LOCAL BOUNDEDNESS OF CATLIN \(q\)-TYPE

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Abstract. In [6], D’Angelo introduced the notion of finite type for points \(p\) of a real hypersurface \(M\) of \(\mathbb{C}^n\) by defining the order of contact \(\Delta_q(M, p)\) of complex analytic \(q\)-dimensional varieties with \(M\) at \(p\). Later, Catlin [4] defined \(q\)-type, \(D_q(M, p)\) for points of hypersurfaces by considering generic \((n - q + 1)\)-dimensional complex affine subspaces of \(\mathbb{C}^n\). We define a generalization of the Catlin’s \(q\)-type for an arbitrary subset \(M\) of \(\mathbb{C}^n\) in a similar way that D’Angelo’s 1-type, \(\Delta_1(M, p)\), is generalized in [13]. Using recent results connecting the D’Angelo and Catlin \(q\)-types in [1] and building on D’Angelo’s work on the openness of the set of points of finite \(\Delta_q\)-type, we prove the openness of the set of points of finite Catlin \(q\)-type for an arbitrary subset \(M \subset \mathbb{C}^n\).

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1. Introduction

Let \(\Omega\) be a pseudoconvex domain in \(\mathbb{C}^n\) with real-analytic boundary. Subellipticity of \(\bar{\partial}\)–Neumann problem at a boundary point \(p\) depends on the order of contact of complex analytic varieties with \(\partial \Omega\) at \(p\). In [12], Kohn proved that subellipticity of \(\bar{\partial}\)–Neumann problem at a boundary point \(p\) on \((0, q)\) forms is equivalent to non-existence of \(q\)-dimensional complex-analytic varieties in \(\partial \Omega\) through \(p\). To measure the order of contact of holomorphic curves with a smooth real hypersurface \(M \subset \mathbb{C}^n\) at \(p\), D’Angelo [6] introduced the notion of type. More precisely, the type of \(M\) at \(p\) is defined by

\[
\Delta(M, p) = \sup_{\gamma \in \mathcal{C}} \frac{\nu(r \circ \gamma)}{\nu(\gamma)}
\]

where \(r\) is a defining function of \(M\), \(\mathcal{C}\) is the set of non-constant holomorphic germs of curves \(\gamma\) at 0 \(\in \mathbb{C}\) so that \(\gamma(0) = p\) and \(\nu(r \circ \gamma)\) denotes the order of vanishing of the function \(r \circ \gamma\) at 0. A point \(p\) is called finite type if \(\Delta(M, p) < \infty\). In [6], D’Angelo proved the crucial property that the set of points of finite type forms an open subset of \(M\). This condition of finite type appeared later to be central in Catlin’s work [4] on subelliptic estimates for the \(\bar{\partial}\)-Neumann problem. See [7, 10, 11] for a more recent discussion of the relationship between finite type and subellipticity.

In [6], D’Angelo defined the \(q\)-type of a hypersurface \(M \subset \mathbb{C}^n\), which possibly contains \(q - 1\) dimensional complex analytic varieties. The \(q\)-type of \(M\) at \(p \in M\) is defined by

\[
\Delta_q(M, p) = \inf\{\Delta(M \cap P, p) : P\ \text{is any} \ n - q + 1 \text{ dimensional complex affine subspace of } \mathbb{C}^n\}.
\]

In this definition, \(M \cap P\) is considered as the germ at \(p\) of a smooth real hypersurfaces in \(\mathbb{C}^{n-q+1}\). We should note that, when \(q = 1\), \(\Delta_1(M, p) = \Delta(M, p)\).
Let $\mathcal{O}_p$ be the ring of germs of holomorphic functions at $p$ in $\mathbb{C}^n$ and $\mathcal{C}^\infty_p$ be the ring of germs of smooth functions at $p$. $\Delta$ can also be defined for ideals in $\mathcal{O}_p$ or $\mathcal{C}^\infty_p$. For a proper ideal $I$ in $\mathcal{O}_p$ or in $\mathcal{C}^\infty_p$, in [6, 8], D’Angelo defined

$$\Delta(I, p) = \sup_{\gamma \in \mathcal{C}} \inf_{\varphi \in I} \frac{\nu(\varphi \circ \gamma)}{\nu(\gamma)},$$

$$\Delta_q(I, p) = \inf_w \Delta((I, w), p),$$

where infimum in the definition of $\Delta_q$ is taken over $q - 1$ linear forms $w = \{w_1, \ldots, w_{q-1}\}$ in $\mathcal{O}_p$ and $(I, w)$ is the ideal generated by $I$ and $w$.

