A note on Wiener-Hopf factorization for Markov Additive processes

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Abstract

We prove the Wiener-Hopf factorization for Markov Additive processes. We derive also Spitzer-Rogozin theorem for this class of processes which serves for obtaining Kendall’s formula and Fristedt representation of the cumulant matrix of the ladder epoch process. Finally, we also obtain the so-called ballot theorem.

1 Introduction

The classical Wiener-Hopf factorization of a probability measure was given by Spitzer (1964) and Feller (1970), and has a strong connection to random walks. This result was generalized by Rogozin (1966), Fristedt (1974) and other authors using approximation based on discrete time skeletons. Greenwood and Pitman (1980) use direct approach which relies on excursion theory for reflected process. For details see Bertoin (1996) and Kyprianou (2006). Presman (1969) and Arjas and Speed (1973) generalize Spitzer identity into different direction, to the class of Markov Additive processes in discrete time (see also Asmussen (2003) and Prabhu (1998)). Later, Kaspi (1982) proves Wiener-Hopf factorization for a continuous time parameter Markov Additive process, where Markovian component has a finite state space and is ergodic. The main weakness of the fluctuation identity given by Kaspi (1982) is that they involve distribution of the inverse local time, which

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is seldom explicitly known. Dieker and Mandjes (2006) investigate discrete-
time Markov Additive processes and use an embedding to relate these to a
continuous-time setting (see also Breuer (2008) and Rogers (1994)).

This paper presents Wiener-Hopf factorization for a special, but none
the less quite general, class of Markov Additive Processes (MAP). For this
class of processes we give short proof of Wiener-Hopf factorization based
on Markov property and additivity. We also express the terms of Wiener-
Hopf factorization directly in terms of the basic data of the process. Finally ,
we derive Spitzer-Rogozin theorem for this class of processes which serves
for obtaining Kendall’s formula and Fristedt representation of the cumulant
matrix of the ladder epoch process. We also present the ballot theorem.

The paper is organized as follows. The Section 2 introduces basic defi-
nitions, facts and properties related with MAPs. In Section 3 we give the
main results of this paper. Finally in Section 4 we prove all theorems.

2 Preliminaries

2.1 Markov Additive Processes.

Before presenting main results we shall simply begin by defining the class of
processes we intend to work with and their properties. Following Asmussen
and Kella (2000) we consider a process $X(t)$, where $X(t) = X^{(1)}(t) + X^{(2)}(t)$,
and the independent processes $X^{(1)}(t)$ and $X^{(2)}(t)$ are specified by the char-
acteristics: $q_{ij}, G_i, \sigma_i, a_i, \nu_i(dx)$ which we shall now define. Let $J(t)$ be a
right-continuous, ergodic, finite state space continuous time Markov chain,
with $\mathcal{I} = \{1, \ldots, N\}$, and with the intensity matrix $Q = (q_{ij})$. We denote
the jumps of the process $J(t)$ by $\{T_n\}$ (with $T_0 = 0$). Let $\{U_n(i)\}$ be i.i.d.
random variables with distribution function $G_i(\cdot)$. Define the jump process by

$$X^{(1)}(t) = \sum_{n \geq 1} \sum_{i \in \mathcal{I}} U_n(i) 1_{\{J(T_n) = i, T_n \leq t\}}.$$ 

For each $j \in \mathcal{I}$, let $X^j(t)$ be a Lévy process with the Lévy-Khinchine ex-
ponent:

$$-\log \mathbb{E}(\exp\{i\alpha X^j(1)\}) = \Psi_j(\alpha)$$

$$= -ia_j \alpha + \frac{\sigma_j^2 \alpha^2}{2} + \int_{-\infty}^{\infty} (1 - e^{i\alpha y} + i\alpha |y| 1_{|y| \leq 1}) \nu_j(dy),$$

where $\int_{-\infty}^{\infty} (1 \wedge |y|^2) \nu_j(dy) < \infty$. By $X^{(2)}(t)$ we denote the process which
behaves in law like $X^j(t)$, when $J(t) = i$. We shall assume that the afore
mentioned class of MAPs is defined on a probability space with probabilities \( \{ P_i : i \in I \} \), where \( P_i(\cdot) = P(\cdot | J(0) = i) \), and right-continuous natural filtration \( \mathbb{F} = \{ \mathcal{F}_t : t \geq 0 \} \). In fact we can consider more general MAP where additional jumps \( U_n^{(i)} \) appearing during the change of the state of \( J(t) \) could also depend on the state \( J(T_{n+1}) \) (so called anticipative MAP). This could be done by considering the vector state space \( I^2 \) and the modified governing Markov process \( J \) on it. If each of the measures \( \nu_i \) are supported on \((-\infty, 0)\) as well as the distributions of each \( U^{(i)} \) then we say that \( X \) is a spectrally negative MAP. These definition and more concerning the basic characterization of MAPs can be found in Chapter XI of Asmussen (2003).

2.2 Time reversal

Predominant in the forthcoming analysis will be the use of the bivariate process \( (\hat{J}, \hat{X}) \), representing the process \( (J, X) \) time reversed from a fixed moment in the future when \( J(0) \) has the stationary distribution \( \pi \). For definitiveness, we mean

\[
\hat{J}(s) = J((t-s)^-) \quad \text{and} \quad \hat{X}(s) = X(t) - X((t-s)^-) , \quad 0 \leq s \leq t
\]

under \( \mathbb{P}_\pi = \sum_{i \in I} \pi_i P_i \). Note that \( \hat{X} \) is also Markov Additive process. The characteristics of \( (\hat{J}, \hat{X}) \) will be indicated by using a hat over the existing notation for the characteristics of \( (J, X) \). Instead of talking about the process \( (\hat{J}, \hat{X}) \) we shall also talk about the process \( (J, X) \) under probabilities \( \{ \hat{P}_i : i \in I \} \). Note also for future use, following classical time reverse path analysis, for \( y \geq 0 \) and \( s \leq t \),

\[
\mathbb{P}_i (G(t) \in ds, -I(t) \in dy | J(t) = j) = \hat{P}_j (\hat{G}(t) \in ds, S(t) - X(t) \in dy | J(t) = i) ,
\]

(1)

where \( I(t) = \inf_{0 \leq s \leq t} X(s) \), \( S(t) = \sup_{0 \leq s \leq t} X(s) \) and \( \hat{G}(t) = \sup\{ s < t : X(s) = S(s) \} \), \( \hat{G}(t) = \sup\{ s < t : X(s) = \hat{I}(s) \} \). (A diagram may help to explain the last identity).

