CLASSIFICATION OF CATEGORICAL SUBSPACES OF LOCALLY NOETHERIAN SCHEMES

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Abstract. We classify the prelocalizing subcategories of the category of quasi-coherent sheaves on a locally noetherian scheme. In order to give the classification, we introduce the notion of a local filter of subobjects of the structure sheaf. The essential part of the argument is given as results on a Grothendieck category with certain properties. We also classify the localizing subcategories, the closed subcategories, and the bilocalizing subcategories in terms of filters.

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1. INTRODUCTION

The aim of this paper is to classify several classes of subcategories of the category QCoh X of quasi-coherent sheaves on a locally noetherian scheme X.

For a Grothendieck category A, a prelocalizing subcategory X of A is a full subcategory of A closed under subobjects, quotient objects, and arbitrary direct sums. A closed subcategory is a prelocalizing subcategory closed under arbitrary direct products. Rosenberg [Ros95] showed that the closed subcategories of the category Mod Λ of right modules over a ring Λ are classified by the two-sided ideals of Λ.

Theorem 1.1 (Rosenberg [Ros95, Proposition III.6.4.1]; [Theorem 11.3]). Let Λ be a ring. Then the map

\{ two-sided ideals of Λ \} \to \{ closed subcategories of Mod Λ \}

given by

I \mapsto \{ M \in Mod Λ \mid MI = 0 \}

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is bijective.

For a scheme $X$, we consider whether the closed subcategories of $\text{Qcoh}_X$ bijectively correspond to the closed subschemes of $X$. Smith [Smi02] showed that this claim holds for a noetherian scheme with an ample line bundle ([Smi02, Theorem 4.1]), and Brandenburg [Bra14] showed the claim for a separated scheme ([Bra14, Proposition 3.18]). These results give a categorical definition of the closed subschemes. In contrast to the case of commutative rings, a Grothendieck category such as $\text{Mod} \Lambda$ for a ring $\Lambda$ does not necessarily have enough closed subcategories (see [Pap02, Example 2.4 (a)]). One approach is to investigate prelocalizing subcategories instead. From this viewpoint, we need to know the structure of prelocalizing subcategories of $\text{Qcoh}_X$ for a scheme $X$.

For a ring $\Lambda$, Gabriel [Gab62] classified the prelocalizing subcategories and localizing subcategories by using classes of filters of right ideals of $\Lambda$. A localizing subcategory of a Grothendieck category is a prelocalizing subcategory closed under extensions.

**Theorem 1.2** ([Gab62, Lemma V.2.1]; [Theorem 9.3] and [Theorem 10.3]). Let $\Lambda$ be a ring. Then the map

$$
\{\text{prelocalizing subcategories of } \text{Mod } \Lambda\} \to \{\text{prelocalizing filters of right ideals of } \Lambda\}
$$

given by

$$
\mathcal{Y} \mapsto \left\{L \subset \Lambda \text{ in } \text{Mod } \Lambda \mid \frac{\Lambda}{L} \in \mathcal{Y}\right\}
$$

is bijective. This map induces a bijection

$$
\{\text{localizing subcategories of } \text{Mod } \Lambda\} \to \{\text{Gabriel filters of right ideals of } \Lambda\}.
$$

A right linear topology of $\Lambda$ is a topology on $\Lambda$ which makes $\Lambda$ a topological ring having an open neighborhood basis of 0 consisting of right ideals. It is known that the prelocalizing filters of right ideals of $\Lambda$ bijectively correspond to the right linear topologies on $\Lambda$ (see [Ste75, section VI.4]).

Gabriel [Gab62] also showed that for a noetherian scheme $X$, the localizing subcategories of $\text{Qcoh}_X$ bijectively correspond to the specialization-closed subsets of the underlying space of $X$ ([Gab62, Proposition VI.2.4 (b)]). This result has been generalized by a number of authors. (For example, [Hov01], [Kra08], [GP08a], [GP08b], [Tak08], [Tak09], [Her97], [Kra97], [Kan12a], and [Kan12b] to some abelian categories. See [GP08a] or [Tak09] for generalizations to derived categories.) We generalize this result to a locally noetherian scheme as [Theorem 1.5].

**Theorem 1.3** ([Corollary 9.4], [Corollary 10.5], and [Theorem 11.3]). Let $R$ be a commutative noetherian ring. Then the map

$$
\{\text{prelocalizing subcategories of } \text{Mod } R\} \to \{\text{filters of ideals of } R\}
$$

given by

$$
\mathcal{Y} \mapsto \left\{I \subset R \text{ in } \text{Mod } R \mid \frac{R}{I} \in \mathcal{Y}\right\}
$$

is bijective. This map induces bijections

$$
\{\text{localizing subcategories of } \text{Mod } R\} \to \{\text{filters of ideals of } R \text{ closed under products}\},$$

and

$$
\{\text{closed subcategories of } \text{Mod } R\} \to \{\text{principal filters of ideals of } R\}.
$$

For a locally noetherian scheme $X$, the prelocalizing subcategories of $\text{Qcoh}_X$ does not bijectively correspond to the filters of quasi-coherent subsheaves of $\mathcal{O}_X$. We need to consider a suitable class of filters, which we call local filters [Definition 9.5]. By using local filters, we give the following classification.
Theorem 1.4 [Theorem 9.1.4 Corollary 10.9, Theorem 11.9, and Theorem 11.11]. Let $X$ be a locally noetherian scheme. Then the map

$$\{\text{prelocalizing subcategories of } \text{Qcoh } X\} \to \{\text{local filters of quasi-coherent subsheaves of } \mathcal{O}_X\}$$

given by

$$\mathcal{Y} \mapsto \left\{ I \subset \mathcal{O}_X \text{ in } \text{Qcoh } X \mid \frac{\mathcal{O}_X}{I} \in \mathcal{Y} \right\}$$

is bijective. This map induces bijections

$$\{\text{localizing subcategories of } \text{Qcoh } X\} \to \left\{ \text{local filters of quasi-coherent subsheaves of } \mathcal{O}_X \right\}$$

and

$$\{\text{closed subcategories of } \text{Qcoh } X\} \to \{\text{principal filters of quasi-coherent subsheaves of } \mathcal{O}_X\}.$$

In particular, there exists a bijection between the closed subcategories of $\text{Qcoh } X$ and the closed subspaces of $X$.

The key part of the proof of Theorem 1.4 is to reduce the problem to open affine subschemes, and it is shown in a purely categorical way (Theorem 8.11). In order to clarify the essential properties of the Grothendieck category $\text{Qcoh } X$, we formulate this part as a fact on a Grothendieck category with certain properties (Setting 8.3). For this purpose, we use the notion of the atom spectrum.

The atom spectrum $\text{ASpec } A$ of a Grothendieck category $A$ is the set of atoms in $A$ which are introduced by Storrer [Sto72] (Definition 3.6). An atom is a generalization of a prime ideal of a commutative ring. Indeed, for every commutative ring $R$, there exists a canonical bijection between $\text{ASpec}(\text{Mod } R)$ and $\text{Spec } R$ (Proposition 3.7). Moreover, we show in this paper that for a locally noetherian Grothendieck category $X$, there exists a canonical bijection between $\text{ASpec}(\text{Qcoh } X)$ and the underlying space of $X$ (Theorem 7.6). Therefore we can regard the atom spectrum as a realization of the underlying space of the Grothendieck category $A$. Notions of commutative rings and locally noetherian schemes are generalized in terms of the atom spectrum as in Table 1.

In Section 3 we recall the definition of the atom spectrum and fundamental notions and results on it. Section 4 is devoted to preliminary results on subcategories and quotient categories by localizing subcategories. In Section 5, we summarize results on the atom spectrum and the localization at an atom. In Section 6 we introduce the class of Grothendieck categories with enough

| Grothendieck category $A$ | Commutative ring $R$ | Locally noetherian scheme $X$ |
|--------------------------|----------------------|-----------------------------|
| Atom spectrum $\text{ASpec } A$ | Prime spectrum $\text{Spec } R$ | Underlying space $|X|$ |
| Atom $\alpha$ in $A$ | Prime ideal $p$ of $R$ | Point $x \in X$ |
| Associated atoms $\text{AAss } M$ | Associated primes $\text{Ass } M$ | Associated points $\text{Ass } M$ |
| Atom support $\text{ASupp } M$ | Support $\text{Supp } M$ | Support $\text{Supp } M$ |
| Open subsets of $\text{ASpec } A$ | Specialization-closed subsets of $\text{Spec } R$ | Specialization-closed subsets of $X$ |
| $\{\alpha\}$ for $\alpha \in \text{ASpec } A$ | $\{q \in \text{Spec } R \mid q \subset p\}$ for $p \in \text{Spec } R$ | $\{y \in X \mid x \in \overline{\{y\}}\}$ for $x \in X$ |
| $\alpha_1 \leq \alpha_2$ | $p_1 \leq p_2$ | $\overline{\{x_1\}} \cap x_2$ |
| Maximal atoms in $A$ | Maximal ideals of $R$ | Closed points in $X$ |
| Open points in $\text{ASpec } A$ | Maximal ideals of $R$ | Closed points in $X$ |
| Minimal atoms in $A$ | Minimal prime ideals of $R$ | Points in $X$ of height 0 |
| (=Closed points in $\text{ASpec } A$) | | |
| Generic point in $\text{ASpec } A$ | Unique maximal ideal of $R$ | Unique closed point in $X$ |
| Injective envelope $E(\alpha)$ | Injective envelope $E(R/p)$ | $j_x E(x)$ |
| Residue field $k(\alpha)$ | Residue field $k(p)$ | Residue field $k(x)$ |
| Atomic object $H(\alpha)$ | Residue field $k(p)$ | $j_x k(x)$ |
| Localization $A_\alpha$ | Localization $R_p$ | Localization $\text{Spec } \mathcal{O}_{X,x}$ |
atoms and show that the localizing subcategories are classified in terms of the atom spectrum for a Grothendieck category with enough atoms. In section 7, we describe the atom spectrum of the Grothendieck category \( \text{QCoh} X \) for a locally noetherian scheme \( X \) and show that \( \text{QCoh} X \) has enough atoms. As a consequence, we obtain the following classification of the localizing subcategories of \( \text{QCoh} X \).

Theorem 1.5. Let \( X \) be a locally noetherian scheme. Then the map
\[
\{ \text{localizing subcategories of } \text{QCoh} X \} \to \{ \text{specialization-closed subsets of } X \}
\]
given by \( X \mapsto \text{Supp} X \) is bijective. The inverse map is given by \( \Phi \mapsto \text{Supp}^{-1} \Phi \).

In section 8, we investigate a Grothendieck category \( \mathcal{A} \) with some properties and relates the prelocalizing subcategories (resp. localizing subcategories) of \( \mathcal{A} \) with the prelocalizing subcategories (resp. localizing subcategories) of quotient categories of \( \mathcal{A} \). For a locally noetherian scheme \( X \), the prelocalizing subcategories, the localizing subcategories, and the closed subcategories of \( \text{QCoh} X \) are classified in section 9, section 10, and section 11 respectively. In section 12, we classify the bilocalizing subcategories of \( \text{QCoh} X \), which are defined as the prelocalizing subcategories closed under both extensions and arbitrary direct products. It is shown that there exists a bijection between the bilocalizing subcategories of \( \text{QCoh} X \) and the subsets of \( X \) which are open and closed (Corollary 12.11).

Conventions 1.6. Throughout this paper, we fix a Grothendieck universe. A set is called small if it is an element of the universe. For every category \( \mathcal{C} \), the collection \( \text{Ob} \mathcal{C} \) (resp. \( \text{Mor} \mathcal{C} \)) of objects (resp. morphisms) in \( \mathcal{C} \) is a set, and \( \text{Hom}_\mathcal{C}(X, Y) \) is supposed to be small for each objects \( X \) and \( Y \) in \( \mathcal{C} \). A category \( \mathcal{C} \) is called skeletally small if there exists a small set \( \mathcal{S} \) of objects in \( \mathcal{C} \) such that each object in \( \mathcal{C} \) is isomorphic to some object belonging to \( \mathcal{S} \). The index set of each limit and colimit is supposed to be skeletally small.

Rings, modules over rings, schemes, and sheaves on schemes are supposed to be small. Every ring is associative and has an identity element.

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3. Atom spectrum

In this section, we recall the definition of the atom spectrum of a Grothendieck category and fundamental results. We start with the definition of a Grothendieck category.

Definition 3.1. (1) An abelian category \( \mathcal{A} \) is called a Grothendieck category if it satisfies the following conditions.

(a) \( \mathcal{A} \) admits arbitrary direct sums (and hence arbitrary direct limits), and for every direct system of short exact sequences in \( \mathcal{A} \), its direct limit is also a short exact sequence.

(b) \( \mathcal{A} \) has a generator \( G \), that is, every object in \( \mathcal{A} \) is isomorphic to a quotient object of the direct sum of some copies of \( G \).

(2) A Grothendieck category is called locally noetherian if it admits a small generating set consisting of noetherian objects.

The exactness of direct limits has the following characterizations.

Proposition 3.2. Let \( \mathcal{A} \) be an abelian category with arbitrary direct sums. Then the following assertions are equivalent.

(1) For every direct system of short exact sequences in \( \mathcal{A} \), its direct limit is also a short exact sequence.
(2) Let $M$ be an object in $\mathcal{A}$. For each subobject $L$ of $M$ and each family $\mathcal{N}$ of subobjects of $M$ such that every finite subfamily of $\mathcal{N}$ has an upper bound in $\mathcal{N}$, we have
\[ L \cap \sum_{N \in \mathcal{N}} N = \sum_{N \in \mathcal{N}} (L \cap N). \]

(3) For every family $\{M_\lambda\}_{\lambda \in \Lambda}$ of objects in $\mathcal{A}$ and every subobject $L$ of $\bigoplus_{\lambda \in \Lambda} M_\lambda$, we have
\[ L = \sum_{\Lambda' \in \mathcal{S}} \left( L \cap \bigoplus_{\lambda \in \Lambda'} M_\lambda \right), \]
where $\mathcal{S}$ is the set of finite subsets of $\Lambda$.

Proof. [Pop73, Theorem 2.8.6].

From now on, we deal with a Grothendieck category $\mathcal{A}$. The atom spectrum of a Grothendieck category is defined by using monoform objects defined as follows.

Definition 3.3. (1) A nonzero object $H$ in $\mathcal{A}$ is called monoform if for each nonzero subobject $L$ of $H$, no nonzero subobject of $H$ is isomorphic to a subobject of $H/L$.

(2) For monoform objects $H_1$ and $H_2$ in $\mathcal{A}$, we say that $H_1$ is atom-equivalent to $H_2$ if there exists a nonzero subobject of $H_1$ which is isomorphic to a subobject of $H_2$.

We recall the definitions of essential subobjects and uniform objects. These are also important notions in a Grothendieck category and related to monoform objects.

Definition 3.4. (1) Let $M$ be an object in $\mathcal{A}$. A subobject $L$ of $M$ is called essential if for every nonzero subobject $L'$ of $M$, we have $L \cap L' \neq 0$.

(2) A nonzero object $U$ in $\mathcal{A}$ is called uniform if every nonzero subobject of $U$ is essential.

In other words, a nonzero object $U$ in $\mathcal{A}$ is uniform if and only if for every two nonzero subobjects $L_1$ and $L_2$ of $U$, we have $L_1 \cap L_2 \neq 0$.

It is easy to show that each nonzero subobject of a uniform object is uniform. This type of result also holds for monoform objects.

Proposition 3.5. (1) Each nonzero subobject of a monoform object is monoform.

(2) Every monoform object is uniform.

(3) Every nonzero noetherian object has a monoform subobject.

Proof. [1] [Kan12a, Proposition 2.2].
[2] [Kan12a, Proposition 2.6].
[3] [Kan12a, Theorem 2.9].

It follows from Proposition 3.5(2) that the atom-equivalence is an equivalence relation on the set of monoform objects in $\mathcal{A}$ ([Kan12a, Proposition 2.8]). The atom spectrum is defined by using this relation.

Definition 3.6. Let $\mathcal{A}$ be a Grothendieck category. Denote by $\text{ASpec} \mathcal{A}$ the quotient set of the set of monoform objects in $\mathcal{A}$ by the atom equivalence. We call it the atom spectrum of $\mathcal{A}$. Each element of $\text{ASpec} \mathcal{A}$ is called an atom in $\mathcal{A}$. For each monoform object $H$ in $\mathcal{A}$, the equivalence class of $H$ is denoted by $\overline{H}$.

It is shown in [Kan13, Proposition 2.7 (2)] that the atom spectrum $\text{ASpec} \mathcal{A}$ of a Grothendieck category $\mathcal{A}$ is in bijection with a small set.

The following result shows that the atom spectrum of a Grothendieck category is a generalization of the prime spectrum of a commutative ring.

Proposition 3.7. Let $R$ be a commutative ring.

(1) ([Sto72, Lemma 1.5]) Let $\mathfrak{a}$ be an ideal of $R$. Then $R/\mathfrak{a}$ is a monoform object in $\text{Mod} R$ if and only if $\mathfrak{a}$ is a prime ideal.
The map $\text{Spec } R \to \text{ASpec}(\text{Mod } R)$ given by $p \mapsto R/p$ is a bijection.

We can also generalize the notions of supports and associated primes in commutative ring theory.

**Definition 3.8.** Let $M$ be an object in $\mathcal{A}$.

1. Define the subset $\text{AAss } M$ of $\text{ASpec } \mathcal{A}$ by
   $$\text{AAss } M = \{ \alpha \in \text{ASpec } \mathcal{A} | \alpha = \overline{H} \text{ for some monoform subobject } H \text{ of } M \}.$$ We call each element of $\text{AAss } M$ an **associated atom** of $M$.

2. Define the subset $\text{ASupp } M$ of $\text{ASpec } \mathcal{A}$ by
   $$\text{ASupp } M = \{ \alpha \in \text{ASpec } \mathcal{A} | \alpha = \overline{H} \text{ for some monoform subquotient } H \text{ of } M \}.$$ We call it the **atom support** of $M$.

**Proposition 3.9.** Let $R$ be a commutative ring, and let $M$ be an $R$-module. Then the bijection $\text{Spec } R \to \text{ASpec}(\text{Mod } R)$ in Proposition 3.7 (2) induces bijections $\text{Ass } M \to \text{AAss } M$ and $\text{Supp } M \to \text{ASupp } M$.

**Proof.** [Kan13, Proposition 2.13].

The following results are generalizations of fundamental results in commutative ring theory.

**Proposition 3.10.** Let $0 \to L \to M \to N \to 0$ be an exact sequence in $\mathcal{A}$. Then we have
$$\text{AAss } L \subset \text{AAss } M \subset \text{AAss } L \cup \text{AAss } N$$
and
$$\text{ASupp } M = \text{ASupp } L \cup \text{ASupp } N.$$

**Proof.** [Kan12a, Proposition 3.5] and [Kan12a, Proposition 3.3].

**Proposition 3.11.**

1. Let $\{ M_\lambda \}_{\lambda \in \Lambda}$ be a family of objects in $\mathcal{A}$. Then we have
   $$\text{AAss } \bigoplus_{\lambda \in \Lambda} M_\lambda = \bigcup_{\lambda \in \Lambda} \text{AAss } M_\lambda$$
   and
   $$\text{ASupp } \bigoplus_{\lambda \in \Lambda} M_\lambda = \bigcup_{\lambda \in \Lambda} \text{ASupp } M_\lambda.$$

2. Let $M$ be an object in $\mathcal{A}$, and let $\{ L_\lambda \}_{\lambda \in \Lambda}$ be a family of subobjects of $M$. Then we have
   $$\text{ASupp } \sum_{\lambda \in \Lambda} L_\lambda = \bigcup_{\lambda \in \Lambda} \text{ASupp } L_\lambda.$$

**Proof.** [1] [Kan13, Proposition 2.12].

[2] Since we have the canonical epimorphism $\bigoplus_{\lambda \in \Lambda} L_\lambda \twoheadrightarrow \sum_{\lambda \in \Lambda} L_\lambda$ and the inclusion $L_\mu \subset \sum_{\lambda \in \Lambda} L_\lambda$ for each $\mu \in \Lambda$, we obtain
$$\text{ASupp } L_\mu \subset \text{ASupp } \sum_{\lambda \in \Lambda} L_\lambda \subset \bigcup_{\lambda \in \Lambda} \text{ASupp } L_\lambda$$
by [1]. Hence the claim follows.

Similarly to the case of commutative rings, we have the following results on the associated atoms of uniform objects and essential subobjects.

**Proposition 3.12.**

1. Let $U$ be a uniform object in $\mathcal{A}$. Then $\text{AAss } U$ consists of at most one element. In particular, for every monoform object $H$ in $\mathcal{A}$, we have $\text{AAss } H = \{ \overline{H} \}$.

2. Let $M$ be an object in $\mathcal{A}$, and let $L$ be an essential subobject of $M$. Then we have $\text{AAss } L = \text{AAss } M$.
Proof. [1] [Kan13] Proposition 2.15 (1). [2] [Kan13] Proposition 2.16. □

We introduce a topology on the atom spectrum.

Definition 3.13. We call a subset Φ of ASpec A a localizing subset if there exists an object M in A such that Φ = ASupp M.

Proposition 3.14. The set of localizing subsets of ASpec A satisfies the axioms of open subsets of ASpec A.

Proof. [Kan12a] Proposition 3.8. □

We call the topology on ASpec A defined by the set of localizing subsets of ASpec A the localizing topology. Throughout this paper, we regard ASpec A as a topological space in this way.

For a commutative ring R, the localizing subsets of ASpec(Mod R) define a different topology from the Zariski topology on Spec R. Recall that a subset Φ of Spec R is said to be closed under specialization if for every p, q ∈ Spec R, the conditions p ∈ Φ and p ⊂ q imply q ∈ Φ.

Proposition 3.15. Let R be a commutative ring, and let Φ be a subset of Spec R. Then the corresponding subset

\[ \left\{ \frac{R}{p} \in \text{ASpec(Mod R)} \mid p \in \Phi \right\} \]

of ASpec(Mod R) is localizing if and only if Φ is closed under specialization.

Proof. [Kan12a] Proposition 7.2 (2). □

For each α ∈ ASpec A, let Λ(α) be the topological closure of \{α\} in ASpec A. We introduce a partial order on the atom spectrum.

Definition 3.16. For α, β ∈ ASpec A, we write α ≤ β if α ∈ Λ(β).

The relation ≤ is called the specialization order on the topological space ASpec A with respect to the localizing topology. This is in fact a partial order on ASpec A since the topological space ASpec A is a Kolmogorov space (Kan13 Proposition 3.5).

By definition, we have \( \Lambda(\beta) = \{ \alpha \in \text{ASpec } A \mid \alpha \leq \beta \} \) for each \( \beta \in \text{ASpec } A \). The partial order has the following descriptions.

Proposition 3.17. Let α, β ∈ ASpec A. Then the following assertions are equivalent.

1. α ≤ β, that is, α ∈ Λ(β).
2. For every object M in A, the condition α ∈ ASupp M implies β ∈ ASupp M.
3. For every monoform object H in A with \( \overline{H} = \alpha \), we have \( \beta \in \text{ASupp } H \).

Proof. [Kan13] Proposition 4.2. □

The following result claims that the partial order ≤ on ASpec A is a generalization of the inclusion relation between prime ideals of a commutative ring.

Proposition 3.18. Let R be a commutative ring and p, q ∈ Spec R. Then we have \( \frac{R}{p} \leq \frac{R}{q} \) in ASpec(Mod R) if and only if p ⊂ q. In other words, the bijection Spec R → ASpec(Mod R) in Proposition 3.7 (2) is an isomorphism between the partially ordered sets \( \text{Spec R, } \subset \) and \( \text{(ASpec(Mod R), ≤)} \).

Proof. [Kan13] Proposition 4.3. □
4. SUBCATEGORIES AND QUOTIENT CATEGORIES

In this section, we show preliminary results on subcategories and quotient categories of a Grothendieck category \( \mathcal{A} \). We start with defining some classes of subcategories, which are the main objects in this paper.

**Definition 4.1.** (1) For full subcategories \( \mathcal{X}_1 \) and \( \mathcal{X}_2 \) of \( \mathcal{A} \), we denote by \( \mathcal{X}_1 \star \mathcal{X}_2 \) the full subcategory of \( \mathcal{A} \) consisting of all objects \( M \) admitting an exact sequence

\[
0 \to M_1 \to M \to M_2 \to 0
\]

in \( \mathcal{A} \), where \( M_i \) belongs to \( \mathcal{X}_i \) for each \( i = 1, 2 \).

(2) We say that a full subcategory \( \mathcal{X} \) of \( \mathcal{A} \) is closed under extension if \( \mathcal{X} \star \mathcal{X} \subset \mathcal{X} \), that is, for every exact sequence \( 0 \to L \to M \to N \to 0 \) in \( \mathcal{A} \), the condition \( L, N \in \mathcal{X} \) implies \( M \in \mathcal{X} \).

(3) A full subcategory \( \mathcal{X} \) of \( \mathcal{A} \) is called a prelocalizing subcategory (or weakly closed subcategory) of \( \mathcal{A} \) if \( \mathcal{X} \) is closed under subobjects, quotient objects, and arbitrary direct sums.

(4) A prelocalizing subcategory \( \mathcal{X} \) of \( \mathcal{A} \) is called a localizing subcategory of \( \mathcal{A} \) if \( \mathcal{X} \) is also closed under extensions.

(5) For a full subcategory \( \mathcal{X} \) of \( \mathcal{A} \), denote by \( \langle \mathcal{X} \rangle_{\text{preloc}} \) (resp. \( \langle \mathcal{X} \rangle_{\text{loc}} \)) the smallest prelocalizing (resp. localizing) subcategory of \( \mathcal{A} \) containing \( \mathcal{X} \). For an object \( M \) in \( \mathcal{A} \), let \( \langle M \rangle_{\text{preloc}} = \{ \{ M \} \}_{\text{preloc}} \) and \( \langle M \rangle_{\text{loc}} = \{ \{ M \} \}_{\text{loc}} \).

**Proposition 4.2.** (1) Let \( \mathcal{X}_1, \mathcal{X}_2, \) and \( \mathcal{X}_3 \) be full subcategories of \( \mathcal{A} \). Then we have

\[
(\mathcal{X}_1 \star \mathcal{X}_2) \star \mathcal{X}_3 = \mathcal{X}_1 \star (\mathcal{X}_2 \star \mathcal{X}_3).
\]

(2) Let \( \mathcal{X}_1 \) and \( \mathcal{X}_2 \) be prelocalizing subcategories of \( \mathcal{A} \). Then \( \mathcal{X}_1 \star \mathcal{X}_2 \) is also a prelocalizing subcategory of \( \mathcal{A} \).

