On piecewise isomorphism of some varieties. *

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Abstract

Two quasi-projective varieties are called piecewise isomorphic if they can be stratified into pairwise isomorphic strata. We show that the \( m \)-th symmetric power \( S^m(\mathbb{C}^n) \) of the complex affine space \( \mathbb{C}^n \) is piecewise isomorphic to \( \mathbb{C}^{mn} \) and the \( m \)-th symmetric power \( S^m(\mathbb{CP}^\infty) \) of the infinite dimensional complex projective space is piecewise isomorphic to the infinite dimensional Grassmannian \( \text{Gr}(m, \infty) \).

Let \( K_0(\mathcal{V}_\mathbb{C}) \) be the Grothendieck ring of complex quasi-projective varieties. This is the Abelian group generated by the classes \([X]\) of all complex quasi-projective varieties \( X \) modulo the relations:

1) \([X] = [Y]\) for isomorphic \( X \) and \( Y \);

2) \([X] = [Y] + [X \setminus Y]\) when \( Y \) is a Zariski closed subvariety of \( X \).

The multiplication in \( K_0(\mathcal{V}_\mathbb{C}) \) is defined by the Cartesian product of varieties: \([X_1] \cdot [X_2] = [X_1 \times X_2]\). The class \([A_1^\mathbb{C}] \in K_0(\mathcal{V}_\mathbb{C})\) of the complex affine line is denoted by \( L \).

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**Definition:** Quasi-projective varieties $X$ and $Y$ are called *piecewise isomorphic* if there exist decompositions $X = \coprod_{i=1}^{s} X_i$ and $Y = \coprod_{i=1}^{s} Y_i$ of $X$ and $Y$ into (Zariski) locally closed subsets such that $X_i$ and $Y_i$ are isomorphic for $i = 1, \ldots, s$.

If the varieties $X$ and $Y$ are piecewise isomorphic, their classes $[X]$ and $[Y]$ in the Grothendieck ring $K_0(\mathcal{V}_C)$ coincide. There exists the conjecture (or at least the corresponding question) that the opposite also holds: if $[X] = [Y]$, then $X$ and $Y$ are piecewise isomorphic: see [8, 9].

It is well-known that the $m$-th symmetric power $S^m \mathbb{C}^n$ of the affine space $\mathbb{C}^n$ is birationally equivalent to $\mathbb{C}^{mn}$: see e.g. [3]. An explicit birational isomorphism between $S^m \mathbb{C}^n$ and $\mathbb{C}^{mn}$ was constructed in [1]. Moreover the class $[S^m \mathbb{C}^n]$ of the variety $S^m \mathbb{C}^n$ in the Grothendieck ring $K_0(\mathcal{V}_C)$ of complex quasi-projective varieties is equal to the class $[\mathbb{C}^{mn}] = \mathbb{L}^{mn}$: see e.g. [4, 3]. The conjecture formulated above means that the varieties $S^m \mathbb{C}^n$ and $\mathbb{C}^{mn}$ are piecewise isomorphic. This is well-known for $n = 1$. Moreover $S^m \mathbb{C}$ and $\mathbb{C}^m$ are isomorphic. The fact that indeed $S^m \mathbb{C}^n$ and $\mathbb{C}^{mn}$ are piecewise isomorphic seems to (or must) be known to specialists. Moreover proofs are essentially contained in [4] (Lemma 4.4 proved by Burt Totaro) and [5] (Statement 3). However this fact is not explicitly reflected in the literature. Here we give a proof of this statement.

In [7], it was shown that the Kapranov zeta function $\zeta_{BC^*}(T)$ of the classifying stack $B\mathbb{C}^* = BGL(1)$ is equal to

$$1 + \sum_{m=1}^{\infty} \frac{\mathbb{L}^{m^2-m}}{(\mathbb{L}^{m} - \mathbb{L}^{m-1})(\mathbb{L}^{m} - \mathbb{L}^{m-2}) \ldots (\mathbb{L}^{m} - 1)} T^m.$$  