2.3 Ladder height process

We start from recalling the representation of the local time given in Kaspi (1982) in formula (3.21). For MAP we say that state \( i \in \{1, \ldots, N\} \) is regular when \( P_i(R = 0) = 1 \) for \( R = \inf\{ t \geq 0 : t \in \mathcal{M} \} \), where \( \mathcal{M} \) is a closure of \( \mathcal{M} = \{ t \geq 0 : X(t) = S(t) \} \). Denote by \( \{ U_n \} \) the stopping times
at which \( R(t-) = 0 \) and \( R(t) > 0 \) for the \( R(t) = \inf\{s > t : S(t) = X(t)\} - t \) and \( J(t) \) is irregular. Denote
\[
S_n = \begin{cases} 
U_n & \text{on } X(U_n) = S(U_n), \\
\infty & \text{otherwise}.
\end{cases}
\]

By Theorem 3.28 of Kaspi (1982) (see also Maissoneuve (1975)), for the MAP we can define the ladder height process:

\[
\{(L^{-1}(t), H(t) = X(L^{-1}(t)), J(L^{-1}(t))), t \geq 0\}
\]

choosing the local time:

\[
L(t) = L^c(t) + \sum_{S_n < t} \lambda(J(U_n))e_1^{(n)},
\]

where \( L^c(t) \) is a continuous additive process that increases only on \( \mathcal{M} \) and \( e_1^{(n)} \) are independent exponential random variables with intensity 1,
\[
\lambda(i) = \mathbb{E}_i \left[ (1 - e^{-R}) \right].
\]

Obviously to make functional (2) measurable we enlarge probability space to include the exponential random variables. One can easily verify that \((L^{-1}(t), H(t), J(L^{-1}(t)))\) is again (bivariate) MAP (see Kaspi (1982, p. 185)). For each moment of time we can define the excursion:

\[
\epsilon_t(s) = \begin{cases} 
X(L^{-1}(t-) + s) - X(L^{-1}(t-)) & \text{for } L^{-1}(t-) < L^{-1}(t) \\
\partial & \text{otherwise},
\end{cases}
\]

where \( \partial \) is a cementary state. Let \( \zeta(\epsilon_t) = L^{-1}(t) - L^{-1}(t-) \) be the length of the excursion. From (2) it follows that the excursion process \( \{(t, \epsilon_t), t \geq 0\} \) is (possibly stopped at the first excursion with infinite length) marked Cox point process with the intensity \( n(J(L^{-1}(t-)), d\epsilon) \) depending on the state process \( J(L^{-1}(t-)) \). Denote by \( \mathcal{E} \) the \( \sigma \)-field on the excursion state space.

### 2.4 Spectrally negative Markov Additive process

Letting \( Q \circ \hat{G}(\alpha) = (q_{ij} \hat{G}_i(\alpha)) \), where \( \hat{G}_i(\alpha) = \mathbb{E} \left\{ \exp(\alpha U(i)) \right\} \), for spectrally negative MAP we can define *cumulant generating matrix* (cgm) of MAP \( X(t) \):

\[
F(\alpha) = Q \circ \hat{G}(\alpha) + \text{diag}(\psi_1(\alpha), \ldots, \psi_N(\alpha)), \quad \alpha \in \mathbb{R}_+.
\]

(3)
Perron-Frobenius theory identifies $F(\alpha)$ as having a real-valued eigenvalue with maximal absolute value which we shall label $\kappa(\alpha)$. The corresponding left and right $1 \times N$ eigenvectors we label $v(\alpha)$ and $h(\alpha)$, respectively. In this text we shall always write vectors in their horizontal form and use the usual $^T$ to mean transpose. Since $v(\alpha)$ and $h(\alpha)$ are given up to multiplying constants, we are free to normalize them such that
\[
v(\alpha) h(\alpha)^T = 1 \quad \text{and} \quad \pi h(\alpha)^T = 1 .
\]
Note also that $h(0) = e$, the $1 \times N$ vector consisting of a row of ones. We shall write $h_i(\alpha)$ for the $i$-th element of $h(\alpha)$.

The eigenvalue $\kappa(\alpha)$ is a convex function (this can also be easily verified) such that $\kappa(0) = 0$ and $\kappa'(0)$ is the asymptotic drift of $X$ in the sense that for each $i \in \mathcal{I}$ we have
\[
\lim_{t \to \infty} E(X(t)|J(0) = i, X(0) = x)/t = \kappa'(0).
\]

