Spin and Statistics and First Principles

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Abstract

It was shown in the early Seventies that, in Local Quantum Theory (that is the most general formulation of Quantum Field Theory, if we leave out only the unknown scenario of Quantum Gravity) the notion of Statistics can be grounded solely on the local observable quantities (without assuming neither the commutation relations nor even the existence of unobservable charged field operators); one finds that only the well known (para)statistics of Bose/Fermi type are allowed by the key principle of local commutativity of observables. In this frame it was possible to formulate and prove the Spin and Statistics Theorem purely on the basis of First Principles.

In a subsequent stage it has been possible to prove the existence of a unique, canonical algebra of local field operators obeying ordinary Bose/Fermi commutation relations at spacelike separations.

In this general guise the Spin - Statistics Theorem applies to Theories (on the four dimensional Minkowski space) where only massive particles with finite mass degeneracy can occur. Here we describe the underlying simple basic ideas, and briefly mention the subsequent generalisations; eventually we comment on the possible validity of the Spin - Statistics Theorem in presence of massless particles, or of violations of locality as expected in Quantum Gravity.

1 What is Statistics?

Most would answer: “look at the commutation/anticommutation relations between spacelike separated field operators”; but field operators usually are not observable quantities, their properties might be mere features of the formalism - or they might even fail to exist.
Otherwise, many would answer: “Construct n particle states, then take the symmetric/antisymmetric part of the tensor product, then...”.

But this means to impose a choice in the construction of free field operators; to do something similar in a generic interacting theory, we must:

(i) prove that there exist a product operation between suitably localised states in the theory which produces other states in the same theory (i.e. expectation functionals on the algebra of observable quantities), which describe the composed states. Such composed states should be independent of the order in which the factors are listed.

(ii) associate to that product of states in a canonical way a product state vector: the latter may depend upon the order, e.g. changing under a permutation of factors by a phase or by a unitary operator which commutes with all observables.

(iii) if all the factors are vector states of a fixed superselection sector, the statistics of that superselection sector can then be defined by the actions of the permutation groups (of all possible orders for the different numbers of factors) obtained as above (provided we can show that the way the product state vector changes under permutations of the order of factors, is indeed described by an action of the permutation group which depends only upon the choice of the superselection sector).

Note that the distinction between integer/half integer spin arises in a similar way: rotations of $2\pi$ leave expectation functional (states) invariant, but may change the phase of the state vectors (only by a sign if the choice is canonical).

It is a remarkable fact that the above approach to statistics can be realized in any Local Quantum Theory, based essentially only on the Locality Principle.

More precisely, suppose the observables are given as bounded operators on a fixed Hilbert space, describing a single superselection sector, the vacuum sector. Their collection is therefore irreducible. The main postulate is that this is the collection of (quasi) local observables \[ \mathfrak{O}(O) \subset B(\mathcal{H}) \] (1)

whose selfadjoint elements are the observables which can be measured in the spacetime region $O$, and such that local commutativity holds, i.e. the measurements of two spacelike separated observables must be compatible,
so that they commute with each other:

$$\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)' \quad \text{if} \quad \mathcal{O}_1 \subset \mathcal{O}_2'$$  \hfill (2)

where the prime on a set of operators denotes its commutant (the set of all bounded operators commuting with all the operators in the given set) and on a set in Minkowski space denotes the spacelike complement. Thus each $\mathfrak{A}(\mathcal{O})$ is included in the intersection of the commutants of all $\mathfrak{A}(\mathcal{O}_n)$, as $\mathcal{O}_n$ runs through all the double cones spacelike to $\mathcal{O}$.

This axiom is strengthened to Duality: each $\mathfrak{A}(\mathcal{O})$ is maximal with the above property, namely that inclusion is actually an equality:

$$\mathfrak{A}(\mathcal{O}) = \mathfrak{A}(\mathcal{O}')',$$  \hfill (3)

where, here and in the following, $\mathfrak{A}(\mathcal{O}')$ denotes the norm closed *subalgebra generated by all the local algebras associated to the various double cones which are spacelike separated from $\mathcal{O}$, i.e. included in $\mathcal{O}'$.

(A weaker form of this assumption is “essential duality”, requiring only that the $\mathfrak{A}(\mathcal{O}')$ locally commute with one another; if the theory is suitably described by Wightman fields, essential duality can be proved to hold \cite{[3]}; the weakening of duality to essential duality indicate the presence of spontaneously broken global gauge symmetries \cite{[4]}).

