ELIMINATING FIELD QUANTIFIERS IN STRONGLY DEPENDENT HENSELIAN FIELDS

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Abstract. We prove elimination of field quantifiers for strongly dependent henselian fields in the Denef-Pas language. This is achieved by proving the result separately for (a generalization of) algebraically maximal Kaplansky fields and for p-valued fields. We deduce that if \((K,v)\) is strongly dependent then so is its henselization.

1. Introduction

This paper stemmed from the need for a complete proof that algebraically maximal Kaplansky fields eliminate field quantifiers (in the sense \(^1\), e.g., of \([20, \text{Definition 1.14}]\)). It quickly became clear that the same methods could be applied to prove elimination of field quantifiers for all strongly dependent henselian fields.

While elimination of field quantifiers for algebraically maximal Kaplansky fields may be folklore, we could not find a proof in the literature, though several closely related theorems do exist. In \([16, \text{Theorem 2.6}]\) Kuhlmann proves that such valued fields admit quantifier elimination relative to a structure he calls an \textit{amc-structure of level 0}. It is well known that this structure is essentially the \textit{RV-structure} (see for instance \([5, \text{Section 3.2}]\)). In this language, Kuhlmann proves that if \(L\) and \(F\) are models of a theory of an algebraically maximal Kaplansky field and \(K\) is a common substructure then

\[ \text{RV}_L \equiv_{\text{RV}_K} \text{RV}_F \implies (L,v) \equiv_{(K,v)} (F,v), \]

where the valued fields are considered, for instance, in the \(L_{\text{div}}\) language. This is proved by showing that every embedding \(\text{RV}_L \hookrightarrow \text{RV}_F\) (over \(\text{RV}_K\)) lifts to an embedding \((L,v) \hookrightarrow (F,v)\) (over \((K,v)\)), provided that \(F\) is \(|L|^{+}\)-saturated.

Using this result Bélair proves that, in equi-characteristic \((p,p)\), every algebraically maximal Kaplansky field eliminates field quantifiers in the Denef-Pas language, the 3-sorted language enriched with an angular component map (see \([1, \text{Lemma 4.3}]\)). It seems, though it is not claimed, that Bélair’s proof may apply to the mixed characteristic case as well.

Using ideas from \([11, \text{Chapter 3}]\), this result may be extended to strongly dependent henselian valued fields. The main results of this paper are the following

\[ \text{Theorem 1. Let } (K,v) \text{ be a henselian valued field, admitting an angular component map, and such that either} \]
\[ (1) \text{ p-valued of rank } d, \]
\[ (2) \text{ algebraically maximal Kaplansky or} \]
\[ (3) \text{ strongly dependent} \]

\(^1\)See also \([19, \text{Appendix A}]\) for a more detailed discussion.
then \((K,v)\) eliminates field quantifiers in the Denef-Pas language.

Shelah’s conjecture ([20]), usually interpreted as stating that strongly dependent fields which are neither real closed nor algebraically closed are hence linear, is our main motivation for carrying out the present research. Strong dependence will, however, be used as a black box, and will never be invoked explicitly. For a more detailed discussion of strongly dependent henselian fields the reader is referred to [8] and references therein.

As a consequence of our main result we deduce a transfer principle, providing a new method for constructing strongly dependent fields:

\[ \text{Theorem 2. Let } (K, v) \text{ be a strongly dependent valued field. Then its henselization } (K^h, v) \text{ is also strongly dependent. If, in addition, } (K, v) \text{ is Kaplansky then also its inertia field } (K^t, v) \text{ is strongly dependent.} \]

In section 5 we show that any field of finite dp-rank admitting a non-trivial henselian valuation is a geometric field, in the sense of [9], and that if it also has geometric elimination of imaginaries it is either algebraically closed or real closed.

In Appendix A we show that the exact same proof gives elimination of field quantifiers for strongly dependent henselian fields in the RV-language.

2. Preliminaries

2.1. Valued Fields. We review some terminology and definitions. For a valued field \((K,v)\) let \(vK\) denote the value group, \(Kv\) the residue field, \(res\) the residue map and \(O_K\) (or \(O\), if the context is clear) the valuation ring.

Valued fields will be considered in the 3-sorted language, with a sort for the base field, the value group and the residue field, with the obvious functions and relations.

For a valued field \((K,v)\) an angular component map is a multiplicative group homomorphism

\[ ac : K^\times \to Kv^\times \]

such that \(ac(a) = res(a)\) whenever \(v(a) = 0\), we extend it to \(ac : K \to Kv\) by setting \(ac(0) = 0\).

\[ \text{Fact 2.1. [23, Corollary 5.18]} \text{ Every valued field } (K, v) \text{ has an elementary extension with an angular component map on it.} \]

In fact, one only requires that \(vK\) be \(\aleph_1\)-saturated.

A valued field, admitting an angular component map will be referred to as an ac-valued field. The 3-sorted language of valued fields augmented by a function symbol \(ac\) for the angular component map, is called the Denef-Pas language.

A rough characterization of strongly dependent henselian fields was given in [11, Theorem 4.3.1] and a little more explicitly in [8, Theorem 5.14]. We make it explicit here, but we first remind some necessary definitions:

**Definition 2.2.** (1) A valued field \((K,v)\) of residue characteristic \(p > 0\) is a Kaplansky field if the value group is \(p\)-divisible, the residue field is perfect and does not admit any finite separable extensions of degree divisible by \(p\).

(2) A valued field \((K,v)\) of residue characteristic 0 is Kaplansky.

(3) \((K,v)\) is algebraically maximal if if does not admit any immediate algebraic extension.

(4) \((K,v)\) is \(p\)-valued of \(p\)-rank \(d\) if

- \(char(Kv) = p > 0\) and \(char(K) = 0\) and
- \(dim_{\mathbb{F}_p} O_K/(p) = d\).
Let \((K, v, \Gamma, k)\) be a valued field with value group \(\Gamma\), residue field \(k\), valuation ring \(O\) and maximal ideal \(M\). Given a convex subgroup \(\Delta \leq \Gamma\), we let \(v^{\Gamma/\Delta} : K \to \Gamma/\Delta\) denote the coarsening of \(v\) with valuation ring \(O^{\Gamma/\Delta} = \{x \in K : \exists \delta \in \Delta, v(x) \geq \delta\}\), whose union encompasses (by Fact 2.4) all strongly dependent fields. The first field quantifiers in the Denef-Pas language for strongly dependent henselian fields, we actually show a bit more. We show it for the two theories given below:

