Online Learning in Fisher Markets: Static Pricing Limits and Adaptive Enhancements

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August 2022

Abstract

In a Fisher market, agents (users) spend a budget of (artificial) currency to buy goods that maximize their utilities while a central planner sets prices on capacity-constrained goods such that the market clears. However, the efficacy of pricing schemes in achieving an equilibrium outcome in Fisher markets typically relies on complete knowledge of users’ budgets and utility functions and requires that transactions happen in a static market wherein all users are present simultaneously.

Motivated by these practical considerations, in this work, we study an online variant of Fisher markets, wherein budget-constrained users with privately known utility and budget parameters, drawn i.i.d. from a distribution $D$, enter the market sequentially. In this setting, we develop an algorithm that adjusts prices solely based on observations of user consumption, i.e., revealed preference feedback, and achieves a regret and capacity violation of $O(\sqrt{n})$, where $n$ is the number of users and the good capacities scale as $O(n)$. Here, our regret measure is the optimality gap in the objective of the Eisenberg-Gale program between an online algorithm and an offline oracle with complete information on users’ budgets and utilities. To establish the efficacy of our approach, we show that any uniform (static) pricing algorithm, including one that sets expected equilibrium prices with complete knowledge of the distribution $D$, cannot achieve both a regret and constraint violation of less than $\Omega(\sqrt{n})$. While our revealed preference algorithm requires no knowledge of the distribution $D$, we show that if $D$ is known, then an adaptive variant of expected equilibrium pricing achieves $O(\log(n))$ regret and constant capacity violation for discrete distributions. Finally, we present numerical experiments to demonstrate the performance of our revealed preference algorithm relative to several benchmarks.

1 Introduction

In a Fisher market, one of the most fundamental models for resource allocation [1], agents (users) spend a budget of (artificial) currency to purchase goods that maximize their utilities while a central planner sets prices on capacity-constrained goods. Since Fisher introduced his framework, a focal point of the Fisher market literature has been in developing methods to compute market equilibria. Most notably, in a seminal work, Eisenberg and Gale [2] developed a convex program that maximizes the (weighted) Nash social welfare objective, i.e., the (weighted) geometric mean of all users’ utilities [4, 5], to compute equilibrium prices and the corresponding allocations for a broad range of utility functions. Despite the many desirable properties of the Eisenberg-Gale program [3, 6], including it being polynomial time solvable [7, 8], computing equilibrium prices via a centralized optimization problem relies on complete information on users’ utilities and budgets, which are typically unavailable in practice.

As a result, there has been a growing interest in developing distributed approaches for market equilibrium computation, e.g., tatonnement [9, 10], proportional response [11, 12], primal-dual [13], alternating direction

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1 A market equilibrium corresponds to a (uniform) price vector (for all users) and an allocation of goods to users such that all goods are sold, and users obtain a utility-maximizing bundle of goods that is affordable under the set prices. We refer to Section 3.1 for a formal definition of an equilibrium price vector.

2 In the special that all buyers have the same budgets, the market equilibrium outcome maximizes the (unweighted) Nash social welfare objective and is known as a competitive equilibrium from equal incomes (CEEI) [3].

3 We refer to Section 3.2 for a detailed discussion of the properties of the optimal allocations of the Eisenberg-Gale program.
method [14], and auction-based [15, 16] approaches. In the distributed setting, the central planner typically updates the prices of the goods in the market until convergence to market equilibria (under certain, often mild, conditions). To enable the convergence of these algorithms to equilibrium prices, existing distributed approaches for Fisher markets, e.g., tatonnement or proportional response approaches, typically involve a simulated setting wherein all users repeatedly interact in a static market. In practice, however, users generally do not repeatedly interact in the market to enable the central planner to learn equilibrium prices and instead tend to arrive into the market sequentially, as users are often not all present at once.

Motivated by the aforementioned practical limitations of centralized and distributed approaches for Fisher markets, in this work, we study a generalization of Fisher markets to the setting of online user arrival wherein users with privately known utilities and budgets arrive sequentially. As opposed to centralized allocation methods, we develop a novel algorithmic approach to adjust prices in the market based solely on users’ revealed preferences, i.e., observations of the bundle of goods users purchase given the set prices. Furthermore, in contrast to traditional distributed approaches for Fisher markets that involve a simulated setting with repeated user interactions, our work considers a real market setting wherein users arrive sequentially over time rather than repeatedly interacting in the market. In the online and incomplete information setting studied in this work, we also establish the performance limitations of uniform (static) pricing, which has been the holy grail in the equilibrium computation literature for Fisher markets. As a result, our work points toward developing novel tools and methods to more deeply understand market equilibria in online variants of Fisher markets and, more generally, highlights the benefit of designing adaptive mechanisms for online resource allocation [17].

1.1 Our Contributions

In this work, we study an online variant of Fisher markets wherein budget-constrained users, with privately known utility and budget parameters, arrive into the market sequentially. In particular, we focus on the setting when users have linear utilities and their budget and utility parameters are independently and identically (i.i.d.) distributed according to some probability distribution $D$. Since traditional methods, e.g., solving the Eisenberg-Gale convex program [2], are not amenable to computing equilibria in this setting, we consider the problem of learning prices online to minimize two performance measures: regret and constraint violation. In this work, regret refers to the optimality gap in the objective of the Eisenberg-Gale convex program between the online allocation and that of an offline oracle with complete information on users’ budget and utility parameters, and constraint violation represents the norm of the excess demand for goods beyond their capacity. For a detailed discussion on these performance measures, we refer to Section 3.

In this online incomplete information setting, we first study the performance limitations of uniform (static) pricing algorithms, wherein the same price vector $p$ applies to all users. For any static pricing algorithm, we establish that its expected regret or constraint violation must be $\Omega(\sqrt{n})$, where $n$ is the number of arriving users and the capacities of the goods scale as $O(n)$. As an immediate consequence of this result, even an algorithm that sets expected equilibrium prices with complete knowledge of the distribution $D$ must have a regret or constraint violation of $\Omega(\sqrt{n})$, which thus serves as a performance benchmark for an algorithm for online Fisher markets.

The performance limitations of static pricing algorithms motivate the design of adaptive (dynamic) pricing algorithms for online Fisher markets. To this end, we first present an adaptive variant of an expected equilibrium pricing approach that achieves an $O(\log(n))$ regret and a constant constraint violation, i.e., independent of the number of users $n$, when the probability distribution $D$ is discrete.

Since the probability distribution $D$, in general, may not be known (and the distribution may be continuous rather than discrete), we develop a simple yet effective approach to set prices that only relies on users’ revealed preferences, i.e., past observations of user consumption. This algorithmic approach not only preserves user privacy as it requires no information on users’ utility and budget parameters but also has a computationally efficient price update step, making it practically implementable. Furthermore, the revealed preference algorithm achieves an expected regret and constraint violation of $O(\sqrt{n})$ when the good capacities scale with the number of users. That is, our revealed preference algorithm achieves an expected regret and

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4An equilibrium price vector in classical Fisher markets is by definition a single (uniform) price vector that applies to all users in the market.
constraint violation (up to constants) that are no more than that of a static expected equilibrium pricing approach (and that of any static pricing algorithm) that has complete knowledge of the distribution $D$.

Finally, we evaluate the performance of our revealed preference algorithm through numerical experiments. Our results validate our theoretical guarantees and highlight a fundamental trade-off between regret and constraint violation, which we observe when comparing our algorithm’s performance to two benchmarks with additional information on users’ utility and budget parameters.

Organization: This paper is organized as follows. Section 2 reviews related literature. We then present our model and performance measures to evaluate the efficacy of an algorithm for online Fisher markets in Section 3. Then, in Section 4, we study the performance limitations of static pricing algorithms and develop an adaptive pricing algorithm that outperforms its static pricing counterpart. Next, we introduce our revealed preference algorithm and bounds on its regret and constraint violation in Section 5 and evaluate its efficacy through numerical experiments in Section 6. Finally, we conclude the paper and provide directions for future work in Section 7.

In the appendix, we provide proofs omitted from the main text and discuss additional numerical results to further demonstrate the efficacy of the algorithms developed in this work.

2 Literature Review

Online resource allocation problems have been widely studied in operations research and computer science, having found applications in domains including online advertising [18], allocation of food donations to food banks [19], and rationing of social goods [20] among others. One of the most well-studied classes of online allocation problems is online linear programming (OLP), wherein columns of the constraint matrix and coefficients of the linear objective are revealed sequentially to an algorithm designer. While the traditional approach to OLP problems has been to develop guarantees for adversarial inputs [21, 22], the often exceedingly pessimistic worst-case guarantees have prompted the study of beyond worst-case methods [23, 24] for such problems [25, 26].

Beyond worst-case approaches for OLP problems have largely focused on designing algorithms under (i) the random permutation and (ii) the stochastic input models. In the random permutation model, the constraints and objective coefficients arrive according to a random permutation of an adversarially chosen input sequence. In this context, [27, 28] develop a two-phase algorithm, which includes training the model on a small fraction of the input sequence and then using the learned parameters to make online decisions on the remaining input sequence. On the other hand, in the stochastic input model, the input sequence is drawn i.i.d. from some potentially unknown probability distribution. In this setting, [29] investigate the convergence of the dual price vector and design algorithms using LP duality to obtain logarithmic regret bounds. Since the algorithms developed in [29] involve solving an LP at specified intervals, [30] developed a gradient descent-based algorithm wherein the dual prices are adjusted solely based on the allocation to users at each time step. More recently, [31] devised an adaptive allocation algorithm with a constant regret bound when the samples are drawn from a discrete probability distribution. As with some of the above works, we develop algorithms for online Fisher markets under the stochastic input model; however, in contrast to these works that assume a linear objective, we develop regret guarantees for a non-linear concave objective function.

Since non-linear objectives tend to arise in many online resource allocation problems, there has been a growing interest in studying online convex optimization (OCO) [32]. In this context, [33] study a general class of OCO problems with concave objectives and convex constraints and develop dual-based algorithms with near-optimal regret guarantees under both the random permutation and stochastic input models. More recently, [34] developed an online mirror descent algorithm with sub-linear regret guarantees by ensuring that enough resources are remaining at the end of the time horizon. The mirror descent approach of [34] was later extended by [35] to a class of regularized objective functions. In line with these works, we also study an online resource allocation problem with a concave objective, which, in the Fisher market context, is the budget-weighted log utility objective [2], i.e., the sum of the logarithm of the utilities of all users weighted by their budgets. However, unlike these works that assume non-negativity and boundedness of the concave objective, we make no such assumption since the Fisher social welfare objective we consider involves
a logarithm, which can be both unbounded and non-negative.

As in our work that studies an online variant of Fisher markets, several other online variants of Fisher markets have also been considered in the literature. For instance, [36, 37, 38] study the setting when goods arrive sequentially and must be allocated irrevocably upon their arrival to a fixed set of users. While [36, 37] investigate this repeated allocation setting in Fisher markets from a competitive analysis perspective, [39] studies this problem from a regret analysis lens and establishes a connection between this setting and first-price auctions. In contrast to these works that focus on the repeated allocation setting wherein goods arrive online, we consider the setting wherein users enter the market sequentially and purchase a fixed set of limited resources.

In the context of online user arrival in Fisher markets, [40] studies the problem of allocating a fixed set of resources to a random number of users that arrive over multiple rounds. While we also consider the setting of online user arrival in Fisher markets, our work differs from that of [40] in several ways. First, [40] consider a setting where users belong to a finite set of types where all users have the same budgets. However, in this work, users’ preferences can, in general, be drawn from a continuous probability distribution, i.e., the number of user types may not be finite, and users’ budgets may not be equal. Next, we introduce and adopt different metrics to evaluate the performance of an algorithm as compared to that considered in [40], and refer to Section 3 for a detailed discussion of our chosen performance measures. Finally, as opposed to the two-phase algorithms that include a prediction and an optimization step proposed in [40], we develop a dual-based algorithm that adjusts prices solely based on users’ revealed preferences.

Our algorithmic approach to adjusting the prices of the goods using users’ revealed preferences, i.e., observed user consumption information, is analogous to price update mechanisms that use information from interactions with earlier buyers to inform pricing decisions for future arriving buyers [41]. While our dual-based price update mechanism is akin to those used in prior work on revealed preferences [42, 43, 44], our work considers a setting with budget-constrained users, unlike the quasi-linear utility setting studied in these works. Prior literature on revealed preference has also considered the setting of budget-constrained users [45, 46, 47, 48], as in this work. However, these works focus on the problem of learning the budgets and valuation functions of users that rationalize their observed buying behavior rather than designing algorithms with good performance using their revealed preferences, which is one of the main focuses of this work.

Finally, our work is also closely related to the line of literature on the design and analysis of artificial currency mechanisms [49, 50]. Such mechanisms have found applications in various resource allocation settings, including the allocation of food to food banks [51], the allocation of students to courses [52], and the allocation of public goods to people [53]. Mechanisms that involve artificial currencies have also been designed for repeated allocation settings [54], as is the main focus of this paper. However, unlike [54] that studies the repeated allocation of goods that arrive online, we investigate the setting of online user arrival.

3 Model and Problem Formulation

In this section, we introduce our modeling assumptions and the individual optimization problem of users (Section 3.1), present the Eisenberg-Gale convex program used to compute equilibrium prices (Section 3.2), and introduce the performance measures used to evaluate the efficacy of an online algorithm (Section 3.3).

3.1 Preliminaries and Individual Optimization Problem

We study the problem of allocating $m$ divisible goods to a population of $n$ users that arrive sequentially over time. Each good $j \in [m]$ has a capacity $c_j = nd_j$, where we denote $\mathbf{c} \in \mathbb{R}^m$ as the vector of good capacities, $\mathbf{d} \in \mathbb{R}^m$ as the vector of good capacities per user, and the set $[m] = \{1, \ldots, m\}$. Each user $t \in [n]$ has a budget $w_t$ of (artificial) currency, and to model users’ preferences over the goods, we assume that each user’s utility is linear in their allocations. In particular, for a vector of allocations $\mathbf{x}_t \in \mathbb{R}^m$, where $x_{tj}$ represents the consumption of good $j$ by user $t$, the utility function $u_t(\mathbf{x}_t) : \mathbb{R}^m \to \mathbb{R}$ is given by $u_t(\mathbf{x}_t) = u^t_1 x_{t1} + \sum_{j=2}^{m} u^t_j x_{tj}$, where $\mathbf{u}_t \in \mathbb{R}^m$ is a vector of utility coefficients and $u^t_j$ is the utility received by user $t$ for consuming one unit of good $j$. Then, for a given price vector $\mathbf{p} \in \mathbb{R}^m$, the individual
optimization problem for user $t$ can be described as

$$\max_{x_t \in \mathbb{R}^m} \ u_t^\top x_t = \sum_{j=1}^{m} u_{ij} x_{ij},$$  \hspace{1cm} (1a)$$

subject to:

$$p^\top x_t \leq w_t,$$  \hspace{1cm} (1b)$$

$$x_t \geq 0,$$  \hspace{1cm} (1c)$$

where (1b) is a budget constraint, and (1c) are non-negativity constraints. Since each user’s utility function is linear, a property of the optimal solution of Problem (1a)-(1c) is that given a price vector $p$, users will only purchase goods that maximize their bang-per-buck. In other words, user $t$ will purchase an affordable bundle of goods in the set $S^*_t(p) = \{j : j \in \arg \max_{x \in [0, \infty]^m} \frac{u_{ij}}{p_j}\}$.

The prices of the goods in the market that users best respond to through the solution of Problem (1a)-(1c) are set by a central planner whose goal is to set equilibrium prices, defined as follows.

**Definition 1.** (Equilibrium Price Vector) A price vector $p^* \in \mathbb{R}_{\geq 0}^m$ is an equilibrium price if there are allocations $x_t^*(p^*) \in \mathbb{R}^m$ for each user $t \in [n]$ such that:

1. The allocation $x_t^*(p^*) \in \mathbb{R}^m$ is a solution of Problem (1a)-(1c) for all users $t \in [n]$ given the price vector $p^*$;

2. The prices of all goods are non-negative and the demand for all goods is no more than their capacity, i.e., $p_j^* \geq 0$ and $\sum_{t=1}^{n} x_{ij}^*(p^*) \leq c_j$ for all goods $j \in [m]$;

3. If the price of a good $j$ is strictly positive, then the total demand for that good is equal to its capacity, i.e., if $p_j^* > 0$ for some good $j \in [m]$, then $\sum_{t=1}^{n} x_{ij}^*(p^*) = c_j$.

The computation of equilibrium prices has been a holy grail in the Fisher market literature as it corresponds to one uniform price vector for all users at which the market clears. In classical (offline) Fisher markets, several methods, such as the Eisenberg-Gale convex program, to compute market equilibria have been developed. However, these approaches assume that the central planner has complete knowledge of users’ budget and utility parameters. For a detailed discussion on offline Fisher markets and the Eisenberg-Gale convex program, we refer to Section 3.2.

Since information on users’ utility and budget parameters are typically not known, and, in real markets, users tend to arrive sequentially over time, we study the online user arrival setting with incomplete information. In this context, we assume that users arrive sequentially with budget and utility parameters drawn i.i.d. from some unknown distribution $D$ with bounded and non-negative support. That is, the budget and utility parameters $(w_t, u_t) \overset{i.i.d.}{\sim} D$ for each user $t$, where the budget $w_t \in [\bar{w}, \tilde{w}]$ for some $\bar{w} > 0$ and the utility vector $u_t \in [\bar{u}, \tilde{u}]$ for some $\bar{u} \geq 0$. In line with the Fisher market literature in the offline setting, we make the following assumption on the distribution $D$.

**Assumption 1** (All Goods have Potential Buyers). The distribution $D$ is such that for each good $j \in [m]$, there exists a positive probability that users have a strictly positive utility for that good, i.e., $\mathbb{P}(w, u : u_j > 0) > 0$.

Assumption 1 is akin to analogous assumptions in classical Fisher markets that require each good to have a potential buyer to guarantee the existence of equilibria [15]. In particular, [15] shows that if each good has a potential buyer and users have linear utilities, then equilibrium prices exist and are positive. Assumption 1 is mild since if no proportion of users had a positive utility for certain goods, those goods can be removed from the market as no user prefers to purchase them.

