Solution to the Navier-Stokes equations with random initial data

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Abstract

We construct a solution to the spatially periodic $d$-dimensional Navier–Stokes equations with a given distribution of the initial data. The solution takes values in the Sobolev space $H^\alpha$, where the index $\alpha \in \mathbb{R}$ is fixed arbitrary. The distribution of the initial value is a Gaussian measure on $H^\alpha$ whose parameters depend on $\alpha$. The Navier–Stokes solution is then a stochastic process verifying the Navier–Stokes equations almost surely. It is obtained as a limit in distribution of solutions to finite-dimensional ODEs which are Galerkin-type approximations for the Navier–Stokes equations. Moreover, the constructed Navier–Stokes solution $U(t, \omega)$ possesses the property:

\[
\mathbb{E}\left[f(U(t, \omega))\right] = \int_{H^\alpha} f(e^{t\nu \Delta} u) \gamma(du),
\]

where $f \in L^1(\gamma)$, $e^{t\Delta}$ is the heat semigroup, $\nu$ is the viscosity in the Navier–Stokes equations, and $\gamma$ is the distribution of the initial data.

1. Introduction

Among the abundant amount of literature on the Navier–Stokes equations just a relatively smaller number of works treat this problem from the infinite dimensional analysis point of view, i.e. by reducing the Navier–Stokes equations to an infinite-dimensional problem. We mention here the papers [1, 2, 3, 9, 10, 12] that use infinite-dimensional approaches.

In the present paper we prove the existence of a Sobolev space valued solution to the Navier–Stokes equations on the torus $\mathbb{T}^d$ with the given Gaussian distribution of the initial data. The result of [1] related to the Euler equation on $\mathbb{T}^2$ follows from our result as a particular case ($d = 2, \nu = 0$). Moreover,
our Sobolev space index $\alpha$ is an arbitrary real number whereas the result of [1] was proved for $\alpha < -\frac{1}{2}$.

Similar to [1], we obtain a solution as the limit of Galerkin-type approximations. At first, we search for a mild solution to the Navier–Stokes equations, and then, under somewhat stronger assumptions, we derive the existence of a strong solution. As basis functions for the Galerkin-type method, we use the basis of divergence-free vector fields on the torus $\mathbb{T}^d$ constructed in [9]. The Galerkin-type equations are then modified by means of the change of variable which multiplies each $k$-th component by $e^{\nu|k|^2}$, $k \in \mathbb{Z}_d$. The latter change of variable is used to obtain the existence of invariant measures for the Galerkin-type approximations which is an important tool in the proof of the main result. Unlike [1], the Galerkin-type equations have time-dependent right-hand sides which does not allow us to solve them by methods of [7] directly. The given distribution of the initial value is a product of finite-dimensional Gaussian measures $\gamma_k$, $k \in \mathbb{Z}_d$, with variances $\frac{1}{|k|^{2\alpha}}$, where the parameter $l$ is a positive integer, and the pair $(\alpha, l)$ should be fixed so that $l > \alpha + \frac{d}{2} + 1$.

2. Representations of the Navier–Stokes equations

Consider the classical spatially-periodic $d$-dimensional Navier-Stokes equations:

$$
\begin{align*}
\frac{\partial y}{\partial t}(t, \theta) &= -(y, \nabla) y(t, \theta) + \nu \Delta y(t, \theta) - \nabla p(t, \theta), \\
\text{div } y(t, \theta) &= 0, \\
y(0, \theta) &= y_0(\theta),
\end{align*}
$$

(1)

where $\theta$ belongs to the $d$-dimensional torus $\mathbb{T}^d$, and $t \in [0, T]$. Below we use the notation introduced in [9]:

$$
\mathbb{Z}_d^+ = \{(k_1, k_2, \ldots, k_d) \in \mathbb{Z}_d : k_1 > 0 \text{ or } k_1 = \cdots = k_{i-1} = 0, k_i > 0, \text{ for } i = 2, \ldots, d\};
$$

if $k = (k_1, \ldots, k_d) \in \mathbb{Z}_d^+$, and $\theta = (\theta_1, \ldots, \theta_d) \in \mathbb{T}^d$,

then $|k| = \sqrt{\sum_{i=1}^{d} k_i^2}$, $k \cdot \theta = \sum_{i=1}^{n} k_i \theta_i$, $\Delta_{\mathbb{T}^d} = \sum_{i=1}^{d} \frac{\partial^2}{\partial \theta_i^2}$. 

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For every $k \in \mathbb{Z}_d^+$, $(\vec{k}^1, \ldots, \vec{k}^{d-1})$ denotes an orthogonal system of vectors of length $|k|$ which is also orthogonal to $k$. According to [9], any divergence-free vector field on $\mathbb{T}^d$ has the following Fourier series representation:

$$
\sum_{p=1}^{d} u_0^p e^p + \sum_{k \in \mathbb{Z}_d^+} \sum_{p=1}^{d-1} \left[ u_k^p A_k^p + v_k^p C_k^p \right] (2)
$$

where

$$
A_k^p = \frac{\sqrt{2}}{(2\pi)^{\frac{d}{2}}} \cos(k \cdot \theta) \frac{\vec{k}^p}{|k|^\frac{2}{2}}, \quad C_k^p = \frac{\sqrt{2}}{(2\pi)^{\frac{d}{2}}} \sin(k \cdot \theta) \frac{\vec{k}^p}{|k|^\frac{2}{2}},
$$

$$
p = 1, \ldots, d-1, \quad k \in \mathbb{Z}_d^+, \quad (3)
$$

and the constant vector fields $e^p$, $p = 1, \ldots, d$, are such that the $p$-th coordinate is $\frac{1}{2\pi}$ and the other coordinates are 0. The system (3) together with the constant vectors $\{e^p\}_{p=1}^d$ is orthonormal in $L^2(\mathbb{T}^d)$. The periodic divergence-free Sobolev space $H^\alpha(\mathbb{T}^d)$, $\alpha \in \mathbb{R}$, is defined as the totality of vector fields of form (2) with

$$
\sum_{k \in \mathbb{Z}_d^+} \sum_{p=1}^{d-1} |k|^{2\alpha} (|u_k^p|^2 + |v_k^p|^2) < \infty.
$$

We will search the Navier–Stokes solution in the form:

$$
y(t, \theta) = \sum_{p=1}^{d} u_0^p(t) e^p + \sum_{k \in \mathbb{Z}_d^+} \sum_{p=1}^{d-1} \left[ u_k^p(t) A_k^p + v_k^p(t) C_k^p \right]. (4)
$$

Define $u(t) = \{u_k^p(t), v_k^p(t), u_0^q(t)\}$, $k \in \mathbb{Z}_d^+$, $p = 1, \ldots, d-1$, $q = 1, \ldots, d$, and consider the following representation of the Navier–Stokes equations:

$$
u(t) = u(0) - \int_0^t B(u(s)) \, ds + \nu \int_0^t \Delta u(s) \, ds, (5)
$$

where $B(u(s)) = \{B_k^{p,\cos}, B_k^{p,\sin}\}$, $p = 1, \ldots, d-1$, $k \in \mathbb{Z}_d^+$, $B_k^{p,\cos}$ and $B_k^{p,\sin}$ are the coordinates of the expansion of $\mathbf{P}[(y, \nabla y)(s, \cdot)]$ with respect to the basis $A_k^p$, $C_k^p$, and $\mathbf{P}$ is the projector onto the divergence-free vector fields, i.e. onto the space spanned by vectors (3) and $\{e^p\}_{p=1}^d$. Note that the vector $B$ does
not have non-zero components along the constant vectors $e^p$, $p = 1, \ldots, d$. By $\Delta u(s)$ we understand the vector with the coordinates 
\[ \{-|k|^2 u^p_k(s), -|k|^2 v^p_k(s), k \in \mathbb{Z}^+_d, p = 1, \ldots, d - 1, u^d_0(s) = 0, q = 1, \ldots, d\}. \]

Navier–Stokes equations (5) can be represented as:

\begin{align*}
  u^p_0(t) &= u^p_0(0), \\
  u^p_k(t) &= u^p_k(0) - \int_0^t B^p_{k, \cos}(u(s)) \, ds - \nu |k|^2 \int_0^t u^p_k(s) \, ds, \\
  v^p_k(t) &= v^p_k(0) - \int_0^t B^p_{k, \sin}(u(s)) \, ds - \nu |k|^2 \int_0^t v^p_k(s) \, ds,
\end{align*}

(6)

where $p = 1, \ldots, d - 1, k \in \mathbb{Z}^+_d$. Define

\begin{align*}
  u_k(t) &= \sum_{p=1}^{d-1} u^p_k(t) \frac{\bar{k}^p}{|k|}, \\
  v_k(t) &= \sum_{p=1}^{d-1} u^p_k(t) \frac{\bar{k}^p}{|k|}, \\
  u_0(t) &= \sum_{p=1}^{d} u^p_0(t) e^p.
\end{align*}

With the above definition, we can search the Navier–Stokes solution $y(t, \theta)$ in the form:

\[ y(t, \theta) = u_0(0) + \sum_{k \in \mathbb{Z}^+_d} \left[ u_k(t) \cos(k \cdot \theta) + v_k(t) \sin(k \cdot \theta) \right]. \]

Analogously, we define

\begin{align*}
  B^p_{k, \cos} &= \sum_{p=1}^{d-1} B^p_{k, \cos} \frac{\bar{k}^p}{|k|}, \\
  B^p_{k, \sin} &= \sum_{p=1}^{d-1} B^p_{k, \sin} \frac{\bar{k}^p}{|k|}.
\end{align*}

Equations (6) take the form:

\begin{align*}
  u_0(t) &= u_0(0), \\
  u_k(t) &= u_k(0) - \int_0^t B^p_{k, \cos}(u(s)) \, ds - \nu |k|^2 \int_0^t u_k(s) \, ds, \\
  v_k(t) &= v_k(0) - \int_0^t B^p_{k, \sin}(u(s)) \, ds - \nu |k|^2 \int_0^t v_k(s) \, ds.
\end{align*}

(7)

We exclude the first equation in (7) and rewrite (7) in the equivalent form:

\begin{align*}
  u_k(t) &= e^{-\nu |k|^2 t} u_k(0) - \int_0^t e^{-(t-s)|k|^2 \nu} B^p_{k, \cos}(u(s)) \, ds \\
  v_k(t) &= e^{-\nu |k|^2 t} v_k(0) - \int_0^t e^{-(t-s)|k|^2 \nu} B^p_{k, \sin}(u(s)) \, ds.
\end{align*}

(8)
3. Galerkin-type approximations

3.1 Change of variable in the Navier–Stokes equations

Consider the Navier–Stokes equations in form (8). The direct computation of $B_{k}^{\sin}(u(s))$ and $B_{k}^{\cos}(u(s))$ gives the formal expression

\[
B_{k}^{\sin}(u(s)) = \frac{1}{\sqrt{2}(2\pi)^{d/2}} \left( \sum_{k_1 + k_2 = k} \sum_{i,j=1}^{d-1} (v_{k_1}^i(s)v_{k_2}^j(s) - u_{k_1}^i(s)u_{k_2}^j(s)) \frac{(\tilde{k}_1, k_2)}{2|k_1||k_2|} \mathbf{P}_k \bar{k}_2^j \right) \]

\[
+ \sum_{k_1 - k_2 = k} \sum_{i,j=1}^{d-1} (u_{k_1}^i(s)u_{k_2}^j(s) + v_{k_1}^i(s)v_{k_2}^j(s)) \frac{(\tilde{k}_1, k_2)}{2|k_1||k_2|} \mathbf{P}_k \bar{k}_2^j \]

\[
+ \sum_{k_2 - k_1 = k} \sum_{i,j=1}^{d-1} (-u_{k_1}^i(s)u_{k_2}^j(s) - v_{k_1}^i(s)v_{k_2}^j(s)) \frac{(\tilde{k}_1, k_2)}{2|k_1||k_2|} \mathbf{P}_k \bar{k}_2^j \]

\[
- \sum_{i=1}^{d} \sum_{j=1}^{d-1} u_{0}^i u_{k}^j(s) (e_i, k) \frac{\bar{k}_2^j}{|k|} \]

where $\mathbf{P}_k$ denotes the orthogonal projection in $\mathbb{R}^d$ onto its $(d-1)$-dimensional subspace generated by the vectors $\bar{k}^p$, $p = 1, \ldots, d-1$. Now let us combine the sums over $k_1$ and $k_2$ into one. For each pair $k, h \in \mathbb{Z}_d^+$, we define the functions:

\[
\pm(k - h) = \begin{cases} k - h, & \text{if } k - h \in \mathbb{Z}_d^+, \\ h - k, & \text{if } h - k \in \mathbb{Z}_d^+ \end{cases}
\]

and

\[
\text{sign}(k - h) = \begin{cases} 1, & \text{if } k - h \in \mathbb{Z}_d^+, \\ -1, & \text{if } h - k \in \mathbb{Z}_d^+ \end{cases}
\]

The both functions are undefined if $k = h$. Also, we introduce

\[
\lambda^+(k, h) = \frac{(k + h, h)}{\sqrt{2}(2\pi)^{d/2} |h||k + h|};
\]

\[
\lambda^-(k, h) = \frac{\pm(k - h)^i, h)}{\sqrt{2}(2\pi)^{d/2} |h| |k - h|} \quad \text{if } k \neq h, \quad \text{and } \lambda^-(h, h) = 0;
\]

\[
\lambda_i(k) = \frac{1}{(2\pi)^{d/2} |k|} (e_i, k).
\]
We obtain:

\[ B^\sin_k(u(s)) = \sum_{h \in \mathbb{Z}_+^1} \sum_{i,j=1}^{d-1} \lambda^+_i(k, h) (u^i_{k+h}(s)u^j_h(s) + v^i_{k+h}(s)v^j_h(s)) P_k \tilde{h}^j \]

\[ + \lambda^-_i(k, h) \left( \text{sign}(k - h)v^i_{\pm(k-h)}(s)v^j_h(s) - u^i_{\pm(k-h)}(s)u^j_h(s) \right) P_k \tilde{h}^j \]

\[ - \sum_{i=1}^{d-1} \lambda_i(k) u^i_0 \sum_{j=1}^{d-1} u^j_k(s) \tilde{k}^j. \]

Analogously, we obtain the formal series for \( B^\cos_k(u(s)) \)

\[ B^\cos_k(u(s)) = \frac{1}{\sqrt{2(2\pi)^2}} \left( \sum_{k_1+k_2=k} \sum_{i,j=1}^{d-1} (u^i_{k_1}(s)u^j_{k_2}(s) + u^i_{k_1}(s)v^j_{k_2}(s)) \frac{(\tilde{k}^i_1, \tilde{k}^i_2)}{2|k_1||k_2|} P_k \tilde{k}^j_2 \right) \]

\[ + \sum_{k_1+k_2=k} \sum_{i,j=1}^{d-1} (u^i_{k_1}(s)v^j_{k_2}(s) - v^i_{k_1}(s)u^j_{k_2}(s)) \frac{(\tilde{k}^i_1, \tilde{k}^i_2)}{2|k_1||k_2|} P_k \tilde{k}^j_2 \]

\[ + \sum_{i=1}^{d-1} \sum_{j=1}^{d-1} u^i_0 v^j_k(s)(e^i, k) \tilde{k}^j. \]

Combining the first two sums into one gives:

\[ B^\cos_k(u(s)) = \sum_{h \in \mathbb{Z}_+^1} \sum_{i,j=1}^{d-1} \lambda^+_i(k, h) (u^i_{k+h}(s)v^j_h(s) - v^i_{k+h}(s)u^j_h(s)) P_k \tilde{h}^j \]

\[ + \lambda^-_i(k, h) (u^i_{\pm(k-h)}(s)v^j_h(s) + \text{sign}(k - h)v^i_{\pm(k-h)}(s)u^j_h(s)) P_k \tilde{h}^j \]

\[ + \sum_{i=1}^{d} \lambda_i(k) u^i_0 \sum_{j=1}^{d} v^j_k(s) \tilde{k}^j. \]

We dropped the time dependence in \( u^i_0 \) since we proved that it does not depend on time. Multiplying equations \((8)\) by \( e^{tv|k|^2} \) and introducing the new variables

\[ \tilde{u}_k(t) = e^{tv|k|^2} u_k(t), \quad \tilde{v}_k(t) = e^{tv|k|^2} v_k(t) \]

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as well as their coordinates $\tilde{u}_k(t), \tilde{v}_k(t)$ with respect to the bases $\left\{ \frac{k^p}{|k|} \right\}$, $p = 1, \ldots, d - 1$, we rewrite equations (8):

$$
\tilde{u}_k(t) = \tilde{u}_k(0) - \int_0^t \tilde{B}_k^\cos(s, \tilde{u}(s)) \, ds,
$$

$$
\tilde{v}_k(t) = \tilde{v}_k(0) - \int_0^t \tilde{B}_k^\sin(s, \tilde{u}(s)) \, ds,
$$

where $\tilde{u}(s)$ is the vector with the coordinates $\tilde{u}_k(t), \tilde{v}_k(t)$, $\tilde{B}_k^\sin(s, \tilde{u}(s)) = e^{s\nu|k|^2} B_k^\sin(u(s))$, and $\tilde{B}_k^\cos(s, \tilde{u}(s)) = e^{s\nu|k|^2} B_k^\cos(u(s))$. Explicitly,

$$\tilde{B}_k^\sin(s, \tilde{u}(s))$$

$$= \sum_{h \in \mathbb{Z}_d^+} \sum_{i,j=1}^{d-1} \lambda_i^+(k, h) e^{-2(k+h, h)\nu} \left( \tilde{u}_k^i(h(s)) \tilde{u}_k^j(h(s)) + \tilde{v}_k^i(h(s)) \tilde{v}_k^j(h(s)) \right) P_{hj} + \lambda_i^-(k, h) e^{2(k-h, h)\nu} \left( \text{sign}(k-h) \tilde{v}_k^i(h(s)) \tilde{v}_k^j(h(s)) - \tilde{u}_k^i(h(s)) \tilde{u}_k^j(h(s)) \right) P_{hj} - \sum_{i=1}^d \lambda_i(k) \sum_{j=1}^{d-1} u_0^i \tilde{u}_k^j(s) \delta_{ij}.$$ 

$$\tilde{B}_k^\cos(s, \tilde{u}(s))$$

$$= \sum_{h \in \mathbb{Z}_d^+} \sum_{i,j=1}^{d-1} \lambda_i^+(k, h) e^{-2(k+h, h)\nu} \left( \tilde{u}_k^i(h(s)) \tilde{u}_k^j(h(s)) - \tilde{v}_k^i(h(s)) \tilde{v}_k^j(h(s)) \right) P_{hj} + \lambda_i^-(k, h) e^{2(k-h, h)\nu} \left( \text{sign}(k-h) \tilde{v}_k^i(h(s)) \tilde{v}_k^j(h(s)) + \text{sign}(k-h) \tilde{v}_k^i(h(s)) \tilde{v}_k^j(h(s)) \right) P_{hj} + \sum_{i=1}^d \lambda_i(k) \sum_{j=1}^{d-1} u_0^i \tilde{v}_k^j(s) \delta_{ij}.$$ 

We mention here once again that the infinite series representing $B_k^\sin(u(s))$ and $B_k^\cos(u(s))$, as well as $\tilde{B}_k^\sin(s, \tilde{u}(s))$ and $\tilde{B}_k^\cos(s, \tilde{u}(s))$ are so far just formal expressions. Below we introduce spaces where the series for $B_k^\sin(u(s))$ and $B_k^\cos(u(s))$ converge. On the other hand, we will only deal with $\tilde{B}_k^{(n),\sin}(s, \tilde{u}(s))$ and $\tilde{B}_k^{(n),\cos}(s, \tilde{u}(s))$ which are obtained from $\tilde{B}_k^\sin(s, \tilde{u}(s))$ and $\tilde{B}_k^\cos(s, \tilde{u}(s))$ by discarding the terms with $|h| > n$. 

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3.2 Spaces of convergence

Here we introduce a family of Sobolev-type spaces with a Gaussian measure on each of them so that the infinite series for $B^\sin_k$ and $B^\cos_k$, i.e.

$$\sum_{h \in \mathbb{Z}_d^+} \sum_{i,j=1}^{d-1} \lambda_i^+(k, h) \left( u^i_{k+h} u^j_h + v^i_{k+h} v^j_h \right) P_k \bar{h}^j \quad \text{and} \quad \lambda_i^-(k, h) \left( \text{sign}(k - h) v^i_{\pm(k-h)} v^j_h - u^i_{\pm(k-h)} u^j_h \right) P_k \bar{h}^j \quad (10)$$

and, respectively,

$$\sum_{h \in \mathbb{Z}_d^+} \sum_{i,j=1}^{d-1} \lambda_i^+(k, h) \left( u^i_{k+h} v^j_h - v^i_{k+h} u^j_h \right) P_k \bar{h}^j \quad \text{and} \quad \lambda_i^+(k, h) \left( u^i_{\pm(k-h)} v^j_h + \text{sign}(k - h) v^i_{\pm(k-h)} u^j_h \right) P_k \bar{h}^j \quad (11)$$

converge. For any $r \in \mathbb{R}$, we define a Hilbert space

$$H^r = \left\{ u : \sum_{k \in \mathbb{Z}_d^+} |k|^{2r} (|u_k|^2 + |v_k|^2) < \infty \right\}.$$

For each $k \in \mathbb{Z}_d^+$, on the space $\mathbb{R}^{2(d-1)}$ of the variables $(u_k, v_k)$ we define the Gaussian measure $\gamma_k$ by its density with respect to the Lebesgue measure on $\mathbb{R}^{2(d-1)}$:

$$\left( \frac{|k|^{2l}}{2\pi} \right)^{(d-1)} \exp \left( -\frac{|k|^{2l}}{2} (|u_k|^2 + |v_k|^2) \right).$$

The number $l$ is a positive integer which is sufficiently large to satisfy certain inequalities which will be obtained below to ensure the convergence of series (10) and (11). Define the measure $\gamma$ on $\times_{k \in \mathbb{Z}_d^+} \mathbb{R}^{2(d-1)}$ by

$$\gamma(du) = \bigotimes_{k \in \mathbb{Z}_d^+} \gamma_k (d(u_k, v_k)).$$

**Proposition 1.** For any $\alpha \in \mathbb{R}$, for any integer $l > \frac{d}{2} + \alpha$, $(H^\alpha, H^1, \gamma)$ is an abstract Wiener space.
Proof. The measure \( \gamma \) is supported on \( H^\alpha \). Indeed,

\[
\int \left\{ \times_{k \in \mathbb{Z}_d^+} \mathbb{R}^{2(d-1)} \right\} \| u \|^2_\alpha ^2 \gamma (du) = \sum_{k \in \mathbb{Z}_d^+} |k|^{2 \alpha} \int_{\mathbb{R}^{2(d-1)}} (|u_k|^2 + |v_k|^2) \gamma_k (d(u_k, v_k))
\]

\[
= 2(d-1) \sum_{k \in \mathbb{Z}_d^+} \frac{1}{|k|^{2(l-\alpha)}}.
\]

The latter series converges, for example, by the integral test of convergence. It is easy to verify that the Fourier transform of the measure \( \gamma \) is

\[
u \mapsto \exp \left( \frac{1}{2} \sum_{k \in \mathbb{Z}_d^+} |k|^{4\alpha - 2l} (|u_k|^2 + |v_k|^2) \gamma_k \right).
\]

(12)

Defining the operator \( K \) as

\[
KA_k = |k|^{2\alpha - 2l} A_k, \quad KC_k = |k|^{2\alpha - 2l} C_k,
\]

we observe that the sum in (12) equals to \((Ku, u)\), where \(( \cdot, \cdot)\) is the scalar product in \( H^\alpha \). This means that \( K \) is the covariance operator for the measure \( \gamma \). Take a \( u \in H^\alpha \). Then,

\[
\| \sqrt{K} u \|^2 = \sum_{k \in \mathbb{Z}_d^+} |k|^{2l} |k|^{2\alpha - 2l} (|u_k|^2 + |v_k|^2) = \| u \|^2_\alpha.
\]

This proves that \( \sqrt{K} H^\alpha = H^l \), and, therefore, \( H^l \) is the Cameron-Martin space of the measure \( \gamma \). \( \square \)

**Lemma 1.** Let \( l > \alpha + \frac{d}{2} + 1 \). Then, \( B_k^{\sin} \) and \( B_k^{\cos} \) are well defined as elements of \( L_2(\gamma, \mathbb{R}^{d-1}) \), i.e. series (10) and (11) converge in \( L_2(\gamma, \mathbb{R}^{d-1}) \). Moreover, \( B \in L_2(\gamma, H^\alpha) \).

