Some results on condition numbers in convex multiobjective optimization

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Abstract

Various notions of condition numbers are used to study some sensitivity aspects of scalar optimization problems. The aim of this paper is to introduce a notion of condition number to study the case of a multiobjective optimization problem defined via \( m \) convex \( C^{1,1} \) objective functions on a given closed ball in \( \mathbb{R}^n \). Two approaches are proposed: the first one adopts a local point of view around a given solution point, whereas the second one considers the solution set as a whole. A comparison between the two notions of well-conditioned problem is developed. We underline that both the condition numbers introduced in the present work reduce to the same condition number proposed by Zolezzi in 2003, in the special case of the scalar optimization problem considered there. A pseudodistance between functions is defined such that the condition number provides an upper bound on how far from a well-conditioned function \( f \) a perturbed function \( g \) can be chosen in order that \( g \) is well-conditioned too. For both the local and the global approach an extension of classical Eckart–Young distance theorem is proved, even if only a special class of perturbations is considered.

Key words. Condition number; Eckart–Young theorem; sensitivity analysis; multiobjective optimization.

Mathematics subject classification. 49K40, 90C31, 90C29

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1 Introduction

The goal of the analysis in parametric optimization is to study how a change in the data of the problem, represented by a vector of parameters, affects the solution of the given problem. A first distinction is usually made between a qualitative and a quantitative approach to the post–optimal analysis: a qualitative approach concerns the continuity properties of the optimal solution map, while a quantitative approach usually involves the evaluation of some kind of derivative of the optimal solution map. The Lipschitzian behaviour of such a map is usually considered as a part of the sensitivity analysis and it is deeply related to the condition number theory. Besides evaluating bounds on the change in the solutions due to perturbations, condition numbers provide an estimation on how large a perturbation can be without disrupting the regular behaviour of the solution map.

The first attempts to consider the numerical implications of Lipschitzian stability properties and of their reformulations in terms of metric regularity were developed in the 70s in the pioneering works by Robinson on generalized equations (see e.g. [14] and the references therein). Later Renegar related the distance from ill–posedness to a notion of condition number for linear programming, thus generalizing the well–known Eckart–Young theorem of numerical linear algebra. The condition number introduced in [12, 13] proved also to be a fundamental tool in the analysis of the rate of convergence of interior point methods in linear programming. An extension of the distance theorem to convex processes was proved in [6].

A few years ago, Zolezzi studied condition numbers in the setting of quadratic optimization [15] and subsequently, more in general, in problems where a differentiable objective function with a Lipschitz continuous gradient is minimized on a ball of a Banach space. The appropriate notion of condition number depends on the class of perturbations of the given problem which preserve solvability. In [17], perturbations by linear continuous functions (the so–called tilt perturbations) are considered, where each perturbed problem has exactly one solution.

Even if condition numbers are widely used both as tools for sensitivity analysis and as instruments that may give precious insights on the numerical aspects of optimization, to our knowledge conditioning techniques have not yet been developed in the field of vector optimization.

The aim of the present work is a first attempt to define an appropriate notion of condition number for multiobjective optimization problems. We limit our study to the case of \( m \) convex differentiable objective functions with Lipschitz continuous gradients and we consider the weakly Pareto efficient solution map on a ball in \( \mathbb{R}^n \) under componentwise uniform tilt perturbations, in the sense that each component of the objective function will receive the same tilt perturbation. We avoid any requirement of uniqueness of the solution, since it is unduly restrictive in the setting of vector optimization. We will consider two different approaches. The first one, the pointwise approach, will focus on a given weakly efficient solution \( \bar{x} \). The pointwise condition number is defined as the Lipschitz modulus of the weakly efficient solution map at \( \bar{x} \) for \( p = 0 \), where
the value \( p = 0 \) characterizes the unperturbed problem.

On the other hand, one can follow a global approach where the whole weakly efficient solution set on a given ball is considered under tilt perturbations. The global condition number is built as a direct sensitivity measure on the weakly efficient solution map of the perturbed problems. Moreover we show that the local and the global definitions of condition numbers introduced here allow us to build a consistent framework. Indeed, the global condition number for a given multiobjective optimization problem is finite if and only if the local one is finite at every weakly efficient solution of the same problem.

In both cases a distance theorem is proved. A pseudodistance between functions is defined such that the condition number provides an upper bound on how far from a well-conditioned function \( f \) a perturbed function \( g \) can be chosen in order that \( g \) is well-conditioned too.

The two notions of condition number considered in the present work, either following the pointwise approach or the global one, coincide with the condition number \( c_2 \) proposed in [17] in the special scalar optimization problem discussed there. They can both be considered as tools to extend the classical Eckart–Young distance theorem to the setting of multiobjective optimization, even if only a special class of perturbations is considered.

It is interesting to remark that in both approaches strong convexity plays the role of a sufficient condition to obtain well-conditioning.

The paper is organized in five sections. Section 2 contains some notations and preliminaries. In Sections 3 and 4 our main results are stated and proved. Indeed, in Section 3 we introduce a condition number for a multiobjective optimization problem, following a local point of view, whereas Section 4 is devoted to the study of a notion of condition number adopting a global point of view. Moreover, in both these sections, we prove an Eckart–Young type theorem related, respectively, to the local and the global condition number.

2 Notations and preliminaries

In order to study the behaviour of the set-valued solution maps of a multiobjective optimization problem, we need to consider some classical notions of continuity and Lipschitz continuity for set-valued maps. For a detailed exposition see, e.g., [2], or [10].

A set-valued map \( T : A \subset \mathbb{R}^n \rightrightarrows \mathbb{R}^m \) is upper semicontinuous at \( x \in A \) if for every open set \( \mathcal{V}(T(x)) \) such that \( T(x) \subseteq \mathcal{V}(T(x)) \) there exists a neighborhood \( \mathcal{U}(x) \) of \( x \) such that for every \( x' \in \mathcal{U}(x) \), \( T(x') \subseteq \mathcal{V}(T(x)) \).

If \( T : A \subset \mathbb{R}^n \rightrightarrows \mathbb{R}^m \) is a closed-valued map with compact values at \( x \in A \), then \( T \) is upper semicontinuous at \( x \) if and only if its graph is closed at \( x \), i.e., for each sequence \( \{x_n\} \subset A, x_n \to x \), and each \( y_n \in T(x_n) \), \( y_n \to y \), then \( y \in T(x) \).

Lipschitzian properties of maps play a crucial role in many aspects of variational analysis. We consider here some well-known generalizations of the notion
of Lipschitz continuity to set–valued maps that will allow us to build a meaningful notion of condition number for multiobjective optimization.

