A 2-Approximation for the Bounded Treewidth Sparsest Cut Problem in FPT Time

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Abstract

In the non-uniform sparsest cut problem, we are given a supply graph $G$ and a demand graph $D$, both with the same set of nodes $V$. The goal is to find a cut of $V$ that minimizes the ratio of the total capacity on the edges of $G$ crossing the cut over the total demand of the crossing edges of $D$. In this work, we study the non-uniform sparsest cut problem for supply graphs with bounded treewidth $k$. For this case, Gupta, Talwar and Witmer [STOC 2013] obtained a 2-approximation with polynomial running time for fixed $k$, and the question of whether there exists a $c$-approximation algorithm for a constant $c$ independent of $k$, that runs in FPT time, remained open. We answer this question in the affirmative. We design a 2-approximation algorithm for the non-uniform sparsest cut with bounded treewidth supply graphs that runs in FPT time, when parameterized by the treewidth. Our algorithm is based on rounding the optimal solution of a linear programming relaxation inspired by the Sherali-Adams hierarchy. In contrast to the classic Sherali-Adams approach, we construct a relaxation driven by a tree decomposition of the supply graph by including a carefully chosen set of lifting variables and constraints to encode information of subsets of nodes with super-constant size, and at the same time we have a sufficiently small linear program that can be solved in FPT time.

1 Introduction

In the non-uniform sparsest cut problem, we are given two weighted graphs $G$ and $D$ on the same set of nodes $V$, such that $G = (V, E_G)$ is the so-called supply graph, and $D = (V, E_D)$ is the so-called demand graph. For every edge $e \in E_G$ we have a positive integer weight $\text{cap}(e)$ called capacity, and for every edge $e \in E_D$ we have a positive integer weight $\text{dem}(e)$ called the demand. An instance $I$ is given by a tuple $(G, D, \text{cap}, \text{dem})$ and we denote by $|I|$ the encoding length of an instance $I$. The goal is to compute a non-empty subset of nodes $S \subseteq V$ that minimizes

$$\phi(S) = \frac{\sum_{e \in \delta_G(S)} \text{cap}(e)}{\sum_{e \in \delta_D(S)} \text{dem}(e)},$$

where $\delta_G(S) = \{e \in E_G : |e \cap S| = 1\}$ and $\delta_D(S) = \{e \in E_D : |e \cap S| = 1\}$. Since this problem is NP-hard [26], the focus has been on the design of approximation algorithms. In this line of work, Agrawal, Klein, Rao and Ravi [20, 21] took the first major step by describing an $O(\log D \log C)$-approximation algorithm, where $D$ is the total sum of the demands and $C$ is the total sum of the capacities. Currently, the best approximation factor is $O(\sqrt{\log n} \log \log n)$ due to Arora, Lee and Naor [2]. The uniform version of the problem, where the demand graph

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is unweighted and complete, has received a lot of attention through the years. The best bound for this problem is slightly better: $O(\sqrt{\log n})$ [3].

The non-uniform sparsest cut problem is hard to approximate within a constant factor, for any constant, under the unique games conjecture [10, 18, 19]. Therefore, the problem has also been studied under the assumption that the supply graph belongs to a specific family of graphs. Most notable examples include planar graphs, graph excluding a fixed minor, and bounded treewidth graphs. In this paper, we focus on the latter (see Section 1.1 for further related work on minor-closed families).

For inputs to the problem where the supply graph has treewidth at most $k$, Chlamtac, Krauthgamer and Raghavendra [12] designed a $C(k)$-approximation algorithm that runs in time $2^{O(k)} |I|^{O(1)}$, where $C$ is a double exponential function of the treewidth $k$. Later, Gupta, Talwar and Witmer designed a 2-approximation algorithm that runs in time $|I|^{O(k)}$ [18]. However, these two results are only complementary: Chlamtac, Krauthgamer and Raghavendra’s algorithm is Fixed-Parameter Tractable (FPT) in the treewidth $k$ of the supply graph, (i.e. $f(k)|I|^{O(1)}$ time for some computable function $f$), while the algorithm of Gupta, Talwar and Witmer is not. The approximation factor achieved by Gupta, Talwar and Witmer is independent of $k$, and furthermore, they show that there is no $(2 - \epsilon)$-approximation algorithm for any $\epsilon > 0$ on graphs with constant treewidth, assuming the Unique Games Conjecture and that there is no $1/\alpha_{GW} - \epsilon$ approximation algorithm for treewidth 2 graphs unless $P = NP$.

This left open the question of whether there exists a 2-approximation algorithm that runs in FPT time, when parameterized by the treewidth. We answer this question in the affirmative and show the following result.

**Theorem 1.** There is an algorithm that computes a 2-approximation for every instance $I = (G, D, \text{cap}, \text{dem})$ of the non-uniform sparsest cut problem in time $2^{O(k)} |I|^{O(1)}$, where $k$ is the treewidth of the supply graph $G$.

Following the argumentation of Gupta et al. [18], for treewidth $k$ graphs our result implies a $2^{O(k)} |I|^{O(1)}$-time 2-approximation to the minimum-distortion $\ell_1$ embedding.

The results obtained in the predecessor papers [12, 18] where based on rounding certain linear programs obtained through the Sherali-Adams lift & project hierarchy [28]. Our approximation algorithm is also based on rounding a linear program with a fractional objective given by the non-uniform sparsest cut value, but we construct this linear program in a different way, with the goal of obtaining a linear program of smaller size, but sufficiently strong in terms of gap. If we followed the classic Sherali-Adams approach, the relaxation of level $\ell$ would be constructed by using a variable encoding the value of any subset of the original variables up to size $\ell$, and it would take $n^{O(\ell)}$ time to solve this relaxation, where $n$ is the number of nodes in the graph. In particular, solving a relaxation of level $\Theta(k)$ would take $n^{O(k)}$ time, which in principle rules out the possibility of achieving FPT running time by applying directly this approach. In order to overcome this problem, we construct a linear programming relaxation driven by a tree decomposition of the supply graph $G$, where the variables are carefully chosen with the goal of encoding information of subsets of nodes with super-constant size, and at the same time the number of variables and constraints is sufficiently small so we can solve the relaxation in FPT time. We show that the relaxation is strong enough to get a 2-approximation by rounding the optimal fractional solution. The construction of our relaxation and the analysis of our algorithm can be found in Section 3.

### 1.1 Related Work

Despite the difficulties in approximating the non-uniform sparsest cut problem in general graphs, there are several other results for restricted families of graphs. The case in which $G$ is
planar has received a lot of attention. Quite recently, Cohen-Addad, Gupta, Klein and Li [13] showed the existence of a quasi-polynomial time $(2 + \varepsilon)$-approximation for the non-uniform sparsest cut problem in the planar case. To get this result they combine a patching lemma approach with linear programming techniques. We remark that for the planar case there is no polynomial time $1/(0.878 + \varepsilon) \approx (1.139 - \varepsilon)$-approximation algorithm under the unique games conjecture [18].

