Backreaction in an analogue black hole experiment

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For many physical systems of interest, there is a natural division between a quasi-stationary background and small perturbations on that background. In curved spacetime scenarios the perturbations can either be of classical or quantum origin, and much progress has been made in understanding the behaviour of such perturbations by assuming a fixed background. Less understood is how these perturbations in turn alter the background structure - a phenomenon known as backreaction. In this letter we report on the first measurement of backreaction in an analogue gravity simulator. We scatter surface waves from a draining bathtub vortex, in analogy with scalar waves scattering from a rotating black hole. We predict and detect a mass flux associated with the surface waves that flows through the analogue event horizon and out the drain. This manifests itself in a measurable decrease in the water height that agrees with our theoretical prediction. Changes in water height correspond to changes in the effective gravitational field, as energy and angular momentum are exchanged between the incident waves and the analogue black hole. Although our experimental findings are constrained to classical systems, our conceptual framework can be generalized to apply to quantum systems. Hence, we argue that analogue quantum simulators of gravitational systems could be used to investigate black hole backreaction due to the processes predicted by Penrose and Hawking.

Introduction. Analogue gravity, pioneered by Unruh in 1981 [1], is a research programme which studies gravitational phenomena using a wide variety of non-gravitational systems (see [2] for a review). Unruh originally considered the propagation of sound waves through a fluid, and showed that if the fluid becomes supersonic in some region, the system exhibits a dumb hole horizon - the analogue of a black hole horizon. More generally, he showed that wave propagation through certain media is described by the Klein-Gordon equation on an effective curved spacetime. Since then, analogues have been investigated in a wide variety of condensed matter systems [3–8], including small surface waves propagating on an incompressible, irrotational, inviscid, shallow fluid [9]. Although the analogy was originally conceived to investigate the trans-plankian problem associated with Hawking radiation [10], analogue gravity has enjoyed a number of other successes: notably surface wave experiments have been used to measure Hawking radiation [11–13] and superradiance [14].

One particularly simple model of a rotating black hole is provided by surface waves propagating on a rotating, draining fluid flow - the so-called draining bathtub vortex (DBT). Much work in the literature has gone into understanding features of this model, see e.g. [15–20]. This model rests on a number of assumptions, specifically that the fluid be incompressible, irrotational and shallow. Modifications have been considered when the last two of these are violated [21, 22], and black hole effects were shown to persist in such scenarios in [14]. Although the analogy to black hole physics arises at the linear level, non-linearities in the fluid equations will cause the waves to induce changes in the background quantities, which enter at second order in a perturbative expansion of the equations. In fluid dynamics this is known as wave-mean interaction theory [23] and, in general, one can obtain corrections to the background by solving the fluid equations perturbed to second order. For a system with an open boundary, like a bathtub containing a drain, we argue that the scale of this change can be estimated by how much mass is pushed out of the system by the waves, ultimately leading to a measurable change in the water height. We perform an experiment to measure precisely this, finding good agreement with our prediction. A change in the water height in the shallow water regime induces correction in the underlying effective metric, which correspond in the gravitational analogy to changes in the black hole energy and angular momentum. Thus, one could in principle use a system of this type to investigate backreaction due to superradiance and Hawking radiation.

Theory. Consider an experiment in which water enters a tank via an inlet and exits at a drain in a continuous cycle, where the water height and velocity field describing the system are given by \((H, \mathbf{V})\). For simplicity, we assume cylindrical symmetry about the drain and adopt polar coordinates \((r, \theta, z)\). The rate of change of mass in the system is given by \(M = - \int_S \mathbf{J} \cdot d\mathbf{A}\), where \(\mathbf{J}\) is the mass flux and \(S\) is the boundary of the system, comprised of an inner and outer boundary denoted
If the system is in equilibrium, the integral of the energy changes over these two surfaces is equal and opposite leading to $\dot{M} = 0$ (mass conservation gives rise to the familiar continuity equation which can also be used to derive our height change expressions, see Appendix A). For the present case, we assume the boundaries are radial surfaces $r = r_{1,2}$.

In the case that waves are present, we consider placing additional mass flux there due to the waves. In the first place, we consider placing additional mass flux there due to the waves. In the case of free surface gravity waves \cite{25, 26}, the mass flux has the same symmetry properties as the norm and energy currents (see Appendix C). For an individual $m$-mode, the average height in this annulus therefore serves as an indicator for whether the mode is superradiant. We note, however, that the initial state in the water tank experiment described here is a Cartesian plane wave, which is composed of a variety of $m$-modes; in this case the water height goes down regardless of superradiating modes.