If an ideal $I$ in $\mathcal{O}_p$ contains $q$ independent linear functions, then it follows from Theorem 2.7 in [6] that

$$\Delta(I, p) \leq \text{mult}(I, p) \leq (\Delta(I, p))^{n-q},$$

where $\text{mult}(I) = \dim_C(\mathcal{O}_p/I)$.

D’Angelo $q$-type for a real hypersurface $M$ of $\mathbb{C}^n$ has an equivalent definition (see p.86 of [8]):

$$\Delta_q(M, p) = \Delta_q(I(M), p)$$

where $I(M)$ is the ideal of smooth germs in $\mathcal{C}^\infty_p$, vanishing on $M$ near $p$.

In [4], Catlin defined $q$-type, $D_q(M, p)$, of a smooth real hypersurface $M \subset \mathbb{C}^n$ at $p$ by considering the intersections of germs at $p$ of $q$ dimensional varieties with generic $(n-q+1)$-dimensional complex affine subspaces of $\mathbb{C}^n$ through $p$. More precisely, Catlin $q$-type is defined by

$$D_q(M, p) = \sup_{V^q} \text{gen.val} \max_{S \in G^{n-q+1}_p} \frac{\nu(r \circ \gamma^k_S)}{\nu(\gamma^k_S)}.$$

Here the supremum is taken over the germs at $p$ of $q$-dimensional complex varieties $V^q$, $G^{n-q+1}_p$ denotes the set of $(n-q+1)$-dimensional complex affine subspaces of $\mathbb{C}^n$ through $p$ and $\gamma^k_S$ denote the germs of the one-dimensional irreducible components of $V^q \cap S$. In [4, Proposition 3.1], Catlin showed that for any germ at $p$ of a $q$-dimensional complex variety $V^q$, there exists an open (hence, dense) subset $W$ of $G^{n-q+1}_p$ such that for any $S \in W$, $V^q \cap S$ has same number of one-dimensional components and

$$\max_{k=1,\ldots,P} \frac{\nu(r \circ \gamma^k_S)}{\nu(\gamma^k_S)}$$

is independent of $S \in W$. Hence, the generic value in the definition of $D_q(M, p)$ is computed by any $S \in W$. Similarly, Catlin also defined $D_q$ type for an ideal $I$ in $\mathcal{O}_p$ by

$$D_q(I, p) = \sup_{V^q} \text{gen.val} \max_{S \in G^{n-q+1}_p} \frac{\nu(r \circ \gamma^k_S)}{\nu(\gamma^k_S)}.$$

In a combination of work [2, 3, 4], Catlin showed that finite $D_q$-type at point $p$ of smooth real hypersurface $M$ is equivalent to subellipticity of the $\bar{\partial}$-Neumann problem for $(0, q)$ forms at $p$. Since subellipticity is an open condition, this result implies in particular that, for a smooth real hypersurface $M$, the set of points of finite $D_q$-type is an open subset of $M$. 
When $q = 1$, 
\[ D_1(M, p) = \Delta_1(M, p) = \Delta(M, p). \]
For a long time, Catlin $q$-type, $D_q(M, p)$, and D’Angelo $q$-type, $\Delta_q(M, p)$, were believed to be equal. In [9], Fassina gave examples of ideals and hypersurfaces to show that these two types can be different when $q \geq 2$.

