A generalization of Minkowski’s inequality by Hahn integral operator

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ABSTRACT

In this paper, we use the Hahn integral operator for the description of new generalization of Minkowski’s inequality. The use of this integral operator definitely generalizes the classical Minkowski’s inequality. Our results with this new integral operator have the abilities to be utilized for the analysis of many mathematical problems as applications of the work.

1. Introduction

The application of fractional calculus in scientific fields like engineering, physics, chemistry and many more has increased the attention of researchers to its different aspects. One of the aspects which are nowadays very much popular among the scientists for research is the integral inequalities with applications. In this area, most of the authors are generalizing the standard results in the available literature by using different types/definitions of the fractional integral operators (FIOs) [1–9].

Here, we consider some work recently published in different fields related to the inequalities. For instance, Khan et al. [1] studied an integral inequality for a class of decreasing n positive functions where \( n \in \mathbb{N} \) for the left- and right-fractional conformable integrals. Khan et al. [2] considered new fixed point theorems by the help of integral inequalities for a class of quadruple self-mappings. Chen and Katugampola [3] obtained fractional integral inequalities called Hermite–Hadamard and Hermite–Hadamard–Fejér which generalize the classical cases. Niezgoda [4] studied Minkowski and Gruss-type inequalities. Khan et al. [5] studied existence results and error analysis for a class of fractional-order differential–integral equations.

Budak and Sarikaya [6] studied Ostrowski-type inequalities for a class of mappings which are possessing bounded variation and the application for quadrature formulation was also presented. Cerone et al. [7] studied Jensen–Ostrowski-type inequalities for a class of functions of \( t \)-divergence measures and provided applications. Erdem et al. [8] studied Ostrowski-type inequalities for a class of mappings whose derivatives of second order are absolutely convex and given some special cases of their inequalities with applications. El-Deeb and Ahmed [9] studied some nonlinear retarded inequalities for Volterra–Fredholm integral equations and provided some applications of their results. Chang and Luor [10] proved retarded integral inequalities and provided its application for the analysis of integral equations.

The applications of inequalities are very much common for fixed point theorems and existence and uniqueness of solutions for differential equations. Here, we highlight two recent results: Baleanu et al. [11] studied existence criteria for the solution of hybrid fractional differential equations and provided some applications using inequalities. Khan et al. [12] used some inequalities and proved existence and Hyers-stability for a class of nonlinear fractional differential equations involving \( p \)-Laplacian operator.

In this paper, we use Hahn FIO [13] for the integrable functions to produce Minkowski’s integral inequalities. Our results are more general and applicable than the classical case. We divide the paper in three main sections. This section is for literature review and includes some useful information about the Hahn calculus from [13]. Here, we highlight some relevant definitions from the available literature. In the second section, we give the proofs of our main results. Finally, we prove some results on fractional integral inequalities.

We now give some basic ideas related to the Hahn FIO.
**Definition 1.1 ([14]):** Let $I$ be any closed interval of $\mathbb{R}$, which contains $a, b, r \in (0, 1), \omega > 0$ and $\omega_0 = \omega/(1 - r)$. Assuming that $h : I \to \mathbb{R}$ is a function, we define $r, \omega$-integral of $f$ from $a$ to $b$ by

$$
\int_a^b h(t) \, df_{\omega,t} = \int_a^b h(t) \, df_{\omega,t} - \int_a^b h(t) \, df_{\omega,t},
$$

where

$$
\int_a^x h(t) \, df_{\omega,t} = f(x) - \omega_0 - \omega + \omega_0 \sum_{k=0}^{\infty} \beta^k h(x^k + \omega(k), x \in I)
$$

such that $h$ is called $r, \omega$-integrable on $[a, b]$, and the sum to the right-hand side of (1) will be called the sum of the Jackson-Nörlund.

**Lemma 1.2 ([13]):** Let $r \in (0, 1), \omega > 0, a, b \in [\omega_0, T]_{r,\omega}$ where

$$
[\omega_0, T]_{r,\omega} := \{r^k T + \omega[k], k \in \mathbb{N} \cup \{0\} \cup \omega_0\},
$$

integrable on $[\omega_0, T]_{r,\omega}$. Then, the following formulas hold:

(i) $\int_a^b h(t) \, df_{\omega,t} = 0$;

(ii) $\int_a^b \beta h(t) \, df_{\omega,t} = \beta \int_a^b h(t) \, df_{\omega,t}, \beta \in \mathbb{R}$;

(iii) $\int_a^b h(t) \, df_{\omega,t} = -\int_b^a h(t) \, df_{\omega,t}$;

(iv) $\int_a^b h(t) \, df_{\omega,t} = \int_a^1 h(t) \, df_{\omega,t} + \int_1^b h(t) \, df_{\omega,t}, c \in [\omega_0, T]_{r,\omega}$

(v) $\int_a^b (h(t) + g(t)) \, df_{\omega,t} = \int_a^b h(t) \, df_{\omega,t} + \int_a^b g(t) \, df_{\omega,t}$;

(vi) $\int_a^b h(t) \, df_{\omega,t} = \int_a^b (h(t) + g(t)) \, df_{\omega,t} - \int_a^b g(t) \, df_{\omega,t}$.