Another reality of the water experiment is that a nonzero flow exists in much of the fluid domain. Since a velocity field changes the rate of mass flowing through a surface, the amount of mass removed from a region where $\mathbf{V} \neq 0$ will differ from that at infinity, leading to a difference in the local height over a length scale $L$ set by the velocity field. However, since large gradients in the free surface are energetically unfavourable, gravity and surface tension will act to average out the height change $\Delta H$ on a timescale $t_{\text{rest}}$, whereas the timescale of the height change is $T \sim \Delta H/H$. The case described above is then valid only when $t_{\text{rest}} \gg T$.

If $t_{\text{rest}} \ll T$, a more realistic estimate for the height change can be found by taking $r_{1}$ closer to the drain. If the fluid becomes supersonic in a region close the drain, the system will exhibit an acoustic horizon $r = r_{H}$ defined by $|\mathbf{V}(r_{H})| = c$ where $c$ is the speed of wave propagation. Evaluating the average height change using the flux in shallow water across $r_{1} = r_{H}$ gives,

$$
\dot{H} = - \frac{1}{A} \int_{0}^{2\pi} \left( h \mathbf{v} \cdot \mathbf{r} \right)^{2} d\theta,
$$

where $h$ is a harmonic mode amplitude for a given azimuthal number $m \in (-\infty, \infty)$.

Depending on the specific choice of annulus $r \in [r_{1}, r_{2}]$, the average height change given by Eq. (3) offers different information about the system. To illustrate this, we briefly discuss two particular cases of interest using results from shallow water theory (see Appendix B). To make contact with the water tank experiment, the rate of mass flux there due to the waves. In the first case, we consider placing $r_{1}$ at a finite radius far away from the drain where the velocity field is small, $\mathbf{V} \approx 0$. In this case we have,

$$
\dot{H} = \frac{\pi \omega^{2}}{A \rho c g} \sum_{m} \mathcal{Z}_{m} |A_{m}^{r}|^{2},
$$

where $\mathcal{Z}_{m} = |A_{m}^{r}|^{2}/|A_{m}^{\perp}|^{2} - 1$ is the amplification and $A_{m}^{r}$ are the amplitudes of the in ($\cdot$) and out-going (+) components of the $m$-modes. It can be shown in this case that the mass flux has the same symmetry properties as the norm and energy currents (see Appendix C). For an individual $m$-mode, the average height in this annulus therefore serves as an indicator for whether the mode is superradiant. We note, however, that the initial state in the water tank experiment described here is a Cartesian plane wave, which is composed of a variety of $m$-modes; in this case the water height goes down regardless of superradiating modes.

Experimental procedure. Three experiments were conducted in a water tank to measure the height change by placing $r_{1}$ at a finite radius far away from the drain where the velocity field is small, $\mathbf{V} \approx 0$. In this case we have,

$$
\dot{H} = \frac{\pi \omega^{2}}{A \rho c g} \sum_{m} \mathcal{Z}_{m} |A_{m}^{r}|^{2},
$$

where $\mathcal{Z}_{m} = |A_{m}^{r}|^{2}/|A_{m}^{\perp}|^{2} - 1$ is the amplification and $A_{m}^{r}$ are the amplitudes of the in ($\cdot$) and out-going (+) components of the $m$-modes. It can be shown in this case that the mass flux has the same symmetry properties as the norm and energy currents (see Appendix C). For an individual $m$-mode, the average height in this annulus therefore serves as an indicator for whether the mode is superradiant. We note, however, that the initial state in the water tank experiment described here is a Cartesian plane wave, which is composed of a variety of $m$-modes; in this case the water height goes down regardless of superradiating modes.
The gradient of the best fit line in is \( \dot{H} = -1.18 \pm 0.02 \times 10^{-5}\text{ms}^{-1} \), computed from a least squares regression with the error calculated from the residuals.

In Fig. 2 we display the results of experiment 2, with incident waves of frequency \( f = 2.3, 4 \text{ Hz} \). Since each frequency is spread randomly about the best fit line to the mean of the three data samples, we deduce that \( \Delta H \) is not sensitive to significant changes in \( f \). This observation is supported by our predictions: in Eq. (3) the factor of \( \omega^2 \) is absorbed when converting to the height field, \( |a_m|^2 = \omega^2 |A_m|^2 / g^2 \), and in Eq. (4) the frequency dependence is weak since for realistic experimental parameters \( \omega \ll m\Omega_H \).

