and \( D_{ij} \) corresponding to layers of uniform thickness with superscript "c", one easily finds
\[ A_{ij}^c = A_{ij} g(x), \]
\[ B_{ij}^c = B_{ij} g(x), \]
\[ D_{ij}^c = D_{ij} g(x), \]  \tag{11}

where
\[ A_{ij}^c = \sum_k \bar{e}_{ij}^{(k)} (z_k - z_{k-1}), \]
\[ B_{ij}^c = \frac{1}{2} \sum_k \bar{Q}_{ij}^{(k)} (z_k^2 - z_{k-1}^2), \]
\[ D_{ij}^c = \frac{1}{3} \sum_k \bar{Q}_{ij}^{(k)} (z_k^3 - z_{k-1}^3), \]  \tag{12}

and \( z_k, z_{k-1} \) are boundaries of the \( k \)-th layer.

The displacement components \( u, v, w \) and shear rotations \( \psi_x \) and \( \psi_y \) are assumed and the nondimensional parameters are introduced. The modified equation in matrix is as follows:
\[ \begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} & L_{15} & U \cr L_{21} & L_{22} & L_{23} & L_{24} & L_{25} & V \cr L_{31} & L_{32} & L_{33} & L_{34} & L_{35} & \Psi_X \cr L_{41} & L_{42} & L_{43} & L_{44} & L_{45} & \Psi_Y \cr L_{51} & L_{52} & L_{53} & L_{54} & L_{55} & W \end{bmatrix} \begin{bmatrix} U \cr V \cr \Psi_X \cr \Psi_Y \cr W \end{bmatrix} = \begin{bmatrix} 0 \cr 0 \cr -0 \cr 0 \cr 0 \end{bmatrix}. \]  \tag{13}

2.6. Method of Solution. The differential equations in equation (13) contain derivatives of second order in \( U(X), V(X), \Psi_X(X), \Psi_Y(X), W(X) \). These functions can be approximated by using cubic spline functions, in the range of \( X \in [0, 1] \).

The displacement functions \( U(X), V(X), \) and \( W(X) \) and the rotational functions \( \alpha \Psi_X(X), \alpha \Psi_Y(X) \) are approximated by splines.

The resulting equations contain \((5N + 5)\) homogeneous system of equations in the \((5N + 15)\) spline coefficients.

The boundary conditions considered on the edges \( x = 0 \) and \( x = a \) are as follows:

(1) \( (C-S) \): one end is clamped and the other end is simply supported
At \( x = 0 \) clamped boundary condition is considered
\[ U = V = W = \psi_x = \psi_y = 0. \]  \tag{14}

At \( x = a \) simply supported boundary condition is considered:
\[ U = W = \psi_y = M_x = N_{xy} = 0. \]  \tag{15}

(2) \( (C-F) \): one end is clamped and the other end is free
At \( x = 0 \) clamped boundary condition is considered:
\[ U = V = W = \psi_x = \psi_y = 0. \]  \tag{16}

At \( x = a \) free boundary condition is considered:
\[ N_x = M_x = Q_x = N_{xy} = M_{xy} = 0. \]  \tag{17}

Each of these cases gives 10 more equations, thus making a total of \((5N + 15)\) equations, in the same number of unknowns. The resulting field and boundary condition equations may be written in the form
\[ [M][q] = \lambda^2 [P][q]. \]  \tag{18}

3. Result and Discussion
The vibration of nonuniform layered plates for clamped-simply supported and clamped-free support conditions is analyzed. Layers are composed of two different materials: Kevlar-49/epoxy (KE) and E-glass epoxy (EGE). Material properties of these two materials are given in the following Table 1.

Figures 2 and 3 are made using Autocad Maya and show the geometry of 2-layered plate with different ply orientation and Figure 3 shows the geometry of 4-layered plate with different ply orientation.

