Abstract—The propagation of arbitrary amplitude ion-acoustic (IA) solitary waves (SWs) is studied in unmagnetized, collisionless, homogeneous electron–positron–ion (e–p–i) plasmas with finite temperature degeneracy of both electrons and positrons. Starting from a set of fluid equations for classical ions and Fermi–Dirac distribution for degenerate electrons and positrons, a linear dispersion relation for IA waves is derived. It is seen that the wave dispersion is significantly modified due to the presence of positron species and the effects of finite temperature degeneracy of electrons and positrons. In the nonlinear regime, Sagdeev’s pseudopotential approach is employed to study the existing domain and the evolution of nonlinear IA-SWs in terms of the parameters that are associated with the finite temperature degeneracy, the background number densities, and the thermal energies of electrons and positrons. It is found that in contrast to classical electron–ion plasmas, both the subsonic and supersonic IA-SWs can exist in a partially degenerate e–p–i plasma.

Index Terms—Electron–positron–ion (e–p–i) plasma, ion-acoustic (IA) wave, partially degenerate plasma, pseudopotential, solitary wave (SW).

I. INTRODUCTION

Electron–Positron (e–p) plasmas typically behave as a fully ionized gas composed of two fermions, namely electrons and positrons having equal mass but opposite charges. Such plasmas are ubiquitous not only in laboratory [1] but also in space and astrophysical environments, for example, in the early universe [2], active galactic nuclei [3], as well as white dwarfs [4]. The e–p pair plasmas can be created due to collisions between particles which are accelerated by the electrostatic or electromagnetic waves with or without the influence of the gravitational force. In view of their potential applications in these environments, various linear and nonlinear wave phenomena have been studied in e–p plasmas (see, e.g., [5], [6], [7]).

The presence of mobile ions in an admixture of electrons and positrons not only modifies the existing high-frequency waves, but also generates a new low-frequency ion-acoustic (IA) wave mode. So, the characteristics of nonlinear waves in e–p plasmas significantly differ from those in electron–positron–ion (e–p–i) plasmas due to the presence of these massive ions. Over the last many years, a number of authors have paid their attention to investigate the nonlinear propagation of solitary waves (SWs) in e–p–i plasmas using the Sagdeev potential approach or the reductive perturbation technique [8], [9], [10], [11], [12], [13], [14], [15], [16]. While the latter is commonly used to study weakly nonlinear small amplitude electrostatic or electromagnetic perturbations [17], [18], the former, on the other hand, is applicable for the nonlinear evolution of arbitrary amplitude SWs [7], [16]. However, an alternative approach of Sagdeev potential has also been developed by McKenzie [19], [20], [21] where the coherent nonlinear structures are examined in their own frames of reference.

The number density of degenerate particles in an extremely dense matter such as cosmic environments, compact astrophysical objects like white dwarfs [22] and active galactic nuclei are so high (∼1028–1034 cm−3) that the average interparticle distance is considerably smaller than the electron (or positron) thermal de-Broglie wavelength and so they obey the Fermi–Dirac (FD) statistics [23]. The degenerate pressure, which depends upon the number density of constituent particles but is independent of its thermodynamic temperature, occurs due to the combined effects of Pauli’s exclusion and Heisenberg’s uncertainty principles. It has been observed that the said compact objects, which support themselves against the gravitational collapse by cold, degenerate pressure of fermions (electrons/positrons), are of two types: Type-I consists of those objects (like white dwarfs) which are supported by the pressure of degenerate electrons or positrons and type-II are those (like neutron stars) which are supported by the pressure due to the combination of nucleon degeneracy and nuclear interactions [5]. The degenerate pressure significantly influences the evolution of electrostatic and electromagnetic perturbations in degenerate matters [24]. The energy distribution of degenerate particles in e–p–i plasmas follows the FD distribution which is usually characterized by two independent parameters: the chemical potential and the thermodynamic temperature. Here, if the thermodynamic temperatures of the electrons (T_e) and positrons (T_p) are comparable to the corresponding Fermi temperature (T_{Fj}, j = e, p), then the usual Maxwell–Boltzmann
distribution is modified to the FD distribution \[23\]. In particular, for theth species particles when the condition \( T_j \gg T_{Fj} \) is fulfilled, the particles are then said to be in the nondegenerate state and their background distribution can be governed by the Maxwell–Boltzmann ones. In the opposite limit, that is, \( T_j \ll T_{Fj} \), the particles are completely degenerate and their distributions follow that of the FD. However, in real situations, the particle’s temperature \( T_j \) may be finite and not all of them are degenerate. In this situation, the partial degeneracy of particles like electrons and positrons come into the picture in which case either \( T_j < T_{Fj} \) or \( T_j > T_{Fj} \), \( (j = e, p) \) holds, that is, electrons and positrons are neither nondegenerate nor completely degenerate. In the nonrelativistic regime of partial degeneracy, the energy \( E \) of electrons and positrons can be considered as \((1/2)mv^2\), where \( m \) is the mass and \( v \) is the velocity. Since the particles are not completely degenerate, there is no strict upper limit of the energy level and one can evaluate the number density and pressure over all the energy levels by extending the velocity to infinity, that is, \( 0 < v < \infty \).

