Classical kinematics for isotropic, minimal Lorentz-violating fermion operators

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Abstract

In this article the classical, relativistic Lagrangian based on the isotropic fermion sector of the Lorentz-violating (minimal) Standard-Model Extension is considered. The motion of the associated classical particle in an external electromagnetic field is studied and the evolution of its spin, which is introduced by hand, is investigated. It is shown that the particle travels along trajectories that are scaled versions of the standard ones. Furthermore there is no spin precession due to Lorentz violation, but the rate is modified at which the longitudinal and transverse spin components transform into each other. This demonstrates that it is practical to consider classical physics within such an isotropic Lorentz-violating framework and it opens the pathway to study a curved background in that context.

Keywords: Lorentz violation, Electron and positron properties, Mechanics; Lagrangian and Hamiltonian, Differential geometry

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1. Introduction

Since CPT- and Lorentz violation was shown to appear in the context of string theory \([1-4]\), the interest in exploring a possible violation of this fundamental symmetry in nature has grown steadily. Subsequently such a violation was also found to occur in loop quantum gravity \([5, 6]\), models of noncommutative spacetimes \([7]\), spacetime foams models \([8, 9]\), and in spacetimes endowed with a nontrivial topology \([10, 11]\). Therefore it can be considered as a window to physics at the Planck scale. A further boom creating a new field of research took place when the minimal Standard-Model Extension (SME) was established \([12]\). The latter provides a powerful effective framework for describing Lorentz violation for energies much smaller than the Planck scale.

Since then the field has been developing largely both concerning experiments and the study of theoretical aspects. There has been a broad experimental search for Lorentz violation (see the data tables \([13]\) and references therein) and there are ongoing studies on the properties of quantum field theories based on the SME \([14-26]\). Recently, also the nonminimal versions of the SME including all higher-dimensional operators of the photon, fermion, and neutrino sector have been constructed \([27-29]\).

Although the SME seems to work very well in flat spacetime, certain issues arise when it is coupled to gravitational fields. Around ten years ago a no-go theorem was proven stating that an explicitly Lorentz-violating field theory cannot be consistently coupled to gravity, because this leads to incompatibilities with the Bianchi identities \([30]\). A coupling is only possible if Lorentz invariance is violated spontaneously, e.g., in a Bumblebee model \([1, 14, 30-34]\).

Note that the incompatibilities mentioned were found in the context of Riemann-Cartan spacetimes, i.e., spacetimes endowed with the Riemannian concept of curvature including torsion. An alternative approach to consider Lorentz violation in gravitational backgrounds is to change the fundamental geometrical concept. Hence instead of Riemann-Cartan geometry one might be tempted to consider Finsler geometry \([39-46]\). Geometrical quantities in Finsler spaces such as curvature do not only depend on the particular point considered in the space but also on the angle that a given line element encloses with an inherent direction in this space. Finsler spaces are based on more general length functionals so they can be considered as Riemannian spaces without the quadratic restriction \([47]\).

For this reason Finsler geometry may be a natural framework to describe preferred directions in a curved spacetime, i.e., Lorentz violation in the presence of gravity. Lately plenty of work has been done to identify Finsler spaces linked to certain sectors of the SME fermion sector, which includes studies of the minimal \([48, 51]\) and also the nonminimal sector \([52]\). In the current article isotropic subsets of the minimal fermion sector will be considered. We will obtain the corresponding Finsler structure and address certain physical problems such as the propagation of a classical, relativistic, pointlike particle in the Lorentz-violating background and the time evolution of the particle spin.

The paper is organized as follows. In Sec. \([2]\) all isotropic coefficients of the minimal SME fermion

\(^1\)Besides, note that certain tensions with the generalized second law of black-hole thermodynamics may occur when particular Lorentz-violating theories are coupled to a black-hole gravitational background. The reason is the multiple-horizon structure, e.g., for photons that arises in such frameworks \([35, 38]\).
sector are identified and the corresponding dispersion relations are computed. In Sec. 3 a generic isotropic dispersion relation is considered and its associated classical, relativistic Lagrangian is derived, which is then promoted to a Finsler structure. Section 4 is dedicated to studying the physics of the classical Lagrangian obtained. First of all the motion of the classical particle in an electromagnetic field will be investigated. Besides the interest also lies in the behavior of the particle spin, which is introduced by hand and treated with the Bargmann-Michel-Telegdi (BMT) equation [53]. Finally the results are summarized and discussed in Sec. 5. Throughout the paper natural units with \( c = \hbar = 1 \) are used unless otherwise stated.

2. Isotropic dispersion laws in the minimal fermion sector

The intention of the current section is to find all isotropic dispersion relations of the minimal SME fermion sector. The full action including both minimal and nonminimal contributions reads as [29]

\[
S = \int_{\mathbb{R}^4} d^4x \, \mathcal{L}, \quad \mathcal{L} = \frac{1}{2} \bar{\psi} \left( \gamma^\mu i \partial_\mu - m_\psi + \hat{Q} \right) \psi + \text{H.c.}, \tag{1a}
\]

\[
\hat{Q} = i \left( \hat{e}^{\mu \alpha_1} \gamma_\mu + \hat{d}^{\mu \alpha_1} \gamma_5 \gamma_\mu + \hat{e}^{\alpha_1} \mathbb{1}_4 + i \hat{f}^{\alpha_1} \gamma_5 + \frac{1}{2} \hat{g}^{\mu \nu \alpha_1 \sigma_{\mu \nu}} \right) \partial_{\alpha_1}
- \left( \hat{m} \mathbb{1}_4 + i \hat{m} \gamma_5 + \hat{a}^{\mu} \gamma_\mu + \hat{b}^{\mu} \gamma_5 \gamma_\mu + \frac{1}{2} \hat{H}^{\mu \nu} \sigma_{\mu \nu} \right). \tag{1b}
\]

Here \( \psi \) is a Dirac spinor field, \( \bar{\psi} \equiv \psi^\dagger \gamma^0 \) its Dirac conjugate, and \( m_\psi \) is the fermion mass. The \( \gamma^\mu \) for \( \mu = 0 \ldots 3 \) are the standard Dirac matrices obeying the Clifford algebra \( \{ \gamma^\mu, \gamma^\nu \} = 2 \eta^{\mu \nu} \mathbb{1}_4 \) and \( \mathbb{1}_4 \) is the unit matrix in spinor space. The operator \( \hat{Q} \) is a collection of all minimal and nonminimal Lorentz-violating composite operators in the pure fermion sector. All fields and operators are defined in Minkowski spacetime with the metric \( (\eta_{\mu \nu}) = \text{diag}(1, -1, -1, -1) \).

