Seiberg-Witten theory and modular $\lambda$-function

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**Abstract:** In this paper, we will apply the tools in number theory and modular forms to the study of Seiberg-Witten theory. We will obtain the transformation properties of the masses of BPS states under $S$-duality and $T$-duality. We will also study the behavior of these masses near the non-purturbative limit using Great Picard’s Theorem.

**Keywords:** Seiberg-Witten theory, modular form, theta function, Great Picard’s Theorem.

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1 Introduction

In Seiberg and Witten’s fundamental paper [6], the prepotential of the $N = 2$ supersymmetric Yang-Mills theory with $SU(2)$ gauge group is completely determined by holomorphy and electric-magnetic duality. The masses of BPS states are given by the periods of a meromorphic oneform on elliptic curves [6]. More precisely, let us consider the following family of elliptic curves

$$E_u : y^2 = (x - 1)(x + 1)(x - u), u \in \mathbb{C} - \{-1, 1\},$$

(1.1)

and for every $E_u$, the meromorphic oneform $\Omega_{SW}$, called the Seiberg-Witten oneform, is defined by

$$\Omega_{SW} = \frac{\sqrt{2}}{2\pi} \left( \frac{xdx}{y} - \frac{udx}{y} \right).$$

(1.2)

The residue of $\Omega_{SW}$ at its pole is zero, therefore it defines an element of the cohomology group $H^1(E_u, \mathbb{C})$ [6]. The homology group $H_1(E_u, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^2$, and suppose $\{\gamma_0^u, \gamma_1^u\}$ form a basis for it. The masses of BPS states for the SU(2) pure gauge theory form a rank-2 lattice

$$\{ma + na_D\}, (m, n) \in \mathbb{Z}^2$$

(1.3)

where $a$ and $a_D$ are given by

$$a = \int_{\gamma_0^u} \Omega_{SW}, \quad a_D = \int_{\gamma_1^u} \Omega_{SW}.$$

(1.4)
In order to use the tools developed in number theory and modular forms, we first transform the family in the formula 1.1 to the Legendre family of elliptic curves. It is straightforward to see that the following transformation

\[ x \mapsto 2x - 1, \; y \mapsto \sqrt{2}y, \; u \mapsto 2\lambda - 1 \]  

sends the family 1.1 to the Legendre family

\[ E_\lambda : y^2 = x(x - 1)(x - \lambda), \; \lambda \in \mathbb{C} - \{0, 1\}, \]  

while the meromorphic oneform \( \Omega_{SW} \) is sent to the oneform

\[ \Omega_{SW} = \frac{1}{\pi} \left( \frac{xdx}{y} - \frac{\lambda dx}{y} \right) . \]  

In this paper, we will study the periods of the new meromorphic oneform \( \Omega_{SW} \) in formula 1.7 by tools from number theory and modular forms, most notably the properties of the modular \( \lambda \)-function and theta functions. We will show how the periods \( \{a_D, a\} \) transform under the S-duality and T-duality. We will also apply the Great Picard’s Theorem to study the behavior of the quotient \( a_D/a \) near the non-perturbative limit \( u = \pm 1 \), which does reveal some pathological properties that has connection with wall-crossing.

The outline of this paper is as follows. In Section 2, we will introduce the Legendre family of elliptic curves and compute its periods and Picard-Fuchs equation. In Section 3, we will discuss the properties of modular \( \lambda \)-function. In Section 4, we will look at the Seiberg-Witten oneform and its periods. In Section 5, we will study the transformation properties of periods of Seiberg-Witten oneform. In Section 6, we will apply Great Picard’s Theorem to study the behavior of the Seiberg-Witten periods near the non-perturbative limit. In Section 7, we will look at the metric of the moduli space of vacua for the \( N = 2 \) supersymmetric Yang-Mills theory with \( SU(2) \) gauge group.

