Some Features of Blown-Up Nonlinear $\sigma$-Models

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Abstract

In terms of the gauged nonlinear $\sigma$-models, we describe some results and implications of solving the following problem: Given a smooth symplectic manifold as target space with a quasi-free Hamiltonian group action, perform the symplectic blowing up of the point singularity and identify the blow-up modes in the corresponding (gauged) $\sigma$-model. Both classical and quantum aspects of the construction are explained, along with illustrating examples from the toric projective space and the Kähler manifold. We also discuss related problems such as the origin of Mirror symmetry and the quantum cohomologies.

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One of the characterizing properties of a string theory is the duality symmetry which exchanges the role of momentum and winding modes. It has recently been discovered that duality symmetry can also be implemented in the gauged version of some nonlinear $\sigma$-model [1]. This appears to be somewhat mysterious. As the notion of the winding modes is unique to strings, the result of [1] seems to suggest a new geometric property undermined in the (gauged) nonlinear $\sigma$-models.

In understanding the origin of the duality symmetry in nonlinear $\sigma$-models, one may compare the case of the orbifold compactifications of the string theory. Owing to the existence of the winding modes in the string excitations, the orbifold singularities are automatically resolved by adding the twisted sectors to the Hilbert space. The resulting low energy effective theory is geometrically equivalent to a nonlinear $\sigma$-model blown-up at the orbifold singular points. This fact is often phrased as ”string theory resolves (orbifold) singularity”. One therefore expects that the blowing up operation in a nonlinear $\sigma$-model would provide additional structures responsible for the stringy properties such as duality symmetry. Further evidence comes from recent studies of the topology changing phenomenon in string theory [2].

Apart from the above ”string-generated” motivation, there are more modest purposes of studying blown-up nonlinear $\sigma$-models both from mathematical and physical standpoints, among them we just mention two: The topological $\sigma$-model [3] is originally proposed to capture some salient features of Floer’s theory of symplectic diffeomorphisms [4], a basic ingredient of the classification theory of the symplectic (as well as complex) manifolds is the notion of birational equivalence. It is both necessary and natural to include equivalence classes of the topological $\sigma$-models obtained by blowing-up; In applying the Duistermaat-Heckman therorem [5] to the singular case, one is led to consider the jump of the cohomology class of the symplectic 2-form at the blowing up points [6]. The natural invariant is the Duistermaat-Heckman polynomials appeared as the coefficients of the expansion of the push-forward-by-moment-map symplectic volume element in terms of the volumes of the subdivided convex polyhedral sets. In a formulation of the blown-up $\sigma$-model, one is able to make calculations which lead to quantities quite similar to the Duistermaat-Heckman polynomials [7].
To describe what a blown-up nonlinear $\sigma$-model looks like, it is necessary to recall the quotient constructions. One of them, which is called symplectic reduction, will be our basic tool. So we begin by giving a brief explanation.

Symplectic reduction is a procedure for obtaining quotient from a symplectic manifold $M$ with Hamiltonian $G$-action, where $G$ is an arbitrary Lie group. The action being Hamiltonian means simply that the interior product of the vector field $X^a$ generating the action with the symplectic 2-form $\omega$, is an exact 1-form:

$$i_{X^a}\omega = d\Phi^a,$$

where $\Phi^a$'s are scalar functions on $M$ the whole set of which is called the moment map. When the action is free of fixed points, one can go to the quotient of the level set of the moment map, i.e. $\Phi^{-1}(m)$ at level $m$, by the isotropy subgroup $G_m$ of the whole isometry group $G$, the resulting manifold $M'$ is a smooth symplectic manifold with reduced symplectic form, and with dimension $2\dim G_m$ smaller than $\dim M$. There exists a construction of symplectic quotient in terms of nonlinear $\sigma$-model by gauging the isometry corresponding to the Hamiltonian $G$-actions. The procedure is roughly as follows. Let $g_{ij}$ be the metric on $M$ which is compatible with the symplectic structure $\omega = \omega_{ij}dx^i \wedge dx^j$, it means that $g_{ij} = \omega^{kl}g_{kl}\omega_{ij}$. The action for the $\sigma$-model with target space $M$ is given by

$$S = \frac{1}{2} \int d^2\sigma g_{ij} \partial_{\mu}x^i \partial_{\mu}x^j.$$

Let $\xi_i^a(x)$ be the Killing vectors generating the isometry transformations, then,

