ENTRYWISE CONVERGENCE OF ITERATIVE METHODS FOR EIGENPROBLEMS

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February 21, 2020

Abstract

Several problems in machine learning, statistics, and other fields rely on computing eigenvectors. For large scale problems, the computation of these eigenvectors is typically performed via iterative schemes such as subspace iteration or Krylov methods. While there is classical and comprehensive analysis for subspace convergence guarantees with respect to the spectral norm, in many modern applications other notions of subspace distance are more appropriate. Recent theoretical work has focused on perturbations of subspaces measured in the $\ell_2\to\infty$ norm, but does not consider the actual computation of eigenvectors. Here we address the convergence of subspace iteration when distances are measured in the $\ell_2\to\infty$ norm and provide deterministic bounds. We complement our analysis with a practical stopping criterion and demonstrate its applicability via numerical experiments. Our results show that one can get comparable performance on downstream tasks while requiring fewer iterations, thereby saving substantial computational time.

1 Introduction & Background

Spectral methods play a fundamental role in machine learning, statistics, and data mining. Methods for foundational tasks such as clustering (Von Luxburg, 2007); semi-supervised learning (Mahoney et al., 2012); linear and nonlinear dimensionality reduction (Belkin and Niyogi, 2002; Friedman et al., 2001; Roweis and Saul, 2000); latent factor models (Gower, 2014) ranking and preference learning (Maystre and Grossglauser, 2015; Vigna, 2016); and covariance estimation all use information about eigenvalues and eigenvectors (or singular values and singular vectors) from an underlying data matrix. Often, spectral approaches outperform their “traditional” counterparts. For example, in spectral clustering applied to a point cloud, one forms a pairwise distance matrix, computes $k$ eigenvectors of the associated graph Laplacian, and applies a method

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like $k$-means clustering in an embedding space defined by the eigenvectors, instead of directly clustering in the original space.

In many of these cases, the relevant information is in the “leading” eigenvectors, i.e., those corresponding to the $k$ algebraically largest eigenvalues for some $k$ (some methods, such as spectral clustering, use the $k$ smallest, but simple shifting and scaling transformations make them the largest for a related matrix). For sufficiently large data sets computing a full eigendecomposition is prohibitively expensive, so we rely on iterative algorithms for computation. Perhaps the most well-known technique is the power method which dates back nearly a century; other common methods are subspace iteration and Krylov methods. Importantly, these methods only produce approximations to the eigenvectors. However, the approximation quality, as measured by subspace distance (effectively equivalent to using the $\ell_2$ norm), is well-understood and there is well-established convergence analysis (Demmel, 1997; Golub and Van Loan, 2013; Parlett, 1998; Saad, 2011).

While spectral norm error analysis has been the standard-bearer for numerical analysis, recent statistical research has considered different subspace distance measures (Cai et al., 2019; Chen et al., 2018; Fan et al., 2018; Xia and Zhou, 2019). The motivation for these changes are statistical, as opposed to numerical: one observes a matrix $\tilde{A} = A + E$, where $E$ is a source of noise and $A = E[\tilde{A}]$ is the “population” version of $A$, containing the desired spectral information. We are then interested in $\|\tilde{u}_i \pm u_i\|_\infty$ as a distance measure between the eigenvectors of $\tilde{A}$ and $A$. Here, the $\ell_\infty$ norm captures “entry-wise” error and is more appropriate when we care about maximum deviation; for example, when entries of the eigenvector are used to rank nodes or provide cluster assignments in a graph. This type of distance is often much smaller the spectral norm and, in contrast to the spectral norm, reveals information about the distribution of error over the entries. Recent theoretical results relate the noise $E$ to the perturbation in the eigenvectors, as measured by $\ell_\infty$ norm or $\ell_{2 \to \infty}$ errors (Cape et al., 2019; Damle and Sun, 2019; Fan et al., 2018; Koltchinskii and Xia, 2016). Moreover, these results are often directly connected to machine learning problems such as clustering random graph models (Abbe et al., 2017; Eldridge et al., 2018).

The message from this body of literature is that when eigenvector entries are interpreted entry-wise, we should really measure our error entry-wise as well. This recent theoretical work shows what we can do if we have eigenvectors satisfying perturbation bounds in a different norm. Actually computing eigenvectors satisfying such error bounds is another question entirely. Current numerical algorithms typically use the $\ell_2$ norm, and the motivation for norms like $\ell_{2 \to \infty}$ is that $\ell_2$ can be a severe overestimate for the relevant approximation quality. Moreover, in contrast to the long history of research into stopping criteria for iterative methods in the unitarily-invariant setting (Arioli et al., 1992; Bai et al., 1993; Bennani and Bracci, 1994; Golub and Meurant, 1997; Lehoucq et al., 1997), there are no generic stopping criteria closely tracking the quality of an approximation in the $\ell_{2 \to \infty}$ norm. For example, downstream tasks that depend on entrywise ordering, such as graph cluster quality obtained with an approximate Fiedler vector (Fairbanks et al., 2016) or spectral ranking via the Katz centrality (Nathan et al., 2017) employ $\ell_2$ bounds, when instead the $\ell_\infty$ norm would constitute a better proxy. Some local spectral graph partitioning methods can be written as iteratively approximating an eigenvector in a (scaled) $\ell_\infty$ norm (Andersen et al., 2006), but these algorithms are far more specialized than general eigensolvers.
Here, we bridge this gap by providing an analysis for the convergence of subspace iteration, a widely-used iterative methods for computing leading eigenvectors, in terms of $\ell_{2\to\infty}$ errors. As part of this, we provide a practical stopping criterion applicable to any iterative method for invariant subspace computation that tracks the $\ell_{2\to\infty}$ error of the approximation. Our results show how, for a given error tolerance, one can perform many fewer subspace iterations to get the same desired performance on a downstream task that uses the eigenvectors (or, more generally, an invariant subspace). This reduction in iterations directly translates to substantial reductions in computation time. We demonstrate our methods with the help of applications involving real-world graph data, including node ranking in graphs, sweep cut profiles for spectral bipartitioning, and general spectral clustering.

1.1 Notation

We use the standard inner product on Euclidean spaces, defined by $\langle x, y \rangle := \text{Tr}(X^TY)$ for vectors/matrices $X, Y$. We write $O_{n,k}$ for the set of matrices $U \in \mathbb{R}^{n \times k}$ such that $U^TU = I_k$, dropping the second subscript when $n = k$. We use standard notation for norms, namely $\|A\|_2 := \sup_{x \in S^{d-1}} \|Ax\|_2$ and $\|A\|_F := \sqrt{\langle A, A \rangle}$. Moreover, we remind the reader that the $\ell_\infty \to \ell_\infty$ operator norm for a matrix $A \in \mathbb{R}^{m \times n}$ is given by $\|A\|_\infty := \max_{i \in [m]} \|A_{i,:}\|_1$, where $A_{i,:}$ denotes the $i^\text{th}$ row of $A$ and $A_{:,i}$ denotes its $i^\text{th}$ column. Finally, the $\ell_{2\to\infty}$ norm is defined by

$$\|A\|_{2\to\infty} := \sup_{x \in S^{d-1}} \|Ax\|_\infty = \max_{i \in [m]} \|A_{i,:}\|_2.$$  

