ON THE FOCK SPACE FOR NONRELATIVISTIC ANYON FIELDS AND BRAIDED TENSOR PRODUCTS

Gerald A. Goldin¹
Departments of Mathematics and Physics
Rutgers University
New Brunswick, NJ 08903, USA
gagoldin@dimacs.rutgers.edu

Shahn Majid²
School of Mathematical Sciences
Queen Mary, University of London, Mile End Rd
London E1 4NS, UK
s.majid@qmw.ac.uk

Abstract We realize the physical $N$-anyon Hilbert spaces, introduced previously via unitary representations of the group of diffeomorphisms of the plane, as $N$-fold braided-symmetric tensor products of the 1-particle Hilbert space. This perspective provides a convenient Fock space construction for nonrelativistic anyon quantum fields along the more usual lines of boson and fermion fields, but in a braided category, and clarifies how discrete (lattice) anyon fields relate to anyon fields in the continuum. We also see how essential physical information is encoded. In particular, we show how the algebraic structure of the anyonic Fock space leads to a natural anyonic exclusion principle related to intermediate occupation number statistics, and obtain the partition function for an idealized gas of fixed anyonic vortices.

1 Introduction

Anyons are particles or excitations in two-dimensional space that obey exchange statistics interpolating those of bosons and fermions. When two identical anyons are exchanged without coincidence along a continuous path in the plane, their relative winding number $m$ (the net number of counterclockwise exchanges) is well-defined, depending only on the homotopy class of the path implementing the exchange. The quantum-mechanical wave function then

¹Professor
²Royal Society University Research Fellow
acquires a relative phase $e^{im\theta}$, where $\theta$ is a real fixed parameter between 0 and $2\pi$. When $\theta = 0$ we have bosons, and $\theta = \pi$ corresponds to fermions.

The possibility of such intermediate statistics was suggested by Leinaas and Myrheim [1] and confirmed by Goldin, Menikoff and Sharp [2], who derived the quantum theory rigorously from representations of local nonrelativistic current algebra and the corresponding diffeomorphism group. They obtained the anyonic shifts in angular momentum and energy spectra, and made connections with configuration space topology and the physics of charged particles circling regions of magnetic flux. The term “anyon” was subsequently introduced by Wilczek [3], who proposed a model for such objects based on charged-particle/flux-tube composites and suggested their association with fractional spin in two dimensions. The idea found some immediate applications to surface phenomena and related areas of physics. [4, 5]

In [6] the braid group $B_N$ was identified as the group whose one-dimensional representations describe the anyonic wave function symmetry. A more extensive discussion of the braid group and anyon statistics followed in [7], where it was argued that only the one-dimensional representations of $B_N$ should occur in quantum mechanics. However, as noted in [8], the diffeomorphism group approach allows also the possibility of quantum systems described by higher-dimensional representations of $B_N$ (particles later termed “plektons”). An overview of that approach, which underlies the present article, may be found in [9].

The extensive development of these ideas that occurred during the 1980s and early 1990s, including their relation to Chern-Simons quantum field theories, their application in describing the integer and fractional quantum Hall effects, and their role in describing possible mechanisms for superconductivity, are reviewed in [10] and [11].

Anyonic systems can also be associated with quantum groups and $q$-deformations of classical Lie algebras [12, 13, 14, 15, 16]. In [17] it was shown that creation and annihilation fields for anyons could be constructed so as to intertwine the $N$-anyon representations of the group of compactly supported diffeomorphisms $\text{Diff}_c(\mathbb{R}^2)$ in the Hilbert space $\mathcal{H}_N$ of $N$-particle states. The assumption that these fields transform consistently with the diffeomorphism group representations dictates the form they should take; and the fact that they obey $q$-commutation relations, where $q$ is the anyonic phase shift, emerges as a consequence of this. In this work, the spaces $\mathcal{H}_N$ were constructed (for each $N$) using ‘topological configurations’ of $N$ points equipped with attached filaments going out to infinity in the plane.

In the present article, we show how to construct such a theory along lines more familiar from bosonic (respectively, fermionic) algebras of canonical commutation (resp. anticommutation) relations—i.e., CCR (resp. CAR) algebras—where the $N$-particle Fock subspaces are built up by symmetrizing or antisymmetrizing sets of 1-particle states. We shall effectively ‘$q$-symmetrize’ [18] using braided category techniques coming out of quantum group theory.
so that

\[ \mathcal{H}_N = \mathcal{H}_1 \otimes_s \mathcal{H}_1 \otimes_s \cdots \otimes_s \mathcal{H}_1 \]

\((N \text{ times})\), where \(\otimes_s\) is the symmetrized tensor product with respect to a certain symmetry \(\Psi_0\). An apparent difficulty in formulating anyonic field theory this way has been the need to work in a \textit{strictly} braided category, with a braiding \(\Psi\) defined by \(q\). The fact that the braid group is infinite would then seem to require an infinite sum of powers of \(q\). But it turns out that the associated operator \(\Psi_0\) obeys the condition \(\Psi_0^2 = \text{id}\). Thus it generates an action of the symmetric group \(S_N\), rather than the braid group \(B_N\). In effect, the nontrivial braid group representation and its inverse conspire with each other to give us a braided tensor category that is actually \textit{symmetric} (up to sets of measure zero).

Moreover, in our construction the full anyonic Fock space \(S_{\Psi_0}(\mathcal{H}_1)\), obtained as the direct sum of the different spaces \(\mathcal{H}_1 \otimes_s \cdots \otimes_s \mathcal{H}_1\), is an algebra with product \(\otimes_s\). This algebra generalizes the algebra of functions on a linear space or superspace in the Bose or Fermi cases, with its respective commutative or anticommutative product. The annihilation and creation operators in our construction then act by pointwise multiplication and (functional) differentiation on this space, exactly in conformity with the usual functional representation in field theory. The difference is that the anyonic case only makes sense in a braided category, with a braiding \(\Psi\). In this category \(S_{\Psi_0}(\mathcal{H}_1)\) is the ‘coordinate ring’ of a braided group, with the braided coproduct \(\Delta\) expressing addition \([12, 19, 20, 21]\). The use here of \textit{two} Yang-Baxter operators \(\Psi\) and \(\Psi_0\) is a general feature of the theory of such ‘braided linear spaces’, and the strictness of the braiding \(\Psi\) is essential for this.

As an important application, we show how the structure of \(S_{\Psi_0}(\mathcal{H}_1)\) as the braided version of the coordinate algebra of a linear space leads to an ‘anyonic exclusion principle’ when \(q\) is a root of unity. Specifically, with \(q^r = 1\), the creation operator \(\psi^*(x)\) cannot occur in a reduced Fock state more than \(r - 1\) times. This important physical fact means that the relevant occupation number statistics is of the nature of Gentile statistics \([22]\), and has some similarities to other algebraic approaches to generalized exclusion principles such as that proposed in \([23]\). The exclusion also applies to states obtained from the smeared field; i.e., \(\psi^*(h)^r = 0\) in our Hilbert space representation for any test function \(h\). Earlier articles have approached the fractional exclusion statistics of a one-dimensional gas, or of anyons in two space dimensions, from quite different perspectives \([24, 25, 26, 27, 28, 29]\); see also \([30]\).

An outline of the present article is as follows. In Section 2 we summarize the topological construction of the spaces \(\mathcal{H}_N\) of \(N\)-anyon states, and write the corresponding current algebra, diffeomorphism group, and anyon creation and annihilation field representations. In Section 3 we carry out our construction using braided symmetric tensor categories, and in Section 4 we show that this indeed leads to the same result as the earlier topological construction. In Section 5 we provide some algebraic consequences, including the anyonic
exclusion principle when \( q \) is a root of unity. Throughout we consider both the physical continuum case and the discrete case, as the latter may be useful for lattice versions of the theory or for anyon fields on a finite number of points. The discrete case turns out to have a number of instructive subtleties. Finally, in Section 6 we offer some further comments of a conceptual nature.

2 Topological state spaces \( \mathcal{H}_N \)

We begin by recalling the set-up for anyon field representations proposed in [17]. For the nonrelativistic quantum theory of identical particles in \( n \) space dimensions, we are interested in representations \( \rho, J \) of the semidirect Lie algebra \( C_\infty^c(\mathbb{R}^n) \rtimes \text{vect}_c(\mathbb{R}^n) \), where \( \rho \) and \( J \) are self-adjoint operator-valued distributions describing the mass and momentum densities respectively; i.e.,

\[
\rho(f) = \int \rho(x)f(x)d^n x, \quad J(v) = \int J(x) \cdot v(x)d^n x,
\]

where the test function \( f \in C_\infty^c(\mathbb{R}^n) \) is a compactly-supported real-valued \( C^\infty \) function on \( \mathbb{R}^n \), and \( v \in \text{vect}_c(\mathbb{R}^n) \) is a compactly-supported (tangent) vector field on \( \mathbb{R}^n \). Then the well-known current algebra

\[
[\rho(f),\rho(g)] = 0, \quad [\rho(f),J(v)] = i\hbar \rho(\nabla_vf), \quad [J(v),J(w)] = -i\hbar J([v,w])
\]  

(1)

represents the bracket in the Lie algebra, where \([v,w]\) is the usual Lie bracket of vector fields. The group-level version is based on the natural semidirect product of the group of compactly-supported functions under addition, with the group of compactly-supported diffeomorphisms of \( \mathbb{R}^n \) under composition: \( G = C_\infty^c(\mathbb{R}^n) \rtimes \text{Diff}_c(\mathbb{R}^n) \), with \((f,\phi) \cdot (g,\psi) = (f + g \circ \phi, \psi \circ \phi)\).

Then in a continuous unitary representation of \( G \), we can write the 1-parameter subgroups \( U(f) \) and \( V(\phi_t) \), where \( \phi_t \) is the flow on \( \mathbb{R}^n \) generated by \( v \). Under appropriate conditions, the self-adjoint generators defined from \( U(f) = e^{i\rho(f)} \) and \( V(\phi_t) = e^{i(\phi_t/\hbar)J(v)} \) represent the current algebra. The idea is that different physical systems in quantum mechanics should correspond to different (unitarily inequivalent) irreducible representations of \( G \).

