Eigenvalue Fluctuations of 1-dimensional random Schrödinger operators

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Abstract

As an extension to the paper by Breuer, Grinshpon, and White [2], we study the linear statistics for the eigenvalues of the Schrödinger operator with random decaying potential with order $O(x^{-\alpha})$ ($\alpha > 0$) at infinity. We first prove similar statements as in [2] for the trace of $f(H)$, where $f$ belongs to a class of analytic functions: there exists a critical exponent $\alpha_c$ such that the fluctuation of the trace of $f(H)$ converges in probability for $\alpha > \alpha_c$, and satisfies a CLT statement for $\alpha \leq \alpha_c$, where $\alpha_c$ differs depending on $f$. Furthermore we study the asymptotic behavior of its expectation value.

1 Introduction

The study of the one-dimensional Schrödinger operator with random decaying potential was initiated by [3] where they found that it has various spectral properties depending on the decay exponent. After the discovery [4] of the connection between the Jacobi matrices and beta-ensemble, there appear many papers studying from the RMT-point of view, e.g., the eigenvalue statistics on the bulk [9 7 10 8], linear statistics [11 2], and eigenfunction statistics [13 12]. In this paper, we consider the following Hamiltonian, to

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extend the work by Breuer, Grinshpon, and White [2].

\[
(H_\alpha u)_n := \begin{cases} 
  u_{n+1} + u_{n-1} + V_\alpha(n)u_n & (n \geq 2) \\
  u_2 + V_\alpha(1)u_1 & (n = 1)
\end{cases}
\]

\[
V_\alpha(n) \equiv \frac{X_n}{n^\alpha}, \quad n \in \mathbb{N}
\]

where \(\{X_n\}\) is a family of i.i.d. random variables. For \(N \in \mathbb{N}\), let \(H_{\alpha,N}\) be the restriction of \(H_\alpha\) on \([1, N]\). Breuer, Grinshpon, and White [2] studied the fluctuation of \(\text{Tr}(P(H_{\alpha,N}))\) and showed that we can find a critical exponent \(\alpha_c\) depending on \(P \in \mathcal{P}_m\) such that the behavior of the fluctuation of \(\text{Tr}(P(H_{\alpha,N}))\) changes drastically at \(\alpha_c\):

1. (\(\alpha > \alpha_c\)) \(\text{Tr}(P(H_{\alpha,N})) - \mathbb{E}[\text{Tr}(P(H_{\alpha,N}))] \xrightarrow{a.s.} Q\)
2. (\(\alpha \leq \alpha_c\)) \(\frac{1}{g_{\alpha/\alpha_c}(N)} \{\text{Tr}(P(H_{\alpha,N})) - \mathbb{E}[\text{Tr}(P(H_{\alpha,N}))]\} \xrightarrow{d} N(0, \sigma(P)^2)\)

where \(Q\) is a random variable with finite variance, and

\[
g_t(N) := \begin{cases} 
  \frac{N^{1-t}}{1-t} & (0 < t < 1) \\
  \log N & (t = 1)
\end{cases}
\]

\(\sigma(P) \geq 0\) is an explicit constant. Moreover, the space \(\mathcal{P}_m\) of polynomials of degree \(m\) has a decomposition \(\mathcal{P}_m = \mathcal{Q}_m \oplus \mathcal{E}_m \oplus \mathcal{Q}_m^\perp\) such that

\[
\alpha_c = \begin{cases} 
  1/2 & (P \in \mathcal{Q}_m) \\
  1/4 & (P \in \mathcal{E}_m) \\
  1/6 & (P \in \mathcal{Q}_m^\perp)
\end{cases}
\]

On the other hand, [11] studied its continuum analogue of \(H_\alpha\) and derived the asymptotics of \(\text{Tr}(1_{(a,b)}(H_{\alpha,N}))\) (\(0 < a < b\)).

1. (\(\alpha > 1/2\)) \(\text{Tr}(1_{(a,b)}(H_{\alpha,N})) - N\mu(a, b) \xrightarrow{a.s.} Q\)
2. (\(\alpha = 1/2\)) \((\log N)^{-1/2} \left\{\text{Tr}(1_{(a,b)}(H_{\alpha,N})) - N\mu(a, b) - (\text{Const.}) \log N\right\} \xrightarrow{a.s.} G(a, b)\)
3. (\(\alpha < 1/2\)) \(N^{-(1/2-\alpha)} \left\{\text{Tr}(1_{(a,b)}(H_{\alpha,N})) - N\mu(a, b) - \sum_{k=1}^{D+1} C_k N^{1-k\alpha}\right\} \xrightarrow{a.s.} G(a, b)\)
where $\mu$ is the IDS (in the continuum, $\mu(a, b) = \pi^{-1}(\sqrt{b} - \sqrt{a})$). $Q$ is a bounded random variable. $D := \min\{d \in \mathbb{N} | \frac{1}{2a} < d + 1\}$, and $\{G(a, b)\}_{0 < a < b}$ is a Gaussian field. $(2)$, $(3)$ holds in the sense of joint distribution w.r.t. $a, b$. In view of $[2, 11]$, we consider the following problems: $(1)$ we consider $Trf(H_{a,N})$ instead of $TrP(H_{a,N})$ where $f$ is an analytic function. $(2)$ For a number of analytic functions $f_1, f_2, \ldots, f_K$, we consider the joint limit of $Trf_j(H_{a,N}) - \mathbb{E}[Trf_j(H_{a,N})]$, $(j = 1, 2, \ldots, K)$, and $(3)$ We study the behavior of $\mathbb{E}[f(H_{a,N})]$. We possibly would like to $Tr1_{(a,b)}(H_{a,N})$ but for that we need to study Prüfer coordinate [6] which is beyond the scope of this paper. It would be difficult to consider general continuous function, so that we work under the following condition in this paper.

**Assumption**

(1) $\{X_n\}$ is a family of i.i.d. bounded random variables, such that $\mathbb{E}[X_n] = 0$, $\eta^2 := \mathbb{E}[X_n^2] > 0$, and $|X_n(\omega)| \leq C_X$, a.s. for a positive constant $C_X$.

(2) $f$ has the Taylor expansion around the origin and its convergence radius satisfies $r(f) > C_X + 2$.

$$f(x) = \sum_{j=0}^{\infty} c_j x^j, \quad |x| < r(f), \quad r(f) > C_X + 2.$$ 

Since $\sigma(H_a) \subset [-2 - C_X, 2 + C_X]$, the assumption on $r(f)$ is a natural one ensuring the convergence of $f(H) = \sum_j c_j H^j$ in the norm topology. We state our results in each subsections below.

**1.1 Analytic extension**

Let $f_k, k = 1, 2, \ldots$ be the polynomial obtained by truncating higher order terms in the Taylor expansion of $f$:

$$f_k(x) = \sum_{j=0}^{k} c_j x^j.$$ 

And let

$$p^k(\beta) := \#\mathcal{P}^k(\beta) \quad (1.1)$$

be the number paths where the initial and terminal points coincide and the number of flat steps on each site is given by a multi-index $\beta$. The precise definition is given in $[2, 11]$. 

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Then we have an analogous statement as in \[2\]: we can find a critical exponent \( \alpha_c \) such that for \( \alpha_c < \alpha \) the fluctuation converges to a random variable, while for \( 0 < \alpha \leq \alpha_c \) the fluctuation is unbounded and satisfies a CLT statement after a suitable scaling.

**Theorem 1**

**Case A**: general case  
(1) \( 1/2 < \alpha \): there exists a random variable \( Q_A \) with \( \text{var}(Q_A) < \infty \) s.t.

\[
\text{Trf}(H_{\alpha,N}) - \mathbb{E}[\text{Trf}(H_{\alpha,N})] \xrightarrow{N \to \infty} Q_A, \quad \text{a.s.}
\]

If \( \alpha > 1 \), \( Q_A \) is bounded.  
(2) \( 0 < \alpha \leq 1/2 \): we have a following CLT statement.

\[
\frac{\text{Trf}(H_{\alpha,N}) - \mathbb{E}[\text{Trf}(H_{\alpha,N})]}{g_{2\alpha}(N)} \xrightarrow{d} N(0, \sigma_A(f)^2),
\]

where

\[
\sigma_A(f)^2 := \sum_{j=0}^{\infty} c_{2j+1} p_j^{2j+1}(\delta) \eta^2, \quad p^k(\delta) = \delta \left(1 - \frac{k-1}{2}\right) 1(k \text{ odd}).
\]

**Case B**: Taylor expansion of \( f \) have even terms only: \( c_{2j-1} = 0 \), \( j \in \mathbb{N} \).  
(1) \( 1/4 < \alpha \): there exists a random variable \( Q_B \) with \( \text{var}(Q_B) < \infty \) s.t.

\[
\text{Trf}(H_{\alpha,N}) - \mathbb{E}[\text{Trf}(H_{\alpha,N})] \xrightarrow{N \to \infty} Q_B \quad \text{in probability.}
\]

(2) \( 0 < \alpha \leq 1/4 \):

\[
\frac{\text{Trf}(H_{\alpha,N}) - \mathbb{E}[\text{Trf}(H_{\alpha,N})]}{g_{4\alpha}(N)} \xrightarrow{d} N(0, \sigma_B(f)^2)
\]

where

\[
\sigma_B(f)^2 := \left(\sum_{j=1}^{\infty} c_j p^j(2\delta)\right)^2 \left(\mathbb{E}[X_1^4] - \eta^4\right) + \sum_{s=1}^{\infty} \left(\sum_{j=1}^{\infty} c_j p^j(\delta + \delta^s)\right)^2 \eta^4.
\]

**Case C**: Taylor expansion of \( f \) have odd terms only: \( c_{2j} = 0 \), \( j \in \mathbb{N} \), and \( f \) has the form of

\[
f(x) = \sum_{j=3}^{\infty} c_j (x^j - p^j(\delta)x) = \sum_{j=3}^{\infty} c_j x^j - \left(\sum_{j=3}^{\infty} p^j(\delta)\right) x.
\]
(1) $1/6 < \alpha$ : there exists a random variable $Q_C$ with $\text{var}(Q_C) < \infty$ s.t.
\[
\text{Trf}(H_{\alpha,N}) - \mathbf{E}[\text{Trf}(H_{\alpha,N})] \overset{N \to \infty}{\to} Q_C \quad \text{in probability.}
\]

(2) $0 < \alpha \leq 1/6$ :
\[
\frac{\text{Trf}(H_{\alpha,N}) - \mathbf{E}[\text{Trf}(H_{\alpha,N})]}{g_{\alpha\alpha}(N)} \overset{d}{\to} N(0, \sigma_C(f)^2)
\]
where
\[
\sigma_C(f)^2 := \lim_{k \to \infty} \sigma_C(f_k)^2.
\]

The expression of $\sigma_C(f_k)$, given in [2](2.11), is so complicated and is omitted. We believe that the convergence in Theorem 1(1) in cases B and C also holds almost surely, not only in probability. In fact, we can show the a.s. convergence if we interchange the order of summation in a random series (Lemma 2.9).

1.2 Joint limit

Suppose $f_t$ $(t = 1, 2, \cdots, d)$ satisfy Assumption. We assume that they belong to same cases among A, B, and C, and set $\alpha_c := 1/2$ (Case A), $1/4$ (Case B), and $1/6$ (Case C), and similarly, $\sigma(f) := \sigma_\sharp(f)$ for Case $\sharp$ $(\sharp = A, B, C)$. We consider the joint limit of the following vector :
\[
F_N := (F_{1,N}, \cdots, F_{d,N})
\]
where
\[
F_{t,N} := \frac{\text{Trf}_t(H_{\alpha,N}) - \mathbf{E}[\text{Trf}_t(H_{\alpha,N})]}{g_{\alpha\alpha_c}(N)}, \quad t = 1, 2, \cdots, d.
\]

Theorem 2 Let $0 < \alpha \leq \alpha_c$
\[
F_N \overset{d}{\to} F, \quad \text{where} \quad F \sim N(0, \Sigma_F), \quad \Sigma_F := (\sigma(f_i)\sigma(f_j))_{i,j}.
\]

1.3 Behavior of the expectation value

Let $m := \max\{k \in \mathbb{N} \mid k\alpha \leq 1\}$. Then we have the asymptotic expansion for the expectation value of the trace of $f(H)$.