First let us recall the notion of Hausdorff distance between two sets. Let $A, B \subseteq \mathbb{R}^n$. The Hausdorff distance $d_H$ between $A$ and $B$ is defined by

$$d_H(A, B) = \max \{ e(A, B), e(B, A) \},$$

where $e(A, B) = \sup_{a \in A} d(a, B) = \sup_{a \in A} \inf_{b \in B} \| a - b \|$ is the excess functional; in particular, $e(\emptyset, A) = 0$, $e(\emptyset, \emptyset) = 0$ and $e(A, \emptyset) = +\infty$.

In the sequel, we will denote by $B(x, r)$ the closed ball in $\mathbb{R}^n$ centered at $x$ and with radius $r$.

**Definition 2.1.** A set–valued map $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ is Lipschitz continuous relative to a nonempty set $A \subseteq \mathbb{R}^n$ if $A \subseteq \text{dom}(T)$, $T$ is closed–valued on $A$, and there exists a Lipschitz constant $k \geq 0$ such that

$$T(x') \subset T(x) + k\| x' - x \| B(0, 1), \quad \forall x, x' \in A. \quad \text{(1)}$$

The infimum of the Lipschitz constants on $A$ is called the Lipschitz modulus of $T$ on $A$ and is denoted by $\text{lip}(T; A)$.

If $k < 1$, the set–valued map $T$ is said to be a contraction of constant $k$.

An equivalent reformulation of condition (1) can be given in terms of the Hausdorff distance as follows:

$$d_H(T(x'), T(x)) \leq k\| x' - x \| \quad \forall x, x' \in A.$$

A local version of Lipschitz continuity of a map is the so called Aubin property.

**Definition 2.2.** A set–valued map $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ is said to have the Aubin property at $\bar{x}$ for $\bar{y} \in T(\bar{x})$ if there exist $k \geq 0$, a neighborhood $U(\bar{x})$ of $\bar{x}$ and a neighborhood $V(\bar{y})$ of $\bar{y}$ such that

$$T(x') \cap V(\bar{y}) \subset T(x) + k\| x - x' \| B(0, 1), \quad \forall x, x' \in U(\bar{x}). \quad \text{(2)}$$

The infimum of $k$ such that (2) holds is called the Lipschitz modulus of $T$ at $\bar{x}$ for $\bar{y}$ and is denoted by $\text{lip}(T; \bar{x}|\bar{y})$.

When $T$ is single–valued on $A$, then the Lipschitz modulus of $T$ on $A$ corresponds to the usual definition:

$$\text{lip}(T; A) = \sup_{x, x' \in A} \frac{\| T(x) - T(x') \|}{\| x - x' \|},$$

while the Lipschitz modulus of $T$ at $\bar{x}$ for $T(\bar{x})$ equals $\text{lip}(T; \bar{x})$, i.e., the usual Lipschitz modulus at $\bar{x}$:

$$\text{lip}(T; \bar{x}) = \lim_{x, x' \to \bar{x}, x \neq x'} \sup \frac{\| T(x) - T(x') \|}{\| x - x' \|}.$$
**Definition 2.3.** Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with $\text{dom}(T) \neq \emptyset$ and let $(\bar{x}, \bar{y}) \in \text{gph} T$. The set-valued map $T$ is said to be *metrically regular* at $\bar{x}$ for $\bar{y} \in T(\bar{x})$ with modulus $\mu \geq 0$ if there exist a neighborhood $\mathcal{U}(\bar{x})$ of $\bar{x}$ and a neighborhood $\mathcal{V}(\bar{y})$ of $\bar{y}$ such that

$$d(x, T^{-1}(y)) \leq \mu d(y, T(x))$$

for all $x \in \mathcal{U}(\bar{x})$ and for all $y \in \mathcal{V}(\bar{y})$. The infimum of $\mu$ such that (3) holds for some $\mathcal{U}(\bar{x})$ and $\mathcal{V}(\bar{y})$ is called the *regularity modulus* for $T$ at $\bar{x}$ for $\bar{y} \in T(\bar{x})$, and is denoted by $\text{reg}(T; \bar{x}|\bar{y})$.

A fundamental characterization of the metric regularity of a given map is the following (see, e.g., Theorem 3E.6 in [2]):

**Proposition 2.4.** Let $(\bar{x}, \bar{y}) \in \text{gph} T$, where $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$. Then $T$ satisfies the Aubin property at $\bar{x}$ for $\bar{y}$ if and only if its inverse $T^{-1} : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is metrically regular at $\bar{y}$ for $\bar{x}$ with the same modulus, i.e., $\text{lip}(T; \bar{x}|\bar{y}) = \text{reg}(T^{-1}; \bar{y}|\bar{x})$.

In order to measure how far a metrically regular map $T$ can be perturbed by a linear operator without losing metric regularity, in [3] Dontchev, Lewis and Rockafellar introduced the notion of radius of metric regularity.

**Definition 2.5.** Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with $\text{dom}(T) \neq \emptyset$ and let $(\bar{x}, \bar{y}) \in \text{gph} T$. The *radius of metric regularity* of $T$ at $\bar{x}$ for $\bar{y} \in T(\bar{x})$ is the value

$$\text{rad}(T; \bar{x}|\bar{y}) = \inf_{G \in \mathcal{L}([\mathbb{R}^n, \mathbb{R}^m])} \{ \|G\| : T + G \text{ is not metrically regular at } \bar{x} \text{ for } (\bar{y} + G(\bar{x})) \}.$$  

In the Euclidean setting the radius $\text{rad}(T; \bar{x}|\bar{y})$ defined above turns out to be meaningful also for a larger class of perturbation maps, namely

$$\text{rad}(T; \bar{x}|\bar{y}) = \min_{G \in \mathcal{G}} \{ \text{lip}(G; \bar{x}) : T + G \text{ is not metrically regular at } \bar{x} \text{ for } (\bar{y} + G(\bar{x})) \}$$

where $\mathcal{G}$ is the collection of all the functions $G : \mathbb{R}^n \to \mathbb{R}^m$ that are Lipschitz continuous at $\bar{x}$.

The radius of metric regularity can be characterized as the reciprocal of the regularity modulus. We recall that the local closedness of a set at a point $x$ means that some ball $B(x, r)$ has closed intersection with the set.

