Planar Analytic Functions

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Abstract

If $a$ is a point in the domain of convergence of a planar power series $f$ in a single variable $x$ one can expand $f$ into a planar power series in the variable $(x-a)$. One arrives at the notion of planar analytic functions on any domain $D$ in the complex plane. It can be described by section $S$ of the sheaf of planar germs. The $k$-ary exponential series $\exp_k(x)$ has infinite radius of convergence. It is possible to define a planar analogue $\mathcal{P}$ of the classical zeta-function. As yet a functional equation for $\mathcal{P}$ has not been obtained.

1 Radius of convergence

Let $\mathcal{C}\{\{x\}\}\mathcal{P}$ be the $\mathcal{C}$-algebra of formal planar power series in a variable $x$ over the field $\mathcal{C}$ of complex numbers, see [G1]. An element $f \in \mathcal{C}\{\{x\}\}\mathcal{P}$ has a unique expansion $f = \sum \gamma_T x^T$

where $\mathcal{P}$ is the set of finite, planar, reduced rooted trees and $\gamma_T \in \mathcal{C}$ for all $T$. The coefficient $\gamma_T$ of is also denoted by $\langle f, x^T \rangle$ and thus $f = \sum \langle f, x^T \rangle x^T$.

In this way $\mathcal{C}\{\{x\}\}\mathcal{P}$ is identified with $\mathcal{C}$-vector space of $\mathcal{C}$-valued functions on $\mathcal{P}$ or on $\{x^T : T \in \mathcal{P}\}$.

Definition: $\text{rad}(f) = \sup \{ r \in \mathbb{R}_{\geq 0} : \sum |\gamma_T| r^{\deg(T)} < \infty \}$ where $\deg(T)$ is the number of leaves of $T$ and $|\gamma_T|$ is the absolute value of $\gamma_T$. Then $\text{rad}(f) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ is called the radius of convergence of $f$.

Let now $(f_n)_{n \geq 0}$ be a sequence of planar power series $f_n \in \mathcal{C}\{\{x\}\}\mathcal{P}$. It is called point wise convergent, if $\lim \langle T,f_n \rangle$ for $n \to \infty$ is convergent in $\mathcal{C}$ for all $T \in \mathcal{P}$.
In this case the power series
\[\lim_{n \to \infty} f_n = \sum_{T \in \mathbb{P}} \langle T, f_n \rangle \cdot x^T\]
is the limit of the sequence \((f_n)_{n \geq 0}\).

## 2 Expansion into powers of \((x - a)\)

Let \(\mathbb{P}_n\) be the set of trees \(T\) in \(\mathbb{P}\) with \(\deg(T) \leq n\).

Let \(\mathcal{C}\{x\} \leq n\) be the \(\mathcal{C}\) - vector pace of planar polynomials in \(x\) of degree \(\leq n\). Then by definition the system \(\{x^T : T \in \mathbb{P}_n\}\) is a \(\mathcal{C}\)-basis of \(\mathcal{C}\{x\} \leq n\).

**Proposition 2.1.** Let \(a \in \mathcal{C}\). Then \((x - a)^T : T \in \mathbb{P}_n\) is a \(\mathcal{C}\) - basis of \(\mathcal{C}\{x\} \leq n\).

The coefficient \(\langle f, (x-a)^T \rangle\) of \(f \in \mathcal{C}\{x\} \leq n\) relative to this basis and to the basis element \((x-a)^T\) is given by the formula
\[\langle f, (x-a)^T \rangle = \sum_{U \in \mathbb{P}_n} \langle f, x^U \rangle \cdot (U/T) \cdot a \cdot (\deg(U) - \deg(T))\]
where \((U/T)\) is the planar binomial coefficient of \(U\) over \(T\), see [G7].

Denote by \(\mathcal{C}\{\mathcal{C}\{x-a\}\}\mathbb{P}\) the \(\mathcal{C}\)-algebra of formal planar power series in the variable \(x-a\).

An element \(f \in \mathcal{C}\{\mathcal{C}\{x-a\}\}\mathbb{P}\) has a unique expansion \(f = \sum_{T \in \mathbb{P}} \gamma_T \cdot (x-a)^T\)
with \(\gamma_T \in \mathcal{C}\). The coefficient \(\gamma_T\) is also denoted by \(\langle f, (x-a)^T \rangle\) and thus
\[f = \sum_{T \in \mathbb{P}} \langle f, (x-a)^T \rangle \cdot (x-a)^T\]

In this way the set of \(\mathcal{C}\{\mathcal{C}\{x-a\}\}\mathbb{P}\) is identified with the set of \(\mathcal{C}\) - valued functions on \((x-a)^T : T \in \mathbb{P}\).