**Proof.** [1][Kan12] Proposition 2.4 (2)], [2][Pop73 Lemma 4.8.11].

**Remark 4.3.** Let \( \mathcal{X} \) be a full subcategory of \( \mathcal{A} \) closed under quotient objects and arbitrary direct sums, and let \( M \) be an object in \( \mathcal{A} \). Since the sum \( L = \sum_{\lambda \in \Lambda} L_{\lambda} \) of all subobjects of \( M \) belonging to \( \mathcal{X} \) is a quotient object of the direct sum \( \bigoplus_{\lambda \in \Lambda} L_{\lambda} \), the subobject \( L \) of \( M \) also belongs to \( \mathcal{X} \). Hence \( L \) is the largest subobject of \( M \) belonging to \( \mathcal{X} \).

The operation in **Remark 4.3** of taking the subobject \( L \) from \( M \) is used throughout this paper. The following result shows that this operation commutes with taking the direct sum.

**Proposition 4.4.** Let \( \mathcal{X} \) be a full subcategory of \( \mathcal{A} \) closed under quotient objects and direct sums, and let \( \{ M_{\lambda} \}_{\lambda \in \Lambda} \) be a family of objects in \( \mathcal{A} \). Let \( L_{\lambda} \) be the largest subobject of \( M_{\lambda} \) belonging to \( \mathcal{X} \) for each \( \lambda \in \Lambda \). Then \( \bigoplus_{\lambda \in \Lambda} L_{\lambda} \) is the largest subobject of \( \bigoplus_{\lambda \in \Lambda} M_{\lambda} \) belonging to \( \mathcal{X} \).

**Proof.** Let \( N \) be the largest subobject of \( \bigoplus_{\lambda \in \Lambda} M_{\lambda} \) belonging to \( \mathcal{X} \). It suffices to show that

\[
N \subset \bigoplus_{\lambda \in \Lambda} L_{\lambda}.
\]

We show the claim in the case where \( \Lambda = \{1, \ldots, n\} \) for some \( n \in \mathbb{Z}_{\geq 1} \). Let \( \pi_i : M_1 \oplus \cdots \oplus M_n \to M_i \) be the projection for each \( i \in \{1, \ldots, n\} \). Since \( \pi_i(N) \) is a quotient object of \( N \), it belongs to \( \mathcal{X} \). By the maximality of \( L_i \), we have \( \pi_i(N) \subset L_i \). Hence it holds that

\[
N \subset \pi_1(N) \oplus \cdots \oplus \pi_n(N) \subset L_1 \oplus \cdots \oplus L_n
\]
as subobjects of \( M_1 \oplus \cdots \oplus M_n \).

In the general case, let \( S \) be the set of finite subsets of \( \Lambda \). Then by **Proposition 3.2**, we have

\[
N = \sum_{A' \in S} \left( N \cap \bigoplus_{\lambda \in A'} M_{\lambda} \right) \subset \sum_{A' \in S} \bigoplus_{\lambda \in A'} L_{\lambda} = \bigoplus_{\lambda \in \Lambda} L_{\lambda}.
\]
For a localizing subcategory $\mathcal{X}$ of $\mathcal{A}$, we have the quotient category $\mathcal{A}/\mathcal{X}$ of $\mathcal{A}$ by $\mathcal{X}$. It is a Grothendieck category together with a canonical (covariant) functor $\mathcal{A} \rightarrow \mathcal{A}/\mathcal{X}$ ([Pop73 Corollary 4.6.2]). We refer the reader to [Kan13 Definition 5.2] for the explicit definition of the quotient category. Instead, we state a universal property of the quotient category.

**Theorem 4.5.** Let $\mathcal{A}$ be a Grothendieck category, and let $\mathcal{X}$ be a localizing subcategory. The canonical functor is denoted by $F: \mathcal{A} \rightarrow \mathcal{A}/\mathcal{X}$.

1. The functor $F: \mathcal{A} \rightarrow \mathcal{A}/\mathcal{X}$ is exact and has a right adjoint $\mathcal{A}/\mathcal{X} \rightarrow \mathcal{A}$. For every object $M$ in $\mathcal{A}$, we have $F(M) = 0$ if and only if $M$ belongs to $\mathcal{X}$.

2. Let $\mathcal{B}$ be an abelian category together with an exact functor $Q: \mathcal{A} \rightarrow \mathcal{B}$ with $Q(M) = 0$ for each object $M$ in $\mathcal{X}$. Then there exists a unique functor $\overline{Q}: \mathcal{A}/\mathcal{X} \rightarrow \mathcal{B}$ such that $\overline{Q}F = Q$. Moreover, the functor $\overline{Q}$ is exact.

**Proof.** [Pop73 Proposition 4.6.3], [Pop73 Theorem 4.3.8], and [Pop73 Lemma 4.3.4].

Every object $M$ in a Grothendieck category $\mathcal{A}$ has the injective envelope $E(M)$ ([Gab62 Theorem II.6.2], see also [Pop73 Theorem 3.10.10]). By definition, the object $M$ is an essential subobject of the injective object $E(M)$. The object $E(M)$ is also denoted by $E_A(M)$ in order to specify the category explicitly.

Let $\mathcal{X}$ be a localizing subcategory of $\mathcal{A}$. An object $M$ in $\mathcal{A}$ is called $\mathcal{X}$-torsionfree if $M$ has no nonzero subobject belonging to $\mathcal{X}$. Note that every subobject of an $\mathcal{X}$-torsionfree object is $\mathcal{X}$-torsionfree.

**Proposition 4.6.** Let $\mathcal{X}$ be a localizing subcategory of $\mathcal{A}$. Let $M$ be an object in $\mathcal{A}$, and let $L$ be the largest subobject of $M$ belonging to $\mathcal{X}$. Then $M/L$ is $\mathcal{X}$-torsionfree.

**Proof.** Assume that $M/L$ is not $\mathcal{X}$-torsionfree. Then there exists a subobject $L'$ of $M$ such that $L \subseteq L'$, and $L'/L$ belongs to $\mathcal{X}$. The subobject $L'$ of $M$ also belongs to $\mathcal{X}$. This contradicts the maximality of $L$. $\square$

For an object $M$ in $\mathcal{A}$, it is also important to consider the torsionfreeness of $E(M)/M$.

**Proposition 4.7.** Let $\mathcal{X}$ be a localizing subcategory of $\mathcal{A}$, and let $L \rightarrow M \rightarrow N \rightarrow 0$ be an exact sequence in $\mathcal{A}$. If $M$ and $E(L)/L$ are $\mathcal{X}$-torsionfree, then $N$ is $\mathcal{X}$-torsionfree.

**Proof.** This can be shown similarly to the proof of [Pop73 Proposition 4.5.5]. $\square$

We state important properties of the canonical functors of quotient categories by using the notion of torsionfreeness.

**Proposition 4.8.** Let $\mathcal{X}$ be a localizing subcategory of $\mathcal{A}$. The canonical functors are denoted by $F: \mathcal{A} \rightarrow \mathcal{A}/\mathcal{X}$ and $G: \mathcal{A}/\mathcal{X} \rightarrow \mathcal{A}$.

1. The counit morphism $\varepsilon: FG \rightarrow 1_{\mathcal{A}/\mathcal{X}}$ is an isomorphism. Hence $F$ is dense, and $G$ is fully faithful.

2. Let $\eta: 1_{\mathcal{A}} \rightarrow GF$ be the unit morphism. Then for each object $M$ in $\mathcal{A}$, the subobject $\ker\eta_M$ of $M$ is the largest subobject belonging to $\mathcal{X}$, the subobject $\im\eta_M$ of $GF(M)$ is essential, and $\operatorname{cok}\eta_M$ belongs to $\mathcal{X}$. The objects $GF(M)$ and $E(GF(M))/GF(M)$ are $\mathcal{X}$-torsionfree.

3. Let $M'$ be an object in $\mathcal{A}/\mathcal{X}$. Then $G(M')$ and $E(G(M'))/G(M')$ are $\mathcal{X}$-torsionfree.

**Proof.** [Pop73 Proposition 4.4.3 (1)].

[2] This follows from [Pop73 Proposition 4.4.3 (2)] and the proof of [Pop73 Proposition 4.4.5].

[3] By [1] there exists an object $M$ in $\mathcal{A}$ such that $F(M)$ is isomorphic to $M'$. Hence the claim follows from [2]. $\square$

The next result is necessary to describe subobjects of an object in a quotient category.
Proposition 4.9. Let $\mathcal{X}$ be a localizing subcategory of $\mathcal{A}$. The canonical functors are denoted by $F: \mathcal{A} \to \mathcal{A}/\mathcal{X}$ and $G: \mathcal{A}/\mathcal{X} \to \mathcal{A}$. Let $M$ be an object in $\mathcal{A}$. For each subobject $L'$ of $F(M)$, there exists a largest subobject $L$ of $M$ satisfying $F(L) \subseteq L'$ as a subobject of $F(M)$. Moreover, it holds that $F(L) = L'$, and the quotient object $M/L$ is $\mathcal{X}$-torsionfree. The quotient object $F(M)/L'$ of $F(M)$ is equal to $F(M/L)$.

Proof. Since $G$ is left exact, the object $G(L')$ can be regarded as a subobject of $GF(M)$. Let $\eta: 1_{\mathcal{A}} \to GF$ be the unit morphism. Then we have the commutative diagram

$$
\begin{array}{ccccccc}
0 & \xrightarrow{} & \eta^{-1}_M(G(L')) & \xrightarrow{} & M & \xrightarrow{} & \eta^{-1}_M(G(L')) & \xrightarrow{} & 0 \\
& & \downarrow{\eta_M} & & \downarrow{\eta_M} & & \downarrow{\eta_M} & & \\
0 & \xrightarrow{} & G(L') & \xrightarrow{} & GF(M) & \xrightarrow{} & GF(M) & \xrightarrow{} & 0
\end{array}
$$

By applying $F$ to this diagram, we obtain the commutative diagram

$$
\begin{array}{ccccccc}
0 & \xrightarrow{} & F(\eta^{-1}_M(G(L'))) & \xrightarrow{} & F(M) & \xrightarrow{} & F(\eta^{-1}_M(G(L'))) & \xrightarrow{} & 0 \\
& & \downarrow{\cong} & & \downarrow{\cong} & & \downarrow{\cong} & & \\
0 & \xrightarrow{} & FG(L') & \xrightarrow{} & FGF(M) & \xrightarrow{} & FGF(M) & \xrightarrow{} & 0 \\
& & \downarrow{\cong} & & \downarrow{\cong} & & \downarrow{\cong} & & \\
0 & \xrightarrow{} & L' & \xrightarrow{} & F(M) & \xrightarrow{} & F(GF(M)/G(L')) & \xrightarrow{} & 0
\end{array}
$$

by Proposition 4.8(1) and Proposition 4.8(2). Hence the subobject $L := \eta^{-1}_M(G(L'))$ of $M$ satisfies $F(L) = L'$, and $F(M)/L' = F(M/L)$. By Proposition 4.7, the object $GF(M)/G(L')$ is $\mathcal{X}$-torsionfree, and hence $M/L$ is also $\mathcal{X}$-torsionfree.

Let $\tilde{L}$ be a subobject of $M$ such that $F(\tilde{L}) \subseteq L'$. Since we have the commutative diagram

$$
\begin{array}{ccccccc}
\tilde{L} & \xrightarrow{} & M & \xrightarrow{} & GF(\tilde{L}) & \xrightarrow{} & GF(M) \\
& & \downarrow{\eta_{\tilde{L}}} & & \downarrow{\eta_M} & & \downarrow{\eta_M} \\
& & GF(\tilde{L}) & \xrightarrow{} & GF(\tilde{L}) & \xrightarrow{} & GF(M)
\end{array}
$$

it holds that $\eta_M(\tilde{L}) \subseteq GF(\tilde{L})$. Therefore we have

$$
\tilde{L} \subseteq \eta^{-1}_M(GF(\tilde{L})) \subseteq \eta^{-1}_M(G(L')) = L.
$$

Several properties of objects are preserved by the canonical functors of a quotient category as in the following results.

Proposition 4.10. Let $\mathcal{X}$ be a localizing subcategory of $\mathcal{A}$. The canonical functors are denoted by $F: \mathcal{A} \to \mathcal{A}/\mathcal{X}$ and $G: \mathcal{A}/\mathcal{X} \to \mathcal{A}$.

1. Let $M'$ be an object in $\mathcal{A}/\mathcal{X}$, and let $L'$ be an essential subobject of $M'$. Then $G(L')$ is an essential subobject of $G(M')$.

2. Let $U'$ be a uniform object in $\mathcal{A}/\mathcal{X}$. Then $G(U')$ is a uniform object in $\mathcal{A}$.

3. Let $H'$ be a monoform object in $\mathcal{A}/\mathcal{X}$. Then $G(H')$ is a monoform object in $\mathcal{A}$.

4. Let $I'$ be an injective object in $\mathcal{A}/\mathcal{X}$. Then $G(I')$ is an injective object in $\mathcal{A}$.
Proposition 4.12. Let \( M' \) be an indecomposable object in \( A/X \). Then \( G(M') \) is an indecomposable object in \( A \).

Proof. (1) \[ \text{Pop73, Corollary 4.4.7}. \]

(2) This follows from Proposition 4.8 \( 1 \) and \( 1 \).

(3) \[ \text{Kan13, Lemma 5.14 (1)}. \]

(4) \[ \text{Pop73, Lemma 4.5.1 (2)}. \]

(5) This follows from Proposition 4.8 \( 1 \).

\[ \square \]

Proposition 4.11. Let \( X \) be a localizing subcategory of \( A \). The canonical functors are denoted by \( F: A \to A/X \) and \( G: A/X \to A \).

(1) Let \( M \) be an \( X \)-torsionfree object in \( A \), and let \( L \) be an essential subobject of \( M \). Then \( F(L) \) is an essential subobject of \( F(M) \).

(2) Let \( U \) be a uniform \( X \)-torsionfree object in \( A \). Then \( F(U) \) is a uniform object in \( A/X \).

(3) Let \( H \) be a monoform \( X \)-torsionfree object in \( A \). Then \( F(H) \) is a monoform object in \( A/X \).

(4) Let \( I \) be an injective \( X \)-torsionfree object in \( A \). Then \( F(I) \) is an injective object in \( A/X \).

Proof. (1) \[ \text{Pop73, Lemma 4.4.6 (3)}. \]

(2) This follows from Proposition 4.9 and \( 1 \).

(3) \[ \text{Kan13, Lemma 5.14 (2)}. \]

(4) \[ \text{Pop73, Lemma 4.5.1 (2)}. \]

\[ \square \]

The prelocalizing subcategories of \( A \) and those of quotient categories are related by the following operations.

Proposition 4.12. Let \( X \) be a localizing subcategory of \( A \). The canonical functors are denoted by \( F: A \to A/X \) and \( G: A/X \to A \).

(1) For each prelocalizing subcategory \( Y' \) of \( A/X \), the full subcategory \( F^{-1}(Y') := \{ M \in A \mid F(M) \in Y' \} \) of \( A \) is a prelocalizing subcategory, and we have \( X \ast F^{-1}(Y') \ast X = F^{-1}(Y') \).

(2) For each prelocalizing subcategory \( Y \) of \( A \), the full subcategory \( F(Y) := \{ N \in A \mid N \cong F(M) \text{ for some } M \in Y \} \) of \( A/X \) is a prelocalizing subcategory.

(3) Let \( Y_1 \) and \( Y_2 \) be prelocalizing subcategories of \( A \). Then we have \( F(Y_1 \ast X \ast Y_2) = F(Y_1) \ast F(Y_2) \).

Proof. (1) Since \( F \) is exact and commutes with arbitrary direct sums, the full subcategory \( F^{-1}(Y') \) is a prelocalizing subcategory. The inclusion \( F^{-1}(Y') \subseteq X \ast F^{-1}(Y') \ast X \) is obvious. By \( \text{Theorem 4.5 (1)} \), we have \( F(X \ast F^{-1}(Y') \ast X) \subseteq F(X) \ast F(F^{-1}(Y')) \ast F(X) \subseteq Y' \).

Hence it holds that \( X \ast F^{-1}(Y') \ast X \subseteq F^{-1}(Y') \).

(2) By \( \text{Proposition 4.9} \), the full subcategory \( F(Y) \) of \( A/X \) is closed under subobjects and quotient objects. It is also closed under arbitrary direct sums since \( F \) commutes with arbitrary direct sums.

(3) Since \( F \) is exact, it holds that \( F(Y_1 \ast X \ast Y_2) \subseteq F(Y_1) \ast F(Y_2) \) by \( \text{Theorem 4.5 (1)} \). Let \( M' \) be an object in \( A/X \) belonging to \( F(Y_1) \ast F(Y_2) \). Then there exists an exact sequence \( 0 \to F(M_1) \to M' \to F(M_2) \to 0 \).
where $M_i$ is an object in $A$ belonging to $\mathcal{Y}_i$ for each $i = 1, 2$. Since $G$ is left exact, we have the exact sequence
\[ 0 \rightarrow GF(M_1) \rightarrow G(M') \rightarrow GF(M_2). \]
Let $\eta: 1_A \rightarrow GF$ be the unit morphism, and let $B$ be the image of the morphism $G(M') \rightarrow GF(M_2)$. Then we obtain a commutative diagram
\[
\begin{array}{c}
0 & \rightarrow & GF(M_1) & \rightarrow & M & \rightarrow & B \cap \text{Im } \eta_{M_2} & \rightarrow & 0, \\
0 & \rightarrow & GF(M_1) & \rightarrow & G(M') & \rightarrow & B & \rightarrow & 0
\end{array}
\]
where $M$ is an object in $A$. Let $N$ be the cokernel of the composite $\text{Im } \eta_{M_1} \hookrightarrow GF(M_1) \hookrightarrow G(M')$. Then we have a commutative diagram
\[
\begin{array}{c}
0 & \rightarrow & \text{Im } \eta_{M_1} & \rightarrow & M & \rightarrow & N & \rightarrow & 0, \\
0 & \rightarrow & GF(M_1) & \rightarrow & M & \rightarrow & B \cap \text{Im } \eta_{M_2} & \rightarrow & 0
\end{array}
\]
By the snake lemma, we have an exact sequence
\[ 0 \rightarrow \text{Cok } \eta_{M_1} \rightarrow N \rightarrow B \cap \text{Im } \eta_{M_2} \rightarrow 0. \]
By Proposition 4.8 (2), the object $\text{Cok } \eta_{M_i}$ belongs to $\mathcal{X}$ for each $i = 1, 2$. Hence we have $F(\text{Cok } \eta_{M_1}) = 0$ and
\[ F \left( \frac{B}{B \cap \text{Im } \eta_{M_2}} \right) \cong F \left( \frac{B + \text{Im } \eta_{M_2}}{\text{Im } \eta_{M_2}} \right) \subset F \left( \frac{GF(M_2)}{\text{Im } \eta_{M_2}} \right) = 0. \]
By applying $F$ to the morphisms $B \cap \text{Im } \eta_{M_1} \hookrightarrow B$ and $\text{Im } \eta_{M_1} \hookrightarrow GF(M_1)$, we obtain $F(B \cap \text{Im } \eta_{M_1}) \cong F(B)$ and $F(\text{Im } \eta_{M_1}) \cong F(GF(M_1)) \cong F(M_1)$. Hence we have the commutative diagram
\[
\begin{array}{c}
0 & \rightarrow & F(\text{Im } \eta_{M_1}) & \rightarrow & F(M) & \rightarrow & F(N) & \rightarrow & 0, \\
0 & \rightarrow & FGF(M_1) & \rightarrow & F(M) & \rightarrow & F(B \cap \text{Im } \eta_{M_2}) & \rightarrow & 0 \\
0 & \rightarrow & FGF(M_1) & \rightarrow & FG(M') & \rightarrow & F(B) & \rightarrow & 0 \\
0 & \rightarrow & F(M_1) & \rightarrow & M' & \rightarrow & F(M_2) & \rightarrow & 0
\end{array}
\]
For each $i = 1, 2$, the quotient object $\text{Im } \eta_{M_i}$ of $M_i$ belongs to $\mathcal{Y}_i$, and hence the object $N$ belongs to $\mathcal{X} \ast \mathcal{Y}_2$. Therefore $M'$ belongs to $F(\mathcal{Y}_1 \ast \mathcal{X} \ast \mathcal{Y}_2)$. \hfill \square

**Proposition 4.13.** Let $\mathcal{X}$ be a localizing subcategory of $A$. The canonical functors are denoted by $F: A \rightarrow A/\mathcal{X}$ and $G: A/\mathcal{X} \rightarrow A$.

1. The map
\[ \begin{cases} \text{prelocalizing subcategories } \mathcal{Y} \text{ of } A \\ \text{satisfying } \mathcal{X} \ast \mathcal{Y} \ast \mathcal{X} = \mathcal{Y} \end{cases} \rightarrow \begin{cases} \text{prelocalizing subcategories of } A/\mathcal{X} \end{cases} \]
given by $\mathcal{Y} \mapsto F(\mathcal{Y})$ is bijective. The inverse map is given by $\mathcal{Y}' \mapsto F^{-1}(\mathcal{Y}')$.

2. For each $i = 1, 2$, let $\mathcal{Y}_i$ be a prelocalizing subcategory of $A$ such that $\mathcal{X} \ast \mathcal{Y}_i \ast \mathcal{X} = \mathcal{Y}_i$. Then we have
\[ F(\mathcal{Y}_1 \ast \mathcal{Y}_2) = F(\mathcal{Y}_1) \ast F(\mathcal{Y}_2). \]
(3) The bijection in (1) induces a bijection
\[
\left\{ \text{localizing subcategories } Y \text{ of } A \right\} \rightarrow \left\{ \text{localizing subcategories of } \frac{A}{N} \right\}.
\]

Proof. \(1\) By Proposition 4.12 \((1)\) and Proposition 4.12 \((2)\) these maps are well-defined. Let \(\eta: 1_A \rightarrow GF\) be the unit morphism.

Let \(Y\) be a prelocalizing subcategory of \(A\) satisfying \(X \ast Y \ast X = Y\). It is obvious that \(Y \subset F^{-1}F(Y)\). Let \(M\) be an object in \(A\) belonging to \(F^{-1}F(Y)\). Then there exists an object \(N\) in \(A\) belonging to \(Y\) such that \(F(M) \cong F(N)\). We have the exact sequence
\[
0 \rightarrow \text{Im } \eta N \rightarrow GF(N) \rightarrow \text{Cok } \eta N \rightarrow 0.
\]
The quotient object \(\text{Im } \eta N\) of \(N\) belongs to \(Y\). By Proposition 4.8 \((2)\), the object \(\text{Cok } \eta N\) belongs to \(X\). Hence \(GF(M) \cong GF(N)\) belongs to \(Y \ast X\). By Proposition 4.2 \((2)\) the subobject \(\text{Im } \eta M\) of \(GF(M)\) belongs to \(Y \ast X\). We have the exact sequence
\[
0 \rightarrow \text{Ker } \eta M \rightarrow M \rightarrow \text{Im } \eta M \rightarrow 0,
\]
where \(\text{Ker } \eta M\) belongs to \(X\). Therefore \(M\) belongs to \(X \ast Y \ast X = Y\). This shows that \(F^{-1}F(Y) \subset Y\).

Let \(Y'\) be a prelocalizing subcategory of \(A/X\). It is obvious that \(FF^{-1}(Y') \subset Y'\). Let \(M'\) be an object in \(A/X\) belonging to \(Y'\). Then by Proposition 4.8 \((1)\) there exists an object \(M\) in \(A\) such that \(F(M) \cong M'\). Since \(M\) belongs to \(F^{-1}(Y')\), the object \(M' \cong F(M)\) belongs to \(FF^{-1}(Y')\). This shows that \(Y' \subset FF^{-1}(Y')\).

(2) By Proposition 4.12 \((3)\) we have
\[
F(Y_1 \ast Y_2) = F(Y_1 \ast X \ast Y_2) = F(Y_1) \ast F(Y_2).
\]

(3) This follows from \((2)\). \(\square\)

Remark 4.14. In the setting of Proposition 4.12 \((3)\) the assertion \(F(Y_1 \ast X \ast Y_2) = F(Y_1 \ast Y_2)\) does not necessarily hold. The next example gives a counter-example.

Example 4.15. Let \(K\) be a field, and let \(\Lambda\) be the ring
\[
\Lambda = \begin{bmatrix}
    K & 0 & 0 \\
    K & K & 0 \\
    K & K & K
\end{bmatrix}
\]
of \(3 \times 3\) lower triangular matrices. Define simple \(\Lambda\)-modules \(S_i\) for each \(i = 1, 2, 3\) by
\[
S_1 = \begin{bmatrix}
    K & 0 & 0
\end{bmatrix},
\]
\[
S_2 = \begin{bmatrix}
    K & K & 0 \\
    K & 0 & 0
\end{bmatrix},
\]
\[
S_3 = \begin{bmatrix}
    K & K & K \\
    K & K & 0
\end{bmatrix},
\]
and let \(X_i\) the localizing subcategory of \(\text{Mod } \Lambda\) consisting of arbitrary direct sums of copies of \(S_i\).

Let \(F: A \rightarrow A/X_2\) and \(G: A/X_2 \rightarrow A\) denote the canonical functors. Since the \(\Lambda\)-module
\[
M = \begin{bmatrix}
    K & K & K
\end{bmatrix}
\]
belongs to \(X_1 \ast X_2 \ast X_3\), it follows that \(M \cong GF(M)\) belongs to \(GF(X_1 \ast X_2 \ast X_3)\).

On the other hand, every \(\Lambda\)-module belonging to \(X_1 \ast X_3\) is the direct sum of some object in \(X_1\) and some object in \(X_3\). Since \(\text{Mod } \Lambda\) is a locally noetherian Grothendieck category, by \(\text{Pop73} \) Proposition 5.8.12, the functor \(G\) commutes with arbitrary direct sums. Hence every \(\Lambda\)-module belonging to \(GF(X_1 \ast X_3)\) is the direct sum of some object in \(GF(X_1) = X_1 \ast X_2\) and some object in \(GF(X_3) = X_3\). Since \(M\) is indecomposable and belongs to neither \(X_1 \ast X_2\) nor \(X_3\), the \(\Lambda\)-module \(M\) does not belong to \(GF(X_1 \ast X_3)\). This shows that \(F(X_1 \ast X_2 \ast X_3) \not\subset F(X_1 \ast X_3)\).

The following result gives a characterization of a quotient category.
Proposition 4.16. Let \( A \) and \( B \) be Grothendieck categories, and let \( Q : A \to B \) be an exact functor with a fully faithful right adjoint \( B \to A \). Then the full subcategory
\[
\mathcal{X} = \{ M \in A \mid Q(M) = 0 \}
\]
of \( A \) is a localizing subcategory, and there exists a unique equivalence \( \mathcal{Q} : A/\mathcal{X} \to B \) such that \( \mathcal{Q}F = Q \), where \( F : A \to A/\mathcal{X} \) is the canonical functor.

