Open Topological String Amplitudes and BPS Invariants on Complete Intersection Calabi-Yau Threefolds

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This work is dedicated to the memory of our dear supervisor Prof. Fu-Zhong Yang, who passed away while this paper was being prepared.

Abstract

Open topological string partition function on compact Calabi-Yau threefolds satisfies the extended holomorphic anomaly equation. By direct integration, we solve these equations and obtain partition functions for first several genus and boundaries on complete intersection Calabi-Yau threefolds. Complemented by the unoriented worldsheet contribution, the annulus functions encode the genus one BPS invariants.

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1 Introduction

The study of topological strings on Calabi-Yau threefolds has led to important insights in various aspects of string theory and supersymmetric gauge theories. It has also benefited the development of enumerative problem on Calabi-Yau threefold by mirror symmetry\cite{14, 38, 39, 40, 50}.

For the closed string theory, the string amplitudes are captured by the holomorphic anomaly equation \cite{9, 10}, which makes it possible to recursively determine the partition function at each genus up to a holomorphic ambiguity. Physically, the holomorphic anomaly equation can be interpreted as the realization of quantum background independence of the topological string, known as the wavefunction interpretation in \cite{67}. The non-holomorphic part of the topological string partition function is generated by finitely many generators\cite{68}, which is applied to solve high genus Gromov-Witten invariants on one-modulus calabi-Yau threefolds\cite{35, 36}.

The topological string partition functions in the presence of boundaries are first explored in local Calabi-Yau. The spacetime description of open string in terms of string field theory can be reduced to a matrix model\cite{17, 18, 19}, which is generalized into the toric Calabi-Yau case in \cite{12, 13, 52, 53}. It has developed into a mathematical technique appearing in asymptotic expansion of many integrable system and enumerative problem, called topological recursion\cite{21, 22, 23, 24}.

The open mirror symmetry relates topological A-model on a Calabi-Yau threefold with A-branes as special Lagrangian submanifolds to topological B-model on a Calabi-Yau threefold with B-branes as holomorphic submanifolds. Homological mirror symmetry implies the equivalence between the A-branes category, Fukaya category, and the B-branes category, derived category of coherent sheaves\cite{43, 44}. A-branes defined as the fixed locus of anti-holomorphic involution admit two vacua separated by a domainwall and the instantons are maps from the disk to the Lagrangian A-branes. The tree level data of open topological string theory is described by the domainwall tension, or the superpotential change between two domainwalls\cite{66}. It is the solution of the inhomogeneous Picard-Fuchs differential equation\cite{65}, as well as a Poincare normal function\cite{56}, and encodes genus zero real BPS invariants\cite{41, 45, 57, 62}. The extended holomorphic anomaly equation for compact geometry with D-branes was proposed in \cite{64}, which can be solved by direct integration using special geometry relation\cite{64}, Feymann rule method\cite{16}, and polynomial structure of the partition function \cite{3, 42}. The high genus partition functions give rise to the prediction of high genus open Gromov-Witten invariants and real BPS invariants\cite{41, 46, 59, 63}. 

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In this article, we solve the high genus open topological partition function by directly integrating the extended holomorphic anomaly equation on one-modulus complete intersection Calabi-Yau (CICY) threefolds. In the large volume point, we extract the genus one real BPS invariants and list the several amplitudes for low boundaries and genus. The organization of this article are as follows: In Section 2, we review the background knowledge about special geometry, partition functions, and holomorphic anomaly equation for closed and open topological string theory, including the geometry of moduli space of the B-model, domain-walls tension, and the partition functions at tree, one-loop, and two-loop. In Section 3, the extended holomorphic anomaly equations are solved on four one-modulus CICY threefolds $(X_{4,4}, X_{6,6}, X_{3,4}, X_{4,6})$, and genus one BPS invariants are extracted by including Klein bottle contribution to cancel tadpoles. The last section is a brief summary and further discussion. In Appendix A, the BPS invariants of genus one are summarized. In Appendix B and C, the amplitudes determined by the domain-walls tensions that are different from the domain-wall tension given in section 3.1 and 3.2 are calculated.

2 Special Geometry, Partition Functions, and Holomorphic Anomaly Equation

2.1 Special Geometry

The moduli space of the B-model is the complex structure moduli space $M_{CS}(X)$ of $X$. The holomorphic $(3,0)$-form $\Omega$ defines a line bundle $\mathcal{L}$ over $M_{CS}(X)$ with the metric,

$$||\Omega||^2 = i \int \bar{\Omega} \wedge \Omega.$$ 

The Zamolodchikov metric $G_{ij}$ on $M_{CS}(X)$ [69], identified with the Weil-Peterson metric, is a Kahler metric given as the curvature of $\mathcal{L}$,

$$G_{ij} = \partial_i \partial_j K,$$

with Kahler potential

$$K = -\log ||\Omega||^2.$$ 

In addition, there is a holomorphic symmetric tensor $C_{jkl}$ with coefficients in $\mathcal{L}^2$ satisfying,

$$\partial_i C_{jkl} = 0, \quad D_i C_{jkl} = D_j C_{kli},$$

$$2\pi,$$
s.t., the Riemann curvature of $G_{ij}$ reads,

$$R^l_{ijk} \equiv -\partial_j \Gamma^l_{ki} = G_{kj} \delta^l_i + G_{ij} \delta_k^l - e^{2K} C_{ikn} G_{m\bar{n}} G_{\bar{j}m\bar{l}}.$$

Equation 2.1, 2.2 and 2.3 are known as the $N=2$ special geometry relations on $M_{CS}$.

$G_{ij}$ can also be written as,

$$G_{ij} = \frac{g_{ij}}{g_{0\bar{0}}},$$

Here $g_{ij}$ is the $tt^*$-metric on the vacuum bundle $\mathcal{V} \to X$,

$$g_{ij} = g(e_j, e_i) = \langle \Theta j | i \rangle,$$

with $\Theta$ is the CPT operator acting on the ground states. It induces the $tt^*$-connection, i.e., the connection compatible with the metric and the holomorphic structure on $\mathcal{V}$, with connection matrix, $D_i(e_j) = (A_i)^k e_k$ and $A_i = g^{-1} \partial_i g$,

$$A_i = \begin{pmatrix} g^{0\bar{0}} \partial_i g^{0\bar{0}} & 0 & 0 & 0 \\ 0 & g^{j\bar{l}} \partial_i g^{m\bar{j}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_{\bar{i}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & g^{\bar{j}k} \partial_{\bar{i}} g_{\bar{k}m} & 0 \\ 0 & 0 & 0 & g^{0\bar{0}} \partial_{\bar{i}} g_{0\bar{0}} \end{pmatrix}$$

The $tt^*$-connection and the chiral ring multiplication satisfy the $tt^*$-equations,

$$[D_i, D_j] = [D_i, C_j] = [D_i, C_{\bar{j}}] = 0,$$

$$[D_i, C_j] = [D_j, C_i], \quad [D_{\bar{i}}, C_{\bar{j}}] = [D_{\bar{j}}, C_{\bar{i}}], \quad [D_i, D_{\bar{j}}] = -[C_i, C_{\bar{j}}],$$

which is equivalent to the flatness of the Gauss-Manin connection,

$$\nabla_i = D_i - \alpha C_i, \quad \nabla_{\bar{i}} = D_{\bar{i}} - \alpha^{-1} C_{\bar{i}},$$

Here $C_i$ is the action of the chiral fields over $\mathcal{V}$, with matrix representation,

$$C_i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \delta^i_j & 0 & 0 & 0 \\ 0 & C_{im\bar{l}} & 0 & 0 \\ 0 & 0 & G_{im\bar{n}} & 0 \end{pmatrix}, \quad C_{\bar{i}} = \begin{pmatrix} 0 & G_{\bar{j}m} & 0 & 0 \\ 0 & 0 & C_{\bar{j}m\bar{l}} & 0 \\ 0 & C_{im\bar{l}} & 0 & \delta^i_j \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
The B-model on $X$ is determined by the variation of the complex structure of $X$,
\[ H^{3-p,q}(X) \cong H^q(\wedge^p TX). \] (2.5)

The cohomology groups $H^3(X)$ decomposes as follow,
\[ H^{3,0}(X) \oplus H^{2,1}(X) \oplus H^{1,2}(X) \oplus H^{0,3}(X) \]
which lead to a Hodge filtration on $H^3(X)$,
\[ H^{3,0}(X) = F^3H^3(X) \subset F^2H^3(X) \subset F^1H^3(X) \subset F^0H^3(X), \]
with
\[ F^qH^3(X) = \oplus_{q' \geq q} H^{q'-3+q'}(X), \]

On $M_{CS}(X)$, there are the flat coordinates defined by a symplectic basis of homology 3-cycles, $\{\alpha_i, \beta^j\}_{i=0,1,...,h^{1,2}}$. The period integrals of $\Omega$ are,
\[ \omega^i = \int_{\alpha_i} \Omega, \quad F_i = \int_{\beta^i} \Omega, \] (2.6)

Note that $F_i$’s are homogeneous functions of $\omega^i$’s and can be expressed as derivatives of a single function $F(\omega)$, i.e.,
\[ F_i(\omega) = \frac{\partial F(\omega)}{\partial \omega^i} \] (2.7)
where $F(\omega)$ is a homogeneous function of $\omega^i$’s of weight 2, called the prepotential. The flat coordinates of $M_{CS}$ is defined as the ratios of $\omega^i$’s,
\[ t^i = \frac{\omega^i}{\omega^0}, \quad i = 1, \ldots, h^{1,2}, \] (2.8)
which plays an important role in defining the mirror maps. Here the generating function of the solutions of GKZ-system is used,
\[ \sigma(z; \rho) = \sum_{n \geq 0} \frac{\Gamma[1 - l_0(n + \rho)]}{\prod_{l>0} \Gamma[n + \rho + l]} z^{n+\rho}, \]
\[ \omega^0 = \sigma(z; 0), \]
\[ \omega^1 = \partial_\rho \varphi_i(z; \rho)|_{\rho=0}, \]
\[ \ldots \]
The prepotentials and the flat coordinates can be obtained by solving the Picard-Fuchs equation governing the variation of Hodge structure.

The notion of special geometry has been generalized to the open string case, known as the N=1 special geometry [47, 48, 49, 54]. In the open-closed B-model, the chiral ring is isomorphic to the relative cohomology group $H^3(X, D)$, with $D$ determined by the D-brane geometry. The bulk sector is identified with the section of $H^{3,3-p}(X)$ as Equation 2.5, and the boundary sector is given by the isomorphism,

$$H^0(D, N_D) \cong H^{2,0}(D), \quad H^1(D, T|D \wedge N_D) \cong H^{1,1}(D),$$

which can be interpreted as a Poincare residue for $\Omega$. The deformation of the open-closed B-model over the open-closed moduli space is described by the variation of mixed Hodge structure on $H^3(X, D)$. The connection on the vacuum bundle is the Gauss-Manin connection on the relative cohomology bundle $H^3(X, D)$, whose integrability and flatness determines the flat coordinates and superpotentials by Picard-Fuchs equations [25, 26].

