Unobstructed symplectic packing by ellipsoids for tori and hyperkähler manifolds

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Abstract

Let \( M \) be a closed symplectic manifold of volume \( V \). We say that the symplectic packings of \( M \) by ellipsoids are unobstructed if any collection of disjoint symplectic ellipsoids (possibly of different sizes) of total volume less than \( V \) admits a symplectic embedding to \( M \). We show that the symplectic packings by ellipsoids are unobstructed for all even-dimensional tori equipped with Kähler symplectic forms and all closed hyperkähler manifolds of maximal holonomy, or, more generally, for closed Campana simple manifolds (that is, Kähler manifolds that are not unions of their complex subvarieties), as well as for any closed Kähler manifold which is a limit of Campana simple manifolds in a smooth deformation. The proof involves the construction of a Kähler resolution of a Kähler orbifold with isolated singularities and relies on the results of Demailly-Paun and Miyaoka on Kähler cohomology classes.

Contents

1 Introduction 2

2 Main results 3
  2.1 Preliminaries 3
  2.2 Symplectic packing of tori and IHS hyperkähler manifolds by ellipsoids 4
  2.3 Symplectic packing of arbitrary Campana simple manifolds 5
  2.4 Idea of the proof of Theorem 2.4 and plan of the paper 6

3 Orbifolds, weighted blow-ups and symplectic packing by ellipsoids 7
  3.1 Basics of orbifolds 7
  3.2 Resolution of orbifolds and Kähler classes 12
  3.3 Construction of a Kähler resolution 13
  3.4 Weighted blow-ups and symplectic embeddings of ellipsoids 17

4 Demailly-Paun theorem and the Kähler cone 21

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1 Introduction

The symplectic packing problem is one of the central problems of symplectic topology – it concerns the existence of symplectic embeddings of a union of disjoint copies of various (possibly different) sizes of a particular standard shape (ball, ellipsoid, polydisk etc.) into a given $2n$-dimensional symplectic manifold $(M, \omega)$. An immediate obstruction to such symplectic embeddings is given by the symplectic volume. It has been known since the pioneering work by Gromov [Gro] that there might be additional obstructions coming from pseudo-holomorphic curves in $(M, \omega)$ – a symplectic rigidity phenomenon. In [EV] we prove an opposite, symplectic flexibility, claim for the symplectic packing of Kähler manifolds by balls: in the case when $(M, \omega)$ is a Kähler manifold admitting “really few” (genuinely, not pseudo-) holomorphic subvarieties (such a Kähler manifold is called Campana simple), or if it can be approximated by Campana simple manifolds, then there are no obstructions to symplectic embeddings of disjoint unions of balls into $(M, \omega)$ apart from the volume. In this paper we extend this flexibility result to symplectic packings of Kähler manifolds by ellipsoids.

Let us say a few words about the method of the proof. In the case of the symplectic packings by balls, McDuff and Polterovich [McDP] reduced the question about symplectic embeddings of unions of $k$ balls into a symplectic manifold $(M, \omega)$ to a question about the structure of the symplectic cone in the cohomology of a blow-up of $M$ at $k$ points. In the same paper they showed that symplectic packings of Kähler manifolds by balls are deeply related to algebraic geometry that allows sometimes to describe the shape of the Kähler cone in the cohomology of a Kähler manifold. In [EV] we proved the above-mentioned flexibility result for the symplectic packings by balls using the results of McDuff-Polterovich along with several strong results from complex geometry – in particular, the Demailly-Paun theorem [DP] describing completely the Kähler cone of a closed Kähler manifold. (A similar approach was previously used by Latschev-McDuff-Schlenk [LMcDS] in the case when $(M, \omega)$ is a Kähler torus of real dimension 4).

The problem with extending this kind of argument to the study of symplectic packing by ellipsoids is that instead of the usual blow-ups of a symplectic manifold one has to consider weighted blow-ups which, unlike the usual blow-ups, produce not smooth manifolds but orbifolds, where the results and the techniques used to describe the symplectic/Kähler cone of the usual blow-up do not apply. (An orbifold version of the Demailly-Paun
theorem seems to be true but its proof is not published, as far as we can ascertain). Therefore we use an indirect argument where the Demailly-Paun theorem is applied not to the orbifold but to its Kähler resolution which is a smooth manifold.

Let us note that McDuff [McD] invented an approach to the study of symplectic packing of balls by ellipsoids in real dimension 4 which avoids dealing with orbifolds and which is based on an ingenious trick reducing the problem about symplectic embeddings of an ellipsoid to the problem about symplectic embeddings of a certain disjoint union of balls. F.Schlenk has independently proved the flexibility results for symplectic packings of tori and hyperkähler manifolds by ellipsoids in dimension 4 [Sch] by combining McDuff’s method with our flexibility results for symplectic packing by balls [EV]. The proof that we give below works in all dimensions.

2 Main results

2.1 Preliminaries

Symplectic and complex structures. We view complex structures as tensors, that is, as integrable almost complex structures.

We say that an almost complex structure $J$ and a differential 2-form $\omega$ on a smooth manifold $M$ are compatible with each other if $\omega(\cdot, J\cdot)$ is a $J$-invariant Riemannian metric on $M$.

The compatibility between a complex structure $J$ and a symplectic form $\omega$ means exactly that $\omega(\cdot, J\cdot) + i\omega(\cdot, \cdot)$ is a Kähler metric on $M$.

We call a symplectic form Kähler, if it is compatible with some complex structure.

A degree-2 real cohomology class of a complex manifold $(M, J)$ is called Kähler (with respect to $J$) if it can be realized by a Kähler form compatible with $J$. Such classes form an open cone that will be denoted by $\text{Kah}(M, J) \subset H^2(M; \mathbb{R})$.

We will say that a complex structure is of Kähler type if it is compatible with some symplectic form.

Symplectic forms on tori. Consider a torus $T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ and let $\pi : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}/\mathbb{Z}^{2n} = T^{2n}$ be the natural projection.

The Kähler forms on $T^{2n}$ are exactly the ones that can be mapped by a diffeomorphism of $T^{2n}$ to a symplectic form whose lift by $\pi$ to $\mathbb{R}^{2n}$ has constant coefficients with respect to the standard coordinates on $\mathbb{R}^{2n}$ (see e.g. [EV, Proposition 6.1]).

Hyperkähler manifolds. There are several equivalent definitions of a
hyperkähler manifold. Since we study hyperkähler manifolds from the symplectic viewpoint, here is a definition which is close in spirit to symplectic geometry: A hyperkähler manifold is a manifold equipped with three complex structures $I_1, I_2, I_3$ satisfying the quaternionic relations and three symplectic forms $\omega_1, \omega_2, \omega_3$ compatible, respectively, with $I_1, I_2, I_3$, so that the three Riemannian metrics $\omega_i(\cdot, I_i \cdot)$, $i = 1, 2, 3$, coincide. Such a collection of complex structures and symplectic forms on a manifold is called a hyperkähler structure and will be denoted by $\mathfrak{h} = \{I_1, I_2, I_3, \omega_1, \omega_2, \omega_3\}$.

We will say that a symplectic form is hyperkähler and a complex structure is of hyperkähler type, if they appear in some hyperkähler structure. In particular, any hyperkähler symplectic form is Kähler and any complex structure of hyperkähler type is also of Kähler type.

A hyperkähler manifold $(M, \mathfrak{h})$ is called irreducible holomorphically symplectic (IHS) if $\pi_1(M) = 0$ and $\dim \mathbb{C}H^{2,0}_I(M; \mathbb{C}) = 1$, where $I$ is any of the three complex structures appearing in $\mathfrak{h}$ and $H^{2,0}_I(M; \mathbb{C})$ is the $(2,0)$-part in the Hodge decomposition of $H^2(M; \mathbb{C})$ defined by $I$ (for all three complex structures in $\mathfrak{h}$ the space $H^{2,0}_I(M; \mathbb{C})$ has the same dimension). K3-surfaces, as well as the Hilbert schemes of points for K3-surfaces, are IHS. Any closed hyperkähler manifold admits a finite covering which is the product of a torus and several IHS hyperkähler manifolds [Bo]. The IHS hyperkähler manifolds are also called hyperkähler manifolds of maximal holonomy, because the holonomy group of a hyperkähler manifold is $\text{Sp}(n)$ (the group of invertible quaternionic $n \times n$-matrices) – and not its proper subgroup – if and only if it is IHS [Bes].

### 2.2 Symplectic packing of tori and IHS hyperkähler manifolds by ellipsoids

By Vol we will always denote the symplectic volume of a symplectic manifold.

A closed ellipsoid in $\mathbb{C}^n$ is defined as a set

$$\{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid \sum_{i=1}^n a_i |z_i|^2 \leq r\}$$

for some $a_1, \ldots, a_n, r > 0$.

Let $(M, \omega)$, $\dim_{\mathbb{R}} M = 2n$, be a closed connected symplectic manifold. We say that the symplectic packings of $(M, \omega)$ by ellipsoids are unobstructed, if any finite collection of pairwise disjoint closed ellipsoids in the standard symplectic $\mathbb{R}^{2n}$ of total volume less than $\text{Vol}(M, \omega)$ has an open neighborhood that can be symplectically embedded into $(M, \omega)$.
Theorem 2.1:
Let $M$ be either a torus $T^{2n}$ with a Kähler form $\omega$ or an IHS hyperkähler manifold with a hyperkähler symplectic form $\omega$. Then the symplectic packings of $(M, \omega)$ by ellipsoids are unobstructed.

Theorem 2.1 follows from a similar result (see Theorem 2.4) for a wider class of Kähler manifolds as explained in Section 2.3 below.

2.3 Symplectic packing of arbitrary Campana simple manifolds

If $J$ is a complex structure of Kähler type on a closed connected manifold $M$, then the union $\mathcal{U}$ of all complex subvarieties $Z \subset M$ satisfying $0 < \dim \mathbb{C} Z < \dim \mathbb{C} M$ either has measure zero or is the whole $M$ (see [EV, Remark 4.2]).

If $\mathcal{U}$ has measure zero, $J$ is called Campana simple and the points of $M \setminus \mathcal{U}$ are called Campana-generic.

We say that $(M, J)$ is a Campana simple complex manifold, if $J$ is a Campana simple complex structure (of Kähler type) on $M$.

Remark 2.2:
Campana simple manifolds are non-algebraic. According to a conjecture of Campana (see [Cam, Question 1.4], [CDV, Conjecture 1.1]), any Campana simple manifold is bimeromorphic to a hyperkähler orbifold or a finite quotient of a torus.

We say that a complex structure $J$ of Kähler type on $M$ can be approximated by Campana-simple complex structures (in a smooth deformation) if there exists a smooth family $\{J_t\}_{t \in B^{2m}}, J_0 = J$, of complex structures $J_t$ on $M$ and a sequence $\{t_i\} \to 0$ in $B^{2m}$ so that each $J_{t_i}$ is Campana simple. (Here $B^{2m} \subset \mathbb{C}^m$ is an open ball centered at 0). Note that it follows from a version of Kodaira-Spencer stability theorem [KoSp] (see [EV, Theorem 5.6] for more details), that if $J$ is of Kähler type, then $J_t$ for $t \in B^{2m}$ sufficiently close to 0.

