Geometry of non-commutative orbits
related to Hecke symmetries

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Abstract

To some braiding $R$ of Hecke type (a Hecke symmetry) we put into correspondence an associative algebra called the modified Reflection Equation Algebra (mREA). We construct a series of matrices $L(m)$, $m = 1, 2, \ldots$ with entries belonging to such an algebra so that each of them satisfies a version of the Cayley-Hamilton identity with central coefficients.

We also consider some quotients of the mREA which are called the non-commutative orbits. For each of these orbits we construct a large family of projective modules. In such a family we introduce an algebraic structure which is close to that of $K^0(\text{Fl}(\mathbb{C}^n))$. This algebraic structure respects an equivalence relation motivated by a “quantum” trace compatible with the initial Hecke symmetry $R$. For a subclass of non-commutative orbits we compute the spectrum of central elements $\text{Tr}_R L_k^k$, $k \in \mathbb{N}$ of the mREA

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1 Introduction

As was shown in [Se], the category of locally trivial vector bundles over an affine regular algebraic variety $X$ is equivalent to that of finitely generated projective $\mathbb{K}(X)$-modules where $\mathbb{K}(X)$ is the coordinate algebra of $X$. (In what follows the ground field $\mathbb{K}$ is assumed to be $\mathbb{C}$ or $\mathbb{R}$, the latter case will be specified each time.) A similar equivalence is valid for compact smooth varieties (cf. [Ro]). This is the reason why in the non-commutative geometry finitely generated projective modules over non-commutative algebras are considered as appropriate analogs of vector bundles. Thus, the problem of constructing and classifying such $\mathcal{A}$-modules for a given non-commutative (NC) algebra $\mathcal{A}$ is of great interest. Unfortunately, besides $\mathbb{C}^*$-algebras very few examples of algebras with a significant family of projective modules are known.

In [GS1] we have suggested a way of constructing projective $\mathcal{A}$-modules via an NC version of the Cayley-Hamilton (CH) identities. The existence of these identities (as well as the Newton identities) is the very remarkable property of the so-called reflection equation algebras (REA) or their modified versions (mREA). The algebras of this type can be associated to a large class of braidings (solutions of the quantum Yang-Baxter equation) of the Hecke type.
In the present paper we consider some quotients of the mREA — the non-commutative (NC) orbits. The terminology is motivated by a close connection of these quotients with the coordinate algebras of orbits in \( gl(n)^* \).

For each generic NC orbit \( \mathcal{A} \) we establish certain combinatorial relations among its central elements. The relations can be interpreted as the higher NC counterparts of the Newton identities. Recall, that the classical Newton relations connect the elementary symmetric functions of some commutative variables and the power sums of the same variables. Besides, we construct a large family of projective \( \mathcal{A} \)-modules (in what follows all modules are assumed to be finitely generated and one-sided).

Also, we introduce an algebra \( Q(\mathcal{A}) \) which is an analog of \( K^0(\text{Fl}(\mathbb{C}^n)) \), where \( \text{Fl}(\mathbb{C}^n) \) is a flag variety, and define a \( q \)-analog of the Euler characteristic of line bundles on the flag variety which is well defined on \( Q(\mathcal{A}) \).

A particular case of the NC orbits is the set of the so-called quantum orbits arising from the quantization of a certain Poisson pencil on semisimple orbits in \( gl(n)^* \) (i.e., \( GL(n) \) orbits of semisimple elements of \( gl(n)^* \)). The Poisson pencil is generated by the two brackets: the Kirillov bracket and another one, related to a classical \( r \)-matrix. One of the main properties of the algebra \( \mathcal{A} \) arising from the quantization of the Poisson pencil is that the product \( \mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \) is equivariant (covariant) with respect to the action of the quantum group \( U_q(\mathfrak{sl}(n)) \)

\[
U \triangleright \mu(a \otimes b) = \mu(U \triangleright a \otimes U \triangleright b), \quad a, b \in \mathcal{A}, \quad U \in U_q(\mathfrak{sl}(n)),
\]

where \( \Delta(U) = U_1 \otimes U_2 \). In this case (the \( U_q(\mathfrak{sl}(n)) \) case for short) at the limit \( q \to 1 \) we get the \( SL(n) \)-equivariant (or, if we pass to the compact form, \( SU(n) \)-equivariant) algebras which are also called the fuzzy orbits.

Let us describe the algebras in question in more detail. We start from the definition of the reflection equation algebra connected with a braiding of the Hecke type. The initial data for its construction is a braiding \( R \)

\[
R : \quad V^{\otimes 2} \to V^{\otimes 2},
\]

which is a solution of the quantum Yang-Baxter equation \([3,1]\) satisfying additionally the second order equation \([3,5]\). Here \( V \) is a finite dimensional vector space, \( \dim V = n \). Such a braiding will be called the Hecke symmetry. A well-known example is connected with the quantum group \( U_q(\mathfrak{sl}(n)) \) when the corresponding Hecke symmetry is the image of the universal \( R \)-matrix in the fundamental vector representation of \( U_q(\mathfrak{sl}(n)) \).

In \([3]\) there were constructed other examples of Hecke symmetries such that the Hilbert-Poincaré series of the associated "symmetric" and "skew-symmetric" subalgebras of the tensor algebra \( T(V) \) differ from the classical ones. Below we shall additionally assume the Hilbert-Poincaré series of the "skew-symmetric" subalgebra to be a monic polynomial. Such a Hecke symmetry will be called even and the degree \( p \) of this polynomial will be called the symmetry rank of \( R \).

Consider now an associative unital algebra generated by \( n^2 \) indeterminates \( l^i_j, 1 \leq i, j \leq n \) satisfying the following relation

\[
R L_1 R L_1 - L_1 R L_1 R = h(R L_1 - L_1 R), \quad L_1 \equiv L \otimes I,
\]

where \( h \) is a formal parameter, \( I \) is an \( n \times n \) unit matrix and \( L = ||l^i_j|| \) is a matrix with entries \( l^i_j \). If \( h = 0 \), we call this algebra the reflection equation algebra (REA) and denote it \( \mathcal{L}_q \); if \( h \neq 0 \), we call it the modified reflection equation algebra (mREA) and denote it \( \mathcal{L}_{h,q} \).

In the \( U_q(\mathfrak{sl}(n)) \) case, being specialized at \( q = 1 \), the mREA coincides with the enveloping algebras \( U(\mathfrak{gl}(n)_h) \) (hereafter, given a Lie algebra \( \mathfrak{g} \) with the bracket \([\cdot, \cdot]\), the symbol \( \mathfrak{g}_h \) will denote the Lie algebra with the bracket \( h[\cdot, \cdot] \)). In some sense the mREA is a "braided" analog of the enveloping algebra \( U(\mathfrak{gl}(n)_h) \). For a motivation of this treatment, see section \([4]\).
Now we list some properties of the mREA. First of all, the definition implies that the generators $l_i^j$ of any mREA obey the quadratic-linear commutation relations. Second, the category $\mathcal{L}_{h,q} \text{-} \text{Mod}$ of equivariant finite dimensional representations of the mREA corresponding to an even Hecke symmetry is close to the category $U(gl(p)) \text{-} \text{Mod}$, where $p$ is the symmetry rank of the Hecke symmetry $R$ (see section [4]). For example, simple objects of the category $\mathcal{L}_{h,q} \text{-} \text{Mod}$ can be labelled by signatures (partitions) $\lambda = (\lambda_1, \ldots, \lambda_p)$, $\lambda_1 \geq \ldots \geq \lambda_p$, similarly to the category of $gl(p)$-modules. Besides, the Grothendieck rings of these categories are isomorphic ($GLS$).

To explain the term “equivariant” we restrict ourselves to the $U_q(sl(n))$-case. Consider an $U_q(sl(n))$-module $V$. Then, as is well known, the space $\text{End}(V)$ of the internal endomorphisms of $V$ is endowed with an $U_q(sl(n))$-action, too. The mREA can as well be equipped with an $U_q(sl(n))$-action satisfying (1.1). In this case a representation $\pi : \mathcal{L}_{h,q} \to \text{End}(V)$ is called equivariant if $\pi$ commutes (as a mapping) with the $U_q(sl(n))$-action.

The next important property of the mREA is the existence of a monic polynomial $CH(x)$ of degree $p$ such that the matrix $L$ entering formula (1.2) satisfies the Cayley-Hamilton (CH) identity

$$CH(L) = \sum_{k=0}^{p} (-L)^{p-k} \sigma_k(L) \equiv 0, \quad \sigma_0(L) = \text{id}_L$$

where the coefficients $\sigma_k(L)$, $0 \leq k \leq p$ are linear independent generators of the center $Z(\mathcal{L}_{h,q})$ of the mREA. This identity was proved in $GPS$ for the non-modified REA (the $U_q(sl(n))$ case was previously considered in $[PS]$). The above CH identity for the mREA can be established by a linear change of generators ($GLS$).

Let $\chi : Z(\mathcal{L}_{h,q}) \to \mathbb{K}$ be a character of the center. The character $\chi$ is completely fixed by its values on $\sigma_k$

$$\chi(\sigma_k) = \sum_{1 \leq i_1 < \ldots < i_k \leq p} \mu_{i_1} \ldots \mu_{i_k} \chi(z)$$

where the numbers $\mu_i \in \mathbb{K}$ are assumed to be distinct. The quotient of the algebra $\mathcal{L}_{h,q}$ modulo the ideal generated by the elements $z - \chi(z)$, where $z \in Z(\mathcal{L}_{h,q})$, will be called an NC orbit and denoted $\mathcal{L}_{h,q}^\chi$ (although as explained in remark [4] such an ”NC orbit” in the $U_q(sl(n))$ case can arise from a quantization of a union of some orbits in $gl(n)^*$).

Observe that, being switched to the algebra $\mathcal{L}_{h,q}^\chi$, the CH identity for $L$ takes the form

$$\prod_{i}(L - \mu_i I) = 0.$$  

(Thus, the numbers $\mu_i$ are thought of as eigenvalues of the matrix $L$ with entries from $\mathcal{L}_{h,q}^\chi$.)

A great importance of the CH identity is occasioned by the fact that it allows us to construct a family of idempotents from $\text{Mat}(\mathcal{L}_{h,q}^\chi)$ and therefore projective modules. Moreover, the family of idempotents (and the corresponding projective modules) can be essentially enlarged. For this purpose, we construct a series of higher order matrices $L_{(m)}$ and polynomials $CH_{(m)}$, $m = 2, 3, \ldots$ with central coefficients such that the higher order CH identities $CH_{(m)}(L_{(m)}) = 0$ are satisfied. We also set $L_{(1)} = L$, $CH_{(1)} = CH$.

Upon restricting to the NC orbit $\mathcal{L}_{h,q}^\chi$, we come to the series of polynomials $CH_{(m)}^\chi$ with the numerical coefficients. Assuming the roots of these polynomials to be distinct for any $m$ we find $p_m = \deg CH_{(m)}$ idempotents $e_k(m) \in \text{Mat}(\mathcal{L}_{h,q}^\chi)$, each corresponding to a projective $\mathcal{L}_{h,q}^\chi$-module.

One of the main aims of this paper is to prove the existence of the higher order CH identities and to compute the coefficients of the polynomials $CH_{(m)}$. This is rigorously done for the mREA associated with any even Hecke symmetries of rank $p = 2$. For $p \geq 3$ we present an explicit formula for these polynomials as a conjecture.

Our other aim is to compute the values of the central elements $Tr_R L_{(m)}^s$, $s \geq 1$, for a generic NC orbit. Here $Tr_R : \text{Mat}(\mathcal{L}_{h,q}) \to \mathcal{L}_{h,q}$ is the trace defined by the initial Hecke symmetry $R$. It is
closely related to the categorical trace which is discussed in section 4 (see [GLS1] for more detail). For example, in the $U_q(sl(n))$ case $\text{Tr}_R$ is the well-known quantum trace which is a weighed sum of the diagonal entries, but in the general case it can be more complicated. We express $\text{Tr}_R L^k_m$ (up to the aforementioned conjecture) in terms of the eigenvalues of the matrix $L$. For $m = 1$ we treat these expressions as a parametric resolution of the Newton relations. For $m > 1$ they are thought of as higher analogs of the Newton relations.

As a byproduct we compute the value of $\text{Tr}_R e_k(m)$ on generic NC orbits. In contrast with the usual orbits when these quantities are always equal to 1 (since the related projective modules correspond to line bundles), for the NC orbits they are more informative. We show that the assignment $M_k(m) \mapsto \text{Tr}_R e_k(m)$, where the module $M_k(m)$ corresponds to the idempotent $e_k(m)$, can be considered as an analog of the Euler characteristic of a line bundle over a flag variety. Developing the analogy with a flag variety, we can introduce a multiplicative structure on the set of the modules $M_k(m)$ and construct an algebra $Q(\mathcal{L}_{h,q})$ playing the role of $K^0$ of the flag variety.

Completing the Introduction, we consider the $U_q(sl(n))$ case in more detail. As we have already mentioned, in this case the NC orbits arise from a quantization of a Poisson pencil on some algebraic varieties. So, it is natural to consider the problem of quantization of the vector bundles over these varieties in terms of projective modules. In this case the projective modules $M_k(m)$ over the quantum orbits are nothing but deformations of the corresponding modules over the coordinate rings of the initial varieties.

The problem of quantization of semisimple orbits in $g^*$ was considered in numerous papers. In [DMI] the full solution of this problem was given in terms of the mREA and its appropriate quotients related to the minimal polynomials of orbits in question. In [DM2] this method of quantization is compared in the $U(sl(n))$ case with an approach based on the so-called generalized Verma modules. In particular, the authors produce a formula describing the eigenvalues $\mu_i$ to be functions of the corresponding generalized Verma module weight $\lambda_i$. We find a q-analog of this formula by using different methods. (Note that in the $U(sl(n))$ case this formula is obtained by means of the coproduct, see section 2, whereas in the mREA we did not find any convenient coproduct.)

The paper is organized as follows. In the next section we consider the NC orbits arising from the quantization of the Kirillov bracket on some semisimple orbits in $gl(n)$ (or $su(n)$). We call them the fuzzy orbits, the term is motivated by numerous papers devoted to “the fuzzy physics”. In contrast with all those papers, we present a general scheme of constructing a large family of projective modules via the CH identities. In subsequent sections we generalize this scheme to the mREA associated with a large class of Hecke symmetries and thereby we construct a similar family of projective modules for their appropriate quotients (“q-fuzzy orbits”).

Note that the methods of section 2 are close to those going back to the pioneering paper [KO] and having been used in numerous works devoted to the characteristic identities (which are nothing but a specialization of the CH identities to concrete representations of the algebras in question, cf. [GZ, O] and references therein).

In section 3 we introduce the REA and list its basic properties in detail.

In section 4 we give reasons allowing us to treat the mREA as a braided analog of the enveloping algebra.

Section 5 is devoted to the proof of a parametric resolution of the basic Newton identity.

\footnote{In fact, this formula is well-known, cf. for example, [BR] where the eigenvalues of the Casimir elements $\text{Tr}_R L^k$ in finite dimensional representations are computed.}

\footnote{Note that there are known q-analogs of such characteristic identities related to the quantum groups (cf. [GZB]). However, the quantum groups are not deformations of commutative algebras and are not convenient objects for constructing projective modules and the related Newton identities. A set of specific Newton-Cayley-Hamilton identities exists in algebras dual to the quantum groups, cf. [OP]. However, they are useless for constructing the projective modules over these algebras.}
In section 6 we give a proof of the relation between eigenvalues of the matrix $L$ and those of the highest matrices $L_{(m)}$. At the end of the section we compute the values of the central elements $Tr R L_{(m)}^k$. As a byproduct we get the aforementioned formula featuring the eigenvalues of the matrix $L$ as a function of a partition $\lambda$.

Finally, in section 7 we introduce the algebra $Q(L_{h,q}^k)$ and define a $q$-analog of the Euler characteristic.

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# 2 Fuzzy orbits

In this section we are dealing with algebras arising from the quantization of the Kirillov bracket alone. First, we describe our initial object — the coordinate algebra of a generic classical orbit. Let us fix a diagonal matrix

$$M = \text{diag}(\mu_1, \ldots, \mu_n), \quad \mu_i \in \mathbb{C} \quad (2.1)$$

with simple $\mu_i$. We treat this matrix as an element of $\mathfrak{g}^* = gl(n)^*$ (identified with $\mathfrak{g} = gl(n)$) and consider its orbit $O_M$ with respect to the $GL(n)$-action

$$GL(n) \ni g : M \mapsto M_g = g^{-1} M g. \quad (2.2)$$

The orbit $O_M$ is a closed affine algebraic variety and its coordinate algebra $\mathbb{K}(O_M)$ can be described as follows. Let $\mathbb{K}(\mathfrak{g}^*)$ be the polynomial algebra in $n^2$ commutative indeterminates $l_{ij}^k$, $1 \leq i, j \leq n$, which are nothing but the coordinate functions. Then we have $\mathbb{K}(O_M) = \mathbb{K}(\mathfrak{g}^*)/\mathcal{I}$ where $\mathcal{I} \subset \mathbb{K}(\mathfrak{g}^*)$ is the ideal generated by the elements

$$Tr L^k - \beta_k, \quad k = 1, 2, \ldots, n. \quad (2.3)$$

Here the matrix $L = \|l_{ij}^k\|$ is composed of the indeterminates $l_{ij}^k$ ($i$ labels the rows and $j$ — the columns) and

$$\beta_k = \sum_{i=1}^{n} \mu_i^k, \quad k = 1, 2, \ldots, n. \quad (2.4)$$

**Remark 1** Note that if $O_M$ is semisimple but not generic (i.e., the eigenvalues $\mu_i$ of a given diagonal matrix $M$ are not simple) the coordinate algebra is the quotient $\mathbb{K}(O_M) = \mathbb{K}(\mathfrak{g}^*)/\mathcal{I}'$, where the ideal $\mathcal{I}'$ is of the following form. Let

$$\mathcal{P}(M) = (M - \mu_1 I) \ldots (M - \mu_r I)$$

be the minimal polynomial of the matrix $M$ ($I$ stands for the $n \times n$ unit matrix). Then $n^2$ entries of the matrix $\mathcal{P}(L)$ are polynomials in generators $l_{ij}^k$. The ideal $\mathcal{I}'$ is generated by these polynomials and by elements $\{2.3\}$ with $k = 1, \ldots, r - 1$. Note, that if we disregard the latter elements we get a union of all semisimple orbits possessing the same minimal polynomial.