Other families with constant factor approximation algorithms are outerplanar graphs [27], series-parallel [17, 11, 22], $k$-outerplanar graphs [10], graphs obtained by 2-sums of $K_4$ [8] and graphs with constant pathwidth [23]. The impact of the treewidth parameter has also been studied in the context of polynomial optimization [4]. Finally, we mention that the Sherali-Adams hierarchy has been useful to design algorithms in other minor-free and bounded treewidth graph problems, including independent set and vertex cover [24, 5], and also in several recent results on scheduling and clustering [29, 16, 25, 15, 1].

Independent of our work, Chalermsook et al. [9] obtained a $O(k^2)$-approximation algorithm for sparsest cut in treewidth $k$ graphs, with running time $2^{O(k)} \cdot \text{poly}(n)$ and, for arbitrary $\varepsilon > 0$, an $O(1/\varepsilon^2)$-approximation algorithm with running time $2^{O(k^{1+\varepsilon})} \cdot \text{poly}(n)$. Observe that these results are incomparable with our result: they obtain an asymptotically lower running time, whereas the obtained (constant) approximation ratio is considerably larger than 2. Similar to our result, they build on the techniques from [12, 18]. However, their approach is based on a new measure for tree decompositions which they call the combinatorial diameter.

## 2 Preliminaries: Tree Decompositions

A tree decomposition of a graph $G = (V, E)$ is a pair $(\mathcal{X}, \mathcal{T})$ where $\mathcal{T} = (\mathcal{X}, E_\mathcal{T})$ is a tree and $\mathcal{X}$ is a collection of subsets of nodes in $V$ called bags. Each bag is a node in the tree $\mathcal{T}$. Furthermore, the pair $(\mathcal{X}, \mathcal{T})$ satisfies the following conditions.

1. Every node in $V$ is in at least one bag, that is, $\bigcup_{X \in \mathcal{X}} X = V$.
2. For every edge $\{u, v\} \in E$ there exists a bag $X \in \mathcal{X}$ such that $\{u, v\} \subseteq X$.
3. For every node $u \in V$ the bags containing $u$ induce a subtree of $\mathcal{T}$.

The \textit{width} of the tree decomposition $(\mathcal{X}, \mathcal{T})$ corresponds to the size of the largest bag in the tree decomposition, minus one. The treewidth of $G$ is the minimum possible width of a tree decomposition for $G$. We typically consider the tree $\mathcal{T}$ to be rooted, and we denote its root by $\mathcal{R}$. We denote by $\text{depth}(\mathcal{T})$ the depth of the tree $\mathcal{T}$ and we say that a bag $X$ is at level $\ell$ if the distance from the root $\mathcal{R}$ to $X$ in the tree $\mathcal{T}$ is equal to $\ell$. We denote by $\mu(X)$ the parent of $X$ in the tree $\mathcal{T}$. The intersection between a non-root bag $X$ and the parent bag, $\mu(X) \cap X$, is called the \textit{adhesion} of the bag $X$. We say that a bag $Y$ is a \textit{descendant} of $X$ if $X \neq Y$ and the bag $X$ belongs to the unique path in $\mathcal{T}$ from $Y$ to the root, and in this case we say that $X$ is an \textit{ancestor} of $Y$.

The following result due to Bodlaender [6, Theorem 4.2] states the existence of tree decompositions with a particular structure that is useful for our algorithm.

\textbf{Lemma 1} ([6]). Let $G$ be a graph with $n$ nodes and treewidth $k$. Then, there exists a tree decomposition $(\mathcal{X}, \mathcal{T})$ of $G$ satisfying the following:

(a) $\mathcal{T}$ is a binary tree and $\text{depth}(\mathcal{T}) \in O(\log n)$.

(b) For every $X \in \mathcal{X}$ we have that $|X| \leq 3k + 3$.

The tree decomposition $(\mathcal{X}, \mathcal{T})$ can be computed in time $2^{O(k^3)} n$. 

3
3 The LP Relaxation and the Rounding Algorithm

Our algorithm is based on rounding the optimal solution of a linear programming relaxation for the non-uniform sparsest cut problem. In Section 3.1 we provide the construction of our linear programming relaxation and in Section 3.2 we provide the rounding algorithm and the proof of Theorem 1. In the following lemma we show the existence of a tree structure that we use to construct the linear program (see also Fig. 1).

**Lemma 2.** Let $G$ be a graph with treewidth $k$ and let $\ell$ be a positive integer. Then, there exists a tree decomposition $(\mathcal{Y}, \mathcal{E})$ of $G$ such that the following holds:

(a) The width of $(\mathcal{Y}, \mathcal{E})$ is $O(2^\ell k)$ and $\text{depth}(\mathcal{E}) \in O(\log(n)/\ell)$.

(b) For every non-root bag $Y \in \mathcal{Y}$, the size of the adhesion of $Y$ is $O(k)$.

The decomposition $(\mathcal{Y}, \mathcal{E})$ can be found in $2^{O(k^2)} n$ time.

**Proof.** Consider a tree decomposition $(X, \mathcal{T})$ satisfying the conditions guaranteed by Lemma 1. That is, the tree $\mathcal{T}$ is binary, $\text{depth}(\mathcal{T}) \in O(\log(n))$ and $|X| \leq 3k + 3$ for every bag $X \in X$. Let $X'_i$ be the set of all bags in $X$ at level $i(\ell - 1)$ with $i \in \{0, 1, \ldots, \text{depth}(\mathcal{T})/(\ell - 1)\}$. For every bag $X \in X'_i$ with level $i(\ell - 1)$ consider the set $Y_X$ given by the union of $X$ with all its descendant bags at level $i(\ell - 1) + j$ with $j \in \{0, 1, \ldots, \ell - 1\}$. Since $\mathcal{T}$ is binary we have that $|Y_X| \leq 2^i(3k + 3) = O(2^\ell k)$ and we define $\mathcal{Y} = \{Y_X : X \in X'_i\}$. We define the edges of the tree $\mathcal{E}$ with nodes $\mathcal{Y}$ as follows: For every bag $X$ at level $i(\ell - 1)$ with $i \in \{0, 1, \ldots, \text{depth}(\mathcal{T})/(\ell - 1)\} - 1$, we have the edge $\{Y_X, Y_{X'}\}$ in the tree if $X'$ is a descendant of $X$ and the level of $X'$ is $(i + 1)(\ell - 1)$ (See Figure 1). In particular, the depth of $\mathcal{E}$ is at most $O(\text{depth}(\mathcal{T})/\ell) = O(\log(n)/\ell)$. This proves condition (a).

