Improved Error Bounds for Dirichlet-to-Neumann Absorbing Boundaries

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Abstract

It has long been known how to construct radiation boundary conditions for the time dependent wave equation. Although arguments suggesting that they are accurate have been given, it is only recently that rigorous error bounds have been proved. Previous estimates show that the error caused by these methods behaves like $\epsilon C_\gamma e^{\gamma t}1$ for any $\gamma > 0$. We improve these results and show that the error behaves like $C_\gamma t^2$.

1 Introduction

Numerical solution of time dependent wave equations is an important problem in physics, engineering and mathematics. To solve the wave equation on $\mathbb{R}^{3+1}$, one must truncate the domain to a finite region due to the limited memory of most computers. Of course, on a finite region, boundary conditions must be specified in such a way as to minimize spurious reflections. Boundary conditions of this form were first described in [1, 2, 3, 4, 5], although rigorous error bounds would wait until more recently [6, 7].

In [6], a family of absorbing boundary conditions based on rational function approximation to the Dirichlet-to-Neumann operator in the frequency domain are reviewed. Boundary conditions for the half-space (a boundary at $x = 0$), as well as cylindrical and spherical coordinates are also constructed. Error bounds are proved for this family by inverting the Fourier-Laplace transform for both the true solution and its approximation and bounding the difference. Due to poles on the imaginary line in $s$ ($s$ being the variable dual to $t$), the difference is bounded on a contour separated from the singular points, namely a line in the right half plane $\gamma + i\mathbb{R}$. This shows that the error is bounded by $C_\gamma e^{\gamma t}$, with $C_\gamma$ left implicit.

A careful examination of the poles of the rational function reveal that they approximate the branch cut of the true solution in the sense of hyperfunctions [8]. Instead of using the machinery of hyperfunctions, we take an elementary approach. The true solution can be represented as a certain integral over a compact region. The approximate solution, after we collect the residues associated to the poles on the imaginary line, turns out to be a quadrature for this integral.
By computing the difference between the quadrature and the true integral, we can compute an optimal error bound.

Let us now state our results precisely. Let \( u(x, y, t) \) solve:

\[
\frac{\partial^2}{\partial t^2} u(x, y, t) = \frac{\partial^2}{\partial x^2} u(x, y, t) + \Delta_y u(x, y, t) \tag{1}
\]

where \( x \in \mathbb{R} \) and \( y \in \mathbb{R}^{N-1} \) (\( x \) is the normal direction, \( y \) the tangential directions).

We wish to solve (1) on \( \mathbb{R}^{N+1} \). The boundary will be taken to be the surface \( x = 0 \), and thus the approximation region will be the region \( \{(x, y, z, t) : x \geq 0\} \).

We let \( u_b(x, y, t) \) be the approximation, solving (1) on the half-space. The boundary conditions imposed are Hagstrom’s:

\[
\prod_{j=1}^{n} \left( \cos \left( \frac{j\pi}{n+1} \right) \partial_t - \partial_x \right) u_b(x, y, t) = 0 \tag{2}
\]

The main theorem is the following:

**Theorem 1.** Let \( u(x, y, t) \) solve (1) on \( \mathbb{R}^{N+1} \), and \( u_b(x, y, t) \) solve (1) with boundary conditions (2). Then we have the following error bound:

\[
|u(x, y, t) - u_b(x, y, t)| \leq \frac{K_{\text{max}}}{3} \frac{\pi^4}{(n+1)^3} M(x) \left( 2nt^2 + 9nt + nK_{\text{max}} + 8n + 3 \right) = O \left( \frac{K_{\text{max}}}{n^2} (K_{\text{max}} + t^2) \right) \tag{3}
\]

**2 Proof**

**2.1 The Exact Boundary**

We begin by reviewing the exact boundary conditions described in [6]. Applying the Laplace transform of (1) with respect to time (letting \( s \) be dual to \( t \)) and the Fourier transform with respect to \( y \) (with \( k \) dual to \( y \)) yields:

\[
s^2 \hat{\pi} = \partial_x^2 \hat{\pi} - k^2 \hat{\pi} \tag{4}
\]

The solution to (4) is:

\[
\hat{\pi}(x) = A(s, k)e^{\sqrt{s^2 + k^2}x} + B(s, k)e^{-\sqrt{s^2 + k^2}x} \tag{5}
\]

The solutions with nonzero \( A(s, k) \) are nonphysical, since they correspond to a wave coming from infinity to the object. Thus our boundary conditions must imply \( A(s, k) = 0 \). Such a boundary condition is (in the frequency domain):

\[
\partial_x \hat{\pi}(x, k, s) + \sqrt{s^2 + |k|^2} \hat{\pi}(x, k, s) = 0 \tag{6}
\]
Of course, the operator $\sqrt{s^2 + |k|^2}$ is non-local in time and space, so we will approximate it.

