Replication-Robust Payoff-Allocation with Applications in Machine Learning Marketplaces

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Abstract

The ever-increasing take-up of machine learning techniques requires ever-more application-specific training data. Manually collecting such training data is a tedious and time-consuming process. Data marketplaces represent a compelling alternative, providing an easy way for acquiring data from potential data providers. A key component of such marketplaces is the compensation mechanism for data providers. Classic payoff-allocation methods such as the Shapley value can be vulnerable to data-replication attacks, and are infeasible to compute in the absence of efficient approximation algorithms. To address these challenges, we present an extensive theoretical study on the vulnerabilities of game theoretic payoff-allocation schemes to replication attacks. Our insights apply to a wide range of payoff-allocation schemes, and enable the design of customised replication-robust payoff-allocations. Furthermore, we present a novel efficient sampling algorithm for approximating payoff-allocation schemes based on marginal contributions. In our experiments, we validate the replication-robustness of classic payoff-allocation schemes and new payoff-allocation schemes derived from our theoretical insights. We also demonstrate the efficiency of our proposed sampling algorithm on a wide range of machine learning tasks.

1 Introduction

Training well performing machine learning (ML) models typically requires large volumes of high quality training data. As a consequence, many ML models are trained for standard benchmark problems on carefully collected (public) datasets such as ImageNet [9] or datasets from the UCI Machine Learning Repository [11]. However, when training ML models for custom applications there is often an under-supply of high-quality public data as these applications may require specialized, up-to-date structured training data of sufficient volume and fine-grained categories [25]. Obtaining such training data can be a critical bottleneck in ML [22] which typically relies on data discovery (e.g., searching) or data generation (e.g., manual collection, crowd-sourced gathering or labelling). In many cases, multiple potential data providers can be identified, and the data would then have to be acquired separately from each data provider. To simplify the data acquisition process and to accurately valuate the exchanged data, there is a demand for ML data markets which can readily connect data collectors (buyers) with data providers (sellers) [15].

A naïve implementation of such a market in the form of direct data exchange is likely to fail in practice due to the following reasons: (i) Data can be freely replicated, and hence can be easily resold by a buyer. (ii) Acquiring ownership of a large dataset may exceed the budget of the buyer. Nevertheless, both issues can be alleviated by considering the data exchange as an integral part of an ML platform as illustrated in Figure 1 as the buyer’s goal is only to use the data for training an ML model towards a custom application.

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This platform could bring together data from multiple sellers and return an ML model trained specifically towards the application of the buyer. The buyer will then pay a fee according to the performance of the model. A key question then arises in this setting: How can the market allocate the payoff among the data sellers?

Fortunately, payoff-allocation has been extensively studied in the cooperative game theory literature, the most common one being the Shapley value \cite{1,5,7,15}. Despite its fairness axioms, the Shapley value can suffer from undesirable properties when applied in the ML market: due to the free replicability of data and properties of common machine learning models, the Shapley value can be vulnerable to replication attacks from malicious data sellers—a seller can exploit properties of the payoff-allocation and gain significantly higher-payoffs by replicating its data and acting under multiple identities.

In this paper, we study replication-robustness properties of payoff-allocation, going beyond the Shapley value, and their efficient computation with the vision of enabling ML data markets. In particular, we make the following contributions:

1. We present the first extensive theoretical study of the properties of payoff-allocation under replication attacks (Section 4.1). Using these theoretical insights, we show that while the Shapley value is prone to replication attacks, we can define a whole family of replication-robust payoff-allocations, including the Banzhaf value and Leave-one-out (Section 4.2).
2. We demonstrate how future market designers can leverage our theoretical insights for designing customized robust payoff allocations (Section 4.3).
3. We introduce a computationally efficient sampling algorithm for approximating a wide range of payoff-allocations, which significantly outperforms the baseline methods (Section 5).
4. We present empirical validation of our assumptions underlying our theoretical insights, followed by the robustness properties across a range of payoff-allocation schemes, and finally demonstrate the strong performance of our proposed sampling algorithm (Section 6).

Our theoretical results have implications for game-theoretic payoff-allocation in general, such as contribution valuation for data sharing scenarios and ML feature importance computations.

2 Background

We model the market as a cooperative game, and start by defining the basic notations and concepts.

**Cooperative Games.** A cooperative game \cite{5} is defined as a tuple \(G = (N, v)\), where \(N = \{1, ..., n\}\) is the set of players of the game. The subsets of \(N\) are referred to as **coalitions** and \(v: 2^N \to \mathbb{R}\) is a **characteristic function**, assigning a real-valued valuation to each coalition \(S \subseteq N\). The **marginal contribution** of player \(i\) towards coalition \(S\) is defined as \(MC_i(S) := v(S \cup \{i\}) - v(S)\).

**Solution Concepts.** A solution concept \cite{5} is a function which assigns a payoff \(\varphi_i(N, v) \in \mathbb{R}\) to each player \(i\), commonly axiomatized through a collection of **natural properties** such as the following:

(A1) **Symmetry:** Two players \(i\) and \(j\) who have the same marginal contribution in any coalition have the same payoff, i.e., \((\forall S \subseteq N \setminus \{i, j\}: v(S \cup \{i\}) = v(S \cup \{j\})) \to \varphi_i(N, v) = \varphi_j(N, v)\).

(A2) **Efficiency:** The payoff values of all players sum to \(v(N)\), i.e., \(v(N) = \sum_{i \in N} \varphi_i(N, v)\).

(A3) **Null-player:** a player whose marginal contribution is zero in any coalition has zero payoff, i.e., \((\forall S \subseteq N: v(S \cup \{i\}) = v(S)) \to \varphi_i(N, v) = 0\).

(A4) **Linearity:** Given two cooperative games \(G^1 = (N, v^1)\) and \(G^2 = (N, v^2)\), then for any player \(i \in N\), \(\varphi_i(N, v^1 + v^2) = \varphi_i(N, v^1) + \varphi_i(N, v^2)\).

(A5) **2-Efficiency** \cite{18}: \(\varphi_i(N, v) + \varphi_j(N, v) = \varphi_{ij}(N', v')\) characterises neutrality of collusion, where \(\varphi_{ij}(N', v')\) is player \(ij\)’s payoff in a game in which players \(i\) and \(j\) merged as a single player.

(A6) **Anonymity** \cite{10}: For every \((N, v)\) and every permutation \(\pi: N \to N\) of the players, it holds that the players’ payoffs are invariant under permutations of the players.

In the following we review some common solution concepts:

**Shapley Value** is a common solution concept for reward division, defined as the average marginal contribution of a player towards its predecessors in any permutation of \(N\):

\[
\varphi_i^S(N, v) := \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(N-|S| - 1)!}{N!} MC_i(S).
\]

It is the unique value that satisfies (A1)(A4).
We consider the class of solution concepts based on marginal contributions, i.e., a player's payoff can only depend on the size of the coalition and not on the players' identities inside the coalition, which can be leveraged to design the countermeasures.

In the later sections of this paper, we will show that due to the nature of data, the four Shapley fairness axioms can be insufficient for defining suitable payoff-allocations for the machine learning market.

3 Data Market Game and Data Replication

Data, as a novel type of digital good, is freely replicable. A malicious party could exploit this property to cheat the market with the goal of maximizing its payoff. In this section, we will first define the data market game and the submodularity property of characteristic functions inspired by real-world machine learning models (Assumption 3.1), which will bring about the motivation behind the replication attack. Then, we will define the attack model and its deficiency (Assumption 3.2) which can be leveraged to design the countermeasures.

3.1 The Market Game

We model the data market as a cooperative game $G = (N, v)$, where the players $i$ are the data sellers $N = \{1, \ldots, n\}$ each holding a dataset $D_i$. At each round of interaction, a buyer will provide a classification task (regression tasks can be treated equivalently), specified by a validation dataset $D_{val}$. The data from all players jointly contribute towards training a model $M(\cup_{i \in N} D_i)$. Therefore, a natural characteristic function of a coalition $S$ is the accuracy $\bar{G}(M, D_{val})$ achieved by the model $M$ trained on the data held by players in the coalition, i.e., $v(S) := \bar{G}(M(\cup_{i \in S} D_i), D_{val})$.

The behaviour of the participants in the market will depend on the characteristics of $v$ and therefore on the properties of the accuracy as a function of the model trained on a subset of all data available in the market. Submodularity is often a good model for approximating properties of this accuracy—the value of additional training datasets typically diminishes with growing data size [16].

Assumption 3.1 (Submodularity). A characteristic function $v$ is submodular if and only if

$$\forall S \subseteq S' \subseteq N \setminus \{i\} \quad MC_i(S) \geq MC_i(S').$$

Experiments substantiating the (approximate) submodularity of the accuracy of common machine learning models can be found in the appendix. Unlike many other studies on cooperative games, the data market game with submodular characteristic functions has certain peculiar properties: for a fixed buyer, each seller party individually favours to participate in a market which is as small as possible, where he/she can obtain a payoff close to its characteristic value $v(\{i\})$. This opens the door for misusage by malicious players. As we will show and as observed in [11], under payoff-allocation by Shapley value, a malicious player can increase its payoff by replication. On the other hand, in many ML models, adding replicated data does not contribute additionally to the model performance. We will show in the later sections that this property can help design replication-robust payoff-allocations.

3.2 Solution Concepts in the Anonymous Market

We consider the class of solution concepts based on marginal contributions, i.e., a player’s payoff can be expressed as a weighted sum of marginal contributions towards coalitions of other players:

$$\varphi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \alpha_S MC_i(S),$$

In the anonymous market, the weight $\alpha_S$ for player $i$’s marginal contribution towards coalition $S$ will only depend on the size of the coalition and not on the players’ identities inside the coalition, i.e., $\forall S_1, S_2 \subseteq N \setminus \{i\}, |S_1| = |S_2| \implies \alpha_{S_1} = \alpha_{S_2}$, as a result of the anonymity axiom (A6). We can hence define the data market game solution concepts as a weighted sum of the average marginal contributions towards coalitions that share the same size, which we will denote by $z_i(c)$:

$$\varphi_i = \sum_{c=0}^{N-1} \alpha_c z_i(c), \quad z_i(c) := \frac{1}{\binom{N-1}{c}} \sum_{S \subseteq N \setminus \{i\}, |S| = c} MC_i(S)$$

where $\alpha_c = \sum_{S \subseteq N \setminus \{i\}, |S| = c} \alpha_S = \binom{N-1}{c} \alpha_S \forall S \subseteq N \setminus \{i\}, |S| = c$. In particular, when $\sum_{c=0}^{N-1} \alpha_c = 1$, the class of solutions are denoted as semivalues [12][10]. Many solution concepts such as the Shapley value, Banzhaf value, and LOO, can be represented in the form of Equation [1] e.g., the Shapley value can be expressed as $\varphi_i^S = \frac{1}{N} \sum_{c=0}^{N-1} z_i(c)$. Due to space constraints all proofs are deferred to the appendix.
### 3.3 Data Replication

Having defined a general market solution concept, we can introduce the malicious player who replicates its data and acts under multiple identities.

**Definition 3.1 (Data Replication).** In the market game $G = (N, v)$, a malicious player $i$ may execute a replication action $k$ times on its data $D_i$ and act as $k+1$ players $S^R = \{i_0, i_1, \ldots, i_k\}$ each holding one replica of $D_i$. Denote the induced market game as $G^R = (N^R, v^R)$ where $N^R = N \setminus \{i\} \cup S^R$. By replicating, the player receives a total payoff value which is the sum of its own value and that of all its replicas: $\varphi_i^{\text{tot}}(k) = \sum_{i=0}^{k} \varphi_i(N^R, v^R)$.

