TWO FORMULAE FOR INVERSE KAZHDAN-LUSZTIG POLYNOMIALS IN $S_n$

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Abstract. Let $w_0$ denote the permutation $[n, n-1, \ldots, 2, 1]$. We give two new explicit formulæ for the Kazhdan-Lusztig polynomials $P_{w_0 w, w_0 x}$ in $S_n$ when $x$ is a maximal element in the singular locus of the Schubert variety $X_w$. To do this, we utilize a standard identity that relates $P_{x,w}$ and $P_{w_0 w, w_0 x}$.

1. Introduction

Kazhdan-Lusztig (KL) polynomials were introduced by Kazhdan and Lusztig in [12] in their study of the representations of Hecke algebras of Coxeter groups. Since then, these polynomials have been discovered to have many important interpretations in the context of Lie theory and Schubert varieties (see [3, 12, 13]). However, their combinatorial structure is far from clear even though numerous people have results in specific cases (see [2] for an overview of such results). In this paper we give explicit formulæ for the KL polynomials $P_{x,w}$ related to certain maximal singular points in the singular locus of the Schubert variety $X_w$. These singular points correspond to the first points where the KL polynomials are non-trivial. To state our results precisely, we first introduce the following four families of permutations:

Definition 1.

For $k, m \geq 1$, define

$$x_{k,m} = [k, \ldots, 1, k + m, \ldots, k + 1],$$

$$w_{k,m} = [k + m, k, \ldots, 2, k + m - 1, \ldots, k + 1].$$

For $k \geq 1$, define

$$y_{k,m} = [k, \ldots, 1, k + 2, k + 1, k + 2 + m, \ldots, k + 2 + 1],$$

$$v_{k,m} = [k + 2, k, \ldots, 2, k + m + 2, 1, k + 2 + (m - 1), \ldots, k + 2 + 1, k + 1].$$

Writing $w_0$ for the permutation $[n, n-1, \ldots, 1]$, our main results are the following two formulæ:

Theorem 2.

1. For $k, m \geq 1$,

$$P_{w_0 x_{k,m}, w_0 x_{k,m}} = \sum_{r=0}^{\min(k-1, m-1)} \binom{k-1}{r} \binom{m-1}{r} q^r.$$

2. For $k, m \geq 1$,

$$P_{w_0 y_{k,m}, w_0 y_{k,m}} = 1 + (k + m - 1)q.$$

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The pairs \((x_{k,m}, w_{k,m})\) and \((y_{k,m}, v_{k,m})\) correspond (using Theorem \([11]\)) to two of the three types of irreducible components of the singular loci of Schubert varieties in \(SL(n)/B\). While our combinatorial techniques do not easily extend to the third type, the analogue of Theorem \(\mathbb{3}\) for this third type can be found in \([15]\).

Remark 3. Let \(u, v\) be elements in an arbitrary Coxeter group and set \(c(u, v)\) to be the number of coatoms in the Bruhat interval \([u, v]\). Brenti shows in \([3]\) that the coefficient of \(q\) in \(P_{u,v}\) is bounded above by \(c(u, v) - 1\). The intervals \([w_0 w_{k,k}, w_0 x_{k,k}]\) afford a class of intervals in \(S_n\) for which the coefficient of \(q\) in \(P_{w_0 w_{k,k}, w_0 x_{k,k}}\) asymptotically approaches \(c(w_0 w_{k,k}, w_0 x_{k,k})\). This confirms the asymptotic tightness of Brenti’s bound.

Section \(\mathbb{2}\) contains the necessary preliminaries and Section \(\mathbb{3}\) contains our proof of Theorem \(\mathbb{2}\).

2. Preliminaries

We now introduce the necessary background on KL polynomials and \(S_n\). The reader is referred to \([1]\) for a more leisurely introduction to most of combinatorial material presented here. A good reference for Schubert varieties and the information about them encoded by KL polynomials is \([2]\).