(1)

Subspace distances. Given two orthogonal matrices $V, \tilde{V} \in O_{n,r}$ inducing subspaces $\mathcal{V}, \mathcal{V}$, their so-called subspace distance is formally defined as

$$\text{dist}_2(V, \tilde{V}) := \|VV^T - \tilde{V}\tilde{V}^T\|_2$$  

(2)

and there are several equivalent definitions, e.g., via the concept of principal angles, or via $\|V_\perp \tilde{V}\|_2$, where $V_\perp$ is a basis for the subspace orthogonal to $\mathcal{V}$. Here, in contrast, we will consider a slightly different notion of distance between subspaces with respect to $\|\cdot\|_{2\to\infty}$ defined as

$$\text{dist}_{2\to\infty}(V, \tilde{V}) := \inf_{O \in O_{r,r}} \|V - \tilde{V}O\|_{2\to\infty}.$$  

(3)

This metric allows us to control errors in a “row-wise” or “entry-wise” sense; for example, in the case where $r = 1$ this reduces to infinity norm control over the differences between eigenvectors.

2 Convergence of subspace iteration

In this section, we analyze the convergence of subspace iteration with respect to the $\ell_{2\to\infty}$ distance. In particular, we assume that we are working with a symmetric matrix $A$ with eigenvalue decomposition

$$A = V\Lambda V^T + V_\perp \Lambda_\perp V_\perp^T,$$  

(4)

where $\Lambda, \Lambda_\perp$ are diagonal matrices containing the $r$ largest and $n - r$ smallest eigenvalues of $A$. Throughout, we have the “blanket” assumption that we are interested in an incoherent subspace.
Algorithm 1 Subspace iteration

\begin{algorithm}
\textbf{Input:} $V^{(0)} \in \mathbb{O}_{n,k}$, symmetric matrix $A$, iterations $T$

$Q_0 := V^{(0)}$

\textbf{for} $t = 1, 2, \ldots, T$ \textbf{do}

$V^{(t)} := AQ_{t-1}$

$Q_t, R_t = V^{(t)} \triangleright$ QR decomposition

\textbf{end for}

\textbf{return} $Q_T$
\end{algorithm}

Assumption 1. The subspace of interest $V \in \mathbb{O}_{n,r}$ is $\mu$-incoherent:

\[
\max_{i \in [n]} \|VV^T e_i\|_2 \leq \mu \sqrt{r/n}.
\]  

Subspace iteration with a fixed number of steps is given in Algorithm 1. For simplicity, we assume that the eigenvalues of the symmetric input $A$ satisfy $\lambda_1(A) \geq \cdots \geq \lambda_r(A) > \lambda_{r+1}(A) \geq \cdots \lambda_n(A)$ and, furthermore, that $\min_{k=1,\ldots,r} |\lambda_k(A)| > \max_{k=r+1,\ldots,n} |\lambda_k(A)|. \footnote{Our results hold for the largest magnitude eigenvalues assuming one defines the eigenvalue gap appropriately later, the simplification to the $r$ algebraically largest eigenvalues being the largest in magnitude avoids burdensome notation without losing anything essential.}$

The following result shows that $\text{dist}_{2 \to \infty}(Q_t, V)$ can be considerably smaller than $\text{dist}_{2 \to \infty}(Q_0, V)$. Unfortunately, the final result involves the unwieldy term $\|V_\perp \Lambda_t \perp V_\perp^T\|_\infty$, which is nontrivial to upper bound to obtain a better rate than that obtain using norm equivalence. To circumvent this, we impose an additional assumption.

Assumption 2. For the matrix of interest, $V_\perp$ satisfies

\[
\|V_\perp \Lambda_t \perp V_\perp^T\|_\infty \leq C \cdot \lambda_{r+1}^t \|V_\perp V_\perp^T\|_\infty,
\]  

for a small constant $C$ and all powers $t \in \mathbb{N}$.

Proposition 1. Suppose Assumptions 1 and 2 hold. The iterates $\{Q_t\}$ produced by Algorithm 1 with initial guess $Q_0$ satisfy

\[
\text{dist}_{2 \to \infty}(Q_t, V) \leq \left(\frac{\lambda_{r+1}}{\lambda_r}\right)^t \mu \sqrt{\frac{2r}{n}} \frac{d_0}{\sqrt{1 - d_0^2}} + \left(\frac{\lambda_{r+1}}{\lambda_r}\right)^t \frac{C(1 + \mu \sqrt{r})}{\sqrt{1 - d_0^2}} \text{dist}_{2 \to \infty}(Q_0, V),
\]

where $d_0 := \|Q_0^T V_\perp\|_2 \equiv \text{dist}_2(Q_0, V)$ and $r = \dim(V)$.

Proof. Starting with the definition of the $2 \to \infty$ distance, we have

\[
\text{dist}_{2 \to \infty}(Q_t, V) = \inf_{Z \in \mathbb{O}_r} \|Q_t - V Z\|_{2 \to \infty} = \inf_{Z \in \mathbb{O}_r} \|(VV^T + V_\perp V_\perp^T)(Q_t - V Z)\|_{2 \to \infty}
\]

\[
\leq \sqrt{2} \text{ dist}_2(Q_t, V) + \|V_\perp V_\perp^T(Q_t - V Z)\|_{2 \to \infty}.
\]
where (‡) is due to the triangle inequality, followed by combining Lemmas A.2 and A.3. At this point, note that standard convergence results (Golub and Van Loan, 2013; Saad, 2011) state that

\[
\text{dist}_2(Q_t, V) \leq \left( \frac{\lambda_{r+1}}{\lambda_r} \right)^t \frac{d_0}{\sqrt{1 - d_0^2}},
\]

and additionally Assumption 1 implies that \( \|VV^\top\|_{2 \to \infty} \leq \mu \sqrt{r/n} \).

For the remainder, let us first recall a fact from the analysis of subspace iteration; the \( t \)th iterate \( Q_t \) satisfies

\[
Q_t R_t = A^t V^{(0)}, \text{ with } R_t \text{ invertible}
\]

\[
\Rightarrow V_\perp^\top Q_t = V_\perp^\top A^t V^{(0)} R_t^{-1} = \Lambda_\perp^t V_\perp^\top V^{(0)} R_t^{-1}.
\] (9)

Then, notice that \( V_\perp^\top V = 0 \) and therefore we can rewrite the second term in (8) as

\[
\|V_\perp V_\perp^\top Q_t\|_{2 \to \infty} \overset{(*)}{=} \|V_\perp \Lambda_\perp^t V_\perp^\top Q_t R_t^{-1}\|_{2 \to \infty} \overset{(b)}{=} \inf_{Z \in \mathbb{O}_r} \|V_\perp \Lambda_\perp^t V_\perp^\top (Q_0 - V Z) R_t^{-1}\|_{2 \to \infty}
\]

\[
\overset{(z)}{\leq} \inf_{Z \in \mathbb{O}_r} C \|V_\perp V_\perp^\top\|_{\infty} \lambda_{r+1}^t \|(Q_0 - V Z) R_t^{-1}\|_{2 \to \infty}
\]

\[
\leq C \|V_\perp V_\perp^\top\|_{\infty} \lambda_{r+1}^t \inf_{Z \in \mathbb{O}_r} \|Q_0 - V Z\|_{2 \to \infty} \|R_t^{-1}\|_2,
\]

where (\( )^* \) follows from Eq. (9), (\( b \)) holds since we can reintroduce \( V Z \) for any \( Z \), as \( V_\perp^\top V \), (\( z \)) holds after combining Eq. (29) and Assumption 2, and the last inequality is Eq. (28). Notice that \( \|R_t^{-1}\|_2 = \frac{1}{\sqrt{1 - d_0^2}} \lambda_r^{-t} \), by tracing the proof of (Golub and Van Loan, 2013, Theorem 8.2.2).

