A direct product decomposition of the automorphism group of Cayley graphs generated by transposition sets

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Abstract

Let $S$ be a set of transpositions generating the symmetric group $S_n$, where $n \geq 3$. It is shown that if the girth of the transposition graph of $S$ is at least 5, then the automorphism group of the Cayley graph Cay$(S_n, S)$ is the direct product $S_n \times \text{Aut}(T(S))$, where $T(S)$ is the transposition graph of $S$; the direct factors are the right regular representation of $S_n$ and the image of the left regular action of $\text{Aut}(T(S))$ on $S_n$. This strengthens a previous result of the author, where the automorphism group was factored as a semidirect product.

Index terms — Cayley graphs; transposition sets; automorphisms of graphs; direct products; normal Cayley graphs.

1. Introduction

Let $X = (V, E)$ be a simple undirected graph. The (full) automorphism group of $X$, denoted $\text{Aut}(X)$, is the set of permutations of the vertex set that preserve adjacency: $\text{Aut}(X) := \{g \in \text{Sym}(V) : E^g = E\}$. Let $H$ be a group and let $S$ be a subset of $H$. The Cayley graph of $H$ with respect to $S$, denoted Cay$(H, S)$, is the graph with vertex set $H$ and arc set $\{(h, sh) : h \in H, s \in S\}$. When $S$ satisfies the condition $1 \notin S = S^{-1}$, the Cayley graph Cay$(H, S)$ has no self-loops and can be considered to be undirected.

Given a group $H$, let $R$ be the action of $H$ on itself by right multiplication, so that $R(h) : x \mapsto xh$, for all $h \in H$. The right regular representation of $H$, denoted $R(H)$, is the set of permutations $\{R(h) : h \in H\}$. Similarly, the left regular representation of $H$ consists of the set of permutations $\{L(h) : h \in H\}$, where $L(h) : x \mapsto h^{-1}x$. The product action $U$ of $H \times H$ on itself is defined by rule $U(h, g) : x \mapsto h^{-1}xg$. It can be shown that both $L(H)$ and $R(H)$ are normal subgroups of $U(H \times H) = L(H)R(H)$.

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(cf. [2, p. 8]), though in general the two regular subgroups do not intersect trivially since the cardinality of their intersection is the order of the center of the group.

A Cayley graph $\text{Cay}(H, S)$ is vertex-transitive since the right regular representation $R(H)$ acts as a group of automorphisms of the Cayley graph \cite{1, 2}. A Cayley graph $X := \text{Cay}(H, S)$ is said to be normal if the right regular representation $R(H)$ is a normal subgroup of $\text{Aut}(X)$, or equivalently, if $\text{Aut}(X) = R(S_n) \rtimes \text{Aut}(S_n, S)$. Let $S$ be a set of transpositions generating the symmetric group $S_n$. The transposition graph of $S$, denoted $T(S)$, is defined to be the graph with vertex set $\{1, \ldots, n\}$, and with two vertices $i$ and $j$ being adjacent in $T(S)$ whenever $(i, j) \in S$. It is known that if the girth of the transposition graph $T(S)$ is at least 5, then the automorphism group of the Cayley graph $\text{Cay}(S_n, S)$ is the semidirect product $R(S_n) \rtimes \text{Aut}(S_n, S)$, where $\text{Aut}(S_n, S)$ is the set of automorphisms of $S_n$ that fixes $S$ setwise (cf. \cite{4}). It is also known that $\text{Aut}(S_n, S) \cong \text{Aut}(T(S))$ (cf. \cite{3}).

Given a set $S$ of transpositions generating $S_n$, let $G := \text{Aut}(\text{Cay}(S_n, S))$. In the instances where $G = R(S_n) \rtimes G_e$, the factor $G_e \cong \text{Aut}(T(S))$ is in general not a normal subgroup of $G$, so that the semidirect product cannot be written as a direct product. In the present paper, it is shown that $R(S_n)$ has another complement in $G$ which is a normal subgroup of $G$. Recall that any two complements of a normal subgroup are isomorphic to each other (cf. \cite{3, p. 65}), so that a normal complement of $R(S_n)$, if one exists, would have to be isomorphic to $G_e \cong \text{Aut}(T(S))$. In the proof below, we show that the image of $\text{Aut}(T(S))$ under the left regular action of $S_n$ on itself is a normal complement of $R(S_n)$ in $G$. Thus, the direct factor $\text{Aut}(T(S))$ that arises in $G \cong R(S_n) \times \text{Aut}(T(S))$ is not $G_e$ but is obtained by considering the left regular action.

The main result of this paper is the following:

**Theorem 1.** Let $S$ be a set of transpositions generating $S_n$, $n \geq 3$. If the girth of the transposition graph $T(S)$ is at least 5, then the automorphism group of the Cayley graph $\text{Cay}(S_n, S)$ is the direct product $S_n \rtimes \text{Aut}(T(S))$.

