I. INTRODUCTION

Modern microwave quantum engineering exploits efficient detection of low power microwave signals \[1, 2\] requiring linear amplification with ultra-low added noise. Nowadays, superconducting parametric amplifiers, exhibiting quantum limited sensitivity, have become the most favourable implementation in practical devices \[3\].

The superconducting parametric amplifiers that have been demonstrated so far can be divided in two classes: the resonant parametric amplifier \[4–7\] and the traveling wave parametric amplifier (TWPA) \[8–11\]. A resonant parametric amplifier works as a nonlinear resonator, which provides energy transfer from a strong pump tone to the signal to be amplified \[12\]. The finite interaction time of the waves providing amplification is enhanced by a high quality factor of the nonlinear resonator, which in turn limits the bandwidth. Typically, the required nonlinearity is achieved by integrating the resonator with an appropriate array of Josephson junctions (JJs) \[13\].

In a TWPA, the 3-wave mixing (3WM) (in the presence of DC bias) or the 4-wave mixing (4WM) process occurs, as waves propagate in a relatively long nonlinear transmission line (TL) \[14\]. The performance of the TWPA is usually described by coupled mode (CM) equations in the standard form for three types of waves: pump, signal, and idler, analogously to the fiber optics theory \[15\]. Standard CM equations predicts that the bandwidth is limited by nonlinear phase modulations only. However, this phase modulations can be controlled by means of dispersion engineering. For instance, in Ref. \[16\] smoothed broadband amplification from 4 to 8 GHz has been achieved via 4WM and in \[16\] from 3.5 to 5.5 GHz under 3WM.

Analysis based on the conventional CM, however, assumes that the impedance of the nonlinear TL corresponds to the standard value of 50 $\Omega$. In experiments, an impedance mismatch results in modulation of the amplifier transmission (ripples) \[17, 18\]. This gain modulation can be significant, often more than 5 dB \[8, 9\], which limits some of the practical applications of these amplifiers. So far, these ripples are described as Fabry-Pérot-like resonances \[11, 19\] whose bandwidth is inversely proportional to the length of the waveguide. Despite having important influence on the amplifier’s transmission, these resonances, to our best knowledge, have not been described satisfactorily.

Modifications of CM theory, required for photonic crystal engineering, account for reflections inside the photonic medium composing TWPA \[20–22\]. In this paper, we generalize the conventional CM theory for both 3WM and 4WM processes, by taking into account the reflections at the ends of the unmatched TWPA. This enables us to properly describe the transmission of broadband TWPs, including the gain ripples. We have fabricated two coplanar waveguides, one made of high kinetic inductance (KI) superconductor, one consisting of 2000 JJs, both connected to 50 $\Omega$ input and output lines. They were tested in 3WM and 4WM regime, respectively. Due to the impedance mismatch, both produced amplifiers operate in the intermediate regime between resonant and traveling wave regimes. A reasonable agreement between theory and experiment is demonstrated.

II. COUPLED MODE THEORY

Non-linear media are commonly exploited for parametric amplification. In our case, such medium is provided by the middle wire of the TL, which is formed either by an array of JJs (top in Fig. 3) or by a high kinetic inductance superconductor. Dependencies of voltage \(V(z, t)\) and current \(I(z, t)\) on the coordinate \((z)\) and time \((t)\) in the transmission line are described by nonlinear telegra-
inductance per unit length enhanced by the DC bias is
sion line into account (see Appendix A), the RF current
\( I \)
the RF current and
different parts of the transmission line are present. This
however, impedance mismatches at interconnections of
along an ideally matched transmission line. In practice,
Commonly, it is assumed that such waves propagate
and 4WM idler

\[ V \]
\[ I \]
where \( L_t \) and \( C_t \) denote the respective inductance and capacitance per unit length of the transmission line. Here
\( I_* \) relates to the critical current \( I_c \): \( I_* = \sqrt{2I_c} \) for JJ TWPA and kinetic inductance TWPA, respectively. As the superconducting medium does not exhibit DC losses, the DC bias current \( I_D \) does not contribute to the voltage \( V \), however, it alters the nonlinear inductance. Therefore, in the presence of \( I_D \) (2) becomes

\[ \frac{\partial I(z,t)}{\partial t} = -(L_t^0(1 + I(z,t)^2/I_*^2))^{-1} \frac{\partial V(z,t)}{\partial z}, \]

where \( \epsilon = 2I_D/(I_D^2 + I_*^2) \), \( \xi = 1/(I_D^2 + I_*^2) \) [10] and the inductance per unit length enhanced by the DC bias is \( L_t^0 = L_t(1 + I_D^2/I_*^2) \). In the following, \( I(z,t) \) denotes only the RF current and \( I_D \) is fixed. Equivalently, equations [1] and [3] can be presented as:

\[ \frac{\partial^2 I(z,t) }{\partial z^2 } - \frac{\partial^2 I(z,t) }{\partial t^2 } = \frac{\partial^2}{\partial t^2} \left[ \frac{1}{2} \epsilon I(z,t)^2 + \frac{1}{3} \xi I(z,t)^3 \right], \]

where \( v = 1/\sqrt{L_t^0 C_t} \) is the phase velocity in the waveguide.

