ON THE SECTION CONJECTURE OF GROTHENDIECK

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ABSTRACT. For a given arithmetic scheme, in this paper we will introduce and discuss the monodromy action on a universal cover of the étale fundamental group and the monodromy action on an $sp$-completion constructed by the graph functor, respectively; then by these results we will give a proof of the section conjecture of Grothendieck for arithmetic schemes.

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INTRODUCTION

The section conjecture in anabelian geometry is originally from [12], the so called “Anabelian Letter to Faltings ”, a letter to Faltings written by Grothendieck in 1983.

“To Grothendieck’s disappointment, Faltings never responded to this letter. However, Faltings’ student Shinichi Mochizuki picked up the subject years later and proved Grothendieck’s anabelian conjecture for hyperbolic curves ”(see [19]).

See [15] for a formal introduction to this topic on anabelian geometry. Many people have proved the section conjecture for the various cases of algebraic curves.

2000 Mathematics Subject Classification. Primary 14F35; Secondary 11G35.

Key words and phrases. anabelian geometry, arithmetic scheme, étale fundamental group, section conjecture.
Now let $X$ be an arithmetic scheme surjectively over $\text{Spec}(\mathbb{Z})$ of finite type. In this paper we will prove the section conjecture of Grothendieck for arithmetic schemes.

Here are the key points to overcome in the paper:

Naturally and fortunately, we will have the monodromy action of the group $\text{Aut}(X_{\Omega \text{et}}/X)$ on the universal cover $X_{\Omega \text{et}}$ for the étale fundamental group $\pi_1^{\text{et}}(X)$ and the monodromy action of the group $\text{Aut}(X_{\text{sp}}/X)$ on the $\text{sp}$-completion $X_{\text{sp}}$ constructed by the graph functor $\Gamma$.

Then by these monodromy actions of groups on integral schemes, we will obtain a bijection between the sets of homomorphisms which are considered.

Acknowledgment. The author would like to express his sincere gratitude to Professor Li Banghe for his advice and instructions on algebraic geometry and topology.

1. Statements of the Main Theorems

1.1. Notation. In this paper, an arithmetic variety is an integral scheme $X$ satisfying the conditions:

- $\dim X \geq 1$.
- There is a surjective morphism $f : X \to \text{Spec}(\mathbb{Z})$ of finite type.

For a number field $K$ (i.e., a finite extension of $\mathbb{Q}$), let $\mathcal{O}_K$ denote the ring of algebraic integers of $K$.

For an integral scheme $Z$, put

- $k(Z) \triangleq$ the function field of an integral scheme $Z$;
- $\pi_1^{\text{et}}(Z) \triangleq$ the étale fundamental group of $Z$ for a geometric point of $Z$ over a separable closure of the function field $k(Z)$.

In particular, for a field $L$, we set

$$\pi_1^{\text{et}}(L) \triangleq \pi_1^{\text{et}}(\text{Spec}(L)).$$

1.2. Outer homomorphisms. Let $G, H, \pi_1, \pi_2$ be four groups with homomorphisms $p : G \to \pi_1$ and $q : H \to \pi_2$, respectively. The outer homomorphism set $\text{Hom}_{\pi_1, \pi_2}^{\text{out}}(G, H)$ is defined to be the set of the maps $\sigma$ from the quotient $\pi_1/p(G)$ into the quotient $\pi_2/q(H)$ given by a group homomorphism $f : G \to H$ in such a manner:

$$\sigma : x \cdot p(G) \mapsto f(x) \cdot q(H)$$

for any $x \in \pi_1$.

In fact, if $G$ and $H$ are normal subgroups of $\pi_1$ and $\pi_2$, respectively, $\text{Hom}_{\pi_1, \pi_2}^{\text{out}}(G, H)$ can be regarded as a subset of $\text{Hom}(\text{Out}(G), \text{Out}(H))$. Here, $\text{Out}(G) \triangleq \text{Aut}(G)/\text{Inn}(G)$ and $\text{Out}(H) \triangleq \text{Aut}(H)/\text{Inn}(H)$ are the outer automorphism groups.
However, in general, it is not true that
\[ \text{Hom}_{\pi_1,\pi_2}^\text{out} (G, H) = \text{Hom} (\text{Out}(G), \text{Out}(H)) \]
holds.