Now let $\{Y_X, Y_{X'}\}$ be any edge of the tree $\mathcal{E}$ and consider a node $u \in Y_X \cap Y_{X'}$. Since $(X, \mathcal{T})$ is a tree decomposition, we have that the set of bags in $X$ that contain the node $u$ induces a subtree of $\mathcal{T}$. Furthermore, $X'$ is the unique bag of $X$ in the intersection of the set of bags defining $Y_X$ and $Y_{X'}$, and therefore the connectivity of this subtree implies that $u \in X'$. Since $|X'| \leq 3k + 3$, we conclude that the size of the adhesion of any bag $Y \in \mathcal{Y}$ is $O(k)$. This shows that condition (b) holds. The fact that $(\mathcal{Y}, \mathcal{E})$ is a tree decomposition holds since the construction preserves conditions (1)-(3) from $(X, \mathcal{T})$. 

**Definition 1.** Given a graph $G$, we say that a tree decomposition $\Theta = (\mathcal{Y}, \mathcal{E})$ satisfying properties (a)-(b) is a $(k, \ell)$-decomposition of $G$. 

![Figure 1: Tree decomposition from Lemma 2 for $\ell = 3$. The gray nodes form the set $X'_i$.](image-url)
Given a bag \( Y \in \mathcal{Y} \), we denote by \( \mathcal{P}^Y_\Theta \) the subset of bags that belong to the path from \( Y \) to the root \( R \) in the tree \( \mathcal{E} \). We denote by \( \mathcal{J}_Y \) the adhesion of \( Y \). Furthermore, let

\[
\mathcal{Y}^Y_\Theta = \bigcup_{Z \in \mathcal{P}^Y_\Theta} \mathcal{J}_Z,
\]

and for every pair of non-root bags \( Y, Z \in \mathcal{Y} \) let \( \mathcal{S}_\Theta(Y, Z) \) be the power set of \( (Y \cup \mathcal{Y}^Y_\Theta) \cup (Z \cup \mathcal{Y}^Z_\Theta) \). Finally, let

\[
\mathcal{S}_\Theta = \bigcup_{Y, Z \in \mathcal{Y}} \mathcal{S}_\Theta(Y, Z).
\]

Observe that for every bag \( Y \in \mathcal{Y} \), the size of \( Y \cup \mathcal{Y}^Y_\Theta \) is \( O(2^k k + k \log(n)/\ell) \).

### 3.1 The LP Relaxation

Consider a positive integer \( \ell \) and an instance \((G, D, \text{cap}, \text{dem})\) where \( G \) has treewidth \( k \). Let \( \Theta = (\mathcal{Y}, \mathcal{E}) \) be a \((k, \ell)\)-decomposition of the supply graph \( G \). In what follows we describe our LP relaxation, inspired by the Sherali-Adams hierarchy [28] and the predecessor works [12, 18]. In this linear program there are two types of variables. The variable \( x(S, T) \), with \( S \in \mathcal{S}_\Theta \) and \( T \subseteq S \), indicates that the cut solution \( C \) satisfies that \( C \cap S = T \). The variable \( y(\{u, v\}) \) for \( u, v \in V \) with \( u \neq v \), indicates whether the nodes \( u \) and \( v \) fall in different sides of the cut. For notation simplicity, we sometimes denote the union between a set \( A \) and a singleton \( \{a\} \) by \( A + a \). Consider the following linear fractional program:

\[
\begin{align*}
\text{minimize} \quad & \frac{\sum_{e \in E_G} \text{cap}(e) y(e)}{\sum_{e \in E_G} \text{dem}(e) y(e)} & (1) \\
\text{subject to} \quad & x(\{u, v\}, u) + x(\{u, v\}, v) = y(\{u, v\}) & \text{for every } u, v \in V \text{ with } u \neq v, \quad (2) \\
& \sum_{A \subseteq S} x(S, A) = 1 & \text{for every } S \in \mathcal{S}_\Theta, \quad (3) \\
& x(S, A) \geq 0 & \text{for every } S \in \mathcal{S}_\Theta \text{ and } A \subseteq S, \quad (4) \\
& x(S + u, A) + x(S + u, A + u) = x(S, A) & \text{for every } S \subseteq V, u \notin S \text{ such that } S + u \in \mathcal{S}_\Theta \text{ and } A \subseteq S. \quad (5)
\end{align*}
\]

The feasible region of this linear program is a polytope encoding the cuts in \( V \). Indeed, given any cut \( C \), define \( U_j = 1 \) if \( j \in C \) and zero otherwise. For every \( S \in \mathcal{S}_\Theta \) and \( A \subseteq S \), define \( x(S, A) = \prod_{j \in A} U_j \prod_{i \in S \setminus A} (1 - U_i) \) and \( y(\{u, v\}) = U_u(1 - U_v) + U_v(1 - U_u) \). The solution \((x, y)\) satisfies conditions \((2)-(5)\). We remark that \((5)\) is valid for every cut since given a subset \( S \) and a node \( u \notin S \), the intersection between \( S + u \) and a cut \( C \) is either \( C \cap S \) or \((C \cap S) + u \), which are the two possibilities in the left hand side of \((5)\). Since for every bag \( Y \in \mathcal{Y} \) the size of \( Y \cup \mathcal{Y}^Y_\Theta \) is \( O(2^k k + k \log(n)/\ell) \), we get

\[
|\mathcal{S}_\Theta(Y, Z)| = 2^O(k(2^k + \log(n)/\ell)) \quad \text{for any pair of bags } Y, Z \in \mathcal{Y},
\]

\[
|\mathcal{S}_\Theta| \leq \sum_{Y, Z \in \mathcal{Y}} |\mathcal{S}_\Theta(Y, Z)| = n^2 2^O(k(2^k + \log(n)/\ell)),
\]

and therefore the number of variables and constraints in the linear fractional program is

\[
O\left(|\mathcal{S}_\Theta| \cdot 2^{\max\{|S|: S \in \mathcal{S}_\Theta\}}\right) = n^2 2^O(k(2^k + \log(n)/\ell)).
\]

By using a standard equivalent reformulation the linear fractional program \((1)-(5)\) can be solved by a linear program with one additional variable and constraint [7].
3.2 The Rounding Algorithm

In this section we describe our algorithm for the non-uniform sparsest cut problem. Before stating the algorithm, we introduce an object that will be used in the analysis. Recall that \( G \) is of treewidth \( k \) and \( \Theta = (\mathcal{Y}, \mathcal{E}) \) is a \((k, \ell)\)-decomposition of \( G \).

