On Invariant Measures
for $Diff(S^1)$

Doug Pickrell
Mathematics Department
University of Arizona
Tucson, Arizona 85721

Abstract. In [Pi] we constructed biinvariant measures (possibly having values in a line bundle) for a unitary loop group $LK$ acting on the formal completion of its complexification $LG$. One motivation for this was to find a geometric construction for the unitary structure of the positive energy representations of $LK$. In this paper we pursue an analogous construction for $Diff(S^1)$. 
§0. Introduction.

The complex Virasoro algebra is the universal central extension of the Lie algebra of complex trigonometric vector fields on the circle. As a vector space

\[ Vir = \left( \sum_{n \in \mathbb{Z}} \mathbb{C} L_n \right) \oplus \mathbb{C} \kappa, \]  

(0.1)

where \( L_n = i e^{in\theta} \frac{d}{d\theta} = -z^{n+1} \frac{d}{dz} \); the bracket is determined by the relations

\[ [L_n, L_m] = (m - n)L_{n+m} + \frac{1}{12} n(n^2 - 1) \delta(n + m) \kappa, \quad [L_n, \kappa] = 0. \]  

(0.2)

The universal central extension \( \hat{D}_{an} \) of \( D_{an} \), the analytic, orientation-preserving diffeomorphisms of \( S^1 \), is a global real form for \( Vir \).

The Virasoro algebra has a triangular decomposition, in the technical sense of [MP], where

\[ n^\pm = \sum_{\pm n > 0} \mathbb{C} L_n, \quad \text{and} \quad \hat{h} = \mathbb{C} L_0 \oplus \mathbb{C} \kappa, \]  

(0.3)

so that \( Vir \) and \( \hat{D}_{an} \) are, in some limited respects, similar to a rank two affine Kac-Moody Lie algebra \( g(A) \) and a global real form \( K(A) \). In particular the highest weight representations can be realized via a Borel-Weil type construction ([KY]), and the unitarizable modules can be exponentiated to \( D_{an} \). For this reason, following [Pi], it is natural to inquire whether there exist invariant (bundle-valued) measures for \( \hat{D}_{an} \), acting on a completion of \( D_{an} \). This question arises in string theory as well, where the appropriate completion is \( D_{qs} \), the group of quasisymmetric homeomorphisms of \( S^1 \) (see the “physicist’s wish-list” in §5 of [Pe]). In this paper we will construct (bundle-valued) measures, using basically the same technique as in Part III of [Pi]. We conjecture that these measures are invariant, and we would like to believe that they are supported on \( D_{qs} \), but we are very far from being able to prove this rigorously.

Only fragments of the general setup in [Pi] carry over to the Virasoro context. In particular there is not a complex algebraic Virasoro group \( \hat{D}^C \), so that it is unlikely that there exists a good analogue of the formal completion, \( \hat{D}^C_{\text{formal}} \). Despite this we can consider the top stratum

\[ \mathcal{N}^- \times \hat{H} \times \mathcal{N}^+ \subset \hat{D}^C_{\text{formal}}, \]  

(0.4)
where $\mathcal{N}^\pm$ are the simply connected profinite nilpotent Lie groups corresponding to the Lie algebras

$$n^\pm_{\text{formal}} = \prod_{\pm k > 0} C L_k,$$

(0.5)

respectively, and $\hat{\mathcal{H}}$ is the simply connected group corresponding to $\hat{\mathfrak{h}}$. Unlike the situation for Kac-Moody groups, there is an injection

$$\hat{\mathcal{D}} \to \mathcal{N}^- \times \hat{\mathcal{H}} \times \mathcal{N}^+.$$

(0.6)

In concrete terms this amounts to the classical fact that each (quasisymmetric) $\phi \in \mathcal{D}$ can be uniquely factored as a composition of maps

$$\phi = l \circ \text{diag} \circ u$$

(0.7)

where

$$u = z(1 + \sum_{n \geq 1} u_n z^n)$$

(0.8)

is a univalent holomorphic function on the disk, $\text{diag}$ is rotation by a complex number of absolute value $\leq 1$, the mapping inverse to $l$,

$$l^{-1} = z + \sum_{n \geq 0} b_n z^{-n},$$

(0.9)

is a univalent holomorphic function on the disk about infinity, and the compatibility condition

$$\text{diag}(u(S^1)) = l^{-1}(S^1)$$

(0.10)

is satisfied (see for example page 100 of [L]). Similarly there does not exist a formal flag space (in the sense of [Pi]), but because of the inclusion (0.6), and the Bieberbach-de Branges inequalities ([B]), we can reasonably define the formal completion of $\text{Rot}(S^1) \backslash \mathcal{D}_{an}$, viewed as a subset of $\mathcal{N}^+$, as

$$(\text{Rot}(S^1) \backslash \mathcal{D})_{\text{formal}} = \{ u = z(1 + \sum_{n \geq 1} u_n z^n) : |u_n| < n + 1 \},$$

where we are viewing $u$ simply as a formal power series. The Lie algebra $\mathfrak{vir}$ acts holomorphically on these spaces in a natural way.

As in the theory of loop groups, the elemental matrix coefficients can be formally written in terms of Toeplitz determinants

$$\sigma_{c,h}(\hat{\phi}) = \det(A_a(\hat{\phi}))^{c-s_h} \det(A_p(\hat{\phi}))^{sh},$$

(0.11)
\[ = \det(A_a(\hat{\phi}))^c \text{diag}(\phi)^{8h} \]

where \( \phi \) denotes a diffeomorphism of \( S^1 \), \( \hat{\phi} \) is in the universal central extension, and the subscripts refer to a choice of periodic or antiperiodic spin structure for \( S^1 \). To make sense of the formal measure

\[ |\text{diag}(\phi)|^{16h} \det|A_a(\phi)|^{2c} \mathcal{D}(\phi), \quad (0.12) \]

we proceed as in the case of loop groups, by first constructing a regularization with parameter

\[ d\nu_{\beta,c,h}(\phi) = \frac{1}{E} |\text{diag}(\phi)|^{16h} \det|A_a(\phi)|^{2c} d\nu_\beta(\phi), \quad (0.13) \]

where \( \nu_\beta \) denotes a (slight variation of a) left quasi-invariant measure on \( \mathcal{D}_{C^1} \) constructed by the Malliavins in [MM] and Shavgulidze in [Sh]. One can make sense of this regularization by essentially the same method used in the loop group case (for \( c,h \geq 0 \)). From a geometric point of view, one should think of \( \sigma_{c,h} \) as a section of a line bundle \( L^c_{c,h} \rightarrow \mathcal{D} \); the expression (0.12) is heuristically

\[ \langle \sigma_{c,h}, \sigma_{c,h} \rangle_{\mathcal{D}^\phi}, \quad (0.14) \]

the density for the norm squared of \( \sigma_{c,h} \) as a section. From this point of view, \( \nu_{\beta,c,h} \) is a coordinate expression for a measure having values in the line bundle

\[ |\mathcal{L}_{c,h}|^2 = |\text{Det}A_a|^{-2(c-8h)} \otimes |\text{Det}A_p|^{-16h} \rightarrow \mathcal{D} \quad (0.15) \]

It is very likely that the bundle-valued measure \( \nu_{\beta,c,h} \) is asymptotically invariant as \( \beta \downarrow 0 \), with respect to \( \mathcal{D}_{an} \) acting from the left. This is an interesting issue, because it turns out to be intimately related to stationary phase approximations for Brownian paths. The measure \( \nu_{\beta,c,h} \) is not quasi-invariant for the right action, although it does heuristically appear to be invariant in the limit as \( \beta \downarrow 0 \).

The next step is to take the limit of \( \nu_{\beta,c,h} \) as the inverse temperature \( \beta \) tends to zero. The Bieberbach-de Branges inequalities (or just inequalities coming from the area theorem) immediately imply that weak limits exist. Unfortunately in the Virasoro case there is not enough structure to easily parlay this into a statement about the existence of an invariant measure \( \mu \) with values in \( |\mathcal{L}_{c,h}|^2 \). What we lack, in contrast to the Kac-Moody case, is a priori knowledge of certain elemental distributions (the first nontrivial coefficients \( u_1 \) and \( b_0 \) for the univalent functions \( u \) and \( l \) above, which correspond to the simple roots).
If we do succeed in proving that the limit

\[ \mu_{c,h} = \lim_{\beta \downarrow 0} \nu_{\beta,c,h} \]  

exists and is $D_{an}$-invariant (as a bundle-valued measure), then we will have an appealing geometric way of understanding why the lowest weight module corresponding to $(c,h)$ is unitary. It is intuitively clear that this should work for only a continuous set of parameters, and unfortunately our work so far does not pinpoint where the constraint $c > 1$ should arise (we suspect that it arises in the calculation of the distribution for the $u_1$ coefficient - this probably has an absolutely continuous distribution only for $c > 1$).

§1. Completions and Classical Analysis.

Virtually all the material in this section is well-known. In constructing the loop group analogue of the measure (0.7), Peller’s work on the Schatten properties of Hankel operators for multiplication operators played a pivotal role. The analogous results for homeomorphisms of $S^1$ have been established by Coifman, Meyer, Peller, Rochberg and Semmes. In §1.2 we formulate their fundamental work in the context of this paper. In §1.5 we discuss embeddings into the top stratum of the (nonexistent) formal completion, which involves the Ahlfors-Beurling theory of quasi-symmetric homeomorphisms of $S^1$.

§1.1. Notation.

We will denote a group of orientation preserving homeomorphisms of $S^1$ by $D$, where a subscript will indicate the order of smoothness of the elements of the group. The universal covering will be denoted by $\tilde{D}$, and it will be identified with the set of $\Phi : \mathbb{R} \to \mathbb{R}$ such that

\[ \Phi(s + 2\pi) = \Phi(s) + 2\pi, \quad \Phi \text{ is increasing}, \]  

(1.1.1)

and $\Phi$ and $\Phi^{-1}$ satisfy the appropriate smoothness condition. We will frequently write a homeomorphism of $S^1$ as

\[ \phi(e^{i\theta}) = e^{i\Phi(\theta)}, \]  

(1.1.2)

without further comment.
The group \( \mathcal{D}_{C^\infty} \) acts naturally and unitarily on half-densities on \( S^1 \). We identify these densities with functions in the usual way,

\[
L^2(S^1) \to \Omega_{L^2}^{1/2}(S^1) : f \mapsto f|d\theta|^{1/2}.
\]

Then \( \phi \in \mathcal{D}_{C^\infty} \) corresponds to a unitary operator \( U_\phi \) of \( L^2(S^1) \), where

\[
U_{\phi^{-1}} : f \mapsto (\Phi')^{1/2} f \circ \phi.
\]

Relative to the Hardy space polarization of \( L^2(S^1) \), we will write

\[
U_\phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]

In the next subsection we will be interested in intrinsic characterizations of certain induced topologies on \( \mathcal{D}_{C^\infty} \). A standard prototype is the following standard

(1.1.7)Proposition. The closure of \( \mathcal{D}_{C^\infty} \) in the strong operator topology for \( U(L^2(S^1)) \) is

\[
\mathcal{D}_{W^1_{L^1}} = \{ \phi \in \mathcal{D} : \phi \text{ and } \phi^{-1} \text{ are absolutely continuous} \}.
\]

Thus \( U_{\phi_n} \to U_\phi \) strongly if and only if \( \phi_n \to \phi \) and \( \psi_n \to \psi \) in \( W^1_{L^1} \), where \( \psi = \phi^{-1} \).

