THE BINARY RETURNS!

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Abstract. Consider the spatial Newtonian three body problem at fixed negative energy and fixed angular momentum. The moment of inertia $I$ provides a measure of the overall size of a three-body system. We will prove that there is a positive number $I_0$ depending on the energy and angular momentum levels as well as the masses such that every solution at these levels passes through $I \leq I_0$ at some instant of time. Motivation for this result comes from trying to prove the impossibility of realizing a certain syzygy sequence in the zero angular momentum problem. 3-body problem and lunar problem and syzygy sequences and perturbation methods

1. Introduction

The spatial 3-body problem concerns three point masses in space moving according to Newton’s equations of gravitation. The point of this article is to prove that there exist no periodic solutions to this problem which “hang out near infinity”.

The conserved quantities for the problem are the energy $H$, angular momentum $J$ and linear momentum. As is standard, we may, without loss of generality, assume that the linear momentum is zero and the origin of space coincides with the center of mass of the three bodies. If $m_i$ denote the masses and $q_i \in \mathbb{R}^3$ the positions of the bodies, then the standard measure of size is $\|q\| = \sqrt{I(q)}$ where $q = (q_1, q_2, q_3)$ and $I = \sum m_i |q_i|^2$ is known as the total moment of inertia. Neighborhoods of infinity are regions of the form $\{q : I(q) \geq I_0\}$. As $I_0 \to \infty$ the neighborhood converges to infinity. Our main theorem is:

Theorem 1. For $H < 0$ there exists $I_0(m_j, H, J) > 0$ such that any orbit at these energy and momentum levels beginning in the region $I > I_0$ enters the region $I \leq I_0$ in forwards or backwards time.

Motivation. The motivation behind our result came from the problem of which syzygy sequences are realized in the zero angular momentum planar three body problem (see [13], [14], [15]). The term syzygy is from astronomy and refers to when the three bodies are in eclipse, that is collinear. Each syzygy has a ‘type’ 1, 2, or 3, according
to the label of the mass in the middle. Then the syzygy sequence of an orbit is this list of syzygy types in temporal order. A first open problem is whether or not the periodic sequence of repeating 1212’s is realized by a periodic solution to the zero angular momentum problem. One imagines such a motion as consisting of masses 1 and 2 going around each other in a near circular orbit, very far from mass 3, and the center of mass of the \( m_1 \) and \( m_2 \) orbit slowly going around mass 3, like the Earth-Moon-Sun system. The action over such solutions decreases as the distance of the earth moon system to the sun goes to infinity i.e. minimizing the action forces the solution to slide off into a neighborhood of infinity, see \[3\]. The theorem excludes the existence of such solutions “near infinity” i.e in the region \( I \geq I_0(m_j, H, 0) \).

**Remark.** In the theorem we may either exclude orbits having a binary collision singularity or we may pass through them using Levi-Civita regularization. Can we prove an analogous result to Theorem 1 for \( N \geq 4 \)? The proof here breaks down in proposition \[5\] where the neighborhoods of infinity fail to split into connected components characterized by a far body with suitable Jacobi coordinates. The connectedness of the neighborhoods of infinity due to these spread out clusters of tight binaries is utilized for Jeff Xia’s orbits realizing infinity in finite time singularities where \( N \geq 5 \). Can these infinity in finite time orbits provide counterexamples to Theorem 1 for \( N \geq 5 \)?

**Remark.** In \[11\] comet-like periodic orbits for the \( N \)-body problem are established in a region \( I \geq I_C \) for \( I_C \) large. These orbits do not contradict our theorem because as \( I_C \to \infty \) their orbits angular momentum \( |J| \to \infty \).

## 2. Related Results.

The behavior of \( I(t) \) has long been studied to gain some qualitative understanding of the N-body problem. Sundman, \[19\], showed for the three-body problem that non-zero angular momentum implies no orbits suffer triple collision i.e. \( I > 0 \) for all orbits. Namely there exists a positive lower bound, \( I_S(m_i, H, J, I(0), \dot{I}(0)) \), for orbits at such levels. That is \( I(t) > I_S > 0 \) over the solutions with energy \( H \) and angular momentum \( J \neq 0 \) and with initial conditions at \( I(0), \dot{I}(0) \). Hadamard, \[5\] pg. 259, gave an explicit formula for such an \( I_S \) and G.D. Birkhoff \[2\] ch. IX §§8 studied escape conditions in the non-zero angular momentum case by showing for example (pg. 282) that \( I \) sufficiently small (near zero) at some instant, \( t_0 \), implies \( I \) becomes infinite as \( t \) goes to...
infinity. One might paraphrase Birkhoff’s result as ‘no hanging out in neighborhoods of triple collision.’