## 2 The Legendre Family of Elliptic Curves

The Legendre family of elliptic curves is a family of curves in \( \mathbb{P}^2 \) defined by the equation

\[ E_\lambda : Y^2Z = X(X - Z)(X - \lambda Z), \; \lambda \in \mathbb{P}^1. \]  

The singular fibers of this family are over the points \( \lambda = 0, 1, \infty \). When \( \lambda = 0, 1 \), this family degenerates to a nodal cubic with a double point. In literature, it is usually denoted by its affine open subset where \( Z \neq 0 \), which becomes

\[ E_\lambda : y^2 = x(x - 1)(x - \lambda) . \]  

When \( Z = 0 \), equation 2.1 implies \( X = 0 \), thus we deduce that the compact curve in the formula 2.1 is the compactification of the affine curve in the formula 2.2 by the infinity point is \( (0, 1, 0) \). Let us now recall the construction of elliptic curves by branch cuts and gluing. We will follow the book [2] closely, but we will choose a different branch cut. First cut the complex plane at two lines: 0 to \( \lambda \) and 1 to \( \infty \), and next take a second copy of the complex plane and cut it at the same lines. Then we glue the two copies together along the branch cuts. In this way, we obtain a torus, which has a complex structure defined by the equation 2.2. The readers are referred to the book [2] for more details.
2.1 Periods of the Legendre family

Let $\Omega^1_{E_\lambda}$ be the sheaf of algebraic oneforms on $E_\lambda$. In fact, it is a trivial sheaf, and there exists a nowhere vanishing oneform $\Omega$ that induces a trivialization of $\Omega^1_{E_\lambda}$. On the affine open subset of $E_\lambda$ where $y \neq 0$, $\Omega$ is given by

$$\Omega = \frac{dx}{2\pi y}. \quad (2.3)$$

It is a straightforward exercise to check that $dx/(2\pi y)$ actually extends to a nowhere vanishing oneform on $E_\lambda$ [2]. Before we come to the computations of the periods of $\Omega$, let us first construct a basis $\{\gamma_0, \gamma_1\}$ for the homology group $H_1(E_\lambda, \mathbb{Z}) \simeq \mathbb{Z}^2$. Let $\gamma_0$ be the cycle on one copy of the complex plane $\mathbb{C}$ that encircles the branch line $(1, \infty)$, while let $\gamma_1$ be the circle that is the composite of two lines: the line from $1$ to $\lambda$ on the first copy of the complex plane and the line from $\lambda$ to $1$ on the second copy. The integration of $\Omega$ over this basis gives us two periods

$$\varpi_0(\lambda) = \int_{\gamma_0} \Omega, \quad \varpi_1(\lambda) = \int_{\gamma_1} \Omega. \quad (2.4)$$

The computations of these two periods are certainly very well-known and have a very long history. Up to a sign, they are given by the integrals

$$\varpi_0(\lambda) = \frac{1}{\pi} \int_1^\infty \frac{dx}{\sqrt{x(x-1)(x-\lambda)}},$$
$$\varpi_1(\lambda) = \frac{1}{\pi} \int_1^\lambda \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}. \quad (2.5)$$

After a change of variable by $x = 1/z$, the first integral in formula 2.5 becomes

$$\varpi_0(\lambda) = \frac{1}{\pi} \int_0^1 \frac{dz}{\sqrt{z(1-z)(1-\lambda z)}}. \quad (2.6)$$

When $\lambda$ is in a small neighborhood of 0, we can expand the factor $(1 - \lambda z)^{-1/2}$ to a power series in $\lambda z$. Then the integration over $z$ gives us

$$\varpi_0(\lambda) = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^2 \frac{\lambda^n}{n!}. \quad (2.7)$$