(1.1.8)Remarks. (a) It is not the case that \( \phi \) absolutely continuous implies that its inverse \( \psi \) is absolutely continuous. For example, let \( S \) denote a generalized Cantor set of positive measure. Then the complement \( S^c \) is open and dense, hence

\[
\Phi(x) = \int_0^x \chi_{S^c} ds
\]

is strictly increasing and absolutely continuous. But \( S \) has positive measure while \( \Phi(S) \) has measure zero, hence its inverse is not absolutely continuous. This illustrates why we generally need to impose conditions on both \( \phi \) and its inverse. In a vague way, this explains why the conditions that arise in the next subsection are expressed in terms of \( \log \Phi' \).

(b) A second, but less interesting, topology on \( \mathcal{D}_{W^1_{L^1}} \) is that induced by the operator norm topology on \( U(L^2) \). In this case the induced topology is the discrete topology.

§1.2. The \( p \)-Kac-Peterson completion and restricted groups.
Given a polarized separable Hilbert space, \( H = H_+ \oplus H_- \), and a symmetrically normed ideal \( \mathcal{I} \), there is an associated Banach \(*\)-algebra, \( \mathcal{L}(\mathcal{I}) \), which consists of bounded operators on \( H \), represented as two by two matrices with respect to the polarization, as in (1.1.5), such that the norm
\[
\left| \begin{pmatrix} A & D \\ C & B \end{pmatrix} \right|_{\mathcal{I}} + \left| \begin{pmatrix} A & D \\ C & B \end{pmatrix} \right|_{\mathcal{I}}
\]
(1.2.1)
is finite. The \(*\)-operation is the usual adjoint operation. The corresponding unitary group is
\[ U(\mathcal{I}) = U(H) \cap \mathcal{L}(\mathcal{I}); \]
it is referred to as a restricted unitary group in [PS]. Geometrically this group is the group of automorphisms of a Grassmannian (Finsler) symmetric space modelled on \( \mathcal{I} \). There are two obvious topologies on \( U(\mathcal{I}) \). The first is the induced Banach topology, and in this topology \( U(\mathcal{I}) \) has the additional structure of a Banach Lie group. The second is the Polish topology \( \tau_{KM} \) for which convergence means that for \( g_n, g \in U(\mathcal{I}) \), \( g_n \to g \) if and only if \( g_n \to g \) strongly and
\[
\begin{pmatrix} C_n & B_n \\ C & B \end{pmatrix} \to \begin{pmatrix} C & B \\ C_n & B_n \end{pmatrix} \text{ in } \mathcal{I}.
\]
For \( \mathcal{I} = \mathcal{L}_2 \), Hilbert-Schmidt operators, the identity component of \( (U(\mathcal{I}), \tau_{KM}) \) is the Kac-Peterson completion of the (infinite classical) Kac-Moody group of type \( A_{2\infty} \), modulo its center (see the introduction to Part II of [Pi]).

When \( H = L^2(S^1, \mathbb{C}^n) \) with the Hardy polarization, and \( \mathcal{I} = \mathcal{L}_p \), there is a beautiful intrinsic description of the (metric isomorphism class of the) induced norm on the subalgebra
\[
\text{Map}(S^1, \mathcal{L}(\mathbb{C}^n)) \subset \mathcal{L}(\mathcal{L}_p);
\]
(1.2.2)
in this case Peller has shown that the norm (1.2.1) is equivalent to
\[
|F|_{L^\infty} + |F|_{B^{1/p}};
\]
(1.2.3)
where the Besov \( p \)-norm is defined by (3.3.3) of Part III (if \( p = \infty \), then the Besov space is replaced by \( VMO \), and the result in this case is due to Hartmann; to my knowledge it not known how to intrinsically characterize the induced norm corresponding to other ideals of interest, such as the Dixmier trace class). This implies that the map of Banach Lie groups
\[
L_{L^\infty \cap B^{1/p}} U(n) \to U(\mathcal{L}_p)
\]
(1.2.4)
is a homeomorphism onto its image (in finite dimensions, it would automatically follow that this map is an embedding, but this does not appear to be true here, because for the map of Lie algebras, the image does not appear to have a complement). For the Kac-Peterson type topology,

$$L_{B^{1/p}} U(n) \rightarrow (U_{\mathcal{L}_p}, \tau_{KM})$$

is a homeomorphism onto its image (when \( p = \infty \), we replace the Besov class by \( VMO \)), and when \( p = 2 \) this describes the Kac-Peterson completion of the loop group.

To formulate the analogue of this result for \( \mathcal{D} \), we will have to work exclusively with the Kac-Moody type topologies, because the inclusion

$$\mathcal{D}_{C^\infty} \rightarrow U_{\mathcal{L}_p}$$

defined by (1.1.2-5) is not continuous for the Banach topology (as in (b) of (1.1.8), the induced topology is the discrete topology). From now on it will be understood that \( U_{\mathcal{L}_p} \) is equipped with the topology \( \tau_{KM} \). We will refer to the completion of the smooth diffeomorphism group in \( U_{\mathcal{L}_2} \) as the Kac-Peterson completion, since this is the analogue of the corresponding object for Kac-Moody groups. The problem then is to intrinsically characterize the Kac-Peterson completion and its \( p \)-generalization.

The following fundamental theorem (with \( \mathbb{R} \) in place of \( S^1 \)) first appeared in full generality in the dissertation of Semmes (Theorem 3 of [S]).

**Theorem.** (a) For \( p = \infty \),

$$\mathcal{D}_{W^{1,1}_{L_1}} \cap U_{\mathcal{L}_\infty} = \{ \phi \in \mathcal{D} : \log \Phi' \in VMO \}.$$

(b) For \( 1 \leq p < \infty \),

$$\mathcal{D}_{W^{1,1}_{L_1}} \cap U_{\mathcal{L}_p} = \{ \phi \in \mathcal{D} : \log \Phi' \in B^{1/p} \}.$$

The direction of critical importance to us is containment. The proof proceeds as follows. Suppose that \( \log \Phi' \in B^{1/p} \) (or \( VMO \) if \( p = \infty \)). We must show that

$$U_{\phi} \circ j \circ U^{-1}_{\phi} - j \in \mathcal{L}_p.$$

(1.2.7)

Coifman and Meyer considered the homotopy

$$\Phi_t(\theta) = \Phi(0) + \int_0^\theta \Phi'(\tau) d\tau,$$
from a rigid rotation to $\Phi$. They differentiated the corresponding deformation of (1.2.7) to obtain

$$U_\phi \circ j \circ U_\phi^{-1} - j = \int_0^1 (U_{\phi_t} \circ L_2 (\log \Phi' \circ \Phi_t^{-1}) \circ U_{\phi_t}^{-1}) dt$$

(1.2.8)

where $\log \Phi' \circ \Phi_t^{-1} \in B^{1/p}, \forall t$, and

$$L_2(b) = [B \frac{d}{d\theta}, j] + \frac{1}{2} [b, j], \quad B' = b.$$ 

(1.2.9)

Rochberg and Peller (Coifman and Meyer, and Hartmann, resp.) obtained the necessary $L_p$ estimates on the first and second terms of $L_2$, respectively, for $p < \infty$ (for $p = \infty$, resp.).

§1.3. Spin structures.

The circle $S^1$ has two distinct real spin structures, periodic (or trivial) and antiperiodic (or Mobius). In the latter case there is a natural action by $D^{(2)}$, the double cover. The complexification of the antiperiodic spin structure is trivial, but not equivariantly trivial. In each case there is a natural Hilbert space structure for half-forms, denoted by $H_p$ and $H_a$, respectively. We will identify both of these spaces with $H$ as follows. In the periodic case the identification is simply

$$H \to H_p : f \to f(d\theta)^{1/2}.$$ 

(1.3.1)

In the antiperiodic case there is a polarization

$$H_a = H_a^+ \oplus H_a^-,$$ 

(1.3.2)

where $H_a^\pm$ is the closure of holomorphic sections of the spin bundle for the disk $\bar{D}^\pm$, respectively. There is an isomorphism of polarized spaces

$$H \to H_a : f \to f(dz)^{1/2}.$$ 

(1.3.3)

We let

$$D^{(2)}_{W_{1,1}} \to U(H) : \tilde{\phi} \to V_{\tilde{\phi}} = \begin{pmatrix} A_{\phi}(\tilde{\phi}) & B_{\phi}(\tilde{\phi}) \\ C_{\phi}(\tilde{\phi}) & D_{\phi}(\tilde{\phi}) \end{pmatrix}$$ 

(1.3.4)

denote the induced action on $H$. The same chain of arguments as in the previous subsection (one simply systematically replaces $d\theta$ by $dz$) implies the following
(1.3.5) **Theorem.** For each \( p \geq 1 \),

\[
V_\phi \in U(p) \Rightarrow U_\phi \in U(p) \Leftrightarrow \log \Phi' \in B^{1/p}.
\]

(1.3.6) **Remark.** In the next section we will see that

\[
det|A_p(\phi)| \leq det|A_a(\tilde{\phi})|
\]

for all \( \phi \). It is natural ask whether this actually follows from an inequality of the operators, i.e. is

\[
|A_p(\phi)| \leq |A_a(\tilde{\phi})|?
\]

For later purposes it is convenient to briefly discuss the implication

\[
\log \Phi' \in B^{1/2} \Rightarrow V_\phi \in U(2),
\]

by considering the square-integrability of the kernel for \([j,V_\phi]\). Using the fact that

\[
proj_{H_\pm} : H \rightarrow H_\pm : f \rightarrow \frac{\pm 1}{2\pi i} \int_{S_1} f(\zeta) \frac{1}{\zeta - z} d\zeta, \quad |z| < 1 \quad (\text{resp. } |z| > 1),
\]

a direct computation shows that the kernel for \(\frac{1}{2}[j,V_\phi]\) is given by

\[
-\frac{1}{2\pi i} \left(\frac{1}{z - \phi(\zeta)} \left(\frac{d\phi}{d\zeta}\right)^{1/2} - \frac{1}{\psi(z) - \zeta} \left(\frac{d\psi}{dz}\right)^{1/2}\right) d\zeta,
\]

where \( \psi = \phi^{-1} \). To check that this kernel is \( L^2 \), it is equivalent and slightly easier to check that the kernel for \( V_{\phi^{-1}} \circ \frac{1}{2}[j,V_\phi] \) is \( L^2 \), and this is given by

\[
K_\phi(s,t) \frac{d\zeta}{2\pi i} = -\frac{1}{2\pi i} \left(\frac{\Phi'(t)\phi(\zeta)}{\phi(z)} \frac{1}{\Phi'(s)} \frac{1}{\zeta - z} \frac{d\zeta}{\zeta - \phi(\zeta)}\right) d\zeta
\]

\[
= -\frac{1}{2\pi i} \left(\frac{\Phi'(t)\phi(\zeta)}{\phi(z)} \frac{1}{\Phi'(s)} \frac{1}{\zeta - z} \frac{d\zeta}{\zeta - \phi(\zeta)} - 1\right)
\]

\[
= \frac{1}{2\pi i} \left(\frac{\Phi'(s)\Phi'(t)}{\zeta - z} \frac{\sin((t-s)/2)}{\sin((\Phi(t)-\Phi(s))/2)} - 1\right) d\zeta.
\]

\[
= \frac{1}{2\pi i} \exp\left(\frac{B(t)+B(s)}{2}\right) - \ln\left(\frac{\sin(\frac{1}{2} \int_s^t e^B)}{\sin((t-s)/2)}\right) - 1 d\zeta
\]

(1.3.9)
where $z = e^{it}$, $\zeta = e^{is}$, and $B(t) = ln\Phi'(t)$. The proof of (3.3.6) below implicitly indicates how to directly show this is an $L^2$ kernel, assuming that $B \in B^{1/2}$. It would be interesting to know if this follows from a general result.