A sequence \((f_n)_{n \geq 0}\) in \(\mathcal{C}\{\mathcal{C}\{x\}\}\mathbb{P}\) is pointwise convergent,
if \(\lim_{n \to \infty} \langle f_n, (x-a)^T \rangle\) exists in \(\mathcal{C}\) for all \(T \in \mathbb{P}\).

Let \(f \in \mathcal{C}\{\mathcal{C}\{x\}\}\mathbb{P}\) and \(a \in \mathcal{C}\).

Let \(f_n := \sum_{T \in \mathbb{P}_n} \langle f, x^T \rangle \cdot x^T\) and
\[g_n := \sum_{T \in \mathbb{P}_n} \langle f_n, (x-a)^T \rangle \cdot (x-a)^T\]

Then \(g_n = f_n\) and \(\lim_{n \to \infty} f_n = f\) in \(\mathcal{C}\{\mathcal{C}\{x\}\}\mathbb{P}\).

Now we consider the sequence \((g_n)_{n \geq 0}\) in \(\mathcal{C}\{\mathcal{C}\{x-a\}\}\mathbb{P}\).
**Proposition 2.2.** (a) Assume that \( f \in \mathcal{C}\{x\}_\mathcal{P} \) has radius of convergence \( r > 0 \) and that \( a \in \mathcal{C} \) with \( |a| < r \). Then for all \( T \in \mathcal{P} \) the sequence \( (g_n, (x-a)^T)_{n \geq 0} \) is convergent in \( \mathcal{C} \). If its limit is denoted by \( \gamma_T(a) \), then \( f = \sum \gamma_T(a) \cdot (x-a)^T \in \mathcal{C}\{x-a\}_\mathcal{P} \). It is called the expansion of \( f \) around the place \( a \).

Moreover \( \gamma_T(a) = \langle g, (x-a)^T \rangle = \sum \langle f, x^U \rangle \cdot (U/T) \cdot a^{(\deg(U) - \deg(T))} \).

and the expansion of \( f \) around \( a \) has radius of convergence \( \geq r - |a| \).

**Special cases.**

If \( T = 1 = \text{empty tree} \), then \( (U/T) = 1 \) for all \( n \in \mathcal{P} \) and \( \gamma_1(a) = \sum \gamma_U \cdot a^{\deg(U)} = f(a) \).

If \( T = x \), then \( (U/x) = \deg(U) \), if \( n \neq 1 \) and

\[ \gamma_x(a) = \sum \gamma_U \cdot \deg(U) \cdot a^{(\deg(U) - 1) / \deg(U)} \geq 1 \]

### 3 The notion of planar analytic functions.

Let \( D \) be a region in the complex plane \( \mathcal{C} \) and \( F = (f_a)_{a \in D} \) a collection of planar power series \( f_a \) in the variable \( x - a \) for all \( a \in G \).

**Definition:** \( F \) is called planar analytic function on \( D \) if

(i) \( f_a \) is a convergent planar power series which means that the radius of convergence of \( f_a \) is \( > 0 \).

(ii) if \( a, b \in G \) and if the radius of convergence \( \rad(f_a) > |b-a| \), then the expansion of \( f_a \) around \( b \) is equal to \( f_b \).

Also \( f_a \) is called the germ of \( F \) at \( a \). There is a sheaf \( \mathfrak{S}_{\mathcal{P},D} \) on \( D \) for which the stalk in \( a \in D \) consists of the convergent planar power series in the variable \( (x-a) \).

The sections in \( \Gamma(D, \mathfrak{S}_{\mathcal{P},D}) \) are the planar analytic functions on \( D \).

**Proposition (3.1).** Let \( f \in \mathcal{C}\{x\}_\mathcal{P} \), \( r = \rad(f) > 0 \) and \( G = \{a \in \mathcal{C} : |a| < r\} \).

Let \( f_a \) be the expansion of \( f \) around \( a \) and \( F = (f_a) \) \( a \in G \).

Then \( F \) is a planar analytic function on \( G \).