2.1. The symmetric group. We will view elements of \(S_n\) as permutations on \([1, \ldots, n]\) with elements \(s_i\) of the generating set \(S = \{s_i\}_{i \in [1, \ldots, n-1]}\) associated with the adjacent transpositions \((i, i + 1)\). \(t_{i,j}\) will denote the transposition \((i, j)\). We have a one-line notation for a permutation \(w\) given by writing the image of \([1, \ldots, n]\) under the action of \(w\): \([w(1), w(2), \ldots, w(n)]\). The length function for \(S_n\) is given by

\[
l(w) = |\{1 \leq i < j \leq n : w(i) > w(j)\}|.
\]

We will denote the ordered pair of permutations \(x\) and \(w\) by \((x, w)\). Finally, let \(w_0 = [n, n - 1, \ldots, 1] \in S_n\) denote the element of maximal length in \(S_n\).

We say that \(w \in S_n\) is \(v\)-avoiding for \(v \in S_k\) if we cannot find \(1 \leq i_1 < \cdots < i_k \leq n\) with \(w(i_1), \ldots, w(i_k)\) in the same relative order as \(v(1), \ldots, v(k)\); i.e., no submatrix of \(\text{mat}(w)\) on rows \(i_1, \ldots, i_k\) and columns \(w(i_1), \ldots, w(i_k)\) is the permutation matrix of \(v\). There are many properties pertaining to \(S_n\) and Schubert varieties that can be characterized efficiently in terms of pattern avoidance (see, e.g., \([1, 3, 8, 10, 1\])).

For \(w \in S_n\) and \(1 \leq i_1 < \cdots < i_k \leq n\) for \(k \leq n\), define \(\text{fl}[w(i_1), w(i_2), \ldots, w(i_k)]\) to be the unique permutation \([v(1), \ldots, v(k)]\) \(\in S_k\) such that \(v(j) < v(k)\) if and only if \(w(i_j) < w(i_k)\).

We now introduce an important partial order on \(S_n\). The characterization we give in Definition \(\mathbb{4}\) is non-standard, and requires the following definition, but it is equivalent to more common descriptions such as the Tableau Criterion.

Definition 4. Let \(x, w \in S_n\), \(p, q \in \mathbb{Z}\). Define \(r_w(p, q) := |\{i \leq p : w(i) \geq q\}|\) and \(d_{x,w}(p, q) := r_w(p, q) - r_x(p, q)\).

Definition 5. We define the Bruhat partial order “\(\leq\)” on \(S_n\) by setting \(x \leq w\) if and only if \(d_{x,w}(p, q) \geq 0\) for all \(p, q\).

Lemma 6. If \(x \leq y \leq w\), then \(d_{x,w} - d_{y,w}\) is everywhere non-negative.
We can now introduce our pictorial version of the Bruhat order.

**Definition 7.** A Bruhat picture for \( x, w \in S_n \) is an overlay of some of the rows and columns of their permutation matrices that is augmented by shading the regions on which \( d_{x,w} \geq 1 \).

Figure 1 displays examples of this notation. Let mat(\( w \)) refer to the permutation matrix for \( w \). Entries of mat(\( x \)) (resp., mat(\( w \))) are denoted by black disks (resp., open circles). Positions corresponding to 1’s of both mat(\( x \)) and mat(\( w \)) are denoted by a black disk and a larger concentric circle.

![Figure 1](image)

Figure 1. 1) Bruhat picture for \( x = [3, 1, 5, 2, 4, 6] \), \( w = [6, 3, 4, 2, 5, 1] \). 2) and 3) give visualizations of the permutations defined in Definition 1.

Our proof of Theorem 2 is inductive and reduces the calculation of \( P_{x,w} \) to a related polynomial \( \tilde{P}_{x,\tilde{w}} \). We define \( \tilde{x} \) and \( \tilde{w} \) now:

**Definition 8.** Let
\[
\Delta(x, w) = \{ i : x(i) \neq w(i) \text{ or } d_{x,w}(i, x(i)) \neq 0 \}.
\]
For \( \Delta(x, w) = \{ d_1, d_2, \ldots, d_k \} \) with \( d_i < d_j \) for \( i < j \), set
\[
\tilde{x} = \text{fl}(x(d_1), x(d_2), \ldots, x(d_k)) \quad \text{and} \quad \tilde{w} = \text{fl}(w(d_1), w(d_2), \ldots, w(d_k)).
\]
Note that \( \tilde{x} \) and \( \tilde{w} \) are permutations in \( S_k \).

