Information-Theoretic Caching: Sequential Coding for Computing
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Abstract

Under the paradigm of caching, partial data is delivered before the actual requests of the users are known. In this paper, this problem is modeled as a canonical distributed source coding problem with side information, where the side information represents the users’ requests. For the single-user case, a single-letter characterization of the optimal rate region is established, and for several important special cases, closed-form solutions are given, including the scenario of uniformly distributed user requests. In this case, it is shown that the optimal caching strategy is closely related to total correlation and Wyner’s common information. For the two-user case, five representative subproblems are considered, which draw connections to existing source coding problems in the literature: the Gray–Wyner system, distributed successive refinement, and the Kaspi/Heegard–Berger problem.

Index Terms

Coded caching, function computation, multi-terminal source coding, source coding with side information.

I. INTRODUCTION

Consider a sports event filmed simultaneously by many cameras. After the game, a sports aficionado would like to watch a customized video sequence on his mobile device that shows him, in every moment, the best angle on his favorite player in the game. Of course, he would like that video as soon as possible. To meet such demand, the provider could choose to use the
paradigm of caching: even before knowing the precise camera angles of interest, cleverly coded partial data is delivered to the end user device. If all goes well, that partial data is at least partially useful, and hence, at delivery time, a much smaller amount of data needs to be downloaded, leading to faster (and possibly cheaper) service. To make matters even more interesting, there might be several users in the same mobile cell with the same wish, except that they probably have different favorite players. Now, the caching technique could be employed at the base station of the mobile cell, and the goal is to design cache contents such that at delivery time, almost all users experience a faster download speed.

In the present paper, we model this situation in a canonical information-theoretic fashion. We model the data as a long sequence $X_1, X_2, X_3, \cdots$, where the subscript may represent the time index. That is, in the above example, $X_i$ would represent the full collection of video images acquired at time $i$. Furthermore, in our model, the user defines a separate request for each time instant $i$. Hence, the user’s requests are also modeled as a sequence $Y_1, Y_2, Y_3, \cdots$, whose length we assume to be identical to the length of the data sequence. That is, in the above example, $Y_i$ represents the user’s desired camera angle at time $i$. There are two encoders: The cache encoder and the update encoder. The cache encoder only gets to observe the data sequence, and encodes it using an average rate of $R_c$ bits. The update (or data delivery) encoder gets to observe both the data sequence and the request sequence, and encodes them jointly using an average rate of $R_u$ bits. At the decoding end, to model the user’s desired data, we consider a per-letter function $g(X_i, Y_i)$ which needs to be recovered losslessly for all $i$. For example, we may think of $X_i$ as being a vector of a certain length, and of $Y_i$ as the index of the component of interest to the user at time $i$. Then, $g(X_i, Y_i)$ would simply return the component indexed by $Y_i$ from the vector $X_i$. The goal of this paper is to characterize the set of those rate pairs $(R_c, R_u)$ that are sufficient to enable the user to attain his/her goal of perfectly recovering the entire sequence $g(X_1, Y_1), g(X_2, Y_2), g(X_3, Y_3), \cdots$.

When there are multiple users, we allow different end users to have possibly different requests and functions, denoted by $Y_i^{(\ell)}$ and $g_\ell(\cdot, \cdot)$, respectively, for end user $\ell$. Moreover, we allow different end users to share caches and updates. However, in this work we make the simplification that each user’s request is known to all users. That is, denoting by $Y_i = \{Y_i^{(\ell)}\}$ the collection of requests, we assume that $Y_i$ is globally known except to the cache encoder. Thus, we denote $f_\ell(X_i, Y_i) = g_\ell(X_i, Y_i^{(\ell)})$ and focus on the functions $\{f_\ell(\cdot, \cdot)\}$ hereafter.

An important inspiration for the work presented here are the pioneering papers of Maddah-
Ali and Niesen [1], [2]. Their work emphasizes the case when there is a large number of users and develops clever caching strategies that centrally leverage a particular multi-user advantage: Cache contents is designed in such a way that each update (or delivery) message is simultaneously useful for as many users as possible. Our present work places the emphasis on the statistical properties of the data and requests, exploiting these features in a standard information-theoretic fashion by coding over long blocks. Furthermore, besides the network with only private caches and one common update studied in [1], [2], we take advantage of our small-scale analysis to explore various cache and update configurations.

On the modeling side, one could relate the Maddah-Ali–Niesen model to our model in two different ways: a “one-shot” version of our model or a related “single-request” model. The detail of the “single-request” interpretation is given in Appendix A, and the “one-shot” interpretation is as follows. There is only a single data $X$, rather than a sequence of data, and this $X$ is composed of $N$ files, each containing $F$ bits. Moreover, there is only a single request for each user, and this request identifies one out of the $N$ files. In this sense, the caching strategies developed in [1], [2] can be applied to some of the scenarios considered in the present paper as well. In standard information-theoretic parlance, they correspond to memoryless (or letter-by-letter) coding strategies, and can sometimes be outperformed by coding over long blocks, but not always. Ample results are available for the Maddah-Ali–Niesen model at this point: the worst-case analysis [1], the average-case analysis [2], decentralized [3], delay-sensitive [4], online [5], multiple layers [6], request of multiple items [7], secure delivery [8], wireless networks [9], [10], etc. In addition, some improved order-optimal results for the average case can be found in [11], [12].

The Maddah-Ali–Niesen model fits well with applications in which the users’ requests remain fixed over the entire time period of interest, e.g., on-demand video streaming of a movie from a given database. By contrast, our model may be an interesting fit for applications in which the users’ requests change over time, such as the multi-view video system example in the beginning. Furthermore, our model fits well with sensor network applications. In many cases, only the sensor data (modeled as $X$) with certain properties (modeled as $Y$) are of interest and the desired properties may vary over a timescale of minutes, hours, or even days. In terms of coding strategies, we also take a different approach from [1], [2] and the follow-up works, in which the core schemes are based on linear network coding. By contrast, by formulating the caching problem into a multi-terminal source coding problem, our main tools are standard information-
theoretic arguments, including joint typicality encoding/decoding, superposition coding, and binning.

A. Related Works

The structure of many source coding problems studied in the literature can be captured by our formulation. Let us denote by $L$ the number of users and start with the case $L = 1$, as shown in Figure 1. Denote by $R_c$ and $R_u$ as the rates of the cache and the update, respectively. Depending on the availability of the cache and the update, each configuration can be seen as a special case or a straightforward extension of

1) $(0, R_u)$: lossless source coding with side information [13];
2) $(R_c, 0)$: lossy source coding with side information [14] or lossless coding for computing with side information [15];
3) $(R_c, R_u)$: lossless source coding with a helper [16, 17].

Now consider the case $L = 2$. There are three classes of source coding problems that are related to our problem setup: the Kaspi/Heegard–Berger problem, the Gray–Wyner system, and the problem of successive refinement.
In the Kaspi/Heegard–Berger problem [18], [19], each decoder observes distinct side information and wants to recover a distinct representation of the source. A system diagram in the setup where the requests are locally known is shown in Figure 2. This problem remains open in general, even for the case of interest where the requests $Y_1, Y_2$ are independent. Nevertheless, for the following special cases, the optimal rates are known:

1) both decoders wish to recover the same function losslessly [20],
2) the side information is physically degraded [19],
3) the side information is conditionally less noisy [21].

The case where the encoder also knows the side information was studied in [22], [23]. The problem has also been extended to multiple decoders [24].

In the Gray–Wyner system, each decoder receives a private message and a common message. A system diagram of the considered setup is shown in Figure 3. The optimal rate–distortion region was characterized in [25]. The extension to include distinct side information at the decoders was studied in [26] and [27, Section V].

In the problem of successive refinement [28], [29], one of the decoders has no private message, i.e., all of its received messages are also available at the other decoder. A system diagram in the considered setup is shown in Figure 4. The optimal rate–distortion region was characterized
in [30]. The extension to include distinct side information at the decoders was studied in [31]–[33]. The problem of successive refinement can also be extended to multiple sources [34], [35]. One special case of the multi-source extension is the problem of sequential coding of correlated sources [36].

Finally consider the case $L = 3$. The most well-known configuration in this case is the problem of multiple description coding: The encoder generates two messages each of which is received by a distinct user and there is another user who receives both messages. A system diagram in the considered setup is shown in Figure 5. This problem remains open in general. The achievabilities and the optimality for some special cases can be found in [37]–[39]. The problem was extended to the general $K$ messages and there are $2^K - 1$ users receiving a distinct subset of the $K$ messages [40]. Our formulation can be seen as an extension of [40] to the case of sequential coding with two encoders. However, instead of fixing the number of messages, we first fix the number of users and then consider all valid configurations allowing lossless recovery of every desired function.

**B. Summary of Results**

Although the general $L$-user caching problem remains open, in this work we make some progress for $L \leq 2$. For the single-user case, the results are summarized as follows:

- Theorem 1 provides a single-letter characterization of the optimal rate region.
- Propositions 1 and 2 give the exact optimal rate regions for the cases of independent components and nested components, respectively, and confirm that some intuitive caching strategies are indeed optimal.
Proposition 3 shows that if the components are uniformly requested, then the optimal caching strategy is to cache a description of the data that minimizes the conditional total correlation.

For the two-user case, a single-letter characterization of the optimal rate region remains unknown. To gain some insight, we consider five representative configurations:

1) one common cache, two private caches, and two private updates (Figure 7);
2) one common cache, one common update, and two private updates (Figure 8);
3) one common cache, one common update, one private cache for User 2, and one private update for User 2 (Figure 9);
4) one private cache for User 1, one common update, and one private update for User 2 (Figure 10);
5) two private caches and one common update (Figure 11).

Configurations 1, 2, 3 correspond to the configurations with the largest number of messages for which we are able to characterize the optimal rate regions, which are given in Theorems 2, 3, and 4, respectively. Configurations 4, 5 are the configurations with the smallest number of messages whose optimal rate regions are still unknown. We remark that Configurations 1 and 2 are extensions of the Gray–Wyner system, Configuration 3 is a special case of distributed successive refinement, and Configuration 5 captures the structure of the system studied in [1].

The paper is organized as follows. In Section II we provide the problem formulation for the two-user case. Sections III and IV are devoted to the single-user case and the two-user case, respectively. Finally, we conclude in Section V.

C. Notations

Random variables and their realizations are represented by uppercase letters (e.g., $X$) and lowercase letters (e.g., $x$), respectively. We use calligraphic symbols (e.g., $\mathcal{X}$) to denote sets. Denote by $| \cdot |$ the cardinality of a set. We denote $[1 : L] := \{1, 2, \ldots, L\}$ and $A \setminus B := \{x \in A \mid x \notin B\}$. We denote $X^k := (X_1, X_2, \ldots, X_k)$ and $X^{[i:j]} := (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_k)$. Throughout the paper, all logarithms are to base two. Let $h_2(p) := -p \log(p) - (1-p) \log(1-p)$ for $p \in [0,1]$ and $0 \log(0) := 0$ by convention. We denote $(x)^+ := \max\{x, 0\}$ and follow the $\epsilon$–$\delta$ notation in [41]. The adopted notion of typicality follows from [15] (see also [41]). For $X \sim p_X$ and $\epsilon \in (0,1)$, the set of typical sequences of length $k$ with respect to the probability
distribution \( p_X \) and the parameter \( \epsilon \) is denoted by \( T_\epsilon^{(k)}(X) \), which is defined as
\[
T_\epsilon^{(k)}(X) := \left\{ x^k \in \mathcal{X}^k : \left| \frac{\#(a|x^k)}{k} - p_X(a) \right| \leq \epsilon p_X(a), \forall a \in \mathcal{X} \right\},
\]
where \( \#(a|x^k) \) is the number of occurrences of \( a \) in \( x^k \).