To prove this statement we have to show the following: Assume that \( a, b \in \mathcal{C} \) with \( |a| < R, |b| < R \) where \( R \) is the radius of convergence of a planar series \( f \) in \( x \), Assume more over that \( |b-a| \) is smaller than the radius of convergence of the expansion \( f_a \) of \( f \) around \( a \). Then the expansion \( f_b \) of \( f \) around \( b \) coincides with the expansion of \( f_a \) around \( b \). It means that the expansions do not depend on the “paths” within the circle of convergence.
Thus we have to show that
\[ \gamma_T(b) := \sum_{U \in \mathcal{P}} \gamma_U(U/T) \ b^{(\deg(U) - \deg(T))} \]
is equal to
\[ \delta_T(b) := \sum_{S \in \mathcal{P}} \gamma_U(a) (S/T) (b-a)^{\deg(S) - \deg(T)}. \]
Substituting \( \gamma_T(a) \) into this expression gives
\[ \delta_T(b) = \sum_{S \in \mathcal{P}} \sum_{S \in \mathcal{P}} \gamma_U(U/T) \ (S/U) \ a^{\deg(U) - \deg(T)} (b-a)^{\deg(S) - \deg(T)}. \]
The crucial observation in proving \( \gamma_T(b) = \delta_T(b) \) is the following relation:
Let \( S, T \in \mathcal{P} \), \( \deg(S) = n \), \( \deg(T) = m \), and \( a, b \in \mathbb{C} \). Then
\[ \sum_{U \in \mathcal{P}} (S/U) \ (U/T) \ a^{\deg(U) - \deg(T)} (b-a)^{\deg(U) - m} = (S/T) \ b^{n-m}. \]
The left hand side is equal to
\[ \sum_{k=0}^{n-m} \sum_{\deg(U) = m+k} (S/U) \ (U/T) \ a^{n-m-k} (b-a)^k. \]
and
\[ \sum_{\deg(U) = m+k} (S/U) \ (U/T) = (S/T) \ (n-m/k). \]
The left hand side of this expression is \# \( M \) with \( M := \{(J,I) : J, I \leq L(S), \# I = m, \# J = m + k, \text{the contraction } S|I \text{ is equal to } T\} \) which is equal to \( (S/T) \ (n-m/k) \) because for fixed \( I \) with \( S|I = T \) there are \( (n-m/k) \) subsets \( J \) with \((J, I) \in M\).

It follows that the left hand side is equal
\[ \sum_{k=0}^{n-m} (S/T) \ (n-m/k) \ a^{n-m-k} \ (b-a)^k. \]
which is equal to \( (S/T) \ b^{n-m} \).

Let \( F = (f_a)_{a \in D} \) be a planar analytic function on \( D \).
For \( T \in \mathcal{P} \) the function \( h_T : D \to \mathbb{C} \) given by \( h_T(a) := (f_a, (x-a)^T) \) is analytic, it is called the \( T \)-th coefficient \( \langle F, T \rangle \) of \( F \).

4 Exponential functions

Let \( \exp_k(x) \in \mathbb{C} \{\{x\}\} \mathcal{P} \) be the \( k \)-ary exponential series where \( k \in \mathbb{N} \geq 2 \), see [G1].
Then from the recursion formula for the coefficients \( A_T(k) \) of \( \exp_k(x) \), see [G1], p. 352, it follows that all coefficients \( A_T(k) \) of \( \exp_k(x) \) are real and positive and
\[ \sum A_{kT} = 1/n \]
for \( T \in \mathcal{P} \).
where $\mathcal{P}(n)$ is the set of trees in $\mathcal{P}$ of degree $n$.

It follows that the radius of convergence of $\exp_k(x)$ is $\infty$.

**Remark 4.1.** It is not obvious that the radius of convergence of $\log_k(1+x)$ is equal to 1.

Let $\lambda \in \mathcal{C}$. Then $f = \exp_k(\lambda + x)$ is a planar power series in $\mathcal{C} \\{ \{x + \lambda\}\} \mathcal{P}$ which has infinite radius of convergence. Thus $f$ has expansion around 0, and $f(\lambda + x) = \sum a_{k,T}(\lambda) x^T$.

with $a_{k,T}(\lambda) = \sum (U/T) a_{k,n} \lambda^{(\deg(U) - \deg(T))}$

$U \in \mathcal{P}$

**Proposition 4.2.** $\exp_k(\lambda + x) = \exp_k(x) \cdot e^\lambda$

Proof: Proceed as in the proof of Proposition 5.1 in [G5]

Work in $A = \mathcal{C} \langle \lambda \rangle \otimes \mathcal{C} \{ \{x\}\} \mathcal{P}$

Let $A_n := \mathcal{C} \cdot \text{vector space generated by} \{ \lambda^i \cdot x^T : i + \deg(T) = n\}$ Each $f \in A$ has a unique expansion

$$f = \sum_{n=0}^{\infty} f_n,$$

with $f_n \in A_n$.

