Classification of stable solutions for non-homogeneous higher-order elliptic PDEs

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Abstract

Under some assumptions on the nonlinearity \(f\), we will study the nonexistence of nontrivial stable solutions or solutions which are stable outside a compact set of \(\mathbb{R}^n\) for the following semilinear higher-order problem:

\[
(-\Delta)^k u = f(u) \quad \text{in } \mathbb{R}^n,
\]

with \(k = 1, 2, 3, 4\). The main methods used are the integral estimates and the Pohozaev identity. Many classes of nonlinearity will be considered; even the sign-changing nonlinearity, which has an adequate subcritical growth at zero as for example \(f(u) = -mu + \lambda |u|^\theta - 1 u - \mu |u|^{p-1} u\), where \(m \geq 0, \lambda > 0, \mu > 0, p, \theta > 1\). More precisely, we shall revise the nonexistence theorem of Berestycki and Lions (Arch. Ration. Mech. Anal. 82:313-345, 1983) in the class of smooth finite Morse index solutions as the well-known work of Bahri and Lions (Commun. Pure Appl. Math. 45:1205-1215, 1992). Also, the case when \(f(u)\) is a nonnegative function will be studied under a large subcritical growth assumption at zero, for example \(f(u) = |u|^m u(1 + |u|^q)\) or \(f(u) = |u|^m u e^{|u|^q}\), \(\theta > 1\) and \(q > 0\). Extensions to solutions which are merely stable are discussed in the case of supercritical growth with \(k = 1\).

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1 Introduction

This paper is devoted to the study of solutions, possibly unbounded and sign-changing, of the semilinear partial differential equation,

\[
(-\Delta)^k u = f(u) \quad \text{in } \mathbb{R}^n,
\]

where \(k = 1, 2, 3, 4\), \(n \geq 1\) and \(f \in C^1(\mathbb{R})\). Under some assumptions on the nonlinearity \(f\), we will show that this problem does not possess a nontrivial solution with finite Morse index.

In the last decades, problems related to the nonexistence of finite Morse index solutions for second-, fourth- and sixth-order Lane-Emden equation on unbounded domains of \(\mathbb{R}^n\) have received a lot of attention (see [2–12]).
We now list some known results. We start with the second-order Lane-Emden equation

$$-\Delta u = |u|^{p-1} u, \quad \text{in } \mathbb{R}^n, p > 1,$$

(F.1)

Farina [6] proved that nontrivial finite Morse index solutions of (F.1) exist if and only if $p \geq p_*(n)$ and $n \geq 11$, or $p = \frac{2n}{n-4}$ and $n \geq 3$, where $p_*(n)$ is the so-called Joseph-Lundgren exponent. Also, in [13] several Liouville-type theorems are presented for stable solutions, where $f > 0$ is a general convex, nondecreasing function. Extensions to solutions which are merely stable outside a compact set are discussed.

For the fourth-order Lane-Emden problem

$$\Delta^2 u = |u|^{p-1} u, \quad \text{in } \mathbb{R}^n, p > 1,$$

(F.2)

the subcritical case has been studied by Ramos and Rodriguez for finite Morse index sign-changing solutions (see [14]). The supercritical case is more complicated and there are several new approaches dealing with (F.2). The first approach is to use the test function $v = -\Delta u$. To this end, one has to use Souplet’s inequality [15], this will give an exponent $\frac{n}{n-4} + \epsilon_n$ for some $\epsilon_n > 0$; see [16]. These results were improved in [12] by adapting Farina’s approach with the restriction on the power $q < \frac{n}{2}$. The second approach was obtained by Cowan and Ghoussoub [3], Dupaigne et al. [17] and further exploited by Hajlaoui, Ye and one of the authors [7]. This approach improves the first upper bound $\frac{n}{n-4} + \epsilon_n$, but it again fails to catch the fourth-order Joseph-Lundgren exponent computed by Gazzola and Grunau [18]. It should be remarked that by combining these two approaches one can show that stable positive solutions to (F.2) do not exist when $n \leq 12$ and $p > 1$; see [7].

Finally in [5], Dávila et al. employed a monotonicity formula-based approach and gave a complete classification of stable and finite Morse index (positive or sign-changing) solutions to (F.2). A remarkable outcome of this third approach is that it gives the optimal exponent. The main tool of [5] is a monotonicity formula, used to perform a blow-down analysis and reduce the nonexistence of nontrivial entire solutions for the problem (F.2), to that of nontrivial homogeneous solutions.

Thanks to the Liouville-type theorem with finite Morse index in [8], the authors proved the nonexistence result of sign-changing solutions for the sixth-order problem

$$-\Delta^3 u = |u|^{p-1} u, \quad \text{in } \mathbb{R}^n, p > 1.$$

(L.1)

Let us give a brief sketch of their method. They proved various classification theorems and Liouville-type results for $C^6$-solutions belonging to one of the following classes: stable solutions, solutions which are stable outside a compact set of $\mathbb{R}^n$. These results apply to the subcritical case using the Pohozaev identity. In the supercritical case, motivated by the monotonicity formula established in [19], they reduced the nonexistence of nontrivial entire solutions for the problem (L.1), to that of nontrivial homogeneous solutions.

Through this approach, they gave a complete classification of stable solutions and those finite Morse indices, whether positive or sign-changing. Also, this analysis reveals the existence of a new critical exponent called the sixth-order Joseph-Lundgren exponent, also they gave the explicit value of this exponent.

In this work, we are concerned with Liouville-type theorems for the nonlinear elliptic equation (F.1) for $k = 1, 2, 3, 4$. We prove Liouville-type theorems for solutions (whether
positive or sign-changing) belonging to one of the following classes: stable solutions and solutions which are stable outside a compact set. Our proof is based on a combination of the integral estimates and the Pohozaev-type identity.

The paper is organized as follows. In Section 2 we state our main results, which are then proved in Section 4. Section 3 contains some important auxiliary tools, which are used in the proofs of the main theorems.

2 Statement of the main results
In order to state our results, we present first some assumptions on the nonlinearity \( f \):

- **H1**: There exists a constant \( \theta > 1 \) such that
  \[
  f'(s)s^2 - \theta f(s)s \geq 0, \quad \forall s \in \mathbb{R}.
  \]

- **H2**: There exist constants \( s_0 > 0, \theta > 1 \) and \( C_0 > 0 \) such that
  \[
  C_0 |s|^{\theta+1} \leq f(s)s, \quad \forall |s| \leq s_0.
  \]

- **H3**: There exists a constant \( 0 < \alpha_0 < 1 \) such that
  \[
  \frac{2n}{n-2k} F(s) - (1 + \alpha_0)f(s)s \geq 0, \quad \forall s \in \mathbb{R},
  \]
  where \( F(s) = \int_0^s f(t) \, dt \).

**Remark 2.1** (1) \( H_1 \) implies \( H_1' \): There exist constants \( s_0 > 0, \theta > 1 \) and \( C_0 > 0 \) such that

  \[
  C_0 |s|^{\theta+1} \leq f(s)s, \quad \forall |s| \geq s_0.
  \]

  Indeed, by \( H_1 \), we have \( f'(s)|s| \) is nondecreasing function for all \( |s| \geq s_0 \). This implies that

  \[
  C_0 |s|^{\theta+1} \leq f(s)s, \quad \forall |s| \geq s_0.
  \]

(2) \( H_1 \) implies the Ambrosetti-Rabinowitz condition (A-R): there exist constants \( \tilde{\theta} > 2 \) and \( s_0 > 0 \) such that

  \[
  f(s)s \geq \tilde{\theta} F(s) > 0, \quad \text{for } |s| > s_0.
  \]

**Examples** We easily verify that the following functions satisfy \( H_1 \) and \( H_2 \).

1. \( f(s) = C_0(1 + |s|^q)|s|^{\theta-1}s, \theta > 1, q > 0 \) and \( C_0 > 0 \).
2. \( f(s) = |s|^{\theta-1} \exp(s), \theta > 1 \) and \( q > 1 \).
3. \( f(s) = \sum_{i=1}^{i_0} c_i |s|^{\theta_i-1}s, \) with \( \theta_i > 1 \) \( \forall i = 1, 2, \ldots, i_0 \) and \( c_i > 0 \) \( \forall i = 1, 2, \ldots, i_0 \). In this example we choose \( \theta = \min_{1 \leq i \leq i_0} (\theta_i) \).