**Definition 2.** Given a feasible solution \((x, y)\) satisfying \((2)-(5)\), we define the function given by

\[
 f_{R, \Theta}^{x, y}(A) = x(R, A) \quad \text{for every } A \subseteq R,
\]

where \( R \) is the root bag of \( \mathcal{E} \). Furthermore, given any non-root bag \( Y \in \mathcal{Y} \) and a subset \( T \subseteq V^Y_\Theta \) such that \( x(V^Y_\Theta \cup T) > 0 \), we define the function given by

\[
 f_{x, \Theta}^{T, Y}(A) = \frac{x(V^Y_\Theta \cup Y, T \cup A)}{x(V^Y_\Theta, T)}
\]  

for every \( A \subseteq Y \setminus \mu(Y) \), where \( \mu(Y) \) is the parent of \( Y \) (see Figure 2).

The functions introduced in Definition 2 have a probabilistic interpretation that will be at the basis of our rounding algorithm. The structure provided by constraints \((3)-(5)\) induces probability distributions over subsets of a bag in the decomposition \( \Theta \). For a bag \( Y \), the value \((6)\) can be interpreted as a conditional probability given the choice of \( T \subseteq V^Y_\Theta \). The following proposition summarizes these properties.

**Proposition 1.** Consider an instance \((G, D, \text{cap}, \text{dem})\) with \( G \) of treewidth \( k \) and let \( \ell \) be a positive integer. Let \( \Theta = (\mathcal{Y}, \mathcal{E}) \) be a \((k, \ell)\)-decomposition of the graph \( G \) and let \((x, y)\) be a solution satisfying \((2)-(5)\). Then, the following holds:

(a) Let \( L, I \in S_\Theta \) such that \( L \subseteq I \). Then, for every \( C \subseteq L \), we have \( x(L, C) = \sum_{I' \subseteq I \setminus L} x(I, C \cup I') \).

(b) \( \sum_{A \subseteq R} f_{x, \Theta}^{R}(A) = 1 \).

(c) For every non-root bag \( Y \in \mathcal{Y} \) and every \( T \subseteq V^Y_\Theta \), we have \( \sum_{A \subseteq Y \setminus \mu(Y)} f_{x, \Theta}^{T, Y}(A) = 1 \).

**Proof.** We prove part (a) by induction on the size of \( I \setminus L \). When \( I \setminus L = \{u\} \) for some node \( u \in V \), since \( x \) satisfies condition \((5)\) we have

\[
 x(L, C) = x(L + u, C) + x(L + u, C + u) = x(I, C) + x(I, C + u) = \sum_{I' \subseteq I \setminus L} x(I, C \cup I'),
\]
where the second equality holds since \( I = L + u \). Now consider any pair \( L, I \in S_\Theta \) with \( L \subseteq I \) such that the size of \( I \setminus L \) is larger than one, consider any node \( v \in I \setminus L \). Then, by condition (5) and the inductive step we get

\[
x(L, C) = x(L + v, C) + x(L + v, C + v)
\]

\[
= \sum_{H \subseteq I \setminus (L + v)} x(I, C \cup H) + \sum_{H \subseteq I \setminus (L + v)} x(I, (C + v) \cup H)
\]

\[
= \sum_{H \subseteq I \setminus (L + v)} x(I, C \cup H) + \sum_{H \subseteq I \setminus (L + v)} x(I, C \cup (H + v))
\]

\[
= \sum_{I' \subseteq I \setminus L} x(I, C \cup I'),
\]

which concludes the proof for this part. Part (b) follows directly since \( x \) satisfies (3) for \( S = \mathcal{R} \) and therefore

\[
\sum_{A \subseteq \mathcal{R}} f_{x,\Theta}^R(A) = \sum_{A \subseteq \mathcal{R}} x(\mathcal{R}, A) = 1.
\]

To show part (c), by applying part (a) with \( L = \mathcal{V}_\Theta^Y \) and \( I = \mathcal{V}_\Theta^Y \cup Y \) we get that

\[
x(\mathcal{V}_\Theta^Y, T) = \sum_{A \subseteq \mathcal{V}_\Theta^Y \setminus \mathcal{V}_\Theta^Y} x(\mathcal{V}_\Theta^Y \cup Y, T \cup A) = \sum_{A \subseteq \mathcal{V}_\Theta^Y \setminus \mu(Y)} x(\mathcal{V}_\Theta^Y, T) \cdot f_{x,\Theta}^{T \setminus Y}(A),
\]

where we used that \( Y \setminus \mathcal{V}_\Theta^Y = Y \setminus \mu(Y) \). That concludes the proof.

We first design a randomized algorithm to show the existence of 2-approximation by rounding an optimal solution of the linear fractional program (1)-(5) defined by a \((k, \ell)\)-decomposition \( \Theta \). We start by constructing a solution at the root level, and then by conditioning on this assignment we construct a solution for the children, and we continue this propagation process until we recover an integral solution. Theorem 1 is finally obtained by optimizing the running time of our algorithm as a function of \( \ell \), and by performing a derandomization to get a deterministic 2-approximation algorithm. We provide the detailed randomized algorithm below.

**Algorithm 1 Randomized Rounding**

**Input:** An instance \((G, D, \text{cap, dem})\) with \( G \) of treewidth \( k \) and a positive integer number \( \ell \).

**Output:** A cut in the nodes \( V \).

1. Compute a \((k, \ell)\) decomposition \( \Theta = (\mathcal{Y}, \mathcal{E}) \) of \( G \).
2. Let \((x, y)\) be an optimal solution of (1)-(5).
3. Sample a subset \( B_\mathcal{R} \subseteq \mathcal{R} \) according to the probability distribution \( f_{x,\Theta}^R \) and let \( H_\mathcal{R} = \emptyset \).
4. for \( \ell = 1 \) to \( \text{depth}(\mathcal{E}) \) do
5. For every bag \( Y \) of level \( \ell \) in the tree \( \mathcal{E} \), let \( H_Y = H_{\mu(Y)} \cup (B_{\mu(Y)} \cap \mathcal{J}_Y) \).
6. Sample a subset of nodes \( B_Y \subseteq Y \setminus \mu(Y) \) according to the probability distribution \( f_{x,\Theta}^{H_Y \setminus Y} \).
7. Return \( B = \bigcup_{Y \in \mathcal{Y}} B_Y \).

For a bag \( Y \in \mathcal{Y} \), the set \( B_{\mu(Y)} \cap \mathcal{J}_Y \) is a subset of the adhesion of \( Y \), and the set \( H_Y \) collects the union of these subsets in the path of \( \Theta \) that goes from the root to \( Y \). Then, the set \( B_Y \subseteq Y \setminus \mu(Y) \) is sampled according to a conditional probability that depends on \( H_Y \). The output of Algorithm 1 is a random subset of nodes in \( V \) and we denote by \( P_{x,\Theta} \) the probability measure induced by this random set-valued variable. The following lemmas summarize some properties of the algorithm.

