SYMPECTIC RIGIDITY, SYMPECTIC FIXED POINTS AND
GLOBAL PERTURBATIONS OF HAMILTONIAN SYSTEMS.

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ABSTRACT. In this paper we study a generalized symplectic fixed point problem, first considered by J. Moser in [19], from the point of view of some relatively recently discovered symplectic rigidity phenomena. This problem has interesting applications concerning global perturbations of Hamiltonian systems.

1. INTRODUCTION

Let \((M, \omega)\) be a symplectic manifold. We recall that a submanifold \(N\) of \(M\) is called \(\textit{coisotropic}\) if at any point \(x \in N\) we have that \((T_xN)^{\omega} \subseteq T_xN\), where \((T_xN)^{\omega} = \{ v \in T_xM \mid \omega(v, u) = 0, u \in T_xN \}\). The distribution, \((T_xN)^{\omega}\), on \(N\) is integrable (see [18, 19]), and therefore gives rise to a foliation on \(N\). Denote by \(L_xN\) the leaf of this foliation through \(x \in N\). We are interested in the following geometric problem.

\textbf{Problem 1.} Given a symplectomorphism \(\phi\) of \(M\) (i.e., \(\phi^* \omega = \omega\)), under what conditions on \(\phi\) and possibly on \(N\), there exists a point \(x \in N\) so that its image \(\phi(x)\) lies on a leaf through \(x\), i.e., \(\phi(x) \in L_xN\).

In this paper we are going to study the above problem for a special class of coisotropic submanifolds, following Ph. Bolle, [2], we present the following:

\textbf{Definition 1.} Let \(N\) be a \(k\)-codimensional compact coisotropic submanifold of a symplectic manifold \((M^{2n}, \omega)\) and \(1 \leq k \leq n\). \(N\) is called \(\textit{\(k\)-contact}\) if there exist \(k\) 1-forms \(\alpha_1, \ldots, \alpha_k\) defined on \(N\) so that

1. \(d\alpha_i = \omega|_N\) for \(i = 1, \ldots, k\).
2. For all \(x \in N\), \(\alpha_1 \wedge \ldots \wedge \alpha_k \wedge \omega^{n-k}(x) \neq 0\).

Equivalently one can state the second condition as follows, for all \(x \in N\), the restrictions of \(\alpha_1(x), \ldots, \alpha_k(x)\) to \(\text{Ker}\omega|_N\) are linearly independent.

Our main result is:

\textbf{Theorem 1.} Let \(N\) be a compact submanifold of \((\mathbb{R}^{2n}, \omega_0)\) of \(k\)-contact type. Let \(\phi\) be the time-1 map of a compactly supported Hamiltonian \(H\) on \([0, 1] \times \mathbb{R}^{2n}\) such that \(E(\phi) < c_{FH}(N)\). Then there exists \(x \in N\) such that \(\phi(x) \in L_xN\).

Here \(c_{FH}\) stands for the Floer-Hofer capacity as defined in [15], see Section 3.3. \(\omega_0 = -d\lambda_0\) with \(\lambda_0 = 1/2 \sum_{j=1}^n (y_j dx_j - x_j dy_j)\), is the standard symplectic structure on \(\mathbb{R}^{2n} = \mathbb{C}^n\), and the energy \(E(\phi)\) is defined as follows. Denote by \(\mathcal{F}\) the space of all smooth functions \(H : [0, 1] \times M \rightarrow \mathbb{R}\) with compact support. To every such
function one can associate a symplectic map \( \phi_H = \varphi^t \), where \( \varphi^t \) is the flow of the Hamiltonian vector field, \( X_H \), defined by the equation \( \omega(H, \cdot) = -dH(\cdot) \). We call a symplectic map \( \phi \) Hamiltonian if \( \phi = \phi_H \) for some function \( H \in \mathcal{F} \). Following Hofer, \cite{16}, we define the norm of \( H \) to be \( \|H\| = \sup H - \inf H \) and the energy of a Hamiltonian map,

\[
E(\phi) = \inf_{H \in \mathcal{F}} \{ \|H\| \mid \phi = \phi_H \}
\]

Problem 1 was first considered in \cite{19} and J. Moser proved that Problem 1 has a solution if \((M, \omega = d\alpha)\) is a simply connected, exact symplectic manifold, \( N \) is a compact coisotropic submanifold of \( M \) and \( \phi \) is an exact symplectomorphism of \( M \), (that is \( \phi^*\alpha - \alpha \) is exact), which is \( C^1 \) close to the identity. Obviously Moser’s result is of local nature. In 1989, I. Ekeland and H. Hofer derived more global versions of this theorem, for the case where \( N \) is a compact hypersurface of restricted contact type in \((\mathbb{R}^{2\mathfrak{n}}, \omega_0)\) and the map \( \phi \) is a Hamiltonian symplectomorphism. They presented various conditions on the map \( \phi \) for which the problem above has a solution, see \cite{7} for details. Here we recall that a hypersurface \( N \) in a symplectic manifold \((M^{2\mathfrak{n}}, \omega)\) is called of contact type if it is \( 1 \)-contact in terms of Definition \ref{1}. \( N \) is said to be of restricted contact type if, in addition, the form \( \alpha_1 \) can be extended to \( M \) satisfying \( d\alpha_1 = \omega \). In \cite{16}, H. Hofer, proved a very surprising result stating that the problem above has a solution if \( N \) is a compact hypersurface of restricted contact type in \((\mathbb{R}^{2\mathfrak{n}}, \omega_0)\), and \( \phi \) is a time-1 map of a compactly supported Hamiltonian, provided the energy of \( \phi \) is bounded by the Ekeland-Hofer capacity of \( N \), (defined in \cite{8,9}), i.e.

\[
E(\phi) \leq c_{EH}(N).
\]

Theorem \ref{1} extends, in a way, Hofer’s result to coisotropic submanifolds of higher codimension and even in codimension one we do not assume \( N \) to be of restricted contact type. We point out that the results in \cite{7,16} were obtained by using variational methods which are somehow restricted to the Euclidean case. Another limitation of the variational approach, even in the Euclidean case, is that it does not allow us to gain the needed control of the gradient trajectories of the Hamiltonian action functional, defined in Section \ref{3} \cite{11}. On the other hand, mixing the variational approach with pseudo-holomorphic curve methods in the spirit of Floer homology, allows us to regain this control from a geometric or rather topological perspective. Namely the idea behind the proof of the main theorem is to foliate a small neighborhood of \( N \) into diffeomorphic images of \( N \). Then we consider the critical points of a special action functional and establish the existence of a critical point which is a closed trajectory for a special Hamiltonian and consists of two arcs one is \( \psi^t x \) s.t. \( \psi^1 = \phi \) and the other arc connects \( x \) and \( \phi(x) \) through a path which is on the leaf through \( x \) on a nearby image of \( N \). We do this by studying the symplectic homology groups of this neighborhood. The existence of the closed trajectory of the type described above is a consequence of the non-vanishing of certain Floer homology groups filtered by the action. Taking smaller and smaller neighborhoods of \( N \), and repeating the previous step we get a family of closed trajectories of this type and we want to take a limit of these which will be a solution of Problem 1. The subtle part is to show that the lengths of the arcs which are on the leaves of the nearby images of \( N \) are uniformly bounded. We achieve this by getting some bounds on the action of the critical point which comes automatically from the fact that we work with filtered Floer homology groups plus some additional information.
coming from the functorial properties of the symplectic (Floer) homology i.e. that the critical point is a deformation of the constant solution of a certain Hamiltonian. This information is impossible to be detected by the variational approach and that is the reason, which directs us to work with the Floer-Hofer capacity which is based on the symplectic homology of Floer and Hofer, [11]. Perhaps it is worth mentioning that the contact condition is significant for the Hofer’s theorem. We refer to a recent paper of V. Ginzburg, [13], for a discussion about the significance of the contact condition in the various existence and almost existence results of periodic orbits on hypersurfaces and its importance for the validity of the Weinstein conjecture. Since the methods we are going to employ are reminiscent to some of the methods used to prove certain cases of the Weinstein conjecture, we must impose some sort of a contact type condition on $\mathcal{N}$, and this justifies our choice of the $k$-contact condition. We postpone the discussion on what are the right conditions on $\mathcal{N}$ and the consideration of Problem 1 on more general symplectic manifolds, most notably cotangent bundles, to our forthcoming paper [6].

As an almost immediate application of Theorem 1 we consider the Hamiltonian system describing the motion of $n$ independent harmonic oscillators on $\mathbb{R}^{2n}$, with Hamiltonian

$$H_0 = \frac{1}{2} \sum_{j=1}^{n} m_j (x_j^2 + y_j^2).$$

It is well-known that this system is integrable with first integrals $G_j = x_j^2 + y_j^2$. Consider for suitable positive constants $c, c_1, \ldots, c_{k-1}$, where $2 \leq k \leq n$, the level manifold

$$N(c, c_1, \ldots, c_{k-1}) = \{ H_0 = c, G_j = c_j, j = 1, \ldots, k-1 \}.$$ 

It is not hard to see that $N(c, c_1, \ldots, c_{k-1})$ is a compact, coisotropic, $k$-codimensional submanifold of $(\mathbb{R}^{2n}, \omega_0)$, see [19]. In fact we shall see that it is of $k$-contact type. In polar coordinates

$$x_j - iy_j = r_j e^{i\theta_j}$$

one has

$$r_j^2 = c_j$$

$$\frac{1}{2} \sum_{j=k}^{n} m_j r_j^2 = c - \frac{1}{2} \sum_{j=1}^{k-1} m_j c_j > 0.$$ 

The flow generated by $G_j$ is given by $r_j \to r_j$ and $\theta_j \to \theta_j - \delta_j l \tau_l$, where $\delta_j l$ is the Kronecker symbol. The leaves through a point $(r^*, \theta^*)$ are given by

$$r_j = r_j^*; \theta_j = \theta_j^* + \sum_{l=1}^{k-1} \delta_j l \tau_l + m_j \tau_k$$

where $j = 1, \ldots, n$ and $\tau_1, \ldots, \tau_k$ are the $k$ parameters on the leaf.

