Well–Poised Macdonald Functions $W_\lambda$ and Jackson Coefficients $\omega_\lambda$ On $BC_n$

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Abstract. The very well–poised elliptic Macdonald functions $W_{\lambda/\mu}$ in $n$ independent variables are defined and their properties are investigated. The $W_{\lambda/\mu}$ are generalized by introducing an extra parameter to the elliptic Jackson coefficients $\omega_{\lambda/\mu}$ and their properties are studied. $BC_n$ multivariable Jackson sums in terms of both $W_\lambda$ and $\omega_\lambda$ functions are proved.

1. Introduction

A family of symmetric elliptic functions $W_\lambda$ associated to the root system $BC_n$ are introduced which extend the symmetric Macdonald polynomials $M_\lambda$ and Okounkov’s $O_\lambda$ symmetric interpolation polynomials. These $W_\lambda$ functions satisfy a number of identities which generalize important identities for classical very-well-poised basic hypergeometric series, such as the Jackson or $q$-Dougall summation theorem and Bailey’s $10\phi_9$ transformation. The symmetric elliptic $\omega_\lambda$ function which extends $W_\lambda$ by an additional parameter is also defined in this paper. However, the basic (trigonometric) version of $\omega_\lambda$ functions, together with the basic $10\phi_9$ transformation are first given in $[C1]$. Some of these results are proved independently and by different methods by Rains $[R1]$.

The genesis of the $W_\lambda$ functions goes back to a family of symmetric interpolation polynomials defined by Biedenharn and Louck $[BL]$. The proof given by Biedenharn and Louck for the symmetry of their polynomials depends on a well-known hypergeometric transformation formula for balanced terminating $4F_3$ hypergeometric series. By using Bailey’s $10\phi_9$ transform, it was possible to include two additional parameters to their functions in the two variable case, obtaining the two variable (trigonometric) $W_\lambda$ functions. The general $W_\lambda$ functions were then studied as a means to generalize these classical hypergeometric methods to the context of multivariate hypergeometric series. By using the elliptic version of the Bailey’s $10\phi_9$ transformation as given in Frenkel and Turaev $[FT]$, it was easily seen that the same approach generalized to the elliptic case. This extension of the trigonometric $W_\lambda$ and $\omega_\lambda$ to the elliptic case is conjectured first in $[C1]$. Following

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Frenkel and Turaev, there has been much work done recently in studying different generalizations of hypergeometric series involving elliptic functions, e.g. [HR], [SZ] and [W2].

The initial motivation for the basic (trigonometric) version of $\omega_\lambda$ was to prove the $BC_n$ Bailey Lemmas given in [C1]. Bailey Lemma is a powerful iterative method classically used for proving basic hypergeometric series identities (see [AAB], [B1], [AAR], [GIS], [LM], [Sp], [W1], [CG] and [C2] for example). However, as described in Theorem 3.3 properties of each function $W_\lambda$ or $\omega_\lambda$ are used in the proof of the properties of the other. For example, that $W_\lambda(x_1, \ldots, x_n; q, p, t, a, b)$ is symmetric follows from the symmetry of $\omega_\lambda(x_1, \ldots, x_n; r, q, p, t, a, b)$ in $x_i$ variables.

The paper is organized as follows. At the start of §2 we give a definition (Definition 2.1) characterizing the symmetric elliptic $W_\lambda$ functions as solutions of a multivariate interpolation problem. This definition is analogous to that given by Okounkov [O2] for the symmetric interpolation polynomials associated to the root system $C_n$. In [A1], a combinatorial definition of the elliptic $W_\lambda$ functions is given in definitions (2.11), (2.12), and (2.14). Some of the basic properties of these combinatorial $W_\lambda$ functions is also proved. In [A2], a definition (Def. 2.11) of multiple very-well-poised elliptic hypergeometric series is given. The definition of the elliptic $\omega_\lambda$ or more generally the $\omega_{\lambda/\mu}$ functions $[E1]$ is also given in this section and some of their properties are proved. In particular, $\omega_\lambda$ function is characterized in terms of certain multiplication and shift operators, analogous to the characterization of Biedenharn and Louck’s factorial Schur functions given by Goulden and Hamel [GH]. It is also shown that $\omega_\lambda$ is independent of different representations of $\lambda$.

In [A2] Theorem 3.3 is proved by induction in nine steps. The first steps (2.5)–(2.10) establish that the combinatorial $W_\lambda$ functions satisfy the conditions given in Definition 2.1. Other steps include the proof of the $W$–Jackson sum (3.6), a $BC_n$ generalization of elliptic Bailey’s $10\varphi_9$ transformation (3.31), and an elliptic cocycle identity for the $\omega$ functions. There is also a discussion of a $BC_n$ generalization of the Bailey Transform, which is a key algorithm used to generate classical hypergeometric series identities (see [An], [AAR], [B2], [LM], [CG] and [C2]). Last few results mentioned here are proved by easily extending the methods developed for the basic (trigonometric) case in [C1] to the elliptic case.

Finally, [A2] gives a proof of the $\omega$–Jackson sum (3.12) which extends the $W$–Jackson sum by an additional parameter $r \in \mathbb{C}$. The proof uses the symmetry and vanishing properties of $\omega_{\lambda/\mu}$ and a key identity (3.10) for $\omega_{\lambda/\mu}(x \delta(m); r, q, p, t; a, b)$. The $\omega$–Jackson sum reduces to the $W$–Jackson sum in the limit as $r \to t$. The basic (trigonometric) case of $\omega$–Jackson sum (3.12) is proved in [C1]. We finish this section by listing further important properties of the $\omega_\lambda$ function.

2. Elliptic Macdonald Functions And Elliptic Jackson coefficients

We recall some terminology from the theory of basic hypergeometric series. For $a, q \in \mathbb{C}, |q| < 1$ and integer $m$, we define the basic factorial $(a; q)_\infty := \prod_{i=0}^{\infty}(1-aq^i)$ and $(a; q)_m := (a; q)_\infty/(aq^m; q)_\infty$. We also define an elliptic analogue of the basic factorial as follows. For $x, p \in \mathbb{C}$ and $|p| < 1$, let the elliptic function be given by

\begin{equation}
E(x) = E(x;p) := (x;p)_\infty(p/x;p)_\infty
\end{equation}
and for \( a \in \mathbb{C} \), and a positive integer \( m \) define

\[
(a; q, p)_m := \prod_{k=0}^{m-1} E(aq^m) 
\]

The definition is extended to negative \( m \) by setting \((a; q, p)_m = 1/(aq^m; q, p)^{-m} \). Note also that when \( p = 0 \), \((a; q, p)_m \) reduces to standard (trigonometric) \( q \)-Pochhammer symbol.

For any partition \( \lambda = (\lambda_1, \ldots, \lambda_n) \) and \( t \in \mathbb{C} \) we can also define

\[
(a)_{\lambda} = (a; q, p, t)_{\lambda} := \prod_{k=1}^{n} (at^{1-i}; q, p)_{\lambda_i}.
\]

Note that when \( \lambda = (\lambda_1) \) is a single part partition, then \((a; q, p, t)_{\lambda} = (a; q, p)_{\lambda_1} = (a)_{\lambda_1} \). We’ll also use the notation

\[
(a_1, \ldots, a_k)_{\lambda} = (a_1, \ldots, a_k; q, p, t)_{\lambda} := (a_1)_{\lambda} \cdots (a_k)_{\lambda}.
\]

Let \( x_1, \ldots, x_n \in \mathbb{C} \) be a set of variables and \( \mathbb{F} = \mathbb{C}(q, p, t, a, b) \) field of rational functions in the parameters \( q, p, t, a, b \in \mathbb{C} \). For positive integers \( i, k \) with \( 1 \leq i \leq n \), let \( V(i, k) \) be the vector space of functions in \( x_1, \ldots, x_n \) spanned over \( \mathbb{F} \) by 1 and

\[
\{1/E(bq^m x_i), 1/E(bq^m a^{-1} x_i^{-1}) | 1 \leq m \leq k \}
\]

Set \( V(i, 0) = \mathbb{F} \) and \( V(i) = \bigcup_{k \geq 0} V(i, k) \). Also let \( V_n[k] \) be the vector space over \( \mathbb{F} \) generated by the product \( V(1, k) V(2, k) \cdots V(n, k) \) and \( V_n \) the vector space generated by \( V(1) V(2) \cdots V(n) \).

Let \( \mathcal{S} \) be the hyperoctahedral group of symmetries generated by the permutations of the variables \( x_1, \ldots, x_n \) together with the inversions \( x_i \rightarrow 1/ax_i \) for \( 1 \leq i \leq n \). Then \( V^S_n[k] \) (respectively \( V^S_n \)) is the subspace of \( V_n[k] \) \((V_n)\) invariant under \( \mathcal{S} \).

At various times it may be necessary to assume that \( p, q, t, a, b \) do not take on special values or that \(|t| < 1 \). It will generally be clear when this is required.

Note the effect on \( V(i, k) \) of setting \( p = 0 \). Then \( V(i, k) \) reduces to a subspace of the field of rational functions \( \mathbb{C}(x_1, \ldots, x_n) \) over \( \mathbb{C}(q, t, a, b) \) spanned by 1 and

\[
\{1/(1-bq^m x_i), 1/(1-bq^m a^{-1} x_i^{-1}) | 1 \leq m \leq k \}
\]

After setting \( p = 0 \), we may also consider the limits \( b \rightarrow \infty \) or \( b \rightarrow 0 \). The effect on \( V(i) \) is essentially to shift the poles of \( 1/(1-bq^m x_i) \) and \( 1/(1-bq^m a^{-1} x_i^{-1}) \) to 0 or \( \infty \). Consequently, the elements of \( V^S_n \) will asymptotically tend toward symmetric polynomials in the variables \( x^{-1}_i \) and \( ax_i \).

We will now characterize a special basis of \( V^S_n \). This basis will extend, up to a change of variables and normalization, the \( BC_n \) type interpolation polynomials \( P^*_n(x; q, t, s) \) of Okounkov [1] and consequently also extend the shifted Schur functions of F. Knop [Kn], A. Okounkov [O2], G. Olshanski [OO] and S. Sahi [Sa]. These in turn extend the homogeneous Macdonald polynomials \( P_\lambda(x; q, t) \) [MH] and Schur functions.

**Definition 2.1**. Let \( \delta(n) = (t^{n-1}, t^{n-2}, \ldots, 1) \) and for \( n \)-part partition \( \lambda \) \( q^\lambda = (q^{\lambda_1}, \ldots, q^{\lambda_n}) \). Also define \( -\lambda = (-\lambda_1, -\lambda_{n-1}, \ldots, -\lambda_1) \) and set \( q^{-\lambda} \delta(n) = (q^{-\lambda_1} t^{n-1}, q^{-\lambda_{n-1}} t^{n-2}, \ldots, q^{-\lambda_1}) \). Define \( W_\lambda(x_1, \ldots, x_n; p, q, t, a, b) \) to be the element of \( V^S_n \) satisfying the following conditions:

\[
W_\lambda(x; q, p, t, a, b) \in V^S_n[\lambda_1]
\]
(2.6) \[ W_\lambda(q^{\mu}t^{\delta(n)}; q, p, t, a, b) = 0 \]
if \( \lambda \nsubseteq \nu \).

Setting

(2.7) \[ \tilde{W}_\lambda(x; q, p, t, a, b) = W_\lambda(x; q, p, t, a, b) \prod_{i=1}^{n} \frac{(q^{bx_i}, q^{ba-1}x_i^{-1})_{\lambda_i}}{(x_i^{-1}, ax_i)_{\lambda_i}}. \]

Then

(2.8) \[ \tilde{W}_\lambda(b^{-1}q^{-\nu}t^{\delta(n)}; q, p, t, a, b) = 0 \]
if \( \nu \nsubseteq \lambda \) and \( \nu \subseteq (\lambda) \).

(2.9) \[ W_\lambda(q^{\lambda}t^{\delta(n)}; q, p, t, a, b) = N(\lambda, n; q, p, t, a, b) \]
is a normalization constant to be specified below.

Neither the existence nor the uniqueness of the \( W_\lambda \) are obvious. We will construct the \( W_\lambda \) functions below satisfying conditions (1)–(4). The set of the \( W_\lambda \) functions as \( \lambda \) ranges over all \( n \)-partitions \( \lambda \) will form a basis of \( V_n^\mathbb{C} \) over \( \mathbb{F} \). The uniqueness of the \( W_\lambda \) functions will then follow from conditions (2)–(4).

Remark 2.2. The definition of the \( W_\lambda \) functions is similar to to Okounkov’s \( P_\lambda^* \) functions \cite{Okounkov2000}. The difference is that conditions (1) and (3) above are replaced by a multidegree condition \( \deg P_\lambda^*(x; q, t, s) \leq |\lambda| \) or \( \deg P_\lambda^*(x; q, t) \leq |\lambda| \) where \( |\lambda| = \lambda_1 + \ldots + \lambda_n \) is the weight of the \( n \)-part partition.