Translation and Lorentz covariance, Spectrum Condition play no role in this analysis, except for a mild technical consequence, proven long ago by Borchers, that we called the Property $B$ \cite{[5]}, which can just be assumed as an additional axiom besides duality. Most of the analysis requires nothing more.

The collection $\mathfrak{A}$ of quasilocal observables will be the operator norm closure of the union of all the $\mathfrak{A}(\mathcal{O})$, that is, due to (1) and (2), their norm closed inductive limit. Thus $\mathfrak{A}$ is a norm closed * subalgebra of $B(\mathcal{H})$ (i.e. a $C^*$ Algebra of operators on $\mathcal{H}$) which is irreducible. The physical states of the theory are described by normalised positive linear functionals (in short: states) of $\mathfrak{A}$, i.e. are identified with the corresponding expectation functionals.

A general comment on locality is in order: it is often claimed that the Einstein, Podolski and Rosen “paradox” shows that Quantum Mechanics is

\footnote{Property B: If $\mathcal{O}_1$ and $\mathcal{O}_2$ are double cones and the second includes the closure of the first, then any selfadjoint projection $E$ localised in the first is of the form $E = WW^*$, where $W^*W = I$ and $W$ is localised in the second.

(We could even choose $W$ in the same algebra $\mathfrak{A}(\mathcal{O}_1)$ if the latter were a so called type III factor, which is most often the case by general theorems \cite{[5]}).}
“non local”. What does this statement mean? Folklore says that at least it does not mean that we can use EPR to transmit information. We would like to stress that certainly it is not in contradiction with the notion of locality just recalled here. EPR shows that there will be states with long range correlations; but such states can be shown to exist in any theory which fulfills locality. In particular, the local algebras of free field theory provide mathematically precise sharp examples of this scenario.

Contrasts may well arise, however, between the EPR picture and a truly local picture of the measurement process [48].

Unit vectors in $H_0$ induce pure states all belonging to the same superselection sector, identified with the vacuum superselection sector; among these pure states a reference vector state $\omega_0$, induced by the unit vector $\Omega_0$, will be called the Vacuum State (resp. the Vacuum State Vector).

In general, there will be a maze of other pure states (by the so called GNS construction, all appearing as vector states of other inequivalent irreducible representations of the algebra $A$).

To implement the program outlined at the beginning of this section, we must define the “suitably localised states” in the theory. This will select, from all irreducible representations, those which describe superselection sectors (we must exclude the enormous family of mathematically possible but physically not significant representations, in the same way as in Quantum Mechanics we select from all the representations of the Heisenberg relations only those which are integrable into representations of the Weyl relations; but we must exclude also physically meaningful states which are not related to superselection sectors, such as the pure states describing the homogeneous equilibrium at finite constant densities and absolute zero temperature). In other words, we ought to consider representations describing elementary perturbations of the vacuum.

For the sake of simplicity of the exposition, we adopt here the restrictive notion of double cone localisation adopted in [6, 13]; it was recognised later ([7], see also [9]) that the analysis goes through for a wider class (spacelike cone localisation) which was shown in [7] to cover all superselection sectors in any massive theory (but QED is left out by both).

A state $\omega$ is strictly localised in a double cone $O$ if the expectation value in $\omega$ of any local observable which can be measured in the spacelike complement of $O$ coincides with the expectation value in the vacuum. In intuitive terms, we will select representations which, among their vector states, have sufficiently many strictly localised states, with all possible double cone localisations.

More precisely, it turns out that this is achieved by the following:
selection criterion: the representations $\pi$ of $\mathfrak{A}$ describing elementary perturbations of the vacuum are those whose restriction to $\mathfrak{A}(O')$, for each double cone $O$, is unitarily equivalent to the restriction to $\mathfrak{A}(O')$ of the vacuum representation. This means that they describe “superselection charges” which can be localised exactly in any tiny region of spacetime (note that an electric charge cannot be localised in this sense, as a result of Gauss theorem [6]).

It is important to note that such representations need not to be irreducible; the unitary equivalence classes of the irreducible representations fulfilling the criterion will be the superselection sectors of the theory; their collection is thus determined by the vacuum sector together with the algebraic structure of the collection of all local observables.