Theorem 2.3: [EV, Theorem 4.5]
(A) Any complex structure of Kähler type on $T^{2n}$ can be approximated by

1 The measure is defined by means of a volume form on $M$. One can easily see that if a set is of measure zero with respect to some volume form, then it is of measure zero with respect to any other volume form.
complex structures $J$ such that $(T^{2n}, J)$ does not admit any proper complex subvarieties of positive dimension and, in particular, is Campana simple.

(B) Let $(M, h)$, $\dim_{\mathbb{R}} M \geq 4$, be a closed connected IHS hyperkähler manifold and let $I$ be a complex structure appearing in $h$. Then $I$ can be approximated by Campana simple complex structures.

**Theorem 2.4:**
Let $(M, I, \omega)$ be a Kähler manifold and assume that $I$ can be approximated in a smooth deformation by Campana simple complex structures. Then the symplectic packings of $(M, \omega)$ by ellipsoids are unobstructed.

For the proof of Theorem 2.4 see Section 5.

**Remark 2.5:**
In case $\dim_{\mathbb{R}} M = 4$ Theorem 2.4 follows immediately from the analogous result for the symplectic packings by balls proved in [EV] and an observation by McDuff [McD] that $(M, \omega)$ admits a symplectic embedding of a 4-dimensional ellipsoid if and only if it admits a symplectic embedding of the disjoint union of a number of equal balls of the same total volume as the ellipsoid. However, the proof of Theorem 2.4 that we give below works in all dimensions.

**Proof of Theorem 2.1:**
Theorem 2.1 follows from Theorem 2.4 and Theorem 2.3.

2.4 Idea of the proof of Theorem 2.4 and plan of the paper

The idea of the proof of Theorem 2.4 is as follows. For simplicity we outline it in the case when $(M, I)$ is Campana simple.

By a result on simultaneous Diophantine approximation (see Proposition 6.1), any ellipsoid can be approximated by an ellipsoid of the form $\{ \pi \sum_{i=1}^{n} |a_i z_i|^2 \leq r \}$ where all $a_i$ are pairwise coprime positive integers. (We call such an ellipsoid simple). Thus, it suffices to prove that a disjoint union of $k$ simple ellipsoids of total volume less than $\text{Vol}(M, \omega)$ admits a symplectic embedding to $(M, \omega)$.

Recall that McDuff-Polterovich in [McDP] have shown that the problem of symplectic packing of a symplectic manifold $M$ by symplectic balls can be interpreted as a problem about the shape of the symplectic cone of the symplectic blow-up of $M$. There is a similar interpretation for symplectic packing by symplectic ellipsoids where the role of the blow-ups is played by the weighted blow-ups (Subsection 5.3).
By an extension of the McDuff-Polterovich results (for symplectic embeddings of balls) to the ellipsoid case, it suffices to show that a certain degree-2 real cohomology class \( \tilde{\alpha} \) of the weighted blow-up \( \tilde{M} \) of \( M \) at some \( k \) points \( x_1, \ldots, x_k \) (with the weights given by the coefficients \( a_i \) from the equations of the ellipsoids) is Kähler. The weighted blow-up \( \tilde{M} \) is a complex orbifold with isolated singularities due to the fact that the ellipsoids are simple. Moreover, by an extension of a result of McDuff-Polterovich (for symplectic embeddings of balls) to the ellipsoid case, the complex orbifold \( \tilde{M} \) admits a Kähler structure. We construct a Kähler resolution \( \pi: \hat{N} \to \tilde{M} \) (our construction uses the fact that \( \tilde{M} \) has only isolated singularities) and consider a cohomology class of the form \( \pi^*\tilde{\alpha} + \delta b \), where \( \delta > 0 \) is small and \( b \in H^2(\hat{N}; \mathbb{R}) \) has the property \( \pi_*b = 0 \) and \( b \cup \pi^*\tilde{\alpha} = 0 \). Using the Demailly-Paun theorem, describing the Kähler cone of \( \hat{N} \), and the fact that the points \( x_1, \ldots, x_k \) are Campana generic we show that for a sufficiently small \( \delta > 0 \) the class \( \pi^*\pi^*\tilde{\alpha} + \delta b \) is Kähler. Then, using a result of Miyaoka on the extension of a Kähler form over an isolated puncture, we show that the class \( \pi_* (\pi^*\tilde{\alpha} + \delta b) = \tilde{\alpha} \) is Kähler which finishes the proof.

The plan of the paper is as follows.

In Section 3.1 we recall basic facts about orbifolds.

In Section 3.2 we state the result on the existence of a Kähler resolution of a closed Kähler orbifold with isolated singularities and prove that the pushforward of a Kähler class of the resolution is a Kähler class of the base.

In Section 3.3 we construct the resolution and prove the existence of a Kähler form on it. The construction uses plurisubharmonic functions – we recall basic facts about them in the beginning of the same section.

In Section 3.4 we recall the basics concerning weighted blow-ups in the complex and symplectic category and discuss the relation between the symplectic/Kähler classes on weighted blowups and symplectic embeddings of ellipsoids.

In Section 4 we apply the Demailly-Paun theorem to prove that the needed cohomology class of the resolution is Kähler.

In Section 5 we combine the previous results together and prove Theorem 2.4.

In the appendix (Section 6) we prove the above-mentioned result on the simultaneous Diophantine approximation.

3 orbifolds, weighted blow-ups
and symplectic packing by ellipsoids

Here we recall a few basic facts on orbifolds. Orbifolds were originally introduced under the name “V-manifolds” by I.Satake in 1950s [Sa]. The name was changed to “orbifolds” in W.Thurston’s seminar in the 1970s [Thu].
3.1 Basics of orbifolds

**Definition 3.1:** A real (complex) orbifold chart (also known as a locally uniformizing system) on a topological space $N$ consists of the following objects:

- $U_\alpha$ – an open connected subset of $N$.
- $\tilde{U}_\alpha$ – an open connected neighborhood of the origin 0 in $\mathbb{R}^n$ (in $\mathbb{C}^n$).
- $\Gamma_\alpha$ – a finite (possibly trivial) group acting effectively on $\mathbb{R}^n$ (on $\mathbb{C}^n$) by linear real (complex) transformations so that $\tilde{U}_\alpha$ is invariant under the action, 0 is a fixed point of the action and the set of points of $\tilde{U}_\alpha$ with a non-trivial stabilizer is of real codimension 2 (complex codimension 1) or more.
- $\phi_\alpha : U_\alpha \to \tilde{U}_\alpha/\Gamma_\alpha$ – a homeomorphism.

The number $n$ is called the real (complex) dimension of the chart. If $x \in U_\alpha$, we call the stabilizer of a pre-image of $\phi_\alpha(x) \in \tilde{U}_\alpha/\Gamma_\alpha$ in $\tilde{U}_\alpha$ under the action of $\Gamma_\alpha$ the stabilizer of $x$ in $U_\alpha$ and denote it by $\Gamma_{\alpha,x}$.

A real (complex) orbifold atlas on $N$ is a collection

$$\{(U_\alpha, \tilde{U}_\alpha, \Gamma_\alpha, \phi_\alpha)\}$$

of $n$-dimensional real (complex) orbifold charts on $N$ with the following properties:

- $N = \bigcup_\alpha U_\alpha$.
- Any finite intersection of sets from the collection $\{U_\alpha\}$ is a union of sets from the collection.
- If $U_\alpha \subset U_\beta$, then there exists an injective homomorphism $f_{\alpha\beta} : \Gamma_\alpha \to \Gamma_\beta$ a smooth (complex analytic) embedding $\tilde{\phi}_{\alpha\beta} : \tilde{U}_\alpha \to \tilde{U}_\beta$ equivariant with respect to the actions of $\Gamma_\alpha, \Gamma_\beta$ (related by $f_{\alpha\beta}$) and covering the inclusion $U_\alpha \hookrightarrow U_\beta$ (where $U_\alpha, U_\beta$ are identified with $\tilde{U}_\alpha/\Gamma_\alpha$ and $\tilde{U}_\beta/\Gamma_\beta$, respectively, by $\phi_\alpha$ and $\phi_\beta$).

A real (complex) orbifold of real (complex) dimension $n \geq 2$ is a Hausdorff paracompact topological space $N$ equipped with a maximal real (complex) orbifold atlas formed by $n$-dimensional orbifold charts. Such a maximal atlas is called an orbifold structure.

Given a point $x$ in an orbifold $N$, the stabilizers $\Gamma_{\alpha,x}$ of $x$ in different orbifold charts $U_\alpha$ on $N$ containing $x$ are all isomorphic. We will denote any
of these stabilizers by $\Gamma_x$, or by $\Gamma^N_x$, if we want to emphasize that it is the stabilizer of $x$ in $N$.

The **tangent space** of an $n$-dimensional orbifold $N$ at $x \in N$ is defined as the $n$-dimensional representation of $\Gamma_x \cong \Gamma_{\alpha,x}$ on $T_{\tilde{\phi}_{\alpha}(x)}\tilde{U}_\alpha$, where $U_\alpha$ is an orbifold chart containing $x$. The representations of $\Gamma_x$ coming from different orbifold charts $U_\alpha, U_\beta$ containing $x$ are isomorphic (as representations of isomorphic groups $\Gamma_{\alpha,x} \cong \Gamma_{\beta,x}$).

If the stabilizer of a point in $N$ is trivial, then it is called a **regular point** of $N$; otherwise it is called a **singular point**. The set of singular points of $N$ is called the **singular locus** of $N$ and its complement the **regular part** of $N$. The regular part of $N$ is a smooth manifold – it admits a smooth atlas formed by some of the orbifold charts from the maximal orbifold atlas on $N$. We say that $N$ is an **orbifold with isolated singularities** if the singular locus of $N$ is a discrete set.

Of course, any smooth real (complex) manifold is also a real (complex) orbifold. We will say that a real (complex) orbifold is **smooth**, if its orbifold structure contains a smooth real (complex) subatlas or, equivalently, its singular locus is empty.

Most differential-geometric objects (smooth/complex analytic maps and their differentials, vector fields and their flows, differential forms, the differential of a differential form, Lie derivative along a vector field, almost complex structures, Riemannian metrics, vector bundles etc.) can be generalized to orbifolds in a straightforward way: first, one considers a $\Gamma_\alpha$-invariant (or equivariant) version of an object on each $\tilde{U}_\alpha$ and then requires that the maps $\tilde{\phi}_{\alpha\beta}$ glue the objects on all $\tilde{U}_\alpha$ and $\tilde{U}_\beta$. In order to distinguish between the objects on smooth manifolds and their counterparts on orbifolds we will use the prefix “orbifold” for the latter counterparts: orbifold smooth functions, orbifold smooth vector fields etc. Note that the restriction of an orbifold smooth function, orbifold smooth vector field etc. on an orbifold $N$ to the regular part $N^{reg}$ of $N$ is a smooth function, smooth vector field etc. in the usual sense on the smooth manifold $N^{reg}$.

