An Effective Two-component Entanglement in Double-well Condensation

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We propose a spin-half approximation method for two-component condensation in double wells to discuss the quantum entanglement of two components. This approximation is presented to be valid under stationary tunneling effect for odd particle number of each component. The evolution of the entanglement is found to be affected by the particle number both quantitatively and qualitatively. In detail, the maximal entanglement are shown to be hyperbolic like with respect to tunneling rate and time. To successively obtain large and long time sustained entanglement, the particle number should not be large.

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I. INTRODUCTION

The experimental achievement of atomic Bose-Einstein condensation (BEC) [1] has opened fascinating possibilities for studying quantum properties of a macroscopic number of cold quantum atoms [2] and remains one of the most active research areas in recent years. Certainly, theoretical attentions are directed towards the underlying quantum correlation properties of the condensed atoms [3]. Due to the macroscopic natural characteristics of BEC, it should be an ideal system for describing some quantum phenomena related to coherence. Correspondingly, although the Bose-Einstein condensate is well represented by mean-field theory, it has many aspects that can be represented in a quantum picture containing some proper description of correlations. The essential ultimate physics can be realized with the study of a simpler double-well BEC in the well-known quantum two-mode approximation.

As is well known, entanglement has come to be regarded as a physical resource which can be utilized to perform numerous tasks in quantum information processing. Also, apart from the fundamental physical interest in entanglement, the whole field of quantum computing and quantum information is based upon the ability to create and control entangled states [4]. In recent times, the study of the entanglement characteristics of various condensed matter system [5] is focused on a pair of tunnel-coupled Bose-Einstein condensates (BEC’s). In the simplest model, bosons are restricted to occupy one of two modes, each of which is in a BEC [6].

A dynamical scheme of engineering many-particle entanglement in BEC has been proposed by several authors, such as, Khan W. Mahmud et al. [7]. They introduced a quantum phase-space dynamics to generate tunable entangled number states using the Husimi projection method [8]. However, the notion of entanglement in macroscopic ensembles allows to investigate the boundary between quantum physics and classical physics and, possibly, could also give some insight into the measurement process [2, 10, 11]. In this paper we will investigate in detail a scheme that measures entanglement of \( N \) two-component particles trapped in a double-well under a kind of reasonable approximate assumption. This approximation is presented to be valid under stationary tunneling effect for odd particle number of each component.

II. THE MODEL

Our system consists of a double-well and two component condensate trapped in it as depicted in FIG. 1. Initially, the atoms of the component A are trapped in the left side and the atoms of the component B are trapped in the right side.

FIG. 1: A sketch of the double-well system with two-component condensate trapped in. Initially, the atoms of the component A are trapped in the left side and the atoms of the component B are trapped in the right side.

\[
\hat{H} = \int dr \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(r) \right] \hat{\psi}(r) + \frac{g}{2} \int dr \hat{\psi}^\dagger(r) \hat{\psi}^\dagger(r) \hat{\psi}(r) \hat{\psi}(r),
\]

where \( \hat{\psi}(r) \) and \( \hat{\psi}^\dagger(r) \) are the boson annihilation and creation field operators, \( m \) is the particle mass and \( g = \frac{4\pi a_s^2 \hbar^2}{m} \), with \( a_s \) the s-wave scattering length. In studies of double-well BEC or two-component condensates, the low-energy many-body Hamiltonian in Eq. (1) can be simplified in the well-known two-mode approximation [12, 13], which

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has been used widely in studying the double-well condensate. Under this approximation, the system is modeled by the Hamiltonian ($\hbar = 1$) [12].

\[
\hat{H} = \frac{\Omega_A}{2}(a_i^* a_R + a_i^* a_L) + \frac{\Omega_B}{2}(b_i^* b_R + b_i^* b_L) + \kappa (a_i^* a_i b_i^* b_L + a_i^* b_i^* a_L b_i)
\]

\[
+ \kappa_A (a_L^* a_i b_i^* b_R + a_R^* b_i^* a_i b_L) + \frac{\kappa_A}{2}[(a_L^* a_L)^2 + (a_R^* a_R)^2]
\]

\[
+ \frac{\kappa_B}{2}[(b_L^* b_L)^2 + (b_R^* b_R)^2],
\]