Theorem 3

\[ E[Tr f(H_{\alpha,N})] = \left( \sum_l c_l C_{0,l} \right) N + \left( \sum_l c_l C_{2,l} \right) S_2(N) + \cdots + \left( \sum_l c_l C_{m,l} \right) S_m(N) + C_N(f) \]

where

\[ S_j(N) := \sum_{n=1}^N \frac{1}{n^{j\alpha}} = \begin{cases} \frac{N^{1-j\alpha}}{1-j\alpha} + O(1) & (j\alpha < 1) \\ \log N + O(1) & (j\alpha = 1) \end{cases} \]

for \( j = 1, 2, \cdots, m \), \( C_{0,l}, \cdots, C_{m,l} \) are explicit constants, and \( C_N(f) \) is convergent as \( N \to \infty \).

In later sections, we prove these theorems. In Section 2, we first recall the argument in [2] which relates \( Tr(H^k) \) to the path counting on \( Z \). Then we prepare some lemmas to prove Theorem 1. In Section 3, we prove Theorem 1. Proof of the CLT statement is a simple application of a limit theorem on the triangular array of random variables. However, to show Theorem 1(1) in Cases B and C, one needs complicated computations and estimates of the variance. In Section 4, we prove Theorem 2 using Cramer-Wald’s method. In Section 5 (resp. in Section 6), we prove Theorem 3 for \( f(x) = x^k \) (resp. for analytic functions).

2 Preliminaries

2.1 Paths with flat steps

Since \( Tr H_{\alpha,N}^k \) is a polynomial of \( V(1), \cdots, V(N) \), it is important to derive the coefficients of that. In [2], they express them in terms of the lattice path on \( Z \) with certain conditions. We briefly recall this argument in [2], not only for completeness but also to fix the notation.

Definition 2.1

(1) A multi-index is a map \( \beta : \mathbb{Z} \to \mathbb{N} \cup \{0\} \) such that \( \beta(h) = 0 \) but finitely many \( h \)'s. We sometimes write \( \beta_h \) instead of \( \beta(h) \). For a multi-index \( \beta \), let
$V^\beta$ be the monomial given by
\[ V^\beta := \prod_{h \in \mathbb{Z}} V(h)^{\beta(h)}. \]
We set $V(h) = 0$ for $h \leq 0$, and also set $0^n := 1$ for convenience. The multiplicity of $V^\beta$ is denoted by $|\beta| := \sum_{j=1}^N \beta(j)$. The translation $\beta^i$ of $\beta$ for $i \in \mathbb{Z}$ is defined by
\[ (\beta^i)(h) := \beta(h - i), \quad h \in \mathbb{Z}. \]

(2) $\delta = \delta_0$ is a multi-index with
\[ \delta(h) := 1(h = 0). \]

**Definition 2.2**

(1) Path on $\mathbb{Z}$ of length $k$ is a sequence $y := (y_0, y_1, \ldots, y_k)$ ($y_i \in \mathbb{Z}$) such that $|y_t - y_{t-1}| \leq 1$. We say a pair $(y_{t-1}, y_t)$ is a up step (resp. down step, flat step) if $y_t - y_{t-1} = 1$ (resp. $y_t - y_{t-1} = -1$, $y_t - y_{t-1} = 0$). We say $(y_{t-1}, y_t)$ is a flat step of level $h \in \mathbb{Z}$ if $y_t = y_{t-1} = h \in \mathbb{Z}$. Let $Q^k$ be the set of paths of length $k$ starting from the origin:
\[ Q^k := \{ y = (y_0, y_1, \ldots, y_k) \mid y_t \in \mathbb{Z}, \ y_t - y_{t-1} = \pm 1, 0 \ (t = 1, 2, \ldots, k), \ y_0 = 0 \}. \]

For a given multi-index $\beta$, let $Q^k(\beta)$ be the set of paths in $Q^k$ such that the number of flat steps of which is counted by a suitable translation of $\beta$:
\[ Q^k(\beta) := \{ y = (y_0, y_1, \ldots, y_k) \in Q^k \mid \exists i \in \mathbb{Z} \ s.t. \ \beta^i(h) = \sharp\{ t \mid y_{t-1} = y_t = h \}, \ \forall h \}. \]
We note that, for $y \in Q^k(\beta)$, the corresponding $i \in \mathbb{Z}$ is uniquely determined.

We aim to relate each terms in $Tr(H^k_{\alpha,N})$ (as a polynomial of $V$) to the paths in $Q^k$. In order for that, we write
\[ H_{\alpha,N} = S_N + V_{\alpha,N} + S_N^* \]
where $S_N$ is the right shift on $\ell^2([1, N])$. Expanding $(S_N + V_{\alpha,N} + S_N^*)^k$ and taking each terms, we have
\[ H^k_{\alpha,N} = (S_N + V_{\alpha,N} + S_N^*)^k = \sum_{M \in \mathcal{M}} M \]
where \( \mathcal{M} \) is the set of strings of \( S_N, S_N^*, V_{\alpha,N} \) of length \( k \):

\[
\mathcal{M} := \{ M_1M_2\cdots M_k \mid M_t = S_N, S_N^*, V_{\alpha,N}, \ t = 1, 2, \cdots, k \}.
\]

For \( M = M_1M_2\cdots M_k \in \mathcal{M} \), we set the corresponding path \( y = (y_0, y_1, \cdots, y_k) \in \mathcal{Q}^k \) as

\[
y_0 := 0, \quad y_t - y_{t-1} := \begin{cases} 
1 & (M_t = S_N) \\
-1 & (M_t = S_N^*) \\
0 & (M_t = V_{\alpha,N})
\end{cases}
\]

which gives a bijection between \( \mathcal{M} \) and \( \mathcal{P}^k \). We henceforth write \( y(M) \in \mathcal{Q}^k \) is a path corresponding to \( M \in \mathcal{M} \) and vice versa.

**Lemma 2.1**

For \( M = M_1M_2\cdots M_k \in \mathcal{M} \), let \( y(M) = (y_0, y_1, \cdots, y_k) \in \mathcal{Q}^k \) be the corresponding path. For \( i, j \in [N] \), we have

\[
M_{ij} = V^\beta \cdot 1 \left( \beta, y(M) \text{ satisfies condition } (C)^\beta_{ij} \right)
\]

where \( 1(P) \) is the indicator function associated to a proposition \( P \) and we say a multi-index \( \beta \) and \( y \in \mathcal{P}^k \) satisfies condition \( (C)^\beta_{ij} \) if

\[
(C_1) \ y + i \text{ lies in } [1, N]: \quad 1 \leq y_t \leq N, \quad 1 \leq t \leq k.
\]

\[
(C_2) \ \beta^{-1} \text{ counts the number of flat steps of } y: \quad \sharp \{ t \mid y_t = y_{t-1} = h \} = \beta(h + i), \quad \forall h \in \mathbb{Z}.
\]

\[
(C_3) \ j - i = y_k(M).
\]

Note that, the condition \( (C_2) \) determines \( \beta \) uniquely, and for each \( i \in [N] \), there is at most one \( j \) such that the condition \( (C)^\beta_{ij} \) is satisfied.

**Proof.** For simplicity of notation, we write \( U := S_N, \ D := S_N^* \). On the products in \( M = M_1M_2\cdots M_k \in \mathcal{M} \), we group together the products of \( U \)'s
which are divided by an appearance of $V$, and denote by $W_1, W_2, \ldots$ the products of $U$’s in each group:

$$M = M_1 M_2 \cdots M_k = (UD \cdots U) V (DU \cdots D) V \cdots V (DU \cdots U)$$

$$= W_1 V W_2 V \cdots W_f V W_{f+1}$$

where the number of flat steps is equal to $f$ in $\mathbf{y}(M)$. For each $W_l$ ($l = 1, 2, \cdots, f + 1$), let $s_l := \sharp U - \sharp D$ be the differences between the number of $U$’s and that of $V$’s. Letting $y_t (n_l \leq t \leq m_l)$ be the part of $\mathbf{y}(M)$ associated to $W_l$, we have

$$(W_l)_{ij} = 1 \ (1 \leq i + y_t \leq N, \ n_l \leq t \leq m_l) \cdot 1 \ (j = i + s_l)$$

so that

$$M_{ij} = V(i + s_1) V(i + s_1 + s_2) \cdots V(i + s_1 + s_2 + \cdots + s_f) \times$$

$$\times 1 \ (1 \leq i + y_t \leq N, \ 1 \leq t \leq k) \cdot 1 \ (j = i + y_k)$$

Because the set $\{i + s_1, i + s_1 + s_2, \cdots, i + s_1 + \cdots + s_f\}$ coincides as a multi-set with that of levels $\{h \mid \exists t \in \{0, 1, \cdots, k - 1\}, \text{s.t. } y_{t-1} = y_t = h\}$ of flat steps of $\mathbf{y}(M)$, we have

$$M_{ij} = V^\beta \cdot 1 \ (1 \leq i + y_t \leq N, \ 1 \leq t \leq k, \ j = i + y_k)$$

where $\beta(h + i) = \sharp \{t \mid y_t = y_{t-1} = h\}, \ \forall h \in \mathbb{Z}.$

□

In what follows, we consider the diagonal element of $H_{\alpha,N}^k$ and set $i = j$, in the corresponding paths of which the number of up steps is equal to that of down steps. We introduce the subset of $Q^k$ returning to the origin satisfying $(C_1)_{i,i}^\beta$, and those satisfying $(C_2)$ only:

$$C_i^k(\beta) := \left\{ y \in Q^k \mid y_0 = y_k = 0, \ \beta, y \text{ satisfies } (C_1)_{i,i}^\beta \right\}$$

$$P_i^k(\beta) := \left\{ y \in Q^k \mid y_0 = y_k = 0, \ \sharp \{t \mid y_t = y_{t-1} = h\} = \beta(h + i), \ \forall h \in \mathbb{Z} \right\}$$

$$\mathcal{P}^k(\beta) := \bigcup_i \mathcal{P}_i^k(\beta)$$

By definition, $C_i^k(\beta) \subset \mathcal{P}_i^k(\beta)$ and $\mathcal{P}^k(\beta) = \bigcup_i \mathcal{P}_i^k(\beta)$ is a disjoint union. And let

$$p^k(\beta) := \sharp \mathcal{P}^k(\beta)$$

(2.1)
be the number paths in $P^k(\beta)$, which is the number of paths where the number of flat steps on each site is given by a multi-index $\beta$. For a polynomial $P$, we denote by $[x^k]P$ the coefficient of $x^k$ in $P$.

**Lemma 2.2**

1. $[V^\beta](H^k_{\alpha,N})_{ii} = \sharp C_i^k(\beta)$,
2. $\frac{k}{2} \leq i \leq N - \frac{k}{2} \implies [V^\beta](H^k_{\alpha,N})_{ii} = \sharp P_i^k(\beta)$.

