Classification of Canonical Bases for \((n-2)\)-Dimensional Subspaces of \(n\)-Dimensional Vector Space

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Abstract

Famous K. Gauss introduced reduced row echelon forms for matrices approximately 200 years ago to solve systems of linear equations but the number of them and their structure has been unknown until 2016 when it was determined at first in the previous article given up to \((n-1)\times n\) matrices. The similar method is applied to find reduced row echelon forms for \((n-2)\times n\) matrices in this article, and all canonical bases for \((n-2)\)-dimensional subspaces of \(n\)-dimensional vector space are found also.

Keywords: Vector space; Subspaces; Canonical bases

Introduction

The canonical bases for \((n-2)\)-dimensional subspaces of \(n\)-dimensional vector space are introduced in the article, and all nonequivalent of them are classified. Canonical bases for \((n-1)\)-dimensional subspaces of \(n\)-dimensional vector space were classified in the previous article [2] of the same author. This new case of \((n-2)\)-dimensional subspaces is interesting to be studied because some \(n\)-dimensional Lie algebras haven't any \((n-1)\)-dimensional subalgebras. For example, in the article [3], it was proved that 6-dimensional Lie algebra of Lorentz group doesn't have any 5-dimensional subalgebra but this Lie algebra has 4-dimensional subalgebras. We start to introduce the necessary definitions.

Let \(V\) be an \(n\)-dimensional vector space with its standard basis \(e_1, e_2, \ldots, e_n\). Suppose that \(a_1, a_2, \ldots, a_{n-2}\) are \((n-2)\) linearly independent vectors in the vector space \(V\), where

\[
\begin{align*}
 a_1 &= a_1 e_1 + a_2 e_2 + \cdots + a_{n-2} e_{n-2} + a_{n-1} e_{n-1} + a_n e_n, \\
 a_2 &= a_1 e_1 + a_2 e_2 + \cdots + a_{n-2} e_{n-2} + a_{n-1} e_{n-1} + a_n e_n, \\
 &\vdots \\
 a_{n-2} &= a_1 e_1 + a_2 e_2 + \cdots + a_{n-2} e_{n-2} + a_{n-1} e_{n-1} + a_n e_n. 
\end{align*}
\]

(1)

The vectors (1) describe all possible bases for any \((n-2)\)-dimensional subspace \(S\) of \(V\). This description contains too many arbitrary components; their number is \((n-2)\times n\). Instead of that, we introduce canonical bases with much smaller number of arbitrary components in each of them (maximum 2\((n-2)\)).

Definition 1: Two bases are called equivalent if they generate the same two subspaces of \(V\), and they are called nonequivalent if they generate two different subspaces of \(V\).

We will associate the following \((n-2)\times n\) matrix \(M\) with a basis (I)

\[
M = \begin{bmatrix}
 a_{11} & a_{12} & \cdots & a_{1n} \\
 a_{21} & a_{22} & \cdots & a_{2n} \\
 & \vdots & \ddots & \vdots \\
 a_{n-2,1} & a_{n-2,2} & \cdots & a_{n-2,n}
\end{bmatrix}
\]

(II)

Definition 2: Two matrices are called row equivalent (or just equivalent) if they have the same reduced row echelon form, and they are called nonequivalent if they have different reduced row echelon forms.

About reduced row echelon forms of matrices, see for example [1].

Definition 3: The basis (I) is called canonical if its vectors \(a_1, a_2, \ldots, a_{n-2}\) are the corresponding rows in some reduced row echelon form of the matrix \(M\).

Thus, there is one-to-one correspondence between nonequivalent canonical bases for \((n-2)\)-dimensional subspaces of \(n\)-dimensional vector space and nonequivalent reduced row echelon forms for \((n-2)\times n\) matrix \(M\) of the rank\((n-2)\).

Part I. Basic Examples

Consider two examples of nonequivalent canonical bases for \((n-2)\)-dimensional subspaces of \(n\)-dimensional vector spaces where \(n=4\) and \(n=6\).