For a smooth real hypersurface $M$ in $\mathbb{C}^n$, Brinzanescu and Nicoara [1] introduced another definition of $q$-type at a point $p \in M$ by 
\[ \tilde{\Delta}_q(M, p) = \text{gen.val} \Delta((I(M), w), p), \]
where the generic value is taken over all $q - 1$ linear forms $w = \{w_1, \ldots, w_{q-1}\}$ in $\mathcal{O}_p$ and $(I(M), w)$ is the ideal generated by $I(M)$ and $w$ in $\mathcal{C}_p^\infty$. A similar definition is given for a proper ideal $I$ in $\mathcal{O}_p$ by 
\[ \tilde{\Delta}_q(I, p) = \text{gen.val} \Delta((I, w), p). \]

In the same paper, they showed that 
\[ D_q(M, p) = \tilde{\Delta}_q(M, p), \]
\[ D_q(I, p) = \tilde{\Delta}_q(I, p). \]
Using these equalities, a relation is given between $D_q(I, p)$ and $\Delta_q(I, p)$ in [9] and [1], which implies that $\Delta_q(I, p)$ and $D_q(I, p)$ are simultaneously finite.

Recently, in their study of the $C^\infty$ regularity problem for CR maps between smooth CR manifolds, Lamel and Mir [13] considered finite D’Angelo-type points for arbitrary subsets $M \subset \mathbb{C}^n$. In [14], the author showed that, for any subset $M$ of $\mathbb{C}^n$, the set of finite D’Angelo-type points is open in $M$.

In a similar way that D’Angelo type, $\Delta(M, p)$, is generalized in [13], we define a generalization of the Catlin’s $q$-type for an arbitrary subset $M \subset \mathbb{C}^n$ by
\[ D_q(M, p) = \sup_{V_q} \inf_{r \in I(M)} \text{gen.val} \max_{S \in G_{n-q+1}} \nu(\gamma^k_S) \frac{\nu(r \circ \gamma^k_S)}{\nu(\gamma^k_S)}. \]

In this note, by combining D’Angelo’s arguments with the ideas in the proof the equality (1.1), we establish the local boundedness of Catlin $q$-type for an arbitrary subset of $\mathbb{C}^n$. More precisely, our main result is the following.

**Theorem 1.1.** Let $M$ be a subset of $\mathbb{C}^n$ and $p_0$ be a point of finite Catlin $q$-type, that is, $D_q(M, p_0) < \infty$. Then there is a neighborhood $V$ of $p_0$ so that
\[ D_q(M, p) \leq 2^{(n-q+1)^2+n-q+2}D_q(M, p_0)^{(n-q+1)^2} \]
for all $p \in V$. In particular, the set of points of finite Catlin $q$-type is an open subset of $M$.

In [5], D’Angelo gave an example of hypersurface on which $\Delta(M, p)$ fails to be upper-semi continuous. In a similar way, the following example shows that $D_2(M, p)$ is not upper-semi continuous.
Example 1.2. Let $M$ be the real hypersurface in $\mathbb{C}^4$ with the defining function

$$r(z) = \Re z_4 + |z_1^2 - z_2 z_3|^2 + |z_2|^4.$$ 

Let $w = az_1 + bz_2 + cz_3 + z_4$ be a generic linear form where $a, b, c$ are all non-zero and consider the curve $\gamma_1(t) = (t, \frac{-a}{b} t, 0, 0) \subset \{w = 0\}$. Then $\nu(r \circ \gamma_1) = 4$ and hence $D_2(M, 0) = \tilde{\Delta}_2(M, 0) = 4$.

Let $p = (0, 0, \epsilon, 0)$ be a point on $M$ and $w = az_1 + bz_2 + z_3 + cz_4 - \epsilon$ be a linear form. By solving $w = 0$, $z_1^2 - z_2 z_3$ and $z_4 = 0$ together, we obtain that

$$z_2 = \frac{(\epsilon - az_1) \pm (\epsilon - az_1) \sqrt{1 - \frac{4bz_1^2}{(\epsilon - az_1)^2}}}{2b}.$$ 

In local parametrization, we write the curve $\gamma_2$ as

$$\gamma_2(t) = (t, \frac{t^2}{\epsilon - at} + O(t^4), \epsilon - at - \frac{bt^2}{\epsilon - at} + O(t^4), 0).$$

Here we choose the root for $z_2$ with the minus sign. Then $\nu(r \circ \gamma_2) = 8$, which implies that $D_2(M, p) = \tilde{\Delta}_2(M, p) = 8$. Hence $D_2(M, p)$ is not an upper-semi continuous function of $p$.

