Orthogonal polynomials of several discrete variables and the 3nj-Wigner symbols: applications to spin networks

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Abstract. The use of orthogonal polynomials for integral models on the lattice is applied to the 3nj-symbols that appear in the coupling of several angular momenta. These symbols are connected to the Ponzano-Regge method to solve the Einstein equations on a discrete Riemannian manifold.

1. Classical orthogonal polynomials of one discrete variable

These polynomials satisfy a difference equation of hypergeometric type such that the difference derivatives of the some polynomial satisfy a similar equation. The discrete variable can be consider of two types.

a) On homogeneous lattice: \( x = 0, 1, 2, \ldots \)

The corresponding polynomials satisfy a difference equation

\[
\sigma(x) \Delta \nabla p_n(x) + \tau(x) \Delta p_n(x) + \lambda_n p_n(x) = 0
\]

with \( \Delta f(x) = f(x+1) - f(x), \quad \nabla f(x) = f(x) - f(x-1), \quad \sigma(x) \) and \( \tau(x) \) are functions of second and first order respectively;

an orthogonality relation

\[
\sum_{x=a}^{b} p_n(x) p_m(x) \rho(x) = d_n^2 \delta_{nm}
\]

with \( \rho(x) \) a weight function and \( d_n \) a normalization constant. To these polynomials correspond the Meixner, Kravchuk, Charlier and Hahn polynomials [1]

b) On non homogeneous lattice: \( x = x(s) = s(s+1) \)

\( x = q^s \) or \( x = q^{is} + q^{-is} \), we have the \( q \)-Kravchuk, \( q \)-Meixner, \( q \)-Charlier and \( q \)-Hahn polynomials.

For \( x(s) = q^s + q^{-s} \) or \( q^{is} + q^{-is} \), we have the \( q \)-Racach and \( q \)-dual Hahn polynomials [2].
2. Generalized Clebsch-Gordon coefficients and generalized 3nj-Wigner symbols

If two angular momentum operators are coupled to give a total angular momentum \( J = J_1 + J_2 \) the new basis can be expressed in terms of the old ones

\[
| j_1 j_2 j m \rangle = \sum_{m_1 + m_2 = m} \langle j_1 j_2 m_1 m_2 \mid j_1 j_2 m j \rangle | j_1 j_2 m_1 m_2 \rangle.
\]

The symmetry properties of the Clebsch-Gordon coefficients in this expansion are more patent if one substitutes them by the Wigner symbols

\[
\langle j_1 j_2 m \mid j_1 j_2 m_1 m_2 \rangle = (-1)^{j_1 - j_2 + j - m} \sqrt{2j + 1} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix}.
\]

Similarly if we couple three angular momentum operators we obtain a new basis in terms of the old ones:

\[
| j_1 j_2 j_3 j_{12} j m \rangle = \sum \langle j_1 j_2 j_3 m_1 m_2 m_3 \mid j_1 j_2 j_3 j_{12} j m \rangle | j_1 j_2 j_3 m_1 m_2 m_3 \rangle
\]

for the coupling \((J_1 + J_2) + J_3 = J\),

\[
| j_1 j_2 j_3 j_{23} j m \rangle = \sum \langle j_1 j_2 j_3 m_1 m_2 m_3 \mid j_1 j_2 j_3 j_{23} j m \rangle | j_1 j_2 j_3 m_1 m_2 m_3 \rangle
\]

for the coupling \(J_1 + (J_2 + J_3) = J\).

Both bases are related by some matrix \( U(j_{12}, j_{23}) \) that can be written in terms of generalized 6j-Wigner symbol

\[
U(j_{12}, j_{23}) = (-1)^{j_1 + j_2 + j_3 + j} \sqrt{(2j_{12} + 1)(2j_{23} + 1)} \begin{pmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{pmatrix}
\]

In similar fashion can be written the generalized Clebsch-Gordon coefficients and generalized 3nj-Wigner symbols [3]. The algebraic properties of these symbols can be represented by geometrical graphs [3].

3. 3nj-symbols as orthogonal polynomials of several discrete variable

The 6j-symbols are proportional to the Racah polynomials through the following relation [2]

\[
(-1)^{j_1 + j_2 + j_{23}} \sqrt{(2j_{12} + 1)(2j_{23} + 1)} \frac{\rho(x)}{d_n} u_n^{(\alpha, \beta)}(x, a, b)
\]

with

\[
x(s) = s(s + 1), \quad s = j_{23}, \quad a = j_3 - j_2, \quad b = j + j_3 + 1, \quad n = j_{12} - j_1 + j_2 - j, \quad \alpha = j_1 - j_2 - j_3 + j, \quad \beta = j_1 - j_2 + j_3 - j.
\]

Using the asymptotic limit of the Racah polynomials and the connections between the Jacobi polynomials and the Wigner little functions one can prove the following approximation of the 6j-symbols when \( j_1 \sim j_2 \sim j_3 \sim j \gg j_{12} \)

\[
\begin{pmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{pmatrix} \sim \frac{(-1)^{j_2 + j_3 + j_{23}}}{\sqrt{j_1 + j_2 + 1} \sqrt{j_3 + j + 1}} u_{j_1 - j_2, j_3 - j}(\theta)
\]

(1)
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with
\[
\cos \theta = \frac{(2j_{23} + 1)^2 - (j_1 + j_2 + 1)^2 - (j_3 + j + 1)^2}{2(j_1 + j_2 + 1)(j_3 + j + 1)}
\]