It can be checked that under the following Girsanov change of measure
\[
\frac{dP^\gamma}{dP_i} \bigg|_{F_t} := e^{\gamma X(t) - \kappa(\gamma)t} \frac{h_{J(t)}(\gamma)}{h_i(\gamma)}, \quad \text{for } \gamma \text{ such that } \kappa(\gamma) < \infty
\]
the process $(X, P^\gamma)$ is again a spectrally negative MAP whose intensity matrix $F_\gamma(\alpha)$ is well defined and finite for $\alpha \geq -\gamma$. Generally for all quantities calculated for $P^\gamma$ we will add subscript $\gamma$. Further, if $F_\gamma(\alpha)$ has largest eigenvalue $\kappa_\gamma(\alpha)$ and associated right eigenvector $h_\gamma(\alpha)$, the triple $(F_\gamma(\alpha), \kappa_\gamma(\alpha), h_\gamma(\alpha))$ is related to the original triple $(F(\alpha), \kappa(\alpha), h(\alpha))$ via
\[
F_\gamma(\alpha) = \Delta_h(\gamma)^{-1} F(\alpha + \gamma) \Delta_h(\gamma) - \kappa(\gamma) I
\]
and
\[
\kappa_\gamma(\alpha) = \kappa(\alpha + \gamma) - \kappa(\gamma),
\]
where $I$ is the $N \times N$ identity matrix and
\[
\Delta_h(\gamma) := \text{diag}(h_1(\gamma), ..., h_N(\gamma)).
\]

Similarly, the time reversed process $\tilde{X}(t)$ is the spectrally negative MAP with the characteristics $\tilde{F}$, $\tilde{h}$, $\tilde{\kappa}$. To relate them to the original ones, recall that the intensity matrix of $\tilde{J}$ must satisfy
\[
\tilde{Q} = \Delta_\pi^{-1} Q^T \Delta_\pi,
\]
where $\Delta_\pi$ is the diagonal matrix whose entries are given by the vector $\pi$. Hence according to (3) we find that:
\[
\tilde{F}(\alpha) = \Delta_\pi^{-1} F(\alpha)^T \Delta_\pi.
\]
Moreover, \( \hat{\kappa}(\alpha) = \kappa(\alpha) \) and \( \Delta_{\pi} \hat{h}(\alpha)^T = v(\alpha)^T \) (see Kyprianou and Palmowski (2008) for details).

The spectrally negative MAP is easier to analyze since its ladder height process \( (L^{-1}(t), H(t), J(L^{-1}(t))) \) has explicit matrix cumulant generating matrix \( \Xi(q, \alpha) \). Let

\[
\tau_a^+ := \inf\{t \geq 0 : X(t) \geq a\},
\]

where \( a \geq 0 \). Denote the generator of the Markov process \( \{J(\tau_a^+), a \geq 0\} \) by \( \Lambda(q) \) on \( \mathbb{P}^\Phi(q) \). It solves equation:

\[
F_{\Phi(q)}(-\Lambda(q)) = 0;
\]

see Pistorius (2005) and Ivanovs et al. (2008). Note that ladder height process can be identified as \( \{J(\tau_a^+, X(\tau_a^+) = a, J(\tau_a^+)), a \geq 0\} \). It is a bivariate Markov additive process with the cumulant generating matrix:

\[
\Xi(q, \alpha) = \Delta_h(\Phi(q))(\Phi(q)I - \Lambda(q))\Delta_h(\Phi(q))^{-1} + \alpha = \Delta_h(\Phi(q))((\Phi(q) + \alpha)I - \Lambda(q))\Delta_h(\Phi(q))^{-1}
\]

for \( \alpha, q > 0 \). Above could be deduced from the equalities:

\[
e^{-\Xi(q, \alpha)a} = E\left(e^{-q\tau_a^+ - \alpha X(\tau_a^+)}; J(\tau_a^+)\right) = E\left(e^{-q\tau_a^+ - \alpha a}; J(\tau_a^+)\right)
\]

and the Theorem 1 of Kyprianou and Palmowski (2008) stating that

\[
E(e^{-q\tau_x^+}; \tau_x^+ < e_q; J(\tau_x^+)) = E\left(e^{-(q+\xi)\tau_x^+}1_{(\tau_x^+ < \infty)}; J(\tau_x^+)\right)
\]

\[
= \Delta_h(\Phi(q + \xi))e^{-(\Phi(q+\xi)I - \Lambda(q+\xi)x)}\Delta_h(\Phi(q + \xi))^{-1}
\]

\[
= \exp\{-\Delta_h(\Phi(q + \xi))(\Phi(q + \xi)I - \Lambda(q + \xi))\Delta_h(\Phi(q + \xi))^{-1}x\}
\]

3 Main results

As much as possible, from now on, we shall prefer to work with matrix notation. For a random variable \( Y \) and (random) time \( \tau \), we shall understand \( E(Y; J(\tau)) \) to be the matrix with \((i, j)\)-th elements \( E_i(Y; J(\tau) = j) \). For an event, \( A \), \( P(A; J(\tau)) \) will be understood in a similar sense. Here and throughout we work with the definition that \( e_q \) is random variable which is exponentially distributed with mean \( 1/q \) and independent of \((J, X)\). Let \( I_{ij}(q) = \mathbb{P}_{i,0}(J(e_q) = j) \), in other words

\[
I(q) = q(qI - Q)^{-1}.
\]
From now on we assume that *none of the processes* $X^i$ *are downward subordinators and compound Poisson process*. To include compound Poisson process $X^{(i)}(t)$ in Theorem 1(i) on the event $\{J(G(e_q)) = i\}$ it is necessary to work with the new definition $\hat{G}(t) = \inf \{ s < t : X(s) = I(s) \}$ instead the previous one.

**Theorem 1.** (i) For a general MAP the random vectors $(S(e_q), \hat{G}(e_q))$ and $(S(e_q) - X(e_q), e_q - \hat{G}(e_q))$ are conditionally on $J(\hat{G}(e_q))$ independent.