**Proof.** (1) Letting $i = j$ in Lemma 2.1, we have

$$M_{ii} = V^\beta \cdot 1 \left( y(M) \in C_i^k(\beta) \right).$$

We thus have

$$(H^k_{\alpha,N})_{ii} = \sum_{y \in \mathcal{P}^k} M_{ii}(y) = \sum_{\beta} \sum_{y \in C_i^k(\beta)} M_{ii}(y) = \sum_{\beta} \sum_{y \in C_i^k(\beta)} V^\beta = \sum_{\beta} \sharp C_i^k(\beta)V^\beta.$$

(2) If $\frac{k}{2} \leq i \leq N - \frac{k}{2}$ then the condition $(C_1)$ is always satisfied and $C_i^k(\beta) = P_i^k(\beta)$. \qed

For a multi-index $\beta$, let

$$\iota(\beta) := \min \text{ supp } \beta$$

$$a_N^k(\beta) := [V^\beta]Tr(H^k_{\alpha,N}).$$

**Lemma 2.3** If $\beta \neq 0$, and $\iota(\beta) \in [k, N - k]$, then $a_N^k(\beta) = p^k(\beta)$.

**Proof.** We note that, if $\iota(\beta) \in [k, N - k]$, then the corresponding $i \in \mathbb{Z}$, such that $\beta^{-i}$ counts the number of flat steps starting at the origin, satisfies $i \in \left[\frac{k}{2}, N - \frac{k}{2}\right]$. We decompose $Tr H^k_{\alpha,N}$ as

$$Tr H^k_{\alpha,N} = \sum_i (H^k_{\alpha,N})_{ii} = \sum_i \sum_{\beta} \sharp C_i^k(\beta)V^\beta$$

$$= \left( \sum_{\iota(\beta) \in [k, N - k]} + \sum_{\iota(\beta) \in [k, N - k]^c} \right) \sum_i \sharp C_i^k(\beta)V^\beta$$

$$=: (1) + (2).$$

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Since $P^k(\beta) = \bigcup_{i \in \mathbb{Z}} P^k_i(\beta)$ is a disjoint union, $p^k(\beta) = \sum_i \#P^k_i(\beta)$ so that we have

\[(1) = \sum_{i(\beta) \in [k,N-k]} \sum_i \#P^k_i(\beta)V^\beta = \sum_{i(\beta) \in [k,N-k]} p^k(\beta)V^\beta.\]

At the end of this subsection, we give some estimates on the number of the specific paths.

**Lemma 2.4**

1. $\sum_{\beta : |\beta| = j} p^l(\beta) \leq \binom{l}{j} \left( \binom{l-j}{\frac{l-j}{2}} \right)$

2. $\sum_{j=0}^{l} \sum_{\beta : |\beta| = j} p^l(\beta)C_X^j \leq (C_X + 2)^l$

3. $a^k(\delta) \leq p^k(\delta) = \frac{1}{\sqrt{2\pi}} \sqrt{k} \cdot 2^k(1 + o(1)), \quad k \to \infty$

**Proof.** (1)

\[\sum_{\beta : |\beta| = j} p^l(\beta) \leq \#\{y = (y_0, y_1, \ldots, y_l) \in P^l \mid y_0 = y_l = 0, \#\{\text{flat steps}\} = j\} \leq \binom{l}{j} \left( \binom{l-j}{\frac{l-j}{2}} \right)\]

where $\binom{l}{j}$ is equal to the number of ways of choosing the flat steps, and $\binom{l-j}{\frac{l-j}{2}}$ is equal to the number of path of length $l - j$ starting from and coming back to the origin. (2) We use (1) and the multinomial theorem to compute

\[\sum_{j=0}^{l} \sum_{\beta : |\beta| = j} p^l(\beta)C_X^j \leq \sum_{j=0}^{l} \frac{l!}{j! \left( \frac{l-j}{2} \right)!} C_X^j \leq \sum_{i,j,k} \frac{l!}{j! \cdot i! \cdot k!} 1^i \cdot 1^k \cdot C_X^j \]

\[= (1 + 1 + C_X)^l = (C_X + 2)^l.\]
The first inequality follows from $C_k^i(\beta) \subset P_k^i(\beta)$. For the second one, we use the following equations

$$p^k(\delta) = \frac{k!}{\left(\frac{k-1}{2}\right)!}, \quad k: \text{odd}$$

$$\left(\begin{array}{c} n \\ n/2 \end{array}\right) = \frac{1}{\sqrt{2\pi}} \frac{2^{n+1}}{\sqrt{n}} (1 + o(1)), \quad n: \text{even}, n \to \infty.$$
Then by Lemma 2.4, we have
\[
|c_j| \sum_{\beta \in [1,N]} |a_N^j(\beta) - p^j(\beta)| \left| V_\beta - E[V_\beta] \right|
\]
\[
= |c_j| \left( \sum_{i=1}^{N/2} \sum_{i=N-j}^{N} |a_N^i(\beta) - p^i(\beta)| \left| V_\beta^i - E[V_\beta^i] \right| \right)
\]
\[
\leq |c_j| j \left( \frac{j}{\ell} \right) \frac{2C_X}{\alpha} \rightarrow 0.
\]
Therefore the conclusion follows from the dominated convergence theorem.

\[\square\]

**Lemma 2.6**

\[
\sum_{j=0}^{\infty} c_j \sum_{\beta \in [1,N]} \left| a_N^j(\beta) - p^j(\beta) \right| \left( V_\beta - E[V_\beta] \right)
\]
\[
\rightarrow \sum_{j=0}^{\infty} c_j \sum_{\beta \in [1,j/2]} \left| a^j(\beta) - p^j(\beta) \right| \left( V_\beta - E[V_\beta] \right), \text{ a.s.}
\]
Proof. It suffices to make sure that the condition to apply the dominated convergence theorem is satisfied: then we sum up the result of Lemma 2.5 with respect to \( \ell \geq \ell_0 \). The point is that, for fixed \( j \), \( \sum_{\ell \geq \ell_0} \) becomes a finite sum. In fact,

\[
|c_j| \sum_{\ell=\ell_0}^{j} \sum_{|\beta| \geq \ell_0 \atop \iota(\beta) \in [1, N]} |a_N^j(\beta) - p^j(\beta)| |V^\beta - \mathbf{E}[V^\beta]| \leq |c_j| \sum_{\ell=\ell_0}^{j} j \left( \frac{j - \ell}{\ell} \right) 2C_X \leq 2|c_j|j^2(2 + C_X)^j.
\]

\( \square \)

Lemma 2.7

Suppose \( \alpha \ell_0 > 1 \). Then

\[
\sum_{j=0}^{\infty} c_j \sum_{\iota(\beta) \in [1, N] \atop |\beta| \geq \ell_0} p^j(\beta) (V^\beta - \mathbf{E}[V^\beta])
\]

converges as \( N \to \infty \) almost surely.

Proof. Since

\[
\sum_{j=0}^{\infty} c_j \sum_{\iota(\beta) \in [1, N] \atop |\beta| \geq \ell_0} p^j(\beta) (V^\beta - \mathbf{E}[V^\beta]) = \sum_{j=0}^{\infty} \sum_{i=1}^{N} c_j \sum_{\iota(\beta) = i \atop |\beta| \geq \ell_0} p^j(\beta) (V^{\beta_i} - \mathbf{E}[V^{\beta_i}]),
\]
it suffices to show that RHS converges absolutely. In fact,

\[
\sum_{j=0}^{\infty} \sum_{i=1}^{\infty} |c_j| p^j(\beta) \left| V^\beta \right| \leq \sum_{j=0}^{\infty} \sum_{i=1}^{\infty} |c_j| \sum_{\ell=\ell_0}^{j} p^j(\beta) \frac{2C_X^\ell}{i^{\alpha \ell}} \\
\leq \sum_{j=0}^{\infty} \sum_{i=1}^{\infty} |c_j| \sum_{\ell=\ell_0}^{j} \binom{j}{\ell} \binom{j-\ell}{\ell/2} \frac{2C_X^\ell}{i^{\alpha \ell_0}} \\
\leq 2 \sum_{j=0}^{\infty} |c_j|(2 + C_X)^j \sum_{i=1}^{\infty} \frac{1}{i^{\alpha \ell_0}} < \infty.
\]

\[\square\]

**Lemma 2.8**

*Suppose \(\alpha \ell > \frac{1}{2}\). Then*

\[
\sum_{i(\beta) \in [1, N]}^{\infty} \sum_{|\beta| = \ell} c_j p^j(\beta) (V^\beta - \mathbb{E}[V^\beta])
\]

*converges as \(N \to \infty\) in probability.*
Proof. We show that the quantity in question is Cauchy in probability.

\[
\begin{align*}
\mathbb{E} & \left[ \sum_{\substack{i(\beta) \in [N+1,M] \atop |\beta| = \ell}} \sum_{j=0}^{\infty} c_j \breve{p}^j(\beta) \left( V^\beta - \mathbb{E}[V^\beta] \right) \right]^2 \\
& = \mathbb{E} \left[ \sum_{i=N+1}^{M} \sum_{\substack{i(\beta) = 0 \atop |\beta| = \ell}} \sum_{j=0}^{\infty} c_j \breve{p}^j(\beta) \left( V^{\beta_i^1} - \mathbb{E}[V^{\beta_i^1}] \right) \right]^2 \\
& = \mathbb{E} \left[ \sum_{i_1=N+1}^{M} \sum_{i_2=N+1}^{M} \sum_{\substack{i(\beta) = 0 \atop |\beta| = \ell}} \sum_{\substack{i(\gamma) = 0 \atop |\gamma| = \ell}} \left( \sum_{j=0}^{\infty} c_j \breve{p}^j(\beta) \right) \left( \sum_{j' \geq 0} c_{j'} \breve{p}^{j'}(\gamma) \right) \right. \\
& \quad \times \left. \left( V^{\beta_i^1} - \mathbb{E}[V^{\beta_i^1}] \right) \left( V^{\gamma_i^2} - \mathbb{E}[V^{\gamma_i^2}] \right) \right] \\
& = \sum_{i_1=N+1}^{M} \sum_{i_2=N+1}^{M} \sum_{\substack{i(\beta) = 0 \atop |\beta| = \ell}} \sum_{\substack{i(\gamma) = 0 \atop |\gamma| = \ell}} \left( \sum_{j=0}^{\infty} c_j \breve{p}^j(\beta) \right) \left( \sum_{j' \geq 0} c_{j'} \breve{p}^{j'}(\gamma) \right) \text{Cov} \left( V^{\beta_i^1}, V^{\gamma_i^2} \right).
\end{align*}
\]

We remark that, the estimate

\[
\breve{p}^j(\beta) \leq \left( \frac{j}{\ell} \right) \left( \frac{j - \ell}{j - \ell} \right) \leq \left( \frac{j}{\ell} \right) \cdot 2^{j-\ell}
\]
guarantee the absolute convergence of $\sum_{j\geq 0} c_j \beta^j(\beta)$. We further compute

$$\begin{align*}
&= \sum_{\ell(\beta)=0 \atop |\beta|=\ell} \sum_{\ell(\gamma)=0 \atop |\gamma|=\ell} \left( \sum_{j_1 \geq 0} c_{j_1} \beta^{j_1}(\beta) \right) \left( \sum_{j_2 \geq 0} c_{j_2} \beta^{j_2}(\gamma) \right) \frac{M}{i_1=N+1} \frac{M}{i_2=N+1} \text{Cov} \left( V^{\beta_{i_1}}, V^{\gamma_{i_2}} \right) \\
&= \sum_{\ell(\beta)=0 \atop |\beta|=\ell} \sum_{\ell(\gamma)=0 \atop |\gamma|=\ell} \left( \sum_{j_1 \geq 0} c_{j_1} \beta^{j_1}(\beta) \right) \left( \sum_{j_2 \geq 0} c_{j_2} \beta^{j_2}(\gamma) \right) \times \left( \sum_{i_1=N+1}^M \sum_{i_2=N+1}^M + \sum_{i_2=N+1}^M \sum_{i_1=N+1}^M \sum_{i_2=N+1}^M \right) \text{Cov} \left( V^{\beta_{i_1}}, V^{\gamma_{i_2}} \right) \\
&\leq 2 \sum_{\ell(\beta)=0 \atop |\beta|=\ell} \sum_{\ell(\gamma)=0 \atop |\gamma|=\ell} \left( \sum_{j_1 \geq 0} |c_{j_1}| \beta^{j_1}(\beta) \right) \left( \sum_{j_2 \geq 0} |c_{j_2}| \beta^{j_2}(\gamma) \right) \frac{M}{i_1=N+1} \frac{M}{i_2=N+1} \frac{Cov \left( X^{\beta_{i_1}}, X^{\gamma_{i_2}} \right)}{\prod_h h^{-2(\alpha_h^{\beta_h}+\alpha_h^{\gamma_h})}} \\
\end{align*}$$