Ex. 1: Let \(V\) be 4-dimensional vector space with its standard basis \(e_1, e_2, e_3, e_4\). Any 2-dimensional subspace \(S\) of \(V\) can be described as \(S = \text{Span}\{a_1, a_2\}\) where,

\[
\begin{align*}
 a_1 &= a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 ; \\
 a_2 &= b_1 e_1 + b_2 e_2 + b_3 e_3 + b_4 e_4.
\end{align*}
\]

This arbitrary basis is equivalent to one and only one canonical basis from the next list:

\[
\begin{align*}
 (1) a &= e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 ; \\
 (2) a &= e_2 + a_1 e_1 + a_3 e_3 + a_4 e_4 ; \\
 (3) a &= e_3 + a_1 e_1 + a_2 e_2 + a_4 e_4 ; \\
 (4) a &= e_4 + a_1 e_1 + a_2 e_2 + a_3 e_3 ; \\
 (5) a &= e_1, \quad b = e_1.
\end{align*}
\]

Details of evaluation are omitted because it is similar (but easier) to the evaluation in the example 2. The last canonical bases generate the following 6 matrices associated with them:

\[
\begin{align*}
 \begin{bmatrix}
 1 & 0 & a_1 & a_2 & a_3 & a_4 \\
 0 & 1 & a_1 & a_2 & a_3 & a_4 \\
 0 & 0 & a_1 & a_2 & a_3 & a_4 \\
 0 & 0 & 0 & a_1 & a_2 & a_3 \\
 0 & 0 & 0 & 0 & a_1 & a_2 \\
 0 & 0 & 0 & 0 & 0 & a_1
\end{bmatrix}
\end{align*}
\]

Ex. 2: Let \(V\) be 6-dimensional vector space with its standard basis \(e_1, e_2, e_3, e_4, e_5, e_6\). Any 4-dimensional subspace \(S\) of \(V\) can be described as \(S = \text{Span}\{a, b, c, d\}\) where,

\[
\begin{align*}
 a &= a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6 ; \\
 b &= b_1 e_1 + b_2 e_2 + b_3 e_3 + b_4 e_4 + b_5 e_5 + b_6 e_6.
\end{align*}
\]

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Received September 20, 2016; Accepted November 18, 2016; Published November 29, 2016

Citation: Shhtukar U (2016) Classification of Canonical Bases for (n-2)-Dimensional Subspaces of n-Dimensional Vector Space. J Generalized Lie Theory Appl 10: 245. doi:10.4172/1736-4337.1000245

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As the result, the following canonical basis is obtained:
\[ a = e_1 + a_1 e_1 + a_2 e_1 + a_3 e_1 + a_4 e_1, \quad \bar{a} = e_1 + b_1 e_1 + b_2 e_1 + b_3 e_1 + b_4 e_1, \]
\[ c = c_1 e_1 + c_2 e_1 + c_3 e_1 + c_4 e_1, \quad \bar{c} = d_1 e_1 + d_2 e_1 + d_3 e_1 + d_4 e_1. \] (3)

Remark 1: The first components of vectors \( \bar{a}, \bar{b}, \bar{c}, \bar{d} \) are changed as the result of operations performed but all other components of them still have the same notations just for the common convenience. This idea will be used also in all steps of the procedure below.

1. Suppose now that at least one coefficient from \( b_2, c_2, d_2 \) at the basis (a) is not zero. Without any loss in generality, let \( b_2 \neq 0 \). Perform the linear operations: first \( b \rightarrow b/\bar{b} \), and then \( a \rightarrow a - b \bar{a} \), \( c \rightarrow c - \bar{c} \), \( d \rightarrow d - \bar{d} \). As the result, the following transformed basis is obtained:
\[ a = e_1 + a_1 e_1 + a_2 e_1 + a_3 e_1 + a_4 e_1, \quad \bar{a} = e_1 + b_2 e_1 + b_3 e_1 + b_4 e_1, \]
\[ c = c_2 e_1 + c_3 e_1 + c_4 e_1, \quad \bar{c} = d_2 e_1 + d_3 e_1 + d_4 e_1. \] (4)

2. Suppose that at least one coefficient among \( c_2, d_2 \) at the basis (1) is not zero. Again, without any loss in generality, let \( c_2 \neq 0 \). Perform the operation \( c \rightarrow c/\bar{c} \), first, and then operations \( a \rightarrow a - c \bar{a}, \ b \rightarrow b - c \bar{b}, \ d \rightarrow d - c \bar{d} \). As the result, the following basis is done:
\[ a = e_1 + a_2 e_1 + a_3 e_1 + a_4 e_1 + a_5 e_1, \quad \bar{a} = e_1 + b_2 e_1 + b_3 e_1 + b_4 e_1 + b_5 e_1, \]
\[ c = c_2 e_1 + c_3 e_1 + c_4 e_1 + c_5 e_1, \quad \bar{c} = d_2 e_1 + d_3 e_1 + d_4 e_1 + d_5 e_1. \] (5)