Hence for $\alpha \in \mathbb{R}$, $\xi \geq 0$,

$$
\mathbb{E} \left[ e^{i\alpha X(e_q) - \xi e_q}; J(e_q) \right]
= \mathbb{E} \left[ e^{i\alpha S(e_q) - i\xi \hat{G}(e_q)}; J(\hat{G}(e_q)) \right] \Delta^{-1}_\pi \mathbb{E} \left[ e^{i\alpha I(e_q) - \xi \hat{G}(e_q)}; J(\hat{G}(e_q)) \right]^T \Delta \pi .
$$

(ii) For the spectrally negative MAP and $\alpha, \xi \geq 0$,

$$
\mathbb{E} \left[ e^{-\alpha S(e_q) - \xi \hat{G}(e_q)}; J(\hat{G}(e_q)) \right] = \Xi(q + \xi, \alpha)^{-1} \text{diag} \left( \Xi(q, 0) I(q) e^T \right),
$$

(iii) For the spectrally negative MAP and $\alpha, \xi \geq 0$,

$$
\mathbb{E} \left[ e^{\alpha I(e_q) - \xi \hat{G}(e_q)}; J(\hat{G}(e_q)) \right]
= q \left( (q + \xi) I - F(\alpha) \right)^{-1} \Delta^{-1}_\pi \hat{\Xi}(q + \xi, -\alpha)^T \text{diag} \left( \hat{\Xi}(q, 0) I(q) e^T \right)^{-1} \Delta \pi ,
$$

$$
\mathbb{E} \left[ e^{\alpha I(e_q) - \xi \hat{G}(e_q)}; J(e_q) \right]
= q \left( (q + \xi) I - F(\alpha) \right)^{-1} \Delta^{-1}_\pi \hat{\Xi}(q + \xi, -\alpha)^T \left[ \hat{\Xi}(q, 0) - 1 \right]^T \Delta \pi .
$$

**Remark 2.** Applying Theorem 1(i) to the reversed process derives similar conclusion for the infimum functional. Namely, processes $\{(X(t), \hat{G}(t)), 0 \leq t < \hat{G}(e_q)\}$ and $\{(X(\hat{G}(e_q) + t) - X(\hat{G}(e_q)), J(\hat{G}(e_q) + t)), t \geq 0\}$ are conditionally on $J(\hat{G}(e_q))$ independent.
Remark 3. For $N = 1$ (hence $\Lambda(q) = 0$, $I(q) = 1$) above theorem gives well-known identities for the spectrally negative Lévy process:

\[
\mathbb{E} \left[ e^{-\alpha S(q)} \right] = \frac{\Phi(q)}{\Phi(q + \xi + \alpha)}, \\
\mathbb{E} \left[ e^{\alpha I(q)} \right] = \frac{q(\Phi(q + \xi) - \alpha)}{\Phi(q)(q + \xi - \psi(\alpha))},
\]

where $\psi(\theta) = -\Psi(-i\theta)$ is a Laplace exponent of $X$. Finally, for $\xi = 0$ above theorem gives already known identity for spectrally negative MAP (see Kyprianou and Palmowski (2008)):

\[
\mathbb{E} \left( e^{\alpha I(q)}; J(q) \right) = q \Delta_x (\Phi(q)) \left[ \alpha (\Phi(q) - \hat{\Lambda}(q))^{-1} - 1 \right] \Delta_x (\Phi(q))^{-1},
\]

which was derived using Asmussen-Kella martingale.

We prove also the following counterpart of Spitzer-Rogozin version of Wiener-Hopf factorization and the Fristedt theorem:

**Theorem 4.** Assume that the matrix $\mathbb{E} \exp\{i\alpha X(1)\}$ has distinct eigenvalues and that for any $t, s \geq 0$:

\[
\mathbb{E} \left[ e^{i\alpha X(t)} 1_{\{X(t) \geq 0\}}; J(t) \right] \mathbb{E} \left[ e^{i\alpha X(s)} 1_{\{X(s) < 0\}}; J(s) \right] = \mathbb{E} \left[ e^{i\alpha X(s)} 1_{\{X(s) < 0\}}; J(s) \right] \mathbb{E} \left[ e^{i\alpha X(t)} 1_{\{X(t) \geq 0\}}; J(t) \right].
\]

Then

\[
\mathbb{E} \left[ e^{-\alpha S(q)} \right] = \exp \left\{ \int_0^\infty dt \int_{[0,\infty)} \left( e^{-\xi t - \alpha x} - 1 \right) t^{-1} e^{-qt} \mathbb{P}(X(t) \in dx; J(t)) \right\} I(q)
\]

and

\[
\mathbb{E} \left[ e^{\alpha I(q)} \right] = \exp \left\{ \int_0^\infty dt \int_{(-\infty,0)} \left( e^{-\xi t + \alpha x} - 1 \right) t^{-1} e^{-qt} \mathbb{P}(X(t) \in dx; J(t)) \right\} I(q).
\]
The assumption (17) is satisfied for example for Markov modulated Brownian motion $X(t) = \sigma(J(t))B(t)$, where $\sigma$ is a positive function.

The following generalization of the Kendall’s identity and the ballot theorem also hold.

**Theorem 5.** Consider the spectrally negative Markov Additive process $X(t)$. If there exist distinct $q_1, q_2, \ldots, q_N$ such that vectors $h(\Phi(q_1)), h(\Phi(q_2)), \ldots, h(\Phi(q_N))$ are independent, then

$$tP(\tau_{x}^{+} \in dt; J(t))dx = xP(X(t) \in dx; J(t))dt.$$ 

**Theorem 6.** Let $X(t) = ct - \sigma(t)$, where $\{\sigma(t), t \geq 0\}$ is a Markov Additive Subordinator without drift component. Under the assumptions of the Theorem 5 the following identity holds:

$$P(X(t) \in dx, I(t) = 0; J(t)) = \frac{x}{ct}P(X(t) \in dx; J(t)).$$

One can straightforward check that the assumptions of the Theorem 5 are satisfied e.g. for $X(t) = ct - J(t)N(t)$, where $N(t)$ is a Poisson process and $J(t)$ is a two-state birth-death process.