**Lemma 3.** Consider \((G, D, \text{cap, dem})\) with \( G \) of treewidth \( k \) and let \( \ell \) be a positive integer. Let \( \Theta = (\mathcal{Y}, \mathcal{E}) \) be a \((k, \ell)\) decomposition of \( G \) and let \((x, y)\) be a solution satisfying (2)-(5). Then, the following holds:
(a) For every $Y \in \mathcal{Y}$ and every $S \subseteq \mathcal{Y} \cup \mathcal{Y}_o$, we have $P_{x,\Theta}(B \cap S = T) = x(S, T)$ for every $T \subseteq S$.

(b) For every edge $e \in E_G$ in the supply graph, we have $P_{x,\Theta}(|e \cap B| = 1) = y(e)$.

Proof. We prove part (a) by induction on the level of the bag $Y$. Suppose first that $Y = \mathcal{R}$, that is the root bag of $\Theta$. In this case we have, by construction, that $\mathcal{V}_o^\mathcal{R} = \emptyset$. Given $S \subseteq \mathcal{R}$, we have that $P_{x,\Theta}(B_R = S) = f_{x,\Theta}^\mathcal{R}(S) = x(\mathcal{R}, S)$. On the other hand, for any $T \subseteq S$, we have

$$P_{x,\Theta}(B \cap S = T) = P_{x,\Theta}(B_R \cap S = T) = \sum_{T' \subseteq \mathcal{R} \setminus S} P_{x,\Theta}(B_R = T \cup T') = \sum_{T' \subseteq \mathcal{R} \setminus S} x(\mathcal{R}, T \cup T') = x(S, T),$$

where the last equality holds by Proposition 1 (a). Now consider any non-root bag $Y \in \mathcal{Y}$ and let $Z$ be its parent bag. In particular, we have that $\mathcal{V}_o^Y \subseteq \mathcal{V}_o^Z \cup Z$. Let $S \subseteq \mathcal{V}_o^Y \cup Y$ and consider $S_Y = S \cap \mathcal{V}_o^Y$ and $\bar{S} = S \setminus \mathcal{V}_o^Y$. Then, we have

$$P_{x,\Theta}(B_Y \cup H_Y = S) = P_{x,\Theta}(H_Y = S_Y, B_Y = \bar{S}) = P_{x,\Theta}(H_Y = S_Y) \cdot P_{x,\Theta}(B_Y = \bar{S} | H_Y = S_Y) = P_{x,\Theta}(B \cap \mathcal{V}_o^Y = S_Y) \cdot P_{x,\Theta}(B_Y = \bar{S} | H_Y = S_Y) = x(\mathcal{V}_o^Y, S_Y) \cdot f_{x,\Theta}^Y(\bar{S}) = x(\mathcal{V}_o^Y \cup Y, S_Y \cup \bar{S}) = x(\mathcal{V}_o^Y \cup Y, S),$$

where the fourth equality holds since $\mathcal{V}_o^Y \subseteq \mathcal{V}_o^Z \cup Z$ together with the inductive step and the choice of $B_Y$ in Algorithm 1. Given $S \subseteq \mathcal{V}_o^Y \cup Y$ and $T \subseteq S$, we have

$$P_{x,\Theta}(B \cap S = T) = P_{x,\Theta}((B_Y \cup H_Y) \cap S = T) = \sum_{T' \subseteq (\mathcal{V}_o^Y \cup Y) \setminus S} P_{x,\Theta}(B_Y \cup H_Y = T \cup T') = \sum_{T' \subseteq (\mathcal{V}_o^Y \cup Y) \setminus S} x(\mathcal{V}_o^Y \cup Y, T \cup T') = x(S, T),$$

where the last equality holds by Proposition 1 (c). We now prove part (b). Consider any edge $e = \{u, v\} \in E_G$ and let $Y \in \mathcal{Y}$ be such that $e \in Y$. We know this bag exists since $\Theta$ is a tree decomposition by Lemma 2. Since $x$ satisfies (2), together with part (a) we have that $P_{x,\Theta}(|e \cap B| = 1) = P_{x,\Theta}(e \cap B = \{u\}) + P_{x,\Theta}(e \cap B = \{v\}) = x(e, \{u\}) + x(e, \{v\}) = y(e)$.

\qed

Lemma 4. Consider $(G, D, \text{cap}, \text{dem})$ with $G$ of treewidth $k$ and let $\ell$ be a positive integer. Let $\Theta = (\mathcal{Y}, \mathcal{E})$ be a $(k, \ell)$-decomposition of $G$ and let $(x, y)$ be a solution satisfying (2)-(5). Then, for every edge $e \in E_D$ in the demand graph we have $P_{x,\Theta}(|e \cap B| = 1) \geq y(e)/2$.

Proof. Let $e = \{s, t\} \in E_D$ be a demand edge. When $e \in E_G$ we are done since $P_{x,\Theta}(|e \cap B| = 1) = y(e)$ by Lemma 3 (b). Suppose in what follows that $e \notin E_G$, and let $Y_s$ and $Y_t$ be the least depth bags in the tree $\mathcal{E}$ such that $s \in Y_s$ and $t \in Y_t$. Furthermore, let $Y$ be the lowest common ancestor of the bags $Y_s, Y_t$ in the tree $\mathcal{Y}$. Let $C_e = (Y_s \cup \mathcal{V}_o^Y_s) \cup (Y_t \cup \mathcal{V}_o^Y_t)$. For every $T \subseteq C_e$ consider the value $g_e(T) = x(C_e, T)$. Since $x$ satisfies (3), we have that $\sum_{T \subseteq C_e} g_e(T) = 1$ and therefore $g_e$ defines a probability mass function over $\mathcal{S}_o(Y_s, Y_t)$, which is the power set of $C_e$. Consider the set-valued random variable $W$ distributed according to $g_e$ and let $Q_e$ the
probability measure induced by this random variable. Then, we have

\[
Q_e(\{e \cap W = 1\}) = Q_e(\{e \cap W = \{s\}\}) + Q_e(\{e \cap W = \{t\}\}) \\
= \sum_{C' \subseteq C \setminus \{e\}} x(C_e, s + C') + \sum_{C' \subseteq C \setminus \{e\}} x(C_e, t + C') \\
= x(e, s) + x(e, t) = y(e),
\]

where the third equality holds by Proposition 1 (a) and the last equality holds since \(x\) satisfies condition (2). Let \(Z_s\) and \(Z_t\) be the children bags of \(Y\) such that \(Z_s\) belongs to unique path from \(Y_s\) to the root \(R\) and \(Z_t\) belongs to unique path from \(Y_t\) to the root \(R\), in the tree \(E\). Define the set \(\Lambda = Y^c_{\Theta} \cup J_{Z_s} \cup J_{Z_t}\). Observe that

\[
P_{x,\Theta}(\{e \cap B\} = 1) = \sum_{T \subseteq \Lambda} P_{x,\Theta}(\{e \cap B\} = 1 \mid B \cap \Lambda = T) \cdot P_{x,\Theta}(B \cap \Lambda = T)
\]

where the last equality holds by Lemma 3 (a) and the fact that \(\Lambda \subseteq V^c_{\Theta} \cup Y\). On the other hand, for any \(L \subseteq W\) and every \(I \subseteq L\) we have

\[
Q_e(W \cap L = I) = \sum_{C' \subseteq C \setminus \Lambda} x(C_e, I \cup C') = x(L, T),
\]