Orbifolds admit partitions of unity (see e.g [BDD Theorem B.12]). This allows to equip any orbifold with an orbifold Riemannian metric and to define the integral of an orbifold differential form over an oriented orbifold (use a partition of unity subordinated to orbifold charts and for an $n$-dimensional chart $\{(U_\alpha, \tilde{U}_\alpha, \Gamma_\alpha, \phi_\alpha)\}$ and an orbifold form $\Omega$ of degree $n$ supported in $U_\alpha$ define $\int_{\tilde{U}_\alpha} \Omega$ as $1/|\Gamma_\alpha| \int_{\tilde{\Omega}} \tilde{\Omega}$, where $\tilde{\Omega}$ is the lift of $\Omega$ to $\tilde{U}_\alpha$).

A **suborbifold** $L$ of a real (complex) orbifold $N$ is then defined as a subset $L \subset N$ equipped with an orbifold structure so that the inclusion $L \to N$ is an orbifold smooth (analytic) map. In particular, this means that $\Gamma^L_x$ injects into $\Gamma^N_x$ for every $x \in L$. A smooth suborbifold of an orbifold $N$ will be called a **submanifold** of $N$. 
Definition 3.2:
A pairwise coprime vector is an ordered tuple of pairwise coprime positive integers.

Given a vector $\vec{a} = (a_1, \ldots, a_n)$ set
$$\langle \vec{x} \rangle := a_1 \cdot \ldots \cdot a_n.$$

Example 3.3:
Let $\vec{a} := (a_0, \ldots, a_n)$ be a pairwise coprime vector. The weighted projective space $\mathbb{C}P^n(a_0, \ldots, a_n)$ (which we will also denote by $\mathbb{C}P^n(\vec{a})$) is defined as the quotient of $\mathbb{C}^{n+1}$ by the action of $\mathbb{C}^\ast$ given by
$$\lambda : (z_0, \ldots, z_n) \mapsto (\lambda^{a_0} z_0, \ldots, \lambda^{a_n} z_n). \quad (3.1)$$

The weighted projective space $\mathbb{C}P^n(\vec{a})$ can be equipped with the structure of a complex orbifold (see e.g. [Go]). Since $a_0, \ldots, a_n$ are pairwise coprime, the singular locus of $\mathbb{C}P^n(\vec{a})$ is discrete.

The integral homology/cohomology of $\mathbb{C}P^n(\vec{a})$ is isomorphic (as a group) to that of $\mathbb{C}P^n$, while the multiplicative structure of $H^\ast(\mathbb{C}P^n(\vec{a}); \mathbb{Q})$ may differ from the case of $\mathbb{C}P^n$ (see [Tru], [Kaw]) and does depend on $\vec{a}$. In particular, in our case, when all $a_0, \ldots, a_n$ are pairwise coprime, the map
$$\mathbb{C}P^n \to \mathbb{C}P^n(\vec{a}),$$
given by
$$[z_0 : \ldots : z_n] \mapsto [z_0^{a_0} : \ldots : z_n^{a_n}],$$
induces a ring isomorphism
$$H^\ast(\mathbb{C}P^n(\vec{a}); \mathbb{Z}) \to H^\ast(\mathbb{C}P^n; \mathbb{Z})$$
which is the multiplication by $\langle \vec{a} \rangle$ in each degree. For each $i = 0, \ldots, n$ denote by $\alpha_i$ the generator of $H^{2i}(\mathbb{C}P^n(\vec{a}); \mathbb{Z})$ mapped by this ring isomorphism to the positive multiple of the standard generator of $H^{2i}(\mathbb{C}P^n; \mathbb{Z})$. Then $\alpha_i \alpha_j = \langle \vec{a} \rangle \alpha_{i+j}$ for all $i, j$, $i + j \leq n$. In particular, $\alpha_1^n = \langle \vec{a} \rangle^{n-1} \alpha_n$.

The hyperplane section $z_0 = 0$ is a suborbifold $L$ of $\mathbb{C}P^n(\vec{a})$ which is orbifold diffeomorphic to $\mathbb{C}P^{n-1}(a_1, \ldots, a_n)$. Then $\alpha_1 \in H^2(\mathbb{C}P^n(\vec{a}); \mathbb{Z})$ is Poincaré-dual to $[L]$ and
$$\langle \alpha_1^n, [\mathbb{C}P^n(\vec{a})] \rangle = \langle \vec{a} \rangle^{n-1}.$$

The space $\mathbb{C}P^n(\vec{a})$ can be equipped with an orbifold symplectic structure. Namely, consider the Hamiltonian $H(z_0, \ldots, z_n) = \pi \sum_{i=0}^n a_i |z_i|^2$ on the standard symplectic $\mathbb{C}^{n+1}$. It defines an $S^1 = \mathbb{R}/\mathbb{Z}$-action:
$$t : (z_0, \ldots, z_n) \mapsto (e^{2\pi \sqrt{-1} a_0 t} z_0, \ldots, e^{2\pi \sqrt{-1} a_n t} z_n). \quad (3.2)$$
which is the restriction of the $\mathbb{C}^*$-action (5.1) to $S^1 \subset \mathbb{C}^*$. For $r > 0$ consider the reduced space $H^{-1}(r)/S^1$ – it has the structure of a real $2n$-dimensional orbifold (see e.g. [Go]) and is naturally identified (by an orbifold diffeomorphism) with $\mathbb{C}P^n(\overline{a})$. The reduction induces an orbifold symplectic form on $H^{-1}(r)/S^1$ and hence on $\mathbb{C}P^n(\overline{a})$. We will denote this orbifold form by $\Omega_{\overline{a},r}$. Set $\Omega_{\overline{a}} := \Omega_{\overline{a},1}$ – we will call it the Fubini-Study symplectic form on $\mathbb{C}P^n(\overline{a})$. One can check that the form $\Omega_{\overline{a}}$ is Kähler, $\Omega_{\overline{a},r} = r \Omega_{\overline{a}}$ and

$$[\Omega_{\overline{a},r}] = \frac{r}{\langle \overline{a} \rangle} \alpha_1,$$

$$\int_{\mathbb{C}P^n(\overline{a})} \Omega^n_{\overline{a},r} = \frac{r^n}{\langle \overline{a} \rangle} \text{Vol}_{2n},$$

where $\text{Vol}_{2n}$ is the volume of the Euclidean $2n$-dimensional unit ball.

Poincaré duality (over $\mathbb{Q}$), de Rham and Hodge theorems for closed manifolds extend to closed (=compact, without boundary) orbifolds – see [Sa] and [Bai]. This allows to obtain an orbifold version of Moser’s theorem [Mos] – its proof literally repeats the proof for the smooth case and we write it here as an example of a straightforward generalization of a result for smooth manifolds to orbifolds.

**Theorem 3.4:**
Let $N$ be a closed symplectic orbifold, and $\omega_t$ a smooth family of orbifold symplectic forms, parameterized by $t \in [0, 1]$ and lying in the same (de Rham) cohomology class. Then there exists a smooth family of orbifold diffeomorphisms $\Psi_t : N \to N$ such that $\Psi_t^* \omega_0 = \omega_t$.

**Proof.**
The orbifold form $\dot{\omega}_t$ is exact and therefore there exists an orbifold 1-form $\eta_t$ satisfying $d\eta_t = \dot{\omega}_t$. The Hodge theorem allows to choose the orbifold forms $\eta_t$ so that they depend on $t$ smoothly. Let $X_t$ be an orbifold vector field satisfying $\omega_t \lrcorner X_t = \eta_t$. Then $\text{Lie}_{X_t} \omega_t = \eta_t = \dot{\omega}_t$ by Cartan’s formula (which also extends to orbifolds). The orbifold vector field $X_t$ defines a flow of orbifold diffeomorphisms $\Psi_t$ on $N$. Then $\text{Lie}_{X_t} \omega_t = \dot{\omega}_t$ implies $\Psi_t^* (\omega_0) = \omega_t$.

**Remark 3.5:**
Similarly one can prove orbifold versions of various symplectic neighborhood theorems – they are all based on Moser’s method as above and easily generalize to orbifolds.
3.2 Resolution of orbifolds and Kähler classes

Given a smooth map $F: M \to N$ between closed oriented smooth manifolds, we define the pushforward $F_*b \in H^*(N; \mathbb{R})$ of a cohomology class $b \in H^*(M; \mathbb{R})$ using the Poincaré-duality on $M$ and $N$ and the pushforward of homology classes.

Similarly to the smooth case, a real $(1, 1)$-cohomology class of a closed complex orbifold $(N, J)$ is called Kähler if it can be realized by an orbifold Kähler form. Denote by $\text{Kah}(N, J)$ the cone in $H^2(N; \mathbb{R})$ formed by the Kähler cohomology classes.

**Theorem 3.6:**

Let $N$, $\dim_{\mathbb{C}} N = n \geq 2$, be a closed complex orbifold with isolated singularities. Denote its singular locus by $\Sigma = \{y_1, \ldots, y_m\}$.

A. Let $N'$, $\dim_{\mathbb{C}} N = n$, be a closed complex manifold and let $P: N' \to N$ be a surjective smooth map such that $P: N' \setminus P^{-1}(\Sigma) \to N \setminus \Sigma$ is a biholomorphism and $P^{-1}(\Sigma)$ has zero volume (with respect to a volume form on $N'$).

Then the pushforward of any Kähler cohomology class on $N'$ is a Kähler cohomology class on $N$.

B. There exist
- a smooth closed complex manifold $\hat{N}$ of the same dimension as $N$;
- a holomorphic map $\pi: \hat{N} \to N$ such that $\pi: \hat{N} \setminus \pi^{-1}(\Sigma) \to N \setminus \Sigma$ is a biholomorphism;
- cohomology classes $b_i \in H^2(\hat{N}; \mathbb{R})$, $i = 1, \ldots, m$, so that $\pi_* b_i = 0$ and $b_i \cup \pi^* u = 0$ for any $i$ and any $u \in H^*(N)$, $\deg u > 0$;

so that for any $v \in \text{Kah}(N, J)$ and any sufficiently small $\delta_1, \ldots, \delta_m > 0$,

$$\pi^* v + \sum_{i=1}^m \delta_i b_i \in \text{Kah}(\hat{N}, \hat{J}).$$

The construction of $\hat{N}$ and $\pi$ in part B of **Theorem 3.6** amounts to resolving the isolated singularities of a Kähler orbifold $N$ in the Kähler category. In particular, this yields the following corollary.

**Corollary 3.7:** Any closed Kähler orbifold with isolated singularities admits a Kähler resolution.

Part A of **Theorem 3.6** is proved below.

For the proof of part B of **Theorem 3.6** see Section 3.3.
Proof of part A of (Theorem 3.6).

Let \( \theta \) be a Kähler form on \( N' \). Then \( P^* \theta \) is a Kähler form on the smooth complex manifold \( N \setminus \Sigma \).