A few comments about our modeling assumptions are in order. First, we assume that each agent’s utility function is linear in their allocations, a commonly used and well-studied utility function in classical Fisher markets [2, 55, 15]. Next, as in the Fisher market literature, we assume that the goods are divisible, and thus fractional allocations are possible. Furthermore, in the online setting, we study the arrival of users under the stochastic input model, which has been widely studied in the online linear programming [29, 30] and the online convex optimization [34] literature.
In the offline setting, when complete information on the budgets and utilities of all users is known, the central planner can compute equilibrium prices through the dual variables of the capacity constraints of the following Eisenberg-Gale convex program [2]

\[
\max_{x_t \in \mathbb{R}^m, \forall t \in [n]} U(x_1, ..., x_n) = \sum_{t=1}^n w_t \log \left( \sum_{j=1}^m u_{tj} x_{tj} \right),
\]

subject to

\[
\sum_{t=1}^n x_{tj} \leq c_j, \quad \forall j \in [m],
\]

\[
x_{tj} \geq 0, \quad \forall t \in [n], j \in [m],
\]

where (2b) are capacity constraints, (2c) are non-negativity constraints, and the Objective (2a) represents a budget-weighted geometric mean of buyer’s utilities and is closely related to the Nash social welfare objective [5, 56]. If the prices in the market are set based on the dual variables of the capacity Constraints (2b), then the optimal allocations of each user’s individual optimization Problem (1a)-(1c) can be shown to be equal to that of the social optimization Problem (2a)-(2c) [2]. That is, the dual variables of the capacity Constraints (2b) correspond to equilibrium prices.

**Properties of the Eisenberg-Gale Convex Program and its Optimal Allocations:** The Eisenberg-Gale convex program has several computational advantages that make it practically feasible. In particular, the computational complexity of solving Problem (2a)-(2c) is identical to that of a linear program [57], i.e., it can be solved in polynomial time [7, 8] with the same complexity as that of maximizing a linear objective function. Furthermore, in the special case that the utility coefficients, budgets, and good capacities are all rational, the optimal solution of Problem (2a)-(2c) is also rational [58], i.e., the solution to Problem (2a)-(2c) can be exactly computed.

In addition to its computational advantages, the weighted geometric mean objective has several desirable properties compared to optimizing other social welfare objectives. First, the weighted geometric mean objective results in an allocation that satisfies both Pareto efficiency, i.e., no user can be made better off without making another user worse off, and envy-freeness, i.e., each user prefers their allocation compared to that of other users. However, other social welfare objectives often only satisfy one of these desirable properties, e.g., the utilitarian welfare (weighted sum of user’s utilities) and egalitarian welfare (maximizing the minimum utility) objectives only achieve Pareto efficiency. Next, maximizing the weighted geometric mean of users’ utilities achieves a natural compromise between the utilitarian and egalitarian objectives [56], thereby resulting in a simultaneously efficient and fair allocation. In particular, compared to maximizing utilitarian welfare, which may result in unfair allocations [59] since some users may obtain zero utilities, under Objective (2a) all users receive a strictly positive utility. Furthermore, compared to the egalitarian objective that may result in highly inefficient outcomes, optimizing the geometric mean of users’ utilities is more robust since it provides a lower bound on the utilitarian welfare.

### 3.3 Algorithm Design and Performance Measures in Online Setting

While the offline allocations corresponding to the Eisenberg-Gale program have several desirable properties, achieving such allocations is generally not possible in the online setting when the central planner does not have access to information on users’ utility and budget parameters. As a result, we focus on devising algorithms that achieve good performance relative to an offline oracle with complete information on users’ utilities and budgets. In particular, we evaluate the efficacy of an online allocation policy through two metrics: (i) expected regret, i.e., the optimality gap in the social welfare Objective (2a) of this allocation policy relative to the optimal offline allocation, and (ii) expected constraint violation, i.e., the degree to which the goods are over-consumed relative to their capacities. Here the expectation is taken with respect to the distribution \( D \) from which users’ budget and utility parameters are drawn. In this section, we first present the class of online policies (algorithms) we focus on in this work (Section 3.3.1) and then formally define the regret and constraint violation metrics to evaluate the performance of these online algorithms (Section 3.3.2).
3.3.1 Algorithm Design in Online Fisher Markets

In online Fisher markets, a central planner needs to make an allocation \( x_t \) (or a pricing decision \( p^t \)) instantaneously upon the arrival of each user \( t \in [n] \). The allocation or pricing decisions made by the central planner depend on the information set \( I_t \) available to it at the time of arrival of each user. Some examples of the information sets \( I_t \) include the distribution \( D \) from which users’ budget and utility parameters are drawn, the history of past user allocations, i.e., \( \{x_{t'}\}_{t'=1}^{t-1} \), and the history of budget and utility parameters of users that have arrived prior to user \( t \), i.e., \( \{w_{t'}, u_{t'}\}_{t'=1}^{t-1} \). Under a given information set \( I_t \), the allocations \( x_t \) (or pricing decisions \( p^t \)) for each user are specified by a policy \( \pi^t = (\pi_1, \ldots, \pi_n) \) (or \( \pi^p = (\pi_1, \ldots, \pi_n) \)), where \( x_t = \pi^A(I_t) \) (or \( p^t = \pi^P(I_t) \)). Here the superscript “\( A \)” refers to an allocation-based policy, while the superscript “\( P \)” refers to a pricing-based policy. Note that when the central planner makes pricing decisions \( p^t = \pi^P(I_t) \) for each user, the corresponding allocations \( x_t \) are given by the optimal solution to Problem (1a)-(1c) given the price vector \( p^t \). For the remainder of this work, since we focus on designing pricing policies, we drop the superscript in the notation for conciseness and use \( \pi \) to refer to a pricing policy. However, we do mention that our pricing policies have corresponding allocation-based analogues, as elucidated for one of our designed algorithms (Algorithm 1) in Section 4.2, and that both pricing and allocation policies are closely connected to each other by duality.

While a range of information sets \( I_t \) have been investigated in the study of online resource allocation [29, 28], of particular interest in this work is the privacy-preserving setting where each user’s parameters are private information, known only to the users. As a result, we focus on designing a revealed preference dual-based algorithm (see Section 5) that adjust the prices of the goods in the market solely based on past user consumption data, which is directly observable to the central planner. In particular, we focus on devising a pricing policy \( \pi = (\pi_1, \ldots, \pi_n) \) that sets a sequence of prices \( p^1, \ldots, p^t \) such that \( p^t = \pi_t(I_t) \), where \( x_t \) is an optimal consumption vector given by the solution of Problem (1a)-(1c) for user \( t \) given the price vector \( p^t \). We note that our focus on designing a dual-based algorithm in Section 5 that only relies on users’ revealed preferences is in contrast to traditional primal algorithms that require information on the attributes of users that have previously arrived in the system to make subsequent allocation decisions [30].

3.3.2 Performance Measures

We now detail the regret and constraint violation performance measures used to evaluate the efficacy of an online pricing policy.

**Regret:** We evaluate the regret of any online algorithm (pricing policy) \( \pi \) through the difference between the optimal objective of Problem (2a)-(2c) and that corresponding to the allocations resulting from the online pricing policy \( \pi \). For a given set of utility and budget parameters for all users \( t \in [n] \), let \( U_n \) denote the optimal Objective (2a), i.e., \( U_n(x_1^*, \ldots, x_n^*) \), where \( x_1^*, \ldots, x_n^* \) are the optimal offline allocations corresponding to the solution of Problem (2a)-(2c). Further, let \( x_{t} \) be the solution to the individual optimization Problem (1a)-(1c) given the price vector \( p^t \) corresponding to the policy \( \pi \) for each user \( t \in [n] \). Then, the social welfare Objective (2a) obtained by the policy \( \pi \) is \( U_n(\pi) = U(x_1, \ldots, x_n) \), and the corresponding expected regret of an algorithm \( \pi \) is given by

\[
R_n(\pi) = E_D[U_n - U_n(\pi)],
\]

where the expectation is taken with respect to the distribution \( D \) from which the budget and utility parameters are drawn. In the rest of this work, with a slight abuse of notation, we drop the subscript \( D \) in the expectation and assume all expectations are with respect to \( D \), unless stated otherwise.

While the regret measure is defined with respect to the objective of the social optimization Problem (2a)-(2c), we note that regret guarantees derived for Objective (2a) directly translate into corresponding guarantees for the Nash social welfare objective, defined as \( \text{NSW}(x_1, \ldots, x_n) = \left( \prod_{t=1}^n u_t(x_t^*) \right)^{\frac{1}{n}} \). In particular, if the regret of an algorithm \( \pi \) is \( o(n) \), then the ratio of the Nash social welfare objective of the algorithm \( \pi \) approaches that of the optimal offline oracle as \( n \) becomes large, i.e., if \( U_n - U_n(\pi) \leq o(n) \) for some algorithm \( \pi \), then \( \frac{\text{NSW}(x_1^*, \ldots, x_n^*)}{\text{NSW}(x_1, \ldots, x_n)} \to 1 \) as \( n \to \infty \). We present a detailed discussion of this fundamental connection between the above-defined regret measure, which applies to Objective (2a), and the ratio between
the Nash social welfare objective of the optimal offline oracle and that corresponding to an online algorithm in Appendix A.

**Constraint Violation:** We evaluate the constraint violation of algorithm through the norm of the expected over-consumption of the goods beyond their capacity. In particular, for the consumption bundles $x_1, \ldots, x_n$ corresponding to the pricing policy $\pi$, the vector of excess demands is given by $v(x_1, \ldots, x_n) = (\sum_{t=1}^{n} x_t - c)_+$, and the corresponding expected norm of the constraint violation is

$$V_n(x_1, \ldots, x_n) = \mathbb{E}[\|v(x_1, \ldots, x_n)\|_2].$$

A few comments about the above regret and constraint violation metrics are in order. First, we reiterate that we define our regret metric based on the budget-weighted geometric mean Objective (2a). Our choice of this objective, as opposed to, for example, a utilitarian welfare one, stems from the superior properties of the allocations corresponding to the Eisenberg-Gale convex program elucidated in Section 3.2. Next, observe that the budget-weighted geometric mean Objective (2a) is nonlinear and unbounded. As a result, our regret metric differs from that considered in the online linear programming and online convex optimization literature that either assumes a linear or a concave objective that is bounded and non-negative. Finally, while we defined our constraint violation metric with the $L_2$ norm, by norm-equivalence, any constraint violation guarantees obtained with the $L_2$ norm can be extended to any $p$-norm, e.g., the $L_\infty$ norm.

In this work, we jointly optimize for regret and constraint violation, as in [29, 44], for two primary reasons. First, designing strictly feasible algorithms in online Fisher markets can result in infinite regret due to the logarithmic objective, as all the resources may be exhausted before the arrival of some users who would receive zero utility. Second, we optimize for both regret and constraint violation metrics since achieving good performance on either is typically easy. In particular, setting the prices of all goods to be very low will result in low regret but potentially lead to constraint violations since users will purchase large quantities of goods at lower prices. On the other hand, setting exceedingly large prices will have the opposite effect, in that the constraint violations will be low, but the regret is likely to be high. Due to this fundamental trade-off between regret and constraint violation [29], we focus on optimizing both metrics in this work.

### 4 Static Pricing Limits and Adaptive Pricing Enhancements

In classical Fisher markets, the central planner determines one uniform price vector that applies to all users, i.e., an equilibrium price vector, which can be computed through the solution of the Eisenberg-Gale convex program. As a result, we begin our study of online Fisher markets by establishing the performance limitations of static pricing in this online incomplete information setting. In particular, we show that the expected regret or constraint violation of any static pricing algorithm that sets prices $p_t = p_{t'}$ for all $t, t' \in [n]$ must be $\Omega(\sqrt{n})$ in the online incomplete information setting.

Theorem 1. Suppose that users’ budget and utility parameters are drawn i.i.d. from a distribution $D$. Then, there exists a market instance for which either the expected regret or expected constraint violation of any static pricing algorithm is $\Omega(\sqrt{n})$, where $n$ is the number of arriving users.
Proof (Sketch). To prove this claim, we consider a setting with \( n \) users (with a budget of one for all users) and two goods, each with a capacity of \( n \). Furthermore, let the utility parameters of users be drawn i.i.d. from a probability distribution, where the users have utility \((1, 0)\) and a utility of \((0, 1)\), each with probability 0.5. For this instance, to derive a lower bound on the regret, we first lower bound the expected optimal social welfare objective, i.e., Objective (2a), which we obtain by utilizing the property that users’ utility distribution is binomial. Next, to prove the desired lower bound on the expected regret and constraint violation, we consider two cases: (i) the price of either of the two goods is at most 0.5, and (ii) the price of both goods is strictly greater than 0.5. In the first case, we use the central limit theorem to establish that the expected constraint violation is \( \Omega(\sqrt{n}) \). In the second case, we show that either the expected constraint violation or the expected regret is \( \Omega(\sqrt{n}) \) utilizing both the central limit theorem and the obtained lower bound on the expected optimal social welfare objective, which establishes our claim.

For a complete proof of Theorem 1, see Appendix B. Theorem 1 establishes the limitations of static pricing algorithms in online Fisher markets and is in stark contrast to the efficacy of equilibrium pricing, wherein one uniform price vector is set for all users, in classical Fisher markets. This result points toward developing novel methods for understanding market equilibria in online Fisher markets and augments the literature in online resource allocation where static pricing or allocation approaches have limited performance [17].

While the lower bound result in Theorem 1 does not apply to adaptive pricing algorithms, Theorem 1 provides a benchmark for the performance of any algorithm for online Fisher markets as static pricing includes an expected equilibrium pricing approach with complete knowledge of the distribution \( D \). Since Theorem 1 establishes a lower bound on the expected regret and constraint violation of all static pricing algorithms, it, in particular, implies that even with complete information on the distribution \( D \), setting expected equilibrium prices will result in either an expected regret or constraint violation of \( \Omega(\sqrt{n}) \), as highlighted by the following corollary.

Corollary 1. Suppose that the budget and utility parameters of users are drawn i.i.d. from a distribution \( D \). Then, there exists a market instance for which the expected constraint violation of an algorithm that sets equilibrium prices based on the expected number of user arrivals, i.e., the algorithm has complete information on the distribution \( D \), is \( \Omega(\sqrt{n}) \).

We reiterate that the static expected equilibrium pricing policy \( \pi \) utilizes distributional information, and thus \( p^t = \pi_t(D) \) for all users \( t \in [n] \). As this price vector is uniform across all users, Corollary 1 follows as an immediate consequence of Theorem 1, and thus we omit its proof.

4.2 Adaptive Variant of Expected Equilibrium Pricing

Motivated by the performance limitations of static pricing, we now turn to develop adaptive pricing algorithms for online Fisher markets. To this end, in this section, we introduce an adaptive variant of expected equilibrium pricing for discrete probability distributions \( D \) (Section 4.2.1), as in the counterexample used to prove Theorem 1, and show that it achieves an \( O(\log(n)) \) regret and constant constraint violation (Section 4.2.2), thereby highlighting the benefit of adaptivity in algorithm design for online Fisher markets.

4.2.1 Adaptive Expected Equilibrium Pricing Algorithm

To present the adaptive expected equilibrium pricing algorithm, we first introduce some notation and the certainty equivalent problem used to set static equilibrium prices. In particular, we assume that the utility and budget parameters \((w, u)\) of users are drawn i.i.d. from a discrete probability distribution with known and finite support \((\tilde{w}_k, \tilde{u}_k)_{k=1}^K\), where the support size \( K \in \mathbb{N} \), and the probability of a user having budget and utility parameters \((\tilde{w}_k, \tilde{u}_k)\) is given by \( q_k \). That is, \( P((w_t, u_t) = (\tilde{w}_k, \tilde{u}_k)) = q_k \) for all \( k \in [K] \), where \( q_k \geq 0 \) and \( \sum_{k \in [K]} q_k = 1 \). Then, to set static expected equilibrium prices, we can define the following
certainty equivalent formulation of the Eisenberg-Gale program

max \quad \frac{U(z_1, \ldots, z_K)}{\text{subject to}} \quad \sum_{k=1}^{K} q_k \hat{w}_k \log \left( \sum_{j=1}^{m} \hat{u}_{kj} z_{kj} \right),

where \((3a)\) represents the objective of the Eisenberg-Gale program weighted by the probability of occurrence of the corresponding utility and budget parameters, and \((3b)\) represents the capacity constraints wherein the allocations are weighted by their corresponding probabilities. For succinctness, we denote Problem \((3a)-(3c)\) as \(CE(d)\) for a vector of average resource capacities \(d\).

Observe that the optimal dual variables of the capacity constraints of this certainty equivalent problem correspond to the static expected equilibrium prices. As an example, for the two-good counterexample used to prove Theorem 1, the distribution support size \(K = 2\), where the budget and utility parameters \((w, u)\) are \((1, (1, 0))\) and \((1, (0, 1))\), each with probability \(q_1 = q_2 = 0.5\), and the average resource capacity of each good per user is one, i.e., \(d_1 = d_2 = 1\). For these parameters, the optimal dual variables of the capacity constraints of the certainty equivalent Problem \((3a)-(3c)\) correspond to a price vector of \((0.5, 0.5)\), which are the static expected equilibrium prices for the market instance described in the proof of Theorem 1.

However, as observed in Corollary 1, an issue with static expected equilibrium prices is that it can result in large constraint violations as some goods may be consumed too early, i.e., well before the arrival of the last user, due to the stochasticity in user arrivals. To circumvent this issue, we design an adaptive variant of the aforementioned expected equilibrium pricing approach (see Algorithm 1) that increases the prices (relative to the static expected equilibrium prices) of goods that have been over-consumed and vice-versa. In particular, Algorithm 1 keeps track of the average “remaining” capacity of all goods at the time of arrival (relative to the static expected equilibrium prices) of goods that have been over-consumed and vice-versa.