The normalization we will use for the \( W_\lambda \) functions is given as follows. Let \( \lambda = (\lambda_1, \ldots, \lambda_n) \) be a partition, then

(2.10) \[ N(\lambda, n; q, p, t, a, b) = W_\lambda(q^{\lambda}t^{\delta(n)}; q, p, t, a, b) \]
\[ = \prod_{k=1}^{n} \left\{ \frac{(q^{bt^{n-k}}, q^{t^{n-k}})_{\lambda_k}(at^{2n-2k})_{2\lambda_k}}{((a/b)t^{n-k}, at^{n-k})_{\lambda_k}(q^{bt^{n+1-2k}})_{2\lambda_k}} \right\} \]
\[ \cdot \frac{1}{a/(qb)} |\lambda| \prod_{1 \leq i < j \leq n} \frac{(qt^{j-1})_{\lambda_i-\lambda_j}}{(qt^{j-1})_{\lambda_i}} \frac{(q^{t^{j-1}})_{\lambda_i-\lambda_j}}{(qt^{j-1})_{\lambda_i}} \]

2.1. Combinatorial Definition of \( W_\lambda \) Functions. For positive integer \( n \), let \( \lambda = (\lambda_1, \ldots, \lambda_n) \), \( \mu = (\mu_1, \ldots, \mu_n) \) be partitions such that the skew partition \( \lambda/\mu \) is a horizontal strip; i.e. \( \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \lambda_n \geq \mu_n \). Also set \( \lambda_{n+1} = \mu_{n+1} = 0 \). For \( \lambda \) and \( \mu \) as above and \( b \in \mathbb{C} \), we define

(2.11) \[ H_{\lambda/\mu}(q, p, t, b) \]
\[ := \prod_{1 \leq i < j \leq n} \left\{ \frac{(q^{\mu_i-\mu_j-1}t^{j-i})_{\mu_i-1-\lambda_j}}{(q^{\mu_i-\mu_j-1}t^{j-i})_{\mu_j-1-\lambda_i}} \frac{(q^{\lambda_i+\lambda_j+1}t^{i-j})_{\mu_j-1-\lambda_j}}{(q^{\lambda_i+\lambda_j+1}t^{i-j})_{\mu_j-1-\lambda_i}} \right\} \prod_{1 \leq i < (j-1) \leq n} \frac{(q^{\mu_i+\lambda_j+1}t^{i-j-1})_{\mu_j-1-\lambda_j}}{(q^{\mu_i+\lambda_j+1}t^{i-j-1})_{\mu_j-1-\lambda_i}}. \]
and also for \( x \in \mathbb{C} \),

\[
W_{\lambda/\mu}(x; q, p, t, a, b) := H_{\lambda/\mu}(q, p, t, b) - \sum_{\nu < \lambda} W_{\lambda/\mu}(y; q, p, t, a, b) W_{\nu/\mu}(z_1, \ldots, z_\ell; q, p, t, a, b).
\]

We also set \( W_{\lambda/\mu}(x; q, p, t, a, b) = W_{\lambda}(x; q, p, t, a, b) \) where \( x \in \mathbb{C}^n \).

Note that when \( \lambda \) has only one part and \( x \in \mathbb{C} \), then

\[
W_{\lambda}(x; q, p, t, a, b) := \frac{(x^{-1}, ax)_{\lambda}}{(qbx, qb/(ax))_{\lambda}}
\]

is independent of the parameter \( t \).

Notice also the strange property that for any \( n \)-part partition \( \lambda \neq 0 \), and \( x \in \mathbb{C} \), \( W_{\lambda}(x; q, p, t, a, b) \) is not identically 1. In fact, \( W_{\lambda}(x; q, p, t, a, b) \) is not even a constant.

We’ll now discuss some properties of the combinatorially defined \( W_{\lambda} \) functions. The first proposition is an easy consequence of the definitions [11] and [12].

**Proposition 2.3.** Let \( \lambda \) be an \( n \)-part partition with \( \lambda_n \neq 0 \) and \( 0 \leq k \leq \lambda_n \) for some integer \( k \). Let \( x = (x_1, \ldots, x_n) \in \mathbb{C}^n \), then

\[
W_{\lambda_k}(x; q, p, t, a, b) = \prod_{j=1}^{[n/2]} \frac{(qbt^{n-2j})_{2k}}{(qbt^{-1-n+2j})_{2k}} \prod_{i=1}^{n} \frac{(x_i^{-1})_{k}(ax_i)_{k}}{(qbx_i)_{k}(qb/(ax_i))_{k}} \cdot W_{\lambda^-k}(xq^{-k}; q, p, t, aq^{2k}, bq^{2k})
\]

where \( k^n \) is the \( n \)-part partition all of whose parts equal the integer \( k \).

**Corollary 2.4.**

\[
W_{\lambda_k}(x; q, p, t, a, b) = \prod_{j=1}^{[n/2]} \frac{(qbt^{n-2j})_{2\lambda_j}}{(qbt^{-1-n+2j})_{2\lambda_j}} \prod_{i=1}^{n} \frac{(x_i^{-1})_{\lambda_i}(ax_i)_{\lambda_i}}{(qbx_i)_{\lambda_i}(qb/(ax_i))_{\lambda_i}}
\]
Remark 2.5. An alternative normalization of the $W_\lambda$ functions are given as follows:

\begin{equation}
W_\lambda(x; q, p, t, a, b) = \prod_{1 \leq i < j \leq n} \left\{ \frac{(t^{j-i})_{\lambda_i - \lambda_j} (qbt^{n-i-j})_{\lambda_i + \lambda_j}}{(t^{j-i+1})_{\lambda_i - \lambda_j} (qbt^{n-i-j+1})_{\lambda_i + \lambda_j}} \right\} W_\lambda(x; q, p, t, a, b)
\end{equation}

One advantage of this normalization is that it simplifies the above formula. We have

\begin{equation}
W_\lambda(x; q, p, t, a, b) = \prod_{i=1}^{n} \frac{(x_i^{-1})_k (ax_i)_k}{(qbx_i)_k (qbx_i/a)_k} W_{\lambda-k,n}(xq^{-k}; q, p, t, aq^{2k}, bq^{2k})
\end{equation}

Another property is the vanishing theorem for the combinatorially defined $W_\lambda$ function.

**Theorem 2.6.** Let $\pi = (\pi_1, \ldots, \pi_n)$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$ be $n$-partitions such that $\lambda \not\subseteq \pi$. We then have

\begin{equation}
W_\lambda(q^\pi t^\delta; q, p, t, a, b) = 0.
\end{equation}

**Proof.** By induction on $n$ and using the recursive definition (2.11) of the $W$ functions

\begin{equation}
W_\lambda(q^\pi t^\delta; q, p, t, a, b) = \sum_\nu W_{\lambda/\nu}(y_1; q, p, t, at^{2(n-l+1)}, bt^{n-l+1}) \cdot W_\nu(y_2; q, pt, a, b),
\end{equation}

where $y_1 = (q^{\pi_1 t^{l-2}}, q^{\pi_2 t^{l-2}}, \ldots, q^{\pi_{l-1}})$ and $y_2 = (q^{\pi_{l+1} t^{n-l-1}}, q^{\pi_{l+2} t^{n-l-1}}, \ldots, q^{\pi_n})$ and $l, 1 \leq l \leq n$ is chosen so that $\lambda_l > \pi_l$ (since $\lambda \not\subseteq \pi$).

Corollary 2.7. If $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a partition with $\lambda_{n-k+1} = \ldots = \lambda_n = 0$, where $1 \leq k \leq n$, and $x = (x_1, \ldots, x_{n-k}, t^\delta(k)) \in C^n$ and $t^\delta(k) = (t^{k-1}, \ldots, 1)$, then set $\hat{\lambda} = (\lambda_1, \ldots, \lambda_{n-k})$ and $\hat{x} = (x_1, \ldots, x_{n-k})$. We have

\begin{equation}
W_\lambda(x; q, p, t, a, b) = W_{\lambda}(\hat{x}t^{-k}; q, p, t, at^{2k}, bt^k).
\end{equation}

**Proof.** Use the recurrence relation

\begin{equation}
W_\lambda(x; q, p, t, a, b) = \sum_{\mu < \lambda} W_{\lambda/\mu}(\hat{x}t^{-k}; q, p, t, at^{2k}, bt^k) W_\mu(t^\delta(k); q, p, t, a, b).
\end{equation}

By Theorem 2.6, the only non–vanishing term in (2.23) is when $\mu = 0^k$. The result follows after observing that $W_{0^k}(t^{-\delta(k)}; q, p, t, a, b) = 1$ and

\[W_{\lambda}/0^k(\hat{x}t^{-k}; q, p, t, at^{2k}, bt^k) = W_{\lambda}(\hat{x}t^{-k}; q, p, t, at^{2k}, bt^k).\]

A further property of the combinatorially defined $W_\lambda$ function corresponds to a property of Okounkov’s BC-type interpolation polynomials (11):

\begin{equation}
P_\mu^*(1/x_1, \ldots, 1/x_n; 1/q, 1/t, 1/s) = s^{2|\mu|} t^{2n-2|\mu|} P_\mu^*(x_1, \ldots, x_n; q, t, s).
\end{equation}

We have
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**Proposition 2.8.**

(2.25) $W_\lambda(x_1^{-1}, \ldots, x_n^{-1}; q^{-1}, p, t^{-1}, a^{-1}, b^{-1})
= a^{-2|\lambda|} b^{2|\lambda|} q^{2|\lambda|} t^{2n(\lambda) - 2(n-1)|\lambda|} W_\lambda(x_1, \ldots, x_n; q, p, t, a, b)$

**Proof.** One checks that for skew $W$ functions of one variable $x \in \mathbb{C}$, we have

(2.26) $W_{\lambda/\mu}(x^{-1}; q^{-1}, p, t^{-1}, a^{-1}, b^{-1})
= (qba^{-1})^{2|\lambda|-2|\mu|} t^{2n(\lambda)-2n(\mu)-2|\mu|} W_{\lambda/\mu}(x; q, p, t, a, b)$

and the result follows by the induction using the recursive definition (2.14). \qed

The next result is an analog for the $W_\lambda$ functions of the reversal formulas for basic hypergeometric series.

**Proposition 2.9.** Let $\lambda \subseteq N^n$ be a partition and $x \in \mathbb{C}^n$. Let $\tilde{\lambda} = N^n - \lambda^r = (N - \lambda_n, \ldots, N - \lambda_1)$. Then

(2.27) $W_\lambda(x; q, p, t, a, b) = \prod_{1 \leq i < j \leq n} \frac{(qbt^{n-i-j+1}, q^{1-N}b^{-1}t^{n-i-j})_{\lambda_i+\lambda_j}}{(qbt^{n-i-j}, q^{1-N}b^{-1}t^{n-i-j+1})_{\lambda_i+\lambda_j}}
\cdot (qba^{-1})^{2|\lambda|} \prod_{i=1}^n \frac{(x_i^{-1}, ax_i)_N}{(qbx_i, qb/(ax_i))_N} W_\lambda(bq^N x; q, p, t, ab^{-2}q^{-2N}, b^{-1}q^{-2N})$

which using the alternate normalization can be written as

(2.28) $W_\lambda^*(x; q, p, t, a, b) = W_{\lambda/n}(x; q, p, t, a, b)
\cdot (qb/2a)^{2|\lambda|} W_\lambda(bq^N x; q, p, t, ab^{-2}q^{-2N}, b^{-1}q^{-2N})$

**Proof.** Direct computation. \qed

We finally prove the following

**Proposition 2.10.** $W_\lambda(x; q, p, t, a, b)$, where $x = (x_1, \ldots, x_n) = q^u = (q^{u_1}, \ldots, q^{u_n}) \in \mathbb{C}^n$, is elliptic in each of the variables $u_1, \ldots, u_n \in \mathbb{C}^n$ and $0 < q < 1$

**Proof.** It will suffice to show that for any variable $x_i, 1 \leq i \leq n$ substituting $px_i$ in place of $x_i$ leaves the $W_\lambda(x; q, p, t, a, b)$ function invariant. From the definition (2.14) of the $W$ function this reduces to considering the skew function in one variable $W_{\lambda/\mu}$. Using the definition of $W_{\lambda/\mu}$ given in (2.12) and (2.11), consider the result of replacing the variable $x$ by $px$ in the factor

(2.29) $\frac{((px)^{-1}, apx)_\lambda}{(qbp, qb/(apx))_\lambda} = \frac{((x)^{-1}, ax)_\lambda}{(qbx, qb/(ax))_\lambda}$

and a similarly

(2.30) $\frac{(qbx/t, qb/(apxt))_\mu}{((px)^{-1}, apx)_\mu} = \frac{(qbx/t, qb/(axt))_\mu}{((x)^{-1}, ax)_\mu}$

Hence $W_{\lambda/\mu}$ is invariant under replacing $x$ by $px$ and $W_\lambda(x; q, p, t, a, b)$ is invariant under replacing $x_i$ by $px_i$. \qed
2.2. Multiple Elliptic Hypergeometric Series. We’ll now define an analogue of terminating very–well–poised elliptic hypergeometric series on $BC_n$.

**Definition 2.11.** Let $a_i$ be non–zero complex parameters for $i \in [k-1]$ where $k \geq 5$ is an integer, and $\mu$ and $\nu$ be partitions such that $\ell(\nu) \leq n$ and $\mu \subseteq \nu$. Then set

$$
(2.31) \quad \Phi_k^{\mu}[a_1, a_2, \ldots, a_{k-3}; a_{k-2}, \nu]_{q,p,t} = \sum_{\lambda} K_\lambda(a_1) \frac{[a_1 t^{1-n}, a_2, \ldots, a_{k-3}]_\lambda}{[a_{k-1}, t^{1-n}, t^{-1}, q a_1 a_2, \ldots, q a_1 a_{k-3}]\lambda} \cdot W_{\lambda}(q^\nu t^\delta(n); q, p, t, a_{k-2} t^{2-2n}, a_{k-1} t^{1-n}) \cdot W_{\mu}(q^\lambda t^\delta(n); q, p, t, a_1 t^{2-2n}, a_{k-1} t^{1-n})
$$

where

$$
(2.32) \quad K_\lambda(a_1) := \prod_{i=1}^{n} \left\{ \frac{E(a_1 t^{2-2i} q^2\mu)}{E(a_1 t^{2-2i})} (q t^{2i-2})^\mu_i \right\} \cdot \prod_{1 \leq i < j \leq n} \left\{ \frac{(q t^{2i} t^{-i-j}) A_{\lambda_i - \lambda_j} (a_1 t^{2i-j})}{(q t^{2j-i-1}) A_{\lambda_i - \lambda_j} (a_1 t^{2j-i-1})} \right\}
$$

and $\subseteq$ denotes the partial inclusion ordering defined by

$$
(2.33) \quad \mu \subseteq \lambda \iff \mu_i \leq \lambda_i, \quad \forall i \geq 1.
$$

We’ll sometimes suppress the $q, p, t$ dependence. We occasionally drop $n$ in the notation as well, for the series does not depend on $n$ as we verify below. In particular when $\mu = 0$, we simplify the notation for the left hand side of $(2.31)$ and write

$$
(2.34) \quad \Phi_k[a_1, a_2, \ldots, a_{k-3}; a_{k-2}, \nu] := \Phi_k^{\mu}[a_1, a_2, \ldots, a_{k-3}; a_{k-2}, \nu]_{q,p,t}
$$

Note that the series $(2.31)$ and $(2.34)$ are very well–poised, in the sense that the denominator parameters have the form $qa_1/a_i$, where $a_i$ are the numerator parameters, and the very–well–poised factor is located inside the factor $K_\lambda(a_1)$. We’ll say that the series $(2.31)$ and $(2.34)$ are balanced, if the product of the denominator parameters is $q t^{n-1}$ times that of numerator parameters.