In particular consider a family \( \mathcal{H}_N \) of Hilbert spaces, where \( N \in \mathbb{N} \), along with annihilation operators \( \psi(h) \) and creation operators \( \psi^*(h) \), where the test functions \( h \) belong to a domain in \( \mathcal{H}_1 \). Thus

\[
\psi(h): \mathcal{H}_{N+1} \to \mathcal{H}_N, \quad \psi^*(h): \mathcal{H}_N \to \mathcal{H}_{N+1}.
\]

Suppose we have representations \( U_N, V_N \) in \( \mathcal{H}_N \) of the group \( G = C_\infty^c(\mathbb{R}^n) \rtimes \text{Diff}_c(\mathbb{R}^n) \), for each \( N \), intertwined by \( \psi \) and \( \psi^* \) in such a way that

\[
U_{N+1}(f)\psi^*(h) = \psi^*(U_1(f)h)U_N(f), \quad V_{N+1}(\phi)\psi^*(h) = \psi^*(V_1(\phi)h)V_N(\phi).
\]

(2)
Then the $U_N, V_N$ ($N = 1, 2, 3, \ldots$) are interpreted as a hierarchy of representations of $G$ describing systems of $N$ particles (or quantum excitations) of the species created and annihilated by the field operators. At the level of the algebra, the corresponding requirements are

$$[\rho(f), \psi^*(h)] = \psi^*(\rho_1(f)h), \quad [J(v), \psi^*(h)] = \psi^*(J_1(v)h).$$  \hspace{1cm} (3)

Here $\psi^*$ is the adjoint of $\psi$; but we note that later, when we consider the discrete anyonic case, that will be modified.

When $n = 2$, one has the possibility in this general framework of anyonic representations of $G$, and corresponding fields satisfying Eqs. (2) and (3). Then the representation $U_N, V_N$ of $G$ acts in the Hilbert space

$$\mathcal{H}_N = L^2_{B_N}(\tilde{\Delta}_N),$$

defined as follows. The configuration space $\Delta_N$ for $N$ identical anyons is the space of (unordered) $N$-point subsets of $\mathbb{R}^2$; thus $\gamma \in \Delta_N$ is given by $\gamma = \{x_1, \ldots, x_N\} \subset \mathbb{R}^2$. The fundamental group $\pi_1(\Delta_N)$ is $B_N$, the braid group on $N$-strands. We denote by $\tilde{\Delta}_N$ the universal covering space of $\Delta_N$, which has infinitely many sheets; for $\tilde{\gamma} \in \tilde{\Delta}_N$, we have the projection map $p : \tilde{\gamma} \rightarrow \gamma$. The braid group then acts on $\tilde{\Delta}_N$; writing this action as $\tilde{\gamma} \cdot b$ for $b \in B_N$, we have $p(\tilde{\gamma} \cdot b) = p(\tilde{\gamma})$. The elements of $\mathcal{H}_N$ are now wave functions $\tilde{\Phi}$ on $\tilde{\Delta}_N$, taking values in an inner product space $V$ that carries a unitary representation $T(b)$ of $B_N$. We shall consider only the case $V = \mathbb{C}$ (scalar-valued wave functions), and 1-dimensional representations of $B_N$. Such a representation is specified by choosing a fixed phase $q = \exp i\theta$, and setting $T(b) = q$ when $b$ is the crossing of one strand over another in a forward (left over right) direction. The wave functions are required to be equivariant under $T$, in the sense that for all $b \in B_N$,

$$\tilde{\Phi}(\tilde{\gamma} \cdot b) = T(b)\tilde{\Phi}(\tilde{\gamma}).$$

In other words, $\tilde{\Phi} \in \mathcal{H}_N$ is an equivariant section of a vector bundle over $\tilde{\Delta}_N$. When $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ satisfy the same such equivariance condition, the product $\overline{\tilde{\Phi}_1(\tilde{\gamma})}\tilde{\Phi}_2(\tilde{\gamma})$ depends only on $\gamma = p(\tilde{\gamma})$, and not on the particular choice of $\tilde{\gamma}$ within $p^{-1}(\gamma)$ at which $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ are evaluated. Thus we may write the integral of this product with respect to a (local) Lebesgue measure $dx_1 \cdots dx_N$ on $\Delta_N$. Finally, we take $\mathcal{H}_N$ to consist of the square-integrable functions, so that for any pair $\tilde{\Phi}_1, \tilde{\Phi}_2$,

$$\langle \tilde{\Phi}_1, \tilde{\Phi}_2 \rangle = \int_{\Delta_N} \overline{\tilde{\Phi}_1(\tilde{\gamma})}\tilde{\Phi}_2(\tilde{\gamma}) \; dx_1 \cdots dx_N < \infty$$  \hspace{1cm} (4)

defines the inner product of $\tilde{\Phi}_1$ with $\tilde{\Phi}_2$ in $\mathcal{H}_N$.

Given any diffeomorphism $\phi$ of $\mathbb{R}^2$, let $\phi$ act on $\Delta_N$ in the obvious way. This action lifts to an action on $\tilde{\Delta}_N$ compatible with the projection map; i.e., if $p(\tilde{\gamma}) = \gamma$ then $p(\phi\tilde{\gamma}) = \phi\gamma$. 
The $N$-anyon representation in $\mathcal{H}_N$ is then defined by

$$U_N(f)\tilde{\Phi}(\tilde{\gamma}) = e^{i\sum_{j=1}^{N} f(x_j) \tilde{\Phi}(\tilde{\gamma})}, \quad V_N(\phi)\tilde{\Phi}(\tilde{\gamma}) = \tilde{\Phi}(\phi \tilde{\gamma}) \prod_{j=1}^{N} \sqrt{J_\phi(x_j)}, \quad (5)$$

where $J_\phi$ is the Jacobian of $\phi$. The factor of $\prod_{j=1}^{N} \sqrt{J_\phi(x_j)}$ means that $V_N$ transforms $\tilde{\Phi}_1(\tilde{\gamma})\tilde{\Phi}_2(\tilde{\gamma}) \prod_{j=1}^{N} dx_j$ to $\tilde{\Phi}_1(\phi \tilde{\gamma})\tilde{\Phi}_2(\phi \tilde{\gamma}) \prod_{j=1}^{N} J_\phi(x_j) dx_j$, and thus does not change the value of the inner product.

One may construct these representations more explicitly as follows. We describe an element of $\tilde{\Delta}_N$ by a set of non-intersecting paths $\Gamma = \{\Gamma_1, \cdots, \Gamma_N\}$ in the plane, extending from infinity in (let us say) the negative $y$-direction, and terminating in the unordered set $\gamma$ of $N$ distinct points in $\mathbb{R}^2$. Then $\tilde{\Delta}_N$ is the set of homotopy classes of such $\Gamma$, with the projection map given by mapping $\Gamma$ to the set of its endpoints. We shall call the homotopy class of $\Gamma$ a “topological configuration.” Moreover, we can lift any configuration $\gamma \in \Delta_N$ (with the exception of a measure zero set) to the element $\Gamma_0$ belonging to $\tilde{\Delta}_N$, given by taking paths that go vertically downward (in the negative $y$-direction) from each point in $\gamma$. This defines a sheet $\Gamma_0(\Delta_N) \subset \tilde{\Delta}_N$ that we conventionally associate with the identity element of $B_N$. Now diffeomorphisms in Diff$_c(\mathbb{R}^2)$ act as the identity at $y = -\infty$ since they are compactly supported, and so they lift from $\Delta_N$ to act on the space of topological configurations.

Consider the subgroup Diff$_c^\gamma(\mathbb{R}^2)$ of diffeomorphisms that take the set of endpoints $\gamma$ of a fixed topological configuration $\Gamma$ into itself. This is the stability subgroup for the point $\gamma \in \Delta_N$. A diffeomorphism $\phi \in$ Diff$_c^\gamma(\mathbb{R}^2)$ then determines an element of $B_N$ by its action on $\Gamma_0$, denoted by $b = h_\gamma(\phi)$. Moreover, given any topological configuration $\Gamma$ with endpoints $\gamma$, we can obtain it by starting with $\Gamma_0$ (with the same endpoints $\gamma$), and applying a diffeomorphism $\phi \in$ Diff$_c^\gamma(\mathbb{R}^2)$, so that $\phi \Gamma_0 = \Gamma$. Then $h_\gamma$ is a surjective homomorphism, from Diff$_c^\gamma(\mathbb{R}^2)$ onto $B_N$. A topological configuration $\Gamma$ with end points $\gamma$ is conventionally identified with the pair $(\gamma, b)$; when $b$ is the identity element, we have $\Gamma_0$. The equivariance condition in this description becomes

$$\tilde{\Phi}(\Gamma) = T(b)\tilde{\Phi}(\Gamma_0).$$

Thus it is enough to specify $\tilde{\Phi}$ on the sheet $\Gamma_0(\Delta_N)$; the equivariance condition then defines it almost everywhere in $\tilde{\Delta}_N$.

