Hamiltonian Analysis of Gauged $CP^1$ Model, the Hopf term, and fractional spin

B. Chakraborty$^1$ and A. S. Majumdar$^2$

S. N. Bose National Centre for Basic Sciences
Block-JD, Sector-III, Salt Lake, Calcutta-700091, India.

Recently it was shown by Cho and Kimm that the gauged $CP^1$ model, obtained by gauging the global $SU(2)$ group and adding a corresponding Chern-Simons term, has got its own soliton. These solitons are somewhat distinct from those of pure $CP^1$ model as they cannot always be characterised by $\pi_2(CP^1) = \mathbb{Z}$. In this paper, we first carry out a detailed Hamiltonian analysis of this gauged $CP^1$ model. This reveals that the model has only $SU(2)$ as the gauge invariance, rather than $SU(2) \times U(1)$. The $U(1)$ gauge invariance of the original (ungauged) $CP^1$ model is actually contained in the $SU(2)$ group itself. Then we couple the Hopf term associated to these solitons and again carry out its Hamiltonian analysis. The symplectic structures, along with the structures of the constraints of these two models (with or without Hopf term) are found to be essentially the same. The model with a Hopf term is shown to have fractional spin which, when computed in the radiation gauge, is found to depend not only on the soliton number $N$, but also on the nonabelian charge. We then carry out a reduced (partially) phase space analysis in a different physical sector of the model where the degrees of freedom associated with the $CP^1$ fields are transformed away. The model now reduces to a $U(1)$ gauge theory with two Chern-Simons gauge fields getting mass-like terms and one remaining massless. In this case the fractional spin is computed in terms of the dynamical degrees of freedom and shown to depend purely on the charge of the surviving abelian symmetry. Although this reduced model is shown to have its own solitonic configuration, it turns out to be trivial.

$^1$e-mail: biswajit@boson.bose.res.in
$^2$e-mail: archan@boson.bose.res.in
1. Introduction

Recently there has been an upsurge of interest in the study of physics of 2 + 1-dimensional systems. Particularly because of the strange nature of the Poincare group ISO(2, 1) in 2 + 1-dimension, in contrast to ISO(3, 1), there arises possibilities of nontrivial configuration space Q and the associated fractional spin and statistics. These possibilities can be realised in practice by adding topological terms like the Chern-Simons (CS) or Hopf term in the model[1,2]. Fractional spin and Galilean/Poincare’ symmetry in these various models have been exhibited in detail in the literature, where both path integral[2,3,4] and the canonical analysis[2,5-9] have been performed.

The CS term (abelian) is a local expression (∼ $\epsilon^{\mu\nu\lambda}a_\mu \partial_\nu a_\lambda$) involving a “photon-less” gauge field $a_\mu$[5]. This gauge field is basically introduced to mimic, in the manner of Aharanov-Bohm, the phase acquired by the system in traversing a nontrivial loop in the configuration space[2]. On the other hand, the Hopf term is usually constructed by writing the conserved topological current $j^\mu (\partial_\mu j^\mu = 0)$ of a model as a curl of a ‘fictitious’ gauge field $a_\mu$:

$$j^\mu = \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda$$ (1.1)

and then contracting $j^\mu$ with $a_\mu$ to get the Hopf term $H$ as,

$$H \sim \int d^3x j^\mu a_\mu$$ (1.2)

Written entirely in terms of $a_\mu$ (using (1.1)), this Hopf term has also the appearance of CS term. However there is a subtle difference. In the case of CS term, the gauge field $a_\mu$ should be counted as an independent variable in the configuration space[7-9]. This is despite the fact that the gauge field is “photon-less”. On the other hand, the $a_\mu$ in (1.1) is really a ‘fictitious’ gauge field (as we have mentioned above) and is not an independent variable in the configuration space. It has to be, rather, determined by inverting (1.1), by making use of a suitable gauge fixing condition. Once done that, the Hopf term (1.2) represents a non-local current-current interaction. It should however be mentioned that this distinction, in terminology, has not always been maintained in the literature (see for example [4]).

It is well known that the Hopf term has geometrical significance in certain cases. For example, consider the $O(3)$ non-linear sigma model(NLSM). The model has solitons[10] characterised by a conserved topological charge $N$. There exists a topological current $j^\mu$ satisfying $\partial_\mu j^\mu = 0$ such that $N = \int d^2x j^0(x)$. In this case the Hopf term provides a representation of the fundamental group of the configuration space $Q(\pi_1(Q))$. Note that the configuration space for NLSM is basically given as the space $Q = Map(S^2, S^2)$, so that $\pi_1(Q) = \pi_3(S^2) = Z$. As mentioned earlier, here too the Hopf term has an inherent non-locality. Nevertheless, it is possible to write a local version of the Hopf term in the equivalent $CP^1$ model[10,11,12] as this is a $U(1)$ gauge theory having an enlarged phase space. However it should be mentioned that this trick
of enlarging the phase space and writing a local expression of the Hopf term may not work all the time, as we shall see later in this paper.

That the Hopf term can impart fractional spin, was demonstrated initially by Wilczek and Zee[3], in the context of the NLSM, using path integral technique. It has been found to depend on the soliton number. This result was later corroborated by Bowick et al.[13], using canonical quantization. On the other hand, the fractional spin obtained in the models involving abelian(nonabelian) CS term, have been found to depend on the total abelian (nonabelian) charge of the system.

This is an important observation, considering the fact that NLSM has become almost ubiquitous in physics, appearing in various circumstances where the original $O(3)$ symmetry is broken spontaneously. For example, in particle physics, the model is considered a prototype of QCD, as the model is asymptotically free in $(1 + 1)$ dimension. On the other hand, in condensed matter physics, this model can describe antiferromagnetic spin chain in its relativistic version[14]. And in its nonrelativistic version, it can describe a Heisenberg ferromagnetic system in the long wavelength limit, i.e. the Landau-Lifshitz(LL) model[15,16]. Besides, the Hopf term can arise naturally in this NLSM, when one quantizes a $U(1)$ degree of freedom hidden in the configuration space $Q$, as has been shown recently by Kobayashi et. al.[17]. Further, it has been shown recently in [12], that the Hopf term can alter the spin algebra of the LL model drastically.

This NLSM has global $O(3)$ symmetry. Recently Nardelli[18] has shown that if this $O(3)$ group is gauged by adding an $SO(3)$ CS term, then the resulting model also has got its own soliton. This work was later extended by Cho and Kimm[19] for the general $CP^N$ model, where one has to gauge the global $SU(N + 1)$ group and add a corresponding CS term. These solitons are somewhat distinct from those of pure $CP^N$ model, in the sense that these are not always characterised by the second homotopy group ($\pi_2(CP^N) = \mathbb{Z}$) of the manifold, unlike the pure $CP^1$ model[10].

The purpose of the paper is to investigate ($N = 1$ case), whether a Hopf term associated with this new soliton number can be added to the model to obtain fractional spin. The question is all the more important, as the model has already got an nonabelian $SU(2)$ CS term, needed for the very existence of these new type of solitons. And, as we have mentioned earlier, the CS term, is likely to play its own role in imparting fractional spin to the model. We find that the fractional angular momentum is given in terms of both the soliton number and the nonabelian charge, where the radiation gauge condition is used. This corresponds to one particular sector of the theory. Apart from this sector, one can also consider a different physical sector in which all the degrees of freedom associated with the $CP^1$ variables are transformed away using the local $SU(2)$ gauge transformation. The physical differences between these two sectors arise from the fact that the model is invariant under only those gauge transformations which tend to a constant at infinity (the spacetime infinity is mapped into one point on the group manifold). For example, because of the constraint on the magnitude of the $CP^1$ variables, one can expect
the CS gauge fields to acquire mass-like terms a la the Higgs mechanism in the standard model. This, on turn, is expected to affect the asymptotic properties of the fields, with non-trivial effects on the physical observables, particularly, the angular momentum. We find that one $U(1)$ gauge symmetry survives in the Lagrangian of this partially reduced configuration space, and the angular momentum is now given in terms of the abelian charge. This shows that the computation of fractional spin yields different results in different physical sectors associated with different asymptotic behaviour of the gauge variant fields. Lastly, we shall investigate the role of the Hopf term in this latter gauge. In doing so, we find that although this model admits a static minimum energy configuration, the solitonic charge in this case is given by the Noether charge, with vanishing Hopf term.

To that end, we organise the paper as follows. In section 2, we carry out the Hamiltonian analysis of the gauged $CP^1$ model. The Hopf term is introduced in section 3 and again the Hamiltonian analysis of the resulting model is performed. In section 4, we compute the fractional spin of the model. We carry out a reduced phase space analysis corresponding to a different physical sector of the theory in section 5 and calculate the fractional spin in terms of the surviving dynamical degrees of freedom. Finally we conclude in section 6.