Finally, since the certainty equivalent problem \(CE(d)\) is only well-defined for an average capacity vector \(d\), in Algorithm 1 we adopt two pricing mechanisms depending on the difference between the average remaining capacity \(d\) and the initial average capacity \(d_1 = \frac{s}{n}\). In particular, if the average remaining capacity \(d\) does not deviate too far from the initial average capacity \(d_1 = \frac{s}{n}\), i.e., \(d_1 \in [d - \Delta, d + \Delta]\) for some constant vector \(0 < \Delta < d\), e.g., \(\Delta = \frac{d}{2}\), then prices are set based on the dual variables of the above certainty equivalent Problem \(CE(d)\), which is well defined for any such \(d_1 \in [d - \Delta, d + \Delta]\). After the first time \(\tau\) that \(d_1 \notin [d - \Delta, d + \Delta]\), the static expected equilibrium prices are set in the market. Note that the pricing policy \(\pi\) in Algorithm 1 depends on the distribution \(D\) and the history of past allocations, i.e., \(p^t = \pi_t(D, \{x_{t'}\}_{t'=1}^{t-1})\). In response to the set prices, users consume their optimal bundle of goods given by the solution of Problem \((1a)-(1c)\).

We note that Algorithm 1 is similar in spirit to the adaptive allocation algorithm in [31]; however, in contrast to the algorithm in [31], which applies for online linear programs, our algorithm applies for online Fisher markets with a non-linear objective. As with the allocation-based algorithm in [31], there is an analogous allocation-based variant of Algorithm 1. In particular, for a remaining average good capacity \(d_t\), where \(z_k^1, \ldots, z_k^K\) are the optimal solutions the certainty equivalent problem \(CE(d_t)\), the allocation made to user \(t\) can given by \(x_t = z_t^k\) if user \(t\) has the budget and utility parameters \((\hat{w}_k, \hat{u}_k)\). Note here that \(z_t^k\) is one of the optimal consumption vectors given the price \(p^t\) for a user of type \(k \in [K]\). Furthermore, note that as compared to the adaptive expected equilibrium pricing, which only requires information on the distribution \(D\) and the history of past allocations, this allocation-based algorithm additionally requires information on the budget and utility parameters of the user for which an allocation decision needs to be made, i.e., \(x_t = \pi_t^d(D, \{x_{t'}\}_{t'=1}^{t-1}, (w_t, u_t))\).
Algorithm 1: Adaptive Expected Equilibrium Pricing

**Input**: Initial Good Capacities $c$, Number of Users $n$, Threshold Parameter Vector $\Delta$, Support of Probability Distribution $\{w_k, u_k\}_{k=1}^K$, Occurrence Probabilities $\{q_k\}_{k=1}^K$

Initialize $c_1 = c$ and the average remaining good capacity to $d_1 = \frac{c}{n}$

for $t = 1, 2, ..., n$ do
  **Phase I: Set Price**
  if $d'_t \in [d - \Delta, d + \Delta]$ for all $t' \leq t$ then
    Set price $p^t$ as the dual variables of the capacity constraints of the certainty equivalent problem $CE(d_t)$ with capacity $d_t$
  else
    Set price $p^t$ using the dual variables of the capacity constraints of the certainty equivalent problem $CE(d)$ with capacity $d = d_1$
  end

  **Phase II: Observed User Consumption and Update Available Good Capacities**
  User purchases optimal bundle $x_t$ by solving Problem (1a)-(1c) given price $p^t$
  Update the available good capacities $c_{t+1} = c_t - x_t$
  Compute the average remaining good capacities $d_{t+1} = \frac{c_{t+1}}{n-t}$

end

4.2.2 Regret and Constraint Violation Guarantee of Algorithm 1

We now show that Algorithm 1 achieves an $O(\log(n))$ regret and a constant constraint violation. To this end, we require the following assumption, which imposes a stability restriction on the change in the dual prices of the certainty equivalent problem $CE(d_t)$ for small changes in the average remaining capacities $d_t$.

**Assumption 2** (Price Stabilization). Let two average capacity vectors $\tilde{d}, d' > 0$ be such that $\|\tilde{d} - d'\|_2 \leq O(\frac{1}{n-t})$ for a given $t \in [n-1]$. Then, the optimal price vectors $\tilde{p}, p'$ of the certainty equivalent problems $CE(d)$ and $CE(d')$, respectively, satisfy $\|\tilde{p} - p'\|_2 \leq O(\frac{1}{n-t})$.

Assumption 2 implies that small changes in the average capacity vector will result in only small changes in the optimal dual prices of the certainty equivalent Problem (3a)-(3c). Such a stability assumption on the dual prices aligns with prior work on adaptive allocation algorithms [30, 31], wherein stability and uniformity assumptions are required to establish regret guarantees for an algorithm that allocates goods based on the average remaining good capacities at each step. We also numerically validate Assumption 2 through experiments in Appendix G.1.

We now establish that Algorithm 1 achieves a constant constraint violation, i.e., independent of the number of users $n$, and, under Assumption 2, an $O(\log(n))$ expected regret, as is elucidated by the following theorem.

**Theorem 2** (Regret and Constraint Violation Bounds for Algorithm 1). Suppose that the budget and utility parameters of users are drawn i.i.d. from a discrete probability distribution $\mathcal{D}$ satisfying Assumption 1, and let $\pi$ denote the online pricing policy described by Algorithm 1, with $p, \tilde{p} > 0$ as the lower and upper bounds, respectively, for the prices $p^t_j$ for all goods $j$ and for all users $t \in [n]$. Furthermore, let $x_1, x_2, ..., x_n$ be the allocations for the $n$ users, where $x_1$ is an optimal solution for that user corresponding to the certainty equivalent problem $CE(d_t)$ for $t \leq \tau$, where $\tau$ is the first time at which $d_t \notin [d - \Delta, d + \Delta]$, and $x_t$ is an optimal solution to $CE(d)$ for $t > \tau$. Then, the constraint violation $V_n(\pi) \leq O(1)$ and, under Assumption 2, the regret $R_n(\pi) \leq O(\log(n))$.

**Proof (Sketch)**. We prove this claim using three intermediate lemmas, presented in Appendix C. Our first lemma establishes a generic upper bound on the regret of an algorithm for online Fisher markets using convex programming duality. This regret upper bound is composed of two terms: (i) the first term is akin to the constraint violation of the algorithm and, in particular, accounts for the loss corresponding to over (or under-consuming) goods, and (ii) the second term accounts for the loss of setting prices that deviate far from the static expected equilibrium prices. To bound the first term of the generic regret bound (and the
constraint violation of Algorithm 1, we first note that the expected constraint violation is upper bounded by $O(\mathbb{E}[n - \tau])$, where $\tau$ is the stopping time at which the average remaining resource capacity vector $d_t \notin [d - \Delta, d + \Delta]$. Then, using concentration inequalities and arguments analogous to those in [31], we show that $O(\mathbb{E}[n - \tau])$ is a constant, establishing our desired constraint violation bound. Finally, to upper bound the second term in the generic regret bound, we apply Assumption 2 and use induction on the optimal dual prices of the certainty equivalent problem $CE(d_t)$ to bound the difference between the adaptive and static expected equilibrium prices for each user. Our analysis of the second term of the generic regret bound gives the desired $O(\log(n))$ regret upper bound, which establishes our claim.

For a complete proof of Theorem 2, see Appendix C. As mentioned in the statement of Theorem 2, we reiterate that the prices $p^*_j$ are strictly positive and bounded throughout the operation of Algorithm 1. To see this, first note that the boundedness of the optimal dual prices of the certainty equivalent problem at each step follows directly from the restriction that the vector of average remaining capacities $d_t \in [d - \Delta, d + \Delta]$ for some $\Delta < d$. In particular, the optimal dual prices of the certainty equivalent problem $CE(d_t)$ for any vector $d_t > 0$ remain bounded as long as users’ budgets and utilities are bounded. Next, the optimal dual prices of the certainty equivalent problem are guaranteed to be positive under Assumption 1 [15], i.e., as long as the distribution $D$ is such that for each good $j$, there is at least one type $k \in [K]$ with probability $q_{k} > 0$ such that the utility $u_{k,j} > 0$. In other words, the positivity of the prices during the operation of Algorithm 1 follows for any distribution $D$ where some proportion of users have a positive utility for each good.

We also note in the statement of Theorem 2 that the allocation $x_t$ is a solution to the corresponding certainty equivalent problem at each step. That is, if user $t$ is of type $k \in [K]$, then the allocation $x_t = z^*_k$ as in the case of the allocation-based analogue of Algorithm 1 presented in Section 4.2.1, where $z^*_k$ is a solution to the certainty equivalent problem $CE(d_t)$. Observe that such an allocation corresponds to one of the optimal consumption vectors for each arriving user given the price $p^*_k$, which is the dual price of that certainty equivalent problem. Furthermore, in some special cases, e.g., the counter-example in the proof of Theorem 1 where users only have utility for one good, there is only one optimal consumption vector, characterized by the solution of the certainty equivalent problem, for each user for any price vector $p^*_j > 0$. Given this observation and noting that the utility accrued by a user remains the same at all optimal consumption vectors given a price $p^*_j$, focusing on the optimal solution to the corresponding certainty equivalent problem is without loss of generality. Our purpose for doing so is that it guarantees that the expected consumption at each step is equal to the average remaining good capacity, i.e., $\mathbb{E}[x_t] = d_t$ for all $t \leq \tau$, which we require to analyse the constraint violation in the proof of Theorem 2.

Theorem 2 implies that Algorithm 1 is feasible, up to constants, with respect to the capacity constraints and achieves an $O(\log(n))$ regret, which significantly improves upon the $\Omega(\sqrt{n})$ lower bound on either the expected regret or constraint violation of any static pricing algorithm obtained in Theorem 1. As a result, Theorem 2 highlights the benefit of adaptivity in online Fisher markets and motivates the further development of adaptive pricing algorithms for this novel problem setting. To further highlight the advantages of Algorithm 1 as compared to static pricing approaches, we present numerical experiments in Appendix G.2, which show that Algorithm 1 achieves a low regret with almost no constraint violation even for large problem instances with $n = 20,000$ users.

Despite the significant advantages of Algorithm 1 as compared to static pricing approaches, it has its limitations in applications when the distribution $D$ is non-discrete and knowledge of the distribution $D$ is not readily available (in which case the certainty equivalent problem $CE(d_t)$ cannot be solved at each step). To address these concerns of Algorithm 1, in the next section, we develop a revealed preference algorithm that does not use any distributional information when making pricing decisions and is applicable for general (non-discrete) probability distributions.

5 Revealed Preference Algorithm and Regret Guarantees

In this section, we present a revealed preference algorithm for online Fisher markets and its corresponding regret and constraint violation guarantees. In particular, this algorithm solely utilizes observations of past user consumption to inform pricing decisions for future arriving users without requiring any information on users’ utility and budget parameters, thereby preserving user privacy. We further show that this algorithm
achieves a regret and constraint violation of $O(\sqrt{n})$. Note that this guarantee matches, up to constants, the corresponding lower bound on the regret and constraint violation of a static pricing algorithm with complete distributional information on the budget and utility parameters of users (see Theorem 1 and Corollary 1). To motivate the revealed preference algorithm, we first present the dual of the Eisenberg-Gale convex Program (2a)-(2c) (Section 5.1), which also plays an essential role in our upper bound regret analysis. Then, we present the revealed preference algorithm (Section 5.2), which follows from performing gradient descent on the dual of the Eisenberg-Gale convex program. Finally, in Section 5.3, we establish an upper bound on both the regret and constraint violation of the revealed preference algorithm.

5.1 Dual Formulation of Eisenberg-Gale Program

Letting the price $p_j$ be the dual variable of the capacity Constraint (2b) corresponding to good $j$, the dual of the Problem (2a)-(2c) is

$$
\min_p \sum_{t=1}^{n} w_t \log(w_t) - \sum_{t=1}^{n} w_t \log \left( \min_{j \in [m]} \frac{p_j}{u_{tj}} \right) + \sum_{j=1}^{m} p_j c_j - \sum_{t=1}^{n} w_t. \quad (4)
$$

For a derivation of the above dual using the Lagrangian of Problem (2a)-(2c), we refer to Appendix D. We note that the above dual problem is the unconstrained version of the dual problem presented in [60] with the additional terms $\sum_{t=1}^{n} w_t \log(w_t)$ and $-\sum_{t=1}^{n} w_t$ in the objective. Observe that these terms in the objective are independent of the prices and thus do not influence the optimal solution of the dual problem but are necessary to analyze the regret of the algorithm we develop.

Since users’ budget and utility parameters are drawn i.i.d. from the same distribution, this dual problem achieves a regret and constraint violation of $O(\sqrt{n})$. Note that this guarantee matches, up to constants, the corresponding lower bound on the regret and constraint violation of a static pricing algorithm with complete distributional information on the budget and utility parameters of users (see Theorem 1 and Corollary 1). To motivate the revealed preference algorithm, we first present the dual of the Eisenberg-Gale convex Program (2a)-(2c) (Section 5.1), which also plays an essential role in our upper bound regret analysis. Then, we present the revealed preference algorithm (Section 5.2), which follows from performing gradient descent on the dual of the Eisenberg-Gale convex program. Finally, in Section 5.3, we establish an upper bound on both the regret and constraint violation of the revealed preference algorithm.

5.2 Revealed Preference Algorithm

In this section, we present a revealed preference algorithm to dynamically update the prices of the goods in the market solely based on observations of user consumption. In particular, we devise a pricing policy $\pi = (\pi_1, \ldots, \pi_n)$ that sets a sequence of prices $p^1, \ldots, p^n$ such that the pricing decision at each step only depends on the observed history of user consumption at the previous steps, i.e., $p^t = \pi_t((x_{t'})_{t' \leq t-1})$, where the allocations $x_t$ are given by the optimal solutions to Problem (1a)-(1c) given the price vector $p^t$. Our algorithm adjusts the prices in the market when a user arrives based on whether the previous arriving user consumed more or less than their respective market share of each good. In particular, the price of a good $j$ is increased (decreased) if the previous arriving user consumed more (less) than the average good capacity $d_j = \frac{c_j}{n}$ units of good $j$. The prices are updated using a step size $\gamma_t$, which we specify in Section 5.3. This process of updating the prices based on the observed optimal consumption of users is presented formally in Algorithm 2.

A few comments about Algorithm 2 are in order. First, Algorithm 2 is akin to several revealed preference approaches in the literature [42, 43, 44]. However, unlike prior approaches that focus on the setting when users have quasi-linear utilities, Algorithm 2 applies in the setting when users are budget-constrained, as in the context of Fisher markets. Next, since Algorithm 2 relies on users’ revealed preferences, the price update step does not require any information on the budgets and utilities of users and thus preserves user privacy. Furthermore, Algorithm 2 is practically implementable with low computational overhead since the computational complexity of the price updates is only $O(m)$ at each time a user arrives. Note here that Phase I of Algorithm 2, wherein each arriving user solves their individual optimization problem, is a distributed step, and thus the central planner only incurs a cost when performing the price updates in Phase II. Finally,
for each user \( t \), the price update step follows from performing gradient descent on the \( t \)’th term of the dual Problem (5). In particular, if the optimal consumption set \( S^*_t \) for user \( t \), given the price vector \( p^t \), consists of one good, then the sub-gradient of the \( t \)’th term of the dual Problem (5) is given by

\[
\partial_p \left( \sum_{j=1}^{m} p_j d_j + w_t \log (w_t) - w_t \log \left( \min_{j \in [m]} \frac{p_j}{u'_{tj}} \right) - w_t \right) \bigg|_{p=p^t} = d - x_t,
\]

where \( x_t \) is an optimal bundle corresponding to the solution of the individual optimization Problem (1a)-(1c) of agent \( t \). Note here that \( x_{tj^*} = \frac{w_t}{p^*_j} \) for the good \( j^* \) in its optimal consumption set \( S^*_t \), which is of cardinality one, and \( x_{tj} = 0 \) for all goods \( j \neq j^* \).

Given the connection between gradient descent and the price updates in Algorithm 2, we note that other price update steps could also have been used in Algorithm 2 based on mirror descent. For instance, instead of adjusting the prices through an additive update, as in Algorithm 2, prices can be modified through a multiplicative update using the following widely studied [10, 34] update rule

\[
p^{t+1} \leftarrow p^t e^{-\gamma (d - x_t)}.
\]  

(6)

In Appendix G.3, we present a comparison between the regret and constraint violation of Algorithm 2 with the additive price update step and the corresponding algorithm with a multiplicative price update step through numerical experiments. For our theoretical analysis in Section 5.3, we focus on the additive price update step in Algorithm 2 and defer an exploration of the regret and constraint violation guarantees resulting from the multiplicative price update steps to future research. To this end, we do mention that this mirror descent-based multiplicative price update rule achieves \( O(\sqrt{n}) \) regret guarantees in [34] for bounded and non-negative concave utilities and believe that some of their techniques can be extended to the budget-weighted log utility objective, i.e., Objective (2a) that can be negative and is unbounded, studied in this work.

5.3 Regret and Constraint Violation Upper Bound

We now present the main result of this section, which establishes an \( O(\sqrt{n}) \) upper bound on both the expected regret and constraint violation of Algorithm 2.

**Theorem 3** (Regret and Constraint Violation Bounds for Algorithm 2). Suppose that the budget and utility parameters of users are drawn i.i.d. from a distribution \( D \) satisfying Assumption 1. Furthermore, let \( \pi \) denote the online pricing policy described by Algorithm 2, \( x_1, \ldots, x_n \) be the corresponding allocations for the \( n \) users, and suppose that the price vector \( p^t \) corresponding to Algorithm 2 is such that \( 0 < p^t \leq p^t \leq \bar{p} \) for all users \( t \in [n] \). Then, for a step size \( \gamma = \gamma_t = \frac{\bar{D}}{\sqrt{n}} \) for some constant \( \bar{D} > 0 \) for all users \( t \in [n] \), the regret \( R_n(\pi) \leq O(\sqrt{n}) \) and the constraint violation \( V_n(\pi) \leq O(\sqrt{n}) \).