The examples (1) and (2) show that \( f \) can have an exponential growth at infinity. Therefore, clearly an adequate behavior of \( f \) at zero is needed to obtain the Liouville theorem. The unique and important nonexistence result for stable solutions of the non-homogeneous second-order equation (1.1) has been recently obtained in [13]. It is shown
there, among other things, that (1.1) does not admit nontrivial stable or stable outside a compact set solution provided that $f$ is regular, positive, nondecreasing and convex function in $(0, +\infty)$. More precisely, under a mere nonnegativity assumption on the nonlinearity, the authors begin this work by stating that up to space dimension $n = 4$, bounded stable solutions of (1.1) are trivial. For the next series of results, they restricted themselves to the following class of nonlinearities:

$$f \in C^0(\mathbb{R}_+) \cap C^2(\mathbb{R}_+^*), \quad f > 0 \text{ is nondecreasing and convex in } \mathbb{R}_+. \tag{2.1}$$

In order to relate the nonlinearity $f$ and the below exponents (2.3) and (2.4), they introduced a quantity $q$ defined for $u \in \mathbb{R}_+^*$ by $q(u) = \frac{f^2}{f''}(u)$, whenever $ff''(u) \neq 0$ and $q(u) = +\infty$ otherwise. They assumed that $q(u)$ converges as $u \to 0^+$ and denote its limit by

$$q_0 = \lim_{u \to 0^+} q(u) \in \mathbb{R}. \tag{2.2}$$

Define now $p_0 \in \mathbb{R}$ the conjugate exponent of $q_0$, by $\frac{1}{p_0} + \frac{1}{q_0} = 1$. The exponent $p_0$ must be understood as a measure of the 'flatness' of $f$ at 0. However, we establish their following theorem.

**Theorem A** [13] Assume that $f$ satisfies (2.1) and (2.2). Assume that $u \in C^2(\mathbb{R}^n)$ is stable solution of (1.1) with $k = 1$. Then $u \equiv 0$ if any one of the following conditions holds:

1. $1 \leq n < 9$ and $1 < p_\infty$,
2. $n = 10$, $p_0 < +\infty$ and $1 < p_\infty$,
3. $n \geq 11$, $p_0 < p_\infty(n)$ and $1 < p_\infty(n) < p_\infty(n)$,

where $p_\infty \in \mathbb{R}$ be defined by $q_\infty = \limsup_{u \to +\infty} q(u)$, $\frac{1}{p_\infty} + \frac{1}{q_\infty} = 1$.

A typical example of nonlinearity function $f$ satisfying the above conditions (2.1) and (2.2) is $f(u) = |u|^{\theta-1}u + |u|^{p-1}u$, where $\theta > 0$. A simple calculation, we get $p_0 = \theta$ and $p_\infty = p$. We use this nonlinearity function to establish some new Liouville-type theorems. Our method is different from (and complementary to) the one used in [13]. It exploits the attractive character of the difference between $f'(u)u^2 - \theta f(u)u \geq 0$, if $p > \theta$, that is, $f$ satisfies $H_1$ and $H_2$. It will be shown in Theorem 2.1 that problem (1.1) does not possess nontrivial stable solutions if and only if $1 < \theta < p_\infty(n), \forall p \geq \theta$. Also, we may consider nonlinearities with exponential growth at infinity, i.e. $p_\infty = +\infty$ satisfying $H_1$ and $H_2$, as for example $f(u) = |u|^{p-1}ue^{|u|^\theta}$, $\theta > 1$ and $q > 0$; therefore, in view again of Theorem 2.1, one has $u \equiv 0$. Furthermore, the present paper is motivated by the interesting work [1], we shall revise the nonexistence theorem of Berestycki and Lions [1] if one substitutes their assumption, which is

$$\int_{\mathbb{R}^n} |\nabla u|^2 + \int_{\mathbb{R}^n} f(u)u < +\infty,$$

by assuming that $u$ is stable or stable outside a compact set. Therefore sign-changing nonlinearities will also be considered and we do not require that $f'(0) = 0$ as the instructive example given by Berestycki and Lions [1] is $f(u) = -mu + \lambda|u|^{p-1}u - \mu|u|^{p-1}u$, where $\lambda, \mu$ are positive constants, $m \geq 0$ and $1 < \theta, p$. Observe that the above nonlinearity satisfies ($H_1$), thus we shall prove that equation (1.1) does not possess a nontrivial stable solution
provided $1 < p \leq \frac{n+2k}{n-2k}$ and $p < \theta$, also if $u$ is bounded solution to (1.1) and $m > 0$, then $u \equiv 0$, for any $\theta \geq p$. If $p \leq \frac{n+2k}{n-2k} \leq \theta$ and $m > 0$, it follows from the Pohozaev identity that there cannot exist a nontrivial solution of (1.1) which is stable outside a compact set. This result is similar to [1] for $k = 1$. To conclude, this work completes the study of Dupaigne and Farina [13] since here we do not assume that $f$ is positive and convex function. Therefore, to be more concrete in our analysis of nonexistence, we will distinguish between stable and stable outside a compact set. We provide some elliptic decay estimates that we use frequently later in the proofs. Deriving the right decay estimates for solutions of (1.1) plays a fundamental role in most our proofs. On the other hand, we shall also consider the question of the nonexistence of stable solutions (positive or sign-changing) in the supercritical case of a second-order equation.

In order to state our results we need to recall the following.

**Definition 2.1** A solution $u$ of (1.1) belonging to $C^{2k}(\mathbb{R}^n)$

- is said to be stable if

$$Q_u(\psi) := \int_{\mathbb{R}^n} \left| D^k \psi \right|^2 dx - \int_{\mathbb{R}^n} f'(u) \psi^2 dx \geq 0, \quad \forall \psi \in C^k_c(\mathbb{R}^n),$$

where

$$D^k = \begin{cases} \Delta^k & \text{for } k = 2, 4, \\ \nabla \Delta^{k-1} & \text{for } k = 1, 3, \end{cases}$$

- is stable outside a compact set $K \subset \mathbb{R}^n$, if $Q_u(\psi) \geq 0$ for any $\psi \in C^k_c(\mathbb{R}^n \setminus K)$.

More generally, the Morse index of a solution is defined as the maximal dimension of all subspaces $E$ of $C^k_c(\mathbb{R}^n)$ such that $Q_u(\psi) < 0$ in $E \setminus \{0\}$. Clearly, a solution is stable if and only if its Morse index is equal to zero.

**Remark 2.2** It is well known that any finite Morse index solution $u$ is stable outside a compact set $K \subset \mathbb{R}^n$. Indeed, there exist $K \geq 1$ and $X_K := \text{Span}\{\phi_1, \ldots, \phi_K\} \subset C^k_c(\mathbb{R}^n)$ such that $Q_u(\phi) < 0$ for any $\phi \in X_K \setminus \{0\}$. Hence, $Q_u(\psi) \geq 0$ for every $\psi \in C^k_c(\mathbb{R}^n \setminus K)$, where $K := \bigcup_{j=1}^K \text{supp}(\phi_j)$.

To state the following result we need to introduce some notation. Let two critical exponents play an important role, namely the classical Sobolev exponent

$$p_s(n, k) = \begin{cases} +\infty & \text{if } n \leq 2k, \\ \frac{n+2k}{n-2k} & \text{if } n > 2k, \end{cases}$$

and the Joseph-Lundgren exponent

$$p_s(n) = \begin{cases} +\infty & \text{if } n \leq 10, \\ \frac{(n-2)^2 - 4n + 8\sqrt{n-1}}{(n-2)(n-10)} & \text{if } n \geq 11. \end{cases}$$

Note that the exponent $p_s(n)$ is larger than the classical critical Sobolev exponent $p_s(n, 1)$, $n \geq 11$.

Now we can state our main nonexistence results.
Theorem 2.1 Let $u \in C^2(\mathbb{R}^n)$ be a stable solution of (1.1). Assume that $f$ satisfies $H_1$ and $H_2$. If $1 < \theta \leq p_1(n,k)$, then $u \equiv 0$.

Theorem 2.2 Let $u \in C^2(\mathbb{R}^n)$ be a solution of (1.1) which is stable outside a compact set. Assume that $f$ satisfies $H_1, H_2$ and $H_3$. If $1 < \theta < p_1(n,k)$, then $u \equiv 0$.

The next result concerns the complete classification of entire stable solutions of the second-order equation (1.1) in the supercritical case.

Theorem 2.3 Let $u \in C^2(\mathbb{R}^n)$ be a stable solution of (1.1) with $k = 1$. Assume that $f$ satisfies $H_1$ and $H_2$. If $\frac{n+2}{n-2} < \theta < p_1(n)$, then $u \equiv 0$.

2.1 Berestycki and Lions Liouville-type theorem

Now, we fix in this subsection

$$ f(u) = -mu + \lambda |u|^{\theta-1}u - \mu |u|^{p-1}u, \tag{2.5} $$

where $\lambda, \mu$ are positive constants, $m \geq 0$ and $1 < \theta, p$. We will show that $u = 0$ is the unique solution of equation (1.1) under some assumptions on the parameter $m, \theta$ and $p$. Also, we observe that $f$ is neither convex nor positive function in $\mathbb{R}^n$. Then we have the following.

Theorem 2.4 Let $u \in C^2(\mathbb{R}^n)$ be a stable solution of (1.1) with $f$ satisfies (2.5).

1. If $u$ is bounded and $m > 0$, then $u \equiv 0$, for any $\theta \geq p > 1$.
2. If $1 < p < \theta$ and $1 < p \leq p_1(n,k)$, then $u \equiv 0$.

Remark 2.3 Clearly, if $u$ is unbounded stable solution to (1.1) with $f(u) = -mu + \lambda |u|^{\theta-1}u - \mu |u|^{p-1}u$ and $m > 0$, then $u \equiv 0$, for any $\theta \geq p > 1$ and $n < 2p$.