Now let us consider a nonautonomous, compactly supported, Hamiltonian perturbation $H_1(t, x) : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}$, such that $\text{supp} H_1 \subset [0, 1] \times K$, for some compact subset $K$ of $\mathbb{R}^{2n}$. We have the following theorem.
Theorem 2. Assume that

\[ \| H_1 \| < \min \left\{ \min_{p=1, \ldots, k-1} \{ \pi c_p \}, \min_{p=k, \ldots, n} \left\{ \pi \left( 2c - \sum_{j=1}^{k-1} m_j c_j \right) \right\} \right\} \]

then there exists a solution \( y \) of the perturbed system with Hamiltonian \( H_0 + H_1 \) which aside from phase shifts \( \tau_1, \ldots, \tau_k \) of \( \theta_1, \ldots, \theta_{k-1} \), \( t \) returns to the continuation of the unperturbed orbit. In particular the integrals \( H_0, G_1, \ldots, G_{k-1} \) have the same value for \( t \in (\infty, 0) \cup (1, \infty) \).

The paper is organized as follows. In Section 3 we review the definition and some of the properties of the Floer homology and the symplectic homology respectively as well as the definition of the symplectic capacities and the Floer-Hofer capacity in particular. In Section 2 we review some consequences of Definition 1. The proof of the main theorem is done in Section 4 and Section 5 is devoted to the proof of Theorem 2.

2. Consequences of the contact definition.

In this section we review some useful results from [2]. Denote by \( B^k \varepsilon \) the ball with center 0 and radius \( \varepsilon \) in \( \mathbb{R}^k \). Then we have the following lemma.

Lemma 1. Let \( N \) be a smooth, compact, connected coisotropic submanifold of a symplectic manifold \( (M, \omega) \) which is of \( k \)-contact type. Then there exists \( \varepsilon > 0 \), an open neighborhood \( U \) of \( N \) in \( M \) and a diffeomorphism \( \psi : N \times B^k \varepsilon \rightarrow U \) such that:

i) For all \( x \in N \) we have \( \psi(x, 0) = x \);

ii) \( \psi^* \omega = (1 + \sum_{j=1}^k y_j q^*(\omega|_N)) + \sum_{j=1}^k dy_j \wedge q^*(\alpha_j) \);

where the 1-forms \( \alpha_j \) are the ones from Definition 1, \( q : N \times B^k \varepsilon \rightarrow N \) is the projection onto the first factor and \( y_1, \ldots, y_k \) are coordinates in \( B^k \varepsilon \).

We have some useful consequences. We set the notation,

\[ r = q \circ \psi^{-1} : U \rightarrow N \]

(4)

\[ \beta_j = \psi^{-1*}(q^*(\alpha_j)) = r^* \alpha_j \]

\[ W = r^* (\omega|_N) \]

(5)

\[ z_j = y_j \circ \psi^{-1} \]

With this notation we have from the lemma that in \( U \) the following is true:

\[ \omega = (1 + \sum_{j=1}^k z_j)W + \sum_{j=1}^k dz_j \wedge \beta_j \]

(6)

Denote by \( X_{z_j} \) the Hamiltonian vector field associated to \( z_j \), i.e., \( \omega(X_{z_j}, \cdot) = -dz_j(\cdot) \). Then by the above lemma we have

\[ dz_j(X_{z_i}) = 0 \]

(7)

\[ \beta_j(X_{z_i}) = \delta_{ij} \]

(8)
for \( i, j = 1, \ldots, k \). It follows from (7), that the functions \( z_1, \ldots, z_k \) are in involution. Set for \( \nu = (\nu_1, \ldots, \nu_k) \in B_k \), \( N(\nu) = \cap_{j=1}^k z_j^{-1}(\nu_j) \). Then \( \tau_{N(\nu)} \) is a diffeomorphism from \( N(\nu) \) onto \( N \). Moreover it is not hard to see that \( N(\nu) \) is a coisotropic submanifold of \( M \) and \((TN(\nu))^\omega\) is spanned by \( X_{z_1}, \ldots, X_{z_k} \). From this follows that if we have a trajectory \( x(t) \) satisfying the equation

\[
\dot{x} = \sum_{j=1}^k \gamma_j X_{z_j}(x(t))
\]

for some coefficients \( \gamma_j \), then \( x(t) \) will be on the leaf through \( x(0) \) of \( N(\nu) \) where \( \nu = (z_1(x(t)), \ldots, z_k(x(t))) \). This observation will play a significant role in the proof of Theorem 1. We conclude this section by noticing (due to Lemma 1), that we can foliate a neighborhood of a \( k \)-contact submanifold \( N \) into coisotropic images of \( N \) in \( M \).

3. Review of the symplectic homology and the definition of \( c_{FH} \).

In this section we review briefly the definition and the properties of the Floer homology and the symplectic homology. Based on the properties of the symplectic homology we will present a very useful symplectic invariant called the Floer-Hofer capacity, in the terminology of D. Hermann, \[15\].

3.1. Floer Homology for the Hamiltonian action functional. The Floer homology is an infinite-dimensional equivalent to the Morse theory. In other words it can be thought as a version of Morse theory for the Hamiltonian action functional. Here we recall the definition and the properties of Floer homology. Details can be found in \[17, 21\] or in the A. Floer original paper \[10\].

Let \((M^{2n}, \omega)\) be a closed symplectic manifold, which is symplectically aspherical, that is,

\[
\omega|_{\pi_2(M)} = 0 \quad \text{and} \quad c_1(TM)|_{\pi_2(M)} = 0
\]

where \( c_1(TM) \) is the first Chern class of the tangent bundle of \( M \). Let \( H \in \mathcal{F} \) be a time-dependent function on \( M \) and \( X_H \) be its Hamiltonian vector field. Denote by \( \mathcal{P}(H) \) the set of contractible one-periodic orbits of \( X_H \). Let \( \Lambda M \) be the space of smooth contractible loops in \( M \). We define the Hamiltonian action functional, \( \mathcal{A}_H : \Lambda M \to \mathbb{R} \), associated with \( H \in \mathcal{F} \), as follows,

\[
\mathcal{A}_H(x) = \int_D u^*\omega - \int_0^1 H(t, x(t))dt
\]

with \( D \) being the closed unit disc \( (\partial D = S^1) \), and \( u : D \to M \) an extension of \( x \) so that \( u|_{\partial D} = x \). This functional is well defined because of our assumption that \( M \) is symplectically aspherical. As we mentioned above the Floer homology may be viewed as a Morse theory on \( \Lambda M \). To be precise we denote by \( \mathcal{J}_M \) the space of \( \omega \)-compatible almost complex structures on \( M \), i.e. the space of all \( J : TM \to TM \) such that \( J^2 = -Id \) and

\[
\omega(\xi, J\eta) = g_J(\xi, \eta) \quad \text{for all} \quad \xi, \eta \in TM
\]
so that \( g_J \) is a Riemannian metric on \( M \). Now consider the \( L_2 \)-metric induced on \( \Lambda M \) by \( g_J \). Then the gradient of \( A_H \) is given by

\[
\nabla_J A_H(x) = -J \dot{x} - \nabla H(t, x)
\]

In view of (13) and the fact that \( X_H = J \nabla H \), we notice that the critical points of \( A_H \), are exactly the one-periodic solutions of the Hamiltonian equations \( \dot{x} = X_H(x), x(0) = x(1) \) i.e. the elements of \( \mathcal{P}(H) \). The set of critical values of \( A_H \) is called the action spectrum of \( H \) and denoted by \( \Sigma(H) \). Of course, (13) does not define a flow on \( \Lambda M \) but despite that we are going to consider the gradient lines of \( \nabla A_H \) as the solutions of the following elliptic equation of Cauchy-Riemann type:

\[
\partial_u + J(t, u) \partial_t + \nabla H(t, u) = 0 \quad \text{for} \quad u \in C^\infty(\mathbb{R} \times S^1, M)
\]

Given two critical points \( x^+, x^- \in \mathcal{P}(H) \) of \( A_H \) we consider the space of solutions \( \mathcal{M}(x^-, x^+, J, H) \) of (14) connecting \( x^- \) and \( x^+ \),

\[
\mathcal{M}(x^-, x^+, J, H) = \{ u \in C^\infty(\mathbb{R} \times S^1, M) | \partial_u = 0 \} \text{ and } \lim_{s \to \pm \infty} u(s, t) = x^\pm(t) \}
\]

An element, \( u \), of \( \mathcal{M}(x^-, x^+, J, H) \) will be called a Floer trajectory. In this situation the difference of the actions between the ends is given by the energy, \( E_J(u) \) of the Floer trajectory \( u \), defined as follows,

\[
A_H(x^+) - A_H(x^-) = \int_{\mathbb{R} \times S^1} J(t, u) \frac{\partial u}{\partial s} ds dt = E_J(u) \geq 0
\]

Notice that the action is increasing along the gradient trajectory, that is, \( \frac{\partial A_H(u(s, \cdot))}{\partial s} = \| \nabla A_H(u(s, \cdot)) \|^2 \geq 0 \). It is not hard to see that if \( E_J(u) = 0 \), then \( u \) is independent of \( s \), one-periodic solution of the Hamiltonian equations for \( H \). Assume that the elements of \( \mathcal{P}(H) \) are non-degenerate, which means that if \( x(t) = x(t + 1) \in \mathcal{P}(H) \), then

\[
\det(Id - d\varphi_H^1(x(0))) \neq 0
\]

where \( \varphi_H^1 \) is the flow of \( X_H \). With this assumption and utilizing our assumption that \( c_1(TM)|_{\Sigma(M)} = 0 \), the elements of \( \mathcal{P}(H) \) are graded by their Conley-Zehnder index, \( \mu_{CZ} \); see [22]. The key result concerning moduli spaces \( \mathcal{M}(x^-, x^+, J, H) \) is the following, see [21],

**Theorem 3.** For generic choices of \( J \) and \( H \), the moduli spaces \( \mathcal{M}(x^-, x^+, J, H) \) are compact, finite dimensional, manifolds, of dimension \( \mu_{CZ}(x^+) - \mu_{CZ}(x^-) \).