In momentum space the Lorentz-violating operators are decomposed in momenta and Lorentz-violating component coefficients, cf. Eqs. (5), (6) in [29]. The transformation properties of the operators with respect to (proper and improper) observer Lorentz transformations and charge conjugation are stated in Table 1 of the latter reference. Both the scalar \( \hat{m} \) and the pseudoscalar operator \( \hat{m}_5 \) only appear in the nonminimal sector, i.e., the analysis will be restricted to the vector operators \( \hat{a}^{\mu}, \hat{b}^{\mu}, \hat{c}^{\mu}, \hat{d}^{\mu} \), the scalar operators \( \hat{e}, \hat{f} \), and the tensor operators \( \hat{g}^{\mu \nu}, \hat{H}^{\mu \nu} \). The following calculations will be based on Eq. (39) of [29], which gives the general dispersion relation of the SME fermion sector including all minimal and nonminimal contributions. The dispersion relation involves the operators \( \hat{S}, \hat{P}, \hat{V}, \hat{A}, \hat{T}^{\mu \nu} \) defined by Eqs. (2), (7) and the operators \( \hat{S}_\pm, \hat{V}_\pm, \hat{T}_\pm^{\mu \nu} \) given by Eq. (35) in the latter reference.

First of all the vector operators \( \hat{a}^{\mu} \equiv a^{(3)\mu} \) and \( \hat{b}^{\mu} \equiv b^{(3)\mu} \) shall be considered. They are contained in the operators \( \hat{V}^{\mu} \) and \( \hat{A}^{\mu} \), respectively, and they contribute to \( \hat{V}_\pm \). For \( \hat{S}_\pm = -m_\psi, \hat{V}_\pm = p^{\mu} + \hat{V}^{\mu}, \) and \( \hat{T}_\pm^{\mu \nu} = 0 \) the dispersion relation results in:

\[
p^2 + 2p \cdot \hat{V} + \hat{V}^2 - m_\psi^2 = 0, \tag{2}
\]
with the fermion four-momentum $p^\mu$. Setting $\hat{V}^\mu = -\hat{a}^\mu$ the second term on the left-hand side of the latter equation cannot be isotropic for any choice of $\hat{a}^\mu$ besides $(a^{(3)}{}^\mu) = (a^{(3)}{}^0, 0, 0)^T$. The corresponding dispersion relation is then given by

$$(p_0)^+ = a^{(3)}{}^0 + \sqrt{p^2 + m_\psi^2},$$

where $p$ is the particle three-momentum. Here $(p_0)^+$ denotes the positive-energy dispersion law. This result is encoded in Eq. (94) of [29]. Note that a nonzero coefficient $a^{(3)}{}^0$ just leads to an unobservable shift of the particle energy, which reminds us of the fact that the coefficients $a^{(4)}{}^\alpha_1$ can be removed by a phase redefinition [29]. As a next step we consider the operator $\hat{b}^\mu$. From $\hat{S}_\pm = -m_\psi, \hat{V}_\pm = p^\mu \pm \hat{A}^\mu$, and $\hat{T}_\pm = 0$ we obtain:

$$(p^2 + 2p \cdot \hat{A} + \hat{A}^2)(p^2 - 2p \cdot \hat{A} + \hat{A}^2) - 2m_\psi^2(p^2 - \hat{A}^2) + m_\psi^4 = 0.$$  

For $\hat{A}^\mu = -\hat{b}^\mu$ the term $p \cdot \hat{A}$ can only be isotropic, if $(b^{(3)}{}^\mu) = (b^{(3)}{}^0, 0, 0, 0)^T$. Then there are two different dispersion relations that read as

$$(p_0)^+_{1,2} = \sqrt{p^2 + m_\psi^2 + (b^{(3)}{}^0)^2 \pm 2|b^{(3)}{}^0||p|} \approx \sqrt{p^2 + m_\psi^2} \left(1 \pm \frac{|b^{(3)}{}^0|}{p^2 + m_\psi^2}\right).$$

Due to Lorentz violation the energies of fermion states with different spin projections are no longer degenerate. This behavior resembles a birefringent vacuum for the photon sector. The expansion here and all subsequent ones are understood to be valid for a sufficiently small Lorentz-violating coefficient.

The situation is slightly similar for the vector operators $\hat{c}^\mu \equiv c^{(4)}{}^{\mu\alpha_1}p_{\alpha_1}$ and $\hat{d}^\mu \equiv d^{(4)}{}^{\mu\alpha_1}p_{\alpha_1}$ consisting of second-rank tensor coefficients that are contracted with one additional four-momentum. We consider $\hat{V}^\mu = c^{(4)}{}^{\mu\alpha_1}p_{\alpha_1}$ at first. To end up with an isotropic dispersion relation, the coefficients $c^{(4)}{}^{\mu\alpha_1}$ must be chosen such that $p \cdot \hat{V} = p_\mu c^{(4)}{}^{\mu\alpha_1}p_{\alpha_1}$ in Eq. (2) is isotropic. This is only the case if all off-diagonal components vanish and $c^{(4)}{}^{11} = c^{(4)}{}^{22} = c^{(4)}{}^{33}$. Since $c^{(4)}{}^{\mu\alpha_1}$ is traceless, that heavily restricts the possibilities of choices for the coefficients, with only one remaining:

$$(c^{(4)}{}^{\mu\nu}) = c^{(4)}{}^{00} \begin{pmatrix} 1 & 1/3 & 1/3 & 1/3 \end{pmatrix}.$$  

Then even $p_\mu c^{(4)}{}^{\mu\alpha_1}p_{\alpha_1} = 0$, which makes the dispersion relation manifestly isotropic. The coefficients $d^{(4)}{}^{\mu\alpha_1}$ behave in a similar manner. Setting $\hat{A}^\mu = d^{(4)}{}^{\mu\alpha_1}p_{\alpha_1}$ the expression $p \cdot \hat{A} = p_\mu d^{(4)}{}^{\mu\alpha_1}p_{\alpha_1}$ in Eq. (11) must be isotropic. With an analogous argument this leads to

$$(d^{(4)}{}^{\mu\nu}) = d^{(4)}{}^{00} \begin{pmatrix} 1 & 1/3 & 1/3 & 1/3 \end{pmatrix}.$$  

For the choice of Eq. (17) it can be checked that $p_\mu d^{(4)}{}^{\mu\alpha_1}p_{\alpha_1} = 0$, from which results an isotropic dispersion relation. Now the modified dispersion laws in case of nonvanishing coefficients $c^{(4)}{}^{00}$ and $d^{(4)}{}^{00}$ read as
follows:
\[
(p_0)^{+}_{1,2} = \frac{\sqrt{3 + (2 - e^{(4)0})e^{(4)00} + (d^{(4)00})^2}}{3 \left[(1 + e^{(4)00})^2 - (d^{(4)00})^2\right] \frac{p^2}{2}} \approx \sqrt{p^2 + m_\psi^2 \left[1 - e^{(4)00} \left(\frac{4/3}{p^2 + m_\psi^2}\right)\right]} \pm \frac{4}{3} d^{(4)00}|p|.
\]
(8)

For \(d^{(4)00} = 0\) there is a single dispersion relation for both spin projections of the fermion. At first order in \(e^{(4)00}\) (and for \(m_\psi = 0\)) this modification corresponds to the isotropic sector of the CPT-even extension of the photon sector, since both sectors are related by a coordinate transformation (see [54] and references therein). The result is confirmed by Eq. (95) in [29]. For \(d^{(4)00} \neq 0\) there exist two distinct isotropic dispersion relations.

