Solitary waves in one-dimensional pre-stressed lattice and its continual analog

Vsevolod Vladimirov†, Sergii Skuratovskyi‡

†Faculty of Applied Mathematics, AGH University of Science and Technology, Al. Mickiewicza 30, 30-059 Krakow, Poland
‡Subbotin Institute of Geophysics, NAS of Ukraine, Bohdan Khmelnytskyi str. 63-G, Kyiv, Ukraine

Abstract: One of the most interesting phenomena occurring in nonlinear media models is the existence of wave patterns, such as kinks, solitons, compactons, peakons and many others. There are known numerous nonlinear evolutionary PDEs, supporting soliton (multi-soliton) and compacton traveling wave (TW) solutions. Unfortunately, the vast majority of the models, with the exception of completely integrable ones, do not enable to analyze the properties of solitary waves interaction using only qualitative methods. Therefore it is instructive, when dealing with the non-integrable PDEs, to combine the qualitative treatment with numerical simulations. In this report we are going to present the results of studying compacton solutions in the continual models for granular pre-stressed chains. The model is shown to possess a pair of compacton TW solutions which are the bright and dark compactons. First we consider the stability properties of the compacton solutions and show that both the bright and the dark compactons pass the stability test. Next we analyze the dynamics of the compactons, simulating numerically the temporal evolution of a single compacton, as well as the interaction of pairs of compactons, including bright-bright, dark-dark and bright-dark pairs. To be able to simulate the evolution of interacting compactons, we have modified the numerical scheme built by J. de Frutos, M. A. Lopez-Marcos, and J. M. Sanz-Serna. Results of simulations are compared with that of evolution of corresponding impulse in the granular pre-stressed chain.

Keywords: granular chain; travelling wave solutions; solitary waves; stability of travelling waves

Introduction

This paper deals with some nonlinear evolutionary PDEs associated with dynamics of one-dimensional chains of pre-stressed granules. Since Nesterenko’s pioneering works, propagation of pulses in such media has been a subject of a great number of experimental studies and numerical works. We consider a nonlinear evolutionary PDE associated with an ODE systems describing the interaction of the adjacent elements of the chain with the forces depending on the relative displacement of the centers of mass of the granules. The PDE in question is obtained by means of the passage to the continuum limit, followed by the formal multi-scale decomposition.

We perform in this paper qualitative and numerical study of dark and bright compacton traveling wave solutions, supported by the PDE, paying attention to the stability and dynamical features of the compactons’ solutions. The paper is arranged as follows. In section 2 we introduce the continual analog of the granular pre-stressed media with the specific interaction allowing for the propagation of the wave of compression as well as the wave of rarefaction. In section 3 we construct the Hamiltonian representations for the model, and show that the compacton traveling wave solutions, satisfying factorised equations, fulfill

1Corresponding author: vsevolod.vladimirov@gmail.com
2Corresponding author: skurserg@gmail.com
necessary conditions of extrema for some Lagrange functionals. Next we perform stability
tests for compacton solutions, basing on the approach developed in [2, 6, 7], and show that
both the dark and the bright compactons pass the stability test. The results of qualitative
analysis are backed and partly supplemented by the numerical study. Numerical simulations
show that the compacton solutions completely reestablish their shapes after the mutual col-
lisions. Finally, in section 4 we present the results of numerical simulation of the Cauchy
problem for discrete chains and compare the results of the numerical simulation with the
analogous simulations performed within the continual analog.

1 Evolutionary PDEs associated with the granular pre-
stressed chains

Unusual features of the solitons associated with the celebrated Korteweg-de Vries (KdV)
equation, as well as other completely integrable models [3], are often prescribed to the
existence of higher symmetries and (or) infinite set of conservation laws. However, there are
known non-integrable equations possessing the localised TW solutions with quite similar
features. As a well-known example, the so called \(K(m, n)\) hierarchy [4] can be presented:

\[
K(m, n) = u_t + (u^m)_x + (u^n)_{xxx} = 0, \quad m \geq 2, \quad n \geq 2.
\]

Members of this hierarchy are not completely integrable at least for the generic values of the
parameters \(m, n\) [14] and yet possess the compactly-supported TW solutions demonstrating
the solitonic features [1, 4].

