Polynomial-Time Amoeba Neighborhood Membership and Faster Localized Solving

Eleanor Anthony, Sheridan Grant, Peter Gritzmann, and J. Maurice Rojas

Abstract We derive efficient algorithms for coarse approximation of algebraic hypersurfaces, useful for estimating the distance between an input polynomial zero set and a given query point. Our methods work best on sparse polynomials of high degree (in any number of variables) but are nevertheless completely general. The underlying ideas, which we take the time to describe in an elementary way, come from tropical geometry. We thus reduce a hard algebraic problem to high-precision linear optimization, proving new upper and lower complexity estimates along the way.

Dedicated to Tien-Yien Li, in honor of his birthday.

1 Introduction

As students, we are often asked to draw, hopefully without a calculator, real zero sets of low degree polynomials in few variables. As scientists and engineers, we are often asked to count or approximate, hopefully with some computational assistance, real and complex solutions of arbitrary systems of polynomial equations in many variables. If one allows sufficiently coarse approximations, then the latter problem is as easy as the former. Our main results clarify this transition from hardness to easiness. In particular, we significantly speed up certain queries involving distances between points and algebraic hypersurfaces (see Theorems 1.4–1.6 and Remark 1.9 below).

Polynomial equations are ubiquitous in numerous applications, such as algebraic statistics [HRS13], chemical reaction kinetics [MFRCSD13], discretization of partial differential equations [HHHLSZ13], satellite orbit design [NAM11], circuit complexity [KPR13], and cryptography [BFP13]. The need to solve larger and larger equations, in applications as well as for theoretical purposes, has helped shape algebraic geometry and numerical analysis for centuries. More recent work in algebraic complexity tells us that many basic questions involving polynomial equations are \( \text{NP} \)-hard (see, e.g., [Pla84, Ko96, BL07, BS09]). This is by no means an excuse to consider polynomial equation solving hopeless: computational scientists solve problems of near-exponential complexity every day.

More to the point, thanks to recent work on Smale’s 17th Problem [BP09, BC10], we have learned that randomization and approximation are key to avoiding the bottlenecks present in hard deterministic questions involving roots

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of polynomial systems. Smale’s 17th Problem concerns the complexity of approximating a single complex root of a random polynomial system and is well-discussed in [Sma98, Sma00, SS92a, SS92b, SS93, SS96, SS94]. Our ultimate goal is to extend this philosophy to the harder problem of localized solving: estimating how far the nearest root of a given system of polynomials (or intersection of several zero sets) is from a given point. We make some initial steps by first approximating the shape of a single zero set, and we then outline a tropical-geometric approach to localized solving in Section 3.

Toward this end, let us first recall the natural idea [Vir01] of drawing zero sets on log-paper. In what follows, we let \( \mathbb{C}^* \) denote the non-zero complex numbers and write \( \mathbb{C} [x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \) for the ring of Laurent polynomials with complex coefficients, i.e., polynomials with negative exponents allowed.

**Definition 1.1** We use the abbreviations \( x := (x_1, \ldots, x_n) \) and \( \log |x| := (\log |x_1|, \ldots, \log |x_n|) \), and, for any \( f \in \mathbb{C} [x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \), we define \( \text{Amoeba}(f) := \{ \log |x| \mid f(x) = 0, \ x \in (\mathbb{C}^*)^n \} \). We call \( f \) an \( n \)-variate \( t \)-nomial when we can write \( f(x) = \sum_{i=1}^n c_i x_i^{a_i} \) with \( c_i \neq 0 \), \( a_i := (a_{1i}, \ldots, a_{ni}) \), and \( x_i^{a_i} := x_1^{a_{1i}} x_2^{a_{2i}} \cdots x_n^{a_{ni}} \) for all \( i \). Finally, we define the Archimedean tropical variety of \( f \), denoted \( \text{ArchTrop}(f) \), to be the set of all \( w \in \mathbb{R}^n \) for which \( \max_i |c_i e^{u_i w}| \) is attained for at least \( 2 \) distinct indices \( i \).\(^1\)

**Example 1.2** Taking \( f(x) = 1 + x_1^2 + x_2^2 - 3x_1x_2 \), an illustration of \( \text{Amoeba}(f) \) and \( \text{ArchTrop}(f) \), truncated to \([-7, 7]^2 \), appears to the right. \( \text{Amoeba}(f) \) is lightly shaded, while \( \text{ArchTrop}(f) \) is the piecewise-linear curve.\(^\diamond\)

One may be surprised that \( \text{Amoeba}(f) \) and \( \text{ArchTrop}(f) \) are so highly structured: \( \text{Amoeba}(f) \) has tentacles reminiscent of a living amoeba, and \( \text{ArchTrop}(f) \) is a polyhedral complex, i.e., a union of polyhedra intersecting only along common faces. One may also be surprised that \( \text{Amoeba}(f) \) and \( \text{ArchTrop}(f) \) are so closely related: for our example above, one set is strictly contained in the other, every point of one set is close to some point of the other, and both sets have topologically similar complements (4 open connected components, exactly one of which is bounded).

Proving that \( \text{Amoeba}(f) \) and \( \text{ArchTrop}(f) \) are in fact equal when \( f \) has two or fewer monomial terms is a simple exercise (see Proposition 2.1 below). More generally, to quantify exactly how close \( \text{Amoeba}(f) \) and \( \text{ArchTrop}(f) \) are, one can recall the Hausdorff distance, denoted \( \Delta(U, V) \), between two subsets \( U, V \subseteq \mathbb{R}^n \): it is defined to be the maximum of \( \sup_{u \in U} \inf_{v \in V} |u - v| \) and \( \sup_{v \in V} \inf_{u \in U} |u - v| \). We then have the following recent result of Avendaño, Kogan, Nisse, and Rojas.

**Theorem 1.3** [AKNR13] For any \( n \)-variate \( t \)-nomial \( f \) we have \( \Delta(\text{Amoeba}(f), \text{ArchTrop}(f)) \leq (2t - 3) \log(t - 1) \). In particular, we also have \( \sup_{u \in \text{Amoeba}(f)} \inf_{v \in \text{ArchTrop}(f)} |u - v| \leq \log(t - 1) \). Finally, for any \( t > n \geq 1 \), there is an \( n \)-variate \( t \)-nomial \( f \) with \( \Delta(\text{Amoeba}(f), \text{ArchTrop}(f)) \geq \log(t - 1) \).\(^\blacksquare\)

Note that the preceding upper bounds are completely independent of the coefficients, degree, and number of variables of \( f \). We conjecture that an \( O(\log t) \) upper bound on the above Hausdorff distance is possible. More practically, as we will see in later examples, \( \text{Amoeba}(f) \) and \( \text{ArchTrop}(f) \) are often much closer than guaranteed by any proven upper bound.

Given the current state of numerical algebraic geometry and algorithmic polyhedral geometry, the preceding metric result suggests that it might be useful to apply Archimedean tropical varieties to speed up polynomial system solving. Our first two main results help set the stage for such speed-ups. Recall that \( \mathbb{Q}[\sqrt{-1}] \) denotes those complex numbers whose real and imaginary parts are both rational. Our complexity results will all be stated relative to the classical Turing (bit) model, with the underlying notion of input size clarified below in Definition 1.7.

**Theorem 1.4** Suppose \( f \in \mathbb{C} [x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \) and \( w \in \mathbb{R}^n \). Then

\[
- \log(t - 1) \leq \Delta(\text{Amoeba}(f), w) - \Delta(\text{ArchTrop}(f), w) \leq (2t - 3) \log(t - 1).
\]

In particular, if we also assume that \( n \) is fixed and \( (f, w) \in \mathbb{Q}[\sqrt{-1}] [x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \times \mathbb{Q}^n \) with \( f \) a \( t \)-nomial, then we can compute polynomially many bits of \( \Delta(\text{ArchTrop}(f), w) \) in polynomial-time, and there is a polynomial-time algorithm that declares either (a) \( \Delta(\text{Amoeba}(f), w) \leq (2t - 2) \log(t - 1) \) or (b) \( w \notin \text{Amoeba}(f) \) and \( \Delta(\text{Amoeba}(f), w) \geq \Delta(\text{ArchTrop}(f), w) - \log(t - 1) > 0 \).

