\textbf{G-COALGEBRAS DETERMINE FUNDAMENTAL GROUPS}

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\textbf{ABSTRACT.} In this paper, we extend earlier work by showing that if \(X\) and \(Y\) are simplicial complexes (i.e. simplicial sets whose nondegenerate simplices are determined by their vertices), an isomorphism \(N(X) \cong N(Y)\) of \(G\)-coalgebras implies that the 3-skeleton of \(X\) is weakly equivalent to the 3-skeleton of \(Y\), also implying that \(\pi_1(X) = \pi_1(Y)\).

1. Introduction

In [13], the author defined the functor \(C(\ast)\) on simplicial sets — essentially the chain complex equipped with the structure of a coalgebra over an operad \(G\). This coalgebra structure determined all Steenrod and other cohomology operations. Since these coalgebras are not nilpotent\(^1\) they have a kind of “transcendental” structure that contains much more information.

In section A, we define a variant of the \(C(\ast)\)-functor, named \(N(\ast)\). It is defined for simplicial complexes — semi-simplicial sets whose simplices are uniquely determined by their vertices. The script-N emphasizes that its underlying chain-complex is \textit{normalized} and \(C(\ast)\) can be viewed as an extension of \(N(X)\) to general simplicial sets (see [12]).

In [12], we showed that if \(X\) and \(Y\) are pointed, reduced simplicial sets, then a quasi-isomorphism \(C(X) \to C(Y)\) induces one of their \(Z\)-completions \(\mathbb{Z}_\infty X \to \mathbb{Z}_\infty Y\). It follows that that the \(C(\ast)\)-functor determine a nilpotent space’s weak homotopy type.

In the present paper, we extend this by showing:

Corollary 4.4 If \(X\) and \(Y\) are simplicial complexes with the property that there exists an isomorphism

\[ g : N(X) \to N(Y) \]

\(^1\)In a nilpotent coalgebra, iterated coproducts of elements “peter out” after a finite number of steps. See [10, chapter 3] for the precise definition.
then their 3-skeleta are isomorphic and
\[ \pi_1(X) \cong \pi_1(Y) \]

This implies that the functors \( \mathcal{C}(*) \) and \( \mathcal{N}(*) \) encapsulate “non-abelian” information about a simplicial set — such as its (possibly non-nilpotent) fundamental group. The requirement that \( g \) be an isomorphism is stronger than needed for this (but quasi-isomorphism is not enough).

The proof actually requires \( X \) and \( Y \) to be simplicial complexes rather than general simplicial sets. The author conjectures that the \( \mathcal{C}(*) \)-functor determines the integral homotopy type of an arbitrary simplicial set.

Since the transcendental portion of \( \mathcal{C}(X) \) can be mapped to a power series ring (see the proof of lemma B.1), the analysis of this data may require methods of analysis and algebraic geometry.

2. DEFINITIONS AND ASSUMPTIONS

Definition 2.1. Let \( \text{Ch} \) denote the category of \( \mathbb{Z} \)-graded \( \mathbb{Z} \)-free chain complexes and let \( \text{Ch}_0 \subset \text{Ch} \) denote the subcategory of chain complexes concentrated in positive dimensions.

If \( c \in \text{Ch} \),
\[ C \otimes^n = C \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} C \]

We also have categories of spaces:

Definition 2.2. Let \( \text{SS} \) denote the category of simplicial sets and \( \text{SC} \) that of simplicial complexes. A simplicial complex is a simplicial set without degeneracies (i.e., a semi-simplicial set) with the property that simplices are uniquely determined by their vertices.

Remark. Following \[11]\), we can define a simplicial set to have \textit{Property A} if every face of a nondegenerate simplex is nondegenerate. Theorem 12.4.4 of \[11]\) proves that simplicial sets with property A have \textit{second subdivisions} that are simplicial complexes. The bar-resolution \( R S_2 \) is an example of a simplicial set that does not have property A.

On the other hand, it is well-known that \textit{all} topological spaces are weakly homotopy equivalent to simplicial complexes — see, for example, Theorem 2C.5 and Proposition 4.13 of \[?\].

We make extensive use of the Koszul Convention (see \[8]\)) regarding signs in homological calculations:
Definition 2.3. If \( f: C_1 \to D_1, \ g: C_2 \to D_2 \) are maps, and \( a \otimes b \in C_1 \otimes C_2 \) (where \( a \) is a homogeneous element), then \((f \otimes g)(a \otimes b)\) is defined to be \((-1)^{\deg(g) \cdot \deg(a)} f(a) \otimes g(b)\).

Remark 2.4. If \( f_i, g_i \) are maps, it isn’t hard to verify that the Koszul convention implies that \((f_1 \otimes g_1) \circ (f_2 \otimes g_2) = (-1)^{\deg(f_2) \cdot \deg(g_1)} (f_1 \circ f_2 \otimes g_1 \circ g_2)\).

Definition 2.5. Given chain-complexes \( A, B \in \text{Ch} \) define
\[
\text{Hom}_Z(A, B)
\]
to be the chain-complex of graded \( Z \)-morphisms where the degree of an element \( x \in \text{Hom}_Z(A, B) \) is its degree as a map and with differential
\[
\partial f = f \circ \partial_A - (-1)^{\deg f} \partial_B \circ f
\]
As a \( Z \)-module \( \text{Hom}_Z(A, B)_k = \prod_j \text{Hom}_Z(A_j, B_{j+k}) \).

Remark. Given \( A, B \in \text{Ch}^{S_n} \), we can define \( \text{Hom}_{ZS_n}(A, B) \) in a corresponding way.

Definition 2.6. Let \( \alpha_i, i = 1, \ldots, n \) be a sequence of nonnegative integers whose sum is \( |\alpha| \). Define a set-mapping of symmetric groups
\[
T_{\alpha_1, \ldots, \alpha_n}: S_n \to S_{|\alpha|}
\]
as follows:

(1) for \( i \) between 1 and \( n \), let \( L_i \) denote the length-\( \alpha_i \) integer sequence:

(2) where \( A_i = \sum_{j=1}^{i-1} \alpha_j \) — so, for instance, the concatenation of all of the \( L_i \) is the sequence of integers from 1 to \( |\alpha| \);

(3) \( T_{\alpha_1, \ldots, \alpha_n}(\sigma) \) is the permutation on the integers 1, \ldots, \( |\alpha| \) that permutes the blocks \( \{L_i\} \) via \( \sigma \). In other words, \( \sigma \) s the permutation
\[
\begin{pmatrix}
1 & \ldots & n \\
\sigma(1) & \ldots & \sigma(n)
\end{pmatrix}
\]
then \( T_{\alpha_1, \ldots, \alpha_n}(\sigma) \) is the permutation defined by writing
\[
\begin{pmatrix}
L_1 & \ldots & L_n \\
L_{\sigma(1)} & \ldots & L_{\sigma(n)}
\end{pmatrix}
\]
and regarding the upper and lower rows as sequences length \( |\alpha| \).

Remark. Do not confuse the \( T \)-maps defined here with the transposition map for tensor products of chain-complexes. We will use the special notation \( T_i \) to represent \( T_{1, \ldots, 2, \ldots, 1} \), where the 2 occurs...
in the \(i^{th}\) position. The two notations don’t conflict since the old notation is never used in the case when \(n = 1\). Here is an example of the computation of \(T_{2,1,3}((1, 3, 2)) = T_{2,1,3} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}: L_1 = \{1\}, L_2 = \{3\}, L_3 = \{4, 5, 6\}. The permutation maps the ordered set \(\{1, 2, 3\}\) to \(\{3, 1, 2\}\), so we carry out the corresponding mapping of the sequences \(\{L_1, L_2, L_3\}\) to get \( \left( \begin{array}{ccc} L_1 & L_2 & L_3 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{array} \right) \) (or \((1, 4)(2, 5)(3, 6))\), in cycle notation).