1.3. **Statements of the main theorems.** For anabelian geometry of arithmetic schemes, we have the following results, which are the main theorems in the present paper.

**Theorem 1.1.** Let \( X \) and \( Y \) be two arithmetic varieties such that \( k(Y) \) is contained in \( k(X) \). Then there is a bijection
\[ \text{Hom}(X, Y) \cong \text{Hom}_{\pi_1^\text{et}(k(X)),\pi_1^\text{et}(k(Y))} (\pi_1^\text{et}(X), \pi_1^\text{et}(Y)) \]
between sets.

**Theorem 1.2.** Let \( X \) be an arithmetic variety and let \( K \) be a number field. Suppose that there is a surjective morphism from \( X \) onto \( \mathcal{O}_K \). Then there is a bijection
\[ \Gamma(X/\text{Spec}(\mathcal{O}_K)) \cong \text{Hom}_{\pi_1^\text{et}(K),\pi_1^\text{et}(k(X))} (\pi_1^\text{et}(\text{Spec}(\mathcal{O}_K)), \pi_1^\text{et}(X)) \]
between sets.

Now fixed a function field \( L \) over a number field \( K \). Set
- \( G(L) \triangleq \) the absolute Galois group \( \text{Gal}(L^{al}/L) \);
- \( G(L)^{un} \triangleq \) the Galois group \( \text{Gal}(L^{un}/L) \) of the maximal unramified extension \( L^{un} \) of \( L \) (see Definition 2.8).

Using Galois groups of fields, we have the following versions of the main theorems above, respectively.

**Theorem 1.3.** Let \( X \) and \( Y \) be two arithmetic varieties such that \( k(Y) \) is contained in \( k(X) \). Then there is a bijection
\[ \text{Hom}(X, Y) \cong \text{Hom}_{G(k(X)),G(k(Y))} (G(k(X))^{un}, G(k(Y))^{un}) \]
between sets.

**Theorem 1.4.** Let \( X \) be an arithmetic variety and let \( K \) be a number field. Suppose that there is a surjective morphism from \( X \) onto \( \mathcal{O}_K \). Then there is a bijection
\[ \Gamma(X/\text{Spec}(\mathcal{O}_K)) \cong \text{Hom}_{G(K),G(k(X))} (G(K)^{un}, G(k(X))^{un}) \]
between sets.

We will prove the main theorems above in §8 after preparations are made in §§2-7.

**Remark 1.5.** In a similar manner, we can prove that the two theorems still hold for the case of projective schemes.
2. Preliminaries

2.1. **Convention.** For an integral domain $D$, let $Fr(D)$ denote the field of fractions of $D$. In particular, $Fr(D)$ will be assumed to be contained in $\Omega$ if $D$ is contained in a field $\Omega$.

By an **integral variety** we will always understand an integral scheme over $Spec(\mathbb{Z})$ by a surjective morphism (not necessarily of finite type).

2.2. **Galois extension.** Let $L$ be an extension of a field $K$. Note that $L$ is not necessarily algebraic over $K$. Let $Gal(L/K)$ be the Galois group of $L$ over $K$.

Recall that $L$ is said to be **Galois** over $K$ if $K$ is the invariant subfield of $Gal(L/K)$, that is, if $K = \{x \in L : \sigma(x) = x\}$ holds for any $\sigma \in Gal(L/K)$.

2.3. **Quasi-galois extension.** Let $L$ be an extension of a field $K$ (not necessarily algebraic).

**Definition 2.1.** $L$ is said to be **quasi-galois** over $K$ if each irreducible polynomial $f(X) \in F[X]$ that has a root in $L$ factors completely in $L[X]$ into linear factors for any subfield $F$ with $K \subseteq F \subseteq L$.

Let $D \subseteq D_1 \cap D_2$ be three integral domains. Then $D_1$ is said to be **quasi-galois** over $D$ if $Fr(D_1)$ is quasi-galois over $Fr(D)$.

