ANALYSIS ON A DIFFUSIVE SEI EPIDEMIC MODEL WITH/WITHOUT IMMIGRATION OF INFECTED HOSTS

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ABSTRACT. In this paper, we study a reaction-diffusion SEI epidemic model with/without immigration of infected hosts. Our results show that if there is no immigration for the infected (exposed) individuals, the model admits a threshold behaviour in terms of the basic reproduction number, and if the system includes the immigration, the disease always persists. In each case, we explore the global attractivity of the equilibrium via Lyapunov functions in the case of spatially homogeneous environment, and investigate the asymptotic behavior of the endemic equilibrium (when it exists) with respect to the small migration rate of the susceptible, exposed or infected population in the case of spatially heterogeneous environment. Our results suggest that the strategy of controlling the migration rate of population can not eradicate the disease, and the disease transmission risk will be underestimated if the immigration of infected hosts is ignored.

1. Introduction. The COVID-19 pandemic has swept the globe, causing huge losses of human lives and economy. It is a disease caused by a new virus and needs a model taking into account its known specific characteristics. In particular, the COVID-19 pandemic has a long incubation period and some of its victims move from one area to another, making the new area infectious. This clearly shows that the importation of infection from one region to another can be unstoppable and mathematical models to be proposed to model its transmission should take into account the migration of infected individuals.
account the immigration of infected hosts. Along this research direction, Mccluskey [28] proposed the following SEI ODE model of infectious disease with immigration:

\[
\begin{align*}
\dot{S} &= \Lambda - \beta SI - \mu S, & t > 0, \\
\dot{E} &= W_E + (1 - q)\beta SI - (\mu + k)E, & t > 0, \\
\dot{I} &= W_I + q\beta SI + kE - (\mu + d)I, & t > 0,
\end{align*}
\] (1)

where \(S, E\) and \(I\) represent the number of individuals who are susceptible, exposed and infected, respectively. The parameters \(q \in [0, 1]\), \(\Lambda, \mu, \beta, k, d\) are positive constants and the constants \(W_E, W_I \geq 0\). Furthermore, \(\Lambda\) is the birth rate, \(\mu\) is the death rate, \(\beta\) is the transmission rate, \(k\) is the rate with which the exposed population moves into the infective class, \(\mu + d\) is the removal rate (including the mortality rate), \(q\) is the rate with which the susceptible population move into the infective class and \(1 - q\) is the rate with which the susceptible population move into the exposed class. In addition, \(W_E\) represents the importation of an alien exposed individual and \(W_I\) represents the importation of an alien infective individual.

In the face of infectious diseases, many mathematical models have been proposed, especially the susceptible-infected-susceptible (SIS) epidemic reaction-diffusion models; refer to [3, 6, 8, 10, 15, 18, 19, 20, 25, 21, 23, 24, 33, 36, 49, 53] without advection term, and [4, 5, 14] with advection term as well as [35, 37, 38, 50] in the time-periodic environment. In the more commonly studied situation, we usually ignore the immigration of infected individuals. These disease models exhibit a threshold dynamics that depends on the basic reproduction number \(R_0\): the disease-free equilibrium is locally stable and the disease dies out if \(R_0 < 1\) while disease-free equilibrium is unstable and the disease persists if \(R_0 > 1\). To take into account the inhomogeneous distribution of the population in different spatial locations within a fixed bounded domain \(\Omega \subset \mathbb{R}^n (n \geq 1)\) at any given time (see for example, [7, 16, 17, 22, 26, 32, 42, 43, 44, 46, 51, 55, 56]), and the natural tendency of each class of population to diffuse to areas of smaller population concentration, and ignoring the immigration of infected individuals (that is, \((W_E, W_I) = (0, 0)\) in (1)), then the system (1) can be extended to the following PDE system of reaction-diffusion type:

\[
\begin{align*}
\frac{\partial S}{\partial t} - d_S \Delta S &= \Lambda(x) - \beta(x)SI - \mu(x)S, & x \in \Omega, t > 0, \\
\frac{\partial E}{\partial t} - d_E \Delta E &= (1 - q)\beta(x)SI - (\mu(x) + k(x))E, & x \in \Omega, t > 0, \\
\frac{\partial I}{\partial t} - d_I \Delta I &= q\beta(x)SI + k(x)E - (\mu(x) + d(x))I, & x \in \Omega, t > 0, \\
\frac{\partial S}{\partial \nu} = \frac{\partial E}{\partial \nu} = \frac{\partial I}{\partial \nu} &= 0, & x \in \partial \Omega, t > 0, \\
S(x, 0) &= S_0(x), E(x, 0) = E_0(x), I(x, 0) = I_0(x), & x \in \Omega,
\end{align*}
\] (2)

where \(\Omega\) is a bounded smooth domain of \(\mathbb{R}^n\), \(I(x, t), E(x, t), S(x, t)\) are the densities of infected, exposed and susceptible individuals at location \(x\) and time \(t\), respectively. The positive constants \(d_S, d_E\) and \(d_I\) are the movement (or diffusive) rates of susceptible, exposed and infected individuals, respectively. In the current situation, \(\Lambda(x), \mu(x), \beta(x), k(x), d(x)\) are positive Hölder continuous functions and have the same implications as in (1). Throughout the paper, we assume that the initial
conditions satisfy $S(x, 0), E(x, 0), I(x, 0)$ are continuous functions, and

$$S(x, 0), E(x, 0), I(x, 0) \geq 0 \quad \text{for} \quad x \in \overline{\Omega} \quad \text{and} \quad \int_{\Omega} I(x, 0) dx > 0. \quad (3)$$

Clearly, the steady state problem corresponding to (2) is governed by the following elliptic system:

$$\begin{cases}
-d_S \Delta S = \Lambda(x) - \beta(x)SI - \mu(x)S, & x \in \Omega, \\
-d_E \Delta E = (1 - q)\beta(x)SI - [\mu(x) + k(x)]E, & x \in \Omega, \\
-d_I \Delta I = q\beta(x)SI + k(x)E - [\mu(x) + d(x)]I, & x \in \Omega, \\
\frac{\partial S}{\partial \nu} = \frac{\partial E}{\partial \nu} = \frac{\partial I}{\partial \nu} = 0, & x \in \partial \Omega.
\end{cases} \quad (4)$$

In system (4), $S(x), E(x)$ and $I(x)$ denote the densities of susceptible, exposed and infected populations, respectively, at location $x$. If $(S, E, I)$ satisfies (4) and $S(x), E(x) \geq 0, I(x) \not\equiv 0$ on $\overline{\Omega}$, we call the solution $(S, E, I) \in C^2(\overline{\Omega}) \times C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$ to (4) as the endemic equilibrium (EE). By the strong maximum principle and Hopf lemma for elliptic equations, we can obtain that $S(x), E(x), I(x) > 0$ on $\overline{\Omega}$.

However, as mentioned before, the migration of infected individuals can not be ignored in certain diseases, such as the COVID-19 pandemic. To incorporate the migration of exposed and infected individuals (i.e., $(W_E, W_I) \neq (0, 0)$) and spatial random diffusion, the system (1) is extended to the following reaction-diffusion system:

$$\begin{cases}
\frac{\partial S}{\partial t} - d_S \Delta S = \Lambda(x) - \beta(x)SI - \mu(x)S, & x \in \Omega, t > 0, \\
\frac{\partial E}{\partial t} - d_E \Delta E = W_E + (1 - q)\beta(x)SI - [\mu(x) + k(x)]E, & x \in \Omega, t > 0, \\
\frac{\partial I}{\partial t} - d_I \Delta I = W_I + q\beta(x)SI + k(x)E - [\mu(x) + d(x)]I, & x \in \Omega, t > 0, \\
\frac{\partial S}{\partial \nu} = \frac{\partial E}{\partial \nu} = \frac{\partial I}{\partial \nu} = 0, & x \in \partial \Omega, t > 0, \\
S(x, 0) = S_0(x), E(x, 0) = E_0(x), I(x, 0) = I_0(x), & x \in \Omega.
\end{cases} \quad (5)$$

We also assume that the initial conditions (3) hold.

From our theory for parabolic equation and our assumption ([2, 40]), system (2) or (5) admits a unique classical solution $S, E, I \in C^{2,1}(\overline{\Omega} \times (0, \infty))$. Obviously, it follows from the strong maximum principle for parabolic equation that $S(x, t), E(x, t) > 0$ and $I(x, t) > 0$ for all $x \in \overline{\Omega}$ and $t > 0$. In addition, the steady state solution (i.e., EE) satisfies system (79) below. In fact, once $(W_E, W_I) \neq (0, 0)$, (5) always admits at least one EE. If immigration is added into the models, it would be interesting to know whether the approach of Lyapunov functions works or not to derive the global attractivity of equilibrium. In [28], by using several examples, the author proved a general result via Lyapunov functions that apply in models with infected host migration; other related studies can be found, for example, in [29, 31, 39, 42, 41, 58].

In this paper, we will investigate the qualitative properties of systems (2) and (5). On the one hand, one will see that the technique of Lyapunov functions still works to establish the global attractivity of the equilibrium even for the reaction-diffusion systems (2) and (5). On the other hand, once the disease is persistent in space,
by controlling one of the migration rates $d_S$, $d_E$, $d_I$ to be sufficiently small, we will determine the spatial profile of population distribution of the disease. The obtained theoretical findings may provide the decision-makers with useful information on the disease control. In particular, we will compare our results for systems (2) and (5) and reveal the effect of immigration of infected hosts on the disease transmission. One may refer to the last section for more discussions.

This paper is organized as follows. In section 2, we introduce and study some properties of the basic reproduction number $R_0$ associated with system (2). In Section 3, we consider the disease model (2) without immigration, and apply the global Lyapunov functions to obtain attractivity of the disease-free equilibrium and EE, and determine the asymptotic profiles of the EE with small $d_S$, $d_E$, $d_I$. In Section 4, we deal with system (5) with immigration, and investigate the attractivity and asymptotic profiles of the EE. Section 5 ends the paper with some conclusions.

2. The basic reproduction number $R_0$ associated with (2). Basic reproduction number is one of the most important concepts in the study of the spread of infectious diseases. In this section, we will introduce and investigate the properties of the basic reproduction number $R_0$ associated with (2). We have to point out that model (5) does not have the basic reproduction number.

To our aim, let us first consider the elliptic problem

$$-d_S \Delta S = \Lambda(x) - \mu(x) S, \quad x \in \Omega; \quad \frac{\partial S}{\partial \nu} = 0, \quad x \in \partial \Omega.$$

Obviously, it admits a unique positive solution $\hat{S} > 0$. Then we define $X_0 = (\hat{S}, 0, 0)$ as a steady state of (2), which is usually called the disease-free equilibrium (DFE for short).

The linearization of system (2) at $X_0$ is given by

$$\begin{align*}
\frac{\partial \hat{S}}{\partial t} &= d_S \Delta \hat{S} - \mu(x) \hat{S} - \beta(x) \hat{S} \hat{T}, \quad x \in \Omega, t > 0, \\
\frac{\partial \hat{E}}{\partial t} &= d_E \Delta \hat{E} + (1 - q) \beta(x) \hat{S} \hat{T} - [\mu(x) + k(x)] \hat{E}, \quad x \in \Omega, t > 0, \\
\frac{\partial \hat{T}}{\partial t} &= d_I \Delta \hat{T} + [q \beta(x) \hat{S} (x) - (\mu(x) + d(x))] \hat{T} + k(x) \hat{E}, \quad x \in \Omega, t > 0, \\
\frac{\partial \hat{S}}{\partial \nu} - \frac{\partial \hat{E}}{\partial \nu} - \frac{\partial \hat{T}}{\partial \nu} &= 0, \quad x \in \partial \Omega, t > 0.
\end{align*}$$

Note that the infected compartments are $E$ and $I$ in system (2). Besides, $L$, $F(x)$ and $V(x)$ in [48] can be defined as $L = \text{diag}(-d_E \Delta, -d_I \Delta)$ with

$$F(x) = \begin{pmatrix} 0 & (1 - q) \beta(x) \hat{S}(x) \\ 0 & q \beta(x) \hat{S}(x) \end{pmatrix}, \quad V(x) = \begin{pmatrix} \mu(x) + k(x) & 0 \\ -k(x) & \mu(x) + d(x) \end{pmatrix}.$$

By adopting the theory developed in [45, 48], we can define the basic reproduction number of system (2), denoted by $R_0$. Moreover, we have the following conclusion.

**Lemma 2.1.** The linear eigenvalue problem

$$\begin{align*}
-d_E \Delta \varphi_E + [\mu(x) + k(x)] \varphi_E = g(1 - q) \beta(x) \hat{S}(x) \varphi_I, & \quad x \in \Omega, \\
-d_I \Delta \varphi_I + [\mu(x) + d(x)] \varphi_I - k(x) \varphi_E = g q \beta(x) \hat{S}(x) \varphi_I, & \quad x \in \Omega, \\
\frac{\partial \varphi_E}{\partial \nu} - \frac{\partial \varphi_I}{\partial \nu} &= 0, & \quad x \in \partial \Omega
\end{align*}$$


admits a unique positive eigenvalue, denoted by $\varrho_0$, with a positive eigenfunction. Moreover, the basic reproduction number $R_0$ of system (2) satisfies

$$R_0 = \frac{1}{\varrho_0}. \quad (8)$$

Proof. By Theorem 5.1 in [30] (or [12]), (7) has a positive eigenvalue with a positive eigenfunction. To prove the uniqueness, we assume that there exists a positive eigenvalue $\varrho_1$ with positive eigenfunction $\varphi_1 = (\varphi_{E,1}, \varphi_{I,1})^T$ and positive eigenvalue $\varrho_2$ with positive eigenfunction $\tilde{\varphi}_2 = (\tilde{\varphi}_{E,2}, \tilde{\varphi}_{I,2})^T$ such that

$$L\varphi_1 + V\varphi_1 = \varrho_1 F\varphi_1 \quad \text{in } \Omega, \quad \frac{\partial \varphi_1}{\partial \nu} |_{\partial \Omega} = 0 \quad (9)$$

and

$$L\tilde{\varphi}_2 + V^T\tilde{\varphi}_2 = \varrho_2 F^T\tilde{\varphi}_2 \quad \text{in } \Omega, \quad \frac{\partial \tilde{\varphi}_2}{\partial \nu} |_{\partial \Omega} = 0. \quad (10)$$

We now multiply the equation in (9) by $(\tilde{\varphi}_2)^T$ and the equation in (10) by $\varphi_1^T$, subtract the two resulting equations, and integrate by parts over $\Omega$ to give

$$(\varrho_1 - \varrho_2) \int_{\Omega} \left[ (1 - q)\beta(x)\tilde{S}\varphi_{I,1}\tilde{\varphi}_{E,2} + q\beta(x)\tilde{S}\varphi_{I,1}\tilde{\varphi}_{I,2} \right] dx = 0.$$ 

Since $q, 1 - q, \beta(x), \tilde{S}, \varphi_{I,1}, \tilde{\varphi}_{I,2}$ and $\tilde{\varphi}_{E,2}$ are positive, we obtain $\varrho_1 = \varrho_2$. This establishes the uniqueness. In view of Theorem 3.2 in [48] and the uniqueness of positive eigenvalue with a positive eigenfunction for (7), we easily obtain (8). \qed

Next we consider the eigenvalue problem

$$\begin{cases} 
- \Delta \tilde{\varphi}_E - (1 - q)\beta(x)\tilde{S}(x)\varphi_I + [\mu(x) + k(x)]\varphi_E = \lambda \varphi_E, \quad x \in \Omega, \\
- \Delta \tilde{\varphi}_I + [\mu(x) + d(x) - q\beta(x)\tilde{S}(x)]\varphi_I - k(x)\varphi_E = \lambda \varphi_I, \quad x \in \Omega, \\
\frac{\partial \tilde{\varphi}_E}{\partial \nu} = \frac{\partial \varphi_E}{\partial \nu} = 0, \quad \frac{\partial \varphi_I}{\partial \nu} = \frac{\partial \tilde{\varphi}_I}{\partial \nu} = 0, \quad x \in \partial \Omega. 
\end{cases} \quad (11)$$

By the Krein-Rutman theorem ([13]), the eigenvalue problem (11) has a unique principal eigenvalue $\lambda_1$, that is, a real and simple eigenvalue with positive eigenfunction $\varphi_E, \tilde{\varphi}_I$, and it is strictly less than the real parts of all other eigenvalues.