The period $\varpi_1(\lambda)$ can also be computed by similar methods, and it admits an expansion with leading term given by

$$\varpi_1(\lambda) = -\frac{1}{\pi i} \int_1^{\lambda} \frac{dx}{\sqrt{x(1-x)(x-\lambda)}} = -\frac{1}{\pi i} \left(4 \log 2 - \log \lambda + \cdots\right), \quad (2.8)$$

where under $\lambda \to 0$, the limit of terms in $\cdots$ is zero. Thus we deduce that the period $\varpi_1(\lambda)$ must be of the form

$$\varpi_1(\lambda) = \frac{1}{\pi i} \left(\varpi_0(\lambda) \log \lambda + h(\lambda)\right) - \frac{\log 16}{\pi i} \varpi_0(\lambda), \quad (2.9)$$

where $h(\lambda)$ is a power series with first several terms given by [2]

$$h(\lambda) = \frac{1}{2} \lambda + \frac{21}{64} \lambda^2 + \frac{185}{768} \lambda^3 + \cdots. \quad (2.10)$$
2.2 Picard-Fuchs equation

The derivative of \(\Omega\) with respect to \(\lambda\) is given by
\[
\frac{d\Omega}{d\lambda} = \frac{1}{4\pi} \frac{dx}{\sqrt{x(x-1)(x-\lambda)^3}},
\]
which is a rational oneform with a double pole at the point \((x = \lambda, y = 0)\). However its residue at this pole is zero, therefore it also defines an element of \(H^1(\mathcal{E}_\lambda, \mathbb{C}) \simeq \mathbb{C}^2\). Moreover, \(\{\Omega, d\Omega/d\lambda\}\) form a basis of \(H^1(\mathcal{E}_\lambda, \mathbb{C})\). The second derivative of \(\Omega\), i.e. \(d^2\Omega/d\lambda^2\), also defines an element of \(H^1(\mathcal{E}_\lambda, \mathbb{C})\), hence must be a linear combination of \(\Omega\) and \(d\Omega/d\lambda\).

From Chapter 1 of [2], the oneform \(\Omega\) satisfies
\[
\lambda(\lambda - 1) \frac{d^2\Omega}{d\lambda^2} + (2\lambda - 1) \frac{d\Omega}{d\lambda} + \frac{1}{4} \Omega = 0,
\]
which is called the Picard-Fuchs equation of \(\Omega\). From it, we deduce that the period \(\varpi_i(\lambda)\) also satisfies this Picard-Fuchs equation
\[
\lambda(\lambda - 1) \varpi_i''(\lambda) + (2\lambda - 1) \varpi_i'(\lambda) + \frac{1}{4} \varpi_i(\lambda) = 0, \quad i = 0, 1.
\]
In fact, \(\{\varpi_0(\lambda), \varpi_1(\lambda)\}\) form a basis of the solution space since \(\{\gamma_0, \gamma_1\}\) form a basis of \(H_1(\mathcal{E}_\lambda, \mathbb{Z})\).

3 Modular \(\lambda\)-function

From the computation in last section, the monodromy of the periods \(\{\varpi_0(\lambda), \varpi_1(\lambda)\}\) at \(\lambda = 0\) is given by
\[
\varpi_0(\lambda) \rightarrow \varpi_0(\lambda), \quad \varpi_1(\lambda) \rightarrow \varpi_1(\lambda) + 2\varpi_0(\lambda).
\]

If we form a column vector \((\varpi_0, \varpi_1)^\top\), then the monodromy matrix is
\[
T_0 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.
\]
The monodromy matrices of the Legendre family at the three singular points \(\{0, 1, \infty\}\) generate the modular group \(\Gamma(2)\), and the readers are referred to the book [2] for more details.