**Remark 2.12.** The definition $(2.31)$ is a direct extension of the multiple basic hypergeometric series given in [C1] to the elliptic case.

Just as for the trigonometric version given in [C1], the elliptic Jackson coefficients $\omega_\lambda(x; r, q, p, t, a, b)$ are first defined explicitly in the case of one variable $\omega_\lambda/\mu$ function associated to a skew partition $\lambda/\mu$. Then the multivariable $\omega_\lambda$ function is defined in terms of the $\Omega$ algebra generated by certain shift and multiplication operators on the $\mathbb{Z}$–space $V$ of infinite lower–triangular matrices defined in

**Definition 2.13.** Let $F = \mathbb{C}(q, p, t, r, a, b)$ and let $F_\infty = F(X)$ denote the extended field of of rational functions over $F$ in the infinite set of indeterminates $X = \{x_1, x_2, \ldots\}$. We use $F(z)$ for the case when $z = (x_1, \ldots, x_n)$ is arbitrary
number of variables \( n \in \mathbb{Z}_{\geq 0} \). Then \( V \) denotes the \( \mathbb{Z} \)-space of all infinite lower-triangular matrices indexed by partitions, whose entries are from \( \mathbb{F}_{\infty} \). The condition that \( u \in V \) is lower triangular with respect to the partial inclusion ordering can be phrased in the form

\[
(2.35) \quad u_{\lambda \mu} = 0, \quad \text{when } \mu \not\subseteq \lambda
\]

The addition operation is defined in the standard form as

\[
(2.36) \quad (u + v)_{\lambda \mu} := u_{\lambda \mu} + v_{\lambda \mu},
\]

and the multiplication operation is defined by the relation

\[
(2.37) \quad (uv)_{\lambda \mu} := \sum_{\mu \subseteq \nu \subseteq \lambda} u_{\lambda \nu} v_{\nu \mu}
\]

for \( u, v \in V \). This operation is clearly associative since the same matrix entries enter into the double sum either way we sum it.

We now give the definition of the elliptic Jackson coefficients. Recall that if \( \mu \subseteq \lambda \), the set theoretic difference \( \lambda - \mu \) is called a skew (diagram) partition. A skew partition is called a horizontal strip if it satisfies the betweenness condition \( \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \).

**Definition 2.14 (Elliptic Jackson Coefficients).** Let \( \lambda \) and \( \mu \) be partitions of at most \( n \)-parts such that \( \lambda/\mu \) is a skew partition. Then the Jackson coefficients \( \omega_{\lambda/\mu} \) are defined by

\[
(2.38) \quad \omega_{\lambda/\mu}(x; r, q, p, t; a, b) := \frac{(x^{-1}, ax)_\lambda}{(qbx, qb/ax)_\lambda} \frac{(qbr^{-1}x, qb/a_x)_\mu}{(x^{-1}, ax)_\mu} \\
\cdot \frac{(r, br^{-1}t^j^1-n)_\mu}{(qbr^{-2}t, qt^{-n-1})_\mu} \prod_{i=1}^{n} \left\{ \frac{E(br^{-1}t^{2-2i}q^{2\mu_i})}{E(br^{-1}t^{2-2i})} \right\} \\
\cdot \prod_{1 \leq i < j \leq n} \left\{ \frac{(qt^{j-i-1})_{\mu_i-\mu_j}}{(qt^{-j-i})_{\mu_i-\mu_j}} \right\} W_{\mu}(q^\lambda \delta(n); q, p, t, bt^{2-2n}, br^{-1}t^1-n)
\]

where \( x, r, q, p, t, a, b \in \mathbb{C} \) are free complex parameters.

Note that \( \omega_{\lambda/\mu}(x; r; a, b) = \omega_{\lambda/\mu}(x; r, q, p, t; a, b) \) is defined when the skew partition \( \lambda/\mu \) is not a horizontal strip.

Defining \( \mathbb{W}(x_i) = \mathbb{W}(x_i; r, q, p, t, a, b) \) for each \( i \in [n] \) as the infinite matrix whose \( (\lambda \mu) \)-entry is \( \omega_{\lambda/\mu}(x_i; r, q, p, t, a, b) \) we introduce the \( \Omega \) algebra that produces the recursion formula for the Jackson coefficients \( \omega_{\lambda} \). The recurrence formula we obtain is completely analogous to that for \( W_{\lambda} \) functions.

**Definition 2.15.** With notation as above, let \( x_i \) denote the linear operator on \( V \) defined as the right matrix multiplication by \( \mathbb{W}(x_i) \), for each indeterminate \( x_i \), and let \( \mathcal{I} \) denote the identity operator on the same space. Also define the linear operator \( \mathfrak{M} \) on \( V \) by

\[
(2.39) \quad (\mathfrak{M}v(X; r, q, p, t, a, b))_{\lambda \mu} := v(r^{-1}X; r, q, p, t, ar^2, br)_\lambda
\]

for any \( v \in V \) where \( r^{-1}X = \{r^{-1}x_1, r^{-1}x_2, \ldots \} \). Then \( \Omega^n \) is the algebra generated by the composite operators

\[
(2.40) \quad \mathbb{W}(x_i) := x_i \mathfrak{M}
\]

for all \( i \in [n] \) and \( \mathcal{I} \), where the multiplication operation is composition.
Consider the algebra representation $\pi : \Omega^n \mapsto \text{End} V$ of $\Omega^n$ on the space $V$ defined by
\begin{equation}
\pi(f)v = fv
\end{equation}
for $f \in \Omega^n$ and $v \in V$. Then for every linear combination (which is a polynomial due to the symmetry proved below) $f$ in operators $\mathfrak{M}(x_1), \mathfrak{M}(x_2), \ldots, \mathfrak{M}(x_n)$ in $\Omega^n$, there exists an infinite lower triangular matrix associated to it, namely $\pi(f)I = fI$.

This leads to our definition of the recursion formula for $\omega_\lambda$.

**Definition 2.16.** With the notation as above and $z = (x_1, \ldots, x_n)$, we set
\begin{equation}
\mathbb{W}(z) := \mathfrak{M}(z^t) \cdot I = \mathfrak{M}(r_1 \cdot x_1) \cdots \mathfrak{M}(r_1 \cdot x_{n-1}) \mathfrak{M}(x_n)
\end{equation}
where $z^t = (x_n, x_{n-1}, \ldots, x_1)$ and $\mathfrak{M}(z^t) := \prod_{i=1}^n \mathfrak{M}(x_{n-i+1})$.

Entrywise writing of (2.42) gives the recursion formula for the Jackson coefficients
\begin{equation}
\omega_{\lambda/\tau}(y, z; r; a, b) := \sum_\mu \omega_{\lambda/\mu}(r^{-k}y; r; ar^{2k}, br^k) \omega_{\mu/\tau}(z; r; a, b)
\end{equation}
where $y = (x_1, \ldots, x_{n-k}) \in \mathbb{C}^{n-k}$ and $z = (x_{n-k+1}, \ldots, x_n) \in \mathbb{C}^k$.

Using the recurrence relation (2.43) we can extend the definition of $\omega_{\lambda/\mu}(x; a, b)$ from the single variable $x \in \mathbb{C}$ case to the multivariable $\omega_{\lambda/\mu}(z; r; a, b)$ case with arbitrary number of variables $z = (x_1, \ldots, x_n) \in \mathbb{C}^n$. The fact that multivariable $\omega_{\lambda/\mu}(z; r; a, b)$ is symmetric will be proved below.

We set $\omega_\lambda(z; r; a, b) := \omega_{\lambda/0}(z; r; a, b)$. In the case of one variable $x \in \mathbb{C}$ we get
\begin{equation}
\omega_\lambda(x; r; a, b) := \omega_{\lambda/0}(x; r, q, p, t; a, b) = \frac{(x^{-1})_\lambda(ax)_\lambda}{(qbz)_\lambda(qb/ax)_\lambda}
\end{equation}
which is independent of $r$.

The definition of the elliptic Jackson coefficients directly implies the following properties.

**Lemma 2.17.** For $n$–part partitions $\lambda$ and $\mu$, if $\mu \not\subset \lambda$, then
\begin{equation}
\omega_{\lambda/\mu}(x; r; a, b) = 0.
\end{equation}

**Proof.** The proof follows from the vanishing property (2.20) of $W_\lambda$ function which says that for an $n$–part partitions $\lambda$, $W_\mu(q^n \Delta_\lambda^{\mu}); q, p, t, a, b) = 0$ if $\mu \not\subset \lambda$. \(\square\)

We next establish the fact that the definition of $\omega_{\lambda/\mu}$ is invariant under any representation of $\lambda$. More precisely, we have

**Lemma 2.18.** Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be an $n$–part partition with $\lambda_{n-m+1} \neq 0$ and $\lambda_{n-m+1} = \cdots = \lambda_n = 0$ where $1 \leq m \leq n$. Then we have
\begin{equation}
\omega_{\lambda/\mu}(x; r; a, b) = \omega_{\hat{\lambda}/\hat{\mu}}(x; r; a, b).
\end{equation}
where $\hat{\lambda}$ and $\hat{\mu}$ denotes the $(n-m)$–part partition obtained by dropping the last $m$ zero parts from $\lambda$ and $\mu$ respectively. That is, $\hat{\lambda} = (\lambda_1, \ldots, \lambda_{n-m})$ and $\hat{\mu} = (\mu_1, \ldots, \mu_{n-m})$.

**Proof.** We first recall (2.22) that for an $n$–part partition $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ with $\nu_{n-m+1} = \cdots = \nu_n = 0$ where $1 \leq m \leq n$, we have
\begin{equation}
W_\nu(z; q, p, t, a, b) = W_{\hat{\nu}}(yt^{-m}; q, p, t, at^{2m}, bt^m)
\end{equation}
where \( \nu' = (\nu_1, \ldots, \nu_{n-m}) \) and \( z = (y, t^{(m)}) \in \mathbb{C}^n \) with \( y \in \mathbb{C}^{n-m} \).

Assume that \( \mu \subseteq \lambda \), for otherwise \( \omega_{\lambda/\mu} \) will vanish by Lemma \((2.17)\). Then if we set \( z = q^{\lambda} t^{(n)} \) and \( y = (q^{\lambda_1} t^{n-1}, \ldots, q^{\lambda_{n-m}} t^m) \) and use the property discussed above, we get

\[
W_{\mu}(q^{\lambda} t^{(n)}; q, p, t, bt^2 - 2n, br^{-1} t^{1-n}) = W_{\mu}((y, t^{(m)}); q, p, t, bt^2 - 2n, br^{-1} t^{1-n}) = W_{\mu}(q^{\lambda} t^{(n-m)}; q, p, t, bt^2 - 2(n-m), br^{-1} t^{1-(n-m)})
\]

It remains to show that the factor

\[
\prod_{1 \leq i < j \leq n} \left\{ \frac{(qt^{j-i})_{\mu_i - \mu_j}}{(qt^{i-j})_{\mu_i - \mu_j}} \frac{(br^{-1} t^{3-i-j})_{\mu_i + \mu_j}}{(br^{-1} t^{2-i-j})_{\mu_i + \mu_j}} \right\}
\]

is also invariant under any representation of \( \lambda \). It is clear that for \( k < n \) we have

\[
\prod_{1 \leq i < j \leq n} c_{ij} = \prod_{1 \leq i < j \leq k} c_{ij} \prod_{1 \leq k < j \leq n} c_{ij} \prod_{k < i < j \leq n} c_{ij}
\]

for an array of objects \( c_{ij} \). If we set \( k = n - m \) and

\[
c_{ij} := \left\{ \frac{(qt^{j-i})_{\mu_i - \mu_j}}{(qt^{i-j})_{\mu_i - \mu_j}} \frac{(br^{-1} t^{3-i-j})_{\mu_i + \mu_j}}{(br^{-1} t^{2-i-j})_{\mu_i + \mu_j}} \right\}
\]

we see that the last factor on the right hand side of the product \((2.50)\) drops, because \( \mu_i - \mu_j = \mu_i + \mu_j = 0 \) for all \( k < i < j \leq n \). For the second factor in \((2.50)\) we have \( \mu_i - \mu_j = \mu_i \), since \( \mu_j = 0 \) when \( k < j \leq n \). Therefore, this middle factor in the right hand side of \((2.50)\) becomes

\[
\prod_{1 \leq i \leq k < j \leq n} \left\{ \frac{(qt^{j-i})_{\mu_i}}{(qt^{i-j})_{\mu_i}} \frac{(br^{-1} t^{3-i-j})_{\mu_i}}{(br^{-1} t^{2-i-j})_{\mu_i}} \right\} = \frac{(br^{-1} t^{1-k})_{\mu} (qt^{n-1})_{\mu}}{(qt^{k-1})_{\mu} (br^{-1} t^{1-n})_{\mu}}
\]

so that we have

\[
\prod_{1 \leq i < j \leq n} c_{ij} = \prod_{1 \leq i < j \leq k} c_{ij} \cdot \frac{(br^{-1} t^{1-k})_{\mu} (qt^{n-1})_{\mu}}{(qt^{k-1})_{\mu} (br^{-1} t^{1-n})_{\mu}}
\]

Therefore the factor

\[
\frac{(br^{-1} t^{1-n})_{\mu}}{(qt^{n-1})_{\mu}} \prod_{1 \leq i < j \leq n} c_{ij} = \frac{(br^{-1} t^{1-k})_{\mu}}{(qt^{k-1})_{\mu}} \prod_{1 \leq i < j \leq k} c_{ij}
\]

is also invariant under any representation of \( \lambda \). Other factors of the form \((u)_{\lambda}\) or \((u)_{\mu}\) and the very well-poised factor depend only on the length \( \ell(\lambda) \) and \( \ell(\mu) \) of \( \lambda \) and \( \mu \) respectively, and not on \( n \).