**Definition 2.7.** A sequence of differential graded \(\mathbb{Z}\)-free modules, \(\{\mathcal{V}_i\}\), will be said to form an *operad* if:

1. there exists a *unit map* (defined by the commutative diagrams below)
   \[ \eta: \mathbb{Z} \rightarrow \mathcal{V}_1 \]
2. for all \(i > 1\), \(\mathcal{V}_i\) is equipped with a left action of \(S_i\), the symmetric group.
3. for all \(k \geq 1\), and \(i_s \geq 0\) there are maps
   \[ \gamma: \mathcal{V}_{i_1} \otimes \cdots \otimes \mathcal{V}_{i_k} \otimes \mathcal{V}_k \rightarrow \mathcal{V}_i \]
   where \(i = \sum_{j=1}^{k} i_j\).

The \(\gamma\)-maps must satisfy the conditions:

**Associativity:** the following diagrams commute, where \(\sum j_t = j\), \(\sum i_s = i\), and \(g_\alpha = \sum_{t=1}^{j} j_\ell\) and \(h_\beta = \sum_{s} i_\gamma + 1\):

\[
\begin{array}{ccc}
\bigotimes_{s=1}^{i} \mathcal{V}_{i_s} & \otimes & \bigotimes_{t=1}^{k} \mathcal{V}_{j_t} \\
\downarrow \text{shuffle} & & \downarrow \gamma \\
\bigotimes_{t=1}^{j} \mathcal{V}_{j_t} & \otimes & \mathcal{V}_k \\
\end{array}
\]

\[
\begin{array}{ccc}
\bigotimes_{i=1}^{k} \mathcal{V}_i \\
\downarrow \gamma \\
\bigotimes_{t=1}^{j} \mathcal{V}_{j_t} & \otimes & \mathcal{V}_k \\
\end{array}
\]

**Units:** the diagrams

\[
\begin{array}{ccc}
\mathbb{Z}^k & \otimes & \mathcal{V}_k \\
\eta^k \otimes \text{Id} & \downarrow \gamma & \downarrow \gamma \\
\mathcal{V}_1^k & \otimes & \mathcal{V}_k \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{V}_k & \otimes & \mathbb{Z} \\
\text{Id} \otimes \eta & \downarrow \gamma & \downarrow \gamma \\
\mathcal{V}_k & \otimes & \mathcal{V}_1 \\
\end{array}
\]

commute.
Equivariance: the diagrams

\[
\begin{array}{c}
V_{j_1} \otimes \cdots \otimes V_{j_k} \otimes V_k \\
\downarrow \sigma^{-1} \otimes \sigma
\end{array} \quad \gamma \quad \begin{array}{c}
V_{j_1} \otimes \cdots \otimes V_{j_k} \otimes V_k \\
\downarrow T_{j_1, \ldots, j_k}(\sigma)
\end{array}
\]

commute, where \( \sigma \in S_k \), and the \( \sigma^{-1} \) on the left permutes the factors \( \{V_{j_i}\} \) and the \( \sigma \) on the right simply acts on \( V_k \). See [2.6] for a definition of \( T_{j_1, \ldots, j_k}(\sigma) \).

\[
\begin{array}{c}
V_{j_1} \otimes \cdots \otimes V_{j_k} \otimes V_k \\
\downarrow \tau_1 \otimes \cdots \otimes \tau_k \otimes \text{Id}
\end{array} \quad \gamma \quad \begin{array}{c}
V_{j_1} \otimes \cdots \otimes V_{j_k} \otimes V_k \\
\downarrow \tau_1 \otimes \cdots \otimes \tau_k
\end{array}
\]

where \( \tau_s \in S_{j_s} \) and \( \tau_1 \oplus \cdots \oplus \tau_k \in S_j \) is the block sum.

Remark. The alert reader will notice a discrepancy between our definition of operad and that in [9] (on which it was based). The difference is due to our using operads as parameters for systems of maps, rather than \( n \)-ary operations. We, consequently, compose elements of an operad as one composes maps, i.e. the second operand is to the \( \text{left} \) of the first. This is also why the symmetric groups act on the \( \text{left} \) rather than on the right.

Definition 2.8. An operad, \( \mathcal{V} \), will be called \textit{unital} if \( \mathcal{V} \) has a 0-component \( \mathcal{V}_0 = \mathbb{Z} \), concentrated in dimension 0 and augmentations

\[
\epsilon_n: \mathcal{V}_0 \otimes \cdots \otimes \mathcal{V}_0 \otimes \mathcal{V}_n = \mathcal{V}_n \rightarrow \mathcal{V}_0 = \mathbb{Z}
\]

induced by their structure maps.

Remark. The term “unital operad” is used in different ways by different authors. We use it in the sense of Kriz and May in [9], meaning the operad has a 0-component that acts like an arity-lowering augmentation under compositions.

We will frequently want to think of operads in other terms:

Definition 2.9. Let \( \mathcal{V} \) be an operad, as defined above. Given \( i \leq k_1 > 0 \), define the \( i^{\text{th}} \) composition

\[
\circ_i: \mathcal{V}_{k_2} \otimes \mathcal{V}_{k_1} \rightarrow \mathcal{V}_{k_1 + k_2 - 1}
\]
as the composite
\[
\begin{array}{c}
Z \otimes \cdots \otimes Z \otimes V_{k_2} \otimes Z \otimes \cdots \otimes Z \otimes V_{k_1}
\end{array}
\]
\[
\xrightarrow{i\text{th factor}}
\begin{array}{c}
V_1 \otimes \cdots \otimes V_1 \otimes V_{k_2} \otimes V_1 \otimes \cdots \otimes V_1 \otimes V_{k_1} \rightarrow V_{k_1+k_2-1}
\end{array}
\]
where the final map on the right is $\gamma$.

These compositions satisfy the following conditions, for all $a \in \mathcal{U}_n$, $b \in \mathcal{U}_m$, and $c \in \mathcal{U}_i$:

- **Associativity:** $(a \circ_i b) \circ_j c = a \circ_{i+j-1} (b \circ_j c)$
- **Commutativity:** $a \circ_{i+m-1} (b \circ_j c) = (-1)^{mn} b \circ_j (a \circ_i c)$
- **Equivariance:** $a \circ_{\sigma(i)} (\sigma \cdot b) = T_{\sigma_1,\ldots,\sigma_n,1}(\sigma) \cdot (a \circ_i b)$

**Remark.** I am indebted to Jim Stasheff for pointing out to me that operads were originally defined this way and called composition algebras. Given this definition of operad, we recover the $\gamma$ map in definition 2.7 by setting:
\[
\gamma(u_{i_1} \otimes \cdots \otimes u_{i_k} \otimes u_k) = u_{i_1} \circ_1 \cdots \circ_{k-1} u_{i_k} \circ_k u_k
\]
(where the implied parentheses associate to the right). It is left to the reader to verify that the two definitions are equivalent (the commutativity condition, here, is a special case of the equivariance condition).

Given a unital operad, we can use the augmentation maps to recover the composition operations.

A simple example of an operad is:

**Example 2.10.** For each $n \geq 0$, $X$, the operad $\mathcal{G}_n$ has $\mathcal{G}_0(n) = \mathbb{Z} S_n$, concentrated in dimension 0, with structure-map induced by
\[
\gamma_{\alpha_1,\ldots,\alpha_n} : S_{\alpha_1} \times \cdots \times S_{\alpha_n} \times S_n \rightarrow S_{\alpha_1+\cdots+\alpha_n}
\]
\[
\sigma_{\alpha_1} \times \cdots \times \sigma_{\alpha_n} \times \sigma_n \mapsto T_{\alpha_1,\ldots,\alpha_n}(\sigma_n) \circ (\sigma_{\alpha_1} \oplus \cdots \oplus \sigma_{\alpha_n})
\]
In other words, each of the $S_{\alpha_i}$ permutes elements within the subsequence $\{\alpha_1 + \cdots + \alpha_{i-1} + 1, \ldots, \alpha_1 + \cdots + \alpha_i\}$ of the sequence $\{1, \ldots, \alpha_1 + \cdots + \alpha_n\}$ and $S_n$ permutes these $n$ blocks.

