LARGE GAPS BETWEEN PRIMES IN ARITHMETIC PROGRESSIONS

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Abstract. For \((M, a) = 1\), put

\[ G(X; M, a) = \sup_{p'_n \leq X} (p'_{n+1} - p'_n), \]

where \(p'_n\) denotes the \(n\)-th prime that is congruent to \(a \pmod{M}\). We show that for any positive \(C\), provided \(X\) is large enough in terms of \(C\), there holds

\[ G(MX; M, a) \geq (C + o(1)) \frac{\log X \log_2 X \log_4 X}{(\log_3 X)^2}, \]

uniformly for all \(M \leq \kappa (\log X)^{1/5}\) that satisfy

\[ \omega(M) \leq \exp \left( \frac{\log_2 M \log_4 M}{\log_3 M} \right). \]

1. Introduction

Denote by

\[ (1.1) \quad G(X) = \sup_{p_n \leq X} (p_{n+1} - p_n) \]

the largest gap between consecutive primes up to \(X\). The study of how large \(G(X)\) can be has a long history. Westzynthius [11] was the first to show that \(G(X)\) can be arbitrarily large compared to the average gap \((1 + o(1)) \log X\). Erdős [1] and Rankin [8] showed

\[ G(X) \geq (c + o(1)) \frac{\log X \log_2 X \log_4 X}{(\log_3 X)^2} \]

for some positive constant \(c\), where \(\log_\nu\) denotes the \(\nu\)-fold iterated logarithm. Subsequent years saw the constant improved from Rankin’s 1/3 by various authors—Schönhage [10], Rankin [9], Maier and Pomerance [5] among others—with the best constant \(c = 2e^{\gamma}\) due to Pintz [7]. After the emergence of the Maynard-Tao method from the study of the small gaps between primes, the method was also applied to the large gap problem by Maynard [6] and Ford, Green, Konyagin and Tao [3] independently to show that \((1.2)\) holds with \(c\) arbitrarily large. Later [2] the five authors were able to quantify this by proving that

\[ (1.3) \quad G(X) \gg \frac{\log X \log_2 X \log_4 X}{\log_3 X} \]

holds.

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To discuss the corresponding question for primes in an arithmetic progression, given a modulus $M$ and and a reduced residue class $a$ (mod $M$), put
\begin{equation}
G(X; M, a) = \sup_{p'_n \leq X} (p'_{n+1} - p'_n),
\end{equation}
where $p'_i$ denotes the $i$-th prime that is congruent to $a$ (mod $M$). Zaccagnini [13] showed that given any positive $C < 1$, uniformly for $M$ satisfying
\begin{equation}
\omega(M) \leq \exp\left(C \log_2 M \frac{\log_4 M}{\log_3 M}\right),
\end{equation}
there holds
\begin{equation}
G(MX; M, a) \geq (e^\gamma + o(1))\varphi(M) \frac{\log X \log_2 X \log_4 X}{(\log_3 X)^2}.
\end{equation}
The improvements that led to the breakthrough developments in the study of large gaps between primes naturally lend themselves to the setting of arithmetic progressions. The present work follows Maynard’s paper [6] on large gaps between primes to derive the analogous result for the case of primes in arithmetic progressions, giving a lower bound that is uniform in terms of the moduli. Our main result is

**Theorem 1.** Let $C > 0$ be given. There is an absolute constant $\kappa > 0$ such that if $X > X_0(C)$ is large enough, we have uniformly for $M \leq \kappa \log X^{1/5}$ satisfying
\begin{equation}
\omega(M) \leq \exp\left(\frac{\log_2 M \log_4 M}{\log_3 M}\right),
\end{equation}
and all reduced residues $a$ (mod $M$), we have
\begin{equation}
G(X; M, a) \geq (C + o(1))\varphi(M) \frac{\log X \log_2 X \log_4 X}{(\log_3 X)^2}.
\end{equation}

2. Setup and the Erdős-Rankin construction

Recall that a set of primes $\mathcal{P}$ is said to sieve out an interval $I$ if there is a choice of residue classes $a_p$ (mod $p$) for each $p \in \mathcal{P}$, such that for all $n \in I$ there is a $p \in \mathcal{P}$ such that $n \equiv a_p$ (mod $p$). Our aim is to show, along the lines of the classical Erdős-Rankin construction, that if $M$ is an integer $\leq cx^{1/5}$, then the primes $p \leq x$, $p \mid M$ can sieve out the interval $[1, U]$, while taking $U$ as large as possible with respect to $x$.

We will write $\mathfrak{P}$ to denote all primes and $\mathfrak{P}_x$ to denote those that don’t exceed $x$, and denote by $\mathfrak{P}_x^{(M)}$ and $\mathfrak{P}_x^{(M)}$ the same sets with prime divisors of $M$ excluded. Put $P_M(x)$ and $P(x)$ for products of primes in $\mathfrak{P}_x^{(M)}$ and $\mathfrak{P}_x$ respectively.

Suppose that $\mathfrak{P}_x^{(M)}$ can sieve out $[1, U]$, so that corresponding to each $p \in \mathfrak{P}_x^{(M)}$, there exists a residue class $a_p$ (mod $p$), such that each number $n = 1, \ldots, U$ satisfies $n \equiv a_p$ (mod $p$) for some $p$.

By the Chinese Remainder Theorem, in any block of $P_M(x)$, integers, there is a $U_0$ such that $U_0 \equiv -a_p$ (mod $p$) for each $p \mid P_M(x)$. Let $j \in [1, U]$, and let $p$ be a prime in $\mathfrak{P}_x^{(M)}$ such that $j \equiv a_p$ (mod $p$). Let $r$ be such that $Mr \equiv -1$ (mod $P_M(x)$). Then for any reduced residue $a$ (mod $M$),
\begin{equation}
M(U_0 + ar + j) + a \equiv M(U_0 + j) - a + a
\equiv M(-a_p + a_p) \equiv 0 \pmod{p},
\end{equation}
so $M(U_0 + ar + j) + a$ is composite provided it is greater than $x$, which is the case if $\frac{a}{M} \leq U_0$. We would also like to ensure the existence of a prime in the arithmetic progression preceding our block of composites. That would follow from the best known result on Linnik’s constant \[12\] if $MU_0 \geq c_0 M^5$, i.e. $U_0 \geq c_0 M^4$. Here and throughout, $\kappa$ denotes $c_0^{-5}$, where $c_0$ is the constant for which Linnik’s theorem with exponent 5 is valid. We impose the condition $M \leq \kappa x^{1/5}$ so that $U_0 \geq c_0 M^4$ is implied by $x M \leq U_0$. So with $U_0 \in \left[ \frac{x}{M^r}, \frac{x}{M^r} + P_M(x) \right]$, we find a prime $p_n \leq M(U_0 + ar)$ such that $p_{n+1} - p_n \geq MU$.