(7)

where the last equality holds by Proposition 1 (a). Therefore, we have

\[
y(e) = Q_e(\{e \cap W = 1\}) = \sum_{T \subseteq \Lambda} Q_e(\{e \cap W = 1 \mid W \cap \Lambda = T\}) \cdot Q_e(W \cap \Lambda = T)
\]

where the last equality holds by applying (7) with \(L = \Lambda\). Then, in order to conclude the lemma it is sufficient to show that \(Q_e(\{e \cap W = 1 \mid W \cap \Lambda = T\}) \leq 2 \cdot P_{x,\Theta}(\{e \cap B\} = 1 \mid B \cap \Lambda = T)\).

Given \(T \subseteq \Lambda\), consider the random variable \(\omega_{s,T} \in \{0,1\}\) that indicates whether \(s \in W\) given \(W \cap \Lambda = T\), and let \(\beta_{s,T} \in \{0,1\}\) be the random variable that indicates whether \(s \in B\) given \(B \cap \Lambda = T\). We define analogously the random variables \(\omega_{t,T}\) and \(\beta_{t,T}\). Since \(s, t \not\in \Lambda\), we observe that for any \(T \subseteq \Lambda\) and \(v \in \{s, t\}\) it holds that \(v \in W\) and \(W \cap \Lambda = T\) if and only if \(W \cap (\Lambda + v) = T + v\). Therefore, for every \(T \subseteq \Lambda\), we have that

\[
Q_e(\omega_{s,T} = 1) = \frac{Q_e(\{s \in W, W \cap \Lambda = T\})}{Q_e(W \cap \Lambda = T)} = \frac{x(\Lambda + s, T + s)}{x(\Lambda, T)} = P_{x,\Theta}(\beta_{s,T} = 1),
\]

\[
Q_e(\omega_{t,T} = 1) = \frac{Q_e(\{t \in W, W \cap \Lambda = T\})}{Q_e(W \cap \Lambda = T)} = \frac{x(\Lambda + t, T + t)}{x(\Lambda, T)} = P_{x,\Theta}(\beta_{t,T} = 1),
\]

where, in both cases, the first equality comes from the above observation and (7) and the second equality is a consequence of the above observation and Proposition 1 (a). We conclude that for every \(T \subseteq \Lambda\) the random variables \(\omega_{s,T}\) and \(\beta_{s,T}\) are identically distributed, for \(v \in \{s, t\}\). Furthermore, by construction in Algorithm 1, the random variables \(\omega_{s,T}\) and \(\beta_{s,T}\) are independent.

Claim 1. Suppose we have two random variables \(G\) and \(K\) taking values in \(\{0,1\}\). Then, we have that \(Pr(G \neq K) \leq 2(Pr(G = 1) Pr(K = 0) + Pr(G = 0) Pr(K = 1))\).

We show how to conclude the lemma using the claim. Taking \(G = \omega_{s,T}\) and \(K = \omega_{l,T}\), we have

\[
Q_e(\{e \cap W = 1 \mid W \cap \Lambda = T\}) = Q_e(\omega_{s,T} \neq \omega_{l,T}) \\
\leq 2(Q_e(\omega_{s,T} = 1) Q_e(\omega_{l,T} = 0) + Q_e(\omega_{s,T} = 0) Q_e(\omega_{l,T} = 1)) \\
= 2(P_{x,\Theta}(\beta_{s,T} = 1) P_{x,\Theta}(\beta_{l,T} = 0) + P_{x,\Theta}(\beta_{s,T} = 0) P_{x,\Theta}(\beta_{l,T} = 1)) \\
= 2 \cdot P_{x,\Theta}(\beta_{s,T} \neq \beta_{l,T}) = 2 \cdot P_{x,\Theta}(\{e \cap B\} = 1 \mid B \cap \Lambda = T),
\]

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which concludes the lemma. We now show how to prove the claim. Let \( a = \Pr(G = 1) \) and \( b = \Pr(K = 1) \). Then, we have that \( \Pr(G \neq K) \leq a(1 - b) + b(1 - a) + \min\{ab, (1 - a)(1 - b)\} \). We prove next that \( \min\{ab, (1 - a)(1 - b)\} \leq (1 - b) + b(1 - a) \) for every \( a, b \in [0, 1] \). Having this inequality, we get that \( \Pr(G \neq K) \leq 2a(1 - b) + 2b(1 - a) \) and therefore the claim holds. To prove the inequality consider two cases. Suppose first that \( a + b \leq 1 \). In particular, we have that \( ab \leq 1 - a - b + ab = (1 - a)(1 - b) \). Furthermore, \( ab \leq a^2 + b^2 \leq a(1 - b) + b(1 - a) \), where the last inequality holds since \( a, b \in [0, 1] \) and \( a + b \leq 1 \) implies that \( a \leq 1 - b \) and \( b \leq 1 - a \). Now suppose that \( a + b > 1 \). In particular, we have \( ab > 1 - a - b + ab = (1 - a)(1 - b) \). On the other hand, \( (1 - a)(1 - b) \leq (1 - a)^2 + (1 - b)^2 < b(1 - a) + (a + b) \), where the last inequality holds since \( a, b \in [0, 1] \) and \( a + b > 1 \) implies that \( a > 1 - b \) and \( b > 1 - a \). This concludes the proof of the inequality. \( \Box \)

**Definition 3.** Let \( G \) be a graph of treewidth \( k \), let \( \Theta = (\mathcal{Y}, \mathcal{E}) \) be a \((k, \ell)\)-decomposition of \( G \), and consider a node \( u \in V \). Let \( X \) be the least depth bag in the tree containing the node \( u \). Given a bag \( Z \in \mathcal{P}_\Theta^X \) and \( H \subseteq \mathcal{Y}_\Theta^Z \cup Z \), we say that a pair \((M, N)\) is an \( H \)-extension for the node \( u \) if the following holds:

(i) \( N \subseteq X \setminus \mathcal{V}_\Theta^X \) and \( u \in N \),

(ii) \( M = (H \cap \mathcal{V}_\Theta^X) \cup L \) where \( L \subseteq \mathcal{V}_\Theta^X \setminus (\mathcal{V}_\Theta^Z \cup Z) \).