For the purposes of this paper, the main example of an operad is

**Definition 2.11.** Given any $C \in \text{Ch}$, the associated coendomorphism operad, $\text{CoEnd}(C)$ is defined by
\[
\text{CoEnd}(C)(n) = \text{Hom}_\mathbb{Z}(C, C^{\otimes n})
\]
Its structure map
\[ \gamma_{\alpha_1, \ldots, \alpha_n} : \text{Hom}_\mathbb{Z}(C, C^\otimes n) \otimes \text{Hom}_\mathbb{Z}(C, C^\otimes \alpha_1) \otimes \cdots \otimes \text{Hom}_\mathbb{Z}(C, C^\otimes \alpha_n) \to \text{Hom}_\mathbb{Z}(C, C^\otimes \alpha_1 + \cdots + \alpha_n) \]

simply composes a map in \( \text{Hom}_\mathbb{Z}(C, C^\otimes n) \) with maps of each of the \( n \) factors of \( C \).

This is a non-unital operad, but if \( C \in \text{Ch} \) has an augmentation map \( \varepsilon : C \to \mathbb{Z} \) then we can regard \( \varepsilon \) as the only element of \( \text{Hom}_\mathbb{Z}(C, C^\otimes n) = \text{Hom}_\mathbb{Z}(C, C^\otimes 0) = \text{Hom}_\mathbb{Z}(C, \mathbb{Z}) \).

Morphisms of operads are defined in the obvious way:

**Definition 2.12.** Given two operads \( \mathcal{V} \) and \( \mathcal{W} \), a *morphism* 
\[ f : \mathcal{V} \to \mathcal{W} \]

is a sequence of chain-maps 
\[ f_i : \mathcal{V}_i \to \mathcal{W}_i \]

commuting with all the diagrams in 2.7.

Verification that this satisfies the required identities is left to the reader as an exercise.

**Definition 2.13.** Let \( \mathcal{S} \) denote the operad defined in [13] — where \( \mathcal{S}_n = R \mathcal{S}_n \) is the normalized bar-resolution of \( \mathbb{Z} \) over \( \mathbb{Z} \mathcal{S}_n \) for all \( n > 0 \). This is similar to the Barratt-Eccles operad defined in [1], except that the latter is composed of *unnormalized* bar-resolutions. See [13] or appendix A of [12], for the details.

Appendix A of [12] contains explicit computations of some composition-operations in \( \mathcal{S} \).

Now we are ready to define the all-important concept of coalgebras over an operad:

**Definition 2.14.** A chain-complex \( C \) is a *coalgebra over the operad* \( \mathcal{V} \) if there exists a morphism of operads 
\[ \mathcal{V} \to \text{CoEnd}(C) \]

**Remark.** A coalgebra, \( C \), over an operad, \( \mathcal{V} \), is a sequence of maps 
\[ f_n : \mathcal{V}_n \otimes C \to C^\otimes n \]

for all \( n > 0 \), where \( f_n \) is \( \mathbb{Z} \mathcal{S}_n \)-equivariant and \( \mathcal{S}_n \) acts by permuting factors of \( C^\otimes n \). The maps, \( \{ f_n \} \), are related in the sense that they fit
into commutative diagrams:

\[
\begin{array}{cccccc}
V_n \otimes V_m \otimes C & \xrightarrow{\phi_{n,m}} & V_{n+m-1} \otimes C & \xrightarrow{f_{n+m-1}} & C^\otimes n+m-1 \\
V_n \otimes V_m \otimes C & \xrightarrow{1 \otimes \cdots \otimes f_n} & V_n \otimes C^\otimes m & \xrightarrow{Z_{i-1}} & V_n \otimes C \otimes C^\otimes m-i
\end{array}
\]

for all \( n, m \geq 1 \) and \( 1 \leq i \leq m \). Here \( Z_{i-1} : V_n \otimes C^m \to C^\otimes i-1 \otimes V_n \otimes C \otimes C^\otimes m-i \) is the map that shuffles the factor \( V_n \) to the right of \( i-1 \) factors of \( C \). In other words: The abstract composition-operations in \( V \) exactly correspond to compositions of maps in \( \{ \text{Hom}_\mathbb{Z}(C, C^\otimes n) \} \).

We exploit this behavior in applications of coalgebras over operads, using an explicit knowledge of the algebraic structure of \( V \).

The structure of a coalgebra over an operad can also be described in several equivalent ways:

1. \( f_n : V(n) \otimes C \to C^\otimes n \)
2. \( g : C \to \prod_{n=0}^\infty \text{Hom}_{\mathbb{Z}S_n}(V(n), C^\otimes n) \)

where both satisfy identities that describe how composites of these maps are compatible with the operad-structure.

**Definition 2.15.** A chain-complex \( C \) is a coalgebra over the operad \( V \) if there exists a morphism of operads

\[ V \to \text{CoEnd}(C) \]

**Remark.** The structure of a coalgebra over an operad can be described in several equivalent ways:

1. \( f_n : V(n) \otimes C \to C^\otimes n \)
2. \( g : C \to \prod_{n=0}^\infty \text{Hom}_{\mathbb{Z}S_n}(V(n), C^\otimes n) \)

where both satisfy identities that describe how composites of these maps are compatible with the operad-structure.

**Definition 2.16.** Using the second description,

\[ \alpha_C : C \to \prod_{n=1}^\infty \text{Hom}_{\mathbb{Z}S_n}(RS_n, C^\otimes n) \]

an \( S \)-coalgebra morphism

\[ f : C \to D \]

is a chain-map that makes the diagram

\[
\begin{array}{ccc}
[C] & \xrightarrow{\alpha_C} & \prod_{n=1}^\infty \text{Hom}_{\mathbb{Z}S_n}(RS_n, [C]^\otimes n) \\
[f] & & \downarrow \prod_{n=1}^\infty \text{Hom}_{\mathbb{Z}S_n}(1, [f]^\otimes n) \\
[D] & \xrightarrow{\alpha_D} & \prod_{n=1}^\infty \text{Hom}_{\mathbb{Z}S_n}(RS_n, [D]^\otimes n)
\end{array}
\]
commute, where \([\ast]\) is the forgetful functor that turns a coalgebra into a chain-complex.

We also need

3. MORPHISMS OF \(\mathcal{S}\)-COALGEBRAS

Proposition A.8 proves that if \(e_n = \underbrace{(1, 2) \cdots (1, 2)}_{n \text{ terms}} \in R S_2\) and \(x \in N(X)\) is the image of a \(k\)-simplex, then

\[
f_2(e_k \otimes x) = \xi_k \cdot x \otimes x
\]

where \(\xi_k = (-1)^{k(k-1)/2}\).

**Definition 3.1.** If \(k, m\) are positive integers, \(C\) is a chain-complex, and \(F_{2,m} = e_m\) and \(F_{k,m} = e_m \circ_1 \cdots \circ_1 e_m \in RS_k\) — compositions in the operad \(\mathcal{S}\) — set

\[
\rho_m = (\xi_m \cdot E_{2,m}, \xi_m^2 \cdot E_{3,m}, \xi_m^3 \cdot E_{4,m}, \ldots) \in \prod_{n=2}^{\infty} R S_n
\]

with \(\xi_m = (-1)^{m(m-1)/2}\) and define

\[
\gamma_m: \prod_{n=2}^{\infty} \hom_{\mathcal{Z}S_n}(RS_n, C \otimes^n) \to \prod_{n=2}^{\infty} C \otimes^n
\]

via evaluation on \(\rho_m\).