In Fig. 3 we display the height change from the last experiment. Since we expect a constant, negative gradient at early times and an exponential tendency to a new equilibrium height at late times, we fit the data with a curve satisfying these properties (see figure caption for details). From this fit we find the initial gradient in the linear regime is \( \dot{H} = -3.89 \pm 0.05 \times 10^{-5}\text{ms}^{-1} \), and the time taken to deviate significantly from linearity is \( 383 \pm 3 \text{s} \). The fit predicts a total height change \( \Delta H_{\text{tot}} = -1.4 \pm 0.1 \text{cm} \) once equilibrium is reached.

Comparing the initial slopes across the three experiments, we see that \( \dot{H} \) depends on \( H(t=0) \) supporting our claim that the long term behaviour should be exponential, which is further evidenced by the late time tail in experiment 3. In all experiments we see that at early times the gradient of the height decrease is well approximated as linear, in agreement with our predictions in the previous section.

**Comparison with previous work.** In [14], a wave scattering experiment was performed to detect superradiant scattering with the same experimental set-up used here. Analysing the data obtained there, we found that in all experiments the height of the water decreased during the time of wave incidence with a gradient of around \( H \sim -2 \pm 1 \times 10^{-5}\text{ms}^{-1} \). The large error is the result of only recording for 13s, hence a wider range of gradients provides an adequate fit due to inherent noise in the measurement apparatus. The flow parameters of [14] are similar to those of experiment 3 in the present work and correspondingly, we see that the order of magnitude of \( \dot{H} \) is in agreement.

We can estimate the height change using our predictions from the previous section with the scattering amplitudes taken from the experiment at \( f = 4\text{Hz} \) of [14]. To do this, we summed over \( |m| \leq 5 \) since higher \( m \) modes were not resolvable within our window of observation. Eq. (3) gives \( \dot{H} = -2.5 \pm 0.5 \times 10^{-6}\text{ms}^{-1} \) and from Eq. (4) we have \( \dot{H} = -2.3 \pm 0.6 \times 10^{-5}\text{ms}^{-1} \). To compute the error, we created a distribution for each of the parameters appearing in \( \dot{H} \), taking the uncertainty on the parameter as the distributions standard deviation.
We then create a data set for $\dot{H}$ by sampling randomly from these distributions, whose mean and standard deviation give the values quoted above.

The order of magnitude obtained using Eq. (3) does not match experiment 3 or $\dot{H}$ obtained from 14, the reason being that in a small system there is no reason to expect $t_{\text{rest}} \gg T$. In other words, information that more water has drained from the region where $V \neq 0$ is transmitted to the region where $V \approx 0$ before significant changes to the background occur, resulting in a uniform height change. In this regime, the correct estimate to use is Eq. (4), which is in agreement (within error estimates) with data from 14 and $\dot{H}$ in experiment 3.

To improve the agreement between theory and experiment, a theory of wave propagation accounting for dispersion, dissipation, vorticity and gradients in the free surface is required (the former two being important throughout the flow and the latter two arising close to the drain). The reason for this is that our predictions depend on the precise form of the solution near the drain, which will be influenced by all of these effects (see [9, 15, 21, 22] for attempts to account for these phenomena). However, the estimate provided by the simple irrotational, shallow water theory is already gives the correct order of magnitude. This is a good indication that the proposed mechanism is responsible for our observations.

**Conclusion and Outlook.** Our results demonstrate that surface waves interacting with an initially stationary vortex will trigger the evolution of the background into a new equilibrium state. Due to the flow being externally driven, it was previously unclear whether such backreaction could be observed in analogue gravity simulators. Our findings show that backreaction is indeed observable, which indicates that the system does in fact have freedom to re-distribute energy and angular momentum between the incident waves and the analogue black hole.

This realisation is important for a number of reasons. Firstly, one must ensure that any wave effects (e.g. superradiance and QNM resonance) are measured on a timescale much shorter than the time it takes for the height to drop, so that the assumption of a stationary background is not violated. Secondly, it is feasible that a similar system can be constructed using density perturbations in a Bose-Einstein condensate (BEC), where increased mass flux across the horizon would cause the background density to change. Since quantum fluctuations are also relevant in determining the evolution of the system, this presents the opportunity to experimentally investigate the effects of the quantum backreaction on an analogue black hole spacetime. Indeed, the measurement of analogue Hawking radiation in a BEC has attracted much attention in recent years [5, 27]. Therefore, in addition to being the first observation of classical backreaction in an analogue gravity system, this work represents a natural first step towards experimentally probing the effects of backreaction due to quantum fluctuations. The next step is to determine the extent to which the extent to which the gravitational analogy is applicable to backreacting quantum systems [28], and the approach described herein offers a way to address this question.