Also, we will show, with very few restrictions, that there exists a necessary and sufficient condition for the nonexistence solutions which are stable outside a compact set of problem like (1.1).

Theorem 2.5 Let $u \in C^2(\mathbb{R}^n)$ be a solution of (1.1) which is stable outside a compact set with $f$ satisfies (2.5).

1. If $m > 0$ and $1 < p \leq \frac{n+2}{n-2} \leq \theta$, then $u \equiv 0$.
2. If $m = 0$, $1 < p \leq \frac{n+2}{n-2} \leq \theta$ and $(p, \theta) \neq (\frac{n+2}{n-2}, \frac{n+2}{n-2})$, then $u \equiv 0$.

3 Auxiliary results

In this section we prove the following lemmas and propositions, which will have a crucial role in the proof of Theorems 2.1, 2.2, 2.3, 2.4 and 2.5. Denote $B_R = \{x \in \mathbb{R}^n : |x| < R\}$. The letter $C$ will be used throughout to denote a generic positive constant, which may vary from line to line and only depends on arguments inside the parentheses or arguments which are otherwise clear from the context.

First, define a cut-off function $\varphi_R \in C^4_c(\mathbb{R}^n)$ such that $\varphi_R \equiv 1$ in $B_R$, $\varphi_R \equiv 0$ in $\mathbb{R}^n \setminus B_{2R}$, $0 \leq \varphi_R \leq 1$ in $\mathbb{R}^n$ and $|\nabla^\tau \varphi_R| \leq CR^{\tau-1}$ for $\tau \leq 4$ in $A_R = \{x \in \mathbb{R}^n, R \leq |x| \leq 2R\}$.

Lemma 3.1 For any $v \in C^8(\mathbb{R}^n), m > 4$ and $\epsilon > 0$ arbitrary small number, there exists a constant $C_{v,m,\epsilon} > 0$ such that
An application of Young’s inequality yields

\[
R^{-4} \int_{B_{2R}} (\Delta v) \psi_{\frac{m-4}{2}}^2 dx 
\leq \epsilon^2 \int_{B_{2R}} (\Delta^2 v)^2 \psi_{\frac{m}{2}}^2 dx + \epsilon^2 R^{-2} \int_{B_{2R}} |\nabla (\Delta v)|^2 \psi_{\frac{m-2}{2}}^2 dx 
+ \frac{R^2}{2} \int_{B_{2R}} (\Delta v)^2 \psi_{\frac{m-4}{2}}^2 dx 
+ C_{\epsilon,m} R^{-8} \int_{B_{2R}} v^2 \psi_{\frac{m-8}{2}}^2 dx.
\]

Inserting the latter inequality into (3.1), we obtain

\[
R^{-4} \int_{B_{2R}} (\Delta v)^2 \psi_{\frac{m-4}{2}}^2 dx 
\leq \epsilon^2 \int_{B_{2R}} (\Delta^2 v)^2 \psi_{\frac{m}{2}}^2 dx + \epsilon^2 R^{-2} \int_{B_{2R}} |\nabla (\Delta v)|^2 \psi_{\frac{m-2}{2}}^2 dx 
+ \frac{R^2}{2} \int_{B_{2R}} (\Delta v)^2 \psi_{\frac{m-4}{2}}^2 dx 
+ C_{\epsilon,m} R^{-8} \int_{B_{2R}} v^2 \psi_{\frac{m-8}{2}}^2 dx.
\]

Proof of 2. Integrating by parts and using again Young’s inequality, we obtain

\[
R^{-2} \int_{B_{2R}} |\nabla (\Delta v)|^2 \psi_{\frac{m-2}{2}}^2 dx 
= -R^{-2} \int_{B_{2R}} \Delta v \Delta v \psi_{\frac{m-2}{2}}^2 dx - R^{-2} \int_{B_{2R}} \Delta v \nabla (\Delta v) \nabla (\psi_{\frac{m-2}{2}}^2) dx 
\leq \epsilon \int_{B_{2R}} (\Delta^2 v)^2 \psi_{\frac{m}{2}}^2 dx + \frac{2R^{-4}}{\epsilon} \int_{B_{2R}} (\Delta v)^2 \psi_{\frac{m-4}{2}}^2 dx + C_{\epsilon,m} R^{-8} \int_{B_{2R}} |\nabla (\Delta v)|^2 \psi_{\frac{m-2}{2}}^2 dx.
\]

Inserting (3.2) into the latter, we derive

\[
R^{-2} \int_{B_{2R}} |\nabla (\Delta v)|^2 \psi_{\frac{m-2}{2}}^2 dx 
\leq \epsilon \int_{B_{2R}} (\Delta^2 v)^2 \psi_{\frac{m}{2}}^2 dx + C_{\epsilon,m} R^{-8} \int_{B_{2R}} v^2 \psi_{\frac{m-8}{2}}^2 dx.
\]

Proof of 3. Integrating by parts, we obtain

\[
\int_{B_{2R}} |\nabla v|^2 \psi_{\frac{m-6}{2}}^2 dx = \frac{R^{-6}}{2} \int_{B_{2R}} \Delta (v^2) \psi_{\frac{m-6}{2}}^2 dx - R^{-6} \int_{B_{2R}} v \Delta \psi_{\frac{m-6}{2}}^2 dx 
\leq \epsilon R^{-4} \int_{B_{2R}} (\Delta v)^2 \psi_{\frac{m-4}{2}}^2 dx + C_{\epsilon,m} R^{-8} \int_{B_{2R}} v^2 \psi_{\frac{m-8}{2}}^2 dx.
\]
From (3.2) and (3.3), we deduce
\[ R^{-6} \int_{B_{2R}} |\nabla v|^2 \psi^{2m-6}_R \, dx \leq \epsilon^3 \int_{B_{2R}} (\Delta^2 v)^2 \psi^{2m}_R \, dx + C_{r,m} R^{-8} \int_{B_{2R}} v^2 \psi^{2m-8}_R \, dx. \] (3.4)

**Proof of 4.** Integrating by parts, we obtain
\[ R^{-4} \int_{B_{2R}} |\nabla^2 v|^2 \psi^{2m-4}_R \, dx \]

Using Young’s inequality and from (3.3) and (3.4), we obtain
\[ R^{-4} \int_{B_{2R}} |\nabla^2 v|^2 \psi^{2m-4}_R \, dx \leq R^{-2} \int_{B_{2R}} |\nabla (\Delta v)|^2 \psi^{2m-2}_R \, dx + CR^{-6} \int_{B_{2R}} |\nabla v|^2 \psi^{2m-6}_R \, dx \]

\[ \leq \epsilon^3 \int_{B_{2R}} (\Delta^2 v)^2 \psi^{2m}_R + C_{r,m} R^{-8} \int_{B_{2R}} v^2 \psi^{2m-8}_R. \] (3.5)

**Proof of 5.** Integrating by parts, we get
\[ R^{-2} \int_{B_{2R}} |\nabla^3 v|^2 \psi^{2m-2}_R \, dx \]

where \( f_i = \frac{\partial f}{\partial x_i}, f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}, \) and \( f_{ijk} = \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}. \) (Here and in the sequel, we use the Einstein summation convention: an index occurring twice in a product is to be summed from 1 up to the space dimension.)

Using Young’s inequality of the above, we deduce
\[ R^{-2} \int_{B_{2R}} |\nabla^3 v|^2 \psi^{2m-2}_R \, dx \leq CR^{-2} \int_{B_{2R}} |\nabla (\Delta v)|^2 \psi^{2m-2}_R \, dx + CR^{-4} \int_{B_{2R}} |\nabla^2 v|^2 \psi^{2m-4}_R \, dx, \] (3.6)

which gives the desired conclusion. \( \square \)

**Lemma 3.2** For any \( m > 4 \) and \( \epsilon > 0 \) arbitrary small number, there exists a constant \( C_{r,m} > 0 \) such that
\[ (\Delta^2 (u \psi^m_R))^2 \leq (1 + \epsilon)(\psi^m_R \Delta^2 u)^2 + C_{r,m} B(u, \psi_R, m), \] (3.7)

where \( B(u, \psi_R, m) = (R^{-4} |\Delta u|^2 \psi^{2m-4}_R + R^{-2} |\nabla (\Delta u)|^2 \psi^{2m-2}_R + R^{-6} |\nabla u|^2 \psi^{2m-6}_R + R^{-8} u^2 \psi^{2m-8}_R + R^{-4} |\nabla^2 u|^2 \psi^{2m-4}_R). \)

**Proof** Let \( \psi_R \in C^4_0(\mathbb{R}^n) \) be defined as above and \( m > 4 \). Direct calculation yields
\[ \Delta^2 (u \psi^m_R) = \psi^m_R \Delta^2 u + A(u, \psi^m_R), \] (3.8)

where \( A(u, \psi^m_R) = 2 \Delta u \Delta \psi^m_R + 4 \nabla u \nabla (\Delta \psi^m_R) + u \Delta^2 \psi^m_R + 4 \nabla (\Delta u) \nabla (\psi^m_R) + 4 u \psi^m_{ij} \).
Thus,
\[
(\Delta^2(u\varphi_R^m))^2 = (\varphi_R^m \Delta^2 u)^2 + A^2(u, \varphi_R^m) + 2A(u, \varphi_R^m)\varphi_R^m \Delta^2 u.
\]

