Bounds on Correlation Functions of Quantum Rotators

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Abstract

We derive a McBryan-Spencer bound to the correlation function of a one-dimensional array of quantum rotators in the Villain approximation of the cosine interaction. We obtain the partition function of the system in the gas representation and establish a lower bound on the external charges correlation function. We also discuss the possible existence of a Kosterlitz-Thouless phase for the quantum rotators in the Villain approximation.

Key words: Quantum Rotators, Villain Action, Correlation Estimates, Disorder Operator

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1 Introduction

We study the ground state of a system of coupled quantum rotators in a one-dimensional lattice $\Lambda \subset \mathbb{Z}$ given by the finite-volume Hamiltonian

$$H_\Lambda = \frac{1}{2I} \sum_{x \in \Lambda} \frac{\partial^2}{\partial \theta^2} + J \sum_{x \in \Lambda} \cos(\theta(x) - \theta(x + 1))$$  \hspace{1cm} (1.1)

$x \in \mathbb{Z}$, where $I$ and $(J > 0)$ are constant.

The Hamiltonian operator, taken with periodic boundary conditions, acts on the Hilbert space of

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square integrable functions on the interval \([-\pi, \pi]\).

We use the Lie-Trotter representation to map our one-dimensional model into a two-dimensional system of classical rotators with an extra time direction. For this we prove a McBryan-Spencer bound on the correlation function in the Villain approximation of the cosine interaction \((1.1)\).

Next we use a duality transformation \([1]\) to obtain the partition function in the sine-Gordon and charge representations, which is a Villain gas partition function with anisotropic Gaussian measure.

We establish a lower bound on the external charges correlation function through Jensen inequality in the charge variables.

Our estimates are for the infinite system in the continuum limit in time direction.

### 2 The Lie-Trotter Representation

We derive a path integral representation for \(\text{Tr}e^{-\beta H_\Lambda}\) by the Lie-Trotter formula

\[
e^{-\beta H_\Lambda} = \lim_{n \to \infty} \left(e^{-\frac{\beta H_\Lambda}{n}} e^{-\frac{\beta V(\theta)}{n}}\right)^n \tag{2.1}
\]

where

\[
H_0 = \frac{1}{2I} \sum_x \frac{\partial^2}{\partial \theta^2} \quad \text{and} \quad V = J \sum_x \cos[\theta(x) - \theta(x+1)]
\]

In computing the trace we take a basis that diagonalizes \(V\) and insert between each two factors of (2.1) the decomposition of the identity \(I = \sum_\theta |\theta(x)\rangle \langle \theta(x)|\) with \(|\theta(x)\rangle\) eigenvectors of \(V\). We get

\[
Z = \lim_{n \to \infty} \int_{-\pi}^\pi e^{-H_{\Lambda,\delta}} \prod_{x,t} d\theta(x,t) \tag{2.2}
\]

where \(e^{-H_{\Lambda,\delta}}\), with \(\delta = \beta/n\), is the Gibbs weight of a configuration \(\theta = \{\theta(x,t), x \in \Lambda, t \in \delta\mathbb{Z} \cap [-\beta/2, \beta/2]\}\), given by

\[
e^{-\beta H_{\Lambda,\delta}} = \sum_{m_1} e^{-\frac{\lambda}{2\pi} \sum_{x,t} (\theta(x,t) - \theta(x,t+\delta) + 2\pi m_1(x,t))^2} e^{-\frac{\lambda}{4} \sum_{x,t} \cos(\theta(x,t) - \theta(x+1,t))}
\]

where \(m_1 = \{m_1(x,t), x \in \Lambda, t \in \delta\mathbb{Z} \cap [-\beta/2, \beta/2]\}\).

We want to replace the cosine form of the interaction by the Villain form \([3]\)

\[
e^{z \cos \theta} \approx \sum_{m \in \mathbb{Z}} e^{-\frac{z}{2}(\theta+2\pi m)^2}
\]
which is valid for \( z \gg 1 \).

In our case \( z = \delta J/2 \) violates that condition. However, we change our original model and define a new one for which the Gibbs weight is taken with the Villain action:

\[
e^{-H_{\Lambda, \delta}} = \sum_{m_1, m_2} \prod_{x,t} e^{-\left\{ \frac{J}{2\pi} [\theta(x,t) - \theta(x,t+\delta) + 2\pi m_1(x,t)]^2 + \frac{2}{\pi} [\theta(x,t) - \theta(1,t) + 2\pi m_2(x,t)]^2 \right\}}
\] (2.3)

where \( m_2 = \{ m_2(x,t) \mid x \in \Lambda, t \in \delta \mathbb{Z} \cap [-\beta/2, \beta/2] \} \).