Proof (Sketch). To establish this result, we proceed in three steps. First, we prove an \( O(\sqrt{n}) \) upper bound on the constraint violation. To do so, we sum the price update equation in Algorithm 2 across all users to establish that the excess demand for any good \( j \) is upper bounded by \( \frac{p^{t+1}_j}{\gamma} \), i.e., \( \sum_{t=1}^n x_{tj} - c_j \leq \frac{p^{t+1}_j}{\gamma} \). Using
this relation and the fact that the prices are upper bounded by $\bar{p}$ and the step size $\gamma = O\left(\frac{1}{\sqrt{n}}\right)$, we obtain the $O(\sqrt{n})$ upper bound on the constraint violation. Next, we derive a generic upper bound on the regret (different from that in the proof of Theorem 2) of any online algorithm $\pi$ using duality (see Section 5.1), and show that $E[U^* - U_n(\pi)] \leq E \left[ \sum_{t=1}^{n} \sum_{j=1}^{m} p_t^j d_j - w_t \right]$. Finally, we apply the price update rule in Algorithm 2 with a step size $\gamma = O\left(\frac{1}{\sqrt{n}}\right)$ to establish an $O(\sqrt{n})$ upper bound on the term $E \left[ \sum_{t=1}^{n} \sum_{j=1}^{m} p_t^j d_j - w_t \right]$, i.e., the right hand side of the generic regret bound, which establishes our claim.

For a complete proof of Theorem 3, we refer to Appendix E. Theorem 3 establishes that the expected regret and constraint violation of Algorithm 2 are sub-linear in the number of users $n$. Note that Theorems 1 and 3 jointly imply that Algorithm 2, while preserving user privacy, achieves expected regret and constraint violation guarantees, up to constants, that are no more than that of an expected equilibrium pricing approach (see Corollary 1) with complete information on the distribution from which users’ budget and utility parameters are drawn. On the other hand, as compared to Algorithm 1 that achieves a constant constraint violation and an $O(\log(n))$ regret (see Theorem 2), Algorithm 2 achieves a higher regret and constraint violation of $O(\sqrt{n})$. We defer the problem of closing the performance gap between these algorithms as a topic for future research.

While the regret and constraint violation bounds of Algorithms 1 and 2 highlight that there is a performance loss in algorithm design for online Fisher markets in the absence of distributional information, we reiterate that Algorithm 2 has several advantages to Algorithm 1. First, Algorithm 2 is applicable for a broader range of probability distributions compared to Algorithm 1, which only applies for discrete probability distributions $D$. Next, the price update step in Algorithm 2 has a very low computational overhead, while updating prices in Algorithm 1 involves solving a convex program at each step. Finally, Algorithm 2 is more likely to be practically viable as it only relies on users’ revealed preferences, while Algorithm 1 requires complete knowledge of the distribution $D$.

Finally, a few comments about the assumption in the statement of Theorem 3 that the price vector $p^t$ is strictly positive and bounded during the operation of Algorithm 2 are in order. In particular, first note that if the price vector $p^t$ at each iteration of Algorithm 2 is bounded below by some vector $\tilde{p}$, then the price vector also remains bounded above by $\bar{p}$, where $\tilde{p} > 0$ is a constant, as we show in Lemma 5 in Appendix F. In other words, the positivity of prices during the operation of Algorithm 2 implies the boundedness of the prices; however, we still retain the boundedness assumption in the statement of Theorem 3 to simplify the exposition in its proof presented in Appendix E. Next, under Assumption 1 on the distribution $D$, it holds that users with a positive utility for each good will arrive with a high probability for a large number of users $n$. As a result, for a small constant $\bar{D} > 0$ for the price updates in Algorithm 2, the prices of any good will not drop below a small constant $\bar{p} > 0$, as eventually, users will purchase large quantities of that good (at lower prices) to drive the prices of that good up through the price update in Algorithm 2. We present some more formalism behind this intuition by using concentration inequalities in Appendix F to further motivate the positivity of the price vector $p^t$ during the operation of Algorithm 2. Furthermore, we validate the positivity of prices through the operation of Algorithm 2 using numerical experiments presented in Appendix G.4.

6 Numerical Experiments

We now compare the performance of Algorithm 2 to several benchmarks on both regret and constraint violation metrics. The results of our experiments not only validate the theoretical bounds obtained in Theorem 3 but also demonstrate the efficacy of Algorithm 2 as compared to two benchmarks that have access to additional information on users’ utility and budget parameters. In this section, we introduce two benchmarks to which we compare Algorithm 2 (Section 6.1), describe the implementation details of Algorithm 2 and the benchmarks (Section 6.2), and present results to demonstrate the performance of Algorithm 2 (Section 6.3).
6.1 Benchmarks

In our experiments, we compare Algorithm 2 to two benchmarks, akin to several classical algorithms developed in the online resource allocation literature [30, 28], with additional information on users’ utility and budget parameters. In particular, the first benchmark assumes knowledge of the distribution $D$ from which the budget and utility parameters are drawn, as is the case for an algorithm that sets expected equilibrium prices. The second benchmark assumes that users’ utility and budget parameters are revealed to the central planner when they enter the market and can be used to set prices for subsequent users. We mention that these algorithms are solely for benchmark purposes, and thus we do not discuss the practicality of the corresponding informational assumptions of these benchmarks. We also reiterate that, as opposed to these benchmarks, the price updates in Algorithm 2 only rely on users’ revealed preferences rather than relying on additional information on their budget and utility parameters.

Stochastic Program: We begin with the benchmark wherein the distribution $D$ from which the budget and utility parameters are generated i.i.d. is known. In this case, the SAA Problem (5) is related to the following stochastic program

$$
\min_{p} \quad D(p) = \sum_{j=1}^{m} p_j d_j + \mathbb{E}_{(w, u) \sim D} \left[ \left( w \log(w) - w \log \left( \min_{j \in [m]} \frac{p_j}{u_j} \right) - w \right) \right],
$$

which can be solved to give an optimal price vector $p^*$. Note that this price vector $p^*$ corresponds to the static expected equilibrium price, as it takes an expectation over the distribution $D$. The corresponding pricing policy $\pi$ only depends on the distribution $D$ is thus given by $p^* = \pi_t(D)$ for all users $t \in [n]$. Given the price vector $p^*$, all arriving users will purchase an affordable utility-maximizing bundle of goods by solving their individual optimization Problem (1a)-(1c). Note here that the price vector $p^*$ is computed before the online procedure, which is possible due to the prior knowledge of the distribution $D$. For numerical implementation purposes, we consider a sample average approximation to compute the expectation in Problem (7), as elucidated in Section 6.2.

Dynamic Learning using SAA: In this benchmark, we consider the setting wherein users’ budget and utility parameters are revealed to the central planner each time a user arrives. In this context, the prices are set based on the dual variables of the capacity constraints of the sampled Eisenberg-Gale program with the observed budget and utility parameters of agents that have previously arrived. That is, the pricing policy $\pi$ depends on the history of users’ budget and utility parameters, i.e., $p^t = \pi_t((w_t, u_t)_{t=1}^{t-1})$. We note that to improve on the computational complexity, we update the dual prices at geometric intervals, as in earlier work [30, 28]. Users arriving in each interval observe the corresponding price vector for that interval and solve their individual optimization problems to obtain their most favorable goods under the set prices. We formally present the algorithm for this benchmark in Appendix H.

6.2 Implementation Details

To numerically evaluate the performance of Algorithm 2 and the benchmarks, we consider a market instance of $m = 5$ goods, each with a capacity of $10n$ when there are $n$ users in the market. Each arriving user’s budget and utility parameters are generated i.i.d. from a probability distribution $D$, specified as follows. In particular, each user’s budget can take on one of three values: 2, 5, or 10, which represent users with low, medium, and high budgets, and a user can have either of these budgets with a probability of $\frac{1}{3}$. Furthermore, each user’s utility for the goods is independent of their budget, and their utility for each good is drawn uniformly at random between the range $[5, 10]$. We choose a continuous utility distribution to validate the theoretical regret and constraint violation guarantee in Theorem 3, as Algorithm 2 applies for general (non-discrete) probability distributions. Since this utility distribution is continuous, the adaptive expected equilibrium pricing algorithm (Algorithm 1) does not apply in this setting. Thus, we focus on comparing Algorithm 2 to the two benchmarks in this section and refer to Appendix G.2 for numerical experiments comparing Algorithm 1 to a static expected equilibrium pricing approach.

Under the above defined market instance, we implement Algorithm 2 using a step size of $\gamma = \gamma_t = \frac{1}{1000\sqrt{n}}$ for all users $t \in [n]$. Furthermore, to implement the stochastic programming benchmark, we compute the
solution to Problem (7) using a sample average approximation with 5000 samples of budget and utility parameters generated from the above described distribution \( \mathcal{D} \) to evaluate the expectation.

### 6.3 Results

**Assessment of Theoretical Bounds:** We first assess the theoretical bounds on the regret and constraint violation obtained in Theorem 3. To this end, Figure 1 depicts the infinity norm of the constraint violation (right) and a log-log plot of the regret of Algorithm 2. As expected, the black dots representing the empirically observed regret of Algorithm 2 on the market instance described in Section 6.2 are very close to the theoretical \( O(\sqrt{n}) \) upper bound, represented by a line with a slope of 0.5 on the log-log plot. On the other hand, the empirical results for the constraint violation (right of Figure 1) of Algorithm 2 indicate that no good capacities are violated. As a result, the \( O(\sqrt{n}) \) upper bound on the constraint violation is also satisfied.

![Figure 1: Validation of theoretical regret and constraint violation upper bounds of Algorithm 2 on market instance described in Section 6.2. The regret of Algorithm 2 is presented on a log-log plot (left), and the empirically observed performance, represented by the black dots, is very close to the theoretical \( O(\sqrt{n}) \) bound, represented by a line of slope 0.5. The infinity norm of the excess demand is zero for all instances and thus trivially satisfies the \( O(\sqrt{n}) \) upper bound on the constraint violation.](image)

**Comparisons between Algorithm 2 and Benchmarks:** We now compare Algorithm 2 and the two benchmarks on regret and constraint violation metrics. The left of Figure 2 depicts the ratio of the regret and the optimal offline objective of the three algorithms, and the right of Figure 2 depicts the ratio between their constraint violation and the capacities of the goods. From this figure, we observe that while Algorithm 2 incurs a higher regret than the two benchmarks, it does not violate any capacity constraints. On the other hand, both the benchmarks violate capacity constraints to achieve an overall lower regret. This observation highlights the fundamental trade-off between regret and constraint violation and underscores the practical viability of Algorithm 2 as it achieves a social welfare efficiency loss (i.e., the ratio between the regret and the optimal offline objective) of only about 5\% for a market with 5000 users while not violating capacity constraints. We further reiterate that Algorithm 2 achieves the performance depicted in Figure 2 without relying on the additional assumptions on users’ budget and utility parameters the two benchmarks require.

### 7 Conclusion and Future Work

In this work, we studied an online variant of Fisher markets wherein users with linear utilities arrive sequentially and have privately known budget and utility parameters drawn i.i.d. from some distribution \( \mathcal{D} \). Since
classical approaches for market equilibrium computation are not suitable in this online incomplete information setting, we studied the problem of setting prices online to minimize regret and constraint violation. In this setting, we first established that no static pricing algorithm, including an algorithm that sets expected equilibrium prices with complete information of the distribution \(D\), can achieve a regret and constraint violation of less than \(\Omega(\sqrt{n})\) (where \(n\) is the number of users). Given the performance limitations of static pricing, we developed adaptive pricing algorithms with improved performance guarantees. To this end, we first developed an adaptive expected equilibrium pricing approach with \(O(\log(n))\) regret and constant constraint violation for discrete probability distributions \(D\). For the setting of general probability distributions (and when the distribution \(D\) is unknown), we proposed an online learning approach to adjust prices solely based on users’ revealed preferences, i.e., past observations of user consumption, thereby preserving user privacy. Our revealed preference algorithm has a computationally efficient price update rule that makes it practically viable and achieves an \(O(\sqrt{n})\) upper bound on the expected regret and constraint violation. Finally, we used numerical experiments to evaluate the efficacy of our proposed revealed preference algorithm, which highlighted a fundamental trade-off between the regret and constraint violation measures.

There are several directions for future research. First, as we obtained a lower bound on the regret and constraint violation of static pricing algorithms (Theorem 1), it would be worthwhile to develop algorithm-independent lower bounds to characterize the limits of the achievable performance of adaptive pricing algorithms for online Fisher markets. Next, while Algorithm 1 achieved an \(O(\log(n))\) regret and constant constraint violation for discrete probability distributions, it would be interesting to study whether adaptive pricing algorithms can achieve a performance better than the \(O(\sqrt{n})\) regret and constraint violation of the revealed preference algorithm (Algorithm 2) for general probability distributions. There is also immense scope to generalize the model studied in this work and the corresponding results to more general settings. For instance, it would be valuable to investigate more general concave utility functions, e.g., homogeneous degree one utility functions, beyond linear utilities and study settings beyond the stochastic input model of user arrival, e.g., the random permutation model. Finally, it would also be interesting to investigate whether there are bandit problems where the weighted geometric mean objective studied in this work is more applicable than the traditionally used linear objectives.

**Acknowledgements**

We thank Yale Wang and Vladimir Gonzalez Migal for their assistance with the simulation experiments.
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A Regret and Nash Social Welfare

We establish a fundamental connection between the regret measure studied in this work and the ratio between the Nash social welfare objective of the optimum offline oracle and that corresponding to an online algorithm. In particular, we show that if the regret $U^*_n - U_n(\pi) \leq o(n)$ for some algorithm $\pi$, then $\frac{\text{NSW}(x^*_1, \ldots, x^*_n)}{\text{NSW}(x_1, \ldots, x_n)} \to 1$ as $n \to \infty$. Here, $x^*_1, \ldots, x^*_n$ are the optimal offline allocations, and $x_1, \ldots, x_n$ are the optimal consumption vectors given by the solution of Problem (1a)-(1c) under the prices corresponding to the online pricing policy $\pi$. Without loss of generality, consider the setting when the budgets of all users are equal. Note that if the budgets are not equal, then we can just re-scale the utilities of each user based on their budget. In this setting, it holds that

$$\frac{1}{n} U^*_n = \frac{1}{n} \sum_{t=1}^n \log(u_t(x^*_t)) = \frac{1}{n} \log \left( \prod_{t=1}^n u_t(x^*_t) \right) = \log \left( \left( \prod_{t=1}^n u_t(x^*_t) \right)^{\frac{1}{n}} \right) = \log(\text{NSW}(x^*_1, \ldots, x^*_n)), $$

and $\frac{1}{n} U_n(\pi) = \log \left( \left( \prod_{t=1}^n u_t(x_t) \right)^{\frac{1}{n}} \right) = \log(\text{NSW}(x_1, \ldots, x_n))$. Then, it follows that

$$\frac{\text{NSW}(x^*_1, \ldots, x^*_n)}{\text{NSW}(x_1, \ldots, x_n)} = e^{\frac{1}{n} U^*_n} = e^{\frac{1}{n} (U^*_n - U_n(\pi))} = e^{\frac{1}{n} \left( U^*_n - U_n(\pi) \right)} \leq e^{o(n)}. $$

Observe that as $n \to \infty$, the term $e^{o(n)} \to 1$. That is, if the regret of an algorithm $\pi$ is $o(n)$, then the ratio of the Nash social welfare objective of the algorithm $\pi$ approaches that of the optimal offline oracle as $n$ becomes large.

B Proof of Theorem 1

Consider a setting with $n$ users with a fixed budget of one and two goods, each with a capacity of $n$. Further, let the utility parameters of users be drawn i.i.d. from a probability distribution, where the users have utility $(1,0)$ with probability 0.5 and a utility of $(0,1)$ with probability 0.5. That is, users only have utility for
good one or good two, each with equal probability. For this instance, we first derive a tight bound for the expected optimal social welfare objective, i.e., Objective (2a). Then, to establish the desired lower bound, we consider two cases: (i) the price of either of the two goods is at most 0.5, and (ii) the price of both goods is strictly greater than 0.5. In the first case, we establish that the expected constraint violation is \( \Omega(\sqrt{n}) \) while in the second case, we establish that either the expected constraint violation or the expected regret is \( \Omega(\sqrt{n}) \).

### B.1 Tight Bound on Expected Optimal Social Welfare Objective

To obtain a bound on the expected optimal social welfare objective, we first find an expression for the objective value is given by

\[
E = \sum_{i=1}^{n} \log(x_{t1}) + \sum_{t=s+1}^{n} \log(x_{t2}),
\]

subject to

\[
\begin{aligned}
\sum_{t=1}^{n} x_{t1} &\leq n, \\
\sum_{t=1}^{n} x_{t2} &\leq n,
\end{aligned}
\]

\( x_{tj} \geq 0, \quad \forall t \in [n], j \in [2]. \)

If \( 0 < s < n \), then the optimal solution of the above problem is to allocate \( x_t = (\frac{s}{n}, 0) \) to each user \( t \) with a utility of \((1, 0)\) and to allocate \( x_t = (0, \frac{n-s}{n-s}) \) to each user \( t \) with a utility of \((0, 1)\). In this case, the optimal objective value is given by

\[
U^*(s) = s \log \left( \frac{n}{n-s} \right) + (n - s) \log \left( \frac{n}{n-s} \right) = n \log(n) - s \log(s) - (n-s) \log(n-s).
\]

We now develop a tight bound on the expected optimal objective \( U^*(s) \) using the fact that the number of arrivals \( s \) of users with utility \((0, 1)\) is binomially distributed with a probability of 0.5. That is, we seek to develop a tight bound bound for

\[
\mathbb{E}[U^*(s)] = \mathbb{E}[n \log(n) - s \log(s) - (n-s) \log(n-s)],
\]

\[
= n \log(n) - \mathbb{E}[s \log(s)] - \mathbb{E}[(n-s) \log(n-s)].
\]

To this end, we present an upper bound for \( s \log(s) \) and \((n-s) \log(n-s) \), which will yield a lower bound for \( \mathbb{E}[U^*(s)] \).