In this explicit realization, one next defines the creation and annihilation fields intertwining the $N$-anyon representations (5) in accordance with Eqs. (2) or (3). Given an $N$-point subset $\gamma \subset \mathbb{R}^2$, and $x \in \mathbb{R}^2$, let us denote by $\Gamma_x^\gamma$ the element of $\tilde{\Delta}_{N+1}$ that is obtained by adjoining to $\Gamma_0^\gamma$ an additional path, terminating at $x$, that extends toward $y = -\infty$ on the left of all the paths in $\Gamma_0^\gamma$ (this modifies the convention in Ref. [17]). Then we set

$$(\psi(x)\tilde{\Phi})(\Gamma_0^\gamma) = \tilde{\Phi}(\Gamma_x^\gamma).$$
To write the adjoint field, let \( \hat{\gamma}_j = \gamma - \{x_j\} \), where \( j \) refers to some indexing of the elements of \( \gamma \). The topological configuration \( \Gamma_{x_j}^{\hat{\gamma}_j} \) then defines an element of \( \tilde{\Delta}_N \), with the set of terminal points \( \hat{\gamma}_j \cup \{x\} \). Express \( \Gamma_{x_j}^{\hat{\gamma}_j} \) as \( \phi_{\Gamma_{0}^{\hat{\gamma}_j \cup \{x\}}} \), and define \( b_{x,j} = h_{\hat{\gamma}_j \cup \{x\}}(\phi) \). Then

\[
(\psi^*(x)\hat{\Phi})(\Gamma_{0}^{\hat{\gamma}_j}) = \sum_{j=1}^{N} \delta(x - x_j)\hat{\Phi}(\Gamma_{0}^{\hat{\gamma}_j}) T^* (b_{x,j}).
\]

In this realization, one can recover the local current algebra \( (1) \) by defining \( \rho(x) = \psi^*(x)\psi(x) \), \( J(x) = \frac{\hbar}{2i} (\psi^*(x)(\nabla \psi)(x) - (\nabla \psi^*)(x)\psi(x)) \),

where \( \rho \) is the number density of anyons and \( J \) is the momentum density. Then also

\[
[\rho(f), \psi^*(h)] = \psi^*(fh), \quad [J(v), \psi^*(h)] = \frac{\hbar}{2i} \psi^*(\nabla_v h + \nabla \cdot (vh)),
\]

which are precisely Eqs. \( (3) \). Eqs. \( (1), (3), \) and \( (6) \) are the same in the anyonic case as in the usual Bose or Fermi cases. But now we have, in place of the CCR or CAR algebras, the following equal-time \( q \)-commutation relations \[17\]: in the half-space \( x^1 < y^1 \), with \( [A,B]_q = AB - qBA \) (where the phase \( q \) generates the representations \( T(b) \) of \( B_N \)),

\[
[\psi(x), \psi(y)]_q = [\psi^*(x), \psi^*(y)]_q = 0, \quad [\psi(y), \psi^*(x)]_q = \delta(x - y).
\]

In the complementary half-space \( x^1 > y^1 \), \( q \) must be replaced by \( \bar{q} = q^{-1} \). The choice of a half-space, like the definition of the sheet \( \Gamma_0(\Delta_N) \), is conventional and has no physical consequence.

### 3 Anyonic Fock space construction

For the usual bosonic or fermionic representations, we have of course a more conventional construction. Let \( \mathcal{H} = \mathcal{H}_1 = L^2(\mathbb{R}^n) \) be the space of 1-particle states, and \( \mathcal{H}_N = \mathcal{H}^{\otimes s N} \) be a symmetrized or skew symmetrized tensor product. Then \( \psi^*(h) = h \otimes s \) and \( \psi(h) \) is given by the interior product, yielding the usual equal-time commutation or anticommutation relations respectively. In this section we give such a ‘Fock space’ construction, more in line with the usual Bose or Fermi cases but now with nontrivial \( q \)-statistics. We then show in Section 4 that in the continuum case it is isomorphic to the topological construction of Ref. \[17\] described above. We shall use the machinery of braided linear spaces and braided Weyl algebras as described in Refs. \[19\], \[20\], \[21\], and elsewhere.

Let us start with a construction that works for any totally ordered space \((X, <)\). We shall initially take \( X \) to be discrete. Note that we do not assume here that \( \psi^* \) is the adjoint of \( \psi \), although this turns out to be true in the continuum case (see Section 6); a more complicated
relation holds between $\psi$ and $\psi^*$ in the discrete case. Subsequently we consider $X = \mathbb{R}^2$, with $x < y$ if $x^1 < y^1$, or if $x^1 = y^1$ then $x^2 < y^2$.

Let $\mathcal{H}$ denote a space of functions on $X$, and $\{\delta_x\}$ a basis of $\delta$-functions. We also define the functions

$$\epsilon_0(x,y) = \begin{cases} 1 & \text{if } x < y \\ 0 & x = y \\ -1 & x > y \end{cases} \quad , \quad \epsilon(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ -1 & x > y \end{cases} .$$

These are almost, but not quite, the same. Even in the continuum case they are not quite the same, because of the existence of distributions with support on the set $x = y$.

In what follows, we derive a version of the equal time $q$-commutation relations (8) as:

$$\psi(x)\psi(y) = q^{\epsilon_0(x,y)}\psi(y)\psi(x) \quad , \quad \psi^*(x)\psi^*(y) = q^{\epsilon_0(x,y)}\psi^*(y)\psi^*(x) \quad (9)$$

where as noted $\psi^*$ is not the same as the adjoint operator $\psi^\dagger$. Eqs. (9) are the refinement of Eqs. (8) that comes out of our Fock space construction, and are consistent with the representations of the fields described using the topological configurations above.

### 3.1 Discrete version

Let $(X,\prec)$ be a discrete totally ordered space, $\mathcal{H}$ a space of functions on $X$, and $\delta_x$ the function that is 1 on $x$ and 0 elsewhere in $X$. We define the generalized flip (or braiding) operators $\Psi, \Psi_0 : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ by

$$\Psi_0(\delta_x \otimes \delta_y) = q^{\epsilon_0(x,y)}\delta_y \otimes \delta_x \quad , \quad \Psi(\delta_x \otimes \delta_y) = q^{\epsilon(x,y)}\delta_y \otimes \delta_x . \quad (10)$$

These both obey the familiar braid or Yang-Baxter relations

$$\Psi_{12}\Psi_{23}\Psi_{12} = \Psi_{23}\Psi_{12}\Psi_{23} ,$$

where the numerical subscripts refer to the position in $\mathcal{H} \otimes 3$. They also satisfy the cross-compatibility conditions

$$(\Psi + \text{id})(\Psi_0 - \text{id}) = 0 \quad , \quad (\Psi_0)_{12}\Psi_{23}\Psi_{12} = \Psi_{23}\Psi_{12}(\Psi_0)_{23} \quad , \quad \Psi_{12}\Psi_{23}(\Psi_0)_{12} = (\Psi_0)_{23}\Psi_{12}\Psi_{23} . \quad (11)$$

In addition $\Psi_0^2 = \text{id}$, but this fact and the braid relations for $\Psi_0$ are not essential here.

One can write these operators in an $R$-matrix form: $\Psi(\delta_x \otimes \delta_y) = \delta_b \otimes \delta_a R^{a \ b}_{\ x \ y} (\text{summing over the repeated variables } a, b)$, with $R^{a \ b}_{\ x \ y} = \delta^\alpha_x \delta^\beta_y q^{\epsilon(x,y)}$; and similarly for $\Psi_0$, in terms of a matrix $R'$. Associated to any such $R, R'$-matrices [19, Th. 10.2.1] is a ‘braided linear
space’ which in our case we denote $S_{\Psi_0}(\mathcal{H})$. It is defined as the quadratic algebra generated by formal products of the $\{\delta_x\}$ generators, modulo the relations

$$\delta_x \otimes \delta_y = \delta_x (\Psi_0(\delta_x \otimes \delta_y)) \equiv \delta_y \otimes \delta_x q^{a(x,y)},$$

(12)

where we denote the product in $S_{\Psi_0}(\mathcal{H})$ by $\otimes_s$. The actual braiding $\Psi$ enters in the braided coproduct $\Delta: S_{\Psi_0}(\mathcal{H}) \rightarrow S_{\Psi_0}(\mathcal{H}) \otimes S_{\Psi_0}(\mathcal{H})$, that makes $S_{\Psi_0}(\mathcal{H})$ a braided group or a Hopf algebra in a braided category. On generators, this is just the linear braiding given by $\Delta \delta_x = \delta_x \otimes 1 + 1 \otimes \delta_x$, but it extends to products under $\otimes_s$ via $\Psi$; thus

$$\Delta(\delta_x \otimes \delta_y) = (\delta_x \otimes \delta_y) \otimes 1 + \delta_x \otimes \delta_y + q^{(x,y)} \delta_y \otimes \delta_x + 1 \otimes (\delta_x \otimes \delta_y).$$

(13)

As explained in [19], for the theory of braided spaces one needs not one but two operators. One of these operators controls the ‘internal’ noncommutativity of the algebra, and the other controls the ‘external’ noncommutativity or braid statistics with other independent copies. In our case, the physics actually dictates the use of $\Psi$ rather than $\Psi_0$ in the second role; as only this choice in Eq. (13) correctly reduces to the constant minus sign in the flip for fermions when $q = -1$.