3. Suppose now that the coefficient \( d_2 \) at the basis (2) is not zero. Perform the operation \( d \rightarrow d/\bar{d} \), first, and then operations \( a \rightarrow a - d \bar{a} - b \bar{d} \), \( c \rightarrow c - d \bar{c} \). As the result, the following canonical basis is obtained:
\[ a = e_1 + a_2 e_1 + a_3 e_1 + a_4 e_1 + a_5 e_1, \quad \bar{a} = e_1 + b_2 e_1 + b_3 e_1 + b_4 e_1 + b_5 e_1, \]
\[ c = c_2 e_1 + c_3 e_1 + c_4 e_1 + c_5 e_1, \quad \bar{c} = d_2 e_1 + d_3 e_1 + d_4 e_1 + d_5 e_1. \] (a)

4. Suppose now that both coefficients \( c_2, d_2 \) at the basis (1) are zero. We have:
\[ a = e_1 + a_2 e_1 + a_3 e_1 + a_4 e_1 + a_5 e_1, \quad \bar{a} = e_1 + b_2 e_1 + b_3 e_1 + b_4 e_1 + b_5 e_1, \]
\[ c = c_2 e_1 + c_3 e_1 + c_4 e_1, \quad \bar{c} = d_2 e_1 + d_3 e_1 + d_4 e_1 + d_5 e_1. \] (b)

Consider coefficients \( c, d \) in the basis (4). Suppose that at least one of them is not zero. Let \( c \neq 0 \). Perform operation \( c \rightarrow c/\bar{c} \), first, and perform operations \( a \rightarrow a - c \bar{c}, \ b \rightarrow b - c \bar{b}, \ d \rightarrow d - c \bar{d} \) after the first one. The following basis is obtained,
\[ a = e_1 + a_2 e_1 + a_3 e_1 + a_4 e_1 + a_5 e_1, \quad \bar{a} = e_1 + b_2 e_1 + b_3 e_1 + b_4 e_1 + b_5 e_1, \]
\[ c = e_1 + c_2 e_1 + c_3 e_1 + c_4 e_1, \quad \bar{c} = d_1 e_1 + d_2 e_1 + d_3 e_1 + d_4 e_1 + d_5 e_1. \] (a)

Remark 2: If \( d \neq 0 \) then doing similarly we obtain the following canonical basis:
\[ a = e_1 + a_2 e_1 + a_3 e_1 + a_4 e_1 + a_5 e_1, \quad \bar{a} = e_1 + b_2 e_1 + b_3 e_1 + b_4 e_1 + b_5 e_1, \]
\[ c = c_2 e_1 + c_3 e_1 + c_4 e_1, \quad \bar{c} = d_1 e_1 + d_2 e_1 + d_3 e_1 + d_4 e_1. \] (b)

Remark 3: If \( d_2 \neq 0 \) in the basis (4), then there will be obtained the bases that are equivalent to (a) and (b).

5. Suppose now that both coefficients \( c_2, d_2 \) in the basis (4) are zero. We obtain:
\[ a = e_1 + a_2 e_1 + a_3 e_1 + a_4 e_1 + a_5 e_1, \quad \bar{a} = e_1 + b_2 e_1 + b_3 e_1 + b_4 e_1 + b_5 e_1, \]
\[ c = c_2 e_1 + c_3 e_1 + c_4 e_1, \quad \bar{c} = d_2 e_1 + d_3 e_1 + d_4 e_1 + d_5 e_1. \] (a)