To reduce the dependence to a single variable, we make the substitution $z = s/|k|$, yielding:

$$\partial_x \hat{u}(x, k, s) + |k|\sqrt{1 + z^2} \hat{u}(x, k, s) = 0$$

This boundary condition can be rewritten as:

$$\partial_x \hat{u}(x, k, s) + |k| \left( z + \frac{1}{z + \sqrt{1 + z^2}} \right) \hat{u}(x, k, s) = 0 \quad (7)$$

Let $h(z) \equiv |k|/(z + \sqrt{1 + z^2})$. We will invert the Laplace transform, and shift the contour to surround the singularities of $h(z)$. The following lemma summarizes the necessary analyticity properties of $h(z)$:

**Lemma 1.** The function $h(z)$ is analytic on $\mathbb{C} \setminus [-i, i]$. In addition, the difference across the branch cut is given by:

$$\lim_{\epsilon \to 0} (h(z + \epsilon) - h(z - \epsilon)) = 2|k|\sqrt{1 + z^2} \quad (8)$$

**Proof.** The function $h(z)$ is well defined and analytic for $\Re z > 0$. It is strictly imaginary on $\{z : \Re z = 0$ and $|z| > 1\}$. By the Schwartz reflection principle, it can be analytically continued to the left half plane, with a discontinuity along the line $[-i, i]$. An explicit calculation shows (8). \qed

We now reconstruct $u(x, y, t)$. This is done by inverting the Laplace transform:

$$u(x, y, t) = \frac{1}{(2\pi)^{(N+1)/2}} \int_{a+i\mathbb{R}} e^{st} \int_{\mathbb{R}^{N-1}} e^{iy \cdot k} \hat{u}(x, k, s) dk ds \quad (9a)$$

$$\hat{u}(x, k, t) = \frac{1}{2\pi} \int_{a+i\mathbb{R}} e^{st} \hat{u}(x, k, s) ds \quad (9b)$$

And so, the integral we must approximate is

$$\int_{-i}^{i} 2|k|\sqrt{1 + z^2} f(z) e^{zt} dz.$$  

### 2.2 The Approximation

We review the approximation itself, and how (2) was derived. Our description follows [6] quite closely. We approximate $|k|\sqrt{1 + z^2}$ by:

$$|k|\sqrt{1 + z^2} = |k| \left( z + \frac{1}{z + \sqrt{1 + z^2}} \right) \approx |k| \left( z + \frac{1}{2z + \frac{1}{z + 2z}} \right) \quad (10)$$
where the right hand side is the $n$’th iteration of the continued fraction.

A straightforward computation shows that in the frequency domain,

\[ \partial_x \hat{u}(x, k, s) + |k| \left( z + \frac{1}{2z + \frac{1}{\ddots + \frac{1}{2z}}} \right) \hat{u}(x, k, s) = 0 \]

corresponds to the boundary condition \( (2) \). We simplify this:

**Lemma 2.** Let \( \theta_j = j\pi/(n+1) \). Then we have the following formula:

\[
\frac{1}{2z + \frac{1}{\ddots + \frac{1}{2z}}} = \sum_{j=1}^{n} \frac{\sin^2 \theta_j}{(n+1)(z - i \cos \theta_j)} \tag{11}
\]

**Proof.** Let \( U_n(x) \) be the \( n \)th Chebyshev Polynomial of the 2nd kind and \( P_n(z) \) be the successive numerators of the sequence of finite continued fractions \( \{P_0(z) = 1, P_1(z) = 2z, \ldots\} \).

If we take \( U_n(iz) \) then for \( n \) odd we get \( iP_n(z) \) and for \( n \) even we get \( P_n(z) \), and the sequence of finite continued fractions is \( (P_{n-1}(z))/(P_n(z)) \) for \( n \geq 1 \).