**Proposition 1.** The following relationship holds:

$$\text{sign}(1 - R_0) = \text{sign}(\lambda_1). \quad (12)$$

Proof. Note that $\lambda_1$ is also the principal eigenvalue of the adjoint problem of (11), i.e.,

$$\begin{cases} 
- \Delta \tilde{\varphi}_E - [\mu(x) + k(x)]\varphi_I + \lambda \varphi_E, \quad x \in \Omega, \\
- \Delta \tilde{\varphi}_I + [\mu(x) + d(x) - q\beta(x)\tilde{S}(x)]\varphi_I = \lambda \varphi_I, \quad x \in \Omega, \\
\frac{\partial \tilde{\varphi}_E}{\partial \nu} = \frac{\partial \varphi_I}{\partial \nu} = 0, \quad x \in \partial \Omega. 
\end{cases} \quad (13)$$

Multiplying the first equation in (7) by $\tilde{\varphi}_E$ and the first equation in (13) by $\varphi_E$ and subtracting the two resulting equations, we then integrate by parts to deduce

$$\lambda_1 \int_{\Omega} \tilde{\varphi}_E \varphi_E dx = \int_{\Omega} \left[ \frac{1}{R_0} (1 - q)\beta(x)\tilde{S}(x)\varphi_I - k \varphi_I \varphi_E \right] dx. \quad (14)$$
Now, multiplying the second equation in (7) by $\dot{\varphi}_I$ and multiplying the second equation in (13) by $\varphi_I$, subtracting the two resulting equations and integrating by parts, we have
\[
\lambda_1 \int_{\Omega} (\frac{1}{R_0} - 1)q\beta \hat{S}(x)\dot{\varphi}_I + k\dot{\varphi}_I \varphi_E - (1 - q)\beta(x)\hat{S}(x)\dot{\varphi}_E \varphi_I dx. \tag{15}
\]
Adding two equations (14) and (15) yields
\[
\lambda_1 \int_{\Omega} (\hat{\varphi}_E \varphi_E + \dot{\varphi}_I \varphi_I) dx = \frac{1 - R_0}{R_0} \int_{\Omega} [(1 - q)\beta(x)\hat{S}(x)\dot{\varphi}_E + q\beta \hat{S}(x)\dot{\varphi}_I] dx. \tag{16}
\]
Since $q \in [0,1)$, $\hat{\varphi}_E$, $\varphi_E$, $\dot{\varphi}_I$, $\varphi_I$, $\beta$, $\hat{S}$ are positive, we have $\text{sign}(1 - R_0) = \text{sign}(\lambda_1)$. \hfill \Box

By Lemma 2.1, $\frac{1}{\bar{R}_0}$ is the unique principal eigenvalue of (7). Thus we have
\[
\begin{align*}
-d_E \Delta \varphi_E + [\mu(x) + k(x)]\varphi_E &= \frac{1}{R_0} (1 - q)\beta(x)\hat{S}(x)\varphi_I, \quad x \in \Omega, \\
-d_I \Delta \varphi_I + [\mu(x) + d(x)]\varphi_I - k(x)\varphi_E &= \frac{1}{R_0} q\beta(x)\hat{S}(x)\varphi_I, \quad x \in \Omega, \\
\frac{\partial \varphi_E}{\partial \nu} &= \frac{\partial \varphi_I}{\partial \nu} = 0, \quad x \in \partial \Omega.
\end{align*}
\tag{17}
\]
Moreover, $\frac{1}{\bar{R}_0}$ is the unique principal eigenvalue of the adjoint problem of (7), i.e.,
\[
\begin{align*}
-d_E \Delta \hat{\varphi}_E + [\mu(x) + k(x)]\hat{\varphi}_E - k(x)\hat{\varphi}_I &= 0, \quad x \in \Omega, \\
-d_I \Delta \hat{\varphi}_I + [\mu(x) + d(x)]\hat{\varphi}_I - q\beta(x)\hat{\varphi}_E &= \frac{1}{R_0} [(1 - q)\beta(x)\hat{S}(x)\hat{\varphi}_E + q\beta(x)\hat{S}(x)\hat{\varphi}_I], \quad x \in \Omega, \\
\frac{\partial \hat{\varphi}_E}{\partial \nu} &= \frac{\partial \hat{\varphi}_I}{\partial \nu} = 0, \quad x \in \partial \Omega,
\end{align*}
\tag{18}
\]
where $(\hat{\varphi}_E, \hat{\varphi}_I)$ is an eigenfunction corresponding to the unique principal eigenvalue of the adjoint problem of (7). Next, we will investigate the asymptotic properties of $R_0$ with respect to $d_S, d_E, d_I$.

**Proposition 2.** (i) Fix $d_E > 0, d_I > 0$. Then $R_0 \to \bar{R}_0$ as $d_S \to 0$, where $\bar{R}_0$ is the unique principal eigenvalue of the problem
\[
\begin{align*}
-d_E \Delta \varphi_E + [\mu(x) + k(x)]\varphi_E &= \frac{1}{R_0} (1 - q)\beta(x)\frac{\Lambda(x)}{\mu(x)} \varphi_I, \quad x \in \Omega, \\
-d_I \Delta \varphi_I + [\mu(x) + d(x)]\varphi_I - k(x)\varphi_E &= \frac{1}{R_0} q\beta(x)\frac{\Lambda(x)}{\mu(x)} \varphi_I, \quad x \in \Omega, \\
\frac{\partial \varphi_E}{\partial \nu} &= \frac{\partial \varphi_I}{\partial \nu} = 0, \quad x \in \partial \Omega.
\end{align*}
\tag{19}
\]

(ii) Fix $d_S > 0, d_I > 0$. Then $R_0 \to \frac{1}{\varphi_1}$ as $d_E \to 0$, where $\varphi_1$ is the principal eigenvalue of the problem:
\[
-d_I \Delta \varphi_I + [\mu(x) + d(x)]\varphi_I = \varphi \left(\frac{k(x) + q\mu(x)}{\mu(x) + k(x)}\frac{\beta(x)\hat{S}(x)}{\mu(x)} \varphi_I \right) \text{ in } \Omega, \quad \frac{\partial \varphi_I}{\partial \nu} \big|_{\partial \Omega} = 0. \tag{19}
\]

(iii) Assume $\Lambda, \mu, d, \beta$ are positive constants. Fix $d_S > 0, d_E > 0$. Then $R_0 \to \frac{1}{\varphi_2}$ as $d_I \to 0$, where $\varphi_2$ is the principal eigenvalue of the problem:
\[
-d_E \Delta \varphi_E + (\mu + k)\varphi_E = \varphi \left(\frac{(1-q)k \beta \Lambda}{\mu + d - q \beta \Lambda} \varphi_E \right) \text{ in } \Omega, \quad \frac{\partial \varphi_E}{\partial \nu} \big|_{\partial \Omega} = 0. \tag{20}
\]
Proof. We first consider the case $d_S \to 0$; indeed (i) can be easily proved from the continuous dependence of the principal eigenvalue on the weight function and the fact of $\hat{S}(x) \to \frac{\partial u}{\partial x}$ uniformly on $\Omega$ as $d_S \to 0$.

Next we consider the case $d_E \to 0$. Give any $\epsilon \in (0, 1)$, since $A = \{u \in C^2(\Omega) | \frac{\partial u}{\partial \nu} = 0\}$ is dense in $C(\Omega)$, we can choose $\bar{\beta}_1(x), \bar{\beta}_2(x) \in A$ such that
\[
\frac{\beta(x)}{1+\epsilon} > \bar{\beta}_2(x) > \beta(x) > \bar{\beta}_1(x) > \frac{\beta(x)}{1-\epsilon}.
\]

Set
\[
(\hat{\varphi}_E, \hat{\varphi}_I) = \left(\frac{q_1(1-q)\bar{\beta}_1\hat{S}\varphi_I}{\mu + k}, \varphi_I\right), \quad (\check{\varphi}_E, \check{\varphi}_I) = \left(\frac{q_1(1-q)\bar{\beta}_2\hat{S}\varphi_I}{\mu + k}, \varphi_I\right).
\]

For any $\epsilon \in (0, 1)$, there exists $\delta$ such that $0 < d_E < \delta$,
\[
- d_E \Delta \hat{\varphi}_E + (\mu + k)(1 - \frac{\beta}{\bar{\beta}_1(1+\epsilon)})\hat{\varphi}_E \geq 0, \ x \in \Omega; \quad \frac{\partial \hat{\varphi}_E}{\partial \nu} = 0, \ x \in \partial \Omega
\]
and
\[
- d_E \Delta \check{\varphi}_E + (\mu + k)(1 - \frac{\beta}{\bar{\beta}_2(1-\epsilon)})\check{\varphi}_E \leq 0, \ x \in \Omega; \quad \frac{\partial \check{\varphi}_E}{\partial \nu} = 0, \ x \in \partial \Omega.
\]

It follows from (19) and (21) that
\[
\begin{cases}
- d_E \Delta \hat{\varphi}_E + (\mu + k)\hat{\varphi}_E \geq \frac{q_1}{1+\epsilon}(1-q)\beta\hat{S}\hat{\varphi}_I, \ x \in \Omega, \\
- d_I \Delta \check{\varphi}_I + (\mu + d)\check{\varphi}_I - k\check{\varphi}_E \geq \frac{q_1}{1+\epsilon}q\beta\hat{S}\check{\varphi}_I, \ x \in \Omega, \\
\frac{\partial \hat{\varphi}_E}{\partial \nu} = \frac{\partial \check{\varphi}_I}{\partial \nu} = 0, \quad x \in \partial \Omega.
\end{cases}
\]

Multiplying the first equation in (23) by $\hat{\varphi}_E$ and the first equation in (18) by $\hat{\varphi}_E$, subtracting the resulting functions and integrating the results by parts over $\Omega$ yield
\[
\int_{\Omega} \left[ k\check{\varphi}_I \hat{\varphi}_E - \frac{q_1}{1+\epsilon}(1-q)\beta\hat{S}\hat{\varphi}_I \hat{\varphi}_E \right] dx \geq 0.
\]

Similarly, multiplying the second equation in (23) by $\hat{\varphi}_I$ and the second equation in (18) by $\hat{\varphi}_I$, subtracting the resulting functions and integrating the results by parts over $\Omega$, we have
\[
\int_{\Omega} \frac{1}{R_0} \left[ (1-q)\beta\hat{S}\hat{\varphi}_E \hat{\varphi}_I + q\beta\hat{S}\hat{\varphi}_I \hat{\varphi}_I \right] - k\hat{\varphi}_I \hat{\varphi}_E dx \geq \int_{\Omega} \frac{q_1}{1+\epsilon}q\beta\hat{S}\hat{\varphi}_I \hat{\varphi}_I dx.
\]

By (24) and (25), we get
\[
\left( \frac{1}{R_0} - \frac{q_1}{1+\epsilon} \right) \int_{\Omega} [(1-q)\beta\hat{S}\hat{\varphi}_E \hat{\varphi}_I + q\beta\hat{S}\hat{\varphi}_I \hat{\varphi}_I] dx \geq 0,
\]
which implies that $R_0 \leq \frac{1+\epsilon}{q_1}$. Similar procedures yield
\[
\left( \frac{1}{R_0} - \frac{q_1}{1-\epsilon} \right) \int_{\Omega} [(1-q)\beta\hat{S}\hat{\varphi}_E \hat{\varphi}_I + q\beta\hat{S}\hat{\varphi}_I \hat{\varphi}_I] dx \leq 0,
\]
from which we obtain $R_0 \geq \frac{1-\epsilon}{q_1}$. This proves that $R_0 \to \frac{1}{q_1}$ as $d_E \to 0$, and so (ii) holds.
Finally, we consider the case $d_1 \to 0$. Assume that $\Lambda, \mu, d, \beta$ are positive constants. Thus, we rewrite (18) as

\[
\begin{align*}
-d_E \Delta \hat{\varphi}_E + (\mu + k)\hat{\varphi}_E - k\hat{\varphi}_I &= 0, & x \in \Omega, \\
-d_I \Delta \hat{\varphi}_I + (\mu + d)\hat{\varphi}_I = \frac{1}{R_0} [(1 - q)\beta \frac{\Lambda}{\mu} \hat{\varphi}_E + q\beta \frac{\Lambda}{\mu} \hat{\varphi}_I], & x \in \Omega, \\
\frac{\partial \hat{\varphi}_E}{\partial \nu} = \frac{\partial \hat{\varphi}_I}{\partial \nu} &= 0, & x \in \partial \Omega.
\end{align*}
\]

(26)

We first need to prove the existence of $\partial_2$. Let $\hat{\lambda}$ be the principal eigenvalue of the elliptic eigenvalue problem

\[-d_E \Delta W + (\mu + k)W = \lambda k W, \quad \frac{\partial W}{\partial \nu} = 0, \quad x \in \partial \Omega\]

(27)

associated with the principal eigenfunction $\hat{W}$. Letting

\[\hat{\lambda} = \partial_2 \frac{(1 - q)\beta \frac{\Lambda}{\mu}}{\mu + d - \theta_2 q\beta \frac{\Lambda}{\mu}},\]

we have

\[\partial_2 = \frac{-\hat{\lambda}(\mu + d)}{\beta \frac{\Lambda}{\mu}[q\lambda + 1 - q]}.
\]

This clearly shows the existence of $\partial_2$, sharing the same principal eigenfunction $\hat{W}$ as $\hat{\lambda}$.