3.1. Comparative Study. Table 2 shows the comparative study of the present results with Liew et al. [54] for simply supported plates. Fundamental frequency parameter values agreed very well with values of Liew et al. [54].
| Elastic property | Density \( \times 10^3 \) N m\(^2\)/s\(^2\) | Yong modulus \( E_x \times 10^{10} \) N/m\(^2\) | Yong modulus \( E_y \times 10^{10} \) N/m\(^2\) | Shear modulus \( G_{xz} \times 10^{10} \) N/m\(^2\) | Shear modulus \( G_{yz} \times 10^{10} \) N/m\(^2\) | Shear modulus \( G_{xy} \times 10^{10} \) N/m\(^2\) | Major Poisson ratio \( \nu_{xy} \) |
|------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|-------------------|
| EGE              | 1440                            | 5.52                            | 86.19                           | 2.07                            | 1.72                            | 2.07                            | 0.34               |
| AGE              | 2550                            | 11.72                           | 42.75                           | 4.14                            | 3.45                            | 4.14                            | 0.27               |
3.2. Geometrical Influences and Support Conditions on Frequency Variations

3.2.1. Effect of Various Thickness Variations

(1) Effect of Linear Thickness Variation. The impact of taper ratio on the fundamental frequency parameter is shown in Tables 3 and 4. Table 3 shows the discrepancy of frequency of 2- and 4-layered antisymmetric angle-ply plates consisting of stacking sequence as 30°/−30°, 45°/−45°, and 60°/−60° having EGE material, whereas four-layered plates have stacking sequence as 45°/−30°/30°/−45°, 45°/−60°/60°/−45°, and 60°/−30°/30°/−60° (KE/EGE/EGE/KE) are considered. The C-S support condition is considered in Table 3, which shows that frequency differs as the taper ratio increases, whereas frequency parameter value is 30°/−30° < 45°/−45° < 60°/−60° and for four-layered plates, it is 45°/−30°/30°/−45° < 60°/−30°/30°/−60° < 45°/−60°/60°/−45°. In Table 4 the values of the frequency of 2- and 4-layered plates under two different support conditions are shown. It is seen that plates having C-F support condition show lesser frequency value than plates with C-S support condition for both two- and four-layered plates.

(2) Effect of Exponential Thickness Variation. In Tables 5 and 6 the influence of exponential variation in thickness on the frequency is shown. It is seen from Table 5 that in case of two-layered plates having 30°/−30° stacking sequence depicts the least frequency and 60°/−60° plates show the highest frequency. Moreover, 45°/−30°/30°/−45° plates show the least...
frequency and $45^\circ/60^\circ/60^\circ/45^\circ$ plates show the highest fundamental frequency value. Table 6 shows that plates with C-S support conditions show higher fundamental frequency value than C-F support conditions.

(3) **Effect of Sinusoidal Thickness Variation.** The influence of sinusoidal variation in thickness on the frequency is shown in Tables 7 and 8. It is seen from Table 7 that in 2-layered plates frequency value increases with the increase of angle, whereas in four-layered plates $45^\circ/60^\circ/60^\circ/45^\circ$ stacking sequence plates showed the highest frequency value and $45^\circ/30^\circ/30^\circ/45^\circ$ stacking sequence plates showed the lowest frequency value. Table 8 shows that plates with C-S support conditions show higher fundamental frequency value than C-F support conditions.

3.2.2. **Effect of Side-to-Thickness Variation and Support Conditions.** The two-layered plates with antisymmetric angle-ply orientation $30^\circ/-30^\circ$, $45^\circ/-45^\circ$, and $60^\circ/-60^\circ$ having EGE material are considered to examine the influence of side-to-thickness ratio ($10 \leq a/h \leq 90$) on the value of fundamental frequency parameter $(\lambda)$ in Figures 4(a)–4(c). The C-S support condition is considered. Also, taper ratio