On the other hand, the ions are typically nondegenerate as their mass is heavier than that of electrons or positrons. In the classical limit, the energy distributions of electrons and positrons are described by the Maxwell–Boltzmann distributions which typically depend on the thermodynamic temperatures, whereas in the fully degenerate limit, the energy distributions depend only on the chemical potential. So, an intermediate regime exists in which both the parameters, namely the thermodynamic temperature and the chemical potential, come into the picture. The present work mainly focuses on this regime of interest.

In view of the important and interesting consequences in the degenerate regimes, many authors have studied IA wave excitation in degenerate plasmas both theoretically and experimentally \[16\], \[25\], \[26\], \[27\], \[28\], \[29\], \[30\], \[31\], \[32\]. To mention a few, Haas and Mahmood \[26\], \[33\], \[34\] investigated the linear and nonlinear propagation of IA waves in various plasma environments with arbitrary degeneracy of electrons. However, their studies were limited to small-amplitude perturbations and plasmas without the positron species. The dispersion properties of electrostatic waves in degenerate plasmas were studied by Melrose and Mushtaq \[35\] in the linear regime. In another investigation, it has been found that the presence of hot electron and positron species gives rise to higher phase velocity, amplitude, and width of small-amplitude positron acoustic waves in a quantum degenerate plasma \[36\]. In a recent review, Misra and Brodin \[37\] presented the theoretical background of the wave–particle interactions and the physical mechanism of linear and nonlinear Landau damping in the propagation of electrostatic waves in degenerate and nondegenerate quantum plasmas. Furthermore, the existence of three different kinds of waves and their nonlinear evolution in degenerate spin-polarized plasmas have been studied by Iqbal and Andreev \[28\].

In this article, our aim is to consider the intermediate degenerate plasma regime, that is, in the partially degenerate e–p–i plasmas and study the existing domain and the formation of both large- and small-amplitude IA SWs in unmagnetized, collisionless, homogeneous e–p–i plasmas with finite temperature degeneracy of both electrons and positrons. This article is organized in the following order: In Section II, we derive the expressions for the number densities of degenerate electrons and positrons in terms of polylogarithmic function and present the basic set of normalized equations for e–p–i plasmas. The linear dispersion relation is derived and analyzed in Section III. The nonlinear analysis is performed in Section IV which comprises the derivation for the Sagdeev pseudopotential as well as the analyses for large and small amplitude IA waves. Finally, Section V is left to summarize the results.

II. BASIC EQUATIONS

We consider the propagation of IA SWs in an e–p–i plasma with finite temperature degeneracy of inertialess electrons and positrons and classical inertial cold ions. We assume that the thermodynamic temperatures of electrons and positrons, to be denoted by, \( T_j \) (with the suffix \( j = e \) for electrons and \( j = p \) for positrons) is slightly larger than their Fermi temperatures \( T_{Fj} \equiv E_{Fj}/k_B \), that is, \( T_j > T_{Fj} \). Here, \( k_B \) is Boltzmann’s constant and \( E_{Fj} \) is the Fermi energy for \( j \)-species particles. In the degenerate regime, the electron and positron number densities \( n_j(\mu_j, T_j) \), and the scalar pressure \( p_j(\mu_j, T_j) \) can be obtained using the FD distribution as \[38\], \[39\].

\[
n_j = \frac{L_{i3/2}}{L_{i3/2} - \exp(\xi_{\mu_j})} \left( \frac{\exp(\xi_{\mu_j})}{\exp(\xi_{\mu_j})} \right)^{\frac{1}{2}}
\]

\[
p_j = \frac{L_{i5/2}}{L_{i5/2} - \exp(\xi_{\mu_j})} \left( \frac{\exp(\xi_{\mu_j})}{\exp(\xi_{\mu_j})} \right)^{\frac{1}{2}}
\]

where \( L_i(z) \) denotes the polylogarithm function of \( z \) with index \( v \). Also, \( \xi_{\mu_j} = \mu_j/k_B T_j \) and \( \xi_{\mu_j, \mu_{j0}} = \mu_{j0}/k_B T_j \) are the degeneracy parameters corresponding to the perturbed and unperturbed chemical potentials \( \mu_j \) and \( \mu_{j0} \), respectively. Furthermore, the number density \( n_j \) and the scalar pressure \( p_j \) are, respectively, normalized by the unperturbed values \( n_{j0} = n_j(\mu_{j0}, T_j) \) and \( n_{j0} k_B T_j \) for \( j = e, p \). The Pauli exclusion principle allows at most one fermion to occupy each possible state, which in the firm condition (where the kinetic energy, \( E \sim 0 \)) results in \[25\].

\[
\sum_{j=e,p} \left[ 1 + \exp\left( -\frac{\mu_j}{k_B T_j} \right) \right] \leq 1.
\]

The equilibrium chemical potential \( \mu_{j0} \) is related to the equilibrium density \( n_{j0} \), given by

\[
-\frac{n_{j0}}{L_{i3/2} - \exp(\xi_{\mu_{j0}})} = \left( \frac{m}{2\pi \hbar^2} \right)^{3/2} = 2 \left( \frac{m}{2\pi \hbar^2} \right)^{3/2}
\]

or the ratio \( T_{Fj}/T_j \) in which case \( \xi_{\mu_{j0}} \) is given by

\[
-\frac{L_{i3/2}}{-\exp(\xi_{\mu_{j0}})} = \frac{4}{3\sqrt{\pi}} \left( \frac{T_{Fj}}{T_j} \right)^{3/2}.
\]