**Proof.** Note that each member of series (10) and (11) indexed by \( h \) is a finite sum in \( i, j \) whose summands are, in turn, sums of four other terms. Clearly, we can exchange the order of summation in \( h \in \mathbb{Z}_d^+ \) and \( i, j \), and, moreover, apply the summation in \( h \) to each of the subsequent four terms. Let us investigate the convergence of the series

\[
\sum_{h \in \mathbb{Z}_d^+} \lambda^+_i(h, k) u^{i}_{h} u^{j}_{h} P_k h^j
\]

(13)
in $L_2(\gamma, \mathbb{R}^{d-1})$. We have:

\[
\mathbb{E}_\gamma \left| \sum_{|h|<N} \lambda^+_i (k, h) u^i_{k+h} u^j_h P_k \bar{h}^j \right|_{\mathbb{R}^{d-1}}^2 \\
= \sum_{|h|<N, |h'|<N} \lambda^+_i (k, h) \lambda^+_i (k, h') \mathbb{E}_\gamma (u^i_{k+h} u^i_{k+h'} u^j_h u^j_{h'}) (P_k \bar{h}^j, P_k \bar{h'}^j)_{\mathbb{R}^{d-1}} \\
= \sum_{|h|<N} \lambda^+_i (k, h) \frac{1}{|k+h|^2} \frac{1}{|h|^2} |P_k \bar{h}^j|_{\mathbb{R}^{d-1}}^2 \leq \sum_{h \in \mathbb{Z}^d_+} \frac{1}{|k+h|^2} \frac{1}{|h|^{2l-2}}. \tag{14}
\]

In the sequel we will show the convergence of the series

\[
\sum_{k \in \mathbb{Z}^d_+} |k|^{2\alpha} \sum_{h \in \mathbb{Z}^d_+} \frac{1}{|k+h|^2} \frac{1}{|h|^{2l}} = \sum_{h \in \mathbb{Z}^d_+} \frac{1}{|h|^{2l-2}} \sum_{k \in \mathbb{Z}^d_+} |k|^{2\alpha} \tag{15}
\]

which, in turn, will imply the convergence of

\[
\sum_{k \in \mathbb{Z}^d_+} |k|^{2\alpha} (|B_{k \sin}^i|^2 + |B_{k \cos}^i|^2) \tag{16}
\]

and, therefore, the statement of the lemma. For the moment we assume that the series on the right hand side of (14) converges, and show that the sequence of partial sums for (13) is a Cauchy sequence. Indeed, for any integers $M < N$,

\[
\mathbb{E}_\gamma \left| \sum_{M<|h|<N} \lambda^+_i (k, h) u^i_{k+h} u^j_h P_k \bar{h}^j \right|_{\mathbb{R}^{d-1}}^2 \leq \sum_{M<|h|<N} \frac{1}{|k+h|^2} \frac{1}{|h|^{2l-2}} \to 0, \quad \text{as} \quad M, N \to \infty.
\]

This proves that series (13) converges in $L_2(\gamma, \mathbb{R}^{d-1})$. Analogously, the other
\[8(d - 1)^2 - 1 \text{ series}\]
\[
\sum_{h \in \mathbb{Z}_+^d} \lambda_i^+(k, h) v_{k+h}^i \mathbf{P}_k \bar{h}^j, \quad \sum_{h \in \mathbb{Z}_+^d} \lambda_i^-(k, h) \text{sign}(k - h) v_{k-h}^i \mathbf{P}_k \bar{h}^j, \\
\sum_{h \in \mathbb{Z}_+^d} \lambda_i^-(k, h) u_{k+h}^i u_h^j \mathbf{P}_k \bar{h}^j, \quad \sum_{h \in \mathbb{Z}_+^d} \lambda_i^+(k, h) u_{k-h}^i \mathbf{P}_k \bar{h}^j, \\
\sum_{h \in \mathbb{Z}_+^d} \lambda_i^+(k, h) v_{k+h}^i \mathbf{P}_k \bar{h}^j, \quad \sum_{h \in \mathbb{Z}_+^d} \lambda_i^-(k, h) u_{k-h}^i \mathbf{P}_k \bar{h}^j, \\
\sum_{h \in \mathbb{Z}_+^d} \lambda_i^-(k, h) \text{sign}(k - h) u_{k-h}^i \mathbf{P}_k \bar{h}^j.
\]

\(i, j = 1, \ldots, d\), which are summands in (10) and (11), converge in \(L_2(\gamma, \mathbb{R}^{d-1})\).

Therefore, series (10) and (11) converge in \(L_2(\gamma, \mathbb{R}^{d-1})\) as sums of a finite number of converging series. Hence, \(B^\text{lin}_k\) and \(B^\text{cos}_k\), \(k \in \mathbb{Z}_+^d\), are well defined as elements of \(L_2(\gamma, \mathbb{R}^{d-1})\). Let us prove that series (16) converges and, therefore, \(B \in L_2(\gamma, H^\alpha)\). First we prove that the series \(\sum_{k \in \mathbb{Z}_+^d} |k|^{2\alpha} \mathbf{P}_k \bar{h}^j\) converges for each fixed \(h \in \mathbb{Z}_+^d\) and estimate its sum. We have:
\[
\sum_{k \in \mathbb{Z}_+^d} \frac{|k|^{2\alpha}}{|k + h|^{2d}} \leq \sum_{k \in \mathbb{Z}_+^d} \frac{\left(\sum_{i=1}^d |k_i|\right)^{2\alpha}}{\left(\sum_{i=1}^d |k_i - |h_i||\right)^{2d}} \mathbb{I}_{\{k_i \neq |h_i|\}} \\
\leq d \sum_{k_1=0}^{2^{d-1}} \cdots \sum_{k_d=0}^{2^{d-1}} \frac{\left(\sum_{i=1}^d k_i\right)^{2\alpha}}{\left(\sum_{i=1}^d |k_i - |h_i||\right)^{2d}} \mathbb{I}_{\{k_i \neq |h_i|\}}, \quad (17)
\]

Note that multiplying by the indicator \(\mathbb{I}_{\{k_i \neq |h_i|\}}\) implies that the denominators of the fraction on the right-hand side are always bigger than 1. Next, we split each sum over \(k_i\) going from 0 to \(\infty\) into two: from 0 to \(|h_i| - 1\) and from \(|h_i|\) to \(+\infty\), i.e. for the series on the right-hand side we obtain:
\[
\left[\sum_{k_1=0}^{h_1 - 1} + \sum_{k_1=|h_1|}^{+\infty}\right] \left[\sum_{k_2=0}^{h_2 - 1} + \sum_{k_2=|h_2|}^{+\infty}\right] \cdots \left[\sum_{k_d=0}^{h_d - 1} + \sum_{k_d=|h_d|}^{+\infty}\right] \frac{\left(\sum_{i=1}^d k_i\right)^{2\alpha}}{\left(\sum_{i=1}^d |k_i - |h_i||\right)^{2d}} \mathbb{I}_{\{k_i \neq |h_i|\}}
\]

which equals to a finite sum of series of the form
\[
\sum_{k_1=|h_1|}^{+\infty} \cdots \sum_{k_m=|h_m|}^{+\infty} \sum_{k_{m+1}=0}^{h_{m+1} - 1} \sum_{k_d=0}^{h_d - 1} \frac{\left(\sum_{i=1}^d k_i\right)^{2\alpha}}{\left(\sum_{i=1}^d |k_i - |h_i||\right)^{2d}} \mathbb{I}_{\{k_i \neq |h_i|\}}
\]
where \( \{i_1, \ldots, i_d\} \) is a perturbation of \( \{1, \ldots, d\} \). It suffices to investigate the convergence of the series

\[
\sum_{k_1=|h|}^{+\infty} \cdots \sum_{k_d=|h|}^{+\infty} \frac{\left(\sum_{i=1}^{d} k_i\right)^{2\alpha}}{\left(\sum_{i=1}^{d} |k_i - |h_i||\right)^{2l}} \mathbb{I}\{k_i \neq |h_i|\}.
\]

Note that we can consider that \( k \neq 0 \) since the term with \( k = 0 \) is always zero. Therefore the numerator and the denominator of all members of the series are bigger than 1. Consider the function:

\[
f(x_1, \ldots, x_d) = \frac{\left(\sum_{i=1}^{d} x_i\right)^{2\alpha}}{\left(\sum_{i=1}^{d} |x_i - |h_i||\right)^{2l}}.
\]

It is an increasing function in each of the variable \( x_{m+1}, \ldots, x_d \) on the intervals \([0, |h_{m+1}|], \ldots, [0, |h_d|]\), respectively, when the rest of the variables is fixed. Therefore,

\[
\sum_{k_{m+1}=0}^{|h_{m+1}|-1} \cdots \sum_{k_d=0}^{|h_d|-1} \frac{\left(\sum_{i=1}^{d} k_i\right)^{2\alpha}}{\left(\sum_{i=1}^{d} |k_i - |h_i||\right)^{2l}}
\leq \int_{0}^{|h_{m+1}|} dx_{m+1} \cdots \int_{0}^{|h_d|} dx_d \frac{\left(\sum_{i=1}^{m} k_i + \sum_{i=m+1}^{d} x_i\right)^{2\alpha}}{\left(\sum_{i=1}^{m} |k_i - |h_i|| + \sum_{i=m+1}^{d} |h_i| - x_i\right)^{2l}}.
\]

The integral on the right-hand side of (18) can be computed or estimated from above by integration by parts. Performing the integration by parts once, we decrease the powers of the numerator and the denominator of the integrand by 1. When the power of the numerator becomes \( 2\alpha - [2\alpha] \) we estimate the numerator from above by replacing \( 2\alpha - [2\alpha] \) with 1. If \( 2\alpha \) is an integer, the integral can be computed explicitly. Our goal is to show that the sum of the series on the right-hand side of (17) is equivalent to \( |h|^\xi \) for some \( \xi \in \mathbb{R} \). So we discard the terms when it is already clear that they are of orders of \( |h| \) smaller than the maximal. Integration by parts in (18) implies that the higher order term is smaller than

\[
\frac{\left(\sum_{i=1}^{m} k_i + \sum_{i=m+1}^{d} |h_i|\right)^{2\alpha}}{\left(\sum_{i=1}^{m} |k_i - |h_i|| + 1\right)^{2l-d+2m}}.
\]

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up to a multiplicative constant. Next, we apply the integral test of convergence to investigate the convergence of the above series in \(k_i\) going from \(|h_i|\) to \(+\infty\). Note that the function

\[
\frac{(\sum_{i=1}^{m} x_i + \sum_{i=m+1}^{d} |h_i|)^{2\alpha}}{(\sum_{i=1}^{m} |x_i - |h_i|| + 1)^{2l-d+2m}}
\]

is decreasing in each \(x_i \in [|h_i|, \infty)\) provided that all other \(x_j\)'s are bigger than \(|h_j|\). Therefore, we can apply the integral test of convergence to each one variable series with summation in \(k_m, k_{m-1}, \ldots, k_1\) subsequently. We obtain:

\[
\sum_{k_m=|h_m|}^{\infty} \frac{(\sum_{i=1}^{m} k_i + \sum_{i=m+1}^{d} |h_i|)^{2\alpha}}{(\sum_{i=1}^{m} (k_i - |h_i|) + 1)^{2l-d+2m}} \leq \frac{(\sum_{i=1}^{m-1} k_i + \sum_{i=m}^{d} |h_i|)^{2\alpha}}{(\sum_{i=1}^{m-1} (k_i - |h_i|) + 1)^{2l-d+2m}}
+ \int_{|h_m|}^{\infty} dx_m \frac{(\sum_{i=1}^{m-1} k_i + x_m + \sum_{i=m+1}^{d} |h_i|)^{2\alpha}}{(\sum_{i=1}^{m-1} (k_i - |h_i|) + x_m - |h_m| + 1)^{2l-d+2m}}.
\]

Integration by parts implies that the higher order term is

\[
\frac{(\sum_{i=1}^{m-1} k_i + \sum_{i=m}^{d} |h_i|)^{2\alpha}}{(\sum_{i=1}^{m-1} |k_i - |h_i|| + 1)^{2l-d+2m-1}}
\]

up to a multiplicative constant. The same argument implies the convergence of each one variable series with summation in \(k_{m-1}, \ldots, k_1\), and that the higher order term is

\[
(\sum_{i=1}^{d} |h_i|)^{2\alpha}
\]

(up to a multiplicative constant) which is equivalent to \(|h|^{2\alpha}\). Note that to ensure the convergence of all integrals that appear as a result of the integration by parts formula, we have to require that \(2l - [2\alpha] - 1 - d > 0\) which is the case by the assumption. Therefore, we proved that the sum of the series on the left-hand side of (17) is of the order \(|h|^{2\alpha}\). It remains to investigate the convergence of the series

\[
\sum_{h \in \mathbb{Z}_d^+} \frac{1}{|h|^{2l-2\alpha-2}}.
\]

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By the above argument, this series convergence if and only if the series below converges.

\[
\sum_{h_1=0}^{+\infty} \cdots \sum_{h_d=0}^{+\infty} \frac{1}{(\sum_{i=1}^{d} h_i)^{2l-2\alpha-2}} \mathbb{I}_{\{h \neq 0\}} = d \sum_{h_1=1}^{+\infty} \sum_{h_2=0}^{+\infty} \cdots \sum_{h_d=0}^{+\infty} \frac{1}{(\sum_{i=1}^{d} h_i)^{2l-2\alpha-2}}.
\]

To investigate the convergence of the series on the right-hand side we can apply the integral test of convergence subsequently to each one variable series to conclude that this series converge when \(2l - 2\alpha - d - 2 > 0\) which is the case by the assumption. Hence, series (16) converges, and therefore the series on the right-hand side of (14) converges. This proves that \(B_k^{\sin}, B_k^{\cos} \in L_2(\gamma, \mathbb{R}^{d-1})\) for every \(k \in \mathbb{Z}^+_d\) and that \(B \in L_2(\gamma, H^\alpha)\).

### 3.3 Galerkin-type equations

Let us consider the finite-dimensional spaces \(E^{(n)}\) of the variables \((u_k, v_k)\), \(|k| \leq n\). For every integer \(n\), we introduce the Gaussian measure on \(E^{(n)}\):

\[
\gamma^{(n)}(du) = \bigotimes_{|k| \leq n} \gamma_k(d(u_k, v_k)).
\]

For \(|k| \leq n\), let \(\tilde{B}_k^{(n),\sin}(s, u)\) and \(\tilde{B}_k^{(n),\cos}(s, u)\) be obtained from \(\tilde{B}_k^{\sin}(s, u)\) and \(\tilde{B}_k^{\cos}(s, u)\) by restricting the summation only over those \(h\) whose absolute values are not bigger than \(n\). If \(|k| > n\), we set \(\tilde{B}_k^{(n),\sin}(s, u) = \tilde{B}_k^{(n),\cos}(s, u) = 0\). Analogously we define \(B_k^{(n),\sin}(u)\) and \(B_k^{(n),\cos}(u)\). Now let

\[
\tilde{B}^{(n)} = \sum_{k \in \mathbb{Z}^+_d} \tilde{B}_k^{(n),\cos} \cos(k \cdot \theta) + \tilde{B}_k^{(n),\sin} \sin(k \cdot \theta),
\]

\[
B^{(n)} = \sum_{k \in \mathbb{Z}^+_d} B_k^{(n),\cos} \cos(k \cdot \theta) + B_k^{(n),\sin} \sin(k \cdot \theta).
\]

Consider the ODE:

\[
\begin{align*}
\frac{d}{ds} \tilde{u}(s, u) &= \tilde{B}^{(n)}(s, \tilde{u}(s, u)), \\
\tilde{u}(0, u) &= u
\end{align*}
\]

where \(u \in H^\alpha\). Let, as before, \(\tilde{u}_k(s, u)\) and \(\tilde{v}_k(s, u)\) be \(\mathbb{R}^{d-1}\)-valued components of \(\tilde{u}(s, u)\), and \(u_k, v_k\) be \(\mathbb{R}^{d-1}\)-valued components of \(u\). Clearly, if
\[ |k| > n, (\bar{u}_k(s, u), \bar{v}_k(s, u)) = (u_k, v_k). \]

Hence, replacing in (19) the variables \((\bar{u}_k(s, u), \bar{v}_k(s, u))\), \(|k| > n\), with \((u_k, v_k)\) and discarding equations of the type
\[
\bar{u}_k(s, u) = u_k, \quad \bar{v}_k(s, u) = v_k
\]
for \(|k| > n\), we obtain an ODE in finite dimensions. Namely, let \(\Pi_n u\) be the orthogonal projection of \(u\) onto \(E^{(n)}\), and let \(\Pi_n^\perp u = u - \Pi_n u\). For every \(k \in \mathbb{Z}_d^+, |k| \leq n\), we define the functions \(\tilde{B}_k^{(n), \text{sin}}(s, \Pi_n u, a)\) and \(\tilde{B}_k^{(n), \text{cos}}(s, \Pi_n u, a)\).

They are obtained from \(\tilde{B}_k^{(n), \text{sin}}(s, u)\) and resp. \(\tilde{B}_k^{(n), \text{cos}}(s, u)\) by fixing the variables \(u_{k+h}, v_{k+h}, u_{\pm (k-h)}, v_{\pm (k-h)}\) with \(|k + h| > n\) or \(|k - h| > n\). The symbol \(a\) denotes the vector of all fixed variables. Define \(\tilde{B}^{(n)}\) as an \(E^{(n)}\)-valued vector with the \((d - 1)\)-dimensional components \(\tilde{B}_k^{(n), \text{cos}}\) and \(\tilde{B}_k^{(n), \text{sin}}\).

The components of \(\tilde{B}^{(n)}\) with respect to the basis \(A_k^p, C_k^p\) will be denoted by \(\tilde{B}_k^{(n), \text{cos}, p}\) and \(\tilde{B}_k^{(n), \text{sin}, p}\), \(|k| \leq n, p = 1, \ldots, d - 1\). The finite-dimensional equation equivalent to (19) looks like this:
\[
\bar{u}(t, \Pi_n u) = \Pi_n u + \int_0^t \tilde{B}^{(n)}(s, \bar{u}(s, \Pi_n u), \Pi_n^\perp u) ds. \tag{20}
\]

**Lemma 2.** Equation (20) has a unique solution for \(\gamma^{(n)}\)-almost all initial conditions. The solution to (20) keeps the measure \(\gamma^{(n)}\) invariant.

For the proof of Lemma 2 we need Lemma 3 below.

**Lemma 3.** For every \(s \geq 0\), for any \(a \in H^a \setminus E^{(n)}, \delta_{\gamma^{(n)}} \hat{B}^{(n)}(s, \cdot, a) = 0\).

**Proof.** It was proved in [7] (see also [6]) that for every \(s \in [0, T], u \in E^{(n)}\), the divergence of \(\hat{B}^{(n)}(s, u, a)\) with respect to the measure \(\gamma^{(n)}\) equals to
\[
\delta_{\gamma^{(n)}} \hat{B}^{(n)}(s, u, a) = (\hat{B}^{(n)}(s, u, a), u)_I
\]
\[- \sum_{|k| \leq n} \sum_{p=1}^{d-1} \frac{\partial}{\partial u_k^p} \hat{B}_k^{(n), \text{cos}, p}(s, u, a) - \sum_{|k| \leq n} \sum_{p=1}^{d-1} \frac{\partial}{\partial v_k^p} \hat{B}_k^{(n), \text{sin}, p}(s, u, a), \tag{21}
\]
where \((\cdot, \cdot)_I\) is the scalar product in \(H^I\). Note that by formulas (11) and (10), the last two summands in formula (21) equal to zero. Indeed, for every \(k \in \mathbb{Z}_d^+, |k| \leq n\), sum (10) representing \(\hat{B}_k^{(n), \text{sin}, p}(s, u, a)\) has exactly one summand that contains the variable \(v_k^p\). However, the coefficient \(\lambda^+_p(k, k)\) at \(v_k^p\) is 0 for all \(p\). This proves that \(\frac{\partial}{\partial v_k^p} \hat{B}_k^{(n), \text{sin}, p}(s, u, a) = 0\). Analogously,
\[ \frac{\partial}{\partial u_k} \hat{B}_k^{(n),\cos}(s, u, a) = 0. \]

Let us show that \( (\hat{B}^{(n)}(s, u, a), u)_t = 0. \) Remember that \( y(t, \theta) \) has representation (4) and that \( \hat{B}(n) \) can be viewed as a function of \( \theta \), i.e.

\[ \hat{B}^{(n)} = \hat{B}_k^{(n),\cos}(s, u, a) \cos(k \cdot \theta) + \hat{B}_k^{(n),\sin}(s, u, a) \sin(k \cdot \theta) \]

(for simplicity of notations we use the same symbol for this function). Furthermore, \( \Delta_{T^d} \) denotes the Laplacian defined in Section 2. We obtain:

\[ (\hat{B}^{(n)}(s, u, a), u)_t = (\Delta_{T^d} \hat{B}^{(n)}, y)_{L^2(T^d, \mathbb{R}^d)} = 4 \int_{T^d} \Delta_{T^d} (\Delta_{T^d} \hat{B}^{(n)}, y) \, d\theta = 0. \]

This proves that \( \delta_{\gamma^{(n)}} \hat{B}^{(n)}(s, \cdot, a) = 0. \)

**Proof of Lemma 2.** Suppose for a moment that the function \( \hat{B}^{(n)} \) on the right-hand side of (20) does not depend on \( s \). Then, in [7] (see also [8] and [6]) the existence of a flow solution was established for \( \gamma^{(n)} \)-almost all initial conditions. In particular, the above mentioned results imply that this flow solution keeps the measure \( \gamma^{(n)} \) invariant.

We prove the existence of a solution to (20) and the fact that it keeps the measure \( \gamma^{(n)} \) invariant by the approximation of the exponential factors in \( \hat{B}^{(n)} \) by stepwise functions. Consider the sequence of partitions \( \mathcal{P}_N = \{0 = s_0 \leq s_1 \leq \cdots \leq s_N = T\} \) of the interval \([0, T]\). Let \( |k| \leq n \) and \( |h| \leq n \), and let

\[ I_0 + \sum_{i=1}^{N} e^{-2(k + h, h)s_i \nu} I_{[s_{i-1}, s_i]} \quad \text{and resp.} \quad I_0 + \sum_{i=1}^{N} e^{2(k - h, h)s_i \nu} I_{[s_{i-1}, s_i]} \]

(22)

be uniform approximations of the functions \( e^{-2(k + h, h)s \nu} \) and \( e^{2(k - h, h)s \nu} \) on the interval \([0, T]\). Denote \( u_{(n)} = \Pi_n u \). Define \( \tilde{u}_{n}^{-1}(0) = u_{(n)} \). Let for every integer \( i = 0, \ldots, N - 1 \), \( \tilde{u}_n^i(s) \) be the solution of the Cauchy problem

\[ \begin{cases} \frac{d}{ds} \tilde{u}_n^i(s) = \hat{B}^{(n)}(s, \tilde{u}_n^i(s), \Pi_n u), \\ \tilde{u}_n^i(s_i) = \tilde{u}_{n}^{-1}(s_i). \end{cases} \]

(23)

on the interval \([s_i, s_{i+1}]\). The latter solution exists by the results [6, 7, 8]. Let \( \hat{B}^N \) be obtained from \( \hat{B}^{(n)} \) by replacing the factors \( e^{-2(k + h, h)s \nu} \) and \( e^{2(k - h, h)s \nu} \) with their approximations (22) that correspond to the partition \( \mathcal{P}_N \). For
simplicity, in the definition of \( \hat{B}^N \) we skip the index \( (n) \). Let us glue the solutions on every interval \([s_i, s_{i+1}]\) and obtain the solution to

\[
\begin{aligned}
\frac{d}{ds} \tilde{u}(s, \Pi_n u) &= \hat{B}^N(s, \tilde{u}(s, \Pi_n u), \Pi_n^+ u), \\
\tilde{u}(0, \Pi_n u) &= \Pi_n u.
\end{aligned}
\]  

(24)

We show that the solution \( \tilde{u}^N(s, u_{(n)}) \) exists on \([0, T]\) for \( \gamma^{(n)} \)-almost all \( u_{(n)} \in E^{(n)} \) and keeps the measure \( \gamma^{(n)} \) invariant. By the argument in Lemma 3, \( \frac{\partial}{\partial u_k} \hat{B}_k^{\text{sin}}(s, u_{(n)}) = \frac{\partial}{\partial u_k} B_k^{\text{cos}}(s, u_{(n)}) = 0 \). This means that \( \nabla \hat{B}(s, u_{(n)}) = 0 \) for all \( s \in [0, T] \). Again, by results of [6] or [7], the measure \( \gamma^{(n)} \) is invariant with respect to the flow \( \tilde{u}^N(s, u_{(n)}) \) with \( \tilde{u}^N(s_i, u_{(n)}) = u_{(n)} \) on every partition interval \([s_i, s_{i+1}]\). Now let \( E_i \subset E^{(n)} \), \( i = 0, \ldots, N \), be the sets of full \( \gamma^{(n)} \)-measure such that for every \( u_{(n)} \in E_i \), the solution to (23) on \([s_i, s_{i+1}]\) with the initial condition \( u_{(n)} \) exists. Now let \( E'_{i} = E_1 \cap \tilde{u}^N(s_1, E_0), E'_2 = E_2 \cap \tilde{u}^N(s_2, E'_1), \) etc. \( E'_{N-1} = E_{N-1} \cap \tilde{u}^N(s_{N-1}, E'_{N-2}) \). Since on every partition interval \([s_i, s_{i+1}]\), the solution \( \tilde{u}_n^i(s, \cdot) \) keeps the measure \( \gamma^{(n)} \) invariant, we obtain that \( \gamma^{(n)}(E'_i) = 1 \) for every \( i = 1, \ldots, N - 1 \). Now let

\[
E'_0 = \tilde{u}^N(s_1, \cdot)^{-1} \circ \cdots \circ \tilde{u}^N(s_{N-1}, \cdot)^{-1} E'_{N-1}.
\]