§1.4. The symplectic action.

Let $V$ denote the symplectic vector space

$$V = (Map_{W^{1/2}}(S^1, \mathbb{R})/\mathbb{R}, \omega), \text{ where } \omega(f, g) = \int fdg. \quad (1.4.1)$$

In analogy with (1.1.7), Nag and Sullivan ([NS]) have proven the following

(1.4.2)Theorem. With respect to the natural action of $\mathcal{D}$ on measurable functions $(C_\phi : f \to f \circ \phi^{-1})$, we have

$$\mathcal{D} \cap Sp(V) = \mathcal{D}_{qs},$$

the group of quasi-symmetric homeomorphisms.

An important consequence of this is the following. With respect to the Hardy polarization of $V^C$, where $V^C_\pm$ are the conformally invariant Hilbert spaces

$$V^C_\pm = \{ f \in H^0(\{|z| < 1\}) : \frac{i}{2} \int df \wedge \bar{d}f < \infty \}/\mathbb{C} \quad (1.4.3)$$

we can write the composition operator $C_\phi$ as a two by two matrix as in (1.1.6). Since $C_\phi$ is symplectic, $A(C_\phi)$ is always invertible.

Since the Hardy polarization is positive, we can form the restricted symplectic group

$$Sp(\mathcal{I}) = \{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(V) : B, C \in \mathcal{I} \}, \quad (1.4.4)$$

for any symmetrically normed ideal (see [Se1]). A corollary of (1.3.5) is that

$$\mathcal{D} \cap Sp(\mathcal{I}_p) = \{ \phi : ln\Phi' \in L^\infty \cap B^{1/p} \}. \quad (1.4.5)$$

§1.5. Mapping $\mathcal{D}$ into the Formal Completion $\mathcal{D}_{\text{formal}}^C$.

The formal completion $\mathcal{D}_{\text{formal}}^C$, if it did exist, would contain as a dense open subset the product space

$$\mathcal{N}^- \cdot \mathbb{C}^* \cdot \mathcal{N}^+ \subset \mathcal{D}_{\text{formal}}^C, \quad (1.5.1)$$
where $\mathcal{N}^\pm$ are the simply connected profinite nilpotent groups corresponding to $n^\pm_{\text{formal}}$, respectively, and $\mathbb{C}^*$ is identified with the complexification of $\text{Rot}(S^1)$. In this subsection, after some preliminary discussion, we will show that various analytical completions of $\mathcal{D}$ (and also the Neretin-Segal semigroup of annuli) can be naturally mapped into $\mathcal{N}^- \cdot \mathbb{C}^* \cdot \mathcal{N}^+$.

First we must realize the group $\mathcal{N}^+ = \exp(n^+_{\text{formal}})$ in a concrete way. There is a natural Lie algebra representation of $n^+_{\text{formal}}$ by continuous derivations of the algebra $\mathbb{C}[[z]]$,

$$n^+_{\text{formal}} \times \mathbb{C}[[z]] \to \mathbb{C}[[z]] : (V, f) \to v \cdot f',$$  \hspace{1cm} (1.5.2)

where $V = v \frac{\partial}{\partial z}$, $v = O(z^2)$ (continuous means $\sigma(\sum f_n z^n) = \sum f_n \sigma(z^n)$). We claim that this can be exponentiated to a group action of $\mathcal{N}^+$ by continuous algebra automorphisms

$$\mathcal{N}^+ \times \mathbb{C}[[z]] \to \mathbb{C}[[z]].$$  \hspace{1cm} (1.5.3)

To see this, we compute that

$$e^V f = \sum_{n,m \geq 0} \frac{1}{n!} f_m (v \frac{\partial}{\partial z})^n (z^m).$$  \hspace{1cm} (1.5.4)

By induction it is easy to see that

$$(v \frac{\partial}{\partial z})^n = \sum_{1 \leq l \leq n} w_{n,l} \left( \frac{\partial}{\partial z} \right)^l, \text{ where } w_{n,l} = O(z^{n+l}).$$  \hspace{1cm} (1.5.5)

This implies that

$$(v \frac{\partial}{\partial z})^n (z^m) = O(z^{n+l+m-l}) = O(z^{n+m}),$$  \hspace{1cm} (1.5.6)

and hence the right hand side of (1.5.4) is a well-defined formal power series. Given that $e^V$ is well-defined, it is routine to check that $e^V$ is a continuous automorphism, and

$$\mathcal{N}^+ \to \text{Aut}_{\mathbb{C}^0}(\mathbb{C}[[z]])$$  \hspace{1cm} (1.5.7)

is a faithful representation.

The image of (1.5.7) can be described geometrically in the following way.

\textbf{(1.5.8) Proposition.} With reference to the group action (1.5.3),

$$\mathcal{B}^+ = \mathbb{C}^* \cdot \mathcal{N}^+ = \text{Aut}_{\mathbb{C}^0}(\mathbb{C}[[z]]) = \{ u \in \mathbb{C}[[z]] : u = \lambda z (1 + \sum_{n \geq 1} u_n z^n), \lambda \neq 0 \},$$
where the multiplication in the latter group is composition of formal power series, and $u$ acts by

$$u : \mathbb{C}[[z]] \to \mathbb{C}[[z]] : f \to f \circ u^{-1}.$$ 

Proof of (1.5.8). Given a continuous automorphism $\sigma$, we must produce $\lambda \in \mathbb{C}^*$, $V = v \frac{\partial}{\partial z} \in n_{\text{formal}}^+$ and $u$ such that

$$\lambda e^V = \sigma = (\cdot) \circ u^{-1}. \quad (1.5.9)$$

Since $\sigma$ is continuous, it is determined by what it does to $z$, so that clearly $u$ is determined by $\sigma(z) = u^{-1}$. To produce $\lambda$ and $V$, note that for $n > 1$, in the expansion (1.5.5),

$$w_{n,1} = (v \frac{\partial}{\partial z})^{n-1}(v). \quad (1.5.10)$$

Applying $\lambda e^V$ to $z$, we obtain the equation

$$\lambda(z + \sum_{n>0} \frac{1}{n!} (v \frac{\partial}{\partial z})^{n-1}(v)) = u^{-1} = c_1 z + c_2 z^2 + .... \quad (1.5.11)$$

This is equivalent to relations

$$\lambda = c_1, \quad \lambda v_2 = c_2, \quad \lambda(v_3 + v_2^2) = c_3, \quad \lambda(v_4 + \frac{5}{2} v_3 v_2 + \frac{1}{3} v_2^3) = c_4, \quad (1.5.12)$$

and so on, from which it is clear that we can obtain $V$ (although I do not see how to write these relations down in simple closed form).//

(1.5.13) Remark. There is a similar identification

$$\mathcal{B}^- = \mathbb{C}^* \cdot \mathcal{N}^- = \text{Aut}_{C^0}(\mathbb{C}[[z^{-1}]])$$

Now we consider $\mathcal{D}$, and more generally the Neretin-Segal semigroup of annuli $\mathcal{A}$. The appropriate smoothness class is the Ahlfors-Beurling group of quasisymmetric orientation-preserving homeomorphisms of $S^1$, $D_{qs}$. The corresponding smoothness condition for Segal’s category of compact Riemann surfaces with parameterized boundaries is the following. An object $\Sigma$ should be an oriented compact $C^0$ surface with boundary, with a compatible complex structure in the interior, and a continuous parameterization of each of its boundary components. In addition for each point $q \in \partial \Sigma$, there should be a chart

$$z : U \to \{|z| \leq 1, \Re(z) \geq 0\}$$
which is holomorphic in the interior, such that each parameterization is quasi-
symmetric when expressed in any pair of such charts. In this context sewing of
such objects is well-defined.

Now consider the semigroup of annuli $A_{qs}$. Given $A \in A_{qs}$, there is a unique
holomorphic isomorphism

$$
\bar{D}^{-} \circ A \circ \bar{D}^{+} = \hat{C}
$$

which is characterized by the condition that restriction induces maps satisfying

$$
\lambda u : \bar{D}^{+} \to \hat{C}, \quad u(z) = z(1 + \sum_{n \geq 1} u_{n}z^{n})
$$

$$
l^{-1} : \bar{D}^{-} \to \hat{C}, \quad l^{-1}(z) = z + \sum_{n \geq 0} b_{n}z^{-n},
$$

where $\lambda \in \mathbb{C}^*$. \hfill (1.5.14)

\textbf{Proposition.} \hspace{1em} There is an equivariant inclusion

$$
A_{qs} \to \mathcal{N}^{-} \cdot \mathbb{C}^* \cdot \mathcal{N}^{+} : A \to (l, \text{diag}, u),
$$

where $l$ is the mapping inverse to $l^{-1}$, and diag = Rot($\lambda$). By passing to the portion
of the boundary where $\lambda u(S^1) = l^{-1}(S^1)$, this also induces an equivariant inclusion

$$
\mathcal{D}_{qs} \to \mathcal{N}^{-} \cdot \mathbb{C}^* \cdot \mathcal{N}^{+} : \phi \to (l, \text{diag}, u),
$$

where as a mapping $\phi(z) = l \circ \text{diag} \circ u(z)$, $z \in S^1$, and $l$ and $u$ have quasi-conformal
extensions to $\hat{C}$. This factorization of $\phi$ is unique. \hfill (1.5.17)

\textbf{Remarks.} (a) The meaning of equivariant is the following. Although $\mathcal{D}_{an}$
does not act on $\mathcal{N}^{-} \cdot \mathbb{C}^* \cdot \mathcal{N}^{+}$, the Lie algebra of polynomial vector fields on $S^1$ does
act by holomorphic vector fields from both the left and the right. The formulas are
exactly the same as in the Kac-Moody case. These are written down in §1.6 of Part
I of [Pi] (see especially (1.6.6)). Similarly there is an action of $\text{vect}_{\text{poly}}$ on the space
of annuli (see §2 of [Se2]). These actions commute with the maps in (1.5.17).

(b) The uniqueness assertion can be understood in the following way. Suppose
that $\phi \in \mathcal{D}_{qs}$. Then we know that $\phi \in Sp(V)$, as in §1.4, and hence $A(C_{\phi})$ is an
invertible operator. Since $l$ must have the form $l = z + \sum_{n \geq 0} l_{n}z^{-n}$, it follows that
we must have

$$
\text{diag} \cdot u = A(C_{\phi})^{-1}(z).
$$

\hfill (1.5.18)
(c) Note that this argument also proves a weak version of the existence of the factorization. For given (1.5.19), \( u \) has \( W^{1/2} \) boundary values, and we also obtain \( l^{-1} \), which also has \( W^{1/2} \) boundary values, and on \( S^1 \)

\[
l^{-1} = \text{diag} \cdot u \circ \phi^{-1}.
\]  \hfill (1.5.20)

To see that \( \text{diag} \cdot u \) is univalent, suppose that \( |z_0| < 1 \). Then because \( u(z_0) \) is not in \( \frac{1}{\text{diag}} l^{-1}(\hat{C} \setminus D) \),

\[
\frac{1}{2\pi i} \int_{S^1} d\log(u - u(z_0)) = \frac{1}{2\pi i} \int_{S^1} d\log(\frac{1}{\text{diag}} l^{-1} \circ \phi - u(z_0))
\]

\[
= \frac{1}{2\pi i} \int_{S^1} d\log(\frac{1}{\text{diag}} l^{-1} - u(z_0)) = 1.
\]

Thus \( u \) is univalent, and similarly \( l^{-1} \) is univalent. Unfortunately this simple argument does not explain why \( u \) and \( l^{-1} \) are continuous on \( S^1 \), let alone that they have quasi-conformal extensions.