We consider the case case where \( n \) is even; in this case, the finite continued fraction is \( (iP_{n-1}(z))/(P_n(z)) \). Thus, ratios of Chebychev polynomials of the 2nd kind only differ from the finite continued fraction by multiplication by \( i \), and will therefore have the same zeros. The continued fraction will have poles where \( P_n(z) \) is zero, for \( n \) even. That is, when \( U_n(iz) = 0 \).

We will take \( z = i \cos \theta \). Thus, we are looking for zeroes of \( U_n(-\cos \theta) \) where \( U_{n-1}(-\cos \theta) \neq 0 \).

\[
U_n(-\cos \theta) = U_n(\cos \theta) = U_n(x) = \frac{\sin(n+1)\theta}{\sin \theta}, \quad \theta \neq 0, \pi, 2\pi, \ldots
\]

So \( \sin(n+1)\theta = 0 \) are our solutions, and that is \( \theta = \frac{j\pi}{n+1} = \theta_j \) as claimed.

Thus, \( U(i \cos \theta_j) = 0 \) and hence \( z = i \cos \theta_j \) are the only poles of the continued fraction approximation. A direct computation shows that the residues at the pole \( i \cos \theta_j \) is \( (\sin^2 \theta_j)/(n+1) \). \( \square \)

As the continued fraction is a close approximation to \( \sqrt{1 + z^2} \), we can use it to approximate an integral involving \( \sqrt{1 + z^2} \) by substituting the approximation, which is a rational function. And so, in evaluating the integral around the branch cut, a finite sum which approximates this integral is given by the sum of the residues at the poles of the rational function above.

**2.3 The Error Bound**

First, we make the definition \( |k|g(s/|k|) = \hat{u}(s, k) \).
Proposition 1. The following error bound holds.

\[
2|k| \left| \int_{-i}^{i} \sqrt{1 + z^2 e^{z t}} g(z) dz - 2\pi i |k| \sum_{j=1}^{n} g(z_j) e^{z_j t} \alpha_j \right| \leq \frac{K_{\max}}{3} \frac{\pi^4}{(n + 1)^3} M(x)(2nt^2 + 9nt + nK_{\max} + 8n + 3)
\]  

(12)

Here, \( z_j = i \cos \theta_j \) are the positions of the poles, \( \alpha_j \) the residue at \( \theta_j \), \( K_{\max} \) the maximal frequency under consideration, \( M(x) \) is a pointwise upper bound on \( \overline{\pi}, \partial_{\theta} \overline{\pi} \) and \( \partial_{\theta}^2 \overline{\pi} \) and \( n \) is the order of the continued fraction approximation.

We will need the following lemma

Lemma 3.

\[
\left| \int_{0}^{\Delta x} f(x) dx - f(0) \Delta x \right| \leq \Delta x^2 f'(\xi), \quad \xi \in [0, \Delta x]
\]

Proof. This follows immediately from the Intermediate Value Theorem.

We will prove Lemma 1 above in several intermediate steps.

Proposition 2.

\[
2k \left| \int_{-i}^{i} \sqrt{1 + z^2 e^{z t}} g(z) dz - 2\pi ik \sum_{j=1}^{n} g(z_j) e^{z_j t} \alpha_j \right| \leq \frac{2}{3} |k| \Delta \theta^3 \left( 3 \left| \max_{s \in [i|k|, i|k| \cos \Delta \theta]} g \left( \frac{s}{|k|}, |k| \right) \right| + 3 \left| \max_{s \in [-i|k| \cos \Delta \theta, -i|k| \cos \Delta \theta]} g \left( \frac{s}{|k|}, |k| \right) \right| + \sum_{j=1}^{n} \left| \max_{s \in [i|k| \cos (\theta_j - \Delta \theta), i|k| \cos (\theta_j + \Delta \theta)]} e^{\pi i s} g \left( \frac{s}{|k|}, |k| \right) \right| \right.