By using

$$\prod_h l_h^{\alpha_h^{\beta_h}+\alpha_h^{\gamma_h}} \geq l_1^{\alpha(|\beta|+|\gamma|)} = l_1^{2\alpha\ell}, \quad \left| \text{Cov} \left( X^{\beta_{i_1}}, X^{\gamma_{i_2}} \right) \right| \leq (2C_X)^{2\ell}$$

we have

$$\begin{align*}
\leq & \sum_{\ell(\beta)=0 \atop |\beta|=\ell} \sum_{\ell(\gamma)=0 \atop |\gamma|=\ell} \sum_{j_1 \geq 0} |c_{j_1}| \beta^{j_1}(\beta) \sum_{j_2 \geq 0} |c_{j_2}| \beta^{j_2}(\gamma) \frac{M}{i_1=N+1} \frac{M}{i_2=N+1} \frac{\sum_{i_1=N+1}^M \sum_{i_2=N+1}^M 4C_X^{2\ell}}{l_1^{2\alpha\ell}} \\
\leq & 8 \left( \sum_{j_1 \geq 0} |c_{j_1}| (j_1+1) \sum_{\ell(\beta)=0 \atop |\beta|=\ell} \beta^{j_1}(\beta) \right) \left( \sum_{j_2 \geq 0} |c_{j_2}| \sum_{\ell(\gamma)=0 \atop |\gamma|=\ell} \beta^{j_2}(\gamma) \right) C_X^{2\ell} \sum_{i_1=N+1}^M \frac{1}{l_1^{2\alpha\ell}}. \\
\end{align*}$$

Here we notice that

$$\sum_{j_1 \geq 0} |c_{j_1}| (j_1+1) \sum_{\ell(\beta)=0 \atop |\beta|=\ell} \beta^{j_1}(\beta) \leq \sum_{j_1 \geq 0} |c_{j_1}| (j_1+1) (\frac{j_1}{\ell})^{2j_1-\ell} < \infty.$$
Thus by Chebyshev’s inequality,

\[
\mathbb{P} \left( \left| \sum_{j=0}^{\infty} c_j \sum_{\substack{\iota(\beta) \in [1,N] \\ |\beta| = \ell}} p^j(\beta) (V^\beta - \mathbb{E}[V^\beta]) - \sum_{j=0}^{\infty} c_j \sum_{\substack{\iota(\beta) \in [1,M] \\ |\beta| = \ell}} p^j(\beta) (V^\beta - \mathbb{E}[V^\beta]) \right| \geq \epsilon \right) \\
\leq \epsilon^{-2} \mathbb{E} \left[ \sum_{j=0}^{\infty} c_j \sum_{\substack{\iota(\beta) \in [1,N] \\ |\beta| = \ell}} p^j(\beta) (V^\beta - \mathbb{E}[V^\beta]) - \sum_{j=0}^{\infty} c_j \sum_{\substack{\iota(\beta) \in [1,M] \\ |\beta| = \ell}} p^j(\beta) (V^\beta - \mathbb{E}[V^\beta]) \right]^2
\]

\[
\rightarrow 0, \quad N, M \to \infty
\]

\[
\square
\]

Before ending this section, we show below that the “limit” of the random variable in Lemma 2.8 really converges as a random series. This fact is not used in this paper, but we include here for completeness.

**Lemma 2.9** Suppose \( \alpha \ell_0 > \frac{1}{2} \). Then the random series

\[
\sum_{j=0}^{\infty} c_j \sum_{\substack{\iota(\beta) \in [1,\infty] \\ |\beta| \geq \ell_0}} p^j(\beta) (V^\beta - \mathbb{E}[V^\beta])
\]

converges and define a random variable.

**Remark** Lemma 2.9 says that, the random series

\[
\sum_{j=0}^{\infty} c_j \sum_{i=1}^{\infty} \sum_{\substack{\iota(\beta) = 0 \\ |\beta| \geq \ell_0}} p^j(\beta) (V^{\beta_i} - \mathbb{E}[V^{\beta_i}])
\]

converges almost surely, if one takes \( \sum_{i=1}^{\infty} \) first and then \( \sum_{j=0}^{\infty} \). Since we are not able to show the absolute convergence, we do not know whether a.s. convergence holds if we change the order of summation.
Proof. We first note that, by Lemma 4.2 in [2], The random series
\[ \sum_{\tau(\beta) \in [1, N]} p^j(\beta) (V^\beta - E[V^\beta]) = \sum_{i=1}^{N} \sum_{\tau(\beta) = 0} p^j(\beta) (V^\beta_i - E[V^\beta_i]) \]
converges a.s. and define a random variable.
\[ \sum_{\tau(\beta) \in [1, \infty)} p^j(\beta) (V^\beta - E[V^\beta]) = \sum_{i=1}^{\infty} \sum_{\tau(\beta) = 0} p^j(\beta) (V^\beta_i - E[V^\beta_i]) \]
with finite variance of which we estimate below.
\[
Var \left( \sum_{i=1}^{\infty} \sum_{\tau(\beta) = 0} p^j(\beta) (V^\beta_i - E[V^\beta_i]) \right) \\
\leq \liminf_{N \to \infty} E \left[ \left( \sum_{i=1}^{N} \sum_{\tau(\beta) = 0} p^j(\beta) (V^\beta_i - E[V^\beta_i]) \right)^2 \right] \\
\leq \sum_{\tau(\beta) = 0} p^j(\beta) \sum_{\tau(\gamma) = 0} p^j(\gamma) \limsup_{N \to \infty} \left| \sum_{i=1}^{N} \sum_{i' = 1}^{N} Cov \left( (V^\beta_i - E[V^\beta_i]), (V^\gamma_{i'} - E[V^\gamma_{i'}]) \right) \right| \\
\leq \sum_{i=1}^{N} \sum_{i' = 1}^{N} \left| Cov \left( (V^\beta_i - E[V^\beta_i]), (V^\gamma_{i'} - E[V^\gamma_{i'}]) \right) \right| \leq \sum_{i=1}^{N} j \cdot \frac{4C_{X}^{\beta |\beta| + |\gamma|}}{2^{2\alpha_{0}}} \\
\]
so that by Lemma 2.4,
\[
\sum_{\tau(\beta) = 0} p^j(\beta) \sum_{\tau(\gamma) = 0} p^j(\gamma) \limsup_{N \to \infty} \sum_{i=1}^{N} \sum_{i' = 1}^{N} Cov \left( (V^\beta_i - E[V^\beta_i]), (V^\gamma_{i'} - E[V^\gamma_{i'}]) \right) \\
\leq 4 \sum_{\tau(\beta) = 0} p^j(\beta) \sum_{\tau(\gamma) = 0} p^j(\gamma) jC_{X}^{\beta |\beta| + |\gamma|} \sum_{i=1}^{\infty} \frac{1}{i^{2\alpha_{0}}} \leq 4j(C_{X} + 2)^{2j} \sum_{i=1}^{\infty} \frac{1}{i^{2\alpha_{0}}}.
\]
By Chebyshev’s inequality,
\[
P \left( \left| \sum_{i=1}^{\infty} \sum_{|\beta| \geq t_0} p^j(\beta) \left( V^{\beta^i} - E[V^{\beta^i}] \right) \right| > j^\delta (C_X + 2)^j \right) \leq 4j^{-2\delta} \sum_{i=1}^{\infty} \frac{1}{j^{2\alpha t_0}}
\]
which is summable with respect to \( j \) if \( \delta > 1 \). Thus by Borel-Cantelli lemma,
\[
\left| \sum_{i=1}^{\infty} \sum_{|\beta| \geq t_0} p^j(\beta) \left( V^{\beta^i} - E[V^{\beta^i}] \right) \right| \leq j^\delta (C_X + 2)^j
\]
for sufficiently large \( j \), almost surely. Therefore
\[
\sum_{j=0}^{\infty} c_j \sum_{i=1}^{\infty} \sum_{|\beta| \geq t_0} p^j(\beta) \left( V^{\beta^i} - E[V^{\beta^i}] \right)
\]
converges almost surely. \( \square \)

3 Proof of Theorem 1

We prove A(1), A(2), \( \cdots \), C(2) in Theorem 1 respectively in each subsection below.

3.1 Proof of A(1)

Since \( \sigma(H_{a,N}) \subset [-C_X - 2, C_X + 2] \), and since \( r(f) > C_X + 2 \), the power series \( f(H) = \sum_{i=0}^{\infty} a_i H^i \) converges in the operator norm topology. Hence \( Tr f(H) = \sum_{i=0}^{\infty} c_i Tr(H^i) \) is absolutely convergent so that by Fubini theorem,
\[
E[Tr f(H)] = \sum_{i=0}^{\infty} c_i E[Tr(H^i)].
\]
Letting $a^j_\beta := [V^\beta](Tr(H^j_{\alpha,N}))$, we compute

$$Trf(H_{\alpha,N}) - E[Trf(H_{\alpha,N})] = \sum_{j=0}^\infty c_j \{Tr(H^j_{\alpha,N}) - E[Tr(H^j_{\alpha,N})]\}$$

$$= \sum_{j=0}^\infty c_j \left( \sum_{|\beta|=1} + \sum_{|\beta|\geq 2} \right) a^j_\beta (V^\beta - E[V^\beta])$$

$$=: I + II.$$

Since $|\beta| = 1$ implies $\beta = \delta^n$, for some $n = 1, 2, \ldots, N$, we have

$$I = \sum_{n=1}^N \sum_{j=0}^\infty c_j a^j_{\delta^n} X_n^{n\alpha}. \quad \text{Here we use}$$

**Lemma 3.1 ([5] Theorem 2.5.6)**

*Suppose $\{Y_n\}_n$ are independent with $E[Y_n] = 0$. If $\sum_n Var(Y_n) < \infty$, then $\sum_{n=1}^\infty Y_n$ is convergent a.s.*

Put $Y_n := \sum_{j=0}^\infty c_j a^j_{\delta^n} X_n^{n\alpha}$. Then by Lemma 2.4(3),

$$Var(Y_n) \leq \left( \sum_{j=0}^\infty c_j a^j_{\delta^n} \right)^2 \frac{n^2}{n^{2\alpha}} \leq (\text{Const.}) \left( \sum_{j=0}^\infty c_j (2 + C_X)^j \right)^2 \frac{n^2}{n^{2\alpha}}$$

so that $\sum_n Var(Y_n) < \infty$ implying $I = \sum_{n=1}^\infty Y_n$ is convergent a.s.