A player may be able to gain a higher total payoff due to its growing number of replicas. This can be true even in cases in which replicas do not change the machine learning model’s performance when used together with other replicas. We characterize this property of many ML models as follows:

**Assumption 3.2 (Replication Redundancy).** A replica does not contribute additional value to coalitions which contain another replica: $\forall i, j \in S^R: (i \in S) \rightarrow MC_j(S) = 0$

With this assumption, we can write the replicating player’s total payoff as a weighted combination of average marginal contributions $z_i(c)$ in the original game, s.t. $z_i(c)$ are invariant under replication:

**Theorem 3.1.** Let $G = (N, v)$ be a market game where $v$ is replication redundant. A malicious player $i$ by replicating $k$ times will receive a total payoff in the induced game $G^R = (N^R, v^R)$ of

$$\varphi_i^{\text{tot}}(k) = \sum_{c=0}^{N-1} \alpha_c^k z_i(c), \text{ where } z_i(c) := \frac{1}{\binom{N}{c}} \sum_{S \subseteq N \setminus \{i\}, |S| = c} MC_i(S)$$

Hence $\alpha_c^k$ is a key factor which can characterise the replication-robustness of solution concepts. Equation 1 can be seen as a special case of Theorem 3.1 with no replicas, i.e., $\alpha_c = \alpha_c^0$. In the proof we show that $\forall S \subseteq N \setminus \{i\}, |S| = c, \alpha_c^k = (k+1)(N_k-1)\alpha_c^0$, where $\alpha_c^0$ is the weight of a player’s marginal contribution towards coalition $S$ as defined by $\varphi$ in the induced game. The next corollary presents some common solution concepts expressed in terms of $\alpha_c^k$:

**Corollary 3.1.** The total payoff of a malicious player $i$ after $k$-replications can be expressed as $\varphi_i^{\text{tot}}(k) = \sum_{c=0}^{N-1} \alpha_c^k z_i(c)$ such that for Leave-one-out $\alpha_c^k = 1_{c=N-1,k=0}$, for the Shapley value $\alpha_c^k = \frac{(k+1)(N_k-1)}{N+k\binom{N-1}{c}}$, and the Banzhaf value $\alpha_c^k = \frac{(k+1)(N_k-1)}{2^{N-1}\binom{N-1}{c}}$

Figure 2 illustrates the changes of $\alpha_c^k$ under different number of replicas $k$. As shown in Lemma A.1 (presented in the Appendix), for any number of replicas $k$, the weights $\alpha_c^k$ of Shapley value always sum to 1, and shift towards smaller coalition sizes as $k$ increases. Moreover, by Lemma A.1, $z_i(c)$ monotonically decreases over coalition size $c$ due to submodularity. Consequently, the weight shift of the Shapley value causes $\varphi_i^{\text{tot}}$ to be monotonically increasing, as we will prove rigorously in section 4.2. In contrast, for Banzhaf value, the first replication yields no changes due to 2-efficiency $(\text{A5})$ afterwards, $\alpha_c^k$ decreases across all sizes with each replica added.

### 4 Replication-Robustness Conditions

In this section, we define replication-robustness in the data market game, and present useful conditions which characterise robustness properties of solution concepts. Given these conditions, we analyze common solution concepts in Section 4.2.

#### 4.1 Definitions and General Results

**Definition 4.1 (Robustness Against Replication).** A solution concept $\varphi$ is replication-robust if the payoff of any player $i$ in the original game $G$ is no less than the total payoff of player $i$ when replicated $k$ times in the induced game $G^R$, i.e., $\forall k > 0, \varphi_i(N, v) \geq \sum_{i=0}^{k} \varphi_i(N^R, v^R)$.

The next two theorems provide conditions on the weights $\alpha_c^k$ and their implications for guaranteeing replication-robustness.

**Theorem 4.1.** Assuming submodularity and replication-redundancy of the accuracy function, a solution concept as defined in Eq.(1) is replication-robust iff for any number of replicas $k$,

$$0 \leq p \leq N - 1, \sum_{c=0}^{p} \alpha_c^k \geq \sum_{c=0}^{p} \alpha_c^k$$

(2)
The total value converges to player $i$’s characteristic value, i.e., $\lim_{k \to \infty} \phi^\text{tot}_i(k) = v^i$.

**Theorem 4.2.** Under the same assumptions as Theorem 3.7, a solution concept is replication-robust and the total value of the malicious player decreases monotonically if for any number of replicas $k$,

$$\forall 0 \leq p \leq N - 1, \sum_{c=0}^{p} \alpha_c^k \geq \sum_{c=0}^{p} \alpha_c^{k+1} \quad (\implies \phi^\text{tot}_i(k) \leq \phi^\text{tot}_i(k+1)).$$

Note that the condition stated in Theorem 4.2 is stricter than that in Theorem 4.1 but additionally ensures that the total payoff of a replicating player monotonically decreases with the number of replicas.

### 4.2 Robustness Properties of Common Solution Concepts

We now use the robustness conditions to characterise common solution concepts in terms of replication-robustness. The results indicate that the Shapley value of a malicious player is monotonic increasing with growing number of replicas $k$, converging to its characteristic value as $k$ approaches infinity. In contrast, the Banzhaf value and Leave-one-out are robust against replication.

**Theorem 4.3.** Let $G = (N, v)$ be a market game where $v$ is replication-redundant and submodular, the Shapley value is not replication-robust, in particular the total value of a replicating player $i$ is monotonic increasing over growing number of replicas. That is, $\forall i \in N, \forall k \geq 0$,

$$\phi^\text{tot}_i(k) \leq \phi^\text{tot}_i(k+1)$$

The total value converges to player $i$’s characteristic value, i.e., $\lim_{k \to \infty} \phi^\text{tot}_i(k) = v^i$.

**Theorem 4.4.** The Banzhaf value and Leave-one-out are robust against replication. Under both payoff allocations, the limit of the total value of the replicating player $\lim_{k \to \infty} \phi^\text{tot}_i(k) = 0$.

**Discussion** The marginal value-based solution concepts balance between a player’s individual value and complementary value. This can be characterised by the weights assigned to the player’s average marginal contributions $\bar{z}_c$ towards different sized coalitions. For example, the Shapley value $\phi^i = \frac{1}{N} \sum_{c=0}^{N-1} \bar{z}_c$ assigns uniform weights over sizes $c$, Banzhaf value $\phi^i = \frac{1}{N^N} \sum_{c=0}^{N-1} \binom{N-1}{c} \bar{z}_c$ assigns larger weights for mid-sized coalitions, while leave-one-out only includes size $N-1$ coalitions.

A solution concept favours the individual value over complementary value if more weights are assigned on the smaller sized coalitions, and vice-versa. As a result of replication-redundancy, the solution concepts which emphasize the complementary value tend to be more replication-robust.

### 4.3 Designing Customized Replication-robust Payoff-Allocations

Having discussed the robustness conditions, we now describe how to apply these conditions to design new robust payoff-allocation solution concepts and illustrate this with an example.

**Corollary 4.1.** To satisfy the robustness conditions in Eq. (2) or (3), it suffices to satisfy one of the following for each summand of index (size) $c$:

$$\alpha^0_c \geq \alpha^k_c \quad (\implies \text{Eq. (2)})$$

or monotonicity:

$$\alpha^k_c \geq \alpha^{k+1}_c \quad (\implies \text{Eq. (3)})$$

One particular point to note when designing replication-robust solutions is the partial information: in an anonymous market, the identity of the replicating player and the number of replicas $k$ are unavailable to the market.

We now derive a robust solution by down-weighing the Shapley value using these two observations. Our solution will take the following form, where $\gamma$ is a function of the total number of players $N$ and coalition size $|S|$, but not of the number of replicas $k$:

$$\tilde{\phi}_i(v) := \sum_{S \subseteq N \setminus \{i\}} \gamma^{|S|} \alpha_S M^c_i(S),$$

where $\alpha_S = \frac{|S|!(N-|S|-1)!}{N!}$ are the Shapley coefficients.

**Theorem 4.5.** (Replication-robust Shapley value) Eq. (5) with

$$\gamma^{|S|} = \begin{cases} \frac{N-1}{2} \frac{1}{|S|!(N-|S|-1)!} & \text{if } |S| < \lfloor \frac{N-1}{2} \rfloor, \\ 1 & \text{otherwise}. \end{cases}$$

defines a replication-robust solution concept.
Algorithm 1 Payoff Allocation Approximation

Input: $T$: number of samples, $N$: the set of players, $q(k)$: any sampling distribution over coalition sizes $k$, $\alpha, \omega$: weights of $v$ as defined in Equation 1
Output: estimated payoff value $\hat{v}_i$ of each player

Step 1: Sampling coalitions and computing utilities

for $t = 1, 2, \ldots, T$ do
  Draw coalition size $k_t \sim q(k)$
  Uniformly sample coalition $S_t \subseteq N$ of size $k_t$
  Compute $v(S_t)$
  Add $v(S_t)$ to sets $U_i^+(k_t), U_i^-(k_t)$ for $i \in S_t$
end for

Step 2: Approximating players' values

Compute $U_c \leftarrow \frac{1}{|S_t| \cdot c} \sum_{S_{t,c} \in S_c} v(S_{t,c})$ for all $c$

Approximate players' values $\tilde{\phi}_k \leftarrow \sum_{c} \frac{\Gamma}{N} \sum_{u \in U_i^-} u; \phi_i^+ \leftarrow \sum_{c} \frac{\Gamma}{N} \sum_{u \in U_i^+} u$

Compute pairwise differences $\Delta \tilde{\phi}_{ij} \leftarrow \tilde{\phi}_i - \tilde{\phi}_j$

Compute sum $\tilde{\phi}_{all} \leftarrow N \sum_{c} \alpha \phi_{c,S_t = U_{c+1} - U_{c}}$

Find $\hat{\phi}_i$ by solving a feasibility program (see Appendix) with constraints $\sum_{i} \hat{\phi}_i = \tilde{\phi}_{all}, |(\hat{\phi}_i - \hat{\phi}_j) - \Delta \tilde{\phi}_{ij}| \leq \varepsilon$ for all $i, j \in N$

5 Efficient Computation

The computation of the so far considered solution concepts is computationally very demanding. It requires the evaluation of $v(\cdot)$ for a large number of coalitions, each involving the training of ML models. As this quickly becomes infeasible, methods for approximating the solution concepts are crucial. In this section, we introduce a sampling algorithm which efficiently approximates a wide range of solution concepts.

5.1 Baselines

Prior approximation methods mainly focused on Shapley value and specific game types [13, 2].

Random Sampling [11] samples $T$ permutations $\pi^t$ of players, iteratively computes the marginal contribution $\mathcal{MC}_i^\pi$ of each player $i$ towards the preceding players in $\pi^t$, and approximates the Shapley value of each player by averaging over the samples $\hat{\phi}_i \leftarrow \frac{1}{T} \sum_{t} \mathcal{MC}_i^\pi$.

Stratified Sampling [20] approximates for each player the average marginal contributions $z_i(c)$ over size-$c$ coalitions, and then computes the Shapley value by averaging over $z_i(c)$’s.

Group Testing [15] shares the sampled coalitions among players. In each turn $t$, it draws a coalition $S_t$ of a sampled size $k_t$. By estimating players’ pairwise differences $\Delta \tilde{\phi}_{ij} \leftarrow \sum_{c} \alpha \phi_{c,S_t = U_{c+1} - U_{c}}$, and using the sum of their values $v(N)$, the Shapley values can be obtained via a feasibility program.

Random sampling and group testing cannot be used to approximate solution concepts beyond the Shapley value without modification: the former performs uniform permutation sampling, while the latter requires knowledge of the total allocated payoff, which can only be efficiently obtained for the Shapley value as $v(N)$ due to the efficiency axiom (A2). Stratified sampling can be extended beyond the Shapley value through Equation 1, however, with a growing number of players, separate assignment of samples to players can be suboptimal.

5.2 A Novel Sampling Algorithm

Motivated by the above considerations, we propose a novel sampling algorithm (Algorithm 1) which applies to any marginal contribution-based solution concept as defined by Equation 1. Our algorithm improves sample efficiency by re-writing the solution concepts as in Theorem 5.1, which enables sample sharing among players. Furthermore, the algorithm extends to solution concepts beyond the Shapley value by approximating the total allocated payoff as in Theorem 5.2.

Theorem 5.1. (Approximate payoff) Let $\varphi$ be a solution concept defined as $\varphi_i = \sum_{c=0}^{N-1} \alpha_c z_i(c)$, the payoff of a player $i$ can be computed by $\varphi_i = \sum_{c=0}^{N-1} \alpha_c \frac{N}{N-c} (\mathbb{E}[S|c,S \subseteq S N[v(S \cup \{i\})] - U_c)$, where $U_c = \binom{N}{c}^{-1} \sum_{S \subseteq N, |S| = c} v(S)$.