2.2. KL Polynomials. While Kazhdan and Lusztig define the KL polynomials for general Coxeter groups via the associated Hecke algebra, there is a purely combinatorial description which we now give for the case of \( S_n \). In order to give this definition succinctly, we let \( [q^k]P_{x,w} \) denote the coefficient of \( q^k \) in the polynomial \( P_{x,w} \), set
\[
\mu(x, w) = [q^{(l(w) - l(x) - 1)/2}]P_{x,w},
\]
and define \( c_s(x) = 1 \) if \( xs < x \); \( c_s(x) = 0 \) if \( xs > x \).

**Theorem 9 ([12]).** There is a unique set of polynomials \( \{ P_{x,w} \}_{x,w \in S_n} \) such that, for all \( x, w \in S_n \):
1. \( P_{w,w} = 1 \)
2. \( P_{x,x} = 0 \) when \( x \not\leq w \).
3. \( \deg(P_{x,w}) \leq (l(w) - l(x) - 1)/2 \) when \( x < w \)
4. For $s \in S$ with $ws < w$,

\[ P_{x,w} = q^{e_s(x)}P_{x,ws} + q^{1-e_s(x)}P_{x,z} - \sum_{z < z < ws} \mu(z, ws) q^{l(z)-l(w)}P_{x,z}. \]

The proof, while not difficult, is intricate and we refer the interested reader to the original paper \[12\] of Kazhdan and Lusztig or to the more detailed exposition in Humphreys \[11\]. We note that $\mu(x, w)$ is the coefficient of the highest possible power of $q$ in $P_{x,w}$.

The KL polynomials satisfy many properties that are not immediately apparent from the definition. We list these properties without proof below. Properties 1 and 2 are standard and are due to Kazhdan and Lusztig \[12\]. (The polynomials $P_{w_0z, w_0x}$ are referred to as “inverse” KL polynomials.) Property 3 is due to Lakshmibai and Sandhya \[14\] using results of Carrell \[7\]. Properties 4 and 5 can be found in \[4\].

**Theorem 10.**

1. For $s \in S$, $P_{x,w} = P_{x', w'}$ (resp., $P_{x,w} = P_{x', w'}$) if $ws < w$ (resp., $sw < w$).
2. $\sum_{x \leq z \leq w} (-1)^{l(z)+l(w)}P_{z,w}P_{w_0z, w_0x} = \delta_{x,w}$ ($\delta$ is the Kronecker delta).
3. $P_{x,w} = 1$ for all $x \leq w$ if and only if $w$ is 3412- and 4231-avoiding.
4. $P_{x,w} = P_{x,w}$.
5. $\deg(P_{x,w}) = 0$ if there do not exist $i < j < k < l$ such that

\[
(\text{fl}([x(i), x(j), x(k), x(l)]), \text{fl}([w(i), w(j), w(k), w(l)]))
\]

has a Bruhat picture of one of the two forms given in Figure 2.

![Figure 2](image-url)

**Figure 2. Requirements for $\deg(P_{x,w}) > 0$.**

We also have the following formulae for KL polynomials of certain irreducible components of the singular loci of Schubert varieties. The formulae are due to the author and Billey \[8\] and, independently, to both Manivel \[17\] and Cortez \[8\].

**Theorem 11.**

1. For $k, m \geq 1$,

\[ P_{x_k, w_k, m} = 1 + q + \cdots + q^{\min(k-1, m-1)} \]

and $P_{z, w_k, m} = 1$ for $x_k < z \leq w_k$.

2. For $k, m \geq 1$,

\[ P_{y_k, v_k, m} = 1 + q \]

and $P_{z, v_k, m} = 1$ for $y_k < z \leq v_k$.

