STRING CONE AND SUPERPOTENTIAL COMBINATORICS FOR
FLAG AND SCHUBERT VARIETIES IN TYPE A

L. BOSSINGER AND G. FOURIER

ABSTRACT. We show that every weighted string cone for a reduced expression of
the longest word in $S_{n+1}$ is unimodularly equivalent to the cone obtained through
the tropicalization of a Landau-Ginzburg superpotential in an appropriate seed of
the cluster algebra $\mathbb{C}[G_{e,w_0}]$, the coordinate ring of the double Bruhat cell $G_{e,w_0}$.
We provide, using Gleizer-Postnikov paths in pseudoline arrangements, a combi-
natorial model to describe the superpotential cones (non-recursively). Our results
implies that any toric degeneration of the flag variety obtained through string
polytopes, appears also in the framework of potentials on cluster varieties as con-
jectured by Gross, Hacking, Keel, and Kontsevich.

We extend this result also to Schubert varieties, i.e. the weighted string cone of
every Schubert variety in type A can be obtained through restricting the superpoten-
tial in an appropriate way.

INTRODUCTION

Let $w_0 \in S_{n+1}$ be the longest word in the symmetric group and $w_0 = s_{i_1} \cdots s_{i_N}$
a reduced expression in terms of simple transpositions. To $w_0$, one associates a
pseudoline arrangement of $n + 1$ lines with intersection points determined by $w_0$.
Gleizer and Postnikov introduced in [17] rigorous paths in these arrangements, by
fixing an orientation and considering all paths from sources to sinks satisfying certain
conditions. Each such path is described by its turning points (e.g. intersection points
of pseudolines). We fix a basis of $\mathbb{R}^N$ indexed by the intersection points, then each
rigorous path determines a vector in $\mathbb{R}^N$ given by its characteristic function.

In [17] a polyhedral cone is introduced, using all these vectors for a fixed reduced
expression $w_0$ as normal vectors of the facets. We will call this cone the Gleizer-
Postnikov cone $C_{w_0} \subset \mathbb{R}^N$ (see Definition 5). By adding natural weight conditions,
we obtain a weighted cone $C_{w_0} \subset \mathbb{R}^{N+n}$ (see Definition 6).

We introduce in this paper another polyhedral cone associated to a pseudoline
arrangement. For this let $p$ be a rigorous path and $A_p$ the area in the pseudoline
arrangement enclosed by $p$. This time we label a basis of $\mathbb{R}^{N+n}$ by the faces of the
pseudoline arrangement and consider the normal vector associated to $p$ defined by
the characteristic function on $A_p$. Taking all these vectors we obtain a cone $S_{w_0}$.
Again, adding natural weight vectors, we define the weighted cone $S_{w_0} \subset \mathbb{R}^{N+n}$ using
all these vectors as facet normals (see Definition 8).

Let $\pi, \tau : \mathbb{R}^{N+n} \to \mathbb{R}^n$ be appropriate projections onto these additional weight
coordinates (see Equation (2.1) in Section 2.4 for details). Our first theorem is

Theorem. For each $w_0$, the two cones $C_{w_0}$ and $S_{w_0}$ are unimodularly equivalent
and the lattice-preserving linear map is given by the duality of faces and vertices in
the pseudoline arrangement. Moreover, this linear map restricts to linear bijections
between the polytopes
\[ \pi^{-1}(\lambda) \cap C_{w_0} \cong \tau^{-1}(\lambda) \cap S_{w_0} \]
and the cones
\[ S_{w_0} \cong C_{w_0}. \]

In fact, we show a stronger version for every \( w \in S_{n+1} \). We introduce the notion of Gleizer-Postnikov paths for \( w \), a reduced expression of \( w \). Then we extend the construction of the two polyhedral cones to arbitrary \( w \in S_{n+1} \) and show that \( C_w \) and \( S_w \) are unimodularly equivalent via a lattice-preserving linear map (see Theorem 1).

We explain how this result can be interpreted in the interplay of Representation Theory and Cluster Theory. One of the results of [17] is that \( C_{w_0} \) is the string cone defined by Littelmann [19] and Berenstein-Zelevinsky [5] in the context of canonical basis of quantum groups. We extend this result: the weighted string cone (defined in [19]) is the weighted Gleizer-Postnikov cone \( C_{w_0} \). The lattice points in \( C_{w_0} \) parametrize a basis of \( \mathbb{C}[SL_{n+1}/U^+] \) for a maximal unipotent subgroup \( U^+ \) of the positive Borel subgroup \( B^+ \) of \( SL_{n+1}(\mathbb{C}) \). This coordinate ring is isomorphic to the direct sum of the dual spaces of all Demazure modules corresponding to \( U^+ \). There are constructions of \( C_{w_0} \) provided in [19] and [5], both give a set of inequalities defining the cone. Moreover, the results in [19] extend to the closure of the Schubert cell \( B^+wB^+/B^+ \) (and \( B^+wB^+/U^+ \), the Schubert cell extended by the torus \( T = B^+ \cap U^+ \)). For any reduced expression \( w \) of an element \( w \in S_{n+1} \), [19] provides a description of a polyhedral cone (the weighted string cone) whose lattice points parametrize a basis of the coordinate ring of \( B^+wB^+/U^+ \). This construction is again via the identification of the coordinate ring with the dual space of all Demazure modules corresponding to \( w \). The double Bruhat cell \( G^{e,w} := B^- \cap B^+wB^+ \) can be identified with an open subset of \( B^+wB^+/U^+ \).

Caldero showed in [6] that the string cone for the Schubert variety \( X(w) = B^+wB^+/B^+ \) can be obtained from a specific string cone of \( SL_{n+1}/B^+ \). Let \( w_0 \) be an extension of the reduced expression \( w = s_{i_1} \cdots s_{i_p} \) to \( w_0 = ws_{{i_p}+1} \cdots s_{i_N} \), then the string cone for the Schubert variety \( X(w) \) is the intersection of the string cone for \( w_0 \) with \( \mathbb{R}^p \times \{0\}^{N-p} \). One can extend this to \( B^+wB^+/U^+ \) and the weighted string cone. We then show that this weighted string cone is the weighted Gleizer-Postnikov cone \( C_w \) (see Theorem 3.1).

The cone \( S_{w_0} \) appears in the framework of superpotentials of cluster varieties. Let \( X \) be the Fock-Goncharov dual of the \( A \)-cluster variety \( G^{e,w_0} \) and let \( s_0 = s_{w_0} \) be the seed of the cluster algebra \( \mathbb{C}[G^{e,w_0}] \) corresponding to the reduced expression \( w_0 = s_1 s_2 s_1 \cdots s_n s_2 s_1 \). Let \( W \) be the Landau-Ginzburg superpotential defined by the sum of the \( \vartheta \)-functions for every frozen variable in \( s_0 \). Then \( W^{\text{trop}} \) denotes the tropicalization of the superpotential. Magee has shown in [20] that
\[
S_{w_0} = \{ x \in \mathbb{R}^{N+n} \mid W^{\text{trop}} |_{X_{w_0}}(x) \geq 0 \} =: \Xi_{s_0}.
\]
We show that the mutation of the pseudoline arrangement and hence of the cone \( S_{w_0} \), is compatible with the mutation of the superpotential [11]. We obtain

**Theorem.** Let \( w_0 \) be an arbitrary reduced expression of \( w_0 \in S_{n+1} \) and \( s_{w_0} \) be the seed corresponding to the pseudoline arrangement, \( X_{w_0} \) the toric chart of the seed \( s_{w_0} \). Then
\[
S_{w_0} = \{ x \in \mathbb{R}^{N+n} \mid W^{\text{trop}} |_{X_{w_0}}(x) \geq 0 \} =: \Xi_{s_{w_0}},
\]
the polyhedral cone defined by the tropicalization of $W$ on the seed $s_{w_0}$.

**Remark.** In [15] a similar result is obtained for the (non-weighted) string cone. Namely using Gleizer-Postnikov paths they defined a superpotential whose induced cone is the string cone for every seed $s_{w_0}$ and then show that this superpotential is isomorphic to $W$, which also implies that $S_{w_0} \cong C_{w_0}$.

Combing both theorems, we show that every string polytope for $SL_{n+1}/B^+$ arises via superpotentials on $X$ in an appropriate seed. Our results imply that every toric degeneration of $SL_{n+1}/B^+$ induced from a string polytope can be obtained through the framework of cluster varieties as conjectured by Gross, Hacking Keel and Kontsevich in [12]. See [8] for an overview on toric degenerations.

Consider $w \in S_{n+1}$ arbitrary and $\underline{w}$ a reduced expression of $w$. Let $W$ be the superpotential on $X$ as above, and consider its restriction $\text{res}_{\underline{w}}(W|_{X_{\underline{w}}})$ to the Fock-Goncharov dual of the $A$-cluster variety $G_{e,w}$. Let $s_{\underline{w}}$ be the corresponding seed in the cluster algebra, see Definition 3. Then the tropicalization of the restriction yields again a cone $\Xi_{s_{\underline{w}}}$.

In the last section we compute an explicit example showing that the restriction of the superpotential $\text{res}_{\underline{w}}(W|_{X_{\underline{w}}})$ for $G_{e,w}$ is different from a GHKK-type superpotential for $G_{e,w}$.

**Theorem.** Let $\underline{w} \in S_{n+1}$, $\underline{w}_0$ an extension of $w$. Let $s_{\underline{w}}$ (resp. $s_{\underline{w}_0}$) be the corresponding seed in the cluster algebra of the double Bruhat cell $G_{e,w}$ (resp. $G_{e,w_0}$). Then $S_{\underline{w}}$ is the cone $\Xi_{s_{\underline{w}}}$ defined by the tropicalization of the restricted superpotential $\text{res}_{\underline{w}}(W|_{X_{\underline{w}}})$.

The paper is organized as follows: We recall briefly the combinatorics of pseudoline arrangements in Section 1. Then in Section 2 we introduce the two weighted cones and show their unimodular equivalence. Section 3 connects to the well-known string cones and Section 4 to cluster varieties and superpotentials.