2. Hamiltonian Analysis of Gauged $CP^1$ model

We are going to carry out the Hamiltonian analysis of the gauged $CP^1$ model, as introduced by Cho and Kimm[19]. The model is given by,

$$\mathcal{L} = (D_\mu Z)^\dagger (D^\mu Z) + \theta \epsilon^{\mu\nu\lambda} [A^a_\mu \partial_\nu A^a_\lambda + \frac{g}{3} \epsilon^{abc} A^a_\mu A^b_\nu A^c_\lambda] - \lambda (Z^\dagger Z - 1)$$

where $Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ is an $SU(2)$ doublet satisfying

$$Z^\dagger Z = 1$$

and enforced by the Lagrange multiplier $\lambda$ in (2.1). The covariant derivative operator $D_\mu$ is given as

$$D_\mu = \partial_\mu - ia_\mu - igA^a_\mu T^a$$

with $T^a = \frac{1}{2} \sigma^a$ ($\sigma^a$, the Pauli matrices), representing the $SU(2)$ generators and $g$ a coupling constant.$\theta$ represents the CS parameter. And finally $a_\mu$ and $A^a_\mu$ represent the $U(1)$ and $SU(2)$ gauge fields respectively. Note that there is no dynamical CS term for the the $a_\mu$ field.

The canonically conjugate momenta variables corresponding to the configuration space variables $(a_\mu, A^a_\mu, z_\alpha, z_\alpha^*)$ are given as,

$$\pi^\mu = \frac{\delta L}{\delta a_\mu} = 0$$
\[ \Pi^{ia} = \frac{\delta L}{\delta A_i^a}, \Pi^{0a} = \frac{\delta L}{\delta A_0^a} = 0 \]  
(2.4)

\[ \pi_\alpha = \frac{\delta L}{\delta \dot{z}_\alpha} = (D_0 Z)_\alpha^*, \pi_\alpha^* = \frac{\delta L}{\delta \dot{z}_\alpha^*} = (D_0 Z)_\alpha \]

where \( L = \int d^2 x \mathcal{L} \) is the Lagrangian.

The Legendre transformed Hamiltonian

\[ H = \pi^\mu \dot{a}_\mu + \Pi^{\mu a} \dot{A}_\mu^a + \pi_\alpha \dot{z}_\alpha + \pi_\alpha^* \dot{z}_\alpha^* - \mathcal{L} \]  
(2.5a)

when expressed in terms of the phase space variables (2.4), gives

\[ H = \pi_\alpha^* \pi_\alpha + ia_0 (\pi_\alpha z_\alpha - \pi_\alpha^* z_\alpha^*) + \frac{1}{2} i g A_0^a [\pi_\alpha (\sigma^a Z) - \pi_\alpha^* (Z^\dagger \sigma^a)_\alpha] \]

\[ -2 \theta A_0^a B^a + (D_i Z)^\dagger (D_i Z) + \lambda (Z^\dagger Z - 1) \]  
(2.5b)

where

\[ B^a \equiv F_{12}^a = (\partial_1 A_2^a - \partial_2 A_1^a + g e^{abc} A_1^b A_2^c) \]  
(2.5c)

is the non-abelian \( SU(2) \) magnetic field.

Clearly the fields \( a_0, A_0^a \) and \( \lambda \) play the role of Lagrange multipliers, which enforce the following constraints,

\[ G_1(x) \equiv i (\pi_\alpha(x) z_\alpha(x) - \pi_\alpha^* (x) z_\alpha^*(x)) \approx 0 \]  
(2.6)

\[ G_2^a(x) \equiv \frac{ig}{2} [\pi_\alpha(x) (\sigma^a Z(x))_\alpha - \pi_\alpha^*(x) (Z^\dagger(x) \sigma^a)_\alpha] - 2 \theta B^a(x) \approx 0 \]  
(2.7)

\[ \chi_1(x) \equiv Z^\dagger(x) Z(x) - 1 \approx 0. \]  
(2.8)

Apart from all these constraints, we have yet another primary constraint,

\[ \chi_2(x) \equiv (\pi_\alpha(x) z_\alpha(x) + \pi_\alpha^*(x) z_\alpha^*(x)) \approx 0 \]  
(2.9)

Also the preservation of the primary constraint \( \pi^i(x) \equiv 0 \) (2.4) yield the following secondary constraint,

\[ 2i Z^\dagger \partial_j Z + 2a_j + g A_j^a M^a \approx 0 \]  
(2.10)

Here

\[ M^a = Z^\dagger \sigma^a Z \]  
(2.11)

is a unit 3-vector, obtained from the \( CP^1 \) variables using the Hopf map. We are left with a pair of primary constraints from the CS gauge field sector in (2.4),

\[ \xi^{ia} \equiv \Pi^{ia} - \theta \epsilon^{ij} A_j^a \approx 0 \]  
(2.12)
This pair of constraints can be implemented strongly by the bracket,

$$\{A^a_i(x), A^b_j(y)\} = \frac{1}{2\hbar} \varepsilon_{ij} \delta^{ab} \delta(x - y)$$

(2.13)

obtained either by using Dirac method[20] or by the symplectic technique of Faddeev-Jackiw[21].

Also note that the constraint $\pi^i \approx 0$ (2.4) is conjugate to the constraint (2.10) and can again be strongly implemented by the Dirac bracket (DB),

$$\{\pi^i(x), a_j(y)\} = 0$$

(2.14)

With this the 'weak' equality in (2.10) is actually rendered into a strong equality and the field $a_i$ ceases to be an independent degree of freedom.

Finally the constraints $\chi_1$ (2.8) and $\chi_2$ (2.9) are conjugate to each other and are implemented strongly by the following DBs,

$$\{z_\alpha(x), z_\beta(y)\} = \{z_\alpha(x), z_\beta^*(y)\} = 0$$

$$\{z_\alpha(x), \pi_\beta(y)\} = (\delta_{\alpha\beta} - \frac{1}{2} z_\alpha z_\beta^*) \delta(x - y)$$

$$\{z_\alpha(x), \pi_\beta^*(y)\} = -\frac{1}{2} z_\alpha z_\beta \delta(x - y)$$

(2.15)

$$\{\pi_\alpha(x), \pi_\beta(y)\} = -\frac{1}{2} (z_\alpha^* \pi_\beta - z_\beta^* \pi_\alpha) \delta(x - y)$$

$$\{\pi_\alpha(x), \pi_\beta^*(y)\} = -\frac{1}{2} (z_\alpha^* \pi_\beta^* - z_\beta \pi_\alpha) \delta(x - y)$$

Precisely the same set of brackets (2.15) are obtained in the case of $CP^1$ model also[22]. We are thus left with the constraints (2.6) and (2.7) and are expected to be the Gauss constraints generating $U(1)$ and $SU(2)$ gauge transformations respectively. The fact that this is indeed true will be exhibited by explicit computations. But before we proceed further, let us note that the constraints (2.8),(2.9) and (2.10) hold strongly now. In view of this, the constraint $G_1$ (2.6) can be simplified as,

$$G_1(x) = 2i \pi_\alpha(x) z_\alpha(x) \approx 0$$

(2.16)

At this stage, one can substitute $\pi_\alpha = (D_0 Z)_\alpha^\dagger$ from (2.4) and solve for $a_0$ to get,

$$a_0 = -i Z^\dagger \partial_0 Z - \frac{1}{2} g A_0^a M^a$$

(2.17)

Clearly this is not a constraint equation, as it involves time derivative. It is nevertheless convenient to club it with the expression of $a_i$, obtained from (2.10) and write covariantly as,

$$a_\mu = -i Z^\dagger \partial_\mu Z - \frac{1}{2} g A_\mu^a M^a$$

(2.18)
Here the first term \((-iZ^\dagger \partial_\mu Z)\) is the pullback, onto the spacetime, of the $U(1)$ connection on the $\mathbb{CP}^1$ manifold[16]. The second term on the other hand has nothing to do with $\mathbb{CP}^1$ connection and arises from the presence of the CS gauge field $A^a_\mu$.