Unrigorously speaking this can be interpreted as the class $[S^m B\mathbb{C}^*]$ of the “$m$-th symmetric power” of the classifying stack $B\mathbb{C}^*$ in the Grothendieck ring $K_0(\mathcal{Stck}_C)$ of algebraic stacks of finite type over $\mathbb{C}$ is equal to $\mathbb{L}^{m^2-m}$ times the class $[BGL(m)] = 1/(\mathbb{L}^{m} - \mathbb{L}^{m-1})(\mathbb{L}^{m} - \mathbb{L}^{m-2}) \ldots (\mathbb{L}^{m} - 1)$ of the classifying stack $BGL(m)$. The natural topological analogues of the classifying stacks $B\mathbb{C}^*$ and $BGL(m)$ are the infinite-dimensional projective space $\mathbb{CP}^\infty$ and the infinite Grassmannian $\text{Gr}(m, \infty)$. We show that the $m$-th symmetric power $S^m \mathbb{CP}^\infty$ of $\mathbb{CP}^\infty$ and $\text{Gr}(m, \infty)$ are piecewise isomorphic in a natural sense.

**Theorem 1** The varieties $S^m \mathbb{C}^n$ and $\mathbb{C}^{mn}$ are piecewise isomorphic.
Proof. The proofs which we know in any case are not explicit, we do not know the necessary partitions of $S^m \mathcal{C}^n$ and $\mathcal{C}^{mn}$. Therefore we prefer to use the language of power structure over the Grothendieck semiring $S_0(\text{Var}_\mathbb{C})$ of complex quasi-projective varieties invented in [5]. This language sometimes permits to substitute somewhat envolved combinatorial considerations by short computations (or even to avoid them at all, as it was made in [6]). Since the majority of statements in [5] (including those which could be used to prove Theorem 1) are formulated and proved in the Grothendieck ring $K_0(\mathcal{V}_\mathbb{C})$ of complex quasi-projective varieties, we repeat a part of the construction in the appropriate setting.

The Grothendieck semiring $S_0(\text{Var}_\mathbb{C})$ of complex quasi-projective varieties is the semigroup generated by isomorphism classes $\{X\}$ of such varieties modulo the relation $\{X\} = \{X - Y\} + \{Y\}$ for a Zariski closed subvariety $Y \subset X$. The multiplication is defined by the Cartesian product of varieties: $\{X_1\} \cdot \{X_2\} = \{X_1 \times X_2\}$. Classes $\{X\}$ and $\{Y\}$ of two varieties $X$ and $Y$ in $S_0(\text{Var}_\mathbb{C})$ are equal if and only if $X$ and $Y$ are piecewise isomorphic. Let $\mathbb{L} \in S_0(\text{Var}_\mathbb{C})$ be the class of the affine line. If $\pi : E \to B$ is a Zariski locally trivial fibre bundle with fibre $F$, one has $\{E\} = \{F\} \cdot \{B\}$. For example if $\pi : E \to B$ is a Zariski locally trivial vector bundle of rank $s$, one has $\{E\} = \mathbb{L}^s \{B\}$.

A power structure over a semiring $R$ is a map $(1 + T \cdot R[[T]]) \times R \to 1 + T \cdot R[[T]]$: $(A(T), m) \mapsto (A(T))^m$, which possesses the properties:

1. $(A(T))^0 = 1$,
2. $(A(T))^1 = A(T)$,
3. $(A(T) \cdot B(T))^m = (A(T))^m \cdot (B(T))^m$,
4. $(A(T))^{m+n} = (A(T))^m \cdot (A(T))^n$,
5. $(A(T))^{mn} = ((A(T))^n)^m$,
6. $(1 + T)^m = 1 + mT + \text{terms of higher degree},$
7. $(A(T^\ell))^m = (A(T))^m |_{T \to T^\ell}, \ell \geq 1.$
In [5], there was defined a power structure over the Grothendieck semiring $S_0(\text{Var}_C)$. Namely, for $A(\{T\}) = 1 + \{A_1\} T + \{A_2\} T^2 + \ldots$ and $\{M\} \in S_0(\text{Var}_C)$, the series $(A(\{T\}))^{\{M\}}$ is defined as