### 2.2 Holomorphic Anomaly Equation

The closed topological string partition function of genus one (torus) is found to be represented by a generalized index, [15],

$$\mathcal{F}^{(1)} = \frac{1}{2} \int \frac{d^2 \tau}{\tau^2} \text{Tr}_{\text{closed}} \left[ (-1)^F F L F R e^{2\pi i (\tau L_0 - \bar{\tau} L_0)} \right],$$

where the integral is defined over the fundamental domain of the action of $SL(2, \mathbb{Z})$ on the upper half-plane. The torus one-point function can be obtained by the holomorphic differentiation of $\mathcal{F}^{(1)}$,

$$\partial_j \mathcal{F}^{(1)} = \frac{1}{2} \int \text{Tr}_{\text{closed}} (-1)^F \left[ \int \mu G^- \int \bar{\mu} \tilde{G}^- \phi_j(0) e^{2\pi i (\tau L_0 - \bar{\tau} \bar{L}_0)} \right],$$

which satisfy the following holomorphic anomaly equation at one-loop,

$$\partial_i \partial_j \mathcal{F}^{(1)} = \frac{1}{2} Tr C_i C_j - \frac{1}{24} \text{Tr}_{\text{closed}} (-1)^F G_{ij}.$$

The closed topological string partition function of genus $g \geq 2$ is defined as the integral over the moduli space $M^{(g)}$ of Riemann surface $\Sigma^{(g)}$,

$$\mathcal{F}^{(g)} = \int_{M^{(g)}} [dm \left( \prod_{a=1}^{3g-3} (\int \mu_a G^-)(\int \bar{\mu}_a \tilde{G}^-) \right)_{\Sigma^{(g)}}, \quad g \geq 2$$
where \( \mu_a, a = 1, \ldots, 3g - 3 \) are the Beltrami differentials. It satisfies the holomorphic anomaly equation \([10]\) as follow,

\[
\partial_j \mathcal{F}(g) = \frac{1}{2} \sum_{g_1 + g_2 = g} C^{ij}_{i} \mathcal{F}^{(g_1)}(j) \mathcal{F}^{(g_2)}_{k} + \frac{1}{2} C^{jk}_{i} \mathcal{F}^{(g - 1)}_{jk}, \quad g \geq 2,
\]

with \( G^{ij}_{k} \equiv C^{ijk}_{i} g^{j} \bar{g}^{k} = C^{ijk}_{i} e^{2K} \bar{g}^{j} \bar{g}^{k} \). On the right handside of the equation, the first term originates from the closed string degeneration in which the Riemann surface splits into two components, of genus \( g_1 > 0 \) and \( g_2 > 0 \), and the second term comes from the pinching of a handle that reduces the genus by one.

The amplitudes with insertion of chiral fields,

\[
\mathcal{F}(g)_{i_1, \ldots, i_n} = \int_{M(\mathbb{Z}_0)} [dm] \left[ \int \phi^{(2)}_{i_1} \cdots \int \phi^{(2)}_{i_n} \prod_{a=1}^{3g-3} \left( \int \mu_a G^-(\int \mu_a \bar{G}^-) \right) \right] \Sigma_g, \quad g \geq 2
\]

can be obtained by covariant differentiation\([20]\),

\[
\mathcal{F}_{i_1, \ldots, i_{n+1}}^{(g)} = D_{i_{n+1}} \mathcal{F}_{i_1, \ldots, i_n}^{(g)}
\]

where \( D \) is the Zamolodchikov-Kahler derivative on \( \text{Sym}^n T^* M \otimes \mathcal{L}^{2g-2} \).

For open topological string, the partition function of one genus and zero boundaries satisfies the holomorphic anomaly equation that is slightly different to the closed string case,

\[
\partial_j \mathcal{F}_{j}^{(1,0)} = \frac{1}{2} C_{jkl} C_{i}^{kl} + (1 - \frac{\chi}{24}) G_{ji},
\]

where the 1 in the second term comes from the propagation of the unique ground state of zero charge \((q, \bar{q}) = (0, 0)\). The partition function of zero genus and two boundaries (annulus or cylinder amplitude) is defined in \([10]\),

\[
\mathcal{F}^{(0,2)} = \int_{0}^{\infty} \frac{dL}{L} \text{Tr} \left[ (-1)^F e^{-LH} \right],
\]

which satisfies the holomorphic anomaly equation\([64]\),

\[
\partial_j \mathcal{F}_{j}^{(0,2)} = -\Delta_{jk} \Delta_{i}^{k} + \frac{N}{2} G_{ji},
\]

where \( N \) is the rank of a bundle over the moduli space in which the charge zero ground states of the open string live.
The open topological string partition function \( \mathcal{F}^{(g,h)}(2g + h - 2 > 0) \) is defined as an integral over the moduli space \( M^{(g,h)} \) of Riemann surface \( \Sigma_{g,h} \) of \( g \) genus and \( h \) boundaries,

\[
\mathcal{F}^{(g,h)} = \int_{M^{(g,h)}} [dm][dl] \langle \prod_{a=1}^{3g+h-3} \mu_a G^-(\int \mu_a G^-) \prod_{b=1}^{h} \lambda_b (G^- + \bar{G}^-) \rangle \Sigma_{g,h},
\]

where \( l^b \) is the length moduli and \( m^a \) is the coordinates on \( M^{(g,h)} \), and \( \mu_a, \mu_a, a = 1, \ldots, 3g + 3 - h \) and \( \lambda_b, b = 1, \ldots, h \) are Beltrami differentials. The holomorphic anomaly equation in the presence of D-branes is as follow,

\[
\partial_i \mathcal{F}^{(g,h)} = \frac{1}{2} \sum_{\substack{g_1 + g_2 = g \\ b_1 + b_2 = h}} C^{-jk}_i \mathcal{F}^{(g_1,h_1)} \mathcal{F}^{(g_2,h_2)} + \frac{1}{2} C^{-jk}_i \mathcal{F}^{(g-1,h),} - \Delta_i \mathcal{F}^{(g,h-1)},
\]

with \( \Delta_i \equiv \Delta_{ij} g^{ij} = \Delta_{ij} e^{2K} G^{ij} \). Again, the first two terms come from closed string degenerations, while the last comes from the shrinking of a boundary component to zero size. Degenerations in the open string channel do not contribute generically. The amplitudes with insertion is defined as,

\[
\mathcal{F}_{i_1, \ldots, i_n}^{(g,h)} = \int_{M^{(g,h)}} \langle \int \phi_{i_1}^{(2)} \cdots \int \phi_{i_n}^{(2)} \prod_{a=1}^{3g+h-3} \mu_a G^- \rangle \langle \prod_{b=1}^{h} \lambda_b (G^- + \bar{G}^-) \rangle \Sigma_{g,h},
\]

again given by the covariant differentiation

\[
\mathcal{F}_{i_1, \ldots, i_n, i_{n+1}}^{(g,h)} = D_{i_{n+1}} \mathcal{F}_{i_1, \ldots, i_n}^{(g,h)}.
\]

The equation (2.12) is solved by Feynmann rules. In this case, the propagators \( S, S', S^{ij} \) relate to the Yukawa coupling,

\[
\partial_i S^{ij} = C_i, \quad \partial_i S = G_{ii} S^{ij}, \quad \partial_i S = G_{ii} S^i,
\]

which are sections of the bundles \( \mathcal{L}^{-2} \otimes \text{Sym}^m T \), and \( \Delta, \Delta^i \) relate to the disk function,

\[
\partial_i \Delta^i = \Delta^i, \quad \partial_i \Delta = G_{ii} \Delta^i.
\]

which are sections of the bundles \( \mathcal{L}^{-1} \otimes \text{Sym}^m T \). The vertices of the Feynmann rules are given by the correlation function \( \mathcal{F}_{i_1, \ldots, i_n}^{(g,h)} \).
In this paper, we study one-parameter models in weighted projective space. The propagators in the holomorphic limits are as follow,

\[ S^{zz} = \frac{1}{C^{zzz}} \partial_z \log (G^z \bar{z} (ze^K)^2), \]

\[ S^z = \frac{1}{C^{zzz}} \left[ \left( \partial_z \log (ze^K) \right)^2 - D \partial_z \log (ze^K) \right], \]

\[ S = \left[ S^z - \frac{1}{2} D \partial_z \log (ze^K) \right] + \frac{1}{2} D S^z + \frac{1}{2} S^z S C_{zzz} \]

\[ \Delta^z = -\Delta_{zz} C_{zzz}, \]

\[ \Delta = D \Delta^z. \]

\[ (2.15) \]

2.3 Open String at Tree Level

The BPS domainwall tension plays an important role in the open topological string theory at tree level. and open mirror symmetry relates the domainwall tension in the A-model to the domainwall tension in the B-model,

\[ \mathcal{T}_A(t) = \mathcal{A}_0(z(t))^{-1} \mathcal{T}_B(z(t)). \]

In the A–model, \( \mathcal{T}_A(t) \) is defined as the generating functional counting holomorphic disks ending on the A-brane \( L_\alpha \), with the form

\[ \mathcal{T}_A(t) = \frac{t}{2} + \mathcal{T}_{\text{classical}} + \sum_{D \in H_2(X,L_\alpha,Z)} \tilde{n}_D q^{\text{Area}(D)}, \]

\[ (2.16) \]

where \( H_2(X,L_\alpha,Z) \) is the relative cohomology group labeling the classes \( D \) of the image of the holomorphic discs.

In the B–model, \( \mathcal{T}_B(z) \) is the superpotential change on two domainwalls formed by two D5-branes wrapped on two distinct holomorphic curves \( C_+ \) and \( C_- \) in the same homology class \( [66] \),

\[ \mathcal{T} = \int_\Gamma \Omega, \]

where \( \Gamma \) is a three chain with boundary \( \partial \Gamma = C_+ - C_- \). Mathematically speaking, it is known as a Abel-Jacobi map \([1, 2, 51]\), and a Poincare Poincare normal function that should be viewed as a holomorphic section of the Griffiths intermediate
Jacobian fibration over $M_{CS}(X)$\[28, 60, 61\]. This domainwall tension satisfies the inhomogeneous Picard-Fuchs equation,

$$\mathcal{L}_{PF} \mathcal{B}(z) = f(z),$$  \hspace{1cm} (2.17)

where $\mathcal{L}_{PF}$ is the Picard-Fuchs equation from variation of Hodge structure [55, 58] by Griffiths-Dwork method [29, 31] or GKZ-system [32, 33, 34, 37].

