A FAST MODULO PRIMES ALGORITHM FOR SEARCHING PERFECT CUBOIDS AND ITS IMPLEMENTATION.

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Abstract. A perfect cuboid is a rectangular parallelepiped whose all linear extents are given by integer numbers, i.e. its edges, its face diagonals, and its space diagonal are of integer lengths. None of perfect cuboids is known thus far. Their non-existence is also not proved. This is an old unsolved mathematical problem.

Three mathematical propositions have been recently associated with the cuboid problem. They are known as three cuboid conjectures. These three conjectures specify three special subcases in the search for perfect cuboids. The case of the second conjecture is associated with solutions of a tenth degree Diophantine equation. In the present paper a fast algorithm for searching solutions of this Diophantine equation using modulo primes sieve is suggested and its implementation on 32-bit Windows platform with Intel-compatible processors is presented.

1. Introduction.

Conjecture 1.1 (Second cuboid conjecture). For any two positive coprime integer numbers \( p \neq q \) the tenth-degree polynomial

\[
Q_{pq}(t) = t^{10} + (2q^2 + p^2)(3q^2 - 2p^2)t^8 + (q^8 + 10p^2q^6 + 4p^4q^4 + 10p^6q^2 + p^8)t^6 - (p^2q^2)(q^8 - 14p^2q^6 + 4p^4q^4 - 14p^6q^2 + p^8)t^4 - (p^6q^6)(3p^2 - 2q^2)t^2 - q^{10}p^{10}
\]

is irreducible over the ring of integers \( \mathbb{Z} \).

Theorem 1.1. A perfect cuboid associated with the polynomial (1.1) does exist if and only if for some positive coprime integer numbers \( p \neq q \) the Diophantine equation \( Q_{pq}(t) = 0 \) has a positive solution \( t \) obeying the inequalities

\[
t > p^2, \quad t > pq, \quad t > q^2, \quad (p^2 + t)(pq + t) > 2t^2.
\]

Theorem 1.1 can be found in [1]. It stems from the results of [2] and [3]. As for the perfect cuboid problem itself, it has a long history reflected in [4–51]. There are also two series of ArXiv publications. The first of them [52–54] continues the research on cuboid conjectures. The second one [55–67] relates perfect cuboids with multisymmetric polynomials.

2000 Mathematics Subject Classification. 11D41, 11D72, 68U99, 65-04.
The scope of perfect cuboids in the case of the second cuboid conjecture is restricted by the following theorem derived from [1].

**Theorem 1.2.** In the case of the second cuboid conjecture there are no perfect cuboids outside the region given by the inequalities

\[ \min \left( \frac{\sqrt{\frac{p}{9}}, \frac{p}{59}}{9} \right) \leq q \leq 59p. \] (1.2)

In [1] the region given by the inequalities (1.2) was presented as a union of two regions which were called the linear and the nonlinear regions respectively. In this paper we present an algorithm for searching cuboids in the region (1.2).

2. A MODULO PRIMES SEIVE.

Let \( p, q, \) and \( t \) be a triple of integer numbers satisfying the Diophantine equation \( Q_{pq}(t) = 0 \) with the polynomial (1.1) and let \( r \) be some prime number. Then we can pass from \( \mathbb{Z} \) to the quotient ring \( \mathbb{Z}_r = \mathbb{Z}/r\mathbb{Z} \) and denote

\[
\tilde{p} = p \mod r, \quad \tilde{q} = q \mod r, \quad \hat{t} = t \mod r. \quad (2.1)
\]

The numbers \( \tilde{p}, \tilde{q}, \) and \( \hat{t} \) are interpreted as division remainders after dividing \( p, q, \) and \( t \) by the prime number \( r \). They obey the quotient equation

\[
Q_{\tilde{p}\tilde{q}}(\hat{t}) \mod r = 0. \quad (2.2)
\]

Once \( r \) is given there are only a finite number of remainders (2.1):

\[
\tilde{p} = 0, \ldots, r - 1, \quad \tilde{q} = 0, \ldots, r - 1, \quad \hat{t} = 0, \ldots, r - 1.
\]

The values in the left hand side of the equation (2.2) for them can be precomputed. They can be either zero or nonzero modulo \( r \). We can use them as a fast computed test for sweeping away those values of \( p, q, \) and \( t \), where \( Q_{pq}(t) \neq 0 \).