**Definition 2.6.** Let \(\Gamma\) be a valued field with value group \(\Gamma\), residue field \(k\), valuation ring \(O\) and maximal ideal \(M\). Given a convex subgroup \(\Delta \leq \Gamma\), we let \(v^{\Gamma/\Delta} : K \to \Gamma/\Delta\) denote the valuation given by

\[
v^{\Delta}(a + M^{\Gamma/\Delta}) = \begin{cases} v(a) & \text{if } a \in O^{\Gamma/\Delta} \setminus M^{\Gamma/\Delta} \\ \infty & \text{otherwise.} \end{cases}
\]

It has valuation ring \(O^{\Delta} = \{a + M^{\Gamma/\Delta} : v(a) \geq 0\}\) with maximal ideal \(M^{\Delta} = \{a + M^{\Gamma/\Delta} : v(a) > 0\}\) and residue field \(k^{\Delta}\). It is well known (and easy to check) that:

**Fact 2.3.** The map \(k \to k^{\Delta}\) given by \(a + M \mapsto (a + M^{\Gamma/\Delta}) + M^{\Delta}\) is a field isomorphism.

If \((K, v)\) is a valued field of mixed characteristic \((0, p)\) the core field of \(K\) is the valued field \((K_1, v^{\Gamma/\Delta})\), where \(\Delta_{p}\) is the minimal convex subgroup containing \(v(p)\). It is also a valued field of mixed characteristic and if \((K, v)\) is henselian then so is the core field. Notice that \((K, v^{\Gamma/\Delta})\) is of equi-characteristic \((0, 0)\).

**Fact 2.4.** \([11\text{ Theorem } 4.3.1]\) \([8\text{ Theorem } 5.13]\) If \((K, v)\) is a henselian valued field with \(K\) strongly dependent then

- if \((K, v)\) is of equi-characteristic \((p, p)\) then \((K, v)\) is an algebraically maximal Kaplansky field,
- if \((K, v)\) is of mixed characteristic \((0, p)\) then
  - (1) if \(Kv\) is infinite then the core field of \((K, v)\) is algebraically maximal Kaplansky,
  - (2) if \(Kv\) is finite then \((K, v)\) is a \(p\)-valued field (the core field is a \(p\)-adically closed field).

### 2.2. Two theories of valued fields.

Although our main goal is to show elimination of field quantifiers in the Denef-Pas language for strongly dependent henselian fields, we actually show a bit more. We show it for the two theories given below, whose union encompasses (by Fact 2.4) all strongly dependent fields. The first theory we consider is a generalization of a theory given in [11 Section 3.2]. The second theory is that of \(p\)-valued fields.

#### 2.2.1. The first theory.

We borrow the following terminology from [11]:

**Definition 2.5.** A valuation \(v : K \to \Gamma\) is roughly \(p\)-divisible if \([-v(p), v(p)] \subseteq p\Gamma\), where

\[
[-v(p), v(p)] = \begin{cases} \{0\} & \text{in pure characteristic } 0 \\
\Gamma & \text{in pure characteristic } p \\
[-v(p), v(p)] & \text{in mixed characteristic} \end{cases}
\]

**Remark.** In the mixed characteristic case, if \(vK\) is roughly \(p\)-divisible and \(\Delta_p\) is the minimal convex subgroup containing \(v(p)\), the coarsening \((K, v^{\Gamma/\Delta_p})\) is of equi-characteristic \(0\) and \(v^{\Delta_p}\), the induced valuation \(Kv^{\Gamma/\Delta_p}\), then \((Kv^{\Gamma/\Delta_p}, v^{\Delta_p})\), is of mixed characteristic \((0, p)\) with a \(p\)-divisible value group.

**Definition 2.6.** Let \(T_1\) be the theory of valued fields stating:

- the valued field is henselian and defectless,
- the base field and the residue field are perfect,
- the valuation is roughly \(p\)-divisible,
- every finite field extension of the residue field has degree prime to \(p\).
Remark. (1) Perfection of the residue field follows, in fact, from the requirement that every finite field extension of the residue field has degree prime to $p$. 

(2) Perfection of the base fields also follows from the other axioms (see below).

(3) Every strongly dependent henselian valued field whose residue field is infinite is a model of $T_1$.

(4) If $(K, v) \models T_1$ and $vK$ is not $p$-divisible then, by perfection of $K$ (or rough $p$-divisibility), it is necessarily of mixed characteristic.

Lemma 2.7. Let $(K, v)$ be a strongly dependent henselian field. If $Kv$ is infinite then $(K, v) \models T_1$.

Proof. By Fact 2.4 if $\text{char}(K) = \text{char}Kv$ then $(K, v)$ is algebraically maximal Kaplansky, and the lemma follows from the above remark. So we are reduced to the case where $(K, v)$ is of mixed characteristic $(0, p)$. By [8, Corollary 5.15] $(K, v)$ is defectless. By Fact 2.4 since $Kv$ is infinite, the core field of $(K, v)$ is algebraically maximal Kaplansky. In particular, the convex sub-group $\Delta_p \leq vK$ generated by $v(p)$ is $p$-divisible, so $(K, v)$ is roughly $p$-divisible. Since $Kv$ is strongly dependent (e.g., [8, Proposition 5.2]) it has no finite extensions of degree divisible by $p$. \qed

We collect a few results, essentially, due to Johnson ([13, Section 3.2]). Johnson states these results under the stronger assumption that the residue field is algebraically closed. We repeat the proofs, sometimes verbatim, only to emphasize that this requirement is inessential. We start with an immediate application of henselianity:

Fact 2.8. [13] Remark 3.2.2] Let $(L, v)/(K, v)$ be an extension of valued fields. Suppose $(L, v)$ is henselian and $K$ is relatively separably closed in $L$. Then $Kv$ is relatively separably closed in $L_v$.

The main properties of models of $T_1$ are collected in the next proposition:

Proposition 2.9. [13] Proposition 3.2.3] Let $(F, v) \models T_1$, of residue characteristic $p$. Then

1. If $vF$ is $p$-divisible then any finite field extension of $F$ has degree prime to $p$.
2. $a \in F^p$ if and only if $v(a) \in p \cdot vF$.
3. If $K$ is relatively algebraically closed in $F$, then $K \models T_1$.

Proof. 1. Let $L/F$ be a finite extension. Since $F$ is henselian and defectless $[L : F] = (vL : vF)[Lv : Fv]$, but $(vL : vF)$ is prime to $p$ since $vF$ is $p$-divisible and $[Lv : Fv]$ is prime to $p$ by assumption.

2. In case $vF$ is $p$-divisible if $a \in F \setminus F^p$ then the polynomial $x^p - a$ is irreducible, contradicting (1).