Clearly, \( E'_0 \subset E_0 \) and, by the invariance of the measure \( \gamma^{(n)} \), \( \gamma^{(n)}(E'_0) = 1 \). Therefore, the solution \( u^N(s) \) exists for all initial conditions \( u_{(n)} \in E'_0 \) and keeps the measure \( \gamma^{(n)} \) invariant. Clearly, for every fixed \( u_{(n)} \in E^{(n)} \), for every fixed \( n \), \( \hat{B}^N(s, u_{(n)}) \rightarrow \hat{B}(s, u_{(n)}) \) uniformly in \( s \in [0, T] \) as \( |\mathcal{P}_N| \rightarrow 0 \). Now let \( \tilde{u}^N \) and \( \tilde{u}^M \) be solutions to (24) that correspond to \( \hat{B}^N \) and, respectively, \( \hat{B}^M \), and let \( \tilde{u}^N_{k,i}, \tilde{v}^N_{k,i} \) and \( \tilde{u}^M_{k,i}, \tilde{v}^M_{k,i} \) be the coordinates of \( \tilde{u}^N \) and \( \tilde{u}^M \). We prove that \( \tilde{u}^M - \tilde{u}^N \) converges to zero in the space \( L_2(\gamma^{(n)}) \). Also, without loss of generality we can assume that \( \mathcal{P} \subset \mathcal{P}^N \) because we can always introduce the third partition \( \mathcal{P}^N \cup \mathcal{P}^M \) which is a refinement of both. Let the norm \( |\cdot|_n \) denote the Euclidean norm of the vector with real components numbered by \( h \in \mathbb{Z}^+_d \) with \( |h| \leq n \), and the norm \( |\cdot|_{n,d} \) denote the Euclidean norm in \( E^{(n)} \). We would like to prove that \( \{ \tilde{u}^N(s, u_{(n)}) \}_{n=1}^{\infty} \) is a Cauchy sequence with respect to the norm \( \mathbb{E}_{\gamma^{(n)}} \sup_{s \in [0, T]} |\cdot|_{n,d} \). We have:

\[
|\tilde{u}^M(t, \cdot) - \tilde{u}^N(t, \cdot)|_{n,d} \leq \int_0^t \left| \hat{B}^M(s, \tilde{u}^M(s, \cdot)) - \hat{B}^N(s, \tilde{u}^N(s, \cdot)) \right|_{n,d} ds \\
+ \int_0^t \left| \hat{B}^N(s, \tilde{u}^N(s, \cdot)) - \hat{B}^N(s, \tilde{u}^N(s, \cdot)) \right|_{n,d} ds.
\]  

(25)
In the estimate below, we take into account that $|\lambda^+_i (k, h)|$, $|\lambda^-_i (k, h)|$, and $|\lambda_i (k)|$ are always smaller than 1, and that the exponents are bounded because $|h| \leq n$, $|k| \leq n$, and $s \in [0, T]$. Let $\tilde{B}^{N,\cos}_k$ be the $(d-1)$-dimensional $\cos(k\cdot \theta)$-component of $\tilde{B}^N$. We observe that there exists a constant $L_{n,T}$ such that

$$
\left| \tilde{B}^{N,\cos}_k(s, \tilde{u}^M) - \tilde{B}^{N,\cos}_k(s, \tilde{u}^N) \right| \leq L_{n,T} \sum_{|h| \leq n} \sum_{i,j=1}^{d-1} \left| \tilde{u}^{M,j}_h - \tilde{u}^{N,j}_h \right| (|\tilde{u}^{M,i}_{\pm (k-h)}| + |\tilde{u}^{M,i}_{k+h}|)
+ |\tilde{u}^{M,j}_h - \tilde{u}^{N,j}_h| \left( (|\tilde{u}^{M,i}_{k+h}| + |\tilde{u}^{N,i}_{k+h}|) + |\tilde{u}^{M,i}_h - \tilde{u}^{N,i}_h| \right)
+ |\tilde{u}^{N,j}_h| \left( |\tilde{u}^{M,i}_{k+h}| - \tilde{u}^{N,i}_{k+h} | + |\tilde{u}^{N,i}_h - \tilde{u}^{N,i}_{k+h}| \right).
$$

We keep in mind that in this estimate, every coordinate on the right-hand side depends on $s$. However, for simplicity of notations we skip this dependence. Let $\tilde{u}^{M,i}_h$ and $\tilde{u}^{N,i}_h$ denote the vectors with the coordinates $\tilde{u}^{M,j}_h$ and resp. $\tilde{u}^{N,j}_h$, $h \in \mathbb{Z}_d^+$, $|h| \leq n$. Further denote by $\tilde{u}^{M,i}_{k+h}$, $\tilde{u}^{N,i}_{k+h}$, $\tilde{u}^{M,i}_{k+}$, $\tilde{u}^{N,i}_{k+}$, $u^{M,i}_{k+}$, $u^{N,i}_{k+}$, $v^{i+}$, the vectors with the coordinates $\tilde{u}^{M,i}_{k+h}$, $\tilde{u}^{N,i}_{k+h}$, $\tilde{u}^{M,i}_{k+}$, $\tilde{u}^{N,i}_{k+}$, $u^{M,i}_{k+}$, $u^{N,i}_{k+}$, $v^{i+}$, respectively, $h \in \mathbb{Z}_d^+$, $|h| \leq n$. Finally let $u^{M,i}_{k+}$ be the vector with the $(d-1)$-dimensional components $u^{M,i}_{k+}$, $v^{i+}$, $|h| \leq n$. First we exchange the summations over $h$ and over $i, j$. Then we notice that sums of the type

$$
\sum_{|h| \leq n} \left| \tilde{u}^{M,j}_h - \tilde{u}^{N,j}_h \right| |\tilde{u}^{M,i}_{k+h} v^{i+}|
$$

can be viewed as the scalar products of the vectors with the components $\{|\tilde{u}^{M,j}_h - \tilde{u}^{N,j}_h|\}_{|h| \leq n}$ and $\{|\tilde{u}^{M,i}_{k+h}|\}_{|h| \leq n}$. Also, note that the absolute values of these vectors are $|\tilde{u}^{M,j}_h - \tilde{u}^{N,j}_h|_{n}$ and $|\tilde{u}^{M,i}_{k+h}|_{n}$ respectively. Next, note that

$$
|\tilde{u}^{M,i}_{k+} n \leq |\tilde{u}^{M,i}_{n} + |v^{i+}_{k+}| n.
$$

Finally we take into account that

$$
|\tilde{u}^{M,i}_{k+} - \tilde{u}^{N,i}_{k+}|_{n} \leq |\tilde{u}^{M,i} - \tilde{u}^{N,i}|_{n},
|\tilde{u}^{M,i}_{\pm (k-\cdot)} - \tilde{u}^{N,i}_{\pm (k-\cdot)}|_{n} \leq |\tilde{u}^{M,i} - \tilde{u}^{N,i}|_{n}.
$$

The latter two inequalities hold because $\tilde{u}^{M,i}$ and $\tilde{u}^{N,i}$ have the same initial condition. Clearly, the all three above inequalities hold when $v$ is replaced
by \( u \). We obtain:

\[
|\hat{B}^{N,\cos}_k(s, \tilde{u}^M) - \hat{B}^{N,\cos}_k(s, \tilde{u}^N)| \leq 2L_{n,T} \\
\times \left[ \sum_{i=1}^{d-1} |\tilde{u}^{M,i} - \tilde{u}^{N,i}|_n \sum_{j=1}^{d-1} (|\tilde{u}^{M,j}|_n + |\tilde{u}^{N,j}|_n + |u^{j+}_k|_n) \\
+ \sum_{i=1}^{d-1} |\tilde{u}^{M,i} - \tilde{u}^{N,i}|_n \sum_{j=1}^{d-1} (|\tilde{u}^{M,j}|_n + |\tilde{u}^{N,j}|_n + |u^{j+}_k|_n) \right]
\leq \tilde{L}_{n,T} (|\tilde{u}^M(s)|_{n,d} + |\tilde{u}^N(s)|_{n,d} + |u_{k+}|_{n,d}) |\tilde{u}^M(s) - \tilde{u}^N(s)|_{n,d}
\]

where \( \tilde{L}_{n,T} \) is a modified constant. Clearly, the estimate for \( |\hat{B}^{N,\sin}_k(s, \tilde{u}^M) - \hat{B}^{N,\sin}_k(s, \tilde{u}^N)| \) will be the same. Therefore, modifying the constant \( L_{n,T} \) we obtain that

\[
|\hat{B}^{N}(s, \tilde{u}^M) - \hat{B}^{N}(s, \tilde{u}^N)|_{n,d} \leq L_{n,T} (|\tilde{u}^M(s)|_{n,d} + |\tilde{u}^N(s)|_{n,d} + |u_{2n,d}|) |\tilde{u}^M(s) - \tilde{u}^N(s)|_{n,d}.
\]

The latter inequality and inequality (25) imply

\[
|\tilde{u}^M(t) - \tilde{u}^N(t)|_{n,d} \leq \int_0^t \left[ |\hat{B}^{M}(s, \tilde{u}^M(s)) - \hat{B}^{N}(s, \tilde{u}^M(s))|_{n,d} + L_{n,T} \int_0^t (|\tilde{u}^M(s)|_{n,d} + |\tilde{u}^N(s)|_{n,d} + |u_{2n,d}|) |\tilde{u}^M(s) - \tilde{u}^N(s)|_{n,d} \right] ds.
\]

By Gronwall’s lemma,

\[
|\tilde{u}^M(t) - \tilde{u}^N(t)|_{n,d} \leq \int_0^t \left[ |\hat{B}^{M}(s, \tilde{u}^M(s)) - \hat{B}^{N}(s, \tilde{u}^M(s))|_{n,d} \right] ds
\times \exp \left\{ L_{n,T} \int_0^t (|\tilde{u}^M(s)|_{n,d} + |\tilde{u}^N(s)|_{n,d} + |u_{2n,d}|) ds \right\}.
\]

Taking supremum over \([0, T]\) and then the expectation of the both sides, using the \( \gamma^{(n)} \)-invariance in the first integral, and making the variable exchange in the integral under the exponent sign, we obtain that there is a constant \( K_{n,T} \) such that:

\[
\left( \mathbb{E}_{\gamma^{(n)}} \sup_{t \in [0, T]} |\tilde{u}^M(t) - \tilde{u}^N(t)|_{n,d} \right)^2 \leq K_{n,T} \int_0^T \mathbb{E}_{\gamma^{(n)}} |\hat{B}^{M}(s, \tilde{u}) - \hat{B}^{N}(s, \tilde{u})|^2_{n,d} ds
\times \mathbb{E}_{\gamma^{(n)}} \exp \left\{ 2L_{n,T} T \int_0^1 (|\tilde{u}^M(Ts)|_{n,d} + |\tilde{u}^N(Ts)|_{n,d}) ds \right\}. \tag{26}
\]

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The first integral clearly converges to zero. Let us prove that the last multiplier remains bounded as $M$ and $N$ tend to infinity. Clearly, it suffices to prove that $\mathbb{E}_{\gamma(n)} \exp \int_0^1 |\tilde{u}^M(Ts)|_{n,d} ds$ is bounded as $M \to \infty$. We use the Taylor expansion for the exponent, Hölder’s inequality, and the inequality

$$\left( \int_0^1 |\tilde{u}^M(Ts)|_{n,d} ds \right)^m \leq \int_0^1 |\tilde{u}^M(Ts)|^m_{n,d} ds$$

to conclude that

$$\exp \int_0^1 |\tilde{u}^M(Ts)|_{n,d} ds \leq \int_0^1 \exp |\tilde{u}^M(Ts)|_{n,d} ds.$$

We already proved that $\tilde{u}^M(Ts)$ keeps the measure $\gamma(n)$ invariant. Therefore,

$$\mathbb{E}_{\gamma(n)} \exp \int_0^1 |\tilde{u}^M(Ts)|_{n,d} ds \leq \int_0^1 \mathbb{E}_{\gamma(n)} \exp |\tilde{u}^M(Ts)|_{n,d} ds = \mathbb{E}_{\gamma(n)} \exp |\Pi_n u|_{n,d} < \infty.$$

By (26), we can find a subsequence of $\tilde{u}^N(s)$ (for simplicity, the subsequence will be also denoted by $\tilde{u}^N(s)$) so that for $\gamma(n)$-almost all initial conditions $u(n)$, there exists a uniform limit $\hat{U}^{(n)}(s, u(n)) = \lim_{N \to \infty} \tilde{u}^N(s, u(n))$. The limit $\hat{U}^{(n)}(s, u(n))$ keeps the measure $\gamma(n)$ invariant. Indeed, let $f$ be a bounded continuous function $E(n) \to \mathbb{R}$. By Lebesgue’s theorem,

$$\mathbb{E}_{\gamma(n)}[f(\hat{U}^{(n)}(t, \cdot))] = \lim_{N \to \infty} \mathbb{E}_{\gamma(n)}[f(\tilde{u}^N(t, \cdot))] = \mathbb{E}_{\gamma(n)}[f(\cdot)].$$

Let us prove that $\hat{U}^{(n)}$ is a solution to (20). We have:

$$\mathbb{E}_{\gamma(n)} \int_0^t \left| \dot{\hat{B}}^{(n)}(s, \hat{U}^{(n)}(s)) - \hat{B}^N(s, \tilde{u}^N(s)) \right|_{n,d} ds \leq \int_0^t \mathbb{E}_{\gamma(n)} \left| \dot{\hat{B}}^{(n)}(s, \hat{U}^{(n)}(s)) - \dot{\hat{B}}^N(s, \tilde{u}^N(s)) \right|_{n,d} ds + \int_0^t \mathbb{E}_{\gamma(n)} \left| \dot{\hat{B}}^{(n)}(s, \tilde{u}^N(s)) - \dot{\hat{B}}^{(n)}(s, \hat{U}^{(n)}(s)) \right|_{n,d} ds. \quad (27)$$

The first term on the right-hand side converges to zero since the measure $\gamma(n)$ is invariant with respect to $\tilde{u}^N(t)$. Let us prove that the second term converges.
to zero. Let \( \varepsilon \) be fixed arbitrary, and let \( F \) be a continuous bounded function on \([0, T] \times E^{(n)}\) such that

\[
\int_0^T \mathbb{E}_{\gamma^{(n)}} \left| \hat{B}^{(n)}(s, \cdot) - F(s, \cdot) \right| ds < \varepsilon.
\]

Since \( F \) is bounded and \( \hat{U}^{(n)}(s, u^{(n)}) \) converges to \( \tilde{u}^N(s, u^{(n)}) \) uniformly in \( s \) and \( \gamma^{(n)} \)-a.s. in \( u^{(n)} \),

\[
\lim_{N \to \infty} \int_0^t \mathbb{E}_{\gamma^{(n)}} \left| F(s, \tilde{u}^N(s, \cdot)) - F(s, \hat{U}^{(n)}(s, \cdot)) \right|_{n,d} ds = 0
\]

by the Lebesgue theorem. By the invariance of the measure \( \gamma^{(n)} \) with respect to \( \tilde{u}^N(s, \cdot) \) and \( \hat{U}^{(n)}(s, \cdot) \),

\[
\int_0^t \mathbb{E}_{\gamma^{(n)}} \left| \hat{B}^{(n)}(s, \tilde{u}^N(s, \cdot)) - F(s, u^N(s, \cdot)) \right| ds < \varepsilon.
\]

The latter inequality holds also if we replace \( \tilde{u}^N(s, \cdot) \) with \( \hat{U}^{(n)}(s, \cdot) \). This proves that the second term in (27) converges to zero as \( N \to \infty \). Thus, we can find a subsequence of the sequence

\[
\left\{ \int_0^t \mathbb{E}_{\gamma^{(n)}} \left| \hat{B}^{(n)}(s, \hat{U}^{(n)}(s, u^{(n)})) - \hat{B}^N(s, \tilde{u}^N(s, u^{(n)})) \right|_{n,d} ds \right\}_N
\]

which converges to zero \( \gamma^{(n)} \)-a.s. in \( u^{(n)} \in E^{(n)} \). For simplicity of notation, we will think of (28) as of the a.s.-converging subsequence. Therefore, (20) is fulfilled for \( \gamma^{(n)} \)-almost all initial conditions.

The uniqueness of the solution is a classical result of the theory of ODEs. The theorem is proved. \( \square \)

**Corollary 1.** *ODE (19) has a unique solution for \( \gamma \)-almost all initial conditions \( u \in H^\alpha \). This solution keeps the measure \( \gamma \) invariant.*

**Proof.** From our construction it follows that the infinite-dimensional solution \( \hat{U}^{(n)} \) to (19) is obtained from the finite-dimensional solution \( \hat{U}^{(n)} \) to (20) as follows:

\[
\hat{U}^{(n)}(t, u) = \hat{U}^{(n)}(t, \Pi_n u) + \Pi_{n^1}^* u.
\]
This solution is unique since $\tilde{U}^{(n)}(t, \Pi_n u)$ is unique. Further, let $f \in L_1(\gamma)$, and let the measure $\gamma^{(n)}_\perp$ be such that $\gamma = \gamma^{(n)} \otimes \gamma^{(n)}_\perp$. Then, by the invariance of $\gamma^{(n)}$, we obtain:

\[
\int_{H^\alpha} f(\tilde{U}^{(n)}(t, u)) \gamma(du)
= \int_{H^\alpha \otimes E^{(n)}} \gamma^{(n)}_\perp (d\Pi_n u) \int_{E^{(n)}} f\left(\tilde{U}^{(n)}(t, \Pi_n u) + \Pi_n u\right) \gamma^{(n)}(d\Pi_n u)
= \int_{H^\alpha} f(u) \gamma(du).
\]

To show that the solution exists for $\gamma$-almost all $u \in H^\alpha$ note that if $u_1 = \Pi_n u$ and $u_2 = \Pi_n^\perp u$, then by Lemma 2, the solution to (19) exists for $\gamma^{(n)}$-almost all $u_1$ and for all $u_2$, and therefore for $\gamma$-almost all $u$.

4. Navier–Stokes solution as the limit of Galerkin-type approximations

Let $\tilde{U}^{(n)}_k(t, u)$ and $\tilde{V}^{(n)}_k(t, u)$ be $(d-1)$-dimensional components of the solution $\tilde{U}^{(n)}(t, u)$ to (19). Define

\[
U^n_k(t, u) = e^{-t|k|^2\nu} \tilde{U}^{(n)}_k(t, u), \quad V^n_k(t, u) = e^{-t|k|^2\nu} \tilde{V}^{(n)}_k(t, u).
\]

Consider the triple $(H^\alpha, B, \gamma)$, where $B$ is the $\sigma$-algebra of Borel subsets of $H^\alpha$, as a probability space, and $U^n(t, u) = \{U^n_k(t, u), V^n_k(t, u)\}$, $k \in \mathbb{Z}^+_d$, as a stochastic process on it. Note that by (29), it is clear that $U^n(t, u)$ takes values in $H^\alpha$.

**Lemma 4.** There exists a subset $H' \subset H^\alpha$ of full $\gamma$-measure such that for all $u \in H'$, for all $t \in [0, T]$, $U^n(t, u)$ verifies the equation

\[
U^n(t, u) = e^{tu\Delta}u + \int_0^t e^{(t-s)\nu\Delta} B^{(n)}(U^n(s, u)) ds.
\]

Moreover, for any function $f \in L_1(\gamma)$ it holds that

\[
\mathbb{E}_\gamma[f(U^n(t, u))] = \mathbb{E}_\gamma[f(e^{tu\Delta}u)],
\]

where $\mathbb{E}_\gamma$ is the expectation with respect to the measure $\gamma$. 

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Proof. Note that (30) follows from (19) by applying the operator $e^{tu\Delta}$ to the both parts and taking into account that $e^{tu\Delta} B^{(n)}(s, \tilde{U}^n(s, u)) = B^{(n)}(U^n(s, u))$. Equality (31) follows from the invariance of the measure $\gamma$ with respect to the solution $\tilde{U}^n(t, u)$:

$$E_\gamma[f(U^n(t, u))] = E_\gamma[f(e^{tu\Delta} \tilde{U}^n(t, u))] = E_\gamma[f(e^{tu\Delta} u)].$$

□

Let $\nu^n$ be the law of $U^n$ on $C([0, T], H^\alpha)$, i.e. for any Borel subset $G \subset C([0, T], H^\alpha)$,

$$\nu^n(G) = \gamma\{u \in H^\alpha : U^n(\cdot, u) \in G\}.$$

We prove that the sequence of measures $\nu^n$ contains a weakly convergent subsequence. For this we need versions of the tightness criterium for probability measures and Arzelá-Ascoli’s theorem. Since we do not know a reference where these theorems are given in the form suitable for our purpose, we prove these results.

**Proposition 2** (Tightness criterium for probability measures). Let $E$ be a Banach space. The sequence of probability measures $P_n$ on $C([0, T], E)$ is tight if and only if

(i) for each $\varepsilon > 0$ and $t_R \in [0, T] \cap \mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers, there exists a compact $K \subset E$ such that

$$P_n(y : y(t_R) \notin K) < \varepsilon \quad \forall n;$$

(ii) for each $\varepsilon > 0$ and $\rho > 0$, there exist a $\delta$, with $0 < \delta < 1$, so that

$$P_n(y : \sup_{0 \leq t' \leq T} \sup_{t'-t < \delta} \|y(t) - y(t')\|_E > \rho) < \varepsilon \quad \forall n.$$

**Proposition 3** (Arzelá-Ascoli’s theorem). Let $E$ be a Banach space. A family of functions $\Phi \subset C([0, T], E)$ is relatively compact if and only if

(i) for every $t_R \in [0, T] \cap \mathbb{Q}$, the set $\{y(t_R), y \in \Phi\}$ is relatively compact in $E$;

(ii) the family $\Phi$ is equicontinuous.
Proof of Proposition 3. Assume that $\Phi$ is relatively compact. By the version of Arzelá-Ascoli’s theorem from [11] (Theorem 3.1 p. 55), $\Phi$ satisfies conditions (i) and (ii), and the necessity of the statement follows.

Let us assume that $\Phi$ satisfies conditions (i) and (ii). To satisfy the assumption of Arzelá-Ascoli’s theorem from [11] we have to prove that the set $\{f(t), t \in \Phi\}$ is relatively compact for each $t \in [0,T]$, i.e. that condition (i) is fulfilled for all $t \in [0,T]$ including irrational numbers. Let a number $t \in [0,T]$ and a sequence $\{f_n\} \subset \Phi$ be fixed arbitrary. Further let $\{t_n\} \subset [0,T] \cap \mathbb{Q}$ be such that $\lim_{n \to \infty} t_n = t$. By condition (i), $\{f_n(t_1)\}$ contains a subsequence $\{f^{1n}(t_1)\}$ that converges in $E$ to a limit $a_1$. Furthermore, we can choose the subsequence $\{f^{1n}(t_1)\}$ so that $\|f^{1n}(t_1) - a_1\|_E \downarrow 0$ as $n \to \infty$. Let $\{\varepsilon_n\}$ be a sequence of real numbers such that $\varepsilon_n \downarrow 0$ as $n \to \infty$ and

$$ \|f^{1n}(t_1) - a_1\|_E < \varepsilon_n. $$

Suppose we found subsequences $\{f^{2n}\}_{n=1}^{\infty}, \ldots, \{f^{(i-1)n}\}_{n=1}^{\infty}$ such that each $\{f^{jn}\}, j \leq i - 1$, converges at points $t_1, \ldots, t_j$. Moreover, if $a_j = \lim_{n \to \infty} f^{jn}(t_j), j = 1, 2, \ldots, i - 1$, then $\|f^{jn}(t_j) - a_j\|_E \downarrow 0$ and $\|f^{jn}(t_j) - a_j\|_E < \varepsilon_n$ for all $n$. By condition (i), we find a subsequence $\{f^{in}\}$ of $\{f^{(i-1)n}\}$ that converges at point $t_i$. Next, we find another subsequence of the subsequence $\{f^{in}\}$ (for convenience, use the same symbol $\{f^{in}\}$ for this subsequence) such that $\|f^{in}(t_i) - a_i\|_E \downarrow 0$ as $n \to \infty$, and, moreover,

$$ \|f^{in}(t_i) - a_i\|_E < \varepsilon_n. \quad (32) $$

Finally, let us take the diagonal subsequence $\{f^{in}\}$. Clearly, $\lim_{n \to \infty} f^{in}(t_i) = a_i$ for all $i$ and

$$ \|f^{in}(t_i) - a_i\|_E < \varepsilon_n \quad (33) $$

for all $i \leq n$. Let us show that $\{f^{nn}(t_n)\}$ is a Cauchy sequence in $E$. Indeed, fix an $\varepsilon > 0$. Let the number $N$ be such that for $n > m > N, \|f(t_n) - f(t_m)\|_E < \frac{\varepsilon}{2}$ for all $f \in \Phi$, and $\|f^{nn}(t_n) - f^{nn}(t_m)\|_E < \frac{\varepsilon}{2}$. The latter inequality holds by (33). This implies that for $n > m > N, \|f^{nn}(t_n) - f^{nn}(t_m)\|_E < \varepsilon$. Thus, $\{f^{nn}(t_n)\}$ is a Cauchy sequence in $E$, and therefore, it has a limit $a$ in $E$. Let us prove that $a$ is also a limit of $f^{nn}(t)$. Indeed,

$$ \|f^{nn}(t) - a\|_E \leq \|f^{nn}(t) - f^{nn}(t_n)\|_E + \|f^{nn}(t_n) - a\|_E. $$
The first term on the right-hand side converges to zero by the equicontinuity of functions from $\Phi$. Thus we found a subsequence $\{f^{m_n}\} \subset \{f_n\}$ that converges at point $t$. This proves that the set $\{f(t), f \in \Phi\}$ is relatively compact. By the version of Arzelà-Ascoli’s theorem from [11], the family $\Phi$ is relatively compact.