Finally, by Lemma A.1, \( \|V_\perp V_\perp^\top\|_{\infty} \leq 1 + \mu \sqrt{r} \). □

When \( \lambda_{r+2} \ll \lambda_{r+1} \), a slight modification of the above proof yields a potentially refined upper bound.

**Proposition 2.** Under Assumption 1, the iterates \( \{Q_t\} \) produced by Algorithm 1 with initial guess \( Q_0 \) satisfy

\[
\text{dist}_{2 \to \infty}(Q_t, V) \leq \left( \frac{\lambda_{r+1}}{\lambda_r} \right)^t \mu \sqrt{\frac{2r}{n}} \cdot \frac{d_0}{\sqrt{1 - d_0^2}}
\]

\[
+ \left( \frac{\lambda_{r+2}}{\lambda_r} \right)^t \frac{d_0}{\sqrt{1 - d_0^2}} + \|v_{r+1} v_{r+1}^\top\|_{\infty} \left( \frac{\lambda_{r+1}}{\lambda_r} \right)^t \cdot \text{dist}_{2 \to \infty}(Q_0, V).
\]

(11)

**Proof.** For simplicity, let us define \( \tilde{V} := [V \ v_{r+1}] \in \mathbb{R}^{n \times (r+1)} \) and \( \tilde{V}_\perp \) for the remaining \( n - r - 1 \) eigenvectors forming a basis of \( \mathbb{R}^n \). Similarly, let \( \tilde{\Lambda}_\perp = \text{diag}(\lambda_{r+2}, \ldots, \lambda_n) \). From the definition of the \( 2 \to \infty \) distance, we have

\[
\text{dist}_{2 \to \infty}(Q_t, V) = \inf_{Z \in \mathbb{O}_r} \|Q_t - V Z\|_{2 \to \infty} = \inf_{Z \in \mathbb{O}_r} \|(VV^\top + V_\perp V_\perp^\top)(Q_t - V Z)\|_{2 \to \infty}
\]

\[
\overset{(z)}{\leq} \sqrt{2} \|VV^\top\|_{2 \to \infty} \text{dist}_2(Q_t, V) + \|V_\perp V_\perp^\top(Q_t - V Z)\|_{2 \to \infty},
\]

(12)
where (7) follows from Lemma A.2 in the main text and the fact that \( \inf_{Z \in \mathcal{O}_r} \| Q_t - V Z \|_2 \leq \sqrt{2} \text{dist}_2(Q_t, V) \). Now we may rewrite the second term as

\[
\| (v_{r+1} v_{r+1}^T + \tilde{V}_\perp \tilde{V}_\perp^T) Q_t \|_2 \rightarrow \infty \leq \| v_{r+1} v_{r+1}^T Q_t \|_2 \rightarrow \infty + \| \tilde{V}_\perp \|_2 \rightarrow \infty \| Q_t R_t^{-1} \|_2 \rightarrow \infty + \| \tilde{A}_\perp^T \tilde{V}_\perp Q_0 R_t^{-1} \|_2 \rightarrow \infty .
\]

Pulling \( \lambda_{r+1}^t \) out of the first norm in (13) yields

\[
\| \lambda_{r+1}^t v_{r+1}^T (Q_0 - V Z) \|_2 \rightarrow \infty \| R_t^{-1} \|_2 \leq \| v_{r+1} v_{r+1}^T \|_\infty \text{dist}_2(Q_0, V) \cdot \frac{\lambda_{r+1}^t}{\sqrt{1 - \delta_0^2}}
\]

after using Lemma A.2 and the fact that \( \| R_t^{-1} \|_2 \leq \frac{\lambda_{r+1}^t}{\sqrt{1 - \delta_0^2}} \), while the second norm in (13) can be upper bounded by

\[
\| \tilde{V}_\perp \tilde{A}_\perp^t \|_2 \| \tilde{V}_\perp^T Q_0 \|_2 \| R_t^{-1} \|_2 = \left( \frac{\lambda_{r+1}^t}{\lambda_r} \right)^t \frac{\text{dist}_2(Q_0, \tilde{V})}{\sqrt{1 - \delta_0^2}}
\]

but as the respective subspaces satisfy \( V \subset \tilde{V} \) we have \( \text{dist}_2(Q_0, \tilde{V}) \leq \text{dist}_2(Q_0, V) \). Combining all the ingredients above completes the proof.

Propositions 1 and 2 show that we can achieve significant practical improvements, especially in the “typical” regime where \( \text{dist}_2(Q_0, V) \ll \text{dist}_2(Q_0, V) \). Section 4 illustrates this concept in practical examples.

3 Stopping criteria

In this section, we propose and analyze a stopping criterion applicable when the desired convergence is with respect to the \( 2 \rightarrow \infty \) norm. Notably, this stopping criterion is generic and applicable to any iterative method for computing an invariant subspace.\(^2\) Suppose that we have

\[
A V - V S = E, \quad \| E \|_2 \leq \varepsilon, \quad V \in \mathcal{O}_{n,r}, \quad S = S^T.
\]

Then it is well-known (e.g., (Golub and Van Loan, 2013, Theorem 8.1.13)) that there exist \( \mu_1, \ldots, \mu_r \in \Lambda(A) \) such that

\[
| \mu_k - \lambda_k(S) | \leq \sqrt{2} \varepsilon, \quad \forall k \in [r]. \tag{14}
\]

This provides a handy criterion for testing convergence of eigenvalues, by setting \( S = D_t \), the diagonal matrix of Ritz values at the \( t \)th step and \( V = Q_t \), the orthogonal matrix of Ritz vectors. One can, in fact, show the following:

\(^2\)This includes Algorithm 1 and other common methods such as (block) Lanczos.
**Lemma 1.** Suppose that $A = A^T \in \mathbb{R}^{n \times n}$ satisfies
\[ AV - VS = E, \quad V^T V = I_r, \] (15)
for some diagonal matrix $S$. Then $V$ is an invariant subspace of the matrix $A - EV^T$.