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H = at late times. Using this fit, we extract the initial gradient at early times and exponentially decaying to a constant value (see Table I in Appendix D for more details). We choose a 4 parameter heuristic fit which is linear from a linear decrease to decay is approximately 3.83 ± 0.05 cm. The time taken to switch from a linear decrease to decay is approximately 383 ± 4 ms.

The equilibrium is reached 1 cm in 3s.

FIG. 3. Long term evolution of the height in experiment 3. The incident wave had a frequency 4 Hz and the initial water height was 6.5cm (see Table I in Appendix D for more details). We choose a 4 parameter heuristic fit which is linear at early times and exponentially decaying to a constant value at late times. Using this fit, we extract the initial gradient $\dot{H} = -3.89 \pm 0.05 \times 10^{-3}$ ms$^{-1}$, the decay time at late times 94 ± 4s and the total prediction height change once a new equilibrium is reached 1.4 ± 0.1cm. The time taken to switch from a linear decrease to decay is approximately 383 ± 4s.

The $R^2$ value of the model is 0.9997 when high frequency oscillations are averaged out of the data.

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Appendix A

In this appendix we expand on the general derivation in the main text based on considerations of mass conservation, and then present an alternative derivation from the shallow water equations. Ultimately, the two methods are the same as it is mass conservation that leads to the shallow water continuity equation.

**Mass conservation.** Here we present a derivation of Eq. (2) from Eq. (1). Using the fact that the mass flux of a fluid is the same as its momentum density $\mathbf{j} = \rho \mathbf{v}$, Eq. (1) becomes

$$\dot{H} = -\frac{1}{A} \int_0^{2\pi} \int_0^H (\mathbf{V} + \mathbf{v}) \cdot \mathbf{r} \, r^2 \, dz \, d\theta. \quad (5)$$

To avoid notational clutter, we deal with the integral and then insert it back into the main result later. Separating
the integrals up to the perturbed surface are reduced to a multiplication in the limit of $h \ll H$. As we assume a stationary background at zeroth order, the first term vanishes since the contributions from the two boundaries cancel. The second and third terms in the second equality are oscillatory, being linear in the perturbations. Therefore, only the final quadratic term can cause large scale height changes. This term $j_h = h v$ is the volume flux - the shallow water analogue of the mass flux. Using the decomposition $(h, v) = \int \sum_m (h_m, v_m) e^{i m \theta - i \omega t} d\omega$ for perturbations to an axisymmetric, stationary background, we can evaluate this as,

$$
\int_0^{2\pi} h v d\theta = \int_0^{2\pi} \int \sum_{m,n} \left\{ [h^R_m(\omega) \cos(m\theta - \omega t) - h^I_m(\omega) \sin(m\theta - \omega t)] [v^R_n(\omega') \cos(m'\theta - \omega' t) - v^I_n(\omega') \sin(m'\theta - \omega' t)] \right\} d\omega' d\theta = \cdots
$$

Where, for compactness, superscript RI denote the real and imaginary components respectively. Evaluating the $\theta$ integral using,

$$
\int_0^{2\pi} \cos(m\theta - \omega t) \cos(m'\theta - \omega' t) d\theta = \pi \delta_{m,m'} \cos(\omega - \omega') t
$$

$$
\int_0^{2\pi} \sin(m\theta - \omega t) \sin(m'\theta - \omega' t) d\theta = \pi \delta_{m,m'} \sin(\omega - \omega') t
$$

$$
\int_0^{2\pi} \sin(m\theta - \omega t) \cos(m'\theta - \omega' t) d\theta = -\pi \delta_{m,m'} \sin(\omega - \omega') t,
$$

this expression simplifies to,

$$
\cdots = \pi \int \sum_m \left\{ \text{Re}[h^*_m v_m] \cos(\omega - \omega') t + \text{Im}[h^*_m v_m] \sin(\omega - \omega') t \right\} d\omega' d\omega.
$$

Inserting back into the integral in the equation for $\dot{H}$, we have,

$$
\dot{H} = -\frac{\pi}{A} \sum_m h^*_m v_m \cdot r^{\perp 2}_r + \text{oscillations}, \quad (7)
$$

where the only non-oscillatory term comes from the $\omega = \omega'$ terms, as quoted in Eq.2. Once integrated, these oscillatory terms are accompanied by a prefactor $1/(\omega - \omega')$ and when this factor is small, the oscillatory terms can in principle be large. However, if the wave is sharply peaked on a single frequency $\omega$, then the $\omega' \neq \omega$ terms will be inherently lower amplitude and the factor $1/(\omega - \omega')$ will not be sufficient to compensate. Thus, any oscillations in the mean height will be dominated by the first order terms which oscillate at frequency $\omega$. This is confirmed experimentally by a peak in the fourier transform of $\dot{H}$ at the excitation frequency $f$. Even in circumstances where the oscillations are large, the quadratic term is the only one that can grow in time and hence, it will always come to dominate after sufficient time has elapsed.

