On the classification of reflexive polyhedra

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ABSTRACT

Reflexive polyhedra encode the combinatorial data for mirror pairs of Calabi-Yau hypersurfaces in toric varieties. We investigate the geometrical structures of circumscribed polytopes with a minimal number of facets and of inscribed polytopes with a minimal number of vertices. These objects, which constrain reflexive pairs of polyhedra from the interior and the exterior, can be described in terms of certain non-negative integral matrices. A major tool in the classification of these matrices is the existence of a pair of weight systems, indicating a relation to weighted projective spaces. This is the cornerstone for an algorithm for the construction of all dual pairs of reflexive polyhedra that we expect to be efficient enough for an enumerative classification in up to 4 dimensions, which is the relevant case for Calabi-Yau compactifications in string theory.

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1 Introduction

In the framework of toric geometry, it is possible to encode properties of algebraic varieties in terms of fans or polyhedra defined on integer lattices. In particular, it has been shown by Batyrev that the Calabi–Yau condition for hypersurfaces of toric varieties is equivalent to reflexivity of the underlying polyhedron [1]. Moreover, the duality of reflexive polyhedra corresponds to the mirror symmetry of the resulting class of Calabi–Yau manifolds (see for example [2–5] and references therein). This is the main motivation for the interest in a classification of 4-dimensional reflexive polyhedra in the context of string theory.

It is known that the total number of reflexive polyhedra is finite in any given dimension, because various bounds on the volume and the number of points have been derived as a function of the dimension and the number of interior lattice points [6–8]. The case of \( n = 2 \) dimensions is the easiest because all polygons with one interior point are reflexive (this is no longer true for \( n > 2 \)). There are 16 such polygons, which were constructed in [9, 10] (we will rederive this result in the last section to illustrate the application of our tools). In 4 dimensions we expect at least some \( 10^4 \) reflexive pairs and the known bounds for general lattice polytopes [6–8] are not very useful for explicit constructions. What we need is an efficient algorithm, which probably should rely on reflexivity in an essential way. It is the purpose of the present paper to provide such an algorithm.

Our approach is partly motivated by experience with transversal polynomials in weighted projective spaces [11–14] and by the orbifold construction of mirror pairs [15–18], but this will become clear only at a later stage. The basic strategy is to find minimal integral polytopes \( M \) that are spanned by vertices of \( \Delta \) and that still have \( 0 \) in the interior (the generic case is a simplex). By duality, \( M^* \) bounds \( \Delta^* \) and its facets carry facets of \( \Delta^* \). If we have minimal polytopes \( M \) and \( M^* \) for \( \Delta \) and \( \Delta^* \), respectively, then the pairing matrix of the respective vertices turns out to be strongly constrained. Such a matrix encodes the structures of \( M \) and \( M^* \), which bound \( \Delta \) from the interior and the exterior.

The final step in the classification is the reconstruction of the complete pairing matrix of vertices of \( \Delta \) and \( \Delta^* \). The pairings of all vertices characterize the reflexive pair up to a finite number of possible choices of dual pairs of lattices. In the simplex case the barycentric coordinates of the interior point correspond to the weights in the context of weighted projective spaces. Indeed, the authors of [19] tried to interpret toric Calabi-Yau manifolds as non-transverse hypersurfaces in weighted \( \mathbb{P}^4 \). Our results imply that, even without transversality, only a finite number of weight systems makes sense in the toric context. Moreover, the large ambiguity in the generalized transposition rule of [19] is constrained by our rules for the selection of vertices, which may be regarded as rules about which transpositions make sense.

In section 2 we give some basic definitions and deduce geometrical properties of minimal polytopes. In general we may need a number of lower dimensional simplices containing \( 0 \) in the interior to span a neighborhood of \( 0 \). Then we have several weight systems and the toric variety can only be related to sort of a (non-direct) product of weighted spaces. In section 3 we discuss the properties of (minimal) pairing matrices and the relations among pairings in higher-dimensional lattices that we use to embed a reflexive pair. We illustrate our concepts using an example of a 4-dimensional polyhedron that was analysed in the toric context in [3]. This completes the setup that we need in section 4 to state the classification algorithm and to prove its finiteness. As an illustration we rederive the 2-dimensional case.
2 Reflexive polyhedra and minimal polytopes

We first recall some elementary definitions about polytopes [20]. A rational polyhedron is an intersection of finitely many halfspaces \( \{ x \in \mathbb{Q}^n : a_ix^i \geq b \} \) with \( a_i, b \in \mathbb{Z} \). A polytope is a bounded polyhedron or, equivalently, the convex hull of a finite number of points. A lattice (or integral) polytope is a polytope whose vertices belong to some lattice \( \Gamma \cong \mathbb{Z}^n \). We will identify \( \mathbb{Q}^n \) with the rational extension \( \Gamma_\mathbb{Q} \) of \( \Gamma \), i.e. \( \mathbb{Q}^n \cong \Gamma_\mathbb{Q} = \Gamma \otimes_\mathbb{Z} \mathbb{Q} \). The distance of a lattice point \( x \in \Gamma \) to a lattice hyperplane \( H(a_i, b) = \{ x \in \Gamma_\mathbb{Q} : a_ix^i = b \} \) where the integers \( a_i \) have greatest common divisor 1, is defined by \( d(H, x) := |a_ix^i - b| \). This number is 1 plus the number of lattice hyperplanes between \( x \) and \( H \). These definitions are invariant under changes of the lattice basis, so we can write \( \langle x, y \rangle \) instead of \( a_ix^i \) whenever we do not want to refer to a specific basis. Then the condition that the \( a_i \) have no common divisor means that \( a \in \Gamma^* \) is primitive.