Following Floer, we define the Morse-Witten complex associated with \( H \) as a graded \( \mathbb{Z}_2 \)-vector space

\[
CF(H) = \bigoplus_{x \in \mathcal{P}(H)} \mathbb{Z}_2 x
\]

We proceed by defining the *Floer boundary operator*, by

\[
\partial^{H,J} x = \sum_{y \in \mathcal{P}(H) : \mu_{CZ}(x) - \mu_{CZ}(y) = 1} \nu(x, y) y
\]

where \( \nu(x, y) \) stands for the number (mod 2) of the elements in \( \tilde{\mathcal{M}}(x^-, x^+, J, H) = \mathcal{M}(x^-, x^+, J, H) / \mathbb{R} \). In the last expression, we observe that \( \mathbb{R} \) acts freely by translation on the Floer trajectories, and we mod out its action. The operator \( \partial^{H,J} \), satisfies \( \partial^{H,J} \circ \partial^{H,J} = 0 \), thus allowing us to define the Floer homology groups,

\[
HF_\ast(H, J) = \ker \partial^{H,J} / \im \partial^{H,J}
\]
It turns out that these groups are independent of the generic choice of $J$, $HF_*(H) = HF_*(H, J)$. Later on it will be useful to consider the Floer homology groups filtered by the action and we take a moment to review their construction. Let $-\infty < a \leq b < \infty$ be two numbers, so that $a, b \notin \Sigma(H)$. Then we define for $a$ (respectively $b$), $\mathcal{P}^a(H) = \{ x \in \mathcal{P}(H) | A_H(x) < a \}$ and $CF^a(H) = \bigoplus_{x \in \mathcal{P}^a(H)} \mathbb{Z}_2 x$.

Then $CF^a(H)$ is a subcomplex of $CF^b(H)$ and we consider the quotient complex $CF^a(H) = CF^a(H)/CF^b(H)$. The filtered Floer homology groups, $HF^a(H)$, are the homology groups of $CF^a(H)$ with the induced boundary operator.

### 3.2. Symplectic Homology

There are several different versions of the symplectic homology. Originally it was introduced by A. Floer and H. Hofer for bounded, open sets in $\mathbb{R}^{2n}$, by further developing the idea behind the Floer theory and combining that with ideas of I. Ekeland and H. Hofer about using the Hamiltonian dynamics to study the symplectic rigidity. Later on versions of the symplectic homology, concerning relatively compact sets in symplectic manifolds with contact type boundary, and symplectic manifolds with contact type boundary, were developed. Here we are going to use the original version of the symplectic homology from [11], with $\mathbb{Z}_2$-coefficients, and refer the interested reader to the survey paper of A. Oancea, where the different versions of the symplectic homology are compared.

Let $U$ be a bounded open set in $(\mathbb{R}^{2n} = \mathbb{C}^n, \omega_0)$. Next we define the set of admissible Hamiltonian functions, $\mathcal{H}_{\text{ad}}(U)$.

**Definition 2.** A function $H : S^1 \times \mathbb{C}^n \to \mathbb{R}$ is called admissible, $H \in \mathcal{H}_{\text{ad}}(U)$, if:

1. $H|_{\bar{U}} < 0$ for all $t \in S^1$;
2. There is a positive-definite matrix $A$ so that $\frac{|H'(t, z) - Az|}{|z|} \to 0$ as $|z| \to \infty$, uniformly for $t \in S^1$;
3. there is a constant $c > 0$ so that $\|H''(t, z)\| \leq c$ and $|\frac{\partial}{\partial t} H(t, z)| \leq c(1 + |z|)$;
4. the system $-iv = Av$, admits no nontrivial 1-periodic solutions.

Before we proceed, let us comment on the conditions in the above definition. The first condition restricts a function $H \in \mathcal{H}_{\text{ad}}(U)$ on the set $U$, and $H$ is allowed to increase fast near the boundary of $U$. The second condition, determines the asymptotic behavior of $X_H$, which combined with the fourth condition, allows us to conclude that all 1-periodic orbits of $H$ are contained in a compact set together with their connecting (Floer) trajectories. The third condition is for technical purposes and allows one to do the necessary estimates needed for the well-definedness of the Floer homology in this situation, i.e. in the case of open symplectic manifolds. Denote by $\mathcal{H}_{\text{reg}}(U)$ the set of admissible Hamiltonians with non-degenerate 1-periodic orbits and by $\mathcal{J}$ the set of almost complex structures, compatible with the standard symplectic structure $\omega_0$, which are equal to the standard complex structure $i$ outside of a compact set. In [11], the transversality of the Floer’s equation, is established for a dense subset of $\mathcal{H}_{\text{reg}}(U) \times \mathcal{J}$. Following the discussion
of the previous section, one can define the Floer homology groups, filtered by the action, for a regular pair \((H,J)\). Symplectic homology arises from certain functorial properties of Floer homology. Given regular pairs \((H_1,J_1)\) and \((H_2,J_2)\), such that \(H_1 \leq H_2\), on \(S^1 \times \mathbb{C}^n\), we consider a monotone homotopy connecting them. That is a homotopy \((L(s,t,z), \tilde{J}(s,t,z))\) such that:

- \((L(s,t,z), \tilde{J}(s,t,z)) = (H_2(t,z), J_2(t,z))\) for \(s \leq -s_0\);
- \((L(s,t,z), \tilde{J}(s,t,z)) = (H_1(t,z), J_1(t,z))\) for \(s \geq s_0\);
- \(\frac{\partial L}{\partial s} \leq 0\) on \(\mathbb{R} \times S^1 \times \mathbb{C}^n\);
- There is a smooth path \(A(s)\) of positive matrices so that \(A(s) = A(-s_0)\) for \(s \leq -s_0\) and \(A(s) = A(s_0)\) for \(s \geq s_0\) and

\[
\lim_{|z| \to \infty} \frac{|L'(s,t,z) - A(s)z|}{|z|} \to 0
\]

plus we require that if the system \(-i\dot{v} = A(s)v\) has a non-trivial 1-periodic solution for some \(s = s'\) then \(\frac{\partial L}{\partial s}|_{s=s'} A(s)\) is positive definite.

Consider the parametrized version of the Floer equations \((14)\),

\[
\frac{\partial u}{\partial s} + \tilde{J}(s,t,u) \frac{\partial u}{\partial t} = \tilde{J}(s,t,u) X_{L(s)}(t,u) \quad \text{for} \quad u \in C^\infty(\mathbb{R} \times S^1, \mathcal{M})
\]

with asymptotic conditions,

\[
\lim_{s \to \pm \infty} u(s,t) = x_\pm
\]

where \(x_-\) and \(x_+\) are 1-periodic orbits for \(H_2\) and \(H_1\) respectively. Because of the conditions imposed, the solutions of \((16,17)\), stay in a compact set. Generically the moduli spaces \(\mathcal{M}(x_-,x_+)\) are manifolds of dimension \(\mu_{CZ}(x_+) - \mu_{CZ}(x_-)\). Unlike the solutions of \((14)\), the solutions of \((16)\) are no longer \(\mathbb{R}\)-invariant and therefore the 0-dimensional moduli spaces are no longer empty. Notice that the action \(A_{L(s)}(u(s,\cdot))\) is increasing along a solution of \((16)\). Indeed,

\[
\frac{d}{ds} A_{L(s)} = \|u_s\|^2_{g_{L(s)}} - \int_{S^1} \frac{\partial L}{\partial s} (s,t,u(s,t)) dt \geq 0
\]

This allows us to define a map, \(m\), between the chain complexes

\[
m : CF^a(H_1,J_1) \to CF^a(H_2,J_2)
\]

\[
m(x_+) = \sum_{\mu_{CZ}(x_+) = \mu_{CZ}(x_-)} \# \mathcal{M}(x_-,x_+)(x_-).
\]

The map \(m\) preserves the grading and commutes with the differential. It descends to a morphism in the homology and is called the monotypic homomorphism, \(m(H_1,H_2)\),

\[
m(H_1,H_2) : HF^{[a,b]}_*(H_1,J_1) \to HF^{[a,b]}_*(H_2,J_2)
\]

**Remark 1.** Standard arguments as in \((11,4)\) show that the monotonicity map, \(m(H_1,H_2)\), is independent of the choice of the monotone homotopy used to define it.