Heuristically, since each $a \equiv 0 \pmod{p}$ removes an element with probability $1/p$, it is reasonable to expect that the integers we can sieve out using primes that don’t divide $M$ will be less numerous by a factor of $\prod_{p|M} (1 - 1/p)$ than those we can sieve out using all primes. Accordingly we put

$$U = Cu \frac{\phi(M)}{M} x \frac{\log y}{\log \log x},$$

where

$$y = \exp \left( (1 - \varepsilon) \frac{\log x \log \log \log x}{\log \log x} \right),$$

and $C_U$ is a constant to be specified later. Now putting $x = (1 - \varepsilon) \log X$ large enough depending only on $\varepsilon$ yields

$$G(X; M, a) \geq G(M(U_0 + ar); M, a) \geq MU = (C_U + o(1)) \frac{x \log_2 X \log_4 X}{(\log_3 X)^2}.$$

Thus our task is to show that $\Psi_x(M)$ can sieve out $[1, U]$ while taking $C_U$ arbitrarily large in (2.2).

We take $a_p \equiv 0 \pmod{p}$ for primes $p \in \Psi(M)$, $y < p \leq z$, where

$$z = \frac{x}{\log \log x}.$$

The set that remains after this sieving is

$$\{m \leq U : m \text{ is } y\text{-smooth}\} \cup \{mp \leq U : p > z, \text{ } m \text{ is } y\text{-smooth}\} \cup \bigcup_{\substack{p|M \\ y < p \leq z}} \{n \leq U : p \mid n\}.$$

Denote the last union over $p \mid M$ by $E_1$. Then

$$|E_1| \leq \frac{U}{y} \omega(M) \ll \frac{U \log M}{y \log_2 M} \ll \frac{x \log y \log x}{y (\log \log x)^2} = o\left( \frac{x}{\log x} \right).$$
For the second sieving we use residue classes \( a_p \equiv 1 \pmod{p} \) for all \( p \in \mathfrak{P}_y(M) \). Then what remains is
\[
\{ m \leq U : m \text{ is } y\text{-smooth}, (m-1, P_M(y)) \} \\
\cup \{ mp \leq U : p > z, m \text{ is } y\text{-smooth}, (mp-1, P_M(y)) = 1 \} \\
= \mathcal{R}^{(M)}_0 \cup \mathcal{R}^{(M)} \cup \mathcal{E}_2,
\]
where \( \mathcal{E}_2 \) is the result of sieving \( \mathcal{E}_1 \), so \( |\mathcal{E}_2| \leq |\mathcal{E}_1| \).

We split \( \mathcal{R}(M) \) according to the integer \( m \), and write
\[
(2.8) \quad \mathcal{R}(M)_m = \{ z < p \leq \frac{U}{m} : (mp-1, P_M(y)) = 1 \}.
\]

Note that if both \( M \) and \( m \) are odd, then this set is vacuous. So we posit the following restriction.
\[
(2.9) \quad 2 \nmid M \Rightarrow 2 \mid m.
\]

In the sequel, \( m \) will be understood to satisfy \((2.10)\). We have the following estimate on the size of \( \mathcal{R}(M)_m \).

**Lemma 1.** Uniformly for \( z + z/\log x \leq V \leq x(\log x)^2, M \leq \kappa x^{1/5} \) and \( m \leq x \) satisfying \((2.10)\), there holds
\[
(2.11) \quad \# \{ z < p \leq V : (mp-1, P_M(y)) = 1 \} = \frac{V - z}{\log x} \left( \prod_{\substack{p \leq y \atop p \mid M}} \frac{p-2}{p-1} \right) \left( 1 + O(\exp(-\log x^{1/2})) \right).
\]

In particular, uniformly for \( m \leq U(1 - 1/\log x)/z \),
\[
(2.12) \quad |\mathcal{R}(M)_m| = \frac{2e^{-\gamma}U(1 + o(1))}{m(\log x)(\log y)} \frac{M}{\varphi(M)} \left( \prod_{p > 2 \atop p \mid M} \frac{p(p-2)}{(p-1)^2} \right) \left( \prod_{p > 2 \atop p \mid m} \frac{p-1}{p-2} \right).
\]

**Proof.** This is almost identical to Lemma 3 of [6], the only difference being that in our case the primes which divide \( M \) are excluded from the sieving process in the application of the fundamental lemma, effecting the constraint \( p \nmid M \) in \((2.11)\). Note that \((2.10)\) ensures that \( p = 2 \) does not occur in the product. \( \square \)

**Lemma 2.** For any \( K \geq 2 \), we have
\[
(2.13) \quad \sum_{U/(zK) \leq m < U/z} |\mathcal{R}(M)_m| \ll \frac{UM \log K}{(\log x)(\log y)\varphi(M)}.
\]

In particular,
\[
(2.14) \quad \sum_{U/(z(\log x)^2) \leq m < U/z} |\mathcal{R}(M)_m| = o \left( \frac{CUx}{\log x} \right), \quad \sum_{1 \leq m < U/(z(\log x)^2)} |\mathcal{R}(M)_m| = O \left( \frac{CUx}{\log x} \right).
\]

**Proof.** Put \( w_1 = U/(zK) \) and \( w_2 = U(1 - 1/\log x)/x \). For \( m \geq w_2 \), we use the trivial bound \( |\mathcal{R}(M)_m| \ll U/(m \log x) \) to see that the contribution from \( w_2 \leq U/z \) is \( O(U/(\log x)^2) \).
We regroup the factors in (2.12) to separate the effect of $M$.

(2.15) \[
\left( \prod_{\substack{p > 2 \\ p|M}} p \right) \left( \prod_{\substack{p > 2 \\ p|M}} \frac{p - 1}{p - 2} \right) \leq \left( \prod_{\substack{p > 2 \\ p|M}} \frac{(p - 1)^2}{p(p - 2)} \right) \left( \prod_{\substack{p > 2 \\ p|M}} \frac{p}{p - 2} \right) \left( \prod_{\substack{p > 2 \\ p|M}} \frac{p - 1}{p - 2} \right).
\]

The first product on the right hand side can be estimated as

(2.16) \[
\prod_{\substack{p > 2 \\ p|M}} \left( 1 + \frac{1}{p(p - 2)} \right) \leq \prod_{\substack{p > 2 \\ p|M}} \left( 1 + \frac{3}{p^2} \right) \leq \zeta(2)^3,
\]

whence we have

(2.17) \[
|R^{(M)}_m| \ll \frac{U(1 + o(1))}{m(\log x)(\log y)\varphi(M)} \left( \prod_{\substack{p > 2 \\ p|M}} \frac{p - 1}{p - 2} \right).
\]

Now for $w_1 \leq m < w_2$, we use the bound

(2.18) \[
\prod_{\substack{p > 2 \\ p|M}} \frac{p - 1}{p - 2} \ll \prod_{\substack{p > 2 \\ p|M}} \frac{p + 1}{p} \leq \sum_{d|m} \frac{1}{d},
\]

and obtain

(2.19) \[
\sum_{w_1 \leq m < w_2} |R^{(M)}_m| \ll \frac{UM}{\log x(\log y)\varphi(M)} \sum_{w_1 \leq m < w_2} \frac{1}{m} \sum_{d|m} \frac{1}{d}
\]

\[
= \frac{UM}{\log x(\log y)\varphi(M)} \sum_{d<w_2} \frac{1}{d^2} \sum_{w_1/d \leq m < w_2/d} \frac{1}{m}
\]

\[
\ll \frac{UM(\log K + O(1))}{\log x(\log y)\varphi(M)},
\]

and substituting the definitions of $U$ and $y$ yields the particular cases. \hfill \Box

We also have a bound for $R^{(M)}_0$.