Due to $1 > \frac{\hat{\lambda}}{q\lambda + (1 - q)\frac{\beta \Lambda}{\mu}}$, thus $\mu + d - \theta_2 q\beta \frac{\Lambda}{\mu} > 0$. Give any $\epsilon \in (0, 1)$ such that

\[\frac{k}{1+\epsilon} > k > \frac{k}{1+\epsilon} \quad \text{and denote}
\]

\[(\hat{\varphi}_E, \hat{\varphi}_I) = \left(\frac{k(1 + \epsilon)}{\mu + d - \theta_2 q\beta \frac{\Lambda}{\mu}}, \frac{k(1 - \epsilon)}{\mu + d - \theta_2 q\beta \frac{\Lambda}{\mu}}\right), \quad (\hat{\varphi}_E, \hat{\varphi}_I) = \left(\frac{k(1 + \epsilon)}{\mu + d - \theta_2 q\beta \frac{\Lambda}{\mu}} \hat{\varphi}_E\right).
\]

For any $\epsilon \in (0, 1)$, there exists $\delta$ such that $0 < d_1 < \delta$,

\[-d_I \Delta \hat{\varphi}_I + (\mu + d - \theta_2 q\beta \frac{\Lambda}{\mu})(1 - k(1 + \epsilon))\hat{\varphi}_I \geq 0, \quad x \in \Omega, \quad \frac{\partial \hat{\varphi}_I}{\partial \nu} = 0, \quad x \in \partial \Omega.
\]

Hence, it follows that

\[-d_I \Delta \hat{\varphi}_I + (\mu + d)\hat{\varphi}_I \geq k\hat{\varphi}_E + \theta_2 q\beta \frac{\Lambda}{\mu} \hat{\varphi}_I \geq k\hat{\varphi}_E + \frac{\theta_2}{1 + \epsilon} q\beta \frac{\Lambda}{\mu} \hat{\varphi}_I.
\]

(28)

We infer from (28) and (20) that

\[
\begin{align*}
-d_I \Delta \hat{\varphi}_I + (\mu + d)\hat{\varphi}_I &\geq k\hat{\varphi}_E + \frac{\theta_2}{1 + \epsilon} q\beta \frac{\Lambda}{\mu} \hat{\varphi}_I, & x \in \Omega, \\
-d_E \Delta \hat{\varphi}_E + (\mu + k)\hat{\varphi}_E &\geq \frac{\theta_2}{1 + \epsilon} (1 - q)\beta \frac{\Lambda}{\mu} \hat{\varphi}_I, & x \in \Omega, \\
\frac{\partial \hat{\varphi}_E}{\partial \nu} = \frac{\partial \hat{\varphi}_I}{\partial \nu} &= 0, & x \in \partial \Omega.
\end{align*}
\]

(29)

Multiplying the first equation in (29) by $\hat{\varphi}_I$ and the second equation in (26) by $\hat{\varphi}_I$, subtracting the resulting functions and integrating the results by parts over $\Omega$ yield

\[
\int_{\Omega} \frac{1}{R_0} [(1 - q)\beta \frac{\Lambda}{\mu} \hat{\varphi}_I + q\beta \frac{\Lambda}{\mu} \hat{\varphi}_I \hat{\varphi}_I] dx \geq \int_{\Omega} k\hat{\varphi}_E \hat{\varphi}_I + \frac{\theta_2}{1 + \epsilon} q\beta \frac{\Lambda}{\mu} \hat{\varphi}_I \hat{\varphi}_I dx.
\]

(30)
Similarly, multiplying the second equation in (29) by \( \tilde{\varphi}_E \) and the first equation in (26) by \( \hat{\varphi}_E \), subtracting the resulting functions and integrating the results by parts over \( \Omega \), we have
\[
\int_{\Omega} k \tilde{\varphi}_I \hat{\varphi}_E dx \geq \int_{\Omega} \frac{\varrho_2}{1 + \epsilon} (1 - q) \frac{\Lambda}{\mu} \hat{\varphi}_E \tilde{\varphi}_I dx.
\] (31)

By (30) and (31), we get
\[
(1 - R_0) - \frac{\varrho_2}{1 + \epsilon} \int_{\Omega} (1 - q) \frac{\Lambda}{\mu} \hat{\varphi}_E \tilde{\varphi}_I + q \frac{\Lambda}{\mu} \hat{\varphi}_I \tilde{\varphi}_I dx \geq 0,
\]
which implies that \( R_0 \leq \frac{1 + \epsilon}{\varrho_2} \). Similar procedures yield
\[
(1 - R_0) - \frac{\varrho_2}{1 - \epsilon} \int_{\Omega} (1 - q) \frac{\Lambda}{\mu} \hat{\varphi}_E \tilde{\varphi}_I + q \frac{\Lambda}{\mu} \hat{\varphi}_I \tilde{\varphi}_I dx \leq 0,
\]
from which it follows that \( R_0 \geq \frac{1 - \epsilon}{\varrho_2} \). This proves that \( R_0 \to 1 - \frac{\varrho_2}{\epsilon} \) as \( d_I \to 0 \) and thus (iii) holds.

3. Analysis on model (2) without immigration.

3.1. Uniform boundedness and uniform persistence to (2). From now on, for notational simplicity, we denote
\[
F^* = \max_{x \in \Omega} F(x) \quad \text{and} \quad F_* = \min_{x \in \Omega} F(x).
\]

We first state a result of uniform bounds of the solution to (2) as follows.

**Lemma 3.1.** There exists a positive constant \( C_1 \) depending on initial data such that the solution \((S, E, I)\) of (2) satisfies
\[
\|S(\cdot, t)\|_{L^\infty(\Omega)} + \|E(\cdot, t)\|_{L^\infty(\Omega)} + \|I(\cdot, t)\|_{L^\infty(\Omega)} \leq C_1, \quad \forall t \geq 0.
\]
Furthermore, there exists a positive constant \( C_2 \) independent of initial data such that
\[
\|S(\cdot, t)\|_{L^\infty(\Omega)} + \|E(\cdot, t)\|_{L^\infty(\Omega)} + \|I(\cdot, t)\|_{L^\infty(\Omega)} \leq C_2, \quad \forall t \geq T,
\]
for some large time \( T \geq 0 \).

**Proof.** Define
\[
L(t) = \int_{\Omega} \left[ S(x, t) + E(x, t) + I(x, t) \right] dx, \quad \forall t \geq 0.
\] (32)

Obviously, we have, for \( t \geq 0 \),
\[
\frac{dL}{dt} = \int_{\Omega} \Lambda(x) dx - \int_{\Omega} \mu(x) S dx - \int_{\Omega} \left[ \mu(x) + d(x) \right] I dx - \int_{\Omega} \mu(x) E dx
\]
\[
\leq \int_{\Omega} \Lambda(x) dx - \mu_* L(t).
\]

Thus, it holds that
\[
\frac{dL(t)}{dt} + \mu_* L(t) \leq \int_{\Omega} \Lambda(x) dx \leq |\Omega| \Lambda^*, \quad \forall t \geq 0,
\]
which gives
\[
L(t) \leq L(0) e^{-\mu_* t} + \frac{|\Omega| \Lambda^*}{\mu_*} (1 - e^{-\mu_* t}), \quad \forall t \geq 0.
\] (33)
By this inequality (33) and Lemma 2.1 of [8] with $\sigma = \rho_0 = 1$, together with the positiveness of $S, E$ and $I$, we obtain

$$\|S(\cdot, t)\|_{L^\infty(\Omega)} + \|E(\cdot, t)\|_{L^\infty(\Omega)} + \|I(\cdot, t)\|_{L^\infty(\Omega)} \leq C_1, \quad \forall t \geq 0.$$ 

In addition, by (33), we can deduce that

$$\limsup_{t \to \infty} \int_{\Omega} \left[ S(x, t) + E(x, t) + I(x, t) \right] dx \leq \frac{\| \Omega \| \Lambda^*}{\mu_s}.$$

Hence, by Lemma 2.1 of [8] again, one can conclude that there exists a positive constant $C_2$ independent of initial data such that

$$\|S(\cdot, t)\|_{L^\infty(\Omega)} + \|E(\cdot, t)\|_{L^\infty(\Omega)} + \|I(\cdot, t)\|_{L^\infty(\Omega)} \leq C_2, \quad \forall t \geq T,$$

for some large time $T \geq 0$. Therefore, the proof is complete. \qed

We also have the following assertion.

**Proposition 3.** The disease-free equilibrium $X_0$ in system (2) is locally asymptotically stable if $R_0 < 1$, and is unstable if $R_0 > 1$.

The proof of Proposition 3 is similar to that of Lemma 2.4 of [42] and hence the details are omitted.

Based on Lemma 3.1, we can establish the uniform persistence of (2) when $R_0 > 1$. More precisely, we can conclude the following theorem.

**Theorem 3.2.** If $R_0 > 1$ and $\Lambda, \beta, \mu, k, d$ are positive functions, then there exists a real number $\eta > 0$ independent of the initial data, such that any solution $(S, E, I)$ satisfies

$$\liminf_{t \to \infty} S(x, t) \geq \eta, \quad \liminf_{t \to \infty} E(x, t) \geq \eta, \quad \liminf_{t \to \infty} I(x, t) \geq \eta,$$

uniformly for $x \in \Omega$, and hence, the disease persists uniformly. Furthermore, (2) admits at least one EE.

**Proof.** Let $U = C(\overline{\Omega})$ and $E$ be the usual positive cone in $U$:

$$E = \{ p \in U : p(x) \geq 0, \forall x \in \overline{\Omega} \}.$$ 

We also set $\hat{E} = E \setminus \{0\}$, and

$$X^0 = E \times E \times E, \quad X = \overline{X^0} = E \times E \times E.$$

We will apply Theorem 2.1 of [57] to complete our proof.

Notice that given initial data $(S_0, E_0, I_0)$, system (2) generates a semiflow, say $\Gamma(t)$, from $X$ to $X$:

$$\Gamma(t)(S_0, E_0, I_0) = (S(\cdot, t), E(\cdot, t), I(\cdot, t)), \quad \forall t \geq 0,$$

where $(S(\cdot, t), E(\cdot, t), I(\cdot, t))$ is the unique solution of (2). In view of Lemma 3.1, $\Gamma(t)$ is point dissipative in $X$. Furthermore, it follows from standard parabolic $L^p$-theory and embedding theorems that $\Gamma(t)$ is compact from $X$ to $X$ for any fixed $t > 0$. Observe that $X^0$ is invariant and relatively open in $X$. When $I_0 \equiv 0$, then it is easily seen that the solution of $(S, E, I)$ of (2) satisfies $I(x, t) \equiv 0$ for all $(x, t) \in \overline{\Omega} \times [0, \infty)$, and $S(x, t), E(x, t) > 0$ for all $(x, t) \in \overline{\Omega} \times [0, \infty)$ due to the parabolic strong maximum principle and Hopf boundary lemma. This implies that $\partial X^0 = E \times E \times \{0\}$ is also invariant.
In what follows, we will claim that the disease-free equilibrium \((\tilde{S}, 0, 0)\) attracts \((S_0, E_0, 0)\) with \(S_0, E_0 \in \bar{E}\). As mentioned above, since \(I_0 = 0\), clearly, \((S, E)\) solves

\[
\begin{align*}
\frac{\partial S}{\partial t} - d_S \Delta S &= \Lambda(x,t) - \mu(x,t) S, & x \in \Omega, t > 0, \\
\frac{\partial S}{\partial \nu} &= 0, & x \in \partial \Omega, t > 0, \\
S(x, 0) &= S_0(x), & x \in \Omega,
\end{align*}
\]

(34)

and

\[
\begin{align*}
\frac{\partial E}{\partial t} - d_E \Delta E &= -[\mu(x,t) + k(x,t)] E, & x \in \Omega, t > 0, \\
\frac{\partial E}{\partial \nu} &= 0, & x \in \partial \Omega, t > 0, \\
E(x, 0) &= E_0(x), & x \in \Omega.
\end{align*}
\]

(35)

It is easily seen that \(\max_{\Omega} E_0 e^{-t \min_{\bar{X}^0} \mu} \) is a supersolution of (35). Thus, we have

\[
0 \leq E(x, t) \leq \max_{\Omega} E_0 e^{-t \min_{\bar{X}^0} \mu} \rightarrow 0 \text{ uniformly on } \Omega, \text{ as } t \rightarrow \infty.
\]

(36)

Clearly, by the equation of \(S\), we can get

\[
S(x, t) \rightarrow \tilde{S} \text{ uniformly on } \Omega, \text{ as } t \rightarrow \infty.
\]

We notice that the omega limit set of the semiflow on \(\partial X^0\), is just \(\{(\tilde{S}, 0, 0)\}\). Now, to apply Theorem 2.1 of [57], it remains to check that the stable set of \(\{(\tilde{S}, 0, 0)\}\), denoted by \(\Sigma\), does not intersect \(X^0\), that is, \(\Sigma \cap X^0 = \emptyset\). Suppose that there is some \((S_0, E_0, I_0)\) lying in the stable set of \(\{(\tilde{S}, 0, 0)\}\), so that the unique solution \((S, E, I)\) satisfies

\[
\lim_{t \to \infty} (S(x, t) \to \tilde{S}) = 0, \quad \lim_{t \to \infty} (E(x, t), I(x, t)) = 0 \text{ uniformly on } \Omega.
\]

By a comparison argument similarly as before, for given \(\epsilon > 0\), we can find \(T_1\) such that

\[
\tilde{S} - \epsilon \leq S(x, t) \leq \tilde{S} + \epsilon, \quad \forall (x, t) \in \Omega \times [T_1, \infty).
\]

(37)

Note that \(I(x, t)\) is a supersolution to the following problem

\[
\begin{align*}
\frac{\partial \psi}{\partial t} - d_I \Delta \psi &= q \beta(x,t)(\tilde{S} - \epsilon) \psi - [\mu(x,t) + d(x,t)] \psi, & x \in \Omega, t > T_1, \\
\frac{\partial \psi}{\partial \nu} &= 0, & x \in \partial \Omega, t > T_1, \\
\psi(x, T_1) &= I(x, T_1) > 0, & x \in \Omega.
\end{align*}
\]

(38)

By Proposition 1 and \(R_0 > 1\), we have \(\lambda_1 < 0\), where \(\lambda_1\) is the unique principal eigenvalue of (11). Take \(\epsilon\) to be small enough so that \(\lambda_1 + \epsilon < 0\). Denote by \(\psi_1\) the eigenfunction corresponding to \(\lambda_1\) which is positive on \(\Omega \times [0, T]\). Then by some elementary computations, one can choose a constant \(q > 0\) to be sufficiently small so that \(I(x, T_1) \geq q \psi_1(x, T_1)\) and \(ge^{-(\lambda_1 + \epsilon)t} \psi_1(x, T_1)\) is a subsolution to (38). Hence, we obtain

\[
I(x, t) \geq ge^{-(\lambda_1 + \epsilon)t} \psi_1(x, T_1) \to \infty \text{ uniformly on } \Omega, \text{ as } t \to \infty.
\]

which arrives at a contradiction with Lemma 3.1. Now, Theorem 2.1 of [57] can be applied to conclude the uniform persistence. In addition, from Theorem 1.3.7
of [58], the uniform persistence implies the existence of an EE. This completes the proof.