**Proof of Proposition 2.** Basically the proof follows the lines of Theorem 8.2 from [5] (p.55) but it is adapted to our case. Let us assume that the family $\{P_n\}$ is tight. Then for every $\varepsilon \in (0, 1)$ there exists a compact $\Phi \subset C([0, T], E)$ such that $P_n(\Phi) \geq 1 - \varepsilon$ for all $n$. Fix a $t_R \in [0, T] \cap \mathbb{Q}$, and let $K = \{y(t_R) : y \in \Phi\}$. By Proposition 3, $K$ is a compact in $E$. Since $\{y : y(t_R) \notin K\} \subset \Phi^c$, where $\Phi^c$ is the complement of $\Phi$ in $C([0, T], E)$, then $P_n(y : y(t_R) \notin K) < \varepsilon$ for all $n$. Condition $(ii)$ can be verified in exactly the same way as in Theorem 8.2 from [5].

Now let us assume that Conditions $(i)$ and $(ii)$ of Proposition 2 are fulfilled, and prove that the sequence $\{P_n\}$ is tight. Let $\{t_m\}_{m=1}^{\infty}$ be a sequence of all rational numbers of the interval $[0, T]$. Fix an $\varepsilon > 0$ and choose a compact $K_m$ so that $P_n(y : y(t_m) \notin K_m) < \frac{\varepsilon}{2^{m+1}}$ for all $n$. Further let $w_y(\delta)$ be the modulus of continuity of $y \in C([0, T], E)$. Choose a sequence $\{\delta_{m'}\}_{m'=1}^{\infty}$ so that $P_n(y : w_y(\delta_{m'}) < \frac{1}{m'}) > 1 - \frac{\varepsilon}{2^{m+1}}$. Let

$$\Phi = \bigcap_{m=1}^{\infty} \{y : y(t_m) \in K_m\} \bigcap_{m'=1}^{\infty} \{y : w_y(\delta_{m'}) < \frac{1}{m'}\}.$$ 

Then $P_n(\Phi) > 1 - \varepsilon$ for all $n$, and, by Proposition 3, $\Phi$ is a compact in $C([0, T], E)$.

Our main result is the following theorem:

**Theorem 1.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $l > \frac{d}{2} + \alpha + 1$. Then, there exist an $H^\alpha$-valued stochastic process $U(t, \omega)$ and a subset $\Omega' \subset \Omega$ of full $\mathbb{P}$-measure such that for all $\omega \in \Omega'$ and for all $t \in [0, T],$

$$U(t, \omega) = e^{t\Delta_\nu}U(0, \omega) - \int_0^t e^{(t-s)\Delta_\nu}B(U(s, \omega))\,ds, \quad (34)$$

and the law of $U(0, \omega)$ on $H^\alpha$ is the measure $\gamma$. Moreover, for every function $f \in L_1(\gamma)$ and for all $t \in [0, T]$ it holds that

$$\mathbb{E}_\mathbb{P}[f(U(t, \omega))] = \int_{H^\alpha} f(e^{\nu \Delta u})\gamma(du). \quad (35)$$

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Taking into account that $U^n(t, u)$ verifies (30), we obtain:

\[
\nu^n\left(\sup_{\substack{0 \leq t \leq t' \leq T \\ t' - t < \delta}} |y(t) - y(t')| \geq \rho\right) \leq \frac{1}{\rho^2} \int_{\mathcal{C}([0, T], H^\alpha)} \sup_{\substack{0 \leq t \leq t' \leq T \\ t' - t < \delta}} \|y(t) - y(t')\|_{\alpha}^2 \nu^n(dy)
\]

\[
= \frac{1}{\rho^2} \int_{H^\alpha} \sup_{\substack{0 \leq t \leq t' \leq T \\ t' - t < \delta}} \|U^n(t, u) - U^n(t', u)\|_{\alpha}^2 \gamma(du)
\]

\[
\leq \frac{3}{\rho^2} \mathbb{E}_\gamma \sup_{\substack{0 \leq t \leq t' \leq T \\ t' - t < \delta}} \left[ \|e^{t' \Delta} - e^{t \Delta}\|^2_{\alpha} + \left\| \int_t^{t'} e^{(t-s)\Delta} B(n) (U^n(s, u)) \, ds \right\|^2_{\alpha}
\]

\[
+ \left\| \int_0^{t'} e^{(t-s)\Delta} (e^{(t'-t)\Delta} - I) B(n) (U^n(s, u)) \, ds \right\|^2_{\alpha}. \tag{36}
\]

Using (31), we can estimate the right-hand side of (36) by

\[
\frac{3}{\rho^2} \left[ \mathbb{E}_\gamma \| (I - e^{\delta \nu \Delta}) u \|^2_{\alpha} + \delta \int_0^T \mathbb{E}_\gamma \| B(n) (e^{s \nu \Delta} u) \|^2_{\alpha} \, ds \right.
\]

\[
\left. + T \int_0^T \mathbb{E}_\gamma \| (I - e^{\delta \nu \Delta}) B(n) (e^{s \nu \Delta} u) \|^2_{\alpha} \, ds \right]. \tag{37}
\]

The first term in (37) converges to zero by the Banach-Steinhaus theorem, since $\|I - e^{\delta \nu \Delta}\| \leq 2$ and the convergence clearly holds for those $u$ that have only a finite number of non-zero coordinates. Next, by the results of Section 3.2, $\mathbb{E}_\gamma \| B(n) (e^{s \nu \Delta} u) \|^2_{\alpha}$ is bounded by the sum of series (15) (multiplied by $8(d - 1)^2$) uniformly in $s$ and $n$ which proves that the second term in (37) also converges to zero. Finally, if we define the $H^\alpha$-vector $L$ whose $k$-th sin- and cos-components are $8(d - 1)^2 \sum_{h \in \mathbb{Z}^d} \frac{1}{|k + h|^2} \frac{1}{|h|^{2-\alpha}}$, we obtain that

\[
\mathbb{E}_\gamma \| (I - e^{\delta \nu \Delta}) B(n) (e^{s \nu \Delta} u) \|^2_{\alpha} \leq \| (I - e^{\delta \nu \Delta}) L \|^2_{\alpha}
\]
where the right-hand side converges to zero by the Banach-Steinhaus theorem. Thus, (37) converges to zero, and, therefore, Condition (ii) is verified.

Let us verify Condition (i). Fix an arbitrary \( \varepsilon > 0 \), and find a compact \( K \subset H^\alpha \) of the form

\[
K = B(r_1) \times \ldots \times B(r_k) \times \ldots,
\]

where \( B(r_k) \) is the \( 2(d-1) \)-dimensional ball of radius \( r_k \) centered at zero in the space of the variables \( (u_k, v_k), k \in \mathbb{Z}^+_d \), so that

\[
\gamma(H^\alpha \backslash K) < \varepsilon. \tag{39}
\]

The compact \( K \) of form (38) with property (39) exists. Indeed, let \( r_k, k \in \mathbb{Z}^+_d \), be a sequence of real numbers satisfying

\[
\sum_{k \in \mathbb{Z}^+_d} |k|^{2\alpha} r_k^2 < \infty. \tag{40}
\]

To show that a set of form (38) is a compact in \( H^\alpha \), take a sequence \( \{f_n\}_{n=1}^\infty \subset K \), and find its subsequence \( \{f_{n_i}\}_{i=1}^\infty \) so that its \( 2(d-1) \)-dimensional components \( f_{n_i}^k, k \in \mathbb{Z}^+_d \), which are pairs of \( (d-1) \)-dimensional cos- and sin-components, have limits \( a_k = \lim_{i \to \infty} f_{n_i}^k \). Since \( |a_k| \leq r_k \), then (40) implies that that \( \{f_{n_i}\}_{i=1}^\infty \) has a limit in \( H^\alpha \). Thus, \( K \) defined by (38) with \( r_k \) satisfying (40) is a compact. Let us show that we can choose numbers \( r_k \) with property (40) so that \( \gamma(K) > 1 - \varepsilon \). Choose a number \( \eta \) so that \( l > \eta > \alpha + \frac{d}{2} \), and set \( r_k = \frac{1}{|k|^\eta} \) for \( |k| > N \) where the number \( N \) is sufficiently large and we choose it based on the arguments below. Clearly, series (40) converges for this choice of \( r_k \). On the other hand,

\[
\gamma_k(B(r_k)) = \left( \frac{|k|^{2l}}{2\pi} \right)^{d-1} \int_{B(r_k)} e^{-\frac{|k|^{2l}}{2}(|u_k|^2+|v_k|^2)} d(u_k, v_k)
\]

\[
> 1 - C |k|^{2(l-\eta)(d-1)} e^{-\frac{|k|^{2(l-\eta)}}{2}},
\]

where \( C \) is a constant. We choose the number \( N \) big enough so that for \( |k| > N \) the right-hand side of the above inequality is bigger than \( 1 - \frac{1}{|k|^{d+1}} \). We obtain

\[
\prod_{|k| > N} \gamma_k(B(r_k)) > \prod_{|k| > N} \left(1 - \frac{1}{|k|^{d+1}}\right) > e^{-2\sum_{|k| > N} \frac{1}{|k|^{d+1}}} > 1 - 2 \sum_{|k| > N} \frac{1}{|k|^{d+1}}.
\]
The series on the right-hand side clearly converges by the arguments from Section 3.2. Let the number $N$ be big enough so that
\[ 2 \sum_{|k|>N} \frac{1}{|k|^{d+1}} < \frac{\varepsilon}{3}. \]
Next, we choose the numbers $r_k$ with $|k| \leq N$ sufficiently big so that
\[ \prod_{|k| \leq N} \gamma_k(B(r_k)) > 1 - \frac{\varepsilon}{3}. \]
Summing up, we obtain:
\[ \gamma(K) = \prod_{k \in \mathbb{Z}_d^+} \gamma_k(B(r_k)) > \left( 1 - \frac{\varepsilon}{3} \right)^2 > 1 - \varepsilon. \]
Take a $t_R \in [0, T] \cap \mathbb{Q}$. By the invariance of the measure $\gamma$ with respect to $U^n(t, u)$, we obtain:
\[ \nu^n(y : y(t_R) \notin K) = \gamma \left( u : (U^n_k(t_R, u), V^n_k(t_R, u)) \notin B(r_k), k \in \mathbb{Z}_d^+ \right) = \gamma \left( u : (\tilde{U}_k^n(t_R, u), \tilde{V}_k^n(t_R, u)) \notin B(e^{tR\nu|k|^2}r_k), k \in \mathbb{Z}_d^+ \right) = \gamma \left( u_k, v_k \right) \notin B(e^{tR\nu|k|^2}r_k), k \in \mathbb{Z}_d^+ \right) \leq \gamma(H^\alpha \setminus K) < \varepsilon. \]
Thus, we verified Conditions (i) and (ii) of the tightness criterium (Proposition 2). Hence, there exists a subsequence of $\nu^n$ (for simplicity we use the same symbol for this subsequence) which converges weakly to a measure $\nu$ on $C([0, T], H^\alpha)$. Next, we apply Skorohod’s representation on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ proved in [4]. In particular, this theorem states the following:

**Skorohod’s representation theorem on a given probability space:** Suppose the measure $\mathbb{P}$ is nonatomic, $\mu_n$ and $\mu$ are separable probabilities on $(S, \mathcal{B})$, where $S$ is a metric space and $\mathcal{B}$ is the Borel $\sigma$-algebra on $S$, and $\mu_n \rightharpoonup \mu$ weakly. Then, if $\mu$ and each $\mu_n$ are tight, there are $S$-valued random variables $X_n$ and $X$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with distributions $\mu_n$ and, respectively, $\mu$, such that $X_n \rightharpoonup X$ a.s..

Set the metric space $S$ to be $C([0, T], H^\alpha)$. By the above version of Skorokhod’s theorem, we conclude that there are $H^\alpha$-valued stochastic processes

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$U^n(t, \omega), \omega \in \Omega$, with the laws $\nu^n$ and, respectively, $\nu$ on $C([0, T], H^\alpha)$ such that $U^n(\cdot, \omega) \to U(\cdot, \omega)$ $\mathbb{P}$-a.s. and with respect to the norm of $C([0, T], H^\alpha)$. Let us prove that for each $n$, the process $U^n(t, \omega)$ verifies equation (30) $\mathbb{P}$-a.s.. Consider the function:

$$F : C([0, T], H^\alpha) \to C([0, T], H^\alpha), f_t \mapsto f_t - \int_0^t e^{(t-s)\Delta \nu} B^{(n)}(f_s) \, ds - e^{t\Delta \nu} f_0.$$ 

It is easy to verify that $F$ is continuos. This implies that the random variables $F(U^n(\cdot, u))$ and $F(U^n(\cdot, \omega))$ have the same distribution $\nu^n \circ F^{-1}$. Therefore,

$$\mathbb{P}(\omega : F(U^n(\cdot, \omega)) = 0) = \gamma(u : F(U^n(\cdot, u)) = 0) = 1.$$ 

The latter equality holds by Lemma 4. This implies that there exists a set $\Omega' \subset \Omega$ of full $\mathbb{P}$-measure so that for all $t \in [0, T]$ and for all $\omega \in \Omega'$,

$$U^n(t, \omega) = e^{t\Delta \nu} U^n(0, \omega) + \int_0^t e^{(t-s)\Delta \nu} B^{(n)}(U^n(s, \omega)) \, ds. \quad (41)$$

Next,

$$\mathbb{E}_\mathbb{P} \sup_{t \in [0, T]} \left\| \int_0^t e^{(t-s)\Delta \nu} \left( B^{(n)}(U^n(s, \omega)) - B(U(s, \omega)) \right) \, ds \right\|_\alpha \leq \mathbb{E}_\mathbb{P} \int_0^T \left\| B^{(n)}(U^n(s, \omega)) - B(U^n(s, \omega)) \right\|_\alpha \, ds$$

$$+ \mathbb{E}_\mathbb{P} \int_0^T \left\| B(U^n(s, \omega)) - B(U(s, \omega)) \right\|_\alpha \, ds. \quad (42)$$

The first summand on the right-hand side of (42) equals to

$$\int_0^T \mathbb{E}_\gamma \left\| B^{(n)}(U^n(s, u)) - B(U^n(s, u)) \right\|_\alpha \, ds$$

$$= \int_0^T \mathbb{E}_\gamma \left\| B^{(n)}(e^{s\nu \Delta} u) - B(e^{s\nu \Delta} u) \right\|_\alpha \, ds.$$ 

The latter tends to zero as $n \to \infty$ since by the results of Section 3.2, $\mathbb{E}_\gamma \| B^{(n)} - B \|^2_\alpha \to 0$. The presence of the semigroup $e^{s\nu \Delta}, s > 0$, just improves
this convergence due to the multipliers $e^{-sv^2}$ applied to each pair $(u_k, v_k)$.

The above convergence to zero specifically follows from the argument below:

$$
\mathbb{E}_\gamma \sum_{k \in \mathbb{Z}_d^+} |k|^{2\alpha} (|B_k^{\sin} - B_k^{(n), \sin}|^2 + |B_k^{\cos} - B_k^{(n), \cos}|^2) \leq \sum_{|k| > n} |k|^{2\alpha} (|B_k^{\sin}|^2 + |B_k^{\cos}|^2)
$$

$$
\leq 4(d - 1)^2 \sum_{|h| > n} \frac{1}{|h|^{2\alpha - 2}} \sum_{k \in \mathbb{Z}_d^+} \left( \frac{|k|^{2\alpha}}{|k + h|^{2\alpha}} + \frac{|k|^{2\alpha}}{|k - h|^{2\alpha}} \right) \to 0, \quad n \to \infty.
$$

For each $t \in [0, T]$, we define the Hilbert space $H_\alpha^t = e^{i\nu \Delta} H_\alpha$ and the measure $\gamma_t = \gamma \circ e^{-i\nu \Delta}$. Clearly, $\gamma_t$ is concentrated on $H_\alpha^t$. Extend the measure $\gamma_t$ to $H_\alpha$ by setting $\gamma_t(H_\alpha \setminus H_\alpha^t) = 0$ and $\sigma$-additivity. Let us show that for each fixed $t \in [0, T]$, the distribution of $U(t, \omega)$ is $\gamma_t$. Indeed, for every bounded continuous function $f : H_\alpha \to \mathbb{R}$, we have:

$$
\mathbb{E}_\pi[f(U^{(n)}(t, \omega))] = \mathbb{E}_\gamma[f(U^{(n)}(t, u))] = \mathbb{E}_\gamma[f(e^{i\Delta \nu} u)] = \mathbb{E}_{\gamma_t}[f(u)].
$$

Passing to the limit on the left-hand side, by Lebesgue’s theorem, we obtain:

$$
\mathbb{E}_\pi[f(U(t, \omega))] = \mathbb{E}_\pi[f(U^{(n)}(t, \omega))] = \mathbb{E}_{\gamma_t}[f] \tag{43}
$$

which proves that $\gamma_t$ is the distribution of $U(t, \omega)$ and $U^{(n)}(t, \omega)$ on $H_\alpha$.

Define $B_{N,M}(u) = B^{(N)}(u) I_{\{\|B^{(N)}\|_\alpha < M\}}(u)$, where the latter multiplier is the indicator function. Let us prove that $\mathbb{E}_{\gamma_t} \|B_{N,M} - B\|_\alpha \to 0$ as $M, N \to \infty$. Indeed,

$$
\mathbb{E}_{\gamma_t} \|B^{(N)} - B\|_\alpha = E_{\gamma_t} \|B^{(N)}(e^{i\Delta \nu} u) - B(e^{i\Delta \nu} u)\|_\alpha \to 0, \quad N \to \infty.
$$

Next, we have:

$$
\mathbb{E}_{\gamma_t} \|B_{N,M} - B\|_\alpha \leq \mathbb{E}_{\gamma_t} \|B_{N,M} - B^{(N)}\|_\alpha + \mathbb{E}_{\gamma_t} \|B^{(N)} - B\|_\alpha.
$$

For the first summand on the right-hand side we obtain:

$$
\mathbb{E}_{\gamma_t} \|B_{N,M} - B^{(N)}\|_\alpha = \mathbb{E}_{\gamma_t} \|B^{(N)}(u) I_{\|B^{(N)}\|_\alpha \geq M}\|_\alpha \leq \left( \mathbb{E}_{\gamma_t} \|B^{(N)}\|^2_\alpha \right)^\frac{1}{2} \gamma_t(\|B^{(N)}\|_\alpha \geq M)^\frac{1}{2} \leq \frac{1}{M} \mathbb{E}_{\gamma_t} \|B^{(N)}\|^2_\alpha. \tag{44}
$$

By the results of Section 3.2, $\mathbb{E}_{\gamma_t} \|B^{(N)}\|^2_\alpha$ is bounded uniformly in $N$ and $t \in [0, T]$ by the sum of the series

$$
4(d - 1)^2 \sum_{h \in \mathbb{Z}_d^+} \frac{1}{|h|^{2\alpha - 2}} \sum_{k \in \mathbb{Z}_d^+} \left( \frac{|k|^{2\alpha}}{|k + h|^{2\alpha}} + \frac{|k|^{2\alpha}}{|k - h|^{2\alpha}} \right),
$$

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and, therefore, the right-hand side of (44) converges to zero as \( M \to \infty \) uniformly in \( N \). Clearly, \( B_{N,M}(u) \) is a bounded and continuous function. It is continuous since \( B^{(N)}(u) \) is a polynomial that depends only on a finite number of coordinates \((u_k, v_k)\). Hence, for each fixed pair of integers \( N, M > 0 \), by Lebesgue’s theorem, we obtain:

\[
\mathbb{E}_{\mathbb{P}} \| B_{N,M}(U'(t, \omega)) - B_{N,M}(U(t, \omega)) \|_\alpha \to 0, \quad n \to \infty.
\]

On the other hand, uniformly in \( n \),

\[
\mathbb{E}_{\mathbb{P}} \| B_{N,M}(U(t, \omega)) - B(U(t, \omega)) \|_\alpha = \mathbb{E}_{\mathbb{P}} \| B_{N,M}(U'(t, \omega)) - B(U'(t, \omega)) \|_\alpha = \mathbb{E}_{\mathbb{P}} \| B_{N,M}(u) - B(u) \|_\alpha \to 0, \quad M, N \to \infty.
\]

This proves that the second summand in (42), and, therefore, the left-hand side of (42), tends to zero as \( n \to \infty \). Choosing a \( \mathbb{P} \)-a.s. converging subsequence, we can pass to the limit as \( n \to \infty \) in (41). The convergence of the both sides holds \( \mathbb{P} \)-a.s. in the space \( C([0, T], H^\alpha) \). Therefore, we conclude that (34) holds for all \( t \in [0, T], \mathbb{P} \)-a.s.. Equality (35) is implied by (43). Furthermore, since we already proved that the distribution of \( U(t, \omega) \) on \( H^\alpha \) is the measure \( \gamma_t \) (see (43)), it implies, in particular, that the distribution of the initial value \( U(0, \omega) \) is the measure \( \gamma \).

\[ \square \]

**Remark 1.** By the Skorohod representation theorem on a given probability space [4], \((\Omega, \mathcal{F}, \mathbb{P})\) is always a space of our choice. In particular, we can choose it to be \((H^\alpha, \mathcal{B}, \gamma)\).

**Remark 2.** On the probability space \((H^\alpha, \mathcal{B}, \gamma)\), we can define the stochastic process \( X(t, u) = e^{\nu \Delta} u \). Then, \( U(t, \omega) \) and \( X(t, u) \) have the same law on \( C([0, T], H^\alpha) \) because their finite-dimensional distributions coincide.

**Corollary 2.** Let \( l > \frac{d}{2} + \alpha + 2 \). Then, there exists a subset \( \Omega'' \subset \Omega \) of full \( \mathbb{P} \)-measure so that the stochastic process \( U(t, \omega) \) constructed in Theorem 1 on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) verifies the equation

\[
U(t, \omega) = U(0, \omega) + \int_0^t (\nu \Delta U(s, \omega) - B(U(s, \omega))) \, ds \quad (45)
\]

for all \( t \in [0, T] \) and for all \( \omega \in \Omega'' \).
Proof. We show that if \( l > \frac{d}{2} + \alpha + 2 \), the stochastic process solving (34) verifies (45). Indeed, equation (34) implies that for \( \mathbb{P} \)-almost all \( \omega \), for all \( t \in [0, T] \), and for each \( k \in \mathbb{Z}^d_+ \) it holds that

\[
U_k(t, \omega) = U_k(0, \omega) - \int_0^t \left( B_k^\cos(U(s, \omega)) + \nu |k|^2 U_k(s, \omega) \right) ds.
\]

It can be obtained by writing (34) in the coordinate form, and then, by differentiating in \( t \) the both parts of each coordinate equation. To prove that (45) holds in \( C([0, T], H^\alpha) \), it suffices to show that for \( \mathbb{P} \)-almost all \( \omega \), \( \int_0^T \Delta U(s, \omega) \) takes values in \( C([0, T], H^\alpha) \). Indeed,

\[
\left( \mathbb{E}_\mathbb{P} \sup_{t \in [0, T]} \left\| \int_0^t \Delta U(s, \omega) ds \right\|_{\alpha} \right)^2 \leq \int_0^T ds \mathbb{E}_\mathbb{P} \| \Delta U(s, \omega) \|_\alpha^2
\]

\[
= \int_0^T ds \sum_{k \in \mathbb{Z}^d_+} |k|^{2\alpha+4} \mathbb{E}_\mathbb{P} \left( |U_k(s, \omega)|^2 + |V_k(s, \omega)|^2 \right)
\]

\[
\leq \int_0^T ds \sum_{k \in \mathbb{Z}^d_+} |k|^{2\alpha+4} e^{-s|k|^2} \gamma \left( |u_k|^2 + |v_k|^2 \right) \leq T \sum_{k \in \mathbb{Z}^d_+} \frac{1}{|k|^{2l-2\alpha-4}}.
\]

The series on the right-hand side converges since \( l > \frac{d}{2} + \alpha + 2 \). This proves that \( \sup_{t \in [0, T]} \| \int_0^t \Delta U(s, \omega) ds \|_{\alpha} \) is finite \( \mathbb{P} \)-a.s. \( \square \)

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Solution to the Navier–Stokes equations with random initial data

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Abstract

We construct a solution to the spatially-periodic $d$-dimensional Navier–Stokes equations with a given distribution of the initial data. The solution takes values in the Sobolev space $H^\alpha$, where the index $\alpha \in \mathbb{R}$ is fixed arbitrary. The distribution of the initial value is a Gaussian measure on $H^\alpha$ whose parameters depend on $\alpha$. The Navier–Stokes solution is then a stochastic process verifying the Navier–Stokes equations almost surely. It is obtained as a limit in distribution of solutions to finite-dimensional ODEs which are Galerkin-type approximations for the Navier–Stokes equations. Moreover, the constructed Navier–Stokes solution $U(t, \omega)$ possesses the property:

$$
\mathbb{E}\left[f(U(t, \omega))\right] = \int_{H^\alpha} f(e^{t\nu \Delta} u) \gamma(du),
$$

where $f \in L_1(\gamma)$, $e^{t\Delta}$ is the heat semigroup, $\nu$ is the viscosity in the Navier–Stokes equations, and $\gamma$ is the distribution of the initial data.

