A Semi-Analytical Study of Texture Collapse

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ABSTRACT

This study presents a simplified approach to studying the dynamics of global texture collapse. We derive equations of motion for a spherically symmetric field configuration using a two parameter ansatz. Then we analyse the effective potential for the resulting theory to understand possible trajectories of the field configuration in the parameter space of the ansatz. Numerical results are given for critical winding and collapse time in spatially flat non-expanding, and flat expanding universes. In addition, the open non-expanding and open-expanding cases are studied.
1. Introduction

Global texture theory is a cosmological model of large-scale structure formation. Texture is a semi-topological defect in a theory where a global symmetry group $G$ is broken to a group $H$ such that $\pi_3(G/H) \neq 1$. We can take as a toy model a theory with a complex doublet of scalar fields and a Mexican hat potential generalized to four dimensions, in which case the vacuum manifold is $S^3$.

Texture is formed when the field wraps around enough of $S^3$ to cause unwinding. Energy lumping of the field configuration causes accretion of matter, thus forming large-scale structure\cite{1}\cite{2}.

To understand cosmological structure formation, we need to have information about the texture distribution. The texture distribution depends crucially on the critical winding\cite{3} (i.e. the winding above which collapse occurs). The formation probability for texture with a given winding distribution depends on the critical winding. More or fewer textures are formed if the critical winding is low or high, respectively.

Critical winding is determined by the dynamics of the field configuration as uncorrelated regions of the field come into the horizon. Thus, we need to have a good understanding of field dynamics in the texture model to understand the texture distribution.

Dynamics of texture field configurations has been analyzed in a variety of studies. A self-similar $\sigma$-model solution has been found for the flat, non-expanding universe with winding $w = 1$\cite{1}. For this solution collapse proceeds at the speed of light. This study is limited by the fact that dynamics at the point of unwinding are not obtainable with the approximations used. Also, solutions for non-integer winding are not known.

Numerical studies have been able to get past the above limitations\cite{4}\cite{5}\cite{6}. The full equations have been integrated for spherically symmetric and ellipsoidal configurations in flat, expanding universes; non-integer winding and the dynamics at the
point of collapse were also included in these studies. Furthermore, full cosmological simulations have also been done, including baryonic matter effects\cite{7}.

Some analytical work has been done in the adiabatic approximation using the $\sigma$-model approach\cite{8}. These studies have approximated the Hamiltonian for large $R$, where $R$ is a physical cutoff understood to be half the intertexture separation, and used a two parameter ansatz for a spherically symmetric field configuration.

This study extends the above analytical work to include kinetic energy. Here we investigate a constrained Hamiltonian, where the field configuration is (as in the above study) approximated by a two parameter ansatz.

First, we investigate the shape of the effective potential due to gradient energy to understand what sort of trajectories can be expected. Then, we integrate the Hamiltonian equations of motion to find the trajectory followed by the field ansatz in phase space.

Analysis of the effective potential for this theory results in a clear understanding of why trajectories in phase space evolve as they do. A saddlepoint is found lying on the barrier between expanding and collapsing trajectories explaining the existence of trajectories which appear initially to be headed for collapse, but then expand, and trajectories which appear to be headed toward expansion, yet finally collapse (see section 3).

Critical windings were found for fields uncorrelated on scales larger than the horizon for intertexture separation $2R$, where $R$ is in units of the horizon size, to be:

For $R = 2.0$ (For $R = 1.5$) (errors are plus or minus the difference to the next closest calculated trajectory.)

- $0.6562 \pm 0.0001 \ (0.6967 \pm 0.0001)$ (flat case),
- $0.6693 \pm 0.0001 \ (0.7471 \pm 0.0001)$ (radiation era),
- $0.667 \pm 0.007 \ (0.7543 \pm 0.0001)$ (matter era).

Collapse time was found to increase at critical winding when approached from
high winding. This phenomenon is also explained by the saddlepoint found in the analysis of the effective potential. In fact, critical winding should, according to arguments based on the shape of the effective potential, go to infinity at critical winding.

2. Derivation of Equations of Motion

As was mentioned in section 1, global texture can be described by a theory with a complex doublet of scalar fields with a Mexican hat potential. The action for this theory is

$$S = \int d^4x [\partial^\mu \Phi^a \partial_\mu \Phi^a - \lambda(|\Phi|^2 - \eta^2)]\sqrt{-g}$$

where $a = 1, ..., 4$ and $\eta$ is the scale of symmetry breaking.

