What happens to spin during the $SO(3) \rightarrow SE(2)$ contraction?
(On spin and extended structures in quantum mechanics)

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As is known, a Lie algebra of a little group of a timelike four vector is not equal to so(3) unless spacelike components of the vector vanish. In spite of this fact the algebra can still be interpreted as the angular momentum algebra, as can be shown with the explicit example of the Dirac equation. The angular momentum corresponds to the even part of the Dirac spin operator. Its eigenvalues in directions perpendicular to momentum decay to zero in the infinite momentum/massless limit. This explains physically why only extremal helicities survive the massless limit. The effect can be treated as a result of a Lorentz contraction of an extended particle. A natural measure of this extension is introduced for massless particles of any spin. It is shown that such particles can be interpreted as circular strings whose classical limit is described by Robinson congruence. Finally, as an application of the even spin, we formulate the Einstein-Podolsky-Rosen-Bohm Gedankenexperiment for Dirac electrons.

I. INTRODUCTION

It is known that the transition from the rotation group $SO(3)$ to the group $SE(2)$ of two-dimensional Euclidean motions can be obtained as a continuous In" onu-Wigner contraction of respective Lie algebras. This contraction can be explained in physical terms simply as an abstract counterpart of the relativistic Lorentz contraction of a moving body and it can be also shown that the same effect is obtained both in the massless ($m \rightarrow 0$) and infinite momentum ($|\vec{p}| \rightarrow \infty$) limits. This result is intuitively quite natural since a light cone is the large-momentum asymptotics of any $m > 0$ mass hyperboloid. Still, even though the $SO(3) \rightarrow SE(2)$ transition is now well understood from a mathematical point of view, it seems that no definite physical interpretation of this fact is generally accepted.

We know that the Lie algebra so(3) corresponds physically to an angular momentum. On the other hand there exist at least two physical interpretations of the translation subalgebra of $e(2)$. First, the generators of translations can be naturally associated with the two-component position operator for massless fields localizing massless particles in a plane perpendicular to their momentum. Second, if one considers the action of a four-dimensional nonunitary representation of $SE(2)$ on the electromagnetic four-potential, it can be shown that the “translation” operators generate gauge transformations. Such results are rather confusing as there is no clear explanation for the continuous deformation of the angular momentum into position or a generator of gauge transformations. Of course, it is possible that for massless fields various physically different observables may satisfy the $e(2)$ algebra since massless unitary representations of the Poincaré group are generated by one-dimensional representations of $E(2)$ so that not much room is left for different possible algebraic structures.

We would like to show in this paper that it is possible to interpret the contraction so(3) → $e(2)$ also as the continuous deformation of the angular momentum of a relativistic particle. The Lie algebra corresponding to the intermediate momentum $\vec{p}$, with $0 < |\vec{p}| < \infty$, is neither so(3) nor $e(2)$ but, as we shall see, there is a natural way of treating it as the relativistically generalized angular momentum algebra.

To make our analysis as explicit as possible we shall discuss the mentioned limits in the context of the Dirac equation. It will be shown that relativistic spin can be naturally represented by an “even” spin operator which reduces to ordinary nonrelativistic spin for a particle at rest and whose components in directions perpendicular to momentum decay to zero as the momentum increases. In the infinite-momentum/massless limit the operator “points” in the momentum direction. The three components of the new spin commute with the free Hamiltonian so that one can consider projections of spin in any direction even for a particle moving with some well defined and nonvanishing momentum. This property will be used for deriving a relativistic version of the Einstein-Podolsky-Rosen-Bohm paradox for two spin-1/2 particles. We shall also discuss the relationship between the even spin operator and the Pauli-Lubanski vector, showing that the infinite-momentum and massless limits are equivalent for the first one but not for the latter.