Further, the monotonicity homomorphism satisfies,

\[
m(H_2,H_3) \circ m(H_1,H_2) = m(H_1,H_3) \quad \text{for} \quad H_1 \leq H_2 \leq H_3
\]
Now we are ready to define the symplectic homology groups of a nonempty open set $U \subset \mathbb{C}^n$, as the direct limit of the Floer homology of regular pairs $(H, J)$:

\[
S_*^{[a,b)}(U) = \lim_{\to} HF_*^{[a,b)}(H, J)
\]

In what follows, in this subsection, we will outline some results and constructions concerning the symplectic homology, which will be important in the proof of our main result. Given $-\infty < a \leq b \leq c \leq \infty$, we have an exact sequence of chain complexes given by inclusions,

\[
0 \to C_*^{[a,b)}(H, J) \to C_*^{(a,c)}(H, J) \to C_*^{[b,c]}(H, J) \to 0
\]

and this generates an exact triangle $\Delta_{a,b,c}(H, J)$ in the homology,

\[
HF_*^{[a,b)}(H, J) \to HF_*^{(a,c)}(H, J) \to HF_*^{[b,c]}(H, J) \to HF_*^{[a,b)}(H, J).
\]

$\Delta_{a,b,c}(H, J)$ commutes with the monotonicity homomorphism, $\hat{\sigma}$, and gives rise to an exact triangle, $\Delta_{a,b,c}(U)$ in symplectic homology,

\[
S_*^{[a,b)}(U) \to S_*^{(a,c)}(U) \to S_*^{[b,c]}(U) \to S_*^{[a,b)}(U).
\]

Given triplets $-\infty < a \leq b \leq c \leq \infty$ and $-\infty < a' \leq b' \leq c' \leq \infty$ with $a \leq a', b \leq b', c \leq c'$ we consider first the natural map, given by inclusions,

\[
C_*^{[a,b)}(H, J) \to C_*^{[a',b')}(H, J),
\]

which gives rise to a map $\sigma$ in homology,

\[
\sigma : HF_*^{[a,b)}(H, J) \to HF_*^{[a',b')}(H, J)
\]

The map $\sigma$ is compatible with the monotonicity homomorphism and generates a map $\hat{\sigma}$ in the symplectic homology,

\[
\hat{\sigma} : S_*^{[a,b)}(U) \to S_*^{[a',b')}(U).
\]

The map $\hat{\sigma}$ commutes with with the triangle $\Delta_{a,b,c}(U)$ and generates homomorphisms,

\[
\Delta_{a,b,c}(U) \to \Delta_{a',b',c'}(U).
\]

Given two open and bounded subsets of $\mathbb{C}^n$, $U \subset V$, we have $H_*^{ad}(V) \subset H_*^{ad}(U)$. This observation together with the monotonicity homomorphisms gives an inclusion morphism, $i_{U,V}$,

\[
i_{U,V} : S_*^{[a,b)}(V) \to S_*^{[a,b)}(U)
\]

For $U \subset V \subset W$, we have,

\[
i_{U,W} = i_{U,V} \circ i_{V,W}
\]

Consider a regular pair $(H, J)$, and let $c \geq 0$ be a constant. From (22) we get a map

\[
\sigma(H, c) : HF_*^{[a-c,b-c]}(H, J) \to HF_*^{[a,b)}(H, J)
\]

Now observe that the action functionals associated with $H$ and $H - c$ are related via $A_{H-c} = A_H + c$. This equality translates into an isomorphism,

\[
\phi(H - c, H) : HF_*^{[a,b)}(H - c, J) \to HF_*^{[a-c,b-c]}(H, J)
\]

Composing the last two maps, we get a map,

\[
\hat{\mu}(H - c, H) = \sigma(H, c) \circ \phi(H - c, H) : HF_*^{[a,b)}(H - c, J) \to HF_*^{[a,b)}(H, J)
\]
On the other hand we have from \cite{13}, the monotonicity homomorphism \( m(H - c, H) \). The following lemma, proven in \cite{15}, will be useful.

**Lemma 2.** For any constant \( c \geq 0 \), \( \hat{m}(H - c, H) = m(H - c, H) \).

We conclude this subsection by outlining a way to compute the symplectic homology groups for given open set \( U \). For this we need the notion of a cofinal (exhausting) family.

**Definition 3.** A family of functions \( \{H_\lambda\}_{\lambda \in \Lambda} \), where \( \Lambda \subset \mathbb{R} \) is unbounded from above, is called a cofinal family for \( U \) if for every \( K \in \mathcal{H}_{ad}(U) \) there exists a number \( \lambda' \) s.t. \( H_\lambda \geq K \) for \( \lambda > \lambda' \).

Once we have a cofinal family \( \{H_\lambda\}_{\lambda \in \Lambda} \), we pair each \( H_\lambda \) with a compatible almost complex structure \( J_\lambda \). Then one perturbs the family \( (H_\lambda, J_\lambda) \) to get a regular cofinal family or argues as in \cite{1}, Section 4, and the symplectic homology groups are computed, as,

\[
S_*^{[a,b)}(U) = \lim_{\lambda \to \infty} HF_*^{[a,b)}(H_\lambda, J_\lambda)
\]

For examples of such computations we refer to \cite{12, 5, 1, 3, 14}.

### 3.3. The definition of the capacity \( c_{FH} \).

First recall the definition of a symplectic capacity on \( (\mathbb{R}^{2n} = \mathbb{C}^n, \omega_0 = -d\lambda_0) \).

**Definition 4.** A symplectic capacity is a map which associates to a given set \( U \subset \mathbb{C}^n \) a number \( c(U) \) with the following properties,

1. Monotonicity: If \( U \subset V \) then \( c(U) \leq c(V) \),
2. Symplectic invariance: \( c(\phi(U)) = c(U) \), for any symplectomorphism \( \phi \) of \( \mathbb{C}^n \),
3. Homogeneity: \( c(aU) = a^2c(U) \) for any real number \( a \).
4. Normalization: \( c(B^{2n}(1)) = c(Z(1)) = \pi \), where \( B^{2n}(1) \) is the unit ball in \( \mathbb{C}^n \), centered at the origin and \( Z(1) = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n | |z_1| < 1\} \)

**Remark 2.** Notice that it is sufficient to find such map \( c \) with the above properties on open and bounded subsets of \( \mathbb{C}^n \), afterwards we can extend it to any open set as follows,

\[
c(U) = \sup\{c(V) \mid V \text{ is bounded and connected and } V \subset U\}
\]

and to any subset by:

\[
c(E) = \inf\{c(U) \mid U \text{ is open and } E \subset U\}
\]

Now we are ready to review the definition of the Floer-Hofer capacity as in \cite{15}. It is based on the computations of the symplectic homology groups for open balls in \cite{12}. We have

**Lemma 3.** The symplectic homology groups of an open ball of radius \( R \), \( B_R = B^{2n}(R) \subset \mathbb{C}^n \), satisfy

\[
S_n^{[a,b)}(B^{2n}(R)) = \mathbb{Z}_2 \text{ for } a \leq 0 < b \leq \pi R^2, \text{ and } 0 \text{ otherwise.}
\]

\[
S_{n+1}^{[a,b)}(B^{2n}(R)) = \mathbb{Z}_2 \text{ for } 0 < a \leq \pi R^2 < b, \text{ and } 0 \text{ otherwise.}
\]

\[
S_k^{[a,b)}(B^{2n}(R)) = 0 \text{ for } k < n \text{ or } n < k < 3n
\]
Let $U$ be an open and bounded subset of $\mathbb{C}^n$ and let $r > 0$ be a number such that $B^{2n}(r) \subset U$. Pick numbers $\varepsilon > 0$ such that $\varepsilon < \pi r^2$ and a number $b > \pi r^2$. Originally, in [12], the following capacity function was defined. With the inclusion morphism,

$$\sigma^b_U : S_{n+1}^{[\varepsilon, b]}(U) \to S_{n+1}^{[\varepsilon, b]}(B^{2n}(R)) = \mathbb{Z}_2$$

we define a capacity function $c'(U)$ as

$$c'(U) = \inf \{ b \mid \sigma^b_U \text{ is onto} \}$$

D. Hermann, was able to extract another capacity from the symplectic homology which he called the Floer-Hofer capacity and we adopted his terminology, (see [15]). Observe that for large $b$, the natural map,

$$\mathbb{Z}_2 = S_n^{(0, \varepsilon)}(B^{2n}(\rho)) \to S_n^{(0, b)}(B^{2n}(\rho))$$

vanishes, (see [24]). Let $R$ be sufficiently large so that $B_r = B^{2n}(r) \subset U \subset B^{2n}(R) = B_R$, then we have

$$\mathbb{Z}_2 = S_n^{(0, \varepsilon)}(B_R) \xrightarrow{i_R} SH_n^{(0, \varepsilon)}(U) \xrightarrow{i_r} S_n^{(0, \varepsilon)}(B_r) = \mathbb{Z}_2$$

Since the composition $i_R \circ i_r$ is an isomorphism, it follows that $0 \neq \alpha_U = i_R(1) \in S_n^{(0, \varepsilon)}(U)$. One then considers the natural map

$$i^b_U : S_n^{(0, \varepsilon)}(U) \to S_n^{(0, b)}(U)$$

and the Floer-Hofer capacity is defined as

$$(26) \quad c_{FH}(U) = \inf \{ b \mid i^b_U(\alpha_U) = 0 \}$$

The next proposition, relates the capacities $c'$ and $c_{FH}$. It is proven in [15] but we sketch a part of the proof for convenience and better understanding of the nature of the two capacities.

**Proposition 1.** The maps $c'$ and $c_{FH}$ are symplectic capacities and $c' \leq c_{FH}$.

**Proof.** Consider the following diagram,

$$\begin{array}{cccccc}
S_{n+1}^{[\varepsilon, b]}(B_R) & \longrightarrow & S_{n+1}^{[\varepsilon, b]}(B_R) & \longrightarrow & S_n^{(0, \varepsilon)}(B_R) & = \mathbb{Z}_2 & \longrightarrow & S_n^{(0, b)}(B_R) \\
\downarrow & & \downarrow & & \downarrow i_R & & \\
S_n^{(0, b)}(U) & \longrightarrow & S_n^{(0, b)}(U) & \longrightarrow & S_n^{(0, \varepsilon)}(U) & \longrightarrow & S_n^{(0, b)}(U) \\
\downarrow & & \downarrow \sigma^b_U & & \downarrow \iota_r & & \\
S_{n+1}^{(0, b)}(B_r) = 0 & \longrightarrow & S_{n+1}^{(0, b)}(B_r) & = \mathbb{Z}_2 & \longrightarrow & S_n^{(0, \varepsilon)}(B_r) & = \mathbb{Z}_2 & \longrightarrow & S_n^{(0, b)}(B_r) \\
\end{array}$$

Here the horizontal arrows are the exact triangles $\Delta_{0, \varepsilon, b}$ and the vertical ones are the inclusion morphisms. We have that $i_r(\alpha_U) = 1$ and $\iota_r$ is an isomorphism. If $i^b_U(\alpha_U) = 0$ then there is $\beta \in S_{n+1}^{(0, b)}(U)$, such that $\alpha_U = \iota_U(\beta)$. We deduce that $\iota_r(\sigma^b_U(\beta)) = 1$ and therefore $\sigma^b_U$ is onto, implying $c'(U) \leq c_{FH}(U)$. For the fact that $c'$ and $c_{FH}$ are symplectic capacities we refer to [13] [12].