Proof. \cite[Theorem 4.4.9]{Pop73}. \( \square \)

We state some facts on the image of a localizing subcategory in a quotient category.

Proposition 4.17. Let \( X \) and \( Y \) be localizing subcategories of \( A \). The canonical functors are denoted by \( F : A \to A/\mathcal{X} \) and \( G : A/\mathcal{X} \to A \).

(1) It holds that \( \langle F(Y) \rangle_{\text{loc}} \subseteq F(\langle X \cup Y \rangle_{\text{loc}}) \).

(2) If \( X \subseteq Y \), then the composite \( Y \to A \to A/\mathcal{X} \) induces an equivalence \( \mathcal{Y} \mathcal{X} \to F(Y) \).

(3) If \( X \subseteq Y \), then the composite \( A \to A/\mathcal{X} \to A/\mathcal{Y} \) induces an equivalence \( A \mathcal{Y} \to A/\mathcal{F}(Y) \).

Proof. (1) It is obvious that \( \langle F(Y) \rangle_{\text{loc}} \subseteq F(\langle X \cup Y \rangle_{\text{loc}}) \). Since \( F \) is exact and commutes with arbitrary direct sums, we have
\[
F(\langle X \cup Y \rangle_{\text{loc}}) \subseteq F(\langle X \cup Y \rangle)_{\text{loc}} = (F(X) \cup F(Y))_{\text{loc}} = (F(Y))_{\text{loc}}
\]
by Theorem 4.5 (1).

[2] The equivalence follows from the construction of \( A/\mathcal{X} \) (see \cite[p. 365]{Gab62} or \cite[Definition 5.2]{Kan13}).

[3] By Proposition 4.13 (3) the full subcategory \( F(Y) \) of \( A/\mathcal{X} \) is a localizing subcategory, and we have \( F^{-1}F(Y) = Y \). By Proposition 4.8 (1) the composite is an exact functor with a fully faithful right adjoint. Hence by Proposition 4.16 it induces an equivalence
\[
\mathcal{A} \mathcal{Y} = \mathcal{A} \mathcal{F}^{-1}F(Y) \to A/\mathcal{F}(Y).
\]

5. Atom spectra of quotient categories and localization

Throughout this section, let \( A \) be a Grothendieck category. We recall a description of the atom spectrum of a quotient category of \( A \) and fundamental results on the localization of \( A \) at an atom. We start with relating localizing subcategories of \( A \) and localizing subsets of \( \text{ASpec} \) \( A \).

Definition 5.1. (1) For a full subcategory \( \mathcal{X} \) of \( A \), define the subset \( \text{ASupp} \mathcal{X} \) of \( \text{ASpec} A \) by
\[
\text{ASupp} \mathcal{X} = \bigcup_{M \in \mathcal{X}} \text{ASupp} M.
\]

(2) For a subset \( \Phi \) of \( \text{ASpec} A \), define the full subcategory \( \text{ASupp}^{-1} \Phi \) of \( A \) by
\[
\text{ASupp}^{-1} \Phi = \{ M \in A \mid \text{ASupp} M \subseteq \Phi \}.
\]

Proposition 5.2. (1) For every full subcategory \( \mathcal{X} \) of \( A \), the subset \( \text{ASupp} \mathcal{X} \) of \( \text{ASpec} A \) is a localizing subset.

(2) For every subset \( \Phi \) of \( \text{ASpec} A \), the full subcategory \( \text{ASupp}^{-1} \Phi \) of \( A \) is a localizing subcategory.
Proof. [1] Recall that \(\text{ASpec} A\) is in bijection with a small set. For each \(\alpha \in \text{ASupp} X\), choose an object \(M(\alpha)\) in \(A\) belonging to \(X\) such that \(\alpha \in \text{ASupp} M(\alpha)\). Then we have
\[
\text{ASupp} \bigoplus_{\alpha \in \text{ASupp} X} M(\alpha) = \bigcup_{\alpha \in \text{ASupp} X} \text{ASupp} M(\alpha) = \text{ASupp} X
\]
by Proposition 3.11 (1). [2] This follows from Proposition 3.10 and Proposition 3.11 (1)

The following result shows that a localizing subset of \(\text{ASpec} A\) is determined by the corresponding localizing subcategory of \(A\).

**Proposition 5.3.** For every localizing subset \(\Phi\) of \(\text{ASpec} A\), it holds that
\[
\text{ASupp}(\text{ASupp}^{-1} \Phi) = \Phi.
\]

**Proof.** This follows from the proof of [Kan12a, Theorem 4.3].

If \(A\) is a locally noetherian Grothendieck category, we also have \(\text{ASupp}^{-1}(\text{ASupp} X) = X\) for every localizing subcategory \(X\) of \(A\), and these correspondences establish a bijection between the localizing subcategories of \(A\) and the localizing subsets of \(\text{ASpec} A\) ([Kan12a, Theorem 5.5]). We generalize this result later as Theorem 6.8.

We describe the atom spectrum of the quotient category by a localizing subcategory.

**Theorem 5.4.** Let \(A\) be a Grothendieck category, and let \(X\) be a localizing subcategory of \(A\). Let \(F: A \to A/X\) and \(G: A/X \to A\) denote the canonical functors. Then the map \(\text{ASpec} A \setminus \text{ASupp} X \to \text{ASpec}(A/X)\) given by \(H \mapsto F(H)\) is a homeomorphism. The inverse map is given by \(H' \mapsto G(H')\).

**Proof.** [Kan13, Theorem 5.17].

**Remark 5.5.** Every localizing subcategory \(X\) of \(A\) is a Grothendieck category, and \(\text{ASpec} X\) is homeomorphic to the localizing subset \(\text{ASupp} X\) of \(\text{ASpec} A\) by the correspondence \(H \mapsto F(H)\) ([Kan13, Proposition 5.12]). We identify \(\text{ASpec} X\) with \(\text{ASupp} X\), and \(\text{ASpec}(A/X)\) with \(\text{ASpec} A \setminus \text{ASupp} X\) via the homeomorphism in Theorem 5.4. Then we have
\[
\text{ASpec} A = \text{ASpec} X \cup \text{ASpec} A/X
\]
and
\[
\text{ASpec} X \cap \text{ASpec} A/X = \emptyset.
\]

We describe atom supports and associated atoms in a quotient category.

**Proposition 5.6.** Let \(X\) be a localizing subcategory of \(A\). The canonical functors are denoted by \(F: A \to A/X\) and \(G: A/X \to A\).

1. For every object \(M'\) in \(A/X\), we have
\[
\text{AAss} G(M') = \text{AAss} M'
\]
and
\[
\text{ASupp} G(M') \setminus \text{ASupp} X = \text{ASupp} M'.
\]
2. For every object \(M\) in \(A\), we have
\[
\text{AAss} F(M) \supset \text{AAss} M \setminus \text{ASupp} X
\]
and
\[
\text{ASupp} F(M) = \text{ASupp} M \setminus \text{ASupp} X.
\]

**Proof.** These follow from [Kan13, Lemma 5.16]. By considering Proposition 4.8 (3) the assertion \(\text{AAss} G(M') = \text{AAss} M'\) also follows.
The atom spectrum of the image of a localizing subcategory in a quotient category is described as follows.

**Proposition 5.7.** Let \( X \) and \( Y \) be localizing subcategories of \( A \). Let \( F: A \to A/X \) and \( G: A/X \to A \) denote the canonical functors. Then we have

\[
\text{ASpec} \left( F(Y) \right)_{\text{loc}} = \text{ASpec} F(\langle X \cup Y \rangle_{\text{loc}})
\]

\[
= \text{ASpec} Y \cap \text{ASpec} \frac{A}{X}
\]

\[
= \text{ASpec} Y \setminus \text{ASpec} X
\]

and

\[
\text{ASpec} \left( \frac{A}{X} \right)_{\text{loc}} = \text{ASpec} \frac{A}{\langle X \cup Y \rangle_{\text{loc}}}
\]

\[
= \text{ASpec} \frac{A}{X} \cap \text{ASpec} \frac{A}{Y}
\]

\[
= \text{ASpec} A \setminus (\text{ASpec} A \cup \text{ASpec} Y).
\]

**Proof.** This follows from \( \text{ASupp} \langle X \cup Y \rangle_{\text{loc}} = \text{ASupp} X \cup \text{ASupp} Y \) and Proposition 4.17. \( \square \)

**Definition 5.8.** Let \( A \) be a Grothendieck category and \( \alpha \in \text{ASpec} A \). Define a localizing subcategory \( X(\alpha) \) of \( A \) by \( X(\alpha) = \text{ASupp}^{-1}(\text{ASpec} A \setminus \Lambda(\alpha)) \). Define the localization \( A_{\alpha} \) of \( A \) at \( \alpha \) by \( A_{\alpha} = A/X(\alpha) \). The canonical functor \( A \to A_{\alpha} \) is denoted by \((\cdot)_{\alpha}\).

In Definition 5.8, the subset \( \text{ASpec} A \setminus \Lambda(\alpha) \) of \( \text{ASpec} A \) is localizing. By Proposition 5.3, it holds that \( \text{ASupp} X(\alpha) = \text{ASpec} A \setminus \Lambda(\alpha) \). Therefore we have the following result.

**Theorem 5.9.** Let \( A \) be a Grothendieck category and \( \alpha \in \text{ASpec} A \). Then we have \( \text{ASpec} A_{\alpha} = \Lambda(\alpha) \). In particular, the partially ordered set \( \text{ASpec} A \) has the largest element \( \alpha \).

**Proof.** [Kan13, Proposition 6.6 (1)]. \( \square \)

We obtain the following description of atom supports.

**Proposition 5.10.**

1. For every \( \alpha \in \text{ASpec} A \), we have
   
   \[
   X(\alpha) = \{ M \in A \mid \alpha \notin \text{ASupp} M \}
   \]

2. For every object \( M \) in \( A \), we have
   
   \[
   \text{ASupp} M = \{ \alpha \in \text{ASpec} A \mid M_{\alpha} \neq 0 \}.
   \]

**Proof.** [Kan13, Proposition 6.2]. \( \square \)

We show that the localization of a Grothendieck category at an atom is “local” in the following sense.

**Definition 5.11.** Let \( A \) be a Grothendieck category.

1. We say that \( A \) is local if there exists a simple object in \( A \) such that \( E(S) \) is a cogenerator of \( A \).

2. A localizing subcategory \( X \) of \( A \) is called prime if \( A/X \) is a local Grothendieck category.

**Theorem 5.12.** Let \( A \) be a Grothendieck category. Then the following assertions are equivalent.

1. \( A \) is local.

2. \( \text{ASpec} A \) has a largest element.

3. There exists \( \alpha \in \text{ASpec} A \) such that for every nonzero object \( M \) in \( A \), we have \( \alpha \in \text{ASupp} M \).

4. There exists \( \alpha \in \text{ASpec} A \) such that the canonical functor \( A \to A_{\alpha} \) is an equivalence.

In particular, all simple objects in a local Grothendieck category are isomorphic to each other.

**Proof.** [Kan13, Corollary 6.5], [Kan13, Proposition 6.6 (2)], and [Kan13, Proposition 6.4 (2)]. \( \square \)
In the case of where $\mathcal{A}$ is a locally noetherian Grothendieck category, the localness of $\mathcal{A}$ is characterized as follows.

**Proposition 5.13.** Let $\mathcal{A}$ be a locally noetherian Grothendieck category. Then $\mathcal{A}$ is local if and only if all simple objects in $\mathcal{A}$ are isomorphic to each other.

**Proof.** [Kan13, Proposition 6.4 (2)]. □

**Theorem 5.12** shows that the localizing subcategory $\mathcal{X}(\alpha)$ is prime for every $\alpha \in \text{ASpec} \mathcal{A}$. This correspondence gives the following bijection.

**Theorem 5.14.** Let $\mathcal{A}$ be a Grothendieck category. Then the map

$$\text{ASpec} \mathcal{A} \to \{\text{prime localizing subcategories of } \mathcal{A}\}$$

given by $\alpha \mapsto \mathcal{X}(\alpha)$ is bijective. For each $\alpha, \beta \in \text{ASpec} \mathcal{A}$, we have $\alpha \leq \beta$ if and only if $\mathcal{X}(\alpha) \supset \mathcal{X}(\beta)$.

**Proof.** [Kan13, Theorem 6.8]. □

We consider the localization of a quotient category.

**Proposition 5.15.** Let $\mathcal{X}$ be a localizing subcategory of $\mathcal{A}$ and $\alpha \in \text{ASpec} \mathcal{A} \setminus \text{ASupp} \mathcal{X}$. Then the composite of the canonical functors $\mathcal{A} \to \mathcal{A}/\mathcal{X}$ and $\mathcal{A}/\mathcal{X} \to (\mathcal{A}/\mathcal{X})_\alpha$ induces an equivalence $\mathcal{A}_\alpha \sim \to (\mathcal{A}/\mathcal{X})_\alpha$.

**Proof.** By [Proposition 5.10 (1)], we have $\mathcal{X} \subset \mathcal{X}(\alpha)$. Hence the claim follows from [Proposition 5.6 (2)] and [Proposition 4.17 (3)]. □

In the setting of **Proposition 5.15**, we identify $\mathcal{A}_\alpha$ and $(\mathcal{A}/\mathcal{X})_\alpha$.

The following result shows that the localization of a Grothendieck category at an atom is a generalization of the localization a commutative ring at a prime ideal.

**Proposition 5.16.** Let $\mathcal{R}$ be a commutative ring.

1. Let $\mathcal{R}/p \in \text{Spec} \mathcal{R}$. Denote by $\alpha$ the corresponding atom $\mathcal{R}/p$ in $\text{Mod} \mathcal{R}$. Then the functor $- \otimes \mathcal{R}/p: \text{Mod} \mathcal{R} \to \text{Mod} \mathcal{R}/p$ induces an equivalence $(\text{Mod} \mathcal{R})_\alpha \sim \to \text{Mod} \mathcal{R}/p$.

2. The Grothendieck category $\text{Mod} \mathcal{R}$ is local if and only if the commutative ring $\mathcal{R}$ is local.

**Proof.**

1. [Kan13, Proposition 6.9].

2. This follows [Theorem 5.12] and [Proposition 3.18]. □

### 6. Grothendieck Categories with Enough Atoms

The purpose of this paper is to investigate the category $\text{QCoh } X$ of quasi-coherent sheaves on a locally noetherian scheme $X$. In general, the category $\text{QCoh } X$ is a Grothendieck category but not necessarily locally noetherian (see [Remark 7.5]). In this section, we introduce the notion of a Grothendieck category with enough atoms and investigate its properties. It is shown later that $\text{QCoh } X$ is a Grothendieck category with enough atoms.

Let $\mathcal{A}$ be a Grothendieck category. Recall that every monoform object in $\mathcal{A}$ is uniform. We say that uniform objects $U_1$ and $U_2$ in $\mathcal{A}$ are equivalent (denoted by $U_1 \sim U_2$) if there exists a nonzero subobject of $U_1$ which is isomorphic to a subobject of $U_2$. The equivalence between monoform objects is exactly the same as the atom-equivalence defined in [Definition 3.3 (2)].

**Proposition 6.1.** Let $U_1$ and $U_2$ be uniform objects in $\mathcal{A}$. Then $U_1$ is equivalent to $U_2$ if and only if $E(U_1)$ is isomorphic to $E(U_2)$.

**Proof.** [Kra03, Lemma 2]. □
Since every indecomposable injective object in $\mathcal{A}$ is uniform (Ste75, Proposition V.2.8]), the map
\[
\{\text{uniform objects in } \mathcal{A}\} \sim \rightarrow \{\text{indecomposable injective objects in } \mathcal{A}\}
\]
induced by the correspondence $U \mapsto E(U)$ is bijective. We consider the restriction of this bijection to $\text{ASpec } \mathcal{A}$.

**Definition 6.2.** Let $\mathcal{A}$ be a Grothendieck category. For $\alpha \in \text{ASpec } \mathcal{A}$, define the injective envelope $E(\alpha)$ of $\alpha$ by $E(\alpha) = E(H)$, where $H$ is a monoform object in $\mathcal{A}$ satisfying $H = \alpha$.

Proposition 6.1 implies that the isomorphism class of $E(\alpha)$ in Definition 6.2 does not depend on the choice of the representative $H$.

**Definition 6.3.** We say that a Grothendieck category $\mathcal{A}$ has enough atoms if $\mathcal{A}$ satisfies the following conditions.

1. Every injective object in $\mathcal{A}$ has an indecomposable decomposition.
2. Each indecomposable injective object in $\mathcal{A}$ is isomorphic to $E(\alpha)$ for some $\alpha \in \text{ASpec } \mathcal{A}$.

Note that an indecomposable decomposition of an injective object is unique in the following sense.

**Theorem 6.4.** Let $\mathcal{A}$ be a Grothendieck category, and let $I$ be an injective object with
\[
I \cong \bigoplus_{\lambda \in \Lambda} I_\lambda \cong \bigoplus_{\mu \in \Lambda'} I'_\mu,
\]
where $I_\lambda$ and $I'_\mu$ are indecomposable for each $\lambda \in \Lambda$ and $\mu \in \Lambda'$. Then there exists a bijection $\varphi: \Lambda \rightarrow \Lambda'$ such that $I_\lambda$ is isomorphic to $I'_\varphi(\lambda)$ for each $\lambda \in \Lambda$.

**Proof.** This follows from Krull-Remak-Schmidt-Azumaya’s theorem (Pop73, Theorem 5.1.3) and the fact that the endomorphism ring of each indecomposable injective object in $\mathcal{A}$ is local (Pop73, Lemma 4.20.3). □

The following result shows that a Grothendieck category with enough atoms is a generalization of a locally noetherian Grothendieck category.

**Proposition 6.5.** Every locally noetherian Grothendieck category has enough atoms.

**Proof.** This follows from Proposition 3.5 (3) and Ste75, Proposition V.4.5 since every nonzero object in a locally noetherian Grothendieck category has a nonzero noetherian subobject. □

We show that every quotient category of a Grothendieck category with enough atoms has enough atoms.

**Proposition 6.6.** Let $\mathcal{A}$ be a Grothendieck category, and let $\mathcal{X}$ be a localizing subcategory of $\mathcal{A}$.

1. If every injective object in $\mathcal{A}$ has an indecomposable decomposition, then every injective object in $\mathcal{A}/\mathcal{X}$ has an indecomposable decomposition.
2. If $\mathcal{A}$ has enough atoms, then $\mathcal{A}/\mathcal{X}$ has enough atoms.

**Proof.** The canonical functors are denoted by $F: \mathcal{A} \rightarrow \mathcal{A}/\mathcal{X}$ and $G: \mathcal{A}/\mathcal{X} \rightarrow \mathcal{A}$.

1. Let $I'$ be an injective object in $\mathcal{A}/\mathcal{X}$. By Proposition 4.10 (4), the object $G(I')$ in $\mathcal{A}$ is injective. Hence $G(I')$ has an indecomposable decomposition $G(I') = \bigoplus_{\lambda \in \Lambda} I_\lambda$.

We obtain $I' \cong FG(I') \cong \bigoplus_{\lambda \in \Lambda} F(I_\lambda)$.

By Proposition 4.8 (3), Proposition 4.11 (2) and Proposition 4.11 (4), the object $F(I_\lambda)$ is an indecomposable injective object in $\mathcal{A}/\mathcal{X}$ for each $\lambda \in \Lambda$. 

Let $I'$ be an indecomposable injective object in $A/X$. Then by Proposition 4.10[5] and Proposition 4.11[4], the object $G(I')$ in $A$ is indecomposable and injective. Hence there exists $\alpha \in \text{ASpec} A$ such that $G(I') \cong E_A(\alpha)$. We obtain $I' \cong FG(I') \cong F(E_A(\alpha))$. Let $H$ be a monomorphic subobject of $E_A(\alpha)$. By Proposition 4.8[3], the object $H$ is $X$-torsionfree. By Proposition 4.11[3], the object $I'$ has the monomorphic subobject $F(H)$. This implies that $I' \cong E(F(H)) = E_A/X(\alpha)$. □

A Grothendieck category $A$ is called locally uniform[1] if every nonzero object in $A$ has a uniform subobject. It is shown that this holds whenever $A$ has enough atoms.

**Proposition 6.7.** Let $A$ be a Grothendieck category with enough atoms. Then every nonzero object in $A$ has a monoform subobject. In particular, the Grothendieck category $A$ is locally uniform.

**Proof.** Let $M$ be a nonzero object in $A$. Then there exists a family $\{\alpha_\lambda\}_{\lambda \in \Lambda}$ of atoms in $A$ such that

$$E(M) \cong \bigoplus_{\lambda \in \Lambda} E(\alpha_\lambda).$$

Hence $E(M)$ has a monoform subobject $H$. Since $M$ is an essential subobject of $E(M)$, the subobject $H \cap M$ of $M$ is monoform by Proposition 3.5[1]. The last assertion follows from Proposition 5.3[2]. □

The classification of the localizing subcategories by the atom spectrum we mentioned after Proposition 5.3 is generalized to a Grothendieck category with enough atoms.

**Theorem 6.8.** Let $A$ be a Grothendieck category with enough atoms. Then the map

$$\{ \text{localizing subcategories of } A \} \rightarrow \{ \text{localizing subsets of } \text{ASpec} A \}$$

given by $X \mapsto \text{ASupp} X$ is bijective. The inverse map is given by $\Phi \mapsto \text{ASupp}^{-1} \Phi$.

**Proof.** By Proposition 5.2 and Proposition 5.3, it suffices to show that $\text{ASupp}^{-1}(\text{ASupp} X) = X$ for each localizing subcategory $X$ of $A$. The inclusion $X \subset \text{ASupp}^{-1}(\text{ASupp} X)$ holds obviously. Let $M$ be an object in $A$ belonging to $\text{ASupp}^{-1}(\text{ASupp} X)$, and let $L$ be the largest subobject of $M$ belonging to $X$. If $M/L$ is nonzero, then by Proposition 6.7 there exists a monoform subobject $H$ of $M/L$. Since we have $\prod \in \text{ASupp} M \subset \text{ASupp} X$, there exists a nonzero subobject $H'$ of $H$ belonging to $X$. Let $L' = L'/L \subset M/L$. Since $L$ and $L'/L$ belong to $X$, the subobject $L'$ of $M$ also belongs to $X$. This contradicts the maximality of $L$. Therefore $\text{ASupp}^{-1}(\text{ASupp} X) = X$. □

We show that every localizing subcategory is the intersection of some family of prime localizing subcategories.

**Corollary 6.9.** Let $A$ be a Grothendieck category with enough atoms. For every localizing subcategory $X$ of $A$, it holds that

$$X = \bigcap_{\alpha \in \text{ASpec} A \setminus \text{ASupp} X} X(\alpha).$$

**Proof.** By Proposition 5.10[1] and Theorem 6.8, we have

$$\bigcap_{\alpha \in \text{ASpec} A \setminus \text{ASupp} X} X(\alpha) = \{ M \in A \mid \alpha \notin \text{ASupp} M \text{ for each } \alpha \in \text{ASpec} A \setminus \text{ASupp} X \}
= \{ M \in A \mid \text{ASupp} M \subset \text{ASupp} X \}
= \text{ASupp}^{-1}(\text{ASupp} X)
= X.$$ □

---

1In [Pop73] p. 330, it is called locally coirreducible since a uniform object is called a coirreducible object.
Let $\mathcal{A}$ be a Grothendieck category and $\alpha \in \text{ASpec}\mathcal{A}$. It is shown in the proof of \cite[Theorem 2.5]{Kan2b} that the injective envelope $E(\alpha)$ has a largest monoform subobject $H(\alpha)$. The object $H(\alpha)$ is called the atomic object corresponding to $\alpha$. It is straightforward to show that no object in $\mathcal{A}$ has a proper essential subobject isomorphic to $H(\alpha)$.

The atomic objects correspond to the simple objects in the localizations.

**Proposition 6.10.** Let $\mathcal{A}$ be a Grothendieck category and $\alpha \in \text{ASpec}\mathcal{A}$. The canonical functors are denoted by $F_\alpha : \mathcal{A} \to \mathcal{A}_\alpha$ and $G_\alpha : \mathcal{A}_\alpha \to \mathcal{A}$. Let $S'$ be the simple object in $\mathcal{A}_\alpha$.

1. $S'$ is the atomic object corresponding to the atom $\mathfrak{a}$ in $\mathcal{A}_\alpha$.
2. $G_\alpha(S')$ is isomorphic to the atomic object $H(\alpha)$.
3. The ring $\text{End}_{\mathcal{A}}(H(\alpha))$ is isomorphic to the skew field $\text{End}_{\mathcal{A}_\alpha}(S')$.

**Proof.**

1. We have $S' \subset H(\mathfrak{a}) \subset E(\mathfrak{a}) = E(S')$. If $S' \subsetneq H(\mathfrak{a})$, then by Proposition 6.12 we have $S' \in \text{ASupp}(H(\mathfrak{a})/S')$, and hence there exist a subobject $L$ of $H(\mathfrak{a})$ with $S' \subset L$ and a subobject of $H(\mathfrak{a})$ which is isomorphic to $S'$. This contradicts the monoformness of $H(\mathfrak{a})$.

2. By Theorem 5.4, the object $F_\alpha(H(\alpha))$ is a monoform object in $\mathcal{A}_\alpha$, and $G_\alpha F_\alpha(H(\alpha))$ is a monoform object in $\mathcal{A}$. By 1 we have $F_\alpha(H(\alpha)) \cong S'$. Since $H(\alpha)$ is $\mathcal{A}(\alpha)$-torsionfree, by Proposition 4.8(2) the canonical morphism $H(\alpha) \to G_\alpha F_\alpha(H(\alpha))$ is a monomorphism, and $H(\alpha)$ is essential as a subobject of $G_\alpha F_\alpha(H(\alpha))$. Therefore the morphism $H(\alpha) \to G_\alpha F_\alpha(H(\alpha))$ is an isomorphism, and we have $G_\alpha(S') \cong G_\alpha F_\alpha(H(\alpha)) \cong H(\alpha)$.

3. By 2 and Proposition 4.8(1) we have

$$\text{End}_{\mathcal{A}}(H(\alpha)) \cong \text{End}_{\mathcal{A}_\alpha}(S').$$

This gives a ring isomorphism $\text{End}_{\mathcal{A}}(H(\alpha)) \cong \text{End}_{\mathcal{A}_\alpha}(S')$. \hfill $\square$

The skew field $\text{End}_{\mathcal{A}}(H(\alpha))$ is called a residue field of $\alpha$ and denoted by $k(\alpha)$.