3.2. Global attractivity of the DFE and EE. In this subsection, we always assume that \( \Lambda, \beta, \mu, k, d \) are positive constants, and discuss the global attractivity of the DFE and EE respectively.

As in [1], we know that in the current situation, the basic reproduction number \( R_0 \) can be explicitly expressed as

\[
R_0 = \frac{\Lambda \beta (k + q \mu)}{\mu (\mu + d)(k + \mu)}.
\]

As in [1], the long-time dynamics of system (1.2) is expected to be determined by the basic reproduction number \( R_0 \). If \( R_0 \leq 1 \), then the only equilibrium is the DFE \( X_0 = (\frac{\Lambda}{\mu}, 0, 0) \). If \( R_0 > 1 \), it is obvious that the EE of (1.2) satisfies:

\[
\begin{align*}
\Lambda - \beta SI - \mu S &= 0, \\
(1 - q)\beta SI - (\mu + k)E &= 0, \\
q\beta SI + kE - (\mu + d)I &= 0.
\end{align*}
\]

We define the EE as \( X^{**} = (S^{**}, E^{**}, I^{**}) \). From the above three equations, we can conclude that (1.2) has a unique EE with

\[
S^{**} = \frac{\Lambda}{\mu R_0}, \quad E^{**} = \frac{\Lambda(1 - q)}{(\mu + k)R_0} (R_0 - 1), \quad I^{**} = \frac{\mu}{\beta} (R_0 - 1).
\]

Our main result of this subsection reads as follows.

**Theorem 3.3.** The following assertions hold.

(i) If \( R_0 \leq 1 \) and \( \Lambda, \beta, \mu, k, d \) are positive constants, the DFE \( X_0 \) is globally attractive.

(ii) If \( R_0 > 1 \) and \( \Lambda, \beta, \mu, k, d \) are positive constants, the EE \( X^{**} \) is globally attractive.

**Proof.** First of all, we will verify (i). Given any positive solution \((S, E, I)\) of (1.2), let us define the following Lyapunov function:

\[
W(t) = \int_{\Omega} \left[ L(S(x,t), E(x,t), I(x,t)) \right] dx, \quad \forall t > 0,
\]

where

\[
L(S, E, I) = S - S_0 \ln S + \frac{k}{k + q \mu} E + \frac{k + \mu}{k + q \mu} I.
\]

For the sake of convenience, from now on let us denote

\[
\begin{align*}
f_1(S, E, I) &= \Lambda - \beta SI - \mu S, \\
f_2(S, E, I) &= (1 - q)\beta SI - (\mu + k)E, \\
f_3(S, E, I) &= q\beta SI + kE - (\mu + d)I.
\end{align*}
\]

The basic calculation shows

\[
\begin{align*}
L_S(S, E, I) f_1(S, E, I) + L_E(S, E, I) f_2(S, E, I) + L_I(S, E, I) f_3(S, E, I) \\
&= \left(1 - \frac{S_0}{S}\right) (\Lambda - \beta SI - \mu S) + \frac{k}{k + q \mu} \left[(1 - q)\beta SI - (\mu + k)E\right] \\
&\quad + \frac{k + \mu}{k + q \mu} \left[q\beta SI + kE - (\mu + d)I\right] \\
&= \mu S_0 \left(2 - \frac{S_0}{S}\right) - \frac{S}{S_0} + \frac{(k + \mu)(\mu + d)}{k + q \mu} I(R_0 - 1).
\end{align*}
\]
Thus it holds that
\[
W'(t) = \int_{\Omega} \left( L_S(S,E,I)S_t + L_E(S,E,I)E_t + L_I(S,E,I)I_t \right) dx
= \int_{\Omega} \left( \left( 1 - \frac{S_0}{S} \right) \left( d_S \Delta S + \Lambda - \beta SI - \mu S \right) + \frac{k}{k+q_t} \left( d_E \Delta E + (1-q) \beta SI - (\mu + k) E \right) + \frac{k+\mu}{k+q_t} \left( d_I \Delta I + q \beta SI + kE - (\mu + d) I \right) \right) dx
- \int d_S \frac{S_0}{S^2} \nabla S^2 dx + \int \mu S_0 \left( 2 - \frac{S_0}{S} - \frac{S}{S_0} \right) dx + \int \frac{(k+\mu)(\mu+d)}{k+q_t} I(R_0 - 1) dx.
\]

Clearly, \((2 - \frac{S_0}{S} - \frac{S}{S_0}) \leq 0\) holds for all \(S > 0\). Because of \(I > 0, R_0 < 1\), we can get \(\frac{d}{dt} W(t) \leq 0\), where the equality \(\frac{d}{dt} W(t) = 0\) holds only if \(S = S_0\) and \(R_0 = 1\) simultaneously. Hence, combined with Lemma 3.1, by some standard arguments we obtain that
\[
\left( S(x,t), E(x,t), I(x,t) \right) \to \left( \frac{\Lambda}{\mu}, 0, 0 \right) \text{ in } [L^\infty(\Omega)]^3, \text{ as } t \to \infty.
\]

Thus, when \(R_0 < 1\), we can conclude that the DFE \(X_0\) is globally attractive, and (i) is proved.

Next, we will verify (ii) by assuming \(R_0 > 1\). Consider the following Lyapunov function
\[
W(t) = \int_{\Omega} \left[ L((S(x,t), E(x,t), I(x,t)) \right] dx, \quad t > 0,
\]
where
\[
L(S, E, I) = S - S^{**} \ln S + A(E - E^{**} \ln E) + B(I - I^{**} \ln I)
\]
with the coefficients \(A, B > 0\) to be determined later.

Recall that \(f_1(S^{**}, E^{**}, I^{**}) = 0, f_2(S^{**}, E^{**}, I^{**}) = 0, f_3(S^{**}, E^{**}, I^{**}) = 0\). Thus we have
\[
\frac{dW(t)}{dt} = \int \frac{\partial L}{\partial S} \frac{dS}{dt} dx + \int \frac{\partial L}{\partial E} \frac{dE}{dt} dx + \int \frac{\partial L}{\partial I} \frac{dI}{dt} dx
= \int \left( 1 - \frac{S^{**}}{S} \right) \left[ (d_S \Delta S + \Lambda - \beta SI - \mu S) - (\Lambda - \beta S^{**} I^{**} - \mu S^{**}) \right] dx
+ \int A \left( 1 - \frac{E^{**}}{E} \right) \left( d_E \Delta E + (1-q) \beta SI - (\mu + k) E \right) dx
- \int A \left( 1 - \frac{E^{**}}{E} \right) \frac{E}{E^{**}} \left[ (1-q) \beta S^{**} I^{**} - (\mu + k) E^{**} \right] dx
+ \int B \left( 1 - \frac{I^{**}}{I} \right) d_I \Delta I + q \beta SI + kE - (\mu + d) I \right) dx
- \int B \left( 1 - \frac{I^{**}}{I} \right) \frac{I}{I^{**}} \left[ q \beta S^{**} I^{**} + kE^{**} - (\mu + d) I^{**} \right] dx
= - \int \left( d_S \frac{S^{**}}{S^2} \nabla S^2 + Bd_I \frac{I^{**}}{I^2} |\nabla I|^2 + Ad_E \frac{E^{**}}{E^2} |\nabla E|^2 \right) dx.
\]
By (40), we find that
\[
A = \frac{k}{k + q\mu}, \quad B = \frac{k + \mu}{k + q\mu}.
\]

Then we have
\[
1 = A(1 - q) + Bq, \quad BkE^* = A(1 - q)\beta S^* I^*.
\]

By (40), we find that
\[
\frac{dW(t)}{dt} = -\int_\Omega \left\{ dS \|\nabla S\|^2 + BkI^* \|\nabla I\|^2 + AdE \|\nabla E\|^2 \right\} dx
+ \int_\Omega \left[ \mu S^* \left( 2 - \frac{S^*}{S} - \frac{S}{S^*} \right) + Bq\beta S^* I^* \left( 2 - \frac{S^*}{S} - \frac{S}{S^*} \right) \right.
+ A(1 - q)\beta S^* I^* \left( 3 - \frac{S^*}{S} - \frac{SIE^*}{S^*I^*E} - \frac{EI^*}{E^*I} \right) \right] dx, \quad t > 0.
\]

Observe that \( \left( 2 - \frac{S^*}{S} - \frac{S}{S^*} \right) \leq 0 \) and \( \left( 3 - \frac{S^*}{S} - \frac{SIE^*}{S^*I^*E} - \frac{EI^*}{E^*I} \right) \leq 0 \) hold for all \( S, E, I > 0 \), and \( S^*, E^*, I^* > 0 \). Thus, since the EE \( X^* \) exists, \( \frac{d}{dt}W(t) \leq 0 \), where the equality \( \frac{d}{dt}W(t) = 0 \) holds only if \( S = S^*, I = I^* \) and \( E = E^* \) simultaneously. By the same argument as in (i), we can conclude that
\[
(S(x,t), E(x,t), I(x,t)) \rightarrow (S^*, E^*, I^*) \quad \text{in} \quad [L^\infty(\Omega)]^3, \quad \text{as} \quad t \rightarrow \infty.
\]

Hence, the EE \( X^* \) is globally attractive. This complete the proof of (ii).

3.3. Asymptotic profiles of the EE of (2). In this section, we are concerned with the asymptotic behavior of the EE of (2). We will treat three cases: \( d_S \to 0, d_E \to 0 \) and \( d_I \to 0 \). Before going further, we recall a useful lemma from [34].

Lemma 3.4. Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^n \). Let \( M \) be a non-negative constant and suppose that \( z \in W^{1,2}(\Omega) \) is a non-negative weak solution of the inequalities
\[
0 \leq -\Delta z + Mz \quad \text{in} \quad \Omega, \quad \partial_\nu z \leq 0 \quad \text{on} \quad \partial\Omega.
\]

Then, for any \( p \in [1,n/(n-2)) \), there exists a positive constant \( C_0 \), depending only on \( p, \alpha \) and \( \Omega \), such that
\[
\|z\|_p \leq C_0 \inf_\Omega z.
\]

Theorem 3.5. Assume that \( \bar{R}_0 > 1 \). Fix \( d_E > 0, d_I > 0 \) and let \( d_S \to 0 \). Then any positive solution \( S, E, I \) of (1.3) satisfies (up to a subsequence of \( d_S \to 0 \)) that \( (S, E, I) \to (\bar{S}, E, I) \) uniformly on \( \Omega \), where
\[
\bar{S}(x) = \frac{\Lambda(x)}{\beta(x)I(x) + \mu(x)},
\]

\( 0 \leq -\Delta z + Mz \quad \text{in} \quad \Omega, \quad \partial_\nu z \leq 0 \quad \text{on} \quad \partial\Omega. \) Then, for any \( p \in [1,n/(n-2)) \), there exists a positive constant \( C_0 \), depending only on \( p, \alpha \) and \( \Omega \), such that
\[
\|z\|_p \leq C_0 \inf_\Omega z.
\]
and \((\tilde{E}, \tilde{I})\) is a positive solution to

\[
\begin{cases}
-d_E \Delta \tilde{E} = (1 - q)\beta(x)\tilde{S} \tilde{I} - [\mu(x) + k(x)]\tilde{E}, & x \in \Omega, \\
-d_I \Delta \tilde{I} = q(x)\beta(x)\tilde{S} \tilde{I} + k(x)\tilde{E} - [\mu(x) + d(x)]\tilde{I}, & x \in \Omega, \\
\partial \tilde{E} / \partial \nu = \partial \tilde{I} / \partial \nu = 0, & x \in \partial \Omega.
\end{cases}
\] (42)

\textbf{Proof.} Our proof consists of three steps as follows.

\textbf{Step 1.} Estimates of upper and lower bounds. Let \(S(x_1) = \max_{x \in \Omega} S(x)\) for some \(x_1 \in \Omega\). By Proposition 2.2 of [27], we have from the first equation in (1.3) that

\[
\Lambda(x_1) \geq \beta(x_1)S(x_1)I(x_1) + \mu(x_1)S(x_1) \geq \mu(x_1)S(x_1),
\]

from which it follows that

\[
S(x) \leq \frac{\Lambda^*}{\mu_*}, \quad \forall x \in \Omega.
\] (43)

Let \(W(x) = d_S S + d_E E + d_I I\). Then

\[
-\Delta (d_S S + d_E E + d_I I) = \Lambda(x) - \mu(x)S - \mu(x)E - [\mu(x) + d(x)]I.
\]

We set \(W(x_0) = \max_{x \in \Omega} W(x)\), and so

\[
\Lambda^* \geq \Lambda(x_0) \geq \mu(x_0)[S(x_0) + E(x_0) + I(x_0)].
\]

Thus it holds that

\[
\max_{x \in \Omega} W(x) = W(x_0) \leq \max\{d_S, d_E, d_I\} \frac{\Lambda^*}{\mu_*}.
\]

Without loss of generality, we assume that \(0 < d_S < 1\). Then one has

\[
I(x) \leq \max_{x \in \Omega} I(x) < \max \frac{\{1, d_E, d_I\}}{d_I} \frac{\Lambda^*}{\mu_*},
\] (44)

\[
E(x) \leq \max_{x \in \Omega} E(x) < \max \frac{\{1, d_E, d_I\}}{d_E} \frac{\Lambda^*}{\mu_*}.
\] (45)

This shows that there is a positive constant \(C\) such that \(E(x), I(x) \leq C\). Hereafter, the positive constant \(C\) does not depend on \(d_S > 0\) but allows to vary from place to place.