We begin our analysis with a brief summary of properties of little algebras corresponding to any timelike or null four-momentum.
II. LITTLE ALGEBRAS FOR ANY $P$ WITH $P^0 P_0 \geq 0$

A Lorentz transformation

$$\Lambda(\vec{\mu}, \vec{\nu}) = e^{-i\vec{\mu} \vec{J} - i\vec{\nu} \vec{K}} \equiv e^{-iL(\vec{\mu}, \vec{\nu})},$$

(1)

where $\vec{J}$ and $\vec{K}$ are, respectively, the generators of rotations and boosts, leaves a four-vector $p$ unchanged if

$$L(\vec{\mu}, \vec{\nu})p = i(\vec{\nu} \cdot \vec{p}, \vec{\mu} \times \vec{p}) = 0.$$  

(2)

If $p_0 = 0$ then either $p$ is massless and $p = 0$ or $p$ is space-like. Both cases can be regarded as physically meaningless. So let us assume that $p_0 \neq 0$. (3) is then satisfied if

$$\vec{\nu} p_0 = -\vec{\mu} \times \vec{p}.$$  

(3)

Substituting (3) into (1) we find that the little group is a three-parameter and generated by

$$\vec{L} = \vec{J} - \frac{\vec{p}}{p_0} \times \vec{K},$$

(4)

satisfies the algebra

$$\vec{L} \times \vec{L} = i\vec{J} - i\frac{\vec{p}}{p_0}(\vec{p} \cdot \vec{K}).$$

(5)

Let now $\vec{n} = \vec{p}/|\vec{p}|$, $\vec{m}$ be orthogonal to $\vec{n}$, $|\vec{m}| = 1$, and $\tilde{\vec{L}} = \vec{n} \times \vec{m}$ and let $A_1 = \vec{A} \cdot \vec{m}$, $A_2 = \vec{A} \cdot \tilde{\vec{L}}$ and $A_3 = \vec{A} \cdot \vec{n}$ for any three-component vector $\vec{A}$. We get

$$L_1 = J_1 + K_2 \frac{|\vec{p}|}{p_0},$$

$$L_2 = J_2 - K_1 \frac{|\vec{p}|}{p_0},$$

$$L_3 = J_3$$

(6)

and (3) implies

$$[L_1, L_2] = i\frac{\vec{p} \cdot \vec{p}_0}{p_0} L_3$$

$$[L_3, L_1] = iL_2$$

$$[L_2, L_3] = iL_1.$$  

(7)

Eqs. (1) and (4) mean that the Lie algebra of the little group is parametrized by $\vec{p}$. In the massive case we, in general, do not obtain $so(3)$; in fact this is the case only if $|\vec{p}| = 0$, that is when we consider the little group of a rest frame four-momentum. Such a rest frame always exists for $m > 0$. For $m = 0$ we have

$$p_0 = \pm |\vec{p}|.$$  

(8)

and (3) is indeed the Lie algebra $e(2)$. Note also that $\frac{\vec{p}}{p_0} \rightarrow \pm 1$ in the infinite momentum limit and, as expected, the algebra is again $e(2)$. The same conclusions follow from the $m \rightarrow 0$ and $|\vec{p}| \rightarrow \infty$ contractions of (4).

The two contractions are equivalent and the contracted algebra is $e(2)$. A reader interested in a geometrical interpretation of these contractions is referred to the book by Kim and Noz (p. 200 in [3]).

The form (6) of the generators leads to a difficulty in a direct physical interpretation, as $L_1$ and $L_2$ are not Hermitian for finite dimensional representations of $SL(2, C)$ and nonvanishing $\vec{p}$. For photons represented by a four-potential these operators generate gauge transformations. For the rest frame four-momentum of massive particles such representations correspond to spin. However, what is a physical meaning of this algebra for particles that are not at rest? In order to answer this question we must first understand in what way the spin operator enters relativistic quantum mechanics.

III. SPIN OF THE ELECTRON AND THE DIRAC EQUATION

The Uhlenbeck and Goudsmith idea of an internal angular momentum of the electron was formally introduced to quantum mechanics by Pauli in 1927 [3]. Pauli added a new interaction term to the Schrödinger equation in order to explain a behavior of electrons in a magnetic field. This approach was successful but, in fact, no other justification of the concept of spin existed at that time. A year later Dirac formulated his relativistic wave equation [3]. He assumed that the equation (1) has a Schrödinger form $i\partial_t \Psi = H \Psi$ where $H$ does not contain time derivatives, (2) factorizes the Klein-Gordon equation, (3) is relativistically covariant, and found that no single-component wave function can satisfy such requirements. The additional degrees of freedom present in the multicomponent wave function could be interpreted physically by a non-relativistic approximation where positive energy solutions of the Dirac equation were shown to satisfy the equation postulated by Pauli. The Pauli spin operator was found to be an “internal” part of the generator of rotations restricted to “large” components of a bispinor.