### 7. The Atom Spectra of Locally Noetherian Schemes

In this section, we describe the atom spectrum of the category of quasi-coherent sheaves on a locally noetherian scheme. Let $X$ be a locally noetherian scheme with the underlying topological space $|X|$ and the structure sheaf $\mathcal{O}_X$. It is known that the category $\text{Mod}\mathcal{O}_X$ of $\mathcal{O}_X$-modules and the category $\text{QCoh}\mathcal{X}_X$ of quasi-coherent sheaves on $X$ are Grothendieck categories (see \cite[Theorem II.7.8]{Har66} and \cite[Lemma 2.1.7]{Con00}). For a commutative ring $R$, we identify $\text{QCoh}(|\text{Spec}\, R|)$ with $\text{Mod}\, R$.

**Proposition 7.1.** Let $U$ be an open affine subscheme of $X$, and let $i : U \hookrightarrow X$ be the immersion. Then the functor $i_* : \text{Mod}\, U \to \text{Mod}\, X$ and its left adjoint $i^* : \text{Mod}\, X \to \text{Mod}\, U$ induce the functor $i_* : \text{QCoh}\, U \to \text{QCoh}\, X$ and its left adjoint $i^* : \text{QCoh}\, X \to \text{QCoh}\, U$.

**Proof.** \cite[0.4.4.3.1]{Gro60} and \cite[Proposition 1.9.4.2 (ii)]{Gro60}. \hfill $\square$

In the rest of this paper, every quasi-coherent sheaf $M$ on $X$ is always regarded as an object in $\text{QCoh}\, X$, not in $\text{Mod}\, X$. Hence a subobject of $M$ means a quasi-coherent subsheaf of $M$.

For an open affine subscheme $U$ of $X$ with the immersion $i : U \hookrightarrow X$, the functor $i^* : \text{QCoh}\, X \to \text{QCoh}\, U$ is also denoted by $(-)|_U$. The category $\text{QCoh}\, U$ is realized as a quotient category of $\text{QCoh}\, X$ through this functor.

**Proposition 7.2.** Let $U$ be an open affine subscheme of $X$. Then the functor $(-)|_U : \text{QCoh}\, X \to \text{QCoh}\, U$ induces an equivalence $(\text{QCoh}\, X)/\mathcal{X}_U \simeq \text{QCoh}\, U$, where $\mathcal{X}_U$ is a localizing subcategory of $\text{QCoh}\, X$ defined by

$$\mathcal{X}_U = \{ M \in \text{QCoh}\, X \mid M|_U = 0 \}.$$
Theorem 7.6. Let \( i: U \hookrightarrow X \) be the immersion. Since the counit functor \( i^*i_* \to 1 \text{QCoh}\,U \) is an isomorphism, the functor \( i_* \) is fully faithful. The functor \( i^* \) is exact. Hence the claim follows from Proposition 4.16. \( \square \)

For each object \( M \) in \( \text{QCoh}\,X \), the subset \( \text{Supp}\,M \) of \( X \) is defined by
\[
\text{Supp}\,M = \{ x \in X \mid M_x \neq 0 \}.
\]

For each \( x \in X \), let \( j_x: \text{Spec}\,\mathcal{O}_{X,x} \to X \) be the canonical morphism. Note that \( j_x^* \) is equal to the localization \( (-)_x: \text{QCoh}\,X \to \text{Mod}\,\mathcal{O}_{X,x} \). The category \( \text{Mod}\,\mathcal{O}_{X,x} \) is realized as a quotient category of \( \text{QCoh}\,X \) through this morphism.

**Proposition 7.3.** For every \( x \in X \), the full subcategory
\[
\mathcal{X}(x) := \{ M \in \text{QCoh}\,X \mid x \notin \text{Supp}\,M \} = \{ M \in \text{QCoh}\,X \mid M_x = 0 \}
\]
of \( \text{QCoh}\,X \) is a prime localizing subcategory. The functor \( (-)_x: \text{QCoh}\,X \to \text{Mod}\,\mathcal{O}_{X,x} \) induces an equivalence \( \text{QCoh}\,\mathcal{X}(x) \to \text{Mod}\,\mathcal{O}_{X,x} \).

**Proof.** Let \( i: U \hookrightarrow X \) be the immersion of an open affine subscheme with \( x \in U \). Then the functor \( (-)_x: \text{QCoh}\,X \to \text{Mod}\,\mathcal{O}_{X,x} \) is equal to the composite of \( (-)_x : \text{QCoh}\,X \to \text{QCoh}\,U \) and \( \mathcal{O}_U \to \text{Mod}\,\mathcal{O}_{X,x} \). By Proposition 5.16 and Proposition 5.16[1], these two functors are exact functors with fully faithful right adjoints. Hence we obtain the equivalence by Proposition 4.16 by Proposition 5.16[2]. The localization subcategory \( \mathcal{X}(x) \) is prime. \( \square \)

For each \( x \in X \), denote the unique maximal ideal of \( \mathcal{O}_{X,x} \) by \( m_x \), the residue field of \( x \) by \( k(x) = \mathcal{O}_{X,x}/m_x \), and the injective envelope of \( k(x) \) in \( \text{Mod}\,\mathcal{O}_{X,x} \) by \( E(x) = E_{\mathcal{O}_{X,x}}(k(x)) \). We state that every injective object in \( \text{QCoh}\,X \) is a direct sum of indecomposable injective objects of this form.

**Theorem 7.4** (Hartshorne [Har66]). Let \( X = (|X|, \mathcal{O}_X) \) be a locally noetherian scheme.

1. For every family \( \{ I_x \}_{x \in X} \) of injective objects in \( \text{QCoh}\,X \), the direct sum \( \bigoplus_{x \in X} I_x \) is also injective.
2. Every injective object in \( \text{QCoh}\,X \) has an indecomposable decomposition.
3. The map
\[
|X| \to \{ \text{indecomposable injective objects in } \text{QCoh}\,X \}
\]
given by \( x \mapsto j_x^*E(x) \) is bijective.

**Proof.** Let \( \text{Con}03 \) Lemma 2.1.5. \( \square \)

**Remark 7.5.** In [Har66, p. 135], it is shown that there exists a locally noetherian scheme \( X \) such that the Grothendieck category \( \text{QCoh}\,X \) is not locally noetherian. By combining Theorem 7.4[1] and [Pop73, Theorem 5.8.7], we deduce that \( \text{QCoh}\,X \) is not even (categorically) locally finitely generated. On the other hand, it holds that the set of coherent sheaves on \( X \) generates \( \text{QCoh}\,X \) [Gro60, Corollary I.9.4.9]. Consequently, a coherent sheaf on \( X \) is not necessarily a finitely generated object in \( \text{QCoh}\,X \).

We give a description of the atom spectrum of \( \text{QCoh}\,X \).

**Theorem 7.6.** Let \( X = (|X|, \mathcal{O}_X) \) be a locally noetherian scheme.

1. For each \( x \in X \), the set \( \text{Ass}\,j_x^*E(x) \) consists of one element, say \( \alpha_x \). The injective envelope of \( \alpha_x \) is \( E(\alpha_x) = j_x^*E(x) \). The atomic object is \( H(\alpha_x) = j_x^*k(x) \). The residue field is \( k(\alpha_x) \cong k(x) \).
2. The map \( |X| \to \text{ASpec}(\text{QCoh}\,X) \) given by \( x \mapsto \alpha_x \) is bijective. Moreover, the Grothendieck category \( \text{QCoh}\,X \) has enough atoms.
Proof. (1) By Proposition 7.3 and Proposition 5.6 (1) we have
\[ A\text{Ass}_j \epsilon E(x) = A\text{Ass} E(x) = \{ k(x) \}. \]
Since \( j_x, E(x) \) is an indecomposable injective object by Theorem 7.4 (3) it is the injective envelope of each of its nonzero subobjects. Hence we have \( E(\alpha_x) = j_x, E(x) \).

By Proposition 6.10 (2), we have \( H(\alpha_x) = j_x, k(x) \). By Proposition 6.10 (3) we have \( k(\alpha_x) \cong \text{End}_{\text{QCoh}}(x, k(x)) \cong k(x) \).

(2) The bijection in Theorem 7.4 (3) is the composite of the map
\[ |X| \to \text{ASpec} (\text{QCoh} X) \]
given by \( x \mapsto \alpha_x \) and the injection
\[ \text{ASpec} (\text{QCoh} X) \to \{ \text{indecomposable injective objects in } \text{QCoh} X \} \]
given by \( \alpha \mapsto E(\alpha) \). Hence these maps are also bijective. By Theorem 7.4 (2), the Grothendieck category \( \text{QCoh} X \) has enough atoms.

A subset \( \Phi \) of \( X \) is said to be closed under specialization if for every \( x \in \Phi \), we have \( \{ x \} \subset \Phi \). Atom supports and related notions in \( \text{QCoh} X \) are described as follows.

Corollary 7.7. (1) Let \( M \) be an object in \( \text{QCoh} X \). Then the bijection \( |X| \to \text{ASpec} (\text{QCoh} X) \) in Theorem 7.6 (2) induces a bijection \( \text{Supp} M \to \text{ASupp} M \).

(2) For each \( x \in X \), we have \( X(\alpha_x) = X(x) \). The canonical functor \( \text{QCoh} X \to \text{Mod} \mathcal{O}_{X,x} \) induces an equivalence \( (\text{QCoh} X)_{\alpha_x} \cong \text{Mod} \mathcal{O}_{X,x} \).

(3) For each subset \( \Phi \) of \( X \), the corresponding subset
\[ \{ \alpha_x \in \text{ASpec} (\text{QCoh} X) \mid x \in \Phi \} \]
of \( \text{ASpec} (\text{QCoh} X) \) is localizing if and only if \( \Phi \) is closed under specialization.

(4) Let \( x, y \in X \). Then we have \( \alpha_x \leq \alpha_y \) if and only if \( y \in \{ x \} \).

Proof. (1) For each \( x \in X \), by Proposition 7.3 and Proposition 5.6 (2) we have \( \alpha_x \in \text{ASupp} M \) if and only if \( k(x) \in \text{ASupp} j_x^* M \). By Proposition 3.9 this is equivalent to \( m_x \in \text{Supp} j_x^* M \), which means \( M_x = j_x^* M \neq 0 \).

(2) By (1) and Proposition 5.10 (1) we have \( X(\alpha_x) = X(x) \). The equivalence follows from Proposition 7.3.

(3) By (2) it suffices to show that \( \Phi \) is closed under specialization if and only if there exists an object \( M \) in \( \text{QCoh} X \) satisfying \( \Phi = \text{Supp} M \). For every object \( M \) in \( \text{QCoh} X \), it is straightforward to show that \( \text{Supp} M \) is closed under specialization.

Assume that \( \Phi \) is closed under specialization. For each \( x \in \Phi \), we have \( \text{Supp} j_x, k(x) = \{ x \} \). Hence it holds that
\[ \text{Supp} \bigoplus_{x \in \Phi} j_x, k(x) \subseteq \bigcup_{x \in \Phi} \text{Supp} j_x, k(x) = \Phi. \]

(4) This follows from (3). \( \square \)

We specialize Theorem 6.8 to the case of \( \text{QCoh} X \). For a full subcategory \( \mathcal{X} \) of \( \text{QCoh} X \), define the specialization-closed subset \( \text{Supp} \mathcal{X} \) of \( X \) by
\[ \text{Supp} \mathcal{X} = \bigcup_{M \in \mathcal{X}} \text{Supp} M. \]

For a subset \( \Phi \) of \( X \), define the localizing subcategory \( \text{Supp}^{-1} \Phi \) of \( \text{QCoh} X \) by
\[ \text{Supp}^{-1} \Phi = \{ M \in \text{QCoh} X \mid \text{Supp} M \subset \Phi \}. \]

Theorem 7.8. Let \( X \) be a locally noetherian scheme. Then the map
\[ \{ \text{localizing subcategories of } \text{QCoh} X \} \to \{ \text{specialization-closed subsets of } X \} \]
given by \( \mathcal{X} \mapsto \text{Supp} \mathcal{X} \) is bijective. The inverse map is given by \( \Phi \mapsto \text{Supp}^{-1} \Phi \).
Proposition 7.10, it holds that

In Theorem 7.6 (2), we showed that the Grothendieck category QCoh \( X \) has enough atoms and described \( \text{ASpec(QCoh } X) \). Hence the claim follows from Theorem 6.8 and Corollary 7.3 (3). □

Definition 7.9. Let \( X \) be a locally noetherian scheme, and let \( M \) be an object in QCoh \( X \). The subset \( \text{Ass } M \) of \( X \) is defined by

\[
\text{Ass } M = \{ x \in X \mid m_x \in \text{Ass}_{O_{X,x}} M_x \}.
\]

Each element of \( \text{Ass } M \) is called an associated point of \( M \).

In order to show that associated atoms are generalizations of associated points defined in Definition 7.9, we need the following results.

Proposition 7.10. Let Spec \( R \) be an open affine subscheme of \( X \), and let \( i : \text{Spec } R \hookrightarrow X \) be the immersion. For every \( R \)-module \( M \), we have \( \text{Ass } i_! M = i(\text{Ass } R M) \).

Proof. [Gro65 Proposition 3.1.13] and [Gro65 Proposition 3.1.2]. □

Lemma 7.11. For each \( x \in X \), we have \( \text{Ass } j_x E(x) = \{ x \} \) and \( \text{Supp } j_x E(x) = \{ x \} \).

Proof. Let \( i : \text{Spec } R \hookrightarrow X \) be the immersion of an open affine subscheme such that \( x = i(p) \) for some \( p \in \text{Spec } R \). Then the morphism \( j_x \) is the composite of \( j : \text{Spec } O_{X,x} \cong \text{Spec } R_p \hookrightarrow \text{Spec } R \) and \( i : \text{Spec } R \hookrightarrow X \). By [Mat89 Theorem 18.4 (vi)], we have \( j_x E(x) = E_R(R/p) \). By Proposition 7.10 it holds that

\[
\text{Ass } j_x E(x) = \text{Ass } i_* E_R \left( \frac{R}{p} \right) = i_!(\text{Ass } R E_R \left( \frac{R}{p} \right)) = i_!(\{ p \}) = \{ x \}.
\]

By the argument in [Mat89 p. 150], for each \( q \in \text{Spec } R \), we have \( E_R(R/p)_q = E_R((R/p) q) \). Hence we obtain

\[
\text{Supp } E_R \left( \frac{R}{p} \right) = \{ q \in \text{Spec } R \mid p \subseteq q \}
\]

and

\[
\text{Supp } j_x E(x) = \text{Supp } i_! \left( E_R \left( \frac{R}{p} \right) \right) = \{ x \}.
\]

Proposition 7.12. Let \( M \) be an object in QCoh \( X \). Then the bijection \( |X| \rightarrow \text{ASpec(QCoh } X) \) in Theorem 7.6 (2) induces a bijection \( \text{Ass } M \rightarrow \text{AAss } M \).

Proof. Assume that \( \alpha_x \in \text{AAss } M \), and let \( i : U \hookrightarrow X \) be the immersion of an open affine subscheme with \( x \in U \). By Proposition 7.2 and Proposition 5.6 (2), we have \( \alpha_x \in \text{AAss } i_* M \). By Proposition 3.9 and Proposition 7.10, we obtain \( x \in \text{Ass } i_* i^* M \). Since the canonical morphism \( M \rightarrow i_* i^* M \) induces an isomorphism \( M_x \cong i_* i^* M_x \), we deduce that \( x \in \text{Ass } M \).

Conversely, assume that \( x \in \text{Ass } M \). By Theorem 7.3 (2) and Theorem 7.3 (3), there exists a family \( \{ x_{\lambda} \}_{\lambda \in \Lambda} \) of points of \( X \) such that

\[
E(M) \cong \bigoplus_{\lambda \in \Lambda} j_{x_{\lambda}*} E(x_{\lambda}).
\]

By [Gro65 Proposition 3.1.7], it holds that

\[
x \in \text{Ass } M \subset \text{Ass } E(M) = \bigcup_{\lambda \in \Lambda} \text{Ass } j_{x_{\lambda}*} E(x_{\lambda}).
\]

Hence there exists \( \lambda \in \Lambda \) such that \( x \in \text{Ass } j_{x_{\lambda}*} E(x_{\lambda}) \). By Lemma 7.11 we have \( x_{\lambda} = x \). By Proposition 3.12 (2) we deduce that

\[
\alpha_x \in \text{AAss } j_{x_{\lambda}*} E(x) \subset \text{AAss } E(M) = \text{AAss } M.
\]
8. Localization of Prelocalizing Subcategories and Localizing Subcategories

In order to classify the prelocalizing subcategories of $\text{QCoh} \, X$ for a locally noetherian scheme $X$, we show that they are determined by their restrictions to open affine subschemes of $X$. In this section, we prove this claim in a categorical setting (Setting 8.3). We start with two lemmas, which show the setting includes the case of $\text{QCoh} \, X$.

Lemma 8.1. Let $X$ be a locally noetherian scheme, and let $M$ be an object in $\text{QCoh} \, X$. Then for each $y \in \text{Supp} \, M$, there exists $x \in \text{Ass} \, M$ with $y \in \{x\}$.

Proof. By Theorem 7.4 [2] and Theorem 7.4 [3], there exists a family $\{x_\lambda\}_{\lambda \in \Lambda}$ of points in $X$ such that $E(M) \cong \bigoplus_{\lambda \in \Lambda} j_{x_\lambda}^* E(x_\lambda)$.

Then it holds that $y \in \text{Supp} \, M \subset \text{Supp} \, E(M) = \bigcup_{\lambda \in \Lambda} \text{Supp} \, j_{x_\lambda}^* E(x_\lambda)$.

By Lemma 7.11, we have $y \in \text{Supp} \, j_{x_\lambda}^* E(x_\lambda) = \{x_\lambda\}$ for some $\lambda \in \Lambda$. By Proposition 7.12 and Proposition 3.12 [2], we obtain

$$\text{Ass} \, M = \text{Ass} \, E(M) = \bigcup_{\lambda \in \Lambda} \text{Ass} \, j_{x_\lambda}^* E(x_\lambda) = \{x_\lambda \mid \lambda \in \Lambda\}.$$ 

Therefore the claim follows. 

Lemma 8.2. Let $R$ be a commutative ring, and let $S$ be a multiplicatively closed subset of $R$. Let $M$ be an $R$-module. Then the $R$-module $M_S$ is a quotient object of the direct sum of some copies of $M$. In particular, for every $p \in \text{Spec} \, R$, the $R$-module $M_p$ belongs to the prelocalizing subcategory $\langle M \rangle_{\text{preloc}}$ of $\text{Mod} \, R$.

Proof. For each $s \in S$, the image of the $R$-homomorphism $M \to M_S$ given by $x \mapsto xs^{-1}$ is $M_S^{-1}$. Hence the $R$-submodule $M_S^{-1}$ of $M_S$ is a quotient $R$-module of $M$. Since we have $\bigoplus_{s \in S} M_S^{-1} = \sum_{s \in S} M_S^{-1} = M_S$,

the claim follows.

In the rest of this section, we investigate localizations of prelocalizing subcategories in the following setting.

Setting 8.3. Let $\mathcal{A}$ be a Grothendieck category with enough atoms, and let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of localizing subcategories of $\mathcal{A}$. For each $\lambda \in \Lambda$, let $\mathcal{U}_\lambda = \mathcal{A}/X_\lambda$. Denote the canonical functors by

- $F_\lambda : \mathcal{A} \to \mathcal{U}_\lambda$ and $G_\lambda : \mathcal{U}_\lambda \to \mathcal{A}$ for each $\lambda \in \Lambda$,
- $F^\lambda_\mu : \mathcal{U}_\lambda \to \mathcal{U}_\mu$ and $G^\lambda_\mu : \mathcal{U}_\mu \to \mathcal{U}_\lambda$ for each $\lambda, \mu \in \Lambda$ with $\mathcal{U}_\mu \subset \mathcal{U}_\lambda$,
- $F^\alpha : \mathcal{A} \to \mathcal{A}_\alpha$ and $G^\alpha : \mathcal{A}_\alpha \to \mathcal{A}$ for each $\alpha \in \text{ASpec} \, \mathcal{A}$,
- $F^\lambda_\alpha : \mathcal{U}_\lambda \to (\mathcal{U}_\lambda)_\alpha$ and $G^\lambda_\alpha : (\mathcal{U}_\lambda)_\alpha \to \mathcal{U}_\lambda$ for each $\lambda \in \Lambda$ and $\alpha \in \text{ASpec} \, \mathcal{U}_\lambda$. (Note that $(\mathcal{U}_\lambda)_\alpha = \mathcal{A}_\alpha$.)

We assume the following properties.

(1) It holds that $\text{ASpec} \, \mathcal{A} = \bigcup_{\lambda \in \Lambda} \text{ASpec} \, \mathcal{U}_\lambda$.

Moreover, for each $\lambda_1, \lambda_2 \in \Lambda$ and $\alpha \in \text{ASpec} \, \mathcal{U}_{\lambda_1} \cap \text{ASpec} \, \mathcal{U}_{\lambda_2}$, there exists $\mu \in \Lambda$ such that $\alpha \in \text{ASpec} \, \mathcal{U}_\mu \subset \text{ASpec} \, \mathcal{U}_{\lambda_1} \cap \text{ASpec} \, \mathcal{U}_{\lambda_2}$.

In other words, the family $\{\text{ASpec} \, \mathcal{U}_\lambda\}_{\lambda \in \Lambda}$ satisfies the axiom of open basis of $\text{ASpec} \, \mathcal{A}$.

\[ \text{However, we regard ASpec} \, \mathcal{A} \text{ as a topological space only by the localizing topology.} \quad \text{(See Proposition 3.14)} \]
In particular, for every \( \eta \in \Lambda \), and let \( M' \) be an object in \( \mathcal{U}_M \) and \( \alpha \in \text{ASupp} \). Then the object \( G^\lambda_\alpha F^\lambda_\alpha(M') \) belongs to \( \langle M'_\text{preloc} \rangle \).

For a locally noetherian scheme \( X \), let \( \{U_\lambda\}_{\lambda \in \Lambda} \) be an open affine basis of \( X \). Then \[ \text{Lemma 8.2} \]
and \[ \text{Lemma 8.2} \]
show that the Grothendieck category \( \text{QCoh} \) together with \( \{\text{QCoh}U_\lambda\}_{\lambda \in \Lambda} \) satisfies the conditions in \( \text{Setting 8.3} \).

We assume \( \text{Setting 8.3} \) in the rest of this section.

We show that every quotient category of \( \mathcal{A} \) also satisfies the same conditions.

**Proposition 8.4.** Let \( X \) be a localizing subcategory of \( \mathcal{A} \), and let \( F: \mathcal{A} \to \mathcal{A}/X \) and \( G: \mathcal{A}/X \to \mathcal{A} \) denote the canonical functors. Then the Grothendieck category \( \mathcal{A}/X \) together with the family \( \{(F(X_\lambda))_{\text{loc}}\}_{\lambda \in \Lambda} \) of localizing subcategories of \( \mathcal{A}/X \) also satisfies the conditions in \( \text{Setting 8.3} \).

In particular, for every \( \alpha \in \text{ASpec} \), the Grothendieck category \( \mathcal{A}_\alpha \) together with \( \{\langle (X_\lambda)_\alpha \rangle_{\text{loc}}\}_{\lambda \in \Lambda} \) satisfies the conditions in \( \text{Setting 8.3} \).

**Proof.** By \[ \text{Proposition 6.6} \]
(2) the Grothendieck category \( \mathcal{A}/X \) has enough atoms.

(1) By \[ \text{Proposition 5.7} \] we have
\[
\text{ASpec} \frac{\mathcal{A}/X}{(F(X_\lambda))_{\text{loc}}} = \text{ASpec} \frac{\mathcal{A}}{\mathcal{A}/X} \cap \text{ASpec} \frac{\mathcal{A}}{X}.
\]

(2) Let \( M' \) be an object in \( \mathcal{A}/X \), and let \( \beta \in \text{ASupp}M' \). By \[ \text{Proposition 5.6} \]
(1) we have \( \beta \in \text{ASupp}G(M') \). Hence there exists \( \alpha \in \text{Ass} \) \( \text{Ass} \) \( \mathcal{A} \) with \( \alpha \leq \beta \).

(3) Let \( \lambda \in \Lambda \). By \[ \text{Proposition 4.17} \]
(1) and \[ \text{Proposition 4.17} \]
(3) we have
\[
\frac{\mathcal{A}/X}{(F(X_\lambda))_{\text{loc}}} \cong \frac{\mathcal{A}}{\mathcal{X}_{\lambda}} \cap \frac{\mathcal{A}}{\mathcal{X}}.
\]

Let \( U'_\lambda := U_\lambda/(F(X_\lambda))_{\text{loc}} \). Denote the canonical functors by \( F': U_\lambda \to U'_\lambda \), \( G': U'_\lambda \to U_\lambda \), \( F'_\alpha: U'_\lambda \to \mathcal{A}_\alpha \), and \( G'_\alpha: \mathcal{A}_\alpha \to U'_\lambda \). Let \( M'' \) be an object in \( U'_\lambda \), and let \( \alpha \in \text{ASpec} \). Then by the assumption, the object \( G^\lambda_\alpha F^\lambda_\alpha G'(M'') \) belongs to \( \langle G'(M'') \rangle_{\text{preloc}} \). Since \( F' \) is exact, the object \( F' F^\lambda_\alpha G'(M'') \) belongs to \( \langle G'(M'') \rangle_{\text{preloc}} \). Hence the claim follows.

Under the assumptions of \( \text{Setting 8.3} \) we can show a complemental fact on associated atoms in a quotient category.