**Remark 3.** D. Hermann, [13], proves also that the two capacities are equal on open sets with restricted contact type boundary.
Let $\epsilon > 0$ be the number given by Lemma 11 we may assume in addition that $1 > \epsilon > 0$. Fix $\epsilon'$ such that $\epsilon > \epsilon' > 0$. For $0 < \tau \leq \epsilon$, denote by

$$V_\tau = \psi(N \times B^k) = \{ x \in U \mid \sum_{j=1}^k z^2_j(x) < \tau^2 \}$$

Consider the 1-forms $B_j$ defined on $\mathbb{R}^n$ by $B_j = f \beta_j$, where $f$ is a smooth function on $\mathbb{R}^n$ such that $f = 1$ on $V_\tau$ and $f = 0$ on $\mathbb{R}^n \setminus V_\tau$ and $\beta_j$ are given by (4). This way we get $k$ one-forms defined on $\mathbb{R}^n$ such that

(27) $B_j = \beta_j$ on $V_\epsilon$, and

(28) $B_j = 0$ on $\mathbb{R}^n \setminus V_\epsilon$

Now, fix $0 < \delta < \epsilon'$, and consider the set $V_{\delta/2}$. Using the properties of the capacity $c_{FH}$ we have that

$$c_{FH}(N) \leq c_{FH}(V_{\delta/2}).$$

Next we want to construct a cofinal family $H_\lambda$ for $V_{\delta/2}$ for fixed $\delta$. In what follows the parameter $\lambda$ should be thought as a sufficiently large number since we will be interested in taking the limit as $\lambda \to \infty$ and so we assume that $\lambda > 16/\delta$. We mention that the family we will construct is the one considered by D. Hermann in [15], but adapted for our purposes. Consider smooth functions $g$ and $h$ on $\mathbb{R}^+$ so that,

- $h'(t) = \lambda$ for $t \in [\delta/2 + \lambda^{-1}, \delta/2 + \lambda^{-1/2}]$,
- $h(t) = -\lambda^{-1}$ for $t \in [0, \delta/2 - \lambda^{-1}]$,
- $h(t) = -\lambda^{-1} + \lambda^{1/2}$ for $t \geq \delta/2 + \lambda^{-1} + \lambda^{-1/2}$,
- $h$ is convex on $[\delta/2 - \lambda^{-1}, \delta/2 + \lambda^{-1}]$ and concave on $[\delta/2 + \lambda^{-1/2}, \delta/2 + \lambda^{-1/2} + \lambda^{-1}]$,
- $h(\delta/2) < 0$,
- $g(t) = -\lambda^{-1} + \lambda^{1/2}$ for $t < (\lambda^{1/6} + 1)^2 - \lambda^{-1}$,
- $g'(t) = \mu/2$ for $t > (\lambda^{1/6} + 1)^2$,
- $g$ is convex on $[(\lambda^{1/6} + 1)^2 - \lambda^{-1}, (\lambda^{1/6} + 1)^2]$.

Here $\mu \sim \lambda^{1/6}$ and $\mu \notin \pi\mathbb{Z}$. Now define $H_\lambda$ as follows,

- $H_\lambda(x) = h(\sum_{j=1}^k z_j^2(x))$ for $x \in \bigcup_{|\nu|^2 < \delta/2 + \lambda^{-1} + \lambda^{-1/2}} N(\nu)$,
- $H_\lambda(x) = g(|x|^2)$ for $|x| > \lambda^{1/6}$,
- $H_\lambda(x) = -\lambda^{-1} + \lambda^{1/2}$ for $x \in B^{2n}(\lambda^{1/6}) \setminus \bigcup_{|\nu|^2 < \delta/2 + \lambda^{-1} + \lambda^{-1/2}} N(\nu)$.

Obviously $H_\lambda$ is a cofinal family for $V_{\delta/2}$. Before we proceed we would like to perturb each $H_\lambda$ where it is negative to create non-degenerate critical points. We do this as follows. Let $z_0 \in N$, we will create a small “dimple” at $z_0$. Let $\rho > 0$ be such that $B_p(z_0) \subset V_{\delta/4}$. Consider a smooth cutoff function $\chi$, such that $\chi(0) = 0$, $\chi(s) = 1/2$, for $s \geq \rho/2$ and $\chi'(s) > 0$ for $s > 0$. Denote by $p(x)$ the function $\chi(r^2(x, z_0))$, where $r(x, z_0)$ is the distance function. Glue smoothly to $p$ a smooth function $q(x)$ so that $q(x) = 0$ on $B_p(z_0)$ and $q(x) = q_\lambda(\sum_{j=1}^k z_j^2(x))$ for $x$ outside of $V_{\delta/4}$ and $q_\lambda$ is a smooth function on $[\delta/4, \infty)$, such that it is equal to 1 on $(\delta/2 - \lambda^{-1}, \infty)$ and $q_\lambda' > 0$ on $(\delta/4, \delta/2 - \lambda^{-1})$. Call the new function $\tilde{q}_\lambda$. We assume that it has the following properties:
Now perturb each $H_{\lambda}$ by adding $\lambda^{-2}(\tilde{q}_{\lambda}(x) - 1)$. This way we get a family $\tilde{H}_{\lambda}(x) = H_{\lambda}(x) + \lambda^{-2}(\tilde{q}_{\lambda}(x) - 1)$. We will abuse the notation and call the new family $H_{\lambda}$. Again it is a cofinal family for $V_{\delta/2}$. This way we ensure that for sufficiently small $\varepsilon > 0$ there is a large $\lambda$ so that the only critical points of $A_{\lambda}$ with action in the interval $[0, \varepsilon)$ are the critical points of $\tilde{q}_{\lambda}(x)$ which are non-degenerate. We have that the Conley-Zehnder indices of these critical points, as critical points of $A_{\lambda}$, satisfy:

$\mu_{CZ}(x) = m(x) - n$ for $x \in \text{Crit}(\tilde{q}_{\lambda}) \subset \text{Crit}(H_{\lambda})$

where $m(x)$ is the Morse index of $x$, we refer to [22], for this and other facts concerning the properties of the Conley-Zehnder index. That is to say that for sufficiently small $\varepsilon > 0$ and large $\lambda$, and $x_0$ - a critical point of $H_{\lambda}$ with Morse index $l$, then $\mathbb{Z}_2(x_0) \subset CF_{l-n}(H_{\lambda})$. In particular if $x_0$ is a local minimum of $H_{\lambda}$, then $\mathbb{Z}_2(x_0) \subset CF^{0,0}(H_{\lambda})$.

Next we pair each $H_{\lambda}$ with a compatible almost complex structure $J_{\lambda}$. We can perturb $J_{\lambda}$ if necessary to have that the gradient of the function $\tilde{q}_{\lambda}(x)$ with respect to the metric $g_{\lambda}$ is Morse-Smale, see [22], Theorem 8.1. Notice that the critical points of $A_{\lambda}$ may not be non-degenerate. In fact there are degenerate critical points coming from the region on which $H_{\lambda} = -\lambda^{-1} + \lambda^{1/2}$. In this situation we can argue as in [1], Section 4, that the groups $HF_{*}^{a,b}(H_{\lambda}, J_{\lambda})$ are well-defined as long as $a, b \notin \Sigma(H_{\lambda})$, see especially Remark 4.4.1 in [1].

Consider the function $\tilde{H}_{\lambda}(t, x)$, defined as follows,

$\tilde{H}_{\lambda}(t, x) = 0$ for $0 \leq t < 1/2$; and $\tilde{H}_{\lambda}(t, x) = 2H_{\lambda}(x)$ for $1/2 \leq t < 1$

If we consider the action functional, associated with $\tilde{H}_{\lambda}$, it has the form:

$A_{\tilde{H}_{\lambda}} = -\int_{S^1} x^*\lambda_0 - \int_{1/2}^1 2H_{\lambda}(x(t))dt.$

Straightforward computations show that $A_{\tilde{H}_{\lambda}}$ and $A_{\tilde{H}_{\lambda}}$ have the same critical points with the same critical values and Conley-Zehnder indices. In fact, they generate the same Floer homology groups.

**Proposition 2.** $HF_{*}^{a,b}(H_{\lambda}) \cong HF_{*}^{a,b}(\tilde{H}_{\lambda})$ for all $-\infty < a \leq b \leq \infty$.

Observe that the function $\tilde{H}_{\lambda}$ is not smooth. Despite that it has well defined Floer homology. The reason is that the set up for the Floer homology involves Sobolev spaces of the type $W^{1,p}$ and all the analysis is carried over initially in a weak sense and then elliptic “bootstrapping” arguments are applied for the smoothness of the solutions. The same type of analysis can be carried for piecewise smooth functions. Besides, the critical points of $A_{\tilde{H}_{\lambda}}$ are smooth loops. So, in a way Floer homology “forgives” slight irregularities of the Hamiltonians. Now observe that the functions $H_{\lambda}$ and $\tilde{H}_{\lambda}$ generate the same time 1-maps. In that situation Proposition 2 is a consequence of the discussion in [23], Section 4.