We have

**Corollary 3.2.** If \(X\) is a simplicial set and \(c \in C(X)\) is an element generated by an \(n\)-simplex, then the image of \(c\) under the composite

\[
N(X)_n \xrightarrow{\alpha e(X)} \prod_{k=1}^{\infty} \hom_{\mathcal{Z}S_k}(RS_k, N(X) \otimes^k) \xrightarrow{\gamma_n} \prod_{k=1}^{\infty} N(X) \otimes^k
\]

is

\[
e(c) = (c, c \otimes c, \ldots)
\]

**Proof.** This follows immediately from proposition A.8 and the fact that operad-compositions map to compositions of coproducts. \(\square\)

Lemma B.1 implies that

**Corollary 3.3.** Let \(X\) be a simplicial set and suppose

\[
f: N^n = N(\Delta^n) \to N(X)
\]

is a \(\mathcal{S}\)-coalgebra morphism. Then the image of the generator \(\Delta^n \in N(\Delta^n)\) is a generator of \(N(X)\) defined by an \(n\)-simplex of \(X\).
Proof. Suppose

\[ f(\Delta^n) = \sum_{k=1}^{t} c_k \cdot \sigma^n_k \in N(X) \]

where the \( \sigma^n_k \) are images of \( n \)-simplices of \( X \). If \( f(\Delta^n) \) is not equal to one of the \( \sigma^n_k \), lemma B.1 implies that its image is linearly independent of the \( \sigma^n_k \), a contradiction. The statement about sub-simplices follows from the main statement.

We also conclude that:

Corollary 3.4. If \( f : N(\Delta^n) \to N(\Delta^n) \) is

1. an isomorphism of \( \mathcal{S} \)-algebras in dimension \( n \) and
2. an endomorphism in lower dimensions

then \( f \) must be an isomorphism. If \( n \leq 3 \), then \( f \) must also be the identity map.

Remark. The final statement actually works for some larger values of \( n \), but the arguments become vastly more complicated (requiring the use of higher coproducts). It would have extraordinary implications if it were true for all \( n \).

Proof. We first show that \( f \) must be an isomorphism. We are given that \( f \) is an isomorphism in dimension \( n \). We use downward induction on dimension to show that it is an isomorphism in lower dimensions:

Suppose \( f \) is an isomorphism in dimension \( k \) and \( \Delta^k \subset \Delta^n \) is a simplex. The boundary of \( \Delta^k \) is a linear combination of \( k + 1 \) distinct faces which corollary B.3 implies must map to \( k - 1 \)-dimensional simplices with the same coefficients (of \( \pm 1 \)). The Pigeonhole Principal and the fact that \( f \) is a chain-map imply that all of the \( k + 1 \) distinct faces of \( f(\Delta^k) \) must be in the image of \( f \) so that \( f \) induces a 1-1 correspondence between faces of \( \Delta^k \) and those of \( f(\Delta^k) \). It follows that \( f|N(\Delta^k) \) is an isomorphism in dimension \( k - 1 \). Since \( \Delta^k \) was arbitrary, it follows that \( f \) is an isomorphism in dimension \( k - 1 \).

It follows that \( f \) is actually an automorphism of \( N(\Delta^n) \). Now we assume that \( n \leq 3 \) and show that \( f \) is the identity map:

If \( n = 1 \) then corollary 3.3 implies that \( f|N(\Delta^1) \) is 1. Since the \( 0 \)-simplices must map to \( 0 \)-simplices (by corollary 3.3) with a \( +1 \) sign it follows that the only possible non-identity automorphism of \( N(\Delta^1) \) swaps the ends of \( \Delta^1 \) — but this would violate the condition that \( f \) is a chain-map.

In dimension 2, let \( \Delta^2 \) be a 2-simplex. Similar reasoning to that used in the one-dimensional case implies that a non-identity automorphism of \( N(\Delta^2) \) would (at most) involve permuting some of its
faces. Since
\[ \partial \Delta^2 = F_0 \Delta^2 - F_1 \Delta^2 + F_2 \Delta^2 \]
the only non-identity permutation compatible with the boundary map
swaps \( F_0 \Delta^2 \) and \( F_2 \Delta^2 \). But the coproduct of \( \Delta^2 \) is given by
\[ | |_2 \otimes \Delta^2 \mapsto \Delta^2 \otimes F_0 F_1 \Delta^2 + F_2 \Delta^2 \otimes F_0 \Delta^2 + F_1 F_2 \Delta^2 \otimes \Delta^2 \]
(see proposition 4.6) where \(| | \) is the 0-dimensional generator of \( R S_2 \)
— the bar-resolution of \( \mathbb{Z} \) over \( \mathbb{Z} S_2 \). It follows that swapping
\( F_0 \Delta^2 \) and \( F_2 \Delta^2 \) would violate the condition that \( f \) must preserve coproducts. The case where \( n = 1 \) implies that the vertices cannot be permuted.

When \( n = 3 \), we have
\[ \partial \Delta^3 = F_0 \Delta^3 - F_1 \Delta^3 + F_2 \Delta^3 - F_3 \Delta^3 \]
so, in principal, we might be able to swap \( F_0 \Delta^3 \) and \( F_2 \Delta^3 \) or \( F_1 \Delta^3 \) and \( F_3 \Delta^3 \). The coproduct does not rule any of these actions out since it
involves multiple face-operations. The first “higher coproduct” does, however — see 4.7:
\[
\begin{align*}
  f_2([[(1, 2)] \otimes \Delta^3]) &= F_1 F_2 \Delta^3 \otimes \Delta^3 - F_2 \Delta^3 \otimes F_0 \Delta^3 \\
  &+ \Delta^3 \otimes F_0 F_1 \Delta^3 - \Delta^3 \otimes F_0 F_3 \Delta^3 \\
  &- F_1 \Delta^3 \otimes F_3 \Delta^3 - \Delta^3 \otimes F_2 F_3 \Delta^3
\end{align*}
\]
(3.1)
The two terms with two-dimensional factors are \(- F_2 \Delta^3 \otimes F_0 \Delta^3 \) and
\(- F_1 \Delta^3 \otimes F_3 \Delta^3 \) and these would be altered by the permutation mentioned above. It follows that the only automorphism of \( N(\Delta^3) \) is the
identity map. The lower-dimensional cases imply that the 1-simplices
and vertices cannot be permuted either. \( \square \)

A similar line of reasoning implies that:

**Corollary 3.5.** Let \( X \) be a simplicial complex and let
\[ f: N(\Delta^n) \to N(X) \]
map \( \Delta^n \) to a simplex \( \sigma \in \mathcal{C}(X) \) defined by the inclusion \( \iota: \Delta^n \to X. \)
Then
\[ f(N(\Delta^n)) \subset N(\iota)(N(\Delta^n)) \]
so that \( f = \alpha \circ N(\iota) \), where \( \alpha: N(\Delta^n) \to N(\Delta^n) \) is an automorphism. If \( n \leq 3 \), then \( f = N(\iota) \).

**Proof.** Since \( X \) is a simplicial complex, the map \( \iota \) is an inclusion.
Suppose \( \Delta^k \subset \Delta^n \) and \( f(N(\Delta^k))_k \subset N(\Delta^k)_k. \) Since the boundary
of \( \Delta^k \) is an alternating sum of \( k + 1 \) faces, and since they must map to
\( k - 1 \)-dimensional simplices of \( N(f(\Delta^k)) \) with the same signs (so no
cancellations can take place) we must have \( f(F_k) \subseteq N(f(\Delta^k)) \) and the conclusion follows by downward induction on dimension. The final statements follow immediately from corollary \( \text{[3.4]} \).

