PREHEATING IN FRW UNIVERSES

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The nonlinear time evolution of the quantum fields is studied in the \( O(N) \) model for large \( N \) in a radiation dominated FRW universe, with a view towards the phenomenon of explosive particle production due to either spinodal instabilities or parametric amplification, i.e. preheating. Quantum backreaction effects due to the produced particles are included consistently within the large \( N \) approximation. We find that preheating persists when the expansion is included, although the amount of particle production is reduced compared to the values found in Minkowski space. We also see that the behavior of the evolving zero mode is very different from that in Minkowski space, though the late time behavior in all cases is determined by a sum rule that implies the existence of Goldstone bosons in the final state.

I. INTRODUCTION

A viable model of inflation must be able to reheat the Universe to high enough temperatures to regain the tested features of the standard Big-Bang model, such as nucleosynthesis. However, it has recently been noted \cite{1-3} that the theory of reheating as originally conceived \cite{4} requires revision due to abundant particle production originating from the coupling of the zero mode of the inflaton field to other modes of this same field or modes of a different field (see also refs. \cite{5-8}).

Preheating is an inherently non-equilibrium process and as such, any field theoretical treatment must take this into account. Preheating is also a non-perturbative process; if \( \lambda \) is either the self-coupling of the given field or the coupling of this field to another, then, in typical scenarios for which the initial energy in the zero mode is \( \mathcal{O}(1/\lambda) \), the number of particles produced is typically \( \mathcal{O}(1/\lambda) \). Thus any perturbative treatment will necessarily be incomplete.

In the Minkowski space analysis of refs. \cite{3,9}, these constraints were dealt with by applying the so-called closed time path formalism (CTP) \cite{10} of quantum field theory, which describes the non-equilibrium time evolution of the density matrix describing the state of the field theory, to both the Hartree approximation of a quartically self-coupled scalar theory as well as the large \( N \) approximation to the \( O(N) \) vector model \cite{11,12}. These analyses clearly displayed the dissipational effects of backreaction due to the produced particles as the zero mode evolved towards its final equilibrium state. They also showed that the zero mode can execute extremely interesting and unexpected behavior, depending upon the couplings as well as the initial conditions \cite{5}.

In this work, we ask the question: does the inclusion of the Hubble expansion affect the preheating process? Some partial answers to this question were given in ref. \cite{13,14}.

We consider the \( O(N) \) model embedded in a fixed radiation dominated FRW background spacetime. Again, we include the effects of backreaction to leading order in \( 1/N \) and calculate the behavior of the zero mode, as well as the particle production per mode. Our results are that the expansion does \textit{not} prevent an epoch of preheating from occurring, but will in general reduce the number of particles produced per physical volume compared to the flat spacetime results. Furthermore, the energy lost from the zero mode due to redshift will typically prevent some of the unexpected behavior found for the zero mode in Minkowski space \cite{3,9} from occurring in an expanding universe.

II. THE FORMALISM AND THE MODEL

The correct way to describe the evolution of quantum fields in non-equilibrium situations is the CTP formalism \cite{10} which describes the time evolution of the density matrix of field theory. Details of the applications to field theory are given in ref. \cite{13,14}.

A controllable yet non-perturbative approximation scheme is the large \( N \) approximation of the \( O(N) \) vector model, which has been generalized for non-equilibrium situations \cite{11}. The lowest order term in this approximation corresponds to mean field theory, but, at least in principle, the next to leading terms that would include effects such as
The leading contribution is obtained by neglecting the $\phi$ for $g$ fluctuations operator where $\lambda$ scattering, could be computed. Such corrections will be important in any discussion of how the particles produced during preheating actually thermalize, and in particular for any estimates of the reheating temperature.