II. PROBLEM STATEMENT

Here we state the problem for the two-user case and then the single-user case follows naturally as a special case. We remark that the general \( L \)-user case can also be formulated in a straightforward manner. A pair of discrete memoryless sources \((X, Y)\) are drawn from the finite alphabets \( \mathcal{X} \) and \( \mathcal{Y} \), respectively, with a joint probability mass function \( p_{X,Y} \) over \( \mathcal{X} \times \mathcal{Y} \). The two sources jointly generate an independent and identically distributed (i.i.d.) random process \( \{(X_i, Y_i)\} \). There are two encoding terminals and two decoding terminals. The cache encoder observes the source sequence \( X^k \), the update encoder observes the source sequences \( (X^k, Y^k) \), and both decoders observe \( Y^k \). Decoder \( \ell \in \{1, 2\} \) wishes to recover an element-wise function \( f_\ell(x, y) \) losslessly. The cache encoder generates three messages \( M_{c,(1)} \), \( M_{c,(2)} \), and \( M_{c,(1,2)} \) of rate \( R_{c,(1)} \), \( R_{c,(2)} \), and \( R_{c,(1,2)} \), respectively. Similarly, the update encoder generates three messages \( M_{u,(1)} \), \( M_{u,(2)} \), and \( M_{u,(1,2)} \) of rate \( R_{u,(1)} \), \( R_{u,(2)} \), and \( R_{u,(1,2)} \), respectively. We note that some of these rates may be zero. Then, Decoder \( \ell \in \{1, 2\} \) receives the set of messages \((M_{c,(\ell)}, M_{c,(1,2)}, M_{u,(\ell)}, M_{u,(1,2)})\). The messages \( M_{c,(\ell)} \) and \( M_{u,(\ell)} \) are called private cache content and private update content, respectively. Besides, the messages \( M_{c,(1,2)} \) and \( M_{u,(1,2)} \) are called common cache content and common update content, respectively.

Denote \( n_a = \lceil 2^{kR_a} \rceil \) for any subscript \( a \). Also, for convenience we denote
\[
\mathbf{n} := (n_{c,(1)}, n_{c,(2)}, n_{c,(1,2)}, n_{u,(1)}, n_{u,(2)}, n_{u,(1,2)}),
\]
\[
\mathbf{R} := (R_{c,(1)}, R_{c,(2)}, R_{c,(1,2)}, R_{u,(1)}, R_{u,(2)}, R_{u,(1,2)}).
\]

Then, an \((\mathbf{n}, k)\) distributed multiple description code consists of \((\mathcal{A} \in \{\{1\}, \{2\}, \{1, 2\}\})\)

- two encoders, where the cache encoder assigns 3 indices \( m_{c,(\mathcal{A})}(x^k) \in [1 : n_{c,(\mathcal{A})}] \) to each sequence \( x^k \in \mathcal{X}^k \) and the update encoder assigns 3 indices \( m_{u,(\mathcal{A})}(x^k, y^k) \in [1 : n_{u,(\mathcal{A})}] \) to each pair of sequences \((x^k, y^k) \in \mathcal{X}^k \times \mathcal{Y}^k\);
- two decoders, where Decoder \( \ell \in \{1, 2\} \) assigns an estimate \( \hat{s}_\ell^k \) to each tuple \((m_{c,(\ell)}, m_{c,(1,2)}, m_{u,(\ell)}, m_{u,(1,2)}, y^k)\).
A rate tuple $\mathbf{R}$ is said to be achievable if there exists a sequence of $(n, k)$ codes with

$$\lim_{k \to \infty} P \left( \bigcup_{\ell \in \{1,2\}} \bigcup_{i \in [1:k]} \left\{ \hat{S}_{\ell,i} \neq f_\ell(X_i, Y_i) \right\} \right) = 0.$$  

The optimal rate region $\mathcal{R}^*$ is the closure of the set of achievable rate tuples.

We are also interested in the subsets of $\mathcal{R}^*$, in which some of the rate components are set to zero. Let $\mathcal{A}_c$ and $\mathcal{A}_u$ be any subsets of $\{\{1\}, \{2\}, \{1, 2\}\}$. We use the notation $(\mathcal{A}_c | \mathcal{A}_u)$, termed configuration, to specify the available caches and updates in the system. For example, Configuration $(\mathcal{A}_c | \mathcal{A}_u) = (\{1\}, \{2\}|\{1, 2\})$ says that each user has his/her own private cache and there is an update common to both users. Then, we define

$$\mathcal{R}^*(\mathcal{A}_c | \mathcal{A}_u) := \{ \mathbf{R} \in \mathcal{R}^* : R_{c,B_c} = 0, R_{u,B_u} = 0 \text{ for all } B_c \notin \mathcal{A}_c, B_u \notin \mathcal{A}_u \}.$$  

III. THE SINGLE-USER CASE

In this section, we consider the single-user case. For notational convenience, in this section we drop the decoder index, e.g., $f_1(x, y)$ is simply denoted by $f(x, y)$. This setup is a special case of the problem of lossy source coding with a helper and receiver side information, which remains open in general. We establish the optimal rate region of the considered setup in the following theorem.

**Theorem 1:** The optimal rate region $\mathcal{R}^*$ of the single-user caching problem is the set of rate pairs $(R_c, R_u)$ such that

$$R_c \geq I(X; V|Y),$$  

$$R_u \geq H(f(X, Y)|V, Y),$$  

for some conditional probability mass function (pmf) $p_{V|X}$, where $|V| \leq |X| + 1$.

**Proof:** The proof of achievability follows from standard random coding arguments as in the problem of lossless source coding with a helper. Here we provide a high-level description of the coding scheme. First, the cache encoder applies Wyner–Ziv coding on $x^k$ given side information $y^k$ so that the decoder learns $v^k$, a quantized version of $x^k$. Then the update encoder applies Slepian–Wolf coding on $\{f(x_i, y_i)\}_{i \in [1:k]}$ given side information $(v^k, y^k)$. 
The converse is also straightforward. Denote \( S_i = f(X_i, Y_i), i \in [1 : k] \). First, we have

\[
kR_c \geq H(M_c)
\]
\[
\geq H(M_c|Y^k)
\]
\[
= I(X^k; M_c|Y^k)
\]
\[
= \sum_{i=1}^{k} I(X_i; M_c|X^{i-1}, Y^k)
\]
\[
(a) = \sum_{i=1}^{k} I(X_i; M_c, X^{i-1}, Y^{[1:k]\{i}\} | Y_i)
\]
\[
\geq \sum_{i=1}^{k} I(X_i; V_i | Y_i),
\]

where \((a)\) follows since \((X_i, Y_i)\) is independent of \((X^{i-1}, Y^{[1:k]\{i}\})\). For the last step, we define \( V_i := (M_c, S^{i-1}, Y^{[1:k]\{i\}}) \). Note that \( V_i \dasharrow X_i \dasharrow Y_i \) form a Markov chain.

Next, we have

\[
kR_u \geq H(M_u)
\]
\[
\geq H(M_u|M_c, Y^k)
\]
\[
= H(S^k, M_u|M_c, Y^k) - H(S^k|M_c, M_u, Y^k)
\]
\[
(a) \geq H(S^k|M_c, Y^k) - k\epsilon_k
\]
\[
= \sum_{i=1}^{k} H(S_i|M_c, S^{i-1}, Y^k) - k\epsilon_k
\]
\[
= \sum_{i=1}^{k} H(S_i|V_i, Y_i) - k\epsilon_k,
\]

where \((a)\) follows from the data processing inequality and Fano’s inequality. The rest of the proof follows from the standard time-sharing argument and then letting \( k \to \infty \). The cardinality bound on \( V \) can be proved using the convex cover method [41, Appendix C].

Note that the optimal rate region \( \mathcal{R}^* \) is convex. As can be seen from the achievability, even if the update encoder is restricted to access only the sequence of functions \( \{ f(x_i, y_i) \} \), instead of \((x^k, y^k)\), the rate region remains the same.

A simple consequence of Theorem 1 is the following lower bound on the sum rate.

**Corollary 1:** \( R_c + R_u \geq H(f(X, Y)|Y) \) for all \((R_c, R_u) \in \mathcal{R}^*\).
Proof: The corollary can be proved by a simple cut-set argument. Alternatively, from Theorem [I], we observe that
\[ R_c + R_u \geq I(X; V|Y) + H(f(X, Y)|V, Y) \]
\[ \geq I(f(X, Y); V|Y) + H(f(X, Y)|V, Y) \]
\[ = H(f(X, Y)|Y). \]

Let us briefly mention the two extreme cases. First, when \( R_c = 0 \), the optimal update rate is \( R_u^* = H(f(X, Y)|Y) \). Second, when the update rate \( R_u = 0 \), the minimum cache rate is
\[ R_c^* = \min_{p_{V|X}} I(X; V|Y). \]

Thus, we recover the result of lossless coding for computing with side information [15]. Note that the same result can be inferred by borrowing the results from rate-distortion theory.

In general, the optimal sum rate can only be achieved at the point \( (R_c, R_u) = (0, H(f(X, Y)|Y)) \), i.e., the update encoder does all the work. Nevertheless, for the class of partially invertible functions, one can arbitrarily distribute the work load without compromising the sum rate.

Corollary 2: If the function \( f \) is partially invertible, i.e., \( H(X|f(X, Y), Y) = 0 \), then \( R_c^* = R_u^* = H(X|Y) \).

Proof: First, Theorem [I] implies that \( R_u^* \leq R_c^* \leq H(X|Y) \). Then, the corollary is an easy consequence of the property of the function \( f \):
\[ H(f(X, Y)|Y) = H(f(X, Y), X|Y) = H(X|Y). \]

In other words, Corollary 2 says that for partially invertible functions, e.g., arithmetic sum and modulo sum, the side information \( Y \) at the update encoder is useless in lowering the compression rate and thus in this case the cache encoder is as powerful as the update encoder. More generally, it can be shown that \( R_c^* = R_u^* \) if and only if there exists a conditional pmf \( p_{V|X} \) such that
1) \( H(V|f(X, Y), Y) = H(V|X, Y) \), and
2) \( H(f(X, Y)|V, Y) = 0 \).
For most of the problems, it is challenging to find a closed-form expression for the optimal rate region $R^*$. That is, we do not know the optimal caching strategy in general. In the following, we consider three cases where $X$ and $Y$ are independent, which implies that $I(X; V|Y) = I(X; V)$. For the first two cases, we are able to show that some intuitive caching strategies are indeed optimal. In the last case, we provide some guidance for the optimal caching strategy. Without loss of generality, we assume that $Y = [1 : N]$. Besides, we will find it convenient to denote $x^{(y)} := f(x, y)$.

A. Independent Source Components

In this subsection, we consider the case where $H(X^{(1)}, \ldots, X^{(N)}) = \sum_{n=1}^{N} H(X^{(n)})$. Recall that $X^{(n)} = f(X, n)$. Without loss of generality, we assume that $p_Y(1) \geq p_Y(2) \geq \cdots \geq p_Y(N)$. Then, we have the following proposition.

**Proposition 1**: If $X$ and $Y$ are independent and $H(X^{(1)}, \ldots, X^{(N)}) = \sum_{n=1}^{N} H(X^{(n)})$, then the optimal rate region $R^*$ is the set of rate pairs $(R_c, R_u)$ such that

\[
R_c \geq r,
R_u \geq \sum_{n=1}^{N} (p_Y(n) - p_Y(n + 1)) \left( \sum_{j=1}^{n} H(X^{(j)}) - r \right)^+,
\]

for some $r \geq 0$, where $p_Y(N + 1) = 0$.

**Proof**: We start with the converse part. Consider any conditional pmf $p_{V|X}$ and let $r = I(X; V|Y)$. Then, we have for all $n \in [1 : N]$,

\[
r = I(X; V|Y) \\
= I(X; V) \\
\geq I(X^{(1)}, \ldots, X^{(n)}; V) \\
\geq \sum_{j=1}^{n} H(X^{(j)}) - \sum_{j=1}^{n} H(X^{(j)}|V),
\]

A caching strategy is said to be (Pareto) optimal if its achievable rate pair lies on the boundary of the optimal rate region $R^*$.\(^1\)
where (a) follows since $X$ and $Y$ are independent. Next we show that $R_u$ can be lower bounded as in (3):

\[
R_u \geq H(f(X, Y)|V, Y)
= \sum_{j=1}^{N} p_Y(j)H(X^{(j)}|V)
\geq p_Y(N) \left( \sum_{j=1}^{N} H(X^{(j)}) - r - \sum_{j=1}^{N-1} H(X^{(j)}|V) \right)^+ + \sum_{j=1}^{N-1} p_Y(j)H(X^{(j)}|V)
\geq p_Y(N) \left( \sum_{j=1}^{N} H(X^{(j)}) - r \right)^+ + \sum_{j=1}^{N-1} (p_Y(j) - p_Y(N))H(X^{(j)}|V),
\]

where (a) follows from (4) with $n = N$ and $H(X^{(N)}|V) \geq 0$ and (b) follows since $(u - v)^+ \geq (u)^+ - v$ for all $v \geq 0$. The term on the right-hand side can be lower bounded as

\[
\sum_{j=1}^{N-1} (p_Y(j) - p_Y(N))H(X^{(j)}|V)
\geq (p_Y(N - 1) - p_Y(N)) \left( \sum_{j=1}^{N-1} H(X^{(j)}) - r - \sum_{j=1}^{N-2} H(X^{(j)}|V) \right)^+ + \sum_{j=1}^{N-2} (p_Y(j) - p_Y(N - 1))H(X^{(j)}|V),
\]

where (a) follows from (4) with $n = N - 1$ and $H(X^{(N-1)}|V) \geq 0$. At this point, it is clear that we can apply the same argument for another $N - 2$ times and arrive at

\[
R_u \geq \sum_{n=1}^{N} (p_Y(n) - p_Y(n + 1)) \left( \sum_{j=1}^{n} H(X^{(j)}) - r \right)^+, \quad (5)
\]

where $p_Y(N + 1) = 0$.