**Lemma 3.** We have

(2.20) \[
|R^{(M)}_0| \ll \frac{x}{(\log x)^{1+\varepsilon}}.
\]

**Proof.** This is Theorem 5.3 in \[5\], again with the only difference being that prime divisors of $M$ are excluded from the sieving process in the invocation of Theorem 4.2 of \[4\], again contributing a factor $\ll M/\varphi(M)$. By our restriction on the size of $M$, this can be absorbed in the $(\log x)^{-\varepsilon}$ factor. \hfill \Box

With these estimates, we will use the key proposition below to prove our main result.

**Proposition 1.** Let $\delta > 0$ be given, and $x > x_0(\delta)$ be large enough. For each $m < Uz^{-1}(\log_2 x)^{-2}$ satisfying (2.10), let $I_m \subseteq [x/2, x]$ be an interval of length at least $\delta |R^{(M)}_m| \log x$. Then there exists a choice of residue classes $a_q \pmod{q}$ for each prime $q \in I_m$ such that for all $p \in R^{(M)}_m$ there is a $q \in I_m$ such that $p \equiv a_q \pmod{q}$.
Proof of Theorem 7 assuming Proposition 7: By Lemma 2 we have
\begin{equation}
\sum_{m < U/(z\log x)^2} \delta |R_m^{(M)}| \log x \ll \delta C_U x.
\end{equation}

Thus, if \( \delta \) is small enough, we can choose the \( I_m \subseteq [x/2, x] \) for even \( m < U z^{-1}(\log x)^{-2} \) to be disjoint. By Proposition 1, the primes in those intervals are enough to sieve out all the primes in the \( R_m^{(M)} \). By lemmas 1, 2, and 3 this shows that we can cover all but \( o(x/\log x) \) numbers in \( R_n^{(M)} \cup R^{(M)} \cup E_2 \) using primes in \([x/2, x]\). So using one residue class each for the primes in \([z, x]\) is sufficient to cover what remains. This proves that \( \mathfrak{P}_x^{(M)} \) can sieve out \([1, U]\).

The proof of Proposition 1 is probabilistic. Assume that for \( q \in I_m \), we pick a residue class \( a \) (mod \( q \)) with probability \( \mu_{m, q}(a) \). Then the probability that a given \( p_0 \in R_m^{(M)} \) is not picked for any \( q \in I_m \) is
\begin{equation}
\prod_{q \in I_m} \left( 1 - \mu_{m, q}(p_0) \right) \leq \exp\left( - \sum_{q \in I_m} \mu_{m, q}(p_0) \right).
\end{equation}

If we show that the sum on the right hand side can be made arbitrarily large, then we can deduce that there’s a choice of residue classes \( a_q \) (mod \( q \)) such that an arbitrarily small portion of the primes in \( R_m^{(M)} \) is left out.

We put
\begin{equation}
\omega_{m, q}(p) = \# \{ 1 \leq n \leq p : n + h_i q \equiv 0 \pmod{p} \}
\end{equation}

or \( m(n + h_i q) \equiv 1 \pmod{p} \) for some \( i = 1, \ldots, k \), where \( \mathcal{H} = h_1, \ldots, h_k \) with \( h_i = p_{\pi(k) + i} P(w) \) is an admissible \( k \)-tuple (recall that \( \{h_i\} \) is called admissible if \( |\{h_i \pmod{p}\}| < p \) for all primes \( p \)). Also let \( \varphi_{m, q} \) be the multiplicative function defined on primes by \( \varphi_{m, q}(p) = p - \omega_{m, q}(p) \). With this, we define the singular series
\begin{equation}
\mathfrak{S}_{m, q}^{(M)} = \prod_{\substack{p \leq y \\ p \nmid M}} \left( 1 - \frac{\omega_{m, q}(p)}{p} \right) \left( 1 - \frac{1}{p} \right)^{-k} \prod_{\substack{p \leq y \\ p \nmid M}} \left( 1 - \frac{1}{p} \right)^{-k} \prod_{\substack{p \leq w \\ p \nmid M}} \left( 1 - \frac{1}{p} \right)^{1-k}.
\end{equation}

We will define \( \mu_{m, q} \) by
\begin{equation}
\mu_{m, q}(a) = \alpha_{m, q} \sum_{\substack{n \leq U/m \\ n \equiv \varphi_{m, q}(p) \pmod{p^{\pi(w)}} \pmod{p^{\pi(w)}}}} \left( \sum_{d_1 \ldots d_k \mid (n+h_i q), \sum \epsilon_{i, e_i, \ldots, e_k}} \lambda_{d_1, \ldots, d_k, e_1, \ldots, e_k} \right)^2,
\end{equation}

where \( \alpha_{m, q} \) is a normalizing constant, \( w = \log_4 x \) and the \( \lambda \) are given by
\begin{equation}
\lambda_{d_1, \ldots, d_n, e_1, \ldots, e_k} = \left( \prod_{i=1}^{k} \mu(d_i) \mu(e_i) \sum_{j=1}^{J} \left( \prod_{\ell=1}^{k} F_{i,j} \frac{\log d_\ell}{\log x} G(\frac{\log \epsilon_\ell}{\log x}) \right) \right).
\end{equation}

for some smooth nonnegative functions \( F_{i,j}, G : [0, \infty) \to \mathbb{R} \) which are not identically zero. These functions and the parameter \( J \) may depend on \( k \) but not on \( x \) or
q. Thus $|\lambda_{d_1, \ldots, d_k, e_1, \ldots, e_k}| \ll k 1$. Also for each $j = 1, \ldots, J$, we require

\begin{equation}
(2.27) \quad \sup\{\sum_{i=1}^{k} u_i : F_{i,j}(u_i) \neq 0\} \leq 1/10,
\end{equation}

and restrict $G$ to be supported on $[0,1]$. Also put

\begin{equation}
(2.28) \quad F(t_1, \ldots, t_k) = \sum_{j=1}^{J} \prod_{\ell=1}^{k} F'_{\ell,j}(t_{\ell}),
\end{equation}

and assume that $F_{i,j}$ are chosen so that $F$ is symmetric.

Two things are different in (2.25) compared to \([6]\). Firstly, we have the weaker corresponding condition on the definition of $R_{m}^{(M)}$. Also, we require that $(e_i, M) = 1$ to simplify certain divisibility conditions that will arise.