(d) The area of the annulus \( A \) is given by

\[
\text{area}(A) = \frac{i}{2} \int_{S^1} l^{-1} d\bar{l}^{-1} - \frac{i}{2} \int_{S^1} |\text{diag}|^2 u d\bar{u} = \pi (1 - \sum_{n \geq 0} n(|b_n|^2 + |\text{diag}|^2 |u_n|^2))
\]  \hfill (1.5.22)

where \( u_0 = 1 \), so that

\[
|b_n|^2 + |\text{diag} \cdot u_n|^2 \leq 1/n,
\]  \hfill (1.5.23)

and in particular \( |\text{diag}| \leq 1 \).

§2. The extension \( \hat{D} \) and determinant formulas.

§2.1. The universal central extension, \( \hat{D} \).

The group \( D_{an} \) has a universal central extension

\[
0 \to \mathbb{Z} \oplus i\mathbb{R} \to \hat{D}_{an} \to D_{an} \to 0.
\]  \hfill (2.1.1)

As is well-known ([S1],[KY]), the group \( \hat{D}_{an} \) can be realized in the following explicit way. As a manifold

\[
\hat{D}_{an} = \tilde{D}_{an} \times i\mathbb{R}.
\]  \hfill (2.1.2)

In these coordinates the multiplication is given by

\[
(\Phi, it) \cdot (\Psi, is) = (\Phi \circ \Psi, it + is + iC(\phi, \psi)),
\]  \hfill (2.1.3)
where $C$ is the $\mathbb{R}$-valued cocycle given by

$$C(\phi, \psi) = \frac{1}{48\pi \mathbb{R}} \int_{S^1} \log(\frac{\partial \phi}{\partial z} \circ \psi) d(\log(\frac{\partial \psi}{\partial z}))$$

The corresponding Lie algebra is the real form of $\text{vir}$ which as a vector space equals

$$\text{vect}(S^1) \oplus i\mathbb{R},$$

with the bracket given by (0.2).

(2.1.4)Remarks. (a) Note that the natural domain of $C$ is the Kac-Peterson completion, \{\Phi : \ln \Phi' \in B^{1/2}\}.

(b) To check the assertion about the Lie algebra, recall that one obtains the corresponding Lie algebra cocycle via

$$c(\vec{\xi}, \vec{\eta}) = \frac{\partial}{\partial s \partial t} |_{s=t=0}(C(e^{s\xi}, e^{t\eta}) - C(e^{t\eta}, e^{s\xi}))$$

$$= \frac{i}{24\pi} \int_{S^1} \frac{\partial \xi}{\partial z} d(\frac{\partial \eta}{\partial z}) = \frac{i}{24\pi} \int_0^{2\pi} (\tilde{\xi}'''(\theta) + \tilde{\xi}'(\theta))\tilde{\eta}(\theta) d\theta,$$

where $\tilde{\xi} = \xi(z) \frac{d}{dz} = \tilde{\xi}(\theta) \frac{d}{d\theta}$. This gives the commutation relations in (0.2).

The homomorphism

$$\mathcal{D}_{an}^{(2)} \rightarrow U_{(2)} : \tilde{\phi} \rightarrow V_{\tilde{\phi}},$$

induces a unique homomorphism of central extensions,

$$\hat{\mathcal{D}}_{an} \rightarrow \hat{U}_{(2)},$$

where, as in chapter 6 of [PS],

$$\hat{U}_{(2)} = \{(g, q) \in U_{(2)} \times U(H^+_a) : A_a(g) - q \in L_1 \}/U(H^+_a)_{1}. $$

It is not known how to write down this homomorphism explicitly. However, it is known that if this homomorphism is composed with the basic (lowest weight) representation of $\hat{U}_{(2)}$ (i.e. the representation described in chapter 10 of [PS]), then the module generated by the vacuum is the lowest weight module for $\text{vir}$ corresponding to the parameters $c = 1, h = 0$. Similarly, if we use the homomorphism $\phi \rightarrow U_{\phi}$, then we obtain the module corresponding to $c = 1$ and $h = 1/8$ (this is calculated for example in section 7 of [S2]). It follows that the matrix coefficient
corresponding to the vacuum in the representation corresponding to \((c, h)\) is given by (0.11).

There is an inclusion
\[
\mathcal{D}_{an} \to N^- \cdot \hat{\mathcal{H}} \cdot N^+ : \hat{\phi} \to l \cdot \text{diag} \cdot u,
\]  

where (as a consequence of (0.11))
\[
\text{diag} = (\det A_a(\hat{\phi}))^\kappa (\frac{\det A_a(\hat{\phi})}{\det A_p(\hat{\phi})})^{8L_0},
\]

and we are interpreting \(\det A_a\) and \(\det A_p\) as the pullbacks of the canonical section; this covers the factorization of \(\phi\) as a composition of functions
\[
\phi = l \circ \text{diag} \circ u, \quad \text{diag} = (\frac{\det A_a(\hat{\phi})}{\det A_p(\hat{\phi})})^{8L_0} = \text{Rot}(\frac{\det A_p(\hat{\phi})}{\det A_a(\hat{\phi})})^8
\]

\(u \in N^+, \ l \in N^-\), and \(\hat{\mathcal{H}}\) is the simply connected group corresponding to \(\hat{h} = CL_0 \oplus \mathbb{C}\kappa\). Note that by (1.5.23) we always have
\[
|\text{diag}| \leq 1, \quad \text{hence} \quad |\det A_p(\phi)| \leq |\det A_a(\phi)|.
\]

These formulas have interesting analytical consequences. For each \(n > 0\), there is an embedding
\[
di_n : sl(2, \mathbb{C}) \to \text{vir} : f \to f_n = -\frac{1}{n} L_{-n}, \ h \to h_n = \frac{2}{n} L_0 + \frac{1}{12} n(n^2 - 1) \kappa, \ e \to e_n = \frac{1}{n} L_n.
\]

Geometrically this corresponds to the following. The group of projective transformations of \(\hat{\mathcal{C}}\) which map the circle to itself is the subgroup \(PSU(1, 1) \subset PSL(2, \mathbb{C})\), where
\[
\left( \begin{array}{cc} a & b \\ \bar{b} & \bar{a} \end{array} \right) \cdot z' = \frac{\bar{b} + \bar{a}z'}{a + bz'}
\]

For \(n \geq 1\) there is an \(n\)-fold covering map,
\[
S^1 \to S^1 : z \to z' = z^n,
\]

and the diffeomorphisms of \(z\) which cover the projective transformations of \(z'\) form a group \(PSU(1, 1)^{(n)}\) which is an \(n\)-fold covering
\[
0 \to \mathbb{Z}_n \to PSU(1, 1)^{(n)} \to PSU(1, 1) \to 0
\]
The map \( \hat{d}_n \), modulo the center, is the complexification of the differential of the embedding

\[
i_n : PSU(1,1)^{(n)} \to \mathcal{D}. \tag{2.1.19}
\]

Suppose that \( \phi \in PSU(1,1)^{(n)} \subset \hat{\mathcal{D}} \) covers \( i_1 \left( \begin{array}{ll} a & b \\ \bar{b} & \bar{a} \end{array} \right) \in PSU(1,1) \subset \mathcal{D} \).

Then there is a unique factorization

\[
\phi = e^{-\frac{1}{n}L_0}a^\frac{2}{n}L_0 + \frac{1}{12}n(n^2-1)\kappa e^{-\frac{1}{n}L_0}a^\frac{2}{n}L_0 + 1 \in \mathcal{N}^- \cdot \hat{\mathcal{H}} \cdot \mathcal{N}^+ \tag{2.1.20}
\]

When we project \( \phi \) into \( \mathcal{D} \), we obtain a factorization of \( \phi \) as a composition of functions

\[
\phi = z(1 + \bar{b}a^{-1}z^{-n})^{1/n} \circ \text{Rot}(a^{-\frac{2}{n}}) \circ z(1 + a^{-1}bz^n)^{-1/n} \tag{2.1.21}
\]

(This is the rigorous expression corresponding to the heuristic expression

\[
\phi = (\frac{\bar{b} + \bar{a}z^n}{a + bz^n})^{1/n}; \tag{2.1.22}
\]

one can also obtain (2.1.21) from (2.1.20) by using formulas from [FLM], especially Prop. 8.3.10, page 186.).

In terms of the determinant formulas above, we have

\[
da^\frac{2}{n}L_0 + \frac{1}{12}n(n^2-1)\kappa = \text{det}A_\alpha(\phi)^\kappa(\frac{\text{det}A_\alpha(\phi)}{\text{det}A_\beta(\phi)})^8L_0 \tag{2.1.23}
\]

This implies

\[
\text{det}|A_\alpha(\phi)|^2 = (1 - r^2)^\frac{1}{12}n(n^2-1)
\]

\[
\text{det}|A_\beta(\phi)|^2 = \text{det}|A_\alpha(\phi)|^2(1 - r^2)^{1/4n} \tag{2.1.24}
\]

where \( r = |a^{-1}b| \). As a consequence, if \( \psi = \left( \begin{array}{ll} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{array} \right) \), then

\[
\text{det}(A_\alpha(\phi)A_\alpha(\psi)A_\alpha(\phi \circ \psi)^{-1}) = (1 + w\zeta)^\frac{1}{12}n(n^2-1), \tag{2.1.25}
\]

where \( w = a^{-1}b \), \( \zeta = \bar{\beta}\alpha^{-1} \). It is a highly nontrivial matter to verify (2.1.24) directly.