In the tree level, the most fundamental amplitudes of open topological string are the Yukawa $C_{ijk}$ and the disk amplitudes $\Delta_{ij}$. The Yukawa couplings is defined as,

$$C_{ijk} = -\langle \Omega, \nabla_i \nabla_j \nabla_k \Omega \rangle = -\langle \Omega, \partial_i \partial_j \partial_k \Omega \rangle.$$  

where $\Omega(z) \in H^{3,0}(X)$ is the holomorphic three form on the Calabi-Yau three-fold $X$, $\nabla$ the Gauss-Manin connection satisfying the Griffiths transversality, and $\langle \cdot, \cdot \rangle = \int_X \cdot \land \cdot$ the symplectic pairing on $H^3(X)$. The Yukawa coupling is holomorphic and can be written as the covariant differentiation of the genus zero partition function $\mathcal{F}^{(0)}$,

$$\partial \bar{C}_{ijk} = 0, \quad C_{ijk} = D_i D_j D_k \mathcal{F}^{(0)}.$$  

In particular, the Yuakawa couplings of the one-modulus models in this article have the form [8],

$$C_{zzz} = W(0) \frac{dz}{1 - \mu z} \partial_0 \left( \frac{dz}{z} \right)^3.$$  \hspace{1cm} (2.18)

Meanwhile, the disk amplitude is identified with the Griffiths infinitesimal invariant [27, 30],

$$\mathcal{F}^{(0,1)}_{ij} = \Delta_{ij} = \langle \Omega, \nabla_i \nabla_j \nu \rangle$$

with $\nu$ a normal function defined by a three-chain $\Gamma \subset X$. By the Griffiths transversality condition of the normal function $\nu$, i.e., $\langle \Omega, \nabla \nu \rangle = 0$, $\Delta_{ij}$ can be rewritten by covariant differentiation,

$$\Delta_{ij} = D_i D_j \mathcal{F} - C_{ijk} g^{\bar{k}} D_\bar{k} \mathcal{F}.$$  

Instead of being holomorphic, $\Delta_{ij}$ satisfies the following equations,

$$\partial_\bar{k} \Delta_{ij} = -C_{jkl} \Delta_{i\bar{l}}, \quad \Delta_{i\bar{j}} = \Delta_{ij} e^K g^{\bar{k}j}.$$  \hspace{1cm} (2.19)

and in the holomorphic limit,

$$\lim_{\bar{z} \to 0} \Delta_{ij} = \lim_{z \to 0} D_i D_j \mathcal{F} = \partial_\bar{z} \partial_z \mathcal{F}.$$  \hspace{1cm} (2.20)
2.4 Open String at One-loop

The torus amplitudes can be solved by equation 2.13,

\[ \partial_i \mathcal{F}^{(1,0)} = \frac{1}{2} C_{jkl} C_{kl}^i + (1 - \frac{\chi}{24}) G_{ji} \]
\[ = \frac{1}{2} C_{jkl} \partial_i S^{kl} + (1 - \frac{\chi}{24}) \partial_j \partial_i K \]
\[ = \partial_i (\frac{1}{2} C_{jkl} S^{kl} + (1 - \frac{\chi}{24}) K_i) \]

i.e.,

\[ \mathcal{F}^{(1,0)} = \frac{1}{2} C_{jkl} S^{kl} + (1 - \frac{\chi}{24}) K_i + \text{hol.amb}, \]

which has the following general form after inserting the propagators \( S^{kl} \),

\[ \mathcal{F}^{(1,0)} = \frac{1}{2} \log[\det G_{ij}^{-1} e^{K(3 + n - \frac{1}{2} \chi)} |\text{hol.amb.}|^2], \]

In the holomorphic limit, this amplitude at large volume on one-parameter is,

\[ \mathcal{F}^{(1,0)} \xrightarrow{\text{hol.lim.}} \frac{1}{2} \log[\frac{q}{z} \frac{d}{dq} \mathcal{O}_0^{-\chi} z^{\chi} - \chi (\text{discirminant})^{-\frac{1}{2}}] \quad (2.21) \]

with \( \chi \) the Euler number and \( c_2 \) the second chern class of the Calabi-Yau threefolds.

Similarly, the annulus amplitudes can be solved by equation 2.14,

\[ \partial_i \partial_j \mathcal{F}^{(2,0)} = \partial_i (-\Delta^j_k \Delta^k + \frac{N}{2} \partial_j K) - C_{jkl} \Delta^l \Delta^k \]
\[ = \partial_i (-\Delta^j_k \Delta^k - \frac{1}{2} C_{jkl} \Delta^k \Delta^l + \frac{N}{2} \partial_j K) \]
\[ = \partial_i (-\frac{1}{2} (\Delta^j + f_{jk}) \Delta^k + \frac{N}{2} \partial_j K) \]

i.e.,

\[ \mathcal{F}^{(2,0)} = -\frac{1}{2} \Delta^j K^j + \frac{N}{2} K_j + \text{hol.amb.} \]

or

\[ \mathcal{F}^{(2,0)} = -\frac{1}{2} \Delta^j \Delta^j + \text{hol.amb.} \quad (2.22) \]
for one-parameter models. Then, $\mathcal{F}^{(0,2)}(z)_B$ can be obtained in the B-model, and relates to $\mathcal{F}^{(0,2)}_A$ by

$$\mathcal{F}^{(0,2)}_A(t) = \mathcal{F}^{(0,2)}_B(z),$$

where $\mathcal{F}^{(0,2)}_A(t)$ is defined as the generating functional counting holomorphic maps of annuli ending on $L$, with the form,

$$\mathcal{F}^{(0,2)}_A = \sum_{A \in H_2(X, L, \mathbb{Z})} \tilde{n}_A q^{\text{Area}(A)},$$

where $\tilde{n}_A$ are not integer in general. Because the A- and B-model only decouple if the tadpoles are cancelled, the integrality of BPS state counting can be assured only when unoriented worldsheets are included. Specifically, the generating functional $\mathcal{K}$ for the holomorphic maps of Klein bottles in the A-model is

$$\mathcal{K}_A(t) = \sum_{K \in H_2(X, L, \mathbb{Z})} n_K q^{\text{Area}(K)}.$$  

The corresponding B-model Klein bottle partition function satisfies the holomorphic anomaly equation,

$$\partial_i \partial_j \mathcal{K} = \frac{1}{2} C_{ij}^{kl} C_{jkl} - G_{ij},$$

which can be solved by special geometry relation 2.3,

$$\mathcal{K} = \frac{1}{2} \log (\det G_{ij}^{-1} e^{K(n-1) |\text{hol.amb.}|^2}).$$

It turns out that under holomorphic limit, above equation can be rewritten for one-parameter models as,

$$\mathcal{K} \to \frac{1}{2} \log \left[ (\frac{q}{z} \frac{dz}{dq}) (\text{discriminant})^{-1/4} \right], \quad (2.23)$$

with the expansion at the large volume point,

$$\mathcal{K} \xrightarrow{\text{hol. lim.}} \sum_{d \text{ even}} \tilde{n}_d^{(1),k} q^{d/2}. \quad (2.24)$$

The one-loop open/unoriented partition function receive non-trivial contribution from annulus partition functions $\mathcal{A} = \mathcal{F}^{(0,2)}(z)$ and Klein bottle partition functions $\mathcal{K}$, underlying the real BPS invariants $n_d^{(1,\text{real})}$,

$$\mathcal{A} + \mathcal{K} = 2 \sum_{d \text{ even}} \frac{1}{k} n_d^{(1,\text{real})} q^{d/2}. \quad (2.24)$$

1Here A/B refer to A/B-model, not a insertion.
2.5 Open String at Two-loop

Similarly, for \((g,h) = (0,3)\), we use equation (2.19)

\[
\partial_i \mathcal{F}^{(0,3)} = -\Delta_i^{(0,2)}
\]

\[
= -\partial_i(\delta^j \mathcal{F}_j^{(0,2)}) + (-\Delta_{jk} \Delta_i^k + \frac{N}{2} G_{ji}) \Delta^j
\]

\[
= -\partial_i(\delta^j \mathcal{F}_j^{(0,2)}) - \Delta_{jk} \Delta_i^k + \frac{N}{2} G_{ji} \Delta^j
\]

\[
= -\partial_i(\delta^j \mathcal{F}_j^{(0,2)}) - \frac{1}{2} \partial_i(\Delta_{jk} \Delta^j) - \partial_i \Delta_{jk} \Delta^k + \frac{N}{2} G_{ji} \Delta^j \partial_j \Delta
\]

\[
= -\partial_i(\delta^j \mathcal{F}_j^{(0,2)}) - \frac{1}{2} \partial_i(\Delta_{jk} \Delta^j) - C_{jkl} \Delta_i^k \Delta^j + \frac{N}{2} \partial_i \Delta
\]

\[
= \partial_i(\delta^j \mathcal{F}_j^{(0,2)}) + \frac{N}{2} \Delta - \frac{1}{2} \Delta_{jk} \Delta^k \Delta^l - \frac{1}{6} C_{jkl} \Delta_i^k \Delta^j \Delta^l
\]

i.e.,

\[
\mathcal{F}^{(0,3)} = -\mathcal{F}_j^{(0,2)} \Delta^j + \frac{N}{2} \Delta - \frac{1}{2} \Delta_{jk} \Delta^j \Delta^k - \frac{1}{6} C_{jkl} \Delta_i^k \Delta^j \Delta^l + \text{hol.amb.}
\]

or

\[
\mathcal{F}^{(0,3)} = -\mathcal{F}_z^{(0,2)} \Delta^z - \frac{1}{3} \Delta_{zz} \Delta^z \Delta^z + \text{hol.amb.} \quad (2.25)
\]

for one-parameter models

The oriented one-loop partition function is given by,

\[
\partial_i \mathcal{F}^{(1,1)} = \frac{1}{2} C_{i}^{jk} \Delta_{jk} - \mathcal{F}^{(1,0)} \Delta_i^l
\]

\[
= \partial_i \left( \frac{1}{2} S^{jk} \Delta_{jk} - \mathcal{F}_j^{(1,0)} \Delta^j + \frac{1}{2} S^{jk} C_{jkl} \Delta_i^l + \left( \frac{1}{2} C_{jkl} C_{i}^{k} - \left( \frac{\chi}{24} - 1 \right) G_{ji} \right) \Delta^j \right)
\]

\[
= \partial_i \left( \frac{1}{2} S^{jk} \Delta_{jk} - \mathcal{F}_j^{(1,0)} \Delta^j + \frac{1}{2} C_{jkl} S^{kl} \Delta^l - \left( \frac{\chi}{24} - 1 \right) \Delta \right)
\]

i.e.,

\[
\mathcal{F}^{(1,1)} = \frac{1}{2} S^{jk} \Delta_{jk} - \mathcal{F}_j^{(1,0)} \Delta^j + \frac{1}{2} C_{jkl} S^{kl} \Delta^l - \left( \frac{\chi}{24} - 1 \right) \Delta + \text{hol.amb.}
\]

or

\[
\mathcal{F}^{(1,1)} = -\mathcal{F}_z^{(1,0)} \Delta^z - \left( \frac{\chi}{24} - 1 \right) \Delta + \text{hol.amb.} \quad (2.26)
\]

for one-parameter models
In the two-loop level, the non-orientable diagram contribution satisfies the holomorphic anomaly equation,
\[ \partial_r \mathcal{K}^{(1,1)} = \frac{1}{2} C_{i}^{jk} \Delta_{jk} - \mathcal{K}_{j}^{j} \]
\[ = \partial_r \left( \frac{1}{2} S^{jk} \Delta_{jk} - \mathcal{K}_{j}^{j} \right) + \frac{1}{2} S^{jk} C_{jkl} \Delta_{l} + \left( \frac{1}{2} C_{jkl} C_{i}^{kl} - G_{ij} \right) \Delta_{i} \]
\[ = \partial_r \left( \frac{1}{2} S^{jk} \Delta_{jk} - \mathcal{K}_{j}^{j} + \frac{1}{2} C_{jkl} \Delta_{j} - \Delta \right) \]
i.e.,
\[ \mathcal{K}^{(1,1)} = \frac{1}{2} S^{jk} \Delta_{jk} - \mathcal{K}_{j}^{j} + \frac{1}{2} C_{jkl} \Delta_{j} - \Delta + \text{hol.amb.} \]
and,
\[ \mathcal{K}^{(1,1)} = - \mathcal{K}_{z}^{z} - \Delta + \text{hol.amb.} \quad (2.27) \]
for one-parameter models.