The \(K(m, n)\) family was introduced in years 90th of the XX century as a formal
generalisation of the KdV hierarchy, without referring to its physical context. Earlier V.F.
Nesterenko [9] considered the dynamics of a chain of preloaded granules described by the
following ODE system:

\[
\ddot{Q}_k(t) = F(Q_k - Q_{k-1}) - F(Q_k - Q_{k+1}), \quad k \in \{0, \pm 1, \pm 2, \ldots\}
\]

where \(Q_k(t)\) is the displacement of granule \(k\) centre of mass from its equilibrium position,

\[
F(z) = Az^n, \quad n > 1.
\]

A passage to the continual analog of the above discrete model is attained by the substitution:

\[
Q_k(t) = u(t, ka) = u(t, x),
\]

where \(a\) is the average distance between granules. Inserting this formula, together with the
identities

\[
Q_{k \pm 1} = u(t, x \pm a) = e^{\pm a D_x} u(t, x) = \sum_{j=0}^{n+3} \frac{(\pm a)^j}{j!} \frac{\partial^j}{\partial x^j} u(t, x) + O(|a|^{n+4}),
\]

into (2) and dropping out terms of the order \(O(|a|^{n+4})\) and higher, we get the equation:

\[
u_{tt} = -C \left\{ (-u_x)^n + \beta (-u_x)^{n-1} \left[ (-u_x)^{n+1} \right]_x \right\}_x,
\]

where

\[
C = Aa^{n+1}, \quad \beta = \frac{n a^2}{6(n+1)}.
\]
Differentiating the above equation with respect to \( x \) and employing the new variable \( S = (-u_x) \), we obtain the Nesterenko’s equation \([11]\):

\[
S_{tt} = C \left\{ S^n + \beta S^{\frac{n-1}{2}} \left[ S^{\frac{n+1}{x}} \right]_{xx} \right\}_x.
\]  

(7)

Eq. (7) describes dynamics of strongly preloaded media in which the propagation of acoustic waves is impossible (the effect of “sonic vacuum”). Nesterenko had shown \([11]\) that this equation possesses a one parameter family of compacton TW solutions describing the propagation of the waves of compression. Unfortunately, he did not pay much attention to the investigation of their dynamical features. Our preliminary study show that the compacton solutions supported by Eq. (7) are unstable. This situation is absolutely analogous to that with processing the celebrated Fermi-Pasta-Ulam problem \([3]\) which takes the form of system (2) in which the interaction force has the form \( F(z) = A z^2 + B z \) with \(|A| = O(|B|)\). Passing to the continual analog by means of the substitution (4) one obtains the Boussinesq equation, possessing unstable soliton-like solutions. The famous KdV equation is extracted from the Boussinesq equation by means the multi-scale decomposition \([3]\).

Our proceedings to the "proper" compacton-supporting equation is following. We start from the discrete system (2) in which the interaction force has the form

\[
F(z) = A z^n + B z.
\]  

(8)

In addition, we assume that \( B = \gamma a^{n+3} \), \(|\gamma| = O(|A|)\). Making in the formula (2) the substitution (4), (5) we get the equation

\[
u_{tt} = -C \left\{ (-u_x)^n + \beta (-u_x)^{\frac{n+1}{2}} \left[ (-u_x)^{\frac{n-1}{2}} \right]_x \right\}_x + \gamma a^{n+3} (-u_x)_x.
\]  

(9)

Differentiating the above equation with respect to \( x \) and introducing the new variable \( S = (-u_x) \), we obtain the following equation:

\[
S_{tt} = C \left\{ S^n + \beta S^{\frac{n+1}{2}} \left[ S^{\frac{n-1}{x}} \right]_{xx} \right\}_x + \gamma a^{n+3} S_{xx}.
\]  