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1. Pseudoline arrangements and Gleizer-Postnikov paths

1.1. Preliminaries. Let $S_{n+1}$ denote the symmetric group on $n+1$ letters. We fix generators $s_i = (i, i+1)$, $1 \leq i \leq n$, the simple transpositions. Then for every $w \in S_{n+1}$, we denote by $\ell(w)$ the minimal length of $w$ as a word in the generators $s_i$. Further $\underline{w}$ denotes a reduced expression

$$\underline{w} = s_{i_1} \cdots s_{i_{\ell(w)}}.$$

Such an expression is not unique. For any two reduced expressions of $w$ there is a sequence of local transformations leading from one to the other. These local transformations are either swapping orthogonal reflections $s_i s_j = s_j s_i$ if $|i - j| > 1$ or exchanging consecutive $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.

1.2. Pseudoline arrangements and mutations.

**Definition 1.** A pseudoline arrangement associated to a reduced expression $\underline{w} = s_{i_1} \cdots s_{i_{\ell(w)}}$ is a diagram consisting of $n+1$ horizontal pseudolines $l_1, \ldots, l_{n+1}$ (or short
lines) from bottom to top labelled at the left end, whose crossings are indicated by the reduced expression. A reflection $s_i$ indicates a crossing at level $i$.

For a given reduced expression $w = s_{i_1} \cdots s_{i_l}$, we associate to each $s_{i_j}$ a pair $\beta_{i_j} = (k, m)$, $k < m$, which represents the two lines intersecting at position $i_j$. As two lines $l_k, l_m$ cross at most once, there is at most one position with label $(k, m)$. The intersection point is a vertex in the diagram and will be labelled $v_{(k,m)}$. Further the right end of a pseudoline $l_i$ is a vertex labelled $L_i$.

**Definition 2.** A *mutation* of a pseudoline arrangement is a change of consecutive $s_is_{i+1}s_i$ to $s_{i+1}s_is_{i+1}$ or vice versa (see Figure 1). A face $X$ of the pseudoline arrangement is called *mutable* if it has exactly three vertices corresponding to consecutive simple reflections $s_is_{i+1}s_i$ or $s_{i+1}s_is_{i+1}$. The corresponding mutation will be denoted $\mu_X$. In Figure 1, the local structure of a mutable face is given.

![Figure 1. Mutation of pseudoline arrangements.](image)

### 1.3. The quiver of a pseudoline arrangement.

**Definition 3.** [4, Definition 2.2] The *quiver* $\Gamma_w$ associated to the pseudoline arrangement for $w$ has vertices associated to the faces of the diagram and arrows of two kinds. Firstly, if two faces are at the same level separated by a crossing then there is an arrow from left to right (see Figure 2a). Secondly, if two faces are on consecutive levels separated by two crossing then there is an arrow from right to left (either upwards or downwards, see Figure 2b, 2c). Vertices corresponding to unbounded faces are called *frozen* and arrows between those can be disregarded. All the other vertices are called *mutable*.

Note, that not every mutable vertex corresponds to a mutable face of the pseudoline arrangement (in the sense of Definition 2).

![Figure 2. Arrows of the quiver arising from the pseudoline arrangement.](image)

Consider $w_0 = s_1s_2s_1s_3s_2s_1 \cdots s_ns_{n-1} \cdots s_3s_2s_1$ as reduced expression for the quiver $\Gamma_{w_0}$. We label the vertices corresponding to faces of the pseudoline arrangement for $w_0$ by $v_{(i,j)}$ in correspondence to the crossing of lines $l_i$ and $l_j$ to their left. In particular, the frozen vertices at the right boundary are labelled $v_{n,n+1}, \ldots, v_{1,n+1}$ from...
bottom to top. Referring to their level, the frozen vertices on the left boundary are labelled by $v_1, \ldots, v_n$ from bottom to top. In the following example we describe the quiver corresponding to this initial reduced expression $w_0$ for $n = 5$.

**Example 1.** For $SL_5$ the initial reduced expression is $w_0 = s_1 s_2 s_1 s_3 s_1 s_4 s_3 s_2 s_1$. The pseudoline arrangement and the corresponding quiver are as in Figure 3.

![Figure 3. Pseudoline arrangement and quiver for $w_0 = s_1 s_2 s_1 s_3 s_1 s_4 s_3 s_2 s_1$.](image)

1.4. Orientation and paths. For every pair $(l_i, l_{i+1})$ with $1 \leq i \leq n$ we give an orientation to a pseudoline arrangement by orienting lines $l_1, \ldots, l_i$ from right to left and lines $l_{i+1}, \ldots, l_n$ from left to right, see Figure 4. Consider an oriented path with three adjacent vertices $v_{k-1} \rightarrow v_k \rightarrow v_{k+1}$ belonging to the same pseudoline $l_i$. Then $v_k$ is the intersection of $l_i$ with some line $l_j$. If either $i < j$ and both lines are oriented to the left, or $i > j$ and both lines are oriented to the right, the path is called *non-rigorous*. Figure 5 shows these two situations. A path is *rigorous* if it is not non-rigorous.

**Definition 4.** Let $w$ be a fixed reduced expression of $w \in S_{n+1}$. A *Gleizer-Postnikov path* (or short *GP-path*) is a rigorous path $p$ in the pseudoline arrangement associated to $w$ with orientation $(l_i, l_{i+1})$. It has source $L_p$ and sink $L_q$ for $p \leq i, q \geq i + 1$ and $w(i + 1) \leq w(p), w(q) \leq w(i)$. We then say $p$ is of shape $(l_i, l_{i+1})$. The set of all GP-paths for all orientations in the pseudoline arrangement associated to $w$ is denoted by $P_w$.

Note that if $w(i) < w(i + 1)$ there are no GP-paths of shape $(l_i, l_{i+1})$.

![Figure 4. Pseudoline arrangement corresponding to $w_0 = s_1 s_2 s_1$ for $SL_3$ with orientation for $(l_1, l_2)$.](image)

![Figure 5. The two red arrows are forbidden in a rigorous path.](image)
1.5. Properties of Gleizer-Postnikov paths.

Proposition 1. Let $p$ be a GP-path of shape $(l_i, l_{i+1})$, then $p$ is either the empty path or does not cross the lines $l_{i+1}$ and $l_i$, i.e. $p$ does not leave the area in the pseudoline arrangement that is bounded by $l_i$ and $l_{i+1}$ to the left.

**Proof.** Fix an orientation for the pseudoline arrangement corresponding to the lines $l_i$ and $l_{i+1}$. All lines $l_k$ for $k \leq i$ are oriented to the left and all lines $l_j$ for $j \geq i + 1$ are oriented to the right.

Suppose now that $p$ is a GP-path of shape $(l_i, l_{i+1})$ partially to the right of $l_{i+1}$ with respect to the orientation of $l_{i+1}$. Then there is a line $l_k$ crossing $l_{i+1}$ from left to right. There are two cases: first if $k > i + 1$, then $l_k$ is also oriented to the right. So $p$ is passing through the crossing of $l_k$ and $l_{i+1}$ on line $l_k$, but this is a contradiction to being rigorous. If $k < i$ the right end of $l_k$ is above the right end of $l_{i+1}$. So the two lines have to intersect once again near the right end, which is a contradiction.

The case of the path $p$ being partially to the right of the line $l_i$ can be treated similarly. Lastly we consider a path that travels along $l_{i+1}$ beyond the crossing with $l_i$. Using the property that a path cannot cross $l_i$ from right to left, we see that the path $p$ has to cross $l_{i+1}$ from left to right to continue to $l_i$. In this case, $p$ would form a loop with contradicts the GP-path condition. 

There are only four possible local orientations of pseudoline arrangement around a mutable face $X$. Depending on the shape of $X$, they are shown in Figure 6. We are presenting for each orientation $(l_i, l_{i+1})$ all GP-paths that pass this area and how they change under mutation. For the orientation (III) we have

![Figure 6](image-url)
The GP-paths passing through locally with orientation (lrr) are

\[
b_{k-1} \rightarrow v_1 \rightarrow a_{k+1} \leftrightarrow b_{k+1} \rightarrow v_2 \rightarrow v_3 \rightarrow a_{k+1}
\]

\[
b_k \rightarrow v_3 \rightarrow v_1 \rightarrow a_{k+1} \leftrightarrow \begin{cases} b_k \rightarrow v_2 \rightarrow v_3 \rightarrow a_{k+1} \\ b_k \rightarrow v_2 \rightarrow v_1 \rightarrow v_3 \rightarrow a_{k+1} \end{cases}
\]

\[
b_k \rightarrow v_3 \rightarrow v_1 \rightarrow v_2 \rightarrow a_k \leftrightarrow \begin{cases} b_{k-1} \rightarrow v_2 \rightarrow v_1 \rightarrow v_3 \rightarrow a_k \\ b_{k+1} \rightarrow v_1 \rightarrow v_3 \rightarrow a_k \end{cases}
\]

\[
b_{k+1} \rightarrow v_3 \rightarrow v_1 \rightarrow v_2 \rightarrow a_k \leftrightarrow \begin{cases} b_{k+1} \rightarrow v_1 \rightarrow v_3 \rightarrow a_k \\ b_{k+1} \rightarrow v_1 \rightarrow a_{k-1} \end{cases}
\]

The GP-paths passing through locally with orientation (lrr) are

\[
b_k \rightarrow v_3 \rightarrow b_{k+1} \leftrightarrow \begin{cases} b_k \rightarrow v_2 \rightarrow v_3 \rightarrow v_1 \rightarrow b_{k+1} \\ b_k \rightarrow v_2 \rightarrow v_1 \rightarrow b_{k+1} \end{cases}
\]

Similarly, for orientation (lrr) we have

\[
b_{k-1} \rightarrow v_1 \rightarrow v_3 \rightarrow b_k \leftrightarrow \begin{cases} b_{k-1} \rightarrow v_2 \rightarrow v_3 \rightarrow b_k \\ b_{k-1} \rightarrow v_2 \rightarrow v_3 \rightarrow b_k \end{cases}
\]