Then, the two-loop open/unoriented partition function has the expansion,
\[ i(\mathcal{F}^{(0,3)} + \mathcal{F}^{(1,1)} + \mathcal{K}^{(1,1)}) = 2 \sum_{d \text{ odd}} d_{\text{even}} \sum_{k \text{ odd}} d_{\text{even}} (n_{2}^{(2,\text{real})} - \frac{1}{24} n_{d}^{(0,\text{real})} ) d^{kd/2} , \quad (2.28) \]
counting real BPS invariants.

### 3 Amplitudes and BPS Invariants on CICY Threefolds

In this section, we study open topological string amplitudes and BPS invariants on CICY threefolds in Weighted Projective Space. The Calabi-Yau threefolds can be realized by Batyrev-Borisov toric mirror construction [6, 7, 11]. Relevant geometric data are listed in the following table [5, 8].
\[ \omega_0 = \sum_{n=0}^{\infty} \frac{(4n!)^2}{(n!)^4(2n)!^2} \varphi^n \]

\[ W(0) = 4^2 \]

\[ \mu = 2^{12} \]

\[ C_3 = -144 \]

\[ C_2 \cdot H = 40 \]

| \( X \) | \( \omega_0 \) | \( W(0) \) | \( \mu \) | \( C_3 \) | \( C_2 \cdot H \) |
|---|---|---|---|---|---|
| \( X_{4,4} \subset \mathbb{P}(1,1,1,1,2,2) \) | \( \sum_{n=0}^{\infty} \frac{(4n!)^2}{(n!)^4(2n)!^2} \varphi^n \) | 4 | 2^{12} | -144 | 40 |
| \( X_{6,6} \subset \mathbb{P}(1,1,2,2,3,3) \) | \( \sum_{n=0}^{\infty} \frac{(6n!)^2}{(n!)^6(2n)!^2} \varphi^n \) | 1 | 2^{8}3^{6} | -120 | 22 |
| \( X_{3,4} \subset \mathbb{P}(1,1,1,1,1,2) \) | \( \sum_{n=0}^{\infty} \frac{(3n!)^4(4n!)^2}{(n!)^8(2n)!^2} \varphi^n \) | 6 | 2^{6}3^{3} | -156 | 48 |
| \( X_{4,6} \subset \mathbb{P}(1,1,1,2,2,3) \) | \( \sum_{n=0}^{\infty} \frac{(6n!)^2(4n!)^2}{(n!)^8(2n)!^2} \varphi^n \) | 2 | 2^{10}3^{3} | -156 | 32 |

Table 1: Geometric Data of Complete Intersection in Weighted Projective Space

\( \omega_0 \) are the fundamental periods of the manifolds \( X \). \( W(0) \) and \( \mu \) are related to the Yukawa couplings \([2,18]\). \( C_3 \) and \( C_2 \cdot H \) are the characteristic classes.

### 3.1 \( X_{4,4} \)

The A-model manifold \( X_{4,4}^* \) is given by the complete intersection of two quartics in the weighted projective space \( \mathbb{P}^5_{(1,1,2,1,1,2)} \). \( X_{4,4}^* \) is described by a five dimensional polyhedron \( \Delta^* \) with vertices,

\[
\nu_1^* = (-1, -2, -1, -1, -1), \quad \nu_2^* = (1, 0, 0, 0, 0), \quad \nu_3^* = (0, 1, 0, 0, 0), \\
\nu_4^* = (0, 0, 1, 0, 0), \quad \nu_5^* = (0, 0, 0, 1, 0), \quad \nu_6^* = (0, 0, 0, 0, 1),
\]

and a inner point \( \nu_0^* = (0, 0, 0, 0, 0) \). The nef-partition, \( E_1 = \{ \nu_1^*, \nu_2^*, \nu_3^* \} \), \( E_2 = \{ \nu_4^*, \nu_5^*, \nu_6^* \} \), leads to a linear relation between vertices,

\[
l = (-4, -4, 1, 1, 2, 1, 1, 2).
\]

and the equations of the mirror Calabi-Yau \( X_{4,4} \),

\[
P_1 = a_{1,0} - a_1 (X_2X_3X_4X_5X_6^2)^{-1} - a_2 X_2 - a_3 X_3, \\
P_2 = a_{2,0} - a_4 X_4 - a_5 X_5 - a_6 X_6,
\]

or

\[
P_1 = x_1^4 + x_2^4 + x_3^3 + \psi x_4x_5x_6 \\
P_2 = x_4^4 + x_5^4 + x_6^2 + \psi x_1x_2x_3,
\]

after coordinate transform.

The period integral satisfies the Picard-Fuchs equation,

\[
\mathcal{L}_{PF} = \theta^4 - 2\theta^2(4\theta + 1)^2(4\theta + 3)^2,
\]
where \( z = \psi^{-4} \) is the coordinate of \( M_{CS}(X_{4,4}) \), \( \theta = z \frac{d}{dz} \) is the logarithmic derivative. The logarithmic solution underlies the mirror map from the complex structure moduli space of \( X_{4,4}, M_{CS}(X_{4,4}) \) to the Kahler moduli space of \( X^*_4, M_K(X^*_4) \)

\[
z(q) = q - 960q^2 + 213600q^3 - 160471040q^4 - 136981068240q^5 + \ldots
\]

The curve \( C \) wrapped by D-brane is defined as the intersection of \( X_{4,4} \) with two planes,

\[
h_1 = \{x_1 + \alpha x_2 = 0\}, \quad h_2 = \{x_4 + \beta x_5 = 0\}, \quad \alpha^4 = \beta^4 = -1
\]

which splits into

\[
C_+ = P_1 \cap P_2 \cap \{x_3 = 0, x_6 = 0\},
\]

\[
C_- = P_1 \cap P_2 \cap \{x_3^2 - \psi \beta x_5^2 x_6 = 0, x_6^2 - \psi \alpha x_2^2 x_3 = 0, x_3^4 + \psi \beta^2 \alpha x_2^2 x_4^2 = 0\},
\]

The tree level has been solved in \([62]\). The domainwall tension is given by shifting \( n \rightarrow n + \frac{1}{2} \) in the fundamental period,

\[
\mathcal{T}(z) = \mathcal{G}(z; 1/2),
\]

with \( \mathcal{G}(z; \rho) \) the generating function of the solutions of GKZ-system, and satisfies the inhomogeneous Picard-Fuchs equation,

\[
\mathcal{L}_{PF} \mathcal{T} = \frac{4}{(2\pi i)^2} z^{1/2}, \quad (3.1)
\]

Given the domainwall tension,

\[
\mathcal{T}(q) = 64q^{1/2} + \frac{50176}{9} q^{3/2} + \frac{116721664}{25} q^{5/2} + \frac{275837288448}{49} q^{7/2}
\]

\[
+ \frac{671623092863488}{81} q^{9/2} + \ldots,
\]

the disk two point function \( \Delta_{zz} \) is obtained by equation \([2.20]\)

\[
- i \Delta_{zz} = 16q^{1/2} + 12544q^{3/2} + 29180416q^{5/2} + 68959322112q^{7/2} + \ldots \quad (3.2)
\]

Another crucial physical quantity is the Yukawa coupling given by equation \([2.18]\)

\[
C_{zzz} = 4 + 3712q + 7863424q^2 + 18453913600q^3 + \ldots \quad (3.3)
\]
Inserting $\Delta_{zz}$ and $C_{zzz}$ into (2.22), the amplitudes $\mathcal{F}_{zz}^{(0,2)}$ with one insertion is obtained,

$$\mathcal{F}_{zz}^{(0,2)} = -32q - 20480q^2 - 54477824q^3 - 128899612672q^4 + \ldots$$

and the partition function with zero genus and two boundaries is solved by direct integration,

$$\mathcal{F}^{(0,2)} = -32q - 10240q^2 - \frac{54477824}{3}q^3 - 32224903168q^4 + \ldots$$

In one-loop level, the Klein bottle partition function $\mathcal{K}$ contribution has to be considered to cancel the tadpole. In the holomorphic limit, $\mathcal{K}$ is given by equation (2.23) with discriminant $1 - 2^{12}z$ from table [1]

$$\mathcal{K} = 32q + 79456q^2 + \frac{597450752}{3}q^3 + 467690079328q^4 + \ldots$$

The sum of $\mathcal{F}^{(0,2)}$ and $\mathcal{K}$ encodes the genus one BPS invariants by equation (2.24). The first a few invariants are listed in the following table [2].

The next loop level, $\mathcal{F}^{(0,3)}, \mathcal{F}^{(1,1)},$ and $\mathcal{K}^{(1,1)}$ need to be considered. To begin with, $\mathcal{F}^{(0,3)}$ is solved by equation (2.25)

$$\mathcal{F}^{(0,3)} = -\frac{128}{3}q^{3/2} - \frac{63488}{3}q^{5/2} - \frac{205033472}{3}q^{7/2} - \frac{481914650624}{3}q^{9/2} - \frac{1180059100638208}{3}q^{11/2} - 979023701838217216q^{13/2} - \frac{7394674719238803193856}{3}q^{15/2} + \ldots$$

Here $\mathcal{F}_{zz}^{(0,2)}$ is from equation (3.4) and $\Delta^z$ is from (2.15).

Secondly, $\mathcal{F}^{(1,1)}$ can be obtained from equation (2.26)

$$\mathcal{F}^{(1,1)} = -\frac{62}{3}q^{1/2} + 13792q^{3/2} + 35328q^{5/2} - \frac{4535729152}{3}q^{7/2} - 9352462105744q^{9/2} - 31075040152947456q^{11/2} - \frac{273946358860176045056}{3}q^{13/2} - 255818331892779093696512q^{15/2} + \ldots$$

Here we use the formula of $\mathcal{F}^{(1,0)}$ under the holomorphic limit (2.21). The discriminant, Euler characteristic, and second Chern class can be read from table [1]
Moreover, the unoriented contribution $F^{(1,1)}$ is given by solving equation 2.27.

\[
F^{(1,1)} = 2q^{1/2} - 736q^{3/2} + 532736q^{5/2} + 2120819712q^{7/2} + 6893532191312q^{9/2} + 2053350404232928q^{11/2} + 58911806918364210q^{13/2} + 165605905671703558q^{15/2} + \ldots
\]

The domainwall tension has another form when the D-brane wraps different curve. It leads to different partition function that are summarized in Appendix B.