A great deal of analysis on $I$ has followed these two tracks around small $I$ values. See for example [7], [10] on the greatest lower bound of $I$ for bounded orbits and [8], [10], [18] for efficient tests of escape in a variety of cases. The book [9] ch. 11 is a detailed reference for the qualitative study of $I$.

For each orbit let $I_m$ be the minimum value of $I$ over this orbit, sharpening Sundman leads one to seek a (greatest) lower bound of $I_m$ over classes of orbits. An analogous question here is instead to seek a (least) upper bound of $I_m$ over all orbits.

While most focus in the literature so far appears on the greatest lower bound and escape this upper bound question has not entirely escaped notice. A statement similar to Theorem 1 appears in [9] pg. 468 where an upper bound is given in a remark about a class of equal mass cases (those with $H|J|^2 = -\frac{4}{3\pi}$) and the least upper bound is conjectured to be attained over the Brouke-Henon orbit ([9] pg. 469). Here we give a new motivation to this question as to the existence of the 1212... solution in the zero angular momentum case and use a different method than that of [9]. Additionally we observe that both methods give upper bounds in a general case rather than just treating an equal mass case (see also [9] pg. 483). Moreover the method we use here offers hope of lowering the upper bound if the perturbation step (propositions [10], [15], [18]) is dealt with more effectively. In the appendix we give some comparison of the two methods.

3. Structure of proof.

For $H < 0$ as we let $I_0$ increase, eventually the domain \{\(I \geq I_0\)} splits into three components each component characterized by the selection of one of the three masses. The two remaining masses stay close to each other while this third selected mass, stays relatively far away from either member of this pair (see figure 1). We fix attention on one of these regions, supposing, after relabeling, that the close masses are $m_1$ and $m_2$. In this region, we use the standard Jacobi coordinates $\xi_1, \xi_2$. See figure 2.

When written in these coordinates, Newton’s differential equations becomes a perturbation of two uncoupled Kepler problems, one for each Jacobi vector, with the perturbation term getting arbitrarily small as $I_0 \to \infty$. We focus attention on the long Jacobi vector, which connects the center of mass of the $m_1$ and $m_2$ system to the 3rd mass. When we drop the perturbation term of this perturbed Kepler system, we get an
Figure 1. For the planar 3-body problem the shape space is $\mathbb{R}^3$ where $I$ is the distance from the origin. The admissible configurations at fixed $H < 0$ are interior to a pair of pants where each leg of the pants is asymptotic to a cylinder around a binary collision ray. See [12], or [16] for details.

Figure 2. A tight binary configuration, set $r := |\xi_1|$, $\rho := |\xi_2|$. 
exact solvable Kepler problem whose solutions we call “the osculating solutions”.

The Kepler parameters (energy, angular momentum, Laplace or Runge-Lenz vector) for the osculating system can be bounded using that $H, J$, the masses, are fixed and the fact that $I_0 >> 0$. Now here comes the key observation, due to Chenciner. Consider a family of solutions to Kepler’s equation having fixed energy and bounded angular momentum. If, along the solutions of this family the initial distance from the origin tends to infinity then these orbits become extremely eccentric, and thus must come close to the origin. Thus the osculating orbits cannot “hang out near infinity”. Said slightly differently, since large circular orbits for the Kepler problem have large angular momentum and since our total angular momentum is fixed, large near circular motions for osculating system are excluded and this excludes orbits of the type of our Earth-Moon-Sun cartoon described above.

Here is the strategy of proof then. Show that for sufficiently large $I_0$ all of the osculating solutions starting in $\{I \geq I_0\}$ are extremely eccentric, enough so to enter the region $\{I \leq I_0\}$ (see proposition 9). Next show that the real solutions do not vary too much from these osculating solutions, as long as they stay in the region $I \geq I_0$, and for bounded times (indeed for times of order $O(I_0^{3/2})$, proposition 10). It follows that if the osculating orbit enters the region $I \leq I_0$ within the time $O(I_0^{3/2})$ (which we expect by Kepler’s third law) then the true orbit must also enter into that region. Finally, (proposition 18) we verify that there is indeed sufficient time: the time scale over which the approximation of the true motion by the osculating motion is valid is long enough that the true motions must follow their osculating leads into a region $I \leq I_0$.

4. Set-up and Notation

In the spatial 3-body problem, we consider the motion of three point masses $m_1, m_2, m_3$ under Newton’s gravitational attraction. We will denote the configurations by

$q = (q_1, q_2, q_3) \in (\mathbb{R}^3)^3 \backslash \{(x_1, x_2, x_3) : x_i = x_j \text{ some } i \neq j\}$.