Next step: up to unitary equivalence, the representations $\pi$ of $\mathfrak{A}$ fulfilling the criterion can be more conveniently described by “localised morphisms” of $\mathfrak{A}$ into itself.

For, if the unitary operator $U$ implements the equivalence of $\pi$ and the vacuum representation when both are restricted to $\mathfrak{A}(O')$ for a chosen double cone $O$, we can realize the representation $\pi$ in question on the same Hilbert space as the vacuum representation, carrying it back with $U^{-1}$. The representation $\rho$ we obtain this way is now the identity map on $\mathfrak{A}(O')$: 

$$\rho(A) = A \quad \text{if} \quad A \in \mathfrak{A}(O')$$

(4)

and the duality postulate implies that it must map $\mathfrak{A}(O)$ into itself; if $O$ is replaced by any larger double cone, $\mathfrak{A}(O')$ is replaced by a smaller algebra, hence the forgoing applies, showing that any larger local algebra is mapped into itself; hence $\rho$ is an endomorphism of $\mathfrak{A}$.

Since the choice of $O$ was arbitrary up to unitary equivalence, our localised morphisms are endomorphisms of $\mathfrak{A}$ which, up to unitary equivalence, can be localised in the sense of [1] in any double cone.

Unitary equivalence, inclusion or reduction of representations are decided studying their intertwining operators $T : T \pi(A) = \pi'(A)T, A \in \mathfrak{A}$. Duality implies that the intertwining operators between two localised morphisms must be local observables, in particular they belong to $\mathfrak{A}$. Hence localised morphisms act on their intertwiners.

Now the composition of maps of two localised morphisms produces a localised morphism, their product; one can easily prove that:

morphisms localised in mutually spacelike double cones commute.

If $\rho_n$ are morphisms unitary equivalent to $\rho$ and localised in double cones $O_n$, which

\[ \rho_n(A) = A \quad \text{if} \quad A \in \mathfrak{A}(O'_n) \quad (4) \]
Composing a localised morphism with the vacuum state produces a state which is strictly localised, a vector state in a superselection sector if our morphism is irreducible. Now we can define the product of such states! For, if \( \omega_j = \omega_0 \circ \rho_j, \ j = 1, 2, \ldots, n \) are such states and the morphisms used to create them from the vacuum are localised in pairwise spatially separated double cones, we can define

\[
\omega_1 \times \omega_2 \times \cdots \times \omega_n \equiv \omega_0 \circ \rho_1 \rho_2 \cdots \rho_n
\]

a product state, since it will be independent of the order of factors thanks to the local commutativity of the localised morphisms, and will agree with \( \omega_j \) when tested with a local observable localised in a double cone which is spacelike to the localisation regions of all our morphisms except the \( j \)th one.

This defines a commutative composition law among the classes of our representations, which can be interpreted as the composition of superselection charges; but the composition of irreducible representations might well be reducible.

(In mathematical terms, our localised morphisms \( \rho, \sigma, \ldots \), and their intertwiners \( R \in (\rho, \rho') \) form a tensor category, tensor products of objects being the composition of morphisms, and that of two arrows, say \( R \in (\rho, \rho') \), \( S \in (\sigma, \sigma') \), being given by:

\[
R \times S \equiv R \rho(\sigma, \rho' \sigma').
\]

(6) Now the structure we described so far allows us to define STATISTICS.

If the morphisms in (5) are all equivalent to a given \( \rho \), and say \( U_j \) in \( \mathcal{A} \) are the associated (local!) unitary intertwiners, then the product \( \rho_1 \rho_2 \cdots \rho_n \) is equivalent to \( \rho^n \) and

\[
U_1 \times U_2 \times \cdots \times U_n \in (\rho^n, \rho^n \rho_1 \rho_2 \cdots \rho_n).
\]

run away to spacelike infinity, the unitary intertwiners \( U_n \) in \( (\rho_n, \rho) \) are easily seen to induce automorphisms which converge to \( \rho \). Choosing \( \mathcal{O}_n \) contained together with \( \mathcal{O} \) in some double cone spacelike to the support of \( \sigma \), we have that, for all \( n \), \( \sigma(U_n) = U_n \), hence in the limit \( \rho_n \) and \( \sigma \) commute.