$U_c$ is the average value of all size-$c$ coalitions, and can be approximated by all sampled coalitions of size-$c$. The details on approximating $\mathbb{E}[S|c,S \subseteq S N[v(S \cup \{i\})]$ are provided in the appendix.

Theorem 5.2. (Total allocated payoff) Let $\phi$ be a solution concept defined as $\phi_i = \sum_{c=0}^{N-1} \alpha_c z_i(c)$, the total allocated payoff to all players $N$ can be computed by $\phi_{all} = \sum_i \varphi_i = N \sum_{c=0}^{N-1} \alpha_c (U_{c+1} - U_c)$. 

Corollary 4.2. Let $G = (N, v)$ be a market game with payoff allocation satisfying the replication-robustness condition in Theorem 4.5. Then the loss for a malicious player $i$ by replicating $k$ times is at least: $\varphi_i^{(k)}(0) - \varphi_i^{(k)}(k) \geq \frac{1}{k} \sum_{c=0}^{N-1} (1 - \frac{k+1}{2^c}) \gamma_{N} z_i(c)$.

The robust Shapley value satisfies the following axioms: symmetry (A1), null-player (A3), linearity (A4). Additionally, the total allocated payoff does not exceed the characteristic value $v(N)$.
Algorithm 1 works as follows: It first samples coalitions according to a user-defined distribution \( q(c) \) over sizes \( k \), then approximates the payoff \( \hat{\varphi}_i \) of each player according to Theorem 5.1. Here we can either output the values, or go through a feasibility program which makes use of the players’ pairwise differences computed from the last step, and an additional sum constraint, provided by approximating the total allocated payoff as Theorem 5.2. A detailed explanation is presented in the appendix.

The sampling algorithm reduces the number of characteristic value evaluations from \( 2^N \) to the number of samples. All other calculations incur negligible time. Orthogonal to a sampling based approach, approximations to the characteristic values \( v(S) \) can be applied to further improve efficiency, but this is not the focus of this paper.

6 Experiments

In this section, we empirically justify our assumptions on the properties of machine learning models’ performances. Then we compare the replication-robustness of a range of solution concepts. Finally, we show significantly improved sample complexity of our sampling algorithm over the baselines.

Datasets. We use three datasets of varied sizes: (a) Covertype [11]: using the 10 continuous attributes, 7 classes. 5 honest players each holds 1000 datapoints, 5 replicas share the same 1000 datapoints. (b) CIFAR-100 [17]: 32x32x3 images of 20 superclasses, 100 subclasses. 4 sets of experiments are performed with varied data assignments where the players have data of all/disjoint/mixed classes. (c) Tiny ImageNet [19]: 64x64x3 images of 20 random classes. 3 honest players each holding 2000 datapoints and 3 replicas holding the same 2000 datapoints.

Models. We used a 4-layer (512 units) fully-connected neural network for the Covertype classification. For CIFAR-100 and Tiny-ImageNet, we used the VGG-16 network [23]. More details on data assignment (Table 2) and training (e.g., the optimization algorithm) can be found in the appendix.

6.1 Results

Properties of the Accuracy We empirically validate assumption 3.1 on submodularity and assumption 3.2 on replication redundancy. Figure 3 shows the average marginal contributions \( z_i(c) \) for each player over coalition sizes \( c \). Observe that \( z_i(c) \) is monotonic decreasing, which according to Lemma A.1 is a result of the submodularity of the accuracy. The curves validates replication redundancy with \( z_i(c) \approx 0 \) for the replicas when \( c \) exceeds the number of honest players. Moreover, the standard deviation of \( z_i(c) \) is high for the replicas, as a result of the difference in their marginal contributions when joining a coalition with/without another replica.

Replication Robustness Figure 4 compares the replication-robustness properties of various solution concepts. The curves show the change in total payoffs of the replicating player as a percentage of the total allocated payoffs, over growing number of replicas. The Covertype, CIFAR, and Tiny ImageNet start with 5,4,3 honest players respectively and 1 malicious player, and gradually increase the number of replicas. In all settings, the Shapley value is vulnerable to replication, and the total share of value gained by the replica player increases. Both Banzhaf and Robust Shapley value are replication-robust. The Robust Shapley value is furthermore budget-efficient, i.e., the total allocated payoffs do not exceed the value of the grand coalition. Leave-one-out only includes a player’s marginal contribution.
We considered the problem of replication-robust payoff-allocation in machine learning data markets. The concept of a machine learning data market has been explored in the literature. For instance, Table 1 across 10 seeds. Empty entries (-) indicate the algorithm cannot be applied to approximate the values of single training datapoints. The former performs approximations during efficient computation of the Shapley value has been studied in various settings, e.g., random sampling. To further test the algorithms on a larger scale, we compare the algorithms on a facility location function[8], where the ground truth Shapley value (128 players) and Banzhaf value (64 players) can be efficiently computed (the random function and facility location function are defined in the appendix). We present the mean and standard deviation of the relative error on the total allocated payoff. Compared with all baselines, our algorithm significantly reduces the relative error and standard deviation on all tasks and datasets.

7 Related Work

The concept of a machine learning data market has been explored in the literature. For instance, Agarwal et al. [11] introduced an algorithmic framework for data markets and addressed issues arising from free replicability of data. In particular, they considered the data replication problem specifically for the Shapley value. To overcome this problem, they rely on similarity metrics for replica detection; this however is challenging in practice because no good metrics are known for many applications and similarity metrics may fail to detect replicas that are also slightly transformed. In contrast, our robustness guarantees avoid replica-detection and arise naturally from replication-redundancy in the accuracy. Our results apply to a wide range of solution concepts, and accompanied by thorough empirical validations. Ohrimenko et al. [21] addresses the replication attack through building a collaborative market where each player must participate both as seller and buyer. This naturally discourages replication, and may apply to alternative application settings.

Efficient computation of the Shapley value has been studied in various settings, e.g., random sampling [3] and [13] for weighted voting games. Later, stratified sampling was proposed such as in [20, 4]. A more recent line of work is applying Shapley value to valuating machine learning features, such as [7, 6]. Most relevant to our work on valuating ML training data are [14, 15], both tend to focus on approximating values of single training datapoints. The former performs approximations during training, which may not represent the Shapley value without convergence of the training algorithm at each point. The latter proposed a series of algorithms including group testing, as we described and tested empirically, while other approaches rely on properties of single data points.

8 Conclusions

We considered the problem of replication-robust payoff-allocation in machine learning data market-places. In particular, we studied properties and remedies for replication attacks on such data markets when the Shapley value is commonly employed for payoff-allocation. We characterized the causes of vulnerabilities to replication attacks and derived conditions for avoiding these vulnerabilities. To make the payoff-allocation practically feasible, we introduced a novel approximation algorithm for the payoffs which can be applied whenever these payoffs are defined through marginal contributions. We validated our findings in a large set of experiments and empirically demonstrated the low sample complexity of our approximation algorithm.

Table 1: Relative error of sampling algorithms. Bold font: best result for each dataset & # samples.

| Table 1: Relative error of sampling algorithms. Bold font: best result for each dataset & # samples. |
|-----------------|-----------------|-----------------|-----------------|-----------------|
|                  | Random Sampling | Stratified Sampling | Group Testing | Ours           |
|                  | # samples | 64 | 128 | 64 | 128 | 64 | 128 | 64 | 128 |
| Covertype        |          |   |     |    |     |    |     |    |     |
| Shapley          | 0.93 ± 0.16 | 0.63 ± 0.11 | 1.30 ± 0.53 | 0.89 ± 0.12 | 2.98 ± 0.26 | 2.04 ± 0.46 | 0.96 ± 0.02 | 0.04 ± 0.01 | 0.31 ± 0.12 | 0.25 ± 0.11 |
| Banzhaf          | -         | - |     |    |     |    |     |    |     |
| Robust Shapley   | -         | - |     |    | 0.16 ± 0.03 | 0.53 ± 0.10 |       |       |
| CIFAR-100        |          |   |     |    |     |    |     |    |     |
| Shapley          | 0.23 ± 0.12 | 0.16 ± 0.06 | 0.30 ± 0.14 | 0.22 ± 0.11 | 1.08 ± 0.44 | 0.67 ± 0.28 | 0.13 ± 0.05 | 0.08 ± 0.03 | 0.20 ± 0.10 | 0.10 ± 0.04 |
| Banzhaf          | -         | - |     |    | 0.31 ± 0.24 | 0.27 ± 0.25 |       |       |
| Robust Shapley   | -         | - |     |    | 0.26 ± 0.15 | 0.16 ± 0.08 |       |       |
| Tiny ImageNet    |          |   |     |    |     |    |     |    |     |
| Shapley          | 0.38 ± 0.14 | 0.24 ± 0.12 | 0.44 ± 0.19 | 0.24 ± 0.15 | 1.34 ± 0.39 | 0.91 ± 0.29 | 0.12 ± 0.06 | 0.09 ± 0.03 | 0.42 ± 0.12 | 0.26 ± 0.09 |
| Banzhaf          | -         | - |     |    | 0.63 ± 0.39 | 0.53 ± 0.23 |       |       |
| Robust Shapley   | -         | - |     |    | 0.88 ± 0.46 | 0.67 ± 0.35 |       |       |
| Random Location  |          |   |     |    |     |    |     |    |     |
| Shapley          | 0.46 ± 0.11 | 0.36 ± 0.07 | 0.40 ± 0.21 | 0.21 ± 0.15 | 2.00 ± 0.35 | 1.40 ± 0.33 | 0.12 ± 0.02 | 0.08 ± 0.02 | 0.51 ± 0.50 | 0.23 ± 0.05 |
| Banzhaf          | -         | - |     |    | 1.06 ± 0.45 | 0.63 ± 0.20 |       |       |
| Robust Shapley   | -         | - |     |    | 1.09 ± 0.38 | 0.83 ± 0.16 |       |       |
| Facility Location|          |   |     |    |     |    |     |    |     |
| Shapley (N = 128) | 1.57 ± 0.04 | 0.95 ± 0.06 | 5.73 ± 0.61 | 0.08 ± 0.02 | 0.72 ± 0.10 | 0.40 ± 0.02 |
| Banzhaf (N = 64) | -         | 1.50 ± 0.99 (±0.99) | 14.99 ± 9.99 | 0.72 ± 0.10 | 0.40 ± 0.02 | 0.40 ± 0.02 |
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Appendix

We will provide proofs for all theoretical results in Appendix A and provide additional experimental details in Appendix B. Below is the list of the contents:

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A Proofs

A.1 Proofs for Section 3.2 Anonymous Market Solution Concepts:

Proof for Equation 1

$$\varphi_i = \sum_{S \subseteq N \setminus \{i\}} \alpha_S \mathcal{MC}_i(S)$$

Proof. By grouping the marginal contributions of player $i$ towards equal-sized coalitions, the payoff of the player $\varphi_i$ can be re-written as a weighted average over the mean marginal contributions to each coalition size.

Let $\alpha_S$ be the weight of marginal contribution towards coalition $S$, as defined by the solution concept, whereas $\alpha_c$ is the average of the marginal contribution towards all size-$c$ coalitions. Note also that $\alpha_S$ only depends on coalition $S$ by the coalition size $|S|$.

$$\varphi_i = \sum_{S \subseteq N \setminus \{i\}} \alpha_S \mathcal{MC}_i(S)$$

$$= \sum_{c=0}^{N-1} \sum_{|S|=c, S \subseteq N \setminus \{i\}} \alpha_S \mathcal{MC}_i(S)$$

$$= (i) \sum_{c=0}^{N-1} \phi(c) \sum_{|S|=c, S \subseteq N \setminus \{i\}} \mathcal{MC}_i(S)$$

$$= (ii) \sum_{c=0}^{N-1} \binom{N-1}{c} \phi(c) \frac{1}{\binom{N-1}{c}} \sum_{|S|=c, S \subseteq N \setminus \{i\}} \mathcal{MC}_i(S)$$

$$= \sum_{c=0}^{N-1} \alpha_c z_i(c) \text{ by letting } \alpha_c = \binom{N-1}{c} \phi(c)$$

(i): since $\alpha_S$ only depends on $S$ by the size $|S|$, let $\phi(c) := \alpha_S \forall S \subseteq N \setminus \{i\}, |S| = c$

(ii): $z_i(c) := \frac{1}{\binom{N-1}{c}} \sum_{|S|=c, S \subseteq N \setminus \{i\}} \mathcal{MC}_i(S)$ is defined as the average of player $i$’s marginal value towards size-$c$ coalitions (excluding player $i$), where $\frac{1}{\binom{N-1}{c}}$ is the total number of size-$c$ coalitions excluding player $i$.