1. Introduction

We continue developing infinite dimensional and stochastic analysis approaches to deeper understand the Navier–Stokes equations. Different methods of infinite dimensional and stochastic analysis in connection to the Navier–Stokes problem were considered in [1, 2, 3, 9, 10, 12]. Here we follow an approach which is somewhat similar to [1], but adapted to treat the Navier–Stokes equations in $d$ dimensions.

The Navier–Stokes equations are a set of partial differential equations that describe the flow of incompressible fluids. They model a lot of gas and fluid flow phenomena such as motion of air in the atmosphere, currents in oceans as well as they are the basis for weather forecasts.
In the current work we search for an $H^\alpha$-valued Navier–Stokes solution in the form of the Fourier series with respect to a complete orthonormal divergence-free system of vector fields on the $d$-dimensional torus. In Section 2, the original Navier–Stokes system is reduced to an ordinary differential equation in $H^\alpha$ where the latter is written with respect to the Fourier series coefficients. In Section 3 we study the Galerkin-type approximations for this equation. Applying results of [7], we show that the solution to the finite-dimensional Galerkin-type equation has no blow-up time. Also, in Section 3 we introduce conditions that imply the convergence of the series representing the non-linear term in the Navier–Stokes equations. Namely, we introduce a Gaussian measure $\gamma$ on $H^\alpha$ whose variance depends on a positive integer bigger than $\alpha + \frac{d}{2} + 1$. The latter condition, in particular, implies the convergence of the above mentioned series in the space $L_2(H^\alpha, \gamma)$. Since the solution to the Galerkin-type equation, known as Galerkin-type approximation, depends on the initial condition $u \in H^\alpha$ and the time $t \in [0, T]$, we consider $u$ as a random variable on the probability space $(H^\alpha, \mathcal{B}, \gamma)$ and study the distribution of this solution on $C([0, T], H^\alpha)$. As it is shown in Section 4, the tightness criteria for probability measures implies the convergence in distribution of the Galerkin-type approximations. The Navier–Stokes solution is then obtained as the limit of Galerkin-type approximations, and its existence, as of a stochastic process, follows from the Skorokhod theorem. In [1], the authors prove the existence of the solution to the spatially-periodic 2-dimensional Euler equation with random initial data. The result of [1] follows from our result as a particular case ($d = 2, \nu = 0$). Moreover, our Sobolev space index $\alpha$ is an arbitrary real number whereas the result of [1] was proved for $\alpha < -\frac{1}{2}$.

Let us describe our method in more detail. At first, we search for a mild solution to the Navier–Stokes equations, and then, under somewhat stronger assumptions, we derive the existence of a strong solution. As basis functions for the Galerkin-type method, we use the basis of divergence-free vector fields on the torus $\mathbb{T}^d$ constructed in [9]. The Galerkin-type equations are then modified by means of the change of variable which multiplies each $k$-th component by $e^{\nu|k|^2}$, $k \in \mathbb{Z}_d$. The latter change of variable is used to obtain the existence of invariant measures for the Galerkin-type approximations which is an important tool in the proof of the main result. Unlike [1], the Galerkin-type equations have time-dependent right-hand sides which does not allow us to solve them by methods of [7] directly. The given distribution of the initial value is a product of finite-dimensional Gaussian measures $\gamma_k$, where
\( k \in \mathbb{Z}_d \), with variances \( \frac{1}{|k|^2} \), where the parameter \( l \) is a positive integer, and the pair \((\alpha, l)\) should be fixed so that \( l > \alpha + \frac{d}{2} + 1 \).

2. Representations of the Navier–Stokes equations

Consider the classical spatially-periodic \( d \)-dimensional Navier-Stokes equations:

\[
\frac{\partial y}{\partial t}(t, \theta) = -(y, \nabla) y(t, \theta) + \nu \Delta y(t, \theta) - \nabla p(t, \theta),
\]
\[
\text{div} y(t, \theta) = 0,
\]
\[
y(0, \theta) = y_0(\theta),
\]

where \( \theta \) belongs to the \( d \)-dimensional torus \( \mathbb{T}^d \), and \( t \in [0, T] \). Below we use the notation introduced in [9]:

\[
\mathbb{Z}^+_d = \{ (k_1, k_2, \ldots, k_d) \in \mathbb{Z}_d : k_1 > 0 \text{ or } k_1 = \cdots = k_{i-1} = 0, k_i > 0, \ i = 2, \ldots, d \};
\]

if \( k = (k_1, \ldots, k_d) \in \mathbb{Z}^+_d \), and \( \theta = (\theta_1, \ldots, \theta_d) \in \mathbb{T}^d \),

then \( |k| = \sqrt{\sum_{i=1}^{d} k_i^2} \), \( k \cdot \theta = \sum_{i=1}^{n} k_i \theta_i \), \( \Delta_{\mathbb{T}^d} = \sum_{i=1}^{d} \frac{\partial^2}{\partial \theta_i^2} \).

For every \( k \in \mathbb{Z}^+_d \), \((\bar{k}^1, \ldots, \bar{k}^{d-1})\) denotes an orthogonal system of vectors of length \(|k|\) which is also orthogonal to \( k \). According to [9], any divergence-free vector field on \( \mathbb{T}^d \) has the following Fourier series representation:

\[
\sum_{p=1}^{d} u_0^p e^p + \sum_{k \in \mathbb{Z}^+_d} \sum_{p=1}^{d-1} \left[ u_k^p A_k^p + v_k^p C_k^p \right]
\]

where

\[
A_k^p = \frac{\sqrt{2}}{(2\pi)^{d/2}} \cos(k \cdot \theta) \frac{\bar{k}^p}{|k|}, \quad C_k^p = \frac{\sqrt{2}}{(2\pi)^{d/2}} \sin(k \cdot \theta) \frac{\bar{k}^p}{|k|},
\]

\[
p = 1, \ldots, d - 1, \quad k \in \mathbb{Z}^+_d,
\]

and the constant vector fields \( e^p, p = 1, \ldots, d \), are such that the \( p \)-th coordinate is \( \frac{1}{(2\pi)^{d/2}} \) and the other coordinates are 0. The system (3) together
with the constant vectors \( \{e^p\}_{p=1}^d \) is orthonormal in \( L_2(\mathbb{T}^d) \). The periodic divergence-free Sobolev space \( \dot{H}^\alpha(\mathbb{T}^d), \alpha \in \mathbb{R} \), is defined as the totality of vector fields of form (2) with

\[
\sum_{k \in \mathbb{Z}_d^+} \sum_{p=1}^{d-1} |k|^{2\alpha} (|u^p_k|^2 + |v^p_k|^2) < \infty.
\]

We will search the Navier–Stokes solution in the form:

\[
y(t, \theta) = \sum_{p=1}^d u^p_0(t) e^p + \sum_{k \in \mathbb{Z}_d^+} \sum_{p=1}^{d-1} u^p_k(t) A^p_k + v^p_k(t) C^p_k.
\]

Define \( u(t) = \{u^p_k(t), v^p_k(t), u_0^q(t)\}, k \in \mathbb{Z}_d^+, p = 1, \ldots, d-1, q = 1, \ldots, d \), and consider the following representation of the Navier–Stokes equations:

\[
u(t) = u(0) - \int_0^t B(u(s)) \, ds + \nu \int_0^t \Delta u(s) \, ds,
\]

where \( B(u(s)) = \{B^p_k, \cos, B^p_k, \sin\}, p = 1, \ldots, d-1, k \in \mathbb{Z}_d^+ \), \( B^p_k, \cos \) and \( B^p_k, \sin \) are the coordinates of the expansion of \( \mathcal{P}[\langle y, \nabla \rangle y](s, \cdot) \) with respect to the basis \( A^p_k, C^p_k \), and \( \mathcal{P} \) is the projector onto the divergence-free vector fields, i.e. onto the space spanned by vectors (3) and \( \{e^p\}_{p=1}^d \). Note that the vector \( B \) does not have non-zero components along the constant vectors \( e^p, p = 1, \ldots, d \). By \( \Delta u(s) \) we understand the vector with the coordinates

\[\{-|k|^2 u^p_k(s), -|k|^2 v^p_k(s), k \in \mathbb{Z}_d^+, p = 1, \ldots, d-1, u_0^q(s) = 0, q = 1, \ldots, d\}.
\]

Navier–Stokes equations (5) can be represented as:

\[
u(t)
=
u(0)

-\int_0^t B^p_k, \cos(u(s)) \, ds - \nu|k|^2 \int_0^t u^p_k(s) \, ds,
\]

\[
nu(t)
=
u(0)

-\int_0^t B^p_k, \sin(u(s)) \, ds - \nu|k|^2 \int_0^t v^p_k(s) \, ds,
\]

where \( p = 1, \ldots, d-1, k \in \mathbb{Z}_d^+ \). Define

\[
u_k(t)
=\sum_{p=1}^{d-1} u^p_k(t) \frac{k^p}{|k|},
\]

\[
nv_k(t)
=\sum_{p=1}^{d-1} v^p_k(t) \frac{k^p}{|k|},
\]

\[
u_0(t)
=\sum_{p=1}^d u^p_0(t) e^p.
\]
With the above definition, we can search the Navier–Stokes solution \( y(t, \theta) \) in the form:

\[
y(t, \theta) = u_0(0) + \sum_{k \in \mathbb{Z}_d^+} \left[ u_k(t) \cos(k \cdot \theta) + v_k(t) \sin(k \cdot \theta) \right].
\]

Analogously, we define

\[
B^\text{cos}_k = \sum_{p=1}^{d-1} B^p \cos \frac{\bar{k}^p}{|k|}, \quad B^\text{sin}_k = \sum_{p=1}^{d-1} B^p \sin \frac{\bar{k}^p}{|k|}.
\]

Equations (6) take the form:

\[
\begin{align*}
    u_0(t) &= u_0(0), \\
    u_k(t) &= u_k(0) - \int_0^t B^\text{cos}_k(u(s)) \, ds - \nu |k|^2 \int_0^t u_k(s) \, ds, \\
    v_k(t) &= v_k(0) - \int_0^t B^\text{sin}_k(u(s)) \, ds - \nu |k|^2 \int_0^t v_k(s) \, ds.
\end{align*}
\]

We exclude the first equation in (7) and rewrite (7) in the equivalent form:

\[
\begin{align*}
    u_k(t) &= e^{-t|\nu|} u_k(0) - \int_0^t e^{-(t-s)|\nu|} B^\text{cos}_k(u(s)) \, ds \\
    v_k(t) &= e^{-t|\nu|} v_k(0) - \int_0^t e^{-(t-s)|\nu|} B^\text{sin}_k(u(s)) \, ds.
\end{align*}
\]
3. Galerkin-type approximations

3.1 Change of variable in the Navier–Stokes equations

Consider the Navier–Stokes equations in form (8). The direct computation of \( B_k^{\text{sin}}(u(s)) \) and \( B_k^{\text{cos}}(u(s)) \) gives the formal expression

\[
B_k^{\text{sin}}(u(s)) = \frac{1}{\sqrt{2}(2\pi)^{d/2}} \left( \sum_{k_1+k_2=k} \sum_{i,j=1}^{d-1} (v_i^{k_1}(s)v_j^{k_2}(s) - u_i^{k_1}(s)u_j^{k_2}(s)) \frac{(\bar{k}_i^i, k_j^j)}{2|k_1||k_2|} P_{k} \bar{k}_j^j \right)
\]

\[
+ \sum_{k_1-k_2=k} \sum_{i,j=1}^{d-1} (u_i^{k_1}(s)u_j^{k_2}(s) + v_i^{k_1}(s)v_j^{k_2}(s)) \frac{(\bar{k}_i^i, k_j^j)}{2|k_1||k_2|} P_{k} \bar{k}_j^j 
\]

\[
+ \sum_{k_2-k_1=k} \sum_{i,j=1}^{d-1} (-u_i^{k_1}(s)u_j^{k_2}(s) - v_i^{k_1}(s)v_j^{k_2}(s)) \frac{(\bar{k}_i^i, k_j^j)}{2|k_1||k_2|} P_{k} \bar{k}_j^j 
\]

\[
- \sum_{i=1}^{d} \sum_{j=1}^{d-1} u_i^i u_j^j(s) (e_i^i, k) \bar{k}_j^j,
\]

where \( P_k \) denotes the orthogonal projection in \( \mathbb{R}^d \) onto its \((d-1)\)-dimensional subspace generated by the vectors \( \bar{k}_p^p, p = 1, \ldots, d-1 \). Now let us combine the sums over \( k_1 \) and \( k_2 \) into one. For each pair \( k, h \in \mathbb{Z}_d^+ \), we define the functions:

\[
\pm(k-h) = \begin{cases} 
  k-h, & \text{if } k-h \in \mathbb{Z}_d^+, \\
  h-k, & \text{if } h-k \in \mathbb{Z}_d^+
\end{cases}
\]

and

\[
\text{sign}(k-h) = \begin{cases} 
  1, & \text{if } k-h \in \mathbb{Z}_d^+, \\
  -1, & \text{if } h-k \in \mathbb{Z}_d^+
\end{cases}
\]

The both functions are undefined if \( k = h \). Also, we introduce

\[
\lambda^+_i(k,h) = \frac{(k+h, \bar{k}_i^i)}{\sqrt{2}(2\pi)^{d/2} |h| |k+h|};
\]

\[
\lambda^-_i(k,h) = \frac{(\pm(k-h), \bar{k}_i^i)}{\sqrt{2}(2\pi)^{d/2} |h| |k-h|}, \quad \text{if } k \neq h, \text{ and } \lambda^-_i(h,h) = 0;
\]

\[
\lambda_i(k) = \frac{1}{(2\pi)^{d/2}} \frac{(e_i^i, k)}{|k|}.
\]
We obtain:

\[
B_k^\sin(u(s)) = \sum_{h \in \mathbb{Z}_+^d} \sum_{i,j=1}^{d-1} \lambda_i^+(k, h) \left( u_{k+h}(s) u_{i}^j(s) + v_{k+h}(s) v_{i}^j(s) \right) P_k \bar{h}^j \\
+ \lambda_i^-(k, h) \left( \text{sign}(k - h) v_{\pm(k-h)}(s) v_{i}^j(s) - u_{\pm(k-h)}(s) u_{i}^j(s) \right) P_k \bar{h}^j \\
- \sum_{i=1}^{d} \lambda_i(k) u_0^i \sum_{j=1}^{d-1} u_{k}^j(s) \bar{k}^j.
\]

Analogously, we obtain the formal series for \( B_k^\cos(u(s)) \)

\[
B_k^\cos(u(s)) = \frac{1}{\sqrt{2(2\pi)^2}} \left( \sum_{k_1+k_2=k, i,j=1}^{d-1} \sum_{k_1 \neq k_2} (u_{k_1}(s) u_{k_2}(s) + u_{k_1}(s) u_{k_2}(s)) \frac{(\bar{k}_1^j, \bar{k}_2^j)}{2 |k_1| |k_2|} P_k \bar{k}^j \\
+ \sum_{k_1 \neq k_2} (u_{k_1}(s) v_{k_2}(s) - v_{k_1}(s) u_{k_2}(s)) \frac{(\bar{k}_1^j, \bar{k}_2^j)}{2 |k_1| |k_2|} P_k \bar{k}^j \\
+ \sum_{i=1}^{d} \sum_{j=1}^{d-1} u_{k}^j(s) (e^i, k) \bar{k}^j \right).
\]

Combining the first two sums into one gives:

\[
B_k^\cos(u(s)) = \sum_{h \in \mathbb{Z}_+^d} \sum_{i,j=1}^{d-1} \lambda_i^+(k, h) \left( u_{k+h}(s) v_{i}^j(s) - v_{k+h}(s) u_{i}^j(s) \right) P_k \bar{h}^j \\
+ \lambda_i^-(k, h) \left( u_{\pm(k-h)}(s) v_{i}^j(s) + \text{sign}(k - h) u_{\pm(k-h)}(s) u_{i}^j(s) \right) P_k \bar{h}^j \\
+ \sum_{i=1}^{d} \lambda_i(k) u_0^i \sum_{j=1}^{d-1} v_{k}^j(s) \bar{k}^j.
\]

We dropped the time dependence in \( u_0^i \) since we proved that it does not depend on time. Multiplying equations (8) by \( e^{t|k|^2} \) and introducing the new variables

\[
\tilde{u}_k(t) = e^{t|k|^2} u_k(t), \quad \tilde{v}_k(t) = e^{t|k|^2} v_k(t)
\]
as well as their coordinates $\tilde{u}_k^p(t), \tilde{v}_k^p(t)$ with respect to the bases $\{k^p_k\}$, $p = 1, \ldots, d - 1$, we rewrite equations (8):

\[
\begin{align*}
\tilde{u}_k(t) &= \tilde{u}_k(0) - \int_0^t \tilde{B}_k^\cos(s, \tilde{u}(s)) \, ds, \\
\tilde{v}_k(t) &= \tilde{v}_k(0) - \int_0^t \tilde{B}_k^\sin(s, \tilde{u}(s)) \, ds,
\end{align*}
\]

where $\tilde{u}(s)$ is the vector with the coordinates $\tilde{u}_k^p(t), \tilde{v}_k^p(t)$, $\tilde{B}_k^\sin(s, \tilde{u}(s)) = e^{s\nu|k|^2} B_k^\sin(u(s))$, and $\tilde{B}_k^\cos(s, \tilde{u}(s)) = e^{s\nu|k|^2} B_k^\cos(u(s))$. Explicitly,

\[
\begin{align*}
\tilde{B}_k^\sin(s, \tilde{u}(s)) &= \sum_{h \in \mathbb{Z}_d^+} \sum_{i,j=1}^{d-1} \lambda_i^+(k, h) e^{-2(k+h,h)s\nu} \left( \tilde{u}_{k+h}^i(s) \tilde{u}_h^j(s) + \tilde{v}_{k+h}^i(s) \tilde{v}_h^j(s) \right) P_{k^i h^j} \\
&+ \lambda_i^-(k, h) e^{2(k-h,h)s\nu} \left( \text{sign}(k - h) \tilde{u}_{k-h}^i(s) \tilde{v}_h^j(s) - \tilde{u}_{k-h}^i(s) \tilde{u}_h^j(s) \right) P_{k^i h^j} \\
&- \sum_{i=1}^d \lambda_i(k) \sum_{j=1}^{d-1} u_0^i \tilde{u}_h^j(s) \tilde{k}_h^j.
\end{align*}
\]

\[
\begin{align*}
\tilde{B}_k^\cos(s, \tilde{u}(s)) &= \sum_{h \in \mathbb{Z}_d^+} \sum_{i,j=1}^{d-1} \lambda_i^+(k, h) e^{-2(k+h,h)s\nu} \left( \tilde{u}_{k+h}^i(s) \tilde{v}_h^j(s) - \tilde{v}_{k+h}^i(s) \tilde{u}_h^j(s) \right) P_{k^i h^j} \\
&+ \lambda_i^-(k, h) e^{2(k-h,h)s\nu} \left( \tilde{u}_{k-h}^i(s) \tilde{v}_h^j(s) + \text{sign}(k - h) \tilde{v}_{k-h}^i(s) \tilde{u}_h^j(s) \right) P_{k^i h^j} \\
&+ \sum_{i=1}^d \lambda_i(k) \sum_{j=1}^{d-1} u_0^i \tilde{v}_h^j(s) \tilde{k}_h^j.
\end{align*}
\]

We mention here once again that the infinite series representing $B_k^\sin(u(s))$ and $B_k^\cos(u(s))$, as well as $\tilde{B}_k^\sin(s, \tilde{u}(s))$ and $\tilde{B}_k^\cos(s, \tilde{u}(s))$ are so far just formal expressions. Below we introduce spaces where the series for $B_k^\sin(u(s))$ and $B_k^\cos(u(s))$ converge. On the other hand, we will only deal with $\tilde{B}_k^{\sin(s, \tilde{u}(s))}$ and $\tilde{B}_k^{\cos(s, \tilde{u}(s))}$ which are obtained from $B_k^\sin(s, \tilde{u}(s))$ and $B_k^\cos(s, \tilde{u}(s))$ by discarding the terms with $|h| > n$. 

8
3.2 Spaces of convergence

Here we introduce a family of Sobolev-type spaces with a Gaussian measure on each of them so that the infinite series for $B^\sin_k$ and $B^\cos_k$, i.e.

$$
\sum_{h \in \mathbb{Z}_d^+} \sum_{i,j=1}^{d-1} \lambda^+_i(k, h) \left( u^i_{k+h} u^j_h + v^i_{k+h} v^j_h \right) P^i_h
$$

$$
+ \lambda^-_i(k, h) \left( \text{sign}(k - h) v^i_{\pm(k-h)} v^j_h - u^i_{\pm(k-h)} u^j_h \right) P^i_h, \quad (10)
$$

and, respectively,

$$
\sum_{h \in \mathbb{Z}_d^+} \sum_{i,j=1}^{d-1} \lambda^+_i(k, h) \left( u^i_{k+h} v^j_h - v^i_{k+h} u^j_h \right) P^i_h
$$

$$
+ \lambda^-_i(k, h) \left( u^i_{\pm(k-h)} v^j_h + \text{sign}(k - h) v^i_{\pm(k-h)} u^j_h \right) P^i_h, \quad (11)
$$

converge. For any $r \in \mathbb{R}$, we define a Hilbert space

$$
H^r = \left\{ u : \sum_{k \in \mathbb{Z}_d^+} |k|^{2r} (|u_k|^2 + |v_k|^2) < \infty \right\}.
$$

For each $k \in \mathbb{Z}_d^+$, on the space $\mathbb{R}^{2(d-1)}$ of the variables $(u_k, v_k)$ we define the Gaussian measure $\gamma_k$ by its density with respect to the Lebesgue measure on $\mathbb{R}^{2(d-1)}$:

$$
\left( \frac{|k|^{2l}}{2\pi} \right)^{(d-1)} \exp \left( - \frac{|k|^{2l}}{2} (|u_k|^2 + |v_k|^2) \right).
$$

The number $l$ is a positive integer which is sufficiently large to satisfy certain inequalities which will be obtained below to ensure the convergence of series (10) and (11). Define the measure $\gamma$ on $\bigotimes_{k \in \mathbb{Z}_d^+} \mathbb{R}^{2(d-1)}$ by

$$
\gamma(du) = \bigotimes_{k \in \mathbb{Z}_d^+} \gamma_k \left( d(u_k, v_k) \right).
$$

**Proposition 1.** For any $\alpha \in \mathbb{R}$, for any integer $l > \frac{d}{2} + \alpha$, $(H^\alpha, H^1, \gamma)$ is an abstract Wiener space.
Proof. The measure $\gamma$ is supported on $H^\alpha$. Indeed,

$$
\int \left\{ \times \right\}_{k \in \mathbb{Z}_d^+} \mathbb{R}^{2(d-1)} \| u \|_\alpha^2 \gamma(du) = \sum_{k \in \mathbb{Z}_d^+} |k|^{2\alpha} \int_{\mathbb{R}^{2(d-1)}} (|u_k|^2 + |v_k|^2) \gamma_k(d(u_k, v_k))
$$

$$= 2(d-1) \sum_{k \in \mathbb{Z}_d^+} \frac{1}{|k|^{2(l-\alpha)}}.$$

The latter series converges, for example, by the integral test of convergence. It is easy to verify that the Fourier transform of the measure $\gamma$ is

$$u \mapsto \exp \left( \frac{1}{2} \sum_{k \in \mathbb{Z}_d^+} |k|^{4\alpha - 2l} (|u_k|^2 + |v_k|^2) \right).$$

(12)

Defining the operator $K$ as

$$KA^i_k = |k|^{2\alpha - 2l} A^i_k, \quad KC^i_k = |k|^{2\alpha - 2l} C^i_k,$$

we observe that the sum in (12) equals to $(K u, u)_\alpha$, where $( \cdot, \cdot )_\alpha$ is the scalar product in $H^\alpha$. This means that $K$ is the covariance operator for the measure $\gamma$. Take a $u \in H^\alpha$. Then,

$$\| \sqrt{K} u \|_\alpha^2 = \sum_{k \in \mathbb{Z}_d^+} |k|^{2\alpha - 2l} (|u_k|^2 + |v_k|^2) = \| u \|_\alpha^2.$$

This proves that $\sqrt{K} H^\alpha = H^l$, and, therefore, $H^l$ is the Cameron-Martin space of the measure $\gamma$. 

Lemma 1. Let $l > \alpha + \frac{d}{2} + 1$. Then, $B^\text{sin}_k$ and $B^\text{cos}_k$ are well defined as elements of $L_2(\gamma, \mathbb{R}^{d-1})$, i.e. series (10) and (11) converge in $L_2(\gamma, \mathbb{R}^{d-1})$. Moreover, $B \in L_2(\gamma, H^\alpha)$.