4. THE FUNCTOR \( \text{hom}_n(\star, \ast) \)

We define a complement to the \( N(\ast) \)-functor:

**Definition 4.1.** Define a functor
\[
\text{hom}_n(\star, \ast): \mathcal{I}_0 \to \text{SS}
\]
to the category of semi-simplicial sets, as follows:

If \( C \in \mathcal{I}_0 \), define the \( n \)-simplices of \( \text{hom}_n(\star, C) \) to be the \( \mathcal{S} \)-coalgebra morphisms
\[
N^\ast \to C
\]
where \( N^\ast = N(\Delta^n) \) is the normalized chain-complex of the standard \( n \)-simplex, equipped with the \( \mathcal{S} \)-coalgebra structure defined in theorem \( \text{[A.3]} \).

Face-operations are duals of coface-operations
\[
d_i: [0, \ldots, i, i+1, \ldots n] \to [0, \ldots, n]
\]
with \( i = 0, \ldots, n \) and vertex \( i \) in the target is not in the image of \( d_i \).

**Remark.** Compare this to the functor \( \text{hom}(\star, \ast) \) defined in \( \text{[12]} \). The subscript \( n \) emphasizes that we do not take degeneracies into account.

**Proposition 4.2.** If \( X \) is a simplicial complex (i.e., its simplices are uniquely determined by their vertices) there exists a natural map
\[
u_X: X \to \text{hom}_n(\star, N(X))
\]

**Proof.** To prove the first statement, note that any simplex \( \Delta^k \) in \( X \) comes equipped with a canonical inclusion
\[
\iota: \Delta^k \to X
\]
The corresponding order-preserving map of vertices induces an \( \mathcal{S} \)-coalgebra morphism
\[
N(\iota): N(\Delta^k) = N^k \to N(X)
\]
so \( u_X \) is defined by
\[
\Delta^k \mapsto N(\iota)
\]
It is not hard to see that this operation respects face-operations.

**Theorem 4.3.** If \( X \in \text{SC} \) is a simplicial complex then the canonical map
\[
u_X: X \to \text{hom}_n(\star, N(X))
\]
defined in proposition \( \text{[4.2]} \) is an isomorphism of the 3-skeleton.
Proof. This follows immediately from corollary 3.3 which implies that simplices map to simplices and corollary 3.5 which implies that these maps are unique.

□

Corollary 4.4. If $X$ and $Y$ are simplicial complexes with the property that there exists an isomorphism

$$
N(X) \rightarrow N(Y)
$$

then their 3-skeleta are weakly equivalent and

$$
\pi_1(X) \cong \pi_1(Y)
$$

Proof. Any morphism $g: N(X) \rightarrow N(Y)$ induces a morphism of simplicial sets

$$
\text{hom}(\star, g): \text{hom}_n(\star, N(X)) \rightarrow \text{hom}_n(\star, N(Y))
$$

and this is an isomorphism (and homeomorphism) of simplicial complexes if $g$ is an isomorphism. The conclusion follows from theorem 4.3 which implies that the canonical maps

$$
u_X: X \rightarrow \text{hom}_n(\star, N(X))$$
$$
u_Y: Y \rightarrow \text{hom}_n(\star, N(Y))$$

are isomorphisms of the 3-skeleta, and the fact that fundamental groups are determined by the 2-skeleta. □

APPENDIX A. THE FUNCTOR $N(\star)$

We begin with the elementary but powerful Cartan Theory of Constructions, originally described in [5, 2, 3, 6]:

Lemma A.1. Let $M_i$, $i = 1, 2$ be DGA-modules, where:

1. $M_1 = A_1 \otimes N_1$, where $N_1$ is $\mathbb{Z}$-free and $A_1$ is a DGA-algebra (so $M_1$, merely regarded as a DGA-algebra, is free on a basis equal to a $\mathbb{Z}$-basis of $N_1$)

2. $M_2$ is a left DGA-module over a DGA-algebra $A_2$, possessing
   (a) a sub-DG-module, $N_2 \subseteq M_2$, such that $\partial_{M_2}|N_2$ is injective,
   (b) a contracting chain-homotopy $\varphi: M_2 \rightarrow M_2$ whose image lies in $N_2 \subseteq M_2$.

Suppose we are given a chain-map $f_0: M_1 \rightarrow M_2$ in dimension 0 with $f_0(N_1) \subseteq N_2$ and want to extend it to a chain-map from $M_1$ to $M_2$, subject to the conditions:

- $f(N_1) \subseteq N_2$
- $f(a \otimes n) = g(a) \cdot f(n)$, where $g: A_1 \rightarrow A_2$ is some morphism of DG-modules such that $a \otimes n \rightarrow g(a) \cdot f(n)$ is a chain-map.
Then the extension \( f: M_1 \to M_2 \) exists and is unique.

**Remark.** In applications of this result, the morphism \( g \) will often be a morphism of DGA-algebras, but this is not necessary.

The existence of \( f \) immediately follows from basic homological algebra; the interesting aspect of it is its uniqueness (not merely uniqueness up to a chain-homotopy). We will use it repeatedly to prove associativity conditions by showing that two apparently different maps satisfying the hypotheses must be identical.

The Theory of Constructions formed the cornerstone of Henri Cartan’s elegant computations of the homology and cohomology of Eilenberg-MacLane spaces in [4].

**Proof.** The uniqueness of \( f \) follows by induction and the facts that:

1. \( f \) is determined by its values on \( N_1 \)
2. the image of the contracting chain-homotopy, \( \varphi \), lies in \( N_2 \subset M_2 \).
3. the boundary map of \( M_2 \) is injective on \( N_2 \) (which implies that there is a unique lift of \( f \) into the next higher dimension).

□

Now construct a contracting cochain on the normalized chain-complex of a standard simplex:

**Definition A.2.** Let \( \Delta^k \) be a standard \( k \)-simplex with vertices \( \{[0], \ldots, [k]\} \) and \( j \)-faces \( \{[i_0, \ldots, i_j]\} \) with \( i_0 < \cdots < i_j \) and let \( s^k \) denote its normalized chain-complex with boundary map \( \partial \). This is equipped with an augmentation

\[
\epsilon: s^k \to \mathbb{Z}
\]

that maps all vertices to \( 1 \in \mathbb{Z} \) and all other simplices to 0. Let

\[
\iota_k: \mathbb{Z} \to s^k
\]

denote the map sending \( 1 \in \mathbb{Z} \) to the image of the vertex \([n]\). Then we have a contracting cochain

\[
\varphi_k([i_0, \ldots, i_t]) = \begin{cases} 
(-1)^{t+1}[i_0, \ldots, i_t, k] & \text{if } i_t \neq k \\
0 & \text{if } i_t = k 
\end{cases}
\]

and \( 1 - \iota_k \circ \epsilon = \partial \circ \varphi_k + \varphi_k \circ \partial \).

**Theorem A.3.** The normalized chain-complex of \([i_0, \ldots, i_k] = \Delta^k \) has a \( \mathcal{S} \)-coalgebra structure that is natural with respect to order-preserving mappings of vertex-sets

\[
[i_0, \ldots, i_k] \to [j_0, \ldots, j_k]
\]
with \( j_0 \leq \cdots \leq j_\ell \) and \( \ell \geq k \). This \( \mathcal{G} \)-coalgebra is denoted \( \mathcal{N}^k \).

If \( X \) is a simplicial complex (a semi-simplicial set whose simplices are uniquely determined by their vertices), then the normalized chain-complex of \( X \) has a natural \( \mathcal{G} \)-coalgebra structure
\[
\mathcal{N}(X) = \lim_{\longrightarrow} \mathcal{N}^k
\]
for \( \Delta^n \in \Delta \downarrow X \) — the simplex category of \( X \).

**Remark.** The author has a Common LISP program for computing \( f_n(x \otimes C(\Delta^k)) \) — the number of terms is exponential in the dimension of \( x \).

Compare this with the functor \( C(\cdot) \) defined in [13] and [12]. For simplicial complexes, \( C(X) = \mathcal{N}(X) \).