**Definition 2.2.** $D_1$ is said to be a **conjugation** of $D_2$ over $D$ if there is an $F$–isomorphism $\tau : Fr(D_1) \to Fr(D_2)$ such that $\tau(D_1) = D_2$, where $F \triangleq k(\Delta)$, $k \triangleq Fr(D)$, $\Delta$ is a transcendental basis of the field $Fr(D_1)$ over $k$, and $F$ is contained in $Fr(D_1) \cap Fr(D_2)$.

2.4. **Affine covering with values.** Let $X$ be a scheme. An **affine covering** of $X$ is a family $\mathcal{C}_X = \{(U_\alpha, \phi_\alpha; A_\alpha)\}_{\alpha \in \Delta}$, where for each $\alpha \in \Delta$, $\phi_\alpha$ is an isomorphism from an open set $U_\alpha$ of $X$ onto the spectrum $Spec A_\alpha$ of a commutative ring $A_\alpha$.

Each $(U_\alpha, \phi_\alpha; A_\alpha) \in \mathcal{C}_X$ is called a **local chart**. For the sake of brevity, a local chart $(U_\alpha, \phi_\alpha; A_\alpha)$ will be denoted by $U_\alpha$ or $(U_\alpha, \phi_\alpha)$.

An affine covering $\mathcal{C}_X$ of $(X, \mathcal{O}_X)$ is said to be **reduced** if $U_\alpha \neq U_\beta$ holds for any $\alpha \neq \beta$ in $\Delta$.

Let $\mathfrak{Comm}$ be the category of commutative rings with identity. For a given field $\Omega$, let $\mathfrak{Comm}(\Omega)$ be the category consisting of the subrings of $\Omega$ and their isomorphisms.

**Definition 2.3.** Let $\mathfrak{Comm}_0$ be a subcategory of $\mathfrak{Comm}$. An affine covering $\{(U_\alpha, \phi_\alpha; A_\alpha)\}_{\alpha \in \Delta}$ of $X$ is said to be **with values** in $\mathfrak{Comm}_0$ if for each $\alpha \in \Delta$ there are $\mathcal{O}_X(U_\alpha) = A_\alpha$ and $U_\alpha = Spec(A_\alpha)$, where $A_\alpha$ is a ring contained in $\mathfrak{Comm}_0$. 
In particular, an affine covering $\mathcal{C}_X$ of $X$ with values in $\mathbf{Comm}(\Omega)$ is said to be with values in the field $\Omega$.

By an affine covering with values in a field, it is seen that an affine open set of a scheme is measurable, at the same time, the non-affine open sets are unmeasurable.

If we ignore the non-affine open sets, almost all properties of the scheme will be still preserved. Hence, we have the following notions.

Let $\mathcal{O}_X$ and $\mathcal{O}'_X$ be two structure sheaves on the underlying space of an integral scheme $X$. The two integral schemes $(X, \mathcal{O}_X)$ and $(X, \mathcal{O}'_X)$ are said to be essentially equal provided that for any open set $U$ in $X$, we have

\[ U \text{ is affine open in } (X, \mathcal{O}_X) \iff \text{ so is } U \text{ in } (X, \mathcal{O}'_X) \]

and in such a case, $D_1 = D_2$ holds or there is $Fr(D_1) = Fr(D_2)$ such that for any nonzero $x \in Fr(D_1)$, either

\[ x \in D_1 \bigcap D_2 \]

or

\[ x \in D_1 \setminus D_2 \iff x^{-1} \in D_2 \setminus D_1 \]

holds, where $D_1 = \mathcal{O}_X(U)$ and $D_2 = \mathcal{O}'_X(U)$.

Two schemes $(X, \mathcal{O}_X)$ and $(Z, \mathcal{O}_Z)$ are said to be essentially equal if the underlying spaces of $X$ and $Z$ are equal and the schemes $(X, \mathcal{O}_X)$ and $(X, \mathcal{O}_Z)$ are essentially equal.

**Definition 2.4.** An affine covering $\{(U_\alpha, \phi_\alpha; A_\alpha)\}_{\alpha \in \Delta}$ of $X$ is said to be an affine patching of $X$ if the map $\phi_\alpha$ is the identity map on $U_\alpha = \text{Spec}A_\alpha$ for each $\alpha \in \Delta$.

Evidently, an affine patching is reduced.

