BINOMIAL EDGE IDEALS OF COGRAPHS

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Abstract. We determine the Castelnuovo–Mumford regularity of binomial edge ideals of complement reducible graphs (cographs). On $n$ vertices the maximum regularity is essentially $2n/3$. Independently of the number of vertices, we also bound the regularity by graph theoretic invariants. Finally, we construct a family of counterexamples to a recent conjecture of Hibi and Matsuda.

1. Introduction

Let $G = ([n], E)$ be a simple undirected graph on the vertex set $[n] = \{1, \ldots, n\}$. Let $X = (x_1 \cdots x_n)$ be a generic $2 \times n$ matrix and $S = k[x_1 \cdots x_n]$ the polynomial ring whose indeterminates are the entries of $X$ and with coefficients in a field $k$. The binomial edge ideal is $J_G = \langle x_i y_j - y_i x_j : \{i, j\} \in E \rangle \subseteq S$, the ideal of $2 \times 2$ minors whose columns are indexed by the edges of the graph. Since their inception in [5, 12], connecting combinatorial properties of $G$ with algebraic properties of $J_G$ or $S/J_G$ has been a popular activity. Particular attention has been paid to the free resolution of $S/J_G$ as a standard $\mathbb{N}$-graded $S$-module [3, 9]. The data of a minimal free resolution is encoded in its graded Betti numbers $\beta_{i,j}(S/J_G) = \dim_k \text{Tor}^i(S/J_G, k)_j$ and the Castelnuovo–Mumford regularity is $\text{reg}(S/J_G) = \max\{j - i : \beta_{i,j}(S/J_G) \neq 0\}$. It is a complexity measure as low regularity implies favorable properties like vanishing of local cohomology. Binomial edge ideals have square-free initial ideals by [5, Theorem 2.1] and, using [1], this implies that the extremal Betti numbers and regularity can also be derived from those initial ideals. In this paper we rely on recursive constructions of graphs rather than Gröbner deformations.

Our starting point are the following bounds due to Matsuda and Murai [11].

Theorem 1.1. Let $\ell$ be the maximum length of an induced path in a graph $G$. Then

$$\ell \leq \text{reg}(S/J_G) \leq n - 1.$$ 

One aim of the second author’s MSc thesis was to investigate families of graphs for which the lower bound is constant. We picked the family of graphs with no induced path of length 3. These are the complement reducible graphs (cographs). They have

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been characterized in [2, Theorem 2] as graphs such that for every connected induced subgraph with at least two vertices, the complement of that subgraph is disconnected. Cographs are hereditary in the sense that every induced subgraph of a cograph is a cograph [2, Lemma 1]. The graphs $G$ for which $\text{reg}(S/J_G) \leq 2$ are all cographs by the characterization in [14, Theorem 3.2].

One can quickly find that among (connected) cographs arbitrarily high regularity is possible (Corollary 2.2), but the upper bound becomes stricter than that in Theorem 1.1. Our Theorem 2.7 shows that for cographs the regularity is essentially bounded by $2n/3$. The experimental results and also the proof methods leading to the $2n/3$ bound lead us to study regularity bounds in terms of other graph invariants. Theorem 2.11 bounds $\text{reg}(S/J_G)$ by the independence number $\alpha(G)$, the number of maximal independent sets $s(G)$, and the number of maximal cliques $c(G)$. Whether $c(G)$ bounds the regularity of $S/J_G$ in general is an open question [13, p. 12]. Interestingly, for cographs the mentioned invariants are computable in linear time, while in general they are hard to compute.

The investigations for this paper started with a large scale experiment in which we tabulated properties of binomial edge ideals and the corresponding graphs for many small graphs. To this end we developed a database of algebraic properties of $S/J_G$ and the necessary tools to extend the database. Our source code is available at

https://github.com/kruesemann/graph_ideals.

In the course of this work we found a counterexample to a recent conjecture of Hibi and Matsuda. We present this in Section 3. This last section leaves the class of cographs.