Remark 4: If \( d_3 \neq 0 \) then doing similarly we obtain the following canonical basis:
\[ a = e_1 + a_2 e_1 + a_3 e_1 + a_4 e_1 + a_5 e_1, \quad \bar{a} = e_1 + b_2 e_1 + b_3 e_1 + b_4 e_1 + b_5 e_1, \]
\[ c = c_3 e_1 + c_4 e_1, \quad \bar{c} = d_3 e_1 + d_4 e_1 + d_5 e_1 + d_6 e_1 + d_7 e_1. \] (b)
Let $d_i \neq 0$. Perform the operation $\overline{a} / d_i$ first, and the operations $\overline{a - a_i, b - b_i, c - c_i, d}$ after the first one. We obtain the new canonical basis:

$$\overline{\overline{a}} = \overline{a_i} + a_i \overline{e} + a_i \overline{e} + a_i \overline{e} = \overline{e}_i + b_i \overline{e}_i + b_i \overline{e}_i + b_i \overline{e}_i,$$  

$$\overline{c} = c_i \overline{e}_i + c_i \overline{e}_i + d_i \overline{e}_i + d_i \overline{e}_i.$$  

(9)

If $d_i = 0$ in the basis (9), at least one coefficient from $\overline{c}_i, \overline{d}_i$ is not zero. Otherwise, vectors $\overline{c}, \overline{d}$ are linearly dependent. Let $\overline{c} \neq 0$. Perform the operation $\overline{c} / c_i$ first, and the operations $\overline{a - a_i, b - b_i, c - c_i, d - d_i}$ after the first one. We obtain the following basis:

$$\overline{\overline{a}} = a_i \overline{e}_i + a_i \overline{e}_i + a_i \overline{e}_i + a_i \overline{e}_i = \overline{e}_i + b_i \overline{e}_i + b_i \overline{e}_i + b_i \overline{e}_i,$$  

$$\overline{c} = c_i \overline{e}_i + c_i \overline{e}_i + c_i \overline{e}_i + c_i \overline{e}_i = d_i \overline{e}_i + d_i \overline{e}_i + d_i \overline{e}_i.$$  

(10)

In the last basis, at least one coefficient from $\overline{c}_i, \overline{d}_i$ is not zero. Otherwise, vectors are linearly dependent but it’s impossible for any basis. Let $\overline{b} \neq 0$ (without any loss in the generality). Perform the operation $\overline{b} / b_i$ first, and the operations $\overline{a - a_i, b - b_i, c - c_i, d - d_i}$ after the first one. We obtain the following basis:

$$\overline{\overline{a}} = a_i \overline{e}_i + a_i \overline{e}_i + a_i \overline{e}_i + a_i \overline{e}_i = \overline{e}_i + b_i \overline{e}_i + b_i \overline{e}_i + b_i \overline{e}_i,$$  

$$\overline{c} = c_i \overline{e}_i + c_i \overline{e}_i + c_i \overline{e}_i + c_i \overline{e}_i = d_i \overline{e}_i + d_i \overline{e}_i + d_i \overline{e}_i.$$  

In the basis (8), at least one coefficient among $\overline{a}_i, \overline{b}_i, \overline{c}_i, \overline{d}_i$ is not zero. Otherwise, vectors $\overline{c}, \overline{d}$ are linearly dependent but it’s impossible for any basis. Let $\overline{c} \neq 0$. Perform the operation $\overline{c} / c_i$ first, and the operations $\overline{a - a_i, b - b_i, c - c_i, d - d_i}$ after the first one. We obtain the following basis:

$$\overline{\overline{a}} = a_i \overline{e}_i + a_i \overline{e}_i + a_i \overline{e}_i + a_i \overline{e}_i = \overline{e}_i + b_i \overline{e}_i + b_i \overline{e}_i + b_i \overline{e}_i,$$  

$$\overline{c} = c_i \overline{e}_i + c_i \overline{e}_i + c_i \overline{e}_i + c_i \overline{e}_i = d_i \overline{e}_i + d_i \overline{e}_i + d_i \overline{e}_i + d_i \overline{e}_i.$$  

(11)

If $d_i = 0$ in the basis (11), then the basis is equivalent to $(a_0)$ will be obtained. We have analyzed all possibilities in the situation A.

B. Suppose now that all coefficients $a_i, b_i, c_i, d_i$ are zero in (III). The following basis is obtained:

$$\overline{\overline{a}} = a_i \overline{e}_i + a_i \overline{e}_i + a_i \overline{e}_i + a_i \overline{e}_i = \overline{e}_i + b_i \overline{e}_i + b_i \overline{e}_i + b_i \overline{e}_i,$$  

$$\overline{c} = c_i \overline{e}_i + c_i \overline{e}_i + c_i \overline{e}_i + c_i \overline{e}_i = d_i \overline{e}_i + d_i \overline{e}_i + d_i \overline{e}_i.$$  

(12)

If $d_i = 0$ in the basis (12), then the basis is equivalent to $(a_0)$ will be obtained. We have analyzed all possibilities in the situation A.
Consider coefficients \( a, b, c, d \) in the basis \( (b) \). Suppose now (in opposition to the step 1) that all coefficients \( a, b, c, d \) are zero.