Proof for Shapley value written as Equation 1

The Shapley value, for example, can be rewritten as above:

$$\varphi_i^S = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|(N-|S|-1)}{N!} \mathcal{MC}_i(S)$$

$$= \sum_{c=0}^{N-1} \binom{N-1}{c} \frac{c(N-c-1)}{N!} \frac{1}{\binom{N-1}{c}} \sum_{|S|=c, S \subseteq N \setminus \{i\}} \mathcal{MC}_i(S)$$

$$= (i) \sum_{c=0}^{N-1} \frac{1}{N} z_i(c)$$

(i) Since $\phi(c) = \frac{\binom{N-c-1}{c}}{\binom{N-1}{c}} = \frac{1}{\binom{N}{c}} \cdot \frac{1}{N}$, by grouping the equal-sized coalitions $S$, we arrive at $\alpha_c = \binom{N-1}{c} \alpha_S = \frac{1}{N}$, and $z_i(c)$ by definition.
Proof for Lemma A.1

**Lemma A.1.** Given a submodular characteristic function, the average marginal contribution by size is monotonically decreasing: \( \forall 0 \leq c < N - 1, z_i(c) \geq z_i(c + 1). \)

**Proof.** Recall \( z_i(c) := \frac{1}{(N-1)} \sum_{S \subseteq N \setminus \{i\}, |S| = c} MC_i(S) \) is the average marginal contribution of player i towards size-c coalitions, as defined in Equation 1.

**Proof Sketch:** The idea is that each size-c coalition \( S_1 \) can be mapped to \((N-1-c)\) number of size-\((c+1)\) coalitions \( S_2 \) where \( S_1 \subseteq S_2 \), by choosing to add one of remaining \((N-1-c)\) elements. Conversely, each \( S_2 \) can be mapped to \((c+1)\) size-\(c\) subsets \( S_1 \), by removing any one of the elements \( j \in S_2 \). Then, in order to compare \( z_i(c) \) and \( z_i(c + 1) \), partition \( MC_i(S_1) \) into \((N-1-c)\) equal parts, and for each \( S_2 \), sum up the \((c+1)\) parts from each \( S_1 \subseteq S_2 \) to compare with \( MC_i(S_2) \).

Let \( S^c := \{ S \subseteq N \setminus \{ i \} \mid |S| = c \} \)

\[
z_i(c + 1) - z_i(c) = \sum_{S_2 \in S^{c+1}} \frac{MC_i(S_2)}{N-1} - \sum_{S_1 \in S^c} \frac{MC_i(S_1)}{N-1}
= \sum_{S_2 \in S^{c+1}} \left( \frac{MC_i(S_2)}{N-1} - \frac{1}{N-1-c} \frac{MC_i(S_1)}{N-1} \right)
= \sum_{S_2 \in S^{c+1}} \left( \frac{MC_i(S_2)}{N-1} - \frac{1}{N-1-c} \frac{MC_i(S_1)}{N-1} \right)
= \sum_{S_2 \in S^{c+1}} \left( \frac{MC_i(S_2)}{N-1} - \frac{1}{N-1-c} \frac{MC_i(S_2)}{N-1} \right)
= \sum_{S_2 \in S^{c+1}} \left( \frac{MC_i(S_2)}{N-1} - \frac{1}{N-1-c} \frac{MC_i(S_2)}{N-1} \right)
= 0 \]

By submodularity: \( \forall S_1 \subseteq S_2 \subseteq N \setminus \{ i \}, MC_i(S_1) \geq MC_i(S_2) \)

\[
\sum_{S_2 \in S^{c+1}} \left( \frac{MC_i(S_2)}{N-1} - \frac{1}{N-1-c} \frac{MC_i(S_1)}{N-1} \right)
= \sum_{S_2 \in S^{c+1}} \left( \frac{MC_i(S_2)}{N-1} - \frac{1}{N-1-c} \frac{MC_i(S_2)}{N-1} \right)
= 0 \]

\( \square \)
A.2 Proofs for Section 3.3: Solution Concepts under Replication

**Theorem 3.1.** Let \( G = (N, v) \) be a market game where \( v \) is replication redundant. A malicious player \( i \) by replicating \( k \) times will receive a total payoff in the induced game \( G^R = (N^R, v^R) \) of

\[
\varphi_{tot}^k(k) = \sum_{c=0}^{N-1} \alpha^k_c z_i(c), \quad \text{where } z_i(c) := \frac{1}{\binom{c}{k}} \sum_{S \subseteq \bar{N}\setminus\{i\}, |S| = c} MC_i(S)
\]

We write the total payoff of the replicating player \( i \) in the induced game \( G^R = (N^R, v^R) \) as a weighted combination of the average size-\( c \) marginal contributions \( z_i(c) \) in the original game \( G = (N, v) \). Since \( z_i(c) \) are defined w.r.t the honest players \( N \setminus \{i\} \) in the original game, \( z_i(c) \) are invariant under replication. Hence their weights \( \alpha^k_c \) characterise the change in total payoff.

**Proof.** In the induced game \( G^R \), \( S^R = \{i_1, \ldots, i_{k+1}\} \) is the set of replicas in the induced game (including the original of the malicious player). We let \( \alpha^k_S \) denote the coefficients defined by a solution concept for the marginal contributions towards a coalition in the induced game.

\[
\varphi_{tot}^k(k) = \sum_{i \in S^R} \varphi_i
\]

\[
= (k+1) \varphi_i \text{ by the symmetry axiom}
\]

\[
= (k+1) \sum_{S \subseteq \bar{N}\setminus\{i\}} \alpha^k_S MC_i(S)
\]

\[
= (k+1) \sum_{S \subseteq \bar{N}\setminus\{i\}} \alpha^k_S MC_i(S) + \sum_{S \subseteq \bar{N}\setminus\{i\}, S \neq \emptyset} \alpha^k_S MC_i(S)
\]

\[
= (k+1) \sum_{S \subseteq \bar{N}\setminus\{i\}} \alpha^k_S MC_i(S)
\]

\[
= \sum_{c=0}^{N-1} (k+1) \binom{N-1}{c} \phi^k(c) z_i(c), \quad \text{where } \phi^k(c) := \alpha^k_S \forall S \subseteq N \setminus \{i\}, |S| = c
\]

\[
= \sum_{c=0}^{N-1} \alpha^k_c z_i(c), \quad \text{where } \alpha^k_c := (k+1) \binom{N-1}{c} \phi^k(c)
\]

In the next corollary, we rewrite the total value of replicating player \( i \) under common solution concepts, as a function of average marginal contributions \( z_i(c) \), according to Theorem 3.1.

**Corollary 3.1.** The total payoff of a malicious player \( i \) after \( k \)-replications can be expressed as \( \varphi_{tot}^k(k) = \sum_{c=0}^{N-1} \alpha^k_c z_i(c) \) such that for Leave-one-out \( \alpha^k_c = \frac{1}{c+1} \), for the Shapley value \( \alpha^k_c = \frac{(k+1) \binom{N-1}{c}}{(N+k) \binom{N}{c}} \), and the Banzhaf value \( \alpha^k_c = \frac{(k+1) N \binom{N-1}{c}}{2^N+1} \).

**Proof.** (a) Shapley Value:

\[
\varphi_{tot}^k(k) = (k+1) \sum_{S \subseteq \bar{N}\setminus\{i\}} \frac{|S|!(N+|S| - |S| - 1)!}{(N+k)!} MC_i(S)
\]

\[
= (k+1) \sum_{S \subseteq \bar{N}\setminus\{i\}} \frac{|S|!(N+|S| + k - 1)!}{(N+k)!} MC_i(S) + \sum_{S \subseteq \bar{N}\setminus\{i\}, S \neq \emptyset} |S|!(N+|S| - |S| - 1)! MC_i(S)
\]

\[
= (k+1) \sum_{S \subseteq \bar{N}\setminus\{i\}} \frac{|S|!(N+|S| + k - 1)!}{(N+k)!} MC_i(S)
\]

\[
= (k+1) \sum_{c=0}^{N-1} \frac{1}{(N+k) \binom{N+k-1}{c}} \sum_{S \subseteq \bar{N}\setminus\{i\}, |S| = c} MC_i(S)
\]

\[
= \sum_{c=0}^{N-1} \alpha^k_c z_i(c)
\]
(b) Banzhaf Value:

\[
\phi_{\text{Banzhaf}}(k) = \frac{(k + 1)}{2^{N+1}} \sum_{S \subseteq N \setminus \{i\}} MC_{ik}(S)
\]

\[
= \frac{(k + 1)}{2^{N+1}} \sum_{S \subseteq N \setminus \{i\}} MC_{ik}(S) = \frac{(k + 1)}{2^{N+1}} \sum_{S \subseteq N \setminus \{i\}} MC_{ik}(S)
\]

\[
= \frac{(k + 1)}{2^{N+1}} \sum_{S \subseteq N \setminus \{i\}} MC_{ik}(S)
\]

where (1) is because of replication-redundancy and (2) by the definition of \(z_i(c)\).

(c) Leave-one-out:

(i) \(\alpha^k_c = 1\) if \(c = N - 1, k = 0\): The Leave-one-out value assigns to each player the marginal contribution towards all other players.

(ii) \(\alpha^k_c = 0\) otherwise: By replication-redundancy, as long as there exists one replica, i.e., \(k \geq 0\), the marginal contribution of this player towards all other players is 0.

The following lemma shows some interesting properties for the Shapley value 1. for any number of replications \(k\), the weights \(\alpha^k_c\) always sum up to 1; 2. the weights \(\alpha^k_c\) gradually shift towards smaller coalition sizes \(c\) when \(k\) increases; and 3. additionally for interested readers, the change in weights is the most significant when the player replicates for the first time, and then gradually decreases. These properties will be useful for explaining the vulnerabilities of Shapley value under replication attacks.

**Lemma A.2.** Under payoff allocation by the Shapley value, the weights \(\alpha^k_c\) of the total payoff of a malicious player after \(k\)-replications satisfy the following three properties: \(\forall 0 \leq p \leq N - 1\)

\[
\sum_{c=0}^{N-1} \alpha^k_c = 1 \quad (6a)
\]

\[
\sum_{c=0}^{p} \alpha^k_c \leq \sum_{c=0}^{p} \alpha^{k+1}_c \quad (6b)
\]

\[
\sum_{c=0}^{p} \alpha^{k+1}_c - \sum_{c=0}^{p} \alpha^k_c \geq \sum_{c=0}^{p} \alpha^{k+2}_c - \sum_{c=0}^{p} \alpha^{k+1}_c \quad (6c)
\]

**Proof.** (1) Proof for Eq. (6a): Eq. (6a) shows that the sum of \(\alpha^k_c\) stays constant under changing \(k\).

\[
\sum_{c=0}^{N-1} \alpha^k_c = \sum_{c=0}^{N-1} \frac{(k + 1)(N - 1)}{(N + k)(N - 1)}
\]

\[
= (k + 1) \sum_{c=0}^{N-1} \frac{(N - 1)(N + k - 1 - c)!}{(N - 1 - c)(N + k)!}
\]

\[
= (k + 1) \sum_{c=0}^{N-1} \frac{(N - 1 - c)!}{(N + k)!}
\]

\[
= \frac{1}{(N + k + 1)} \sum_{c=0}^{N-1} \binom{N + k - 1 - c}{k}
\]

\[
= \frac{1}{(N + k + 1)} \sum_{c=0}^{N-1} \binom{N + k - 1}{k}
\]

\[
= (1) \frac{1}{(N + k + 1)} \binom{N + k}{k+1}
\]

\[
= 1,
\]

where (1) is by substituting \(i = N + k - 1 - c\) and (2) by the Hockey-Stick identity.