We now show the achievability. Note that the lower bound (5) is equivalent to saying that

1) if $r \geq \sum_{n=1}^{N} H(X^{(n)})$, then $R_u \geq 0$, and
2) if $\sum_{j=1}^{n-1} H(X^{(j)}) \leq r < \sum_{j=1}^{n} H(X^{(j)})$ for some $n \in [1 : N]$, then

\[
R_u \geq p_Y(n) \left( \sum_{j=1}^{n} H(X^{(n)}) - r \right)^+ + \sum_{j=n+1}^{N} p_Y(j)H(X^{(j)}).
\]
Therefore, for all \( n \in [0 : N] \), substituting \( V = (X^{(1)}, \ldots, X^{(n)}) \) in (1) and (2) shows that the rate pair
\[
(R_c, R_u) = \left( \sum_{j=1}^{n} H(X^{(j)}), \sum_{j=n+1}^{N} p_Y(j)H(X^{(j)}) \right)
\]
is achievable, which corresponds to a corner point in the region described by \( R_c \geq r \) and (5). Since the rest of points on the boundary can be achieved by time sharing, the proposition is established.

Thus, when relating to the motivating example, Proposition 1 indicates that the best caching strategy for independent views is to cache the most popular ones, no matter how different the video qualities are (see also [2] and the references therein).

**B. Nested Source Components**

Again using the shorthand notation \( X^{(n)} = f(X, n) \), in this subsection we assume that \( H(X^{(n)}|X^{(n+1)}) = 0 \) for all \( n \in [1 : N - 1] \). Then, we have the following proposition.

**Proposition 2:** If \( X \) and \( Y \) are independent and \( H(X^{(n)}|X^{(n+1)}) = 0 \) for all \( n \in [1 : N - 1] \), then the optimal rate region \( \mathcal{R}^* \) is the set of rate pairs \( (R_c, R_u) \) such that
\[
R_c \geq r, \quad R_u \geq \sum_{n=1}^{N} p_Y(n) \left( H(X^{(n)}) - r \right)^+, \quad (6)
\]
for some \( r \geq 0 \).

**Proof:** We start with the converse part. Consider any conditional pmf \( p_{V|X} \) and let \( r = I(X; V|Y) \). Then, we have for all \( n \in [1 : N] \),
\[
r = I(X; V|Y) \\
\overset{(a)}{=} I(X; V) \\
\geq I(X^{(1)}, \ldots, X^{(n)}; V) \\
\overset{(b)}{=} H(X^{(n)}) - \sum_{j=1}^{n} H(X^{(j)}|V, X^{(j-1)}), \quad (7)
\]
where \((a)\) follows since \(X\) and \(Y\) are independent and \((b)\) follows from the assumption that 
\[H(X^{(n)}|X^{(n+1)}) = 0\] for all \(n \in \{1 : N - 1\}\). Next, we show that \(R_u\) can be lower bounded as in (6):

\[
R_u \geq H(f(X, Y)|V, Y)
\]

\[
= \sum_{n=1}^{N} p_Y(n) H(X^{(n)}|V)
\]

\[
= \sum_{n=1}^{N} p_Y(n) H(X^{(1)}, \cdots, X^{(n)}|V)
\]

\[
= \sum_{n=1}^{N} p_Y(n) \sum_{j=1}^{n} H(X^{(j)}|V, X^{(j-1)})
\]

\[
= \sum_{j=1}^{N} \left( \sum_{n=j}^{N} p_Y(n) \right) H(X^{(j)}|V, X^{(j-1)}),
\]

where \((a)\) and \((b)\) follow from the assumption that 
\[H(X^{(n)}|X^{(n+1)}) = 0\] for all \(n \in \{1 : N - 1\}\). For notational convenience, let us denote 
\(s_j = \sum_{n=j}^{N} p_Y(n)\) and 
\(q_j = H(X^{(j)}|V, X^{(j-1)})\). Then, we have

\[
R_u \geq \sum_{j=1}^{N} s_j q_j
\]

\[
\geq s_N \left( H(X^{(N)}) - r - \sum_{j=1}^{N-1} q_j \right) + \sum_{j=1}^{N-1} s_j q_j
\]

\[
\geq s_N \left( H(X^{(N)}) - r \right) + \sum_{j=1}^{N-1} (s_j - s_N) q_j
\]

\[
= p_Y(N) \left( H(X^{(N)}) - r \right) + \sum_{j=1}^{N-1} (s_j - s_N) q_j
\]

\[
\geq p_Y(N) \left( H(X^{(N)}) - r \right) + (s_{N-1} - s_N) \left( H(X^{(N-1)}) - r - \sum_{j=1}^{N-2} q_j \right)
\]

\[
+ \sum_{j=1}^{N-2} (s_j - s_N) q_j
\]

\[
\geq p_Y(N) \left( H(X^{(N)}) - r \right) + (s_{N-1} - s_N) \left( H(X^{(N-1)}) - r \right) + \sum_{j=1}^{N-2} (s_j - s_{N-1}) q_j
\]

\[
= \sum_{n=N-1}^{N} p_Y(n) \left( H(X^{(n)}) - r \right) + \sum_{j=1}^{N-2} (s_j - s_{N-1}) q_j,
\]
where (a) and (c) follow from (7) and \( H(X^{(n)}|V,X^{(n-1)}) \geq 0 \) with \( n = N \) and \( n = N - 1 \), respectively, and (b) and (d) follow since \((u - v)^+ \geq (u)^+ - v\) for all \( v \geq 0 \). At this point, it is clear that we can apply the same argument for another \( N - 2 \) times and arrive at

\[
R_u \geq \sum_{n=1}^{N} p_Y(n) \left( H(X^{(n)}) - r \right)^+. \tag{8}
\]

We now show the achievability. Note that the lower bound (8) is equivalent to saying that

1) if \( r \geq H(X^{(N)}) \), then \( R_u \geq 0 \), and
2) if \( H(X^{(j-1)}) \leq r < H(X^{(j)}) \) for some \( j \in [1 : N] \), where \( H(X^{(0)}) := 0 \), then

\[
R_u \geq \sum_{n=j}^{N} p_Y(n) \left( H(X^{(n)}) - r \right)^+.
\]

Therefore, for all \( n \in [0; N] \), substituting \( V = X^{(n)} \) in (1) and (2) shows that the rate pair

\[
(R_c, R_u) = \left( H(X^{(n)}), \sum_{j=n+1}^{N} p_Y(j)H(X^{(j)}|X^{(n)}) \right),
\]

is achievable, which corresponds to a corner point in the region described by \( R_c \geq r \) and (8). Since the rest of points on the boundary can be achieved by time sharing, the proposition is established.

If we think of \( X^{(1)}, \ldots, X^{(N)} \) as representations of the same view but with different levels of quality, then Proposition 2 indicates that the best caching strategy is to cache the coarsest versions up to the cache size.

C. Arbitrarily Correlated Components with Uniform Requests

Here we assume that the request is uniformly distributed, i.e., \( p_Y(n) = \frac{1}{N} \) for all \( n \in [1 : N] \), but \( X^{(1)}, \ldots, X^{(N)} \) can be arbitrarily correlated. Recall that \( X^{(n)} = f(X,n) \). Although we cannot give a closed-form expression of the optimal rate region, we provide a necessary and sufficient condition on the auxiliary random variable which characterizes the boundary of the optimal rate region.

\textbf{Proposition 3:} If \( X \) and \( Y \) are independent and \( p_Y(n) = \frac{1}{N} \) for all \( n \in [1 : N] \), then all points \((R_c, R_u)\) on the boundary of the optimal rate region \( R^* \) can be expressed as

\[
R_c = r,
\]

\[
R_u = \frac{1}{N} \left( H(X) - r + \min_{p_V|X \ s.t. \ I(X;V)=r} C(X|V) \right),
\]
for some \( r \in [0, H(X)] \), where \( \overline{X} := (X^{(1)}, X^{(2)}, \ldots, X^{(N)}) \) and
\[
C(\overline{X}|V) := \left[ \sum_{n=1}^{N} H(X^{(n)}|V) \right] - H(X^{(1)}, \ldots, X^{(N)}|V).
\]

**Proof:** Denote by \( \mathcal{R} \) the set of rate pairs \((R_c, R_u)\) such that
\[
R_c \geq I(\overline{X}; V|Y),
\]
\[
R_u \geq H(f(X, Y)|V, Y),
\]
for some conditional pmf \( p_{V|X} \). Since \( I(X; V|Y) \geq I(\overline{X}; V|Y) \), we have \( \mathcal{R}^* \subseteq \mathcal{R} \). Also, it can be shown that the rate region \( \mathcal{R} \) is achievable, so we conclude that \( \mathcal{R}^* = \mathcal{R} \). By using the assumptions that \( I(X; Y) = 0 \) and that \( Y \) is uniformly distributed, we can simplify the rate expressions as
\[
R_c \geq I(\overline{X}; V),
\]
\[
R_u \geq \frac{1}{N} \sum_{n=1}^{N} H(X^{(n)}|V).
\]

Now denote by \( p_{V|\overline{X}} \) the conditional pmf induced by the conditional pmf \( p_{V|X} \). As can be checked, both \( I(\overline{X}; V) \) and \( \{H(X^{(n)}|V)\}_{n \in [1:N]} \) can be completely determined by the induced conditional pmf \( p_{V|\overline{X}} \). Thus, it suffices to consider the space of all conditional pmfs \( p_{V|\overline{X}} \). Finally, noting that
\[
\sum_{n=1}^{N} H(X^{(n)}|V) = H(\overline{X}) - I(\overline{X}; V) + C(\overline{X}|V),
\]
it holds that if \( R_c = r \in [0, H(\overline{X})] \), then
\[
\min\{R_u|R_c = r, (R_c, R_u) \in \mathcal{R}^*\} = \frac{1}{N} \left( H(\overline{X}) - r + \min_{p_{V|\overline{X}} \text{ s.t. } I(\overline{X}; V) = r} C(\overline{X}|V) \right).
\]

If \( N = 2 \), we have
\[
C(\overline{X}|V) = I(X^{(1)}; X^{(2)}|V),
\]
so the term \( C(X^{(1)}, \ldots, X^{(N)}|V) \) can be interpreted as a generalization of conditional mutual information. In fact, the term \( C(X^{(1)}, \ldots, X^{(N)}) = \left[ \sum_{n=1}^{N} H(X^{(n)}) \right] - H(X^{(1)}, \ldots, X^{(N)}) \) was first studied by Watanabe [42] and given the name **total correlation**. Following this convention, we refer to \( C(X^{(1)}, \ldots, X^{(N)}|V) \) as **conditional total correlation**. Proposition [3] indicates that
an optimal caching strategy is to cache a description of the data that minimizes the conditional total correlation.

When the cache rate is large enough, there exists a conditional pmf $p_{V|X}$ such that the conditional total correlation is zero and thus we have the following corollary.

**Corollary 3:** The boundary of the region $\{(R_c, R_u) \in \mathcal{R}^* | R_c \leq R_c \leq H(\overline{X})\}$ is a straight line $R_c + NR_u = H(\overline{X})$, where

$$R_c = \min_{p_{V|X} \text{ s.t. } C(X|V) = 0} I(\overline{X}; V).$$

Note that when $N = 2$, $R_{crit}$ is Wyner’s common information \[43\].