Next, we define the operators $\psi^*_x$ and $\psi^x$ on $S_{\Psi_0}(\mathcal{H})$, by left multiplication and braided differentiation respectively. Let $[N+1,\Psi] = id + \Psi_{12} + \Psi_{123} + \ldots + \Psi_{12\ldots N,N+1}$ be the braided integer matrix [19] [21], where for example $\Psi_{12}$ denotes $\Psi$ acting in the (1,2) pair of copies of $\mathcal{H}$. Then:

$$\psi^*_x(\delta_x \otimes \cdots \otimes \delta_{x_N}) = \delta_x \otimes \delta_{x_1} \otimes \cdots \otimes \delta_{x_N},$$

(14)

and

$$\psi^x(\delta_{x_1} \otimes \cdots \otimes \delta_{x_{N+1}}) = \delta_{x_2} \otimes \cdots \otimes \delta_{x_{N+1}} [N+1,\Psi]_{x_1x_2\ldots x_{N+1}}^{x_1x_2\ldots x_{N+1}}$$

$$= \delta^x_{x_1} \delta_{x_2} \otimes \cdots \otimes \delta_{x_{N+1}} + q^{(x_1,x)} \delta^x_{x_2} \delta_{x_3} \otimes \cdots \otimes \delta_{x_{N+1}}$$

$$+ \cdots + q^{(x_1,x)+\cdots+\epsilon(x_N,x)} \delta^x_{x_{N+1}} \delta_{x_1} \otimes \cdots \otimes \delta_{x_{N'}}.$$  

(15)

Clearly the braided derivative $\psi^x$ (or $\partial^x$ in the notation of [19]) is given by the evaluation or interior product pairing with $\mathcal{H}$, but extended to the whole of $S_{\Psi_0}(\mathcal{H})$ as a braided derivation. This is like a super-derivation, but using $\Psi$ to braid $\psi^x$ past elements of $\mathcal{H}$. These operators of braided differentiation arise as infinitesimal translations on the braided space, as expressed through the linear braided coproduct. From the braided Leibniz rule for these operators, one has easily the following relations [19]:

$$\psi^*_x \psi^*_y = \psi^*_y \psi^*_x R^{x.a.b}_{y.a.b} = \psi^*_y \psi^*_x q^{a(x,y)}, \quad \psi^x \psi^y = R^{x.a.y}_{y.a.b} \psi^b \psi^a = q^{a(x,y)} \psi^y \psi^x,$$

(16)
and

\[ \delta^x_y = \psi^x \psi^*_y - \psi^*_a R^a_y \psi^b = \psi^x \psi^*_y - q^{(y,x)} \psi^*_y \psi^x, \]  

just as in Eqs. (12). Notice that the \( \psi^x \) operators again obey the algebra \( S_{\Psi_0}(H) \), as do (more obviously) the \( \psi^*_x \), but now we have also the cross relation (17).

To sum up, the space \( S_{\Psi_0}(H) \), which should be viewed geometrically as the algebra of \( \Psi_0 \)-symmetric functions (on the dual space \( H \)), first decomposes as a vector space. We have \( S_{\Psi_0}(H) = \bigoplus_N H \otimes_s \cdots \otimes_s H \) (\( N \) copies) consist of \( \Psi_0 \)-symmetric functions of degree \( N \). We shall study these components more explicitly in the next section. Secondly, the same algebra acts by ‘pointwise multiplication’ and ‘infinitesimal translation’ on itself, generating a braided Weyl algebra and satisfying the relations (9) in our discrete setting. This construction generalizes both the usual CCR and CAR algebras, along with their usual realizations on symmetric or antisymmetric product algebras. All of this is an easy case of the general theory of braided spaces and braided derivatives [21], for our particular \( R, R' \)-matrices.

Note also that although we have carefully distinguished \( \delta_x \) (the basis of the space \( H \) generating \( S_{\Psi_0}(H) \)) from the operators \( \psi^*_x \) and \( \psi^x \) that act on \( S_{\Psi_0}(H) \), we can also identify \( H \) with the space generated by the \( \psi^*_x \) acting on 1 by multiplication. Thus we can identify the whole “acted-upon” copy of \( S_{\Psi_0}(H) \) with the copy of \( S_{\Psi_0}(H) \) generated by the \( \psi^*_x \).

The vacuum state is \( |0\rangle = 1 \). From the geometrical point of view, the \( \psi^x \) act by braided differentiation. From the “Fock space” point of view, they act via the commutation relations and the condition that \( \psi^x |0\rangle = 0 \).

Finally, let us specialize \( X \) to a finite ordered set, writing \( X = \{1, \ldots, n\} \). Then using Eqs. (15), we have the following results for \( S_{\Psi_0}(H) \) and the corresponding braided Weyl algebra:

\[ \psi^*_i \psi^*_j = q \psi^*_j \psi^*_i, \quad \psi^i \psi^j = q \psi^j \psi^i, \quad \psi^i \psi^*_j = q^{-1} \psi^*_j \psi^i, \quad \psi^j \psi^*_i = q \psi^*_i \psi^j, \quad (\forall i < j) \]

\[ \psi^j \psi^*_i - q \psi^*_i \psi^j = 1, \quad (\forall i) \]  

(18)

where

\[ \psi^*_i |m_1, \ldots, m_n\rangle = q^{-\sum_{j<i} m_j} |m_1, \ldots, m_i + 1, \ldots, m_n\rangle, \]

\[ \psi^i |m_1, \ldots, m_n\rangle = q^{\sum_{j<i} m_j} [m_i; q] |m_1, \ldots, m_i - 1, \ldots, m_n\rangle. \]

(19)

Here

\[ |m_1, \ldots, m_n\rangle = (\psi^*_1)^{m_1} \cdots (\psi^*_n)^{m_n} |0\rangle, \]

while

\[ [m; q] = 1 + q + \cdots + q^{m-1} = (1 - q^m)/(1 - q) \]
is a ‘q-integer’. These equations yield, for example, the “density operator” as:

\[ \rho_i = \psi_i^\dagger \psi_i, \quad \rho_i | m_1, \cdots, m_n \rangle = | m_i; q \rangle | m_1, \cdots, m_n \rangle. \]  

(20)

We note again that the final equation of [Eq. (15)] implies \( \psi_i^\dagger \) cannot be the adjoint of \( \psi_i \) unless \( q \) is real. In general the \( q \)-integers in Eq. (20) are complex, and the operator \( \rho_i \) is not self-adjoint.

Although we are taking the above “quantum-mechanical harmonic oscillator” point of view, the same mathematical structures can also be viewed as spacetime position and momentum generators. Then we would denote the generators \( \delta_i \) by \( X^i \), and regard them as the coordinates of a noncommutative spacetime, with the usual “\( q \)-plane” relations, \( X_i X_j = q X_j X_i \) for \( i < j \). Similarly, we would write \( \psi^*_i = \hat{X}_i \) for the operation of left-multiplication by \( X_i \), and \( \psi^j = \partial^j \) for the braided differentiation

\[ \partial^j (X_1^{m_1} \cdots X_n^{m_n}) = q^{\sum_{j<i} m_j} [m_i; q] X_1^{m_1} \cdots X_i^{m_i-1} \cdots X_n^{m_n}. \]

This is the “geometrical” point of view to which we alluded above. Here \( \partial^i \) acts on the \( X_i \) variable by \( q \)-differentiation, with an additional braiding \( q \)-factor as \( \partial^j \) moves past the \( X_j \), \( j < i \), in order to reach the \( X_i \). Note that the \( q \)-plane is usually associated with more complicated \( R \)-matrices related to quantum groups \( SL_q(n) \) but this is not the case here. Indeed, many different \( R \)-matrices can give the same quantum plane.

### 3.2 Functional version

We next proceed to a functional version of the above, more suitable for the continuum case. Clearly, on general functions \( h, g \in \mathcal{H} \), the braiding is given by

\[ \Psi_0(h \otimes g)(x, y) = g(x)h(y)q^{-\alpha(x, y)}, \]

(21)

where \( \Psi_0(h \otimes g) = \int dx \, dy \, \Psi_0(h \otimes g)(x, y) \delta_x \otimes \delta_y \) (the integrals here and below should be interpreted as sums in the discrete case). Next, a basis for \( \mathcal{H} \otimes_s \cdots \otimes_s \mathcal{H} \) is given by “normal ordering”; i.e., we choose \( \{ \delta_{x_1} \otimes_s \cdots \otimes_s \delta_{x_N} | x_1 \leq x_2 \leq \cdots \leq x_N \} \). We let the coefficients in this basis be partially defined functions \( f(x_1, x_2, \cdots, x_N) \) so that

\[ f = \int_{x_1 \leq \cdots \leq x_N} dx_1 \cdots dx_N f(x_1, x_2, \cdots, x_N) \delta_{x_1} \otimes_s \cdots \otimes_s \delta_{x_N}. \]

Then, defining \( \psi^*(h) = \int dx \, h(x) \hat{\delta}_x \), we compute:

\[ \psi^*(h) f = \int_{x \leq x_1 \leq \cdots \leq x_N} dx \, dx_1 \cdots dx_N h(x) f(x_1, \cdots, x_N) \delta_x \otimes_s \delta_{x_1} \otimes_s \cdots \otimes_s \delta_{x_N} \]

\[ + q^{-1} \int_{x_1 < x \leq x_2 \leq \cdots \leq x_N} dx_1 \cdots dx_N h(x) f(x_1, \cdots, x_N) \delta_{x_1} \delta_x \otimes_s \delta_{x_2} \otimes_s \cdots \otimes_s \delta_{x_N} \]

\[ + \cdots + q^{-N} \int_{x_1 \leq \cdots \leq x_N < x} dx_1 \cdots dx_N dx \, h(x) f(x_1, \cdots, x_N) \delta_{x_1} \otimes_s \cdots \otimes_s \delta_{x_N} \delta_x, \]
where we have used the relations (12). After relabeling, we read off the coefficients as

\[
(\psi^*(h)f)(x_1, \cdots, x_{N+1}) = h(x_1)f(x_2, \cdots, x_{N+1}) + q^{-1}h(x_2)f(x_1, x_3, \cdots, x_{N+1}) + \cdots + q^{-N}h(x_{N+1})f(x_1, \cdots, x_N),
\]

for distinct “normally ordered” arguments. This is our representation of \(\psi^*\) in the continuum case.

Similarly, we linearly extend the definition (15) of \(\partial_x\), doing in each case one of the integrals and relabeling the integration variables \(x_1, \cdots, x_N\). This gives

\[
\psi(x)f = \int_{x_1 \leq \cdots \leq x_N} dx_1 \cdots dx_N f(x, x_1, \cdots, x_N) \delta_{x_1 \cdots x_N} + q \int_{x_1 \leq x \leq x_2 \leq \cdots \leq x_N} dx_1 \cdots dx_N f(x_1, x, x_2, \cdots, x_N) \delta_{x_1 \cdots x_N} + \cdots + q^N \int_{x_1 \leq \cdots \leq x_N} dx_1 \cdots dx_N f(x_1, \cdots, x_N, x) \delta_{x_1 \cdots x_N},
\]

which yields

\[
(\psi(x)f)(x_1, \cdots, x_N) = q^{m} f(x_1, \cdots, x_m, x_{m+1}, \cdots, x_N) \quad \text{for} \quad x_m < x < x_{m+1},
\]

with distinct, “normally ordered” \(x_1, \cdots, x_N\) (using the usual conventions for \(m = 0\) or \(m = N\)). This is our representation of \(\psi\) in the continuum case.