To extract the gain as a function of circuit parameters, typically four planar waves

\[ I(z,t) = \sum_n \frac{1}{2} (I_n(z)e^{i(k_n z - \omega_n t)} + c.c.) \]

are substituted into the equation [4]. Here \( \omega_n \) is the circular frequency and \( k_n = \omega_n/v \) denotes wave vector, where the index \( n \) indicates the type of the wave; namely \( n = p, s, i_3, i_4 \) stands for pump, signal, and two idlers, respectively. The idler \( i_3 \) entering the 3WM satisfies

\[ \omega_p = \omega_s + \omega_{i_3} \]

and 4WM idler \( i_4 \) obeys

\[ 2\omega_p = \omega_s + \omega_{i_4}. \]

Commonly, it is assumed that such waves propagate along an ideally matched transmission line. In practice, however, impedance mismatches at interconnections of different parts of the transmission line are present. This leads to partial reflection of the propagating waves, characterized by the reflection coefficient \( \Gamma_n \) determined by the impedance mismatch at the frequency \( \omega_n \). Taking such reflections at both ends of the non-linear transmission line into account (see Appendix A), the RF current in the transmission line can be expressed as

\[ I(z,t) = \sum_n \frac{1}{2} \left( I_n(z) t_n(e^{i(k_n z + \Gamma_n e^{-i\omega_n t}}) e^{-i\omega_n t} + c.c.) \right), \]

where \( t_n = 1/(1 - \Gamma_n^2 e^{2ik_n}) \) is the transmission amplitude at the frequency \( \omega_n \) and \( l \) is the length of the transmission line. Therefore, the transmission coefficient can be expressed

\[ T_n = (1 - \Gamma_n^2)^2|t_n|^2 = \frac{(1 - \Gamma_n^2)^2}{1 + \Gamma_n^2 - 2\Gamma_n \cos(2k_n l)}. \]

To reconstruct the functions \( I_n(x) \), \( I_{i_3}(x) \) and \( I_{i_4}(x) \), the expression [8] is substituted into the wave equation [4]. By making use of standard approximations, differential equations for the spatial evolution of these amplitudes are found (for details refer to Appendix A). The resulting equations for the signal gain under 3WM and 4WM are obtained in an approximate form in the limit \( \Gamma \ll 1 \) (see Appendix B). Finally, a solution formally identical to the well-known equation for signal gain [15] is obtained from [17]:

\[ G_{3,4}(l) \equiv \left| \frac{I_{i_3}(l)}{I_{i_3}(0)} \right|^2 = \cos^2(g_{3,4} l) + \frac{\beta_{3,4}^2}{4g_{3,4}^2} \sinh^2(g_{3,4} l), \]

where \( \beta_{3,4} \) are the parameters of the phase mismatch and \( g_{3,4} \) are the gain factors for 3WM or 4WM indicated by the indices. It is important to note that expression [10] is generalized to include reflections, which results in the following correction to the phase mismatches \( \beta_{3,4} \) and the gain factors \( g_{3,4} \).

The phase mismatch for 3WM process \( \beta_3 \) takes the form:

\[ \beta_3 = \Delta k_3 (1 + 2\gamma (1 + \Gamma_p^2)) - k_p \gamma (1 - \Gamma_p^2), \]

where \( \Delta k_3 = k_p - k_s - k_{i_3} \) is the mismatch of wave vectors and

\[ \gamma = \frac{|t_p I_p|^2}{8(I_D^2 + I_*^2)} \]

is the strength of the nonlinearity. Now, the phase matching condition, i.e. \( \beta_3 = 0 \), dictates:

\[ \Delta k_3 = k_p \gamma (1 - \Gamma_p^2) 1 + 2\gamma (1 + \Gamma_p^2). \]

Obviously, the presence of reflections suppresses the phase mismatch caused by the nonlinearity. Moreover, \( g_3 \) is increased, compared to a perfectly impedance-matched system with \( \Gamma_p = 0 \):

\[ g_3 = \sqrt{k_s k_{i_3} \gamma \frac{8I_D^2}{(I_D^2 + I_*^2)} (1 + \Gamma_p^2) - \frac{\beta_3^2}{4}}, \]
The 4WM is obtained for zero DC bias, thus all waves participating in the mixing are partially reflected at the ends of the TWPA. This leads to even more radical changes. The phase mismatch $\beta_4$ is:

$$\beta_4 = \Delta k (1 + 2\gamma (1 + \Gamma_p^2)) - 2k_p \gamma (1 - \Gamma_p^2),$$

(15)

where $\Delta k_4 = 2k_p - k_s - k_i$ is the mismatch of wave vectors under 4WM. The enhancement of the gain factor for 4WM is

$$g_4 = \sqrt{k_s k_i \gamma^2 (1 + 4\Gamma_p^2) - \frac{\beta_4^2}{4}}.$$  

(16)

Both current amplitudes - the amplified signal and the pump - influence the $\gamma$ coefficient (Eq. 12), which is also modulated by the Fabry-Pérot-like transmission amplitude $t_n$. Therefore, the gain of the unmatched amplifier reads

$$G_{3,4}(l) \equiv \left| \frac{I_s(l)}{I_s(0)} \right|^2 = G_{3,4}(l)T_s.$$  

(17)

Here, we emphasise that not only the gain increase with $\Gamma_p$, but also reversely $\Gamma_p$ increases with the gain (see Appendix C). This positive feedback results in so-called gain ripples, observed by many groups dealing with TWPA, even for nearly impedance-matched devices (aiming for $\Gamma_n \rightarrow 0$) [11, 18]. Moreover to obtain a high gain of the TWPA, sufficiently long nonlinear waveguides have to be fabricated. However, this leads to the narrowing of the gain ripples, whereas a periodic metamaterial filter has to be implemented to ensure the phase-matching condition. By utilizing a nonlinear waveguide with an intermediate length, the ripples in the amplifiers transmission become wider. Thus, the amplifier is in the transition regime between travelling-wave and resonant parametric amplifier. Furthermore, making use of the resonance by avoiding impedance matching elements allows higher gain. Additionally, the omission of any periodic filter design enables a wide range of settings for the working frequency of the amplifier, compensating the narrower bandwidth of the amplifier in this regime.