Finally, let us consider an example which covers all the mentioned cases.

**Example 1:** Fix $q \in [0, \frac{1}{2}]$ and denote by $\oplus$ the modulo-two sum. Consider $Y \sim \text{Uniform}\{1, 2\}$ and $X = (X^{(1)}, X^{(2)})$, where $X^{(1)}, X^{(2)} \sim \text{Bernoulli}(1/2)$ and $X^{(1)} \oplus X^{(2)} \sim \text{Bernoulli}(q)$. Assume that $X$ and $Y$ are independent. We first consider two extreme cases.

1) If $q = 1/2$, then the two components are independent and $\mathcal{R}^* = \{(R_c, R_u) | R_c \geq 0, R_u \geq 0, R_c + 2R_u \geq 2\}$.

2) If $q = 0$, then the two components are nested and $\mathcal{R}^* = \{(R_c, R_u) | R_c \geq 0, R_u \geq 0, R_c + R_u \geq 1\}$.

Now consider $0 < q < \frac{1}{2}$. Wyner’s common information of $(X^{(1)}, X^{(2)})$ is known as \[43\]

$$R_{crit} = 1 + h_2(q) - 2h_2(q'),$$

where $q' = \frac{1}{2}(1 - \sqrt{1 - 2q})$. Thus, from Corollary 3 we have

$$\min\{R_u | R_c \geq R_{crit}, (R_c, R_u) \in \mathcal{R}^*\} = \frac{1}{2}(1 + h_2(q) - R_c).$$

Note that $R_u \geq \frac{1}{2}(1 + h_2(q) - R_c)^+$ is also a valid lower bound for all $R_c \geq 0$. Besides, from Corollary 1 we have $R_c + R_u \geq 1$.

As for the case $0 < q < \frac{1}{2}$ and $R_c < R_{crit}$, we do not have a complete characterization. Let us consider the following choice of the auxiliary random variable $V$. We set

$$V = \begin{cases} 
X^{(1)} \oplus U & \text{if } X^{(1)} \oplus X^{(2)} = 0, \\
W & \text{if } X^{(1)} \oplus X^{(2)} = 1,
\end{cases}$$

(9)

where $\oplus$ denotes modulo-two sum, $U, W \in \{0, 1\}$ are independent of $(X, Y)$, and furthermore $W \sim \text{Bernoulli}(1/2)$. It can be checked that setting

$$p_U(1) = \frac{1}{2} - \frac{\sqrt{1 - 2q}}{2(1 - q)}$$
achieves Wyner’s common information $R_{\text{crit}}$. Figure 6 plots the resulting inner bound and the combined outer bound $R_u \geq \max\{\frac{1}{2}(1 + h_2(q) - R_c)^+, (1 - R_c)^+\}$, with $q = 0.1$. The blue diamond point corresponds to $(R_c, R_u) = (R_{\text{crit}}, 1 + h_2(q) - R_{\text{crit}})$. The inner bound is plotted by evaluating all $p_U(1) \in [0, 0.5]$ and finding the convex hull of the resulting points and the point $(R_c, R_u) = (1 + h_2(q), 0)$.

IV. THE TWO-USER CASE

In this section, we consider the two-user case. In this case, there are totally six messages $M_{c,\{1\}}, M_{c,\{2\}}, M_{c,\{1,2\}}, M_{u,\{1\}}, M_{u,\{2\}}, M_{u,\{1,2\}}$. A complete characterization of the optimal rate region $R^*$ is unavailable. Usually, we take a top-down approach to develop the most general achievability involving every message and then specialize the achievable rate region to different subproblems. However, here we find it more informative to take a bottom-up approach. In particular, we consider five representative configurations. The first three configurations are

\[ (A_c|A_u) = \]

\[ (\{1\}, \{2\}, \{1,2\}|\{1\}, \{2\}), \tag{10} \]

\[ (\{1,2\}|\{1\}, \{2\}, \{1,2\}), \tag{11} \]

\[ (\{2\}, \{1,2\}|\{2\}, \{1,2\}), \tag{12} \]

which correspond to the configurations with the largest number of messages for which we are able to characterize the optimal rate regions. The last two configurations are $(\{1\}|\{2\}, \{1,2\})$
and \((\{1\}, \{2\}|\{1, 2\})\), which are the configurations with the smallest number of messages whose optimal rate regions are still unknown.

To provide some interesting insights, we find it convenient to classify our coding strategies via a concept that we will refer to as “decoding order.” More precisely, in all achievable schemes considered here, the decoders will proceed in multiple steps according to a successive decoding logic: A first description of the source is recovered and then used as side information in the recovery of the second description, and so on. In fact, we have already encountered this in the single-user case, resolved in Theorem 1. In the scheme used to prove the achievability part of the theorem, the decoder first recovers the description sent by the cache encoder, and then recovers the desired function. For this case, we would thus say that the decoding order is “cache → update.” In this particular case, the converse proof of Theorem 1 implies that employing this decoding order is without loss of optimality. A second example is the classical work of Gray and Wyner [25], which is a special case of the setup considered in the present paper. As can be seen in [25, Theorem 8] (and more generally in Subsection IV-A below), for this scenario, each decoder first recovers the common description, and then leverages the private description in order to fully decode the object of interest. Here, we would thus say that the decoding order is “common → private.” Again, for this special case, [25, Theorem 8] shows that employing this decoding order is without loss of optimality. In general, how should we prioritize the decoding order among private cache, common cache, private update, and common update? In the sequel, we will thus classify our achievabilities in terms of their decoding order. For a subset of the cases, we will also be able to show optimality.

A. Extension of the Gray–Wyner System

We first consider Configuration \((A_c|A_u) = (\{1\}, \{2\}, \{1, 2\}|\{1\}, \{2\})\) and Configuration \((A_c|A_u) = (\{1, 2\}|\{1\}, \{2\}, \{1, 2\})\), depicted in Figures 7 and 8 respectively. It can be seen that these configurations include the Gray–Wyner system [25] as a special case. To see this, in Configuration \((\{1\}, \{2\}, \{1, 2\}|\{1\}, \{2\})\) set \(R_{u,\{1\}} = R_{u,\{2\}} = 0\), and in Configuration \((\{1, 2\}|\{1\}, \{2\}, \{1, 2\})\) set \(R_{c,\{1,2\}} = 0\). We follow the principle “common → private” and establish the optimal rate regions of the two configurations in the following theorems.

**Theorem 2:** The rate region \(\mathcal{R}^* = (\{1\}, \{2\}, \{1, 2\}|\{1\}, \{2\})\) is the set of rate tuples \(R\) such that \(R_{u,\{1,2\}} = 0\),

\[
R_{c,\{1,2\}} \geq I(X; V_c|Y),
\]
Theorem 3: The rate region $R^*(\{1, 2\}, \{1\}, \{2\})$ is the set of rate tuples $R$ such that $R_{c,1} = R_{c,2} = 0$,
\[
\begin{align*}
R_{c,1} &\geq I(X; V_1 | V_c, Y), \\
R_{c,2} &\geq I(X; V_2 | V_c, Y), \\
R_{u,1} &\geq H(f_1(X, Y) | V_c, V_1, Y), \\
R_{u,2} &\geq H(f_2(X, Y) | V_c, V_2, Y),
\end{align*}
\]
for some conditional pmf $p_{V_c | X} p_{V_1 | V_c, X} p_{V_2 | V_c, X}$ satisfying $|V_c| \leq |X| + 4$, $|V_j| \leq |V_c||X| + 1$, $j \in \{1, 2\}$. 

Fig. 7. The source network with Configuration $(A_c | A_u) = (\{1\}, \{2\}, \{1, 2\}, \{1\}, \{2\})$.

Fig. 8. The source network with Configuration $(A_c | A_u) = (\{1, 2\}, \{1\}, \{2\}, \{1, 2\})$. 

\[
\begin{align*}
R_{c,1} &\geq I(X; V_1 | V_c, Y), \\
R_{c,2} &\geq I(X; V_2 | V_c, Y), \\
R_{u,1} &\geq H(f_1(X, Y) | V_c, V_1, Y), \\
R_{u,2} &\geq H(f_2(X, Y) | V_c, V_2, Y),
\end{align*}
\]
for some conditional pmf $p_{V_c | X} p_{V_1 | V_c, X} p_{V_2 | V_c, X}$ satisfying $|V_c| \leq |X| + 3$, $|V_u| \leq |V_c||X||Y| + 2$. 

$\text{Fig. 7. The source network with Configuration } (A_c | A_u) = (\{1\}, \{2\}, \{1, 2\}, \{1\}, \{2\}).$ 

$\text{Fig. 8. The source network with Configuration } (A_c | A_u) = (\{1, 2\}, \{1\}, \{2\}, \{1, 2\}).$
As in the Gray–Wyner system, the optimal order is to first recover the common descriptions at both decoders first, and then each decoder recovers their respective private descriptions. Moreover, to account for the additional update encoder with private links connected to the decoders in Configuration $(\mathcal{A}_c|\mathcal{A}_u) = ((\{1\}, \{2\}, \{1, 2\})|\{1\}, \{2\})$, the private descriptions are decoded successively. Similarly, to account for the additional cache encoder that has a common link to both decoders in Configuration $(\{1\}, \{2\}, \{1, 2\})|\{1\}, \{2\})$, the common descriptions are decoded successively. In summary, the rate expressions in Theorems 2 and 3 are established with the following optimal decoding orders:

1) common cache $\rightarrow$ private cache $\rightarrow$ private update, and
2) common cache $\rightarrow$ common update $\rightarrow$ private update.

The same principle can be extended to the general $L$-user case. For all configurations such that there is no conflict between the two decoding orders “cache $\rightarrow$ update” and “common $\rightarrow$ private”, a single-letter characterization of the optimal rate region can be found. For the rest of the section, we consider three configurations in which there is a conflict between “cache $\rightarrow$ update” and “common $\rightarrow$ private”.

B. Sequential Successive Refinement

Configuration $(\mathcal{A}_c|\mathcal{A}_u) = ((\{2\}, \{1, 2\})|\{2\}, \{1, 2\})$ (see Figure 9) can be seen as a special case of the distributed successive refinement problem. In the first stage, the cache encoder and the private encoder each send a coarse description to both decoders, and then in the second stage they each send a refined description only to Decoder 2. Here we can see a conflict between “cache $\rightarrow$ update” and “common $\rightarrow$ private”. It is not clear in the first place whether Decoder 2 should decode the common update content first or the private cache content first.
As shown in the following theorem, it turns out that the following decoding order is optimal for Decoder 2:

common cache → common update → private cache → private update.

**Theorem 4:** The rate region \( \mathcal{R}^*([2], [1, 2]|[2], [1, 2]) \) is the set of rate tuples \( \mathbf{R} \) such that \( R_{c,(1)} = R_{u,(1)} = 0 \),

\[
R_{c,(1,2)} \geq I(X; V_c|Y),
\]

\[
R_{c,(1,2)} + R_{c,(2)} \geq I(X; V_c|Y) + I(X; V_2|f_1(X, Y), V_c, Y),
\]

\[
R_{u,(1,2)} \geq H(f_1(X, Y)|V_c, Y),
\]

\[
R_{u,(1,2)} + R_{u,(2)} \geq H(f_1(X, Y)|V_c, Y) + H(f_2(X, Y)|V_2, f_1(X, Y), V_c, Y),
\]

for some conditional pmf \( p_{V_2, V_c|X} \) satisfying \( |V_c| \leq |X| + 3 \) and \( |V_2| \leq |V_c||X| + 1 \).

**Proof:** **Achievability:** The achievability can be proved by applying Theorem 1 and its straightforward extension. Here we provide a high-level description. Consider a simple two-stage coding. In the first stage, we use a multiple description code which follows the achievability for the single-user case and each encoder sends its generated message through its common link. Since both messages \( (M_{c,(1,2)}, M_{c,(2)}) \) are also received by Decoder 2, Decoder 2 can also learn \( (v_c^k, \{f_1(x_i, y_i)\}_{i \in [1:k]}) \). Then, in the second stage, we use another multiple description code which follows the achievability for the single-user case but with the augmented side information \( (v_c^k, \{f_1(x_i, y_i)\}_{i \in [1:k]}, y^k) \). Once the messages are generated, each encoder can divide its message of the second stage into two parts, one of which is sent through the common link and the other is sent through the private link.

