SPECTRAL PROPERTIES OF NON-SELF-ADJOINT OPERATORS IN THE SEMI-CLASSICAL REGIME

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Abstract

We give a spectral description of the semi-classical Schrödinger operator with a piecewise linear, complex valued potential. Moreover, using these results, we show how an arbitrarily small bounded perturbation of a non-self-adjoint operator can completely change the spectrum of the operator.

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1 Introduction

This work was motivated by the paper of Shkalikov [4] concerning the analysis of the semi-classical Airy operator, together with our computer simulations of the associated discrete problem using the numerical package Matlab. Specifically, we examined the operator $H_{\delta,h}$ given formally by

\[ H_{\delta,h} := -h^2 \frac{d^2}{dx^2} + V_{\delta} \quad \text{on } L^2(-1, 1) \] (1)

where

\[ V_{\delta}(x) := \begin{cases} i(x + \delta) & \text{for } x > 0 \\ i(x - \delta) & \text{for } x < 0 \end{cases} \]

and both $\delta > 0$, $h > 0$ are small. In [4] it is shown rigorously that as $h \to 0$ the spectrum of $H_{0,h}$ becomes dense inside an arbitrarily small neighbourhood of the Y-shaped subset of $\mathbb{C}$ defined by

\[ [i, 1/\sqrt{3}], [-i, 1/\sqrt{3}] \quad \text{and} \quad [1/\sqrt{3}, \infty), \]

where we use $[\alpha, \beta]$ to denote the line segment joining $\alpha, \beta \in \mathbb{C}$ (see figure 2).
This paper confirms our surprising numerical results, which suggested that when an arbitrarily small jump discontinuity is inserted into the otherwise linear (purely imaginary) potential, and then the semi-classical parameter $h$ allowed to go to 0, the asymptotic spectrum of $H_{\delta,h}$ turns out to be completely different from that of $H_{0,h}$ (see figure 1 and Corollary 4).

Several papers [5, 3, 4] have been written about the operator $H_{0,h}$ - a major motivation being that it is a model operator for the ‘Orr-Sommerfeld’ problem [3]. The operator also defines the ‘Squire model for the Couette flow’ in hydrodynamics; and in its own right, defines the semigroup which is the solution of the so-called ‘Torrey equation’ [5], related to the diffusion of magnetic fields. Thus, although the spectrum of this non-self-adjoint operator displays sometimes strange and singular behaviour, it must not be dismissed as a ‘pathological’ example from pure mathematics since it has important applications - for example, in magnetic resonance imaging devices.

It is well known that a basis for solutions of the so-called ‘Airy equation’

$$-f''(z) + zf(z) = 0$$

is given by any two of the Airy functions $Ai(z)$, $Ai(e^{-2\pi i/3}z)$ and $Ai(e^{2\pi i/3}z)$ (see [2]). Thus, the analytic investigation of the eigenvalues will involve examining the asymptotic behaviour of certain Airy functions, where the expressions we use are WKB-type approximations. We will show that the eigenvalues lie inside a certain subset of the complex plane, which is intimately related to the Stokes’ lines (or principal curves) of the problem (see [2]). The proof will depend upon showing that for all $\lambda$ outside this subset, the eigenvalue problem

$$H_{h}f(x) = \lambda f(x)$$

has a well-defined Green’s function. As in [4], our analysis uses the concept of the characteristic determinant which we will describe in Section 2.

In Section 3 we analyse the spectral behaviour in the semi-classical limit $h \to 0$ for a general complex-valued, piecewise linear potential. The surprising result is that each linear segment of the potential gives rise to a characteristic ‘Y-shaped’ set of eigenvalues; and the spectrum of the operator is contained within the superposition of these ‘Y-shaped’ sets. From the point of view of applications, this means that whilst the asymptotic spectrum is theoretically computable for an idealised linear potential; in practice, any arbitrarily small deviation from the ideal can completely change the spectrum. One way of expressing this for an operator $H$ is in terms of the pseudospectral sets:

$$\text{Spec}_\epsilon(H) := \text{Spec}(H) \cup \{z \in \mathbb{C} : \| (H - z)^{-1} \| \geq \epsilon^{-1}\},$$

i.e. the contour sets of the resolvent norm, with the convention that $z \in \text{Spec}(H)$ implies

$$\| (H - z)^{-1} \| := \infty.$$
For many non-self-adjoint operators it has been demonstrated \cite{1, 3, 6} that the pseudospectral sets become very large as some parameter varies, even though $z$ may be far from the spectrum of the operator. This is equivalent to saying that the spectrum is computationally unstable. Our aim in this paper is to demonstrate for a relatively transparent case, the mechanism behind this phenomenon; we believe the results to be capable of extension to a more general class of piecewise analytic potentials.

In Section 5 we provide an analysis of the simultaneous limit as $h \to 0$ and $\delta \to 0$ together.

2 The Characteristic Determinant

In this section we describe the characteristic determinant of the operator $H_h$ defined by

$$H_h f(x) := -h^2 \frac{d^2 f(x)}{dx^2} + V(x) f(x)$$

acting on $L^2(-1, 1)$ with Dirichlet boundary conditions, $h > 0$ small, and $V(x)$ the complex valued, $n$-times piecewise linear function

$$V(x) := \begin{cases} m_1 x + l_1 & x_0 \leq x < x_1 \\ m_2 x + l_2 & x_1 < x < x_2 \\ \vdots & \vdots \\ m_n x + l_n & x_{n-1} < x \leq x_n \end{cases}$$

with $-1 = x_0 < x_1 < \ldots < x_n = 1$, and the $m_i, l_i \ i = 1, \ldots, n$ complex constants. The domain of the operator is given precisely by

$$\text{Dom}(H_h) = \{ f \in C[-1, 1] : f(-1) = f(1) = 0, f' \in C[-1, 1] \text{ and } f'' \in L^2(-1, 1) \}$$

where the primes $'$ denote differentiation with respect to $x$, and $f''$ is initially to be interpreted in the distributional sense. A direct substitution shows that a basis of solutions for the differential equation

$$-h^2 f''(x) + (V(x) - \lambda) f(x) = 0$$

where $V(x) := mx + l$; and $l, m$ are complex constants, is given by any two of the Airy functions $Ai(w)$ and $Ai(e^{\pm 2\pi i/3} w)$, where