$$1 + \sum_{k=1}^{\infty} \left( \sum_{\{i\}: \sum \frac{1}{\prod S_{k_i}} \right) \left( \frac{\prod \frac{1}{\prod A_{k_i}}}{\prod \frac{1}{\prod S_{k_i}}} \right) \cdot T^k,$$

where $\Delta$ is the “large diagonal” in $M^{\Sigma k_i} = \prod_i M^{k_i}$ which consists of $(\sum k_i)$-tuples of points of $M$ with at least two coinciding ones, the group $S_{k_i}$ of permutations on $k_i$ elements acts by permuting corresponding $k_i$ factors in $\prod_i M^{k_i} \supset (\prod_i M^{k_i}) \setminus \Delta$ and the spaces $A_i$ simultaneously. The action of the group $\prod_i S_{k_i}$ on $(\prod_i M^{k_i}) \setminus \Delta$ is free. The properties 1–7 are proved in [5, Theorem 1].

Special role is played by the Kapranov zeta function in the Grothendieck semiring $S_0(\text{Var}_C)$:

$$\zeta_{\{M\}}(T) := 1 + \sum_{k=1}^{\infty} \{S^k M\} T^k,$$

where $S^k M$ is the $k$-th symmetric power $M^k/S_k$ of the variety $M$. In terms of the power structure one has $\zeta_{\{M\}}(T) = (1 + T + T^2 + \ldots)^{\{M\}}$. Theorem 1 is equivalent to the fact that

$$\zeta_{L, m}(T) = (1 + \sum_{i=1}^{\infty} L_i \cdot T^i).$$

**Lemma 1** Let $A_i$ and $M$ be complex quasi-projective varieties, $A(\{T\}) = 1 + \{A_1\} T + \{A_2\} T^2 + \ldots$. Then, for any integer $s \geq 0$,

$$(A(L^s T))^{\{M\}} = (A(T))^{\{M\}} \big|_{T \Rightarrow L^s T}.$$

**Proof.** The coefficient at the monomial $T^k$ in the power series $(A(T))^{\{M\}}$ is a sum of the classes of varieties of the form

$$V = \left( \frac{(\prod_i M^{k_i}) \setminus \Delta \times \prod_i A_{k_i}^i}{\prod_i S_{k_i}} \right).$$
with \( \sum i k_i = k \). The corresponding summand \( \{ \widetilde{V} \} \) in the coefficient at the monomial \( T^k \) in the power series \( (A(L^s T))^{(M)} \) has the form

\[
\widetilde{V} = \left( ((\prod_i M_i^{k_i}) \setminus \Delta) \times \prod_i (L^{s_i} \times A_i)^{k_i} \right) / \prod_i S_i.
\]

The natural map \( \widetilde{V} \to V \) is a Zariski locally trivial vector bundle of rank \( sk \): see e.g. [10, Section 7, Proposition 7]. This implies that \( \{ \widetilde{V} \} = L^{sk} \cdot \{ V \} \). □

One has \( \zeta_L(T) = (1 + LT + L^2 T^2 + \ldots) \). For all \( A_i \) being points, i.e. \( \{ A_i \} = 1 \), one gets

\[
\zeta_{L\{M\}}(T) = (1 + T + T^2 + \ldots)^{L\{M\}} = ((1 + T + T^2 + \ldots)^{L})^{(M)} = (1 + LT + L^2 T^2 + \ldots)^{(M)}.
\]

Equation (3) implies that

\[
\zeta_{L\{M\}}(T) = (1 + LT + L^2 T^2 + \ldots)^{(M)} = \zeta_{\{M\}}(LT).
\]

Assuming (2) holds for \( m < m_0 \) and applying the equation above to \( m = m_0 - 1 \) one gets

\[
\zeta_{L^{m_0}}(T) = \zeta_{L^{m_0-1}}(LT) = (1 + L^{m_0-1}T + L^{2(m_0-1)}T^2 + \ldots)|_{T \to LT} = (1 + L^{m_0}T + L^{2m_0}T^2 + \ldots).
\]