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\(^1\) Throughout this paper, for any two vectors \( u := (u_1, \ldots, u_N) \) and \( v := (v_1, \ldots, v_N) \) in \( \mathbb{R}^N \), we use \( u \cdot v \) to denote the standard dot product \( u_1 v_1 + \cdots + u_N v_N \).
Theorem 1.5 Suppose $n$ is fixed. Then there is a polynomial-time algorithm that, for any input $(f, w) \in \mathbb{Q}[^{\sqrt{-1}}]_1 \times \mathbb{Q}^n$ with $f$ a $t$-nomial, outputs the closure of the unique cell $\sigma_w$ of $\varnothing \backslash \text{ArchTrop}(f)$ (or $\text{ArchTrop}(f)$) containing $w$, described as an explicit intersection of $O(t^2)$ half-spaces.

The importance of Theorem 1.4 is that deciding whether an input point $w$ lies in an input Amoeba $(f)$, even restricting to the special case $n = 1$, is already $\text{NP}$-hard [AKNR13]. Theorem 1.5 enables us to find explicit regions, containing a given query point $w$, where $f$ can not vanish. As we will see later in Sections 2.2 and 2.3, improving Theorems 1.4 and 1.5 to polynomial dependence in $n$ leads us to deep questions in Diophantine approximation and the complexity of linear optimization.

It is thus natural to speculate that tropical varieties can be useful for localized polynomial system solving, i.e., estimating how far the nearest root of a given system of $n$-variate polynomials $f_1, \ldots, f_k$ is from an input point $x \in \mathbb{C}$. Our framework indeed enables new positive and negative results on this problem.

Theorem 1.6 Suppose $n$ is fixed. Then there is a polynomial-time algorithm that, for any input $k$ and $(f_1, \ldots, f_k, w) \in (\mathbb{Q}[^{\sqrt{-1}}]_1 \times \mathbb{Q}^n) \times \mathbb{Q}^n$, outputs the closure of the unique cell $\sigma_w$ of $\varnothing \backslash \bigcup_{i=1}^k \text{ArchTrop}(f_i)$ (or $\text{ArchTrop}(f_i)$) containing $w$, described as an explicit intersection of half-spaces. However, if $n$ is allowed to vary, then deciding whether $\sigma_w$ has a vertex in $\bigcap_{i=1}^n \text{ArchTrop}(f_i)$ is $\text{NP}$-hard.

We will see in Section 3 how the first assertion is useful for finding special start-points for Newton Iteration and Homotopy Continuation that sometimes enable the approximation of just the roots with norm vector near $(e^{w_1}, \ldots, e^{w_n})$.

The second assertion can be considered as a refined tropical analogue to a classical algebraic complexity result: deciding whether an arbitrary input system of polynomials equations (with integer coefficients) has a complex root is $\text{NP}$-hard [GJ79]. However, in light of the recent partial solutions to Smale’s 17th Problem [BP09, BC10] (showing that randomization and approximation help us evade $\text{NP}$-hardness for average-case inputs), we suspect that an analogous speed-up is possible in the tropical case as well.

On the practical side, we point out that the algorithms underlying Theorems 1.4–1.6 are quite easily implementable. (A preliminary MatLab implementation of our algorithms is available upon request.) Initial experiments, discussed in Section 3 below, indicate that a large-scale implementation could be a worthwhile companion to existing polynomial system solving software.

Theorems 1.4, 1.5, and 1.6 are respectively proved in Sections 5, 4, and 6. Before moving on to the necessary technical background, let us first clarify our underlying input size and point out some historical context.

Definition 1.7 We define the size of a polynomial $f \in \mathbb{Z}[x_1, \ldots, x_n]$, written $f(x) = \sum_{i=1}^t c_i x_1 a_i$, to be $\text{size}(f) := \sum_{i=1}^t \log \left(2 + |c_i| \prod_{j=1}^t (2 + |a_{ij}|)\right)$, where $a_i = (a_{i1}, \ldots, a_{in})$ for all $i$. Similarly, we define the input size of a point $(v_1, \ldots, v_n) \in \mathbb{Q}^n$ as the sum of sizes of the numerators and denominators of the $v_i$ (written in lowest terms), and thus extend the notion of input size to polynomials in $\mathbb{Q}[x_1, \ldots, x_n]$. Considering real and imaginary parts, and summing the respect sizes, we then extend the definition of input size further still to polynomials in $\mathbb{Q}[^{\sqrt{-1}}]_1 [x_1, \ldots, x_n]$. 

Remark 1.8 Note that $\text{size}(f)$ is, up to a bounded multiple, the sum of the bit-sizes of all the coefficients and exponents of $f$. Put even more simply, assuming we write integers as usual in some fixed base, and we write rational numbers as fractions in lowest terms, $\text{size}(f)$ is asymptotically the same as the amount of ink needed to write out $f$ as a sum of monomial terms. We extend our definition of size to a system of polynomials $F := (f_1, \ldots, f_k)$ in the obvious way by setting $\text{size}(F) := \sum_{i=1}^k \text{size}(f_i)$. Thus, for example, the size of an input in Theorem 1.6 is $\text{size}(w) + \sum_{i=1}^k \text{size}(f_i)$.

Via a slight modification of the classical Horner’s Rule [CKS99], it is easy to see that the number of ring operations needed to evaluate an arbitrary $f$ at an arbitrary $x \in \mathbb{C}^n$ easily admits an $O(\text{size}(f)^2)$ upper bound.²

Remark 1.9 The definition of input size we use implies that our preceding algorithms yield a significant speed-up over earlier techniques: for an $n$-variate $t$-nomial $f$ of degree $d$, with $n$ and $t$ fixed, our algorithms have complexity polynomial in $\log d$. The best previous techniques from computational algebra, including recent advances on Smale’s 17th Problem [BP09, BC10], have complexity polynomial in $\frac{(d+n)!}{d!n!} \geq \min\{d^n, n^d\}$. 

² When just counting ring operations we can in fact ignore the contribution of the coefficient sizes.
Historical Notes Using convex and/or piecewise-linear geometry to understand solutions of algebraic equations can be traced back to work of Newton around 1676 [New76]. The earliest precursor we know to the $n = 1$ case of the metric estimate of Theorem 1.3 can be found in work of Ostrowski from around 1940 [Ost40, Cor. IX, pg. 143].

More recently, tropical geometry [EKL06, LS09, IMS09, BR10, MS13] has emerged as a rich framework for reducing deep questions in algebraic geometry to more tractable questions in polyhedral and piecewise-linear geometry. For instance, the combinatorial structure of amoebae was first observed by Gelfand, Kapranov, and Zelevinsky around 1994 [GKZ94].

Remark 1.10 The reader may wonder why we have not considered the phases of the root coordinates and focussed just on norms. The phase analogue of an amoeba is the co-amoeba, which has only recently been studied [HHP08, NP10, NS13, NS14]. While it is known that the phases of the coordinates of the roots of polynomial systems satisfy certain equidistribution laws (see, e.g., [Kho91, Thm. 1 (pp. 82–83), Thm. 2 (pp. 87–88), and Cor. 3′ (pg. 88)] and [AGS13]), there does not yet appear to be a phase analogue of $\text{ArchTrop}(f)$. Nevertheless, we will see in Section 3 that our techniques sometimes allow us to approximate actual complex roots, in addition to norms.

2 Background

2.1 Convex, Piecewise-Linear, and Tropical Geometrical Notions

Let us first recall the origin of the phrase “tropical geometry”, according to [Pin98]: the tropical semifield $\mathbb{R}_{\text{trop}}$ is the set $\mathbb{R} \cup \{-\infty\}$, endowed with with the operations $x \odot y := x + y$ and $x \oplus y := \max\{x, y\}$. The adjective “tropical” was coined by French computer scientists, in honor of Brazilian computer scientist Imre Simon, who did pioneering work with algebraic structures involving $\mathbb{R}_{\text{trop}}$. Just as algebraic geometry relates geometric properties of zero sets of polynomials to the structure of ideals in commutative rings, tropical geometry relates the geometric properties of certain polyhedral complexes (see Definition 2.7 below) to the structure of ideals in $\mathbb{R}_{\text{trop}}$.