Proof. Note that each member of series (10) and (11) indexed by $h$ is a finite sum in $i, j$ whose summands are, in turn, sums of four other terms. Clearly, we can exchange the order of summation in $h \in \mathbb{Z}_d^+$ and $i, j$, and, moreover, apply the summation in $h$ to each of the subsequent four terms. Let us investigate the convergence of the series

$$\sum_{h \in \mathbb{Z}_d^+} \sum_{k \in \mathbb{Z}_d^+} \lambda^+_i(k, h) u^i_{k+h} u^j_h P_k \bar{h}^j$$

(13)
in $L_2(\gamma, \mathbb{R}^{d-1})$. We have:

$$
\mathbb{E}_\gamma \left| \sum_{|h|<N} \lambda_i^+(k, h) u_{k+h}^i u_{h}^j P_k \bar{h}^j \right|^2_{\mathbb{R}^{d-1}}
= \sum_{|h|<N, |h'|<N} \lambda_i^+(k, h) \lambda_i^+(k, h') \mathbb{E}_\gamma (u_{k+h}^i u_{k+h'}^i u_{h}^j u_{h'}^j) (P_k \bar{h}^j, P_k \bar{h'}^j)_{\mathbb{R}^{d-1}}
= \sum_{|h|<N} \lambda_i^+(k, h) \sum_{|h|<N} \frac{1}{|k+h|^{2l}} \frac{1}{|h|^{2l-2}} \left| P_k \bar{h}^j \right|^2_{\mathbb{R}^{d-1}} \leq \sum_{h \in \mathbb{Z}_d^+} \frac{1}{|k+h|^{2l}} \frac{1}{|h|^{2l-2}}. \tag{14}
$$

In the sequel we will show the convergence of the series

$$
\sum_{k \in \mathbb{Z}_d^+} |k|^{2\alpha} \sum_{h \in \mathbb{Z}_d^+} \frac{1}{|k+h|^{2l}} \frac{1}{|h|^{2l-2}} = \sum_{h \in \mathbb{Z}_d^+} \frac{1}{|h|^{2l-2}} \sum_{k \in \mathbb{Z}_d^+} \frac{|k|^{2\alpha}}{|k+h|^{2l}} \tag{15}
$$

which, in turn, will imply the convergence of

$$
\sum_{k \in \mathbb{Z}_d^+} |k|^{2\alpha} (|B_k^{\sin}|^2 + |B_k^{\cos}|^2) \tag{16}
$$

and, therefore, the statement of the lemma. For the moment we assume that the series on the right hand side of (14) converges, and show that the sequence of partial sums for (13) is a Cauchy sequence. Indeed, for any integers $M < N$,

$$
\mathbb{E}_\gamma \left| \sum_{M<|h|<N} \lambda_i^+(k, h) u_{k+h}^i u_{h}^j P_k \bar{h}^j \right|^2_{\mathbb{R}^{d-1}} \leq \sum_{M<|h|<N} \frac{1}{|k+h|^{2l}} \frac{1}{|h|^{2l-2}} \to 0,
\text{ as } M, N \to \infty.
$$

This proves that series (13) converges in $L_2(\gamma, \mathbb{R}^{d-1})$. Analogously, the other
Therefore, series \((8(d - 1)^2 - 1)\) series

\[
\sum_{h \in \mathbb{Z}_d^+} \lambda_i^+(k, h) v_{k+h}^i P^j, \quad \sum_{h \in \mathbb{Z}_d^+} \lambda_i^-(k, h) \text{sign}(k-h) v_{k-h}^i P^j,
\]

\[
\sum_{h \in \mathbb{Z}_d^+} \lambda_i^+(k, h) u_{k-h}^i P^j, \quad \sum_{h \in \mathbb{Z}_d^+} \lambda_i^-(k, h) u_{k+h}^i P^j.
\]

\[
\sum_{h \in \mathbb{Z}_d^+} \lambda_i^+(k, h) v_{k+h}^i P^j, \quad \sum_{h \in \mathbb{Z}_d^+} \lambda_i^-(k, h) u_{k+(h-1)}^i P^j.
\]

\[
\sum_{h \in \mathbb{Z}_d^+} \lambda_i^-(k, h) \text{sign}(k-h) u_{k-h}^i P^j.
\]

\(i, j = 1, \ldots, d\), which are summands in \((10)\) and \((11)\), converge in \(L_2(\gamma, \mathbb{R}^{d-1})\). Therefore, series \((10)\) and \((11)\) converge in \(L_2(\gamma, \mathbb{R}^{d-1})\) as sums of a finite number of converging series. Hence, \(B_k^\text{in} \) and \(B_k^\text{cos} \), \(k \in \mathbb{Z}_d^+\), are well defined as elements of \(L_2(\gamma, \mathbb{R}^{d-1})\). Let us prove that series \((16)\) converges and, therefore, \(B \in L_2(\gamma, H^\alpha)\). First we prove that the series \(\sum_{k \in \mathbb{Z}_d^+} \frac{|k|^{2\alpha}}{|k+h|^{2\alpha}}\) converges for each fixed \(h \in \mathbb{Z}_d^+\) and estimate its sum. We have:

\[
\sum_{k \in \mathbb{Z}_d^+} \frac{|k|^{2\alpha}}{|k+h|^{2\alpha}} \leq d \sum_{k \in \mathbb{Z}_d^+} \frac{(\sum_{i=1}^d |k_i|)^{2\alpha}}{(\sum_{i=1}^d |k_i - |h_i||)^{2\alpha} \mathbb{I}\{k_i \neq |h_i|\}}
\]

\[
\leq d 2^{d-1} \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} \frac{(\sum_{i=1}^d k_i)^{2\alpha}}{(\sum_{i=1}^d |k_i - |h_i||)^{2\alpha} \mathbb{I}\{k_i \neq |h_i|\}}. \quad (17)
\]

Note that multiplying by the indicator \(\mathbb{I}\{k_i \neq |h_i|\}\) implies that the denominators of the fraction on the right-hand side are always bigger than 1. Next, we split each sum over \(k_i\) going from 0 to \(\infty\) into two: from 0 to \(|h_i| - 1\) and from \(|h_i|\) to \(+\infty\), i.e. for the series on the right-hand side we obtain:

\[
\left[ \sum_{k_1=1}^{[h_1]-1} + \sum_{k_1=|h_1|}^{+\infty} \right] \left[ \sum_{k_2=0}^{[h_2]-1} + \sum_{k_2=|h_2|}^{+\infty} \right] \cdots \left[ \sum_{k_d=0}^{[h_d]-1} + \sum_{k_d=|h_d|}^{+\infty} \right] \frac{(\sum_{i=1}^d k_i)^{2\alpha}}{(\sum_{i=1}^d |k_i - |h_i||)^{2\alpha} \mathbb{I}\{k_i \neq |h_i|\}}
\]

which equals to a finite sum of series of the form

\[
\sum_{k_1=|h_1|}^{+\infty} \cdots \sum_{k_m=|h_m|}^{+\infty} \sum_{k_{m+1}=0}^{[h_{m+1}]-1} \sum_{k_{m+2}=0}^{+\infty} \cdots \sum_{k_d=0}^{[h_d]-1} \frac{(\sum_{i=1}^d k_i)^{2\alpha}}{(\sum_{i=1}^d |k_i - |h_i||)^{2\alpha} \mathbb{I}\{k_i \neq |h_i|\}}
\]

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where \( \{i_1, \ldots, i_d\} \) is a perturbation of \( \{1, \ldots, d\} \). It suffices to investigate the convergence of the series

\[
\sum_{k_1=|h_1|}^{+\infty} \cdots \sum_{k_l=|h_l|}^{+\infty} \sum_{k_{m+1}=0}^{|h_{m+1}|-1} \cdots \sum_{k_d=0}^{|h_d|-1} \frac{(\sum_{i=1}^d k_i)^{2\alpha}}{(\sum_{i=1}^d |k_i - |h_i||)^{2l} \mathbb{I}\{k_i \neq |h_i|\}}.
\]

Note that we can consider that \( k \neq 0 \) since the term with \( k = 0 \) is always zero. Therefore the numerator and the denominator of all members of the series are bigger than 1. Consider the function:

\[
f(x_1, \ldots, x_d) = \frac{(\sum_{i=1}^d x_i)^{2\alpha}}{(\sum_{i=1}^d |x_i - |h_i||)^{2l}}.
\]

It is an increasing function in each of the variable \( x_{m+1}, \ldots, x_d \) on the intervals \([0, |h_{m+1}|], \ldots, [0, |h_d|]\), respectively, when the rest of the variables is fixed. Therefore,

\[
\sum_{k_{m+1}=0}^{|h_{m+1}|-1} \cdots \sum_{k_d=0}^{|h_d|-1} \frac{(\sum_{i=1}^d k_i)^{2\alpha}}{(\sum_{i=1}^d |k_i - |h_i||)^{2l}} \leq \int_0^{|h_{m+1}|-1} dx_{m+1} \cdots \int_0^{|h_d|-1} dx_d \frac{(\sum_{i=1}^m k_i + \sum_{i=m+1}^d x_i)^{2\alpha}}{(\sum_{i=1}^m |k_i - |h_i|| + \sum_{i=m+1}^d |h_i| - x_i)^{2l}}.
\]

The integral on the right-hand side of (18) can be computed or estimated from above by integration by parts. Performing the integration by parts once, we decrease the powers of the numerator and the denominator of the integrand by 1. When the power of the numerator becomes \( 2\alpha - [2\alpha] \) we estimate the numerator from above by replacing \( 2\alpha - [2\alpha] \) with 1. If \( 2\alpha \) is an integer, the integral can be computed explicitly. Our goal is to show that the sum of the series on the right-hand side of (17) is equivalent to \( |h|^\xi \) for some \( \xi \in \mathbb{R} \). So we discard the terms when it is already clear that they are of orders of \( |h| \) smaller than the maximal. Integration by parts in (18) implies that the higher order term is smaller than

\[
\frac{(\sum_{i=1}^m k_i + \sum_{i=m+1}^d |h_i|)^{2\alpha}}{(\sum_{i=1}^m |k_i - |h_i|| + 1)^{2l-d+2m}}.
\]

13
up to a multiplicative constant. Next, we apply the integral test of convergence to investigate the convergence of the above series in $k_i$ going from $|h_i|$ to $+\infty$. Note that the function

$$ \frac{\left(\sum_{i=1}^{m} x_i + \sum_{i=m+1}^{d} |h_i|\right)^{2\alpha}}{\left(\sum_{i=1}^{m} |x_i - |h_i|| + 1\right)^{2l-d+2m}} $$

is decreasing in each $x_i \in [|h_i|, \infty)$ provided that all other $x_j$'s are bigger than $|h_j|$. Therefore, we can apply the integral test of convergence to each one variable series with summation in $k_m, k_{m-1}, \ldots, k_1$ subsequently. We obtain:

$$ \sum_{k_m=|h_m|}^{\infty} \frac{\left(\sum_{i=1}^{m} k_i + \sum_{i=m+1}^{d} |h_i|\right)^{2\alpha}}{\left(\sum_{i=1}^{m} (k_i - |h_i|) + 1\right)^{2l-d+2m}} \leq \frac{\left(\sum_{i=1}^{m-1} k_i + \sum_{i=m}^{d} |h_i|\right)^{2\alpha}}{\left(\sum_{i=1}^{m-1} (k_i - |h_i|) + 1\right)^{2l-d+2m}} $$

$$ + \int_{|h_m|}^{\infty} dx_m \frac{\left(\sum_{i=1}^{m-1} k_i + x_m + \sum_{i=m+1}^{d} |h_i|\right)^{2\alpha}}{\left(\sum_{i=1}^{m-1} (k_i - |h_i|) + x_m - |h_m| + 1\right)^{2l-d+2m}}. $$

Integration by parts implies that the higher order term is

$$ \frac{\left(\sum_{i=1}^{m-1} k_i + \sum_{i=m}^{d} |h_i|\right)^{2\alpha}}{\left(\sum_{i=1}^{m-1} (k_i - |h_i|) + 1\right)^{2l-d+2m-1}} $$

up to a multiplicative constant. The same argument implies the convergence of each one variable series with summation in $k_{m-1}, \ldots, k_1$, and that the higher order term is

$$ \left(\sum_{i=1}^{d} |h_i|\right)^{2\alpha} $$

(up to a multiplicative constant) which is equivalent to $|h|^{2\alpha}$. Note that to ensure the convergence of all integrals that appear as a result of the integration by parts formula, we have to require that $2l - 2\alpha - 1 - d > 0$ which is the case by the assumption. Therefore, we proved that the sum of the series on the left-hand side of (17) is of the order $|h|^{2\alpha}$. It remains to investigate the convergence of the series

$$ \sum_{h \in \mathbb{Z}_d^+} \frac{1}{|h|^{2l-2\alpha-2}}. $$
By the above argument, this series convergence if and only if the series below converges.

\[ \sum_{h_1=0}^{+\infty} \cdots \sum_{h_d=0}^{+\infty} \frac{1}{(\sum_{i=1}^{d} h_i)^{2l-2\alpha-2}} \mathbb{I}_{\{h \neq 0\}} = d \sum_{h_1=1}^{+\infty} \sum_{h_2=0}^{+\infty} \cdots \sum_{h_d=0}^{+\infty} \frac{1}{(\sum_{i=1}^{d} h_i)^{2l-2\alpha-2}}. \]

To investigate the convergence of the series on the right-hand side we can apply the integral test of convergence subsequently to each one variable series to conclude that this series converge when \(2l - 2\alpha - d - 2 > 0\) which is the case by the assumption. Hence, series (16) converges, and therefore the series on the right-hand side of (14) converges. This proves that \(B_k^{\sin}, B_k^{\cos} \in L_2(\gamma, \mathbb{R}^{d-1})\) for every \(k \in \mathbb{Z}^+_d\) and that \(B \in L_2(\gamma, H^\alpha)\).

### 3.3 Galerkin-type equations

Let us consider the finite-dimensional spaces \(E^{(n)}\) of the variables \((u_k, v_k)\), \(|k| \leq n\). For every integer \(n\), we introduce the Gaussian measure on \(E^{(n)}\):

\[ \gamma^{(n)}(du) = \bigotimes_{|k| \leq n} \gamma_k(d(u_k, v_k)). \]

For \(|k| \leq n\), let \(\tilde{B}_k^{(n),\sin}(s, u)\) and \(\tilde{B}_k^{(n),\cos}(s, u)\) be obtained from \(\tilde{B}_k^{\sin}(s, u)\) and \(\tilde{B}_k^{\cos}(s, u)\) by restricting the summation only over those \(h\) whose absolute values are not bigger than \(n\). If \(|k| > n\), we set \(\tilde{B}_k^{(n),\sin}(s, u) = \tilde{B}_k^{(n),\cos}(s, u) = 0\). Analogously we define \(B_k^{(n),\sin}(u)\) and \(B_k^{(n),\cos}(u)\). Now let

\[ \tilde{B}^{(n)} = \sum_{k \in \mathbb{Z}^+_d} \tilde{B}_k^{(n),\cos} \cos(k \cdot \theta) + \tilde{B}_k^{(n),\sin} \sin(k \cdot \theta), \]

\[ B^{(n)} = \sum_{k \in \mathbb{Z}^+_d} B_k^{(n),\cos} \cos(k \cdot \theta) + B_k^{(n),\sin} \sin(k \cdot \theta). \]

Consider the ODE:

\[
\begin{aligned}
\frac{d}{ds} \tilde{u}(s, u) &= \tilde{B}^{(n)}(s, \tilde{u}(s, u)), \\
\tilde{u}(0, u) &= u
\end{aligned}
\]

where \(u \in H^\alpha\). Let, as before, \(\tilde{u}_k(s, u)\) and \(\tilde{v}_k(s, u)\) be \(\mathbb{R}^{d-1}\)-valued components of \(\tilde{u}(s, u)\), and \(u_k, v_k\) be \(\mathbb{R}^{d-1}\)-valued components of \(u\). Clearly, if
\(|k| > n, (\tilde{u}_k(s, u), \tilde{v}_k(s, u)) = (u_k, v_k)\). Hence, replacing in (19) the variables 

\( (\tilde{u}_k(s, u), \tilde{v}_k(s, u)), |k| > n, \) with \((u_k, v_k)\) and discarding equations of the type 

\[ \tilde{u}_k(s, u) = u_k, \quad \tilde{v}_k(s, u) = v_k \]

for \(|k| > n\), we obtain an ODE in finite dimensions. Namely, let \(\Pi_n u\) be the orthogonal projection of \(u\) onto \(E^{(n)}\), and let \(\Pi_n^\perp u = u - \Pi_n u\). For every \(k \in \mathbb{Z}_d^+, |k| \leq n\), we define the functions \(\tilde{B}_k^{(n), \sin}(s, \Pi_n u, a)\) and \(\tilde{B}_k^{(n), \cos}(s, \Pi_n u, a)\). They are obtained from \(\tilde{B}_k^{(n), \sin}(s, u)\) and resp. \(\tilde{B}_k^{(n), \cos}(s, u)\) by fixing the variables \(u_{k+h}, v_{k+h}, u_{\pm(k-h)}, v_{\pm(k-h)}\) with \(|k+h| > n\) or \(|k-h| > n\). The symbol \(a\) denotes the vector of all fixed variables. Define \(\tilde{B}^{(n)}\) as an \(E^{(n)}\)-valued vector with the \((d-1)\)-dimensional components \(\tilde{B}_k^{(n), \cos}\) and \(\tilde{B}_k^{(n), \sin}\).

The components of \(\tilde{B}^{(n)}\) with respect to the basis \(A_k^{p}, C_k^{p}\) will be denoted by \(\tilde{B}_k^{(n), \cos, p}\) and \(\tilde{B}_k^{(n), \sin, p}\), \(|k| \leq n, p = 1, \ldots, d-1\). The finite-dimensional equation equivalent to (19) looks like this:

\[ \tilde{u}(t, \Pi_n u) = \Pi_n u + \int_0^t \tilde{B}^{(n)}(s, \tilde{u}(s, \Pi_n u), \Pi_n^\perp u) ds. \]  

(20)

**Lemma 2.** Equation (20) has a unique solution for \(\gamma^{(n)}\)-almost all initial conditions. The solution to (20) keeps the measure \(\gamma^{(n)}\) invariant.

For the proof of Lemma 2 we need Lemma 3 below.

**Lemma 3.** For every \(s \geq 0\), for any \(a \in H^d \setminus E^{(n)}\), \(\delta_{\gamma^{(n)}} \tilde{B}^{(n)}(s, \cdot, a) = 0\).

**Proof.** It was proved in [7] (see also [6]) that for every \(s \in [0, T], u \in E^{(n)}\), the divergence of \(\tilde{B}^{(n)}(s, u, a)\) with respect to the measure \(\gamma^{(n)}\) equals to

\[ \delta_{\gamma^{(n)}} \tilde{B}^{(n)}(s, u, a) = (\tilde{B}^{(n)}(s, u, a), u)_l \]

\[ - \sum_{|k| \leq n} \sum_{p=1}^{d-1} \frac{\partial}{\partial u_k^p} \tilde{B}_k^{(n), \cos, p}(s, u, a) \]

\[ - \sum_{|k| \leq n} \sum_{p=1}^{d-1} \frac{\partial}{\partial v_k^p} \tilde{B}_k^{(n), \sin, p}(s, u, a), \]  

(21)

where \((\cdot, \cdot)_l\) is the scalar product in \(H^l\). Note that by formulas (11) and (10), the last two summands in formula (21) equal to zero. Indeed, for every \(k \in \mathbb{Z}_d^+, |k| \leq n\), sum (10) representing \(\tilde{B}_k^{(n), \sin, p}(s, u, a)\) has exactly one summand that contains the variable \(v_k^p\). However, the coefficient \(\lambda^{+}_p(k, k)\) at \(v_k^p\) is 0 for all \(p\). This proves that \(\frac{\partial}{\partial v_k^p} \tilde{B}_k^{(n), \sin, p}(s, u, a) = 0\). Analogously,
\[
\frac{\partial}{\partial u_k} \hat{B}^{(n),\cos,p}(s,u,a) = 0.
\]
Let us show that \( \hat{B}^{(n)}(s,u,a), u \) \(_{l} = 0 \). Remember that \( y(t, \theta) \) has representation (4) and that \( \hat{B}^{(n)} \) can be viewed as a function of \( \theta \), i.e.

\[
\hat{B}^{(n)} = \hat{B}^{(n),\cos}(s,u,a) \cos(k \cdot \theta) + \hat{B}^{(n),\sin}(s,u,a) \sin(k \cdot \theta)
\]

(for simplicity of notations we use the same symbol for this function). Furthermore, \( \Delta_{d} \) denotes the Laplacian defined in Section 2. We obtain:

\[
(\hat{B}^{(n)}(s,u,a), u)_{l} = (\Delta_{d}^{l} \hat{B}^{(n)}, y)_{L_{2}(\mathbb{T}^{d};\mathbb{R}^{d})} = 4\int_{\mathbb{T}^{d}} \Delta_{d}^{l-1} \hat{B}^{(n)}, y_{d} d\theta = 0.
\]

This proves that \( \delta_{\gamma(n)} \hat{B}^{(n)}(s, \cdot, a) = 0 \). \( \Box \)

**Proof of Lemma 2.** Suppose for a moment that the function \( \hat{B}^{(n)} \) on the right-hand side of (20) does not depend on \( s \). Then, in [7] (see also [8] and [6]) the existence of a flow solution was established for \( \gamma^{(n)} \)-almost all initial conditions. In particular, the above mentioned results imply that this flow solution keeps the measure \( \gamma^{(n)} \) invariant.