**Shallow water equations.** Another derivation of Eq.2 can be formulated by considering the continuity equation in shallow water. For an irrotational flow, the governing equations are,

$$
\partial_t h_{\text{tot}} + \nabla \cdot (h_{\text{tot}} v_{\text{tot}}) = 0
$$

$$
\partial_t v_{\text{tot}} + \nabla \left( \frac{1}{2} v_{\text{tot}}^2 + g h_{\text{tot}} + \Psi \right) = 0 \quad (8)
$$

where $f = \nabla \Psi$ allows for the inclusion of an additional forcing term on the background. Fluid quantities perturbed to second order are,

$$
h_{\text{tot}} = H' + \epsilon h + \epsilon^2 \eta + O(\epsilon^3)
$$

$$
v_{\text{tot}} = V' + \epsilon v + \epsilon^2 u + O(\epsilon^3)
$$

The leading order equations describe the background $(H', V')$, where the primes denote the part of the background which is time independent rather than the unprimed versions in the main text which include the second order corrections. The next to leading order equations give the usual wave equation. The shallow water equations at $O(\epsilon^2)$ are,

$$
\partial_t \eta + \nabla \cdot (v \eta + V' \eta + u H') = 0 \quad (9)
$$

$$
\partial_t u + \nabla \left( \frac{1}{2} v^2 + V' \cdot u + g \eta \right) = 0. \quad (10)
$$

Since we search for a solution at early times, we can write,

$$
\eta = \eta_0 + \eta_0' t + O(t^2),
$$

where prime denotes derivative with respect to $t$ and zero indicates a quantity is evaluated at $t = 0$. We define $t = 0$ to be the instant when the backreaction begins, hence $\eta_0 = 0$. In Eq.8, the quadratic term $h v$ has a contribution which is constant in time, hence at leading order in the $t$ expansion, the terms proportional to $\eta$ and $u$ do not contribute, allowing us to solve for the initial slope at $t = 0$. This approach breaks down once enough time has elapsed and these extra terms become important.

Using the divergence theorem, the remaining terms can be cast in the form,

$$
\int_A \partial_t \eta \, dA + \oint_{\partial A} h v \cdot dl = 0 \quad (12)
$$

where the first term is an integral over the area $A$ of the system and we have used divergence theorem to express...
the second term as an integral over the boundary $\delta A$. The first term is simply the definition of the average $\bar{\nabla} \eta$ over the spatial region $A$, and using our choice of boundary in the main text, we can rewrite the second term to obtain,
\[
\bar{A} \bar{\nabla} \eta + \int_{r_{in}}^{r_{2\pi}} h v \cdot r |^{2} d \theta = 0 \tag{13}
\]
Alternatively, if we had kept the second order terms under the integral, for an approximately uniform change the difference between contributions at the two boundaries is small compared to the quadratic term and can be neglected. Eq. (13) is identical to the relevant term in Eq. (2) if we incorporate the $O(\epsilon^2)$ height change back into the $O(1)$ variable, i.e. $H = H + \epsilon^2 \eta$. This scheme of iteratively resolving the equations using a new background state comprised of the old background plus quadratic corrections is routinely applied in general relativity [29], where the effective energy-momentum tensor sourcing the corrections is small compared to the quadratic term and can be neglected.

\[\text{Appendix B}\]