Notation. By the regularity of $G$ we mean $\text{reg}(S/J_G)$. The complement of a graph is a graph on the same vertex set, but with the edge set $E(G) = \{(i,j) : i \neq j, (i,j) \notin E(G)\}$. The path of length $\ell - 1$ has $\ell$ vertices and is denoted by $P_\ell$.

2. Regularity for cographs

Our first aim is to show that $\text{reg}(S/J_G)$ can take arbitrarily large values, even if $G$ is restricted to connected cographs and the lower bound of Theorem 1.1 does not apply. If one allows disconnected graphs, this follows from the simple observation that regularity is additive under the disjoint union $G \sqcup H$ of two graphs:

$$\text{reg}(S/J_{G \sqcup H}) = \text{reg}(S/J_G) + \text{reg}(S/J_H).$$

See [3, proof of Theorem 2.2]. To see that arbitrary regularity is possible for connected cographs, we employ the join of two simple undirected graphs $G$ and $H$ which is

$$G \ast H = (V(G) \sqcup V(H), E(G) \cup E(H) \cup \{(v,w) : v \in V(G), w \in V(H)\}).$$

Here and in the following, $V(G)$ and $E(G)$ denote, respectively, the vertex set and the edge set of an undirected simple graph $G$. The join also behaves nicely with regularity as shown by Kiani and Madani [14, Theorem 2.1].
Theorem 2.1. Let $G$ and $H$ be simple, undirected graphs and not both complete. Then
\[ \text{reg}(S/J_{G*H}) = \max\{\text{reg}(S/J_G), \text{reg}(S/J_H), 2\}. \]

The join of two cographs is a cograph as the path $P_4$ of length 3 is not a join itself. It follows that the regularity can be made arbitrarily high by forming cones, that is, joins with a single vertex graph.

Corollary 2.2. For any $r \geq 1$ there is a connected cograph with $\text{reg}(S/J_G) = r$.

Proof. Let $\text{cone}(G)$ be the join of a graph $G$ with a single vertex. Theorem 2.1 implies that, unless $G$ is complete, $\text{reg}(S/J_{\text{cone}(G)}) = \max\{\text{reg}(S/J_G), 2\}$. Now take a cograph $G$ with desired regularity. Then $\text{cone}(G)$ is a connected cograph with the same or higher regularity. \qed

With the lower bound settled, we aim for a stricter upper bound on the regularity of cographs. For this we employ the original definition of complement reducible graphs: they are constructed recursively by taking complements and disjoint unions of cographs, starting from the single-vertex graph. As a result, there is an unusual one-to-one-relationship between connected and disconnected cographs of the same order.

Lemma 2.3. Let $G$ be a cograph with at least 2 vertices. Then $G$ is connected if and only if $\overline{G}$ is disconnected.

Proof. Let $\overline{G}$ be disconnected and $v, w \in V(G)$. If $v$ and $w$ are in different connected components of $\overline{G}$, then $\{v, w\} \in E(G)$, so there is a path from $v$ to $w$ in $G$. If $v$ and $w$ are in the same connected component of $\overline{G}$, then there exists another vertex $u \in V(G)$ in a different connected component of $\overline{G}$. In particular, $\{v, u\}, \{w, u\} \not\in E(\overline{G})$ and so $\{v, u\}, \{w, u\} \in E(G)$, so $(v, u, w)$ is a path from $v$ to $w$ in $G$. Thus $G$ is connected.

The other implication follows from the stronger fact, that every induced subgraph of a cograph with more than one vertex has disconnected complement [2, Theorem 2]. \qed

In Lemma 2.3, the implication $\overline{G}$ disconnected $\Rightarrow$ $G$ connected holds for any graph, not just cographs. In the same generality, the join of graphs can be expressed using complement and disjoint union.

Lemma 2.4. Let $G_1$ and $G_2$ be simple undirected graphs. Then $G_1 * G_2 = \overline{G_1 \sqcup G_2}$.