For $II$, the relevant paths have more than one flat steps which produces the convergent factor. In fact, by Lemma 2.4(2),

$$|II| \leq \sum_{j=0}^\infty |c_j| \sum_{i=1}^N \sum_{|\beta|\geq 2} a^j_{\delta^n} \frac{|X^\beta - E[X^\beta]|}{\prod_{n \in \mathbb{N}} n^\alpha^{\beta_n}}$$

$$\leq \sum_{j=0}^\infty |c_j| \sum_{i=1}^N \sum_{|\beta|\geq 2} a^i_j(\beta) \frac{2(C_X)^{|\beta|}}{i^{2\alpha}}$$

$$\leq (\text{Const.}) \sum_{j=0}^\infty |c_j|(C_X + 2)^j \sum_{i=1}^N \frac{1}{i^{2\alpha}} < \infty$$

Therefore $II$ converges a.s.
3.2 Proof of A(2)

The key lemma is:

Lemma 3.2 ([1] Theorem 25.5) Suppose that \( \{Y_N\}, \{X_{N,k}\}, N, k \in \mathbb{N} \) are random variables satisfying

(i) \( X_{N,k} \xrightarrow{d} X_k, N \to \infty, \) for any fixed \( k \)
(ii) \( X_k \xrightarrow{d} X, k \to \infty \)
(iii) For any \( \epsilon > 0, \lim_{k \to \infty} \limsup_{N \to \infty} P(|Y_N - X_{N,k}| > \epsilon) = 0. \)

Then \( Y_N \xrightarrow{d} X. \)

We set

\[
X_{N,k} := \frac{Trf_k(H_{\alpha,N}) - E[Trf_k(H_{\alpha,N})]}{g_{2\alpha}(N)}, \quad X_k := N(0, \sigma_A(P_k)^2)
\]

\[
Y_N := \frac{Trf(H_{\alpha,N}) - E[Trf(H_{\alpha,N})]}{g_{2\alpha}(N)}, \quad X := N(0, \sigma_A(f)^2).
\]

to apply Lemma 3.2, where \( f_k(x) := \sum_{j=0}^{k} c_j x^j \) is the polynomial by truncating the Taylor expansion of \( f. \) To finish the proof, it suffices to check the three conditions in Lemma 3.2

(i) By Theorem 1.1 in [2], we have \( X_{N,k} \xrightarrow{d} X. \)
(ii) \( X_k \xrightarrow{d} X \) is obvious. In fact, the characteristic function of \( X_k \) converges to that of \( X : \exp\left(-\frac{1}{2} \sigma_A(f_k)^2 t^2\right) \)
\( \xrightarrow{k \to \infty} \exp\left(-\frac{1}{2} \sigma_A(f)^2 t^2\right). \)
(iii) As in the proof for Theorem 1(1), we can show

\[
Var(Y_N - X_{N,k}) = E\left[ \left( \sum_{j \geq k+1} c_j \frac{\sum_{|\beta| \geq 1} a_{j}^{\beta} (V_{\alpha}^{\beta} - E[V_{\alpha}^{\beta}])}{g_{2\alpha}(N)} \right)^2 \right] (3.1)
\]

We further compute

\[
E\left[ \left\{ \sum_{j \geq k+1} c_j \sum_{|\beta| \geq 1} a_{j \beta} (V_{\alpha}^{\beta} - E[V_{\alpha}^{\beta}]) \right\}^2 \right] = \sum_{j_1 \geq k+1} \sum_{j_2 \geq k+1} c_{j_1} c_{j_2} \sum_{|\beta| \geq 1} \sum_{|\gamma| \geq 1} a_{j_1 \beta} a_{j_2 \gamma} \cdot Cov(V_{\alpha}^{\beta}, V_{\alpha}^{\gamma}) (3.2)
\]
and moreover
\[
\sum_{|\beta|,|\gamma|\geq 1} a_{\beta}^j a_{\gamma}^j \cdot |\text{Cov}(V_{\alpha}^\beta, V_{\alpha}^\gamma)|
\leq \sum_{i=1}^{N} \sum_{i'} \sum_{i''} \sum_{i'''} \sum_{i'} \sum_{i''} \sum_{i'''} |p^{j_1}(\beta)p^{j_2}(\gamma)| |\text{Cov}(V_{\alpha}^\beta, V_{\alpha}^\gamma)|
\]
\[
= \sum_{\ell=-\infty}^{\infty} |\text{Cov}(X_{\beta}^{\ell}, X_{\gamma}^{\ell})| \sum_{i\in I_{N,\ell}} \prod_{n\in \mathbb{N}} 1_{n}^{\alpha(\beta_{n-1}+\gamma_{n-i-\ell})}
\]

where we set \(I_{N,\ell} := \{i \mid 1 \leq i, i + \ell \leq N\}\). The covariance and the sum of the products are bounded by
\[
|\text{Cov}(X_{\beta}^{\ell}, X_{\gamma}^{\ell})| \leq \sqrt{V(X_{\beta}) \cdot V(X_{\gamma})} \leq C_{X}^{|\beta|+|\gamma|} \leq C_{X}^{j_{1}+j_{2}}
\]
\[
\sum_{i\in I_{N,\ell}} \prod_{n\in \mathbb{N}} 1_{n}^{\alpha(\beta_{n-1}+\gamma_{n-i-\ell})} \leq \sum_{i\in I_{N,\ell}} \frac{1}{i^{2\alpha}} \leq \sum_{i=1}^{N} \frac{1}{i^{2\alpha}}.
\]

On the other hand, since \(\text{supp } \beta \subset [0, \frac{j_{1}}{2}]\), \(\text{supp } \gamma \subset [0, \frac{j_{2}}{2}]\), the number of \(\ell\)'s such that \(\text{Cov}(X_{\beta}^{\ell}, X_{\gamma}^{\ell}) \neq 0\) is at most \((j_{1} + j_{2})(\leq j_{1} \cdot j_{2})\). Using all these estimates and Lemma 2.4 yields
\[
\sum_{|\beta|,|\gamma|\geq 1} a_{\beta}^j a_{\gamma}^j \cdot |\text{Cov}(V_{\alpha}^\beta, V_{\alpha}^\gamma)|
\leq \sum_{\ell=-\infty}^{\infty} |\text{Cov}(X_{\beta}^{\ell}, X_{\gamma}^{\ell})| \sum_{i\in I_{N,\ell}} \prod_{n\in \mathbb{N}} 1_{n}^{\alpha(\beta_{n-1}+\gamma_{n-i-\ell})}
\]
\[
\leq j_{1}(C_{X} + 2)^{j_{1}} \cdot j_{2}(C_{X} + 2)^{j_{2}} \sum_{i=1}^{N} \frac{1}{i^{2\alpha}}.
\]

Plugging this one to (3.2) and then (3.1), we have
\[
E \left[ \left( \sum_{j\geq k+1} c_{j} \sum_{|\beta|\geq 1} \alpha_{\beta}^{j}(V_{\alpha}^{\beta} - E[V_{\alpha}^{\beta}]) \right)^{2} \right] \leq \left( \sum_{j=k+1}^{\infty} |c_{j}|(C_{X} + 2)^{j} \right)^{2} \sum_{i=1}^{N} \frac{1}{i^{2\alpha}},
\]
\[
\text{Var } (Y_{N} - X_{N,k}) \leq D_{k}^{2} \sum_{i=1}^{N} \frac{1}{i^{2\alpha}} g_{2\alpha}(N).
\]

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where we set $D_k := \sum_{j=k+1}^{\infty} |c_j| (C_X + 2)^j$. Therefore,

$$\lim_{k \to \infty} \limsup_{N \to \infty} P(|X_{N,k} - Y_N| \geq \epsilon) \leq \lim_{k \to \infty} \limsup_{N \to \infty} D_k^2 \frac{\sum_{i=1}^{N} \frac{1}{g_{2\alpha}(N)}}{g_{2\alpha}(N)} = 0.$$ 

3.3 Proof of B(1)

By assumption $c_j = 0$ for $j$ : odd so that if $|\beta| \notin 2\mathbb{N}, \mathcal{C}_{i,j}(\beta) = \emptyset$, $\forall i$. We then compute

$$\sum_{j=0}^{\infty} c_j \{\text{Tr}(H_{\alpha,N}^j) - \mathbb{E}[\text{Tr}(H_{\alpha,N}^j)]\}$$

$$= \sum_{j=0}^{\infty} c_j \sum_{\alpha(\beta) \in [1,N]} \sum_{|\beta| = 2} p^j(\beta) (V^\beta - \mathbb{E}[V^\beta]) + \sum_{j=0}^{\infty} c_j \sum_{\alpha(\beta) \in [1,N]} \sum_{|\beta| \geq 4} p^j(\beta) (V^\beta - \mathbb{E}[V^\beta])$$

$$+ \sum_{j=0}^{\infty} c_j \sum_{\alpha(\beta) \in [1,N]} \sum_{|\beta| \geq 2} \left(a_{j,N}^\beta - p^j(\beta)\right) (V^\beta - \mathbb{E}[V^\beta])$$

$$=: I + II + III.$$ 

Here we apply Lemma 2.8 to $I$, Lemma 2.7 to $II$, and Lemma 2.6 to $III$, to have the desired result. []

3.4 Proof of B(2)

Proof of B(2) and C(2) is similar to that of A(1) : let

$$X_{N,k} := \frac{\text{Tr} f_k(H_{\alpha,N}) - \mathbb{E}[\text{Tr} f_k(H_{\alpha,N})]}{g_{4\alpha}(N)}, \quad X_k := N(0, \sigma_B(f_k)^2)$$

$$Y_N := \frac{\text{Tr} f(H_{\alpha,N}) - \mathbb{E}[\text{Tr} f(H_{\alpha,N})]}{g_{4\alpha}(N)}, \quad X := N(0, \sigma_B(f)^2)$$

and show that the conditions to apply Lemma 3.2 are satisfied.

(i) $X_{N,k} \xrightarrow{d} X_k$, $N \to \infty$ has been proved in [2].

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(ii) To show $X_k \xrightarrow{d} X$, let $\varphi_k(t)$ be the characteristic function of $X_k$. Then

$$\varphi_k(t) = \exp \left( -\frac{1}{2} \sigma_B(f_k)^2 t^2 \right)$$

$$= \exp \left[ -\frac{1}{2} \left\{ \left( \sum_{j=2}^{k} c_j p^j(2\delta) \right)^2 \left( \mathbb{E}[X_1^4] - \eta^4 \right) + \sum_{s=1}^{\infty} \left( \sum_{j=2}^{k} c_j p^j(\delta + \delta^s) \right)^2 \eta^4 \right\} t^2 \right].$$

To estimate that the second term in the exponential factor in RHS, we use

$$p^j(2\delta) = j^{2j-3},$$

$$p^j(\delta + \delta^s) \leq j(\delta + \delta^s) \leq j^2 \frac{1}{2} \frac{1}{(j-2)/2} = 2^{j-1}(1 + o(1)) \leq (\text{const.})j^{3/2}2^{j-1},$$

and thus

$$\sum_{s=1}^{\infty} \left( \sum_{j=2}^{k} c_j p^j(\delta + \delta^s) \right)^2 = \left( \sum_{j=2}^{k} c_j \cdot \sum_{j'=2}^{k} c_{j'} \cdot \sum_{s=1}^{\infty} p^j(\delta + \delta^s)p^{j'}(\delta + \delta^s) \right)$$

$$\leq \left( \sum_{j=2}^{k} c_j j^{5/2}2^{j-1} \right)^2.$$ 

Therefore the RHS converges as $k \to \infty$ so that

$$\varphi_k(t) \xrightarrow{k \to \infty} \varphi(t) = \exp \left[ -\frac{1}{2} \sigma_B(f)^2 t^2 \right].$$