**Proposition 2.6.** (see Theorem 1.5 in [3]). Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with $\text{dom}(T) \neq \emptyset$ and let $(\bar{x}, \bar{y}) \in \text{gph} T$ be a point such that $\text{gph} T$ is locally closed at $(\bar{x}, \bar{y})$. Then

$$\text{rad}(T; \bar{x}|\bar{y}) = \frac{1}{\text{reg}(T; \bar{x}|\bar{y})}.$$  

### 2.1 Parametric multiobjective problems

Now we introduce a parametric multiobjective optimization problem that will be considered in the sequel in order to develop a suitable notion of condition number. In the present work we restrict to the special case of componentwise uniform tilt perturbations, in the sense that we perturb all the components of
the objective function with the same linear term. Although some of the results obtained could be stated in a more general setting, the main results require some regularity of the data. This explain why we will confine ourselves to a more restrictive framework.

We denote by $C^{1,1}(B(0, r))$ the set of all vector–valued functions $f : B(0, r) \subset \mathbb{R}^n \to \mathbb{R}^m$ such that $f_i$, $i = 1, 2, \ldots, m$, is differentiable at each interior point of $B(0, r)$, and whose gradient $\nabla f_i$ can be extended to the closed ball $B(0, r)$ in such a way that it is Lipschitz continuous on the whole set $B(0, r)$. Let $\mathcal{U}(0)$ be a neighborhood of the origin in $\mathbb{R}^n$, and let $p \in \mathcal{U}(0)$. We denote by $f^p$ the perturbation of $f$ defined by

$$f^p(x) = f(x) - [p] \cdot x,$$

where $[p]$ denotes a matrix with $m$ rows and $n$ columns such that all the rows are equal to $p$.

Given a function $f \in C^{1,1}(B(0, r))$ we consider the following parametric multiobjective optimization problem

$$\min_{x \in B(0, r)} f^p(x),$$

where the ordering cone in $\mathbb{R}^m$ is given by the nonnegative orthant

$$\mathbb{R}^m_+ = \{x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : x_i \geq 0, i = 1, 2, \ldots, m\}.$$

We will denote by $WE_f$ the set–valued map $WE_f : \mathcal{U}(0) \subset \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ of the weakly efficient solutions of problem (VO$_p$), i.e., $WE_f(p)$ is the set of points $x \in B(0, r)$ such that

$$(f^p(x) - \text{int}(\mathbb{R}_+^m)) \cap f^p(B(0, r)) = \emptyset.$$

Since the set $f^p(B(0, r))$ is compact, then the set $WE_f(p) \subseteq B(0, r)$ is nonempty and closed, for every $p \in \mathcal{U}(0)$ (see, for instance, Theorem 6.5 in [4]). Furthermore, the set–valued map $WE_f$ turns out to be upper semicontinuous at 0 (see Corollary 4.6, Ch. 4 in [8]).

First order optimality conditions play a key role in the development of a condition number theory. Indeed, in the present work the sensitivity of the solution map $WE_f$ will be investigated by means of the effect that a perturbation of the objective function produces on the inclusion that is equivalent to the first order optimality conditions.

To any function $f \in C^{1,1}(B(0, r))$, we associate the set–valued map $H_f : B(0, r) \rightrightarrows \mathbb{R}^n$ defined as

$$x \mapsto H_f(x) = \left\{ \sum_{i=1}^m \lambda_i \nabla f_i(x), \quad \sum_{i=1}^m \lambda_i = 1, \lambda_i \geq 0 \right\}.$$

It is worthwhile noticing that this map has nonempty, compact and convex values; moreover, it can be easily proved that it has a closed graph. Set $s_f(x) = d(0, H_f(x))$. 
In the next proposition the Lipschitz continuity of the maps $H_f$ and $s_f$ is proved (for more details on $H_f(x)$ and $s_f(x)$ we refer to [9]).

**Proposition 2.7.** Let $f \in \mathcal{C}^{1,1}(B(0,r))$. Then, the set-valued map $H_f$ and the function $s_f$ are Lipschitz continuous on $B(0,r)$ with the same Lipschitz constant given by $K = \max_{i=1,\ldots,m} \text{lip(} \nabla f_i; B(0,r))$. 

*Proof.* Let $x, y \in B(0,r)$; then
\[
\nabla f_i(x) - \nabla f_i(y) \in \text{lip}(\nabla f_i; B(0,r)) \quad B(0,\|x-y\|) \subseteq K \quad B(0,\|x-y\|).
\]

By considering the convex hull of the first term, we obtain
\[
\sum_{i=1}^{n} \lambda_i(\nabla f_i(x) - \nabla f_i(y)) \in K \quad B(0,\|x-y\|),
\]
whenever $\sum_{i=1}^{m} \lambda_i = 1$, $\lambda_i \geq 0$. This implies that
\[
H_f(x) \subseteq H_f(y) + K \quad B(0,\|x-y\|),
\]
i.e., $H_f$ is Lipschitz continuous on $B(0,r)$ with Lipschitz constant $K$. This yields that $s_f : B(0,r) \to \mathbb{R}$ is Lipschitz continuous on $B(0,r)$ with the same Lipschitz constant $K$ (see [18], p. 368). 

The above–mentioned map $H_f$ is an essential tool to deal with useful first order conditions. As a matter of fact it is well–known that any interior solution $\bar{x} \in WE_f(p) \cap \text{int}(B(0,r))$ of problem $(\text{VO}_p)$ satisfies the inclusion $0 \in H_{f_p}(\bar{x})$ or, equivalently, $s_{f_p}(\bar{x}) = 0$. It is easy to see that
\[
H_{f_p}(x) = H_f(x) - \{p\},
\]
hence $0 \in H_{f_p}(x)$ if and only if $p \in H_f(x)$. Therefore, the following inclusion holds:
\[
WE_f(p) \cap \text{int}(B(0,r)) \subseteq H_1^{-1}(p).
\]

If $f$ is an $\mathbb{R}_+^n$–convex function on $B(0,r)$, i.e., all its components $f_i$, $i = 1, 2, \ldots, m$, are convex functions on $B(0,r)$, and $\bar{\tau} \in \text{int}(B(0,r))$, then $s_{f_p}(\bar{\tau}) = 0$, and hence $0 \in H_{f_p}$ if and only if $\bar{\tau} \in WE_f(p)$. Therefore $\tau \in WE_f(p) \cap \text{int}(B(0,L))$ if and only if $\tau \in H_1^{-1}(p)$. Moreover, if $WE_f(p) \subset \text{int}(B(0,r))$, the following characterization of the weakly efficient solution map holds:
\[
WE_f(p) = H_1^{-1}(p).
\]