It is now quite trivial to show that $G_1(x)$ (2.16) generates $U(1)$ gauge transformation on the $Z$ fields

$$\delta Z(x) = \int d^2y f(y)\{Z(x), G_1(y)\} = if(x)Z(x) \quad (2.19)$$

but leaves the CS gauge field $A^a_\mu$ unaffected

$$\delta A^a_\mu(x) = \int d^2y f(y)\{A^a_\mu(x), G_1(y)\} = 0 \quad (2.20)$$

Consequently $M^a = Z^\dagger \sigma^a Z$ (2.11) remains invariant under this transformation and hence $a_\mu$ (2.18) undergoes the usual gauge transformation

$$\delta a_\mu(x) = \int d^2y f(y)\{a_\mu(x), G_1(y)\} = \partial_\mu f(x) \quad (2.21)$$

Here in the equations (2.19-2.21) we have taken $f(x)$ to be an arbitrary differentiable functions with compact support.

Proceeding similarly, one can show that the constraints $G^a_2(x)$ (2.7) generates $SU(2)$ gauge transformation,

$$\delta Z(x) \equiv \int d^2y f^a(y)\{Z(x), G^a_2(y)\} = igf^a(x)(T^a Z(x)) \quad (2.22a)$$

$$\delta A^a_\mu(x) \equiv \int d^2y f^b(y)\{A^a_\mu(x), G^b_2(y)\} = \partial_\mu f^a(x) - g\epsilon^{abc} f^b(x)A^c_\mu(x) \quad (2.22b)$$

Using these one can also show that,

$$\delta B^a = -g\epsilon^{abc} f^b B^c$$

$$\delta M^a = -g\epsilon^{abc} f^b M^c \quad (2.23a)$$

but $(M^a B^a)$ is an $SU(2)$ scalar as

$$\delta (M^a B^a) = 0 \quad (2.23b)$$

It also follows from (2.23a) and (2.18) that $a_\mu$ remains unaffected by this $G^a_2$,

$$\delta a_\mu(x) = \int d^2y f^a(y)\{a_\mu(x), G^a_2(y)\} = 0 \quad (2.24)$$

just as $A^a_\mu(x)$ remains unaffected by $G_1$ (2.20).

The fact that $G_1(x)$ and $G^a_2(x)$ are indeed the first class constraints of the model can be easily seen. Firstly one has to just rewrite $G_1(2.16)$ using (2.4) as

$$G_1(x) = 2i(D_0 Z)^\dagger Z \approx 0 \quad (2.25)$$
to see that this is manifestly invariant under the $SU(2)$ gauge transformation generated by $G^a_2(x)$ (2.22). We thus have,
\[
\{G_1(x), G^a_2(y)\} = 0 \tag{2.26}
\]

It also follows after a straightforward algebra that $G^a_2$'s satisfy an algebra isomorphic to $SU(2)$ Lie algebra and thus vanishes on the constraint surface,
\[
\{G^a_2(x), G^b_2(y)\} = 2\epsilon^{abc}G^c_2(x)\delta(x - y) \approx 0 \tag{2.27}
\]

At this stage one can observe that the infinitesimal gauge transformation generated by $G_1$ (2.19) can be integrated to get the following transformation
\[
Z(x) \rightarrow Z'(x) = e^{if(x)}Z(x) \tag{2.28}
\]

Here $f$ is taken to be a finite quantity. Although such a transformation matrix \(\begin{pmatrix} e^{if} & 0 \\ 0 & e^{-if} \end{pmatrix} \in U(2)\) is not an element of $SU(2)$, it nevertheless generates an orbit in $S^3$, taking a point $Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ on $S^3$ to another point $e^{if}Z$ on the same manifold. On the other hand, we also know that the group $SU(2)$ acts transitively on $S^3$. Indeed, one can check that the following $SU(2)$ action on $Z$ is identical to the $U(1)$ transformation (2.28):
\[
\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = e^{if} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{2.29}
\]

with
\[
\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} |z_1|^2e^{if} + |z_2|^2e^{-if} \\ z_1z_2^*e^{if} - z_1^*z_2e^{-if} \end{pmatrix} \tag{2.30}
\]

so that \(\begin{pmatrix} a \\ -b^* \\ a^* \end{pmatrix} \in SU(2)\). Considering again the case where \(|f(x)| \ll 1\), one can get an element
\[
\begin{pmatrix} (1 + ifM_3) & if(M_1 - iM_2) \\ if(M_1 + iM_2) & (1 - ifM_3) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + ifM^a\sigma^a \in SU(2) \tag{2.31}
\]

close to the identity of $SU(2)$. The associated $SU(2)$ Lie algebra element is therefore $fM^a\sigma^a$. On the other hand, the corresponding $U(1)$ Lie algebra element, from (2.28), is simply $fI$, where $I$ is the $2 \times 2$ unit matrix. Since the Gauss constraints $G^a_2$ and $G_1$ satisfy algebra isomorphic to the $SU(2)$ and $U(1)$ Lie algebra (2.26, 2.27) respectively, one can expect the following relation
\[
G_1(x) = M^a(x)G^a_2(x) \tag{2.32}
\]

to hold. To prove that this is indeed the case, first consider the quantity $M^aB^a$ which we have shown to be an $SU(2)$ scalar (2.23b). Therefore, it can be evaluated in any gauge of our choice.
Choosing $Z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we get $M^a = -\delta^a \delta^3$. Thus, the constraint $G_2^3 \approx 0 \ (2.7)$ reduces to $B_3^a \approx 0$, and it follows that

$$M^a B^a = 0 \quad (2.33)$$

Now using the definitions of $\pi_\alpha$ and $\pi_\alpha^*$ (2.4) in (2.7), it is easy to show that (2.32) is satisfied. This shows that $G_1 \ (2.6)$ is not an independent constraint. In other words, the $U(1)$ symmetry transformation generated by (2.6) is not a different kind of transformation, but can be generated by a suitable combination (2.32) of $G_3^3 \ (2.7)$ itself. Thus, strictly speaking the model has only an $SU(2)$ gauge invariance. In this context, it is useful to distinguish the transformation generated by this $U(1)$ from the transformation generated by the $U(1)$ subgroup of $SU(2)$. The latter acts as

$$Z \rightarrow Z' = \begin{pmatrix} e^{if} & 0 \\ 0 & e^{-if} \end{pmatrix} Z \quad (2.34)$$

and the corresponding generator is $G_2^3$.

Finally note that (2.18) really corresponds to the Euler-Lagrange’s equation for the $a_\mu$ field. The corresponding equations for $Z$ and $A_\alpha^\lambda$ are given by,

$$D_\mu D^\mu Z + \lambda Z = 0 \quad (2.35)$$

$$\theta \epsilon^{\mu\nu\lambda} F_{\nu\lambda}^a = ig [(D^\mu Z)^\dagger T^a Z - Z^\dagger T^a (D^\mu Z)] \quad (2.36)$$

respectively.

3. Introducing the Hopf term

In order to introduce the Hopf term, it will be convenient to provide a very brief review of some of the essential features of these new solitons. For this we essentially follow [19]. The symmetric expression for the energy-momentum(EM) tensor, as obtained by functionally differentiating the action $S (= \int d^3 x L)$ with respect to the metric, is given by

$$T_{\mu\nu} = (D_\mu Z)^\dagger (D_\nu Z) + (D_\nu Z)^\dagger (D_\mu Z) - g_{\mu\nu} (D_\rho Z)^\dagger (D^\rho Z) \quad (3.1)$$

The energy functional

$$E = \int d^2 x T_{00} = \int d^2 x [2(D_0 Z)^\dagger (D_0 Z) - (D_\mu Z)^\dagger (D^\mu Z)] \quad (3.2)$$

can be expressed alternatively as,

$$E = \int d^2 x (|D_0 Z|^2 + |(D_1 \pm i D_2) Z|^2) \pm 2\pi N \quad (3.3a)$$

where

$$N = \frac{1}{2\pi i} \int d^2 x \epsilon^{ij} (D_i Z)^\dagger (D_j Z) \quad (3.3b)$$
is the soliton charge.

It immediately follows that the energy functional satisfy the following inequality,

\[ E \geq 2\pi |N| \] (3.4)

The corresponding saturation conditions are,

\[ |D_0 Z|^2 = |(D_1 \pm iD_2) Z|^2 = 0 \] (3.5)

For static configuration \((\dot{Z} = 0)\), this yields

\[ A_0^a = kM^a \] (3.6)

where \(k\) is an arbitrary constant.