In our setting, we work with a particular kind of tropical variety that, thanks to Theorem 1.3, approximates $\text{Amoeba}(f)$ quite well. For example, one can see directly that $\text{Amoeba}(0) = \text{ArchTrop}(0) = \mathbb{R}^n$ and, for any $c \in \mathbb{C}^*$ and $a \in \mathbb{Z}^n$, $\text{Amoeba}(cx^a) = \text{ArchTrop}(cx^a) = \emptyset$. The binomial case is almost as easy.

Proposition 2.1 For any $a \in \mathbb{Z}^n$ and non-zero complex $c_1$ and $c_2$, we have

$$\text{Amoeba}(c_1 + c_2x^a) = \text{ArchTrop}(c_1 + c_2x^a) = \{ w \in \mathbb{R}^n \mid a \cdot w = \log |c_1/c_2| \}.$$ 

Proof: If $c_1 + c_2x^a = 0$ then $|c_2x^a| = |c_1|$. We then obtain $a \cdot w = \log |c_1/c_2|$ upon taking logs and setting $w = \text{Log}|x|$. This proves that $\text{Amoeba}(c_1 + c_2x^a)$ is exactly the stated affine hyperplane. Similarly, since the definition of $\text{ArchTrop}(c_1 + c_2x^a)$ implies that we are looking for $w$ with $|c_2x^aw| = |c_1|$, we see that $\text{ArchTrop}(c_1 + c_2x^a)$ defines the same hyperplane.

While $\text{ArchTrop}(f)$ and $\text{Amoeba}(f)$ are always metrically close, $\text{ArchTrop}(f)$ need not be contained in, nor even have the same homotopy type as $\text{Amoeba}(f)$, in general.

Example 2.2.
Letting $f := 1 + x_1^2 + x_2^2 + x_1x_2 + x_1^2x_2 + x_1^2x_2^2 + x_1^3$ and $g := 0.1 + 0.2x_1^2 + 0.1x_1^4 + 10x_1x_2^2 + 0.001x_1x_2 + 0.01x_1^4x_2 + 0.1x_2^3 + 0.000005x_1^3$

we obtain the amoebae and tropical varieties (and more lightly shaded neighborhoods), restricted to $[-11, 11] \times [-9, 9]$, as respectively drawn on the left and right above. The outermost shape in the left-hand (resp. right-hand) illustration is a neighborhood of ArchTrop($f$) (resp. Amoeba($g$)).

We thus see that every point of Amoeba($f$) (resp. ArchTrop($g$)) lies well within a distance of $0.65$ (resp. $0.49$) of some point of ArchTrop($f$) (resp. Amoeba($g$)), safely within the distance $\log 7 < 1.946$ (resp. $13 \log 7 < 25.3$) guaranteed by the second (resp. first) bound of Theorem 1.3. Note in particular that ArchTrop($g$) has two holes while Amoeba($g$) has only a single hole.\footnote{For our purposes, a hole of a subset $S \subseteq \mathbb{R}^n$ will simply be a bounded connected component of the complement $\mathbb{R}^n \setminus S$.}

Given any $f$, one can always easily construct a family of deformations whose amoebae tend to ArchTrop($f$) in a suitable sense. This fact can be found in earlier papers of Viro and Mikhalkin, e.g., [Vir01, Mik04]. However, employing Theorem 1.3 here, we can give a 4-line proof.

**Theorem 2.3** For any $n$-variate $t$-nomial $f$ written $\sum_{i=1}^t c_i x^{a_i}$, and $s > 0$, define $f^{\ast s}(x) := \sum_{i=1}^t c_i^s x^{a_i}$. Then $\Delta(\frac{1}{s}\text{Amoeba}(f^{\ast s}), \text{ArchTrop}(f)) \to 0$ as $s \to +\infty$.

**Proof:** By Theorem 1.3, $\Delta(\text{Amoeba}(f^{\ast s}), \text{ArchTrop}(f^{\ast s})) \leq (2t - 3) \log(t - 1)$ for all $s > 0$. Since $|c_i e^{a_i \cdot w}| = |c_i e^{a_i \cdot w}| \iff |c_i e^{a_i \cdot w}| = |c_i e^{a_i \cdot w}|$, and similarly when “$\leq$” is replaced by “$>$”, we immediately obtain that ArchTrop($f^{\ast s}$) $= s\text{ArchTrop}(f)$. So then $\Delta(\text{Amoeba}(f^{\ast s}), \text{ArchTrop}(f^{\ast s})) = s\Delta(\frac{1}{s}\text{Amoeba}(f^{\ast s}), \text{ArchTrop}(f))$ and thus $\Delta(\frac{1}{s}\text{Amoeba}(f^{\ast s}), \text{ArchTrop}(f)) \leq \frac{(2t - 3) \log(t - 1)}{s}$ for all $s > 0$. \hfill $\blacksquare$

To more easily link ArchTrop($f$) with polyhedral geometry we will need two variations of the classical Newton polygon. First, let us use Conv($S$) to denote the convex hull of $S$ a subset $S \subseteq \mathbb{R}^n$. $O := (0, \ldots, 0)$, and $|N| := \{1, \ldots, N\}$. Recall also that a polytope is the convex hull of a finite point set, a (closed) half-space is any set of the form $\{w \in \mathbb{R}^n \mid a \cdot w \leq b\}$ (for some $b \in \mathbb{R}$ and $a \in \mathbb{R}^n \setminus \{O\}$), and a (closed) polyhedron is any finite intersection of (closed) half-spaces. It is a basic fact from convex geometry that every polytope is a polyhedron, but not vice-versa [Gru03, Zie95].

**Definition 2.4** Given any $n$-variate $t$-nomial $f$ written $\sum_{i=1}^t c_i x^{a_i}$, we define its (ordinary) Newton polytope to be Newt($f$) := Conv($\{a_i\}_{i \in |N|}$), and the Archimedean Newton polytope of $f$ to be ArchNewt($f$) := Conv($\{(a_i, -\log|c_i|)\}_{i \in |N|}$).

Also, for any polyhedron $P \subset \mathbb{R}^N$ and $v \in \mathbb{R}^N$, we define the face of $P$ with outer normal $v$ to be $P^v := \{x \in P \mid v \cdot x$ is maximized$\}$. The dimension of $P$, written dim$P$, is simply the dimension of the smallest affine linear subspace containing $P$. Faces of $P$ of dimension 0, 1, and dim$P - 1$ are respectively called vertices, edges, and facets. ($P$ is called the improper face of $P$ and we set dim$\emptyset = -1$.) Finally, we call any face of $P$ lower if and only if it has an outer normal $\langle w_1, \ldots, w_N \rangle$ with $w_N < 0$, and we let the lower hull of ArchNewt($f$) be the union of the lower faces of ArchNewt($f$). \hfill $\diamond$

Note that ArchNewt($f$) usually has dimension 1 greater than that of Newt($f$). ArchNewt($f$) enables us to relate ArchTrop($f$) to linear programming, starting with the following observation.

**Proposition 2.5** For any $n$-variate $t$-nomial $f$, ArchTrop($f$) also has the equivalent definition

\{w \in \mathbb{R}^n \mid \langle w, -1 \rangle$ is an outer normal of a positive-dimensional face of ArchNewt($f$)\}.  

**Proof:** The quantity $|c_i e^{a_i \cdot w}|$ being maximized at at least two indices is equivalent to the linear form with coefficients $\langle w, -1 \rangle$ being maximized at at least two difference points in $\{(a_i, -\log|c_i|)\}_{i \in |N|}$. Since a face of a polytope is positive-dimensional if and only if it has at least two vertices, we are done. \hfill $\blacksquare$

**Example 2.6** The Newton polytope of our first example, $f = 1 + x_1^2 + x_2^2 - 3x_1x_2$, is simply the convex hull of the exponent vectors of the monomial terms: Conv($\{(0, 0), (3, 0), (0, 2), (1, 1)\}$). For the Archimedian Newton polytope, we take the coefficients into account via an extra coordinate: ArchNewt($f$) = Conv($\{(0, 0, 0), (3, 0, 0), (0, 2, 0), (1, 1, -\log 3)\}$).