### III. EXPERIMENT

#### A. Kinetic inductance TWPA

In order to investigate the validity of the model described above, we fabricated the KI-TWPA device shown in Fig. 1. The design represents a waveguide where an inductive element - the central high-KI strip - is coupled to the ground plane via fractalized capacitors $C_F$ (Fig. 1 bottom inset). The central strip is 27 mm long and 1 µm wide and has a sheet kinetic inductance of $\sim 4.2$ pH/□. The combination of a highly inductive central strip and increased capacitance of fractalized capacitors results in slow propagation velocity ($\sim 2\%$ of the speed of light), thereby reducing the length of the transmission line required to get sufficient amplification [23, 24].

The KI-TWPA chip was measured in a Helium gas-flow cryostat with a base temperature of $\sim 2$ K. The input and output terminals of the device were bonded onto sampling lines in a printed circuit board (PCB) and the microwave transmission was measured using a Vector Network Analyzer (VNA). In order to eliminate spurious ground plane resonances, the ground plane of the chip was carefully bonded to the PCB ground around the perimeter of the sample.

---

**FIG. 1:** Schematic of the KI-TWPA with chip dimensions. The right inset shows an optical micrograph of a few sections of quasi-fractal waveguide. The bottom inset is a magnified SEM image of the fractal structure. The width of the central high KI line is 1 µm.
The Fig. 2 presents the amplifier gain measured in 3WM operation regime for DC bias current $I_D = 1 \text{ mA}$ and the pump current amplitude $I_p \approx 1 \text{ mA}$ at frequency of 10.38 GHz. The measured gain profile shows an average gain of 9.15 dB in the frequency range of 3-7 GHz, along with the presence of ripples indicating an impedance mismatch. As seen in Fig. 2 (inset), the ripples can be reasonably fit with the proposed model. The reflection coefficient $\Gamma$ can be precisely estimated from the fit, and are presented in Tab. I along with the values of $I_P$ and $I_D$. A fit to the standard theory returns higher values of $I_D$ and $I_P$. The parameter $I_s$ was estimated in Ref. 25 as $I_s = I_c/0.27$ for superconducting nanowires.

![Graph of measured amplification profile](image)

**FIG. 2:** Measured amplification of kinetic inductance TWPA (blue data). Red curve is a fit by the corrected theory. The inset shows how the Fabry-Pérot oscillations were matched to the measured transmission, allowing the extraction of phase velocity and reflection coefficient utilized to reconstruct the frequency profile of the gain. The orange curve is a fit of the data by standard theory.

| $f$ (GHz) | Gain (dB) |
|----------|-----------|
| 0.0     | 1.6       |
| 2.5     | 1.8       |
| 5.0     | 2.0       |
| 7.5     | 2.2       |
| 10.0    | 2.4       |
| 12.5    | 2.6       |

**TABLE 1:** Parameters of the KI-TWPA. The length $l$ is given by the design, the other parameters are obtained from the fit of the amplification profile.

| $l$ [mm] | $v_p [c]$ | $L_s[l_c]$ | $I_p$ | $I_p/l_c$ |
|----------|-----------|------------|-------|-----------|
| standard theory | 27 | 0.018 | 0.5 | 0.00 | 0.25 |
| with correction | | | 0.48 | 0.52 | 0.1 |

**B. Josephson Junction TWPA**

To show that the gain of a TWPA with an even higher reflection coefficient ($\Gamma_p \approx 0.7$) can also be described by the introduced model, a high impedance CPW with nonlinear inductance was studied (see Fig. 3). The middle wire, 11 mm long, is formed out of 2000 niobium-based Josephson junctions (JJs), see Appendix [2] as well as Ref. 29. The Josephson inductance, the inductance per unit length can be estimated by the relation $L_{iJ}/2000 = \Phi_0/2\pi l_c$, where $\Phi_0$ is the magnetic flux quantum. Here, the critical current $I_c$ is estimated from the BCS relation for the product of $I_c$ and the normal state resistance of the junction $R_n$ [27]. The ground capacitance is estimated both by the standard formula (see Ref. 28), and an EM simulation in Sonnet software. These values for the estimation of the phase velocity $1/\sqrt{L_s/C_s}$ and the characteristic impedance $\sqrt{L_s/C_s}$ are listed in Tab. II.

The sample was installed in a copper box and its 50 Ω contacting pads were contacted by indium to the SMA connectors. Prior to the actual measurement, the transmission has been calibrated by using a 50 Ω TL connected instead of the TWPA (for details on calibration, see Ref. 29). The transmission was measured by a VNA in a pulse-tube refrigerator at a temperature of 3.5 K.

![Schematic of Josephson Junction TWPA](image)

**FIG. 3:** Top: lumped element model of the JJ transmission line terminated by 50 Ω coaxial cables. Estimated values of inductance $L_s$ and capacitance $C_s$ are listed in table II. 

Bottom: schematic of the TWPA chip with dimensions.

Measurements of the transmission of the unmatched transmission line were performed at low signal power, where nonlinearities are negligible (see blue points in Fig. 4). The measured transmission exhibits resonances (dashed line in Fig. 4 a), with peaks at frequencies $f_{n_1} = n_1 v/2l$, where $n_1$ is an integer, and $l$ is the length of the waveguide. As the unmatched waveguide creates a stepped impedance resonator, the transmission can be described by Eq. (4). However, additional narrower ripples are present, with peaks at frequencies $f_{n_2} = n_2 v/2l$, where $n_2$ is an integer. As the separation of these narrow peaks $\Delta f_2$ (indicated in Fig. 4 a) is smaller, the length $l_c$ of this step impedance resonator is
TABLE II: Parameters of the JJ-TWPA waveguide. The length \( l \) and the number of junctions \( N \) is set by design. \( NR_a \) is the room temperature resistance of the chain of \( N \) JJs. Other parameters are obtained from the fit of the weak signal transmission by Eq. (18) and the amplified signal transmission by the presumed model.