As is standard, we may take the center of mass zero coordinates ($\sum m_i q_i = 0$) and will now define the Jacobi coordinates in which the splitting into two perturbed Kepler problems will be clear (see Figure 2 as well as [17] 2.7, [4], or [6]):

$$\xi_1 = q_2 - q_1,$$
\[ \xi_2 = q_3 - (m_1 + m_2)^{-1}(m_1q_1 + m_2q_2) = \frac{m_1 + m_2 + m_3}{m_1 + m_2}q_3. \]

We set
\[ r = |\xi_1| \text{ and } \rho = |\xi_2|. \]

For reference we record here in one place the mass constants that will be used throughout:

**Mass constants:**

\[
\begin{align*}
\mu &= m_1 + m_2 \\
M &= m_1 + m_2 + m_3 \\
\alpha_1 &= m_1 m_2 \mu^{-1} \\
\alpha_2 &= m_3 \mu M^{-1} \\
\beta_1 &= \mu \alpha_1 \\
\beta_2 &= M \alpha_2
\end{align*}
\]

Then in these coordinates we find:

(1) \[ I := \sum m_i |q_i|^2 = \alpha_1 r^2 + \alpha_2 \rho^2 \]

(2) \[ J := \sum m_i (q_i \times \dot{q_i}) = \alpha_1 \xi_1 \times \dot{\xi}_1 + \alpha_2 \xi_2 \times \dot{\xi}_2 = J_1 + J_2 \]

for the moment of inertia, and angular momentum respectively. Also the energy splits into
\[ H = H_{kep} + g \]

where
\[ H_{kep} = \frac{1}{2} \alpha_1 |\dot{\xi}_1|^2 - \frac{\beta_1}{r} + \frac{1}{2} \alpha_2 |\dot{\xi}_2|^2 - \frac{\beta_2}{\rho} = H_1 + H_2 \]

is an energy for two uncoupled Kepler problems and
\[ g = \frac{\beta_2}{\rho} \left( \frac{m_1 m_3}{|\xi_2 + m_2 \mu^{-1} \xi_1|} - \frac{m_2 m_3}{|\xi_2 - m_1 \mu^{-1} \xi_1|} \right) \]

is a perturbation term with \( g = O(r^2/\rho^3), \ g_{\xi_1} = O(r/\rho^3) \) and \( g_{\xi_2} = O(r^2/\rho^4) \).

The equations of motion are then the two perturbed Kepler problems

(3) \[ \alpha_i \ddot{\xi}_i = -\beta_i \frac{\xi_i}{|\xi_i|^3} - g_{\xi_i} \]

**Definition 4.** A solution to the unperturbed Kepler problems satisfying the same initial conditions as a solution to these perturbed Kepler problems (eq. 3) will be called an osculating orbit (see [17], 1.16).
5. Proof of Main Theorem

Fix the masses, angular momentum, negative energy \( H < 0 \), linear momentum zero and a parameter \( \lambda > 0 \) and only consider orbits at these energy and momentum levels in appropriate Jacobi coordinates. We will use \( \bar{I} \) for a placeholder constant.

**Proposition 5.** For \( H < 0 \), there exists \( I^*(m_i, H, J) > 0 \) such that the region \( I > I^* \) consists of three connected components \( B_1, B_2, B_3 \). Moreover relabeling if necessary to fix our attention to \( B_3 \) (where \( q_3 \) is the far body) with appropriate Jacobi coordinates we have the bounds:

\[
|g| \leq c_g(r^2/\rho^3), \quad |g_{\xi_2}| \leq c_{g_2}(r^2/\rho^4)
\]

(6)

\[
|J_2| \leq \alpha_2 c_J
\]

(7)

\[
r \leq c_r
\]

(8)

on the perturbation term \( g \) angular momentum \( J_2 \) and short Jacobi vector \( r \) throughout \( B_3 \) for some constants \( c_g, c_{g_2}, c_J, c_r \) depending on masses, energy and angular momentum.

See [12], [4], [6], [9] regarding these well known lunar regions.

**Proposition 9.** Take \( I^{**} = \max\{I^*, \alpha_1 c_r^2 + \alpha_2 c_{J_2}^4/M^2\} \) where \( I^*, c_r, c_{J_2} \) are from Proposition 5. Then any osculating orbit with initial condition in \( I > I^{**} \) falls in forwards or backwards time into the region \( I \leq I^{**} \). Moreover the time to fall into the region \( I \leq I^{**} \) is less than or equal to the time to reach pericenter.