A reflexive polyhedron \( \Delta \) is a polytope with one interior point \( P \) whose bounding hyperplanes are all at distance 1 from \( P \). If an arbitrary convex set \( \Delta \) contains the origin in its interior we define the dual (or polar) set

\[
\Delta^* := \{ y \in \Gamma_\mathbb{Q}^* : \langle y, x \rangle \geq -1 \ \forall x \in \Delta \}.
\]

Assuming that the interior point \( P = 0 \) is the origin it is easy to see that a polytope \( \Delta \) is reflexive if and only if \( \Delta^* \) is integral, i.e. if all vertices of \( \Delta^* \) belong to \( \Gamma^* \).

Consider an \( n \)-dimensional reflexive pair of polyhedra \( \Delta \) and \( \Delta^* \) defined on lattices \( \Gamma \) and \( \Gamma^* \). For each of these polyhedra we choose a set of \( k \) (\( \overline{k} \)) hyperplanes \( H_i, i = 1, \ldots, k \) (\( \overline{H}^j, j = 1, \ldots, \overline{k} \)) carrying facets in such a way that these hyperplanes define a bounded convex body \( Q \) (\( \overline{Q} \)) containing the original polyhedron and that \( k \) and \( \overline{k} \) are minimal. We define a redundant coordinate system where the \( i^{th} \) coordinate of a point is given by its integer distance to \( H_i \) (nonnegative on the side of the polyhedron). This is just the degree of the homogeneous coordinate [21] corresponding to \( H_i \) in the monomial determined by the point. Note that the vertices of \( Q \) and \( \overline{Q} \) need not have integer coordinates. All coordinates of the interior points are equal to 1, each coordinate of any point of a polyhedron is nonnegative. Whenever we use this sort of coordinate system we will label the interior points by \( \mathbf{1} \) and \( \mathbf{1}^\overline{\mathbf{1}} \). Note that \( \mathbf{1} \) is the only integer point in the interior of \( Q \). For all other points one coordinate must be smaller than one so that they belong to some \( H_i \). We have thus shown that any such polytope \( Q \) has all lattice points of \( \Delta \), except for \( \mathbf{1} \), at its boundary.

The duals of these hyperplanes are two collections of \( \overline{k} \) (\( k \)) vertices \( V_j \) (\( \overline{V}^j \)) spanning polyhedra \( M = \overline{Q} \) and \( \overline{M} = Q^* \) that contain the interior points of \( \Delta \) (\( \Delta^* \)). \( M \) and \( \overline{M} \) are minimal in the sense that there are no collections of less than \( \overline{k} \) (\( k \)) vertices of \( \Delta \) (\( \Delta^* \)) containing \( \mathbf{1} \) (\( \mathbf{1}^\overline{\mathbf{1}} \)) in the interior.

Let us first obtain some information on the general structure of minimal polytopes. Here we will not use the affine structure (labelling the interior point by \( \mathbf{1} \)), but instead we will use a linear structure, calling the interior point \( \mathbf{0} \) and identifying vertices \( V \) with vectors. Then the fact that \( M \) has \( \mathbf{0} \) in its interior is equivalent to the fact that any point in \( \mathbb{Q}^n \) can be written as a nonnegative linear combination of vertices. Considering all triangulations where every simplex contains a specific vertex \( V \) of \( M \), we see that there is at least one simplex of dimension \( n \) with this vertex containing \( \mathbf{0} \), i.e. \( \mathbf{0} \) lies in the interior of this simplex or one of its simplicial faces containing \( V \). So we have a collection of vertices and a collection of subsets...
of this set of vertices defining lower dimensional simplices with 0 in their interiors (we will call such simplices “good simplices”), in such a way that each vertex belongs to at least one good simplex. Now we note that if we have a collection of good simplices, then 0 is also in the interior of the polytope spanned by all the vertices of these simplices (of course, “interior” here means interior w.r.t. the linear subspace spanned by these vertices).

**Lemma 1:** A minimal polytope $M = \text{ConvexHull}\{V_1, \ldots, V_s\}$ in $\mathbb{R}^n$ is either a simplex or contains an $n'$-dimensional minimal polytope $M' := \text{ConvexHull}\{V_1, \ldots, V_{s'}\}$ and a good simplex $S := \text{ConvexHull}(R \cup \{V_{s'+1}, \ldots, V_s\})$ with $R \subset \{V_1, \ldots, V_{s'}\}$ such that $\overline{k} - k' = n - n' + 1$ and $\dim S \leq n'$.