For low temperatures relative to the phase transition and times before un-winding we can fix the scalar field at the minimum of the potential and treat the potential as a constraint. This gives us a $\sigma$-model action

$$S = \int d^4x [\partial^\mu \Phi^a \partial_\mu \Phi^a] \sqrt{-g}$$

with the constraint

$$|\Phi|^2 = \eta^2.$$

We can use the spherically symmetric ansatz

$$\Phi^a = \eta(\sin \chi \sin \theta \sin \phi, \sin \chi \sin \theta \cos \phi, \sin \chi \cos \theta, \cos \chi),$$

where $\chi = \chi(r,t)$ is a radial field and the background Friedmann- Robertson-
Walker metric

\[ g_{\mu\nu} = \text{diag}(1, -\frac{a^2}{1-kr^2}, -a^2r^2, -a^2r^2\sin^2\theta), \]

to obtain the action

\[ S = \int d^4x [\dot{\chi}^2 - \frac{\chi'^2 (1-kr^2)}{a^2} - \frac{2\sin^2\chi}{a^2r^2} \frac{a^3r^2}{\sqrt{1-kr^2}}]. \]

In the flat \((k = 0)\) non-expanding case \((a(t) = 1)\), the solution to the equations of motion is

\[ \chi = 2 \arctan(-\frac{r}{\gamma}) \text{ for } t < 0. \]

We can mimic the exact solution with the simple ansatz (Figure 1)

\[ \chi = \alpha\pi r \text{ for } 0 < r < x \]
\[ \chi = \xi\pi \text{ for } x < r < R \]

and vary \(\xi\) to obtain non-integer winding. In this ansatz \(\alpha = \alpha(t)\) and \(\xi = \xi(t)\), \(x\) is the point of intersection of the two segments of the ansatz and \(R\) is a cutoff. \(x\) is initially taken to be the horizon size (the scale on which the field is correlated) and \(R\) is taken to be \(1/2\) the intertexture separation, a fixed multiple of \(H^{-1}\). The initial horizon \(H(t_0)\) is set to be equal to 1.

This ansatz is expected to be good for understanding precollapse dynamics, especially around the time when the field configuration enters the horizon.

With this ansatz in the above action, we derive the Hamiltonian

\[ H = p_{\alpha}^2 (4a^3\pi^2 I_1)^{-1} + p_{\xi}^2 (4a^3\pi^2 I_4)^{-1} \]
\[ + a\alpha^2 \pi^2 I_2 + 2a I_3 + 2a \sin^2(\xi\pi) I_5 \]

where \(p_\alpha\) and \(p_\xi\) are momenta canonical to \(\alpha\) and \(\xi\) respectively, and \(I_i\) are integrals over the radial variable \(r\).
From this Hamiltonian we find the equations of motion in phase-space \((p_\alpha, p_\xi, \alpha, \xi)\) to be

\[
\dot{\alpha} = p_\alpha (2a^3\pi^2 I_1)^{-1}
\]
\[
\dot{\xi} = p_\xi (2a^3\pi^2 I_4)^{-1}
\]
\[
\dot{p}_\alpha = p_\alpha^2 (4a^3\pi^2)^{-1} I_1^{-2} \frac{\partial I_1}{\partial \alpha} + p_\xi^2 (4a^3\pi^2)^{-1} I_4^{-2} \frac{\partial I_4}{\partial \alpha} - 2a \alpha^2 \pi^2 \frac{\partial I_2}{\partial \alpha} - 2a \frac{\partial I_3}{\partial \alpha}
\]
\[
-2a \alpha^2 \pi^2 I_2 - a \alpha^2 \pi^2 \frac{\partial I_2}{\partial \alpha} - 2a \frac{\partial I_3}{\partial \alpha}
\]
\[
-2a \sin^2(\xi \pi) \frac{\partial I_5}{\partial \alpha}
\]
\[
\dot{p}_\xi = p_\alpha^2 (4a^3\pi^2)^{-1} I_1^{-2} \frac{\partial I_1}{\partial \xi} + p_\xi^2 (4a^3\pi^2)^{-1} I_4^{-2} \frac{\partial I_4}{\partial \xi} - a \alpha^2 \pi^2 \frac{\partial I_2}{\partial \xi} - 2a \frac{\partial I_3}{\partial \xi}
\]
\[
-4a \pi \sin(\xi \pi) \cos(\xi \pi) I_5 - 2a \sin^2(\xi \pi) \frac{\partial I_5}{\partial \xi},
\]

and the integrals \(I_i\) are

\[
I_1 = \int_0^x dr \frac{r^4}{\sqrt{1 - kr^2}}
\]
\[
I_2 = \int_0^x dr r^2 \sqrt{1 - kr^2}
\]
\[
I_3 = \int_0^x dr \sin^2(\alpha \pi r) \frac{1}{\sqrt{1 - kr^2}}
\]
\[ I_4 = \int_x^R dr \frac{r^2}{\sqrt{1 - kr^2}}. \]

\[ I_5 = \int_x^R dr \frac{dr}{\sqrt{1 - kr^2}}. \]