(iii) We note that if $|\beta| \notin 2\mathbb{N}$, $C_{i,j}(\beta) = \emptyset$, $\forall i$. In particular $a_N^1(\beta) = 0$ for $|\beta| = 1$. Hence

$$\text{Var} \left( \sum_{j \geq k+1} c_j \frac{\text{Tr}(H_{\alpha,N}^j) - \mathbb{E}[\text{Tr}(H_{\alpha,N}^j)]}{g_{4\alpha}(N)} \right)$$

$$= \mathbb{E} \left[ \left\{ \sum_{j \geq k+1} c_j \sum_{|\beta| \geq 2} a_N^j(\beta) \left( V^{\beta} - \mathbb{E}[V^{\beta}] \right) \frac{g_{4\alpha}(N)}{g_{4\alpha}(N)} \right\}^2 \right]$$

$$= \frac{1}{g_{4\alpha}(N)^2} \sum_{j_1 \geq k+1} \sum_{j_2 \geq k+1} c_{j_1} \cdot c_{j_2} \sum_{|\beta| \geq 2} \sum_{|\gamma| \geq 2} a_N^{j_1}(\beta)a_N^{j_2}(\gamma) \text{Cov}(V^{\beta}, V^{\gamma}).$$
Then its absolute value is bounded from above by

\[
\frac{1}{g_4(\alpha)(N)^2} \sum_{j_1 \geq k+1} \sum_{j_2 \geq k+1} |c_{j_1}| \cdot |c_{j_2}| \sum_{|\beta| \geq 2} \sum_{|\gamma| \geq 2} a_{i}^{2j}(|\beta|)a_{j}^{2j}(|\gamma|) |\text{Cov}(V^\beta, V^\gamma)|
\]

\[
\leq \frac{1}{g_4(\alpha)(N)^2} \sum_{j_1 \geq k+1} \sum_{j_2 \geq k+1} |c_{j_1}| \cdot |c_{j_2}| \sum_{i(\beta) = 0} \sum_{i(\gamma) = 0} p^{\beta}_{j_1}(\beta)p^{\gamma}_{j_2}(\gamma)
\]

\[
\times \sum_{\ell \in \mathbb{Z}} |\text{Cov}(X^\beta, X^\gamma)| \sum_{i \in I_{N,\ell}} \prod_{n \in \mathbb{N}} \frac{1}{N^{\alpha(\beta_n + \gamma_n - i)}} \sum_{i = 1}^{N} \frac{1}{i^{4\alpha}}
\]

\[
\leq \frac{2}{g_4(\alpha)(N)^2} \sum_{j \geq k+1} |c_{j}| \left( \sum_{|\beta| \geq 2} \sum_{|\gamma| \geq 2} p^{\beta}_{j}(\beta)C_{X}^{\beta} \right) \left( \sum_{|\beta| \geq 2} \sum_{|\gamma| \geq 2} p^{\gamma}_{j}(\gamma)C_{X}^{\gamma} \right) \sum_{i = 1}^{N} \frac{1}{i^{4\alpha}}
\]

\[
\leq \frac{2}{g_4(\alpha)(N)^2} \left( \sum_{j \geq k+1} |c_{j}|j(C_{X} + 2)^{j} \right)^{2} \sum_{i = 1}^{N} \frac{1}{i^{4\alpha}} \frac{1}{\epsilon^{2}}.
\]

Take \( \epsilon > 0 \) arbitrary small. Then by Chebyshev’s inequality,

\[
P \left( \left| \sum_{j \geq k+1} c_{j} \frac{\text{Tr}(H_{\alpha,N}^{j}) - E[\text{Tr}(H_{\alpha,N}^{j})]}{g_4(\alpha)(N)} \right| \geq \epsilon \right)
\]

\[
\leq \frac{2}{g_4(\alpha)(N)^2} \left( \sum_{j \geq k+1} |c_{j}|j(C_{X} + 2)^{j} \right)^{2} \sum_{i = 1}^{N} \frac{1}{i^{4\alpha}} \frac{1}{\epsilon^{2}}.
\]

Therefore by the definition of \( g_4(\alpha)(N) \), we have

\[
\lim_{N} \sup_{k} P \left( \left| \sum_{j \geq k+1} c_{j} \frac{\text{Tr}(H_{\alpha,N}^{j}) - E[\text{Tr}(H_{\alpha,N}^{j})]}{g_4(\alpha)(N)} \right| \geq \epsilon \right) \leq \frac{2}{\epsilon^{2}} \left( \sum_{j \geq k+1} |c_{j}|j(C_{X} + 2)^{j} \right)^{2}
\]

\[
\lim_{N} \sup_{k} P \left( \left| \sum_{j \geq k+1} c_{j} \frac{\text{Tr}(H_{\alpha,N}^{j}) - E[\text{Tr}(H_{\alpha,N}^{j})]}{g_4(\alpha)(N)} \right| \geq \epsilon \right) = 0.
\]
3.5 Proof of C(1)

The idea is the same as that of the proof of B(1). In fact,

\[ Trf(H_{\alpha,N}) - E[Trf(H_{\alpha,N})] = \sum_{j=0}^{\infty} c_j \left\{ (Trf(H_{\alpha,N}^j) - E[Trf(H_{\alpha,N}^j)]) - (p^j(\delta)Trf(H_{\alpha,N}) - E[p^j(\delta)Trf(H_{\alpha,N})]) \right\} \]

\[ = \sum_{j=0}^{\infty} \sum_{\nu(\beta) \in [1,N]} p^j(\beta) (V^\beta - E[V^\beta]) + \sum_{j=0}^{\infty} c_j \sum_{\nu(\beta) \in [1,N]} p^j(\beta) (V^\beta - E[V^\beta]) \]

\[ + \sum_{j=0}^{\infty} \sum_{\nu(\beta) \in [1,N]} p^j(\beta) (V^\beta - E[V^\beta]) + \sum_{j=0}^{\infty} c_j \sum_{\nu(\beta) \in [1,N]} (a_{j,N}^\beta - p^j(\beta)) (V^\beta - E[V^\beta]) \]

\[ =: I + II + III + IV. \]

We apply Lemma 2.8 to I, II, Lemma 2.7 to III, and Lemma 2.6 to IV to obtain the desired conclusion.

3.6 Proof of C(2)

As in the proof of B(2), we set

\[ X_{N,k} := \frac{Trf_k(H_{\alpha,N}) - E[Trf_k(H_{\alpha,N})]}{g_{6\alpha}(N)}, \quad X_k := N(0, \sigma_C(f_k)^2) \]

\[ Y_N := \frac{Trf(H_{\alpha,N}) - E[Trf(H_{\alpha,N})]}{g_{6\alpha}(N)}, \quad X := N(0, \sigma_C(f)^2) \]

and we show that the three conditions (i), (ii), (iii) to apply Lemma 3.2 are satisfied. (i) has been done in [2].

(ii) We shall show that, if \(0 < \alpha \leq 1/6\), the characteristic function \(\varphi_k(t)\) of \(N(0, \sigma_C(f_k)^2)\) satisfies

\[ \varphi_k(t) = \exp \left( -\frac{1}{2} \sigma_C(f_k)^2 \right) \rightarrow \exp \left( -\frac{1}{2} \sigma_C(f)^2 t^2 \right). \]

In fact, we have

\[ \sigma_C(f_k)^2 = \langle f_k, C_C^{(k)} f_k \rangle = \sum_{3 \leq j \leq k, 3 \leq j' \leq k, j \text{ odd}, j' \text{ odd}} c_j c_{j'} M_{j,j'} \]
where

\[ M_{j',j} := \lim_{N \to \infty} \text{Cov} \left( \frac{\text{Tr} P_3(H_{\alpha,N}) - \mathbb{E}[\text{Tr} P_3(H_{\alpha,N})]}{g_{6\alpha}(N)}, \frac{\text{Tr} P_{j'}(H_{\alpha,N}) - \mathbb{E}[\text{Tr} P_{j'}(H_{\alpha,N})]}{g_{6\alpha}(N)} \right). \]

Since

\[ |M_{j',j}| \leq \lim_{N \to \infty} \left| \text{Cov} \left( \frac{\text{Tr} P_3(H_{\alpha,N}) - \mathbb{E}[\text{Tr} P_3(H_{\alpha,N})]}{g_{6\alpha}(N)}, \frac{\text{Tr} P_{j'}(H_{\alpha,N}) - \mathbb{E}[\text{Tr} P_{j'}(H_{\alpha,N})]}{g_{6\alpha}(N)} \right) \right| \]

\[ \leq \sum_{|\beta| \geq 3} \sum_{\ell \in \mathbb{Z}} p^j(\beta)p^{j'}(\gamma) \sum \text{Cov}(X^\beta, X^{\gamma}) \leq \sum_{|\beta| \geq 3} \sum_{\ell \in \mathbb{Z}} p^j(\beta)p^{j'}(\gamma) j \cdot j' \cdot (C_X + 2)^j(C_X + 2)^{j'} \]

we have

\[ \sigma_C(f_k)^2 \leq \sum_{3 \leq j < k} \sum_{\substack{3 \leq j' < k \leq j' \text{ even} \ V \ \ 3 \leq j' < k \text{ odd}}} |c_j| \cdot |c_{j'}| \cdot j \cdot j' \cdot (C_X + 2)^j(C_X + 2)^{j'} \]

which ensures the convergence of \( \lim_{k \to \infty} \sigma_C(f_k)^2 \).

(iii) By assumption, \( c_j = 0 \) for \( j \) even so that \( C_{i,j}(\beta) = \emptyset, \forall i \) for \( |\beta| \notin 2N - 1 \).

We compute

\[ \text{Var} \left( \sum_{j \geq k+1} c_j \left\{ \frac{\text{Tr}(H_{\alpha,N}^j) - \mathbb{E}[\text{Tr}(H_{\alpha,N}^j)]}{g_{6\alpha}(N)} - p^j(\delta) \frac{\text{Tr}(H_{\alpha,N}) - \mathbb{E}[\text{Tr}(H_{\alpha,N})]}{g_{6\alpha}(N)} \right\} \right) \]

\[ = \frac{1}{g_{6\alpha}(N)^2} \text{Var} \left( \sum_{j \geq k+1} c_j \left\{ \sum_{|\beta| = 1} a_N^j(\beta)(V^\beta - \mathbb{E}[V^\beta]) + \sum_{|\beta| \geq 3} a_N^j(\beta)(V^\beta - \mathbb{E}[V^\beta]) \right. \right. \]

\[ \left. \left. - p^j(\delta) \sum_{i=1}^N \left( V^{\delta^i} - \mathbb{E}[V^{\delta^i}] \right) \right\} \right) \]

\[ = \frac{1}{g_{6\alpha}(N)^2} \text{Var} \left( \sum_{j \geq k+1} c_j \left\{ \sum_{i=1}^N \left( a_N^j(\delta^i) - p^j(\delta) \right) \left( V^{\delta^i} - \mathbb{E}[V^{\delta^i}] \right) + \sum_{|\beta| \geq 3} a_N^j(\beta)(V^\beta - \mathbb{E}[V^\beta]) \right\} \right) \]

\[ \leq \frac{2}{g_{6\alpha}(N)^2} \left\{ \text{Var} \left( \sum_{j \geq k+1} c_j \sum_{|\beta| \geq 3} a_N^j(\beta)(V^\beta - \mathbb{E}[V^\beta]) \right) \right. \]

\[ + \text{Var} \left( \sum_{j \geq k+1} c_j \left\{ \sum_{i=1}^N \left( a_N^j(\delta^i) - p^j(\delta) \right) \left( V^{\delta^i} - \mathbb{E}[V^{\delta^i}] \right) \right\} \right) \]

\[ =: 2(I + II). \]
For $I$,

$$I = E \left[ \left\{ \sum_{j \geq k+1} c_j \sum_{\|\beta\| \geq 3} \frac{\alpha^j_N(\beta) (V^\beta - E[V^\beta])}{g_{6\alpha}(N)} \right\}^2 \right]$$

$$\leq \frac{1}{g_{6\alpha}(N)^2} \sum_{j_1 \geq k+1} \sum_{j_2 \geq k+1} |c_{j_1}| \cdot |c_{j_2}| \sum_{i(\beta)=0, i(\gamma)=0} |p^{j_1}(\beta)p^{j_2}(\gamma)| \sum_{\ell \in \mathbb{Z}} |Cov(X^\beta X^\gamma)|$$

$$\times \sum_{i \in I_{\alpha, \ell}} \prod_{n \in \mathbb{N}} \frac{1}{\eta^{(\beta_{n-i}+\gamma_{n-i-\ell})}}$$

$$\leq \frac{1}{g_{6\alpha}(N)^2} \sum_{j_1 \geq k+1} \sum_{j_2 \geq k+1} |c_{j_1}| \cdot |c_{j_2}| \sum_{i(\beta)=0, i(\gamma)=0} |p^{j_1}(\beta)p^{j_2}(\gamma)| \cdot j_1 \cdot j_2 \cdot 4C^{|\beta|+|\gamma|}_X \cdot \sum_{i=1}^{N} \frac{1}{i^{6\alpha}}$$

$$\leq 4 \sum_{i=1}^{N} i^{-6\alpha} \left( \sum_{j \geq k+1} |c_j| \cdot j \cdot (C_X + 2)^j \right)^2.$$