The 3nj-symbols of the first and second kind can be written in terms of 6j-symbols, and therefore in terms of product of Racah polynomials, giving rise to orthogonal polynomials of several discrete variables. To illustrate this take, f.i., the 12j-symbol of the second kind as a combination of 6j-symbols.

\[
\begin{align*}
\{ j_1 & j_2 j_3 j_4 \\ l_1 & l_2 l_3 l_4 \\ k_1 & k_2 k_3 k_4 \} = \sum_x (2x + 1) (-1)^{R_n + 4x} \begin{Bmatrix} j_1 & k_1 & x \\ l_1 & k_2 & j_2 \\ k_3 & j_3 & l_2 \end{Bmatrix} \begin{Bmatrix} j_3 & k_3 & x \\ j_4 & k_4 & l_3 \end{Bmatrix} \begin{Bmatrix} j_4 & k_4 & x \\ j_1 & k_1 & l_4 \end{Bmatrix} \\
\end{align*}
\]

Here \( R_n = \sum_{i=1}^{4} (j_i + l_i + k_i) \). Substituting each 6j-symbol for the corresponding Racah polynomial we obtain:

\[
\begin{align*}
\{ j_1 & j_2 j_3 j_4 \\ l_1 & l_2 l_3 l_4 \\ k_1 & k_2 k_3 k_4 \} = \sum_x \frac{1}{2x + 1} \prod_{i=1}^{4} \frac{\sqrt{\rho(l_i)}}{d_{n_i}} u_n^{(\alpha_i, \beta_i)} (l_i) \equiv p_n (l_1 l_2 l_3 l_4)
\end{align*}
\]

which is a polynomial of four discrete variables.

For the asymptotic limit we find

\[
\begin{align*}
\{ j_1 & j_2 j_3 j_4 \\ l_1 & l_2 l_3 l_4 \\ k_1 & k_2 k_3 k_4 \} \approx \sum_x (2x + 1) \prod_{i=1}^{4} \frac{1}{j_i + k_i + 1} d_p^{(j_i, j_{i+1} - k_i + 1)} (\vartheta_i)
\end{align*}
\]

These formulas can be easily generalized to any 3nj-symbols of first and second kind.

4. Application to spin networks and to Ponzano-Regge integral action

Penrose has proposed a model for the space and time in which the underlying structure is given by a set of interactions between elementary units that satisfy the coupling of angular momentum operators, called spin networks [4]. One particular case of these networks can be described by the graphs of 3nj-symbols. From different point of view Regge has proposed a method to calculate Einstein action by the approximation of curved riemannian manifold by a polyedron built up of triangles. Later Ponzano and Regge applied the properties of 6j-symbols to calculate the sum action over this triangulation [5].

Let \( M \) be a riemannian manifold that is approximated by a polyedron with boundary \( D \) and it is decomposed into \( p \) tetrahedra \( T_k \) represented by 6j-symbols.

The polyedron give rise to triangular faces \( f \), represented by 3j-symbols, and to \( q \) internal edges \( x_i \), as well as to external ones \( l_i \) with respect to the boundary \( D \).

Ponzano and Regge define the sum

\[
S = \sum_{x_i} \prod_{k=1}^{p} T_k (-1)^q \prod_{i=1}^{q} (2x_i + 1)
\]

When \( l_i \to \infty \), \( \hbar \to 0 \), \( \hbar l_i \to \) finite we recovered the continuous manifold. In order to compute the 6j-symbols in the classical limit, we uses the asymptotic formula [2]
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\[ d_{m'm'}^{j}(\theta) \approx (-1)^{m-m'} \sqrt{\frac{2}{\pi (j-m)}} \left( \frac{2j+m-m'+1}{2j-m+m'+1} \right)^{\frac{m+m'}{2}} \]

\[ \cos \left[ \left( j + \frac{1}{2} \right) \theta - \left( m - m' + \frac{1}{2} \right) \frac{\pi}{2} \right] \]

at \( m \sim m' \sim 1, j >> 1 \). Substituting this expression in (1) with \( m = j_1 - j_2, m' = j_4 - j_5, j = j_6 \) and taking the edges of the tetrahedra \( j_1 + \frac{1}{2}, \ldots, j_6 + \frac{1}{2} \), very large except \( j_6 \), we have

\[
\begin{align*}
\{ j_1 & \quad j_2 & \quad j_3 \\
\{ j_4 & \quad j_5 & \quad j_6 \}
\} \approx \frac{1}{\sqrt{12\pi V}} \cos \left\{ \left( j_6 + \frac{1}{2} \right) \theta - \left( j_1 + \frac{1}{2} \right) \frac{\pi}{2} + \\
& \left( j_2 + \frac{1}{2} \right) \frac{\pi}{2} - \left( j_4 + \frac{1}{2} \right) \frac{\pi}{2} + \left( j_5 + \frac{1}{2} \right) \frac{\pi}{2} + \frac{\pi}{4} \right\} = \\
& \frac{1}{\sqrt{12\pi V}} \cos \left\{ \sum_{i=1}^{6} \left( j_i + \frac{1}{2} \right) \theta_i + \frac{\pi}{4} \right\}
\end{align*}
\]

where \( \theta_i \) is the dihedral angle for the edge \( j_i \) and \( V = \frac{1}{6} \left( j_1 + \frac{1}{2} \right) \left( j_4 + \frac{1}{2} \right) \left( j_6 + \frac{1}{2} \right) \sin \theta \).

Note the formula (3) has been proved rigorously by Roberts [5]. Introducing formula (3) in formula (2), Ponzano and Regge proved that it leads in the continuous limit to the integral action of the general relativity.

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