This shows that the apparent \( n \) dependency of the definition \((2.33)\) is not essential in the sense that \( n \) could be any integer such that \( n \geq \ell(\lambda) \). Other properties of \( \omega_{\lambda/\mu} \) will be proved after the proof of the main theorem in the next section.
3. W–Jackson sum

We will prove in this section a number of the important properties of the elliptic combinatorial $W$ functions. We begin with a result that gives a special evaluation of the $W$ function, corresponding to a special evaluation of Okounkov’s $P^* \lambda$ functions [O1].

**Proposition 3.1.** For any $n$-part partition $\lambda$ and $a, b, q, p, t \in \mathbb{C}$ we have

\[(3.1) \quad W_\lambda(q^{\lambda} t^{\delta(n)}; q, p, t, a, b) = N(\lambda, n; q, p, t, a, b)\]

**Proof.** Equation (3.1) is a direct consequence of repeated applications of identities (2.16) and (2.22). □

In the proof of the main theorem in this section, we’ll use a generalization of a series transformation identity known as Bailey Transform as given in [C1]. It can be stated as follows: Let $\alpha, \delta$, and $m \in V$ where $V$ is defined as in Definition (2.13). That is, $\alpha$, $\delta$, and $m$ are infinite lower triangular matrices (doubly indexed array of objects) whose rows and columns are indexed by partitions with respect to the partial inclusion ordering (2.33). Denote by $\beta$ and $\gamma$ the product matrices $\beta = m \alpha$ and $\gamma = \delta m$, where the matrix multiplication in $V$ is defined by (2.37). Then it is obvious that $\gamma \alpha = \delta \beta$. Since this identity involves three matrices on both sides, we have two ways of summing the series due to the associativity of matrix multiplication. This observation is stated as a change of summation or der in

**Lemma 3.2 (Bailey Transform).** Suppose we are given three infinite lower triangular matrices $\alpha$, $\delta$ and $m$ in $V$. If we define $\beta$ and $\gamma$ to be matrices with entries

\[(3.2) \quad \beta_\lambda = \sum_{\tau \subseteq \mu \subseteq \lambda} m_{\lambda \mu} \alpha_{\mu \tau}, \quad \text{and} \quad \gamma_{\nu \lambda} = \sum_{\mu \subseteq \nu \subseteq \lambda} \delta_{\nu \mu} m_{\mu \lambda}\]

then we have

\[(3.3) \quad \sum_{\tau \subseteq \lambda \subseteq \nu} \gamma_{\nu \lambda} \alpha_{\lambda \tau} = \sum_{\tau \subseteq \lambda \subseteq \nu} \delta_{\nu \lambda} \beta_{\lambda \tau}\]

**Proof.** We have that

\[(3.4) \quad \sum_{\tau \subseteq \lambda \subseteq \nu} \gamma_{\nu \lambda} \alpha_{\lambda \tau} = \sum_{\tau \subseteq \lambda \subseteq \nu} \alpha_{\lambda \tau} \sum_{\mu \subseteq \nu \subseteq \lambda} \delta_{\nu \mu} m_{\mu \lambda}\]

\[= \sum_{\tau \subseteq \mu \subseteq \nu} \delta_{\nu \mu} \sum_{\tau \subseteq \lambda \subseteq \nu} m_{\mu \lambda} \alpha_{\lambda \tau} = \sum_{\tau \subseteq \mu \subseteq \nu} \delta_{\nu \mu} \beta_{\mu \tau}\]

where the second step follows from the associativity of matrix multiplication as explained above. □

In particular, if we set $\tau = 0$ and send $\nu \rightarrow \infty$ in (3.2) and (3.3), we get, in the one dimensional case, the statement of the classical Bailey Transform. Because of the limits involved, we require that certain convergence conditions are satisfied so that $\gamma$ is well–defined and the change of summation makes sense.

We now state one of the main theorems of the paper: a nine step theorem proved by induction on the number of parts (including zero parts at the end) of the partition $\lambda$.  

Well-poised Macdonald functions $W_\lambda$ and Jackson coefficients $\omega_\lambda$ on $BC_3$

**Theorem 3.3.** In the following assume partitions $\lambda$, $\nu$, $\tau$, $\mu$ have at most $n$ parts, $x \in \mathbb{C}$, and $p$, $q$, $t$, $r$, $s$, $a$, $a'$, $b$, $c$, $d$, $e$, $f$, $g \in \mathbb{C}$.

1. The combinatorial $W_\lambda$ function satisfy conditions $(3.4)$. 
2. A $W$ function symmetry identity:

\[
W_\lambda \left( k^{-1} q^{i'} t^\delta; q, p, t, k^2 a, k b \right) \frac{(q b t^{n-1})_\lambda (q b/a)_\lambda}{(k)_\lambda (k a t^{n-1})_\lambda} \prod_{1 \leq i < j \leq n} \left\{ \begin{array}{l}
(t^{j-i})_{\lambda_i - \lambda_j} (q a t^{2n-i-j-1})_{\lambda_i + \lambda_j} \\
(t^{j-i+1})_{\lambda_i - \lambda_j} (q a t^{2n-i-j})_{\lambda_i + \lambda_j}
\end{array} \right\} = W_\nu \left( h^{-1} q^{\lambda^2}; q, p, t, h^2 a', h b \right) \frac{(q b t^{n-1})_\nu (q b/a')_\nu}{(h)_\nu (h a t^{n-1})_\nu} \prod_{1 \leq i < j \leq n} \left\{ \begin{array}{l}
(t^{j-i})_{\nu_i - \nu_j} (q a t^{2n-i-j-1})_{\nu_i + \nu_j} \\
(t^{j-i+1})_{\nu_i - \nu_j} (q a t^{2n-i-j})_{\nu_i + \nu_j}
\end{array} \right\}
\]

where $k = a't^{n-1}/b$ and $h = at^{n-1}/b$.

3. A $W$ function Jackson sum:

\[
W_\lambda(s^{-1} x; q, p, t, a t^{-2n} s^2, b t^{-n}) = \frac{(s)_\lambda (a s t^{-n-1})_\lambda}{(q b t^{-1})_\lambda (q b t^{n})_\lambda} \prod_{1 \leq i < j \leq n} \left\{ \begin{array}{l}
(t^{j-i+1})_{\lambda_i - \lambda_j} (q b s t^{i-j+1})_{\lambda_i + \lambda_j} \\
(t^{j-i})_{\lambda_i - \lambda_j} (q b s t^{i-j})_{\lambda_i + \lambda_j}
\end{array} \right\} \sum_{\mu \leq \lambda} \frac{(bt^{-n})_\mu (qt^{n-1})_\mu}{(as t^{-n-1})_\mu} \prod_{i=1}^n \left\{ \frac{E(bt^{-1} q^{2n_i})}{E(bt^{1-2i})} (qt^{2i-2})_{\mu_i} \right\} \prod_{1 \leq i < j \leq n} \left\{ \begin{array}{l}
(t^{j-i})_{\mu_i - \mu_j} (qt^{j-i})_{\mu_i + \mu_j} \\
(q t^{j-i})_{\mu_i - \mu_j} (t^{j-i+1})_{\mu_i + \mu_j} (bt^{i-j+1})_{\mu_i + \mu_j} (qt^{i-j+1})_{\mu_i + \mu_j}
\end{array} \right\} W_\mu \left( q^{\lambda^2(n)}; q, p, t, b s t^{-2n}, b t^{-n} \right) \cdot W_\mu \left( x; q, p, t, at^{-2n}, bt^{-n} \right)
\]

4. A cocycle identity for $\omega$ functions:

\[
\omega_{\nu/\tau}((r s)^{-1}; r s; a(r s)^2, b r s) = \sum_{\tau \leq \lambda \leq \nu} \omega_{\nu/\lambda}(s^{-1}; s; a(r s)^2, b r s) \cdot \omega_{\lambda/\tau}(r^{-1}; r a r^2, b r)
\]

5. A $BC_n$ generalization of Bailey’s $10\Phi_9$ transformation

\[
(q b/g, q b/f, q r, q g/f g)_{\nu} (\gamma c/b, \gamma c/b)_{\nu} = \sum_{(c, e)} (q b, q b/f, q r, q g/f g)_{\nu} (\gamma c/b, \gamma c/b)_{\nu} \cdot 10\Phi_9 \left[ b, c, d, e, f, g; q r b, \nu \| b/d, \tau \right] = 10\Phi_9 \left[ \gamma, \gamma c/b, \gamma d/b, \gamma e/b, f, g; q r b/f g, \nu \| b/d, \tau \right]
\]

where $\gamma = q b^2/c d e$.

6. The multivariable $\omega_{\lambda/\mu}$ function is symmetric. Namely,

\[
\omega_{\lambda/\mu}(x_1, \ldots, x_k; r; a, b) = \omega_{\lambda/\mu}(x_{\sigma(1)}, \ldots, x_{\sigma(k)}; r; a, b)
\]

for any $\sigma \in S$, the symmetric group.
(7) A skew $\omega$ function special evaluation identity:

\[
\omega_{\lambda/\mu}(xr^{\delta(m)}; r; a, b) = \frac{(x^{-1}, axr^{m-1})_{\lambda}}{(br^{-1}x, qbx/(axr^{m}))_{\mu}} \cdot \prod_{i=1}^{n} \left\{ \frac{E(br^{-1}t^{2i-2}q^{2\mu_i})}{E(br^{-1}t^{2i-2})} \right\} \left\{ \frac{E(br^{-1}t^{2i-2-\mu_i})}{E(br^{-1}t^{2i-2})} \right\}
\]

where $x \in \mathbb{C}$ and $x^{r^{\delta(m)}} = (x_{r^{m-1}} \ldots, x, x)$ for a positive integer $m$.

(8) The skew $W$ function $W_{\lambda/\mu}(x, y; q, p, t, a, b)$ is symmetric in the variables $x, y \in \mathbb{C}$.

(9) An “extended” $W$ Jackson sum: i.e. the identity (3.3) for $n+1$ part partitions $\lambda = (\lambda_1, \ldots, \lambda_{n+1})$ with the last part $\lambda_{n+1} = 0$.

**Proof.** When $\lambda = (0)$, the identities in Theorem (3.3) are trivially true. We now do induction on $n$, the number of parts of $\lambda$ assuming that at least one part is nonzero.

Observe that in step one above, identities (2.6) and (2.10) have already been proven. Identity (2.6) is simply Theorem (2.6). Identity (2.10) is a consequence of the $W$ reversal identity (2.14) and Theorem (2.6). Finally Identity (2.9) is proved in Proposition (3.1). This leaves identity (2.5) to be proved. The combinatorial definition of the $W$ function given in (2.11), (2.12), and (2.14) shows that the function $W_{\lambda}(x_1, \ldots, x_n; q, p, t, a, b) \in V_n[\lambda_1]$. To further show that $W_{\lambda} \in V_n^{\mathbb{C}}[\lambda_1]$, we need to demonstrate that the $W_{\lambda}$ function is invariant in the variables $\{x_1, \ldots, x_n\}$ under the action of the group $S$. The difficult condition here is to show that the $W_{\lambda}$ function is invariant under the transpositions

\[x_i \leftrightarrow x_j, \quad 1 \leq i < j \leq n.\]

Using induction on $n$ and the $y \in \mathbb{C}^2$ case of the generalized $W$ recurrence identity

\[
W_{\lambda/\mu}(y, z_1, z_2, \ldots, z_\ell; q, p, t, a, b) = \sum_{\nu \subseteq \lambda} W_{\lambda/\nu}(yt^{-\ell}, q, p, t, at^{2\ell}, bt^{\ell}) W_{\nu/\mu}(z_1, \ldots, z_\ell; q, p, t, a, b),
\]

it will suffice to show that $W_{\lambda}(x_1, \ldots, x_n; q, p, t, a, b)$ is invariant under the transposition $x_1 \leftrightarrow x_2$. When $n = 1$ there is nothing to prove, so we assume $n \geq 2$. We can also assume that $\lambda_n = 0$, because if $\lambda_n > 0$ then by the $W_{\lambda}$ reduction identity (2.10) we can reduce to the case $\lambda_n = 0$. Using the recurrence identity (3.11), the symmetry of the $W_{\lambda}$ function in $x_1$ and $x_2$ follows from a proof of the symmetry of the function $W_{\lambda/\nu}(x_1 t^{-(n-2)}, x_2 t^{-(n-2)}; q, p, t, at^{2(n-1)}, bt^{n-1})$ in the variables $x_1$ and $x_2$, where assume that $\lambda_n = 0$ and hence $\nu_n = 0$. If $\lambda$ and $\nu$ are the $n$-part partitions obtained by dropping the $n$th part of $\lambda$ and $\nu$, we observe that

\[
W_{\lambda/\nu}(x_1 t^{-(n-2)}, x_2 t^{-(n-2)}; q, p, t, at^{2(n-2)}, bt^{n-2}) = W_{\lambda/\nu}(x_1 t^{-(n-2)}, x_2 t^{-(n-2)}; q, p, t, at^{2(n-2)}, bt^{n-2}).
\]
Applying the step seven (n-1 part partitions) inductive assumption to the right-hand-side of (3.13), we finish the proof that \( W_\lambda(x_1, \ldots, x_n; q, pt, a, b) \in V^S_\alpha[\lambda_1] \) and step one of the n part partitions case of Theorem 3.3.