In total theorems given here might be seen as a fundamental of the fluctuation theory for the (spectrally negative) MAP and might serve for the deriving counterparts of well-known identities for the Lévy processes.

4 Proofs

4.1 Proof of Theorem 1

(i) Sampling MAP process $(X(t), J(t))$ up to exponential random time $e_q$ corresponds to the sampling the marked Cox point process (double Poisson point process) of the excursions up to time $L(e_q)$. Moreover, since conditioning on realization of the process $J(t)$ the point process $(t, \epsilon_t)$ is a non-homogeneous marked Poisson process, we know that, conditioning on $J(L^{-1}(\sigma^A -))$ for

$$\sigma^A = \inf\{t \geq 0 : \epsilon_t \in A\},$$

the point process $\{(t, \epsilon_t), t < \sigma^A\}$ is independent of $\epsilon_{\sigma^A}$.

Consider now

$$\sigma_1 = \inf\left\{t \geq 0 : \int_0^{L^{-1}(t)} 1_{\{X(s) = S(s)\}} \, ds > e_q\right\}$$
and

\[ \sigma_2 = \inf \{ t \geq 0 : \zeta(\epsilon_t) > e_\eta^t \}, \]

where \( e_\eta^t \) is the independent exponential random variable with intensity \( q \) if \( \epsilon_t \neq \partial \) and \( e_\eta^t = \partial \) otherwise. Note that \( \sigma_2 \) is \( \sigma^A \) for the \( A = \{ \zeta(\epsilon) > e_\eta^t \} \).

If \( \sigma_2 < \sigma_1 \), then conditioning on \( J(L^{-1}(\sigma_1 \wedge \sigma_2)) = J(L^{-1}(\sigma_2)) \) the process

\[ \{ (t, \epsilon_t), \ t < \sigma_1 \wedge \sigma_2 \ \text{and} \ \epsilon_t \neq \partial \} \]  

is independent of \( \epsilon_{\sigma_2} = \epsilon_{\sigma_1 \wedge \sigma_2} \). If \( \sigma_1 < \sigma_2 \), then \( \epsilon_{\sigma_1} = \epsilon_{\sigma_1 \wedge \sigma_2} = \partial \) and is also independent of the process \( J \). Hence conditioning on \( J(L^{-1}(\sigma_1 \wedge \sigma_2)) \) the excursion \( \epsilon_{\sigma_1 \wedge \sigma_2} \) is independent of the process \( J \). Note also that

\[ G(e_q) = L^{-1}((\sigma_1 \wedge \sigma_2)^-), \quad S(e_q) = H((\sigma_1 \wedge \sigma_2)^-) \]

and the last excursion \( \epsilon_{\sigma_1 \wedge \sigma_2} \) occupies the final \( e_q - G(e_q) \) units of time in the interval \( [0, e_q] \) and reaches the depth \( X(e_q) - S(e_q) \). This completes the proof of the first part of the Theorem 1(i).

Note that \( e_q - G(e_q) \) has the same law like \( \hat{G}(e_q), \hat{I}(e_q) \). The second part of the Theorem 1(i) follows now from the first part applied for the reversed process.

(ii) To prove Theorem 1(ii) we follow Bertoin (2000). Fix \( n \in \mathbb{N} \) setting

\[ i_n^+ = [nS(e_q)]/n, \]

where \([\cdot]\) stands for integer parts. Applying the strong Markov property at time \( \tau_{k/n}^+ \) and using (21) yields:

\[
\begin{align*}
\mathbb{E} \left[ e^{-\xi_{i_n^+}^+ - \alpha i_n^+} ; J(e_q) \right] \\
= \sum_{k=0}^{\infty} \mathbb{E} \left[ e^{-\xi_{k/n}^+ - \alpha k/n} ; k/n \leq S(e_q) < (k + 1)/n ; J(e_q) \right] \\
= \sum_{k=0}^{\infty} e^{\Xi(q + \alpha)k/n} P \left( e_q < \tau_{1/n}^+ ; J(e_q) \right) \\
= \left[ n(I - e^{-\Xi(q + \alpha)1/n}) \right]^{-1} nP \left( e_q < \tau_{1/n}^+ ; J(e_q) \right). 
\end{align*}
\]

Taking \( n \to \infty \) we have \( i_n^+ \to S(e_q) \) and \( \tau_{i_n^+}^+ \to \overline{G}(e_q) \) and hence the left hand side of above equation converges by the dominated convergence theorem. Thus also right hand side converges. Note that for any matrix \( A \)

\[ I - e^{-A/n} = \frac{1}{n} A + o(1/n). \]

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Thus
\[
E \left[ e^{-\xi G(e_q) - \alpha S(e_q)}; J(e_q) \right] = \Xi(q + \xi, \alpha)^{-1} B
\]
for some matrix \( B \) using fact that matrix \((\Phi(q + \xi) + \alpha)I - \Lambda(q + \xi)\) is invertible for \( q > 0 \). Taking \( \xi = \alpha = 0 \) we obtain
\[
B = \Xi(q, 0)I(q)
\]
which completes the proof of (13). Similarly, from (21),
\[
E_i \left[ e^{-\xi G(e_q) - \alpha S(e_q)}; J(e_q) \right] = E \left[ e^{-\xi G(e_q) - \alpha S(e_q)}; J(G(e_q)) \right] C,
\]
for
\[
C = \lim_{n \to \infty} \text{diag} \left( \frac{1}{n} \right) P \left( e_q < \tau_{1/n}^+ ; J(e_q) \right)
\]
and therefore
\[
E \left[ e^{-\xi G(e_q) - \alpha S(e_q)}; J(e_q) \right] = E \left[ e^{-\xi G(e_q) - \alpha S(e_q)}; J(G(e_q)) \right] \Delta^{-1} \hat{P} \left( J(G(e_q)) \right)^{T} \Delta,
\]
which completes the proof of (15) in view of (12).
4.2 Proof of Theorem 4

