Affine SISO Feedback Transformation Group and Its Faà di Bruno Hopf Algebra

W. Steven Gray¹,² and Kurusch Ebrahimi-Fard¹

¹Instituto de Ciencias Matemáticas, Consejo Superior de Investigaciones Científicas, C/ Nicolaús Cabrera, no. 13-15, 28049 Madrid, Spain
²On leave from Old Dominion University, Norfolk, Virginia 23529, USA

This paper describes a transformation group for the class of nonlinear single-input, single-output (SISO) systems that can be represented in terms of Chen-Fliess functional expansions. There is no a priori requirement that these input-output systems have state space realizations, so the results presented here are independent of any particular state space coordinate system or state space embedding when realizations are available. The group is referred to as the affine feedback transformation group since it can always represent the input-output feedback linearization law of any control affine state space realization having a well defined relative degree. It can also be viewed as a generalization of the output feedback group developed earlier by the first author and collaborators. The corresponding Hopf algebra of coordinate maps is also presented in order to facilitate the computation of the group inverse. Finally, the Lie algebra of the group is described as well as some of the invariants of the group action.

1. Introduction

Let $G$ be a group and $S$ a given set. $G$ is said to act as a transformation group on the right of $S$ if there exists a mapping $\phi : S \times G \to S : (h, g) \mapsto hg$ such that:

i. $h1 = h$, 1 is the identity element of $G$;
ii. $h(g_1g_2) = (hg_1)g_2$ for all $g_1, g_2 \in G$. 
The action is said to be free if \( hg = h \) implies that \( g = 1 \). Transformation groups have been used extensively in system theory since its inception. The early work of Brockett, Krener and others in the case of linear systems \([3,4]\) and nonlinear state space systems \([2,25]\) has been important in understanding the role of invariance under feedback and coordinate transformations. More recently in \([15,21]\), an output feedback transformation group was presented for the class of nonlinear single-input, single-output (SISO) systems that can be represented in terms of Chen-Fliess functional expansions, so called Fliess operators \([9,10]\). There was no a priori requirement that these input-output systems have state space realizations, so the results presented there are independent of any particular state space coordinate system or state space embedding when realizations are available. In particular, it was shown that this output feedback transformation group leaves a certain subseries of an operator’s generating series invariant. The order, \( r \), of this invariant subseries corresponds to the notion of relative degree (defined purely from an input-output point of view) when it is well defined. Such a subseries does not, however, coincide with the transfer function of the Brunovsky form, \( 1/s^r \), unless the generating series has a stronger notion of relative degree referred to as extended relative degree. It was mentioned in \([15]\) that this fact hints at the possibility of a larger feedback transformation group for this class of input-output systems whose invariants do correspond exactly to Brunovsky forms. In this paper, that group is presented. It is referred to as affine since it can always represent the input-output feedback linearization law of any control affine state space realization having a well defined relative degree in the usual sense \([24]\). The generalization requires a nontrivial extension of the approach taken in \([15,21]\), as well as generalizations of the combinatorial tools used in \([12,14,16,19,20]\) to characterize system interconnections. A preliminary version of this part of the paper appeared in \([13]\).

The next part of the paper is devoted to a characterization of the Faà di Bruno Hopf algebra of coordinate maps for the group, as was done for the output feedback transformation group in \([11,12,14,16]\) for the SISO case and in \([18]\) for the multivariable case. Such combinatorial Hopf algebras provide explicit and powerful computational tools for finding the group inverse via the antipode of the algebra. This is useful in applications such as computing the generating series of a closed-loop system \([14,18]\) and analytical system inversion \([19]\). The Hopf algebra presented here is commutative, graded and connected and contains as a subalgebra the Hopf algebra of the output feedback transformation group. It will be shown that its antipode can be computed in a fully recursive manner. In addition, the Lie algebra of the group is described and shown to be induced by a pre-Lie product analogous to what was found for the output feedback transformation group in \([21]\). Overall, the focus here is on the SISO case since the multivariable extension of the affine feedback group and its Hopf algebra do not appear to be as straightforward as in the earlier work.

The final part of the paper is dedicated to describing the invariants of the affine feedback group action for series which have well defined relative degree. The computation tools developed in the previous part are demonstrated on a few examples. Specifically, it is shown how input-output feedback linearization can be performed in a coordinate-free manner using only formal power series operations.

The paper is organized as follows. In the next section, a few key preliminary concepts are briefly outlined to establish the notation and make the presentation more self-contained. In Section 3, the new transformation group is described in detail. In the subsequent section, the Hopf algebra is developed. In Section 5, the associated Lie algebra is described, and the invariance theory is presented is the next section. The paper’s conclusions are given in the final section.

2. Preliminaries

A finite nonempty set of noncommuting symbols \( X = \{x_0, x_1, \ldots, x_m\} \) is called an alphabet. Each element of \( X \) is called a letter, and any finite sequence of letters from \( X \), \( \eta = x_i_1 \cdots x_i_k \), is called a word over \( X \). The length of \( \eta, |\eta| \), is the number of letters in \( \eta \). The set of all words with length \( k \) is denoted by \( X^k \). The set of all words including the empty word, \( \emptyset \), is designated by \( X^* \). It
A causal \( m \)-input, \( \ell \)-output operator, \( F_c \), is defined in the following manner. Let \( p \geq 1 \) and \( t_0, t_1 \) be given. For a Lebesgue measurable function \( u : [t_0, t_1] \to \mathbb{R}^m \), define \( \|u\|_p = \max\{\|u_i\|_p : 1 \leq i \leq m\} \), where \( \|u_i\|_p \) is the usual \( L^p \)-norm for a measurable real-valued function, \( u_i \), defined on \( [t_0, t_1] \). Let \( L^p_{\text{loc}}([t_0, t_1]) \) denote the set of all measurable functions defined on \( [t_0, t_1] \) having a finite \( \|u\|_p \) norm and \( B^p_{\text{loc}}(R)|[t_0, t_1]| := \{u \in L^p_{\text{loc}}([t_0, t_1]) : \|u\|_p \leq R\} \). Assume \( C[0, t_1] \) is the subset of continuous functions in \( L^1_{\text{loc}}([0, t_1]) \). Define inductively for each \( \eta \in X^* \) the map \( E_\eta : L^p_{\text{loc}}[t_0, t_1] \to C[0, t_1] \) by setting \( E_\emptyset[u] = 1 \) and letting
\[
E_{x, \eta}[u](t, t_0) = \int_{t_0}^t u_i(\tau)E_\emptyset[u](\tau, t_0) d\tau,
\]
where \( x_i \in X \), \( \eta \in X^* \), and \( u_0 = 0 \). The input-output operator corresponding to \( c \) is the Flies operator
\[
F_c[u](t) = \sum_{\eta \in X^*} (c, \eta) E_\eta[u](t, t_0)
\]
[9,10].

When Flies operators \( F_c \) and \( F_d \) are connected in a parallel-product fashion, it was shown in [9] that \( F_cF_d = F_{c \cdot d} \). If \( F_c \) and \( F_d \) with \( c \in \mathbb{R}^f(\langle X \rangle) \) and \( d \in \mathbb{R}^m(\langle X \rangle) \) are interconnected in a cascade manner, the composite system \( F_c \circ F_d \) has the Flies operator representation \( F_{c \circ d} \), where \( c \circ d \) denotes the composition product of \( c \) and \( d \) as described in [6,7]. This product is associative and \( \mathbb{R} \)-linear in its left argument \( c \). In the event that two Flies operators are interconnected to form a feedback system, the closed-loop system has a Flies operator representation whose generating

\[\text{Eq. 25}\]

For notational convenience, \( p = (p, \emptyset) \) is written as \( p = (p, \emptyset) \).
series is the feedback product of \( c \) and \( d \), denoted by \( c \circ d \) \[6,20\]. Consider, for example, the SISO case where \( X = \{x_0, x_1\} \) and \( \ell = 1 \). Define the set of operators

\[ \mathcal{F}_I = \{ I + F_c : c \in R(\langle X \rangle) \}, \]

where \( I \) denotes the identity map. It is convenient to introduce the symbol \( \delta \) as the (fictitious) generating series for the identity map. That is, \( F_c := I \) such that \( I + F_c := F_{\delta + c} = F_c \delta \). The set of all such generating series for \( \mathcal{F}_I \) is denoted by \( R(\langle X \rangle) \). \( \mathcal{F}_I \) forms a group under the composition

\[ F_c \circ F_d = (I + F_c) \circ (I + F_d) = F_{c \circ d}, \]

where \( c \circ d := \delta + c \circ d \), and \( \tilde{\delta} \) denotes the modified composition product \[14\]. It is of central importance that the corresponding group \( R(\langle X \rangle) \circ \delta \) has a dual that forms a Faà di Bruno type Hopf algebra. In which case, the group (composition) inverse \( c_{\circ}^{-1} \) can be computed efficiently by a recursive algorithm \[5,17\]. This inverse is also key in describing the feedback product as shown in the following theorem.

**Theorem 2.1.** \[14\] For any \( c, d \in R(\langle X \rangle) \) it follows that \( \tilde{c} \circ \tilde{d} = c \circ (\delta - d \circ c)^{-1} \).

(b) **Shuffle Product Operations on Ultrametric Spaces**

The \( R \)-vector space \( R(\langle X \rangle) \) with the distance between two series defined as \( \text{dist}(c, d) = \sigma^{\text{ord}(c-d)} \) for some arbitrary but fixed \( 0 < \sigma < 1 \) is a complete ultrametric space \[1\]. In this section two lemmas are presented which describe how the distance between two series are altered by operations involving the shuffle product. These results are employed in Section 3 to prove the existence of a group inverse.

**Lemma 2.1.** For any series \( c_i, d_i \in R(\langle X \rangle) \), \( i = 1, 2 \),

\[ \text{dist}(c_1 \shuffle d_1, c_2 \shuffle d_2) \leq \max(\sigma^{\text{ord}(c_1)} \text{dist}(d_1, d_2), \sigma^{\text{ord}(d_2)} \text{dist}(c_1, c_2)). \]

**Proof.** First observe that

\[ \text{dist}(c_1 \shuffle d_1, c_2 \shuffle d_2) = \sigma^{\text{ord}(c_1 \shuffle (d_1 - d_2))} = \sigma^{\text{ord}(c_1) + \text{ord}(d_1 - d_2)} = \sigma^{\text{ord}(c_1)} \text{dist}(d_1, d_2). \]

In which case, from the ultrametric triangle inequality it follows that

\[ \text{dist}(c_1 \shuffle d_1, c_2 \shuffle d_2) \leq \max(\text{dist}(c_1 \shuffle d_1, c_1 \shuffle d_2), \text{dist}(c_1 \shuffle d_2, c_2 \shuffle d_2)) \]

\[ = \max(\sigma^{\text{ord}(c_1)} \text{dist}(d_1, d_2), \sigma^{\text{ord}(d_2)} \text{dist}(c_1, c_2)). \]

**Corollary 2.1.** For a fixed \( c \in R(\langle X \rangle) \), the mapping \( d \mapsto c \shuffle d \) is an ultrametric contraction if \( c \) is proper and an isometry on \( (R(\langle X \rangle), \text{dist}) \), otherwise.

