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CHARACTERISTIC FUNCTIONS ON THE BOUNDARY OF A PLANAR DOMAIN NEED NOT BE TRACES OF LEAST GRADIENT FUNCTIONS

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Abstract. Given a smooth bounded planar domain Ω, we construct a compact set on the boundary such that its characteristic function is not the trace of a least gradient function. This generalizes the construction of Spradlin and Tamasan [3] when Ω is a disc.

1. Introduction

We let Ω be a bounded $C^2$ domain of $\mathbb{R}^2$. For a function $h \in L^1(\partial \Omega, \mathbb{R})$, the least gradient problem with boundary datum $h$ consists in deciding whether

$$\inf \left\{ \int_{\Omega} |Dw| ; w \in BV(\Omega) \text{ and } \text{tr}_{\partial \Omega} w = h \right\}$$

(1.1)

is achieved or not.

In the above minimization problem, $BV(\Omega)$ is the space of functions of bounded variation. It is the space of functions $w \in L^1(\Omega)$ having a distributional gradient $Dw$ which is a bounded Radon measure.

If the infimum in (1.1) is achieved, minimal functions are called functions of least gradient.

Sternberg, Williams and Ziemmer proved in [4] that if $h : \partial \Omega \rightarrow \mathbb{R}$ is a continuous map and if $\partial \Omega$ satisfies a geometric properties then there exists a (unique) function of least gradient. For further use, we note that the geometric property is satisfied by Euclidean balls.

On the other hand, Spradlin and Tamasan [3] proved that, for the disc $\Omega = \{ x \in \mathbb{R}^2 : |x| < 1 \}$, we may find a function $h_0 \in L^1(\partial \Omega)$ which is not continuous such that the infimum in (1.1) is not achieved. The function $h_0$ is the characteristic function of a Cantor type set $K \subset S^1 = \{ x \in \mathbb{R}^2 : |x| = 1 \}$

The goal of this article is to extend the main result of [3] to a general $C^2$ bounded open set $\Omega \subset \mathbb{R}^2$.

We prove the following theorem.

THEOREM 1.1. — Let $\Omega \subset \mathbb{R}^2$ be a bounded $C^2$ open set. Then there exists a measurable set $K \subset \partial \Omega$ such that the infimum

$$\inf \left\{ \int_{\Omega} |Dw| ; w \in BV(\Omega) \text{ and } \text{tr}_{\partial \Omega} w = \mathbb{1}_K \right\}$$

(1.2)

is not achieved.

The calculations in [3] are specific to the case $\Omega = \mathbb{D}$. The proof of Theorem 1.1 relies on new arguments for the construction of the Cantor set $K$ and the strategy of the proof.

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2. Strategy of the proof

2.1. The model problem. We illustrate the strategy developed to prove Theorem 1.1 on the model case $Q = (0, 1)^2$. Clearly, this model case does not satisfy the $C^2$ assumption.

Nevertheless, the flatness of $\partial Q$ allows to get a more general counterpart of Theorem 1.1. Namely, the counterpart of Theorem 1.1 (see Proposition 2.1 below) is no more an existence result of a set $K \subset \partial Q$ such that Problem (1.2) is not achieved. It is a non existence result of a least gradient function for $h = \mathbb{1}_M$ for any measurable domain $M \subset [0, 1] \times \{0\} \subset \partial Q$ with positive Lebesgue measure.

We thus prove the following result whose strategy of the proof is due to Petru Mironescu.

**Proposition 2.1** (P. Mironescu). — Let $\tilde{M} \subset [0, 1]$ be a measurable set with positive Lebesgue measure. Then the infimum in

$$\inf \left\{ \int_Q |Du| : w \in BV(Q) \text{ and } \tr_{\partial Q} w = \mathbb{1}_{\tilde{M} \times \{0\}} \right\}$$  

(2.1)

is not achieved.

This section is devoted to the proof of Proposition 2.1. We fix a measurable set $\tilde{M} \subset [0, 1]$ with positive measure and we let $h = \mathbb{1}_{\tilde{M} \times \{0\}}$. We argue by contradiction: we assume that there exists a minimizer $u_0$ of (2.1). We obtain a contradiction in 3 steps.

**Step 1.** Upper bound and lower bound

This first step consists in obtaining two estimates. The first estimate is the upper bound

$$\int_Q |Du_0| \leq \|\mathbb{1}_{\tilde{M} \times \{0\}}\|_{L^1(\partial Q)} = H^1(\tilde{M}).$$  

(2.2)

Here, $H^1(\tilde{M})$ is the length of $\tilde{M}$.

Estimate (2.2) follows from Theorem 2.16 and Remark 2.17 in [2]. Indeed, by combining Theorem 2.16 and Remark 2.17 in [2] we may prove that for $h \in L^1(\partial \Omega)$ and for all $\varepsilon > 0$ there exists a map $u_\varepsilon \in BV(\Omega)$ such that

$$\int_{\Omega} |Du_\varepsilon| \leq (1 + \varepsilon)\|h\|_{L^1(\partial \Omega)} \quad \text{and} \quad \tr_{\partial \Omega} u_\varepsilon = h.$$ 

The proof of this inequality when $\Omega$ is a half space is presented in [2]. It is easy to adapt the argument when $\Omega = Q = (0, 1)^2$. The extension for a $C^2$ set $\Omega$ is presented in Appendix E.

**Step 2.** Optimality of (2.2) (see (2.3))

The optimality of (2.2) is obtained via the following lemma.

**Lemma 2.2.** — For $u \in BV(Q)$ we have

$$\int_Q |D_2u| \geq \int_0^1 |\tr_{\partial Q} u(\cdot, 0) - \tr_{\partial Q} u(\cdot, 1)|.$$ 

Here, for $k \in \{1, 2\}$ we denoted

$$\int_Q |D_k u| = \sup \left\{ \int_Q u \partial_k \xi : \xi \in C^1_c(Q) \text{ and } |\xi| \leq 1 \right\}.$$
where $C^1_c(Q)$ are the set of real valued $C^1$-functions with compact support included in $Q$.

Lemma 2.2 is proved in Appendix B.1.

From Lemma 2.2 we get
\[
\int_Q |D_2u_0| \geq \int_0^1 |\text{tr}_{\partial Q}u_0(\cdot,0) - \text{tr}_{\partial Q}u_0(\cdot,1)| = \int_0^1 \mathbb{1}_{\tilde{M} \times \{0\}} = \mathcal{H}^1(\tilde{M}).
\]

Since we have
\[
\int_Q |Du_0| := \sup \left\{ \int_Q u \text{div}(\xi) : \xi = (\xi_1, \xi_2) \in C^1_c(Q, \mathbb{R}^2) \text{ and } \xi_1^2 + \xi_2^2 \leq 1 \right\}
\]
\[
\geq \int_Q |D_2u_0| \geq \mathcal{H}^1(\tilde{M}),
\]
we get the optimality of (2.2).

**Step 3.** A transverse argument

From (2.2) and (2.3) we may prove
\[
\int_Q |D_1u_0| = 0.
\]

Equality (2.4) is a direct consequence of the following lemma.

**LEMMA 2.3.** — Let $\Omega$ be a planar open set. If $u \in BV(\Omega)$ is such that
\[
\int_{\Omega} |Du| = \int_{\Omega} |D_2u|,
\]
then $\int_{\Omega} |D_1u| = 0$.

Lemma 2.3 is proved in Appendix B.2.

In order to conclude we state an easy lemma.

**LEMMA 2.4** (Poincaré inequality). — For $u \in BV(Q)$ satisfying $\text{tr}_{\partial Q}u = 0$ in $\{0\} \times [0,1]$ we have
\[
\int_Q |u| \leq \int_Q |D_1u|.
\]

Lemma 2.4 is proved in Appendix B.3.

Hence, from (2.4) and Lemma 2.4 we have $u_0 = 0$ which is in contradiction with $\text{tr}_{\partial Q}u_0 = \mathbb{1}_{\tilde{M} \times \{0\}}$ with $\mathcal{H}^1(\tilde{M}) > 0$.

2.2. **Outline of the proof of Theorem 1.1.** The idea is to adapt the above construction and argument to the case of a general $C^2$ domain $\Omega$. If $\Omega$ has a flat or concave part $\Gamma$ of the boundary $\partial \Omega$, then a rather straightforward variant of the above proof shows that $\mathbb{1}_M$, where $M$ is a non trivial part of $\Gamma$, is not the trace of a least gradient function.

**Remark 2.5.** — Things are more involved when $\Omega$ is convex. For simplicity we illustrate this fact when $\Omega = \mathbb{D} = \{ x \in \mathbb{R}^2 : |x| < 1 \}$. Let $M \subset S^1 \cap \{ (x, y) \in \mathbb{R}^2 : x < 0 \}$ be an arc whose endpoints are symmetric with respect to the $x$-axis. We let $(x_0, -y_0)$ and $(x_0, y_0)$ be the endpoints of $M$ (here $x_0 \leq 0$ and $y_0 > 0$).
We let \( C \) be the chord of \( \mathcal{M} \). On the one hand, if \( u \in C^1(\mathcal{D}) \cap W^{1,1}(\mathcal{D}) \) is such that \( \text{tr}_{\mathcal{M}} u = \mathbb{I}_{\mathcal{M}} \) then, using the Fundamental Theorem of calculus, we have for \(-y_0 < y < y_0\)

\[
\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} |\partial_x u(x, y)| \geq 1.
\]

Thus we easily get

\[
\int_\mathcal{D} |\nabla u| \geq \int_\mathcal{D} |\partial_x u| \geq \int_{-y_0}^{y_0} dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} |\partial_x u(x, y)| \geq 2y_0 = \mathcal{H}^1(C).
\]

Consequently, with the help of a density argument (e.g. Lemma A.1 in Appendix A) we obtain

\[
\inf \left\{ \int_\mathcal{D} |Du| ; u \in BV(\mathcal{D}) \text{ and } \text{tr}_{\mathcal{M}} u = \mathbb{I}_{\mathcal{M}} \right\} \geq \mathcal{H}^1(C).
\]

On the other hand we let \( \omega := \{(x, y) \in \mathbb{R}^2 : x < x_0\} \). It is clear that \( u_0 = \mathbb{I}_\omega \in BV(\mathcal{D}) \) and \( \text{tr}_{\mathcal{M}} u_0 = \mathbb{I}_{\mathcal{M}} \). Moreover

\[
\int_\mathcal{D} |Du_0| = \mathcal{H}^1(C).
\]

Consequently \( u_0 \) is a function of least gradient. We may do the same argument for a domain \( \Omega \) as soon as we have a chord entirely contained in \( \Omega \). This example suggest that for a convex set \( \Omega \), the construction of a set \( K \subset \partial \Omega \) such that (1.2) is not achieved has to be 'sophisticated'.

The strategy to prove Theorem 1.1 consists of constructing a special set \( K \subset \partial \Omega \) (of Cantor type) and to associate to \( K \) a set \( B_\infty \) (the analog of \( \mathcal{M} \times (0,1) \) in the model problem) which "projects" onto \( K \) and such that, if \( u_0 \) is a minimizer of (1.1), then

\[
\int_{B_\infty} |\vec{X} \cdot Du_0| \geq \mathcal{H}^1(K),
\]

(2.5)

Here, \( \vec{X} \) is a vector field satisfying \(|\vec{X}| \leq 1\). It is the curved analog of \( \vec{X} = e_2 \) used in the above proof.

By (2.5) (and Proposition E.1 in Appendix E), if \( u_0 \) is a minimizer, then

\[
\int_{\Omega \setminus B_\infty} |Du_0| + \int_{B_\infty} (|Du_0| - |\vec{X} \cdot Du_0|) = 0.
\]

(2.6)

We next establish a Poincaré type inequality implying that any \( u_0 \) satisfying (2.6) and \( \text{tr}_{\partial \Omega \setminus K} u = 0 \) is 0, which is not possible.

The heart of the proof consists of constructing \( K, B_\infty \) and \( \vec{X} \) (see Sections 4 and 5).