**Lemma 8.5.** Let \( X \) be a localizing subcategory of \( \mathcal{A} \). The canonical functors are denoted by \( F: \mathcal{A} \to \mathcal{A}/X \) and \( G: \mathcal{A}/X \to \mathcal{A} \). For every object \( M \) in \( \mathcal{A} \), we have
\[
\text{Ass} F(M) = \text{Ass} M \setminus \text{Ass} \mathcal{X}.
\]

In particular, for every \( \alpha \in \text{ASpec} \), we have
\[
\text{Ass} M_\alpha = \text{Ass} M \cap \Lambda(\alpha).
\]

**Proof.** By \[ \text{Proposition 5.6} \] we have
\[
\text{Ass} G F(M) = \text{Ass} F(M) \supset \text{Ass} M \setminus \text{Ass} \mathcal{X}.
\]

Let \( \eta: 1_\mathcal{A} \to G F \) be the unit morphism and \( \beta \in \text{Ass} G F(M) \). Note that \( \beta \notin \text{Ass} \mathcal{X} \). By \[ \text{Proposition 4.8} \]
(2), the subobject \( L := \text{Ker}_{\text{M}} \) of \( M \) belongs to \( \mathcal{X} \), and \( \text{Im} \eta_{\text{M}} \) is an essential subobject of \( \text{GF}(M) \). By \[ \text{Proposition 3.12} \]
(2) we have \( \beta \in \text{Ass}(\text{Im} \eta_{\text{M}}) = \text{Ass}(M/L) \). Hence there exists a subobject \( L' \) of \( M \) with \( L \subset L' \) such that \( L'/L \) is a monomorphism representing \( \beta \). Since we have \( \beta \in \text{Ass} L' \), by \[ \text{Setting 8.3} \]
(2) there exists \( \alpha \in \text{Ass} L' \) with \( \alpha \leq \beta \). Since we have \( \beta \notin \text{Ass} \mathcal{X} \), it holds that \( \alpha \notin \text{Ass} L' \) by \[ \text{Proposition 3.17} \]. By \[ \text{Proposition 3.10} \] and

---

3It is shown in \[ \text{Proposition 8.15} \]
(3) that \( \langle F(X_\lambda) \rangle_{\text{loc}} = F(X_\lambda) \). In particular, it holds that \( \langle (X_\lambda)_\alpha \rangle_{\text{loc}} = (X_\lambda)_\alpha \).
Proposition 3.12 (1) we have $\alpha \in \text{AAss}(L'/L) = \{\beta\}$. Therefore it holds that $\beta = \alpha \in \text{AAss}L' \subset \text{AAss}M$. \hfill \Box

We show two lemmas as parts of the proof of Theorem 8.8. It is useful to determine whether an object belongs to a given prelocalizing subcategory.

Lemma 8.6. Let $\lambda \in \Lambda$, and let $\mathcal{Y}'$ be a prelocalizing subcategory of $\mathcal{U}_\lambda$. Let $U'$ be a uniform object in $\mathcal{U}_\lambda$ with $\text{AAss}U' = \{\alpha\}$. If $U'_\alpha$ belongs to $\mathcal{Y}'_\alpha$, then $U'$ belongs to $\mathcal{Y}'$.

Proof. There exists an object $N'$ in $\mathcal{U}_\lambda$ belonging to $\mathcal{Y}'$ such that $U'_\alpha \cong N'_\alpha$. By Setting 8.3 (3), the object $G^{\lambda}_\alpha F^{\lambda}_\alpha(U') \cong G^{\lambda}_\alpha F^{\lambda}_\alpha(N')$ belongs to $\mathcal{Y}'$. Let $\eta_1: U'_\alpha \to G^{\lambda}_\alpha F^{\lambda}_\alpha$ be the unit morphism. Then by Proposition 4.11 (2) we have $\alpha \notin \text{ASupp}(\ker \eta_1)$. If $\ker \eta_1 \neq 0$, then by Proposition 3.12 (2) we have $\alpha \in \text{AAss}(\ker \eta_1) \subset \text{ASupp}(\ker \eta_1)$. This is a contradiction. Hence $\eta_1$ is a monomorphism. The object $U'$ belongs to $\mathcal{Y}'$. \hfill \Box

Lemma 8.7. Let $\mathcal{Y}$ be a prelocalizing subcategory of $\mathcal{A}$, and let $U$ be a uniform object in $\mathcal{A}$ with $\text{AAss}U = \{\alpha\}$. If $U'_\alpha$ belongs to $\mathcal{Y}'_\alpha$, then $U$ belongs to $\mathcal{Y}'$.

Proof. Let $L$ be the largest subobject of $U$ belonging to $\mathcal{Y}$. Assume that $L \subsetneq U$. Then by Proposition 6.7 there exists $\beta \in \text{AAss}(U/L)$. By Setting 8.3 (2) we have $\alpha \leq \beta$. By Setting 8.3 (1) there exists $\lambda \in \Lambda$ such that $\beta \in \text{ASpec} U_\lambda = \text{ASpec} \mathcal{A} \setminus \text{ASupp} \mathcal{X}_\lambda$. Then by Proposition 3.17 we also have $\alpha \in \text{ASpec} U_\lambda$. By a similar argument to that in the proof of Lemma 8.6, the canonical morphism $U \to G^{\lambda}_\lambda F^{\lambda}_\lambda(U)$ is a monomorphism, and $U$ is $\mathcal{X}_\lambda$-torsionfree.

By Proposition 4.11 (2) the object $F^{\lambda}_\lambda(U)$ is uniform, and $\text{AAss} F^{\lambda}_\lambda(U) = \{\alpha\}$ by Proposition 3.12 (1) Since $F^{\lambda}_\lambda(U) = U_\alpha$ belongs to $\mathcal{Y}'_\alpha = F^{\lambda}_\lambda(\mathcal{Y}_\alpha)$, by Lemma 8.6 the object $F^{\lambda}_\lambda(U)$ belongs to $F^{\lambda}_\lambda(\mathcal{Y})$. We obtain an object $N$ in $\mathcal{A}$ belonging to $\mathcal{Y}$ such that $F^{\lambda}_\lambda(N) \cong F^{\lambda}_\lambda(U)$. Let $V$ be the image of the composite of the canonical morphism $N \to G^{\lambda}_\lambda F^{\lambda}_\lambda(N)$ and $G^{\lambda}_\lambda F^{\lambda}_\lambda(N) \cong G^{\lambda}_\lambda F^{\lambda}_\lambda(U)$. By Proposition 4.8 (2) the object $G^{\lambda}_\lambda F^{\lambda}_\lambda(U)/V$ belongs to $\mathcal{X}_\lambda$. Hence we have

$$
\frac{G^{\lambda}_\lambda F^{\lambda}_\lambda(U) \cap V}{U \cap V} \cong \frac{G^{\lambda}_\lambda F^{\lambda}_\lambda(U)}{U} \oplus \frac{G^{\lambda}_\lambda F^{\lambda}_\lambda(U)}{V} \in \mathcal{X}_\lambda.
$$

Since $U \cap V$ belongs to $\mathcal{Y}$, we have $U \cap V \subset L$ by the maximality of $L$. Hence $U/L$ also belongs to $\mathcal{X}_\lambda$, and $\beta \in \text{ASupp}(U/L) \subset \text{ASupp} \mathcal{X}_\lambda$. This is a contradiction. Therefore we have $L = U$. \hfill \Box

Theorem 8.8. Assume Setting 8.3. Let $\mathcal{Y}$ be a prelocalizing subcategory of $\mathcal{A}$, and let $M$ be an object in $\mathcal{A}$. If $M_\alpha$ belongs to $\mathcal{Y}_\alpha$ for every $\alpha \in \text{AAss} M$, then $M$ belongs to $\mathcal{Y}$.

Proof. Since $\mathcal{A}$ has enough atoms, there exists a family $\{\omega_\alpha\}_{\alpha \in \Omega}$ of elements of $\text{ASpec} \mathcal{A}$ such that

$$
E(M) \cong \bigoplus_{\omega \in \Omega} E(\omega_\alpha).
$$

Let $Z = \langle M \rangle^\preloc$. For each $\omega \in \Omega$, let $L_\omega$ be the largest subobject of $E(\omega_\alpha)$ belonging to $Z$. Then by Proposition 4.4, we have $M \subset \bigoplus_{\omega \in \Omega} L_\omega$. Since $L_\omega$ is uniform for each $\omega \in \Omega$, by Proposition 3.12 it holds that

$$
\{\omega_\alpha\} = \text{AAss}L_\omega \subset \text{AAss}E(M) = \text{AAss}M.
$$

By Proposition 4.12 (2) it is straightforward to show that $Z_{\omega_\alpha} = \langle M_{\omega_\alpha} \rangle^\preloc$. Hence by the assumption, we have $Z_{\omega_\alpha} \subset \mathcal{Y}_{\omega_\alpha}$. Since $L_\omega$ belongs to $Z$, the object $(L_\omega)_{\omega_\alpha}$ belongs to $\mathcal{Y}_{\omega_\alpha}$. By Lemma 8.7, we deduce that $L_\omega$ belongs to $\mathcal{Y}$. Therefore the subobject $M$ of $\bigoplus_{\omega \in \Omega} L_\omega$ also belongs to $\mathcal{Y}$. \hfill \Box

The following results are consequences of Theorem 8.8

Proposition 8.9. Let $\mathcal{X}$ be a localizing subcategory of $\mathcal{A}$. The canonical functors are denoted by $F: \mathcal{A} \to \mathcal{A}/\mathcal{X}$ and $G: \mathcal{A}/\mathcal{X} \to \mathcal{A}$. Then for every object $M$ in $\mathcal{A}$, the object $GF(M)$ belongs to $\langle M \rangle^\preloc$. 

Proof. Let $\eta: 1_A \to GF$ be the unit morphism. Let $\alpha \in \text{AAss } GF(M)$. By Proposition 5.6(1) we have

$$\alpha \in \text{AAss } GF(M) = \text{AAss } F(M) \subset \text{ASpec } \frac{A}{X} = \text{ASpec } A \setminus \text{ASupp } X.$$ 

By Proposition 4.8(2), the objects $\text{Ker } \eta_M$ and $\text{Cok } \eta_M$ belong to $X$. By applying $(-)_\alpha$ to the exact sequence

$$0 \to \text{Ker } \eta_M \to M \to GF(M) \to \text{Cok } \eta_M \to 0,$$

we obtain the isomorphism $M_\alpha \cong GF(M)_\alpha$. Hence $GF(M)_\alpha$ belongs to $\langle (M)_{\text{preloc}} \rangle$. By Theorem 8.8 we deduce that $GF(M)$ belongs to $\langle M \rangle_{\text{preloc}}$.

**Proposition 8.10.** Let $\mathcal{Y}$ be a prelocalizing subcategory of $A$ and $\alpha \in \text{ASpec } A$. Then $\alpha \in \text{ASupp } \mathcal{Y}$ if and only if $H(\alpha)$ belongs to $\mathcal{Y}$.

Proof. If $H(\alpha)$ belongs to $\mathcal{Y}$, then $\alpha = \overline{H(\alpha)} \in \text{ASupp } \mathcal{Y}$.

Assume $\alpha \in \text{ASupp } \mathcal{Y}$. Then there exists a monomorphism $H$ in $A$ with $\overline{H} = \alpha$ such that $H$ belongs to $\mathcal{Y}$. By Proposition 8.9 the object $G_\alpha F_\alpha(H)$ belongs to $\mathcal{Y}$. By the proof of Proposition 6.10(2) the object $G_\alpha F_\alpha(H)$ is isomorphic to $H(\alpha)$.

We show the main result in this section.

**Theorem 8.11.** Assume Setting 8.3. Then there exist bijections between the following sets.

1. The set of prelocalizing subcategories of $A$.
2. The set of families $\{\mathcal{Y}_\lambda \subset U_\lambda\}_{\lambda \in \Lambda}$ of prelocalizing subcategories such that $F_\mu \mathcal{Y}_\lambda = \mathcal{Y}_\mu$ for each $\lambda, \mu \in \Lambda$ such that $\text{ASpec } U_\mu \subset \text{ASpec } U_\lambda$.
3. The set of families $\{\mathcal{Y}(\alpha) \subset A\alpha\}_{\alpha \in \text{ASpec } A}$ of prelocalizing subcategories such that $\mathcal{Y}(\beta) = \mathcal{Y}(\alpha)$ for each $\beta, \alpha \in \text{ASpec } A$ with $\alpha \leq \beta$.

The correspondences are given as follows.

\[ \begin{align*}
(1) \mathcal{Y} & \mapsto \bigg\{ \mathcal{Y}_\lambda \bigg\}_{\lambda \in \Lambda} \\
(2) \bigg\{ \mathcal{Y}_\lambda \bigg\}_{\lambda \in \Lambda} & \mapsto \bigg\{ \bigg( \bigcup_{\lambda \in \Lambda} F_\lambda^{-1}(\mathcal{Y}_\lambda) \bigg) \bigg\}_{\lambda \in \Lambda} \\
(3) \bigg\{ \mathcal{Y}(\alpha) \bigg\}_{\alpha \in \text{ASpec } A} & \mapsto \bigg\{ \bigg( \bigcup_{\alpha \in \text{ASpec } A} F_\alpha^{-1}(\mathcal{Y}(\alpha)) \bigg) \bigg\}_{\lambda \in \Lambda}.
\end{align*} \]

Proof. $(1) \leftrightarrow (2)$. Let $\mathcal{Y}$ be a prelocalizing subcategory of $A$. It is obvious that $\mathcal{Y} \subset \bigcap_{\lambda \in \Lambda} F_\lambda^{-1}F_\lambda(\mathcal{Y})$. Let $M$ be an object in $A$ belonging to $\bigcap_{\lambda \in \Lambda} F_\lambda^{-1}F_\lambda(\mathcal{Y})$. For each $\alpha \in \text{AAss } M$, by Setting 8.3(1) there exists $\lambda \in \Lambda$ such that $\alpha \in \text{ASpec } U_\lambda$. Then it holds that

$$M_\alpha = F_\lambda(M)_\alpha \in F_\lambda(\mathcal{Y}) = \mathcal{Y}.$$ 

By Theorem 8.8 the object $M$ belongs to $\mathcal{Y}$. We obtain $\mathcal{Y} = \bigcap_{\lambda \in \Lambda} F_\lambda^{-1}F_\lambda(\mathcal{Y})$.

Let $\{\mathcal{Y}_\lambda\}_{\lambda \in \Lambda}$ be an element of $(2)$ and $\mathcal{Y} := \bigcap_{\lambda \in \Lambda} F_\lambda^{-1}(\mathcal{Y}_\lambda)$. It is obvious that $F_\lambda(\mathcal{Y}) \subset \mathcal{Y}_\lambda$. Let $M'$ be an object in $U_\lambda$ belonging to $\mathcal{Y}_\lambda$. We show that $F_\lambda G_\lambda(M')$ belongs to $\mathcal{Y}_\lambda$ for each $\lambda'. \alpha \in \text{AAss } F_\lambda G_\lambda(M')$, by Lemma 8.5 and Proposition 6.6(1) we have

$$\beta \in \text{AAss } M' \cap \text{ASpec } U_{\lambda'} \subset \text{ASpec } U_{\lambda} \cap \text{ASpec } U_{\lambda'}. $$

Hence there exists $\mu \in \Lambda$ such that

$$\beta \in \text{ASpec } U_{\mu} \subset \text{ASpec } U_{\lambda} \cap \text{ASpec } U_{\lambda'}.$$
Since the object
\[ F_{\lambda}G_{\lambda}(M')_\beta = G_{\lambda}(M')_\beta = F_{\lambda}G_{\lambda}(M')_\beta = M'_\beta \]
belongs to
\[ (Y_\alpha)_\beta = F^\lambda_\alpha(Y_\lambda)_\beta = (Y_\mu)_\beta = F^\lambda_\mu(Y_\lambda)_\beta = (\Lambda_\lambda)_\beta, \]
by Proposition 8.4 and Theorem 8.8 the object \( F_{\lambda}G_{\lambda}(M') \) belongs to \( Y'_\lambda \). Hence \( G_{\lambda}(M') \) belongs to \( Y \), and \( M' \cong F_{\lambda}G_{\lambda}(M') \) belongs to \( F_{\lambda}(Y') \). We obtain \( F_{\lambda}(Y') = Y_\lambda \).

(Well-definedness of (2)) Let \( \{ Y_\lambda \}_{\lambda \in \Lambda} \) be an element of (2) and \( \alpha \in \text{ASpec} \mathcal{A} \). Let \( \lambda_1, \lambda_2 \in \Lambda \) such that \( \alpha \in \text{ASpec} \mathcal{U}_{\lambda_i} \) for each \( i = 1, 2 \). Then by Setting 8.3(1) there exists \( \mu \in \Lambda \) such that
\[ \alpha \in \text{ASpec} \mathcal{U}_\mu \subset \text{ASpec} \mathcal{U}_{\lambda_1} \cap \text{ASpec} \mathcal{U}_{\lambda_2}. \]

Hence we have
\[ F^\lambda_\mu(Y_\lambda)_\alpha = F^\lambda_\mu(Y_{\lambda_1})_\alpha = (Y_{\lambda_2})_\alpha = (Y_\lambda)_\alpha. \]

(2) Let \( \{ Y_\lambda \}_{\lambda \in \Lambda} \) be an element of (2). For each \( \lambda \in \Lambda \), let
\[ \tilde{Y}_\lambda := \bigcap_{\alpha \in \text{ASpec} \mathcal{U}_\lambda} (F^\lambda_\alpha)^{-1}(F^\lambda_\alpha)(Y_\lambda). \]
Then we have \( Y_\lambda \subset \tilde{Y}_\lambda \). Let \( M' \) be an object in \( \mathcal{U}_\lambda \) belonging to \( \tilde{Y}_\lambda \). For each \( \alpha \in \text{AAss} M' \), we have \( M'_\alpha \subset Y_\lambda \). Hence by Proposition 8.4 and Theorem 8.8 the object \( M' \) belongs to \( Y_\lambda \), and we obtain \( Y_\lambda = \tilde{Y}_\lambda \).

Let \( \{ Y(\alpha) \}_{\alpha \in \text{ASpec} \mathcal{A}} \) be an element of (3). For each \( \lambda \in \Lambda \), let
\[ Y_\lambda := \bigcap_{\alpha \in \text{ASpec} \mathcal{U}_\lambda} (F^\lambda_\alpha)^{-1}(Y(\alpha)). \]
For each \( \alpha \in \text{ASpec} \mathcal{A} \), by Setting 8.3(1) there exists \( \mu \in \Lambda \) such that \( \alpha \in \text{ASpec} \mathcal{U}_\mu \). It is obvious that \( F^\lambda_\mu(Y_\lambda) \subset Y(\alpha) \). Let \( M'' \) be an object in \( \mathcal{A}_\mu \) belonging to \( Y(\alpha) \). We show that \( F^\lambda_\mu G^\lambda_\alpha(M'') \) belongs to \( Y(\beta) \) for each \( \beta \in \text{ASpec} \mathcal{U}_\lambda \). For each \( \gamma \in \text{AAss} F^\lambda_\beta G^\lambda_\alpha(M'') \), by Lemma 8.3 and Proposition 5.6(1) we have
\[ \gamma \in \text{AAss} M'' \cap \Lambda(\beta) \subset \Lambda(\alpha) \cap \Lambda(\beta). \]

Since the object
\[ F^\lambda_\beta G^\lambda_\alpha(M'')_\gamma = G^\lambda_\alpha(M'')_\gamma = F^\lambda_\beta G^\lambda_\alpha(M'')_\gamma = M''_\gamma \]
belongs to
\[ Y(\alpha)_\gamma = Y(\gamma) = Y(\beta)_\gamma, \]
the object \( F^\lambda_\beta G^\lambda_\alpha(M'') \) belongs to \( Y(\beta) \). Hence \( G^\lambda_\alpha(M'') \) belongs to \( Y_\lambda \), and we have \( M'' \cong F^\lambda_\alpha G^\lambda_\alpha(M'') \in F^\lambda_\alpha(Y_\lambda) \). We obtain \( F^\lambda_\beta(Y_\lambda) = Y(\beta). \)

For a family \( \{ Y^\omega \}_{\omega \in \Omega} \) of prelocalizing subcategories of \( \mathcal{A} \), we can consider the smallest prelocalizing subcategory \( \langle \bigcup_{\omega \in \Omega} Y^\omega \rangle_{\text{preloc}} \) containing \( Y^\omega \) for every \( \omega \in \Omega \) and the intersection \( \bigcap_{\omega \in \Omega} Y^\omega \). These are described in terms of prelocalizing subcategories of quotient categories in the following ways.

**Proposition 8.12.** Assume that the following elements correspond to each other by the bijections in Theorem 8.11 for each \( \omega \in \Omega \).

1. \( Y^\omega \).
2. \( \{ Y^\lambda \}_{\lambda \in \Lambda} \).
3. \( \{ Y^\alpha(\alpha) \}_{\alpha \in \text{ASpec} \mathcal{A}} \).

Then the following elements correspond to each other by the bijections.

1. \( \langle \bigcup_{\omega \in \Omega} Y^\omega \rangle_{\text{preloc}} \).
2. \( \{ (\bigcup_{\omega \in \Omega} Y^\lambda)_{\text{preloc}} \}_{\lambda \in \Lambda} \).
3. \( \{ (\bigcup_{\omega \in \Omega} Y^\omega(\alpha))_{\text{preloc}} \}_{\alpha \in \text{ASpec} \mathcal{A}} \).
Proof. For each $\lambda \in \Lambda$, we have
\[
F_\lambda \left( \bigcup_{\omega \in \Omega} \mathcal{Y}_\omega \right)_{\text{preloc}} = \left( \bigcup_{\omega \in \Omega} F_\lambda (\mathcal{Y}_\omega) \right)_{\text{preloc}} = \left( \bigcup_{\omega \in \Omega} \mathcal{Y}_\omega \right)_{\text{preloc}}.
\]
It is shown similarly that $(\bigcup_{\omega \in \Omega} \mathcal{Y}_\omega)_{\text{preloc}} = \bigcup_{\omega \in \Omega} \mathcal{Y}_\omega(\alpha)_{\text{preloc}}$ for each $\alpha \in \text{ASpec } \mathcal{A}$. □

**Proposition 8.13.** Assume that the following elements correspond to each other by the bijections in **Theorem 8.11** for each $\omega \in \Omega$.

1. $\mathcal{Y}_\omega$.
2. $\{ \mathcal{Y}_\omega \}_{\lambda \in \Lambda}$.
3. $\{ \mathcal{Y}_\omega(\alpha) \}_{\alpha \in \text{ASpec } \mathcal{A}}$.

Then the following elements correspond to each other by the bijections.

1. $\bigcap_{\omega \in \Omega} \mathcal{Y}_\omega$.
2. $\{ \bigcap_{\omega \in \Omega} \mathcal{Y}_\omega \}_{\lambda \in \Lambda}$.
3. $\{ \bigcap_{\omega \in \Omega} \mathcal{Y}_\omega(\alpha) \}_{\alpha \in \text{ASpec } \mathcal{A}}$.

**Proof.** Let $\lambda \in \Lambda$. It is obvious that $F_\lambda (\bigcap_{\omega \in \Omega} \mathcal{Y}_\omega) \subset \bigcap_{\omega \in \Omega} F_\lambda (\mathcal{Y}_\omega) = \bigcap_{\omega \in \Omega} \mathcal{Y}_\omega(\lambda)$. Let $M'$ be an object in $\mathcal{U}_\lambda$ belonging to $\bigcap_{\omega \in \Omega} \mathcal{Y}_\omega(\lambda)$. Then for each $\omega \in \Omega$, there exists an object $M_\omega$ in $\mathcal{A}$ belonging to $\mathcal{Y}_\omega$ such that $F_\lambda (M_\omega) \cong M'$. By **Proposition 8.9**, the object $G_\lambda (M') \cong G_\lambda F_\lambda (M_\omega)$ belongs to $\bigcap_{\omega \in \Omega} \mathcal{Y}_\omega$. Hence $G_\lambda (M')$ belongs to $\bigcap_{\omega \in \Omega} \mathcal{Y}_\omega(\lambda)$, and we have $M' \cong F_\lambda G_\lambda (M') \in F_\lambda (\bigcap_{\omega \in \Omega} \mathcal{Y}_\omega(\lambda))$. This shows that $F_\lambda (\bigcap_{\omega \in \Omega} \mathcal{Y}_\omega(\lambda)) = \bigcap_{\omega \in \Omega} \mathcal{Y}_\omega(\lambda)$. It is shown similarly that $(\bigcap_{\omega \in \Omega} \mathcal{Y}_\omega(\alpha))_{\alpha \in \text{ASpec } \mathcal{A}}$ have the following characterization.

Families in **Theorem 8.11** have the following assertions are equivalent.

1. There exists a prelocalizing subcategory $\mathcal{V}$ of $\mathcal{A}$ satisfying $\mathcal{V}_\alpha = \mathcal{V}(\alpha)$ for each $\alpha \in \text{ASpec } \mathcal{A}$.
2. For each $\alpha \in \text{ASpec } \mathcal{A}$, there exist $\lambda \in \Lambda$ with $\alpha \in \text{ASpec } \mathcal{U}_\lambda$ and a prelocalizing subcategory $\mathcal{V}'$ of $\mathcal{U}_\lambda$ satisfying $\mathcal{V}'(\beta) = \mathcal{V}(\beta)$ for each $\beta \in \text{ASpec } \mathcal{U}_\lambda$.
3. For each $\alpha, \beta \in \text{ASpec } \mathcal{A}$ with $\alpha \leq \beta$, it holds that $\mathcal{V}(\beta, \alpha) = \mathcal{V}(\alpha)$.

**Proof.** This can be shown straightforwardly by using **Theorem 8.11**. □

In order to investigate the localizing subcategories of $\text{QCoh } \mathcal{X}$, we improve **Proposition 4.12** under the assumptions of **Setting 8.3**.

**Proposition 8.15.** Let $\mathcal{X}$ be a localizing subcategory of $\mathcal{A}$. The canonical functors are denoted by $F: \mathcal{A} \to \mathcal{A}/\mathcal{X}$ and $G: \mathcal{A}/\mathcal{X} \to \mathcal{A}$.

1. Let $\mathcal{V}$ be a prelocalizing subcategory of $\mathcal{A}$. Then it holds that $\mathcal{V} \ast \mathcal{X} \subset \mathcal{X} \ast \mathcal{V}$.
2. Let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be prelocalizing subcategories of $\mathcal{A}$. Then we have
\[
F(\mathcal{Y}_1 \ast \mathcal{Y}_2) = F(\mathcal{Y}_1) \ast F(\mathcal{Y}_2).
\]
3. Let $\mathcal{Y}$ be a localizing subcategory of $\mathcal{A}$. Then $F(\mathcal{Y})$ is a localizing subcategory of $\mathcal{A}/\mathcal{X}$.

**Proof.** (1) Let $M$ be an object in $\mathcal{A}$ belonging to $\mathcal{Y} \ast \mathcal{X}$. Then there exists an exact sequence
\[
0 \to L \to M \to N \to 0
\]
Proof. This follows from Theorem 8.16.

By Proposition 4.8 (2) the object $\text{Ker} \eta_M$ belongs to $\mathcal{X}$. The subobject $\text{Im} \eta_M$ of $GF(M)$ belongs to $\mathcal{Y}$. Therefore $M$ belongs to $\mathcal{X} \ast \mathcal{Y}$.

(2) By Proposition 4.12 (3) we have

$$F(\mathcal{Y}_1 \ast \mathcal{Y}_2) \subset F(\mathcal{X} \ast \mathcal{Y}_1 \ast \mathcal{Y}_2) \subset F(\mathcal{X}) \ast F(\mathcal{Y}_1 \ast \mathcal{Y}_2) = F(\mathcal{Y}_1 \ast \mathcal{Y}_2).$$

(3) By (2) we have

$$F(\mathcal{Y}) \ast F(\mathcal{Y}) = F(\mathcal{Y} \ast \mathcal{Y}) = F(\mathcal{Y}).$$

\[ \square \]

Theorem 8.16. Assume that the following elements correspond to each other by the bijections in Theorem 8.11 for each $i = 1, 2$.