2.5. **Quasi-galois closed affine covering.** Let $f : X \to Y$ be a surjective morphism of integral schemes. Fixed an algebraic closure $\Omega$ of the function field $k(X)$.

**Definition 2.5.** A reduced affine covering $\mathcal{C}_X$ of $X$ with values in $\Omega$ is said to be quasi-galois closed over $Y$ by $f$ if there exists a local chart $(U'_\alpha, \phi'_\alpha; A'_\alpha) \in \mathcal{C}_X$ such that $U'_\alpha \subseteq \varphi^{-1}(V_\alpha)$ holds

- for any $(U_\alpha, \phi_\alpha; A_\alpha) \in \mathcal{C}_X$;
- for any affine open set $V_\alpha$ in $Y$ with $U_\alpha \subseteq f^{-1}(V_\alpha)$;
- for any conjugate $A'_\alpha$ of $A_\alpha$ over $B_\alpha$,

where $B_\alpha$ is the canonical image of $\mathcal{O}_Y(V_\alpha)$ in $k(X)$ via $f$. 
2.6. **Quasi-galois closed scheme.** Let \( f : X \to Y \) be a surjective morphism between integral schemes. Let \( \text{Aut}(X/Y) \) denote the group of automorphisms of \( X \) over \( Y \).

A integral scheme \( Z \) is said to be a **conjugate** of \( X \) over \( Y \) if there is an isomorphism \( \sigma : X \to Z \) over \( Y \).

**Definition 2.6.** \( X \) is said to be **quasi-galois closed** over \( Y \) by \( f \) if there is an algebraically closed field \( \Omega \) and a reduced affine covering \( \mathcal{C}_X \) of \( X \) with values in \( \Omega \) such that for any conjugate \( Z \) of \( X \) over \( Y \) the two conditions are satisfied:

- \( (X, \mathcal{O}_X) \) and \( (Z, \mathcal{O}_Z) \) are essentially equal if \( Z \) has a reduced affine covering with values in \( \Omega \).
- \( \mathcal{C}_Z \subseteq \mathcal{C}_X \) holds if \( \mathcal{C}_Z \) is a reduced affine covering of \( Z \) with values in \( \Omega \).

**Remark 2.7.** In the above definition, \( \Omega \) is in deed an algebraic closure of the function field \( k(X) \); \( \mathcal{C}_X \) is the unique maximal affine covering of \( X \) with values in \( \Omega \) (see [8]).

2.7. **Unramified extension.** Let us recall the definition for unramified extensions of function fields over a number field in several variables.

**Definition 2.8.** Let \( L_1 \) and \( L_2 \) be two extensions over a number field \( K \) such that \( L_1 \subseteq L_2 \).

(i) \( L_2 \) is said to be a **finite unramified Galois** extension of \( L_1 \) if there are two arithmetic varieties \( X_1 \) and \( X_2 \) and a surjective morphism \( f : X_2 \to X_1 \) such that

- \( k(X_1) = L_1, k(X_2) = L_2 \);
- \( X_2 \) is a finite étale Galois cover of \( X_1 \) by \( f \).

(ii) \( L_2 \) is said to be a **finite unramified** extension of \( L_1 \) if there is a field \( L_3 \) over \( K \) such that \( L_2 \) is contained in \( L_3 \) and \( L_3 \) is a finite unramified Galois extension of \( L_1 \).

(iii) \( L_2 \) is said to be an **unramified** extension of \( L_1 \) if the field \( L_1(\omega) \) is a finite unramified extension of \( L_1 \) for each element \( \omega \in L_2 \). In such a case, the element \( \omega \) is said to be **unramified** over \( L_1 \).

**Remark 2.9.** It is seen that there exists the geometric model \( X_2/X_1 \) for the extension \( L_2/L_1 \) (see [4]). In deed, we can take the valuation rings \( A_1 \subseteq A_2 \) of \( L_1 \subseteq L_2 \), respectively; then put \( X_1 = \text{Spec}(A_1) \) and \( X_2 = \text{Spec}(A_2) \).

**Remark 2.10.** Let \( L_1 \subseteq L_2 \subseteq L_3 \) be function field over a number field \( K \). Suppose that \( L_2/L_1 \) and \( L_3/L_2 \) are unramified extensions. Then \( L_3 \) is unramified over \( L_1 \).
Remark 2.11. It is seen that for the case of an algebraic extension, the unramified extension defined in Definition 2.8 coincides exactly with that in algebraic number theory.