**Definition 1.1.** A pair of integer numbers \( 0 \leq \tilde{p} \leq r - 1 \) and \( 0 \leq \tilde{q} \leq r - 1 \) is called solvable modulo \( r \) if there is at least one integer number \( 0 \leq \hat{t} \leq r - 1 \) such that \( Q_{\tilde{p}\tilde{q}}(\hat{t}) \mod r = 0 \). Otherwise it is called unsolvable.

We can represent solvable and unsolvable pairs in the form of bit-arrays \( u_r \):

\[
u_r(\tilde{p}, \tilde{q}) = \begin{cases} 
0 & \text{if } (\tilde{p}, \tilde{q}) \text{ is solvable}; \\
1 & \text{if } (\tilde{p}, \tilde{q}) \text{ is unsolvable}. 
\end{cases} \quad (2.3)
\]

The value \( u_r(\tilde{p}, \tilde{q}) \) of the function (2.3) is called the unsolvability bit. Bit-arrays of the form (2.3) can be stored as tables. For \( r = 2 \) this table looks like

| \( u_2(\tilde{p}, \tilde{q}) \) | \( \tilde{q} = 0 \) | \( \tilde{q} = 1 \) |
|---|---|---|
| \( p=0 \) | 0 | 0 |
| \( p=1 \) | 0 | 0 |
As we see in (2.4), the values of the function $u_2(\tilde{p}, \tilde{q})$ are identically zero. The same is true for the functions $u_3(\tilde{p}, \tilde{q})$, $u_5(\tilde{p}, \tilde{q})$, and $u_7(\tilde{p}, \tilde{q})$ associated with the prime numbers $r = 3$, $r = 5$, and $r = 7$. The case of $r = 11$ is different:

$$
\begin{array}{cccccccccccc}
    u_{11} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
    \hline
    p=0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    p=1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
    p=2 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
    p=3 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
    p=4 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
    p=5 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
    p=6 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
    p=7 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
    p=8 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
    p=9 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
    p=10 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
$$

(2.5)

In the memory of a computer bit-arrays like (2.5) are packed into byte-arrays with 8 bits per 1 byte, e.g. the array $u_{11}$ looks like

$$
\begin{array}{cccccccccccc}
00000000 & 11100000 & 10011111 & \ldots & 11111111 & 00000000
\end{array}
$$

(2.6)

Note that bits in a byte are written in the reverse order — the highest bit is the leftmost. This is because bytes are designed to represent binary numbers. Note also that the last byte of the table (2.5) in (2.6) is incomplete. It is appended with zero bits which are shown in blue.

Bytes associated with the prime number $r = 11$ can be written into some linear locus of memory. Similarly, bytes associated with several other prime numbers can be written into adjacent loci. Altogether they constitute a bit sieve. Accessing a proper bit of this sieve, we can easily decide whether for a certain pair of integer numbers $p$ and $q$ the equation $Q_{pq}(t) = 0$ is unsolvable modulo some prime number $r$ enclosed in the sieve. Then it is unsolvable in the ring of integers $\mathbb{Z}$ as well. Quickening the search algorithm is reached through sweeping away those $(p, q)$ pairs that do not go through the bit sieve for several prime numbers. Indeed, it is clear that calculating the remainders

$$
\tilde{p} = p \mod r, \quad \tilde{q} = q \mod r.
$$

and then addressing bits in a memory locus are much faster operations than factoring a polynomial with numeric coefficients.

In our particular case we use the bit sieve for 96 consecutive prime numbers from 11 to 541. This bit sieve is stored in the binary file Cuboid_pq_bit_tables.bin.
In order to access effectively bit-tables for each particular prime number from 11 to 541 one should know their offsets within this file. These offsets are written to the separate binary file \texttt{Cuboid\_primes.bin}. They are enclosed in the structures described as follows in C++ language:

\begin{verbatim}
struct primes_item
{
    short prime; // prime number
    unsigned int p_offset; // prime bit-table offset
};
\end{verbatim}

In our implementation the values of prime numbers are restricted not only by \texttt{short=2\,bytes} data format used for them. Each prime number \( r \) is associated with the \( r \times r \) bit-table that occupies \( r^2/8 \) bytes in memory. Using \texttt{unsigned\,int=4\,bytes} format for offsets, we have the following restriction:

\[
\sum_{i=5}^{N} \frac{r_i^2}{8} < 2^{32}, \quad \text{where} \quad r_5 = 11, \; r_6 = 13, \ldots.
\]  

From (2.7) we derive \( N < 1198 \) and \( r_N < 9697 \). These inequalities fit the 4 Gb RAM (random access memory) limit. Actually we have chosen \( N = 100 \) in which case 1.5 Mb RAM is sufficient.