In particular, Newt($f$) is a triangle and ArchNewt($f$) is a triangular pyramid with base Newt($f$) $\times \{0\}$ and apex lying beneath Newt($f$) $\times \{0\}$. Note also that the image of the orthogonal projection of the lower hull of ArchNewt($f$) onto $\mathbb{R}^2 \times \{0\}$ naturally induces a triangulation of Newt($f$), as illustrated to the right. \hfill $\diamond$
Our last example motivates us to consider more general subdivisions and duality. (An outstanding reference is [dLRS10].) Recall that a \(k\)-simplex is the convex hull of \(k+1\) points in \(\mathbb{R}^N\) not lying in any \((k-1)\)-dimensional affine linear subspace of \(\mathbb{R}^N\). A simplex is then simply a \(k\)-simplex for some \(k\).

**Definition 2.7** A polyhedral complex is a collection of polyhedra \(\Sigma = \{\sigma_i\}\), such that for all \(i\) we have (a) every face of \(\sigma_i\) is in \(\Sigma\) and (b) for all \(j\) we have that \(\sigma_i \cap \sigma_j\) is a face of both \(\sigma_i\) and \(\sigma_j\). (We allow empty and improper faces.) The \(\sigma_i\) are the cells of the complex, and the underlying space of \(\Sigma\) is \(|\Sigma| := \bigcup_i \sigma_i\).

A polyhedral subdivision of a polyhedron \(P\) is then simply a polyhedral complex \(\Sigma = \{\sigma_i\}\), with \(|\Sigma| = P\). We call \(\Sigma\) a triangulation if and only if every \(\sigma_i\) is a simplex. Given any finite subset \(A \subset \mathbb{R}^n\), a polyhedral subdivision of \(A\) is then just a polyhedral subdivision of \(\text{Conv}(A)\) where the vertices of the \(\sigma_i\) all lie in \(A\). Finally, the polyhedral subdivision of \(\text{Newt}(f)\) induced by \(\text{ArchNewt}(f)\), denoted \(\Sigma_f\), is simply the polyhedral subdivision whose cells are \(\{\pi(Q) \mid Q\ \text{is a lower face of ArchNewt}(f)\}\), where \(\pi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n\) denotes the orthogonal projection forgetting the last coordinate.

Recall that a (pointed polyhedral) cone is just the set of all nonnegative linear combinations of a finite set of points. Such cones are easily seen to always be polyhedra [Grü03, Zie95]. Recall also that a bijection, \(\phi\), between two finite sets \(A\) and \(B\) is just a function \(\phi : A \rightarrow B\) such that the cardinalities of \(A\), \(B\), and \(f(A)\) are all equal.

**Example 2.8** The illustration from Example 2.6 shows a triangulation of the point set \(\{(0, 0), (3, 0), (0, 2), (1, 1)\}\) which happens to be \(\Sigma_f\) for \(f = 1 + x_1^3 + x_2^3 - 3x_1x_2\). More to the point, it is easily checked that the outer normals to a face of dimension \(k\) of \(\text{ArchNewt}(f)\) form a cone of dimension \(3 - k\). In this way, thanks to the natural partial ordering of cells in any polyhedral complex by inclusion, we get an order-reversing bijection between the cells of \(\Sigma_f\) and pieces of \(\text{ArchTrop}(f)\).

That \(\text{ArchTrop}(f)\) is always a polyhedral complex follows directly from Proposition 2.5 above. It is then easy to show that there is always an order-reversing bijection between the cells \(\Sigma_f\) and the cells of \(\text{ArchTrop}(f)\) — an incarnation of polyhedral duality [Zie95].

**Example 2.9** We illustrate the preceding order-reversing bijection of cells through our first three tropical varieties, and corresponding subdivisions \(\Sigma_f\) of \(\text{Newt}(f)\), below:

Note that the vertices of \(\text{ArchTrop}(f)\) correspond bijectively to the 2-dimensional cells of \(\Sigma_f\), and the 1-dimensional cells of \(\text{ArchTrop}(f)\) correspond bijectively to the edges of \(\Sigma_f\). (In particular, the rays of \(\text{ArchTrop}(f)\) are perpendicular to the edges of \(\text{Newt}(f)\).) Note also that the vertices of \(\Sigma_f\) correspond bijectively to connected components of the complement \(\mathbb{R}^2 \setminus \text{ArchTrop}(f)\). We have taken the liberty of slightly distorting the right-most illustration to make the bijections clearer.

### 2.2 The Complexity of Linear Programming

Let us first point out that [Pap95, AB09, Sip12] are outstanding references for further background on the classical Turing model and NP-completeness. Let us now focus on some well-known late-20th century results on the complexity of linear optimization. These results are covered at much greater length in [Sch86, GLS93].
**Definition 2.10** Let \( \mathbb{R}^N_+ := \{ (x_1, \ldots, x_N) \in \mathbb{R}^n \mid x_1, \ldots, x_N \geq 0 \} \) denote the nonnegative orthant. Given \( M \in \mathbb{R}^{k \times N} \) with linearly independent rows, \( c = (c_1, \ldots, c_N) \in \mathbb{R}^N \), and \( b = (b_1, \ldots, b_k) \in \mathbb{R}^k \), the (standard form) linear optimization problem \( \mathcal{L}(M, b, c) \) is the following problem:

\[
\begin{align*}
\text{Maximize } & \quad c \cdot x \\
\text{subject to: } & \quad Mx = b \\
& \quad x \in \mathbb{R}^N_+ 
\end{align*}
\]

We then define \( \text{size}(\mathcal{L}(M, b, c)) := \text{size}(M) + \text{size}(b) + \text{size}(c) \). The set of all \( x \in \mathbb{R}^N_+ \) satisfying \( Mx = b \) is the feasible region of \( \mathcal{L}(M, b, c) \). We call \( \mathcal{L}(M, b, c) \) feasible if and only if there is no \( x \in \mathbb{R}^N_+ \) satisfying \( Mx = b \). Finally, if \( \mathcal{L}(M, b, c) \) is feasible but does not admit a well-defined maximum, then we call \( \mathcal{L}(M, b, c) \) unbounded. ☐

**Theorem 2.11** Given any linear optimization problem \( \mathcal{L}(M, b, c) \) as defined above, we can decide infeasibility, unboundedness, or (if \( \mathcal{L}(M, b, c) \) is feasible) find an optimal solution \( x^* \), all within time polynomial in size(\( \mathcal{L}(M, b, c) \)).

In particular, if \( \mathcal{L}(M, b, c) \) is feasible, we can find an optimal solution \( x^* \) of size polynomial in size(\( \mathcal{L}(M, b, c) \)). ☐

Theorem 2.11 goes back to work of Khachiyan in the late 1970s on the Ellipsoid Method, building upon earlier work of Shor, Yudin, and Nemirovski [Sch86]. Since then, Interior Point Methods have emerged as one of the most practical methods attaining the complexity bound asserted in Theorem 2.11. For simplicity, we will not focus on the best current complexity bounds, since we simply want to prove polynomiality for our algorithms in this paper. Further discussion on improved complexity bounds for linear optimization can be found in [MT02].

Any system of linear inequalities, at the expense of a minor increase in size, is essentially equivalent to the feasible region of some \( \mathcal{L}(M, b, c) \). In what follows, \( Mx \leq b \) is understood to mean that \( M_1 \cdot x \leq b_1, \ldots, M_k \cdot x \leq b_k \) all hold, where \( M_i \) denotes the \( i \)-th row of \( M \).

**Proposition 2.12** Given \( M \in \mathbb{R}^{k \times N} \) and any collection of inequalities of the form \( M_i \cdot x \leq b_i \), there is a standard form linear optimization problem \( \mathcal{L}(M, b, c) \), satisfying \( \text{size}(\mathcal{L}(M, b, c)) \leq 2(\text{size}(M) + \text{size}(b)) + k \), that is feasible if and only if \( \{ x \in \mathbb{R}^n \mid Mx \leq b \} \) is non-empty. ☐

There is thus no loss of generality in restricting to standard form.

We will frequently work with polyhedra given explicitly in the form \( P = \{ x \in \mathbb{R}^n \mid Mx \leq b \} \) (usually called \( \mathcal{H} \)-polytopes), and use Proposition 2.12 and Theorem 2.11 together to rapidly decide various basic questions about \( P \). For instance, we call a constraint \( M_i \cdot x \leq b_i \) of \( Mx \leq b \) redundant if and only if the corresponding row of \( M \) can be deleted from \( M \) without affecting \( P \).