For $II$,

$$II = \frac{1}{g_{6\alpha}(N)^2} Var \left( \sum_{j \geq k+1} c_j \sum_{i=1}^{N} \left( \alpha^j_N(\delta^i) - p^j(\delta) \right) (V^{\delta^i} - E[V^{\delta^i}]) \right)$$

$$= \frac{1}{g_{6\alpha}(N)^2} \sum_{i=1}^{N} \left\{ \sum_{j \geq k+1} c_j \left( \alpha^j_N(\delta^i) - p^j(\delta) \right) \right\}^2 \frac{E[X^2_i]}{i^{2\alpha}}$$

$$= \frac{\eta^2}{g_{6\alpha}(N)^2} \sum_{j_1 \geq k+1} \sum_{j_2 \geq k+1} c_{j_1} c_{j_2} \sum_{i=1}^{N} \left( \alpha^j_N(\delta^i) - p^j(\delta) \right) \left( \alpha^j_N(\delta^i) - p^j(\delta) \right) \frac{1}{i^{2\alpha}}.$$

Here we decompose

$$\sum_{j_1 \geq k+1} \sum_{j_2 \geq k+1} = \sum_{j_1=k+1}^{N} \sum_{j_2=k+1}^{N} + \sum_{j_1=k+1}^{N} \sum_{j_2=N+1}^{\infty} + \sum_{j_1=N+1}^{\infty} \sum_{j_2=k+1}^{N} + \sum_{j_1=N+1}^{\infty} \sum_{j_2=N+1}^{\infty}$$

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Since $a_N^i(\delta^i) = p^i(\delta)$ for $\frac{i}{2} \leq i \leq N - \frac{i}{2}$,

\[
II = \frac{\eta^2}{g_{6\alpha}(N)^2} \sum_{j_1=k+1}^{N} \sum_{j_2=k+1}^{N} c_{j_1} c_{j_2} \left( \min\left(\frac{j_1-1}{j_2-1}\right) + \sum_{i=1}^{N} \sum_{i=N-\min\left(\frac{j_1-1}{j_2-1}\right)}^{N} \right) \\
\quad \times \left( a_N^{j_1}(\delta^i) - p^{j_1}(\delta) \right) \left( a_N^{j_2}(\delta^i) - p^{j_2}(\delta) \right) \frac{1}{i^{2\alpha}} \\
+ \frac{\eta^2}{g_{6\alpha}(N)^2} \left( \sum_{j_1=k+1}^{N} \sum_{j_2=N+1}^{\infty} + \sum_{j_1=N+1}^{\infty} \sum_{j_2=k+1}^{N} + \sum_{j_1=N+1}^{\infty} \sum_{j_2=N+1}^{\infty} \right) c_{j_1} c_{j_2} \\
\quad \times \sum_{i=1}^{N} \left( a_N^{j_1}(\delta^i) - p^{j_1}(\delta) \right) \left( a_N^{j_2}(\delta^i) - p^{j_2}(\delta) \right) \frac{1}{i^{2\alpha}}
\]

\[
\leq \frac{\eta^2}{g_{6\alpha}(N)^2} \sum_{j_1=k+1}^{N} \sum_{j_2=k+1}^{N} |c_{j_1}| |c_{j_2}| \min(j_1, j_2) \cdot j_1 2^{j_1} \cdot j_2 2^{j_2} \\
+ \frac{\eta^2}{g_{6\alpha}(N)^2} \left( \sum_{j_1=k+1}^{N} \sum_{j_2=N+1}^{\infty} + \sum_{j_1=N+1}^{\infty} \sum_{j_2=k+1}^{N} + \sum_{j_1=N+1}^{\infty} \sum_{j_2=N+1}^{\infty} \right) |c_{j_1}| |c_{j_2}| \cdot N \cdot j_1 2^{j_1} \cdot j_2 2^{j_2}.
\]

Here we used the following estimates.

\[
\sum_{i=1}^{N} \frac{1}{i^{2\alpha}} \leq N, \quad |a_N^{j_1}(\delta^i) - p^{j_1}(\delta)| \leq p^{j_2}(\delta) \leq j \cdot 2^j.
\]

We note that $N \leq j_2$ in the 2nd term, $N \leq j_1$ in the 3rd term, and $N \leq j_1$ in the 4th term. Therefore

\[
II \leq \frac{\eta^2}{g_{6\alpha}(N)^2} \left( \sum_{j=k+1}^{\infty} |c_j| \cdot j^2 \cdot 2^j \right)^2.
\]
Plugging these estimates for $I, II$ above into Chebyshev’s inequality, we have

\[
P\left(\left|\sum_{j \geq k+1} c_j \left(\frac{TrH_{\alpha,N}^j - E[TrH_{\alpha,N}^j]}{g_{6\alpha}(N)} - p^j(\delta) \frac{TrH_{\alpha,N} - E[TrH_{\alpha,N}]}{g_{6\alpha}(N)}\right)\right| > \epsilon\right) \leq \frac{2(I + II)}{\epsilon^2}
\]

\[
\leq \frac{2}{\epsilon^2} \left[\frac{4}{g_{6\alpha}(N)^2} \sum_{i=1}^{N} i^{-6\alpha} \left( \sum_{j \geq k+1} |c_j| j(C_X + 2)^j \right)^2 + \frac{\eta^2}{g_{6\alpha}(N)^2} \left( \sum_{j=k+1}^{\infty} |c_j| \cdot j^2 \cdot 2^j \right)^2\right]
\]

which yields the conclusion.

\[N \to \infty \quad \frac{2}{\epsilon^2} \left( \sum_{j \geq k+1} |c_j| j(C_X + 2)^j \right)^2 k \to \infty 0\]

\[4 \quad \text{Proof of Theorem 2}\]

Take any $\theta := (\theta_1, \cdots, \theta_d) \in \mathbb{R}^d$ and let $\tilde{F}_{\theta,N} := \theta \cdot F_N$, $f_\theta := \sum_1^d \theta_t f_t$, $\tilde{F}_{\theta} \sim N(0, \sigma(f_\theta)^2)$. By Theorem 1, we have

\[
\tilde{F}_{\theta,N} := \sum_{t=1}^d \theta_t F_{t,N} = \frac{1}{g_{6\alpha}(N)} (Tr(f_\theta(H_{\alpha,N})) - E[Tr(f_\theta(H_{\alpha,N}))]) \overset{d}{\to} N(0, \sigma(f_\theta)^2)
\]

which shows $\tilde{F}_{\theta,N} \overset{d}{\to} \tilde{F}_{\theta}$. We denote by $\varphi_X(t)$ the characteristic function of the random variable $X$. Since $\tilde{F}_{\theta} \sim N(0, \sigma(f_\theta)^2)$ and since $r(f) > C_X + 2$, we have

\[
\varphi_{\tilde{F}_{\theta}}(\xi) = \exp \left[-\frac{1}{2} \sigma(f_\theta)^2 \xi^2\right] = \exp \left[-\frac{1}{2} \xi^2 \left( \sum_{t=1}^d \theta_t \sigma(f_t) \right)^2\right] = \varphi_{\theta \cdot F}(\xi)
\]

thus $\tilde{F}_{\theta} \overset{d}{=} \theta \cdot F$. By the method of Cramer-Wald, this implies $F_N \overset{d}{\to} F$, completing the proof of Theorem 2.

\[5 \quad \text{Proof of Theorem 3 : Polynomial case}\]

In this section we prove Theorem 3 when $f$ is a polynomial of degree $k$. General case will be considered in the next section. Our goal in this section
Theorem 5.1

\[ \mathbb{E}[Tr(H_{\alpha,N}^k)] = A_{k,1}N + A_{k,0} + B^k(N) + \sum_{j=1}^k C_{j,k} \sum_{i=1}^N \frac{1}{i^{\alpha_j}} + D^k(N) \]  

(5.1)

where

\[ A_{k,1} = \left( \frac{k}{k/2} \right) 1(k: \text{even}), \quad A_{k,0} = - \sum_{y \in \mathcal{R}^k} (\max y - \min y) \]

\[ \mathcal{R}^k := \{ y = (y_0, \cdots, y_k) \in \mathcal{P}^k \mid y_0 = y_k = 0, \ y \ \text{has no flat steps} \} \]

\[ C_{j,k} := \sum_{\beta: \iota(\beta)=0 \atop |\beta|=j} p^k(\beta)\mathbb{E}[X^{\beta}] \]

\[ B^k(N) := \sum_{\iota(\beta) \in [k,N-k]^c} (a_{\beta} - p^k(\beta))\mathbb{E}[V_{\alpha}^{\beta}] \]

\[ D^k(N) = \sum_{j=1}^k \sum_{\iota(\beta)=0 \atop |\beta|=j} p^k(\beta)\mathbb{E}[X^{\beta}] \sum_{i=1}^N \left( \prod_{l \in \mathbb{N}} \frac{1}{l^{\alpha_{\beta_l-i}}} - \frac{1}{l^{\alpha_{\beta_l}}} \right) \]

\[ D^k(N) \xrightarrow{N \to \infty} D^k, \quad D^k := \sum_{j=1}^k \sum_{\iota(\beta)=0 \atop |\beta|=j} p^k(\beta)\mathbb{E}[X^{\beta}] \sum_{i=1}^\infty \left( \prod_{l \in \mathbb{N}} \frac{1}{l^{\alpha_{\beta_l-i}}} - \frac{1}{l^{\alpha_{\beta_l}}} \right) \]

\[ |D^k| \le \sum_{j=1}^k \sum_{\iota(\beta)=0 \atop |\beta|=j} p^k(\beta)\mathbb{E}[|X^{\beta}|] \sum_{i=1}^k \frac{1}{i^{\alpha_{|\beta|}}} \]

(i) \( A_{k,1}N + A_{k,0} \) is the contribution from the paths with no flat steps, which can be derived explicitly. (ii) \( B^k(N) \) is a boundary effect. (iii) \( \sum_{j=1}^k C_{j,k} \sum_{i=1}^N \frac{1}{i^{\alpha_j}} = \mathcal{O}(N^{1-j\alpha}) \) is the main term. The asymptotic behavior for \( Tr(P(H_{\alpha,N})) \) for a polynomial \( P \) can be derived by summing up eq. (5.1).