Now by the Young inequality, for any \( \epsilon > 0 \), there exists \( C_\epsilon \) a constant such that
\[
(\Delta^2(u\varphi_R^m))^2 \leq (1 + \epsilon)(\varphi_R^m \Delta^2 u)^2 + C_\epsilon A^2(u, \varphi_R^m). \tag{3.9}
\]

For the second term on the right hand side of inequality (3.9), one obtains
\[
A^2(u, \varphi_R^m) \leq C_\epsilon \left( |\Delta u|^2 |\varphi_R^m|^2 + |\nabla u|^2 |\nabla (\Delta \varphi_R^m)|^2 + |u|^2 |\Delta^2 \varphi_R^m|^2 \right.
+ \left. |\nabla (\Delta u)|^2 |\nabla (\varphi_R^m)|^2 + |u_{ij}|^2 |(\varphi_R^m_{ij})|^2 \right)
\leq C_{\epsilon, m}(R^{-4} |\Delta u|^2 |\varphi_R^m|^2 + R^{-2} |\nabla (\Delta u)|^2 |\varphi_R^m|^2 + R^{-6} |\nabla u|^2 |\varphi_R^m|^2
+ R^{-8} u^2 |\varphi_R^m|^2 + R^{-4} |\nabla^2 u|^2 |\varphi_R^m|^2),
\]
which gives the desired inequality (3.7).

Using the previous lemmas, we obtain the following results.

**Proposition 3.1** Let \( u \in C^{2k}(\mathbb{R}^n) \) be a stable solution of (1.1). Assume that \( f \) satisfies \( H_1 \) and \( H_2 \). Then there exists a constant \( C > 0 \) such that, for any \( R > 0 \), we have
\[
\int_{B_R} \left( |u|^{p+1} + |D^k u|^2 \right) dx \leq CR^{n-2k \frac{p+4}{p+1}} \quad \text{and} \quad \int_{B_R} f(u) u \, dx \leq CR^{n-2k \frac{p+4}{p+1}}.
\]

When attempting to prove the nonexistence of the nontrivial solution which is stable outside a compact set of (1.1) in the subcritical case, we need first to establish the following proposition.

**Proposition 3.2** Let \( u \in C^{2k}(\mathbb{R}^n) \) be a solution of (1.1) which is stable outside a compact set. Assume that \( f \) satisfies \( H_1 \) and \( H_2 \). Then there exists a constant \( C > 0 \) such that, for any \( R > 0 \), we have
\[
\int_{B_R} \left( |u|^{p+1} + |D^k u|^2 \right) dx \leq C(1 + R^{-2k \frac{p+4}{p+1}}) \quad \text{and} \quad \int_{B_R} f(u) u \, dx \leq C(1 + R^{-2k \frac{p+4}{p+1}}).
\]

**Proof of Proposition 3.1** The proof of the case \( k = 1, 2, 3 \), bears resemblance to an argument found in [5, 6, 8]. For more details, please see the proof of proposition 4 in [6] for the case \( k = 1 \), the proof of Lemma 4.2 in [5] for the case \( k = 2 \) and the proof of Proposition 1.2 in [8] for the case \( k = 3 \). For this reason, we omit the details.

**Proof of the case \( k = 4 \)**. Let \( \varphi_R \in C_c^k(\mathbb{R}^n) \) defined as above, let \( u \) be a solution of equation (1.1). The function \( uw_R^m \) belongs to \( C_c^1(\mathbb{R}^n) \), and thus it can be used as a test function in the quadratic form \( Q_u \). Hence, the stability assumption on \( u \) gives
\[
\int_{B_{2R}} f(u) u^2 \varphi_R^{2m} \, dx \leq \int_{B_{2R}} |\Delta^2(u \varphi_R^m)|^2 \, dx.
\]
Applying Lemma 3.2, we obtain
\[
\int_{B_{2R}} f'(u)u^2 \psi_R^{2m} \, dx \leq (1 + \epsilon) \int_{B_{2R}} (\psi_R^m \Delta^2 u)^2 \, dx + C_{\epsilon} \int_{B_{2R}} B(u, \varphi_R, m) \, dx. \tag{3.10}
\]

In view of Lemma 3.1, we get
\[
\int_{B_{2R}} f'(u)u^2 \psi_R^{2m} \, dx \leq (1 + \epsilon) \int_{B_{2R}} (\psi_R^m \Delta^2 u)^2 + C_{\epsilon} R^8 \int_{B_{2R}} u^2 \psi_R^{2m-8} \, dx. \tag{3.11}
\]

Multiplying equation (1.1) by \(u \psi_R^{2m}\) and integrating by parts, we get
\[
\int_{B_{2R}} \Delta^2 u \Delta^2 (u \psi_R^{2m}) \, dx = \int_{B_{2R}} f(u)u \psi_R^{2m} \, dx.
\]

From (3.8), we derive
\[
\int_{B_{2R}} \Delta^2 u \Delta^2 (u \psi_R^{2m}) \, dx = \int_{B_{2R}} \Delta^2 u A(u, \psi_R^{2m}) \, dx.\tag{3.12}
\]

Then, using Young’s inequality, we derive
\[
\int_{B_{2R}} ((\Delta^2 u)^2 \psi_R^{2m} - f(u)u \psi_R^{2m}) \, dx \leq \epsilon \int_{B_{2R}} \Delta^2 u \psi_R^{2m} \, dx + C_{\epsilon, m} \int_{B_{2R}} B(u, \varphi_R, m) \, dx.
\]

Applying again Lemma 3.1, we have
\[
\int_{B_{2R}} \Delta^2 u \psi_R^{2m} \, dx - \int_{B_{2R}} f(u)u \psi_R^{2m} \, dx \leq \epsilon \int_{B_{2R}} \Delta^2 u \psi_R^{2m} \, dx + C_{\epsilon, m} \int_{B_{2R}} u^2 \psi_R^{2m-8} \, dx. \tag{3.13}
\]

Multiplying (3.13) by \(\theta\) and combining it with (3.11), we derive
\[
\int_{B_{2R}} [f'(u)u^2 - \theta f(u)u] \psi_R^{2m} \, dx + \left[\theta (1 - \epsilon) - (1 + \epsilon)\right] \int_{B_{2R}} \Delta^2 u \psi_R^{2m} \, dx \leq CR^8 \int_{B_{2R}} u^2 \psi_R^{2m-8} \, dx.
\]

From (H1) and for \(\epsilon\) sufficiently small such that \(\epsilon < \frac{\theta - 1}{\theta + 1}\), we deduce
\[
\int_{B_{2R}} (\Delta^2 u)^2 \psi_R^{2m} \, dx \leq CR^{-8} \int_{B_{2R}} u^2 \psi_R^{2m-8} \, dx. \tag{3.14}
\]
By Young’s inequality, we have
\[
\int_{B_{2R}} (\Delta^2 u)^2 \psi_R^{2m} \, dx \leq \frac{2}{\theta + 1} \int_{B_{2R}} |u|^{|\theta + 1|} \psi_R^{(\theta + 1)(m - 4)} \, dx + CR^{n-\frac{\theta + 1}{m-\frac{1}{2}}}. \tag{3.15}
\]
As above, we find from (3.13) that
\[
\int_{B_{2R}} f(u)u^2 \psi_R^{2m} \, dx \leq (1 + \epsilon) \int_{B_{2R}} (\Delta^2 u)^2 \psi_R^{2m} \, dx + C_{\epsilon,m} R^{-8} \int_{B_{2R}} u^2 \psi_R^{2m-8} \, dx.
\]
Using (3.14) in the latter, we obtain
\[
\int_{B_{2R}} f(u)u^2 \psi_R^{2m} \, dx \leq \frac{2}{\theta + 1} \int_{B_{2R}} |u|^{|\theta + 1|} \psi_R^{(\theta + 1)(m - 4)} \, dx + CR^{n-\frac{\theta + 1}{m-\frac{1}{2}}}. \tag{3.16}
\]
From \((H_1)\) and \((H_2)\), we get
\[
C_0 \int_{B_{2R}} |u|^{|\theta + 1|} \psi_R^{2m} \, dx \leq \int_{B_{2R}} f(u)u^2 \psi_R^{2m} \, dx \leq \frac{2}{\theta + 1} \int_{B_{2R}} |u|^{|\theta + 1|} \psi_R^{(\theta + 1)(m - 4)} \, dx + CR^{n-\frac{\theta + 1}{m-\frac{1}{2}}},
\]
if \((\theta + 1)(m - 4) = 2m\), then
\[
\int_{B_{2R}} |u|^{|\theta + 1|} \psi_R^{2m} \, dx \leq CR^{n-\frac{\theta + 1}{m-\frac{1}{2}}}. \tag{3.17}
\]
From (3.15), (3.16) and (3.17), we deduce that
\[
\int_{B_{2R}} (\Delta^2 u)^2 \psi_R^{2m} \, dx \leq CR^{n-\frac{\theta + 1}{m-\frac{1}{2}}}, \quad \text{and} \quad \int_{B_{2R}} f(u)u^2 \psi_R^{2m} \, dx \leq CR^{n-\frac{\theta + 1}{m-\frac{1}{2}}}.
\]
Since \(\psi_R \equiv 1\) in \(B_R\), we have
\[
\int_{B_R} (|u|^{|\theta + 1|} + (\Delta^2 u)^2) \, dx \leq CR^{n-\frac{\theta + 1}{m-\frac{1}{2}}}, \quad \text{and} \quad \int_{B_R} f(u)u^2 \, dx \leq CR^{n-\frac{\theta + 1}{m-\frac{1}{2}}}.
\]