And finally for the orientation (rrr) the six pairs of GP-paths are

\[
a_{k+1} \rightarrow v_1 \rightarrow b_{k-1} \leftrightarrow a_{k+1} \rightarrow v_2 \rightarrow v_3 \rightarrow b_{k-1}
\]

\[
a_k \rightarrow v_2 \rightarrow v_1 \rightarrow b_{k-1} \leftrightarrow \begin{cases} a_k \rightarrow v_2 \rightarrow v_3 \rightarrow b_{k-1} \\ a_k \rightarrow v_2 \rightarrow v_1 \rightarrow v_3 \rightarrow b_{k-1} \end{cases}
\]

\[
a_k \rightarrow v_2 \rightarrow v_1 \rightarrow v_3 \rightarrow b_k \leftrightarrow \begin{cases} a_k \rightarrow v_2 \rightarrow v_1 \rightarrow v_3 \rightarrow b_k \\ a_{k-1} \rightarrow v_1 \rightarrow v_3 \rightarrow b_k \end{cases}
\]

\[
a_{k-1} \rightarrow v_2 \rightarrow v_1 \rightarrow v_3 \rightarrow b_k \leftrightarrow \begin{cases} a_{k-1} \rightarrow v_1 \rightarrow v_3 \rightarrow b_k \\ a_{k-1} \rightarrow v_2 \rightarrow v_3 \rightarrow b_k \end{cases}
\]

\[
a_{k-1} \rightarrow v_2 \rightarrow v_3 \rightarrow b_{k+1} \leftrightarrow a_{k-1} \rightarrow v_1 \rightarrow b_{k+1}.
\]

2. Two cones arising from Gleizer-Postnikov paths

We define two (weighted) cones that arise from the set of all Gleizer-Postnikov path of a fixed $w$. We relate these two cones in the forthcoming sections, one to the weighted string cones (introduced by Littelmann [19] and Berenstein-Zelevinsky [5]), the other to tropicalizations of a superpotential for a double Bruhat cell (see Magee [20]).
2.1. The first cone. Recall that for \( \mathbf{w} = s_{i_1} \ldots s_{i_{\ell(w)}} \) every \( s_{i_j} \) can be associated to a pair \((k,l)\) and fix the standard basis \( \{c_{(k,l)}\} \) of \( \mathbb{R}^{\ell(w)} \). Let \( \mathbf{p} \) be a GP-path in \( \mathcal{P}_\mathbf{w} \), say
\[
\mathbf{p} = L_p \rightarrow v_1 \rightarrow \cdots \rightarrow v_m \rightarrow L_q, \quad 1 \leq p \leq i < q \leq n + 1.
\]
Every vertex corresponds to the crossing of two lines. Let \( v_{r_1}, \ldots, v_{r_k} \) be those vertices where \( \mathbf{p} \) changes from line \( l_{s_1} \) to \( l_{s_2} \) for \( 1 \leq s \leq k \). We associate to \( \mathbf{p} \) the vector
\[
c_\mathbf{p} = \sum_{s=1}^{k} c_{(s_1,s_2)} \in \mathbb{R}^{\ell(w)},
\]
where we set \( c_{(i,j)} := -c_{(j,i)} \) if \( i > j \) and \( c_{(i,i)} := 0 \).

**Definition 5.** The following polyhedral cone will be called *Gleizer-Postnikov cone* (due to Gleizer-Postnikov [17]):
\[
C_\mathbf{w} = \{(x_{i,j}) \in \mathbb{R}^{\ell(w)} \mid (c_\mathbf{p})^t(x_{i,j}) \geq 0, \forall \mathbf{p} \in \mathcal{P}_\mathbf{w}\}.
\]

**Example 2.** Consider the reduced expression \( \mathbf{w} = \mathbf{w}_0 = s_{i_1}s_{i_2}s_{i_1} \) for \( SL_3/B^+ \). We endow the pseudoline arrangement with the orientation for \((l_1,l_2)\), i.e. \( l_1 \) is oriented to the left and \( l_2, l_3 \) are oriented to the right, see Figure 4. There are two GP-paths in the diagram from \( L_1 \) to \( L_2 \):
\[
\mathbf{p}_1 = L_1 \rightarrow v_{(1,3)} \rightarrow v_{(1,2)} \rightarrow v_{(2,3)} \rightarrow L_2 \quad \text{and} \quad \mathbf{p}_2 = L_1 \rightarrow v_{(1,3)} \rightarrow v_{(2,3)} \rightarrow L_2.
\]
These yield \( c_{\mathbf{p}_1} = c_{(1,2)} \) and \( c_{\mathbf{p}_2} = c_{(1,3)} - c_{(2,3)} \). Similarly the orientation for \((l_2,l_3)\) induces a path \( \mathbf{p}_3 = L_2 \rightarrow v_{(2,3)} \rightarrow L_3 \) with \( c_{\mathbf{p}_3} = c_{(2,3)} \). The cone \( C_{121} \) is given by the inequalities
\[
\{(x_{1,2}, x_{1,3}, x_{2,3}) \in \mathbb{R}^3 \mid x_{1,2} \geq 0, x_{1,3} \geq x_{2,3} \geq 0\}.
\]

In this paper, we are considering a weighted version of this cone to incorporate a natural multigrading. This cone will be defined in \( \mathbb{R}^{\ell(w)+n} \), where the additional basis elements are indexed \( c_1, \ldots, c_n \). By some abuse of notation we will denote by \( c_\mathbf{p} \) also the vector \((c_\mathbf{p},0,\ldots,0) \in \mathbb{R}^{\ell(w)} \times \{0\}^n \subset \mathbb{R}^{\ell(w)+n} \).

Recall that for every \( k \) with \( 1 \leq k \leq \ell(w) \), \( s_{i_k} \) induces a crossing in the pseudoline arrangement for \( \mathbf{w} \). Hence, we identify \( c_{i_k} = c_{(p,q)} \) if the crossing of lines \( l_p \) and \( l_q \) is induced by \( s_{i_k} \). This notation will simplify the definition of weight condition vectors. Let again \( \mathbf{w} \) be the reduced expression \( \mathbf{w} = s_{i_1} \cdots s_{i_{\ell(w)}} \). Consider \( k \) with \( 1 \leq k \leq \ell(w) \). Then \( s_{i_k} = s_i \) for some \( i \). We will define the weight inequality for \( k \) by the following vector
\[
d_{i,k} := c_i - c_{i_k} - 2 \sum_{p > k, s_{i_p} = s_i} c_{i_p} + \sum_{j > k, s_{i_j} = s_{i_{k+1}}} c_{i_j}.
\]

**Definition 6.** The *weighted Gleizer-Postnikov cone* \( C_{\mathbf{w}} \subset \mathbb{R}^{\ell(w)+n} \) is defined as
\[
C_{\mathbf{w}} = \left\{(x_{p,q}, x_l) \in \mathbb{R}^{\ell(w)+n} \left| \begin{array}{l}
x_l \geq 0, \quad \forall 1 \leq l \leq n, \\
(c_\mathbf{p})^t(x_{p,q}) \geq 0, \quad \forall \mathbf{p} \in \mathcal{P}_\mathbf{w}, \\
(d_{i,k})^t(x_{p,q}, x_l) \geq 0, \quad \forall 1 \leq k \leq \ell(w) \end{array} \right. \right\}.
\]
Example 3. Consider $w = w_0$ the longest word and fix the reduced expression

$$w_0 = s_1 s_2 s_1^3 s_2 s_1 \cdots s_n s_{n-1} \cdots s_3 s_2 s_1.$$  
For $1 \leq i \leq n$ all GP-paths in the diagram with orientation $(l_i, l_{i+1})$ are of form

$$p = L_i \to c_{(i,n+1)} \to c_{(i,n)} \to \cdots \to c_{(i,j)} \to c_{(i+1,j)} \to \cdots \to c_{(i+1,n)} \to L_{i+1}.$$  
In particular, the GP-cone $C_{w_0}$ is described by the normal vectors $c_{(i,j+1)} - c_{(i+1,j+1)}$ and $c_{(i,i+1)}$ for $1 \leq i \leq n$ and $i + 1 < j \leq n$.

The weight restriction vectors are (for all $i < j$):

$$c_{j-i} - c_{(i,j)} = 2 \sum_{k=1}^{n-j} c_{(i+k,j+k)} + \sum_{k=0}^{n-j-1} c_{(i+k,j+1+k)} + \sum_{k=0}^{n-j} c_{(i+1+k,j+k)}.$$

2.2. The second cone. We associate to the set of all GP-paths $P_w$ a second cone. In this setup, the standard basis of $\mathbb{R}^{\ell(w) + n}$ is indexed by the faces of the pseudoline arrangement. Namely, there are basis vectors associated to faces $F_{(i,j)}$ bounded to the left by crossing and to faces $F_\ell$ unbounded to the left for every level. Let $p \in P_w$. We denote by $A_p$ the area on the left side of $p$, i.e., the area enclosed by $p$. Note that for non-trivial $p$, $A_p$ is a non-empty union of faces $F$ in the pseudoline arrangement. We associate to $p$ the vector

$$e_p = - \sum_{F \subset A_p} e_F \in \mathbb{R}^{\ell(w) + n}.$$  

With a little abuse of notation we denote also by $e_p$ also the vector in $\mathbb{R}^{\ell(w)}$ obtained by projecting onto the first $\ell(w)$ coordinates (forgetting the coordinates belonging to the faces that are unbounded to the left, these coordinates are equal to 0 in $e_p$ anyway).

Definition 7. For a reduced expression $w \in S_{n+1}$, we define the second cone

$$S_w = \{(x_F) \in \mathbb{R}^{\ell(w)} | (e_p)^t(x_F) \geq 0, \forall p \in P_w\}.$$  

Again, we are interested in a weighted cone for the pseudoline arrangement. For this, we associate to every level $1 \leq i \leq n$ a sequence of faces. $F_i$ is the unbounded area at the left end of the diagram at level $i$. Denote by $F_{(i,j_1)}, \ldots, F_{(i,j_n)}$, read from left to right in the pseudoline arrangement, all faces at this level. We define $A_i := F_i \cup \bigcup_{r=1}^{n-1} F_{(i,j_r)}$, then $A_i \cap A_{i'} = \emptyset$ if $i \neq i'$. It is called the weight area associated to the level $i$. For each $k$ with $0 \leq k \leq n_i - 1$, we denote

$$f_{i,k} = -e_{F_i} - \sum_{r=1}^{k} e_{F_{(i,j_r)}} \in \mathbb{R}^{\ell(w) + n}.$$  

Note that $f_{i,0} = -e_{F_i}$ as we set the empty sum to be zero.

Definition 8. The cone $S_w \subset \mathbb{R}^{\ell(w) + n}$ is defined as

$$S_w = \left\{ (x_F) \in \mathbb{R}^{\ell(w) + n} \left| \begin{array}{c}
(e_p)^t(x_F) \geq 0, \forall p \in P_w, \\
(f_{i,k})^t(x_F) \geq 0, \forall 1 \leq i \leq n \text{ and } 0 \leq k \leq n_i - 1
\end{array} \right. \right\}.$$  

The additional inequalities induced by the $f_{i,k}$ will be called weight inequalities.