$$w := h^{-2/3} m^{-2/3} (V(x) - \lambda).$$

It follows that, in order to construct an eigenfunction of the operator $H_h$, we seek constants $\alpha_{i1}, \alpha_{i2} \ i = 1, \ldots, n$ such that the function

$$f(x) := \begin{cases} \alpha_{11} u_{11}(x) + \alpha_{12} u_{12}(x) & x_0 \leq x < x_1 \\ \alpha_{21} u_{21}(x) + \alpha_{22} u_{22}(x) & x_1 < x < x_2 \\ \vdots & \vdots \\ \alpha_{n1} u_{n1}(x) + \alpha_{n2} u_{n2}(x) & x_{n-1} < x \leq x_n \end{cases}$$
satisfies all of the domain conditions (3), where

\[ u_{i1}(x) := Ai \left( e^{-2k\pi i/3}h^{-2/3}m_i^{-2/3}((m_ix + l_i) - \lambda) \right) \quad (3) \]

with \( k \in \{-1, 0, 1\} \). For each \( i = 1, \ldots, n \), the functions \( u_{i2} \) are defined similarly, except that a different choice of \( k \) is to be taken from \( \{-1, 0, 1\} \).

In addition to satisfying the boundary conditions \( f(-1) = f(1) = 0 \), \( f \) must also be continuously differentiable, even at the points \( x_i \). From the power series definition [2 p.54], it is clear that the Airy functions \( Ai \) are analytic on the whole of \( \mathbb{C} \), and so the requirement that \( f \) be continuously differentiable reduces to the \( 2(n-1) \) simultaneous ‘matching’ conditions

\[
\alpha_{i1}u_{i1}(x_i-) + \alpha_{i2}u_{i2}(x_i-) - \alpha_{(i+1)1}u_{(i+1)1}(x_{i+}) - \alpha_{(i+1)2}u_{(i+1)2}(x_{i+}) = 0
\]

and

\[
\alpha'_{i1}u_{i1}(x_i-) + \alpha'_{i2}u_{i2}(x_i-) - \alpha'_{(i+1)1}u'_{(i+1)1}(x_{i+}) - \alpha'_{(i+1)2}u'_{(i+1)2}(x_{i+}) = 0.
\]

The boundary conditions \( f(-1) = f(1) = 0 \) demand that

\[
\alpha_{11}u_{11}(-1) + \alpha_{12}u_{12}(-1) = 0
\]

and

\[
\alpha_{n1}u_{n1}(1) + \alpha_{n2}u_{n2}(1) = 0.
\]

Thus finding a solution of the differential equation

\[-h^2f''(x) + V(x)f(x) = \lambda f(x)\]

which satisfies all of the domain conditions (3), involves solving the matrix equation

\[
\begin{pmatrix}
  u_{11}(-1) & u_{12}(-1) & 0 & 0 & \cdots & 0 \\
u_{11}(x_1) & u_{12}(x_1) & -u_{21}(x_1) & -u_{22}(x_1) & 0 & \cdots & 0 \\
u'_{11}(x_1) & u'_{12}(x_1) & -u'_{21}(x_1) & -u'_{22}(x_1) & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & u_{(n-1)1}(x_{n-1}) & u_{(n-1)2}(x_{n-1}) & -u_{n1}(x_{n-1}) & -u_{n2}(x_{n-1}) \\
0 & \cdots & \cdots & u'_{(n-1)1}(x_{n-1}) & u'_{(n-1)2}(x_{n-1}) & -u'_{n1}(x_{n-1}) & -u'_{n2}(x_{n-1}) \\
0 & \cdots & \cdots & 0 & 0 & u_{n1}(1) & u_{n2}(1)
\end{pmatrix} \times 
\begin{pmatrix}
\alpha_{11} \\
\alpha_{12} \\
\alpha_{21} \\
\alpha_{22} \\
\vdots \\
\alpha_{n1} \\
\alpha_{n2}
\end{pmatrix} = 
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
\vdots \\
0 \\
0
\end{pmatrix}
\quad (4)
\]

for the constants \( \alpha_{i1}, \alpha_{i2} \). Note that we are taking the left- and right-hand limits at the points \( x_i \), although here and subsequently we will abuse the notation in order to add clarity, and simply write \( u_{i1}(x_i) \) etc. It is the determinant of the matrix on the left-hand side of (4) that we shall call the characteristic determinant of the eigenvalue problem defined by \( H_h \).
3 Airy functions and Stokes’ lines

For our proofs in the next section, we will need some notation and basic properties of the Airy functions. In all that follows we let the argument function \( \text{Arg} \) take principal values i.e.

\[
\text{Arg} : \mathbb{C} \rightarrow (-\pi, \pi].
\]

If \( \text{Arg} (\beta - \alpha) := \theta \), the subsets \( Y(\alpha, \beta) \) of \( \mathbb{C} \) are to be constructed as follows: take the lines

\[
\alpha + r e^{2\theta i/3} \quad \text{and} \quad \begin{cases} 
\beta + r e^{2\theta i/3+2\pi i/3} & \text{for } \theta < 0 \\
\beta + r e^{2\theta i/3-2\pi i/3} & \text{for } \theta \geq 0
\end{cases}
\]

as \( r \) ranges in \([0, \infty)\), to their point of intersection, \( \Gamma \) say. Then, from \( \Gamma \) extend the infinite line defined by the set of \( \lambda \in \mathbb{C} \) such that

\[
\text{Re}\{(e^{-2\pi i/3}(\alpha - \lambda))^{3/2}\} = \text{Re}\{(e^{-2\pi i/3}(\beta - \lambda))^{3/2}\}.
\]

The motivation for this set will become clear during our proofs; in fact, it will be seen to comprise a curve asymptotic to the line

\[
\left\{ z \in \mathbb{C} : \text{Im} (z) = \frac{\text{Im} (\alpha) + \text{Im} (\beta)}{2} \right\}.
\]

Note for now, however, that (5) is \( h \)-independent. The \( \varepsilon \)-neighbourhood of any subset \( T \) of \( \mathbb{C} \) will be defined by

\[
\text{Nhd}(T; \varepsilon) := \{ t + z : t \in T \text{ and } |z| < \varepsilon \}.
\]

We will use the well-known [2] asymptotic expansion of the Airy function \( Ai \), giving the WKB-type approximation:

\[
Ai(z) = \frac{z^{-1/4}}{2\sqrt{\pi}} \exp \left( -\frac{2}{3} z^{3/2} \right) (1 + O(z^{-3/2}))
\]

as \( |z| \rightarrow \infty \), valid for \( |\text{Arg} (z)| < \pi \); and where the principal value of \( z^{3/2} \) is taken.