**Proof:** By eqs. (1, 8) in the region \( I > I^{**} \) we have \( \rho^2 > c_{J_2}^4/M^2 \).

The ‘\( \rho \)’ component of the osculating orbit of an initial condition in \( I > I^{**} \) is a solution to the Kepler problem

\[
\ddot{\xi}_{osc} = -M\xi_{osc}/|\xi_{osc}|^3
\]

with \( \rho^{osc}_2(0) = |\xi_{osc}(0)|^2 > c_{J_2}^4/M^2 \) and the restriction from eq. 7

\[
|\xi_{osc} \times \dot{\xi}_{osc}| = \alpha_2^{-1}|J_2(0)| \leq c_J
\]

on the angular momentum. Also from Proposition 5 we have the \( r \) component satisfying \( r \leq c_r \) as long as we remain in the region \( I > I^* \).

We now verify that for all such orbits, \( \xi_{osc} \), the pericenter distance, \( \rho^{pc}_{osc} \) is bounded.

**case 1:** \( J_2 \neq 0 \).
In polar coordinates, any non-collision osculating orbit is (for some $e \geq 0$):

$$\rho_{osc} = \frac{\alpha_2^{-2}|J_2(0)|^2}{M(1 + e \cos \theta)}$$

where $\theta = 0$ corresponds to the pericenter.

Then as $e \geq 0$ and by eq. (7),

$$\rho_{pc}^{pc} = \frac{\alpha_2^{-2}|J_2(0)|^2}{M(1 + e)} \leq \frac{c_{j_2}^2}{M}.$$  

case 2: $J_2 = 0$.

Collision! So the pericenter distance in this case is zero.

Now an osculating orbit starting in $I > I^\ast$ either reaches pericenter or leaves $I^\ast$ before it reaches pericenter. If it reaches pericenter before leaving $I > I^\ast$ then we have $I_{pc} \leq \alpha_1 c_2^2 + \alpha_2 c_2^{4} / M^2 \leq I^\ast$ so in either case we fall into the region $I \leq \max\{I^\ast, \alpha_1 c_2^2 + \alpha_2 c_2^{4} / M^2\} = I^\ast$ in forwards or backwards time which is no more than $t_{pc}$, the time to pericenter.

**Proposition 10.** Let $I \geq \max\{I^\ast, \alpha_1 c_2^2 + \max\{1, (\frac{3c_{j_2}^2}{2M})^2\} \alpha_2\} = \mathcal{T}$. Set $\mathcal{R} = \sqrt{\alpha_2^{-1}(I - \alpha_1 c_2^2)}$ and $\varepsilon = 1/\mathcal{R}$. Then any orbit with initial condition in $I \geq \mathcal{T}$ satisfies:

\begin{align*}
(11) & \quad |\rho(t) - \rho_{osc}(t)| < A_1\varepsilon \\
(12) & \quad |t| \leq B_1\varepsilon^{-3/2}
\end{align*}

throughout the region $I \geq \mathcal{T}$.

Here we may pick the constant $B_1 > 0$ and then define $A_1 = \frac{\alpha}{M} (2 + e\sqrt{2M + 3c_{j_2}^2 B_1})$ where $a = \alpha_2^{-1}((c_{g_2} c_2^2 B_1)^2 + 2c_{j_2} c_{g_2} c_2^2 B_1 + c_{g_2} c_2^2)$.

Proof: First, from eq. (8) any configuration with $I \geq \mathcal{T}$ has $\rho \geq \mathcal{R} \geq \max\{1, \frac{3c_{j_2}^2}{2M}\} \geq \max\{1, \frac{3\alpha_2 c_{j_2}^2}{2M}\}$, in particular our initial condition.

We consider our perturbed Kepler problem for the ‘$\rho$’ motion:

$$\ddot{\xi}_2 = -\frac{M\xi_2}{\rho^3} + F(\xi_2, t)$$

Where the time dependence in the perturbation term $F = -\alpha_2^{-1}g\xi_2$ is due to the interaction of the motion of masses 1 and 2.