**Converse:** Denote \( S_{1i} = f_1(X_i, Y_i) \) and \( S_{2i} = f_2(X_i, Y_i) \) for \( i \in [1:k] \). The rates \( R_{c,(1,2)} \) and \( R_{u,(1,2)} \) can be lower bounded in the same manner as the single-user case and thus the details are omitted. Denote \( V_{ci} = (M_{c,(1,2)}, S_{i-1,1}, Y^{[1:k] \setminus \{i\}}) \). Now consider the bounds on \( R_{c,(1,2)} + R_{c,(2)} \) and \( R_{u,(1,2)} + R_{u,(2)} \). First, we have

\[
k(R_{c,(1,2)} + R_{c,(2)}) \geq H(M_{c,(1,2)}, M_{c,(2)}|Y^k)
\]

\[
= I(X^k; M_{c,(1,2)}, M_{c,(2)}|Y^k)
\]

\[
= I(S_{1,1}^k, X^k; M_{c,(1,2)}, M_{c,(2)}|Y^k)
\]
the proof follows from the standard time-sharing argument, letting
where
\[ a_i \leq H(x_i) = a \]
follows since
\[ H(x_i) \geq H(x_{i+1}) + H(y_i) + H(u_i) + H(x_{i+1} | u_i, x_i) \]
form a Markov chain. Second, we have
\[
k(R_{u,1}^{(2)})
\geq H(M_{u,1}, M_{c,1}^{(2)}, Y^k) + H(M_{u,2})
\geq H(S_1^k, M_{u,1}, M_{c,1}, Y^k) + H(M_{u,2}) - k\epsilon_k
\geq \sum_{i=1}^{k} H(S_1^i | M_{c,1}, S_1^{i-1}, Y^k) + H(M_{u,1} | S_1^k, M_{c,1}, Y^k) + H(M_{u,2}) - k\epsilon_k
\geq \sum_{i=1}^{k} H(S_1 | V_i, Y_i) + H(M_{u,1}, M_{u,2} | S_1^k, M_{c,1}, M_{c,2}, Y^k) - k\epsilon_k'
\geq \sum_{i=1}^{k} H(S_1^i | V_i, Y_i) + H(S_2^k, M_{u,1}, M_{u,2} | S_1^k, M_{c,1}, M_{c,2}, Y^k) - k(\epsilon_k' + \epsilon_k'')
\geq \sum_{i=1}^{k} H(S_1^i | V_i, Y_i) + H(S_2^k | S_1^k, M_{c,1}, M_{c,2}, Y^k) - k(\epsilon_k' + \epsilon_k'')
\geq \sum_{i=1}^{k} H(S_1^i | V_i, Y_i) + \sum_{i=1}^{k} H(S_2^i | S_2^{i-1}, M_{c,1}, M_{c,2}, Y^k) - k(\epsilon_k' + \epsilon_k'')
\geq \sum_{i=1}^{k} H(S_1^i | V_i, Y_i) + \sum_{i=1}^{k} H(S_2^i | S_1^i, V_i, V_i, Y_i) - k(\epsilon_k' + \epsilon_k''),
\]
where (a) follows since \((X_i, Y_i, S_{1i})\) is independent of \((X_{i-1}, S_1^{i-1}, Y_{i-1})\). For the last step, we define \(V_{2i} := (M_{c,1}, X_{i-1}, S_{1i+1}^k)\). Note that \((V_i, V_{2i}) \rightarrow X_i \rightarrow Y_i\) form a Markov chain. Second, we have
\[
k(R_{u,1} + R_{u,2})
\geq H(M_{u,1}, M_{c,1}^{(2)}, Y^k) + H(M_{u,2})
\geq H(S_1^k, M_{u,1}, M_{c,1}, Y^k) + H(M_{u,2}) - k\epsilon_k
\geq \sum_{i=1}^{k} H(S_1^i | M_{c,1}, S_1^{i-1}, Y^k) + H(M_{u,1} | S_1^k, M_{c,1}, Y^k) + H(M_{u,2}) - k\epsilon_k
\geq \sum_{i=1}^{k} H(S_1^i | V_i, Y_i) + H(M_{u,1}, M_{u,2} | S_1^k, M_{c,1}, M_{c,2}, Y^k) - k\epsilon_k'
\geq \sum_{i=1}^{k} H(S_1^i | V_i, Y_i) + H(S_2^k, M_{u,1}, M_{u,2} | S_1^k, M_{c,1}, M_{c,2}, Y^k) - k(\epsilon_k' + \epsilon_k'')
\geq \sum_{i=1}^{k} H(S_1^i | V_i, Y_i) + H(S_2^k | S_1^k, M_{c,1}, M_{c,2}, Y^k) - k(\epsilon_k' + \epsilon_k'')
\geq \sum_{i=1}^{k} H(S_1^i | V_i, Y_i) + \sum_{i=1}^{k} H(S_2^i | S_2^{i-1}, M_{c,1}, M_{c,2}, Y^k) - k(\epsilon_k' + \epsilon_k'')
\geq \sum_{i=1}^{k} H(S_1^i | V_i, Y_i) + \sum_{i=1}^{k} H(S_2^i | S_1^i, V_i, V_i, Y_i) - k(\epsilon_k' + \epsilon_k''),
\]
where (a) and (b) follow from the data processing inequality and Fano’s inequality. The rest of the proof follows from the standard time-sharing argument, letting \(k \to \infty\), and the fact that
\[
I(f_1(X,Y); V|Y) + I(X; V, V_2| f_1(X,Y), Y)
\]
Fig. 10. The source network with Configuration $\mathcal{A}_c,A_u = (\{1\}|\{2\}, \{1, 2\})$.

$$= I(X; V_c|Y) + I(X; V_2|f_1(X, Y), V_c, Y).$$

The cardinality bounds on $\mathcal{V}_c$ and $\mathcal{V}_2$ can be proved using the convex cover method [41], Appendix C.

C. Configuration $(\{1\}|\{2\}, \{1, 2\})$

In this configuration (see Figure 10), User 1 has a private cache, User 2 has a private update, and additionally they both receive a common update. Again, there is a conflict at Decoder 1 because it receives both private cache and common update. Unfortunately, the optimal rate region $\mathcal{R}^*=(\{1\}|\{2\}, \{1, 2\})$ is unknown. We first present an inner bound and an outer bound with the principle “cache $\rightarrow$ update”.

**Proposition 4 (Inner Bound “cache $\rightarrow$ update”):** A rate tuple $\mathbf{R}$ belongs to $\mathcal{R}^*=(\{1\}|\{2\}, \{1, 2\})$ if its elements satisfy $R_{c,\{2\}} = R_{c,\{1,2\}} = R_{u,\{1\}} = 0$, and

$$R_{c,\{1\}} > I(X; V|Y),$$

$$R_{u,\{1,2\}} > I(X; U|V,Y),$$

$$R_{u,\{1,2\}} + R_{u,\{2\}} > I(V, X; U|Y) + H(f_2(X, Y)|U,Y),$$

for some conditional pmf $p_{V|X}p_{U|V,X,Y}$ such that $H(f_1(X, Y)|U,V,Y) = 0$.

**Proof:** Fix the conditional pmf $p_{V|X}p_{U|V,X,Y}$ such that $H(f_1(X, Y)|U,V,Y) = 0$. Denote $s_{1i} = f_1(x_i, y_i)$ and $s_{2i} = f_2(x_i, y_i), i \in [1 : k]$. Since $H(f_1(X, Y)|U,V,Y) = 0$, there exists a function $g$ such that $g(U,V,Y) = f_1(X, Y)$ almost surely.

**Codebook generation:** Randomly and independently generate $[2^{kR_{c,\{1\}}}][2^{kR_1}]$ sequences $u^k(\ell_v, \ell_1), \ell_v \in [1 : [2^{kR_{c,\{1\}}}]]$ and $\ell_1 \in [1 : [2^{kR_1}]]$, each according to $\prod_{i=1}^k p_V(v_i)$. Also, for each $y^k \in \mathcal{Y}^k$, randomly and independently generate $[2^{kR_2}][2^{kR_3}]$ sequences $u^k(\ell_2, \ell_3, y^k)$,
and independently assign a bin index \( m(s_2^k) \) to each sequence \( s_2^k \in S_2^k \) according to a uniform pmf over \([1 : 2^{kR_2}]\). The codebooks are revealed to all nodes.

**Encoding:** Given \( x^k \), the cache encoder finds an index pair \((\ell_v, \ell_1)\) such that \((x^k, v^k(\ell_v, \ell_1)) \in T_{c}^{(k)}(X, V)\). If there is more than one such index, it selects the one that minimizes \( \ell_v [2^{kR_1}] + \ell_1 \). If there is no such index, it sets \((\ell_v, \ell_1) = (1, 1)\). Then, the cache encoder sends the index \( \ell_v \) to Decoder 1.

Given \((x^k, y^k)\), the update encoder first finds \( v^k(\ell_v, \ell_1) \) as done by the cache encoder. Then, the update encoder finds an index pair \((\ell_2, \ell_3)\) such that \((v^k(\ell_v, \ell_1), x^k, u^k(\ell_2, \ell_3, y^k)) \in T_{u}^{(k)}(V, X, U|y^k)\). Similarly, if there is more than one such index, it selects the one that minimizes \( \ell_2 [2^{kR_3}] + \ell_3 \). If there is no such index, it sets \((\ell_2, \ell_3) = (1, 1)\). Then, the update encoder sends the index \( \ell_2 \) to Decoders 1 and 2 through the common update link. Finally, the update encoder sends the indices \((\ell_3, m(s_2^k))\) to Decoder 2 through the private update link and the common update link. Therefore, we have the conditions \( R_{u,\{1,2\}} \geq R_2 \) and \( R_{u,\{1,2\}} + R_{u,\{2\}} \geq R_2 + R_3 + R_4 \).

**Decoding:** Let \( \epsilon > \epsilon' \). Upon seeing \((\ell_v, \ell_2)\), Decoder 1 first finds the unique index \( \hat{\ell}_1 \) such that \((v^k(\ell_v, \hat{\ell}_1), y^k) \in T_{c}^{(k)}(V, Y)\); otherwise it sets \( \hat{\ell}_1 = 1 \). Then, Decoder 1 finds the unique index \( \hat{\ell}_3 \) such that \((v^k(\ell_v, \hat{\ell}_1), u^k(\ell_2, \hat{\ell}_3)) \in T_{c}^{(k)}(V, U|y^k)\); otherwise it sets \( \hat{\ell}_3 = 1 \). Decoder 1 then computes the reconstruction sequence \( \hat{s}_1^i = g(u_i(\ell_2, \hat{\ell}_3), v_i(\ell_v, \hat{\ell}_1), y_i) \) for \( i \in [1 : k] \).

Upon seeing \((\ell_2, \ell_3, m)\), Decoder 2 declares the estimate \( \hat{s}_2^k \) if it is the unique sequence with bin index \( m \) such that \((\hat{s}_2^k, u^k(\ell_2, \ell_3, y^k)) \in T_{c}^{(k)}(S_2, U|y^k)\); otherwise it declares an error.

**Analysis:** The following can be shown from the standard typicality arguments. Note that each event is conditioned on the success of the previous events.

1) If \( R_{c,\{1\}} + R_1 > I(X; V) \), then the cache encoder finds an index pair \((\ell_v, \ell_1)\) with high probability. Note that it implies that \((x^k, y^k, v^k(\ell_v, \ell_1)) \in T_{c}^{(k)}(X, Y, V)\) with high probability.
2) If \( R_1 < I(Y; V) \), then Decoder 1 identifies the index \( \ell_1 \) with high probability.
3) If \( R_2 + R_3 > I(V, X; U|Y) \), then the update encoder finds an index pair \((\ell_2, \ell_3)\) with high probability.
4) If \( R_3 < I(U; V|Y) \), then Decoder 1 identifies the index \( \ell_3 \) with high probability.
5) If \( R_4 > H(f_2(X, Y)|U, Y) \), then Decoder 2 recovers \( s_2^k \) correctly with high probability.
Finally, conditioned on the event that Decoder 1 recovers the sequences $u^k, v^k$ that are jointly typical with $(x^k, y^k)$, we have

\[
P \left( \bigcup_{i=1}^{k} \{ g(u_i, v_i, Y_i) \neq f_1(X_i, Y_i) \} \middle| (u^k, v^k, X^k, Y^k) \in \mathcal{T}_e^{(k)} \right)
\leq \sum_{i=1}^{k} P \left( \{ g(u_i, v_i, Y_i) \neq f_1(X_i, Y_i) \} \middle| (u^k, v^k, X^k, Y^k) \in \mathcal{T}_e^{(k)} \right)
\leq k(1 + \epsilon) P(g(U, V, Y) \neq f_1(X, Y))
= 0,
\]

where \((a)\) follows from the typical average lemma [41]. The rest of the proof follows from Fourier–Motzkin elimination.