Following the notation of Olver (see [2, p.413]), we define

\[
S_0 := \{ z : |\text{Arg} (z)| < \pi/3 \}
\]

\[
S_1 := \{ z : \pi/3 < \text{Arg} (z) < \pi \}
\]

\[
S_{-1} := \{ z : -\pi/3 > \text{Arg} (z) > -\pi \}
\]

(suffixes enumerated modulo 3). One can check that for all complex \( z \) (and taking principal values), we have

\[
\text{Re}\{(e^{-2\pi i/3}z)^{3/2}\} = \begin{cases} 
-\text{Re}\{(e^{2\pi i/3}z)^{3/2}\} & \text{for } z \in S_1 \cup S_{-1} \\
\text{Re}\{(e^{2\pi i/3}z)^{3/2}\} & \text{for } z \in S_0
\end{cases}
\]
\[ \text{Re} \left\{ (e^{-2\pi i / 3} z)^{3/2} \right\} = \begin{cases} \text{Re} \left\{ (z)^{3/2} \right\} & \text{for } z \in S_1 \cup S_0 \\ \text{Re} \left\{ (z)^{3/2} \right\} & \text{for } z \in S_{-1} \end{cases} \] (8)

and

\[ \text{Re} \left\{ (e^{2\pi i / 3} z)^{3/2} \right\} = \begin{cases} -\text{Re} \left\{ (z)^{3/2} \right\} & \text{for } z \in S_0 \cup S_{-1} \\ \text{Re} \left\{ (z)^{3/2} \right\} & \text{for } z \in S_1. \end{cases} \] (9)

Then, putting

\[ A_i k(z) := A i(e^{-2k\pi i / 3} z) \] (10)

the asymptotics (8) show that as \(|z| \to \infty\), \(|A_i k(z)|\) decreases exponentially for \(z \in S_k\), and increases exponentially for \(z \in S_{k-1} \cup S_{k+1}\). The boundaries of the sectors \(S_k\) i.e. the rays \(te^{\pm \pi i / 3}\) and \(te^{\pi i}\) for \(t \in [0, \infty)\), are known as the Stokes’ lines (or principal curves) of the problem \([2, \text{p.503}]\). Indeed, for the Airy equation

\[-f''(z) + zf(z) = 0\]

the Stokes’ lines are defined to be the values of \(z\) such that

\[ \text{Re} \int_0^z \sqrt{t} \, dt = \text{Re} \left\{ \frac{2}{3} z^{3/2} \right\} = 0, \]

and denote the boundaries of the principal subdomains \(S_1\) etc., inside of which the asymptotic expression (8) is valid for each \(k\).

We will call the suffix \(k\) ‘allowable’ for any given \(z \in \mathbb{C}\), if

\[ \left| \text{Arg} \left( e^{-2k\pi i / 3} z \right) \right| < \pi. \] (11)

4 The Semi-Classical Limit

The following is a generalisation of the argument used in [4] for the potential \(V(x) = ix\), and will form a lemma for our main theorem.

Proposition 1 Let \(V\) be the complex valued linear potential given by

\[ V(x) = mx + l \quad x \in [-1, 1] \]

where \(m\) and \(l\) are complex constants; \(u_{11}(x), u_{12}(x)\) are as defined in (3), and \(a, b \in [-1, 1]\), \(a < b\).

Let \(\varepsilon > 0\) be given and \(\lambda \in \mathbb{C}\). If

\[ \lambda \notin \text{Nhd}(Y(V(a), V(b)); \varepsilon), \]

then

\[ u_{11}(b)u_{12}(a) = o(u_{11}(a)u_{12}(b)) \] (12)

as \(h \to 0\).
Proof. A simple scaling and translation of the operator \( H \) allows us, without loss of generality, to assume that \( a := -1, b := 1 \) and \( l = 0 \). That is, we assume

\[ V(x) := xe^{i\theta}, \quad \text{where } \theta := \text{Arg} \,(m). \]

By elementary trigonometry, one can check that we then have

\[ \Gamma = e^{i\theta} + \frac{4}{\sqrt{3}} \sin \left( \frac{|\theta|}{3} \right) e^{2(\theta - \pi)i/3} \]

or

\[ \Gamma = -e^{i\theta} + \frac{4}{\sqrt{3}} \sin \left( \frac{2\pi}{3} + \frac{|\theta|}{3} \right) e^{2\theta i/3}. \]

Recalling our definition of the Airy functions \( u_{11}(x) \) and \( u_{12}(x) \) (3), we put

\[ z(h, \lambda, x) := h^{2/3} e^{-2\theta i/3} (xe^{\theta i} - \lambda), \]

and can rewrite (3) explicitly in terms of \( h \). Then, taking the modulus we obtain

\[ |\text{Ai}(z(h, \lambda, x))| = h^{1/6} \left| xe^{\theta i} - \lambda \right|^{-1/4} \exp \left( -\frac{2}{3h} \text{Re} \left( e^{-2\theta i/3} (xe^{\theta i} - \lambda)^{3/2} \right) \right) (1 + O(h)) \quad (13) \]

as \( h \to 0 \), valid for \(|\text{Arg} \,(z(h, \lambda, x))| < \pi\). Therefore, in order to estimate the moduli of the Airy functions \( \text{Ai}_k(z(h, \lambda, x)) \) in the limit \( h \to 0 \), it is sufficient to examine the behaviour of the functions

\[ x \mapsto \text{Re} \left\{ (e^{-2k\pi i/3} z(h, \lambda, x))^{3/2} \right\}, \quad x \in \mathbb{R} \]

for \( k = -1, 0, 1 \).