Now we set \(S(x_2) = \min_{x \in \Omega} S(x)\), and so

\[
\Lambda(x_2) \leq \beta(x_2)S(x_2)I(x_2) + \mu(x_2)S(x_2),
\]

equivalently,

\[
\frac{\Lambda_*}{\beta^* C + \mu_*} \leq \frac{\Lambda(x_2)}{\beta(x_2)I(x_2) + \mu(x_2)} \leq S(x_2) \leq S(x).
\]

This then gives

\[
S(x) \geq \min_{x \in \Omega} S(x) = S(x_2) \geq C > 0.
\] (46)
In what follows, we are going to derive the positive lower bound for the component of $E$. We will use the equation of $(E, I)$:

$$
\begin{aligned}
- d_1 \Delta E &= (1 - q) \beta(x) S I - [\mu(x) + k(x)] E, \quad x \in \Omega, \\
- d_1 \Delta I &= q \beta(x) S I + k(x) E - [\mu(x) + d(x)] I, \quad x \in \Omega, \\
\frac{\partial E}{\partial \nu} &= \frac{\partial I}{\partial \nu} = 0, \quad x \in \partial \Omega.
\end{aligned}
$$

We proceed by contradiction and suppose that there is a sequence $d_{S_n}$ satisfying $d_{S_n} \to 0$ as $n \to \infty$ and the corresponding positive solution sequence $(S_n, E_n, I_n)$ of (4) with $d_S = d_{S_n}$, such that $\min_{x \in \Omega} E_n \to 0$ as $n \to \infty$. According to the first equation of (47), we will see

$$
-d_n \Delta E_n + [\mu(x) + k(x)] E_n \geq 0, \quad x \in \Omega, \quad \frac{\partial E_n}{\partial \nu} = 0, \quad x \in \partial \Omega.
$$

Hence by Lemma 3.4 with $p = 1$, we have

$$
\|E_n\|_{L^1(\Omega)} \leq C \inf_{\Omega} E_n.
$$

Thus, $\|E_n\|_{L^1(\Omega)} \to 0$, as $n \to \infty$. Next, integrating the first equation of (47) by parts infers

$$
(1 - q) \beta_n C \int_{\Omega} I_n \, dx \leq \int_{\Omega} (1 - q) \beta(x) S_n I_n \, dx = \int_{\Omega} [\mu(x) + k(x)] E_n \, dx.
$$

Hence, it follows that

$$
\|I_n\|_{L^1(\Omega)} \to 0, \quad \text{as} \quad n \to \infty. \quad (48)
$$

Notice that $I_n$ satisfies

$$
\begin{aligned}
- d_1 \Delta I + [\mu(x) + d(x)] I &= q(x) \beta(x) S I + k(x) E, \quad x \in \Omega, \\
\frac{\partial I}{\partial \nu} &= 0, \quad x \in \partial \Omega.
\end{aligned}
$$

According to (43), (44) and (45), we see that

$$
\|q \beta S_n I_n + k E_n\|_{L^p(\Omega)} \leq C, \quad \forall p \geq 1.
$$

By the well-known $L^p$-theory for elliptic equations, we have

$$
\|I_n\|_{W^{2, p}(\Omega)} \leq C, \quad \forall p \geq 1.
$$

Using the Sobolev embedding theorem, we obtain

$$
\|I_n\|_{C^{1+\alpha}(\overline{\Omega})} \leq C,
$$

for some $0 < \alpha < 1$. As $\{I_n\}_{0 < d_S \leq 1}$ is compact in $C^{1}(\overline{\Omega})$, there is a sequence of $d_S$, denoted by $d_n := d_{S_n}$, satisfying $d_n \to 0$ as $n \to 0$, and a corresponding positive solution $(S_n, E_n, I_n)$ such that $I_n \to \alpha$ as $n \to \infty$. Due to (48), we can get $\alpha \equiv 0$. Otherwise, $\int_{\Omega} I_n \to \int_{\Omega} \alpha \to 0$, this is a contradiction. Thus,

$$
I_n \to 0 \text{ in } C^{1}(\overline{\Omega}), \quad \text{as } n \to \infty.
$$

Consider the first equation of (4) satisfied by $S_n, I_n$:

$$
-d_n \Delta S_n = \Lambda(x) - \beta(x) S_n I_n - \mu(x) S_n, \quad x \in \Omega, \quad \frac{\partial S}{\partial \nu} = 0, \quad x \in \partial \Omega.
$$
By the well-known L-theory for elliptic equations, we have
\begin{equation}
\|1-q\|_{L^p(\Omega)} \leq C, \quad \forall p \geq 1.
\end{equation}
By the well-known L^p-theory for elliptic equations, we have
\begin{equation}
\|E\|_{W^{2,p}(\Omega)} \leq C, \quad \forall p \geq 1.
\end{equation}
Using the Sobolev embedding theorem, we obtain that \(\|E\|_{C^{1+\alpha}(\Omega)} \leq C\) for some \(0 < \alpha < 1\). Thus, \(\{E\}_{0 < d_S \leq 1}\) is compact in \(C^1(\Omega)\). Consequently, there is a sequence of \(d_S\), denoted by \(d_n := d_S, n\), satisfying \(d_n \to 0\) as \(n \to 0\), and a corresponding positive solution \((S_n, E_n, I_n)\) such that
\begin{equation}
E_n \to \bar{E} \quad \text{in} \quad C^1(\Omega), \quad \text{as} \quad n \to \infty,
\end{equation}
where \(\bar{E} > 0\) in \(C^1(\Omega)\) due to (50). Notice that \(I\) satisfies
\begin{equation}
\begin{cases}
-d_I \Delta I + [\mu(x) + d(x)]I = q(x)\beta(x)SI + k(x)E, & x \in \Omega, \\
\frac{\partial I}{\partial \nu} = 0, & x \in \partial \Omega.
\end{cases}
\end{equation}
Similarly as before, we can obtain that \( \|I\|_{C^{1+\alpha}(\Omega)} \leq C \) for some \( 0 < \alpha < 1 \). Thus, \( \{I\}_{0 < d \leq 1} \) is compact in \( C^1(\Omega) \) and there is a sequence of \( d_n \), denoted by \( d_n := d_{S,n} \), satisfying \( d_n \to 0 \) as \( n \to 0 \), and a corresponding positive solution \( (S_n, E_n, I_n) \) such that

\[
I_n \to \tilde{I} \quad \text{in} \ C^1(\Omega), \quad n \to \infty, \tag{53}
\]

where \( \tilde{I} > 0 \) in \( C^1(\Omega) \).

**Step 3.** Convergence of \( S \). Notice that for any \( n \geq 1 \), \( S_n \) solves

\[
\begin{dcases}
-d_n \Delta S_n = \Lambda(x) - \beta(x) S_n I_n - \mu(x) S_n, \quad x \in \Omega, \\
\frac{\partial S_n}{\partial \nu} = 0, \quad x \in \partial \Omega.
\end{dcases}
\]

In view of (53), for any small \( \epsilon > 0 \), it holds that

\[
0 < \tilde{I} - \epsilon \leq I_n(x) \leq \tilde{I} + \epsilon, \quad \forall x \in \overline{\Omega},
\]

for all large \( n \). Hence, taking sufficiently large \( n \), we observe that on \( \Omega \),

\[
\begin{align*}
\Lambda(x) - \beta(x) S_n I_n - \mu(x) S_n & \geq \Lambda(x) - \beta(x) S_n (\tilde{I} + \epsilon) - \mu(x) S_n, \\
\Lambda(x) - \beta(x) S_n I_n - \mu(x) S_n & \leq \Lambda(x) - \beta(x) S_n (\tilde{I} - \epsilon) - \mu(x) S_n.
\end{align*}
\]

Next, for fixed large \( n \), we consider the following auxiliary problem

\[
\begin{dcases}
-d_n \Delta W = \Lambda(x) - \beta(x) W (\tilde{I} + \epsilon) - \mu(x) W, \quad x \in \Omega, \\
\frac{\partial W}{\partial \nu} = 0, \quad x \in \partial \Omega.
\end{dcases}
\]

It is clear that a supersolution of (54) is \( S_n \) and (54) has a unique positive solution, denoted by \( \overline{W} \). Furthermore, using the similar proof of Lemma 2.4 of [37], we can show that

\[
\overline{W} \to \frac{\Lambda(x)}{\mu(x) + \beta(x)(\tilde{I} + \epsilon)} \quad \text{uniformly on} \ \overline{\Omega}, \quad n \to \infty.
\]

Thus \( S_n \) satisfies

\[
\liminf_{n \to \infty} S_n \geq \lim_{n \to \infty} \overline{W} = \frac{\Lambda(x)}{\mu(x) + \beta(x)(\tilde{I} + \epsilon)} \quad \text{on} \ \overline{\Omega}. \tag{55}
\]

The same reasoning as above shows that the following problem

\[
\begin{dcases}
-d_n \Delta W = \Lambda(x) - \beta(x) W (\tilde{I} - \epsilon) - \mu(x) W, \quad x \in \Omega, \\
\frac{\partial W}{\partial \nu} = 0, \quad x \in \partial \Omega.
\end{dcases}
\]

has a unique solution, denoted by \( \underline{W} \), and a subsolution of (56) is \( S_n \). Thus

\[
\limsup_{n \to \infty} S_n(x) \leq \lim_{n \to \infty} \underline{W} = \frac{\Lambda(x)}{\mu(x) + \beta(x)(\tilde{I} - \epsilon)} \quad \text{on} \ \overline{\Omega}. \tag{57}
\]

Due to the arbitrariness of small \( \delta > 0 \), thanks to (55) and (57), we have

\[
S_n(x) \to \hat{S}(x) := \frac{\Lambda(x)}{\mu(x) + \beta(x)\tilde{I}} \quad \text{uniformly on} \ \overline{\Omega}, \quad n \to \infty.
\]

Clearly, \((\hat{E}, \hat{I})\) satisfies (42). Therefore, the proof is complete.

The following result concerns the asymptotic behavior of positive solution of (4) with \( d_E \to 0 \). Our result reads as follows.
Theorem 3.6. Assume that \( \left\{ \frac{[k(x) + q(x)]\beta(x)\tilde{S}(x)}{\mu(x) + k(x)} > \mu(x) + d(x) : x \in \Omega \right\} \) is non-empty. Fix \( d_E > 0 \), \( d_T > 0 \) and let \( d_E \to 0 \). Then any positive solution \((S, E, I)\) of (1.3) satisfies (up to a subsequence of \( d_E \to 0 \)) that \((S, E, I) \to (\tilde{S}, \tilde{E}, \tilde{I})\) uniformly on \( \Omega \), where

\[
\tilde{E}(x) = \frac{(1 - q)\beta(x)\tilde{S}\tilde{I}}{k(x) + \mu(x)},
\]

and \((\tilde{S}, \tilde{I})\) is a positive solution to

\[
\begin{aligned}
-d_S \Delta \tilde{S} &= \Lambda(x) - \beta(x)\tilde{S}\tilde{I} - \mu(x)\tilde{S}, & x &\in \Omega, \\
-d_I \Delta \tilde{I} &= q(x)\beta(x)\tilde{S}\tilde{I} + k(x)\tilde{E} - [\mu(x) + d(x)]\tilde{I}, & x &\in \Omega, \\
\frac{\partial \tilde{S}}{\partial \nu} &= \frac{\partial \tilde{I}}{\partial \nu} = 0, & x &\in \partial \Omega.
\end{aligned}
\]

Proof. We divide the proof into three steps. In the following, \( C \) is a positive constant independent of \( d_E > 0 \).

Step 1. Estimates of upper and lower bounds. Notice that (43) (44) and (46) remain true and so \( I(\x) \leq C, \frac{1}{C} \leq S(\x) \leq C \). We now set \( E(x_1) = \max_{x \in \Omega} E(x) \) for some \( x_1 \in \Omega \). Then it holds that

\[
[\mu(x_1) + k(x_1)]E(x_1) \leq (1 - q)\beta(x_1)S(x_1)J(x_1) \leq C.
\]

Thus, we obtain

\[
E(x) \leq E(x_1) \leq \frac{C}{\mu_1 + k_1}. \tag{59}
\]

Next we show that \( E \) has a positive lower bound. Arguing indirectly, we suppose that there is a sequence \( d_{E_n} \) satisfying \( d_{E_n} \to 0 \) as \( n \to \infty \) and the corresponding positive solution sequence \((S_n, E_n, I_n)\) of (4) with \( d_E = d_{E_n} \), such that \( \min_{x \in \Omega} E_n \to 0 \) as \( n \to \infty \). Similar to Theorem 3.5, we get that

\[
I_n \to 0 \quad \text{in} \ C^1(\Omega), \quad \text{as} \ n \to \infty.
\]

Consider the first equation of (4):

\[
-d_S \Delta S_n = \Lambda(x) - \beta(x)S_nI_n - \mu(x)S_n, \quad x \in \Omega; \quad \frac{\partial S_n}{\partial \nu} = 0, \quad x \in \partial \Omega.
\]

Let \( n \to \infty \), we can easily obtain that

\[
S_n(x) \to \frac{\Lambda(x)}{\mu(x)} \quad \text{uniformly on} \ \Omega, \quad \text{as} \ n \to \infty. \tag{60}
\]

For any \( n \geq 1 \), let us denote \( \lambda_n \) to be the principal eigenvalue of the eigenvalue problem:

\[
\begin{aligned}
-d_n \Delta \phi_E - (1 - q)\beta(x)S_n\phi_I + [\mu(x) + k(x)]\phi_E &= \lambda_n \phi_E, & x &\in \Omega, \\
-d_I \Delta \phi_I + [\mu(x) + d(x) - q\beta(x)S_n]\phi_I - k(x)\phi_E &= \lambda_n \phi_I, & x &\in \Omega, \\
\frac{\partial \phi_E}{\partial \nu} &= \frac{\partial \phi_I}{\partial \nu} = 0, & x &\in \partial \Omega.
\end{aligned}
\]

Clearly, by (47), \( \lambda_n = 0 \) for all \( n \geq 1 \) as \((E_n, I_n)\) is a corresponding eigenfunction.