3. Notation, definitions

The ambient space is the Euclidean plan \( \mathbb{R}^2 \). We let \( \mathcal{B}_{\text{can}} \) be the canonical basis of \( \mathbb{R}^2 \).

a) The open ball centered at \( A \in \mathbb{R}^2 \) with radius \( r > 0 \) is denoted \( B(A, r) \).
b) A vector may be denoted by an arrow when it is defined by its endpoints (e.g. \( \overrightarrow{AB} \)). It may be also denoted by a letter in bold font (e.g. \( \mathbf{u} \)) or more simply by a Greek letter in normal font (e.g. \( \nu \)). We let also \( |\mathbf{u}| \) be the Euclidean norm of the vector \( \mathbf{u} \).

c) For a vector \( \mathbf{u} \) we let \( \mathbf{u}^\perp \) be the direct orthogonal vector to \( \mathbf{u} \), i.e., if \( \mathbf{u} = (x_1, x_2) \) then \( \mathbf{u}^\perp = (-x_2, x_1) \).

d) For \( A, B \in \mathbb{R}^2 \), the segment of endpoints \( A \) and \( B \) is denoted \( [AB] = \{A + t\overrightarrow{AB} : t \in [0, 1]\} \) and \( \text{dist}(A, B) = |\overrightarrow{AB}| \) is the Euclidean distance.

e) For a set \( U \subset \mathbb{R}^2 \), the topological interior of \( U \) is denoted by \( \overset{\circ}{U} \) and its topological closure is \( \overline{U} \).

f) For \( k \geq 1 \), a \( C^k \)-curve is the range of a \( C^k \) injective map from \( (0, 1) \) to \( \mathbb{R}^2 \). Note that, in this article, \( C^k \)-curves are not closed sets of \( \mathbb{R}^2 \).

g) For \( \Gamma \) a \( C^1 \)-curve, \( \mathcal{H}^1(\Gamma) \) is the 1-dimensional Hausdorff measure of \( \Gamma \).

h) For \( k \geq 1 \), a \( C^k \)-Jordan curve is the range of a \( C^k \) injective map from the unit circle \( S^1 \) to \( \mathbb{R}^2 \).

i) For \( \Gamma \) a \( C^1 \)-curve or a \( C^1 \)-Jordan curve, \( \mathcal{C} = [AB] \) is a chord of \( \Gamma \) when \( A, B \in \Gamma \) with \( A \neq B \).

j) If \( \Gamma \) is a \( C^1 \)-Jordan curve then, for \( A, B \in \Gamma \) with \( A \neq B \), the set \( \Gamma \setminus \{A, B\} \) admits exactly two connected components: \( \Gamma_1 \) and \( \Gamma_2 \). These connected components are \( C^1 \)-curves.

By smoothness of \( \Gamma \), it is clear that there is \( \eta_\Gamma > 0 \) such that for \( 0 < \text{dist}(A, B) < \eta_\Gamma \) there always exists a unique smallest connected component: we have \( \mathcal{H}^1(\Gamma_1) < \mathcal{H}^1(\Gamma_2) \) or \( \mathcal{H}^1(\Gamma_2) < \mathcal{H}^1(\Gamma_1) \).

If \( 0 < \text{dist}(A, B) < \eta_\Gamma \) we may define \( \overline{AB} \) by:

\[
\overline{AB} \text{ is the closure of the smallest curve between } \Gamma_1 \text{ and } \Gamma_2. \tag{3.1}
\]

k) In this article \( \Omega \subset \mathbb{R}^2 \) is a \( C^2 \) bounded open set.

4. CONSTRUCTION OF THE CANTOR SET \( \mathcal{K} \)

It is clear that, in order to prove Theorem 1.1, we may assume that \( \Omega \) is a connected set.

We fix \( \Omega \subset \mathbb{R}^2 \) a bounded \( C^2 \) open connected set. The set \( \mathcal{K} \subset \partial \Omega \) is a Cantor type set we will construct below.

4.1. First step: localization of \( \partial \Omega \). From the regularity of \( \Omega \), there exist \( \ell + 1 \) \( C^2 \)-open sets, \( \omega_0, \ldots, \omega_\ell \), such that \( \Omega = \omega_0 \setminus \overline{\omega_1} \cup \cdots \cup \overline{\omega_\ell} \) and

- \( \omega_i \) is simply connected for \( i = 0, \ldots, \ell \),
- \( \overline{\omega_i} \subset \omega_j \) for \( i = 1, \ldots, \ell \),
- \( \overline{\omega_i} \cap \overline{\omega_j} = \emptyset \) for \( 1 \leq i < j \leq \ell \).

We let \( \Gamma = \partial \omega_0 \). The Cantor type set \( \mathcal{K} \) we construct 'lives' on \( \Gamma \). Note that \( \Gamma \) is a Jordan-curve.

Let \( M_0 \in \Gamma \) be such that the inner curvature of \( \Gamma \) at \( M_0 \) is positive (the existence of \( M_0 \) follows from the Gauss-Bonnet formula). Then there exists \( r_0 \in (0, 1) \) such that \( [AB] \subset \overline{\Gamma} \) and \( [AB] \cap \partial \Omega = \{A, B\}, \forall A, B \in B(M_0, r_0) \cap \Gamma \). Note that we may assume \( 2r_0 < \eta_\Gamma \) (where \( \eta_\Gamma \) is defined in Section 3.j).

We fix \( A, B \in B(M_0, r_0) \cap \Gamma \) such that \( A \neq B \). We have:
By the definition of $M_0$ and $r_0$, the chord $\mathcal{C}_0 := [AB]$ is included in $\overline{\Pi}$.

- We let $\overline{AB}$ be the closure of the smallest part of $\Gamma$ which is delimited by $A, B$ (see (3.1)). We may assume that $\overline{AB}$ is the graph of $f \in C^2([0, \eta], \mathbb{R}^+)$ in the orthonormal frame $\mathcal{R}_0 = (A, e_1, e_2)$ where $e_1 = \overline{AB}/|\overline{AB}|$.
- The function $f$ satisfies $f(x) > 0$ for $x \in (0, \eta)$ and $f''(x) < 0$ for $x \in [0, \eta]$.

For further use we note that the length of the chord $[AB]$ is $\eta$ and that for intervals $I, J \subset [0, \eta]$, if $I \subset J$ then

$$\begin{align*}
\left\| f'_I \right\|_{L^\infty(I)} &\leq \left\| f'_J \right\|_{L^\infty(J)} \\
\left\| f''_I \right\|_{L^\infty(I)} &\leq \left\| f''_J \right\|_{L^\infty(J)}
\end{align*}$$

(4.1)

where $f_{|I}$ is the restriction of $f$ to $I$.

Replacing the chord $\mathcal{C}_0 = [AB]$ with a smaller chord of $\overline{AB}$ parallel to $\mathcal{C}_0$, we may assume that

$$0 < \eta < \min\left\{ \frac{1}{2} : \frac{1}{16\|f''\|_{L^\infty([0, \eta])}^2} : \frac{1}{2\left\| f' \right\|_{L^\infty([0, \eta])}\left\| f'' \right\|_{L^\infty([0, \eta])}} \right\}.$$  

(4.2)

We may also assume that

- Letting $D^+_0$ be the bounded open set such that $\partial D^+_0 = [AB] \cup \overline{AB}$ we have $\Pi_{\partial \Omega}$, the orthogonal projection on $\partial \Omega$, is well defined and of class $C^1$ in $D^+_0$.
- We have

$$1 + 4\|f''\|_{L^\infty}^2 \text{diam}(D^+_0) < \frac{16}{9}$$

(4.3)

where $\text{diam}(D^+_0) = \sup\{\text{dist}(M, N) : M, N \in D^+_0\}$. (Here we used (4.1).)

### 4.2. Step 2: Iterative construction

We are now in position to construct the Cantor type set $\mathcal{K}$ as a subset of $\overline{AB}$. The construction is iterative.

The goal of the construction is to get at step $N \geq 0$ a collection of $2^N$ pairwise disjoint curves included in $AB$ (denoted by $\{K^N_1, \ldots, K^N_{2^N}\}$) and their chords (denoted by $\{\mathcal{C}^N_1, \ldots, \mathcal{C}^N_{2^N}\}$).

The idea is standard: at step $N \geq 0$ we replace a curve $\Gamma_0$ included in $\overline{AB}$ by two curves included in $\Gamma_0$ (see Figure 4.1).

**Initialization.** We initialize the procedure by letting $K^0_1 := \overline{AB}$ and $\mathcal{C}^0_1 = \mathcal{C}_0 = [AB]$.

At step $N \geq 0$ we have:

- A set of $2^N$ curves included in $\overline{AB}$, $\{K^N_1, \ldots, K^N_{2^N}\}$. The curves $K^N_k$’s are mutually disjoint. We let $\mathcal{K}_N = \bigcup_{k=1}^{2^N} K^N_k$.
- A set of $2^N$ chords, $\{\mathcal{C}^N_1, \ldots, \mathcal{C}^N_{2^N}\}$ such that for $k = 1, \ldots, 2^N$, $\mathcal{C}^N_k$ is the chord of $K^N_k$.

**Remark 4.1.** — (1) Note that since the $\mathcal{C}^N_k$’s are chords of $\overline{AB}$ and since in the frame $\mathcal{R}_0 = (A, e_1, e_2)$, $\overline{AB}$ is the graph of a function, none of the chords $\mathcal{C}^N_k$ is vertical, i.e., directed by $e_2$.

Since the chords $\mathcal{C}^N_k$ are not vertical, for $k \in \{1, \ldots, 2^N\}$, we may define $\nu_{\mathcal{C}^N_k}$ as the unit vector orthogonal to $\mathcal{C}^N_k$ such that $\nu_{\mathcal{C}^N_k} = \alpha e_1 + \beta e_2$ with $\beta > 0$. 

(2) For $\eta$ satisfying (4.2), if we consider a chord $C^N_k$ and a straight line $D$ orthogonal to $C^N_k$ and intersecting $K^N_k$, then the straight line $D$ intersect $K^N_k$ at exactly one point. This fact is proved in Appendix C.1.

**Induction rules.** From step $N \geq 0$ to step $N + 1$ we follow the following rules:

1. For each $k \in \{1, \ldots, 2^N\}$, we let $\eta^N_k$ be the length of $C^N_k$. Inside the chord $C^N_k$ we center a segment $I^N_k$ of length $(\eta^N_k)^2$.
2. With the help of Remark 4.1.2, we may define two distinct points of $K^N_k$ as the intersection of $K^N_k$ with straight lines orthogonal to $C^N_k$ which pass to the endpoints of $I^N_k$.
3. These intersection points are the endpoints of a curve $\tilde{K}^N_k$ included in $K^N_k$. We let $K^N_{2k-1}$ and $K^N_{2k}$ be the connected components of $K^N_k \setminus \tilde{K}^N_k$. We let also
   - $C^{N+1}_{2k-1}$ and $C^{N+1}_{2k}$ be the corresponding chords;
   - $K_{N+1} = \bigcup_{k=1}^{2^N+1} K^N_{2k}$. 

**Definition 4.2.** — A natural terminology consists in defining the father and the sons of a chord or a curve:

- $\mathcal{F}(C^{N+1}_{2k-1}) = \mathcal{F}(C^{N+1}_{2k}) = C^N_k$ is the father of the chords $C^{N+1}_{2k-1}$ and $C^{N+1}_{2k}$.
- $\mathcal{F}(K^{N+1}_{2k-1}) = \mathcal{F}(K^{N+1}_{2k}) = K^N_k$ is the father of the curves $K^{N+1}_{2k-1}$ and $K^{N+1}_{2k}$.
- $\mathcal{S}(C^N_k) = \{C^{N+1}_{2k-1}, C^{N+1}_{2k}\}$ is the set of sons of the chord $C^N_k$, i.e. $\mathcal{F}(C^{N+1}_{2k-1}) = \mathcal{F}(C^{N+1}_{2k}) = C^N_k$.
- $\mathcal{S}(K^N_k) = \{K^{N+1}_{2k-1}, K^{N+1}_{2k}\}$ is the set of sons of the curve $K^N_k$, i.e., $\mathcal{F}(K^{N+1}_{2k-1}) = \mathcal{F}(K^{N+1}_{2k}) = K^N_k$.

The inductive procedure is represented in Figure 4.1.

![Figure 4.1. Induction step](image)

In Figures 4.2 and 4.3 the two first iterations of the process are represented.

![Figure 4.2. First iteration of the process](image)  ![Figure 4.3. Second iteration of the process](image)

We now define the Cantor type set

$$\mathcal{K} = \bigcap_{N \geq 0} \overline{\mathcal{K}_N}. \quad (4.4)$$
The Cantor type set $K$ is fat:

**Proposition 4.3.** — We have $\mathcal{H}^1(K) > 0$.

This proposition is proved in Appendix C.3.

5. **Construction of a sequence of functions**

A key argument in the proof of Theorem 1.1 is the use of the coarea formula to calculate a lower bound for (1.2). The coarea formula is applied to a function adapted to the set $K$.