We begin by observing that the expectation of the binomial random variable is given by \( \mathbb{E}[s] = \frac{n}{2} \) and its variance is \( \mathbb{E}[(s - \frac{n}{2})^2] = \frac{n}{4} \). Next, letting \( \sigma = \frac{2}{n} (s - \frac{n}{2}) \), which has zero mean and a standard deviation of \( \frac{1}{\sqrt{n}} \), we obtain the following upper bound on the term \( s \log(s) \)

\[
s \log(s) = s \log \left( \frac{n}{2} + s - \frac{n}{2} \right),
\]

\[
= s \log \left( \frac{n}{2} \left( 1 + \frac{2}{n} (s - \frac{n}{2}) \right) \right),
\]

\[
= s \log \left( \frac{n}{2} \right) + s \log \left( 1 + \frac{2}{n} (s - \frac{n}{2}) \right),
\]

\[
= s \log \left( \frac{n}{2} \right) + s \log(1 + \sigma),
\]

\[
\leq s \log \left( \frac{n}{2} \right) + s \sigma.
\]

\[ (9) \]
Similarly, we obtain the following upper bound for \((n - s) \log(n - s)\):

\[
(n - s) \log(n - s) = (n - s) \log \left( \frac{n}{2} \right) + (n - s) \log(1 - \sigma) \leq (n - s) \log \left( \frac{n}{2} \right) - (n - s)\sigma
\]

Adding Equations (9) and (10), we have that

\[
s \log(s) + (n - s) \log(n - s) \leq n \log \left( \frac{n}{2} \right) + (2s - n)\sigma = n \log \left( \frac{n}{2} \right) + n\sigma^2.
\]

As a result, it holds that

\[
U^*(s) = n \log(n) - s \log(s) - (n - s) \log(n - s) \geq n \log(n) - n \log \left( \frac{n}{2} \right) - n\sigma^2 = n \log(2) - n\sigma^2
\]

for all \(0 < s < n\). Next, letting \(q_s\) be the probability of observing \(s\) users with utility \((1, 0)\), it follows that

\[
\mathbb{E}[U^*(s)] = \sum_{s=0}^{n} q_s U^*(s) \overset{(a)}{=} \sum_{s=1}^{n-1} q_s U^*(s) \overset{(b)}{\geq} \sum_{s=1}^{n-1} q_s (n \log(2) - n\sigma^2),
\]

\[
\overset{(c)}{\geq} \left( 1 - \frac{1}{2^{n-1}} \right) n \log(2) - n\mathbb{E}[\sigma^2],
\]

\[
= \left( 1 - \frac{1}{2^{n-1}} \right) n \log(2) - n \mathbb{E} \left[ \left( \frac{2}{n} \left( s - \frac{n}{2} \right) \right)^2 \right],
\]

\[
= \left( 1 - \frac{1}{2^{n-1}} \right) n \log(2) - 1
\]

where \((a)\) follows as \(U^*(0) = 0\) and \(U^*(n) = 0\), \((b)\) follows by Equation (11), \((c)\) follows as \(\sum_{s=1}^{n-1} q_s = 1 - \frac{1}{2^{n-1}}\) and \(\sum_{s=1}^{n-1} q_s \sigma^2 \leq \sum_{s=0}^{n} q_s \sigma^2 = \mathbb{E}[\sigma^2]\).

Finally, using Jensen’s inequality for a concave function, we obtain the following upper bound on the expected optimal social welfare objective:

\[
\mathbb{E}[U^*(s)] \leq U^*(\mathbb{E}(s)) \leq n \log(2).
\]

As a result, we have shown the following tight bound on the expected optimal social welfare objective for the earlier defined instance:

\[
\left( 1 - \frac{1}{2^{n-1}} \right) n \log(2) - 1 \leq \mathbb{E}[U^*(s)] \leq n \log(2).
\]

**B.2 \(\Omega(\sqrt{n})\) bound on Expected Regret and Constraint Violation**

**Case (i):** We first consider the case when the price of either of the two goods is at most 0.5. Without loss of generality, let \(p_1 \leq 0.5\). Then, with \(s\) arrivals of users with utility \((1, 0)\), the expected constraint violation of good one is given by

\[
v_1 = \mathbb{E} \left[ \frac{1}{p_1} \left( s - \frac{n}{2} \right)_{+} \right] \geq 2 \mathbb{E} \left[ \left( s - \frac{n}{2} \right)_{+} \right],
\]

which is \(O(\sqrt{n})\) by the central limit theorem as \(\frac{n}{\sqrt{7}}\) users of each type arrive in expectation. As a result, the norm of the constraint violation \(\Omega(\sqrt{n})\). This establishes that if the price of either of the goods is below 0.5, the expected constraint violation is \(\Omega(\sqrt{n})\).

**Case (ii):** Next, we consider the case when the price of both goods is strictly greater than 0.5. In particular, suppose that \(\mathbf{p} = (p_1, p_2) = (\frac{1}{2\epsilon_1(n)}, \frac{1}{2\epsilon_2(n)})\), where \(\epsilon_1(n), \epsilon_2(n) > 0\) can depend on the number of users \(n\) and are constants for any fixed value of \(n\). We now show that for any choice of \(\epsilon_1(n), \epsilon_2(n) > 0\) that either the expected regret or the expected constraint violation is \(\Omega(\sqrt{n})\).
To this end, first note by the central limit theorem that the expected constraint violation for good one for \( s \) arrivals of users with utility \((1, 0)\) is given by
\[
v_1 = \mathbb{E} \left[ \frac{1}{p_1} \left( s - \frac{n}{2} \right) \right] = \mathbb{E} \left[ (2 - \epsilon_1(n)) \left( s - \frac{n}{2} \right) \right] = (2 - \epsilon_1(n))O(\sqrt{n}). \tag{12}\]
Similarly, the expected constraint violation of good two is given by \((2 - \epsilon_2(n))\Omega(\sqrt{n})\).

Next, using the lower bound on the expected optimal social welfare objective we obtain the following lower bound on the regret of any static pricing policy with \( p = \left( \frac{1}{2 - \epsilon_1(n)}, \frac{1}{2 - \epsilon_2(n)} \right)\):
\[
\text{Regret} \geq \left( 1 - \frac{1}{2n-1} \right) n \log(2) - 1 - \mathbb{E} \left[ \sum_{t=1}^{n} \log \left( \frac{1}{2 - \epsilon(t)} \right) \right],
\]
\[
= \left( 1 - \frac{1}{2n-1} \right) n \log(2) - 1 - \mathbb{E} \left[ \sum_{t=1}^{n} \log(2 - \epsilon(n)) \right],
\]
\[
= \left( 1 - \frac{1}{2n-1} \right) n \log(2) - 1 - \mathbb{E} \left[ \sum_{t=1}^{n} \log \left( 2 - \frac{\epsilon(n)}{2} \right) \right],
\]
\[
\geq - \frac{1}{2n-1} n \log(2) - 1 + \frac{n \epsilon(n)}{2},
\]
where \( \epsilon(n) = \min\{\epsilon_1(n), \epsilon_2(n)\} \).

Finally, to simultaneously achieve the lowest regret and constraint violation, we set \((2 - \epsilon(n))\Omega(\sqrt{n}) = -1 + \frac{n \epsilon(n)}{2} - \frac{1}{2n-1} n \log(2)\). Solving for \(\epsilon(n)\), we get that \(\epsilon(n) = O\left(\frac{1}{\sqrt{n}}\right)\) as \(n\) becomes large. This relation implies that to minimize both regret and constraint violation, \(\epsilon(n)\) needs to be set on the order of \(\frac{1}{\sqrt{n}}\), which will result in a corresponding expected regret and constraint violation of \(\Omega(\sqrt{n})\). Observe that for any other choice of \(\epsilon(n)\), either the regret or the constraint violation must be \(\Omega(\sqrt{n})\) since setting \(\epsilon(n) = O\left(\frac{1}{\sqrt{n}}\right)\) guarantees that both the regret and constraint violation are minimized. This establishes our claim that either the regret or the constraint violation must be \(\Omega(\sqrt{n})\) when the price of both goods is strictly greater than 0.5, which proves our claim.

## C Proof of Theorem 2

We prove Theorem 2 using three intermediate lemmas, which we elucidate below. After presenting the statements of these lemmas, we then present their proofs.

Our first lemma establishes a generic upper bound on the regret of an algorithm for the online Fisher market setting considered in this work. To define this generic regret bound, we first introduce the following stochastic program
\[
\min_{\mathbf{p}} \quad D(\mathbf{p}) = \sum_{j=1}^{m} p_j d_j + \mathbb{E} \left[ \left( w \log (w) - w \log \left( \min_{j \in [m]} \frac{p_j}{d_j} \right) - w \right) \right], \tag{13}\]
which is the stochastic programming formulation of the dual of the Eisenberg-Gale program (see Equation (5)) presented in Section 5.1. Letting \(\mathbf{p}^*\) be the optimal solution to this stochastic Program, we obtain the following generic bound on the regret of any algorithm for online Fisher markets.

**Lemma 1** (Generic Regret Bound). Suppose that the budget and utility parameters of users are drawn i.i.d. from a probability distribution \(\mathcal{D}\). Furthermore, let \(\pi\) denote an online pricing policy, \(\mathbf{x}_1, \ldots, \mathbf{x}_n\) be the corresponding allocations for the \(n\) users, and \(p, \bar{p} > 0\) be the lower and upper bounds, respectively, for the prices \(p_j\) for all goods \(j\) and for all users \(t \in [n]\), where the price upper bound \(\bar{p} \geq \max_{j \in [m]} p_j\). Then, the regret \(R_n(\pi) \leq \frac{2\sqrt{m} u}{\bar{p}} \sum_{t=1}^{n} \mathbb{E} \|\mathbf{p}^* - \mathbf{p}^t\|_2 + \mathbb{E} \left[ \frac{\bar{p}}{\bar{p}} \sum_{j=1}^{m} \left( \sum_{t=1}^{n} x_{tj} - c_j \right) \right].\)
A few comments about Lemma 1 are in order. First, observe that the generic regret bound obtained in Lemma 1 applies to general (non-discrete) probability distributions $\mathcal{D}$. Next, the generic regret bound is composed of two terms: (i) the first term accounts for the loss for setting prices that deviate from the optimal expected prices $p^*$, and (ii) the second term is akin to the constraint violation of the algorithm and, in particular, accounts for the loss corresponding to to over (or under-consuming) certain goods.

As a result, to upper bound the regret of Algorithm 1, we now present two lemmas that present upper bounds to both the terms in the generic regret upper bound. To this end, we first show that the upper bound on the expected constraint violation is constant in the number of arriving users. This result not only establishes the desired constraint violation bound in the statement of Theorem 2 but its analysis also provides a bound on the second term of the generic regret upper bound in Lemma 1.

**Lemma 2 (Constraint Violation Bound of Algorithm 1).** Suppose that the budget and utility parameters of users are drawn i.i.d. from a discrete probability distribution $\mathcal{D}$ and let $\pi$ denote the online pricing policy described by Algorithm 1. Furthermore, let $x_1, \ldots, x_n$ be the corresponding allocations for the $n$ users, where $x_t$ is an optimal solution for that user corresponding to the certainty equivalent problem $CE(\mathbf{d}_t)$ for $t \leq \tau$, where $\tau$ is the first time at which $\mathbf{d}_t \notin [d - \Delta, d + \Delta]$, and $x_t$ is an optimal solution to $CE(\mathbf{d})$ for $t > \tau$. Then, the constraint violation $V_n(\pi) \leq O(1)$.

The proof of Lemma 2 follows from an application of similar techniques to that used in [31]. In this proof, we leverage the fact that the allocations $x_t$ are given by the optimal solution of the certainty equivalent problem $CE(\mathbf{d}_t)$ for $t \leq \tau$, which is one of the optimal consumption vectors corresponding to the price $p^*$. Note that doing so is without loss of generality, since the utility of the users is unchanged for any optimal consumption bundle. Furthermore, recall from Section 4.2.1 that the allocations corresponding to the optimal solution of the certainty equivalent problem $CE(\mathbf{d}_t)$ at each step can be implemented in Algorithm 1 using an allocation-based algorithm, wherein users are given allocations based on their observed type $k \in [K]$.

Having obtained a bound on the constraint violation, we next upper bound the first term in the generic regret upper bound by applying Assumption 2, as is elucidated through the following lemma.

**Lemma 3.** Suppose that the budget and utility parameters of users are drawn i.i.d. from a discrete probability distribution $\mathcal{D}$. Furthermore, let $\pi$ denote the online pricing policy described by Algorithm 1 and let $x_1, \ldots, x_n$ be the corresponding allocations for the $n$ users. Then, under Assumption 2, $\frac{2\sqrt{\text{OPT}}}{\mathcal{E}} \sum_{t=1}^{n} \mathbb{E}[\|p^* - p^t\|_2] \leq O(\log(n))$.

Finally, we combine the results obtained in Lemmas 2 and 3 to obtain the $O(\log(n))$ upper bound on the regret of Algorithm 1.

**Corollary 2 (Regret Upper Bound of Algorithm 1).** Suppose that the budget and utility parameters of users are drawn i.i.d. from a discrete probability distribution $\mathcal{D}$ and let $\pi$ denote the online pricing policy described by Algorithm 1. Furthermore, let $x_1, \ldots, x_n$ be the corresponding allocations for the $n$ users, where $x_t$ is an optimal solution for that user corresponding to the certainty equivalent problem $CE(\mathbf{d}_t)$ for $t \leq \tau$, where $\tau$ is the first time at which $\mathbf{d}_t \notin [d - \Delta, d + \Delta]$, and $x_t$ is an optimal solution to $CE(\mathbf{d})$ for $t > \tau$. Then, under Assumption 2, the regret $R_n(\pi) \leq O(\log(n))$.

Note that Lemma 2 and Corollary 2 jointly imply Theorem 2, which thus proves our claim.

### C.1 Proof of Lemma 1

We now establish a generic bound on the regret of any online algorithm as long as the prices $p^t$ are strictly positive and bounded, i.e., $0 < p < p^*_j \leq \bar{p}$ for all goods $j$ and for all users $t \in [n]$. To establish a generic upper bound on the regret, we first obtain a bound on the expected value of the optimal objective, i.e., Objective (2a), and a relation for the expected value of the objective for any online allocation policy $\pi$. We finally combine both these relations to obtain an upper bound on the regret.

To perform our analysis, we define the function $g(p) = \mathbb{E}[w_t \log(u_t^x x_t)] + \sum_{j=1}^{m} (d_j - x_{tj}(p))p^*_j$, where $p^*$ is the optimal price vector of the stochastic Program (13). Then, by duality we have that the expected primal objective value $\mathbb{E}[U^*_n]$ is no more than the dual objective value with $p = p^*$, which gives the following
upper bound on the optimal objective

$$
\mathbb{E}[U^*_n] \leq \mathbb{E} \left[ \sum_{j=1}^{m} p_j^* c_j + \sum_{t=1}^{n} \left( w_t \log(w_t) - w_t \log \left( \min_{j \in [m]} \frac{p_j^*}{u_{tj}} \right) - w_t \right) \right],
$$

$$= nD(p^*), \tag{14}
$$

by the definition of $D(p)$ in Problem (13). Next, we establish a relation between the function $g(p)$ and the above obtained bound on the expected value of the optimal objective value by noting that

$$g(p^*) = \mathbb{E} [w_t \log(u_t^T x_t^*) + \sum_{j=1}^{m} (d_j - x_{tj}^*(p^*))p_j^*],$$

$$\overset{(a)}{=} \mathbb{E} \left[ w_t \log(w_t) - w_t \log \left( \min_{j \in [m]} \frac{p_j^*}{u_{tj}} \right) + \sum_{j \in [m]} p_j^* d_j - w_t \right],$$

$$= nD(p^*), \tag{15}
$$

where (a) follows by the definition of $g(p)$ and noting that for each agent $t \in [n]$ it holds that $u_t^T x_t = u_{tj}^* \frac{w_t}{p_{j'}^*}$ for some good $j'$ in the optimal bundle for the user $t$, and that $\sum_{j \in [m]} x_{tj}(p^*)p_j^* = w_t$ since each user spends their entire budget when consuming its optimal bundle of goods given the price vector $p^*$. Combining the relations obtained in Equations (14) and (15), we obtain the following upper bound on the expected value of the optimal objective:

$$\mathbb{E}[U^*_n] \leq ng(p^*). \tag{16}
$$

Having obtained an upper bound on the expected optimal objective, we now obtain the following relationship for the true accumulated social welfare objective, i.e., Objective (2a), accrued by any online policy $\pi$ that sets prices $p_1^*, \ldots, p_n^*$ with corresponding allocations $x_1, \ldots, x_n$:

$$\mathbb{E} [U_n(\pi)] = \mathbb{E} \left[ \sum_{t=1}^{n} w_t \log(u_t^T x_t) \right],$$

$$= \mathbb{E} \left[ \sum_{t=1}^{n} w_t \log(u_t^T x_t) + \sum_{j=1}^{m} p_j^* \left( c_j - \sum_{i=1}^{n} x_{tj} \right) - \sum_{j=1}^{m} p_j^* \left( c_j - \sum_{i=1}^{n} x_{tj} \right) \right],$$

$$= \mathbb{E} \left[ \sum_{t=1}^{n} \left( w_t \log(u_t^T x_t) + \sum_{j=1}^{m} p_j^* (d_j - x_{tj}) \right) \right] + \mathbb{E} \left[ \sum_{j=1}^{m} p_j^* \left( \sum_{i=1}^{n} x_{tj} - c_j \right) \right] \tag{17}
$$

We can analyse the first term of the Equation (17) as follows:

$$\mathbb{E} \left[ \sum_{t=1}^{n} \left( w_t \log(u_t^T x_t) + \sum_{j=1}^{m} p_j^* (d_j - x_{tj}) \right) \right] \overset{(a)}{=} \sum_{t=1}^{n} \mathbb{E} \left[ w_t \log(u_t^T x_t) + \sum_{j=1}^{m} p_j^* (d_j - x_{tj}) \right],$$

$$\overset{(b)}{=} \sum_{t=1}^{n} \mathbb{E} \left[ \mathbb{E} \left[ w_t \log(u_t^T x_t) + \sum_{j=1}^{m} p_j^* (d_j - x_{tj}) \mid \mathcal{H}_{t-1} \right] \right],$$

$$\overset{(c)}{=} \sum_{t=1}^{n} \mathbb{E} \left[ g(p_t^*) \right] = \mathbb{E} \left[ \sum_{t=1}^{n} g(p_t^*) \right], \tag{18}
$$

where (a) follows by the linearity of expectation, (b) follows from nesting conditional expectations, where the history $\mathcal{H}_{t-1} = \{w_i, u_i, x_i\}_{i=1}^{t-1}$, and (c) follows from the definition of $g(p)$ and the fact that the allocation $x_{tj}$ depends on the vector of prices $p_t$. 