3. Estimations

To first estimate $\alpha_{m,q}$, we sum (2.25) over a $(\mod q)$ and rearrange sums to obtain

\begin{equation}
(3.1) \quad \alpha_{m,q}^{-1} = \sum_{d_1, \ldots, d_k, e_1, \ldots, e_k} \sum_{d_1', \ldots, d_k', e_1', \ldots, e_k'} \lambda_{d,e} \lambda_{d',e'} \sum_{n \leq U/m} \sum_{(n,P(w))=1} \sum_{(m-1,P_M(w))=1} \sum_{|d_i, d_i'| \mid n+h_i} \sum_{|e_i, e_i'| \mid (n+h_i)q^{-1}} 1.
\end{equation}

Here if $p \mid d_i d_i'$, then $p \mid n+h_i q$, whence $p > w$ since $P(w) \mid h_i$ and $(n, P(w)) = 1$. So we have $(d_i d_i', P(w)) = 1$ for all $i$. Similarly if $p \mid e_i e_i'$, then $p \nmid M$, and we also have $q \mid e_i e_i' P_M(w)$ as well, whence $(e_i e_i', P(w)) = 1$ for all $i$. Also, $d_i d_i'$ and $d_i d_i'$ are relatively prime for $i \neq j$, since a common divisor $p$ of both would have to divide $(h_i - h_j)q$, but this is absurd when $p \mid h_i - h_j$ implies $p \leq w$, but $p \nmid P(w)$ and $p \leq x^{1/10} < q$. Along the same lines, we see also that $e_i e_i'$, $e_i e_i'$ are pairwise relatively prime. Also, we see immediately that if $p \mid (d_i d_i', e_i e_i')$ then $p \mid m q (h_j - h_i) - 1$ and that $(e_i e_i', M) = 1$. Under these restrictions, the inner sum counts the $n$ satisfying

\begin{align}
(3.2) \quad & n \equiv 0 \pmod{p}, & p & \leq w, \\
(3.3) \quad & mn \equiv 1 \pmod{p}, & p & \leq w, p \nmid M, \\
(3.4) \quad & n \equiv -h_i q \pmod{|d_i, d_i'|}, & \forall i, \\
(3.5) \quad & n \equiv m - h_i q \pmod{|e_i, e_i'|}, & \forall i.
\end{align}

By the Chinese Remainder Theorem, the number of such $n$ in a block of $P(w)[d, d', e, e']$ integers is $\varphi_{m,q}(P_M(w)) \varphi\left(\frac{P(w)}{P_M(w)}\right)$. So we have

\begin{equation}
(3.6) \quad \alpha_{m,q}^{-1} = \sum_{d_1, \ldots, d_k, e_1, \ldots, e_k} \sum_{d_1', \ldots, d_k', e_1', \ldots, e_k'} \lambda_{d,e} \lambda_{d',e'} \times \left( \frac{U \varphi_{m,q}(P_M(w)) \varphi\left(\frac{P(w)}{P_M(w)}\right)}{m P(w)[d, d', e, e']} + O \left( \varphi_{m,q}(P_M(w)) \varphi\left(\frac{P(w)}{P_M(w)}\right) \right) \right),
\end{equation}
where $\sum'$ denotes the sums with the aforementioned divisibility conditions. Note that in our case the only extra constraint compared to the original case is $(e_i e'_i, M) = 1$.

Using the fact that $|\lambda_{d,e}| \ll_k 1$, and recalling the support conditions $\prod_i d_i < x^{1/10}$ and $\prod_i e_i, y^k < x^k$, together with the fact that $\varphi_m(q(P_M(w))\varphi(P_m(w)^{-1}) \leq P(w) \ll \log x$, we see that the contribution of the error term is at most $\ll x^{1/2}$.

We expand the $\lambda$ using (2.26), so that we are left to evaluate

$$
(3.7) \quad \sum_{j=1}^J \sum' \prod_{\ell=1}^k \mu(d_\ell)\mu(d'_\ell)\mu(e_\ell)\mu(e'_\ell) H_{\ell,j'}(d_\ell, d'_\ell, e_\ell, e'_\ell),
$$

where

$$
(3.8) \quad H_{\ell,j'}(d_\ell, d'_\ell, e_\ell, e'_\ell) = F_{\ell,j}(\log d_\ell / \log x) F_{\ell,j'}(\log d'_\ell / \log x) G(\log e_\ell / \log x) G(\log e'_\ell / \log y).
$$

The functions $e^t F_{\ell,j}(t)$ can be extended to smooth compactly supported functions on $\mathbb{R}$, so has a Fourier expansion $e^t F_{\ell,j}(t) = \int_\mathbb{R} e^{-it\xi} f_{\ell,j}(\xi) d\xi$, with $f_{\ell,j}(\xi) \ll_k A (1 + |\xi|)^{-A}$ rapidly decreasing. Thus

$$
(3.9) \quad F_{\ell,j}(\log d_\ell / \log x) = \int_\mathbb{R} f_{\ell,j}(\xi) e^{i\xi \log d_\ell / \log x} d\xi,
$$

and similarly for $G$. So we can rewrite the inner two sums in (3.7) as

$$
(3.10) \quad \int_\mathbb{R} \cdots \int_\mathbb{R} \left( \sum' \prod_{\ell=1}^k \mu(d_\ell)\mu(d'_\ell)\mu(e_\ell)\mu(e'_\ell) \right) \prod_{\ell=1}^k \frac{1}{d_\ell^{(1+i\xi_\ell)/\log x}} \frac{1}{d'_\ell^{1+i\tau_\ell}} \frac{1}{e_\ell^{1+i\xi_\ell}} \frac{1}{e'_\ell^{1+i\tau_\ell}}
\times \left( \prod_{\ell=1}^k f_{\ell,j}(\xi_\ell) f_{\ell,j'}(\xi'_\ell) g(\tau_\ell) g(\tau'_\ell) d\xi_\ell d\xi'_\ell d\tau_\ell d\tau'_\ell \right),
$$

and in turn write the sum here as a product $\prod_{p} K_{p}$, where

$$
K_p = \sum' \sum' \prod_{\ell=1}^k \frac{1}{d_\ell^{(1+i\xi_\ell)/\log x}} \frac{1}{d'_\ell^{1+i\tau_\ell}} \frac{1}{e_\ell^{1+i\xi_\ell}} \frac{1}{e'_\ell^{1+i\tau_\ell}}
\times \left( \prod_{\ell=1}^k f_{\ell,j}(\xi_\ell) f_{\ell,j'}(\xi'_\ell) g(\tau_\ell) g(\tau'_\ell) d\xi_\ell d\xi'_\ell d\tau_\ell d\tau'_\ell \right)
\times (1 + O_k(p^{-1/2}\log x),
$$

so that $\prod_{p} K_p \ll (\log x)^{O_k(1)}$. By the rapid decrease of the functions $f,g$ we can truncate the integrals to $|\xi_\ell|, |\xi'_\ell|, |\tau_\ell|, |\tau'_\ell| \leq (\log x)^{1/2}$. We relabel $s_j = (1 + i\xi_j) / \log x, r_j = (1 + i\tau_j) / \log x$, and similarly for $s'_j, r'_j$.