For $N = 0$ we let

- $D_0^-$ be the compact set delimited by $K_0 = \overrightarrow{AB}$ and $\mathcal{C}_0 := [AB]$ the chord of $K_0$.
- We recall that we fixed a frame $\mathcal{R}_0 = (A, e_1, e_2)$ where $e_1 = \overrightarrow{AB} / |\overrightarrow{AB}|$. For $\sigma = (\sigma_1, 0) \in \mathcal{C}_0$, we define:

$I_\sigma$ is the connected component of $\{(\sigma_1, t) \in \Omega : t \leq 0\}$ which contains $\sigma$. (5.1)

$I_\sigma$ is a vertical segment included in $\Omega$.

- $D_0^- = \cup_{\sigma \in \mathcal{C}_0} I_\sigma$.
- We now define the maps

$$\Psi_0 : D_0^- \to \mathcal{C}_0^0 x \mapsto \Pi_{\mathcal{C}_0^0}(x)$$

and

$$\Psi_0 : D_0^- \cup D_0^+ \to \mathcal{C}_0^0 x \mapsto \begin{cases} \Pi_{\partial \Omega}(x) & \text{if } x \in D_0^+ \\ \Pi_{\partial \Omega}[\Psi_0(x)] & \text{if } x \in D_0^- \end{cases}$$

where $\Pi_{\partial \Omega}$ is the orthogonal projection on $\partial \Omega$ and $\Pi_{\mathcal{C}_0^0}$ is the orthogonal projection on $\mathcal{C}_0^0$. Note that, in the frame $\mathcal{R}_0$, for $x = (x_1, x_2) \in D_0^-$, we have $\Pi_{\mathcal{C}_0^0}(x) = (x_1, 0)$.

For $N = 1$ and $k \in \{1, 2\}$ we let:

- $D_k^1$ be the compact set delimited by $K_k^1$ and $\mathcal{C}_k^1$;
- $T_k^1$ be the compact right-angled triangle (with its interior) having $\mathcal{C}_k^1$ as side adjacent to the right angle and whose hypotenuse is included in $\mathcal{C}_0^0$;
- $H_k^1$ be the hypotenuse of $T_k^1$.

We now define $D_1^- = \Psi_0^{-1}(H_1^1 \cup H_2^1)$, $T_1 = T_1^1 \cup T_2^1$ and $D_1^+ = D_1^1 \cup D_2^1$.

We first consider the map

$$\Psi : T_1 \cup D_1^- \to \mathcal{C}_1^1 \cup \mathcal{C}_2^1 x \mapsto \begin{cases} \Pi_{\mathcal{C}_1^1}(x) & \text{if } x \in T_1^1 \\ \Pi_{\mathcal{C}_2^1}(x) & \text{if } x \in D_1^- \end{cases}$$

In Appendix D (Lemma D.1 and Remark D.2), it is proved that the triangles $T_1^1$ and $T_2^1$ are disjoint. Thus the map $\Psi$ is well defined.
By projecting $C_1 \cup C_2$ on $\partial \Omega$ we get

$$
\Psi_1 : \ T_1 \cup D_1^- \cup D_1^+ \to K_1
$$

$$
x \mapsto \begin{cases} 
\Pi_{\partial \Omega}(x) & \text{if } x \in D_1^+ \\
\Pi_{\partial \Omega}(\hat{\Psi}_1(x)) & \text{if } x \in T_1 \cup D_1^-.
\end{cases}
$$

\begin{figure}[h]
\centering
\includegraphics[width=0.6\textwidth]{Figure_5.1}
\caption{The sets defined at Step $N = 1$ and the dashed level line of $\Psi_1$ associated to $\sigma \in K_1$}
\end{figure}

For $N \geq 1$, we first construct $\hat{\Psi}_{N+1}$ and then $\Psi_{N+1}$ is obtained from $\hat{\Psi}_{N+1}$ and $\Pi_{\partial \Omega}$.

For $k \in \{1, \ldots, 2^{N+1}\}$, we let

- $D_k^{N+1}$ be the compact set delimited by $K_k^{N+1}$ and $C_k^{N+1}$ (recall that $C_k^{N+1}$ is the chord associated to $K_k^{N+1}$);
- $T_k^{N+1}$ be the right-angled triangle (with its interior) having $C_k^{N+1}$ as side adjacent to the right angle and whose hypotenuse is included in $F(C_k^{N+1})$.

Here $F(C_k^{N+1})$ is the father of $C_k^{N+1}$ (see Definition 4.2);

- $H_k^{N+1} \subset F(C_k^{N+1})$ be the hypotenuse of $T_k^{N+1}$.

We denote

$$
T_{N+1} = \bigcup_{k=1}^{2^{N+1}} T_k^{N+1}, \quad D_{N+1}^- = \bigcup_{k=1}^{2^{N+1}} H_k^{N+1}, \quad \text{and} \quad D_{N+1}^+ = \bigcup_{k=1}^{2^{N+1}} D_k^{N+1}.
$$

\begin{figure}[h]
\centering
\includegraphics[width=0.6\textwidth]{Figure_5.2}
\caption{Induction. The bold lines correspond to the new iteration}
\end{figure}
Remark 5.1. — It is easy to check that for $N \geq 0$:

1. $T_{N+1} \subset D_N^-$,
2. if $x \in T_N$ then $x \notin T_{N'}$ for $N' \geq N + 1$ (here $T_0 = \emptyset$).

We now define

$$\tilde{\Psi}_{N+1} : T_{N+1} \cup D_{N+1}^- \to \bigcup_{k=1}^{2^{N+1}} C_k^{N+1}$$

$$x \mapsto \begin{cases} 
\Pi_k(x) & \text{if } x \in T_k^{N+1} \\
\tilde{\Psi}_{N+1}(x) & \text{if } x \in \tilde{\Psi}_{N+1}^{-1}(\bigcup_{k=1}^{2^{N+1}} H_k^{N+1}).
\end{cases}$$

In Appendix D (Lemma D.1 and Remark D.2), it is proved that for $N \geq 1$, the triangles $T_k^N$ for $k = 1, \ldots, 2^N$ are mutually disjoint. recursively, we find that all the $\tilde{\Psi}_N$’s are well-defined.

As in the Initialization Step, we get $\Psi_{N+1}$ from $\tilde{\Psi}_{N+1}$ by projecting $\bigcup_{k=1}^{2^{N+1}} C_k^{N+1}$ on $\partial \Omega$:

$$\Psi_{N+1} : T_{N+1} \cup D_{N+1}^- \cup D_{N+1}^+ \to \mathcal{K}_{N+1}$$

$$x \mapsto \begin{cases} 
\Pi_{\partial \Omega}(\tilde{\Psi}_{N+1}(x)) & \text{if } x \in T_{N+1} \cup D_{N+1}^- \\
\Pi_{\partial \Omega}(x) & \text{if } x \in D_{N+1}^+.
\end{cases}$$

It is easy to see that $\Psi_{N+1}(T_{N+1} \cup D_{N+1}^- \cup D_{N+1}^+) = \mathcal{K}_{N+1}$.

6. Basic properties of $B_\infty$ and $\Psi_N$

6.1. Basic properties of $B_\infty$. We set $B_N = T_N \cup D_N^+ \cup D_N^-$. It is easy to check that for $N \geq 0$ we have $B_{N+1} \subset B_N$ and $\mathcal{K} \subset \partial B_N$. Therefore we may define

$$B_\infty = \bigcap_{N \geq 0} \overline{B_N}$$

which is compact and satisfies $\mathcal{K} \subset \partial B_\infty$.

We are going to prove:

**Lemma 6.1.** — The interior of $B_\infty$ is empty.

**Proof of Lemma 6.1.** — From Lemma D.1 (and Remark D.2) in Appendix D combined with Hypothesis (4.2), we get two fundamental facts:

1. The triangles $T_1^N, \ldots, T_{2^N+1}^N$ are mutually disjoint.
2. We have:

$$\mathcal{H}^1(H_k^{N+1}) < \frac{\mathcal{H}^1(\mathcal{F}(C_k^N))}{2}. \quad (6.1)$$

For a non empty set $A \subset \mathbb{R}^2$ we let

$$\text{rad}(A) = \sup\{r \geq 0 : \exists x \in A \text{ such that } \overline{B(x, r)} \subset A\}.$$ 

Note that the topological interior of $A$ is empty if and only if $\text{rad}(A) = 0$.

On the one hand, it is not difficult to check that for sufficiently large $N$

$$\text{rad}(B_N) = \text{rad}(B_N \cap D_N^-). \quad (6.2)$$

On the other hand, using (6.1) we obtain for $N \geq 1$:

$$\text{rad}(B_{N+1} \cap D_{N+1}^-) \leq \frac{\text{rad}(B_N \cap D_N^-)}{2}. \quad (6.3)$$
Consequently, by combining (6.2) and (6.3) we get the existence of $C_0$ such that

$$\text{rad}(B_N) \leq \frac{C_0}{2^N}. \tag{6.4}$$

Since $B_\infty = \cap_{N \geq 0} B_N$, from (6.4) we get that $\text{rad}(B_\infty) = 0$. \hfill \Box

6.2. Basic properties of $\Psi_N$. We now prove the key estimate for $\Psi_N$:

**Lemma 6.2.** — There exists $b_N = o_N(1)$ such that for $N \geq 1$ and $U$ a connected component of $B_N$, the restriction of $\Psi_N$ to $U$ is $(1 + b_N)$-Lipschitz.

**Proof.** — Let $N \geq 1$ and $U$ be a connected component of $B_N$. The restriction of $\Psi_N$ to $U \cap (T_N \cup D_N^\circ)$ is obtained as composition of orthogonal projections on straight lines and thus is $1$-Lipschitz.

There exists $b_N = o_N(1)$ such that the projection $P_N := \Pi_{\partial \Omega}$ defined in $\overline{D_N^\circ}$ is $(1 + b_N)$-Lipschitz. The functions $\Psi_N$ are either the composition of $\Psi_N$ with $P_N$ or $\Psi_N = P_N$. Consequently the restriction of $\Psi_N$ to $U$ is $(1 + b_N)$-Lipschitz. \hfill \Box

In the following we will not use $\Psi_N$ but "its projection" on $\mathbb{R}$. For $N \geq 1$ and $k \in \{1, \ldots, 2^N\}$, we let $B_k^N := \Psi_N^{-1}(K_k^N)$ and we define

$$\Pi_{k,N} : B_k^N \to \mathbb{R} \quad x \mapsto \mathcal{H}^1(\overline{A\Psi_N(x)})$$

where $\overline{A\Psi_N(x)} \subset \overline{AB}$ is defined by (3.1) as the smallest connected component of $\partial \Omega \setminus \{A, \Psi_N(x)\}$ if $\Psi_N(x) \neq A$ and $\overline{A\Psi_N(x)} = \{A\}$ otherwise.

**Lemma 6.3.** — For $N \geq 1$ there exists $c_N \in (0, 1)$ with $c_N = o_N(1)$ such that for $k \in \{1, \ldots, 2^N\}$ the function $\Pi_{k,N} : B_k^N \to \mathbb{R}$ is $(1 + c_N)$-Lipschitz.

**Proof.** — Let $N \geq 1$, $k \in \{1, \ldots, 2^N\}$ and let $x, y \in B_k^N$ be such that $\Psi_N(x) \neq \Psi_N(y)$. It is clear that we have

$$|\Pi_{k,N}(x) - \Pi_{k,N}(y)| = \mathcal{H}^1(\overline{\Psi_N(y)\Psi_N(x)})$$

where $\overline{\Psi_N(y)\Psi_N(x)} \subset K_k^N$ is defined by (3.1) as the smallest connected component of $\partial \Omega \setminus \{\Psi_N(y), \Psi_N(x)\}$.