Proof. Let $V$ denote the common vertex set of both graphs. Let $e = \{v, w\}$ where $v, w \in V$ are arbitrary vertices. If $e \not\subseteq V(G_i)$ for $i \in \{1, 2\}$, then $e$ is an edge in both $\overline{G_1 \sqcup G_2}$ and $G_1 * G_2$. If $e \subseteq V(G_i)$ for one $i \in \{1, 2\}$, then $e \in E(\overline{G_1 \sqcup G_2})$ if and only if $e$ is an edge in $G_i$, which is the case if and only if $e \in E(G_1 * G_2)$. \qed

Lemma 2.5. A connected cograph $G$ is the join of induced subgraphs $G_1, \ldots, G_m$, which are exactly the complements of the connected components of $\overline{G}$. 
Proof. By Lemma 2.3, $\overline{G}$ is disconnected. Let $\overline{G_1}, \ldots, \overline{G_m}$ be the connected components of $\overline{G}$ and $G_1, \ldots, G_m$ their complements. Then, by Lemma 2.4, $G = \overline{G_1} \sqcup \ldots \sqcup \overline{G_m} = G_1 \ast \ldots \ast G_m$ as both operations $\sqcup$ and $\ast$ are associative. \( \square \)

Using Lemma 2.5, we can assume that any connected cograph $G$ is written as a join of the complements of the connected components of its complement. Since any induced subgraph of a cograph is a cograph, the $G_i$ in the lemma are cographs too and as complements of connected cographs they are disconnected or have only one vertex.

**Proposition 2.6.** Let $G$ be a connected cograph that is not complete. Then

$$\operatorname{reg}(S/ J_G) = \max(\{2\} \cup \{\operatorname{reg}(S/ J_{G_i}) : i \in [m]\}),$$

where $G = G_1 \ast \ldots \ast G_m$ as in Lemma 2.5 and the $G_i$ are cographs that each are either disconnected or single vertices.

**Proof.** Since $G$ is not complete, the $G_i$ cannot all be complete, so the equations follows from Lemma 2.5 and Theorem 2.1. \( \square \)

We now have all the ingredients for a recursive computation of $\operatorname{reg}(S/ J_G)$ for any cograph $G$. In the disconnected case, add the regularities of all connected components. In the connected case, compute the maximum regularity of the complements $G_i$ of the connected components $\overline{G_i}$ of $\overline{G}$. We use this to bound the maximum regularity.

**Theorem 2.7.** Let $G$ be a cograph on $3k - a$ vertices, with $k \in \mathbb{N}$ and $a \in \{0, 1, 2\}$. Then

$$\operatorname{reg}(S/ J_G) \leq 2k - a.$$ 

If $G$ is connected, $k > 1$, and $a \in \{0, 1\}$, then $\operatorname{reg}(S/ J_G) \leq 2k - a - 1$.

**Proof.** The proof is by induction over $k$. The cographs with at most 3 vertices are $K_1$, $K_2 = P_1$, $\overline{K_2}$, $K_3$, $\overline{K_3}$, $P_2$ and $\overline{P_2}$. Then $\operatorname{reg}(S/ J_G) \leq 2 - a = \operatorname{reg}(S/ J_{P_3 - a})$.

Now let $k > 1$. By Proposition 2.6, if $G$ is connected, it either has regularity 2 or there is a smaller, disconnected cograph with the same regularity. So it can be assumed that $G$ is disconnected. Let $H$ be a connected component of $G$ with $3k_H - a_H$ vertices and $H' = G \setminus H$ have $3k_{H'} - a_{H'}$ vertices, where $k_H, k_{H'} \in \mathbb{N}$ and $a_H, a_{H'} \in \{0, 1, 2\}$. Then both $H$ and $H'$ have fewer vertices than $G$ and

$$k_H + k_{H'} = \begin{cases} 
  k & \text{if } a_H + a_{H'} = a, \\
  k + 1 & \text{if } a_H + a_{H'} = a + 3.
\end{cases}$$