**Proof of Proposition 3.2** The proof of the case \(k = 1, 2, 3\), bears resemblance to an argument found in [5, 6, 8]. Now, we prove the case \(k = 4\). The proof is the same as the proof of Proposition 3.1. We need only to replace \(\varphi_R\) by \(\varphi_{a,R}\), where \(\varphi_{a,R} \in C^1_c(\mathbb{R}^n)\) satisfies \(0 \leq \varphi_{a,R} \leq 1\) everywhere on \(\mathbb{R}^n\) such that \(\varphi_{a,R}(x) = 0\) for \(|x| < a\) or \(|x| > 2R\), \(\varphi_{a,R}(x) = 1\) for \(2a < |x| < R\) and \(\nabla \varphi_{a,R} \leq CR^{-7}, \tau \leq 4\), for \(R < |x| < 2R\). By the stability assumption on \(u\), there exists \(a_0 > 0\) such that \(Q_u(\varphi_{a,R}^m) \geq 0\) for any \(R > 2a_0\). Hence, by the choice of the test function \(\varphi_{a,R}\), the constant \(C_{a_0}\) depending on \(a_0, \epsilon, m\) and \(u\) appears and the rest of the proof is unchanged. Thus Proposition 3.2 follows.

As in [20], we shall employ a cut-off function with compact support to derive a variant of the Pohozaev identity. This device allows us to avoid the spherical integrals raised in
[21], which are very difficult to control, especially for the polyharmonic situations. For \( k = 1, 2, 3 \), the Pohozaev identity is similar to [7, 8, 20, 22].

**Proposition 3.3** Let \( u \in C^8(\mathbb{R}^n) \) be a solution of (1.1) and \( \psi \in C^4_c(B_R) \), then

\[
\frac{n - 8}{2} \int_{B_R} (\Delta^2 u)^2 \psi \, dx - n \int_{B_R} F(u) \psi \, dx = \int_{B_R} B_4(u, \psi) \, dx,
\]

where

\[
B_4(u, \psi) = F(u)\langle x, \nabla \psi \rangle - \frac{1}{2} (\Delta^2 u)^2 \langle x, \nabla \psi \rangle + 2 \Delta^2 u \nabla \langle \langle x, \nabla (\Delta u) \rangle \rangle \nabla \psi \\
+ \Delta^2 u \{ \langle x, \nabla (\Delta u) \rangle \Delta \psi + 2 \Delta u \Delta \psi \} + \Delta^2 u \{ 4 \nabla (\Delta u) \nabla \psi + \Delta^2 \psi \langle x, \nabla u \rangle \} \\
+ \Delta^2 u \{ \Delta \psi \Delta \langle x, \nabla u \rangle + 2 \nabla (\Delta \psi) \nabla \langle \langle x, \nabla u \rangle \rangle + 2 \Delta \langle \langle x, \nabla u \rangle \rangle \nabla \psi \}.
\]

Thanks to Propositions 3.2 and 3.3, we derive the following.

**Proposition 3.4** Let \( u \in C^{2k}(\mathbb{R}^n) \) be a solution of (1.1) which is stable outside a compact set. Assume that \( f \) satisfies \( H_1 \) and \( H_2 \). If \( 1 < \theta < p_1(n, k) \), then

\[
\int_{\mathbb{R}^n} |D^k u|^2 \, dx = \frac{2n}{n - 2k} \int_{\mathbb{R}^n} F(u) \, dx
\]

and

\[
\int_{\mathbb{R}^n} |D^k u|^2 \, dx = \int_{\mathbb{R}^n} f(u) u \, dx < \infty.
\]

**Proof of Proposition 3.3** Let \( u \in C^8(\mathbb{R}^n) \) be a solution of (1.1) and \( \psi \in C^4_c(B_R) \), we have

\[
\Delta \langle x, \nabla u \rangle \psi = \langle x, \nabla (\Delta u) \rangle \psi + 2 \Delta u \psi + \langle x, \nabla u \rangle \Delta \psi + 2 \nabla \langle \langle x, \nabla u \rangle \rangle \nabla \psi.
\]

Multiplying equation (1.1) by \( \langle x, \nabla u \rangle \psi \) and integrating by parts in \( B_R \), we obtain

\[
\int_{B_R} f(u) \langle x, \nabla u \rangle \psi \, dx = \int_{B_R} \Delta^3 u \Delta \langle x, \nabla u \rangle \psi \, dx.
\]

(3.21)

For the right hand side of (3.21), we integrate by parts to get

\[
\int_{B_R} \Delta^3 u \Delta \langle x, \nabla u \rangle \psi \, dx \\
= \int_{B_R} \Delta^3 u \{ \langle x, \nabla (\Delta u) \rangle \psi + 2 \Delta u \psi + \langle x, \nabla u \rangle \Delta \psi + 2 \nabla \langle \langle x, \nabla u \rangle \rangle \nabla \psi \} \, dx \\
= \int_{B_R} \Delta^2 u \Delta \langle x, \nabla (\Delta u) \rangle \psi \, dx + 2 \int_{B_R} \Delta^2 u \nabla \langle \langle x, \nabla (\Delta u) \rangle \rangle \nabla \psi \, dx + 2 \int_{B_R} (\Delta^2 u)^2 \psi \, dx \\
+ \int_{B_R} \Delta^2 u \{ \langle x, \nabla (\Delta u) \rangle \Delta \psi + 2 \Delta u \Delta \psi + 4 \nabla (\Delta u) \nabla \psi + \langle x, \nabla u \rangle \Delta^2 \psi \} \, dx \\
+ \int_{B_R} \Delta^2 u \{ \Delta \langle x, \nabla u \rangle \Delta \psi + 2 \nabla \langle \langle x, \nabla u \rangle \rangle \nabla (\Delta \psi) \} \, dx \\
+ 2 \int_{B_R} \Delta^2 u \Delta \langle \langle x, \nabla u \rangle \rangle \nabla \psi \, dx.
\]

(3.22)
For the first term on the right hand side of (3.22), we integrate by parts to find
\[ \int_{B_R} \Delta^2 u \Delta [\langle x, \nabla (\Delta u) \rangle] \psi \, dx = \frac{4 - n}{2} \int_{B_R} (\Delta^2 u)^2 \psi \, dx - \frac{1}{2} \int_{B_R} (\Delta^2 u)^2 \langle x, \nabla \psi \rangle \, dx. \quad (3.23) \]

For the term on the left hand side of (3.22), by integrating by parts, we derive
\[ \int_{B_R} f(u) \langle x, \nabla u \rangle \psi \, dx = \int_{B_R} \langle x, \nabla [F(u)] \rangle \psi \, dx \]
\[ = -n \int_{B_R} F(u) \psi \, dx - \int_{B_R} F(u) \langle x, \nabla \psi \rangle \, dx. \quad (3.24) \]

Therefore, the claim follows from (3.21)-(3.24). \( \square \)

Here, we are concerned with the proof of Proposition 3.4.

**Proof of Proposition 3.4** To simplify the proof, we will concentrate on the case \( k = 4 \) which is the most delicate case; even we believe that the results should hold true for \( k = 1, 2, 3 \), for more details, see for example [5, 6, 8, 23]. Let \( R_0 > 0 \). Assume that \( u \) is stable outside \( B_{R_0} \). Let \( 0 < \alpha < \beta \). We begin by defining some smooth compactly supported functions which will be used several times in the sequel. More precisely, we choose \( \phi_R \in C^1_c(\mathbb{R}^n) \) satisfies \( 0 \leq \phi_R \leq 1 \) everywhere on \( \mathbb{R}^n \) such that

\[ \phi_R(x) = \begin{cases} 
1 & \text{for } \alpha R < |x| < \beta R, \\
0 & \text{for } |x| < \frac{\alpha}{2} R \text{ or } |x| > 2\beta R, \\
|\nabla^k \phi_R| \leq C R^{-k} & \text{on } \left\{ \frac{\alpha}{2} R < |x| < 2\beta R \right\}, k = 1, 2, 3, 4.
\end{cases} \]