For irrotational perturbations satisfying $v = \nabla \phi$, the $O(\epsilon)$ shallow water equations combine to give the wave equation,
\[
(\partial_t + V \cdot \nabla)^2 \phi - gH \nabla^2 \phi = 0. \tag{14}
\]
which describes waves $\phi$ propagating at speed $c = \sqrt{gH}$. Note, we use $(H, V)$ rather than $(H', V')$ in Appendix A since the time dependent parts in the former are $O(\epsilon^2)$ so their product with $\phi$ enters only at third order. The solution to the $O(1)$ equations for an approximately spatially uniform $H$ is in polar coordinates,
\[
V = V_r \hat{r} + V_\theta \hat{\theta} = \frac{D}{r} \hat{r} + \frac{C}{r} \hat{\theta}, \tag{15}
\]
where we have defined the circulation and drain constants $C > 0$ and $D < 0$ respectively. This system (widely studied in the literature [15][20]) exhibits a horizon at $r_H = D/c$. Solutions to this equation can be obtained in the asymptotic limits $r \to \infty$ and $r \to r_H$ by defining a new field $\tilde{\psi}_m$ via,
\[
\phi(r, \theta, t) = \sum_m \phi_m(r) e^{i \theta \theta - i \omega t}, \tag{16}
\]
\[
\tilde{\psi}_m(r) = \frac{\psi_m(r)}{\sqrt{r}} \exp \left( -i \int_{\gamma - V_r \tilde{\phi}}^{\gamma} \frac{V_r \tilde{\phi}}{r^2 - V_r^2} dr \right),
\]
where $\tilde{\omega} = \omega - mV_\theta / r$. The wave equation can be rewritten as an equation for $\tilde{\psi}_m$ which in the two asymptotic limits becomes,
\[
\psi_m(r \to \infty) = A_m e^{\omega r / c} + A_m^* e^{-\omega r / c},
\]
\[
\tilde{\psi}_m(r \to r_H) = A_m^* e^{i \tilde{\omega} r}, \tag{17}
\]
where the tortoise coordinate is defined through $dr_s = c dr / c^2 - V_r^2$. We can compute the product $\Re[r h_m v_{r,m}]$ using the solutions above and the relations,
\[
h_m = -i \tilde{\omega} \phi_m + V_r \partial_r \phi_m, \quad v_{r,m} = \partial_r \phi_m
\]
leading to Eq. (3) and (4) in the main text.

\[\text{Appendix C}\]

Here we provide a discussion of the different notions of energy current and their relation to the mass flux. We begin with an action for the perturbations $S = \int \mathcal{L} d^2 x dt$, where the Lagrangian density is given by
\[
\mathcal{L} = -\tilde{g}^{\mu \nu} \partial_\mu \phi \partial_\nu \phi^*, \tag{18}
\]
which is a conserved current $j[\phi]$ is a quantity with components $j^\mu[\phi]$ satisfying,
\[
\partial_\mu j^\mu[\phi] = 0 \Rightarrow \partial_\mu \rho[\phi] + \nabla \cdot j[\phi] = 0, \tag{21}
\]
A conserved current $j[\phi]$ is a quantity with components $j^\mu[\phi]$ satisfying,
\[
\partial_\mu j^\mu[\phi] = 0 \Rightarrow \partial_\mu \rho[\phi] + \nabla \cdot j[\phi] = 0, \tag{21}
\]
where the second form in terms of $(\rho, j)$ splits $j$ into a temporal part - charge - and a spatial part - current (note: the two together are collectively called the 4-current). Square brackets are used to indicate that a quantity is a functional of $\phi$. Conserved currents are derived as follows. Consider an infinitesimal transformation of the field which induces a shift in the Lagrangian,
\[
\phi^0 \rightarrow \phi^0 + \delta \phi \implies \mathcal{L} \rightarrow \mathcal{L} + \delta \mathcal{L}
\]
Noether’s theorem [30][31] states that $j$ is a conserved current if the Lagrangian changes by a total derivative
\[\delta L = \partial_\mu F^{\mu}.\]

The components of the current are given by,
\[j^\mu = \frac{\partial L}{\partial (\partial_\tau \phi^\nu)} \delta \phi^\nu - F^\mu = -\tilde{g}^{\mu\nu}(\partial_\nu \phi^* \partial_\nu \phi + \partial_\nu \phi \partial_\nu \phi^*) - F^\mu\]

where \(a = 1,2\) with \(\phi_1 = \phi\) and \(\phi_2 = \phi^*\). We now consider several different conserved quantities.

**Norm conservation.** We consider first performing a phase rotation on \(\phi\). Since the wave equation is linear in \(\phi\), there is an internal symmetry \(\phi \rightarrow \exp(i\alpha)\phi\) that leaves the equation’s of motion unchanged, where \(\alpha\) is a phase rotation. For infinitesimal \(\alpha\), \(\phi\) changes by \(\delta \phi = i\alpha \phi\) and \(\delta L = 0\). The conserved quantities associated with this transformation are the norm \(\rho_n[\phi]\) and the norm current \(j_n[\phi]\) given by,

\[j_n[\phi] = i\tilde{g}^{\mu\nu}(\phi^* \partial_\nu \phi - \phi \partial_\nu \phi^*)\]
or
\[\rho_n[\phi] = i(\bar{\phi} \partial_\nu D_t \phi - \phi \partial_\nu D_t \phi^*)\]