Again assuming the static case, one can easily show that \(\mu = 0\) component of the Euler-Lagrange equation (2.29) implies that the \(SU(2)\) magnetic field \(B^a\) vanishes,

\[ B^a = 0 \] (3.7)

where use of (3.6) has been made. This in turn implies that \(A_i^a\) is a pure gauge, so that one can write without loss of generality

\[ A_i^a = 0 \] (3.8)

In this gauge, the soliton charge \(N\) (3.3b) reduces to the standard \(CP^1\) soliton charge,

\[ N = \frac{1}{2\pi i} \int d^2x \epsilon^{ij} (D_i Z)^\dagger (D_j Z) \] (3.9a)

where

\[ D_i = D_i|_{A_i^a = 0} = \partial_i - (Z^\dagger \partial_i Z) \] (3.9b)

is the covariant derivative operator for the standard \(CP^1\) model. Thus in this gauge (3.8), the “soliton charge” is essentially characterised by \(\pi_2(CP^1) = Z\). Nonetheless, it is possible to make “large” topology changing gauge transformation, where \(A_i^a\) is no longer zero and one has to make use of (3.3b), rather than (3.9a), to compute the solitonic charge. Of course this will yield the same value for the charge, but the various solitonic sectors will not be characterised by \(\pi_2(CP^1)\) anymore.

To make things explicit, consider a typical solitonic configuration:

\[ Z = \frac{1}{\sqrt{r^2 + \lambda^2}} \left( \begin{array}{c} re^{-i\phi} \\ \lambda \end{array} \right) \] (3.10a)

\[ A_i^a = 0 \] (3.10b)
where \((r, \Phi)\) represents the polar coordinates in the two-dimensional plane and \(\lambda\) is the size of the soliton. The corresponding unit vector \(M^a(2.11)\) takes the form,

\[
M^1 = \sin \Theta \cos \Phi = \frac{2r\lambda}{r^2 + \lambda^2} \cos \Phi
\]

\[
M^2 = \sin \Theta \sin \Phi = \frac{2r\lambda}{r^2 + \lambda^2} \sin \Phi
\]

\[
M^3 = \cos \Theta = \frac{r^2 - \lambda^2}{r^2 + \lambda^2}
\]

We therefore have for the time component of the gauge field \(A^0_a = kM^a(3.6)\).

At this stage, one can make a topology changing transformation,

\[
Z \rightarrow Z' = UZ = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

where,

\[
U = \frac{1}{\sqrt{r^2 + \lambda^2}} \begin{pmatrix} \lambda & -re^{-i\Phi} \\ re^{i\Phi} & \lambda \end{pmatrix} \in SU(2)
\]

so that \(A^0_a\) undergoes the transformation,

\[
A^0_a \rightarrow A'^0_a = -k \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

The spatial components on the other hand, undergoes the transformation,

\[
A_i \rightarrow A'_i = U A_i U^{-1} + \frac{i}{g} U \partial_i U^{-1} = \frac{i}{g} (\partial_i U) U^{-1}
\]

which on further simplification yields the following form for the connection one-form in the cartesian coordinate system,

\[
A'^1 = \frac{2\lambda}{g(r^2 + \lambda^2)} dy
\]

\[
A'^2 = -\frac{2\lambda}{g(r^2 + \lambda^2)} dx
\]

\[
A'^3 = -\frac{2}{g(r^2 + \lambda^2)} (xdy - ydx)
\]

Now it is a matter of straightforward exercise to calculate the soliton charge \('N'\) in either of these gauges (3.10) and (3.12). For example, in the gauge (3.10), this can be computed by using (3.9a) to get,

\[
N = \frac{1}{2\pi} \int d(-iZ'^\dagger dZ) = -1
\]
On the other hand, the same soliton charge can also be computed in the gauge (3.12), but where the use of (3.3b), rather than (3.9a), has to be made. Note that the topological density $j^0$ ($N \equiv \int d^2x j^0$) can be written as,

$$j^0 = \frac{1}{2\pi i} \epsilon^{ij} (D_i Z)^\dagger (D_j Z) = \tilde{j}^0 + \frac{g}{4\pi} \epsilon^{ij} A_i^a (\partial_j M^a + \frac{g}{2} \epsilon^{abc} A_j^b M^c)$$  (3.14a)

where,

$$\tilde{j}^0 = \frac{\epsilon^{ij}}{2\pi i} (D_i Z)^\dagger (D_j Z)$$  (3.14b)

is the expression of topological density in the gauge (3.10). But in the gauge (3.12), this $\tilde{j}^0$ vanishes, and one can rewrite $N$ completely in terms of the CS gauge field as,

$$N = \frac{g^2}{8\pi k} \int d^2x \epsilon^{ij} \epsilon^{abc} A_i^a A_j^b A_j^c$$  (3.15)

The corresponding $Z$ field configuration being trivial ($Z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$), the soliton number $N$ cannot be captured by $\pi_2(CP^1)$. Incidentally, we shall see in section 5 that it is possible to write a reduced form of the model (2.1) by going to the gauge $Z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. However, the corresponding solitons turn out to be trivial with zero solitonic charge.

At this point we wish to make certain clarifications. The $SU(2)$ transformation (3.12b) which takes the $Z$-field configuration (3.10) to $Z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (3.12a) does not tend to a constant asymptotically and hence does not belong to the gauge group of the model (2.1). As mentioned earlier, because of the presence of the $SU(2)$ CS term in (2.1), the gauge group of this model consists of only those elements of $SU(2)$ which become constant asymptotically. The configurations (3.10) and (3.12) are therefore not physically equivalent and represent different physical states, even though both of them are associated with the same solitonic charge. In other words, the transformation (3.12) is, strictly speaking, not a gauge transformation, but rather a transformation that connects two different physical sectors of the theory.

Since ‘$N$’ is a conserved soliton charge, with an associated topological density $j^0 (3.14a)$, one can regard $j^0$ to be the time-component of a conserved topological $3(= 2 + 1)$-current

$$j^\mu = \frac{1}{2\pi i} \epsilon^{\mu\nu\lambda} (D_\nu Z)^\dagger (D_\lambda Z)$$  (3.16)

This can therefore be expressed as the curl of a ‘fictitious’ $U(1)$ gauge field $A_\lambda$:

$$j^\mu = \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda$$  (3.17)
Unlike the case of pure \( CP^1 \) model\[11,12\], this equation cannot be solved trivially for \( \mathcal{A}_\lambda \) in a gauge independent manner\[2,16\]. We therefore find it convenient to follow Bowick et.al.\[13\], to solve (3.17) for \( \mathcal{A}_\lambda \) in the radiation gauge \( (\partial_i \mathcal{A}_i = 0) \), where one can prove the following identity,

\[
\int d^3x j_0 \mathcal{A}_0 = - \int d^3x j_i \mathcal{A}_i
\]

so that the Hopf action

\[
S_{Hopf} = \Theta \int d^3x j^\mu \mathcal{A}_\mu,
\]

\( (\Theta \) being the Hopf parameter and should not be confused with the spherical angles introduced in (3.11)\) simplifies to the following non-local term

\[
S_{Hopf} = -2\Theta \int d^3x j_i \mathcal{A}_i.
\]

Adding this term to the original model (2.1), we get the following model,

\[
\mathcal{L} = (D_\mu Z)^\dagger (D^\mu Z) + \theta \epsilon^{\mu \nu \lambda} [A_\mu \partial_\nu A_\lambda + \frac{g}{3} \epsilon^{abc} A_\mu A_\nu A_\lambda]
\]

\[
+ \frac{2}{\pi} \epsilon^{ij} \mathcal{A}_i [(D_j Z)^\dagger (D_0 Z) - (D_0 Z)^\dagger (D_j Z)] - \lambda (Z^\dagger Z - 1)
\]

In the rest of this section, we shall be primarily concerned with the Hamiltonian analysis of this model. As the Hopf term is linear in time derivative of the \( Z \) variable, the analysis is expected to undergo only minor modification. Indeed we shall verify this by explicit computations.

To begin with, note that the only change in the form of canonically conjugate momenta variables takes place in the variables \( \tilde{\pi}_\alpha \) and its complex conjugates, counterpart of \( \pi_\alpha \) and \( \pi_\alpha^* \) (2.4)-the momenta variables for the model (2.1). They are now given as,

\[
\tilde{\pi}_\alpha = (D_0 Z)_\alpha^* + \frac{\Theta}{\pi} \epsilon^{ij} \mathcal{A}_i (D_j Z)_\alpha^* = \pi_\alpha + \frac{\Theta}{\pi} \epsilon^{ij} \mathcal{A}_i (D_j Z)_\alpha^*
\]

\[
\tilde{\pi}_\alpha^* = (D_0 Z)_\alpha - \frac{\Theta}{\pi} \epsilon^{ij} \mathcal{A}_i (D_j Z)_\alpha = \pi_\alpha^* - \frac{\Theta}{\pi} \epsilon^{ij} \mathcal{A}_i (D_j Z)_\alpha
\]

Rest of the momenta variables undergo no change from that of (2.4).