So we now assume that $vF$ is not $p$-divisible. Hence, for $\Delta_p$ the convex sub-group generated by $v(p)$ we get that $(Fv^{\Gamma/\Delta_p}, v^{\Delta_p}) \models T_1$ and has a $p$-divisible value group. Fix some $a \in F$ such that $v(a) \in p \cdot vF$. By considering $a/p^p$ for $v(a) = pv(b)$, we reduce to the case where $v(a) = 0$. Because $(F, v^{\Gamma/\Delta_p})$ is henselian of residue characteristic $0$, and $v^{\Gamma/\Delta_p}(a) = 0$ we know that $a \in F^p$ if and only if $\text{res}^{\Gamma/\Delta_p}(a) \in (Fv^{\Gamma/\Delta_p})^p$. Since $\Gamma/\Delta_p$ is $p$-divisible, by the previous paragraph $Fv^{\Gamma/\Delta_p} = (Fv^{\Gamma/\Delta_p})^p$, with the desired conclusion.
(3) We first assume that $F$ has a $p$-divisible value group and show that $(K, v) \models T_1$ and $vK$ is $p$-divisible. Since $F$ is henselian and perfect and $K$ is algebraically closed in $F$, also $K$ is henselian and perfect. So $F/K$ is regular, implying that $F$ and $K^{\text{alg}}$ are linearly disjoint over $K$. Thus $p$ does not divide the degree of any finite extension of $K$. Indeed, if $(K(a)/K$ is a finite extension with degree divisible by $p$ then by linear disjointness so is $F(a)/F$.

It follows that $(K, v)$ is defectless, $Kv$ is perfect and $vK$ is $p$-divisible. Since every finite extension of $Kv$ may be lifted to a finite extension $K$, $p$ does not divide the degree of any finite extension of $Kv$.

Assume now that $vF$ is not $p$-divisible. Let $\Delta_p, v^{\Gamma/\Delta_p}$ and $v^{\Delta_p}$ be as before. By Fact 2.8, $K^{\Gamma/\Delta_p}$ is relatively algebraically closed in $F^{\Gamma/\Delta_p}$. As $(F^{\Gamma/\Delta_p}, v^{\Delta_p})$ is a model of $T_1$ with $p$-divisible value group by what we have done above so is $(K^{v^{\Gamma/\Delta_p}}, v^{\Delta_p})$. Thus the place $K \rightarrow Kv$ decomposes into $K \rightarrow K^{v^{\Gamma/\Delta_p}} \rightarrow Kv$ each of them henselian, defectless and roughly $p$-divisible. Thus so is $K \rightarrow Kv$, i.e., $(K, v) \models T_1$.

\[\square\]

**Fact 2.10.** [10] Lemma 3.12 Let $L$ and $F$ be two algebraically maximal Kaplansky fields (and hence defectless) and $K$ a common henselian subfield. Assume that both $vL/vK$ and $vF/vK$ are $p$-torsion groups and both $Lv/Kv$ and $Fv/Kv$ are purely inseparable algebraic extensions. Then the relative algebraic closures of $K$ in $L$ and $F$ are isomorphic.

In order to use the above fact in our setting we will need a result from ramification theory, see [4, Section 5.2] for notation.

**Lemma 2.11.** Let $(K, v)$ be a henselian valued field of residue characteristic 0 and $(L, v)$ and $(F, v)$ two algebraic extensions with $vL = vK = vF$. If $vF$ and $Lv$ are isomorphic over $Kv$ (as fields) then $F$ and $L$ are isomorphic over $K$ (as fields).

**Proof.** Since $(K, v)$ is of residue characteristic 0 and henselian, $Kv$ is perfect and $(K, v)$ is defectless. Since $vL = vK = vF$, by [4, Theorem 5.2.9(1)], $L, F \subseteq K^v$, the inertia field of $K^{\text{alg}}/K$. The result now follows from [4, Theorem 5.2.7(2)].

\[\square\]

The following is an adaptation of [11, Lemma 3.2.4].

**Lemma 2.12.** Let $L, F \models T_1$ and $K$ a common valued subfield. Assume that $vL = vK = vF$ and that both $Lv/Kv$ and $Fv/Kv$ are purely inseparable algebraic extensions. Then the relative algebraic closures of $K$ in $L$ and $F$ are isomorphic.

**Proof.** We may replace $K$ with the perfection if its henselization, thus it is enough to show that the relative algebraic closures are isomorphic as fields over $K$.

If $vK$ is $p$-divisible, then $L$ and $F$ are algebraically maximal Kaplansky fields and result follows from Fact 2.10. Otherwise, $K$, $L$ and $F$ have characteristic 0. By Proposition 2.9(4) the respective relative algebraic closures of $K$ in $L$ and $F$ are also models of $T$. Denote them by $K^L$, $K^F$, respectively.

Let $\Delta_p, v^{\Gamma/\Delta_p}$ and $v^{\Delta_p}$ be as before. Since $vL = vK^L = vK = vF$ we also get $\Delta_p, v^{\Gamma/\Delta_p}$ in $vK^L = v^{\Gamma/\Delta_p}K^L = v^{\Gamma/\Delta_p}F$. As $(K, v^{\Gamma/\Delta_p})$ is a henselian field with residue characteristic 0 we may use Lemma 2.11 and thus $K^L$ and $K^F$ are isomorphic as fields as long as $K^Lv^{\Gamma/\Delta_p}$ and $K^Fv^{\Gamma/\Delta_p}$ are isomorphic extension of $K^{v^{\Gamma/\Delta_p}}$. By Fact 2.8 and Proposition 2.9(3) we may apply the $p$-divisible case on $Lv^{\Gamma/\Delta_p}$ and $Fv^{\Gamma/\Delta_p}$, implying that they are, indeed, isomorphic.

\[\square\]

### 2.2.2. The theory of $p$-valued fields

Let $T_2$ be the theory of henselian $p$-valued fields of $p$-rank $d$. We consider it in the language of valued fields augmented by $d$
constants. Notice that once we named constants, saying that they form an $\mathbb{F}_p$-basis for $\mathcal{O}_K/(p)$ is a universal sentence. Hence every substructure of a model of $T_2$ is again a $p$-valued field of $p$-rank $d$.

We review some facts concerning $p$-valued fields from [17].

Fact 2.13. [17, Section 2.1] $K_v$ and $[0, v(p)]$ are finite and $d = \dim_{\mathbb{F}_p} K_v \cdot [0, v(p)]$.