We prove the existence of a solution to (20) and the fact that it keeps the measure \( \gamma^{(n)} \) invariant by the approximation of the exponential factors in \( \hat{B}^{(n)} \) by stepwise functions. Consider the sequence of partitions \( \mathcal{P}_{N} = \{0 = s_{0} \leq s_{1} \leq \cdots \leq s_{N} = T \} \) of the interval \([0, T]\). Let \( |k| \leq n \) and \( |h| \leq n \), and let

\[
\mathbb{I}_{0} + \sum_{i=1}^{N} e^{-2(k+h,h)_{s_{i},s_{i}}^{s_{i+1},s_{i}}} \mathbb{I}_{(s_{i-1},s_{i})} \quad \text{and resp.} \quad \mathbb{I}_{0} + \sum_{i=1}^{N} e^{2(k-h,h)_{s_{i},s_{i}}^{s_{i+1},s_{i}}} \mathbb{I}_{(s_{i-1},s_{i})}
\]

(22)

be uniform approximations of the functions \( e^{-2(k+h,h)_{s_{i},s_{i}}^{s_{i+1},s_{i}}} \) and \( e^{2(k-h,h)_{s_{i},s_{i}}^{s_{i+1},s_{i}}} \) on the interval \([0, T]\). Denote \( u_{(n)} = \Pi_{n} u \). Define \( \tilde{u}_{n}^{-1}(0) = u_{(n)} \). Let for every integer \( i = 0, \ldots, N - 1 \), \( \tilde{u}_{n}^{i}(s) \) be the solution of the Cauchy problem

\[
\begin{aligned}
\frac{d}{ds} \tilde{u}_{n}^{i}(s) &= \hat{B}^{(n)}(s_{i}, \tilde{u}_{n}^{i}(s), \Pi_{n}^{i} u), \\
\tilde{u}_{n}^{i}(s_{i}) &= \tilde{u}_{n}^{i-1}(s_{i}).
\end{aligned}
\]

(23)

on the interval \([s_{i}, s_{i+1}]\). The latter solution exists by the results [6, 7, 8]. Let \( \hat{B}_{n}^{N} \) be obtained from \( \hat{B}^{(n)} \) by replacing the factors \( e^{-2(k+h,h)_{s_{i},s_{i}}^{s_{i+1},s_{i}}} \) and \( e^{2(k-h,h)_{s_{i},s_{i}}^{s_{i+1},s_{i}}} \) with their approximations (22) that correspond to the partition \( \mathcal{P}_{N} \). For
simplicity, in the definition of $\hat{B}^N$ we skip the index $(n)$. Let us glue the solutions on every interval $[s_i, s_{i+1}]$ and obtain the solution to

$$
\begin{cases}
\frac{d}{ds} \tilde{u}(s, \Pi_n u) = \hat{B}^N(s, \tilde{u}(s, \Pi_n u), \Pi_n^+ u), \\
\tilde{u}(0, \Pi_n u) = \Pi_n u.
\end{cases}
$$

(24)

We show that the solution $\tilde{u}^N(s, u_{(n)})$ exists on $[0, T]$ for $\gamma^{(n)}$-almost all $u_{(n)} \in E^{(n)}$ and keeps the measure $\gamma^{(n)}$ invariant. By the argument in Lemma 3, \[ \frac{\partial}{\partial x_k} \hat{B}_k^{\text{sin}}(s, u_{(n)}) = \frac{\partial}{\partial v_k} B_k^{\text{cos}}(s, u_{(n)}) = 0. \] This means that $\nabla \hat{B}(s, u_{(n)}) = 0$ for all $s \in [0, T]$. Again, by results of [6] or [7], the measure $\gamma^{(n)}$ is invariant with respect to the flow $\tilde{u}^N(s, u_{(n)})$ with $\tilde{u}^N(s_i, u_{(n)}) = u_{(n)}$ on every partition interval $[s_i, s_{i+1}]$. Now let $E_i \subset E^{(n)}$, $i = 0, \ldots, N$, be the sets of full $\gamma^{(n)}$-measure such that for every $u_{(n)} \in E_i$, the solution to (23) on $[s_i, s_{i+1}]$ with the initial condition $u_{(n)}$ exists. Now let $E'_i = E_1 \cap \tilde{u}^N(s_1, E_0), E'_2 = E_2 \cap \tilde{u}^N(s_2, E'_1)$, etc. $E'_{N-1} = E_{N-1} \cap \tilde{u}^N(s_{N-1}, E'_{N-2})$. Since on every partition interval $[s_i, s_{i+1}]$, the solution $\tilde{u}^N_i(s, \cdot)$ keeps the measure $\gamma^{(n)}$ invariant, we obtain that $\gamma^{(n)}(E'_i) = 1$ for every $i = 1, \ldots, N - 1$. Now let

$$
E'_0 = \tilde{u}^N(s_1, \cdot)^{-1} \circ \cdots \circ \tilde{u}^N(s_{N-1}, \cdot)^{-1} E'_{N-1}.
$$

Clearly, $E'_0 \subset E_0$ and, by the invariance of the measure $\gamma^{(n)}$, $\gamma^{(n)}(E'_0) = 1$. Therefore, the solution $u^N(s)$ exists for all initial conditions $u_{(n)} \in E'_0$ and keeps the measure $\gamma^{(n)}$ invariant. Clearly, for every fixed $u_{(n)} \in E^{(n)}$, for every fixed $n$, $\hat{B}^N(s, u_{(n)}) \to \hat{B}^n(s, u_{(n)})$ uniformly in $s \in [0, T]$ as $|P_N| \to 0$. Now let $\tilde{u}^N$ and $\tilde{u}^M$ be solutions to (24) that correspond to $\hat{B}^N$ and, respectively, $\hat{B}^M$, and let $\tilde{u}_{k,i}^N, \tilde{v}_{k,i}^N$ and $\tilde{u}_{k,i}^M, \tilde{v}_{k,i}^M$ be the coordinates of $\tilde{u}^N$ and $\tilde{u}^M$. We prove that $\tilde{u}^M - \tilde{u}^N$ converges to zero in the space $L_2(\gamma^{(n)})$. Also, without loss of generality we can assume that $P \subset P^N$ because we can always introduce the third partition $P^N \cup P^M$ which is a refinement of both. Let the norm $| \cdot |_n$ denote the Euclidean norm of the vector with real components numbered by $h \in \mathbb{Z}_d^+$ with $|h| \leq n$, and the norm $| \cdot |_{n,d}$ denote the Euclidean norm in $E^{(n)}$. We would like to prove that $\{ \tilde{u}^N(s, u_{(n)}) \}_{n=1}^\infty$ is a Cauchy sequence with respect to the norm $E_{\gamma^{(n)}} \sup_{s \in [0, T]} | \cdot |_{n,d}$. We have:

$$
|\tilde{u}^M(t, \cdot) - \tilde{u}^N(t, \cdot)|_{n,d} \leq \int_0^t |\hat{B}^M(s, \tilde{u}^M(s, \cdot)) - \hat{B}^N(s, \tilde{u}^N(s, \cdot))|_{n,d} ds \\
+ \int_0^t |\hat{B}^N(s, \tilde{u}^N(s, \cdot)) - \hat{B}^N(s, \tilde{u}^N(s, \cdot))|_{n,d} ds.
$$

(25)
In the estimate below, we take into account that \(|\lambda^+(k, h)|, |\lambda^-(k, h)|\), and \(|\lambda_i(k)|\) are always smaller than 1, and that the exponents are bounded because \(|h| \leq n, |k| \leq n\), and \(s \in [0, T]\). Let \(\hat{B}_k^{N, \cos}\) be the \((d-1)\)-dimensional \(\cos(k \cdot \theta)\)-component of \(\hat{B}_k^N\). We observe that there exists a constant \(L_{n,T}\) such that

\[
\left| \hat{B}_k^{N, \cos}(s, \hat{u}^M) - \hat{B}_k^{N, \cos}(s, \tilde{u}^N) \right| \leq L_{n,T} \sum_{|h| \leq n} \sum_{i,j=1}^{d-1} \left| \tilde{u}^{M,i}_h - \tilde{u}^N_i \right| \left( |\tilde{v}^{M,i}_{k+h} - \tilde{v}^{M,i}_h| + |\tilde{v}^{M,i}_{k+h} - \tilde{v}^{M,i}_h| \right)
\]

We keep in mind that in this estimate, every coordinate on the right-hand side depends on \(s\). However, for simplicity of notations we skip this dependence. Let \(\tilde{u}^{M,i}\) and \(\tilde{v}^{M,i}\) denote the vectors with the coordinates \(\tilde{u}^{M,i}_h\) and resp. \(\tilde{v}^{M,i}_h\), \(h \in \mathbb{Z}_d^+, |h| \leq n\). Further denote by \(\tilde{u}^{M,i}_{k+}, \tilde{v}^{M,i}_{k+}\), \(\tilde{u}^{M,i}_{\pm(k-h)}, \tilde{v}^{M,i}_{\pm(k-h)}, u^{M,i}_{k+}, \) and \(v^{M,i}_{k+}\), the vectors with the coordinates \(\tilde{u}^{M,i}_{k+}, \tilde{v}^{M,i}_{k+}, \tilde{u}^{M,i}_{\pm(k-h)}, \tilde{v}^{M,i}_{\pm(k-h)}, u^{M,i}_{k+}, \) and \(v^{M,i}_{k+}\), respectively, \(h \in \mathbb{Z}_d^+, |h| \leq n\). Finally let \(u^{M,i}_{k+}\) be the vector with the \((d-1)\)-dimensional components \(u^{M,i}_{k+}, v^{M,i}_{k+}, |h| \leq n\). First we exchange the summations over \(h\) and over \(i, j\). Then we notice that sums of the type

\[
\sum_{|h| \leq n} |\tilde{u}^{M,i}_h - \tilde{v}^N_i| |\tilde{v}^{M,i}_{k+}|^2
\]

can be viewed as the scalar products of the vectors with the components \(\{|\tilde{u}^{M,i}_h - \tilde{v}^{N,i}_h|\}_{|h| \leq n}\) and \(\{|\tilde{v}^{M,i}_{k+}|\}_{|h| \leq n}\). Also, note that the absolute values of these vectors are \(|\tilde{u}^{M,i}_h - \tilde{v}^{N,i}_h|\) and \(|\tilde{v}^{M,i}_{k+}|\), respectively. Next, note that

\[
|v^{M,i}_{k+}| \leq |v^{M,i}_{k+}| \leq |v^{M,i}_{k+}| + |v^{M,i}_{k+}|.
\]

Finally we take into account that

\[
|\tilde{u}^{M,i}_{k+} - \tilde{v}^{N,i}_{k+}| \leq |\tilde{u}^{M,i}_h - \tilde{v}^{N,i}_h|_n,
\]

\[
|\tilde{v}^{M,i}_{\pm(k-h)} - \tilde{v}^{N,i}_{\pm(k-h)}| \leq |\tilde{v}^{M,i}_h - \tilde{v}^{N,i}_h|_n.
\]

The latter two inequalities hold because \(\tilde{v}^{M,i}\) and \(\tilde{v}^{N,i}\) have the same initial condition. Clearly, the all three above inequalities hold when \(v\) is replaced
by \( u \). We obtain:

\[
|\tilde{B}^{N,\cos}(s, \tilde{u}^{M}) - \tilde{B}^{N,\cos}(s, \tilde{u}^{N})| \leq 2L_{n,T} \\
\times \left[ \sum_{i=1}^{d-1} |\tilde{u}^{M,i} - \tilde{u}^{N,i}| \right] _{n} \sum_{j=1}^{d-1} (|\tilde{u}^{M,j}| _{n} + |\tilde{u}^{N,j}| _{n} + |v_{k+1}^{j}| _{n}) \\
+ \sum_{i=1}^{d-1} (|\tilde{u}^{M,i}| _{n} - |\tilde{u}^{N,i}| _{n}) \sum_{j=1}^{d-1} (|\tilde{u}^{M,j}| _{n} - |\tilde{u}^{N,j}| _{n} + |u_{k+1}^{j}| _{n}) \\
\leq \tilde{L}_{n,T} (|\tilde{u}^{M}(s)| _{n,d} + |\tilde{u}^{N}(s)| _{n,d} + |u_{k+1}^{n}| _{n,d}) |\tilde{u}^{M}(s) - \tilde{u}^{N}(s)| _{n,d}
\]

where \( \tilde{L}_{n,T} \) is a modified constant. Clearly, the estimate for \( |\tilde{B}^{N,\sin}(s, \tilde{u}^{M}) - \tilde{B}^{N,\sin}(s, \tilde{u}^{N})| \) will be the same. Therefore, modifying the constant \( L_{n,T} \) we obtain that

\[
|\tilde{B}^{N}(s, \tilde{u}^{M}) - \tilde{B}^{N}(s, \tilde{u}^{N})| _{n,d} \leq L_{n,T} (|\tilde{u}^{M}(s)| _{n,d} + |\tilde{u}^{N}(s)| _{n,d} + |u| _{2n,d}) \\
\times |\tilde{u}^{M}(s) - \tilde{u}^{N}(s)| _{n,d}.
\]

The latter inequality and inequality (25) imply

\[
|\tilde{u}^{M}(t) - \tilde{u}^{N}(t)| _{n,d} \leq \int_{0}^{t} |\tilde{B}^{M}(s, \tilde{u}^{M}(s)) - \tilde{B}^{N}(s, \tilde{u}^{M}(s))| _{n,d} ds \\
+ L_{n,T} \int_{0}^{t} (|\tilde{u}^{M}(s)| _{n,d} + |\tilde{u}^{N}(s)| _{n,d} + |u| _{2n,d}) |\tilde{u}^{M}(s) - \tilde{u}^{N}(s)| _{n,d} ds.
\]

By Gronwall’s lemma,

\[
|\tilde{u}^{M}(t) - \tilde{u}^{N}(t)| _{n,d} \leq \int_{0}^{t} |\tilde{B}^{M}(s, \tilde{u}^{M}(s)) - \tilde{B}^{N}(s, \tilde{u}^{M}(s))| _{n,d} ds \\
\times \exp \left\{ L_{n,T} \int_{0}^{t} (|\tilde{u}^{M}(s)| _{n,d} + |\tilde{u}^{N}(s)| _{n,d} + |u| _{2n,d}) ds \right\}.
\]

Taking supremum over \([0, T]\) and then the expectation of the both sides, using the \( \gamma^{(n)} \)-invariance in the first integral, and making the variable exchange in the integral under the exponent sign, we obtain that there is a constant \( K_{n,T} \) such that:

\[
(\mathbb{E}_{\gamma^{(n)}} \sup_{t \in [0, T]} |\tilde{u}^{M}(t) - \tilde{u}^{N}(t)| _{n,d})^2 \leq K_{n,T} \int_{0}^{T} \mathbb{E}_{\gamma^{(n)}} |\tilde{B}^{M}(s, u)) - \tilde{B}^{N}(s, u)| _{n,d}^2 ds \\
\times \mathbb{E}_{\gamma^{(n)}} \exp \left\{ 2L_{n,T} T \int_{0}^{1} (|\tilde{u}^{M}(Ts)| _{n,d} + |\tilde{u}^{N}(Ts)| _{n,d}) ds \right\}. \quad (26)
\]

20
The first integral clearly converges to zero. Let us prove that the last multiplier remains bounded as $M$ and $N$ tend to infinity. Clearly, it suffices to prove that $\mathbb{E}_{\gamma(n)} \exp \int_0^1 |\tilde{u}^M(Ts)|_{n,d} ds$ is bounded as $M \to \infty$. We use the Taylor expansion for the exponent, Hölder’s inequality, and the inequality

$$\left( \int_0^1 |\tilde{u}^M(Ts)|_{n,d} ds \right)^m \leq \int_0^1 |\tilde{u}^M(Ts)|^m_{n,d} ds$$

to conclude that

$$\exp \int_0^1 |\tilde{u}^M(Ts)|_{n,d} ds \leq \int_0^1 \exp |\tilde{u}^M(Ts)|_{n,d} ds.$$

We already proved that $\tilde{u}^M(Ts)$ keeps the measure $\gamma(n)$ invariant. Therefore,

$$\mathbb{E}_{\gamma(n)} \exp \int_0^1 |\tilde{u}^M(Ts)|_{n,d} ds \leq \int_0^1 \mathbb{E}_{\gamma(n)} \exp |\tilde{u}^M(Ts)|_{n,d} ds = \mathbb{E}_{\gamma(n)} \exp |\Pi_n u|_{n,d} < \infty.$$

By (26), we can find a subsequence of $\tilde{u}^N(s)$ (for simplicity, the subsequence will be also denoted by $\tilde{u}^N(s)$) so that for $\gamma(n)$-almost all initial conditions $u(n)$, there exists a uniform limit $\hat{U}^{(n)}(s, u(n)) = \lim_{N \to \infty} \tilde{u}^N(s, u(n))$. The limit $\hat{U}^{(n)}(s, u(n))$ keeps the measure $\gamma(n)$ invariant. Indeed, let $f$ be a bounded continuous function $E^{(n)} \to \mathbb{R}$. By Lebesgue’s theorem,

$$\mathbb{E}_{\gamma(n)}[f(\hat{U}^{(n)}(t, \cdot))] = \lim_{N \to \infty} \mathbb{E}_{\gamma(n)}[f(\tilde{u}^N(t, \cdot))] = \mathbb{E}_{\gamma(n)}[f(\cdot)].$$

Let us prove that $\hat{U}^{(n)}$ is a solution to (20). We have:

$$\mathbb{E}_{\gamma(n)} \int_0^t |\hat{B}^{(n)}(s, \hat{U}^{(n)}(s)) - \hat{B}^N(s, \tilde{u}^N(s))|_{n,d} ds \leq \int_0^t \mathbb{E}_{\gamma(n)} |\hat{B}^{(n)}(s, \tilde{u}^N(s)) - \hat{B}^N(s, \tilde{u}^N(s))|_{n,d} ds$$

$$+ \int_0^t \mathbb{E}_{\gamma(n)} |\tilde{B}^{(n)}(s, \tilde{u}^N(s)) - \hat{B}^{(n)}(s, \hat{U}^{(n)}(s))|_{n,d} ds. \quad (27)$$

The first term on the right-hand side converges to zero since the measure $\gamma(n)$ is invariant with respect to $\tilde{u}^N(t)$. Let us prove that the second term converges
to zero. Let $\varepsilon$ be fixed arbitrary, and let $F$ be a continuous bounded function on $[0, T] \times E^{(n)}$ such that
\[
\int_0^T \mathbb{E}_{\gamma^{(n)}} \left| \hat{B}^{(n)}(s, \cdot) - F(s, \cdot) \right| ds < \varepsilon.
\]
Since $F$ is bounded and $\hat{U}^{(n)}(s, u^{(n)})$ converges to $\tilde{u}^N(s, u^{(n)})$ uniformly in $s$ and $\gamma^{(n)}$-a.s. in $u^{(n)}$,
\[
\lim_{N \to \infty} \int_0^t \mathbb{E}_{\gamma^{(n)}} \left| F(s, \tilde{u}^N(s, \cdot)) - F(s, \hat{U}^{(n)}(s, \cdot)) \right|_{n,d} ds = 0
\]
by the Lebesgue theorem. By the invariance of the measure $\gamma^{(n)}$ with respect to $\tilde{u}^N(s, \cdot)$ and $\hat{U}^{(n)}(s, \cdot)$,
\[
\int_0^t \mathbb{E}_{\gamma^{(n)}} \left| \hat{B}^{(n)}(s, \tilde{u}^N(s, \cdot)) - \hat{B}^N(s, \tilde{u}^N(s, u^{(n)})) \right|_{n,d} ds < \varepsilon.
\]
The latter inequality holds also if we replace $\tilde{u}^N(s, \cdot)$ with $\hat{U}^{(n)}(s, \cdot)$. This proves that the second term in (27) converges to zero as $N \to \infty$. Thus, we can find a subsequence of the sequence
\[
\left\{ \int_0^t \left| \hat{B}^{(n)}(s, \hat{U}^{(n)}(s, u^{(n)})) - \hat{B}^N(s, \tilde{u}^N(s, u^{(n)})) \right|_{n,d} ds \right\}_N
\]
which converges to zero $\gamma^{(n)}$-a.s. in $u^{(n)} \in E^{(n)}$. For simplicity of notation, we will think of (28) as of the a.s.-converging subsequence. Therefore, (20) is fulfilled for $\gamma^{(n)}$-almost all initial conditions.

The uniqueness of the solution is a classical result of the theory of ODEs. The theorem is proved.

**Corollary 1.** ODE (19) has a unique solution for $\gamma$-almost all initial conditions $u \in H^\alpha$. This solution keeps the measure $\gamma$ invariant.

**Proof.** From our construction it follows that the infinite-dimensional solution $\hat{U}^{(n)}$ to (19) is obtained from the finite-dimensional solution $\hat{U}^{(n)}$ to (20) as follows:
\[
\hat{U}^{(n)}(t, u) = \hat{U}^{(n)}(t, \Pi_n u) + \Pi_n^1 u. \quad (29)
\]
This solution is unique since $\tilde{U}^{(n)}(t, \Pi_n u)$ is unique. Further, let $f \in L_1(\gamma)$, and let the measure $\gamma^{(n)\perp}$ be such that $\gamma = \gamma^{(n)} \otimes \gamma^{(n)\perp}$. Then, by the invariance of $\gamma^{(n)}$, we obtain:

$$
\int_{H^\alpha} f(\tilde{U}^{(n)}(t, u)) \gamma(du)
= \int_{H^\alpha \otimes E^{(n)}} \gamma^{(n)\perp}(d\Pi_n u) \int_{E^{(n)}} f(\tilde{U}^{(n)}(t, \Pi_n u) + \Pi_n u) \gamma^{(n)}(d\Pi_n u)
= \int_{H^\alpha} f(u) \gamma(du).
$$

To show that the solution exists for $\gamma$-almost all $u \in H^\alpha$ note that if $u_1 = \Pi_n u$ and $u_2 = \Pi_n^\perp u$, then by Lemma 2, the solution to (19) exists for $\gamma^{(n)}$-almost all $u_1$ and for all $u_2$, and therefore for $\gamma$-almost all $u$. □

4. Navier–Stokes solution as the limit of Galerkin-type approximations

Let $\tilde{U}^n_k(t, u)$ and $\tilde{V}^n_k(t, u)$ be $(d - 1)$-dimensional components of the solution $\tilde{U}^n(t, u)$ to (19). Define

$$
U^n_k(t, u) = e^{-t|k|^2 \nu} \tilde{U}^n_k(t, u), \quad V^n_k(t, u) = e^{-t|k|^2 \nu} \tilde{V}^n_k(t, u).
$$

Consider the triple $(H^\alpha, \mathcal{B}, \gamma)$, where $\mathcal{B}$ is the $\sigma$-algebra of Borel subsets of $H^\alpha$, as a probability space, and $U^n(t, u) = \{U^n_k(t, u), V^n_k(t, u)\}$, $k \in \mathbb{Z}^+_d$, as a stochastic process on it. Note that by (29), it is clear that $U^n(t, u)$ takes values in $H^\alpha$.

**Lemma 4.** There exists a subset $H' \subset H^\alpha$ of full $\gamma$-measure such that for all $u \in H'$, for all $t \in [0, T]$, $U^n(t, u)$ verifies the equation

$$
U^n(t, u) = e^{tu\Delta} u + \int_0^t e^{(t-s)u\Delta} B^{(n)}(U^n(s, u)) \, ds. \quad (30)
$$

Moreover, for any function $f \in L_1(\gamma)$ it holds that

$$
\mathbb{E}_\gamma[f(U^n(t, u))] = \mathbb{E}_\gamma[f(e^{tu\Delta} u)], \quad (31)
$$

where $\mathbb{E}_\gamma$ is the expectation with respect to the measure $\gamma$. 23
Proof. Note that (30) follows from (19) by applying the operator $e^{t\nu\Delta}$ to the both parts and taking into account that $e^{t\nu\Delta}\tilde{B}^{(n)}(s, U^n(s, u)) = B^{(n)}(U^n(s, u))$. Equality (31) follows from the invariance of the measure $\gamma$ with respect to the solution $\tilde{U}^n(t, u)$:

$$E_\gamma[f(U^n(t, u))] = E_\gamma[f(e^{t\nu\Delta}\tilde{U}^n(t, u))] = E_\gamma[f(e^{t\nu\Delta}u)].$$

\[\square\]

Let $\nu^n$ be the law of $U^n$ on $C([0, T], H^\alpha)$, i.e. for any Borel subset $G \subset C([0, T], H^\alpha)$,

$$\nu^n(G) = \gamma\{u \in H^\alpha : U^n(\cdot, u) \in G\}.$$

We prove that the sequence of measures $\nu^n$ contains a weakly convergent subsequence. For this we need versions of the tightness criterium for probability measures and Arzelá-Ascoli’s theorem. Since we do not know a reference where these theorems are given in the form suitable for our purpose, we prove these results.

**Proposition 2** (Tightness criterium for probability measures). Let $E$ be a Banach space. The sequence of probability measures $P_n$ on $C([0, T], E)$ is tight if and only if

(i) for each $\varepsilon > 0$ and $t_R \in [0, T] \cap \mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers, there exists a compact $K \subset E$ such that

$$P_n(y : y(t_R) \notin K) < \varepsilon \quad \forall n;$$

(ii) for each $\varepsilon > 0$ and $\rho > 0$, there exist a $\delta$, with $0 < \delta < 1$, so that

$$P_n(y : \sup_{\substack{0 \leq t' \leq T \atop |t-t'| < \delta}} \|y(t) - y(t')\|_E > \rho) < \varepsilon \quad \forall n.$$

**Proposition 3** (Arzelá-Ascoli’s theorem). Let $E$ be a Banach space. A family of functions $\Phi \subset C([0, T], E)$ is relatively compact if and only if

(i) for every $t_R \in [0, T] \cap \mathbb{Q}$, the set $\{y(t_R) : y \in \Phi\}$ is relatively compact in $E$;

(ii) the family $\Phi$ is equicontinous.
Proof of Proposition 3. Assume that \( \Phi \) is relatively compact. By the version of Arzelá-Ascoli’s theorem from [11] (Theorem 3.1 p. 55), \( \Phi \) satisfies conditions \((i)\) and \((ii)\), and the necessity of the statement follows.

Let us assume that \( \Phi \) satisfies conditions \((i)\) and \((ii)\). To satisfy the assumption of Arzelá-Ascoli’s theorem from [11] we have to prove that the set \( \{f(t), t \in \Phi\} \) is relatively compact for each \( t \in [0, T] \), i.e. that condition \((i)\) is fulfilled for all \( t \in [0, T] \) including irrational numbers. Let a number \( t \in [0, T] \) and a sequence \( \{f_n\} \subset \Phi \) be fixed arbitrary. Further let \( \{t_n\} \subset [0, T] \cap \mathbb{Q} \) be such that \( \lim_{n \to \infty} t_n = t \). By condition \((i)\), \( \{f_n(t_1)\} \) contains a subsequence \( \{f^{1n}(t_1)\} \) that converges in \( E \) to a limit \( a_1 \). Furthermore, we can choose the subsequence \( \{f^{1n}(t_1)\} \) so that \( \|f^{1n}(t_1) - a_1\|_E \downarrow 0 \) as \( n \to \infty \). Let \( \{\varepsilon_n\} \) be a sequence of real numbers such that \( \varepsilon_n \downarrow 0 \) as \( n \to \infty \) and

\[
\|f^{1n}(t_1) - a_1\|_E < \varepsilon_n.
\]

Suppose we found subsequences \( \{f^{2n}\}_{n=1}^{\infty}, \ldots, \{f^{(i-1)n}\}_{n=1}^{\infty} \) such that each \( \{f^{jn}\}, j \leq i - 1 \), converges at points \( t_1, \ldots, t_j \). Moreover, if \( a_j = \lim_{n \to \infty} f^{jn}(t_j), j = 1, 2, \ldots, i - 1 \), then \( \|f^{jn}(t_j) - a_j\|_E \downarrow 0 \) and \( \|f^{jn}(t_j) - a_j\|_E < \varepsilon_n \) for all \( n \). By condition \((i)\), we find a subsequence \( \{f^{jn}\} \) of \( \{f^{(i-1)n}\} \) that converges at point \( t_i \). Next, we find another subsequence of the subsequence \( \{f^{jn}\} \) (for convenience, use the same symbol \( \{f^{jn}\} \) for this subsequence) such that \( \|f^{jn}(t_i) - a_i\|_E \downarrow 0 \) as \( n \to \infty \), and, moreover,

\[
\|f^{jn}(t_i) - a_i\|_E < \varepsilon_n.
\]

Finally, let us take the diagonal subsequence \( \{f^{jn}\} \). Clearly, \( \lim_{n \to \infty} f^{jn}(t_i) = a_i \) for all \( i \) and

\[
\|f^{jn}(t_i) - a_i\|_E < \varepsilon_n
\]

for all \( i \leq n \). Let us show that \( \{f^{jn}(t_n)\} \) is a Cauchy sequence in \( E \). Indeed, fix an \( \varepsilon > 0 \). Let the number \( N \) be such that for \( n > m > N \), \( \|f(t_n) - f(t_m)\|_E < \frac{\varepsilon}{2} \) for all \( f \in \Phi \), and \( \|f^{jn}(t_m) - f^{jn}(t_m)\|_E < \frac{\varepsilon}{2} \). The latter inequality holds by \((33)\). This implies that for \( n > m > N \), \( \|f^{jn}(t_n) - f^{jn}(t_m)\|_E < \varepsilon \). Thus, \( \{f^{jn}(t_n)\} \) is a Cauchy sequence in \( E \), and therefore, it has a limit \( a \) in \( E \). Let us prove that \( a \) is also a limit of \( f^{jn}(t) \). Indeed,

\[
\|f^{jn}(t) - a\|_E \leq \|f^{jn}(t) - f^{jn}(t_n)\|_E + \|f^{jn}(t_n) - a\|_E.
\]
The first term on the right-hand side converges to zero by the equicontinuity of functions from $\Phi$. Thus we found a subsequence $\{f^{m_n}\} \subset \{f_n\}$ that converges at point $t$. This proves that the set $\{f(t), f \in \Phi\}$ is relatively compact.