Proof of (2.1.24) for \( n = 1 \). The antiperiodic determinant is 1, since \( \phi \) commutes with the polarization. So we need only consider the periodic case.
If we multiply \( \phi \) on the left or right by a diagonal element, then \( \det|A_\phi|^2 \) is unchanged. Hence we can assume that \( \phi = (1 - r^2)^{-1/2} \left( \begin{array}{cc} 1 & r \\ r & 1 \end{array} \right) \), where \( r < 1 \). We have the Riemann-Hilbert factorization

\[
\Psi'(\theta) = \frac{d}{d\theta} \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta)} = (1 - rz^{-1})^{-1}(1 - r^2)(1 - rz)^{-1} \tag{2.1.26}
\]

where \( z = e^{i\theta} \). Now \( U_\phi(\cdot) = M_{\sqrt{\psi}}C_\psi \), where \( C_\psi f = f \circ \psi \). The operator \( C_\psi \) is nearly diagonal with respect to the Hardy polarization of \( H \); to be precise, \( C(C_\psi) = 0 \), and \( B(C_\psi) \) has rank \( \leq 1 \), and its range is contained in \( \mathbb{C}z^0 \). Using this, we see that

\[
A(U_\phi)*A(U_\phi) = A(C_\psi)*A(\sqrt{\Psi'})^2A(C_\psi)
\]

\[
= A(C_\psi \Psi^{-1})A(\sqrt{\Psi'})^2A(C_\psi)
\]

\[
= A(C_\psi)(\Psi^{-1})A(\sqrt{\Psi'})^2A(C_\psi) + B(C_\psi)C(\Psi^{-1})A(\sqrt{\Psi'})^2A(C_\psi)
\]

\[
= A(C_\psi)(1 + B(C_\psi)C(\Psi^{-1})A(\Psi^{-1})^{-1})A(\Psi^{-1})A(\sqrt{\Psi'})^2A(C_\psi)^{-1}. \tag{2.1.27}
\]

Hence

\[
\det|A(U_\phi)|^2 = \det(1 + B(C_\psi)C(\Psi^{-1})A(\Psi^{-1})^{-1})\det(A(\Psi^{-1})A(\sqrt{\Psi'})^2). \tag{2.1.28}
\]

We now have two determinants to evaluate. Using the Riemann-Hilbert factorization (2.1.26), we see that

\[
A(\Psi^{-1}) = A((1 - rz^{-1})(1 - r^2)^{-1}A((1 - rz)), \tag{2.1.29}
\]

and there is a similar decomposition for \( A(\sqrt{\Psi'}) \). By inserting and rearranging terms, it is easy to see that

\[
\det(A(\sqrt{\Psi'})^2A(\Psi^{-1})) = \det(S_1)^3 \tag{2.1.30}
\]

where \( S_1 \) is the group-theoretic commutator

\[
S_1 = A((1 - rz^{-1})^{-1/2})A((1 - rz)^{-1/2})A((1 - rz)A((1 - rz)^{1/2}). \tag{2.1.31}
\]

The Helton-Howe formula for the determinant of such a commutator implies that (2.1.30) equals

\[
\exp(3tr[\log((1 - rz^{-1})^{-1/2}), \log((1 - rz)^{-1/2})]) = \det(S_2)^{3/4}, \tag{2.1.32}
\]
where $S_2$ is the group-theoretic commutator obtained by removing all four square roots in the expression (2.1.31) for $S_1$. We now directly calculate that for $n > 0$, $S_2 z^n = z^n$, and

$$S_2 z^0 = \operatorname{proj}_{H^+}(z^0 + (1 - rz^{-1})^{-1}(1 - rz)^{-1}rz^{-1}) = \frac{1}{1 - r^2} z^0 + O(z). \quad (2.1.33)$$

It follows that $\det(S_2) = (1 - r^2)^{-1}$, and hence (2.1.32) equals $(1 - r^2)^{-3/4}$.

For the other determinant, we calculate

$$A(\Psi'{-1})^{-1} z^0 = \frac{1 - r^2}{1 - rz} \operatorname{proj}_{H^+}(\frac{1}{1 - rz^{-1}}) = \frac{1 - r^2}{1 - rz},$$

$$C(\Psi'{-1}) A(\Psi'{-1})^{-1} z^0 = \operatorname{proj}_{H^-}((1 + r^2 - r(z + z^{-1}))(1 - rz^{-1})) = -rz^{-1},$$

$$B(C_\phi)(rz^{-1}) = (-r) \operatorname{proj}_{H^+}(\frac{1 + rz}{r + z}) = -r^2 z^0. \quad (2.1.34)$$

Since $\operatorname{Range}(B(C_\phi)) \subset \mathbb{C} z^0$, it follows that

$$\det(1 + B(C_\phi) C(\Psi'{-1}) A(\Psi'{-1})^{-1}) = 1 - r^2. \quad (2.1.35)$$

Thus (2.1.24) now follows from (2.1.28), the calculation of (2.1.30) in the preceding paragraph, and (2.1.35), in case $n = 1$. //

§3. The measures $\nu_{\beta,c,h}$, $\beta > 0, c, h \geq 0$.

We begin by recalling the Malliavin-Shavgulidze construction of probability measures $\nu_\beta$ on $D_{C^1}$, indexed by inverse temperature $\beta$, which are quasi-invariant with respect to the left action of $D_{C^3}$. The family of measures we consider differs slightly from the family considered in [K] and [MM], the point being that the family we consider has some chance of being asymptotically invariant as $\beta \downarrow 0$. The quasi-invariance and expression for the Radon-Nikodym derivative follow from the general theory of Gaussian measures, which we recall in the first subsection.

§3.1. Preliminaries on Gaussian measures.

Let $H \to B$ denote an abstract Wiener space, and $\nu_G$ the associated Gaussian measure. It is well-known that the affine automorphisms of $H$ (the Cameron-Martin space for $\nu_G$) fix the measure class of $\nu_G$, and the family of probability measures

$$d\nu_{G,\beta}(b) = d\nu_G(\sqrt{\beta} b) = \frac{1}{Z} e^{-\frac{1}{2} \langle b, b \rangle} \mathcal{D} b^n \quad (3.1.1)$$
is asymptotically invariant with respect to these automorphisms in the following precise sense: given an affine automorphism $\phi \in O(H) \ltimes H,$

$$
\int |\frac{d\phi^*\nu_{G,\beta}}{d\nu_{G,\beta}} - 1|^p d\nu_{G,\beta} \leq 2\Gamma\left(\frac{p+1}{2}\right)(\beta|h_0|^2_H)^{p/2}
$$

(3.1.2)

where $h_0$ is the translational part of $\phi$ (see (4.1.3) of Part III of [Pi]).

We need a nonlinear version of these results. Part of what we need is standard, namely we have the following result on nonlinear transformations of Wiener space: suppose that

(a) $K : B \to H$ is a $C^1$ map with the property that the restriction of the derivative

$$dK|_b : H \to H$$

is Hilbert-Schmidt, for each $b \in B,$ and

(b) $T = 1 + K : B \to B$ is a homeomorphism, and the restriction of the derivative

$$dT|_b = 1 + dK|_b : H \to H$$

is in $GL(H),$ for each $b \in B;$

then $T_*\nu_{G,\beta}$ is equivalent to $\nu_{G,\beta}.$ From the heuristic expression (3.1.1) one calculates that the Radon-Nikodym derivative is given heuristically by the formula

$$\frac{d\nu_{G,\beta}(Tb)}{d\nu_{G,\beta}(b)} = det |dT|_b| exp\left(-\frac{\beta}{2}(2\langle Kb, b \rangle - |Kb|^2_H)\right);$$

(3.1.5)

the correct mathematical expression has the form

$$det_2 |dT|_b| exp\left((-\beta\langle Kb, b \rangle - tr(dK|_b)) - \frac{\beta}{2}|Kb|^2_H\right),$$

(3.1.6)

where the expression in parentheses is interpreted stochastically, and $det_2$ refers to the regularized Hilbert-Schmidt determinant for $dT|_b$ restricted to $H$ (see [R], especially section 4; in our application ordinary determinants suffice, and in this context the original result is due to Gross).

It would be interesting to determine necessary and sufficient conditions under which

$$\int |\frac{d\nu_{G,\beta}(Tb)}{d\nu_{G,\beta}(b)} - 1|^p d\nu_{G,\beta}(b) \to 0 \text{ as } \beta \downarrow 0,$$

(3.1.7)

for each finite $p.$ It’s very plausible that $spec(dK|_b) = \{0\}, a.e.b,$ is a necessary condition, but whether there is a useful general sufficient condition is unclear to me. We will consider this question in our specific context.
§3.2. The Malliavin-Shavgulidze measure $\nu_\beta$.

The definition of $\nu_\beta$ involves a number of identifications. First, for any reasonable smoothness condition, there is a bijection

$$D_1 \times \text{Rot}(S^1) \leftrightarrow D : \psi_1, \text{Rot}(\lambda) \leftrightarrow \psi$$

(3.2.1)

where $\psi^{-1}(1) = \lambda^{-1}$, $\psi_1 = \psi \circ \text{Rot}(\psi^{-1}(1))$, and $D_1$ denotes the stabilizer subgroup at $1 \in S^1$. This bijection induces an identification of $D_1$ with the left-coset space $D/\text{Rot}(S^1)$. The left action is given explicitly by

$$D \times D_1 \to D_1 : \phi, \psi_1 \to (\phi \circ \psi_1)_1 = \phi \circ \psi_1 \circ \text{Rot}(\psi_1^{-1}(\phi^{-1}(1))).$$

(3.2.2)

In turn there is an identification (where we now impose a specific smoothness condition)

$$(D_{C^1})_1 \leftrightarrow \text{Path}^{0,0}_{C_0R} : \psi_1 \leftrightarrow b = b_{\psi_1},$$

(3.2.3)

where

$$b(t) = \ln\Psi_1'(t) - \ln\Psi_1'(0), \quad \Psi_1(t) = \left(\frac{1}{2\pi} \int_0^{2\pi} e^{b} \right)^{-1} \int_0^{t} e^{b(\tau)} d\tau.$$  

(3.2.4)

Below we will routinely view $b$ as a $2\pi$-periodic function. Also it is useful to note that the map $\psi_1 \to b_{\psi_1}$ extends to a map

$$D_{C^1} \to \text{Path}^{0,0}_{C_0R} : \psi \to b_{\psi} = \ln \Psi' - \ln \Psi'(0);$$

(3.2.5)

this extension satisfies the equation

$$b_{\phi \circ \psi} = b_{\phi} \circ \Psi + b_{\psi} - (\ln \Phi'(\Psi(0)) - \ln \Phi'(0)).$$

(3.2.6)

In terms of the identification (3.2.3), the left action (3.2.2) becomes

$$D_{C^1} \times \text{Path}^{0,0}_{C_0R} \to \text{Path}^{0,0}_{C_0R} : \phi, b \to b_{(\phi \circ \psi_1)_1}.$$  

(3.2.7)

In particular for $\phi = \text{Rot}(\lambda)$, $\lambda = e^{is},$

$$\phi : b(t) \to b(t + T) - b(T),$$

(3.2.8)

where $T = T_{\phi}(b) = \Psi_1^{-1}(-s)$, i.e. $T$ is the unique solution of

$$\left(\frac{1}{2\pi} \int_0^{2\pi} e^{b} \right)^{-1} \int_0^{T} e^{b} = 2\pi - s;$$  

(3.2.9)
and for $\phi \in (\mathcal{D}_{\mathcal{C}^1})_1$,
\[ \phi : b(t) \to b(t) + b_\phi \circ \Psi_1. \] (3.2.10)

Let $\nu^{0,0}_\beta$ denote the Brownian bridge probability measure on $B = Path_{0,0}^{0,0}\mathbb{R}$. This is the Gaussian measure $\nu_{G,\beta}$ corresponding to the Cameron-Martin Hilbert space $H = Path_{0,0}^{0,0}\mathbb{R}$, with
\[ \langle x, y \rangle_H = \int_0^{2\pi} x' d\tau y' d\tau. \] (3.2.11)

The following is essentially due to the Malliavins.

(3.2.12) Proposition. The measure $\nu^{0,0}_\beta$ is quasi-invariant with respect to the left action of $\mathcal{D}_{\mathcal{C}^3}$ on $(\mathcal{D}_{\mathcal{C}^1})_1 \equiv Path_{0,0}^{0,0}\mathbb{R}$, and the Radon-Nikodym derivative is given by
\[ \frac{d\nu^{0,0}_\beta(b_{\phi \Psi_1}, \cdot)}{d\nu^{0,0}_\beta(b)} = \exp \left( \frac{-\beta}{2} \int_0^{2\pi} (b'_\phi(\Psi_1(\tau)))^2 - 2b''_\phi(\Psi_1(\tau))) \left\{ \frac{e^{b(\tau)}}{\int_0^{2\pi} e^b} \right\}^2 d\tau \right), \]
where $b = b_{\psi_1}$.