Fact 2.14. [17, Lemma 3.5(iii)] Let $L/K$ be an extension of $p$-valued fields of the same $p$-rank. For every $\gamma \in vL$ such that $a\gamma \in vK$ for some $n \in \mathbb{N}$ there exists $t \in L$ such that $v(t) = \gamma$ and $t^n \in K$.

Fact 2.15. [17, Lemma 3.7] Let $L/K$ be an algebraic extension of $p$-valued fields of the same $p$-rank. If $vL = vK$ then $L = K$.

For future reference we sum up Fact 2.13 and Lemma 2.15.

Lemma 2.16. If $(K, v)$ is a strongly dependent henselian field then $(K, v) \models T_1$ or $(K, v) \models T_2$.

In fact, by [8, Theorem 5.14] we get the following:

Corollary 2.17. Let $K$ be a strongly dependent field. Then $(K, v) \models T_1$ or $(K, v) \models T_2$ for any henselian valuation $v$ on $K$.

3. Extending Embeddings

The following results are proved in [23] for the $(0,0)$ case. We use results from [15] to give the slight generalizations necessary for our needs.

Lemma 3.1. [23, Lemma 5.20] Let $L, F$ be ac-valued fields, $f : L \to F$ an isomorphism of ac-valued fields $g : L' \to F'$ a valued-field isomorphism extending $f$ to some ac-valued field extensions $L'/L$ and $F'/F$ such that $vL = vL'$. Then $g$ commutes with the ac-map.

Proof. Let $x \in L'$, we may write $x = x_1 x_2$ where $x_1 \in L$ and $x_2 \in \mathcal{O}_L^\times$, and thus $ac(x) = ac(x_1) res(x_2)$ and

$$f(ac(x)) = f(ac(x_1)) f(res(x_2)) = ac(f(x_1)) res(f(x_2)) = ac(f(x)).$$

$\square$

If $(K, v)$ is a valued field we say, Following F.-V. Kuhlmann, that $K$ is Artin-Schreier closed if every irreducible polynomial of the form $x^p - x - c$ has a root in $K$ for $p = \text{char}(Kv) > 0$.

Lemma 3.2. Let $L_1$ and $L_2$ be henselian ac-valued fields, $K$ a common henselian ac-valued subfield and $f : K \to L_2$ be the embedding (so $f$ commutes with the ac-map). Further assume that if $\text{char}(Kv) = p > 0$ then $L_1, L_2$ are both either

- Artin-Schreier closed or
- for every $\gamma \in vL_1 \setminus vK$ with $p\gamma \not\in vK$, there exists $a \in L_1$ with $ap \in K$ and $va^p = p\gamma$.

If $vL_1 \equiv_k vL_2$ then for any $\gamma_i \in vL_i$ with $tp(\gamma_i/vK) = tp(\gamma_2/vK)$ there exist $b_i \in L_i$ with $v(b_i) = \gamma_i$ such that $f$ may be lifted to an isomorphism of ac-valued fields $f : K(b_1) \to K(b_2)$, in particular $K(b_1)$ are ac-valued fields.

Proof. By the previous lemma we may, of course, assume that $\gamma_i \not\in vK$. We now break the proof in two:
Case 1: Assume that $n \gamma \notin vK$ for every $n \geq 1$ and let $b_i \in L_i$ be any elements with $v(b_i) = \gamma_i$. Replacing $b_i$, if needed, by $b_i/u_i$ with $u_i \in \mathcal{O}_K^\times$ such that $ac(u_i) = ac(b_i)$, we have $ac(b_i) = 1$.

Because $\gamma$ is not in the divisible hull of $vK$, necessarily $b_i$ is transcendental over $K$. By [23, Lemma 3.23] (or [15, Lemma 6.35]), $v$ extends uniquely to $K(b_i)$. $K(b_i)v = Kv$ and $vK(b_i) = vK \oplus \mathbb{Z}[\gamma]$.

It remains to show that this extension commutes with the ac-map. Let $a \in K(b_i)$. Because $vK(b_i) = vK \oplus \mathbb{Z}[\gamma]$ there exists $c$ with $v(c) \in vK$ such that $v(a) = v(cb_i^n)$ for some $n$. Thus $a = (cb_i^n) \cdot a/(cb_i^n)$ and $v(a/(cb_i^n)) = 0$. So $ac(a) = ac(cb_i^n)res(a/(cb_i^n)) = ac(c)res(a/(cb_i^n))$ and

$$f(ac(a)) = f(ac(c))f(res(a/(cb_i^n))) = ac(f(c)f(res(a/(cb_i^n)))) = ac(f(a)).$$

Case 2: Assume that $q \gamma_i \notin vK$ for some prime $q$.

Case 2.1: Assume that $q \neq \text{char}(Kv)$. Let $d \in K$ with $v(d) = q\gamma$. We may replace $d$ by $d/u$ with $u \in \mathcal{O}_K^\times$ such that $ac(u) = ac(d)$ and then $ac(d) = 1$. Let $a_i \in L_i$ with $v(a_i) = \gamma_i$ and $ac(a_i) = 1$. Then the polynomial $P_i(x) = x^q - d/a_i^q$ is over $\mathcal{O}_K$, satisfies $v(P_i(1)) > 0$ and $v(P_i'(1)) = 0$. Indeed, $v(d/a_i^q) = 0$ and $res(d/a_i^q) = ac(d)/ac(a_i^q) = 1$, so $P_i(1) = 0$. Since $q \neq \text{char}(Kv)$ we automatically get that $P_i$ is separable, so $P_i'(1) \neq 0$.

This gives $u_i \in L_i$ such that $P_i(u_i) = 0$ and $\bar{a}_i = 1$. Now let $b_i = a_i u_i$, clearly $b_i^q = d$ and $ac(b_i) = 1$. By [15, Lemma 6.40], $K(b_i)v = Kv$ and $vK(b_i) = vK \oplus \mathbb{Z}[\gamma]$. Now $f|_{K(b_i)}$ commutes with the ac-map just as in case 1.

Case 2.2: Assume that $q = p = \text{char}(vK) > 0$.

Since both $L_1$ and $L_2$ are either Artin-Schreier closed or have $p$-roots, just as in Case 2.1 we may use [15, Lemma 6.40] to extend the isomorphism (see the statement of [15, Lemma 6.40]).

Proof. It will suffice, by Lemma 3.3, to show that we can extend $f$ to an isomorphism of valued field extensions preserving the value group of $K$. We break into two cases.