If the initial matrix $M$ is such that $Tr M = 0$, then the corresponding orbit is embedded into $sl(n)^*$. Since

$$\mathbb{K}(sl(n)^*) = \mathbb{K}(gl(n)^*)/\{\text{Tr } L\},$$

the coordinate algebra of the corresponding orbit can be realized as above but with $\beta_1 = 0$. (Hereafter $\{S\}$ stands for the ideal generated by a set $S$.)
If all eigenvalues of the matrix $M$ are real, we can consider the matrix $iM$ (here $i = \sqrt{-1}$) as an element of $u(n)^*$ (or $su(n)^*$ if $\text{Tr} M = 0$). Choosing the generators $x_i^j$ for $i \leq j$ and $y_i^j$ for $i < j$ such that

$$l_i^j = x_i^j + iy_i^j \quad \text{for} \quad i < j, \quad l_i^j = x_j^i - iy_j^i \quad \text{for} \quad i > j, \quad \text{and} \quad l_i^i = x_i^i,$$

we get a compact real variety which is an $SU(n)$-orbit (i.e., in [22] we assume that $g \in SU(n)$). Consequently, we consider its coordinate algebra as an $\mathbb{R}$-algebra.

Now let us pass to the fuzzy orbits. In the spirit of "NC affine algebraic geometry" we realize them via some explicit relations on generators. Consider again the matrix $L = \|l_i^j\|$, but now we let the generators $l_i^j$ to satisfy the defining relations of the algebra $U(\mathfrak{g}_h)$ with $\mathfrak{g} = gl(n)$:

$$l_i^j l_m^j - l_m^j l_i^j - h(l_i^j \delta_m^j - l_m^j \delta_i^j) = 0.$$

Then the matrix $L \in \text{Mat}_n(U(\mathfrak{g}_h))$ obeys a polynomial relation

$$C\mathcal{H}(L) = \sum_{k=0}^{n} (-L)^{n-k} \sigma_k(L) = 0,$$

such that the coefficients $\sigma_k(L)$ are central and $\sigma_0(L) = 1$. This fact is well known. An expression of the coefficients $\sigma_k(L)$ in terms of the generators $l_i^j$ can be found in [CS1]. Below we present them in a convenient form for a more general case of NC orbit (see section 5).

It is also well known that the center $Z(U(\mathfrak{g}_h))$ of the algebra $U(\mathfrak{g}_h)$ is generated by $\sigma_k(L)$ for $1 \leq k \leq n$. Another family generating the center is $s_k(L) = \text{Tr} L^k$, $1 \leq k \leq n$. Therefore, any character

$$\chi : Z(U(\mathfrak{g}_h)) \to \mathbb{K}$$

is completely determined by its values on the generators of the center $\chi(\sigma_k(L)) = \alpha_k$ or $\chi(s_k(L)) = \beta_k$, $k = 1, \ldots, n$.

Consider the quotient algebra

$$\mathcal{L}^\chi_h = U(\mathfrak{g}_h)/\{z - \chi(z) \mid z \in Z(U(\mathfrak{g}_h))\},$$

where $\chi$ is a fixed character. Being switched to $\mathcal{L}^\chi_h$, relation (2.5) takes the form

$$C\mathcal{H}^\chi(L) \overset{\text{def}}{=} \sum_{k=0}^{n} (-L)^{n-k} \chi(\sigma_k(L)) = \sum_{k=0}^{n} (-L)^{n-k} \alpha_k = 0.$$  

(2.7)

In what follows the superscript $\chi$ means that we have passed from $U(\mathfrak{g}_h)$ to $\mathcal{L}^\chi_h$ (and similarly, for other algebras below).

Also, consider the associated numerical equation

$$\sum_{k=0}^{n} (-\mu)^{n-k} \alpha_k = 0.$$  

(2.8)

Denoting the roots of this equation by $\mu_i$ we get

$$\alpha_k = \chi(\sigma_k) = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} \mu_{i_1} \mu_{i_2} \cdots \mu_{i_k}.$$  

(2.9)

Hereafter (up to the last section) we assume the numbers $\mu_i \in \mathbb{K}$ to be fixed and the character $\chi$ to be defined by the set of values (2.9).
Definition 2 The roots of equation (2.8) will be called the eigenvalues of the matrix $L$ on the orbit $\mathcal{L}_\hbar^\chi$ (that is, when entries of $L$ belong to the algebra $\mathcal{L}_\hbar^\chi$). The algebra $\mathcal{L}_\hbar^\chi$ will be called the 1-generic fuzzy orbit if the eigenvalues $\mu_i$ are simple.

In what follows we shall only consider the 1-generic fuzzy orbits without specifying it each time. Note, that the 1-generic fuzzy orbits can arise from a quantization of semisimple but not generic orbits (see remark 18 below).

For any fuzzy orbit, we introduce $n$ idempotents:

$$e_j = \prod_{i \neq j} \frac{(L - \mu_i I)}{(\mu_j - \mu_i)}$$

(j = 1, \ldots, n.) (2.10)

Identity (2.7) leads to

$$e_i e_j = \delta_{ij} e_i, \quad \sum_{i=1}^n e_i = I.$$ 

The CH identity (2.5) and all related objects (idempotents (2.10), corresponding projective $\mathcal{L}_\hbar^\chi$-modules, etc.) will be called basic.

Now, we shall describe a regular way of constructing some higher analogs of these objects. To this end, consider the category of finite dimensional representations of the algebra $U(g)$. Its simple objects $V_\lambda$ are labelled by sequences of numbers

$$\lambda = (\lambda_1, \ldots, \lambda_n),$$

where $\lambda_i - \lambda_{i+1}$ are nonnegative integers. Following [W] we call these sequences the signatures. If moreover, $\lambda_i$ themselves are nonnegative integers and $\sum \lambda_i = m$ we call the signature $\lambda$ ordered partition of the integer $m$. Since the algebras $U(\mathfrak{g}_\hbar)$ and $U(\mathfrak{g})$ are isomorphic to each other, the objects $V_\lambda$ can be equipped with an $U(\mathfrak{g}_\hbar)$-action.

Consider a left $U(\mathfrak{g}_\hbar)$-module $V_\lambda$ and let

$$\pi_\lambda : U(\mathfrak{g}_\hbar) \to \text{End} (V_\lambda)$$

be the corresponding left irreducible representation. All representations in question are assumed to be equivariant, i.e., they commute with the $GL(n)$-action, where we suppose $U(\mathfrak{g}_\hbar)$ to be equipped with the adjoint $GL(n)$-action.

Introduce now the map

$$\pi_\lambda^{(2)} = I \otimes \pi_\lambda : U(\mathfrak{g}_\hbar) \otimes U(\mathfrak{g}) \to U(\mathfrak{g}_\hbar) \otimes \text{End} (V_\lambda)$$

(note, that in the second factor we put $\hbar = 1$). On fixing a basis in the space $V_\lambda$, we can identify the spaces $\text{End} (V_\lambda)$ and $\text{Mat}_{n_\lambda}(K)$, where $n_\lambda = \dim V_\lambda$. Consequently, the spaces

$$U(\mathfrak{g}_\hbar) \otimes \text{End} (V_\lambda) \quad \text{and} \quad U(\mathfrak{g}_\hbar) \otimes \text{Mat}_{n_\lambda}(K) = \text{Mat}_{n_\lambda}(U(\mathfrak{g}_\hbar))$$

(2.12)

can be identified. Therefore, the above map $\pi_\lambda^{(2)}$ is of the form

$$\pi_\lambda^{(2)} : U(\mathfrak{g}_\hbar) \otimes U(\mathfrak{g}) \to \text{Mat}_{n_\lambda}(U(\mathfrak{g}_\hbar)).$$

Remark 3 Note that the spaces $V_\lambda$ and $V_\mu$ whose signatures are different by a constant shift

$$\lambda_i - \mu_i = z, \quad 1 \leq \forall i \leq n, \quad z \in K$$
have equal dimensions $n_\lambda = n_\mu$. Making the transformation

$$\pi_\lambda(e_i^j) \mapsto \pi_\lambda(e_i^j) - z \delta_i^j \text{id}_{V_\lambda} \quad \forall z \in \mathbb{K} \tag{2.13}$$

where $e_i^j$ are the $U(gl(n))$ generators, one can convert the $U(gl(n))$-representation $\pi_\lambda$ into $\pi_\mu$. In particular, setting $z = \lambda_n$ we come to the representation $\pi_\lambda$ where $\lambda$ is the partition of the form

$$\hat{\lambda} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_{n-1}, 0), \quad \hat{\lambda}_i = \lambda_i - \lambda_n. \tag{2.14}$$

Note that the objects $V_\lambda$ and $V_{\hat{\lambda}}$ differ as $U(gl(n))$-modules but coincide as $U(sl(n))$-ones (recall that simple $U(sl(n))$-modules are labelled by partitions $\lambda$). Therefore, $\dim V_\lambda = \dim V_{\hat{\lambda}}$.

Moreover, to the space $V_{\hat{\lambda}}^\ast$ (dual to $V_{\hat{\lambda}}$) there corresponds the space $V_\lambda^* \in$ the category of finite dimensional $U(gl(n))$-modules. Here the signature $\lambda^*$ is defined as

$$\lambda^* = (-\lambda_n, \ldots, -\lambda_1). \tag{2.15}$$

Note that in the category of $U(sl(n))$-modules the dual object is labelled by the partition $\hat{\lambda}^\ast$.

Now we apply $\pi_\lambda^{(2)}$ to the split Casimir element

$$\text{Cas} = l_i^1 \otimes l_j^1. \tag{2.16}$$

Hereafter the summation over the repeated indices is understood. For $\hbar = 1$, the image of the split Casimir element under the product map $U(g)^{\otimes 2} \to U(g)$ becomes the usual quadratic Casimir element $s_2 = \text{Tr} L^2 \in U(g)$.

Denote the matrix transposed to $\pi_\lambda^{(2)}(\text{Cas})$ by $L_\lambda$, i.e.

$$L_i^j = \pi_\lambda^{(2)}(\text{Cas}) = l_i^1 \otimes \pi_\lambda(l_j^1).$$

It can be easily seen that if $\lambda_0 = (1, 0, \ldots, 0)$, then $L_{\lambda_0} = L$ provided that the basis $\{x_i\} \subset V$ is fixed in such a way that $\pi_{\lambda_0}(l_i^1) \triangleright x_k = \delta_k^i x_i$.

We emphasize, that the element $\pi_\lambda^{(2)}(\text{Cas})$ actually belongs to the algebra $\text{Mat}_{n_\lambda}(U(\mathfrak{g}_h))$ — the $GL(n)$-invariant subalgebra of $\text{Mat}_{n_\lambda}(U(\mathfrak{gl}_h))$. This follows from the fact that we are working with equivariant representations $\pi_\lambda : U(\mathfrak{gl}_h) \to \text{End}(V_\lambda) = \text{Mat}_{n_\lambda}(\mathbb{K})$. Here we assume that the action of $GL(n)$ on $\text{Mat}_{n_\lambda}(U(\mathfrak{g}_h))$ is defined on the base of identification (2.12). In [K] the algebras $\text{Mat}_{n_\lambda}(U(\mathfrak{g}_h))$ were called the family algebras (classical if $\hbar = 0$ and quantum if $\hbar \neq 0$), see also [R].

Example Let $n = 2$. Then

$$\text{Cas} = a \otimes a + b \otimes c + c \otimes b + d \otimes d, \quad \text{where} \quad a = l_1^1, \ b = l_2^1, \ c = l_2^1, \ d = l_2^2.$$

Set $\lambda = (l + 1, 1, l) \in \mathbb{K}$ and consider the representation $\pi_\lambda$ of $U(gl(2))$. We have

$$\pi_\lambda(a) = \begin{pmatrix} 1 + l & 0 \\ 0 & l \end{pmatrix}, \ \pi_\lambda(b) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ \pi_\lambda(c) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ \pi_\lambda(d) = \begin{pmatrix} l & 0 \\ 0 & 1 + l \end{pmatrix}.$$ 

The corresponding matrix $L_\lambda$ is

$$L_\lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + l(a + d) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

If $l = 0$, we have just the matrix $L$.

Below we shall restrict ourselves to the matrices $L_\lambda$ corresponding to the partitions $\lambda = (m) = (m, 0, 0, \ldots, 0)$. These matrices, as well as related objects, will be denoted $L_{(m)}, \pi_{(m)}, V_{(m)}$, etc.
The matrix $L_{(m)}$ as well as $L = L_{(1)}$ satisfies the CH identity. Now we describe the construction of the corresponding CH polynomial.

In what follows we shall often use the set of all possible (non-ordered) partitions $k$ of the integer $m$ with the length not greater than $n$. By definition, $k$ is the set of $n$ integers $k_i$ with the following properties

$$k = (k_1, \ldots, k_n), \quad k_i \geq 0, \quad |k| = k_1 + \ldots + k_n = m. \quad (2.17)$$

For any partition $k \vdash m$ from the set (2.17), let

$$\mu_k(m) = \sum_{i=1}^{n} k_i \mu_i + h \sum_{1 \leq i < j \leq n} k_i k_j \quad (2.18)$$

where $\mu_i, 1 \leq i \leq n$, are elements of the algebraic closure of the center $Z(U(\mathfrak{g}_h))$. They are the solutions of the set of polynomial relations

$$\sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} \mu_{i_1}\mu_{i_2} \cdots \mu_{i_k} = \sigma_k(L), \quad 1 \leq k \leq n, \quad (2.19)$$

where $\sigma_k(L)$ are the coefficients of the basic CH polynomial (2.5).

Let us define the monic polynomial

$$CH_{(m)}(x) = \prod_{k \vdash m} (x - \mu_k(m)). \quad (2.20)$$

Since its coefficients are symmetric functions in $\mu_i$, they can be expressed via the elementary symmetric functions. By virtue of (2.19) we conclude that the coefficients of $CH_{(m)}$ belongs to $Z(U(\mathfrak{g}_h))$. Besides, $\deg CH_{(m)}(x) = n_m = \dim V_{(m)}$.

**Proposition 4** The polynomial $CH_{(m)}(x)$ is a CH polynomial for $L_{(m)}$, namely, we have

$$CH_{(m)}(L_{(m)}) \equiv 0.$$

When switching to the 1-generic orbit $L_{(h)}(\chi)$ (2.6), we fix the eigenvalues $\mu_i$ of the matrix $L$ (see (2.6)) and thereby fix the values of the quantities $\mu_k(m)$. Similarly to the case $m = 1$, the quantities $\mu_k(m)$ are called eigenvalues of the matrix $L_{(m)}$ on the orbit $L_{(h)}$. Besides, for the sake of uniformity, we put $\mu_k(1) = \mu_i$ for $|k| = 1$ and $k_j = \delta_{i,j}$.

**Proof** In what follows, besides the representations $\pi_\lambda$, we also need the right representations $\bar{\pi}_\lambda$. The representation $\pi_\lambda$ is defined in the space $V_\lambda$ dual to $V_\lambda$

$$\langle X \triangleleft \pi_\lambda(a), Y \rangle = \langle X, \pi_\lambda(a) \triangleright Y \rangle, \quad X \in V_\lambda^*, \ Y \in V_\lambda, \ a \in U(\mathfrak{g}_h).$$

Hereafter, $\triangleleft$ (resp., $\triangleright$) stands for the right (resp., left) action of a given operator on a given vector. Replacing the operators $\pi_\lambda(a)$ by $-\pi_\lambda(a)^*$ where $^*$ is an involution, we get a left representation $\pi_\lambda^*$ of the algebra $U(\mathfrak{g}_h)$ which is called contragradient to $\pi_\lambda$ and is labelled by the signature $\lambda^*$ defined in (2.15) (cf. [W]).

The idea of our proof consists in the following. To verify a relation in the algebra $U(\mathfrak{g}_h)$, we consider the image of this relations in an arbitrary representation $\pi_\lambda$ and prove that this relation is true in all such representations. Then, since any element of the enveloping algebra with trivial image in any representation $\pi_\lambda$ is trivial (cf. [D]), we conclude that our relation is valid in the algebra itself. Moreover, it suffices to check the relation in question for $h = 1$, since by rescaling we can pass to an arbitrary $h \neq 0$.

Let us introduce the following notation:

$$L_{(\lambda,m)} = \pi_\lambda(L_{(m)}), \quad (2.15)$$

The representation $\pi_\lambda$ is defined in the space $V_\lambda$ dual to $V_\lambda$. We consider the image of this relations in an arbitrary representation $\pi_\lambda$ and prove that this relation is true in all such representations. Then, since any element of the enveloping algebra with trivial image in any representation $\pi_\lambda$ is trivial (cf. [D]), we conclude that our relation is valid in the algebra itself. Moreover, it suffices to check the relation in question for $h = 1$, since by rescaling we can pass to an arbitrary $h \neq 0$.
where applying a representation \( \pi_\lambda \) to a matrix means applying it to any entry of the matrix in question. Thus, \( \pi_\lambda(L_{(m)}) \) is a matrix with entries from \( \text{End}(V_\lambda) \) which in turn is identified with \( \text{Mat}_{m\lambda}(K) \).

So, given a fixed \( m \) and an arbitrary \( \lambda \), we should prove the identity \( CH_{(m)}(L_{(\lambda,m)}) \equiv 0 \) with the polynomial \( CH_{(m)}(x) \) defined in (2.20).

For a fixed \( m \), let us first consider a subset of representations \( \pi_\lambda \) corresponding to signatures \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) such that

\[
\lambda_i - \lambda_{i+1} \geq m \quad \text{for} \quad 1 \leq i \leq n - 1. \tag{2.21}
\]

For such signatures, irreducible components in the tensor product \( V_\lambda^* \otimes V_{(m)} \) are of the form \( V_{\lambda+k}^* \), where \( k \) runs over the full set of all possible partitions (2.17). The sum \( \lambda + k \) means the signature with the components

\[
(\lambda + k)_i = \lambda_i + k_i. \tag{2.22}
\]

Since all \( k_i \leq m \), then, due to restriction (2.21), we have \( (\lambda^* + k)_i \geq (\lambda^* + k)_{i+1} \) and the space \( V_{\lambda+k}^* \) is well-defined.

Now, assuming \( \lambda \) to satisfy (2.21), we use the split Casimir element (2.16) in order to calculate the eigenvalues of the matrices \( L_{(\lambda,1)} \) and \( L_{(\lambda,m)} \). With this purpose we insert the split Casimir \( \text{Cas} \in U(g) \otimes U(g) \) between the factors of the tensor product \( V_\lambda \otimes V_{(m)} \) and consider the image

\[
(\pi_\lambda \otimes \pi_{(m)})(V_\lambda^* \otimes \text{Cas} \otimes V_{(m)}) \overset{\text{def}}{=} \sum_{1 \leq i, j \leq n} (V_\lambda^* \otimes \pi_\lambda(l_i^j)) \otimes (\pi_{(m)}(l_j^i)) \triangleright V_{(m)}. \tag{2.23}
\]

Thus, we have represented \( \text{Cas} \) as a linear operator \( \text{Cas}_{(\lambda,m)} \) in the space \( V_\lambda^* \otimes V_{(m)} \)

\[
\text{Cas}_{(\lambda,m)} : V_\lambda^* \otimes V_{(m)} \rightarrow V_\lambda^* \otimes V_{(m)}.
\]

This operator commutes with the action of the group \( GL(n) \) and therefore it is scalar on each irreducible component \( V_{\lambda+k}^* \subset V_\lambda^* \otimes V_{(m)} \). Since in an appropriate basis of \( V_\lambda^* \otimes V_{(m)} \) the matrix of the operator \( \text{Cas}_{(\lambda,m)} \) coincides with \( L_{(\lambda,m)} \), then the eigenvalues of the matrix \( L_{(\lambda,m)} \) are nothing but the eigenvalues of the operator \( \text{Cas}_{(\lambda,m)} \).