**Proof.** By assumption, we have $VS = AV - E$ and therefore
\[ (A - EV^T)V = AV - EV^T V = AV - E = VS, \]
which means that $(A - EV^T)V_{i,i} = S_{i,i}V_{i,i}, \ \forall i \in [r]$, and $V_{i,i}$ are pairwise orthogonal. \(\square\)

We now demonstrate that checking $\|AV - VS\|_{2,2,\rightarrow \infty}$ leads to an appropriate stopping criterion for iterative methods for approximate eigenvectors, which simplifies under standard incoherence assumptions.\(^3\)

**Proposition 3.** Assume that $A$ is symmetric with $V_1$ as its dominant subspace and $V_2$ is the complement of $V_1$, with $V_1 \in \mathbb{O}_{n,r}$; furthermore, suppose that $A$ satisfies the conditions of Lemma 1 for some $V$ and let gap be defined as in Theorem A.1. Then, if $V$ is the leading invariant subspace of $A - EV^T$ and $\|E\|_2 \leq \frac{\text{gap}}{5}$, we have
\[ \text{dist}_{2,\rightarrow \infty}(V_1, V) \leq 8 \|V_1\|_{2,\rightarrow \infty} \left( \frac{\|E\|_2}{\lambda_r - \lambda_{r+1}} \right)^2 + 2 \|V_2 V_2^T\|_{\infty} \|E\|_{2,\rightarrow \infty} \frac{\|E\|_2}{\text{gap}} \left( 1 + \frac{2 \|E\|_2}{\lambda_r - \lambda_{r+1}} \right), \]

**Proof.** The condition on $\|E\|_2$ combined with the assumption that $V$ is the leading invariant subspace of the perturbed matrix $A - EV^T$ allows us to apply Theorem A.1 for the perturbation $EV^T$, from which we deduce that the approximate eigenvector matrix $V$ satisfies
\[ \text{dist}_{2,\rightarrow \infty}(V_1, V) \leq 8 \|V_1\|_{2,\rightarrow \infty} \left( \frac{\|E\|_2}{\lambda_r - \lambda_{r+1}} \right)^2 + 2 \|V_2 V_2^T EV^T V_1\|_{2,\rightarrow \infty} + 4 \|V_2 V_2^T E\|_{2,\rightarrow \infty} \|E\|_2 \frac{\|E\|_2}{\text{gap}} \left( \lambda_r - \lambda_{r+1} \right) \]
with the appropriate definition of gap. Using Lemma A.2, we can upper bound the terms above as
\[ \|V_2 V_2^T EV^T V_1\|_{2,\rightarrow \infty} \leq \|V_2 V_2^T\|_{\infty} \|EV^T V_1\|_{2,\rightarrow \infty} \leq \|V_2 V_2^T\|_{\infty} \|E\|_{2,\rightarrow \infty} \left(\|V^T V_1\|_{2,\rightarrow \infty} \right) \leq 1 \]
and similarly for the term $\|V_2 V_2^T E\|_{2,\rightarrow \infty}$. \(\square\)

**Corollary 1.** Suppose that $V_1 \in \mathbb{O}_{n,r}$ is $\mu$-incoherent and the conditions of Proposition 1 are satisfied with $\|E\|_2 \leq \varepsilon_1$, $\|E\|_{2,\rightarrow \infty} \leq \varepsilon_2$. Then the approximate eigenvector matrix $V$ satisfies
\[ \text{dist}_{2,\rightarrow \infty}(V, V_1) \leq 8 \mu \sqrt{\frac{r}{n}} \left( \frac{\varepsilon_1}{\lambda_r - \lambda_{r+1}} \right)^2 + 2 \frac{1 + \mu \sqrt{r}}{\text{gap}} \left( \varepsilon_2 + 2 \frac{\varepsilon_1 \varepsilon_2}{\lambda_r - \lambda_{r+1}} \right), \]
with gap defined as in Theorem A.1.

\(^3\)As the perturbed matrix is non-normal, an eigengap condition does not suffice to guarantee that $V$ is the leading invariant subspace of the perturbed matrix. To invoke Proposition 3 with the approximate eigenvectors in the place of $V$, one relies on the fact that $V$ approaches $V_1$ by convergence theory of subspace iteration.
The main drawback of using Equation (16) is that the substitutions used above are not accurate via some globally convergent iterative method. We use a proxy for tracking the behavior of the is looser than average in the first few iterations.

The SVD of $V$ orthogonal Procrustes problem an instance of the so-called $\ell_2$ distance. In particular, we use augmenting the “seed” subspace by a constant number of columns and setting $|\lambda_r - \lambda_{r+1}| \approx \hat{\lambda}_r - \hat{\lambda}_{r+1}$, since it is well known that eigenvalue estimates converge at a quadratic rate for symmetric matrices (Stewart, 1969).

In the absence of incoherence information, we can replace all quantities in the residual by estimates. For any compatible $B$, $\|BV\|_{2 \rightarrow \infty} = \|BVV_1^\top\|_{2 \rightarrow \infty}$ by (30), which we may replace with $\|BV\|_{2 \rightarrow \infty}$, as $V_1V_1^\top \approx VV^\top$ after sufficient progress. Similarly, we can write $V_2V_2^\top = I - V_1V_1^\top \approx I - VV^\top$, to obtain the approximated residual (at iteration $t$, with $V \equiv Q_t$):

$$
\text{res}_{2 \rightarrow \infty}(t) \approx 8 \|Q_t\|_{2 \rightarrow \infty} \left( \frac{\|E\|_2}{\lambda_r - \lambda_{r+1}} \right)^2 + 2 \frac{\|I - Q_tQ_t^\top\|}{\text{gap}} E \|_{2 \rightarrow \infty} \left( 1 + 2 \frac{\|E\|_2}{\lambda_r - \lambda_{r+1}} \right)
$$

The main drawback of using Equation (16) is that the substitutions used above are not accurate until $Q_tQ_t^\top$ is sufficiently close to $V_1V_1^\top$. This is observed empirically in Section 4, as $\text{res}_{2 \rightarrow \infty}(t)$ is looser than average in the first few iterations.

Another practical concern is evaluating the quality of the bound in Corollary 1; formally, there is no known method for computing $Z_* := \arg\min_{Z \in \mathbb{O}_r} \|\hat{V} - VZ\|_{2 \rightarrow \infty}$ in closed form or via some globally convergent iterative method. We use a proxy for tracking the behavior of the $\ell_2$ distance. In particular, we use

$$
Z_F = \arg\min_{Z \in \mathbb{O}_r} \|\hat{V} - VZ\|_F,
$$

an instance of the so-called orthogonal Procrustes problem, whose solution can be obtained via the SVD of $V^\top V$ (Higham, 1988). It is then straightforward to show the following:

**Corollary 2.** Let $Z_F$ be the solution of the orthogonal Procrustes problem from (17). The iterates $\{Q_t\}$ produced by Algorithm 1 with initial guess $Q_0$ satisfy (under Assumptions 1 and 2):

$$
\|Q_t - VZ_F\|_{2 \rightarrow \infty} \leq 2\mu \sqrt{n} \frac{r}{\sqrt{d_0}} \left( \frac{\lambda_{r+1}}{\lambda_r} \right)^t \frac{d_0}{\sqrt{1 - d_0^2}} + \left( \frac{\lambda_{r+1}}{\lambda_r} \right)^t C(1 + \mu\sqrt{r}) \inf_{Z \in \mathbb{O}_r} \|Q_t - VZ\|_{2 \rightarrow \infty} \leq 2\mu \sqrt{n} \frac{r}{\sqrt{d_0}} \left( \frac{\lambda_{r+1}}{\lambda_r} \right)^t \frac{d_0}{\sqrt{1 - d_0^2}} + \left( \frac{\lambda_{r+1}}{\lambda_r} \right)^t C(1 + \mu\sqrt{r}) \|Q_t - VZ_F\|_2 + \|V_{\perp}V_{\perp}^\top Q_t\|_{2 \rightarrow \infty},
$$