But this argument shows also that the ”charged” state obtained composing the vacuum state with \( \rho \) is the limit of vector states in the vacuum sector, which are bilocalised in \( \mathcal{O} \) and \( \mathcal{O}_n \), induced by the images of the vacuum vector through the \( U_n \); these states describe the limit state in \( \mathcal{O} \) plus some compensating ”charge” in \( \mathcal{O}_n \) (we got a charge transfer chain). This gives a precise form to the old argument of the particles behind the moon by Haag and Kastler [1].

If we replace the unitaries \( U_n \) by their inverses, we can capture in the limit the compensating charges localised in \( \mathcal{O} \) itself; they will lie in another (conjugate) sector of the same family if statistics, defined below, is finite.
Now obviously our states $\omega_j$ are vector states in the representation $\rho$ induced by the state vectors

$$\Psi_j = U_j^* \Omega_0$$

and we can define a product state vector $\Psi_1 \times \Psi_2 \times \cdots \times \Psi_n$ which induces the state $\omega_1 \times \omega_2 \times \cdots \times \omega_n$ in the representation $\rho^n$ by setting:

$$\Psi_1 \times \Psi_2 \times \cdots \times \Psi_n \equiv (U_1 \times U_2 \times \cdots \times U_n)^* \Omega_0.$$ 

If we change the order $(1, 2, \ldots, n)$ by a permutation $p$, the product state will not change but the product state vector changes to

$$\Psi_{p^{-1}(1)} \times \Psi_{p^{-1}(2)} \times \cdots \times \Psi_{p^{-1}(n)} = (U_{p^{-1}(1)} \times U_{p^{-1}(2)} \times \cdots \times U_{p^{-1}(n)})^* (U_1 \times U_2 \times \cdots \times U_n) \Psi_1 \times \Psi_2 \times \cdots \times \Psi_n \equiv \epsilon^{(n)}_\rho(p) \Psi_1 \times \Psi_2 \times \cdots \times \Psi_n.$$

At first sight, we can only say that the unitary operator relating the two state vectors, the $\epsilon^{(n)}_\rho(p)$ defined by the last relation, belongs to the commutant of $\rho^n$. But it can be proved that:

1. the map $p \mapsto \epsilon^{(n)}_\rho(p)$ is a representation of the permutation group which depends upon $\rho$ only (not on the choice of the $U_j$);

2. if $\rho$ is changed to another localised morphism $\rho'$ by a unitary equivalence, say $U$ in $(\rho, \rho')$, $\epsilon^{(n)}_\rho(p)$ is changed to $\epsilon^{(n)}_{\rho'}(p)$ by a unitary equivalence, implemented by $U \times U \times \cdots \times U$; thus the hierarchy of unitary equivalence classes of the representations $\epsilon^{(n)}_\rho$, $n = 2, 3, \ldots$ depends only upon the unitary equivalence class of $\rho$, a superselection sector if $\rho$ was irreducible.

This hierarchy is then the statistics of that superselection sector.

The main result on statistics says that (as a consequence solely of our assumptions, that is essentially as a consequence of locality alone) the statistics of a superselection sector is uniquely characterised by a “statistics parameter” associated to that sector, which takes values $\pm 1/d$, or $0$, where $d$ is a positive integer. The integer $d$ will be the order of parastatistics, and $+$ or $-$ will be its Bose or Fermi character (no distinction for infinite order, when the parameter vanishes).

More explicitly, let $K$ be a fixed Hilbert space of dimension $d$, and let $\theta^{(d)}_n$ denote the representation of the permutation group of $n$ objects which acts
on the \( n \)th tensor power of \( \mathcal{K} \) shifting the factors; our theorem says that, given a superselection sector, if its statistics parameter \( \lambda \) is \(+1/d\) then, for each \( n \), \( \epsilon^{(n)}_\rho \) is unitarily equivalent to the sum of infinitely many copies of \( \theta_d^n \); if \( \lambda = -1/d \), the same is true provided we further multiply with the sign of the permutation; the latter being irrelevant if \( d = \infty \), i.e. if \( \lambda = 0 \).