Definition 3.1. The coordinate \(\tau\) of the Legendre family is defined by the quotient
\[
\tau = \frac{\varpi_1(\lambda)}{\varpi_0(\lambda)}.
\]
In a small neighborhood of \(\lambda = 0\), \(\tau\) is given by
\[
\tau = \frac{1}{\pi i} \left( \log \lambda + \frac{h(\lambda)}{\varpi_0(\lambda)} \right) - \frac{\log 16}{\pi i}.
\]
The coordinate $\tau$ has a geometric meaning, more precisely the elliptic curve $E_\lambda$ is isomorphic to the quotient of $\mathbb{C}$ by the lattice spanned by $\{1, \tau\}$ [2]. Let $\lambda$ be the coordinate of $\mathbb{P}^1$, then $\tau$ defines a map \[ \tau : \mathbb{P}^1 \to \mathbb{H} \] (3.5)
The inverse of $\tau$ is the famous modular $\lambda$-function. Furthermore, $\lambda$ generates the function field of the modular curve $X(2)$, i.e. it is a Hauptmodul for $X(2)$ [3]. Let us define $q$ by
\[ q := \exp\pi i \tau. \] (3.6)
From formula 3.4, we obtain
\[ q = \frac{1}{16} \lambda \exp(h(\lambda)/\wp_0(\lambda)). \] (3.7)
We can invert this equation order by order, which gives us the power series expansion of $\lambda$ with respective to $q$ [2]. For example, the first two terms of $\lambda(q)$ is given by
\[ \lambda = 16 q - 128 q^2 + \cdots. \] (3.8)
The modular $\lambda$-function can be written as products of $\theta$ functions, and from [3] we have
\[ \lambda(q) = \frac{\theta_4^4(0, q)}{\theta_3^4(0, q)}, \quad 1 - \lambda(q) = \frac{\theta_4^4(0, q)}{\theta_3^4(0, q)}; \] (3.9)
where we have used the Jacobi identity
\[ \theta_3^4(0, q) = \theta_2^4(0, q) + \theta_4^4(0, q). \] (3.10)

**Remark 3.2.** It is interesting to notice that the index of the group $\Gamma(2)$ in $\text{PSL}(2, \mathbb{Z})$ is 6 and the quotient group $\text{PSL}(2, \mathbb{Z})/\Gamma(2)$ is generated by
\[ T : \tau \mapsto \tau + 1, \quad S : \tau \mapsto -\frac{1}{\tau}. \] (3.11)
Therefore $\Gamma(2) \backslash \mathbb{H}$ is a six-fold cover of $\text{PSL}(2, \mathbb{Z}) \backslash \mathbb{H}$.

Follow mirror symmetry [4, 5], let us now define the normalized Yukawa coupling by
\[ \mathcal{Y} := \frac{1}{\wp_0^2(\lambda)} \left( \int_{\delta\lambda} \Omega \wedge \frac{d\Omega}{d\lambda} \right) d\lambda, \] (3.12)
which is a oneform defined on $\mathbb{C} - \{0, 1\}$. The computation of the Yukawa coupling $\mathcal{Y}$ is very simple, and we have
\[ \mathcal{Y} = \frac{1}{\wp_0^2(\lambda)} \left( \wp_0(\lambda) \frac{d\wp_1(\lambda)}{d\lambda} - \wp_1(\lambda) \frac{d\wp_0(\lambda)}{d\lambda} \right) d\lambda = \frac{d(\wp_1(\lambda)/\wp_0(\lambda))}{d\lambda} d\lambda = d\tau. \] (3.13)
Now define $W_k$ by
\[ W_k = \int_{\delta\lambda} \Omega \wedge \frac{d^k\Omega}{d\lambda^k}, \quad k = 0, 1, 2, \cdots, \] (3.14)
and from this definition, we immediately have
\[ W_0 = 0, \quad W_2 = \frac{dW_1}{d\lambda}. \] (3.15)

The Picard-Fuchs equation 2.12 implies
\[ \lambda(\lambda - 1) \frac{dW_1}{d\lambda} + (2\lambda - 1) W_1 = 0. \] (3.16)

Solve this differential equation and we obtain
\[ W_1 = \frac{C}{\lambda(1 - \lambda)}, \] (3.17)
where \( C \) is a nonzero constant, and in fact it is \( \frac{1}{\pi i} \). Hence the normalized Yukawa coupling \( Y \) is also equal to
\[ Y = \frac{1}{\pi i} \frac{d\lambda}{\varpi_0^2(\lambda) \lambda(1 - \lambda)}. \] (3.18)