Example 4. Consider $n = 5$ and the reduced expression $w_5 = s_1 s_2 s_1 s_3 s_2 s_1$. We have seen all GP-paths for this word in Example 3. Take the path $p = L_1 \to c_{(1,5)} \to c_{(1,4)} \to c_{(1,3)} \to c_{(2,3)} \to c_{(2,4)} \to c_{(2,5)} \to L_2$. The area $A_p$
associated to this path for \( i = 1 \) is colored blue in Figure 7. The weight area \( A_2 \) corresponding to level 2 is also shown in Figure 7 in red. The two red horizontal lines indicate level 2.

![Figure 7. The area \( A_p \) for \( p \) as in Example 4 in blue and the weight area \( A_2 \) in red.](image)

**Example 5.** Consider \( \underline{w} = \underline{w}_0 = s_1s_2s_1s_3s_2s_1 \ldots s_n s_{n-1} \ldots s_3 s_2 s_1 \). Recall the GP-path \( p \) from Example 3. The assigned area is \( A_p = F_{(i,j)} \cup F_{(i,j+1)} \cup \cdots \cup F_{(i,n)} \cup F_{(i,n+1)} \) for \( F_{(i,k)} \) the area bounded by \( c_{(i,k)} \) to the left. Hence, the cone \( S_{\underline{w}_0} \) is given by normal vectors

\[
-e_{F_{(i,j)}} - e_{F_{(i,j+1)}} - \cdots - e_{F_{(i,n)}} - e_{F_{(i,n+1)}}.
\]

The additional normal vectors for the weight inequalities to obtain the cone \( S_{\underline{w}_p} \) are of form

\[
-e_{F_{i}} - e_{F_{(i+1)}} - e_{F_{(i+2)}} - \cdots - e_{F_{(i+j)}},
\]

for \( 1 \leq i \leq n \) and \( 1 \leq j \leq n - i + 1 \).

2.3. **Mutation on the second cone.** Let \( X \) be a mutable face of the pseudoline arrangement associated to \( \underline{w} \). Let \( p \in \mathcal{P}_{\underline{w}} \) and \( A_p \) be the enclosed area. Then there are only four different cases, how this area is transformed under the mutation \( \mu_X \). They follow directly from the behavior of the GP-paths under mutation as in Figure 6 and Section 1.5. A mutable face \( X \) has four direct neighbors. These are the faces that are connected to \( X \) by an arrow in the corresponding quiver (see Definition 3). Depending on the direction of the arrow, we call these faces predecessor for incoming arrows, resp. successor for outgoing arrows.

0. Case: The face \( X \) is not contained in \( A_p \) and neither are any predecessors or successors of \( X \), then \( \mu_X(A_p) = A_p \).

1. Case: The number of predecessors and the number of successors of \( X \) in \( A_p \) are equal. If \( X \) is contained in \( A_p \) then \( \mu_X(A_p) = A_p \setminus X \). If \( X \) is not contained in \( A_p \), then \( \mu_X(A_p) = A_p \cup X \).

2. Case: There are less predecessors than successors of \( X \) in \( A_p \). In this case we always have \( X \subset A_p \). After mutation, \( A_p \) splits into two areas, \( \mu_X(A_p) = A_{p_1} \cup A_{p_2} \). Where \( A_{p_1} = A_p \) and \( A_{p_2} = A_p \setminus X \).

3. Case: There are more predecessors than successors of \( X \) in \( A_p \). If \( A_p \) contains \( X \), it does not change under mutation, i.e. \( \mu_X(A_p) = A_p \). If \( X \not\subset A_p \), after mutation the area \( A_p \) does not exist anymore, i.e. \( \mu_X(A_p) = \emptyset \).

The weight areas \( A_i \) for \( 1 \leq i \leq n \) are also changed under mutation. If \( X \) is given by a triple \( s_is_{i+1}s_i \), the level of \( X \) is \( i \), hence \( X \subset A_i \). After mutation \( X \) is given by the triple \( s_{i+1}s_is_{i+1} \) and so \( \mu_X(X) \subset \mu_X(A_{i+1}) \).
2.4. A unimodular equivalence. We will define a map \( \Psi : \mathbb{R}^{|\mathcal{F}|+n} \to \mathbb{R}^{|\mathcal{F}|+n} \) on the basis \( \{-e_F\} \) associated to faces \( F \) of the pseudoline arrangement. Consider a face \( F_{(i,j)} \). It is bounded to the left by the crossing \( (i,j) \) and, if bounded, it is bounded to the right by another crossing of the same level, which we denote \( (i',j') \). If \( (i',j') \) does not exist, set \( c_{(i',j')} = 0 \). By definition of the pseudoline arrangement, there is at least one crossing bounding \( F_{(i,j)} \) from above or below. Let \( (i_k,j_k) \) denote all these crossings for \( 1 \leq k \leq n_{F_{(i,j)}} \). Here \( n_{F_{(i,j)}} \) is the number of such \( (i_k,j_k) \).

For the faces \( F_i \) unbounded to the left, let \( k_i \) be the smallest index such that \( s_{i_k} = s_i \) in \( w \).

**Definition 9.** We define the linear map \( \Psi : \mathbb{R}^{|\mathcal{F}|+n} \to \mathbb{R}^{|\mathcal{F}|+n} \). For every crossing \( (i,j) \) we set

\[
\Psi(-e_{F_{(i,j)}}) := c_{(i,j)} + c_{(i',j')} - \sum_{r=1}^{n_{F_{(i,j)}}} c_{(i_r,j_r)}.
\]

For every level \( i, 1 \leq i \leq n \), we set

\[
\Psi(-e_{F_i}) := d_{i,k_i}.
\]

**Remark 1.** The map \( \Psi \) restricted to \( \mathbb{R}^{|\mathcal{F}|} \) is similar to the Chamber Ansatz due to Berenstein, Fomin, and Zelevinsky in [3] (see also [15]).

**Proposition 2.** For every GP-path \( p \) we have

\[
\Psi(e_p) = c_p.
\]

In particular, \( \Psi \) sends the normal vector \( e_p \) induced by \( A_p \) to the normal vector \( c_p \) induced by \( p \).

**Proof.** Let \( A_p \) be the area bounded by a GP-path \( p \). Then \( A_p \) is the union of faces \( \bigcup_{F \subset A_p} F = A_p \). There are only two possibilities for a path at a crossing: either it changes the line or not. If \( p \) does not change the line at a crossing \( (i,j) \), then the variable \( c_{(i,j)} \) does not appear in \( c_p \). For \( A_p \), this means that two faces, \( F_1 \) and \( F_2 \), are adjacent to \( (i,j) \). Exactly one of the two, say \( F_1 \), is bounded by \( (i,j) \) to the left where for \( F_2 \), \( (i,j) \) is the part of the upper or lower boundary. Hence \( \Psi(e_p) \) contains \( c_{(i,j)} \) once with a positive sign and once with a negative sign and they cancel each other out in \( c_p \).

Assume \( p \) changes the line at the crossing \( (i,j) \). Figure 8 shows the three possible orientations of a crossing, each yielding two possibilities for the path. If in situation 1a, there is only one face \( F \) in \( A_p \) bounded by \( (i,j) \) to the left and so \( c_{(i,j)} \) appears only once with a positive sign in \( \Psi(e_p) \). The path \( p \) changes from \( l_i \) to \( l_j \) with \( i < j \), so \( c_p \) contains \( c_{(i,j)} \) with a positive sign too. In cases 2a and 3a, \( A_p \) contains only one face bounded by \( (i,j) \) below resp. above. Hence \( c_{(i,j)} \) appears only once with a negative sign in \( \Psi(e_p) \). The same is true for \( c_p \) in both cases \( p \) changes from line \( l_j \) to \( l_i \) but \( i < j \). Three cases remain to be checked, 1b, 2b and 3b. In all of them \( A_p \) contains three faces \( F_1, F_2 \) and \( F_3 \) adjacent to \( (i,j) \). In case 1b, \( (i,j) \) bounds one face to the left and the other two from above, resp. below. This implies that \( c_{(i,j)} \) appears with coefficient \(-1\) in \( \Psi(e_p) \). As \( p \) changes from \( l_j \) to \( l_i \) the same is true for \( c_p \). For 2b and 3b we are in the opposite case: two faces in \( A_p \) are bounded to the left, resp. right, by \( (i,j) \) and only one from above, resp. below. Hence, \( \Psi(e_p) \) contains \( c_{(i,j)} \) with a positive sign and the same is true for \( c_p \), as \( p \) changes from line \( l_i \) to line \( l_j \).

□
It remains to show that the weight inequalities are preserved under $\Psi$. Recall the weight area $A_i = F_i \cup \bigcup_{r=1}^{n_i-1} F_{(i_r,j_r)}$ of level $i$ for $1 \leq i \leq n$.

**Proposition 3.** Consider $k$ with $0 \leq k \leq n_i - 1$. Then $\Psi(f_{i,k})$ is a weight inequality of the weighted Gleizer-Postnikov cone.

**Proof.** Define the map $\varphi_i : \{1, \ldots, n_i\} \to \{1, \ldots, \ell(w)\}$ that sends an index $k$ to the index $\varphi_i(k)$, where the simple reflection $s_{i_{\varphi_i(k)}} = s_i$ in $w$ induces the crossing $(i_k, j_k)$.

We will prove by induction $\Psi(f_{i,k}) = d_{i,\varphi_i}(k+1)$.

By definition we have $\Psi(f_{i,0}) = \Psi(-e_{F_i}) = d_{i,\varphi_i}(1)$. Let $1 \leq k < n_i - 1$, then using induction for the third equation, we obtain

\[
\Psi(f_{i,k+1}) = \Psi(-e_{F_i}) + \sum_{r=1}^{k+1} \Psi(-e_{F_{(i_r,j_r)}})
\]

\[
= \Psi(f_{i,k}) + \Psi(-e_{F_{(i_{k+1},j_{k+1})}})
\]

\[
= d_{i,\varphi_i}(k+1) + \Psi(-e_{F_{(i_{k+1},j_{k+1})}})
\]

\[
= d_{i,\varphi_i}(k+1) + c_{(i_{k+1},j_{k+1})} + c_{(i_{k+2},j_{k+2})} - \sum_{s_{i_p}=s_{i,\varphi_i(k+1)} < p < \varphi_i(k+2)} c_{i_p}
\]

\[
= d_{i,\varphi_i}(k+2)
\]

Combining Proposition 2 and Proposition 3 yields the following.