By induction $\operatorname{reg}(S/ J_H) \leq 2k_H - a_H$ and $\operatorname{reg}(S/ J_{H'}) \leq 2k_{H'} - a_{H'}$, and

$$\operatorname{reg}(S/ J_G) \leq 2k_H - a_H + 2k_{H'} - a_{H'}$$

$$= \begin{cases} 
  2k - a & \text{if } a_H + a_{H'} = a, \\
  2k - a - 1 & \text{if } a_H + a_{H'} = a + 3
\end{cases}$$

$$\leq 2k - a.$$
If $G$ is connected and $k > 1$, the regularity is either 2 or at most that of a disconnected cograph with one vertex fewer. So if $G$ has $3k - a$ vertices, with $a \in \{0, 1\}$, then $\text{reg}(S/J_G) \leq 2k - a - 1$. If $G$ has $3k - 2$ vertices, then $\text{reg}(S/J_G) \leq 2(k - 1) = 2k - 2$. □

Looking at disjoint unions of paths on 2 and 3 vertices and at cones over these graphs, it is easy to see that the bounds in Theorem 2.7 can be realized. Even more, in two of three cases these are the only cographs of maximum regularity.

**Theorem 2.8.** Let $G$ be a cograph with $3k - a$ vertices, where $k \in \mathbb{N}$ and $a \in \{0, 1\}$. Then $\text{reg}(S/J_G) = 2k - a$ if and only if $G$ is a disjoint union of $P_3$ and at most one $P_2$.

*Proof.* Since the regularity is additive under disjoint union of graphs, a disjoint union of $P_3$ and at most one $P_2$ has maximum regularity $2k - a$.

Now consider an arbitrary cograph $G$. By Theorem 2.7, there is nothing to prove if $G$ is connected, so we assume it is disconnected. We first show that any connected component has at most 3 vertices. To this end, let $H$ be a connected component of $G$ with $3k_H - a_H$ vertices, $k_H > 1$, and $a_H \in \{0, 1, 2\}$. Let $H' = G \setminus H$ have $3k_{H'} - a_{H'}$ vertices with $a_{H'} \in \{0, 1, 2\}$. Then

$$k_H + k_{H'} = \begin{cases} k & \text{if } a_H + a_{H'} = a, \\ k + 1 & \text{if } a_H + a_{H'} = a + 3. \end{cases}$$

If $a_H \neq 2$, then, by Theorem 2.7,

$$\text{reg}(S/J_G) \leq 2k_H - a_H - 1 + 2k_{H'} - a_{H'}$$

$$= \begin{cases} 2k - a - 1 & \text{if } a_H + a_{H'} = a, \\ 2k - a - 2 & \text{if } a_H + a_{H'} = a + 3 \end{cases} < 2k - a.$$  

If otherwise $a_H = 2$, then since $a \in \{0, 1\}$, $G$ must have another connected component $H'$ with $3k_{H'} - a_{H'}$ vertices and $a_{H'} \in \{1, 2\}$. Let $H'' = G \setminus (H \cup H')$ with $3k_{H''} - a_{H''}$ vertices, where $k_{H''} \in \mathbb{N}_0$ and $a_{H''} \in \{0, 1, 2\}$. Then

$$k_H + k_{H'} + k_{H''} = \begin{cases} k + 1 & \text{if } a_H + a_{H'} + a_{H''} = a + 3, \\ k + 2 & \text{if } a_H + a_{H'} + a_{H''} = a + 6. \end{cases}$$

Therefore Theorem 2.7 implies

$$\text{reg}(S/J_G) \leq 2k_H - a_H + 2k_{H'} - a_{H'} + 2k_{H''} - a_{H''}$$

$$= \begin{cases} 2k - a - 1 & \text{if } a_H + a_{H'} + a_{H''} = a + 3, \\ 2k - a - 2 & \text{if } a_H + a_{H'} + a_{H''} = a + 6 \end{cases} < 2k - a.$$
We have thus shown that if $G$ has a connected component with more than 3 vertices, it cannot have maximum regularity. Since the only cographs of maximum regularity with 3 and 2 vertices are $P_3$ and $P_2$ respectively, it follows that if $G$ has maximum regularity, then it is a disjoint union of 2-paths $P_3$, single edges $P_2$ and isolated vertices. We now analyze these cases separately for two possible values of $a$.