Then formula 3.13 immediately implies
\[ \pi i \frac{d\tau}{d\lambda} = \frac{1}{\varpi_0^2(\lambda) \lambda(1 - \lambda)}, \] (3.19)
which is equivalent to
\[ \frac{1}{\pi i} \frac{d\lambda}{d\tau} = \varpi_0^2(\lambda) \lambda(1 - \lambda). \] (3.20)

**Lemma 3.3.**
\[ \varpi_0(\lambda(q)) = \theta_3^2(0, q). \] (3.21)

*Proof.* From formula 3.20, we immediately have
\[ \varpi_0^2(\lambda) = \frac{1}{\pi i} \frac{d\lambda}{d\tau} \frac{1}{\lambda(1 - \lambda)}. \] (3.22)

The actions of the two generators \( T \) and \( S \) of \( \text{PSL}(2, \mathbb{Z}) \) on \( \lambda \) are given by
\[
T : \tau \mapsto \tau + 1 \quad : \lambda \mapsto \frac{\lambda}{\lambda - 1},
\]
\[
S : \tau \mapsto -\frac{1}{\tau} \quad : \lambda \mapsto 1 - \lambda.
\] (3.23)

From formula 3.22, the actions of \( T \) and \( S \) on \( \varpi_0^2(\lambda) \) are given by
\[
T : \tau \mapsto \tau + 1 \quad : \varpi_0^2(\lambda) \mapsto (1 - \lambda) \varpi_0^2(\lambda),
\]
\[
S : \tau \mapsto -\frac{1}{\tau} \quad : \varpi_0^2(\lambda) \mapsto -\tau^2 \varpi_0^2(\lambda).
\] (3.24)

But the actions of \( T \) and \( S \) on the \( \theta_3^2(0, q) \) are given by
\[
T : \tau \mapsto \tau + 1 \quad : \theta_3^2(0, q) \mapsto \theta_3^2(0, q) = (1 - \lambda) \theta_3^2(0, q),
\]
\[
S : \tau \mapsto -\frac{1}{\tau} \quad : \theta_3^2(0, q) \mapsto -\tau^2 \theta_3^2(0, q); \] (3.25)
where we have used formula 3.9. Since \( \varpi_0(\lambda(q)) \) does not vanish on the upper half plane, the quotient \( \theta_3^2(0, q)/\varpi_0(\lambda(q)) \) is a holomorphic function on \( \mathbb{H} \) that is invariant under the actions of \( \text{PSL}(2, \mathbb{Z}) \). Near the cusp \( i\infty \), the \( q \)-expansion of \( \theta_3^2(0, q)/\varpi_0(\lambda(q)) \) is given by

\[
\theta_3^2(0, q)/\varpi_0(\lambda(q)) = 1 + O(q^4),
\]

hence we deduce \( \theta_3^2(0, q)/\varpi_0(\lambda(q)) \) must be equal to 1.

**Remark 3.4.** This result is already well-known in the 19-th century [1].

To summarize, we have obtained

\[
\theta_2^2(0, q) = \lambda \varpi_0(\lambda(q)), \quad \theta_3^2(0, q) = \varpi_0(\lambda(q)), \quad \theta_4^2(0, q) = (1 - \lambda) \varpi_0^2(\lambda(q)).
\]

### 4 The Seiberg-Witten oneform

The Seiberg-Witten oneform \( \Omega_{SW} \) in the formula 1.7 is a meromorphic oneform on the elliptic curve \( \mathcal{E}_\lambda \) that can also be written as

\[
\Omega_{SW} = 2(x \Omega - \lambda \Omega).
\]

However for the purpose of this paper, it is more convenient to expand it with respect to the basis \( \{\Omega, \Omega'\} \) of \( H^1(\mathcal{E}_\lambda, \mathbb{C}) \). Let the function \( f \) be

\[
f := x^{1/2}(x - 1)^{1/2}(x - \lambda)^{-1/2},
\]

which defines a meromorphic function on the elliptic curve \( \mathcal{E}_\lambda \). Its differential is given by

\[
df = \frac{1}{2} x^{-1/2}(x - 1)^{1/2}(x - \lambda)^{-1/2}dx + \frac{1}{2} x^{1/2}(x - 1)^{-1/2}(x - \lambda)^{-1/2}dx - \frac{1}{2} x^{1/2}(x - 1)^{1/2}(x - \lambda)^{-3/2}dx.
\]