Moreover, from Lemma C.3 in Appendix C.2, we have the existence of $C \geq 1$ independent of $N$ and $k$ such that for $x, y \in B_k^N$ such that $\Psi_N(x) \neq \Psi_N(y)$ we have (denoting $X := \Psi_N(x), Y := \Psi_N(y)$)

$$\text{dist} (X, Y) \leq \mathcal{H}^1 (\overline{XY}) \leq \text{dist} (X, Y) [1 + C \text{dist} (X, Y)]$$

and

$$\mathcal{H}^1(K_k^N) \leq \mathcal{H}^1(\mathcal{E}^N_k) \left[1 + C \mathcal{H}^1(\mathcal{E}^N_k)\right].$$

From Step 1 in the proof of Proposition 4.4 (Appendix C.3) we have

$$\max_{k=1,\ldots,2^N} \mathcal{H}^1(\mathcal{E}^N_k) \leq \left(\frac{2}{3}\right)^N.$$
Thus, letting $a_N := \left(\frac{2}{3}\right)^N \left[1 + C\left(\frac{2}{3}\right)^N\right]$, we have $a_N \to 0$, and since $\overline{XY} \subset K_k^N$ we get:
\[
\text{dist}(X, Y) \leq \mathcal{H}^1(\overline{XY}) \leq \mathcal{H}^1(K_k^N) \\
\leq \mathcal{H}^1(\mathcal{C}_k^N) \left[1 + C\mathcal{H}^1(\mathcal{C}_k^N)\right] \leq a_N(1 + Ca_N).
\]
Thus, letting $\tilde{a}_N = \max\{a_N(1 + Ca_N), |b_N|\}$ where $b_N$ is defined in Lemma 6.2, we get:
\[
\mathcal{H}^1(\overline{XY}) = |\Pi_{k,N}(x) - \Pi_{k,N}(y)| \leq \mathcal{H}^1(\Psi_N(y)\Psi_N(x))(1 + C\tilde{a}_N) \\
\leq (1 + \tilde{a}_N)(1 + C\tilde{a}_N)|x - y|.
\]
Therefore, letting $c_N$ be such that $1 + c_N = (1 + \tilde{a}_N)(1 + C\tilde{a}_N)$ we have $c_N = o_N(1)$, $c_N$ is independent of $k \in \{1, \ldots, 2^N\}$ and $\Pi_{k,N}$ is $(1 + c_N)$-Lipschitz. □

7. Proof of Theorem 1.1

We are now in position to prove Theorem 1.1. This is done by contradiction. We assume that there exists a map $u_0 \in BV(\Omega)$ which minimizes (1.2).

7.1. Upper bound. The first step in the proof is the estimate
\[
\int_\Omega |Du_0| \leq \|\mathbb{1}_K\|_{L^1(\partial\Omega)} = \mathcal{H}^1(K). \tag{7.1}
\]
This estimate is obtained by proving that for all $\varepsilon > 0$ there exists $u_\varepsilon \in W^{1,1}(\Omega)$ such that $\text{tr}_{\partial\Omega}u_\varepsilon = \mathbb{1}_K$ and
\[
\|\nabla u_\varepsilon\|_{L^1(\Omega)} \leq (1 + \varepsilon)\|\text{tr}_{\partial\Omega}u_\varepsilon\|_{L^1(\Omega)} = (1 + \varepsilon)\mathcal{H}^1(K). \tag{7.2}
\]
Proposition E.1 in Appendix E gives the existence of such $u_\varepsilon$’s.
Clearly (7.2) implies (7.1).

7.2. Optimality of the upper bound. In order to have a contradiction we follow the strategy of Spradlin and Tamasan in [3]. We fix a sequence $(u_n)_n \subset C^1(\Omega)$ such that
\[
u_n \in W^{1,1}(\Omega) : u_n \to u \text{ in } L^1(\Omega) : \int_\Omega |\nabla u_n| \to \int_\Omega |Du_0| : \text{tr}_{\partial\Omega}u_n = \text{tr}_{\partial\Omega}u_0. \tag{7.3}
\]
Note that (7.3) implies
\[
\int_F |\nabla u_n| \to \int_F |Du| \text{ for all } F \subset \Omega \text{ relatively closed set.} \tag{7.4}
\]
Such a sequence can be obtained via partition of unity and smoothing; see the proof of Theorem 1.17 in [2]. For the convenience of the reader a proof is presented in Appendix A (see Lemma A.1).

For further use, let us note that the sequence $(u_n)_n$ constructed in Appendix A satisfies the following additional property:
\[
\text{ If } u_0 = 0 \text{ outside a compact set } L \subset \overline{\Omega} \text{ and if } \omega \text{ is an open set such that dist}(\omega, L) > 0 \text{ then, for large } n, u_n = 0 \text{ in } \omega.
\]
For \( x \in B_0 \) we let
\[
V_0(x) = \begin{cases} 
\nu_{\partial \Omega}(x) & \text{if } x \in D_0^+ \\
(0, 1) & \text{if } x \in D_0^-
\end{cases}
\] (7.5)
and for \( N \geq 0, x \in B_{N+1} \) we let
\[
V_{N+1}(x) = \begin{cases} 
V_N(x) & \text{if } x \in B_N \setminus \bar{T}_N^{N+1} \\
\nu_{\partial_k N+1} & \text{if } x \in \bar{T}_k^{N+1}
\end{cases}
\] (7.6)
where, for \( \sigma \in \partial \Omega \), \( \nu_\sigma \) is the normal outward of \( \Omega \) in \( \sigma \) and \( \nu_{\partial_k N+1} \) is defined in Remark 4.1.1.

We now prove the following lemma.

**Lemma 7.1.** — When \( N \to \infty \) we may define \( V_\infty(x) \) a.e. \( x \in B_\infty \) by
\[
V_\infty : B_\infty \to \mathbb{R}^2 \quad x \mapsto \lim_{N \to \infty} V_N(x)
\] (7.7)
Moreover, from dominated convergence, we have:
\[
V_N \mathbb{I}_{B_N} \to V_\infty \mathbb{I}_{B_\infty} \text{ in } L^1(\Omega).
\]

**Proof.** — If \( x \in B_\infty \setminus \bigcup_{N \geq 1} T_N \), then we have \( V_N(x) = V_0(x) \) for all \( N \geq 1 \). Thus \( \lim_{N \to \infty} V_N(x) = V_0(x) \).

For a.e. \( x \in B_\infty \cap \bigcup_{N \geq 1} T_N \) there exists \( N_0 \geq 1 \) such that \( x \in \bar{T}_{N_0} \). Therefore for all \( N > N_0 \) we have \( V_N(x) = V_{N_0}(x) \). Consequently \( \lim_{N \to \infty} V_N(x) = V_{N_0}(x) \). \( \square \)

This section is devoted to the proof of the following lemma:

**Lemma 7.2.** — For all \( w \in C^\infty \cap W^{1,1}(\Omega) \) such that \( \text{tr}_{\partial \Omega} w = \mathbb{I}_K \) we have
\[
\int_{B_\infty \cap \Omega} |\nabla w \cdot V_\infty| \geq \mathcal{H}^1(K)
\]
where \( V_\infty \) is the vector field defined in (7.7).

**Remark 7.3.** — Since \( |V_\infty(x)| = 1 \) for a.e. \( x \in B_\infty \), it is clear that Lemma 7.2 implies that for all \( n \) we have
\[
\int_{B_\infty \cap \Omega} |\nabla u_n| \geq \mathcal{H}^1(K).
\]
From (7.4) we have:
\[
\int_{B_\infty \cap \Omega} |Du_0| \geq \mathcal{H}^1(K).
\]
Section 7.3 is devoted to a sharper argument than above to get
\[
\int_{B_\infty \cap \Omega} |\nabla u_n| \geq \int_{B_\infty \cap \Omega} |\nabla u_n \cdot V_\infty| + \delta
\]
with \( \delta > 0 \) is independent of \( n \). The last estimate will imply \( \int_{B_\infty \cap \Omega} |Du_0| \geq \mathcal{H}^1(K) + \delta \) which will be the contradiction we are looking for.
Proof of Lemma 7.2. — We will first prove that for \( w \in C^\infty \cap W^{1,1}(\Omega) \) such that \( \text{tr}_{\partial \Omega} w = \Pi_K \), we have

\[
\int_{B_N \cap \Omega} |\nabla w \cdot V_N| \geq \frac{\mathcal{H}^1(K)}{1 + o_N(1)}, \tag{7.8}
\]

where \( V_N \) is the vector field defined in (7.5) and (7.6).

Granted (7.8), we conclude as follows: if \( w \in C^\infty \cap W^{1,1}(\Omega) \) such that \( \text{tr}_{\partial \Omega} w = \Pi_K \), then

\[
\int_{B_N \cap \Omega} |\nabla w \cdot V_N| = \lim_{N \to \infty} \int_{B_N \cap \Omega} |\nabla w \cdot V_N| \geq \lim_{N \to \infty} \frac{\mathcal{H}^1(K)}{1 + o_N(1)} = \mathcal{H}^1(K),
\]

by dominated convergence.

It remains to prove (7.8). We fix \( w \in C^\infty \cap W^{1,1}(\Omega) \) such that \( \text{tr}_{\partial \Omega} w = \Pi_K \). Using the Coarea Formula we have for \( N \geq 1 \) and \( k \in \{1, ..., 2^N\} \), with the help of Lemma 6.3, we have

\[
(1 + c_N) \int_{B_N^{(k)} \cap \Omega} |\nabla w \cdot V_N| \geq \int_{B_N^{(k)} \cap \Omega} |\nabla \Pi_{k,N} w| |\nabla w \cdot V_N| = \int_\mathbb{R} dt \int_{\Pi_{k,N}^{-1}(\{t\}) \cap \Omega} |\nabla w \cdot V_N|.
\]

Here, if \( \Pi_{k,N}^{-1}(\{t\}) \) is non trivial, then \( \Pi_{k,N}^{-1}(\{t\}) \) is a polygonal line:

\[
\Pi_{k,N}^{-1}(\{t\}) = I_{\sigma(t,k,N)} \cup I_{k,N,t}^1 \cup \cdots \cup I_{k,N,t}^{N+1},
\]

where

- \( \sigma(t,k,N) \in [AB] \) is such that \([AB] \cap \Pi_{k,N}^{-1}(\{t\}) = \{\sigma(t,k,N)\} \),
- \( I_{\sigma(t,k,N)} \) is defined in (5.1),
- for \( l = 1, ..., N \) we have \( I_{k,N,t} = \Pi_{k,N}^{-1}(\{t\}) \cap T_{N+1-l} \),
- \( I_{k,N,t}^N = \Pi_{k,N}^{-1}(\{t\}) \cap D_N^+ \).

From the Fundamental Theorem of calculus and from the definition of \( V_N \), denoting

- \( I_{\sigma(t,k,N)} = [M_0, M_1] \), where \( M_0 \in \partial \Omega \setminus \overline{AB} \) and \( M_1 = \sigma(t,k,N) \),
- \( I_{k,N,t} = [M_l, M_{l+1}] \), \( l = 1, ..., N + 1 \) and \( M_{N+2} \in K_k^N \),

we have for a.e. \( t \in \Pi_{k,N}(K_k^N) \) and using the previous notation,

\[
\int_{[M_l, M_{l+1}]} |\nabla w \cdot V_N| = |w(M_{l+1}) - w(M_l)|.
\]

Here we used the convention \( w(M_l) = \text{tr}_{\partial \Omega} w(M_l) \) for \( l = 0 \) and \( N + 2 \).

Therefore for a.e \( t \in \Pi_{k,N}(K_k^N) \) we have

\[
\int_{\Pi_{k,N}^{-1}(\{t\}) \cap \Omega} |\nabla w \cdot V_N| \geq |\text{tr}_{\partial \Omega} w(M_{N+2}) - \text{tr}_{\partial \Omega} w(M_0)| = \Pi_K(M_{N+2}).
\]

Since \( K \subset K_N = \bigcup_{k=1}^{2^N} K_k^N \), we may thus deduce that

\[
(1 + c_N) \int_{B_N \cap \Omega} |\nabla w \cdot V_N| = (1 + c_N) \sum_{k=1}^{2^N} \int_{B_N^{(k)} \cap \Omega} |\nabla w \cdot V_N| \geq \int_{AB} \Pi_K = \mathcal{H}^1(K).
\]
The last estimate clearly implies (7.8) and completes the proof of Lemma 7.2. □

7.3. Transverse argument. We assumed that there exists a map \( u_0 \) which solves Problem (1.2).

We investigate the following dichotomy:

- \( u_0 \not\equiv 0 \) in \( \Omega \setminus B_\infty \);
- \( u_0 \equiv 0 \) in \( \Omega \setminus B_\infty \).

We are going to prove that both cases lead to a contradiction.

7.3.1. The case \( u_0 \not\equiv 0 \) in \( \Omega \setminus B_\infty \). We thus have

\[
\int_{\Omega \setminus B_\infty} |u_0| > 0.
\]

In this case, since \((\text{tr} \partial \Omega u_0)|_{\partial \Omega \setminus \partial B_\infty} \equiv 0\), we have

\[
\delta := \int_{\Omega \setminus B_\infty} |Du_0| > 0.
\] (7.9)

Estimate (7.9) is a direct consequence of the following lemma applied on each connected components of \( \Omega \setminus B_\infty \).