In mathematical terms this notion is canonical: the \( \epsilon^{(n)}_\rho \) arise in a standard way from a “symmetry” for our tensor category (that is a map assigning to pairs of morphisms \( \rho, \sigma \) a unitary intertwiner

\[
\epsilon(\rho, \sigma) \in (\rho\sigma, \sigma\rho)
\]

which, in a precise mathematical sense, expresses the rule of commuting factors in the \( \times \) product on arrows) which naturally arises here, since locality propagates to the arrows: \( T \times S = S \times T \) if the sources of \( T \) and \( S \) are mutually spacelike localised morphisms, and the same is true for the targets; this symmetry is unique with the property that it reduces to the identity operator if \( \rho \) and \( \sigma \) are spacelike separated.

Can infinite statistics actually occur? The answer is yes in low dimension \[10, 11, 12\], where anyway the theory above does not apply: in \( 1 + 1 \) dimension our category would not be necessarily symmetric, but only a braided category in general (a similar phenomenon occurs in \( 2 + 1 \) dimensions in the case of the weaker spacelike cone localisation, see below); in \( 3 + 1 \) dimensions, however, it can be proved that, in theories with purely massive particles, statistics is automatically finite; furthermore a slight generalisation of the above scheme (allowing localisation in spacelike cones - appropriate neighbourhoods of a string joining a point to spacelike infinity, suitable to describe topological charges) covers all positive energy representations and the whole theory can be extended to that case (\[7, 8\]; see also \[9, 13\]).

Thus, in a widely general sense, to each superselection sector is associated (an integer, the order of parastatistics, and) a sign, \(+1\) for paraBose and \(-1\) for paraFermi.

In relativistic theories, to each sector another sign is intrinsically attached, \(+1\) for sectors with integer and \(-1\) for those with half integer spin values.

The Spin Statistics Theorem based solely on First Principles states that, for sectors with an isolated point in mass spectrum with finite particle multiplicity, \( \text{those signs must agree} \).

This theorem, first proved for the class of sectors described here \[13\], was then extended to sectors localisable only in spacelike cones \[14\]. More recent variants replaced the assumptions of covariance and finite mass degeneracy.
by that of “modular covariance” \[15\]. It has been generalised even to QFT on some appropriate kinds of curved spacetimes \[16\].

Note that the assumption of finite multiplicity of one particle subrepresentations (either explicit, or, to some extent, implicit in the assumption of modular covariance: which in fact is based on the Bisognano Wichmann property, in turned proved, originally, for Wightman theories with finite tensor character) is an essential condition. For, as shown first by I. Todorov in the sixties, it is easy to construct free field models with the \textit{wrong} connection between spin and statistics, yet fulfilling all the other axioms (Quantum Mechanics, Relativistic Covariance, Spectrum Condition, Locality), if we let in an additional infinite dimensional unitary representation of $SL(2,\mathbb{C})$ acting on the internal degrees of freedom of one particle states.

In a world with only one (or, for sectors which are only localisable in spacelike cones, with only two) space dimensions, as mentioned above, the statistics might be described by a braiding, not necessarily by a symmetry \[17, 18\]; the sign of the statistics parameter is replaced by a phase, as is the sign associated to univalence; again, in this general setting, these phases can be shown to agree (\[19, 20\], and References therein).

The connection between spin and statistics might fail also in a nonrelativistic theory; necessary and sufficient conditions for its validity have been extensively studied \[21\].

Dealing with a theory on the ordinary Minkowski space, but not necessarily assuming Covariance and Spectrum Condition, it is natural to restrict attention to localised morphisms with finite statistics, thus restricting the definition of superselection sectors. They will be described by a tensor category of localised morphisms, where each object can be decomposed into a finite direct sum of irreducibles, and which possesses an additional important piece of structure.

Given two states as in \[15\], the product state $\omega_1 \times \omega_2$ need not to be a pure state even if the factors are pure, i.e. vector states in superselection sectors; it will however be at most a finite convex combination of pure states, vector states in some superselection sectors. Can one of these belong to the vacuum sector (so that there is a channel where the two factors can annihilate one another)? We would expect that this property characterises precisely sectors which are related by conjugation of particle-antiparticle “charge” quantum numbers.

Again, the general principle of locality allows us to prove that any superselection sector in our class has a conjugate in this sense (which can be captured as mentioned in footnote \[2\] with a careful use of the charge transfer chains); more precisely, in mathematical terms, this allows us to
prove that our category of localised morphisms with finite statistics is a “rigid” symmetric tensor C* category, where rigidity tells that to any object (localised morphism) we can assign another one such that the “tensor product” contains the tensor identity (the identity morphism) as a component, with some minimality conditions which make its class unique [6, 13]. To be slightly pedantic, the existence of conjugates identifies with rigidity if we first perform a trivial change, from the symmetry given by locality to another symmetry, changing the sign of its value on pairs of irreducible (para)Fermi morphisms.