(3.2.13) Remarks. (a) Because $b_{\text{Rot}(\lambda)} \circ \phi = b_\phi$ (by (3.2.6)), the formula above asserts that in particular $\nu^{0,0}_\beta$ is strictly invariant with respect to the left action (3.2.8) of $\text{Rot}(S^1)$ on the based loop space.

(b) The analysis of Kosyak in [K] appears to correctly show that the induced representation
\[ (\mathcal{D}_{\mathcal{C}^3})_1 \times L^2((\mathcal{D}_{\mathcal{C}^1})_1, d\nu^{0,0}_\beta) \]
is irreducible.

Proof of (3.2.12). We first prove invariance with respect to a rotation $\phi = \text{Rot}(e^{i\sigma})$. Given a constant $T_0$, the Brownian bridge is invariant with respect to the time translation
\[ (T_0)_* : B \to B : b(t) \to b(t + T_0) - b(T_0), \]
since this defines a unitary transformation of the Cameron-Martin subspace. As an immediate consequence, if we have a function $T : B \to \mathbb{R}$ having a finite range, then the corresponding (random translation of time) transformation,
\[ T_* : B \to B : b(t) \to b(t + T(b)) - b(T(b)), \]
is an isomorphism of measure spaces.
Now consider the transformation (3.2.8) defined by $\phi$. For each $n$ we can define an approximation to $T_\phi$ having a finite number of values, say

$$T_n : B \to \mathbb{R} : b \to k/n,$$

where $2\pi k/n \leq T_\phi(b) \leq 2\pi(k + 1)n$.

Each $(T_n)_*$ is measure-preserving. Given a bounded functional $\Phi$ of the Brownian bridge, we have

$$\int \Phi \circ (T_n)_* \to \int \Phi \circ (T_\phi)_* \quad \text{as} \quad n \to \infty,$$

by dominated convergence. From this it follows that $(T_\phi)_*$ is measure-preserving.

Now suppose that $\phi \in (D_{C^3})_1$. Let $h = b_\phi$, and let $K$ denote the transformation

$$K : B \to H : b \to h \circ \Psi_1,$$  \tag{3.2.14}

where $b = b_\psi$. We calculate that

$$dK|_b(x)(t) = \frac{d}{d\epsilon}|_{\epsilon=0} h((\frac{1}{2\pi}\int_0^{2\pi} e^{b+\epsilon x})^{-1} \int_0^t e^{b+\epsilon x})$$

$$= h'((\frac{1}{2\pi}\int_0^{2\pi} e^{b})^{-1} \int_0^t e^{b}) - (\frac{1}{2\pi}\int_0^{2\pi} e^{b})^{-2} (\frac{1}{2\pi}\int_0^{2\pi} e^{b} x) \int_0^t e^{b})$$

$$= \frac{h'((\Psi_1(t))(\frac{1}{2\pi}\int_0^{2\pi} e^{b})^{-1} \int_0^t e^{b} x - (\frac{1}{2\pi}\int_0^{2\pi} e^{b})^{-2} (\frac{1}{2\pi}\int_0^{2\pi} e^{b} x) \int_0^t e^{b})}{\frac{1}{2\pi}\int_0^{2\pi} e^{b} x - \Psi_1(t)(\frac{1}{2\pi}\int_0^{2\pi} e^{b} x)}.$$  \tag{3.2.15}

From this it is obvious that

$$dK|_b : H \to H$$  \tag{3.2.16}

is a trace class operator. Hence $\nu_{\beta,0}^{0,0}$ is quasi-invariant.

To compute the Radon-Nikodym derivative, we first show that the spectrum for the operator in (3.2.16) is $\{0\}$. To see this, let $\lambda \neq 0$ and suppose that (3.2.15) equals $\lambda x(t)$. It is then straightforward to check that $x$ is a solution of the first-order linear equation

$$\rho^2 e^{b}(x - c) = -\mu \rho' x + \mu \rho x',$$

where $c = (\int_0^{2\pi} e^{b} x)/(\int_0^{2\pi} e^{b})$, $\rho = h' \circ \Psi$, $\mu = \lambda \frac{1}{2\pi} \int_0^{2\pi} e^{b}$. The general solution of () is given by

$$x(t) = (-\frac{C}{\lambda}) \rho(t) e^{\lambda^{-1} h(\Psi(t))} \int_0^t e^{-\lambda^{-1} h(\Psi(\tau))} e^{b(\tau)} d\tau + C \rho(t) e^{\lambda^{-1} h(\Psi(t))}.$$  \tag{3.2.17}
The condition $x(0) = 0$ implies that $C = 0$. The condition $x(2\pi) = 0$ forces either $c = 0$, in which case $x = 0$, or $\rho(2\pi) = 0$, i.e. $h'(2\pi) = 0$. It follows that if $\phi$ is such that $h'(2\pi) \neq 0$ (recall that $h = b \phi$), then the spectrum is \{0\}. But now it follows by the continuous dependence of the spectrum on $\phi$ that the spectrum is always zero.

We can now explicitly compute the Radon-Nikodym derivative from the stochastic expression (3.1.6),

$$
\exp\left(-\frac{\beta}{2} \int_0^{2\pi} \{2(Kb)'db + (Kb)^2d\tau\}\right),
$$

$$
= \exp\left(-\frac{\beta}{2} \int_0^{2\pi} \{2h'(\Psi_b(\tau)) \frac{e^{b(\tau)}}{2\pi} \int_0^{2\pi} e^b \, db + (h'(\Psi_b(\tau)) \frac{e^{b(\tau)}}{2\pi} \int_0^{2\pi} e^b \, db)^2\}\right). \tag{3.2.18}
$$

The meaning of the stochastic integral is that one integrates by parts, and this gives the expression in (3.2.12).//

We now want to discuss the question of whether $\nu_{0,0}^0$ is asymptotically invariant, i.e. we want to evaluate the limit as $\beta \downarrow 0$ of

$$
\int \left| \frac{d\nu_{0,0}^0(b \circ \psi_1)}{d\nu_{0,0}^0(b)} - 1 \right|^p d\nu_{0,0}^0(b)
$$

$$
= \int \exp\left(-\frac{\beta}{2} \int_0^{2\pi} (h'(\Psi_b/\sqrt{\beta})^2 - 2h''(\Psi_b/\sqrt{\beta}) \left\{ \frac{\frac{b}{2\pi} \int_0^{2\pi} e^{\frac{b}{\sqrt{\beta}}} \, db}{\frac{b}{2\pi} \int_0^{2\pi} e^{\frac{b}{\sqrt{\beta}}} \, db} \right\}^2\}\right) - 1 \right|^p d\nu_{1,0}^0(b) \tag{3.2.19}
$$

This limit would be zero if we could answer in the affirmative the following relatively straightforward

\textbf{(3.2.20)Question. For any positive integer $n$, do we have}

$$
\lim_{\beta \to 0} \int \exp(n\beta^2 \frac{\int \frac{e^{2b}}{(\int e^{b/\beta})^2} \, db}{(\int e^{b/\beta})^2} d\nu_{1,0}^0(b) = 1?
$$

Since $h''$ is bounded, an affirmative answer would allow us to apply dominated convergence to (3.2.19), and the limit would be zero.

To make sense of (3.2.20) we first must show that the function

$$
1 \leq \int \exp(n\beta^2 \frac{\int \frac{e^{2b/\beta}}{(\int e^{b/\beta})^2} \, db}{(\int e^{b/\beta})^2} d\nu_{1,0}^0(b) \tag{3.2.21}
$$
is actually finite. Given finiteness, because the function is even, and unbounded as \( \beta \) goes to infinity, it is very plausible that it is an increasing function of \( \beta^2 \), and has a minimum at \( \beta = 0 \).

Is it plausible that (3.2.21) is finite? Suppose that \( H \) is a positive Morse function. Stationary phase implies that

\[
\int e^{-H/T} = \sum_{t: H(t) = \min H} e^{-H(t)/T} \left( \frac{2\pi T}{H''(t)} \right)^{1/2} + O(T^{3/2}).
\]  

(3.2.22)

If we formally apply this to \( H = b - \sup \{b\} \), with \( T = \beta \), then we see that

\[
\int e^{2b/\beta} / (\int e^{b/\beta})^2 = \int e^{2(b - \sup \{b\})/\beta} / (\int e^{(b - \sup \{b\})/\beta})^2 = \beta^{-1/2} \ast \text{const} + O(\sqrt{\beta}),
\]  

(3.2.23)

and hence (3.2.20) seems trivially true. However Brownian paths behave very differently than Morse functions in the vicinity of their (unique) maxima. In fact Pittman and Yor have proven that the random variables

\[
\beta^{-2} \int e^{(b - \sup \{b\})/\beta}
\]  

(3.2.24)

converge in law as \( \beta \downarrow 0 \) ([PY]). This makes it very unclear whether (3.2.21) is actually finite.

Now consider the principal bundle

\[
\begin{align*}
\mathcal{D}_{C^1} &\leftarrow \text{Rot}(S^1) \\
\downarrow &
\end{align*}
\]

(3.2.25)

Since \( \nu_{\beta,0}^0 \), viewed as a measure on the base, is \( \text{Rot}(S^1) \)-invariant, there is a unique \( \text{Rot}(S^1) \)-biinvariant probability measure on \( \mathcal{D}_{C^1}, \nu_{\beta} \), which projects to \( \nu_{\beta,0}^0 \).

(3.2.26)Corollary. The measure \( \nu_{\beta} \) is quasi-invariant with respect to the left action of \( \mathcal{D}_{C^3} \). If the answer to (3.2.20) is affirmative, then we have

\[
\int |d\nu_{\beta}(\phi \circ \psi) / d\nu_{\beta}(\psi)| - 1|^p d\nu_{\beta}(\psi) \to 0 \quad \text{as} \quad \beta \to 0,
\]

for each finite \( p \).

§3.3. Existence of the measures \( \nu_{\beta,c,h} \).
In this subsection our task is to make sense of the measure which can be written heuristically as
\[ d\nu_{\beta;c,h} = \frac{1}{E} \text{det}|A_a(\phi)|^2|\text{diag}(\phi)|^{16h}d\nu_\beta(\phi) \]
and rigorously (as will show below) as
\[ \frac{1}{E} \text{exp}\left(-\frac{c}{8\pi^2} \lim_{\delta \downarrow 0} \int_\delta^\infty \int_\delta^{t-s} (K_\phi(s,t)^2 - E_\beta K_\phi(s,t)^2) \text{det}_2|A_a(\phi)|^{2c}|\text{diag}(\phi)|^{16h}d\nu_\beta \right) \]
where \( E_\beta \) denotes expectation with respect to \( \nu_\beta \), and the kernel \( K_\phi \) is given by (1.3.9).