**Lemma 1.3 ([14] (Fundamental theorem of Hahn calculus)):** Let $h : I \to \mathbb{R}$ be continuous at $\omega_0$. We define

$$
H(x) := \int_{\omega_0}^x h(t) \, df_{\omega,t}, \quad x \in I.
$$

Then, $H$ is continuous at $\omega_0$. Furthermore, $D_{r,\omega}H(x)$ exists for every $x \in I$ and $D_{r,\omega}H(x) = h(x)$.

**Lemma 1.4 ([15] (Leibniz formula of Hahn calculus)):** Let $h : [\omega_0, T]_{r,\omega} \times [\omega_0, T]_{r,\omega} \to \mathbb{R}$. Then

$$
D_{r,\omega} \left[ \int_{\omega_0}^t h(t, s) \, df_{\omega,s} \right] = \int_{\omega_0}^t h(t, s) \, df_{\omega,s} + h(\sigma_{r,\omega}(t), t),
$$

where $D_{r,\omega}$ is Hahn difference with respect to the variable $t$.

The $r$-gamma function is defined by

$$
\Gamma_r(y) := \frac{(1 - r)^{-1}}{(1 - y)^{-1}}, \quad y \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\},
$$

where

$$
\Gamma_r(y) = \prod_{n=0}^{\infty} (1 - n^r + 1)/(1 - r^{n+1}),
$$

$r \neq 0$. The $r, \omega$-forward jump operator is defined by

$$
\sigma_{r,\omega}(t) = rt + \omega.
$$

**Definition 1.5 ([13] (Fractional Hahn integral)):** For $\beta, \omega > 0, r \in (0, 1)$ and $h$ defined on $[\omega_0, T]_{r,\omega}$, the fractional Hahn integral is defined by

$$
I^\beta_r(h(t)) = \frac{1}{\Gamma_r(\beta)} \int_{\omega_0}^t (t - \sigma_{r,\omega}(s))^{\beta-1} h(s) \, df_{\omega,s},
$$

such that $I^\beta_r f(t) = f(t)$.

The following theorems are available in [13].

**Theorem 1.6 ([13]):** For $h : [\omega_0, T]_{r,\omega} \to \mathbb{R}, \beta > 0, r \in (0, 1), \omega > 0$,

$$
I^\beta_r h(t) = \frac{h(\omega_0)}{\Gamma_r(\beta + 1)} (t - \omega)_\beta.
$$

**Theorem 1.7 ([13]):** For $h : [\omega_0, T]_{r,\omega} \to \mathbb{R}, \alpha, \beta > 0, r \in (0, 1), \omega > 0$,

$$
\int_{\omega_0}^b (t - \sigma_{r,\omega}(s))^{\beta-1} h(t) \, df_{\omega,s} = 0.
$$

**Theorem 1.8 ([13]):** For $h : [\omega_0, T]_{r,\omega} \to \mathbb{R}, \alpha, \beta > 0, r \in (0, 1), \omega > 0$, and $b \in [\omega_0, T]_{r,\omega}$

$$
I^\beta_r (\int_{\omega_0}^a h(t) \, df_{\omega,t}) = I^\beta_r (\int_{\omega_0}^b h(t) \, df_{\omega,t}) = I^\beta_r (\int_{\omega_0}^b h(t) \, df_{\omega,t}).
$$

**Theorem 1.9 ([13]):** For $h : [\omega_0, T]_{r,\omega} \to \mathbb{R}, r \in (0, 1); \beta, \omega > 0$,

$$
I^\beta_r \left( D_{r,\omega} h(t) \right) = D_{r,\omega} I^\beta_r h(t) - \frac{(t - \omega)^{\beta-1}}{\Gamma_r(\beta)} h(\omega_0).
$$

**Theorem 1.10 ([13]):** For $\beta, \omega > 0, r \in (0, 1)$, and $h : [\omega_0, T]_{r,\omega} \to \mathbb{R}$,

$$
D_{r,\omega} I^\beta_r h(t) = h(t).
$$

**Theorem 1.11 ([13]):** For $\beta, \omega > 0, r \in (0, 1)$, and $h : [\omega_0, T]_{r,\omega} \to \mathbb{R}$,

$$
I^\beta_r \left( D_{r,\omega} h(t) \right) = h(t) - \sum_{k=0}^{N-1} \frac{(t - \omega)^{\beta - N + k}}{\Gamma_r(\beta - N + k + 1)} \times \left[ D_{r,\omega}^{\beta - N + k} h(\omega_0) \right],
$$

where $N - 1 < \beta < N, N \in \mathbb{N}$. 