$$\nabla_{(i}\xi_{j)\alpha} = 0,$$

the $\sigma$-model fields transform according to

$$\delta x^i = \lambda^a\xi^a_i(x).$$

It is possible to make to transformation local by introducing a gauge field $A^a_\mu$ which transforms as

$$\delta A^a_\mu = \partial_\mu \lambda^a(x) + f^a_{bc}A^b_\mu \lambda^c.$$
Consider the effective action of the form

\[ S'(x) = - \ln Z(x), \quad Z(x) = \int DA D\eta \exp\left\{ - \frac{1}{2} \int d^2\sigma g_{ij} D_\mu x^i D_\mu x^j + \eta_a \Phi^a(x) \right\} \]

where \( D_\mu x^i = \partial x^i - A^a_\mu \xi^i_a(x) \), \( \Phi^a \) is a component of the constraints specified by the moment map of the \( G \)-action, and \( \eta_a \) the Lagrangian multipliers. The effective action \( S' \) therefore describes a manifold \( M' \) which is the symplectic quotient of \( M \).

Although the expression for the \( S' \) above is known to be affiliated with difficulties associated to functional integrations, there are some cases where the tree level approximation seems to be exact as far as only target space geometry is concerned, these include the \( CP^n \) and \( G(n, N) \) (complex Grassmannian) \( \sigma \)-models.

If the \( G \)-action on \( M \) is not free, the naive symplectic reduction fails because the resulting quotient ceases to be a manifold, but rather a singular space. The mathematical studies of the singular symplectic reduction have revealed some interesting structures such as stratification [8], but global properties of the general singular symplectic reduction remain largely unclear. Quite recently, singular reduction has been employed in a beautiful work by Guillemin and Sternberg [9] in their formulating the notion of the symplectic cobordism. In a sense, the results of [9] suggest a role of singular symplectic reduction as a substitute for performing blowing up of singularities in a Hamiltonian \( G \)-space. In our previous paper [10] we have explained how the symplectic blowing up of Guillemin-Sternberg can indeed been realized in terms of the gauged nonlinear \( \sigma \)-models. The basic construction involves viewing the operation of symplectic blowing up as a combination of the local and global reductions. Let us review the main points in [10]. Recall the geometric content of symplectic blowing up [11]. Any point \( x_0 \) in a \( 2m + 2 \) dimensional symplectic manifold \( (M, \omega) \) has a small neighborhood \( U_0 \subset M \) which is isometric to \( C^{m+1} \) with standard linear symplectic form \( \omega_0 \). We cut open \( M \) along a solid ball \( B \) of radius \( \epsilon \) in \( U_0 \) and collapse the boundary \( \partial B \sim S^{2m+1} \) of \( B \) into a complex projective space \( (CP^m, \omega_1) \). On \( CP^m \) there is a canonical line bundle \( p : L \to CP^m \) which is constructed from simple incidence relation as subset of the product \( C^{m+1} \times CP^m \), i.e. \( L = \{(x, y) \in C^{m+1} \times CP^m | xy - yx = 0 \} \). the projection
of \( L \) onto the first factor \( \sigma : L \to C^{m+1} \) is a holomorphic map which has the following properties. \( \sigma \) restricted to the complement of the zero section of \( L \) is bi-holomorphic onto \( C^{m+1} - \{0\} \), and \( \sigma^{-1}(0) = CP^m \). By attaching \( \sigma^{-1}(0) = CP^m \) to \( M - B \), one obtains the blown-up symplectic manifold \( M' \) which is smoothly diffeomorphic to the connected sum \( M \# CP^m \) endowed with a well-defined symplectic form

\[
\omega' = \sigma^*(\omega_0) + \epsilon p^*(\omega_1).
\]

In our \( \sigma \)-model relaxation of symplectic blowing up, the process of cutting open \( M \) is replaced by performing the normal coordinate expansion at the fixed point \( x \in M \) of the quasi-free\footnote{A \( G \)-action is said quasi-free, if the stabilizer groups are either empty or the whole \( G \). This has been imposed mainly to simplify presentation, more general cases can in principle be handled similarly. However, the case of singular submanifold deserves another study which involves blowing up along submanifold and knowing its normal forms. The gauged \( \sigma \)-model realization of the later case requires auxilliary fields coming from the coadjoint orbit of \( G \).} \( G \)-action, and that of collapsing boundary of \( B \subset U_0 \) is replaced by gauging of an Abelian \( U(1) \) isometry of a linear \( \sigma \)-model resulted from normal coordinate expansion. This is the local reduction advocated to above. The global reduction turns out to rely on a use of symplectic diffeomorphism whose fixed point does not coincide with the fixed point of the Hamiltonian \( G \)-action. The role of this symplectic diffeomorphism is to glue the blow-up exceptional divisor back to the complement of the blown-up point in the nonlinear \( \sigma \)-model. Thus we arrive at the conclusion that the blown-up \( \sigma \)-model is a sum of two gauged nonlinear \( \sigma \)-models related by a symplectic diffeomorphism. The classical target space geometry so obtained is a connected sum \( M \# CP^m \), with a well-defined symplectic form. One may view the process as depicted in Figure 1.