The Legendre transformed Hamiltonian \( \tilde{\mathcal{H}} \) can be calculated in a straightforward manner to get,

\[
\tilde{\mathcal{H}} = \mathcal{H} + g\frac{\Theta}{2\pi} A_0^a \epsilon^{ij} \mathcal{A}_i (D_j M)^a
\]

where \( \mathcal{H} \) is just the expression of the Legendre transformed Hamiltonian density (2.5) corresponding to the model (2.1) and \( (D_j M)^a \) is given by,

\[
D_j M^a = \partial_j M^a + g \epsilon^{abc} A_j^b M^c
\]
as can be easily obtained by using the Hopf map (2.11) and the fact that the covariant derivative operator $D_\mu$ boils down, using (2.3) and (2.18) to,

$$
D_\mu Z = \partial_\mu Z - (Z^\dagger \partial_\mu Z)Z + \frac{ig}{2} A^a_\mu (M^a - \sigma^a) Z 
$$

(3.25)

Clearly the structure of all the constraints remain the same, except the $SU(2)$ Gauss constraint. This is clearly given as,

$$
\tilde{G}^a_2 = G^a_2 + \frac{g\Theta}{2\pi} \epsilon^{ij} A_i (D_j M)^a 
$$

(3.26)

where $G^a_2$ is given in (2.7). But we have to rewrite this in terms of $\tilde{\pi}_\alpha$ and $\tilde{\pi}_\alpha^*$. Once we do this, we find that that the $SU(2)$ Gauss constraint $G^a_2$ (2.7) for the model (2.1) is now given by,

$$
G^a_2 = ig\left( [\tilde{\pi}_\alpha (T^a Z)_\alpha - \tilde{\pi}_\alpha^* (Z^\dagger T^a)_\alpha] + \frac{i\Theta}{2\pi} \epsilon^{ij} A_i (D_j M)^a \right) - 2\theta B^a \approx 0 
$$

(3.27)

Substituting this in (3.26), $\tilde{G}^a_2$ is found to have the same form as that of $G^a_2$ (2.7) with the replacement $\pi_\alpha \rightarrow \tilde{\pi}_\alpha$ and $\pi_\alpha^* \rightarrow \tilde{\pi}_\alpha^*$,

$$
\tilde{G}^a_2 = ig(\tilde{\pi}_\alpha (T^a Z)_\alpha - \tilde{\pi}_\alpha^* (Z^\dagger T^a)_\alpha) - 2\theta B^a \approx 0 
$$

(3.28)

The other $U(1)$ Gauss constraint $\tilde{G}_1$ (2.6) can also be seen to take the same form, with identical replacement,

$$
\tilde{G}_1(x) = i(\tilde{\pi}_\alpha (x) z_\alpha(x) - \tilde{\pi}_\alpha^* (x) z_\alpha^*(x)) \approx 0 
$$

(3.29)

where use of the identity $Z^\dagger D_\mu Z = 0$ has been made. Here again, one can show that the relation $\tilde{G}_1 = M^a \tilde{G}^a_2$ (the counterpart of (2.33) holds. This shows that the only basic first class constraints are given by $\tilde{G}^a_2$ (3.27).

Finally note that the constraint (2.9) also preserves its form, i.e.

$$
\chi_2(x) = \tilde{\pi}_\alpha (x) z_\alpha(x) + c.c 
$$

and is again conjugate to $\chi_1$ (2.8). Thus these pair of constraints can be implemented strongly by using the DB (2.15), again taken with the replacement $\pi_\alpha \rightarrow \tilde{\pi}_\alpha$ and $\pi_\alpha^* \rightarrow \tilde{\pi}_\alpha^*$. On the other hand, the pair of second class constraints (2.12) are implemented strongly by the brackets (2.13) in this case also. These set of DB furnishes us with the symplectic structure of the model (3.21).

4. Angular momentum

In this section, we are going to find the fractional spin imparted by the Hopf term. As was done for the models involving the CS[7-9] and Hopf[13] term, the fractional spin was essentially revealed by computing the difference $(J^s - J^N)$ between the expression of angular momentum
\( J^s \), obtained from the symmetric expression of the EM tensor \( T_{\mu \nu} \) (\( \sim \frac{\delta S}{\delta g_{\mu \nu}} \)) and the one \( J^N \), obtained by using Noether’s prescription. It is \( J^s \), which is taken to be the physical angular momentum. This is because it is gauge invariant by construction, in contrast to \( J^N \), which turn out to be gauge invariant only on the Gauss constraint surface and that too usually under those gauge transformations, which tend to identity asymptotically\(^{[8,9]}\).

To that end, let us consider the generator of linear momentum. This is obtained by integrating the \((0i)\) component of the EM tensor (3.1), which undergoes no modification as the metric independent topological (Hopf) term (3.20) is added to the original Lagrangian (2.1) to get the model (3.21).

\[
P^s_i = \int d^2x T^s_{0i} = \int d^2x [(D_0Z)\ddag (D_iZ) + (D_iZ)\ddag (D_0Z)] \tag{4.1}
\]

Expressing this in terms of phase-space variables (3.22), one gets

\[
P^s_i = \int d^2x [\tilde{\pi}_\alpha (D_iZ)\alpha + \tilde{\pi}^\alpha_\alpha (D_iZ)^\alpha + 2\Theta A_i(x)j^0(x)] \tag{4.2}
\]

This can now be re-expressed as,

\[
P^s_i = \int d^2x [\tilde{\pi}_\alpha \partial_i z_\alpha + \tilde{\pi}^\alpha_\alpha \partial_i z^\alpha - 2\theta A^0_i B^a + 2\Theta A_i j^0 - a_i \tilde{G}_1 - A^a_i \tilde{G}_2] \tag{4.3}
\]

However this cannot be identified as an expression of linear momentum, as this fails to generate appropriate translation,

\[
\{Z(x), P^s_i\} \approx D_i Z \tag{4.4}
\]

in contrast to the corresponding expression of linear momentum

\[
P^N_k = \int d^2x T^N_{0k} = \int d^2x [\tilde{\pi}_\alpha \partial_k z_\alpha + \tilde{\pi}^\alpha_\alpha \partial_k z^\alpha - \theta \epsilon^{ij} A^a_i \partial_k A^a_j] \tag{4.5}
\]

obtained through Noether’s prescription, as this generates appropriate translation by construction,

\[
\{Z(x), P^N_k\} = \partial_k Z(x)
\]

\[
\{A^a_i(x), P^N_k\} = \partial_k A^a_i \tag{4.6}
\]

The adjective “appropriate” in this context means that the bracket \( \{\Phi(x), G\} \) is just equal to the Lie derivative \( (L_{V_G}(\Phi(x))) \) of a generic field \( \Phi(x) \) with respect to the vector field \( V_G \), associated to the symmetry generator \( G \). We have not, of course, displayed any indices here. The field \( \Phi \) may be a scalar, spinor, vector or tensor field in general. In this case, it can correspond either to the scalar field \( Z(x) \) or the vector field \( A^a_i(x) \). And \( G \) can be, for example, the momentum(\( P_i \)) or angular momentum (\( J \)) operator generating translation and spatial rotation respectively. The
associated vector fields $V_G$ are thus given as $\partial_i$ and $\partial_\phi$ respectively ($\phi$ being the angle variable in the polar coordinate system in 2-dimensional plane).

Coming back to the translational generator $P_i^a$ (4.3), we observe that the EM tensor (3.1) is not unique by itself. One has the freedom to modify it to $\tilde{T}_{\mu\nu}$ by a linear combination of first class constraint(s), here $\tilde{G}_1$ (3.28) and $\tilde{G}_2^a$ (3.27) with arbitrary tensor valued coefficients $u_{\mu\nu}$ and $v_{\mu\nu}^a$:

$$\tilde{T}_{\mu\nu} = T_{\mu\nu} + u_{\mu\nu}\tilde{G}_1 + v_{\mu\nu}^a\tilde{G}_2^a$$  \hfill (4.7)

Choosing

$$u_0^i = a_i$$
$$v_0^a_i = A_i^a$$ \hfill (4.8)

one can easily see that the corresponding modified expression of momentum

$$\tilde{P}_i = \int d^2x \tilde{T}_{0i} = \int d^2x\left[\tilde{\pi}_\alpha \partial_i z_\alpha + \tilde{\pi}_\alpha^* \partial_i z_\alpha^* - 2\theta A_i^a B^a + 2\Theta A_i j_0\right]$$  \hfill (4.9)

generate appropriate translation,

$$\{Z(x), \tilde{P}_i\} = \partial_i Z(x)$$
$$\{A_k^a(x), \tilde{P}_i\} = \partial_i A_k^a(x)$$ \hfill (4.10)

just like $P_i^N$ (4.6).