(10)

Now we use a series of scaling transformations. Employment of the scaling \( \tau = \sqrt{\gamma a^{n+3}} t \), enables to rewrite the above equation in the form:

\[
S_{\tau \tau} = \frac{C}{\gamma a^{n+3}} \left\{ S^n + \beta S^{\frac{n+1}{2}} \left[ S^{\frac{n-1}{x}} \right]_{xx} \right\}_x + S_{xx}.
\]

Next the transformation \( \hat{T} = \frac{1}{\gamma} a^q \tau, \quad \xi = a^p (x - \tau) \), \( S = a^r W \) is used. If, for example, we make a choice \( q = 1, \quad p = -1, \quad r = 5 \), then the higher order coefficient \( O(a^2) \) will stand at the term with the second derivative with respect to \( \hat{T} \). So, dropping out the terms proportional to \( O(a^2) \), we obtain, after the integration with respect to \( \xi \), the equation:

\[
W_{\hat{T}} + \frac{A}{\gamma} \left\{ W^n + \frac{n}{6(n+1)} W^{\frac{n+1}{2}} \left[ W^{\frac{n-1}{2}} \right]_{\xi \xi} \right\}_\xi = 0.
\]

The scaling \( T = \frac{A}{\gamma} L \hat{T}, \quad X = L \xi, \quad L = \sqrt{\frac{6(n+1)}{n}} \) leads us finally to the target equation:

\[
W_T + \left\{ W^n + W^{\frac{n+1}{2}} \left[ W^{\frac{n-1}{2}} \right]_{XX} \right\}_X = 0.
\]  

(11)

Description of waves of reeference in the case \( n = 2k \) requires the following modification of the interaction force:

\[
F(z) = -Az^{2k} + B z
\]  

(12)
for \( n = 2k + 1 \) the formula \( (8) \) describes automatically both wave of compression and rarefaction. Applying the above machinery to the system \( (2) \) with the interaction \( (12) \), one can get the equation

\[
W_T - \left\{ W^n + W^2 \frac{W^{n+1}}{2} \right\}_X = 0, \quad n = 2k. \tag{13}
\]

Thus, for \( n = 2k \) the universal equation describing waves of compression and rarefaction can be presented in the form

\[
W_T + \text{sgn}(W) \left\{ W^n + W^2 \frac{W^{n+1}}{2} \right\}_X = 0. \tag{14}
\]

Let us note in conclusion that the equations \( (11) - (14) \) are obtained by formal employment of the multiscale decomposition method, which cannot be justified in our case because of negativity of the index \( p \). Nevertheless further investigations of these equations is still of interest because they occur to possess a set of compacton solutions demonstrating interesting dynamical features. As will be shown below, these solutions qualitatively correctly describe propagation of short impulses in the chain of prestressed blocks.

2 Compacton solutions and stability tests

Let us consider the pair of equations \( (11), (13) \), which can be expressed in the common form

\[
W_T + \varepsilon \left\{ W^n + W^2 \frac{W^{n+1}}{2} \right\}_X = 0, \quad \varepsilon = \pm 1. \tag{15}
\]

Since we are interested in the TW solutions \( W = W(z) \equiv W(X - cT) \), it is instructive to make a passage to the traveling wave coordinates \( T \to T, \quad X \to z = X - cT \). Performing this change, we get:

\[
W_T - cW_z + \varepsilon \left\{ W^n + W^2 \frac{W^{n+1}}{2} \right\}_z = 0. \tag{16}
\]

Below we formulate several statements, which are easily verified by direct inspection.