\left. + 2g' \left( \frac{s}{|k|}, |k| \right) \left( \frac{s}{|k|} \right)^2 + 2tg' \left( \frac{s}{|k|}, |k| \right) + 5t^2 g' \left( \frac{s}{|k|}, |k| \right) \right)
\]

(13)

In this equation \( z_j \) are the positions of the poles, \( \alpha_j \) are the residues at the poles, \( \Delta \theta = \frac{\pi}{2(n + 1)} \), and \( \alpha(\theta) = g(i \cos \theta)e^{i \cos \theta} \sin^2 \theta \).
Proof. We first change variables to \( z = i \cos \theta \). Using the fact that
\[
\sqrt{1 + z^2} = \frac{1}{2z + \frac{1}{2z + \ldots}}.
\]
we can approximate \( \sqrt{1 + z^2} \) by taking a truncated continued fraction. This yields:
\[
|k| \left| 2 \int_0^\pi g(i \cos \theta)e^{it \cos \theta} \sin^2 \theta d\theta - 2 \pi i \sum_{j=1}^n g(i \cos \theta_j)e^{it \cos \theta_j} \frac{\sin^2 \theta_j}{n+1} \right| \quad (14)
\]
We define \( f(z) = g(z)e^{iz} \) and \( \Delta \theta = \frac{\pi}{2(n+1)} \) and expand the integral around each pole to obtain
\[
(14) = |k| \left| 2i \int_0^{\Delta \theta} f(i \cos \theta) \sin^2 \theta d\theta + 2i \int_{\pi-\Delta \theta}^{\pi} f(i \cos \theta) \sin^2 \theta d\theta
+ 2i \sum_{j=1}^n \left( \int_{\theta_j - \Delta \theta}^{\theta_j + \Delta \theta} f(i \cos \theta) \sin^2 \theta d\theta - \pi g(i \cos \theta_j) \frac{\sin^2 \theta_j}{n+1} \right) \right| \quad (15)
\]
To simplify further, we substitute \( \alpha(\theta) = f(i \cos \theta) \sin^2 \theta \) cancel terms, and use the fact that \( \int_{\theta_j - \Delta \theta}^{\theta_j + \Delta \theta} \alpha'(\theta_j)(\theta - \theta_j) d\theta = 0 \) to get
\[
(15) = |k| \left| 2i \int_0^{\Delta \theta} \alpha(\theta) d\theta + 2i \int_{\pi-\Delta \theta}^{\pi} \alpha(\theta) d\theta + 2i \sum_{j=1}^n \left( \int_{\theta_j - \Delta \theta}^{\theta_j + \Delta \theta} (\alpha(\theta) - \alpha(\theta_j)
- \alpha'(\theta_j)(\theta - \theta_j) \right) d\theta \right| \quad (16)
\]
By the triangle inequality, and the mean value theorem, we have
\[
(16) \leq 2|k| \left( \int_0^{\Delta \theta} |\alpha(\theta)| d\theta + \int_{\pi-\Delta \theta}^{\pi} |\alpha(\theta)| d\theta
+ \frac{1}{3} \sum_{j=1}^n \left| \max_{\theta_j - \Delta \theta, \theta_j + \Delta \theta} \alpha''(\xi) \Delta \theta^3 \right| \quad (17)
\]
To deal with the ends of the integral, we substitute \( \alpha \) and \( f \) back into the integrals near the endpoints. We then use the fact that \( \int_a^b f \leq \max_{x \in [a,b]} f(x)(b-a) \) and \( |\sin \theta| \leq |\theta| \) and \( |\sin(\pi - \theta)| \leq |\pi - \theta| \) to obtain:
\[
(17) \leq 2|k| \left( \max_{[0,\Delta \theta]} g(i \cos \theta) \Delta \theta^3 + \max_{[\pi-\Delta \theta, \pi]} g(i \cos \theta) \Delta \theta^3
+ \frac{1}{3} \sum_{j=1}^n \left| \max_{\theta_j - \Delta \theta, \theta_j + \Delta \theta} \alpha''(\xi) \Delta \theta^3 \right| \quad (18)
\]
Upon substitution back and simplification, this becomes

\[
\left| \frac{2}{3} |k| \Delta \theta^3 \left( 3 \max_{|i, i \cos \Delta \theta|} g(z) \right) + 3 \max_{[-i \cos \Delta \theta, -i]} g(z) \right| \\
+ \sum_{j=1}^{n} \left| \frac{\max_{[i \cos (\theta_j - \Delta \theta), i \cos (\theta_j + \Delta \theta)]} e^{\delta t} \left( g''(z) z^4 + 2g''(z) z^2 + g''(z) + 2tg'(z) z^4 + 4tg'(z) z^2 + 2tg'(z) z^2 + t^2 g(z) z^2 + t^2 g(z) z^2 + 5z^3 g'(z) + 5g'(z) z + 5z^3 tg(z) \\
+ 5tg(z) z + 4z^2 g(z) + 2g(z) \right) \right| \right| (19) \\
\]

We also know that \( g(z) = g(s/|k|, k) \). All the derivatives in (20) are in \( s/|k| \).