**Theorem 1.12 ([13]):** For $\beta, \omega > 0, r \in (0, 1)$, and $h : [\omega_0, T]_{r,\omega} \to \mathbb{R}$,

$$
I^\beta_r \left( D_{r,\omega} h(t) \right) = h(t) - \sum_{k=0}^{N-1} \frac{(t - \omega)^{\beta - N + k}}{\Gamma_r(\beta - N + k + 1)} \times \left[ D_{r,\omega}^{\beta - N + k} h(\omega_0) \right].
$$

where $N - 1 < \beta < N, N \in \mathbb{N}$.
**Corollary 1.12 ([13]):** Let $\beta, \omega > 0$, $r \in (0, 1)$, and $h : [\omega_0, T]_{I_{r,\omega}} \to \mathbb{R}$,

$$I_{r,\omega}^\beta h(t) = h(t) + k_1(t - \omega_0)^{\beta-1} + \cdots + k_N(t - \omega_0)^{\beta-N},$$

for $k_j \in \mathbb{R}$, $j = 1, 2, \ldots, N$ and $N - 1 < \beta < N, N \in \mathbb{N}$.

### 2. Main results

**Theorem 2.1:** Let $\beta, \omega > 0$ and $p \geq 1$. Let $h, g$ be defined on $[\omega_0, T]_{I_{r,\omega}}$. If $h(t) < \infty$ and $\int_T^th(t)\,dt < \infty$.

If $0 < k \leq h(s)/g(s) \leq m$ for $m, k \in \mathbb{R}_+$ and $s \in [\omega_0, t]_{I_{r,\omega}}$. Then

$$\left(I_{r,\omega}^\beta h(t)\right)^{1/p} + \left(I_{r,\omega}^{\beta-1}g(t)\right)^{1/p} \leq c_1 t^{\beta-1/p} h(t) + (1 + 1/k) g(t) \leq 0.$$

with $c_1 = (m(k + 1) + (m + 1))/((k + 1)(m + 1))$.

**Proof:** Using the condition $h(s)/g(s) \leq m, s \in [\omega_0, t]_{I_{r,\omega}}$ we can write

$$h(s) \leq m(h(s) + g(s)) - mh(s),$$

which implies

$$m^p(h(s) + g(s))^p - (m + 1)^p h^p(s) \leq 0.$$

Let us define a function

$$\psi_1(s) = m^p(h(s) + g(s))^p - (m + 1)^p h^p(s) \geq 0. \quad (3)$$

Multiplying (3) by $I_{r,\omega}^\beta h(s)/\Gamma_r(\beta)$, we get

$$\frac{(t - \sigma_{r,\omega}(s))^{\beta-1}}{\Gamma_r(\beta)} \psi_1(s) = \frac{(t - \sigma_{r,\omega}(s))^{\beta-1}}{\Gamma_r(\beta)} (m^p h(s) + g(s)^p - (m + 1)^p h^p(s)).$$

Integrating estimate (4) from $\omega_0$ to $t$ with respect to the variable $s$, we get

$$\int_{\omega_0}^t \frac{(t - \sigma_{r,\omega}(s))^{\beta-1}}{\Gamma_r(\beta)} \psi_1(s) ds = \int_{\omega_0}^t \frac{(t - \sigma_{r,\omega}(s))^{\beta-1}}{\Gamma_r(\beta)} (m^p h(s) + g(s)^p - (m + 1)^p h^p(s)) ds.$$

This implies

$$I_{r,\omega}^\beta \psi_1(t) = m^p I_{r,\omega}^\beta (h(t) + g(t))^p - (m + 1)^p h^p(t) \geq 0.$$

Then, we have

$$\left(\frac{m}{m + 1}\right)^p I_{r,\omega}^\beta (h(t) + g(t))^p \geq I_{r,\omega}^\beta h^p(t). \quad (5)$$

Taking $1/p$ power for both sides of (5), we have

$$\left(I_{r,\omega}^\beta h^p(t)^{1/p} \leq \frac{m}{m + 1} I_{r,\omega}^\beta (h(t) + g(t))^p \right)^{1/p} \geq I_{r,\omega}^\beta h^p(t). \quad (6)$$

On the other hand, since $kg(s) \leq h(s)$, then we see that

$$\left(1 + \frac{1}{k}\right)^p g^p(s) \leq \left(\frac{1}{k}\right)^p (h(s) + g(s))^p.$$

Hence, we have

$$\left(\frac{1}{k}\right)^p (h(s) + g(s))^p - \left(1 + \frac{1}{k}\right)^p g^p(s) \geq 0.$$