For a general matrix $A$ with the distinct eigenvalues $\lambda_i$ (hence with the independent eigenvectors $s_i$), under assumption that $qI + A$ is invertible, using Frullani integral and the representation $A = S \text{ diag}\{\lambda_i\} S^{-1}$ with $S = (s_1, \ldots, s_N)$, we can derive the following identity:

$$q (qI + A)^{-1} = \exp \left\{ \int_0^\infty (e^{-Ax} - I) \frac{1}{x} e^{-qx} \, dx \right\}. \quad (27)$$

Lemma 7. Under assumption [17] for $\xi \geq 0$,

$$E \left[ e^{i\alpha X(t)}; J(e_q) \right] = \exp \left\{ \int_0^\infty \int_{[0,\infty)} \left( \exp \{-\xi t + i\alpha x\} - 1 \right) \frac{1}{t} e^{-qt} \, P(X(t) \in dx; J(t)) \, dt \right\} \cdot \exp \left\{ \int_0^\infty \int_{(-\infty,0)} \left( \exp \{-\xi t + i\alpha x\} - 1 \right) \frac{1}{t} e^{-qt} \, P(X(t) \in dx; J(t)) \, dt \right\}. \quad (28)$$

Proof. By additivity of process $X(t)$ there exists a matrix $F$ such that $E \exp\{i\alpha X(t)\} = \exp\{F(i\alpha)t\}$ (see Prop. XI.2.2, p. 311 of Asmussen (2003)). Note that this matrix also has distinct eigenvalues. Then we have:

$$E \left[ e^{i\alpha X(e_q) - \xi e_q}; J(e_q) \right] = \int_0^\infty q e^{-qt} \exp\{-\xi (I - F(i\alpha))t\} \, dt$$

$$= \exp \left\{ \int_0^\infty \left( \exp\{-\xi (I - F(i\alpha))t\} - 1 \right) \frac{1}{t} e^{-qt} \, dt \right\}$$

$$= \exp \left\{ \int_0^\infty \int_{R} \left( \exp\{-\xi t + i\alpha x\} - 1 \right) \frac{1}{t} e^{-qt} \, P(X(t) \in dx; J(t)) \, dt \right\}. \quad (28)$$

Note that by identity [17] the matrices

$$\int_0^\infty \int_{[0,\infty)} \left( \exp\{-\xi t + i\alpha x\} - 1 \right) \frac{1}{t} e^{-qt} \, P(X(t) \in dx; J(t)) \, dt$$

and

$$\int_0^\infty \int_{(-\infty,0)} \left( \exp\{-\xi t + i\alpha x\} - 1 \right) \frac{1}{t} e^{-qt} \, P(X(t) \in dx; J(t)) \, dt$$

commutes. This gives the assertion of the lemma by the factorization. \qed
From Lemma 7 and Theorem 1 we have

\[
\exp \left\{ \int_0^\infty \int_{[0,\infty)} \left( \exp \{-\xi t + i\alpha x\} - 1 \right) \frac{1}{t} e^{-qt} P(X(t) \in dx; J(t)) \, dt \right\}
\]

\[
\cdot \exp \left\{ \int_0^\infty \int_{(-\infty,0)} \left( \exp \{-\xi t + i\alpha x\} - 1 \right) \frac{1}{t} e^{-qt} P(X(t) \in dx; J(t)) \, dt \right\}
\]

\[= H(\alpha, \xi)T(\alpha, \xi), \tag{28} \]

where

\[H(\alpha, \xi) = E \left[ e^{i\alpha S(e_q) - \xi \overline{G}(e_q)} ; J(e_q) \right] I(q)^{-1} \tag{29}\]

and

\[T(\alpha, \xi) = \Xi(q, 0)^{-1} \text{diag} (\Xi(q, 0)e^T) \Delta_\pi^{-1} \hat{E} \left[ e^{i\alpha I(e_q) - \xi \overline{G}(e_q)} ; J(G(e_q)) \right] \Delta_\pi. \tag{30} \]

From Theorem 1(i) it follows that matrices \(H(\alpha, \xi)\) and \(T(\alpha, \xi)\) are invertible. Thus,

\[H^{-1}(\alpha, \xi) \exp \left\{ \int_0^\infty \int_{[0,\infty)} \left( \exp \{-\xi t + i\alpha x\} - 1 \right) \frac{1}{t} e^{-qt} P(X(t) \in dx; J(t)) \, dt \right\}
\]

\[= T(\alpha, \xi) \exp \left\{ - \int_0^\infty \int_{(-\infty,0)} \left( \exp \{-\xi t + i\alpha x\} - 1 \right) \frac{1}{t} e^{-qt} P(X(t) \in dx; J(t)) \, dt \right\}. \tag{31} \]

Moreover, each entry of the matrix \(H(\alpha, \xi)\) is analytical in the upper half of the complex plane. The same concerns also the matrix \(H^{-1}(\alpha, \xi)\). Thus each entry of the LHS of (31) extends analytically to the lower half of the complex plane in \(\alpha\) and similarly each entry of the matrix on the RHS of (31) extends analytically to the upper half of the complex plane in \(\alpha\). Hence matrices on both sides of (31) can be defined in the whole \(\alpha\)-plane. Observe that each entry of these matrices is bounded function. Indeed, from definitions (29) and (30) by Jensen inequality it follows that each entry of matrices \(H(\alpha, \xi)\) and \(T(\alpha, \xi)\) is bounded in respective regions. Note that reciprocal of determinant of \(H(\alpha, \xi)\) is also bounded within any bounded circle. Thus on any circle each entry of \(H^{-1}(\alpha, \xi)\) is bounded. Similarly,
one can prove that each entry of the second factors of the RHS and LHS of \(31\) is bounded. Thus by Liouville’s Theorem each entry of \(31\) must be a constant. Putting \(\alpha = \xi = 0\) gives the assertion of the theorem.