**Theorem 2.2.** \[19\] The set of non proper series \( R_{np}(\langle X \rangle) \subset R(\langle X \rangle) \) is a group under the shuffle product. In particular, the shuffle inverse of any such series \( c \) is

\[ c^{\shuffle -1} = ((c, \emptyset)(1 - c'))^{\shuffle -1} := (c, \emptyset)^{-1} (c')^{\shuffle *}, \]

where \( c' := 1 - c/(c, \emptyset) \) is proper and \( (c')^{\shuffle *} := \sum_{k \geq 0} (c')^{\shuffle k}. \)

**Lemma 2.2.** The shuffle inverse is an isometry for any \( c, d \in R_{np}(\langle X \rangle) \) having identical constant terms.

\[2\] The same symbol will be used for composition on \( R(\langle X \rangle) \) and \( R(\langle X_i \rangle) \). As elements in these two sets have a distinct notation, i.e., \( c \) versus \( c_j \), respectively, it will always be clear which product is at play.
Proof. For any \( c, d \in \mathbb{R}^{ap}(\langle X \rangle) \) with \( (c, \emptyset) = (d, \emptyset) \) observe
\[
\text{ord}(c^\omega - d^\omega) = \text{ord} \left( \sum_{k=1}^{\infty} \left( c^\omega \cdot k - (d^\omega \cdot k) \right) \right) = \text{ord}(c - d),
\]
and hence the lemma is proved. \( \square \)

(c) Hopf Algebras

Some definitions and facts concerning Hopf algebras are summarized here for later use [8,23,29].

A coalgebra over \( \mathbb{R} \) consists of a triple \((C, \Delta, \varepsilon)\). The coproduct \( \Delta : C \to C \otimes C \) is coassociative, that is, \((\text{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \text{id}) \circ \Delta \), and \( \varepsilon : C \to \mathbb{R} \) denotes the counit map. A bialgebra \( B \) is both a unital algebra and a coalgebra together with compatibility relations, such as both the algebra product, \( m(x, y) = xy \), and unit map, \( e : \mathbb{R} \to B \), are coalgebra morphisms. This provides, for example, that \( \Delta(xy) = \Delta(x) \Delta(y) \). The unit of \( B \) is denoted by \( 1 = e(1) \). A bialgebra is called graded if there are \( \mathbb{R} \)-vector subspaces \( B_n, n \geq 0 \) such that \( B = \bigoplus_{n \geq 0} B_n \) and \( \Delta B_n \subseteq \bigoplus_{k+l=n} B_k \otimes B_l \). Elements \( x \in B_n \) are given a degree \( \deg(x) = n \). Moreover, \( B \) is called connected if \( B_0 = \mathbb{R}1 \). Define \( B_+ = \bigoplus_{n>0} B_n \). For any \( x \in B_0 \), the coproduct is of the form
\[
\Delta(x) = x \otimes 1 + 1 \otimes x + \Delta'(x) \in \bigoplus_{k+l=n} B_k \otimes B_l,
\]
where \( \Delta'(x) := \Delta(x) - x \otimes 1 - 1 \otimes x \in B_+ \otimes B_+ \) is the reduced coproduct.

Suppose \( A \) is an \( \mathbb{R} \)-algebra with product \( m_A \) and unit \( e_A \), e.g., \( A = \mathbb{R} \) or \( A = B \). The vector space \( L(B, A) \) of linear maps from the bialgebra \( B \) to \( A \) together with the convolution product \( \Phi \ast \Psi := m_A \circ (\Phi \otimes \Psi) \circ \Delta : B \to A \), where \( \Phi, \Psi \in L(B, A) \), is an associative algebra with unit \( 1 := e_A \otimes e \). A Hopf algebra \( H \) is a bialgebra together with a particular \( \mathbb{R} \)-linear map called an antipode \( S : H \to H \) which satisfies the Hopf algebra axioms and has the property that \( S(xy) = S(y)S(x) \). When \( A = H \), the antipode \( S \in L(H, H) \) is the inverse of the identity map with respect to the convolution product, that is,
\[
S \ast \text{id} = \text{id} \ast S := m \circ (S \otimes \text{id}) \circ \Delta = e \otimes e.
\]
A connected graded bialgebra \( H = \bigoplus_{n \geq 0} H_n \) is always a connected graded Hopf algebra.

Suppose \( A \) is a commutative unital algebra. The subset \( g_0 \subseteq L(H, A) \) of linear maps \( \alpha \) satisfying \( \alpha(1) = 0 \) forms a Lie algebra in \( L(H, A) \). The exponential \( \exp^\gamma(\alpha) = \sum_{j \geq 0} \frac{1}{j!} \alpha^j \) is well defined and gives a bijection from \( g_0 \) onto the group \( G_0 = \{ 0 \} + g_0 \) of linear maps \( \gamma \) satisfying \( \gamma(1) = 1_A \). A map \( \Phi \in L(H, A) \) is called a character if \( \Phi(1) = 1_A \) and \( \Phi(xy) = \Phi(x)\Phi(y) \) for all \( x, y \in H \).

A connected character \( \Phi \in G_A \subseteq G_0 \). The neutral element \( 1 := e_A \circ e \) in \( G_A \) is given by \( \iota(1) = 1_A \) and \( \iota(x) = 0 \) for \( x \in \Ker(\varepsilon) = H_+ \). The inverse of \( \Phi \in G_A \) is given by
\[
\Phi^{-1} = \Phi \circ S.
\]

Given an arbitrary group \( G \), the set of real-valued functions defined on \( G \) is a commutative unital algebra. There is a subalgebra of functions known as the representative functions, \( R(G) \), which can be endowed with a Hopf algebra, \( H \). In this case, there is a group isomorphism relating \( G \) to the convolution group \( G_A \), say, \( \Phi : G \to G_A : g \mapsto \Phi_g \). A coordinate map is any \( a : H \to \mathbb{R} \) satisfying
\[
(\Phi_{g_1} \ast \Phi_{g_2})(a) = a(g_1g_2), \ \forall g_1 \in G.
\]

In some sense, the coordinates maps are the generators of \( H \), though they can not always be easily identified in general.
3. Affine Feedback Transformation Group

It will be assumed henceforth that \( X = \{x_0, x_1\} \) and \( \ell = 1 \), which corresponds to a SISO system. The first step in building the affine feedback transformation group is to redefine \( \mathbb{R}\langle\langle X_0\rangle\rangle\) in terms of pairs of series and then to generalize the modified composition product in a consistent fashion.

**Definition 3.1.** Consider a pair of series \( d_3 = (d_L, d_R) \in \mathbb{R}\langle\langle X_0\rangle\rangle \times \mathbb{R}\langle\langle X_3\rangle\rangle =: \mathbb{R}\langle\langle X_3\rangle\rangle \). Define the mixed composition product mapping \( \mathbb{R}\langle\langle X_0\rangle\rangle \times \mathbb{R}\langle\langle X_3\rangle\rangle \) into \( \mathbb{R}\langle\langle X_0\rangle\rangle \) as

\[
\phi_{d} = \sum_{(c, \eta) \in \mathbb{R}\langle\langle X_0\rangle\rangle} (c, \eta) \phi_{d}(\eta)(1),
\]

where \( \phi_{d} \) is the continuous (in the ultrametric sense) algebra homomorphism from \( \mathbb{R}\langle\langle X_0\rangle\rangle \) to \( \text{End}(\mathbb{R}\langle\langle X_0\rangle\rangle) \) uniquely specified by \( \phi_{d}(x_1) = \phi_{d}(x_i) \circ \phi_{d}(\eta) \) with

\[
\phi_{d}(x_0)(e) = x_0 e, \quad \phi_{d}(x_1)(e) = x_0 (d_L \omega e) + x_0 (d_R \omega e)
\]

for any \( e \in \mathbb{R}\langle\langle X_0\rangle\rangle \), and where \( \phi_{d}(\emptyset) \) denotes the identity map on \( \mathbb{R}\langle\langle X_0\rangle\rangle \).

The modified composition product mentioned in Section 2 corresponds here to the special case where \( d_L = 1 \). Some fundamental properties of this product are given next.

**Lemma 3.1.** The mixed composition product on \( \mathbb{R}\langle\langle X_0\rangle\rangle \times \mathbb{R}\langle\langle X_3\rangle\rangle \)

1. is left \( \mathbb{R} \)-linear;
2. satisfies \( c \circ d_3 \circ k = c \circ d_3 \) for any fixed \( d_3 \) if and only if \( c = k \);
3. satisfies \( (x_0 c) \circ d_3 = x_0 (c \circ d_3) \) and \( (x_1 c) \circ d_3 = x_1 (d_0 \omega (c \circ d_3)) + x_0 (d_0 \omega (c \circ d_3)) \);
4. distributes to the left over the shuffle product.

**Proof.**

(1) This fact follows directly from the definition of the mixed composition product.

(2) The claim is immediate since \( \phi_{d}(1, 0)(1) = \eta \).

(3) The only non trivial assertion is that \( c \circ d_3 \circ k = k \) implies \( c = k \). This claim is best handled in a Hopf algebra setting. So this part of the proof will be deferred until Section 4.

(4) Observe:

\[
(x_0 c) \circ d_3 \circ k = \phi_{d}(x_0 c)(1) = \phi_{d}(x_0) \circ \phi_{d}(c)(1) = x_0 (c \circ d_3)
\]

\[
(x_1 c) \circ d_3 \circ k = \phi_{d}(x_1 c)(1) = \phi_{d}(x_1) \circ \phi_{d}(c)(1) = x_1 (d_0 \omega (c \circ d_3)) + x_0 (d_0 \omega (c \circ d_3)).
\]

(5) One can define a shuffle product within \( \text{End}(\mathbb{R}\langle\langle X_0\rangle\rangle) \) via

\[
\phi_{e}(x_1 \eta) \omega \phi_{e}(x_j \xi) = \phi_{e}(x_1) \circ [\phi_{e}(\eta) \omega \phi_{e}(x_j \xi)] + \phi_{e}(x_j) \circ [\phi_{e}(x_1 \eta) \omega \phi_{e}(\xi)].
\]

In which case, \( \phi_{e} \) acts as an algebra map between the shuffle algebra on \( \mathbb{R}\langle\langle X_0\rangle\rangle \) and the shuffle algebra within \( \text{End}(\mathbb{R}\langle\langle X_0\rangle\rangle) \). Specifically, \( \phi_{e}(c \omega d) = \phi_{e}(c) \omega \phi_{e}(d) \). Hence, \( (c \omega d) \circ \varepsilon_3 = \phi_{e}(c \omega d)(1) = \phi_{e}(c)(1) \omega \phi_{e}(d)(1) = (c \circ \varepsilon_3) \omega (d \circ \varepsilon_3) \).

It is easily checked that

\[
\text{dist}(c_3, d_3) := \max(\text{dist}(c_L, d_L), \text{dist}(c_R, d_R))
\]

is an ultrametric on \( \mathbb{R}\langle\langle X_3\rangle\rangle \). The following lemma states that the mixed composition product acts as an ultrametric contraction on this space.