For \( R \) large enough such that \( \frac{\alpha}{2} R > R_0 \), then \( B_{R_0} \cap \left\{ \frac{\alpha}{2} R \leq |x| \leq 2\beta R \right\} = \emptyset \). Then \( u \) is stable in \( A^{2\beta R}_{\frac{\alpha}{2} R} := \left\{ \frac{\alpha}{2} R < |x| < 2\beta R \right\} \). By Proposition 3.1, there exists a constant \( C > 0 \) such that
\[ \int_{A^{2\beta R}_{\frac{\alpha}{2} R}} \left( |u|^2 + (\Delta^2 u)^2 \right) \, dx \leq C R^{n-8} \quad \text{and} \quad \int_{A^{2\beta R}_{\frac{\alpha}{2} R}} f(u) \, dx \leq C R^{n-8}. \quad (3.25) \]

Let \( \psi_R \in C^1_c(\mathbb{R}^n) \) satisfies \( 0 \leq \psi_R \leq 1 \) on \( \mathbb{R}^n \) defined by

\[ \psi_R(x) = \begin{cases} 
1 & \text{for } |x| < \alpha R, \\
0 & \text{for } |x| > \beta R, \\
|\nabla^k \psi_R| \leq C R^{-k} & \text{on } \left\{ \alpha R < |x| < \beta R \right\}, k = 1, 2, 3, 4.
\end{cases} \]

In view of Lemma 3.1 and Proposition 3.1, we have
\[ \int_{B_{\beta R}} \left( |u|^2 + (\Delta^2 u)^2 \right) \psi_R^2 \, dx \leq C R^{n-8\frac{1}{24}}, \quad (3.26) \]
\[ \int_{B_{\beta R}} \left( B(u, \psi_R, m) + R^{-2} |\nabla^3 u|^2 \psi_R^{2m-2} \right) \, dx \leq C R^{n-8\frac{1}{12}}. \quad (3.27) \]

Now, we estimate all terms on the right hand side of (3.18). Take \( \psi = \psi_R^{2m} \) in (3.18), \( m > 4 \).
The second term on the right hand side of (3.18) can be estimated as

\[
\left| \frac{1}{2} \int_{B_{2R}} (\Delta^2 u)^2 (x, \nabla \psi_R^{2m}) \right| \leq \frac{1}{2} \int_{A_{2R}} (\Delta^2 u)^2 (x, \nabla \psi_R^{2m}) \leq C_m \int_{A_{2R}} (\Delta^2 u)^2 \psi_R^{2m-1} dx \leq C R^{e_{d_{H}/2}}. \tag{3.28}
\]

Next

\[
\left| \int_{B_{2R}} (\Delta^2 u) (x, \nabla (\Delta u)) \Delta \psi_R^{2m} + 2 \Delta u \Delta \psi_R^{2m} + 4 \nabla (\Delta u) \nabla \psi_R^{2m} + \Delta^2 \psi_R^{2m} dx \right|
\]

\[
= \left| \int_{A_{2R}} (\Delta^2 u) (x, \nabla (\Delta u)) \Delta \psi_R^{2m} + 2 \Delta u \Delta \psi_R^{2m} + 4 \nabla (\Delta u) \nabla \psi_R^{2m} + \Delta^2 \psi_R^{2m} dx \right|
\]

\[
\leq C_m \int_{A_{2R}} (\Delta^2 u) (x, \nabla (\Delta u)) \psi_R^{2m-1} + 2 R^2 |\Delta u| \psi_R^{2m-1} + 3 |\nabla u| \psi_R^{2m-3} dx
\]

\[
\leq C_m \int_{A_{2R}} (\Delta^2 u) (x, \nabla (\Delta u)) \psi_R^{2m-1} + 2 R^2 |\Delta u| \psi_R^{2m-1} + 3 |\nabla u| \psi_R^{2m-3} dx \tag{3.29}
\]

the last line comes from the fact that \(0 \leq \psi_R \leq 1\), hence \(\psi_R^{2m-1} \leq \psi_R^m\), for any \(t \leq s\).

By applying the Hölder inequality and the Young inequality to (3.29), we have

\[
\left| \int_{B_{2R}} (\Delta^2 u) (x, \nabla (\Delta u)) \Delta \psi_R^{2m} + 2 \Delta u \Delta \psi_R^{2m} + 4 \nabla (\Delta u) \nabla \psi_R^{2m} + \Delta^2 \psi_R^{2m} dx \right|
\]

\[
\leq \left( \int_{A_{2R}} (\Delta^2 u)^2 dx \right)^{1/2} \left( \int_{A_{2R}} (R^2 |\nabla (\Delta u)| \psi_R^{2m-1} + 2 R^2 |\Delta u| \psi_R^{2m-1} + 3 \psi_R^{2m-3} dx \right)^{1/2}
\]

\[
\leq C \left( \int_{A_{2R}} (\Delta^2 u)^2 dx \right)^{1/2} \left( \int_{A_{2R}} (R^2 |\nabla (\Delta u)| \psi_R^{2m-1} + 2 R^2 |\Delta u| \psi_R^{2m-1} + 3 \psi_R^{2m-3} dx \right)^{1/2}. \tag{3.30}
\]

Similarly, we also obtain

\[
\left| \int_{B_{2R}} \Delta^2 u \nabla ((x, \nabla (\Delta u))) \psi_R^{2m} \right|
\]

\[
= \left| \int_{B_{2R}} \Delta^2 u \nabla (\Delta u) \psi_R^{2m} + x_i (\Delta u) \psi_R^{2m} \right|
\]

\[
\leq C_m \int_{A_{2R}} \Delta^2 u |(R^2 |\nabla (\Delta u)| \psi_R^{2m-1} + |(\Delta u) \psi_R^{2m-1} | dx
\]

\[
\leq C \left( \int_{A_{2R}} (\Delta^2 u)^2 dx \right)^{1/2} \left( \int_{A_{2R}} (R^2 |\nabla (\Delta u)| \psi_R^{2m-1} + |(\Delta u) \psi_R^{2m-1} |^2 dx \right)^{1/2}
\]

\[
\leq C \left( \int_{A_{2R}} (\Delta^2 u)^2 dx \right)^{1/2} \left( \int_{A_{2R}} (R^2 |\nabla (\Delta u)| \psi_R^{2m-1} + |(\Delta u) \psi_R^{2m-1} |^2 dx \right)^{1/2}. \tag{3.31}
\]
Integrating by parts and using Young’s inequality, we obtain

\[
\int_{\mathcal{B}_{R}} \left[ R^{-2}\left| \nabla (\Delta u) \right|^{2} \psi_{R}^{4m-2} + \left| (\Delta u)_{ij} \right|^{2} \psi_{R}^{4m-2} \right] dx 
\leq \int_{\mathcal{B}_{R}} \left[ R^{-2}\left| \nabla (\Delta u) \right|^{2} \psi_{R}^{4m-2} + \left| (\Delta u)_{ij} \right|^{2} \psi_{R}^{4m-2} \right] dx 
= \int_{\mathcal{B}_{R}} \left( \Delta^{2} u \right)^{2} \psi_{R}^{4m-2} dx + \int_{\mathcal{B}_{R}} \Delta^{2} u \nabla (\Delta u) \nabla (\psi_{R}^{4m-2}) dx 
+ \int_{\mathcal{B}_{R}} \left| \nabla (\Delta u) \right|^{2} \left[ R^{-2} \psi_{R}^{4m-2} + \frac{1}{2} \Delta (\psi_{R}^{4m-2}) \right] dx 
\leq C_{m} \int_{\mathcal{B}_{R}} \left( \Delta^{2} u \right)^{2} \psi_{R}^{2m} dx + C_{m} R^{-2} \int_{\mathcal{B}_{R}} \left| \nabla (\Delta u) \right|^{2} \psi_{R}^{2m-2} dx. \tag{3.32}
\]

From (3.31) and (3.32), we obtain

\[
\left| \int_{\mathcal{B}_{R}} \Delta^{2} u \nabla (\nabla (\Delta u)) \nabla \psi_{R}^{2m} dx \right| 
\leq C \left( \int_{\mathcal{B}_{R}} \left( \Delta^{2} u \right)^{2} dx \right)^{\frac{1}{2}} \left( \int_{\mathcal{B}_{R}} \left( \left( \Delta^{2} u \right)^{2} \psi_{R}^{2m} + R^{-2} \left| \nabla (\Delta u) \right|^{2} \psi_{R}^{2m-2} \right) dx \right)^{\frac{1}{2}}. \tag{3.33}
\]

The sixth term on the right hand side of (3.18) yields

\[
\left| \int_{\mathcal{B}_{R}} \Delta^{2} u \left( \Delta (\psi_{R}^{2m}) \Delta (\nabla (\Delta u)) + 2 \nabla (\Delta (\psi_{R}^{2m})) \nabla (\nabla (\Delta u)) \right) dx \right| 
= \left| \int_{\mathcal{B}_{R}} \Delta^{2} u \left( \nabla (\Delta (\psi_{R}^{2m})) + 2 \nabla (\nabla (\Delta (\psi_{R}^{2m}))) \right) dx \right| 
\leq C \left( \int_{\mathcal{B}_{R}} \left( \Delta^{2} u \right)^{2} dx \right)^{\frac{1}{2}} \left( \int_{\mathcal{B}_{R}} \left( \left( \Delta^{2} u \right)^{2} \psi_{R}^{4m-2} + R^{-4} \left| \nabla (\Delta u) \right|^{2} \psi_{R}^{4m-4} \right. \right. 
+ \left. \left. R^{-4} \left| \nabla (\nabla (\Delta u)) \right|^{2} \psi_{R}^{2m-6} + R^{-4} \left| \nabla (\Delta u) \right|^{2} \psi_{R}^{2m-6} \right) dx \right)^{\frac{1}{2}}. \tag{3.34}
\]