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Finally, combining the above analysis in Equations (16), (17), and (18) for $\mathbb{E}[U'_n]$ and $\mathbb{E}[U_n(\pi)]$, we obtain the following bound on the regret of any online allocation policy $\pi$ for $\bar{p} \geq \max_{j \in [m]} p'_j$:

$$
\mathbb{E}[U'_n - U_n(\pi)] \leq n g(p^*) - \mathbb{E} \left[ \sum_{t=1}^{n} g(p^t) \right] - \mathbb{E} \left[ \sum_{j=1}^{m} p'_j \left( \sum_{t=1}^{n} x_{tj} - c_j \right) \right],
$$

$$
\leq \mathbb{E} \left[ \sum_{t=1}^{n} (g(p^*) - g(p^t)) \right] + \mathbb{E} \left[ \bar{p} \sum_{j=1}^{m} \left( \sum_{t=1}^{n} x_{tj} - c_j \right) \right].
$$

(19)

Finally, to obtain the desired generic regret bound, we establish that $\mathbb{E}[g(p^*) - g(p^t)] \leq O(\mathbb{E}[\|p^* - p^t\|_2])$. To this end, first observe from the definition of the function $g$ that for the optimal solution $x_t(p)$ of the individual optimization Problem (1a)-(1c) given a price vector $p$ that

$$
g(p^*) - g(p^t) = \mathbb{E} \left[ w_t \log \left( \frac{p'_j}{p_t^j} \right) + \sum_{j=1}^{m} (d_j - x_{tj}(p^*))p'_j \right] - \mathbb{E} \left[ w_t \log \left( \frac{p'_j}{p_t^j} \right) + \sum_{j=1}^{m} (d_j - x_{tj}(p^t))p'_j \right],
$$

$$
= \mathbb{E} \left[ w_t \log \left( \min_{j \in [m]} \left\{ \frac{p'_j}{w_{tj}} \right\} \right) - \frac{1}{\min_{j \in [m]} \left\{ \frac{p'_j}{w_{tj}} \right\}} \right] + \mathbb{E} \left[ \sum_{j=1}^{m} (x_{tj}(p^*) - x_{tj}(p^t))p'_j \right].
$$

Then, letting the good $j' \in \arg \min_{j \in [m]} \left\{ \frac{p'_j}{w_{tj}} \right\}$ and $j^*(p) \in \{j' \}$ be a good in the optimal consumption set of user $t$ given the price $p$, we observe that

$$
g(p^*) - g(p^t) \leq \mathbb{E} \left[ w_t \log \left( \frac{p'_j}{p_t^{j'}} \right) \right] + \mathbb{E} \left[ \sum_{j=1}^{m} \left( \mathbbm{1}_{j = j^*(p')} \frac{w_t}{p'_j} \mathbbm{1}_{j = j^*(p^*)} \frac{w_t}{p'_j} \right) p'_j \right],
$$

$$
\leq \mathbb{E} \left[ w_t \log \left( 1 + \frac{p'_j - p_t^{j'}}{p_t^{j'}} \right) \right] + \mathbb{E} \left[ \sum_{j=1}^{m} \frac{w_t (p'_j - p_t^{j'})}{p'_j p_t^{j'}} (\mathbbm{1}_{j = j^*(p')} - \mathbbm{1}_{j = j^*(p^*)}) p'_j \right],
$$

$$
\leq \mathbb{E} \left[ w_t \frac{p'_j - p_t^{j'}}{p_t^{j'}} \right] + \mathbb{E} \left[ \sum_{j=1}^{m} \frac{w_t (p'_j - p_t^{j'})}{p'_j} \right],
$$

$$
\leq \frac{2\bar{w}}{p} \mathbb{E} \left[ \|p^* - p^t\|_1 \right],
$$

$$
\leq \frac{2\sqrt{mw}}{p} \mathbb{E} \left[ \|p^* - p^t\|_2 \right],
$$

(20)

where (a) follows since $j' \in \arg \min_{j \in [m]} \left\{ \frac{p'_j}{w_{tj}} \right\}$ and $x_t(p)$ corresponds to the optimal solution to the individual optimization Problem (1a)-(1c), (b) follows by rearranging the right hand side of the equation in (a). Next, (c) follows from the fact that $\log(1 + x) \leq x$ for $x > -1$ and that the difference between two indicators can be at most one. Inequality (d) follows by the upper bound on the budgets of users and the lower bound on the price vector. The final inequality (e) follows from the norm equivalence property which holds for the one and two norms.

Finally, using Equations (20) and (19), we obtain the following generic upper bound on the regret of any online algorithm $\pi$:

$$
\mathbb{E}[U'_n - U_n(\pi)] \leq \frac{2\sqrt{mw}}{p} \sum_{t=1}^{n} \mathbb{E} \left[ \|p^* - p^t\|_2 \right] + \mathbb{E} \left[ \bar{p} \sum_{j=1}^{m} \left( \sum_{t=1}^{n} x_{tj} - c_j \right) \right],
$$

(21)

which proves our claim.
C.2 Proof of Lemma 2

To prove this result, we first prove an upper bound on the expected constraint violation in terms of the stopping time $\tau$ of the algorithm. Then, we establish a lower bound on the expected value of the stopping time to establish the constant constraint violation bound.

**Upper Bound on constraint violation in terms of stopping time:** We begin by establishing that the constraint violation of Algorithm 1 is upper bounded by $O(\mathbb{E}[n - \tau])$, where the stopping time $\tau = \min\{t \leq n : d_t \notin [d - \Delta, d + \Delta]\} \cup \{n\}$. To this end, first note by the definition of $\tau$ and that $\Delta < d$ that there are no constraints are violated up until user $\tau$. Furthermore, since consumption $x_{tj} \leq \frac{w}{p}$ for all $t > \tau$, it follows that the constraint violation

$$
\mathbb{E} \left[ \left\| \sum_{t=1}^{n} x_{tj} - c_j \right\|^2 \right] \leq \mathbb{E} \left[ \left\| \sum_{t=\tau+1}^{n} x_{tj} \right\|^2 \right] \leq \mathbb{E} \left[ (n - \tau)\sqrt{\frac{w}{p}} \right] = O(\mathbb{E}[n - \tau]).
$$

(22)

**Bound on Expected Stopping time $\tau$:** From the above analysis, we observed that bounding the expected constraint violation amounts to obtaining a bound on the expected stopping time $\tau$. To this end, we first introduce some notation. In particular, as in [31], we define the following auxiliary process:

$$\tilde{d}_t = \begin{cases} 
  d_t, & t < \tau \\
  d_\tau, & t \geq \tau 
\end{cases}$$

Then, we can obtain a generic bound on the expected stopping time by observing that

$$
\mathbb{E}[\tau] = \sum_{t=1}^{n} t\mathbb{P}(\tau = t) = \sum_{t=1}^{n} \mathbb{P}(\tau \geq t) = \sum_{t=1}^{n} (1 - \mathbb{P}(\tau < t)) \geq \sum_{t=1}^{n} (1 - \mathbb{P}(\tau \leq t)),
$$

$$\overset{(a)}{\geq} \sum_{t=1}^{n} \left[ 1 - \mathbb{P}(d_s \notin [d - \Delta, d + \Delta] \text{ for some } s \leq t) \right],$$

$$\overset{(b)}{\geq} n - \sum_{t=1}^{n} \mathbb{P}\left(d_s \notin [d - \Delta, d + \Delta] \text{ for some } s \leq t \right),$$

where (a) follows by the definition of $\tau$, (b) follows since the auxiliary process $\tilde{d}_s$ is identical to $d_s$ for all $s$ less than $\tau$. The above analysis implies that

$$
\mathbb{E}[n - \tau] \leq \sum_{t=1}^{n} \mathbb{P}\left(d_s \notin [d - \Delta, d + \Delta] \text{ for some } s \leq t \right). \tag{23}
$$

Thus, to obtain an upper bound for $\mathbb{E}[n - \tau]$, we now proceed to finding an upper bound for the term $\mathbb{P}\left(d_s \notin [d - \Delta, d + \Delta] \text{ for some } s \leq t \right)$ for each user $t \in [n]$.

**Upper bound on $\mathbb{P}\left(d_s \notin [d - \Delta, d + \Delta] \text{ for some } s \leq t \right)$:** To obtain an upper bound on this term, we leverage Hoeffding’s inequality:

**Lemma 4.** (Hoeffding’s Inequality [61]) Suppose there is a sequence of random variables, $\{X_t\}^{n}_{t=1}$, adapted to a filtration $\mathcal{H}_t$, and $\mathbb{E}[X_t | \mathcal{H}_{t-1}] = 0$ for all $t \in [n]$, where $\mathcal{H}_0 = \emptyset$. Suppose further that $L_t$ and $U_t$ are $\mathcal{H}_{t-1}$ measurable random variables such that $L_t \leq X_t \leq U_t$ almost surely for all $t \in [n]$. Then, letting $S_t = \sum_{s=1}^{t} X_s$ and $V_t = \sum_{s=1}^{t} (U_s - L_s)^2$, the following inequality holds for any constants $b,c > 0$:

$$
\mathbb{P}\left(|S_t| \geq b, V_t \leq c^2 \text{ for some } t \in \{1, \ldots, T\} \right) \leq 2e^{-\frac{2c^2}{24b}}.
$$

To leverage Lemma 4, we begin by introducing some notation. First define $Y_{tj} := \tilde{d}_{j,t+1} - \tilde{d}_{j,t}$ and $X_{tj} := Y_{tj} - \mathbb{E}[Y_{tj} | \mathcal{H}_{t-1}]$ for $t \geq 1$, where $\mathcal{H}_{t-1} = ((w_1, u_1), \ldots, (w_{t-1}, u_{t-1}))$ is the history of observed budget and utility parameters.
Next, observe for \( t \geq \tau \) that \( \tilde{d}_{j,t+1} = \tilde{d}_{j,t} \) and when \( 1 \leq t < \tau \) we have that:

\[
\tilde{d}_{j,t+1} = d_{j,t+1} = \frac{c_{j,t+1}}{n-t} - \frac{c_{jt}-x_{jt}}{n-t} = d_{jt} - \frac{1}{n-t}(x_{jt} - \tilde{d}_{jt}) = \tilde{d}_{jt} - \frac{1}{n-t}(x_{jt} - \tilde{d}_{jt})
\]

Next, noting that \( \tilde{d}_{jt} \) is \( \mathcal{H}_{t-1} \) measurable, we have that:

\[
|X_{tj}| = \frac{1}{n-t}(x_{jt} - \tilde{d}_{jt}) - \mathbb{E}\left[\frac{1}{n-t}(x_{jt} - \tilde{d}_{jt}) | \mathcal{H}_{t-1}\right] \\
= \frac{1}{n-t}|x_{jt} - \mathbb{E}[x_{jt} | \mathcal{H}_{t-1}]| \leq \frac{\bar{w}}{p(n-t)}
\]

for each \( t \leq n-1 \) due to the boundedness of the allocations \( x_{jt} \). Then, defining \( L_t = -\frac{\bar{w}}{p(n-t)} \) and \( U_t = \frac{\bar{w}}{p(n-t)} \), we obtain that

\[
V_t = \sum_{s=1}^{t}(U_s - L_s)^2 = \sum_{s=1}^{t} \frac{4\bar{w}^2}{p^2(n-s)^2} \leq \frac{4\bar{w}^2}{p^2(n-t-1)},
\]

which holds for all \( t \leq n-2 \).

Then, from a direct application of Hoeffding’s inequality (Lemma 4) for some constant \( \Delta' > 0 \) we have that

\[
P\left(\left|\sum_{i=1}^{s} X_{ij}\right| \geq \Delta' \text{ for some } s \leq t\right) \leq 2e^{-\frac{s\Delta'^2(n-t-1)}{2p^2}}.
\]

Next, we observe that

\[
|X_{tj} - Y_{tj}| = |\mathbb{E}[Y_{tj} | \mathcal{H}_{t-1}]|,
\]

\[
= |\mathbb{E}\left[\tilde{d}_{j,t+1} - \tilde{d}_{j,t} | \mathcal{H}_{t-1}\right]|
\]

\[
= (a) \left|\frac{1}{n-t} \mathbb{E}\left[(x_{jt} - \tilde{d}_{jt}) I(t < \tau) | \mathcal{H}_{t-1}\right]\right|
\]

\[
= 0,
\]

where (a) follows since the probability distribution is exactly known in Algorithm 1 and thus the term \( \mathbb{E}\left[(x_{jt} - \tilde{d}_{jt}) I(t < \tau) | \mathcal{H}_{t-1}\right] = 0 \) for all users \( t < \tau \).

Then, to obtain a bound on \( P(\tilde{d}_s \notin [d - \Delta, d+\Delta] \text{ for some } s \leq t) \), we first note the following key relation for the set \( \{\tilde{d}_s \notin [d - \Delta, d+\Delta] \text{ for some } s \leq t\}\):

\[
\left\{\left|\tilde{d}_{j,s} - d_j\right| > \Delta_j \text{ for some } s \leq t\right\} = (a) \left\{\left|\sum_{i=1}^{s-1} Y_{ij}\right| > \Delta_j \text{ for some } s \leq t\right\}
\]

\[
= \left\{\sum_{i=1}^{s} Y_{ij} > \Delta_j \text{ for some } s \leq t-1\right\},
\]

\[
= (b) \left\{\sum_{i=1}^{s} X_{ij} > \Delta_j \text{ for some } s \leq t-1\right\},
\]

where (a) follows from the definition of \( Y_i \), and (b) follows since \( \sum_{i=1}^{s} X_{ij} = \sum_{i=1}^{s} Y_{ij} \), as proved in Equation (25). Then setting \( \Delta' = \min_{j \in [m]} \Delta_j = \Delta \) in Equation (24), and applying a union bound over all the goods \( j \in [m] \), we obtain that

\[
P(\tilde{d}_s \notin [d - \Delta, d+\Delta] \text{ for some } s \leq t) \leq 2me^{-\frac{s\Delta^2(n-t-1)}{2p^2}},
\]

which holds for all \( t \leq n-2 \).
Constant Bound on Expected Constraint Violation: We have already observed from our earlier analysis that the expected constraint violation is upper bounded by \( O(\mathbb{E}[n - \tau]) \), where

\[
\mathbb{E}[n - \tau] \leq \sum_{t=1}^{n} \mathbb{P}(d_s \notin [d - \Delta, d + \Delta] \text{ for some } s \leq t)
\]

follows from Equation (23). Thus, we now use the obtained upper bound on \( \mathbb{P}(d_s \notin [d - \Delta, d + \Delta] \text{ for some } s \leq t) \) (Equation (26)) for any \( t \leq n - 2 \) to show that \( \mathbb{E}[n - \tau] \) is bounded above by a constant. To see this, observe that

\[
\mathbb{E}[n - \tau] \leq 2 + \sum_{t=1}^{n-2} \mathbb{P}(d_s \notin [d - \Delta, d + \Delta] \text{ for some } s \leq t),
\]

\[
\leq 2 + 2m \sum_{s=1}^{n-2} e^{-\frac{p^2}{2s^2}},
\]

\[
= 2 + 2me^{-\frac{p^2}{2s^2}} (1 - e^{-\frac{p^2}{2s^2}}),
\]

\[
\leq 2 + 2m \left[ \frac{1}{1 - e^{-\frac{p^2}{2s^2}}} \right],
\]

\[
= O(m).
\]

The above analysis for the upper bound on the term \( \mathbb{E}[n - \tau] \) along with Equation (22) establishes the constant upper bound on the expected constraint violation for Algorithm 1, as

\[
\mathbb{E} \left[ \left\| \sum_{t=1}^{n} x_{tj} - c_j \right\|_2 \right] \leq O(\mathbb{E}[n - \tau]) \leq O(m).
\]

This completes the proof of our claim that the constraint violation of Algorithm 1 is bounded by a constant independent of the number of users \( n \).

C.3 Proof of Lemma 3

We analyse the first term in the generic regret bound in Equation (38) for Algorithm 1 and establish that

\[
\frac{2\sqrt{m \nu}}{P} \sum_{t=1}^{n} \mathbb{E}[\|p^* - p^t\|_2] \leq O(\log(n)).
\]

To this end, we first show that \( \mathbb{E}[\|p^t - p^\star\|_2] \leq O(\frac{1}{\sqrt{n + k + 1}}) \) for all \( t = \{1, \ldots, n - \tau\} \) for Algorithm 1. To see this, we proceed by induction. For the base case, take \( t = 1 \), in which case Algorithm 1 initializes the price \( p^1 = p^\star \), as the adaptive expected equilibrium pricing algorithm sets the static expected equilibrium prices at \( t = 1 \) as \( d_1 = \frac{c}{n} \). As a result, it clearly holds that \( \mathbb{E}[\|p^1 - p^\star\|_2] = 0 \leq O(\frac{1}{\sqrt{n}}) \). For the inductive step, we now assume that \( \mathbb{E}[\|p^t - p^\star\|_2] \leq O(\frac{1}{\sqrt{n + k + 1}}) \) for all \( t \leq k \). Then, we have for \( t = k + 1 \) that

\[
\mathbb{E}[\|p^{k+1} - p^\star\|_2] \leq \mathbb{E}[\|p^{k+1} - p^k\|_2] + \mathbb{E}[\|p^k - p^\star\|_2],
\]

\[
\leq \mathbb{E}[\|p^{k+1} - p^k\|_2] + O\left(\frac{1}{n - k + 1}\right),
\]

\[
\leq \mathbb{E}[\|p^{k+1} - p^k\|_2] + O\left(\frac{1}{n - k}\right).
\]

(28)
To bound $\mathbb{E}[\|p^{k+1} - p^k\|_2]$, we note that $d_{k+1} = d_k + \frac{d_k - x_k(p^k)}{n-p^k}$, i.e., $\|d_{k+1} - d_k\| = O\left(\frac{1}{n-k}\right)$. Then, using Assumption 2, it follows that $\mathbb{E}[\|p^{k+1} - p^k\|_2] = O\left(\frac{1}{n-k}\right)$. This inequality, together with Equation (28), implies that

$$
\mathbb{E}[\|p^{k+1} - p^*\|_2] \leq O\left(\frac{1}{n-k}\right), \tag{29}
$$

which establishes our inductive step and thus establishes our claim that $\mathbb{E}[\|p^t - p^*\|_2] \leq O\left(\frac{1}{n-t+1}\right)$ for all $t = \{1, \ldots, n - \tau\}$ for Algorithm 1. Furthermore, observe that since $p^t = p^*$ for $t > \tau$, it holds that $\mathbb{E}[\|p^t - p^*\|_2] \leq O\left(\frac{1}{n-t+1}\right)$ for all $t = \{1, \ldots, n\}$. Using this result, we obtain the following upper bound on the first term of Equation (21)

$$
\frac{2\sqrt{m\bar{w}}}{p} \sum_{t=1}^{n} \mathbb{E}\left[\|p^* - p^t\|_2\right] \leq \frac{2\sqrt{m\bar{w}}}{p} \sum_{t=1}^{n} \mathbb{E}\left[O\left(\frac{1}{n-t+1}\right)\right] \leq \frac{2\sqrt{m\bar{w}}}{p} \sum_{t=1}^{n} \mathbb{E}\left[O\left(\frac{1}{t}\right)\right] \leq O(\log(n)), \tag{30}
$$

which proves our claim.