\(^3\)Using \( \text{dist} \) for both the ultrametric on \( \mathbb{R}\langle\langle X_0\rangle\rangle \) and \( \mathbb{R}\langle\langle X_3\rangle\rangle \) should cause minimal confusion since their arguments are distinct.
Lemma 3.2. For any \( c \in \mathbb{R}(\langle X \rangle) \) and \( d_{\delta,1}, d_{\delta,2} \in \mathbb{R}(\langle X \delta \rangle) \) it follows that
\[
\text{dist}(c \circ d_{\delta,1}, c \circ d_{\delta,2}) \leq \sigma^{\text{ord}(c')} \text{dist}(d_{\delta,1}, d_{\delta,2}),
\]
where \( c = (c, 0) \oplus c' \). In which case, the mixed composition product acts as a contraction from \((\mathbb{R}(\langle X \delta \rangle), \text{dist})\) into \((\mathbb{R}(\langle X \rangle), \text{dist})\).

Proof. For a fixed \( d_{R} \), consider the map \( d_{L} \mapsto c \circ (d_{L}, d_{R}) \). Likewise, for a fixed \( d_{L} \) there is a companion map \( d_{R} \mapsto c \circ (d_{L}, d_{R}) \). It is first shown that on the ultrametric space \((\mathbb{R}(\langle X \rangle), \text{dist})\):
\[
dist(c \circ (d_{L,1}, d_{R}), c \circ (d_{L,2}, d_{R})) \leq \sigma^{\text{ord}(c')} \text{dist}(d_{L,1}, d_{L,2}) \tag{3.2a}
\]
\[
dist(c \circ (d_{L,1}, d_{R}), c \circ (d_{L,2}, d_{R})) \leq \sigma^{\text{ord}(c')} \text{dist}(d_{R,1}, d_{R,2}). \tag{3.2b}
\]
The first step is to verify that (3.2a) holds when \( c = \eta \in X^{*} \). It is shown by induction on the length of \( \eta \) that
\[
\text{ord}(\phi_{d_{1}}(\eta)(1) - \phi_{d_{2}}(\eta)(1)) \geq |\eta| + \text{ord}(d_{L,1} - d_{L,2}), \tag{3.3}
\]
where \( d_{\delta,1} = (d_{L,1}, d_{R}) \) and \( d_{\delta,1} \neq 0 \) (the nondegenerate case). The claim is trivial when \( \eta \) is empty or a single letter. Assume the inequality holds for words up to length \( k \geq 0 \). For any \( x_{1} \eta \) with \( \eta \in X^{k} \), inequality (3.3) follows directly from the induction hypothesis. The case for \( x_{1} \eta \) is handled as follows:
\[
\text{ord}(\phi_{d_{1}}(x_{1} \eta)(1) - \phi_{d_{2}}(x_{1} \eta)(1)) = \text{ord}(x_{1}[d_{L,1} \cup \phi_{d_{1}}(\eta)(1) - d_{L,2} \cup \phi_{d_{2}}(\eta)(1)] + x_{0}[d_{R} \cup (\phi_{d_{1}}(\eta)(1) - \phi_{d_{2}}(\eta)(1))])
\]
\[
= \text{ord}(x_{1}[d_{L,1} \cup \phi_{d_{1}}(\eta)(1) + d_{L,2} \cup (\phi_{d_{1}}(\eta)(1) - \phi_{d_{2}}(\eta)(1))] +
\]
\[
x_{0}[d_{R} \cup (\phi_{d_{1}}(\eta)(1) - \phi_{d_{2}}(\eta)(1))]) \geq 1 + \min(\text{ord}(d_{L,1} - d_{L,2}) \cup \phi_{d_{1}}(\eta)(1)), \text{ord}(d_{L,2} \cup [\phi_{d_{1}}(\eta)(1) - \phi_{d_{2}}(\eta)(1))],
\]
\[
\text{ord}(d_{L} \cup (\phi_{d_{1}}(\eta)(1) - \phi_{d_{2}}(\eta)(1))]) \geq 1 + \min(\text{ord}(d_{L,1} - d_{L,2}) + |\eta|, \text{ord}(\phi_{d_{1}}(\eta)(1) - \phi_{d_{2}}(\eta)(1)))
\]
\[
= |\eta| + 1 + \text{ord}(d_{L,1} - d_{L,2}).
\]
In which case, (3.3) holds for any \( \eta \in X^{*} \). The inequality (3.2a) is now derived. Observe
\[
\text{dist}(c \circ (d_{L,1}, d_{R}), c \circ (d_{L,2}, d_{R})) = \text{dist}(c' \circ (d_{L,1}, d_{R}), c' \circ (d_{L,2}, d_{R})) = \sigma^{\text{ord}(\sum_{a}(c', \eta)(1) - \phi_{d_{2}}(\eta)(1))}
\]
\[
\leq \sigma^{\text{ord}(c')} \text{dist}(d_{L,1}, d_{L,2}) \leq \sigma^{\text{ord}(c')} \text{dist}(d_{L,1}, d_{L,2}).
\]
The proof for (3.2b) is completely analogous. The final step of the proof is to employ the ultrametric triangle inequality in conjunction with (3.2). Observe
\[
\text{dist}(c \circ (d_{L,1}, c \circ (d_{L,2}, d_{R,2})) = \max(\text{dist}(c \circ (d_{L,1}, d_{R,1}), c \circ (d_{L,2}, d_{R,1}), c \circ (d_{L,2}, d_{R,2})), \text{dist}(c \circ (d_{L,2}, d_{R,1}), c \circ (d_{L,2}, d_{R,2}))),
\]
\[
\leq \sigma^{\text{ord}(c')} \max(\text{dist}(d_{L,1}, d_{L,2}), \text{dist}(d_{R,1}, d_{R,2})) = \sigma^{\text{ord}(c')} \text{dist}(d_{\delta,1}, d_{\delta,2}).
\]

Analogous to the special case \( d_{\delta,1} = 1 \) in [14], where the modified composition product is used to define the group product, here the mixed composition product is used to define a
group product on $\mathbb{R} \langle \langle X_0 \rangle \rangle$ as generalized in Definition 3.1. Its basic properties are given in the subsequent lemma.

**Definition 3.2.** The composition product on $\mathbb{R} \langle \langle X_0 \rangle \rangle$ is defined as

$$c_3 \circ d_3 = ((c_L \circ d_3 \circ d_L) \circ d_R + c_R \circ d_3).$$

**Lemma 3.3.** The composition product on $\mathbb{R} \langle \langle X_0 \rangle \rangle$

(1) is left $\mathbb{R}$-linear;

(2) satisfies $(c_3 \circ d_3) \circ e_3 = c_3 \circ (d_3 \circ e_3)$ (mixed associativity);

(3) is associative.

**Proof.**

(1) This claim is a direct consequence of the left linearity of the mixed composition product.

(2) In light of the first item it is sufficient to prove the claim only for $c_3 = \eta \in X^k$, $k \geq 0$. The cases $k = 0$ and $k = 1$ are trivial. Assume the claim holds up to some fixed $k \geq 0$. Then via Lemma 3.1 (4) and the induction hypothesis it follows that

$$(\eta \circ d_3) \circ e_3 = \eta \circ (d_3 \circ e_3) = \eta \circ (d_3 \circ e_3).$$

In a similar fashion, apply the properties in Lemma 3.1 (1), (4), and (5) to get

$$((x_1 \eta) \circ d_3) \circ e_3 = x_1[(d_3 \circ e_3) \circ d_L \circ (\eta \circ d_3)] + x_0[(d_3 \circ e_3) \circ d_R \circ (\eta \circ d_3)] \circ e_3.$$ 

Now employ the induction hypothesis so that

$$((x_1 \eta) \circ d_3) \circ e_3 = x_1[(d_3 \circ e_3) \circ d_L \circ (\eta \circ d_3)] + x_0[(d_3 \circ e_3) \circ d_R \circ (\eta \circ d_3)].$$

Therefore, the claim holds for all $\eta \in X^*$, and the identity is proved.

(3) First apply Definition 3.2 twice, Lemma 3.1 (1) and (5) to get

$$(c_3 \circ d_3) \circ e_3 = ((c_L \circ d_3) \circ d_L \circ d_R + c_R \circ d_3) \circ e_3$$

$$= ((c_L \circ d_3) \circ d_L \circ d_R + c_R \circ d_3 \circ e_3) \circ e_3$$

Now apply the mixed associativity property from the previous item and then recombine terms according to Definition 3.2 so that

$$(c_3 \circ d_3) \circ e_3 = ((c_L \circ d_3) \circ d_L \circ d_R + c_R \circ d_3 \circ e_3) \circ e_3 = (c_L \circ d_3 \circ e_3) \circ (d_L \circ d_R + c_R \circ d_3 \circ e_3).$$
\[
= \left( \left[ c_L \circ (d_\delta \circ e_\delta) \right]_L \right) \cup \left( \left( d_\delta \circ e_\delta \right) \right)_R \cup \left( \left( d_\delta \circ e_\delta \right) \right)_R + \left( c_R \circ (d_\delta \circ e_\delta) \right)
\]
and the lemma is proved. \hfill \square \square \square

For any \( c_\delta \in \mathbb{R}(\langle X_\delta \rangle) \) associate the functional \( F_{c_\delta}[u] := uF_{c_L}[u] + F_{c_R}[u] \). The primary motivation behind the two series products defined above is given in the following theorem.

**Theorem 3.1.** For any \( c \in \mathbb{R}(\langle X \rangle) \) and \( c_\delta, d_\delta \in \mathbb{R}(\langle X_\delta \rangle) \) the following identities hold:

1. \( F_c \circ F_{d_\delta} = F_{c \circ d_\delta} \)
2. \( F_{c_\delta} \circ F_{d_\delta} = F_{c_\delta \circ d_\delta} \).

**Proof.**

(1) It is sufficient to prove the claim for \( c = \eta \in X^* \). This is done by induction on the length of \( \eta \). The case for the empty word is trivial. Assume the identity holds for words \( \eta \in X^k \) up to some fixed length \( k \geq 0 \). Then

\[
E_{x_\delta \eta} \circ F_{d_\delta}[u](t) = \left[ \int_{t_0}^t E_{\eta}[F_{d_\delta}[u]](\tau, t_0) \, d\tau \right] \int_{t_0}^t E_{\eta}(\eta \circ d_\delta)[u](\tau, t_0) \, d\tau = F_{x_\delta(\eta \circ d_\delta)}[u](t)
\]

Similarly,

\[
E_{x_\delta \eta} \circ F_{d_\delta}[u](t) = \left[ \int_{t_0}^t u(\tau) F_{d_\delta}[u](\tau) + F_{d_\delta}[u](\tau) \right] \int_{t_0}^t \eta \circ d_\delta[u](\tau) \, d\tau
\]

Hence, the claim holds for all \( \eta \in X^* \).

(2) Observe

\[
F_{c_\delta} \circ F_{d_\delta}[u] = (uF_{c_\delta}[u] + F_{c_R}[u]) \circ (uF_{c_L}[u] + F_{d_R}[u])
\]

Let \( \mathbb{R}_{np}(\langle X_\delta \rangle) \) denote the subset of \( \mathbb{R}(\langle X_\delta \rangle) \) with the defining property that the left series are not proper. A main result of the paper is given below.