| \( N \) | \( l \) [mm] | \( NR_a \) [\( \Omega \)] | \( I_c \) [\( \mu A \)] |
|---|---|---|---|
| 2000 | 11 | 309 | 12.3 |

| \( L_i \) [\( \mu H/\mu m \)] | \( C_i \) [\( \mu F/\mu m \)] | \( 1/\sqrt{L_iC_i} \) | \( \sqrt{L_i/C_i} \) |
|---|---|---|---|
| 4.68 | 0.13 | 0.14 | 189 |

| \( \Gamma_s \) | \( \Gamma_{sc} \) | \( v / c \) | \( v_c / c \) |
|---|---|---|---|
| 0.425 | 0.405 | 0.14 | 0.75 |

| \( \Gamma_p \) | \( \Gamma_{sc} \) | \( l_c \) [mm] | \( I_p / I_c \) |
|---|---|---|---|
| 0.75 | 0.7 | 740 | 0.81 |

bigger and, assuming a phase velocity \( v_c \approx 0.75c \) of the used coaxial cable, correlates with its length. Thus, these resonances originate from the interplay of the measurement setup with the unmatched TWPA. This is further verified by measuring the transmission of the TWPA with a 10 dB attenuator at the output. In this case, the amplitude of these resonances is suppressed (see Fig. 4b).

In the same way as discussed above, these resonances are also described as Fabry-Pérot oscillations (9) with phase velocity \( v \) and length \( l_c \). The obtained phase velocity \( v \) of the waveguide (solid line in Fig. 4a) can be approximated as

\[
T_s \approx (1 - \Gamma_s^2)^2 |t_s|^2 (1 - \Gamma_{sc}^2)^2 |t_{sc}|^2 = (1 - \Gamma_s^2)^2 (1 - \Gamma_{sc}^2)^2 |1 - \Gamma_s^2 e^{2ikc l}|^2 |1 - \Gamma_{sc}^2 e^{i4\pi f j l_c / v_c}|^2.
\]

(18)

This equation for \( \Gamma_s \approx 0.3 \) fits the experimental data very well. The obtained phase velocity \( v \) and the reflection coefficient \( \Gamma_s \) of the waveguide are consistent with the estimated inductance and capacitance per unit length (see Tab. II). These parameters are used to calculate the gain of the TWPA from Eq. (17), taking \( T_s = T_{sc} \). In addition, the reflection coefficients \( \Gamma_s \) and \( \Gamma_{sc} \) are changed by the amplification (see Appendix C). Therefore, they were replaced by new values \( \Gamma_s' \approx 0.75 \) and \( \Gamma_{sc}' \approx 0.7 \) for the pump tone being on. We have tuned their values in order to obtain agreement with the experiment, as shown in Fig. 5.

The signal transmission with pump tone was measured for a variety of signal and pump powers. The highest amplification was achieved for a pump power of -69.6 dBm and signal power from -100 dBm to -86 dBm at the input of the device. Fig. 5 presents the gain as a function of the signal frequency at pump frequency 3.66 GHz. The measured gain profile shows a region where amplification occurs, clearly corresponding to a peak resulting from resonance caused by impedance mismatches (Fig. 4). When the standard model for matched TL is applied (\( \Gamma_p = 0 \), see orange curve in Fig 5), a weak amplification (< 5 dB) is obtained over a wide bandwidth. Including the effects of reflections by the discussed model and accounting for the correction \( \Gamma_p = \Gamma_p' \) from table II significantly improves the correspondence between theory and experiment. Especially, the ripples in the gain are in good agreement with the proposed model (see Fig. 5).

FIG. 4: (a) Calibrated weak signal transmission without pumping tone (blue dots). Fit of the data by superposition (15) of two Fabry-Pérot oscillations (red curve). The parameters of the fit; \( \Gamma_s, \Gamma_{sc}, v, v_c \) and \( l_c \) are listed in table II. (b) Calibrated transmission of the waveguide with 10 dB attenuator at the output. The attenuator suppressed the narrower ripples, indicating that corresponding resonances occur between sample and a part of the measurement setup.

FIG. 5: The measured gain of the TWPA (blue dots) for pump frequency 3.66 GHz. Orange and red curves are predictions of the uncorrected model and the model with reflections for \( \Gamma_p = 0.75 \), respectively.
IV. CONCLUSION

In this article, we presented a modification of the standard CM theory of parametric amplification, which is normally utilized to analyze TWPs. By considering reflections due to impedance mismatches, we showed that the narrow gain ripples are an inherent property of unmatched TWPs with finite length. We point out that instead of trying to eliminate them they could be used to enhance the gain at smaller TL length. Such TWPA still have reasonable bandwidth centered at multiple frequencies, compared to resonant parametric amplifiers. Moreover, the drawback of limited bandwidth is compensated also by the possibility of changing the range where amplification occurs by tuning the pump frequency, and as it turns out from the theory, this range is further enhanced by the presence of reflections. Furthermore, we demonstrate the usability of our model by analyzing the response of two types of devices: a JJ and a kinetic inductor. It turns out from the theory, this range is further enhanced by the presence of reflections. To solve the nonlinear wave equation (4), the spatially-dependent current amplitudes $I_n$ are defined as

$$I_n(z,t) = \frac{1}{2} I_n(z)(e^{i\omega_n z} + \Gamma_n e^{i\omega_n (1-z)} + \Gamma_n^2 e^{i\omega_n (2l+z)} + ...) e^{-i\omega_n t} + c.c. =$$

$$= \frac{1}{2} I_n(z) (e^{i\omega_n z} + \Gamma_n e^{-i\omega_n z}) e^{-\omega_n t} + c.c.,$$

where in the second line, the following notation is introduced

$$\Gamma_n = \Gamma_n e^{i\omega_n t}, \quad (A2) \quad t_n = \frac{1}{1 - \Gamma_n^2}. \quad (A3)$$