3. Code for generating binary files.

The code for preparing \texttt{Cuboid\_pq\_bit\_tables.bin} and \texttt{Cuboid\_primes.bin} binary files is implemented as a DLL library interacting with a Maple code. The DLL file \texttt{Cuboid\_search\_v01.dll} is generated within the 32-bit x86 makefile project for Microsoft Visual C++ 2005 Express Edition package. The project files

1) \texttt{make.bat}
2) \texttt{makefile}
3) \texttt{Cuboid\_search\_v01.h}
4) \texttt{Cuboid\_search\_v01.cpp}

are supplied as ancillary files to this paper. The C++ file \texttt{Cuboid\_search\_v01.cpp} is the main source file of the project. It comprises a C++ and inline assembly language code for running on 32-bit Windows machines with Intel compatible processors. The DLL library file \texttt{Cuboid\_search\_v01.dll} is produced from this code by running the batch file \texttt{make.bat} in a command prompt window:

\verb|> make.bat|

Along with \texttt{Cuboid\_search\_v01.dll} several other files are generated, including two LOG files \texttt{compiler.log} and \texttt{linker.log}. They can be removed by running the same batch file with the \texttt{clean} option:

\verb|> make.bat clean|

Note that the Visual C++ 2005 Express Edition package should be installed for successfully running the above files. In our implementation it was installed on Windows XP machine with Intel Pentium 4 Prescott CPU 2.80 GHz.

The generated DLL library \texttt{Cuboid\_search\_v01.dll} exports several functions. Their declarations are in the C++ header file \texttt{Cuboid\_search\_v01.h}. Three of
these functions are declared as follows:

```c
extern "C" __declspec(dllexport)
    void __stdcall Open_pq_file_stream();
extern "C" __declspec(dllexport)
    unsigned int __stdcall Write_pq_file_stream(unsigned int rrr);
extern "C" __declspec(dllexport)
    void __stdcall Close_pq_file_stream();
```

These declarations correspond to the following Maple worksheet declarations:

```maple
dll_file:="./Cuboid_search_v01.dll":
Open_pq_file_stream:=define_external('Open_pq_file_stream',
    LIB=dll_file);
Write_pq_file_stream:=define_external('Write_pq_file_stream',
    'rrr'::(integer[4]), RETURN::(integer[4]), LIB=dll_file);
Close_pq_file_stream:=define_external('Close_pq_file_stream',
    LIB=dll_file);
```

Maple worksheets are supplied in XML format as ancillary files to this paper. They can be imported to Maple. Here is the list of these files:

1. Create_binary_seive_files.xml
2. Test_external_DLL_procedures_01.xml
3. Test_external_DLL_procedures_02.xml
4. Search_for_cuboids.xml

The external function `Write_pq_file_stream(r)` imported to the Maple worksheet creates the bit-seive table for a given prime number `r` in its argument and writes it to the binary file `Cuboid_pq_bit_tables.bin`. It returns the integer value equal to the number of bytes written to the file `Cuboid_pq_bit_tables.bin`. The other binary file `Cuboid_primes.bin` is written simultaneously using the Maple worksheet code in `Create_binary_seive_files.xml`.

The external function `Write_pq_file_stream(r)` exploits another external function `Calculate_Pq_mod_prime(p,q,t,r)`. This function returns the value of the polynomial (1.1) modulo prime number `r` taken as its fourth argument. Though its arguments are declared as 32-bit integers, its code is designed to deal with 16-bit unsigned integers only. Due to the restriction `r < 9697` derived from (2.7) its usage in `Write_pq_file_stream(r)` does not require 32-bit integers in its arguments:

\[ 0 \leq p \leq r - 1 < 9797, \quad 0 \leq q \leq r - 1 < 9797. \]

The function `Calculate_Pq_mod_prime(p,q,t,r)` is a delicate part of the project. It is written in assembly language. Therefore it is carefully tested in the Maple worksheet code file `Test_external_DLL_procedures_01.xml`.