**Proof:** If $M$ is a simplex, there is nothing left to prove. Otherwise, consider the set of all good simplices consisting of vertices of $M$. Any subset of this set will define a lower dimensional minimal polytope. Among these, take one (call it $M'$) with the maximal dimension $n'$ smaller than $n$. $\mathbb{Q}^n$ factorizes into $\mathbb{Q}^{n'}$ and $\mathbb{Q}^{n}/\mathbb{Q}^{n'} \cong \mathbb{Q}^{n-n'}$ (equivalence classes in $\mathbb{Q}^n$). The remaining vertices define a polytope $M_{n-n'}$ in $\mathbb{Q}^{n}/\mathbb{Q}^{n'}$. If $M_{n-n'}$ were not a simplex, it would contain a simplex of dimension smaller than $n - n'$ which would define, together with the vertices of $M'$, a minimal polytope of dimension $s$ with $n' < s < n$, in contradiction with our assumption. Therefore $M_{n-n'}$ is a simplex. Because of minimality of $M$, each of the $n - n' + 1$ vertices of $M_{n-n'}$ can have only one representative in $\mathbb{Q}^n$, implying $\overline{k} - k' = n - n' + 1$. The equivalence class of 0 can be described uniquely as a positive linear combination of these vertices. This linear combination defines a vector in $\mathbb{Q}^{n'}$, which can be written as a negative linear combination of $\leq n'$ linearly independent vertices of $M'$. These vertices, together with those of $M_{n-n'}$, form the simplex $S$. By the maximality assumption about $M'$, $\dim S$ cannot exceed $\dim M'$.

**Corollary 1:** A minimal polytope $M = \text{ConvexHull}\{V_1, \ldots, V_s\}$ in $\mathbb{Q}^n$ allows a structure $\{V_j\} = \{V_1, \ldots, \hat{V}_{k_1}, \ldots, \hat{V}_{k_{\lambda-1}+1}, \ldots, \hat{V}_{k_\lambda}\}$ with the following properties:
(a) $M_\mu := \text{ConvexHull}\{V_1, \ldots, \hat{V}_{k_\mu}\}$ is a $(\overline{k}_\mu - \mu)$-dimensional minimal polytope, $\lambda = M$.
(b) For each $\mu$, there is a subset $R_\mu$ of $\{V_1, \ldots, \hat{V}_{k_{\mu-1}}\}$ such that $S_\mu := \text{ConvexHull}(R_\mu \cup \{V_{k_{\mu-1}+1}, \ldots, \hat{V}_{k_\lambda}\})$ defines a simplex with $\dim S_\mu \leq \dim M_{\mu-1}$ for $\mu > 1$.

**Proof:** If $M$ is a simplex, $\lambda = 1$ and $\overline{k} = \overline{k}_1 = n + 1$. Otherwise one can proceed inductively using lemma 1.

**Corollary 2:** $n + 1 \leq \overline{k} \leq 2n$.

**Proof:** If $M$ is a simplex, the lower bound is satisfied. Otherwise, $\overline{k} = \overline{k}_1 + n - n' + 1$ and induction gives $\overline{k} \in \{n + 2, \ldots, n' + n + 1\} \subset \{n + 1, \ldots, 2n\}$.

**Lemma 2:** Denote by $\{S_\lambda\}$ a set of good simplices spanning $M$. Then $S_\mu - \bigcup_{\nu \neq \mu} S_\nu$ never contains exactly one point.

**Proof:** A simplex with 0 in its interior contains line segments $VV'$ with $V' = -\varepsilon V$, where $\varepsilon$ is a positive number. If a simplex $S = \text{ConvexHull}\{V_1, \ldots, V_{s+1}\}$ has all of its vertices except one ($V_{s+1}$) in common with other simplices, then all points in the linear span of $S$ are nonnegative linear combinations of the $V_j$ and the $-\varepsilon_j V_j$ with $j \leq s$, thus showing that $V_{s+1}$ violates the minimality of $M$.

**Example:** $n = 5$, $M = \text{ConvexHull}\{V_1, \ldots, V_8\}$ with

\[
V_1 = (1, 1, 0, 0, 0)^T, \quad V_2 = (1, -1, 0, 0, 0)^T, \quad V_3 = (-1, 0, 1, 0, 0)^T, \quad V_4 = (-1, 0, -1, 0, 0)^T,
\]
\[
V_5 = (-1, 0, 1, 0, 0)^T, \quad V_6 = (-1, 0, 0, -1, 0)^T, \quad V_7 = (1, 0, 0, 0, 1)^T, \quad V_8 = (1, 0, 0, 0, -1)^T.
\]

$M$ contains the good simplices $S_{1234} = V_1 V_2 V_3 V_4$ (in the $x_1 x_2 x_3$–plane), $S_{1256}$ (in the $x_1 x_2 x_4$–plane), $S_{3478}$ (in the $x_1 x_3 x_5$–plane), $S_{5678}$ (in the $x_1 x_4 x_5$–plane) and the 4-dimensional minimal
polytopes $M_{123456}$, $M_{123478}$, $M_{125678}$, $M_{345678}$. Each of these 4-dimensional minimal polytopes spans a hyperplane of codimension 1. In order to span $Q^5$, we need two additional points which may belong to one of two possible simplices. For example, if we choose $M = M_{123456}$, then we require $V_7$ and $V_8$ and may choose $S = S_{3478}$ or $S = S_{5678}$. $M_{123456}$ again has to fulfill lemma 1, and indeed it contains the two simplices $S_{1234}$ and $S_{2567}$. The structure of corollary 1 can be realised, for example, by $S_1 = M_1 = S_{1234}$, $S_2 = S_{1256}$ (implying $M_2 = M_{123456}$), and $S_3 = S_{3478}$. Note that $S_{5678}$ does not occur in this structure and that $S_1 - (S_2 \cup S_3)$ is empty (compare with lemma 2).