To prove step two, identity (3.5), we will do a double induction, first on the last part \( \lambda_n \) of \( \lambda \), and finally on the last part \( \nu_n \) of \( \nu \). By the general inductive assumption for Theorem 3.3, we also assume that identity (3.5) is true when \( \lambda \) and \( \nu \) have only n-1 parts. The general inductive assumption also implies that the “extended” W Jackson sum (3.6) is true for n-part partitions \( \lambda \) with last part zero, \( \lambda_n = 0 \). Substituting \( at^n \) in place of \( a \) and \( bt^n \) in place of \( b \), and setting \( x = q^b t^{\delta(n)} \), \( s = k \), \( k = a't^{n-1}/b \) and \( h = a't^{n-1}/b \) in this “extended” W Jackson sum identity, the right-hand-side of the W Jackson sum (3.6) exhibits a symmetry between \( \lambda \), \( a \), \( k \) and \( \nu \), \( a' \), \( h \). This proves the n-part partition identity (3.5) when \( \lambda_n = \nu_n = 0 \). If \( \lambda_n > 0 \), then using identity (3.5) the left-hand side of (3.5) becomes

\[
(3.13) \quad \prod_{i=1}^{n} \frac{E(kq^{-i}t^{n-i})E(kat^{n-i})E(qbt^{n-i})E(qba^{-1}t^{1-i})}{E(k^{1-i})E(kat^{n-i})E(qbt^{n-i})E(qba^{-1}q^{i-1})} \cdot \prod_{1 \leq i < j \leq n} \frac{(t^{j-i})_{\lambda_i-\lambda_j}(q^3a't^{2n-i-j-1})_{\lambda_i+\lambda_j-2}}{(t^{j-i+1})_{\lambda_i-\lambda_j}(q^3a't^{2n-i-j})_{\lambda_i+\lambda_j-2}} \cdot \frac{(q^2b^t^{n-1})_{\lambda-1^n}(q^2b/a)_{\lambda-1^n}}{(q^k)_{\lambda-1^n}(qkat^{n-1})_{\lambda-1^n}} W_{\lambda-1^n}(k^{-1}q^tq^{-1}; q, p, t, k^2aq^2, kbq^2).
\]

where \( \lambda - 1^n \) is the partition \( (\lambda_1 - 1, \ldots, \lambda_n - 1) \) and we have simplified by using the identity

\[
(3.14) \quad \frac{(qkb(1-2i)^2)(qa't^{n-i-1})_{2}}{(qkb+t^2n-1-i)_{2}} = \frac{(qa't^{n+2i-2})_{2}}{(qa't^{n+i-1})_{2}} = \frac{(q^2b^t^{n-1}; t)_{n-i}(q^2a't^{n-1}; t)_{n-i}}{t_{n-i}} = 1.
\]

By induction on \( \lambda_n \), we can evaluate the \( W_{\lambda-1^n} \) function in (3.13) from the \( \lambda - 1^n \) case of (3.5) with \( b, k, h, a' \) replaced by \( qb, kq, q^{-1}h, qa' \), respectively and \( \nu \) and \( a \) unchanged. We obtain

\[
(3.15) \quad W_{k^{-1}q^tq^{-1}; q, p, \lambda-1^n}(t, k^2aq^2, kbq^2) = \frac{(q^2b^t^{n-1})_{\nu}(b/a')_{\nu}}{(q^{-1}h)_{\nu}(qhat^{n-1})_{\nu}} \prod_{1 \leq i < j \leq n} \frac{(t^{j-i})_{\nu_{i-\nu_j}}(qat^{2n-i-j})_{\nu_{i}+\nu_j}}{(t^{j-i+1})_{\nu_{i-\nu_j}}(qat^{2n-i-j-1})_{\nu_{i}+\nu_j}} W_{\nu}(h^{-1}t^3q^{-1}; q, p, t, h^2a', hb).
\]

Substituting identity (3.15) into the expression (3.5) and simplifying further, we prove identity (3.5) in the case \( \lambda_n > 0 \).

The case with \( \nu_n > 0 \) can be proven in exactly the same way. Therefore we reduce to the previously proven case \( \lambda_n = \nu_n = 0 \). This completes the proof of step two of Theorem 3.3.

We now prove induction step three of Theorem 3.3, the \( W \) function Jackson sum (3.6) in the case when the variable \( x \in \mathbb{C}^n \), treating the partition \( \lambda \) as an
n-tuple. We will do a further induction on $\lambda$ using reverse lexicographical order. We will use the notation $\mu < \lambda$ to denote $\mu$ is less than $\lambda$ in reverse lexicographical order (and similarly $\mu \leq \lambda$ also includes the case $\mu = \lambda$). The inductive assumption is that identity (3.6) is true whenever $\lambda < \gamma$ for some partition $\gamma$ with $l(\gamma) \leq n$. We now consider the case $\lambda = \gamma$ and proceed to show that identity (3.6) is true in this case as well. Note that we can begin our induction in the trivial case of the partition (0). We begin with the following

**Lemma 3.4.**

\begin{equation}
W_{\lambda}(s^{-1}; q, p, t, at^{-2n}s^2, bt^{-n}s) = \sum_{\mu \leq \lambda} c_\mu W_\mu(x; q, p, t, at^{-2n}, b),
\end{equation}

where $x \in \mathbb{C}^n$, $s \in \mathbb{C}$, $c_\mu \in \mathbb{F}$, and the sum is over partitions $\mu$ with $l(\mu) \leq n$ and $\mu \leq \lambda$.

**Proof.** Since $W_\lambda(s^{-1}; q, p, t, at^{-2n}s^2, bt^{-n}s) \in V^S_n[\lambda_1]$, it follows that

\begin{equation}
W_{\lambda}(s^{-1}; q, p, t, at^{-2n}s^2, bt^{-n}s) = \sum_{\mu} c_\mu W_\mu(x; q, p, t, at^{-2n}, bt^{-n}),
\end{equation}

where the sum is over partitions $\mu$ with $l(\mu) \leq n$ and $\mu_1 \leq \lambda_1$. We will denote the residue of any function $f \in V^S_n[\lambda_1]$ at the pole $x_n = (bt^{-n}q^{\lambda_1})^{-1}$ in the variable $x_n$ by $R_{\mu, \lambda_1}(f)$ and consider the residues of both sides of equation (3.17). First observe that $R_{\mu, \lambda_1}$ induces an injection

\begin{equation}
R_{\mu, \lambda_1} : (V^S_n[\lambda_1]/V^S_n[\lambda_1 - 1]) \longrightarrow V^S_{n-1}[\lambda_1].
\end{equation}

Next observe that

\begin{equation}
W^n_{\mu} = I_{n-1, \lambda_2}(W_{\lambda}^{(n-1)}(s^{-1}; q, p, t, at^{-2n}s^2, bt^{-n}s))
\end{equation}

\begin{equation}
= \prod_{i=1}^{n-2} (qbt^{-n}x_i)^{\lambda_2} (qbt^{-n}/(at^{-2n}x_i)^{\lambda_1})(bt^{-n}/(at^{-2n}x_i))^{\lambda_2} \cdot R_{\mu, \lambda_1}(W_{\lambda}^{(n-1)}(s^{-1}; q, p, t, at^{-2n}s^2, bt^{-n}s)) \in V^S_{n-2}[\lambda_2],
\end{equation}

which implies that, in expansion (3.16), the coefficients $c_\mu = 0$ unless $(\mu_1 < \lambda_1)$ or $(\mu_1 = \lambda_1$ and $\mu_2 \leq \lambda_2$). Iterating this procedure then proves expansion (3.16) and completes the proof of Lemma 3.4.

Using Lemma 3.4, the vanishing theorem (Theorem 2.6), and identity (3.6), we can inductively compute the coefficients $c_\nu$ in expansion the (3.16) for $W_{\nu}(s^{-1}; q, p, t, at^{-2n}s^2, bt^{-n}s)$. Let $\nu$ be a partition with $l(\nu) \leq n$ and $\nu < \lambda$. Suppose we have for $\mu < \nu$, that all the coefficients $c_\mu$ are equal to the corresponding coefficients of $W_\mu(x; q, p, t, at^{-2n}, bt^{-n})$ on the right-hand side of expansion (3.6). Setting $x = q^\delta t^{\delta(n)}$ in identity (3.6) will allow us to compute $c_\nu$. By Theorem 2.5, all the $W_\mu(x)$ terms on the right-hand side of (3.6) will vanish when $\nu < \mu$. Then the coefficient $c_\nu$ in the expansion
for $W_{\lambda}(s^{-1}x; q, p, t, at^{-2n}s^2, bt^{-n}s)$ can be computed by the identity \((3.6)\) and the induction hypothesis for \((3.6)\). Identity \((3.22)\) allows us to compute the expansion of $W_{\lambda}(s^{-1}x; q, p, t, at^{-2n}s^2, bt^{-n}s)$ at $x = q^{r^2(s(n))}$ by means of the expansion of $W_{\nu}(h^{-1}x; q, p, t, h^2asbt^{1-n}, hb)$ at $x = q^{\lambda^2(s(n))}$ where $h = at^{n-1}/b$. Since all the non-vanishing terms in $W_{\lambda}$ at $x = q^{r^2(s(n))}$ are known except for the $c_{\nu}$ term and all the terms of the expansion $W_{\nu}(h^{-1}x; q, p, t, h^2asbt^{1-n}, hb)$ are known by the induction hypothesis for \((3.6)\), then we can solve for $c_{\nu}$. Given the symmetry of the expansion of $W_{\lambda}(s^{-1}x; q, p, t, at^{-2n}s^2, bt^{-n}s)$ at $x = q^{r^2(s(n))}$, it is easily seen that the coefficient $c_{\nu}$ agrees with the corresponding coefficient in the $W$-Jackson sum \((3.6)\). By induction on the partitions $\nu$ with $l(\nu) \leq n$ and $\nu < \lambda$, it follows that all the coefficients $c_{\nu}$, $\nu < \lambda$, in the expansion of $W_{\lambda}(s^{-1}x; q, p, t, at^{-2n}s^2, bt^{-n}s)$ agree with the corresponding coefficients on the right-hand side of \((3.6)\). We are left to show that the leading coefficient $c_{\lambda}$ equals the coefficient of $W_{\lambda}(x; q, p, t, at^{-2n}, bt^{-n})$ in \((3.6)\). The last part of the proof of step three of Theorem \((3.3)\) is a direct computation of the coefficient $c_{\lambda}$. This can be done by computing the result $\bar{W}_{\lambda}(0)$ of composing the $n$ maps defined in \((3.19), (3.20), \ldots\), acting on $W_{\lambda}(s^{-1}x; q, p, t, at^{-2n}s^2, bt^{-n}s)$. In order to calculate the residues $R_{n-k, \lambda_k+1}$ for $k = 0, \ldots, n-1$ we apply the combinatorial definition of $W_{\lambda}(s^{-1}x)$ given in definitions \((2.11)-(2.14)\) and repeatedly apply the one-dimensional Jackson sum:

\[
(3.21) \quad \frac{(azs, z^{-1}s, qb, qb/a)_m}{(qb/(az), qbz, as, s)_m} = \sum_{k=0}^{m} \frac{E(q^{2k}b)(az, z^{-1}b, qb/(as), bsq^m, q^{-m})_k}{E(b)(qb/(az), qbz, q, as, q^{1-m}s^{-1}, bq^{1+m})_k} j^k,
\]

for any non-negative integer $m$. The result is, after some simplification,

\[
(3.22) \quad \bar{W}_{\lambda}(0) = s_{\lambda}^{(bt^{-n}, qbt^n/(as))_{\lambda}} \prod_{j=1}^{n} \frac{(bst^{-j}t^{2j})_{2\lambda_j}}{(bt^{-n}, qbt^n/a)_{\lambda}} \cdot \prod_{1 \leq i < j \leq n} \frac{(qbst^{-i-j}, qbt^{-i-j}, bt^{2-i-j}, bst^{-i-j})_{(\lambda_i, \lambda_j)}}{(qbst^{-i-j}, qbst^{-i-j}, bt^{2-i-j}, bst^{-i-j})_{(\lambda_i+\lambda_j)}}.
\]

We obtain the identical result from composing the $n$ maps in the right-hand side of \((3.6)\), using the identity \((3.22)\) applied to the factor $W_{\lambda}(q^{\lambda^2(s(n))})$ as well as the definitions \((2.11)-(2.14)\). This proves that $c_{\lambda}$ equals the coefficient of $W_{\lambda}(x)$ on the right-hand side of \((3.6)\) and completes the proof of step three of Theorem \((3.3)\).