Now for $w \leq p \leq y$ with $p \nmid M \prod_{h,h' \in \mathcal{H}(mq(h-h')) - 1}$, or for $p > y$, we have

$$
(3.12) \quad K_p = \left( 1 + O_k\left( \frac{1}{p^3} \right) \right) \prod_{\ell=1}^k \frac{1 - p^{-1-s_\ell}}{(1 - p^{-1-s_\ell})(1 - p^{-1-s'_\ell})(1 - p^{-1-r_\ell})}(1 - p^{-1-r'_\ell})(1 - p^{-1-r_\ell})
\times (1 - p^{-1-r_\ell-r'_\ell})(1 - p^{-1-r_\ell-r'_\ell}).
$$
If \( w \leq p \leq y \) with \( p \mid \prod_{h,w \in \mathcal{H}} (mq(h - h') - 1) \) and \( p \nmid M \), we will have an extra factor corresponding to products of \( d_j, e_\ell \) if \( p \mid mq(h_\ell - h_j) - 1 \).

\[
(1 + O_k \left( \frac{1}{p^2} \right)) \prod_{j, \ell: p \nmid mq(h_\ell - h_j) - 1} \left(1 + \frac{1}{p} \sum_{T \subseteq \{s_j, s_j', r_\ell, r_\ell'\}} (-1)^{|T|} p^{-\sum_{i \in T} i} \right) \\
\sum_{\substack{\mathcal{T} \subseteq \{s_j, s_j'\} \
\mathcal{T} \cap \{r_\ell, r_\ell'\} \neq \emptyset}} \frac{1}{p} \sum_{\mathcal{T} \cap \{r_\ell, r_\ell'\} \neq \emptyset} \left(1 + O_k \left( \frac{1}{p^2} + \frac{\log p \sqrt{\log x}}{p \log y} \right) \right),
\]

(3.13)

where we used the fact that \( p^{-\sum i} = 1 + O((\log p)(\log x)^{1/2}/(\log y)) \) by the truncation of the variables. The first factor here simplifies to \( 1 - (\omega_{m,q}(p) - 2k)/p \).

If \( p \mid M \), then due to our constraint \((e_i, e_i', M) = 1\), we have no contribution from the \( e_i \)'s, so such \( p \) contribute a factor of

\[
K_p = \left(1 + O_k \left( \frac{1}{p^2} \right) \right) \prod_{\ell=1}^{k} \frac{(1 - p^{-1-s_\ell})(1 - p^{-1-s_i'})(1 - p^{-1-r_\ell})(1 - p^{-1-r_i'})}{(1 - p^{-1-s_\ell-s_i'})(1 - p^{-1-r_\ell-r_i'})}.
\]

(3.14)

Finally, we can supply the same factors as in (3.12) for small primes by noting that

\[
\prod_{p \leq w} \left(1 - \frac{1}{p} \right)^{-2k} \prod_{p \leq w} \prod_{\ell=1}^{k} \frac{(1 - p^{-1-s_\ell})(1 - p^{-1-s_i'})(1 - p^{-1-r_\ell})(1 - p^{-1-r_i'})}{(1 - p^{-1-s_\ell-s_i'})(1 - p^{-1-r_\ell-r_i'})} = \left(1 + o_k(1) \right).
\]

Putting these together, we find that

\[
\prod_{p > w} K_p = \left(1 + o_k(1) \right) \prod_{p \leq w} \left(1 - \frac{1}{p} \right)^{-2k} \prod_{w < p \leq y} \left(1 - \frac{\omega_{m,q}(p) - 2k}{p} \right) \\
\times \prod_{\ell=1}^{k} \frac{\zeta \left(1 + \frac{2+ix_\ell + iy_\ell'}{\log x} \right) \zeta \left(1 + \frac{2+ix_\ell + iy_\ell'}{\log y} \right)}{\zeta \left(1 + \frac{1+ix_\ell}{\log x} \right) \zeta \left(1 + \frac{1+ix_\ell}{\log y} \right) \zeta \left(1 + \frac{1+ix_\ell'}{\log x} \right) \zeta \left(1 + \frac{1+ix_\ell'}{\log y} \right)} \\
\times \prod_{w < p \leq y} \prod_{\ell=1}^{k} \frac{(1 - p^{-1-r_\ell-r_i'})}{(1 - p^{-1-r_\ell})(1 - p^{-1-r_i'})}.
\]

(3.16)

We see that the last product is

\[
\prod_{w < p \leq y} \prod_{\ell=1}^{k} \frac{(1 - p^{-1-r_\ell-r_i'})(1 - p^{-1})}{(1 - p^{-1-r_\ell})(1 - p^{-1-r_i'})} \prod_{w < p \leq y} \prod_{p \mid M} \left(1 - \frac{1}{p} \right)^{-k},
\]

(3.17)

and the double product can be written as

\[
\exp \left( \sum_{w < p \leq y} \sum_{\ell=1}^{k} p^{-1} \left(-p^{-r_\ell-r_i'} + p^{-r_\ell} + p^{-r_i'} - 1 + O(p^{-1}) \right) \right).
\]

(3.18)
Since $|r_j|, |r'_j| < (\log x)^{1/2} / \log y$, we have

\begin{equation}
(3.19) \quad p^{-r-r'} - p^{-r} - p^{-r'} = 1 + O(\log p) (\log x)^{1/2} / \log y),
\end{equation}

so the double sum is

\begin{equation}
(3.20) \quad O_k \left( \frac{(\log x)^{1/2}}{\log y} \sum_{w < p \leq y} \frac{(\log p)^2}{p} + o_k(1) \right).
\end{equation}

By the assumption (4.7), we see that this is $o_k(1)$. Integrating the zeta factors proceeds identically as in [6], so putting everything together and noting that

\begin{equation}
(3.21) \quad \varphi_{m,q}(P_M(w)) \varphi\left( \frac{P_M(w)}{P_M(w)} \right) \prod_{p \leq w} \left( 1 - \frac{1}{p} \right)^{-2k} \prod_{w < p \leq y} \left( \frac{1}{p} \right) ^{-k} = \mathcal{G}^{(M)}_{m,q},
\end{equation}

we obtain

**Lemma 4.** We have

\begin{equation}
(3.22) \quad \alpha_{m,q}^{-1} = (1 + o(1)) \frac{U \mathcal{G}^{(M)}_{m,q}}{m(\log x)^{k}(\log y)^{k}} I_k^{(1)}(F) I_k^{(2)}(G),
\end{equation}

where $\mathcal{G}^{(M)}_{m,q}$ is given by (2.24), and

\begin{equation}
(3.23) \quad I_k^{(1)}(F) = \int \cdots \int_{t_1, \ldots, t_k \geq 0} F(t_1, \ldots, t_k)^2 dt_1, \ldots, dt_k,
\end{equation}

\begin{equation}
(3.23) \quad I_k^{(2)}(G) = \left( \int_0^\infty G'(t) dt \right)^k.
\end{equation}