The $O(N)$ model in an FRW universe can be formulated in the following manner. The Lagrangian density is:

$$\mathcal{L} = a^3(t) \left[ \frac{1}{2} \ddot{\phi}^2(\vec{x}, t) - \frac{1}{2} \frac{(\vec{\nabla} \phi(\vec{x}, t))^2}{a(t)^2} - V(\phi, \phi) \right]$$

$$V(\sigma, \vec{\pi}) = -\frac{1}{2} m^2 \vec{\phi} \cdot \vec{\phi} + \frac{\lambda}{8N}(\vec{\phi} \cdot \vec{\phi})^2$$

(2.1)

for fixed $\lambda$ in the large $N$ limit and $m^2 > 0$ (symmetry breaking). Here $a(t)$ is the FRW scale factor, and $\vec{\phi} = (\sigma, \vec{\pi})$ is an $O(N)$ vector where $\vec{\pi}$ represents the $N-1$ “pions”. For simplicity, we will only consider the case of flat spatial sections, and we will set the renormalized coupling to the Ricci scalar $R$ to zero.

We decompose the $\sigma$ field as

$$\sigma(\vec{x}, t) = \sigma_0(t) + \chi(\vec{x}, t).$$

(2.2)

The large $N$ limit can be implemented consistently with the following Hartree-like factorization (for more details the reader is referred to ref.

$$\vec{\pi}(\vec{x}, t) = \psi(\vec{x}, t) (1, 1, \cdots, 1), \quad \sigma_0(t) = \phi(t) \sqrt{N},$$

(2.3)

with

$$\langle \psi^2 \rangle \approx O(1), \quad \langle \chi^2 \rangle \approx O(1), \quad \phi \approx O(1).$$

(2.4)

The leading contribution is obtained by neglecting the $O(1/N)$ terms in the formal large $N$ limit.

If $V_k(t)$ are the mode functions (for comoving wavenumbers $k$) corresponding to $\psi(\vec{x}, t)$, then the evolution equations for $\phi(t)$ and the $V_k(t)$ are:

$$\ddot{\phi}(t) + 3H \dot{\phi}(t) + \phi(t) \left[ m^2 + \frac{\lambda}{2} \phi^2(t) + \frac{\lambda}{2} \langle \psi^2(t) \rangle \right] = 0$$

(2.5)

$$\left[ \frac{d^2}{dt^2} + 3H \frac{d}{dt} + \frac{k^2}{a^2(t)} + m^2 + \frac{\lambda}{2} \phi^2(t) + \frac{\lambda}{2} \langle \psi^2(t) \rangle \right] V_k(t) = 0,$$

(2.6)

$$\langle \phi^2(t) \rangle = \int \frac{d^3k}{(2\pi)^3} \frac{|V_k(t)|^2}{2W_k}$$

(2.7)

$$W_k = \sqrt{\frac{k^2}{a^2(t_0)} + m^2 + \frac{\lambda}{2} \phi^2(t_0)},$$

(2.8)

where $H \equiv \dot{a}(t)/a(t)$ is the Hubble parameter. The initial conditions are $V_k(t_0) = 1, \dot{V}_k(t_0) = -iW_k$.

After renormalization, we have:

$$\eta''(\tau) + 3 \frac{a'(\tau)}{a(\tau)} \eta'(\tau) + \mathcal{M}^2(\tau) \eta(\tau) = 0,$$

(2.9)

$$\left[ \frac{d^2}{d\tau^2} + 3 \frac{a'(\tau)}{a(\tau)} \frac{d}{d\tau} + \frac{q^2}{a^2(\tau)} + \mathcal{M}^2(\tau) \right] V_q(\tau) = 0,$$

(2.10)

$$\mathcal{M}^2(\tau) = -1 + \eta^2(\tau) + g \Sigma(\tau),$$

(2.11)

where $\tau \equiv mt, q \equiv k/m, \eta^2(\tau) \equiv \lambda \phi^2(\tau)/2m^2$, and $g \equiv \lambda/8\pi^2$. Primes denote derivatives with respect to $\tau$. The fluctuations operator $g \Sigma(\tau)$, which accounts for the back reaction of the produced particles on the modes, is:

$$g \Sigma(\tau) = g \int dq \, q^2 \left\{ \frac{|V_q(\tau)|^2}{W_q} - \frac{1}{qa^2(\tau)} + \frac{\theta(q - \kappa)}{2q^3} \left[ \mathcal{M}^2(\tau) - \frac{1}{6} R(\tau) - \frac{a'^2(\tau)}{a^2(\tau)} \right] \right\},$$

(2.12)
where $\kappa$ is a renormalization scale, $R(\tau)$ is the Ricci scalar in units of $|m^2|$ and $V_q(\tau_0) = 1, V'_q(\tau_0) = -i W_q$.