In this way the spin operator was identified, from a mathematical viewpoint, with the spinor part of the generator of rotations.

The first difficulty met in relativistic interpretation of this operator was the fact that, contrary to the nonrelativistic case, the components of spin were not constants of motion even for a free particle (unless in a rest frame). In the Heisenberg picture the spin operator of the free electron satisfies the following precession equation

$$\vec{S} = \vec{\omega} \times \vec{S},$$  

(9)

where $\vec{\omega} = -2\gamma^{5}\vec{p}$ and only the projection of $\vec{S}$ on the “precession axis” $\vec{p}$, the helicity, is conserved. The total angular momentum $\vec{J}$ also commutes with the Dirac
Hamiltonian and the purely spin part of $\vec{J}$ can be extracted by $\vec{n} \cdot \vec{J} = \vec{n} \cdot \vec{S}$, where $\vec{n} = \vec{p}/|\vec{p}|$. The existence of the conserved helicity is sufficient for representation classification purposes and physical applications in, for instance, the Clebsch-Gordan coefficients problem. The components of spin in directions other than $\vec{n}$ are rarely needed. In the last section of this paper we shall consider one such case, namely the Einstein-Podolsky-Rosen-Bohm Gedankenexperiment and the Bell inequality for relativistic electrons.

The general theory of representations of the Poincaré group shows that the group possesses two Casimir operators: the mass $P^\alpha P_\alpha$ and the square $W^{\alpha}W_\alpha$ of the Pauli-Lubanski vector

$$W_\alpha = \frac{1}{2} \epsilon_{\alpha \beta \gamma \delta} M^\beta \gamma P^\delta$$

where $M^{\beta \gamma}$ are generators of SL$(2, C)$. $W_\alpha$ commutes with $P^\beta$ for all $\beta$ hence, in particular, with $P^0$. The Pauli-Lubanski vector appears naturally in the theory because it, in fact, generalizes the generators of the little group described in Sec. II. The Casimir $W^\alpha W_\alpha$ has eigenvalues $m^2 j(j + 1)$ where $m$ and $j$ are, respectively, the rest mass and the modulus of helicity of the irreducible representation in question.

In standard approaches to relativistic field theories it is often stated that $W_\alpha$ is the covariant generalization of spin. Its square is defined as the following element of the enveloping field of the Poincaré Lie algebra

$$W^{\alpha}W_\alpha = \frac{P^\beta P^\delta}{P^2}.$$  

We have, therefore, two possibilities of introducing the spin operator in Poincaré invariant theories. The Pauli-Lubanski vector has the advantage of having four conserved components and is closely related to the generators of the little group of a momentum four-vector. The dimension of $W_\alpha$ is, however, energy times angular momentum so its relationship to Pauli’s 1927 spin is not evident.

### IV. THE PAULI-LUBANSKI VECTOR AND ELECTRON’S SPIN

In the following discussion we will work in the momentum representation and units are chosen in such a way that $c = 1 = \hbar$. The bispinor parts of generators are $S^{\alpha \beta} = \frac{1}{2} \gamma^{[\alpha} \gamma^{\beta]}$ and the generators of SL$(2, C)$ are

$$M^{\alpha \beta} = x^\alpha p^\beta - x^\beta p^\alpha + S^{\alpha \beta}.$$ 

Let $\vec{S}$, $\vec{\alpha}$, $\vec{J}$ and $\vec{K}$ be defined by

$$S^{kl} = \epsilon^{klm} S^m$$

$$S^{0k} = \frac{i}{2} \alpha^k$$

$$J^m = \epsilon^{mkl} M^{kl}$$

$$K^m = M^{0m}$$

where $\epsilon^{klm}$ is the three dimensional Levi-Civita symbol. The explicit form of the generators of the Poincaré group is

$$P^0 = H = \vec{\alpha} \cdot \vec{p} + m \gamma^0$$

$$\vec{p} = \vec{p}$$

$$\vec{J} = \vec{x} \times \vec{p} + \vec{S}$$

$$\vec{K} = i \vec{p} - \vec{x} H + \frac{i}{2} \vec{\alpha}$$

The Pauli-Lubanski vector is

$$W^0 = \vec{J} \cdot \vec{p}$$

$$\vec{W} = \frac{1}{2} (\vec{S} H + H \vec{S}).$$  \hspace{1cm} (10)

In the rest frame we find $W^0 = 0$ and $\vec{W} = m \vec{S} \gamma^0$. $\vec{W}$ thereby spin times the rest mass operator. In the subspace of the “large”, positive energy two component spinors it is indeed proportional to Pauli’s spin.