Lemma 7.4 (Weak Poincaré lemma). — Let \( \omega \subset \mathbb{R}^2 \) be an open connected set.

Assume that there exist \( x_0 \in \partial \omega \) and \( r > 0 \) such that \( \omega \setminus B(x_0, r) \) is Lipschitz.

If \( u \in BV(\omega) \) satisfies \( \text{tr} \partial_{\omega \cap B(x_0, r)} = 0 \) and \( \int_\omega |Du| = 0 \) then \( u = 0 \).

Lemma 7.4 is proved in Appendix B.4.

Recall that we fixed a sequence \((u_n)_n \subset C^1 \setminus W^{1,1}(\Omega)\) satisfying (7.3).

In particular, for sufficiently large \( n \), we have

\[
\int_{\Omega \setminus B_\infty} |\nabla u_n| > \frac{\delta}{2}.
\]

This implies

\[
\int_{\Omega} |D u_0| = \lim_n \int_{\Omega} |\nabla u_n| \geq \mathcal{H}^1(K) + \frac{\delta}{2},
\]

which is in contradiction with (7.1).

7.3.2. The case \( u_0 \equiv 0 \) in \( \Omega \setminus B_\infty \). We first note that, since \( \text{tr} \partial_{D_\infty^+} u_0 \not\equiv 0 \), there exists a triangle \( T_k^{N_0} \) such that \( \int_{T_k^{N_0}} |u_0| > 0 \). We fix such a triangle \( T_k^{N_0} \) and we let \( \alpha \) be the vertex corresponding to the right angle.

We let \( \tilde{\mathcal{R}} = (\alpha, \tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2) \) be the direct orthonormal frame centered in \( \alpha \) where \( \tilde{\mathbf{e}}_2 = \nu_{\epsilon_k^{N_0}} (\nu_{\epsilon_k^{N_0}} \text{ is defined Remark 4.1.1}), \) i.e., the directions of the new frame are given by the side of the right-angle of \( T_k^{N_0} \).

It is clear that for \( N \geq N_0 \) we have \( V_N \equiv \tilde{\mathbf{e}}_2 \) in \( T_k^{N_0} \).

By construction of \( B_\infty \), \( T_k^{N_0} \cap B_\infty \) is a union of segments parallel to \( \tilde{\mathbf{e}}_2 \), i.e. \( \mathbb{I}_{B_\infty \cap T_k^{N_0}}(s, t) \) depends only on the first variable "s" in the frame \( \tilde{\mathcal{R}} \).
Since $\int_{\mathcal{P}} |u_0| > 0$, in the frame $\mathcal{R}$, we may find $a, b, c, d \in \mathbb{R}$ such that, considering the rectangle (whose sides are parallel to the direction of $\mathcal{R}$)

$$\mathcal{P} := \{ \alpha + s\hat{e}_1 + t\hat{e}_2 : (s, t) \in [a, b] \times [c, d] \} \subset T^N_{k}\n$$

we have

$$\int_{\mathcal{P}} |u_0| > 0.$$

Since from Lemma 6.1 the set $B_\infty$ has an empty interior (and that $\mathbb{I}_{B_\infty}|_{T^N_{k}}(s, t)$ depends only on the first variable in the frame $\mathcal{R}$), we may find $a' < b'$ such that

- $[a', b'] \times [c, d] \subset [a, b] \times [c, d]$,
- $\mathcal{S} \cap B_\infty = \emptyset$ with $\mathcal{S} := \{ \alpha + s\hat{e}_1 + t\hat{e}_2 : (s, t) \in [a', b'] \times [c, d] \}$
- $\delta := \int_{\mathcal{P'}} |u_0| > 0$ with $\mathcal{P'} := \{ \alpha + s\hat{e}_1 + t\hat{e}_2 : (s, t) \in [a', b'] \times [c, d] \}$.

Moreover, since $\mathcal{S}$ and $B_\infty$ are compact sets with empty intersection, we may find $\mathcal{V}$, an open neighborhood of $\mathcal{S}$ such that dist$(\mathcal{V}, B_\infty) > 0$.

Noting that $u_0 \equiv 0$ in $\Omega \setminus B_\infty$, from Lemma A.1 (in Appendix A) it follows that for sufficiently large $n$ we have

- $u_n \equiv 0$ in $\mathcal{S}$,
- $\int_{\mathcal{P'}} |u_n| > \frac{\delta}{2}$.

Consequently, from a standard Poincaré inequality

$$\int_{\mathcal{P'}} |\partial_{e_1} u_n| \geq \frac{2}{b' - a'} \int_{\mathcal{P'}} |u_n| > \frac{\delta}{b' - a'} =: \delta'.$$

Therefore $\int_{\mathcal{P'}} |\partial_{e_1} u_n| > \delta'$, $\int_{\mathcal{P'}} |\partial_{e_2} u_n| \leq 2\mathcal{H}^1(\mathcal{K})$ and then by Lemma 3.3 in [3] we obtain:

$$\int_{\mathcal{P'}} |\nabla u_n| \geq \int_{\mathcal{P'}} |\partial_{e_2} u_n| + \frac{\delta'^2}{4\mathcal{H}^1(\mathcal{K}) + \delta'}.$$

Thus, from Lemma 7.2, for sufficiently large $n$:

$$\int_{\Omega} |\nabla u_n| \geq \mathcal{H}^1(\mathcal{K}) + \frac{\delta'^2}{4\mathcal{H}^1(\mathcal{K}) + \delta'} - o_n(1).$$

From the convergence in $BV$-norm of $u_n$ to $u_0$ we have

$$\int_{\Omega} |Du_0| \geq \mathcal{H}^1(\mathcal{K}) + \frac{\delta'^2}{4\mathcal{H}^1(\mathcal{K}) + \delta'}.$$

Clearly this last assertion contradicts (7.1) and ends the proof of Theorem 1.1.

**Appendices**

**Appendix A. A smoothing result**

We first state a standard approximation lemma for $BV$-functions.

**Lemma A.1.** — Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz open set and let $u \in BV(\Omega)$. There exists a sequence $(u_n)_n \subset C^1(\Omega)$ such that

1. $u_n \overset{\text{strictly}}{\rightarrow} u$ in the sense that $u_n \rightarrow u$ in $L^1(\Omega)$ and $\int_{\Omega} |\nabla u_n| \rightarrow \int_{\Omega} |Du|$, 
2. $\text{tr}_{\partial \Omega} u_n = \text{tr}_{\partial \Omega} u$ for all $n$,
(3) for \( k \in \{1, 2\} \),
\[
\int_\Omega |\partial_k u_n| \to \int_\Omega |D_k u| := \sup \left\{ \int_\Omega u_\partial_k \xi : \xi \in C^1_c(\Omega, \mathbb{R}) \text{ and } |\xi| \leq 1 \right\},
\]

(4) If \( u = 0 \) outside a compact set \( L \subset \Omega \) and if \( \omega \) is an open set such that \( \text{dist}(\omega, L) > 0 \) then, for large \( n \), \( u_n = 0 \) in \( \omega \).

Proof. — The first assertion is quite standard. It is for example proved in [1][Theorem 1]. We present below the classical example of sequence for such approximation result (we follow the presentation of [2][Theorem 1.17]).

Let \( \Omega \subset \mathbb{R}^2 \) be a bounded Lipschitz open set and let \( u \in BV(\Omega) \).

For \( n \geq 1 \), we let \( \varepsilon = 1/n \). We may fix \( m \in \mathbb{N}^* \) sufficiently large such that letting for \( k \in \mathbb{N} \)
\[
\Omega_k = \left\{ x \in \Omega : \text{dist}(x, \partial \Omega) > \frac{1}{m+k} \right\}
\]
we have
\[
\int_{\Omega \setminus \Omega_k} |Du| < \varepsilon.
\]

We fix now \( A_1 := \Omega_2 \) and for \( i \in \mathbb{N} \setminus \{0, 1\} \) we let \( A_i = \Omega_{i+1} \setminus \overline{\Omega_{i-1}} \). It is clear that \((A_i)_{i \geq 1} \) is a covering of \( \Omega \) and that each point in \( \Omega \) belongs to at most three of the sets \((A_i)_{i \geq 1} \).

We let \((\varphi_i)_{i \geq 1} \) be a partition of unity subordinate to the covering \((A_i)_{i \geq 1} \), i.e., \( \varphi_i \in C_c^\infty(A_i) \), \( 0 \leq \varphi_i \leq 1 \) and \( \sum_{i \geq 1} \varphi_i = 1 \) in \( \Omega \).

We let \( \eta \in C_c^\infty(\mathbb{R}^2) \) be such that \( \text{supp}(\eta) \subset B(0, 1), \eta \geq 0, \int \eta = 1 \) and for \( x \in \mathbb{R}^2 \) \( \eta(x) = \eta(|x|) \). For \( t > 0 \) we let \( \eta_t = t^{-2} \eta(\cdot/t) \).

As explained in [2], for \( i \geq 1 \), we may choose \( \varepsilon_i \in (0, \varepsilon) \) sufficiently small such that
\[
\begin{cases}
\text{supp}(\eta_{\varepsilon_i} * (u \varphi_i)) \subset A_i \\
\int_\Omega |\eta_{\varepsilon_i} * (u \varphi_i) - u \varphi_i| < \frac{\varepsilon}{2^i} \\
\int_\Omega |\eta_{\varepsilon_i} * (u \nabla \varphi_i) - u \nabla \varphi_i| < \frac{\varepsilon}{2^i}.
\end{cases}
\]

Here * is the convolution operator.

Define
\[
u_n := \sum_{i \geq 1} \eta_{\varepsilon_i} * (u \varphi_i).
\]

In some neighborhood of each point \( x \in \Omega \) there are only finitely many nonzero terms in the sum defining \( u_n \). Thus \( u_n \) is well defined and smooth in \( \Omega \).

Moreover, we may easily check that
\[
\|u_n - u\|_{L^1(\Omega)} + \int_\Omega |Du| - \int_\Omega |\nabla u_n| < \varepsilon \text{ (here } \varepsilon = 1/n \).
\]

The previous estimate proves that \((u_n)\) satisfies the first assertion, i.e. \( u_n \rightarrow u \) strictly.

As claimed in [2][Remark 2.12] we have \( \text{tr}_{\partial \Omega} u_n = \text{tr}_{\partial \Omega} u \) for all \( n \). Thus the second assertion is satisfied.
We now prove the third assertion. Since \( u_n \to u \) in \( L^1(\Omega) \), by inferior semi continuity we easily get for \( k \in \{1, 2\} \)
\[
\int_\Omega |D_k u| \leq \liminf_{n \to \infty} \int_\Omega |\partial_k u_n|.
\]
We now prove \( \int_\Omega |D_k u| \geq \limsup_{n \to \infty} \int_\Omega |\partial_k u_n| \).