The main ingredients in the proof of Theorem 3 are Theorems 1 and 10.2.
3. KL POLYNOMIALS $P_{w_0,w,w_0,x}$

Suppose $x \leq w$ such that $P_{z,w} = 1$ for all $x < z \leq w$. In this case, Property 2 of Theorem 10 simplifies to

$$P_{w_0,w,w_0,x} = (-1)^{l(x)+l(w)+1}P_{x,w} + \sum_{x < z < w} (-1)^{l(z)+l(w)+1}P_{w_0,z,w_0,x}.$$  

Before utilizing (1) to prove Theorem 2, we first prove two technical lemmas.

3.1. Two technical lemmas.

**Lemma 12.** For $k, m \geq 1$,

$$\sum_{a,b=0}^{a=k-1, b=m-1} (-1)^{a+b+1} \binom{k}{a} \binom{m}{b} \sum_{r=0}^{\min(k-a-1, m-b-1)} \binom{k-a-1}{r} \binom{m-b-1}{r} q^r = (-1)^{k+m+1} \sum_{r=0}^{\min(k-1, m-1)} q^r.$$  

**Proof.** It is convenient to sum $r$ from 0 to $\min(k-1, m-1)$ rather than from 0 to $\min(k-a-1, m-b-1)$. As $\binom{n}{d} = 0$ whenever $0 \leq n < d$, an extension of our summation range in this manner adds only terms equal to 0. So, we rewrite the left hand side of (2) as

$$\sum_{r=0}^{\min(k-1, m-1)} \sum_{a,b=0}^{a=k-1, b=m-1} (-1)^{a+b+1} \binom{k}{a} \binom{m}{b} \binom{k-a-1}{r} \binom{m-b-1}{r} q^r.$$  

It is a standard fact (see, e.g., [10, (5.25)]) that for $r \leq m-1$,

$$\sum_{b=0}^{m-1} (-1)^b \binom{m-b-1}{r} = (-1)^r m^{-r-1} = (-1)^r m^{-r-1}.$$  

Applying this identity twice to (3), we find that it equals

$$\sum_{r=0}^{\min(k-1, m-1)} q^r (-1)^{r+k-1+1} (-1)^{r+m-1} = \sum_{r=0}^{\min(k-1, m-1)} q^r (-1)^{k+m+1}.$$  

Our second lemma is similar in quality to the first, but requires an additional piece of notation. Define

$$f_{k,m}(a,b) = \binom{k}{a} \binom{m}{b} [(-1)^{a+b+1} (1 + (k + m - a - b - 1)q) + 2(-1)^{a+b}].$$  

**Lemma 13.** For $k, m \geq 1$,

$$\sum_{a=0}^{k-1} \sum_{b=0}^{m-1} f_{k,m}(a,b) = (-1)^{k+m}(1 + q).$$
Proof. We can rewrite the left side of (4) as

\[ \sum_{a=0}^{k} \sum_{b=0}^{m} f_{k,m}(a,b) - \sum_{a=0}^{k} f_{k,m}(a,m) - \sum_{b=0}^{m} f_{k,m}(k,b) + f_{k,m}(k,m). \] 

(5)

By definition, \( f_{k,m}(k,m) = (-1)^{k+m}(1 + q). \) Repeated application of the identities

\[ \sum_{i=0}^{n} \binom{n}{i}(-1)^i = 0 \] \( \text{and} \) \[ \sum_{i=0}^{n} i \binom{n}{i}(-1)^i = 0 \quad \text{(valid for} \ n > 0) \]

does not show that the remaining terms in (5) are zero.

3.2. Inverse KL polynomials.

Proof of Theorem 2. As it streamlines the argument, we will prove the theorem for any pair \((x, w)\) for which \((\tilde{x}, \tilde{w})\) equals \((x_{k,m}, w_{k,m})\) or \((y_{k,m}, v_{k,m})\) for some \(k, m\).

Part 1. We will argue by double induction on \(k\) and \(m\). First suppose that \((\tilde{x}, \tilde{w}) = (x_{k,m}, w_{k,m})\) where either \(k = 1\) or \(m = 1\). As can be seen by extrapolating from Figure 3.1, \(w_{k,m}\) is 3412- and 4231-avoiding in these cases. Hence, by Theorem 10, parts 3 and 4, \(P_{x_{k,m},w_{k,m}} = 1\). Now assume we have proven Part 1 for all pairs \((x', w')\) for which \((\tilde{x}', \tilde{w}') = (x_{r,s}, w_{r,s})\) with \(1 \leq r \leq k\), \(1 \leq s \leq m\) and \(r + s < k + m\).