The last term on the right hand side of (3.18) can be estimated as

\[
\int_{\mathcal{B}_{R}} \Delta^{2} u \Delta (\nabla (\nabla (\Delta u))) dx 
= \int_{\mathcal{B}_{R}} \Delta^{2} u \left( 3 \nabla (\Delta u) \nabla (\psi_{R}^{2m}) + \nabla u \nabla (\Delta (\psi_{R}^{2m})) + 2 \times \nabla (u_{i}) \times \nabla (\psi_{R}^{2m}) \right) dx 
+ \int_{\mathcal{B}_{R}} \Delta^{2} u \left( u_{i} \times (\Delta u)_{ij} \times (\psi_{R}^{2m})_{ij} + u_{ij} \times [x_{i} (\Delta (\psi_{R}^{2m}))]_{j} + 2 \psi_{R}^{2m} \right) dx 
+ 2 \int_{\mathcal{B}_{R}} x_{i} \Delta^{2} u \times u_{ijk} \times (\psi_{R}^{2m})_{jk} dx.
\]
By Hölder’s inequality and Young’s inequality, we get
\[
\left| \int_{B_\beta} \Delta^2 u \Delta (\nabla (x, \nabla u)) \nabla (\psi_R^{2m}) \right| \\
\leq C \left( \int_{\Delta^2 u} (\frac{\Delta^2 u}{2}) \, dx \right)^{\frac{1}{2}} \\
\times \left( \int_{B_\beta} (R^{-2} |\Delta u|^2 \psi_R^{4m-2} + R^{-6} |\nabla u|^2 \psi_R^{4m-6} + (\Delta u)^2 \times \psi_R^{4m-2} \\
+ R^{-4} |\nabla u|^2 \times \psi_R^{4m-6} + R^{-2} |\nabla^3 u|^2 \times \psi_R^{4m-4}) \, dx \right)^{\frac{1}{2}} \\
\leq C \left( \int_{\Delta^2 u} (\frac{\Delta^2 u}{2}) \, dx \right)^{\frac{1}{2}} \left( \int_{\Delta^2 u} \left( (\Delta^2 u)^2 \psi_R^{2m} + B(u, \psi_R, m) \right) \\
+ R^{-2} |\nabla^3 u|^2 \psi_R^{2m} \, dx \right)^{\frac{1}{2}}. \tag{3.35}
\]

From hypothesis \( H_1 \), one has \((\theta + 1)f(s) \leq f(s), \forall s \in \mathbb{R}\). Using the latter inequality, (3.25) and \(1 < \theta < p_s(n, 4)\), we get
\[
\int_{B_\beta} F(u) \nabla \psi_R^{2m} \, dx = o(1) \quad \text{as} \quad R \to +\infty. \tag{3.36}
\]

From (3.18), (3.25)-(3.36), and \(1 < \theta < p_s(n, 4)\), we obtain
\[
\int_{\mathbb{R}^n} (\Delta^2 u)^2 \, dx = \frac{2n}{n - 8} \int_{\mathbb{R}^n} F(u) \, dx.
\]

Now, multiplying equation (1.1) by \( u \psi_R^{2m} \) and integrating by parts, we get
\[
\int_{B_\beta} \left( (\Delta^2 u)^2 \psi_R^{2m} - f(u) u \psi_R^{2m} \right) \, dx \\
= - \int_{B_\beta} \Delta^2 u \left( 2 \Delta u \Delta (\psi_R^{2m}) + 4 u_{ij} (\psi_R^{2m})_{ij} + 4 \nabla (\Delta u) \nabla (\psi_R^{2m}) \\
+ 4 \nabla u \nabla (\Delta (\psi_R^{2m})) + u \Delta^2 (\psi_R^{2m}) \right) \, dx.
\]

By the same reasoning as above, we find
\[
\int_{\mathbb{R}^n} (\Delta^2 u)^2 \, dx = \int_{\mathbb{R}^n} f(u) u \, dx < \infty.
\]

4 Proof of Theorems 2.1, 2.2, 2.3, 2.4 and 2.5

Proof of Theorem 2.1 The proof of Theorem 2.1 for the case \( k = 1, 2, 3 \) is exactly the same as in [5, 6, 8]. Now, we prove the case \( k = 4 \). Let \( u \) be a stable solution to (1.1).

Subcritical case: \(1 < \theta < p_s(n, 4)\). By Proposition 3.1, there exists \( C > 0 \) such that
\[
\int_{B_R} |u|^{\theta + 1} \, dx \leq CR^{n-\frac{n+1}{\theta + 1}}, \quad \forall R > 0.
\]
Note that
\[ n - \frac{8}{\theta} + 1 = n - 8 - \frac{16}{\theta - 1} < 0, \quad \forall \theta \in (1, p_s(n, 4)). \]

Then, if \( 1 < \theta < p_s(n, 4) \), after sending \( R \to \infty \), we get \( u \equiv 0 \) in \( \mathbb{R}^n \).

**Critical case:** \( \theta = \frac{4n}{n - 8} \). By Proposition 3.1, we have
\[
\int_{\mathbb{R}^n} \left( (\Delta^2 u)^2 + |u|^\theta \right) \, dx < +\infty.
\]

So,
\[
\lim_{R \to +\infty} \int_{A_R} \left( (\Delta^2 u)^2 + |u|^\theta \right) \, dx = 0. \tag{4.1}
\]

Moreover, if we come back to the proof of Proposition 3.1, we may improve the following integral estimates:
\[
\int_{B_R} \left( (\Delta^2 u)^2 + |u|^\theta \right) \, dx \leq C \int_{A_R} (\Delta^2 u)^2 \, dx + C R^{-8} \int_{A_R} u^2 \, dx.
\]

By Hölder’s inequality, we have
\[
\int_{B_R} \left( (\Delta^2 u)^2 + |u|^\theta \right) \, dx \leq C \int_{A_R} (\Delta^2 u)^2 \, dx + C R^{\frac{n-4}{n+2}} \left( \int_{A_R} |u|^\theta \, dx \right)^{\frac{2}{n+2}}.
\]

Using (4.1), we get
\[
\int_{\mathbb{R}^n} |u|^\theta \, dx = 0.
\]
This implies that \( u \equiv 0 \) in \( \mathbb{R}^n \). \( \square \)

**Proof of Theorem 2.2** We now collect (3.19) and (3.20). By assumption \( H_3 \), if \( u \) is not identically zero, then
\[
\int_{\mathbb{R}^n} |D^k u|^2 \, dx = \frac{2n}{n - 2k} \int_{\mathbb{R}^n} F(u) \, dx \geq (1 + \alpha_0) \int_{\mathbb{R}^n} f(u) u \, dx
\]
\[
> \int_{\mathbb{R}^n} f(u) u \, dx = \int_{\mathbb{R}^n} |D^k u|^2 \, dx.
\]
This is a contradiction. Then \( u \equiv 0 \). The proof of Theorem 2.2 is thus completed. \( \square \)