3 Pairing matrices

Let the elements $A^i_j$ of the integer $k \times k$ matrix $A$ denote the $i$th coordinate of $V_j$ in the coordinate system defined by $Q$, i.e. $A^i_j = (V_j)^i$ is the distance of $V_j$ to the hyperplane $H_i$. Because of reflexivity of $\Delta^*$ (all facets are at distance 1 from $T$) this is related to the pairing of $V_j$ with $V^i = H^*_i$ by $A^i_j = \langle V_i, V_j \rangle$ with $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle + 1$, where $\langle \cdot, \cdot \rangle$ is the original lattice pairing. The definition of the affine pairing $\langle \cdot, \cdot \rangle$ might seem awkward at first sight, but it has two advantages: On the one hand, it is nonnegative for any pairing between $\Delta^*$ and $\Delta$, and on the other hand, we will see later that it is a natural linear pairing for a higher dimensional pair of lattices into which we will embed $\Gamma$ and $\Gamma^*$. By duality $A^i_{-j}$ also denotes the $j$th coordinate of $V_j$ in the coordinate system defined by $Q^*$, i.e. $A^i_{-j} = (\langle \cdot, V \rangle)_j$. In other words, the columns of $A$ correspond to the vertices of $M$ whereas the lines correspond to the vertices of $-M$. We will label all points of $\Delta$ by column vectors and all points of $\Delta^*$ by line vectors, in particular $1 = (1, \ldots, 1)^T$ and $T = (1, \ldots, 1)$.

If $M$ and $M$ are simplices, then $A$ is an $(n + 1) \times (n + 1)$ matrix. We denote by the “weights” $q_i$ and $\overline{q}^j$ the barycentric coordinates of $T$ and $1$, respectively:

$$
\overline{1} = \sum_i q_i \overline{V}^i, \quad \overline{1} = \sum_j \overline{q}^j V_j, \quad \sum_i q_i = \sum_j \overline{q}^j = 1. \quad (3)
$$

This implies

$$
\sum_i q_i A^i_j = 1, \quad \sum_j \overline{q}^j A^i_j = 1, \quad \sum_i q_i (V_j)^i = 1, \quad \sum_j \overline{q}^j (\overline{V}^i)_j = 1. \quad (4)
$$

We can now give a new interpretation to our coordinate systems as coordinates in $(n + 1)$-dimensional lattices $\Gamma^{n+1} \cong \mathbb{Z}^{n+1}$ and $\overline{\Gamma}^{n+1} \cong \mathbb{Z}^{n+1}$ and their rational extensions $\Gamma^{n+1}_Q \cong \mathbb{Q}^{n+1}$ and $\overline{\Gamma}^{n+1}_Q \cong \mathbb{Q}^{n+1}$. Then the equation $\sum_i q_i x^i = 1$ defines an $n$-dimensional affine hyperplane $\Gamma^{n+1}_0$ spanned by the $V_j$, which obviously contains $1$. Linear independence of the $V_j$, i.e. of the columns of $A$, implies regularity of $A$. We can invert Eqs. (4) to get

$$
q_i = \sum_{j=0}^n (A^{-1})^i_j, \quad \overline{q}^j = \sum_{i=0}^n (A^{-1})^i_j. \quad (5)
$$

Defining arbitrary point pairings $\langle \cdot, \cdot \rangle_{n+1}$ between $\Gamma^{n+1}_Q$ and $\overline{\Gamma}^{n+1}_Q$ by

$$
\langle \overline{P}, P \rangle_{n+1} := \overline{P} k (A^{-1})^k P^l \quad (6)
$$
allows us to identify $\Gamma^n_Q$ and $\Gamma^n_Q$ as

$$\Gamma^n_Q \cong \{ P \in \Gamma^n_Q+1 \mid \langle T, P \rangle_{n+1} = 1 \}, \quad \Gamma^n_Q \cong \{ Q \in \Gamma^n_Q+1 \mid \langle Q, 1 \rangle_{n+1} = 1 \}. \quad (7)$$

At this point it is easy to see the relation of our framework to the orbifold mirror construction that works for minimal polynomials in weighted projective spaces $\mathbb{P}$. That construction relates a monomial with exponent vector $W$ to a twist group element whose diagonal action on the homogeneous coordinates is $\exp(2\pi i \text{diag}(WA^{-1}))$. Even in our more general context the lines of $A^{-1}$ provide the phases for generators of the phase symmetry group of the $n + 1$ monomials whose exponents are the columns of the matrix $A$ (this does not mean, however, that all such symmetries can be used for an orbifold construction, because transversality requires additional monomials in the non-minimal case; we will soon give an example for how this manifests itself in the context of toric geometry).

**Lemma 3:** Let $P \in \Gamma^n_Q$, $Q \in \Gamma^n_Q$. Then

(a) $\langle \overline{P}, P \rangle_{n+1} = \langle \overline{P}, P \rangle_+$.