The basic idea of our proof is to show that for \( \lambda \) outside an arbitrarily small \( \varepsilon \)-neighbourhood of \( Y(V(-1), V(1)) \), one can assign allowable values of \( j \) and \( k \) from \( \{-1, 0, 1\} \) (in the sense of (14)) such that the inequalities

\[ \text{Re} \left\{ (e^{-2j\pi i/3} z(h, \lambda, -1))^{3/2} \right\} < \text{Re} \left\{ (e^{-2j\pi i/3} z(h, \lambda, 1))^{3/2} \right\} \quad (14) \]

and

\[ \text{Re} \left\{ (e^{-2k\pi i/3} z(h, \lambda, 1))^{3/2} \right\} < \text{Re} \left\{ (e^{-2k\pi i/3} z(h, \lambda, -1))^{3/2} \right\} \quad (15) \]

hold in the limit \( h \to 0 \). This will then be enough, by (13), to ensure that \( u_{11}(1)u_{12}(-1) \) and \( u_{11}(-1)u_{12}(1) \) are of different orders of magnitude as \( h \to 0 \), thus implying (14).

Using the statements of the previous section; for all values of \( \lambda \) such that

\[ z(h, \lambda, \pm 1) := h^{-2/3} e^{-2\theta i/3} (\pm e^{\theta i} - \lambda) \]

does not lie within an \( \varepsilon \)-neighbourhood of any of the Stokes’ lines, and \( z(h, \lambda, -1), z(h, \lambda, 1) \) lie in different principal domains, one can always obtain (14) and (15).
and the asymptotics (13) will be valid. However, for \( \lambda \) lying in the sector having its apex at \( \Gamma \), and bounded by the rays
\[
\Gamma + re^{\frac{2\theta i}{3}} \quad \text{and} \quad \begin{cases} 
\Gamma + re^{\frac{2\theta i}{3} + 2\pi i/3} & \text{for } \theta < 0 \\
\Gamma + re^{\frac{2\theta i}{3} - 2\pi i/3} & \text{for } \theta \geq 0
\end{cases}
\]
r \in [0, \infty), it is easy to check that \( e^{-2k\pi i/3}z(h, \lambda, \pm 1) \) both lie in the same principal domain, for each \( k \in \{-1, 0, 1\} \). Then it is also straightforward to check that as \( x \) ranges from \(-1\) to \(1\), the function
\[
x \mapsto \text{Re} \left\{ (e^{-2\pi i/3}z(\varepsilon, \lambda, x))^{3/2} \right\}
\]
which has been at the heart of our analysis, has a single maximum/minimum. Together with the identities (14) and (15), this means that there will be values of \( \lambda \) such that equality holds in both (14) and (15) - no matter what choices of \( j \) and \( k \) are made. Thus, (and without loss of generality assuming \( j = k = 0 \),) the set of \( \lambda \) satisfying
\[
\text{Re} \left\{ (e^{-2\pi i/3}(e_{\theta i} - \lambda))^{3/2} \right\} = \text{Re} \left\{ (e^{-2\pi i/3}(-e_{\theta i} - \lambda))^{3/2} \right\}
\]
(16)
lies in \( Y(V(-1), V(1)) \). We now examine this set in more detail. Expanding the Taylor series, we have
\[
(e^{-2 \theta i/3}(e_{\theta i} - \lambda))^{3/2} = -i\lambda^{3/2}e^{-\theta i} + \frac{3}{2}i\lambda^{1/2} - \frac{3}{8}i\lambda^{-1/2}e^{\theta i} - \frac{1}{16}i\lambda^{-3/2}e^{2\theta i} + O(\lambda^{-5/2})
\]
and
\[
(e^{-2 \theta i/3}(-e_{\theta i} - \lambda))^{3/2} = -i\lambda^{3/2}e^{-\theta i} - \frac{3}{2}i\lambda^{1/2} - \frac{3}{8}i\lambda^{-1/2}e^{\theta i} + \frac{1}{16}i\lambda^{-3/2}e^{2\theta i} + O(\lambda^{-5/2})
\]
as \( |\lambda| \to \infty \). Dividing through by \(-i\), this means that (16) will hold if and only if
\[
\text{Im} \left\{ \frac{3}{2}i\lambda^{1/2} - \frac{\lambda^{-3/2}e^{2\theta i}}{16} \right\} = O(\lambda^{-5/2})
\]
as \( |\lambda| \to \infty \). Putting \( \lambda := \rho e^{\phi i} \), this is equivalent to the requirement
\[
\sin \left( \frac{\phi}{2} \right) - \frac{1}{24\rho^2} \sin \left( 2\theta - \frac{3\phi}{2} \right) = O(\rho^{-6})
\]
as \( \rho \to \infty \). But then
\[
\text{Im} (\lambda) = \rho \sin (\phi)
\]
\[
= 2\rho \sin \left( \frac{\phi}{2} \right) \cos \left( \frac{\phi}{2} \right)
\]
\[
= 2\rho \cos \left( \frac{\phi}{2} \right) \left\{ \frac{1}{24\rho^2} \sin \left( 2\theta - \frac{3\phi}{2} \right) + O(\rho^{-6}) \right\}
\]
\[
= O(\rho^{-1})
\]
as $\rho \to \infty$. By our definition of $\Gamma$, $z(h, \Gamma, \pm 1)$ lies at the intersection of two Stokes’ lines, and so
\[
\Re \{ (z(h, \Gamma, \pm 1))^{3/2} \} = 0
\]
showing that $\Gamma$ certainly lies in the set of $\lambda$ satisfying (16). Therefore, we deduce that $Y(V(-1), V(1))$ contains a curve from $\Gamma$ asymptotic to the positive real-axis.

Finally, we must examine what happens when $z$ does lie on one of the Stokes’ lines. Firstly, suppose Arg $(z(h, \lambda, 1)) = \pi / 3$, corresponding to $\lambda$ lying on the ray centred at $e^{\theta_i}$ and passing through $\Gamma$. Then $k = 0, 1$ are allowable, and one checks that if $\lambda$ lies on the segment $[e^{\theta_i}, \Gamma]$, we have $z(h, \lambda, -1) \in S_{-1}$. It follows by (8) that
\[
\Re \{(e^{-2\pi i / 3}z(h, \lambda, -1))^{3/2}\} = \Re \{(z(h, \lambda, -1))^{3/2}\},
\]
and so (14) and (13) cannot hold. However, if $\lambda$ lies on that part of the ray which extends past $\Gamma$ (but not $\lambda = \Gamma$ itself,) then $z(h, \lambda, -1) \in S_1$, and
\[
\Re \{(e^{-2\pi i / 3}z(h, \lambda, -1))^{3/2}\} = -\Re \{(z(h, \lambda, -1))^{3/2}\}
\]
causa (14) and (13) to hold for $j = 0, k = 1$.