**Theorem 3.2.** The set \( \langle \mathbb{R}_{np}(\langle X_\delta \rangle), \circ, (1, 0) \rangle \) is a group.

**Proof.** Using the identities \( 1 \circ c_\delta = 1 \) and \( 0 \circ c_\delta = 0 \), it is straightforward to show that \( c_\delta \circ (1, 0) = (1, 0) \circ c_\delta = c_\delta \). Associativity was established in Lemma 3.3 (3). So the only open issue is the existence of inverses. Suppose \( c_\delta \) is fixed and one seeks a right inverse \( c_\delta^{-1} = (c_L^{\circ -1}, c_R^{\circ -1}) \), that is, \( c_\delta \circ c_\delta^{-1} = (1, 0) \). Then it follows directly from Theorem 2.2 and Definition 3.2 that

\[
c_L^{\circ -1} = (c_L \circ (c_L^{\circ -1}, c_R^{\circ -1})) \cup \eta \quad \text{and} \quad c_R^{\circ -1} = -(c_L^{\circ -1} \cup (c_L^{\circ -1} \circ (c_L^{\circ -1}, c_R^{\circ -1}))).
\]
It is first shown that the mapping
\[
S_R: (e_L, e_R) \mapsto ((c_L \circ \bar{\delta}(e_L, e_R))^\omega^{-1}, -e_L \circ \bar{\delta}(e_L, e_R))
\]
is an ultrametric contraction, and therefore, has a unique fixed point, which by design is a right inverse, \(c_\delta^{-1}\). Note that for any \(e_\delta\) it follows that \((S_R(e_L, e_R))(\emptyset) = (e_L, \emptyset)^{-1} \neq 0\). Thus, the fixed point will always be in the group. Then it is shown that this same series is also a left inverse, that is, \(c_\delta^{-1} \circ c_\delta = (1, 0)\), or equivalently,
\[
\begin{align*}
   c_L &= (c_L^{\circ^{-1}} \circ \bar{\delta}(e_L, e_R))^\omega^{-1} \quad (3.5a) \\
   c_R &= -c_L \circ (c_R^{\circ^{-1}} \circ \bar{\delta}(e_L, e_R)) \quad (3.5b)
\end{align*}
\]
To establish the first claim, observe via Corollary 2.1 and Lemma 2.2 that for arbitrary \(e_\delta, \bar{e_\delta}\)
\[
\text{dist}(S_R(e_\delta), S_R(\bar{e_\delta})) = \max(\text{dist}((c_L \circ \bar{\delta}(e_L, e_R))^\omega^{-1}, (c_L \circ \bar{\delta}(\bar{e_L}, \bar{e_R}))^\omega^{-1}),
\]
\[
dist(-e_L \circ \bar{\delta}(c_R \circ \bar{\delta}(e_L, e_R)), -\bar{e_L} \circ \bar{\delta}(c_R \circ \bar{\delta}(\bar{e_L}, \bar{e_R})),
\]
\[
\leq \max(\text{dist}(c_L \circ \bar{\delta}(e_L, e_R), c_L \circ \bar{\delta}(\bar{e_L}, \bar{e_R})), \text{dist}(c_R \circ \bar{\delta}(e_L, e_R), c_R \circ \bar{\delta}(\bar{e_L}, \bar{e_R}))).
\]
In which case, from Lemma 3.2 it follows that
\[
\text{dist}(S_R(e_\delta), S_R(\bar{e_\delta})) \leq \max(\sigma^{\text{ord}(c_L)} \text{dist}((e_L, e_R), (\bar{e_L}, \bar{e_R})), \sigma^{\text{ord}(c_R)} \text{dist}((e_L, e_R), (\bar{e_L}, \bar{e_R})))
\]
\[
\leq \sigma \text{ dist}(e_\delta, \bar{e_\delta}).
\]
To address the second claim, suppose \(c_\delta^{-1}\) satisfies (3.4a). In which case,
\[
\begin{align*}
   (c_L \circ \bar{\delta}(c_\delta^{-1})) \circ (c_L \circ \bar{\delta}(c_\delta^{-1}))))^\omega^{-1} = 1
\end{align*}
\]
Using Lemma 3.3 (2) and Lemma 3.1 (5) gives
\[
\begin{align*}
   (c_L \circ \bar{\delta}(c_\delta^{-1}) \circ ((c_L \circ \bar{\delta}(c_\delta^{-1}) \circ (c_L \circ \bar{\delta}(c_\delta^{-1}))))^\omega^{-1} = 1
\end{align*}
\]
Applying Lemma 3.1 (3) then yields
\[
\begin{align*}
   c_L \circ (c_L^{\circ^{-1}} \circ \bar{\delta}c_\delta) = 1
\end{align*}
\]
which is (3.5a). If, in addition, \(c_\delta^{-1}\) also satisfies (3.4b), then substituting (3.4a) into (3.4b) gives
\[
\begin{align*}
   c_R^{\circ^{-1}} &= -(c_L \circ \bar{\delta}(c_\delta^{-1})) \circ \bar{\delta}(c_R \circ \bar{\delta}(c_\delta^{-1})).
\end{align*}
\]
Therefore, in a similar fashion
\[
\begin{align*}
   -(c_L \circ \bar{\delta}(c_\delta^{-1})) \circ \bar{\delta}(c_R \circ \bar{\delta}(c_\delta^{-1})) = c_R \circ \bar{\delta}(c_R \circ \bar{\delta}(c_\delta^{-1}))
\end{align*}
\]
\[
\begin{align*}
   (c_R \circ \bar{\delta}(c_R \circ \bar{\delta}(c_\delta^{-1}))) \circ \bar{\delta}(c_R \circ \bar{\delta}(c_\delta^{-1})) = c_R \circ \bar{\delta}(c_\delta^{-1})
\end{align*}
\]
Once again applying Lemma 3.1 (3) gives
\[
\begin{align*}
   c_R \circ (c_R^{\circ^{-1}} \circ \bar{\delta}c_\delta) = 0.
\end{align*}
\]
which is equivalent to (3.5b). \(\square\)
Example 3.1. The subgroup with \( c_L = 1 \) was the main object of study in [14–17]. In this case, \( F_c[u] = u + F_c u \) and (3.4)-(3.5) reduce to
\[
(1, c_R^\circ-1) = (1, c_R \circ \delta (1, c_R^\circ-1)), \quad (1, c_R) = (1, -c_R^\circ-1 \circ \delta (1, c_R)),
\]
respectively.

\[\square\]

Corollary 3.1. The group \( (\mathbb{R}_{np}(\langle X_\delta \rangle), \circ, (1, 0)) \) acts as a right transformation group on \( \mathbb{R}(\langle X \rangle) \) via the action \( c \circ \delta \).

Proof. See Lemma 3.1 (2) and Lemma 3.3 (2).

\[\square\square\]

4. Hopf Algebra of Coordinate Maps for \( \mathbb{R}_{np}(\langle X_\delta \rangle) \)

In order to describe the Hopf algebra of coordinate functions for \( \mathbb{R}_{np}(\langle X_\delta \rangle) \), it is first necessary to restrict the set up to the subset of series having the form \( c_\delta = (1 + c_L', c_R) \), where \( c_L' \) is proper. From a control theory point of view, there is no loss of generality since the generating series of any Fliess operator \( y = F_c[u] \) can assume this form by simply rescaling \( y \). So abusing the notation, in this section \( \mathbb{R}_{np}(\langle X_\delta \rangle) \) will be used to denote only this subset. For any \( \eta \in X^* \) define the left and right coordinate maps as
\[
b_\eta : \mathbb{R}_{np}(\langle X_\delta \rangle) \to \mathbb{R} : c_\delta \mapsto (c_L, \eta), \quad a_\eta : \mathbb{R}_{np}(\langle X_\delta \rangle) \to \mathbb{R} : c_\delta \mapsto (c_R, \eta),
\]
respectively.\(^4\) Let \( V \) denote the \( \mathbb{R} \)-vector space spanned by these maps. Define the corresponding free commutative algebra, \( H \), with product
\[
\mu : h_\eta \otimes h_\xi \mapsto h_\eta h_\xi,
\]
h, \( \tilde{h} \in \{a, b\} \) and unit 1 which maps every \( c_\delta \in \mathbb{R}_{np}(\langle X \rangle) \) to 1. The degree of a coordinate map is taken as
\[
deg(b_\eta) = 2 |\eta|_{x_0} + |\eta|_{x_1}, \quad \deg(a_\eta) = 2 |\eta|_{x_0} + |\eta|_{x_1} + 1,
\]
and \( \deg(1) = 0 \). In which case, \( V \) is a connected graded vector space, that is, \( V = \bigoplus_{n \geq 0} V_n \) with \( V_n \) denoting the span of all coordinate maps of degree \( n \) and \( V_0 = \mathbb{R}1 \). Let \( V_+ = \bigoplus_{n \geq 1} V_n \).

Similarly, \( H \) has the connected graduation \( H = \bigoplus_{n \geq 0} H_n \) with \( H_0 = \mathbb{R}1 \).

Three coproducts are now introduced. The first coproduct is \( \Delta_{\square} (V_+) \subset V_+ \otimes V_+ \), which is isomorphic to \( sh^* \) via the coordinate maps. That is, for any \( h, \tilde{h} \in V_+ \):
\[
\Delta_{\square} h_\theta = h_\theta \otimes \tilde{h}_\theta, \quad \Delta_{\square} \tilde{h}_\theta \circ \theta_k = (\theta_k \otimes 1 + 1 \otimes \theta_k) \circ \Delta_{\square} \tilde{h}_\theta,
\]
where \( \theta_k \) denotes the endomorphism on \( V_+ \) specified by \( \theta_k h_\eta = h_{x_k \eta} \) for \( k = 0, 1 \). Clearly this coproduct can be computed recursively. Next consider for any \( \eta \in X^* \) the coproduct \( \Delta(V_+) \subset V_+ \otimes H \), where
\[
\Delta b_\eta(c_\delta, d_\delta) = (c_L \circ \delta d_\delta, \eta), \quad \Delta a_\eta(c_\delta, d_\delta) = (c_R \circ \delta d_\delta, \eta).
\]
In either case, using the notation of Sweedler [29],
\[
\Delta h_\eta(c_\delta, d_\delta) =: \sum h_\eta(1) (c_\delta) h_\eta(2) (d_\delta) = \sum h_\eta(1) \otimes h_\eta(2) (c_\delta, d_\delta),
\]
where \( h_\eta(1) \in V_+ \) and \( h_\eta(2) \in H \). The sums are taken over all the terms that appear in the respective composition in (4.3), and the specific nature of the factors \( h_\eta(1) \) and \( h_\eta(2) \) is not important here. Like the shuffle coproduct, this coproduct can also be computed inductively as described next.

\(^4\) This terminology will be justified later.
Lemma 4.1. The following identities hold:

(1) \( \tilde{\Delta} h_\emptyset = h_\emptyset \otimes 1 \)
(2) \( \tilde{\Delta} \circ \theta_1 = (\theta_1 \otimes \mu) \circ (\tilde{\Delta} \otimes \text{id}) \circ \Delta^a_0 \)
(3) \( \tilde{\Delta} \circ \theta_0 = (\theta_0 \otimes \text{id}) \circ \tilde{\Delta} + (\theta_1 \otimes \mu) \circ (\tilde{\Delta} \otimes \text{id}) \circ \Delta^a_1 \),

where \( \text{id} \) denotes the identity map on \( H \).

Proof.