In the region $I \geq \mathcal{T}$, we have $|F| \leq \alpha_2^{-1}c_{g_2} c_2^2 \rho^{-4} \leq \alpha_2^{-1} c_{g_2} c_2^2 \varepsilon^4$. We will set

$$A = \alpha_2^{-1}c_{g_2} c_2^2.$$
An estimate for the variation of $c_t^2 := |\xi_2 \times \dot{\xi}_2|^2 = \alpha_2^{-2}|J_2(t)|^2$ will be needed. Since $|\dot{c}| \leq |\alpha_2^{-1}\dot{J}_2| = |\xi_2 \times F| \leq A \rho^{-3}$, we have

$$|\dot{c}| \leq A \varepsilon^3$$

so that

$$|c_t - c_0| \leq A \varepsilon^3 |t|.$$ 

Hence

$$|c_t^2 - c_0^2| \leq A \varepsilon^3 |t|(A \varepsilon^3 |t| + 2c_0) \leq A \varepsilon^3 |t|(A \varepsilon^3 |t| + 2c_2).$$

so that for $|t| \leq B_1 \varepsilon^{-3/2}$ and $I \geq \mathcal{T}$ with $b = (AB_1)^2 + 2c_2 AB_1$ we have

$$|\dot{c}_t| \leq A \varepsilon^3 |t|.$$

provided $\varepsilon \leq 1$ which is guaranteed so long as $\mathcal{T} \geq \alpha_1 c_t^2 + \alpha_2$ as is indeed the case since $\mathcal{T} \geq \mathcal{R}$.

To prove the proposition we’ll use the Sandwich Lemma (see [15] pg. 1942. Note that in [15] there is an unneeded assumption requiring that $F_+ < 0$):

Sandwich Lemma: Given $\ddot{x}_- = F_-(x_-)$, $\ddot{x} = F(x, t)$ and $\ddot{x}_+ = F_+(x_+)$ satisfying $F_-(x) \leq F(x, t) \leq F_+(x)$ and $\frac{\partial F}{\partial x_\pm} \geq 0$ over some time interval, then over this same time interval the solutions to $F_+, F$ satisfying the same initial conditions have:

$$x_- (t) \leq x(t) \leq x_+ (t).$$

Now:

$$\rho_{osc} \dot{\rho}_{osc} + \rho_{osc}^2 = \frac{d}{dt} \rho_{osc} \dot{\rho}_{osc} = \frac{d}{dt} \xi_{osc} \dot{\xi}_{osc} = -M \rho_{osc}^2 \rho_{osc}^{-3} + |\dot{\xi}_{osc}|^2 = -M \rho_{osc}^{-1} \dot{\rho}_{osc}^2 + c_0^2 \rho_{osc}^{-2},$$

so

$$\dot{\rho}_{osc} = c_0^2 \rho_{osc}^{-3} - M \rho_{osc}^{-2}.$$ 

And likewise:

$$\ddot{\rho} = c_1^2 \rho^{-3} - M \dot{\rho}^{-2} + f(t)$$

where $|f(t)| = |\rho(t)^{-1}(\xi_2(t) \cdot F(\xi_2(t), t))| \leq A \rho(t)^{-4}$.

Take $v_1(\rho) = c_0^2 \rho^{-3} - M \rho^{-2}$ and $v_2(\rho, t) = c_1^2 \rho^{-3} - M \rho^{-2} + f$. We view $f$ here as $f(t)$ by plugging the true solutions $\xi_1(t), \xi_2(t)$ into $F, \rho$. 


Now using our $|c_t^2 - c_0^2|$ estimate eq. (13) and our bound on $f$ we get:

$$|v_1 - v_2| \leq b\varepsilon^{9/2} + A\varepsilon^4 \leq a\varepsilon^4$$

for $a = b + A$, or

$$v_1 - a\varepsilon^4 \leq v_2 \leq v_1 + a\varepsilon^4$$

for time $|t| \leq B_1\varepsilon^{-3/2}$ and $I \geq T$.

Now $\rho$ is a solution to $\ddot{\rho} = v_2$ and let $\rho_{\pm}$ be solutions to:

$$\ddot{\rho}_{\pm} = v_1(\rho_{\pm}) \pm a\varepsilon^4 =: F_{\pm}(\rho_{\pm})$$

satisfying the same initial conditions as $\rho$. Throughout the region $I \geq T$ we have $\rho_{\pm} \geq \frac{3a^2}{2M}$ which implies $\frac{\partial F_{\pm}}{\partial \rho_{\pm}} \geq 0$ so that we may apply the Sandwich Lemma throughout the region $I \geq T$ yielding:

$$\rho_{-} \leq \rho \leq \rho_{+}$$

for time $|t| \leq B_1\varepsilon^{-3/2}$ as long as we remain in the region $I \geq T$.

Likewise since $v_1 - a\varepsilon^4 \leq v_1 \leq v_1 + a\varepsilon^4$, we have for $|t| \leq B_1\varepsilon^{-3/2}$ and throughout $I \geq T$ that

$$\rho_{-} \leq \rho_{osc} \leq \rho_{+}$$

holds.