**Proposition 5 (Outer Bound “cache → update”):** If $R \in \mathcal{R}^\ast(\{1\}|\{2\}, \{1, 2\})$, then its elements must satisfy $R_{c,\{2\}} = R_{c,\{1,2\}} = R_{u,\{1\}} = 0$ and the inequalities

\[
R_{c,\{1\}} > I(X; V|Y),
\]
\[
R_{u,\{1,2\}} > I(X; U|V, Y),
\]
\[
R_{u,\{1,2\}} + R_{u,\{2\}} > I(X; U|Y) + H(f_2(X, Y)|U, Y),
\]

for some conditional pmf $p_{V\mid X, P_{U\mid X, Y}}$ such that $H(f_1(X, Y)|U, V, Y) = 0$.

**Proof:** Denote $S_{1i} = f_1(X_i, Y_i)$ and $S_{2i} = f_2(X_i, Y_i)$ for $i \in [1 : k]$. The bounds (13) and (15) follow similar lines as in the converse proof of Theorem 1 and therein we set $V_i = (M_{c,\{1\}}, X^{i-1}, Y^{[1:k]\setminus\{i\}})$ and $U_i = (M_{u,\{1,2\}}, S_2^{i-1}, Y^{[1:k]\setminus\{i\}})$. Now we proceed to prove the bound (14) and the constraint $H(f_1(X, Y)|U, V, Y) = 0$. First, we have

\[
kR_{u,\{1,2\}} \geq H(M_{u,\{1,2\}}|M_{c,\{1\}}, Y^k)
= I(X^k; M_{u,\{1,2\}}|M_{c,\{1\}}, Y^k)
= \sum_{i=1}^{k} I(X_i; M_{u,\{1,2\}}|M_{c,\{1\}}, X^{i-1}, Y^k)
= \sum_{i=1}^{k} I(X_i; M_{u,\{1,2\}}, S_2^{i-1}, Y^{[1:k]\setminus\{i\}}|M_{c,\{1\}}, X^{i-1}, Y^k)
= \sum_{i=1}^{k} I(X_i; U_i|V_i, Y_i).
\]
Second, from the data processing inequality and Fano’s inequality, we have

\[
k\epsilon_k \geq H(S_k^i | \mathcal{M}_{c,\{1\}}, \mathcal{M}_{u,\{1,2\}}, Y^k)
\]

\[
= \sum_{i=1}^{k} H(S_1 | S_1^i, \mathcal{M}_{c,\{1\}}, \mathcal{M}_{u,\{1,2\}}, Y^k)
\]

\[
\geq \sum_{i=1}^{k} H(S_1 | X^i, \mathcal{M}_{c,\{1\}}, \mathcal{M}_{u,\{1,2\}}, Y^k)
\]

\[
= \sum_{i=1}^{k} H(S_1 | U_i, V_i, Y_i) \geq 0.
\]

The rest of the proof follows from the standard time-sharing argument and then letting \(k \to \infty\). 

Note that when the conditional pmf \(p_{V|X} p_{U|V,X,Y}\) is fixed, the inner bound in Proposition 4 differs from the outer bound in Proposition 5 only by the term \(I(U; V|X, Y)\) in the constraint on the sum rate.

Next, we present an inner bound based on the principle “common \(\to\) private”.

**Proposition 6 (Inner Bound “common \(\to\) private”):** A rate tuple \(R\) belongs to \(\mathcal{R}^*(\{1\},\{2\},\{1,2\})\) if its elements satisfy \(R_{c,\{2\}} = R_{c,\{1,2\}} = R_{u,\{1\}} = 0\), and

\[
R_{u,\{1,2\}} > I(X; U|Y, Q),
\]

\[
R_{c,\{1\}} > I(X; V|U, Y, Q),
\]

\[
R_{u,\{2\}} > H(f_2(X, Y)|U, Y, Q),
\]

for some conditional pmf \(p_Q p_{V|X} p_{U|X,Y,Q}\) with \(|Q| \leq 4, |U| \leq |\mathcal{X}||\mathcal{Y}| + 3,\) and \(|V| \leq |\mathcal{X}| + 1\) such that \(H(f_1(X, Y)|U, V, Y, Q) = 0\).

**D. Configuration \((\{1\},\{2\},\{1,2\})\)**

Configuration \((\{1\},\{2\},\{1,2\})\) (see Figure 11) is essentially the setup studied by Maddah-Ali and Niesen [1], where each user has a private cache and both receive a common update. We first present an achievability proof based on the principle “common \(\to\) private”.

Fig. 11. The source network with Configuration $\mathcal{A}_c|\mathcal{A}_u = ([1],[2]|\{1,2\})$. 

**Proposition 7 (Inner Bound “common → private”):** A rate tuple $\mathbf{R}$ belongs to $\mathcal{R}^*(\{1\},\{2\}|\{1,2\})$ if its elements satisfy $R_{c,\{1,2\}} = R_{u,\{1\}} = R_{u,\{2\}} = 0$, and 

$$R_{u,\{1,2\}} > I(X;U|Y,Q),$$

$$R_{c,\{1\}} > I(X;V_{1}|U,Y,Q),$$

$$R_{c,\{2\}} > I(X;V_{2}|U,Y,Q),$$

for some conditional pmf $p_{Q|P_{V_{1}|X,Q}P_{V_{2}|X,Q}P_{U|X,Y,Q}}$ with $|U| \leq |X||Y| + 4$, $|V_j| \leq |X| + 1$, $j \in \{1,2\}$, and $|Q| \leq 5$ such that $H(f_j(X,Y)|U,V_j,Y,Q) = 0$, $j \in \{1,2\}$.

We remark that replacing $p_{V_{1}|X,Q}p_{V_{2}|X,Q}$ by $p_{V_{1},V_{2}|X,Q}$ cannot enlarge the achievable rate region because all the involved terms only depend on the marginal distributions.

When we specialize Proposition 7 to the single-user case, i.e., set either $f_1(X,Y) \equiv 0$ or $f_2(X,Y) \equiv 0$, the decoding order is update first, cache second, which violates the principle “cache → update”. Some discussion on the decoding order “update first, cache second” is given in Appendix B. Since the optimality of “update first, cache second” is in question, we next provide an alternative achievable rate region that prioritize the principle “cache → update”.

**Proposition 8 (Inner Bound “cache → update”):** A rate tuple $\mathbf{R}$ belongs to $\mathcal{R}^*(\{1\},\{2\}|\{1,2\})$ if its elements satisfy $R_{c,\{1,2\}} = R_{u,\{1\}} = R_{u,\{2\}} = 0$, and 

$$R_{c,\{1\}} > I(X;V_{1}|Y),$$

$$R_{c,\{2\}} > I(X;V_{2}|Y),$$

$$R_{c,\{1\}} + R_{c,\{2\}} > I(X;V_{1}|Y) + I(X;V_{2}|Y) + I(V_{1};V_{2}|X),$$

$$R_{u,\{1,2\}} > \max\{I(V_{2},X;U|V_{1},Y), I(V_{1},X;U|V_{2},Y)\} + H(f_1(X,Y)|U,V_1,Y) + H(f_2(X,Y)|U,V_2,Y),$$
for some conditional pmf $p_{V_1, V_2|X} p_{U|V_1, V_2, X, Y}$.

**Proof Outline:** First, we compress $x^k$ into two descriptions $v^k_1$ and $v^k_2$ using joint typicality encoding and binning. Then, Decoder $\ell \in \{1, 2\}$ applies joint typicality decoding to recover the description $v^k_\ell$. The rate constraints (16)-(18) follow from the multivariate covering lemma, the fact that

$$I(\mathbf{X}; \mathbf{V}_1, \mathbf{V}_2) + I(\mathbf{V}_1; \mathbf{V}_2) = I(\mathbf{X}; \mathbf{V}_1) + I(\mathbf{X}; \mathbf{V}_2) + I(\mathbf{V}_1; \mathbf{V}_2|\mathbf{X}),$$

packing lemma, and Fourier–Motzkin elimination.

As Decoders 1 and 2 recover $v^k_1$ and $v^k_2$, respectively, the system can be treated as a special case of the Kaspi/Heegard–Berger problem with an informed encoder and then the achievability follows from [22] (see also [23]).

**Remark 1:** Let us briefly discuss the scenario in which the requests are only locally known (a configuration is depicted in Figure 2). In that scenario, every configuration with at least one common link can be simplified to the Kaspi/Heegard–Berger problem by setting some of the rates to zero. Following the principle “cache $\rightarrow$ update,” for each configuration we can develop an achievability with rate expressions similar to Proposition 8. For example, consider Configuration $R^*\big(\{1\}, \{2\}|\{1, 2\}\big)$ in which the requests are only locally known. It can be shown that the following rate region is achievable: the set of rate tuples $\mathbf{R}$ whose elements satisfy

$$R_{c,\{1, 2\}} = R_{u,\{1\}} = R_{u,\{2\}} = 0, \text{ and}$$

$$R_{c,\{1\}} > I(\mathbf{X}; \mathbf{V}_1|\mathbf{Y}_1),$$

$$R_{c,\{2\}} > I(\mathbf{X}; \mathbf{V}_2|\mathbf{Y}_2),$$

$$R_{c,\{1\}} + R_{c,\{2\}} > I(\mathbf{X}; \mathbf{V}_1|\mathbf{Y}_1) + I(\mathbf{X}; \mathbf{V}_2|\mathbf{Y}_2) + I(\mathbf{V}_1; \mathbf{V}_2|\mathbf{X}),$$

$$R_{u,\{1, 2\}} > \max\{I(\mathbf{V}_2, \mathbf{Y}_2, \mathbf{X}; \mathbf{U}|\mathbf{V}_1, \mathbf{Y}_1), I(\mathbf{V}_1, \mathbf{Y}_1, \mathbf{X}; \mathbf{U}|\mathbf{V}_2, \mathbf{Y}_2)\}$$

$$+ H(f_1(X, Y)|U, \mathbf{V}_1, \mathbf{Y}_1) + H(f_2(X, Y)|U, \mathbf{V}_2, \mathbf{Y}_2),$$

for some conditional pmf $p_{V_1, V_2|X} p_{U|V_1, V_2, X, Y_1, Y_2}$.

Finally, we present a simple outer bound which follows by combining several single-user bounds using a cut-set based argument.
**Proposition 9 (Outer Bound):** If $R \in R^\ast(\{1, 2\}|\{1, 2\})$, then its elements must satisfy $R_{c,(1,2)} = R_{u,(1)} = R_{u,(2)} = 0$ and the inequalities

$$R_{c,1} \geq I(X; V_1|Y),$$
$$R_{c,2} \geq I(X; V_2|Y),$$
$$R_{c,1} + R_{c,2} \geq I(X; V_1, V_2|Y),$$
$$R_{u,(1,2)} \geq \max \{H(f_1(X,Y)|V_1,Y), H(f_2(X,Y)|V_2,Y),$$
$$H(f_1(X,Y), f_2(X,Y)|V_1, V_2, Y)\},$$

for some $p_{V_1,V_2|X}$.

Finally, we consider two examples.

**Example 2 (Independent View Selection):** Let $X = (A, B)$, $Y = (Y_1, Y_2)$, where $A, B, Y_1, Y_2$ are i.i.d. drawn from Bernoulli$(1/2)$. Assume that

$$f_j(X,Y) = \begin{cases} 
  A & \text{if } Y_j = 0, \\
  B & \text{if } Y_j = 1,
\end{cases}$$

where $j \in \{1, 2\}$. Unfortunately, the optimal rate region is unknown, even for the symmetric case, i.e., $R_{c,(1)} = R_{c,(2)} = R_c$. Denote $R_u = R_{u,(1,2)}$. It can be easily checked that $(R_{c}, R_{u}) = (0, 1.5), (2, 0)$ are two extreme points of the symmetry-constrained optimal rate region. Next, borrowing the idea from the achievability in [1, Example 4], we substitute $V_1 = A$, $V_2 = B$, and

$$U = \begin{cases} 
  A & \text{if } (Y_1, Y_2) = (0, 0), \\
  \emptyset & \text{if } (Y_1, Y_2) = (0, 1), \\
  A \oplus B & \text{if } (Y_1, Y_2) = (1, 0), \\
  B & \text{if } (Y_1, Y_2) = (1, 1),
\end{cases}$$

into the rate expressions of Proposition 8. Then, the rate pair $(R_c, R_u) = (1, 0.5)$ is achievable.