To solve the nonlinear wave equation (4), the spatially-dependent current amplitudes $I_n$ are searched, such that the current $\Gamma_n$ satisfies (A1). In the following, the notation $I'_n \equiv \frac{\partial}{\partial z}$ is adopted and $\ast$ denotes complex conjugation. Within a slowly varying envelope approximation (i.e., $|I'_n| \ll \omega_n |I_n|$) one obtains

$$\frac{1}{L_i} \frac{d}{dz} I_n(z) t_n(e^{i\omega_n z} - \Gamma_n e^{-i\omega_n z}) =$$

$$- \frac{\omega_n}{2} \sum_{p,s,i} \left\{ \sum_{a,b} \left( \prod_{m} \frac{1}{2} \frac{\partial}{\partial z} I_m^{(a)} I_m^{(s)} e^{i\omega_m z} + \Gamma_m e^{-i\omega_m z} \right) \times \delta(\omega_n + \omega_a + \omega_b) - \right.$$  

$$\left. - \frac{\xi}{3} \sum_{a,b,c} \left( \prod_{m} \frac{1}{2} \frac{\partial}{\partial z} I_m^{(a)} I_m^{(s)} e^{i\omega_m z} + \Gamma_m e^{-i\omega_m z} \right) \times \delta(\omega_n + \omega_a + \omega_b + \omega_c), \right.$$  

$$\right.$$  

$$\left. \right. \right.$$  

where the first sum on the right hand side describes 3WM terms and the second includes 4WM terms. Here, $\ast$ above $I_m$ and $t_n$ denotes complex conjugation in the case of a minus sign of the corresponding frequency $\omega_m$ in the delta function.

To study the parametric amplification provided by 3 and 4-wave mixing the equation (4) is solved for four waves: strong pump $I_p$, signal $I_s$ and two idlers $I_{i3}, I_{i4}$ such that $I_p \gg I_s, I_{i3,4}$:

$$I'_p = \frac{i k_p}{8} \xi |I|^2 |t|^2 I_p F^{pp}, \quad (A5)$$

$$I'_s = \frac{i k_s}{4} \xi I_s T_{i3} t_{i3} T_{i3} F^{pp}, \quad (A6)$$

$$+ \frac{i k_s}{8} \xi \left( I_s T_{i4} t_{i4} + 2 |I|^2 I_s |t|^2 F^{ps} \right),$$

$$I'_{i3} = \frac{i k_{i3}}{4} \left( \xi I_p T_{i3} t_{i3} F^{pp} + \xi |I|^2 I_{i3} |t|^2 F^{pss} \right), \quad (A7)$$

$$I'_{i4} = \frac{i k_{i4}}{8} \xi \left( I_p T_{i4} t_{i4} F^{pp} + 2 |I|^2 I_{i4} |t|^2 F^{pss} \right). \quad (A8)$$

V. ACKNOWLEDGMENTS

This work was supported by the European Unions Horizon 2020 research and innovation programme under Grant Agreement No. 863313 (SUPERGALAX) and by the QuantERA grant SIUCs. The support from the Slovak Research and Development Agency under the contracts APVV-16-0372, APVV-18-0358, and SAS-MTVS are gratefully acknowledged. The Chalmers group acknowledges the support from the Swedish Research Council (VR) (grant agreements 2016-04828 and 2019-05480), EU H2020 European Microkelvin Platform (grant agreement 824109), Engineering and Physical Sciences Research Council (EPSRC) Grant No. EP/T004088/1 and from Knut and Alice Wallenberg Foundation via the Wallenberg center for Quantum Technology (WACQT). This work has received funding from the Free State of Thuringia under the number 2021 FGI 0049.

Appendix A: Coupled mode equations for waves in a resonator

The field inside a Fabry-Pérot resonator terminated by two identical mirrors with reflection coefficients $\Gamma_n$ consists of an infinite number of reflected waves. Thus the amplitude $I_n$ of the field oscillating at frequency $\omega_n$ is defined as

$$I_n(z,t) = \frac{1}{2} I_n(z)(e^{i\omega_n z} + \Gamma_n e^{i\omega_n (1-z)} + \Gamma_n^2 e^{i\omega_n (2l+z)} + ...) e^{-i\omega_n t} + c.c. =$$

$$= I_n(z) (e^{i\omega_n z} + \Gamma_n e^{-i\omega_n z}) e^{-\omega_n t} + c.c.,$$

where in the second line, the following notation is introduced

$$\Gamma_n = \Gamma_n e^{i\omega_n t}, \quad (A2) \quad t_n = \frac{1}{1 - \Gamma_n^2}. \quad (A3)$$

To solve the nonlinear wave equation (4), the spatially-dependent current amplitudes $I_n$ are searched, such that the current $\Gamma_n$ satisfies (A1). In the following, the notation $I'_n \equiv \frac{\partial}{\partial z}$ is adopted and $\ast$ denotes complex conjugation. Within a slowly varying envelope approximation (i.e., $|I'_n| \ll \omega_n |I_n|$) one obtains

$$\frac{1}{L_i} \frac{d}{dz} I_n(z) t_n(e^{i\omega_n z} - \Gamma_n e^{-i\omega_n z}) =$$

$$- \frac{\omega_n}{2} \sum_{p,s,i} \left\{ \sum_{a,b} \left( \prod_{m} \frac{1}{2} \frac{\partial}{\partial z} I_m^{(a)} I_m^{(s)} e^{i\omega_m z} + \Gamma_m e^{-i\omega_m z} \right) \times \delta(\omega_n + \omega_a + \omega_b) - \right.$$  

$$\left. - \frac{\xi}{3} \sum_{a,b,c} \left( \prod_{m} \frac{1}{2} \frac{\partial}{\partial z} I_m^{(a)} I_m^{(s)} e^{i\omega_m z} + \Gamma_m e^{-i\omega_m z} \right) \times \delta(\omega_n + \omega_a + \omega_b + \omega_c), \right.$$  

$$\right.$$  

$$\right.$$  

where the first sum on the right hand side describes 3WM terms and the second includes 4WM terms. Here, $\ast$ above $I_m$ and $t_n$ denotes complex conjugation in the case of a minus sign of the corresponding frequency $\omega_m$ in the delta function.