### 4.3 Proof of Theorem \(5\)

We will use now the martingale technique to prove Theorem 5 (see Borovkov and Burq (2001) for similar considerations in the case of spectrally negative Lévy processes).

**Lemma 8.** We have,

\[
\int_{y}^{\infty} dx e^{-\Phi(q)x} \frac{1}{\kappa'(q)} = \int_{0}^{\infty} te^{-qt} \int_{y}^{\infty} dx \sum_{j=1}^{N} \frac{h_{j}(\Phi(q))}{h_{i}(\Phi(q))} \mathbb{P}_{i}(\tau_{x}^{+} \in dt; J(t) = j).
\]

**Proof.** From (1) we have

\[
e^{-\Phi(q)x} = \mathbb{E}_{i} e^{-q\tau_{x}^{+}} \frac{h_{j}(\Phi(q))}{h_{i}(\Phi(q))}.
\]

Differentiating with respect to \(q\) and noting that \(\Phi'(q) = 1/\kappa'(q)\) we have

\[
x e^{-\Phi(q)x} \frac{1}{\kappa'(q)} = \int_{0}^{\infty} te^{-qt} \sum_{j=1}^{N} \frac{h_{j}(\Phi(q))}{h_{i}(\Phi(q))} \mathbb{P}_{i}(\tau_{x}^{+} \in dt; J(t) = j).
\]

(32)

The proof completes by dividing left-hand side and right-hand side of (32) by \(x\) and integrating them w.r.t. \(dx\) over \((y, \infty)\).

**Lemma 9.** We have,

\[
\int_{y}^{\infty} dx e^{-\Phi(q)x} \frac{1}{\kappa'(q)} = \int_{0}^{\infty} e^{-qt} \int_{y}^{\infty} dx \sum_{j=1}^{N} \frac{h_{j}(\Phi(q))}{h_{i}(\Phi(q))} \mathbb{P}_{i}(X(t) \in dx; J(t) = j) dt.
\]

**Proof.** From Corollary XI.2.6 of Asmussen (2003) we have

\[
a = \kappa'(\Phi(q)) e^{\Phi(q)} \tau_{\alpha}^{+} + h_{i}(\Phi(q)) - e^{\Phi(q)} h_{j}(\tau_{x}^{+}) \Phi(q)
\]

and hence

\[
\lim_{a \to \infty} \frac{\mathbb{E}_{i}(\Phi(q)) \tau_{\alpha}^{+}}{a} = \frac{1}{\kappa'(\Phi(q))}.
\]

(33)

Let \(T_{A} = \int_{0}^{\infty} I_{A}(X(t)) dt\) be the time spend by our process in the set \(A\). Note that

\[
\tau_{\alpha}^{+} - T_{(-\infty, 0]} \leq T_{(0, a]} \leq \tau_{\alpha}^{+} + T_{(-\infty, 0]},
\]

(34)
where $T^J_{\{-\infty,0\}}$ denotes the time spend in $(-\infty,0]$ by the process $\{(X(t + \tau_a^+) - a, J(\tau_a^+ + t)), t \geq 0\}$. Moreover,

$$\max_i \mathbb{E}_i^{\Phi(q)} T_{\{-\infty,0]\} = \max_i \mathbb{E}_i^{\Phi(q)} \int_0^\infty \mathbf{1}_{\{-\infty,0\}}(X(t)) \, dt$$

$$\leq \max_i \mathbb{E}_i^{\Phi(q)} \int_0^\infty \mathbf{1}_{\{-\infty,0\}}(X(t)) \, dt$$

$$\leq \max_i \mathbb{E}_i^{\Phi(q)} \int_0^\infty e^{-qX(t)} \, dt \leq \max_i \int_0^\infty \mathbb{E}_i^{\Phi(q)} e^{-qX(t)} \, dt$$

$$\leq \max_i \frac{h_j(\Phi(q))}{h_i(\Phi(q))} \int_0^\infty e^{-qt} \, dt = \frac{1}{q} \max_i \frac{h_j(\Phi(q))}{h_i(\Phi(q))} < \infty .$$

Taking expectation from both sides of (34) and dividing by $a$ we derive:

$$\lim_{a \to \infty} \frac{\mathbb{E}_i^{\Phi(q)} T_{(0,a][)} \cdot a}{a} = \frac{1}{\kappa'(\Phi(q))}. \quad (35)$$

Observe that for $0 < a < b < c < \infty$

$$T_{(a,b]} + T_{(b,c]} = T_{(a,c]}$$

and hence

$$\mathbb{E}_i^{\Phi(q)} T_{(a,b]} = c(b - a)$$

which together with (35) gives:

$$\mathbb{E}_i^{\Phi(q)} T_{(0,a]} = \frac{a}{\kappa'(\Phi(q))}. \quad (36)$$

Above can be rewritten in the following way:

$$\frac{a}{\kappa'(\Phi(q))} = \mathbb{E}_i^{\Phi(q)} T_{(0,a]} = \mathbb{E}_i^{\Phi(q)} \int_0^\infty \mathbf{1}_{(0,a]}(X(t)) \, dt$$

$$= \int_0^\infty \mathbb{P}_i^{\Phi(q)}(X(t) \in (0,a]) \, dt$$

$$= \int_0^\infty dt \int_{(0,a]} e^{\Phi(q)x - qt} \sum_{j=1}^N \frac{h_j(\Phi(q))}{h_i(\Phi(q))} \mathbb{P}_i(X(t) \in dx; J(t) = j)$$

$$= \int_{(0,a]} e^{\Phi(q)x} \int_0^\infty e^{-qt} \sum_{j=1}^N \frac{h_j(\Phi(q))}{h_i(\Phi(q))} \mathbb{P}_i(X(t) \in dx; J(t) = j) \, dt .$$
Since this equation holds for any $a > 0$, we have

\[ \frac{dx}{\kappa'(\Phi(q))} = e^{\Phi(q)x} \int_0^\infty e^{-qt} \sum_{j=1}^N \frac{h_j(\Phi(q))}{h_i(\Phi(q))} P_i (X(t) \in dx; J(t) = j) \, dt \]

which completes the proof by integrating them w.r.t. $dx$ over $(y, \infty)$.