Finally, if the basis elements are taken in a different order, the corresponding coefficients are related through Eq. (12); e.g., we have

\[
f(x_2, x_1, \cdots, x_N) = q^{\epsilon} f(x_1, x_2, \cdots, x_N),
\]

and so forth. In this way, any coefficient function defined on “normally ordered” arguments extends uniquely to an element of \(H^\otimes N\) obeying such relations.

Hence, proceeding now in the continuum case, we can characterize \(H^\otimes N\) as the subspace of \(\Psi_0\)-symmetric functions:

\[
H^\otimes N = \{ f \in H^\otimes N \mid f(\sigma(x_1, \cdots, x_N)) = q^{\ell(\sigma)} f(x_1, \cdots, x_N), \forall \sigma \in S_N, x_1 < \cdots < x_N \}
\]

where \(\ell(\sigma)\) is the length of the permutation \(\sigma\). It suffices to impose the symmetrization condition here almost everywhere. For example,

\[
H_0 \otimes H = \{ f \mid f(y, x) = q f(x, y), \forall x < y \},
\]
and
\[ \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} = \{ f \mid f(y, x, z) = f(x, z, y) = qf(x, y, z), \]
\[ f(z, x, y) = f(y, z, x) = q^2f(x, y, z), \ f(z, y, x) = q^3f(x, y, z), \ \forall \ x < y < z \}. \]

Making use of the subspace description \((25)\), there is a corresponding formula for the action of \(\psi^*(h)\) that involves factors of \(q^\epsilon\). For example, letting \(\psi^*(h)\) act on a function \(f \in \mathcal{H} \otimes \mathcal{H}\), we have
\[
(\psi^*(h)f)(x, y, z) = h(x)f(y, z) + q^\epsilon(x,y)h(y)f(x, z) + q^\epsilon(z,x)h(z)f(x, y).
\] (26)

The general formula for degree \(N\) is
\[
\psi^*(h)f = (\text{id} + \Psi_{12} + \Psi_{23}\Psi_{12} + \ldots + \Psi_{N,N+1}\ldots\Psi_{23}\Psi_{12})(h \otimes f),
\] (27)
which up to a normalization is just \(h \otimes f\) followed by total \(\Psi\)-symmetrization (given the assumed \(\Psi\)-symmetry of \(f\)). Similarly, the formula for \(\psi(x)\) may be written in the subspace description. It comes out simply as the interior product
\[
\psi(x)f = f(x, ),
\] (28)
where we evaluate the first argument of \(f\) at \(x\). These are the field operators when we work with \(N\)-particle wave functions as symmetrized functions in \(N\) variables, as in the usual Bose or Fermi Fock spaces.

When there are products of fields, we will typically smear at least some of the variables with test functions to make sense of the distributions. Thus the \(q\)-commutation relations can be written as
\[
\psi(x)\psi(h) = \psi(\theta_0xh)\psi(x), \ \psi^*(x)\psi^*(h) = \psi^*(\theta_0xh)\psi^*(x)
\]
\[
\psi(h)\psi^*(x) = \psi^*(x)\psi(\theta_xh) + h(x),
\] (29)
for \(x \in X\) and \(h\) in a dense domain of \(\mathcal{H}\), where for each fixed \(x\) we define
\[
\theta_0x(y) = q^{\epsilon_0(x,y)}, \ \theta_x(y) = q^{\epsilon(x,y)}
\]
as functions of \(y\) (having modulus 1 when \(q\) is a phase), and where \(\theta_0xh\) and \(\theta_xh\) refer to the action on \(\mathcal{H}\) given by pointwise multiplication. The second equation of \((29)\) can also be written
\[
\psi(x)\psi^*(h) = \psi^*(\bar{\theta}_xh)\psi(x) + h(x),
\]
where \(\bar{\theta}_x(y) = q^{\epsilon(y,x)}\).
As before, essentially the same algebra $S_{\Psi_0}(\mathcal{H})$ leads both to the Hilbert space of the system with components $\mathcal{H} \otimes \cdots \otimes \mathcal{H}$ as above (ignoring its algebra structure), and to the generalized field algebra of the $\psi^*(h)$ together with another copy for the $\psi(h)$ as an algebra of operator-valued distributions obeying (9) and represented according to (23)-(24). The Fock space can also be viewed as generated by the operators $\psi^*(h)$ for a sufficient set of test functions $h$, acting repeatedly on $|0\rangle$.

We also have a geometrical picture as at the end of the last section, with $\psi(h)$ acting now by ‘braided functional differentiation’.

### 3.3 Unitarity considerations

Until now we have worked rather generally, and have taken $q$ to be arbitrary. We now work specifically over $\mathbb{C}$, and take $q$ to be a phase. Addressing first the continuum case, we consider the $L^2$ inner product on $\mathcal{H}$, and verify that all our operator constructions are suitably self-adjoint.

First we see that $\Psi_0$ is self-adjoint, since

\[
(a \otimes b, \Psi_0(h \otimes g)) = \int \int dx \, dy \, \bar{a}(x) \bar{b}(y) g(y) h(x) q^{-\epsilon_0(x,y)} = \int \int dx \, dy \, \bar{\Psi}_0(a \otimes b)(y,x) h(y) g(x) = (\Psi_0(a \otimes b), h \otimes g).
\]

Next we make use of the fact that the spaces $\mathcal{H} \otimes \cdots \otimes \mathcal{H}$ have $L^2$ inner products, when we write their elements as functions $f$ defined on the fundamental domain of “normally ordered” coordinates. We perform the integration over this domain. Thus:

\[
(g, \psi^*(h)f) = \int_{x_1 \leq \cdots \leq x_{N+1}} dx_1 \cdots dx_{N+1} \, g(x_1, \cdots, x_{N+1}) \cdot \sum_{m=0}^{N+1} q^{-m} \hat{h}(x_{m+1}) f(x_1, \cdots, x_{m+1}, \cdots, x_{N+1})
= \sum_{m=0}^{N} \int_{x_1 \leq \cdots \leq x_N} dx_1 \cdots dx_N \cdot \int_{x_m}^{x_{m+1}} dx \, q^m \bar{h}(x) g(x_1, \cdots, x_m, x, x_{m+1}, \cdots, x_N) f(x_1, \cdots, x_N)
= (\psi(\bar{h})g, f),
\]

where we have used the previous results for $\psi^*(h)$ and $\psi(x)$, interpreted $\psi(\bar{h})$ as the integral of $\psi(x)$ times $\bar{h}$, and relabeled the $x_i$ in the calculation. The cases $m = 0$ and $m = N$ are understood in the obvious way (the integration is then taken to $\pm \infty$). In the summations
for $\psi^*(h)f$ (see Eq. 22) some of the inequalities in the region of integration are strict, but we are permitted to ignore this distinction in the present continuum case.

Finally, in our subspace description of $\mathcal{H} \otimes \cdots \otimes \mathcal{H}$ we would like to define the inner product in terms of that on $\mathcal{H}^\otimes N$. Indeed, for $f, g$ extended to $\Psi_0$-symmetric elements of $\mathcal{H}^\otimes N$, we have

$$ (g, f)_{\mathcal{H}^\otimes N} = \sum_{\sigma \in S_N} \int_{x_1 \leq \cdots \leq x_N} dx_1 \cdots dx_N \tilde{g}(\sigma(x_1, \cdots, x_N)) f(\sigma(x_1, \cdots, x_N)) = N! (f, g). $$

That is, the natural inner product with respect to which $\psi$ and $\psi^*$ are mutually adjoint is $N!^{-1}$ times the usual tensor product inner product. Alternatively, if one wished to use the usual tensor product inner product, then one should work with $(N + 1)^{-1}\psi^*(h)$. From Eq. (27) we see that this would then be a true averaging over the symmetric group, and would correspond to the usual normalization in the Bose and Fermi cases.

For the continuum case, we see that $\psi^*$ is the adjoint of $\psi$ as in Ref. [17]. The discrete case requires us perform summations instead of integrations, and to be careful about the domains of summation. Thus, in the second term of Eq. (22), we have $x_1 < x \leq x_2 \leq \cdots \leq x_N$; and similarly for the other terms. Then, as a correction to Eq. (30), we have the following:

$$ (\psi(\tilde{h})g, f) = (g, \psi^*(h)f) + \sum_{x_1 \leq x_2 \leq \cdots \leq x_N} (q^{-1}h(x_1)\tilde{g}(x_1, x_2, \cdots x_N) + \cdots + q^{-N}h(x_N)\tilde{g}(x_1, \cdots, x_{N-1}, x_N, x_N)) \cdot f(x_1, \cdots x_N). \quad (31) $$

Writing $\psi(\tilde{h})^\dagger = \psi^*(h) + T_h^*$, where $\psi(\tilde{h})^\dagger$ is the adjoint of $\psi(\tilde{h})$, we find

$$ T_h^* (\delta_{x_1} \otimes \cdots \otimes \delta_{x_N}) = \sum_{m=1}^{N} q^{-m}h(x_m) \delta_{x_1} \otimes \cdots \otimes \delta_{x_m} \otimes \delta_{x_m} \otimes \cdots \otimes \delta_{x_N} $$

where $\delta_{x_m}$ is duplicated on the right-hand side, and where $x_1 \leq \cdots \leq x_N$.