Note that the identity morphism describes the vacuum sector, hence the tensor identity is irreducible, that is its selfintertwiners reduce to the complex numbers.

Now there is a wide, well known class of mathematical examples of rigid symmetric tensor C* categories with irreducible tensor identity: the unitary continuous finite dimensional representations of compact groups.

Any such categories has an additional feature, with respect to our superselection category: its objects are finite dimensional vector spaces (the representation spaces), and the arrows are subspaces of linear operators between the corresponding representation spaces (the ordinary intertwiners between the representations). The tensor operations here are ordinary tensor products of representations and of intertwiners, the flip of tensor products give the symmetry, and the complex conjugate the conjugation expressing rigidity.

Any rigid symmetric tensor C* categories with irreducible tensor identity, provided it can be faithfully represented in the category of finite dimensional vector spaces, is the dual of a unique compact group (by classical theorems of Tannaka and Krein). What in our more general case? Tannaka and Krein do not help, since our category is not represented in the required way.

This question called for a new duality theory for compact groups, where it was shown that each rigid symmetric tensor C* categories with irreducible tensor identity is the dual of a unique compact group, at the same time proving the existence of the desired faithful representation (as a symmetric tensor category) in the category of finite dimensional vector spaces [22, 23].

Here the main tool was the theory of highly noncommutative C* Algebras; in particular, it required the construction of a “crossed product” of a C* Algebra with centre reduced to the multiples of the identity (the algebra of quasilocal observables in our case) by a rigid symmetric full subcategory of endomorphisms (the superselection structure in our case); the automorphisms of this larger algebra which leave the original one pointwise invariant provide the desired (automatically compact!) group G. The cross product
exists and is unique with some requirements, including the fact that each object $\rho$ of the category (the localised morphisms with finite statistics in our case) become inner in the larger algebra, in the sense that there are sufficiently many operators $\psi$ there such that

$$\psi A = \rho(A)\psi, \quad A \in \mathfrak{A}$$

If $\rho_1, \rho_2$ are irreducible and spatially separated, the corresponding $\psi$’s anti-commute with one another if both the chosen sectors are paraFermi, they commute otherwise. Among other conditions, this is crucial to make the solution unique.

When $\rho$ runs through all our morphisms which are localised in a fixed double cone $\mathcal{O}$, these operators generate the local algebra of field operators in $\mathcal{O}$, and altogether, varying $\mathcal{O}$, the quasilocal field algebra $\mathfrak{F}$; the set of fixed points in $\mathfrak{F}$ under $G$ is precisely $\mathfrak{A}$; the vacuum representation of $\mathfrak{A}$ induces the irreducible vacuum representation of $\mathfrak{F}$, which restricted to $\mathfrak{A}$ gives a reducible representation, the direct sum of all superselection sectors each with multiplicity given by the order of parastatistics.

In short we have constructed ordinary Bose/Fermi field operators, and the global gauge group acting upon them, whose irreducible representations label the superselection sectors, with dimensionality coinciding with the order of parastatistics. All compact groups must arise this way [24].

Algebraic Quantum Field Theory provides also a weak form of the Noether Theorem, local current algebras [25, 26, 27], a generalised Goldstone Theorem [28], and allows us to discuss in mathematically precise terms the scaling limit and the phenomenon of confinement of superselection charges [29, 30, 31, 32].

It is worth noting that the need for an abstract duality theory for compact groups, a problem which arose in Algebraic Quantum Field Theory at the end of the 60’s and was solved at the end of the 80’s, emerged meanwhile in similar terms (for Algebraic Groups) in Mathematics, in the context of Grothendieck Theory of Motives; an independent solution, just slightly later and with slightly different assumptions, was given by Deligne [33]. In recent years, Mueger gave an alternative proof of the Abstract Duality Theorem for Compact groups, following the line of the Deligne approach [34].