(1) Assume \( h = a \) and write \( c_R = x_0 c_R^0 + x_1 c_R^1 + (c_R, \emptyset) \) with \( c_R^0, c_R^1 \in R \langle \langle X \rangle \rangle \). Then from Lemma 3.1 it follows that

\[
\begin{align*}
\tilde{\Delta} \circ d_3 &= (x_0 c_R^0 \circ d_3 + (x_1 c_R^1 \circ d_3 + (c_R, \emptyset) \otimes d_3) + x_0(c_R \otimes (c_R^1 \circ d_3)) + (c_R, \emptyset),
\end{align*}
\]

In which case, \( \Delta a_0(c_\emptyset, d_3) = (c_R \circ d_3, \emptyset) = (c_R, \emptyset) = (a_1 \otimes 1)(c_\emptyset, d_3) \). A similar argument holds when \( h = b \).

(2) If \( h = a \) then

\[
\begin{align*}
(\tilde{\Delta} \circ \theta_1)a_\eta(c_\emptyset, d_3) &= \tilde{\Delta} a_{x_\eta}(c_\emptyset, d_3) = (c_R \circ d_3, x_\eta) \\
&= (\xi^{-1}(x_0 c_R^0 \circ d_3) + x_1(d_L \cup (c_R^1 \circ d_3)) + x_0(d_R \cup (c_R^1 \circ d_3))),
\end{align*}
\]

where \( x_\eta \) is the \( R \)-linear operator specified by \( x_\eta = \eta' \) when \( \eta = x_\eta \) and \( \eta' \in X^* \) and zero otherwise, and \( 1_{xy} \) is the indicator function. So \( 1_{xy} = 1 \) when \( x = y \) and zero otherwise. Letting \( c_R^1 = (c_L, c_R^1) \), it follows that

\[
\begin{align*}
(\tilde{\Delta} \circ \theta_1)a_\eta(c_\emptyset, d_3) &= (d_L \cup (c_R^1 \circ d_3), \eta) \\
&= \sum_{\xi \in X^*} (c_R^1 \circ d_3, \xi)(d_L, \nu)(\xi \cup \nu, \eta) \\
&= \sum_{\xi \in X^*} \tilde{\Delta} a_\xi(c_\emptyset, d_3) b_\nu(d_3)(\xi \cup \nu, \eta) \\
&= \sum_{\xi \in X^*} \sum_{\nu \in X^*} \theta_1(a_\xi(1)) \otimes a_\xi(2)(c_R^1 \circ d_3, \nu, \eta) \\
&= (\theta_1 \otimes \text{id}) \circ \sum_{\xi \in X^*} \tilde{\Delta} a_\xi \otimes b_\nu(c_\emptyset, d_3)(\xi \cup \nu, \eta) \\
&= (\theta_1 \otimes \mu) \circ (\tilde{\Delta} \otimes \text{id}) \circ \Delta^a_1 a_\eta(c_\emptyset, d_3).
\end{align*}
\]

The proof when \( h = b \) is perfectly analogous.

(3) If \( h = a \) then

\[
(\tilde{\Delta} \circ \theta_0)a_\eta(c_\emptyset, d_3) = (c_R^0 \circ d_3, \eta) + (d_R \cup (c_R^1 \circ d_3), \eta).
\]

At this point, the method of proof is exactly the same as that in part (2) modulo the fact that \( \Delta^a_0 \) is used here due to the presence of \( d_R \) instead of \( d_L \) in the shuffle product. \( \square \)

Example 4.1. Applying the identities in Lemma 4.1 gives the first few coproduct terms ordered by degree \( n_\emptyset \) \((h = b)\), \( n_\emptyset \) \((h = a)\):

\[
\begin{align*}
n_\emptyset, n_\emptyset &= 0, 1 : \tilde{\Delta} h_\emptyset = h_\emptyset \otimes 1 \\
n_\emptyset, n_\emptyset &= 1, 2 : \tilde{\Delta} h_{x_2} = h_{x_2} \otimes 1
\end{align*}
\]
\[ n_k, n_a = 2, 3 : \Delta h_{x_0} = h_{x_0} \otimes 1 + h_{x_1} \otimes a_0 \]
\[ n_k, n_a = 2, 3 : \Delta h_{x_1} = h_{x_1} \otimes 1 + h_{x_1} \otimes b_1 \]
\[ n_k, n_a = 3, 4 : \Delta h_{x_0x_1} = h_{x_0x_1} \otimes 1 + h_{x_1} \otimes a_{x_1} + h_{x_1} \otimes a_0 \]
\[ n_k, n_a = 3, 4 : \Delta h_{x_1x_0} = h_{x_1x_0} \otimes 1 + h_{x_1} \otimes b_{x_0} + h_{x_1} \otimes a_0 \]
\[ n_k, n_a = 3, 4 : \Delta h_{x_1} = h_{x_1} \otimes 1 + 3h_{x_1} \otimes b_{x_1} + h_{x_1} \otimes b_{x_1} \]
\[ n_k, n_a = 4, 5 : \Delta h_{x_2} = h_{x_2} \otimes 1 + h_{x_1} \otimes a_{x_0} + h_{x_0x_1} \otimes a_0 + h_{x_1x_0} \otimes a_0 + h_{x_1} \otimes (a_0)^2. \]

The next lemma provides a grading for this coproduct, which is clearly evident in the example above.

**Lemma 4.2.** For any \( h_\eta \in V_n \)

\[
\Delta h_\eta \in \bigoplus_{j+k=n} V_j \otimes H_k =: (V \otimes H)_n.
\] (4.4)

**Proof.** The following facts are essential:

(i) \( \deg(\theta_1 h) = \deg(h) + 1 \)

(ii) \( \deg(\theta_0 h) = \deg(h) + 2 \)

(iii) \( \Delta^b_{j+1} h \in (V \otimes V)_{\deg(h) + 1} \).

The proof is via induction on the length of \( \eta \). When \( |\eta| = 0 \) then clearly \( \Delta h_\eta = h_\eta \otimes 1 \in V_n \otimes H_0 \),
where \( n = \deg(h_\eta) \in \{0, 1\} \) (noting that \( h_\eta \sim 1 \)). Assume now that (4.4) holds for words up to length \( |\eta| \geq 0 \). Let \( n = \deg(h_\eta) \).
There are two ways to increase the length of \( \eta \). First consider \( h_{x_1, \eta} \).
From item i above \( \deg(h_{x_1, \eta}) = n + 1 \).
Now apply item iii, the induction hypothesis, and Lemma 4.1 in that order:

\[
\Delta^b_{j+1} h_\eta \in (V \otimes V)_n
\]
\[
(\Delta \otimes \text{id}) \circ \Delta^b_{j+1} h_\eta \in (V \otimes H \otimes V)_n
\]
\[
(\theta_1 \otimes \mu) \circ (\Delta \otimes \text{id}) \circ \Delta^b_{j+1} h_\eta \in \bigoplus_{j+k=1}^n V_{j+1} \otimes H_k
\]
\[
\Delta h_{x_1, \eta} \in (V \otimes H)_{n+1},
\]

which proves the assertion. Consider next \( h_{x_0, \eta} \).
From item ii above \( \deg(h_{x_0, \eta}) = n + 2 \).
In this case, repeat the first two steps of the previous case and apply item i to get

\[
(\theta_1 \otimes \mu) \circ (\Delta \otimes \text{id}) \circ \Delta^b_{j+1} h_\eta \in \bigoplus_{j+k=1}^{n+1} V_{j+1} \otimes H_k \subset (V \otimes H)_{n+2}.
\]

In addition, from the induction hypothesis and item ii it follows that

\[
(\theta_0 \otimes \text{id}) \circ \Delta h_\eta \in \bigoplus_{j+k=1}^n V_{j+2} \otimes H_k \subset (V \otimes H)_{n+2}.
\]

Thus, applying Lemma 4.1, \( \Delta h_{x_0, \eta} \in (V \otimes H)_{n+2} \), which again proves the assertion and completes the proof. \( \square \)
The final coproduct is described by
\[ \Delta b_\eta(c_3, d_3) = b_\eta(c_3 \circ d_3) = ((c_L \circ d_3) \odot d_L, \eta) \]
\[ \Delta a_\eta(c_3, d_3) = a_\eta(c_3 \circ d_3) = ((c_L \circ d_3) \odot d_R, \eta) + (c_R \circ d_3, \eta). \]

Its coassociativity follows directly from the associativity of the group product on \( \mathbb{R}_{np}(\langle X \rangle) \). The following lemma shows how to compute this coproduct in term of \( \Delta \) and \( \Delta_\cup \).

**Lemma 4.3.** The following identities holds:

1. \( \Delta b_\eta = (\text{id} \otimes \mu) \circ (\Delta \otimes \text{id}) \circ \Delta_\text{\uplus} b_\eta \)
2. \( \Delta a_\eta = (\text{id} \otimes \mu) \circ (\Delta \otimes \text{id}) \circ \Delta_\text{\uplus} a_\eta \).

**Proof.** (1) Observe
\[
\Delta b_\eta(c_3, d_3) = \sum_{\xi, \nu \in X^*} (c_L \circ d_3, (\xi, \nu))(d_L, \eta) = \sum_{\xi, \nu \in X^*} \Delta b_\eta(c_3, d_3)(\xi, \nu, \eta) = (\Delta \otimes \text{id}) \circ \Delta_\text{\uplus} b_\eta(c_3, d_3) = (\text{id} \otimes \mu) \circ (\Delta \otimes \text{id}) \circ \Delta_\text{\uplus} b_\eta(c_3, d_3).
\]

(2) In a similar fashion
\[
\Delta a_\eta(c_3, d_3) = (\Delta \otimes \text{id}) \circ \Delta_\text{\uplus} a_\eta(c_3, d_3) + \Delta a_\eta(c_3, d_3)
\]
\[
= [(\text{id} \otimes \mu) \circ (\Delta \otimes \text{id}) \circ \Delta_\text{\uplus} a_\eta(c_3, d_3)](c_3, d_3).
\]

\[ \square \]

**Example 4.2.** Applying the identities in Lemma 4.3 gives the first few reduced coproduct terms, namely, \( \Delta' h_\eta := \Delta h_\eta - h_\eta \odot 1 - 1 \odot h_\eta \):