where the subscripts L and R denote respectively the localized modes in the left and right potential wells. Because of two modes available for each component, there are four operators $a_j^*$ and $b_j^*$ ($j = L, R$) that denote the creation operators of the components A and B in two wells in the model. The parameters $\Omega_i, \kappa$ with $i = A, B$ and $\kappa$ describe the tunneling rate, self-interaction strength of the component A (B) and the interspecies interaction strength. Especially, $\Omega_i$ depends linearly on particle number $N_i$, so that, $\Omega_i = \epsilon_i R + g T_i (N_i - 1)$, where $\epsilon_i R, T_i$ are the parameters after second quantization and $N_i$ is particle number of component $i$ [7]. For simplicity, we set $\Omega_A = \Omega_B = \Omega$ in following sections.

### III. CLASSIC ANALYSIS

Due to the macroscopic nature of its wave function, BEC should be an ideal system for the generation of macroscopic quantum superposition states (Schrödinger cat states) [7]. Such a system could be, on one hand, analyzed in a classical method from a large particle number point of view and on the other hand studied in a quantum method because of the quantum superposition character.

The classical Hamiltonian that describes the mean-field dynamics of BEC in a double well has been studied in several papers [13, 14]. In a mean-field assumption for the two-mode double well BEC in case of large enough $N$, the operators $\hat{a}_j$ can be replaced by a c-number $\sqrt{N_j} e^{i \theta_j}$, where $j = L, R$. With this assumption and defining population difference (POD) $n = \frac{n_i}{\sqrt{N_i}}$, $\theta = \theta_L - \theta_R$, we analogously introduce c-numbers both for component A and B, that is,

\[
\hat{a}_j \rightarrow \sqrt{N_j} e^{i \theta_j}, \ \hat{b}_j \rightarrow \sqrt{N_j} e^{i \theta_j},
\]

\[
n_{AL} + n_{AR} = N_A,
\]

\[
n_{BL} + n_{BR} = N_B,
\]

\[
n_A = \frac{n_{AL} - n_{AR}}{2},
\]

\[
n_B = \frac{n_{BL} - n_{BR}}{2},
\]

the classical Hamiltonian is then given by

\[
H_{cl} = \frac{\Omega}{2} \sqrt{N_A^2 - (2n_A)^2} \cos \theta_A + \sqrt{N_B^2 - (2n_B)^2} \cos \theta_B
\]

\[
+ \frac{\kappa}{4} N_A N_B + \frac{\kappa_A}{4} N_A^2 + \frac{\kappa_B}{4} N_B^2
\]

\[
+ \kappa_A n_A^2 + \kappa_B n_B^2 + 2 \kappa n_A n_B
\]

and the equations of motion are

\[
\dot{n}_A = \frac{\Omega}{2} \sqrt{N_A^2 - (2n_A)^2} \sin \theta_A,
\]

\[
\dot{n}_B = \frac{\Omega}{2} \sqrt{N_B^2 - (2n_B)^2} \sin \theta_B,
\]

\[
\dot{\theta}_A = - \frac{2 \Omega n_A \cos \theta_A}{\sqrt{N_A^2 - (2n_A)^2}} + 2 \kappa_A n_B + 2 \kappa_B n_A,
\]

\[
\dot{\theta}_B = - \frac{2 \Omega n_B \cos \theta_B}{\sqrt{N_B^2 - (2n_B)^2}} + 2 \kappa_B n_B + 2 \kappa_A n_A.
\]

We plot the amount of the POD for component A and B ($n_A, n_B$) as a function of the tunneling rate $\Omega$ and time $t$ in FIG. 2 and FIG. 3.

FIG. 2: Shown is the population difference for component A with parameter values $\kappa = 20, \kappa_A = 20, \kappa_B = 20$.

FIG. 3: Shown is the population difference for component B for parameter values $\kappa = 20, \kappa_A = 20, \kappa_B = 20$. The POD between two wells of each component presents a behavior of local maximum and minimum. The period is determined by the particle number of the component. The larger the particle number, the smaller
the period. It should be stressed that the peak of the POD of component A between two wells is decreased gradually. In FIG. 3, there have the same conditions for component B but for the negative initial value of POD.