**Proof.** If \( C = s^k = C(\Delta^k) \) — the (unnormalized) chain complex — we can define a corresponding contracting homotopy on \( C \otimes^\mathcal{G} \) via
\[
\Phi = \varphi_k \otimes 1 \otimes \cdots \otimes 1 + \iota_k \circ \epsilon \otimes \varphi_k \otimes \cdots \otimes 1 + \\
\cdots + \iota_k \circ \epsilon \otimes \cdots \otimes \iota_k \circ \epsilon \otimes \varphi_k
\]
where \( \varphi_k, \iota_k, \) and \( \epsilon \) are as in definition [A.2]. Above dimension 0, \( \Phi \) is effectively equal to \( \varphi_k \otimes 1 \otimes \cdots \otimes 1 \). Now set \( M_2 = C \otimes^\mathcal{G} \) and \( N_2 = \text{im}(\Phi) \). In dimension 0, we define \( f_n \) for all \( n \) via:
\[
f_n(A \otimes [0]) = \begin{cases} 
[0] \otimes \cdots \otimes [0] & \text{if } A = [] \\
0 & \text{if } \dim A > 0
\end{cases}
\]
This clearly makes \( s^0 \) a coalgebra over \( \mathcal{G} \).

Suppose that the \( f_n \) are defined below dimension \( k \). Then \( \mathcal{C}(\partial \Delta^k) \) is well-defined and satisfies the conclusions of this theorem. We define \( f_n(a[a_1|\ldots|a_j] \otimes [0,\ldots,k]) \) by induction on \( j \), requiring that:

**Condition A.4 (Invariant Condition).**
\[
(A.2) \quad f_n(A(S_n, 1) \otimes s^k) \subseteq [i_1, \ldots, k] \otimes \text{other terms}
\]
— in other words, the leftmost factor must be in \( \text{im} \varphi_k \). This is the same as the leftmost factors being “rearward” faces of \( \Delta^k \).

Now we set
\[
f_n(A \otimes s^k) = \Phi \circ f_n(\partial A \otimes s^k) + (-1)^{\dim A} \Phi \circ f_n(A \otimes \partial s^k)
\]
where \( A \in A(S_n, 1) \subseteq R S_n \) and the term \( f_n(A \otimes \partial s^k) \) refers to the coalgebra structure of \( \mathcal{C}(\partial \Delta^k) \).

The term \( f_n(A \otimes \partial s^k) \) is defined by induction and diagram [2.1] commutes for it. The term \( f_n(\partial A \otimes s^k) \) is defined by induction on the dimension of \( A \) and diagram [2.1] for it as well.
The composite maps in both branches of diagram 2.1 satisfy condition [A.4] since:

1. any composite of \( f_n \)-maps will continue to satisfy condition [A.4].

2. \( \circ_i (1 \otimes A(S_n, 1) \otimes \cdots \otimes A(S_{m-1}, 1)) \subseteq 1 \otimes A(S_{n+m-1}, 1) \) so that composing an \( f_n \)-map with \( \circ_i \) results in a map that still satisfies condition [A.4].

3. the diagram commutes in lower dimensions (by induction on \( k \)).

Lemma [A.1] implies that the composites one gets by following the two branches of diagram 2.1 must be equal, so the diagram commutes.

We ultimately get an expression for \( f_n(x \otimes [0, \ldots, k]) \) as a sum of tensor-products of sub-simplices of \([0, \ldots, k]\) — given as ordered lists of vertices.

We claim that this \( \mathcal{G} \)-coalgebra structure is natural with respect to ordered mappings of vertices. This follows from the fact that the only significant property that the vertex \( k \) has in equation [A.1] condition [A.4] and equation [A.3] is that it is the highest numbered vertex. □

We conclude this section some computations of higher coproducts:

**Example A.5.** If \([0, 1, 2] = \Delta^2\) is a 2-simplex, then

\[
(A.4) \quad f_2([\ ] \otimes \Delta^2) = \Delta^2 \otimes F_0 F_1 \Delta^2 + F_2 \Delta^2 \otimes F_0 \Delta^2 + F_1 F_2 \Delta^2 \otimes \Delta^2
\]

— the standard (Alexander-Whitney) coproduct — and

\[
f_2([(1, 2)] \otimes \Delta^2) = [0, 1, 2] \otimes [1, 2] - [0, 2] \otimes [0, 1, 2] - [0, 1, 2] \otimes [0, 1]
\]

or, in face-operations

\[
(A.5) \quad f_2([(1, 2)] \otimes \Delta^2) = \Delta^2 \otimes F_0 \Delta^2 - F_1 \Delta^2 \otimes \Delta^2 - \Delta^2 \otimes F_2 \Delta^2
\]

**Proof.** If we write \( \Delta^2 = [0, 1, 2] \), we get

\[
f_2([\ ] \otimes \Delta^2) = [0, 1, 2] \otimes [2] + [0, 1] \otimes [1, 2] + [0] \otimes [0, 1, 2]
\]

To compute \( f_2([(1, 2)] \otimes \Delta^2) \) we have a version of equation [A.3]:

\[
f(e_1 \otimes \Delta^2) = \Phi_2(f_2(\partial e_1 \otimes \Delta^2)) - \Phi_2 f_2(e_1 \otimes \partial \Delta^2)
\]

\[
= -\Phi_2(f_2((1, 2) \cdot [\ ] \otimes \Delta^2)) + \Phi_2(f_2([\ ] \otimes \Delta^2)) - \Phi_2 f_2(e_1 \otimes \partial \Delta^2)
\]
Now
\[
\Phi_2(1, 2) \cdot (f_2([]) \otimes \Delta^2) = (\varphi_2 \otimes 1)([2] \otimes [0, 1, 2] - [1, 2] \otimes [0, 1] \\
+ [0, 1, 2] \otimes [0]) \\
+ (i \circ \epsilon \otimes \varphi_2)([2] \otimes [0, 1, 2] \\
- [1, 2] \otimes [0, 1] + [0, 1, 2] \otimes [0]) \\
= 0
\]
and
\[
\Phi_2(f_2([]) \otimes \Delta^2) = (\varphi_2 \otimes 1)([0, 1, 2] \otimes [2] + [0, 1] \otimes [1, 2] \\
+ [0] \otimes [0, 1, 2]) \\
= [0, 1, 2] \otimes [1, 2] - [0, 2] \otimes [0, 1, 2]
\]
In addition, proposition [A.8] implies that
\[
f_2(e_1 \otimes \partial \Delta^2) = [1, 2] \otimes [1, 2] - [0, 2] \otimes [0, 1, 2] \\
+ [0, 1] \otimes [0, 1]
\]
so that
\[
\Phi_2 f_2(e_1 \otimes \partial \Delta^2) = [0, 1, 2] \otimes [0, 1]
\]
We conclude that
\[
f_2((1, 2) \otimes \Delta^2) = [0, 1, 2] \otimes [1, 2] - [0, 2] \otimes [0, 1, 2] \\
- [0, 1, 2] \otimes [0, 1]
\]
which implies equation [A.5].

We end this section with computations of some “higher coproducts.” We have a \(ZS_2\)-equivariant chain-map
\[
f_2(RS_2 \otimes C) \to C \otimes C
\]

**Proposition A.6.** If \(\Delta^2\) is a 2-simplex, then:

(A.6) \(f_2([] \otimes \Delta^2) = \Delta^2 \otimes F_0 F_1 \Delta^2 + F_2 \Delta^2 \otimes F_0 \Delta^2 + F_1 F_2 \Delta^2 \otimes \Delta^2\)

Here \(e_0 = []\) is the 0-dimensional generator of \(RS_2\) and this is just the standard (Alexander-Whitney) coproduct.