(b) $\langle \overline{P} - \overline{T}, \overline{P} - 1 \rangle_{n+1} = \langle \overline{P}, P \rangle$.

**Proof:** For vertices we have by definition

$$\langle \overline{V}^i, V_j \rangle_{n+1} = \langle \overline{V}^i \rangle_k (A^{-1})^k_i (V_j)^l = A^i_k (A^{-1})^k_i A^l_j = \langle \overline{V}^i, V_j \rangle_+ \phantom{\text{ (9)}} (8)$$

and

$$\langle \overline{V}^i - \overline{T}, V_j - 1 \rangle_{n+1} = \langle \overline{V}^i, V_j \rangle_+ - 1 = 1 = \langle \overline{V}^i, V_j \rangle. \quad (9)$$

For general $P$, $\overline{P}$ (b) follows from linearity in $\Gamma^n_Q$ and $\Gamma^n_Q$ and (a) follows from (b) because $\langle \overline{P}, 1 \rangle_{n+1} = \langle \overline{T}, P \rangle_{n+1} = \langle \overline{T}, 1 \rangle_{n+1} = 1$. \hfill $\Box$

The first statement of this lemma shows us that $\langle \cdot, \cdot \rangle_{n+1}$ is a natural extension of $\langle \cdot, \cdot \rangle_+$ to $\Gamma^n_Q+1 \times \Gamma^n_Q+1$. We will use this fact to define $\langle \cdot, \cdot \rangle_+$ in $\Gamma^n_Q+1 \times \Gamma^n_Q+1$, thus showing that our originally affine pairing is indeed a linear pairing in the higher dimensional context. Let us also define the $n$-dimensional sublattices $\Gamma^n = \Gamma^n_Q \cap \Gamma^n_Q$ and $\overline{\Gamma}^n = \Gamma^n_Q \cap \Gamma^n_Q$ carrying $\Delta$ and $\Delta^*$, respectively.

**Corollary 3:** There is a natural identification $(\Gamma^n)^* \cong \overline{T} + \text{Span}\{\overline{V}^i - \overline{T}\} \subseteq \Gamma^n$.

**Proof:** By the embedding of $\Gamma^n$ into $\mathbb{Z}^{n+1}$ an element of $(\Gamma^n)^*$ becomes an equivalence class of points in the dual lattice $\mathbb{Z}^{n+1}$ modulo $\overline{T}$. Since $\langle \overline{V}^i \rangle_k (A^{-1})^k_i = \delta^i_j$ the vertices $\overline{V}^i$ are representatives of equivalence classes that generate $(\Gamma^n)^*$. Using the mod $\overline{T}$ ambiguity we may always choose a representative in $\overline{T} + \text{Span}\{\overline{V}^i - \overline{T}\}$ because $\langle \overline{T}, P \rangle_{n+1} = 1$. \hfill $\Box$

Given a pairing matrix $A$ for our simplices $M$ and $\overline{M}$, let us see how we can obtain all corresponding dual pairs $\Delta$, $\Delta^*$: First we choose some sublattice $\Gamma \subseteq \Gamma^n$ that contains $\overline{T}$ and all vectors $V_j$. The dual lattice $\Gamma^*$ is a sublattice of $\Gamma^n$, which obviously contains the vectors $\overline{V}^i$. Then

$$Q = \{ P \in \Gamma^{n+1}_Q \mid \langle T, P \rangle_+ = 1 \wedge P^i \geq 0 \forall i \}, \quad \overline{Q} = \{ Q \in \Gamma^{n+1}_Q \mid \langle Q, 1 \rangle_+ = 1 \wedge Q^j \geq 0 \forall j \}. \quad (10)$$

Defining the finite point sets $\Gamma_+ = \{ P \in \Gamma \mid P^i \geq 0 \forall i \}$ and $(\Gamma^n)_+$ and their convex hulls $\Delta_{\text{max}}$ and $\overline{\Delta}_{\text{max}}$, respectively, we may choose polyhedra $\Delta$ and $\overline{\Delta}$ with $\{ V_j \} \subseteq \Delta \subseteq \Delta_{\text{max}}$ and $\{ \overline{V}^i \} \subseteq \overline{\Delta} \subseteq \overline{\Delta}_{\text{max}}$ and check for duality. In practice, the following algorithm will be far more efficient: Calculate all points $P, \overline{P}$ in $\Gamma^n_+$ and $\overline{\Gamma}^n_+$ and the corresponding pairing matrix (w.r.t.
\( (\, , \, \rangle_+ \), which may have rational entries. Then we can create a list of possible vertices \( V \) by noting that any vertex is dual to a hyperplane, i.e. for any vertex \( V \) there must be \( n \) linearly independent points \( \mathcal{P} \) with \( \langle \mathcal{P}, V \rangle_+ = 0 \). Creating a list of possible vertices \( \mathcal{V} \), we use the same argument, working only with our list of possible vertices in \( \Gamma_n^+ \). This procedure may be iterated, reducing the respective lists in each step. In particular we can drop a model whenever our original vertices \( V_j^+ \) or \( \mathcal{V}^+ \) don’t show up in the resulting lists of possible vertices. In a last step we may then choose subsets of these lists, making sure that each coordinate hyperplane contains \( n \) linearly independent vertices. Choosing a particular point \( P \) to be a vertex of \( \Delta \) implies that we can eliminate all points \( \mathcal{P} \) with rational or negative pairings with \( P \) from our list of candidates for vertices of \( \Delta^* \).