We denote by \( \Delta_{\Theta}(H, u) \) the set of \( H \)-extensions for \( u \).

Observe that for any node \( u \) and \( X \) being the least depth bag containing \( u \), for any bag \( Z \in \mathcal{P}_\Theta^X \) and any \( H \subseteq \mathcal{V}_\Theta^Z \cup Z \), the set \( \Delta_{\Theta}(H, u) \) has cardinality at most

\[
2^{O(k(2^\ell + \log(n)/\ell))}
\]

where \( \Theta \) is a \((k, \ell)\)-decomposition. This holds since, by Lemma 2, we have \( |X \setminus \mathcal{V}_\Theta^X| \leq O(2^\ell k) \) and \( |\mathcal{V}_\Theta^X \setminus (\mathcal{V}_\Theta^Z \cup Z)| \leq O(k \log(n)/\ell) \). We need one more lemma before proving Theorem 1.

**Lemma 5.** For every positive real value \( x \geq 4 \), there exists a unique value \( \alpha^* \) such that \( \alpha^* 2^{\alpha^*} = x \), and it satisfies the inequality \( 2^{[\alpha^*]} + x/\lfloor \alpha^* \rfloor \leq 12x/ \log(x) \).

**Proof.** It is well-known that the unique real solution of the equation \( a^{2^a} = x \) is given by \( W_0(x \ln(2))/\ln(2) \), where \( W_0 \) is the principal branch of the Lambert function [14]. Furthermore, we have \( W_0(z) \geq \ln(z) - \ln \ln(z) \geq \ln(z)/2 \) for every \( z \geq e \), and therefore we get

\[
\alpha^* = \frac{W_0(x \ln(2))}{\ln(2)} \geq \frac{\ln(x) + \ln \ln(2)}{2 \ln(2)} \geq 0.4 \ln(x) \geq 0.27 \log(x)
\]

for every \( x \geq 4 \). Then, we have

\[
2^{[\alpha^*]} + \frac{x}{\lfloor \alpha^* \rfloor} \leq 2^{\alpha^* + 1} + \frac{x}{\alpha^*} = \frac{3x}{\alpha^*} < \frac{12x}{\log(x)},
\]

where the equality holds since \( \alpha^* \) solves \( a^{2^a} = x \) and the last inequality holds by (9). \( \Box \)

**Proof of Theorem 1.** Let \( \mathcal{I} = (G, D, c, d) \) be an instance of the non-uniform sparsest cut problem. Recall that we denote by \( n \) the number of nodes in the instance. Let \( a_n^* \) be the unique positive real solution of the equation \( a^{2^a} = \log(n) \) and let \( \ell^* = \lceil \alpha_n^* \rceil \). We run Algorithm 1 over the instance \( \mathcal{I} \), using the value \( \ell^* \), and let \( \Theta \) be the \((k, \ell^*)\)-decomposition computed in step 1 of the algorithm. Let \( (x, y) \) be an optimal solution of the optimization problem (1)-(5) solved in step 2 of the algorithm, and we denote by \( \text{opt}_{LP} \) the optimal value \( \sum_{e \in E_G} \text{cap}(e)y(e)/\sum_{e \in E_D} \text{dem}(e)y(e) \). Let \( B \) be the solution computed by the randomized algorithm. For every pair of nodes \( e = \{u, v\} \subseteq V \), with \( u \neq v \), let \( \xi(e) \) be equal to one if \( |e \cap B| = 1 \) and zero otherwise.
random variable indicates when a pair of nodes is cut by the algorithm solution. Consider $C = \sum_{e \in E_G} \text{cap}(e) \xi(e)$ and $D = \sum_{e \in E_D} \text{dem}(e) \xi(e)$. By Lemmas 3 (b) and 4 we have that

$$E_{x, \Theta}(C) = \sum_{e \in E_G} \text{cap}(e) \cdot E_{x, \Theta}(\xi(e)) = \sum_{e \in E_G} \text{cap}(e) \cdot P_{x, \Theta}(|e \cap B| = 1) = \sum_{e \in E_G} \text{cap}(e) y(e),$$

$$E_{x, \Theta}(D) = \sum_{e \in E_D} \text{dem}(e) \cdot E_{x, \Theta}(\xi(e)) = \sum_{e \in E_D} \text{dem}(e) \cdot P_{x, \Theta}(|e \cap B| = 1) \geq \frac{1}{2} \sum_{e \in E_D} \text{dem}(e) y(e),$$

and therefore we get $E_{x, \Theta}(C) / E_{x, \Theta}(D) \leq 2 \cdot \text{opt}_{LP} \leq 2 \cdot \min_{S \subseteq V} \phi(S)$, where the last inequality holds since the sparsest cut of value $\min_{S \subseteq V} \phi(S)$ defines a feasible solution for (1)-(5). We now show how to derandomize the solution $B$ to get a deterministic 2-approximation. We use the method of conditional expectations. Define the random variable $\Gamma = C - 2 \cdot D \cdot \text{opt}_{LP}$.

Then, we have that $0 \geq E_{x, \Theta}(\Gamma) = E(E_{x, \Theta}(\Gamma|B_R))$ and therefore there exists $R' \subseteq R$ such that $E_{x, \Theta}(\Gamma|B_R = R') \leq 0$. Fix any subset $Y'_1 \subseteq R$ with $E_{x, \Theta}(\Gamma|B_R = Y'_1) \leq 0$ and let $R = Y_1, Y_2, \ldots, Y_{|Y|}$ be the bags visited according to some BFS ordering. Suppose we have computed for some $t \in \{1, \ldots, |Y| - 1\}$ the set $A_t = \cup_{i=1}^t Y'_i \subseteq \cup_{i=1}^t Y_i$, with $Y'_i \subseteq Y_i \setminus \mu(Y_i)$ for each $\ell \in \{1, \ldots, t\}$, and such that $E_{x, \Theta}(\Gamma|B \cap (\cup_{i=1}^t Y_i) = A_t) \leq 0$. Then, we have

$$0 \geq E_{x, \Theta}(\Gamma|B \cap (\cup_{i=1}^t Y_i) = A_t) = \sum_{Y' \subseteq \mu(Y)} E_{x, \Theta}(\Gamma|B \cap (\cup_{i=1}^t Y_i) = A_t, B_{Y_{i+1}} = Y') \cdot P_{x, \Theta}(B_{Y_{i+1}} = Y'),$$

and therefore there exists $Y' \subseteq Y_{t+1} \setminus \mu(Y_{t+1})$ such that $E_{x, \Theta}(\Gamma|B \cap (\cup_{i=1}^{t+1} Y_i) = A_t \cup Y') \leq 0$. Fix any of these subsets and we denote it by $Y_{t+1}$. By the end of this process, let $A$ be the union of $Y_1, \ldots, Y_{|Y|}$. By construction, we have recovered a solution such that $E_{x, \Theta}(\Gamma|B = A) \leq 0$ and therefore $A$ is a 2-approximation.