1. $\mathcal{Y}^i$.
2. $\{\mathcal{Y}_\lambda^i\}_{\lambda \in \Lambda}$.
3. $\{\mathcal{Y}(\alpha)^i\}_{\alpha \in \text{ASpec} \mathcal{A}}$.

Then the following elements correspond to each other by the bijections.

1. $\mathcal{Y}^1 \ast \mathcal{Y}^2$.
2. $\{\mathcal{Y}_\lambda^1 \ast \mathcal{Y}_\lambda^2\}_{\lambda \in \Lambda}$.
3. $\{\mathcal{Y}(\alpha)^1 \ast \mathcal{Y}(\beta)^2\}_{\alpha \in \text{ASpec} \mathcal{A}}$.

Proof. This follows from Proposition 8.15 \[ \square \]

Corollary 8.17. The bijections in Theorem 8.11 induces bijections between following sets.

1. The set of localizing subcategories of $\mathcal{A}$.
2. The set of families $\{\mathcal{Y}_\lambda \subset \mathcal{U}_\lambda\}_{\lambda \in \Lambda}$ of localizing subcategories such that $F^\lambda(\mathcal{Y}_\lambda) = \mathcal{Y}_\mu$ for each $\lambda, \mu \in \Lambda$ with $\text{ASpec} \mathcal{U}_\mu \subset \text{ASpec} \mathcal{U}_\lambda$.
3. The set of families $\{\mathcal{Y}(\alpha) \subset \mathcal{A}_\alpha\}_{\alpha \in \text{ASpec} \mathcal{A}}$ of localizing subcategories such that $\mathcal{Y}(\beta)_{\alpha} = \mathcal{Y}(\alpha)$ for each $\alpha, \beta \in \text{ASpec} \mathcal{A}$ with $\alpha \leq \beta$.

Proof. This follows from Theorem 8.16 \[ \square \]

Prime localizing subcategories of $\mathcal{A}$ are characterized as follows.

Theorem 8.18. Assume Setting 8.3 and let $\mathcal{X}$ be a localizing subcategory of $\mathcal{A}$. Then the following assertions are equivalent.

1. $\mathcal{X}$ is a prime localizing subcategory of $\mathcal{A}$.
2. There exists $\alpha \in \text{ASpec} \mathcal{A}$ such that $\mathcal{X} = \mathcal{X}(\alpha)$.
3. For each family $\{\mathcal{X}_\omega \subset \mathcal{Y}_\omega\}_{\omega \in \Omega}$ of localizing subcategories of $\mathcal{A}$ satisfying $\mathcal{X} = \bigcap_{\omega \in \Omega} \mathcal{X}_\omega$, there exists $\omega \in \Omega$ such that $\mathcal{X} = \mathcal{X}_\omega$.
4. For each family $\{\mathcal{Y}_\omega \subset \mathcal{Y}_\omega\}_{\omega \in \Omega}$ of prelocalizing subcategories of $\mathcal{A}$ satisfying $\mathcal{X} = \bigcap_{\omega \in \Omega} \mathcal{X}_\omega$, there exists $\omega \in \Omega$ such that $\mathcal{X} = \mathcal{Y}_\omega$.

Proof. The equivalence [1] $\iff$ [2] follows from Theorem 5.14.

Let $\{\mathcal{Y}_\omega\}_{\omega \in \Omega}$ be a family of prelocalizing subcategories of $\mathcal{A}$ satisfying $\mathcal{X}(\alpha) = \bigcap_{\omega \in \Omega} \mathcal{Y}_\omega$. Since $H(\alpha)$ does not belong to $\mathcal{X}(\alpha)$, there exists $\omega \in \Omega$ such that $H(\alpha)$ does not belong to $\mathcal{Y}_\omega$. By Proposition 8.10 we have $\alpha \notin \text{ASupp} \mathcal{Y}_\omega$, and hence $\mathcal{Y}_\omega \subset \mathcal{X}(\alpha)$. This shows [2] $\implies$ [4]

The implication [4] $\implies$ [3] is obvious. The implication [3] $\implies$ [2] follows from Corollary 6.9 \[ \square \]
9. Classification of prelocalizing subcategories

Let $X$ be a locally noetherian scheme with the structure sheaf $\mathcal{O}_X$. In this section, we classify the prelocalizing subcategories of $\text{QCoh} X$. Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an open affine basis of $X$. Let $i_\lambda : U_\lambda \hookrightarrow X$ be the immersion for each $\lambda \in \Lambda$, and let $i_{\lambda, \mu} : U_\mu \hookrightarrow U_\lambda$ be the immersion for each $\lambda, \mu \in \Lambda$ with $U_\mu \subset U_\lambda$.

We recall the notion of a filter. This is an essential tool to classify prelocalizing subcategories.

**Definition 9.1.** Let $\mathcal{A}$ be a Grothendieck category, and let $M$ be an object in $\mathcal{A}$.

1. A filter of subobjects of $M$ in $\mathcal{A}$ is a set $\mathcal{F}$ of subobjects of $M$ satisfying the following conditions.
   a. $M \in \mathcal{F}$.
   b. If $L \subset L'$ are subobjects of $M$ with $L \in \mathcal{F}$, then $L' \in \mathcal{F}$.
   c. If $L_1, L_2 \in \mathcal{F}$, then $L_1 \cap L_2 \in \mathcal{F}$.

2. For each subobject $L$ of $M$, denote by $\mathcal{F}(L)$ the filter consisting of all subobjects $L'$ of $M$ with $L \subset L'$. A filter of the form $\mathcal{F}(L)$ is called a principal filter.

**Remark 9.2.** In Definition 9.1 (2), the principal filter $\mathcal{F}(L)$ is closed under arbitrary intersection. Conversely, if a filter $\mathcal{F}$ of $M$ is closed under arbitrary intersection, then $\mathcal{F} = \mathcal{F}(L)$, where $L$ is the smallest element of $\mathcal{F}$.

It is obvious that the map

$$\{\text{subobjects of } M\} \to \{\text{principal filters of } M\}$$

given by $L \mapsto \mathcal{F}(L)$ is bijective.

For a ring $\Lambda$, we say that a filter $\mathcal{F}$ of right ideals of $\Lambda$ is prelocalizing if for each $L \in \mathcal{F}$ and $a \in \Lambda$, the right ideal $a^{-1}L = \{ b \in \Lambda \mid ab \in L \}$ of $\Lambda$ belongs to $\mathcal{F}$. For a ring $\Lambda$, Gabriel [Gab62] gave a classification of the prelocalizing subcategories of $\text{Mod} \, \Lambda$.

**Theorem 9.3** ([Gab62, Lemma V.2.1]). Let $\Lambda$ be a ring. Then the map

$$\{\text{prelocalizing subcategories of } \text{Mod} \, \Lambda\} \to \{\text{prelocalizing filters of right ideals of } \Lambda\}$$

given by

$$\mathcal{Y} \mapsto \left\{ L \subset \Lambda \text{ in } \text{Mod} \, \Lambda \mid \frac{\Lambda}{L} \in \mathcal{Y} \right\}$$

is bijective. The inverse map is given by

$$\mathcal{F} \mapsto \left\{ M \in \text{Mod} \, \Lambda \mid \text{Ann}_\Lambda(x) \in \mathcal{F} \text{ for every } x \in M \right\} = \left\{ \frac{\Lambda}{L} \in \text{Mod} \, \Lambda \mid L \subset \Lambda \text{ in } \text{Mod} \, \Lambda \right\}_{\text{preloc}}.$$

**Proof.** [Pop73, Theorem 4.9.1].

For a commutative ring $R$, every filter $\mathcal{F}$ of $R$ is prelocalizing. Indeed, for $L \in \mathcal{F}$ and $a \in R$, we have $L \subset a^{-1}L$, and hence $a^{-1}L \in \mathcal{F}$. Therefore the following assertion holds.

**Corollary 9.4.** Let $R$ be a commutative ring. Then the map

$$\{\text{prelocalizing subcategories of } \text{Mod} \, R\} \to \{\text{filters of ideals of } R\}$$

given by

$$\mathcal{Y} \mapsto \left\{ I \subset R \text{ in } \text{Mod} \, R \mid \frac{R}{I} \in \mathcal{Y} \right\}$$

is bijective.
is bijective. The inverse map is given by
\[ F \ni \{ M \in \text{Mod} \mathcal{R} \mid \text{Ann}_R(x) \in F \text{ for } x \in M \} \]
\[ = \left\{ \frac{R}{I} \in \text{Mod} \mathcal{R} \mid I \subset R \text{ in } \text{Mod} \mathcal{R} \right\}_{\text{preloc}}. \]

Proof. This is immediate from Theorem 9.3.

In the case of a commutative ring, the case where \( X \) is quasi-compact, there exists an open affine neighborhood \( U \) of \( x \) in \( X \) and \( I' \in F \) such that \( I'|_U \subset I'|_U \) as a subobject of \( O \). Then we have \( I \in F \).

Definition 9.5. Let \( X \) be a locally noetherian scheme. We say that \( F \) is a local filter of \( O_X \) if \( I \) is a subobject of \( O_X \), and assume that for each \( x \in X \), there exist an open affine neighborhood \( U \) of \( x \) in \( X \) and \( I' \in F \) such that \( I'|_U \subset I'|_U \) as a subobject of \( O_U \). Then we have \( I \in F \).

Proposition 9.6. Every principal filter of \( O_X \) is a local filter.

Proof. For every subobject \( I \) of \( O_X \), we show that \( F(I) \) is a local filter. Let \( I' \) be a subobject of \( O_X \) such that for each \( x \in X \), there exist an open affine neighborhood \( U(x) \) of \( x \) in \( X \) and \( J(x) \in F(I) \) such that \( J(x)|_{U(x)} \subset I'|_{U(x)} \). Let \( J := \bigcap_{x \in X} J(x) \) in \( \text{QCoh} X \). Then we have \( J \in F(I) \) and \( J|_{U(x)} \subset I'|_{U(x)} \) for each \( x \in X \). For each open subset \( U \) of \( X \), we have
\[ J(U) = \{ s \in O_X(U) \mid s|_{U(x)} \in J|_{U(x)} \text{ for each } x \in X \} \]
\[ \subset \{ s \in O_X(U) \mid s|_{U(x)} \in I'|_{U(x)} \text{ for each } x \in X \} \]
\[ = I'(U), \]
and hence \( J \subset I' \) follows. This implies that \( I' \in F \).

The next result shows that the local filters of \( O_X \) are exactly the same as the filters of \( O_X \) in the case where \( X \) is quasi-compact. This is the reason that we do not need to consider a local filter in the case of a commutative ring.

Proposition 9.7. If \( X \) is a noetherian scheme, then every filter of \( O_X \) is a local filter.

Proof. Let \( F \) be a filter of \( O_X \). Let \( I \) be a subobject of \( O_X \), and assume that for each \( x \in X \), there exist an open affine neighborhood \( U(x) \) of \( x \) in \( X \) and \( I'(x) \in F \) such that \( I'(x)|_{U(x)} \subset I|_{U(x)} \). Since \( X \) is quasi-compact, there exist \( x_1, \ldots, x_n \in X \) such that \( X = \bigcup_{j=1}^n U(x_j) \). Let \( I := \bigcap_{j=1}^n I'(x_j) \). Then \( I \) belongs to \( F \). Since \( I'|_{U(x_j)} \subset I|_{U(x_j)} \) for each \( j = 1, \ldots, n \), we have \( I \subset I' \), and hence \( I \) also belongs to \( F \).

The following result describes the local filter generated by a set of subobjects of \( O_X \).

Proposition 9.8. Let \( S \) be a set of subobjects of \( O_X \). Let \( F \) be the set consisting of all subobjects \( I \) of \( O_X \) satisfying the following condition: for each \( x \in X \), there exist an open affine neighborhood \( U \) of \( x \) in \( X \) and \( n \in \mathbb{Z}_{\geq 1} \) and \( I_1, \ldots, I_n \in S \) such that
\[ (I_1 \cap \cdots \cap I_n)|_U \subset I|_U. \]
Then \( F \) is the smallest local filter of \( O_X \) including \( S \).

Proof. It is obvious that \( F \) satisfies the conditions [a] and [b] in Definition 9.1(1). We show that [c] is satisfied. Let \( I^{(1)}, I^{(2)} \in F \). Then for each \( j = 1, 2 \) and \( x \in X \), there exist an open affine neighborhood \( U^{(j)} \) of \( x \) in \( X \) and \( n_j \in \mathbb{Z}_{\geq 1} \) and \( I_1^{(j)}, \ldots, I_{n_j}^{(j)} \in S \) such that
\[ (I_1^{(j)} \cap \cdots \cap I_{n_j}^{(j)})|_{U^{(j)}} \subset I^{(j)}|_{U^{(j)}}. \]
Then we have
\[ (I_1^{(1)} \cap \cdots \cap I_{n_1}^{(1)} \cap I_1^{(2)} \cap \cdots \cap I_{n_2}^{(2)})|_{U^{(1)} \cap U^{(2)}} \subset (I^{(1)} \cap I^{(2)})|_{U^{(1)} \cap U^{(2)}}. \]
This shows \( I^{(1)} \cap I^{(2)} \in F \). Hence \( F \) is a filter of \( O_X \).
Let \( I \) be a subobject of \( O_X \) such that for each \( x \in X \), there exist an open affine neighborhood \( U \) of \( x \) in \( X \) and \( I' \in \mathcal{F} \) such that \( I'|_U \subset I|_U \). Let \( x \in X \), and take such \( U \) and \( I' \). Then there exists an open affine neighborhood \( U'' \) of \( x \) in \( X \) and \( n \in \mathbb{Z}_{\geq 1} \) and \( I'_1, \ldots, I'_n \in S \) such that 
\[
(I'_1 \cap \cdots \cap I'_n)|_{U''} \subset I'|_{U''}.
\]
Since we have
\[
(I'_1 \cap \cdots \cap I'_n)|_{U \cap U''} \subset I'|_{U \cap U''} \subset I'|_U \cap I''_U,
\]
it holds that \( I \in \mathcal{F} \). This shows that \( \mathcal{F} \) is a local filter. It is obvious that \( \mathcal{F} \) is the smallest local filter of \( O_X \) including \( S \). \( \square \)

In the setting of Proposition 9.8, the local filter \( \mathcal{F} \) is denoted by \( \langle S \rangle_{\text{loc filt}} \).

We investigate the restriction of a filter to an open affine subscheme and the localization at a point.

**Proposition 9.9.** Let \( \mathcal{F} \) be a filter of \( O_X \).

1. For every \( \lambda \in \Lambda \), the set \( \mathcal{F}|_{U_\lambda} := \{ I|_{U_\lambda} \subset O_{U_\lambda} \mid I \in \mathcal{F} \} \) is a filter of \( O_{U_\lambda} \).
2. For every \( x \in X \), the set \( \mathcal{F}_x := \{ I_x \subset O_{X,x} \mid I \in \mathcal{F} \} \) is a filter of \( O_{X,x} \).

**Proof.**

1. Since \( O_X \in \mathcal{F} \), we have \( O_{U_\lambda} \in \mathcal{F}|_{U_\lambda} \).

Let \( I \subset \tilde{I} \) be subobjects of \( O_{U_\lambda} \) with \( \tilde{I} \in \mathcal{F}|_{U_\lambda} \). By Proposition 4.9 there exists a largest subobject \( I \) (resp. \( I' \)) of \( O_X \) satisfying \( I|_{U_\lambda} \subset \tilde{I} \) (resp. \( I'|_{U_\lambda} \subset \tilde{I} \)), and it holds that \( I|_{U_\lambda} = \tilde{I} \) (resp. \( I'|_{U_\lambda} = \tilde{I} \)). Then \( I \in \mathcal{F} \) and \( I \subset I' \) imply \( I' \in \mathcal{F} \). We deduce that \( \tilde{I} = I'|_{U_\lambda} \in \mathcal{F}|_{U_\lambda} \).

Let \( \tilde{I}_1, \tilde{I}_2 \in \mathcal{F}|_{U_\lambda} \). Then for each \( i = 1, 2 \), there exists \( I_i \in \mathcal{F} \) such that \( I_i|_{U_\lambda} = \tilde{I}_i \). It holds that \( I_1 \cap I_2 \in \mathcal{F} \). Since \((-)|_{U_\lambda} : \text{QCoh } X \to \text{QCoh } U_\lambda \) is an exact functor, we have \( I_1|_{U_\lambda} \cap I_2|_{U_\lambda} = (I_1 \cap I_2)|_{U_\lambda} \in \mathcal{F}|_{U_\lambda} \).

2. This is shown similarly to (1). \( \square \)

We give a characterization of a local filter.

**Proposition 9.10.** Let \( \mathcal{F} \) be a filter of \( O_X \). Then the following assertions are equivalent.

1. \( \mathcal{F} \) is a local filter.
2. Let \( I \) be a subobject of \( O_X \) such that for each \( x \in X \), there exists an open affine neighborhood \( U \) of \( x \) in \( X \) satisfying \( I|_U \in \mathcal{F}|_U \). Then we have \( I \in \mathcal{F} \).

**Proof.** It is obvious that (1) implies (2).

Assume (2). Let \( I \) be a subobject of \( O_X \) such that for each \( x \in X \), there exists an open affine neighborhood \( U \) of \( x \) in \( X \) and \( I' \in \mathcal{F} \) satisfying \( I'|_U \subset I|_U \). Since \( \mathcal{F}|_U \) is a filter of \( O_U \) by Proposition 9.9(1) we have \( I|_U \in \mathcal{F}|_U \). Hence it holds that \( I \in \mathcal{F} \). This shows (1). \( \square \)

The following lemmas show that the bijection in Corollary 9.4 commutes with the restriction to an open affine subscheme and the localization at a point.

**Lemma 9.11.** Let \( \lambda, \mu \in \Lambda \) with \( U_\mu \subset U_\lambda \). Let \( \mathcal{Y} \) be a prelocalizing subcategory of \( \text{QCoh } U_\lambda \), and let \( \mathcal{F} \) be the corresponding filter of \( O_{U_\lambda} \) by the bijection in Corollary 9.4. Then the filter \( \mathcal{F}|_{U_\mu} \) of \( O_{U_\mu} \) corresponds to the prelocalizing subcategory \( \mathcal{Y}|_{U_\mu} \) of \( \text{QCoh } U_\mu \) by the bijection.

**Proof.** Let \( \mathcal{F}' \) be the filter of \( O_{U_\mu} \) corresponding to \( \mathcal{Y}|_{U_\mu} \), that is,
\[
\mathcal{F}' = \left\{ \tilde{I} \subset O_{U_\mu} \mid \frac{O_{U_\mu}}{\tilde{I}} \in \mathcal{Y}|_{U_\mu} \right\}.
\]
It is obvious that $\mathcal{F}|_{U_\mu} \subset \mathcal{F}'$. Let $\tilde{I} \in \mathcal{F}'$. Then there exists an object $M$ in Qcoh $X$ belonging to $\mathcal{Y}$ such that $O_{U_\mu}/I \cong M|_{U_\mu}$. By Proposition 4.9, there exists a subobject $I$ of $O_{U_\lambda}$ such that $I|_{U_\mu} = \tilde{I}$, and $O_{U_\lambda}/I$ is $X'$-torsionfree, where

$$\mathcal{X} = \{ M | M|_{U_\mu} = 0 \}.$$ 

By Proposition 4.8(2), the canonical morphism $O_{U_\lambda}/I \to (i_{\lambda, \mu})_* i_{\lambda, \mu}^* (O_{U_\lambda}/I)$ is a monomorphism. By Proposition 8.9, the object

$$(i_{\lambda, \mu})_* i_{\lambda, \mu}^* \left( \frac{O_{U_\lambda}}{I} \right) \cong (i_{\lambda, \mu})_* \left( \frac{O_{U_\mu}}{I} \right) \cong (i_{\lambda, \mu})_* i_{\lambda, \mu}^* M$$

belongs to $\mathcal{Y}$. Hence $O_{U_\lambda}/I$ also belongs to $\mathcal{Y}$. This shows that $I \in \mathcal{F}$ and that $\tilde{I} = I|_{U_\mu} \in \mathcal{F}|_{U_\mu}$. Therefore we have $\mathcal{F}|_{U_\mu} = \mathcal{F}'$.

Lemma 9.12. Let $x, y \in X$ with $y \in \{ x \}$. Let $\mathcal{Y}$ be a prelocalizing subcategory of $\text{Mod} O_{X,y}$, and let $\mathcal{F}$ be the corresponding filter of $O_{X,y}$ by the bijection in Corollary 9.4. Then the filter $\mathcal{F}_x$ of $O_{X,x}$ corresponds to the prelocalizing subcategory $\mathcal{Y}_x$ of $\text{Mod} O_{X,x}$ by the bijection.

Proof. This is shown similarly to Lemma 9.11.

We show a lemma to glue filters on open affine basis to a local filter of $O_X$.

Lemma 9.13. (1) For every local filter $\mathcal{F}$ of $O_X$, it holds that

$$\mathcal{F} = \{ I \subset O_X \text{ in Qcoh } X | I|_{U_\lambda} \in \mathcal{F}|_{U_\lambda} \text{ for each } \lambda \in \Lambda \}.$$ 

(2) Let $\mathcal{F}_\lambda$ be a filter of $O_{U_\lambda}$ for each $\lambda \in \Lambda$, and assume that $\mathcal{F}_\lambda|_{U_\mu} = \mathcal{F}_\mu$ for each $\lambda, \mu \in \Lambda$ with $U_\mu \subset U_\lambda$. Then there exists a unique local filter $\mathcal{F}$ of $O_X$ satisfying $\mathcal{F}|_{U_\lambda} = \mathcal{F}_\lambda$ for each $\lambda \in \Lambda$.

Proof. (1) This follows from Proposition 9.10. (2) The uniqueness follows from (1). Let

$$\mathcal{F} := \{ I \subset O_X \text{ in Qcoh } X | I|_{U_\lambda} \in \mathcal{F}_\lambda \text{ for each } \lambda \in \Lambda \}.$$ 

It is straightforward to show that $\mathcal{F}$ is a filter of $O_X$ satisfying $\mathcal{F}|_{U_\lambda} \subset \mathcal{F}_\lambda$ for each $\lambda \in \Lambda$.

Let $I$ be a subobject of $O_X$ such that for each $x \in X$, there exists an open affine neighborhood $U$ of $x$ in $X$ satisfying $I|_U \in \mathcal{F}|_U$. For each $\lambda \in \Lambda$ and $y \in U_\lambda$, there exists an open affine neighborhood $U'$ of $y$ in $X$ satisfying $I|_{U'} \in \mathcal{F}|_{U'}$. Take $\mu \in \Lambda$ satisfying $y \in U_\mu \subset U_\lambda \cap U'$. Then we have $(I|_{U'})|_{U_\mu} = (I|_{U'})|_{U_\lambda} \subset (\mathcal{F}|_{U'})|_{U_\mu} = (\mathcal{F}|_{U'})|_{U_\lambda}$. Since $\mathcal{F}|_{U_\lambda}$ is a local filter by Proposition 9.9(1) and Proposition 9.7, we have $I|_{U_\lambda} \in \mathcal{F}|_{U_\lambda} \subset \mathcal{F}_\lambda$. This shows that $I \in \mathcal{F}$. By Proposition 10, the filter $\mathcal{F}$ is a local filter.

We show that $\mathcal{F}_\lambda \subset \mathcal{F}|_{U_\lambda}$. Let $J \in \mathcal{F}_\lambda$. By Proposition 4.9, there exists a subobject $J$ of $O_X$ such that $J|_{U_\lambda} = \tilde{J}$, and $O_X/J$ is $X_\lambda$-torsionfree (see Proposition 7.2). It suffices to show that $J \in \mathcal{F}$, that is, $J|_{U_\nu} \in \mathcal{F}_\nu$ for each $\nu \in \Lambda$. Denote by $\mathcal{Y}_\lambda$ and $\mathcal{Y}_\mu$ the prelocalizing subcategories of Qcoh $U_\lambda$ and Qcoh $U_\mu$, respectively. We show that the object $O_{U_\mu}/J|_{U_\mu}$ belongs to $\mathcal{Y}_\nu$. Let $x \in \text{Ass}_{U_\mu}(O_{U_\mu}/J|_{U_\mu})$. By Lemma 8.5, we have

$$x \in \text{Ass}_{U_\mu}(O_{U_\mu}|_{U_\mu} \cap U_\mu \subset U_\lambda \cap U_\mu).$$

Hence it holds that

$$(O_{U_\mu}|_{U_\mu})_x = \frac{O_{U_\lambda}}{J|_{U_\lambda}}_x = \frac{O_{U_\lambda}}{J|_{U_\lambda}}_x = (\mathcal{Y}_\lambda)_x.$$ 

Take $\nu \in \Lambda$ such that $x \in \mathcal{Y}_\nu \subset U_\lambda \cap U_\mu$. Then we have $O_{U_\nu}|_{U_\mu} = (\mathcal{Y}_\nu|_{U_\mu})_x = (\mathcal{Y}_\nu|_{U_\mu})_x = (\mathcal{Y}_\nu)_x$. Hence by Theorem 8.8, the object $O_{U_\nu}/J|_{U_\nu}$ belongs to $\mathcal{Y}_\nu$. This shows that $J|_{U_\mu} \in \mathcal{F}_\mu$. □
The following theorem is the main result in this section, which gives a classification of the prelocalizing subcategory of $\text{QCoh} X$.

**Theorem 9.14.** Let $X$ be a locally noetherian scheme, and let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an open affine basis of $X$. Then there exist bijections between the following sets.

1. The set of prelocalizing subcategories of $\text{QCoh} X$.
2. The set of families $\{\mathcal{Y}_\lambda \subset \text{QCoh} U_\lambda\}_{\lambda \in \Lambda}$ of prelocalizing subcategories such that $\mathcal{Y}_\lambda|_{U_\mu} = \mathcal{Y}_\mu$ for each $\lambda, \mu \in \Lambda$ with $U_\mu \subset U_\lambda$.
3. The set of families $\{\mathcal{Y}(x) \subset \text{Mod} O_{X,x}\}_{x \in X}$ of prelocalizing subcategories such that $\mathcal{Y}(y)_x = \mathcal{Y}(x)$ for each $x, y \in X$ with $y \in \{x\}$.
4. The set of local filters of $O_X$.
5. The set of families $\{\mathcal{F}_\lambda\}_{\lambda \in \Lambda}$, where $\mathcal{F}_\lambda$ is a filter of $O_{U_\lambda}$ for each $\lambda \in \Lambda$, such that $\mathcal{F}_\lambda|_{U_\mu} = \mathcal{F}_\mu$ for each $\lambda, \mu \in \Lambda$ with $U_\mu \subset U_\lambda$.
6. The set of families $\{\mathcal{F}(x)\}_{x \in X}$, where $\mathcal{F}(x)$ is a filter of $O_{X,x}$ for each $x \in X$, such that $\mathcal{F}(y)_x = \mathcal{F}(x)$ for each $x, y \in X$ with $y \in \{x\}$.