From Lemma 8 and 9 we have the following equality of Laplace transforms:

\[
\int_0^\infty t e^{-qt} \int_0^\infty dx \, \sum_{j=1}^N \frac{h_j(\Phi(q))}{h_i(\Phi(q))} P_i (\tau_x^+ \in dt; J(t) = j) \\
= \int_0^\infty e^{-qt} \int_0^\infty dx \, \sum_{j=1}^N \frac{h_j(\Phi(q))}{h_i(\Phi(q))} P_i (X(t) \in dx; J(t) = j) \, dt
\]

that implies the equality of the following measures:

\[
\sum_{j=1}^N h_j(\Phi(q)) \left( t \frac{dx}{x} P_i (\tau_x^+ \in dt; J(t) = j) - P_i (X(t) \in dx; J(t) = j) \right) dt = 0 \tag{37}
\]

for each $i = 1, \ldots, N$. This is equivalent to:

\[
(tdx P(\tau_x^+ \in dt; J(t)) - x P(X(t) \in dx; J(t)) \, dt) \, h(\Phi(q)) = 0.
\]

Choosing now the independent vectors $h(\Phi(q_1)), h(\Phi(q_2)), \ldots, h(\Phi(q_N))$ completes the proof.

### 4.4 Proof of Theorem 6

By Kendall’s identity given in Theorem 5 it suffices to prove that

\[ P(X(t) \in dx; I(t) = 0; J(t)) dt = \frac{1}{c} P(\tau_x^+ \in dt; J(t)) dx \]

or that for all $q > 0$ and sufficiently large $s > 0$:

\[
q \int_0^\infty e^{-qt} dt \int_0^\infty e^{sx} P(X(t) \in dx, I(t) = 0; J(t)) \\
= \frac{q}{c} \int_0^\infty e^{-qt} \int_0^\infty e^{sx} dx P(\tau_x^+ \in dt; J(t)) \tag{38}
\]

that is equivalent to

\[
\lim_{\alpha \to \infty} E \left[ e^{sX(\epsilon(x)) + \alpha I(\epsilon(x))}; J(\epsilon(x)) \right] = \frac{q}{c} \int_0^\infty e^{sx} E \left[ e^{-q\tau_x^+}; J(\tau_x^+) \right] dx. \tag{39}
\]
We prove (39) passing from its left-hand side to its right-hand side. Let \( \overline{q} = q - \kappa(s) \). The change of measure \((4)\) and Wiener-Hopf factorization given in Theorem \((1)\) gives:

\[
\lim_{\alpha \to \infty} \mathbb{E} \left[ e^{sX(e_q) + \alpha I(e_q)} ; J(e_q) \right] = \lim_{\alpha \to \infty} \Delta_h(s) E^s \left[ e^{\alpha I(e_q) - \kappa(s)e_q} ; J(e_q) \right] \Delta_h(s)^{-1} \\
= \lim_{\alpha \to \infty} \frac{q}{\overline{q}} \Delta_h(s) E^s \left[ e^{\alpha I(e_q)} ; J(e_q) \right] \Delta_h(s)^{-1} \\
= \lim_{\alpha \to \infty} \Delta_h(s) q \left( \frac{\overline{q}}{\alpha} - F_s(\alpha)/\alpha \right)^{-1} \Delta_{\pi_s}^{-1} \frac{1}{\alpha} \hat{E}_s(\overline{q}, -\alpha)^T \hat{E}_s(0, -\alpha)^T \Delta_{\pi_s} \Delta_h(s)^{-1}.
\]

Note that
\[
\lim_{\alpha \to \infty} \left( \frac{\overline{q}}{\alpha} - F_s(\alpha)/\alpha \right)^{-1} = (-cI)^{-1} = -\frac{1}{c} I
\]

and from \((7)\)
\[
\lim_{\alpha \to \infty} \frac{1}{\alpha} \hat{E}_s(\overline{q}, -\alpha)^T = -I.
\]

Hence
\[
\lim_{\alpha \to \infty} \mathbb{E} \left[ e^{sX(e_q) + \alpha I(e_q)} ; J(e_q) \right] = \frac{q}{c} \Delta_h(s) \Delta_{\pi_s}^{-1} \hat{E}_s(0, -\alpha)^T \Delta_{\pi_s} \Delta_h(s)^{-1}.
\]

Using classical arguments for reversed process we can proceed as follows:

\[
\frac{q}{c} \Delta_h(s) \Delta_{\pi_s}^{-1} \hat{E}_s(0, -\alpha)^T \Delta_{\pi_s} \Delta_h(s)^{-1} \\
= \frac{q}{c} \Delta_h(s) \int_0^{\infty} \Delta_{\pi_s}^{-1} \hat{E}^s \left[ e^{-\tau^+_x} ; J(\tau^+_x) \right]^T \Delta_{\pi_s} \Delta_h(s)^{-1} dx \\
= \frac{q}{c} \int_0^{\infty} \Delta_h(s) \mathbb{E} \left[ e^{-\tau^+_x} ; J(\tau^+_x) \right] \Delta_h(s)^{-1} dx \\
= \frac{q}{c} \int_0^{\infty} \mathbb{E} \left[ e^{-(\overline{q} + \kappa(s))\tau^+_x + sx} ; J(\tau^+_x) \right] dx,
\]

which is the right-hand side of (39).

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