**Proof:** Let $X := \text{Cay}(S_n, S)$. Since the girth of the transposition graph $T(S)$ is at least 5, the Cayley graph $X$ is a normal Cayley graph with automorphism group $\text{Aut}(X) = R(S_n) \rtimes \text{Aut}(S_n, S)$, where $\text{Aut}(S_n, S) \cong \text{Aut}(T(S))$. Let $L$ be the left regular action of $S_n$ on itself, and let $L(\text{Aut}(T(S)))$ denote the image of $\text{Aut}(T(S))$ under this action, i.e. $L(\text{Aut}(T(S))) := \{L(a) : a \in \text{Aut}(T(S))\} \leq \text{Sym}(S_n)$.

We first show that the elements in $L(\text{Aut}(T(S)))$ are automorphisms of $X$. Let $a \in \text{Aut}(T(S))$. We show that $(h, g) \in E(X)$ if and only if $(h, g)^{L(a)} \in E(X)$. Suppose $(h, g) \in E(X)$. Then $g = sh$ for some $s = (i, j) \in S$. We have that $(h, g)^{L(a)} = (h, sh)^{L(a)} = (h^{L(a)}, (sh)^{L(a)}) = (a^{-1}h, a^{-1}sh) = (a^{-1}h, (a^{-1}sa)a^{-1}h)$. Now $a^{-1}sa = a^{-1}(i, j)a = (i^a, j^a) \in S$ since $a$ is an automorphism of the graph $T$ that has edge set $S$. Thus, $(h, sh)^{L(a)} \in E(X)$. Conversely, suppose $(h, g)^{L(a)} \in E(X)$. Then $a^{-1}h = sa^{-1}g$ for some $s \in S$. Hence $h = (a^{-1})g$. As before, $asa^{-1} \in S$, so that $h$ is adjacent to $g$. Thus, $L(\text{Aut}(T(S)))$ is a subgroup of $\text{Aut}(X)$.

The left and right regular representations of a group have as their intersection the image of the center of the group. The center of $S_n$ is trivial, whence $L(\text{Aut}(T(S)))$
and $R(S_n)$ have a trivial intersection. Since $X$ is a normal Cayley graph, $\text{Aut}(X) = R(S_n) \rtimes \text{Aut}(T(S))$, where $\text{Aut}(S_n, S) \cong \text{Aut}(T(S))$, and it follows from cardinality arguments that $R(S_n) L(\text{Aut}(T(S)))$ exhausts all the elements of $\text{Aut}(X)$. Thus, $R(S_n)$ and $L(\text{Aut}(T(S)))$ are complements of each other in $\text{Aut}(X)$ and every element in $\text{Aut}(X)$ can be expressed uniquely in the form $R(a)L(b)$ for some $a \in S_n$ and $b \in \text{Aut}(T(S))$.

Suppose $g \in \text{Aut}(X)$ and $c \in \text{Aut}(T(S))$. Then $g = R(a)L(b)$ for some $a \in S_n, b \in \text{Aut}(T(S))$. Hence, $g^{-1}L(c)g = (R(a)L(b))^{-1}L(c)(R(a)L(b))$, which maps $x \in S_n$ to $b^{-1}c^{-1}bxa^{-1}a = b^{-1}c^{-1}bx$. Since $b, c \in \text{Aut}(T(S))$, $d^{-1} := b^{-1}c^{-1}b \in \text{Aut}(T(S))$. Thus, $g^{-1}L(c)g = L(d) \in L(\text{Aut}(T(S)))$. Hence $L(\text{Aut}(T(S)))$ is a normal subgroup of $\text{Aut}(X)$.

As an example, consider the special case where $S$ is the set of $n$ cyclically adjacent transpositions, where $n \geq 5$. The corresponding Cayley graph $X := \text{Cay}(S_n, S)$ is called the modified bubble sort graph of dimension $n$. It was known previously that $\text{Aut}(X) \cong R(S_n) \rtimes D_{2n}$. By Theorem $\Box$ $\text{Aut}(X) \cong S_n \times D_{2n}$.

While the theorem above assumes that the transposition graph of $S$ has girth at least 5, it can be seen that the theorem holds for arbitrary transposition sets as long as the Cayley graph is normal. For any transposition set $S$, the permutation group image of $\text{Aut}(T(S))$ under the left regular action $L$ of $S_n$ on itself is a subgroup of $\text{Aut}(\text{Cay}(S_n, S))$. Thus, if the Cayley graph $X := \text{Cay}(S_n, S)$ is such that $\text{Aut}(X)$ is the semidirect product $R(S_n) \rtimes \text{Aut}(S_n, S) \cong R(S_n) \rtimes \text{Aut}(T(S))$, then $\text{Aut}(X)$ is the direct product $R(S_n) \times \text{Aut}(T(S))$.

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