So finally the corresponding expression of angular momentum can be written as,

$$J^s = \int d^2 x \epsilon^{ij} x_i \tilde{T}_{0j} = \int d^2 x \epsilon^{ij} x_i \left[\tilde{\pi}_\alpha \partial_j z_\alpha + \tilde{\pi}_\alpha^* \partial_j z_\alpha^* - 2\theta A_j^a B^a + 2\Theta A_j j_0\right]$$  \hfill (4.11)

$$J^N = \int d^2 x \epsilon^{ij} x_i \left[\tilde{\pi}_\alpha \partial_j z_\alpha + \tilde{\pi}_\alpha^* \partial_j z_\alpha^* - \theta \epsilon^{kl} A_k^a \partial_j A_l^a - \theta A_j j_0\right]$$ \hfill (4.12)

Just like the case the case of linear momentum, here too one can show that both $J^s$ and $J^N$ generate appropriate spatial rotation,

$$\{Z(x), J^s\} = \{Z(x), J^N\} = \epsilon^{ij} x_i \partial_j Z(x)$$
$$\{A_k^a(x), J^s\} = \{A_k^a(x), J^N\} = \epsilon^{ij} x_i \partial_j A_k^a(x) + \epsilon_{ki} A^a_i(x)$$ \hfill (4.13)

(Again the adjective “appropriate” has been used in the sense, mentioned above.) However, they are not identical and the difference $J_f \equiv (J^s - J^N)$ is given as,

$$J_f = \theta \int d^2 x \partial_i \left[x_j A^a_j A_i^a - x_i A_j^a A^a_j\right] + 2\Theta \int d^2 x \epsilon^{ij} x_i A_j j_0$$ \hfill (4.14)
The first $\theta$-dependent boundary term has occurred earlier in [8,9], where some of its properties were studied in detail. For example, it was noted that this term is gauge invariant under only those gauge transformations which tend to identity asymptotically[9]. As mentioned earlier, only the set of such transformations constitute the gauge group of the model (2.1). Thus $J_f$ is actually gauge invariant under the complete gauge group of (2.1). To evaluate it in a rotationally symmetric configuration therefore, one can use the radiation gauge ($\partial_i A_i^a = 0$) condition. (Clearly, this corresponds to one particular sector of the theory). To this end, let us rewrite the Gauss constraint (3.27) as,

$$j^a_0 \approx 2 \theta B^a$$  \hspace{2cm} (4.15a)

with

$$j^a_0 = ig(\tilde{\pi}_\alpha(T^a Z)_\alpha - \tilde{\pi}^*_\alpha(Z^a T^a)_\alpha)$$  \hspace{2cm} (4.15b)

Note that the global $SU(2)$ invariance of the model (3.21) yields the following conserved 3(= 2 + 1)-current

$$J^a_\mu = ig[(D^\mu Z)^\dagger T^a Z - Z^a (D^\mu Z)] - g\theta \epsilon^{\mu\nu\lambda} \epsilon^{abc} A^b_\nu A^c_\lambda$$

$$+ \frac{\Theta g}{\pi} \epsilon^{ij} A_i [(Z^a T^a (D_j Z) + (D_j Z)^\dagger T^a Z) \delta^a_0 - (Z^a T^a (D_0 Z) + (D_0 Z)^\dagger T^a Z) \delta^a_j]$$  \hspace{2cm} (4.16)

satisfying $\partial_\mu J^\mu_\alpha = 0$. The time component $J^a_0$, when simplified using (3.22) and (4.15), yields

$$J^a_0 = 2 \theta b^a$$  \hspace{2cm} (4.17a)

where

$$b^a = \partial_1 A^a_2 - \partial_2 A^a_1$$  \hspace{2cm} (4.17b)

is the ‘abelianized’ part of the nonabelian magnetic field $B^a$ (2.5c). This shows that the associated $SU(2)$ conserved charges $Q^a(\equiv \int d^2x J^a_0)$ are related to the triplet of ‘abelianized’ fluxes $\Phi^a_{\text{abelian}} = \int d^2x b^a$ by

$$Q^a \approx 2 \theta \Phi^a_{\text{abelian}}$$  \hspace{2cm} (4.18)

rather then the flux of the nonabelian magnetic field ($\Phi = \int d^2x B^a$).

Proceeding as in [8], we can write the following configuration of the $SU(2)$ gauge field,

$$A^a_i = -\frac{Q^a}{4\pi \theta} \epsilon^{ij} \frac{x^j}{r^2}$$  \hspace{2cm} (4.19)

in the radiation gauge ($\partial_i A^a_i = 0$). Using this, one can easily show that the first $\theta$-dependent term in (4.14) yields $\frac{Q^a Q^a}{4\pi \theta}$ and the second $\Theta$-dependent term yields, following [13], $\Theta N^2$. So finally, we have from (4.14),

$$J_f = \frac{Q^a Q^a}{4\pi \theta} + \Theta N^2$$  \hspace{2cm} (4.20)
(Incidentally, it turns out that in gauge (4.19), \( Q^a = \int d^2 x J^a_0 = \int d^2 x j^a_0 \). We thus see that the classical expression of fractional angular momentum contains two terms. One depends on the soliton number \( N \) and the other on the nonabelian charge \( Q^a \). The former is just as in the model, where Hopf term is coupled to NLSM [3,13]. On the other hand, the latter is a typical nonabelian expression, as in [8]. It needs to be mentioned here that for preservation of invariance of the action (2.1) or (3.21) involving the nonabelian \((SU(2))\) CS term under a homotopically nontrivial gauge transformation, the CS coefficient \( \theta \) is quantized \((\theta = n/8\pi)\) with \( n \in \mathbb{Z} \) being an integer [23]. In the next section we shall compute \( J_f \) (for \( \Theta = 0 \)) for a physically different sector of the theory.

5. Reduced phase space analysis of the model

In this section we shall consider a reduced phase space version of the model (2.1) associated with a different physical sector of the theory defined by (3.12). At first we perform its Hamiltonian analysis in terms of the surviving dynamical degrees of freedom. Upon using (3.12), i.e.,

\[
Z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]  

(5.1)

the degrees of freedom corresponding to the \( CP^1 \) matter fields \( Z \) get completely eliminated. This configuration can be obtained by making use of the local \( SU(2) \) gauge invariance of the model (2.1). With this choice, we have broken the \( SU(2) \) gauge symmetry of the model (2.1). Thus with the condition (5.1), the configuration space and hence the phase space get reduced partially. As will become apparent subsequently, this still leaves a residual \( U(1) \) symmetry. We shall obtain the generator of angular momentum and also the expression for fractional spin in this model. Before concluding this section, we shall investigate whether one can have any topological solitons, and hence the Hopf term in this case also.

The substitution of (5.1) in the model (2.1) yields the gauge fixed Lagrangian

\[
\mathcal{L} = \frac{g^2}{4} A^a_\mu A^{a,\mu} + \theta e^{\mu \nu \lambda} (A^a_\mu \partial_\nu A^a_\lambda + \frac{g}{3} e^{abc} A^a_\mu a^b \partial_\nu A^c_\lambda)
\]  

(5.2)

where the group index \( \alpha = 1, 2 \). (Henceforth, the first two Greek indices \( \alpha \) and \( \beta \) will take values 1 or 2 only. All the other symbols will retain their usual meanings.) This form of the Lagrangian clearly shows that the original \( SU(2) \) symmetry could not be broken entirely. The \( U(1) \) subgroup of \( SU(2) \) (see 2.34) survives as a gauge symmetry. Thus \( G_1^1 \) and \( G_2^2 \) of (2.7) correspond to the broken generators of \( SU(2) \), and \( G_2^3 \) corresponds to the surviving \( U(1) \) symmetry. (Note, that we do not bother about the other \( U(1) \) symmetry generator \( G_1 \) (2.6) since it is not an independent constraint (2.32).) Correspondingly, there are mass-like terms for \( A^1_\mu \) and \( A^2_\mu \) in the Lagrangian (5.2), whereas, \( A^3_\mu \) remains massless. To understand better the survival of the residual \( U(1) \) symmetry even after making the gauge choice (5.1), note that had
we chosen a configuration of the $Z$-field as $Z(x) = \begin{pmatrix} 0 \\ e^{i\phi(x)} \end{pmatrix}$, the Lagrangian would still have taken the form (5.2). Actually, all these $Z$-field configurations correspond to the same $M^a$ field configuration ($M^a = -\delta^{a3}$), so that $\vec{M}$ points towards the south pole in the isospin space of unit radius. The surviving $U(1)$ symmetry (2.34) is the $SO(2)$ rotational symmetry around the $z$-axis. The situation is somewhat analogous to the Higgs mechanism of the standard model where broken symmetry generators provide masses for the gauge fields. The constraint (2.2) on the $Z$-fields here plays the role of the vacuum expectation value of the Higgs field in the standard model.