Case 1: Assume that $a_i \in L_i v$ are transcendental over $Kv$. Pick any $a_i \in \mathcal{O}_{L_i}$ such that $\bar{a}_i = \alpha_i$. They are necessarily transcendental over $K$. Either by [15, Lemma 6.35] or by [23, Lemma 3.22], $v$ extends uniquely to $K(a_i)$ and $vK(a_i) = vK$.

Case 2: If $\alpha_i$ is algebraic, let $P(X) \in \mathcal{O}_K[x]$ be monic such that $P(X)$ is the minimal polynomial of $\alpha_i$ over $Kv$. Pick $b_i \in \mathcal{O}_L$ such that $b_i = \alpha_i$. As the $\alpha_i$ are separable over $Kv$, $P$ is separable and since $v(P(b_i)) > 0$ and $v(P'(b_i)) = 0$ we may find $a_i \in \mathcal{O}_L$ such that $P(a_i) = 0$ and $\bar{a}_i = \alpha_i$. Either by [15, Lemma 6.41] or [23, Lemma 3.21], $v$ extends uniquely to $K(a_i)$ and $vK(a_i) = vK$.

\[\text{[L: extending residue]}\]
4. ELIMINATING FIELD QUANTIFIERS

In this section all fields are assumed to be ac-valued considered in the Denef-Pas language. We use the results from Section 3 and meld them with the proof from [16, Section 3].

Recall that a valuation transcendence basis \( \mathcal{T} \) for an extension \( L/K \) is a transcendence basis for \( L/K \) of the form

\[ \mathcal{T} = \{ x_i, y_j : i \in I, j \in J \} \]

such that \( \{ vx_i : i \in I \} \), forms a maximal system of values in \( vL \) which are \( \mathbb{Q} \)-linearly independent over \( vK \) and the residues \( \{ y_j : j \in J \} \), form a transcendence basis of \( Lv/Kv \).

Recall also that an algebraic extension of henselian fields \( L/K \) is tame if for every finite subextension \( K'/K \):

1. \( K'v/Kv \) is separable.
2. if \( p = \text{char}(Kv) > 0 \) then \( (vK' : vK) \) is prime to \( p \).
3. \( K'/K \) is a defectless extension.

\( K \) will be called tame if it is henselian and every algebraic extension is a tame field.

{L:emb-tame-of-transc}

Lemma 4.1. Let \( L \) and \( F \) be henselian ac-valued fields and \( K \) a common ac-valued subfield. Assume that \( L \) is a tame algebraic extension of the henselization of \( K \). Then for every embedding \( \tau = (\tau_L, \tau_v) : (vL, Lv) \hookrightarrow (vF, Fv) \) over \( K \), there is an embedding of \( L \) in \( F \) over \( K \) inducing \( \tau \) and commuting with the ac-map.

Proof. By the uniqueness of the henselization of \( K \), we may extend the embedding \( K \hookrightarrow F \) to the henselization of \( K \). Since the henselization is an immediate extension, by Lemma 3.1 the embedding respects the ac-map. Since every algebraic extension is the union of its finite sub-extensions we may further assume that \( L/K \) is finite.

Since \( Lv/Kv \) is finite and separable it is simple (recall \( L/K \) is tame). Assume that \( Kv(\alpha) = Lv \). By Lemma 3.3 Case 2 there exists \( c \in L \) with \( \bar{c} = \alpha \), \( K(c)v = Lv \) and \( vK(c) = vK \) such that we may extend the embedding of \( K \hookrightarrow F \) to an embedding \( K(c) \hookrightarrow F \) respecting the ac-map.

Set \( K' := K(c) \). Since \( L/K' \) is finite, the group \( vL/vK' \) is a finite torsion group:

\[ vL/vK' = (\gamma_1 + vK)\mathbb{Z} \oplus \cdots \oplus (\gamma_r + vK)\mathbb{Z}, \]

and by tameness the order of each \( \gamma_i \) is prime to \( p = \text{char}(Kv) \). Using Lemma 3.2 Case 2 repeatedly there exist \( d_1, \ldots, d_r \in L \) with \( v(d_i) = \gamma_i \) such that \( L/K'(d_1, \ldots, d_r) \) is an immediate extension and we may extend the embedding of \( K' \hookrightarrow F \) to an embedding \( K'(d_1, \ldots, d_r) \hookrightarrow F \) preserving the ac-map.

Finally, \( K'(d_1, \ldots, d_r) \) is a tame field since it is an algebraic extension of a tame field, thus \( L/K'(d_1, \ldots, d_r) \) is defectless and immediate. Since henselian defectless fields are algebraically maximal, it follows that \( K'(d_1, \ldots, d_r) = L \) and we are done. \( \square \)

{L:emb-tame-of-transc}

Lemma 4.2. Let \( L \) and \( F \) by henselian ac-valued field and \( K \) a common ac-valued subfield. Assume that \( L \) admits a valuation transcendence basis \( \mathcal{T} \) such that \( L \) is a tame extension of \( K(\mathcal{T}) \). Then for every embedding \( \tau = (\tau_L, \tau_v) : (vL, Lv) \hookrightarrow (vF, Fv) \) over \( K \), there is an embedding of ac-valued fields \( L \hookrightarrow F \) over \( K \) inducing \( \tau \).

Proof. By using Lemma 3.2 Case 1 and Lemma 3.3 Case 1 repeatedly we may extend the embedding \( K \hookrightarrow F \) to an embedding \( K(\mathcal{T}) \hookrightarrow F \) over \( K \) respecting

\footnote{If they are models of \( T_2 \), we consider the structures in the augmented language.}
the ac-map, and inducing the embedding $\tau$. The result now follows from Lemma 4.1. \hfill \Box

The following embedding theorem is, as usual, the main result:

**Theorem 4.3.** Let $L$ and $F$ be ac-valued fields. Assume that $F$ is $|L|^+$-saturated and $K$ a common ac-valued substructure in the Denef-Pas language. If both $L$ and $F$ are models of either $T_1$ or $T_2$, then for every embedding $\tau = (\tau_L, \tau_F) : (vL, Lv) \hookrightarrow (vF, Fv)$ over $K$, there is an embedding of $L$ in $F$ over $K$ inducing $\tau$ and preserving the ac-map.

**Proof.** In order to embed $L$ in $F$ (over $K$) it suffices to embed every finitely generated sub-extension. So we may assume that $L/K$ is a finitely generated extension.

Let $T = \{x_i, y_j : i \in I, j \in J\}$ be such that $\{x_i : i \in I\}$ is a maximal system of $Q$-linearly independent values in $vL$ over $vK$ and the residues $\{y_j : j \in J\}$ are a transcendence basis of $Lv/Kv$. By [10] Lemma 6.30 $T$ is algebraically independent over $K$.