Let us define a function

$$\psi_2(s) = \frac{1}{k} (h(s) + g(s))^p - \left(1 + \frac{1}{k}\right)^p g^p(s). \quad (7)$$

Multiplying (7) by $$(t - \sigma_{r,\omega}(s))^{\beta-1}/\Gamma_r(\beta),$$ we get

$$\frac{(t - \sigma_{r,\omega}(s))^{\beta-1}}{\Gamma_r(\beta)} \psi_2(s) = \frac{(t - \sigma_{r,\omega}(s))^{\beta-1}}{\Gamma_r(\beta)} \left(\left(\frac{1}{k}\right)^p (h(s) + g(s))^p - \left(1 + \frac{1}{k}\right)^p g^p(s)\right). \quad (8)$$

Integrating (8) from $\omega_0$ to $t$ with respect to the variable $s$, we get

$$\int_{\omega_0}^t \frac{(t - \sigma_{r,\omega}(s))^{\beta-1}}{\Gamma_r(\beta)} \psi_2(s) ds = \int_{\omega_0}^t \frac{(t - \sigma_{r,\omega}(s))^{\beta-1}}{\Gamma_r(\beta)} (h(s) + g(s))^p ds \geq I_{r,\omega}^\beta g^p(t). \quad (9)$$

This implies that

$$\left(\frac{1}{1 + k}\right)^p I_{r,\omega}^\beta (h(t) + g(t))^p \geq I_{r,\omega}^\beta g^p(t). \quad (10)$$

Taking $1/p$ power for both sides of inequality (10), we have

$$I_{r,\omega}^\beta g^p(t)^{1/p} \leq \frac{1}{1 + k} I_{r,\omega}^\beta (h(t) + g(t))^p \leq I_{r,\omega}^\beta g^p(t)^{1/p}. \quad (11)$$

Adding (6) and (11), we get

$$I_{r,\omega}^\beta h^p(t)^{1/p} + I_{r,\omega}^\beta g^p(t)^{1/p} \leq c_1 I_{r,\omega}^\beta (h(t) + g(t))^p \leq I_{r,\omega}^\beta g^p(t)^{1/p}. \quad (12)$$

**Theorem 2.2:** Let $\beta, \omega > 0$ and $p \geq 1$. Let $h, g$ be defined on $[\omega_0, T]_{I_{r,\omega}}$. If $h^p(t) < \infty$ and $I_{r,\omega}^\beta h^p(t) < \infty$. If $0 < k \leq$
Using the Minkowski’s inequality on the right side of (18), we have

\[
\left( \int_{t_0}^{t} (h(t)+g(t))^p \, dt \right)^{1/p} \leq \left( \int_{t_0}^{t} h(t)^p \, dt \right)^{1/p} + \left( \int_{t_0}^{t} g(t)^p \, dt \right)^{1/p}.
\]

From (13), we conclude that

\[
\left( \frac{(m+1)(k+1)}{m} \right)^{1/p} \left( \int_{t_0}^{t} h(t)^p \, dt \right)^{1/p} \leq \left( \int_{t_0}^{t} g(t)^p \, dt \right)^{1/p} \leq \left( \int_{t_0}^{t} h(t)^p \, dt \right)^{1/p} + \left( \int_{t_0}^{t} g(t)^p \, dt \right)^{1/p}.
\]

3. Other fractional integral inequalities

**Theorem 3.1:** Let \( \beta, \omega > 0 \) and \( p > 1, q > 1 \), \( 1/p + 1/q = 1 \). Let \( h, g \) be defined on \([\omega_0, T] \) such that \( \int_{t_0}^{T} h(t)^p \, dt < \infty \) and \( \int_{t_0}^{T} g(t)^p \, dt < \infty \). If \( 0 < k \leq h(s)/g(s) \leq m \) for \( h, g \) on \([0, \infty) \) and \( s \in [\omega_0, t] \), then

\[
\left( \int_{t_0}^{t} h(t)^{1/p} g(t)^{1/q} \, dt \right)^{1/q} \leq \left( \int_{t_0}^{t} h(t)^{1/p} g(t)^{1/q} \, dt \right)^{1/q}.
\]

**Proof:** Using the condition \( h(s)/g(s) \leq m, s \in [\omega_0, t] \), we have

\[
h(s) \leq mg(s) \Rightarrow g^{1/q}(s) \geq m^{-1/q}h^{1/q}(s).
\]

Multiplying by \( h^{1/p}(s) \) both sides of (15), we obtain

\[
\frac{h^{1/p}(s)g^{1/q}(s)}{m^{1-q/p}} \geq m^{-1/q}h^{1/q}(s).
\]