Now we will show $\rho_{+}$ and $\rho_{-}$ remain close to finish the proof. Set $\eta = \rho_{+} - \rho_{-} \geq 0$.

Note that $v_1$ is Lipschitz in the region $\rho \geq \bar{\rho}$ with

$$|v_1(x) - v_1(y)| \leq \omega|x - y|$$

for $x, y \geq \bar{\rho}$ and $\omega = (2M + 3c_0^2)\varepsilon^3 = k\varepsilon^3$.

Then $\ddot{\eta} = v_1(\rho_{+}) - v_1(\rho_{-}) + 2a\varepsilon^4 \Rightarrow |\ddot{\eta}| \leq \omega|\eta| + 2a\varepsilon^4 = \omega\eta + 2a\varepsilon^4$ so

$$|\ddot{\eta}| \leq \omega\eta + 2a\varepsilon^4.$$

Let $F = v_1(\rho_{+}) - v_1(\rho_{-}) + 2a\varepsilon^4$ then we have $0 \leq F \leq \omega\eta + 2a\varepsilon^4$ provided $\rho_{-} \leq \rho_{+}$ and $\frac{\partial v_1}{\partial \rho} > 0$; which indeed holds throughout the region $I \geq T$ for time $|t| \leq B_1\varepsilon^{-3/2}$. Now the Sandwich Lemma with $F_{\pm}(\eta) = \omega\eta + 2a\varepsilon^4$ and $F_{-} = 0$ gives:

$$0 \leq \eta(t) \leq \frac{2a\varepsilon^4}{\omega}(\cosh \sqrt{\omega}t - 1)$$

and since $\omega = k\varepsilon^3$ where $2M \leq k \leq 2M + 3c_0^2$ we have

$$|\rho(t) - \rho_{osc}(t)| \leq \rho_{+}(t) - \rho_{-}(t) = \eta(t) \leq \frac{2a\varepsilon}{k}(2 + e^{\sqrt{\omega}|t|}) \leq A_1\varepsilon.$$
for time $|t| \leq B_1 \epsilon^{-3/2}$ as long as we are in the region $I \geq T$ and where we set $A_1 = \frac{a}{M} (2 + e \sqrt{2M + 3a^2 B_1})$. \hfill $\square$

**Proposition 15.** Set $R = \max \{ \overline{R}, I^{**}, 4\alpha_1 c^2 \}$ where $\overline{R}$ is from proposition 10. For $\bar{T} \geq R$ set

$$T^+ = 4(I - \alpha_1 c^2) > T.$$  

Then for any orbit with an initial condition in the strip

$$I \leq I \leq I^+$$

we have that eq. (11) holds with $B_1 = 2^{3/2} \pi \sqrt{M}$ until the osculating orbit enters the region $I \leq T$.

Proof: First consider orbits with initial condition in $I \geq I$ for some $I \geq \max \{ I^{**}, \overline{R} \}$ and with $\varepsilon = 1/\overline{p}$ defined as in proposition 10 and recall that $I \geq \bar{T}$ implies that $\rho \geq \overline{p}$. For osculating collision orbits with $J_2(0) = 0$, some energy $H_2$ and $\rho(0) = \rho_{osc}(0) > \overline{p}$ the time to collision in forwards time (or time from expulsion in backwards time) $t_c$, satisfies:

$$t_c \leq \pi (8M)^{-1/2} \rho_{osc}(0)^{3/2}. \tag{16}$$

We will use Lambert’s Theorem (see [1]) to compare time to pericenter for general osculating orbits to these collision times. Lambert says that for Kepler orbits, the time of travel between two points, $a_1, a_2$ on the orbit is a function of the energy, chord length $d = |a_1 - a_2|$ and $|a_1| + |a_2| = r_1 + r_2$ (where the origin is at the focus, see figure 3). Namely, for equivalent configurations (those having the same energy, same chord length $d$, and $r_1 + r_2 = s_1 + s_2$) the time of travel from $a_2$ to $a_1$ is the same as the time of travel from $b_2$ to $b_1$. Figure 3 is how we will choose our equivalent configurations:

For a general osculating orbit $\rho_{osc}$, take $r_1 = \rho_{osc}^{pc}$, $r_2 = \rho_{osc}(0) = \rho(0) > \overline{p}$ and then $s_1, s_2$ are determined by $s_2 - s_1 = d = |a_2 - a_1|$ and $s_1 + s_2 = r_1 + r_2$. By Lambert’s theorem and eq. (16) we have the time to pericenter, $t_{pc}$, satisfies

$$t_{pc} \leq \pi (8M)^{-1/2} s_2^{3/2}. \tag{17}$$

And since $r_2 \geq r_1$ (as we are in $I \geq I^{**}$) we have:

$$2s_2 - (r_1 + r_2) = s_2 - s_1 = d \leq r_1 + r_2 \Rightarrow$$

$$s_2 \leq r_1 + r_2 \leq 2r_2.$$  

So continuing with eq. (17)

$$t_{pc} \leq \pi M^{-1/2} r_2^{3/2}. \tag{17}$$
Figure 3. Two equivalent configurations.