**Statement 1.** Eq. \( (10) \) admits the following representation

\[
\frac{\partial}{\partial T} W = \frac{\partial}{\partial z} \frac{\delta (\varepsilon H + cQ)}{\delta W}, \tag{17}
\]

where

\[
H = \int \left[ \frac{n + 1}{4} W^{n-1} W_X^2 - \frac{1}{n + 1} W^{n+1} \right] d z, \quad Q = \frac{1}{2} \int W^2 d z.
\]

**Statement 2.** The functionals \( H, Q \) are conserved in time.

**Statement 3.** Consider the following functions:

\[
W_{c}^{\varepsilon}(z) = \varepsilon W_{c}(z) = \begin{cases} \varepsilon M \cos(\gamma (B z)), & \text{if } |K z| < \frac{\pi}{2}, \\ 0, & \text{elsewhere}, \end{cases} \tag{18}
\]

where

\[
M = \left[ \frac{c(n + 1)}{2} \right]^{\frac{1}{n+1}}, \quad K = \frac{n - 1}{n + 1}, \quad \gamma = \frac{2}{n - 1}.
\]

If \( n = 2k + 1, \quad k \in \mathbb{N} \), then the functions \( W_{c}^{\pm}(z) \) are the generalized solutions to the equation

\[
\delta (H + cQ) / \delta W|_{W=W_{c}^{\pm}} = 0. \tag{19}
\]
If \( n = 2k, \ k \in \mathbb{N} \), then the function \( W_c^\epsilon(z) \) satisfies the equation

\[
\delta \left( \epsilon H + cQ \right) / \delta W_{W-W_c^\epsilon} = 0. \tag{20}
\]

So, the TW solutions (18) are the critical points of either the Lagrange functional \( \Lambda[\beta] = (H + \beta Q) \) (case \( n = 2k + 2 \)) or \( \Lambda^\epsilon[\beta] = (\epsilon H + \beta Q) \) (case \( n = 2k \)) with the common Lagrange multiplier \( \beta = c \). A necessary and sufficient condition for \( \Lambda[\beta] \) (\( \Lambda^\epsilon[\beta] \)) to attain the minimum on the compacton solution can be formulated in terms of the positivity of the second variation of the corresponding functional, which, in turn, guarantees the orbital stability of the TW solution, [5]. Here we do not touch upon the problem of strict estimating the signs of the second variations. Instead of this, we follow the approach suggested in [2,6,7], which enables to test a mere possibility of the local minimum appearance on a selected sets of perturbations of TW solutions.

Let us consider a family of perturbations

\[
W_c^\epsilon(z) \rightarrow \lambda^\alpha W_c^\epsilon(\lambda z). \tag{21}
\]

By choosing \( \alpha = 1/2 \) we guarantee that

\[
Q[\lambda] = \frac{1}{2} \int_{-\pi/2}^{\pi/2} \left[ \lambda^{\frac{1}{2}} W_c^\epsilon(\lambda z) \right]^2 dz = Q[1]. \tag{22}
\]

Imposing this condition, we reject the ”longitudinal” perturbations, associated with symmetry \( T_\gamma [W_c^\epsilon(z)] = W_c^\epsilon(z + \gamma) \). Indeed, since the equations (19), (20) are invariant under the shift \( z \rightarrow z + \gamma \), then \( T_\gamma W_c^\epsilon(z) \) belongs to the set of solutions as well, while formally the transformation \( W_c^\epsilon(z) \rightarrow W_c^\epsilon(z + \gamma) \) can be treated as a perturbation. In order to exclude the perturbations of this sort, the orthogonality condition is posed. Introducing the representation for the perturbed solution

\[
W_c^\epsilon(z)[\lambda] = W_c^\epsilon(z)[1] + v(z),
\]

and using the condition (22), we get

\[
0 = Q[\lambda] - Q[1] = \int_{-\pi/2}^{\pi/2} W_c^\epsilon(z) v(z) dz + O \left( \|v(z)\|^2 \right).
\]