### C.4 Proof of Corollary 2

We now use the generic bound on the regret derived in Equation (21) to obtain a $O(\log(n))$ bound on the regret of Algorithm 1. In particular, we upper bound both the terms on the right hand side of Equation (21) using the analysis performed in Lemmas 2 and 3 to establish that

$$
\mathbb{E}[U_n^* - U_n(\pi)] \leq \frac{2\sqrt{m\bar{w}}}{p} \sum_{t=1}^{n} \mathbb{E}\left[\|p^* - p^t\|_2\right] + \mathbb{E}\left[\bar{p}\left| \sum_{j=1}^{m} \left(\sum_{i=1}^{n} x_{ij} - c_j\right)\right|\right] \leq O(\log(n)).
$$

To establish the above claim, we first observe by Lemma 3 that the first term of right hand side of the generic regret bound, i.e., Equation (21), is upper bounded by $O(\log(n))$. Next, noting that the second term on the right hand side of Equation (21) is analogous to the constraint violation of Algorithm 1, we observe that

$$
\mathbb{E}\left[\left| \sum_{j=1}^{m} \left(\sum_{i=1}^{n} x_{ij} - c_j\right)\right|\right] \overset{(a)}{\leq} \mathbb{E}\left[\bar{p}\left| \sum_{j=1}^{m} \left(\sum_{i=t+\tau}^{n} x_{ij}\right)\right|\right] \overset{(b)}{\leq} \bar{p}\mathbb{E}\left[m(n - \tau)\frac{\bar{w}}{p}\right] \overset{(c)}{\leq} \frac{m\bar{w}\bar{p}}{p} O(m), \tag{31}
$$

where (a) follows since no constraints are violated up until the stopping time $\tau$, (b) follows as $x_{ij} \leq \frac{\bar{w}^j}{\bar{p}}$, and (c) follows from Equation (27). As a result, we have established that the second term in the generic regret bound is bounded above by a constant (and thus is also bounded above by $O(\log(n))$) for Algorithm 1, which thus proves our claim.

### D Derivation of Dual of Social Optimization Problem

In this section, we derive the dual of the social optimization Problem (2a)-(2c). To this end, we first consider the following equivalent primal problem

$$
\max_{x_t \in \mathbb{R}^m, u_t} \quad U(x_1, \ldots, x_n) = \sum_{t=1}^{n} w_t \log(u_t), \tag{32a}
$$

s.t.

$$
\sum_{i=1}^{n} x_{ij} \leq c_j, \quad \forall j \in [m], \tag{32b}
$$

$$
x_{ij} \geq 0, \quad \forall t \in [n], j \in [m], \tag{32c}
$$

$$
u_t = \sum_{j=1}^{m} u_{tj} x_{ij}, \quad \forall t \in [n], \tag{32d}$$

32
where we replaced the linear utility \( \sum_{j=1}^{m} u_{tj} x_{tj} \) in the objective with the variable \( u_t \) and added the constraint \( u_t = \sum_{j=1}^{m} u_{tj} x_{tj} \). Observe that the optimal solution of this problem is equal to that of the social optimization Problem (2a)-(2c). We now formulate the Lagrangian of this problem and derive the first order conditions of this Lagrangian to obtain the dual Problem (4).

To formulate the Lagrangian of Problem (32a)-(32d), we introduce the dual variables \( p_j \) for each good \( j \in [m] \) for the capacity Constraints (32b), \( \lambda_{ij} \) for each user \( t \in [n] \) and good \( j \in [m] \) for the non-negativity Constraints (32c), and \( \beta_t \) for each user \( t \in [n] \) for the linear utility Constraints (32d). For conciseness, we denote \( \mathbf{p} \in \mathbb{R}^m \) as the vector of dual variables of the capacity Constraints (32b), \( \Lambda \in \mathbb{R}^{n \times m} \) as the matrix of dual variables of the non-negativity Constraints (32c), and \( \beta \) as the vector of dual variables of the linear utility Constraints (32d). Then, we have the following Lagrangian:

\[
\mathcal{L}(\{x_t, u_{tj}\}_{t=1}^{n}, \mathbf{p}, \Lambda, \beta) = \sum_{t=1}^{n} w_t \log(u_t) - \sum_{j=1}^{m} p_j \left( \sum_{t=1}^{n} x_{tj} - c_j \right) - \sum_{t=1}^{n} \sum_{j=1}^{m} \lambda_{ij} x_{tj} - \sum_{t=1}^{n} \beta_t (u_t - \sum_{j=1}^{m} u_{tj} x_{tj})
\]

Next, we observe from the first order derivative condition of the Lagrangian that

\[
\frac{\partial \mathcal{L}}{\partial u_t} = \frac{w_t}{u_t} - \beta_t = 0, \quad \forall t \in [n], \text{ and}
\]

\[
\frac{\partial \mathcal{L}}{\partial x_{tj}} = -p_j - \lambda_{ij} + \beta_t u_{tj} = 0, \quad \forall t \in [n], j \in [m].
\]

Note that we can rearrange the first equation to obtain that \( u_t = \frac{w_t}{\beta_t} \) for all \( t \in [n] \). Furthermore, by the sign constraint that \( \lambda_{ij} \leq 0 \) for all \( t \in [n], j \in [m] \) it follows from the second equation that \( \beta_t u_{tj} \leq p_j \) for all \( t \in [n], j \in [m] \). Using the above equations, we can write the following dual problem:

\[
\min_{\mathbf{p}, \beta} \sum_{t=1}^{n} w_t \log(u_t) - \sum_{t=1}^{n} w_t \log(\beta_t) + \sum_{j=1}^{m} p_j c_j - \sum_{t=1}^{n} w_t - \beta_t u_{tj} \leq p_j, \quad \forall t \in [n], j \in [m]
\]

Note that at the optimal solution to the above problem \( \beta_t = \min_{j \in [m]} \{ \frac{p_j}{u_{tj}} \} \). Using this observation, we can rewrite the above problem as

\[
\min_{\mathbf{p}} \sum_{t=1}^{n} w_t \log(w_t) - \sum_{t=1}^{n} w_t \log \left( \min_{j \in [m]} \{ \frac{p_j}{u_{tj}} \} \right) + \sum_{j=1}^{m} p_j c_j - \sum_{t=1}^{n} w_t,
\]

which is the dual Problem (4).

E Proof of Theorem 3

To establish this result, we proceed in three steps. First, we first prove an \( O(\sqrt{n}) \) upper bound on the constraint violation for the price update rule in Algorithm 2. Then, to establish an upper bound on the regret, we establish a generic bound on the regret (different from that in Lemma 1 in the proof of Theorem 2) of any online algorithm as long as the prices \( \mathbf{p}_t \) are strictly positive and bounded for all users \( t \in [n] \). Finally, we apply the price update rule in Algorithm 2 to establish an \( O(\sqrt{n}) \) upper bound on the regret for \( \gamma = \gamma_t = \frac{\bar{D}}{\sqrt{n}} \) for all users \( t \in [n] \) for some constant \( \bar{D} > 0 \).

**Expected Constraint Violation Bound:** To establish an \( O(\sqrt{n}) \) upper bound on the constraint violation, we utilize the price update rule in Algorithm 2 where \( \gamma_t = \frac{\bar{D}}{\sqrt{n}} \) for some constant \( \bar{D} > 0 \). In particular, the price update step

\[
p_{j+1} = p_j - \frac{\bar{D}}{\sqrt{n}} (d_j - x_{tj})
\]
in Algorithm 2 can be rearranged to obtain

\[ x_{tj} - d_j = \frac{\sqrt{n}}{D} (p_j^{t+1} - p_j^t). \]

Summing this equation over all arriving users \( t \in [n] \), it follows that

\[ \sum_{t=1}^{n} x_{tj} - c_j \leq \frac{\sqrt{n}}{D} \sum_{t=1}^{n} (p_j^{t+1} - p_j^t) = \frac{\sqrt{n}}{D} (p_j^{n+1} - p_j^1) \leq \frac{\sqrt{n}}{D} p_j^{n+1} \leq \bar{p} D \sqrt{n}, \]

where the last inequality follows since \( p_j^{n+1} \leq \bar{p} \) by the boundedness assumption on the price vector. Using this relation, the norm of the constraint violation can be bounded as

\[ \left\| \left( \sum_{t=1}^{n} x_t - c \right) \right\|_2 \leq \left\| \sum_{t=1}^{n} x_t - c \right\|_2 = \sqrt{\sum_{j=1}^{m} \left( \sum_{t=1}^{n} x_{tj} - c_j \right)^2} \leq \sqrt{\sum_{j=1}^{m} \left( \frac{\bar{p}}{D} \right)^2 n} = \sqrt{m \left( \frac{\bar{p}}{D} \right)^2} n \leq O(\sqrt{n}). \]

Taking an expectation of the above quantity, we obtain a \( O(\sqrt{n}) \) upper bound on the expected constraint violation, where \( \mathbb{E}[V_t(x_1, \ldots, x_n)] \leq \frac{\bar{p}}{D} \sqrt{mn} = O(\sqrt{n}). \)

**Generic Bound on the Regret:** We now turn to establishing a generic bound on the regret of any online algorithm for which the price vector \( \mathbf{p}^t \) is strictly positive and bounded for each user \( t \in [n] \). To perform our analysis, let \( \mathbf{p}^* \) be the optimal price vector for the following stochastic program

\[ \min_{\mathbf{p}} D(\mathbf{p}) = \sum_{j=1}^{m} p_j^* d_j + \mathbb{E} \left[ \left( w \log(w) - w \log \left( \min_{j \in [m]} \frac{p_j^*}{u_{tj}} \right) - w \right) \right]. \tag{35} \]

Then, by duality we have that the primal objective value \( U_n^* \) is no more than the dual objective value with \( \mathbf{p} = \mathbf{p}^* \), which gives the following upper bound on the optimal objective

\[ U_n^* \leq \sum_{j=1}^{m} p_j^* c_j + \sum_{t=1}^{n} \left( w_t \log(w_t) - w_t \log \left( \min_{j \in [m]} \frac{p_j^*}{u_{tj}} \right) - w_t \right). \]

Then, taking an expectation on both sides of the above inequality, it follows that

\[ \mathbb{E}[U_n^*] \leq \mathbb{E} \left[ \sum_{j=1}^{m} p_j^* c_j + \sum_{t=1}^{n} \left( w_t \log(w_t) - w_t \log \left( \min_{j \in [m]} \frac{p_j^*}{u_{tj}} \right) - w_t \right) \right], \]

\[ = n D(\mathbf{p}^*), \]

by the definition of \( D(\mathbf{p}) \) in Problem (35). Finally, noting that \( \mathbf{p}^* \) is the optimal solution to the stochastic Program (35), it follows that

\[ \mathbb{E}[U_n^*] \leq n D(\mathbf{p}^*) \overset{(a)}{\leq} \sum_{t=1}^{n} \mathbb{E} \left[ D(\mathbf{p}^t) \right] \overset{(b)}{=} \sum_{t=1}^{n} \mathbb{E} \left[ \sum_{j=1}^{m} p_j^t d_j + w_t \log(w_t) - w_t \log \left( \min_{j \in [m]} \frac{p_j^t}{u_{tj}} \right) - w_t \right], \]

\[ \overset{(c)}{=} \mathbb{E} \left[ \sum_{t=1}^{n} \left( \sum_{j=1}^{m} p_j^t d_j + w_t \log(w_t) - w_t \log \left( \min_{j \in [m]} \frac{p_j^t}{u_{tj}} \right) - w_t \right) \right], \]

where (a) follows by the optimality of \( \mathbf{p}^* \) for the stochastic Program (35), (b) follows by the definition of \( D(\mathbf{p}^t) \), and (c) follows from the linearity of expectations.
Next, let \( j_t \) be a good in the optimal consumption set \( S^*_t \) for user \( t \) given the price vector \( \mathbf{p}^t \). Then, the true accumulated social welfare objective under an algorithm \( \pi \) can be expressed as

\[
U_n(\pi) = \sum_{t=1}^{n} w_t \log \left( \sum_{j=1}^{m} u_{tj} x_{tj} \right),
\]

\[
= \sum_{t=1}^{n} w_t \log \left( \sum_{j=1}^{m} u_{tj} \mathbb{I}_{j=j_t} \frac{w_t}{p_{j_t}} \right),
\]

which follows since the utility when consuming any feasible bundle of goods in their optimal consumption set equals their utility when purchasing \( \frac{w_t}{p_{j_t}} \) units of good \( j_t \in S^*_t(\mathbf{p}^t) \). Finally combining the upper bound on the expected optimal objective and the above obtained relation on the accumulated objective under an algorithm \( \pi \), we obtain the following upper bound on the expected regret

\[
E[U^*_n - U_n(\pi)] \leq E \left[ \sum_{t=1}^{n} \left( \sum_{j=1}^{m} p_j^t d_j + w_t \log(w_t) - w_t \log \left( \min_{j \in [m]} \frac{p_j^t}{u_{tj}} \right) - w_t \right) \right]
\]

\[
= E \left[ \sum_{t=1}^{n} \sum_{j=1}^{m} p_j^t d_j - w_t \right].
\]

**Square Root Regret Bound:** We now use the generic regret bound derived in Equation (38) for any online algorithm with bounded prices that are always strictly positive for each \( t \in [n] \) to obtain an \( O(\sqrt{n}) \) upper bound on the regret of Algorithm 2. In particular, we use the price update equation in Algorithm 2 to derive the \( O(\sqrt{n}) \) regret bound. We begin by observing from the price update equation that

\[
\|\mathbf{p}^{t+1}\|_2^2 = \|\mathbf{p}^t - \frac{D}{\sqrt{n}} (\mathbf{d} - \mathbf{x}_t)\|_2^2.
\]

Expanding the right hand side of the above equation, we obtain that

\[
\|\mathbf{p}^{t+1}\|_2^2 \leq \|\mathbf{p}^t\|_2^2 - \frac{2D}{\sqrt{n}} \left( \sum_{j=1}^{m} p_j^t d_j - \sum_{j=1}^{m} p_j^t x_{tj} \right) + \frac{D^2}{n} \|\mathbf{d} - \mathbf{x}_t\|_2^2.
\]

We can then rearrange the above equation to obtain

\[
\sum_{j=1}^{m} p_j^t d_j - \sum_{j=1}^{m} p_j^t x_{tj} \leq \frac{\sqrt{n}}{2D} \left( \|\mathbf{p}^t\|_2^2 - \|\mathbf{p}^{t+1}\|_2^2 \right) + \frac{D}{2\sqrt{n}} \|\mathbf{d} - \mathbf{x}_t\|_2^2.
\]

Finally, summing both sides of the above equation over \( t \in [n] \), we get

\[
\sum_{t=1}^{n} \frac{\sqrt{n}}{2D} \sum_{j=1}^{m} \left( \|\mathbf{p}^t\|_2^2 - \|\mathbf{p}^{t+1}\|_2^2 \right) + \sum_{t=1}^{n} \frac{D}{2\sqrt{n}} \|\mathbf{d} - \mathbf{x}_t\|_2^2,
\]

\[
= \left( \sum_{t=1}^{n} \frac{\sqrt{n}}{2D} \left( \|\mathbf{p}^t\|_2^2 - \|\mathbf{p}^{t+1}\|_2^2 \right) + \sum_{t=1}^{n} \frac{D}{2\sqrt{n}} \|\mathbf{d} - \mathbf{x}_t\|_2^2, \right)
\]

\[
\leq \sqrt{n} \left( \|\mathbf{p}^1\|_2^2 + \frac{D m}{2} \left( \max_{j \in [m]} d_j + \frac{\bar{w}}{\bar{p}} \right)^2 \right),
\]

\[
\leq O(\sqrt{n}),
\]
where the (a) follows by the boundedness of the consumption vector for each agent, since the prices are strictly positive and bounded below by $p > 0$. Finally, noting that all agents completely spend their budget at the optimal solution of the individual optimization problem, i.e., $\sum_{j \in [M]} p^t_j x_{ij} = w_i$, we obtain from the generic regret bound in Equation (38) that

$$
\mathbb{E}[U^*_n - U_n(\pi)] \leq \mathbb{E} \left[ \sum_{i=1}^n \sum_{j=1}^m p^t_j d_{ij} - w_i \right] = \sum_{i=1}^n \mathbb{E} \left[ \sum_{j=1}^m p^t_j d_{ij} - \sum_{i=1}^n \sum_{j=1}^m p^t_j x_{ij} \right],
$$

\begin{align*}
&\leq \sqrt{n} \left( \frac{\|p^t\|^2}{2D} + \frac{\bar{D}m}{2} \left( \max_{j \in [m]} \frac{\bar{\omega}}{\bar{w}} + \frac{\bar{d}}{\bar{p}} \right)^2 \right), \\
&= O(\sqrt{n}),
\end{align*}

where (a) follows from Equation (41). Thus, we have proven the $O(\sqrt{n})$ upper bound on the expected regret of Algorithm 2 under the assumed conditions on the price vectors $p^t$ for all $t \in [n]$.

\[\square\]

### F Remarks on the Positivity and Boundedness of Prices in Algorithm 2

In this section, we present a few remarks regarding the assumption in the statement of Theorem 3 that the price vector $p^t$ is strictly positive and bounded during the operation of Algorithm 2 for all $t \in [n]$. In particular, we first show that the strict positivity of prices during the operation of Algorithm 2 implies that the prices are bounded for all $t \in [n]$ in Appendix F.1. Next, in Appendix F.2, we present further motivation for why the price vector in Algorithm 2 will remain strictly positive with high probability as the number of users $n$ grows large.