Next pick a compactly supported Hamiltonian function $K_{\delta}$, which generates $\phi$ and such that $\|K_{\delta}\| < c < c_{\delta} = c_{FH}(V_{\delta/2})$, where $c$ is some positive number.
satisfying the previous inequality, see (29). Denote by $H_{\lambda}^\sharp K_\delta$ the following function

$$H_{\lambda}^\sharp K_\delta(t, x) = 2(K_\delta(2t, x) - \sup K_\delta) \quad \text{for } 0 \leq t < 1/2;$$

$$H_{\lambda}^\sharp K_\delta(t, x) = 2H_\lambda(x) \quad \text{for } 1/2 \leq t < 1.$$  

We will be interested in the critical points of the action functional associated with $H_{\lambda}^\sharp K_\delta$.

$$A_{H_{\lambda}^\sharp K_\delta}(x) = -\int_{S^1} x^*\lambda_0 - 2\int_0^{1/2} (K_\delta(2t, (x)) - \sup K_\delta)dt - 2\int_{1/2}^1 H_\lambda(x)dt$$

To be more precise we are going to show that this functional possesses a critical point (i.e., a 1-periodic orbit of $H_{\lambda}^\sharp K_\delta$) with action in the interval $[0, c_\delta]$, for sufficiently large $\lambda$. Observe that any critical point of $A_{H_{\lambda}^\sharp K_\delta}$ consists of two arcs, one is a trajectory of the flow of $K_\delta$, followed by a trajectory of $H_\lambda$. Notice that we have

$$\bar{H}_\lambda - c \leq H_{\lambda}^\sharp K_\delta \leq \bar{H}_\lambda$$

and

$$A_{\bar{H}_\lambda} \leq A_{H_{\lambda}^\sharp K_\delta} \leq A_{\bar{H}_\lambda} + c$$

The next lemma is a modification of Corollary 5.9 in [14], but notice that we assume less in our case.

**Lemma 4.** Let $c < c_\delta$ be as above. Then for sufficiently large $\lambda$, $H_{\lambda}^\sharp K_\delta$ has a 1-periodic orbit with action in the interval $[0, c_\delta]$.

**Proof:** Pick a sufficiently small $\varepsilon$ so that $0 < \varepsilon < c_\delta - c$. Let $B_r$ and $B_R$ be balls centered at $z_0 \in N \subset \mathbb{C}^n$ with radii $r$ and $R$ respectively so that $B_r \subset V_{\delta/2} \subset B_R$. We know from Lemma 3 that

$$S_n^{[-c, \varepsilon]}(B_r) \simeq \mathbb{Z}_2 \simeq S_n^{[-c, \varepsilon]}(B_R)$$

Moreover we can easily construct cofinal families for $B_r$ and $B_R$, respectively with a single “dimple”, i.e. unique local minimum at $z_0$ for both families in the spirit of what we did with $H_\lambda$. Our arguments above show that then the generator of the symplectic homology groups $S_n^{[-c, \varepsilon]}(B_r)$ and $S_n^{[-c, \varepsilon]}(B_R)$ is the class of the constant solution, i.e. $[z_0]$. Consider the following diagram for sufficiently large $R$,

$$
\begin{array}{ccc}
Z_2 = S_n^{[-c, \varepsilon]}(B_R) & \xrightarrow{\sigma_R} & S_n^{[0, c+\varepsilon]}(B_R) = \mathbb{Z}_2 \\
\downarrow & & \downarrow \\
\{0\} \neq S_n^{[-c, \varepsilon]}(V_{\delta/2}) & \xrightarrow{\sigma'} & S_n^{[0, c+\varepsilon]}(V_{\delta/2}) \neq \{0\} \\
\downarrow & & \\
Z_2 = S_n^{[-c, \varepsilon]}(B_r)
\end{array}
$$

where the vertical arrows are the inclusion morphisms and the fact that $S_n^{[0, c+\varepsilon]}(V_{\delta/2}) \neq \{0\}$ follows from the definition of the Floer-Hofer capacity, [K]. This diagram, together with the definition of the Floer-Hofer capacity, implies that the map $\sigma'$
is nonzero. In fact, the map $\sigma'$ keeps “alive” the class of $z_0$. Next consider the commutative diagram.

\[
\begin{array}{ccc}
HF_n^{[c]}(H_\lambda, J_\lambda) & \xrightarrow{\sigma(H_\lambda, c)} & HF_n^{[0, c]}(H_\lambda, J_\lambda) \\
\{0\} & \xrightarrow{\sigma'} & S_n^{(0, c)}(V_{\delta/2})
\end{array}
\]

Here the vertical arrows are the direct limit morphisms, which are surjective for sufficiently large $\lambda$. This diagram implies that the map $\sigma(H_\lambda, c)$ must be nonzero. But then from (25), we have that the map

\[
\hat{m}(H_\lambda - c, H_\lambda) : HF_n^{[0, c]}(H_\lambda - c, J_\lambda) \rightarrow HF_n^{[0, c]}(H_\lambda, J_\lambda)
\]

is nonzero. From Lemma 2, we know that $\hat{m}(H_\lambda - c, H_\lambda) = m(H_\lambda - c, H_\lambda)$. And this shows that $m(H_\lambda - c, H_\lambda) \neq 0$. Denote by $\hat{m}(H_\lambda - c, H_\lambda)$ the monotonicity map between $HF_n^{[0, c]}(H_\lambda - c, J_\lambda)$ and $HF_n^{[0, c]}(H_\lambda, J_\lambda)$. It is not hard to see that $\hat{m}$ agrees with the map induced by $m$ through the isomorphism of Proposition 2. Then this map is nonzero. If $H_\lambda\sharp K_{\delta}$ did not have a 1-periodic orbit with action in $[0, cs]$, then the Floer homology group $HF_n^{[0, c]}(H_\lambda\sharp K_{\delta}, J_\lambda)$ would be well-defined and equal to zero. But then the monotonicity map $\hat{m}(H_\lambda - c, H_\lambda) = \hat{m}(H_\lambda - c, H_\lambda\sharp K_{\delta})\circ\hat{m}(H_\lambda\sharp K_{\delta}, \hat{H}_{\lambda})$, would have been zero, which is a contradiction.

The previous Lemma gives the existence of a 1-periodic orbit, $x_\lambda(t)$ of $H_\lambda\sharp K_{\delta}$ with bounded action for sufficiently large $\lambda$ and moreover that this solution is a deformation of the constant class of $0 \neq [z_0] \in HF_n^{[0, c]}(H_\lambda - c, J_\lambda)$ or in other words,

\[
x_\lambda = \hat{m}(H_\lambda - c, H_\lambda\sharp K_{\delta})(z_0)
\]

and

\[
0 \neq \hat{m}(H_\lambda - c, H_\lambda\sharp K_{\delta})([z_0]) \in HF_n^{[0, c]}(H_\lambda\sharp K_{\delta}, J_\lambda)
\]

This observation will be important later on. The periodic orbit $x_\lambda(t)$, satisfies the equations,

\[
\dot{x}_\lambda(t) = 2X_{K_\delta(2\lambda)}(x(t)) \text{ for } t \in (0, 1/2)
\]

\[
\dot{x}_\lambda(t) = 2X_{H_\lambda}(x(t)) \text{ for } t \in (1/2, 1)
\]

Denote by $\phi^x_\delta$ the flow of $K_{\delta}$ and by $\phi^x_\lambda$ the flow of $H_\lambda$.

Next we claim that for sufficiently large $\lambda$, $x_\lambda(0) \in N_\nu$, where $\nu = (\nu_1, \ldots, \nu_k)$ and $\sum_{j=1}^k \nu_j^2 < \delta/2 + \lambda^{-1} + \lambda^{-1/2}$. Indeed, if we assume that this is not the case then we have two possibilities: either $x_\lambda(t) = x_\lambda(1/2)$ for $t \in [1/2, 1]$ or (perhaps after taking sufficiently large $\lambda$ so large that the ball $B(\lambda^{1/6}) \supset \text{supp} K_{\delta}$), $x_\lambda(t) = x_\lambda(0)$ for $t \in [0, 1/2]$. In the former case we have that $x_\lambda(t) = \phi^x_\delta(x_\lambda(0))$ is a 1-periodic solution for $K_{\delta}$, then its action satisfies,

\[
\mathcal{A}_{H_\lambda\sharp K_{\delta}}(x_\lambda(t)) = \mathcal{A}_{K_{\delta}}(x_\lambda(t)) - \sup K_{\delta} - C(\lambda)
\]

where $C(\lambda) = \lambda^{1/2} - \lambda^{-1}$. Since $K_{\delta}$ is compactly supported, the critical values of $\mathcal{A}_{K_{\delta}}(x(t)) - \sup K_{\delta}$ are bounded and therefore for large $\lambda$, the right hand side of (38) will be very negative, which is a contradiction with the fact that $\mathcal{A}_{H_\lambda\sharp K_{\delta}}(x_\lambda(t)) \geq 0$. 

Similarly in the latter case, \( x_\lambda(t) \) is a 1-periodic orbit for \( H_\lambda \), satisfying the equation
\[
- i \dot{x}_\lambda = \mu x_\lambda.
\]
Then we have for the action of the periodic orbit \( x_\lambda \),
\[
\mathcal{A}_{H_\lambda \sharp \mathcal{K}_\delta}(x_\lambda) = \int_0^1 (\mu |x_\lambda(t)|^2 - g(|x_\lambda|^2)) dt + c \leq \mu (\lambda^{1/6} + 1)^2 - C(\lambda) + c
\]
where \( C(\lambda) \) is given before. Because of our choice of \( \mu \) and since \( c \) is bounded, for large \( \lambda \) this action will be very negative, which is a contradiction.

