We find spatial anisotropy in the asymptotic correlations of two-dimensional Ising models under non-equilibrium phase-ordering. Anisotropy is seen for critical and off-critical quenches and both conserved and non-conserved dynamics. We argue that spatial anisotropy is generic for scalar systems (including Potts models) with an anisotropic surface tension. Correlation functions will not be universal in these systems since anisotropy will depend on, e.g., temperature, microscopic interactions and dynamics, disorder, and frustration.

Stress-free Spatial Anisotropy in Phase-Ordering

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We find spatial anisotropy in the asymptotic correlations of two-dimensional Ising models under non-equilibrium phase-ordering. Anisotropy is seen for critical and off-critical quenches and both conserved and non-conserved dynamics. We argue that spatial anisotropy is generic for scalar systems (including Potts models) with an anisotropic surface tension. Correlation functions will not be universal in these systems since anisotropy will depend on, e.g., temperature, microscopic interactions and dynamics, disorder, and frustration.

Naively, it is reasonable to expect anisotropic correlations in phase-ordering. The 2d Ising model below $T_c$, for example, is spatially anisotropic at arbitrarily large scales in equilibrium correlations, in the resulting surface tension, and hence in the interface dynamics. In fact, we demonstrate below that spatial anisotropy is generic in scalar systems for quenches into an ordered phase with an anisotropic surface tension. Isotropic theories are inadequate for these systems.

For a coarse-grained scalar order parameter, $\phi$, in the continuum limit, we define an effective free energy in momentum space $F[\{\phi\}] = \int d^d k ((k^2 + D_k)\phi k \delta k - V_k)$, where $V_k(\{\phi\})$ is the Fourier transform of local potential terms and the differential operator $D_k$ includes anisotropic higher-order gradients. We define an angle dependent free-energy density $\epsilon(\mathbf{n})$ by restricting the integral in $F$ to momenta in a given direction $\mathbf{n}$ and averaging (denoted by angle brackets) over the random initial conditions. To do the restricted momentum integral in $(F)$, at late enough times for domain walls to be well defined, we use the anisotropic Porod law for the structure factor $S(\mathbf{k}) = \langle \phi \delta \mathbf{k} \rangle$ in general dimension $d$ (generalizing $2d$):

$$S(\mathbf{k}) \simeq 2^{d+2} \pi^{d-1} Ak^{-(d+1)} P(\mathbf{k}/k),$$

which holds for $L^{-1} \ll |\mathbf{k}| \ll \xi(\mathbf{n})^{-1}$, where $L(t)$ is the characteristic growing length-scale of the system and $\xi(\mathbf{n})$ is the domain wall width for domain walls with normal orientation $\mathbf{n} = \mathbf{k}/k$. $A(t) \sim L^{-1}$ is the average area density of domain wall and $P(\mathbf{n})$ is the angle distribution function of domain wall orientation [with $P(\mathbf{n}) = P(-\mathbf{n})$ and $\int d\mathbf{n} P(\mathbf{n}) = 1$]. We find

$$\epsilon(\mathbf{n}) = A\sigma(\mathbf{n}) P(\mathbf{n}),$$

where the leading $k^2$ and potential terms make isotropic contributions to $\sigma(\mathbf{n})$, while $D_k$ makes anisotropic contributions both directly and through $\xi(\mathbf{n})^{-1}$. From Eqn. (2), we identify $\sigma(\mathbf{n})$ as the effective angle dependent surface-tension for domain walls with normal direction $\mathbf{n}$.

Now consider a dissipative quench to $T = 0$, with no thermal noise. The dynamics will be given by $\dot{\phi}_k = -\Gamma(k)\delta F/\delta \phi_{-k}$, where the dot indicates a time-derivative, $\Gamma = \text{const.}$ corresponds to non-conserved dynamics and $\Gamma = \Gamma_2 k^2$ to conserved dynamics. The time rate of change of the energy-density is then simply

$$\dot{\epsilon}(\mathbf{n}) = \int dk k^{d-1} \delta F/\delta \phi_{-k} \dot{\phi}_k = -\int dk k^{d-1} \Gamma(\mathbf{k}) \dot{\phi}_{\mathbf{-k}},$$

where $\mathbf{k} \equiv \mathbf{k} n$. We have used the dynamics of $\phi$ to replace the functional derivative of the free-energy so that the resulting expression for $\dot{\epsilon}(\mathbf{n})$ has no explicit dependence on $F[\{\phi\}]$.