2. PROOF OF THE MAIN RESULT

Let $M$ be a subset of $\mathbb{C}^n$. In this section, we will show that $D_q(M, p)$ is locally bounded by above which, in particular, implies that the set of points of finite Catlin $q$-type is an open subset of $M$. We need give some more definitions analogous to the ones in [6].

For any $r \in I(M)$ and $k \in \mathbb{Z}^+$, we denote by $r_k$ the Taylor polynomial of $r$ of order $k$ at $p$. $I(M_k)$ denotes the ideal generated by the set $\{r_k : r \in I(M)\}$ in $C_\infty^\infty$.

$\Delta_q(M_k, p)$ is defined by

$$\Delta_q(M_k, p) = \Delta_q(I(M_k), p).$$

Similarly, $D_q(M_k, p)$ is defined by

$$D_q(M_k, p) = \sup_{V_q} \inf_{r \in I(M_k)} \max_{S \in G^\infty_{p^n-q+1}} \frac{\nu(r \circ \gamma^j_S)}{\nu(\gamma^j_S)}.$$ 

By Proposition 3.1 in [6], we can decompose the polynomial $r_k$ as

$$r_k = \Re (h^k) + \sum_{j=1}^{N} |f^k_j|^2 - \sum_{j=1}^{N} |g^k_j|^2,$$

where $N = N_k$ depends only on $n, k, h$ is a holomorphic function, $(f^k_j)^N$ and $(g^k_j)^N$ are holomorphic mappings. Let $U(N_k)$ denote the group of unitary matrices on $\mathbb{C}^{N_k}$. For any $U \in U(N_k)$, we denote by $I(r, U, k, p)$, the ideal generated by $h^k$ and $f^k - U g^k$. We should note that the decomposition of $r_k$ is not unique and $I(r, U, k, p)$ depends on the choice of decomposition. Here, we use the one in the proof of Proposition 3.1 in [6]. By $I(U, k, p)$, we denote the ideal generated by functions $h^k$ and $f^k - U g^k$ corresponding to the decomposition of $r_k$ for all $r \in I(M)$. 

Lemma 2.1. Let $M$ be a subset of $\mathbb{C}^n$. If $D_q(M_k, p) < k$ for some $k \in \mathbb{Z}^+$, then $D_q(M_k, p) = D_q(M, p)$.

Proof. For any germ at $p$ of $q$-dimensional variety $V^q$ and small enough $\epsilon > 0$, there exists an $r^0 \in I(M)$ such that

$$\max_{j=1,\ldots,P} \frac{\nu(r^0_k \circ \gamma^j_S)}{\nu(\gamma^j_S)} < D_q(M_k, p) + \epsilon < k.$$ 

Let $W$ and $W_k$ be open dense subsets of $G^{n-q+1}$ in which the generic values are assumed in the definitions of $D_q(M, p)$, $D_q(M_k, p)$ for $V^q$, respectively. Then for all $S \in W \cap W_k$,

$$\max_{j=1,\ldots,P} \frac{\nu(r^0_k \circ \gamma^j_S)}{\nu(\gamma^j_S)} < D_q(M_k, p) + \epsilon < k,$$

where $\gamma^j_S$’s are connected components of $V^q \cap S$. As in the proof of Lemma 4.5 in [6], we have

$$\frac{\nu(r^0_k \circ \gamma^j_S)}{\nu(\gamma^j_S)} = \frac{\nu(r^0_k \circ \gamma^j_S)}{\nu(\gamma^j_S)},$$