We are now ready to prove the induction step four, the cocycle identity \((3.23)\), mentioned above. We first note that the Jackson sum \((3.6)\) can be written as

\[
(3.23) \quad W_{\lambda}(r^{-1}z; q, p, t, ar^2, br) = \sum_{\mu \subseteq \lambda} c_{\lambda/\mu}(r, a, b) W_{\mu}(z; q, p, t, a, b)
\]
where
\begin{equation}
(3.24)
c_{\lambda/\mu}(r, a, b) := \frac{[r]!}{[q]!} \frac{[a r t^{n-1}]_{\lambda}}{[q b r t^{n-1}]_{\lambda}} \frac{[b]_{\mu}}{[q b r t^{n-1}]_{\lambda}} \frac{[u, v, a b]}{[q t^{n-1}]_{\mu}} W_{\mu}(q^{\lambda\nu}(n); q, p, t, b r t^{1-n}, b)
\end{equation}

\begin{equation}
(3.26)
W_{\lambda}((uv)^{-1}; x; q, p, t, a (uv)^2, buv)
= \sum_{\mu \subseteq \lambda \subseteq \nu} c_{\lambda/\mu}(v^{-1}; a (uv)^2, buv) W_{\mu}(u^{-1}; x; q, p, t, au^2, bu)
\end{equation}

Proof. We start with the Jackson sum (3.23) in the form
\begin{equation}
(3.25)
c_{\nu/\mu}((uv)^{-1}; a (uv)^2, buv) = \sum_{\mu \subseteq \lambda \subseteq \nu} c_{\nu/\lambda}(v^{-1}; a (vu)^2, buv) c_{\lambda/\mu}(u^{-1}; au^2, bu)
\end{equation}

for any \( r \in \mathbb{C} \). We note that the identity (3.23) could be iterated repeatedly. A double iteration gives

Lemma 3.5. With the notation as above and \( u, v \in \mathbb{C} \), we have
\begin{equation}
(3.27)
\sum_{\tau} c_{\lambda/\tau}((uv)^{-1}; a (uv)^2, buv) W_{\tau}(x; q, p, t, a, b)
\end{equation}

Proof. Switch the order of summation and compare the coefficients of the basis functions \( W_{\tau}(x; a, b) \) to get the identity to be proved. 

Note that some factors in the expansion (3.27) cancel. We now rewrite the above theorem (3.25) in terms of the \( \omega_{\lambda/\mu} \) function in a way involving the essential factors and independent of different representations of \( \nu \) in

Corollary 3.6 (Cocycle Identity). Let \( \lambda \) be a partition. With the notation as above and \( u, v \in \mathbb{C} \), we have
\begin{equation}
(3.28)
\omega_{\nu/\mu}((uv)^{-1}; uv; a (uv)^2, buv)
= \sum_{\mu \subseteq \lambda \subseteq \nu} \omega_{\nu/\lambda}(v^{-1}; v; a (vu)^2, buv) \omega_{\lambda/\mu}(u^{-1}; u; au^2, bu)
\end{equation}

Proof. It suffices to note that
\begin{equation}
(3.29)
\omega_{\lambda/\mu}(u^{-1}; u; au^2, bu) = P_{\lambda}(b) c_{\lambda/\mu}(u, au^{-2} t^{-1}, bu^{-1} t^{-1}) P_{\mu}^{-1}(bu^{-1})
\end{equation}

where
\begin{equation}
(3.30)
P_{\lambda}(b) := \prod_{1 \leq i < j \leq n} \frac{(t^{i-j})_{\lambda_i-\lambda_j} (q b t^{j-i})_{\lambda_i+\lambda_j}}{(t^{j-i})_{\lambda_i-\lambda_j} (q b t^{-1+j})_{\lambda_i+\lambda_j}}
\end{equation}
The diagonal factor $P_\lambda(b)$ cancels in the expansion, and we get the identity to be proved after a simple reparametrization.

**Remark 3.7.** An important application of the cocycle identity is the proof of a $BC_n$ $10\varphi_9$ transformation which we now obtain in step six. The $10\varphi_9$ transformation is an important application of the Bailey Transform (3.3). In fact, the $10\varphi_9$ transformation formula it is referred to as the Bailey Transform in the literature.

The $10\varphi_9$ transformation formula is proved by summing both inner series above in the Bailey Transform (3.3) (i.e. computing $\beta$ and $\gamma$) by means of the Jackson sum. We use the cocycle identity for $\omega_\lambda$ to prove a more general version of the classical $10\varphi_9$ transformation in

**Lemma 3.8.** For any partitions $\nu$ and $\tau$ and complex parameters $b, c, d, e, f$ and $g$ we have

$$
\omega_{\lambda/\tau}(sr)^{-1}; sr; as^2, bs) = \sum_\mu \omega_{\lambda/\mu}(s^{-1}; s; as^2, bs) \omega_{\mu/\tau}(r^{-1}; r; a, b)
$$

We will use this identity to compute the inner sum (say $\beta_\lambda$) in one side of the Bailey Transform (3.3) and the reparametrized version

$$
\omega_{\nu/\lambda}(us)^{-1}; us; a'u^2, bsu) = \sum_\mu \omega_{\nu/\mu}(u^{-1}; u; a'u^2, bsu) \omega_{\mu/\lambda}(s^{-1}; s; a', bs)
$$

of this identity to compute the inner sum (say $\gamma_\lambda$) on the other side of (3.3).

Now we set

$$
m_{\lambda\mu} = W_\mu(q^{\lambda t^{(n)}}; q, p, t, bst^{2-2n}, bt^{1-n})
$$

and

$$
\alpha_{\mu\tau} = \frac{(qb/as, bt^{1-n}, ra^{-1})_\mu}{(as, qt^{n-1}, qbr^{-1}, quq/a)_\mu} W_\tau(q^{\mu t^{(n)}}; q, p, t, bt^{2-2n}, br^{-1}t^{1-n})
$$

$$
\prod_{i=1}^n \left\{ \frac{E(bt^{2-2i}q^{2\mu})}{E(bt^{2-2i})} (qt^{2i-2})^{\mu_i} \right\} \prod_{1 \leq i < j \leq n} \left\{ \frac{(qt^{j-i})_{\mu_i - \mu_j} (bt^{3-j-i})_{\mu_i + \mu_j}}{(qt^{j-i-1})_{\mu_i - \mu_j} (bt^{2j-i-1})_{\mu_i + \mu_j}} \right\}
$$

and also set

$$
\delta_{\nu\mu} = \frac{(qb/as, bst^{1-n}, s, a's^{-1})_\mu}{(a'u, qt^{n-1}, qbr/a')_\mu} W_\nu(q^{\mu t^{(n)}}; q, p, t, bsu^{2-2n}, bst^{1-n})
$$

$$
\prod_{i=1}^n \left\{ \frac{E(bst^{2-2i}q^{2\mu})}{E(bst^{2-2i})} (bt^{2i-2})^{\mu_i} \right\} \prod_{1 \leq i < j \leq n} \left\{ \frac{(qt^{j-i})_{\mu_i - \mu_j} (bst^{3-j-i})_{\mu_i + \mu_j}}{(qt^{j-i-1})_{\mu_i - \mu_j} (bst^{2j-i-1})_{\mu_i + \mu_j}} \right\}
$$
Using the first version of the cocycle identity we compute

\[
(3.37) \quad \beta_{\lambda\nu} = \frac{(rs, asr^{-1}, qb, qb/a)_{\lambda}(qb/as, ar^{-1})_{\tau}}{(qbr^{-1}, qbr'/a, s, as)_{\lambda}(asr^{-1}, qbr/a)_{\tau}} \cdot W_{\tau}(q^{\lambda}t^{\delta(n)}; q, p, t, bst^{2-2n}, br^{-1}t^{1-n})
\]

and by means of the reparametrized version of (3.33) we compute \(\gamma_{\lambda\nu}\), which becomes

\[
(3.38) \quad \gamma_{\nu\lambda} = \frac{(su, a'u'us^{-1}, qbs, qbs/a')_{\nu}(qbs/a'u, a's^{-1})_{\lambda}}{(qb, qbs^2/a', u, a'u')_{\nu}(a'u'us^{-1}, qbs/a')_{\lambda}} \cdot W_{\lambda}(q^{\nu}t^{\delta(n)}; q, p, t, bst^{2-2n}, bt^{1-n})
\]

So, by using the Bailey Transform we get

\[
(3.39) \quad \frac{(su, a'u'us^{-1}, qbs, qbs/a')_{\nu}(asr^{-1}, qb/a)_{\tau}}{(qb, qbs^2/a', u, a'u')_{\nu}(qb/as, ar^{-1})_{\tau}} \cdot \sum_{\mu \leq \lambda \leq \nu} \prod_{i=1}^{n} \left\{ \frac{E(bst^{2-2i}q^{\lambda_i})}{E(bst^{2-2i})} (qt^{2i-2})_{\lambda_i} \right\} \prod_{1 \leq i < j \leq n} \left\{ \frac{(qt^{-1})_{\lambda_i - \lambda_j}}{(qt^{-1})_{\lambda_i - \lambda_j}} \frac{(bt^{3-i-j})_{\lambda_i + \lambda_j}}{(bt^{3-i-j})_{\lambda_i + \lambda_j}} \right\} \\
\cdot W_{\lambda}(q^{\nu}t^{\delta(n)}; q, p, t, bst^{2-2n}, bt^{1-n}) W_{\tau}(q^{\lambda}t^{\delta(n)}; q, p, t, bst^{2-2n}, br^{-1}t^{1-n})
\]

\[
= \sum_{\mu \leq \lambda \leq \nu} \prod_{i=1}^{n} \left\{ \frac{E(bst^{2-2i}q^{\lambda_i})}{E(bst^{2-2i})} (qt^{2i-2})_{\lambda_i} \right\} \prod_{1 \leq i < j \leq n} \left\{ \frac{(qt^{-1})_{\lambda_i - \lambda_j}}{(qt^{-1})_{\lambda_i - \lambda_j}} \frac{(bst^{3-i-j})_{\lambda_i + \lambda_j}}{(bst^{3-i-j})_{\lambda_i + \lambda_j}} \right\} \\
\cdot W_{\lambda}(q^{\nu}t^{\delta(n)}; q, p, t, bst^{2-2n}, bst^{1-n}) W_{\tau}(q^{\lambda}t^{\delta(n)}; q, p, t, bst^{2-2n}, br^{-1}t^{1-n})
\]

A simple reparametrization \(b = b, c = qb/as, d = r, e = ar^{-1}, f = qbs/a'u, g = a's^{-1}\) in the above identity completes the proof. Using the definition we write this result in the form given above.

The identity or (3.30) reduces to a \(BC_n\) generalization of the Bailey’s classical \(10\varphi_9\) transformation when we set \(\tau = 0\).

In step seven we prove the symmetry using the elliptic \(10\varphi_9\) transformation formula. Note that the \(\omega_{\lambda}(z; r, a, b)\), by definition, has also the built-in symmetry \(x_i \mapsto (ax_i)^{-1}\) for each \(i \in [k]\). Therefore we have

**Lemma 3.9.** Let \(\lambda, \mu\) be partitions and \(z = (x_1, \ldots, x_k) \in \mathbb{C}^k\). Then

\[
(3.40) \quad \omega_{\lambda/\mu}(x_1, \ldots, x_k; r, a, b) = \omega_{\lambda/\mu}(x_{\sigma(1)}, \ldots, x_{\sigma(k)}; r, a, b)
\]

for any \(\sigma \in S\), the hyperoctahedral group, where the inversions are as defined above.

**Proof.** First we show that, for \(x_1, x_2 \in \mathbb{C}\) we have

\[
(3.41) \quad \omega_{\lambda/\mu}(x_1, x_2; r, a, b) = \omega_{\lambda/\mu}(x_2, x_1; r, a, b)
\]

which follows from a double application of the elliptic \(BC_n\) generalization of Bailey’s \(10\varphi_9\) transformation (3.33). Namely, we first set \(b = b, c = x_2^{-1}, d = ax_2, e =\)
complex parameters. Then we have (3.3).

□

The result to be proved.

that the identity (3.10) holds for all $k < m$ by setting $b = \gamma_1, c = qb/a, d = qbx_1/r, e = qb/arx_1, f = qbx_2/r, g = qb/arx_2$ this time where we compute $\gamma_2 = b$. This gives the symmetry for the two variable case mentioned above, and implies the commutativity of the operators $\mathfrak{W}(x_1)$ and $\mathfrak{W}(x_2)$.

The rest of the proof follows by induction. Assuming that the symmetry holds for $m$ variables where $m \leq k$, we show that it also holds for $k + 1$ variables. For $z = (x_1, \ldots, x_k) \in \mathbb{C}^k$ and $x_{k+1} \in \mathbb{C}$ we have

(3.42) $\mathfrak{W}(z, x_{k+1})\lambda_\mu = (\mathfrak{W}(x_{k+1}, z_i) \cdot I)_{\lambda\mu}$

= $(\mathfrak{W}(r^{-2}x_1, \ldots, r^{-2}x_{k-1})\mathfrak{W}(x_n, x_{k+1}))_{\lambda\mu}$

= $(\mathfrak{W}(r^{-2}x_1, \ldots, r^{-2}x_{k-1})\mathfrak{W}(x_{k+1}, x_n))_{\lambda\mu}$

= $(\mathfrak{W}(r^{-1}x_1, \ldots, r^{-1}x_{k-1}, r^{-1}x_{k+1})\mathfrak{W}(x_n))_{\lambda\mu}$

by the definition of the multiplication rule and the first part of the proof. But $\mathfrak{W}(r^{-1}x_1, \ldots, r^{-1}x_{k-1}, r^{-1}x_{k+1})$ is symmetric by hypothesis which concludes the induction step. This is because of the fact that the symmetric group $S_{k+1}$ in $(k+1)$ letters is generated by the permutation $(12 \ldots k)$ and the transposition $(k(k+1))$. Therefore, $\mathfrak{W}(z) = \mathfrak{W}(\sigma(z))$ for any $\sigma \in S_k$ as required.