In order to understand the physical meaning of the Pauli-Lubanski vector let us multiply $\vec{S} \gamma_0$ from the right by $H^{-1}$ (this operator is well defined for massive fields; for massless fields it exists in the subspace of nonzero momenta). We get

$$\vec{S} p = \vec{W} H^{-1} = \frac{1}{2} (\vec{S} + \lambda \vec{S} \lambda)$$

$$= \Pi_+ \vec{S} \Pi_+ + \Pi_- \vec{S} \Pi_-$$

where $\lambda$ is the sign of energy operator and $\Pi_{\pm} = \frac{1}{2} (1 \pm \lambda)$ project on positive (+) or negative (−) energy solutions. The operator $\vec{S} p$ is therefore the so-called even part of the generator of rotations $\vec{S}$. The decomposition of operators into even and odd parts is well known in first quantized approaches to the Dirac equation [4,5]. The even parts of operators are effectively the parts that contribute to average values of observables calculated in states of a definite sign of energy. The even spin operator occurs naturally in the context of Zitterbewegung and the magnetic-moment operator of the Dirac electron [4,5].

All these facts suggest that the even spin operator, which is somehow in between the two notions discussed above, might be the correct candidate for the electron’s spin. All the three components of $\vec{S} p$ commute with $H$ so a projection of $\vec{S} p$ in any direction is a constant of motion.

The explicit form of $\vec{S} p$ for free electrons moving with momentum $\vec{p}$ is the following

$$\vec{S} p = \frac{m^2}{p^0} \vec{S} + \frac{|\vec{p}|^2}{2 p^0} (\vec{n} \cdot \vec{S}) \vec{n} + \frac{im}{2 p^0} \vec{p} \times \vec{\gamma}$$  \hspace{1cm} (12)

where $m^2 = p^\alpha p_\alpha$. Its components

$$\vec{n} \cdot \vec{S} p = \frac{m^2}{p^0} \vec{n} \cdot \vec{S} - \frac{im |\vec{p}|}{2 p^0} \vec{n} \times \vec{\gamma} = S_{p1}$$

$$\vec{i} \cdot \vec{S} p = \frac{m^2}{p^0} \vec{i} \cdot \vec{S} + \frac{im |\vec{p}|}{2 p^0} \vec{i} \times \vec{\gamma} = S_{p2}$$

$$\vec{n} \cdot \vec{S} p = \vec{n} \cdot \vec{S} = S_{p3}$$  \hspace{1cm} (13)
satisfy the Lie algebra \[ \{ \vec{S}_p, \vec{S}_q \} = i \frac{m^2}{\vec{p}.\vec{q}} \vec{S}_{pq} \]

\[
\begin{align*}
[S_{p1}, S_{p2}] &= i \frac{m^2}{\vec{p}.\vec{q}} S_{p3} \\
[S_{p3}, S_{p1}] &= i S_{p2} \\
[S_{p2}, S_{p3}] &= i S_{p1}.
\end{align*}
\]

(14)

It follows that the even spin operator \(|13\rangle\) is a Hermitian representation of the algebra \(|\vec{a}\rangle\) although a direct substitution of generators of \(\{ \vec{a}, 0 \}\) to \(|\vec{a}\rangle\) would not lead to Hermitian matrices. The eigenvalues of \(\vec{a} \cdot \vec{S}_p\), for any unit \(\vec{a}\), are

\[
s_{\vec{a}} = \pm \frac{1}{2} \sqrt{(\vec{p} \cdot \vec{a})^2 + m^2} \tag{15}\]

and the corresponding eigenvector in a standard representation is

\[
\Psi_{\pm} = N \left( \frac{\sqrt{\vert \vec{p} \vert} + m (\pm s_{\vec{a}} + i \vec{a} \cdot \vec{n}) w_{\pm}}{\sqrt{\vert \vec{p} \vert} - m (\pm s_{\vec{a}} + i \vec{a} \cdot \vec{n}) w_{\mp}} \right) \]

where \(w_{\pm}\) satisfies \(\vec{a} \cdot \vec{w}_{\pm} = \pm w_{\pm}\).