Let $L'$ be the maximal tame algebraic extension of the henselization $K(T)^h$ in $L$. By definition it is an algebraic extension so $T$ is a transcendence basis for $L'/K$ and it is a tame extension of $K(T)^h$. We may use Lemma 4.2 and embed $L'$ in $F$ over $K$, this embedding induces $\tau$ and commutes with the ac-map.

Since $L'$ is the maximal tame algebraic extension of $K(T)^h$, necessarily $vL/vL'$ is a $p$-group and $Lv/L'v$ is a purely inseparable algebraic extension.

The next step breaks into two cases:

**Case 1:** Assume that $L, F \models T_1$. By Lemma 2.12 (Case 2.2) we may repeatedly apply Lemma 5.1 (Case 2.2) and extend the embedding of $L' \hookrightarrow F$ to an intermediary ac-valued field $L' \subseteq L'' \subseteq L$ with $vL'' = vL'$ (since $vL/vL'$ is a $p$-group) and $Lv/L''v$ purely inseparable algebraic extension, in such a way that it commutes with the ac-map.

By Lemma 2.12 we may extend the embedding to $L'''$, the relative algebraic closure of $L''$ in $L$. The embedding $L''' \hookrightarrow F$ preserves the ac-map, by Lemma 5.1 since $vL'' = vL'''$. By Proposition 2.9 $L'''$ is an ac-valued field which is a model of $T_1$, and hence it is algebraically maximal. Moreover, notice that $L/L'''$ is immediate.

**Case 2:** Assume that $L, F \models T_2$. Recall that $Lv$ is finite so $Lv = L'v$. By applying Fact 2.14 and Lemma 5.1 (Case 2.2) repeatedly we may extend the embedding to a subfield $L''' \hookrightarrow F$ preserving the ac-map, and where $L/L'''$ is an immediate extension. By Fact 2.15 $L'''$ is algebraically maximal so $L'''$ is algebraically closed in $L$.

The rest of the proof is the same for both cases. Let $a \in L \setminus L'''$. Since $L'''(a)/L'''$ is an immediate extension, by [14] Theorem 1 there exists a pseudo-cauchy sequence with $a$ as a pseudo-limit but with no pseudo-limit in $L'''$. Since $L'''$ is algebraically maximal it must be of transcendental type. Since $F$ is $|L|^+$-saturated it is also $|L|^+$-pc-complete. By [14] Theorem 2, we may thus find a pc-limit $a' \in F$ and the map $a \mapsto a'$ extends the embedding to an embedding of valued fields, it preserves the ac-map by Lemma 5.1. Doing this repeatedly we may embed $L$ in $F$ over $K$ as ac-valued fields. \hfill \Box

**Corollary 4.4.** Let $L$ and $F$ be ac-valued fields and $K$ a common substructure in the Denef-Pas language. If both $L$ and $F$ are models of either $T_1$ or $T_2$ then

$$(vL, Lv) \equiv_{(\sigma_K, K\sigma)} (vF, Fv) \implies L \equiv_K F.$$

As a result, $T_1$ and $T_2$ eliminate field quantifiers and the value group and residue field are stably embedded as pure structures.
Proof. There exist elementary extensions $L^*$ and $F^*$ of $L$ and $F$, respectively, such that $vL^* \cong_{Kv} vF^*$ and $L^*v \cong_{Kv} F^*v$. Since $L^* \cong_{K} F^*$ implies $L \cong_{K} F$, we may assume from the start that $\tau = (\tau_T, \tau_h) : (vL, Lv) \cong_{(vL, Lv)} (vF, Fv)$. The rest is standard and follows from the previous theorem. □

As special cases of the above corollary consider $K = (\mathbb{Q}, \mathbb{Q}, 0)$ if $L$ is of equi-characteristic $(0,0)$; let $K = (\mathbb{F}_{p}, \mathbb{F}_{p}, 0)$ if $L$ is of equi-characteristic $(p,p)$ (for $p > 0$) and in mixed characteristic take $K = (\mathbb{Q}, \mathbb{F}_{p}, v(p)\mathbb{Z})$ allowing us to conclude:

Corollary 4.5. The theories $T_1$ and $T_2$ are complete once the complete theories of the value group and the residue field are fixed, and $v(p)$ is specified.

Every algebraically maximal Kaplansky field is a model of $T_1$ so

Corollary 4.6. The theory of any algebraically maximal Kaplansky field eliminates field quantifiers in the Denef-Pas language.

By definition of $T_2$:

Corollary 4.7. The theory of any henselian $p$-valued field of $p$-rank $d$ eliminates field quantifiers in the Denef-Pas language (augmented by $d$ constants).

Combining these results with Lemma 2.15 and the fact that it is known for the $(0,0)$ case, we have shown:

Corollary 4.8. Let $(K,v)$ be a strongly dependent henselian field. Then $\text{Th}(K,v)$ eliminates field quantifiers in the Denef-Pas language.

The above corollaries give Theorem 1 of the introduction. We now proceed to some applications.

We remind ([8, Proposition 3.4]) that strongly dependent ordered abelian groups have quantifier elimination in the language

$$L = L_{oag} \cup \{(x =_H y, y + k G/H), k \in \mathbb{Z}, i < \alpha, (x \equiv_{m,H} y, y + k G/H), k \in \mathbb{Z}, m \in \mathbb{N}, i < \alpha\},$$

where

- $L_{oag}$ is the language of ordered groups,
- for each $k \in \mathbb{Z}$, “$x =_H y + k G/H$” is defined by $\pi(x) = \pi(y) + k G/H$ for $\pi : G \to G/H$ and $G/H$ denotes $k$ times the minimal positive element of $G/H$, if it exists, and 0 otherwise.

Thus, the above corollary implies that a strongly dependent henselian field $(K,v)$ has quantifier elimination modulo $Kv$ in the Denef-Pas language augmented by the new predicates in $L$. In particular, if $Kv$ has explicit quantifier elimination (e.g., $Kv$ is an algebraically closed field or a real closed field) we get complete quantifier elimination for $(K,v)$. This strengthens [11, Theorem 3.2.16].

We can also give an alternative proof of the following:

Corollary 4.9. [8, Theorem 5.14] Let $K$ be a strongly dependent field. Assume that $v$ is a henselian valuation on $K$ then $(K,v)$ is strongly dependent.

Proof. By [8, Proposition 5.2] and [8, Proposition 5.7], $vK$ and $Kv$ are strongly dependent. Since $(K,v)$ eliminates field quantifiers in the Denef-Pas language, the result follows from [21, Claim 1.17(2)]. □

We end by showing that elimination of field quantifiers of the henselization can be deduced from strong dependentness of the valued field.