Under the assumption of isotropic correlations, we generically obtain different anisotropies in Eqns. (2) and (3) at arbitrarily late times — a contradiction since $\dot{\epsilon} \equiv \partial_t \epsilon$. Apart from possible anisotropies in the correlations, the anisotropy of $\epsilon(\mathbf{n})$ in Eqn. (2) is determined by the statics, through $\sigma(\mathbf{n})$. On the other hand the anisotropy of $\dot{\epsilon}(\mathbf{n})$ in Eqn. (3) is determined by the dynamics, through $\Gamma(\mathbf{k})$, in addition to possible contributions by the effective UV cutoff $\xi(\mathbf{n})^{-1}$. Since $\sigma$ and $\Gamma$ are independent, the anisotropies will not generally be equal unless anisotropic correlations make up the difference. For the general case, anisotropy in $\sigma(\mathbf{n})$ implies anisotropy in $S(\mathbf{k})$ even in the scaling limit.
The renormalization-group (RG) approach to phase-ordering [16] is easily generalized to include anisotropy. The only change is to note that any anisotropy of either $F[\phi]$ or $\Gamma(k)$ will be renormalized by microscopic details. (An illustration of this renormalization is the temperature dependence of the effective surface tension [13].) The demonstration that thermal noise will be asymptotically irrelevant for quenches to below $T_c$ will still apply, with the caveat that the effective $T = 0$ dynamics will include the surface-tension at the quench temperature. We then apply our above argument that predicts anisotropy with noise free dynamics. As a result, we expect anisotropy for all scalar quenches below $T_c$ [17]. Anisotropy may be renormalized by temperature, disorder, the details of the local interactions in the system (such as frustration), and even by global conservation laws that are “irrelevant” [16] in terms of growth laws. Anisotropy may also depend on the details of the microscopic dynamics.

In principle, we could try to choose the anisotropy of $\Gamma(k)$ to allow isotropic correlations despite an anisotropic $\sigma(n)$. This fine-tuning of the dynamics ($\Gamma$) with respect to the statics ($\sigma$) will not generically occur. The RG approach [14] shows that $\Gamma(k)$ will only be renormalized analytically, i.e. anisotropy will only enter at $O(k^4)$ and higher. For conserved dynamics, these contributions are subdominant in Eqn. (3) since the integral converges in the UV [14]. Thus neither the anisotropy of $\Gamma(k)$ nor the anisotropy of the core scale $\xi(n)$ will affect $\tilde{\epsilon}(n)$ through Eqn. (3). For conserved dynamics, even fine tuning of $\Gamma$ cannot eliminate anisotropic correlations. With non-conserved dynamics, the UV regime dominates the energy-dissipation integral [14] and both $\Gamma(k)$ and $\xi(n)$ make anisotropic contributions to $\tilde{\epsilon}(n)$. In principle the anisotropy of $\Gamma(k)$ could be renormalized to compensate $\sigma(n)$ — allowing isotropic correlations within our argument. However, we find anisotropy numerically for $T > 0$ even for non-conserved dynamics. We deduce that fine tuning of the kinetic prefactor does not occur.

For a system defined on a lattice, anisotropies will be present in the surface tension at $T = 0$, because lattice interactions are not rotationally invariant. Our argument then implies anisotropic correlations in the scaling limit. Anisotropy is only implied in some of the correlation functions and not necessarily in two-point correlations. In practice we find, in all of our numerical studies, significant anisotropy effects in the two-point correlations.

For the remainder of this letter, we explore 2d Ising models with nearest-neighbor interactions on a square lattice under quenches from random initial conditions. We find anisotropic correlations quite generally — for a variety of temperatures below $T_C$, of initial magnetizations, and for all of globally conserved, non-conserved, and locally conserved dynamics. These anisotropies do not decrease at late times, as would be expected for transient effects introduced by the dynamics at earlier times.

We measure the normalized correlations $C(r, t) = ((\phi(r)\phi(0)) - \langle \phi \rangle^2)/((\phi(0^+)\phi(0)) - \langle \phi \rangle^2)$, which ranges from 1 at short distances to 0 at infinity [15]. The anisotropic length-scale $L(n, t)$ of a system is defined by the scale in direction $n$ at which $C = 0.5$ for non-conserved dynamics, and by the first zero of $C$ for conserved dynamics. We scale correlations in all directions by the length-scale in the diagonal direction. A natural measure of anisotropy is $\chi = (L_{max}/L_{min} - 1)/(\sqrt{2} - 1)$, where $L_{max}$ is the maximum length-scale at a given time, $L_{min}$ is the minimum, and the normalization makes $\chi$ run from 0 (circle) to 1 (square) for convex contours of $C(r)$.

We first consider off-critical quenches with a global conservation law to prevent the magnetization from saturating. We couple the system to a Creutz spin reservoir of size 2 [16]: each randomly chosen spin is updated by a Metropolis algorithm, subject to an additional micro-canonical constraint that any spin change ($\pm 2$) fits in the spin reservoir. We study size $1024^2$ systems with $\langle \phi \rangle = 0.4$. A snapshot from a quench to $T = 0.2T_c$ in Fig. 2 illustrates the strong anisotropy even above the roughening transition [14]. We show some contour plots of the scaled correlations in Fig. 2. It is clear that the anisotropies are not limited to the small $r$ regime. The anisotropy is increasing at late times (see inset of Fig. 3). In the same regime, the spherically averaged correlations scale well. The latest anisotropies, at $t = 2049$ MCS, before finite-size effects entered were $\chi = 0.45$ ($T = 0$), 0.38 ($T = 0.2T_c$), and 0.12 ($T = 0.4T_c$). [Statistical error bars, with at least 30 samples in each case, are less than $\pm 0.001$.]
FIG. 2. Anisotropic contours of scaled correlations $C(r/L) = 0.9, 0.8, 0.7, 0.6, \text{and } 0.5$ (from the center) for an off-critical quench to $T = 0$ with $\langle \phi \rangle = 0.4$ and globally conserved Creutz dynamics. The times are $t = 513$ MCS (dotted), 1025 (dashed), and 2049 (solid). Also shown, scaled by 1.4 for clarity, are the $C = 0.5$ contours of quenches to $T/T_c = 0, 0.2, 0.4, \text{and } 0.9$ (solid, dashed, dot-dashed, and dotted lines, respectively) at $t = 1025$ MCS. The length-scale $L$ is such that, along the diagonal direction, $C(L) = 0.5$.