FIG. 2 suggests that the stationary state of population would be in particle number equilibrium. Similar result has been presented in Ref. [12], where the expectation value of the defined inversion operator \( \langle a_i^+ a_l - a_l^+ a_i \rangle \) was found to tend to zero. Our solution is based on a continuous analysis of Equ. [4-7]. While, in physical system scenario, the population number must be discrete since there are \( N+1 \) orthogonal Fork states of each component. That is, for even component particle number \( N_i \), the stationary population in each well would be \( \frac{N}{2} \). While for odd \( N_i \), the stationary POD would be \( \pm \frac{1}{2} \). Whatever be the initial population number of components in each well, a minimal POD-0 or 1 can be obtained due to the tunneling effect and repulsive interaction.

In the following section, we shall see how to generate and measure a mode entangled state under such an approximate situation.

IV. GENERATING AND MEASURING OF ENTANGLEMENT UNDER SPIN-HALF APPROXIMATION ASSUMPTION

The quantum phase-space model presented in Ref. [7] points to a way that an entangled state can be generated with a single-component BEC in double well. The authors showed the generation of tunable entangled states in phase space using Husimi distribution function. We now introduce another method to prepare entangled states. Schwinger has developed an entire angular-momentum algebra in terms of two sets (up and down) of creation and annihilation operators for uncorrelated harmonic oscillator constructing the angular-momentum operators as [12]

\[
J_+ = J_x + i J_y = a_+^* a_-, \\
J_- = J_x - i J_y = a_-^* a_+, \\
J_z = \frac{1}{2}(a_+^* a_+ - a_-^* a_-).
\]

(8)

This construction then satisfies the standard angular-momentum commutation relation \( [J_i, J_j] = i \epsilon_{ijn} J_n \). Now, we regard the double-well as a two-state system and similarly introduce an analogous-angular-momentum algebra to express the Hamiltonian in Eq. [2]. For the two-component condensate trapped in a double well, we define

\[
S_i^+ = \Lambda_i^+ \Lambda_R, \\
S_i^- = \Lambda_i \Lambda_i^+, \\
S_i^z = \frac{1}{2}(\Lambda_i^+ \Lambda_R - \Lambda_R^+ \Lambda_i),
\]

(9)

where \( i = A, B \), \( \Lambda = a, b \) (for components A and B respectively). \( S_i^+, S_i^- \) satisfy the angular-momentum commutation relations: \( [S_i^+, S_i^-] = 2 S_i^z \). With these descriptions, we, in fact, consider the left and right wells as two states \( |L \rangle \) and \( |R \rangle \) of the particles. As for \( N \)-particle system, the collective spin is \( S_z = \sum_{i=1}^{N} S_i^z \). So, there would be \( N_i \) “excited states” for component \( i \) with particle number \( N_i \) as illustrated in FIG. 4. Now, we assume the initial particle number of each component is odd. The stationary POD, from the analysis in the last section, must be 1. So, the collective spin of each component should be \( \pm \frac{1}{2} \). This assumption suggests us to eliminate adiabatically “high energy” states of each component, thus each component can be treated as a pseudo-particle with expectation value \( \pm \frac{1}{2} \) of spin.

Then, the raising and lowering operators of this pseudo-particle \( S_i^+ (S_i^-) \) can be expressed in a \( 2 \times 2 \) dimensional Hilbert space by

\[
S_A^+ \rightarrow |L\rangle_A \langle R|, \quad S_A^- \rightarrow |R\rangle_A \langle L|, \\
S_B^+ \rightarrow |L\rangle_B \langle R|, \quad S_B^- \rightarrow |R\rangle_B \langle L|.
\]

(10)

After the operators \( \hat{a}_i^\dagger (\hat{a}_i) \) replaced by the \( \hat{S}_i^\dagger (\hat{S}_i^-) \), where \( j = A, B, i = L, R \), the global Hamiltonian in Equ. [2] can be spanned on the basis of \( |L\rangle_A \otimes |L\rangle_B, |L\rangle_A \otimes |R\rangle_B, |R\rangle_A \otimes |L\rangle_B, |R\rangle_A \otimes |R\rangle_B \). We obtain the effective Hamiltonian of our system as

\[
H = \begin{pmatrix}
K_1 & K & K & 0 \\
K & K_2 & 0 & K \\
K & 0 & K_2 & K \\
0 & K & K & K_1
\end{pmatrix}
\]