The next step is to consider the scalar operators \(\tilde{e} \equiv e^{(4)\alpha_1} p_\alpha\) and \(\tilde{f} \equiv f^{(4)\alpha_1} p_\alpha\). For the operator \(\tilde{e}\) it holds that \(\hat{S} = \tilde{e}\), \(\hat{S}_\pm = -m_\psi \pm \tilde{e}\), \(\hat{V}_\mu = p_\mu\), and \(\hat{T}_{\mu\nu} = 0\), which is subsequently inserted in Eq. (39) in [29] to give
\[
(p^2 - (m_\psi - \tilde{e})^2) = 0.
\]
(9)

The latter can only be isotropic for \((e^{(4)\alpha_1}) = (e^{(4)0}, 0, 0, 0)^T\) resulting in the dispersion relation
\[
(p_0)^{+} = \sqrt{\frac{1 - (e^{(4)0})^2}{1 - (e^{(4)0})^2}} \approx \sqrt{p^2 + m_\psi^2 - e^{(4)0}m_\psi^2}.
\]
(10)

The result corresponds to the observation that \(a_{\text{eff}}^{(5)00}\) is isotropic (see Eq. (97) in [29]) where this effective dimension-5 coefficient also contains \(e^{(4)0}\) according to the first of Eqs. (27) in [29]. A similar investigation can be carried out for \(\tilde{f}\) where \(\hat{S}_\pm = -m_\psi \pm i\hat{P}\), which leads to
\[
(p^2 - (m_\psi + \tilde{f})^2) = 0.
\]
(11)

Also this result can only be isotropic for \((f^{(4)\alpha_1}) = (f^{(4)0}, 0, 0, 0)^T\) leading to
\[
(p_0)^{+} = \sqrt{\frac{p^2 + m_\psi^2}{1 - (f^{(4)0})^2}} \approx \sqrt{p^2 + m_\psi^2 \left(1 + \frac{1}{2}(f^{(4)0})^2\right)}.
\]
(12)

Note that by a spinor redefinition the coefficients \(f^{(4)\alpha_1}\) can be transferred to the \(\tilde{e}\) operator [55].

Last but not least the tensor coefficients \(\hat{H}^{\mu\nu} \equiv H^{(3)\mu\nu}\) and \(\tilde{g}^{\mu\nu} \equiv g^{(4)\mu\nu\alpha_1} p_\alpha\) will be investigated. They are both contained in the tensor operator \(\hat{T}_{\mu\nu} = \tilde{g}^{\mu\nu} - \hat{H}^{\mu\nu}\). The special case obtained from the general dispersion relation of Eq. (39) in [29] by setting \(\hat{S}_\pm = -m_\psi\), \(\hat{V}_\mu^\pm = p_\mu\) is given by:
\[
0 = (m_\psi^2 - \tilde{T}_-^{\mu\nu} \tilde{T}_-^{\mu\nu}) \left(m_\psi^2 - \tilde{T}_+^{\rho\sigma} \tilde{T}_+^{\rho\sigma}\right) + p^4
- 2p_\mu \left(-m_\psi \eta^{\mu\nu} + 2i \tilde{T}_-^{\mu\nu}\right) \left(-m_\psi \eta_{\rho\sigma} - 2i \tilde{T}_+^{\rho\sigma}\right) p^\rho,
\]
(13a)
with the convenient definition

\[ \hat{T}_{\pm}^{\mu\nu} = \frac{1}{2} \left( \hat{T}^{\mu\nu} \pm i\tilde{T}^{\mu\nu} \right), \quad \tilde{T}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \hat{T}_{\rho\sigma}. \]  

(13b)

The latter involves the dual of \( \hat{T}^{\mu\nu} \), which is denoted with an additional tilde and formed by contraction of \( \hat{T}^{\mu\nu} \) with the four-dimensional Levi-Civita symbol \( \epsilon^{\mu\nu\rho\sigma} \) where \( \epsilon^{0123} = 1 \). Now Eq. (13a) can be further simplified by using the properties of \( \hat{T}^{\mu\nu} \). When adding the operators defined by Eq. (13b), the dual is eliminated. Furthermore the square of \( \hat{T}^{\mu\nu} \) corresponds to the square of its dual with an additional minus sign. Eventually, \( \hat{T}^{\mu\nu} \) contracted with two four-momenta vanishes:

\[ \hat{T}^{\mu\nu}_{+} - \hat{T}^{\mu\nu}_{-} = i\tilde{T}^{\mu\nu}, \quad \tilde{T}^{\mu\nu}\hat{T}^{\mu\nu} = -\tilde{T}^{\mu\nu}\tilde{T}^{\mu\nu}, \quad p_{\mu}\tilde{T}^{\mu\nu}p_{\nu} = 0. \]  

(14)

The second and third relationship follow from the antisymmetry of \( \hat{T}^{\mu\nu} \). By using these relations, Eq. (13a) can be further simplified:

\[ p^4 - 2m_{\psi}^2p^2 + \left( m_{\psi}^2 - \frac{1}{2} \hat{T}^{\mu\nu}\hat{T}_{\mu\nu} \right)^2 + \frac{1}{4}(\hat{T}^{\mu\nu}\tilde{T}_{\mu\nu})^2 - 8p_{\mu}\hat{T}^{\mu\nu}\hat{T}_{\nu\nu\nu}\nu p^0 = 0. \]  

(15)

For the tensor operator \( \tilde{H}^{\mu\nu} \) the only term that may lead to anisotropy is the last one on the left-hand side of the latter equation. By explicitly inserting \( \tilde{H}^{\mu\nu} \) it can be demonstrated that no choice of the coefficients of \( H^{(3)\mu\nu} \) leads to an isotropic expression. In [29] it was shown that no dimension-3 but only the dimension-5 coefficients \( \tilde{H}^{(5)0j0j} \) produce an isotropic dispersion law (see Eq. (97) in [29]). According to the fourth of Eqs. (27) in [29] these effective coefficients contain \( \tilde{H}^{(5)0j0j} \) where \( \tilde{H}^{(5)\mu\nu\alpha\alpha} \) are the dual coefficients of \( H^{(5)\mu\nu\alpha\alpha} \). Furthermore, by symmetry arguments they also comprise \( d^{(4)00} \). This explains the isotropic dispersion laws of Eq. (8) following from a nonzero coefficient \( d^{(4)00} \).