Proof. As in the proof of Theorem 1, let \( a_{\beta} := [V_{\alpha}^{\beta}](Tr(H_{\alpha,N}^k)) \) be the
coefficient of $V^\beta$, $\beta \neq 0$ in $\text{Tr}(H^k_{\alpha,N})$. Then by Lemma 2.3,

$$\mathbb{E}[\text{Tr}H^k_{\alpha,N}] = \sum_{|\beta| \geq 1} a_\beta \mathbb{E}[V^\beta_{\alpha,N}] + A^k(N) =: B^k(N) + C^k(N)$$

where $A^k(N)$ is the contribution of the paths with no flat steps which we write

$$A^k(N) = A_{k,1}N + A_{k,0}$$

and

$$B^k(N) := \sum_{i(\beta) \in [k,N-k]} (a_\beta - p^k(\beta)) \mathbb{E}[V^\beta_{\alpha,N}]$$

$$C^k(N) := \sum_{i(\beta) \in [1,N]} p^k(\beta) \mathbb{E}[V^\beta_{\alpha,N}]$$

We compute these terms separately below.

(A) We compute $A_k(N) = A_{k,1}N + A_{k,0}$. On the expansion of $H^k$,

$$H^k = (V + U + D)^k = \sum_{M \in \mathcal{M}} M$$

we consider the terms $M = \prod_t M_t \in \mathcal{M}$ such that $M_t \neq V$ and $\text{Tr} M \neq 0$. Since we must have $\#U = \#D$, in $\{M_t\}$, $k$ must be even. In that case, such terms are expressed as

$$U^{u_1}D^{d_1} \cdots U^{u_j}D^{d_j}, \quad u_1 + \cdots + u_j = d_1 + \cdots + d_j = k/2$$

while the set of corresponding paths is

$$\mathcal{R}^k := \{y = (y_0, \cdots, y_k) \in \mathcal{P}^k \mid y \text{ has no flat steps} \}$$

By an explicit computation, we see that, in the corresponding $M \in \mathcal{M}$, $\max y(M)$ components in the leftmost diagonal part and $-\min y(M)$ components in the rightleast diagonal part are all zero. Thus for even $k$,

$$\text{Tr} ((U + D)^k) = \sum_{y(M)} \{N - (\max y(M) - \min y(M))\}$$

$$= N \left( \frac{k}{k/2} \right) - \sum_{y \in \mathcal{R}^k} (\max y - \min y).$$
(B) Since $V_\alpha$ is decaying, $B^k(N)$ is bounded and convergent:

$$B^k(N) \xrightarrow{N \to \infty} B^k, \quad B^k := \sum_{\iota(\beta) \in [1,k]} (a_\beta - p^k(\beta)) E[V_\alpha^\beta]$$

(C) Using $E[V_\alpha^\beta] = \prod_l \left[ \frac{X^\beta_l}{i_\alpha^l} \right] = E[X^\beta] \prod_{l \in \mathbb{N}} \frac{1}{i_\alpha^l}$, we have

$$C^k(N) = \sum_{j=1}^{k} \sum_{i=1}^{N} \sum_{\iota(\beta) = i}^{\iota(\beta) = j} \sum_{|\beta| = j}^{p^k(\beta) E[X^\beta] \prod_{l \in \mathbb{N}} \frac{1}{i_\alpha^l}}$$

Since $\iota(\beta) = i$, $|\beta| = j$, $p^k(\beta) \neq 0$, $\beta_l > 0$ implies $l \in [i, i + k]$ \supset [i, i + k], and thus

$$\frac{1}{(i + k)^{|\beta|}_\alpha} = \prod_{l \in \mathbb{N}} \frac{1}{i_\alpha^l} \leq \prod_{l \in \mathbb{N}} \frac{1}{i_\alpha^l} \leq \prod_{l \in \mathbb{N}} \frac{1}{j_\alpha^l} = \frac{1}{j_\alpha^l}$$

We replace $\prod_{l \in \mathbb{N}} \frac{1}{i_\alpha^l}$ by $\frac{1}{j_\alpha^l}$ in (5.2), and denote the error by $D^k(N)$:

$$C^k(N) = \sum_{j=1}^{k} \sum_{i=1}^{N} \sum_{\iota(\beta) = i}^{\iota(\beta) = j} p^k(\beta) E[X^\beta] \prod_{l \in \mathbb{N}} \frac{1}{i_\alpha^l} + D^k(N)$$

By some change of variables, we have

$$D^k(N) = \sum_{j=1}^{k} \sum_{i=1}^{N} \sum_{\iota(\beta) = 0}^{\iota(\beta) = j} p^k(\beta) E[X^\beta] \left( \prod_{l \in \mathbb{N}} \frac{1}{i_\alpha^l} - \frac{1}{j_\alpha^l} \right) = \sum_{j=1}^{k} \sum_{\iota(\beta) = 0}^{\iota(\beta) = j} p^k(\beta) E[X^\beta] (-b_N(\beta))$$

where $b_N(\beta) := \sum_{i=1}^{N} \left( \frac{1}{i_\alpha^l} - \prod_{l \in \mathbb{N}} \frac{1}{i_\alpha^l} \right)$

Since $b_N(\beta)$ is monotonically increasing and bounded,

$$\lim_{N \to \infty} b_N(\beta) = b(\beta) := \sum_{i=1}^{\infty} \left( \frac{1}{i_\alpha^l} - \prod_{l \in \mathbb{N}} \frac{1}{i_\alpha^l} \right).$$

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Therefore

\[ D^k(N) \xrightarrow{N \to \infty} D^k, \quad D^k := \sum_{j=1}^{k} \sum_{i \in \mathbb{N}} p^k(\beta)E[X^\beta] \sum_{i=1}^{\infty} \left( \prod_{l \in \mathbb{N}} \frac{1}{|\alpha_{l-1}|} - \frac{1}{i|\alpha|} \right) \]

\[ |D^k| \leq \sum_{j=1}^{k} \sum_{i \in \mathbb{N}} p^k(\beta)E[|X|^\beta] \sum_{i=1}^{k} \frac{1}{i|\alpha|}. \]

On the other hand,

\[ C^k(N) - D^k(N) = \sum_{j=1}^{k} C_{j,k} \sum_{i=1}^{N} \frac{1}{i|\alpha|}, \quad C_{j,k} := \sum_{i \in \mathbb{N}, |\beta|=j} p^k(\beta)E[X^\beta]. \]

Here we note that \( C_{1,k} = 0 \), since \( \{ \beta \mid i(\beta) = 0, |\beta| = 1 \} = \{ \delta \} \). The proof of Theorem 5.1 is now complete. \( \square \)

6 Proof of Theorem 3: general case

In this section, we use Theorem 5.1 to finish the proof of Theorem 3. To be more concrete, we show

Theorem 6.1

\[ E[Trf(H)] = \left( \sum_{l} c_l C_{0,l} \right) N + \left( \sum_{l} c_l C_{2,l} \right) S_2(N) + \cdots + \left( \sum_{l} c_l C_{m,l} \right) S_m(N) + C_N(f) \]

\[ C_N(f) := \left( \sum_{l} c_l A_{t,0} \right) + \sum_{l \geq m+1} c_l \sum_{j=m+1}^{l} C_{j,t} S_j(N) + \sum_{l} c_l B^l(N) + \sum_{l} c_l D^l(N) \]
where

$$\lim_{N \to \infty} C_N(f) = \left( \sum_{l} c_l A_{l,0} \right) + \sum_{l \geq m+1} c_l \sum_{j=m+1}^{l} C_{j,l} S_j + \sum_{l} c_l B^l + \sum_{l} c_l D^l$$

$$S_j := \sum_{n=1}^{\infty} \frac{1}{n^j}, \quad j \geq m+1$$

$$B^k := \sum_{\nu(\beta) \in [1,k]} (a_\beta - p^k(\beta)) E[V_{\alpha}^\beta]$$

$$D^k := \sum_{l=1}^{k} \sum_{\nu(\beta) = 0 \mid \beta = j} p^k(\beta) E[X_{\alpha}^\beta] \sum_{i=1}^{\infty} \left( \prod_{l \in N} \frac{1}{l^{\alpha_{\beta_l} - 1}} - \frac{1}{l^{\alpha_{\beta_l}}} \right)$$

Proof. By Theorem 5.1,

$$E[Tr H^k] = C_{0,k} N + C_{2,k} S_2(N) + \cdots + C_{m,k} S_m(N) + C_{m+1,k} S_{m+1}(N) + \cdots + C_{k,k} S_k(N)$$

$$+ B^k(N) + D^k(N) + A_{k,0}$$

(6.1)

where $C_{0,k} = A_{k,1}$, $S_{m+1}(N), \cdots, S_k(N)$ are bounded w.r.t. $N$. If $k \leq m$, we do not have terms of the form $C_{m+1,k} S_{m+1} + \cdots + C_{k,k} S_k$. As we mentioned in the proof of Theorem 1, $f(H) = \sum_{l \geq 0} a_l H^l$ is norm convergent, so that by Fubini theorem,

$$E[Tr f(H)] = \sum_{l \geq 0} c_l E[Tr (H^l)].$$

Plugging it into (6.1) yields

$$E[Tr f(H)]$$

$$= \sum_{l} c_l E[Tr (H^l)]$$

$$= \sum_{l} c_l \left( C_{0,l} N + A_{l,0} + C_{2,l} S_2(N) + \cdots + C_{m,l} S_m(N)$$

$$+ C_{m+1,l} S_{m+1}(N) + \cdots + C_{l,l} S_l(N) + B^l(N) + D^l(N) \right)$$

Here we would like to change the order of summation. In order for that, it
suffices to show that,

\[(i) \sum_{l \geq j} c_l C_{j,l}, \quad (ii) \sum_{l \geq m+1} c_l \sum_{j=m+1}^l C_{j,l} S_j(N), \quad (iii) \sum_{l} c_l B^l(N)\]

\[(iv) \sum_{l} c_l D^l(N), \quad (v) \sum_{l} c_l A_{l,0}\]

are absolutely convergent, and the quantities in (ii), (iii), (iv) converge as \(N \to \infty\).

(i) By Lemma 2.4 (2), we have \(|C_{j,l}| \leq (C_X + 2)^l\) so that since \(r(f) > C_X + 2\), (i) is absolutely convergent.

(ii) That (ii) is absolutely convergent is shown similarly as (i). Since

\[S(N) := \max_{j \geq m+1} S_j(N) < \infty, \quad \lim_{N \to \infty} S_j(N) = S_j := \sum_{n=1}^{\infty} \frac{1}{\eta^{j\alpha}}, \quad j \geq m + 1\]

we have

\[\lim_{N \to \infty} \sum_{l \geq m+1} c_l \sum_{j=m+1}^l C_{j,l} S_j(N) = \sum_{l \geq m+1} c_l \sum_{j=m+1}^l C_{j,l} S_j.\]

(iii) Absolute convergence is similar as (i), (ii), and it is easy to see

\[\lim_{N \to \infty} \sum_{l} c_l B^l(N) = \sum_{l} c_l B^l.\]

(iv) We have only to use the following estimate :

\[|D^l(N)| \leq \sum_{j=1}^l \sum_{\beta \in \mathbb{Z}, |\beta|=j} p^l(\beta) C_X^j (\text{Const.}) \max\{1, l^{1-\alpha|\beta|}\}\]

\[\leq (C_X + 2)^l (\text{Const.}) \max\{1, l^{1-\alpha|\beta|}\}.\]

(v) By Stirling’s formula, we have

\[|A_{k,0}| \leq \sum_{y \in \mathcal{R}_k} (\max y - \min y) \leq k \left(\frac{k}{k^2}\right) \leq (\text{Const.}) \sqrt{k} 2^k\]
so that $\sum_i c_i \cdot A_{l,0}$ is absolutely convergent. The proof of Theorem 6.1 is now complete. \square

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