In the rest frame the eigenvalues \(s_{\vec{a}}\) are \(\pm \frac{1}{2}\) for any \(\vec{a}\). \(s_{\vec{a}}\) tend to 0 for both \(m \to 0\) and \(\vert \vec{p} \vert \to \infty\), if \(\vec{a} \cdot \vec{p} = 0\).

The transition from \(\vert \vec{p} \vert = 0\) to \(\vert \vec{p} \vert = \infty\) deforms continuously \(su(2)\) into \(e(2)\) and the spin operator \(\vec{S}_p\) becomes parallel to the momentum direction. The latter phenomenon can be deduced from either \(|12\rangle\) and \(|13\rangle\) or the discussed limits of \(|15\rangle\).

The above limits must be understood in terms of Lie algebra contractions. Physically the infinite momentum limit is more reasonable than \(m \to 0\). It means that the greater velocity of a particle, the less “fuzzy” are the components of spin in directions perpendicular to momentum. Intuitively, the particle becomes flattened by the Lorentz contraction so that contributions to the momentum. Intuitively, the particle becomes flattened by the Lorentz contraction so that contributions to the intrinsic angular momentum from rotations around directions perpendicular to \(\vec{p}\) become smaller the greater is the flattening.

For \(m\) equals exactly zero, two of the three components of spin vanish which agrees with the fact that the only self-adjoint finite dimensional representations of \(e(2)\) are one dimensional. Physically this effect can be again explained by the Lorentz contraction: A massless particle is completely flattened and its “intrinsic” angular momentum can result only from rotations in the plane perpendicular to \(\vec{p}\).

Equations \(|11\rangle\) and \(|15\rangle\) imply that the eigenvalues of \(\vec{a} \cdot \vec{W}\) are

\[
w_{\vec{a}} = p_0 s_{\vec{a}} = \pm \frac{1}{2} \frac{p_0}{\vert \vec{p} \vert} \sqrt{(\vec{p} \cdot \vec{a})^2 + m^2}
\]

and the eigenvectors are identical to those of \(\vec{a} \cdot \vec{S}_p\). For \(\vec{W}\) the massless and infinite momentum limits are not equivalent. Indeed, let \(\vec{a} \cdot \vec{p} = 0\). Then \(w_{\vec{a}} = \pm \frac{1}{2} m \neq 0\) for any \(\vec{p}\) and \(w_{\vec{a}} = 0 = 0\). It follows that the Pauli-Lubanski vector, as opposed to \(\vec{S}_p\), cannot be used for a unified treatment of spin in both massive and massless cases. The same concerns the seemingly natural and covariant choice of \((W_0/m, \vec{W}/m)\) as the relativistic spin four-vector. This property of \(\vec{W}\) explains the following apparent paradox. The “polarization density matrix” (normalized by \(\text{Tr} \rho = 2m\)) for the Dirac ultra-relativistic electron can be written as \(|16\rangle\)

\[
\rho = \frac{1}{2} \rho^\mu \gamma_\mu \left( 1 - \gamma_5 (\vec{\zeta}_\perp \cdot \vec{\gamma}_\perp) \right)
\]

where \(\vec{\gamma}_\perp = \frac{2}{m} (\vec{W} - (\vec{W} \cdot \vec{n}) \vec{n})\), and \(\langle \cdot \rangle\) denotes an average. For helicity eigenstates \(\vec{\zeta}_\perp = 0\) and \(\vec{\gamma}_\perp\) equals twice the helicity so that

\[
\rho = \frac{1}{2} \rho^\mu \gamma_\mu (1 \pm \gamma_5).
\]

\(|16\rangle\) is identical to the expression for the density matrix of the Dirac neutrino. However, for superpositions of different helicities \(\vec{\zeta}_\perp \neq 0\) which seems to suggest that even in the infinite momentum limit some “remains” of spin’s components in directions perpendicular to the momentum may be found. Still, there is no contradiction with our analysis if we treat \(\vec{S}_p\) and not \(\vec{W}\) as the relativistic spin operator. The remains are those of \(\vec{W}\) and not of \(\vec{S}_p\). The “transverse polarization” vector \(\vec{\zeta}_\perp\) has to be treated as a measure of superposition of the two helicities.