| $\eta$ | $30^\circ/30^\circ$ | $45^\circ/45^\circ$ | $60^\circ/60^\circ$ | $45^\circ/30^\circ/30^\circ/45^\circ$ | $45^\circ/60^\circ/60^\circ/45^\circ$ | $60^\circ/30^\circ/30^\circ/60^\circ$ |
|--------|----------------------|----------------------|----------------------|-------------------------------------|-------------------------------------|-------------------------------------|
| 0.5    | 0.515640             | 0.624937             | 0.720619             | 0.564178                            | 0.659166                            | 0.624451                            |
| 0.9    | 0.529232             | 0.635048             | 0.731629             | 0.572529                            | 0.667124                            | 0.635186                            |
| 1.3    | 0.535721             | 0.640301             | 0.737287             | 0.577036                            | 0.671602                            | 0.640514                            |
| 1.7    | 0.542518             | 0.646041             | 0.743439             | 0.582046                            | 0.676662                            | 0.646196                            |
| 2.1    | 0.549668             | 0.652301             | 0.750120             | 0.587583                            | 0.682324                            | 0.652256                            |

Table 3: Fundamental frequency parameter value with taper ratio for different stacking sequences.

| $\eta$ | $45^\circ/45^\circ$ | $60^\circ/60^\circ$ | $45^\circ/30^\circ/30^\circ/45^\circ$ | $45^\circ/60^\circ/60^\circ/45^\circ$ | $60^\circ/30^\circ/30^\circ/60^\circ$ |
|--------|----------------------|----------------------|-------------------------------------|-------------------------------------|-------------------------------------|
| 0.5    | 0.624937             | 0.305586             | 0.720619                            | 0.564178                            | 0.624451                            |
| 0.9    | 0.635048             | 0.327285             | 0.731629                            | 0.572529                            | 0.635186                            |
| 1.3    | 0.640301             | 0.338762             | 0.737287                            | 0.577036                            | 0.640514                            |
| 1.7    | 0.646041             | 0.350424             | 0.743439                            | 0.582046                            | 0.646196                            |
| 2.1    | 0.652301             | 0.362868             | 0.750120                            | 0.587583                            | 0.652256                            |

Table 4: Fundamental frequency parameter value with taper ratio for different support conditions.

| $C_e$ | $30^\circ/30^\circ$ | $45^\circ/45^\circ$ | $60^\circ/60^\circ$ | $45^\circ/30^\circ/30^\circ/45^\circ$ | $45^\circ/60^\circ/60^\circ/45^\circ$ | $60^\circ/30^\circ/30^\circ/60^\circ$ |
|-------|----------------------|----------------------|----------------------|-------------------------------------|-------------------------------------|-------------------------------------|
| -0.2  | 0.534893             | 0.639618             | 0.736553             | 0.576446                            | 0.671011                            | 0.639829                            |
| -1    | 0.530827             | 0.636317             | 0.732998             | 0.573610                            | 0.668190                            | 0.636486                            |
| 0     | 0.527439             | 0.636411             | 0.730107             | 0.571337                            | 0.665956                            | 0.633734                            |
| 0.1   | 0.534893             | 0.639618             | 0.736553             | 0.576446                            | 0.671011                            | 0.639829                            |

Table 5: Fundamental frequency parameter value with exponential variation in thickness for different stacking sequences.

| $C_e$ | $45^\circ/45^\circ$ | $60^\circ/60^\circ$ | $45^\circ/30^\circ/30^\circ/45^\circ$ | $45^\circ/60^\circ/60^\circ/45^\circ$ | $60^\circ/30^\circ/30^\circ/60^\circ$ |
|-------|----------------------|----------------------|-------------------------------------|-------------------------------------|-------------------------------------|
| -0.2  | 0.639618             | 0.337357             | 0.736553                            | 0.427408                            | 0.576446                            | 0.293166                            | 0.639829                            | 0.374635                            |
| -0.1  | 0.641315             | 0.340844             | 0.738376                            | 0.430882                            | 0.577916                            | 0.296811                            | 0.641528                            | 0.378052                            |
| 0     | 0.636317             | 0.330494             | 0.732998                            | 0.420582                            | 0.573610                            | 0.285953                            | 0.636486                            | 0.367884                            |
| 0.1   | 0.636411             | 0.324839             | 0.730107                            | 0.414969                            | 0.571337                            | 0.279968                            | 0.633734                            | 0.362296                            |
| 0.2   | 0.631978             | 0.321276             | 0.728306                            | 0.411439                            | 0.569939                            | 0.276179                            | 0.632004                            | 0.358766                            |