Multiplying inequality (16) by \( (t - \sigma_{t_0}(s))^{-1} / \Gamma(\beta) \), it is seen that

\[
\frac{(t - \sigma_{t_0}(s))^{-1} / \Gamma(\beta)}{m^{1-q/p}} \leq m^{-1/q} (t - \sigma_{t_0}(s))^{-1} / \Gamma(\beta) \leq h(s).
\]

Integrating (17) from \( \omega_0 \) to \( t \) with respect to the variable \( s \), we arrive at

\[
\int_{\omega_0}^{t} \frac{(t - \sigma_{t_0}(s))^{-1} / \Gamma(\beta)}{m^{1-q/p}} \, ds \leq \int_{\omega_0}^{t} \frac{(t - \sigma_{t_0}(s))^{-1} / \Gamma(\beta)}{m^{1-q/p}} \, ds.
\]

This implies

\[
\int_{\omega_0}^{t} h(t) \leq \int_{\omega_0}^{t} g(t)^{1/q} \, dt.
\]

Taking \( 1/p \) power on both sides of inequality (18), we get

\[
\int_{\omega_0}^{t} h(t)^{1/p} \leq \int_{\omega_0}^{t} g(t)^{1/q} \, dt.
\]

On the other hand, we find

\[
k^{1/q} g^{1/q} \leq k^{1/q} h^{1/q}.
\]

Multiplying by \( g^{1/q} \) both sides of (20), we have

\[
k^{1/q} g \leq k^{1/q} h.
\]

Multiplying (21) by \( (t - \sigma_{t_0}(s))^{-1} / \Gamma(\beta) \), we see that

\[
k^{1/q} (t - \sigma_{t_0}(s))^{-1} / \Gamma(\beta) \leq \int_{\omega_0}^{t} \frac{(t - \sigma_{t_0}(s))^{-1} / \Gamma(\beta)}{m^{1-q/p}} \, ds.
\]

Integrating inequality (22) from \( \omega_0 \) to \( t \) with respect to the variable \( s \), we have

\[
\int_{\omega_0}^{t} \frac{(t - \sigma_{t_0}(s))^{-1} / \Gamma(\beta)}{m^{1-q/p}} \, ds \leq \int_{\omega_0}^{t} \frac{(t - \sigma_{t_0}(s))^{-1} / \Gamma(\beta)}{m^{1-q/p}} \, ds.
\]

Taking \( 1/q \) power on both sides of estimate (23), we obtain

\[
\int_{\omega_0}^{t} h(t) \leq \int_{\omega_0}^{t} g(t) \, dt.
\]

Theorem 3.2: Let \( \beta, \omega > 0 \) and \( p > 1, q > 1, 1/p + 1/q = 1 \) and \( h, g \) be defined on

\[
[h(s)/g(s) < \infty, h(t)^p < \infty, g(t)^q < \infty]
\]
Multiplying inequality (26) by \( \frac{1}{\Gamma_1(\beta)} \), we have
\[
I_{\tau_0}^{\beta} h^p(t) g^q(t) \leq c_3 I_{\tau_0}^{\beta} (h(t) + g(t))^q 
\]
with \( c_3 = 2^{b-1} m^p / p(m + 1)^p \) and \( c_4 = 2^{d-1} / q(k + 1)^q \).

**Proof:** Using the hypothesis, we have the following identity:
\[
(m + 1)^p h^p(s) \leq m^p (h(s) + g(s))^p. \tag{26}
\]

Multiplying (26) by \((t - \sigma_{\tau_0}(s))^{\beta-1} / \Gamma_1(\beta)\), we get
\[
(m + 1)^p \left( \frac{(t - \sigma_{\tau_0}(s))^{\beta-1}}{\Gamma_1(\beta)} \right) h^p(s) \leq m^p \left( \frac{(t - \sigma_{\tau_0}(s))^{\beta-1}}{\Gamma_1(\beta)} \right) (h(s) + g(s))^p. \tag{27}
\]

Integrating (27) from \( \omega_0 \) to \( t \) with respect to the variable \( s \), we have
\[
(m + 1)^p \int_{\omega_0}^{t} \left( \frac{(t - \sigma_{\tau_0}(s))^{\beta-1}}{\Gamma_1(\beta)} \right) h^p(s) \, ds \leq m^p \int_{\omega_0}^{t} \left( \frac{(t - \sigma_{\tau_0}(s))^{\beta-1}}{\Gamma_1(\beta)} \right) (h(s) + g(s))^p \, ds.
\]

This implies
\[
I_{\tau_0}^{\beta} h^p(t) \leq \frac{m^p}{(m + 1)^p} I_{\tau_0}^{\beta} (h(t) + g(t))^p. \tag{28}
\]

Multiplying both sides of (28) by \( 1 / p \), we have
\[
\frac{1}{p} I_{\tau_0}^{\beta} h^p(t) \leq \frac{m^p}{p(m + 1)^p} I_{\tau_0}^{\beta} (h(t) + g(t))^p. \tag{29}
\]