We already know that \(|\text{diag}(\phi)|\) is a well-defined function of \( \phi \in D_{qs} \), and it is bounded by 1, by (1.5.17) and (1.5.23). Thus providing that \( h \geq 0 \),
\[ |\text{diag}|^{16h} \]
is a well-defined random variable, because \( \nu_\beta \) is supported on \( D_{C^\alpha} \), provided that \( \alpha < 3/2 \), and this is certainly contained in \( D_{qs} \). The Hilbert determinant is taken care of by the following

(3.3.3) Corollary of (1.3.5). The function
\[ \phi \rightarrow \text{det}_2|A_a(\phi)|^2 \]
is well-defined, continuous and bounded by 1 on \( \{ \phi \in D_{C^1} : \log\Phi \in B^{1/4} \} \).

Proof. The function is well-defined and continuous by (1.3.5). Because \(|A|^2 + |C|^2 = 1\), we have \( 0 \leq |C|^2 \leq 1 \). It follows from this that
\[ 0 \leq \text{det}_2|A|^2 = \text{det}((1 - |C|^2)e^{C^2}) \leq 1, \]
for the function \( \lambda \rightarrow (1 - \lambda)\text{exp}(\lambda) \) is bounded by 1 for \( \lambda \geq 0 \). //

Thus the Hilbert determinant in (3.3.3) is a well-defined bounded random variable with respect to \( \nu_\beta \). Now consider
\[ \int \int_{|s-t|>\delta} |K_\phi(s,t)|^2 - E_\beta \int \int_{|s-t|>\delta} |K_\phi(s,t)|^2, \quad (3.3.5) \]
This is a well-defined random variable with respect to \( \nu_\beta \), for each \( \delta > 0 \).
**Proposition.** (3.3.5) has a limit in probability \([\nu_\beta]\) as \(\delta \to 0\).

**Remark.** The basic intuition is that, by Theorem (1.2.6), \(|K_\phi|^2\) and \(|b(t) - b(s)|^2\) have essentially the same singular behavior along the diagonal. Hence the integral of one can be regularized if and only if the integral of the other can be regularized, by subtracting out the expectation. If we write \(b\) in terms of its sine series, \(b(t) = \sum_{n>0} b_n \sin(n/2 t)\), (3.3.8) then \(\nu_{0,0}^\beta\) corresponds to the product measure

\[
\prod_{n>0} \sqrt{\frac{\beta n^2}{2\pi}} e^{-\frac{\alpha^2}{4} n^2} dB_n,
\]

and

\[
\lim_{\delta \downarrow 0} \iint_{|e^{is} - e^{it}| > \delta} |b(t) - b(s)|^2 = \sum_{n>0} n(b_n^2 - E_\beta(b_n^2)),
\]

where the latter sum is absolutely convergent in the \(L^2\) sense. Moreover, by Kolmogorov’s theorem, it converges for \(a.e.\) \(b\) (see §3.1 of Part III of [Pi]).

We will reduce the proof of (3.3.6) to a linear calculation which is essentially equivalent to this.

**Proof of (3.3.6).** Since \(\beta\) does not play an important role, we fix it such that \(E(b(s)b(t)) = s(2\pi - t), \ s \leq t, \) where \(E = E_\beta\). It is convenient to introduce the following notation. Let \(A = (b(t) + b(s))/2, \Delta = t - s, \ I = \int_s^t e^b, \) \(\alpha = (\frac{1}{2\pi} \int_0^{2\pi} e^b)^{-1}, \) and

\[
F = F(s,t;b) = A - \ln \left( \frac{\sin(\frac{1}{2} \alpha I)}{\alpha \sin(\frac{1}{2} \Delta)} \right).
\]

By (1.4.8) we have

\[
|K_\phi|^2 = \frac{|e^F - 1|^2}{2 - 2\cos \Delta}.
\]

Therefore

\[
(e^F - 1)^2 - F^2 = \sum_{n\geq3} \frac{1}{n!} (2^n - 2) F^n,
\]

\[
|K_\phi|^2 - \frac{F^2}{2 - 2\cos \Delta} = \sum_{n\geq3} \frac{1}{n!} (2^n - 2) F^n.
\]

We now want to estimate the expected value of this difference. Note that

\[
\ln \left( \frac{\sin(\frac{1}{2} \alpha I)}{\alpha \sin(\frac{1}{2} \Delta)} \right) = b(u)
\]

for some \(u\) between \(s\) and \(t\). We will use the following standard fact several times,
(3.3.15) **Lemma.** There is a constant $c$ such that
\[ E\left( \sup_{s \leq u \leq t} \left| \frac{b(s) + b(t)}{2} - b(u) \right|^n \right) \leq c 2^n (n!!) (\Delta (2\pi - \Delta))^{n/2}. \]

Proof of (3.3.15). For a standard Brownian motion $x(t)$, we have
\[
\text{Prob}\{ \sup_{s \leq u \leq t} |x(s) - x(u)| > \lambda \} = \frac{2}{\sqrt{2\pi(t-s)}} \int_{\lambda}^{\infty} e^{-\frac{y^2}{2(t-s)}} dy \tag{3.3.16}
\]
\[
E \sup_{s \leq u \leq t} |x(s) - x(u)|^n = \lambda^n d\left\{ \frac{2}{\sqrt{2\pi(t-s)}} \int_{\lambda}^{\infty} e^{-\frac{y^2}{2(t-s)}} dy \right\} = \left( \frac{n!!(t-s)^{n/2}}{\sqrt{2/\pi n!!(t-s)^{n/2}}} \right) \text{ for } n \text{ even}, \quad \left( \frac{n!!\Delta^{n/2}}{\sqrt{2/\pi n!!(t-s)^{n/2}}} \right) \text{ for } n \text{ odd}. \tag{3.3.17}
\]
(see Thm 2.5 of [Kn]). Similarly
\[ E \sup_{s \leq u \leq t} |x(t) - x(u)|^n \leq n!! \Delta^{n/2}. \tag{3.3.18} \]

Since $b(t) = x(t) - tx(2\pi)$, we have
\[
\sup_{s \leq u \leq t} |b(s) - b(u)|^n = 2^n \sup_{u} |x(s) - x(u) + (u-s)x(2\pi)|^n
\]
\[
= 2^{n-1} (\sup_{u} |x(s) - x(u)|^n + \Delta^n |x(2\pi)|^n). \tag{3.3.19}
\]

Taking the expected value and using (3.3.17), we obtain
\[ E \sup_{s \leq u \leq t} |b(s) - b(u)|^n \leq 2^{n-1} (n!! \Delta^{n/2} + n!! \Delta^n) \tag{3.3.20} \]

There is a similar estimate for $E \sup_{s \leq u \leq t} |b(t) - b(u)|^n$ (using (3.3.18)). We also get similar estimates in terms of $(2\pi - \Delta)$ (using the periodicity of $b$). It follows that
\[
E \sup_{s \leq u \leq t} \left| \frac{b(s) + b(t)}{2} - b(u) \right|^n \leq E \frac{1}{2} (\sup_{u} |b(s) - b(u)|^n + \sup_{u} |b(t) - b(u)|^n)
\]
\[
\leq c 2^n n!! (\Delta (2\pi - \Delta))^{n/2}. // \tag{3.3.21}
\]

It follows from (3.3.14) and (3.3.15) that $E|F|^n \leq cn!! (\Delta (2\pi - \Delta))^{n/2}$. Hence
\[
E(\int_{0 \leq s,t \leq 2\pi} |K_0|^2 - \frac{F^2}{2 - 2\cos \Delta}) \leq \sum_{n \geq 3} \frac{2^n - 2}{n!} E \int \int \frac{|F|^n}{2 - 2\cos \Delta}
\]
Thus to prove (a) it suffices to prove that
\[
\lim_{\delta \to 0} \iint_{|1-e^{i\Delta}| > \delta} \frac{F^2 - E(F^2)}{2 - 2\cos \Delta} \]  
exists. In the remainder of the proof, we will split the region of integration into two pieces, \(\{ \Delta \leq \pi \} \) and the complement. We will write out the estimates for the case \(\Delta \leq \pi \) only, since they are essentially the same for the complement, using periodicity.

We write
\[
F^2 = (A - \ln \frac{1}{\Delta} I) - \ln \frac{\sin(\frac{\alpha I}{2})}{\frac{\alpha I}{2}})^2 
= (A - \ln \frac{1}{\sin(\frac{\alpha I}{2})})^2 - 2(A - \ln \frac{1}{\sin(\frac{\alpha I}{2})}) \ln \frac{\sin(\frac{\alpha I}{2})}{\frac{\alpha I}{2}} + (\ln \frac{\sin(\alpha I)}{\frac{\alpha I}{2}})^2. \tag{3.3.24}
\]
By (3.3.15), and the fact that \(I\) involves integrating (not averaging) over an interval of length \(\Delta\),
\[
E\{(A - \ln \frac{1}{\sin(\frac{\Delta}{2})})^2 \} \leq c\Delta, \quad E\{(\ln \frac{\sin(\alpha I)}{\frac{\alpha I}{2}})^2 \} \leq c\Delta^4. \tag{3.3.25}
\]
Hence using (3.3.14), we have
\[
E \iint_{\Delta \leq \pi} \frac{|A - \ln \frac{1}{\sin(\frac{\Delta}{2})}|^2 - |A - \ln \frac{\sin(\alpha I)}{\frac{\alpha I}{2}}|^2}{2 - 2\cos \Delta} < \infty.
\]
Using similar estimates to replace \(\sin(\Delta/2)\) by \(\Delta/2\), we see that it suffices to show that
\[
\lim_{\delta \downarrow 0} \iint_{|1-e^{i\Delta}| > \delta, \Delta \leq \pi} \frac{|A - \ln(\frac{1}{\Delta} I)|^2 - E|A - \ln(\frac{1}{\Delta} I)|^2}{2 - 2\cos \Delta} dsdt \tag{3.3.26}
\]
exists for a.e. \(b\).

Now we can write
\[
A - \ln(\frac{1}{\Delta} I) = \ln(\frac{1}{\Delta} \int_s^t e^{A-b(\tau)} d\tau) = \frac{1}{\Delta} \int_s^t (A - b) + \ln(f) \quad f = \frac{1}{\Delta} \int_s^t e^{A-b},
\]
\[
(A - \ln(\frac{1}{\Delta} I))^2 = (\frac{1}{\Delta} \int_s^t (A - b))^2 + 2(\frac{1}{\Delta} \int_s^t (A - b))\ln(f) + \ln^2(f). \tag{3.3.28}
\]
The important points are that the first term in (3.3.28) is linear in \(b\), and the nonlinear term \(f\) is now relatively tractable. The following lemma states that we can regularize the first term in (3.3.28); it is essentially equivalent to (3.3.10).
(3.3.29) Lemma. The

\[ \lim_{\delta \to 0} \int_{\delta < \Delta < \pi} \frac{\frac{1}{\Delta} \int_s^t (A - b) |^2 - E \frac{1}{\Delta} \int_s^t (A - b) |^2}{\Delta^2} \]

exists in the \( L^2 \) sense.