Suppose first that $a = 0$. If $G$ has a connected component $H$ with one or two vertices, that is $3 - a_H$ vertices and $a_H \in \{1, 2\}$, then $G$ must have another connected component $H'$ with $3 - a_{H'}$ vertices, where $a_{H'} \in \{1, 2\}$. With a similar computation as above we find that $\text{reg}(S / J_G) < 2k$. Therefore if $G$ has $3k$ vertices and maximal regularity, each connected component must have exactly 3 vertices and be equal to $P_3$.

Finally, consider the case $a = 1$. If $G$ has an isolated vertex, then $G$ must have another connected component with fewer than 3 vertices. Then $G$ cannot have maximal regularity as isolated vertices contribute no regularity and a disjoint union of an edge and a vertex has regularity 1. If there are two isolated edges, then the subgraph on these 4 vertices contributes regularity only 2 and thus $G$ cannot have maximal regularity.

\[\square\]

**Remark 2.9.** The remaining graphs with $3k - 2$ vertices and maximal regularity do not have a simple characterization. The class contains cones over disjoint unions of 2-paths as well as other types of joins and disjoint unions of joins, paths, and isolated vertices.

Combining the join decomposition from Lemma 2.5 with Proposition 2.6 and Theorem 2.7 shows that cones maximize regularity among connected cographs.

**Corollary 2.10.** Let $G$ be a connected cograph with maximum regularity among connected cographs on $3k - a + 1$ vertices with $k > 1$ and $a \in \{0, 1\}$. Then $G$ is a cone.

In both Theorem 2.7 and Corollary 2.10 the maximizer is unique.

The results so far give a fairly clear picture of the regularity of cographs. Nevertheless, if the exact structure of a cograph is unknown, it can be useful to bound regularity using graph-theoretic invariants. We consider here $\alpha(G)$ – the size of the largest independent set, $s(G)$ – the number of maximal independent sets in $G$, and $c(G)$ – the number of maximal cliques of $G$. The recursive construction of cographs yields bounds because these invariants satisfy simple formulas under disjoint union and join:

\[(i)\quad s(G \sqcup H) = s(G) s(H),\]

\[(ii)\quad s(G \star H) = s(G) + s(H),\]

\[(iii)\quad \alpha(G \sqcup H) = \alpha(G) + \alpha(H),\]

\[(iv)\quad \alpha(G \star H) = \max\{\alpha(G), \alpha(H)\} .\]

We find the following bounds which are independent of the number of vertices and give an affirmative answer to the question on p. 12 of [13].
Theorem 2.11. Let $G$ be a cograph. Then $\operatorname{reg}(S/J_G) \leq \min\{c(G), s(G), \alpha(G)\}$.

Proof. We only need to show the bounds for $s(G)$ and $\alpha(G)$ as $\alpha(G) \leq c(G)$. This holds because two vertices in an independent set cannot be in the same clique.

The proof is by induction on the number of vertices of $G$. An isolated vertex has exactly one maximal independent set with one vertex and $\operatorname{reg}(S/J_{K_1}) = 0$, so the statement holds. Now let $G$ be any cograph. If $G$ is connected, it is the join of smaller cographs $G_1, \ldots, G_m$ and by induction

$$\operatorname{reg}(S/J_{G_i}) \leq \min\{s(G_i), \alpha(G_i)\} \quad \text{for all } i = 1, \ldots, m.$$  

If $G$ is complete, both inequalities are trivial. In the other case, it follows by Theorem 2.1 and (ii) that

$$\operatorname{reg}(S/J_G) = \max\{2\} \cup \{\operatorname{reg}(S/J_{G_i}) : i = 1, \ldots, m\} \leq \sum_{i=1}^m s(G_i) = s(G)$$  

and, since $\alpha(G) = 1$ if and only if $G$ is complete, (iv) leads to

$$\operatorname{reg}(S/J_G) = \max\{2\} \cup \{\operatorname{reg}(S/J_{G_i}) : i = 1, \ldots, m\} \leq \max\{\alpha(G_i) : i = 1, \ldots, m\} = \alpha(G).$$  