**Remark 1.** In forcibly approaching our model to a classical lattice Villain model we have in mind proving a Kosterlitz-Thouless transition for it. Since classical lattice and also continuous Coulomb gases exhibit a Kosterlitz-Thouless phase, it seems natural to expect our anisotropic Villain gas to display such a phase as well.

The McBryan-Spencer bound we derive below guarantees that our model does not have a first order transition, but says nothing about transitions of higher order.

We are now in position to obtain a McBryan-Spencer bound on the correlation function.

**Theorem 1.** Let

\[
G_{\Lambda, \delta}((y,s), (y', s')) = \langle e^{i(\theta(y,s) - \theta(y', s'))} \rangle_{\Lambda, \delta}
\]

where \( \langle \cdot \rangle_{\Lambda, \delta} \) is the mean value with respect to the Gibbs weight (2.3) for all \( \Lambda, \delta \).

Let \( G((y,s), (y', s')) \) be the limit of \( G_{\Lambda, \delta}((y,s), (y', s')) \) when \( \Lambda, n, \beta \to \infty \).

Then

\[
G((y,s), (y', s')) \leq e^{- \sqrt{\frac{I J}{2\pi\beta}} \sqrt{(y-y')^2 + (J/I)(s-s')^2}}
\]

**Proof.** Let \( a(x,t) \) be a \( C^{(2)} \)-function. We make the shift \( \theta(x,t) \to \theta(x,t) + a(x,t) \), where

\[
a(x,t) = C(x - y, t - s) - C(x - y', t - s'),
\]

and \( C(x,t) \) is the Green’s function of the finite difference Laplacian in two dimensions, given by

\[
C(x - y, t - s) = \frac{1}{|\Lambda|} \frac{1}{\beta} \sum_{p, q} \exp(i(p(x - y) + q(t - s)) / J(2 - 2 \cos p) + I \delta^{-2}(2 - 2 \cos q \delta)) \] (2.5)

with \( p = 2\pi / |\Lambda| \cap [-|\Lambda|/2, |\Lambda|/2] \) and \( q = 2\pi / \beta \cap [-n/2, n/2] \).

We obtain a quadratic form

\[
\langle e^{i(\theta(y,s) - \theta(y', s'))} \rangle \leq e^{-\langle a(y,s) - a(y', s') \rangle} e^{-\frac{I}{2\pi}(a, -\Delta a)}
\] (2.6)
where \(-\Delta = J\partial_x^1\partial_1 + I\partial_y^\delta\partial_\delta\), with the usual definitions \(\partial_y f(x,t) = \delta^{-1}[f(x,t+\delta) - f(x,t)]\), and scalar product \((f,g) = \sum_{x,t} f(x,t)g(x,t)\).

Thus we get from (2.6) \[
\langle e^{\text{(\theta(y,s) - \theta(y',s'))}} \rangle_{\Lambda,\delta} \leq e^{C(y-y',s-s') - C(0,0)}
\] (2.7)

The asymptotic behavior of the difference of Green’s functions in (2.7) in the limit \(\Lambda, n, \beta \to \infty\) is:

\[
C(y - y', s - s') - C(0, 0) \approx -\sqrt{\frac{1}{4\pi^2 IJ}} \ln \sqrt{(y - y')^2 + (J/I)(s - s')^2}
\]

for large \(|y - y'|, |s - s'|\).

It follows that

\[
\langle e^{\text{(\theta(y,s) - \theta(y',s'))}} \rangle \leq e^{-\sqrt{IJ/4\pi^2} \ln \sqrt{(y-y')^2 + (J/I)(s-s')^2}} \quad \square
\]

This should be compared with a similar result in [2].

3 Duality Transformation

Duality transformation is the representation of the partition function obtained by performing the Fourier transform on its angle variables \(\{\theta(x,t)\}\). For the partition function (2.2) with the Gibbs weight (2.3) the transformation is trivial. After duality the partition function reads\[
Z = \int \prod_{x,t} \sum_{m \in \mathbb{Z}} \delta(\phi(x,t) - m(x,t))d\mu(\phi)
\] (3.1)

where \(\{\phi(x,t)\}\) are the variables of the dual lattice, and \(d\mu(\phi)\) is the discrete Gaussian measure

\[
\prod_{x,t} e^{-\frac{J}{2}(\phi(x,t) - \phi(x+1,t))^2 - \frac{I}{2}(\phi(x,t) - \phi(x,t+\delta))^2} d\phi(x,t)
\] (3.2)