We now employ the cocycle identity (3.28) to compute a closed expression (3.10), step seven, for $\omega_{\lambda/\mu}(x^{r\delta(m)}; r; a, b)$. We get

**Lemma 3.10.** Let $\lambda$ and $\mu$ be partitions, $m \in \mathbb{Z}_{\geq 0}$, and $q, p, t, a, b, r, x \in \mathbb{C}$ be complex parameters. Then we have

(3.43) $\omega_{\lambda/\mu}(x^{r\delta(m)}; r, a, b) = \frac{(x^{-1}, axr^{m-1})_{\lambda}}{(qbr^{-1}x, qb/(axr^m))_{\mu}} \cdot \frac{(x^{-1}, axr^{m-1})_{\lambda}}{(qbr^{-1}x, qb/(axr^m))_{\mu}} \cdot \frac{(r^m, qbr^{-m-2})_{\lambda}}{(qbr^{-1}, r)_\lambda} \cdot \frac{(r, br^{-1}t^{2i-2}q^{2\mu_i})}{(qbr^{-1}, q^{2\mu_i})} \cdot \prod_{1 \leq i < j \leq n} \frac{(qt^{j-i})_{\mu_i+\mu_j}}{(qt^{j-i-1})_{\mu_i+\mu_j}} \cdot W_{\mu}(q^\lambda t^{\delta(n)}; q, p, t, br^{-1}t^{2i-2}n, br^{-1}t^{1-n})$

where $xr^{\delta(m)} = (x, axr^{m-1}, \ldots, axr, x)$ and $n$ is a positive integer such that $n \geq \ell(\lambda)$.

**Proof.** The $m = 1$ case of (3.10) reduces to the definition (2.28). Assuming that the identity (3.10) holds for all $k < m$, we expand the left hand side of (3.10) using the recurrence relation (2.43) and verify that it is summable by the cocycle identity (3.28) giving the right hand side of (3.10). Finally, an induction on $m$ gives the result to be proved.

The next lemma will be crucial in proving the last two inductive steps in Theorem 3.3.
Lemma 3.11. Let \( x, q, p, t, a, b \in \mathbb{C} \), then

\[
(3.44) \quad W_{\lambda/\mu}(x; q, p, t, a, b) = \lim_{r \to 0} \left\{ \prod_{1 \leq i < j \leq n} \frac{(t^{i-j+1})_{\lambda_i-\lambda_j} (t^{j-i+1})_{\mu_i-\mu_j} (qbt^{2i-j})_{\lambda_i+\lambda_j} (qbt^{2j-i})_{\mu_i+\mu_j}}{(t^{i-j})_{\lambda_i-\lambda_j} (t^{j-i})_{\mu_i-\mu_j} (qbt^{1i-j})_{\lambda_i+\lambda_j} (qbt^{1j-i})_{\mu_i+\mu_j}} \cdot \frac{(r)_{\lambda} (qbr^{-n})_{\lambda} (r^n)_{\mu} (qbr^{-2n})_{\mu}}{(r^n)_{\lambda} (qbr^{-1})_{\lambda} (r)_{\mu} (qbr^{-n-1})_{\mu}} \omega_{\lambda/\mu}(x; r, a, b) \right\}
\]

Proof. We will prove Lemma 3.11 in three parts. In the first part we reduce to proving the case when \( \mu_n = 0 \). If \( \mu_n \neq 0 \), then we can express

\[
(3.45) \quad W_{\lambda/\mu}(x; q, p, t, a, b) = f_1 W_{\hat{\lambda}/\hat{\mu}}(xq^{-1}; q, p, t, aq^2, bq^2)
\]

and

\[
(3.46) \quad \omega_{\lambda/\mu}(x; r, q, p, t; a, b) = f_2 \omega_{\hat{\lambda}/\hat{\mu}}(xq^{-1}; r, q, p, t; aq^2, bq^2)
\]

where \( \hat{\lambda} = (\lambda_1 - 1, \ldots, \lambda_n - 1) \), \( \hat{\mu} = (\mu_1 - 1, \ldots, \mu_n - 1) \), and \( f_1, f_2 \) are (complicated) factors. A tedious computation shows that \( f_1 = f_2 \) and, after iteration, we reduce to the case \( \mu_n = 0 \).

Using the n-part Jackson sum (3.49), we now will give an explicit formula for

\[
W_{\lambda/\mu}(t^{-n}; q, p, t, at^{2n}, bt^n)
\]

in the case where \( l(\lambda) = n \) and \( l(\mu) < n \). For \( s \in \mathbb{C} \) and \( z = (z_1, \ldots, z_{n-1}) \in \mathbb{C}^{n-1} \), we consider

\[
W_{\lambda}(s, z_1, \ldots, z_{n-1}, 1; q, p, t, a, b) = W_{\lambda}(s, z, 1; q, p, t, a, b).
\]

By identity (3.49), we have

\[
(3.47) \quad W_{\lambda}(s, z, 1; q, p, t, a, b) = W_{\lambda}(st^{-1}, t^{-1}z; q, p, t, at^{2n}, bt^n).
\]

The property (3.49) for n-part partitions implies that

\[
(3.48) \quad W_{\lambda}(st^{-1}, t^{-1}z; q, p, at^{2n}, bt^n) = W_{\lambda}(t^{-1}z, st^{-1}; q, p, t, at^{2n}, bt^n).
\]

Expanding the right-hand side of equation (3.48), we obtain

\[
(3.49) \quad W_{\lambda}(r, z, 1; q, p, t, a, b)
\]

\[
= \sum_{\mu \leq \lambda} \frac{(s^{-1}t)_{\lambda} (ast^n)_{\lambda}}{(qbst^{n-1})_{\lambda} (qbt^2)_{\lambda}} \prod_{1 \leq i < j \leq n} \frac{(t^{i-j+1})_{\lambda_i-\lambda_j} (qbt^{2i-j})_{\lambda_i+\lambda_j}}{(t^{i-j})_{\lambda_i-\lambda_j} (qbt^{2i-j})_{\lambda_i+\lambda_j}} \cdot \frac{(r)_{\lambda} (qbr^{-n})_{\lambda}}{(r^n)_{\lambda} (qbr^{-1})_{\lambda}} \cdot \frac{E(bt^{n+1-2i}q^{2p})}{E(bt^{n+1-2})} \frac{E(2t^{i-2})}{E(2t^{i-2})} \cdot \omega_{\lambda/\mu}(s^{-1}z, 1; q, p, t, as^2, bs)
\]

We can express the last \( W \) factor in expression (3.49) as

\[
(3.50) \quad W_{\mu}(s^{-1}z, 1; q, p, t, as^2, bs) = W_{\mu((st)^{-1}z; q, p, t, a(st)^2, bst})
\]

which forces the partitions \( \mu \) appearing in the sum (3.49) to have \( l(\mu) < n \).
Using the the $W$-recurrence relation (2.14), we also find

\[(3.51) \quad W_\lambda(s, z, p, q, p, t, a, b) = \sum_{\mu \subseteq \lambda} W_{\lambda/\mu}(st^{-n}; q, p, t, at^{2n}, bt^n) W_\mu(z, 1; q, p, t, a, b) = \sum_{\mu \subseteq \lambda} W_{\lambda/\mu}(st^{-n}; q, p, t, at^{2n}, bt^n) W_\mu(t^{-1}z; q, p, t, at^2, bt),\]

where again $l(\mu) < n$. Taking the limits as $s \to 1$ of the right-hand sides of identities (3.49) and (3.50) and comparing the coefficients of the (linearly independent) functions $W_\mu(t^{-1}; q, p, t, at^2, bt)$ for a fixed partition $\mu$, we obtain the relation

\[(3.52) \quad W_{\lambda/\mu}(st^{-n}; q, p, t, at^{2n}, bt^n) = \lim_{s \to 1} \left\{ \frac{(s^{-1} t)_L (a(st^n)_L)}{(qbst^{-n-1})_L} \prod_{1 \leq i < j \leq n} \frac{(t^{j-i+1})_L - (t^{j-i})_L (qbt^{n+2-i-j})_L}{E(bst^{n+1-i-j})_L} \cdot \prod_{1 \leq i < j \leq n} \frac{(t^{j-i+1})_L - (t^{j-i})_L (qbt^{n+2-i-j})_L}{E(bst^{n+1-i-j})_L} W_\mu(q^t L^s(n); q, p, t, bt^{-n}, bs) \right\}.\]

Substitute $r = s^{-1} t$ into relation (3.52) and take the limit as $r \to t$ in place of $s \to 1$. Using the definition of the $\omega_{\lambda/\mu}$ function given by identity (2.38), then one observes that we can replace the right-hand side of (3.53) to find

\[(3.53) \quad W_{\lambda/\mu}(st^{-n}; q, p, t, at^{2n}, bt^n) = \lim_{r \to t} \left\{ \prod_{1 \leq i < j \leq n} \frac{(t^{j-i+1})_L - (t^{j-i})_L (qbt^{n+2-i-j})_L}{E(bst^{n+1-i-j})_L} \frac{(r)_L (qbt^n; r^{-1})_L}{(r)_L (qbt^n; r^{-1})_L} \omega_{\lambda/\mu}(t^{-n}; r; at^{2n}, bt^n) \right\}.\]

Identity (3.53) is simply the $x = t^{-n}$ (and $l(\mu) < n$) case of Lemma 3.11. We complete the proof of Lemma 3.11 by observing that if Lemma 3.11 is true for this special value of $x$, then it must be true for all $x$ in the domains of $W_{\lambda/\mu}$ and $\omega_{\lambda/\mu}$. This is because the factors involving the variable $x$ in $W_{\lambda/\mu}(x; q, p, t, at^{2n}, bt^n)$ by its definition (2.12) and in $\omega_{\lambda/\mu}(t^{-n}; r; at^{2n}, bt^n)$ by its definition (2.38) are identical in the limit $r \to 1$. This means that the two sides of (3.53) agree up to a factor independent of $x$, which must be one. This completes the proof of Lemma 3.11. □

Applying the $W$-recurrence relation (2.14) together with the corresponding $\omega$-recurrence relation (2.43) we obtain an immediate corollary of Lemma 3.11.
Corollary 3.12. Let k be a positive integer and \( z = (z_1, \ldots, z_k) \in \mathbb{C}^k \). Let \( q, p, t, a, b \in \mathbb{C} \), then

\[
W_{\lambda/\mu}(z; q, p, t, a, b)
= \lim_{r \to \infty} \prod_{1 \leq i < j \leq n} \left( \frac{(t^{j-i+1})_{\lambda_i - \lambda_j}(t^{j-i})_{\mu_i - \mu_j} (qbt^{k+1-i-j})_{\lambda_i + \lambda_j}(qbt^{-i-j})_{\mu_i + \mu_j}}{(t^{j-i})_{\lambda_i - \lambda_j}(t^{j-i+1})_{\mu_i - \mu_j} (qbt^{k-i-j})_{\lambda_i + \lambda_j}(qbt^{-i-j})_{\mu_i + \mu_j}} \right)
\times \frac{(r)^{\lambda_q - n + k - 1}(r)_\mu (qbr^{-n-1})_{\mu}}{(r^n)^{\lambda_q - k - 1}(r)_\mu (qbr^{-n-1})_{\mu}} \omega_{\lambda/\mu}(z; r, a, b)
\]

Setting \( z = (x, y) \) or \( z = (y, x) \) for \( x, y \in \mathbb{C} \) in identity (3.54), and using the symmetry of \( \omega_{\lambda/\mu}(x, y; r, a, b) \) in the variables \( x \) and \( y \), it follows that \( W_{\lambda/\mu}(x, y; q, p, t, a, b) \) is also symmetric in the variables \( x \) and \( y \). This proves step seven of Theorem 3.3.

To prove step eight, we will consider the function \( W_{\lambda}(z, t^{\delta(k)}; q, p, t, a, b) \), for \( z \in \mathbb{C}^{n+1}, t(\lambda) \leq n \) and positive integer \( k \). It follows from identities (2.22), (2.23) and step seven, that

\[
W_{\lambda}(zt^{-k}; q, p, t, at^{2k}, bt^k)
= W_{\lambda}(z, t^{\delta(k)}; q, p, t, a, b)
= \sum_{\mu \subseteq \lambda} W_{\lambda/\mu}(t^{n-k}v^{\delta(k)}; q, p, t, at^{2n+2}, bt^{n+1}) \cdot W_{\mu}(z; q, p, t, a, b).
\]

Substituting identities (3.54) and (3.55) into (3.55) and simplifying, we obtain the “extended” W-Jackson sum (3.55) in the case of a \((n=1)\)-tuple variable \( x \) and \( t(\lambda) \leq n \). This completes the proof of step eight and also Theorem 3.3.

4. \( \omega \)-Jackson sum

In this section we prove an extension of the W-Jackson sum (3.5) and obtain further properties of the Jackson coefficients. First note that the symmetry (Lemma 3.10) of the infinite matrix \( \mathcal{W}(x_1, x_2) \) in \( x_1, x_2 \in \mathbb{C} \) implies that

\[
\omega_{\lambda}(1, x_1; r, a, b) = \mathcal{W}(1, x_1)_{\lambda_0} = \mathcal{W}(x_1, 1)_{\lambda_0}
\]

Entry level writing of this relation gives

\[
\omega_{\lambda}(1, x_1; r, a, b) = \sum_{\mu} \omega_{\lambda/\mu}(r^{-1}; r, ar^2, br) \omega_{\mu}(x_1; r, a, b)
= \sum_{\mu} \omega_{\lambda/\mu}(r^{-1}; x_1, ar^2, br) \omega_{\mu}(1; r, a, b) = \omega_{\lambda}(r^{-1}; x_1, ar^2, br)
\]

where the last equality follows from the obvious identity

\[
\omega_{\mu}(1; r, a, b) = \delta_{\mu0}
\]

In other words, setting \( x_2 = 1 \) in \( \omega_{\lambda}(x_2, x_1; a, b) \) will have the effect of shifting the parameters. The identity (4.2) simplifies to give the summation formula

\[
\omega_{\lambda}(r^{-1}; x_1; ar^2, bu) = \sum_{\mu} \omega_{\lambda/\mu}(r^{-1}; r, ar^2, br) \omega_{\mu}(x_1; r, a, b)
\]
which is exactly the one variable version of the $W$–Jackson sum
\begin{equation}
(\frac{sx^{-1}, ax}{qbx, qb/ax})_{\lambda} = \sum_{\mu \leq \lambda} \frac{(s, as)_{\lambda}}{(q, qb/a)_{\lambda}} \frac{(bt^{1-n}, qb/as)_{\mu}}{(qt^{n-1}, as)_{\mu}}
\end{equation}
\begin{equation}
\cdot \prod_{i=1}^{n} \left\{ \frac{E(bt^{2i-2})^{2\mu_{i}}}{E(bt^{2-2i})} \right\} \prod_{1 \leq i < j \leq n} \left\{ \frac{(qt^{j-i})_{\mu_{i} - \mu_{j}} (bt^{1-j})_{\mu_{i} + \mu_{j}}}{(qt^{j-1-i})_{\mu_{i} - \mu_{j}} (bt^{2-j-i})_{\mu_{i} + \mu_{j}}} \right\}
\end{equation}
\begin{equation}
\cdot W_{\mu}(q^\lambda \delta^{(n)}; q, p, t, br^{1-n}) (\frac{x^{-1}, ax}{qbx, qb/ax})_{\mu}
\end{equation}
after a simple reparametrization. We expect to have a similar formulation of the
Jackson sum in the multivariable case.