We also studied non-conserved critical quenches. With heat-bath dynamics and a sublattice update, late times in large lattices could be reached. Even so, the asymmetry remained small. For lattices of size $2048^2$, and a quench to $T = 0$, we show $dC/dx$ vs. $x (x = r/L)$, along the diagonal and lattice directions, in Fig. 3 [20]. We find $\chi = 0.03$ at the latest time ($t = 4097$ MCS, 74 samples, statistical error ±0.0003). The constant but orientation dependent domain wall width, $\xi(n)$, evident in the sharp downturn of $dC/dx$ near $x = 0$ in the diagonal correlations, increases $\chi$ by $O(1/L)$. This is significant for small anisotropies and early times. Directly subtracting this $O(1/L)$ contribution leads to $\chi$ slowly increasing with time (inset of Fig. 3), with a corrected latest value $\chi = 0.02$.

We have also simulated conserved 2d Ising systems with nearest-neighbor Kawasaki exchange dynamics and a Metropolis update. At low enough temperatures for the anisotropy of $\sigma(n)$ to be visible, the activated dynamics slows the simulations considerably. We explored size $256^2$ systems, with $\langle \phi \rangle = 0.4$ and $T = 0.4T_c$, up to times $t = 10^6$ MCS (10 samples). The length scales achieved ($L \lesssim 12$) are so small that $\chi \approx 0$ within numerical accuracy, so in Fig. 4 we plot $C(r)$ against the energy-energy correlation function $C_E(r) = \langle E(r)E(0) \rangle/(\langle E \rangle^2 - 1)$, where $E(r)$ is the number of broken bonds at site $r$ minus the equilibrium bulk average. This shows a significant and increasing difference between correlations in the lattice and diagonal directions.

In summary of our numerical results, we find anisotropic correlations in various quenched 2d Ising models. Anisotropy increases with decreasing temperature and for increasing net magnetization. Anisotropy effects are always slowly increasing at the latest times of our simulations. In all of our simulations, the spherically averaged correlations scale reasonably well while the anisotropy is still evolving. To study the non-zero asymptotic anisotropy in these systems, some sort of acceleration method is needed (see, e.g., [3]) — though in
In disordered and frustrated models, it has been argued that scaling functions will be “universal” — identical to those of the Ising systems that we have studied in this letter. While logarithmic growth is seen, it is thought to come from $L$ dependence in the kinetic prefactor $\Gamma$ — so that scaled correlations are unaffected. However, anisotropy should be renormalized by disorder and frustration, so we do not expect scaling function universality to hold in these systems. In frustrated 2d and 3d Ising models, the fairly large anisotropy seen numerically should remain at arbitrarily late-times. Hopefully, in experimental random-field systems (e.g.,), anisotropy can be measured directly.

We can generalize our argument around Eqs. (1)-(3) to systems with other types of singular defects. Using vector $O(n)$ order-parameters, and a generalized Porod’s law $S(k) = D(k/k)L^{-n}k^{-(d+n)}$, we find that the anisotropic contribution to the energy density is asymptotically negligible for systems without domain walls. However, other systems with dissipative dynamics in which domain walls dominate the asymptotic energetics will be anisotropic if the surface tension is anisotropic, e.g., Potts models (see ).

The growth laws of the characteristic length scale $L(t)$ will remain independent of any anisotropies present, as long as dynamical scaling is maintained. This follows from the Energy-Scaling approach since anisotropy does not change the scaling properties of the energy or the rate of energy dissipation. We would be surprised if the anisotropy affected the dynamical scaling of the correlations (see however ), though the scaling regime seems to be pushed to much later times as the anisotropy slowly develops. It remains an open question whether non-zero anisotropies have implications beyond the scaled correlations, such as in autocorrelation exponents.

In practice, isotropic theories have worked fairly well for spherically averaged correlations. Certainly, lattices, interactions, and dynamics can be chosen to minimize anisotropies. This would be desirable, for instance, in lattice simulations of isotropic fluid or polymer systems. However, the language of an isotropic zero-temperature phase-ordering fixed point is inappropriate for a scalar system with an anisotropic surface tension.

In summary, we expect anisotropy for any scalar lattice system quenched to below $T_c$, including disordered and/or frustrated systems. We expect the anisotropy to depend on the details of the system. Scaled correlation functions of anisotropic systems will not be universal.

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