The periodized \(\delta\)-functions can be expanded in Fourier series

\[
\prod_{x,t} \sum_{m(x,t)} \delta(\phi(x,t) - m(x,t)) = \sum_q e^{i2\pi(\phi,q)}
\]

where \(q = \{q(x,t) \in \mathbb{Z}\}\), and we get

\[
Z = \int d\mu(\phi) \prod_{x,t} \left(1 + 2 \sum_{q=1}^{\infty} \cos 2\pi \phi(x,t)q(x,t)\right).
\] (3.3)
On integrating out the $\phi$-variables in (3.3) we obtain the charge representation of the partition function:

$$Z = \int e^{-\frac{1}{2}(q, (-\Delta)^{-1}q) \prod_{x,t} d\lambda(q(x, t))}$$ \hspace{1cm} (3.4)

where now $-\Delta = I^{-1} \partial_1^x \partial_1 + J^{-1} \partial_1^y \partial_1$, with the measure

$$d\lambda(q) = \sum_{m \in \mathbb{Z}} \delta(q - 2\pi m) dq$$

If we put into the system (3.4) two fractional charges, one $+\xi$ placed at the origin, another $-\xi$ placed at $x$, the external charges correlation function reads

$$G_{\Lambda,\delta}(\xi) = \mathbb{E}_\phi[D_{0,x}(\xi)]$$ \hspace{1cm} (3.5)

where $\rho(y, t) = \xi(\delta_{y0} - \delta_{yx})\delta_{t0}$ is the density of external charges.

Let $G^\xi(x)$ denote the limit of $G_{\Lambda,\delta}(\xi)$ when $\Lambda, n, \beta \to \infty$. We have

**Theorem 2.** The external charges correlation function (3.5) is bounded below by

$$G^\xi(x) \geq C_\xi e^{-\xi^2 \sqrt{\frac{|x|}{4\pi^2}} \ln|x|}$$

with $0 < C_\xi < \infty$.

**Proof.** Immediate by the application of Jensen inequality in the $q$-variables.

### 4 The Disorder Operator

When the duality transformation is applied to the correlation function (2.4), its expectation value for the correlation between the points 0 and $(x, 0)$ can be written as $G_{\Lambda,\delta}(0, x) = \langle D_{0,x}^\xi \rangle$ where $\langle D_{0,x}^\xi \rangle$ is the expectation value, taken with the measure (3.2), of the so-called disorder operator

$$D_{0,x}^\xi(\phi) = \frac{v(\partial_2^x \phi + \xi f^x)}{v(\partial_2^x \phi)}$$ \hspace{1cm} (4.1)

where

$$v(\partial_2^x \phi) = e^{-\frac{\delta}{2\beta} \sum_{y,t} (\phi(y,t) - \phi(y, t + \delta))^2}$$ \hspace{1cm} (4.2)

and $f^x$ is defined by

$$f^x(y, t) = \begin{cases} 1, & \text{if } 1 \leq y \leq x, \ t = 0; \\ 0, & \text{otherwise}. \end{cases}$$ \hspace{1cm} (4.3)
From (4.1) and (3.3) we get

\[
Z\langle \hat{D}_0^{\xi_0} \phi \rangle = e^{-\frac{\delta^2}{2J} |x|^2} \int d\mu(\phi) e^{\frac{\delta^2}{2J} \phi(\partial_\phi f_x)} \prod_{y,t} \left( 1 + 2 \sum_{q=1}^{\infty} \cos 2\pi \phi(y, t) q(y, t) \right)
\]

The real shift $\phi \to \phi + \sigma$, where $\sigma(y, t) = (\xi/J)(-\Delta^{-1}) \partial_\phi f_x(y, t)$ leads to

\[
Z\langle \hat{D}_0^{\xi_0} \phi \rangle = e^{g(x)} Z(\sigma)
\]

where

\[
Z(\sigma) = \int \prod_{y,t} \left[ 1 + 2 \sum_{q=1}^{\infty} \cos(2\pi (\phi(y, t) + \sigma(y, t)) q(y, t)) \right] d\mu(\phi) \tag{4.4}
\]

and

\[
g(x) = -\frac{\delta^2}{2J} (C(0,0) - C(x,0))
\]

is the Gaussian contribution to the expectation value, which vanishes in the limit when $\Lambda, n, \beta \to \infty$.

**Remark 2.** The application of Jensen inequality to obtain the decay of the correlation function is the first step towards proving a Kosterlitz-Thouless transition for the quantum rotators. The second step consists in obtaining a lower bound to the expectation value of the disorder operator. Starting from (4.4) expectations in the gas are written as convex combinations of expectations of dilute gases of neutral multipoles. The falloff of correlation intended to imply the transition is obtained by an inductive procedure over multipoles of various sizes. In each step of the process the entropy of the multipoles increases by a factor proportional to the diameter of the support of charges. This entropy increase must be offset by the extraction of the self-energy of charged multipoles in order to renormalize the activity of the gas. The energy estimate yields roughly a factor $e^{-C(0,0)}$.

Naive application of these techniques to the partition function (4.4) shows that the entropic cost grows as the inverse of the spacing $\delta^{-1}$, while the self-energy factor grows as $\exp \delta$, so that a more careful analysis is required to render the balance energy-entropy favorable.

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