**Proof.** Tracing the proofs of Proposition 1, we bound the $\ell_2$ distance above by

$$
\inf_{Z \in \mathbb{O}_r} \|Q_t - VZ\|_{2 \rightarrow \infty} \leq \|Q_t - VZ_F\|_{2 \rightarrow \infty} \leq \mu \sqrt{n} \frac{r}{\sqrt{d_0}} \left( \frac{\lambda_{r+1}}{\lambda_r} \right)^t \frac{d_0}{\sqrt{1 - d_0^2}} + \left( \frac{\lambda_{r+1}}{\lambda_r} \right)^t C(1 + \mu\sqrt{r}) \|Q_t - VZ_F\|_2 + \|V_{\perp}V_{\perp}^\top Q_t\|_{2 \rightarrow \infty}.
$$
where the second term can be analyzed as before. For the first term, let us denote $Z_2 := \arg\min_{Z \in O_r} \|Q_t - VZ\|_2$ and observe that we can upper bound
\[\|Q_t - VZ_F\|_2 \leq \|Q_t - VZ_F\|_F \leq \|Q_t - VZ_2\|_F \leq \sqrt{2r} \|Q_t - VZ_2\|_2\]
\[
\leq 2\sqrt{r} \cdot \|Q_t - VZ_2\|_F
= 2\sqrt{r} \left( \frac{\lambda_{r+1}}{\lambda_r} \right)^t \frac{d_0}{\sqrt{1 - d_0^2}},
\]
where (1) is since $\|X\|_2 \leq \|X\|_F$, $\forall X$, (2) is by the definition of $Z_F$, (3) uses the fact that rank($Q_t - VZ$) $\leq 2r$ by definition and norm equivalence, and step (4) follows from Lemma A.3.

Therefore, the oft-used proxy $\|\hat{V} - VZ_F\|_2 \to \infty$ enjoys a similar convergence guarantee with an additional multiplicative factor that is typically small compared to $n$ (a similar rate can be shown for Proposition 2 using the same argument as above). This is precisely the $2 \to \infty$ distance we report in the forthcoming experiments and is in alignment with the approach of Damle and Sun (2019), who study the effect of perturbations on the $\ell_2 \to \infty$ subspace distance via the solution to (17).

4 Applications

In this section, we present a set of numerical experiments illustrating the results of our analysis in practice, as well as the advantages of the proposed stopping criterion. Importantly, in our applications, entry-wise error is the natural criterion, often because what matters for the downstream task is an ordering induced by computed eigenvectors.

4.1 Synthetic examples

To verify our theory and get a sense of the tightness of our bounds on convergence rates, we first test on synthetic data. To this end, we implemented Algorithm 1 in Julia (Bezanson et al., 2017) and generated matrices as follows, given a matrix dimension $n$ and subspace dimension $r$:

1. Sample a random matrix from $O_{n,n}$ uniformly at random (see (Mezzadri, 2007) for details);
2. select $r$ of its columns uniformly at random to form $Q$.
3. Form $A = [Q \quad Q_\perp] \Lambda [Q \quad Q_\perp]^T$, where $Q_\perp$ is any matrix in $O_{n,n-r}$ orthogonal to $Q$. We initialize $Q_\perp$ to be a random subset of the columns of the identity matrix, and orthogonalize it against $Q$.

We compare distances and residuals for synthetic examples with $n = 5000$ and $r = 50$ and various stopping thresholds $\varepsilon$ for the residuals (Figure 1). While the $\ell_2$ norm residual closely tracks the corresponding distance, the residual from Equation (16) overshoots by a small multiplicative factor, suggesting that the large constants in Proposition 3 may only be necessary in pathological cases and could be reduced in practice. Moreover, the $\ell_2 \to \infty$ norm residual can substantially overestimate the actual distance in the first few iterations, as the estimate of
Equation (16) depends on $Q_t Q_t^\top$ not being “too far” from $V V^\top$. The gap narrows after a few dozen iterations.

In addition, we examine the looseness of the bounds from Propositions 1 and 2 for the same experiment (Figure 2). We evaluate the following rates:

\begin{align}
\text{rate}_1(t) & := \left( \frac{\lambda_{r+1}}{\lambda_r} \right)^t \frac{\text{dist}_{2\to\infty}(Q_0, V)}{\sqrt{1 - d_0^2}}, \\
\text{rate}_2(t) & := \text{rate from Proposition 2}, \\
\text{rate}_3(t) & := \text{rate from Proposition 1}, \\
\text{rate}_{\text{naive}}(t) & := \left( \frac{\lambda_{r+1}}{\lambda_r} \right)^t \frac{d_0}{\sqrt{1 - d_0^2}}.
\end{align}

Here, $\text{rate}_1$ is an idealized rate that we would like to hold as an analog of the classical convergence results for the $\ell_2$ norm (Golub and Van Loan, 2013, Theorem 8.2.2). Via Proposition 1, $\text{rate}_3$ uses incoherence and a decay relationship involving spectral projectors and $\lambda_{r+1}$, whereas $\text{rate}_2$ only uses incoherence but depends on $\lambda_{r+2}/\lambda_r$. The naive rate uses an $\ell_2$ subspace distance. In all the synthetic examples we generated, Assumption 2 was verified to hold with constant $C < 1.5$.

Remarkably, for a range of dimensions $n$ and $r$ we find that $\text{rate}_3$ (which uses Proposition 1) closely tracks the “idealized” $\text{rate}_1$ on these synthetic matrices (Fig. 2). Also, $\text{rate}_2$ (which uses Proposition 2) is a looser upper bound. This agrees with our theoretical analysis, as $\lambda_{r+2}$ is only moderately smaller than $\lambda_{r+1}$ in our synthetic matrix construction. Finally, as expected, the naive rate is the loosest bound.

### 4.2 Eigenvector centrality

Next, we develop an experiment for network centrality, where the task is to measure the influence of nodes in a graph (Newman, 2008). Each node is assigned a score, which is a function of the graph topology, and a typical underlying (recursive) assumption is that a node with a high score contributes a larger influence to other nodes to which it is connected. Here, we consider eigenvector centrality, which is one the standard measures in network science. Given a graph...
Figure 2: Distance (solid lines) and convergence rates from Equation (21) for matrix and subspace dimensions \((n, r) = (1000, 10)\) (left); \((3500, 15)\) (middle); and \((8000, 20)\) (right). Here, rate_1 is an idealized rate based on an analog of the classical convergence, which is the best one could hope to achieve. Our rate_2 from Proposition 1 tracks this idealized closely in the synthetic data examples.