Now we can consider the sum

\begin{equation}
(3.24) \quad \sum_{q \in \mathcal{I}_m \text{ prime}} \mu_{m,q}(p_0)
= \sum_{q \in \mathcal{I}_m \text{ prime}} \alpha_{m,q} \sum_{n \leq U/m} \sum_{\substack{n \equiv p_0 \pmod{q} \backslash (n,p_M(w)) = 1 \backslash (mn - 1, P_M(w)) = 1}} \lambda_{d_1, \ldots, d_k, e_1, \ldots, e_k}^2,
\end{equation}

where $p_0 \in \mathcal{R}^{(M)}_m$. We remark that even though our sifting primes must not divide $M$, since $\mathcal{I}_m \subseteq \left[ x/2, x \right]$, the $q$ under consideration are larger than $M$, so we needn’t impose $q \mid M$ explicitly.
We minorize this sum by dropping all terms except when \(n = p_0 - hq\) for some \(h \in \mathcal{H}\) which are clearly in the sum. Thus

\[
(3.25) \quad \sum_{q \in \mathcal{I}_m \text{ prime}} \mu_{m,q}(p_0) \geq \sum_{h \in \mathcal{H}} \sum_{q \in \mathcal{I}_m \text{ prime}} \alpha_{m,q} \left( \sum_{d_1, \ldots, d_k \in H} \sum_{\theta \in H} \lambda_{d_1, \ldots, d_k, \theta} \right)^2.
\]

In turn, we split the sum over \(q\) into residue classes modulo \(P(w)\) and obtain

\[
(3.26) \quad \sum_{h \in \mathcal{H}} \sum_{q \equiv 0 \pmod{P(w)}} \alpha_{m,q} \left( \sum_{d_1, \ldots, d_k \in H} \sum_{\theta \in H} \lambda_{d_1, \ldots, d_k, \theta} \right)^2.
\]

We now replace \(\alpha_{m,q}\) by an expression with less dependence on \(q\). We note that for \(p \leq w\), we have \(\omega_{m,q}(p) = 1\) or \(2\) according as \(p \mid m\) or not, and for \(w < p \leq y\), we have \(\omega_{m,q}(p) = 2k\) if

\[
(3.27) \quad p \nmid h, h' \in \mathcal{H}, (m \mid (h - h') - 1).
\]

So,

\[
(3.28) \quad \left( \mathcal{E}_{m,q}^{(M)} \right)^{-1} = \prod_{p \leq y} \left( 1 - \frac{\omega_{m,q}(p)}{p} \right)^{-1} \left( 1 - \frac{1}{p} \right)^{2k} \prod_{p \leq y} \left( 1 - \frac{1}{p} \right)^k \prod_{p \leq w} \left( 1 - \frac{1}{p} \right)^{k-1},
\]

\[
(3.29) \quad \geq (1 + o_k(1)) \mathcal{E}_{m}^{(M)} \prod_{w < p \leq y} \left( 1 - \frac{1}{p} \right)^{2k} \prod_{p \mid M} \left( 1 - \frac{1}{p} \right)^{2k} \prod_{p \mid (m \mid (h - h')) - 1} \left( 1 - \frac{1}{p} \right)^{k-1},
\]

where

\[
(3.30) \quad \mathcal{E}_{m}^{(M)} = 2^{-(2k-1)} \left( \varphi(M) \right)^k \prod_{2 < p \leq \phi(M)} \left( \frac{p-2}{p-1} \right) \prod_{2 < p \leq w} \left( \frac{p-2}{p-1} \right)^2 \left( \frac{1}{p} \right)^{-1}.
\]

We restrict the primes occurring in the last product in \((3.28)\) to be less than \(z_0 = \log x \log_2 x \log_3 x / \log_2 x\) at a cost of \((1 + o_k(1))\) and expand the product to obtain

\[
(3.31) \quad \left( \mathcal{E}_{m,q}^{(M)} \right)^{-1} \geq (1 + o_k(1)) \mathcal{E}_{m}^{(M)} \sum_{a_1, \ldots, a_k \mid \phi(M)/\phi(M) = 1} \frac{(-2k)^{\omega([a])}}{[a]},
\]

\(a_{1,2, \ldots, a_k \mid \phi(M)/\phi(M) = 1} \).

\(a_{1,2, \ldots, a_k \mid \phi(M)/\phi(M) = 1} \).
where \([a] = [a_1, \ldots, a_{k,k-1}]\). Substituting this and (3.22) in (3.20), we obtain

\[
\sum_{q \in \mathcal{I}_m \text{prime}} \mu_m(q, p_0) \geq \frac{(1 + o(1)) \mathcal{E}_m^{(M)}(p_0 \log x)^k (\log y)^k}{U^{(1)}(F) I_k^{(2)}(G)} \times \sum_{h \in \mathcal{H}} \sum_{(w_0, P(w)) = 1} \left( \sum_{d_1, \ldots, d_k, e_1, \ldots, e_k} \lambda_{d_1, \ldots, d_k, e_1, \ldots, e_k} \right)^2.
\]

We consider the sum over \(q\). We suppose \(h\) in the outer sum is \(h_k\), without any loss of generality. By the support of \(F\) and the fact that \(p_0 > x\), we must have \(d_k = 1\), and similarly \(e_k = 1\). We expand the square and rearrange the sums to find that the sum over \(q\) equals

\[
\sum_{d_1, \ldots, d_k, e_1, \ldots, e_k} \lambda_{d, e} \lambda_{d', e'} \sum_{q \equiv w_0 (\text{mod } P(w))} \sum_{d_i = d_i', e_i = e_i'} \left( \sum_{(w_0, P(w)) = 1} \frac{(-2k)^{|d, e|}}{|a|} \sum_{d_j, e_j} \right)^2.
\]

We see that if \(p \mid d_i d_j'\) for some \(p \leq w\), then \([d_i, d_j'] \mid p_0 + (h_i - h_k)q\) would imply \(p \mid p_0\), an absurdity. Thus we have \(d, d'\) relatively prime to \(P(w)\) for all \(i\). Also, if \(p \mid (d_i d_j, d_j d_j')\), this would imply \(p \mid q(h_i - h_j)\), but \(h_i - h_j\) only has prime divisors not exceeding \(w\), and \(p < x^{1/10} < q\), so we also have \((d_i d_j, d_j d_j') = 1\) for all \(i \neq j\). Similarly, if \(p \mid e_i e_j\) with \(p \leq w\), then since \(p \mid h_i - h_k\), then necessarily \(p \mid m p_0 - 1\), but \(p_0\), being in \(\mathcal{R}_m^{(M)}\), is relatively prime to \(P_M(w)\), so since by assumption \(p \mid M\), we have \((e_i e_j, P(w)) = 1\) for all \(i\). Similarly it is easy to see that \((e_i e_j, e_j e_j') = 1\) for \(i \neq j\). Plainly \((a_{i,j}, m) = 1\) for all \(i \neq j\), and any prime dividing \((a_{i,j}, a_{i',j'})\) would have to divide \(q(h_i + h_j) - (h_i + h_j)\), but the latter has no prime divisors in \([w, z_0]\), so the \(a_{i,j}\) are also pairwise coprime. We have the compatibility conditions