Now fix a very large \( \lambda = \lambda(\delta) \) so that \( B(\lambda^{1/6}) \supset V_\epsilon \), \( \lambda^{-1} + \lambda^{-1/2} < \delta/2 \) and the 1-periodic orbit of \( H_\lambda \sharp \mathcal{K}_\delta, x_\lambda, \) satisfies \( x_\lambda(0) \in N(\nu) \) for some \( \nu = \nu(\delta) = (\nu_1, \ldots, \nu_k) \), with \( (\sum_{j=1}^k \nu_j^2)^{1/2} < \delta/2 + \lambda^{-1} + \lambda^{-1/2} < \delta \). From (47), follows that we can write \( x_{\lambda(\delta)}(t) = \phi^{s_1}_{\lambda}(x_{\lambda(\delta)}(0)) \), for \( t \in [0, 1/2] \) and \( x_{\lambda(\delta)}(t) = \phi^{s_2t-1}_{\lambda}(x_{\lambda(\delta)}(1/2)) \), for \( t \in [1/2, 1] \). Then we have,
\[
\phi^{s_1}_{\lambda}(x_{\lambda(\delta)}(0)) = x_{\lambda(\delta)}(1/2) = (\varphi^{s_1}_{\lambda})^{-1}(x_{\lambda(\delta)}(0))
\]
We argue that \((\varphi^{s_1}_{\lambda})^{-1}(x_{\lambda(\delta)}(0)) \in \mathcal{L}_{x_{\lambda(\delta)}(0)}N(\nu(\delta)) \). Indeed, if \( x \in N(\nu(\delta)) \) then the flow \( \varphi^{s_1}_{\lambda} \) of \( H_\lambda \), satisfies an equation of the form (10), with coefficients
\[
\lambda_j = 2h' \left( \sum_{i=1}^k \nu_i^2 \right) \nu_j = 2\lambda(\delta) \nu_j
\]
and therefore the flow of \( H_\lambda \) is on the leaf through \( x \). The flow \((\varphi^{s_1}_{\lambda})^{-1} \) is generated by the Hamiltonian \( \tilde{H}_\lambda(x) = -H_\lambda(\varphi^{s_1}_{\lambda}x) \). From this it is not hard to see that the flow \((\varphi^{s_1}_{\lambda})^{-1} \) on \( N(\nu(\delta)) \) satisfies an equation of the form,
\[
\dot{x}(t) = \sum_{j=1}^k \gamma_j X_{\lambda_j}(x(t))
\]
and this shows that \((\varphi^{s_1}_{\lambda})^{-1}(x_{\lambda(\delta)}(0)) \in \mathcal{L}_{x_{\lambda(\delta)}(0)}N(\nu(\delta)) \) for any \( t \) and in particular for \( t = 1 \). To summarize we demonstrated that \( x_{\lambda(\delta)}(0) \in N(\nu(\delta)) \) satisfies
\[
\phi(x_{\lambda(\delta)}(0)) = \phi^{s_1}_{\lambda}(x_{\lambda(\delta)}(0)) \in \mathcal{L}_{x_{\lambda(\delta)}(0)}N(\nu(\delta))
\]
The next lemma is crucial since it will allow us to take a limit as \( \delta \) goes to 0.

**Lemma 5.** The length of the arc \( l((\varphi^{s_1}_{\lambda})^{-1}(x_{\lambda(\delta)}(0))) \) is bounded independently of \( \delta \).

**Proof:** In view of (10) this statement is equivalent to showing that each of the coefficients \( \gamma_j, j = 1, \ldots, k \) is uniformly bounded. Recall from (39), that the periodic orbit \( x_{\lambda(\delta)}(t) \) is a deformation of the constant solution of \( H_\lambda - c \) through a monotone homotopy. From Remark 1, we know that the map \( \bar{m}(H_\lambda - c, H_\lambda \sharp \mathcal{K}_\delta) \) is independent of the choice of the monotone homotopy of Hamiltonians used to define it. This allows us to choose a particular regular monotone homotopy \( (L(s), \bar{J}(s)) \) which realizes \( \bar{m}(H_\lambda - c, H_\lambda \sharp \mathcal{K}_\delta) \). We pick \( L \) of the form
\[
L(s, t, u(s, t)) = (1 - \kappa(s))\bar{H}_\lambda(t, u) - c + \kappa(s)H_\lambda \sharp \mathcal{K}_\delta(t, u)
\]
where \( \kappa(s) \) is a smooth function on \( \mathbb{R} \) so that \( \kappa(s) = 0 \) for \( s \leq -s_0; \kappa(s) = 1 \), for \( s \geq s_0 \) and \( \kappa'(s) \geq 0 \) on \( (-s_0, s_0) \). Of course we assume that \( \bar{J}(s, t) \) is a regular homotopy of families of almost complex structures so that \( \bar{J}(s, t) = \bar{J}(\lambda(\delta))(t, \cdot) \), for \( s \in (-\infty, -s_0] \cup [s_0, \infty) \). Consider now the equation,
\[
u_s + \bar{J}(s, t, u(s, t))(u_t - X_{L(s, t, u(s, t))}) = 0
\]
Our arguments imply that it possesses a solution \( u(s, t) \), such that

\[
\lim_{s \to -\infty} u(s, t) = z_0
\]

and

\[
\lim_{s \to \infty} u(s, t) = x_{\lambda(\delta)}(t)
\]

In view of this and integrating (18) over \( \mathbb{R} \), we get, for our particular case, the following inequality,

\[
(42) \quad A_{H_{\lambda} \cdot c}(z_0) - A_{H_{\lambda} \cdot K_\delta}(x_{\lambda}(t)) \geq \frac{1}{2} \int_{\mathbb{R} \times S^1} (\|u_s\|^2_{g_{J(s)}} + \|u_t - X_L\|^2_{g_{J(s)}}) ds dt
\]

The left-hand side of (42) is bounded from above by \( c_\delta = c_{FH}(V_{\delta/2}) \). We are going to work with the right-hand side. Recall that \( X_{H_{\lambda} \cdot K_\delta} = 2X_{H_{\lambda}} \) for \( t \in (1/2, 1) \). Using this we get the following inequality for the right-hand of (42),

\[
(43) \quad \frac{1}{2} \int_{\mathbb{R} \times S^1} (\|u_s\|^2_{g_{J(s)}} + \|u_t - X_L\|^2_{g_{J(s)}}) ds dt \geq \frac{1}{2} \int_{\mathbb{R} \times [1/2, 1]} (\|u_s\|^2_{g_{J(s)}} + \|u_t - X_L\|^2_{g_{J(s)}}) ds dt
\]

Now recall the 1-forms \( B_j \), \( j = 1, \ldots, k \), which we introduced in the beginning of this section, see (27, 28). We claim that \( B_j(X_{H_{\lambda}} \cdot \cdot) = 0 \). This is easy to be seen, to be the case on \( \mathbb{R}^2n \setminus V_\varepsilon \), since there \( B_j = 0 \). On \( V_\varepsilon \setminus V_{\varepsilon'} \), it is true because \( H_{\lambda} = \text{const} \), there. On \( V_{\varepsilon'} \), \( B_j = \beta_j \), and on that region \( X_{H_{\lambda}} \) is a linear combination of \( \{X_{z_j}\}_{j=1}^k \), and our claim follows from (3). Choose a constant \( C_1 > 0 \) so that for \( j = 1, \ldots, k \) and all \( \xi, \eta \in \mathbb{R}^2n \) we have

\[
(44) \quad |dB_j(\xi, \eta)| \leq C_1|\xi||\eta|
\]

Consider the space of all almost complex structures \( J \) on \( C^n \), compatible with \( \omega_0 \). Denote, as before, by \( g_J \) the corresponding metric, i.e., \( g_J(\cdot, \cdot) = \omega_0(\cdot, J\cdot) \). Since the set \( V_\varepsilon \) is a compact subset of \( C^n \), there is a constant \( C_2 > 0 \) so that on \( V_\varepsilon \) we have that,

\[
\|\xi\|_{g_J} \geq \sqrt{C_2}\|\xi\|_{g_{\omega_0}} = \sqrt{C_2}|\xi|
\]
for any $\xi \in \mathbb{C}^n$. Here $J_0 = i$ is the standard complex structure on $\mathbb{C}^n$. In view of our discussion above and (42, 43) we obtain,

$$c_3 \geq A_{H_{\xi-c}}(z_0) - A_{H_{\xi+c}\mathbb{K}_x}(x_{\lambda(\delta)}(t))$$

$$\geq \frac{1}{2} \int_{\mathbb{R} \times S^1} (\|u_s\|^2_{g_{J(\epsilon)}} + \|u_t - X_L\|^2_{g_{J(\epsilon)}}) ds dt$$

$$\geq \frac{1}{2} \int_{\mathbb{R} \times [1/2,1]} (\|u_s\|^2_{g_{J(\epsilon)}} + \|u_t - 2X_{H_s}\|^2_{g_{J(\epsilon)}}) ds dt$$

$$\geq \frac{1}{2} \int_{\mathbb{R} \times [1/2,1]} (\|u_s\|^2_{g_{J(\epsilon)}} + \|u_t - 2X_{H_s}\|^2_{g_{J(\epsilon)}}) ds dt$$

$$\geq \frac{1}{2} \int_{\mathbb{R} \times [1/2,1]} (\|u_s\|_{g_{J(\epsilon)}} \|u_t - 2X_{H_s}\|_{g_{J(\epsilon)}}) ds dt$$

$$\geq \int_{\mathbb{R} \times [1/2,1]} (\|u_s\|_{g_{J(\epsilon)}} \|u_t - 2X_{H_s}\|_{g_{J(\epsilon)}}) ds dt$$

$$\geq \frac{C_2}{C_1} \int_{\mathbb{R} \times [1/2,1]} |dB_j(u_s, u_t - 2X_{H_s})| ds dt$$

$$= \frac{C_2}{C_1} \int_{\mathbb{R} \times [1/2,1]} |dB_j(u_s, u_t)| ds dt = \frac{C_2}{C_1} \int_{\mathbb{R} \times [1/2,1]} |dB_j(u_s, u_t)| ds dt$$

$$\geq \frac{C_2}{C_1} \int_{\mathbb{R} \times [1/2,1]} dB_j(u_s, u_t) ds dt = \frac{C_2}{C_1} \int_{1/2}^1 x_{\lambda(\delta)}(t)^* B_j dt - \int_{1/2}^1 z_0^* B_j dt$$

$$= \frac{C_2}{C_1} \int_{1/2}^1 x_{\lambda(\delta)}(t)^* B_j dt = \frac{C_2}{C_1} |\gamma_j|$$

