Higher Order Corrections to Density and Temperature of Fermions from Quantum Fluctuations

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A novel method to determine the density and temperature of a system based on quantum Fermionic fluctuations is generalized to the limit where the reached temperature \(T\) is large compared to the Fermi energy \(\varepsilon_f\). Quadrupole and particle multiplicity fluctuations relations are derived in terms of \(\frac{T}{\varepsilon_f}\). The relevant Fermi integrals are numerically solved for any values of \(\frac{T}{\varepsilon_f}\) and compared to the analytical approximations. The classical limit is obtained, as expected, in the limit of large temperatures and small densities. We propose simple analytical formulas which reproduce the numerical results, valid for all values of \(\frac{T}{\varepsilon_f}\). The entropy can also be easily derived from quantum fluctuations and give important insight for the behavior of the system near a phase transition. A comparison of the quantum entropy to the entropy derived from the ratio of the number of deuterons to neutrons gives a very good agreement especially when the density of the system is very low.

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The availability of heavy-ion accelerators which provide colliding nuclei from a few MeV/nucleon to GeV/nucleon has fueled a field of research loosely referred to as Nuclear Fragmentation. The characteristics of the fragments produced depend on the beam energy and the target-projectile combinations which can be externally controlled\textsuperscript{1,2}. Fragmentation experiments could provide informations about the nuclear matter properties and constrain its EOS\textsuperscript{4}. Long ago, W. Bauer stressed the crucial influence of the Pauli blocking in the momentum distributions of nucleons emitted in heavy ion collisions near the Fermi energy\textsuperscript{5}. We have recently proposed a method to estimate the density and temperature based on fluctuations estimated from an event by event determination of fragments arising after the energetic collision\textsuperscript{6}. A similar approach has also been applied to observe experimentally the quenching of fluctuations in a trapped Fermi gas\textsuperscript{7}. We go beyond the method of\textsuperscript{7} by including quadrupole fluctuations as well to have a direct measurement of densities and temperatures for subatomic systems. In this paper, we extend the method to derive the entropy of the system and we show how to recover the classical limit when the temperatures are large compared to the Fermi energy. We apply the proposed method to the microscopic CoMD approach\textsuperscript{8} which includes Fermionic statistics. The resulting energy densities and temperatures calculated using protons and neutrons display a rapid increase around 3 MeV temperature which is an indication of a first order phase transition. This result is confirmed by the rapid increase of the entropy per unit volume in the same temperature region. Similar results are found from the entropy density derived from the ratio of the number of produced deuterons to nucleons. Some differences between the numerical estimates and the \(\frac{T}{\varepsilon_f}\) expansions are found.

A method for measuring the temperature was proposed in\textsuperscript{9} based on momentum fluctuations of detected particles at temperature \(T\), assumed which gives: \(\sigma_{xy}^2 = \int d^3p (p_x - p_y)^2 n(p)\) where \(n(p)\) is the momentum distribution of particles. In\textsuperscript{9} a classical Maxwell-Boltzmann distribution of particles at temperature \(T\) was assumed which gives: \(\sigma_{xy}^2 = \int d^3p (p_x - p_y)^2 \frac{1}{\varepsilon_f} \frac{\varepsilon_f}{\sqrt{2\pi}} e^{-\frac{\varepsilon_f}{\varepsilon_f}}\)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1}
\caption{(Top) \(\frac{T}{\varepsilon_f}\) versus multiplicity fluctuations using different approximations. Full line gives the numerical solution of Eq. (1), full dots are the lowest order approximation discussed in ref.\textsuperscript{6}; (Bottom) Entropy per particle \(S\) (in units of \(\hbar\)) versus multiplicity fluctuations. Full line gives the numerical solution of Eq. (1), full triangles are the Sackur-Tetrod results.}
\end{figure}
where $\bar{N}m^2T^2$, $m$ is the mass of the fragment, $\bar{N}$ is the average number of particles. In heavy ion collisions, the produced particles do not follow classical statistics thus the correct distribution function must be used in Eq. (1). Protons(p), neutrons(n), tritium etc. follow the Fermi statistics. In this work we will concentrate on fermions only and in particular p and n which are abundantly produced in the collisions thus carrying important informations on the densities and temperatures reached. Using a Fermi-Dirac distribution $n(p)$

$$\langle \sigma^2_{xy} \rangle = \frac{4}{15} \int_0^\infty \int_0^\infty dp d\rho \langle n(p) \rangle$$

$$= \frac{4}{15} \frac{(2m)^2}{2} \int_0^\infty d\rho \int_0^\infty d\rho \frac{1}{(e^{\rho/\rho_f} + 1)}$$

$$= \frac{4}{15} \frac{(2m)^2}{2} \int_0^\infty dy d\rho \frac{1}{(e^{\rho/\rho_f} + 1)}$$