**Example:** The following example is motivated by the non-degenerate Landau–Ginzburg potential

\[
W = x_1^{25} + x_2^8 x_1 + x_3^3 x_5 + x_4^3 x_2 + x_5^3 x_1 + \lambda x_2^3 x_5^2,
\]  
(11)
to which we assign the matrix

\[
A = \begin{pmatrix}
25 & 1 & 0 & 0 & 1 \\
0 & 8 & 0 & 1 & 0 \\
0 & 0 & 3 & 0 & 0 \\
0 & 0 & 1 & 0 & 3 \\
\end{pmatrix}
\]  
(12)

with \( q = \frac{1}{75}(3, 9, 17, 22, 24) \) and \( \mathcal{q} = \frac{1}{36}(1, 3, 12, 12, 8) \). It is easy to construct all 33 points allowed by the \( q \) system (the points in \( \Gamma^+_1 \)) and the 100 points in \( \Gamma^+_4 \). With the help of

\[
A^{-1} = \begin{pmatrix}
\frac{1}{25} & -\frac{1}{2} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{75} \\
0 & \frac{1}{8} & 0 & -\frac{1}{24} & 0 \\
0 & 0 & \frac{1}{3} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{3} & 0 \\
0 & 0 & -\frac{1}{9} & 0 & \frac{1}{3} \\
\end{pmatrix}
\]  
(13)

we get the \( 33 \times 100 \) matrix of point pairings, which turns out to have half-integer entries. After eliminating all lines and columns with less than 4 zeroes we get the pairing matrix for candidates for vertices shown in table 1. The first five lines and columns indicate coordinates w.r.t. the coordinate systems defined by the \( q \) and \( \mathcal{q} \) system. All entries (\( i \)'th line, \( j \)'th column) are \( \sum_{k,l=1}^{5} (\mathcal{P}^k) (A^{-1})^k (P_j)^l \). The occurrence of half integers means that we still have a \( \mathbb{Z}_2 \) freedom in choosing sublattices. Eliminating all columns with non–integer entries corresponds to choosing \( \Gamma = \Gamma^4/\mathbb{Z}_2 \) and \( \Gamma^* = \Gamma^4 \), whereas eliminating all lines with non–integer entries corresponds to choosing \( \Gamma = \Gamma^4 \) and \( \Gamma^* = \Gamma^4/\mathbb{Z}_2 \). In the first case we would eliminate \( P_6 \) and \( P_7 \) which would result in a first line with only two entries of 0, in contradiction with our requirement that \( \mathcal{V}^1 \) is a vertex of \( \Delta^* \). Transversality of the polynomial (\( \prod \)) requires the presence of its last monomial \( x_3^3 x_5^2 \), which is not invariant under this \( \mathbb{Z}_2 \) twist (see the column corresponding to \( P_6 \) in table 1). In our context, the \( \mathbb{Z}_2 \) twist is forbidden by the requirement that the vertices of \( \mathcal{M} \) remain vertices of \( \Delta^* \) (dropping this requirement, the \( \mathbb{Z}_2 \) twist may and does lead to reflexive pairs). In the case \( \Gamma = \Gamma^4 \) the full matrix of point pairings is a \( 33 \times 52 \) matrix. The convex hulls of the points \( P_j \) and \( \mathcal{P} \) are polytopes \( \Delta_{\text{max}} \) and \( \mathcal{A}_{\text{max}} \), respectively, which are obviously not dual to one another, as the entries \( -1 \) show. We can now choose any subset of our candidates of vertices (containing the vertices in \( A \), thus defining some polytope
Table 1: Pairing matrix for candidates for vertices
Then we have to eliminate all points of $\Sigma_{\text{max}}$ which have negative pairings with vertices of $\Delta$, resulting in $\Gamma^* \cap \Delta^* \subseteq \Delta^*$. Then $\Delta$ is reflexive if and only if each of its vertices has pairings of 0 with 4 linearly independent points of $\Gamma^* \cap \Delta^*$. If we keep all points of $\Delta_{\text{max}}$, for example, we have to delete all lines containing $-1$. It turns out that this indeed leads to a reflexive pair $[3]$. In fact, it was checked numerically for all transversal polynomials that $\Delta_{\text{max}}$ is reflexive $[19,25]$. In $[26]$ there is also an explicit proof (again, for $n \leq 4$) that $\Delta_{\text{max}}$ is always reflexive even for a larger class of weight systems.