(11)

where \( K = \frac{\Theta}{2}, K_1 = k + \frac{\Omega}{2}, K_2 = k - \frac{\Omega}{2} \). The eigenvalues and eigenvectors of the Hamiltonian can be found explicitly,

\[
E_1 = K_2, \quad \phi_1 = -\frac{1}{\sqrt{2}}[(|L\rangle_A |R\rangle_B - |R\rangle_A |L\rangle_B)]; \\
E_{2,3} = \frac{K_2 + K_1 \pm \Theta}{2}, \\
\phi_{2,3} = \frac{1}{\sqrt{2}}[(|L\rangle_A |R\rangle_B + |R\rangle_A |L\rangle_B)] \pm \frac{-K_2 + K_1 \mp \Theta}{4K}[(|L\rangle_A |R\rangle_B + |R\rangle_A |L\rangle_B) + |R\rangle_A |R\rangle_B)]; \\
E_4 = K_1, \quad \phi_4 = \frac{1}{\sqrt{2}}[(|L\rangle_A |L\rangle_B - |R\rangle_A |R\rangle_B)].
\]

(12)
where $\xi_2$, $\xi_3$ are normalized factors and $\Theta = \sqrt{K_2^2 - 2K_1K_2 + K_1^2 + 16K^2}$. We assume that, initially, the POD of component A is $\frac{1}{2}$, and the POD of component B is $\frac{1}{2}$, so that, the initial state of system is

$$|\psi(0)\rangle = |L\rangle_A \otimes |R\rangle_B.$$  \hfill (13)

Wootters Concurrence has been widely used in measuring the entanglement of bipartite two-state system, which is defined as \cite{15}.

$$C(\rho) = \max \{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\},$$  \hfill (14)

where $\lambda_i$ are the square roots of the eigenvalues of the non-Hermitian matrix $\rho \hat{\rho}$ with $\hat{\rho} = (\sigma_y \otimes \sigma_y) \rho (\sigma_y \otimes \sigma_y)$ in decreasing order. Wootters Concurrence gives an explicit expression for the entanglement of formation, which quantifies the resources needed to create entangled state.

Then the behavior of entanglement in this system described by spin-half assumption Hamiltonian is illustrated in FIG. 5 and FIG. 6. In this situation when there is no decoherence, entangled states can be generated because of the overlap of the wave packet of each component in two wells. We note that the entanglement presents a periodic distribution for any value of $t$ falling in the region, that is, the behavior of the amount of entanglement between the two modes is non-monotonic, the maximum entanglement can be achieved if we control the tunneling time at about $t = \frac{k^2}{2}(\frac{\pi}{k} + \frac{\pi}{k})$. The corresponding maximal entanglement for different $k$ described by FIG. 6 is shown to be hyperbolic like with respect to tunneling rate and time. The half width of the peak of entanglement $\tau$ can be defined as a parameter to represent the quantity and the quality of the maximal entanglement. As an example, we show the entanglement when $\Omega = 1$ in FIG. 6. At the peak position of the entanglement, $t = \frac{k^2}{2}(\frac{\pi}{k} + \frac{\pi}{k})$, the half width is $\tau \approx 31.369$. While for $\Omega = 1.5$, the half width is $\tau \approx 13.899$, which suggests that the larger the $\Omega$, the smaller the half width of the peak. Then, we can easily obtain large and long time sustained entanglement successively for $1 \leq \Omega \leq 1.5$, which presents a broad banding distribution.

V. CONCLUDING REMARKS

The creation of many particle entangled states in macroscopic systems is one of the major goals in the studies on fundamental aspects of quantum theory. In this paper, the studies we have performed show a spin-half approximation method for two-component condensation in double wells to discuss the quantum entanglement of two components, which is presented to be valid under stationary tunneling effect for odd particle number of each component. The evolution of the entanglement is found to be affected by the particle number both quantitatively and qualitatively. In detail, the maximal
entanglement are shown to be hyperbolic like with respect to tunneling rate and time. To successively obtain large and long time sustained entanglement, the particle number should not be large. Thus we can transform an exact many-body problem which is difficult even for single-component condensate to a bipartite two-state problem, similarly, we could measure the entanglement simply and conveniently. There also refer to the comparison of classical and quantum dynamics, great progress has been made in this subject [7], which is interesting and sensitive recently and is good for further studies. Our entanglement states are specified with the particle spatial location in left or right well rather than internal energy levels, this may be operational in physical applications such as quantum entangled particles distribution and quantum measurement. Moreover, the method developed here may find applications in the studies of entanglement of other BEC.

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