for all $j = 1, \ldots, P$. Thus

$$\max_{s \in G^{n-q+1}_p, j=1,\ldots,P} \frac{\nu(r^0 \circ \gamma^j_S)}{\nu(\gamma^j_S)} = \max_{j=1,\ldots,P} \frac{\nu(r^0 \circ \gamma^j_S)}{\nu(\gamma^j_S)} = \max_{j=1,\ldots,P} \frac{\nu(r^0 \circ \gamma^j_S)}{\nu(\gamma^j_S)} < D_q(M_k, p) + \epsilon,$$

which implies that

$$D_q(M, p) \leq D_q(M_k, p) + \epsilon.$$ 

As $\epsilon > 0$ is arbitrary, $D_q(M, p) \leq D_q(M_k, p)$.

Since $D_q(M, p) \leq D_q(M_k, p) < k$, following the above argument, for any germ at $p$ of $q$-dimensional variety $V^q$ and small enough $\epsilon > 0$, there exists an $r' \in I(M)$ such that

$$\max_{s \in G^{n-q+1}_p, j=1,\ldots,P} \frac{\nu(r' \circ \gamma^j_S)}{\nu(\gamma^j_S)} < D_q(M, p) + \epsilon < k.$$ 

Let $W$ and $W_k$ be open dense subsets of $G^{n-q+1}$ in which the generic values are assumed in the definitions of $D_q(M, p)$ and $D_q(M_k, p)$ for $V^q$, respectively. Then by the similar argument above, for all $S \in W \cap W_k$,

$$\max_{s \in G^{n-q+1}_p, j=1,\ldots,P} \frac{\nu(r'_k \circ \gamma^j_S)}{\nu(\gamma^j_S)} = \max_{j=1,\ldots,P} \frac{\nu(r'_k \circ \gamma^j_S)}{\nu(\gamma^j_S)} = \max_{j=1,\ldots,P} \frac{\nu(r'_k \circ \gamma^j_S)}{\nu(\gamma^j_S)} < D_q(M, p) + \epsilon,$$

which implies that

$$D_q(M_k, p) \leq D_q(M, p).$$ 

Lemma 2.2. Let $M \subset \mathbb{C}^n$ be a subset. Then

$$\sup_U \Delta_q(I(U, k, p)) \leq D_q(M_k, p) \leq 2 \sup_U (\Delta_q(I(U, k, p)))^{n-q+1}.$$
Proof. For any \( r \in I(M) \) and \( k \in \mathbb{Z}^+ \), as in the proof of Theorem 3.4 in \cite{7}, we have that
\[
\inf_{\varphi \in I(r,U,k,p)} \nu(\varphi \circ \gamma) \leq \nu(r \circ \gamma)
\]
for any curve \( \gamma \in \mathcal{C} \). Thus for any set of \( q - 1 \) linear forms \( w \),
\[
\inf_{\varphi \in I(r,U,k,p),w} \frac{\nu(\varphi \circ \gamma)}{\nu(\gamma)} \leq \inf_{\varphi \in (r_k,w)} \frac{\nu(\varphi \circ \gamma)}{\nu(\gamma)},
\]
where \((r_k,w)\) is the ideal generated by \( r_k \) and \( w \) in \( C_p^\infty \).

By taking infimum over \( r \), supremum over \( \gamma \) and infimum over \( w \), we obtain that
\[
\Delta_q(I(U, k, p)) \leq \inf \sup_{w} \inf_{\varphi \in (I(M_k),w)} \frac{\nu(\varphi \circ \gamma)}{\nu(\gamma)}.
\]
Let \( w_0 \) be a set of \( q-1 \) linear forms at which right hand side of (2.1) attains its infimum. If a curve \( \gamma \) is not contained in the set \( \{w_0 = 0\} \) then there exists a linear function \( w_0' \) so that \( \nu(w_0' \circ \gamma) = \nu(\gamma) \).