There is a unique $f \in A$ such that $f = 1 + x + \sum_{n=2}^{\infty} f_n$ $f_n \in A$

$$f^k(\lambda, x) = f(\lambda, k \cdot x)$$

One can show that both $e^\lambda \cdot \exp_k(x)$ and $\exp_k(\lambda + x)$ do satisfy both conditions. This shows that they are equal.

**Corollary 4.3.** Let $n = \deg(T)$. Then

$$\left(\alpha_T / m!\right) = \sum_{U \in \mathcal{P}_{n+m}} (U/T) a_U$$

**Example 4.4.**

$T = x \cdot x^2$, $m = 1$:

$\mathcal{P}_4 = \{ x \cdot (x \cdot x^2), x \cdot x^2 \cdot x^2, x \cdot x \cdot x^2, (x \cdot x^2) \cdot x, (x^2 x) \cdot x \} = \{ U_1, \ldots, U_5 \}$

and $(U_1/T) = 4$, $(U_2/T) = 3$, $(U_3/T) = 2$, $(U_4/T) = 1$, $(U_5/T) = 0$

Thus $\alpha_T/1! = 4 \alpha_{U_1} + 3 \alpha_{U_2} + 2 \alpha_{U_3} + \alpha_{U_4}$

$\alpha_{U_1} = 1/4! \cdot 1/1$ if $i \neq 3$ and $\alpha_{U_3} = 1/4! \cdot 1/7 \cdot 3$

Thus $\alpha_T = 4 \alpha_{U_1} + 3 \alpha_{U_2} + 2 \alpha_{U_3} + \alpha_{U_4}$

$$= \frac{1}{4! \cdot 7} (4 + 3 + 2 + 2 \cdot 3 + 1) = \frac{14}{4! \cdot 7} = \frac{1}{4!} = \frac{2 \cdot 1 \cdot 1}{4! \cdot 3! \cdot 2}$$
5 Planar zeta function and Gamma function

Fix \( k \in \mathbb{N}_{\geq 2} \) and let \( \exp(s) = \exp_k(s) \) be the planar exponential series of arity \( k \) in the planar variable \( s \).

For an integer \( n \in \mathbb{N}_{\geq 1} \) let \( n^s := \exp(s \cdot \log(n)) \).

If \( \exp(s) = \sum \alpha_T s^T \), one gets
\[
T \in \mathbb{P} \quad n^s = \sum \alpha_T \cdot (\log(n))^{\deg(T)} \cdot s^T
\]

Let \( r \in \mathbb{C} \). Then \( n^{s+r} = n^s \cdot n^r \)

**Proposition 5.1.** There is a planar analytic function \( \zeta_{\mathbb{P}} \) on \( D = \mathbb{C} - \{ \lambda \in \mathbb{R} : \lambda \leq 1 \} \) such that for \( r \in D \), \( \text{Re}(r) > 1 \), one has the expansion \( \zeta_{\mathbb{P}} = \sum_{n=1}^{\infty} n^{-r} (n^{-(s-r)}) \) around \( r \).

More precisely
\[
\zeta_{\mathbb{P}} = \sum_{n=1}^{\infty} \sum_{T \in \mathbb{P}} n^{-r} \alpha_T (\log(n))^{\deg(T)} (s-r)^T
\]

\[
\zeta_{\mathbb{P}} = \sum_{T \in \mathbb{P}} \beta_{T,r} \cdot (s-r)^T
\]

with \( \beta_{T,r} = (\sum_{n=1}^{\infty} n^{-r} (\log(n))^{\deg(T)}) \cdot \alpha_T \)

The radius of convergence of the expansion of \( \zeta_{\mathbb{P}} \) around \( r \) is \( (r-1) \).

Open question: what is the relation between \( \zeta_{\mathbb{P}}(s) \) and \( \zeta_{\mathbb{P}}(1-s) \) ?

The definition of the Gamma function \( \Gamma(s) \) of Euler by the integral
\[
\int_0^{\infty} e^t \cdot t^{s-1} \, dt
\]

has an immediate extension as a planar analytic function in the domain \( \{ a \in \mathbb{C} : \text{Re}(a) > 0 \} \). This can be done as follows:

Let \( r \in \mathbb{C} \), \( \text{Re}(r) > 0 \). We work in the algebras \( \mathbb{C}[[t]] \otimes \mathbb{C} \{ (s-r) \} \).