Let \( \mu_k(\lambda, m) \) be the eigenvalue of \( \text{Cas}_{(\lambda,m)} \) on the component \( V_{\lambda+k}^* \) (if \( m = |k| = 1 \), \( k_j = \delta_{i,j} \) we shall also use the notation \( \mu_i(\lambda) \) for the eigenvalues of \( \text{Cas}_{(\lambda,1)} = L_{(\lambda,1)} \)). One can prove that

\[
\mu_k(\lambda, m) = -\frac{1}{2}(s_2(\lambda^* + k) - s_2(\lambda^*) - s_2((m))),
\]

where \( s_2(\lambda) \) stands for the value of the quadratic Casimir element on the irreducible module \( V_\lambda \). This formula is an immediate consequence of the Leibnitz rule (or in other words, the coproduct in the algebra \( U(g) \)). The negative sign appears due to the passage from the representation \( \pi_\lambda \) to \( \pi_\lambda^* \).

The straightforward calculation gives

\[
s_2(\lambda) = \sum_{1 \leq i \leq n} (\lambda_i^2 + \lambda_i(n + 1 - 2i)).
\]

Therefore, for any \( \lambda \) satisfying (2.21), we have

\[
\mu_k(\lambda, m) = -\sum_{i=1}^n k_i(-\lambda_{n-i+1} - i + 1) + \sum_{1 \leq i < j \leq n} k_ik_j. \tag{2.24}
\]

In particular, if \( m = 1 \), we get the eigenvalues of \( L_{(\lambda,1)} \)

\[
\mu_i(\lambda) = \lambda_{n-i+1} + i - 1, \quad i = 1, \ldots, n. \tag{2.25}
\]
To pass to a generic $\hbar$, we should multiply all these eigenvalues by $\hbar$. Finally, we conclude that for all $\hbar$ and the signatures $\lambda$ satisfying (2.21) the eigenvalues of $L_{(\lambda,m)}$ and $L_{(\lambda,1)}$ are indeed connected by (2.18):

$$\mu_k(\lambda, m) = \sum_i k_i \mu_i(\lambda) + \hbar \sum_{1 \leq i < j \leq n} k_i k_j.$$  

(2.26)

So, if we apply such a representation $\pi_\lambda$ to the matrix $\mathcal{C}H_{(m)}(L_{(m)})$ we obtain 0. Now we must get rid of restriction (2.21).

For this purpose, observe the following. Let $m = 1$. Then once $\lambda_i \geq \lambda_{i+1} + 1$ the eigenvalues $\mu_i(\lambda)$ are given by (2.25). However, this formula is valid even if the condition (2.21) with $m = 1$ is not fulfilled. Indeed, formula (2.33) below gives correct values of central elements $\text{Tr} L^s_{(\lambda,1)}$ if we assume the eigenvalues $\mu_j(\lambda)$ to be always given by (2.26) without any restrictions on $\lambda$. It can be verified through calculating $\text{Tr} L^s_{(\lambda,m)}$ by other means. For instance, it can be done as in [BR] or $\text{Tr} L^s_{(\lambda,1)}$ can be obtained from the $q$-Newton identities below. By inverting the Newton identities we can express the central elements $\sigma_k(L)$ in terms of $\text{Tr} L^s$. This implies that formulae (2.26) are true for any representation $\pi_\lambda$.

Now, we are able to complete the proof. Even if $\lambda$ does not satisfy (2.21) for a given $m$, the eigenvalues of the split Casimir $\text{Cas}_{(\lambda,m)}$ form a subset of the set $\{\mu_k(m)\}$ with $\mu_k(m)$ defined in (2.18). So, if we apply the representation $\pi_\lambda$ to the matrix $\mathcal{C}H(L_{(m)})$ we get 0. This implies the statement.

Remark 5 As we have seen above, the degree of the polynomial $\mathcal{C}H_{(\lambda,1)}$ is always equal to $n$, even though the number of the eigenvalues of the split Casimir $\text{Cas}_{(\lambda,1)}$ can be smaller. This means that not all eigenvalues $\mu_i(\lambda)$ of the matrix $L_{(\lambda,1)}$ can be found from the split Casimir but only those appearing in the minimal polynomial (see remark 18). However, formula (2.33) below is always true since extra eigenvalues which do not come in the split Casimir spectrum have vanishing coefficients in the r.h.s. of (2.33).

Definition 6 For a given natural $m$, a fuzzy orbit will be called $m$-generic if it is 1-generic and the eigenvalues $\mu_k(m)$ are simple. A fuzzy orbit will be called generic if it is $m$-generic for any $m$.

For any $m$-generic fuzzy orbit, we define $n_m = \dim V_{(m)} = \binom{n+m}{m}$ idempotents in the usual way:

$$e_k(m) = \prod_{k' \neq k} \frac{(L_{(m)} - \mu_k(m))I}{(\mu_k(m) - \mu_{k'}(m))}.$$  

(2.27)

The following proposition is an easy corollary of the CH identity $\mathcal{C}H^\chi_{(m)}(L_{(m)}) = 0$ and it is treated as an NC version of the spectral decomposition.

Proposition 7 For any $m$-generic fuzzy orbit $\mathcal{L}^\chi_{\hbar}$ characterized by eigenvalues $\mu_i$, $1 \leq i \leq n$, there exist quantities $d_k(m)$ such that

$$\text{Tr} L^s_{(m)} = \sum_{|k|=m} \mu_k(m)^s d_k(m), \quad s = 1, 2, \ldots,$$

where $\mu_k(m)$ are given by (2.18). Moreover, we have

$$d_k(m) = \text{Tr} e_k(m).$$

The quantities $d_k(m)$ will be called the quantum multiplicities. Their precise values on fuzzy orbits are given by the following proposition.
Proposition 8 For any \( m \)-generic orbit \( \mathcal{L}_m^\chi \) characterized by eigenvalues \( \mu_i, \, 1 \leq i \leq n \) we have
\[
d_k(m) = \prod_{1 \leq i < j \leq n} \frac{\mu_i - \mu_j - (k_i - k_j)\hbar}{\mu_i - \mu_j}.
\] (2.28)

Equivalently, the following "higher Newton identities" are valid
\[
\text{Tr} L_{(m)}^s = \sum_{|k| = m} \left( \sum_{i=1}^{n} k_i \mu_i + \hbar \sum_{1 \leq i < j \leq n} k_i k_j^s \prod_{1 \leq i < j \leq n} \frac{\mu_i - \mu_j - (k_i - k_j)\hbar}{\mu_i - \mu_j} \right). 
\] (2.29)

Proof The proof is based on the following observation. It is not difficult to see that the quantities \( \text{Tr} L_{(m)}^s \) in (2.29) are (symmetric) polynomials in \( \mu_i \). To define a polynomial unambiguously, it suffices to fix it at a finite set of values of its arguments.

As such a set we take the values \( \mu_i(\lambda) \) connected with the finite dimensional representations parametrized by signatures \( \lambda \). However, a subtle point here is that not any representation \( \pi_\lambda \) is admissible for our purpose. In the proof we use the split Casimir element, therefore we should take a signature \( \lambda \) in such a way that the corresponding eigenvalues \( \mu_k(\lambda, m) \) would be all present in the spectrum of \( \text{Cas}_{(\lambda, m)} \) and, besides, they would be all distinct. A simple analysis of the structure of \( \mu_i(\lambda) \) and \( \mu_k(\lambda, m) \) shows that these conditions can always be met if the differences \( \lambda_i - \lambda_{i+1} \) are sufficiently large. Let us introduce the corresponding notion.

Definition 9 A representation \( \pi_\lambda \) (and the corresponding signature \( \lambda \)) will be called \( m \)-admissible if condition (2.24) is fulfilled and the eigenvalues \( \mu_k(\lambda, m) \) (2.26) are all distinct.

So, in an \( m \)-admissible representation \( \pi_\lambda \) we have
\[
\pi_\lambda(\text{Tr} L_{(m)}^s) = \sum_{|k| = m} \mu_k(\lambda, m)^s d_k(\lambda, m) \text{Id}_{V_\lambda^s}
\] (2.30)

with some multiplicities \( d_k(\lambda, m) \). The identity operator in the right hand side appears since \( \text{Tr} L_{(m)}^s \in \mathcal{Z}(U(g_h)) \) and \( \pi_\lambda \) is an irreducible representation.

Now, on the one hand, the multiplicities \( d_k(\lambda, m) \) should be specializations of \( d_k(m) \) at \( \mu_i = \mu_i(\lambda) \). On the other hand, \( d_k(\lambda, m) \) can be computed independently by means of the split Casimir element. To prove the proposition, we have to show that these two ways of computation give the same results.

So, we proceed in the way analogous to proof of proposition \[4\]. We apply the representation \( \pi_\lambda \) to the matrix \( L_{(m)}^s \) passing thereby to \( L_{(\lambda, m)}^s \). Then, the transposed matrix \( (L_{(\lambda, m)}^s)^t \) is nothing but the matrix of \( \text{Cas}_{(\lambda, m)}^s \), i.e., the \( s \)-th power of the linear operator \( \text{Cas}_{(\lambda, m)} \) constructed via the split Casimir element (for detail, see \[GLS2\]).

Applying the transposition operator to the CH identities \( \mathcal{CH}_{(m)}^\chi(L_{(\lambda, m)}) = 0 \) where \( \chi \) is fixed by the values of \( \mu_i \), we conclude that the set of eigenvalues of the matrix \( L_{(\lambda, m)} \) coincides with that of the operator \( \text{Cas}_{(\lambda, m)} \).

Since the operator \( \text{Cas}_{(\lambda, m)} \) is scalar on irreducible components \( V_\chi + k \), we get
\[
\text{Tr} \text{Cas}_{(\lambda, m)}^s = \sum_{|k| = m} \mu_k(\lambda, m)^s \text{dim} V_\chi + k.
\] (2.31)

Here \( \text{Tr} = \text{Tr} \otimes \text{Tr} : \text{End}(V_\lambda) \otimes \text{End}(V_{(m)}) \to \mathbb{K} \). Taking the traces in (2.30) and comparing the result with (2.31) we conclude that
\[
d_k(\lambda, m) = \frac{\text{dim} V_\chi + k}{\text{dim} V_\chi}.
\] (2.32)
At last, using the well known Frobenius formula
\[
\dim V_{\lambda} = \prod_{i<j} (\lambda_i - \lambda_j - i + j)
\]
and taking into account (2.25), we find that \(d_k(\lambda, m)\) are indeed given by (2.28) with the substitution \(\mu_i = \mu_i(\lambda)\) and \(\hbar = 1\). The passage to a generic \(\hbar \neq 0\) is standard.

So, formulae (2.28) and (2.29) have been proved for any \(m\)-admissible representation. But since the family of \(m\)-admissible signatures \(\lambda\) is a large enough (actually infinite) set, then by virtue of reasons discussed at the beginning of the proof we conclude that (2.29) is true in general. □

If \(m = 1\), then (2.29) reduces to
\[
\text{Tr} L^k = \sum_{j=1}^n \mu_j^s \prod_{i \neq j} \frac{\mu_j - \mu_i - \hbar}{\mu_j - \mu_i}. \tag{2.33}
\]
As we have said above, at \(\hbar = 1\) and \(\mu_i = \mu_i(\lambda)\) this formula is equivalent to that from [BR].

Formula (2.33) together with (2.19) can be treated as an \(n\)-parametric resolution of the Newton relations in the algebra \(U(\mathfrak{g}_\hbar)\). Indeed, the right hand side of (2.33) is a symmetric polynomial in \(\mu_i\), and therefore it can be expressed as a polynomial in elementary symmetric functions in \(\mu_i\). Upon replacing these functions by \(\sigma_k\), we find explicit relations between two central families \(\{\sigma_k\}\) and \(\{s_k = \text{Tr} L^k\}\) in the algebra \(U(\mathfrak{g}_\hbar)\). The variables \(\mu_i\) can be thought of as the elements of the algebraic closure of the center of \(U(\mathfrak{g}_\hbar)\).

In the same sense we consider (2.29) as the higher order counterparts of the Newton relations. In section 6 we present the \(q\)-analogs of these relations.

3 Reflection Equation Algebra: definition and basic properties

This section is devoted to the detailed description of the Hecke symmetries \(R\) and corresponding reflection equation algebras which were shortly outlined in the Introduction.

Let \(V\) be a vector space over \(K\), \(\dim V = n\) and let \(R \in \text{End} (V^{\otimes 2})\) be an endomorphism. We call \(R\) braiding if it satisfies the Yang-Baxter equation
\[
R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}, \tag{3.1}
\]
where \(R_{12} = R \otimes \text{id}_V\) and \(R_{23} = \text{id}_V \otimes R\) are treated as elements of \(\text{End} (V^{\otimes 3})\). On fixing a basis \(\{x_i\} \in V, 1 \leq i \leq n\), in the space \(V\), we can realize the endomorphism \(R\) as a numerical \(n^2 \times n^2\) matrix \(R\) for which we use the same notation and call it \(R\)-matrix:
\[
(x_i \otimes x_j) \triangleleft R = R_{ij}^{kl} x_k \otimes x_l. \tag{3.2}
\]

In what follows we shall deal with a special case of Hecke type \(R\)-matrices satisfying the four additional conditions listed below under the items C1) – C4).

C1) First of all, the \(R\)-matrix should obey the Hecke condition
\[
(R - qI)(R + q^{-1}I) = 0, \tag{3.3}
\]
where \(q\) is a fixed nonzero number from the ground field \(K\) with the only constraint
\[
q^m \neq 1, \quad \forall m \in \mathbb{N}. \tag{3.4}
\]
As a consequence, the \(q\)-analogs of all integers are nonzero
\[
m_q \equiv \frac{q^m - q^{-m}}{q - q^{-1}} \neq 0, \quad \forall m \in \mathbb{Z}. \tag{3.5}
\]
In some cases it proves to be convenient to consider $q$ as a formal parameter and extend $\mathbb{K}$ to the field of rational functions in the indeterminate $q$. We shall always bear in mind this extension when considering the classical limit $q \to 1$.

**C2** To formulate the further restriction on $R$, we need to recall some properties of the Hecke algebras of $A_{m-1}$ series and to describe their relation to the Hecke type $R$-matrices.

Fix a nonzero number $q \in \mathbb{K}$. The *Hecke algebra of $A_{m-1}$ series* $(m \geq 2)$ is an associative algebra $H_m(q)$ over the field $\mathbb{K}$ generated by the unit element $1_H$ and $m-1$ generators $\sigma_i$ subject to the following relations:

\[
\begin{align*}
\sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} \\
\sigma_i\sigma_j &= \sigma_j\sigma_i \quad \text{if } |i-j| \geq 2 \\
(\sigma_i - q^{1_H})(\sigma_i + q^{-1}1_H) &= 0
\end{align*}
\]

$i = 1, 2, \ldots, m-1$.

If the parameter $q$ satisfies (3.4), then for any positive integer $m$ the Hecke algebra $H_m(q)$ is isomorphic to the group algebra $\mathbb{K}S_m$ of the $m$-th order permutation group $S_m$. As a consequence, $H_m(q)$ is isomorphic to the following direct sum of matrix algebras

\[
H_m(q) \cong \bigoplus_{\lambda \vdash m} \text{Mat}_{d_\lambda}(\mathbb{K})
\]  

(3.6)

where the summation goes over all possible ordered partitions $\lambda$ of the integer $m$. The parameter $d_\lambda$ is equal to the number of all standard Young tableaux $\lambda(a)$ which can be constructed for the given partition $\lambda$.

As is known, the associative $\mathbb{K}$-algebra $\text{Mat}_k(\mathbb{K})$ possesses the linear basis of $k^2$ generators (matrix units) $E_{ab}$ $1 \leq a, b \leq k$ with the multiplication law

\[
E_{ab}E_{cd} = \delta_{bc}E_{ad}.
\]

Due to isomorphism (3.3) in the Hecke algebra $H_m(q)$ one can choose a system of generators $Y_{aa}^{\lambda}(\sigma) \in H_m(q)$, $\lambda \vdash m$, $1 \leq a, b \leq d_\lambda$, which form a linear basis in $H_m(q)$ and satisfy the following multiplication rule

\[
Y_{aa}^{\lambda}(\sigma)Y_{cd}^{\mu}(\sigma) = \delta^{\lambda\mu}\delta_{bc}Y_{ad}^{\lambda}(\sigma).
\]  

(3.7)

The diagonal "matrix units" $Y_{aa}^{\lambda}(\sigma)$ are the primitive idempotents of the Hecke algebra. Evidently, they are in one-to-one correspondence with the set of standard Young tableaux $\lambda(a)$ constructed for the given partition $\lambda$. Below we shall use the compact notation $Y_{aa}^{\lambda} \equiv Y_{\lambda(a)}$ for the primitive idempotents. The idempotent, corresponding to the partition $(1^m)$ (the one-column Young diagram) will be called the $q$-antisymmetrizer and denoted $A^{(m)}(\sigma)$.