An entirely similar argument holds when Arg $(z(h, \lambda, -1)) = \pi$, corresponding to $\lambda$ lying on the ray centred at $-e^{\theta_i}$ and passing through $\Gamma$ using (9), with $j = 1$ and $k = -1$.

Finally, the case where Arg $(z(h, \lambda, \pm 1)) = -\pi / 3$ is taken care of using (8), which shows that we may use allowable values $-1$ and $0$ to obtain (14) and (13).

This completes the proof.

In the case $\theta = \pi / 2$; $\Gamma = 1 / \sqrt{3}$ lies on the real-axis, and the figure $Y(-i, i)$ has three linear ‘arms’. When $\theta = 0$, the symmetric case, $Y(-1, 1)$ is the semi-infinite interval $[-1, \infty)$, as is well-known from the theory of self-adjoint operators.

We now give our main result.

**Theorem 2** Let
\[
H_h f(x) := -h^2 \frac{d^2 f(x)}{dx^2} + V(x)f(x)
\]
act on $L^2(-1, 1)$ with Dirichlet boundary conditions, where $h > 0$ is small, and $V(x)$ is the complex valued $n$-times piecewise linear function

\[
V(x) := \begin{cases} 
  m_1x + l_1 & x_0 \leq x < x_1 \\
  m_2x + l_2 & x_1 < x < x_2 \\
  \vdots & \vdots \\
  m_nx + l_n & x_{n-1} < x \leq x_n
\end{cases}
\]

with $-1 = x_0 < x_1 < \ldots < x_n = 1$ and the $m_i$, $l_i$, $i = 1, \ldots, n$ complex constants.

We assume for each $i$ that if $m_ix_i + l_i = m_{i+1}x_i + l_{i+1}$, then $m_i \neq m_{i+1}$. Put $\theta_i := \text{Arg} (m_i)$, and, using our earlier notation
\[
T := \bigcup_{i=1}^{n} Y(V(x_i), V(x_{i+1})).
\]
Let $\varepsilon > 0$ and $N \in \mathbb{Z}^+$ be given. Then

$$\text{Spec}(H_h) \cap \{z : |z| \leq N\} \subset \text{Nh}(T; \varepsilon)$$

for all small enough $h > 0$.

Proof. Our proof involves an analysis of the behaviour of the characteristic-determinant, i.e. the left-hand side of (1), as $h \to 0$. We give a proof for the case $n = 3$; the general case follows by a similar argument. For $n = 3$, the characteristic-determinant is given by

$$
\begin{vmatrix}
  u_{11}(-1) & u_{12}(-1) & 0 & 0 & 0 & 0 \\
  u_{11}(x_1) & u_{12}(x_1) & -u_{21}(x_1) & -u_{22}(x_1) & 0 & 0 \\
  u'_{11}(x_1) & u'_{12}(x_1) & -u'_{21}(x_1) & -u'_{22}(x_1) & 0 & 0 \\
  0 & 0 & u_{21}(x_2) & u_{22}(x_2) & -u_{31}(x_2) & -u_{32}(x_2) \\
  0 & 0 & u'_{21}(x_2) & u'_{22}(x_2) & -u'_{31}(x_2) & -u'_{32}(x_2) \\
  0 & 0 & 0 & 0 & u_{31}(1) & u_{32}(1)
\end{vmatrix}

(17)

and we must prove that for certain values of $\lambda \in \mathbb{C}$, this determinant does not vanish as $h \to 0$. Expanding (17), one obtains

$$
\left\{ (u_{11}(-1)u_{12}(x_1) - u_{12}(-1)u_{11}(x_1))(u'_{22}(x_1)u_{21}(x_2) - u'_{21}(x_1)u_{22}(x_2)) \times
\right.

\left. (u_{31}(1)u'_{32}(x_2) - u'_{31}(x_2)u_{32}(1)) \right\} -

\left\{ (u_{11}(-1)u_{12}(x_1) - u_{12}(-1)u_{11}(x_1))(u'_{22}(x_1)u'_{21}(x_2) - u'_{21}(x_1)u'_{22}(x_2)) \times
\right.

\left. (u_{31}(1)u_{32}(x_2) - u_{31}(x_2)u_{32}(1)) \right\}

+ \left\{ (u_{11}(-1)u'_{12}(x_1) - u_{12}(-1)u'_{11}(x_1))(u_{22}(x_1)u'_{21}(x_2) - u_{21}(x_1)u'_{22}(x_2)) \times
\right.

\left. (u_{31}(1)u_{32}(x_2) - u_{31}(x_2)u_{32}(1)) \right\} -

\left\{ (u_{11}(-1)u'_{12}(x_1) - u_{12}(-1)u'_{11}(x_1))(u_{22}(x_1)u_{21}(x_2) - u_{21}(x_1)u_{22}(x_2)) \times
\right.

\left. (u_{31}(1)u'_{32}(x_2) - u'_{31}(x_2)u_{32}(1)) \right\}

(18)

where so far, no asymptotics are involved.

Now, let $\varepsilon > 0$ and $N \in \mathbb{Z}^+$ be as given in the statement of the theorem. Taking any

$$\lambda \in \{z : |z| \leq N\} \setminus \text{Nh}(T; \varepsilon),$$

we can use the results of Proposition [1] to show that (17) is non-zero in the limit as $h \to 0$. Indeed, by the proof of Proposition [1], we can ensure that the asymptotic estimates

$$u_{12}(-1)u_{11}(x_1) = o(u_{11}(-1)u_{12}(x_1)),$$

$$u_{21}(x_1)u_{22}(x_2) = o(u_{22}(x_1)u_{21}(x_2))$$

and hence

$$u_{12}(-1)u_{11}(x_1) \to 0$$

$$u_{21}(x_1)u_{22}(x_2) \to 0$$

as $h \to 0$. This completes the proof.
and
\[ u_{31}(x_2)u_{32}(1) = o(u_{32}(x_2)u_{31}(1)) \]
hold, as \( h \to 0 \). Then, using the standard asymptotic expansions of the Airy functions [2], which give