**Theorem 1.** For every reduced expression $w$ of $w \in S_{n+1}$, the cones $C_w$ and $S_w$ are unimodularly equivalent via the linear map $\Psi$.

Let $\pi : \mathbb{R}^{\ell(w)+n} \to \mathbb{R}^n$ be the projection onto the last $n$ coordinates, called weight coordinates. For each $\lambda \in \mathbb{R}^n$ we consider the preimage under $\pi$ in the cone $C_w$. Let $\tau : \mathbb{R}^{\ell(w)+n} \to \mathbb{R}^n$ be the map

\[
(\lambda_F)_{F} \mapsto \left( \sum_{\text{level}(F)=1}^{\text{level}(F)} \lambda_F, \ldots, \sum_{\text{level}(F)=n}^{\text{level}(F)} \lambda_F \right).
\]

For each $\lambda \in \mathbb{R}^n$ we consider the preimage under $\tau$ in the cone $S_w$.

Since the map $\Psi$ is weight preserving, e.g. sends the weight condition vectors of $S_w$ to the weight condition vectors in $C_w$, we deduce the following.
Corollary 1. For any \( \lambda \in \mathbb{Z}^n \), the map \( \Psi \) induces a unimodular equivalence between the polytopes

\[
\tau^{-1}(\lambda) \cap S_\omega \text{ and } \pi^{-1}(\lambda) \cap C_\omega.
\]

By Proposition 2 we also obtain the following corollary for the non-weighted cones.

Corollary 2. The maps \( \Psi \) induces a linear transformation of the polyhedral cones \( S_\omega \) and \( C_\omega \) for every reduced expression \( \omega \) of an element \( \omega \in S_{n+1} \).

3. The first cone is the string cone

Let \( g = \mathfrak{sl}_{n+1}(\mathbb{C}) \). We fix the triangular decomposition \( g = n^+ \oplus \mathfrak{h} \oplus n^- \), where \( b^+ = n^+ \oplus \mathfrak{h} \) denotes the Borel subalgebra of upper triangular matrices. We denote the corresponding subgroups of \( SL_{n+1}(\mathbb{C}) \) as \( B^+ \) and \( B^- \) the Borel subgroups, as \( U^+ \) and \( U^- \) the maximal unipotent subgroups and as \( T \) the torus. For any Lie algebra \( \mathfrak{a} \), we denote by \( U(\mathfrak{a}) \) the universal enveloping algebra. Let \( \Delta^+ \) be the set of positive roots, \( \alpha_{i,j} := \alpha_i + \ldots + \alpha_j \) where \( \alpha_i \) denotes the simple roots. Let \( W = S_{n+1} \) be the Weyl group with \( s_i \) being the reflection at \( \alpha_i \). We denote by \( f_\alpha \) a root vector for the root \( -\alpha \), \( e_\alpha \) a root vector for \( \alpha \) and \( h_\alpha = [e_\alpha, f_\alpha] \). The PBW theorem states that ordered monomials in \( f_\alpha \) form a basis of \( U(n^-) \).

Let \( P \subset \mathfrak{h}^* \) be the lattice of integral weights spanned by the fundamental weights \( \omega_1, \ldots, \omega_n \) and \( P^+ \) the monoid of dominant integral weights. For \( \lambda \in P^+ \) let \( V(\lambda) \) be the irreducible highest weight module of highest weight \( \lambda \) and \( v_\lambda \) a highest weight vector. Then \( V(\lambda) = U(n^-).v_\lambda \). For \( \omega \in W \) and \( \lambda \in P^+ \), \( v_\omega(\lambda) \) is an extremal weight vector of weight \( \omega(\lambda) \) in \( V(\lambda) \). The Demazure submodule \( V_\omega(\lambda) \) is defined as \( V_\omega(\lambda) = U(b^+).v_\omega(\lambda) \).

Let \( \omega = s_{i_1} \cdots s_{i_{\ell(\omega)}} \) be a reduced expression. Then for any \( \lambda \in P^+ \)

\[
\{ f^{s_{i_{1}}} \cdots f^{s_{i_{\ell(\omega)}}} v_\lambda \in V(\lambda) \mid s_{i_j} \geq 0 \}
\]

forms a spanning set of \( V_\omega(\lambda) \) as a vector space. Littelmann [19] introduced in the context of quantum groups and crystal bases the so called (weighted) string cones and string polytopes \( Q_\omega(\lambda) \). The idea is to define a polytope in \( \mathbb{R}^{\ell(\omega)} \) whose lattice points parametrize a monomial basis of \( V_\omega(\lambda) \). A lattice point \((m_{i_1}) \in \mathbb{R}^{\ell(\omega)} \) is mapped to the vector \( f^{m_{i_1}} \cdots f^{m_{i_{\ell(\omega)}}} v_\lambda \). These polytopes depend on the reduced expression. The transformation map from \( Q_\omega(\lambda) \) to \( Q_\omega(\lambda) \) for two reduced expression of the same Weyl group element, is piecewise linear only.

In [19] a recursive formula for \( Q_\omega(\lambda) \) is provided (another non-recursive formula for \( Q_\omega(\lambda) \) can be found in [5], using subword sequences in \( \omega_0 \)). The string cone for \( \omega \) is then the convex hull of all \( Q_\omega(\lambda) \) and the weighted string cone is

\[
\text{conv} \left( \bigcup_{\lambda \in P^+} Q_\omega(\lambda) \times \lambda \right) \subset \mathbb{R}^{\ell(\omega)} \times \mathbb{R}^n.
\]

The string polytope \( Q_\omega(\lambda) \) is the intersection of the weighted string cone with the hyperplane defined by \( \lambda \in \mathbb{R}^n \). The lattice points in the weighted string cone parametrize a basis of \( \mathbb{C}[SL_{n+1}/U^+] \).

Gleizer and Postnikov develop in [17] a combinatorial model on pseudoline arrangements associated with \( \omega_0 \) and show that the string cone can be described using
Gleizer-Postnikov paths. They show that the string cone is $C_{w_0}$. Now using the description from [19], one can extend their construction to obtain the weighted string cone for $SL_{n+1}/U^+$ as $C_{\lambda}$ (Definition 5).

### 3.1. String cones for Schubert varieties

The picture is less complete for Schubert varieties. Let $w \in S_{n+1}$ and consider the Schubert variety $X(w) = B^+ w B^+ / B^+$. Then the global sections of $X(w)$ on an ample line bundle $L_\lambda$ can be identified with the dual of the Demazure submodule $V_w(\lambda) \subset V(\lambda)$.

Let $\underline{w} = s_{i_1} \ldots s_{i_{l(w)}}$ be a reduced expression of $w \in S_{n+1}$. We extend $\underline{w}$ to the right to obtain a reduced expression $\underline{w}_0 = s_{i_1} \ldots s_{i_{l(w)}} \ldots s_{i_N}$. This extension is not unique but the results are independent of the extension. Caldero realizes the string polytope for the reduced expression $\underline{w}$ for the Demazure module $V_w(\lambda)$ as a face of the string polytope $Q_{\lambda}(\lambda)$ in [6]. He proves the following theorem.

**Theorem.** The string cone for the Schubert variety $X(w)$ and the reduced expression $\underline{w}$ is equal to the face of $C_{\underline{w}_0}$ defined by setting the coordinates corresponding to the simple reflections $s_{i_{l(w)+1}} \ldots s_{i_N}$ to 0.

We show that this face is actually $C_{\underline{w}}$ (Theorem 2). Even more, Calderos result can be extended to the weighted string cone. Then the lattice points parametrize a basis of $C[B^+ w B^+ / U^+]$. We show that the face in $C_{\underline{w}_0}$ defined by setting the additional coordinates to 0, is the weighted string cone $C_{\underline{w}}$.

In order to do so we need to show that there is a bijection between the faces of $C_{\underline{w}}$ and the intersection of the faces of $C_{\underline{w}_0}$ with the subspace defined by $x_{i_{l(w)+1}} = \ldots = x_{i_N} = 0$.

Using Proposition 4 we will deduce that the weighted Gleizer-Postnikov cone $\mathcal{C}_{\underline{w}}$ is the weighted string cone for $X(w)$ with reduced expression $\underline{w}$. From now on $p_{\underline{w}}$ denotes a GP-path for $\underline{w}$.

**Definition 10.** Let $w$ be a reduced expression and $\underline{w}_0$ an extension to the right of $\underline{w}$. Let $p_{\underline{w}}$ be a GP-path for $\underline{w}_0$. We define $\text{res}_{\underline{w}}(p_{\underline{w}_0})$ to be the restriction of $p_{\underline{w}_0}$ to the pseudoline arrangement for $\underline{w}$. This restriction is obtained by setting all $c_\alpha = 0$ if $w(\alpha) > 0$. To an open endpoint on line $l_i$ we associate the vertex $\tilde{L}_i$.

The following procedure will allow us to define an induced GP-path for $\underline{w}_0$ for a given GP-path $p_{\underline{w}}$. Consider $\tilde{p}_{\underline{w}} = \tilde{L}_{i_{i-1}} \rightarrow v_{r_1} \rightarrow \ldots \rightarrow v_{r_m} \rightarrow \tilde{L}_{i_{i+m}}$ in a pseudoline arrangement for $\underline{w}$ with orientation $(l_i, l_{i+1})$. Extend $\underline{w}$ to the right to $\underline{w}_0$, i.e. extend the pseudoline arrangement for $\underline{w}$ to the right to an arrangement for $\underline{w}_0$ and keep the orientation.

Starting from $\tilde{L}_{i+m}$ follow line $l_{i+m}$ to the next crossing with a line $l_{i+m-k_1}$ for some $1 \leq k_1 \leq m - 1$. Then follow the $l_{i+m-k_1}$ until the next crossing with a line $l_{i+m-k_1-k_2}$ ($1 \leq k_2 \leq m - k_1 - 1$) and proceed always with changing to lines closer to $l_{i+1}$. After a finite number of changes there is a crossing with $l_{i+1}$ by Proposition 1. Continue on $l_{i+1}$ to $L_{i+1}$. For the path from $L_i$ to $\tilde{L}_{i-l}$ we apply a similar procedure going against the orientation of the lines and always changing to lines $l_{i-l+n_r}$ closer to $l_i$ until line $l_i$.

**Definition 11.** The induced GP-path $\text{ind}_{\underline{w}}(p_{\underline{w}})$ from a path $p_{\underline{w}}$ is the GP-path for $\underline{w}_0$ obtained from the above construction.