Given a Hecke type $R$-matrix, we can construct a *local representation* of $H_m(q)$ in $V^\otimes m$ by the following rule

\[
\sigma_i \mapsto \rho_R(\sigma_i) = R_{ii+1} = I^\otimes(i-1) \otimes R \otimes I^\otimes(m-i-1) \in \text{End}(V^\otimes m).
\]  

(3.8)

In the local representation (3.8) the idempotents $Y_{\lambda(a)}(\sigma)$ are realized as some projection operators in $V^\otimes m$. With respect to the action of these projectors the space $V^\otimes m$ splits into the direct sum of subspaces $V_{\lambda(a)}$: \[
V^\otimes m = \bigoplus_{\lambda \vdash m} \bigoplus_{a=1}^{d_\lambda} V_{\lambda(a)}, \quad V_{\lambda(a)} = Y_{\lambda(a)}(R) \triangleright V^\otimes m.
\]  

(3.9)

The projector $Y_{\lambda(a)}(R) = \rho_R(Y_{\lambda(a)})$ is given by a polynomial in matrices $R_{ii+1}$. For a detailed treatment of these questions, explicit formulae for $q$-projectors and the extensive list of original papers, see [OP].
So, we shall assume the Hecke symmetry in question to be even. By definition this means that there exists an integer \( p > 0 \) such that the image of the \( q \)-antisymmetrizer \( A^{(p+1)}(\sigma) \) in the local \( R \)-matrix representation \( \rho_R \) identically vanishes while the image of the \( q \)-antisymmetrizer \( A^{(p)}(\sigma) \) is a projector of the unit rank in the space \( V^\otimes m \) (for any \( m > p \))

\[
\exists p \in \mathbb{N} : \begin{cases} 
A^{(p+1)}(\sigma) \mapsto A^{(p+1)}(\rho) \equiv 0, \\
A^{(p)}(\sigma) \mapsto A^{(p)}(\rho), \quad \text{rank } A^{(p)}(\rho) = 1.
\end{cases}
\]

(3.10)

Such a number \( p \) will be called the symmetry rank of the matrix \( R \). For example, the symmetry rank of the \( R \)-matrix connected with the quantum universal enveloping algebra \( U_q(sl(n)) \) is equal to \( n \). Examples of \( n^2 \times n^2 \) \( R \)-matrices with \( p < n \) (for \( n \geq 3 \)) can be found in [G].

**C3)** Besides, we assume the \( R \)-matrix to be skew-invertible; that is there exists an \( n^2 \times n^2 \) matrix \( \Psi \) with the property

\[
\sum_{a,b} R_{ia}^{jb} \Psi_{bk}^{as} = \delta_i^s \delta_j^k = \sum_{a,b} \Psi_{ia}^{jb} R_{bk}^{as}.
\]

In the compact notations the above formula reads

\[
\text{Tr}_{(2)} R_{12} \Psi_{23} = P_{13} = \text{Tr}_{(2)} \Psi_{12} R_{23},
\]

(3.11)

where the symbol \( \text{Tr}_{(2)} \) means applying the trace in the second space and \( P \) is the permutation matrix. Note, that this condition does not depend on a choice of basis, therefore we can treat \( \Psi \) as an endomorphism as well.

Let now \( B \) and \( C \) be two endomorphisms of the space \( V \) represented by the following \( n \times n \) matrices

\[
B = \text{Tr}_{(2)} \Psi_{21}, \quad C = \text{Tr}_{(2)} \Psi_{12},
\]

(3.12)

where \( \Psi \) is defined in (3.11). If the \( R \)-matrix has the symmetry rank \( p \), we have

\[
B \cdot C = \frac{1}{q^{2p}} I
\]

(cf. [C]). This implies that the matrices \( B \) and \( C \) are invertible. Moreover, we have

\[
\text{Tr} B = \text{Tr} C = \frac{pq}{q^p}.
\]

(3.14)

Note that **C3)** is in fact a consequence of **C2)** since any even Hecke symmetry is skew-invertible automatically.

**C4)** As the last requirement on the \( R \)-matrix we shall assume, that the space \( V^* \) dual to \( V \) can be identified with the \( (p-1) \)-th wedge \( q \)-power of \( V \)

\[
V^* = \wedge_q^{p-1} V.
\]

Explicitly this identification is constructed as follows. Since \( A^{(p)}(\rho) \) is the unit rank projector then in the basis \( x_{i_1} \otimes \ldots \otimes x_{i_p} \) (see (3.2)) its matrix can be represented in the form

\[
A^{(p)}_{i_1i_2\ldots i_p}^{j_1j_2\ldots j_p} = u_{i_1i_2\ldots i_p}^{j_1j_2\ldots j_p}
\]

with some structure tensors \( u \) and \( v \). One can show that the vectors

\[
x^i = v^{ia_2\ldots a_p} x_{a_2} \otimes \ldots \otimes x_{i_p}
\]

(3.15)
are linear independent and by definition form the basis of the space \( \wedge_q^{p-1}V \subset V^{\otimes(p-1)} \). We shall not explicitly describe the property of \( R \) which allows us to identify \( V^* \) and \( \wedge_q^{p-1}V \). The thorough treatment of this problem is presented in [GLST].

In the particular case \( p = 2 \) which will be studied in detail below the components of structure tensor \( v^{ij} \) form a nondegenerated matrix [G]. Therefore, as follows from \( 3.15 \), the space \( V^* \) is isomorphic to \( V \) itself

\[
V^* \cong V \quad \text{at} \quad p = 2. \tag{3.16}
\]

As the next step, we define the reflection equation algebra. Consider a unital associative \( \mathbb{K} \)-algebra \( \mathcal{L}_{h,q} \) generated by \( n^2 \) elements \( l_i^j \), \( 1 \leq i, j \leq n \) satisfying the following relations

\[
RL_1 RL_1 - L_1 RL_1 R = h(RL_1 - L_1 R), \quad L_1 \equiv L \otimes I, \tag{3.17}
\]

where \( h \) is a numerical parameter and \( L = \| l_i^j \| \) is a matrix composed of \( l_i^j \). The algebra \( \mathcal{L}_{h,q} \) is said to be the modified reflection equation algebra (mREA). In the particular case \( h = 0 \) this algebra will be called (non-modified) REA and denoted \( \mathcal{L}_q \).

**Remark 10** Similarly to the algebra \( U(g_0) \) the mREA corresponding to different nonzero parameters \( h \) are isomorphic — one can easily pass from one \( h \neq 0 \) to another \( h' \neq 0 \) by a trivial renormalization of generators.

Moreover, at \( q \neq \pm 1 \) the generators \( l_i^j \) of the algebra \( \mathcal{L}_{h,q} \) is connected with the generators \( \hat{l}^{ij}_i \) of \( \mathcal{L}_q \) via a linear shift by the unit element \( \text{id}_\mathcal{L} \)

\[
l_i^j = \hat{l}^{ij}_i + \frac{h}{\zeta} \delta_i^j \text{id}_\mathcal{L}, \quad \zeta = q - q^{-1}, \tag{3.18}
\]

and therefore these two kinds of REA are isomorphic, too. Nevertheless, their classical limits are different, since the above isomorphism is broken as \( q \to 1 \).

Let us now define an important map \( \text{Tr}_R : \text{Mat}_n(\mathcal{L}_{h,q}) \to \mathcal{L}_{h,q} \)

\[
\text{Tr}_R(X) \overset{\text{def}}{=} \text{Tr}(C \cdot X), \quad X \in \text{Mat}_n(\mathcal{L}_{h,q}). \tag{3.19}
\]

For the \( U_q(sl(n)) \) R-matrix, such a map is called the quantum trace [PRT] and is often denoted \( \text{Tr}_q \). Note that in the sequel we apply the trace \( 3.19 \) only to the matrices from the space \( \text{Mat}_n(\mathcal{L}_{h,q}) \) which are in some sense invariant.

Similarly to the classical case, a generating set for the center of the algebra \( \mathcal{L}_{h,q} \) consists of the elements:

\[
s_0 \equiv \text{id}_\mathcal{L}, \quad s_k = q^{Tr_R L_k}, \quad 1 \leq k \leq p, \tag{3.20}
\]

where the factor \( q \) is chosen for the future convenience. These elements become scalar operators on each irreducible module over the mREA (see section [1]).

One of the basic properties of the REA (modified or not) consists in the following. In this algebra, similarly to \( U(su(n)) \) (and to some other Lie algebras, cf. [Go]), one can find a series of the Cayley-Hamilton identities. Let us discuss the first of them, which will be called the basic \( CH \) identity.

As was shown in [GPS], the matrix \( \hat{L} = \| \hat{l}^{ij}_i \| \) of generators of REA satisfies the CH identity of the form

\[
\sum_{k=0}^{p} (-\hat{L})^{p-k} \sigma_k(\hat{L}) = 0, \quad \sigma_0(\hat{L}) \equiv \text{id}_\mathcal{L}, \quad \hat{L}^0 \equiv I \tag{3.21}
\]

where \( \{ \sigma_k \} \) is another set of generators of the center of REA connected with the set \( \{ s_k \} \) by means of the \( q \)-Newton relations \( 5.6 \) and \( p \) is the symmetry rank of the corresponding \( R \).
Applying the shift (3.18) it is possible to get analogous relations for the matrix \( L \) with entries belonging to the corresponding mREA:

\[
\sum_{k=0}^{p} (-L)^{p-k} \sigma_k(L) = 0, \quad \sigma_0(L) \equiv \text{id}_L.
\]

As above the coefficients \( \sigma_k(L) \) are central element of the algebra \( L_{\hbar,q} \).

An explicit form of these coefficients was obtained in [GS1]. However, this form is somewhat cumbersome. Here we only present an example of such a CH identity for the mREA algebra related to a Hecke symmetry with the symmetry rank \( p = 2 \):

\[
L^2 - \left( q \text{Tr}_R L + \frac{\hbar}{q} \right) L + \left( \frac{q^2}{2q} (q(\text{Tr}_R L)^2 - \text{Tr}_R L^2) + \hbar \frac{q}{2q} \text{Tr}_R L \right) I = 0. \tag{3.22}
\]

In this formula the coefficients \( \sigma_i(L) \) are expressed via the quantities \( \text{Tr}_R (L^k) \). In what follows we will find some inverse relations expressing the latter quantities via \( \sigma_i(L) \) but in a parametric form.

By analogy with the case of fuzzy orbit (see definition 2 and formula (2.9)) we define a character \( \chi : Z(L_{\hbar,q}) \to K \) of the center of the mREA \( L_{\hbar,q} \) by fixing its values on the central elements \( \sigma_k \)

\[
\chi(\sigma_k(L)) = \alpha_k = \sum_{1 \leq i_1 < ... < i_k \leq p} \mu_{i_1} \cdots \mu_{i_k}, \tag{3.23}
\]

where the numbers \( \mu_i \), \( 1 \leq i \leq p \) are assumed to be all distinct.

Then we define a 1-generic NC orbit \( L_{\hbar,q}^\chi \) as the quotient of the mREA (3.17) modulo the two sided ideal \( \mathcal{I}^\chi \)

\[
L_{\hbar,q}^\chi = L_{\hbar,q} / \mathcal{I}^\chi \tag{3.24}
\]

where \( \mathcal{I}^\chi \) is generated by the set of relations

\[
\sigma_k(L) - \alpha_k, \quad \text{for} \quad 1 \leq k \leq p. \tag{3.25}
\]

Finally, we will compute the quantities \( \text{Tr}_R (L^k) \) restricted on the NC orbit \( L_{\hbar,q}^\chi \) in terms of \( \mu_i \).

Following the pattern of the case considered in section 2 we will also compute the quantities \( \text{Tr}_R (L_{(m)}^k) \) for some higher extensions \( L_{(m)} \) of the matrix \( L \). The crucial role in this computing is played by a split Casimir element which is defined for the mREA in the following way

\[
\text{Cas} = q^{2p} l_1^k \otimes l_1^j C_j^i \in L_{\hbar,q} \otimes L_{\hbar,q}. \tag{3.26}
\]

As in the classical case considered in section 2, this element allows us to introduce the aforementioned extensions \( L_{(m)} \) of the matrix \( L \) and to construct the higher CH identities for them. Also note, that in the \( U_q(sl(n)) \) case at the limit \( q = 1 \) we get just the split Casimir element (2.16) described above.

## 4 mREA as a braided enveloping algebra

In this section we give a short review of the representation theory of REA which is necessary for the subsequent sections. Besides, we adduce some arguments which allow us to consider the mREA as a “braided” analog of the enveloping algebra.

First of these arguments originates from the consideration of the mREA for an involutive \( R \)-matrix: \( R^2 = I \). This is a particular case of the Hecke condition (3.3) corresponding to \( q = 1 \). The mREA (3.17) with involutive \( R \)-matrix turns out to be the enveloping algebra of some generalized Lie algebra.
The notions of generalized Lie algebra and its enveloping algebra were introduced in [GLS1] (see also the references therein). Given an involutive $R$-matrix (treated as an endomorphism of $V^{\otimes 2}$), the generalized enveloping algebra is defined as the quotient of the free tensor algebra of the space $\text{End}(V)$ over the two-sided ideal generated by the relations

$$X \otimes Y - R_{\text{End}(V)}(X \otimes Y) = \circ X \otimes Y - \circ R_{\text{End}(V)}(X \otimes Y), \quad X, Y \in \text{End}(V),$$

where $\circ : \text{End}(V)^{\otimes 2} \to \text{End}(V)$ is the usual product in the space of endomorphisms and $R_{\text{End}(V)}$ is the extension of the initial braiding $R$ to the space $\text{End}(V)$ (which is well defined as $R$ is skew-invertible).

If we realize $\text{End}(V)$ as the space of left endomorphisms, fix a natural basis $\{h_i^j\}$ such that $h_i^j \circ h_k^l = \delta_i^k h_l^j$ and compute $R_{\text{End}(V)}$ in this basis, we recover the enveloping algebra of the generalized Lie algebra in terms of generators $h_i^j$. Note, that $h_i^j$ can be identified with $x_i \otimes x^j$ where $\{x^j\}$ is the basis of the left dual space to $V$, see [GLS1] for detail.

The point is that in $\text{End}(V)$ we can choose another basis $\{l_i^j\}$ such that $l_i^j \circ l_k^l = l_l^i B_i^j$, where $B_i^j$ is the matrix element of the endomorphism $B$ (3.12) written in the basis $\{x_i\}$. Being expressed in terms of the new generators $l_i^j$, the enveloping algebra in question turns into the mREA with $\hbar = 1$. The elements $l_i^j$ can be identified with $x_i \otimes x^j$ where $\{x^j\}$ is the basis of the right dual space to $V.$ The details are left to the reader.

Let us also mention that in the $U_q(sl(n))$ case (in contrast with quantum groups corresponding to other simple Lie algebras) the mREA is a two parameter deformation of the symmetric algebra of $gl(n)$ (a close treatment of this algebra is given in [IP]). The corresponding Poisson structure is the aforementioned pencil which, in fact, is well defined on the whole $gl(n)^*.$

The most important property of the mREA which enable us to treat this algebra as a braided analog of the enveloping algebra is that the category of its equivariant finite dimensional representations is close to that of $U(gl(p))\text{-Mod}$ where $p$ is the symmetry rank of the $R$-matrix.

Let us first consider the quotient of the mREA $\mathcal{L}_{h,q}$ over the ideal generated by the relation $\text{Tr}_R L = 0.$ We denote this quotient algebra as $\mathcal{S}\mathcal{L}_{q,h}$. This is an analog of the $U(sl(p))$ subalgebra in the $U(gl(p))$. The category of finite dimensional completely reducible modules over the $\mathcal{S}\mathcal{L}_{q,h}$ is the quasitensor Schur-Weyl category introduced in [GLS1]. Its simple objects (irreducible modules) are labelled by the partitions $\lambda$ whose height (the number of nonzero parts) are not greater than $p - 1$. Besides, the Grothendieck ring of the Schur-Weyl category is isomorphic to that of the category $U(sl(p))\text{-Mod}$.

The representations of $\mathcal{L}_{h,q}$ are labelled by the partitions $\lambda$ whose height is not greater than $p$ and a number $z \in \mathbb{K}$ which is analog of the shift (2.13) of the $U(gl(p))$ representations (see remark 14 below). In full analogy with the classical case discussed in remark 3 the vector spaces $V_{\lambda,z}$ and $V_{\hat{\lambda}}$ are isomorphic as $\mathcal{S}\mathcal{L}_{q,h}$ modules, $\hat{\lambda}$ being constructed from $\lambda$ in accordance with (2.14).

The map $\text{Tr}_R$ defined in (3.19) is closely related to the categorical trace $\text{Tr}_{V_\lambda} : \text{End}(V_\lambda) \to \mathbb{K}$. The categorical trace is a morphism of the Schur-Weyl category which plays the same role as the usual trace does in the category $U(sl(p))\text{-Mod}$. In particular, the categorical trace allows one to define the notion of the $q$-dimension

$$\dim_q V_\lambda = \text{Tr}_{V_\lambda}(\text{id}_{V_\lambda}),$$

(4.1)

which is a multiplicative-additive functional on the Grothendieck ring of the Schur-Weyl category. Namely we have

$$\dim_q (V_\lambda \otimes V_\mu) = \dim_q (V_\lambda) \dim_q (V_\mu),$$

$$\dim_q (V_\lambda \oplus V_\mu) = \dim_q (V_\lambda) + \dim_q (V_\mu)$$

(4.2)

and the $q$-dimensions of the isomorphic spaces are equal to each other (see [GLS1] for detail).
Now let us give a short review of some facts from the representation theory of the mREA. Their detailed description can be found in [S].

The mREA $\mathcal{L}_{h,q}$ possesses a profound representation theory. We shall confine ourselves to considering the finite dimensional, completely reducible, equivariant modules over $\mathcal{L}_{h,q}$ (the term "equivariant" will be explained at the end of the section). Besides, we can set $\hbar = 1$ by virtue of the isomorphisms mentioned in Remark 10.

First of all, we define the so-called left fundamental module of $B$ type. Let $V$ be an $n$-dimensional vector space with a fixed basis $\{x_i\}, 1 \leq i \leq n$. Putting $h = 1$, we consider the homomorphism $\pi : \mathcal{L}_{1,q} \rightarrow \text{End}(V)$ defined as follows

$$\pi(l^j_i) \triangleright x_k = x_k B^j_k,$$

where the matrix $B$ is introduced in (3.12). Since $B$ is invertible (see (3.13)) the representation $\pi$ is irreducible.

The tensor power $V^\otimes m$, $m \in \mathbb{N}$, can be endowed with the structure of a (reducible) mREA module. The corresponding homomorphism $\rho_m : \mathcal{L}_{1,q} \rightarrow \text{End}(V^\otimes m)$ is of the form

$$\rho_m(l^j_i) = \pi_1(l^j_i) + R_{12}^{-1} \pi_1(l^j_i) R_{12}^{-1} + \ldots + R_{m-1,m}^{-1} \pi_1(l^j_i) R_{m-1,m}^{-1} \ldots R_{12}^{-1} \pi_1(l^j_i) R_{12}^{-1},$$

where

$$\pi_1 = \pi \otimes I^{\otimes (m-1)}, \quad R_{k,k+1} = I^{k-1} \otimes R \otimes I^{m-k-1} \quad 1 \leq k \leq m-1.$$  

Here $R$ is an automorphism of $V^\otimes 2$ connected with the matrix $R$ of the Hecke symmetry by the following definition

$$R \triangleright (x_i \otimes x_j) = \sum_{r,s} R_{ij}^{rs} x_k \otimes x_l.$$  

The representation (4.1) is reducible. In accordance with (3.9) it decomposes into the direct sum of mREA submodules $V_{\lambda(a)}$, where $\lambda$ is an ordered partition of $m$. The representations $\pi_{\lambda(a)}$ are extracted from $\rho_m$ by the action of the corresponding projectors $Y_{\lambda(a)}(R)$.