We have the basis:

\[
a = a_e + a_e + a_e + a_e, \quad b = b_e + b_e + b_e + b_e, \\
c = c_e + c_e + c_e + c_e, \quad d = d_e + d_e + d_e + d_e.
\]

We consider coefficients \( a, b, c, d \) in the last basis. At least one of them is not zero. Otherwise, vectors \( b, c, d \) are linearly dependent but it’s impossible for any basis. Let \( a \neq 0 \) (without any loss in generality). Perform the operation \( a/ a \) first, and the operations \( b-b, c-c, a-d, a-d \) after the first step. We obtain the following basis:

\[
a = a_e + a_e + a_e + a_e, \quad b = b_e + b_e + b_e + b_e, \\
c = c_e + c_e + c_e + c_e, \quad d = d_e + d_e + d_e + d_e.
\]

Consider coefficients \( a, b, c, d \) in the last basis. At least one of them is not zero. Otherwise, vectors \( b, c, d \) are linearly dependent but it’s impossible for any basis. Let \( b \neq 0 \). Perform the operation \( b-b \) first, and the operations \( b-b, c-c, a-d, a-d \) after the first step. We obtain:

\[
a = a_e + a_e + a_e + a_e, \quad b = b_e + b_e + b_e + b_e, \\
c = c_e + c_e + c_e + c_e, \quad d = d_e + d_e + d_e + d_e.
\]

We continue the procedure; we will obtain the following canonical basis at the end:

\[
a = a_e, \quad b = b_e, \quad c = c_e, \quad d = d_e. \quad (b)
\]

All other subcases in the step 6 give the same basis \( (b) \).

The total list of all canonical bases that are found at the situations A and B is done here:

\[
a = a_e + a_e + a_e + a_e, \quad b = b_e + b_e + b_e + b_e, \\
c = c_e + c_e + c_e + c_e, \quad d = d_e + d_e + d_e + d_e. \quad (b)
\]

\[
a = a_e + a_e + a_e + a_e, \quad b = b_e + b_e + b_e + b_e, \\
c = c_e + c_e + c_e + c_e, \quad d = d_e + d_e + d_e + d_e. \quad (a)
\]

\[
a = a_e + a_e + a_e + a_e, \quad b = b_e + b_e + b_e + b_e, \\
c = c_e + c_e + c_e + c_e, \quad d = d_e + d_e + d_e + d_e. \quad (a)
\]

\[
a = a_e + a_e + a_e + a_e, \quad b = b_e + b_e + b_e + b_e, \\
c = c_e + c_e + c_e + c_e, \quad d = d_e + d_e + d_e + d_e. \quad (a)
\]

\[
a = a_e + a_e + a_e + a_e, \quad b = b_e + b_e + b_e + b_e, \\
c = c_e + c_e + c_e + c_e, \quad d = d_e + d_e + d_e + d_e. \quad (a)
\]

\[
a = a_e + a_e + a_e + a_e, \quad b = b_e + b_e + b_e + b_e, \\
c = c_e + c_e + c_e + c_e, \quad d = d_e + d_e + d_e + d_e. \quad (a)
\]

\[
a = a_e + a_e + a_e + a_e, \quad b = b_e + b_e + b_e + b_e, \\
c = c_e + c_e + c_e + c_e, \quad d = d_e + d_e + d_e + d_e. \quad (a)
\]

\[
a = a_e + a_e + a_e + a_e, \quad b = b_e + b_e + b_e + b_e, \\
c = c_e + c_e + c_e + c_e, \quad d = d_e + d_e + d_e + d_e. \quad (a)
\]

\[
a = a_e + a_e + a_e + a_e, \quad b = b_e + b_e + b_e + b_e, \\
c = c_e + c_e + c_e + c_e, \quad d = d_e + d_e + d_e + d_e. \quad (a)
\]