To proceed further, it is convenient (though not essential) to use the simplified notation for the set $X = \{1, \cdots, n\}$. We then find

$$ T_i^* ((\delta_1)^{m_1} \cdots (\delta_n)^{m_n}) = q^{-\sum j < i, m_j} (q^{-1} + \cdots + q^{-m_i}) (\delta_1^{m_1}) \cdots (\delta_i^{m_i+1}) \cdots (\delta_n^{m_n}) $$

(where we have omitted the symbols $\otimes_s$ for the symmetrized tensor product). Hence

$$ T_i^* |m_1, \cdots, m_n\rangle = q^{-\sum j < i, m_j} [m_i; q^{-1}] |m_1, \cdots, m_i + 1, \cdots m_n\rangle, $$

and

$$ \psi^\dagger |m_1, \cdots, m_n\rangle = (1 + q^{-1}[m_i; q^{-1}])\psi_i^* |m_1, \cdots, m_n\rangle = [m_i + 1; q^{-1}] \psi_i^* |m_1, \cdots, m_n\rangle. $$
Noting that $\rho_i^\dagger$ has values which are the complex conjugates of the values of the diagonal operator $\rho_i$ in Eq. (20), we see that

$$\psi_i^\dagger = \rho_i^\dagger \psi_i^*, \quad \psi^i = (\psi_i^*)^\dagger \rho_i. \quad (32)$$

It is worth remarking that

$$\psi_i^\dagger \psi^i | m_1, \ldots, m_n \rangle = \rho_i^\dagger \rho_i = | [m_i; q] |^2 | m_1, \ldots, m_n \rangle = \frac{1 - \cos m_i \theta}{1 - \cos \theta} | m_1, \ldots, m_n \rangle \quad (33)$$

if $q = e^{i\theta}$.

While the density operators $\rho_i$, are not self-adjoint in the discrete case, they have some nice properties. For example

$$[\rho_i, \psi_j^*] = 0 = [\rho_i, \psi^j], \quad (\forall i \neq j)$$

but

$$\rho_i \psi_i^* - q \psi_i^* \rho_i = \psi_i^*, \quad \psi^i \rho_i - q \rho_i \psi^i = \psi^j, \quad (34)$$

with similar relations for $\rho_i^\dagger$; also $[\rho_i, \rho_i^\dagger] = 0$. The results differ from the continuum case, where we already know that Eq. (7) holds with the usual commutator, not the $q$-commutator of Eq. (34).

The same conclusions apply for any discrete set $X$. The origin of the differences from the corresponding equations in the continuum case is that in the discrete case, we have treated the $\delta$-function in Eq. (10) as a Kronecker-$\delta$ with values 0 and 1. This makes it the “same size” as the $\delta$-functions arising in the combinatorics of the summations. Such a treatment leads to elegant $q$-deformation formulas, as obtained above. For a $\mathbb{Z}^n$-lattice theory that correctly converges in the zero-spacing limit to the continuum theory, one could use $\delta_{x,y}/\Delta^n$ in the right hand side of Eq. (9), where $\Delta$ is the lattice spacing.

4 Identification with the topological picture

It remains to identify the Hilbert spaces $\mathcal{H}_N$ in the previous section with the braided-symmetrized tensor products we have obtained. Let the map $\pi : B_N \to S_N$ be the natural homomorphism defined by identifying a crossing with its inverse crossing. As in Section 2, we describe a topological configuration $\tilde{\gamma}$ by the pair $(\gamma, b)$, where $p(\tilde{\gamma}) = \gamma \in \Delta_N(\mathbb{R}^2)$, and where $b \in B_N$ (the fundamental group of $\Delta_N$). Then with $\gamma = \{x_1, \ldots, x_N\}$,

$$\mathcal{H}_N \cong \mathcal{H} \otimes \cdots \otimes \mathcal{H}, \quad \Phi(\gamma, b) = f(\pi(b)(x_1, \ldots, x_N)), \quad \forall x_1 < x_2 < \cdots < x_N.$$

While $\gamma$ is unordered, we introduce the conventional lexicographic ordering in indexing its elements.
For example if \( x < y < z \) in \( \mathbb{R}^2 \), then
\[
\tilde{\Phi}\({\{x, y, z\}, e}\) = f(x, y, z), \quad \tilde{\Phi}\({\{x, y, z\}, b_{23}\}) = f(x, z, y) = q \tilde{\Phi}\({\{x, y, z\}, e}\) = q f(x, y, z)
\]
by equivariance, where \( b_{23} \) is the braid group generator braiding strand 2 with strand 3. In general, if \( x_1 < \cdots < x_N \) then
\[
f(\sigma(x_1, \ldots, x_N)) = \tilde{\Phi}\({\{x_1, \ldots, x_N\}, i(\sigma)\}) = q^\ell(\sigma)\tilde{\Phi}\({\{x_1, \ldots, x_N\}, e}\) = q^\ell(\sigma) f(x_1, \ldots, x_N);
\]
where \( i(\sigma) \) is the braid defined as follows: Let \( \sigma = s_{j_1} \cdots s_{j_{\ell(\sigma)}} \) be a reduced expression for \( \sigma \) in terms of simple exchanges \( s_j = (j, j+1) \). Let \( b_j \) be the braid group generator braiding strands \( j \) and \( j+1 \). Then \( i(\sigma) = b_{i_1} \cdots b_{i_{\ell(\sigma)}} \). Note that \( i(\sigma) \) does not define a group homomorphism. Eq. (35) is just as required; our symmetry condition on \( f \) corresponds to the equivariance under \( B_N \) of \( \tilde{\Phi} \).

Likewise, using the definitions in Section 2 and the diagrammatic notation as in Ref. [17], one obtains
\[
(\psi^*(h)\tilde{\Phi})({\{x, y, z\}, e}) = q^{-\#(y, z < x)} h(x)\tilde{\Phi}({\{y, z\}, e}) + q^{-\#(x, z < y)} h(y)\tilde{\Phi}({\{x, z\}, e}) + q^{-\#(x, y < z)} h(z)\tilde{\Phi}({\{x, y\}, e}),
\]
as well as
\[
(\psi(z)\tilde{\Phi})({\{x, y\}, e}) = q^{\#(x, y < z)} \tilde{\Phi}({\{x, y, z\}, e}),
\]
where \( \#(x, y < z) \) denotes the number of points in the set \( \gamma = \{x, y\} \) to the left of \( z \) in \( \mathbb{R}^2 \). These actions are to be compared with (24) and (26). Similarly one proves in general that the two constructions coincide; that is, the representation of the anyon creation and annihilation fields in the spaces \( H_N \) of topological configurations is equivalent to their representation in the braided Fock space by \( h \otimes_s \) and the interior product.

5 Roots of unity and the anyonic exclusion principle

In this section we specialize further to the case where \( q = e^{2\pi i/r} \) is a primitive \( r \)th root of unity. Our main observation is that in the continuum case, the explicit representation (23) on \( \Psi_0 \)-symmetric tensor products implies that
\[
\psi(h)^r = 0, \quad \psi^*(h)^r = 0, \quad (\forall h);
\]
that is, we have an anyonic exclusion principle. Here \( h \) is any test-function, and the smearing of \( \psi \) by \( h \) makes sense of the products of distributions. By taking ‘bump functions’ that are increasingly localized at an arbitrary point \( x \), we can also write informally,
\[
\psi(x)^r = 0, \quad \psi^*(x)^r = 0, \quad (\forall x),
\]
a condition which we shall justify directly in the discrete case. Before doing so let us note that conversely, the pointwise condition \((37)\) in the discrete case implies \((36)\), since

\[
\psi(\lambda \delta_x + \mu \delta_y)^r = (\lambda \psi(x) + \mu \psi(y))^r = \sum_{s=0}^{s=r} \lambda^s \mu^{r-s} \psi(x)^s \psi(y)^{r-s} \left[ \frac{r}{s}; q^{\sigma(y,x)} \right] = \lambda^r \psi(x)^r + \mu^r \psi(y)^r,
\]

for any constants \(\lambda, \mu\) and \(x \neq y\). In obtaining \((38)\), we use the \(q\)-binomial theorem for \(q\)-commuting quantities, with \(q\)-binomial coefficients defined in the usual way but with \(q\)-integers in place of integers in the factorials. Since \(q^r = 1\) we have \([r; q^{\pm 1}] = 0\) and hence only \(s = 0\) and \(s = r\) contribute. So the pointwise and smeared versions are formally equivalent, with the smeared version being more suitable in the continuum case.

To prove \((36)\) in the continuum, we use \((23)\) acting on the vacuum (the identity function with no arguments) to deduce that

\[
\psi^*(h)^m(x_1, \cdots, x_m) = [m; q^{-1}] h(x_1) \cdots h(x_m)
\]

for all non-negative integers \(m\). This follows by induction on \(m\). It is true for \(m = 1\); assuming it for \(m\), we have

\[
\psi^*(h)^{m+1}(x_1, \cdots, x_{m+1}) = h(x_1)\psi^*(h)^m(x_2, \cdots, x_{m+1}) + q^{-1} h(x_2)\psi^*(h)^m(x_1, x_3, \cdots, x_{m+1}) + \cdots + q^{-m} h(x_{m+1})\psi^*(h)^m(x_1, \cdots, x_m)
\]

\[
= (1 + q^{-1} + \cdots + q^{-m})[m; q^{-1}] h(x_1) \cdots h(x_{m+1})
\]

as required. The exclusion principle follows when \(q^* = 1\), since then \([r; q^{-1}] = 0\) as already noted above. Next, letting \(\psi^*(h)^r\) act on any Fock state of the form \(\psi^*(h_1) \cdots \psi^*(h_N)|0\rangle\), we move \(\psi^*(h)^r\) to the right until it arrives to act on the vacuum. In doing so, \(h\) will be replaced by a convolution with all the \(h_1, \cdots h_N\) [see Eqs. \((29)\)], giving us a new test function \(h'\). But we have already verified that \(\psi^*(h')^r|0\rangle\) for all test functions; hence \((36)\) holds in the representation. The result for \(\psi(h)^r\) follows, as in the continuum case it is the adjoint.