### 3.2 $X_{6,6}$

The description of A-model geometry $X^*_{6,6}$ in weighted projective space $\mathbb{P}(1, 2, 3, 1, 2, 3)$ is given by the polyhedron $\Delta^*$ with one inner point $v_0^* = (0, 0, 0, 0)$ and the following six vertices,

\[
\begin{align*}
  v_1^* &= (-2, -3, -1, -2, -3), & v_2^* &= (1, 0, 0, 0, 0), & v_3^* &= (0, 1, 0, 0, 0), \\
  v_4^* &= (0, 0, 1, 0, 0), & v_5^* &= (0, 0, 0, 1, 0), & v_6^* &= (0, 0, 0, 0, 1),
\end{align*}
\]

satisfying a linear relation $l$ corresponding to the maximal triangulation of $\Delta^*$.

\[
l = (-6, -6; 1, 2, 3, 1, 2, 3).
\]

By the nef-partition $E_1 = \{v_1^*, v_2^*, v_3^*\}, E_2 = \{v_4^*, v_5^*, v_6^*\}$, the equations of mirror threefold $X_{6,6}$ are obtained,

\[
\begin{align*}
P_1 &= a_{1,0} - a_1(X_2^2X_3^3X_4X_5^2X_6^3)^{-1} - a_2X_2 - a_3X_3, \\
P_2 &= a_{2,0} - a_4X_4 - a_5X_5 - a_6X_6,
\end{align*}
\]

and after homogenization, the equations are,

\[
\begin{align*}
P_1 &= x_1^6 + x_2^3 + x_3^2 + \psi x_4x_5x_6 \\
P_2 &= x_4^6 + x_5^3 + x_6^2 + \psi x_1x_2x_3.
\end{align*}
\]

The Picard-Fuchs equation determines the deformations of the complex structure of $X_{6,6}$,

\[
\mathcal{L}_{PF} = \theta^4 - 2^3 3^2 z(6\theta + 1)^2(6\theta + 5)^2.
\]
with $z = \psi^{-6}$, which solves the mirror map,

$$z = q - 37440q^2 + 84900960q^3 - 15150231951360q^4 + \ldots$$

On $X_{6,6}$, B-brane wraps on the curve $C$ defined by the intersection between $X_{6,6}$ and $h_1, h_2$,

$$h_1 = \{x_1^3 + \alpha x_3 = 0\}, \quad h_2 = \{x_2^3 + \beta x_6 = 0\}, \quad \alpha^2 = \beta^2 = -1.$$  

The domainwall tension is given in [62],

$$\mathcal{T}(q) = 256q^{1/2} + \frac{15683584}{9}q^{3/2} + \frac{1626659168256}{25}q^{5/2} + \frac{186494133791883264}{49}q^{7/2} + \frac{22204094064940507334656}{81}q^{9/2} + \ldots,$$

which satisfies the inhomogeneous Picard-Fuchs equation,

$$\mathcal{L}_{PF} \mathcal{T} = \frac{16}{(2\pi i)^2} z^{1/2} \quad (3.7)$$

The two point function $\Delta_{zz}$ is given by equation [2.20]

$$-i\Delta_{zz} = 64q^{1/2} + 3920896q^{3/2} + 406664792064q^{5/2} + \ldots \quad (3.8)$$

and the Yukawa coupling is given by equation [2.18]

$$C_{zzz} = 1 + 67104q + 6778372896q^2 + 771747702257664q^3 + \ldots \quad (3.9)$$

In the one-loop level, we need to consider $\mathcal{F}^{(0,2)}$ and $\mathcal{X}$. The partition function $\mathcal{F}^{(0,2)}$ is solved by inserting $\Delta_{zz}$ and $C_{zzz}$ into [2.22]

$$\mathcal{F}^{(0,2)} = -2048q - 56754176q^2 - \frac{12214287269888}{3}q^3 + \ldots \quad (3.10)$$

The Klein bottle contribution $\mathcal{X}$ can be obtained by equation [2.23] with discriminant $1 - 2^83^6z$.

$$\mathcal{X} = 4608q + 336967776q^2 + 33802444071936q^3 + \ldots \quad (3.11)$$

The genus one BPS invariants on $X_{6,6}$ are extracted by equation in table [5].
For the two-loop level, $F^{(0,3)}$ is given by equation \ref{eq:225}

$$
F^{(0,3)} = -\frac{131072}{3} q^{3/2} - 2166358016 q^{5/2} - 24501652167264 q^{7/2} \\
- \frac{8516834364311833088}{3} q^{9/2} - 3396141356895699624525824 q^{11/2} \\
- 41387003404873243540420753408 q^{13/2} + \ldots
$$

Here $F^{(0,2)}_z$ is from equation \ref{eq:34} and $\Delta^z$ is from \ref{eq:215}.

Moreover, $F^{(1,1)}$ and $K^{(1,1)}$ can be obtained by solving the corresponding holomorphic anomaly equation \ref{eq:226}-\ref{eq:227},

$$
F^{(1,1)} = -\frac{752}{3} q^{1/2} + \frac{12262144}{3} q^{3/2} + \frac{44974083072}{3} q^{5/2} + 3415441373093888 q^{7/2} \\
+ \frac{960915772990614614656}{3} q^{9/2} + 36672842106979528199706624 q^{11/2} \\
+ 461198289188843900932102619136 q^{13/2} + \ldots
$$

$$
K^{(1,1)} = 32 q^{1/2} - 265728 q^{3/2} + 36233883648 q^{5/2} + 5892952067162112 q^{7/2} \\
+ 856257228798248625408 q^{9/2} + 119070042311934462996393984 q^{11/2} \\
+ 16222224902976953255052967624704 q^{13/2} + \ldots
$$

Another two form of domainwall tension on $X_{6,6}$ are studied in \cite{62}. We summarize the new amplitudes related to the new domainwall tension and omit certain detail in Appendix \ref{appen:3}.

### 3.3 $X_{3,4}$

The A-incarnation $X_{3,4}^* \subset \mathbb{P}^5_{(1,1,1,1,1,2)}$ is related to the polyhedron $\Delta^*$ in the mirror construction. There is one inner point $v_0^* = (0,0,0,0,0)$, and six vertices,

$$
\begin{align*}
v_1^* &= (-1,-1,-1,-1,-2), & v_2^* &= (1,0,0,0,0), & v_3^* &= (0,1,0,0,0), \\
v_4^* &= (0,0,1,0,0), & v_5^* &= (0,0,0,1,0), & v_6^* &= (0,0,0,0,1),
\end{align*}
$$

and the maximal triangulation of $\Delta^*$ corresponds to the charge vector,

$$
l = (-3,-4;1,1,1,1,1,2). \quad (3.12)
$$
The mirror threefold $X_{3,4}$ is defined by the equation,

$$P_1 = a_{1,0} - a_1 (X_2 X_3 X_4 X_5 X_6)^{-1} - a_2 X_2 - a_3 X_3,$$

$$P_2 = a_{2,0} - a_4 X_4 - a_5 X_5 - a_6 X_6,$$

and the period integrals are annihilated by the GKZ-operator obtained from $I^{3,12}$.

$$\mathcal{L} = \prod_{i=1}^5 \frac{\partial}{\partial a_i} \frac{\partial}{\partial a_6} - \left( \frac{\partial}{\partial a_{1,0}} \right)^3 \left( \frac{\partial}{\partial a_{2,0}} \right)^4,$$

In terms of the logarithmic derivatives, $\mathcal{L}$ can be rewritten as,

$$\mathcal{L} = \theta^5 (2\theta)(2\theta - 1) - z \prod_{i=1}^3 \prod_{j=1}^3 (4\theta + i)(3\theta + j),$$

with $z = \frac{a_1 a_2 a_3 a_4 a_5 a_6}{a_{1,0} a_{2,0}}$ the coordinate on $M_{CS}(X_{3,4})$. It can be reduced to the Picard-Fuchs equation,

$$\mathcal{L}_{PF} = \theta^4 - 2^2 3\pi (4\theta + 1)(3\theta + 1)(3\theta + 2)(4\theta + 3)$$

and solves the mirror map,

$$z = q - 420q^2 + 47070q^3 - 12722000q^4 - 3647205075q^5 + \ldots.$$  

At tree level, the domainwall tension satisfies the inhomogeneous Picard-Fuchs equation [4] as before,

$$\mathcal{L}_{PF} \mathcal{T} = - \frac{3}{8\pi^2} z^{1/2},$$

and $\mathcal{T}(q)$ is obtained by restricting the domainwall tension on $X_{12} \subset \mathbb{P}_4^{(1,2,3,3,3)}$ to the point $t_2 = t_3 = 0$,

$$\mathcal{T}(q) = 48q^{1/2} + \frac{4192}{3} q^{3/2} + \frac{13300848}{25} q^{5/2} + \frac{13037063136}{49} q^{7/2} + \ldots,$$

after inserting the mirror map.

The disk amplitude $\Delta_{zz}$ is related to the domainwall tension by equation [2,20]

$$-i\Delta_{zz} = 12q^{1/2} + 3144q^{3/2} + 3325212q^{5/2} + \ldots$$  

(3.14)
and the Yukawa coupling is given by equation 2.18

\[ C_{zzz} = 6 + 1944q + 1790424q^2 + 1748375280q^3 + \ldots \] (3.15)

Given \( \Delta_{zz} \) and \( C_{zzz} \), \( \mathcal{F}^{(0,2)} \) can be solved by 2.22

\[ \mathcal{F}^{(0,2)} = -12q - 1200q^2 - 1038568q^3 - 759633600q^4 + \ldots \] (3.16)

In addition, there are Klein bottle contribution in the one loop level,

\[ \mathcal{K} = 6q + 10674q^2 + 12298308q^3 + 12377069538q^4 + \ldots \] (3.17)

Then, the genus one BPS invariants are extracted from \( \mathcal{F}^{(0,2)} + \mathcal{K} \) in table 4.

At two loop level, the partition functions \( \mathcal{F}^{(0,3)} \), \( \mathcal{F}^{(1,1)} \), and \( \mathcal{K}^{(1,1)} \) can be computed as before by solving the corresponding holomorphic anomaly equations.

\[ \mathcal{F}^{(0,3)} = -8q^{3/2} - 1104q^{5/2} - 1968216q^{7/2} - 1887797648q^{9/2} - 1917394366728q^{11/2} - 1975452738939456q^{13/2} - 2058121403895827040q^{15/2} + \ldots, \]

\[ \mathcal{F}^{(1,1)} = -\frac{23}{2}q^{1/2} + 1967q^{3/2} + \frac{210573}{2}q^{5/2} + 78910897q^{7/2} + 6406184491q^{9/2} + 34970231999763q^{11/2} + \frac{6229034499535317}{2}q^{13/2} + 31248235175950182696q^{15/2} + \ldots, \]

\[ \mathcal{K}^{(1,1)} = q^{1/2} - 174q^{3/2} + 35877q^{5/2} + 64858286q^{7/2} + 90920039958q^{9/2} + 114240749617194q^{11/2} + 137210633017558493q^{13/2} + 160866795930971392080q^{15/2} + \ldots. \]

### 3.4 \( X_{4,6} \)

The A-model geometry \( X_{4,6}^* \) is related to the polyhedron \( \Delta^* \) with vertices,

\[ v_1^* = (-1, -1, -2, -2, -3), \quad v_2^* = (1, 0, 0, 0, 0), \quad v_3^* = (0, 1, 0, 0, 0), \]

\[ v_4^* = (0, 0, 1, 0, 0), \quad v_5^* = (0, 0, 0, 1, 0), \quad v_6^* = (0, 0, 0, 0, 1), \]
in the Batyrev-Borisov construction, which give rise to the mirror geometry described by
\[ a_{1,0} - a_1(X_2X_3X_4^2X_5^2X_6^3)^{-1} + a_2X_2 + a_4X_4 = 0, \]
\[ a_{2,0} - a_3X_3 - a_5X_5 - a_6X_6 = 0. \]
The charge vector,
\[ l = (-4, -6; 1, 1, 1, 2, 2, 3), \]
defines the GKZ operator \( \mathcal{L} \) that can be reduced to Picard-Fuchs operator,
\[ \mathcal{L}_{PF} = \theta^4 - 2^3 3\zeta(6\theta + 1)(4\theta + 1)(4\theta + 3)(6\theta + 5), \]
with \( z = \frac{a_1a_2a_3a_4^2a_6}{a_1^0a_2^0} \). The mirror map is given by the logarithmic solution of \( \mathcal{L}_{PF} \),
\[ z = q - 6144q^2 + 6866784q^3 - 48364795904q^4 - 347475565045200q^5 + \ldots. \]