\[ Ai(z) = \frac{z^{-1/4}}{2\sqrt{\pi}} \exp\left(-\frac{2}{3}z^{3/2}\right) (1 + O(z^{-3/2})) \] (19)

and

\[ Ai'(z) = -\frac{z^{1/4}}{2\sqrt{\pi}} \exp\left(-\frac{2}{3}z^{3/2}\right) (1 + O(z^{-3/2})) \] (20)
as \( |z| \to \infty \), valid for all \( z \) such that \( |\text{Arg}(z)| < \pi \); we see that, if

\[ z(h, \lambda, x_i) := h^{-2/3}m_i^{-2/3}((m_ix_i + l_i) - \lambda), \]

then

\[
\frac{d}{dx} Ai(z) \quad = \frac{dz}{dx} Ai'(z) \\
= -\frac{dz}{dx} \frac{z^{1/4}}{2\sqrt{\pi}} \exp\left(-\frac{2}{3}z^{3/2}\right) (1 + O(z^{-3/2})) \\
= -\frac{dz}{dx} \frac{z^{1/2}z^{-1/4}}{2\sqrt{\pi}} \exp\left(-\frac{2}{3}z^{3/2}\right) (1 + O(z^{-3/2}))
\]
as \( |z| \to \infty \). Comparing this last expression with (19), and using \( f \sim g \) to mean that

\[ f(h) \sim g(h) \quad \text{as} \quad h \to 0, \]

we obtain

\[ \frac{d}{dx} Ai(z(h, \lambda, x_i)) \sim -h^{-1}((m_ix_i + l_i) - \lambda)^{1/2} Ai(z(h, \lambda, x_i)) \quad \text{as} \quad h \to 0. \]

Moreover, similar calculations show that

\[ \frac{d}{dx} Ai(e^{\pm 2\pi i/3}z(h, \lambda, x_i)) \sim h^{-1}((m_ix_i + l_i) - \lambda)^{1/2} Ai(z(h, \lambda, x_i)) \quad \text{as} \quad h \to 0. \]

Reverting to our notation of (3), we will write

\[ u'_{1i}(x_i) \sim h^{-1}c_{1i}(x_i)u_{1i}(x_i) \quad \text{etc.} \] (21)
as \( h \to 0 \), where it is important to note that the \( c_{ij}(x_i) \), \( i = 1, \ldots, (n-1) \), \( j = 1, 2 \) are independent of \( h \). Then, since the constant terms \( c_{ij}(x_i) \) are negligible in magnitude compared with the exponential terms \( u_{ij}(x_i) \) as \( h \to 0 \), the relations (21) imply that we also have the estimates

\[ u_{12}(-1)u'_{11}(x_1) = o(u_{11}(-1)u'_{12}(x_1)), \]
\[ u_{21}(x_1)u_{22}(x_2) = o(u_{22}(x_1)u_{21}(x_2)) \]
\[ u_{31}'(x_2)u_{32}(1) = o(u_{32}'(x_2)u_{31}(1)) \]
\[ u_{21}'(x_1)u_{22}'(x_2) = o(u_{22}'(x_1)u_{21}'(x_2)) \]

and
\[ u_{21}(x_1)u_{22}'(x_2) = o(u_{22}(x_1)u_{21}'(x_2)) \]
as \( h \to 0 \). Returning to (18), we first use the above estimates (since we may ignore the sub-dominant term in each round-bracketed expression), and then the relations (21) again, to obtain the asymptotic estimate on the first of the curly-bracketed terms:

\[
\left\{ (u_{11}(-1)u_{12}(x_1) - u_{12}(-1)u_{11}(x_1))(u_{22}'(x_1)u_{21}(x_2) - u_{21}'(x_1)u_{22}(x_2)) \times \\
\times (u_{31}(1)u_{32}(x_2) - u_{31}'(x_2)u_{32}(1)) \right\} \\
\sim u_{11}(-1)u_{12}(x_1)u_{22}'(x_1)u_{21}(x_2)u_{31}(1)u_{32}'(x_2) \\
\sim u_{11}(-1)u_{12}(x_1)\varepsilon^{-1/2}c_{22}(x_1)u_{22}(x_1)u_{21}(x_2)u_{31}(1)\varepsilon^{-1/2}c_{32}(x_2)u_{32}(x_2) \\
= h^{-2}[c_{22}(x_1)c_{32}(x_2)](u_{11}(-1)u_{12}(x_1)u_{22}(x_1)u_{21}(x_2)u_{31}(1)u_{32}(x_2))
\]
as \( h \to 0 \). Similar estimates apply to each of the remaining three terms in (18) i.e.

\[
\left\{ (u_{11}(-1)u_{12}'(x_1) - u_{12}(-1)u_{11}'(x_1))(u_{22}(x_1)u_{21}'(x_2) - u_{21}(x_1)u_{22}'(x_2)) \times \\
\times (u_{31}(1)u_{32}(x_2) - u_{31}'(x_2)u_{32}(1)) \right\} \\
\sim h^{-2}[c_{22}(x_1)c_{21}(x_2)](u_{11}(-1)u_{12}(x_1)u_{22}(x_1)u_{21}(x_2)u_{31}(1)u_{32}(x_2)), \\
\]

\[
\left\{ (u_{11}(-1)u_{12}'(x_1) - u_{12}(-1)u_{11}'(x_1))(u_{22}(x_1)u_{21}'(x_2) - u_{21}(x_1)u_{22}'(x_2)) \times \\
\times (u_{31}(1)u_{32}(x_2) - u_{31}'(x_2)u_{32}(1)) \right\} \\
\sim h^{-2}[c_{12}(x_1)c_{21}(x_2)](u_{11}(-1)u_{12}(x_1)u_{22}(x_1)u_{21}(x_2)u_{31}(1)u_{32}(x_2))
\]
and

\[
\left\{ (u_{11}(-1)u_{12}'(x_1) - u_{12}(-1)u_{11}'(x_1))(u_{22}(x_1)u_{21}'(x_2) - u_{21}(x_1)u_{22}'(x_2)) \times \\
\times (u_{31}(1)u_{32}(x_2) - u_{31}'(x_2)u_{32}(1)) \right\} \\
\sim h^{-2}[c_{12}(x_1)c_{32}(x_2)](u_{11}(-1)u_{12}(x_1)u_{22}(x_1)u_{21}(x_2)u_{31}(1)u_{32}(x_2))
\]
as \( h \to 0 \). Collecting these estimates together, we see that the characteristic determinant (17) tends asymptotically towards