Let \( \xi \in C^1_c(\Omega, \mathbb{R}) \) with \(|\xi| \leq 1\). Since \( \eta \) is a symmetric mollifier and \( \sum \varphi_i = 1 \) we have
\[
\int_\Omega u_n \partial_k \xi = \sum_{i \geq 1} \int_\Omega \eta_{\xi_i} \ast (u \varphi_i) \partial_k \xi
= \sum_{i \geq 1} \int_\Omega u \varphi_i \partial_k (\eta_{\xi_i} \ast \xi)
= \sum_{i \geq 1} \int_\Omega u \partial_k [\varphi_i (\eta_{\xi_i} \ast \xi)] - \sum_{i \geq 1} \int_\Omega u \partial_k \varphi_i (\eta_{\xi_i} \ast \xi)
= \sum_{i \geq 1} \int_\Omega u \partial_k [\varphi_i (\eta_{\xi_i} \ast \xi)] - \sum_{i \geq 1} \int_\Omega \xi (\eta_{\xi_i} \ast (u \partial_k \varphi_i) - u \partial_k \varphi_i).
\]
On the one hand we have (note that \( \varphi_i (\eta_{\xi_i} \ast \xi) \in C^1_c(\mathbb{A}_i) \) and \(|\varphi_i (\eta_{\xi_i} \ast \xi)| \leq 1\))
\[
\left| \sum_{i \geq 1} \int_\Omega u \partial_k [\varphi_i (\eta_{\xi_i} \ast \xi)] \right| = \left| \int_{\mathbb{A}_i} u \partial_k [\varphi_i (\eta_{\xi_i} \ast \xi)] + \sum_{i \geq 2} \int_{\mathbb{A}_i} u \partial_k [\varphi_i (\eta_{\xi_i} \ast \xi)] \right|
\leq \int_\Omega |D_k u| + \sum_{i \geq 2} \int_{\mathbb{A}_i} |D_k u|
\leq \int_\Omega |D_k u| + 3 \int_{\Omega \setminus \Omega_0} |D_k u|
\leq \int_\Omega |D_k u| + 3 \varepsilon.
\]
Here we used that each point in \( \Omega \) belongs to at most three of the sets \( (\mathbb{A}_i)_{i \geq 1} \), for \( i \geq 2 \) we have \( \mathbb{A}_i \subset \Omega \setminus \Omega_0 \) and
\[
\int_{\Omega \setminus \Omega_0} |D_k u| \leq \int_{\Omega \setminus \Omega_0} |Du| < \varepsilon.
\]
On the other hand, since for \( i \geq 1 \)
\[
\int_\Omega |\eta_{\xi_i} \ast (u \nabla \varphi_i) - u \nabla \varphi_i| < \frac{\varepsilon}{2^i},
\]
we get
\[
\left| \sum_{i \geq 1} \int_\Omega \xi (\eta_{\xi_i} \ast (u \partial_k \varphi_i) - u \partial_k \varphi_i) \right| \leq \sum_{i \geq 1} \int_\Omega |\eta_{\xi_i} \ast (u \partial_k \varphi_i) - u \partial_k \varphi_i| < \varepsilon.
\]
Consequently
\[
\sup \left\{ \int_\Omega u_n \partial_k \xi : \xi \in C^1_c(\Omega, \mathbb{R}) \text{ and } |\xi| \leq 1 \right\} = \int_\Omega |\partial_k u_n| \leq \int_\Omega |D_k u| + 4 \varepsilon
\]
and thus
\[ \limsup_n \int_\Omega |\partial_k u_n| \leq \int_\Omega |D_k u|. \]
This inequality in conjunction with
\[ \liminf_n \int_\Omega |\partial_k u_n| \geq \int_\Omega |D_k u| \]
proves the third assertion of Lemma A.1.

The last assertion of Lemma A.1 is a direct consequence of the definition of the \( u_n \)'s. \( \Box \)

**Appendix B. Proofs of Lemma 2.2, Lemma 2.3, Lemma 2.4 and Lemma 7.4**

**B.1. Proof of Lemma 2.2.** Let \( u \in BV(Q) \). We prove that
\[ \int_Q |D_2 u| \geq \int_0^1 |\text{tr}_\partial Q u(\cdot, 0) - \text{tr}_\partial Q u(\cdot, 1)|. \]
From Lemma A.1, there exists \((u_n)_n \subset C^1(Q)\) with \(\text{tr}_\partial Q u_n = \text{tr}_\partial Q u\) and such that \(u_n \xrightarrow{\text{strictly}} u\) and
\[ \int_Q |\partial_2 u_n| \to \int_Q |D_2 u|. \]
From Fubini’s theorem and the Fundamental theorem of calculus we have
\[ \int_Q |\partial_2 u_n| = \int_0^1 dx_1 \int_0^1 |\partial_2 u_n(x_1, x_2)| dx_2 \]
\[ \geq \int_0^1 dx_1 \int_0^1 \partial_2 u_n(x_1, x_2) dx_2 \]
\[ = \int_0^1 dx_1 |\text{tr}_\partial Q u_n(x_1, 1) - \text{tr}_\partial Q u_n(x_1, 0)| \]
\[ = \int_0^1 |\text{tr}_\partial Q u(\cdot, 1) - \text{tr}_\partial Q u(\cdot, 0)|. \]
Since \(\int_Q |\partial_2 u_n| \to \int_Q |D_2 u|\), Lemma 2.2 is proved.

**B.2. Proof of Lemma 2.3.** Let \( \Omega \) be a planar open set. Let \( u \in BV(\Omega) \) be such that
\[ \int_\Omega |Du| = \int_\Omega |D_2 u|. \]
We prove that \( \int_\Omega |D_1 u| = 0 \). We argue by contradiction and we assume that \( \int_\Omega |D_1 u| > 0 \), i.e., there exists \( \xi \in C^1_c(\Omega) \) such that \(|\xi| \leq 1 \) and
\[ \eta := \int_\Omega u \partial_1 \xi > 0. \]
Let \((\xi_n)_n \subset C^1_c(\Omega)\) be such that \(|\xi_n| \leq 1 \) and
\[ \eta_n := \int_\Omega u \partial_2 \xi_n \to \int_\Omega |D_2 u|. \]
For \((\alpha, \beta) \in \{x \in \mathbb{R}^2 : |x| \leq 1\}\) we let \(\xi^{(n)}_{\alpha, \beta} = (\alpha \xi, \beta \xi_n) \in C^1_c(\Omega, \mathbb{R}^2)\). Clearly, 
\[ |\xi^{(n)}_{\alpha, \beta}| \leq 1 \quad \text{and} \quad \int_{\Omega} |Du| \geq \int_{\Omega} u \text{div}(\xi^{(n)}_{\alpha, \beta}) = \alpha \eta + \beta \eta_n. \quad \text{(B.1)} \]

If we maximize the right-hand side of (B.1) w.r.t. \((\alpha, \beta) \in \{x \in \mathbb{R}^2 : |x| \leq 1\}\), then we find with \((\alpha, \beta) = \left(\frac{\eta}{\sqrt{\eta^2 + \eta_n^2}}, \frac{\eta_n}{\sqrt{\eta^2 + \eta_n^2}}\right)\) that
\[
\int_{\Omega} |Du| \geq \sqrt{\eta^2 + \eta_n^2} \rightarrow \sqrt{\eta^2 + \left(\int_{\Omega} |Du|\right)^2} > \int_{\Omega} |Du|.
\]
This is a contradiction.

B.3. Proof of Lemma 2.4. Let \(u \in BV(Q)\) satisfying \(\text{tr}_{\partial Q} u = 0\) in \(\{0\} \times [0, 1]\). We are going to prove that
\[
\int_Q |u| \leq \int_Q |D_1 u|.
\]

Let \((u_n)_n \subset C^1(\Omega)\) be given by Lemma A.1. Using the Fundamental theorem of calculus we have for \((x_1, x_2) \in Q\) that
\[
|u_n(x_1, x_2)| \leq \int_0^{x_1} |\partial_1 u_n(t, x_2)| dt \leq \int_0^1 |\partial_1 u_n(t, x_2)| dt.
\]

Therefore, from Fubini’s theorem, we get
\[
\int_Q |u_n| \leq \int_Q dx_1 dx_2 \int_0^1 |\partial_1 u_n(t, x_2)| dt = \int_0^1 dx_2 \int_0^1 |\partial_1 u_n(t, x_2)| dt = \int_Q |\partial_1 u_n|.
\]

It suffices to see that \(\int_Q |u_n| \rightarrow \int_Q |u|\) and \(\int_Q |\partial_1 u_n| \rightarrow \int_Q |D_1 u|\) to get the result.

B.4. Proof of Lemma 7.4. Let \(\omega \subset \mathbb{R}^2\) be an open connected set. Assume there exist \(x_0 \in \partial \omega\) and \(r > 0\) such that \(\omega \cap B(x_0, r)\) is Lipschitz.

Let \(u \in BV(\omega)\) satisfying \(\text{tr}_{\partial \omega \cap B(x_0, r)} u = 0\) and \(\int_\omega |Du| = 0\). We are going to prove that \(u = 0\). On the one hand, since \(\int_\omega |Du| = 0\), we get \(u = C\) with \(C \in \mathbb{R}\) a constant. We thus have \(\text{tr}_{\partial \omega \cap B(x_0, r)} u = C\). Consequently \(C = 0\) and \(u \equiv 0\).

Appendix C. Results related to the Cantor set \(K\)

C.1. Justification of Remark 4.1.(1). We prove the following lemma:

Lemma C.1. — Let \(\eta > 0\) and let \(f \in C^2([0, \eta], \mathbb{R})\) be such that
\[
\eta < \frac{1}{2\|f\|_{L^\infty([0,\eta])}\|f''\|_{L^\infty([0,\eta])}}.
\]

We denote \(C_f\) the graph of \(f\) in an orthonormal frame \(\mathcal{R}_0\).

For \(0 \leq a < b \leq \eta\), denoting \(\mathcal{C}\) the chord \([\langle a, f(a)\rangle, \langle b, f(b)\rangle]\), for any straight line \(D\) orthogonal to \(\mathcal{C}\) such that \(D \cap \mathcal{C} \neq \emptyset\), the straight line \(D\) intersects \(C_{f,a,b}\) at exactly one point, where \(C_{f,a,b}\) is the part of \(C_f\) delimited by \([\langle a, f(a)\rangle, \langle b, f(b)\rangle]\).

Remark C.2. — We may state an analog result with \(f \in C^1\) where we use the modulus of continuity of \(f'\) instead of \(\|f''\|_{L^\infty([0,\eta])}\) in the hypothesis.
Proof. — The key point here is uniqueness. Indeed, for $0 \leq a < b \leq \eta$ and $C, D$ as in the lemma, we may easily prove that $C_{f,a,b} \cap D \neq \emptyset$ by solving an equation. (We do not use $\eta < (2\|f'\|_{L^\infty([0,\eta])}\|f''\|_{L^\infty([0,\eta])})^{-1}$ for the existence)

In contrast with the existence of an intersection point, its uniqueness is valid only for $\eta$ not too large. To prove uniqueness we argue by contradiction and we consider $f$ and $\eta$ as in lemma and we assume that there exist two points $0 \leq a < b \leq \eta$ such that $\eta < x < y \leq b$ such that the segments $[(x, f(x)), (y, f(y))]$ and $[(a, f(a)), (b, f(b))]$ are orthogonal. Note that with this hypothesis the straight line $D := ((x, f(x)), (y, f(y)))$ is orthogonal to the chord $C := [(a, f(a)), (b, f(b))]$.

So we get

$$f(y) - f(x) \quad y - x = -\frac{b - a}{f(b) - f(a)}.$$ 

From the Mean Value Theorem, there exist $c \in (x, y)$ and $\tilde{c} \in (a, b)$ such that $f'(c) = -\frac{1}{f'(\tilde{c})}$. Consequently

$$f'(c) \times [f'(\tilde{c}) - f'(c)] = -1 - [f'(c)]^2.$$ \hspace{1cm}(C.1)

From the hypothesis $\eta < (2\|f'\|_{L^\infty([0,\eta])}\|f''\|_{L^\infty([0,\eta])})^{-1}$, we have

$$|f'(\tilde{c}) - f'(c)| \leq \eta\|f''\|_{L^\infty([0,\eta])} < \frac{1}{2\|f'\|_{L^\infty([0,\eta])}}.$$ 

Therefore, we get

$$|f'(c) \times [f'(\tilde{c}) - f'(c)]| < \frac{1}{2}$$ 

which is in contradiction with (C.1). \hspace{1cm} \square

C.2. Two preliminary results. We first prove a standard result which states that the length of a small chord is a good approximation for the length of a curve.

Lemma C.3. — Let $0 < \eta < 1$ and let $f \in C^2([0, \eta], \mathbb{R}^+)$. We fix an orthonormal frame and we denote $C_f$ the graph of $f$ in the orthonormal frame. Let $A = (a, f(a)), B = (b, f(b)) \in C_f$ (with $0 \leq a < b \leq \eta$) and let $C = [AB]$ be the chord of $C_f$ joining $A$ and $B$. We denote $\overline{AB}$ the arc of $C_f$ with endpoints $A$ and $B$.

We have

$$\mathcal{H}^1(C) \leq \mathcal{H}^1(\overline{AB}) \leq \mathcal{H}^1(C) \{1 + (b - a)\|f''\|_{L^\infty} + \|f''\|_{L^\infty}(b - a)]\}.$$ 

Proof. — The estimate $\mathcal{H}^1(C) \leq \mathcal{H}^1(\overline{AB})$ is standard, we thus prove the second inequality.