\[
\begin{align*}
ad = a_e + a_e + a_e + a_e, \quad b = b_e + b_e + b_e + b_e, \\
c = c_e + c_e + c_e + c_e, \quad d = d_e + d_e + d_e + d_e. \quad (a) \\
ad = a_e + a_e + a_e + a_e, \quad b = b_e + b_e + b_e + b_e, \\
c = c_e + c_e + c_e + c_e, \quad d = d_e + d_e + d_e + d_e. \quad (a) \\
ad = a_e + a_e + a_e + a_e, \quad b = b_e + b_e + b_e + b_e, \\
c = c_e + c_e + c_e + c_e, \quad d = d_e + d_e + d_e + d_e. \quad (a) \\
ad = a_e + a_e + a_e + a_e, \quad b = b_e + b_e + b_e + b_e, \\
c = c_e + c_e + c_e + c_e, \quad d = d_e + d_e + d_e + d_e. \quad (a) \\
ad = a_e + a_e + a_e + a_e, \quad b = b_e + b_e + b_e + b_e, \\
c = c_e + c_e + c_e + c_e, \quad d = d_e + d_e + d_e + d_e. \quad (a) \\
ad = a_e + a_e + a_e + a_e, \quad b = b_e + b_e + b_e + b_e, \\
c = c_e + c_e + c_e + c_e, \quad d = d_e + d_e + d_e + d_e. \quad (a)
\end{align*}
\]
For each of the \( n \geq 4 \), this matrix is row equivalent to one and only one of the \( n \times n \) matrices of Part II. General Case:

### Theorem 1

Let \( M \) be a \((n-2) \times n\) matrix (II) of the rank \((n-2)\) where \( n \geq 4 \). This matrix is row equivalent to one and only one of the following matrices:

\[
\begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & a_{n-1} & a_n \\
1 & 0 & 0 & \cdots & 0 & a_{n-2} & a_{n-1} \\
0 & 1 & 0 & \cdots & 0 & a_{n-3} & a_{n-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & a_{n-3} & a_{n-2} \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\end{bmatrix}
\]

### Proof:

We will use the mathematical induction method with respect to the dimension \( n \). This statement is correct in the cases \( n=4 \) and \( n=6 \) according Examples 1 and 2. Suppose that the statement is true for arbitrary \( n \geq 4 \), and prove it for the dimension \((n+1)\). Let \( M \) be a matrix of the size \((n-1) \times (n+1)\):

\[
\begin{bmatrix}
a_{1,1} & a_{1,2} & \cdots & a_{1,n-1} & a_{1,n} \\
a_{2,1} & a_{2,2} & \cdots & a_{2,n-1} & a_{2,n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & a_{n-2,1} \\
0 & 0 & \cdots & 0 & a_{n-2,2} \\
\end{bmatrix}
\]

Consider the \((n-2) \times n\) submatrix \( M' \) located in the upper left corner of the matrix \( M \). According the assumption, this submatrix can be transformed into one of the matrices listed in this statement. We will substitute submatrix \( M' \) by the corresponding matrix, and then transform the special matrix \( M' \) into reduced row echelon form. The standard linear operations with rows (vectors) will be utilized: (a) interchange any two rows, (b) multiply any row by a nonzero constant, and (c) add a multiple of some row to another row.

1. At the first case, we have:

\[
\begin{bmatrix}
0 & 0 & \cdots & 0 & a_{n-2,1} & a_{n-2,2} \\
1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 \\
\end{bmatrix}
\]

Perform linear transformations \( a_{n-2,1} \to a_{n-2,1} - a_{n-2,2} a_{n-2,1} \), \( a_{n-2,2} \to a_{n-2,2} - a_{n-2,1} a_{n-2,2} \), ..., \( a_{n-2,n-1} \to a_{n-2,n-1} - a_{n-2,n-2} a_{n-2,n-1} \).

The result of the operations is the following matrix:

\[
\begin{bmatrix}
1 & 0 & \cdots & 0 & 0 & a_{n-1,n-2} & a_{n-1,n-1} \\
0 & 1 & \cdots & 0 & 0 & a_{n-2,n-2} & a_{n-2,n-1} \\
0 & 0 & \cdots & 1 & 0 & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
After the first one.