**2.2 Condition number for scalar optimization problems**

For the convenience of the reader we repeat some relevant material from [17] without proofs, thus making our exposition self–contained. Let $E$ be a real Banach space and $E^*$ its dual, and let $\langle \cdot, \cdot \rangle$ denote the duality pairing. $\mathcal{C}^{1,1}(B(0,L))$ denotes the class of Fréchet differentiable functions on the closed ball $B(0,L)$.
such that the Fréchet derivative $Df$ is a Lipschitz function on $B(0, L)$. Consider the problem

$$\min_{B(0,L)} f_p = \min_{B(0,L)} (f - (p, \cdot)).$$

Under the assumption that the solution is unique, for every small $p$, we can set

$$m : U(0) \subset E^* \to E, \quad m(p) = \arg\min(B(0, L), f_p).$$

The condition number $\text{cond}(f)$ is the extended real number defined as follows:

$$\text{cond}(f) = \limsup_{p,q \to 0, p \neq q} \frac{\|m(p) - m(q)\|}{\|p - q\|} \quad (5)$$

If $m(0) = \bar{x}$, by Proposition 2.4 the condition number $\text{cond}(f)$ agrees with the regularity modulus of the set–valued map $m^{-1}$ at $(\bar{x}, 0)$, i.e.,

$$\text{cond}(f) = \text{reg}(m^{-1}; \bar{x}|0). \quad (6)$$

The class $C^{1,1}(B(0, r))$ can be endowed with the pseudodistance

$$d_Z(f_1, f_2) = \sup \left\{ \frac{\|Df_1(x) - Df_2(x) - Df_1(x') + Df_2(x')\|}{\|x - x'\|} \right\} \quad (7)$$

where $x, x' \in B(0, L), x \neq x'$.

Denote by $T_1$ the class of functions $f \in C^{1,1}(B(0, L))$ such that

- $\arg\min(B(0, L), f_p) \neq \emptyset$ for small $p$;
- $\arg\min(B(0, L), f) = \{0\}$;
- $p \mapsto \arg\min(B(0, L), f_p)$ is upper semicontinuous at $p = 0$.

The next class $W_1$ can be thought of as the class of “good” functions, giving rise to well–conditioned problems: $f \in W_1$ if

- $\arg\min(B(0, L), f_p)$ is a singleton for $p$ small;
- $\text{cond}(f) < +\infty$.

The ill–conditioned functions $I_1 = \{g \in T_1 : g \notin W_1\}$ satisfy the following result:

**Theorem 2.8.** *(see [17], Theorem 3.1).* Let $f \in T_1 \cap W_1$ with $Df$ one–to–one near 0. Then

$$d_Z(f, I_1) \geq \frac{1}{\text{cond}(f)}.$$  

8
3 Condition number: a pointwise definition

The definition of condition number in the scalar case, and its equivalent formulation give rise to different approaches in the setting of multiobjective optimization. This section is devoted to the analysis of the pointwise conditioning, that will extend the scalar approach summarized in the formula for condition number given in . We will focus on a fixed efficient solution of the multiobjective optimization problem (VO0).

Definition 3.1. Let \( \bar{x} \in W E_f(0) \). The condition number of \( f \) at \( \bar{x} \) is the extended real number

\[
c(\bar{x}, f) = \text{reg}(W E_f^{-1}; \bar{x}|0).
\]

By Proposition 2.4 the regularity modulus of a set–valued map coincides with the Lipschitz modulus of its inverse. Hence we deduce that

\[
c(\bar{x}, f) = \text{lip}(W E_f; 0|\bar{x}).
\]

In force of Proposition 2.6 we can use the condition number \( c(\bar{x}, f) \) to characterize the radius of metric regularity of the map \( W E_f^{-1} \) at \( \bar{x} \) for 0 as follows:

\[
\text{rad}(W E_f^{-1}; \bar{x}|0) = \frac{1}{c(\bar{x}, f)}.
\]

This result, which holds without any assumption of smoothness on \( f \), allows us to establish a first version of distance theorem: the condition number bounds the linear continuous perturbations that can be applied to the inverse of the solution map \( W E_f \) without losing metric regularity. As already remarked in [17], the distance to ill–conditioning defined through \( \text{rad}(W E_f^{-1}; \bar{x}|0) \) cannot be considered as a variational notion since the addition of a linear perturbation to \( W E_f^{-1} \) does not correspond to an additive perturbation of the original objective function.

In order to define well–conditioned problems and to use the pointwise condition number \( c(\bar{x}, f) \) to provide a lower bound of the distance from ill–conditioning of a given well–conditioned multiobjective optimization problem, it is necessary to put some restrictions on \( f \). Indeed, one of the main difficulties to face when dealing with vector optimization is to provide conditions that fully characterize the weakly efficient solutions of the optimization problem. In the scalar case, Zolezzi considered essentially functions with a unique global minimum point at 0, and such that \( Df \) is one–to–one near 0, thereby ensuring that the zeroes of the gradient completely identifies the solutions of the problem. In this framework, the necessary first order conditions for extrema play a crucial role. Clearly, strictly convex functions with a minimum point at 0 would be suitable.

In the vector–valued case we will restrict our analysis to the class of \( \mathbb{R}^m_+ \)–convex functions; the advantage of using functions of this class lies in the fact that the first order conditions completely characterize the weakly efficient solutions. In addition, throughout the sequel of the paper we will assume that

\[
W E_f(0) \subset \text{int}(B(0, r)).
\]
We underline that, since the set–valued map $WE_f$ is upper semicontinuous, the inclusion above implies that there exists a positive real number $\delta_f$ such that

$$WE_f(p) \subset \text{int}(B(0, r)) \quad \text{for every } p \in \mathbb{R}^n, \|p\| < \delta_f.$$  \hfill (8)

Let us given the following

**Definition 3.2.** Let $\bar{x}$ be a point in $WE_f(0)$. A function $f \in C^1(B(0, r))$ belongs to the class $T_1(\bar{x})$ if the following conditions hold:

- $f$ is $\mathbb{R}^m_+$–convex;
- $c(\bar{x}, f) > 0$.

A condition that entails the positivity of the condition number introduced in Definition 3.1 can be established. The following proposition extends a result proved in Lemma 3.2 in [17].

**Proposition 3.3.** Let $\bar{x} \in WE_f(0)$. If there exist $\kappa > 0$ and $\{p_s\} \subset \mathbb{R}^n \setminus \{0\}$, $p_s \to 0$, such that

$$\|p_s\| \leq \kappa d(\bar{x}, WE_f(p_s)),$$

then $c(\bar{x}, f) > 0$.