Assuming this for the moment, we can complete the proof of (3.3.6) by showing that the last two terms in (3.3.28) do not require any regularization, i.e.

\[ \int_{\Delta < \pi} \frac{E((\frac{1}{\Delta} \int_s^t |A - b||ln(f) + ln^2(f))}{\Delta^2} < \infty. \] (3.3.30)

We have

\[ E((\frac{1}{\Delta} \int_s^t (A - b))^2) \leq \frac{1}{\Delta} \int_s^t E(A - b)^2 \leq \Delta. \] (3.3.31)

Because \( f \geq 1, ln^2 f \leq (f - 1)^2 \), hence

\[ E(ln^2(f)) \leq E((f - 1)^2) = 1 + Ef^2 - 2Ef. \] (3.3.32)

Now

\[ f = (1 + \frac{1}{\Delta} \int_s^t (A - b) + \frac{1}{2} \frac{1}{\Delta} \int_s^t (A - b)^2 +..)(1 - \frac{1}{\Delta} \int_s^t (A - b) + \frac{1}{2} \frac{1}{\Delta} \int_s^t (A - b)^2 -..) \] (3.3.33)

Using

\[ E(\frac{1}{\Delta} \int_s^t (A - b)^n) \leq E(\frac{1}{\Delta} \int_s^t (A - b)^n) = \frac{1}{\Delta} \int_s^t E(A - b)^n \leq c\Delta^{n/2}, \] (3.3.34)

we see that

\[ Ef = 1 + \frac{1}{2} E(\frac{1}{\Delta} \int_s^t (A - b)^2 - (\frac{1}{\Delta} \int_s^t (A - b)^2))^2 + O(\Delta^{3/2}). \] (3.3.35)

Similarly,

\[ Ef^2 = 1 + E(\frac{1}{\Delta} \int_s^t (A - b)^2 - (\frac{1}{\Delta} \int_s^t (A - b)^2))^2 + O(\Delta^{3/2}). \] (3.3.36)

It follows that (3.3.32) is \( O(\Delta^{3/2}) \). This together with (3.3.31) implies (3.3.30).//
Proof of (3.3.29). Since this is a linear problem, we will just indicate what is involved. We must show that in the $L^2$ sense,

$$
\int \int_{\delta < \Delta < \epsilon} \frac{1}{\Delta^2} \int_s^t (A - b)^2 - E \frac{1}{\Delta^2} \int_s^t (A - b)^2 \rightarrow 0 \quad \text{as} \quad \delta, \epsilon \rightarrow 0,
$$

i.e.

$$
E((\int \int_{\delta < \Delta < \epsilon} \frac{1}{\Delta^2} \int_s^t (A - b)^2)^2) - (E \int \int_{\delta < \Delta < \epsilon} \frac{1}{\Delta^2} \int_s^t (A - b)^2)^2 \rightarrow 0 \quad \text{in} \quad L^2.
$$

By expanding all the terms, all the expectations involved here can be computed explicitly. //

We can now formulate our main

**Theorem.** For $c, h \geq 0$, (a) there exists an essentially unique measure $\nu_{\beta; c, h}$ which (i) is absolutely continuous with respect to $\nu_{\beta}$, and (ii) has Radon-Nikodym derivative

$$
\frac{d\nu_{\beta; c, h}(\phi \circ \psi)}{d\nu_{\beta}(\psi)} = \frac{d\nu_{\beta}(\phi \circ \psi)}{d\nu_{\beta}(\psi)} \Bigg|_{\sigma_{c, h}(\hat{\phi} \circ \hat{\psi})}^2
$$

for all $\phi \in \mathcal{D}_{C^3}$; and (b) the measure $\nu_{\beta; c, h}$ is finite, so that we can normalize it to be a probability measure.

**Remark.** Note that

$$
|\sigma_{c, h}(\hat{\phi} \circ \hat{\psi})|^2 = \frac{|\text{diag}(\phi \circ \psi)|^{16h} \text{det}|A_a(\phi) + B_a(\phi)Z_a(\psi)|^2}{|\text{diag}(\psi)|^{16h} \text{det}|A_a(\phi)|^2},
$$

where $Z_a = C_a A_a^{-1}$. With probability one $Z_a(\psi) \in \mathcal{L}_p$ for all $p > 2$, by (1.3.5), hence the product $B_a(\phi)Z_a(\psi)$ is trace class, and the determinant is well-defined.

**Proof of (3.3.37).** Let $\nu_{\beta; c, h}$ be the measure given by the last line of (3.3.3), where we interpret the exponential term as a random variable using (3.3.6), and we temporarily set $E' = 1$. It is a simple matter to check that it has the Radon-Nikodym derivative claimed above. It is essentially unique because $\nu_{\beta}$ is ergodic with respect to the left action of $\mathcal{D}_{C^3}$, as established by Kosyak in [K]. //

**Remark.** It is easy to check that $\nu_{\beta}$ is right quasi-invariant only with respect to rotations, with respect to which it is strictly invariant. Hence the same is true for $\nu_{\beta; c, h}$. Thus these regularized Virasoro measures, compared to those for loop groups, are fundamentally asymmetric.
§4. On Existence of Invariant Measures

Via the natural inclusion

$$\mathcal{D}_{qs} \to \mathcal{N}^- \cdot \mathcal{H} \cdot \mathcal{N}^+$$

(4.1.1)

given by (1.5.17), we can view each measure $\nu_{\beta; c, h}$ as a probability measure on $\mathcal{N}^- \cdot \mathcal{H} \cdot \mathcal{N}^+$.

Given $\phi \in \mathcal{D}_{qs}$, we write

$$\phi = l \circ \text{diag} \circ u,$$

(4.1.2)

as in (1.5.17), and

$$u(z) = z(1 + \sum_{n \geq 1} u_n z^n), \quad l^{-1}(z) = z + \sum_{n \geq 0} b_n z^{-n}.$$  

(4.1.3)

By (1.5.23) and the Bieberbach-de Branges inequalities, we know that $|\text{diag}| \leq 1$, $|b_n| \leq 1/n$, and $|u_n| \leq n + 1$.

(4.1.4) Corollary. For each $c, h \geq 0$, the family of measures

$$\{\nu_{\beta; c, h} : \beta > 0\}$$

has weak limits as $\beta \to 0$, with respect to $BC(\mathcal{N}^- \cdot \{|z| \leq 1\} \cdot \mathcal{N}^+)$ (bounded continuous functions). Any such limit is supported on the space

$$\prod\{|b_n| \leq 1/n\} \cdot \{|\text{diag}| \leq 1\} \cdot \prod\{|u_n| \leq n + 1\}.$$

The question now is whether the limit points are $\mathcal{D}_{an}$-invariant, viewed as bundle-valued measures. As it stands, this question is not well-posed, because $\mathcal{D}_{an}$ does not act on the formal completion. One way to proceed would be to first investigate whether the limit points are infinitesimally invariant. However, ultimately one would like to know whether the limit points are supported on $\mathcal{D}_{qs}$, so that the global question would be well-posed.

To relate this to a concrete problem, it is convenient to again refer to the case of loop groups. In that case the analogue of the factorization (4.1.2) is the Riemann-Hilbert factorization

$$g = g_- \cdot g_0 \cdot g_+$$

(4.1.5)
where \( g_0 \in G, \quad g_\pm \in G(\mathcal{O}(D^\pm))_1 \), respectively. If we write
\[
g_+ = 1 + \hat{g}_+(1)z + ... , \tag{4.1.6}
\]
then it is easy to prove that
\[
\nu_\beta^{**}\{|\hat{g}_+(n)| > R\} \to 0 \quad \text{as} \quad n \to \infty, \tag{4.1.7}
\]
for each fixed \( R \). If we could prove that this limit is monotone (which, at least to me, seems intuitively reasonable), then we could describe the support of \( \mu_0 = \lim_{\beta \to \infty} \nu_\beta^{**} \) quite precisely. This depends upon the fact that we can estimate the distribution for the first coefficient \( \hat{g}_+(1) \), using invariance considerations.

For this method to work in the Virasoro case, we would need an estimate for the \( u_1 \) distribution (which we suspect is absolutely continuous only for \( c > 1 \)). Assuming that this has been resolved in some way, we would then need something like the following

\[ \text{(4.1.8)Conjecture.} \quad \text{For each } \beta > 0, \ c \geq 0,
\]
\[
\nu_{\beta,c}\{|c_n| > (n+1)R\} \leq \nu_{\beta,c}\{|c_{n-1}| > nR\}.
\]

This would clearly imply the Bieberbach-de Branges inequalities, since we know that \( |u_1| < 2 \) with probability one. It would follow from this conjecture that we could estimate the distribution of \( u_n \) in terms of the distribution for \( u_1 \), and hopefully this would show that the limit points are supported on a space such as \( D_{qs} \).

Concerning the diagonal distribution, as in the case of loop groups, it should be possible to a priori arrive at a reasonable conjecture for the evaluation of the integral
\[
\int \hat{a}(\phi)^{-i\lambda}d\mu_{c,h}(\phi) = \lim_{\beta \downarrow 0} \int \hat{a}(\phi)^{-i\lambda}d\nu_{\beta,c,h}(\phi), \tag{4.1.9}
\]
where we are writing
\[
\phi = l \cdot \text{diag} \cdot u, \quad \text{diag} = m\hat{a}, \quad 0 < \hat{a} < 1, \tag{4.1.10}
\]
but I have not succeeded at this.

Acknowledgements. I thank Nasser Towghi for many useful conversations, and Jan Wehr for pointing out [PY] to me.

References
[B] L. de Branges, A proof of the Bieberbach conjecture, Acta Mathematica 154 (1985) 137-152.

[FLM] I. Frenkel, J. Lepowsky, and A. Meurman, Vertex Operator Algebras and the Monster, Academic Press (1990).

[K] A.V. Kosyak, Irreducible regular Gaussian representations of the group of the interval and circle diffeomorphism, J. Funct. Anal. 125 (1994) 493-547.

[Kn] F. Knight, Essentials of Brownian Motion and Diffusion, Mathematical Surveys, No. 18, AMS (1981).

[KY] A.A. Kirillov and D.V. Yuriev, Representations of the Virasoro algebra by the orbit method, J. Geom. Phys., Vol. 5, n. 3 (1988).

[L] O. Lehto, Univalent Functions and Teichmuller Spaces, Springer-Verlag (1986).

[MM] M. Malliavin and P. Malliavin, Quasi-invariant measures on the group of diffeomorphisms of the circle, Proceedings of the Lisbonne Conference, ed. by A.B. Cruzeiro, Birkhauser (1991).

[MP] R. Moody and Pianzola, Lie algebras with Triangular Structure, Wiley (1995).

[NS] S. Nag and D. Sullivan, Teichmuller theory and the universal period mapping via quantum calculus and the $H^{1/2}$ space on the circle, Osaka J. Math., 32, no.1 (1995).

[Pe] O. Pekonen, Universal Teichmuller space in geometry and physics, J. Geom. Phys. 15 (1995) 227-251.

[Pi] D. Pickrell, Invariant measures for unitary forms of Kac-Moody Lie groups, Parts I-III, submitted to J. Funct. Anal., funct-an 9510005

[PY] J.W. Pitman and M. Yor, A limit theorem for one-dimensional Brownian motion near its maximum, and its relation to a representation of the 2-dimensional Bessel bridge, preprint.

[PS] A. Pressley and G. Segal, Loop Groups, Oxford University Press (1986).

[R] R. Ramer, On nonlinear transformations of Gaussian measures, J. Funct. Anal. 15 (1974) 166-187.

[Se1] G. Segal, Unitary representations of some infinite dimensional groups, Comm. Math. Phys. 80 (1981), 301-342.

[Se2] ——, The definition of conformal field theory, preprint.

[S] S. Semmes, On the Cauchy integral and related operators on smooth curves, dissertation, Washington University, St. Louis, Missouri (1983)

[Sh] E.T. Shavgulidze, Distributions on infinite dimensional spaces and second quantization in string theories, II, in “V International Vilnius Conference in Probability Theory and Mathematical Statistics” (1989) 359-360.