In particular, for finite temperature degeneracy with \( T_j > T_{Fj} \), \( 5 \) reduces to

\[
\xi_{\mu_{j0}} \approx \ln \left( \frac{4}{3\sqrt{\pi}} \left( \frac{T_{Fj}}{T_j} \right)^{3/2} \right) ; \quad j = e, p.
\]

We note that in the nondegenerate limit \( (T_j \gg T_{Fj}) \), the degeneracy parameter \( \xi_{\mu_j} \) is large but negative, however, in the case of full degeneracy \( (T_j \ll T_{Fj}) \), the parameter \( \xi_{\mu_j} \) is both large and positive \[38\].
Thus, the basic set of normalized equations governing the dynamics of IA waves consists of the ion continuity and momentum balance equations; the momentum equations for inertialess electrons and positrons, and the Poisson equation. These are

\[
\begin{align*}
\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{u}_i) &= 0, \\
\frac{\partial \mathbf{u}_i}{\partial t} + (\mathbf{u}_i \cdot \nabla) \mathbf{u}_i &= -\frac{1}{\beta_e} \nabla \phi,
\end{align*}
\]

with \(\beta_j = \frac{\gamma_j}{\gamma_j - 1}\) and \(\gamma_j = \frac{e^{\beta_j u_0 - 1}}{\exp^{\beta_j - 1}}\). The time and space variables are normalized by the ion plasma period \(\omega_p^{-1} = (4\pi n_0 e^2/m_i)^{-1/2}\) and the Debye length \(\lambda_D = (c_\circ/\omega_p)\) respectively. Furthermore, \(\sigma = T_p/T_e\) and \(c_\circ = 1/(1 - \delta)\), where \(\delta = n_p/n_i\). The temperature ratios \(\delta = T_e/T_p\) and \(\delta = T_p/T_e\) are related by \(\delta = \delta^{2/3} \approx \delta^{2/3}\). In the nondegenerate limit, \(\exp(\xi_{\mu \rho}) \ll 1\), for which (2) and \(\beta_j \approx 1\), we obtain \(c_i = (K_BT_e/m_i)\), \(\lambda_D \approx (K_BT_e/m_i\omega_0^2)^{1/2}\), and from (2) the isothermal pressure law: \(p_j = n_iK_BT_j\), that is, the well-known classical results are recovered. In the opposite case (full degeneracy) with \(\exp(\xi_{\mu \rho}) \gg 1\), for which \(L_i(\exp(\xi_{\mu \rho})) \approx (\xi_{\mu \rho})^\gamma/\Gamma(\gamma + 1)\), \(\mu_j \approx K_BT_j\) and \(\beta_j = (2/3)\tau_\circ\), one obtains \(c_i = \sqrt{(2/3)K_BT_e/m_i}\), \(\lambda_D = (2/3K_BT_e/m_i\omega_0^2)^{1/2}\), and the Fermi pressure law: \(p_j = (2/5)n_j E_{Fj}(n_j/n_{j0})^{5/3}\).

Next, using (1), (2), (9), and (10), we obtain the following reduced expressions for the electron and positron number densities:

\[
\begin{align*}
\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{u}_e) &= 0, \\
\frac{\partial \mathbf{u}_e}{\partial t} + (\mathbf{u}_e \cdot \nabla) \mathbf{u}_e &= -\frac{1}{\beta_e} \nabla \phi,
\end{align*}
\]

with \(\beta_j = \frac{\gamma_j}{\gamma_j - 1}\) and \(\gamma_j = \frac{e^{\beta_j u_0 - 1}}{\exp^{\beta_j - 1}}\). The time and space variables are normalized by the ion plasma period \(\omega_p^{-1} = (4\pi n_0 e^2/m_i)^{-1/2}\) and the Debye length \(\lambda_D = (c_\circ/\omega_p)\) respectively. Furthermore, \(\sigma = T_p/T_e\) and \(c_\circ = 1/(1 - \delta)\), where \(\delta = n_p/n_i\). The temperature ratios \(\delta = T_e/T_p\) and \(\delta = T_p/T_e\) are related by \(\delta = \delta^{2/3} \approx \delta^{2/3}\). In the nondegenerate limit, \(\exp(\xi_{\mu \rho}) \ll 1\), for which (2) and \(\beta_j \approx 1\), we obtain \(c_i = (K_BT_e/m_i)\), \(\lambda_D \approx (K_BT_e/m_i\omega_0^2)^{1/2}\), and from (2) the isothermal pressure law: \(p_j = n_iK_BT_j\), that is, the well-known classical results are recovered. In the opposite case (full degeneracy) with \(\exp(\xi_{\mu \rho}) \gg 1\), for which \(L_i(\exp(\xi_{\mu \rho})) \approx (\xi_{\mu \rho})^\gamma/\Gamma(\gamma + 1)\), \(\mu_j \approx K_BT_j\) and \(\beta_j = (2/3)\tau_\circ\), one obtains \(c_i = \sqrt{(2/3)K_BT_e/m_i}\), \(\lambda_D = (2/3K_BT_e/m_i\omega_0^2)^{1/2}\), and the Fermi pressure law: \(p_j = (2/5)n_j E_{Fj}(n_j/n_{j0})^{5/3}\).