**Proposition 4.** For every path $p_{\underline{w}}$ there exists a path $p_{\underline{w}_0}$ such that, $\text{res}_{\underline{w}}(p_{\underline{w}_0}) = p_{\underline{w}}$. Moreover, we have $\text{res}_{\underline{w}}(\text{ind}_{\underline{w}}(p_{\underline{w}})) = p_{\underline{w}}$. 
Proof. Let \( \mathbf{p}_w = \hat{L}_{i-1} \rightarrow v_{r_1} \rightarrow \cdots \rightarrow v_{r_m} \rightarrow \hat{L}_{i+m} \) for the orientation \( (l_i, l_{i+1}) \). Assume \( \alpha_i \in \{ \alpha \mid w^{-1}(\alpha) \leq 0 \} \) and extend \( w \) to \( \hat{w}_0 \). Consider the pseudoline arrangement for \( \hat{w}_0 \). Note that this is not unique, as it depends on the extension. By construction \( \text{ind}_w(\mathbf{p}_w) \) is a GP-path satisfying \( \text{res}_w(\text{ind}_w(\mathbf{p}_w)) = \mathbf{p}_w \). 

Proposition 5. Let \( \mathbf{p}_w \) be a GP-path for \( w \), then \( \text{res}_w(\mathbf{p}_w) \) is either empty or the union of GP-paths for \( w \).

Proof. The restriction \( \text{res}_w(\mathbf{p}_w) \) decomposes into a union of paths in the pseudoline arrangement for \( w \). As the definition of GP-paths is local, it follows that all paths in this union are rigorous, hence GP-paths for \( w \). 

Let \( \mathbf{p}_{w_0} \) be a GP-path and \( \text{res}_w(\mathbf{p}_{w_0}) = \mathbf{p}_w \cup \mathbf{p}'_w \) for two GP-paths \( \mathbf{p}_w, \mathbf{p}'_w \). We can decompose the normal vector of \( \mathbf{p}_w \) as \( c_{\mathbf{p}_w} = c_1 + c_2 + c_3 + c_4 + c_5 \). The vectors \( c_i \) correspond to segments of \( \mathbf{p}_w \) as illustrated in Figure 9 below. The normal vector for \( \text{res}_w(\mathbf{p}_{w_0}) \) is given by \( c_{\text{res}_w(\mathbf{p}_{w_0})} = c_1 + c_4 \), the normal vector of \( \mathbf{p}_w \) is \( c_{\mathbf{p}_w} = c_2 \) and the normal vector of \( \mathbf{p}'_w \) is \( c_{\mathbf{p}'_w} = c_3 \) (see Example 6). Now the condition given by \( \text{res}_w(\mathbf{p}_{w_0}) \) is redundant, as it is imposed by the conditions for \( \mathbf{p}_w \) and \( \mathbf{p}'_w \), which are both by Proposition 4 obtained by restriction. So we see that the conditions obtained from all restricted GP-paths of \( w \) define the same cone as the conditions obtained from all GP-paths of \( w \). This implies the following theorem.

Theorem 2. For every \( w \in S_{n+1} \) with reduced expression \( w \), the associated weighted string cone is the weighted Gleizer-Postnikov cone \( C_w \).

Example 6. Consider \( w = s_2s_1s_3 \) and extend to \( \hat{w}_0 = s_2s_1s_3s_2s_3s_1 \) for \( SL_4/B^+ \). We draw the pseudoline arrangement and endow it with the orientation for \( (l_2, l_3) \). Figure 9 shows a GP-path \( \mathbf{p}_{\hat{w}_0} \). Its restriction \( \text{res}_w(\mathbf{p}_{\hat{w}_0}) \) consists of two GP-paths for \( w \) shown in blue the figure. Let \( \mathbf{p}_w \) denote the upper blue path and \( \mathbf{p}'_w \) the lower blue path to the left of the dashed line. Each segment of the path \( \mathbf{p}_{\hat{w}_0} \) has an associated normal vector \( c_i \) for \( i = 1, \ldots, 5 \) as shown in the figure.

4. The second cone is the tropicalization of a superpotential

4.1. Cluster algebras and cluster varieties. We refer for the definition of a cluster algebra to the paper by Fomin and Zelevinsky [10]. We denote by \( C \) a
cluster algebra and \( s \) a cluster seed. The initial seed defining a cluster algebra is associated to a fixed initial reduced expression \( \mu \) and the corresponding quiver \( \Gamma_\mu \) is as in Definition 3. The initial seed will be denoted \( s_\mu = s_0 \). For an arbitrary seed \( s \), the associated quiver is denoted by \( \Gamma_s \).

For the definition of \( \mathcal{A} \)- and \( \mathcal{X} \)-cluster varieties we refer to Fock and Goncharov [9]. We will recall necessary notions but not in the most general setup as we are only interested in a particular case where the cluster algebra is associated to a quiver.

Let \( s \) be an arbitrary seed in a cluster algebra \( \mathcal{C} \) with cluster variables \( \alpha_1, \ldots, \alpha_n \). We interpret \( \alpha_i \) as a function on \((\mathbb{C}^*)^n = \mathcal{A}_s\) sending the standard basis vector \( f_i \) to 1 and all the other \( f_j \) to zero if \( j \neq i \). Let \( M \) be the lattice spanned by \( \{f_1, \ldots, f_n\} \) and identify \( \mathcal{A}_s = \text{Spec} \mathbb{C}[M] \). Let \( N \) be the dual lattice with dual basis \( \{e_1, \ldots, e_n\} \) and \( \mathcal{X}_s = \text{Spec} \mathbb{C}[N] \). The mutation of the seed \( s \) to a seed \( \mu_k(s) \) induces birational morphisms between the tori \( \mathcal{A}_s \) and \( \mathcal{A}_{\mu_k(s)} \) resp. \( \mathcal{X}_s \) and \( \mathcal{X}_{\mu_k(s)} \), see Proposition 2.4 in [11]. This enables us to define the following.

**Definition 12.** The \( \mathcal{A} \)-cluster variety (resp. \( \mathcal{X} \)-cluster variety) is glued from the tori \( \mathcal{A}_s \) (resp. \( \mathcal{X}_s \)) along the birational morphisms induced by cluster mutation, i.e.

\[
\mathcal{A} = \bigcup_{s \text{ seed}} \mathcal{A}_s \quad \text{and} \quad \mathcal{X} = \bigcup_{s \text{ seed}} \mathcal{X}_s.
\]

The \( \mathcal{X} \)-cluster variety is also the Fock–Goncharov dual of the cluster variety \( \mathcal{A} \) in the cases we consider.

Let \( \{e_i'\} \) be the basis of \( N \) associated to \( \mathcal{X}_{\mu_k(s)} \). Further let

\[
e_i' = \begin{cases} 
  e_i + \max\{0, e_{i,k}\}e_k & \text{if } i \neq k \\
  -e_k & \text{if } i = k.
\end{cases}
\]

The birational maps \( \mu_k : \mathcal{X}_s \to \mathcal{X}_{\mu_k(s)} \) corresponding to the mutation are defined by the pullback of functions \( \mu_k^* \) given on monomials \( z^n \) for \( n \in N \)

\[
\mu_k(z^n) = z^n(1 + z^{e_k})^{-(n,e_k)}.
\]

**4.2 Double Bruhat cells and the Superpotential.** Let \( \tau \) and \( w \) be a pair of elements in \( S_{n+1} \) and

\[
G^{\tau,w} := B^- \tau B^- \cap B^+ w B^+.
\]

be the corresponding double Bruhat cell. We are particularly interested in the case \( G^{e,w} \), where \( e \) is the identity element and

\[
G^{e,w} = B^- \cap B^+ w B^+.
\]

The reduced double Bruhat cell is

\[
L^{e,w} = U^- \cap B^+ w B^+.
\]
The coordinate ring of any double Bruhat cell $G_{e,w}$ (as well as of the reduced double Bruhat cell) is an upper cluster algebra (see [10], [4], [16], and recently further results from [18]) endowing $G_{e,w}$ with the structure of an $A$-cluster variety. The associated cluster algebra is defined by the quiver $\Gamma_w$ for a fixed initial reduced expression $w$ of $w$. The two cluster algebras for the double Bruhat cell and its reduced double Bruhat cell only differ by their coefficient algebras, see Chapter 5.2 in [16].

Corresponding to the $A$-cluster variety $G_{e,w}$ we consider the Fock–Goncharov dual variety $\mathcal{X}$. We refer to [21, Section 2] for a detailed discussion. The cluster variety $\mathcal{X}$ comes with a function $W: \mathcal{X} \to \mathbb{C}$ called superpotential. Restricted to a torus $\mathcal{X}_s$ for a seed $s$ it can be given explicitly in the variables corresponding to $s$. For the double Bruhat cell $G_{e,w_0}$ Magee gave the description of $W$ for the initial seed corresponding to the reduced expression $w_0 = s_1 s_2 s_1 s_3 s_2 s_1 \ldots s_n \ldots s_1$. We will now focus on $G_{e,w_0}$ and summarize his result.

Following [12] we want to associate $\vartheta$-functions to the frozen variables of the double Bruhat cell corresponding to boundary divisors. For the special case of $G_{e,w_0}$ and the initial seed $s_0$ corresponding to the reduced expression $w_0 = s_1 s_2 s_1 s_3 s_2 s_1 \ldots s_n \ldots s_1$, these $\vartheta$-functions have been computed explicitly in [20]. For every frozen vertex there is a sequence of mutations from $s_0$ to a seed, where this vertex is a sink. We call a seed with this property optimized for this frozen vertex. On optimized seeds the $\vartheta$-functions are given as monomials. To obtain the function on the initial seed, this monomial is pulled back along the reverse mutation sequence using Equation 4.1. Recall the notation of the initial seed quiver from Section 1.3 and Figure 3. The frozen vertices on the right are labelled $v_{(1,n+1)}, \ldots, v_{(n,n+1)}$ and $v_n, \ldots, v_1$ on the left, each from top to bottom. The $\vartheta$-functions on the initial seed $s_0$ are of form (see Corollary 24 in [20])

$$
\vartheta_{i,n+1} = z^{-e_{i,n+1} + e_{i,n+1} - e_{i,n}} + \cdots + z^{-e_{i,n+1} - e_{i,n} - \cdots - e_{i,2}}
$$

$$
\vartheta_i = z^{-e_i + e_{i-1,i+1}} + \cdots + z^{-e_{i-1,i+1} - e_{i+2,i+2} - \cdots - e_{i,n}}.
$$

Then the following definition is a Theorem due to Magee [20].