Thus the supremum above is attained for the curves which are contained in \( \{w_0 = 0\} \). For any such curve, infimum is attained for \( \varphi \in I(M_k) \). Hence, there exists a \( \gamma_0 \subset \{w_0 = 0\} \) such that
\[
\Delta_q(I(U, k, p)) \leq \inf_{r \in I(M_k)} \frac{\nu(r \circ \gamma_0)}{\nu(\gamma_0)}.
\]

Let \( H \) be the linear subspace \( \{w_0 = 0\} \) in \( G_p^{n-q+1} \) and \( Z \) be the \( q-1 \) dimensional hyperplane through \( p \), that is transversal to \( H \). That is, \( \dim_C(H \oplus Z) = n \). By Lemma 3.1 in \cite{1}, there exists a \( q \)-dimensional variety \( C_Z^q \) at \( p \) that contains \( \gamma_0 \) and whose tangent spaces at \( p \) contains \( Z \). As in the proof of Proposition 3.5 in \cite{1}, \( H \in G_p^{n-q+1} \) is one of the hyperplane which gives the generic value in \( D_q(M_k, p) \) for \( C_Z^q \). Thus \( \gamma_0 \subset H \cap C_Z^q \) is one of the curves which enter in the computation of \( D_q(M_k, p) \). That is,
\[
\inf_{r \in I(M_k)} \frac{\nu(r \circ \gamma_0)}{\nu(\gamma_0)} \leq D_q(M_k, p).
\]

Hence by (2.2),
\[
\Delta_q(I(U, k, p)) \leq D_q(M_k, p).
\]
By taking supremum over \( U \in U(N_k) \), the inequality on the left hand side in Lemma 2.2 follows.

There exists a germ at \( p \) of \( q \)-dimensional variety \( V^q \) and an open subset \( W \) of \( G_p^{n-q+1} \), which depends on \( V^q \), such that for all \( S \in W \)
\[
D_q(M_k, p) = \inf_{r \in I(M_k)} \max_{j=1,...,P} \frac{\nu(r \circ \gamma_j^S)}{\nu(\gamma_j^S)} \leq \sup_{\gamma \subset \{w = 0\}} \inf_{r \in I(M_k)} \frac{\nu(r \circ \gamma)}{\nu(\gamma)}
\]
where \( w \) is the set of \( q-1 \) linear forms with \( \{w = 0\} = S \) and \( \gamma_j^S \)'s are the connected components of \( V^q \cap S \).
Lemma 2.3. If \( \nu(r_k \circ \gamma) \leq 2\nu(\varphi \circ \gamma) \) for all \( \varphi \in I(r, U, k, p) \). Dividing both sides by \( \nu(\gamma) \), taking infimum over \( r \in I(M) \), supremum over \( \gamma \subset \{w = 0\} \), we obtain by (2.3) that

\[
D_q(M_k, p) \leq 2 \sup_U \sup_{\gamma \subset \{w = 0\}} \inf_{\varphi \in I(M_k, w_0)} \frac{\nu(\varphi \circ \gamma)}{\nu(\gamma)} = 2 \sup_U \sup_{\gamma \subset \{w = 0\}} \inf_{\varphi \in I(U, k, p, w)} \frac{\nu(\varphi \circ \gamma)}{\nu(\gamma)}
\]

\[
= 2 \sup_U (I(U, k, p, w)) \leq 2 \sup_U \inf_{\varphi \in I(M_k, w_0)} \frac{\nu(r_k \circ \gamma)}{\nu(\gamma)}
\]

where \( \tilde{w}_U \) is the set of \( q - 1 \) linear functions at which \( \Delta_q(I(U, k, p)) \) assumes its infimum. Second and fourth inequalities above follow from Theorem 2.7 in [6]. Third inequality holds since \( \text{mult}(I) \) is an upper semi continuous function of the generators of \( I \).

\[ \square \]

Lemma 2.3. If \( \Delta_q(M_k, p) < k \) then \( \Delta_q(M, p) = \Delta_q(M_k, p) \).