So to compare \( t_{pc} \) with our estimates eq. (12) we want \( t_{pc} \leq B_1 \varepsilon^{-3/2} = B_1 \rho^{3/2} \), which holds when:

\[
\pi M^{-1/2} r_2^{3/2} \leq B_1 \rho^{3/2} \Rightarrow \\
\quad r_2^{3/2} \leq \pi^{-1} M^{1/2} B_1 \rho^{3/2}.
\]

Take \( B_1 = 2^{3/2} \pi / \sqrt{M} \) so that we will be working in the strip:

\[
\rho^{3/2} \leq r_2^{3/2} \leq 2^{3/2} \rho^{3/2}
\]

i.e. (recall that \( r_2 = \rho(0) = \rho_{osc}(0) \))

\[
\rho \leq \rho \leq 2 \rho.
\]

The condition \( \rho \leq 2 \rho \) is ensured (eqs. [1][8] when \( I \leq T^+ := 4 \alpha_2 \rho^2 = 4(\bar{T} - \alpha_1 c_r^2) \).

Also, we ensure \( \bar{T} < T^+ = 4(\bar{T} - \alpha_1 c_r^2) \) provided \( \bar{T} \geq 4 \alpha_1 c_r^2 > \frac{4}{3} \alpha_1 c_r^2 \).

**Proposition 18.** (Main Theorem) Fix a parameter \( \lambda > 0 \). Then there exists \( R_\lambda(m, H, J) > 0 \) such that any orbit with initial condition satisfying \( I(0) \geq R_\lambda \) comes in forwards or backwards time into the region \( I \leq R_\lambda \).

Explicitly, take \( \bar{R}_\lambda = \max \{ R, \alpha_1 c_r^2 + \alpha_2(\frac{\alpha_2 A_r^2}{\lambda}), 2 \alpha_2 A_1 + 4 \alpha_1 c_r^2 + \lambda \} \) and \( R_\lambda = \bar{R}_\lambda + 2 \alpha_2 A_1 + \lambda \) where \( R \) is from proposition 15.

Proof: Take \( \bar{T} \geq \bar{R}_\lambda \) and \( \varepsilon^{-1} = \bar{\rho} = \sqrt{\frac{\alpha_2}{\bar{\lambda}}(\bar{T} - \alpha_1 c_r^2)} \) and consider an orbit with initial condition in \( T^+ \geq I \geq T \) as in proposition 15.

By proposition 9 we can let \( t^* \) be the time the osculating orbit hits
\[ \alpha_1 c_r^2 + \alpha_2 \rho_{osc}(t^*) = T \] i.e. \( \rho_{osc}(t^*) = \overline{\rho} = \varepsilon^{-1} \).

Along the true motion then at \( t^* \) we have by proposition \( 15 \) (eqs. 8, 11) that
\[
I(t^*) \leq \alpha_1 c_r^2 + \alpha_2 (\rho_{osc}(t^*) + A_1 \varepsilon)^2 = T + 2\alpha_2 A_1 + \alpha_2 A_1^2 \varepsilon^2
\]
holds. Moreover due to the condition \( T \geq R_\lambda \geq \alpha_1 c_r^2 + \alpha_2 \left( \frac{A_1^2}{\lambda} \right) \) we have \( \varepsilon^2 \leq \frac{\lambda}{\alpha_2 A_1} \) so that
\[
I(t^*) \leq T + 2\alpha_2 A_1 + \lambda.
\]
Also the condition \( T \geq R_\lambda \geq 2\alpha_2 A_1 + 4\alpha_1 c_r^2 + \lambda > \frac{1}{3} (2\alpha_2 A_1 + 4\alpha_1 c_r^2 + \lambda) \) ensures that \( T + 2\alpha_2 A_1 + \lambda < \overline{T}^+ \).