But what about the limitations imposed on the proofs of the connection between Spin and Statistics by the condition that only particles with positive mass appear in the Theory? And what about locality itself?
2 The paradise lost: nonlocalisability of states and nonlocality of observables

We pointed out in passing that electrically charged states will not be captured by the selection criterion described above (not even by its more general form in terms of spacelike cones). While the theory is believed (and indirectly checked, down to the scale of $10^{-17}$ cm) to be local, those states will not be localised, due to the slow decay of Coulomb fields \[35, 36\]. The relevant family of representations describing superselection sectors will have only asymptotic localisation properties.

It might still be, however, described by a tensor category of morphisms of our algebra of quasilocal observables; this category can at most be expected to be asymptotically Abelian in an appropriate sense; but this might well be enough to derive again a symmetry \[37\].

We are still far from a Spin Statistics Theorem as described in the previous section, which applies to QED; however, in very general terms, it is reasonable to expect that it still holds when not only the charge carrying operations, but even the observables, are only asymptotically local, with sufficiently fast decays. For, in that case, scattering theory is still applicable, and if relativistic covariance holds, and if its validity propagates to the scattering states, the latter would be described by ordinary free fields, which must obey the connection between Spin and Statistics.

These comments however are far from being conclusive. Already in QED, the scattering theory becomes quite subtle \[38\]. But why should we worry about the possible breakdown of the very locality of observables? We are bound to face such a scenario if gravitational forces are taken into account.

At large scales spacetime is a pseudo Riemannian manifold locally modelled on Minkowski space. But the concurrence of the principles of Quantum Mechanics and of Classical General Relativity points at difficulties at the small scales, which make that picture untenable. For, if we try to locate an event in say a spherically symmetric way around the origin in space with accuracy $a$, according to Heisenberg principle an uncontrollable energy $E$ of order $1/a$ has to be transferred, which will generate a gravitational field with Schwarzschild radius $R \simeq E$ ($\hbar = c = G = 1$). Hence $a \gtrsim R \simeq 1/a$ and $a \gtrsim 1$, i.e. in CGS units

$$a \gtrsim \lambda_P \simeq 1.6 \cdot 10^{-33} cm.$$  \hspace{1cm} (7)

However, if we measure one of the space coordinates of our event with great precision $a$, but allow large uncertainties $L$ in the knowledge of the
other coordinates, the energy $1/a$ may spread over a thin disk of radius $L$ and thus generate a gravitational potential that would vanish everywhere as $L \to \infty$.

One has therefore to expect Space Time Uncertainty Relations emerging from first principles, already at a semiclassical level. Carrying through such an analysis [39, 40] one finds indeed that at least the following minimal restrictions must hold

$$\Delta q_0 \cdot \sum_{j=1}^{3} \Delta q_j \gtrsim 1; \quad \sum_{1 \leq j < k \leq 3} \Delta q_j \Delta q_k \gtrsim 1.$$  \hspace{1cm} (8)

Thus points become fuzzy and locality loses any precise meaning. We believe it should be replaced at the Planck scale by an equally sharp and compelling principle, which reduces to locality at larger distances.

The Space Time Uncertainty Relations strongly suggest that spacetime has a Quantum Structure at small scales, expressed, in generic units, by

$$[q_\mu, q_\nu] = i \lambda^2 Q_{\mu\nu}$$ \hspace{1cm} (9)

where $Q$ has to be chosen not as a random toy mathematical model, but in such a way that (8) follows from (9).

To achieve this in the simplest way, it suffices to select the model where the $Q_{\mu\nu}$ are central, and impose the “Quantum Conditions” on the two invariants

$$Q_{\mu\nu} Q^{\mu\nu}$$ \hspace{1cm} (10)

$$[q_0, \ldots, q_3] \equiv \det \begin{pmatrix} q_0 & \cdots & q_3 \\ \vdots & \ddots & \vdots \\ q_0 & \cdots & q_3 \end{pmatrix}$$

$$\equiv \epsilon^{\mu\nu\lambda\rho} q_\mu q_\nu q_\lambda q_\rho = -\frac{1}{2} Q_{\mu\nu}(Q_{\lambda\rho})^{\mu\nu}$$ \hspace{1cm} (11)

whereby the first one must be zero and the square of the second is $I$ (in Planck units; we must take the square since it is a pseudoscalar and not a scalar).

One obtains in this way [39, 40] a model of Quantum Spacetime which implements exactly our Space Time Uncertainty Relations and is fully Poincare’ invariant. In any Lorentz frame, however, the Euclidean distance between two independent events can be shown to have a lower bound of order one in
Planck units. Two distinct points can never merge to a point. However, of course, the state where the minimum is achieved will depend upon the reference frame where the requirement is formulated. (The structure of length, area and volume operators on QST has been studied in full detail \cite{47}).