\[
n = 1: \Delta' b_{x_1} = 0
\]
\[
n = 2: \Delta' b_{x_0} = b_{x_1} \circ a_0
\]
\[
n = 2: \Delta' b_{x_1} = 3b_{x_1} \circ b_{x_1}
\]
\[
n = 3: \Delta' b_{x_0,x_1} = b_{x_0} \circ b_{x_1} + b_{x_1} \circ b_{x_0} + b_{x_1} \circ a_{x_1} + b_{x_1} \circ b_{x_1} \circ a_0 + b_{x_1} \circ a_0
\]
\[
n = 3: \Delta' b_{x_0} = b_{x_0} \circ b_{x_1} + 2b_{x_1} \circ b_{x_0} + b_{x_1} \circ b_{x_1} \circ a_0 + b_{x_1} \circ a_0
\]
\[
n = 3: \Delta' b_{x_1} = 6b_{x_1} \circ b_{x_1} + 4b_{x_1} \circ b_{x_1} \circ a_0 + 3b_{x_1} \circ (b_{x_1})^2
\]
\[
n = 4: \Delta' b_{x_0} = 2b_{x_0} \circ b_{x_0} + b_{x_0} \circ a_{x_0} + 2b_{x_1} \circ b_{x_0} \circ a_0 + b_{x_0,x_0} \circ a_0 + b_{x_0,x_0} \circ a_0
\]
\[
n = 4: \Delta' a_0 = 0
\]
\[
n = 2: \Delta' a_{x_1} = b_{x_1} \circ a_0
\]
\[
n = 3: \Delta' a_{x_0} = b_{x_0} \circ a_0 + b_{x_0} \circ a_0 + (a_0)^2
\]
\[
n = 3: \Delta' a_{x_1} = b_{x_1} \circ b_{x_1} + 2b_{x_1} \circ a_{x_0} + b_{x_1} \circ b_{x_1} \circ a_0 + b_{x_1} \circ a_0
\]
\[
n = 4: \Delta' a_{x_0} = a_{x_1} \circ b_{x_0} + b_{x_0} \circ a_{x_0} + b_{x_0} \circ a_{x_0} + b_{x_1} \circ b_{x_0} \circ a_0 + b_{x_0} \circ a_0 + (a_0)^2
\]
\[
n = 4: \Delta' a_{x_1} = b_{x_1} \circ a_0 + b_{x_1} \circ a_0 + (a_0)^2
\]
\[
n = 4: \Delta' a_{x_0} = b_{x_0} \circ a_0 + 3b_{x_1} \circ b_{x_0} \circ a_0 + b_{x_1} \circ b_{x_0} \circ a_0 + 3b_{x_1} \circ a_{x_0} + 3b_{x_1} \circ b_{x_0} \circ a_0 + 3b_{x_1} \circ b_{x_0} \circ a_0 + 3b_{x_1} \circ a_{x_0} + 3b_{x_1} \circ b_{x_0} \circ a_0 + 3b_{x_1} \circ a_{x_0} + 3b_{x_1} \circ b_{x_0} \circ a_0 + 3b_{x_1} \circ b_{x_0} \circ a_0 + 3b_{x_1} \circ a_{x_0} + 3b_{x_1} \circ b_{x_0} \circ a_0 + 3b_{x_1} \circ a_{x_0} + 3b_{x_1} \circ b_{x_0} \circ a_0 + 3b_{x_1} \circ b_{x_0} \circ a_0 + 3b_{x_1} \circ a_{x_0} + 3b_{x_1} \circ b_{x_0} \circ a_0 + 3b_{x_1} \circ a_{x_0} + 3b_{x_1} \circ b_{x_0} \circ a_0 + 3b_{x_1} \circ b_{x_0} \circ a_0 + 3b_{x_1} \circ a_{x_0} + 3b_{x_1} \circ b_{x_0} \circ a_0 + 3b_{x_1} \circ a_{x_0} + 3b_{x_1} \circ b_{x_0} \circ a_0 + 3b_{x_1} \circ b_{x_0} \circ a_0 + 3b_{x_1} \circ a_{x_0} + 3b_{x_1} \circ b_{x_0} \circ a_0 + 3b_{x_1} \circ a_{x_0} + 3b_{x_1} \circ b_{x_0} \circ a_0 + 3b_{x_1} \circ b_{x_0} \circ a_0 + 3b_{x_1} \circ a_{x_0} + 3b_{x_1} \circ b_{x_0} \circ a_0 + 3b_{x_1} \circ a_{x_0} + 3b_{x_1} \circ b_{x_0} \circ a_0 + 3b_{x_1} \circ b_{x_0} \circ a_0 + 3b_{x_1} \circ a_{x_0} + 3b_{x_1} \circ b_{x_0} \circ a_0 + 3b_{x_1} \circ a_{x_0} + 3b_{x_1} \circ b_{x_0} \circ a_0 + 3b_{x_1} \circ b_{x_0} \circ a_0 + 3b_{x_1} \circ a_{x_0} + 3b_{x_1} \circ b_{x_0} \circ a_0 + 3b_{x_1} \circ a_{x_0} + 3b_{x_1} \circ b_{x_0} \circ a_0 + 3b_{x_1} \circ b_{x_0} \circ a_0 + 3b_{x_1} \circ a_{x_0} + 3b_{x_1} \circ b_{x_0} \circ a_0 + 3b_{x_1} \circ a_{x_0} + 3b_{x_1} \circ b_{x_0} \circ a_0 + 3b_{x_1} \circ b_
Theorem 4.2. \( \Delta H \) is a connected graded commutative unital Hopf algebra.

Proof. From the development above, it is clear that \((H, \mu, \Delta)\) is a connected bialgebra with counit \(\varepsilon\) defined by \(\varepsilon(a_n) = 0\) for all \(n \in X^*\), \(\varepsilon(b_n) = 0\) for all nonempty \(n \in X^*\), and \(\varepsilon(1) = 1\). Here it is shown that this bialgebra is also graded and thus is automatically a Hopf algebra, i.e., has a well defined antipode, \(S\) \[8\]. Specifically, it needs to be shown for any \(n \geq 0\) that \(\Delta H_n \subseteq (H \otimes H)_n\).

It is well known if \(h \in V_n \) that \(\Delta^2 h \in (V \otimes V)_n\). Therefore, it follows directly from Lemmas 4.2 and 4.3 that \(\Delta h \in (V \otimes V)_n\). In which case, via the identity \(\Delta(a_n a_{\ell}) = \Delta a_n \Delta a_{\ell}\), it must hold that \(\Delta H_n \subseteq (H \otimes H)_n, n \geq 0\).

The next theorem says that the antipode of any graded connected Hopf algebra can be computed in a recursive manner once the coproduct is computed.

Theorem 4.3. \( \Delta h \in V_+ \) can be computed by the following algorithm:

1. Recursively compute \(\Delta^k_{\downarrow\downarrow} \) via (4.2).
2. Recursively compute \(\Delta \) via Lemma 4.1.
3. Compute \(\Delta \) via Lemma 4.3.
4. Recursively compute \(S \) via Theorem 4.2.

The first few antipode terms computed via this theorem are:

\(n = 1 : S b_{x_1} = -b_{x_1}\)

\(n = 2 : S b_{x_0} = -b_{x_0} + b_{x_1} a_0\)
Consider now the output is presented next. The output functions $F$ of the inverse mapping these antipode formulas do not depend on the existence of any state space realization. 

Example 4.3. The antipode formulas above can be computed directly in the special case where $F_{cL}: u \mapsto y_L$ and $F_{cR}: u \mapsto y_R$ are realizable by a smooth state space realization

$$\dot{x} = g_0(z) + g_1(z)u, \quad z(0) = z_0, \quad y_L = h_L(z), \quad y_R = h_R(z).$$

where, for example, $(c_L, \eta) = L_{g_0} h_L(z_0)$, and

$$L_{g_0} h_L := L_{g_{i_1}} \cdots L_{g_{i_k}} h_L, \quad \eta = x_{i_k} \cdots x_{i_1},$$

with $L_{g_i} h_L$ denoting the Lie derivative of $h_L$ with respect to $g_i$. Consider now the output $y = u y_L + y_R$. If the solution to the state equation is written in the form $z = F_{c_L}[u]$ for some $c_2 \in \mathbb{R}^{n}(\langle X \rangle)$ then

$$y = uh_L(F_{c_L}[u]) + h_R(F_{c_L}[u]) = uF_{c_L}[u] + F_{c_R}[u] = F_{c\bar{L}}[u],$$

where $c_{\bar{L}} = (c_L, c_R)$. Substitute $u = (y - h_R)/h_L$ into the state equation in (4.5) renders a realization of the inverse mapping $F_{c\bar{L}}^{-1}: y \mapsto u$, namely, $(\bar{g}_0, \bar{g}_1, \bar{h}_L, \bar{h}_R, z_0) = (g_0 - g_1 h_R/h_L, g_1/h_L, 1/h_L, -h_R/h_L, z_0)$. Assuming $h_L(z_0) = 1$, it follows, for example, that

$$(c_{\bar{L}}^{-1}, z_0) = L_{\bar{g}_0} \bar{h}_L(z_0) = -L_{g_0} h_L(z_0) + L_{g_1} h_L(z_0) h_R(z_0) = -(c_L, x_0) + (c_L, x_1)(c_R, 0),$$

which is equivalent to the expression $Sb_{x_0} = -b_{x_0} + b_{x_1} a_0$ computed earlier. But as demonstrated above, these antipode formulas do not depend on the existence of any state space realization.

The deferred proof from Section 2 is presented next.

---

5 The output functions $h_L$ and $h_R$ are not to be confused with elements of $H$. 

---

16
Proof of Lemma 3.1 (3). The only non-trivial claim is that \( c \circ d_k = k \) implies \( c = k \). The proof is by induction on the grading of \( H \). If \( c \circ d_k = k \) then clearly \( k = a_0(c \circ d_k) = \Delta a_0(c_k, d_k) = a_0 c_\delta \) assuming without loss of generality that \( c_\delta = (1, c) \). Therefore, \((c, 0) = k \). Similarly, it follows that \( 0 = a_{x_1}(c \circ d_k) = \Delta a_{x_1}(c_k, d_k) = a_{x_1} c_\delta \). Thus, \((c, x_1) = 0 \). Now suppose \( a_0 c_\delta = 0 \) for all \( a_0 \in H_n \), up to some fixed \( n \geq 2 \). Then for any \( x_j \in X \)

\[
0 = \Delta a_{x_j \eta}(c_\delta, d_\delta) = a_{x_j \eta} c_\delta + \sum_{a_{x_j \eta(1)} \neq 1} a_{x_j \eta(1)}(c_\delta) a_{x_j \eta(2)}(d_\delta),
\]

where in general \( a_{x_j \eta(1)} \neq a_\delta \). Therefore, \( a_{x_j \eta} c_\delta = 0 \), or equivalently, \((c, x_j \eta) = 0 \). In which case \( c = k \).

The section is concluded by some dimensional analysis of the grading of \( V \) and \( H \). Let \( V_{h, k} \) denotes the subspace of \( V_k \) spanned by the coordinate functions \( h \) of degree \( k \) where \( h \in \{a, b\} \). Define \( p_{h,k} = \dim(V_{h,k}) \), \( p_k = \dim(V_k) \) and the corresponding generating functions \( F_{V_k} = \sum_{k \geq 1} p_{h,k} X^k \), \( F_V = \sum_{k \geq 1} p_k X^k \). Analogous definitions apply when \( V \) is replaced by \( H \).