On the other hand, we also have
\[
(k + 1)^q g^q(s) \leq (h(s) + g(s))^q. \tag{30}
\]

Multiplying inequality (26) by \((t - \sigma_{\tau_0}(s))^{\beta-1} / \Gamma_1(\beta)\), we get
\[
(k + 1)^q \left( \frac{(t - \sigma_{\tau_0}(s))^{\beta-1}}{\Gamma_1(\beta)} \right) g^q(s) \leq \left( \frac{(t - \sigma_{\tau_0}(s))^{\beta-1}}{\Gamma_1(\beta)} \right) (h(s) + g(s))^q. \tag{31}
\]

Integrating estimate (31) from \( \omega_0 \) to \( t \) with respect to the variable \( s \), we find
\[
(k + 1)^q \int_{\omega_0}^{t} \left( \frac{(t - \sigma_{\tau_0}(s))^{\beta-1}}{\Gamma_1(\beta)} \right) g^q(s) \, ds \leq \int_{\omega_0}^{t} \left( \frac{(t - \sigma_{\tau_0}(s))^{\beta-1}}{\Gamma_1(\beta)} \right) (h(s) + g(s))^q \, ds.
\]

This implies
\[
I_{\tau_0}^{\beta} h^p(t) g^q(t) \leq \frac{1}{(k + 1)^q} I_{\tau_0}^{\beta} (h(t) + g(t))^q. \tag{32}
\]

Multiplying both sides of estimate (32) by \( 1/q \), we have
\[
\frac{1}{q} I_{\tau_0}^{\beta} h^p(t) g^q(t) \leq \frac{1}{q(k + 1)^q} I_{\tau_0}^{\beta} (h(t) + g(t))^q. \tag{33}
\]

Consider the Young’s Inequality
\[
\frac{h^p}{p} + \frac{g^q}{q} \geq h g. \tag{34}
\]

Multiplying inequality (34) by \((t - \sigma_{\tau_0}(s))^{\beta-1} / \Gamma_1(\beta)\), we get
\[
\frac{(t - \sigma_{\tau_0}(s))^{\beta-1}}{\Gamma_1(\beta)} h(s) g(s) \leq \frac{(t - \sigma_{\tau_0}(s))^{\beta-1}}{\Gamma_1(\beta) p} h^p(s) + \frac{(t - \sigma_{\tau_0}(s))^{\beta-1}}{\Gamma_1(\beta) q} g^q(s). \tag{35}
\]

Integrating inequality (35) from \( \omega_0 \) to \( t \) with respect to the variable \( s \), we have
\[
\int_{\omega_0}^{t} \left( \frac{(t - \sigma_{\tau_0}(s))^{\beta-1}}{\Gamma_1(\beta)} \right) h(s) g(s) \, ds \leq \int_{\omega_0}^{t} \left( \frac{(t - \sigma_{\tau_0}(s))^{\beta-1}}{\Gamma_1(\beta) p} h^p(s) + \frac{(t - \sigma_{\tau_0}(s))^{\beta-1}}{\Gamma_1(\beta) q} g^q(s) \right) \, ds.
\]

This implies
\[
I_{\tau_0}^{\beta} h(t) g(t) \leq \frac{1}{p} I_{\tau_0}^{\beta} h^p(t) + \frac{1}{q} I_{\tau_0}^{\beta} g^q(t). \tag{36}
\]

By using estimates (29), (33) and (36), it follows that
\[
I_{\tau_0}^{\beta} h(t) g(t) \leq \frac{m^p}{p(m + 1)^p} I_{\tau_0}^{\beta} (h(t) + g(t))^p + \frac{1}{q(k + 1)^q} I_{\tau_0}^{\beta} (h(t) + g(t))^q. \tag{37}
\]

Using the fact \((a + b)^r \leq 2^{(r-1)(a^r + b^r)}, r > 1, a, b \geq 0\), we can obtain
\[
I_{\tau_0}^{\beta} (h(t) + g(t))^p \leq 2^{p-1} I_{\tau_0}^{\beta} (h^p(t) + g^p(t)), \tag{38}
\]
\[
I_{\tau_0}^{\beta} (h(t) + g(t))^q \leq 2^{q-1} I_{\tau_0}^{\beta} (h^q(t) + g^q(t)). \tag{39}
\]

Replacing inequalities (38) and (39) at inequality (37), the result follows.