This gives the proof. □

Let \( \mathbb{CP}^\infty = \lim_{\leftarrow} \mathbb{CP}^N \) be the infinite dimensional projective space and let \( \text{Gr}(m, \infty) = \lim_{\leftarrow} \text{Gr}(m, N) \) be the infinite dimensional Grassmannian. (In the both cases the inductive limit is with respect to the natural sequence of inclusions. The spaces \( \mathbb{CP}^\infty \) and \( \text{Gr}(m, \infty) \) are, in the topological sense, classifying spaces for the groups \( \mathbb{C}^* = GL(1; \mathbb{C}) \) and \( GL(m; \mathbb{C}) \) respectively.) The symmetric power \( S^m \mathbb{CP}^\infty \) is the inductive limit of the quasi-projective varieties \( S^m \mathbb{CP}^N \). For a sequence \( X_1 \subset X_2 \subset X_3 \subset \ldots \) of quasi-projective varieties, let \( X = \lim_{\leftarrow} X_i (= \bigcup X_i) \) be its (inductive) limit. A partition of the space \( X \) compatible with the filtration \( \{ X_i \} \) is a representation of \( X \) as a disjoint union.
\[ \bigcap Z_j \] of (not more than) countably many quasi-projective varieties \( Z_j \) such that each \( X_i \) is the union of a subset of the strata \( Z_j \) and each \( Z_j \) is a Zariski locally closed subset in the corresponding \( X_i \).

**Theorem 2** The spaces \( S^m \mathbb{CP}^\infty \) and \( \text{Gr}(m, \infty) \) are piecewise isomorphic in the sense that there exist partitions \( S^m \mathbb{CP}^\infty = \bigsqcup_j U_j \) and \( \text{Gr}(m, \infty) = \bigsqcup_j V_j \) into pairwise isomorphic quasi-projective varieties \( U_j \) and \( V_j \) (\( U_j \cong V_j \)) compatible with the filtrations \( \{S^m \mathbb{CP}^N\}_N \) and \( \{\text{Gr}(m, N)\}_N \).

**Proof.** The natural partition of \( \text{Gr}(m, N) \) consists of the Schubert cells corresponding to the flag \( \{0\} \subset C^1 \subset C^2 \subset \ldots \) see e.g. [2, §5.4]. Each Schubert cell is a locally closed subvariety of \( \text{Gr}(m, N) \) isomorphic to the complex affine space of certain dimension. This partition is compatible with the inclusion \( \text{Gr}(m, N) \subset \text{Gr}(m, N + 1) \) and therefore gives a partition of \( \text{Gr}(m, \infty) \). The number of cells of dimension \( n \) in \( \text{Gr}(m, \infty) \) is equal to the number of partitions of \( n \) into not more than \( m \) summands.

Since \( \mathbb{CP}^\infty = \mathbb{C}^0 \coprod \mathbb{C}^1 \coprod \mathbb{C}^2 \coprod \ldots \) and \( S^p(A \coprod B) = \bigcap_i S^i A \times S^{p-i} B \), one has

\[
S^m \mathbb{CP}^\infty = \bigcap_{\{i_0, i_1, i_2, \ldots \} : i_0 + i_1 + i_2 + \ldots = m} \bigcap_j S^{i_j} C^j = \bigcap_{\{i_1, i_2, \ldots \} : i_1 + i_2 + \ldots \leq m} \bigcap_j S^{i_j} C^j,
\]

where \( i_j \) are non-negative integers. This partition is compatible with the natural filtration \( \{\mathbb{CP}^0\} \subset \mathbb{CP}^1 \subset \mathbb{CP}^2 \subset \ldots \). The number of parts of dimension \( n \) is equal to the number of sequences \( \{i_1, i_2, \ldots \} \) such that \( \sum_j i_j \leq m \), \( \sum_j i_j = n \). Thus it coincides with the number of partitions of \( n \) into not more than \( m \) summands and is equal to the number of \( n \)-dimensional Schubert cells in the partition of \( \text{Gr}(m, \infty) \). Due to Proposition 1 each part \( \prod_j S^{i_j} C^j \) is piecewise isomorphic to the complex affine space of the same dimension. This concludes the proof. \( \square \)

It would be interesting to find explicit piecewise isomorphisms between the spaces in Theorems 1 and 2.
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