To study the parametric amplification provided by 3 and 4-wave mixing the equation (4) is solved for four waves: strong pump $I_p$, signal $I_s$ and two idlers $I_{i3,4}$ such that $I_p \gg I_s, I_{i3,4}$:

$$I'_p = \frac{i k_p}{8} \xi |I|^2 |t|^2 I_p F^{pp}, \quad (A5)$$

$$I'_s = \frac{i k_s}{4} \xi I_s T_{i3} t_{i3} T_{i3} F^{pp}, \quad (A6)$$

$$+ \frac{i k_s}{8} \xi \left( I_s T_{i4} t_{i4} + 2 |I|^2 I_s |t|^2 F^{ps} \right),$$

$$I'_{i3} = \frac{i k_{i3}}{4} \left( \xi I_p T_{i3} t_{i3} F^{pp} + \xi |I|^2 I_{i3} |t|^2 F^{pss} \right), \quad (A7)$$

$$I'_{i4} = \frac{i k_{i4}}{8} \xi \left( I_p T_{i4} t_{i4} F^{pp} + 2 |I|^2 I_{i4} |t|^2 F^{pss} \right). \quad (A8)$$
In the following subsections, the obtained equations are expressed as complex amplitudes. Finally, one obtains the coupled mode equations for the utilising solution (A11), the equations (A6), (A7) and (A8) are simplified by the following transformation:

\[ A_n = \mathcal{I}_n t_n \exp(-2i \int_0^z \kappa_n \mathcal{F}_{pp}^n dx) \quad n = s, i_3, i_4. \tag{A14} \]

Finally, one obtains the coupled mode equations for the complex amplitudes \( A_s \) and \( A_{i_3, i_4} \):

\[ A_s' = i \kappa_s^D A_{i_3}^* \mathcal{F}_{i_3 s}^p e^{ib_3}, \tag{A15} \]

\[ A_{i_3}' = i \kappa_{i_3}^D A_s^* \mathcal{F}_{s i_3}^p e^{ib_3}, \tag{A16} \]

\[ A_{i_4}' = i \kappa_{i_4} A_{s}^* \mathcal{F}_{s i_4}^p e^{ib_4}. \tag{A17} \]

Here, the functions \( b_3(z) \) and \( b_4(z) \) contain contributions to the nonlinear phase modulations and are expressed as follows:

\[ b_3(z) = \int_0^z \left( \kappa_p \mathcal{F}_{pp}^p(x) - 2 \kappa_s \mathcal{F}_{ps}(x) - 2 \kappa_{i_3} \mathcal{F}_{pi_3}^p(x) \right) dx, \tag{A18} \]

\[ b_4(z) = 2 \int_0^z \left( \kappa_p \mathcal{F}_{pp}^p(x) - \kappa_s \mathcal{F}_{ps}(x) - \kappa_{i_3} \mathcal{F}_{pi_3}^p(x) \right) dx. \tag{A19} \]

In the following subsections, the obtained equations are solved in two cases: 1. \( I_D \gg \mathcal{I}_p \) (3-wave mixing) \[16\] \[22\] and 2. \( I_D = 0 \) (4-wave mixing).

1. 3-wave mixing

To study 3WM, let us assume that the DC bias current is much larger than the pump amplitude, which means, according to eqs. (A12, A13), that \( \kappa_{i_3}^D \gg \kappa_n \). If no idler is applied at the frequency \( \omega_{i_3} \) to the input of the device, the 4WM idler is generated proportionally to \( \kappa_{i_3} \). Therefore, the 4WM idler is much weaker than 3WM idler which is proportional to \( \kappa_{i_3}^D \) and the system of equations (A15-A17) is approximated by two coupled equations

\[ A_s' = i \kappa_{i_3}^D A_{i_3}^* \mathcal{F}_{i_3 s}^p e^{ib_3}, \tag{A20} \]

\[ A_{i_3}' = i \kappa_s^D A_s^* \mathcal{F}_{s i_3}^p e^{ib_3}, \tag{A21} \]

which can be easily decoupled. For \( A_s \) the uncoupled equation is

\[ A_s'' - \left( \frac{e^{ib_3} \mathcal{F}_{i_3 s}^p}{\mathcal{F}_{i_3 i_3}^p} \right) A_s' - \kappa_s^D \kappa_{i_3}^D \mathcal{F}_{i_3 i_3}^p \mathcal{F}_{s s}^{i_3} A_s e^{-i 2 \Gamma_m(b_3)} = 0. \tag{A22} \]

2. 4-wave mixing

When no DC bias is applied, pure 4WM is observed and the system (A15-A17) becomes

\[ A_s' = i \kappa_{i_4} A_{i_4}^* \mathcal{F}_{i_4 s}^p e^{ib_4}, \tag{A23} \]

\[ A_{i_4}' = i \kappa_s A_s^* \mathcal{F}_{s i_4}^p e^{ib_4}, \tag{A24} \]

which gives the equation for the spatial evolution of the transformed signal amplitude \( A_s(z) \):

\[ A_s'' - \left( \frac{e^{ib_4} \mathcal{F}_{i_4 i_4}^p}{\mathcal{F}_{i_4 i_4}^p} \right) A_s' - \kappa_s \kappa_{i_4} \mathcal{F}_{i_4 i_4}^p \mathcal{F}_{s s}^{i_4} A_s e^{-i 2 \Gamma_m(b_4)} = 0, \tag{A25} \]

A simple check of our derivation is achieved by identifying equations (A22-A25) as the general form of the standardly presented CM result for \( \Gamma_n = 0 \).