And so we get:

\[
\left| \frac{2}{3} |k| \Delta \theta^3 \left( 3 \max_{s \in [i|k|, i|k| \cos \Delta \theta]} g \left( \frac{s}{|k|}, |k| \right) \right) + 3 \max_{s \in [-i|k| \cos \Delta \theta, -i|k|]} g \left( \frac{s}{|k|}, |k| \right) \right| \\
+ \sum_{j=1}^{n} \left| \frac{\max_{s \in [i|k| \cos (\theta_j - \Delta \theta), i|k| \cos (\theta_j + \Delta \theta)]} e^{\delta t} \left( g'' \left( \frac{s}{|k|}, |k| \right) \right) \left( \frac{s}{|k|} \right)^4 \right) \right| \\
+ 2g'' \left( \frac{s}{|k|}, |k| \right) \left( \frac{s}{|k|} \right)^2 + g'' \left( \frac{s}{|k|}, |k| \right) \left( \frac{s}{|k|} \right)^2 + 2tg' \left( \frac{s}{|k|}, |k| \right) \left( \frac{s}{|k|} \right)^4 \\
+ 4tg' \left( \frac{s}{|k|}, |k| \right) \left( \frac{s}{|k|} \right)^2 + 2tg' \left( \frac{s}{|k|}, |k| \right) \left( \frac{s}{|k|} \right)^2 + t^2 g \left( \frac{s}{|k|}, |k| \right) \left( \frac{s}{|k|} \right)^3 \\
+ 2t^2 g \left( \frac{s}{|k|}, |k| \right) \left( \frac{s}{|k|} \right)^2 + t^2 g \left( \frac{s}{|k|}, |k| \right) \left( \frac{s}{|k|} \right)^2 + 5 \left( \frac{s}{|k|} \right)^3 g' \left( \frac{s}{|k|}, |k| \right) \\
+ 5g' \left( \frac{s}{|k|}, |k| \right) \left( \frac{s}{|k|} \right)^3 + 5g \left( \frac{s}{|k|}, |k| \right) \left( \frac{s}{|k|} \right)^3 + 5tg \left( \frac{s}{|k|}, |k| \right) \left( \frac{s}{|k|} \right)^2 \\
+ 4 \left( \frac{s}{|k|} \right)^2 g \left( \frac{s}{|k|}, |k| \right) + 2g \left( \frac{s}{|k|}, |k| \right) \right| \right| (20) \\
\]

\[
\square
\]

Now we can prove the final bound, and complete the proof of the main theorem. Once Proposition 1 is proven, this implies the main result by (9a) and (9b).

**Proof of Proposition 1**

\[
\partial^1_x \pi(s, k) = \partial_s k g(s/k, k) = kD_1 g(s/k, k). \\
\partial^2_x \pi(s, k) = \partial^2_x (kg(s/k, k)) = k\partial^2_x g(s/k, k) = 1/kD^2_1 g(s/k, k)
\]

So \( D_1 g = \partial^1_x \pi \) and \( D^2_1 g = |k| \partial^2_x \pi \). 

Thus, (13) above can be simplified to:
\[ \frac{2}{3} \Delta \theta^3 \left( \left| \max_{s \in |i| [k], i|k| \cos \Delta \theta} \hat{\pi}(s, |k|) \right| + 3 \left| \max_{s \in [-i]|k|, i|k| \cos \Delta \theta} \hat{\pi}(s, |k|) \right| \right) \]

\[ + \sum_{j=1}^{n} \left| \max_{s \in |i| [k], i|k| \cos (\theta_j - \Delta \theta), i|k| \cos (\theta_j + \Delta \theta)} e^{i \cos (\theta_j - \Delta \theta) t} \left( |k| \partial_t^2 \hat{\pi}(s, |k|) \cos^4 (\theta_j - \Delta \theta) \right) \right. \]

\[ - 2 |k| \partial_t^2 \hat{\pi}(s, |k|) \cos^2 (\theta_j - \Delta \theta) + |k| \partial_t^2 \hat{\pi}(s, |k|) + 2 t \partial_s \hat{\pi}(s, |k|) \cos (\theta_j - \Delta \theta) \]