Assume \( x \in \Sigma \) and \( \Gamma_x \) is its stabilizer. A neighborhood \( U \) of \( x \) in \( N \) is biholomorphic to a neighborhood of zero in \( \mathbb{C}^n / \Gamma_x \) and, since \( x \) is an isolated singularity of \( N \), the form \( P^* \theta |_{U \setminus x} \) lifts under the projection \( \mathbb{C}^n \to \mathbb{C}^n / \Gamma_x \) to a \( \Gamma_x \)-invariant Kähler form \( \zeta \) on \( V \setminus 0 \), where \( V \subset \mathbb{C}^n \) is a \( \Gamma_x \)-invariant open set which is the lift of \( U \). By a result of [Mi], there exists a Kähler form \( \zeta' \) on the whole \( V \) that coincides with \( \zeta \) outside a small neighborhood of 0.

Averaging, if necessary, \( \zeta' \) with respect to the action of \( \Gamma_x \) on \( V \), we can assume that \( \zeta' \) is \( \Gamma_x \)-invariant. Thus, \( \zeta' \) descends to an orbifold Kähler form on \( U \) that coincides with \( P^* \theta \) outside a small neighborhood of \( x \) in \( U \). Thus, \( P^* \theta \) can be extended from \( N \setminus \Sigma \) to an orbifold Kähler form on \( N \). This Kähler form coincides with \( P^* \theta \) outside a finite union of disjoint contractible sets and therefore its cohomology class is \( P^* [\theta] \).

3.3 Construction of a Kähler resolution

We recall a few basic facts about currents and plurisubharmonic functions needed for the proof of (Theorem 3.6). For more details see e.g. [Dem, GH, LG].

Recall that a function \( \varphi \), with values in \( \mathbb{R} \cup \{ -\infty \} \), on an open domain \( U \subset \mathbb{C}^n \) is called plurisubharmonic if it is upper semi-continuous (hence, locally bounded from above), not identically equal to \( -\infty \) on any open set, and for any complex line \( L \) in \( \mathbb{C}^n \) the restriction of \( \varphi \) to \( U \cap L \) is either subharmonic or identically equal to \( -\infty \). Plurisubharmonic functions are locally integrable [LG, Proposition I.9]. A function \( \varphi : U \to \mathbb{R} \cup \{ -\infty \} \) is called strictly plurisubharmonic if for every \( p \in U \) and any sufficiently small \( \epsilon > 0 \) the function \( \varphi - \epsilon |z|^2 \) is plurisubharmonic. (Here and below \( z = (z_1, \ldots, z_n) \) and \( |z|^2 := |z_1|^2 + \ldots + |z_n|^2 \).) A smooth function \( \varphi \) is (strictly) plurisubharmonic if and only if the \((1,1)\)-form \( \sqrt{-1} \partial \bar{\partial} \varphi \) is a (strictly) positive Hermitian form (being strictly positive is equivalent for the form \( \sqrt{-1} \partial \bar{\partial} \varphi \) to being Kähler).

If \( \varphi_1, \varphi_2 \) are two – not necessarily smooth – (strictly) plurisubharmonic functions on \( U \subset \mathbb{C}^n \), then for any \( \epsilon > 0 \) there exists (see [Dem, Lemma 5.18]) a (strictly) plurisubharmonic function \( \max_{\epsilon} \{ \varphi_1, \varphi_2 \} \) on \( U \), called the regularized maximum of \( \varphi_1, \varphi_2 \), such that

- if \( \varphi_1, \varphi_2 \) are smooth near \( x \in U \), then so is \( \max_{\epsilon} \{ \varphi_1, \varphi_2 \} \);
- \( \max \{ \varphi_1, \varphi_2 \} \leq \max_{\epsilon} \{ \varphi_1, \varphi_2 \} \leq \max \{ \varphi_1, \varphi_2 \} + \epsilon \) on \( U \);
A current of degree \((p, q)\) on a complex manifold of complex dimension \(n\) is a continuous linear functional on the space of smooth compactly supported differential complex-valued \((2n - p - q)\)-forms that vanishes on \((k, l)\)-forms as long as \((k, l) \neq (n - p, n - q)\).

A current \(T\) is called real if \(T(\xi) = T(\overline{\xi})\) for all differential forms \(\xi\) (the bar is the complex conjugation).

Each differential \((p, q)\)-form \(\theta\) with locally integrable coefficients on a complex manifold defines a current \(T_\theta\) of degree \((p, q)\): the value of \(T_\theta\) on a smooth compactly supported \((2n - p - q)\)-form \(\xi\) is defined as the integral of \(\theta \wedge \xi\) over the manifold. If two differential forms with continuous coefficients define the same current they coincide everywhere. If \(\theta\) is real, then so is \(T_\theta\).

The currents defined in this way by smooth forms \(\theta\) are called smooth.

A real current \(T\) of degree \((p, p)\) is called positive if \(T(\eta \wedge \overline{\eta}) \geq 0\) for any smooth compactly supported differential \((n - p, 0)\)-form, and strictly positive if \(T - T'\) is positive for some positive current \(T'\) of degree \((p, p)\).

Recall that a real differential \((1, 1)\)-form \(\theta\) is called positive (strictly positive) if \(\theta(v, \overline{v}) \geq 0\) (\(\theta(v, \overline{v}) > 0\)) for any tangent vector \(v \neq 0\). Thus, a positive (strictly positive) real differential form \((1, 1)\)-form \(\theta\) defines a positive (strictly positive) real current \(T\) of degree \((1, 1)\).

The differentials \(d, \partial, \bar{\partial}\) for currents on a complex manifold are defined by duality using the corresponding differentials for smooth complex-valued differential forms. The homomorphism \(\theta \mapsto T_\theta\) induces a homomorphism between the cochain complexes of differential forms and currents (for any of the differentials \(d, \partial, \bar{\partial}\)) that induces an isomorphism between the corresponding cohomologies (see e.g. [GH, p.385]).

A smooth real \((1, 1)\)-form \(\theta\) is Kähler if and only if the smooth current \(T_\theta\) is closed and strictly positive.

By the local \(\partial\bar{\partial}\)-lemma [LG, Theorem 2.28], any (not necessarily smooth) closed real (strictly) positive current \(T\) of degree \((1, 1)\) on a complex manifold can be represented locally as \(T = \sqrt{-1} \partial\bar{\partial}T_\varphi\) for a (strictly) plurisubharmonic function \(\varphi\). The function \(\varphi\) is defined uniquely up to an addition of a harmonic (hence, smooth) function. Thus, if \(T\) is smooth near a point then \(\varphi\) is also smooth near that point.

**Proof of part B of Theorem 3.6**

Let \(\Gamma_1, \ldots, \Gamma_m\) be the stabilizers of the singular points \(y_1, \ldots, y_m\) of \(N\). Consider orbifold charts \((U_i, \bar{U}_i, \Gamma_i, \phi_i)\) on \(N\) such that \(y_i \in U_i\) for all \(i\) and all \(U_i\) are pairwise disjoint.
Pick an arbitrary $i = 1, \ldots, m$. The singular point $y_i$ is isolated and therefore for a smaller neighborhood $U'_i \subset \hat{U}_i$ of $y_i$ the punctured neighborhood $U'_i \setminus y_i$ can be bi-holomorphically identified with $B(R) \setminus 0$, where $B(R) \subset \mathbb{C}^n$ denotes an open ball of radius $R$ centered at zero.

The orbifold $\mathbb{C}^n/\Gamma_i$ is a quasi-projective algebraic variety with a single singularity at the origin. By the famous result of Hironaka [Hir] (see also [BM], [Vil1, Vil2] for more accessible proofs of Hironaka’s result), there exists a resolution $\pi_i : X_i \to \mathbb{C}^n/\Gamma_i$ of the singularity where $X_i$ is a smooth quasi-projective variety. Moreover, the resolution of singularity $\pi_i : X_i \to \mathbb{C}^n/\Gamma_i$ is biholomorphic outside of the singular set of $X_i$. Define $V_i := \pi_i^{-1}(\phi_i(U'_i)) \subset X_i$. Since $\pi_i : X_i \setminus \pi_i^{-1}(0) \to \mathbb{C}^n/\Gamma_i \setminus 0$ is a biholomorphism, the maps $\pi_i^{-1}$ and, respectively, $\pi_i^{-1} \circ \phi_i$ identify $B(R) \setminus 0 \subset \mathbb{C}^n$ and, respectively, $U'_i \setminus y_i \subset N$ biholomorphically with the same open subset $W_i := V_i \setminus \pi_i^{-1}(0)$.

Let us attach all the $V_i$ to $N \setminus \Sigma$ using the identifications $\pi_i^{-1} \circ \phi_i$ between the open sets $U'_i \setminus y_i \subset N$ and $W_i = V_i \setminus \pi_i^{-1}(0)$. As a result we get a complex manifold $\hat{N} := (N \setminus \Sigma) \cup \bigcup_{i=1}^{m} V_i$ along with a holomorphic projection $\pi : \hat{N} \to N$ which is a biholomorphism over $\hat{N} \setminus \pi^{-1}(\Sigma)$ and which coincides with $\pi_i$ over each $V_i$.

Assume $\eta$ is an orbifold Kähler form on $N$ such that $[\eta] = v$. The restriction of $\eta$ to the smooth complex manifold $N \setminus \Sigma$ can be viewed as a usual smooth Kähler form on a smooth complex manifold. In particular, under the identification $\phi_i$ the form $\eta|_{U'_i \setminus y_i}$ is identified with a smooth Kähler form on $B(R) \setminus 0$ that will be also denoted by $\eta$.

As a smooth quasi-projective variety, $X_i$ carries a Kähler form $\omega_i$ (induced by the Fubini-Study form on the projective space of which $X_i$ is a subvariety). By means of the identifications above, $\omega_i$ induces Kähler forms on $B(R) \setminus 0$ and $U'_i \setminus y_i$ that, by an abuse of notation, will be both denoted also by $\omega_i$.

**Lemma 3.8:**

For any sufficiently small $\delta_i > 0$ there exists a Kähler form $\xi_{i,\delta_i}$ on $B(R) \setminus 0$ that equals to $\delta_i \omega_i$ near $0$ and to $\eta$ outside $B(3R/4)$.

Postponing the proof of the lemma let us finish the proof of the theorem.

By means of the identification $\phi_i$, the form $\xi_{i,\delta_i}$ induces a Kähler form on $U'_i \setminus y_i$, that will be also denoted by $\xi_{i,\delta_i}$, which coincides with $\eta$ near the boundary of $U'_i$ and with $\omega_i$ near $y_i$.

It follows that for any sufficiently small $\delta_1, \ldots, \delta_m > 0$, the manifold $\hat{N}$ carries a Kähler form $\hat{\eta}$ which, by definition, is equal to $\eta$ on $N \setminus \bigcup_{i=1}^{m} U'_i$ (identified by $\pi$ with a subset of $\hat{N}$), to $\xi_{i,\delta_i}$ on each $U'_i$ (identified by $\pi_i^{-1} \circ \phi_i$ with $W_i$) and to $\delta_i \omega_i$ on each $V_i$. 