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$$= (2mT)^2 F_{QC}(\nu)$$

Within the same framework we can calculate the fluctuations of the $p$, $n$ multiplicity distributions. These are given by [10]:

$$\langle (\Delta N)^2 \rangle = \frac{3}{2} \frac{T}{\varepsilon_f}$$

The lowest order expansion in $(\frac{T}{\varepsilon_f})$, was also derived in [6] and is given by:

$$\langle (\Delta N)^2 \rangle = 3 \frac{T}{\varepsilon_f}$$

From the above equation (4) we can calculate numerically the multiplicity fluctuations for a given $\nu$ and recover the value of $(\frac{T}{\varepsilon_f})$ from the following equation which is solved numerically:

$$\frac{T}{\varepsilon_f} = \frac{1}{\frac{3}{2} \int_0^\infty dy d\rho \frac{1}{(e^{\rho/\rho_f} + 1)}}$$

In Fig. 1 we plot the quantity $\frac{T}{\varepsilon_f}$ vs the normalized fluctuations obtained by solving numerically eqs.(4) and eqs.(6) while the lowest order approximation, eq.(5), is given by the full dots. Since in experiments or modeling one recovers the normalized fluctuations, it is better to find a relation between the normalized temperatures as function of the normalized fluctuations displayed in the Fig. 1. It is useful to parametrize the numerical results as:

$$\frac{T}{\varepsilon_f} = -0.442 + \frac{0.442}{1 - \frac{3}{2} \int_0^\infty dy d\rho \frac{1}{(e^{\rho/\rho_f} + 1)}^{0.566}}$$

$$+ 0.345 \frac{\langle (\Delta N)^2 \rangle}{\langle N \rangle} - 0.12 \left( \frac{(\Delta N)^2}{\langle N \rangle} \right)^2$$

which is practically indistinguishable from the numerical result (full line) reported in Fig. 1. As expected the approximations contained in eq. (5) reproduce the numerical results (full line) up to $\frac{T}{\varepsilon_f} \approx 0.5$. Since from experimental data or models it is possible to extract directly the normalized fluctuations, one can easily derive the value of $\frac{T}{\varepsilon_f}$ from Eq. (7).

Before proceeding further, it is important to test the validity of the approximations for the quadrupole fluctuations by comparing them to the numerical result solving Eq. (2). In Fig. 2 we plot the quantum correction term $F_{QC}$ versus $\frac{T}{\varepsilon_f}$. The difference with the classical case is again striking (the $F_{QC}$ in Eq. (2) equal to one for a classical perfect gas). For simplicity we can parametrize the numerical result with the simple approximation:

$$F_{QC}\vert_{fit} = 0.2 \left( \frac{T}{\varepsilon_f} \right)^{-1.71} + 1.$$
which is indistinguishable from the numerical result displayed in Fig. 2 (full line). Clearly such an equation converges to one at high $T$ as expected. Notice that the lowest order approximation \[ \frac{1}{N} \ln \rho \] is valid up to a modest $T \approx 0.2$. Eqs. (7) and (8) might be very useful when deriving densities and temperatures from data or models, without worrying if one is in the classical or fully quantum limit, the only constraint is that we are dealing with fermions.