Thus the existence of a minimal length is not at all in contradiction with the Lorentz covariance of the model; note that models where the commutators of the coordinates are just numbers $\Theta$, which appear so often in the literature, arise as irreducible representations of our model; such models, taken for a fixed choice of $\Theta$ rather than for its full Lorentz orbit, necessarily break Lorentz covariance. To restore it as a twisted symmetry is essentially equivalent to going back to the model where the commutators are operators. This point has been recently clarified in great depth \cite{44}.

On the other side, a theory with a fixed, numerical commutator ($a \Theta$ in the sky) can hardly be realistic.

The geometry of Quantum Spacetime and the free field theories on it are fully Poincare' covariant. The various formulation of interaction between fields, all equivalent on ordinary Minkowski space, provide inequivalent approaches on QST; but all of them, sooner or later, meet problems with Lorentz covariance, apparently due to the nontrivial action of the Lorentz group on the centre of the algebra of Quantum Spacetime. On this point in our opinion a deeper understanding is needed.

One can however introduce interactions in different ways, all preserving spacetime translation and space rotation covariance; among these it is just worth mentioning here one of them, where one takes into account, in the very definition of Wick products, the fact that in our Quantum Spacetime two distinct points can never merge to a point. But it turns out that there is a canonical quantum diagonal map which associates to functions of $n$ independent points a function of a single point, evaluating conditional expectation which on functions of the differences takes a numerical value, associated with the minimum of the Euclidean distance (in a given Lorentz frame!).

The “Quantum Wick Product” obtained by this procedure leads to a perturbative Gell'Mann Low formula free of ultraviolet divergences at each term of the perturbation expansion \cite{43}.

The common feature of all approaches is that, due to the quantum nature of spacetime at the Planck scale, locality is broken (even at the level of free fields, for explicit estimates see \cite{39}); in perturbation theory, its breakdown produces a non local kernel, which spreads the interaction vertices \cite{39,41,42}; this forces on us the appropriate modifications of Feynman rules \cite{40}.

But nonlocal effects should be visible only at Planck scales, and vanish
fast for larger separations. If Lorentz invariance can be maintained by interactions, a point quite open at present, then we ought to expect the Spin and Statistics to remain true, as mentioned earlier in this section.

That argument might, however, raise the objection that, in a theory which accounts for gravitational interactions as well, there might be no reasonable scattering theory at all, due to the well known paradox of loss of information, if black holes are created in a scattering process, destroying the unitarity of the S matrix.

Of course, this is an open problem; but one might well take the attitude that a final answer to it will come only from a complete theory, while at the moment we are rather relying on semiclassical arguments. Which might be quite a reasonable guide in order to get indications of local behaviours; but scattering theory involves the limit to infinite past/future times; and it might well be that interchanging these limits with those in which the semiclassical approximations are valid, or with the infinite volume limit in which the thermal behaviour of the vacuum for a uniformly accelerated observer becomes an exact mathematical statement, is dangerous, if not misleading. And whatever theory will account for Quantum Gravity, it should also describe the world of Local Quantum Field Theory as an appropriate approximation.

One might expect that a complete theory ought to be covariant under general coordinate transformations as well. This principle, however, is grounded on the conceptual experiment of the falling lift, which, in the classical theory, can be thought of as occupying an infinitesimal neighbourhood of a point. In a quantum theory the size of a "laboratory" must be large compared with the Planck length, and this might pose limitations on general covariance.

On the other side elementary particle theory deals with collisions which take place in narrow space regions, studied irrespectively of the surrounding large scale mass distributions, which we might well think of as described by the vacuum, and worry only about the short scale effects of gravitational forces.

We are thus lead to consider Quantum Minkowski Space as a more realistic geometric background for Elementary Particle Physics. But the energy distribution in a generic quantum state will affect the Spacetime Uncertainty Relations, suggesting that the commutator between the coordinates ought to depend in turn on the metric field. This scenario could be related to the large scale thermal equilibrium of the cosmic microwave background, and to the non vanishing of the Cosmological Constant \[45, 46\].

This might well be the clue to restore Lorentz covariance in the interac-
tions between fields on Quantum Spacetime.

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