Table 6: Fundamental frequency parameter value with exponential variation in thickness for different support conditions.
In Figures 4(a)–4(c), influence of different stacking sequences of four-layered plates $45^\circ/−30^\circ/30^\circ/−45^\circ$, $45^\circ/−60^\circ/60^\circ/−45^\circ$, and $60^\circ/−30^\circ/30^\circ/−60^\circ$ with KE/EGE/EGE/KE material combination on the relationship between side-to-thickness ratio and frequency parameter is presented for the C-S support condition. The aspect ratio $(a/b = 1)$, taper ratio $(\eta = 0.75)$, exponential thickness variation $(C_\eta = 0.1)$, and sinusoidal thickness variation $(C_\varsigma = 0.4)$ are fixed for Figures 4(a)–4(c), respectively.

In Figures 5(a)–5(c), influence of different stacking sequences of four-layered plates $45^\circ/−30^\circ/30^\circ/−45^\circ$, $45^\circ/−60^\circ/60^\circ/−45^\circ$, and $60^\circ/−30^\circ/30^\circ/−60^\circ$ with KE/EGE/EGE/KE material combination on the relationship between side-to-thickness ratio and frequency parameter is presented for the C-S support condition. The aspect ratio $(a/b = 1)$, taper ratio $(\eta = 0.75)$, exponential thickness variation $(C_\eta = 0.1)$, and sinusoidal thickness variation $(C_\varsigma = 0.4)$ are fixed for Figures 5(a)–5(c), respectively. The frequency parameter value varies as $45^\circ/−30^\circ/30^\circ/−45^\circ < 60^\circ/−30^\circ/30^\circ/−60^\circ < 45^\circ/−60^\circ/60^\circ/−45^\circ$. Moreover, the frequency differs for different thickness variations.
The variations in the value of frequency parameter are shown in Figures 6(a)–6(c) under C-F support condition. Generally, the frequency parameter value decreases as the side-to-thickness ratio increases. Moreover, the frequency value varies by using different ply angles.

Four-layered plates with different lamination scheme and material combination as KE/EGE/EGE/KE are considered to investigate the influence of side-to-thickness ratio on the frequency parameter value by fixing aspect ratio \( a/b = 1 \) for Figures 7(a)–7(c), respectively, under C-F support condition. The value of side-to-thickness ratio increases when the value of the frequency decreases.

The influence of different ply angles and C-S support conditions for 4-layered plates having KE/EGE/EGE/KE material combination is considered to study the variation of the frequency parameter value with respect to side-to-thickness ratio value by fixing aspect ratio \( a/b = 0.4 \) and taper ratio \( \eta = 0.75\) in Figures 8(a)–8(c), respectively. The frequency parameter value is higher for higher modes. The same parameters are considered in Figure 9 for C-F support condition. The results show that C-F support condition significantly lowers the value of the frequency as compared to C-S support condition.

Figure 10 demonstrates the effect of aspect ratio \( a/b = 0.4, 1 \) on relation of frequency parameter and side-to-thickness ratio for four-layered plates. The frequency parameter value decreases as the aspect ratio increases. Similarly, the same trend is seen in Figure 11 and the only difference is the support condition: C-F support condition lowers the frequency.