Next, using (1), (2), (9), and (10), we obtain the following reduced expressions for the electron and positron number densities:

\[
\begin{align*}
\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{u}_e) &= 0, \\
\frac{\partial \mathbf{u}_e}{\partial t} + (\mathbf{u}_e \cdot \nabla) \mathbf{u}_e &= -\frac{1}{\beta_e} \nabla \phi,
\end{align*}
\]

III. LINEAR THEORY

In order to derive the linear dispersion relation for IA waves, we linearize the system of (7)–(11) by considering the first-order perturbation quantities relative to the equilibrium values, that is, \(n_i = 1 + n_{i1}, n_e = 1 + n_{e1}, n_p = 1 + n_{p1}, \mathbf{u}_i = \mathbf{u}_0 + \mathbf{u}_1, \phi = 0 + \phi_1, \xi_{\mu \rho} = \xi_{\mu \rho 0} + \xi_{\mu \rho 1}, \xi_{\mu \rho 0} = \xi_{\mu \rho 0} + \xi_{\mu \rho 1}\). Also, we use the expansion of the polylogarithm function up to the first order, that is,

\[
Li_0 \left[ -\exp(\gamma(x_0 + x_1)) \right] = Li_0 \left[ -\exp(\gamma x_0) \right] + \gamma x_1 Li_{-1} \left[ -\exp(\gamma x_0) \right].
\]

Assuming the perturbations to vary as plane waves of the form \(\sim \exp i(k \cdot r - \omega t)\) with the angular frequency \(\omega\) (normalized by \(\omega_0\)) and the wave vector \(k\) (normalized by \(\lambda_D^{-1}\)), we obtain the following dispersion relation.

\[
\omega^2 = \alpha_0 + \beta_0 + \gamma^2 \square.
\]
e–p–i plasmas with finite temperature degeneracy. To this end, we follow Sagdeev’s pseudopotential approach in which we assume that all the physical quantities depend on a single variable \( \zeta = I_x x + I_y y + I_z z - M_t \), where \( I_x, I_y, \) and \( I_z \) are the direction cosines of a line along the axes and \( M \) is the Mach number (the velocity of the localized wave normalized by \( c_s \)). Integrating (7) and (8), and making use of the boundary conditions, that is, \( u_i, \phi \to 0 \) and \( n_i \to 1 \) as \( \zeta \to \pm \infty \), we obtain the following expression for the ion number density:

\[
n_i = \frac{M}{\sqrt{M^2 - 2\phi / \beta_c}}.
\]

Next, substituting the expressions for the number densities of electrons, positrons and ions from (13), (14) and (18) into (11), we obtain

\[
\frac{d^2 \phi}{d \zeta^2} = \beta_e \left[ \frac{\alpha_e}{L_{5/2}} \left[ - \exp(\phi + \zeta_{\mu,\rho}) \right] \right. \\
\left. - \frac{\alpha_p}{L_{5/2}} \left[ - \exp(-\phi + \zeta_{\mu,\rho}) \right] - \frac{M}{\sqrt{M^2 - 2\phi / \beta}} \right].
\]

Finally, integrating (19) and using the boundary conditions, namely \( \phi \to 0, d\phi/d\zeta \to 0 \), and \( d^2\phi/d\zeta^2 \to 0 \) as \( \zeta \to \pm \infty \), we obtain the following energy-like equation for a pseudo-particle of unit mass with velocity \( d\phi/d\zeta \) and position \( \phi \):

\[
\frac{1}{2} \left( \frac{d\phi}{d\zeta} \right)^2 + V(\phi) = 0
\]

where the pseudopotential \( V(\phi) \) is given by

\[
V(\phi) = \frac{\beta \alpha_e}{L_{5/2}} \left[ - \exp(\zeta_{\mu,\rho}) \right] \\
\times \left[ L_{5/2} \left[ - \exp(\zeta_{\mu,\rho}) \right] - L_{5/2} \left[ - \exp(\phi + \zeta_{\mu,\rho}) \right] \right] \\
+ \frac{\beta \sigma \alpha_p}{L_{5/2}} \left[ - \exp(-\phi + \zeta_{\mu,\rho}) \right] \\
+ \frac{M}{\sqrt{M^2 - 2\phi / \beta}}.
\]

In what follows, the conditions for the existence of arbitrary amplitude IA SWs are [15].

1) \( V(0) = V'(0) = 0 \).
2) \( V''(\phi) < 0 \) at \( \phi = 0 \), so that the fixed point at the origin becomes unstable.
3) \( V(\phi_m) = 0 \) and \( V'(\phi_m) \leq 0 \) according to when the SWs are compressive (with \( \phi > 0 \)) or rarefactive (with \( \phi < 0 \)). Here, \( \phi_m \) represents the amplitude of the SWs or double layers, if they exist.

In Sections IV-A and IV-B, we will verify these conditions and study, in detail, the domain of the existence of arbitrary amplitude IA SWs in the parameter space as well as the properties of IA solitons including those in the limit of small amplitude approximation.