Proof. Let \( \Delta_q(M_k, p) = \sup_{\gamma \subset \{w = 0\}} \inf_{\varphi \in I(M_k, w_0)} \frac{\nu(\varphi \circ \gamma)}{\nu(\gamma)} \) for some set of \( q - 1 \) linear forms \( w_0 \). As in the proof of Lemma 2.2, the supremum above is attained for the curves which are contained in \( \{w_0 = 0\} \). For any such curve \( \gamma \), \( \sup_{\varphi \in I(M_k, w_0)} \inf_{\gamma \subset \{w = 0\}} \frac{\nu(\varphi \circ \gamma)}{\nu(\gamma)} \) is attained for \( \varphi \in I(M_k) \). Hence

\[
\Delta_q(M_k, p) = \sup_{\gamma \subset \{w = 0\}} \inf_{\varphi \in I(M_k)} \frac{\nu(r_k \circ \gamma)}{\nu(\gamma)} < k.
\]

For all \( \gamma \subset \{w = 0\} \), \( \epsilon > 0 \) small enough, there exists \( r_0 \in I(M) \) such that

\[
\frac{\nu(r_0 \circ \gamma)}{\nu(\gamma)} < \Delta_q(M_k, p) + \epsilon < k.
\]

By Lemma 4.5 in [6], \( \nu(r \circ \gamma) = \frac{\nu(\gamma)}{\nu(\gamma)} \nu(\gamma) = \frac{\nu(\gamma)}{\nu(\gamma)} \nu(\gamma) \) for any curve \( \gamma \). By taking infimum over \( r \in I(M) \), supremum over \( \gamma \subset \{w = 0\} \) in \( \frac{\nu(r \circ \gamma)}{\nu(\gamma)} \), we obtain that \( \Delta_q(M, p) \leq \Delta_q(M_k, p) + \epsilon \).

By the same argument above,

\[
\Delta_q(M, p) = \inf_{w} \sup_{\gamma \subset \{w = 0\}} \inf_{r \in I(M)} \frac{\nu(r \circ \gamma)}{\nu(\gamma)} < k.
\]

Let the infimum be attained for some set of \( q - 1 \) linear forms, \( w_1 \). Then for all \( \gamma \subset \{w_1 = 0\} \), there exists \( r_0 \in I(M) \) such that \( \frac{\nu(r_0 \circ \gamma)}{\nu(\gamma)} = \inf_{r \in I(M)} \frac{\nu(r \circ \gamma)}{\nu(\gamma)} < k \). Lemma 4.5 in [6] implies that \( \frac{\nu(r_k \circ \gamma)}{\nu(\gamma)} = \frac{\nu(r_k \circ \gamma)}{\nu(\gamma)} \). By taking infimum over \( r \in I(M) \) and supremum over \( \gamma \subset \{w_1 = 0\} \) in \( \frac{\nu(r_k \circ \gamma)}{\nu(\gamma)} \), we obtain that

\[
\Delta_q(M_k, p) \leq \sup_{\gamma \subset \{w_1 = 0\}} \inf_{r \in I(M)} \frac{\nu(r_k \circ \gamma)}{\nu(\gamma)} = \Delta_q(M, p).
\]
Lemma 2.4. \( \sup_U \Delta_q(I(U, k, p)) \leq \Delta_q(M_k, p) \leq 2 \sup_U \Delta_q(I(U, k, p)) \).

**Proof.** Let \( w_0 \) be a set of linear functions such that

\[
\Delta_q(M_k, p) = \sup_{\gamma} \inf_{\varphi \in (I(M), w_0)} \frac{\nu(\varphi \circ \gamma)}{\nu(\gamma)}.
\]

By the same argument in the proof of Lemma 2.3

\[
\Delta_q(M_k, p) = \sup_{\gamma \subset \{w_0 = 0\}} \inf_{r \in I(M)} \frac{\nu(r_k \circ \gamma)}{\nu(\gamma)}.
\]