On the one hand

$$\mathcal{H}^1(C) = \sqrt{(a - b)^2 + [f(a) - f(b)]^2} = (b - a) \sqrt{1 + \left(\frac{f(a) - f(b)}{a - b}\right)^2}.$$ 

On the other hand

$$\mathcal{H}^1(\overline{AB}) = \int_a^b \sqrt{1 + f'^2}.$$ 

With the help of the Mean Value Theorem, there exists $c \in (a, b)$ such that

$$\frac{f(a) - f(b)}{a - b} = f'(c).$$
Applying once again the Mean Value Theorem (to $f'$), for $x \in [a, b]$ there exists $c_x$ between $c$ and $x$ such that
\[ f'(x) = f'(c_x) + f''(c_x)(x - c). \]

Consequently for $x \in [a, b]$ we have:
\[
\begin{align*}
\sqrt{1 + f'(x)^2} &= \sqrt{1 + [f'(c_x) + f''(c_x)(x - c)]^2} \\
&= \sqrt{1 + f'(c)^2} \sqrt{1 + \frac{2f'(c)f''(c_x)(x - c) + f''(c_x)^2(x - c)^2}{1 + f'(c)^2}} \\
&\leq \sqrt{1 + \left(\frac{f(a) - f(b)}{a - b}\right)^2} \left[1 + 2\|f''\|_{L^\infty} \|f''\|_{L^\infty} (b - a) + \|f''\|^2_{L^\infty} (b - a)^2\right].
\end{align*}
\]
Thus we have
\[
\mathcal{H}^1(AB) = \int_a^b \sqrt{1 + f'(x)^2} \, dx \\
\leq (b - a) \sqrt{1 + \left(\frac{f(a) - f(b)}{a - b}\right)^2} \left[1 + 2\|f''\|_{L^\infty} \|f''\|_{L^\infty} (b - a) + \|f''\|^2_{L^\infty} (b - a)^2\right] = \mathcal{H}^1(\mathcal{C}) \{1 + (b - a)\|f''\|_{L^\infty} [2\|f''\|_{L^\infty} + \|f''\|_{L^\infty} (b - a)]\}. \quad \Box
\]

We now state another technical lemma which gives an upper bound for the height of the curve w.r.t. its chord.

**Lemma C.4.** Let $0 \leq a < b \leq \eta$, $f \in C^3([0, \eta], \mathbb{R}^+) \subset C^2([0, \eta], \mathbb{R}^+)$ be a strictly concave function and let $C_f$ be the graph of $f$ in an orthonormal frame. Let $A = (a, f(a))$ and $B = (b, f(b))$ be two points of $C_f$.

Assume that we have
\[
\eta < \frac{1}{2\|f''\|_{L^\infty([0, \eta])}\|f''\|_{L^\infty([0, \eta])}}
\]
in order to define for $C \in [AB]$ (with the help of Lemma C.1) $\tilde{C}$ as the unique intersection point of $C_f$ with the line orthogonal to $[AB]$ passing by $C$.

We have
\[
\mathcal{H}^1([C \tilde{C}]) \leq \frac{(b - a)^2\|f''\|_{L^\infty}}{8}.
\]

**Proof.** Let $0 \leq a < b \leq \eta$, $f \in C^3([0, \eta], \mathbb{R}^+) \subset C^2([0, \eta], \mathbb{R}^+)$ be as in Lemma C.4.

We consider the function
\[
g : [0, \eta] \to \mathbb{R}, \quad x \mapsto f(x) - \left[\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right].
\]

It is clear that $g$ is non negative since $f$ is strictly concave. For $C \in [AB]$, we let $\tilde{C}$ be as in Lemma C.4. Then we have
\[
\sup_{C \in [AB]} \mathcal{H}^1([C \tilde{C}]) = \max_{[0, \eta]} g.
\]

Thus, it suffices to prove
\[
\max_{[0, \eta]} g \leq \frac{(b - a)^2\|f''\|_{L^\infty}}{8}.
\]
Since \( g \) is \( C^1 \) and \( g(a) = g(b) = 0 \), there exists \( c \in (a, b) \) such that
\[
g(c) = \max_{[0, \eta]} g \quad \text{and} \quad g'(c) = 0.
\]

Let \( t \in \{a, b\} \) be such that \( |t - c| \leq \frac{b - a}{2} \). Using a Taylor expansion, there exists \( \tilde{c} \) between \( c \) and \( t \) such that
\[
0 = g(t) = g(c) + (t - c)g'(c) + \frac{(t - c)^2}{2}g''(\tilde{c}).
\]

Thus
\[
0 \leq \max_{[0, \eta]} g = g(c) = \frac{1}{2}(b - a)^2 \left\| f'' \right\|_{L_\infty}. \tag{C.3}
\]

The last inequality completes the proof. \( \square \)

### C.3. Proof of Proposition 4.4

We prove that
\[
\liminf_{N \to \infty} \mathcal{H}^1(K_N) > 0. \tag{C.2}
\]

#### Step 1. We prove that \( \max_{k=1, \ldots, 2^N} \mathcal{H}^1(\mathscr{C}_k^N) \leq \left(\frac{2}{3}\right)^N \).

For \( N \geq 1 \) we let \( \{K_k^N : k = 1, \ldots, 2^N\} \) be the set of the connected components of \( K_N \). We let \( \mathscr{C}_k^N \) be the chord of \( K_k^N \) and we define \( \mu_N = \max_{k=1, \ldots, 2^N} \mathcal{H}^1(\mathscr{C}_k^N) \).

Note that by \( (4.2) \) we have \( \mu_0 < 1 \).

We first prove that for \( N \geq 0 \) we have
\[
\mu_{N+1} \leq \frac{2}{3} \mu_N. \tag{C.3}
\]

By induction \( (C.3) \) implies (since \( \mu_0 < 1 \))
\[
\mu_N \leq \left(\frac{2}{3}\right)^N. \tag{C.4}
\]

In order to get \( (C.3) \), we prove that for \( N \geq 1 \) and \( K_k^N \) a connected component of \( K_N \) and \( \mathscr{C}_k^N \) its chord, we have
\[
\mathcal{H}^1(\mathscr{C}) \leq \frac{2 \mathcal{H}^1(\mathscr{C}_k^N)}{3} \quad \text{for} \quad \mathscr{C} \in \mathcal{S}(\mathscr{C}_k^N) \tag{C.5}
\]
(see Definition 4.2 for \( \mathcal{S}(\cdot) \), the set of sons of a chord).

Let \( N \geq 1 \). For \( k \in \{1, \ldots, 2^N\} \), we let \( K_k^N \) be a connected component of \( K_N \). We let \( K_{2k-1}^{N+1}, K_{2k}^{N+1} \in \mathcal{S}(K_k^N) \) be the curve obtained from \( K_k^N \) in the induction step.

For \( \tilde{k} \in \{2k - 1, 2k\} \), we let \( \mathscr{C}_{\tilde{k}}^{N+1} \) be the chords of \( K_{\tilde{k}}^{N+1} \).

In the frame \( R_0 \), we may define four points of \( \Gamma \),
\[
(a_1, f(a_1)), (b_1, f(b_1)), (a_2, f(a_2)), (b_2, f(b_2)),
\]
with \( 0 < a_1 < b_1 < a_2 < b_2 < \eta \), such that:
- the endpoints of \( K_{2k-1}^{N+1} \) are \( (a_1, f(a_1)) \) and \( (b_1, f(b_1)) \);
- the endpoints of \( K_{2k}^{N+1} \) are \( (a_2, f(a_2)) \) and \( (b_2, f(b_2)) \);
- the endpoints of \( K_{\tilde{k}}^{N+1} \) are \( (a_1, f(a_1)) \) and \( (b_2, f(b_2)) \).
In the frame $\mathcal{R}_0$ we let $(\alpha_1, \beta_1), (\alpha_2, \beta_2)$ be the coordinates of the points of $\mathcal{C}_k^N$ such that for $l \in \{1, 2\}$, the triangles whose vertices are $\{(a_l, f(a_l)); (b_l, f(b_l)); (\alpha_l, \beta_l)\}$ are right angled in $(\alpha_l, \beta_l)$.

We denote
- $I_1$ the segment $[(b_1, f(b_1)); (\alpha_1, \beta_1)];$
- $I_2$ the segment $[(a_2, f(a_2)); (\alpha_2, \beta_2)].$

From the construction of $K_{2k-1}^{N+1}$ and $K_{2k}^{N+1}$ and from the Pythagorean theorem we have for $l = 1, 2$

$$\mathcal{H}^1(\mathcal{C}_{2k-2+l}^{N+1})^2 = \mathcal{H}^1(I_l)^2 + \left(\frac{\mathcal{H}^1(\mathcal{C}_k^N) - \mathcal{H}^1(\mathcal{C}_k^N)^2}{2}\right)^2.$$

Using Lemma C.4 we get that

$$\mathcal{H}^1(I_l) \leq (b_2 - a_1)\|f''\|_{L^\infty}.$$

On the other hand we have obviously $b_2 - a_1 \leq \mathcal{H}^1(\mathcal{C}_k^N)$. Consequently we get

$$\mathcal{H}^1(\mathcal{C}_{2k-2+l}^{N+1})^2 \leq \mathcal{H}^1(\mathcal{C}_k^N)^4\|f''\|_{L^\infty}^2 + \left(\frac{\mathcal{H}^1(\mathcal{C}_k^N) - \mathcal{H}^1(\mathcal{C}_k^N)^2}{2}\right)^2$$

$$\leq \mathcal{H}^1(\mathcal{C}_k^N)^4\|f''\|_{L^\infty}^2 + \frac{\mathcal{H}^1(\mathcal{C}_k^N)^2}{4}.$$

Therefore

$$\mathcal{H}^1(\mathcal{C}_{2k-2+l}^{N+1}) \leq \frac{\mathcal{H}^1(\mathcal{C}_k^N)}{2} \sqrt{1 + 4\|f''\|_{L^\infty}^2\mathcal{H}^1(\mathcal{C}_k^N)^2},$$

thus using (4.3) we get

$$\mathcal{H}^1(\mathcal{C}_{2k-2+l}^{N+1}) \leq \frac{2\mathcal{H}^1(\mathcal{C}_k^N)}{3}.$$

The last estimate gives (C.5) and thus (C.4) holds.

**Step 2.** We prove that $\liminf_{N \to \infty} \sum_{k=1}^{2N} \mathcal{H}^1(\mathcal{C}_k^N) > 0$.

For $N \geq 1$, we let

$$c_N = \sum_{k=1}^{2N} \mathcal{H}^1(\mathcal{C}_k^N).$$

The main ingredient in this step consists in noting that a son of $\mathcal{C}_k^N$ is a hypothenuse of a right angled triangle which admits a cathetus of length

$$\frac{\mathcal{H}^1(\mathcal{C}_k^N) - \mathcal{H}^1(\mathcal{C}_k^N)^2}{2}.$$

Consequently we have

$$\mathcal{H}^1(\mathcal{C}_{2k-1}^{N+1}) + \mathcal{H}^1(\mathcal{C}_{2k}^{N+1}) \geq \mathcal{H}^1(\mathcal{C}_k^N) - \mathcal{H}^1(\mathcal{C}_k^N)^2.$$
Thus, summing the previous inequality for \( k = 1, \ldots, 2^N \) we get

\[
c_{N+1} = \sum_{k=1}^{2^N} \mathcal{H}^1(\mathcal{C}_{2k-1}^{N+1}) + \mathcal{H}^1(\mathcal{C}_{2k}^{N+1}) \geq \sum_{k=1}^{2^N} \mathcal{H}^1(\mathcal{C}_k^N)[1 - \mathcal{H}^1(\mathcal{C}_k^N)]
\]

\[
\geq c_N (1 - \mu_N) \geq c_N \left[ 1 - \left( \frac{2}{3} \right)^N \right].
\]

By induction for \( N \geq 2 \)

\[
c_N \geq c_1 \prod_{k=1}^{N-1} \left[ 1 - \left( \frac{2}{3} \right)^k \right] = c_1 \times \exp \left[ \sum_{k=1}^{N-1} \ln \left[ 1 - \left( \frac{2}{3} \right)^k \right] \right].
\]

It is clear that \( \lim \inf_N \sum_{l=1}^{N-1} \ln \left[ 1 - \left( \frac{2}{3} \right)^k \right] > -\infty \), thus \( \lim \inf_N c_N > 0 \).

**Step 3.** We prove (C.2).

Since for a connected component \( K_k^N \) of \( K_N \) and its chord \( \mathcal{C}_k^N \) we have

\[
\mathcal{H}^1(\mathcal{C}_k^N) \geq \mathcal{H}^1(\mathcal{C}_k^N),
\]

we obtain (C.2) from Step 2.

**Appendix D. A fundamental ingredient in the construction of the \( \Psi_N \)'s**

In this section we use the notation of Sections 4 and 5.