**Lemma 2.13** Given any system of linear inequalities \( Mx \leq b \) we can, in time polynomial in \( \text{size}(M) + \text{size}(b) + \text{size}(c) \), find a submatrix \( M' \) of \( M \) (and a subvector \( b' \) obtained by deleting the corresponding entries from \( b \)) such that \( \{ x \in \mathbb{R}^n \mid M'x \leq b' \} = \{ x \in \mathbb{R}^n \mid Mx \leq b \} \) and \( M'x \leq b' \) has no redundant constraints. ☐

The new set of inequalities \( M'x \leq b' \) is called an irredundant representation of \( Mx \leq b \).

A deep subtlety underlying linear optimization is whether \( \mathcal{L}(M, b, c) \) can be solved in strongly polynomial-time, i.e., is there an analogue of Theorem 2.11 where we instead count arithmetic operations to measure complexity, and obtain complexity polynomial in \( k + N \)?

One of the first successful algorithms for linear optimization — the Simplex Method — has arithmetic complexity \( O(N^k) \), and there are now variations of the Simplex Method (using sophisticated pivoting rules) that attain arithmetic complexity sub-exponential in \( k \). (It was also discovered in the 1970s by Borgwardt and Smale that the simplex method is strongly polynomial provided one averages over a suitable distribution of inputs [Sch86].) Strong polynomiality remains an important open problem and is in fact Problem 9 on Fields Medalist Steve Smale’s list of mathematical problems for the 21st Century [Sma98, Sma00].

These issues are actually relevant to polynomial system solving since the linear optimization problems we ultimately solve will have irrational “right-hand sides”: \( b \) will usually be a (rational) linear combination of logarithms of integers in our setting.

In particular, as is well-known in Diophantine Approximation [Bak77], it is far from trivial to efficiently decide the sign of such an irrational number. This problem is also easily seen to be equivalent to deciding inequalities of the form \( \alpha_1^{\beta_1} \cdots \alpha_r^{\beta_r} > 1 \), where the \( \alpha_i \) and \( \beta_i \) are integers. Note, in particular, that while the number of arithmetic operations necessary to decide such an inequality is easily seen to be \( O((\sum_{i=1}^r \log |\beta_i|)^2) \) (via the classical binary method of exponentiation), taking bit-operations into account naively results in a problem that appears to have complexity exponential
in \( \log |\beta_1| + \cdots + \log |\beta_n| \). Fortunately, another Fields Medalist, Alan Baker, made major progress on this problem in the late 20th century.

### 2.3 Irrational Linear Optimization and Approximating Logarithms Well Enough

Recall the following result on comparing monomials in rational numbers.

**Theorem 2.14** [BRS09, Sec. 2.4] Suppose \( \alpha_1, \ldots, \alpha_n \in \mathbb{Q} \) are positive and \( \beta_1, \ldots, \beta_n \in \mathbb{Z} \). Also let \( A \) be the maximum of the numerators and denominators of the \( \alpha_i \) (when written in lowest terms) and \( B := \max_i(|\beta_i|) \). Then, within 
\[
O(n30^n \log(B) (\log \log B)^2 \log \log \log (B)(\log(A)(\log \log A)^2 \log \log \log A)^n)
\]
bit operations, we can determine the sign of \( \alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n} - 1 \). □

While the underlying algorithm is a simple application of Arithmetic-Geometric Mean Iteration (see, e.g., [Ber03]), its complexity bound hinges on a deep estimate of Nesterenko [Nes03], which in turn refines seminal work of Matveev [Mat00] and Alan Baker [Bak77] on linear forms in logarithms. Whether the dependence on \( n \) in the bound above can be improved to polynomial is a very deep open question related to the famous abc-Conjecture [Bak98, Nit].

Via the Simplex Method, or even a brute force search through all basic feasible solutions of \( \mathcal{L}(M', b', c') \), we can obtain the following consequence of Theorems 2.11 and 2.14.

**Corollary 2.15** Suppose \( n \) is fixed, \( k \leq n, M \in \mathbb{Q}_{\leq n}^{k \times n} \), and \( b_i := \log |\beta_i| \) with \( \beta_i \in \mathbb{Q}^* \) for all \( i \in [k] \), and we set \( b := (b_1, \ldots, b_k) \). Then we can determine feasibility for \( Mx \leq b \), and compute an irredundant representation \( M'x \leq b' \) for \( Mx \leq b \), in time polynomial in size(\( M \)) + size(\( b \)). □

The key trick behind the proof of Corollary 2.15 is that, after converting to standard form, any basic feasible solution of the underlying linear optimization problem has all its irrationalities concentrated on the right-hand side. In particular, standard linear algebra bounds tell us that the right-hand side involves a linear combination of logarithms with coefficients of size polynomial in the input size.

### 3 Tropical Start-Points for Numerical Iteration and an Example

We begin by outlining a method for picking start-points for Newton Iteration (see, e.g., [BCSS98, Ch. 8] for a modern perspective) and Homotopy Continuation [HL95, SW05, Ver10, LL11, BHSW13]. While we do not discuss these methods for solving polynomial equations in detail, let us point out that Homotopy Continuation (combined with Smale’s \( \alpha \)-Theory for certifying roots [BCSS98, BHSW13]) is currently the fastest and most reliable method for numerically solving polynomial systems in complete generality. Other important methods include Resultants [EC95] and Gröbner Bases [FHP03]. However, while these alternative methods are of great importance in certain algebraic and theoretical applications [AKS13, FGHR13], Homotopy Continuation is currently the method of choice for practical large-scale numerical computation.

While the boxed steps below admit a simple and easily parallelizable brute-force search, they form the portion of the algorithm that is the most challenging to speed up to complexity polynomial in \( n \).

**Algorithm 3.1** (Coarse Approximation to Roots with Log-Norm Vector Near a Query Point)

**Input.** Polynomials \( f_1, \ldots, f_n \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \), with \( f_i(x) = \sum_{j=1}^{b_i} c_{i,j} x^{e_{i,j}(i)} \) a \( 1 \)-nomial for all \( i \), and a query point \( w \in \mathbb{R}^n \).

**Output.** An ordered \( n \)-tuple of sets of indices \( (J_i)_{i=1}^n \) such that \( g_i := \sum_{j \in J_i} c_{i,j} x^{e_{i,j}(i)} \) is a sub-summand of \( f_i \), and the roots of \( G := (g_1, \ldots, g_n) \) are near the roots of \( F := (f_1, \ldots, f_n) \) with log-norm vector near \( w \).

**Description.**

1. Let \( \sigma_w \) be the closure of the unique cell of \( \mathbb{R}^n \setminus \bigcup_{i=1}^n \text{ArchTrop}(f_i) \) or \( \bigcap_{\text{ArchTrop}(f_i) \ni w} \text{ArchTrop}(f_i) \) containing \( w \).

2. If \( \sigma_w \) has no vertices in \( \bigcap_{i=1}^n \text{ArchTrop}(f_i) \) then output an irredundant collection of facet inequalities for \( \sigma_w \), output "There are no roots of \( F \) in \( \sigma_w \)," and STOP.

3. Otherwise, fix a vertex \( v \) of \( \sigma_w \) and, for each \( i \in [n] \), let \( E_i \) be any edge of \( \text{ArchNewt}(f_i) \) generating a facet of...
The connection to Homotopy Continuation is also simple: use the pair equations in the logarithms of the original variables. In particular, an $n$ always results in a collection of roots all sharing a coefficients), all the $g$ are binomials, and binomial systems are particularly easy to solve: they are equivalent to linear equations in the logarithms of the original variables. In particular, an $n \times n$ binomial system output by our algorithm always results in a collection of roots all sharing a single vector of norms.

**Remark 3.2** The output system $G$ is useful because, with probability 1 (for most reasonable distributions on the coefficients), all the $g_i$ are binomials, and binomial systems are particularly easy to solve: they are equivalent to linear equations in the logarithms of the original variables. In particular, an $n \times n$ binomial system output by our algorithm always results in a collection of roots all sharing a single vector of norms.

The connection to Newton Iteration is then easy to state: use any root of $G$ as a start-point $z(0)$ for the iteration $z(n+1) := z(n) - \text{Jac}(F)^{-1}(z(n))F(z(n))$.