The oneform \( \Omega \) and its derivative \( \Omega' := d\Omega/d\lambda \) are of the form

\[
\Omega = \frac{1}{2\pi} x^{-1/2}(x - 1)^{-1/2}(x - \lambda)^{-1/2}dx,
\]

\[
\Omega' = \frac{1}{4\pi} x^{-1/2}(x - 1)^{-1/2}(x - \lambda)^{-3/2}dx,
\]

from which we deduce

\[
\frac{1}{2\pi} df = \frac{1}{2} (x - 1) \Omega + \frac{1}{2} x \Omega - x (x - 1) \Omega'.
\]

To simplify this formula, we will need the identity

\[
(x - \lambda) \Omega' = \frac{1}{2} \Omega,
\]
therefore we have
\[
x(x - 1) \Omega' = x(x - \lambda + \lambda - 1) \Omega'
\]
\[
= x\left(\frac{1}{2} \Omega + (\lambda - 1) \Omega'\right)
\]
\[
= \frac{1}{2} x \Omega + (\lambda - 1)(x - \lambda + \lambda) \Omega'
\]
\[
= \frac{1}{2} x \Omega + (\lambda - 1)\left(\frac{1}{2} \Omega + \lambda \Omega'\right).
\]
(4.7)

Hence formula 4.5 becomes
\[
\frac{1}{2\pi} df = \frac{1}{2} x \Omega - \frac{1}{2} \lambda \Omega - \lambda(\lambda - 1) \Omega',
\]
(4.8)

thus at the level of cohomology we have
\[
x \Omega = \lambda \Omega + 2 \lambda(\lambda - 1) \Omega'.
\]
(4.9)

Now the Seiberg-Witten oneform \(\Omega_{SW}\) 4.1 becomes
\[
\Omega_{SW} = 4 \lambda(\lambda - 1) \Omega',
\]
(4.10)

which is of a very simple form. The periods of \(\Omega_{SW}\) with respect to the homology basis \(\{\gamma_0, \gamma_1\}\) are given by
\[
\varpi_{SW}^0 = \int_{\gamma_0} 4 \lambda(\lambda - 1) \Omega' = 4 \lambda(\lambda - 1) \varpi_0'(\lambda),
\]
\[
\varpi_{SW}^1 = \int_{\gamma_1} 4 \lambda(\lambda - 1) \Omega' = 4 \lambda(\lambda - 1) \varpi_1'(\lambda).
\]
(4.11)

which will be called the Seiberg-Witten periods. The lattice of masses of BPS states becomes
\[
\{m \varpi_0^{SW} + n \varpi_1^{SW} : (m, n) \in \mathbb{Z}^2\}.
\]
(4.12)

From the Picard-Fuchs equation 2.12, we obtain
\[
(\varpi_0^{SW})' = -\varpi_0, \ (\varpi_1^{SW})' = -\varpi_1.
\]
(4.13)

Notice that our notation is different from the original paper [6]. We will further define \(\tau^{SW}\) to be the quotient
\[
\tau^{SW} := \varpi_1^{SW}/\varpi_0^{SW}.
\]
(4.14)

When \(\tau^{SW}\) lies in \(\mathbb{C} - \mathbb{R}\), the lattice \(\{m + n \tau^{SW} : (m, n) \in \mathbb{Z}^2\}\) defines an elliptic curve. While when \(\tau^{SW}\) is real, i.e. \(\tau^{SW} \in \mathbb{R}\), it will no longer defines an elliptic curve. The values of \(\lambda\) for which \(\tau^{SW}\) is real is of crucial importance in the study of BPS states wall-crossing [6].
5 Transformations of the Seiberg-Witten Periods