Appendix B: Approximation of the gain equation

Equations (A22, A25) for slow variation of an envelope of waves propagating and reflecting in a nonlinear Fabry-Pérot resonator are second-order differential equations where \( \mathcal{F} \) and \( b \) are functions of \( z \). To solve the differential equations, we expand these functions up to second order in the reflection coefficient \( \Gamma_n < 1 \) and remove terms with harmonic spatial dependence via the averaging method [20, 22]. This transformation is indicated by the arrows in the equations given below. This procedure yields an equation similar to the result of the standard CM for waves propagating only in one direction:

\[ \mathcal{F}_{pp} \approx 1 + 2 \Gamma_p e^{-ik_p(2z - L)} + 2 \Gamma_p e^{-ik_p(2z - L)} + 3 |\Gamma_p|^2 + 4 \Gamma_p^2 e^{-2ik_p(2z - L)} \rightarrow 1 + 3 |\Gamma_p|^2 \tag{B1} \]
where coefficient $\Gamma_n$ are gain factors for 3WM and 4WM, respectively.

(1) for harmonic components:

$$F_{pp}^{3} * F_{pp}^{3} \rightarrow 1 + |\Gamma_p|^2 ,$$

$$F_{pp}^{4} * F_{pp}^{4} \rightarrow 1 + 4 |\Gamma_p|^2 .$$

The above approximations lead to the equation describing 3WM

$$A''_n - i\beta_3 A'_n - k_s \kappa_{i3}^D (1 + |\Gamma_p|^2) A_n = 0$$

which gives the nonlinear phase mismatch $\beta_3$

$$\beta_3 = \Delta k_3 + \kappa_p (1+3 |\Gamma_p|^2) - 2k_s (1+|\Gamma_p|^2 - 2 \kappa_{i3} (1+|\Gamma_p|^2)),$$

where $\Delta k_3 = k_p - k_s - k_{i4}$ is the deviation from linear dispersion relation.

Similarly, the differential equation for 4WM takes the form

$$A''_n - i\beta_4 A'_n - k_s \kappa_{i4}^D (1 + 4 |\Gamma_p|^2) A_n = 0$$

and the nonlinear phase mismatch $\beta_4$ reads

$$\beta_4 = \Delta k_4 + 2(\kappa_p (1+3 |\Gamma_p|^2) - k_s (1+|\Gamma_p|^2) - 2 \kappa_{i4} (1+|\Gamma_p|^2)),$$

where $\Delta k_4 = 2(k_p - k_s - k_{i4})$.

With the boundary conditions $I_s(z) = 0 = I_{s0}$ and $I_{i3,4}(z = 0) = 0$ the solution takes the compact form

$$I_s(z) = \left( \cosh(g_{3,4}z) - i \frac{\beta_3}{2g_{3,4}} \sinh(g_{3,4}z) \right) \times e^{i\left( \frac{2\kappa_p}{g_{3,4}} + 2k_s (1+|\Gamma_p|^2) \right)z},$$

where

$$g_3 = \sqrt{k_s \kappa_{i3}^D (1 + |\Gamma_p|^2) - \frac{\beta_3}{4}},$$

$$g_4 = \sqrt{k_s \kappa_{i4}^D (1 + 4 |\Gamma_p|^2) - \frac{\beta_4}{4}},$$

are gain factors for 3WM and 4WM, respectively.

Appendix C: Quality increase due to amplification

Following the conventional derivation of the reflection coefficient $\Gamma_n$ at the ends of the TL (see Ref. [31]), we derive below the influence of the amplification on the reflection coefficient and, therefore, on the quality factor of the resonator. Utilizing the first telegrapher equation (for harmonic components:

$$I'_n(z) = -i \omega_n CV_n(z),$$

the voltage amplitude along the waveguide is found by substituting the current amplitude, which was determined in the previous chapter (A). This way the impedance at the end of the waveguide is obtained:

$$Z_n(l) = \frac{V_n(l)}{I_s(l)} \approx Z_0 \left( 1 + i \frac{\Gamma_n(z)'}{|\Gamma_n(z)|} \right),$$

where $Z_0 = \sqrt{\frac{L_0}{C_f}}$ and the nonlinear spatial variation of phase velocity was neglected. Finally, the reflection coefficient can be expressed as

$$\Gamma_n = \frac{Z_0 - Z_L - Z_0 |\Gamma_n(z)'|_{z=0}}{Z + Z_L + Z_0 |\Gamma_n(z)'|_{z=0}},$$

showing that the reflection is sensitive to any spatial variation in the amplitude of the current.

Appendix D: Samples preparation

The 140 nm NbN film for a KI-TWPA was fabricated at Chalmers, following the recipe Ref. [32]. These films were deposited on a 2" sapphire wafer and the desired structure was patterned with e-beam lithography and Ar:Cl_2 plasma etching. The central line was further thinned down to 30 nm, to get a sheet resistance of ~ 48 Ω/□, which corresponds to the desired kinetic inductance of ~ 4 pH/□. The thickness of the ground plane and other elements was left unchanged, in order to keep the KI of these elements low and thereby eliminate self-resonances in fractal structures.