Hence there exists no isotropic dispersion relation for any of the dimension-3 component coefficients \( H^{(3)\mu\nu} \). For the tensor operator \( \tilde{g}^{\mu\nu} \) the situation is different. With \( \hat{T}^{\mu\nu} = g^{(4)\mu\nu\alpha\beta}p_{\alpha\beta} \) it can be checked that there is an isotropic dispersion relation for two different choices of coefficients. The first choice is

\[ g^{(4)123} = g^{(4)231} = g^{(4)312} \equiv g_1, \quad g^{(4)132} = g^{(4)213} = g^{(4)321} = -g_1, \]  

and all others set to zero, which results in two modified dispersion relations:

\[ (p_0)_{1,2}^+ = \sqrt{(1 + g_1^2)p^2 + 2g_1m_{\psi}|p| + m_{\psi}^2} \approx \sqrt{p^2 + m_{\psi}^2 + 1 \pm g_1 \frac{m_{\psi}|p|}{p^2 + m_{\psi}^2}}. \]  

(17)

The nonzero coefficients of Eq. (16) are contained in \( \tilde{g}^{(4)0j0j}_{\text{eff}} \) of Eq. (95) in [29] where \( \tilde{g}^{(4)\mu\nu\alpha\beta} \) denotes the dual of \( g^{(4)\mu\nu\alpha\beta} \). According to the third of Eqs. (27) in [29] these effective coefficients also contain \( b^{(4)0} \), which explains the isotropic dispersion relation of Eq. (6). For this particular choice of \( g^{(4)\mu\nu\alpha\beta} \) the last term on the left-hand side of Eq. (15) is isotropic. The second choice of coefficients, which fulfills this condition, is

\[ g^{(4)101} = g^{(4)202} = g^{(4)303} \equiv g_2, \quad g^{(4)011} = g^{(4)022} = g^{(4)033} = -g_2. \]  

(18)
and all remaining ones set to zero. This case gives rise to a single modified dispersion relation:

\[(p_0)^+ = \sqrt{(1 + g_2^2) \mathbf{p}^2 + m_\psi^2} \approx \sqrt{\mathbf{p}^2 + m_\psi^2} \left(1 + \frac{g_2^2}{2} \frac{\mathbf{p}^2}{\mathbf{p}^2 + m_\psi^2}\right).\]  

(19)

Note that Eq. (17) comprises a modification at first order in the Lorentz-violating coefficients, whereas the modification in Eq. (19) is of second order in Lorentz violation. The term \(\hat{T}^{\mu\nu} \hat{T}_{\mu\nu}\) in Eq. (15) differs for both sets of component coefficients leading to distinct dispersion relations.

To summarize, in the minimal fermion sector of the SME an isotropic dispersion relation exists for a particular choice of \(a^{(3)}\mu, b^{(3)}\mu, c^{(4)}\mu\alpha_1, d^{(4)}\mu\alpha_1, e^{(3)}\alpha_1, f^{(3)}\alpha_1,\) and \(g^{(4)}\mu\nu\alpha_1\) component coefficients. Some of these dispersion relations depend on the spin projection of the fermion, which is the analogy of a birefringent vacuum in the photon sector.

3. Construction of the classical Lagrangian and Finsler structure

As of now we intend to consider an isotropic modified dispersion relation of the generic form

\[p_0^2 - \Upsilon^2 \mathbf{p}^2 - m_\psi^2 = 0, \quad (p_0)_{1,2} = \pm \sqrt{\Upsilon^2 \mathbf{p}^2 + m_\psi^2},\]  

(20)

with a dimensionless parameter \(\Upsilon\) where in the standard case \(\Upsilon = 1\). Such a dispersion relation emerges from a particular choice of the \(g^{(4)}\mu\nu\alpha_1\) coefficients, cf. Eq. (19), or for a nonvanishing \(c^{(4)}00\) (see Eq. (8) by setting \(d^{(4)00} = 0\)) when absorbing the global modification before the square root into the fermion mass. This dispersion relation is based on the fermion Lagrangian of the SME, i.e., it is a field theory result.

In what follows, for the particular isotropic dispersion relation of Eq. (20) the Lagrangian \(L\) shall be derived, which describes a classical, relativistic, pointlike particle whose conjugate momentum satisfies the dispersion relation mentioned. It was shown in \([48]\) that such a Lagrangian can, in principle, be obtained from five equations involving the four-momentum components \(p_\mu\) and the four-velocity components \(u^\mu\) of the classical particle. One of these equations is the modified dispersion relation. Furthermore, due to the parameterization invariance of the classical action along a path the Lagrangian must be positive homogeneous of first degree in the velocity. Then it has to be of the following shape, which forms the second equation:

\[L = -u^\mu p_\mu, \quad p_\mu = -\frac{\partial L}{\partial u^\mu}.\]  

(21)

Here \(p_\mu\) is the conjugate momentum of the particle. Note the minus sign in the definition of the latter. If we construct a quantum-mechanical wave packet from the quantum-theoretic free-field equations, its group velocity shall correspond to the velocity of the classical pointlike particle:

\[\frac{\partial p_0}{\partial |\mathbf{p}|} = \Upsilon^2 \frac{\mathbf{p}}{p_0} = \frac{|\mathbf{u}|}{u^0}.\]  

(22)
Because of the assumed isotropy of the Lagrangian the original three conditions, which hold for the spatial momentum components, result in only one equation here for the magnitude \(|\mathbf{p}|\) of the spatial momentum and the magnitude \(|\mathbf{u}|\) of the three-velocity. This single equation can be solved with respect to \(|\mathbf{p}|\):

\[
\frac{\mathcal{Y}^4 p^2}{\mathcal{Y}^2 p^2 + m^2} = \frac{u^2}{(u^0)^2} \Rightarrow |\mathbf{p}| = \frac{m\psi|\mathbf{u}|}{\mathcal{Y} \sqrt{\mathcal{Y}^2(u^0)^2 - u^2}}. \tag{23}
\]

Then the zeroth four-momentum component can be expressed via the velocity as well:

\[
p_0 = \pm \sqrt{\mathcal{Y}^2 p^2 + m^2} = \pm \frac{\mathcal{Y} m\psi|u^0|}{\sqrt{\mathcal{Y}^2(u^0)^2 - u^2}}. \tag{24}
\]