Once the density and the temperature of the system have been determined it is straightforward to derive other thermodynamical quantities. One of such quantities is the entropy:

\[
S \equiv \frac{U - A}{T} = N \left[ \frac{5}{2} \frac{f_{5/2}(z)}{f_{3/2}(z)} - \ln z \right]
\]

(9)

where $f_m(z) = \frac{1}{m!} \int_0^\infty \frac{z^{m-1}e^{-z}}{\sqrt{2\pi}} dz$ and $z = e^{\frac{U}{T}}$ is the fugacity. $U$ and $A$ are the internal and Helmholtz free energy respectively. This equation can be numerically evaluated and the results are plotted in Fig. 1 (bottom panel). For practical purposes it might be useful to have a parametrization of the entropy in terms of the normalized fluctuations, which is physically transparent since entropy and fluctuations are strongly correlated [10]:

\[
\frac{S}{N} \ln \rho = -41.68 + \left(1 - \frac{(\Delta N)^2}{\langle N \rangle}\right)^{0.022}
\]

\[
+ 2.37 \frac{(\Delta N)^2}{\langle N \rangle} - 0.83 \left(\frac{(\Delta N)^2}{\langle N \rangle}\right)^2
\]

(10)

The latter fit is indistinguishable from the numerical result plotted in Fig. 1 (full line-bottom panel) while the Sackur-Tetrod result (full triangles) is valid in the classical limit [10] as confirmed in the figure 1.

To illustrate the strength of our approach we simulated $^{40}\text{Ca} + ^{40}\text{Ca}$ heavy ion collisions at fixed impact parameter $b = 1\text{fm}$ and beam energies $E_{\text{lab}}/A$ ranging from 4 MeV/A up to 100 MeV/A. Collisions were followed up to a maximum time $t = 1000 \text{fm}/c$ in order to accumulate enough statistics. Particles emitted at later times (evaporation) could affect somehow the results and this might be important especially at the lowest beam energies. A complete discussion of these simulations can be found in [6], here we will use the results to compare the different approximations.

In Fig. 3 we plot the temperature vs density as obtained from the quadrupole and multiplicity fluctuations. The top panel refers to protons while the bottom to neutrons. As we can see from the figure, the results obtained using the fit functions, Eqs. (7) and (8), deviate slightly from the lowest order approximations given in Eqs. (9) and (10). This is a signature that we are in the fully quantum regime for the events considered. For comparison, in the same plot we display the classical temperatures which are systematically higher than the quantum one, see Eq.(2) and Fig. 2 [5]. We notice that for a given excitation energy we can derive a classical or a quantum temperature, but the density can be derived for the quantum case only within our approach. Of course other methods could be devised that give both classical temperatures and densities using suitable fragment ratios [11]. We stress that those classical temperatures do not need to coincide with the classical temperatures considered here since we are dealing with protons and neutrons only. Larger fragments could be also included and a discussion on this can be found in [12, 13].

To better summarize the results we plot in Fig. 3 (top panel) the energy density $\varepsilon = \langle \frac{\hbar^2}{2m} \rangle \rho$ versus temperature [6]. Different particle types scale especially at high $T$ where Coulomb effects are expected to be small. A rapid variation of the energy density is observed around $T \approx 2\text{MeV}$ for neutrons and $T \approx 3\text{MeV}$ for protons which indicates a first order phase transition [14]. As we see from the figure, the numerical solution of the Fermi integrals gives small corrections while keeping the relevant features obtained in the lowest approximation intact. This again suggests that in the simulations the system is fully quantal. We also notice that Coulomb effects become negligible at $T \gg 3\text{MeV}$ where the phase transition occurs. The smaller role of the Coulomb field in the phase transition has recently been discussed exper-
The number of deuterons to protons (or neutrons) can be derived from the ratio of the produced number of deuterons to protons (triangles) (neutrons-stars), eq.(11).

The rapid increase of the entropy per unit volume is due to the sudden increase of the number of degrees of freedom (fragments) which is at low density , as plotted in the bottom panel of Fig. 4. The entropy density, are plotted in Figure 4 (bottom panel) with open symbols. We find an overall qualitative good agreement of the entropy density to the quantum results, eq.10, especially for neutrons. Very interesting is the good agreement for neutrons at low T where the particles are emitted from the surface of the nuclei which is at low density, see also Fig.3. Such a feature is not present for the protons due to larger Coulomb distortions. There is a region near the transition ($T \approx 3MeV$), where both ratios do not reproduce the quantum results. However, at large temperatures it seems that all methods converge as expected.