Proposition 4.10. Let $(K,v)$ be a strongly dependent valued field. Then its henselization $(K^h,v)$ is also strongly dependent.
Proof. First we show that \((K^h, v)\) is either a model of \(T_1\) or of \(T_2\). By [11 Theorem 4.2.2], \((K, v)\) is defectless and hence \((K^h, v)\) is algebraically maximal. We break into cases:

- If \(char(K, v) = (0, 0)\) then \((K^h, v)\) is a model of \(T_1\).
- If \(char(K, v) = (p, p)\) then by [11 Theorem 4.3.1] \(K^v\) is finite so \((K^h, v)\) is infinite. \(K^v\) is perfect by strong dependence and \(p\) does not divide the degree of any finite extension of \(K^v\). Perfection of \(K\) implies that \(vK\) is \(p\)-divisible. Thus \((K, v)\) is Kaplansky and, since the henselization is an immediate extension, \((K^h, v)\) is algebraically maximal Kaplansky, so a model of \(T_1\).

Assume \(char(K, v) = (0, p)\). If \(K^v\) is finite then by [11 Theorem 4.3.1], \([0, v(p)]\) is finite so \((K^h, v)\) is a model of \(T_2\). If \(K^v\) is infinite, as before, \((K^h, v)\) is a model of \(T_1\).

Let \((K^h, v) \preceq (L, v)\) be an elementary extension of the henselization supporting an ac-map (Fact 2.1). By Corollary 4.4, \((K^h, v)\) is a model of \(T_2\). If \(K^v\) is infinite, as before, \((K^h, v)\) is a model of \(T_1\).

A similar proof gives the following:

**Proposition 4.11.** Let \((K, v)\) be strongly dependent such that \((K^h, v) \models T_1\). Then the inertia field \((K^t, v)\) of \((K, v)\) is strongly dependent.

**Proof.** The definition and basic properties of the inertia field can be found in e.g. [15 Section 7.4]. Since \((K^h, v) \models T_1\) also \((K^t, v)\) is henselian and defectless. Moreover, \(K^t\) and \(K^t v\) are perfect. Since \(vK^t = vK^h\) in order to show that that \((K^t, v) \models T_1\) it remains to show that the degree of every finite extension of \(K^t v\) is prime to \(p\), but \(K^t v\) being separably closed and perfect it is algebraically closed, so there is nothing to prove.

The rest is as in Proposition 4.10. \(\square\)

This proves Theorem 2 of the introduction.

**Remark.** Notice that a similar proof shows that if \((K^h, v) \models T_1\) then \((L, v)\) admits elimination of field quantifiers whenever \(K^h \subseteq L \subseteq K^t\).

The following example shows that the requirement that \((K^h, v) \models T_1\) is necessary.

**Example 4.12.** Let \((K, v)\) be a strongly dependent field with discrete value group and finite residue field. Then, by Fact 2.1 \((K^t, v)\) is not strongly dependent, despite the fact that \(vK^t = vK\) is strongly dependent and \(K^t v\) is algebraically closed (being the algebraic closure of \(\mathbb{F}_p\)). Note that \((K^t, v)\) is, in addition, henselian and defectless (being an algebraic extension of \((K^h, v)\) which is strongly dependent), so algebraically maximal, with algebraically closed residue field.

We point out the following result:

**Proposition 4.13.** Let \((K, v) \equiv (L, w)\) (in the three-sorted language) be strongly dependent such that \((K^h, v), (L^h, w) \models T_1\). Then \((L^h, w) \equiv (K^h, v)\) and \((L^t, w) \equiv (K^t, v)\).

**Proof.** By Corollary 4.5 the assumption that \((K, v) \equiv (L, w)\) and the fact that the henselisation is an immediate extension, imply that \(v(p) = w(p)\) for \(p = char(Lw) = char(Kv)\), and therefore \((K^h, v) \equiv (L^h, w)\). For the inertia fields, recall that if two fields are elementary equivalent then so are their algebraic closures. \(\square\)
5. Geometric Fields

In this final section, we use arguments from [12, Theorem 5.5] for pure henselian valued fields of characteristic 0, to show that henselian fields of finite dp-rank are geometric fields (see below for the definition). The proof in [12] is duplicated almost verbatim, we give the proof for the sake of completeness.

Proposition 5.1. Let $K$ be a strongly dependent field and $v$ a non-trivial henselian valuation on $K$. Then for every $K \equiv K'$ (in the language of rings) and $A \subseteq K'$, $acl_{K'}(A) = F_0(A)^{alg}$, where $F_0$ is the prime field of $K'$.

Remark. A field satisfying this proposition in called very slim in [12], and by [22, Exercise 4.38] the acl-dimension and the dp-rank coincide.

Proof. If $K$ is separably closed, and hence by perfection algebraically closed, this is known.

Otherwise, by [15, Remark 7.11] $K'$ also admits a definable henselian topology. By [12, Lemma 4.11], it is enough to prove the statement for an elementary extension of $K'$. We, thus, assume that $K'$ is $S_1$-saturated. By [15, Theorem 7.2], $K'$ admits a non-trivial henselian valuation which we will also denote by $v$, and by Fact 2.1 $(K', v)$ admits an angular component map.

Let $\phi(x)$ be an $A$-definable algebraic formula and assume there exists a satisfying $\phi(x)$ which is transcendental over $F_0(A)$.

Claim. For every non constant polynomial $p(x)$ over $A$ such that $p(a) \neq 0$, there exists a formula $\psi_p(x)$ over $A$ satisfying:

1. $\psi_p$ is not algebraic.
2. $a$ satisfies $\psi_p$ and for every $a' \models \psi_p$:
   
   $$rv(p(a)) = rv(p(a'))$$

   where $rv : (K')^\times \to (K')^\times / (1 + M)$ is the natural projection, where $M$ is the maximal ideal corresponding to the valuation $v$.

Proof. For any $x$, $rv(p(a)) = rv(p(x))$ if and only if $v(p(x) - p(a)) > v(p(a))$. $p(a) \neq 0$ and thus this is equivalent to $v(p(x)/p(a) - 1) > 0$. Let $\psi_p(x)$ be this latter formula. □ (claim)

Since $rv(x) = rv(y)$ implies $v(x) = v(y)$ and $ac(x) = ac(y)$, by elimination of field quantifiers (Corollary 4.8), see also [12, Theorem 5.5], and using the claim we may find a non algebraic formula $\psi(x)$ over $A$ such that for every $a'$ satisfying $\psi$, $a'$ satisfies $\phi$ as well. Contradicting the fact that $\phi(x)$ is algebraic. □

Corollary 5.2. [12] Proof of Corollary 5.6] A strongly dependent henselian field has no proper infinite ring-definable subfield.