If the minimal polytope $M$ is not a simplex we define a weight system for each of the lower dimensional simplices $S_\mu$ ($\mu = 1, \ldots, \lambda$) occurring in corollary 1. Then we have lattices $\Gamma^k \cong \mathbb{Z}^k$ and $\Gamma^k \cong \mathbb{Z}^k$ and their rational extensions $\Gamma^n_\mathbb{Q} \cong \mathbb{Q}^k$ and $\Gamma^n_\mathbb{Q} \cong \mathbb{Q}^k$, and we can interpret our coordinate systems as coordinates in $\Gamma^n_\mathbb{Q}$ and $\Gamma^n_\mathbb{Q}$. We get $\lambda = k - n$ equations of the type $\sum \varphi^j x_j = 1$. Due to the structure given in the lemma, we can solve this system by successively eliminating the $x_{\varphi^\mu}$, $\mu = 1, \ldots, \lambda$. Therefore these $k - n$ equations define an $n$-dimensional affine hyperplane $\Gamma^n_\mathbb{Q}$ spanned by the $V^i$, which obviously contains $\Gamma$ again. In the same way we also get $k - n$ equations of the type $\sum q^j x^j = 1$ defining an $n$-dimensional affine hyperplane $\Gamma^n_\mathbb{Q}$ spanned by the $V^j$.

### 4 A classification algorithm

The classification of dual pairs of reflexive polyhedra can be done in 3 steps:

1. Classification of possible structures of minimal polytopes,
2. Classification of weight systems,
3. Construction of complete vertex pairing matrices for dual pairs of polytopes and choice of a lattice.

Let us first discuss the classification of possible structures of minimal polytopes. With the help of lemma 1 of section 2, it is easy to construct all possible structures recursively. For a given dimension $n$, one either has the $n$-dimensional simplex or one has to consider all minimal polytopes of dimension $n'$ with $n/2 \leq n' < n$, add $n - n' + 1$ points and consider all possible structures of $S$ compatible with the lemmata.

For $n = 2$ this allows the triangle $V_1V_2V_3$ and the “1 simplex” $V_1V_2$ with 2 additional points $V'_1, V'_2$, which can only form another 1 simplex. In $n = 3$ dimensions we can either have a 3 simplex $V_1V_2V_3V_4$ or a 2 dimensional minimal polytope with 2 more points. The latter case allows the possibilities $S_1 = V_1V_2V_3$, $S_2 = V'_1V'_2$; $S_1 = V_1V_2V_3$, $S_2 = V_1V'_2V'_3$; $S_1 = V_1V_2$, $S_2 = V'_1V'_2$, $S_3 = V''_1V''_2$.

In $n = 4$ dimensions we can have a 4 simplex, a 3 dimensional minimal polytope with 2 more points or a 2 dimensional minimal polytope with 3 more points defining a 2 simplex. The complete list of possible structures is the following: With 5 points there only is the 4 simplex $M = S_1 = V_1V_2V_3V_4V_5$. With a total of 6 points we have the 4 minimal configurations

$$\{S_1 = V_1V_2V_3V_4, S_2 = V'_1V'_2V'_3, S_3 = V''_1V''_2V''_3\}$$

$$\{S_1 = V_1V_2V_3V_4, S_2 = V'_1V'_2V'_3, S_3 = V''_1V''_2V''_3\}$$

$$\{S_1 = V_1V_2V_3V_4, S_2 = V'_1V'_2V'_3, S_3 = V''_1V''_2V''_3\}$$

With 7 points there are the 3 possibilities

$$\{S_1 = V_1V_2V_3, S_2 = V'_1V'_2, S_3 = V''_1V''_2\}$$

$$\{S_1 = V_1V_2V_3, S_2 = V'_1V'_2V'_3, S_3 = V''_1V''_2V''_3\}$$

$$\{S_1 = V_1V_2V_3, S_2 = V'_1V'_2V'_3, S_3 = V''_1V''_2V''_3\}$$
\{S_1 = V_1V_2V_3, S_2 = V_1V'_2V'_3, S_3 = V''_1V''_2\}, \\
\{S_1 = V_1V_2V_3, S_2 = V_1V'_2V'_3, S_3 = V''_1V''_2\}

and with 8 points we can have only 1 simplices \(S_1 = V_1V_2, S_2 = V'_1V'_2, S_3 = V''_1V''_2\).

The next step in the classification program, namely the classification of weight systems, was
done in a different paper [26]. There, weight systems with up to 5 weights and with the property
that 1 is in the interior of \(\Delta_{\text{max}}\) were completely classified. All weight systems occurring in our
scheme (whether alone or in combination with other weight systems) obviously must have this
"interior point property". The fact that \(Q\) consists of hyperplanes carrying facets of \(\Delta\) leads
to another property of weight systems which we may call the "span property". It asserts that
the facets of \(Q\) must actually be affinely spanned by points of \(\Delta_{\text{max}}\). According to [26],
there are the following weight systems with the interior point property: With two weights, there is
only \((1/2, 1/2)\), which also has the span property; with three weights there are the systems
\((1/3, 1/3, 1/3), (1/2, 1/4, 1/4)\) and \((1/2, 1/3, 1/6)\), which all have the span property as well.
There are 95 systems of four weights (58 of them with the span property), and there are 184026
systems of five weights (38730 with the span property).