By the proof of Theorem 3.4 in \([7]\), for any \( r \in I(M) \) and any curve \( \gamma \),

\[
\inf_{\varphi \in I(r, U, k, p)} \frac{\nu(\varphi \circ \gamma)}{\nu(\gamma)} \leq \nu(r_k \circ \gamma),
\]

for all \( U \in U(N_k) \). By taking infimum over \( r \in I(M) \), we get

\[
\inf_{\varphi \in I(U, k, p)} \frac{\nu(\varphi \circ \gamma)}{\nu(\gamma)} \leq \inf_{r \in I(M)} \frac{\nu(r_k \circ \gamma)}{\nu(\gamma)}.
\]

Thus

\[
\Delta_q(I(U, k, p)) = \inf_U \sup_{\gamma \subset \{w_0 = 0\}} \inf_{\varphi \in I(U, k, p)} \frac{\nu(\varphi \circ \gamma)}{\nu(\gamma)} \leq \sup_{\gamma \subset \{w_0 = 0\}} \inf_{r \in I(M)} \frac{\nu(r_k \circ \gamma)}{\nu(\gamma)} = \Delta_q(M_k, p).
\]

Hence \( \sup_U \Delta_q(I(U, k, p)) \leq \Delta_q(M_k, p) \).

For any set of \( q - 1 \) linear forms \( w \) and any curve \( \gamma \subset \{w = 0\} \), by Theorem 3.5 in \([6]\), there exists a unitary matrix \( U \in U(N_k) \), such that \( \frac{\nu(r_k \circ \gamma)}{\nu(\gamma)} \leq 2 \inf_{\varphi \in I(r, U, k, p)} \frac{\nu(\varphi \circ \gamma)}{\nu(\gamma)} \) for all \( r \in I(M) \). By taking infimum of \( \frac{\nu(r_k \circ \gamma)}{\nu(\gamma)} \), over \( r \in I(M) \), supremum over \( \gamma \subset \{w = 0\} \) and infimum over the set of \( q - 1 \) linear forms \( w \), we obtain that

\[
\Delta_q(M_k, p) \leq 2 \sup_U \inf_{\gamma \subset \{w_0 = 0\}} \inf_{\varphi \in I(U, k, p)} \frac{\nu(\varphi \circ \gamma)}{\nu(\gamma)} = 2 \sup_U \Delta_q(I(U, k, p)).
\]

\[\square\]

Now we can prove our main theorem.

**Proof of Theorem 1.1.** Let \( w_0 \) be the set of \( q - 1 \) linear forms such that \( \Delta_q(M_k, p_0) \) attains its infimum. Then

\[
\Delta_q(M_k, p_0) = \sup_{\gamma \subset \{w_0 = 0\}} \inf_{r \in I(M)} \frac{\nu(r_k \circ \gamma)}{\nu(\gamma)}.
\]
By identifying \( \{ w_0 = 0 \} = \mathbb{C}^{n-q+1} \), Theorem 1.3 in [14] implies that there is a neighborhood \( V \) of \( p_0 \) such that for all \( p \in V \)

\[
\Delta_q(M_k, p) \leq 2(\Delta_q(M_k, p_0))^{n-q+1}.
\]

For all \( p \in V \),

\[
D_q(M_k, p) \leq 2 \sup_U \Delta_q(I(U, k, p))^{n-q+1} \leq 2\Delta_q(M_k, p)^{n-q+1}
\]

\[
\leq 2^{n-q+2} \Delta_q(M_k, p_0)^{(n-q+1)^2} \leq 2^{n-q+2} (2 \sup_U \Delta_q(I(U, k, p)))^{(n-q+1)^2}
\]

\[
\leq 2^{(n-q+1)^2+n-q+2} D_q(M_k, p_0)^{(n-q+1)^2} = 2^{(n-q+1)^2+n-q+2} D_q(M, p_0)^{(n-q+1)^2}.
\]

First and the last inequalities above follow from Lemma 2.2. Second and fourth inequalities follow from Lemma 2.4. Last equality holds for large enough \( k \) by Lemma 2.1. If \( k > 2^{(n-q+1)^2+n-q+2} D_q(M, p_0)^{(n-q+1)^2} \), Lemma 2.1 implies that \( D_q(M_k, p) = D_q(M, p) \) hence the theorem follows.

\[ \square \]

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