In this section, we will use the results obtained in Section 3 to study the transformation properties of Seiberg-Witten periods \( \{\varpi_{0}^{SW}, \varpi_{1}^{SW}\} \) under the action of \( \text{PSL}(2, \mathbb{Z}) \). Notice that \( \varpi_{1}' \) can also be written as

\[
\varpi_{1}' = (\varpi_{0} \tau)' = \frac{d\tau}{d\lambda} \varpi_{0} + \tau \varpi_{0}' = \frac{1}{\pi i} \frac{1}{\lambda(1-\lambda)} \varpi_{0} + \tau \varpi_{0}',
\]

where we have used formula 3.19. Under the action of \( S : \tau \mapsto -1/\tau \), we have

\[
S : \lambda \mapsto 1 - \lambda; \ d\lambda \mapsto -d\lambda; \ \varpi_{0} \mapsto i\tau \varpi_{0}; \ \varpi_{1} = \tau \varpi_{0} \mapsto -\frac{1}{\tau} i\tau \varpi_{0} = -i \varpi_{0};
\]

hence the transformations of \( \{\varpi_{0}^{SW}, \varpi_{1}^{SW}\} \) under \( S \) are

\[
S : \varpi_{0}^{SW} \mapsto -i \varpi_{1}^{SW},
\]

\[
S : \varpi_{1}^{SW} \mapsto i \varpi_{0}^{SW}.
\]

While \( \tau^{SW} \) transforms in the way

\[
S : \tau^{SW} \mapsto -\frac{1}{\tau^{SW}}.
\]

The transformation of \( \{\varpi_{0}^{SW}, \varpi_{1}^{SW}\} \) under \( T \) is however more complicated. Under the action of \( T : \tau \mapsto \tau + 1 \), we have

\[
T : \lambda \mapsto \frac{\lambda}{\lambda - 1}, \ d\lambda \mapsto -\frac{1}{(\lambda - 1)^2} d\lambda, \ \varpi_{0} \mapsto \sqrt{1 - \lambda} \varpi_{0};
\]

from which we obtain

\[
T : \varpi_{0}^{SW} \mapsto \frac{2\lambda}{\sqrt{1 - \lambda}} (2(\lambda - 1)\varpi_{0}' + \varpi_{0}),
\]

\[
T : \varpi_{1}^{SW} \mapsto \frac{2\lambda}{\sqrt{1 - \lambda}} \left( \left( 2(\lambda - 1) \frac{d\tau}{d\lambda} + \tau + 1 \right) \varpi_{0} + 2(\lambda - 1)(\tau + 1)\varpi_{0}' \right).
\]

However under the action of \( T^2 : \tau \mapsto \tau + 2 \), we have

\[
T^2 : \lambda \mapsto \lambda, \ d\lambda \mapsto d\lambda, \ \varpi_{0} \mapsto \varpi_{0},
\]

hence the Seiberg-Witten periods \( \{\varpi_{0}^{SW}, \varpi_{1}^{SW}\} \) transform in the way

\[
T^2 : \varpi_{0}^{SW} \mapsto \varpi_{0}^{SW},
\]

\[
T^2 : \varpi_{1}^{SW} \mapsto \varpi_{1}^{SW} + 2 \varpi_{0}^{SW}.
\]

While \( \tau^{SW} \) transforms in the way

\[
T^2 : \tau^{SW} \mapsto \tau^{SW} + 2.
\]
6 Great Picard’s Theorem and the non-perturbative limit

In this section, we will discuss the behaviors of $\tau^{SW} := \frac{\varpi_1^{SW}}{\varpi_0^{SW}}$ in a small neighborhood of the non-perturbative limit point $\lambda = 0$. First, $\tau^{SW}$ can be written as

$$\tau^{SW} = \frac{\varpi_1'}{\varpi_0'} = \frac{1}{\pi i} \left( \log \lambda + \frac{1}{\lambda} \frac{\varpi_0}{\varpi_0'} + \frac{h'}{\varpi_0'} - \log 16 \right). \quad (6.1)$$