In addition, we have:

(A.7) \(f_2([(1, 2)] \otimes \Delta^2) = \Delta^2 \otimes F_0 \Delta^2 - F_1 \Delta^2 \otimes \Delta^2 - \Delta^2 \otimes F_2 \Delta^2\)
Proof. If we write \( \Delta^2 = [0, 1, 2] \), we get
\[
f_2([\cdot] \otimes \Delta^2) = [0, 1, 2] \otimes [2] + [0, 1] \otimes [1, 2] + [0] \otimes [0, 1, 2]
\]
To compute \( f_2([1, 2]) \otimes \Delta^2 \) we have a version of equation A.11
\[
f_2(e_1 \otimes \Delta^2) = \Phi_2(f_2(\partial e_1 \otimes \Delta^2) - \Phi_2f_2(e_1 \otimes \partial \Delta^2)
\]
\[
= -\Phi_2(f_2((1, 2) \cdot [\cdot] \otimes \Delta^2) + \Phi_2f_2([\cdot] \otimes \Delta^2) - \Phi_2f_2(e_1 \otimes \partial \Delta^2)
\]
Now
\[
\Phi_2(1, 2) \cdot (f_2([\cdot] \otimes \Delta^2) = (\varphi_2 \otimes 1)([2] \otimes [0, 1, 2] - [1, 2] \otimes [0, 1] + [0, 1, 2] \otimes [0])
\]
\[
+ (i \circ \varepsilon \otimes \varphi_2)([2] \otimes [0, 1, 2] - [1, 2] \otimes [0, 1] + [0, 1, 2] \otimes [0])
\]
\[
= 0
\]
and
\[
\Phi_2(f_2([\cdot] \otimes \Delta^2) = (\varphi_2 \otimes 1)([0, 1, 2] \otimes [2] + [0, 1] \otimes [1, 2] + [0] \otimes [0, 1, 2])
\]
\[
= [0, 1, 2] \otimes [1, 2] - [0, 2] \otimes [0, 1, 2]
\]
In addition, proposition A.8 implies that
\[
f_2(e_1 \otimes \partial \Delta^2) = [1, 2] \otimes [1, 2] - [0, 2] \otimes [0, 2] + [0, 1] \otimes [0, 1]
\]
so that
\[
\Phi_2f_2(e_1 \otimes \partial \Delta^2) = [0, 1, 2] \otimes [0, 1]
\]
We conclude that
\[
f_2([1, 2]_2 \otimes \Delta^2) = [0, 1, 2] \otimes [1, 2] - [0, 2] \otimes [0, 1, 2]
\]
\[
- [0, 1, 2] \otimes [0, 1]
\]
which implies equation A.7.

We continue this computation one dimension higher:

**Proposition A.7.** If \( \Delta^3 \) is a 3-simplex, then:
\[
f_2([1, 2] \otimes \Delta^3) = F_1F_2\Delta^3 \otimes \Delta^3 - F_2\Delta^3 \otimes F_0\Delta^3
\]
\[
+ \Delta^3 \otimes F_0F_1\Delta^3 - \Delta^3 \otimes F_0F_3\Delta^3
\]
\[
- F_1\Delta^3 \otimes F_2\Delta^3 - \Delta^3 \otimes F_2F_3\Delta^3
\]

**Proof.** As before, \( \Delta^3 = [0, 1, 2, 3] \), and we have
\[
f_2(e_1 \otimes \Delta^3) = \Phi_3(f_2(\partial e_1 \otimes \Delta^3) - \Phi_3f_2(e_1 \otimes \partial \Delta^3)
\]
\[
= -\Phi_3(f_2((1, 2) \cdot [\cdot] \otimes \Delta^3) + \Phi_3f_2([\cdot] \otimes \Delta^3) - \Phi_3f_2(e_1 \otimes \partial \Delta^3)
\]
and
\[
\Phi_3(f_2((1, 2) \cdot [\cdot] \otimes \Delta^3) = 0
\]
We also conclude
\[
\Phi_3(f_2([] \otimes \Delta^3)) = [0, 3] \otimes \Delta^3 - [0, 1, 3] \otimes [1, 2, 3] \\
+ \Delta^3 \otimes [2, 3]
\]

Now
\[
\partial \Delta^3 = [1, 2, 3] - [0, 2, 3] + [0, 1, 3] - [0, 1, 2]
\]
and equation [A.7] implies that
\[
f_2(e_1 \otimes \partial \Delta^3) = [1, 2, 3] \otimes [2, 3] - [1, 3] \otimes [1, 2, 3] \\
- [1, 2, 3] \otimes [1, 2] \\
- [0, 2, 3] \otimes [2, 3] + [0, 3] \otimes [0, 2, 3] \\
+ [0, 2, 3] \otimes [0, 2] \\
+ [0, 1, 3] \otimes [1, 3] - [0, 3] \otimes [0, 1, 3] \\
- [0, 1, 3] \otimes [0, 1] \\
- [0, 1, 2] \otimes [1, 2] + [0, 2] \otimes [0, 1, 2] \\
+ [0, 1, 2] \otimes [0, 1]
\]

Most of these terms die when one applies \(\Phi_3\):
\[
\Phi_3 f_2(e_1 \otimes \partial \Delta^3) = \Delta^3 \otimes [1, 2] + [0, 2, 3] \otimes [0, 1, 2] \\
- \Delta^3 \otimes [0, 1]
\]

We conclude that
\[
f_2(e_1 \otimes \Delta^3) = [0, 3] \otimes \Delta^3 - [0, 1, 3] \otimes [1, 2, 3] \\
+ \Delta^3 \otimes [2, 3] - \Delta^3 \otimes [1, 2] \\
- [0, 2, 3] \otimes [0, 1, 2] - \Delta^3 \otimes [0, 1]
\]

which implies equation [A.8]. □

With this in mind, note that images of simplices in \(N(*)\) have an interesting property:

**Proposition A.8.** Let \(X\) be a simplicial set with \(C = N(X)\) and with coalgebra structure
\[
f_n: RS_n \otimes N(X) \to N(X)^{\otimes n}
\]
and suppose \(RS_2\) is generated in dimension \(n\) by \(e_n = [(1, 2)(\cdots)(1, 2)]\) \(_n\) terms.

If \(x \in C\) is the image of a \(k\)-simplex, then
\[
f_2(e_k \otimes x) = \xi_k \cdot x \otimes x
\]
where \(\xi_k = (-1)^{(k-1)/2}\).
Remark. This is just a chain-level statement that the Steenrod operation $Sq^0$ acts trivially on mod-2 cohomology. A weaker form of this result appeared in [7].

Proof. Recall that $(R, S_2)_n = \mathbb{Z}[\mathbb{Z}_2]$ generated by $e_n = [(1, 2)| \ldots |(1, 2)]$.

Let $T$ be the generator of $\mathbb{Z}_2$ — acting on $C \otimes C$ by swapping the copies of $C$.

We assume that $f_2(e_i \otimes C(\Delta^j)) \subset C(\Delta^j) \otimes C(\Delta^j)$ so that

$$(A.9) \quad i > j \implies f_2(e_i \otimes C(\Delta^j)) = 0$$

□

As in section 4 of [13], if $e_0 = [] \in R S_2$ is the 0-dimensional generator, we define

$$f_2 : R S_2 \otimes C \to C \otimes C$$

inductively by

$$(A.10) \quad f_2(e_0 \otimes [0, \ldots, k]) = \sum_{i=0}^{k} [0, \ldots, i] \otimes [i, \ldots, k]$$

Let $\sigma = \Delta^k$ and inductively define

$$(A.11) \quad f_2(e_k \otimes \sigma) = \Phi_k(f_2(\partial e_k \otimes \sigma) + (-1)^k \Phi_k f_2(e_k \otimes \partial \sigma) - \Phi_k(f_2(\partial e_k \otimes \sigma))$$

because of equation $A.9$.

Proof. Expanding $\Phi_k$, we get

$$(A.12) \quad f_2(e_k \otimes \sigma) = (\varphi_k \otimes 1)(f_2(\partial e_k \otimes \sigma)) + (i \circ \epsilon \otimes \varphi_k) f_2(\partial e_k \otimes \sigma) = (\varphi_k \otimes 1)(f_2(\partial e_k \otimes \sigma))$$

because $\varphi_k^2 = 0$ and $\varphi_k \circ i \circ \epsilon = 0$.