\[
\begin{align*}
(d_i d_j', e_i e_j') & \mid m p_0 (h_i - h_j) + h_k - h_i & \forall i, j, \\
(d_i d_j', a_{i,j}) & \mid (h_j - h_i) m p_0 + h_i - h_k & \forall i, j, \ell, \\
(e_i e_j', a_{i,j}) & \mid (h_j - h_i) (1 - p_0 m) - h_i + h_k \forall i, j, \ell,
\end{align*}
\]

under which the inner sum counts primes in a single residue class modulo the least common multiple of \(d_1, d_1', e_1, e_1', \ldots, d_k, d_k', e_k, e_k', a_1, \ldots, a_{k,k-1}, P(w)\). With this we see that the inner sum in (3.33) is

\[
\sum_{q \in \mathcal{I}_m \text{prime}} \frac{1}{\varphi(P(w)) \varphi([d, d', e, e', a])} + E(x; P(w)[d, d', e, e', a]).
\]

Summing the error terms is handled by a standard application of the Bombieri-Vinogradov theorem. With this and the fact that the number of primes in \(\mathcal{I}_m\) is
(1 + o(1))|I_m|/ \log x$, we see that \((3.33)\) simplifies to

\[
\frac{(1 + o(1))|I_m|}{\varphi(P(u)) \log x} \sum \sum^* \frac{\lambda_{d, e, \lambda_{d', e'}}}{\varphi([d, d', e, e', a])}
\]

\[
\varphi(P(w)) \log x \sum_{j=1}^{J} \sum_{j'=1}^{J} F_{k,j}(0) F_{k,j'}(0) \sum_{d_1, \ldots, d_{\ell}, \ldots, d_{\ell-1}} \sum_{e_1, \ldots, e_{\ell}, \ldots, e_{\ell-1}} \prod_{i=1}^{k} \mu(d_i) \mu(d'_i) \mu(e_i) \mu(e'_i) F \left( \frac{\log d_i}{\log x} \right) F \left( \frac{\log d'_i}{\log x} \right) G \left( \frac{\log e_i}{\log y} \right) G \left( \frac{\log e'_i}{\log y} \right)
\]

where \(\sum^*\) indicates the divisibility conditions together with the condition \(d_k = d'_k = e_k = e'_k\), and \(\sum^{**}\) the same with the latter dropped. One difference in our case is that the restrictions in the sums include \((e_i e'_i, M) = 1\), and the \(a_{i,j}\) also carry the restriction \((a_{i,j}, M) = 1\).

We handle the sums over the \(d_i, d'_i, e_i, e'_i\) as before by first factorizing into prime factors \(K_p\). The presence of the Euler \(\varphi\)-function in the denominator effects a factor of \((1 + O(p^{-2}))\) in each \(K_p\), so the difference is negligible. Let us first suppose that all the \(a_{i,j}\) are \(1\). This time the primes that contribute mixed factors involving \(p \mid (d_i e_i)\) must divide \(\prod_{h,h' \in H}(mp_h(h - h') + h_k - h)\). So for \(w \leq p \leq y\) with \(p \mid \prod_{h,h' \in H}(mp_h(h - h') + h_k - h)\) and \(p \nmid M\), there is the contribution from products of \(d_j, e_i\) if \(p \mid (mp_h (h_i - h') + h_k - h_j)\).

\[
(1 + O\left( \frac{1}{p^2} \right)) \prod_{j, \ell: p | (mp_h (h_j - h_k) + h_k - h_j)} \left( 1 + \sum_{T \subseteq \{s, s', r, r'\}} \prod_{T \cap \{s, s', r, r'\} = \emptyset} \prod_{T \cap \{r, r'\} \neq \emptyset} \right)
\]

\[
= \left( 1 + \frac{\# \{j, \ell: p \mid (mp_h (h_j - h_k) + h_k - h_j)\}}{p} \right) \left( 1 + O\left( \frac{1}{p^2} + \frac{\log p \sqrt{\log x}}{p \log y} \right) \right),
\]

where this time the first factor simplifies to \(1 - (\omega'_{m,p_0,h_1}(p) - (2k - 2))/p\), with

\[
\omega'_{m,p_0,h} = \# \{1 \leq n \leq p: p_0 + (h_i - h)n \equiv 0 \pmod{p} \}
\]

or \(m(p_0 + (h_i - h)n) \equiv 1 \pmod{p}\) for some \(i\).

The other ranges contribute the same factors as before, so for \(a_{1,2} = \cdots = a_{k,k-1} = 1\), the contribution to \((3.33)\) is

\[
(1 + o(1)) \mathcal{S}^{(M)}_{m,p_0,h_k} |I_m| G(0) |\log x|^k |\log y|^{k-1} \sum_{j=1}^{J} \sum_{j'=1}^{J} F_{k,j}(0) F_{k,j'}(0) \left( \int_{0}^{\infty} G'(t)^2 dt \right)^{k-1}
\]

\[
\times \prod_{\ell=1}^{k-1} \int_{0}^{\infty} F_{\ell,j}(t) F'_{\ell,j'}(t) dt
\]

\[
= (1 + o(1)) \mathcal{S}^{(M)}_{m,p_0,h_k} |I_m| J_{k}^{(1)}(F) J_{k}^{(2)}(G),
\]
where

\[(3.42) \quad \mathcal{S}_{m,p_0,h}^{(M)} = \prod_{p \leq w} \left( 1 - \frac{1}{p} \right)^{-2(k-2)} \prod_{w < p \leq y \atop p | M} \left( 1 - \frac{\omega_{m,p_0,h}(p)}{p} \right) \left( 1 - \frac{1}{p} \right)^{-2(k-2)} \times \prod_{w < p \leq y \atop p | M} \left( 1 - \frac{1}{p} \right)^{-(k-1)}. \]

Now if not all \(a_{i,j} = 1\), the presence of the \(a\) in the denominator means that for any \(p \mid a_{i,j}\), we have \(K_p \ll k_p - 1\). Therefore the contribution of the terms \(a_1, 2, \ldots, a_{k,k-1} \neq 1, \ldots, 1\) is

\[(3.43) \quad \ll_k \sum_{a_1, 2, \ldots, a_{k,k-1} \mid P_M(z_0) / P_M(w)} \frac{(-2k)^{\omega([a])}}{|a|} \times \left( \frac{1 + o(1))\mathcal{S}_{m,p_0,h}^{(M)}|\mathcal{I}_m|}{\varphi(P(w)) (\log x)^k (\log y)^k - \mathcal{J}(1)(F)\mathcal{J}(2)(G) \prod_{p | \Pi a_{i,j}} \frac{O_k(1)}{p} \right) \ll_k \frac{1 + o(1)}{\varphi(P(w)) (\log x)^k (\log y)^k - 1} = o_k \left( \frac{\mathcal{S}_{m,p_0,h}^{(M)}|\mathcal{I}_m|}{\varphi(P(w)) (\log x)^k (\log y)^k} \right), \]