Then \( t^{s-1} = t^{(s-r)} \cdot t^{(r-1)} \) where \( t^{(s-r)} \) is the planar function \( \exp((s-r) \log t) \) and where \( \exp \) is the \( k \)-ary planar exponential series for some \( k \in \mathbb{N}, k \geq 2 \).

Thus
\[
\sum_{T \in \mathbb{P}} (t^{(r-1)} \alpha_T \cdot (\log(t))^{\deg(T)} \cdot (s-r)^T
\]
Now for fixed $T \in \mathbb{P}$ the integral.
\[
\beta_T(r) : = \int_0^\infty e^{-t}t^{r-1} \cdot \alpha_T(\log t)^{\deg(T)} \, dt
\]
is well defined and $\Gamma_\mathbb{P}(s)$ is defined to have the expansion around $r$ given by the planar series
\[
\sum_{T \in \mathbb{P}} \beta_T(r) (s-r)^T
\]
The open problem is again what kind of functional equation holds for $\Gamma_\mathbb{P}$. Certainly $\Gamma_\mathbb{P}(s+1)$ is different from $s \cdot \Gamma(s)$.

6 The multiplicative inverse $F^{-1}$

Let $F = (f_a)_{a \in D}$ be a planar analytic function on $D$ and assume that $f_a(a) \neq 0$ for all $a \in D$. Then the left inverse $f_a^{-1}$ of $f_a$ exists in $\mathcal{C}\{\{x-a\}\}$ which satisfies
\[
f_a^{-1} \cdot f_a = 1
\]

Let $F^{-1} = (f_a^{-1})_{a \in D}$

**Proposition 6.1.** $F^{-1}$ is a planar analytic function on $D$.

The same statement holds for $F^{-1} = (\overline{f_a})_{a \in D}$ where $\overline{f_a}$ is the right inverse of $f_a$.

The planar function $(x^{-1})$

Let $D = \mathcal{C} - \{0\}$, $a \in D$ and $f_a = ((x-a) + a)^{-1}$ be the left inverse of $(x-a) + a$ in $\mathcal{C}\{\{x-a\}\}$ which is $a^{-1}(1 + x-a/a)^{-1} = \sum_{n=0}^{\infty} \frac{1}{a^n} (x-a)^n$

where $(x-a)^{(n+1)} : = (x-a)^n \cdot (x-a) = (x-a)^{C_{n+1}}$

where $C_{n+1}$ is the right comb of degree $n + 1$.

Then $(f_a)_{a + D}$ is a planar analytic function also denoted by $(x^{-1})$
The planar square root function \((\mathbb{P} \sqrt{x})\)

Let \(D = \mathbb{C} - \{\lambda \in \mathbb{R} : \lambda \leq 0\}\) and \(\sqrt{\cdot} : D \to \mathbb{C}\) the analytic function with \(\sqrt{1} = 1\) and \((\sqrt{z})^2 = z\) for all \(z \in D\).

For all \(a \in D\) there is a unique planar series \(f_a \in \mathbb{C} \{\{x-a\}\}\) such that the constant term of \(f_a\) is \(\sqrt{a}\) and \(f_a^2 = a + (z-a)\). One can show that the radius of convergence of \(f_a\) is equal to \(|a|\). The planar analytic function \(\mathbb{P} \sqrt{x} := (f_a)_{a \in D}\) is called the planar square root function.

Let \(g = \sum_{\substack{T \in \mathbb{P}' \\text{deg}(T) \geq 1}} x^T\) where \(\mathbb{P}'\) is the set of trees \(T\) in \(\mathbb{P}\) for each vertex has arity less or equal to 2.

It is easy to show that \(g^2 = g - x\) and thus \((g - \frac{1}{2})^2 = 1/4 - x\)

It follows that \((1-2g)^2 = 1 - 4x\)

Let \(h = g (-1/4 x)\). Then \((1-2h)^2 = 1 + x\) and \(h = \sum_{\substack{T \in \mathbb{P} \\text{deg}(T) \geq 1}} (-1/4)^{\text{deg}(T)} x^T\)

The radius of convergence of \(g\) is equal to 1/4 because all coefficients of \(g\) are positive and the classical series \([\lceil g \rceil] = \frac{1}{2} - \frac{1}{2} \sqrt{1 - 4x}\) and \(\sqrt{1 - 4x}\)

\[= \sum_{n=0}^{\infty} (\frac{1}{2}/n) (-4 x)^n\]

where \((\frac{1}{2}/n)\) is the binomial coefficient of \(\frac{1}{2}\) over \(n\), which has \(\frac{1}{4}\) as radius of convergence.
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