(2) Proof for Equation (6b): Eq. (6b) shows that \(\alpha^k_c\) shift to the smaller coalitions under growing \(k\), which together with Lemma A.1, cause the total payoff of the replicating player to be monotonically
increasing with each added replica.

\[
\sum_{c=0}^{p} \alpha_c^k = \sum_{c=0}^{p} \frac{(k + 1)(N - 1)!(N + k - 1 - c)!}{(N - 1 - c)!(N + k)!}
\]

\[
= \frac{1}{(N + k)!} \sum_{c=0}^{p} \frac{(N + k - 1 - c)!}{(N - 1 - c)!(N + k)!}
\]

\[
= \frac{1}{(N + k)!} \sum_{c=0}^{p} \binom{N + k - 1 - c}{k}
\]

\[
= \frac{1}{(N + k)!} \left( \sum_{c=0}^{N - 1} \binom{N + k - 1 - c}{k} - \sum_{c=p+1}^{N-1} \binom{N + k - 1 - c}{k} \right)
\]

(1)\[ \quad \]\[
= \frac{1}{(N + k)!} \frac{1}{(N - p - 2)!} \frac{1}{(N + k) \cdots (N + k - p)}
\]

where (1) is by Eq. (6a) and (2) is by the Hockey-Stick identity. Similarly,

\[
\sum_{c=0}^{p} \alpha_c^{k+1} = 1 - \frac{(N - 1)!}{(N - p - 2)! (N + k + 1) \cdots (N + k + 1 - p)}
\]

Therefore,

\[
\sum_{c=0}^{p} \alpha_c^{k+1} - \sum_{c=0}^{p} \alpha_c^k = \frac{(N - 1)!}{(N - p - 2)!} \frac{(N + k + 1) - (N + k - p)}{(N + k + 1) \cdots (N + k - p)}
\]

\[
= \frac{(N - 1)!}{(N - p - 2)!} \frac{p + 1}{(N + k + 1) \cdots (N + k - p)} \geq 0
\]

(3) **Proof for Equation (6c)**: Eq. (6c) satisfied by the Shapley value together with Lemma A.1 will lead to the gain of adding one additional replica decreases monotonically with growing \(k\). Denote \(\delta^k := \sum_{c=0}^{p} \alpha_c^{k+1} - \sum_{c=0}^{p} \alpha_c^k\). From the last equation of the proof for Eq. (6b):

\[
\delta^k = \frac{(N - 1)!}{(N - p - 2)!} \frac{p + 1}{(N + k + 1) \cdots (N + k - p)}
\]

\[
\text{Eq. (6c) RHS} - \text{LHS} = \delta^{k+1} - \delta^k
\]

\[
= \frac{(N - 1)!(p + 1)}{(N - p - 2)!} \frac{1}{(N + k + 2) \cdots (N + k + 1 - p)} - \frac{1}{(N + k + 1) \cdots (N + k - p)}
\]

\[
= \frac{(N - 1)!(p + 1)}{(N - p - 2)!} \frac{(N + k - p) - (N + k + 2)}{(N + k + 2) \cdots (N + k - p)}
\]

\[
= \frac{(N - 1)!(p + 1)}{(N - p - 2)!} \frac{-(p + 2)}{(N + k + 2) \cdots (N + k - p)} \leq 0
\]
A.3 Proof of replication-robustness – Theorem 4.1

**Theorem 4.1.** Assuming submodularity and replication-redundancy of the accuracy function, a solution concept as defined in Eq. (1) is replication-robust iif for any number of replicas k,

\[ \forall 0 \leq p \leq N - 1, \sum_{c=0}^{p} \alpha_c^0 \geq \sum_{c=0}^{p} \alpha_c^k. \]  

(2)

**Proof.** **Sufficiency.** We first show that Eq. (2) implies replication-robustness. Following Definition 4.1 and Lemma 3.1, the replication-robustness condition is:

\[ \varphi_i^{\text{tot}}(0) - \varphi_i^{\text{tot}}(k) = \sum_{c=0}^{N-1} (\alpha_c^0 - \alpha_c^k) z_i(c) \geq 0 \]

(i) As shown in Lemma A.4, submodularity results in the average marginal contribution of size-c coalitions decreases monotonically: \( z_i(0) \geq z_i(1) \geq \ldots \geq z_i(N - 1) \).

(ii) By replication-redundancy (Assumption 3.2) and submodularity (Assumption 3.1), for any player \( i \) and coalition \( S \subseteq N \setminus i \), \( MC_i(S) \geq MC_i(S \cup \{i_k\}) = 0 \), where \( i_k \) is a replica of player \( i \). Therefore, the average marginal contributions \( z_i(c) \geq 0 \) for any \( c \).

Denote \( \Delta_c^k := \alpha_c^0 - \alpha_c^k \), Eq. (2) can then be rewritten as \( \sum_{c=0}^{p} \Delta_c^k \geq 0 \forall 0 \leq p \leq N - 1 \). Then

\[ \varphi_i^{\text{tot}}(0) - \varphi_i^{\text{tot}}(k) = \sum_{c=0}^{N-1} \Delta_c^k z_i(c) = z_i(0) \sum_{c=0}^{0} \Delta_c^k + \sum_{c=1}^{N-1} \Delta_c^k z_i(c) \]

\( \geq z_i(1) \sum_{c=0}^{1} \Delta_c^k + \sum_{c=2}^{N-1} \Delta_c^k z_i(c) \)

\( \geq \ldots \)

\[ = z_i(N - 2) \sum_{c=0}^{N-2} \Delta_c^k + \sum_{c=N-1}^{N-1} \Delta_c^k z_i(N - 1) \]

\[ \geq z_i(N - 1) \sum_{c=0}^{N-1} \Delta_c^k \geq 0, \]

(1) is because \( z_i(0) \geq z_i(1) \) and \( \sum_{c=0}^{0} \Delta_c^k \geq 0 \), (2) is because \( z_i(1) \geq z_i(2) \) and \( \sum_{c=1}^{1} \Delta_c^k \geq 0 \).

**Necessity.** We now show that Eq. (2) is also a necessary condition. We will prove by contradiction: Let \( \Delta_c^k := \alpha_c^0 - \alpha_c^k \). Assume the contrary: \( 3Q_m = \{q_0, q_1, \ldots, q_m\} \) s.t. \( q_0 < q_1 < \ldots < q_m \leq N - 1 \), \( \forall q \in Q_m, \sum_{c=0}^{q} \Delta_c^k < 0 \). We can construct \( z_i(c) \)'s which violates replication-robustness. From the contrary assumption:

\[ \sum_{c=0}^{p} \Delta_c^k < 0 \quad \text{if} \quad p = q_0 \]

\[ \sum_{c=0}^{p} \Delta_c^k \geq 0 \quad \text{if} \quad p < q_0 \]

Let \( \gamma < 1 \), such that \( \sum_{c=0}^{q_0-1} \Delta_c^k = \gamma |\Delta_c^k|_{q_0} \). Let \( z_i(c) = 0 \) for \( c > q_0 \) and let \( z_i(q_0) > \gamma z_i(0) \), then

\[ \varphi_i^{\text{tot}}(0) - \varphi_i^{\text{tot}}(k) = \sum_{c=0}^{N-1} \Delta_c^k z_i(c) = \sum_{c=0}^{q_0-1} \Delta_c^k z_i(c) + \sum_{c=q_0}^{q_0-1} \Delta_c^k z_i(c) \]

\[ \leq (\sum_{c=0}^{q_0-1} \Delta_c^k) z_i(0) + \sum_{c=q_0}^{q_0-1} \Delta_c^k z_i(q_0) \]

\[ = |\Delta_c^k|_{q_0} (\gamma z_i(0) - z_i(q_0)) < 0, \] which contradicts the replication-robustness.

Thus we have proven that Eq. (2) is a necessary and sufficient condition for replication-robustness. \( \square \)
Theorem 4.2. Under the same assumptions as Theorem 4.1, a solution concept is replication-robust and the total value of the malicious player decreases monotonically if for any number of replicas \( k \),
\[
\forall 0 \leq p \leq N - 1, \sum_{c=0}^{p} \alpha_{c}^{k} \geq \sum_{c=0}^{p} \alpha_{c}^{k+1} \quad (\implies \varphi_{i}^{\text{tot}}(k) \geq \varphi_{i}^{\text{tot}}(k+1)).
\]

Proof. We show that Eq. (3) implies the monotonicity
\[
\varphi_{i}^{\text{tot}}(k) - \varphi_{i}^{\text{tot}}(k+1) = \sum_{c=0}^{N-1} (\alpha_{c}^{k} - \alpha_{c}^{k+1})z_{i}(c) \geq 0.
\]

Denote \( \delta_{c}^{k} := \alpha_{c}^{k} - \alpha_{c}^{k+1} \). We complete the proof by substituting \( \Delta_{c}^{k} \) with \( \delta_{c}^{k} \) in the proof for the sufficient condition of Theorem 4.1 and following the same steps. \( \square \)

Monotonic Increasing Total Payoff Condition (Additional notes for Shapley value)

We have proven for the case of replication-robust case where the total payoff of the replicating player is monotonic decreasing. Conversely, the below condition leads to monotonic increasing total payoff, and hence the solution concept is not robust.
\[
\forall 0 \leq p \leq N - 1, \sum_{c=0}^{p} \alpha_{c}^{k+1} \geq \sum_{c=0}^{p} \alpha_{c}^{k} \quad (\implies \varphi_{i}^{\text{tot}}(k+1) \geq \varphi_{i}^{\text{tot}}(k))
\]

The proof is similar to the monotonic increasing case above, by letting \( \delta_{c}^{k} := \alpha_{c}^{k+1} - \alpha_{c}^{k} \). The Shapley value satisfies this condition, and is therefore vulnerable to replication, where the total payoff of the replicating player monotonic increases with growing number of replicas.

A.4 Proof for Replication Properties of Shapley value, Banzhaf value and LOO

We will prove that the Shapley value is monotonically increasing with growing number of replicas, and the unit gain of adding one more replica is monotonically decreasing.

Theorem 4.3. Let \( G = (N, v) \) be a market game where \( v \) is replication-redundant and submodular, the Shapley value is not replication-robust, in particular the total value of a replicating player \( i \) is monotonic increasing over growing number of replicas. That is, \( \forall i \in N, \forall k \geq 0, \)
\[
\varphi_{i}^{\text{tot}}(k) \leq \varphi_{i}^{\text{tot}}(k+1)
\]

The total value converges to player \( i \)'s characteristic value, i.e., \( \lim_{k \to \infty} \varphi_{i}^{\text{tot}}(k) = v(\{i\}) \).

Proof. To prove the monotonicity condition, first rewrite the Shapley value after replication in terms of average marginal contributions \( z_{i}(c) \) in Corollary 3.1, then, as shown in Lemma A.2, the coefficients \( \alpha_{c}^{k} \) satisfy Eq. (6b). \( \forall 0 \leq p \leq N - 1, \sum_{c=0}^{p} \alpha_{c}^{k} \leq \sum_{c=0}^{p} \alpha_{c}^{k+1} \). Finally, in Theorem 4.2, we have shown that this condition implies monotonic decreasing total payoff of the replicating player.