For \( \alpha = 1/2 \), we get the following functions to be tested:

\[
\Lambda^\nu[\lambda] = (\nu H + cQ)[\lambda] = \nu \left\{ \lambda^{\frac{\nu+1}{2}} I_n^\epsilon - \lambda^{\frac{\nu-1}{2}} J_n^\epsilon \right\} + cQ, \tag{23}
\]

where

\[
I_n^\epsilon = \frac{n+1}{4} \int_{-\pi/2}^{\pi/2} \left[ W_c^\epsilon \right]^{n-1}(\tau) \left[ W_c^\epsilon \right]^2(\tau) d\tau, \quad J_n^\epsilon = \frac{1}{n+1} \int_{-\pi/2}^{\pi/2} \left[ W_c^\epsilon \right]^{n+1}(\tau) d\tau,
\]

\[
\nu = \epsilon^{n+1} = \begin{cases} +1 & \text{when } n = 2k + 1, \\ \epsilon & \text{when } n = 2k. \end{cases}
\]

If the functional \( \Lambda^\nu = \nu H + cQ \) attains the extremal value on the compacton solution, then the function \( \Lambda^\nu[\lambda] \) has the corresponding extremum in the point \( \lambda = 1 \). The verification of this property is used as a test.

A necessary condition of the extremum \( \frac{d}{d\lambda} \Lambda^\nu[\lambda] \bigg|_{\lambda = 1} = 0 \) gives us the equality:

\[
I_n^\epsilon = \frac{n-1}{n+3} J_n^\epsilon. \tag{24}
\]
Using (24), we can easily get convinced that 
\[
\frac{d^2}{d\lambda^2} A^\nu[\lambda]\bigg|_{\lambda=1} = \nu \frac{n-1}{2} J_n^\nu = \epsilon^2 (k+1) \frac{n-1}{2(n+1)} \int [W_c]^{n+1} (\tau) d\tau > 0
\]
for both \( n = 2k+1 \) and \( n = 2k \).

So the generalized solutions (18) pass the test for stability.

Further information about the properties of the compacton solutions deliver the numerical simulations.

3 Numerical simulations of compactons’ dynamics

Figure 1: The movement of single compacton with the velocity \( D = 1 \) to the right (left panel). The movement of two bright compactons with the velocities \( D = 1 \) and \( D = 1/4 \), respectively (right panel).

The solitary waves’ dynamics is studied by means of direct numerical simulation, based on the finite-difference scheme. To derive the finite-difference scheme, e.g., for the model equation (11), we modify the scheme presented in [1]. In accordance with the methodology proposed in this paper, we introduce the artificial viscosity by adding the term \( \epsilon W_{4x} \), where \( \epsilon \) is a small parameter. Thus, instead of (14) we have in the case \( n = 3 \) the following equation:

\[
W_t + \{W^3\}_x + \{W W^2\}_{xx} + \epsilon W_{4x} = 0.
\] (25)

Let us approximate the spatial derivatives as follows

\[
\frac{1}{120}(\dot{W}_{j-2} + 26\dot{W}_{j-1} + 66\dot{W}_j + 26\dot{W}_{j+1} + \dot{W}_{j+2}) + \\
+ \frac{1}{24h}(-W^3_{j-2} - 10W^3_{j-1} + 10W^3_{j+1} + W^3_{j+2}) + \\
+ \frac{1}{24h}(-L_{j-2} - 10L_{j-1} + 10L_{j+1} + L_{j+2}) + \\
+ \epsilon \frac{1}{h^4}(W_{j-2} - 4W_{j-1} + 6W_j - 4W_{j+1} + W_{j+2}) = 0,
\] (26)
To integrate the system (26) in time, we use the midpoint method. According to this method, the quantities $W_j$ and $\dot{W}_j$ are presented in the form

$$W_j \rightarrow \frac{W_j^{n+1} + W_j^n}{2}, \quad \dot{W}_j \rightarrow \frac{W_j^{n+1} - W_j^n}{\tau}.$$ 

The resulting nonlinear algebraic system with respect to $W_j^{n+1}$ can be solved by iterative methods.

We test the scheme (26) by considering the movement of a single compacton. Assume that the model’s parameters $D_1 = 1, s_0 = 5.5$ and scheme’s parameters $N = 600, h = 30/N, \tau = 0.01, \varepsilon = 10^{-3}$ are fixed. The application of the scheme (26) gives us the figure 1a.