In fact, in the continuum theory one can formally conclude \((36)\) as a statement about operator-valued distributions. We exhibit the reasoning for \(r = 3\) (the general case is similar). We have

\[
\psi^*(h)^3 = \int dxdydz h(x)h(y)h(z)\psi^*(x)\psi^*(y)\psi^*(z)
\]

\[
= \sum_{\sigma \in S_3} \int_{\sigma(x) < \sigma(y) < \sigma(z)} dxdydz h(x)h(y)h(z)q^{-\ell(\sigma)}\psi^*(\sigma(x))\psi^*(\sigma(y))\psi^*(\sigma(z))
\]

\[
= \sum_{\sigma \in S_3} q^{-\ell(\sigma)} \int_{x < y < z} dxdydz \psi^*(x)\psi^*(y)\psi^*(z) = 0,
\]

18
where we have written \( \sigma(x, y, z) = (\sigma(x), \sigma(y), \sigma(z)) \), and where \( \ell(\sigma) \) is the length of the permutation. In effect we have broken up the 3-fold integral into \( 3! \) permutations of the fundamental domain where \( x < y < z \) (not being concerned about coincident points since these form a subset of measure zero); then we have used the \( q \)-commutation relations (9), and finally we have changed variables to give many copies of the integral over the fundamental domain. The result is zero since \( \sum q^{-\ell(\sigma)} = [3; q^{-1}]! = 0 \) when \( q^3 = 1 \). The same argument holds for general \( r \) and for \( \psi(h)^r = 0 \).

The arguments for the discrete case are more subtle, and indeed the formulas \( \psi(x)^r = 0 \) and \( \psi^*(x)^r = 0 \) do not hold automatically in our representation. Rather, we argue that in this case it is natural to impose these according to the algebraic structure. This is similar to ‘truncated versions’ of quantum groups and other \( q \)-algebras in conformal field theory and other settings at roots of unity. Indeed we then have \( \Psi^r = \text{id} \) for the braiding, and we are essentially in the setting that has been called ‘anyonic vector spaces’ in [12] and [19].

From a physical point of view, the key observation is that for \( q \) a primitive \( r \)th root of unity, the operators \( \psi(x)^r \) and \( \psi^*(x)^r \) are in any case central. In fact, from (9) we have

\[
\psi(x)^m \psi(y) = q^{m \epsilon_0(x,y)} \psi(y) \psi(x)^m,
\]

and

\[
\psi^*(x)^m \psi^*(y) = \psi^*(y) \psi(x)^m + \delta(x-y) \psi(x)^{m-1}[m; q^{\epsilon(y,x)}].
\]

Hence, when \( q^r = 1 \), we have \( [r; q^{\pm 1}] = 0 \) and \( \psi(x)^r \) is central. Note that here it is critical that we used \( \epsilon \) and not \( \epsilon_0 \) in the calculation involving \( \psi^*(y) \). Similarly, we have that \( \psi^*(x)^r \) central. Therefore, in an irreducible sector of the theory, these operators should be set to multiples of the identity.

This argument is needed only in the discrete case, but the algebraic structures apply (suitably understood) in both cases; thus we have used a notation applicable to both.

As for which value these operators should be assigned, we have already seen that (37) is suitable for the constraint to be linear (basis independent), in the sense of applying to all \( h \). Clearly zero is the only value with this property. Another way to reach the same conclusion is in terms of the braided coproduct on \( S_{\psi_0}(\mathcal{H}) \). On products it extends by \( \psi \), and one has

\[
\Delta \psi^*(x)^r = \sum_{s=0}^{s=r} \psi^*(x)^s \otimes \psi^*(x)^{r-s} [s; q] = \psi^*(x)^r \otimes 1 + 1 \otimes \psi^*(x)^r.
\]

A similar result holds for \( \psi(x)^r \). Since \( \Delta 1 = 1 \otimes 1 \), only (37) allows the braided coproduct to descend to the reduced algebra; no other constant will do. Moreover, since this \( \Delta \) underlies the braided-differentiation operation \( \partial^F \) in Section 3.1, our representation of \( \psi^* \) (and more obviously \( \psi \)) then descends to an action of the truncated algebra \( S_{\psi_0}(\mathcal{H}) \) (in which (37)
is imposed) on itself, by the same formulas as before. This is in subtle contrast to the continuum case, where (36) already holds.

Next we note that because $\psi^*(x)$ and $\psi^*(y)$ commute up to a factor $q'_{0(x, y)}$, any state formed by a sequence of creation field operators applied to the vacuum, where $r$ or more occurrences of $\psi^*(x)$ are included, must vanish. That is,

$$\psi^*(x)\psi^*(y)\psi^*(z)\cdots \psi^*(w)\cdots \psi^*(x)\cdots |0\rangle = 0$$

if there are $r$ or more instances of $\psi^*(x)$. The distinct $\psi^*(y)$, $\psi^*(z)$ and so forth can equally be annihilation fields $\psi(y)$, $\psi(z)$ and so forth, since these too $q$-commute with $\psi^*(x)$.

More generally, a state formed from the vacuum will be zero if there are $m$ annihilation $\psi(x)$ operators, and $r + m$ or more $\psi^*(x)$. This can be proved by induction on $m$, as follows. The case $m = 0$ is covered above. Suppose it is true for $m - 1$. Given an expression with $m$ annihilation operators, look at the rightmost $\psi(x)$. Applying the $q$-commutation relation with the $\psi^*$ to its right, we move $\psi(x)$ to the right and pick up a second term with a $\delta$-function and with one fewer $\psi(x)$, and at most one fewer $\psi^*(x)$; by our induction hypothesis, this second term vanishes. Meanwhile, the first term has $\psi(x)$ one step to the right; repeating this eventually brings it to act directly on $|0\rangle$, giving zero. This proof makes sense in the discrete case, or with the assumption that the $\delta$-functions are approximated by bounded functions with the limit taken only at the end (in order to treat $\delta(0)$ as a number). More precisely, in terms of annihilation operators smeared with test functions,

$$\psi^*(x)\cdots \psi^*(h_1)\cdots \psi^*(x)\cdots \psi^*(h_m)\cdots \psi^*(x)\cdots |0\rangle = 0$$

if there are $m$ annihilation operators and at least $r + m$ creations $\psi^*(x)$ anywhere in the string. This follows from (29) and a similar proof by induction.

For general $\psi^*(h)$ we always have the exclusion principle (36). But the stronger version, in which some of the $\psi^*(h)$ are not adjacent, is more complicated. The various $\psi^*(h)$ needed are modified by the intervening creation field operators, according to (29). Thus, one has instead exclusion conditions that take the form

$$\psi^*(\theta_{0x_1}\cdots \theta_{0x_m} h)^{n_0} \psi^*(x_1)\psi^*(\theta_{0x_2}\cdots \theta_{0x_m} h)^{n_1}\cdots$$

$$\cdots \psi^*(x_{m-1})\psi^*(\theta_{0x_m} h)^{n_{m-1}}\psi^*(x_m)\psi^*(h)^{n_m} |0\rangle = 0,$$

when $n_0 + \cdots + n_m \geq r$. In the Bose or Fermi cases, the functions $\theta_{0x}$ are constant ($\pm 1$), up to a set of measure zero; but otherwise they must be taken into account. The same complication applies when there are annihilation operators present. This would appear to be a feature of the anyonic theory ($r > 2$), that is not present for fermions.
On the other hand, we do not see this complication if all our smeared fields have disjoint support. Thus, from (29) we find that

\[ \psi^*(h)\psi^*(g) = \psi^*(g)\psi^*(h) \begin{cases} q & \text{if } \text{supp}(h) < \text{supp}(g) \\ q^{-1} & \text{if } \text{supp}(h) > \text{supp}(g) \end{cases} \]

with a similar equation for \( \psi(h)\psi(g) \). When they occur, these have a similar form to (9). Hence, as an application, one may take the \( \psi^*(x) \) as given more precisely by smearing with ‘bump functions’ of small support around the relevant point. As long as these bumps to not touch, the various smeared \( \psi^*(x) \) fields behave as in the discrete case, and may be collected together by similar relations. For such states we have the full exclusion principle again, without any complications when the instances of the smeared field \( \psi^*(x) \) are separated from each other in the product of field operators.

With this last observation in mind, let us give a straightforward application of the exclusion principle to a gas of non-interacting anyonic particles (e.g., vortices) localized in disjoint sets around points \( x_1, \ldots, x_n \) in \( \mathbb{R}^2 \). Let us suppose that each particle carries a fixed unit \( E \) of energy, which does not depend on the positions; the latter will therefore be considered as fixed (or else, we must factor out the resulting degeneracy). As per the discrete version of the theory, the reduced range of states is then:

\[ |m_1, \ldots, m_n \rangle = \psi^*(x_1)^{m_1} \cdots \psi^*(x_n)^{m_n} |0\rangle, \quad m_i = 0, \ldots, r - 1. \]

Let us decompose this reduced Hilbert space as \( \bigoplus_N \mathcal{H}_N \) according to \( N = \sum_{i=1}^n m_i \), which is the value of the occupation number operator

\[ \mathcal{N} = \int dx \rho(x) = \int dx \psi^*(x)\psi(x). \]

The statistical partition function is then

\[ Z_\beta = \text{Trace}(e^{-\beta E N}) = \sum_N e^{-\beta E N} \dim(\mathcal{H}_N) = \sum_{\{m_i\}} e^{-\beta E \sum_i m_i} = [r; e^{-\beta E}]^n. \]  

(39)

The \( q \)-integer again appears, but now at the real value \( e^{-\beta E} \). Each mode \( \psi^*_i \) may be counted separately, so that the computation here is the same as that for 1 particle, but raised to the \( n \)th power. When \( r = 2 \) we recover the usual partition function for fermions, while for \( r = \infty \) we recover the usual result for bosons:

\[ [2; e^{-\beta E}] = e^{-\beta E} + 1; \quad [\infty; e^{-\beta E}] = \frac{1}{1 - e^{-\beta E}}. \]
The general formula \([r; e^{-\beta E}]\) interpolates the two. From the partition function one may then proceed as usual to thermodynamic properties of such a gas \(^{31}\).