Consider now the case of \(x, w\) with \((\tilde{x}, \tilde{w}) = (x_{k,m}, w_{k,m})\). There are two obvious simplifications we can make in the first term of (1). First, we know from Theorem 11 that \(P_{x,w} = 1 + q + \cdots + q^{\min(k-1,m-1)}\). Second, one can check that \(l(w) - l(x) = k + m - 1\). For the sum in (1), we start by noting that for any \(z\) with \(x < z < w\), we must have \((\tilde{x}, \tilde{z}) = (x_{r,s}, w_{r,s})\) for some \(1 \leq r \leq k\), \(1 \leq s \leq m\) and \(r + s < k + m\) (see Figure 3). We see that such a \(z\) can be chosen in \(\sum_{a,b} \binom{k}{a} \binom{m}{b}\) different ways. Fix \(r, s\) and set \(a = k - r\) and \(b = m - s\). It follows that \(a + b = l(w) - l(z)\). This lets us rewrite \((-1)^{l(z)+l(w)+1}\) as \((-1)^{a+b+1}\).
Utilizing the above two facts and the induction hypothesis, we arrive at

\[ P_{w_0w, w_0x} = (-1)^{k+m} \sum_{r=0}^{\min(k-1, m-1)} q^r + \sum_{r=0}^{\min(k-1, m-1)} \binom{k-1}{r} \binom{m-1}{r} q^r + \]

\[ \sum_{a=k-1}^{b=m-1} (-1)^{a+b+1} \binom{k}{a} \binom{m}{b} \sum_{r=0}^{\min(k-a-1, m-b-1)} \binom{k-a-1}{r} \binom{m-b-1}{r} q^r. \]

Note that the second sum in (6) is included to adjust for the fact that we allow \( a = b = 0 \) in the third term (i.e., the case of \( z = w \)). By Lemma 12, the first and third terms cancel. This proves Part 1.

**Part 2.** We will prove by double induction on \( k \) and \( m \). Define \( y_{0,m} = x_{2,m}, y_{k,0} = x_{k,2}, v_{0,m} = w_{2,m} \) and \( v_{k,0} = w_{k,2} \). The base cases of \( k = 0 \) or \( m = 0 \) (not both 0) reduce to Part 1 of this theorem. (In the ensuing induction, we do not use the case \( k = m = 0 \).) So for the remainder of the proof we assume \( k, m \geq 1 \). Now assume we have proven (2) for all pairs \( y', v' \) for which \((\tilde{y}, \tilde{v}) = (y_{r,s}, v_{r,s})\) with \( r \leq k, s \leq m \) and \( 0 < r + s < k + m \).

Consider now the case of \( y, v \) with \((\tilde{y}, \tilde{v}) = (y_{k,m}, v_{k,m})\). Again, we will simplify (6). We know by Theorem 11 that \( P_{z,v} = 1 + q \). Also, one can check that \( l(v) - l(y) = k + m + 1 \). These two facts let us write the first term in (6) as \((-1)^{k+m}(1 + q)\).

We now categorize the \( z \) for which \( y < z < v \) and examine how they contribute to

\[ \sum_{a=k-1}^{b=m-1} (-1)^{a+b+1} \binom{k}{a} \binom{m}{b} \sum_{r=0}^{\min(k-a-1, m-b-1)} \binom{k-a-1}{r} \binom{m-b-1}{r} q^r. \]

Note that by Theorem 11, \( P_{z,v} = 1 \) for each of these \( z \). Such \( z \) fall into four categories. See Figure 4 for examples of the different types.