\(G = (V, E)\); the eigenvector centrality score of a node \(u, x_u > 0\), is defined as a solution to the following equation:

\[
x_u := \frac{1}{\lambda} \sum_{v \in V} A_{uv} x_v, \quad A_{uv} := \begin{cases} 1, & \text{if } u \text{ links to } v \\ 0, & \text{otherwise} \end{cases},
\]

where \(\lambda\) is a proportionality constant. Here, node \(u\)’s scores depend linearly on all of its neighbors’ scores. Under the positivity requirement of \(x_u\) and provided that the graph is connected, rearranging and the Perron-Frobenius theorem show that \(x = v_1\), the eigenvector corresponding to the largest eigenvalue \(\lambda_1\) of \(A\) (up to scaling). In this setting, we are typically interested in the ordering of nodes produced by the centrality score and not the actual scores themselves (e.g., to determine the influential or non-influential nodes). Therefore, the \(\ell_{2,\infty}\) distance, which measures \(\|v_1 - \hat{v}_1\|_{\infty}\), is more appropriate than \(\|v_1 - \hat{v}_1\|_2\) as a proxy for the quality of the estimate \(\hat{v}_1\). To get a correct ranking result, it suffices to have \(\|v_1 - \hat{v}_1\|_{\infty} < (1/2) \cdot \min_{i,j} |v_i - v_j|\). On the other hand, \(\|\hat{v}_1 - v_1\|_2\) does not have an interpretable criterion.

We demonstrate the above principle by comparing two stopping criteria: the criterion from Equation (16) with a specified threshold \(\varepsilon\) against the “naive” way of stopping when \(\|A\hat{v}_1 - \lambda\hat{v}_1\|_2 \leq \lambda\varepsilon\), where \(\lambda\) is the current eigenvalue estimate. If a user specifies a tolerance \(\varepsilon\), we expect that using our \(\ell_{2,\infty}\) error measurements and our corresponding stopping criteria will tell us that we can be confident in our solution much more quickly.

This is indeed the case — using our methodology provides a substantial reduction in computation time on a variety of real-world graphs, whose summary statistics are in Table 1. Figure 3 shows the ratio between the two quantities \(t_{\text{comp}}\) and \(t_{\text{naive}}\), defined as

\[
\begin{align*}
t_{\text{comp}} & := \min \{ t > 0 \mid \text{res}_{2,\infty}(t) \leq \varepsilon \} \\
t_{\text{naive}} & := \min \{ t > 0 \mid \|A\hat{v}_{i,j} - \hat{\lambda}_j\hat{V}_{i,j}\| \leq \varepsilon\hat{\lambda}_j, \forall j \}.
\end{align*}
\]

These are the stopping times for satisfying the residual criterion from Equation (16) and the “naive” \(\ell_2\) residual criterion at a given level, respectively. In the low-to-medium accuracy regimes,
using our stopping method results in least 20 – 40% reductions in the number of iterations needed. In this regime, the ranking induced by the approximate eigenvector had typically already converged to the “true” ordering obtained by computing the eigenvector to machine precision. To measure ranking, we defined

\[
\text{dist}_\tau(v_1, v_2) := 1 - \frac{\tau(v_1, v_2)}{2} \in [0, 1],
\]

where \(\tau(v_1, v_2)\) is Kendall’s \(\tau\)-correlation (Kendall, 1948) between the ranking induced by sorting the entries of \(v_1\) and \(v_2\); it’s straightforward to check that when \(v_1, v_2\) induce the same ranking, \(\text{dist}_\tau(v_1, v_2) = 0\), while \(\text{dist}_\tau(v_1, v_2) = 1\) when the rankings are most dissimilar. Figure 4 (right) shows the behavior of the different distance measures when applying our pipeline to various graphs from the SNAP network collection, where it is apparent that a moderate threshold of roughly \(10^{-4}\) yields the correct ranking in all cases.

4.3 Spectral clustering in graphs

Another downstream task employing invariant subspaces is spectral clustering, which we study here as a way to partition a graph into well-separated “communities” or “clusters”. The standard pipeline is to compute the leading \(r\)-dimensional eigenspace of the normalized adjacency matrix \(A_N = D^{-1/2}AD^{-1/2}\), where \(r\) is the desired number of clusters, \(D\) is the diagonal degree matrix, and \(A\) is the adjacency matrix. The resulting eigenvector matrix provides an \(r\)-dimensional embedding for each node, which is subsequently fed to a point cloud clustering algorithm such as k-means (Von Luxburg, 2007). For our experiment, we employ the deterministic QR-based algorithm from (Damle et al., 2018) on the same set of real-world graphs that we used for eigenvector centrality. Since the algorithm of (Damle et al., 2018) is deterministic, we do not have to worry about randomness pertaining to initialization (e.g. as in k-means++), and only run the experiment once for each configuration of parameters.

In this setup, the eigenvectors (more carefully, a rotation of them) are approximate cluster indicators. Indeed, spectral clustering on graphs is often derived from a continuous relaxation of
Figure 3: Ratio of the number of iterations needed to satisfy \( \text{res}_{2 \to \infty}(t) \leq \varepsilon \cdot t_{\text{comp}} \) over the number of iterations for the “naive” criterion \( \| A \hat{v} - \hat{\lambda} \hat{v} \|_2 \leq \varepsilon \hat{\lambda} \cdot t_{\text{naive}} \), for thresholds \( \varepsilon = 10^{-k} \), in computing the eigenvector centrality to rank the top \( [\sqrt{n}] \) nodes in a graph. Our analysis and stopping criteria enable significantly fewer iterations.

For all the datasets involved, we hand-pick the target number of clusters \( r \) by inspecting the successive ratios of the leading few eigenvalues and setting \( r \) so that the ratio \( \frac{\lambda_{r+1}}{\lambda_r} \) is small, but also taking into account the fact that we don’t want \( r \) to be too small. Additionally, we use the regularized version of the normalized adjacency matrix \( A_\rho \) (Amini et al., 2013), which augments the adjacency and degree matrices \( A, D \) using a regularization parameter \( \rho \):

\[
A_\rho := A + \frac{\rho}{n} 11^T, \quad D_\rho := D + \rho
\]  

Following standard practice (Qin and Rohe, 2013; Zhang and Rohe, 2018), we set \( \rho \) equal to a constant which is near the average degree of the graph and then perform the eigendecomposition of

\[
\tilde{A}_\rho = D_\rho^{-1/2} A_\rho D_\rho^{-1/2} + I,
\]

shifting by \(+I\) to ensure that the algebraically largest eigenvalues are also the largest in magnitude, in order for subspace iteration to be applicable. We summarize the hyperparameter choices for each dataset in Table 2.