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v. 1.2.1, August 18, 2017
The closed 2-forms $\hat{\eta}$ and $\pi^*\eta$ coincide outside $\cup_{i=1}^m V_i$. Hence, the form $\hat{\eta} - \pi^*\eta$ is a sum of closed 2-forms $\delta_i\sigma_i$, $i = 1, \ldots, m$, on $\hat{N}$ so that each $\sigma_i$ is Kähler, supported inside $V_i$ (and therefore $\pi_*[\sigma_i] = 0$) and coincides with the form $\omega_i$ on the analytic subvariety $\pi^{-1}(y_i) \subset V_i$. Each form $\omega_i$ is Kähler on $V_i$ and $\pi^{-1}(y_i)$ is a deformation retract of $V_i$—therefore the cohomology class $b_i := [\sigma_i]$ depends only on the restriction of $\omega_i$ to $\pi^{-1}(y_i)$ (and thus is independent of $\eta$). This immediately yields that $b_i \cup \pi^*u = 0$ for any $i$ and any $u \in H^*(N; \mathbb{R})$, deg $u > 0$, and that

$$[\hat{\eta}] = \pi^*[\eta] + \sum_{i=1}^m \delta_i b_i = \pi^* v + \sum_{i=1}^m \delta_i b_i \in \text{Kah}(\hat{\mathcal{N}}, \hat{J}),$$

which finishes the proof. ■

Proof of Lemma 3.8.

The form $\omega_i$ on $B(R) \setminus 0$ has locally integrable coefficients near 0 and thus defines a real degree-(1, 1) strictly positive current on the whole $B(R)$ which is smooth on $B(R) \setminus 0$.

By the local $\partial\bar{\partial}$-lemma, we may assume, without loss of generality, that $\eta$ can be written on $B(R)$ as $\eta = \sqrt{-1}\partial\bar{\partial} F$ for a smooth strictly plurisubharmonic function $F$ on $B(R)$ and that $T_{\omega_i} = \sqrt{-1}\partial\bar{\partial} T_G$ for some strictly plurisubharmonic function $G$ on $B(R)$ which is smooth on $B(R) \setminus 0$. In particular, this means that $\omega_i = \sqrt{-1}\partial\bar{\partial} G$ on $B(R) \setminus 0$.

Define a function $H$ on $B(R) \setminus 0$ as $G$ if $G(0) = -\infty$ and as $\log |z|^2$ if $G(0) \in \mathbb{R}$. Recall that $\log |z|^2$ is a plurisubharmonic function of $z$ on the whole $\mathbb{C}^n$ which is, of course, smooth on $\mathbb{C}^n \setminus 0$.

Note that $a|z|^2 + b$ is a plurisubharmonic function of $z$ for any $a, b \in \mathbb{R}$, $a > 0$. Choose $a, b \in \mathbb{R}$, $a > 0$, so that

$$\min_{|z| = R/2} a|z|^2 + b > \max_{|z| = R/2} H, \max_{|z| = R/4} a|z|^2 + b < \min_{|z| = R/4} H.$$

Then for a sufficiently small $\epsilon > 0$ the regularized maximum $\max_{\epsilon}\{a|z|^2 + b, H\}$ defines a strictly plurisubharmonic function on a neighborhood of the spherical annulus $\{R/2 \leq |z| \leq R/4\}$ which is equal to $a|z|^2 + b$ on a neighborhood of the sphere $\{|z| = R/2\}$ and to $H$ on a neighborhood of the sphere $\{|z| = R/4\}$. Extending this function outside the sphere $\{|z| = R/2\}$ by $a|z|^2 + b$ and inside the sphere $\{|z| = R/4\}$ by $H$ we get a smooth strictly plurisubharmonic function $K$ on $B(R) \setminus 0$ such that $K(z) = a|z|^2 + b$ outside the ball $B(R/2)$ and $K = H$ on $B(R/4) \setminus 0$.

Since $F$ is strictly plurisubharmonic on $B(R)$ there exists a small $\varepsilon > 0$ such that $F - \varepsilon(a|z|^2 + b)$ is strictly plurisubharmonic on $B(3R/4)$. Thus, $L := F - \varepsilon(a|z|^2 + b) + \varepsilon K$ is a smooth function on $B(R) \setminus 0$ which, being a sum of two plurisubharmonic functions on $B(3R/4) \setminus 0$, is strictly
plurisubharmonic on $B(3R/4) \setminus 0$. Note that $L = F$ outside $B(3R/4)$ and thus is strictly plurisubharmonic also there. Thus, $L$ is a smooth strictly plurisubharmonic function on $B(R) \setminus 0$ equal to $F$ outside $B(3R/4)$. Observe that, since $K = H$ on $B(R/4) \setminus 0$ and $F - \varepsilon(a|z|^2 + b)$ is continuous on $B(R)$, there exists $C_1 > 0$ such that
\[ \varepsilon - C_1 \leq L \leq \varepsilon H + C_1 \text{ on } B(R/4) \setminus 0. \] (3.3)

If $G(0) = -\infty$ and, accordingly, $H = G$, then, in view of (3.3) and since $G$ is continuous on $B(R) \setminus 0$, for any $0 < \delta_i < \varepsilon$ one can find $C_2 > 0$ so that $\delta_i H - C_2 = \delta_i G - C_2 < L$ outside $B(3R/4)$ and $\delta_i G - C_2 < L$ on $B(r) \setminus 0$ for some $0 < r < R/4$. Therefore for a sufficiently small $\varepsilon > 0$ the regularized maximum $\varrho := \max_{i} \{ L, \delta_i G - C_2 \}$ is a smooth strictly plurisubharmonic function on $B(R) \setminus 0$ equal to $L$, and hence to $F$, outside $B(3R/4)$ and to $\delta_i G - C_2$ near $0$.

If $G(0) \in \mathbb{R}$ (meaning that $G$ is bounded from below on a neighborhood of $0$) and, accordingly, $H = \log |z|^2$, then, since $G$ is continuous on $B(R) \setminus 0$, for any $\delta_i > 0$ one can find $C_3 > 0$ so that $\delta_i G - C_3 < F = L$ outside $B(3R/4)$. Since $G$ is bounded from below and $L = H = \log |z|^2$ near $0$ we get that $\delta_i G - C_3 > L$ on some neighborhood of $0$. Therefore for a sufficiently small $\varepsilon > 0$ the regularized maximum $\varrho := \max_{i} \{ L, \delta_i G - C_3 \}$ is a smooth strictly plurisubharmonic function on $B(R) \setminus 0$ equal to $L$, and hence to $F$, outside $B(3R/4)$ and to $\delta_i G - C_3$ near $0$.

Let us sum up: in both cases $(G(0) = -\infty$ and $G(0) \in \mathbb{R})$, for any sufficiently small $\delta_i > 0$ we get a smooth strictly plurisubharmonic function $\varrho$ on $B(R) \setminus 0$ equal to $F$ outside $B(3R/4)$ and to $\delta_i G - C$, for some $C \in \mathbb{R}$, near $0$. Therefore for any sufficiently small $\delta_i > 0$ we get a Kähler form $\xi_{i, \delta_i} := \sqrt{-1} \partial \bar{\partial} \varrho$ on $B(R) \setminus 0$ which equals $\delta_i \omega_i$ near $0$ and $\eta$ outside $B(3R/4)$.

### 3.4 Weighted blow-ups and symplectic embeddings of ellipsoids

We are going to consider weighted blow-ups of smooth manifolds at a point. This operation can be performed both in complex and symplectic categories. Since we are going to compare weighted blow-ups of the same smooth manifold with different complex structures, we will adapt the following point of view.

Fix a complex manifold $M$, $\dim_{\mathbb{C}} M =: n > 1$ and a pairwise coprime vector $\vec{a} := (a_1, \ldots, a_n)$. Note that the vector $(1, \vec{a}) = (1, a_1, \ldots, a_n)$ is then also pairwise coprime.

Topologically, a weighted blow-up of $M$ with the weight $(1, \vec{a})$ is defined as a topological space $\tilde{M}$ obtained by taking the connected sum of $M$ with
the weighted projective space $\mathbb{C}P^n(1, \tilde{a}) := \mathbb{C}P^n(1, a_1, \ldots, a_n)$ near a regular point of $\mathbb{C}P^n(1, \tilde{a})$. Recalling Example 3.3, one easily sees that $\tilde{M}$ can be equipped with the structure of a real orbifold.

A weighted blow-up at a point $x \in M$ can be realized in the complex (or algebraic) category similarly to the usual blow-up (see e.g. [KSC, Sec. 6.38]) – the latter can be viewed as the weighted blow-up with the weight $(1, \ldots, 1)$. Accordingly, any complex structure $I$ on $M$ defines a complex structure $\tilde{I}$ on $\tilde{M}$ and an $(\tilde{I}, I)$-holomorphic map $\Pi_I : \tilde{M} \to M$ so that over $M \setminus x$ the map $\Pi_I$ is a bi-holomorphism, while $\Pi_I^{-1}(x) =: E(I)$ – the exceptional divisor defined by $I$ – is a complex suborbifold biholomorphic (as an orbifold) to $\mathbb{C}P^{n-1}(\tilde{a})$. The complex structure $\tilde{I}$, the map $\Pi_I$ and the exceptional divisor $E(I)$ are defined uniquely, up to a smooth isotopy. The singular locus of $\tilde{M}$ is exactly the singular locus of $E(I)$, that is, a finite collection of points.

If $J$ is another complex structure on $M$, we get another complex structure $\tilde{J}$ on $\tilde{M}$ with another projection $\Pi_J : \tilde{M} \to M$ which is smoothly orbifold isotopic to $\Pi_I$ and therefore induces the same map on cohomology which is independent of the complex structure and will be denoted by $\Pi^*$. The exceptional divisor $E(J)$ defined by $J$ might be different from $E(I)$ but lies in the same homology class which is independent of the complex structure. We will denote the cohomology class that is Poincaré-dual to this homology class by $e \in H^2(\tilde{M}; \mathbb{Z})$.

Now let us briefly recall how the weighted blow-up can be realized in the symplectic category – for more details see e.g. [Go].

For a pairwise coprime $\tilde{a} := (a_1, \ldots, a_n)$ and $r > 0$, denote by $E_\tilde{a}(r)$ the ellipsoid $\pi \sum_{i=1}^n a_i |z_i|^2 \leq r$ in the standard symplectic $\mathbb{C}^n$ with the coordinates $z_1, \ldots, z_n$.

Given a symplectic manifold $(M, \omega)$, $\dim \mathbb{R} M = 2n$, and a symplectic embedding $\iota : E_\tilde{a}(r) \to (M, \omega)$, one can construct an orbifold, which is orbifold diffeomorphic to $\tilde{M}$, by removing $\iota(E_\tilde{a}(r))$ from $M$ and contracting the boundary of the resulting manifold along the fibers of the $S^1$-action induced by $\iota$ from the $S^1$-action on $\partial E_\tilde{a}(r)$ given by $[3,2]$. The form $\omega$ is then extended in a canonical way from $M \setminus \partial(\iota(E_\tilde{a}(r)))$ to a symplectic form on the orbifold $\tilde{M}$ – this procedure is called a symplectic cut (see [Ler], [Go], cf. [NP]).