\[
h^{-2}\{(c_{22}(x_1)-c_{12}(x_1))(c_{32}(x_2)-c_{21}(x_2))\}(u_{11}(-1)u_{12}(x_1)u_{22}(x_1)u_{21}(x_2)u_{31}(1)u_{32}(x_2))
\]
as $h \to 0$. The Airy functions $Ai(z)$ have countably many negative real zeros, \cite{2}; and so by our choice of $\lambda$ outside $\text{Nhd}(T; \varepsilon)$ together with the proof of Proposition 1, we are assured that none of the Airy functions $u_{ij}(x_i)$ vanishes. Therefore, the determinant (17) does not vanish in the limit as $h \to 0$, provided the ‘constant’ terms

$$c_{22}(x_1) \neq c_{12}(x_1) \quad \text{and} \quad c_{32}(x_2) \neq c_{21}(x_2).$$

(22)

Our choice of $\lambda$ ensures that each of the individual constant terms $c_{ij}(x_i)$ is non-zero. Moreover, reviewing the proof of Proposition 1 and the identities (7)-(9), we see that the choices for $j$ and $k$ are not uniquely determined. Therefore, it is always possible to ensure that (22) holds, even when $V$ is continuous at some or all of the $x_i$s. For example, if it happens that $V(x_1-) = V(x_1+)$, then we choose $j$ and $k$ so that the constants $c_{12}(x_1)$ and $c_{22}(x_1)$ take different signs (by the calculations immediately above (21)). Thus, we deduce that such $\lambda$ cannot be an eigenvalue.

It now just requires the following compactness argument to complete the proof. Let $B(z; \varepsilon)$ denote the open ball centred at $z$, with radius $\varepsilon$. Our argument so far shows that for any

$$\lambda \in \{z \in \mathbb{C} : |z| \leq N\}$$

such that

$$B(\lambda; \varepsilon) \cap T = \emptyset$$

we have

$$B(\lambda; \varepsilon) \cap \text{Spec}(H_h) = \emptyset$$

for all $0 < h < E_\lambda$, where $E_\lambda$ is some positive constant dependent upon $\lambda$. Let

$$M := \{z \in \mathbb{C} : |z| \leq N, \text{ and } \text{dist}(z, T) \geq 2\varepsilon\},$$

so that $M$ is compact. Then for all $\lambda \in M$

$$B(\lambda; \varepsilon) \cap \text{Nhd}(T; \varepsilon) = \emptyset$$

and so

$$M \subseteq \bigcup_{\lambda \in M} B(\lambda; \varepsilon).$$

But by compactness this means that there exists a finite sub-covering

$$M \subseteq \bigcup_{r=1}^{n} B(\lambda_r; \varepsilon_{\lambda_r}).$$

Taking $E$ to be $\min(E_{\lambda_1}, \ldots, E_{\lambda_n}) > 0$, we deduce that for all $0 < h < E$ we have

$$\text{Spec}(H_h) \cap M = \emptyset$$

and this is equivalent to the statement of the theorem.
Remark 3 An important but subtle point, to note is that the zeros of
\[(u_{11}(-1)u_{12}(x_1)u_{22}(x_1)u_{21}(x_2)u_{31}(1)u_{32}(x_2))\] as a function of $\lambda$, are not the same as the zeros of \((17)\). However, by a similar argument to that of Shkalikov \([4]\) (i.e. using \((19)\)), one can readily show that along each of the bounded arms of the Y-shaped figures making up $T$, the zeros (eigenvalues) do converge as $h \to 0$ to form a dense set. Finding an asymptotic expression for the density of spectral points along the infinite lines (in the direction of the positive real-axis) appears to be a much more difficult problem; and we have no results yet in that direction.

To illustrate Theorem 2, in figure 3 we show a Matlab plot of the discretised version of the problem where the potential
\[V(x) := \begin{cases} 2ix + i & \text{for } -1 \leq x < 0 \\ (i+1)x & \text{for } 0 < x \leq 1 \end{cases}\]

We now return to the operator $H_{\delta,h}$ defined in the first section.

**Corollary 4** Let $H_{\delta,h}$ be the non-self-adjoint operator defined by
\[H_{\delta,h} := -h^2 \frac{d^2}{dx^2} + V_{\delta}(x)\]
acting on $L^2(-1,1)$ with Dirichlet boundary conditions, $h > 0$, and
\[V_{\delta}(x) := \begin{cases} i(x - \delta) & \text{for } x < 0 \\ i(x + \delta) & \text{for } x > 0 \end{cases}\]
with $\delta > 0$. Define $S \subset \mathbb{C}$ to be the double Y-shaped figure given by the line segments
\[[i\delta, 1/2\sqrt{3} + i(1+2\delta)/2] \\
[i(1+\delta), 1/2\sqrt{3} + i(1+2\delta)/2]\]
together with
\[[1/2\sqrt{3} + i(1+2\delta)/2, +\infty),\]
and
\[[-i\delta, 1/2\sqrt{3} - i(1+2\delta)/2] \\
[-i(1+\delta), 1/2\sqrt{3} - i(1+2\delta)/2]\]
together with
\[[1/2\sqrt{3} - i(1+2\delta)/2, +\infty).\]

Then, given any $\varepsilon > 0$ and $N \in \mathbb{Z}^+$, we have
\[\text{Spec}(H_{\delta,h}) \cap \{z : |z| \leq n\} \subset \text{Nhd}(S; \varepsilon)\]
for small enough $h > 0$ (see figure 1).
By analyticity, however, for fixed $h > 0$ we have

$$\lim_{\delta \to 0} \text{Spec}(H_{\delta,h}) = \text{Spec}(H_{0,h}).$$

Hence, $\lim_{h \to 0} \lim_{\delta \to 0} \text{Spec}(H_{\delta,h})$ is contained within an arbitrarily small neighbourhood of the line segments

$$[i, 1/\sqrt{3}], [-i, 1/\sqrt{3}] \text{ and } [1/\sqrt{3}, \infty)$$

(see figure 2). Thus, the operations of taking limits do not commute, in the sense that as sets

$$\lim_{h \to 0} \lim_{\delta \to 0} \text{Spec}(H_{\delta,h}) \neq \lim_{\delta \to 0} \lim_{h \to 0} \text{Spec}(H_{\delta,h}).$$

5 Simultaneous limits for $H_{\delta,h}$

In response to questions which we have been asked, we give an analysis for the situation in which $\delta$ and $h$ of Corollary 4 are not independent of each other.