Using the fact that the solution is stationary we have \(\partial_\tau \rho_n = 0\). Applying the divergence theorem to \(\nabla \cdot j_n[\phi] = 0\) over the region \(\tau = \{r_H, \infty\}\), we deduce \(\int_0^{2\pi} [r_j]_r \, d\theta = 0\). Using the asymptotic solutions \(A\) in this equation, we obtain

\[\omega(|A_n|^2 - |A_m|^2) = -\bar{\omega}|A_n^T|^2\]

where the left (right) of the equation is the norm current expressed at infinity (the horizon). Restrict our analysis to \(\omega > 0\), notice that if \(|A_n|^2 < |A_m|^2\) the energy current is negative (points towards \(r = r_H\)). However if \(|A_n|^2 > |A_m|^2\) (which is achieved for \(\bar{\omega} < 0\)) the norm current becomes positive and points outward, allowing for the extraction of energy from the system to infinity (this is phenomenon of superradiant scattering). This is understood as the mode carrying a negative energy (indicated by a negative norm \(\rho_n < 0\)) across the horizon, thereby lowering the energy of the system.

**Energy conservation.** Time translation \(t \rightarrow t - \delta t\) induces a change in the field \(\delta \phi^\nu = \delta t \partial_\nu \phi^\nu\) and the Lagrangian \(\delta L = \delta t \partial_\nu L\). This gives rise to conservation of energy current which has components,

\[j_e^\mu[\phi] = -\tilde{g}^{\mu\nu}(\partial_\nu \phi^* \partial_\nu \phi + \partial_\nu \phi \partial_\nu \phi^*)\]
or
\[\rho_e[\phi] = \partial_\tau \phi^* D_t \phi + \partial_\tau \phi D_t \phi^* - L = (\partial_\nu \phi)^2 + c^2(\nabla \phi)^2 - V \cdot \nabla \phi\]

\[j_e[\phi] = V(\partial_\nu \phi^* D_t \phi + \partial_\nu D_t \phi) - c^2(\partial_\nu \phi^* \nabla \phi + \partial_\nu \phi \nabla \phi^*)\]

Defining \(\partial_\tau \phi = \phi\) and \(p = \partial L/\partial \dot{\phi} = D_t \phi^*\) (and similarly for the complex conjugate), one can see that \(\rho_e = H\) is the Hamiltonian density.

To make contact with fluid dynamics, we define \((E, I) = (\rho_e, j_e)/2g\) and also \(\phi \in \mathbb{R}\). Using \(D_t \phi = -gh\) and \(\nabla \phi = v\), we obtain the usual equation for wave energy conservation in shallow water,

\[\partial_t E + \nabla \cdot I = 0\]

where

\[E = \frac{1}{2} gh^2 + \frac{1}{2} H v^2 + h \nabla \cdot v\]

\[I = gh(Hv + Hv)(gh + V \cdot v).\]

In fluid dynamics, these equations are obtained by contracting the \(O(2)\) terms in the shallow water equations \([6]\) with \(gh, Hv^2\) where superscript \(T\) indicates the transpose. Following the same procedure outlined above, we evaluate the expression for \(E\) (using the forms of \(h\) and \(v\) given in Appendix B) to show \(\partial_t E = 0\), applying the divergence theorem we obtain \(\int_2^{2\pi} [r_j]_r \, d\theta = 0\). Evaluating this expression in both limits results again in \(\omega c/g\) multiplied by a factor of \(\omega c/g\), thus making contact between the norm current and the energy current.

As a brief aside, Eq. \((25)\) applied to harmonic field modes becomes,

\[j_e^\mu[\phi] = i\omega \tilde{g}^{\mu\nu}(\phi^* \partial_\nu \phi - \phi \partial_\nu \phi^*).\]

One can then observe that this energy 4-current is \(\omega\) multiplied by the norm 4-current in \((23)\). Moreover, Eq. \((22)\) applied to angular coordinate translations leads to an angular momentum 4-current \(j^\mu_\ell\) satisfying the well-known property \(j^\mu_\ell/j^\mu_\ell = \omega/m\).

Now to make contact with the mass flux, we notice that when \(V = 0\) we have \(I = ghHv\). Once integrated, this is precisely the term appearing in Eq. \((2)\) multiplied by a factor \(c^2\). Therefore, in the absence of a flow, there is a direct correspondence between the energy current and the mass flux given by \(I = J_\ell c^2\) (remiscent of the celebrated result \(E = mc^2\)). When \(V \neq 0\), we see clearly from Eq. \((27)\) that the energy current and mass flux no longer coincide. This explains why Eq. \((1)\) does not have the symmetry properties of \(\omega\).