**Proof.** From the definition of $c(\bar{x}, f)$, for any $\epsilon > 0$ there exist $U(\bar{x})$ and $V(0)$ such that

$$d(x, WE_f(p)) \leq (c(\bar{x}, f) + \epsilon) d(p, WE_f^{-1}(x))$$

for every $x \in U(\bar{x})$ and $p \in V(0)$. Now let us choose $x = \bar{x}$ and $p = p_s$; we obtain, for $s$ big enough,

$$d(\bar{x}, WE_f(p_s)) \leq (c(\bar{x}, f) + \epsilon) \|p_s\| \leq (c(\bar{x}, f) + \epsilon) \kappa d(\bar{x}, WE_f(p_s)).$$

From (9), $d(\bar{x}, WE_f(p_s)) \neq 0$ and the inequality above implies

$$(c(\bar{x}, f) + \epsilon) \kappa \geq 1.$$ 

Since this holds for any $\epsilon > 0$, we conclude that $c(\bar{x}, f) > 0$. \hfill \qed

**Remark 3.4.** Condition (9) weakens the assumption of Lipschitz continuity of the gradient $Df$ at $\bar{x} = 0$ required by Zolezzi in [17]. Indeed, in the scalar case, $Df(0) = 0$ since $x = 0$ is a global minimizer of $f$ in $\text{int}(B(0, r))$, and, thanks to the assumption of upper semicontinuity of the map $m = WE_f$, $WE_f(p)$ is also an internal minimizer of $f_p$ for $p$ small enough, implying that $Df(WE_f(p)) = p$.

Thus the Lipschitz assumption on the gradient gives:

$$\|p\| = \|Df(WE_f(p)) - Df(WE_f(0))\| \leq \kappa \|WE_f(p)\| = \kappa d(0, WE_f(p)).$$

Now we introduce a class of functions that will give rise to well–conditioned problems at $\bar{x}$. 

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**Definition 3.5.** If \( \bar{x} \in WE_f(0) \), a function \( f \in C^{1,1}(B(0, r)) \) is said to belong to the class \( W_1(\bar{x}) \) if \( c(\bar{x}, f) < +\infty \).

A stronger convexity assumption on the objective function will entail a finite condition number. We recall that a function \( f : B(0, r) \subset \mathbb{R}^n \to \mathbb{R} \) is strongly convex if there exists \( \alpha > 0 \) such that
\[
f((1-t)x+tx') \leq (1-t)f(x)+tf(x')-\alpha t\|x-x'\|^2, \quad \forall x, x' \in B(0, r), \quad \forall t \in [0, 1]
\] (see, e.g., \[15\]). If \( f \) is differentiable, strong convexity is equivalent to the strong monotonicity of \( \nabla f \) on \( B(0, r) \), i.e.,
\[
< \nabla f(x) - \nabla f(x'), x - x' > \geq 2\alpha \|x - x'\|^2, \quad \forall x, x' \in B(0, r).
\]

The following proposition holds:

**Proposition 3.6.** Let \( f \in C^{1,1}(B(0, r)) \). If \( f_i \) is strongly convex, for every \( i = 1, 2, \ldots, m \), then \( c(\bar{x}, f) < +\infty \), for every \( \bar{x} \in WE_f(0) \).

**Proof.** From the assumptions, the function \( \nabla f_i \) is strongly monotone for every \( i \), i.e., there exists \( k_i > 0 \) such that
\[
< \nabla f_i(x') - \nabla f_i(x), x' - x > \geq k_i \|x' - x\|^2, \quad \forall x', x \in B(0, r), \quad i = 1, 2, \ldots, m.
\]

It follows easily that the assumptions of Theorem 5.2 in [5] are fulfilled if we take \( F_i(x, p) = \nabla f_i(x) - p \), thereby implying that the map \( p \mapsto WE_f(p) \) is Lipschitz on \( B(0, r) \). \( \square \)

It is interesting to notice that the convexity assumptions in the above proposition cannot be weakened. Indeed, even the strict convexity of the functions \( f_i \) is not enough to ensure the finiteness of \( c(\bar{x}, f) \) at a point \( \bar{x} \in WE_f(0) \).

**Example 3.7.** Let us consider the strictly convex function \( f : [-1, 1] \to \mathbb{R}^2 \) defined by \( f(x) = (x^2, x^4) \). The unperturbed problem has a unique weakly efficient solution \( \bar{x} = 0 \). Let us consider the perturbed problem \((\text{VO}_p)\), where \( p \in \mathbb{R} \) and the objective function is \( f^p(x) = (x^2 - px, x^4 - px) \). The weakly efficient solution set \( WE_f(p) \) is the closed segment \( I(p) = \{x \in [-1, 1] : p/2 \leq x \leq \sqrt{p/4}\} \).

Suppose, by contradiction, that \( c(0, f) = \text{lip}(WE_f; 0) < +\infty \); this means that there exists \( k > 0 \), \( U(0) \) and \( \mathcal{V}(0) \) such that
\[
I(p) \cap \mathcal{V}(0) \subset I(q) + k|p|B(0, 1), \quad \forall p, q \in U(0).
\]

Let \( q = 0 \); then
\[
I(p) \cap \mathcal{V}(0) \subset k|p|B(0, 1), \quad \forall p \in U(0),
\]
or, equivalently, for \( |p| \) small enough,
\[
|\sqrt{p/4}| \leq k|p|, \quad \forall p \in U(0),
\]
which is false.
In the development of a condition number theory for multiobjective optimization a fundamental issue is the possibility to use the condition number to establish a lower bound on the distance to ill-conditioning. How far can we move from a well-conditioned vector function \( f \in W_1(\bar{x}) \cap T_1(\bar{x}) \) to a function \( g \) that is “close” according to a suitable distance, without leaving the class \( W_1(\bar{x}) \)?

In order to tackle this problem, we introduce a pseudodistance \( d^* \) in the space of vector-valued functions \( C^{1,1}(B(0, r)) \) as follows:

\[
d^*(f, g) = \max_{\lambda_i \geq 0, \sum_i \lambda_i = 1} d_Z \left( \sum_i \lambda_i f_i, \sum_i \lambda_i g_i \right),
\]

where \( d_Z \) denotes the pseudodistance defined in (7) for the scalar functions.

In the sequel, we will consider the particular class \( P_f \) of functions obtained by perturbing each component of \( f \) by the same real-valued function \( h : B(0, r) \to \mathbb{R}, h \in C^{1,1}(B(0, r)), \) i.e.,

\[
P_f = \{ g = f + h e, h : B(0, r) \to \mathbb{R}, h \in C^{1,1}(B(0, r)), e = (1, 1, \ldots, 1) \}.
\]

If \( g \in P_f \), the distance \( d^*(f, g) \) turns out to be the following:

\[
d^*(f, g) = \text{lip}(\nabla h; B(0, r)).
\]

The next proposition allows us to establish an upper bound on the regularity modulus of the perturbed function \( g \).