**Definition 14.** Let $G_{e,w_0}$ be the double Bruhat cell with Fock-Goncharov dual $\mathcal{X}$. Then the superpotential $W: \mathcal{X} \to \mathbb{C}$ on a given seed torus $\mathcal{X}_s$ is the sum of all $\vartheta$-functions for the seed $s$, i.e. $W|_{\mathcal{X}_s} = \sum_{i=1}^n \vartheta_{(i,n+1)}|_{\mathcal{X}_s} + \vartheta_i|_{\mathcal{X}_s}$. For the initial seed torus $\mathcal{X}_{w_0}$ we have

$$
W|_{\mathcal{X}_{w_0}} = \sum_{i=1}^n \left( \sum_{k=0}^{n-1} z^{-\sum_{j=0}^k e_{i,n+1-j}} + \sum_{k=0}^{n-1} z^{-e_i - \sum_{j=1}^k e_{j+1}} \right).
$$

**Example 7.** Consider $SL_3/B^+$ and the initial seed with quiver $\Gamma_{s_1 s_2 s_1}$. Then

$$
W|_{\mathcal{X}_{w_0}} = \vartheta_{(1,3)} + \vartheta_{(2,3)} + \vartheta_{1} + \vartheta_{2} = z^{-e_{1,3}} + z^{-e_{1,3} - e_{1,2}} + z^{-e_{2,3}} + z^{-e_{1}} + z^{-e_{1} - e_{1,2}} + z^{-e_{2}}.
$$

Given the pullback on functions by Equation 4.1 and the superpotential on the initial seed $s_0$ for $w_0$, the restriction of the superpotential to any seed $s$ can be computed explicitly once a sequence of mutations from $s$ to $s_0$ is found.
4.3. Cones associated to the Superpotential. Consider a Laurent polynomial $q(x) = \sum a_i x^{m_i} \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}]$ for $m_i \in \mathbb{Z}^r$, $a_i \in \mathbb{C}$ and $x^{m_i} = x_1^{m_1} \cdots x_r^{m_r}$. The tropicalization of $q$ is defined as $q^{\text{trop}}(x) = \min_{a_i \neq 0} \{m_1 x_1 + \cdots + m_r x_r\}$. In particular, $q^{\text{trop}}(x) \geq 0$ translates to $m_1 x_1 + \cdots + m_r x_r \geq 0$ for all $i$ with $a_i \neq 0$. In this manner the tropicalization of a Laurent polynomial defines a polyhedral cone in $\mathbb{R}^r$.

**Definition 15.** Let $G_{c,w_0}$ be the double Bruhat cell. For every seed $s$ the superpotential cone associated to $W|_{X_s} : X_s \rightarrow \mathbb{C}$ is defined as $\Xi_s = \{x \in \mathbb{R}^{N+n} \mid W^{\text{trop}}|_{X_s}(x) \geq 0\}$.

**Proposition 6.** Consider the initial seed $s_0$. Then the superpotential cone $\Xi_{s_0}$ is unimodularly equivalent to the weighted string cone $C_{w_0}$, where $w_0$ is the initial reduced expression of $w_0$.

**Proof.** In [20] a unimodular equivalence between the superpotential cone $\Xi_{s_0}$ and the Gelfand-Tsetlin cone in [14] has been provided. Littelmann proved in [19] an unimodular equivalence between the Gelfand-Tsetlin cone and $C_{w_0}$. Hence, the result follows. \(\square\)

Now suppose we have the cone defined by the tropicalization of a superpotential for a fixed seed. The next proposition describes how the facets of this cone are changed under mutation from $w_0$ to $w'_0$.

**Proposition 7.** Let $s, s'$ be as above and $z^{\mathbf{m}}$ a monomial in $\vartheta|_{X_s}$, where $\mathbf{m} = \sum a_i e_i$, for the basis $\{e_i\}$ of $N$ associated to $X_s$. In fact, $a_i \in \{0, -1\}$ for all $i$. Let $\{e'_i\}$ be the basis of $N$ associated to $X_{s'}$ and let $\mathbf{m}' = \sum a_i e'_i$ with the same coefficients $a_i$ as for $\mathbf{m}$. The following are all possible forms for $\mu_k^{\ast}(z^{\mathbf{m}})$, a monomial in $\mu_k^{\ast}(\vartheta|_{X_s}) = \vartheta|_{X_{s'}}$.

0. Case: The coefficients $a_l$ for $l \in \{i, j, k, p, q\}$ are all zero. Then $\mu_k^{\ast}(z^{\mathbf{m}}) = z^{\mathbf{m}'}$.
1. Case: The number of nonzero coefficients $a_i$ and $a_j$ is the same as the number of nonzero $a_p$ and $a_q$. If $a_k = -1$ then $\mu_k^{\ast}(z^{\mathbf{m}}) = z^{\mathbf{m}' + e'_k}$, otherwise $\mu_k^{\ast}(z^{\mathbf{m}}) = z^{\mathbf{m}' - e'_k}$.
2. Case: The number of nonzero coefficients $a_i$ and $a_j$ is smaller than the number of nonzero $a_p$ and $a_q$. Then $\mu_k^{\ast}(z^{\mathbf{m}}) = z^{\mathbf{m}'} + z^{\mathbf{m}_2}$ with $\mathbf{m}_1 = \mathbf{m}'$ and $\mathbf{m}_2 = \mathbf{m}' - e'_k$.
3. Case: The number of nonzero coefficients $a_i$ and $a_j$ is larger than the number of nonzero $a_p$ and $a_q$. If $a_k = 0$ we have $\mu_k^{\ast}(z^{\mathbf{m}}) = z^{\mathbf{m}'}$, if otherwise $a_k = -1$ then $\mu_k^{\ast}(z^{\mathbf{m}}) = 0$, i.e. there is no such monomial in $\vartheta|_{X_{s'}}$.

We are now ready to prove the following result on cones associated to $G_{c,w_0}$.

**Theorem 3.** Let $w_0$ be an arbitrary reduced expression of $w_0$ and $s_{w_0}$ the associated seed. Then $\Xi_{s_{w_0}} = S_{w_0}$. 

Proof. Recall that the initial reduced expression \( s_1 s_2 s_1 s_3 s_2 s_1 s_n \ldots s_1 \) is associated to the initial seed \( s_0 \). Let \( w_0 \) be an arbitrary reduced expression and \( s = s_{w_0} \) the corresponding seed. We prove the theorem by induction on the number of (braid-)mutations necessary to obtain the seed \( s \) from the initial seed. For \( s_0 = s_{w_0} \), we have seen that

\[
\vartheta_{i,n+1} = z^{-e_i,n+1} + z^{-e_{i,n+1} - e_i,n} + \cdots + z^{-e_{i,n+1} - e_{i,n} - \cdots - e_i,2}
\]

\[
\vartheta_i = z^{-e_i} + z^{-e_i - e_{i+1}} + \cdots + z^{-e_i - e_{i+1} - e_{i+2} - \cdots - e_{n-i,n}}.
\]

Using the notation for the faces of the pseudoline arrangement as in Section 1.3 we see that all areas \( A_p \) between lines \( l_i \) and \( l_{i+1} \) are of form \( F_{(i,j)} \cup F_{(i,j+1)} \cup \cdots \cup F_{(i,n+1)} \). Further, every weight area \( A_i \) induces the successive areas \( F_{(i+1,1)} \cup \cdots \cup F_{(n,i+1)} \subset A_i \). This yields as described in Example 5 the following normal vectors for the cone \( S_{w_0} \), on the initial seed

\[
-e_{F_{(i,j)}} - e_{F_{(i,j+1)}} - \cdots - e_{F_{(i,n)}} - e_{F_{(i,n+1)}} \quad 1 < j \leq n + 1
\]

\[
-e_{F_i} - e_{F_{(i+1,1)}} - e_{F_{(i+1,2)}} - \cdots - e_{F_{(j,i+1)}} \quad 1 \leq j \leq n - i.
\]

Hence, the two cones \( \Xi_{s_0} \) and \( S_{w_0} \) are described by the same facet normals for the initial seed.

In order to have the equality for all seeds \( s \), we have to show that both are transformed in the same way under mutation. But this follows immediately by comparing Section 2.3 with Proposition 7.

Combining Theorem 3 with the unimodular equivalence between the weighted string cone \( C_{w_0} \) and the area cone \( S_{w_0} \) for arbitrary \( w_0 \) in Theorem 1 we can state our main theorem.

**Theorem 4.** Let \( w_0 \in S_{n+1} \) be an arbitrary reduced expression of \( w_0 \) and \( s_{w_0} \) be the seed corresponding to the quiver \( \Gamma_{w_0} \) associated to the pseudoline arrangement for \( w_0 \). Then the weighted string cone \( C_{w_0} \) can be realized as the tropicalization of the superpotential \( W: \mathcal{X} \to \mathbb{C} \) restricted to the seed torus \( \mathcal{X}_{s_{w_0}} \).

### 4.4. Restricted Superpotential and double Bruhat cells

We will now consider the double Bruhat cells of the form \( G^{c,w} \) for \( w \) an arbitrary element of \( S_{n+1} \). We have seen by the extension of Caldero’s result in [6] that the weighted string cone \( C_w \) of the Schubert variety \( X(w) \) is a face of the weighted string cone \( C_{w_0} \), where \( w = s_{i_1} \cdots s_{i_{t(w)}} \) and \( w_0 = s_{i_1} \cdots s_{i_{t(w)}} \cdots s_{i_{n}} \) is an extension of \( w \) (see Theorem in Section 3.1). From now on fix \( w \) and an extension \( w_0 \). As before \( W \) denotes the superpotential for the double Bruhat cell \( G^{c,w_0} \).

**Definition 16.** We define the restricted superpotential \( \text{res}_{w}(W|_{\mathcal{X}_{w_0}}) \) as the function obtained from \( W|_{\mathcal{X}_{w_0}} : \mathcal{X}_{w_0} \to \mathbb{C} \) by restricting to \( \mathcal{X}_{w_0} \). Note that this is equivalent to setting all variables \( e_i \) to zero that don’t correspond to faces of the pseudoline arrangement for \( w \).