### 4. Code for loading and unloading binary files.

Once the binary files `Cuboid_pq_bit_tables.bin` and `Cuboid_primes.bin` are generated, they should be used in searching for perfect cuboids. For this purpose they should be loaded into the memory easily accessible from the DLL library functions. This task is performed by the function `Load_Cuboid_Binaries()` residing within the same DLL library. The opposite task is to unload the binary files, i.e.
to release the memory occupied by them. This task is performed by the function \texttt{Release\_Cuboid\_Binaries()} also residing within the DLL library.

The loading and unloading functions are imported and tested within the Maple worksheet code file \texttt{Test\_external\_DLL\_procedures\_02.xml}.

As an auxiliary test for two generated binary files \texttt{Cuboid\_pq\_bit\_tables.bin} and \texttt{Cuboid\_primes.bin} we visualize the bit-tables from \texttt{Cuboid\_pq\_bitmaps.bin} in the form of the text file \texttt{Cuboid\_bit\_tables.txt}. This text file is written by the code from the Maple worksheet file \texttt{Test\_external\_DLL\_procedures\_02.xml}.

5. Code for searching cuboids.

According to Theorem 1.2 the search for cuboids in the case of the second cuboid conjecture consists in scanning the region given by the inequalities (1.2). For each positive \( p \) these inequalities specify a finite segment of the real axis, which comprises a finite number of integer points. For \( p \leq 151 \) this segment is given by

\[
\frac{p}{59} \leq q \leq 59p. \tag{5.1}
\]

For \( p \geq 152 \) the inequalities are different:

\[
\frac{3\sqrt{p}}{9} \leq q \leq 59p. \tag{5.2}
\]

The inequalities \( 0 < p \leq 151 \) and the inequalities (5.1) outline a finite set of integer points on the coordinate \( pq \)-plane. One can easily verify that these points do not produce perfect cuboids. For this reason the software in the DLL library \texttt{Cuboid\_search\_v01.dll} is designed to search cuboids for \( p \geq 152 \) within each segment specified by the inequalities (5.2). Roughly speaking, it is an infinite loop on \( p \geq 152 \) and an enclosed loop on \( q \) obeying the inequalities (5.2) for each \( p \).

Both loops on \( p \) and on \( q \) are started from within the Maple worksheet file \texttt{Search\_for\_cuboids.txt} by executing the commands

\begin{verbatim}
> Load\_Cuboid\_Binaries();
> Start\_searching(152,3);
\end{verbatim}

Here 152 and 3 are initial values for \( p \) and \( q \) respectively. They should obey the inequalities (5.2). The function \texttt{Start\_searching} is an external function imported from the DLL library \texttt{Cuboid\_search\_v01.dll}. It starts the looping process and returns just immediately with the value 0 indicating that the search is successfully started. The multithreading mechanism is used in the code of this function:

\begin{verbatim}
_thread(Look\_for\_cuboids\_thread,0,(void*)12);
return(0);
\end{verbatim}

Here \texttt{Look\_for\_cuboids\_thread} is an internal function which is not exported from the DLL library. It is executed within a new thread, while the initial function \texttt{Start\_searching} returns control to the Maple worksheet.

You can do anything in the Maple worksheet while the search function \texttt{Look\_for\_cuboids\_thread} is running its infinite loops on \( p \) and \( q \), provided you do not stop the Maple session by closing the worksheet. In particular, you can control the
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process of searching by executing the following function in the Maple worksheet:

\[ \text{Get\_current\_p();} \]

This function returns the current value of the loop variable \( p \) with the discreteness equal to 100. There is another control function:

\[ \text{Get\_current\_r\_max();} \]

This function returns the maximal prime number is used to sieve cuboids within current hundred values of \( p \). The number \( r_{\text{max}} \) is flushed to 1 for each next hundred values of \( p \) and then is recalculated again.

