Numerical Solutions of a Boundary Value Problem for the Anomalous Diffusion Equation with the Riesz Fractional Derivative

Mariusz Ciesielski and Jacek Leszczynski

Institute of Mathematics and Computer Science, Czestochowa University of Technology
ul. Dabrowskiego 73, 42-200 Czestochowa
e-mail: mariusz@imi.pcz.pl, jale@imi.pcz.pl

Abstract

In this paper we present in one-dimensional space a numerical solution of a partial differential equation of fractional order. This equation describes a process of anomalous diffusion. The process arises from the interactions within the complex and non-homogeneous background. We presented a numerical method which bases on the finite differences method. We considered pure initial and boundary-initial value problems for the equation with the Riesz-Feller fractional derivative. In the final part of this paper sample results of simulation were shown.

Keywords: anomalous diffusion, fractional calculus, Riesz-Feller derivative, finite difference method, boundary value problem

1. Introduction

Anomalous diffusion is a phenomenon strongly connected with the interactions within complex and non-homogeneous background. This phenomenon is observed in transport of fluid in porous materials, in the chaotic heat baths, amorphous semiconductors, particle dynamics inside polymer network, two-dimensional rotating flow and also in ecographics. Phenomenon of anomalous diffusion deviates from the standard diffusion behaviour. In opposite to standard diffusion where linear form in the mean unit of measure \([m^α/s]\), According to \([7,11]\) the Riesz-Feller fractional operator for \(0 < α < 2\), \(α \neq 1\) for one-variable function \(u(x)\) is

\[
\frac{∂^α}{∂|x|^α}u(x) = (-d^α) u(x) = -[c_L(α,θ) -∞D^α_0 u(x)]
\]

where

\[
-∞D^α_0 u(x) = \begin{cases} 
-∞I^1_{-α}u(x), & \text{for } 0 < α ≤ 1, \\
-∞I^2_{-α}u(x), & \text{for } 1 < α ≤ 2,
\end{cases}
\]

\[
zD^α_∞ u(x) = \begin{cases} 
zI^1_{-α}u(x), & \text{for } 0 < α ≤ 1, \\
zI^2_{-α}u(x), & \text{for } 1 < α ≤ 2,
\end{cases}
\]

and coefficients \(c_L(α,θ), c_R(α,θ)\) (for \(0 < α ≤ 2, α \neq 1\), and for \(θ ≤ \min (α, 2 - α)\)), are defined as

\[
c_L(α,θ) = \frac{\sin (α - θ) \pi}{\sin (α π)}, \quad c_R(α,θ) = \frac{\sin (α + θ) \pi}{\sin (α π)}.
\]

The fractional operators of order \(α\) \(-∞I^α u(x)\) and \(zI^α u(x)\) are defined as the left- and right-side of Weyl fractional integrals \([6,14,15,16]\) which definitions are

\[
-∞I^α u(x) = \frac{1}{Γ(α)} \int_{-∞}^{x} \frac{u(ξ)}{(x - ξ)^{1-α}} dξ,
\]

\[
zI^α_∞ u(x) = \frac{1}{Γ(α)} \int_{x}^{∞} \frac{u(ξ)}{(ξ - x)^{1-α}} dξ.
\]

Considering Eqn (1) we obtain the classical diffusion equation for \(α = 2\), i.e. the heat transfer equation. If \(α = 1\), and the parameter of skewness \(θ\) admits extreme values in \([5]\), the transport equation is noted. Therefore we assume variations of the
parameter $\alpha$ within the range $0 < \alpha \leq 2$. Analysing behaviour of the parameter $\alpha < 2$ in Eqn (1), we found some combination between transport and propagation processes.

For analytic solution of Eqn (1) we can apply Green functions [6]. We numerically solve Eqn (1) when additional nonlinear term may occur. Some numerical methods used in solution of fractional differential equations can be found in [7]. However they apply the infinite domain.

In this work we will consider Eqn (1) limited for $1 < \alpha \leq 2$ in one dimensional domain $\Omega : L \leq x \leq R$ with the boundary-value conditions of the first kind (the Dirichlet conditions) as

$$
\begin{align*}
    x = L : & \quad C(L, t) = g_L(t), & t > 0, \\
    x = R : & \quad C(R, t) = g_R(t),
\end{align*}
$$

and with the initial-value condition

$$
C(x, t)|_{t=0} = c_0(x).
$$

3. Numerical method

According to the finite difference method [1] we consider a discrete form of Eqn (1) both in time and space. In the previous work [3] we solved numerically the anomalous diffusion equation similar to the Eqn (1) with the time-fractional derivative. We called this method FFDM (Fractional FDM). The problem of solving of Eqn (1) lies in properly approximation of the Riesz-Feller derivative (2) in numerical scheme.