The inner bound plotted in solid blue in Figure 12 follows by time sharing among $(0, 1.5), (1, 0.5)$, and $(2, 0)$. Through some relaxation of the outer bound in Proposition 9 we have that for any $R_c \geq 0$,

$$R_u \geq \max \left\{ \left(1 - \frac{R_c}{2}\right)^+ , \left(\frac{3}{2} - \frac{3}{2}R_c\right)^+ \right\}. $$

The above outer bound is plotted in dashed red in Figure 12.
Example 3 (Complementary View Selection): Let $A, B, Y$ be i.i.d. Bernoulli($1/2$) random variables and $X = (A, B)$. Assume that

$$ (f_1(X, Y), f_2(X, Y)) = \begin{cases} (A, B) & \text{if } Y = 0, \\ (B, A) & \text{if } Y = 1. \end{cases} $$

That is, the desired views of the two users are always complementary to each other. Again, the optimal rate region is unknown, even for the symmetric case, i.e., $R_{c,1} = R_{c,2} = R_c$.

Denote $R_u = R_{u,1,2}$. It is easy to see that $(R_c, R_u) = (0, 2), (2, 0)$ are two extreme points of the symmetry-constrained optimal rate region. Next, we consider the following choice of auxiliary random variables, which borrows the idea from the achievability in [1, Appendix]. Let $A_2, B_1$ be i.i.d. Bernoulli($q$) and denote $A_1 = A \oplus A_2$, $B_2 = B \oplus B_1$. We substitute $V_1 = A_1 \oplus B_1$, $V_2 = A_2 \oplus B_2$, and

$$ U = \begin{cases} (A_2, B_1) & \text{if } Y = 0, \\ (A_1, B_2) & \text{if } Y = 1, \end{cases} $$

into the rate expressions of Proposition 8. It can be verified that the rate pair $(R_c, R_u) = (1, 0.5)$ is achievable with $q = 0$ and $(R_c, R_u) = (0.5, 1)$ is achievable with $q = 1/2$. We remark that when $q = 1/2$, we have $I(V_1, V_2|X) = 1$, i.e., $V_1$ and $V_2$ are not conditionally independent given $X$. The inner bound plotted in solid blue in Figure 13 follows by time sharing among $(0, 2), (0.5, 1), (1, 0.5), (2, 0)$. Through some relaxation of the outer bound in Proposition 9, we have that for any $R_c \geq 0$,

$$ R_u \geq \max \left\{ \left( 1 - \frac{R_c}{2} \right)^+, (2 - 2R_c)^+ \right\}. $$
V. CONCLUSION

In this paper, we have formulated the caching problem as a multi-terminal source coding problem with side information. The key observation is that we can treat the requested data as a function of the whole data and the request. All forms of data and requests are simply modeled by random variables $X$ and $Y$, respectively, and their relation can be simply described by a function $f$. Thanks to the formulation, many coding techniques and insights can be directly borrowed from the well-developed source coding literature. For the single-user case, we have given a single-letter characterization of the optimal rate region and found closed-form expressions for two interesting cases. For the two-user case, we have shown that if the principles “cache $\rightarrow$ update” and “common $\rightarrow$ private” can be applied at the same time, then the optimal rate region of the considered configuration can be characterized. There remain many open problems, including:

1) If there is a conflict between the two principles, is joint decoding necessary? If not, does one of them always have higher priority?

2) We have seen that the two principles are sufficient for optimality, but are they also necessary in some cases?

For the second problem, we have some preliminary results left in Appendix B. We believe that the solutions to these problems will provide a better understanding of the class of source coding problems considered in this paper.
APPENDIX A: THE SINGLE-REQUEST MODEL

Here we present the “single-request” interpretation of the Maddah-Ali–Niesen model. There is a large database that could be denoted as an independent and identically distributed sequence $X_1, X_2, X_3, \ldots$, and each user has only one request, designating the part of the database that is of interest to the user. The database and the requests are independent. The collection of requests is represented by $Y$ and is globally known to all users. Figure 14 depicts the single-user case of the single-request model.

The formal problem statement is the same as in Section II except that $X$ and $Y$ are independent and only one instance is drawn from the pmf $p_Y$. The following theorem implies that under the condition that $X$ and $Y$ are independent, the optimal rate region of the single-request model is the same as the optimal rate region of the model defined in Section II. Furthermore, using similar arguments as in the proof of Theorem 5, it can be shown that all inner and outer bounds derived in this paper are valid for the single-request model.

**Theorem 5:** Consider the single-user case of the single-request model. The optimal rate region $R^*$ is the set of rate pairs $(R_c, R_u)$ such that

$$R_c \geq I(X; V),$$

$$R_u \geq H(f(X, Y)|V, Y),$$

for some conditional pmf $p_{V|X}$, where $|\mathcal{V}| \leq |\mathcal{X}| + 1$.

**Proof:** The proof of achievability follows from random coding arguments. Here we provide a high-level description of the coding scheme. In the first stage, the cache encoder applies joint typicality encoding to find a sequence $v^k$ such that $(x^k, v^k)$ are jointly typical and then sends the index of $v^k$. For the second stage, we prepare $|\mathcal{Y}|$ codebooks. Each codebook $y \in \mathcal{Y}$ is a Slepian–Wolf code for the source $f(X, y)$ with side information $V$. Since the request $y$ is
known to the update encoder and the decoder, the decoder knows the codebook used by the update encoder. Upon seeing \((x^k, y)\), the update encoder first computes \(\{f(x_i, y)\}_{i \in [1:k]}\) and then uses the codebook \(y\) to generate the update message. Then, the decoder uses the received update message and the decoded sequence \(v^k\) from the first stage to recover \(\{f(x_i, y)\}_{i \in [1:k]}\). Finally, it can be shown that the error probability vanishes as \(k \to \infty\) if \(R_c > I(X; V)\) and

\[
R_u > \sum_{y \in \mathcal{Y}} p_Y(y) H(f(X, y) | V)
= \sum_{y \in \mathcal{Y}} p_Y(y) H(f(X, y) | V, Y = y)
= H(f(X, Y) | V, Y),
\]

where \((a)\) follows since \((f(X, y), V)\) is independent of the event \(\{Y = y\}\).

The converse can be established as follows. Denote \(S_i = f(X_i, Y), i \in [1 : k]\). First, we have

\[
k R_c \geq H(M_c)
= I(X^k; M_c)
= \sum_{i=1}^{k} I(X_i; M_c | X^{i-1})
= \sum_{i=1}^{k} I(X_i; M_c, X^{i-1})
\geq \sum_{i=1}^{k} I(X_i; V_i),
\]

where \((a)\) follows since \(X_i\) is independent of \(X^{i-1}\). For the last step, we define \(V_i := (M_c, S^{i-1})\). Note that \(V_i \rightarrow X_i \rightarrow Y\) form a Markov chain. Next, we have

\[
k R_u \geq H(M_u)
\geq H(M_u | M_c, Y)
= H(S^k, M_u | M_c, Y) - H(S^k | M_c, M_u, Y)
\geq H(S^k | M_c, Y) - k \epsilon_k
= \sum_{i=1}^{k} H(S_i | M_c, S^{i-1}, Y) - k \epsilon_k
= \sum_{i=1}^{k} H(S_i | V_i, Y) - k \epsilon_k,
\]
where (a) follows from the data processing inequality and Fano’s inequality. The rest of the proof follows from the standard time-sharing argument and then letting \( k \to \infty \).

**Appendix B: Discussion on the Principle “Cache \( \to \) Update”**

Here we focus on the single-user case and discuss the necessity of the principle “cache \( \to \) update”. First, we observe that the system depicted in Figure [I] can be considered as a special case of distributed lossy compression. To facilitate further discussion, let us consider the distributed compress–bin scheme with successive decoding. There are two possible decoding orders for successive decoding: One of them follows the principle “cache \( \to \) update,” while the other does not.

Denote by \( \mathcal{R}_{c \to u} \) the set of rate pairs \( (R_c, R_u) \) such that
\[
R_c > I(X; V | Y), \\
R_u > I(X; U | V, Y),
\]
for some conditional pmf \( p_{V | X P_U | X, Y} \) such that \( H(f(X, Y) | U, V, Y) = 0 \). Also, denote by \( \mathcal{R}_{u \to c} \), the set of rate pairs \( (R_c, R_u) \) such that
\[
R_u > I(X; U | Y, Q), \\
R_c > I(X; V | U, Y, Q),
\]
for some conditional pmf \( p_{Q | X P_U | X, Y, Q} \) such that \( H(f(X, Y) | U, V, Y, Q) = 0 \). It can be shown that the rate region \( \mathcal{R}_{c \to u} \) is achievable by the distributed compress–bin scheme with successive decoding in the order: cache first, update second. If we set \( U = f(X, Y) \), then it is easy to see that \( \mathcal{R}_{c \to u} = \mathcal{R}^* \). Also, it can be shown that the rate region \( \mathcal{R}_{u \to c} \) is achievable by the distributed compress–bin scheme with successive decoding in the order: update first, cache second. Since \( \mathcal{R}_{u \to c} = \mathcal{R}^* \) implies that the principle “cache \( \to \) update” is unnecessary, we are interested in whether the equality in \( \mathcal{R}_{u \to c} \subseteq \mathcal{R}^* \) holds under all circumstances.

Unfortunately, the problem remains open. In the following, we provide two conditions to check whether \( \mathcal{R}_{u \to c} \subseteq \mathcal{R}^* \) holds with equality. The condition given in Proposition [10] is a necessary and sufficient condition and the condition given in Proposition [11] is a sufficient condition. Before that, we present two lemmas regarding the optimal rate region \( \mathcal{R}^* \).

**Lemma 1:** If \( (R_c, R_u) \in \mathcal{R}^* \), then \( (R_c - a, R_u + a) \in \mathcal{R}^* \) for all \( a \in [0, R_c] \).
**Proof:** Assume that \((R_c, R_u) \in \mathcal{R}^*\) and fix any \(a \in [0, R_c]\). Then, time sharing between \((0, R_u^*)\) and \((R_c, R_u)\) asserts that
\[
\left( R_c - a, R_u + \frac{R_u^* - R_u}{R_c} a \right) \in \mathcal{R}^*.
\]
Then, Corollary 1 implies that
\[
\frac{R_u^* - R_u}{R_c} \leq 1,
\]
so it holds that \((R_c - a, R_u + a) \in \mathcal{R}^*\). \(\blacksquare\)

**Lemma 2:** Let \((R_c, R_u)\) be an extreme point of \(\mathcal{R}^*\) such that \(R_u > 0\). Then, \((R_c + a, R_u - a) \notin \mathcal{R}^*\) for all \(a > 0\).

**Proof:** Recall that \(R_u^* = H(f(X,Y)|Y)\). Assume that \((R_c + a, R_u - a) \in \mathcal{R}^*\). Then, time sharing between \((0, R_u^*)\) and \((R_c + a, R_u - a)\) asserts that
\[
\left( R_c, \frac{aR_u^* + R_c(R_u - a)}{R_c + a} \right) \in \mathcal{R}^*.
\]
However, Corollary 1 implies that
\[
\frac{aR_u^* + R_c(R_u - a)}{R_c + a} \leq R_u,
\]
which contradicts the assumption that \((R_c, R_u)\) is an extreme point. \(\blacksquare\)

**Proposition 10:** We have \(\mathcal{R}_{u\rightarrow c} = \mathcal{R}^*\) if and only if every extreme point in \(\mathcal{R}^*\) can be achieved by some conditional pmf \(p_{V|X}p_{U|X,Y}\) such that \(I(U;V|Y) = 0\).