According to Eq. (22) for \(u^0 \geq 0\) the sign of \(p_0\) has to be chosen as negative. For \(u^0 < 0\) the sign is taken to be positive. However the absolute value of \(u^0\) in Eq. (24) produces an additional minus sign in this case. This leads to:

\[
L = -p_0 u^0 - \mathbf{p} \cdot \mathbf{u} = \frac{\mathcal{Y} m\psi(u^0)^2}{\sqrt{\mathcal{Y}^2(u^0)^2 - u^2}} - \frac{m\psi u^2}{\mathcal{Y} \sqrt{\mathcal{Y}^2(u^0)^2 - u^2}} = m\psi \sqrt{(u^0)^2 - \frac{u^2}{\mathcal{Y}^2}} \nonumber
\]

\[
= m\psi \sqrt{(u \cdot \xi)^2 - \frac{1}{\mathcal{Y}^2} ((u \cdot \xi)^2 - u^2)}, \tag{25}
\]

with the preferred timelike direction \((\xi^\mu) = (1, 0, 0, 0)^T\). If only the positive-energy solution in Eq. (24) is considered, the Lagrangian with a global minus sign must be taken into account as well. It can be checked that Eqs. (20) – (22) are fulfilled by the positive Lagrangian for \(u^0 < 0\) and by the negative Lagrangian for \(u^0 \geq 0\). Since in the remainder of the paper \(u^0 \geq 0\) will be chosen anyhow, the Lagrangian with a global minus sign will be considered from now on. For \(\mathcal{Y} = 1\) one obtains the standard result \(L = \pm m\psi \sqrt{u_\mu u^\mu}\). The Lagrangian itself has an intrinsic metric \(r_{\mu\nu}\) associated to it, which is used to define the scalar products, e.g., \(u \cdot \xi = r_{\mu\nu} u^\mu \xi^\nu\). This intrinsic metric corresponds to the Minkowski metric, i.e., \(r_{\mu\nu} = \eta_{\mu\nu}\).

Now the Lagrangian with a global minus sign, i.e., \(L = -m\psi \sqrt{(u^0)^2 - u^2}/\mathcal{Y}^2\) shall be promoted to a Finsler structure (see [43] for the properties of such a structure). There are two different possibilities of proceeding [44]. The first is to set \(u^0 = 0\), which results in a three-dimensional Finsler structure describing a Euclidean geometry with a global scaling factor:

\[
\hat{F}_{\mathcal{Y}}(y) \equiv \frac{i}{m\psi} L(u^0 = 0, u^i = y^i) = \frac{1}{\mathcal{Y}} \sqrt{r_{ij} y^i y^j}, \quad (r_{ij}) = \text{diag}(1, 1, 1), \quad y \in TM \setminus \{0\} \tag{26},
\]

where \(TM\) is the tangent bundle of the Finsler space. The scalar product of two vectors \(\alpha, \beta\) in the tangent space is given by \(\alpha \cdot \beta = r_{ij} \alpha^i \beta^j\) with the intrinsic metric \((r_{ij}) = \text{diag}(1, 1, 1)\). This structure describes a Euclidian space with all dimensions scaled by \(1/\mathcal{Y}\). A similar space results by applying the same procedure to the Finsler structure of the nonminimal coefficient \(m^{(5)00}\) considered in [52].

The alternative is to perform a Wick rotation leading to the four-dimensional Finsler structure

\[
F_{\mathcal{Y}}(y) \equiv \frac{i}{m\psi} L(u^0 = iy^4, u^i = y^i) = \sqrt{(y^4)^2 + \frac{1}{\mathcal{Y}^2} \sum_{i=1,2,3} (y^i)^2} = \sqrt{(y \cdot \xi)^2 + \frac{1}{\mathcal{Y}^2} [y^2 - (y \cdot \xi)^2]}, \tag{27}
\]

8
where \( y \in TM \setminus \{0\} \). The intrinsic metric here is \((r_{ij}) = \text{diag}(1,1,1,1)\) and \( \zeta = (0,0,0,1)^T \) is a preferred direction where \( \zeta^i \equiv \xi^i \) for \( i = 1 \ldots 3 \) and \( \zeta^4 \equiv \xi^0 \) with the \( \xi^\mu \) used in Eq. (25). The following considerations will concentrate on the second Finsler structure \( F_\Upsilon \). The Finsler metric can be computed via

\[
g_{ij}(y) \equiv \frac{1}{2} \frac{\partial}{\partial y^i} \frac{\partial}{\partial y^j} F_\Upsilon(y)^2, \quad (g_{ij}) = \text{diag} \left( \frac{1}{\Upsilon^2}, \frac{1}{\Upsilon^2}, \frac{1}{\Upsilon^2}, 1 \right),
\]

and the particular result is independent of \( y \). The Finsler structure \( F_\Upsilon \) describes a Euclidean geometry as well. To check this, the Cartan torsion \( C_{ijk} \)

\[
C_{ijk} \equiv \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k} = \frac{1}{4} \frac{\partial^3}{\partial y^i \partial y^j \partial y^k} F_\Upsilon^2, \quad (29)
\]

is needed where its mean is defined as

\[
I \equiv I_i u^i, \quad I_i \equiv g^{jk} C_{ijk}, \quad (g^{ij}) \equiv (g_{ij})^{-1}, \quad (30)
\]

with the inverse Finsler metric \( g^{ij} \). For the special Finsler metric in Eq. (28) the mean Cartan torsion \( I \) vanishes, which according to Deicke’s theorem \( [56] \) shows that the corresponding space is Riemannian. In this space three dimensions are scaled and one dimension remains standard. Therefore the length of a vector in the scaled subspace, which corresponds to the spatial part of the original spacetime, is scaled where the angle between such vectors stays unmodified. However angles between vectors change when they have one component pointing along the \( y^4 \)-axis, which has influence on, e.g., velocities in the corresponding spacetime.

All Finsler spaces in the context of the minimal SME, which have been considered in other references so far, are related to non-Euclidean spaces. This holds for the a-space \([48, 49]\), b-space \([48, 49]\), the bipartite spaces \([50]\), and the spaces considered in \([51]\). A reasonable conjecture is that only isotropic (nonbirefringent) dispersion relations such as the one investigated here lead to Euclidean structures.