3.1. Approximation of the Riesz-Feller derivative

We begin numerical analysis from discrete forms of operators (6) and (7). We introduce homogenous spatial grid $-\infty < \ldots < x_{i-2} < x_{i-1} < x_i < x_{i+1} < x_{i+2} < \ldots < \infty$ with the step $h = x_k - x_{k-1}$ and we denote value of function $u$ in the point $x_k$ as $u_k = u(x_k)$, for $k \in \mathbb{Z}$. In order to simplify notations we take here the function of one variable. For numerical integration scheme we assumed the trapezoidal rule. The integral (6) in point $x_i$ of the grid is replaced by the sum of discrete integrals as

$$
-\infty \int_{x_{i-k}}^{x_i} u_i = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \int_{x_{i-k}}^{x_{i-k-1}} \frac{u_i(\xi)}{(x_i - \xi)^{1+\alpha}} d\xi,
$$

and using linear interpolation of function $u$ in every sub-interval $[x_{i-k-1}, x_{i-k}]$

$$
u_i^*(\xi) = \frac{u_{i-k} - u_{i-k-1}}{h} \xi + \frac{u_{i-k-1}x_{i-k} - u_{i-k}x_{i-k-1}}{h},
$$

we have

$$
-\infty \int_{x_{i-k}}^{x_i} u_i \approx \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \int_{x_{i-k}}^{x_{i-k-1}} \frac{u_i^*(\xi)}{(x_i - \xi)^{1+\alpha}} d\xi = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \left[ (u_{i-k} - u_{i-k-1}) a_{k}^{(\alpha)} + (u_{i-k-1}x_{i-k} - u_{i-k}x_{i-k-1}) b_{k}^{(\alpha)} \right]
$$

where

$$
\begin{align*}
a_k^{(\alpha)} &= h^{\alpha-1} x_1 (k+1)^{\alpha} - k^{\alpha} \\
b_k^{(\alpha)} &= h^{\alpha-1} (k+1)^{\alpha} - k^{\alpha}.
\end{align*}
$$

After next transforms we can write

$$
-\infty \int_{x_i}^{R} u_i \approx h^\alpha \sum_{k=0}^{\infty} u_{i-k} v_k^{(\alpha)}
$$

where

$$
v_k^{(\alpha)} = \frac{1}{\Gamma(2 + \alpha)} \times \begin{cases} 1, & \text{for } k = 0, \\ (k+1)^{1+\alpha} - 2k^{1+\alpha} + (k - 1)^{1+\alpha}, & \text{for } k = 1, \ldots, \infty. \end{cases}
$$

Similar to previous considerations we approximate operator $\int_{-\infty}^{x_i} u(x)$ in the point $x_i$ and finally we obtain

$$
z_i \int_{-\infty}^{x_i} u_i \approx h^\alpha \sum_{k=0}^{\infty} u_{i-k} v_k^{(\alpha)},
$$

where coefficients $v_k^{(\alpha)}$ have identical forms as (16).

In the next step we analyse operator (2). It can be expressed in the form (in order to simplify this we denote $c_L = c_L(\alpha, \theta)$ and $c_R = c_R(\alpha, \theta)$)

$$
z_i D_{x_i}^{\alpha} u_i = - \sum_{k=0}^{\infty} \left[ c_L \frac{d^2}{dx^2} \left[ -\infty \int_{x_i}^{+\infty} u(x) \right] + c_R \frac{d^2}{dx^2} \left[ x \int_{-\infty}^{x_i} u(x) \right] \right].
$$

We used the central difference scheme for the second spatial derivative in the point $x_i$ and we obtain

$$
z_i D_{x_i}^{\alpha} u_i \approx \frac{1}{h^\alpha} \sum_{k=0}^{\infty} u_{i-k} w_k^{(\alpha)},
$$

where coefficients $w_k^{(\alpha)}$ are

$$
w_k^{(\alpha)} = \frac{-1}{\Gamma(4 - \alpha)} \times \begin{cases} ((k+2)^{3-\alpha} - 4(k+1)^{3-\alpha} + 6k^{3-\alpha}) c_L, & \text{for } k \leq -2, \\ (3^{3-\alpha} - 2^{5-\alpha} + 6) c_L + c_R, & \text{for } k = -1, \\ (3^{3-\alpha} - 2^{3-\alpha}) c_R, & \text{for } k = 0, \\ (3^{3-\alpha} - 2^{5-\alpha} + 6) c_R + c_L, & \text{for } k = 1, \\ ((k+2)^{3-\alpha} - 4(k+1)^{3-\alpha} + 6k^{3-\alpha}) c_R, & \text{for } k \geq 2. \end{cases}
$$