In order to confirm the origin of the phase transition, it is useful to derive the entropy density $\Sigma = \frac{S}{N}$ which is plotted in the bottom panel of Fig. 4. The rapid increase of the entropy per unit volume is due to the sudden increase of the number of degrees of freedom (fragments) with increasing T. The entropy can be also derived using the law of mass action from the ratio of the produced number of deuterons to protons (or neutrons) $R_{d,p(n)}$

\[ \frac{S}{N} = 3.95 - \ln R_{d/p(n)} - 1.25 \frac{R_{d/p(n)}}{1 + R_{d/p(n)}} \quad (11) \]

The CoMD results from eq.(11) multiplied by the density, are plotted in Figure 4 (bottom panel) with open symbols. We find an overall qualitative good agreement of the entropy density to the quantum results, eq.10, especially for neutrons. Very interesting is the good agreement for neutrons at low T where the particles are emitted from the surface of the nuclei which is at low density, see also Fig.3. Such a feature is not present for the protons due to larger Coulomb distortions. There is a region near the transition ($T \approx 3MeV$), where both ratios do not reproduce the quantum results. However, at large temperatures it seems that all methods converge as expected.

In conclusion, in this work we have addressed a general method for deriving densities and temperatures of fermions. For high temperatures and small densities the classical result is recovered as expected. However, we have shown in CoMD calculations that the effect of order terms gives small differences in the physical observables considered in this paper but they could become large when approaching the classical limit. To overcome this problem we have produced suitable parameterizations of quadrupole and multiplicity fluctuations which are valid for fermions at all temperatures and densities. The results obtained in this paper are quite general and they could be applied to other systems, for instance trapped Fermi gases [9], to determine the entropy from normalized quantum fluctuations. We have also shown that the quantum entropy can be compared to the one derived from the ratio of the number of deuterons to protons or neutrons produced in the collisions. Especially the neutrons seem to give cleaner results but of course they are more difficult to determine experimentally.

[1] A. Bonasera, F. Gulminelli and J. Molitoris, Phys. Rep. 243, 1 (1994).
[2] G. Bertsch and S. Dasgupta, Phys. Rep. 160, 189 (1988).
[3] A. Bonasera et al., Rivista del Nuovo Cimento 23, 1(2000).
[4] L. P. Csernai, Introduction to Relativistic Heavy Ion Collisions (Wiley, New York 1994).
[5] W. Bauer, Phys. Rev. C51, 803(1995).
[6] H. Zheng and A. Bonasera, Phys. Lett. B696, 178(2011) and ArXiv: 1105.0563 [nucl-th].
[7] T. Mueller et al., Phys. Rev. Lett. 105, 040401(2010); C. Sanner et al., Phys. Rev.Lett. 105, 040402(2010); C. I. Westbrook, Physics 3, 49 (2010).
[8] A. Bonasera, Phys. Rev. C62, 052202(R)(2000); M. Papa, T. Maruyama and A. Bonasera, Phys. Rev. C64, 024612(2001); A. Bonasera, Nucl. Phys. A681, 64c(2001); S. Terranova and A. Bonasera, Phys. Rev. C70, 024906(2004); S. Terranova, D. M. Zhou and A. Bonasera, Eur. Phys. J. A26, 333(2005).
[9] S. Wuenschel et al., Nucl. Phys. A843, 1 (2010).
[10] Landau L. and Lifshits F., Statistical Physics (Pergamon, New York) 1980; Huang K., Statistical Mechanics (J. Wiley and Sons, New York) 1987, 2 n ed.; R. K. Pathria, Statistical Mechanics, Elsevier (Singapore) Pte Ltd, Second edition (2003).
[11] S. Albergo et al., Nuovo Cimento 89, 1(1985).
[12] J. B. Natowitz et al., Phys. Rev. C65, 034618(2002).
[13] J. B. Natowitz et al., Phys. Rev. Lett. 104, 202501(2010).
[14] M. D’Agostino et al., Nucl. Phys. A650, 329 (1999).
[15] A. Bonasera et al., Phys. Rev. Lett. 101, 122702(2008); M. Huang et al., Phys. Rev. C81, 044618(2010); M. Huang et al., Nucl. Phys. A847, (2010) 233.
[16] P.J. Siemens and J.L. Kapusta, Phys.Rev.Lett.43,1486 (1979).