Here \( \mathcal{X}_{w} \) resp. \( \mathcal{X}_{w_0} \) are the tori of the Fock-Goncharov dual \( \mathcal{X} \)-cluster varieties of the \( \mathcal{A} \)-cluster varieties \( G^{c,w} \) resp. \( G^{c,w_0} \) that correspond to the seeds with quivers \( \Gamma_{w} \) resp. \( \Gamma_{w_0} \).
Example 8. Consider $SL_4$ and $\underline{w} = s_1 s_2 s_3 s_2 s_1$ with extension $\underline{w}_0 = s_1 s_2 s_3 s_2 s_1 s_2$. Then

$$W|_{X_{\underline{w}_0}} = (z^{-e_3}) + (z^{-e_2} + z^{-e_2 - e_1, 1, 3} + z^{-e_2 - e_1, 3, 3, 4})$$
$$+ (z^{-e_1} + z^{-e_1 - e_1, 3, 1, 2} + (z^{-e_2, 4} + z^{-e_2, 4 - e_3, 4}) + (z^{-e_2, 3})$$
$$+ (z^{-e_1, 4} + z^{-e_1, 4 - e_1, 3} + (z^{-e_1, 4 - e_1, 3, 3, 4} + z^{-e_1, 4 - e_1, 3, 3, 4 - e_1, 2}).$$

In the pseudoline arrangement corresponding to $\underline{w}$ the face $F_{2, 3}$ does not exist anymore, see Figure 10. Hence

$$\text{res}_{\underline{w}}(W|_{X_{\underline{w}_0}}) = (z^{-e_1}) + (z^{-e_2} + z^{-e_2 - e_1, 1, 3} + z^{-e_2 - e_1, 3, 3, 4})$$
$$+ (z^{-e_1} + z^{-e_1 - e_1, 3, 1, 2} + (z^{-e_2, 4} + z^{-e_2, 4 - e_3, 4})$$
$$+ (z^{-e_1, 4} + z^{-e_1, 4 - e_1, 3} + z^{-e_1, 4 - e_1, 3, 3, 4} + z^{-e_1, 4 - e_1, 3, 3, 4 - e_1, 2}).$$

\[ \text{Figure 10. Restriction/Extension of a pseudoline arrangement.} \]

As before, we can define a polyhedral cone whose facets are given by the tropicalization of the restricted superpotential.

Definition 17. Define $\Xi_{\underline{w}} = \{x \in \mathbb{R}^{l(w) + n} | \text{res}_{\underline{w}}(W|_{X_{\underline{w}_0}})^{\text{trop}}(x) \geq 0\}$.

Now by the identification of $\Xi_{\underline{w}_0}$ and $S_{\underline{w}_0}$ from Theorem 3 we immediately obtain the following.

Theorem 5. Let $\underline{w} = s_{i_1} \ldots s_{i(|\underline{w}|)}$ be a reduced expression of $w \in S_{n+1}$ and $\underline{w}_0 = s_{i_1} \ldots s_{i(|\underline{w}|)} \ldots s_{i_N}$. Then

$$\Xi_{\underline{w}} = S_{\underline{w}}.$$ 

So we see that this tropical cone is nothing but $S_{\underline{w}}$. We strengthen Theorem 4 to the following.

Theorem 6. Let $\underline{w}$ be an arbitrary reduced expression of $w \in S_{n+1}$, then the weighted Gleizer-Postnikov cone $C_{\underline{w}}$ (i.e. the weighted string cone) for the Schubert variety $\mathcal{X}(w)$ is unimodularly equivalent to the cone $\Xi_{\underline{w}}$, defined by $\text{res}_{\underline{w}}(W|_{X_{\underline{w}_0}})^{\text{trop}}$.

4.5. An observation on optimized seeds. We will construct naively in an example a potential, following [12], which we call of GHKK-type. For a fixed seed $s$ in the cluster algebra associated to the double Bruhat cell $G^\circ$, and a frozen vertex $v_f$, we want to find a sequence of mutations from $s$ to an optimized seed for $v_f$. On this optimized seed, we define a $\vartheta$-function $\vartheta_f$ to be $z^{-e_f}$. Using the reverse mutation
sequence we can write \( \vartheta_f \) in our fixed seed \( s \), say \( \vartheta_f^s \). Then the GHKK-type potential on the seed \( s \) would be given by
\[
\text{GHKK} W|_{X_s} = \sum_{f \text{ frozen}} \vartheta_f^s.
\]
If \( w \) is the longest Weyl group element, then this construction yields the superpotential \( W|_{X_w} \) on \( G^{e,w} \). We compute in the following this potential in a particular example to see that \( \text{GHKK} W|_{X_w} \) is not \( \text{res}_w(W|_{X_w}) \), with \( \text{w0} \) being an extension to the right of \( w \).

**Example 9.** Let \( s = s_w \) be the seed of the reduced expression \( w = s_1s_2s_3s_2s_1 \in S_4 \) as in Figure 10. The corresponding quiver is pictured in Figure 11. We proceed as in [20] and compute optimized seeds for every frozen variable. From \( \Gamma_w \) we see that \( s \) is optimized for the frozen variables at \( v_3 \) and \( v_{(2,4)} \), hence we set \( \vartheta_3|_{X_s} = z^{-e_3} \) and \( \vartheta_{2,4}|_{X_s} = z^{-e_{2,4}} \), where \( \{e_{i,j}\} \) denotes the basis associated to \( X_s \).

\[
\begin{array}{c}
V_3 \\
V_2 \\
V_1 \\
\Gamma_w
\end{array}
\begin{array}{c}
\bigodot_{(1,3)} \\
\bigodot_{(1,2)} \\
\bigodot_{(2,4)}
\end{array}
\begin{array}{c}
\mu_{(1,3)}(\Gamma_w) \\
\mu_{(1,2)}(\Gamma_w)
\end{array}
\]

**Figure 11.** The quivers \( \Gamma_w, \mu_{(1,3)}(\Gamma_w) \) and \( \mu_{(1,2)}(\Gamma_w) \) for \( w = s_1s_2s_3s_2s_1 \). The boxes denote frozen variables.

For the other variables we have to find a mutation sequence to an optimized seed. Mutation at \( v_{(1,3)} \) (resp. \( v_{(1,2)} \)) yields the quiver \( \mu_{(1,3)}(\Gamma_w) \) (resp. \( \mu_{(1,2)}(\Gamma_w) \)) in Figure 11. The seed \( \mu_{(1,3)}(s) \) is optimized for \( v_{(1,4)} \) and \( v_2 \), so \( \vartheta_{1,4}|_{X_{\mu_{(1,3)}(s)}} = z^{-e_{1,4}} \) and \( \vartheta_{2}|_{X_{\mu_{(1,3)}(s)}} = z^{-e_2} \). Here \( \{e'_{i,j}\} \) denotes the basis associated to \( X_{\mu_{(1,3)}(s)} \). We pull these back with \( \mu_{(1,3)}^* \) as in Equation 4.1 and obtain \( \vartheta_{1,4}|_{X_s} = z^{-e_{1,4}} + z^{-e_{1,4} - e_{1,3}} \) and \( \vartheta_{2}|_{X_s} = z^{-e_2} + z^{-e_{2} - e_{1,3}} \). Proceeding analogously with \( \mu_{(1,2)}(s) \), which is optimized for \( v_{(3,4)} \) and \( v_1 \), we obtain the following as sum of all \( \vartheta \)-functions on \( X_s \)
\[
\text{GHKK} W|_{X_s} = \left( z^{-e_3} \right) + \left( z^{-e_2} + z^{-e_2 - e_{1,3}} \right) \\
+ \left( z^{-e_1} + z^{-e_{1,2}} \right) + \left( z^{-e_{2,4}} \right) \\
+ \left( z^{-e_3,4} + z^{-e_{3,4} - e_{1,2}} \right) + \left( z^{-e_{1,4}} + z^{-e_{1,4} - e_{1,3}} \right).
\]
We observe that \( \text{GHKK} W|_{X_s} \neq \text{res}_w(W|_{X_w}) \), although both are defined on \( X_s \). Let us compare the cones \( \Sigma_w = \{ x \in \mathbb{R}^{\ell(w)+n} \mid \text{res}_w(W|_{X_w})^{\text{trop}}(x) \geq 0 \} \) and \( \text{GHKK} \Sigma_s = \{ x \in \mathbb{R}^{\ell(w)+n} \mid \text{GHKK} W^{\text{trop}}|_{X_s}(x) \geq 0 \} \). A point \( x \in \mathbb{R}^{\ell(w)+n} = \mathbb{R}^8 \) lies in \( \Sigma_w \) if the following inequalities are satisfied
\[
-x_3 \geq 0, \\
-x_2 \geq 0, -x_2 - x_{1,3} \geq 0, -x_2 - x_{1,3} - x_{3,4} \geq 0 \\
-x_1 \geq 0, -x_1 - x_{1,2} \geq 0, \\
-x_{2,4} \geq 0, -x_{2,4} - x_{3,4} \geq 0, \\
-x_{1,4} \geq 0, -x_{1,4} - x_{1,3} \geq 0, -x_{1,4} - x_{1,3} - x_{3,4} \geq 0, -x_{1,4} - x_{1,3} - x_{3,4} - x_{1,2} \geq 0.
\]
The point \( x \) lies in \( \text{GHKK} \subset \Xi_s \), if it satisfies
\[
-x_3 \geq 0, \\
-x_2 \geq 0, -x_2 - x_{1,3} \geq 0, \\
-x_1 \geq 0, -x_1 - x_{1,2} \geq 0, \\
-x_{2,4} \geq 0, \\
-x_{3,4} \geq 0, -x_{3,4} - x_{1,2} \geq 0, \\
-x_{1,4} \geq 0, -x_{1,4} - x_{1,3} \geq 0.
\]
Comparing the two sets of inequalities one can easily deduce that \( \text{GHKK} \subset \Xi_w \).
The converse is not true as for example the point \((x_1, x_2, x_3, x_{1,2}, x_{1,3}, x_{2,4}, x_{3,4}) = (-1, -1, -1, -1, -1, -1, 1)\) lies in \( \Xi_w \) but not in \( \text{GHKK} \subset \Xi_s \).

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L. Bossinger: University of Cologne, Mathematical Institute, Weyertal 86–90, 50931 Cologne, Germany
E-mail address: lbossing@math.uni-koeln.de

G. Fourier: Leibniz University Hannover, Institute for Algebra, Number Theory and Discrete Mathematics, Welfengarten 1, 30167 Hannover, Germany
E-mail address: fourier@math.uni-hannover.de