Before we proceed to write a multivariable version of the Jackson sum in terms
of $\omega_{\lambda/\mu}$ function, we verify the generalization of the one variable result in

\begin{lemma}
With the notation as above and for any partition $\lambda$, we have
\begin{equation}
\omega_{\lambda}(r^{\delta(m)}; r, a, b) = \delta_{\lambda \lambda}
\end{equation}
where $r^{\delta(m)} = (r^{m-1}, \ldots, r, 1)$ for any $m \in \mathbb{Z}_{\geq}$.\end{lemma}
\begin{proof}
One variable (i.e., $m = 1$) case $\omega_{\lambda}(1; r, a, b) = \delta_{\lambda\lambda}$ is obvious. Assume
that the identity holds for all $k < m$, that is $\omega_{\lambda}(r^{\delta(k)}; r, a, b) = \delta_{\lambda\lambda}$ for such $k$. We’d
like to show that it also holds true for $m$. But,
\begin{equation}
\omega_{\lambda}(r^{\delta(m)}; r, a, b) = \mathcal{W}(r^{\delta(m)})_{\lambda_{0}}
\end{equation}
\begin{equation}
= \sum_{\mu} \mathcal{W}(1)_{\lambda_{0}} \mathcal{W}(r^{\delta(m-1)})_{\mu_{0}} = \mathcal{W}(1; r, ar^{2m-2}, br^{m-1})_{\lambda_{0}}
\end{equation}
by the induction hypothesis. Therefore,
\begin{equation}
\omega_{\lambda}(r^{\delta(m)}; r, a, b) = \omega_{\lambda}(1; r, ar^{2m-2}, br^{m-1}) = \delta_{\lambda\lambda}
\end{equation}
due to the obvious one variable result. \end{proof}

We now generalize the argument used above in one variable case to write a
multivariable version of the Jackson sum in

\begin{lemma}
Let $k, m \in \mathbb{Z}_{\geq}$, $r \in \mathbb{C}$ and $z = (x_{1}, \ldots, x_{k}) \in \mathbb{C}^{k}$, and let $r^{-m} z$
denote $r^{-m} z = (r^{-m} x_{1}, \ldots, r^{-m} x_{k})$ as before. Then, we have
\begin{equation}
\omega_{\lambda}(r^{-m} z; r, ar^{2m}, br^{m}) = \omega_{\lambda}(r^{\delta(m)}; z; r, a, b)
\end{equation}
where $\lambda$ is any partition.\end{lemma}
\begin{proof}
This could be seen from the symmetry property of $\omega_{\lambda}$ which is shown
to be equivalent to the fact that $\mathcal{M}(x_{i})$ commute. For any $m \in \mathbb{Z}_{\geq}$, we have
\begin{equation}
\omega_{\lambda}(r^{\delta(m)}; z; r, a, b) = \mathcal{W}(r^{\delta(m)}; z)_{\lambda_{0}} = \mathcal{W}(z, r^{\delta(m)})_{\lambda_{0}}
\end{equation}
which may be expanded as
\begin{equation}
\omega_{\lambda}(r^{\delta(m)}; z; r, a, b) = \sum_{\mu} \omega_{\lambda/\mu}(r^{-k} r^{\delta(m)}; r, ar^{2k}, br^{k}) \omega_{\mu}(z; r, a, b)
\end{equation}
\begin{equation}
= \sum_{\mu} \omega_{\lambda/\mu}(r^{-m} z; r, ar^{2m}, br^{m}) \omega_{\mu}(r^{\delta(m)}; r, a, b) = \omega_{\lambda}(r^{-m} z; r, ar^{2m}, br^{m})
\end{equation}
where the last equality follows from Lemma. \end{proof}
Using the identity (4.11) and (3.10) we can now write the multivariable Jackson sum explicitly. We have

**Theorem 4.3 (Multivariable ω–Jackson sum).** For a partition λ, and complex parameters q, p, t, r, s, a, b ∈ C, we have

\[
(4.12) \quad \frac{(qb/a, qb)_{\lambda}}{(as, s)_{\lambda}} \frac{\omega_{\lambda}(s^{-1}z; r; ar^{-1}k s^{-2}, br^{-1}k s)}{\omega_{\lambda}(r^{-k}; r; qbr^{-1}, bs)} = \sum_{m=1}^{n} \left\{ \frac{E(bt^{2-2i}q^{-2\mu_i})}{E(bt^{2-2i})} \right\}_{\mu} \prod_{1 \leq i \neq j \leq n} \left\{ \frac{(qt^{j-i})_{\mu_i-\mu_j} (bt^{3-i-j})_{\mu_i+\mu_j}}{(qt^{j-i-1})_{\mu_i-\mu_j} (bt^{2-i-j})_{\mu_i+\mu_j}} \right\}
\]

\[
= \sum_{m=1}^{n} \left\{ \frac{E(bt^{2-2i}q^{-2\mu_i})}{E(bt^{2-2i})} \right\}_{\mu} \prod_{1 \leq i \neq j \leq n} \left\{ \frac{(qt^{j-i})_{\mu_i-\mu_j} (bt^{3-i-j})_{\mu_i+\mu_j}}{(qt^{j-i-1})_{\mu_i-\mu_j} (bt^{2-i-j})_{\mu_i+\mu_j}} \right\}
\]

\[
\cdot \frac{(bt^{1-n}, qb/as)_{\mu}}{(qt^{n-1}, as)_{\mu}} W_{\mu}(q^{\lambda}k^{i(n)}; q, p, t, bst^{-2n}, bt^{1-n}) \omega_{\mu}(z; r; ar^{-1}k, br^{-1}k)
\]

where \( z = (x_1, \ldots, x_k) \in \mathbb{C}^k \) for some \( k \in \mathbb{Z}_{>0} \) and \( n \) is a positive integer such that \( n \geq \ell(\lambda) \).

**Proof.** It follows by induction from the Lemma (4.2) and the key identity (3.10) that

\[
(4.13) \quad \frac{(qb/a, qb, qbr^{-1}, r)_{\lambda}}{(rk, ark^{-1}m^{-1}, rk, qbr^{-1}m^{-2})_{\lambda}} \omega_{\lambda}(r^{-m}z; r; ar^{2m}, br^{m})
\]

\[
= \sum_{m=1}^{n} \left\{ \frac{E(br^{k-1}t^{2-2i}q^{2\mu_i})}{E(br^{k-1}t^{2-2i})} \right\}_{\mu} \prod_{1 \leq i \neq j \leq n} \left\{ \frac{(qt^{j-i})_{\mu_i-\mu_j} (br^{k-1}t^{3-i-j})_{\mu_i+\mu_j}}{(qt^{j-i-1})_{\mu_i-\mu_j} (br^{k-1}t^{2-i-j})_{\mu_i+\mu_j}} \right\}
\]

\[
\cdot \frac{(qb/a, qbr^{-1})_{\mu}}{(rk, qbr^{-1}m^{-2})_{\mu}} \prod_{1 \leq i \neq j \leq n} \left\{ \frac{(qt^{j-i})_{\mu_i-\mu_j} (br^{k-1}t^{3-i-j})_{\mu_i+\mu_j}}{(qt^{j-i-1})_{\mu_i-\mu_j} (br^{k-1}t^{2-i-j})_{\mu_i+\mu_j}} \right\}
\]

\[
\cdot W_{\mu}(q^{\lambda}k^{i(n)}; q, p, t, br^{k+m-1}t^{2-2n}, br^{k-1}t^{1-n}) \omega_{\mu}(z; r; a, b)
\]

where \( z = (x_1, \ldots, x_k) \in \mathbb{C}^k \) and \( n \) is a positive integer such that \( n \geq \ell(\lambda) \).

Both sides of this identity are rational functions in the single parameter \( s \).

We rewrite this identity with some elementary algebra as an identity between two polynomials that are equal at infinitely many values \( s = r^m \) for any positive integer \( m \) by (14.3).

**Remark 4.4.** The identity (4.12) looks like a \( 10\varphi_9 \) summation identity, for it has an extra free parameter \( r \in \mathbb{C} \). However, it reduces to \( W \)–Jackson sum (3.10) if \( z = x_1 \in \mathbb{C} \) or in the case when we send \( r \to t \) and set \( k = n \). Therefore, the identity (4.12) extends \( W \)–Jackson sum to the case where \( r \in \mathbb{C} \) and \( k \in \mathbb{Z}_{>0} \) are arbitrary.

**Remark 4.5.** Besides extending it, the preceding argument also provides an alternative proof to \( W \)–Jackson sum, for the proof of the \( \omega \)–Jackson sum depends only on the cocycle identity (3.28) which can be independently proved by other methods (see the bulk difference equation in (3.1), for example).

We conclude this section by listing some important properties of the \( \omega_{\lambda/\mu} \) functions.

1. \( \omega_{\lambda} \) satisfy the vanishing property

\[
(4.14) \quad \omega_{\lambda}(q^{\tau}r^{\delta(n)}; r, q, p, t; a, b) = 0
\]
where $\pi = (\pi_1, \ldots, \pi_n)$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$ are $n$-part partitions such that $\lambda \not\subseteq \pi^n$. The proof is a slight generalization of that of Lemma 3.1.

(2) Setting $z = r^{i(k)}$ in (4.12) gives

$$\omega_\lambda(xr^{\delta(k)}; r; a, b) = \frac{(x^{-1}, ar^{k-1}x)_\lambda}{(qbr^{k-1}x, qbx/ax)_\lambda} \frac{(r^k, qbr^{k-2})_\lambda}{(qb^{-1}, r)_\lambda}$$

(3) If we set $s = r^{-1}$ in the cocycle identity (4.20) we get the following inversion relation

$$\delta_{\nu, \tau} = \sum_{\tau \leq \lambda \leq \nu} \omega_\nu/\lambda(r; r^{-1}; a, b) \omega_{\lambda/\tau}(r^{-1}; r; ar^2, br)$$

where $\delta_{\nu, \tau} = 1$ when $\nu = \tau$, and $\delta_{\nu, \tau} = 0$ otherwise. The proof is a special case of the proof of Lemma 3.6.

(4) We have the following elliptic transformation identities: For $x \in \mathbb{C}$

$$\omega_{\lambda/\mu}(x; r; a, pb) = (q^2b^2/a)^{|\lambda|-|\mu|}t^{-2n(\lambda)+2n(\mu)}q^{2n(\lambda')-2n(\mu')}\omega_{\lambda/\mu}(x; r; a, b)$$

$$\omega_{\lambda/\mu}(x; r; pa, b) = (qb)^{-|\lambda|+|\mu|}p^{|\lambda|-|\mu|}r^{-2n(\lambda)-2n(\mu)}q^{-2n(\lambda')-2n(\mu')}\omega_{\lambda/\mu}(x; r; a, b)$$

$$\omega_{\lambda/\mu}(x; pr; a, b) = (ar^{-2})^{-|\mu|}p^{2|\mu|}q^{-2n(\lambda')-2n(\mu')}\omega_{\lambda/\mu}(x; r; a, b)$$

Proofs use direct computation and properties like

$$(p^{-1}x)_\lambda = (-1)^{|\lambda|}p^{-|\lambda|}x^{|\lambda|}t^{-n(\lambda)}q^{-n(\lambda')}(x)_\lambda$$

and

$$(px)_\lambda = (-1)^{|\lambda|}x^{-|\lambda|}t^{n(\lambda)}q^{-n(\lambda')}q^{-n(\lambda')}(x)_\lambda.$$  

Note that (4.17) and (4.20) of these identities can be generalized to the multivariable case $x \in \mathbb{C}^n$ using the recurrence formula (2.43) for $\omega_{\lambda/\mu}$.

(5) For $\mu = \lambda$ we get

$$\omega_{\lambda/\lambda}(x; r; a, b) = \frac{(qbr^{-1}x, qbx/arx)_\lambda}{(qb, qbx/arx)_\lambda} \frac{(br^{-1}t^{1-n}, qbr^{-1})_\lambda}{(bt^{1-n}, qbr^{-2})_\lambda}$$

$$\cdot \prod_{i=1}^{n} \left\{ \frac{(bt^{2-i})(2\lambda_i)}{(br^{-1}t^{2-i})2\lambda_i} \right\} \prod_{1 \leq i < j \leq n} \left\{ \frac{(bt^{2-i-j})\lambda_i + \lambda_j}{(br^{-1}t^{2-i-j})\lambda_i + \lambda_j} \right\}$$

The proof follows from Proposition 3.1.

Other properties including a reversal, an inversion and a duality identity for $\omega_{\lambda/\mu}$ will be given in a later publication.
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