**Theorem 3.3:** Let \( \beta, \omega > 0 \). Let \( h, g \) be defined on \([\omega_0, T]_{\tau_0}\), with \( \beta_{\tau_0} h(t) < \infty \) and \( \beta_{\tau_0} g(t) < \infty \). If \( 0 < k \leq h(s)/g(s) \leq m \) for \( k, m \in \mathbb{R}_+ \) and \( \forall s \in [\omega_0, T]_{\tau_0} \). Then
\[
\frac{1}{m} I_{\tau_0}^{\beta} h(t) g(t) \leq \frac{1}{(k + 1)^2} I_{\tau_0}^{\beta} (h(t) + g(t))^2 \leq \frac{1}{k} I_{\tau_0}^{\beta} h(t) g(t). \tag{40}
\]
Multiplying inequalities (41) and (44), we arrive at
\[
(k + 1)g(s) \leq h(s) + g(s) \leq (m + 1)g(s). \tag{41}
\]
Also we can write
\[
\frac{g(s)}{h(s)} \leq \frac{1}{k} \Rightarrow g(s) \leq \frac{1}{k} h(s)
\]
\[
\Rightarrow h(s) + g(s) \leq \left( \frac{1}{k} + 1 \right) h(s). \tag{42}
\]
From inequalities (42) and (43), we get
\[
\frac{1}{m} \leq \frac{g(s)}{h(s)} \Rightarrow \frac{1}{m} h(s) \leq g(s)
\]
\[
\Rightarrow \left( \frac{1}{m} + 1 \right) h(s) \leq h(s) + g(s). \tag{43}
\]
Multiplying inequalities (41) and (44), we arrive at
\[
\frac{1}{m} h(s)g(s) \leq \frac{(h(s) + g(s))^2}{(m + 1)(k + 1)} \leq \frac{1}{k(m + 1)} h(s)g(s). \tag{44}
\]
Multiplying (45) by \((t - \sigma_{r,0}(s))\Gamma_r(\beta)\), we get
\[
\frac{(t - \sigma_{r,0}(s))\Gamma_r(\beta)}{m\Gamma_r(\beta)} h(s)g(s)
\]
\[
\leq \frac{(t - \sigma_{r,0}(s))\Gamma_r(\beta)}{(m + 1)(k + 1)\Gamma_r(\beta)} (h(s) + g(s))^2
\]
\[
= \frac{(t - \sigma_{r,0}(s))\Gamma_r(\beta)}{k\Gamma_r(\beta)} h(s)g(s). \tag{46}
\]
Integrating inequality (46) from \(a_0\) to \(t\) with respect to the variable \(s\), we have
\[
\int_{a_0}^{t} \frac{(t - \sigma_{r,0}(s))\Gamma_r(\beta)}{m\Gamma_r(\beta)} h(s)g(s)ds
\]
\[
\leq \int_{a_0}^{t} \frac{(t - \sigma_{r,0}(s))\Gamma_r(\beta)}{(m + 1)(k + 1)\Gamma_r(\beta)} (h(s) + g(s))^2ds
\]
\[
\leq \int_{a_0}^{t} \frac{(t - \sigma_{r,0}(s))\Gamma_r(\beta)}{k\Gamma_r(\beta)} h(s)g(s)ds.
\]
This implies
\[
\frac{1}{m} \Gamma_r(\beta) h(t)g(t) \leq \frac{1}{(m + 1)(k + 1)} \Gamma_r(\beta) (h(t) + g(t))^2
\]
\[
\leq \frac{1}{k} \Gamma_r(\beta) h(t)g(t).
\]
\[\square\]

**Theorem 3.4:** Let \(\beta, \omega > 0, p > 1\). Let \(h, g\) be defined on \([a_0, T]\), \(p(x) < \infty\) and \(h^p(x) < \infty\). If \(0 < k \leq m\) for \(k, m \in \mathbb{R}_+\) and \(s \in [a_0, T]\), then
\[
\frac{m + 1}{m - c} \Gamma_r(\beta) (h(t) - cg(t))^{1/p} \leq \left( \frac{p}{r_c(t)} h^p(t)^{1/p} + (\frac{p}{r_c(t)} g^p(t))^{1/p} \right)
\]
\[
\leq \frac{k + 1}{k - c} \Gamma_r(\beta) (h(t) - cg(t))^{1/p}.
\]

**Proof:** Using hypothesis \(0 < c < k \leq h(s)/g(s) \leq m\), we have
\[
ck \leq cm \Rightarrow ck + k \leq cm + m
\]
\[
\Rightarrow (m + 1)(k - c) \leq (k + 1)(m - c).
\]
This implies
\[
\frac{m + 1}{m - c} \leq \frac{k + 1}{k - c}.
\]
Also by the same hypothesis, we have
\[
k \leq \frac{h(s)}{g(s)} \Rightarrow (k - c)g(s) \leq h(s) - cg(s), \tag{48}
\]
\[
\frac{h(s)}{g(s)} \leq m \Rightarrow h(s) - cg(s) \leq (m - c)g(s). \tag{49}
\]
From inequalities (48) and (49), we get
\[
(k - c) \leq \frac{h(s) - cg(s)}{g(s)} \leq (m - c). \tag{50}
\]
From (50), we find
\[
\frac{(h(t) - cg(t))^p}{(m - c)^p} \leq g^p(t) \leq \frac{(h(t) - cg(t))^p}{(k - c)^p}. \tag{51}
\]