V. LORENTZ CONTRACTION... OF WHAT?

The decomposition of operators into even and odd parts can be used for rewriting the Dirac Hamiltonian in a form which is rather unusual but especially suitable for investigation of its ultra-relativistic and massless limits. Let us consider the angular velocity operator \(\vec{\omega}\) defined by \(|\vec{a}\rangle\). For \(m \neq 0\) and \(\vec{p} \neq 0\) \(\vec{\omega}\) does not commute with \(H\). Its even part, commuting with \(H\), is given (in ordinary units with \(c \neq 1\)) by

\[
\vec{\Omega} = \frac{c^2 + m^2 \vec{\gamma} \cdot \vec{n}}{c^2 + m^2 \vec{e}^4 / \vec{p}^2} \vec{\omega}.
\]

We can see that \(\vec{\Omega}\) reduces to \(\vec{\omega}\) in both limits. A Hamiltonian of a particle moving with velocity \(\vec{v} = c \vec{\beta}\) can now be expressed as

\[
H = \left( 1 + \frac{m^2 c^4}{c^2 \vec{p}^2} \right) \vec{\Omega} \cdot \vec{S}_p = \vec{\beta}^{-2} \vec{\Omega} \cdot \vec{S} = \vec{\beta}^{-2} \vec{\Omega} \cdot \vec{S}_p
\]

where each of the operators appearing in \(H\) is even and commuting with \(H\). The limiting form \(H = \vec{\omega} \cdot \vec{S}\) is characteristic of all massless fields, where for higher spins the equation \(|\vec{a}\rangle\) is still valid, but angular velocities for a given momentum are smaller the greater the helicity.
The new form of the Hamiltonian leads to the following interesting observation [11]. Notice that for massless fields the Hamiltonian can be written in either of the following two forms

\[ H = \vec{\omega} \cdot \vec{S} \]  
\[ H = \vec{\epsilon} \cdot \vec{p} = \vec{v} \cdot \vec{p} \]

or

\[ H = m_k c^2 = I_k \vec{\omega}^2. \]  

(18)

(19)

(20)

where \( \vec{v} \) is the velocity operator for a general massless field (\( c\vec{a} \) in case of the Dirac equation) and \( \vec{\epsilon} = \frac{(\vec{v} \cdot \vec{p}) \vec{p}}{p^2} \) is its even part. We recognize here the classical mechanical rule for a transition from a point-like description to the extended-object-like one: linear momentum goes into angular momentum, linear velocity into angular velocity, and vice versa. The third part of this rule (mass–moment of inertia) can be naturally postulated as follows

\[ H = m_k c^2 = I_k \vec{\omega}^2. \]

(22)

The equation

\[ I_k = m_k r_s^2 \]

characteristic, by the way, of circular strings (here with mass \( m_k \)) defines some radius which is equal to

\[ r_s = \frac{\hbar s}{|\vec{p}|} \]

(23)

which can be expressed also as an (operator!) form of the “uncertainty principle”

\[ |\vec{p}| r_s = \hbar s. \]

(24)

It is remarkable that this radius occurs also naturally in the twistor formalism [12]. A twistor is a kind of a “square root” of generators of the Poincaré group on a light cone and belongs to a carrier space of a representation of the conformal group. It is known that although spin-0 twistors can be represented geometrically by null straight lines, this does not hold for spin-\( s \), \( s \neq 0 \), twistors [12]. Instead of the straight line we get a congruence of twining, null, shear-free world lines, the so-called Robinson congruence. A three-dimensional projection of this congruence consists of circles, whose radii are given exactly by our formula (23) (cf. the footnote at p. 62 in [13]). The circles propagate with velocity of light in the momentum direction and rotate in the right- or left-handed sense depending on the sign of helicity.