By Proposition 2 (ii), we can see \( R_0^n \to \frac{1}{\hat{\nu}} \) as \( d_E \to 0 \), where \( \hat{\nu}_1 \) is the principal eigenvalue of (19). According to our assumption that \( \left\{ x \in \Omega : \frac{[k(x) + q(x)]\beta(x)\tilde{S}(x)}{\mu(x) + k(x)} > \mu(x) + d(x) \right\} \) is non-empty, it follows that \( \hat{\nu}_1 \in (0, 1) \). Proposition 1 shows that
Arguing similarly as before, we obtain
\[ \| \frac{\partial S_n}{\partial \nu} \|_{L^p(\Omega)} = C, \quad \forall p \geq 1. \]
By the \( L^p \)-theory and embedding theorems, we obtain that \( \| S \|_{C^{1,\alpha}(\Omega)} \leq C \) for some \( 0 < \alpha < 1 \). Thus, there is a sequence of \( d_E \), denoted by \( d_n := d_{E,n} \), satisfying \( d_n \to 0 \) as \( n \to 0 \), and a corresponding positive solution \((S_n, E_n, I_n)\), such that
\[ S_n \to \tilde{S} \text{ in } C^1(\overline{\Omega}), \quad \text{as } n \to \infty, \quad (63) \]
where \( \tilde{S} > 0 \) and \( \tilde{S} \in C^1(\overline{\Omega}) \) due to (46). Notice that \( I \) satisfies
\[ \begin{cases} 
-d_n \Delta I + [\mu(x) + d(x)]I = k(x)E + q\beta(x)SI, & x \in \Omega, \\
\frac{\partial I}{\partial \nu} = 0, & x \in \partial \Omega. 
\end{cases} \quad (64) \]
According to (43), (44) and (59), we see that
\[ \| q\beta SI + kE \|_{L^p(\Omega)} \leq C, \quad \forall p \geq 1. \]
Arguing similarly as before, we obtain \( \| I \|_{C^{1,\alpha}(\Omega)} \leq C \) for some \( 0 < \alpha < 1 \). Thus, there is a sequence of \( d_E \), denoted by \( d_n := d_{E,n} \), satisfying \( d_n \to 0 \) as \( n \to 0 \), and a corresponding positive solution \((S_n, E_n, I_n)\) such that
\[ I_n \to \tilde{I} \text{ in } C^1(\overline{\Omega}), \quad \text{as } n \to \infty, \quad (64) \]
where \( \tilde{I} > 0 \) due to (62) and \( \tilde{I} \in C^1(\overline{\Omega}) \).

**Step 3.** Convergence of \( E \). Similarly, \( E_n \) solves
\[ \begin{cases} 
-d_n \Delta E_n = (1 - q)\beta(x)S_nE_n - [\mu(x) + k(x)]E_n, & x \in \Omega, \\
\frac{\partial E_n}{\partial \nu} = 0, & x \in \partial \Omega. 
\end{cases} \]
In view of (63) and (64), for any small \( \epsilon > 0 \) and all large \( n \), it holds that
\[ 0 < \tilde{I}(x) - \epsilon \leq I_n(x) \leq \tilde{I}(x) + \epsilon, \quad \forall x \in \overline{\Omega}, \]
\[ 0 < \tilde{S}(x) - \epsilon \leq S_n(x) \leq \tilde{S}(x) + \epsilon, \quad \forall x \in \overline{\Omega}. \]
Thus, if $n$ is sufficiently large, we have

$$(1 - q)\beta(x)S_n I_n - [\mu(x) + k(x)]E_n \leq (1 - q)\beta(x)(\bar{S} + \epsilon)(\bar{I} + \epsilon) - [\mu(x) + k(x)]E_n,$$

$$(1 - q)\beta(x)S_n I_n - [\mu(x) + k(x)]E_n \geq (1 - q)\beta(x)(\bar{S} - \epsilon)(\bar{I} - \epsilon) - [\mu(x) + k(x)]E_n.$$ Clearly, for any fixed large $n$, $E_n$ is a subsolution of

$$\begin{cases} -d_n \Delta Q = (1 - q)\beta(x)(\bar{S} + \epsilon)(\bar{I} + \epsilon) - [\mu(x) + k(x)]Q, & x \in \Omega, \\ \frac{\partial Q}{\partial \nu} = 0, & x \in \partial\Omega, \quad (65) \end{cases}$$

and $E_n$ is a supersolution of

$$\begin{cases} -d_n \Delta Q = (1 - q)\beta(x)(\bar{S} - \epsilon)(\bar{I} - \epsilon) - [\mu(x) + k(x)]Q, & x \in \Omega, \\ \frac{\partial Q}{\partial \nu} = 0, & x \in \partial\Omega. \quad (66) \end{cases}$$

Clearly, (65) and (66) has a unique solution, denoted by $\bar{Q}$ and $Q$, respectively. By a simple sub-supersolution argument, combined with the uniqueness, we have $Q \leq E_n \leq \bar{Q}$ on $\bar{\Omega}$ for all large $n$. A similar argument as that of Lemma 2.4 of [37] yields

$$\bar{Q}(x) \rightarrow \frac{(1 - q)\beta(x)(\bar{S} + \epsilon)(\bar{I} + \epsilon)}{\mu(x) + k(x)} \text{ uniformly on } \bar{\Omega}, \text{ as } n \rightarrow \infty,$$

and

$$Q(x) \rightarrow \frac{(1 - q)\beta(x)(\bar{S} - \epsilon)(\bar{I} - \epsilon)}{\mu(x) + k(x)} \text{ uniformly on } \bar{\Omega}, \text{ as } n \rightarrow \infty,$$

from which it follows that

$$\frac{(1 - q)\beta(x)(\bar{S} - \epsilon)(\bar{I} - \epsilon)}{\mu(x) + k(x)} \leq \liminf_{n \rightarrow \infty} E_n(x)$$

and

$$\limsup_{n \rightarrow \infty} E_n(x) \leq \frac{(1 - q)\beta(x)(\bar{S} + \epsilon)(\bar{I} + \epsilon)}{\mu(x) + k(x)}.$$ By the arbitrariness of small $\epsilon > 0$, we obtain

$$E_n \rightarrow \frac{(1 - q)\beta(x)\bar{S}\bar{I}}{\mu(x) + k(x)} \text{ uniformly on } \bar{\Omega}, \text{ as } n \rightarrow \infty,$$

and $(\bar{S}, \bar{I})$ is a positive solution of (58). The proof is complete. \hfill \Box

In the following section, we assume $\Lambda, \mu, k, d, \beta$ are positive constants. Thus, we rewrite the elliptical system in (4) as the following system:

$$\begin{cases} -d_S \Delta S = \Lambda - \beta SI - \mu S, & x \in \Omega, \\ -d_E \Delta E = (1 - q)\beta SI - (\mu + k)E, & x \in \Omega, \\ -d_I \Delta I = q\beta SI + kE - (\mu + d)I, & x \in \Omega, \\ \frac{\partial S}{\partial \nu} = \frac{\partial E}{\partial \nu} = \frac{\partial I}{\partial \nu} = 0, & x \in \partial\Omega. \quad (67) \end{cases}$$

Our main result reads as follows.
Theorem 3.7. Assume that $\Lambda, \mu, d, \beta$ are positive constants and $(k + q\mu)\beta \Lambda > (\mu + k)(\mu + d) > (\mu + k)q\beta \Lambda / \mu$ on $\Omega$. Fix $d_S > 0, d_E > 0$ and let $d_I \to 0$. Then any positive solution $(S, E, I)$ of (67) satisfies (up to a subsequence of $d_I \to 0$) that $(S, E, I) \to (\hat{S}, \hat{E}, \hat{I})$ uniformly on $\bar{\Omega}$, where

$$I(x) = \frac{k\hat{E}}{\mu + d - q\beta \hat{S}(x)},$$

and $(\hat{S}, \hat{E})$ is a positive solution to

$$\begin{cases}
-d_S \Delta \hat{S} = \Lambda - \beta \hat{S}I - \mu \hat{S}, & x \in \Omega, \\
-d_E \Delta \hat{E} = (1-q)\beta \hat{S}I - (\mu + k)\hat{E}, & x \in \Omega, \\
\frac{\partial \hat{S}}{\partial \nu} = \frac{\partial \hat{S}}{\partial \nu} = 0, & x \in \partial \Omega.
\end{cases}$$

(68)

Proof. In the following, $C$ is the positive constant independent of $d_I > 0$. We divide the proof into three steps to derive the conclusion.

Step 1. Estimates of upper and lower bounds. Let $S(x_1) = \max_{x \in \Omega} S(x)$ for some $x_1 \in \Omega$. By Proposition 2.2 of [27], we have from the first equation in (67) that

$$\Lambda \geq \beta S(x_1) I(x_1) + \mu S(x_1) \geq \mu S(x_1).$$

This implies

$$S(x) \leq S(x_1) \leq \frac{\Lambda}{\mu}, \quad \forall x \in \Omega.$$  

(69)

By setting $W(x) = d_S S + d_E E + d_I I$, we have

$$-\Delta(d_S S + d_E E + d_I I) = \Lambda - \mu S - \mu E - (\mu + d) I.$$  

Then we set $W(x_0) = \max_{x \in \Omega} W(x)$ and get $\Lambda \geq S + E + I$. Thus it holds that

$$E(x) \leq \max W(x) = W(x_0) \leq \max\{d_S, d_E, d_I\} \frac{\Lambda}{\mu}.$$  

(70)

Let $I(x_1) = \max_{x \in \Omega} I(x)$ for some $x_1 \in \Omega$. We have

$$q\beta \Lambda \frac{\Lambda}{\mu} I(x_1) + kC \geq q\beta S(x_1) I(x_1) + kE(x_1) \geq (\mu + d) I(x_1).$$

Hence this gives

$$I(x) \leq \frac{kC}{\mu + d - q\beta \Lambda / \mu}.$$  

(71)

We now let $S(x_2) = \min_{x \in \Omega} S(x)$ for some $x_2 \in \Omega$. Then

$$\Lambda \leq \beta S(x_2) I(x_2) + \mu S(x_2),$$

equivalently,

$$\frac{\Lambda}{\beta C + \mu} \leq \frac{\Lambda}{\beta I + \mu} \leq S(x).$$

This then yields

$$S(x) \geq \min_{x \in \Omega} S(x) \geq C > 0.$$  

(72)

Next we will show that $E$ has a positive lower bound. To this end, we use a contradiction argument, and suppose that there is a sequence $d_{I_n}$ satisfying $d_{I_n} \to 0$
as \( n \to \infty \) and the corresponding positive solution sequence \((S_n, E_n, I_n)\) of (67) with 
\[ d_I = d_{I_n}, \]
such that \( \min_{x \in \Omega} E_n \to 0 \) as \( n \to \infty \). Similarly to the proof of Theorem 3.5, we can obtain

\[ I_n \to 0, \quad \text{uniformly on } \Omega, \quad \text{as } n \to \infty. \]

Consider the equation \( S_n \) in (67):

\[ -d_S \Delta S_n = \Lambda - \beta S_n I_n - \mu S_n, \quad x \in \Omega. \]

Letting \( n \to \infty \), we find that

\[ S_n \to \frac{\Lambda}{\mu} \quad \text{uniformly on } \Omega, \quad \text{as } n \to \infty. \]  

(73)

For any \( n \geq 1 \), if denoting by \( \lambda_n \) the principal eigenvalue of the eigenvalue problem:

\[
\begin{cases}
-d_E \Delta \phi_E - (1 - q)\beta S_n \phi_I + (\mu + k)\phi_E = \lambda \phi_E, & x \in \Omega, \\
-d_n \Delta \phi_I + (\mu + d - q S_n) \phi_I - k \phi_E = \lambda \phi_I, & x \in \Omega, \\
\frac{\partial \phi_E}{\partial \nu} = \frac{\partial \phi_I}{\partial \nu} = 0, & x \in \partial \Omega,
\end{cases}
\]

then (47) tells us that \( \lambda_n = 0 \) for all \( n \geq 1 \), because \((E_n, I_n)\) is a corresponding eigenfunction.

By Proposition 2(iii), we can see \( R_0^* \to \frac{1}{\varrho_2} \) as \( d_I \to 0 \), where \( \varrho_2 \) is the principal eigenvalue of (20). Due to \((k + q\mu)\beta \varrho_2 > (\mu + k)(\mu + d)\), we know \( \varrho_2 \in (0, 1) \). On the other hand, Proposition 1 shows that

\[ \text{sign}(1 - R_0^*) = \text{sign}(\lambda_n). \]

Letting \( n \to \infty \), we arrive at a contradiction with \( \lambda_n = 0 \) for all \( n \geq 1 \). This implies that there exists a positive constant \( C \), which is independent of \( d_I \), such that

\[ E(x) \geq C. \]  

(74)

Next, we set \( I(x_1) = \min_{x \in \Omega} I(x) \), for some \( x_1 \in \Omega \). From the third equation of (67), we get that

\[ (\mu + d(x_1))I(x_1) \geq q\beta S(x_1)I(x_1) + k(x_1)E(x_1) \geq k(x_1)E(x_1). \]

By (74), it then follows that

\[ I(x) \geq I(x_1) \geq \frac{Ck_*}{\mu + k_*}. \]  

(75)

**Step 2.** Convergence of \( S \) and \( E \). Since \((S, E)\) solves

\[
\begin{cases}
-d_S \Delta S + \mu S = \Lambda - \beta SI, & x \in \Omega, \\
-d_E \Delta E + (\mu + k)E = (1 - q)\beta SI, & x \in \Omega, \\
\frac{\partial S}{\partial \nu} = \frac{\partial E}{\partial \nu} = 0, & x \in \partial \Omega,
\end{cases}
\]

similarly as in the proof of Theorem 3.6, passing to a sequence of \( d_I \) and a corresponding positive solution \((S_n, E_n, I_n)\), we can assert that

\[ (S_n, E_n) \to (\bar{S}, \bar{E}) \quad \text{in } C(\Omega), \quad \text{as } n \to \infty, \]
where \( \tilde{S}, \tilde{E} > 0 \) on \( \overline{\Omega} \) due to (72) and (74).

Step 3. Convergence of \( I \). We first need to prove \( \mu + d - q\beta \tilde{S}(x) > 0 \). Let \( f(x) = \mu + d - q\beta \tilde{S}(x) \). Assume that \( \min_{\Omega} f(x) \leq 0 \) and \( x_0 = \min_{\Omega} f(x) \), then exist a small ball \( B_0 \subset \Omega \) and a sufficient small \( \epsilon_0 > 0 \) (due to (74)), such that

\[
q\beta S(x_0)I(x_0) + k(x_0)E(x_0) - (\mu + d)I(x_0) \geq k(x_0)E(x_0) \geq \epsilon_0, \quad \text{on } \overline{B_0}
\]

for all small \( d_I > 0 \). Consider the following problem:

\[
\begin{aligned}
-d_I \Delta W &= \epsilon_0, \quad \text{in } B_0, \\
W &= 0, \quad \text{on } \partial B_0.
\end{aligned}
\]  

(76)

Then \( I \) is a supersolution to (76) and \( I \geq W \) in \( B_0 \) for all small \( d_I > 0 \). Clearly, as \( d_I \to 0 \), \( W \to \infty \) locally uniformly in \( B_0 \). Thus, \( I \to \infty \), which is a contradiction. So \( \mu + d - q\beta \tilde{S}(x) > 0 \).