Let us make the following remarks regarding the above result.

1. The parameter \( \epsilon \) is to be interpreted as volume of the exceptional divisors by Poincaré duality. When \( |\epsilon| \) is small, both \( +\epsilon \) and \( -\epsilon \) correspond to regular levels of the \( U(1) \) moment map upon which the symplectic reductions are performed. As \( \epsilon \) approaches zero form either sides, either...
volume of the exceptional divisor becomes infinitely small and finally the exceptional divisor disappears, or reversely the zero limit of the parameter $\epsilon$ signals a nontrivial exceptional divisor, representing the blowing-up mode in the corresponding $\sigma$-model.

2. It is interesting to explore the effects of the instanton associated with the exceptional divisor of the blowing up. One may interpret this blowing up instanton as a small perturbation with parameter $\epsilon$ of the instantons which exist before blowing up. The point is that, since $\epsilon$ can take positive and negative values, a smooth change of topology may occur in which instantons emerge or disappear at some points in the manifold. This has drastic implications for the quantum properties of the model, such as spontaneous symmetry breaking and the solvability of the model in large $N$. This kind of study is being carried out by the author [12].

In what follows we look at the properties of the blowing up operation in some special nonlinear $\sigma$-models.

_Toric $\sigma$-model_

A toric manifold associated with an integral or rational polyhedron can be obtained as a symplectic reduction of $C^N$ by the Hamiltonian action of the subtorus of $T_C^N = (C^*)^N$, at a regular level of the corresponding moment map (see [13] for an explanation). Obviously a toric $\sigma$-model (i.e. a $\sigma$-model with target space being a toric manifold) can be viewed as a suitable symplectic reduction of the linear $\sigma$-model on $C^N$. It is a nontrivial fact that some

Fig. 1. A connected sum resulted from blowing up.
Hamiltonian subtorus actions on the toric manifold can have fixed points when the image of these points are exactly the vertices of the convex polyhedron. Let us take the simplest example of $CP^2$, constructed as a toric manifold whose associated polyhedron is the standard 2-simplex, i.e. a triangle $\Delta \subset R^2$. If $e_1, e_2$ denote the basis vectors of $Z^2 \subset R^2$, which are two edge vectors of $\Delta$, we can form a fan $\Sigma$ whose 1-skeletons (edges of the 2-cones) are all of the form $tx_i, \ 0 \leq t < \infty, \ i = 1, 2, 3$, where $x_i = e_i, x_3 = -(e_1 + e_2)$ (see Figure 2). One can take $x_i$ to be the basis vectors in $Z^3$, thus there exists a natural map $Z^3 \rightarrow Z^2, \ x \mapsto z$ which induces the corresponding map $R^3 \rightarrow R^2$ and the quotient map $T^3 \rightarrow T^2 \rightarrow 0$ with kernel $S^1$. The realization of $CP^2$ as symplectic reduction of $C^3$ is carried out by reducing $C^3$ by the (smooth) Hamiltonian action of this $S^1$. From this construction it is obvious that $T^2 \subset T^3$ acts on $CP^2$ in a Hamiltonian fashion and the image of $CP^2$ under its moment map is exactly $\Delta$.

It is well-known that the nonlinear $\sigma$-model of $CP^2$ can be described by a gauged $\sigma$-model with the gauge field

$$A_\mu = \frac{i}{2} \sum_{i=1}^{3} z^i \partial_\mu \bar{z}^i - z^i \partial_\mu \bar{z}^i,$$

and the action of the form $\int D_z \bar{z}^i D_{\mu} \bar{z}^i, \ D_\mu = \partial_\mu - iA_\mu$. The integral of the symplectic form over a homology cycle in $CP^2$ equals a topological invariant $1/2\pi \int \epsilon_{\mu\nu} \partial_\mu A_\nu$ which is the first Chern number of the tangent bundle. The $CP^2 \sigma$-model has a global $SU(3)$ invariance, of which the maximal torus $T^2$ acts in the Hamiltonian fashion. We know that this $T^2$ action is not free at some points whose image under the moment map are the vertices of $\Delta$. This may cause serious problems in the quantum theory, even though it is harmless classically, as far as one does not perform the quotient (which would be a