The domainwall tension on \( X_{4,6} \) is obtained by restricting the domainwall tension on \( X_{12} \subset \mathbb{P}^4_{(1,2,2,3,4)} \) at the singular locus in \( M_{CS}(X_{12}) \). It is the solution of the inhomogeneous Picard-Fuchs equation[4],
\[ \mathcal{L}_{PF} \mathcal{T} = \frac{4}{(2\pi i)^2}z^{1/2}, \]
and expressed in the \( q \)-coordinate as,
\[ \mathcal{T}(q) = 128q^{1/2} + \frac{874496}{9}q^{3/2} + \frac{13288624128}{25}q^{5/2} + \ldots, \]

The Yukawa coupling \( C_{zzz} \) and the disk two point function \( \Delta_{zz} \),
\[ C_{zzz} = 2 + 15552q + 223248960q^2 + 3614882992128q^3 + \ldots, \]
\[ -i\Delta_{zz} = 32q^{1/2} + 218624q^{3/2} + 3322156032q^{5/2} + \ldots. \]
solves the equation[2,22] and obtains the partition function \( \mathcal{F}^{(0,2)} \),
\[ \mathcal{F}^{(0,2)} = -256q - 753664q^2 - \frac{24806760448}{3}q^3 + \ldots. \]

To extract BPS invariants, the Klein bottle partition function \( \mathcal{K} \) contribution from equation[2,23] has to be considered,
\[ \mathcal{K} = 384q + 5097312q^2 + 79609445376q^3 + \ldots. \]
The first a few genus one BPS invariants are listed in table 5. 
For the two loop level, $\mathcal{F}^{(0,3)}, \mathcal{F}^{(1,1)},$ and $\mathcal{K}^{(1,1)}$ are solved by the corresponding holomorphic anomaly equation,

$$
\mathcal{F}^{(0,3)} = -\frac{4096}{3} q^{3/2} - 6750208q^{5/2} - 124076818432q^{7/2} - 610485294771200q^{9/2} - 34531769181309009920q^{11/2} \\
- 5963796649732493586192q^{13/2} + \ldots
$$

$$
\mathcal{F}^{(1,1)} = -\frac{244}{3} q^{1/2} + \frac{522176}{3} q^{3/2} - 318506752q^{5/2} - \frac{21046447364096}{3} q^{7/2} \\
- \frac{434275724425389088}{3} q^{9/2} - 2749664281128057287168q^{11/2} \\
- \frac{152981476421267688845656064}{3} q^{13/2} + \ldots,
$$

$$
\mathcal{K}^{(1,1)} = 8q^{1/2} - 16512q^{3/2} + 138651648q^{5/2} + 3429512359936q^{7/2} \\
+ 73806308934007104q^{9/2} + 1489674404926065349632q^{11/2} \\
+ 29218644683428988803919872q^{13/2} + \ldots.
$$

## 4 Summary and Conclusion

In this article, we study the open topological string partition function of low genus and boundaries by solving the extended holomorphic anomaly equation. For four one-modulus complete intersection Calabi-Yau threefolds, we compute the partition function, and BPS invariants by including the orientifold plane contribution. In the calculation, we assume that the holomorphic ambiguities are zero in most cases. It seems that the assumption of holomorphic ambiguity as zero does not give rise to the integer BPS invariants for the two loop level. In the future, we will use the localization technique to verify our results, identifies the holomorphic ambiguities, and try to compute BPS invariants of higher genus. Also, it is worthwhile to study the open string amplitudes and BPS invariants on more complicated geometric background, like two-parameter Calabi-Yau hypersurfaces in toric varieties and complete intersection in weighted projective spaces.

Furthermore, we hope to extend the approach of remodeling B-model for local Calabi-Yau threefolds [12] to the compact Calabi-Yau geometry. In the local case,
loading, this method is originally obtained as a solution to the loop equations of matrix models, giving an explicit form for its open and closed amplitudes in terms of residue calculus on the spectral curve of the matrix model unambiguously. We will try to generalize the formalism of mirror curve in the B-model geometry to higher dimensional manifolds so that it applies directly to compact Calabi-Yau threefolds. It may provide some hints or implication on finding holomorphic ambiguity for solving the holomorphic anomaly equation.

A Genus One BPS Invariants

| $d$ | $n_d^{(1,\text{real})}$ |
|-----|------------------------|
| 2   | 0                      |
| 4   | 34608                  |
| 6   | 90495488               |
| 8   | 217732588080           |
| 10  | 519139865625600        |
| 12  | 1244429369521121008    |
| 14  | 3006720894671076040704 |
| 16  | 7321444729803039221389872 |
| 18  | 17953446932877098422752258560 |
| 20  | 44297245727128905301804000758672 |
| 22  | 109886598389085980224169122468220928 |
| 24  | 273879040843960416419367130685174742512 |
| 26  | 685441373723245253419305721507760898859008 |
| 28  | 17217469476331311668003878895071508781112640176 |
| 30  | 4338887285554338333593607708507980232083840516608 |
| 32  | 10965963996574241145002521057990564470746280768723504 |
| 34  | 27787284139820409740273724776125024930464211651719647232 |
| 36  | 70577433742673178934242350825488542706466337629896022356096 |

Table 2: Real BPS Invariants $n_d^{(1,\text{real})}$ on $X_{4,4}^*$
| $d$  | $n_{d}^{(1, \text{real})}$                |
|---|---|
| 2  | 1280          |
| 4  | 140106800     |
| 6  | 14865507490560|
| 8  | 1618366267878984240 |
| 10 | 180879246399441648493312 |
| 12 | 20637943177316354437275713520 |
| 14 | 2392752424285671726708540669594880 |
| 16 | 280938904554424832218336437482840382000 |
| 18 | 33322503987626461636383333029163640299256320 |
| 20 | 3985484966875635872252891080039572436585586076304 |
| 22 | 480000354728744360465611971501493264886844125644120320 |
| 24 | 581506209874274924717087014130835484350552668720148891120 |
| 26 | 7080336576485783736588492713534135020651896830908916304714499840 |
| 28 | 865859554481709720298276080110159012438129283882801758005389628254640 |
| 30 | 106291164929780296502343091642064025525777289377175483312826183882166785280 |
| 32 | 13092038283686665810815987066336234710109058460128602315755261958471721441840 |

Table 3: Real BPS Invariants $n_{d}^{(1, \text{real})}$ on $X_{6,6}$.
| $d$ | $n_d^{(1, real)}$ |
|-----|------------------|
| 2   | -3               |
| 4   | 4737             |
| 6   | 5629871          |
| 8   | 5808717969       |
| 10  | 5833299344640    |
| 12  | 5849047597419579 |
| 14  | 5891041195079975079 |
| 16  | 5967814657032068338641 |
| 18  | 6080507782501626356757812 |
| 20  | 6228358922063613246678825600 |
| 22  | 6410400570932796039941119094847 |
| 24  | 6625980254770789737512663516966507 |
| 26  | 6874895369173424708708048755071562560 |
| 28  | 71574015331487088778224594822468525554579 |
| 30  | 7474184389248964662717428155183980031936826 |
| 32  | 7826326324564347587478671218568566261580459217 |
| 34  | 8215278529651988449131114534693789042732127972935 |
| 36  | 86428411294113777700003000151742474654966182322188388 |

Table 4: Real BPS Invariants $n_d^{(1, real)}$ on $X_{3,4}^*$
| $d$ | $n_d^{(1, \text{real})}$ |
|-----|-------------------|
| 2   | 64                |
| 4   | 2171824           |
| 6   | 35670262592       |
| 8   | 572119847810608   |
| 10  | 9264343811094049984 |
| 12  | 152008070011375287627120 |
| 14  | 2524239861846741083787784768 |
| 16  | 42347361041482559192621064600112 |
| 18  | 716550646088663055133099960033534976 |
| 20  | 12212640014878093727055039426586080874768 |
| 22  | 209432794162057905066976004068843293806233408 |
| 24  | 361055956393932419362664034621338976108612559344 |
| 26  | 62530394810056915742075848828164170928495828847300288 |
| 28  | 10872894717054304714569141400078475721731390116433863314736 |
| 30  | 18972692030691869017569026158060107915758257894036779123617216 |
| 32  | 332100638118912607108732075831745677990230635273178077491923999280 |
| 34  | 582937514397377378627629897451351956438616498273739209349873808170385088 |
| 36  | 102579924595916673344650841965110680957163398068991580485392919975049948288 |
B Partition Functions on $X_{4,4}$

With the D-brane wrapping the following curve,

$$X_{4,4} = \{ P_1 = 0, P_2 = 0 \} \cap h_1 = \{ x_1^2 + \alpha_1 \sqrt{2} x_3 = 0 \} \cap h_2 = \{ x_4^2 + \alpha_2 \sqrt{2} x_6 = 0 \}, \alpha_1^2 = \alpha_2^2 = -1,$$

there is another domainwall $T(z)$ satisfying [62],

$$\mathcal{L}_{PF} T(z) = \frac{1}{(2\pi i)^2} \left( \frac{8}{27} \eta z^{1/3} + \frac{800}{27} \eta^2 z^{2/3} \right)$$

with $\eta^3 = 1$, and $T$ is

$$T(z(q)) = \frac{2}{9} (\tilde{\eta} \sigma(z; 1/3) + \tilde{\eta}^2 \sigma(z, 2/3))$$

$$= 24q^{1/3} + 150q^{2/3} + \frac{2571}{2} q^{4/3} + \frac{417024}{25} q^{5/3} + \ldots,$$

with $\tilde{\eta}^3 = 1$.