A calculation similar to Example 3.3 shows that the cohomology class of

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1 In the algebraic language of modern algebraic geometry (see e.g. [Har]) the weighted blow-up of $M$ at a point $x$ with the weights $(1, a_1, \ldots, a_n)$ can be described as follows: For each $k = 1, \ldots, n$ assign the weight $a_k$ to the $k$-th coordinate $x_k$ on a neighborhood of $x$ in $M$ and let $R^m$ be the ideal of polynomials of $x_1, \ldots, x_k$ of weighted degree $\geq m$. Consider $\bigoplus_{m=0}^{\infty} R^m$ as a graded scheme relative to $M$. The graded spectrum ("Proj") of this sheaf of rings gives the weighted blow-up together with a projective morphism to $M$. 

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\( \tilde{\omega} \) is given by
\[
[\tilde{\omega}] = \Pi^*[\omega] - \frac{r}{(a)}e.
\]

The classical paper of McDuff-Polterovich [McDP] relates symplectic embeddings of a disjoint union of \( k \) balls into a symplectic manifold to the symplectic/Kähler classes of the blow-up of the manifold at \( k \) points. The proof of McDuff and Polterovich’s results is based only on various local and global versions of Moser’s theorem and on local normal forms which generalize in a straightforward way to orbifolds (see Theorem 3.4, Remark 3.5). Together with the construction of weighted blow-ups above this yields the following version of McDuff and Polterovich’s theorem relating symplectic embeddings of a disjoint union of \( k \) ellipsoids to the symplectic/Kähler classes of a weighted blow-up.

**Proposition 3.9:** (cf. [McDP] Proposition 2.1A, 2.1B, 2.1C, Corollary 2.1D)

Let \( M \) be a closed connected manifold equipped with a Kähler form \( \omega \), \( \dim_{\mathbb{C}} M = n \). Let \( \tilde{M} \) be a weighted blow-up of \( M \) at \( k \) points with weights \( \bar{a}_1, \ldots, \bar{a}_k \). Denote by \( \Pi_I : \tilde{M} \to M \) the corresponding projection and by \( e_1, \ldots, e_k \in H^2(\tilde{M}; \mathbb{R}) \) the cohomology classes Poincaré-dual to the homology classes of the corresponding exceptional divisors.

A. For any sufficiently small positive \( r_i, i = 1, \ldots, k \), the symplectic manifold \((M, \omega)\) admits a symplectic embedding of a disjoint union of the ellipsoids \( E_{\bar{a}_i}(r_i), i = 1, \ldots, k \), and for some complex structure \( I \) on \( M \) compatible with \( \omega \) the cohomology class
\[
[\tilde{\omega}] = \Pi^*[\omega] - \sum_{i=1}^{k} \frac{r_i}{(\bar{a}_i)}e_i \in H^2(\tilde{M}; \mathbb{R})
\]
is Kähler with respect to \( \tilde{I} \).

B. Assume there exists a complex structure \( I \) of Kähler type on \( M \) tamed by \( \omega \) and a symplectic form \( \tilde{\omega} \) on \( \tilde{M} \) taming \( \tilde{I} \) so that
\[
[\tilde{\omega}] = \Pi^*[\omega] - \sum_{i=1}^{k} \frac{r_i}{(\bar{a}_i)}e_i
\]
for some \( r_1, \ldots, r_k > 0 \).

Then \((M, \omega)\) admits a symplectic embedding of a disjoint union of the ellipsoids \( E_{\bar{a}_i}(r_i), i = 1, \ldots, k \). \( \blacksquare \)
We will need the following version of Proposition 3.9 part B (cf. [EV, Theorem 8.3]).

**Proposition 3.10:**
Let \((M, I, \omega)\), \(\text{dim}_\mathbb{R} M = 2n\), be a closed connected Kähler manifold. Let \(k \in \mathbb{N}\) and let \(\tilde{M}, \Pi^* : H^2(M; \mathbb{R}) \to H^2(\tilde{M}; \mathbb{R})\), \([E_1], \ldots, [E_k] \in H^2(\tilde{M}, \mathbb{Z})\), \(r_1, \ldots, r_k > 0\) be as above. Assume that there exists a complex structure \(J\) of Kähler type on \(M\) which is tamed by \(\omega\) so that

\[
[\omega] = \Pi^*[\omega] - \sum_{i=1}^{k} \frac{r_i}{\langle a_i \rangle} e_i \in \text{Kah}(\tilde{M}, \tilde{J}).
\]

Then \((M, \omega)\) admits a symplectic embedding of a disjoint union of the ellipsoids \(\mathcal{E}_{a_i}(r_i), i = 1, \ldots, k\).

**Proof:**
The proof virtually repeats the proof of [EV, Theorem 8.3]. We recall it briefly.

Note that \(H^2(\tilde{M}; \mathbb{R}) = H^2(M; \mathbb{R}) \oplus \text{Span}_\mathbb{R} \{[E_1], \ldots, [E_k]\}\) and that the homomorphism \(\Pi^* : H^2(M; \mathbb{R}) \to H^2(\tilde{M}; \mathbb{R})\) acts as an identification of \(H^2(M; \mathbb{R})\) with the first summand. The classes \([E_1], \ldots, [E_k]\) are all of Hodge type \((1, 1)\). The identification preserves the Hodge types.

Since, by our assumption, \(\Pi^*[\omega]_{J, 1} = \sum_{i=1}^{k} \frac{r_i}{\langle a_i \rangle} e_i \in \text{Kah}(\tilde{M}, \tilde{J})\), it can be represented by a Kähler form \(\tilde{\alpha}\) on \((\tilde{M}, \tilde{J})\).

Note that \(\Pi^*[\omega]_{J, 1} \in H^2(\tilde{M}; \mathbb{R})\) is of type \((1, 1)\). Hence, the class \(\Pi^*[\omega] - \Pi^*[\omega]_{J, 1} \in H^2(\tilde{M}; \mathbb{R})\) is of type \((2, 0) + (0, 2)\) and can be represented as \(\Pi^*b\) for a \((2, 0) + (0, 2)\)-class \(b \in H^2(M; \mathbb{R})\). Represent \(b\) by a closed real-valued form \(\beta\) on \(M\) of type \((2, 0) + (0, 2)\) with respect to \(J\). Then the class \(\Pi^*[\omega] - \Pi^*[\omega]_{J, 1}\) is represented by a closed real-valued form \(\Pi^*_J\beta\) on \(\tilde{M}\) of type \((2, 0) + (0, 2)\) with respect to \(\tilde{J}\).

Set \(\tilde{\omega} := \tilde{\alpha} + \Pi^*_J\beta\). The form \(\tilde{\omega}\) is symplectic and tames \(\tilde{J}\). The cohomology class of \(\tilde{\omega}\) can be written as

\[
[\tilde{\omega}] = [\tilde{\alpha}] + [\Pi^*_J\beta] = \Pi^*[\omega]_{J, 1} - \sum_{i=1}^{k} \frac{r_i}{\langle a_i \rangle} e_i + \Pi^*[\omega] - \Pi^*[\omega]_{J, 1} =
\]

\[
= \Pi^*[\omega] - \sum_{i=1}^{k} \frac{r_i}{\langle a_i \rangle} e_i.
\]

Now we can apply Proposition 3.9 with \(J\) instead of \(I\), which yields the needed claim. \(\blacksquare\)
Proposition 3.11:
With the notation as in Proposition 3.9,
\[
\left\langle \left( \Pi^*[\omega] - \sum_{i=1}^{k} \frac{r_i}{\langle \tilde{a}_i \rangle} e_i \right)^n, [\tilde{M}] \right\rangle = \int_{M} \omega^n - \sum_{i=1}^{k} \frac{r_i^n}{\langle \tilde{a}_i \rangle^n} = \int_{M} \omega^n - \operatorname{Vol} \left( \bigcup_{i=1}^{k} \mathcal{E}_{\tilde{a}_i}(r_i) \right).
\]

Proof:
Note that for all \( i = 1, \ldots, k \) we have \( (\Pi^*[\omega]) \cup e_i = 0 \), as well as \( e_i^n = -\langle \tilde{a}_i \rangle^{n-1} \), if \( n \) is even, and \( e_i^n = \langle \tilde{a}_i \rangle^{n-1} \), if \( n \) is odd (see Example 3.3). Also note that \( e_i \cup e_j = 0 \) for all \( i \neq j \). Finally, recall that the symplectic volume of \( \mathcal{E}_{\tilde{a}}(r) \) equals \( r^{2n}/\langle \tilde{a} \rangle^n \). The claim follows directly from these observations.

4 Demailly-Paun theorem and the Kähler cone

Proposition 4.1:
Assume \((M, I)\) is a closed connected Campana simple complex manifold, \( \dim_{\mathbb{C}} M = n \). Let \((\tilde{M}, \tilde{I})\) be a weighted blow-up of \( M \) at \( k \) Campana-generic points \( x_1, \ldots, x_k \) with weights \( \tilde{a}_1, \ldots, \tilde{a}_k \) all of which are pairwise coprime. Define \( \Pi_I : M \to \tilde{M}, E_i := E_i(I) = \Pi_I^{-1}(x_i) \) and \( e_i \in H^2(M; \mathbb{Z}), i = 1, \ldots, k \), as above.

Assume that \( \alpha \in \operatorname{Kah}(M, I) \).

Then, given \( c_1, \ldots, c_k \in \mathbb{R} \), the following claims are equivalent:

(A) The cohomology class \( \tilde{\alpha} := \Pi^*\alpha - \sum_{i=1}^{k} c_i e_i \in H^2(\tilde{M}; \mathbb{R}) \) is Kähler.

(B) The conditions (B1) and (B2) below are satisfied:

(B1) All \( c_i \) are positive.

(B2) \( \langle \tilde{\alpha}^n, [\tilde{M}] \rangle > 0 \).

Recall that \( \tilde{M} \) is, in general, a non-smooth orbifold with isolated singularities. In the case when \( \tilde{M} \) is smooth (that is, in the case of usual, not weighted, blowups) a claim similar to Proposition 4.1 was proved in [EV] using the Demailly-Paun theorem that describes the Kähler cone of a closed Kähler manifold.
Theorem 4.2: (Demailly-Paun, [DP])

Let $X$ be a closed connected Kähler manifold. Let $K(X) \subset H^{1,1}(X; \mathbb{R})$ be the subset consisting of all $(1,1)$-classes $\zeta$ which satisfy $\langle \zeta^*, [Z] \rangle > 0$ for any homology class $[Z]$ realized by a complex subvariety $Z \subset X$ of complex dimension $s > 0$. Then the Kähler cone of $X$ is one of the connected components of $K(X)$.