Fig. 2. A convex polyhedron and its dual fan.
point in this case). A possible resolution is to blow up the fixed point on $CP^2$. Given the combinatoric data determining $\Sigma$ or $\Delta$, it is easy to do blowing up in toric $\sigma$-model. For this purpose let us observe that, the $CP^2$ $\sigma$-model is equivalent to the $SU(3)$ invariant chiral model defined by the action desity

$$S = \frac{1}{2} tr \partial_\mu \phi \partial_\mu \phi$$

where $\phi$ is a $3 \times 3$ Hermitian matrix of trace zero. Indeed, let the torus $T^3$ act on the space of $\phi$ matrices by assigning to the diagonal elements of $\phi$ by the following set of eigenvalues $(\lambda_1, \lambda_2, \lambda_3) = (\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$, this determines $\phi$ uniquely from the eigenvectors $z = (z_1, z_2, z_3)$ as follows

$$\phi_{ab} = \frac{\delta_{ab}}{3} - z_a \bar{z}_b, \quad |z|^2 = 1.$$ 

It can be proved that

$$S = \frac{1}{2} tr \partial_\mu \phi \partial_\mu \phi = \sum_a (\partial_\mu z^a - iA_\mu z^a)(\partial_\mu \bar{z}^a + iA_\mu \bar{z}^a)$$

where $A_\mu$ is as before. From the second last equation, it is clear that the diagonal elements of $\phi$ lie in the convex polyhedron $\Delta$ which is the image of the $T^2 \subset T^3$ moment map. Now recall that blowing up toric manifold at a fixed point just correspond to truncation of a vertex in $\Delta$ (see Figure 3). The dual fan is shown in Figure 4.

Fig. 3. The blow up as truncation of vertex of polyhedron.  
Fig. 4. A fan representing blown-up of $CP^2$.

Using the basic toric methods, one concludes that the $CP^2$ model blown up at a point is obtained by enlarging the rank of $\phi$ by one in such a way
that it becomes a block form with the $T^2 = S^1 \times S^1$ action on the last entry of $\text{diag}(\phi)$ twisted by a sufficiently large integer $a$ (with $\epsilon \sim 1/a$). The symplectic reductions which lead to the blown-up $CP^2$ is reminiscent of our general scenario, but is in this case much easier. The benefit of working with the toric convex data is that it becomes apparent how to identify the fixed points and the blow-up exceptional divisors (see Figure 5). The later are closely related to the important object, rational curves on $CP^2$. We refer to [12] for detailed account of the toric $\sigma$-models along with more results on instanton analysis associated to the holomorphic curves in toric manifolds. Our next example deals with holomorphic Kähler quotient rather than its equivalent symplectic analogue.

Fig. 5. Fixed points and holomorphic curves in the fan of blown-up $CP^2$.

$N=2$ Supersymmetric $\sigma$-model

In this case, there exists a general procedure [14, 1] of performing the $N = 2$ quotient by gauging the (holomorphic) isometries of the $N = 2$ superspace action of the form

$$S = \int D_+ D_- \tilde{D}_+ \tilde{D}_- K(\Phi, \bar{\Phi}, \Lambda, \bar{\Lambda}),$$

with arbitrary chiral and twisted chiral superfield multiplets $\Phi, \Lambda$. The gauged action takes the general form of a new Kähler potential $K'$ which is the original potential $K$ with $\Phi$ and $\Lambda$ minimally coupled to some gauge vector multiplet $V$, plus terms which are trivially gauge invariant, such as the Fayet-Iliopoulos terms which must be included when the isometry group contains $U(1)$ subgroups. In the same spirit as the bosonic $\sigma$-model and its symplectic reduction, we can carry out the symplectic blowing up using the general method of [13]. The basic ingredients are a superspace analogue of the normal coordinate expansion on the one hand, and the identification (and
the interpretation) of the blowing up parameter $\epsilon$ as the coupling constant in front of the Fayet-Iliopoulos terms, on the other hand. We will not describe these points here. But the picture of the blown-up $N = 2$ $\sigma$-model is clear: to each $N = 2$ supersymmetric $\sigma$ model, arising from gauging an appropriate holomorphic isometry group, at any (isolated) configuration which is the fixed point of a subgroup of the isometry group, one can associate a gauged version of the $N = 2$ linear $\sigma$-model. The resulting $N = 2$ model has a bosonic sector which looks exactly like the symplectic blowing up of the type described before.