**Lemma D.1.** — Let \( \gamma \subset AB \) be a curve and let \( \mathcal{C} \) be its chord. We let \( \gamma_1, \gamma_2 \) be the curves included in \( \gamma \) obtained by the induction construction represented Figure 4.1 (section 4.2). For \( l = 1, 2 \), we denote also by \( \mathcal{C}_l \) the chord of \( \gamma_l \) and by \( T_l \) the right-angled triangle having \( \mathcal{C}_l \) as side of the right-angle and having its hypothenuse included in \( \mathcal{C} \).

If \( \mathcal{H}^1(\mathcal{C}) < \min\{2^{-1}, (4\|f''\|_{L^\infty})^{-2}\} \), then the hypothenuses of the triangles \( T_1 \) and \( T_2 \) have their length strictly lower than \( \frac{\mathcal{H}^1(\mathcal{C})}{2} \). In particular the triangles \( T_1 \) and \( T_2 \) are disjoint.

**Remark D.2.** — From (4.2), we know that \( \mathcal{C}_0 = \mathcal{C}_1^0 \) is such that \( \mathcal{H}^1(\mathcal{C}_1^0) < \min\{2^{-1}, (4\|f''\|_{L^\infty})^{-2}\} \). From (C.3) we have that for \( N \geq 1 \) and \( k \in \{1, \ldots, 2^N\} \) we have \( \mathcal{H}^1(\mathcal{C}_k^N) < \mathcal{H}^1(\mathcal{C}_k^0) < \min\{2^{-1}, (4\|f''\|_{L^\infty})^{-2}\} \).

Therefore with the help of Lemma D.1, for \( N \geq 1 \), the triangles \( T_k^N \)'s are pairwise disjoint.

**Proof.** — We model the statement by denoting \( \{M, Q\} \) the set of endpoints of \( \gamma \) and \( N \) and \( P \) are points such that:

- \( M, N \) are the endpoints of \( \gamma_1 \),
- \( P, Q \) are the endpoints of \( \gamma_2 \).

We denote \( \delta := \mathcal{H}^1([MQ]) = \mathcal{H}^1(\mathcal{C}) < \min\{2^{-1}, (4\|f''\|_{L^\infty})^{-2}\} \).

We fix an orthonormal frame \( \mathcal{R} \) with the origin in \( M \), with the \( x \)-axis (\( MQ \)) and such that \( N, P, Q \) have respectively for coordinates \((x_1, y_1), (x_2, y_2) \) and \((x_3, 0) \), where \( 0 < x_1 < x_2 < x_3 \) and \( y_1, y_2 > 0 \).
By construction we have
\[ x_1 = \frac{\delta - \delta^2}{2}, x_2 = \frac{\delta + \delta^2}{2} \text{ and } x_3 = \delta. \]

Moreover, arguing as in the proof of Lemma C.4 we have (recall that \( \overline{AB} \) is the graph of a function \( f \) in an other orthonormal frame):
\[ 0 < y_1, y_2 \leq \delta^2 \| f'' \|_{L^\infty}. \]

From these points, in Section 4.2, we defined two right-angled triangles having their hypothenuses contained in the \( x \)-axis.

The first triangle admits for vertices the origin \((0,0)\), \((x_1, y_1)\) and a point of the \( x \)-axis \((x_4,0)\). This triangle is right angled in \((x_1, y_1)\). In the frame \( \tilde{R} \), one of the side of the right-angle is included in the line parametrized by the cartesian equation \( y = ax \). Since \( \delta \leq 1/2 \), we have
\[ |a| = \left| \frac{y_1}{x_1} \right| \leq \frac{2\delta^2 \| f'' \|_{L^\infty}}{\delta - \delta^2} \leq 4\| f'' \|_{L^\infty} \delta. \]

The second triangle admits for vertices \((x_2, y_2)\), \((x_3,0)\) and a point of the \( x \)-axis \((x_5,0)\). This triangle is right-angled in \((x_2, y_2)\). In the frame \( \tilde{R} \), one of the side of the right-angle is included in the line parametrized by the cartesian equation \( y = \alpha x + \beta \), where
\[ |\alpha| = \left| \frac{y_2}{x_2 - x_3} \right| \leq \frac{2\delta^2 \| f'' \|_{L^\infty}}{\delta - \delta^2} \leq 4\| f'' \|_{L^\infty} \delta. \]

The proof of the proposition consists in obtaining
\[ x_4 < \frac{x_3}{2} \text{ and } x_3 - x_5 < \frac{x_3}{2}. \]

We get the first estimate. With the help of Pythagorean theorem we have
\[ x_1^2 + y_1^2 + (x_1 - x_4)^2 + y_1^2 = x_4^2. \]

By noting that \( y_1 = ax_1 \) we have
\[ x_4 = (1 + a^2)x_1. \]
Thus:

\[ x_4 < \frac{x_3}{2} \iff (1 + a^2) \frac{\delta - \delta^2}{2} < \frac{\delta}{2} \]
\[ \iff (1 + 16\|f''\|_{L^\infty}^2) (1 - \delta) < 1 \]
\[ \iff \delta - \delta^2 < \frac{1}{16\|f''\|_{L^\infty}^2} \]
\[ \iff \delta < \frac{1}{16\|f''\|_{L^\infty}^2}. \]

Following the same strategy we get that if \( \delta < \frac{1}{16\|f''\|_{L^\infty}^2} \) then \( x_3 - x_5 < \frac{x_3}{2} \).

\[ \square \]

**Appendix E. Adaptation of a Result of Giusti in [2]**

In this appendix we present briefly the proof of Theorem 2.16 and Remark 2.17 in [2]. The argument we present below follows the proof of Theorem 2.15 in [2].

**Proposition E.1.** — Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set of class \( C^2 \) and let \( h \in L^1(\partial \Omega) \). For all \( \varepsilon > 0 \) there exists \( u_\varepsilon \in W^{1,1}(\Omega) \) such that \( \operatorname{tr}_{\partial \Omega} u_\varepsilon = h \) and

\[
\| u_\varepsilon \|_{W^{1,1}(\Omega)} := \| u_\varepsilon \|_{L^1(\Omega)} + \| \nabla u_\varepsilon \|_{L^1(\Omega)} \leq (1 + \varepsilon)\|h\|_{L^1(\Omega)}.
\]

**Proof.** — We sketch the proof of Proposition E.1. Let \( h \in L^1(\partial \Omega) \) and let \( \varepsilon > 0 \) be sufficiently small such that

\[
(1 + \varepsilon^2) + \varepsilon^2 + \varepsilon^4 < 1 + \frac{\varepsilon}{2} \text{ and } (1 + \varepsilon^2)\varepsilon^2 < \frac{\varepsilon}{2}.
\]

**Step 1.** We may consider \( \eta > 0 \) sufficiently small such that in \( \Omega_\eta := \{ x \in \Omega : \operatorname{dist}(x, \partial \Omega) < \eta \} \) we have:

1. The function

\[
d : \Omega_\eta \to (0, \eta)
\]
\[
x \mapsto \operatorname{dist}(x, \partial \Omega)
\]

is of class \( C^1 \) and satisfies \( |\nabla d| \geq 1/2 \),

2. The orthogonal projection on \( \partial \Omega_\eta \), \( \Pi_{\partial \Omega} \), is Lipschitz.

We now fix a sequence \( (h_k)_k \subset C^\infty(\partial \Omega) \) such that \( h_k \overset{L^1}{\to} h \). We may assume that (up to replace the first term and to consider an extraction):

1. \( h_0 \equiv 0 \),

2. \( \sum_{k \geq 0} \|h_{k+1} - h_k\|_{L^1} \leq (1 + \varepsilon^2)\|h\|_{L^1} \).

And finally we fix a decreasing sequence \( (t_k)_k \subset \mathbb{R}_+^* \) such that

1. \( t_0 < \min(\eta, \varepsilon^2) \) is sufficiently small such that

\[
4t_0 \max(1; \|\nabla \Pi_{\partial \Omega}\|_{L^\infty}) \times \max(1, \sup_k \|h_k\|_{L^1}) < \min(\varepsilon^2, \varepsilon^2\|h\|_{L^1}),
\]

\[
\text{for } \varphi \in L^1(\partial \Omega) \text{ we have for } s \in (0, t_0)
\]
\[
\int_{d^{-1}(s)} |\varphi \circ \Pi_{\partial \Omega}(x)| \leq (1 + \varepsilon^2) \int_{\partial \Omega} |\varphi(x)|.
\]

2. For \( k \geq 1 \) we have \( t_k \leq \frac{t_0\|h\|_{L^1}}{2^k(1 + \|\nabla h_k\|_{L^\infty} + \|\nabla h_{k+1}\|_{L^\infty})} \).
Step 2. We define a map $u_\varepsilon : \Omega \to \mathbb{R}$ by
$$x \mapsto \begin{cases} \frac{d(x) - t_{k+1}}{t_k - t_{k+1}} h_k \circ \Pi_{\varepsilon \Omega}(x) + \frac{t_k - d(x)}{t_k - t_{k+1}} h_{k+1} \circ \Pi_{\varepsilon \Omega}(x) & \text{if } d(x) \in [t_{k+1}, t_k) \\ 0 & \text{otherwise.} \end{cases}$$

We may easily check that $u_\varepsilon$ is locally Lipschitz and thus weakly differentiable. From the coarea formula and a standard change of variable we have
$$\|u_\varepsilon\|_{L^1} \leq 2 \int_{\{d \leq t_0\}} |u_\varepsilon| |\nabla d|$$
$$\leq 2 \int_0^{t_0} ds \int_{d^{-1}({s})} |u_\varepsilon| dx$$
$$\leq 2 \sum_{k \geq 0} \int_{t_k}^{t_{k+1}} ds \int_{d^{-1}({s})} |u_\varepsilon| dx$$
$$\leq 2 \sum_{k \geq 0} \int_{t_k}^{t_{k+1}} ds \int_{d^{-1}({s})} [h_k \circ \Pi_{\varepsilon \Omega}(x) + h_{k+1} \circ \Pi_{\varepsilon \Omega}(x)]dx$$
$$\leq 2(1 + \varepsilon^2) \sum_{k \geq 0} \int_{t_k}^{t_{k+1}} ds \int_{\partial \Omega} [h_k(x) + h_{k+1}(x)]dx$$
$$\leq 4(1 + \varepsilon^2) t_0 \sup_k \|h_k\|_{L^1}$$
$$\leq (1 + \varepsilon^2) \varepsilon |h|_{L^1}$$
$$\leq \frac{\varepsilon}{2} \|h\|_{L^1}.$$ 

We now estimate $\|\nabla u_\varepsilon\|_{L^1}$. It is easy to check that if $d(x) \in (t_{k+1}, t_k)$ then we have
$$|\nabla u_\varepsilon(x)| \leq |\nabla d(x)| \left[ \frac{|h_k \circ \Pi_{\varepsilon \Omega}(x) - h_{k+1} \circ \Pi_{\varepsilon \Omega}(x)|}{t_k - t_{k+1}} + 2\|\nabla \Pi_{\varepsilon \Omega}\|_{L^\infty} [\|\nabla h_k \circ \Pi_{\varepsilon \Omega}(x) + |\nabla h_{k+1} \circ \Pi_{\varepsilon \Omega}(x)|] \right].$$

Consequently we get
$$\|\nabla u_\varepsilon\|_{L^1} \leq (1 + \varepsilon^2) \sum_{k \geq 0} \left\{ \int_{t_k}^{t_{k+1}} \frac{\|h_{k+1} - h_k\|_{L^1}}{t_k - t_{k+1}} + 2\|\nabla \Pi_{\varepsilon \Omega}\|_{L^\infty} (t_k - t_{k+1})(\|\nabla h_{k+1}\|_{L^1} + \|\nabla h_k\|_{L^1}) \right\}$$
$$\leq (1 + \varepsilon^2)(1 + \varepsilon^2) \|h\|_{L^1} + 2\|\nabla \Pi_{\varepsilon \Omega}\|_{L^\infty} t_0 \|h\|_{L^1}$$
$$\leq (1 + \varepsilon^2)(1 + \varepsilon^2) \|h\|_{L^1}$$
$$\leq (1 + \varepsilon/2) \|h\|_{L^1}.$$ 

Consequently $u_\varepsilon \in W^{1,1}(\Omega)$ and $\|u_\varepsilon\|_{W^{1,1}} \leq (1 + \varepsilon) \|h\|_{L^1}$.
In order to end the proof it suffices to check that $\text{tr}_{\partial \Omega}(u_\varepsilon) = h$. The justification of this property follows the argument of Lemma 2.4 in [2]. □

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