We write down the explicit form of the representation $\pi_m$, corresponding to the partition $(m)$, since it will play an important role in what follows. The subspace $V_m \subset V^\otimes m$ is an image of the $q$-symmetrizer $S^{(m)}(R)$ whose matrix is iteratively defined as follows (see [G])

$$S^{(1)} = I, \quad S^{(m)}_{12...m} = \frac{1}{m_q} S^{(m-1)}_{22...m} (q^{1-m} I + (m-1)_q R_{12}) S^{(m-1)}_{22...m}.$$  

The following proposition holds true [S].

**Proposition 11** Consider an arbitrary tensor power $V^\otimes m$ of the left fundamental module $V$. Its $q$-symmetric subspace $V_m$ is a left $\mathcal{L}_{1,q}$-submodule. On the generators of the mREA the homomorphism $\pi_m : \mathcal{L}_{1,q} \rightarrow \text{End}(V_m)$ reads as follows:

$$\pi_m(l^j_i) = q^{1-m} m_q S^{(m)}(R) \left[ \pi(l^j_i) \otimes I^{\otimes (m-1)} \right] S^{(m)}(R),$$

where $\pi$ and $R$ are defined in (4.3) and (4.5) respectively.

Given the representation (4.3), we can realize the matrix $L$ satisfying the commutation relations (3.17) as an image of the split Casimir element $\text{Cas}$ (3.20) under the map

$$\text{id} \otimes \pi : \mathcal{L}_{h,q} \otimes \mathcal{L}_{1,q} \rightarrow \mathcal{L}_{h,q} \otimes \text{Mat}_n(\mathbb{K}).$$

(Note that in the second factor we put $h = 1$.) Indeed, taking (3.13) into account one easily gets

$$L' = (\text{id} \otimes \pi)(\text{Cas}), \quad L \in \text{Mat}_n(\mathcal{L}_{h,q}).$$  

(4.8)
Hereafter \( L' \) stands for the matrix transposed to \( L \).

By analogy with the \( U(\mathfrak{g}) \) case considered in section 2, we also introduce the symmetric matrix \( L_{(m)} \) as the following image of \( \text{Cas} \)

\[
L'_{(m)} = (\text{id} \otimes \pi_{(m)})(\text{Cas}), \quad L_{(m)} \in \text{Mat}_n (\mathcal{L}_{n,q})
\]

where \( n_m = \dim V_{(m)} \).

In addition to the left \( \mathcal{L}_{1,q} \)-modules we also need the right ones. For a generic \( p \geq 2 \) such a representation can be defined in the dual space \( V^* \). By using the method of the paper [S] we can extend this representation to the tensor power \( (V^*)^\otimes m \) and decompose it into the direct sum of submodules associated with the projectors \( Y_\lambda \). However, if \( p = 2 \) the space \( V^* \) can be identified with \( V \) itself via a categorical pairing arising from the projector of the space \( V^\otimes 2 \) onto its skew-symmetric component (see [3.10]). The spaces \( V_{(k)} \) and \( V_{(k)} \) can also be identified via such a pairing (see [GLS1] for detail). This pairing allows us to equip the space \( V_{(k)} \) with a structure of the right \( \mathcal{L}_{1,q} \)-module. More precisely, the following proposition takes place.

**Proposition 12** Let the symmetry rank of the Hecke symmetry \( R \) is \( p = 2 \). Consider an arbitrary tensor power \( \mathcal{V}^\otimes m \) of the left fundamental module \( \mathcal{V} \). Its \( q \)-symmetric subspace \( V_{(m)} \) is a right \( \mathcal{L}_{1,q} \)-submodule. On the generators of the \( \text{mREA} \) the homomorphism \( \pi_{(m)} : \mathcal{L}_{1,q} \rightarrow \text{End}_r(V_{(m)}) \) reads as follows:

\[
\pi_{(m)}(l^j) = q^{1-m}m q^{(m)}(R) \left[I^\otimes (m-1) \otimes \pi(l^j)\right]S^{(m)}(R),
\]

where the homomorphism \( \pi : \mathcal{L}_{1,q} \rightarrow \text{End}_r(V) \) has the form

\[
x_k \circ \pi(l^j) = \frac{2q}{q^2}A^{(2)}_{ki}l^j x_s, \quad A^{(2)} = \frac{1}{2q} (qI - R),
\]

and the right action of \( R \) is defined in [3.12].

In what follows we shall also need the representations of the generators \( \hat{l}^j_t \) of the \( \text{REA} \) connected with those of \( \text{mREA} \) by shift (3.18). A simple calculation proves the following corollary of the proposition 12.

**Corollary 13** The right representation of the generators \( \hat{l}^j_t \) of the \( \text{REA} \) in the \( q \)-symmetric component \( V_{(m)} \subset V^\otimes m \) is given by the homomorphism

\[
\pi_{(m)}(\hat{l}^j_t) = q^{1-m}m q^{(m)}(R) \left[I^\otimes (m-1) \otimes \pi(\hat{l}^j_t)\right]S^{(m)}(R),
\]

where

\[
x_k \circ \pi(\hat{l}^j_t) = \Phi_{ki}^{sj} l^j x_s, \quad \Phi = q^{1-m}m q^{(m)} I - \frac{2q}{q^2} A^{(2)}.
\]

**Remark 14** Up to now, the representations of \( \text{mREA} \) were labelled by partitions \( \lambda \), whereas in general the finite dimensional representations of \( U(gl(n)) \) are labelled by signatures \( \lambda \). What is an analog of these representations in the \( \text{mREA} \) case? To answer the question, observe the following.

As was mentioned in remark 3, with any finite dimensional \( U(gl(n)) \) representation \( \pi_\lambda \) labelled by a signature \( \lambda \) we can associate the representation \( \pi_\hat{\lambda} \) where the partition \( \hat{\lambda} \) is connected with the signature by relation (2.14). These representations are connected by the unit operator shift (2.13) and the corresponding modules \( V_\lambda \) and \( V_\hat{\lambda} \) are isomorphic as \( U(sl(n)) \)-modules and therefore as vector spaces.
For the mREA representations there exists a transformation analogous to (2.13). Namely, a simple calculation shows that if \( \pi_\lambda \) is a representation of mREA in the space \( V_\lambda \) where \( \lambda \) is a partition then the operators

\[
\pi^z_\lambda(l^j_i) = z \pi_\lambda(l^j_i) + \delta^i_j \frac{1-z}{\zeta} \text{id}_{V_\lambda}, \quad z \in \mathbb{K}\setminus0
\]

(4.13)
also realize an mREA representation in the same space \( V_\lambda \). It can be shown that the representation of the subalgebra \( SL_\hbar, q \) which is a quotient of \( L_\hbar, q \) over the ideal generated by the relation \( T_RL = 0 \) does not change under the shift (4.13). This subalgebra is an analog of \( U(sl(n)) \) in the classical case.

However, in contrast with the classical case discussed in remark 3, we cannot put any signature into correspondence to \( \pi^z_\lambda \) since our approach to the representation theory of the mREA is not based on the technique of the highest weight vectors.

Now we explain the meaning of the statement that the proposed presentation theory of mREA is equivariant.

To any \( R \)-matrix we can assign an associative bialgebra \( T \) generated by the elements \( t^j_i \) subject to the following commutation relations

\[
RT_1T_2 = T_1T_2R, \quad \text{where} \quad T = \|t^j_i\|, \quad T_1 = T \otimes I, \quad T_2 = I \otimes T.
\]

If the \( R \)-matrix is skew-invertible (see (3.11)), then the bialgebra structure can be extended to the Hopf algebra one\(^4\). When \( R \) is the image of the universal \( U_q(sl(n)) \) \( R \)-matrix (in the fundamental vector representation), the Hopf algebra \( T \) is the well known quantization of the algebra of regular functions on the group \( GL(n) \) [FRT]. Since the REA can always be endowed with the structure of the left adjoint comodule over \( T \)

\[
l^j_i \mapsto \sum_{r,s} t^r_i S(t^j_s) \otimes l^s_r;
\]

\( S(t^j_s) \) being the antipode of \( t^j_s \), then REA is also a module over the dual Hopf algebra \( T^* \) (in the \( U_q(sl(n)) \) case this dual is the quantum group \( U_q(gl(n)) \) itself).

All the finite dimensional modules \( V_\lambda \) over the REA are also the modules over \( T^* \). The representations \( \pi_\lambda \) of the REA constructed in [S] commute (as the mappings) with the action of \( T^* \). We call them the equivariant representations precisely in this sense.

At the end of the section we would like to discuss the problem whether the category \( L_\hbar, q - \text{Rep} \) of finite dimensional equivariant representations of the algebra \( L_\hbar, q \) is "big enough".

**Definition 15** Let \( A \) be an algebra and \( A - \text{Rep} \) be the category of its representations. We say that this category is faithful if for any nonzero element \( a \in A \) there exists an object \( V \in A - \text{Rep} \) such that \( \pi_V(a) \neq 0 \) where \( \pi_V \) is the representation corresponding to \( V \).

For an involutive \( R \)-matrix (that is \( R^2 = I \)) the category \( L_\hbar, q - \text{Rep} \) of equivariant representations \( \pi^z_\lambda \) of the algebra \( L_\hbar, q \) is faithful. The same is true for any Hecke symmetry \( R \) which is a flat deformation of an involutive \( R \)-matrix (in particular, for the \( U_q(sl(n)) \) \( R \)-matrix). This can be established by a direct modification of the proof of the analogous classical statement about universal enveloping algebras given in [D]. For an arbitrary Hecke symmetry we shall suppose that the above property of the representation category takes place as a plausible conjecture.

\(^4\)In order to get such a structure it suffices to formally invert the quantum determinant in the extended algebra \( T \). It can be done by an appropriate localization, cf. [C].
5 The basic q-Newton identities

In this section we deal with a non-modified REA and generalize formulae (2.33) to this case.

First, we introduce some convenient notations. Let us denote \( \mathcal{D}(t_1, t_2, \ldots, t_n) \) the following Vandermonde determinant

\[
\mathcal{D}(t_1, t_2, \ldots, t_n) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_n \\ t_1^2 & t_2^2 & \cdots & t_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ t_1^{n-1} & t_2^{n-1} & \cdots & t_n^{n-1} \end{vmatrix} = \prod_{i>j} (t_i - t_j),
\]

where \( t_i \) are some variables. In general, this variables can be elements of a commutative algebra (a ring).

Besides this, we consider the elementary symmetric functions in the variables \( t_i \):

\[
e_0 \equiv 1, \quad e_k = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} t_{i_1} t_{i_2} \cdots t_{i_k}, \quad 1 \leq k \leq n. \tag{5.1}
\]

With each \( e_k \) we associate the series of the following quantities

\[
e_k(\hat{t}_i) \overset{\text{def}}{=} e_k|_{t_i=0}, \quad e_k(\hat{t}_i, \hat{t}_j) \overset{\text{def}}{=} e_k|_{t_i=0, t_j=0}, \quad 1 \leq i, j \leq n, \text{ and so on.} \tag{5.2}
\]

It is evident, that the quantity \( e_k(\hat{t}_i) \) is an elementary symmetric function in the set of \( (n-1) \) variables \( t_j, j \neq i \), etc.

Note some useful properties of the above quantities (their proof is a simple exercise)

\[
e_k = e_k(\hat{t}_i) + t_i e_{k-1}(\hat{t}_i), \tag{5.3}
\]

\[
e_k(\hat{t}_i) - e_k(\hat{t}_j) = (t_j - t_i) e_{k-1}(\hat{t}_i, \hat{t}_j) \tag{5.4}
\]

\[
k e_k = \sum_{i=1}^{n} t_i e_{k-1}(\hat{t}_i). \tag{5.5}
\]

Here we assume \( 1 \leq k \leq n \) and, besides, \( 1 \leq i, j \leq n \) in the first two lines.

The following lemma is easy to verify.

Lemma 16

\[
\begin{vmatrix} 1 & 1 & \cdots & 1 \\ e_1(\hat{t}_1) & e_1(\hat{t}_2) & \cdots & e_1(\hat{t}_n) \\ e_2(\hat{t}_1) & e_2(\hat{t}_2) & \cdots & e_2(\hat{t}_n) \\ \vdots & \vdots & \ddots & \vdots \\ e_{n-1}(\hat{t}_1) & e_{n-1}(\hat{t}_2) & \cdots & e_{n-1}(\hat{t}_n) \end{vmatrix} = \prod_{i<j} (t_i - t_j) \equiv \mathcal{D}(t_n, t_{n-1}, \ldots, t_1).
\]

Proof The lemma is proved by induction in the size of the determinant. Being based on (5.4), the induction proceeds in the same way as when calculating the Vandermonde determinant.

Consider now two sets of independent central elements of REA \( \mathcal{L}_q \)

\[
\sigma_k(\hat{L}) = q^k \text{Tr}_{R(12\ldots k)} A^{(k)} \hat{L}_{1,1} \ldots \hat{L}_{k}, \quad \text{and} \quad s_k(\hat{L}) = q \text{Tr}_R(\hat{L}^k), \quad 1 \leq k \leq p,
\]

were \( \hat{L}_k \) is defined as follows:

\[
\hat{L}_1 = L_1, \quad \hat{L}_k = R_{k-1} \hat{L}_{k-1} R_{k-1}^{-1}, \quad k \geq 2.
\]
We also set by definition
\[ \sigma_0(\hat{L}) = s_0(\hat{L}) = \text{id}_L. \]

These two sets of central elements are connected by the Newton identities \[GPS\] \[PS\]
\[ s_1 = \sigma_1 \]
\[ -s_2 + s_1\sigma_1 = 2q^{-1}\sigma_2 \]
\[ s_3 - s_2\sigma_1 + s_1\sigma_2 = 3q^{-2}\sigma_3 \]
\[ \ldots \]
\[ \ldots \]
\[ (-1)^{p-1}s_p + (-1)^{p-2}s_{p-1}\sigma_1 + \ldots + s_1\sigma_{p-1} = pq^{1-p}\sigma_p \]

Using (5.6) one can in principle express the quantities \(s_k\) (and, therefore, \(\text{Tr}_R(\hat{L}^k)\)) in terms of \(\sigma_i, \ 1 \leq i \leq k\). But the corresponding expressions are very cumbersome and provide no advantage in working with \(\text{Tr}_R(\hat{L}^k)\). On the other hand, there exists a useful and handy parametric resolution of the system of Newton identities.

Namely, we shall assume the central elements \(\sigma_k(\hat{L})\) to be represented in the form, analogous to (5.1) (see also (2.19))
\[ \sigma_k(\hat{L}) = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq p} \mu_{i_1}\mu_{i_2}\ldots\mu_{i_k}, \quad 1 \leq k \leq p. \] (5.7)

The elements \(\mu_i, \ 1 \leq i \leq p\), belong to the algebraic closure of the center of the REA. On passing to an orbit \(L_{\chi_q}\), the quantities \(\mu_i\) take numerical values from the ground field. In this case we assume that all these numbers are distinct pairwise. So, we have a 1-generic orbit. (Hereafter, all the notions are used by analogy with those introduced in section 2.)

The above mentioned parametric resolution of (5.6) is given in the following proposition.

**Proposition 17** Let the central elements \(\sigma_k(\hat{L})\) be parametrized by (5.7). Then
\[ q^{-1}s_k(\hat{L}) \equiv \text{Tr}_R(\hat{L}^k) = q^{-p}\sum_{i=1}^{p} \mu_i^k d_i \] (5.8)

where
\[ d_i = \prod_{j \neq i}^p \frac{q\mu_i - q^{-1}\mu_j}{\mu_i - \mu_j}. \] (5.9)

**Proof** Let us denote
\[ x_i = q^{1-p}d_i = \prod_{j \neq i}^p \frac{\mu_i - q^{-2}\mu_j}{\mu_i - \mu_j}. \]

Then we should prove that \(s_k = \sum \mu_i^k x_i\) is a solution of (5.6), provided that \(\sigma_k\) is given by (5.7).

First of all, note the following representation for \(x_i\): \[ x_i = q^{(p-1)(p-2)} \frac{\mathcal{D}(q^{-2}\mu_1, q^{-2}\mu_2, \ldots, \mu_i, \ldots, q^{-2}\mu_p)}{\mathcal{D}(\mu_1, \mu_2, \ldots, \mu_p)}. \] (5.10)

It is a direct consequence of the explicit form of \(x_i\).

Now we substitute the ansatz \(s_k = \sum \mu_i^k x_i\) into the set of Newton identities and prove that the corresponding system of linear equation in the variables \(x_i\) has a unique solution which coincides with (5.10).

Using (5.3) and (5.7), we transform (5.6) to the following system of linear equations:
\[ \sum_{i=1}^{p} \mu_i \sigma_{k-1}(\hat{\mu}_i)x_i = k_q q^{1-k} \sigma_k, \quad 1 \leq k \leq p, \] (5.11)
where the quantities $\sigma_k(\hat{\mu}_i)$ have the same meaning as $e_k(\hat{t}_i)$ in (5.2).

The determinant of the system is

$$\Delta(\mu) = \begin{vmatrix} \mu_1 & \mu_2 & \cdots & \mu_p \\ \mu_1\sigma_1(\hat{\mu}_1) & \mu_2\sigma_1(\hat{\mu}_2) & \cdots & \mu_p\sigma_1(\hat{\mu}_p) \\ \mu_1\sigma_2(\hat{\mu}_1) & \mu_2\sigma_2(\hat{\mu}_2) & \cdots & \mu_p\sigma_2(\hat{\mu}_p) \\ \cdots & \cdots & \cdots & \cdots \\ \mu_1\sigma_{p-1}(\hat{\mu}_1) & \mu_2\sigma_{p-1}(\hat{\mu}_2) & \cdots & \mu_p\sigma_{p-1}(\hat{\mu}_p) \end{vmatrix}.$$  

Taking into account Lemma [16] one can rewrite the above determinant in an equivalent form:

$$\Delta(\mu) = \left( \prod_{i=1}^p \mu_i \right) D(\mu, \mu_{p-1}, \ldots, \mu_1). \quad (5.12)$$

Since $\Delta(\mu) \neq 0$, the system (5.11) has a unique solution. To find the solution we use the Cramer’s formula. Evidently, it suffices to find the value of $x_1$ say, since the values of other variables can be obtained from the letter one by simple permutation of $\mu_i$.