The integrals are all trivial except \( I_3 \) in the open case \((k \neq 0)\). For this case, we expand \( \sin^2 r \) to sufficient order and calculate the resulting integrals. To solve the equations of motion in the open case, we need expressions for \( a(t) \), the scale factor of the universe. We use the fact that the universe is close to flat up to its current point of evolution and use the flat space radiation- and matter-dominated expanding universe scale factors as approximations to the actual scale factor.

Finally, the expression relating \( \xi \) to winding \( w \) is

\[ w(\xi) = \xi - \frac{\sin(2\pi\xi)}{2\pi}. \]

3. Results

The terms in \( H \) independent of momenta \( p_\alpha \) and \( p_\xi \) form the effective potential due to gradient energy in the field. Here (Figure 2) we plot a portion of it for the flat non-expanding case with \( R = 1.5 \).

There are two attractors in the potential, one in the upper righthand corner (large \( \alpha \), large \( \xi \)) to which collapsing configurations approach; and one in the lower lefthand corner (small \( \alpha \), small \( \xi \)) to which expanding configurations approach.

Notice the saddlepoint. To the lower left (small \( \alpha \), small \( \xi \)) and upper right (large \( \alpha \), large \( \xi \)) are the attractors for collapsing and expanding configurations. To the upper left (small \( \alpha \), large \( \xi \)) and lower right (large \( \alpha \), small \( \xi \)) we have regions of high potential.
If $R$ is increased, the saddlepoint moves leftward to small values of $\alpha$ with $\xi$ remaining approximately the same, and its size shrinks (that is the gradient gets steeper on all sides). If $R$ is decreased, the saddlepoint moves to the right to larger values of $\alpha$ with $\xi$ remaining approximately the same, and its size increases (gradients get less steep on all sides). The gradients around the saddlepoint get large for small $R$ because the total energy of the configuration increases as $R$ gets large and thus the height of the barrier increases.

We take initial conditions for $\xi$ and $\alpha$ based on the physical constraint $x_{initial} = 1$. This means that the initial $\xi$ and $\alpha$ obey the relation $\xi = \alpha$. For intertexture separation $2R = 4$, this implies that for winding near critical the starting point of the field evolution trajectory is on the upper left region (hill) of the saddlepoint.

For our initial conditions, $x_{initial} = 1$ and thus $\dot{\alpha} \sim p_\alpha$ and $\dot{\xi} \sim p_\xi$, so near the saddlepoint, the field momenta can be thought of as ‘rolling’ on the effective potential (farther away momentum behavior becomes more complex).

Therefore the field will roll down the hill and up the barrier on the opposite side of the saddle. Depending on whether the initial configuration is above or below critical winding the trajectory will scatter off the upper or lower side of the righthand barrier. If the trajectory moves to the basin in the upper right of the potential, the configuration collapses and unwinds. But if the trajectory scatters off the lower part of the righthand barrier, the configuration collapses for some time, but then, as the configuration falls to the lower left attractor, the configuration expands.

For large $R$, the opposite effect occurs. Initially trajectories start on the lower right hill, then they roll across toward the opposite barrier and scatter to one side or the other depending on initial winding. These trajectories were discovered by Perivolaropoulos\cite{8} in the large $R$ limit.

Trajectories that don’t start on the saddlepoint fall more or less directly into the attractors on their respective sides of the saddlepoint.
For large $R$, there are fewer trajectories that start on or near the saddlepoint because its size shrinks.

Monte Carlo simulations suggest that [3], if we assume random correlations on length scales larger than the horizon, intertexture separation should be around two times the horizon size. This indicates that, usually, few textures will expand then collapse, but many will first tend towards collapse then expand, due to the location of the saddlepoint with respect to typical initial conditions.

The following figures show integrated trajectories with $x_{initial} = 1$ for flat non-expanding, flat expanding and open expanding universes. The trajectories are plotted in $(\xi, \alpha)$-space.

Critical winding can be found either by distinguishing which trajectories finally expand and which finally collapse, or by calculating the intersection of the line of initial conditions and the line of intersection of the tangent surface to the top of the ridge in which the saddlepoint lies with the ridge. We found the former method more convenient.