We now study the running time of the derandomization, and more specifically, the running time that we need to compute the conditional expectations. Let $t \in \{1, \ldots, |Y|\}$ and let $T \subseteq \cup_{i=1}^t Y_i$. To compute the value of the expectation $E_{x, \Theta}(\Gamma|B \cap (\cup_{i=1}^t Y_i) = T)$, it is sufficient to compute the probability value $P_{x, \Theta}(|e \cap B| = 1|B \cap (\cup_{i=1}^t Y_i) = T)$ for any $e \in E_G$ or $e \in E_D$. Furthermore, when $e \subseteq \cup_{i=1}^t Y_i$ the value of the probability is determined and equal to one or zero. Then, we suppose that $e = \{u, v\}$ is not contained in $\cup_{i=1}^t Y_i$. For every node $a \in V \setminus (Y_1 \cup \cdots \cup Y_t)$ let $X_a$ be the least depth bag in $\Upsilon$ that contains $a$. In particular, we have that $X_a \notin \{Y_1, \ldots, Y_t\}$ and let $Z_a$ be the lowest bag in $\{Y_1, \ldots, Y_t\}$ such that $Z_a$ belongs to the path from $X_a$ to the root. For every $a \in V \setminus (Y_1 \cup \cdots \cup Y_t)$ consider the quantity

$$g_a = P_{x, \Theta}(a \in B \mid B \cap (Y_a^Z \cup Z_a) = T \cap (Y_a^Z \cup Z_a)).$$

**Case 1.** Suppose that $u \notin \cup_{i=1}^t Y_i$ and $v \notin \cup_{i=1}^t Y_i$ and that $Z_u \neq Z_v$. Then, $P_{x, \Theta}^{Z_u}$ and $P_{x, \Theta}^{Z_v}$ are contained in the subtree induced by the bags $Y_1, \ldots, Y_t$. By construction in Algorithm 1 we have that $P_{x, \Theta}(|e \cap B| = 1|B \cap (\cup_{i=1}^t Y_i) = T) = g_a(1 - g_b) + g_b(1 - g_a)$. Furthermore, by denoting $T_a = T \cap (Y_a^Z \cup Z_a)$, we have

$$g_a = \sum_{(M,N) \in A_{\Delta}(T_a, u)} f_{X_a}(N) = \sum_{(M,N) \in A_{\Delta}(T_a, u)} \frac{x(Y_a^X \cup X_a, M \cup N)}{x(Y_a^X, M)},$$

for each $a \in \{u, v\}$. By the observation in (8), $g_a$ and $g_v$ can be computed in time $2^{O(k(2^m + \log(n)/\ell^*))}$. 

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Case 2. Suppose that \( u \notin \cup_{\ell=1}^I Y_\ell \) and \( v \notin \cup_{\ell=1}^I Y_\ell \), and that \( Z_u = Z_v = Z \). Let \( W \) be the lowest common ancestor of \( Y_u \) and \( Y_v \). In particular, \( Z \) is an ancestor of \( W \) and \( W \notin \{Y_1, \ldots, Y_I\} \). For every \( H \subseteq V_\Theta^W \setminus (V_\Theta^Z \cup Z) \) and \( K \subseteq W \setminus \mu(W) \) consider the quantity

\[
\beta(H, K) = \frac{x(V_\Theta^W \cup W, (T \cap V_\Theta^W) \cup H \cup K)}{x(V_\Theta^W, (T \cap V_\Theta^W) \cup H)}.
\]

Furthermore, for every \( H \subseteq V_\Theta^W \setminus (V_\Theta^Z \cup Z) \), \( K \subseteq W \setminus \mu(W) \) and \( a \in \{u, v\} \) let

\[
\gamma_a(H, K) = \sum_{(M, N) \in A_\Theta((T \cap V_\Theta^W) \cup H \cup K)} \frac{x(V_\Theta^W \cup Y_a, M \cup N)}{x(V_\Theta^W, M)}.
\]

Then, we have that \( \sum_{H \subseteq V_\Theta^W \setminus (V_\Theta^Z \cup Z)} \sum_{K \subseteq V_\Theta \setminus \mu(W)} \beta(H, K) \left( \gamma_u(H, K) (1 - \gamma_v(H, K)) + \gamma_v(H, K) (1 - \gamma_u(H, K)) \right) \) is equal to

\[
2^O(k(2r^+ + \log(n)/r^+))
\]

As before, the above summation can be computed in time \( 2^O(k(2r^+ + \log(n)/r^+)) \).

Case 3. Suppose that \( u \in \cup_{\ell=1}^I Y_\ell \) and \( v \notin \cup_{\ell=1}^I Y_\ell \) (the other case is symmetric). In this case, we have that \( \sum_{(M, N) \in A_\Theta((T \cap V_\Theta^W) \cup H \cup K)} \frac{x(V_\Theta^W \cup Y_a, M \cup N)}{x(V_\Theta^W, M)} \left( \gamma_u(H, K) (1 - \gamma_v(H, K)) + \gamma_v(H, K) (1 - \gamma_u(H, K)) \right) \) is equal to \( 1 - g_v \), and therefore we can compute it in time \( 2^O(k(2r^+ + \log(n)/r^+)) \).

As we observe at the end of Section 3.1, the optimization problem (1)-(5) can be solved in time \( 2^O(k(2r^+ + \log(n)/r^+)) |T|^{O(1)} \).

On the other hand, for every \( n \geq 16 \), by Lemma 5 we have

\[
k2^r + \frac{k \log(n)}{r^+} = k2^{\alpha_n^+} + \frac{k \log(n)}{\alpha_n^+} \leq 12k \log(n) \frac{\log \log(n)}{\log \log(n)}
\]

and therefore, the randomized algorithm and the derandomization can be all performed in time

\[
2^O(k \frac{\log \log(n)}{\log \log(n)}) |T|^{O(1)} = 2^{2O(k)} |T|^{O(1)}.
\]

To finish the proof, we verify the above equality by considering two cases. If \( k < \log \log(n) \), we have \( k \log(n) / \log \log(n) < \log(n) \) and the equality holds. Otherwise, if \( k \geq \log \log(n) \) and \( n \geq 4 \) we have \( k \log(n) / \log \log(n) \leq k \log(n) = 2 \log(k) + \log \log(n) \leq 2 \log(k) + k = 2^O(k) \).

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