More generally, we may take Hamiltonians with interaction terms, including those that arise from the field theory rather than the harmonic oscillator point of view (i.e., with kinetic and current-current interaction terms). Such applications will be developed elsewhere.

6 Concluding remarks

We have obtained a generalized exclusion principle for anyons—an important physical result—not from analysis of the statistical mechanics of anyons, but from the explicit representation of nonrelativistic creation and annihilation fields in an appropriate braided tensor product space. As one would expect in this context, the principle holds when the anyonic phase shift is a root of unity. It applies both to smeared fields \(\psi(h), \psi^*(h)\) and to (unsmeread) operator-valued distributions \(\psi(x), \psi^*(x)\) satisfying \(q\)-commutation relations; but it takes a rather cleaner form in the latter case (a subtle distinction that disappears for bosons and fermions).

On the other hand, the discrete version in which we work directly with points (rather then with increasingly peaked bump functions) turns out to be different and algebraically more complicated; with \(\psi^i\) and \(\psi^*_i\) no longer adjoint to each other. The fact that one has a different theory from the continuum limit is an interesting feature of our analysis. We believe this subject, and its relation to Gentile statistics and to generalized harmonic oscillators, deserves some renewed attention.

Let us also mention some related conceptual aspects of interest. Our construction of \(S_{\Psi_i}(\mathcal{H})\) in Section 2 is manifestly dependent on an ordering, since this enters in the braiding. This is true for quantum planes (see the remarks at the start of Section 5), where \(X = \{1, 2, \cdots, n\}\) is the indexing set; and it remains true when we apply our formalism to the second quantization of nonrelativistic fields (so that \(X\) denotes physical space). The Fock space construction might then seem to be strongly dependent on the somewhat artificial lexicographical total ordering on \(\mathbb{R}^2\) used in Section 3. However the isomorphism in Section 4, with the spaces \(\mathcal{H}_N\) described by means of topological configurations, tells us that in fact the underlying diffeomorphism invariance remains as far as the physics is concerned. The lexicographical ordering places the physical system into an algebraic form described by the symmetric tensor products \(\otimes_s\), but this is for mathematical convenience only.

This suggests an answer to a certain puzzle in \(q\)-deformed physics—how to physically interpret noncommutative tensor products (as generated by non-cocommutative quantum groups). That is, if \(A\) and \(B\) are two physically equivalent systems, what is the difference between \(A \otimes_s B\) and \(B \otimes_s A\), and which is the correct description of the joint system? In
our anyonic model this is an unphysical distinction that is needed to work algebraically; just as (physically) the points \( \gamma = \{x_1, \cdots, x_N\} \) in \( \Delta_N \) are intrinsically unordered, but it can nevertheless be helpful (mathematically) to order them. This is the difference between diffeomorphisms of the manifold \( \mathbb{R}^2 \) acting on subsets of \( \mathbb{R}^2 \), and the coordinate description of their lifting to the universal covering space of \( N \)-identical-particle configuration space.

Another interesting feature is the way that the strictness of the braiding \( \Psi \) comes about from the structure of the singularities at coincident points (expressed here as Kronecker or Dirac \( \delta \)-functions). In general one has diagonal singularities when multiplying operator fields, but in some cases (such as in conformal field theory) these can be controlled (e.g., by the operator product expansion). It would be interesting to see how braidings that arise in conformal field theory relate to diffeomorphism group ideas and to \( q \)-Fock space ideas along the lines of the present paper. In some situations, such as the Wess-Zumino-Witten model, there is also a topological path picture leading to the quantum group \( U_q(su_2) \).

Finally, we note that the basic ideas described here apply also to plektons—particles associated with higher-dimensional (non-Abelian) unitary representations of the braid group. Here \( T(b) \) in Section 2 is not simply a phase, but a finite-dimensional unitary operator acting on a multicomponent wave function; and we work with a multiplet of operator fields. Essentially, we then have an \( R \)-matrix for the linear braid group representation, in place of \( q \) in the formulas above; and we must also be more careful about ordering. Thus the creation and annihilation fields act no longer by multiplication by powers of \( q \) representing the number of crossings in the resulting braid, but matrix operation on the multiplet. In place of \( q^{\varepsilon(x,y)} \) and \( q^{\varepsilon_0(x,y)} \), we define \( \Psi \) and \( \Psi_0 \), using \( R \) or \( R^{-1} \) according to the ordering—but with the same conceptual picture that we have used in the anyonic case.

References

[1] J. M. Leinaas and J. Myrheim, On the theory of identical particles. *Nuovo Cimento* **37B**, 1 (1977). See also J. M. Leinaas, *Nuovo Cimento* **4A**, 19 (1978) and *Fort. Phys.* **28**, 579 (1980).

[2] G. A. Goldin, R. Menikoff, and D. H. Sharp, Particle statistics from induced representations of a local current group. *J. Math. Phys.* **21**, 650-664 (1980); Representations of a local current algebra in non-simply connected space and the Aharonov-Bohm effect. *J. Math. Phys.* **22**, 1664-1668 (1981).

[3] F. Wilczek, *Phys. Rev. Lett.* **48**, 1144 (1982); *Phys. Rev. Lett.* **49**, 957 (1982).

[4] R. B. Laughlin, *Phys. Rev. Lett.* **50**, 1395 (1983).

[5] B. I. Halperin, *Phys. Rev. Lett.* **52**, 1583 (1984).
[6] G. A. Goldin and D. H. Sharp, Rotation generators in two-dimensional space and particles obeying unusual statistics. Phys. Rev. D 28, 830-832 (1983). See also G. A. Goldin, R. Menikoff, and D. H. Sharp, Diffeomorphism groups, gauge groups, and quantum theory. Phys. Rev. Lett. 51, 2246-2249 (1983).

[7] Y. S. Wu, General theory for quantum statistics in two dimensions. Phys. Rev. Lett. 52, 2103 (1984).

[8] G. A. Goldin, R. Menikoff, and D. H. Sharp, Phys. Rev. Lett. 54, 603 (1985).

[9] G. A. Goldin and D. H. Sharp, The diffeomorphism group approach to anyons. In F. Wilczek (ed.), Fractional Statistics in Action (special issue), Int. J. Mod. Physics B 5, 2625-2640 (1991).

[10] F. Wilczek, Fractional Statistics and Anyon Superconductivity. Singapore: World Scientific (1990).

[11] A. Khare, Fractional Statistics and Quantum Theory. Singapore: World Scientific (1997).

[12] S. Majid, Anyonic quantum groups. In Z. Oziewicz et al. (eds.), Spinors, Twistors, Clifford Algebras and Quantum Deformations (Procs. of the 2nd Max Born Symposium, Wroclaw, Poland, 1992). Dordrecht: Kluwer (1992), pp. 327-336.

[13] A. Lerda and S. Sciuto, Nucl. Phys. B 401, 613 (1993).

[14] M. Frau, M. A. R.-Monteiro, and S. Sciuto, J. Phys. A 27, 801 (1994).

[15] M. Frau, A. Lerda, and S. Sciuto, Anyons and deformed Lie algebras. arXiv: hep-th/9407161 (July 1994).

[16] A. Lignori and M. Mintchev, Lett. Math. Phys. 33, 283 (1995); Commun. Math. Phys. 169, 635 (1995).

[17] G. A. Goldin and D. H. Sharp, Diffeomorphism groups and anyon fields. In J.-P. Antoine et al. (eds.), Quantization, Coherent States, and Complex Structures (Procs. of the 13th Workshop on Geom. Methods in Physics, Bialowieza, Poland, 1994). New York: Plenum (1995), pp. 43-54. G. A. Goldin and D. H. Sharp, Diffeomorphism groups, anyon fields and $q$-commutators. Phys. Rev. Lett. 76, 1183-1187 (1996).

[18] P. Mitra, Structure of multi-anyon wave functions. Phys. Lett. B 345, 473-476 (1995).

[19] S. Majid, Foundations of Quantum Group Theory. Cambridge, UK: Cambridge University Press (1995).

[20] S. Majid, C-statistical quantum groups and Weyl algebras. J. Math. Phys. 33, 3431-3444 (1992).

[21] S. Majid, Free braided differential calculus, braided binomial theorem and the braided exponential map. J. Math. Phys. 34, 4843-4856 (1993).

[22] G. Gentile, Nuovo Cim. 17 (1940), 493; Nuovo Cim. 19, 109 (1942).
[23] M. Daoud, Y. Hassouni, and M. Kibler, The $k$-fermions as objects interpolating between fermions and bosons. In: B. Gruber and M. Ramek (eds.), Symmetries in Science X. New York: Plenum Press (1998), pp. 63-77.

[24] F. D. M. Haldane, “Fractional statistics” in arbitrary dimensions: a generalization of the Pauli principle. Phys. Rev. Lett. 67, 937-940 (1991).

[25] Y.-S. Wu, Statistical distribution for generalized ideal gas of fractional-statistics particles. Phys. Rev. Lett. 73, 922-925 (1994).

[26] M. V. N. Nayak and F. Wilczek, Exclusion statistics: Low-temperature properties, fluctuations, duality, and applications. Phys. Rev. Lett. 73, 2740-2743 (1994).

[27] M. V. N. Murthy and R. Shankar, Thermodynamics of a one-dimensional ideal gas with fractional exclusion statistics. Phys. Rev. Lett. 73, 3331-3334 (1994).

[28] G. S. Canright and M. D. Johnson, Fractional statistics: $\alpha$ to $\beta$. J. Phys. A: Math. Gen. 27, 3579-3598 (1994).

[29] M. V. Medvedev, Properties of particles obeying ambiguous statistics. Phys. Rev. Lett. 78, 4147-4150 (1997).

[30] P. Mitra, Exclusion statistics and many-particle states. arXiv: hep-th/9411236 (November 1994).

[31] R. Acharya and P. N. Swamy, Statistical mechanics of anyons. J. Phys. A: Math. Gen. 27, 7247-7263 (1994).