Noting that $\partial e_k = (1 + (-1)^k T)e_{k-1} \in R S_2$, we get

$$f_2(e_k \otimes \sigma) = (\varphi_k \otimes 1)(f_2(e_{k-1} \otimes \sigma)) + (-1)^k(\varphi_k \otimes 1) \cdot T \cdot f_2(e_{k-1} \otimes \sigma) = (-1)^k(\varphi_k \otimes 1) \cdot T \cdot f_2(e_{k-1} \otimes \sigma)$$
again, because $\varphi_k^2 = 0$ and $\varphi_k \circ \iota_k \circ \epsilon = 0$. We continue, using equation [A.12] to compute $f(e_{k-1} \otimes \sigma)$:

$$f_2(e_k \otimes \sigma) = (-1)^k (\varphi_k \otimes 1) \cdot T \cdot f_2(e_{k-1} \otimes \sigma)$$

$$= (-1)^k (\varphi_k \otimes 1) \cdot T \cdot (\varphi_k \otimes 1) \left( f_2(\partial e_{k-1} \otimes \sigma) \right)$$

$$+ (-1)^{k-1} f_2(e_{k-1} \otimes \partial \sigma)$$

$$= (-1)^k \varphi_k \otimes \varphi_k \cdot T \cdot \left( f_2(\partial e_{k-1} \otimes \sigma) \right)$$

$$+ (-1)^{k-1} f_2(e_{k-1} \otimes \partial \sigma)$$

If $k - 1 = 0$, then the left term vanishes. If $k - 1 = 1$ so $\partial e_{k-1}$ is 0-dimensional then equation [A.10] gives $f(\partial e_1 \otimes \sigma)$ and this vanishes when plugged into $\varphi_k \otimes \varphi_k$. If $k - 1 > 1$, then $f_2(\partial e_{k-1} \otimes \sigma)$ is in the image of $\varphi_k$, so it vanishes when plugged into $\varphi_k \otimes \varphi_k$.

In all cases, we can write

$$f_2(e_k \otimes \sigma) = (-1)^k \varphi_k \otimes \varphi_k \cdot T \cdot (-1)^{k-1} f_2(e_{k-1} \otimes \partial \sigma)$$

$$= -\varphi_k \otimes \varphi_k \cdot T \cdot f_2(e_{k-1} \otimes \partial \sigma)$$

If $f_2(e_{k-1} \otimes \Delta^{k-1}) = \xi_{k-1} \Delta^{k-1} \otimes \Delta^{k-1}$ (the inductive hypothesis), then

$$f_2(e_{k-1} \otimes \partial \sigma) =$$

$$\sum_{i=0}^k \xi_{k-1} \cdot (-1)^i [0, \ldots, i-1, i+1, \ldots k] \otimes [0, \ldots, i-1, i+1, \ldots k]$$

and the only term that does not get annihilated by $\varphi_k \otimes \varphi_k$ is

$$(-1)^k [0, \ldots, k-1] \otimes [0, \ldots, k-1]$$

(see equation [A.1]). We get

$$f_2(e_k \otimes \sigma) = \xi_{k-1} \cdot \varphi_k \otimes \varphi_k \cdot T \cdot (-1)^{k-1} [0, \ldots, k-1] \otimes [0, \ldots, k-1]$$

$$= \xi_{k-1} \cdot \varphi_k \otimes \varphi_k (-1)^{(k-1)^2+k-1} [0, \ldots, k-1] \otimes [0, \ldots, k-1]$$

$$= \xi_{k-1} \cdot (-1)^{(k-1)^2+2(k-1)} \varphi_k [0, \ldots, k-1] \otimes \varphi_k [0, \ldots, k-1]$$

$$= \xi_{k-1} \cdot (-1)^{k-1} [0, \ldots, k] \otimes [0, \ldots, k]$$

$$= \xi_k \cdot [0, \ldots, k] \otimes [0, \ldots, k]$$

where the sign-changes are due to the Koszul Convention. We conclude that $\xi_k = (-1)^{k-1} \xi_{k-1}$. \qed
Lemma B.1. Let $C$ be a free abelian group, let

$$\hat{C} = \mathbb{Z} \oplus \prod_{i=1}^{\infty} C^\otimes i$$

Let $e: C \to \hat{C}$ be the function that sends $c \in C$ to

$$(1, c, c \otimes c, c \otimes c \otimes c, \ldots) \in \hat{C}$$

For any integer $t > 1$ and any set $\{c_1, \ldots, c_t\} \in C$ of distinct, nonzero elements, the elements

$$\{e(c_1), \ldots, e(c_t)\} \in \mathbb{Q} \otimes \hat{C}$$

are linearly independent over $\mathbb{Q}$. It follows that $e$ defines an injective function

$$\bar{e}: \mathbb{Z}[C] \to \hat{C}$$

Proof. We will construct a vector-space morphism

(B.1) $$f: \mathbb{Q} \otimes \hat{C} \to V$$

such that the images, $\{f(e(c_i))\}$, are linearly independent. We begin with the “truncation morphism”

$$r_t: \hat{C} \to \mathbb{Z} \oplus \bigoplus_{i=1}^{t-1} C^\otimes i = \hat{C}_{t-1}$$

which maps $C^\otimes 1$ isomorphically. If $\{b_i\}$ is a $\mathbb{Z}$-basis for $C$, we define a vector-space morphism

$$g: \hat{C}_{t-1} \otimes \mathbb{Q} \to \mathbb{Q}[X_1, X_2, \ldots]$$

by setting

$$g(c) = \sum_{\alpha} z_\alpha X_\alpha$$

where $c = \sum_\alpha z_\alpha b_\alpha \in C \otimes \mathbb{Q}$, and extend this to $\hat{C}_{t-1} \otimes \mathbb{Q}$ via

$$g(c_1 \otimes \cdots \otimes c_j) = g(c_1) \cdots g(c_j) \in \mathbb{Q}[X_1, X_2, \ldots]$$

The map in equation (B.1) is just the composite

$$\hat{C} \otimes \mathbb{Q} \xrightarrow{r_{t-1} \otimes 1} \hat{C}_{t-1} \otimes \mathbb{Q} \xrightarrow{g} \mathbb{Q}[X_1, X_2, \ldots]$$

It is not hard to see that

$$p_i = f(e(c_i)) = 1 + f(c_i) + \cdots + f(c_i)^{t-1} \in \mathbb{Q}[X_1, X_2, \ldots]$$
for $i = 1, \ldots, t$. Since the $f(c_i)$ are linear in the indeterminates $X_i$, the degree-$j$ component (in the indeterminates) of $f(c_i)$ is precisely $f(c_i)^j$. It follows that a linear dependence relation
\[
\sum_{i=1}^{t} \alpha_i \cdot p_i = 0
\]
with $\alpha_i \in \mathbb{Q}$, holds if and only if
\[
\sum_{i=1}^{t} \alpha_i \cdot f(c_i)^j = 0
\]
for all $j = 0, \ldots, t - 1$. This is equivalent to $\det M = 0$, where
\[
M = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
f(c_1) & f(c_2) & \cdots & f(c_t) \\
\vdots & \vdots & \ddots & \vdots \\
f(c_1)^{t-1} & f(c_2)^{t-1} & \cdots & f(c_t)^{t-1}
\end{bmatrix}
\]
Since $M$ is the transpose of the Vandermonde matrix, we get
\[
\det M = \prod_{1 \leq i < j \leq t} (f(c_i) - f(c_j))
\]
Since $f \mid C \otimes_{\mathbb{Z}} \mathbb{Q} \subset \hat{C} \otimes_{\mathbb{Z}} \mathbb{Q}$ is injective, it follows that this only vanishes if there exist $i$ and $j$ with $i \neq j$ and $c_i = c_j$. The second conclusion follows. \qed

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