In the same manner as the eigenvector centrality experiment, we compare the ratio of iteration counts: \( t_{\text{comp}} \) over \( t_{\text{naive}} \), as defined in Equation (23) (Figure 5, left). In this case, we see even larger savings. For \( \varepsilon \) around \( 10^{-2} \), our stopping criterion results in 70–80% savings in computation time. While this approximation level may seem crude at first, we can measure
Figure 4: Distance metrics for approximating the eigenvector centrality of several graphs to rank their top $\lfloor \sqrt{n} \rfloor$ nodes. **Datasets** (clockwise): CA-HEPHE, CA-ASTROPH, COM-LIVEJOURNAL, GEMSEC.

the performance of the algorithms in terms of the normalized cut metric, for which spectral clustering is a continuous relaxation (Von Luxburg, 2007). Given a partition of the vertex set $V$ into $S_1, \ldots, S_k$, which correspond to the different communities, we define

$$\text{ncut}(S_1, \ldots, S_k) := \frac{1}{2} \sum_{i=1}^{k} \phi(S_i), \quad \phi(S_i) := \frac{\sum_{i \in S, j \notin S} A_{ij}}{\sum_{i \in S} \sum_{j \in V} A_{ij}}$$

We find that by the time we reach residual level $\varepsilon = 10^{-2}$, the cut value found using the approximate subspace is essentially the same as the one using the subspace computed to numerical precision (see right of Figure 5).

---

4Note that the definition of $\phi$ used here is slightly different than the one used in the experiments of Section 4.4.
Figure 5: Spectral clustering on various datasets. Left: Ratio of iteration counts needed to satisfy $\text{res}_{2 \rightarrow \infty}(t) \leq \varepsilon \cdot \lceil t_{\text{comp}} \rceil$ over number of iterations for the “naive” criterion $\|\hat{A} - \hat{\lambda} \hat{v}\|_2 \leq \varepsilon \hat{\lambda}$ [t naïve]. Right: Normalized cut metric at different levels $\varepsilon$. Our analysis and stopping criteria enable significantly fewer iterations without sacrificing performance in the underlying task.

Table 2: Parameters for spectral clustering

| Dataset         | # of clusters | $r$ | $\rho$ |
|-----------------|---------------|-----|-------|
| CA-HEPPh        | 17            | 1.0 |       |
| CA-ASTROPH      | 6             | 1.0 |       |
| GEMSEC          | 12            | 1.0 |       |
| DBLP            | 28            | 5.0 |       |

4.4 Spectral bipartitioning and sweep cuts

Another spectral method for finding clusters in graphs is spectral bipartitioning, which aims to find a single cluster of nodes $S$ with small conductance $\phi(S)$:

$$\phi(S) := \frac{\sum_{i \in S, j \notin S} A_{ij}}{\min(A(S), A(S^c))}, \quad A(S) := \sum_{i \in S} \sum_{j \in V} A_{ij}.$$  

The conductance objective is a standard measure for identifying a good cluster of nodes (Leskovec et al., 2008; Schaeffer, 2007): if $\phi(S)$ is small, there are not many edges leaving $S$ and there are many edges contained in $S$.

Minimizing $\phi(S)$ is NP-hard, but a spectral method provides guarantees. The method computes the eigenvector $v_2$ corresponding to the second largest eigenvalue of the normalized adjacency matrix, which is often called the Fiedler vector (Fiedler, 1973). To find the a set with small conductance, the method uses the so-called “sweep cut”. After scaling $v_2$ by the inverse square root of degrees, we sort the nodes by their value in the eigenvector, and then consider the top-$k$ nodes as a candidate set $S$ for all values of $k$. The value of $k$ that gives the smallest conductance produces a set $S$ satisfying $\phi(S) \leq 2N \min_{S' \subseteq S} \phi(S')$, which is the celebrated Cheeger inequality (Chung, 1997).
Figure 6: Sweep cut profile (cut conductance vs. cardinality) for COM-DBLP. For a fixed $\varepsilon$, our $\ell_{2,\rightarrow\infty}$ stopping criterion leads to faster convergence. Increasing the tolerance for $\ell_2$ by the norm equivalence factor produces lower-quality solutions. Here $t_{\text{comp}} = 1135$ vs. $t_{\text{naive}} = 1378$ iterations.

As in the case of eigenvector centrality, what matters is the ordering induced by the eigenvector. Thus, a $\ell_{2,\rightarrow\infty}$ stopping criterion is again more appropriate than a standard $\ell_2$ one. As a heuristic, one might consider just making $\ell_2$ tolerance larger. We have seen that the spectral norm distance can be a factor of $\sqrt{n}$ loose compared to the $\ell_{2,\rightarrow\infty}$ distance. Thus, we could use $\varepsilon$ for $\ell_{2,\rightarrow\infty}$ and $\sqrt{n}\varepsilon$ for $\ell_2$. However, this can substantially reduce the solution quality; moreover, we expect such effects to become even more pronounced as the problem dimension $n$ increases.

We illustrate this in Fig. 6, where we plot the conductance values obtained in the sweep cut as a function of the size of the set on COM-DBLP. This is a sweep cut approximation of a network community profile plot (Benson et al., 2016; Leskovec et al., 2008), which visualizes cluster structure at different scales. Using the naive $\ell_2$ stopping criterion provides the same solution quality but requires more iterations. In the case of $\varepsilon = 10^{-4}$ in Fig. 6, our methods produce 20% computational savings. Finally, the heuristic $\sqrt{n}\varepsilon$ tolerance for the $\ell_2$ stopping criterion produces a cruder solution and finds a set with larger conductance.

5 Conclusions

The broad applicability of spectral methods, coupled with the prevalence of entry-wise interpretations of eigenvectors (or row-wise interpretations to invariant subspaces) strongly motivates imbuing our computational methods with appropriate stopping criteria. Our theoretical results demonstrate just how much smaller the $\ell_{2,\rightarrow\infty}$ subspace distance can be than traditional measures, an observation supported by experiment. In fact, the accuracy with which we compute eigenvectors can have a non-trivial impact on downstream applications — if we would like use fewer
iterations to save time we must do so carefully, and our new stopping criterion provides an easy to implement way to do this that comes at essentially no cost and with strong guarantees.

Acknowledgments

This research was supported by NSF Award DMS-1830274, ARO Award W911NF19-1-0057, and ARO MURI.

A Auxiliary Results

Lemma A.1 (Incoherence). Consider a subspace \( \mathcal{V} \) of dimension \( r \) and its spectral projector \( \mathcal{V}V^T \). If \( \mathcal{V} \) is \( \mu \)-incoherent, i.e., \( \| \mathcal{V}V^T \|_{2\rightarrow\infty} \leq \mu \sqrt{\frac{r}{n}} \), then for its complementary subspace \( \mathcal{V}_\perp \) it holds that

\[
\| \mathcal{V}_\perp \mathcal{V}_\perp^T \|_{\infty} \leq (1 + \mu \sqrt{r/n}).
\]

Proof. Observe that \( \|A\|_\infty \leq \sqrt{n} \|A\|_{2\rightarrow\infty} \). Thus,

\[
\| \mathcal{V}_\perp \mathcal{V}_\perp^T \|_{\infty} = \| I - \mathcal{V}V^T \|_{\infty} \leq 1 + \| \mathcal{V}V^T \|_{\infty} \\
\quad \leq 1 + \sqrt{n} \| \mathcal{V}V^T \|_{2\rightarrow\infty} \leq 1 + \sqrt{n} \mu \sqrt{r/n}.
\]



The next theorem, originally stated without assuming symmetry, is adapted for the case of a symmetric initial matrix.