The connection to Homotopy Continuation is also simple: use the pair $(G, \xi)$ (for any root $\xi$ of $G$) to start a path converging (under the usual numerical conditioning assumptions on whatever predictor-corrector method one is using) to a root of $F$ with log-norm vector near $w$. Note that it is safer to do the extra work of Homotopy Continuation, but there will be cases where the tropical start-points from Algorithm 3.1 are sufficiently good that Newton Iteration is enough to converge to a true root.

Note in particular that we have the freedom to follow as few start-points, or as few paths, as we want. When our start-points (resp. paths) indeed converge to nearby roots, we obtain a tremendous savings over having to follow all start-points (resp. paths).

**Definition 3.3** Following the notation of Theorem 1.6 and Algorithm 3.1, we call a vertex $v$ of $\sigma_w$, mixed if and only if it lies in $\bigcap_{i=1}^n \text{ArchTrop}(f_i)$.

**Example 3.4** Let us make a $2 \times 2$ polynomial system out of our first and third examples:

\[
\begin{align*}
  f_1 &:= 1 + x_1^3 + x_2^2 - 3x_1x_2 \\
  f_2 &:= 0.1 + 0.2x_2^3 + 0.1x_2^2 + 10x_1x_2^2 + 0.001x_1x_2^3 + 0.01x_1^3x_2 + 0.1x_1^2x_2^2 + 0.000005x_1^3
\end{align*}
\]

The system $F := (f_1, f_2)$ has exactly 12 roots in $\mathbb{C}^2$, the coordinate-wise log-norms of which form the small clusters near certain intersections of $\text{ArchTrop}(f_1)$ and $\text{ArchTrop}(f_2)$.

In particular, there is a heptagonal cell, which we have magnified, with 2 vertices close to the log-norm vectors of some of the roots. This cell, which looks hexagonal because it has a pair of vertices that are too close to distinguish visually, happens to be $\sigma_w$ for $w = (2, 1)$. Note that $\sigma_w$ has exactly 2 mixed vertices.

Applying Algorithm 3.1 to our $(f_1, f_2, w)$ we then have 2 possible outputs, depending on which mixed vertex of $\sigma_w$ we pick. The output corresponding to the circled vertex is the pair of index sets $(\{2, 3\}, \{3, 4\})$. More concretely, Algorithm 3.1 alleges that the system

\[
G := (g_1, g_2) := (x_1^3 + x_2^2, 0.1x_1^2 + 10x_1x_2^2)
\]

has roots with log-norm vector near a log-norm vector of a root of $F$ that is in turn close to $w$. Indeed, the sole log-norm vector coming from the roots of $G$ is $\text{log10}(10, 0.9 \text{log} 10)$ and the roots themselves are $\{(\pm 10, \sqrt{1000})\}$ (with both values of the square root allowed). All 4 roots in fact converge (under Newton iteration, with no need for Homotopy

---

5 The root count was verified via an exact Gröbner basis calculation using the commercial software package Maple 14. Numerical approximation of the log-norm vectors to accuracy $10^{-4}$ per coordinate was then done via the publically available package Bertini [BHSW13], using default settings. Both calculations took a fraction of a second. The cell $\sigma_w$ was computed via Matlab 7.11.0 (R2010b).
Continuation) to true roots of $F$: $(-10, 1000^{1/2})$ and $(-10, -1000^{1/2})$ respectively converge to the roots of $F$ with closest and third closest log-norm vector to $w$. The other two roots of $G$ converge to a conjugate pair of roots of $F$ with log-norm vector $(2.4139, 3.5103)$ (to 4 decimal places) lying in the small circle. ◦

**Remark 3.5** The cell $\sigma_w$ from Step 1 can be found in polynomial-time, thanks to Theorem 1.5, and its underlying algorithm contained in Corollary 2.15.

As for Steps 2 and 3, thanks to duality, the facets of ArchTrop($f_i$) correspond exactly to lower edges of ArchNewt($f_i$). So, to find the vertex $v$ (or decide that it doesn’t exist), it suffices to do a brute-force search through all $n$-tuples of lower edges, one coming from each of ArchNewt($f_1$), …, ArchNewt($f_n$). This particular kind of geometric computation has its origins in the algorithmic study of mixed volume [EC95, LL11]. There are various ways of speeding up this search and there is much interesting computational geometry to be studied in this direction. ◦

Let us be clear that we have not yet proved a metric guarantee for Algorithm 3.1 in the spirit of Theorem 1.3. Rigorous results in this direction, as well as a broad experimental understanding of our techniques, are of the utmost importance and we hope to address these points in the near future.

**Remark 3.6** We have intentionally written Algorithm 3.1 in terms of a more general class of inputs than necessary for our examples. For such general inputs, it makes more sense to measure complexity in terms of arithmetic operations instead of bit operations. ◦

**Remark 3.7** The reader should be aware that while we have relied upon Diophantine approximation and subtle aspects of the Simplex Method to prove our bit-complexity bounds, one can certainly be more flexible when using our approach in practical, floating-point computations. For instance, heuristically, it appears that one can get away with less accuracy than stipulated by Theorem 2.14 when comparing linear combinations of logarithms. Similarly, one should feel free to use the fastest (but still reasonably accurate) algorithms for linear optimization when applying our methods to large-scale polynomial systems. ◦

### 4 Proof of Theorem 1.5

Using $t - 1$ comparisons, we can isolate all indices $i$ such that $\max |c_i e^{w_i}|$ is attained. Thanks to Theorem 2.14, this can be done in polynomial-time. We then obtain, say, $J$ equations of the form $a_i \cdot w = -\log |c_i|$ and $K$ inequalities of the form $a_i \cdot w > -\log |c_i|$ or $a_i \cdot w < -\log |c_i|$.

Thanks to Lemma 2.13, combined with Corollary 2.15, we can determine the exact cell of ArchTrop($f$) containing $w$ if $J \geq 2$. Otherwise, we obtain the unique cell of $\mathbb{R}^n \setminus$ ArchTrop($f$) containing $w$. Note also that an $(n - 1)$-dimensional face of either kind of cell must be the dual of an edge of ArchNewt($f$). Since every edge has exactly 2 vertices, there are at most $(t - 1)/2$ such $(n - 1)$-dimensional faces, and thus $\sigma_w$ is the intersection of at most $(t - 1)/2$ half-spaces. So we are done. □

**Remark 4.1** Theorem 1.5 also generalizes an earlier complexity bound for deciding membership in ArchTrop($f$) from [AKNR13]. ◦

### 5 Proof of Theorem 1.4

Since ArchTrop($f$) and Amoeba($f$) are closed, $\Delta(w, \text{ArchTrop}(f)) = |w - v|$ for some point $v \in \text{ArchTrop}(f)$ and $\Delta(w, \text{Amoeba}(f)) = |w - u|$ for some point $u \in \text{Amoeba}(f)$.

Now, by the second upper bound of Theorem 1.3, there is a point $v' \in \text{ArchTrop}(f)$ within distance $\log(t - 1)$ of $u$. Clearly, $|w - v| \leq |w - v'|$. Also, by the Triangle Inequality, $|w - v'| \leq |w - u| + |u - v'|$. So then,

$$\Delta(w, \text{ArchTrop}(f)) \leq \Delta(w, \text{Amoeba}(f)) + \log(t - 1),$$

and thus $\Delta(w, \text{Amoeba}(f)) - \Delta(w, \text{ArchTrop}(f)) \geq -\log(t - 1)$.

Similarly, by the first upper bound of Theorem 1.3, there is a point $u' \in \text{Amoeba}(f)$ within distance $(2t - 3)\log(t - 1)$ of $v$. Clearly, $|w - u| \leq |w - u'|$. Also, by the Triangle Inequality, $|w - u'| \leq |w - v| + |v - u'|$. So then, $\Delta(w, \text{Amoeba}(f)) \leq \Delta(w, \text{ArchTrop}(f)) + (2t - 3)\log(t - 1)$, and thus
\[ \Delta(w, \text{Amoeba}(f)) - \Delta(w, \text{ArchTrop}(f)) \leq (2t - 3) \log(t - 1). \]

So our first assertion is proved.