### 4. Charged relativistic particle in an electromagnetic field

After clarifying the mathematical foundations of the modified Lagrangian in the last section, its physical properties shall be investigated. In what follows, particle trajectories shall be parameterized such that \( u^0 = c \) and \( u = v \) where \( c \) is the speed of light and \( v \) the ordinary three-velocity of the particle. Note that natural coordinates are used with \( c = 1 \). If the particle moves freely, the trajectory will be
the same straight line such as in the standard case without any Lorentz violation. Hence to understand
the modified physics, the classical particle is assigned an electric charge \( q \) and its propagation in an
electromagnetic field shall be studied. Therefore a four-potential \((A^\mu) = (\phi, A)\) is introduced and the
charged, classical particle is described by the following Lagrangian:

\[
L_{\text{em}} = -m_\psi \sqrt{1 - \frac{v^2}{\upsilon^2}} + qv \cdot A - q\phi,
\]

(31)

with the scalar potential \( \phi \) and the vector potential \( A \). The equations of motion are obtained from the
Euler-Lagrange equations (with the position vector \( x \)), which for the particular Lagrangian of Eq. (31)
read as follows:

\[
\frac{d}{dt} \frac{\partial L_{\text{em}}}{\partial v} = \frac{\partial L_{\text{em}}}{\partial x},
\]

(32a)

\[
\frac{d}{dt} \left( \frac{m_\psi v^2}{\upsilon^2} \right) + qA = -q\nabla \phi + q\nabla (v \cdot A).
\]

(32b)

The total time derivative of the vector potential

\[
\frac{dA}{dt} = -v \times (\nabla \times A) + \nabla \cdot (v \cdot A) + \frac{\partial A}{\partial t},
\]

(33)

is used to express the right-hand side of Eq. (32b) via the physical electric and magnetic fields \( E, B \):

\[
\frac{dp}{dt} = qv \times (\nabla \times A) + q \left( -\nabla \phi - \frac{\partial A}{\partial t} \right) = qv \times B + qE.
\]

(34)

For the zeroth four-momentum component, i.e., the particle energy, a further equation can be derived
directly from the equations of motion for the spatial momentum components:

\[
\frac{dp_0}{dt} = \frac{\upsilon^2}{p_0} p \cdot \frac{dp}{dt} = \frac{1}{\gamma_\tau m_\psi} \gamma_\tau m_\psi v \cdot \frac{dp}{dt} = qv \cdot E.
\]

(35)

Now a relativistic momentum and energy can be introduced via

\[
p = \frac{\gamma_\tau m_\psi v}{\upsilon^2}, \quad p_0 = \gamma_\tau m_\psi, \quad \gamma_\tau = \frac{1}{\sqrt{1 - v^2/\upsilon^2}}.
\]

(36)

with a modified Lorentz factor \( \gamma_\tau \). In fact, with the free Lagrangian \( L \) it can be cross-checked that

\[
p_0 = -\frac{\partial L}{\partial w^0} = m_\psi \frac{u^0}{\sqrt{(u^0)^2 - u^2/\upsilon^2}}, \quad p_i = -\frac{\partial L}{\partial u^i} = -m_\psi \frac{u^i/\upsilon^2}{\sqrt{(u^0)^2 - u^2/\upsilon^2}},
\]

(37)

which corresponds to Eq. (36) for \( u^0 = c = 1, u = v \) and taking into account that \( p \) is the spatial
momentum of the contravariant momentum four-vector. Therefore raising the index on \( p_i \) produces an
additional minus sign. With the modified proper time \( d\tau \equiv dt/\gamma \), the equations of motion \( (34), (35) \) can be written in a covariant form:

\[
\frac{d\tilde{u}^\alpha}{d\tau} = \frac{q}{m_\psi} F^{\alpha\beta} u_\beta, \quad (\tilde{u}^\alpha) = \gamma (\frac{1}{v/\gamma^2}), \quad (u^\alpha) = \gamma (\frac{1}{v}),
\]

(38)

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the electromagnetic field strength tensor. Note that the four-velocity \( \tilde{u}^\alpha \) used on the left-hand side of the latter equation involves both modifications in the Lorentz factor and the spatial velocity components, whereas the four-velocity \( u^\alpha \) on the right-hand side only involves a modified Lorentz factor. The reason for this is that the particle kinematics is modified by the Lorentz-violating background field in contrast to its coupling to the electromagnetic field.

Now the modified equations of motion shall be solved for particular cases to understand how their solutions are affected by Lorentz violation. First, consider the case of a vanishing electric field, \( E = 0 \), where the particle moves perpendicularly to a magnetic field \( B = B\hat{e}_z \), i.e., its initial velocity and position shall be given by \( v(0) = v\hat{e}_y \) and \( x(0) = R\hat{e}_x \), respectively. The equations of motion in the laboratory frame read

\[
\frac{d}{dt} \left( \frac{m_\psi v/\gamma^2}{\sqrt{1 - v^2/\gamma^2}} \right) = qv \times B \quad \Leftrightarrow \quad \gamma m_\psi \frac{dv}{dt} = qv \times B,
\]

(39)

where \( \gamma \) is time-independent, since the magnitude of the velocity does not change in a magnetic field. The latter differential equations with the initial conditions above are satisfied by the following time-dependent particle position and velocity:

\[
x(t) = \begin{pmatrix} R \cos(\omega t) \\ R \sin(\omega t) \\ 0 \end{pmatrix}, \quad v(t) = R\omega \begin{pmatrix} -\sin(\omega t) \\ \cos(\omega t) \\ 0 \end{pmatrix},
\]

(40a)

\[
v = R|\omega|, \quad \omega = -\frac{\gamma^2 C}{\sqrt{1 + \gamma^2 C^2}} = -\gamma^2 C \left( 1 - \frac{1}{2} \gamma^2 C^2 + \ldots \right), \quad C = \frac{qBR}{m_\psi}.
\]

(40b)

This describes a circular movement with radius \( R \) and angular frequency \( \omega \) such as in the standard case where the sign gives the rotational direction. However additional scaling factors \( \Upsilon \) appear that can be explained as follows. It must be taken into account that the magnetic field strength \( B \), the velocity \( v \), and the speed of light (which has been set equal to 1) each gets one power of \( \Upsilon \) due to the scaling of the spatial dimensions. The radius \( R \) of the circle stays constant when \( B \rightarrow \Upsilon B \) and \( v \rightarrow \Upsilon v \). Since the magnitude of the velocity is constant, \( p_0 \) is time-independent. Due to \( E = 0 \) the first of Eq. \( (35) \) is satisfied as well, for consistency.