The JJ devices were fabricated by making use of the so-called cross-type Josephson junction technology. Here a trilayer of Nb/AlO_x/Nb with a critical current density of about 1.7 kA/cm^2 is deposited on an oxidized 4 inch silicon wafer of 500 μm thickness. Thermal oxide thickness on the wafer was about 600 nm. Inside a meander shaped Nb groundplane with a 9 μm slit, an array of Josephson junctions form the center conductor. In total, 2000 Josephson junctions with a nominal junction size of (0.9 x 0.9) μm^2 are fabricated on a single chip, with dimensions of (10800 x 15000) μm^2. By means of Fiske step measurements, the specific junction capacitance has been determined to be around 60 fF/μm for this critical current density. The junctions have been arranged in blocks of 160 Josephson junctions. In between these blocks, inductance elements have been placed with varying length of 14 μm, 22 μm and 36 μm. For sample fabrication, electron beam lithography has been used. Nb patterning was done by making use of reactive ion etching based on CF_4.
[1] P. Krantz, M. Kjaergaard, F. Yan, T. P. Orlando, S. Gustavsson, and W. D. Oliver, Applied Physics Reviews 6, 021318 (2019).
[2] E. Il'ichev, A. Smirnov, M. Grajcar, A. Izmalkov, D. Born, N. Oukhanski, T. Wagner, W. Krech, H.-G. Meyer, and A. Zagoskin, Low Temp. Phys. 30, 620 (2004).
[3] M. Devore and R. Ananda, Comptes Rendus Physique 17, 740 (2016).
[4] B. Yurke, JOSA B 4, 1551 (1987).
[5] B. Yurke, L. Corruccini, P. Kaminsky, L. Rupp, A. Smith, A. Silver, R. Simon, and E. Whittaker, Physical Review A 39, 2519 (1989).
[6] B. Yurke, M. Roukes, M. Roukes, M. Movshovich, and A. Pargellis, Applied Physics Letters 69, 3078 (1996).
[7] A. Zagoskin, E. Il'ichev, M. McCutcheon, J. Young, and F. Nori, Phys. Rev. Lett 101, 253602 (2008).
[8] B. H. Eom, P. K. Day, H. G. LeDuc, and J. Zmuidzinas, Nature Physics 8, 623 (2012).
[9] T. White, J. Mutus, I.-C. Hoi, R. Barends, B. Campbell, Y. Chen, Z. Chen, B. Chiaro, A. Dunsworth, E. Jeffrey, et al., Applied Physics Letters 106, 242601 (2015).
[10] C. Macklin, K. O'brien, D. Hover, M. Schwartz, V. Bolkhovsky, X. Zhang, W. Oliver, and I. Siddiqi, Science 350, 307 (2015).
[11] L. Planat, A. Ranadive, R. Dassonneville, J. P. Martinez, S. Léger, C. Naud, O. Buisson, W. Hasch-Guichard, D. M. Basko, and N. Roch, Physical Review X 10, 021021 (2020).
[12] E. A. Tholén, A. Ergül, E. M. Doherty, F. M. Weber, F. Grégis, and D. B. Haviland, Applied physics letters 90, 253509 (2007).
[13] M. Castellanos-Beltran and K. Lehnert, Applied Physics Letters 91, 083509 (2007).
[14] M. R. Vissers, R. P. Erickson, H.-S. Ku, L. Vale, X. Wu, G. Hilton, and D. P. Pappas, Applied physics letters 108, 012601 (2016).
[15] G. P. Agrawal, Nonlinear Fiber Optics (Academic Press, San Diego, Ca., 525 B Street, Suite 1900, 2001), 3rd ed., ISBN 0-12-045143-3.
[16] M. Malnou, M. Vissers, J. Wheeler, J. Aumentado, J. Hubmeyr, J. Ullom, and J. Gao, PRX Quantum 2, 010302 (2021).
[17] S. Chaudhuri, D. Li, K. Irwin, C. Bockstiegel, J. Hubmeyr, J. Ullom, M. Vissers, and J. Gao, Applied Physics Letters 110, 152601 (2017).
[18] S. Goldstein, N. Kirsh, E. Svetitsky, Y. Zamir, O. Hachmo, C. E. M. de Oliveira, and N. Katz, Applied Physics Letters 116, 152602 (2020).
[19] S. Zhao and S. Withington, Journal of Physics D: Applied Physics 54, 365303 (2021).
[20] W.-P. Huang, JOSA A 11, 963 (1994).
[21] T. Christopoulous, O. Tsilipakos, G. Sinatkas, and E. E. Kriezis, Physical Review B 98, 235421 (2018).
[22] R. P. Erickson and D. P. Pappas, Physical Review B 95, 104506 (2017).
[23] S. d. Graaf, A. Danilov, A. Adamyan, T. Bauch, and S. Kubatkin, Journal of Applied Physics 112, 123905 (2012).
[24] A. Adamyan, S. De Graaf, S. Kubatkin, and A. Danilov, Journal of Applied Physics 119, 083901 (2016).
[25] A. V. Semenov, I. A. Devyatov, M. P. Westig, and T. M. Klapwijk, Physical Review Applied 13, 024079 (2020).
[26] S. Anders, M. Schmelz, L. Fritzsch, R. Stolz, V. Zaksarenko, T. Schönau, and H. Meyer, Superconductor Science and Technology 22, 064012 (2009).
[27] V. Ambegaokar and A. Baratoff, Physical Review Letters 10, 486 (1963).
[28] M. Göppl, A. Fragner, M. Baur, R. Bianchetti, S. Filip, J. M. Fink, P. J. Leek, G. Puebla, L. Steffen, and A. Wallraff, Journal of Applied Physics 104, 113904 (2008).
[29] S. Kern, P. Neillinger, D. Manca, E. Il’ichev, M. Schmelz, J. Kunert, G. Oelsner, R. Stolz, and M. Grajcar, in Proceedings of the 32th International Conference on Applied Physics of Condensed Matter (FEI STU, Bratislava, 2021).
[30] J. A. Sanders, F. Verhulst, and J. Murdock, Averaging methods in nonlinear dynamical systems, vol. 59 (Springer, 2007).
[31] D. M. Pozar, Microwave engineering (John Wiley & sons, 2011).
[32] S. Mahashabde, E. Otto, D. Montemurro, S. de Graaf, S. Kubatkin, and A. Danilov, Physical Review Applied 14, 044040 (2020).