Assuming $\alpha = 2$ and $\theta = 0$ we have $c_L(2,0) = c_R(2,0) = \frac{1}{h}$ and we obtain

$$
w_k^{(2)} = \begin{cases} 0, & \text{for } k \leq -2, \\ 1, & \text{for } k = -1, \\ -2, & \text{for } k = 0, \\ 1, & \text{for } k = 1, \\ 0, & \text{for } k \geq 2. \end{cases}
$$

These coefficients are identical as for wide known the central difference scheme for the second derivative. Also when $\alpha \to 1^+$
and \( \theta = 0 \) after arduous calculations of limits we obtain coefficients

\[
\begin{align*}
    w_k^{(1+)} &= \frac{1}{2\pi} \times \\
    &\left\{ \begin{array}{ll}
    \ln \frac{(k+1)^{4(k+1)^2} (k-1)^{4(k-1)^2}}{(k+2)^{4(k+2)^2} (k-2)^{4(k-2)^2}}, & \text{for } k \leq -2, \\
    16 \ln 2 - 9 \ln 3, & \text{for } k = -1, \\
    -8 \ln 2, & \text{for } k = 0, \\
    16 \ln 2 - 9 \ln 3, & \text{for } k = 1, \\
    \ln \frac{(k+1)^{4(k+1)^2} (k-1)^{4(k-1)^2}}{(k+2)^{4(k+2)^2} (k-2)^{4(k-2)^2}}, & \text{for } k \geq 2.
\end{array} \right.
\end{align*}
\]

In literature didn’t find exact values of approximating coefficients. When \( \alpha = 1 \) the Riesz-Feller operator is singular, hence the problem. Numerous works of Gorenflo and Mainardi i.e. [6, 7] propose various ways which determine values of the coefficients \( w_k^{(0)} \) (i.e. based on the Grünwald-Letnikov discretization) but they don’t provide continuity in the interval \( \alpha \in (1, 2] \). The coefficients (23) can approximate the Cauchy process when we use (23) in numerical calculations.

3.2. Fractional FDM

While discretization of the Riesz-Feller derivative in space is done, in this subsection we describe the finite difference method for the equation of anomalous diffusion (1). Here we restrict this solution to one dimensional space. In comparison with the standard diffusion equation where discretization of the second derivative in space can be approximated by the central difference of second order, we will use generalized scheme given by formula (20). The differences appear in setting of boundary conditions.

We shall introduce a temporal grid \( 0 = t_0 < t_1 < \ldots < t_l < t_{l+1} < \ldots < \infty \) with the step \( \Delta t = t_{i+1} - t_i \) and we denote value of the function \( C(x,t) \) at the moment of time \( t_i \) as \( C_k = C(x_i,t_i) \) for \( k \in \mathbb{Z} \) and \( f \in \mathbb{N} \).

3.2.1. Pure initial value problem

In the explicit scheme of the FDM we replaced Eqn (1) by the following formula

\[
\begin{align*}
    C_{i+1}^2 - C_i^2 &= K_0 \frac{1}{h^2} \sum_{k=\infty}^{\infty} C_{i+k}^2 w_k^{(0)}.
\end{align*}
\]

After simplification finally we obtained

\[
\begin{align*}
    C_{i+1}^2 &= \sum_{k=-\infty}^{\infty} C_{i+k}^2 p_k^{(0)},
\end{align*}
\]

where coefficients \( p_k^{(0)} \) are

\[
\begin{align*}
    p_k^{(0)} &= \begin{cases} 
    1 + K_0 \frac{\Delta t}{h^2} w_0^{(0)}, & \text{for } k = 0, \\
    K_0 \frac{\Delta t}{h^2} w_k^{(0)}, & \text{for } k \neq 0.
    \end{cases}
\end{align*}
\]

Using simple calculations one may proof, that arise the following relationship

\[
\begin{align*}
    \sum_{k=-\infty}^{\infty} p_k^{(0)} = 1.
\end{align*}
\]