If $G$ is disconnected, it has connected components $G_1, \ldots, G_m$ which by induction satisfy

$$\operatorname{reg}(S/J_{G_i}) \leq \min\{s(G_i), \alpha(G_i)\} \quad \text{for all } i = 1, \ldots, m.$$  

In this case, since $G$ is the disjoint union of its connected components, and since regularity is additive, with (i) we have

$$\operatorname{reg}(S/J_G) = \sum_i \operatorname{reg}(S/J_{G_i}) \leq \prod_i s(G_i) = s(G).$$  

Finally, by (iii) we have

$$\operatorname{reg}(S/J_G) = \sum_i \operatorname{reg}(S/J_{G_i}) \leq \sum_i \alpha(G_i) = \alpha(G). \quad \square$$  

The method of bounding by $s(G)$ seems coarse as the maximum and sum over a set of integers are, respectively, replaced by the sum and the product over those integers. Nevertheless $s(G)$ can be a good bound as discussed in Remark 2.14.

Our last bound uses the maximum vertex degree $\delta(G)$ of a connected cograph $G$.

Proposition 2.12. Let $G$ be a connected cograph. Then

$$\operatorname{reg}(S/J_G) \leq \delta(G) = \max\{\delta(v) : v \in V(G)\}.$$  

Proof. If $G = K_n$ is complete, then $\operatorname{reg}(S/J_G) \leq 1$, $\delta(G) = n - 1$, and the inequality holds. If $G$ is not complete but connected, is is the join of induced subgraphs
Let $n_{\text{max}} = \max\{n_1, \ldots, n_m\}$. Then $\text{reg}(S/J_G) \leq n_{\text{max}}$, since $G$ is not complete and thus $n_{\text{max}} \geq 2$. Since $G$ is a join of the $G_i$. The maximum vertex degree satisfies $n_{\text{max}} \leq \max\{\delta(v) : v \in V(G)\}$ and we conclude. \hfill \Box

**Remark 2.13.** The bound in Proposition 2.12 does not give anything new for cone graphs since in this case it agrees with Theorem 1.1.

**Remark 2.14.** One can ask if one of the bounds in this section is generally preferable over the other bounds. Table 1 shows that any bound can beat any other bound, with the exception of $\alpha(G) \leq c(G)$. On the other hand, if one asks for the best bound, it can be confirmed that among the 2341 cographs in our database, for 505 $s(G)$ is strictly the best bound and for 724 $\alpha(G)$ is strictly the best bound. No other bound is ever strictly the best and for all remaining graphs there is a tie for the best bound.

| order bound | order bound | $c(G)$ | $s(G)$ | $\alpha(G)$ | max deg |
|-------------|-------------|--------|--------|-------------|---------|
| $c(G)$      | 0           | 968    | 968    | 146         | 1090    |
| $s(G)$      | 918         | 0      | 1049   | 0           | 724     |
| $\alpha(G)$ | 920         | 1050   | 0      | 514         | 837     |
| max deg     | 1830        | 1139   | 1522   | 0           | 1150    |

Table 1. Comparisons of five regularity bounds for all 2341 cographs in our database. The numbers given are the numbers of cographs for which the bound in the left-most column is strictly better than the bound in the top row respectively. ‘Order bound’ stands for the bound in Theorem 2.7, ‘max deg’ denotes the maximum vertex degree in Proposition 2.12. Comparisons with ‘max deg’ are made only for the 1171 connected cographs.

**Remark 2.15.** The questions about regularity in this paper can also be asked about the regularity of $S/I_G$ where $I_G$ is the **parity binomial edge ideal** of [8]. Using our database we observed the following inequality, slightly weaker than Theorem 1.1,

$$\ell \leq \text{reg}(S/I_G) \leq n.$$

Based on our computations we conjecture that the maximum regularity is achieved exactly for disjoint unions of odd cycles. The free resolutions of parity binomial edge ideals contain many interesting patterns that remain to be investigated. Already explaining the free resolution of $S/I_{K_n}$ is open. We conjecture that $\text{reg}(S/I_{K_n}) = 3$. 
3. Regularity versus \( h \)-polynomials