We consider: (i) slow roll initial conditions with $\eta(\tau_0) \simeq 0$. In this case some of the modes display spinodal instabilities which then drive the production of particles; (ii) chaotic initial conditions, $\eta(\tau_0) \gg 1$. The particle production in this case is due to parametric amplification of the non-zero modes due to the evolution of the zero mode. In both these cases, we will take as our background a radiation dominated FRW universe, with scale factor given by $a(t) = ((t + t_0)/t_0)^{1/2}$. This analysis can be repeated for a matter dominated universe with similar results, while the analysis of the de Sitter case is quite different [16].

### A. Slow-Roll Initial Conditions

We take $\eta(\tau_0)$, to be very near the origin: $\eta(\tau_0) = 10^{-5}$, $\eta'(\tau_0) = 0$. The coupling is fixed at $g = 10^{-12}$ as would befit an inflationary theory. Larger couplings will tend to speed up the dynamics, but will leave the qualitative aspects unchanged.

Fig.(1a) shows the behavior of $\eta(\tau)$ for an initial (dimensionless) Hubble parameter $h_0 = 0.1$. We see that it makes one excursion out to near the classical turning point and then tries to return to its initial value. However, by this time, the expansion of the universe has redshifted enough energy from the zero mode that it cannot come back to the origin and begins to execute damped oscillations about the tree-level minimum at $\eta = 1$.

Fig.(1b) shows the quantum fluctuations operator $D(\tau) = g \Sigma(\tau)$ during the evolution. We see that $D(\tau)$ grows due to spinodal instabilities as $\eta(\tau)$ makes its first excursion; this is when particle production occurs. However, at late times $D(\tau)$ tends towards zero.

Fig.(1c) shows the number of “pions” per unit physical volume, in units of $1/g$, produced as a function of comoving dimensionless wavenumber $q$ at various times. An important feature to notice here is that this distribution has support at $q = 0$.

In Minkowski space where $h_0 = 0$ [9], the zero mode finds a stable minimum near the origin, while the fluctuations grow to $D(\tau \to \infty) \to 1$. Furthermore, the momentum distribution of the produced pions has essentially no support near $q = 0$ in this situation. Thus the addition of expansion induces a qualitative change in the dynamics of the system and the power spectrum of fluctuations.

Increasing $h_0$ yields essentially the same features, the only differences being in the time scale involved. Thus for larger $h_0$, $\eta$ begins to oscillate about the tree-level minima sooner, and the fluctuations begin their decrease towards zero sooner.

### B. Chaotic Initial Conditions

We take $\eta(\tau_0) = 4$, $\eta'(\tau_0) = 0$, $h_0 = 0.1$ as our chaotic initial conditions, again with $g = 10^{-12}$. In this case, the zero mode has enough energy to be able to sample both minima for a period of time until enough energy is lost both due to the redshifting as well as to particle production to restrict it to small oscillations about one of the tree level minima, in this case the one at $\eta = 1$ (Fig.(2a)). This behaviour is different than the Minkowski case [9] where the
zero mode keeps oscillating over the two minima showing that the symmetry is unbroken if the initial energy is larger than $V(\eta = 0)$.

![Figure 2a](image)

**Figure 2a** The zero mode $\eta(\tau)$ vs. $\tau$, (b) the quantum fluctuation operator $D(\tau)$ vs. $\tau$, and (c) the number of pions $N_q$ vs $q$ at $\tau = 40$ (dashed line), $\tau = 200$ (solid line) for the parameter values $\eta_0 = 4, \eta'_0 = 0, g = 10^{-12}, h_0 = 0.1$.