**Proposition 3.8.** Let \( f \in T_1(\bar{x}) \cap W_1(\bar{x}) \), where \( \bar{x} \in W E_f(0) \), and let \( g \) be a function in \( P_f \), \( g = f + h e \) with \( h \) such that \( \text{lip}(\nabla h; \bar{x}) \cdot c(\bar{x}, f) < 1 \). Then

\[
\text{reg}(H_g; \bar{x}|\nabla h(\bar{x})) \leq \frac{c(\bar{x}, f)}{1 - c(\bar{x}, f) \cdot \text{lip}(\nabla h; \bar{x})}.
\]

**Proof.** Let us consider the set-valued map \( H_f : B(0, r) \ni \mathbb{R}^n \) and the function \( \nabla h : B(0, r) \to \mathbb{R}^n \). Since \( \bar{x} \in W E_f(0) \), we have that \( (\bar{x}, 0) \in \text{gph}(H_f) \), where \( \text{gph}(H_f) \) is closed. In addition, by the \( \mathbb{R}^n \)-convexity of \( f \), it holds

\[
c(\bar{x}, f) = \text{reg}(WE_f^{-1}; \bar{x}|0) = \text{reg}(H_f; \bar{x}|0).
\]

The equality \( H_g(x) = H_f(x) + \nabla h(x) \) holds for every \( x \in B(0, r) \). Therefore, by Theorem 3F.1 in [2], the assertion is proved.

Now we can reformulate the former result as a distance theorem for the proposed notion of pointwise condition number.

**Theorem 3.9.** Let \( f \in T_1(\bar{x}) \cap W_1(\bar{x}) \), where \( \bar{x} \in W E_f(0) \), and \( g \in P_f \cap T_1(\bar{x}) \) be such that

\[
d^*(g, f) < \frac{1}{c(\bar{x}, f)}.
\]

If \( \nabla h(\bar{x}) = 0 \), then \( g \in W_1(\bar{x}) \).
Proof. We see at once that \( \text{lip}(\nabla h; x) \leq \text{lip}(\nabla h; B(0, r)) \), for every \( x \in B(0, r) \). Therefore, by Proposition 4.8 we get
\[
\text{reg}(H_g; \bar{x}|0) \leq \frac{c(\bar{x}, f)}{1 - c(\bar{x}, f) \cdot \text{lip}(\nabla h; \bar{x})} \leq \frac{c(\bar{x}, f)}{1 - c(\bar{x}, f) \cdot \text{lip}(\nabla h; B(0, r))}.
\]
Since, by assumptions, \( g \in T_1(\bar{x}) \), the condition \( 0 \in H_g(\bar{x}) \) is equivalent to \( \bar{x} \in W E_g(0) \); therefore, \( \text{reg}(H_g; \bar{x}|0) = c(\bar{x}, g) \). From (12), the conclusion follows. \( \square \)

4 Condition number: a global definition

In the present section we introduce a global definition of condition number that extends the notion given in 5 for the scalar problem. It turns out to be a measure of the sensitivity of the whole weakly efficient solution set with respect to the tilt perturbations on the objective function.

**Definition 4.1.** Let \( f \) be a function in \( C^{1,1}(B(0, r)) \), and \( WE_f(p) \) be the set of weakly efficient solutions of problem \((\text{VO} p)\). The condition number \( c^*(f) \) is defined as follows:
\[
c^*(f) = \limsup_{p, q \to 0, p \neq q} \frac{d_H(W E_f(p), W E_f(q))}{\|p - q\|}. \tag{13}
\]
The global notion of condition number \( c^*(f) \) is consistent with the pointwise approach to conditioning introduced in the former section through \( c(\bar{x}, f) \). Indeed, the next proposition holds:

**Proposition 4.2.** Let \( f \in C^{1,1}(B(0, r)) \). Then, \( c(\bar{x}, f) < +\infty \) for every \( \bar{x} \in W E_f(0) \) if and only if \( c^*(f) < +\infty \).

Proof. From the definition of \( c(\bar{x}, f) \), for every \( \bar{x} \in W E_f(0) \) there exists \( k_{\bar{x}} > 0 \), \( \mathcal{V}(\bar{x}) \) and \( U_{\bar{x}}(0) \) such that
\[
WE_f(p) \cap \mathcal{V}(\bar{x}) \subset W E_f(q) + k_{\bar{x}}\|p - q\|B(0, r), \tag{14}
\]
for every \( p, q \in U_{\bar{x}}(0) \). The family of sets \( \{\mathcal{V}(\bar{x})\}_{\bar{x} \in W E_f(0)} \) is an open covering of the compact set \( W E_f(0) \), therefore there exist \( \{\bar{x}_i\}_{i=1}^k \) such that \( W E_f(0) \subset \bigcup_{i=1}^k \mathcal{V}(\bar{x}_i) \). Set \( U(0) = \bigcap_{i=1}^k U_{\bar{x}_i}(0) \), and \( \bar{k} = \max_i k_{\bar{x}_i} \). We have that, for every \( i = 1, 2, \ldots, k \), and for every \( p, q \in U(0) \),
\[
WE_f(p) \cap \mathcal{V}(\bar{x}_i) \subset W E_f(q) + k_{\bar{x}_i}\|p - q\|B(0, r) \subset W E_f(q) + \bar{k}\|p - q\|B(0, r).
\]
Taking the union of the l.h.s. for \( i = 1, 2, \ldots, k \), we get
\[
WE_f(p) \cap (\bigcup_{i=1}^k \mathcal{V}(\bar{x}_i)) \subset W E_f(q) + \bar{k}\|p - q\|B(0, r).
\]
Since the set \( \bigcup_{i=1}^k \mathcal{V}(\bar{x}_i) \) is a neighborhood of \( W E_f(0) \) and the map \( p \mapsto W E_f(p) \) is upper semicontinuous, if \( p \) is small enough we obtain that
\[
WE_f(p) \cap (\bigcup_{i=1}^k \mathcal{V}(\bar{x}_i)) = W E_f(p),
\]

thereby showing that, for every \( p, q \in U(0) \),

\[
WE_f(p) \subset WE_f(q) + \bar{k}\|p - q\|B(0, r),
\]
i.e., \( d_H(WE_f(p), WE_f(q)) \leq \bar{k}\|p - q\| \). The converse is trivial.