Several amplitudes are listed as follow,

$$F_{zz}^{0,1} = 3q^{1/3} + 75q^{2/3} + 2571q^{4/3} + 52128q^{7/3} + 5677584q^{7/3} + 131074059q^{8/3}$$

$$+ 13380832800q^{10/3} + 310078975968q^{11/3} + 32533714689024q^{13/3}$$

$$+ 756271662397200q^{14/3} + 80669533303699467q^{16/3} + 1878508762455982080q^{17/3}$$

$$+ 202613133782293403616q^{19/3} + \ldots,$$

$$F_{z}^{0,2} = -\frac{27}{16} q^{2/3} - \frac{225}{4} q - \frac{16875}{32} q^{4/3} - \frac{10611}{20} q^{5/3} - \frac{140409}{8} q^2 - \frac{974700}{7} q^{7/3}$$

$$- \frac{126814971}{64} q^{8/3} - \frac{13362659451}{4} q^3 - \frac{14555738232}{40} q^{10/3} - \frac{11}{8} q^{11/3}$$

$$- \frac{899642497977}{16} q^{13/3} - \frac{7818600973032}{13} q^{13/3} - \frac{17720520118920}{7} q^{14/3}$$

$$- 109860797109960q^5 + \ldots.$$
\[ F^{0,3} = -\frac{9}{32} q - \frac{675}{32} q^{4/3} - \frac{16875}{32} q^{5/3} - \frac{36765}{8} q^{2} - \frac{186651}{16} q^{7/3} - \frac{659475}{32} q^{8/3} - \frac{48068955}{32} q^{3} \]
\[ - \frac{553202811}{16} q^{10/3} - \frac{13023069651}{16} q^{11/3} - \frac{61206596433}{8} q^{4} - \frac{1305111161691}{16} q^{13/3} \]
\[ - \frac{30239942059107}{16} q^{14/3} - \frac{34877455807311}{2} q^{5} + \ldots , \]

\[ F^{1,1} = -3 q^{1/3} - \frac{475}{4} q^{2/3} + \frac{7431}{4} q^{4/3} + 90040 q^{5/3} + 78480 q^{7/3} - \frac{19724437}{4} q^{8/3} - 398429040 q^{10/3} - 3346434248 q^{11/3} - 194661693968 q^{13/3} - 38405520795668 q^{14/3} + \ldots \]

\section{Partition Functions on $X_{6,6}$}

\subsection{Domainwall Tension II}

When the D-brane wraps the curve,

\[ X_{6,6} = \{ P_1 = 0, P_2 = 0 \} \cap h_1 = \{ x_1^2 + 2^{1/3} \alpha_1 x_2 = 0 \} \cap h_2 = \{ x_4^2 + \alpha_2 2^{1/3} x_5 = 0 \}, \alpha_1^3 = \alpha_2^3 = -1 \]

the domainwall tension satisfies the inhomogeneous Picard-Fuchs equation \cite{62},

\[ \mathcal{L}_{PF} \mathcal{G}(z) = \frac{1}{(2\pi i)^2} (2\hat{\eta} z^{1/3} + 216 \hat{\eta}^2 z^{2/3}) \]

where \( \hat{\eta} \) dependens on a combination of \( \alpha_1 \) and \( \alpha_2 \). Then, after inserting the mirror map, \( \mathcal{G}(q)(\hat{\eta} = 1) \) is,

\[ \mathcal{G}(z(q)) = \frac{2}{9} (2 \hat{\eta} \mathcal{O}(z; 1/3) + \hat{\eta}^2 \mathcal{O}(z, 2/3)) \]
\[ = 54 q^{1/3} + \frac{2187}{2} q^{2/3} + \frac{1733643}{8} q^{4/3} + \frac{252362304}{25} q^{5/3} + \ldots , \]

with \( \hat{\eta}^3 = 1 \)

The relevant amplitudes are listed as follow.
\[ F_{zz}^{0,1} = 6q^{1/3} + 486q^{2/3} + 385254q^{4/3} + 28040256q^{5/3} + 39380410656q^{7/3} + 2970341239014q^{8/3} + 4499651224412736q^{10/3} + 34182958283713216q^{11/3} + 535032321621406746624q^{13/3} + 4076004793259671669024q^{14/3} + 64986912884772303566448870q^{16/3} + 4958366424970308770413707264q^{17/3} + 799919071732482292670846353088q^{19/3} + \ldots, \]

\[ F_{zz}^{0,2} = -27q^{2/3} - 2916q - \frac{177147}{2}q^{4/3} - \frac{3310956}{5}q^{5/3} - 980289702999702q^{6/3} - \frac{39356189371608192}{7}q^{8/3} - 227301205949499102336q^{10/3} + \frac{238001239476988416}{13}q^{11/3} + \frac{238001239476988416}{5}q^{12} + \ldots, \]

\[ F_{zz}^{0,3} = -36q - 8748q^{4/3} - 708588q^{5/3} - 329073688q^{7/3} - 32164935084q^{8/3} - 965557032876q^{9/3} - 50254848648792q^{10/3} - 3807431288304408q^{11/3} - 122006194369498512q^{12/3} - 5813866460445646680q^{13/3} - 441980260922360950488q^{14/3} - 14242379576529165875136q^{15} + \ldots, \]

\[ F_{zz}^{1,1} = -\frac{35}{2}q^{1/3} - \frac{4779}{2}q^{2/3} - \frac{381189}{2}q^{4/3} + 52806816q^{5/3} + 3085343928q^{7/3} + \frac{741480466149}{2}q^{8/3} + 227008394824256q^{10/3} + 34734669284988288q^{11/3} + 23806991100654596112q^{13/3} + 302167750505907600792q^{14/3} + \ldots, \]

### C.2 Domainwall Tension III

The equation 3.7 has another solution,

\[ \mathcal{T}(z(q)) = \frac{1}{4}(\tilde{\eta} \sigma(z; 1/4) + \tilde{\eta}^2 \sigma(z, 1/2) + \tilde{\eta}^3 \sigma(z; 3/4)) \]

\[ = 32q^{1/4} + 256q^{1/2} + \frac{25088}{9}q^{3/4} + \frac{2092032}{25}q^{5/4} + \ldots, \]

with \( \tilde{\eta}^4 = 1 \), which is independent to the domainwall tension in Section 3.2.
The relevant amplitudes are listed as follows.

\[ F_{zz}^{0,1} = 2q^{1/4} + 64q^{1/2} + 1568q^{3/4} + 130752q^{5/4} + 3920896q^{3/2} + 87386112q^{7/4} + 1329760308q^{9/4} + 406664792064q^{5/2} + 9382033536768q^{11/4} + 151726624562272q^{13/4} + 46623533447970816q^{7/2} + 1081732875754733568q^{15/4} + 180314551186106647608q^{17/4} + 555102351235126833664q^{9/2} + 129090812443883028381312q^{19/4} + \ldots, \]

\[ F_{0,2}^{0.2} = -4q^{1/2} - \frac{512}{3}q^{3/4} - 5184q - \frac{401408}{5}q^{5/4} - \frac{2713216}{3}q^{3/2} - \frac{30482432}{7}q^{7/4} - 141430784q^2 - \frac{20026621952}{9}q^{9/4} - 27029641408q^{5/2} - \frac{319214466258}{11}q^{11/4} - 10211216431104q^3 - \frac{225817004231884}{13}q^{13/4} - \frac{16057665232558080}{7}q^{7/2} - \frac{36669651197018112}{15}q^{15/4} - 8839898177637360q^4 - \frac{2615718938681164648448}{17}q^{17/4} - \frac{1872789489451577109856}{9}q^{9/2} - \frac{43691386492446963795968}{19}q^{19/4} + \ldots, \]

\[ F_{0,3}^{0.3} = -4q^{3/4} - 128q - 7232q^{5/4} - \frac{733184}{3}q^{3/2} - 5752448q^{7/4} - 86075392q^2 - \frac{3079180288}{3}q^{9/4} - 11832852480q^{5/2} - 254801522976q^{11/4} - 3813556254720q^3 - 63257958588928q^{13/4} - 1354065828249600q^{7/2} - 30856030594393088q^{15/4} - 485588407814225920q^4 - 7934104744264269824q^{17/4} - \frac{470970231779873521664}{3}q^{9/2} - 3586148378350321933936q^{19/4} + \ldots, \]

\[ F_{1,1}^{1.1} = -\frac{29}{6}q^{1/4} - \frac{752}{3}q^{1/2} - \frac{25480}{3}q^{3/4} + 40992q^{5/4} + \frac{12262144}{3}q^{3/2} + 212925696q^{7/4} + \frac{2451671612}{3}q^{9/4} + 44974083072q^{5/2} + 1110117057984q^{11/4} + 63019510378496q^{13/4} + 3415441373093888q^{7/2} + 127380480523266048q^{15/4} + 7104502724515200414q^{17/4} + \frac{960915772990614614656}{3}q^{9/2} + \ldots. \]
References

[1] Mina Aganagic, Albrecht Klemm, and Cumrun Vafa. “Disk instantons, mirror symmetry and the duality web”. In: Z. Naturforsch. A 57 (2002), pp. 1–28. arXiv:hep-th/0105045

[2] Mina Aganagic and Cumrun Vafa. “Mirror symmetry, D-branes and counting holomorphic discs”. In: (Dec. 2000). arXiv:hep-th/0012041

[3] Murad Alim and Jean Dominique Lange. “Polynomial Structure of the (Open) Topological String Partition Function”. In: JHEP 10 (2007), p. 045. arXiv:0708.2886 [hep-th]

[4] Murad Alim et al. “Type II/F-theory Superpotentials with Several Deformations and N=1 Mirror Symmetry”. In: JHEP 06 (2011), p. 103. arXiv:1010.0977 [hep-th]

[5] Gert Almkvist et al. “Tables of Calabi–Yau equations”. In: arXiv preprint math/0507430 (2005).

[6] V. V. Batyrev and L. A. Borisov. “Dual cones and mirror symmetry for generalized Calabi-Yau manifolds”. In: AMS/IP Stud. Adv. Math. 1 (1996). Ed. by B. Greene and Shing-Tung Yau, pp. 71–86.

[7] Victor V. Batyrev and Lev A. Borisov. “On Calabi-Yau complete intersections in toric varieties”. In: (Dec. 1994). arXiv:alg-geom/9412017

[8] Victor V. Batyrev and Duco van Straten. “Generalized hypergeometric functions and rational curves on Calabi-Yau complete intersections in toric varieties”. In: Communications in Mathematical Physics 168.3 (Apr. 1995), pp. 493–533.

[9] M. Bershadsky et al. “Holomorphic anomalies in topological field theories”. In: Nucl. Phys. B 405 (1993). Ed. by B. Greene and Shing-Tung Yau, pp. 279–304. arXiv:hep-th/9302103

[10] M. Bershadsky et al. “Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes”. In: Commun. Math. Phys. 165 (1994), pp. 311–428. arXiv:hep-th/9309140

[11] Lev Borisov. “Towards the Mirror Symmetry for Calabi-Yau Complete intersections in Gorenstein Toric Fano Varieties”. In: arXiv e-prints (Oct. 1993). eprint:alg-geom/9310001(math.AG).

[12] Vincent Bouchard et al. “Remodeling the B-Model”. In: Communications in Mathematical Physics 287.1 (Sept. 2008), pp. 117–178.
[13] Vincent Bouchard et al. “Topological Open Strings on Orbifolds”. In: *Communications in Mathematical Physics* 296.3 (Mar. 2010), pp. 589–623.

[14] Philip Candelas et al. “A Pair of Calabi-Yau manifolds as an exactly soluble superconformal theory”. In: *Nucl. Phys. B* 359 (1991), pp. 21–74.

[15] Sergio Cecotti and Cumrun Vafa. “Ising model and N=2 supersymmetric theories”. In: *Communications in Mathematical Physics* 157.1 (Oct. 1993), pp. 139–178.

[16] Paul L. H. Cook, Hirosi Ooguri, and Jie Yang. “Comments on the Holomorphic Anomaly in Open Topological String Theory”. In: *Phys. Lett. B* 653 (2007), pp. 335–337. arXiv:0706.0511 [hep-th].

[17] Robbert Dijkgraaf and Cumrun Vafa. “A Perturbative window into nonperturbative physics”. In: (Aug. 2002). arXiv:hep-th/0208048.