In the above formulas the last couple of equalities follow from Stokes’ Theorem and (40) and $\int^*$ means integrating over the part of the trajectory which is contained in $V_r$. So far, we obtained that for each $j = 1, \ldots, k$, the coefficients $\gamma_j$ are bounded by $c_3C_1/C_2$. Notice that $c_3 \leq c_{FH}(V_r)$ and so it is bounded by a constant independent of $\delta$ and so are the coefficients $\gamma_j$. All this shows that the length of the arc $l(x_{\lambda(\delta)}(t)|_{t \in [1/2,1]})$ is bounded independently of $\delta$. □

Repeating the arguments above for any $\delta \in (0, \epsilon')$ and applying the Arzela-Ascoli Theorem, we can find a sequence $\{\delta_m\}_{m=1}^\infty$ converging to 0 so that

$$\lim_{m \to \infty} x_{\lambda(\delta_m)}(0) = x_0 \in N$$

$$\phi(x_{\lambda(\delta_m)}(0)) \to \phi(x_0)$$

and $x_0$ and $\phi(x_0)$ are connected by an arc which is contained in the leaf $L_{x_0}N$. This proves Theorem 11 □

5. PROOF OF THEOREM 2

Theorem 2 is a consequence of Theorem 11 and the following lemma.

Lemma 6. The level submanifold $N_{c_1, c_2, \ldots, c_{k-1}}$ is of $k$-contact type in $(\mathbb{R}^{2n}, \omega_0)$.

Proof: First we notice that a symplectic change of coordinates does not change the property of a submanifold to be of $k$-contact type. Making a symplectic change of coordinates $(x_j, y_j) \to (I_j, \theta_j)$, where $I_j = r_j^2/2$ and as before $x_j - iy_j = r_j e^{i\theta_j}$. In these coordinates $\omega_0 = d\alpha_0$ with $\alpha_0 = -I_1 d\theta_1 - \ldots - I_n d\theta_n$ and $N_{c_1, c_2, \ldots, c_{k-1}} = \{(I_j, \theta_j)| I_1 = c_1/2, \ldots, I_{k-1} = c_{k-1}/2, \sum_{j=1}^{k} m_j I_j = c - 1/2 \sum_{j=1}^{k-1} m_j c_j\}$. For $1 \leq j \leq k - 1$, consider the one-forms $\alpha_j = \alpha_0 - d\theta_j$. 
Obviously we have $da_j = \omega_0$ for $0 \leq j \leq k - 1$. Next we see that $\text{Ker}\omega_0|_N = \text{span}(X_0 = \sum_{j=1}^n -m_j \frac{\partial}{\partial y_j}, X_1 = -\frac{\partial}{\partial y_{n+1}}, \ldots, X_{k-1} = -\frac{\partial}{\partial y_{k-1}})$. We want to show that the restrictions of $\alpha_0, \alpha_1, \ldots, \alpha_{k-1}$ to $\text{Ker}\omega_0|_N$ are linearly independent on $N = N_{c,c_1,\ldots,c_{k-1}}$ and we check that on the basis $X_0, X_1, \ldots, X_{k-1}$. Denote by $A$ the $k \times k$ matrix with entries $a_{i,j} = \alpha_{j-1}(X_{i-1})$ for $1 \leq i, j \leq k$. Then we have that $a_{1,1} = c$ and $a_{1,j} = c + m_{j-1}$ for $2 \leq j \leq k$, $a_{i,j} = c_{i-1}/2 + \delta_{i,j}$ for $2 \leq i, j \leq k$ where $\delta_{i,j}$ denotes the Kronecker symbol. It is not hard to compute that det $A = c - \frac{1}{2} \sum_{j=1}^{k-1} m_j c_j > 0$, because of our assumption (3). This completes the proof of the lemma.

**Lemma 7.** The Floer-Hofer capacity, $c_{FH}(N_{r_1,\ldots,r_n}^k) = \min_p \{\pi r_p^2\}$, where $N_{r_1,\ldots,r_n}^k = \{ | z_1 | = r_1, \ldots, | z_{k-1} | = r_{k-1}, \sum_{j=1}^n \frac{|z_j|^2}{r_j^2} = 1 \}$ and $z_j = x_j + iy_j$.

**Proof:** First observe that $c_{FH}(N_{r_1,\ldots,r_n}^k) \leq \min_p \{\pi r_p^2\}$. Indeed we have that $N_{r_1,\ldots,r_n}^k \subset Z_{r_j}$, for $j = 1, \ldots, n$, where $Z_{r_j} = \{ z \in \mathbb{C}^n | |z_j| < r_j \}$ and the claim follows from the properties of the capacity.

Next we are going to argue that $c_{FH}(N_{r_1,\ldots,r_n}^k) \geq \min_p \{\pi r_p^2\}$. For this we use arguments similar to those in [12], where the symplectic homology of ellipsoids and polydisks is computed. Because of that we will be somewhat sketchy. Essentially the idea is to exploit the product structure of $(\mathbb{R}^{2n} = \mathbb{C}^n, \omega_0)$. For sufficiently small $\varepsilon > 0$ consider a neighborhood $V_\varepsilon$ of $N_{r_1,\ldots,r_n}^k$ of the form.

$$V_\varepsilon = \left\{ z \in \mathbb{C}^n | 1 - \varepsilon < \frac{|z_j|^2}{r_j^2} < 1 + \varepsilon \text{ for } j = 1, \ldots, k-1 \text{ and } 1 - \varepsilon < \sum_{j=k}^n \frac{|z_j|^2}{r_j^2} < 1 + \varepsilon \right\}$$

For $V_\varepsilon$ we are going to build a cofinal family of Hamiltonians of the form:

$$H_\lambda(z_1, \ldots, z_n) = \sum_{j=1}^{k-1} \rho_\lambda \left( \frac{|z_j|^2}{r_j^2} \right) + \rho_\lambda \left( \sum_{j=k}^n \frac{|z_j|^2}{r_j^2} \right)$$

where the functions $\rho_\lambda : \mathbb{R} \rightarrow \mathbb{R}$ satisfy,

- $\rho_\lambda$ is symmetric with respect to 1, i.e. $\rho_\lambda(1+s) = \rho_\lambda(1-s)$ and has unique non-degenerate minimum at 1;
- $\rho'_\lambda(s) = \rho'_\lambda(\infty) = \text{const}$ for $s \geq s_0(\lambda) > 0$;
- $\rho'_\lambda(s) > 0$ for $s > 1$;
- $\rho'_\lambda(s) < 0$ for $s \in [1 - \varepsilon, 1 + \varepsilon]$;
- for each $\lambda$ the equations $-i\dot{z} = \rho'_\lambda(\infty)z$ have no non-trivial 1-periodic solutions;
- for $\lambda > \lambda'$, $\rho_\lambda > \rho_{\lambda'}$.

Then one perturbs perturbs $H_\lambda$ by small perturbation $\Delta_\lambda$ so that $H_\lambda + \Delta_\lambda \in \mathcal{H}_{reg}(V_\varepsilon)$ and the actions of 1-periodic orbits of $H_\lambda + \Delta_\lambda$ are near the actions of the 1-periodic orbits of $H_\lambda$. We abuse the notation and denote the perturbed family again by $H_\lambda$. Then Proposition 5, in [12], tells us that a minimal non-negative action of periodic orbit of $H_\lambda$, of Conley-Zehnder index $n+1$ will be greater than $\min_p \{\pi r_p^2 (1-\varepsilon)\} - \tau(\lambda)$ for some $\tau(\lambda) > 0$ and such that $\lim_{\lambda \to \infty} \tau(\lambda) = 0$. This immediately gives us,

$$c_{FH}(V_\varepsilon) \geq \min_p \{\pi r_p^2 (1-\varepsilon)\}.$$
Passing to the limit as $\varepsilon \to 0$ we get

$$c_{FH}(N_{r_1,\ldots,r_n}) \geq \min_p \{\pi r_p^2\}.$$ 

This completes the proof of the lemma. □

**Proof of Theorem 2.** Denote by $\phi$ the time-one map of the Hamiltonian $H_0$ given by (2), and by $\psi$ the time-one map of $H_0 + H_1$. Since $\phi(L_N(x)) = L_N(x)$, we have to show that there exists $x \in N$ such that

$$\phi^{-1} \circ \psi(x) \in L_N(x)$$

The map $\phi^{-1} \circ \psi$ is the time-one map of the flow generated by the Hamiltonian $H_0(\psi^t(x)) + H_1(t, \psi^t(x)) - H_0(\psi^t(x)) = H_1(t, \psi^t(x))$. By the preceding lemmata and the properties of the capacity $c_{FH}$, we know that $N_{c,c_1,\ldots,c_{k-1}}$ is of $k$-contact type and

$$c_{FH}(N_{c,c_1,\ldots,c_{k-1}}) = \min \left\{ \min_{p=1,\ldots,k-1} \left\{ \pi c_p \right\}, \min_{p=k,\ldots,n} \left\{ \pi \frac{2c - \sum_{j=1}^{k-1} m_j c_j}{m_p} \right\} \right\}$$

Thus we have $E(\phi^{-1} \circ \psi) \leq ||H_1(t, \psi^t(x))|| < c_{FH}(N_{c,c_1,\ldots,c_{k-1}})$. Now Theorem 1 yields easily Theorem 2. □

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