#### F.1 Positivity of Prices Implies Boundedness

We show through the following lemma that if the price vector $p^t$ is bounded below by some vector $\bar{p}$ at each iteration of Algorithm 2, then the price vector also remains bounded above by $\bar{p}$, where each component of $\bar{p}$ is a constant.

**Lemma 5** (Positivity Implies Price Boundedness in Algorithm 2). Suppose that the budget and utility parameters of users are drawn i.i.d. from a distribution $D$ satisfying Assumption 1, and the price vector $p^t \geq \bar{p} > 0$ for all users $t \in [n]$. Then, the price vector $p^t$ corresponding to Algorithm 2 is bounded at each time an agent $t \in [n]$ arrives, i.e., $p^t \leq \bar{p}$ for all $t \in [n]$ for some vector $\bar{p} \geq \bar{p}$, when the step-size $\gamma = \gamma_t = \frac{D}{\sqrt{n}}$ for some $0 < D \leq 1$.

**Proof.** We establish that the prices of all goods are always bounded above at each step of Algorithm 2 if the prices of the goods are bounded below by $\bar{p}$ at each step. To show that the prices are bounded above, we consider the settings when (i) $\|p^t\|_2 \geq \frac{m(\bar{d} + \bar{w})^2 + 2\bar{w}}{2d}$, and (ii) $\|p^t\|_2 \leq \frac{m(\bar{d} + \bar{w})^2 + 2\bar{w}}{2d}$, where $\bar{d} = \max_{j \in [m]} d_j$ and $\bar{d} = \min_{j \in [m]} d_j > 0$. In case (i), we observe that

$$
\|p^{t+1}\|_2 = \|p^t - \gamma (d - x^t)\|_2 = \|p^t\|_2 - 2\gamma (p^t)^\top (d - x^t) + \gamma^2 \|d - x^t\|_2^2,
$$

\begin{align*}
&\leq \|p^t\|_2^2 + 2\gamma \bar{w} + \gamma^2 m \left( \bar{d} + \frac{\bar{w}}{\bar{p}} \right)^2 - 2\gamma (p^t)^\top d, \\
&\leq \|p^t\|_2^2 + 2\gamma \bar{w} + \gamma^2 m \left( \bar{d} + \frac{\bar{w}}{\bar{p}} \right)^2 - 2\gamma d \|p^t\|_1, \\
&\leq \|p^t\|_2^2 + 2\gamma \bar{w} + \gamma^2 m \left( \bar{d} + \frac{\bar{w}}{\bar{p}} \right)^2 - 2\gamma d \|p^t\|_2, \\
&\leq \|p^t\|_2^2,
\end{align*}

36
where (a) follows from the fact that $(p^t)^\top x^t = w_t \leq \bar{w}$ and $x^t \leq \frac{\bar{w}}{\gamma} \mathbf{1}$, where $\mathbf{1}$ is an $m$-dimensional vector of all ones, (b) follows from the norm equivalence relation between the one and the two norms, and (c) follows for any step-size $\gamma \leq 1$.

Next, in case (ii) it holds that
\[
\|p^{t+1}\|_2 = \|p^t - \gamma(d - x_t)\|_2 \leq \|p^t\|_2 + \gamma \|d - x_t\|_2 \leq \|p^t\|_2 + \gamma \|d - x_t\|_1,
\]
\[
\leq \frac{m(d + \bar{w})^2 + 2\bar{w}}{2\bar{d}} + m\left(\bar{d} + \frac{\bar{w}}{\gamma}\right),
\]
where (a) follows by the triangle inequality, (b) follows from the norm equivalence relation between the one and to norms, and (c) holds since $\|p^t\|_2 \leq \frac{m(d + \bar{w})^2 + 2\bar{w}}{2\bar{d}}$ and $x^t \leq \frac{\bar{w}}{\gamma} \mathbf{1}$.

From the above inequalities, we observe that $\|p^t\|_2 \leq \frac{m(d + \bar{w})^2 + 2\bar{w}}{2\bar{d}} + m\left(\bar{d} + \frac{\bar{w}}{\gamma}\right)$ for all $t$. This relation implies that the price vector of Algorithm 2 is always bounded above when the price vector of Algorithm 2 is bounded below by $p$ at each step, which completes the proof of Lemma 5.\hfill $\Box$

### F.2 Price Positivity During Operation of Algorithm 2

In this section, we present further motivation for why the price vector will remain strictly positive throughout the operation of Algorithm 2. To this end, consider the following natural assumption, which states that the expected consumption of a good by any user is strictly greater than their market share of that good if the price of the good is small.

**Assumption 3.** There exists $\bar{p}$ such that if $p_j < \bar{p}$ for any good $j$, then the distribution $D$ is such that the expected consumption of that good is at least $\frac{d_j}{1-\delta}$ for some $\delta > 0$.

We note that Assumption 3 imposes a mild restriction on the set of allowable distributions from which the utility parameters of users are drawn. In particular, the assumption on the distribution $D$ implies that for each good there are a certain fraction of the arriving users with a sufficiently high utility for that good. As a result if the price of a good drops too low then a certain fraction of users will purchase large quantities of that good that is far greater than their market share for that good. For instance, the distribution $D$ constructed in the counterexample in the proof of Theorem 1 satisfies Assumption 3, as the expected consumption of each good is strictly greater than each user’s market share $d_j$ of that good if its price drops strictly below 0.5. As a result, Assumption 3 intuitively implies that the price of any good cannot drop “too far” below some specified price $\bar{p}$ during the operation of Algorithm 2.

We can also apply the Chernoff bound in combination with Assumption 3. In particular, as in Assumption 3, let $\bar{p}$ be a constant and let $0 < \bar{p} \leq (1 - \epsilon)\bar{p}$ for small $\epsilon > 0$. We now provide a bound on the probability that the price $p^t_j$ of some good $j$ for some user $t$ drops below $\bar{p}$ during the operation of Algorithm 2. Here, we assume that the initial price $p^1_j$ in Algorithm 2 is sufficiently higher than $p$. We now suppose that $t$ is the first time step at which the price of some good $j$ falls below $\bar{p}$, and that the price of that good stays below $\bar{p}$ for another $k$ steps. Then, we can upper bound the probability that the price $p^t_j$ of some good $j$ for some user $t$ drops below $\bar{p}$ as follows
\[
P[p^t_j \leq \bar{p} \text{ for some good } j \text{ for some user } t] \leq \mathbb{P}[p^t_j \leq (1 - \epsilon)\bar{p} \text{ for some good } j \text{ for some user } t],
\]
\[
\leq \mathbb{P}[\bar{p} - \gamma kd_j + \gamma \sum_{t'=t}^{t+k-1} x_{t'j} \leq (1 - \epsilon)\bar{p} \text{ for some } j, t],
\]
\[
\leq \mathbb{P}[\gamma \sum_{t'=t}^{t+k-1} x_{t'j} \leq -\epsilon \bar{p} + \gamma kd_j \text{ for some } j, t],
\]
\[
\leq \mathbb{P}[\sum_{t'=t}^{t+k-1} x_{t'j} \leq kd_j \text{ for some } j, t],
\]
where (a) follows as \( p \leq (1 - \epsilon)\hat{p} \) for small \( \epsilon > 0 \), (b) follows by the price update rule in Algorithm 2, (c) follows by rearranging the terms in the inequality, and (d) follows as \(-\epsilon \hat{p} < 0\). To upper bound the right hand side term \( P[\sum_{t'=t}^{t+k-1} x_{t'j} \leq kd_j \text{ for some } j, t] \), we begin by noting that the user consumption \( x_{t'j} \) is not i.i.d. since user’s consumption bundles depend on the price, which is inherently dependent on the budget and utility parameters of earlier users by the price update equation of Algorithm 2. However, we can upper bound this probability by defining another set of variables \( y_{t'j} \) that are drawn i.i.d. and serves as a lower bound for \( x_{t'j} \) when \( p'_{tj} \leq \hat{p} \). To this end, let \( E[y_{t'j}] = \mu \geq \frac{d_j}{1-\epsilon} \). Then, using the multiplicative Chernoff bound it follows that

\[
P[\sum_{t'=t}^{t+k-1} x_{t'j} \leq kd_j \text{ for some } j, t] \leq P[\sum_{t'=t}^{t+k-1} y_{t'j} \leq k\mu(1-\delta) \text{ for some } j, t] \leq e^{-\frac{\delta^2 k}{2}}
\]

where (a) follows by Assumption 3 and the aforementioned definition of \( y_{t'j} \), and (b) follows by using the multiplicative Chernoff bound. Finally, noting that \( k \) is the number of steps for which the price of a good \( j \) remains below \( \hat{p} \) and that the step size is \( O(\frac{1}{\sqrt{n}}) \), it follows that for a constant reduction in the price of good \( j \), i.e., for \( p'_{tj} \leq (1 - \epsilon)\hat{p} \), it must hold that \( k = O(\sqrt{n}) \). However, combining the above set of inequalities, this observation implies that

\[
P[p'_{tj} \leq \hat{p} \text{ for some } j \text{ for some user } t] \leq e^{-\frac{\delta^2 k}{2}},
\]

which goes to zero as \( n \to \infty \). Thus, for large \( n \), it follows that the price of each good will always remain bounded below by \( \frac{1}{2} \) with high probability.

### G Additional Numerical Experiments

#### G.1 Numerical Validation of Assumption 2

In this section, we present the results of a numerical experiment to validate Assumption 2. In particular, we consider the instance described in the proof of Theorem 1 with \( n = 10,000 \) users, where all users have a fixed budget of one, and two goods, each with a capacity of \( c_j = n = 10,000 \). The utility parameters of users are drawn i.i.d. from a distribution \( D \), where users have an equal 0.5 probability of having the utility \((1,0)\) or \((0,1)\).

Figure 3 depicts the change in the dual prices of the certainty equivalent problem between subsequent iterations of Algorithm 1 for this instance. To see that Assumption 2 is satisfied, first note that the norm of the difference between the average remaining resource capacities between subsequent time steps is \( O(\frac{1}{n^2}) \), i.e., \( \|d_{t+1} - d_t\| \leq O(\frac{1}{n^2}) \) as \( d_{t+1} = d_t + \frac{d_t - x_{t}(p_t)}{n_t} \). Then, Figure 3 implies that the two norm of the change in the dual prices of the certainty equivalent problem, i.e., \( E[\|p^{t+1} - p^t\|] \) is always upper bounded by \( O(\frac{1}{n^2}) \) for all \( t \in [n-1] \), which thus implies that Assumption 2 is satisfied. We note that we present the results on a log plot for readability purposes.

#### G.2 Numerical Experiments Comparing Static and Adaptive Variants of Expected Equilibrium Pricing

In this section, we numerically evaluate the performance of the static expected equilibrium pricing algorithm and its dynamic counterpart (Algorithm 1) on the counterexample in the proof of Theorem 1. In particular, we considered a setting of \( n \) users, where all users have a fixed budget of one, and two goods, each with a capacity of \( n \). The utility parameters of users are drawn i.i.d. from a distribution \( D \), where users have an equal 0.5 probability of having the utility \((1,0)\) or \((0,1)\). For the experiments, we let the number of users \( n \) range between 100 to 20,000 users and average the regret and constraint violation over 300 instances.

Figure 4 depicts both the constraint violation and the regret of the two algorithms. From the figure, it can be observed that the static expected equilibrium pricing approach achieves negative regret for a large constraint violation, while Algorithm 1 achieves a small positive regret for almost no constraint violation. Recall here from the proof of Theorem 1 that the expected optimal social welfare objective \( E[U^*_n] \in [n \log(2) - \ldots] \)
Figure 3: Validation of Assumption 2 for an instance with \( n = 10,000 \) users, where all users have a fixed budget of one, and two goods, each with a capacity of \( c_j = n = 10,000 \).

\[ 1, n \log(2) \], and thus a regret of less than 5 for 20,000 users is negligible. As a result, Figure 4 clearly depicts the benefit of adaptivity in online Fisher markets.

We also note that the results for the static expected equilibrium pricing approach align with those obtained in the proof of Theorem 1 and Corollary 1. In particular, we first note from Figure 4 that the constraint violation is larger than \( \Omega(\sqrt{n}) \). Furthermore, the regret of the static expected equilibrium pricing algorithm is in the range \([-1, 0]\), as the accumulated online objective is \( n \log(2) \), as each user obtains two units of the good for which they have positive utility under the static expected equilibrium prices of \((0.5, 0.5)\) for this instance. As a result, observe that the numerically observed regret in the range \([-1, 0]\) aligns with the tight bound for the expected optimal social welfare objective, i.e., \( n \log(2) - 1 \leq \mathbb{E}[U^*_n] \leq n \log(2) \).

Figure 4: Comparison between the static expected equilibrium pricing algorithm and its dynamic counterpart (Algorithm 1) on regret and constraint violation metrics.
G.3 Numerical Comparison between the Additive and Multiplicative Price Updates in Algorithm 2

We now compare Algorithm 2 that has an additive price update step to a corresponding algorithm with a multiplicative price update step, as in Equation (6). To this end, we consider the market instance described in Section 6.2 and two different step sizes for both the price update steps - (i) \( \gamma = \frac{1}{\sqrt{n}} \) and (ii) \( \gamma = \frac{1}{100 \sqrt{n}} \).

Figures 5 and 6 depict the regret and constraint violation for algorithms with the two price update steps for a step-size of \( \gamma = \frac{1}{\sqrt{n}} \) (Figure 5) and \( \gamma = \frac{1}{100 \sqrt{n}} \) (Figure 6). We can observe from Figure 5 that for a larger step size of \( \gamma = \frac{1}{\sqrt{n}} \), Algorithm 2 with an additive price update rule has a much higher regret and constraint violation as compared to the corresponding algorithm with a multiplicative price update rule. This observation implies the efficacy of the multiplicative price update rule in achieving good regret and constraint violation guarantees and motivates a deeper study of the regret and constraint violation bounds under the multiplicative price update rule.

As opposed to the results obtained for a step-size \( \gamma = \frac{1}{\sqrt{n}} \) in Figure 5, we observe from Figure 6 that for a smaller step-size of \( \gamma = \frac{1}{100 \sqrt{n}} \) the regret of Algorithm 2 is smaller than that of the corresponding algorithm with a multiplicative price update rule. As a result, Figures 5 and 6 show that the choice of the step size \( \gamma \) can be critical to the performance of both Algorithm 2 with an additive price update rule and the corresponding algorithm with a multiplicative price update rule.

![Figure 5: Comparison between Algorithm 2 that has an additive price update step to a corresponding algorithm with a multiplicative price update step, as in Equation (6), on regret and constraint violation metrics. Here the step-size of both the price update steps is \( \gamma = \frac{1}{\sqrt{n}} \).](image)

G.4 Numerical Validation of Positivity of Prices in Algorithm 2

In this section, we present the results of a numerical experiment to validate that the prices remain strictly positive throughout the operation of Algorithm 2. To this end, we consider two market settings: (i) the setting described in the proof of Theorem 1, and (ii) the setting described in Section 6.2. For the experiments, we let the number of users \( n \) range between 100 to 5,000 users, consider a step-size of the price updates as \( \gamma = \frac{1}{100 \sqrt{n}} \), as in Section 6.2, and compute the minimum prices across all goods for 300 instances. In particular, Figure 7 depicts the minimum prices of all goods across 300 instances, which validates the positivity of the prices during the operation of Algorithm 2.
Figure 6: Comparison between Algorithm 2 that has an additive price update step to a corresponding algorithm with a multiplicative price update step, as in Equation (6), on regret and constraint violation metrics. Here the step-size of both the price update steps is $\gamma = \frac{1}{100\sqrt{n}}$.

Figure 7: Numerical validation of the positivity of prices during the operation of Algorithm 2 in two market settings: (i) the market instance in the proof of Theorem 1 (left), and (ii) the market instance described in Section 6.2 (right). The y-axis denotes the minimum price across all goods across 300 problem instances, i.e., 300 runs of Algorithm 2 on different instances drawn from the specified distribution corresponding to each market setting.

H Dynamic Learning SAA Algorithm

We now formally present the dynamic learning benchmark introduced in Section 6.1. In this benchmark, the price vector $p^t$ at each time a user $t$ arrives is obtained based on the dual variables of the capacity constraints of the sampled Eisenberg-Gale program given the observed budget and utility parameters of previous users. We update the prices at geometric intervals, as in earlier work [28, 29], and this price vector is then shown to...
all users arriving in the corresponding interval. Furthermore, each user solves their individual optimization problem to obtain an affordable utility-maximizing bundle of goods under the set prices. This process is presented formally in Algorithm 3.

Algorithm 3: Dynamic Learning SAA Algorithm

\textbf{Input :} Vector of Capacities $c$

Set $\delta \in (1,2]$ and $L > 0$ such that $\lfloor \delta^L \rfloor = n$

Let $t_k = \lfloor \delta^k \rfloor$, $k=1,2,\ldots,L-1$ and $t_L = n+1$

Initialize $p^{t_1} > 0$

Each user $t \in [t_1]$ purchases an optimal bundle of goods $x_t$ by solving Problem (1a)-(1c) given the price $p^{t_1}$

\begin{algorithm}
\textbf{for} $k = 1,\ldots,L-1$ \textbf{do}

\textbf{Phase I: Set Price for Geometric Interval}

Set price $p^{t_k}$ based on dual variables of the capacity constraints of the sampled social optimization problem:

$$
\max_{x_t \in \mathbb{R}^m, \forall t \in [t_k]} U(x_1,\ldots,x_{t_k}) = \sum_{t=1}^{t_k} w_t \log \left( \sum_{j=1}^{m} u_{tj} x_{tj} \right),
$$

\text{s.t.}

$$
\sum_{t=1}^{t_k} x_{tj} \leq \frac{t_k}{n} c_j, \quad \forall j \in [m],
$$

$$
x_{tj} \geq 0, \quad \forall t \in [t_k], j \in [m],
$$

\textbf{Phase II: Each User in Interval Consumes Optimal Bundle}

Each user $t \in \{t_k+1,\ldots,t_{k+1}\}$ purchases an optimal bundle of goods $x_t$ by solving Problem (1a)-(1c) given the price $p^{t_k}$
\end{algorithm}

end