Coming to the Hamiltonian analysis of the model, the canonically conjugate momenta corresponding to the surviving physical variables in the reduced configuration space ($A^a_i, A^a_0$) are given by

$$\pi^{ia} = \frac{\delta L}{\delta A^a_i}, \pi^0a = \frac{\delta L}{\delta A^a_0} = 0$$ (5.3)

The Legendre transformed Hamiltonian can be written as

$$\mathcal{H} = -\frac{1}{4} g^2 A^a_i A^{\mu a} - 2\theta A^a_0 B^a$$ (5.4)

It is apparent that $A^3_0$ is just a Lagrange multiplier enforcing the Gauss constraint

$$G \equiv -2\theta B^3 \approx 0$$ (5.5)

The fact that $G$ generates the appropriate $U(1)$ gauge transformation will be demonstrated later. Now note that preservation of the pair of constraints $\pi^{0a} \approx 0$ (5.3) in time yield the following pair of secondary constraints:

$$\frac{1}{2} g^2 A^a_0 + 2\theta B^a \approx 0$$ (5.6)

It can be checked that the constraint $\pi^{0a} \approx 0$ (5.3) together with (5.6) form two pairs of second class constraints which are strongly implemented by the following DB’s:

$$\{A^a_i(x), A^b_j(y)\} = \frac{1}{2\theta} \epsilon_{ij} \delta^{ab} \delta(x-y)$$ (5.7a)

$$\{A^a_0(x), A^b_0(y)\} = 0$$ (5.7b)

$$\{A^a_0(x), A^b_i(y)\} = \frac{2}{g^2} \delta^{a0} \delta^{bo} \partial_i(x) \delta(x-y) - \frac{2}{g} \delta^{a0} \epsilon^{abc} A^c_i(x) \delta(x-y)$$ (5.7c)

This leaves the first class constraint (5.5)

$$G = -2\theta \epsilon^{ij} (\partial_i A^3_j + \frac{g}{2} \epsilon^{a\beta} A^a_i A^\beta_j) \approx 0$$ (5.8)
as the generator of the surviving $U(1)$ gauge symmetry with respect to the DB’s (5.7), yielding
\[
\delta A^3_i(x) = \int d^2 y f(y) \{ A^3_i(x), G(y) \} = \partial_i f(x)
\] (5.9)
as expected. Furthermore, the action of $G$ on the ‘massive’ fields $A^\alpha_\mu$ yields
\[
\delta A^\alpha_\mu(x) = \int d^2 y f(y) \{ A^\alpha_\mu(x), G(y) \} = g f(x) \epsilon^{\alpha\beta} A^\beta_\mu(x)
\] (5.10)
which is just a rotation in the two-dimensional internal space spanned by $\alpha$ and $\beta$.

We now define the energy-momentum tensor $T^N_{\mu\nu}$ using the Noether prescription, from which it follows that the expression of momentum $P^N_j$ is given by
\[
P^N_j = \int d^2 x T^N_{0j} = -\theta \int d^2 x [\epsilon^{ij} A^\alpha_\mu(x) \partial_j A^\alpha_\mu(x)]
\] (5.11)
From here the Noether angular momentum $J^N$ is computed to be
\[
J^N = -\theta \int d^2 x [\epsilon^{ij} x_i \epsilon^{kl} A^\alpha_\mu(x) \partial_j A^\alpha_\mu(x) + A^\alpha_j A^\alpha_l]
\] (5.12)
It is easy to check that both the momentum $P^N_j$ (5.11) and angular momentum $J^N$ (5.12) respectively generate appropriate translation defined in (4.6), and appropriate rotation defined in (4.13) on the dynamical variables $A^\alpha_i$ and $A^\alpha_0$.

Next we write down the symmetric energy momentum tensor
\[
T^s_{\mu\nu} = \frac{1}{2} g^2 A^\alpha_\mu A^\alpha_\nu - \frac{1}{4} g^2 g_{\mu\nu} A^\alpha_\rho A^\alpha_\rho
\] (5.13)
as obtained by functionally differentiating their action (5.2) with respect to the metric. (Note that the expression (5.13) can also be obtained from (3.1) by substituting (5.1) in it.) Just as in (4.4), here too the expression of momentum obtained from (5.13) fails to generate appropriate translation. It thus requires to be modified by adding a suitable combination of first class constraint(s), here only (5.8):
\[
\tilde{T}_{\mu\nu} = T^s_{\mu\nu} + w_{\mu\nu} G
\] (5.14)
It follows that by choosing $w_{0j} = A^3_j$, one gets the desired expression for symmetric momentum
\[
\tilde{P}^s_j = \int d^2 x \tilde{T}^s_{0j} = \int d^2 x (\frac{g^2}{2} A^\alpha_0 A^\alpha_j + GA^3_j)
\] (5.15)
which generates the appropriate translations defined in (4.10) on $A^\alpha_i$ and $A^\alpha_0$. Upon using the relation (5.6), it follows that (5.15) reduces to a simple expression
\[
\tilde{P}^s_j = -2\theta \int d^2 x B^a A^a_j
\] (5.16)
From here the symmetric angular momentum $J^s$ is given by

$$J^s = -2\theta \int d^2x \epsilon^{ij} x_i B^a A^i_j = -2\theta \int d^2x \epsilon^{ij} \epsilon^{lk} x_i A^a_j \partial_l A^a_k$$

which, again, generates appropriate rotations.

Before proceeding to compute the fractional spin given in this case by the difference between $J^s$ (5.17) and $J^N$ (5.12), let us now consider the expression for energy $E$ obtained by integrating $T^s_{\mu\nu}$ (from 5.13) to get

$$E = \int d^2x T^s_{00} = \int d^2x \frac{g^2}{4} [(A^a_0)^2 + (A^a_i)^2]$$

Clearly, if one demands that the energy $E$ should be finite, it is necessary that $A^a_\mu$ vanish at infinity. This is consistent with the presence of masslike terms for the fields $A^a_\mu$ in Lagrangian (5.2).

The expression for fractional spin is given by

$$J_f = J^s - J^N = \theta \int d^2x \partial_i (x_j A^{j\alpha} A^{i\alpha} - x_i A^{j\alpha} A^{j\alpha} + x_j A^{i3} A^{j3} - x_i A^{3j} A^{j3})$$

which is identical to the first term of (4.14). The first two terms on the right hand side correspond to boundary values of 'massive' fields $A^a_\mu$, and hence vanish identically. One is thus left with

$$J_f = \theta \int d^2x \partial_i (x_j A^{i3} A^{j3} - x^i A^i_3 A^j_3)$$

where $A^3_i$ is the surviving $U(1)$ gauge field.

As in the previous section, the global $U(1)$ invariance of the model (5.2) yields the current

$$J^\mu = \theta g \epsilon^{\mu\lambda} A^{\alpha}_\nu A^{\beta}_\lambda$$

satisfying $\partial_\mu J^\mu = 0$. The corresponding conserved charge is

$$Q = \int d^2x J^0 = \theta g \int d^2x \epsilon^{ij} \epsilon^{\alpha\beta} A^{i\alpha}_i A^{j\beta}_j$$

Using the Gauss constraint (5.8), $Q$ can be written, on the constraint surface, entirely in terms of the gauge field $A^3_i$ as

$$Q \approx 2\theta \epsilon^{ij} \int d^2x \partial_i A^3_j$$

It is easy to verify that the form of $Q$ (5.22), rather than (5.23), generates the transformation (5.10)

$$\{A^\alpha_\mu(x), Q\} = g \epsilon^{\alpha\beta} A^{\beta}_\mu$$

This is expected since (5.23) is only a 'weak' equality.
Now one can invoke the radiation gauge condition ($\partial_i A^3_i = 0$) for this surviving $U(1)$ gauge field $A^3_i$, which yields the following asymptotic form for it:

$$A^3_i = -\frac{Q}{4\pi\theta}\varepsilon^i_j x^j r^2$$

(5.25)

Using this form for the gauge field, the fractional angular momentum $J_f$ (5.20) is computed to be

$$J_f = \frac{Q^2}{4\pi\theta}$$

(5.26)

One can see clearly that this is different from (4.20) in the absence of the Hopf term ($\Theta = 0$). This difference stems from the fact that the radiation gauge condition ($\partial_i A^a_i = 0$) and the condition ($Z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$) (5.1) correspond to different physical sectors of the theory. As we have seen, these conditions are associated with different asymptotic properties of the CS fields here, and this in turn has different consequences on the physics of the system in this case.