Now, consider the equation satisfied by \( I_n \):

\[
\begin{aligned}
-d_n \Delta I_n &= (q\beta S_n - \mu - d)I_n + kE_n, \quad x \in \Omega, \\
\frac{\partial I_n}{\partial \nu} &= 0, \quad x \in \partial \Omega.
\end{aligned}
\]

For any small \( \epsilon > 0 \) and all large \( n \), it holds that

\[
0 < \tilde{S}(x) - \epsilon \leq S_n(x) \leq \tilde{S}(x) + \epsilon, \quad \forall x \in \overline{\Omega},
\]

\[
0 < \tilde{E}(x) - \epsilon \leq E_n(x) \leq \tilde{E}(x) + \epsilon, \quad \forall x \in \overline{\Omega}.
\]

Thus, if \( n \) is sufficiently large, we have

\[
[q\beta S_n - (\mu + d)]I_n + kE_n \leq [q\beta (\tilde{S} + \epsilon) - (\mu + d)]I_n + k(\tilde{E} + \epsilon), \quad \forall x \in \overline{\Omega},
\]

\[
[q\beta S_n - (\mu + d)]I_n + kE_n \geq [q\beta (\tilde{S} - \epsilon) - (\mu + d)]I_n + k(\tilde{E} - \epsilon), \quad \forall x \in \overline{\Omega}.
\]

Clearly, for any fixed large \( n \), \( I_n \) is a subsolution of

\[
\begin{aligned}
-d_n \Delta Z &= [q\beta (\tilde{S} + \epsilon) - (\mu + d)]Z + k(\tilde{E} + \epsilon), \quad x \in \Omega, \\
\frac{\partial Z}{\partial \nu} &= 0, \quad x \in \partial \Omega,
\end{aligned}
\]  

(77)

and \( I_n \) is a supersolution of

\[
\begin{aligned}
-d_n \Delta Z &= [q\beta (\tilde{S} - \epsilon) - (\mu + d)]Z + k(\tilde{E} - \epsilon), \quad x \in \Omega, \\
\frac{\partial Z}{\partial \nu} &= 0, \quad x \in \partial \Omega.
\end{aligned}
\]  

(78)

By a similar argument to that of Lemma 2.4 of [37], it can be proved that

\[
Z \to \frac{k(\tilde{E} + \epsilon)}{\mu + d - q\beta (\tilde{S} + \epsilon)}, \quad Z \to \frac{k(\tilde{E} - \epsilon)}{\mu + d - q\beta (\tilde{S} - \epsilon)} \quad \text{uniformly on } \overline{\Omega}, \quad \text{as } n \to \infty.
\]

The standard sub-supersolution theory yields that \( I_n \) satisfies

\[
Z \leq \liminf_{n \to \infty} I_n \leq \limsup_{n \to \infty} I_n \leq Z.
\]

Thus we deduce that

\[
I_n \to \frac{k\tilde{E}}{\mu + d - q\beta \tilde{S}} \quad \text{uniformly on } \overline{\Omega}, \quad \text{as } n \to \infty,
\]
4. Analysis on model (5) with immigration.

4.1. Global attractivity of EE to (5). As mentioned before, if one of $W_E$ and $W_I$ is positive, then there is no disease-free equilibrium. In this subsection, in the case that $\Lambda, \beta, \mu, k, d$ are positive constants, we will derive the global attractivity of EE; in the general case that $\Lambda, \beta, \mu, k, d$ are positive functions, we will establish the uniform persistence of solutions.

We start with the uniform boundedness of solutions to (5).

Lemma 4.1. There exists a positive constant $C$ depending on initial data such that the solution $(S, E, I)$ of (5) satisfies

$$
\|S(\cdot, t)\|_{L^\infty(\Omega)} + \|E(\cdot, t)\|_{L^\infty(\Omega)} + \|I(\cdot, t)\|_{L^\infty(\Omega)} \leq C, \quad \forall t \geq 0.
$$

Furthermore, there exists a positive constant $C$ independent of initial data such that

$$
\|S(\cdot, t)\|_{L^\infty(\Omega)} + \|E(\cdot, t)\|_{L^\infty(\Omega)} + \|I(\cdot, t)\|_{L^\infty(\Omega)} \leq C.
$$

for some large time $T \geq 0$.

Proof. The proof is similar to that of Lemma 3.1 and hence the details are omitted.\qed

In the case that $\Lambda, \beta, \mu, k, d$ are positive constants and $(W_E, W_I) \neq (0, 0)$, it is obvious that the equilibrium point of system (1.4) satisfies

$$
\Lambda - \beta SI - \mu S = 0,
$$

$$
W_E + (1-q)\beta SI - (\mu + k)E = 0,
$$

$$
W_I + q\beta SI + kE - (\mu + d)I = 0.
$$

From [28], we know that exists a unique EE, denoted by $X_{**} = (S_{**}, E_{**}, I_{**})$. We can use the method of Lyapunov function to investigate the global attractivity of EE.

Theorem 4.2. The following assertions hold.

(i) Assume that $\Lambda, \beta, \mu, k, d$ are positive constants and $(W_E, W_I) \neq (0, 0)$, the EE $X_{**}$ is globally asymptotically attractive.

(ii) Assume that $\Lambda, \beta, \mu, k, d$ are positive functions and $(W_E, W_I) \neq (0, 0)$, then there exists a real number $\eta > 0$ independent of the initial data, such that any solution $(S, E, I)$ satisfies

$$
\liminf_{t \to \infty} S(x, t) \geq \eta, \quad \liminf_{t \to \infty} E(x, t) \geq \eta, \quad \liminf_{t \to \infty} I(x, t) \geq \eta,
$$

uniformly for $x \in \Omega$, and hence, the disease persists uniformly. Furthermore, (5) admits at least one EE.

Proof. We first verify (i). For our purpose, define the following Lyapunov function:

$$
W(t) = \int_\Omega \left[ L(S(x, t), E(x, t), I(x, t)) \right] dx, \quad \forall t > 0,
$$

where

$$
L = S - S_{**} \ln S + A(E - E_{**} \ln E) + B(I - I_{**} \ln I),
$$

and

$$
\frac{\partial L}{\partial S} = \frac{\partial L}{\partial E} = \frac{\partial L}{\partial I} = 0.
$$
with A and B being chosen the same as in (40). Thus, basic calculations show
\[
\frac{dW(t)}{dt} = \int_{\Omega} \left( \frac{\partial L}{\partial S} \frac{\partial S}{\partial t} + \frac{\partial L}{\partial E} \frac{\partial E}{\partial t} + \frac{\partial L}{\partial I} \frac{\partial I}{\partial t} \right) dx
\]
\[
= \int_{\Omega} \left( 1 - \frac{S_*}{S} \right) \left( d_S \Delta S + \Lambda - \beta S I - \mu S - (\Lambda - \beta S_* I_* - \mu S_*) \right) dx
\]
\[
+ \int_{\Omega} A \left( 1 - \frac{E_*}{E} \right) \left\{ [d_E \Delta E + W_E + (1 - q)\beta SI - (\mu + k)E] \right\} dx
\]
\[
- \frac{E}{E_*} \left( W_E + (1 - q)\beta S_* I_* - (\mu + k)E_* \right) dx
\]
\[
+ \int_{\Omega} B \left( 1 - \frac{I_*}{I} \right) \left\{ [d_I \Delta I + W_I + q\beta SI + kE - (\mu + d)I] \right\} dx
\]
\[
- \frac{I}{I_*} \left( W_I + q\beta S_* I_* + kE_* - (\mu + d)I_* \right) \right\} dx
\]
\[
= - \int_{\Omega} \left\{ d_S \frac{S_*}{S} |\nabla S|^2 + B d_I \frac{I_*}{I} |\nabla I|^2 + A d_E \frac{E_*}{E} |\nabla E|^2 \right\} dx
\]
\[
+ \int_{\Omega} \left\{ 2 \frac{s_*}{s} - \frac{S}{S_*} + Bq\beta S_* I_* \left( 2 \frac{s_*}{s} - \frac{S}{S_*} \right) \right\} dx
\]
\[
+ A(1 - q)\beta S_* I_* \left( 3 \frac{s_*}{s} - \frac{S}{S_*} \right) \left( \frac{S}{S_*} \right) \right\} dx
\]
\[
+ A W_E \left( 2 \frac{E_*}{E} - \frac{E}{E_*} \right) + B W_I \left( 2 \frac{I_*}{I} - \frac{I}{I_*} \right) \right\} dx, \quad t > 0.
\]
Observe that \((2 - \frac{s_*}{s} - \frac{S}{S_*}), (2 - \frac{s_*}{s} - \frac{S}{S_*}), (2 - \frac{s_*}{s} - \frac{S}{S_*})\) and \((3 - \frac{s_*}{s} - \frac{S}{S_*}), (1 - q)\beta S_* I_* \left( 3 \frac{s_*}{s} - \frac{S}{S_*} \right) \right\} dx, \quad t > 0
\]
and the inequality holds if and only if \((S, E, I) = (S_*, E_*, I_*)\). Combined with Lemma 4.1, some standard arguments imply that
\[
(S(x, t), E(x, t), I(x, t)) \to (S_*, E_*, I_*) \quad \text{in} \quad [L^\infty(\Omega)]^3, \quad \text{as} \quad t \to \infty.
\]
Therefore, \(X_*\) is globally attractive. \(\square\)

4.2. Asymptotic profiles of the EE. The EE problem corresponding to (5) is governed by the following elliptic system:
\[
\begin{cases}
-d_S \Delta S = \Lambda(x) - \beta(x) S I - \mu(x) S, & x \in \Omega, \\
-d_E \Delta E = W_E + (1 - q)\beta(x) S I - [\mu(x) + k(x)] E, & x \in \Omega, \\
-d_I \Delta I = W_I + q\beta(x) S I + k(x) E - [\mu(x) + d(x)] I, & x \in \Omega, \\
\frac{\partial S}{\partial \nu} = \frac{\partial E}{\partial \nu} = \frac{\partial I}{\partial \nu} = 0, & x \in \partial \Omega.
\end{cases}
\tag{79}
\]
Our first result concerns the case of \(d_S \to 0\), and reads as follows.

**Theorem 4.3.** Fix \(d_E > 0, d_I > 0\) and let \(d_S \to 0\). Then any positive solution \((S, E, I)\) of (79) satisfies (up to a subsequence of \(d_S \to 0\)) that \((S, E, I) \to (\tilde{S}, \tilde{E}, \tilde{I})\) uniformly on \(\Omega\), where
\[
\tilde{S}(x) = \frac{\Lambda(x)}{\beta(x) I(x) + \mu(x)},
\tag{80}
\]
and \((\tilde{S}, \tilde{I})\) is a positive solution to
\[
\begin{cases}
-d_E \Delta \tilde{E} = W_E + (1 - q) \beta(x) \tilde{S} \tilde{I} - [\mu(x) + k(x)] \tilde{E}, & x \in \Omega, \\
-d_I \Delta \tilde{I} = W_I + q \beta(x) \tilde{S} \tilde{I} + k(x) \tilde{E} - [\mu(x) + d(x)] \tilde{I}, & x \in \Omega, \\
\frac{\partial \tilde{E}}{\partial \nu} = \frac{\partial \tilde{I}}{\partial \nu} = 0, & x \in \partial \Omega.
\end{cases}
\] (81)

**Proof.** In the following, \(C\) is a positive constant independent of \(d_S > 0\).

**Step 1.** Estimates of upper bounds of \(S, E, I\). Let \(S(x_1) = \max_{x \in \Omega} S(x)\) for some \(x_1 \in \Omega\). By Proposition 2.2 of [27], we have from the first equation in (79) that
\[
\Lambda(x_1) \geq \beta(x_1) S(x_1) I(x_1) + \mu(x_1) S(x_1) \geq \mu(x_1) S(x_1).
\]
Thus, we get
\[
S(x) \leq S(x_1) \leq \frac{\Lambda^*}{\mu_*}, \quad \forall x \in \Omega.
\] (82)

Define \(W(x) = d_S S + d_E E + d_I I\). Then we have
\[
-\Delta (d_S S + d_E E + d_I I) = \Lambda(x) + W_E + W_I - \mu(x) S - \mu(x) E - [\mu(x) + d(x)] I.
\]
We can set \(\max_{x \in \Omega} W(x) = W(x_1)\)
\[
\Lambda(x_1) + W_E + W_I \geq \mu(x_1) S(x_1) + \mu(x_1) E(x_1) + [\mu(x_1) + d(x_1)] I(x_1).
\]
Let \(\max_{x \in \Omega} W(x) = W(x_1) = \max\{d_S, d_E, d_I\} \Lambda^* + W_E + W_I.\) Thus it holds that
\[
I(x) \leq \max_{x \in \Omega} I(x) \leq \max \left\{ \frac{d_S, d_E, d_I}{d_I} \Lambda^* + W_E + W_I, \right\},
\] (83)
\[
E(x) \leq \max_{x \in \Omega} E(x) \leq \max \left\{ \frac{d_S, d_E, d_I}{d_E} \Lambda^* + W_E + W_I. \right\}.
\] (84)

**Step 2.** Estimates of lower bounds. We now set \(S(x_2) = \min_{x \in \Omega} S(x)\) for some \(x_2 \in \Omega\) and obtain
\[
\Lambda(x_2) \leq \beta(x_2) S(x_2) I(x_2) + \mu(x_2) S(x_2),
\] (85)
equivalently,
\[
\frac{\Lambda(x_2)}{\beta(x_2) C + \mu(x_2)} \leq \frac{\Lambda(x_2)}{\beta(x_2) I(x_2) + \mu(x_2)} \leq S(x_2) \leq S(x).
\]
This then gives
\[
S(x) \geq \min_{x \in \Omega} S(x) = S(x_2) \geq C > 0.
\] (86)

In order to establish the positive lower bounds of \(E, I\), we need to treat three cases as follows.