So, we shall find $x_1$. In accordance with the Cramer’s rule, it is equal to the ratio of the following determinants:

$$x_1 = \frac{1}{\Delta(\mu)} \begin{vmatrix} z_1 & \mu_2 & \cdots & \mu_p \\ 2q^{-1}\sigma_2 & \mu_2\sigma_1(\hat{\mu}_2) & \cdots & \mu_p\sigma_1(\hat{\mu}_p) \\ 3q^{-2}\sigma_3 & \mu_2\sigma_2(\hat{\mu}_2) & \cdots & \mu_p\sigma_2(\hat{\mu}_p) \\ \cdots & \cdots & \cdots & \cdots \\ p_q^{-p}\sigma_p & \mu_2\sigma_{p-1}(\hat{\mu}_2) & \cdots & \mu_p\sigma_{p-1}(\hat{\mu}_p) \end{vmatrix} \equiv \frac{\Delta_1(\mu)}{\Delta(\mu)}.$$  

Let us now identically transform the numerator of the above expression — the determinant $\Delta_1(\mu)$.

First of all, from the first column of $\Delta_1(\mu)$ we subtract the sum of all other columns. Using (5.3) and (5.4), one gets for the general element of the first column

$$k_q^{-k}\sigma_k - \sum_{i=2}^p \mu_i\sigma_{k-1}(\hat{\mu}_i) = z_k \sigma_k(\hat{\mu}_1) + z_{k-1} \mu_1\sigma_{k-1}(\hat{\mu}_1) + q^{(1-k)}\mu_1 \sigma_{k-1}(\hat{\mu}_1). \quad (5.13)$$

where for the sake of compactness we have introduced a notation

$$z_n = n_q^{-1}n - n.$$  

So, we find that each element of the first column of $\Delta_1(\mu)$ is the sum of several terms and therefore one can expand $\Delta_1(\mu)$ into the sum of determinants

$$\Delta_1(\mu) = \Delta'_1(\mu) + \Delta''_1(\mu),$$

where the $k$-th element of the first column of $\Delta'_1(\mu)$ is equal to $q^{(1-k)}\mu_1\sigma_{k-1}(\hat{\mu}_1)$ while the $k$-th element of the first column of $\Delta''_1(\mu)$ contains the sum $\eta_k(\mu)$ of two rest terms in the right hand side of (5.13)

$$\eta_k(\mu) \equiv z_k \sigma_k(\hat{\mu}_1) + z_{k-1} \mu_1\sigma_{k-1}(\hat{\mu}_1).$$

First, consider the determinant

$$\Delta''_1(\mu) = \prod_{i=2}^p \mu_i \begin{vmatrix} \eta_2 & \sigma_1(\hat{\mu}_2) & \sigma_1(\hat{\mu}_3) & \cdots & \sigma_1(\hat{\mu}_p) \\ \eta_3 & \sigma_2(\hat{\mu}_2) & \sigma_2(\hat{\mu}_3) & \cdots & \sigma_2(\hat{\mu}_p) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \eta_p & \sigma_{p-1}(\hat{\mu}_2) & \sigma_{p-1}(\hat{\mu}_3) & \cdots & \sigma_{p-1}(\hat{\mu}_p) \end{vmatrix}.$$
We shall prove that $\Delta_1''(\mu) = 0$.

Subtracting the second column consecutively from the third one, the fourth one and so on, and taking into account (5.4), one gets

$$
\Delta_1''(\mu) = \prod_{i=2}^p \mu_i \prod_{j=3}^p (\mu_2 - \mu_j) \begin{vmatrix}
0 & 1 & 0 & \ldots & 0 \\
\eta_2 & \sigma_1(\hat{\mu}_2) & 1 & \ldots & 1 \\
\eta_3 & \sigma_2(\hat{\mu}_2) & \sigma_1(\hat{\mu}_2, \hat{\mu}_3) & \ldots & \sigma_1(\hat{\mu}_2, \hat{\mu}_p) \\
\eta_4 & \sigma_3(\hat{\mu}_2) & \sigma_2(\hat{\mu}_2, \hat{\mu}_3) & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\eta_p & \sigma_{p-1}(\hat{\mu}_2) & \sigma_{p-2}(\hat{\mu}_2, \hat{\mu}_3) & \ldots & \sigma_{p-2}(\hat{\mu}_2, \hat{\mu}_p)
\end{vmatrix}
$$

Then we repeat this procedure subtracting the third column from each $j$-th column with $j > 3$ and so on. As a result we come to

$$
\Delta_1''(\mu) = N(\mu) = \prod_{i=2}^p \mu_i \prod_{2 \leq j < k \leq p} (\mu_j - \mu_k).
$$

The result obtained admits the further simplification. We multiply the last column by $\mu_p$ and subtract it from the $(p - 1)$-th column, then multiply the last column by $\mu_{p-1}\mu_p$ and subtract it from the $(p - 2)$-th column, etc. Then we repeat the procedure starting from the $(p - 1)$-th column of the resulting determinant and so on. It is not difficult to see that we end up with the following form of $\Delta_1''(\mu)$

$$
\Delta_1''(\mu) = N(\mu) = \begin{vmatrix}
0 & 1 & 0 & \ldots & 0 & 0 \\
\eta_2 & \sigma_1(\hat{\mu}_2) & 1 & 0 & \ldots & 0 \\
\eta_3 & \sigma_2(\hat{\mu}_2) & \sigma_1(\hat{\mu}_2, \hat{\mu}_3) & 1 & \ldots & 0 \\
\eta_4 & \sigma_3(\hat{\mu}_2) & \sigma_2(\hat{\mu}_2, \hat{\mu}_3) & \sigma_1(\hat{\mu}_2, \hat{\mu}_4) & \ldots & 1 \\
\eta_5 & \sigma_4(\hat{\mu}_2) & \sigma_3(\hat{\mu}_2, \hat{\mu}_3) & \sigma_2(\hat{\mu}_2, \hat{\mu}_4) & \ldots & \mu_1 \\
\eta_p & \sigma_{p-1}(\hat{\mu}_2) & \sigma_{p-2}(\hat{\mu}_2, \hat{\mu}_3) & \sigma_{p-3}(\hat{\mu}_2, \hat{\mu}_4) & \ldots & \sigma_2(\hat{\mu}_2, \ldots, \mu_1)
\end{vmatrix}
$$

where we have restored the explicit form of $\eta_k$ and have taken into account that $\sigma_p(\hat{\mu}_1) \equiv 0$.

At last, we multiply the third column by $z_2\sigma_2(\hat{\mu}_1)$, the fourth one by $z_3\sigma_3(\hat{\mu}_1)$, etc., and then subtract all these columns from the first one. We get all elements of the first column of $\Delta_1''(\mu)$ to be zero, therefore $\Delta_1''(\mu) = 0$.

Turn now to the determinant

$$
\Delta_1'(\mu) = \prod_{i=1}^p \mu_i \begin{vmatrix}
1 & q^{-2}\sigma_1(\hat{\mu}_1) & q^{-4}\sigma_2(\hat{\mu}_1) & q^{2(1-p)}\sigma_{p-1}(\hat{\mu}_1) \\
1 & \sigma_1(\hat{\mu}_2) & \sigma_2(\hat{\mu}_2) & \sigma_{p-1}(\hat{\mu}_2) \\
1 & \sigma_1(\hat{\mu}_3) & \sigma_2(\hat{\mu}_3) & \sigma_{p-1}(\hat{\mu}_3) \\
1 & \sigma_1(\hat{\mu}_p) & \sigma_2(\hat{\mu}_p) & \sigma_{p-1}(\hat{\mu}_p)
\end{vmatrix}
$$

With the same operations which were applied to $\Delta_1'(\mu)$ we convert the determinant into the form

$$
\Delta_1'(\mu) = \prod_{i=1}^p \mu_i \prod_{2 \leq j < k \leq p} (\mu_j - \mu_k) \begin{vmatrix}
1 & 1 & 0 & \ldots & 0 \\
q^{-2}\sigma_1(\hat{\mu}_1) & \mu_1 & 1 & \ldots & 0 \\
q^{-4}\sigma_2(\hat{\mu}_1) & 0 & \mu_1 & \ldots & 0 \\
q^{2(1-p)}\sigma_{p-1}(\hat{\mu}_1) & 0 & 0 & \mu_1 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
q^{2(1-p)}\sigma_{p-1}(\hat{\mu}_1) & 0 & 0 & \ldots & \mu_1
\end{vmatrix}
$$

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Let us introduce new parameters
\[ \nu_1 = \mu_1, \quad \nu_i = q^{-2} \mu_i, \quad 2 \leq i \leq p. \]

Since the function \( \sigma_k(\hat{\mu}_1) \) is a homogeneous polynomial of the \( k \)-th order in the variables \( \mu_i, i \geq 2 \), then
\[ q^{-2k} \sigma_k(\hat{\mu}_1) = \sigma_k(\hat{\nu}_1) \]
and therefore
\[ \Delta'_1 = \left( q^{2p-1} \prod_{i=1}^p \nu_i \right) q^{(p-1)(p-2)} \]
\[ \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \sigma_1(\hat{\nu}_1) & \sigma_1(\hat{\nu}_2) & \cdots & \sigma_1(\hat{\nu}_p) \\ \sigma_2(\hat{\nu}_1) & 1 & \sigma_2(\hat{\nu}_2) & \cdots & \sigma_2(\hat{\nu}_p) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_{p-1}(\hat{\nu}_1) & \sigma_{p-1}(\hat{\nu}_2) & \cdots & \sigma_{p-1}(\hat{\nu}_3) & \sigma_{p-1}(\hat{\nu}_p) \end{vmatrix}. \]

Applying Lemma 16 and changing the set of parameters \{\nu_i\} back to the set of \{\mu_i\} we get
\[ \Delta_1(\mu) = \Delta'_1(\mu) = q^{(p-1)(p-2)} \left( \prod_{i=1}^p \mu_i \right) \mathcal{D}(q^{-2} \mu_p, q^{-2} \mu_{p-1}, \ldots, q^{-2} \mu_2, \mu_1). \]

Using the value (5.12) of the determinant \( \Delta(\mu) \), one comes to the final result
\[ x_1 = \frac{\Delta_1(\mu)}{\Delta(\mu)} = q^{(p-1)(p-2)} \frac{\mathcal{D}(q^{-2} \mu_p, q^{-2} \mu_{p-1}, \ldots, \mu_1)}{\mathcal{D}(\mu_p, \mu_{p-1}, \ldots, \mu_1)} \]
which is obviously equivalent to (5.10).

**Remark 18** In [DM1] a way of quantization of semisimple (but not necessary generic) orbit in \( gl(n)^* \) was suggested and in this connection another form of the basic \( q \)-Newton identity was given. Let us briefly describe the quantization procedure from [DM1] and compare two forms of basic \( q \)-Newton identities. (The normalization of the quantum trace in [DM1] differs from that accepted in the present paper.) Here we restrict ourselves to the \( U_q(sl(n)) \) case.

Consider the \( GL(n) \) orbit \( O_M \) of an arbitrary semisimple matrix \( M \in gl(n)^* \) characterized by \( r \leq n \) pairwise distinct eigenvalues \( \mu_i \) with the multiplicities \( m_i \geq 1 \)
\[ m_1 + m_2 + \ldots + m_r = n \]
(see Remark 1).

Let \( P(x) = \prod_{i=1}^r (x - \mu_i) \) be the degree \( r \) minimal polynomial of this orbit (i.e. each \( \mu_i \) is a simple root of \( P(x) \)). Then the quotient of the mREA over the ideal generated by the entries of the matrix \( P(L) = \prod_{i=1}^r (L - \mu_i I) \) and by the elements
\[ \text{Tr}_R L^k - \tilde{\beta}_k, \quad k = 1, \ldots, r - 1 \]
with appropriate values of \( \tilde{\beta}_k \) is a flat deformation (quantization) of the commutative algebra \( \mathbb{K}(O_M) \). This fact was established in [DM1] where the exact values of \( \tilde{\beta}_k \) were expressed in terms of the roots of the minimal polynomial. However, the same values of these quantities can be obtained from our parametric resolution of the basic \( q \)-Newton identities given in (5.3)–(5.4).

For this purpose, we associate to any root \( \mu = \mu_i \) of the minimal polynomial the following string of \( m_i \) quantities
\[ \nu_1 = \mu, \quad \nu_2 = q^{-2} \nu_1 + q^{-1} h, \quad \nu_3 = q^{-2} \nu_2 + q^{-1} h, \quad \ldots, \quad \nu_{m_i} = q^{-2} \nu_{m_i-1} + q^{-1} h. \]
(5.16)
Consider the set of all $\nu_k$ belonging to strings (5.16) and define the character $\chi : \mathbb{Z}(\mathcal{L}_{h,q}) \to \mathbb{K}$ on central elements $\sigma_i$ as in (3.23) but with eigenvalues running over all $\nu$ from the mentioned set. Let us pass to the corresponding NC orbit $\mathcal{L}^{\chi}_{h,q}$ (3.24). Emphasize that the NC orbit $\mathcal{L}^{\chi}_{h,q}$ thus obtained turns out to be bigger than the result of quantization of the Poisson pencil (see Introduction) on the initial $GL(n)$ orbit $\mathcal{O}_M$. In order to get the genuine quantum orbit we should quotient $\mathcal{L}^{\chi}_{h,q}$ over the ideal generated by the entries of the matrix $P(L)$ constructed from the minimal polynomial of the orbit $\mathcal{O}_M$. So, assuming $|q-1|$ and $h$ to be small enough in order to avoid any casual coincidence of the elements from the above union of the strings, we get $n$ pairwise distinct eigenvalues $\nu_i$ of the matrix $L$ with entries considered as elements of $\mathcal{L}^{\chi}_{h,q}$. This is well coordinated with the empirical principle that the quantization decreases the degeneracy.

Finally, we have got a 1-generic orbit $\mathcal{L}^{\chi}_{h,q}$ and the quantities $\text{Tr}_R L^k$ can be computed via (5.8)–(5.9). However, the multiplicities $d_i$ corresponding to extra eigenvalues (i.e. those which are not roots of the minimal polynomial) vanish.

So, given a 1-generic NC orbit $\mathcal{L}^{\chi}_{h,q}$, we can be sure that it is a quantization of a classical generic orbit iff the set of eigenvalues of the matrix $L$ corresponding to this NC orbit contains no string. If it is not the case, we construct the minimal polynomial $P(x)$ taking the first element of each string as its simple root and consider a two sided ideal in $\mathcal{L}^{\chi}_{h,q}$ generated by the entries of the matrix $P(L)$. Then, quotienting the given 1-generic NC orbit over this ideal, we get a quantization of a semisimple but not generic orbit whose eigenvalues are the first elements of strings and the multiplicity of each eigenvalue is equal to the length of the corresponding string.

6 The higher Cayley-Hamilton and Newton identities

Consider a 1-generic quantum orbit $\mathcal{L}^{\chi}_{h,q}$ defined by relations (3.23)–(3.25) where $p$ is the symmetry rank of the Hecke $R$-matrix.

In [GLS2] the following conjecture was formulated.

**Conjecture 19** On a 1-generic NC orbit $\mathcal{L}^{\chi}_{h,q}$, the matrix $L_{(m)}$ (4.7) satisfies the Cayley-Hamilton identity

$$\mathcal{CH}^{\chi}_{(m)}(L_{(m)}) = 0,$$

where the degree of the polynomial $\mathcal{CH}^{\chi}_{(m)}$ is

$$\deg \mathcal{CH}^{\chi}_{(m)} = \binom{m + p - 1}{m}$$

(6.1)

and its roots $\mu_k(m)$ are

$$q^{m-1} \mu_k(m) = \sum_{i=1}^{p} \frac{(k_i)_q}{q^{m-k_i}} \mu_i + h \xi_p(k_1, \ldots, k_p),$$

(6.2)

where $k$ is a partition (2.17) of the integer $m$

$$k = (k_1, \ldots, k_p), \quad k_i \geq 0, \quad |k| = k_1 + \ldots + k_p = m$$

and $\xi_p(k_1, \ldots, k_p)$ is the symmetric function in $k_i$ of the form

$$\xi_p(k_1, \ldots, k_p) = \sum_{s=2}^{p} q^{k_1+k_2+\ldots+k_{s-1}-m}(k_s)_q(k_1 + k_2 + \ldots + k_{s-1})q.$$
Remark 20 The fact that $\xi_q$ is a symmetric function in $k_i$ can be easily verified upon expanding all $q$-numbers in accordance with their definition (3.5).

Note that (6.2) is a generalization of the analogous formula (2.18). In the classical limit $q \to 1$ the above formula transforms into (2.18) for the 1-generic fuzzy orbit in $U(gl(p))$.

The Conjecture is justified by explicit calculations for small values of $m$ but we still have no general proof of it. Here we present a proof for the particular case $p = 2$.

Since at $q \neq \pm 1$ the mREA $L_{h,q}$ is isomorphic to the nonmodified REA $L_q$, we first prove the Conjecture for $L_q$ and then pass to the corresponding result for $L_{h,q}$ by means of the shift of generators.

Proposition 21 Let the symmetry rank $p$ of the $R$-matrix be equal to 2. Then the roots $\hat{\omega}_s$ $0 \leq s \leq m$ of the CH polynomial for the symmetric matrix $\hat{L}_m$ are given by

$$q^{1-m}\hat{\omega}_s = q^{s-m}s_q\hat{\mu}_1 + q^{-s}(m-s)s_q\hat{\mu}_2, \quad s = 0,1,\ldots,m$$

(6.3)

where $\hat{\mu}_i$ are the roots of the basic CH polynomial for the matrix $\hat{L}_1 = ||\hat{I}_2||$ composed of the generators of the REA $L_q$

$$(\hat{L}_1 - \hat{\mu}_1 I_e)(\hat{L}_1 - \hat{\mu}_2 I_e) = 0, \quad I_e = id_L \otimes I.$$ 

Proof Let us shortly outline the strategy of the proof. We shall use the same approach as in proposition 21. Namely, we prove the claim in each finite dimensional representation $V(k)$ of the REA. This means that, for a given fixed $m$, we associate to $\hat{L}_m$ a series of numerical matrices $\hat{L}_{(k,m)}$ of the form

$$\hat{L}_{(k,m)} = (\pi_{(k)} \otimes \pi_{(m)})(\text{Cas}) \quad k = m, m+1,\ldots$$

(6.4)

and prove the claim for each value of $k \geq m$. We should consider the values $k \geq m$ since at $k < m$ the matrix $\hat{L}_{(k,m)}$ can satisfy the CH identity of an order lower than (6.2). This is a peculiarity of low dimensional representations connected with the combinatorics of the Young diagrams.