A. Arbitrary Amplitude Wave

It is easy to verify that the condition: 1) is satisfied and the condition 2) gives the critical Mach number \( M_I \) (i.e., the lower limit of \( M \)) for the existence of large-amplitude SWs, given by

\[
M_I = \frac{\alpha_p}{\beta_c} \left( \frac{1 - \delta}{\delta} \right). \quad (22)
\]

Such critical Mach number must be the same as the phase velocity of linear IA waves to be obtained from (16) in the limit of \( k \ll 1 \). In the nondegenerate limit, (22) reduces to

\[
M_I \approx \sqrt{\frac{\sigma(1 - \delta)}{\sigma + \delta}}. \quad (23)
\]

which is in agreement with the results of [14] and [16]. From (22), it is clear that the lower limit of the Mach number exists for \( 0 < \delta < 1 \) and as \( \delta \) increases in this interval, the value of \( M_I \) decreases. In the limit of \( \delta \to 0 \), we have \( M_I \to 1 \), that is, the nonlinear wave speed equals the ion sound speed. However, the nonlinear IA wave does not exist in the limit of \( \delta \to 1 \).

Since the pseudopotential \( V(\phi) \) is a real-valued function and the polylogarithm function \( \text{Li}_i(z) \) converges for \( |z| \leq 1 \), we must have the interval for \( \phi \) : \( \phi_c^+ \leq \phi \leq \phi_c^- \), where \( \phi_c^+ = \min(\beta_c M^2/2, z_{\mu,\rho}) \) and \( \phi_c^- = \sigma z_{\mu,\rho} \). It follows that for a given set of values of \( M \) and \( \tau_c \) or \( \sigma \), the IA wave amplitude \( \phi_m \) either lies in \( 0 < \phi_m \leq \phi_c^- \) or \( \phi_c^- \leq \phi_m < 0 \) according to when the wave potential is positive (\( \phi > 0 \)) or negative (\( \phi < 0 \)). In the interval \( \phi_c^- \leq \phi_m < 0 \), there does not exist any local minimum of \( V(\phi) \), however, for \( 0 < \phi_m \leq \phi_c^- \), it is possible to find a local minimum of \( V(\phi) \) for some particular values of the parameters \( \delta, \sigma \), and \( \tau_c \) or \( \tau_p \) [since the chemical potentials can be obtained from (6) in terms of \( \tau_c \)]. As a consequence, IA SWs or double layers with negative potential do not exist. So, we focus on the existence of SWs and/or double layers with positive potential only. To this end, we need \( V(\phi_c^-) > 0 \) and so the upper limit of the Mach number \( M_u \) can be obtained by solving \( V(\phi_c^-) = 0 \).

In what follows, we note that for the existence of IA SWs, we must have the values of \( M \) lying in the interval \( M_I < M < M_u \). Since an explicit expression of \( M_u \) is difficult to obtain, one can find its values numerically in terms of the parameters \( \delta, \tau_c \) or \( \tau_p \) and \( \sigma \). The typical values of the basic plasma parameters can be considered as [40] \( n_e \sim 2 \times 10^{24} \text{ cm}^{-3} \) and \( T_e \sim 10^9 \text{ K} \), and use the relations \( \rho_{\mu,\rho} = \delta \rho_{\mu,\rho}, T_p = \sigma T_c, \tau_c = T_{Fe}/T_c, \) and \( \tau_p = \delta^{2/3} \tau_c/\sigma \) such that \( 0 < \delta < 1 \) and \( 0 < \tau_c, \tau_p < 1 \). The values of \( M_I \) and \( M_u \), so obtained, are displayed in Fig. 2. In the latter, the profiles of the Mach numbers are shown against \( \delta \) and \( \tau_c \) for three different values of \( \sigma \) . The solid, dashed and dotted curves, respectively, correspond to the values of \( \sigma > 1 \), \( \sigma = 1 \), and \( \sigma < 1 \). While the values of \( M_I \), the lower limit of the Mach number, are obtained directly from its analytic expression [(22)], those of \( M_u \) (the upper limit of the Mach number) are obtained by a numerical approach. Fig. 2 shows that irrespective of the values of \( \tau_c \) (\( < 1 \)) and \( \sigma \), both the lower and upper limits of the Mach number tend to decrease below the unity with increasing values of \( \delta \). As a result, the range of values of \( M \) for the existence of IA SWs becomes
TABLE I

| $\mathcal{Y}$ | $\delta$ | $M$ ($\sigma = 0.7 < 1$) | $M$ ($\sigma = 1$) | $M$ ($\sigma = 1.5 > 1$) |
|--------------|--------|----------------|----------------|----------------|
| 0.1          | 0.1    | 0.89–1.31     | 0.90–1.51     | 0.92–1.52     |
| 0.1          | 0.3    | 0.70–1.31     | 0.73–1.33     | 0.76–1.35     |
| 0.3          | 0.5    | 0.54–1.07     | 0.58–1.11     | 0.61–1.15     |
| 0.3          | 0.6    | 0.46–0.94     | 0.50–0.98     | 0.53–1.02     |
| 0.5          | 0.7    | 0.39–0.79     | 0.42–0.84     | 0.45–0.88     |
| 0.5          | 0.8    | 0.31–0.63     | 0.33–0.67     | 0.36–0.72     |
| 0.6          | 0.9    | 0.21–0.44     | 0.23–0.47     | 0.25–0.51     |