3.2.3. Effect of Aspect Ratio. The influence of aspect ratio on the variation of frequency value is studied for 2-layered antisymmetric angle-ply plates in Figure 12 for C-S support conditions. The value of side-to-thickness ratio \( a/h = 10 \), taper ratio \( \eta = 0.75\), exponential thickness variation \( C_s = 0.1\), and sinusoidal thickness variation \( C_s = 0.4\) are fixed for Figures 12(a)–12(c), respectively. The frequency value increases as the aspect ratio increases. Examining the effect of different ply angles, it is concluded that the frequency increases with the increase of ply angle but the effect of different thickness variations is marginal on the value of the frequency.

Antisymmetric angle-ply four-layered plates are studied to examine the influence of different ply angles and aspect ratio under C-S boundary conditions in Figure 13. The side-to-thickness ratio \( a/h = 10 \), taper ratio \( \eta = 0.75\), exponential thickness variation \( C_s = 0.1\), and sinusoidal thickness variation \( C_s = 0.4\) are fixed for Figures 13(a)–13(c), respectively. The frequency increases with the increase of aspect ratio. The value of frequency varies with respect to ply orientation as \( 45^\circ/30^\circ/30^\circ/45^\circ < 45^\circ/60^\circ/60^\circ/45^\circ < 60^\circ/30^\circ/30^\circ/60^\circ\).

Two- and 4-layered plates of different lamination scheme are studied in Figures 14 and 15 under C-F support conditions to observe the effect of aspect ratio on the frequency. It is seen that the frequency value increases slowly with the increase of aspect ratio. Further, the lamination angles affect the value of frequency. Also C-F support conditions.
significantly lower the value of fundamental frequency when compared with C-S conditions.

Figures 16 and 17 show the effect of four-layered plates on the frequency under C-S and C-F support conditions. The characteristics pattern of the curves is similar for both graphs. The frequency is higher for higher modes and C-F support conditions significantly lower the value of the frequency.

**Figure 6:** Influence of side-to-thickness ratio on frequency parameter value of 2-layered plates: EGE/EGE.

**Figure 7:** Influence of side-to-thickness ratio on frequency parameter value of 4-layered plates: KE/EGE/EGE/KE.
Figure 8: Influence of side-to-thickness ratio on frequency parameter value of 4-layered plates under C-S support condition.

Figure 9: Influence of side-to-thickness ratio on frequency parameter value of 4-layered plates under C-F support condition.
Figure 10: Influence of side-to-thickness ratio on frequency parameter value of 4-layered plates under C-S support condition.

Figure 11: Impact of side-to-thickness ratio on frequency parameter value of 4-layered plates under C-F support condition.

Figure 12: Influence of aspect ratio on frequency parameter value of 3-layered plates under C-S support condition.
Figure 13: Influence of aspect ratio on frequency parameter value of 4-layered plates under C-S support condition.

Figure 14: Influence of aspect ratio on frequency parameter value of 2-layered plates under C-F support condition.
Figure 15: Influence of aspect ratio on frequency parameter value of 4-layered plates under C-F support condition.

Figure 16: Influence of aspect ratio on frequency parameter value of 4-layered plates under C-S support condition.
4. Conclusion

The investigation of layered plates for different geometric properties and support conditions for FSDT is considered. The vibration of layered plates consisting of 2- and 4-layered plates under clamped-simply supported and clamped-free support conditions is analyzed. It is concluded that as the number of layers are increased the frequency value decreases. Moreover, it is discovered that frequency value is higher for C-S boundary condition as compared to C-F boundary condition. It is determined that geometric parameters and layer constituents affect the frequency, whether this effect is significant or marginal. As can be seen from the results, the frequency parameter rises with the increase of aspect ratio; therefore, the rise in frequency shows that the rigidity of the structure will increase. Meanwhile, frequency parameter reduces with the rise of aspect ratio; therefore, the reductions in frequency show that the stiffness of the structure will decrease. Consequently, decrease or increase in the frequency value results in decrease or increase in the stiffness.

Data Availability

The data supporting the results of this study are included in the manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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