To study the interaction of two bright compactons, we combine the previous compacton and another one with the lower amplitude taking the velocity $D_2 = D_1/4$ and $s_0 = 15.5$. The result of modelling is presented in fig. 1b. The interaction of two dark compactons has the similar properties and is depicted in fig. 2.

4 Comparison of numerical evolution of compactons with the numerical solution of the granular media, subjected to similar initial conditions

We’ve performed the comparison of the evolution of compacton solutions with corresponding solutions of the finite (but long enough) discrete system. The discrete analogs to the field $S(t, x) = -\frac{\partial u(t, x)}{\partial x}$ are the ”stresses” $R_k = Q_{k-1} - Q_k$, satisfying the system

$$\ddot{R}_1(t) = 0, \quad \ddot{R}_k(t) = A \left[ R_{k-1}^n - 2 R_k^n + R_{k+1}^n \right], \quad \ddot{R}_m(t) = 0$$

with the initial conditions

$$R_k(0) = \begin{cases} M \cos \gamma |Bak - I| & \text{if } |Bak - I| < \pi/2 \\ 0 & \text{if otherwise,} \end{cases}$$

and

$$\dot{R}_k(0) = \begin{cases} -M \varepsilon \cos \gamma^{-1} |Bak - I| \sin |Bak - I| & \text{if } |Bak - I| < \pi/2 \\ 0 & \text{if otherwise,} \end{cases}$$

Figure 2: The movement of two dark-compactons with the velocities $D = 1$ and $D = 1/4$, respectively.

where $L_j = W_j \frac{W_{j-2}^2 - 2W_j^2 + W_{j+2}^2}{h^2}$. 

The resulting nonlinear algebraic system with respect to $W_j^{n+1}$ can be solved by iterative methods.

We test the scheme (26) by considering the movement of a single compacton. Assume that the model’s parameters $D_1 = 1, s_0 = 5.5$ and scheme’s parameters $N = 600, h = 30/N, \tau = 0.01, \varepsilon = 10^{-3}$ are fixed. The application of the scheme (26) gives us the figure 1a.

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where \( I \) is a constant phase, \( k = 2, 3, ..., m - 1 \). The result of comparison for a single compacton is shown in fig. 3. In fig. 4, evolution of two initially separated compactons is shown for both continual and discrete models.

5 Conclusion and discussion

In this paper compacton solutions are studied, supported by the continual analogs of the dynamical systems describing one-dimensional chains of prestressed particles. The equation (10) obtained without resorting to the method of multi-scaled decomposition possesses the compacton solutions which fail to pass the stability test. Numerical experiments show that the compacton solutions are destroyed in a very short time.

Contrary, the equations (11), (13) which are obtained with the help of formal multi-scale decomposition, possess families of bright and dark compacton solutions, correspondingly, which occur to be stable. This is backed both by the stability test and results of the numerical simulations.

A characteristic feature of the equation (11) connected with the decomposition used during its derivation is that it describes a processes with the "long" temporal and "short" spatial scales. So it is rather questionable if this equation can adequately describe a localised pulse propagation in a discrete media in which the characteristic sizes of the particles are comparable with compacton’s width \( \Delta x \). In fact, making the backward transformations \( X \to \xi \to x \) we get the following formula for the width of the compacton solution (18) in the initial coordinate:

\[
\Delta x = \pi a \sqrt{\frac{n(n + 1)}{6(n - 1)^2}}.
\]

For \( n = 3/2 \), corresponding to the Hertzian force between spherical particles, we get \( \Delta x \approx 4.96 \ a \). It is then curious to know that the same results for the particles with the spherical geometry were obtained during the numerical work, and experimental studies [8–10, 12, 13].
Let us remark in conclusion that some of the present results, in particular those concerning the stability study, are only preliminary. The full investigations of stability of compacton solutions supported by (11)-(14) will be published elsewhere.

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