**Proposition 5** Defining the operator $H_{\delta,h}$ as above, and putting

$$\delta := h^{1/p}$$

we have

$$\lim_{h \to 0} \text{Spec}(H_{h^{1/p},h}) = \lim_{h \to 0} \text{Spec}(H_{0,h}) \quad \text{if } 0 < p < 1$$

and

$$\lim_{h \to 0} \text{Spec}(H_{h^{1/p},h}) = \lim_{\delta \to 0} \lim_{h \to 0} \text{Spec}(H_{\delta,h}) \quad \text{if } p \geq 1.$$ 

**Proof** Referring to (4) and expanding, we see that the characteristic determinant of $H_{\delta,h}$ is given by

$$\begin{align*}
\{(u_{11}(-1)u_{12}(0) - u_{12}(-1)u_{11}(0))(u'_{22}(0)u_{21}(1) - u'_{21}(0)u_{22}(1))\} - \\
- \{(u_{11}(-1)u'_{12}(0) - u_{12}(-1)u'_{11}(0))(u_{22}(0)u_{21}(1) - u_{21}(0)u_{22}(1))\},
\end{align*}$$

whereas the characteristic determinant of $H_{0,h}$ is given by

$$u_{21}(1)u_{12}(-1) - u_{22}(1)u_{11}(-1).$$

(24)

Now, putting

$$u_{12}(0) := Ai_k(h^{-2/3}e^{-pi/3}(-i\delta - \lambda)) \text{ and } u_{22}(0) := Ai_k(h^{-2/3}e^{-pi/3}(i\delta - \lambda))$$

it is clear by analyticity, that

$$u_{12}(0) \sim u_{22}(0) \quad \text{and} \quad u_{11}(0) \sim u_{21}(0)$$

(26)
as \( \delta \to 0 \). Moreover, the calculations preceding (24) show that

\[
u'_1(0) \sim -\delta^{-p}(-\lambda)^{1/2}u_1(0) \quad \text{and} \quad \nu'_2(0) \sim \delta^{-p}(-\lambda)^{1/2}u_2(0) \quad (27)
\]
as \( \delta \to 0 \), for \( i = 1, 2 \). Therefore, using first (27) and then (24), the characteristic determinant (24) tends asymptotically towards

\[
2\delta^{-p}(-\lambda)^{1/2}\left(u_{11}(-1)u_{12}(0)u_{21}(0)u_{22}(1) - u_{12}(-1)u_{11}(0)u_{22}(0)u_{21}(1)\right)
\]

\[
\sim 2\delta^{-p}(-\lambda)^{1/2}\left(u_{22}(1)u_{11}(-1) - u_{21}(1)u_{12}(-1)\right)
\]
as \( \delta \to 0 \). Then the zeros of (24) tend asymptotically toward the zeros of (25) by Rouché’s theorem, explaining the behaviour of

\[
\lim_{h \to 0} \lim_{\delta \to 0} \text{Spec}(H_{\delta,h}).
\]

Substituting \( \delta = h^{1/p} \), the character of \( \lim_{h \to 0} \text{Spec}(H_{h^{1/p},h}) \) therefore depends upon the range of \( p \) for which

\[
\frac{u_{22}(0)}{u_{12}(0)} \to 1 \quad \text{and} \quad \frac{u_{21}(0)}{u_{11}(0)} \to 1
\]
as \( h \to 0 \). Now, without loss of generality, and using our earlier notation, let

\[
\frac{u_{22}(0)}{u_{12}(0)} : = \frac{Ai(z_1)}{Ai(z_2)}
\]

where

\[
z_1 : = h^{-2/3}e^{\pi i/3}(ih^{1/p} - \lambda) \quad \text{and} \quad z_2 : = h^{-2/3}e^{\pi i/3}(-ih^{1/p} - \lambda)
\]

so that, using the standard asymptotics (3)

\[
\frac{u_{22}(0)}{u_{12}(0)} = \frac{z_1^{-1/4} \exp\left(-\frac{2}{3}z_1^{3/2}\right) (1 + O(z_1^{-3/2}))}{z_2^{-1/4} \exp\left(-\frac{2}{3}z_2^{3/2}\right) (1 + O(z_2^{-3/2}))}
\]

\[
= \left(\frac{z_1}{z_2}\right)^{-1/4} \exp\left(-\frac{2}{3}z_1^{3/2} - z_2^{3/2}\right) (1 + O(z_1^{-3/2}))
\]

\[
\sim \exp\left(-\frac{2}{3}z_1^{3/2} - z_2^{3/2}\right)
\]
as \( h \to 0 \). But

\[
z_1^{3/2} - z_2^{3/2} = h^{-1}e^{\pi i/3}\left\{(ih^{1/p} - \lambda)^{3/2} - (-ih^{1/p} - \lambda)^{3/2}\right\}
\]

\[
= h^{-1}e^{\pi i/3}(-\lambda)^{3/2}\left\{1 - (ih^{1/p}/\lambda)^{3/2} - (1 + ih^{1/p}/\lambda)^{3/2}\right\}
\]

\[
= h^{-1}e^{\pi i/3}(-\lambda)^{3/2}\left\{(1 - 3ih^{1/p}/2\lambda + \ldots) - (1 + 3ih^{1/p}/2\lambda + \ldots)\right\}
\]

\[
= h^{-1}e^{\pi i/3}(-\lambda)^{3/2}(-3ih^{1/p}/\lambda + \ldots)
\]

\[
\to 0
\]
as $h \to 0$ if and only if $0 < p < 1$. So, provided $0 < p < 1$

\[
\frac{u_{22}(0)}{u_{12}(0)} \to 1 \quad \text{as } h \to 0
\]

and a similar calculation shows that we then also have

\[
\frac{u_{21}(0)}{u_{11}(0)} \to 1 \quad \text{as } h \to 0,
\]

as required.

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$h = 0.001 \quad \delta = 0$
$h = 0.00015$

Figure 3