Theorem A.1 (Theorem 5.1 in (Damle and Sun, 2019)). Suppose \( \tilde{A} = A + E \) with \( A \) symmetric, having eigenvalue decomposition \( A = V_1 \Lambda_1 V_1^T + V_2 \Lambda_2 V_2^T \), where \( V_1 \in \mathbb{R}^{n \times r} \), \( V_2 \in \mathbb{R}^{n \times (n-r)} \) have orthonormal columns. Moreover, let

\[
gap := \min \{ \lambda_r - \lambda_{r+1}, \text{sep}_{2\rightarrow\infty, V_2}(\Lambda_1, V_2 \Lambda_2 V_2^T) \},
\]

with \( \text{sep}_{2\rightarrow\infty, V_2} \) defined as the following quantity:

\[
\text{sep}_{2\rightarrow\infty, V_2}(A, B) := \inf \{ \| A Z - Z B \|_{2\rightarrow\infty} \mid Z \in \text{ran}(V_2), \| Z \|_{2\rightarrow\infty} = 1 \}
\]

If \( \| E \|_2 \leq \frac{\gap}{5} \), then the leading invariant subspace of \( \tilde{A}, \tilde{V}_1 \), satisfies

\[
\inf_{O \in O_r} \| \tilde{V}_1 - V_1 O \|_{2\rightarrow\infty} \leq 8 \| V_1 \|_{2\rightarrow\infty} \left( \frac{\| E \|_2}{\lambda_r - \lambda_{r+1}} \right)^2 \\
+ 2 \frac{\| V_2 V_2^T E V_1 \|_{2\rightarrow\infty}}{\gap} + 4 \frac{\| V_2 V_2^T E \|_{2\rightarrow\infty}}{\gap} \| E \|_2.
\]

(27)

Lemma A.2 (Cape et al. (2019)). We have

\[
\| AB \|_{2\rightarrow\infty} \leq \| A \|_{2\rightarrow\infty} \| B \|_2
\]

(28)

\[
\| AB \|_{2\rightarrow\infty} \leq \| A \|_\infty \| B \|_{2\rightarrow\infty}
\]

(29)

Moreover, for any matrix \( V \) with orthonormal columns, it holds that

\[
\| AV^T \|_{2\rightarrow\infty} = \| A \|_{2\rightarrow\infty}.
\]

(30)
While we believe the result of Lemma A.3 may be folklore, we were unable to locate a reference for it, and provide a proof for completeness.

**Lemma A.3.** We have \( \inf_{z \in O_r} \| \tilde{V} - VZ \|_2 \leq \sqrt{2} \text{dist}_2(V, \tilde{V}) \).

**Proof.** Recall the solution of the orthogonal Procrustes problem, given by the SVD of \( V^T \tilde{V} \), \( U \Sigma W^T \). Since \( UW^T \in O_r \), with \( U^T U = UU^T = W^T W = WW^T = I_r \), we have

\[
\begin{align*}
\inf_{z \in O_r} \| \tilde{V} - VZ \|_2 & \leq \| \tilde{V} - VUW^T \|_2 = \sqrt{\sup_x \langle x, (\tilde{V} - VUW^T)^T (\tilde{V} - VUW^T) \rangle} \\
& = \sqrt{\sup_x \langle x, (I - \tilde{V}^T VUW^T - WU^T V^T \tilde{V} + I) x \rangle} \\
& \overset{(\sharp)}{=} \sqrt{\sup_x \langle x, 2(I - W \Sigma W^T) x \rangle} = \sqrt{2 \| I - W \Sigma W^T \|_2} \\
& = \sqrt{2 \| I - \Sigma \|_2} = \sqrt{2} \sqrt{1 - \sigma_r(V^T \tilde{V})} \\
& \overset{(\sharp)}{\leq} \sqrt{2} \sqrt{1 - \sigma_{\min}^2(V^T \tilde{V})} = \sqrt{2} \| V^T \tilde{V} \|_2,
\end{align*}
\]

where (\#) follows after replacing \( V^T \tilde{V} = U \Sigma W^T \) in the expression and gathering terms, while (\#) simply uses the fact that \( \sigma_r(V^T \tilde{V}) \leq 1 \) to upper bound the expression inside the square root.

Finally, we use the fact that:

\[
1 - \sigma_{\min}^2(V^T \tilde{V}) = \| V^T \tilde{V} \|_2^2 = \text{dist}_2^2(V, \tilde{V}).
\]

\[\square\]

**Discussion: eigenvalue localization issues.** We briefly address the issue of when we can safely assume that the approximate invariant subspace \( V \), utilized in Proposition 3, is the leading invariant subspace of the perturbed matrix \( A - EV^T \). While the matrix of Ritz values, \( S \), is within \( \sqrt{2} \| E \|_2 \) distance of a set of \( r \) eigenvalues of \( A \), we do not know whether or not these eigenvalues correspond to the largest (in magnitude) eigenvalues of \( A - EV^T \).

In this case, one has to appeal to algorithm-specific arguments. Recall that \( A \) has spectral decomposition \( A = V_1 \Lambda_1 V_1^T + V_2 \Lambda_2 V_2^T \), where \( \Lambda_1 \) contains the dominant \( r \) eigenvalues. Let \( V_{\perp} \in O_{n,n-r} \) be orthogonal to the approximate eigenvector matrix \( V \in O_{n,r} \). Then the following

\[
\begin{bmatrix}
V^T \\
V_{\perp}^T
\end{bmatrix}
(A - EV^T) \begin{bmatrix}
V \\
V_{\perp}
\end{bmatrix} = \begin{bmatrix}
S & V^T(A - EV^T)V_{\perp} \\
V_{\perp}^T VS & V_{\perp}^T(A - EV^T)V_{\perp}
\end{bmatrix} = \begin{bmatrix}
S & V^T AV_{\perp} \\
0 & V_{\perp}^T AV_{\perp}
\end{bmatrix}
\]

is a Schur decomposition of \( A - EV^T \), with its eigenvalues being the union \( S \cup \Lambda(V_{\perp}^T AV_{\perp}) \) – the objective becomes showing that \( \| \Lambda(V_{\perp}^T AV_{\perp}) \|_2 \) is sufficiently small, after enough progress of the algorithm. By the variational characterization of eigenvalues for symmetric matrices, we
have

$$\lambda_1(V_\perp^T AV_\perp) = \sup_{x \in S^{n-1}} |\langle x, V_\perp^T AV_\perp x \rangle| = \sup_{x \in S^{n-1}} |\langle x, V_\perp^T V_1 \Lambda_1 V_1^T V_\perp x \rangle + \langle x, V_\perp^T V_2 \Lambda_2 V_2^T V_\perp x \rangle|$$

\[\leq \lambda_1 \|V_\perp^T V_1\|_2^2 + \lambda_{r+1} \|V_\perp^T V_2\|_2^2 \leq 1\]  

(33)

Therefore, as soon as \(\text{dist}_2(V, V_1) \leq \sqrt{\varepsilon}\), we know that \(\Lambda(V_\perp^T AV_\perp) \leq \lambda_1 \varepsilon + \lambda_{r+1}\); thus when both \(\|E\|_2\) and \(\varepsilon\) are small enough, we can “match” \(S\) with the leading invariant subspace of \(A - EV^T\), via the leading eigenvalues of \(A\) itself.

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