By the version of Arzelá-Ascoli’s theorem from [11], the family $\Phi$ is relatively compact.

**Proof of Proposition 2.** Basically the proof follows the lines of Theorem 8.2 from [5] (p.55) but it is adapted to our case. Let us assume that the family $\{P_n\}$ is tight. Then for every $\varepsilon \in (0, 1)$ there exists a compact $\Phi \subset C([0, T], E)$ such that $P_n(\Phi) \geq 1 - \varepsilon$ for all $n$. Fix a $t_r \in [0, T] \cap \mathbb{Q}$, and let $K = \{y(t_r) : y \in \Phi\}$. By Proposition 3, $K$ is a compact in $E$. Since $\{y : y(t_r) \notin K\} \subset \Phi^c$, where $\Phi^c$ is the complement of $\Phi$ in $C([0, T], E)$, then $P_n(y : y(t_r) \notin K) < \varepsilon$ for all $n$. Condition (ii) can be verified in exactly the same way as in Theorem 8.2 from [5].

Now let us assume that Conditions (i) and (ii) of Proposition 2 are fulfilled, and prove that the sequence $\{P_n\}$ is tight. Let $\{t_m\}_{m=1}^\infty$ be a sequence of all rational numbers of the interval $[0, T]$. Fix an $\varepsilon > 0$ and choose a compact $K_m$ so that $P_n(y : y(t_m) \notin K_m) < \frac{\varepsilon}{2^m + 1}$ for all $n$. Further let $w_n(\delta)$ be the modulus of continuity of $y \in C([0, T], E)$. Choose a sequence $\{\delta_m\}_{m=1}^\infty$ so that $P_n(y : w_n(\delta_m') < \frac{1}{m'}) > 1 - \frac{\varepsilon}{2^m + 1}$. Let

$$
\Phi = \bigcap_{m=1}^\infty \{y : y(t_m) \in K_m\} \bigcap_{m'=1}^\infty \{y : w_n(\delta_m') < \frac{1}{m'}\}.
$$

Then $P_n(\Phi) > 1 - \varepsilon$ for all $n$, and, by Proposition 3, $\Phi$ is a compact in $C([0, T], E)$.

Our main result is the following theorem:

**Theorem 1.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $l > \frac{d}{2} + \alpha + 1$. Then, there exist an $H^\alpha$-valued stochastic process $U(t, \omega)$ and a subset $\Omega' \subset \Omega$ of full $\mathbb{P}$-measure such that for all $\omega \in \Omega'$ and for all $t \in [0, T]$,

$$
U(t, \omega) = e^{t \nu} U(0, \omega) - \int_0^t e^{(t-s) \nu} B(U(s, \omega)) \, ds,
$$

and the law of $U(0, \omega)$ on $H^\alpha$ is the measure $\gamma$. Moreover, for every function $f \in L_1(\gamma)$ and for all $t \in [0, T]$ it holds that

$$
\mathbb{E}_\mathbb{P}[f(U(t, \omega))] = \int_{H^\alpha} f(e^{\nu} \Delta u) \gamma(du).
$$

(35)
Proof. By Prokhorov’s theorem ([5]), the family of measures \( \{\nu^n\} \) (defined on \( C([0, T], \mathbb{H}^\alpha) \)) is sequentially compact in the space of all probability measures on \( C([0, T], \mathbb{H}^\alpha) \) if and only if it is tight. Therefore, we have to verify Conditions (i) and (ii) of Proposition 2. We start with Condition (ii), i.e. we prove that for all \( \rho > 0 \),
\[
\lim_{\delta \to 0} \sup_{n} \nu^n \left( \sup_{0 \leq t < t' \leq T} \| y(t) - y(t') \| \geq \rho \right) = 0.
\]
Taking into account that \( U^n(t, u) \) verifies (30), we obtain:
\[
\nu^n \left( \sup_{0 \leq t < t' \leq T} |y(t) - y(t')| \geq \rho \right) \leq \frac{1}{\rho^2} \int_{C([0, T], \mathbb{H}^\alpha)} \sup_{0 \leq t < t' \leq T} \| y(t) - y(t') \|^2 \nu(dy)
\]
\[
= \frac{1}{\rho^2} \int_{\mathbb{H}^\alpha} \sup_{0 \leq t < t' \leq T} \| U^n(t, u) - U^n(t', u) \|^2 \gamma(du)
\]
\[
\leq \frac{3}{\rho^2} \mathbb{E}_\gamma \sup_{0 \leq t < t' \leq T} \left[ \| (e^{t\Delta} - e^{t'\Delta})u \|^2 + \left\| \int_{t}^{t'} e^{(t-s)\nu} B(n)(U^n(s, u)) ds \right\|_\alpha^2 \right.
\]
\[
\left. + \left\| \int_{0}^{t'} e^{(t-s)\nu} (e^{(t'-s)\nu} - I) B(n)(U^n(s, u)) ds \right\|_\alpha^2 \right]. \tag{36}
\]
Using (31), we can estimate the right-hand side of (36) by
\[
\frac{3}{\rho^2} \left[ \mathbb{E}_\gamma \| (I - e^{\delta\Delta})u \|^2_\alpha + \delta \int_{0}^{T} \mathbb{E}_\gamma \| B(n)(e^{s\Delta}u) \|^2_\alpha ds \right.
\]
\[
\left. + T \int_{0}^{T} \mathbb{E}_\gamma \| (I - e^{\delta\Delta})B(n)(e^{s\Delta}u) \|^2_\alpha ds \right]. \tag{37}
\]
The first term in (37) converges to zero by the Banach-Steinhaus theorem, since \( \| I - e^{\delta\Delta} \| \leq 2 \) and the convergence clearly holds for those \( u \) that have only a finite number of non-zero coordinates. Next, by the results of Section 3.2, \( \mathbb{E}_\gamma \| B(n)(e^{s\Delta}u) \|^2_\alpha \) is bounded by the sum of series (15) (multiplied by \( 8(d-1)^2 \)) uniformly in \( s \) and \( n \) which proves that the second term in (37) also converges to zero. Finally, if we define the \( \mathbb{H}^\alpha \)-vector \( L \) whose \( k \)-th sin- and cos-components are \( 8(d-1)^2 \sum_{h \in \mathbb{Z}_d^+} \frac{1}{|k+h|^2} \frac{1}{|h|^{d-2}} \), we obtain that
\[
\mathbb{E}_\gamma \| (I - e^{\delta\Delta})B(n)(e^{s\Delta}u) \|^2_\alpha \leq \| (I - e^{\delta\Delta})L \|^2_\alpha
\]
27
where the right-hand side converges to zero by the Banach-Steinhaus theorem. Thus, \((37)\) converges to zero, and, therefore, Condition \((ii)\) is verified.

Let us verify Condition \((i)\). Fix an arbitrary \(\varepsilon > 0\), and find a compact \(K \subset H^\alpha\) of the form

\[
K = B(r_1) \times \ldots \times B(r_k) \times \ldots ,
\]

where \(B(r_k)\) is the \(2(d - 1)\)-dimensional ball of radius \(r_k\) centered at zero in the space of the variables \((u_k, v_k), k \in \mathbb{Z}_d^+\), so that

\[
\gamma(H^\alpha \setminus K) < \varepsilon .
\]

The compact \(K\) of form \((38)\) with property \((39)\) exists. Indeed, let \(r_k, k \in \mathbb{Z}_d^+\), be a sequence of real numbers satisfying

\[
\sum_{k \in \mathbb{Z}_d^+} |k|^{2\alpha} r_k^2 < \infty .
\]

To show that a set of form \((38)\) is a compact in \(H^\alpha\), take a sequence \(\{f^n\}_{n=1}^\infty \subset K\), and find its subsequence \(\{f^{n_i}\}_{i=1}^\infty\) so that its \(2(d - 1)\)-dimensional components \(f^{n_i}_{k}, k \in \mathbb{Z}_d^+\), which are pairs of \((d - 1)\)-dimensional cos- and sin-components, have limits \(a_k = \lim_{i \to \infty} f^{n_i}_k\). Since \(|a_k| \leq r_k\), then \((40)\) implies that that \(\{f^{n_i}\}_{i=1}^\infty\) has a limit in \(H^\alpha\). Thus, \(K\) defined by \((38)\) with \(r_k\) satisfying \((40)\) is a compact. Let us show that we can choose numbers \(r_k\) with property \((40)\) so that

\[
\prod_{|k| > N} \gamma_k(B(r_k)) > \prod_{|k| > N} \left(1 - \frac{1}{|k|^{d+1}}\right) > e^{-2\sum_{|k| > N} \frac{1}{|k|^{d+1}}} > 1 - 2\sum_{|k| > N} \frac{1}{|k|^{d+1}} .
\]

\(28\)
The series on the right-hand side clearly converges by the arguments from Section 3.2. Let the number \( N \) be big enough so that 
\[
2 \sum_{|k|>N} \frac{1}{|k|^{d+1}} < \frac{\varepsilon}{3}.
\]

Next, we choose the numbers \( r_k \) with \(|k| \leq N\) sufficiently big so that 
\[
\prod_{|k|\leq N} \gamma_k(B(r_k)) > 1 - \frac{\varepsilon}{3}.
\]

Summing up, we obtain:
\[
\gamma(K) = \prod_{k \in \mathbb{Z}^+_d} \gamma_k(B(r_k)) > \left(1 - \frac{\varepsilon}{3}\right)^2 > 1 - \varepsilon.
\]

Take a \( t_R \in [0,T] \cap \mathbb{Q} \). By the invariance of the measure \( \gamma \) with respect to \( \tilde{U}_n(t,u) \), we obtain:
\[
\nu^n(y : y(t_R) \notin K) = \gamma\left(u : (U^n_k(t_R,u),V^n_k(t_R,u)) \notin B(r_k), k \in \mathbb{Z}^+_d\right)
= \gamma\left(u : (\tilde{U}^n_k(t_R,u),\tilde{V}^n_k(t_R,u)) \notin B(e^{\varepsilon |r_k|^2} r_k), k \in \mathbb{Z}^+_d\right)
= \gamma\left(u : (u_k,v_k) \notin B(e^{\varepsilon |r_k|^2} r_k), k \in \mathbb{Z}^+_d\right) \leq \gamma(H^\alpha \setminus K) < \varepsilon.
\]

Thus, we verified Conditions (i) and (ii) of the tightness criterium (Proposition 2). Hence, there exists a subsequence of \( \{\nu^n\} \) (for simplicity we use the same symbol for this subsequence) which converges weakly to a measure \( \nu \) on \( C([0,T],H^\alpha) \). Next, we apply Skorohod’s representation on a given probability space \( (\Omega,F,P) \) proved in [4]. In particular, this theorem states the following:

**Skorohod’s representation theorem on a given probability space:** Suppose the measure \( P \) is nonatomic, \( \mu_n \) and \( \mu \) are separable probabilities on \( (S,B) \), where \( S \) is a metric space and \( B \) is the Borel \( \sigma \)-algebra on \( S \), and \( \mu_n \to \mu \) weakly. Then, if \( \mu \) and each \( \mu_n \) are tight, there are \( S \)-valued random variables \( X_n \) and \( X \) on \( (\Omega,F,P) \) with distributions \( \mu_n \) and, respectively, \( \mu \), such that \( X_n \to X \) a.s.

Set the metric space \( S \) to be \( C([0,T],H^\alpha) \). By the above version of Skorokhod’s theorem, we conclude that there are \( H^\alpha \)-valued stochastic processes
$U^m(t, \omega)$ and $U(t, \omega)$, $\omega \in \Omega$, with the laws $\nu^n$ and, respectively, $\nu$ on $C([0, T], H^\alpha)$ such that $U^m(\cdot, \omega) \rightarrow U(\cdot, \omega)$ $\mathbb{P}$-a.s. and with respect to the norm of $C([0, T], H^\alpha)$. Let us prove that for each $n$, the process $U^m(t, \omega)$ verifies equation (30) $\mathbb{P}$-a.s.. Consider the function:

$$F : C([0, T], H^\alpha) \rightarrow C([0, T], H^\alpha), \, f_t \mapsto f_t - \int_0^t e^{(t-s)\Delta \nu} B^{(n)}(f_s) \, ds - e^{t\Delta \nu} f_0.$$ 

It is easy to verify that $F$ is continuous. This implies that the random variables $F(U_n(\cdot, u))$ and $F(U^m(\cdot, \omega))$ have the same distribution $\nu^n \circ F^{-1}$. Therefore,

$$\mathbb{P}(\omega: F(U^m(\cdot, \omega)) = 0) = \gamma(u: F(U^n(\cdot, u)) = 0) = 1.$$ 

The latter equality holds by Lemma 4. This implies that there exists a set $\Omega' \subset \Omega$ of full $\mathbb{P}$-measure so that for all $t \in [0, T]$ and for all $\omega \in \Omega'$,

$$U^m(t, \omega) = e^{t\Delta \nu} U^m(0, \omega) + \int_0^t e^{(t-s)\Delta \nu} B^{(n)}(U^m(s, \omega)) \, ds. \quad (41)$$

Next,

$$\mathbb{E}_{\mathbb{P}} \sup_{t \in [0, T]} \left\| \int_0^t e^{(t-s)\Delta \nu} \left( B^{(n)}(U^m(s, \omega)) - B(U(s, \omega)) \right) \, ds \right\|_\alpha \leq \mathbb{E}_{\mathbb{P}} \int_0^T \left\| B^{(n)}(U^m(s, \omega)) - B(U^m(s, \omega)) \right\|_\alpha \, ds + \mathbb{E}_{\mathbb{P}} \int_0^T \left\| B(U^m(s, \omega)) - B(U(s, \omega)) \right\|_\alpha \, ds. \quad (42)$$

The first summand on the right-hand side of (42) equals to

$$\int_0^T \mathbb{E}_{\gamma} \left\| B^{(n)}(U^n(s, u)) - B(U^n(s, u)) \right\|_\alpha \, ds = \int_0^T \mathbb{E}_{\gamma} \left\| B^{(n)}(e^{sv\Delta} u) - B(e^{sv\Delta} u) \right\|_\alpha \, ds.$$ 

The latter tends to zero as $n \rightarrow \infty$ since by the results of Section 3.2, $\mathbb{E}_{\gamma} \| B^{(n)} - B \|_\alpha^2 \rightarrow 0$. The presence of the semigroup $e^{sv\Delta}$, $s > 0$, just improves
this convergence due to the multipliers $e^{-\nu|k|^2}$ applied to each pair $(u_k, v_k)$.

The above convergence to zero specifically follows from the argument below:

$$
\mathbb{E}_\gamma \sum_{k \in \mathbb{Z}_d^2} |k|^{2\alpha} (|B_k^{\sin} - B_k^{(n),\sin}|^2 + |B_k^{\cos} - B_k^{(n),\cos}|^2) \leq \sum_{|k| > n} |k|^{2\alpha} (|B_k^{\sin}|^2 + |B_k^{\cos}|^2)
$$

$$
\leq 4(d - 1)^2 \sum_{|h| > n} \frac{1}{|h|^{2\alpha}} \sum_{k \in \mathbb{Z}_d^2} \left( \frac{|k|^{2\alpha}}{|k + h|^{2\alpha}} + \frac{|k|^{2\alpha}}{|k - h|^{2\alpha}} \right) \to 0, \quad n \to \infty.
$$

For each $t \in [0, T]$, we define the Hilbert space $H^\alpha_t = e^{\nu t} H^\alpha$ and the measure $\gamma_t = \gamma \circ e^{-\nu t} \Delta$. Clearly, $\gamma_t$ is concentrated on $H^\alpha_t$. Extend the measure $\gamma_t$ to $H^\alpha$ by setting $\gamma_t(H^\alpha \setminus H^\alpha_t) = 0$ and $\sigma$-additivity. Let us show that for each fixed $t \in [0, T]$, the distribution of $U(t, \omega)$ is $\gamma_t$. Indeed, for every bounded continuous function $f : H^\alpha \to \mathbb{R}$, we have:

$$
\mathbb{E}_{\mathbb{P}}[f(U(t, \omega))] = \mathbb{E}_\gamma[\mathbb{E}_{\mathbb{P}}[f(U(t, \omega))] = \mathbb{E}_\gamma[f(e^{t\Delta} u)] = \mathbb{E}_{\gamma_t}[f(u)].
$$

Passing to the limit on the left-hand side, by Lebesgue’s theorem, we obtain:

$$
\mathbb{E}_{\mathbb{P}}[f(U(t, \omega))] = \mathbb{E}_{\mathbb{P}}[f(U(t, \omega))] = \mathbb{E}_{\gamma_t}[f] \quad (43)
$$

which proves that $\gamma_t$ is the distribution of $U(t, \omega)$ and $U^{(n)}(t, \omega)$ on $H^\alpha$. Define $B_{N,M}(u) = B^{(N)}(u) I_{\{B^{(N)} \leq M\}}(u)$, where the latter multiplier is the indicator function. Let us prove that $\mathbb{E}_{\gamma_t} \|B_{N,M} - B\|_\alpha \to 0$ as $M, N \to \infty$. Indeed,

$$
\mathbb{E}_{\gamma_t} \|B^{(N)} - B\|_\alpha = \mathbb{E}_{\gamma_t} \|B^{(N)}(e^{t\Delta} u) - B(e^{t\Delta} u)\|_\alpha \to 0, \quad N \to \infty.
$$

Next, we have:

$$
\mathbb{E}_{\gamma_t} \|B_{N,M} - B\|_\alpha \leq \mathbb{E}_{\gamma_t} \|B_{N,M} - B^{(N)}\|_\alpha + \mathbb{E}_{\gamma_t} \|B^{(N)} - B\|_\alpha.
$$

For the first summand on the right-hand side we obtain:

$$
\mathbb{E}_{\gamma_t} \|B_{N,M} - B^{(N)}\|_\alpha = \mathbb{E}_{\gamma_t} \|B^{(N)}(u) I_{\|B^{(N)}\|_\alpha \leq M}(u)\|_\alpha
$$

$$
\leq \left( \mathbb{E}_{\gamma_t} \|B^{(N)}\|_\alpha^2 \right)^{\frac{1}{2}} \mathbb{E}_{\gamma_t} \|B^{(N)}\|_\alpha \geq M \right)^{\frac{1}{2}} \leq \frac{1}{M} \mathbb{E}_{\gamma_t} \|B^{(N)}\|_\alpha^2 \quad (44)
$$

By the results of Section 3.2, $\mathbb{E}_{\gamma_t} \|B^{(N)}\|_\alpha^2$ is bounded uniformly in $N$ and $t \in [0, T]$ by the sum of the series

$$
4(d - 1)^2 \sum_{h \in \mathbb{Z}_d^2} \frac{1}{|h|^{2\alpha}} \sum_{k \in \mathbb{Z}_d^2} \left( \frac{|k|^{2\alpha}}{|k + h|^{2\alpha}} + \frac{|k|^{2\alpha}}{|k - h|^{2\alpha}} \right).
$$

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and, therefore, the right-hand side of (44) converges to zero as $M \to \infty$ uniformly in $N$. Clearly, $B_{N,M}(u)$ is a bounded and continuous function. It is continuous since $B^{(N)}(u)$ is a polynomial that depends only on a finite number of coordinates $(u_k, v_k)$. Hence, for each fixed pair of integers $N, M > 0$, by Lebesgue’s theorem, we obtain:

$$\mathbb{E}_\mathbb{P}\|B_{N,M}(U'^{(n)}(t, \omega)) - B_{N,M}(U(t, \omega))\|_\alpha \to 0, \quad n \to \infty.$$ 

On the other hand, uniformly in $n$,

$$\mathbb{E}_\mathbb{P}\|B_{N,M}(U(t, \omega)) - B(U(t, \omega))\|_\alpha = \mathbb{E}_\mathbb{P}\|B_{N,M}(U'^{(n)}(t, \omega)) - B(U'^{(n)}(t, \omega))\|_\alpha = \mathbb{E}_\mathbb{P}\|B_{N,M}(u) - B(u)\|_\alpha \to 0, \quad M, N \to \infty.$$ 

This proves that the second summand in (42), and, therefore, the left-hand side of (42), tends to zero as $n \to \infty$. Choosing a $\mathbb{P}$-a.s. converging subsequence, we can pass to the limit as $n \to \infty$ in (41). The convergence of the both sides holds $\mathbb{P}$-a.s. in the space $C([0, T], H^\alpha)$. Therefore, we conclude that (34) holds for all $t \in [0, T]$, $\mathbb{P}$-a.s.. Equality (35) is implied by (43). Furthermore, since we already proved that the distribution of $U(t, \omega)$ on $H^\alpha$ is the measure $\gamma_t$ (see (43)), it implies, in particular, that the distribution of the initial value $U(0, \omega)$ is the measure $\gamma$.

**Remark 1.** By the Skorohod representation theorem on a given probability space [4], $(\Omega, \mathcal{F}, \mathbb{P})$ is always a space of our choice. In particular, we can choose it to be $(H^\alpha, \mathcal{B}, \gamma)$.

**Remark 2.** On the probability space $(H^\alpha, \mathcal{B}, \gamma)$, we can define the stochastic process $X(t, u) = e^{\nu \Delta u}$. Then, $U(t, \omega)$ and $X(t, u)$ have the same law on $C([0, T], H^\alpha)$ because their finite-dimensional distributions coincide.

**Corollary 2.** Let $l > \frac{d}{2} + \alpha + 2$. Then, there exists a subset $\Omega'' \subset \Omega$ of full $\mathbb{P}$-measure so that the stochastic process $U(t, \omega)$ constructed in Theorem 1 on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ verifies the equation

$$U(t, \omega) = U(0, \omega) + \int_0^t (\nu \Delta U(s, \omega) - B(U(s, \omega))) \, ds \quad (45)$$

for all $t \in [0, T]$ and for all $\omega \in \Omega''$. 

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Proof. We show that if \( l > \frac{d}{2} + \alpha + 2 \), the stochastic process solving (34) verifies (45). Indeed, equation (34) implies that for \( \mathbb{P} \)-almost all \( \omega \), for all \( t \in [0, T] \), and for each \( k \in \mathbb{Z}_+^d \) it holds that

\[
U_k(t, \omega) = U_k(0, \omega) - \int_0^t \left( B^\cos_k(U(s, \omega)) + \nu |k|^2 U_k(s, \omega) \right) ds.
\]

It can be obtained by writing (34) in the coordinate form, and then, by differentiating in \( t \) the both parts of each coordinate equation. To prove that (45) holds in \( C([0, T], H^\alpha) \), it suffices to show that for \( \mathbb{P} \)-almost all \( \omega \), \( \int_0^t \Delta U(s, \omega) \) takes values in \( C([0, T], H^\alpha) \). Indeed,

\[
\left( \mathbb{E}_\mathbb{P} \sup_{t \in [0, T]} \left\| \int_0^t \Delta U(s, \omega) ds \right\|_\alpha \right)^2 \leq \int_0^T ds \mathbb{E}_\mathbb{P} \| \Delta U(s, \omega) \|_\alpha^2
\]

\[
= \int_0^T ds \sum_{k \in \mathbb{Z}_+^d} |k|^{2\alpha+4} \mathbb{E}_\mathbb{P} \left( |U_k(s, \omega)|^2 + |V_k(s, \omega)|^2 \right)
\]

\[
= \int_0^T ds \sum_{k \in \mathbb{Z}_+^d} |k|^{2\alpha+4}e^{-s|k|^2 \nu} \mathbb{E}_\gamma \left( |u_k|^2 + |v_k|^2 \right) \leq T \sum_{k \in \mathbb{Z}_+^d} \frac{1}{|k|^{2l-2\alpha-4}}.
\]

The series on the right-hand side converges since \( l > \frac{d}{2} + \alpha + 2 \). This proves that \( \sup_{t \in [0, T]} \left\| \int_0^t \Delta U(s, \omega) ds \right\|_\alpha \) is finite \( \mathbb{P} \)-a.s. \( \square \)

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