In [9, Remark 2.10], Hrushovski-Pillay define the notion of a geometric field, it is a field $K$ satisfying:

- $K$ is perfect.
- For every $K' \equiv K$ (in the language of rings) and $A \subseteq K'$,
  
  $$acl_{K'}(A) = F_0(A)^{alg}$$

  where $F_0$ is the prime field,

- Every $K' \equiv K$ eliminates $\exists \infty$.

These kind of fields, when sufficiently saturated, enjoy a nice group configuration theorem, see [9, Section 3].

Since every strongly dependent field is perfect, combined with Proposition 5.1 and [2, Corollary 2.2] we get
**Proposition 5.3.** Every henselian field of finite dp-rank is a geometric field.

Recall the following:

**Definition 5.4.** A structure $M$ has geometric elimination of imaginaries if every $b \in M^{eq}$ is interalgebraic with some finite tuple from $M$.

**Proposition 5.5.** Let $K$ be a strongly dependent field admitting a non-trivial henselian valuation $v$. If $K$ does not admit any non-trivial definable valuation then $K$ is either algebraically closed or real closed.

As a result, if $K$ has geometric elimination of imaginaries, or more specifically if it is surgical (condition (E) in [9, Definition 2.4]) then it is either algebraically closed or real closed.

**Proof.** By strong dependence $K$ is perfect, and by [7, Proposition 5.2] the residue field $K_v$ is also perfect. By [7, Proposition 5.5] the value group $vK$ is divisible. By the proof of [10, Proposition 2.4], $K_v$ is either real closed or algebraically closed (the proof shows the existence of a non-trivial definable valuation, the last line of the proof proves the statement of [10, Proposition 2.4]).

By [7, Corollary 5.15], $(K, v)$ is algebraically maximal, thus if $K_v$ is algebraically closed then, since $vK$ is divisible, so is $K$. Otherwise, $K_v$ is real closed and thus necessarily so is $K$.

As for the last statement, first recall that if it is surgical it must have geometric elimination of imaginaries (for instance, [6, Corollary 3.6]). If $K$ admits a non-trivial definable valuation $u$ then the formula $u(x) = u(y)$ defines a definable equivalence relation with infinite classes, with each class infinite and hence of dimension 1. This contradicts the fact that it is surgical. □

We note that the first part of the above proposition is also true, more generally, in the strictly dependent case by [3, Corollary 1.3]

**APPENDIX A. ELIMINATION OF FIELD QUANTIFIERS IN THE RV-LANGUAGE**

We conclude by showing that strongly dependent fields (actually models of $T_1$ and $T_2$) eliminate field quantifiers in the RV-language. We briefly review some definitions, see also, e.g. [5].

Let $(K, v)$ be a valued field. The group $1 + \mathcal{M}_K$ of 1-units in a subgroup of $K^\times$. Set $RV_K := K^\times/(1 + \mathcal{M}_K)$ and let

$$rv_K : K^\times \to RV_K$$

be the natural quotient homomorphism. We may extend this map to all of $K$ by adding a new symbol for $rv_K(0)$. $(Kv)^\times$ embeds in $RV_K$ and we have the following exact sequence

$$1 \longrightarrow (Kv)^\times \longrightarrow RV_K \longrightarrow vK \longrightarrow 0.$$  

$RV_K$ also inherits an image of the the addition from $K$ denoted by $\oplus$, see [5] for more information. We consider $RV_K$ as a structure in the language $\{\cdot^{-1}, \oplus, 1, v\}$.

The RV-structure for the valued field $K$ is a two sorted structure $(K, RV_K)$ together with the map $rv_K$.

We need the following well known extension of Lemma 3.2:

**Lemma A.1.** Let $(K, v)$ be a valued field with $\text{char}(Kv) = p > 0$ and $a$ an element of the divisible hull of $vK$ with $pa \in vK$, $\alpha \notin vK$. For any element $a \in K^{alg}$, the algebraic closure of $K$, with $a^p \in K$ and $v(a^p) = pa$, $v$ extend uniquely from $K$ to $K(a)$. Moreover, for any other $a' \in K^{alg}$, the map $a \mapsto a'$ determines, uniquely, the isomorphism of $RV_{K(a)}$ and $RV_{K(a')}$. over $RV_K$.
Proof. The uniqueness of the extension of the valuation is [15 Lemma 6.40], also it is shown there that
\[ v_K(a) = v + Z_a \text{ and } K(a)v = Kv. \]
Let \( x \in K(a) \), by the above there exist \( d \in K \) such that
\[ v(x^{-1}da^n) = 0, \]
and thus \( c \in K^\times \) with \( v(c) = 0 \) such that the value of \( x^{-1}da^n c \) is 0 and its residue is 1. Hence
\[ rv_{K(a)} = rv_K(d) \cdot rv_{K(a)}(a)^n \cdot \iota(\bar{c}), \]
where \( \iota \) is the embedding of \( Kv = K(a)v \) in \( RV_K \).

\[ \square \]

**Theorem A.2.** Let \( L \) and \( F \) be models of either \( T_1 \) or \( T_2 \) in the RV-language, with \( F \mid L \) \( + \)-saturated, and \( K \) a common substructure. Then for every embedding \( \tau : RV_L \hookrightarrow RV_F \) over \( RV_K \), there is an embedding of \( L \) in \( F \) over \( K \) inducing \( \tau \).

**Proof.** As in the proof of Theorem 4.3, let \( L' \) be the maximal tame algebraic extension of \( K(T)^h \), where \( T \) is as in the proof. By [16 Lemma 3.7], we may embed \( L' \) in \( F \) over \( K \), this embedding induces \( \tau \). Necessarily \( vL/vL' \) is a \( p \)-group and \( Lv/Lv' \) is a purely inseparable algebraic extension.

Using Lemma A.1 and Proposition 2.9(2) for \( T_1 \), or Fact 2.14 for \( T_2 \), we may extend the embedding to an intermediary field \( L'' \subseteq L'' \subseteq L \) with \( vL'' = vL \) and \( Lv/Lv'' \) a purely inseparable algebraic extension. The rest of the proof is as in Theorem 4.3. \[ \square \]

**Corollary A.3.** \( T_1 \) and \( T_2 \), and hence any strongly dependent henselian field, eliminate field quantifiers in the RV-language.

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