At least one components among $a_{n-1,x}$, $a_{x-2}$, $a_{x-1,x}$ is not zero but all other components of the $(n-1)$ row are zero. Let $a_{n-1,x} \neq 0$. Perform the operation $a_{x-1} / a_{n-1,x}$ first, and the operations $a_{x}-a_{x-1}a_{n-1,x}, a_{x}-a_{x-2}a_{n-1,x}, \ldots, a_{x}-a_{2}a_{n-1,x}, a_{1}$ after the first one. We obtain:

$$M = \begin{bmatrix}
1 & 0 & 0 & \ldots & 0 & a_{n-2} & 0 & a_{x} & a_{x+1} \\
0 & 1 & 0 & \ldots & 0 & a_{n-3} & 0 & a_{x} & a_{x+1} \\
0 & 0 & 1 & \ldots & 0 & a_{n-4} & 0 & a_{x} & a_{x+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & a_{n-3,x} & 0 & a_{x} & a_{x+1} \\
0 & 0 & 0 & \ldots & 1 & a_{n-2,x} & 0 & a_{x} & a_{x+1} \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

If interchange rows $a_{n-2}$ and $a_{n-1}$ in the last matrix, we obtain the matrix of the first type as we need. Let $a_{n-2}=0$, and $a_{n-1} \neq 0$. Perform the operation $a_{n-1} / a_{n-2}$ first, and the operations $a_{n-1}-a_{n-2}a_{n-1,x}, a_{n-1}-a_{n-2}a_{n-1,x}, \ldots, a_{n-1}-a_{2}a_{n-1,x}, a_{1}$ after the first one. We obtain:

$$M = \begin{bmatrix}
1 & 0 & 0 & \ldots & 0 & a_{n-2} & 0 & a_{x} & a_{x+1} \\
0 & 1 & 0 & \ldots & 0 & a_{n-3} & 0 & a_{x} & a_{x+1} \\
0 & 0 & 1 & \ldots & 0 & a_{n-4} & 0 & a_{x} & a_{x+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & a_{n-3,x} & 0 & a_{x} & a_{x+1} \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

It is the matrix of the $(n-1)$ type from the list above. The statement is proved for the 1st case.

2. At the second case, we have:

$$M = \begin{bmatrix}
1 & 0 & 0 & \ldots & a_{n-2} & 0 & a_{x} & a_{x+1} \\
0 & 1 & 0 & \ldots & a_{n-3} & 0 & a_{x} & a_{x+1} \\
0 & 0 & 1 & \ldots & a_{n-4} & 0 & a_{x} & a_{x+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & a_{n-3,x} & 0 & a_{x} & a_{x+1} \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 1
\end{bmatrix}$$

Perform linear transformations $a_{n-1}-a_{n-2}a_{n-1,x}, a_{n-1}-a_{n-2}a_{n-1,x}, \ldots, a_{n-1}-a_{2}a_{n-1,x}, a_{1}$ and $a_{2}-a_{1}a_{2,x}$. The result of the operations is the following matrix:

$$M = \begin{bmatrix}
1 & 0 & 0 & \ldots & a_{n-2} & 0 & a_{x} & a_{x+1} \\
0 & 1 & 0 & \ldots & a_{n-3} & 0 & a_{x} & a_{x+1} \\
0 & 0 & 1 & \ldots & a_{n-4} & 0 & a_{x} & a_{x+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & a_{n-3,x} & 0 & a_{x} & a_{x+1} \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 1
\end{bmatrix}$$
\[ M = \begin{bmatrix}
1 & 0 & 0 & \cdots & a_{n-3} & 0 & a_{n-1} & 0 & a_{n+1} \\
0 & 1 & 0 & \cdots & a_{n-4} & 0 & a_{n-2} & 0 & a_{n+2} \\
0 & 0 & 1 & \cdots & a_{n-5} & 0 & a_{n-3} & 0 & a_{n+3} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
1 & 0 & 0 & \cdots & a_{n-3} & 0 & a_{n-1} & 0 & a_{n+1} \\
0 & 0 & 0 & \cdots & a_{n-4} & 0 & a_{n-2} & 0 & a_{n+2} \\
\end{bmatrix} \]