That is taking any \( T \geq R_\lambda \) and setting \( \lambda' = 2\alpha_2 A_1 + \lambda \) then all orbits with initial condition in the strip
\[
\overline{T} + \lambda' \leq I(0) \leq \overline{T}^+
\]
come in forwards or backwards time into the region
\[
I \leq T + \lambda'
\]

In particular by setting
\[
T_s = R_\lambda + s
\]
for \( s \geq 0 \), we may exhaust the region \( I \geq R_\lambda = R_\lambda + \lambda' = T_0 + \lambda' \) with the strips
\[
T_s + \lambda' \leq I \leq T_s^+.
\]
Note that \( s > s' \) implies \( T_s^+ - T_s > T_{s'}^+ - T_{s'} \). Hence any orbit with initial condition \( I(0) \geq R_\lambda \) will be forced to jump back along the strips (see figure 4).

\[ \text{Figure 4. Jumping back along the strips.} \]

Finally in Theorem 1 we can take \( I_0 = R_\lambda \) for any choice of \( \lambda \) (for instance \( I_0 = \min_{\lambda \in (0,1)} R_\lambda \)). \( \square \)
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Appendix A. Comparison to Marchal’s equal mass case

We shall now compare our results to [9] by examining the case:

\[ m_i = \frac{1}{3}, \quad H = -\frac{1}{6}, \quad |J| = \frac{\sqrt{8}}{9}. \]

As \( I^* \) corresponds to the apocenter of the collinear Euler motion (where \( U|_{I=1} \) has a saddle point), we have

\[ I^* = \frac{32}{27}. \]

Here Marchal observed in [9] pg. 468 that \( \ddot{\rho} < 0 \) for \( I > I_M \) where

\[ I_M = \frac{1}{3}(2.709629...)^2 = 2.447363... \]

so that every orbit will enter the region \( I \leq I_M \) at some instant (of course excluding or passing through any binary collisions). Moreover it was conjectured that in fact all orbits pass below the minimal inertia of the Henon-Brouke orbit (see [9] pg. 469) which is approximately 2.402035....and resembles an \( e = 0 \) earth-moon sun cartoon type orbit.

Applying our final result (proposition 18) in this case we obtain an \( I_0 > I_M \). However our bound only becomes larger when we apply our perturbation arguments; in this case and in general we obtain a lower pre-perturbation \( I^{**} < I_M \) (of proposition 9). This gives hope that if our perturbation methods are improved (propositions 10, 15, 18) then the Marchal’s bound of \( I_M \) could be lowered. Now we outline how \( I^{**} < I_M \) in general.

Note that Marchal’s observation on the negativity of \( \ddot{\rho} \) works not just in this equal mass case but lends to a shorter proof of Theorem 1 by using eq. 14. We set \( \mu_i = m_i \mu^{-1}, \lambda = r/\rho, \gamma = \angle(\xi_1, \xi_2) \) and rewrite eq. 14 as

\[ \rho^3 \ddot{\rho} = c^2 - \rho M \phi(\lambda, \gamma) \]

where

\[ \phi(\lambda, \gamma) = \mu_1 \frac{1 + \mu_2 \cos(\gamma) \lambda}{(1 + 2 \mu_2 \cos(\gamma) \lambda + \mu_2^2 \lambda^2)^{3/2} + \mu_2} \left[ \frac{1 - \mu_1 \cos(\gamma) \lambda}{(1 - 2 \mu_1 \cos(\gamma) \lambda + \mu_1^2 \lambda^2)^{3/2}} \right]. \]
Note that $\phi \geq \delta > 0$ throughout $I \geq I^*$ for a $\delta \leq \phi(\lambda, \pi/2) < 1$ dependent on the masses. Hence a $\rho_M$ corresponding to Marchal’s $I_M$ is (in general)

$$\rho_M = \frac{c_J^2}{M\delta}.$$ 

However although eq. 14 leads to a simpler proof, following an orbit to pericenter rather than over the region where $\dot{\rho} < 0$ has the potential to yield lower upper bounds as for Keplerian orbits we have

$$\rho^\text{pc} = \frac{c^2}{M(1 + e)} \leq \frac{c_J^2}{M} < \rho_M.$$ 

Thus the $I^{**}$ of proposition 9 satisfies

$$I^{**} < I_M.$$ 

So our strategy of proof provides hope of lowering the bound towards Marchal’s conjectured Henon-Broucke value in this case and a lower upper bound in general provided the techniques in the perturbation steps propositions 10, 15, 18 are improved to follow the orbits past the $\dot{\rho} < 0$ regime. Can they be improved? Perhaps in some non-equal mass cases or for some (outer) eccentricity $e > 0$ orbits above Henon-Broucke? I am optimistic that taking advantage of the sharper bounds and techniques of the literature they can be improved at least for large classes of orbits. Especially so as many bounds of the perturbation steps here are not the sharpest (as the original motivation here was merely the existence of some upper bound in general specifically the zero angular momentum case).

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