The above matrix $\hat{L}_{(k,m)}$ coincides with that of a linear operator $\text{Cas}_{(k,m)} \in \text{End}_{e}(V(k)) \otimes \text{End}_{e}(V(m))$ which is obtained from $\text{Cas}_{(k,m)}$ in the full analogy with the fuzzy case construction (2.25). Recall, that at $p = 2$ the spaces $V_{(k)}$ and $V_{(k)}$ are isomorphic (see section 3). We consider the decomposition of the tensor product $V_{(k)} \otimes V_{(m)}$ into the direct sum of subspaces $V_{(\nu_1,\nu_2)}$ and prove that the restriction of $\text{Cas}_{(k,m)}$ on each subspace $V_{(\nu_1,\nu_2)}$ is a multiple of the unit operator. The corresponding factors will be the roots of the CH polynomial for $\hat{L}_{(k,m)}$ and we should verify that they are given by (6.3).

Step 1. Let us fix an arbitrary integer $k \geq m$ and find the values $\hat{\mu}_1(k)$ and $\hat{\mu}_2(k)$ of the roots of the basic CH polynomial for $\hat{L}_{(1,k)} = ||\hat{\pi}_{(k)}(\hat{I}_2)||$. For this purpose we substitute the matrix $\hat{L}_{(1,k)}$ into identity (3.21)\footnote{Please note the explicit form (4.12) of the right representation $\hat{\pi}_{(k)}$ one can show}

$$\hat{L}^2 - \sigma_1(\hat{L})\hat{L} + \sigma_2(\hat{L})I = 0$$

and calculate the spectrum of the central elements $\sigma_i$. Since

$$\sigma_1(\hat{L}) = q\text{Tr}_R\hat{L} \quad \sigma_2(\hat{L}) = \frac{q^2}{2q}(q(\text{Tr}_R\hat{L})^2 - \text{Tr}_R\hat{L}^2),$$

then basing on the explicit form (4.12) of the right representation $\hat{\pi}_{(k)}$ one can show

$$\sigma_1(\hat{L}_{(k,1)}) = (1 + q^{-2k-2}) S^{(k)}(R) \equiv (1 + q^{-2k-2}) id_{V_{(k)}}$$

and

$$\sigma_2(\hat{L}_{(k,1)}) = q^{-2k-2}S^{(k)}(R) \equiv q^{-2k-2} id_{V_{(k)}}.$$
So, on the subspace \( V(k) \subset V^{\otimes k} \) the Cayley-Hamilton identity for \( \hat{L}_{(k,1)} \) takes the form
\[
(\hat{L}_{(k,1)} - I)(\hat{L}_{(k,1)} - q^{-2k-2}I) = 0
\]
that is
\[
\hat{\mu}_1(k) = 1, \quad \hat{\mu}_2(k) = q^{-2k-2}.
\] (6.5)

**Step 2.** Now we pass to the matrix \( \hat{L}_{(k,m)} \) treated as that of a linear operator from \( \text{End}_r(V(k)) \otimes \text{End}_l(V(m)) \). In the case \( p = 2 \) the general decomposition (6.3) reduces to
\[
V(k) \otimes V(m) = \bigoplus_{s=0}^m V(k+s,m-s).
\]
The subspace \( V(k+s,m-s) \) can be represented as an image of the operator \( \mathcal{P}_s \in \text{End}(V^{\otimes (k+m)}) \)
\[
\mathcal{P}_s = S_{1\ldots k}^{(m)} Y_{(k+s,m-s)}^{(k)} \equiv 1, \ldots, k + s | k + s + 1, \ldots, k + m.
\] (6.6)
Here in the last equality we have introduced a more convenient notation for an explicit enumeration of a tableau corresponding to the two row partition \( \lambda = (k + s, m - s), 0 \leq s \leq m \).

Due to the fact that
\[
S_{1\ldots k}^{(k)} Y_{(k+s,m-s)}^{(k)} = Y_{(k+s,m-s)}^{(k)} S_{1\ldots k}^{(k)} = Y_{(k+s,m-s)}^{(k)}
\]
the expression for \( \mathcal{P}_s \) can be simplified to
\[
\mathcal{P}_s = S_{k+1\ldots k+m}^{(m)} Y_{(k+s,m-s)}^{(k)} S_{k+1\ldots k+m}^{(m)}.
\] (6.7)
The operator \( \mathcal{P}_s \) is a projector up to a normalizing factor
\[
\mathcal{P}_s^2 = \gamma_s \mathcal{P}_s,
\]
the exact value of \( \gamma_s \) is not important for us.

To prove the proposition it is suffice to show that the matrix \( \hat{L}_{(k,m)} \) commutes with \( \mathcal{P}_s \) for any \( 0 \leq s \leq m \) and that the following relation holds
\[
\hat{L}_{(k,m)} \mathcal{P}_s = \hat{\omega}_s \mathcal{P}_s = \mathcal{P}_s \hat{L}_{(k,m)}.
\] (6.8)

**Step 3.** Consider the product \( \hat{L}_{(k,m)} \mathcal{P}_s \) in detail. On the base of definition (6.4) with the explicit form of representations \( \pi_{(m)} \) and \( \pi_{(k)} \) given in propositions [11] and [13] we find the following expression for the matrix \( \hat{L}_{(k,m)} \)
\[
q^{m-1} \hat{L}_{(k,m)} = m_q \frac{\xi q(k+1)q}{q^{k+2}} S_{1\ldots k}^{(k)} S_{k+1\ldots k+m}^{(m)} + \frac{\xi q(k+1)q}{q^{k+1}} S_{k+1\ldots k+m}^{(m)} \cdots S_{k+1\ldots k+m}^{(k)}. \] (6.9)
where we have extracted the overall factor \( q^{m-1} \) to simplify the subsequent calculations. In deriving this formula one should use the relation
\[
S_{1\ldots k}^{(k)} A_{kk+1}^{(2)} S_{1\ldots k}^{(k)} = \frac{(k+1)q}{2q^k} (S_{1\ldots k}^{(k)} I_{k+1} - S_{1\ldots k+1}^{(k+1)}).
\]
Now we are to multiply the above expression for \( \hat{L}_{(k,m)} \) on the matrix \( \mathcal{P}_s \) from the right. The multiplication of the first summand in (6.9) results in \( q^{-2k-2}m_q \mathcal{P}_s \) and the main difficulty concentrates in the second term

\[
\frac{\zeta m_q(k + 1)}{q^{k+1}} S^{(m)}_{k+1...k+m} S^{(k+1)}_{k+1...k+m} S^{(m)}_{k+1...k+m} Y_{(k+s,m-s)} S^{(m)}_{k+1...k+m} = \frac{\zeta m_q(k + 1)}{q^{k+1}} \Omega_s
\]

(6.10)

were the left hand side is a definition of the symbol \( \Omega_s \). At \( s = m \) this is obviously the multiple of \( \mathcal{P}_m \equiv S^{(k+m)} \) therefore below we shall suppose \( 0 \leq s \leq m - 1 \).

To prove that \( \Omega_s \) is a multiple of \( \mathcal{P}_s \) one should somehow get rid of the \( q \)-symmetrizer \( S^{(k+1)}_{k+1...k+1} \) in its expression. For this purpose we decompose the projector \( S^{(m)} \) standing between \( S^{(k+1)}_{k+1...k+1} \) and \( Y_{(k+s,m-s)} \) into the product of the factors (4.6)

\[
S^{(m)}_{k+1...k+m} = \frac{(m - 1)}{m_q} S^{(m-1)}_{k+2...k+m} (\frac{q^{1-m}}{(m-1)} I + R_{k+1}) S^{(m-1)}_{k+2...k+m}.
\]

The first \( S^{(m-1)}_{k+1...k+m} \) in the right hand side of the above relation commutes with \( S^{(k+1)}_{k+1...k+1} \) and can be absorbed into the most left \( S^{(m)}_{k+1...k+m} \) in (6.10)

\[
S^{(m)}_{k+1...k+m} S^{(m-1)}_{k+2...k+m} = S^{(m)}_{k+1...k+m}.
\]

As for the second \( S^{(m-1)}_{k+1...k+m} \), we shall continue its decomposition in the same way until descending to \( S^{(m-s)}_{k+1...k+m} \) which is absorbed by the projector \( Y_{(k+s,m-s)} \). So, we come to the result

\[
\Omega_s = \frac{(m - s)}{m_q} S^{(m)}_{k+1...k+m} (\frac{q^{1-m}}{(m-1)} I + R_{k+1}) \cdots (\frac{q^{s-m}}{(m-s)} I + R_{k+s}) Y_{(k+s,m-s)} S^{(m)}_{k+1...k+m}.
\]

The next step is to draw the factors of the type \((z I + R)\) through \( Y_{(k+s,m-s)} \) and then absorb them into the right \( S^{(m)}_{k+1...k+m} \) on the base of the following property of the \( q \)-symmetrizer

\[
R_{k+i} S^{(m)}_{k+1...k+m} = q S^{(m)}_{k+1...k+m} \quad 1 \leq i \leq m - 1.
\]

The commutation of the linear in \( R \)-matrix terms with \( Y_{(k+s,m-s)} \) is done on the base of the identity [OP]

\[
\left( \frac{q^{-i}}{i_q} I + R_i \right) \begin{pmatrix} 1 \ldots i \ldots \i \ldots \i + 1 \ldots \i + 1 \ldots \i \end{pmatrix} = \begin{pmatrix} 1 \ldots i + 1 \ldots \i + 1 \ldots \i \end{pmatrix} \left( R_i - \frac{q^i}{i_q} I \right)
\]

(6.11)

On applying such like formulae \( s \) times we get the following result for \( \Omega_s \)

\[
\Omega_s = \frac{(m - s)}{(k + s)} \sum_{r=1}^{s} \frac{(k - m + 2r)}{(m - r)} \frac{q^r}{(m-r+1)} S^{(m)}_{k+1...k+m} Y_{(k+s,m-s)}^{[r]} S^{(m)}_{k+1...k+m}, \quad 0 \leq s \leq m - 1.
\]

(6.12)

Here the \( q \)-projector \( Y_{(k+s,m-s)}^{[r]} \) corresponds to the following Young tableau

\[
Y_{(k+s,m-s)}^{[r]} \leftrightarrow [1, \ldots, k+r, k+r+2, \ldots, k+s+1 \mid k+r+1, k+s+2, \ldots, k+m].
\]

The unwanted \( q \)-symmetrizer \( S^{(k+1)}_{k+1...k+1} \) has been absorbed into \( Y_{(k+s,m-s)}^{[r]} \)

\[
S^{(k+1)}_{k+1...k+1} Y_{(k+s,m-s)}^{[r]} = Y_{(k+s,m-s)}^{[r]}, \quad 1 \leq r \leq s.
\]

And at last, we transform all \( Y_{(k+s,m-s)}^{[r]} \) in (6.12) back to \( Y_{(k+s,m-s)} \). This can be done by the multiple successive application of the following consequence of (6.11)

\[
\begin{pmatrix} 1 \ldots i + 1 \ldots \i \ldots \i + 1 \ldots \i \end{pmatrix} = \frac{i_q^2}{(i-1)q(i+1)q} \left( \frac{q^{-i}}{i_q} I + R_i \right) \begin{pmatrix} 1 \ldots i \ldots \i \ldots \i \end{pmatrix} \left( \frac{q^{-i}}{i_q} I + R_i \right).
\]
The brackets with $R$-matrices appearing in this way are absorbed into $S^{(m)}$ which gives rise to the accumulation of an overall numerical factor. The final result for $\Omega_s$ reads

$$\Omega_s = \beta_s S^{(m)} Y_{(k+s,m-s)} S^{(m)}, \quad 0 \leq s \leq m - 1,$$

where

$$\beta_s = (m-s)_q(k+s+1)_q \sum_{r=1}^{s} \frac{(k-m+2r)_q}{(k+r)_q(k+r+1)_q(m-r)_q(m-r+1)_q}.$$

The sum in this relation can be easily calculated by induction in $s$ and one gets

$$\beta_s = \frac{s_q(k+s+1-m)_q}{m_q(k+1)_q}.$$

Now, gathering together the results from the both terms in the right hand side of \eqref{eq:6.7}, we find

$$q^{m-1} \hat{L}_{(k,m)} P_s = \left( \frac{m_q}{q^{2k+2}} + \frac{\zeta}{q^{k+1}} m_q(k+1)_q \beta_s \right) P_s = \left( \frac{m_q}{q^{2k+2}} + \frac{\zeta}{q^{k+1}} s_q(k+s+1-m)_q \right) P_s.$$

Moreover, since the expressions for $\hat{L}_{(k,m)}$ \eqref{eq:6.7} and $P_s$ \eqref{eq:6.7} are symmetric with respect to the $q$-projectors, the same result can be obtained for $P_s \hat{L}_{(k,m)}$ that is $P_s$ and $\hat{L}_{(k,m)}$ commute.

The last step is to show that the coefficient in the right hand side of the above expression for $\hat{L}_{(k,m)} P_s$ is equal to $\hat{\omega}_s$. It is a matter of a short straightforward calculation to verify the identity

$$\frac{m_q}{q^{2k+2}} + \frac{\zeta}{q^{k+1}} s_q(k+s+1-m)_q = \frac{s_q}{q^{m-s}} + \frac{(m-s)_q}{q^{2k+2+s}}.$$

Taking into account the values of the roots $\hat{\mu}_i(k)$ \eqref{eq:6.3} we come to the desired result

$$\frac{s_q}{q^{m-s}} + \frac{(m-s)_q}{q^{2k+2+s}} = q^{s-m} s_q \mu_1(k) + q^{-s} (m-s) q \mu_2(k) = q^{m-1} \hat{\omega}_s$$

and therefore

$$\hat{L}_{(k,m)} P_s = \hat{\omega}_s P_s.$$

As was mentioned above, the same result is valid for $P_s \hat{L}_{(k,m)}$. \hfill $\blacksquare$

The roots of the CH polynomial for the symmetric matrix $L_{(m)}$ composed of the generators $l_i^j$ of the mREA can be found as a simple corollary of the proposition \ref{prop:21}.

**Corollary 22** Let the symmetry rank $p$ of the $R$-matrix be equal to 2. Then the roots $\omega_s, 0 \leq s \leq m$ of the CH polynomial for the symmetric matrix $L_{(m)}$ are given by

$$q^{1-m} \omega_s = q^{s-m} s_q \mu_1 + q^{-s} (m-s) q \mu_2 + \hbar s_q (m-s)_q, \quad s = 0, 1, \ldots, m \quad \text{(6.13)}$$

where $\mu_i$ are the roots of the basic CH polynomial for the matrix $L_{(1)} = \|l_i^j\|$ composed of the generators of the mREA $L_{h,q}$

$$(L_{(1)} - \mu_1 I_e)(L_{(1)} - \mu_2 I_e) = 0, \quad I_e = \text{id}_L \otimes I.$$  

**Proof** Consider in detail how the shift of generators presented in \eqref{eq:3.18} affects the CH identity. If the matrix $\hat{L} = \|l_i^j\|$ of the REA generators satisfies the CH identity

$$(\hat{L} - \hat{\mu}_1 I_e)(\hat{L} - \hat{\mu}_2 I_e) = 0,$$

then the matrix $L$ of the mREA generators satisfies the same identity but with $\mu_i = \hat{\mu}_i + \hbar \zeta^{-1}$. This is a trivial consequence of \eqref{eq:3.18}.
As for the higher order CH identity for symmetrical matrix $L_{(m)}$ \cite{4.31} the modification is as follows. Suppose, we know the CH identity for $L_{(m)}$

$$
\prod_{s=0}^{m}(\hat{L}_{(m)} - \hat{\omega}_s I_e^{(m)}) = 0, \quad q^{m-1}\hat{\omega}_s = q^{-m} s q \hat{\mu}_1 + q^{-s}(m-s) \hat{\mu}_2,
$$

where $I_e^{(m)} = id_{\mathcal{L}} \otimes I_{V_{(m)}}$. To pass to the mREA case we should take into account the connection of the matrices $\hat{L}_{(m)}$ and $L_{(m)}$

$$
\hat{L}_{(m)} = L_{(m)} - \frac{h}{\zeta}q^{1-m}m q I_e^{(m)}.
$$

This relation follows from \cite{3.48}, \cite{4.9} and the explicit form of the representation $\pi_{(m)}$ given in proposition \cite{11}. Besides, one should take into account the shift from $\hat{\mu}_i$ to $\mu_i$ described above. Therefore, we come to the following result

$$
\hat{L}_{(m)} - \hat{\omega}_s I_e^{(m)} = L_{(m)} - q^{1-m}(\frac{h}{\zeta} m q + q^{-m} s q (\mu_1 - \frac{h}{\zeta}) + q^{-s}(m-s) q (\mu_2 - \frac{h}{\zeta})) I_e^{(m)}
\quad = L_{(m)} - q^{1-m}(q^{-m} s q \mu_1 + q^{-s}(m-s) q \mu_2 + h s q (m-s) q) I_e^{(m)}
\quad = L_{(m)} - \omega_s I_e^{(m)}.
$$

This completes the proof.

To sum up, we conclude that on any 1-generic NC orbit $\mathcal{L}_{\hbar,q}^{\chi}$ \cite{3.25} the extended matrix $L_{(m)}$ satisfies the $(m+1)$-th order polynomial identity with the roots \cite{6.13} ($p = 2$). Fixing the finite dimensional representation $\pi_{(k)}$ affects only the particular form of the roots $\mu_i$ of the basic CH polynomial for the matrix $L$.

In full analogy with the constructions of section \cite{2} we can introduce idempotents $e_k(m)$ on any $m$-generic NC orbit. Moreover, the following proposition holds true.

**Proposition 23** For any $m$-generic NC orbit $\mathcal{L}_{\hbar,q}^{\chi}$ defined by \cite{6.24}–\cite{6.25} we have

$$
\text{Tr}_R L_{(m)}^s = q^{-p} \sum_{|k|=m} \mu_k(m)^s d_k(m) \tag{6.14}
$$

with $\mu_k$ introduced in \cite{6.2}. If the symmetry rank $p$ of $R$ is equal to 2, then

$$
d_k(m) = \prod_{1 \leq i < j \leq p} \frac{q^{k_i - k_j} \mu_i - q^{k_j - k_i} \mu_j - h (k_i - k_j) q}{\mu_i - \mu_j}. \tag{6.15}
$$

If the symmetry rank of $R$ $p > 2$ then the above expression for $d_k(m)$ is valid provided that the category of finite dimensional equivariant representations of the corresponding mREA $\mathcal{L}_{\hbar,q}$ is faithful (see definition \cite{12}).

**Proof** The proof of \cite{6.14} is straightforward. As for the proof of \cite{6.15}, it is sufficient to establish this formula for $h = 0$. Similarly to the proof of proposition \cite{8} we consider $\pi_{\chi}(L_{(m)})$, compute the corresponding multiplicities $d_k(\lambda, m) = \pi_{\chi}(d_k(m))$ and prove relation \cite{6.15} with $\mu_i = \mu_i(\lambda)$. Then, due to the faithfulness of the representation category, we conclude that \cite{6.15} takes place at the level of the algebra itself.