which will be negligible. Using (3.41) in (3.32), we obtain

\[(3.44) \quad \sum_{q \in \mathcal{I}_m \text{prime}} \mu_{m,q}(p_0) \geq \frac{(1 + o(1))m(\log y)|\mathcal{I}_m| \mathcal{J}(1)(F)\mathcal{J}(2)(G)}{U \varphi(P(w)) \mathcal{I}\mathcal{J}_k(F)\mathcal{I}\mathcal{J}_k(G) \prod_{p | \Pi a_{i,j}} \frac{O_k(1)}{p}} \times \sum_{h \in \mathcal{H}} \mathcal{S}_m^{(M)} \mathcal{S}_{m,p_0,h}^{(M)} \sum_{w_0 \equiv p_0 \equiv P(w)} \sum_{w_0 \equiv P(w)} 1. \]
The inner sum is clearly $\varphi(P(w))$, and we have the bound $\omega_{m,p_0,h}(p) \leq 2k - 2$ uniformly in $h$. Thus
\begin{equation}
\psi_{m,p_0,h}^{(M)} \mathcal{E}_m^{(M)} = 2^{-1}(\psi(M)/M)^k \prod_{p \in M} \frac{p}{p-1} \prod_{2 < p \leq w} \left(1 - \frac{1}{p}\right)^{(2k-2)} \prod_{2 < p \leq w} \frac{1 - \frac{1}{p}}{1 - \frac{2}{p}} \prod_{w < p \leq y} \frac{1 - \frac{1}{p}}{1 - \frac{1}{p} - \frac{1}{p}}
\prod_{m \leq p \leq M} \left(1 - \frac{\omega_{m,p_0,h}(p)}{p}\right) \left(1 - \frac{1}{p}\right)^{(2k-2)} \prod_{m \leq p \leq M} \left(1 - \frac{1}{p}\right)^{(k-1)}
\gtrsim (1 + o_k(1)) \frac{\varphi(M)}{M} \prod_{p \in M} \frac{p}{p-1} \prod_{2 < p \leq w} \frac{p - 2}{p - 1}
\gtrsim (1 + o(1)) \frac{2e^{-\gamma}U}{m(\log x)(\log y)|\mathcal{R}_m^{(M)}|}
\end{equation}
by Lemma 4. This gives

**Lemma 5.** Let $m < Uz^{-1}(\log x)^{-2}$, and let $p_0 \in \mathcal{R}_m^{(M)}$ with $h_kx < p_0 < U/m - h_kx$. Then
\begin{equation}
\sum_{q \in \mathcal{I}_m \text{ prime}} \mu_{m,q}(p_0) \gtrsim (1 + o(1)) k|\mathcal{I}_m| \frac{J_k^{(1)}(F)J_k^{(2)}(G)}{(\log x)|\mathcal{R}_m^{(M)}|} \frac{J_k^{(1)}(F)J_k^{(2)}(G)}{(\log x)|\mathcal{R}_m^{(M)}|}
\end{equation}
where
\begin{equation}
J_k^{(1)}(F) = \int \cdots \int_{t_1, \ldots, t_{k-1} \geq 0} \left(\int_{t_k \geq 0} F(t_1, \ldots, t_k)dt_k\right)^2 dt_1, \ldots, dt_{k-1},
\end{equation}
\begin{equation}
J_k^{(2)}(G) = G(0)^2 \left(\int_0^\infty G'(t)dt\right)^{k-1}.
\end{equation}

We reproduce here Lemma 8 of [10]:

**Lemma 6.** There exists a choice of smooth functions $F$, $G$ such that
\begin{equation}
\frac{kJ_k^{(1)}(F)J_k^{(2)}(G)}{I_k^{(1)}(F)I_k^{(2)}(G)} \gg \log k.
\end{equation}

This gives us all the ingredients required to prove the main proposition.

**Proof of Proposition** Suppose $\mathcal{I}_m \subseteq [x/2, x]$ is an interval of length at least $\delta|\mathcal{R}_m^{(M)}| \log x$. By lemmas 3 and 4 we see that any prime $p_0 \in \mathcal{R}_m^{(M)}$ with $h_kx < p_0 < U/m - h_kx$ has the expected number of times it is chosen $\sum_q \mu_{m,q}(p_0) \gg \delta \log k$. By (2.1) we see that when $m < Uz^{-1}(\log x)^{-2}$ we have $|\mathcal{R}_m^{(M)}| \gg x(\log x)^{-2}/\log x$, so the number of primes in $\mathcal{R}_m^{(M)}$ which are not considered is $o_k(|\mathcal{R}_m^{(M)}|)$. Given $\varepsilon$ and $\delta$, we choose $k$ sufficiently large so that $\sum_q \mu_{m,q}(p_0) > -\log \varepsilon$. Then the probability that $p_0$ is not in any of the chosen residue classes is
\begin{equation}
\prod_{q \in \mathcal{I}_m \text{ prime}} (1 - \mu_{m,q}(p_0)) \leq \exp\left(-\sum_{q \in \mathcal{I}_m \text{ prime}} \mu_{m,q}(p_0)\right) \leq \varepsilon.
\end{equation}
So the expected number of primes in $\mathcal{R}_m^{(M)}$ which are not chosen is $\varepsilon|\mathcal{R}_m^{(M)}|$. Then there is at least one choice of residue classes which leaves out at most $\varepsilon|\mathcal{R}_m^{(M)}|$ primes. If we now append to $\mathcal{I}_m$ an interval of length $2\varepsilon|\mathcal{R}_m^{(M)}|\log x$, for each prime in the appended integral we can use the residue class of one of the $\varepsilon|\mathcal{R}_m^{(M)}|$ primes that were left out. This shows that we can cover $|\mathcal{R}_m^{(M)}|$ using primes from the interval $\mathcal{I}_m \subseteq [x/2, x]$ which has length $(\delta + 2\varepsilon)|\mathcal{R}_m^{(M)}|\log x$. Since $\delta$ and $\varepsilon$ were arbitrary, we obtain the result by relabeling. \hfill $\square$

4. Discussion

The constraint $M \leq \kappa(\log X)^{1/5}$ seems rather severe. Zaccagnini adopts the convention that $G(X; M, a) = X$ if there’s no prime below $X$ in the progression. We adopted our restriction to avoid having to engage with such degenerate cases. We would have to make similarly severe restrictions in any case. Following (3.24), had the magnitude of $M$ not been already restricted, we would have to explicitly impose $(q, M) = 1$. In turn, $M$ would have to be a factor in the moduli when we count the primes in (3.37), which would necessitate a similarly severe restriction (as well as further exclusions depending on the divisibility of $M$ by exceptional moduli) anyway for the Bombieri-Vinogradov theorem to be applicable. With little gain, we opted to make the restriction upfront and to at least ensure that the gaps we detect are nontrivial, in the sense that they are blocks of composites that indeed fall between two primes.

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