[18] Robbert Dijkgraaf and Cumrun Vafa. “Matrix models, topological strings, and supersymmetric gauge theories”. In: *Nucl. Phys. B* 644 (2002), pp. 3–20. arXiv:hep-th/0206255.

[19] Robbert Dijkgraaf and Cumrun Vafa. “On geometry and matrix models”. In: *Nucl. Phys. B* 644 (2002), pp. 21–39. doi:10.1016/S0550-3213(02)00764-2. arXiv:hep-th/0207108.

[20] Robbert Dijkgraaf, Herman L. Verlinde, and Erik P. Verlinde. “Topological strings in d < 1”. In: *Nucl. Phys. B* 352 (1991), pp. 59–86.

[21] B Eynard. “A short overview of the 'Topological recursion'”. In: (Dec. 2014). arXiv:1412.3286 [math-ph].

[22] Bertrand Eynard, Marcos Marino, and Nicolas Orantin. “Holomorphic anomaly and matrix models”. In: *JHEP* 06 (2007), p. 058. arXiv:hep-th/0702110.

[23] Bertrand Eynard and Nicolas Orantin. “Computation of Open Gromov–Witten Invariants for Toric Calabi–Yau 3-Folds by Topological Recursion, a Proof of the BKMP Conjecture”. In: *Commun. Math. Phys.* 337.2 (2015), pp. 483–567. arXiv:1205.1103 [math-ph].

[24] Bertrand Eynard and Nicolas Orantin. “Invariants of algebraic curves and topological expansion”. In: *Commun. Num. Theor. Phys.* 1 (2007), pp. 347–452. arXiv:math-ph/0702049.

[25] Brian Forbes. “Open string mirror maps from Picard-Fuchs equations on relative cohomology”. In: (July 2003). arXiv:hep-th/0307167.
[26] Brian Forbes and Masao Jinzenji. “Extending the Picard-Fuchs system of local mirror symmetry”. In: J. Math. Phys. 46 (2005), p. 082302. arXiv: hep-th/0503098

[27] Mark L. Green. “Griffiths’ infinitesimal invariant and the Abel-Jacobi map”. In: Journal of Differential Geometry 29.3 (1989), pp. 545–555.

[28] Mark L. Green. “Infinitesimal Methods in Hodge Theory”. In: Algebraic Cycles and Hodge Theory: Lectures given at the 2nd Session of the Centro Internazionale Matematico Estivo (C.I.M.E.) held in Torino, Italy, June 21-29, 1993. Ed. by Fabio Bardelli and Alberto Albano. Berlin, Heidelberg: Springer Berlin Heidelberg, 1994, pp. 1–92. ISBN: 978-3-540-49046-3.

[29] Philip A. Griffiths. “On the Periods of Certain Rational Integrals: I”. In: Annals of Mathematics 90.3 (1969), pp. 460–495.

[30] Phillip A. Griffiths. “Infinitesimal variations of hodge structure (III) : determinantal varieties and the infinitesimal invariant of normal functions”. In: Compositio Mathematica 50.2-3 (1983), pp. 267–324.

[31] Phillip A. Griffiths. “On the Periods of Certain Rational Integrals: II”. In: Annals of Mathematics 90.3 (1969), pp. 496–541.

[32] S. Hosono, B. H. Lian, and Shing-Tung Yau. “GKZ generalized hypergeometric systems in mirror symmetry of Calabi-Yau hypersurfaces”. In: Commun. Math. Phys. 182 (1996), pp. 535–578. arXiv: alg-geom/9511001.

[33] S. Hosono et al. “Mirror symmetry, mirror map and applications to Calabi-Yau hypersurfaces”. In: Commun. Math. Phys. 167 (1995), pp. 301–350. arXiv: hep-th/9308122

[34] S. Hosono et al. “Mirror symmetry, mirror map and applications to complete intersection Calabi-Yau spaces”. In: Nucl. Phys. B 433 (1995). Ed. by B. Greene and Shing-Tung Yau, pp. 501–554. arXiv: hep-th/9406055.

[35] Shinobu Hosono and Yukiko Konishi. “Higher genus Gromov-Witten invariants of the Grassmannian, and the Pfaffian Calabi-Yau threefolds”. In: Adv. Theor. Math. Phys. 13.2 (2009), pp. 463–495. arXiv:0704.2928 [math.AG].

[36] Min-xin Huang, Albrecht Klemm, and Seth Quackenbush. “Topological string theory on compact Calabi-Yau: Modularity and boundary conditions”. In: Lect. Notes Phys. 757 (2009), pp. 45–102. arXiv: hep-th/0612125.
[37] Gel’fand Im, Mikhail M. Kapranov, and Andrei Zelevinsky. “Generalized Euler integrals and A-hypergeometric functions”. In: Advances in Mathematics 84 (1990), pp. 255–271.

[38] Shamit Kachru et al. “Mirror symmetry for open strings”. In: Physical Review D 62.12 (Nov. 2000).

[39] Shamit Kachru et al. “Open string instantons and superpotentials”. In: Physical Review D 62.2 (June 2000).

[40] Sheldon H. Katz and Chiu-Chu Melissa Liu. “Enumerative geometry of stable maps with Lagrangian boundary conditions and multiple covers of the disc”. In: Adv. Theor. Math. Phys. 5 (2001). Ed. by David Auckly and Jim Bryan, pp. 1–49. DOI: 10.2140/gtm.2006.8.1. arXiv: math/0103074.

[41] Johanna Knapp and Emanuel Scheidegger. “Towards Open String Mirror Symmetry for One-Parameter Calabi-Yau Hypersurfaces”. In: Adv. Theor. Math. Phys. 13.4 (2009), pp. 991–1075. arXiv: 0805.1013 [hep-th].

[42] Yukiko Konishi and Satoshi Minabe. “On solutions to Walcher’s extended holomorphic anomaly equation”. In: Commun. Num. Theor. Phys. 1 (2007), pp. 579–603. arXiv: 0708.2898 [math.AG].

[43] Maxim Kontsevich. “Homological Algebra of Mirror Symmetry”. In: (Nov. 1994). arXiv: alg-geom/9411018.

[44] Maxim Kontsevich and Yan Soibelman. “Homological mirror symmetry and torus fibrations”. In: KIAS Annual International Conference on Symplectic Geometry and Mirror Symmetry. Nov. 2000, pp. 203–263. arXiv: math/0011041.

[45] Daniel Krefl and Johannes Walcher. “Real Mirror Symmetry for One-parameter Hypersurfaces”. In: JHEP 09 (2008), p. 031. arXiv: 0805.0792 [hep-th].

[46] Daniel Krefl and Johannes Walcher. “The Real Topological String on a local Calabi-Yau”. In: (Feb. 2009). arXiv: 0902.0616 [hep-th].

[47] W. Lerche and P. Mayr. “On N=1 mirror symmetry for open type 2 strings”. In: (Nov. 2001). arXiv: hep-th/0111113.

[48] W. Lerche, P. Mayr, and N. Warner. “Holomorphic N=1 special geometry of open - closed type II strings”. In: (July 2002). arXiv: hep-th/0207259.

[49] W. Lerche, P. Mayr, and N. Warner. “N=1 special geometry, mixed Hodge variations and toric geometry”. In: (Aug. 2002). arXiv: hep-th/0208039.
[50] Jun Li and Yun S. Song. “Open string instantons and relative stable morphisms”. In: Adv. Theor. Math. Phys. 5 (2001). Ed. by David Auckly and Jim Bryan, pp. 67–91. arXiv: [hep-th/0103100]

[51] Si Li, Bong H. Lian, and Shing-Tung Yau. “Picard-Fuchs Equations for Relative Periods and Abel-Jacobi Map for Calabi-Yau Hypersurfaces”. In: (Oct. 2009). arXiv: [0910.4215 [math.AG]]

[52] Marcos Marino. “Les Houches lectures on matrix models and topological strings”. In: Oct. 2004. arXiv: [hep-th/0410165]

[53] Marcos Mariño. “Open string amplitudes and large order behavior in topological string theory”. In: Journal of High Energy Physics 2008.03 (Jan. 2008), pp. 060–060.

[54] Peter Mayr. “N=1 mirror symmetry and open / closed string duality”. In: Adv. Theor. Math. Phys. 5 (2002), pp. 213–242. DOI: [10.4310/ATMP.2001.v5.n2.a1] arXiv: [hep-th/0108229]

[55] David R Morrison. “Mirror symmetry and rational curves on quintic threefolds: a guide for mathematicians”. In: Journal of the American Mathematical Society 6.1 (1993), pp. 223–247.

[56] David R. Morrison and Johannes Walcher. “D-branes and Normal Functions”. In: Adv. Theor. Math. Phys. 13.2 (2009), pp. 553–598. arXiv: [0709.4028 [hep-th]]

[57] R. Pandharipande, J. Solomon, and J. Walcher. “Disk enumeration on the quintic 3-fold”. In: Journal of the American Mathematical Society 21.4 (Feb. 2008), pp. 1169–1209.

[58] Wilfried Schmid. “Variation of Hodge Structure: The Singularities of the Period Mapping.” In: Inventiones mathematicae 22 (1973), pp. 211–320.

[59] Masahide Shimizu and Hisao Suzuki. “Open mirror symmetry for pfaffian Calabi-Yau 3-folds”. In: Journal of High Energy Physics 2011.3 (Mar. 2011).

[60] Claire Voisin. Hodge Theory and Complex Algebraic Geometry I. Ed. by LeilaTranslator Schneps. Vol. 1. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2002.

[61] Claire Voisin. Hodge Theory and Complex Algebraic Geometry II. Ed. by LeilaTranslator Schneps. Vol. 2. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2003.
[62] Johannes Walcher. “Calculations for Mirror Symmetry with D-branes”. In: *JHEP* 09 (2009), p. 129. arXiv: [0904.4905](https://arxiv.org/abs/0904.4905) [hep-th].

[63] Johannes Walcher. “Evidence for Tadpole Cancellation in the Topological String”. In: *Commun. Num. Theor. Phys.* 3 (2009), pp. 111–172. arXiv: [0712.2775](https://arxiv.org/abs/0712.2775) [hep-th].

[64] Johannes Walcher. “Extended holomorphic anomaly and loop amplitudes in open topological string”. In: *Nucl. Phys. B* 817 (2009), pp. 167–207. arXiv: [0705.4098](https://arxiv.org/abs/0705.4098) [hep-th].

[65] Johannes Walcher. “Opening Mirror Symmetry on the Quintic”. In: *Communications in Mathematical Physics* 276.3 (Oct. 2007), pp. 671–689.

[66] Edward Witten. “Braves and the dynamics of QCD”. In: *Nuclear Physics B* 507.3 (Dec. 1997), pp. 658–690.

[67] Edward Witten. “Quantum background independence in string theory”. In: *Conference on Highlights of Particle and Condensed Matter Physics (SALAM-FEST)*. June 1993. arXiv: [hep-th/9306122](https://arxiv.org/abs/hep-th/9306122).

[68] Satoshi Yamaguchi and Shing-Tung Yau. “Topological string partition functions as polynomials”. In: *JHEP* 07 (2004), p. 047. arXiv: [hep-th/0406078](https://arxiv.org/abs/hep-th/0406078).

[69] A. B. Zamolodchikov. “Irreversibility of the Flux of the Renormalization Group in a 2D Field Theory”. In: *JETP Lett.* 43 (1986), pp. 730–732.