\[ - 4 t \partial_s \hat{\pi}(s, |k|) \cos^2 (\theta_j - \Delta \theta) + 2 t \partial_s \hat{\pi}(s, |k|) \cos (\theta_j - \Delta \theta) \]

\[ - 2 t \partial_s \hat{\pi}(s, |k|) \cos^2 (\theta_j - \Delta \theta) + 2 t \partial_s \hat{\pi}(s, |k|) \cos (\theta_j - \Delta \theta) \]

\[ - 5 \partial_s \hat{\pi}(s, |k|) \cos (\theta_j - \Delta \theta) + 5 t \partial_s \hat{\pi}(s, |k|) \cos (\theta_j - \Delta \theta) \]

\[ + 5 \partial_s \hat{\pi}(s, |k|) \cos (\theta_j - \Delta \theta) - 5 \partial_s \hat{\pi}(s, |k|) \cos (\theta_j - \Delta \theta) + 5 t \partial_s \hat{\pi}(s, |k|) \cos (\theta_j - \Delta \theta) \]

\[ - 4 \cos^2 (\theta_j - \Delta \theta) \hat{\pi}(s, |k|) + 2 \hat{\pi}(s, |k|) \right) \left| \right| \]

\[ (21) \]

If we find the maximum for each term independently, we will obtain an upper bound for this. Noting that \( \cos \theta \) is monotonic decreasing on \([0, \pi]\), we obtain:

\[ \text{[21]} \leq \frac{2}{3} \Delta \theta^3 \left( \left| \max_{s \in |i| [k], i|k| \cos \Delta \theta} \hat{\pi}(s, |k|) \right| + 3 \left| \max_{s \in [-i]|k|, i|k| \cos \Delta \theta} \hat{\pi}(s, |k|) \right| \right) \]

\[ + \sum_{j=1}^{n} \left| \max_{s \in |i| [k], i|k| \cos (\theta_j - \Delta \theta), i|k| \cos (\theta_j + \Delta \theta)} e^{i \cos (\theta_j - \Delta \theta) t} \left( |k| \partial_t^2 \hat{\pi}(s, |k|) \cos^4 (\theta_j - \Delta \theta) \right) \right. \]

\[ - 2 |k| \partial_t^2 \hat{\pi}(s, |k|) \cos^2 (\theta_j - \Delta \theta) + |k| \partial_t^2 \hat{\pi}(s, |k|) + 2 t \partial_s \hat{\pi}(s, |k|) \cos (\theta_j - \Delta \theta) \]

\[ - 4 t \partial_s \hat{\pi}(s, |k|) \cos^2 (\theta_j - \Delta \theta) + 2 t \partial_s \hat{\pi}(s, |k|) \cos (\theta_j - \Delta \theta) \]

\[ - 2 t \partial_s \hat{\pi}(s, |k|) \cos^2 (\theta_j - \Delta \theta) + 2 t \partial_s \hat{\pi}(s, |k|) \cos (\theta_j - \Delta \theta) \]

\[ - 5 \partial_s \hat{\pi}(s, |k|) \cos (\theta_j - \Delta \theta) + 5 t \partial_s \hat{\pi}(s, |k|) \cos (\theta_j - \Delta \theta) \]

\[ + 5 \partial_s \hat{\pi}(s, |k|) \cos (\theta_j - \Delta \theta) - 5 \partial_s \hat{\pi}(s, |k|) \cos (\theta_j - \Delta \theta) + 5 t \partial_s \hat{\pi}(s, |k|) \cos (\theta_j - \Delta \theta) \]

\[ - 4 \cos^2 (\theta_j - \Delta \theta) \hat{\pi}(s, |k|) + 2 \hat{\pi}(s, |k|) \right) \left| \right| \]

\[ (22) \]

We also know that \( \hat{\pi}(s, x, k) \) is bounded, so let \( U(x) \) be an upper bound, let \( U'(x) \) be an upper bound of \( \partial_s \hat{\pi} \) and \( U''(x) \) and upper bound of \( \partial_t^2 \hat{\pi} \). Then let \( M(x) \) be the maximum of these functions. Now, things simplify further to:
Applying the triangle inequality, we obtain the following as a bound.