In order to determine stability of the explicit scheme the coefficient (26) for \( k = 0 \) in formula (25) should be positive

\[
\begin{align*}
    p_0^{(0)} &= 1 + K_0 \frac{\Delta t}{h^2} w_0^{(0)} > 0.
\end{align*}
\]

Hence we fixed the maximum length of the step \( \Delta t \) as

\[
\begin{align*}
    \Delta t < \frac{-h^2}{K_0 w_0^{(0)} - K_0 (2^{2-\alpha} - 4) [\Gamma (4 - \alpha) (1 - \alpha)]}.
\end{align*}
\]

The initial condition (9) is introduced directly to every grid nodes at the first step \( t = t^0 \). This determines values of the function \( C^{(0)} \) as

\[
\begin{align*}
    C_i^{(0)} = c_0 (x_i).
\end{align*}
\]

In unbounded domains the implicit method isn’t easily applicable because it generates infinite dimensions of all matrices. Thus one usually seeks improved difference equations within the explicit scheme.

3.2.2. Boundary-initial value problem

Presenting numerical solution (25) with included unbounded domain \( -\infty < x < \infty \) has no practical implementations in computer simulations.

Now, we present solution of this problem on the finite domain \( \Omega : L \leq x \leq R \) with boundary conditions (8). We divide this domain \( \Omega \) into \( N \) sub-domains with \( L = (R - L)/N \). Figure 1 shows modified spatial grid.

![Figure 1: The nodes grid over space](image)

Here we can observe additional ‘virtual’ points in the grid placed outside of the domain \( \Omega \). In order to introduce the Dirichlet boundary conditions we proposed treatment which bases on assumption that values of the function \( C \) in outside points are identical as values in the boundary nodes \( x_0 \) or \( x_N \).

\[
\begin{align*}
    C(x_0,t) &= \begin{cases} 
    C(x_0,t) = g_L(t), & \text{for } k < 0, \\
    C(x_N,t) = g_R(t), & \text{for } k > N.
    \end{cases}
\end{align*}
\]

On the base of previous considerations we modify expression (20) for discretization of the Riesz-Feller derivative. Thus we have

\[
\begin{align*}
    x_i D_{+\alpha} C(x_i,t) &\approx \frac{1}{h^2} \sum_{k=-i}^{N-i} C_{x+i+k}^{(0)} w_k^{(0)} \\
    &+ g_L(t) s_{L_i}^{(0)} + g_R(t) s_{R_i}^{(0)},
\end{align*}
\]

for \( i = 1, \ldots, N - 1 \), where

\[
\begin{align*}
    s_{L_i}^{(0)} &= \sum_{k=-i}^{1} w_k^{(0)} = \frac{-1}{\Gamma (4 - \alpha)} \times \\
    &\left[ - (i + 2)^{3-\alpha} + 3 (i + 1)^{3-\alpha} - 3 (i - 1)^{3-\alpha} + (i - 1)^{3-\alpha} \right] c_L, \\
    s_{R_i}^{(0)} &= \sum_{k=N-i+1}^{\infty} w_k^{(0)} = \frac{-1}{\Gamma (4 - \alpha)} \times \\
    &\left[ - (N - i + 2)^{3-\alpha} - 3 (N - i)^{3-\alpha} + (N - i - 1)^{3-\alpha} \right] c_R.
\end{align*}
\]

Putting this expression to Eqn (1) we obtain a finite difference scheme depending on weighting factor \( \sigma \). Here we assumed

\[
\begin{align*}
    g_{L_i}^{(f)} &= g_L \left( t_{i+1/2}^{(f)} \right) = g_L \left( \Delta t \left( f + \frac{1}{2} \right) \right), \\
    g_{R_i}^{(f)} &= g_R \left( t_{i+1/2}^{(f)} \right) = g_R \left( \Delta t \left( f + \frac{1}{2} \right) \right).
\end{align*}
\]
in order to simplify the numerical scheme. For internal nodes \( x_i, \ i = 1, \ldots, N - 1 \) we have
\[
\frac{C_i^{t+1} - C_i^t}{\Delta t} = K_\alpha \sum_{k=1}^{N-i} \left( \sigma C_i^{k+1} + (1 - \sigma) C_i^{k+1} \right) w_k^{(\alpha)} + g_L^{(\alpha)} s_{L_i}^{(\alpha)} + g_R^{(\alpha)} s_{R_i}^{(\alpha)},
\]
and for the boundary nodes \( x_0 \) and \( x_N \):
\[
C_0^{t+1} = g_L^{(\alpha)} + \frac{1}{2}, \quad C_N^{t+1} = g_R^{(\alpha)}. \tag{38}
\]

The method is explicit for \( \sigma = 1 \) and partially implicit for \( 0 < \sigma < 1 \) and with \( \sigma = 0 \) being fully implicit. In literature this method is known as the \( \sigma \)-method for parabolic equations.