Next, we derive the limit \( \lim_{k \to \infty} \varphi_{i}^{\text{tot}}(k) \): By Corollary 3.1
\[
\varphi_{i}^{\text{tot}}(k) = \sum_{c=0}^{N-1} \alpha_{c}^{k}z_{i}(c) = \sum_{c=0}^{N-1} \frac{k+1}{N+k} N^{\frac{N-1}{c}} z_{i}(c)
\]
\[
\lim_{k \to \infty} \varphi_{i}^{\text{tot}}(k) = \lim_{k \to \infty} \sum_{c=0}^{N-1} \frac{k+1}{N+k} N^{\frac{N-1}{c}} z_{i}(c)
\]
\[
= \sum_{c=0}^{N-1} \frac{(N-1)}{c} z_{i}(c) \lim_{k \to \infty} \frac{k+1}{N+k} \lim_{k \to \infty} N^{\frac{1}{c}}
\]
\[
= \sum_{c=0}^{N-1} \frac{(N-1)}{c} z_{i}(c) \lim_{k \to \infty} \frac{1}{N+k}
\]
\[
= \sum_{c=0}^{N-1} \frac{(N-1)}{c} z_{i}(c) \mathcal{MC}_{i}(\emptyset) = v(i)
\]
\( \square \)
The unit gain in total payoff by adding one replica is further monotonic decreasing as a function of growing number of replicas. We have shown that the coefficients satisfy Eq. (6c) in Lemma A.2, which implies this property: Denote \( \delta_c^k := \delta_c^{k+1} - \delta_c^k \) where \( \delta_c^k := \alpha_{c+1}^k - \alpha_c^k \).

\[
\delta_c^k := \delta_c^{k+1} - \delta_c^k + [\alpha_{c+1}^k - \alpha_c^k] - [\alpha_{c+2}^k - \alpha_c^k]
= 2\alpha_{c+1}^k - \alpha_{c+2} - \alpha_c^k
\]

By Eq. (6c), \( \forall p \leq N - 1, \sum_{i=0}^p \delta_c^k \geq 0. \) Substituting \( \Delta_k^i \) with \( \delta_c^k \) in the proof for the sufficient condition of theorem 4.1, we can show the following:

\[
[\psi_i^{\text{tot}}(k + 1) - \psi_i^{\text{tot}}(k)] - [\psi_i^{\text{tot}}(k + 2) - \psi_i^{\text{tot}}(k + 1)] = \sum_{i=N-1}^{N-1} \delta_c^k \geq 0
\]

This statement says that for the Shapley value, the unit gain of adding one replica will be monotonically decreasing with growing number of replications \( k \). Therefore, the maximum unit gain of adding one replica happens at the first replication.

**Theorem 4.4.** The Banzhaf value and Leave-one-out are robust against replication. Under both payoff allocations, the limit of the total value of the replicating player \( \lim_{k \to \infty} \psi_i^{\text{tot}}(k) = 0. \)

**Proof.** We prove that the coefficients \( \alpha_c^k \) satisfy Eq. (4) \( \forall k \geq 0, \frac{\alpha_c^k}{\alpha_{c+1}^k} \geq 1 \) for both Banzhaf value and Leave-one-out, which then yield replication-robustness.

(i) Banzhaf value:

\[
\begin{align*}
\alpha_c^k &= \frac{(k+1)}{2} \binom{N-1}{c} \\
\alpha_c^{k+1} &= \frac{(k+2)}{2} \binom{N}{c+1}
\end{align*}
\]

\[
\Rightarrow \frac{\alpha_c^k}{\alpha_c^{k+1}} = \frac{2(k+1)}{(k+2)} \geq 1
\]

The limit \( \lim_{k \to \infty} \psi_i^{\text{tot}}(k) = \lim_{k \to \infty} \sum_{i=0}^{N-1} \alpha_c^k z_i(c) = \sum_{i=0}^{N-1} \binom{N-1}{c} z_i(c) \lim_{k \to \infty} \frac{k+1}{2N+k+1} = 0 \)

(ii) Leave-one-out: \( \forall k \geq 0, \frac{\alpha_c^k}{\alpha_{c+1}^k} = 0, \) hence \( \frac{\alpha_c^k}{\alpha_{c+1}^k} \geq 1, \) and \( \lim_{k \to \infty} \psi_i^{\text{tot}} = 0 \) for \( \forall k > 0 \)

#### A.5 Proof of Robust Shapley Value - Theorem 4.5

**Theorem 4.5.** (Replication-Shapley value) Eq. (4) with

\[
\gamma_{c}^{|S|} = \begin{cases} 
\frac{\binom{N-k-2}{|S|} |S|!}{|S|!(N-|S|-1)!} & \text{if } |S| < \left\lfloor \frac{N-1}{2} \right\rfloor, \\
1 & \text{otherwise}.
\end{cases}
\]

defines a replication-robust solution concept.

**Proof.** We now prove that this solution concept satisfies Eq. (4): \( \forall k \geq 0, \frac{\alpha_c^k}{\alpha_{c+1}^k} \geq 1. \) We consider the following 3 possible cases.

**Case (i):** \( c < \left\lfloor \frac{N+k-k-1}{2} \right\rfloor \leq \left\lfloor \frac{N+k}{2} \right\rfloor \)

Both \( \alpha_c^k \) and \( \alpha_c^{k+1} \) are down-weighed from the Shapley coefficients where \( \gamma_{c}^{|N+k|} = \frac{\binom{N+k-1}{|S|} |S|!}{c!(N+k-c-1)!} \):

\[
\begin{align*}
\alpha_c^k &= \gamma_{c}^{|N+k|} = (k+1) \binom{N-k-1}{c} \\
\alpha_c^{k+1} &= \gamma_{c}^{|N+k+1|} = (k+2) \binom{N-k-2}{c}
\end{align*}
\]

\[
\Rightarrow \frac{\alpha_c^k}{\alpha_c^{k+1}} = \frac{k+1}{k+2} \frac{\binom{N-k-1}{c}}{\binom{N-k-2}{c}} \geq \frac{1}{2} \times 2 = 1
\]

For example \( \left\lfloor \frac{N+k}{2} \right\rfloor = 7, \frac{\alpha_7^k}{\alpha_8^{k+1}} = 1 \)

\[
N+k = 8, \frac{\alpha_8^k}{\alpha_9^{k+1}} > 1
\]
Case (ii): $c \geq \lfloor \frac{N+k}{2} \rfloor$; \quad \text{and} \quad \lfloor \frac{N+k-1}{2} \rfloor $

Both $\tilde{\alpha}_c^k$ and $\tilde{\alpha}_c^{k+1}$ will take the original form of Shapley coefficients after replication, i.e., $\gamma_N^c = 1$:

$$\tilde{\alpha}_c^k = \alpha_c^k = (k+1)^{(N-1)} \frac{c!(N+k-1-c)!}{(N+k)!}$$

$$\tilde{\alpha}_c^{k+1} = \alpha_c^{k+1} = (k+2)^{(N-1)} \frac{c!(N+k-c)!}{(N+k+1)!}$$

$$\frac{\tilde{\alpha}_c^k}{\tilde{\alpha}_c^{k+1}} = \frac{k+1}{k+2} \frac{N+k+1}{N+k-c} \geq 2 \frac{k+1}{k+2} \geq 1. \text{(since } c \geq \lfloor \frac{N+k}{2} \rfloor \text{)}$$

Case (iii): $\lfloor \frac{N+k-1}{2} \rfloor \leq c < \lfloor \frac{N+k}{2} \rfloor$

In this case, $\tilde{\alpha}_c^k$ will take the original form, while $\tilde{\alpha}_c^{k+1}$ will take the down-weighted form. Moreover, $N + k$ must be even, hence $c = \lfloor \frac{N+k-1}{2} \rfloor$.

$$\tilde{\alpha}_c^k = \alpha_c^k = (k+1)^{(N-1)} \frac{c!(N+k-1-c)!}{(N+k)!}$$

$$\tilde{\alpha}_c^{k+1} = \gamma_N^c \alpha_c^{k+1} = (k+2)^{(N-1)} \frac{c!(N+k-c)!}{(N+k+1)!}$$

$$\frac{\tilde{\alpha}_c^k}{\tilde{\alpha}_c^{k+1}} = \frac{k+1}{k+2} \frac{N+k+1}{N+k-c} \geq 2 \frac{k+1}{k+2} \geq 1 \quad \Box$$

**Corollary 4.2.** Let $G = (N, v)$ be a market game with payoff allocation satisfying the replication-robustness condition in Theorem 4.5. Then the loss for a malicious player $i$ by replicating $k$ times is at least: $\varphi_i^{\text{tot}}(0) - \varphi_i^{\text{tot}}(k) \geq \frac{1}{N} \sum_{c=0}^{N-1} \left( 1 - \frac{k+1}{2^c} \right) \gamma_N z_i(c)$

**Proof.** To derive the loss, we use the properties of the coefficients, which was proven as intermediate results for the Robust Shapley value in Theorem 4.5, $\forall k \geq 0$, $k+2 \frac{\tilde{\alpha}_c^k}{\tilde{\alpha}_c^{k+1}} \geq 2$:

$$\varphi_i^{\text{tot}}(0) = \sum_{c=0}^{N-1} \tilde{\alpha}_c^0 z_i(c) := \frac{1}{N} \sum_{c=0}^{N-1} \gamma_N z_i(c)$$

$$\varphi_i^{\text{tot}}(k) = \sum_{c=0}^{N-1} \tilde{\alpha}_c^k z_i(c)$$

$$= (k+1) \sum_{c=0}^{N-1} \frac{\tilde{\alpha}_c^k}{k+1} z_i(c), \text{ and since } \forall k \geq 0, \frac{\tilde{\alpha}_c^k}{\tilde{\alpha}_c^{k+1}} \geq (k+1) \geq 2$$

$$\leq (k+1) \sum_{c=0}^{N-1} \frac{1}{2} \frac{\tilde{\alpha}_c^k}{k+1} z_i(c) \leq \ldots \leq (k+1) \sum_{c=0}^{N-1} \frac{1}{2^k} \frac{\tilde{\alpha}_c^0}{k+1} z_i(c)$$

$$= \left( \frac{k+1}{2^k} \right) \frac{1}{N} \sum_{c=0}^{N-1} \gamma_N z_i(c)$$

**Hence** $\varphi_i^{\text{tot}}(0) - \varphi_i^{\text{tot}}(k) \geq \frac{1}{N} \sum_{c=0}^{N-1} \left( 1 - \frac{k+1}{2^c} \right) \gamma_N z_i(c) \quad \Box$
A.6 Proof of Theorem 5.1 (including details of Algorithm 1), and Theorem 5.2

Theorem 5.1 and Theorem 5.2 provide unbiased estimators for each player’s payoff and the total allocated payoffs across a wide range of solution concepts.

**Theorem 5.1.** (Approximate payoff) Let $\varphi$ be a solution concept defined as $\varphi_i = \sum_{c=0}^{N-1} \alpha_c z_i(c)$, the payoff of a player $i$ can be computed by $\varphi_i = \sum_{c=0}^{N-1} \alpha_c \frac{N}{N-c} (E_{|S|=c} [v(S \cup \{i\})] - U_c)$, where $U_c = \left( \begin{array}{c} N \\ c \end{array} \right)^{-1} \sum_{|S|=c, |S|=N} v(S)$. 

**Proof.** As in Eq. [1] for any solution concept defined as a weighted sum of the average marginal contributions, i.e.,

$$z_i(c) = \frac{1}{\left( \begin{array}{c} N-1 \\ c \end{array} \right)} \sum_{|S|=c, S \subseteq N \setminus \{i\}} MC_i(S)$$

$$= \frac{1}{\left( \begin{array}{c} N-1 \\ c \end{array} \right)} \sum_{|S|=c, S \subseteq N \setminus \{i\}} v(S \cup \{i\}) - v(S).$$

We rewrite the average marginal contribution using the below identity:

$$\sum_{|S|=c, S \subseteq N \setminus \{i\}} MC_i(S) + \sum_{|S|=c, S \subseteq N \setminus \{i\}} MC_i(S) = \sum_{|S|=c, S \subseteq N} MC_i(S)$$

Since $v(S \cup \{i\}) - v(S) = 0$ if $i \in S$, the second term on the left hand side is 0. Therefore:

$$\sum_{|S|=c, S \subseteq N \setminus \{i\}} MC_i(S) = \sum_{|S|=c, S \subseteq N} MC_i(S) \quad (7)$$

We let $z'_i(c)$ denote the average marginal contribution of player $i$ towards any size-$c$ coalition. (Note the difference with $z_i(c)$, which is for any coalition of size $c$ without $i$):

$$z'_i(c) := \frac{1}{\left( \begin{array}{c} N \\ c \end{array} \right)} \sum_{|S|=c, S \subseteq N} MC_i(S)$$

$$= \frac{1}{\left( \begin{array}{c} N \\ c \end{array} \right)} \sum_{|S|=c, S \subseteq N} v(S \cup \{i\}) - v(S)$$

$$z'_i(c) = \left( \begin{array}{c} N-1 \\ c \end{array} \right) z_i(c) \quad \text{due to Eq.}[7]$$

$$\varphi_i := \sum_{c=0}^{N-1} \alpha_c z'_i(c)$$

$$= \sum_{c=0}^{N-1} \alpha_c \frac{N}{N-c} z'_i(c)$$

$$= \sum_{c=0}^{N-1} \alpha_c \frac{N}{N-c} \left( \sum_{|S|=c, S \subseteq N \setminus \{i\}} v(S \cup \{i\}) - v(S) \right)$$

$$= \sum_{c=0}^{N-1} \alpha_c \frac{N}{N-c} \left( E_{|S|=c, S \subseteq N} [v(S \cup \{i\})] - U_c \right)$$

$$= \sum_{c=0}^{N-1} \alpha_c \frac{N-c}{N} \left( E_{|S|=c, S \subseteq N} [v(S \cup \{i\})] - U_c \right) \quad \square$$

**Approximating the values $E_{|S|=c, S \subseteq N} [v(S \cup \{i\})]$ and $U_c$ in Algorithm 1:**

We have now proven the formula for computing the players’ payoffs. Next we will illustrate how to approximate the values $E_{|S|=c, S \subseteq N} [v(S \cup \{i\})]$ and $U_c$’s in the algorithm:

(i) When approximating the shared baselines $U_c$ for each size $c$, we can simply take the mean over the values of all size-$c$ sampled coalitions.
(ii) When approximating $\mathbb{E}_{\{S\} = c, S \subseteq N}[v(S \cup \{i\})]$ for each player $i$, taking the mean value over all size-$(c + 1)$ sampled coalitions which contain player $i$ can be incorrect, because if $S$ contained $i$, then $|S \cup \{i\}| = c$. A simple correction can compensate for this issue.