\[ \left( 6 + \sum_{j=1}^{n} \left| e^{i \cos(\theta_j - \Delta \theta) t} \right| \left| k \right| \cos^4(\theta_j - \Delta \theta) - 2 \left| k \right| \cos^2(\theta_j - \Delta \theta) + |k| 
+ 2t \cos^4(\theta_j - \Delta \theta) - 4t \cos^2(\theta_j - \Delta \theta) + 2t + t^2 \right| \cos(\theta_j - \Delta \theta) - 2t^2 \cos^2(\theta_j - \Delta \theta) + t^2 
- 5i \cos^3(\theta_j - \Delta \theta) + 5i \cos(\theta_j - \Delta \theta) - 5i \cos^3(\theta_j - \Delta \theta)t + 5t \cos(\theta_j - \Delta \theta) 
- 4 \cos^2(\theta_j - \Delta \theta) + 2 \right) \] (23)

Introducing \( \beta_j = |\cos(\theta_j - \Delta \theta)| \), this can be written as

\[ \left( 6 + \sum_{j=1}^{n} \left( |k| \beta_j^4 - 2 |k| \beta_j^2 + |k| + 2t \beta_j^4 - 4t \beta_j^2 + 2t + t^2 \beta_j 
- 2t^2 \beta_j^2 + 5 \beta_j^3 + 5 \beta_j^3 t + 5t \beta_j - 4 \beta_j^2 + 2 \right) \) (25)

Now, we integrate in \( k \) over the circle of radius \( K_{\text{max}} \). This translates to integrating \( |k| \) from 0 to \( K_{\text{max}} \) and multiplying by \( 2\pi \). This gives us

\[ \left( 6K_{\text{max}} + \frac{K_{\text{max}}^{2}}{2} \sum_{j=1}^{n} \left( K_{\text{max}} \beta_j^4 - 2K_{\text{max}}^{2} \beta_j^2 
+ K_{\text{max}} + 4t \beta_j^4 - 8t \beta_j^2 + 4t + 2t^2 \beta_j - 4t^2 \beta_j^2 
+ 2t^2 - 10 \beta_j^3 + 10 \beta_j - 8 \beta_j^2 + 4 \right) \) (26)
And this becomes

\[
\begin{align*}
(26) & \leq \frac{2K_{\text{max}}\pi}{3} \Delta \theta^3 M(x) \left[ 12 + 4n + 2nt^2 + 4nt + K_{\text{max}}n \\
& + \sum_{j=1}^{n} \left( (K_{\text{max}} + 4t)\beta_j^4 - 10(t + 1)\beta_j^3 \\
& - 2(2t^2 + 4t + K_{\text{max}} + 4)\beta_j^2 + 2(t^2 + 5t + 5)\beta_j \right) \right] \quad (27)
\end{align*}
\]

Breaking up the sum yields:

\[
(27) = \frac{2K_{\text{max}}\pi}{3} \Delta \theta^3 M(x) (12 + 4n + 2nt^2 + 4nt + K_{\text{max}}n \\
+ (K_{\text{max}} + 4t) \sum_{j=1}^{n} \beta_j^4 - 10(t + 1) \sum_{j=1}^{n} \beta_j^3 \\
- 2(2t^2 + 4t + K_{\text{max}} + 4) \sum_{j=1}^{n} \beta_j^2 + 2(t^2 + 5t + 5) \sum_{j=1}^{n} \beta_j) \quad (28)
\]

Now, we notice that \( \beta_j = |\cos \phi| \) for some \( \phi \) and that \( |\cos \phi| \leq 1 \). This finally allows us to remove the \( j \) dependence of the terms inside the sum, and we obtain, after substituting \( \Delta \theta \) back in:

\[
(28) \leq \frac{K_{\text{max}}}{3} \frac{\pi^4}{(n + 1)^3} M(x) \left( 2nt^2 + 9nt + nK_{\text{max}} + 8n + 3 \right) \quad (29)
\]

#### 2.4 Improving the quadrature

The result we describe here depends on the following idea: \( \sqrt{1 + z^2} \) has a branch cut on the region \([-i, i]\). The rational function approximation can be expanded as a sum of first order poles, as per (11). Integrating an analytic function against this sum of poles (around a contour encircling \([-i, i]\) yields a sum of the form \( \sum_n w_n f(z_n) \), which approximates the integral of \( f(z)\sqrt{1 + z^2} \) around a contour encircling \([-i, i]\). In particular, this is a second order quadrature.

A natural line of inquiry is to ask is whether higher order quadratures can be used, simply by discarding the rational function approximation, and merely choosing a sum of poles according to some appropriate quadrature rule. We conjecture that this can be done.

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