Multiplying inequality (51) by \((t - \sigma_{r,0}(s))\Gamma_r(\beta)\), we arrive at
\[
\frac{(t - \sigma_{r,0}(s))\Gamma_r(\beta)}{(m - c)^p} (h(s) - cg(s))^p
\]
\[
\leq \frac{(t - \sigma_{r,0}(s))\Gamma_r(\beta)}{(k - c)^p} (h(s) - cg(s))^p. \tag{52}
\]
Integrating inequality (52) from \(a_0\) to \(t\) with respect to the variable \(s\), we have
\[
\int_{a_0}^{t} \frac{(t - \sigma_{r,0}(s))\Gamma_r(\beta)}{(m - c)^p} (h(s) - cg(s))^p ds
\]
\[
\leq \int_{a_0}^{t} \frac{(t - \sigma_{r,0}(s))\Gamma_r(\beta)}{(k - c)^p} (h(s) - cg(s))^p ds
\]
\[
\leq \int_{a_0}^{t} (t - \sigma_{r,0}(s))\Gamma_r(\beta) (h(s) - cg(s))^p ds.
\]
This implies

\[
\frac{1}{m-c} (h(t) - cg(t))^p \leq I_{[a,b]}^p h^p(t) \leq \frac{1}{k-c} (h(t) - cg(t))^p.
\]

(53)

Taking \(1/p\) power for both sides of inequality (53), we have

\[
\frac{1}{m-c} (h(t) - cg(t))^p \leq (I_{[a,b]}^p h^p(t))^{1/p} \leq \frac{1}{k-c} (h(t) - cg(t))^p.
\]

(54)

Since \(1/m \leq (s)/h(s) \leq 1/k\), then we have

\[
\frac{1}{m} \leq \frac{g(s)}{h(s)} \Rightarrow \left(1 - \frac{c}{m}\right) h(s) \geq h(s) - cg(s),
\]

(55)

\[
\frac{g(s)}{h(s)} \leq \frac{1}{k} \Rightarrow \left(1 - \frac{c}{k}\right) h(s) \leq h(s) - cg(s).
\]

(56)

From inequalities (55) and (56), we get

\[
\frac{k - c}{k} \leq \frac{h(s) - cg(s)}{h(s)} \leq \frac{m - c}{m}.
\]

(57)

In view of inequality (57), it follows that

\[
\frac{m^p (h(s) - cg(s))^p}{(m-c)^p} \leq h^p(s) \leq \frac{k^p (h(s) - cg(s))^p}{(k-c)^p}.
\]

(58)

Multiplying inequality (58) by \((t - \sigma_{a,b}(s))^{\beta - 1}/\Gamma(\beta)\), we obtain

\[
\frac{m^p (t - \sigma_{a,b}(s))^{\beta - 1}/\Gamma(\beta)}{(m-c)^p} \leq \frac{h^p(s)}{\Gamma(\beta)} \leq \frac{k^p (t - \sigma_{a,b}(s))^{\beta - 1}/\Gamma(\beta)}{(k-c)^p}.
\]

(59)

Integrating (59) from \(a\) to \(b\) with respect to the variable \(s\), we have

\[
\int_a^b \frac{m^p (t - \sigma_{a,b}(s))^{\beta - 1}/\Gamma(\beta)}{(m-c)^p} ds \leq \int_a^b \frac{h^p(s)}{\Gamma(\beta)} ds \leq \int_a^b \frac{k^p (t - \sigma_{a,b}(s))^{\beta - 1}/\Gamma(\beta)}{(k-c)^p} ds.
\]

This implies

\[
\frac{m^p}{(m-c)^p} I_{[a,b]}^p (h(t) - cg(t))^p \leq \frac{k^p}{(k-c)^p} I_{[a,b]}^p (h(t) - cg(t))^p.
\]

(60)

Taking \(1/p\) power for both sides of inequality (60), we have

\[
\frac{m}{m-c} \frac{I_{[a,b]}^p (h(t) - cg(t))^p}{1/p} \leq \frac{k}{k-c} \frac{I_{[a,b]}^p (h(t) - cg(t))^p}{1/p}.
\]

(61)

adding inequalities (54) and (61), we get the required result.

\[\blacksquare\]

4. Conclusion

In this paper, we have used HIO for the generalization of classical Minkowski’s inequalities. The use of this integral operator definitely extends the classical Minkowski’s inequality. Our results with this new integral operator have the abilities to be utilized for the analysis of many mathematical problems as applications of the work including existence and stability results for the fractional-order differential equations.

Contribution

All the authors have equal contribution in this paper and there is no competing interest.

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