With these informations, there are essentially two ways to construct all reflexive polyhedra
for a given \(n\). We can either pick a specific structure and a combination of weight systems both
for \(M\) and \(\overline{M}\). Then it is easy to write a computer program that finds all \(k \times k\) matrices \(A\) that
are compatible with such structures, and we can proceed as in the previous section. Alternatively,
we may give up the symmetry between \(\Gamma\) and \(\Gamma^*\) and simply construct the polyhedron
\(\Delta_{\text{max}}\) corresponding to some combination of weight systems. Next, we would consider all of
its subpolyhedra \(\Delta\) such that the facets of \(Q\) are affinely spanned by points of \(\Delta\). Finally, we
must classify all sublattices of \(\Gamma^n\) that contain all the vertices of \(\Delta\) and check for reflexivity
of \(\Delta\) w.r.t. any of these lattices. In both approaches it is important to calculate and store
the pairing matrices for the vertices of \(\Delta\) and \(\Delta^*\) because this is the information required for
identifying or distinguishing polyhedra.

As an illustration for our concepts and methods, we will now rederive the well-known (see,
e.g., [3][10]) classification of reflexive polyhedra for \(n = 2\) in the "asymmetric" approach.

In two dimensions there are only two minimal polytopes, namely the triangle \(V_1V_2V_3\) and
the parallelogram \(V_1V_2V'_1V'_3\). Thus we either have a single weight system of one of the types
\((1/3, 1/3, 1/3), (1/2, 1/4, 1/4), (1/2, 1/3, 1/6)\), or the combination of weight systems \((1/2,
1/2, 0, 0; 0, 0, 1/2, 1/2)\). The points (except 1) allowed in the systems of type \((q_1, q_2, q_3)\) can
be arranged as columns of the matrices

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 2 & 3 & 2 & 1 \\
0 & 1 & 2 & 3 & 2 & 1 & 0 & 0 & 0 \\
3 & 2 & 1 & 0 & 0 & 0 & 0 & 1 & 2
\end{pmatrix},
\]

(14)

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\
0 & 1 & 2 & 3 & 4 & 2 & 0 & 0 \\
4 & 3 & 2 & 1 & 0 & 0 & 0 & 2
\end{pmatrix}
\]

(15)

and

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 2 & 1 \\
0 & 1 & 2 & 3 & 0 & 0 \\
6 & 4 & 2 & 0 & 0 & 3
\end{pmatrix}
\]

(16)

respectively (see fig. 1).
Fig. 1: The bounding simplices $Q$ for the weight systems $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ and $(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$.

In the first case there is a $\mathbb{Z}_3$ sublattice defined by $x^1 = x^2 \mod 3$, which reduces the set of allowed points to
\[
\begin{pmatrix}
0 & 0 & 3 \\
0 & 3 & 0 \\
3 & 0 & 0
\end{pmatrix},
\]
(17)

and in the second case there is a $\mathbb{Z}_2$ sublattice defined by $x^2 = x^3 \mod 4$, which reduces the allowed points to
\[
\begin{pmatrix}
0 & 0 & 0 & 2 \\
0 & 2 & 4 & 0 \\
4 & 2 & 0 & 0
\end{pmatrix}
\]
(18)

(see fig. 2), whereas in the case of $(1/2, 1/3, 1/6)$ there is no allowed sublattice.

Fig. 2: Alternative lattices for the weight systems $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$.

For $(1/3, 1/3, 1/3)$ we get the triangle $P_1P_4P_7$ twice (both on the original and the reduced lattice), and in addition we get, on the original lattice, the polygons $P_1P_4P_6P_8$, $P_1P_4P_6P_9$, $P_5P_3P_5P_7P_8$, $P_2P_3P_6P_7P_8$ and $P_2P_3P_6P_7P_8$ (of course, there are more polygons which are related to the ones given above by the permutation symmetry in the coordinates).

For $(1/2, 1/4, 1/4)$ we get the triangle $P_1P_5P_7$ twice (both on the original and the reduced lattice), and in addition we get the polygons $P_1P_4P_6P_7$, $P_1P_3P_6P_7$, $P_1P_2P_6P_7$, $P_2P_4P_6P_7$ and $P_2P_3P_6P_7$.

For $(1/2, 1/3, 1/6)$ we get the polygons $P_1P_4P_5$, $P_2P_4P_5P_6$ and $P_3P_4P_5P_6$. 

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The only case we have not considered so far is the case of two $q$ systems with $q_1 = q_2 = 1/2$. Allowed points can be encoded by

$$
\begin{pmatrix}
0 & 0 & 0 & 1 & 2 & 2 & 2 & 1 \\
2 & 2 & 2 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 2 & 2 & 2 & 1 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 & 1 & 2 & 2
\end{pmatrix}
$$

(19)

(see fig. 3). If we drop any of the vertices $P_1, P_3, P_5, P_7$, we can find a triangle containing $\Delta$, so we get only two new polygons, namely $P_1P_3P_5P_7$ on the original lattice and on the sublattice defined by $x^1 = x^3 \mod 2$ (see fig. 4)

![Fig. 3: 2 + 2 full lattice](image)

![Fig. 4: 2 + 2 alternative lattice](image)

We have constructed some polygons more than once. For example, $P_1P_2P_6P_7$ in the $(1/2, 1/4, 1/4)$ system is (up to a reflection) equivalent to $P_2P_4P_5P_6$ in the $(1/2, 1/3, 1/6)$ system. Here this can be seen by inspection. In our approach this redundancy will be sorted out when we bring the complete pairing matrices into a normal form by permutations of columns and lines. Taking this into account, we arrive at the known 16 reflexive polygons [9, 10].

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