Theorem 4.2 cannot be directly applied to orbifolds (its orbifold version seems to be true but its proof is not published, as far as we can ascertain) and therefore in order to prove Proposition 4.1 we use an indirect argument where Theorem 4.2 is applied not to $\tilde{M}$ but to its Kähler resolution constructed in part B of Theorem 3.6.

Proof of Proposition 4.1.

Proof of (A) $\Rightarrow$ (B).

The implication (A) $\Rightarrow$ (B2) is obvious. To prove (A) $\Rightarrow$ (B1) note that, since $\tilde{\alpha}$ is Kähler, for each $i = 1, \ldots, k$ we have

$$0 < \int_{E_i} \tilde{\alpha}^{n-1} = \int_{E_i} (-c_i e_i)^{n-1},$$

and since the restriction of $-e_i$ to $E_i$ is a positive multiple of the cohomology class of the restriction of the Fubini-Study form $\Omega_{\tilde{\alpha}}$ to the exceptional divisor $E_i$ and the integral of the exterior power of the latter form over $E_i$ is positive, we readily get that $c_i > 0$.

Proof of (B) $\Rightarrow$ (A).

Assume (B1) and (B2) are satisfied.

Note that $\tilde{M}$ is a closed complex orbifold with isolated singularities and, by part A of Proposition 3.9, the complex structure $\tilde{I}$ on $\tilde{M}$ is of Kähler type. Denote by $y_1, \ldots, y_m \in \tilde{M}$ the singular points of $\tilde{M}$ — each of them lies in some exceptional divisor $E_i$. For each $i = 1, \ldots, k$ denote by $S_i$ the set of $j$ such that $y_j \in E_i$.

Applying part B of Theorem 3.6 to $N := \tilde{M}$ we get a Kähler resolution $\pi : (\tilde{N}, \tilde{I}) \to (\tilde{M}, \tilde{I})$ of $N = M$ and the cohomology classes $b_1, \ldots, b_m \in H^2(\tilde{N}; \mathbb{R})$ corresponding to $y_1, \ldots, y_m$.

Consider the map $\tilde{\pi} := \Pi_I \circ \pi : (\tilde{N}, \tilde{I}) \to (M, I)$. It is a biholomorphism over $M \setminus \{x_1, \ldots, x_k\}$. 
Consider the following family of cohomology classes of $\hat{N}$:

$$\hat{\pi}^*[\omega] - \lambda \sum_{i=1}^{k} c_i \pi^* e_i + \delta(b_1 + \ldots + b_m) = [\beta_{\lambda,\delta}] \in H^2(\hat{N}; \mathbb{R}), \quad (4.1)$$

where $\lambda, \delta \geq 0$ and $\beta_{\lambda,\delta}$ is a smooth closed 2-form on $\hat{N}$. By part A of Proposition 3.9 $\Pi^*[\omega] - \lambda \sum c_i e_i \in \text{Kah}(\hat{M}, \hat{I})$ for any sufficiently small $\lambda > 0$ and therefore, by part B of Theorem 3.6 for any sufficiently small $\lambda, \delta > 0$ the class $[\beta_{\lambda,\delta}]$ is Kähler for $(\hat{N}, \hat{I})$ and the form $\beta_{\lambda,\delta}$ can be assumed to be Kähler.

Consider the cohomology class $\beta_{1,\delta} = \pi^* \alpha + \delta(b_1 + \ldots + b_m)$. We claim that $\beta_{1,\delta} \in \mathcal{K}(\hat{N})$ for sufficiently small $\delta > 0$.

Indeed, let $Z$, $\dim Z =: s > 0$, be a complex subvariety of $(\hat{N}, \hat{I})$.

There are 2 cases to consider: $Z \subseteq \hat{N}$ (Case I) and $Z = \hat{N}$ (Case II).

Since $x_i$ are Campana-generic, any connected proper complex subvariety $Z$, $\dim Z =: s > 0$, of $(\hat{N}, \hat{I})$ either does not intersect any of the sets $\hat{\pi}^{-1}(x_i)$ (Case Ia) or is contained in $\hat{\pi}^{-1}(x_i)$ for some $i$ (Case Ib).

**Case Ia:**

If $Z$ does not intersect any of the sets $\hat{\pi}^{-1}(x_i)$, then

$$\langle \beta_{1,\delta}^s \rangle = \langle \alpha^s, Z \rangle > 0$$

for all $\delta$, since $\alpha \in \text{Kah}(M, I)$. This finishes the verification of the claim in the case (Ia).

**Case Ib:**

Assume $Z \subset \hat{\pi}^{-1}(x_i)$ for some $i$. Note that

$$\hat{\pi}^{-1}(x_i) = \pi^{-1}(E_i) = \pi^{-1}(E_i \setminus \bigcup_{j \in S_i} \{y_j\}) \cup \bigcup_{j \in S_i} \pi^{-1}(y_j).$$

Also note that $\hat{\pi}^* \alpha$, all $e_j$, $j \neq i$, and all $b_j$ that correspond to $y_j \notin E_i$ vanish on $\hat{\pi}^{-1}(x_i)$. Hence

$$\langle \beta_{1,\delta}^s, [Z] \rangle = \left\langle \left( c_i \pi^*(-e_i) + \delta \sum_{j \in S_i} b_j \right)^s, Z \right\rangle. \quad (4.2)$$

The restriction of the class $-e_i$ to $E_i$ can be represented by the form $\Omega_{i|E_i}$, where the $\Omega_i$ is the Fubini-Study orbifold Kähler form on $\mathbb{C}P^n(1, a_i)$ (recall that $\hat{M}$ is obtained as a connected sum of $M$ with the weighted projective spaces $\mathbb{C}P^n(1, a_i)$, $i = 1, \ldots, k$). Accordingly, the restriction of the class $\pi^*(-e_i)$ to $\pi^{-1}(E_i)$ can be represented by the smooth form $\pi^*(\Omega_{i|E_i})$. The latter form is Kähler outside $\pi^{-1}(\bigcup_{j \in S_i} \{y_j\})$ and its restriction to $\pi^{-1}(y_j)$
is zero for any $y_j \in E_i$. Thus, for any $i$ we have $\pi^*(\Omega_i|_{E_i}) \geq 0$ on $\pi^{-1}(E_i)$, hence

$$\langle (-e_i)^s, [Z] \rangle \geq 0.$$ 

Recall that for any sufficiently small $\lambda, \delta > 0$ the form $\beta_{\lambda, \delta}$ is Kähler on $(\hat{N}, \hat{I})$. Therefore for any sufficiently small $\lambda, \delta > 0$ (independent of $Z$) the form $c_i\pi^*(\Omega_i|_{E_i}) + \beta_{\lambda, \delta}$ is positive on $\pi^{-1}(E_i)$ for any $i$ and therefore so is its restriction to the complex subvariety $Z \subset \pi^{-1}(E_i)$. Hence,

$$0 < \langle [c_i\pi^*(\Omega_i|_{E_i}) + \beta_{\lambda, \delta}]^s, [Z] \rangle.$$

On the other hand, in view of (4.1), the cohomology class of the restriction of $c_i\pi^*(\Omega_i|_{E_i}) + \beta_{\lambda, \delta}$ to $\pi^{-1}(E_i)$ can be written as

$$[c_i\pi^*(\Omega_i|_{E_i}) + \beta_{\lambda, \delta}] = (1 + \lambda)c_i\pi^*(-e_i) + \delta \sum_{j \in \mathcal{S}_i} b_j.$$

Thus, for any sufficiently small $\lambda, \delta > 0$ (independent of $Z$)

$$0 < \left\langle \left(1 + \lambda \right)c_i\pi^*(-e_i) + \delta \sum_{j \in \mathcal{S}_i} b_j \right]^s, [Z] \right\rangle.$$

Note that, by the properties of the classes $b_j$ given by part B of Theorem 3.6, $\pi^*(-e_i)b_j = 0$ for any $i, j$. Therefore for any sufficiently small $\lambda, \delta > 0$ (independent of $Z$)

$$0 < \left\langle \left(1 + \lambda \right)c_i\pi^*(-e_i) + \delta \sum_{j \in \mathcal{S}_i} b_j \right]^s, [Z] \right\rangle =$$

$$\left\langle \left(c_i\pi^*(-e_i) + \delta \sum_{j \in \mathcal{S}_i} b_j \right)^s, [Z] \right\rangle + \left(\sum_{j=1}^{s} \binom{s}{j} \lambda^j \left\langle (-e_i)^s, [Z] \right\rangle + \sum_{j=1}^{s} \binom{s}{j} \lambda^j \right\rangle.$$

Since $\lambda^j\langle (-e_i)^s, [Z] \rangle \geq 0$ for any $\lambda > 0$ and any $j \in \mathbb{N}$, we get, by (4.2), that for any sufficiently small $\delta > 0$ (independent of $Z$)

$$\langle \beta_{1, \delta}^s, [Z] \rangle = \left\langle \left(c_i\pi^*(-e_i) + \delta \sum_{j \in \mathcal{S}_i} b_j \right)^s, Z \right\rangle > 0.$$

This finishes the verification of the claim in the case (Ib).

Case II:

Assume $Z = \hat{N}$. Then, by (B2) and since $\pi$ is of degree 1,

$$\langle (\pi^*\alpha)^n, [\hat{N}] \rangle = \langle \alpha^n, [\hat{M}] \rangle > 0.$$
Therefore, since $\beta_{1,\delta} \to \pi^*\tilde{\alpha}$ as $\delta \to 0$, for any sufficiently small $\delta > 0$

$$\langle \beta_{1,\delta}^n, \hat{N} \rangle > 0.$$ 

This finishes the verification of the claim in the case (II).

Thus we have proved that $\beta_{1,\delta} \in \mathcal{K}(\hat{N})$ for a sufficiently small $\delta > 0$.

Let us now show that there exists a Kähler class in the connected component of $\mathcal{K}(\hat{N})$ containing $\beta_{1,\delta}$.

Indeed, similarly to Proposition 3.11 one gets that (B2) is equivalent to the condition

$$\sum_{i=1}^{k} c_i^n < \langle \alpha^n, [M] \rangle,$$

If this condition holds for $c_1, \ldots, c_k > 0$, it also holds for $\epsilon c_1, \ldots, \epsilon c_k$ for any $\epsilon \in (0,1]$. For any such $\epsilon$ the numbers $\epsilon c_1, \ldots, \epsilon c_k$ are still positive and therefore, by the argument above, for any sufficiently small $\delta > 0$ and any $\epsilon \in (0,1]$ the class $\hat{\pi}^*\alpha - \epsilon \sum_{i=1}^{k} c_i \pi^* e_i + \epsilon \delta (b_1 + \ldots + b_m)$ also lies in $\mathcal{K}(\hat{N})$.