**Proof of Theorem 2.3** The proof of Theorem 2.3 is similar to proof of Proposition 4 in [6]. Let \( \gamma \in (1, 2\theta - 1 + 2\sqrt{\theta(\theta - 1)}) \). Multiply equation (1.1) by \( |u|^\gamma u_\varphi^2 \Delta \) and integrate by parts to find
\[
\int_{B_R} f(u) u |u|^\gamma \varphi_\varphi^2 \, dx = \frac{4\gamma}{(\gamma + 1)^2} \int_{B_R} |\nabla (|u|^\frac{\gamma}{2} u)|^2 \varphi_\varphi^2 \, dx - \frac{1}{\gamma + 1} \int_{B_R} |u|^\gamma \Delta (\varphi_\varphi^2) \, dx. \tag{4.2}
\]
The function \(|u|^{\frac{\gamma-1}{2}} u \varphi_R \in C^1_c(\mathbb{R}^n)\), and thus it can be used as a test function in the quadratic form \(Q_u\). Hence, the stability assumption on \(u\) gives
\[
\int_{B_{2R}} f(u)|u|^{\gamma+1} \varphi_R^2 \, dx \\
\leq \int_{B_{2R}} \nabla(|u|^{\frac{\gamma-1}{2}} u)^2 \varphi_R^2 \, dx + \int_{B_{2R}} |u|^{\gamma+1} |\nabla \varphi_R|^2 \, dx - \frac{1}{2} \int_{B_{2R}} |u|^{\gamma+1} \Delta(\varphi_R^2) \, dx.
\]
Using (4.2) in the latter, we obtain
\[
\int_{B_{2R}} \left( f(u)u^2 - \theta f(u)u \right) |u|^{\gamma-1} \, dx + \left( \frac{4\gamma \theta}{(\gamma + 1)^2} - 1 \right) |\nabla(|u|^{\frac{\gamma-1}{2}} u)|^2 \varphi_R^2 \, dx \\
\leq C_1(\gamma, \theta) \int_{B_{2R}} |u|^{\gamma+1} \Delta(\varphi_R^2) \, dx + \int_{B_{2R}} |u|^{\gamma+1} |\nabla \varphi_R|^2 \, dx,
\]
where \(C_1(\gamma, \theta) = \left( \frac{\theta}{\gamma + 1} - \frac{1}{2} \right)\). By hypothesis \(H_1\), we obtain
\[
\left( \frac{4\gamma \theta}{(\gamma + 1)^2} - 1 \right) \int_{B_{2R}} |\nabla(|u|^{\frac{\gamma-1}{2}} u)|^2 \varphi_R^2 \, dx \\
\leq C_1(\gamma, \theta) \int_{B_{2R}} |u|^{\gamma+1} \Delta(\varphi_R^2) \, dx + \int_{B_{2R}} |u|^{\gamma+1} |\nabla \varphi_R|^2 \, dx.
\]
Since \(\theta > 1\) and \(\gamma \in [1, 2\theta - 1 + 2\sqrt{\theta(\theta - 1)}]\), we have \(\frac{4\gamma \theta}{(\gamma + 1)^2} - 1 > 0\) and
\[
\int_{B_{2R}} |\nabla(|u|^{\frac{\gamma-1}{2}} u)|^2 \varphi_R^2 \, dx \leq C(\gamma, \theta) \int_{B_{2R}} |u|^{\gamma+1} \left( |\Delta(\varphi_R^2)| + |\nabla \varphi_R|^2 \right) \, dx.
\]
Using again (4.2), we get
\[
\int_{B_{2R}} f(u)u|u|^{\gamma-1} \varphi_R^2 \, dx \leq C'(\gamma, \theta) \int_{B_{2R}} |u|^{\gamma+1} \left( |\nabla \varphi_R|^2 + |\Delta(\varphi_R^2)| \right) \, dx.
\]
First, we replace \(\varphi_R\) by \(\varphi_R^m\) in the latter inequality, for any \(m > 2\), we derive
\[
\int_{B_{2R}} f(u)u|u|^{\gamma-1} \varphi_R^{2m} \, dx \leq C(\gamma, \theta, m) \int_{B_{2R}} |u|^{\gamma+1} \varphi_R^{2m-2} \left( |\nabla \varphi_R|^2 + |\Delta \varphi_R| \right) \, dx \\
\leq \frac{C}{R^2} \int_{B_{2R}} |u|^{\gamma+1} \varphi_R^{2m-2} \, dx.
\]
By \(H_1\) and \(H_2\), we get
\[
\int_{B_{2R}} |u|^{\gamma+1} \varphi_R^{2m} \, dx \leq \frac{C}{R^2} \int_{B_{2R}} |u|^{\gamma+1} \varphi_R^{2m-2} \, dx.
\]
An application of Young’s inequality yields
\[
\int_{B_{2R}} |u|^{\gamma+1} \varphi_R^{2m} \, dx \leq CR^{\frac{\gamma+1}{\gamma+\theta}} + \frac{\gamma + 1}{\gamma + \theta} \int_{B_{2R}} |u|^{\gamma+1} \varphi_R^{\frac{\gamma+1}{\gamma+\theta}} \, dx.
\]
Thus
\[
\int_{B_R} |u|^{\theta+\gamma} \, dx \leq C R^{n-\frac{n\gamma}{\theta+1}}.
\]

As in Farina’s work we readily deduce, by letting \( R \to +\infty \), that there is no nontrivial stable solution of (1.1), in the special case \( 1 < \theta < p_c(n) \).

\[\square\]

Proof of Theorem 2.4 We proceed as in the proof of Proposition 2.1. From (3.11) and (3.13), we deduce by replacing \( f(u) \) by \(-mu + \lambda |u|^{\theta+1}-\mu |u|^{p+1}u\) that
\[
(1-\epsilon) \int_{B_{2R}} (\Delta^2 u)^2 \psi_R^{2m} \, dx - \int_{B_{2R}} (-mu^2 + \lambda |u|^{\theta+1}-\mu |u|^{p+1}) \psi_R^{2m} \, dx
\]
\[
\leq C \epsilon R^{-8} \int_{B_{2R}} u^2 \psi_R^{2m-8} \, dx
\]
(4.3)
and
\[
\int_{B_{2R}} (-mu^2 + \lambda |u|^{\theta+1}-p\mu |u|^{p+1}) \psi_R^{2m} \, dx
\]
\[
\leq (1+\epsilon) \int_{B_{2R}} (\psi_R^m \Delta^2 u)^2 + C \epsilon R^{-8} \int_{B_{2R}} u^2 \psi_R^{2m-8} \, dx.
\]
(4.4)
Multiplying (4.3) by \( \theta \) and combining it with (4.4), we derive
\[
m(\theta-1) \int_{B_{2R}} u^2 \psi_R^{2m} \, dx + \mu (\theta-p) \int_{B_{2R}} |u|^{p+1} \psi_R^{2m} \, dx
\]
\[
+ [\theta(1-\epsilon) - (1+\epsilon)] \int_{B_{2R}} (\Delta^2 u)^2 \psi_R^{2m} \, dx
\]
\[
\leq CR^{-8} \int_{B_{2R}} u^2 \psi_R^{2m-8} \, dx.
\]
For \( \epsilon \) sufficiently small, we deduce
\[
m(\theta-1) \int_{B_{2R}} u^2 \psi_R^{2m} \, dx + \mu (\theta-p) \int_{B_{2R}} |u|^{p+1} \psi_R^{2m} \, dx + \int_{B_{2R}} (\Delta^2 u)^2 \psi_R^{2m} \, dx
\]
\[
\leq CR^{-8} \int_{B_{2R}} u^2 \psi_R^{2m-8} \, dx.
\]
(4.5)

Proof of 1. If \( m > 0 \) and \( \theta \geq p \), then from (4.5), we deduce that
\[
\int_{B_R} u^2 \, dx \leq CR^{-8} \int_{B_{2R}} u^2 \, dx.
\]
Let \( J(R) := \int_{B_R} u^2 \, dx \). If we iterate the above inequality, then we get
\[
J(R) \leq CR^{-8(k+1)}(2^{k+1} R).
\]
(4.6)
We deduce from the boundedness of \( u \) that the right hand side of (4.6) is of order \( R^M \) with \( M = -8(k+1) + n \to 0 \) as \( k \to +\infty \). Hence, we can choose \( k \) large enough such that \( M < 0 \).
Then it follows from (4.6) that \( f(R) \rightarrow 0 \), as \( R \rightarrow +\infty \). So we get

\[
\int_{\mathbb{R}^n} u^2 \, dx = 0.
\]

Then \( u \equiv 0 \).

**Proof of 2.** If \( \theta > p, m \geq 0 \), then from (4.5) and by Young’s inequality, we get

\[
\int_{B_{2R}} |u|^{p+1} \psi_R^{2m} \, dx + \int_{B_{2R}} (\Delta u)^2 \psi_R^{2m} \, dx \leq \frac{2}{p+1} \int_{B_{2R}} |u|^{p+1} \psi_R^{(p+1)(m-4)} \, dx + CR^{n-8 \frac{m+1}{m-4}}.
\]

Choosing \( 2m = (p+1)(m-4) \), thus

\[
\int_{B_{2R}} |u|^{p+1} \psi_R^{2m} \, dx + \int_{B_{2R}} (\Delta u)^2 \psi_R^{2m} \, dx \leq CR^{n-8 \frac{p+1}{m-4}}.
\]

Consequently

\[
\int_{B_R} |u|^{p+1} \, dx + \int_{B_R} (\Delta u)^2 \, dx \leq CR^{n-8 \frac{p+1}{m-4}}.
\]

The result then follows in a similar way to that in the proof of Theorem 2.1. This completes the proof of Theorem 2.4. \( \square \)

**Proof of Theorem 2.5** We can proceed as in the proof of Proposition 3.4, we get

\[
\int_{\mathbb{R}^n} |D^k u|^2 = \frac{2n}{n-2k} \int_{\mathbb{R}^n} \left( -\frac{m}{2} u^2 + \frac{\lambda}{\theta + 1} |u|^{\theta+1} - \frac{\mu}{p+1} |u|^{p+1} \right)
\]

and

\[
\int_{\mathbb{R}^n} |D^k u|^2 = \int_{\mathbb{R}^n} \left( -mu^2 + \lambda |u|^{\theta+1} - \mu |u|^{p+1} \right).
\]

Thus

\[
\frac{2mk}{n-2k} \int_{\mathbb{R}^n} u^2 \, dx + \lambda \left( 1 - \frac{2n}{(n-2k)(\theta + 1)} \right) \int_{\mathbb{R}^n} |u|^{\theta+1} \, dx + \mu \left( \frac{2n}{(n-2k)(p+1)} - 1 \right) \int_{\mathbb{R}^n} |u|^{p+1} \, dx = 0.
\]

This concludes the proof of Theorem 2.5. \( \square \)

**Competing interests**

The authors declare that they have no competing interests.

**Authors’ contributions**

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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