**Proof:** Without loss of generality, we can set \(Q = \emptyset\) since we consider only the extreme points. The key observation is that
\[
I(U;V|Y) = I(X;V|Y) - I(X;V|U,Y) = I(X;U|Y) - I(X;U|V,Y).
\] \(\text{(19)}\)

\((\Rightarrow):\) It is clear that \((0, R_u^*)\) is an extreme point of both \(\mathcal{R}_{u\rightarrow c}\) and \(\mathcal{R}_{c\rightarrow u}\) and can be achieved with \(I(U;V|Y) = 0\). Next, consider any extreme point \((R_c, R_u)\) of \(\mathcal{R}^*\) with \(R_c > 0\). Since we assume that \(\mathcal{R}_{u\rightarrow c} = \mathcal{R}^*\), the point \((R_c, R_u)\) is also an extreme point of \(\mathcal{R}_{u\rightarrow c}\). Therefore, there exists a conditional pmf \(p_{V|X}p_{U|X,Y}\) such that \((I(X;V|U,Y), I(X;U|Y)) = (R_c, R_u)\). Then, Expression \((\text{19})\) implies that \((R_c + I(U;V|Y), R_u - I(U;V|Y)) \in \mathcal{R}_{c\rightarrow u} = \mathcal{R}^* = \mathcal{R}_{u\rightarrow c}\). Since \((R_c, R_u)\) is an extreme point, Lemma 2 implies that \(I(U;V|Y) = 0\).
(⇐): It suffices to show that every extreme point of \( \mathcal{R}^* \) also lies in \( \mathcal{R}_{u \rightarrow c} \). Let \((R_c, R_u)\) be any extreme point of \( \mathcal{R}^* \). From the assumption, there exists a conditional pmf \( p_{V|X}p_{U|X,Y} \) achieving \((R_c, R_u)\) such that \( I(U; V|Y) = 0 \). Then, it implies that

\[
(R_c, R_u) = (I(X; V|Y), I(X; U, V, Y)) = (I(X; V|U, Y), I(X; V|Y))
\]

lies in \( \mathcal{R}_{u \rightarrow c} \).

**Proposition 11:** If every extreme point in \( \mathcal{R}^* \) can be achieved by some conditional pmf \( p_{V|X}p_{U|X,Y} \) such that \( H(V|X) = 0 \), then \( \mathcal{R}_{u \rightarrow c} = \mathcal{R}^* \).

**Proof:** It suffices to show that every extreme point in \( \mathcal{R}^* \) lies in \( \mathcal{R}_{u \rightarrow c} \). Consider any extreme point \((R_c, R_u) \in \mathcal{R}^* \). Since \( \mathcal{R}^* = \mathcal{R}_{c \rightarrow u} \), there exists a conditional pmf \( p_{V|X}p_{U|X,Y} \) such that

\[
R_c = I(X; V|Y),
\]

\[
R_u = I(X; U|V, Y).
\]

Now consider any \( y \in \mathcal{Y} \). Conditioned on \( Y = y \), the functional representation lemma [41, Appendix B] says that there exists a random variable \( U^{(y)} \) of cardinality \( |U^{(y)}| \leq |\mathcal{V}|(|\mathcal{U}| - 1) + 1 \) such that

1) \( H(U^{(y)}|U, Y = y) = 0 \),
2) \( I(U^{(y)}; V|Y = y) = 0 \),
3) \( I(X; U^{(y)}|U, V, Y = y) = 0 \).

Therefore, we have \( I(U^{(y)}; V|Y) = 0 \) and \( I(X; U|V, Y) = I(X; U^{(y)}|V, Y) \). Together with (19), it implies that

\[
R_c = I(X; V|U^{(y)}, Y),
\]

\[
R_u = I(X; U^{(y)}|Y).
\]

Finally, from the assumption \( H(V|X) = 0 \), we have \( p_{U^{(y)}, V|X,Y} = p_{V|X}p_{U^{(y)}|X,Y} \) and thus \((R_c, R_u) \in \mathcal{R}_{u \rightarrow c} \).

We remark that if \( X \) and \( Y \) are independent, then for the cases of independent and nested source components, Proposition [11] implies that \( \mathcal{R}_{u \rightarrow c} = \mathcal{R}^* \).
REFERENCES

[1] M. A. Maddah-Ali and U. Niesen, “Fundamental limits of caching,” *IEEE Trans. Inf. Theory*, vol. 60, pp. 2856–2867, May 2014.
[2] U. Niesen and M. A. Maddah-Ali, “Coded caching with nonuniform demands,” in *arXiv:1308.0178[cs.IT]*, Mar. 2014.
[3] M. A. Maddah-Ali and U. Niesen, “Decentralized coded caching attains order-optimal memory-rate tradeoff,” in *arXiv:1301.5848[cs.IT]*, Mar. 2014.
[4] U. Niesen and M. A. Maddah-Ali, “Coded caching for delay-sensitive content,” in *arXiv:1407.4489[cs.IT]*, Jul. 2014.
[5] R. Pedarsani, M. A. Maddah-Ali, and U. Niesen, “Online coded caching,” in *Proc. IEEE Int. Conf. Commun. (ICC)*, Sydney, Australia, Jun. 2014.
[6] N. Karamchandani, U. Niesen, M. A. Maddah-Ali, and S. Diggavi, “Hierarchical coded caching,” in *arXiv:1403.7007[cs.IT]*, Jun. 2014.
[7] M. Ji, A. M. Tulino, J. Llorca, and G. Caire, “Caching and coded multicasting: Multiple groupcast index coding,” in *Proc. IEEE Global Conf. Signal Info. Processing (GlobalSIP)*, Atlanta, GA, Dec. 2014.
[8] A. Sengupta, R. Tandon, and T. C. Clancy, “Fundamental limits of caching with secure delivery,” *IEEE Trans. Inf. Forensics Security*, vol. 10, pp. 355–370, Feb. 2015.
[9] M. Ji, G. Caire, and A. F. Molisch, “Wireless device-to-device caching networks: Basic principles and system performance,” in *arXiv:1305.5216[cs.IT]*, Apr. 2014.
[10] J. Hachem, N. Karamchandani, and S. Diggavi, “Coded caching for heterogeneous wireless networks with multi-level access,” in *arXiv:1404.6560[cs.IT]*, Apr. 2014.
[11] M. Ji, A. M. Tulino, J. Llorca, and G. Caire, “Order-optimal rate of caching and coded multicasting with random demands,” in *arXiv:1502.03124[cs.IT]*, Feb. 2015.
[12] J. Zhang, X. Lin, and X. Wang, “Coded caching under arbitrary popularity distributions,” in *Proc. Information Theory and Applications Workshop (ITA)*, San Diego, CA, Feb. 2015.
[13] D. Slepian and J. Wolf, “Noiseless coding of correlated information sources,” *IEEE Trans. Inf. Theory*, vol. IT-19, pp. 471–480, Jul. 1973.
[14] A. D. Wyner and J. Ziv, “The rate-distortion function for source coding with side information at the decoder,” *IEEE Trans. Inf. Theory*, vol. IT-22, pp. 1–10, Jan. 1976.
[15] A. Orlitsky and J. R. Roche, “Coding for computing,” *IEEE Trans. Inf. Theory*, vol. 47, pp. 903–917, Mar. 2001.
[16] A. D. Wyner, “On source coding with side information at the decoder,” *IEEE Trans. Inf. Theory*, vol. IT-21, pp. 294–300, May 1975.
[17] R. Ahlswede and J. Körner, “Source coding with side information and a converse for degraded broadcast channels,” *IEEE Trans. Inf. Theory*, vol. IT-21, pp. 629–637, Nov. 1975.
[18] A. Kaspi, “Rate-distortion function when side-information may be present at the decoder,” *IEEE Trans. Inf. Theory*, vol. 40, pp. 2031–2034, Nov. 1994.
[19] C. Heegard and T. Berger, “Rate distortion when side information may be absent,” *IEEE Trans. Inf. Theory*, vol. IT-31, pp. 727–734, Nov. 1985.
[20] A. Sgarro, “Source coding with side information at several decoders,” *IEEE Trans. Inf. Theory*, vol. IT-23, pp. 179–182, Mar. 1977.
[21] R. Timo, T. J. Oechtering, and M. Wigger, “Source coding problems with conditionally less noisy side information,” *IEEE Trans. Inf. Theory*, vol. 60, pp. 5516–5532, Sep. 2014.
[22] E. Perron, S. N. Diggavi, and I. E. Telatar, “On the role of encoder side-information in source coding for multiple receivers,” in Proc. IEEE Int. Symp. Information Theory (ISIT), Seattle, WA, Jul. 2006.

[23] T. Laich and M. Wigger, “Utility of encoder side information for the lossless Kaspi/Heegard-Berger problem,” in Proc. IEEE Int. Symp. Information Theory (ISIT), Istanbul, Turkey, Jul. 2013.

[24] R. Timo, T. Chan, and A. Grant, “Rate distortion with side-information at many decoders,” IEEE Trans. Inf. Theory, vol. 57, pp. 5240–5257, Aug. 2011.

[25] R. M. Gray and A. D. Wyner, “Source coding for a simple network,” Bell Syst. Tech. J., vol. 53, pp. 1681–1721, Nov. 1974.

[26] R. Timo, A. Grant, T. Chan, and G. Kramer, “Source coding for a simple network with receiver side-information,” in Proc. IEEE Int. Symp. Information Theory (ISIT), Toronto, Canada, Jul. 2008.

[27] O. Shayevitz and M. Wigger, “On the capacity of the discrete memoryless broadcast channel with feedback,” IEEE Trans. Inf. Theory, vol. 53, pp. 1681–1721, Mar. 2013.

[28] V. N. Koshelev, “Hierarchical coding of discrete sources,” Probl. Pered. Inform., vol. 16, no. 3, pp. 31–49, 1980.

[29] W. H. R. Equitz and T. M. Cover, “Successive refinement of information,” IEEE Trans. Inf. Theory, vol. 37, pp. 269–275, Mar. 1991.

[30] B. Rimoldi, “Successive refinement of information: Characterization of the achievable rates,” IEEE Trans. Inf. Theory, vol. 40, pp. 253–259, Jan. 1994.

[31] Y. Steinberg and N. Merhav, “On successive refinement for the Wyner–Ziv problem,” IEEE Trans. Inf. Theory, vol. 50, pp. 1636–1654, Aug. 2004.

[32] S. N. Diggavi and C. Tian, “Side-information scalable source coding,” IEEE Trans. Inf. Theory, vol. 54, pp. 5591–5608, Dec. 2008.

[33] E. Akyol, U. Mitra, E. Tuncel, and K. Rose, “On scalable coding in the presence of decoder side information,” in Proc. IEEE Int. Symp. Information Theory (ISIT), Honolulu, HI, Jun. 2014.

[34] C.-Y. Wang and M. C. Gastpar, “On distributed successive refinement with lossless recovery,” in Proc. IEEE Int. Symp. Information Theory (ISIT), Honolulu, HI, Jun. 2014.

[35] E. Akyol, U. Mitra, E. Tuncel, and K. Rose, “Source coding in the presence of exploration-exploitation tradeoff,” in Proc. IEEE Int. Symp. Information Theory (ISIT), Honolulu, HI, Jun. 2014.

[36] H. Viswanathan and T. Berger, “Sequential coding of correlated sources,” IEEE Trans. Inf. Theory, vol. 46, pp. 236–246, Jan. 2000.

[37] A. El Gamal and T. M. Cover, “Achievable rates for multiple descriptions,” IEEE Trans. Inf. Theory, vol. IT-28, pp. 851–857, Nov. 1982.

[38] Z. Zhang and T. Berger, “New results in binary multiple-descriptions,” IEEE Trans. Inf. Theory, vol. IT-33, pp. 502–521, Jul. 1987.

[39] F.-W. Fu and R. W. Yeung, “On the rate-distortion region for multiple descriptions,” IEEE Trans. Inf. Theory, vol. 48, pp. 2012–2021, Jul. 2002.

[40] R. Venkataramani, G. Kramer, and V. K. Goyal, “Multiple description coding with many channels,” IEEE Trans. Inf. Theory, vol. 49, pp. 2106–2114, Sep. 2003.

[41] A. El Gamal and Y.-H. Kim, Network Information Theory. New York: Cambridge University Press, 2011.

[42] S. Watanabe, “Information theoretical analysis of multivariate correlation,” IBM Journal of Research and Development, vol. 4, pp. 66–82, Jan. 1960.

[43] A. D. Wyner, “The common information of two dependent random variables,” IEEE Trans. Inf. Theory, vol. IT-21, pp. 163–179, Mar. 1975.