1. For any \( 0 \leq a < k \) and \( 0 \leq b < m \), there are \( \binom{k}{a} \binom{m}{b} \) different permutations \( z \) with \( y < z \leq v \) and \((\tilde{y}, \tilde{z}) = (y_{k-a,m-b}, v_{k-a,m-b})\). We know by the induction hypothesis that \( P_{y_{0a}z, w_{0b}j} = 1 + (k+m-a-b-1)q \) for such \( z \). Furthermore, it is easily checked that \((-1)^{l(z) + l(v)} = (-1)^{a+b} \). We can write the contribution of these permutations \( z \) succinctly in terms of \( f_{k,m}(a,b) \). However, as we are only interested in those \( z < v \), we add in a corrective term to account for the fact that we allow \( a = b = 0 \) in our sum:

\[ \sum_{a=k-1}^{b=m-1} (-1)^{a+b+1} \binom{k}{a} \binom{m}{b} \sum_{r=0}^{\min(k-a-1, m-b-1)} \binom{k-a-1}{r} \binom{m-b-1}{r} q^r. \]
Incorporating the above knowledge into (1), we see that
\[
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\]

Similarly, the permutations
\[
3. \text{There are} z
\]

For each
\[
P_{k,1 + (w)} = \sum_{a,b=0}^{a=k-1 \atop b=m-1} \binom{k}{a} \binom{m}{b} \left[ (-1)^{a+b+1} (1 + (k + m - a - b - 1)q) \right].
\]

2. For each \( z \) as in the previous case, \( zt_{i,l} \) and \( zt_{j,k} \) also lie strictly below \( v \) and above \( y \) (see Figure 4). By Theorem 10, parts 3 and 5, we know that \( P_{w_0,zt_{i,l},w_0y} = P_{w_0,zt_{j,k},w_0y} = 1 \). Since \( l(zt_{i,l}) + l(w) = l(zt_{j,k}) + l(v) = l(z) + l(v) + 1 \), we get a total contribution of:
\[
\sum_{a,b=0}^{a=k-1 \atop b=m-1} 2(-1)^{a+b+1}.
\]

Note that we don’t correct for \( a = b = 0 \) as \( vt_{i,l}, vt_{j,k} < v \).

3. There are \( \binom{k}{a} \binom{m}{b} \) permutations \( z \) for which \( (y, z) = (y_{2-a, m-b}, v_{2-a, m-b}) \) for some \( a \) and \( b \) with \( 0 \leq a \leq 1 \) and \( 0 < b < m \). The cumulative contribution of these permutations \( z \) to the sum in (3) can be determined using Theorem 10.2.

In particular, permutations of this form are precisely those lying in the interval \( y < z \leq v' \) where \( v' = wt_{12}t_{23} \cdots t_{k-1,k}t_{k,k+2} \). We know from Theorem 10.2 that
\[
\sum_{y \leq z \leq v'} (-1)^{l(z)} P_{z,v'} P_{w_0z,w_0y} = 0.
\]

By Theorem 10.3, \( P_{z,v'} = 1 \) for \( y < z \leq v' \) and, by the first part of this theorem, \( P_{y,v'} = 1 + q \). Hence, we see that these permutations contribute
\[
(-1)^{l(v)+1} \sum_{y < z \leq v'} (-1)^{l(z)} P_{w_0z,w_0y} = (-1)^{l(v)+1} (-1)^{l(y)+1} (1 + q)= (-1)^{k+m+1} (1 + q).
\]

4. Similarly, the permutations \( z \) for which \( (y, z) = (y_{k-a,2-b}, v_{k-a,2-b}) \) also cumulatively contribute \( (-1)^{k+m+1} (1 + q) \).

Incorporating the above knowledge into (4), we see that
\[
\begin{align*}
P_{w_0v,w_0y} &= (-1)^{k+m} P_{y,v} + \text{(Type 1)} + \text{(Type 2)} + \text{(Type 3)} + \text{(Type 4)} \\
&= (-1)^{k+m} (1 + q) + \\
&\quad \left( 1 + (k + m - 1)q + \sum_{a,b=0}^{a=k-1 \atop b=m-1} f_{k,m}(a,b) \right) + \\
&\quad (-1)^{k+m+1} (1 + q) + (-1)^{k+m+1} (1 + q) \\
&= 1 + (k + m - 1)q + (-1)^{k+m+1} (1 + q) + \sum_{a=0}^{k-1} \sum_{b=0}^{m-1} f_{k,m}(a,b) \\
&= 1 + (k + m - 1)q.
\end{align*}
\]
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