The infinite loop on \( p \) cannot ever terminate by itself. Therefore it is terminated manually. This can be done at any time by executing the following function in the Maple worksheet that initiated the thread with this loop:

\[ \text{Stop\_searching();} \]

Upon doing it you can read the time stamp and the exit values of \( p \), \( q \), and \( r_{\text{max}} \) at the end of the file Cuboid\_search\_report.txt, e.g. it could be

2016-2-20 21:36
Stop with \( p=112618 \), \( q=5691455 \), \( r_{\text{max}}=131 \)

Then you can restart the search from this point on by executing the command

\[ \text{Start\_searching(112618,5691455);} \]

Or you can terminate the session by executing the command

\[ \text{Release\_Cuboid\_Binaries();} \]

and then closing the Maple worksheet.

Normally the function \text{Start\_searching(p,q)} returns 0 indicating that the search is successfully started. It returns 1 if the search is already running. So you cannot initiate several search threads running simultaneously with this software. This limitation is planned to be removed in further versions of the DLL library Cuboid\_search\_v01.dll.

The function \text{Start\_searching(p,q)} returns 2 if \( p < 152 \) (see (5.1) and (5.2) above for explanation). This function returns 3 if \( p > 72796055 \), which breaks the 32-bit limit for \( q = 59 \) \( p \) in (5.2).

The function \text{Start\_searching(p,q)} returns 4 if \( q \) is below the lower limit set by the inequalities (5.2). Similarly, it returns 5 if \( q \) is above the upper limit set by the inequalities (5.2).

The function \text{Start\_searching(p,q)} returns 6 if it is invoked before the binary files Cuboid\_pq\_bit\_tables.bin and Cuboid\_primes.bin are loaded into the memory by the function \text{Load\_Cuboid\_Binaries().}

The function \text{Load\_Cuboid\_Binaries()} normally returns the number of bytes loaded from the file Cuboid\_primes.bin, i.e. the size of this file. However, if it is invoked when the binary files Cuboid\_pq\_bit\_tables.bin and Cuboid\_primes.bin are already loaded, it does not load them again and returns 0.

The function \text{Release\_Cuboid\_Binaries()} normally returns 0. However it returns 6 if the binary files Cuboid\_pq\_bit\_tables.bin and Cuboid\_primes.bin are not loaded. This function cannot release the memory occupied by the bit-tables if the search is running. In this case it returns 1.
The function `Stop_searching()` normally returns 0. However, if this function is invoked when the search is not on, it returns 1. Thus the functions

```c
> Load_Cuboid_Binaries();
> Start_searching(p,q);
> Stop_searching();
> Release_Cuboid_Binaries();
```

should be invoked in the above order. Otherwise they signal misuse, but do not lead to a crash. These four functions constitute a toolkit we used in the present numerical research of perfect cuboids.

# 6. Results.

At present date 01.04.2016 the values of $p$ from 1 to 154000 are scanned. For each such $p$ all values of $q$ limited by the inequalities (1.2) are scanned. This stands for about 700 billions $(p, q)$ pairs that have been tested. Indeed, we have

$$N \approx 154000 \sum_{p=1}^{59} p = 699626543000 \approx 0.7 \cdot 10^{12}.$$ 

None of these $(p, q)$ pairs produces a perfect cuboid. Moreover, none of them goes through our primes sieve composed by 96 consecutive primes from 11 to 541. Actually this sieve is so dense that the maximal depth reached thus far is 29th prime in our sequence, which is equal to 137. This result is negative in the sense of finding a perfect cuboid. However it shows that the Second cuboid conjecture 1.1 is rather firm for to believe that it might be valid.

The total time spent for the above computations is 4130 minutes, i.e. about 69 hours. Then we can calculate the time per one $(p, q)$ pair:

$$\Delta t = \frac{4130}{699626543000} \text{min} \approx 3.54 \cdot 10^{-7} \text{sec}.$$ 

The upper limit for $p$ with our 32-bit code is given by the formula

$$p_{\text{max}} = \frac{2^{32}}{59} \approx 72796055.$$ 

Here is the estimate for the time needed to reach this limit:

$$t = \Delta t \sum_{p=1}^{72796055} 59p = 5.53 \cdot 10^{10} \text{sec} \approx 1755 \text{years}.$$ 

This estimate means that our code should be improved not only at the expense of multithreading and multiprocessing. Some fresh theoretical ideas are required.

# 7. Acknowledgement.

The authors are grateful to A. A. Gubarev for helpful advices on linking C++ code with a Maple worksheet session.
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