Above scheme described by expressions (37)-(39) can be written in matrix form as
\[
A \cdot C^{t+1} = B, \tag{40}
\]
where
\[
A = \begin{bmatrix}
1 & 1 + a_1 & a_1 & \cdots & a_{N-3} & a_{N-2} & a_N & 0 \\
0 & 1 & a_2 & \cdots & a_{N-4} & a_{N-3} & a_N & 0 \\
0 & 0 & 1 & \cdots & a_{N-5} & a_{N-4} & a_N & 0 \\
0 & 0 & 0 & \cdots & 1 & a_1 & a_2 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1
\end{bmatrix}
\]
\[
B = \begin{bmatrix}
g_L^{(\alpha)} \\
b_1 \\
b_2 \\
b_3 \\
\vdots \\
b_{N-2} \\
b_{N-1} \\
g_R^{(\alpha)}
\end{bmatrix}, \tag{41}
\]

with
\[
a_j = (\sigma - 1) K_\alpha \frac{\Delta t}{h^\alpha} w_j^{(\alpha)}, \quad \text{for } j = N-1, \ldots, 1, \tag{43}
\]
\[
b_j = C_j^t + K_\alpha \frac{\Delta t}{h^\alpha} \left[ g_L^{(\alpha)} s_{L_j}^{(\alpha)} + g_R^{(\alpha)} s_{R_j}^{(\alpha)} + \sigma \sum_{k=j}^{N-j} C_k^{t} w_k^{(\alpha)} \right], \quad \text{for } j = 1, \ldots, N-1. \tag{44}
\]

and \( C^{t+1} \) is the vector of unknown function’s values \( C \) at the time \( t^{t+1} \).

Particular case of above scheme (37) is the explicit scheme (for \( \sigma = 1 \)) which may be simplified to
\[
C_i^{t+1} = \begin{cases}
g_L^{(\alpha)} & \text{for } i = 0, \\
K_\alpha \frac{\Delta t}{h^\alpha} \left[ g_L^{(\alpha)} s_{L_i}^{(\alpha)} + g_R^{(\alpha)} s_{R_i}^{(\alpha)} \right] + \sigma \sum_{k=1}^{N-i} C_i^{k+1} p_k^{(\alpha)}, & \text{for } i = 1, \ldots, N-1, \\
g_R^{(\alpha)} & \text{for } i = N,
\end{cases} \tag{45}
\]

with \( p_k^{(\alpha)} \) defined by formula (26).

We can observe that boundary conditions influence to all values of the function in every node. In opposite to the second derivative over space which is approximated locally, the characteristic feature of Riesz-Feller and other fractional derivatives is dependence on values of all domain points. For \( \alpha = 2 \) and \( \theta = 0 \) our scheme is identically as wide known and used the forward difference in time and central difference in space scheme (FTCS) [12,14].

The skewness parameter \( \theta \) has great significance influence on the solution. For \( \alpha \to 1^+ \) and \( \theta \to \pm 1^+ \) one can obtain the classical hyperbolic equation, i.e. the first order wave equation (the transport equation). In this case our scheme tends to the known Euler’s forward time and central space (FTCS) approximation of Eqn (1). Unfortunately this is unconditionally unstable and therefore this is disadvantage this method.

Proposed numerical scheme makes a bridge between Gaussian and Cauchy processes. Our scheme is also a bridge between diffusion and transport phenomena.

4. Simulation results

In this section we present results of calculation. In all presented simulations we assumed \( k_\alpha = 1 m^\alpha / s \) and the length of 1D domain \( l = 1 m \). Figure 1 shows two charts over space (one in the logarithmic scale) with absorbing boundary \( C(x,t)|_{x=0} = C(x,t)|_{x=1} = 0 \). On these plots solutions for different values of parameter \( \alpha \in \{1.01, 1.5, 2\} \) at time \( t = 0, 0.01, 0.3s \) for \( \theta = 0 \) are presented. Figure 2 presents another example of the solution.