Note that critical winding increases from the non-expanding (Figure 3) to the radiation-dominated (Figure 4) to the matter-dominated cases (Figure 5). This can be understood physically. The expanding background introduces an extra pull on the texture configuration, thus causing more configurations to expand, and thus pushing critical winding higher than in the non-expanding universe. The matter-dominated background expands more quickly than the radiation-dominated background, thus pushing critical winding even higher in the matter-dominated case.

The effect of extra pull due to the background manifests itself in the effective potential such that the barrier region, including the saddlepoint moves to a higher value of $\xi$.

In an open background, the area of concentric spheres increases more quickly as we leave the origin than in a flat background. This effect also tends to add a pull to the configuration, and push critical winding upwards.
As $R$ increases (Figure 6), the saddlepoint in the effective potential moves to
the left in the figures. This causes the line of initial conditions to move across
the saddlepoint and critical winding (the point of intersection of the line of initial
conditions and the top of the barrier) goes down, since the saddlepoint is in a
skewed orientation with the lefthand region of high potential at larger $\xi$ than the
righthand region of high potential.

Critical windings were found for fields uncorrelated on scales larger than the
horizon ($x_{\text{initial}} = 1$) for intertexture separation $2R$, where $R$ is in units of horizon
size, to be:

For $R = 2.0$ (for $R = 1.5$) (errors are plus or minus the difference to the next
closest calculated trajectory.)

\begin{align*}
0.6562 \pm 0.0001 & \text{ (} 0.6967 \pm 0.0001 \text{) (flat case),} \\
0.6693 \pm 0.0001 & \text{ (} 0.7471 \pm 0.0001 \text{) (radiation era),} \\
0.667 \pm 0.007 & \text{ (} 0.7543 \pm 0.0001 \text{) (matter era).}
\end{align*}

Collapse time is affected by the length of time the configuration takes to roll
off the saddle. Configurations very near to the saddlepoint take longer to roll off,
since the saddlepoint is a point of unstable equilibrium. For configurations starting
at the unstable equilibrium point, collapse time should be infinite.

Also, there are trajectories below critical winding that evolve for a substantial
period of time as if they would collapse, but eventually expand. Although there
is no definable collapse time for these configurations, there might be appreciable
matter accretion possible.

Collapse time, defined as the time it takes for $x$, the joining point of the two
segments in the ansatz, to reach half its initial value, for the flat non-expanding
universe is plotted in Figure 7.
4. Discussion and Concluding Remarks

We have used a two parameter spherically symmetric ansatz in the low temperature $\sigma$-model approximation to the texture action to obtain a Hamiltonian and equations of motion for texture dynamics.

From the Hamiltonian we have isolated the effective potential and found a saddlepoint that explained the behavior of the field configuration. The saddlepoint explains configurations which expand then collapse, and configurations which collapse then expand, as well as simpler trajectories in which the field configuration simply collapses or simply expands.

From the Hamiltonian we derived equations of motion which were ordinary differential equations, in contrast to partial differential equations. The ODEs were easy to integrate on the computer, and results were obtained $\sim 1000$ times faster than integrating the PDEs.

Critical windings were found for fields uncorrelated on scales larger than the horizon ($x_{\text{initial}} = 1$) for intertexture separation $2R$, where $R$ is in units of horizon size, to be:

For $R = 2.0$ (For $R = 1.5$) (errors are plus or minus the difference to the next closest calculated trajectory.)

\[
\begin{align*}
0.6562 \pm 0.0001 & \ (0.6967 \pm 0.0001) \ \text{(flat case)}, \\
0.6693 \pm 0.0001 & \ (0.7471 \pm 0.0001) \ \text{(radiation era)}, \\
0.667 \pm 0.007 & \ (0.7543 \pm 0.0001) \ \text{(matter era)}. 
\end{align*}
\]

Collapse time was found to increase as winding approached critical from above (for collapsing configurations).

This study is also interesting, in general, as an approach to studying the $\sigma$-model for non-integer windings where there is a cutoff scale $R$.

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FIGURE CAPTIONS

1) The ansatz for $\chi$ as a function of $r$

2) A contour plot of the effective potential.

3) Trajectories are projected on $(\alpha, \xi)$-space. In this figure trajectories are plotted for the spatially flat non-expanding universe with $R = 1.5$, $x = 1.0$. All trajectories start on the line $\xi = \alpha$.

4) Trajectories are plotted for the radiation-dominated universe with $R = 1.5$ and $x = 1.0$.

5) Trajectories are plotted for the matter-dominated universe with $R = 1.5$ and $x = 1.0$.

6) Trajectories are plotted for the flat case with $R = 2.0$ and $x = 1.0$.

7) Collapse time, defined as the time it takes for $x$ to reach half its initial value, is plotted versus winding, for a non-expanding universe.