In case of an \( \mathbb{R}^m_+ \)-convex function \( f \), the set of weakly efficient solutions \( WE_f(p) \) can be recovered by the inverse image of \( p \) via \( H_f \). Consequently, the global condition number can be defined by

\[
c^*(f) = \limsup_{p, q \to 0, p \neq q} \frac{d_H(H_f^{-1}(p), H_f^{-1}(q))}{\|p - q\|}. \tag{15}
\]

The next proposition shows that, for \( \mathbb{R}^m_+ \)-convex functions, the condition number is strictly positive. A similar result holds in the scalar case (see Lemma 3.2 in [17]).

**Proposition 4.3.** Let \( f \) be an \( \mathbb{R}^m_+ \)-convex function in \( C^{1,1}(B(0, r)) \). Then \( c^*(f) > 0 \).

**Proof.** By contradiction, let us suppose that \( c^*(f) = 0 \). Now we have that

\[
0 = \limsup_{p, q \to 0, p \neq q} \frac{d_H(WE_f(p), WE_f(q))}{\|p - q\|} \\
\geq \limsup_{p \to 0} \frac{d_H(WE_f(p), WE_f(0))}{\|p\|} \\
\geq \limsup_{p \to 0} \frac{e(WE_f(p), WE_f(0))}{\|p\|};
\]

hence, for every choice of \( x_p \in WE_f(p) \), we have

\[
\limsup_{p \to 0} \frac{d(x_p, WE_f(0))}{\|p\|} = 0 = \lim_{p \to 0} \frac{d(x_p, WE_f(0))}{\|p\|}. \tag{16}
\]

Since the set \( WE_f(0) \) is closed and contained in the open set \( \text{int}B(0, r) \), we can always choose a sequence \( \{x_s\} \subset \text{int}B(0, r) \setminus WE_f(0) \) and a point \( x_0 \in WE_f(0) \) such that \( x_s \to x_0 \). Let us consider the sequence \( \{v_f(x_s)\} \subset \mathbb{R}^n \), where \(-v_f(x_s)\) is the minimal norm element of \( H_f(x_s) \). We remark that \( v_f(x_s) \neq 0 \) by the convexity of \( f \). By the continuity of \( v_f \) (see, for instance, [1]), we have that \( v_f(x_s) \to v_f(x_0) = 0 \).

For every \( s \in \mathbb{N} \setminus \{0\} \), set

\[
p_s = -v_f(x_s),
\]

and consider the function \( f_{p_s} : B(0, r) \to \mathbb{R}^m \) defined by

\[
f_{p_s}(x) = f(x) - [p_s] \cdot x.
\]
Hence the minimal norm element of the set

\[ H_{f_p}(x_s) = \left\{ \sum_{i=1}^{m} \lambda_i \nabla f_i(x_s) + v_f(x_s) : \sum_{i=1}^{m} \lambda_i = 1, \lambda_i \geq 0, i = 1, \ldots, m \right\} \]

is 0. Then, by the convexity assumption, we have that \( x_s \in WE_f(p_s) \). Choose a point \( x_s^0 \in WE_f(0) \) such that

\[ d(x_s, WE_f(0)) = \| x_s - x^0_s \|. \]

Hence, by (16), taking \( p = p_s = -v_f(x_s) \), we get

\[ \lim_{p_s \to 0} \frac{d(x_s, WE_f(0))}{\|p_s\|} = \lim_{s \to +\infty} \frac{\|x_s - x^0_s\|}{\|v_f(x_s)\|} = 0. \quad (17) \]

From (17) and taking into account that \( s_f(x) = \|v_f(x)\| \), we have

\[ \frac{s_f(x_s)}{\|x_s - x^0_s\|} = \frac{\|v_f(x_s)\|}{\|x_s - x^0_s\|} \to +\infty \text{ as } s \to +\infty. \quad (18) \]

Remark 4.4. The proposition above holds under weaker assumptions on \( f \). Indeed, the proof requires only that \( f \) is \( \mathbb{R}^m_+ \)-convex, in the class \( C^1 \), and \( s_f \) is Lipschitz continuous.

In the sequel of the section we would like to extend to the global case the distance theorem already considered in Corollary 3.9. Let us first introduce the class of functions giving rise to the well-conditioned problems:

**Definition 4.5.** Let \( f \) be an \( \mathbb{R}^m_+ \)-convex function in \( C^1(B(0, r)) \). Then \( f \) is said to belong to the class \( W^*_+ \) if \( c^*(f) < +\infty \).

Likewise the pointwise approach, strong convexity of the objective functions entails the finiteness of \( c^*(f) \). Indeed the following proposition easily follows by Theorem 5.2 in [5].

**Proposition 4.6.** Let \( f \in C^{1,1}(B(0, r)) \). If \( f_i \) is strongly convex, for every \( i = 1, 2, \ldots, m \), then \( c^*(f) < +\infty \).

Let us consider a function \( f \in W^*_+ \), and let \( g \) be a function sufficiently "close" to \( f \), according to the distance \( d^* \) defined in (10). We will prove that under appropriate assumptions, the perturbed function \( g \) will give rise to a well-conditioned problem. Moreover, the condition number of the perturbed function will be bounded from above by the reciprocal of the distance between \( f \) and \( g \).

As in the previous section we will consider the class \( P_f \) of functions defined in (11) obtained by perturbing each component of \( f \) with the same real-valued function \( h \).
Our approach is based on the study of the fixed points of a suitable set–valued map. The next theorem collects two known results: one of them concerns the existence of fixed points for contractions with closed values, and the other one provides an upper bound that can be established on the distance between the fixed points of two contractions with closed values. This theorem will play a key role in the proof of the main result of this section.

Let us denote by $F(S)$ the set of the fixed points of the set–valued map $S : X \rightrightarrows X$, i.e.

$$F(S) = \{ x \in X : x \in S(x) \}.$$

**Theorem 4.7.** (see Theorem 5 in [11] and Lemma 1 in [7]) Let $X$ be a complete metric space, and let $S_1, S_2 : X \rightrightarrows X$ be two contractions with constant $\theta$ and closed values. Then $F(S_1)$ and $F(S_2)$ are nonempty sets; moreover,

$$d_H (F(S_1), F(S_2)) \leq \frac{1}{1 - \theta} \sup_{x \in X} d_H (S_1(x), S_2(x)).$$

The following lemma characterizes, for functions $g \in Pf$, the set–valued map $H^{-1}g$, in terms of the fixed points of a suitable map.