As a next example consider the particle motion in a vanishing magnetic field, \( B = 0 \), where the particle moves perpendicularly to the electric field \( E = E\hat{e}_z \) with the initial conditions \( v(0) = v_0\hat{e}_y \) and \( r(0) = 0 \). The equations of motion in the laboratory frame then read as follows:

\[
\frac{d}{dt} \left( \frac{m_\psi v/\gamma^2}{\sqrt{1 - v^2/\gamma^2}} \right) = qE.
\]

(41)
An integration with respect to \( t \) using the initial condition \( \mathbf{v}(0) = v_0 \hat{e}_y \) leads to:

\[
\frac{\mathbf{v}}{\sqrt{1 - \mathbf{v}^2/\Upsilon^2}} - \frac{v_0 \hat{e}_y}{\sqrt{1 - v_0^2/\Upsilon^2}} = \frac{q \Upsilon^2}{m_\psi} E t .
\]

This is a system of algebraic equations for \( v_y \) and \( v_z \) \((v_x \equiv 0)\), which can be solved to give:

\[
\begin{align*}
v_y(t) &= \frac{v_0}{\sqrt{1 + \Upsilon^2 C^2 t^2}} , & v_z(t) &= \frac{\Upsilon^2 \tilde{C} t}{\sqrt{1 + \Upsilon^2 C^2 t^2}} , & \tilde{C} &= \frac{q E}{m_\psi} \sqrt{1 - v_0^2/\Upsilon^2} , \\
y(t) &= \frac{v_0}{\Upsilon \tilde{C}} \ln \left( \Upsilon \tilde{C} t + \sqrt{1 + \Upsilon^2 C^2 t^2} \right) , & z(t) &= \frac{\Upsilon^2 \tilde{C}^2 t^2}{1 + \sqrt{1 + \Upsilon^2 C^2 t^2}} .
\end{align*}
\]

Here the trajectory again involves additional scalings with \( \Upsilon \). The behavior can be understood when considering that besides the particle velocity and the speed of light also the electric field strength gets one power of \( \Upsilon \). The velocity component in \( y \)-direction goes to zero starting from its initial value \( v_0 \). This is due to the relativistic increase in mass, since there is no force along the \( y \)-direction compensating for this effect. Therefore the distance traveled in \( y \)-direction grows logarithmically, i.e., very slowly. The velocity in \( z \)-direction steadily increases to reach its maximum value \( v_z(t = \infty) = \Upsilon \) as expected. For large times the particle then travels with the practically constant velocity \( v_z \) for \( t \to \infty \). This behavior shows as well that the Lorentz group, which the classical particle transforms under, is modified such that the speed of light \( c \) is replaced by \( \Upsilon c \) everywhere. For consistency, the first of Eq. (38) is fulfilled when inserting the electric field vector and the velocity components of Eq. (43a).

4.1. Introduction of particle spin

Since spin is a manifestly quantum-mechanical concept, the classical particle studied in the previous sections does not have any spin associated to it, although it shall be based on a Lorentz-violating fermion. However it is possible to introduce spin for a classical particle according to the lines of [53]. The authors of the latter reference derive a relativistic equation of motion (often denoted as the BMT equation according to the authors’ second names) for the spin of a classical particle of electric charge \( q \) and mass \( m_\psi \) in an electromagnetic field:

\[
\frac{d s^\alpha}{d \tau} = \frac{g q}{2 m_\psi} \left[ F^{\alpha \beta} s_\beta + (F^{\beta \gamma} s_\beta u_\gamma) u^\alpha \right] - \left( \frac{d u^\beta}{d \tau} s_\beta \right) u^\alpha .
\]

Here \( g \) is the Landé factor of the particle, \( u^\mu \) is the particle velocity, \((s^\mu) = (s^0, \mathbf{s})\) the spin four-vector, and \( \tau \) the proper time. Now let us apply this equation to the Lorentz-violating situation considered in the current article. If the particle spin is introduced as an external quantity analogously to the argumentation in [53], there is no direct spin coupling to the Lorentz-violating background field. For this reason the Lorentz group for the spin is standard and \( d \tau = dt/\gamma \) with the standard Lorentz factor \( \gamma = 1/\sqrt{1 - v^2} \).

However Lorentz violation may still have an influence on the particle spin due to the second term on the right-hand side of Eq. (44), which is linked to particle kinematics. It involves the four-acceleration,
which allows us to use the particle equations of motion. Now the isotropic Lorentz-violating coefficients are assumed to be much smaller than one, i.e., \( \Upsilon = 1 + \chi \) with a generic, dimensionless, isotropic Lorentz-violating coefficient \( \chi \). The modified four-velocity \( \tilde{u}^\alpha \) of the particle, which has been employed on the left-hand side of Eq. (38), is then expanded around a zero coefficient \( \chi \):

\[
(\tilde{u}^\alpha) = \gamma \left( \frac{1}{v} \right) - \gamma^3 \left( \frac{v^2}{2 - v^2} \right) \chi, \tag{45}
\]

with the magnitude \( v \equiv |\mathbf{v}| \) of the three-velocity \( \mathbf{v} \). Therefore for a small Lorentz-violating coefficient \( \chi \) the equations of motion of the classical particle involve the standard terms plus an additional contribution on the right-hand side, which is linear in \( \chi \):

\[
\frac{du^\alpha}{d\tau} = \frac{q}{m_\psi} F_{\alpha\beta} u_\beta + \frac{d}{d\tau} \left[ \gamma^3 \left( \frac{v^2}{2 - v^2} \right) \right] \chi. \tag{46}
\]

The nonrelativistic version of this equation is obtained by expanding all quantities with respect to \( v^2 \ll 1 \):

\[
\frac{dv}{dt} \left( \frac{1 + v^2/2}{v} \right) = \frac{q}{m_\psi} F_{\alpha\beta} \left( \frac{1}{-v} \right)_{\beta} + \frac{d}{dt} \left( \frac{v^2}{2v} \right)^\alpha \chi. \tag{47}
\]

This is a set of four nonrelativistic equations where the first one gives a relation for the nonrelativistic kinetic energy of the particle in the electromagnetic field and the remaining ones give the acceleration caused by the Lorentz force. For \( \chi \ll 1 \) they are given by

\[
\frac{dv^2}{dt} = (1 + 2\chi) \frac{q}{m_\psi} \mathbf{E} \cdot \mathbf{v}, \tag{48a}
\]

\[
\frac{d\mathbf{v}}{dt} = (1 + 2\chi) \frac{q}{m_\psi} (\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{48b}
\]

and they will be needed shortly. Then the modified evolution equations for the particle spin can be obtained by inserting Eq. (47) in Eq. (44), neglecting all contributions of the order of \( v^2 \ll 1 \) and assuming \( |v\dot{v}| \ll |\dot{\mathbf{v}}| \):

\[
\frac{ds^\alpha}{dt} = \frac{gq}{2m_\psi} \left[ F^{\alpha\beta} s_\beta + (F^{\beta\gamma}s_\beta u_\gamma)u^\alpha \right] - \frac{q}{m_\psi} (F^{\beta\gamma}s_\beta u_\gamma)u^\alpha - \left( \frac{0}{2\dot{\mathbf{v}} \chi} \right)^\beta s_\beta u^\alpha \\
= \frac{g}{2m_\psi} \left[ \frac{g}{2} F^{\alpha\beta} s_\beta + \left( \frac{g}{2} - 1 \right) (F^{\beta\gamma}s_\beta u_\gamma)u^\alpha \right] + 2\chi \mathbf{v} \cdot \mathbf{s} u^\alpha. \tag{49}
\]