Now, if \( f \) has coefficients with rational real and imaginary parts, Theorem 1.5 tells us that we have an explicit description of \( \sigma_w \) as the intersection of a number of half-spaces polynomial in the input size. Moreover, the bit-sizes of the coefficients of the underlying inequalities are also polynomial in the input size. So we can compute the distance \( D \) from \( w \) to ArchTrop(\( f \)) by finding which facet of \( \sigma_w \) has minimal distance to \( w \). The distance from \( w \) to any such facet can be computed in polynomial-time via the classical formula for distance between a point and an affine hyperplane, and Theorem 2.14:

\[ \Delta(w, \{ x \mid \alpha \cdot x = \beta \}) = \frac{|\alpha w| - \text{sign}(\alpha w)\beta}{|\alpha|} \]

In particular, we may efficiently approximate \( D \) by efficiently approximating the underlying square-roots and logarithms. The latter can be accomplished by Arithmetic-Geometric Iteration, as detailed in [Ber03]. So our statement on leading bits is proved.

The final assertion then follows easily: we merely decide whether \( \Delta(w, \text{ArchTrop}(f)) \) strictly exceeds \( \log(t - 1) \) or not, via the algorithm we just outlined. Thanks to our initial observations using the Triangle Inequality, it is clear that Output (b) or Output (a) occurs according as \( \Delta(w, \text{ArchTrop}(f)) > \log(t - 1) \) or not. \( \blacksquare \)

6 Proving of Theorem 1.6

6.1 Fast Cell Computation: Proof of the First Assertion

First, we apply Theorem 1.5 to \((f_i, w)\) for each \( i \in [k] \) to find which ArchTrop(\( f_i \)) contain \( w \).

If \( w \) lies in no ArchTrop(\( f_i \)), then we simply use Corollary 2.15 (as in our proof of Theorem 1.5) to find an explicit description of the closure of the cell of \( \mathbb{R}^n \setminus \bigcup_{i=1}^k \text{ArchTrop}(f_i) \) containing \( w \). Otherwise, we find the cells of ArchTrop(\( f_i \)) (over those \( i \) with ArchTrop(\( f_i \)) containing \( w \)) that contain \( w \). Then, applying Corollary 2.15 once again, we find the unique cell of \( \bigcap_{\text{ArchTrop}(f_i) \ni w} \text{ArchTrop}(f_i) \) containing \( w \).

Assume that \( f_i \) has exactly \( t_i \) monomial terms for all \( i \). In either of the preceding cases, the total number of half-spaces involved is no more than \( \sum_{i=1}^k t_i(t_i - 1)/2 \). So the over-all complexity of our redundancy computations is polynomial in the input size and we are done. \( \blacksquare \)

6.2 Hardness of Detecting Mixed Vertices: Proving the Second Assertion

It will clarify matters if we consider a related NP-hard problem for rational polytopes first, before moving on to cells with irrationalities.

6.2.1 Preparation over \( \mathbb{Q} \)

In the notation of Definition 3.3, let us first consider the following decision problem. We assume all polyhedra are given explicitly as finite collections of rational linear inequalities, with size defined as in Section 2.2.

MIXED-VERTEX:
Given \( n \in \mathbb{N} \) and polyhedra \( P_1, \ldots, P_n \) in \( \mathbb{R}^n \), does \( P := \bigcap_{i=1}^n P_i \) have a mixed vertex? \( \blacksquare \)

While MIXED-VERTEX can be solved in polynomial time when the dimension is fixed, we will show that, for \( n \) varying, the problem is NP-complete, even when restricting to the case where all polytopes are full-dimensional and \( P_1, \ldots, P_{n-1} \) are axes-parallel bricks.

Let \( e_i \) denote the \( i \)-th standard basis vector in \( \mathbb{R}^n \). Also, given \( \alpha \in \mathbb{R}^n \) and \( \beta \in \mathbb{R} \), we will use the following notation for certain hyperplanes and halfspaces in \( \mathbb{R}^n \) determined by \( \alpha \) and \( \beta \):

\[ H(\alpha, \beta) := \{ x \in \mathbb{R}^n \mid \alpha \cdot x = \beta \} \quad \text{and} \quad H^\geq(\alpha, \beta) := \{ x \in \mathbb{R}^n \mid \alpha \cdot x \leq \beta \}. \]
For \( i \in [n] \), let \( n, s_i \in \mathbb{N} \),

\[
M_i := [m_{i,1}, \ldots, m_{i,t_i}]^T \in \mathbb{Z}^{t_i \times n}, \quad \beta_i := (\beta_{i,1}, \ldots, \beta_{i,t_i}) \in \mathbb{Z}^{t_i}, \quad \text{and} \quad P_i = \{ x \in \mathbb{R}^n \mid M_i x \leq b_i \}.
\]

Since linear programming can be solved in polynomial-time (in the cases we consider) we may assume that the presentations \((s_i, M_i, b_i)\) are irredundant, i.e., \( P_i \) has exactly \( s_i \) facets and the sets \( P_i \cap H_{(a_{ij}, \beta_{ij})} \), for \( j \in [s_i] \), are precisely the facets of \( P_i \) for all \( i \in [n] \).

Now set \( P := \bigcap_{i=1}^n P_i \) and let \( v \in \mathbb{Q}^n \). Note that \( \text{size}(P) \) is thus linear in \( \sum_{i=1}^n \text{size}(P_i) \).

**Lemma 6.1** **Mixed-Vertex \( \in \text{NP}.**

**Proof:** Since the binary sizes of the coordinates of the vertices of \( P \) are bounded by a polynomial in the input size, we can use vectors \( v \in \mathbb{Q}^n \) of polynomial size as certificates. We can check in polynomial-time whether such a vector \( v \) is a vertex of \( P \) simply by exhibiting \( n \) facets (with linearly independent normal vectors), one from each \( P_i \), containing \( v \). If this is not the case, \( v \) cannot be a mixed-vertex of \( P \). Otherwise, \( v \) is a mixed-vertex of \( P \) if and only if for each \( i \in [n] \) there exists a facet \( F_i \) of \( P_i \) with \( v \in F_i \). Since the facets of the polytopes \( P_i \) admit polynomial-time descriptions as \( \mathbb{R}^d \)-polytopes, this can be checked by a total of \( m_1 + \ldots + m_n \) polytope membership tests.

So, we can check in polynomial-time whether a given certificate \( v \) is a mixed-vertex of \( P \). Hence **Mixed-Vertex** is in \( \text{NP} \). ■

Since, in fixed dimensions we can actually list all vertices of \( P \) in polynomial-time, one by one, it is clear that **Mixed-Vertex** can be solved in polynomial-time when \( n \) is fixed. When \( n \) is allowed to vary we obtain hardness:

**Theorem 6.2** **Mixed-Vertex is \( \text{NP-hard}.**

Recall that \( \sqcup \) denotes disjoint union. The proof of Theorem 6.2 will be based on a transformation from the following decision problem:

**Partition**

Given \( d \in \mathbb{N}, \alpha_1, \ldots, \alpha_d \in \mathbb{N} \), is there a partition \( d = I \sqcup J \) such that \( \sum_{i \in I} \alpha_i = \sum_{j \in J} \alpha_j \)? ■

Recall that **Partition** was on the original list of \( \text{NP} \)-complete problems from [Kar72].

Let an instance \((d; \alpha_1, \ldots, \alpha_d)\) of **Partition** be given, and set \( \alpha := (\alpha_1, \ldots, \alpha_d) \). Then we are looking for a point \( x \in \{ -1, 1 \}^d \) with \( \alpha \cdot x = 0 \).

We will now construct an equivalent instance of **Mixed-Vertex**. With \( n := d + 1 \), \( x := (\xi_1, \ldots, \xi_{n-1}) \) and \( \mathbf{1}_n := (1, \ldots, 1) \in \mathbb{R}^n \) let

\[
P_i := \left\{ \begin{bmatrix} x \\ \xi_i \end{bmatrix} \mid -1 \leq \xi_i \leq 1, \quad -2 \leq \xi_j \leq 2 \text{ for all } j \in [n] \setminus \{ i \} \right\}
\]

for \( i \in [n-1] \),

\[
P_n := \left\{ \begin{bmatrix} x \\ \xi_n \end{bmatrix} \mid -2 \cdot \mathbf{1}_{n-1} \leq x \leq 2 \cdot \mathbf{1}_{n-1}, \quad 1 \leq \xi_n \leq 1, \quad 0 \leq 2 \alpha \cdot x \leq 1 \right\},
\]

and set \( P := \bigcap_{i=1}^n P_i \), \( \mathbf{\hat{a}} := \begin{bmatrix} \alpha \\ 0 \end{bmatrix} \).