According to our sampling algorithm, when a coalition $S' = S \cup \{i\}$ that contains $i$ is drawn, the value is recorded both for $S = S'$ and $S = S' \setminus \{i\}$, i.e. set $U_i^-(k)$ and set $U_i^+(k-1)$ respectively in the algorithm. Hence, for any $c$, the value $\mathbb{E}_{\{S\} = c, S \subseteq N}[v(S + i)]$ can be approximated by a weighted average of the mean values $\gamma_c^+ U_i^+(c) + \gamma_c^- U_i^-(c)$, where $\gamma_c^+$ and $\gamma_c^-$ are chosen according to the original probability of length-$c$ coalitions with/without $i$:

$$
\begin{align*}
\gamma_c^+ &\leftarrow \frac{\binom{N-1}{c}}{\binom{N}{c}} = \frac{N-c}{N} \\
\gamma_c^- &\leftarrow \frac{\binom{N}{c} - \binom{N-1}{c}}{\binom{N}{c}} = \frac{c}{N}
\end{align*}
$$

This then leads to $\varphi_i \leftarrow \sum_{c=0}^{N-1} \alpha_c \frac{N-c}{N} \left( \frac{N-c}{N} U_i^+(c) + \frac{c}{N} U_i^-(c) \right)$ in Step 2 of Algorithm 1.

**Theorem 5.2.** (Total allocated payoff) Let $\varphi$ be a solution concept defined as $\varphi_i = \sum_{c=0}^{N-1} \alpha_c z_i(c)$, the total allocated payoff to all players $N$ can be computed by

$$
\varphi_{all} = \sum_i \varphi_i = N \sum_{c=0}^{N-1} \alpha_c (U_{c+1} - U_c).
$$

**Proof.**

$$
\begin{align*}
\varphi_{all} &= \sum_i \sum_{c=0}^{N-1} \alpha_c \frac{\binom{N}{c}}{\binom{N}{c}} \mathbb{E}_{S \subseteq N, |S| = c} [MC_i(S)] \\
&= \sum_{c=0}^{N-1} \alpha_c \frac{\binom{N}{c}}{\binom{N}{c}} \sum_i \mathbb{E}_{S \subseteq N, |S| = c} [MC_i(S)] \\
&= \sum_{c=0}^{N-1} \alpha_c \frac{\binom{N}{c}}{\binom{N}{c}} \mathbb{E}_{S \subseteq N, |S| = c} [v(S)] \\
&= \sum_{c=0}^{N-1} \alpha_c \frac{\binom{N}{c}}{\binom{N}{c}} \left( N-c \mathbb{E}_{S \subseteq N, |S| = c} [v(S)] \right) \\
&= \sum_{c=0}^{N-1} \alpha_c \frac{\binom{N}{c}}{\binom{N}{c}} \left( N-c \mathbb{E}_{S \subseteq N, |S| = c} [v(S)] \right) \\
&= N \sum_{c=0}^{N-1} \alpha_c (U_{c+1} - U_c)
\end{align*}
$$

((i)) To compute $\mathbb{E}_{i \in N, S \subseteq N, |S| = c}[v(S \cup \{i\})]$, when adding a random player $i \in N$ to a size-$c$ coalition $S$, there is probability $p = \frac{c}{N}$ that $i \in S$, hence $|S \cup \{i\}| = c+1$. Therefore, the expectation can be expressed as a weighted sum of $U_c$ and $U_{c+1}$, which are the expected values of coalitions of size $c$ and $c+1$:

$$
\mathbb{E}_{i \in N, S \subseteq N, |S| = c}[v(S \cup \{i\})] = \frac{c}{N} U_{c+1} + \frac{N-c}{N} U_c
$$

\[\Box\]
A.7 Facility Location Function (with Analytic Solution of Shapley value and Banzhaf value)

In this section, we introduce the facility location function and compute an efficient solution of the Shapley value and Banzhaf value.

**Definition A.1 (Facility Function).** Let $D$ be a set of customers and $V$ a set of facility locations. Define a utility function $w : V \times D \to \mathbb{R}_{\geq 0}$, represented by a matrix $W \in \mathbb{R}^{|V| \times d}$, where each $w_{id} \in W$ is the utility of customer $d$ for facility $i$. The facility location function $\text{Fac}(S) = \sum_{d} \max_{i \in S} w_{id}$.

**Lemma A.3.** As shown by [24]

$$\sum_{k=0}^{m} \binom{m}{k} = \frac{n + 1}{n + 1 - m}$$  \hspace{1cm} (8)

**Theorem A.4.** The Shapley value of a facility location function can be computed as

$$\varphi_i = \sum_{i=1}^{d} \left[ w_{id} - \sum_{t=1}^{ \left| V_{id} \right| } \left( n - \left| V_{id} \right| + t - 1 \right) \left( n - \left| V_{id} \right| + t - 1 \right) w_{id} \right],$$

where $V_{id} = \{ j \in V \mid w_{jd} \leq w_{id} \}$ and $e^t_i$ is the $t$-th largest element after element $i$ for dimension $d$.

**Proof.** We let $v(S)$ denote facility function $\text{Fac}(S)$ and $n := |V|$. Observe that

$$\varphi(i) = \sum_{S \subseteq V - i} \alpha_S (v(S \cup \{i\}) - v(S))$$  \hspace{1cm} (9)

$$= \sum_{i=1}^{p} \left[ \sum_{S \subseteq V_{id}} \alpha_S w_{id} - \sum_{S \subseteq V_{id}} \alpha_S \max_{j \in S} w_{jd} \right],$$  \hspace{1cm} (10)

where (1) is because the marginal gain for dimension $d$ is zero unless $i$ is the largest element for that dimension and $V_{id} = \{ j \in V \mid w_{jd} \leq w_{id} \}$ is the set of all elements which have smaller weights in the $d$-th dimension than element $i$. And by definition of the Shapley value $\alpha_S := \frac{1}{n} \binom{n-1}{|S|}^{-1}$.

Intuitively, along each dimension $d$, (#1) is a weighted sum over sets $S$ where $i$ is the largest element; and (#2) sums up for each $j \in V_{id}$ over all sets $S \subseteq V_{id}, j \in S$ where $j$ is the largest element.

For (#1) we have:

$$= \sum_{i=1}^{d} \sum_{S \subseteq V_{id}} \alpha_S w_{id}$$

$$= \sum_{i=1}^{d} w_{id} \sum_{S \subseteq V_{id}} \alpha_S$$

$$= \sum_{i=1}^{d} w_{id} \sum_{c=0}^{ \left| V_{id} \right| - 1} \binom{n}{c - 1} \left( \frac{1}{n} \right)^{c} \binom{ \left| V_{id} \right| - 1}{c}$$

$$= \sum_{i=1}^{d} w_{id} \frac{1}{n} \binom{n}{ \left| V_{id} \right| - 1}$$

$$= \sum_{i=1}^{d} w_{id} \frac{1}{n} \binom{n}{ \left| V_{id} \right| - 1},$$

where (1) is by using Lemma A.3.
For (#2) we have: (for simplicity, we let $+, -$ denote the set operations $S \cup \{e\}, S \setminus \{e\}$)

\[
(\#2) = \sum_{S \subseteq V_{id}} \alpha_S \max_{j \in S} w_jd
\]

\[
= \sum_{S \subseteq V_{id} \setminus (e_{i1}^d)} \alpha_{S + e_{i1}^d} w_{e_{i1}^d} + \sum_{S \subseteq V_{id} \setminus (e_{i1}^d + e_{i2}^d)} \alpha_{S + e_{i1}^d} w_{e_{i2}^d} + \cdots,
\]

\[
= w_{e_{i1}^d} \sum_{S \subseteq V_{id} \setminus (e_{i1}^d)} \alpha_{S + e_{i1}^d} + w_{e_{i2}^d} \sum_{S \subseteq V_{id} \setminus (e_{i1}^d + e_{i2}^d)} \alpha_{S + e_{i1}^d} + \cdots
\]

where $e_{it}^d$ is $t$-th largest element (after the element $i$) in the $d$-th dimension.

Note that

\[
\beta_i = \sum_{S \subseteq V_{id} \setminus (e_{i1}^d + \cdots + e_{it}^d)} \alpha_{S + e_{i1}^d} = \sum_{c=0}^{\frac{|V_{id}| - t}{n}} \sum_{S \subseteq V_{id} \setminus (e_{i1}^d + \cdots + e_{it}^d), |S| = c} \alpha_{S + e_{i1}^d}
\]

\[
= \frac{1}{n} \sum_{c=0}^{\frac{|V_{id}| - t}{n}} \binom{n-1}{c} \left( \binom{|V_{id}| - t}{c} - \binom{|V_{id}| - t + 1}{c+1} \right)
\]

\[
= \frac{1}{n} \sum_{x=1}^{\frac{|V_{id}| - t + 1}{n}} \binom{n-1}{x} \left( \binom{|V_{id}| - t}{x} - \binom{|V_{id}| - t + 1}{x+1} \right)
\]

\[
= \frac{1}{n} \left[ \frac{n}{n - |V_{id}| + t - 1} - 1 + \frac{n}{n - |V_{id}| + t + 1} \right]
\]

\[
= \frac{1}{\gamma + \gamma^2},
\]

where (1) is by Pascal’s identity, (2) by substituting $x = c + 1$, (3) by Lemma A.3 and observing that $\binom{n}{k}$ is zero for $k > n$, and where $\gamma = n - |V_{id}| + t - 1$.

Hence, $\varphi_i = \sum_{i=1}^{d} \left[ \frac{w_{id}}{n - |V_{id}|} \frac{1}{\frac{|V_{id}|}{n - |V_{id}| + t - 1} + \frac{|V_{id}|}{n - |V_{id}| + t + 1}} \right]$.

**Theorem A.5.** The Banzhaf value of a facility location function can be computed as

\[
\varphi_i = \frac{1}{2n-1} \sum_{i=1}^{d} \left[ 2|V_{id}|w_{id} - \sum_{t=1}^{\frac{|V_{id}|}{n - |V_{id}| + t - 1}} 2|V_{id}| - tw_{e_{id}} \right],
\]

where $V_{id} = \{ j \in V \mid w_{jd} \leq w_{id} \}$ and $e_{it}^d$ is the $t$-th largest element after element $i$ for dimension $d$.