Figure 2: Solution over space for \( \alpha \in \{1.01, 1.5, 2\} \)

(a) normal scale; (b) logarithmic scale.
which differs from example presented by the Fig. 1 (boundary conditions $C(x, t)|_{x=0} = C(x, t)|_{x=1} = 100$ and initial condition $C(x, t)|_{t=0} = 0$). In both cases we observe diffusion process arising in different way. The last example reflects case when the parameter of skewness is $\theta = 0.5$ and $\alpha = 1.4$. Figure 3 shows a diffusion transport process over space at different moments of time.

Figure 3: Solution over space for $\alpha = 1.01, 1.5, 2$.

Figure 4: Solution over space for $\alpha = 1.4$ and $\theta = 0.5$.

5. Conclusions

In summary, we proposed the fractional finite difference method for fractional diffusion equation with the Riesz-Feller fractional derivative which is extension to the standard diffusion. We analysed a linear case of diffusion equation and in the future we will work on non-linear cases. We obtained the implicit and explicit FDM schemes which generalise classical schemes of FDM for the diffusion equation. Moreover, for $\alpha = 2$ our solution equals to the classical finite difference method.

Analysing plots included in this work, we can see that in the case $\alpha < 2$ (the Levy flight) diffusion is slower then the standard diffusion (Brownian motion) in the initial time. Nevertheless, when we analyse the probability density function we observe a long tail of distribution in the long time limit. In this way we can simulate same rare and extreme events which are characterized by arbitrary very large values of particle jumps.

Analysing changes in the skewness parameter $\theta$ we observed interesting behaviour in solution. For $\alpha \to 1^+$ and for $\theta \to \pm 1^+$ we obtained the first order wave equation. For $\theta \in (0, 1)$ (with restrictions to order $\alpha$) we generate a class of non-symmetric probability density functions.

References

[1] Ames W.F., Numerical Methods for Partial Differential Equations, Academic Press, 3rd ed., 1992.
[2] Carpinteri A., Mainardi F. (eds.), Fractals and Fractional Calculus in Continuum Mechanics, Springer Verlag, Vienna - New York, 1997.
[3] Ciesielski M., Leszczyński J., Numerical simulations of anomalous diffusion, 15th International Conference on Computer Methods in Mechanics CMM-2003, Gliwice-Wisła, June 3-6, 2003. (proceeding on CD-ROM)
[4] Frank L.S., Difference operators in convolution, Soviet Math. Dokl. 9, pp. 831-834, 1968.
[5] Frank L.S., Spaces of network functions, Math. USSR Sbornik 15, pp. 183-226, 1971.
[6] Gorenflo R., Mainardi F., Fractional diffusion processes: Probability Distributions and Continuous Time Random Walk, Springer-Verlag LNP621, Berlin, pp. 148-166, 2003.
[7] Gorenflo R., Mainardi F., Random walk models for space-fractional diffusion processes, Fractional Calculus and Applied Analysis, Vol. 1 (2), pp. 167-191, 1998.
[8] Hilfer R., Applications of Fractional Calculus in Physics, World Scientific Publ. Co., Singapore, 2000.
[9] Hoffman J.D., Numerical Methods for Engineers and Scientists, McGraw-Hill, 1992.
[10] Leszczynski J., Ciesielski M., A numerical method for solution of ordinary differential equations of fractional order, PPAM 2001 Conf., Springer-Verlag, LNCS 2328, pp. 695-702, 2002.
[11] Mainardi F., Luchko Yu., Pagnini G., The fundamental solution of the space-time fractional diffusion equation, Fractional Calculus and Applied Analysis, Vol. 4, No 2, pp. 153-192, 2001.
[12] Majchrzak E., Mochnacki B., Metody numeryczne, Podstawy teoretyczne, Aspekty praktyczne i algorytmy, Wydawnictwo Politechniki Śląskiej (in Polish), Gliwice, 1996.
[13] Metzler R., Klafter J., The random walk’s guide to anomalous diffusion: a fractional dynamics approach, Phys. Rep. 339, pp. 1-70, 2000.
[14] Oldham K., Spanier J., The fractional Calculus, Academic Press, New York and London, 1974.
[15] Podlubny I., Fractional Differential Equations, Academic Press, San Diego, 1999.
[16] Samko S. G., Kilbas A. A., Marichev O. I., Integrals and derivatives of fractional order and some of their applications, Gordon and Breach, London, 1993.