The intermediate result is that there appears an additional term on the right-hand side of the spin evolution equations, which is proportional to the Lorentz-violating coefficient and describes a coupling between the spin vector and the ordinary particle three-acceleration \( \mathbf{v} \). Introducing the electromagnetic
fields leads to
\[
\frac{d}{dt} \left( s^0 s \right) = \frac{q}{m_\psi} \left\{ \frac{g}{2} \left( \frac{E \cdot s}{E s^0 + s \times B} \right) + \left( \frac{g}{2} - 1 \right) \left[ (E \cdot v)s^0 - (E + v \times B) \cdot s + \frac{m_\psi}{q} \tilde{\chi} \tilde{v} \cdot s \right] \left( \frac{1}{v} \right) \right\},
\]
where the coefficient \( \tilde{\chi} \) has been introduced for convenience. Using the equations of motion (48b), the Lorentz-violating contribution can be combined with the coupling term between the electromagnetic fields and the spatial spin vector:
\[
-(E + v \times B) \cdot s + \frac{m_\psi}{q} \tilde{\chi} \tilde{v} \cdot s = -(1 - \tilde{\chi}) |E + v \times B| \cdot s.
\]
For a vanishing electric field, \( E = 0 \), the spin evolution equations then give
\[
\frac{d}{dt} \left( s^0 s \right) = \frac{q}{m_\psi} \left\{ \frac{g}{2} \left( \frac{0}{s \times B} \right) + \left( 1 - \frac{g}{2} \right) (1 - \tilde{\chi}) |v \times B| \cdot s \left( \frac{1}{v} \right) \right\}.
\]
To make a physical prediction, a particular Ansatz for the spin four-vector is inserted, which was introduced in [53]:
\[
(s^0) = \sqrt{-s^2} (\hat{e}_l \cos \phi + \hat{e}_t \sin \phi), \quad \hat{e}_l = \left( \frac{v}{\tilde{v}} \right), \quad \hat{e}_t = \left( \frac{0}{\tilde{n}} \right), \quad \tilde{v} \equiv \frac{v}{|v|}, \quad |\tilde{n}| = 1.
\]
The Ansatz is chosen such that \( s^2 = 1 \) and \( s \cdot u = 0 \). Both \( \hat{e}_l \) and \( \hat{e}_t \) are normalized four-vectors where the additional \( \gamma \)-factor before \( \hat{e}_l \) has been omitted, since the nonrelativistic regime is considered. The spatial part of the first vector is chosen to point along the particle velocity where the spatial part of the second vector is assumed to be perpendicular to the velocity, i.e., \( \tilde{n} \cdot \tilde{v} = 0 \). Hence \( s^0 \) is decomposed into a longitudinal and a transverse part. Since \( s^2 \) is assumed to be constant, a change of the angle \( \phi \) describes how the longitudinal part is transformed into the transverse part and vice versa. Taking into account that \( v \) does not change in the presence of a magnetic field, one obtains by inserting Eq. (53) in Eq. (52):
\[
\Omega \equiv \dot{\phi} = \frac{q}{m_\psi} \left( \frac{g}{2} - 1 \right) (1 - \tilde{\chi}) \tilde{v} \cdot (B \times \tilde{n}).
\]
The structure of the latter result corresponds to Eq. (9) in [53] for a vanishing electric field, but the global prefactor is modified by Lorentz violation. This alters the rate at which the transverse spin component is transformed into a longitudinal one (and vice versa). The modification is indirectly caused by the modified particle kinematics where the spin itself does not couple directly to the Lorentz-violating background field.

Note that the treatment of the particle spin is different compared to, e.g., in [29, 57]. In the latter references the time evolution of the spin expectation value was obtained from the expectation value of
the commutator of the spin operator and the Lorentz-violating Hamiltonian. A Larmor-like precession of the particle spin is then induced by the birefringent Lorentz-violating coefficients, which are subsets of \( \hat{b}^\mu, \hat{d}^\mu, \hat{H}^{\mu\nu}, \) and \( \hat{g}^{\mu\nu} \). This behavior is reminiscent of the standard case when spin precession occurs for the valence electron of a hydrogen atom in an external magnetic field accompanied by a splitting of its energy levels. Therefore, if all birefringent coefficients vanish, no spin precession is expected to occur due to Lorentz violation, which is exactly what is also observed within the treatment of spin in the current paper.

5. Discussion and outlook

In this article the properties of a generic isotropic dispersion relation of the SME fermion sector were on the focus. The corresponding classical, relativistic Lagrangian was determined and it was promoted to a Finsler structure. It was shown that the associated Finsler space is Riemannian.

The classical particle was then assigned an electric charge and it was coupled to an external electromagnetic field. By doing so, the equations of motion were determined and solved for particular cases. The resulting particle trajectories were shown to be very similar to the standard ones with the difference that some quantities are scaled due to the presence of the isotropic Lorentz-violating background field.

Subsequently the goal was to understand the behavior of the particle spin. Since spin is a quantum theoretical concept, for the classical particle it had to be introduced by hand. Its time evolution was derived by considering a modified version of the BMT equation. The result is that for a nonvanishing magnetic field the rate is modified at which the transverse component is transferred to the longitudinal one and vice versa. However within our approach Lorentz violation does not have any influence on spin precession in the magnetic field. A modification is expected to occur for a birefringent Lorentz-violating theory exhibiting dispersion relations depending on the spin projection. However those dispersion relations were not considered here.

The paper shows that classical calculations within an isotropic, fermionic framework are feasible, which supports to consider isotropic models at first before delving into more complicated frameworks based on the b-structure, for example [48, 49]. A reasonable conclusion is to associate the properties of the modified physics, e.g., scaled particle trajectories in electromagnetic fields and a scaling of the transition rate between transverse and longitudinal spin components, with Euclidean Finsler structures. A next step might be to promote the flat intrinsic metric \( r_{\mu\nu} \) to a curved metric \( g_{\mu\nu}(x) \) and the constant coefficient \( \Upsilon \) to a spacetime-dependent function \( \Upsilon(x) \). Studying the particle trajectories in such a spacetime may be a further step towards a better understanding of Lorentz violation in the context of gravity.

\(^{3}\)Possible isotropic subspaces, e.g., of the b-structure may be treatable on the same level of complexity, though.
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