**Lemma 4.8.** Let $f : B(0, r) \rightarrow \mathbb{R}^m$ and $h : B(0, r) \rightarrow \mathbb{R}$ be differentiable functions on $B(0, r)$. Then

$$H^{-1}_g (p) = F \left( H^{-1}_f (p - \nabla h(\cdot)) \right),$$

where $g = f + he$.

**Proof.** Let $x \in H^{-1}_g (p)$. Then

$$\sum_{i=1}^{m} \lambda_i \nabla f_i (x) + \nabla h(x) = p,$$

for some nonnegative $\lambda_i$, $i = 1, ..., m$, such that $\sum_{i=1}^{m} \lambda_1 = 1$. Therefore $p - \nabla h(x) \in H_f (x)$ and hence $x \in H^{-1}_f (p - \nabla h(\cdot))$, i.e. $x \in F \left( H^{-1}_f (p - \nabla h(\cdot)) \right)$.

In a similar way we can prove that $H^{-1}_g (p) \supseteq F \left( H^{-1}_f (p - \nabla h(\cdot)) \right).$ $\Box$

We are now in the position to state our main result.

**Theorem 4.9.** Let $f$ be a function in the class $W^*_1$, and let $g = f + he \in Pf$ be such that

i) $g$ is $\mathbb{R}_+^m$–convex and satisfies $WE_g (0) \subset \text{int}(B(0, r));$

ii) $\max_{x \in B(0, r)} \| \nabla h(x) \| < \delta_f$, where $\delta_f$ is as in (5);

iii) $d^* (f, g) < \frac{1}{\varepsilon (f)}$.

Then $g \in W^*_1$. 

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Proof. We divide the proof into a sequence of three steps.

Step 1. We show that the set–valued map $H^{-1}_f(p - \nabla h(\cdot))$ is a contraction of $B(0, r)$ for every small $p$. First of all, the upper semicontinuity of $H_f$ entails that $H^{-1}_f(p - \nabla h(\cdot))$ has closed values. Furthermore, given a positive real number $\eta$, let us consider the quantity

$$K(\eta) = \sup \left\{ \frac{d_H(WE_f(p), WE_f(q))}{\|p - q\|} : p \neq q, \|p\| < \eta, \|q\| < \eta \right\}.$$  \hfill (19)

It is easy to observe that $\lim_{\eta \to 0} K(\eta) = c^*(f)$. Since, by assumption,

$$c^*(f) = \limsup_{p, q \to 0, p \neq q} \frac{d_H(WE_f(p), WE_f(q))}{\|p - q\|} < +\infty,$$

there exists a real number $\gamma_1 > 0$ such that $K(\eta) < +\infty$ for every $0 < \eta < \gamma_1$. By assumption ii) there exists a real number $\gamma_2 > 0$ such that

$$\|p - \nabla h(x)\| < \delta_f$$

for every $x \in B(0, r)$ and for every $p$ such that $\|p\| < \gamma_2$. Therefore, by (8) and by the convexity assumptions, the following inequality holds:

$$H^{-1}_f(p - \nabla h(x)) = WE_f(p - \nabla h(x))$$

for every $p$, $\|p\| < \gamma_2$. Therefore, from the equality above, we have

$$d_H \left( H^{-1}_f(p - \nabla h(x)), H^{-1}_f(p - \nabla h(y)) \right) \leq K(\gamma_3) \|\nabla h(x) - \nabla h(y)\|$$

$$\leq K(\gamma_3) \text{lip}(\nabla h; B(0, r)) \|x - y\|$$  \hfill (20)

for every $x, y \in B(0, r)$ and for every $p$ such that $\|p\| < \gamma_3$, where $\gamma_3 = \min\{\gamma_1, \gamma_2\}$.

From the equality $d^*(f, g) = \text{lip}(\nabla h; B(0, r))$ and the assumption iii), there exists a real number $\gamma_4 > 0$ such that for all $\gamma < \gamma_4$ we have

$$K(\gamma_4)d^*(f, g) < 1.$$

By (20), we conclude that the set–valued map $H^{-1}_f(p - \nabla h(\cdot))$ is a contraction with constant $K(\gamma)d^*(f, g)$ for every $p$ such that $\|p\| < \gamma$, where $\gamma = \min\{\gamma_3, \gamma_4\}$.

Step 2. From (19), and the since $c^*(f) < +\infty$, for every $p, q, p \neq q$, $\|p\|, \|q\| < \gamma$, and for every $x \in B(0, r),

$$d_H \left( H^{-1}_f(p - \nabla h(x)), H^{-1}_f(q - \nabla h(x)) \right) \leq K(\gamma)\|p - q\|.$$  \hfill (21)

From Step 1 and Theorem 4.7 we have

$$d_H \left( \mathcal{F} \left( H^{-1}_f(q_1 - \nabla h(\cdot)) \right), \mathcal{F} \left( H^{-1}_f(q_2 - \nabla h(\cdot)) \right) \right) \leq$$
\begin{align}
\leq \frac{1}{1 - K(\gamma) d^*(f, g)} \sup_{x \in B(0, r)} d_H \left( H_f^{-1}(q_1 - \nabla h(x)), H_f^{-1}(q_2 - \nabla h(x)) \right)
\end{align}

(22)

for every $q_1, q_2$ such that $\|q_1\|, \|q_2\| < \gamma$. By (21) and Lemma 4.8 from the last inequality we deduce that

$$d_H \left( H_g^{-1}(q_1), H_g^{-1}(q_2) \right) \leq \frac{K(\gamma)}{1 - K(\gamma) d^*(f, g)} \|q_1 - q_2\|$$

for every $q_1, q_2$ such that $\|q_1\|, \|q_2\| < \gamma$. Hence, by the definition of condition number, it follows that

$$c^*(g) \leq \frac{c^*(f)}{1 - c^*(f) d^*(f, g)}.$$

5 Conclusions

In this work we proposed two approaches for the notion of conditioning for a multiobjective optimization problem. To our knowledge in the recent literature on vector optimization there are no results on this topic. We limit our investigation to the Euclidean setting, and we consider essentially multiobjective problems involving differentiable convex functions. The main obstacle one has to overcome when dealing with this problem is the lack of conditions that fully characterize the solution set. This explains our choice to restrict our analysis to the framework of vector–valued convex functions. Moreover, following our approach we can prove a distance–type theorem only for a special class of perturbed functions. As a matter of fact, a relevant problem is the evaluation of the effect of a more general class of perturbations on the corresponding solution map.

For these reasons we deem that to deal with condition numbers in multiobjective optimization taking into account general perturbations is not an easy task and it may require a completely new approach.

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