**Proof.** We let $v(S)$ denote facility function $Fac(S)$ and $n := |V|$. Similar to the proof of the Theorem A.4 for the Shapley value:

\[
\varphi_i = \sum_{S \subseteq V \setminus \{i\}} \alpha_S (v(S \cup \{i\}) - v(S))
\]

\[
= \left(\frac{1}{d}\sum_{i=1}^{d} \left[ \sum_{S \subseteq V_{id}} \alpha_S w_{id} - \sum_{S \subseteq V_{id}} \alpha_S \max_{j \in S} w_{jd} \right] \right).
\]

\[
= \left(\frac{1}{d}\sum_{i=1}^{d} \left[ \sum_{S \subseteq V_{id}} \alpha_S w_{id} - \sum_{S \subseteq V_{id}} \alpha_S \max_{j \in S} w_{jd} \right] \right).
\]
where (1) is because the marginal gain for dimension $d$ is zero unless $i$ is the largest element for that dimension and $V_{id} = \{ j \in V \mid w_{jd} \leq w_{id} \}$. By definition of the Banzhaf value $\alpha_S := \frac{1}{2n-1}$.

For (#1) we have:

\[
(#1) = \sum_{i=1}^{d} \sum_{S \subseteq V_{id}} \alpha_S w_{id} \\
= \sum_{i=1}^{d} \sum_{S \subseteq V_{id}} \alpha_S w_{id} \\
= \sum_{i=1}^{d} \sum_{c=0}^{\lfloor |V_{id}| \rfloor} \sum_{S \subseteq V_{id} \setminus \{S \mid |S| = c \}} \alpha_S \\
= \sum_{i=1}^{d} \frac{w_{id}}{2^{n-1}} \sum_{c=0}^{\lfloor |V_{id}| \rfloor} \binom{|V_{id}|}{c} \\
= \frac{1}{2^{n-1}} \sum_{i=1}^{d} 2^{\lfloor |V_{id}| \rfloor} w_{id}
\]

For (#2) we have: (for simplicity, we let $+$, $-$ denote the set operations $S \cup \{e\}, S \setminus \{e\}$)

\[
(#2) = \sum_{S \subseteq V_{id}} \alpha_S \max_{j \in S} w_{jd} \\
= \sum_{S \subseteq V_{id} - (e_{1i}^d)} \alpha_S + e_{1j}^d w_{e_{1j}^d d} + \sum_{S \subseteq V_{id} - (e_{1i}^d + e_{12}^d)} \alpha_S + e_{12}^d w_{e_{12}^d d} + \cdots \\
= w_{e_{1j}^d d} + \sum_{S \subseteq V_{id} - (e_{1i}^d + e_{12}^d)} \alpha_S + e_{12}^d w_{e_{12}^d d} + \cdots \\
= w_{e_{1j}^d d} + \sum_{\lfloor |V_{id}| \rfloor = \beta_1} \alpha_S + e_{12}^d w_{e_{12}^d d} + \cdots \\
= w_{e_{1j}^d d} + \sum_{\lfloor |V_{id}| \rfloor = \beta_2} \alpha_S + e_{12}^d w_{e_{12}^d d} + \cdots
\]

where $e_{1j}^d$ is $t$-th largest element (after the element $i$) in the $d$-th dimension.

\[
\beta_1 = \sum_{S \subseteq V_{id} - (e_{1i}^d + \cdots + e_{1j}^d)} \alpha_S + e_{1j}^d \\
= \sum_{c=0}^{\lfloor |V_{id}| \rfloor - t} \sum_{S \subseteq V_{id} - (e_{1i}^d + \cdots + e_{1j}^d) \setminus \{S \mid |S| = c \}} \alpha_S + e_{1j}^d \\
= \frac{1}{2^{n-1}} \sum_{c=0}^{\lfloor |V_{id}| \rfloor - t} \binom{|V_{id}| - t}{c} \\
= \frac{1}{2^{n-1}} 2^{\lfloor |V_{id}| \rfloor - t}
\]

Hence, $\varphi_i = \frac{1}{2^{n-1}} \sum_{i=1}^{d} 2^{\lfloor |V_{id}| \rfloor} w_{id} - \sum_{i=1}^{d} 2^{\lfloor |V_{id}| \rfloor - t} w_{e_{1j}^d d}$. □

The Shapley value and Banzhaf value of the facility location function can be computed efficiently using Theorem A.4 and A.4 respectively. This enables us to perform large scale experiments for the sampling algorithms.
B Additional Details on Experiments

B.1 Training Data

Covertype Dataset: For the Covertype data, we use the dataset (provided by Kaggle) which consist of \( \sim 15000 \) training datapoints, uniformly distributed in the 7 output classes. We use the 10 continuous features (elevation, aspect, slope, horizontal distance to hydrology, vertical distance to hydrology, horizontal distance to roadways, hillshade 9am, hillshade noon, hillshade 3pm, horizontal distance to fire points). The preprocessing steps include normalization of each feature to \([0, 1]\).

CIFAR-100 Dataset: The dataset specifications are given in the main text. Preprocessing includes normalizing the 32x32x3 color images to \([0, 1]\). We carried out 4 sets of experiments with varied data assignments, as shown in Table 2. The Combination experiment does not contain replica players, and is used for analysing the properties of machine learning accuracy functions (Figure 1). The Uniform, Disjoint, Mixed experiments all contain replicas and they differ by the way of data assignment to the players.

Table 2: Experimental settings for CIFAR-100: training and validation data assignments. The Combination experiment uses 5 random superclasses \( C_{sup} \), each containing 4 subclasses \( C_{sub} \). All players are honest, each two hold training data of the same superclass \( C_{sup} \) but different subclasses. The Uniform, Disjoint, Mixed experiments use all 20 superclasses and their 100 subclasses. Players 1-4 are honest players while 5-7 are replicas that hold the same data as malicious player 0.

| N | Superclass \( C_{sup} \) | Subclass \( C_{sub} \) | Training Data Assignment | Validation Task | Validation Datasets |
|---|---|---|---|---|---|
| Combination | 10 | 4 | Each player assigned 2 \( C_{sub} \) of the same \( C_{sup} \) | Predict \( C_{sup} \) | Combinations of \( C_{sup} \)'s validation data |
| Uniform | 8 | 20 (All) | Players 0 - 4 assigned data from 100 \( C_{sup} \) uniformly | Predict \( C_{sup} \) | All \( C_{sup} \)'s validation data |
| Disjoint | 8 | 20 (All) | Players 0 - 4 each assigned 20 \( C_{sub} \) | Predict \( C_{sup} \) | All \( C_{sup} \)'s validation data |
| Mixed | 8 | 20 (All) | Players 0 - 4 each assigned varied portions of each \( C_{sub} \) | Predict \( C_{sup} \) | All \( C_{sup} \)'s validation data |

Training Details: We use the Adam optimizer for training the models. For the Covertype classification, we use learning rate of 0.0001, minibatch size 128, and train for 20000 steps. For the CIFAR-100 experiment, we use learning rate of 0.001, minibatch size 64, and train for 15000 steps. For the Tiny ImageNet task, we use learning rate of 0.001, minibatch size 64, and train for 10000 steps.

B.2 Feasibility Program in Algorithm 1

The feasibility program in our algorithm (Algorithm 1) is solved using the Python PuLP package and is formulated as follows, as presented in [15]:

\[
\begin{align*}
\text{minimize} & \quad 0 \\
\text{subject to} & \quad \sum_{i=1}^{N} \phi_i' = \hat{\phi}_{all} \\
& \quad \phi_i' - \phi_j' - \Delta \hat{\phi}_{ij} \leq \epsilon, \quad i, j = 1, \ldots, N \\
& \quad \phi_i' - \phi_j' - \Delta \hat{\phi}_{ij} \geq -\epsilon, \quad i, j = 1, \ldots, N
\end{align*}
\]

We have shown previously that Theorem 5.1 outputs unbiased estimates of the players’ payoffs. However, in the small sample regime, i.e., when the samples provide insufficient coverage of the different coalition sizes, the feasibility program can improve the estimates by adding a sum constraint and thereby interpolating the values among the players. Figure 5 illustrates this effect on the experiment Uniform for approximating the Shapley value of 8 players. Figures 5a and 5b show the change in the approximated values as a result of the feasibility program. This is reflected in the approximations without the feasibility program in Figure 5a where the (honest) players 1, 2, 3, 4 are below the true Shapley value. With the feasibility program, each player’s value is interpolated from other players’ values, as shown in Figure 5b. In this example, a better coverage of the coalition sizes is shown for sample size 64, and approximations before applying the feasibility program are improved.
We validate that submodularity holds approximately for the accuracy as a function of data used for training a machine learning model. In Figure 7, we show violations of the diminishing returns property (on the CIFAR dataset), a defining characterization of submodularity. The validation datasets for training a machine learning model. In Figure 7, we show violations of the diminishing returns property of the accuracy gain function:

\[ v(S) := (1 - e^{-|S|}) + \mathcal{N}(\mu, \sigma^2) \]

\[ \mu = 0.01 \sum_{i \in S} w_i, \quad w_i = i, \quad \sigma = 0.05 \]

**Algorithm 2 Random Set Function**

**Input:** \( N \): the set of all players  
**Output:** \( v : 2^N \rightarrow \mathbb{R} \): characteristic function of all possible coalitions of the players  
**for** coalition \( S \subseteq N \) **do**  
\[ \mu \leftarrow 0.01 \sum_{i \in S} w_i, \quad \sigma = 0.05 \]  
Assign utility to the coalition \( v(S) \leftarrow (1 - e^{-|S|}) + \mathcal{N}(\mu, \sigma^2) \)  
**end for**

We assign the utility values to all possible coalitions iteratively, according to Algorithm 2. Each player is given a weight \( w_i \) proportional to its index, and we add noise to coalition utility \( 1 - e^{-|S|} \) according to the sum of weights of its participating players. The average marginal contributions by coalition sizes \( z_i(c) \) is shown in Figure 6 where the players differ by their values.

**B.4 Accuracy gain function graphs**

![Figure 6: Average marginal contributions \( z_i(c) \) of the simulated random set function](image)

![Figure 7: Violations of the diminishing returns property of the accuracy gain function: \( \max_{S \in \mathcal{N}(A)} MC_i(B) - MC_i(A), A \subseteq B \). Red cells indicate a violation, and all irrelevant cells \( (A \supset B) \) are grey. The indices on the x and y axis represent the coalitions B and A respectively.](image)

We validate that submodularity holds approximately for the accuracy as a function of data used for training a machine learning model. In Figure 7, we show violations of the diminishing returns property (on the CIFAR dataset), a defining characterization of submodularity. The validation datasets
consist of data from 3 superclasses $C_{sup}$, each 2 of the 6 players holds training data from one $C_{sup}$. Violations are small when data are useful for the validation task (Fig. 7b) and slightly larger otherwise.

(a) Valid. data of $C_{sup}$ 1
(b) Valid. data of $C_{sup}$ 1-2
(c) Valid. data of $C_{sup}$ 1-3
(d) Valid. data of $C_{sup}$ 1-4

Figure 8: Valid. accuracy vs. number of players, from the Combination experiment on CIFAR-100

(a) number of replicas = 0
(b) number of replicas = 1
(c) number of replicas = 2
(d) number of replicas = 3

Figure 9: Valid. accuracy vs. #replicas (w.o. original), from the Uniform experiment on CIFAR-100

Figure 8 plots accuracy curves of all possible permutations $\pi$ of the players on the Combination experiment on CIFAR-100 with 4 superclasses $C_{sup}$, we plot for 8 players where each 2 players hold different data from the same $C_{sup}$. Along the x-axis, we increase the number of players $c$ and each point on a curve represents the accuracy of the model trained on the data held by coalition $\pi_{0:c}$. And the four plots represent different validation datasets. For example, with validation datasets of classes $C_{sup} = 1, 2$, there are 4 players holding relevant training data. The curve shows that when a player joins a coalition of other players, the training data relevant to the validation task and complementary to the other players’ data provides most accuracy gain. And the overall trend of accuracy gain is submodular. Similarly in Figure 9, we plot the accuracy curves of all possible permutations of the players on the Uniform experiment on CIFAR-100. All players hold useful data towards the validation task, and the subplots show varied number of replicas.

Figure 10: Sampling approximations versus actual values on Covertype experiment with 10 players

Figure 10 is a illustration of the sampling algorithms’ results on the Covertype classification experiments, as presented in Table 1