Fig. 2. Lower ($M_l$) and upper ($M_u$) limits of the Mach number $M$ are plotted against the parameters $\delta$ [with a fixed $\tau_e = 0.3$; subplots (a) and (b)] and $\tau_e$ [with a fixed $\delta = 0.3$; subplots (c) and (d)] for three different values of $\sigma$ as in the legend.

narrower. In particular, for $\delta = 0$, one finds the range of $M$ as $1 \leq M < 1.61$ which is in accordance with the well-known classical result $1 \leq M < 1.585$ for electron–ion plasmas with Boltzmann distribution of electrons and $\tau_e \ll 1$. From Fig. 2, an interesting point is to be noted. While the value of $M_l$ decreases slowly with an increasing value of $\tau_e$ in the interval $0 < \tau_e < 1$, there exists a critical value $\tau_{ec}$ of $\tau_e$ such that $M_u$ increases slowly in the interval $0 < \tau_e < \tau_{ec}$, however, it decreases in rest of the interval. Thus, the domain of $M$ for the existence of IA SWs increases with increasing value of $\tau_e$ in $0 < \tau_e < \tau_{ec}$. Moreover, for $\tau_{ec} < \tau_e < 1$, since the upper limit of the Mach number starts decreasing, the existence domain for $M$ becomes narrower. The peculiarities of the existence domains of $M$ for all the three cases, namely $\sigma > 1$, $\sigma = 1$, and $\sigma < 1$ remain similar. From Fig. 2, it is also noticed that the variation of $M$ with $\sigma$ has only a quantitative effect on its upper and lower limits, that is, both of them increase with increasing values of $\sigma$. Furthermore, inspecting the regions of the Mach number, we find that the IA SWs can exist with the Mach numbers greater than or less than the unity. It turns out that the IA waves can propagate in the form of compressive type SWs with subsonic or supersonic speed depending on the parameter regions we consider. Also, there exists a critical value of $\delta$ (say $\delta_c$) above which only subsonic SWs exist. Such critical value of $\delta$ can be estimated, in particular, for $\sigma = 1$ and $\tau_e = 0.3$, as $\delta_c \simeq 0.58$. It is found that in the case of degenerate e–i plasmas with $\delta \sim 0$ only the supersonic IA SWs exist. However, in degenerate e–p–i plasmas with $0 < \delta \leq 0.58$, both the subsonic and supersonic SWs, and only the subsonic SWs exist in the regime $0.58 \lesssim \delta < 1$.

It is to be mentioned that for the existence of IA double layers with positive potential, we require, in addition to the conditions stated in Section IV, the condition $V(\phi_m) = V'(\phi_m) = 0$ with $0 < \phi_m < \phi_m^*$. However, a numerical investigation over a wide range of values of the parameters indicates that such $\phi = \phi_m$ does not exist in $0 < \phi < \phi_m^*$. The domains of the Mach number with different values of the plasma parameters for which the IA SWs exist in partially degenerate e–p–i plasmas are computed as presented in Table I.

Having obtained various parameter regimes for the existence of IA SWs with positive potential, we move on to study the profiles of the SWs, especially we focus on the characteristics of the wave amplitudes and widths with the effects of the plasma parameters. It is, therefore, pedagogic to examine the profiles of the Sagdeev potential as well as the corresponding SW solution of (20). The results are displayed in Figs. 3 and 4.
We find that for certain parameter values $V(\phi)$ crosses the $\phi$-axis at $\phi = \phi_m$ and $dV/d\phi < 0$ for $0 < \phi < \phi^+_m$ (see the top panels of Figs. 3 and 4). Such $\phi_m$ represents the amplitude of the IA SW which we can verify from the profiles of $\phi(\zeta)$ (see the bottom panels of Figs. 3 and 4). The width of the SW can be obtained either from the profile of $\phi(\zeta)$ or by computing the value of $|\phi_m|/|V_{\text{min}}|$, where $|V_{\text{min}}|$ denotes the absolute minimum value of $V(\phi)$. Here, we recast the key parameters as $M$, $\tau_c$, $\tau_p$, $\delta$ and $\sigma$ with $0 < \tau_c$, $\tau_p < 1$ and $0 < \phi < \phi^+_m$. For a fixed value of $\sigma = 1$ and different sets of values of $M$, $\tau_c$ and $\delta$ that fall in the existence domain of IA SWs, the profiles of the pseudopotential $V(\phi)$ and the corresponding solitary potential are shown in Fig. 3. It is found that the depth of the potential well increases by having cut-offs at higher values of $\phi$ with an increasing value of each of the parameters $\tau_c$, $M$, and $\delta$ [upper panel of Fig. 3]. Such increments are significant even with a small variation of the parameters and are correlated with the increments of both the amplitude and width of the SWs [lower panel of Fig. 3] except for the variation with $\delta$ in which case even though the amplitude of the SW increases, its width gets significantly reduced with a small increment of $\delta$. On the other hand, Fig. 4 shows the profiles of $V(\phi)$ (upper panel) and $\phi(\zeta)$ (lower panel) with the variation of the temperature ratio $\sigma$. It is seen that although the depth of the potential well increases, it has cut-offs nearly at the same value of $\phi = \phi_m$ implying that an increase of the wave amplitude is not significant, however, its width increases with increasing values of $\sigma$. 