The next lemma shows that \( P_n \cap \{ -1, 1 \}^n \) still captures the solutions of the given instance of partition.

**Lemma 6.3** \((d; \alpha_1, \ldots, \alpha_d)\) is a “no”-instance of **Partition** if and only if \( P_n \cap \{ -1, 1 \}^n \) is empty.

**Proof:** Suppose, first, that \((d; \alpha_1, \ldots, \alpha_d)\) is a “no”-instance of **Partition**. If \( P_n \) is empty there is nothing left to prove. So, let \( y \in P_n \) and \( w \in \{ -1, 1 \}^{n-1} \times \mathbb{R} \). Since \( \alpha \in \mathbb{N}^d \) we have \( |\mathbf{\hat{a}} \cdot w| \geq 1 \). Hence, with the aid of the Cauchy-Schwarz inequality, we have

\[
1 \leq |\mathbf{\hat{a}} \cdot w| = |\mathbf{\hat{a}} \cdot y + \mathbf{\hat{a}} \cdot (w - y)| \leq |\mathbf{\hat{a}} \cdot y| + |\mathbf{\hat{a}} \cdot (w - y)|
\]

and thus \( |w - y| \geq \frac{1}{2|\alpha|} > 0 \). Therefore \( P_n \cap \{ -1, 1 \}^{n-1} \times \mathbb{R} \) is empty.

Now, let \( P_n \cap \{ -1, 1 \}^n = \emptyset \). Since \( \mathbf{\hat{a}} \in \mathbb{R}^{n-1} \times \{ 0 \} \) we have \( P_n \cap \{ -1, 1 \}^n = \emptyset \). ■

The next lemma reduces the possible mixed-vertices to the vertical edges of the standard cube.
Lemma 6.4 Following the preceding notation, let \( v \) be a mixed-vertex of \( P \). Then \( v \in \{-1, 1\}^{n-1} \times [-1, 1] \).

Proof: First note that \( Q := \bigcap_{i=1}^{n-1} P_i = [-1, 1]^{n-1} \times [-2, 2] \). Therefore, for each \( i \in [n-1] \), the only facets of \( P_i \) that meet \( Q \) are those in \( H_{(\epsilon_i, \pm 2)} \) and \( H_{(\epsilon_i, \pm 2)} \). Since \( P \subset [-1, 1]^n \), and for each \( i \in [n-1] \) the mixed-vertex \( v \) must be contained in a facet of \( P_i \), we have

\[
v \in [-1, 1]^n \cap \bigcap_{i=1}^{n-1} \left( \bigcup_{\delta \in \{-1, 1\}} H_{(\epsilon_i, \delta)} \right) = [-1, 1]^{n-1} \times [-1, 1],
\]

which proves the assertion. \( \blacksquare \)

The next lemma adds \( P_n \) to the consideration.

Lemma 6.5 Let \( v \) be a mixed-vertex of \( P \). Then \( v \in \{-1, 1\}^n \).

Proof: By Lemma 6.4, \( v \subset \{-1, 1\}^{n-1} \times [-1, 1] \). Since the hyperplanes \( H_{(\epsilon_n, \pm 2)} \) do not meet \([-1, 1]^n \),

\[
v \notin H_{(\epsilon_i, -1)} \cup H_{(\epsilon_i, 2)} \quad \text{for all } i \in [n-1].
\]

Hence, \( v \) can only be contained in the constraint hyperplanes \( H_{(\alpha, 0)} \cap H_{(\epsilon, -1)} \cup H_{(\epsilon, 1)} \). Since \( \hat{\alpha} \in \mathbb{R}^{n-1} \times \{0\} \), the vector \( \hat{\alpha} \) is linearly dependent on \( e_1, \ldots, e_{n-1} \). Hence, \( v \in H_{(\epsilon, -1)} \cup H_{(\epsilon, 1)} \), i.e., \( v \in \{-1, 1\}^n \). \( \blacksquare \)

Now we can prove the \( \text{NP} \)-hardness of \( \text{Mixed-Vertex} \).

Proof of Theorem 6.2: First, let \( (d; \alpha_1, \ldots, \alpha_d) \) be a “yes”-instance of \( \text{Partition} \), let \( x^* := (\xi_1^*, \ldots, \xi_{m-1}^*) \in \{-1, 1\}^{n-1} \) be a solution, and set

\[
\xi_m^* := 1, \quad v := \left[ x^* \right]_m, \quad F_i := H_{(\epsilon_i, \xi_i^*)} \cap P_i \quad \text{for all } i \in [n], \text{ and } \hat{F}_n := H_{(\alpha, 0)} \cap P_n.
\]

Then \( v \in \hat{F}_n \subset P_n \), hence \( v \in P \) and, in fact, \( v \) is a vertex of \( P \). Furthermore, \( F_i \) is a facet of \( P_i \) for all \( i \in [n] \), \( v \in \bigcap_{i=1}^n F_i \), and thus \( v \) is a mixed-vertex of \( P \).

Conversely, let \( (d; \alpha_1, \ldots, \alpha_d) \) be a “no”-instance of \( \text{Partition} \), and suppose that \( v \in \mathbb{R}^n \) is a mixed-vertex of \( P \).

By Lemma 6.5, \( v \in \{-1, 1\}^n \). Furthermore, \( v \) lies in a facet of \( P \). Hence, in particular, \( v \in P_n \), i.e., \( P_n \cap \{-1, 1\}^n \) is empty. Therefore, by Lemma 6.3, \( (d; \alpha_1, \ldots, \alpha_d) \) is a “yes”-instance of \( \text{Partition} \). This contradiction shows that \( P \) does not have a mixed-vertex.

Clearly, the transformation works in polynomial-time. \( \blacksquare \)

6.3 Proof of the Second Assertion of Theorem 1.6

We call a polyhedron \( \ell \)-rational if and only if it is of the form \( \{ x \in \mathbb{R}^n \mid Mx \leq b \} \) with \( M \in \mathbb{Q}^{k \times n} \) and \( b = (b_1, \ldots, b_k) \) satisfying \( b_1 = \beta_1 \log |\alpha_1| + \cdots + \beta_k \log |\alpha_k| \), with \( \beta_i, \alpha_j \in \mathbb{Q} \) for all \( i \) and \( j \). We measure the size of such a polyhedron as \( \text{size}(M) + \text{size}([b_{i,j}]) + \sum_{i=1}^k \text{size}(\alpha_i) \). Clearly, it suffices to show that the following variant of \( \text{Mixed-Vertex} \) is \( \text{NP} \)-hard:

**\text{Logarithmic-Mixed-Vertex}:**

Given \( n \in \mathbb{N} \) and \( \ell \)-rational polyhedra \( P_1, \ldots, P_n \subset \mathbb{R}^n \), does \( P := \bigcap_{i=1}^n P_i \) have a mixed vertex? \( \blacksquare \)

Via an argument completely parallel to the last section, the \( \text{NP} \)-hardness of \( \text{Logarithmic-Mixed-Vertex} \) follows immediately from the \( \text{NP} \)-hardness of the following variant of \( \text{Partition} \):

**\text{Logarithmic-Partition}**

Given \( d \in \mathbb{N}, \alpha_1, \ldots, \alpha_d \in \mathbb{N} \setminus \{0\} \), is there a partition \( d = I \cup J \) such that \( \sum_{i \in I} \log \alpha_i = \sum_{j \in J} \log \alpha_j \)? \( \blacksquare \)

We measure size in \( \text{Logarithmic-Partition} \) just as in the original \( \text{Partition} \) Problem: \( \sum_{i=1}^d \log \alpha_i \). Note that \( \text{Logarithmic-Partition} \) is equivalent to the obvious variant of \( \text{Partition} \) where we ask for a partition making the two resulting products be identical. The latter problem is easily seen to be \( \text{NP} \)-hard as well, via an argument mimicking the original proof of the \( \text{NP} \)-hardness of \( \text{Partition} \) in [Kar72]. \( \blacksquare \)
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