So, assuming the representation $\pi_{\chi}$ to be $m$-admissible (see definition \cite{9}) we find

$$
d_k(\lambda, m) = \frac{\dim_q(V_{\chi}^* \otimes V_{(m)})_k}{\dim_q V_{\chi}^*} \tag{6.16}
$$
where \((V_\lambda^* \otimes V_{(m)})_k\) is the irreducible component with the label \(k\) in the tensor product \(V_\lambda^* \otimes V_{(m)}\) and the \(q\)-dimension is defined in (4.11). This formula is the \(q\)-analog of (2.32) and can be obtained by the same method.

Now let us take into account that the decomposition rules for the tensor product of the \(L_{h,q}\) modules \(V_\lambda\) are the same as those for the \(U(gl(p))\) modules, \(p\) being the symmetry rank of \(R\) (see the beginning of section 2). Bearing in mind the property C4 of the Hecke symmetry \(R\) (section 3), we conclude that \(V_\lambda^*\) is isomorphic to \(V_\lambda^*\) as a vector space where \(\lambda\) is defined in (2.13) and (2.14). And secondly, \((V_\lambda^* \otimes V_{(m)})_k \cong V_{\lambda+\lambda+k}^*\) (as vector spaces). Since we are interested in calculating the \(q\)-dimensions which are the same for isomorphic spaces, then in all formulae we can change the spaces \(V_\lambda^*\) and \((V_\lambda^* \otimes V_{(m)})_k\) for the corresponding isomorphic spaces.

On the other hand, by virtue of (5.9) and the isomorphism mentioned above, we have in the case \(m = 1\)

\[
d_j(\lambda, 1) = \prod_{i \neq j} \frac{q \mu_j(\lambda) - q^{-1} \mu_i(\lambda)}{\mu_j(\lambda) - \mu_i(\lambda)} = \frac{\dim_q(V_\lambda^* \otimes V_j)}{\dim_q V_\lambda^*}, \quad j = 1, \ldots, p. \tag{6.17}
\]

The explicit form of the \(q\)-dimension reads

\[
\dim_q V_\lambda = s_\lambda(q^{p-1}, q^{p-3}, \ldots, q^{-p+3}, q^{-p+1}) = \prod_{1 \leq i < j \leq p} \frac{(\lambda_i - \lambda_j - i + j)_q}{(j - i)_q} \tag{6.18}
\]

where \(s_\lambda\) is the Schur function corresponding to the partition \(\lambda\). The first equality in (6.18) was established in [GLS1]. The second one can be proved by a straightforward calculations. Its equivalent form can be also found in [Ma].

So, on taking into account (6.17) and (6.18) we come to the system of \(p\) equations

\[
\prod_{i \neq j} \frac{q \mu_j(\lambda) - q^{-1} \mu_i(\lambda)}{\mu_j(\lambda) - \mu_i(\lambda)} = \prod_{i \neq j} \frac{(\lambda_{p-i+1} - \lambda_{p-j+1} + i - j + 1)_q}{(\lambda_{p-i+1} - \lambda_{p-j+1} + i - j)_q}, \quad 1 \leq j \leq p. \tag{6.19}
\]

Let us find \(\mu_i(\lambda)\) from this system. It is easy to see that there exist a solution of the form

\[
\mu_i(\lambda) = \eta(\lambda) q^{-2(\lambda_{p-i+1} + 1)} \tag{6.20}
\]

where \(\eta(\lambda)\) is an arbitrary nonzero multiplier. Indeed, given such \(\mu_i(\lambda)\) and taking into account definition (5.5) of the \(q\)-numbers, we get

\[
\frac{q \mu_j(\lambda) - q^{-1} \mu_i(\lambda)}{\mu_j(\lambda) - \mu_i(\lambda)} = \frac{(\lambda_{p-i+1} - \lambda_{p+1-j} + i - j + 1)_q}{(\lambda_{p+1-i} - \lambda_{p+1-j} + i - j)_q}.
\]

At \(p = 2\) the above solution (6.20) is unique since the system (6.19) is actually linear in \(\mu_i(\lambda)\).

**Conjecture 24** At an arbitrary value of \(p\) the solution (6.20) of the system (6.14) is unique.

Up to this Conjecture we can extend relation (6.15) for an arbitrary \(p \geq 2\) (at \(p = 2\) it is true rigorously). Indeed, from (6.16) it follows that

\[
d_k(\lambda, m) = \frac{\dim_q V_{\lambda+k}}{\dim_q V_\lambda}.
\]

On the other hand, taking \(\mu_i\) as in (6.20) we find

\[
\frac{q^{k_i-k_j} \mu_i(\lambda) - q^{k_i-k_j} \mu_j(\lambda)}{\mu_i(\lambda) - \mu_j(\lambda)} = \frac{(\lambda_{p+1-j} - \lambda_{p+1-i} + k_i - k_j - i + j)_q}{(\lambda_{p+1-j} - \lambda_{p+1-i} + i + j)_q}.
\]
This relation together with above form of \( d_k(\lambda, m) \) and (6.18) entails result (6.15) for \( h = 0 \). A passage to the general case can be performed by the shift of generators (3.18).

To complete the proof, we point out, that since the category of finite dimensional equivariant representation of \( L_{\hbar, q} \) is faithful, then relation (6.15), being valid in all representations, must take place at the level of the algebra itself.

The above relation (6.20) shows which values of \( \mu_i \) (up to a normalizing factor) correspond to a finite dimensional representation \( \pi_\lambda \) of the NC orbit \( L_{\chi, q} \). To fix the factor \( \eta(\lambda) \) one should consider the value of \( \pi_\lambda(\text{Tr}_R L) \) for the mREA \( L_{\hbar, q} \) and use the connection (3.18). The following corollary of proposition 23 holds true.

**Corollary 25** Let \( L_{\chi, q}^1 \) be the NC orbit defined by (3.23)–(3.25). In the finite dimensional representation \( \pi_\lambda \) of the mREA \( L_{\chi, q}^1 \) the eigenvalues \( \bar{\mu}_i(\lambda) \) of the matrix \( L \) of the mREA generators on the orbit \( L_{\chi, q}^1 \) are as follows

\[
\bar{\mu}_i(\lambda) = \frac{(\lambda_{p-i+1} + i - 1)_q}{q^{\lambda_{p-i+1}+i-1}}.
\]

Here the eigenvalues \( \bar{\mu}_i \) are the roots of the CH polynomial for

\[
\prod_{i=1}^{p} (L - \bar{\mu}_i I) = 0.
\]

**Proof** As was found in [S], the central element \( \text{Tr}_R L \) has the following form in the representation \( \pi_\lambda \)

\[
\pi_\lambda(\text{Tr}_R L) = \chi_1 \text{id}_{V_\lambda}, \quad \chi_1 = q^{-2p} \sum_{r=1}^{p} q^{2r-1-\lambda_r}(\lambda_r)_q.
\]

On the other hand, by virtue of (5.8), (5.7) and (5.6) the R-trace of the matrix \( \hat{L} \) of the generators of non-modified REA can be presented as follows

\[
\pi_\lambda(\text{Tr}_R \hat{L}) = \frac{1}{q} \sum_{i=1}^{p} \mu_i \text{id}_{V_\lambda}
\]

were \( \mu_i \) are given by (6.20). Taking into account the above expressions for the traces and the fact that \( L \) and \( \hat{L} \) are connected by means of shift (3.18), we find

\[
q^{-1} \eta(\lambda) \sum_{r=1}^{p} q^{-2(\lambda_{p-i+1}+i)} = q^{-2p} \sum_{r=1}^{p} q^{2r-1-\lambda_r}(\lambda_r)_q - \frac{pq}{q^p(q-q^{-1})}.
\]

This entails

\[
\eta(\lambda) = -\frac{q^2}{(q-q^{-1})}.
\]

And at last, using the connection \( \bar{\mu}_i = \mu_i + (q-q^{-1})^{-1} \) and values (6.20), we come to (6.21) which is the \( q \)-generalization of the classical result (2.25).

7  q-Euler characteristic and group \( Q(\mathcal{L}^\chi_{h, q}) \)

In what follows we shall consider the multiplicities \( d_k(m) \) on the set of all generic NC orbits as functions in \( \mu_i \). As was pointed out in remark 18, there exist orbits on which some of the basic multiplicities \( d_i \) can vanish. But even if all \( d_i \) are nonzero, some of the higher multiplicities \( d_k(m) \) can vanish as well. We shall restrict ourselves to the set of generic NC orbits such that \( d_k(m) \neq 0 \) for all \( m = 1, 2, \ldots \) and all for all partitions \( k \vdash m \). Such NC orbits will be called strictly generic.
However, in considering the multiplicities as functions in $\mu_i$ it does not matter whether we exclude a low-dimensional subset of values of $\mu_i$ (corresponding to non-strictly generic orbits).

Let $M_k = M_k(m)$ be the left (for the definiteness) projective module corresponding to the idempotent $e_k(m)$ defined on a NC orbit $L_{h,q}^k$ in full analogy with \[227\].

For a strictly generic orbit we consider the set of projective modules $M_k$ with the assignment

$$M_k \mapsto \text{Tr}_R e_k(m).$$

(7.1)

In the classical limit $q = 1$, $\hbar = 0$ of the $U_q(sl(n))$ case this assignment is out of interest (its value is identically equal to 1 since the modules $M_k$ correspond to line bundles). But for generic $q$ and $\hbar$ it is not so. Let us show that the characteristic (7.1) is close to the Euler characteristic of a line bundle over the flag variety $\text{Fl}(\mathbb{C}^n)$.

It is known that the set of $SU(n)$-equivariant line bundles over the flag variety is in the one-to-one correspondence with the set of holomorphic one-dimensional representations of the torus $T \subset SU(n)$. Therefore, the bundles can be labelled by the vectors $k = (k_1, \ldots, k_n), k_i \in \mathbb{Z}$ since any such a representation is of the form $T \ni (t_1, \ldots, t_n) \mapsto \prod t_i^{k_i}$. Two vectors $k = (k_1, \ldots, k_n)$ and $k' = (k'_1, \ldots, k'_n)$ give rise to the same representation and hence to the same line bundle iff $k_i = k'_i + a$ with an integer $a$ (shortly iff $k = k' + a$) since $\prod t_i = 1$. So, applying (if necessary) a shift by an integer we can assume that the labels $k$ of these line bundles obey the condition $k_i \geq 0$.

It can be easily shown (cf. \[1\]) that the Euler characteristic of the line bundle corresponding to $k = (k_1, \ldots, k_n)$ equals

$$\prod_{i<j} \frac{(k_i - k_j + i - j)}{(i - j)}.$$  

(7.2)

Now let us pass to the NC orbits. Recall the notion of the $q$-index introduced in \[GLS2\]. It is defined as the following paring

$$\langle M_k, \overline{\pi}_\lambda \rangle = \text{Tr}_R \overline{\pi}_\lambda (e_k(m))$$

of a projective $L_{h,q}^k$ module $M_k$ and a finite dimensional equivariant representation $\overline{\pi}_\lambda$. (Note that the character $\overline{\chi}$ depends on the representation $\overline{\pi}_\lambda$.) Here $\text{Tr}$ stands for the categorical trace applied to the space $\text{End}(V_\lambda) \otimes \text{End}(V_{\pi}(m))$. (In contrast with the quantities $\text{Tr}_R L_{(m)}^k$ considered above where $\text{Tr}_R$ is applied to the second factor only.)

As follows from constructions of the previous section, for any $m$-admissible $\lambda$ we have

$$\langle M_k(m), \overline{\pi}_\lambda \rangle = \prod_{i<j} \frac{(\lambda_i - \lambda_j + k_i - k_j + i - j)_q}{(i - j)_q}.$$  

(7.3)

Putting $\lambda_i = 0$ in (7.3) we get the quantity

$$\chi_q(M_k) = \prod_{i<j} \frac{(k_i - k_j + i - j)_q}{(i - j)_q}$$

which is a $q$-analog of (7.2). We call it the $q$-Euler characteristic.

The characteristic $\chi_q(M_k)$ does not depend on a concrete form of the initial braiding $R$. Note that $\chi_q(M_k)$ is introduced without any holomorphic structure which is usually employed in the definition of the Euler characteristic of a line bundle since the operator $\partial$ is essentially involved in the definition (cf., e.g. \[II\]).

In this connection we recall that a generic orbit in $su(n)^*$ (treated as a real algebraic variety) can be equipped with different complex structures. These structures are labelled by elements of the Weyl group and the Euler characteristic of a given line bundle essentially depends on the choice of such an element. In our setting the quantity $\chi_q(M_k)$ depends on the ordering of the eigenvalues $\mu_i$. These orderings are also labelled by elements of the Weyl group.

\[\text{Since } \lambda = (0,0,\ldots,0) \text{ is not an } m\text{-admissible partition we cannot directly apply the formula (7.3) to the corresponding representation. However, we consider the extension of (7.3) to this point.}\]
Remark 26 Considering a semisimple orbit $O$ in $su(n)^*$ to be a real algebraic variety, we treat its coordinate ring as an $R$-algebra. The algebra arising from the quantization of the Kirillov bracket on such a real variety can be realized as a quotient of the enveloping algebra $U(su(n))$ (cf. [GS2]). It is also treated as an $R$-algebra. However, for the corresponding quotients of the mREA it is not possible to get rid of complex numbers (cf. [DGH] where an example of the quantum sphere is considered). That is why we treat the above quotient as a $C$-algebra and consider it as the quantization of the complexification of the orbit $O$.

Considering the $q$-Euler characteristic as a function in $\mu_i$ we can see that

$$\chi_q(M_k) = \chi_q(M_{k'}) \iff \exists a \in \mathbb{Z} : k = k' + a.$$  

Being motivated by this property, we introduce the following equivalence relation on the set of our projective modules

$$M_k \sim M_{k'} \iff \exists a \in \mathbb{Z} : k = k' + a.$$  

The equivalence class of the module $M_k$ will be denoted $[M_k]$.

Remark 27 In NC geometry one usually employs the following type of equivalence of projective modules (cf. [Ro]). Two $A$-modules are called equivalent if the corresponding idempotents (being extended by 0 if necessary) are similar. This equivalence is compatible with the usual trace. Namely, for two such idempotents we have $\text{Tr}(\pi(e_1)) = \text{Tr}(\pi(e_2))$ where $\pi$ is any finite dimensional representation of the algebra $A$. However, such an equivalence is not compatible with the categorical trace $\text{Tr}_R$. The matter is that the categorical trace is not invariant with respect to a similarity transformation $M \to P^{-1}MP$ since the matrices $B$ and $C$ (and their higher extensions) entering definition (3.19) of the categorical trace are not invariant under this map (if $R$ differs from the usual flip).

Now, taking the flag variety as a pattern, we define a product on the set of the projective modules $M_k$:

$$M_k \cdot M_{k'} = M_{k+k'}.$$  

It should be pointed out that the above relation does not mean any product of elements of the modules in question.

It is evident that thus defined product of modules factorizes to the set of equivalence classes. This set equipped with the product becomes a group. The role of the unity is played by the class of the trivial module. This group is an analog of the Picard group of the flag variety.

In a similar way we can introduce an analog of $G$-equivariant K-theory. Usually the algebra $K_G(pt)$ of a point is defined as an algebra of all virtual representations of the group $G$. Let $[V]$ be the equivalence class of a finite dimensional module $V$ over the algebra $L_{h,q}$. Consider the $\mathbb{Z}_q$-algebra additively generated by all these classes where $\mathbb{Z}_q$ is in turn a $\mathbb{Z}$-algebra generated by all $q$-numbers $m_q$, $m \in \mathbb{Z}$. As the sum of equivalence classes we take $[U] + [V] = [U \oplus V]$ and as the product $[U] \cdot [V] = [U \otimes V]$. The quotient of this algebra over the subalgebra generated by $[V] - \text{dim}_qV [1]$ will be denoted $Q(pt)$. Here $[1]$ is the class corresponding to the trivial object $V = \mathbb{K}$. Thus, the algebra $Q(pt)$ is smaller than $K_G(pt)$ and it is somewhat similar to the group $K(pt)$ in the non-equivariant K-theory.

Turning to the NC orbit $L_{h,q}^\chi$ we treat the class $[V]$ as that of the free $L_{h,q}^\chi$ module $L_{h,q}^\chi \otimes V$. The corresponding idempotent is the identity operator $I_V$ and its trace equals $\text{dim}_qV$.

Note, that a direct sum of two projective modules is a projective module as well. The corresponding idempotent is a direct sum of idempotents related to the summands. However, the
modules $M_k(m)$ and $M_{k'}(m)$ with the same $m$ admit the usual sum. The corresponding idempotents are added as equal size matrices. So, we get

$$\sum_{|k|=m} e_k(m) = I_{V(m)}.$$  

Regardless of the addition type we associate to the sum of two or more modules the sum of their equivalence classes.

Now, we consider the vector space spanned by the classes $[M_k]$ and $[1]$ with coefficients from $\mathbb{Z}_q$. Its quotient over subspace generated by

$$\sum_{|k|=m} [M_k] - \dim_q V_{(m)} [1]$$

will be denoted $Q(L^\chi_{h,q})$. Moreover, defining $[M_k] \cdot [1]$ to be $[M_k]$ we equip $Q(L^\chi_{h,q})$ with a $\mathbb{Z}_q$-algebra structure. Thus, we have $[M_k] \cdot [V] = \dim_q V [M_k]$ (i.e. we define the class of the tensor product $M_k \otimes V$ to be $\dim_q V [M_k]$).

Proofs of the following two propositions are easy and left to the reader.

**Proposition 28** The algebra $Q(L^\chi_{h,q})$ is generated by the unity $[1]$ and $p$ generators $x_i = [M_i]$, $1 \leq i \leq p$ subject to the following relations

$$\sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq p} x_{i_1} \cdot \ldots \cdot x_{i_k} = \dim_q V_{(1^k)} [1] = \frac{p_q!}{k_q!(p-k)_q!} [1]. \quad (7.4)$$

Here as usual $p_q! = 1_q 2_q \ldots p_q$.

Canceling $[1]$ in this formula we get the relations close to those arising in quantum cohomology theory in a completely different context.

**Example 29** Let $p = 3$ and $x$, $y$, $z$ be the generators of the algebra $Q(L^\chi_{h,q})$. Then they satisfy the system

$$x + y + z = 3_q, \quad x \cdot y + x \cdot z + y \cdot z = 3_q, \quad x \cdot y \cdot z = 1.$$  

**Proposition 30** The $q$-Euler characteristic being extended at $[1]$ by $\chi_q([1]) = 1$ gives rise to a linear map

$$\chi_q : Q(L^\chi_{h,q}) \rightarrow \mathbb{Z}_q.$$  

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