**Theorem 4.4.** The following identities hold:

\[
F_{V_\alpha} = \frac{X}{1 - X - X^2}
\]

\[
= X + X^2 + 2X^3 + 3X^4 + 5X^5 + 8X^6 + 13X^7 + 21X^8 + 34X^9 + \cdots
\]

\[
F_{V_\beta} = \frac{X}{1 - X - X^2}
\]

\[
= X + 2X^2 + 3X^3 + 5X^4 + 8X^5 + 13X^6 + 21X^7 + 34X^8 + 55X^9 + \cdots
\]

\[
F_V = F_{V_\alpha} + F_{V_\beta} = \frac{2X + X^2}{1 - X - X^2}
\]

\[
= 2X + 3X^2 + 5X^3 + 8X^4 + 13X^5 + 21X^6 + 34X^7 + 55X^8 + 89X^9 + \cdots
\]

\[
F_{H_\alpha} = \sum_{k=1}^{\infty} \frac{1}{(1 - X^k)^{p_{a,k}}}
\]

\[
= 1 + X + 2X^2 + 4X^3 + 8X^4 + 15X^5 + 30X^6 + 56X^7 + 108X^8 + 203X^9 + \cdots
\]

\[
F_{H_\beta} = \sum_{k=1}^{\infty} \frac{1}{(1 - X^k)^{p_{b,k}}}
\]

\[
= 1 + X + 3X^2 + 6X^3 + 14X^4 + 28X^5 + 61X^6 + 122X^7 + 253X^8 + 505X^9 + \cdots
\]

\[
F_H = \sum_{k=1}^{\infty} \frac{1}{(1 - X^k)^{p_k}} = F_{H_\alpha} F_{H_\beta}
\]

\[
= 1 + 2X + 6X^2 + 15X^3 + 38X^4 + 89X^5 + 210X^6 + 474X^7 + 1065X^8 + 2339X^9 + \cdots
\]

Proof. The identity for \( F_{V_\alpha} \) is proved in [12, Proposition 8], the proof for \( F_{V_\beta} \) is perfectly analogous. The identity for \( F_V \) follows directly from the fact that \( V = V_\alpha \oplus V_\beta \). It is worth noting that the coefficients of all three series come from the Fibonacci sequence. The identity for \( F_{H_\alpha} \) was also proved in [12], and again the proof for \( F_{H_\beta} \) is very similar. The factorization of \( F_H \) is a consequence of the fact that \( p_k = p_{a,k} + p_{b,k} \). In this case, the coefficients of \( F_{H_\alpha} \) and \( F_{H_\beta} \) are integer sequences A166861 and A200544, respectively, in [28], while the sequence for \( F_H \) appears to be new.
5. The Lie Group $\mathbb{R}_{np}\langle\langle X_δ\rangle\rangle$

In this section, the group $\mathbb{R}_{np}\langle\langle X_δ\rangle\rangle$ is considered as an infinite dimensional Lie group. It is convenient in this case to identify $c_δ = (c_L, c_R) \in \mathbb{R}_{np}\langle\langle X_δ\rangle\rangle$ with $c_δ = δc_L + c_R$, so that the symbol $δ$ is treated more like a letter in $X$. The first goal is to describe the left-invariant vector field on $\mathbb{R}_{np}\langle\langle X_δ\rangle\rangle$, which for a Lie group uniquely identifies the Lie bracket [22]. The left translation of $d_δ$ by $c_δ$ is

$$c_δ \circ d_δ = δ[(c_L \circ d_δ) \cdot d_L] + [(c_L \circ d_δ) \cdot d_R + c_R \circ d_δ].$$

Since composition is left linear, there is no loss of generality in setting $c_δ = ξ_δ := δξ_L + ξ_R$, $ξ_L, ξ_R \in X^*$. The differential of $(ξ_δ) : \mathbb{R}_{np}\langle\langle X_δ\rangle\rangle \rightarrow \mathbb{R}_{np}\langle\langle X_δ\rangle\rangle$ at the identity element $δ$ is the linear map $(ξ_δ)^\ast : T_δ\mathbb{R}_{np}\langle\langle X_δ\rangle\rangle \rightarrow T_δ\mathbb{R}_{np}\langle\langle X_δ\rangle\rangle)$. Consider for some $ε > 0$ a differentiable path $γ : (-ε, ε) \rightarrow \mathbb{R}_{np}\langle\langle X_δ\rangle\rangle : t \rightarrow d_δ(t)$ such that $d_δ(0) = δ$. Define the velocity vector at $t = 0$ as the series in $\mathbb{R}\langle\langle X_δ\rangle\rangle$ of the form

$$v_δ = d_δ(0) = δd_L(0) + δd_R(0) = δv_L + v_R.$$

Then specifically the differential of $ξ_δ \circ δ$ at $δ$ in the direction of $v_δ$ is

$$(ξ_δ \circ δ)(v_δ) = \frac{d}{dt} ξ_δ \circ d_δ(t)\bigg|_{t=0}$$

$$= \frac{d}{dt} δ[(ξ_δ \circ d_δ(t) \cdot d_L(t)) + (ξ_δ \circ d_δ(t) \cdot d_R(t) + ξ_R \circ d_δ(t)]\bigg|_{t=0}$$

$$= δ \left[ \frac{d}{dt} ξ_δ \circ d_δ(t)\bigg|_{t=0} + ξ_L \cdot v_L \right] + ξ_L \cdot v_R + \frac{d}{dt} ξ_R \circ d_δ(t)\bigg|_{t=0}.$$

The time derivative of the product $ξ \circ d_δ$ is computed inductively. It is clearly zero when $ξ = 0$. Otherwise, using Lemma 3.1 (4),

$$\frac{d}{dt} (x_0ξ \circ d_δ(t))\bigg|_{t=0} = x_0 \frac{d}{dt} ξ \circ d_δ(t)\bigg|_{t=0}$$

$$\frac{d}{dt} (x_1ξ \circ d_δ(t))\bigg|_{t=0} = x_1 \frac{d}{dt} (d_L(t) \cdot (ξ \circ d_δ(t))) + x_0(d_R(t) \cdot (ξ \circ d_δ(t)))\bigg|_{t=0}$$

$$= x_1 \left( v_L \cdot ξ + \frac{d}{dt} ξ \circ d_δ(t)\bigg|_{t=0} \right) + x_0(v_R \cdot ξ).$$

Therefore,

$$\frac{d}{dt} ξ \circ d_δ(t)\bigg|_{t=0} = ξ \cdot v_δ,$$

where $0 \cdot v_δ = 0$ and

$$x_0ξ \cdot v_δ = x_0(ξ \cdot v_δ)$$

$$x_1ξ \cdot v_δ = x_1(v_L \cdot ξ + ξ \cdot v_δ) + x_0(v_R \cdot ξ).$$

So the differential is

$$(ξ_δ \circ δ)(v_δ) = δ[ξ_L \cdot v_δ + ξ_L \cdot ξ_R \cdot v_R + ξ_R \cdot v_δ] = (δξ_L) \cdot v_δ + ξ_R \cdot v_δ = ξ_δ \cdot v_δ,$$

where the definition in (5.1) is extended to treat the letter $δ$ as

$$(δξ) \cdot v_δ = δ(v_L \cdot ξ + ξ \cdot v_δ) + (v_R \cdot ξ).$$

In which case, the left-invariant vector field on $\mathbb{R}_{np}\langle\langle X_δ\rangle\rangle$ is

$χ^v_δ : \mathbb{R}_{np}\langle\langle X_δ\rangle\rangle \rightarrow T_δ\mathbb{R}_{np}\langle\langle X_δ\rangle\rangle) : c_δ \mapsto c_δ \cdot v_δ.$

The corresponding Lie bracket is then

$$[v_δ^1, v_δ^2] = [χ^v_δ, χ^v_δ]|_δ = \partial χ^v_δ(c_δ \cdot v_δ^1) - \partial χ^v_δ(c_δ \cdot v_δ^1)|_{c_δ=δ} = v_δ^1 \cdot v_δ^2 - v_δ^1 \cdot v_δ^2,$$
where $\partial x^v : e \mapsto e \cdot v$. This analysis gives the following theorem.

**Theorem 5.1.** The Lie algebra of the Lie group $(R_{op}(\langle X \rangle), \circ, \delta)$ is the smallest $R$-vector subspace of $R(\langle X \rangle)$ closed under the bracket $[v^1_1, v^2_2] = v^3_3 \cdot v^1_1 - v^1_1 \cdot v^3_3$.

Recall that the mixed composition product is left linear, and in light of (4.4) the corresponding coproduct satisfies $\Delta V \subseteq V \otimes H$. Hence, $H$ is a commutative, right-sided Hopf algebra in the sense of [26, Theorem 5.8], and therefore $V$ must inherit a pre-Lie product. The following result is not unexpected.

**Lemma 5.1.** The bilinear product $\bullet$ is a right pre-Lie product, i.e., it satisfies

\[(v^1_1 \bullet v^2_2) \cdot v^3_3 - v^1_1 \cdot (v^2_2 \bullet v^3_3) = (v^1_1 \bullet v^3_3) \cdot v^2_2 - v^1_1 \cdot (v^3_3 \bullet v^2_2) \quad (5.2)\]

for all $v^i_i \in R(\langle X \rangle)$.

**Proof.** The identity can be verified directly using the distributive property

\[(\eta \cdot v_0) \cdot v_3 = (\eta \bullet v_0) \cdot v_3 + \eta \cdot (v_0 \bullet v_3),\]

which can be proved by induction on the sum of the lengths of $\eta, \xi \in X^*$. This also implies that $R(\langle X \rangle)$ is a com-pre-Lie algebra in the sense of [11,12].

**Example 5.1.** In the special case where $c_\delta = \delta + c_R$ and $d_\delta = \delta + d_R$, the corresponding subspace of $T_0 R_{op}(\langle X \rangle)$ is spanned by vectors of the form $v_0 = \delta 0 + v_R$. Thus, the pre-Lie product and Lie bracket above reduce to those described in [12] and [21], respectively.

**Example 5.2.** Consider (5.2) where $v^1_1 = \delta x_1$, $v^2_2 = x_1$ and $v^3_3 = \delta x_0$. Then $\delta x_1 \cdot x_1 = \delta x_0 x_1 + 2x^2_1$, $x_1 \cdot \delta x_0 = x_1 x_0$, $\delta x_1 \cdot \delta x_0 = \delta (2x_1 x_0 + x_0 x_1)$, $\delta x_0 \cdot x_1 = x_0 x_1 + x_1 x_0$, and both sides of (5.2) equal $\delta (2x^2_1 x_0 + x_0 x_1 x_0) + 2x^2_1 x_0 + x_1 x_0 x_1$.

6. **Relative Degree and Group Invariants**

The relationship between relative degree and invariants under the transformation group $R_{op}(\langle X \rangle)$ is described in this section. The following definition describes relative degree from a generating series point of view. It reduces to the usual definition in a state space setting [24]. It uses the notion of a linear word, that is, any word in the language

$$L = \{ \eta \in X^* : \eta = x_0^n x_1 x_0^m, n_1, n_0 \geq 0 \}.$$

Furthermore, note that every $c \in R(\langle X \rangle)$ can be decomposed into its natural and forced components, that is, $c = c_N + c_F$, where $c_N := \sum_{k \geq 0} (c, x_0^k) x_0^k$ and $c_F := c - c_N$.

**Definition 6.1.** [19] Given $c \in R(\langle X \rangle)$, let $r \geq 1$ be the largest integer such that $\text{supp}(c_F) \subseteq x_0^{r-1} X^*$. Then $c$ has relative degree $r$ if the linear word $x_0^{r-1} x_1 \in \text{supp}(c)$, otherwise it is not well defined.

Observe that $c$ having relative degree $r$ is equivalent to saying that

$$c = c_N + c_F = c_N + K x_0^{r-1} x_1 + x_0^{r-1} e \quad (6.1)$$

for some $K \neq 0$ and some proper $e \in R(\langle X \rangle)$ with $x_1 \not\in \text{supp}(e)$.

The main result of this section is given next.
Theorem 6.1. A series $c$ has relative degree $r$ if and only if it is on the orbit of $c_N + x_0^{r-1}x_1$ under $\mathbb{R}_{np}(\langle X_\delta \rangle)$.