### B. Small-Amplitude Wave

So far, we have studied the existing domain and the propagation characteristics of arbitrary amplitude IA SWs in terms of different plasma parameters that are relevant in dense plasma environments. One particular interest in this context is to investigate the profiles of SWs when their amplitudes are no longer arbitrary but small ($|\phi| < 1$). In this case, the pseudopotential $V(\phi)$ can be Taylor expanded about $\phi = 0$ up to a reasonable order of $\phi$ to obtain a soliton solution from the energy integral. Thus, (20) reduces to

$$\frac{1}{2} \left( \frac{d\phi}{d\zeta} \right)^2 + A\phi^2 + B\phi^3 = 0 \quad (24)$$

where

$$A = \frac{1}{2} \frac{-\beta_p a_p}{L_{1/2}} \left[ - \exp(\xi_{p,0}) \right] L_{1/2} \left[ - \exp(\xi_{p,1}) \right] - \frac{\beta_p a_p}{2\sigma L_{1/2}} \left[ - \exp(\xi_{p,0}) \right] L_{1/2} \left[ - \exp(\xi_{p,1}) \right] + \frac{1}{2 M^2}$$

$$B = \frac{1}{6} \frac{-\beta_p a_p}{L_{1/2}} \left[ - \exp(\xi_{p,0}) \right] L_{-1/2} \left[ - \exp(\xi_{p,1}) \right] + \frac{\beta_p a_p}{6\sigma^2 L_{1/2}} \left[ - \exp(\xi_{p,0}) \right] L_{-1/2} \left[ - \exp(\xi_{p,1}) \right] + \frac{1}{2 \beta M^2}.$$  

(25)

(26)

Next, integrating (24) and using the boundary conditions stated before, we obtain the following soliton solution:

$$\phi = \phi_m \sech^2(\zeta/\omega) \quad (27)$$

where $\phi_m = -A/B$ and $\omega = (-2/A)^{1/2}$ are, respectively, the amplitude and width of the IA soliton. Inspecting the coefficients $A$ and $B$, and the expressions for $\phi_m$ and $\omega$, we note that for a real soliton solution to exist, one must have $\phi_m < 0$. This leads to the condition $M > M_t$ where $M_t$ is given by (28). Since the small-amplitude approximation is valid only for a small deviation from the linear phase velocity, we assume the Mach number as $M = M_t + \varepsilon M_0$, where $\varepsilon$ is some small positive scaling parameter and $M_0$ is the deviation from the linear phase velocity $M_t$. It is to be noted that at $M = M_t$, the IA waves become dispersionless and the corresponding SWs behave like Korteweg–de Vries (KdV) solitons. Since $\phi_m < 0$ is noted, the polarity of the soliton depends only on the sign of $B$. Thus, the SWs with positive (negative) potential exists if $B > 0$ ($B < 0$). However, the possibility of $B < 0$ may be ruled out since in the previous section IV-A we have found only the compressive type SWs with arbitrary amplitudes. Furthermore, $B < 0$ leads to the condition $M > M_k$, where

$$M_k = \frac{3^{1/4}}{\sqrt{\beta c}} \left[ \alpha_p L_{-1/2} \left[ - \exp(\xi_{p,0}) \right] - \alpha_p L_{-1/2} \left[ - \exp(\xi_{p,1}) \right] \right]^{-1/4} \quad (28)$$

implying that $M_k - M_t < \varepsilon M_0$. However, there is no parameter region for which this condition may be satisfied. So, the propagation of small-amplitude IA SWs with negative potential may not exist. On the other hand, by choosing some $M_0$ satisfying $\varepsilon M_0 < M_k - M_t$, the propagation of small-amplitude IA SWs
with positive potential is possible. The qualitative features of the solitons with the variations of the parameters will remain the same as for arbitrary amplitude waves. However, since the Mach number has some deviation from $M_1$, its effects (with some new values) on the soliton profiles are to be noticed (see Fig. 5). It is found that in contrast to the large amplitude waves, as the Mach number slightly increases the amplitude of the soliton decreases but its width increases.

V. CONCLUSION

We have studied the propagation characteristics of arbitrary amplitude SWs in multicomponent plasma with finite temperature degeneracy of both electrons and positrons and classical ions. Starting from a set of fluid equations for classical ions and FD distribution for electrons and positrons, a linear dispersion relation for IA waves is derived. It is seen that the wave frequency is significantly reduced due to the presence of positron species, however, it can be increased by increasing the relative temperature of positrons compared to electrons. On the other hand, the nonlinear theory of IA SWs is studied using Sagdeev’s pseudopotential approach. Different domains in parameter space for the existence of IA SWs are obtained and analyzed. While the lower limit of the Mach number $M_l$ is obtained analytically from the Sagdeev’s condition $(ii)$, its upper limit $M_u$ is obtained numerically from the condition $V(\phi^+_{\text{c}}) > 0$, where $\phi^+_{\text{c}} = \min(\beta_e, M^2/2, \zeta_{\text{c}})$. Although the degeneracy effect has no direct influence on the linear wave mode, it expands the existence domain of the Mach number significantly until $T_{Fe}/T_e \lesssim 0.6$; $j = e, p$. In contrast, the presence of the positron species reduces the existence regions of $M$ by increasing the density ratio $\delta$. The main results can be summarized as follows.