**Case (i).** \(W_E > 0, W_I > 0\). In this case, Set \(E(x_2) = \min_{x \in \Omega} E(x)\) for some \(x_2 \in \Omega\). Similarly as before, by the second equation in (79), we have
\[
W_E \leq W_E + (1 - q) \beta(x_2) S(x_2) I(x_2) \leq [\mu(x_2) + k(x_2)] E(x_2).
\]
Thus, there holds
\[
E(x) \geq \frac{W_E}{\mu_* + k_*},
\] (87)
Next we set $I(x_2) = \min_{x \in \Omega} I(x)$ for some $x_2 \in \Omega$ to infer that

$$W_I \leq W_I + q\beta(x_2)S(x_2)I(x_2) + k(x_2)E(x_2) \leq [\mu(x_2) + d(x_2)]I(x_2).$$

As a result, one gets

$$I(x) \geq \frac{W_I}{\mu^* + d^*}. \quad (88)$$

**Case (ii).** $W_E > 0$, $W_I = 0$. It is easy to see that (86) and (87) are still valid. It remains to derive the lower bound of $I$. Let $I(x_2) = \min_{x \in \Omega} I(x)$ for some $x_2 \in \Omega$. Then we have

$$k_*C \leq q\beta(x_2)S(x_2)I(x_2) + k(x_2)E(x_2) \leq [\mu(x_2) + d(x_2)]I(x_2).$$

This gives

$$I(x) \geq \frac{k_*C}{\mu^* + d^*}. \quad (89)$$

**Case (iii).** $W_E = 0$, $W_I > 0$. Note that (86) and (88) remain valid. We only need to derive the lower bound of $E$. Let $E(x_2) = \min_{x \in \Omega} E(x)$. One then finds that

$$(1-q)\beta_*C \leq (1-q)\beta(x_2)S(x_2)I(x_2) \leq [\mu(x_2) + k(x_2)]E(x_2).$$

By (86) and (88), we have

$$E(x) \geq \frac{(1-q)\beta_*C}{\mu^* + k^*}.$$

By the above analysis, we can conclude that

$$\frac{1}{C} \leq S(x), \ E(x), \ I(x) \leq C, \ \forall x \in \Omega. \quad (90)$$

**Step 3.** Convergence of $S, E, I$. In light of (90), we obtain

$$\|S\|_{L^p(\Omega)}, \ |E|_{L^p(\Omega)}, \ |I|_{L^p(\Omega)} \leq C, \ \forall 1 \leq p < \infty. \quad (91)$$

It follows from that

$$\|E\|_{W^{1,p}(\Omega)}, \ |I|_{W^{1,p}(\Omega)} \leq C, \ \forall 1 \leq p < \infty.$$}

Then taking sufficiently large $p$, the standard embedding theorem enables us to conclude that, up to a sequence of $d_S \to 0$, denoted by $d_i := d_{S,i}$, with $d_i \to 0$ as $i \to \infty$, the corresponding positive solution sequence $(S_i, E_i, I_i := S_{S,i}, E_{S,i}, I_{S,i})$ of (79) with $d_{S,i} = d_S$ fulfills

$$E_i \to \tilde{E}, \ I_i \to \tilde{I} \ \text{in} \ C^1(\overline{\Omega}), \ \text{as} \ i \to \infty, \quad (92)$$

where $\tilde{E}, \tilde{I} \in C^1(\overline{\Omega})$ and $\tilde{E}, \tilde{I} > 0$ on $\overline{\Omega}$ due to (90). Note that $S_i$ satisfies

$$\begin{cases} -d_i\Delta S_i = \Lambda(x) - \beta(x)S_iI_i - \mu(x)S_i, & x \in \Omega, \\ \partial S_i \big/ \partial \nu = 0, & x \in \partial \Omega. \end{cases} \quad (93)$$

Because of (92) and (93), one can apply a simple sub-supersolution comparison as in Theorem 3.5 to conclude that $S_i \to \tilde{S}$ uniformly on $\overline{\Omega}$, as $i \to \infty$, where $\tilde{S}$ is given by (80). It is clear that $(\tilde{E}, \tilde{I})$ satisfies (81) and Theorem 4.3 is proved.

Our second result concerns the case of $d_E \to 0$ and can be stated as follows.
Fix $d_S > 0, d_I > 0$ and let $d_E \to 0$. Then any positive solution $(S,E,I)$ of (79) satisfies (up to a subsequence of $d_E \to 0$) that $(S,E,I) \to (\tilde{S},\tilde{E},\tilde{I})$ uniformly on $\Omega$, where
\[
\tilde{E}(x) = \frac{W_E + (1-q)\beta(x)\tilde{S}\tilde{I}}{k(x) + \mu(x)},
\]
and $(\tilde{S},\tilde{I})$ is a positive solution to
\[
\begin{cases}
-d_S \Delta \tilde{S} = \Lambda(x) - \beta(x)\tilde{S}\tilde{I} - \mu(x)\tilde{S}, & x \in \Omega, \\
-d_I \Delta \tilde{I} = W_I + q\beta(x)\tilde{S}\tilde{I} + k(x)\tilde{E} - [\mu(x) + d(x)]\tilde{I}, & x \in \Omega, \\
\frac{\partial \tilde{S}}{\partial \nu} = \frac{\partial \tilde{I}}{\partial \nu} = 0, & x \in \partial \Omega.
\end{cases}
\]

Proof. A simple check shows that the lower bound of $S,E,I$ is exactly the same as in step 2 of the proof of Theorem 4.3. It is easy to see that (82) and (83) are still valid. So we just have to find an upper bound on $E(x)$. Now we set $E(x_1) = \max_{x \in \Omega} E(x)$ for some $x_1 \in \Omega$. Then
\[
[\mu(x_1) + k(x_1)]E(x_1) \leq W_E + (1-q)\beta(x_1)S(x_1)I(x_1),
\]
Thus,
\[
E(x) \leq E(x_1) \leq \frac{W_E + (1-q)\beta^* C}{\mu_0 + k_*}.
\]
Therefore, there exists a positive constant $C$, independent of $d_E > 0$, such that
\[
\frac{1}{C} \leq S(x), E(x), I(x) \leq C, \quad \forall x \in \Omega.
\]
By (97), we can get that
\[
\|S\|_{L^p(\Omega)}, \|E\|_{L^p(\Omega)}, \|I\|_{L^p(\Omega)} \leq C, \quad \forall 1 \leq p < \infty.
\]
It then follows that
\[
\|S\|_{W^{2,p}(\Omega)}, \|E\|_{W^{2,p}(\Omega)} \leq C, \quad \forall 1 \leq p < \infty.
\]
Then taking sufficiently large $p$, the standard embedding theorem enables us to conclude that, up to a sequence of $d_E \to 0$, denoted by $d_i := d_{E,i}$, with $d_i \to 0$ as $i \to \infty$, the corresponding positive solution sequence $(S_i, E_i, I_i := S_{E,i}, E_{E,i}, I_{E,i})$ of (79) with $d_{E,i} = d_E$ fulfills
\[
S_i \to \tilde{S}, I_i \to \tilde{I} \quad \text{in} \quad C^1(\Omega), \quad \text{as} \quad i \to \infty,
\]
where $\tilde{S}, \tilde{I} \in C^1(\Omega)$ and $\tilde{S}, \tilde{I} > 0$ on $\Omega$ due to (97). Note that $E_i$ satisfies
\[
\begin{cases}
-d_i \Delta E_i = W_E + (1-q)\beta(x)S_i I_i - [\mu(x) + k(x)]I_i, & x \in \Omega, \\
\frac{\partial E_i}{\partial \nu} = 0, & x \in \partial \Omega.
\end{cases}
\]
Making use of (98) and (99), one can apply a simple sub-supersolution comparison as Theorem 3.6 to conclude that $E_i \to \tilde{E}$, as $i \to \infty$, where $\tilde{E}$ is given by (94). It is clear that $(\tilde{S}, \tilde{I})$ satisfies (95). Thus, Theorem 4.4 is proved.

At last, we are concerned with the asymptotic behavior of $E$ as $d_I \to 0$. 


\[\]
Theorem 4.5. Assume that \( \min\{\mu_*, d_*\} > \max\{q\beta^\Omega/\rho\} \). Fix \( d_S > 0, d_E > 0 \) and let \( d_L \to 0 \). Then any positive solution \((S,E,I)\) of (79) satisfies (up to a subsequence of \( d_L \to 0 \)) that \((S,E,I) \to (\tilde{S}, \tilde{E}, \tilde{I})\) uniformly on \( \Omega \), where

\[
\tilde{I}(x) = \frac{W_I + k(x)\tilde{E}}{\mu(x) + d(x) - q\beta(x)\tilde{S}},
\]

and \((\tilde{S}, \tilde{E})\) is a positive solution to

\[
\begin{aligned}
-d_S \Delta \tilde{S} &= \Lambda(x) - \beta(x)\tilde{S}\tilde{I} - \mu(x)\tilde{S}, \quad x \in \Omega, \\
d_E \Delta \tilde{E} &= W_E(x) + (1 - q)\beta(x)\tilde{S}\tilde{I} - [\mu(x) + k(x)]\tilde{E}, \quad x \in \Omega, \\
\frac{\partial \tilde{S}}{\partial \nu} &= \frac{\partial \tilde{E}}{\partial \nu} = 0, \quad x \in \partial \Omega.
\end{aligned}
\]

Proof. By a simple check, we can see that (82) and (84) are still valid. Letting \( I(x_1) = \max_{x \in \Omega} I(x) \) for some \( x_1 \in \Omega \), we get that

\[
W_I + k^* C \geq W_I + k(x_1)E(x_1) \geq [\mu(x_1) + d(x_1) - q\beta(x_1)S(x_1)]I(x_1).
\]

By (82), we further have

\[
\frac{W_I + k^* C}{\mu_* + d_* - q\beta} \geq I(x_1) \geq I(x).
\]

We now set \( S(x_2) = \min_{x \in \Omega} S(x) \) for some \( x_2 \in \Omega \) to deduce

\[
\Lambda(x_2) \leq \beta(x_2)S(x_2)I(x_2) + \mu(x_2)S(x_2),
\]

equivalently,

\[
\frac{\Lambda(x_2)}{\beta(x_2)C + \mu(x_2)} \leq \frac{\Lambda(x_2)}{\beta(x_2)I(x_2) + \mu(x_2)} \leq S(x_2) \leq S(x).
\]

This gives

\[
S(x) \geq \min_{x \in \Omega} S(x) = S(x_2) \geq C > 0.
\]

We can use the same argument as in the proof of Theorem 4.3 to derive the lower bound of \( E, I \). Hence, there exists a positive constant \( C \) independent of \( d_L > 0 \), such that

\[
\frac{1}{C} \leq S(x), \quad E(x), \quad I(x) \leq C, \quad \forall x \in \Omega.
\]

(104)

Obviously, \( \mu(x) + d(x) - q\beta(x)\tilde{S} > 0 \) through a similar discussion as in the proof of Theorem 3.7. By (104), it follows that

\[
\|S\|_{W^{2,p}(\Omega)}, \quad \|I\|_{W^{2,p}(\Omega)} \leq C, \quad \forall 1 \leq p < \infty.
\]

Then taking sufficiently large \( p \), one can apply a standard compactness to assert that, up to a sequence of \( d_L \to 0 \), denoted by \( d_i := d_{L,i} \), with \( d_i \to 0 \) as \( i \to \infty \), the corresponding positive solution sequence \((S_i, E_i, I_i) := S_{L,i}, E_{L,i}, I_{L,i}\) of (79) with \( d_{L,i} = d_L \) fulfills

\[
S_i \to \tilde{S}, \quad E_i \to \tilde{E} \quad \text{in} \quad C^1(\Omega), \quad \text{as} \quad i \to \infty,
\]

where \( \tilde{S}, \tilde{E} \in C^1(\Omega) \) and \( \tilde{S}, \tilde{E} > 0 \) on \( \Omega \) due to (104). Since \( I_i \) satisfies

\[
\begin{aligned}
-d_I \Delta I_i &= W_I + q\beta(x)S_iI_i + k(x)E_i - [\mu(x) + d(x)]I_i, \quad x \in \Omega, \\
\frac{\partial I_i}{\partial \nu} &= 0, \quad x \in \partial \Omega.
\end{aligned}
\]

As \( \tilde{I} \) is a positive solution to (101), we can apply the maximum principle to assert that \( \tilde{I} \) is bounded on \( \Omega \) and \( \tilde{I} \) is also a positive solution to (101). Then taking sufficiently large \( p \), one can apply a standard compactness to assert that, up to a sequence of \( d_L \to 0 \), denoted by \( d_i := d_{L,i} \), with \( d_i \to 0 \) as \( i \to \infty \), the corresponding positive solution sequence \((S_i, E_i, I_i) := S_{L,i}, E_{L,i}, I_{L,i}\) of (79) with \( d_{L,i} = d_L \) fulfills

\[
S_i \to \tilde{S}, \quad E_i \to \tilde{E} \quad \text{in} \quad C^1(\Omega), \quad \text{as} \quad i \to \infty,
\]

where \( \tilde{S}, \tilde{E} \in C^1(\Omega) \) and \( \tilde{S}, \tilde{E} > 0 \) on \( \Omega \) due to (104). Since \( I_i \) satisfies

\[
\begin{aligned}
-d_I \Delta I_i &= W_I + q\beta(x)S_iI_i + k(x)E_i - [\mu(x) + d(x)]I_i, \quad x \in \Omega, \\
\frac{\partial I_i}{\partial \nu} &= 0, \quad x \in \partial \Omega.
\end{aligned}
\]

(106)
one can use (105) and (106), together with a simple sub-supersolution comparison, to deduce that $I_i \to \tilde{I}$, as $i \to \infty$, where $\tilde{I}$ is given by (100). Obviously, $(\tilde{S}, \tilde{E})$ satisfies (101). This completes the proof of Theorem 4.5.

5. Conclusion. In this paper, we have studied two SEI reaction-diffusion epidemic models, that is, (2) and (5). The only difference between (2) and (5) is that the former ignores the immigration of infected (exposed) individuals, while the latter takes such a factor into account, thus better describing the transmission dynamics of certain diseases including, for example, the COVID-19 pandemic [28] and the Wolbachia infection [11, 52, 54].

For system (2), we have shown that the basic reproduction number $R_0$ serves as a threshold value in the sense that the disease-free equilibrium state is globally attractive if $R_0 \leq 1$ and the positive equilibrium is globally attractive if $R_0 > 1$ when the system parameters are positive constants, and the disease uniformly persist in space if $R_0 > 1$ when the system parameters are positive functions; see Theorems 3.2 and 3.3. For system (5), due to the presence of the immigration of infected or exposed individuals, the disease will always uniformly persist in space as shown by Theorem 4.2. These results suggest that the risk of the disease modelled by (2) may be underestimated provided that the immigration of infected or exposed population is ignored.

To understand how small immigration rate of the susceptible, infected or exposed population affects the spatial distribution of the disease modelled by (2) and (5), as in [6, 22, 32, 36, 47, 51, 55, 56], we have studied the asymptotic behavior of the EE (when it exists) as the immigration rate $d_S$, $d_I$ or $d_E$ tends to zero. To guarantee the existence of EE for (4) with respect to small $d_S$, $d_I$ or $d_E$, we are led to determine the limits of the the basic reproduction number $R_0$; see Proposition 2. Our results show that as the mobility of the susceptible or exposed or infected population goes to zero, the disease will always be present in the entire habitat, and as a consequence, the disease described by (2) and (5) cannot be eliminated by controlling the mobility of susceptible, exposed, and infected populations alone.

In view of the above discussion, we may conclude that the decision-makers must prevent the importation of foreign infected people in order to achieve the goal of eliminating the disease.

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