The correspondences are given as follows.

\[
\begin{align*}
\mathcal{Y} & \mapsto \begin{cases} 
\{I \subset O_X \text{ in } \text{QCoh} X \mid O_X^I \subset \mathcal{Y}\} 
\end{cases} \\
\mathcal{F} & \mapsto \begin{cases} 
\{I \subset O_X \text{ in } \text{QCoh} X \mid I\}_{\text{preloc}} \subset \mathcal{F}\end{cases}
\end{align*}
\]

Proof. Theorem 8.11 gives bijections between (1) and (2), Corollary 9.4 and Lemma 9.11 give a bijection between (3) and (5) (resp. (3) and (6)). Lemma 9.13 gives a bijection between (4) and (5).

For a family of prelocalizing subcategories of $\text{QCoh} X$, the supremum and the intersection are described in terms of local filters as follows.

**Proposition 9.15.** Assume that the following elements correspond to each other by the bijections in Theorem 9.14 for each $\omega \in \Omega$.

1. $\mathcal{Y}_\omega$.
2. $\{\mathcal{Y}_\lambda\}_{\lambda \in \Lambda}$.
3. $\{\mathcal{Y}(x)\}_{x \in X}$.
4. $\mathcal{F}_\omega$.
5. $\{\mathcal{F}_\lambda\}_{\lambda \in \Lambda}$.
6. $\{\mathcal{F}(x)\}_{x \in X}$.

Then the following elements correspond to each other by the bijections.

1. $\bigcup_{\omega \in \Omega} \mathcal{Y}_\omega$.
2. $\bigcup_{\omega \in \Omega} \mathcal{Y}_\lambda$.
3. $(\bigcup_{\omega \in \Omega} \mathcal{Y}_\lambda)_{\lambda \in \Lambda}$.
4. $(\bigcup_{\omega \in \Omega} \mathcal{F}_\omega)_{\text{locfilt}}$.
5. $(\bigcup_{\omega \in \Omega} \mathcal{F}_\lambda)_{\text{locfilt}}$.
6. $(\bigcup_{\omega \in \Omega} \mathcal{F}(x))_{\text{locfilt}}$.

Proof. This follows from Proposition 8.12.

**Proposition 9.16.** Assume that the following elements correspond to each other by the bijections in Theorem 9.14 for each $\omega \in \Omega$.
Then the following elements correspond to each other by the bijections.

\[
(1) \bigcap_{\omega \in \Omega} \mathcal{Y}^\omega.
\]
\[
(2) \{ \bigcap_{\omega \in \Omega} \mathcal{Y}^\omega \}_{\lambda \in \Lambda}.
\]
\[
(3) \{ \mathcal{Y}^\omega(x) \}_{x \in X}.
\]
\[
(4) \mathcal{F}^\omega.
\]
\[
(5) \{ \mathcal{F}^\omega \}_{\lambda \in \Lambda}.
\]
\[
(6) \{ \mathcal{F}^\omega(x) \}_{x \in X}.
\]

Proof. This follows from \textbf{Proposition 8.13}. \hfill \Box

We demonstrate a calculation of the prelocalizing subcategories by using \textbf{Theorem 9.14}.

**Example 9.17.** Let \( k \) be an algebraically closed field, and consider the polynomial ring \( k[x] \) with a variable \( x \). For each \( a \in k \), let \( p_a := (x - a) \subset k[x] \) and \( m_a := p_a k[x]_{p_a} \). We have

\[
\text{Spec}k[x] = \{ p_a \mid a \in k \} \cup \{ 0 \}.
\]

Since \( k[x]_{p_a} \) is a discrete valuation ring, the set of ideals of \( k[x]_{p_a} \) is

\[
\{ m_a^i \mid i \in \mathbb{Z}_{\geq 0} \} \cup \{ 0 \},
\]

where \( m_0 = k[x]_{p_a} \). For each \( n \in \mathbb{Z}_{\geq 0} \), define the filter \( \mathcal{F}_a^n \) of \( k[x]_{p_a} \) by

\[
\mathcal{F}_a^n = \{ m_a^i \mid 0 \leq i \leq n \},
\]

and let

\[
\mathcal{F}_a^\infty := \{ m_a^i \mid i \in \mathbb{Z}_{\geq 0} \}, \quad \mathcal{F}_a := \{ m_a^i \mid i \in \mathbb{Z}_{\geq 0} \} \cup \{ 0 \}.
\]

Then the set of filters of \( k[x]_{p_a} \) is

\[
\{ \mathcal{F}_a^n \mid n \in \mathbb{Z}_{\geq 0} \cup \{ \infty \} \} \cup \{ \mathcal{F}_a \}.
\]

Since \( k[x]_0 = k(x) \) is a field, the set of the filters of \( k(x) \) consists of \( \mathcal{F}^\infty = \{ k(x) \} \) and \( \mathcal{F} = \{ 0, k(x) \} \). For each \( a \in k \) and \( n \in \mathbb{Z}_{\geq 0} \cup \{ \infty \} \), we have

\[
(\mathcal{F}_a^n)_0 = \mathcal{F}^\infty, \quad (\mathcal{F}_a)_0 = \mathcal{F}.
\]

Hence the set

\[
\left\{ \left( \mathcal{F}_a^n \right)_{a \in k}, \mathcal{F}^\infty \right\}_{r = \{ r(a) \}_{a \in k} \in \prod_{a \in k} \left( \mathbb{Z}_{\geq 0} \cup \{ \infty \} \right)} \cup \{ (\mathcal{F}_a)_{a \in k}, \mathcal{F} \}
\]

is the set of families of filters which are compatible with localizations. By \textbf{Theorem 9.14} the set of prelocalizing subcategories of \( \text{Mod} k[x] \) is

\[
\left\{ \mathcal{Y}_r \mid r \in \prod_{a \in k} \left( \mathbb{Z}_{\geq 0} \cup \{ \infty \} \right) \right\} \cup \{ \text{Mod} k[x] \},
\]

where

\[
\mathcal{Y}_r = \{ M \in \text{Mod} k[x] \mid M_{p_a} m_0^{r(a)} = 0 \text{ for each } a \in k \text{ with } r(a) \neq \infty \}
\]

for each \( r \in \prod_{a \in k} \left( \mathbb{Z}_{\geq 0} \cup \{ \infty \} \right) \).
10. Classification of localizing subcategories

In this section, we investigate extensions of prelocalizing subcategories (Definition 4.1) in terms of local filters and classify the localizing subcategories of QCoh\(_X\) for a locally noetherian scheme \(X\). The classification is given as a restriction of Theorem 9.14. We start with recalling Gabriel’s classification of the localizing subcategories of Mod\(\Lambda\) for a ring \(\Lambda\).

**Definition 10.1.** Let \(\Lambda\) be a ring.

1. For prelocalizing filters \(\mathcal{F}_1\) and \(\mathcal{F}_2\) of right ideals of \(\Lambda\), define the product \(\mathcal{F}_1 \ast \mathcal{F}_2\) as follows: we have \(L \in \mathcal{F}_1 \ast \mathcal{F}_2\) if and only if there exists \(L_1 \in \mathcal{F}_1\) satisfying \(a^{-1}L \in \mathcal{F}_2\) for every \(a \in L_1\).
2. A prelocalizing filter \(\mathcal{F}\) of right ideals of \(\Lambda\) is called a Gabriel filter if \(\mathcal{F} \ast \mathcal{F} \subset \mathcal{F}\) holds.

**Proposition 10.2.** Let \(\Lambda\) be a ring. Then for each prelocalizing filters \(\mathcal{F}_1\) and \(\mathcal{F}_2\) of right ideals of \(\Lambda\), we have \(\mathcal{F}_1 \subset \mathcal{F}_1 \ast \mathcal{F}_2\) and \(\mathcal{F}_2 \subset \mathcal{F}_1 \ast \mathcal{F}_2\).

**Proof.** Let \(L_1 \in \mathcal{F}_1\). Then we have \(a^{-1}L_1 = \Lambda \in \mathcal{F}_2\) for each \(a \in L_1\). This shows that \(\mathcal{F}_1 \subset \mathcal{F}_1 \ast \mathcal{F}_2\).

Let \(L_2 \in \mathcal{F}_2\). Then \(\Lambda \in \mathcal{F}_1\), and \(a^{-1}L_2 \in \mathcal{F}_2\) for each \(a \in \Lambda\). This shows that \(\mathcal{F}_2 \subset \mathcal{F}_1 \ast \mathcal{F}_2\). \(\square\)

**Theorem 10.3.** (Gabriel, Lemma V.2.1). Let \(\Lambda\) be a ring.

1. For each \(i = 1, 2\), let \(\mathcal{Y}_i\) be a prelocalizing subcategory of Mod\(\Lambda\), and let \(\mathcal{F}_i\) be the prelocalizing filter of right ideals of \(\Lambda\) corresponding to \(\mathcal{Y}_i\) by the bijection in Theorem 9.3. Then \(\mathcal{Y}_1 \ast \mathcal{Y}_2\) corresponds to \(\mathcal{F}_2 \ast \mathcal{F}_1\) by the bijection.
2. The bijection in Theorem 9.3 induces a bijection
   \[
   \{\text{localizing subcategories of Mod}\(\Lambda\)\} \rightarrow \{\text{Gabriel filters of right ideals of } \Lambda\}.
   \]

**Proof.** [Ste75, Theorem VI.5.1]. \(\square\)

The classification is given as a restriction of Theorem 9.14. We start with recalling:

1. Let \(R\) be a commutative noetherian ring. Then the bijection in Corollary 9.4 induces a bijection
   \[
   \{\text{localizing subcategories of Mod}\(R\)\} \rightarrow \{\text{filters of } R \text{ closed under products}\}.
   \]

**Corollary 10.5.** Let \(R\) be a commutative noetherian ring. Then the bijection in Corollary 9.4 induces a bijection
   \[
   \{\text{localizing subcategories of Mod}\(R\)\} \rightarrow \{\text{filters of } R \text{ closed under products}\}.
   \]

**Proof.** This follows from Theorem 10.3 (2) and Proposition 10.4 (2). \(\square\)
In the rest of this section, let $X$ be a locally noetherian scheme, and let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an open affine basis of $X$. For an object $M$ in $\text{QCoh} X$ and a subobject $I$ of $\mathcal{O}_X$, the subobject $MI$ of $M$ is defined as the image of the canonical morphism $M \otimes \mathcal{O}_X I \to M$ in $\text{QCoh} X$.

**Definition 10.6.** (1) Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be local filters of $\mathcal{O}_X$. We define the product $\mathcal{F}_1 \ast \mathcal{F}_2$ by

$$\mathcal{F}_1 \ast \mathcal{F}_2 = \langle I_1 I_2 \subset \mathcal{O}_X \mid I_i \in \mathcal{F}_i \text{ for each } i = 1, 2 \rangle_{\text{locfilt}}.$$

(2) We say that a local filter $\mathcal{F}$ is closed under products if $\mathcal{F} \ast \mathcal{F} \subset \mathcal{F}$ holds.

Note that a local filter $\mathcal{F}$ is closed under products if and only if $I_1, I_2 \in \mathcal{F}$ implies $I_1 I_2 \in \mathcal{F}$. Products of local filters of $\mathcal{O}_X$ commute with the restriction to an open affine subscheme and the localization at a point.

**Lemma 10.7.** Let $\mathcal{F}_i$ be a local filter of $\mathcal{O}_X$ for each $i = 1, 2$.

(1) For every $\lambda \in \Lambda$, we have

$$(\mathcal{F}_1 \ast \mathcal{F}_2)|_{U_\lambda} = \mathcal{F}_1|_{U_\lambda} \ast \mathcal{F}_2|_{U_\lambda}.$$

(2) For every $x \in X$, we have

$$(\mathcal{F}_1 \ast \mathcal{F}_2)_x = (\mathcal{F}_1)_x \ast (\mathcal{F}_2)_x.$$

**Proof.** [1] Let $J \in (\mathcal{F}_1 \ast \mathcal{F}_2)|_{U_\lambda}$. Then there exists $I \in \mathcal{F}_1 \ast \mathcal{F}_2$ such that $I|_{U_\lambda} = J$. For each $x \in U_\lambda$, there exist an open affine neighborhood $U$ of $x$ in $X$ and $I_1 \in \mathcal{F}_1$ and $I_2 \in \mathcal{F}_2$ such that $(I_1 I_2)|_U \subset I|_U$. Hence we have

$$(I_1|_{U_\lambda} I_2|_{U_\lambda})_{|_{U \cap U}} = (I_1 I_2)|_{|_{U \cap U}} \subset I|_{U \cap U} = J|_{U \cap U}.$$

This shows that $J \in (\mathcal{F}_1|_{U_\lambda} \ast \mathcal{F}_2|_{U_\lambda})$.

Conversely, assume $J \in (\mathcal{F}_1|_{U_\lambda} \ast \mathcal{F}_2|_{U_\lambda})$. Then for each $x \in U_\lambda$, there exist an open affine neighborhood $V$ of $x$ in $U$ and $I_1 \in \mathcal{F}_1|_{U_\lambda}$ and $I_2 \in \mathcal{F}_2|_{U_\lambda}$ such that $(I_1 I_2)|_V \subset J|_V$. For each $i = 1, 2$, there exists $I_i \in \mathcal{F}_i$ such that $I_i|_{U_\lambda} = J_i$. Then we have $(I_1 I_2)|_{U_\lambda} \in (\mathcal{F}_1 \ast \mathcal{F}_2)|_{U_\lambda}$, and

$$(I_1 I_2)|_{U_\lambda} = (J_1 J_2)|_V \subset J|_V.$$

Since $(\mathcal{F}_1 \ast \mathcal{F}_2)|_{U_\lambda}$ is a local filter by Proposition 9.9 [1] and Proposition 9.7 we obtain $J \in (\mathcal{F}_1 \ast \mathcal{F}_2)|_{U_\lambda}$.

[2] This can be shown similarly to [1].

We describe extensions of prelocalizing subcategories of $\text{QCoh} X$ in terms of products of local filters.

**Theorem 10.8.** Assume that the following elements correspond to each other by the bijections in [Theorem 9.14] for each $i = 1, 2$.

1. $\mathcal{Y}^i$.
2. $\{\mathcal{Y}_\lambda^i\}_{\lambda \in \Lambda}$.
3. $\{\mathcal{Y}(x)\}_{x \in X}$.
4. $\mathcal{F}^i$.
5. $\{\mathcal{F}_\lambda^i\}_{\lambda \in \Lambda}$.
6. $\{\mathcal{F}(x)\}_{x \in X}$.

Then the following elements correspond to each other by the bijections.

1. $\mathcal{Y}^1 \ast \mathcal{Y}^2$.
2. $\{\mathcal{Y}_\lambda^1 \ast \mathcal{Y}_\lambda^2\}_{\lambda \in \Lambda}$.
3. $\{\mathcal{Y}(x) \ast \mathcal{Y}(x)\}_{x \in X}$.
4. $\mathcal{F}^1 \ast \mathcal{F}^2$.
5. $\{\mathcal{F}_\lambda^1 \ast \mathcal{F}_\lambda^2\}_{\lambda \in \Lambda}$.
6. $\{\mathcal{F}(x) \ast \mathcal{F}(x)\}_{x \in X}$.

**Proof.** This follows from Theorem 8.16 Theorem 10.3 [1] and Lemma 10.7.

**Corollary 10.9.** The bijections in [Theorem 9.14] induce bijections between following sets.

1. The set of localizing subcategories of $\text{QCoh} X$. 


(2) The set of families \( \{ X_\lambda \subset \text{QCoh} U_\lambda \}_{\lambda \in \Lambda} \) of localizing subcategories such that \( X_\lambda |_{U_\mu} = X_\mu \) for each \( \lambda, \mu \in \Lambda \) with \( U_\mu \subset U_\lambda \).

(3) The set of families \( \{ \mathcal{X}(x) \subset \text{Mod} \mathcal{O}_{\mathcal{X}, x} \}_{x \in X} \) of localizing subcategories such that \( \mathcal{X}(y)_x = \mathcal{X}(x) \) for each \( x, y \in X \) with \( y \in \{ x \} \).

(4) The set of local filters of \( \mathcal{O}_X \) closed under products.

(5) The set of families \( \{ \mathcal{F}_\lambda \}_{\lambda \in \Lambda} \) where \( \mathcal{F}_\lambda \) is a filter of \( \mathcal{O}_{U_\lambda} \) closed under products for each \( \lambda \in \Lambda \), such that \( \mathcal{F}_\lambda |_{U_\mu} = \mathcal{F}_\mu \) for each \( \lambda, \mu \in \Lambda \) with \( U_\mu \subset U_\lambda \).

(6) The set of families \( \{ \mathcal{F}(x) \}_{x \in X} \) where \( \mathcal{F}(x) \) is a filter of \( \mathcal{O}_{\mathcal{X}, x} \) closed under products for each \( x \in X \), such that \( \mathcal{F}(y)_x = \mathcal{F}(x) \) for each \( x, y \in X \) with \( y \in \{ x \} \).

Proof. This follows from Theorem 10.8.

We apply Corollary 10.9 to Example 9.17.

Example 10.10. In the setting of Example 9.17, we have

\[
\mathcal{F}_a^m \ast \mathcal{F}_a^n = \mathcal{F}_a^{m+n}
\]

for each \( a \in k \) and \( m, n \in \mathbb{Z}_{\geq 0} \cup \{ \infty \} \). Hence by Corollary 10.9, the set of localizing subcategories of \( \text{Mod} k[x] \) is

\[
\left\{ y_r \ \bigg| \ r \in \prod_{a \in k} \{ 0, \infty \} \right\} \cup \{ \text{Mod} k[x] \}.
\]

In Theorem 7.8, we showed that there exists a bijection between the localizing subcategories of \( \text{QCoh} X \) and the specialization-closed subsets of \( X \). For a local filter \( \mathcal{F} \) of \( \mathcal{O}_X \) closed under products, the corresponding specialization-closed subset of \( X \) is \( \{ x \in X \mid \mathcal{F}_x \neq \{ \mathcal{O}_{\mathcal{X}, x} \} \} \).

Prime localizing subcategories of \( \text{QCoh} X \) are characterized in terms of local filters as follows.

Theorem 10.11. Let \( \mathcal{F} \) be a local filter of \( \mathcal{O}_X \) closed under products. Then the following assertions are equivalent.

1. By the bijection in Theorem 9.14, the local filter \( \mathcal{F} \) corresponds to a prime localizing subcategory of \( \text{QCoh} X \).
2. There exists \( x \in X \) such that

\[
\mathcal{F} = \{ I \subset \mathcal{O}_X \text{ in } \text{QCoh} X \mid I_x = \mathcal{O}_{\mathcal{X}, x} \}.
\]

3. For each family \( \{ \mathcal{F}_\omega \}_{\omega \in \Omega} \) of local filters of \( \mathcal{O}_X \) closed under products satisfying \( \mathcal{F} = \bigcap_{\omega \in \Omega} \mathcal{F}_\omega \), there exists \( \omega \in \Omega \) such that \( \mathcal{F} = \mathcal{F}_\omega \).
4. For each family \( \{ \mathcal{F}_\omega \}_{\omega \in \Omega} \) of local filters of \( \mathcal{O}_X \) satisfying \( \mathcal{F} = \bigcap_{\omega \in \Omega} \mathcal{F}_\omega \), there exists \( \omega \in \Omega \) such that \( \mathcal{F} = \mathcal{F}_\omega \).

Proof. This follows from Theorem 8.18.

11. Classification of closed subcategories

In this section, we investigate the closed subcategories of \( \text{QCoh} X \) for a locally noetherian scheme \( X \), whose definition is as follows.

Definition 11.1. Let \( \mathcal{A} \) be a Grothendieck category. A prelocalizing subcategory \( \mathcal{X} \) of \( \mathcal{A} \) is called a closed subcategory of \( \mathcal{A} \) if \( \mathcal{X} \) is closed under arbitrary direct products.

Note that every Grothendieck category has arbitrary direct products ([Pop73, Corollary 3.7.10]).

Closed subcategories are characterized as follows.

Proposition 11.2. Let \( \mathcal{A} \) be a Grothendieck category, and let \( \mathcal{Y} \) be a prelocalizing subcategory of \( \mathcal{A} \). Then the following assertions are equivalent.

1. \( \mathcal{Y} \) is a closed subcategory of \( \mathcal{A} \).
2. The inclusion functor \( \mathcal{Y} \hookrightarrow \mathcal{A} \) has a left adjoint.
(3) For each object $M$ in $\mathcal{A}$, there exists a smallest subobject $L$ of $M$ satisfying $M/L \in \mathcal{Y}$.

Proof. [Bra14, Lemma 3.16].

For a ring $\Lambda$, Rosenberg [Ros95] showed that there exists a bijection between the closed subcategories of $\text{Mod } \Lambda$ and the two-sided ideals of $\Lambda$. This result can be unified into Theorem 9.3 as follows.

**Theorem 11.3** (Gabriel [Gab62, Lemma V.2.1] and Rosenberg [Ros95, Proposition III.6.4.1]). Let $\Lambda$ be a ring. Then there exist bijections between the following sets.

1. The set of closed subcategories of $\text{Mod } \Lambda$.
2. The set of principal prelocalizing filters of right ideals of $\Lambda$.
3. The set of two-sided ideals of $\Lambda$.

The bijection between (1) and (2) is induced by the bijection in Theorem 9.3.

The bijection between (1) and (3) is given by

$$(1) \rightarrow (3): \mathcal{Y} \mapsto \bigcap_{M \in \mathcal{Y}} \text{Ann}_\Lambda(M),$$

$$(3) \rightarrow (1): I \mapsto \{ M \in \text{Mod } \Lambda \mid MI = 0 \} = \left( \frac{\Lambda}{I} \right)_{\text{preloc}}.$$

Proof. We show that for each right ideal $L$ of $\Lambda$, the principal filter $\mathcal{F}(L)$ of right ideals of $\Lambda$ is prelocalizing if and only if $L$ is a two-sided ideal of $\Lambda$. Assume that $\mathcal{F}(L)$ is prelocalizing. Then for each $a \in \Lambda$, we have $a^{-1}L \in \mathcal{F}(L)$. This implies $L \subseteq a^{-1}L$, and hence $aL \subseteq L$. Therefore $L$ is a two-sided ideal of $\Lambda$. The converse is obvious. The bijection between (2) and (3) follows from Remark 9.2.

Let $\mathcal{Y}$ be a prelocalizing subcategory of $\mathcal{A}$, and let $\mathcal{F}$ be the corresponding prelocalizing filter of right ideals of $\Lambda$. If $\mathcal{Y}$ is a closed subcategory of $\mathcal{A}$, then by Proposition 11.2 there exists a smallest element of $\mathcal{F}$. Hence $\mathcal{F}$ is principal.

Conversely, assume that $\mathcal{F}$ is principal. Then $\mathcal{F} = \mathcal{F}(I)$ for some two-sided ideal $I$ of $\Lambda$. Since we have

$$\mathcal{Y} = \{ M \in \text{Mod } \Lambda \mid I \subseteq \text{Ann}_\Lambda(x) \text{ for each } x \in M \}$$

$$= \{ M \in \text{Mod } \Lambda \mid MI = 0 \},$$

the prelocalizing subcategory $\mathcal{Y}$ of $\mathcal{A}$ is also closed under arbitrary direct products. □

The aim of this section is to generalize Theorem 11.3 to a locally noetherian scheme $X$. Let $\{ U_\lambda \}_{\lambda \in \Lambda}$ be an open affine basis of $X$.

We show a lemma on gluing of subobjects on open affine subschemes.

**Lemma 11.4.** Let $M$ be an object in $\text{QCoh } X$, and let $L_\lambda$ be a subobject of $M|_{U_\lambda}$ for each $\lambda \in \Lambda$. Assume that $L_|_{U_\mu} = L_\mu$ for each $\lambda, \mu \in \Lambda$ with $U_\mu \subseteq U_\lambda$. Then there exists a unique subobject $L$ of $M$ such that $L|_{U_\lambda} = L_\lambda$ for each $\lambda \in \Lambda$.

Proof. (Existence) Define a subsheaf $L$ of $M$ by

$$L(U) = \{ s \in M(U) \mid s|_{U_\lambda} \in L_\lambda(U_\lambda) \text{ for each } \lambda \in \Lambda \text{ with } U_\lambda \subseteq U \}$$

for each open subset $U$ of $X$. It is straightforward to show that $L$ is a subsheaf of $M$ satisfying $L|_{U_\lambda} = L_\lambda$ for each $\lambda \in \Lambda$. In particular, the sheaf $L$ is quasi-coherent.

(Uniformity) Let $L'$ be a subobject of $M$ in $\text{QCoh } X$ such that $L'|_{U_\lambda} = L_\lambda$ for each $\lambda \in \Lambda$. Then we have

$$L'(U) = \{ s \in M(U) \mid s|_{U_\lambda} \in L'(U_\lambda) \text{ for each } \lambda \in \Lambda \text{ with } U_\lambda \subseteq U \} = \{ s \in M(U) \mid s|_{U_\lambda} \in L(U_\lambda) \text{ for each } \lambda \in \Lambda \text{ with } U_\lambda \subseteq U \}$$

for each open subset $U$ of $X$. □
The following lemma shows that for a principal filter of $\mathcal{O}_X$, its restriction to an open affine subscheme and its localization at a point are also principal filters.

**Lemma 11.5.** Let $I$ be a subobject of $\mathcal{O}_X$.

1. For every $\lambda \in \Lambda$, we have $F(I)|_{U_\lambda} = F(I_{U_\lambda})$.
2. For every $x \in X$, we have $F(I)_x = F(I_x)$.

**Proof.** (1) For each $J' \in F(I)|_{U_\lambda}$, there exists $J \in F(I)$ such that $J|_{U_\lambda} = J'$. Since we have $I \subseteq J$, it holds that $I_{U_\lambda} \subseteq J_{U_\lambda} = J'$. This shows that $F(I)|_{U_\lambda} \subseteq F(I)$ by **Proposition 9.9**. (1)

If it follows from $I \in F(I)$ that $I_{U_\lambda} \subseteq F(I)$, since $F(I)|_{U_\lambda}$ is a filter of $\mathcal{O}_{U_\lambda}$ by **Proposition 9.9** (1), we have $F(I)|_{U_\lambda} = F(I_{U_\lambda})$.

(2) This is shown similarly by using **Proposition 9.9** (2).

Conversely, if the restriction of a local filter of $\mathcal{O}_X$ to each open affine subscheme $U_\lambda$ is principal, then the local filter is principal.