We would like to make some observations concerning related issues of quantum cohomology and mirror symmetry which have attracted much attention recently.

It is believed that some $N = 2$ nonlinear $\sigma$-models admit equivalent Landau-Ginsburg descriptions. For the $N = 2$ Landau-Ginsburg model of the complex Grassmannian $G(m, m+n)$ [15], the Kähler form $X^{(1,1)}$ serves as perturbation of the ordinary cohomology ring. With the potential perturbed to the form

$$W_{m+n+1}^{m}(X^{(1,1)}) + (-1)^{m} \beta X^{(1,1)},$$

the corresponding quantum cohomology ring is

$$X^{m}Y^{n} = \beta,$$

where $\beta$ can be taken as the (exponential of the) volume of the exceptional divisor corresponding to an instanton. If the instanton comes from our blow up operation, its volume can be both positive and negative. We thus conclude that, with blow up modes included, some (derived) subring of the quantum cohomology ring becomes $\mathbb{Z}^{2}$-graded. It is an interesting question to find the physical degree which is responsible to this grading.

Finally, the origin of the mirror symmetry seems to be connected [10] to the Abelian duality symmetry mentioned at the beginning. If the blown-up $\sigma$-model indeed explains the origin of the duality symmetry, one would be a step closer to understanding the origin of the mirror symmetry. Of course much work must be done before any definite assertion can be made. In this regard, however, the following remark might be helpful. As Roan has shown
in [17] (see also recent work of [18]), some mirror pairs found in [19] admit interpretation as living over the dual lattices determining the corresponding manifolds. Our treatment of the toric example above suggests that a chain of chiral fields defined on the 2-dimensional lattice whose links coincide with some subset of the fan $\Sigma^{(1)} \subset \Sigma$, serves to provide an analogue of the mirror map for a pair of toric manifolds. Details will be reported elsewhere.

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References

[1] M. Rocek and E. Verlinde, Nucl. Phys. B373(1992)630.

[2] E. Witten, Nucl. Phys. B403(1993)185; P. Aspinwall, B. Green and D. Morrison, Phys. Lett 303B(1993)249.

[3] E. Witten, Commun. Math. Phys. 118(1988)411.

[4] A. Floer, Commun. Math. Phys. 120(1989)575.

[5] J.J. Duistermaat and G. Heckman, Invent. Math. 69(1982)259.

[6] V. Guillemin, E. Lerman and S. Sternberg, J. Geom. Phys. 5(1990)721; M. Audin, The Topology of Torus Actions on Symplectic Manifolds, Progress in Mathematics, Vol. 93, Birkhäuser-Verlag (Basel, 1991).

[7] H.B. Gao and H. Römer, "A Class of Toric $\sigma$-Models Determined by the Chiral Chain Fields", preprint to appear.

[8] R. Sjamaar and E. Lerman, Ann. Math. 134(1991)375.

[9] V. Guillemin and S. Sternberg, Invent. Math. 97(1989)485.

[10] H.B. Gao and H. Römer, prepint #, hep-th/9310037.
[11] M. Gromov, Partial Differential Relations, Springer-Verlag, (Berlin-Heidelberg, 1986), p. 340.

[12] H.B. Gao and H. Römer, "Instanton Analysis and the Blowing Up Modes in Toric $\sigma$-Models", preprint to appear.

[13] M. Audin, The Topology of Torus Actions on Symplectic Manifolds, Progress in Mathematics, Vol. 93, Birkhäuser-Verlag (Basel, 1991).

[14] C.M. Hull, A. Karlhede, U. Lindström and M. Rocek, Nucl. Phys. B266(1986)1.

[15] K. Intriligator, Mod. Phys. Lett. A6(1991)3543.

[16] C. Vafa, in Essays in Mirror Symmetry, ed. S.T. Yau, (Hong Kong, 1992).

[17] S.S. Roan, Int. J. Math. 2(1991)439.

[18] V.V. Batyrev, ”Dual Polyhedra and Mirror Symmetries for Calabi-Yau Hypersurfaces in Toric Variety”, preprint Uni-Essen, 1992.

[19] P. Candelas, X.C. de la Ossa, P.S. Green and L. Parkes, Nucl. Phys. B359(1991)21.
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