For the reader who is more familiar with the language of electromagnetism, we draw some comparisons here. The form \(\text{Re}(h_m u_m)\) appearing in Eq. \((2)\) is reminiscent of the complex Poynting vector \((S)\), whose real part describes the flux of power in electromagnetic waves, in terms of the phasor decomposition of the electric \((E)\) and magnetic \((H = B/\mu_0)\) fields, see e.g. \([32]\).

\[S = \frac{1}{2} E \times H^*.\]
The correspondence between $\mathbf{I}$ and $\mathbf{S}$ becomes exact upon defining,

$$
\frac{h}{H} = \frac{\mathbf{B}}{B_0} \cdot \mathbf{\hat{z}} \quad \frac{\nu}{c} = \frac{\mathbf{E}}{E_0} \times \mathbf{\hat{z}}
$$

where $E_0 = cB_0$ are reference electric/magnetic fields respectively and the correspondence requires $H \rightarrow \varepsilon_0$ and $gH \rightarrow 1/\mu_0 \varepsilon_0$. Indeed, one can show using this correspondence that the vacuum Maxwell equations are equivalent to the shallow water system of equations in an irrotational, quiescent fluid.

### Appendix D

In this appendix, we detail our experimental method. To observe the height change resulting from impinging waves, we illuminate the free surface from above using a Yb-doped laser, mean wavelength 457nm, which is converted into a thin laser sheet (thickness ~ 2mm). This appears on the free surface as a line spanning nearly the full length of the tank, thereby allowing us to see the free surface in our data. We filmed this line from the side with a high-speed Phantom Miro Lab camera at 24fps. The line of sight of the camera was at an angle of $\Theta$ to the free surface such that $\cos \Theta = 0.910 \pm 0.010$, which was necessary to avoid shadows of waves passing in between the laser-sheet and the camera, which obscured our vision of the free surface. This induces an uncertainty on the measured height change which we include in our error estimates of $\dot{H}$. We recorded the height before sending any waves to confirm that the background was steady, and then monitored the water height whilst exciting waves of frequency $f$ ($= \omega/2\pi$) and amplitude $a$ over a time window $\Delta t$. The free surface was identified by finding the pixel of maximum intensity in each image and interpolating using adjacent points to determine the maximum to a sub pixel accuracy.

Once the free surface in each image is determined as a function of spatial coordinate, we extract the zero wavenumber contribution to the signal using a spatial Fourier transform. This gave a measurement of the average height across the observed region at each time step, which we correct for the camera angle $\Theta$, obtaining the variation of the background height over time. In all experiments, the initial height was determined to be sufficiently steady (less than 10% of the total change over the experiment) and any slight variation was due to the difficulty in maintaining constant $Q$ over the experiment. We corrected for this (as well as the slightly different initial heights) by fitting the curve prior to the start of the waves with a straight line and subtracting this from the entire data set to give the height change $\Delta H$. This ensured that the height change measured was the result of the perturbations.

In all experiments, we observed a small amplitude oscillation about the linear behaviour at the frequency $f$ (confirmed by a peak in the Fourier transform of $\Delta H$) corresponding the oscillatory (linear) terms we dropped in Eq. (6). Furthermore, the amplitude of this oscillation decreased with $f$ which is expected, since integrating $\dot{H}$ in time brings out a factor $1/f$. We also observe a sharp increase in height immediately before the height starts to decrease. This is because our wave generator panel is initially fully retracted and must move forward to it’s equilibrium position, thereby reducing the area of the system and increasing mean height. In accordance with this explanation, a sharper increase was observed for larger piston amplitudes.

Finally, the flow and wave parameters pertaining to each experiment can be found in Table I.

| Exp. | $Q$[l/min] | $H(t = 0)$[m] | $f$[Hz] | $a$[m] | $\Delta t$[s] |
|------|-----------|-------------|--------|-------|----------|
| 1    | $14.0 \pm 0.4$ | $0.020 \pm 0.001$ | $4$ | $1.6 \pm 0.1 \times 10^{-3}$ | 120 |
| 2    | $14.0 \pm 0.4$ | $0.019 \pm 0.001$ | $2.3, 4$ | $2.3 \pm 0.5 \times 10^{-3}$ | 120 |
| 3    | $29.4 \pm 0.4$ | $0.065 \pm 0.001$ | $4$ | $2.1 \pm 0.5 \times 10^{-3}$ | 600 |

TABLE I. Details of the different experiments performed; $Q$ is the average flow rate over the course of the experiment, $H(t = 0)$ is the initial water height, $f$ and $a$ are the incident wave frequency and amplitude respectively and $\Delta t$ is the window of wave incidence.