Perform linear transformations \( a_{n-1} = \cdots = a_{n+1} = 0 \). The result of the operations is the following matrix:

\[ M = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & a_{n-1} & 0 & a_{n+1} \\
0 & 1 & 0 & \cdots & 0 & a_{n-2} & 0 & a_{n+2} \\
0 & 0 & 1 & \cdots & 0 & a_{n-3} & 0 & a_{n+3} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
1 & 0 & 0 & \cdots & 0 & a_{n-1} & 0 & a_{n+1} \\
0 & 0 & 0 & \cdots & 0 & a_{n-2} & 0 & a_{n+2} \\
\end{bmatrix} \]

At least one component among \( a_{1,2}, a_{2,3}, a_{3,4}, \ldots, a_{n-2,1} \) is not zero but all other components of the (n-1) row are zero. Let \( a_{3,1} = 0 \). Perform the operation \( a_{1,2} = a_{1,2} - a_{1,1} a_{2,1} \), and remove the new last row into the first position. We obtain:

\[ M = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \]

If we interchange the rows \( (n-1) \) and \( (n-2) \) in this matrix, we obtain the matrix of the 2\( n - 3 \) type from the list as we need. Let \( a_{n-3,1} = 0 \), and \( a_{n-1,1} \neq 0 \). Perform the operation \( a_{n-3,1} = a_{n-3,1} \), and remove the new last row into the first position. We obtain:

\[ M = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \]

It is the matrix of the (2\( n - 3 \)) type from the list. The case \( \frac{n(n-1)}{2} \) is proved.

Case \( \frac{n(n-1)}{2} \). At this case, we have the following matrix of \( (n-1) \times (n+1) \) size.

\[ M = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\end{bmatrix} \]

Perform the operations \( a_{n-1,1} = a_{n-1,1} - a_{n-1,2} a_{2,1} \), \( a_{n-2,1} = a_{n-2,1} - a_{n-2,2} a_{3,1} \), \ldots, \( a_{n-3,1} = a_{n-3,2} a_{3,1} \). We obtain:

\[ M = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\end{bmatrix} \]

It is the matrix of the \( (2n-2) \) type from the list as we need. Let \( a_{n-3,1} = 0 \), and \( a_{n-1,1} \neq 0 \) in the previous matrix. Perform the operation \( a_{n-3,1} = a_{n-3,1} \), and remove the new last row into the first position. We obtain:

\[ M = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\end{bmatrix} \]

It is the matrix of the type \( \frac{n(n-3)}{2} + 2 \) from the list as we need. Let \( a_{n-1,1} = 0 \), and \( a_{n-3,1} \neq 0 \) in the previous matrix. Perform the operation \( a_{n-1,1} = a_{n-1,1} \), and the operations \( a_{1,2} = a_{1,2} - a_{2,1} a_{1,1} \), \( a_{2,3} = a_{2,3} - a_{3,1} a_{2,1} \), \ldots, \( a_{n-2,1} = a_{n-2,1} - a_{n-1,1} a_{n-2,1} \) after the first one. We have:
It is the matrix of the $\left(\frac{n(n-1)}{2}\right)$ type from the list as we need. This case is proved, and the total proof is done.

**Remark 2:** Of course, the list of matrices in Theorem 1 doesn’t contain all of them. But any missed matrix can be restored using Ladder Principle. For each subsequence of matrices (between ;) sings in the list, imagine the ladder from the lower right corner to the upper left corner. Take the leftmost columns with arbitrary components, and make 1 step up along the ladder bringing this column up and to the left of the previous position. Fix elements 0 and only one element 1 at the corresponding positions in the released column. The next matrix from the list will be done.

As an obvious consequence of Theorem 1, we obtain the following statement.

**Theorem 2:** Each basis for $(n-2)$-dimensional subspaces of an $n$-dimensional vector space $(\mathbb{R}^n)$ is equivalent to one and only one canonical basis from the following list.

1. $a_1 = c_1 + a_{3,1} c_{3,1} + a_{5,1} c_{5,1} + \ldots + a_{n,1} c_{n,1}$
2. $a_1 = c_1 + a_{2,1} c_{2,1} + a_{3,1} c_{3,1} + \ldots + a_{n,1} c_{n,1}$

Conclusion

Results of this article are ready to be used at any research concerning subalgebras and ideals of noncommutative algebras. Classification of canonical bases for $(n-2)$-dimensional subspaces is very effective to study reductive subalgebras and reductive pairs of any $n$-dimensional Lie algebra.

References

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