But, as we have already mentioned above, for any sufficiently small $\epsilon > 0$ the class

$$\hat{\pi}^*\alpha - \epsilon \sum_{i=1}^{k} c_i \pi^* e_i + \epsilon \delta (b_1 + \ldots + b_m) = [\beta_{\epsilon,\delta}]$$

is Kähler. Thus, for any sufficiently small $\delta > 0$ the class $\beta_{1,\delta}$ lies in the same connected component of $\mathcal{K}(\hat{N})$ as a Kähler class $[\beta_{\epsilon,\delta}]$. Therefore, by Theorem 4.2 the class

$$\beta_{1,\delta} = \pi^*\tilde{\alpha} + \delta (b_1 + \ldots + b_m).$$

is Kähler on $(\hat{N}, \hat{I})$. Since $\pi$ is a biholomorphism outside $\pi^{-1}\{y_1, \ldots, y_m\}$ and $\pi_* b_j = 0$ for any $j$, by part A of Theorem 3.6 $\pi_* \beta_{1,\delta} = \tilde{\alpha}$ is a Kähler class on $(\hat{M}, \hat{I})$, which finishes the proof. ■

5 Proof of Theorem 2.4

Let us say that a closed ellipsoid is simple if it is of the form $E_{\bar{a}}(r)$ for a pairwise coprime vector $\bar{a}$ and some $r > 0$.

Any vector with positive coordinates can be approximated by vectors proportional to pairwise coprime vectors (see Proposition 6.1 below) and therefore any open neighborhood of any closed ellipsoid contains a simple closed ellipsoid. Thus it suffices to prove that if there is a collection of disjoint simple closed ellipsoids $E_{\bar{a}_i}(r_i)$, $i = 1, \ldots, k$, whose total volume is
less than $\text{Vol} M$, then there exists a symplectic embedding of their union into $(M, \omega)$.

Consider a disjoint union $\bigsqcup_{i=1}^{k} E_{\bar{a}_i}(r_i)$ whose total symplectic volume is less than the symplectic volume of $M$, that is,

$$
\sum_{i=1}^{k} r_i^n / \langle \bar{a}_i \rangle^n < \text{Vol}(M) = \langle [\omega]^n, [M] \rangle.
$$

(5.1)

We need to show that it admits a symplectic embedding into $(M, \omega)$.

It follows from the Kodaira-Spencer stability theorem [KoSp] (see [EV, Theorem 5.6] for more details) and the hypothesis of the theorem that there exists a Campana simple complex structure $J$ on $M$ sufficiently close to $I$ with the following properties:

1. $[\omega]^{1,1}_J \in \text{Kah}(M, J)$ (this follows from the Kodaira-Spencer stability theorem – see [EV, Theorem 5.6]),

2. $$
\sum_{i=1}^{k} r_i^n / \langle \bar{a}_i \rangle^n < \langle ([\omega]^{1,1}_J)^n, [M] \rangle.
$$

(5.2)

(This is possible by (5.1)).

3. $J$ is tamed by $\omega$ (this is possible because $I$ is tamed by $\omega$).

Choose $k$ distinct Campana-generic points $x_1, \ldots, x_k \in (M, J)$, and consider the weighted blow-up $(\tilde{M}, \tilde{J})$ of $(M, J)$ at those points with the weights $\tilde{a}_1, \ldots, \tilde{a}_k$. By Proposition 4.1 applied to the Kähler class $[\omega]^{1,1}_J$, the cohomology class $\Pi^*[\omega]^{1,1}_J - \sum_{i=1}^{k} r_i / \langle \bar{a}_i \rangle e_i$ is Kähler with respect to $\tilde{J}$ (note that, by Proposition 3.11, the condition (B2) in Proposition 4.1 is equivalent to (5.2)). Therefore, by Proposition 3.10 $(M, \omega)$ admits a symplectic embedding of $\bigsqcup_{i=1}^{k} E_{\bar{a}_i}(r_i)$. ■

6 Appendix: Approximation by pairwise coprime vectors

In this appendix we will prove that any vector with positive coordinates can be approximated by vectors proportional to pairwise coprime vectors. The result is probably known but we have been unable to find it in the literature. In fact, below we present a proof (due to Uri Shapira) of a stronger claim.
Proposition 6.1:
Any vector with positive coordinates in $\mathbb{R}^n$ can be approximated by vectors proportional to vectors whose coordinates are pairwise different primes.

Proof (Uri Shapira):
Consider a vector $(x_1, \ldots, x_n) \in \mathbb{R}^n$ with positive coordinates. Take an arbitrary $\varepsilon > 0$ and let us find a vector at the $l_\infty$-distance $\leq \varepsilon$ from $(x_1, \ldots, x_n)$ which is proportional to a vector whose coordinates are pairwise different primes.

Choose a vector $(y_1, \ldots, y_n)$ with positive pairwise different rational coordinates so that $\max_i |y_i - x_i| \leq \varepsilon/2$. Let

$$c := \min_{i,j,i \neq j} |y_i - y_j| > 0,$$

$$C := \max_i y_i > 0$$

According to a theorem of Hoheisel [H0], there exists $0 < \vartheta < 1$ and $l_0 \in \mathbb{N}$ so that for any $l \geq l_0$ the interval $(l, l + l^\vartheta)$ contains a prime. Choose $l_1 \geq l_0$ so that

$$l^\vartheta < l \min \left\{ \frac{\varepsilon}{2C}, c \right\}$$

for any $l \geq l_1$.

Choose a sufficiently large $N \in \mathbb{N}$ so that $Ny_1, \ldots, Ny_n$ are integers greater than $l_1$. Then the intervals $(Ny_i, Ny_i + (Ny_i)^\vartheta)$, $i = 1, \ldots, n$, are pairwise disjoint and each of them contains a prime: $p_i \in (Ny_i, Ny_i + (Ny_i)^\vartheta)$. Since the intervals are disjoint, the primes $p_1, \ldots, p_n$ are pairwise different, and since $p_i \in (Ny_i, Ny_i + (Ny_i)^\vartheta)$, we get

$$\max_i \left| p_i/N - y_i \right| \leq \max_i \frac{(Ny_i)^\vartheta}{N} \leq \max_i \frac{(\varepsilon/2C)Ny_i}{N} \leq \varepsilon/2.$$

Thus, the vector $(p_1/N, \ldots, p_n/N)$ is proportional to the vector $(p_1, \ldots, p_n)$ whose coordinates are pairwise different primes and

$$\max_i |p_i/N - x_i| \leq \max_i |p_i/N - y_i| + \max_i |y_i - x_i| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

as required. \(\blacksquare\)

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References

[Bai] Baily, W.L., *The decomposition theorem for V-manifolds*, Amer. J. Math. **78** (1956), 862-888.

[BDD] Barletta, E., Dragomir, S., Duggal, K.L., *Foliations in Cauchy-Riemann geometry*, AMS, Providence, RI, 2007.

[Bes] Besse, A., *Einstein Manifolds*, Springer-Verlag, New York, 1987.

[BM] Bierstone, E., Milman, P., *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Invent. Math. **128** (1997), 207-302.

[Bo] Bogomolov, F. A., *On the decomposition of Kähler manifolds with trivial canonical class*, Math. USSR-Sb. **22** (1974), 580-583.

[Cam] Campana, F., *Isotrivialité de certaines familles Kählériennes de variétés non projectives*, Math. Z. **252** (2006), 147-156.

[CDV] Campana, F., Demailly, J.-P., Verbitsky, M., *Compact Kähler 3-manifolds without non-trivial subvarieties*, Algebr. Geom. **1** (2014), 131-139.

[Dem] Demailly, J.-P., *Complex analytic and differential geometry*, 2012. Available at [https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf](https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf).

[DP] Demailly, J.-P., Paun, M., *Numerical characterization of the Kähler cone of a compact Kähler manifold*, Ann. of Math. **159** (2004), 1247-1274.

[EV] Entov, M., Verbitsky, M., *Unobstructed symplectic packing for tori and hyper-Kähler manifolds*, J. Topol. Anal. **8** (2016), 589-626.

[GH] Griffiths, Ph., Harris, J., *Principles of algebraic geometry*. John Wiley & Son], New York, 1978.

[Go] Godinho, L., *Blowing up symplectic orbifolds*, Ann. Global Anal. Geom. **20** (2001), 117-162.

[Gro] Gromov, M., *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), 307-347.

[Har] R. Hartshorne, *Algebraic Geometry*, GTM **52**, Springer, 1977.

[Hir] Hironaka, H., *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II*, Ann. of Math. **79** (1964), 109-203, 205-326.

[Ho] Hoheisel, G., *Primzahlprobleme in der Analysis*, Sitzungsber. Preuss. Akad. Wiss., Phys.-Math. Kl. (1930), 580-588.

[Kaw] Kawasaki, T., *Cohomology of twisted projective spaces and lens complexes*, Math. Ann. **206** (1973), 243-248.

[KoS] Kodaira, K., Spencer, D.C., *On deformations of complex analytic structures. III. Stability theorems for complex structures*, Ann. of Math. **71** (1960), 43-76.

[KSC] Kollár, J., Smith, K.E., Corti, A., *Rational and nearly rational varieties*. Cambridge University Press, Cambridge, 2004.

[LMcDS] Latschev, J., McDuff, D., Schlenk, F., *The Gromov width of 4-dimensional tori*, Geom. and Topol. **17** (2013), 2813-2853.

[LG] Lelong, P., Gruman, L., *Entire functions of several complex variables*. Springer-Verlag, Berlin, 1986.
M. Entov, M. Verbitsky

Symplectic packing by ellipsoids

[Le] Lerman, E., Symplectic cuts, Math. Res. Lett. 2 (1995), 247-258.

[McD] McDuff, D., Symplectic embeddings of 4-dimensional ellipsoids, J. Topol. 2 (2009), 122, and 8 (2015), 11191122 (corrigendum).

[McDP] McDuff, D., Polterovich, L., Symplectic packings and algebraic geometry. With an appendix by Yael Karshon, Invent. Math. 115 (1994), 405-434.

[Mi] Miyaoka, Y., Extension theorems for Kähler metrics, Proc. Japan Acad. 50 (1974), 407-410.

[Mos] Moser, J., On the volume elements on a manifold, Trans. AMS 120 (1965), 288-294.

[NP] Niederkrüger, K., Pasquotto, F., Resolution of symplectic cyclic orbifold singularities, J. Symplectic Geom. 7 (2009), 337-355.

[Sa] Satake, I., On a generalization of the notion of manifold, Proc. Nat. Acad. Sci. USA 42 (1956), 359-363.

[Sch] Schlenk, F., Symplectic embedding problems, old and new, preprint, to appear in the Bulletin of the AMS, 2017.

[Thu] Thurston, W., The Geometry and Topology of Three-Manifolds, electronic version 1.1, MSRI, 2002. Available at http://library.msri.org/books/gt3m/

[Tra] Tramer, H.J., The cohomology ring of pseudo-projective spaces, PhD Thesis, Johns Hopkins University, 1965.

[Vil1] Villamayor, O., Constructiveness of Hironaka’s resolution, Ann. Sci. Ecole Norm. Sup. 22 (1989), 1-32.

[Vil2] Villamayor, O., Patching local uniformizations, Ann. Sci. Ecole Norm. Sup. 25 (1992), 629-677.

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