Now define $q^{SW}$ by

$$q^{SW} := \exp \pi i \tau^{SW}, \quad (6.2)$$

and $\tau^{SW}$ is real if and only if the modulus of $q^{SW}$ is one. Formula 6.1 implies

$$16 q^{SW} = \lambda \exp(h'/\varpi_0') \exp \left(\frac{1}{\lambda} \frac{\varpi_0}{\varpi_0'} - 4\right). \quad (6.3)$$

The series expansion of $\varpi_0/\varpi_0'$ in a small neighborhood of $\lambda = 0$ is of the form

$$\varpi_0/\varpi_0' = 4 - \frac{7}{2} \lambda + O(\lambda^2), \quad (6.4)$$

from which we deduce

$$16 q^{SW} = F(\lambda) \exp(4/\lambda), \quad F(\lambda) := \lambda \exp(h'/\varpi_0') \exp \left(\frac{1}{\lambda} \frac{\varpi_0}{\varpi_0'} - 4\right). \quad (6.5)$$

The function $F(\lambda)$ is holomorphic in a neighborhood of $\lambda = 0$ and it has a series expansion with the first term given by

$$F(\lambda) = \lambda + O(\lambda^2). \quad (6.6)$$

The crucial observation is that the function $F(\lambda) \exp(4/\lambda)$ is holomorphic in a small punctured neighborhood of 0, while the point $\lambda = 0$ is an essential singularity. Now we need the following important theorem of Picard.

**Great Picard’s Theorem:** If $G(z)$ is a holomorphic function and $z_0$ is an essential singularity of $G(z)$, then on any punctured neighborhood of $z_0$, $G$ takes on all possible complex values, with at most a single exception, infinitely many times.

In particular, the **Great Picard’s Theorem** implies that in every punctured neighborhood of $\lambda = 0$, which is assumed to be small enough to avoid other singularities of $F(\lambda) \exp(4/\lambda)$, the function $F(\lambda) \exp(4/\lambda)$ takes on all possible values in $\{\exp \pi i \tau^{SW} : \tau^{SW} \in \mathbb{R}\}$, with at most a single exception, infinitely many times. Therefore the behavior of $\tau^{SW}$ in a small punctured neighborhood of $\lambda = 0$ is very complicated.

7 Metric of the moduli space

In this section, we will look at the metric of the moduli space of vacua for the $N = 2$ supersymmetric Yang-Mills theory with $SU(2)$ gauge group. First, since the masses $\{a_D, a\}$ are periods of the Seiberg-Witten oneform $\Omega^{SW}$, therefore there is an $SL(2, \mathbb{Z})$ transformation between $\{a_D, a\}$ and $\{\varpi_0^{SW}, \varpi_1^{SW}\}$. After studying the branch cuts, we deduce that [6]

$$a_D = \varpi_1^{SW}, \quad a = \varpi_1^{SW} + \varpi_0^{SW}. \quad (7.1)$$
The metric on the moduli space of vacua is given by [6]

\[(ds)^2 = \text{Im}(da_D d\bar{a}) = -\frac{i}{2}(da_D d\bar{a} - dad\bar{a}_D).\]  

(7.2)

Using the above relation, we obtain

\[(ds)^2 = -\frac{i}{2}(\varpi_1 \varpi_0 - \varpi_0 \varpi_1) d\lambda d\bar{\lambda},\]  

(7.3)

where we have used the formula

\[d\varpi_i^{\text{SW}} / d\lambda = -\varpi_i, \ i = 1, 2.\]  

(7.4)

We immediately recognize that this metric can also be written as

\[(ds)^2 = \left( \frac{i}{2} \int_{\xi_\lambda} \Omega \wedge \bar{\Omega} \right) d\lambda d\bar{\lambda}.\]  

(7.5)

The pull back of this metric to the upper half plane is given by

\[|\varpi_0 \frac{d\lambda}{d\tau}|^2 (\text{Im}\tau) d\tau d\bar{\tau},\]  

(7.6)

which is of a rather simple and explicit form.

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