Proof. If $c$ has well defined relative degree $r$ then it can be decomposed as in (6.1), where without loss of generality $c = x_0e_0 + x_1e_1$ with $e_1$ proper. Then, setting $\epsilon_\delta := (K + e_1, e_0) \in \mathbb{R}_{np}(\langle X_\delta \rangle)$ (since $K + e_1$ is not proper), it follows from (3.1) that

$$c = c_N + x_0^{r-1}x_1(K + e_1) + x_0^r e_0 = c_N + \phi_0(x_0^{r-1}x_1)(1) = (c_N + x_0^{r-1}x_1) \delta \epsilon_\delta.$$  

In which case, $c \circ \delta \epsilon_\delta^{-1} = c_N + x_0^{r-1}x_1$, or equivalently, $c$ is on the orbit of $c_N + x_0^{r-1}x_1$ under $\mathbb{R}_{np}(\langle X_\delta \rangle)$. The converse holds since all the steps above are reversible. □□□

Another consequence of relative degree is given below.

Theorem 6.2. The transformation group $\mathbb{R}_{np}(\langle X_\delta \rangle)$ acts freely on the subset of $\mathbb{R}(\langle X \rangle)$ having well defined relative degree.

Proof. Assume $c$ has relative degree $r$. Without loss of generality let $c_N = 0$. Then there exists an $\epsilon_\delta \in \mathbb{R}_{np}(\langle X_\delta \rangle)$ such that $c \circ \delta \epsilon_\delta^{-1} = x_0^{r-1}x_1$. So if $c \circ \delta \epsilon_\delta = c$ for some $\epsilon_\delta \in \mathbb{R}_{np}(\langle X_\delta \rangle)$, then it follows immediately that

$$\begin{align*}
(c \circ \delta \epsilon_\delta) \circ \delta \epsilon_\delta^{-1} &= c \circ \delta \epsilon_\delta^{-1} \\
(c \circ \delta \epsilon_\delta^{-1}) \circ \delta \epsilon_\delta &= c \circ \delta \epsilon_\delta^{-1}.
\end{align*}$$

where $\delta \epsilon_\delta$ corresponds to the conjugate action $\epsilon_\delta \circ \delta \epsilon_\delta^{-1}$. In which case,

$$\begin{align*}
x_0^{r-1}x_1 \circ \delta \epsilon_\delta &= x_0^{r-1}x_1 \\
x_0^{r-1}x_1 \circ \delta \epsilon_\delta &= x_0^{r-1}x_1,
\end{align*}$$

and therefore, $d_\delta := (d_{L\epsilon_\delta}, d_{R\epsilon_\delta}) = (1, 0)$, the identity element of $\mathbb{R}_{np}(\langle X_\delta \rangle)$. Thus, $\epsilon_\delta \circ d_\delta \circ \epsilon_\delta^{-1} = (1, 0)$, which gives the desired conclusion that $d_\delta = (1, 0)$. □□□

Theorem 6.1 is a generalization of the well known result stating that the relative degree of a finite dimensional control-affine state space realization is invariant under static state feedback [24]. If $(f, g, h, z_0)$ has relative degree $r$ in the classical sense, then the input-output system is put into the form $g^{(r)} = v$ by the state feedback law

$$u = \frac{v - L_f^{(r)}h(z)}{L_g L_f^{(r)}h(z)}.$$  

If the solution to the state equation is written in the form $z = F_{c_2}[u]$ for some $c_2 \in \mathbb{R}^n(\langle X \rangle)$, then this feedback law is equivalent to

$$v = uL_g L_f^{(r)}h(F_{c_2}[u]) + L_f^{(r)}h(F_{c_2}[u]) = \langle uF_{c_2}[u], F_{c_2}[u] = F_{c_2}[u].$$  

The relative degree assumption here, as above, ensures that $e_L$ is not proper, thus $e_\delta \in \mathbb{R}_{np}(\langle X_\delta \rangle)$. It follows directly from the proof of Theorem 6.1 that $u = F_{e_\delta^{-1}}[v]$ has the property

$$g = F_{c}[u] = F_{c}[F_{e_\delta^{-1}}[v]] = F_{c} \circ \delta e_\delta^{-1}[v] = F_{c_N + x_0^{r-1}x_1[v]},$$

as expected.
Example 6.1. Consider the series \( c = x_1 + x_1^2 \), which has relative degree 1. Observe that

\[
c \hat{\circ} (1, e_R) = x_1 + x_1^2 + x_0 e_R + x_1 x_0 e_R + x_0 (e_R \cdot (x_1 + x_0 e_R)).
\]

Since the monomial \( x_1^2 \) cannot be removed by any choice of \( e_R \), there is no element from the output feedback group which will linearize this system. But it is clear that,

\[
x_1 \hat{\circ} \bar{e} = x_1 \bar{e}_L + x_0 \bar{e}_R = c
\]

when \( \bar{e} = (1 + x_1, 0) \). Therefore,

\[
c \hat{\circ} (1 + x_1, 0)^{-1} = x_1,
\]

where

\[
(1 + x_1, 0)^{-1} = ((1 + x_1 S b x_1 + x_1^3 S b x_2 + x_1^3 S b x_3 + \cdots) (1 + x_1), 0)
\]

\[
= (1 - x_1 + 3x_1^2 - 15x_1^3 + \cdots, 0)
\]

using the antipode formulas from Section 4. Thus, \( \bar{e}^{-1} \) is the group element from \( \mathbb{R}_{np}\langle\langle X_\delta \rangle\rangle \) that linearizes the corresponding input-output system \( F_c \).

7. Conclusions

The affine SISO feedback transformation group was described for the class of nonlinear systems that can be represented in terms of Chen-Fliess functional expansions. The corresponding Hopf algebra of coordinate maps was then presented and contains as a subalgebra the Hopf algebra of the output feedback group. Of particular importance for applications is the fact that the antipode of this Hopf algebra can be computed in a fully recursive fashion. In addition, the Lie algebra of the group was described in terms of a pre-Lie product. This has significance for future study of the underlying combinatorial structures. Finally, it was shown that relative degree, defined purely in an input-output setting, is an invariant of the group action. This is not unexpected in light of the classical theory of feedback linearization.

Acknowledgments

The first author was supported by grant SEV-2011-0087 from the Severo Ochoa Excellence Program at the Instituto de Ciencias Matemáticas in Madrid, Spain. The second author was supported by Ramón y Cajal research grant RYC-2010-06995 from the Spanish government. This research was also support by a grant from the BBVA Foundation.

References

1. J. Berstel and C. Reutenauer, Rational Series and Their Languages, Springer–Verlag, Berlin, 1988.
2. R. W. Brockett, Feedback invariants for nonlinear systems, Proc. 7th IFAC World Congress, Helsinki, 1978, pp. 1115–1120.
3. ——, Linear feedback systems and the groups of Galois and Lie, Linear Algebra Appl., 50 (1983) 45–60.
4. R. W. Brockett and P. S. Krishnaprasad, A scaling theory for linear systems, IEEE Trans. Automat. Contr., AC-25 (1980) 197–207.
5. L. A. Duffaut Espinosa, K. Ebrahimi-Fard, and W. S. Gray, A combinatorial Hopf algebra for nonlinear output feedback control systems, under review, arXiv version: http://lanl.arxiv.org/abs/1406.5396.
6. A. Ferrera, Combinatoire du Monoïde Libre Appliquée à la Composition et aux Variations de Certaines Fonctionnelles Issues de la Théorie des Systèmes, Doctoral Dissertation, University of Bordeaux I, 1979.
7. —, Combinatoire du monoïde libre et composition de certains systèmes non linéaires, *Astérisque*, 75–76 (1980) 87–93.
8. H. Figueroa and J. M. Gracia-Bondía, Combinatorial Hopf algebras in quantum field theory I, *Rev. Math. Phys.*, 17 (2005) 881–976.
9. M. Fliess, Fonctionnelles causales non linéaires et indéterminées non commutatives, *Bull. Soc. Math. France*, 109 (1981) 3–40.
10. —, Réalisation locale des systèmes non linéaires, algèbres de Lie filtrées transitives et séries génératrices non commutatives, *Invent. Math.*, 71 (1983) 521–537.
11. L. Foissy, A prelie algebra associated to a linear endomorphism and related algebraic structures, http://arxiv.org/abs/1309.5318v1, 2013.
12. —, The Hopf algebra of Fliess operators and its dual pre-Lie algebra, http://arxiv.org/abs/1304.1726v3, 2014.
13. W. S. Gray, Affine feedback transformation group for nonlinear SISO systems, *Proc. 21st Inter. Symp. on the Mathematical Theory of Networks and Systems*, Groningen, The Netherlands, 2014, pp. 297–302.
14. W. S. Gray and L. A. Duffaut Espinosa, A Faà di Bruno Hopf algebra for a group of Fliess operators with applications to feedback, *Systems Control Lett.*, 60 (2011) 441–449.
15. —, Feedback transformation group for nonlinear input-output systems, *Proc. 52nd IEEE Conf. on Decision and Control*, Florence, Italy, 2013, pp. 2570–2575.
16. —, A Faà di Bruno Hopf algebra for analytic nonlinear feedback control systems, in *Faà di Bruno Hopf Algebras, Dyson-Schwinger Equations, and Lie-Butcher Series*, K. Ebrahimi-Fard and F. Faouvet, Eds., IRMA Lect. Math. Theor. Phys., Eur. Math. Soc., Strasbourg, France, 2014, to appear.
17. W. S. Gray, L. A. Duffaut Espinosa, and K. Ebrahimi-Fard, Recursive algorithm for the antipode in the SISO feedback product, *Proc 21st Inter. Symp. on the Mathematical Theory of Networks and Systems*, Groningen, The Netherlands, 2014, pp. 1088–1093.
18. —, ‘Faà di Bruno Hopf algebra of the output feedback group for multivariable Fliess operators, *Systems Control Lett.*, to appear, arXiv version: http://lanl.arxiv.org/abs/1406.5378.
19. W. S. Gray, L. A. Duffaut Espinosa, and M. Thitsa, Left inversion of analytic nonlinear SISO systems via formal power series methods, *Automatica*, 50 (2014) 2381–2388.
20. W. S. Gray and Y. Li, Generating series for interconnected analytic nonlinear systems, *SIAM J. Control Optim.*, 44 (2005) 646–672.
21. W. S. Gray, M. Thitsa, and L. A. Duffaut Espinosa, Pre-Lie algebra characterization of SISO feedback invariants, *Proc. 53nd IEEE Conf. on Decision and Control*, Los Angeles, California, 2014, to appear.
22. B. C. Hall, *Lie Groups, Lie Algebras, and Representations, An Elementary Introduction*, Springer–Verlag, New York, 2003.
23. G. P. Hochschild, *Basic Theory of Algebraic Groups and Lie Algebras*, Springer-Verlag, New York, 1981.
24. A. Isidori, *Nonlinear Control Systems*, 3rd Ed., Springer–Verlag, London, 1995.
25. A. J. Krener, On the equivalence of control systems and the linearization of nonlinear systems, *SIAM J. Control Optim.*, 11 (1973) 670–676.
26. J.-L. Loday and M. Ronco, Combinatorial Hopf algebras, *Quanta of Maths*, E. Blanchard, D. Ellwood, M. Khalkhali, M. Marcolli, H. Moscovici, and S. Popa, Eds., AMS, Clay Mathematics Institute, Providence, Rhode Island, 2010, pp. 347–383.
27. C. Reutenauer, *Free Lie Algebras*, Oxford University Press, New York, 1993.
28. N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, available at http://www.research.att.com/~njas/sequences.
29. M. E. Sweedler, *Hopf Algebras*, W. A. Benjamin, Inc., New York, 1969.