**Lemma 11.6.** Let $F$ be a local filter of $\mathcal{O}_X$. Then $F$ is a principal filter if and only if the filter $F|_{U_\lambda}$ of $\mathcal{O}_{U_\lambda}$ is principal for every $\lambda \in \Lambda$.

**Proof.** If $F$ is a principal filter, then $F|_{U_\lambda}$ is a principal filter for every $\lambda \in \Lambda$ by **Lemma 11.5** (1).

Assume that there exists a subobject $I_\lambda$ of $\mathcal{O}_{U_\lambda}$ such that $F|_{U_\lambda} = F(I_\lambda)$ for each $\lambda \in \Lambda$. For each $\lambda, \mu \in \Lambda$ with $U_\mu \subseteq U_\lambda$, we have

$$F(I_{U_\mu} \cap I_{U_\lambda}) = F(I_{U_\lambda}) = (F|_{U_\lambda})_\mu = F|_{U_\mu} = F(I_\mu).$$

Hence it holds that $I_{|U_\mu} = I_\mu$. By **Lemma 11.4**, there exists a subobject $I$ of $\mathcal{O}_X$ such that $I|_{U_\mu} = I_\mu$ for each $\lambda \in \Lambda$. Since we have $F(I)|_{U_\lambda} = F(I_{U_\lambda}) = F(I_\lambda) = F_\lambda$ for each $\lambda \in \Lambda$, it follows from **Lemma 9.13** (2) that $F(I) = F$.

**Remark 11.7.** Let $F$ be a local filter of $\mathcal{O}_X$. Even if $F_x$ is a principal filter of $\mathcal{O}_{X,x}$ for each $x \in X$, the local filter $F$ is not necessarily a principal filter. A counter-example is given in **Example 11.12**.

We characterize closed subcategories of $\text{QCoh} X$ in terms of local filters.

**Lemma 11.8.** Let $Y$ be a prelocalizing subcategory of $\text{QCoh} X$, and let $F$ be the corresponding local filter of $\mathcal{O}_X$ by the bijection in **Theorem 9.14**. Then $Y$ is a closed subcategory of $\text{QCoh} X$ if and only if $F$ is a principal filter. If $F = F(I)$ for a subobject $I$ of $\mathcal{O}_X$, then $I$ is the smallest subobject of $\mathcal{O}_X$ satisfying $\mathcal{O}_X/I \in Y$, and we have

$$Y = \{ M \in \text{QCoh} X \mid MI = 0 \}.$$ 

**Proof.** Assume that $Y$ is a closed subcategory of $\text{QCoh} X$. Then by **Proposition 11.2**, there exists a smallest subobject $I$ of $\mathcal{O}_X$ satisfying $\mathcal{O}_X/I \in Y$. Hence we have $F = F(I)$.

Conversely, assume that $F = F(I)$ for some subobject $I$ of $\mathcal{O}_X$. Then for each $\lambda \in \Lambda$, we have $F|_{U_\lambda} = F(I|_{U_\lambda})$ by **Lemma 11.5** (1) and hence

$$Y|_{U_\lambda} = \{ M' \in \text{QCoh} U_\lambda \mid M'|I_{U_\lambda} = 0 \}$$

by **Theorem 11.3**. By **Theorem 9.14**, we have

$$Y = \{ M \in \text{QCoh} X \mid M|_{U_\lambda} \in Y|_{U_\lambda} \text{ for every } \lambda \in \Lambda \}$$

$$= \{ M \in \text{QCoh} X \mid M|_{U_\lambda} I_{U_\lambda} = 0 \text{ for every } \lambda \in \Lambda \}$$

$$= \{ M \in \text{QCoh} X \mid MI = 0 \}.$$ 

For each object $M$ in $\text{QCoh} X$, the subobject $MI$ of $M$ is the smallest among the subobjects $L$ of $M$ satisfying $(M/L)I = 0$. Therefore $Y$ is a closed subcategory of $\text{QCoh} X$.

As in **Remark 11.7**, the same type of theorem as **Corollary 10.9** does not hold for the closed subcategories. For this reason, we use the characterization in **Proposition 8.14** in order to obtain a generalization to the closed subcategories.
Theorem 11.9. Let $X$ be a locally noetherian scheme, and let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an open affine basis of $X$. Then there exist bijections between the following sets.

1. The set of closed subcategories of $\text{QCoh } X$.
2. The set of families $\{Y_\lambda \subset \text{QCoh } U_\lambda\}_{\lambda \in \Lambda}$ of closed subcategories such that $Y_\lambda | U_\mu = Y_\mu$ for each $\lambda, \mu \in \Lambda$ with $U_\mu \subset U_\lambda$.
3. The set of families $\{\mathcal{V}(x) \subset \text{Mod } O_X \}_{x \in X}$ of closed subcategories such that for each $x \in X$, there exist $\lambda \in \Lambda$ with $x \in U_\lambda$ and a closed subcategory $\mathcal{V}'$ of $\text{QCoh } U_\lambda$ satisfying $\mathcal{V}_y' = \mathcal{V}(y)$ for each $y \in U_\lambda$.
4. The set of principal filters of $O_X$.
5. The set of families $\{F_\lambda\}_{\lambda \in \Lambda}$, where $F_\lambda$ is a principal filter of $O_{U_\lambda}$ for each $\lambda \in \Lambda$, such that $F_\lambda | U_\mu = F_\mu$ for each $\lambda, \mu \in \Lambda$ with $U_\mu \subset U_\lambda$.
6. The set of families $\{F(x)\}_{x \in X}$, where $F(x)$ is a principal filter of $O_{X,x}$ for each $x \in X$, such that for each $x \in X$, there exist $\lambda \in \Lambda$ with $x \in U_\lambda$ and a principal filter $F'$ of $O_{U_\lambda}$ satisfying $F'_y = F(y)$ for each $y \in U_\lambda$.
7. The set of subobjects of $O_X$.
8. The set of families $\{I_\lambda\}_{\lambda \in \Lambda}$, where $I_\lambda$ is a subobject of $O_{U_\lambda}$ for each $\lambda \in \Lambda$, such that $I_\lambda | U_\mu = I_\mu$ for each $\lambda, \mu \in \Lambda$ with $U_\mu \subset U_\lambda$.
9. The set of families $\{I(x)\}_{x \in X}$, where $I(x)$ is an ideal of $O_{X,x}$ for each $x \in X$, such that for each $x \in X$, there exist $\lambda \in \Lambda$ with $x \in U_\lambda$ and a subobject $I'$ of $O_{U_\lambda}$ satisfying $I'_y = I(y)$ for each $y \in U_\lambda$.

The bijections between the sets (1), (2), (3), (4), (5), (6) are induced by Theorem 9.14. The bijections (7) and (8) are defined by the bijection $L \mapsto f(L)$ in Remark 9.3.

Proof. This follows from Theorem 9.14 Theorem 11.3 Lemma 11.6 and Lemma 11.8.

We establish a bijection between the closed subcategories of $\text{QCoh } X$ and the closed subschemes of $X$ by using the following fact.

Proposition 11.10. The map

$$\{\text{subobjects of } O_X\} \rightarrow \{\text{closed subschemes of } X\}$$

given by $I \mapsto (\text{Supp}(O_{X/I}), i^{-1}(O_X/I))$, where $i : \text{Supp}(O_{X/I}) \hookrightarrow X$ is the immersion, is bijective. For each closed subscheme $Y$ of $X$, the corresponding subobject $I$ of $O_X$ is given by the exact sequence

$$0 \rightarrow I \rightarrow O_X \rightarrow i_* O_Y \rightarrow 0,$$

where $i : Y \hookrightarrow X$ is the immersion, and $O_X \rightarrow i_* O_Y$ is the canonical morphism.

Proof. [Har77] Proposition II.5.9.

Theorem 11.11. Let $X$ be a locally noetherian scheme. Then there exists a bijection between

1. The set of closed subcategories of $\text{QCoh } X$.
2. The set of closed subschemes of $X$.

For each closed subscheme $Y$ of $X$ with the immersion $i : Y \hookrightarrow X$, the functor $i_* : \text{QCoh } Y \rightarrow \text{QCoh } X$ is fully faithful and induces an equivalence between $\text{QCoh } Y$ and the closed subcategory of $\text{QCoh } X$ corresponding to $Y$.

Proof. The bijection is obtained by [Theorem 11.9] and [Proposition 11.10] By [Gro60] 0.5.1.4, [Gro60] Proposition I.5.5.1 (ii), and [Gro60] Corollary I.9.2.2 (a)], we have the functor $i^* : \text{QCoh } X \rightarrow \text{QCoh } Y$ and its right adjoint $i_* : \text{QCoh } Y \rightarrow \text{QCoh } X$. It is straightforward to show that the counit morphism $i^* i_* \rightarrow 1_{\text{QCoh } Y}$ is an isomorphism. Hence $i_*$ is fully faithful. An object $M$ in $\text{QCoh } X$ is isomorphic to the image of an object in $\text{QCoh } Y$ by $i_*$ if and only if the canonical morphism $M \rightarrow i_* i^* M$ is an isomorphism. Let $I$ be the subobject of $O_X$ corresponding to $Y$. Since we have the exact sequence

$$0 \rightarrow MI \rightarrow M \rightarrow i_* i^* M \rightarrow 0,$$
$M \rightarrow i_i^* M$ is an isomorphism if and only if $MI = 0$. Therefore the claim follows. □

**Example 11.12.** We follow the notations in [Example 9.17](#) and [Example 10.10](#). Each nonzero proper ideal $I$ of $k[x]$ is generated by an element of the form $(x - a_1)^{r_1} \cdots (x - a_l)^{r_l}$, where $l \in \mathbb{Z}_{\geq 1}$, $a_1, \ldots, a_l$ are distinct elements of $k$, and $r_1, \ldots, r_l \in \mathbb{Z}_{\geq 1}$. We have

$$I_p = \left\{ \frac{m_{a_i}^i}{k[x]_p} \text{ if } a = a_i \text{ for some } i \in \{1, \ldots, l\}, \quad \frac{a}{k[x]_p} \text{ if } a \in k \setminus \{a_1, \ldots, a_l\} \right\}.$$ 

For each $r \in \prod_{a \in k} (\mathbb{Z}_{\geq 0} \cup \{\infty\})$, the object $k[x]/I$ belongs to $\mathcal{Y}_r$ if and only if $r_i \leq r(a_i)$ for every $i = 1, \ldots, l$. Hence the corresponding filter of $k[x]$ to $\mathcal{Y}_r$ is

$$\left\{ (x - a_1)^{r_1} \cdots (x - a_l)^{r_l} \subset k[x] \right\} \bigcup \{k[x]\},$$

This is equal to

$$\left\{ (x - a)^r \subset k[x] \mid a \in k, r \in \mathbb{Z}_{\geq 1}, r \leq r(a) \right\}_{\text{locfilt}},$$

and we have the description

$$\mathcal{Y}_r = \left\{ \frac{k[x]}{(x - a)^r} \mid a \in k, r \in \mathbb{Z}_{\geq 1}, r \leq r(a) \right\}_{\text{preloc}}.$$ 

By [Theorem 11.9](#) the set of closed subcategories of $\text{Mod } k[x]$ is

$$\left\{ \mathcal{Y}_r \mid r \in \bigoplus_{a \in k} \mathbb{Z}_{\geq 0} \right\} \cup \{\text{Mod } k[x]\}.$$ 

Let $r \in (\prod_{a \in k} \mathbb{Z}_{\geq 0}) \setminus (\bigoplus_{a \in k} \mathbb{Z}_{\geq 0})$. Then for every $p \in \text{Spec } k[x]$, the prelocalizing subcategory $(\mathcal{Y}_r)_p$ of $\text{Mod } k[x]_p$ is a closed subcategory, and $\mathcal{F}_p$ is a principal filter of $k[x]_p$. However, the prelocalizing subcategory $\mathcal{Y}_r$ of $\text{Mod } k[x]$ is not a closed subcategory, and the corresponding local filter of $k[x]$ is not a principal filter.

### 12. Classification of Bilocalizing Subcategories

Let $X$ be a locally noetherian scheme. We investigate extensions of closed subcategories. The following lemma shows that products of principal filters are also principal.

**Lemma 12.1.** Let $I_1$ and $I_2$ be subobjects of $O_X$. Then it holds that $\mathcal{F}(I_1) * \mathcal{F}(I_2) = \mathcal{F}(I_1 I_2)$.

**Proof.** This follows from [Proposition 9.6](#). □

Extensions of closed subcategories are described in terms of products of principal filters.

**Theorem 12.2.** Assume that the following elements correspond to each other by the bijections in [Theorem 11.9](#) for each $i = 1, 2$.

\begin{align*}
(1) \ & \mathcal{Y}^i, \\
(2) \ & \{\mathcal{Y}^i_1 \}_{\lambda \in \Lambda}, \\
(3) \ & \{\mathcal{Y}^i(x)\}_{x \in X}, \\
(4) \ & \mathcal{F}^i, \\
(5) \ & \{\mathcal{F}^i_1 \}_{\lambda \in \Lambda}, \\
(6) \ & \{\mathcal{F}^i(x)\}_{x \in X}, \\
(7) \ & I^i, \\
(8) \ & \{I^i_1 \}_{\lambda \in \Lambda}, \\
(9) \ & \{I^i(x)\}_{x \in X}.
\end{align*}

Then the following elements correspond to each other by the bijections.

\begin{align*}
(1) \ & \mathcal{Y}^1 \ast \mathcal{Y}^2, \\
(2) \ & \{\mathcal{Y}^1_1 \ast \mathcal{Y}^2_1 \}_{\lambda \in \Lambda}, \\
(3) \ & \{\mathcal{Y}^1(x) \ast \mathcal{Y}^2(x)\}_{x \in X}, \\
(4) \ & \mathcal{F}^1 \ast \mathcal{F}^2, \\
(5) \ & \{\mathcal{F}^1_1 \ast \mathcal{F}^2_1 \}_{\lambda \in \Lambda}, \\
(6) \ & \{\mathcal{F}^1(x) \ast \mathcal{F}^2(x)\}_{x \in X}, \\
(7) \ & I^1 I^2, \\
(8) \ & \{I^1_1 I^2_1 \}_{\lambda \in \Lambda}, \\
(9) \ & \{I^1(x) I^2(x)\}_{x \in X}.
\end{align*}

**Proof.** This follows from [Theorem 10.8](#) and [Lemma 12.1](#). □

As a corollary of [Theorem 12.2](#) we obtain a classification of the bilocalizing subcategories of $\text{QCoh } X$. They are defined as follows.
Definition 12.3. Let $\mathcal{A}$ be a Grothendieck category. A prelocalizing subcategory $\mathcal{X}$ of $\mathcal{A}$ is called a bilocalizing subcategory of $\mathcal{A}$ if $\mathcal{X}$ is both localizing and closed.

Bilocalizing subcategories have the following characterization.

Proposition 12.4. Let $\mathcal{A}$ be a Grothendieck category, and let $\mathcal{X}$ be a localizing subcategory of $\mathcal{A}$. Then $\mathcal{X}$ is a bilocalizing subcategory of $\mathcal{A}$ if and only if the canonical functor $\mathcal{A} \to \mathcal{A}/\mathcal{X}$ has a left adjoint.

Proof. [Pop73 Theorem 4.21.1]. □

For a ring $\Lambda$, the bilocalizing subcategories of $\text{Mod} \, \Lambda$ are classified by the idempotent two-sided ideals of $\Lambda$.

Definition 12.5. Let $\Lambda$ be a ring. A two-sided ideal $I$ of $\Lambda$ is called idempotent if $I^2 = I$ holds.

Proposition 12.6. Let $\Lambda$ be a ring.

1. For each $i = 1, 2$, let $\mathcal{Y}_i$ be a closed subcategory of $\text{Mod} \, \Lambda$, and let $I_i$ be the corresponding two-sided ideal of $\Lambda$ by the bijection in Theorem 11.9. Then $\mathcal{Y}_1 \ast \mathcal{Y}_2$ corresponds to $I_2I_1$ by the bijection.
2. The bijection in Theorem 11.9 induces a bijection

\[
\{ \text{bilocalizing subcategories of } \text{Mod} \, \Lambda \} \to \{ \text{idempotent two-sided ideals of } \Lambda \}.
\]

Proof. \([1]\) For two-sided ideals $I_1$ and $I_2$ of $\Lambda$, it is straightforward to show that $\mathcal{F}(I_1) \ast \mathcal{F}(I_2) = \mathcal{F}(I_1I_2)$. Therefore the claim follows from Theorem 11.9(1), (2). This follows from \([1]\) □

A subobject $I$ of $\mathcal{O}_X$ is called idempotent if $I^2 = I$ holds. We classify the bilocalizing subcategories of $\text{QCoh} \, X$ as follows.

Corollary 12.7. The bijections in Theorem 11.9 induces bijections between following sets.

1. The set of bilocalizing subcategories of $\text{QCoh} \, X$.
2. The set of families $\{ \mathcal{Y}_\lambda \subseteq \text{QCoh} \, U_\lambda \}_{\lambda \in \Lambda}$ of bilocalizing subcategories such that $\mathcal{Y}_\lambda|_{U_\mu} = \mathcal{Y}_\mu$ for each $\lambda, \mu \in \Lambda$ with $U_\mu \subseteq U_\lambda$.
3. The set of families $\{ \mathcal{Y}(x) \subseteq \text{Mod} \, \mathcal{O}_{X,x} \}_{x \in X}$ of bilocalizing subcategories such that for each $x \in X$, there exist $\lambda \in \Lambda$ with $x \in U_\lambda$ and a bilocalizing subcategory $\mathcal{Y}'$ of $\text{QCoh} \, U_\lambda$ satisfying $\mathcal{Y}'|_y = \mathcal{Y}(y)$ for each $y \in U_\lambda$.
4. The set of principal filters of $\mathcal{O}_X$ closed under products.
5. The set of families $\{ \mathcal{F}_\lambda \}_{\lambda \in \Lambda}$, where $\mathcal{F}_\lambda$ is a principal filter of $\mathcal{O}_{U_\lambda}$ closed under products for each $\lambda \in \Lambda$, such that $\mathcal{F}_\lambda|_{U_\mu} = \mathcal{F}_\mu$ for each $\lambda, \mu \in \Lambda$ with $U_\mu \subseteq U_\lambda$.
6. The set of families $\{ \mathcal{F}(x) \}_{x \in X}$, where $\mathcal{F}(x)$ is a principal filter of $\mathcal{O}_{X,x}$ closed under products for each $x \in X$, such that for each $x \in X$, there exist $\lambda \in \Lambda$ with $x \in U_\lambda$ and a principal filter of subobjects $\mathcal{F}'$ of $\mathcal{O}_{U_\lambda}$ which is closed under products and satisfies $\mathcal{F}'|_y = \mathcal{F}(y)$ for each $y \in U_\lambda$.
7. The set of idempotent subobjects of $\mathcal{O}_X$.
8. The set of families $\{ \mathcal{I}_\lambda \}_{\lambda \in \Lambda}$, where $\mathcal{I}_\lambda$ is an idempotent subobjects of $\mathcal{O}_{U_\lambda}$ for each $\lambda \in \Lambda$, such that $\mathcal{I}_\lambda|_{U_\mu} = \mathcal{I}_\mu$ for each $\lambda, \mu \in \Lambda$ with $U_\mu \subseteq U_\lambda$.
9. The set of families $\{ \mathcal{I}(x) \}_{x \in X}$, where $\mathcal{I}(x)$ is an idempotent ideal of $\mathcal{O}_{X,x}$ for each $x \in X$, such that for each $x \in X$, there exist $\lambda \in \Lambda$ with $x \in U_\lambda$ and an idempotent subobject $\mathcal{I}'$ of $\mathcal{O}_{U_\lambda}$ satisfying $\mathcal{I}'|_y = \mathcal{I}(y)$ for each $y \in U_\lambda$.

Proof. This follows from Theorem 12.2 □

Example 12.8. In the setting of Example 11.12 the set of bilocalizing subcategories of $\text{Mod} \, k[x]$ is

\[
\{ \mathcal{Y}_r \mid r = \{ 0 \}_{a \in k} \} \cup \{ \text{Mod} \, k[x] \} = \{ 0, \text{Mod} \, k[x] \}.
\]
Lemma 12.9. Let $R$ be a commutative noetherian ring, and let $I$ be an idempotent ideal of $R$. Then there exists an ideal $J$ of $R$ such that $R = I \oplus J$ in Mod$R$. In particular, the subset $\text{Supp}(R/I)$ of $\text{Spec} R$ is open and closed.

Proof. By Nakayama’s lemma ([Mat89, Theorem 2.2]), there exists $a \in R$ such that $aI = 0$ and $1 - a \in I$. Then we have $a^2 = a$ and $aR = J$. By letting $J = (1 - a)R$, we obtain $R = I \oplus J$, and $\text{Spec} R$ is the disjoint union of the closed subsets $V(I)$ and $V(J)$ determined by $I$ and $J$, respectively. \qed

The idempotence of a subobject of $\mathcal{O}_X$ is characterized in terms of the corresponding closed subscheme.

Lemma 12.10. Let $X$ be a locally noetherian scheme. Let $I$ be a subobject of $\mathcal{O}_X$, and let $Y$ be the corresponding closed subscheme of $X$ by the bijection in Proposition 11.10. Then $I$ is idempotent if and only if $Y$ is also an open subscheme of $X$.

Proof. Assume that $I$ is idempotent. For each open affine subscheme $U$ of $X$, the subobject $I|_U$ of $\mathcal{O}_U$ is idempotent. By Lemma 12.9 the subset $\text{Supp}(\mathcal{O}_U/I|_U)$ of $U$ is open and closed. Since we have

$$\text{Supp} \frac{\mathcal{O}_U}{I|_U} = U \cap \text{Supp} \frac{\mathcal{O}_X}{I},$$

the underlying space $\text{Supp}(\mathcal{O}_X/I)$ of $Y$ is an open subset of $X$. For each $y \in Y$, the ideal $I_y$ of $\mathcal{O}_{X,y}$ is idempotent, and $(\mathcal{O}_X/I)_y \neq 0$. Hence $I_y = 0$. It follows that $\mathcal{O}_Y = (\mathcal{O}_X/I)|_Y = \mathcal{O}_X|_Y$.

Conversely, assume that $Y$ is also an open subscheme. Let $i: Y \hookrightarrow X$ be the immersion. Then we have the exact sequence

$$0 \to I \to \mathcal{O}_X \to i_*((\mathcal{O}_X)|_Y) \to 0.$$ 

For each $x \in X$, it holds that

$$I_x = \begin{cases} 0 & \text{if } x \in Y \\ \mathcal{O}_{X,x} & \text{if } x \notin Y, \end{cases}$$

and hence $I_x$ is idempotent. It follows that $I$ is idempotent. \qed

Corollary 12.11. Let $X$ be a locally noetherian scheme. Then there exist bijections between the following sets.

1. The set of bilocalizing subcategories of QCoh$X$.
2. The set of idempotent subobjects of $\mathcal{O}_X$.
3. The set of closed subschemes of $X$ which are also open subschemes.
4. The set of subsets of $X$ which are open and closed.

The bijection $(1) \leftrightarrow (2)$ is in Corollary 12.7. The bijection $(2) \leftrightarrow (3)$ is induced by the bijection in Proposition 11.10. For each element $Y$ of (3) the corresponding element of (4) is the underlying space of $Y$.

Proof. This follows from Corollary 12.7 and Lemma 12.10. \qed

By using the classification of the prelocalizing (resp. localizing, closed) subcategories of Mod$k[x]$, we can obtain a classification of the prelocalizing (resp. localizing, closed) subcategories for the projective line.

Example 12.12. Let $k$ be an algebraically closed field, and consider the projective line $X = \mathbb{P}^1_k$. For each $r \in \prod_{x \in X} (\mathbb{Z}_{\geq 0} \cup \{\infty\})$, we define a prelocalizing subcategory $\mathcal{V}_r$ of QCoh$X$ by

$$\mathcal{V}_r = \{ M \in \text{QCoh} X \mid M_x m_x^{r(x)} = 0 \text{ for each } x \in X \text{ with } r(x) \neq \infty \}. $$
Then by the main results (Theorem 9.14, Corollary 10.9, Theorem 11.9, and Corollary 12.7) and the examples on Spec \( k[x] \) (Example 9.17, Example 10.10, Example 11.12, and Example 12.8), the set of prelocalizing subcategories of QCoh \( X \) is

\[
\{ \mathcal{Y}_r \mid r \in \prod_{x \in X} (\mathbb{Z}_{\geq 0} \cup \{\infty\}) \} \cup \{ \text{QCoh} \, X \},
\]

the set of localizing subcategories of QCoh \( X \) is

\[
\{ \mathcal{Y}_r \mid r \in \prod_{x \in X} \{0, \infty\} \} \cup \{ \text{QCoh} \, X \},
\]

the set of closed subcategories of QCoh \( X \) is

\[
\{ \mathcal{Y}_r \mid r \in \bigoplus_{x \in X} \mathbb{Z}_{\geq 0} \} \cup \{ \text{QCoh} \, X \},
\]

and the set of bilocalizing subcategories of QCoh \( X \) is

\[
\{ \mathcal{Y}_r \mid r = \{0\}_{x \in X} \} \cup \{ \text{QCoh} \, X \} = \{0, \text{QCoh} \, X\}.
\]

**Example 12.13.** For each \( i \in \mathbb{Z} \), let \( k_i \) be a field, and let \( U_i := \text{Spec} \, k_i \). Consider the disjoint union \( X := \bigsqcup_{i \in \mathbb{Z}} U_i \). For each subset \( B \) of \( \mathbb{Z} \), define a prelocalizing subcategory \( \mathcal{Y}_B \) of QCoh \( X \) by

\[
\mathcal{Y}_B = \{ M \in \text{QCoh} \, X \mid M|_{U_i} = 0 \text{ for each } i \in \mathbb{Z} \setminus B \}.
\]

Then by Theorem 9.14, Corollary 10.9, Theorem 11.9, and Corollary 12.7, the set \( \{ \mathcal{Y}_B \mid B \subset \mathbb{Z} \} \) is the set of prelocalizing subcategories of QCoh \( X \), and every prelocalizing subcategory of QCoh \( X \) is bilocalizing. Therefore every local filter of \( \mathcal{O}_X \) is a principal filter. For each subset \( B \) of \( \mathbb{Z} \), let \( I_B \) be the idempotent subobject of \( \mathcal{O}_X \) corresponding to the bilocalizing subcategory \( \mathcal{Y}_B \). Then the filter

\[
\mathcal{F} = \{ I_B \mid B \supset \mathbb{Z} \ \text{a finite set} \}
\]

of \( \mathcal{O}_X \) is not a local filter since \( \mathcal{F} \) is not a principal filter.

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