As in the slow-roll case, the fluctuations are again driven to zero at late times as the zero mode reaches its minimum (Fig.(2b)). However, the behavior of the momentum distribution of particles (Fig.(2c)) is quite different, looking much more like the Minkowski slow-roll space distribution than the slow-roll one above.

Changing the initial value of $\eta(\tau_0)$ will change the number of times both ground states are sampled; for $\eta(\tau_0) > 4$, we have more oscillations back and forth before enough energy is drained away to make the zero mode settle into one of the vacua. If the initial Hubble parameter is increased, for fixed $\eta(\tau_0)$, the energy of the zero mode is redshifted away faster, so that fewer large amplitude oscillations are executed and fewer particles per physical volume are produced.

### III. LATE TIME ANALYSIS

An interesting by-product of our numerical analysis concerns the late time behavior of the field. The damping term in the zero mode equation suggests that there will be stationary solutions at late times. Such solutions must obey the large N Ward identity

$$\eta_\infty(-1 + \eta_\infty^2 + g\Sigma_\infty) = 0,$$

where the subscript $\infty$ denotes the late time value of the relevant quantity. Our numerical results confirm that eq.(3.1) is satisfied to one part in $10^{10}$. The same sum rule was found to be satisfied in the Minkowski case as well. For $\eta_\infty \neq 0$ the term inside the bracket must vanish, leading to a mode equation for massless Goldstone bosons, which can be solved exactly in a radiation dominated universe:

$$V_q(\tau) = \frac{1}{a(\tau)} \left[ c_q \exp \left( 2i\eta_0^{1/2}(\tau + \tau_0)^{1/2} \right) + d_q \exp \left( -2i\eta_0^{1/2}(\tau + \tau_0)^{1/2} \right) \right],$$

where $c_q$ and $d_q$ are determined by the initial conditions and the dynamical evolution. For $q \neq 0$ and late times, $\tau \gg 1/(q^2\tau_0)$, the phases average out so that $|V_q(t)|^2 \sim a(\tau)^{-2}$ and the fluctuation contribution is driven to

$$g\Sigma(\tau) \approx \frac{C}{\tau + \tau_0} \propto a(\tau)^{-2},$$

where $C$ is a constant which depends on the dynamics of the evolution and initial conditions. Using eq.(3.1), the zero mode then behaves as:

$$\eta^2(\tau) \approx \frac{\tau + \tau_0 - C}{\tau + \tau_0}.$$

If $\eta(\tau_0) \neq 0$, we find that $V_q(\tau)$ tends to a constant at late times. This explains the appearance of the peak in $N_q$ at $q = 0$. Since $\eta(\tau_0) = 0, \eta'(\tau_0) = 0$ is a fixed point of the zero mode equation, eq.(3.1) is satisfied with $g\Sigma_\infty = 1$. In this case only the $q \to 0$ modes contribute substantially to the fluctuations and the power spectrum of the fluctuations
becomes both peaked at zero momentum and narrower at longer times. This analysis reproduces our numerical results to high accuracy.

There is a striking contrast between the radiation dominated and the de Sitter cases. In the latter, for \( \eta(\tau_0) \neq 0 \), there are consistent stationary solutions for which \( \eta_{\infty} \) is not a minimum of the tree level potential and the fluctuation contribution \( g \Sigma_{\infty} \) is driven to a nonzero constant \([16]\).

IV. CONCLUSIONS

We have generalized our previous results on the quantum dynamics of the \( O(N) \) model in Minkowski space to FRW universes. In particular, we find that preheating can still occur in an expanding universe, though the actual amount of particle production depends sensitively on the size of the initial Hubble parameter relative to the mass of the field.

We also showed that the late time dynamics is determined by the sum rule in eq.(3.1). This, together with the behavior of the modes, drives the system to satisfy the sum rule by damping the fluctuations to zero and allowing the zero mode to find one of the tree level minima. This indicates that the symmetry is broken at late times \([8]\). We see no evidence of symmetry restoration \([8]\).

The next step in this program is clearly to attempt to allow the scale factor to evolve dynamically and understand how the back reaction of the produced particles influences the dynamics of \( a(t) \). We are currently setting up the formalism to tackle this problem.

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