Finally, let us consider the effect of addition of the Hopf term, if any, to the model (5.2). For this we have to at first look for a solitonic configuration in this model. To that end, note that the expression for energy $E$ (5.18) can be written as

$$E = \frac{g^2}{4} \int d^2x [(A^1_b)^2 + (A^2_b)^2 + (A^1_2 \pm A^2_2)^2 + (A^1_1 \mp A^2_1)^2 \pm 2\epsilon^{ij}\epsilon^{\alpha\beta}A^\alpha_i A^\beta_j]$$

(5.27)

(Again, the expression (5.27) can also be obtained from (3.3a) by making the substitution $Z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (5.1) in it). Using the expression of Noether charge (5.22), this yields the following Bogomol’nyi type inequality:

$$E \geq \frac{g|Q|}{2\theta}$$

(5.28)

The saturation condition corresponds to the static field configurations satisfying

$$
A^1_b = A^2_b = 0 \\
A^1_2 \pm A^2_2 = 0 \\
A^1_1 \mp A^2_1 = 0
$$

(5.29)

This describes the solitonic configuration in the model. But note that the local minimum of energy is now given by the Noether charge $Q$ which here plays the role of topological charge. With condition that $A^\alpha_\mu$ vanish at infinity, the two dimensional plane gets effectively compactified to the 2-sphere $S^2$, and the ‘weak’ equality (5.23) allows one to identify $Q$, albeit ‘weakly’, with the first Chern class. Thus the field configuration (5.29) can be identified ‘weakly’ with a
topological soliton [4]. However, it turns out that this solitonic configuration is a rather trivial one. This can be seen clearly from (5.28). Note that the current $J^\mu$ (5.21) has vanishing spatial components ($J^i = 0$), which in turn implies that the Hopf term (3.20) vanishes [13].

6. Conclusions

In this paper, we have carried out a detailed classical Hamiltonian analysis of the gauged $CP^1$ model of Cho and Kimm[19]. This model is obtained by gauging the global $SU(2)$ group of the $CP^1$ model which is already a $U(1)$ gauge theory. We find that contrary to our expectation, the gauge group of this gauged $CP^1$ model turns out to be only $SU(2)$, rather than $SU(2) \times U(1)$. As was shown in [19], the model has got its own solitons, the very existence of which depends crucially on the presence of $SU(2)$ CS term. These solitons are somewhat more general than that of NLSM[10]. We use the adjective “general” to indicate that these solitons can be characterised by $\pi_2(CP^1) = Z$ only for the gauge $A^a_i = 0$ (Note that $A^a_i$ is a pure gauge). One can make a topology changing transformation and thereby make $A^a_i \neq 0$, without changing the soliton number. However, such a transformation does not become constant asymptotically, and therefore does not belong to the group of invariance, viz., gauge group of the model. We then constructed the Hopf term associated to these solitons and again carried out the Hamiltonian analysis of the model(3.21), obtained by adding Hopf term to (2.1), to find that the sympletic structure and the structure of the constraints undergo essentially no modification, despite the fact that the form of the momenta variables conjugate to $z_\alpha$ and $z^*_\alpha$ undergo changes. We then calculated the fractional angular momentum by computing the difference between $J^s$ and $J^N$, the expressions of angular momenta obtained from the symmetric expression of energy-momentum tensor and the one obtained through Noether’s prescription respectively. We find that this fractional angular momentum consists of two pieces, one is given in terms of the soliton number and the other is given in terms of the nonabelian ($SU(2)$) charge. In absence of the Hopf term ($\Theta = 0$) (i.e. for the model (2.1)), only this latter term will contribute. Again as in [8], this term can be shown to consist of two pieces, one which involves a direct product in the isospin space and characterises a typical nonabelian feature, while the other contains the abelian charge defined in a nonabelian theory.

Subsequently, by making use of the local $SU(2)$ gauge invariance of the model, we can essentially eliminate all the degrees of freedom associated with the $Z$ fields. In fact, with the choice $Z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (5.1), the $CP^1$ fields are frozen out, and one is considering a distinct physical sector from the one considered earlier. With this choice, we perform a (partially) reduced phase space Hamiltonian analysis of the resultant model. Here we see that the $SU(2)$ symmetry is only partly broken with a residual abelian $U(1)$ symmetry. Correspondingly, masslike terms are generated for the two $A^a_\mu$ fields, whereas, $A^3_\mu$ survives as a $U(1)$ gauge field. This situation is somewhat akin to the standard model where the gauge symmetry is partially broken and corresponding mass terms for the gauge fields are generated by the Higgs mechanism. The
role of the vacuum expectation value of the Higgs field is played here by the constraint on the magnitude of the $Z$-fields. We next calculate the expressions of fractional spin through the two angular momenta in our model. We find that the value of fractional spin depends in this case purely on the abelian charge of the surviving $U(1)$ symmetry. This result of fractional spin is different from the one obtained earlier using the radiation gauge condition. This indicates that different physical sectors of the theory are associated with different fractional angular momentum. Finally, we use a Bogomol’nyi type inequality to find the static minimum energy configuration for this model. This defines a solitonic configuration for the model. The resultant solitonic charge turns out to be equal to the Noether charge here. The solitonic configuration however, turns out to be of a trivial nature, in the sense that the corresponding Hopf term vanishes.
References

1. F. Wilczek (Ed.) “Fractional Statistics and Anyonic Superconductivity”, (World Scientific, Singapore, 1990)

2. S. Forte, Rev. Mod. Phys. 64 (1992) 193.

3. F. Wilczek and A. Zee, Phys. Rev. Lett. 51 (1983) 2250.

4. A. P. Balachandran, G. Marmo, B. S. Skagerstam and A. Stern, “Classical Topology and Quantum States”, (World Scientific, Singapore, 1991).

5. C. R. Hagen, Ann. Phys. (N.Y.) 157 (1984) 342.

6. P. Panigrahi, S. Roy and W. Scherer, Phys. Rev. Lett. 61 (1988) 2827.

7. R. Banerjee, Phys. Rev. Lett. 69 (1992) 17; Phys. Rev. D 41 (1993) 2905; Nucl. Phys. B 390 (1993) 681; R. Banerjee and B. Chakraborty, Phys. Rev. D 49 (1994) 5431.

8. R. Banerjee and B. Chakraborty, Ann. Phys. (N.Y.) 247 (1996) 188.

9. B. Chakraborty, Ann. Phys. (N.Y.) 244 (1995) 312; B. Chakraborty and A. S. Majumdar, Ann. Phys. 250 (1996) 112.

10. R. Rajaraman, “Solitons and Instantons” (North Holland, 1982); C. Rebbi and G. Soliani, “Solitons and Particles” (World Scientific, Singapore, 1984); V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zhakharov, Phys. Rep. 116 (1984) 103.

11. Y. S. Wu and A. Zee, Phys. Lett. 147B (1984) 325.

12. B. Chakraborty and T. R. Govindarajan, Mod. Phys. Lett. A 12 (1997) 619.

13. M. Bowick, D. Karabali and L. C. R. Wijewardhana, Nucl. Phys. B 271 (1986) 417.

14. J. Zinn-Justin, “Quantum Field Theory and Critical Phenomena” (Clarendon Press, Oxford, 1990); E. Fradkin, “Field Theory of Condensed matter systems” (Addison-Wesley, 1991).

15. L. Landau and E. Lifshitz, Phys. A. 8 (1935) 153; A. Kosevich, B. Ivanov and A. Kovalev, Phys. Rep. 194 (1990) 117.

16. R. Banerjee and B. Chakraborty, Nucl. Phys. B 449 (1995) 317.

17. H. Kobayashi, I. Tsutsui and S. Tanimura, “Quantum Mechanically Induced Hopf Term in the O(3) non-linear sigma model”, hep-th/9705183.
18. G.Nardelli, Phys.Rev.Lett. **73**(1994)2524.

19. Y.Cho and K.Kimm, Phys.Rev.**D52**(1995)7325.

20. P.A.M.Dirac, “Lectures on Quantum Mechanics”, Belfar Graduate School of Science, Yeshiva University, New York, 1964; A.Hanson, T.Regge and C.Teitelboim “Constraint Hamiltonian Analysis”, (Academia Nazionale dei Lincei, Rome, 1976).

21. L.Faddeev and R.Jackiw, Phys.Rev.Lett.**60**(1988)1692.

22. R.Banerjee, Phys.Rev.**D49**(1994)2133.

23. S.Deser, R.Jackiw and S.Templeton, Ann. Phys. (N.Y.) **140** (1982) 372.