Remark 2.12. Note that we have defined another unramified extension in [7] for the case of algebraic schemes, which is only a formally abstract definition and is different from the above one in Definition 2.8.

Let \( L \) be an arbitrary extension over a number field \( K \). Set
- \( L^{al} \triangleq \) an algebraical closure of \( L \);
- \( L^{un} \triangleq \) the union of all the finite unramified subextensions over \( L \) contained in \( L^{al} \).

3. Universal Covers

3.1. Facts on quasi-galois closed schemes. Here there are several known results on quasi-galois schemes which will be used in the remainder of the paper (see \([2]-[8]\)).

Lemma 3.1. (Tuning scheme \([5]\)) For any integral variety \( X \), there is an integral variety \( Z \) satisfying the conditions:
- \( k(X) = k(Z) \);
- \( X \cong Z \) are isomorphic;
- \( Z \) has a reduced affine covering with values in \( k(X)^{al} \).

Lemma 3.2. (Geometric model \([6]\)) Let \( f : X \to Y \) be a surjective morphism of integral varieties. Suppose that \( X \) is quasi-galois closed over \( Y \) by \( f \) and that \( k(X) \) is canonically Galois over \( k(Y) \). Then \( f \) is affine and there is a group isomorphism
\[
\text{Aut}(X/Y) \cong \text{Gal}(k(X)/k(Y)).
\]

Lemma 3.3. (Quotient \([6]\)) Let \( X \) and \( Y \) be integral varieties such that \( X \) is quasi-galois closed over \( Y \) by a surjective morphism \( \phi \). Then there is a natural isomorphism
\[
O_Y \cong \phi_*(O_X)^{\text{Aut}(X/Y)}.
\]

Here \((O_X)^{\text{Aut}(X/Y)}(U)\) denotes the invariant subring of \( O_X(U) \) under the natural action of \( \text{Aut}(X/Y) \) for any open subset \( U \) of \( X \).

Lemma 3.4. (Geometric model \([2],[3]\)) Let \( X \) and \( Y \) be arithmetic varieties such that \( X \) is quasi-galois closed over \( Y \) by a surjective morphism \( f \) of finite type. Then
- \( f \) is affine;
- \( k(X) \) is canonically Galois over \( k(Y) \);
there is a group isomorphism
\[ \text{Aut} \left( \frac{X}{Y} \right) \cong \text{Gal} \left( k \left( \frac{X}{k(Y)} \right) \right). \]

In particular, let \( \dim X = \dim Y \). Then \( X \) is a pseudo-galois cover of \( Y \) in the sense of Suslin-Voevodsky (see [16, 17] for definition).

Lemma 3.5. (Criterion) Let \( X, Y \) be integral schemes and let \( f : \frac{X}{Y} \) be a surjective morphism. Suppose that the function field \( k(Y) \) is contained in \( \Omega \). The following statements are equivalent:

- The scheme \( X \) is quasi-galois closed over \( Y \) by \( f \).
- There is a unique maximal affine patching \( C_X \) of \( X \) with values in \( \Omega \) such that \( C_X \) is quasi-galois closed over \( Y \) by \( f \).

Proof. It is easily proved in a manner similar to [8]. \( \square \)

3.2. A universal cover for the Étale fundamental group. For convenience, let’s recall the universal cover for an étale fundamental group of an arithmetic variety.

Fixed an arithmetic variety \( X \). Let \( \Omega \) be an algebraic closure of the function field \( k(X) \). In the following we will construct an integral variety \( X_{\Omega_{\text{et}}} \) and a morphism \( p_X : X_{\Omega_{\text{et}}} \to X \) such that \( X_{\Omega_{\text{et}}} \) is quasi-galois closed over \( X \).

For brevity, put \( K = k(X) \) and \( L = K^{\text{un}} \subseteq \Omega \). By Lemma 3.1, without loss of generality, assume that \( X \) has a reduced affine covering \( C_X \) with values in \( \Omega \). We choose \( C_X \) to be maximal (in the sense of set inclusion).