1) The wave frequency and hence the phase velocity of IA waves is reduced due to the presence of positron species. However, it can be increased with an increase in the positron-to-electron temperature ratio. Such a reduction of the wave frequency may be desirable for IA waves not to be strongly damped in a wave–particle interactions.

2) A numerical investigation indicates that the IA double layers do not exist. However, the SWs exist only of the compressive type and in contrast to classical electron–ion plasmas they can propagate with subsonic or supersonic speed depending on the parameter regimes we consider. Typically, for $\delta \equiv n_0/e_0 \sim 0.1$, $\tau_e \equiv T_{Fe}/T_e \sim 0.1$, the domain of $M$ can be estimated as $0.9 \lesssim M \lesssim 1.5$, or, in dimension, $8 \times 10^6 \text{cm/s} \lesssim M \lesssim 1.35 \times 10^7 \text{cm/s}$ for $T_e \sim T_p \sim 10^6 \text{K}$.

3) The lower ($M_l$) and upper ($M_u$) limits of the Mach number are highly dependent on the positron to electron number density $\delta$ and the degeneracy parameter $\tau_e$ or $\tau_p$. Their values decrease with increasing values of $\delta$, thereby reducing the domain of $M$ for the existence of IA SWs. In the limits of $\delta \to 0$ and $\tau_e, \tau_p \to 0$, the well-known classical results for electron–ion plasmas are recovered, that is, $1 < M < 1.585$.

4) Due to finite temperature degeneracy of electrons and positrons, there exists a critical value of $\tau_e$ or $\tau_p$, where the upper limit of the Mach number $M_u$ can turn over, tend to decrease and then close to the value of $M_l$.

5) Both the amplitude and width of the SWs are increased due to enhancements of $M$ and $\tau_e$ or $\tau_p$. However, the amplitude increases but the width decreases with increasing values of $\delta$.

To conclude, multicomponent degenerate plasmas are not only ubiquitous in astrophysical environments, but also in inertial confinement fusion (ICF) experiments with particle density ranging from $10^{24}$ to $10^{27}$ cm$^{-3}$ and the thermodynamic temperature $T_e \sim T_p \sim 10^6$–$10^8$ K [40]. Also, during ultra-intense short-pulse laser irradiation of solid-density targets, quantum statistical effects tend to become more prominent than mechanical (quantum diffraction) effects. Furthermore, laboratory simulation of astrophysical phenomena in dense plasma environments better fits with the intermediate classical-quantum regime [41]. Thus, our theoretical results should be useful for understanding the localization of arbitrary amplitude IA SWs that are candidates in these environments.

REFERENCES

[1] G. Sarri, K. Poder, and J. E. A. Cole, “Generation of neutral and high-density electron-positron pair plasmas in the laboratory,” Nature Commun., vol. 6, p. 6747, 2015, doi: 10.1038/ncomms7747.

[2] G. W. Gibbons, S. W. Hawking, and S. T. C. Siklos, Eds., The Very Early Universe: Proceedings of the Nuffield Workshop, Cambridge 21 June to 9 July, 1982. Cambridge Univ. Press, Aug. 1983, p. 480.

[3] H. R. Miller and P. J. Wiita, Eds., Active Galactic Nuclei: Proceedings of a Conference Held at the Georgia State University, Atlanta, Georgia October 28-30, 1987. Springer, 1988.

[4] S. L. Shapiro and S. A. Teukolsky, Black Holes, White Dwarfs, and Neutron Stars: The Physics of Compact Objects. Hoboken, NJ, USA: Wiley, 2008.

[5] W. F. El-Taibany and A. A. Mamun, “Nonlinear electromagnetic perturbations in a degenerate ultrarelativistic electron-positron plasma,” Phys. Rev. E, Stat. Phys. Plasmas Fluids Relat. Interdiscip. Top., vol. 85, no. 2, pp. 101–107, Feb. 2012.

[6] D. Chatterjee and A. P. Misra, “Nonlinear Landau damping and modulation of electrostatic waves in a nonextensive electron-positron-pair plasma,” Phys. Rev. E, Stat. Phys. Plasmas Fluids Relat. Interdiscip. Top., vol. 92, no. 6, Dec. 2015, Art. no. 063110, doi: 10.1103/PhysRevE.92.063110.

[7] G. Banerjee, S. Dutta, and A. P. Misra, “Large amplitude electromagnetic solitons in a fully relativistic magnetized electron-positron-pair plasma,” Adv. Space Res., vol. 66, no. 9, pp. 2265–2273, Nov. 2020.

[8] A. Rahman, I. Kourakis, and A. Qamar, “Electrostatic solitary waves in relativistic degenerate electron-positron-ion plasma,” IEEE Trans. Plasma Sci., vol. 43, no. 4, pp. 974–984, Apr. 2015.

[9] M. Akbari-Moghani-Soufi, “Propagation of arbitrary-amplitude nonlinear quantum ion-acoustic waves in electron–ion plasmas: Dimensionality effects,” IEEE Trans. Plasma Sci., vol. 38, no. 12, pp. 3336–3341, Dec. 2010.