The Robinson congruence picture is typical of classical twistors. It suggests that classical massless fields may be related naturally to classical strings whose radii would have to be different for different inertial observers. The quantized twistor formalism does not have such a pictorial representation since even for a spin-0 particle whose momentum is given no world line exists, but additionally because of the difficulties with the relativistic position operator.

The string-like picture of massless fields resulting from the moment of inertia formulas and from their agreement with the classical Robinson congruence is a physical indication that a fundamental role should be played in this context by the conformal group. Indeed, a transition from one inertial reference frame to another not only transforms the particle’s four-momentum, but simultaneously rescales the radius \( r_s \) of the congruence.

Finally, the fact that the massive Dirac Hamiltonian written in terms of even operators has the “energy of precession” form [7] suggests that some kind of an extended structure can be associated also with massive spinning particles. Structures of this type were constructed explicitly by Barut and collaborators [12,13,14].

VI. AN APPLICATION OF \( \vec{S}_\rho \): THE BELL THEOREM FOR DIRAC’S ELECTRONS

Let us consider two electrons with opposite momenta \( \vec{p}_1 \) and \( \vec{p}_2 = -\vec{p}_1 \) (we choose, in this way, a center of mass reference frame).

The squared total even spin operator

\[ (\vec{S}_{\rho_1} \otimes 1 + 1 \otimes \vec{S}_{\rho_2})^2 \]

in the helicity basis is given by the matrix

\[
\begin{pmatrix}
1 + \frac{m^2}{p_0} & 0 & 0 & 0 \\
0 & \frac{m^2}{p_0} & \frac{m^2}{p_0} & 1 \\
0 & \frac{m^2}{p_0} & \frac{m^2}{p_0} & 1 \\
0 & 0 & 0 & 1 + \frac{m^2}{p_0} \end{pmatrix}.
\]

(25)

where 1 is the \( 4 \times 4 \) identity matrix and \( p_0 \) is the energy of one of the particles. Its eigenvalues are: \( 1 + \frac{m^2}{p_0} \), \( 2 \frac{m^2}{p_0} \), and 0. The first two correspond to the non-relativistic triplet state and the third one to the singlet (degeneracies of the eigenvalues are, respectively, 8, 4 and 4). An important property of the definition (23) is the usage of squared even operators (this is not the same as the even part of the ordinary squared two-particle spin operator).

The singlet state takes in the helicity basis \( \Psi \) the usual form

\[ \Psi = \frac{1}{\sqrt{2}} (\Psi_+ \otimes \Psi_- - \Psi_- \otimes \Psi_+) \].


In order to prove the Bell theorem we must calculate the singlet state average of an analog of the nonrelativistic operator $\vec{a} \cdot \vec{\sigma} \otimes \vec{b} \cdot \vec{\sigma}$. Here we find

$$\langle \Psi | \vec{a} \cdot \vec{S}_{p_1} \otimes \vec{b} \cdot \vec{S}_{p_2} | \Psi \rangle = -\frac{1}{4} \frac{m^2}{p_0} \vec{a}_\parallel \cdot \vec{b}_\parallel + \frac{m^2}{p_0} \vec{a}_\perp \cdot \vec{b}_\perp$$

(26)

where the symbols $\parallel$ and $\perp$ denote projections on, respectively, the momentum direction and the plane perpendicular to it. For $\vec{a}$ and $\vec{b}$ perpendicular to $\vec{p}_1$ (24) equals $-\vec{a} \cdot \vec{b}$, the formula known from the nonrelativistic quantum mechanics, and the Bell theorem can be formulated. For other directions the formula (26) differs from the nonrelativistic one so might be used for an experimental verification of the even spin concept [17].

VII. CONCLUSIONS

We have shown that the algebra of a little group of any physical four momentum is isomorphic to the algebra of the even spin operator. This result explains qualitatively the fact that massless fields can exist only in extremal helicity states since eigenvalues of the even spin’s components perpendicular to momentum tend to zero in both infinite momentum and massless limits. A physical origin of this phenomenon can be explained by the Lorentz flattening of the Dirac particle provided the particle is extended. For massless fields (or ultra-relativistic electrons) the flattened picture can be naturally associated with the classical Robinson congruence of null worldlines, leading to a string-like classical limit of spinning particles. In this way we have returned to the old problem of localization of spinning particles [18] and have found another argument for their extended structure and usage of noncommuting position operators.

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