As a standard graded \( k \)-algebra, the Hilbert series of \( S/\mathcal{J}_G \) takes the form \( \frac{h_G(t)}{(1-t)^d} \) where \( d \) is the Krull dimension. The numerator \( h_G \) is known as the \( h \)-polynomial. In [6, Conjecture 0.1] it is conjectured that for binomial edge ideals its degree bounds the regularity from above. We found a minimal counterexample on 8 vertices:

**Example 3.1.** Let \( G \) be the graph in Figure 1, that is the graph on the vertex set \( \{1, \ldots, 8\} \) with edges \( \{1, 8\}, \{2, 6\}, \{3, 7\}, \{3, 8\}, \{4, 5\}, \{4, 8\}, \{5, 6\}, \{5, 7\}, \{6, 7\}, \{6, 8\}, \{7, 8\} \). Then \( \text{reg}(S/\mathcal{J}_G) = 4 \) and \( \deg(h_G) = 3 \).

![Figure 1](image)

**Figure 1.** A graph with \( \text{reg}(S/\mathcal{J}_G) > \deg(h_G) \).

At the time of writing, our database contains 39 counterexamples and none shows a difference greater than 1 between \( \text{reg}(S/\mathcal{J}_G) \) and \( \deg(h_G) \). However, gluing two copies of the counterexample in Figure 1 at vertex 1 yields a graph \( G \) (visible in Figure 2) which satisfies \( \text{reg}(S/\mathcal{J}_G) = 8 \) and \( \deg(h_G) = 6 \). We now show that the difference can be made arbitrarily large. To this end we employ the following two theorems that explain the behaviour of the regularity and the Hilbert series upon gluing two graphs \( G_1 \) and \( G_2 \) over a vertex which is a free vertex in both graphs. If \( G \) is a gluing like this, then \( G_1 \) and \( G_2 \) are a split of \( G \).

**Theorem 3.2** ([7, Theorem 3.1]). Let \( G_1 \) and \( G_2 \) be a split of a graph \( G \) at a vertex \( v \). If \( v \) is a free vertex in both \( G_1 \) and \( G_2 \), then \( \text{reg}(S/\mathcal{J}_G) = \text{reg}(S/\mathcal{J}_{G_1}) + \text{reg}(S/\mathcal{J}_{G_2}) \).

**Theorem 3.3** ([10, Theorem 3.2]). Let \( G_1 \) and \( G_2 \) be the decomposition of a graph \( G \) at vertex \( v \). If \( v \) is a free vertex in both \( G_1 \) and \( G_2 \), then

\[
\text{Hilb}_{S/\mathcal{J}_G}(t) = (1-t)^2\text{Hilb}_{S/\mathcal{J}_{G_1}}(t)\text{Hilb}_{S/\mathcal{J}_{G_2}}(t).
\]

**Theorem 3.4.** Let \( k \in \mathbb{N} \). Then there exists a graph \( G \) such that

\[
\text{reg}(S/\mathcal{J}_G) = \deg(h_G) + k.
\]
Proof. Let $G_1$ be the graph in Figure 1. The reduced Hilbert series of $S/J_{G_1}$ can be computed with MACAULAY2 [4] as
\[
\text{Hilb}_{S/J_{G_1}}(t) = \frac{1 + 7t + 17t^2 + 13t^3}{(1-t)^9}
\]
and its regularity as $\text{reg}(S/J_{G_1}) = 4$. Since $G_1$ has two free vertices 1, 2, we can glue a chain of $k$ copies of $G_1$ along free vertices (see Figure 2 for the case $k = 2$ in which the vertices 2 and 9 are available for further gluing). By Theorem 3.3, the Hilbert series of the resulting graph is
\[
\text{Hilb}_{S/J_{G_2}}(t) = \frac{(1 + 7t + 17t^2 + 13t^3)^k}{(1-t)^{7k+2}}
\]
and, by Theorem 3.2, $\text{reg}(S/J_{G_2}) = 4k$. Thus $\text{reg}(S/J_G) - \deg(h_G) = k$. \hfill \Box

Figure 2. A graph with $\text{reg}(S/J_G) > \deg(h_G) + 1$.

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