We will proceed in several steps:

- Fixed a set \( \Delta \) of generators of the field \( L \) over \( K \). Put \( G = \text{Gal} \left( L/K \right) \).
- For any local chart \((V, \psi_V, B_V) \in C_X\), define \( A_V = B_V \left[ \Delta_V \right] \), i.e., the subring of \( L \) generated over \( B_V \) by the set \( \Delta_V = \{ \sigma(x) \in L : \sigma \in G, x \in \Delta \} \). Set \( i_V : B_V \to A_V \) to be the inclusion.
- Define
  \[ \Sigma = \bigcoprod_{(V, \psi_V, B_V) \in C_X} \text{Spec}(A_V) \]
  to be the disjoint union. Let \( \pi_X : \Sigma \to X \) be the projection induced by the inclusions \( i_V \).
- Define an equivalence relation \( R_\Sigma \) in \( \Sigma \) in such a manner:
  For any \( x_1, x_2 \in \Sigma \), we say \( x_1 \sim x_2 \) if and only if \( j_{x_1} = j_{x_2} \) holds in \( L \).
  Here \( j_x \) denotes the corresponding prime ideal of \( A_V \) to a point \( x \in \text{Spec}(A_V) \).
Let $X_{\Omega_{et}}$ be the quotient space $\Sigma/\sim$ and let $\pi_{\Omega_{et}} : \Sigma \to X_{\Omega_{et}}$ be the projection of spaces.

- Define a map $p_X : X_{\Omega_{et}} \to X$ of spaces by $\pi_{\Omega_{et}}(z) \mapsto \pi_X(z)$ for each $z \in \Sigma$.
- Define $\mathcal{C}_{X_{\Omega_{et}}} = \{(U_V, \varphi_V, A_V) \in \mathcal{C}_X \}$ where $U_V = \pi_X^{-1}(V)$ and $\varphi_V : U_V \to \text{Spec}(A_V)$ is the identity map for each $(V, \psi_V, B_V) \in \mathcal{C}_X$.

Hence, there is a scheme, namely $X_{\Omega_{et}}$, obtained by gluing the affine schemes $\text{Spec}(A_V)$ for all $(U_V, \varphi_V, A_V) \in \mathcal{C}_X$ with respect to the equivalence relation $R_\Sigma$ (see [10, 13]). Naturally, $p_X$ becomes a morphism of schemes.

This completes the construction.

It follows that we have the following lemma.

**Lemma 3.6. (Universal cover)** For an arithmetic variety $X$, there is an integral variety $X_{\Omega_{et}}$ and a surjective morphism $p_X : X_{\Omega_{et}} \to X$ satisfying the conditions:

- $k(X_{\Omega_{et}}) = k(X)^{un}$;
- $p_X$ is affine;
- $k(X_{\Omega_{et}})$ is Galois over $k(X)$;
- $X_{\Omega_{et}}$ is quasi-galois closed over $X$ by $p_X$.

Such an integral variety $X_{\Omega_{et}}$ is called a **universal cover** over $X$ for the étale fundamental group $\pi_1^{et}(X)$, denoted by $(X_{\Omega_{et}}, p_X)$.

**Proof.** Let $K$ be the function field $k(X)$. Take any $\omega$ in the field $K^{un}$. By **Definition 2.4** and **Lemma 3.4** it is easily seen that every conjugate of $\omega$ over $K$ is also contained in $K^{un}$. This proves that $K^{un}$ is a Galois extension of $K$.

It is seen that $\mathcal{C}_{X_{\Omega_{et}}}$ is the unique maximal affine patching of the scheme $X_{\Omega_{et}}$. From **Lemma 3.5** it is seen that $X_{\Omega_{et}}$ is quasi-galois closed over $X$. Then it is immediate from **Lemma 3.2**. \qed

### 4. Monodromy Action, I

We have the following computations of the étale fundamental group of an arithmetic variety.

**Lemma 4.1. ([5])** Fixed any arithmetic variety $X$. Then there exists an isomorphism

$$\pi_1^{et}(X) \cong \text{Gal}(k(X)^{un}/k(X)).$$
Lemma 4.2. For any arithmetic variety $X$, there is an isomorphism
\[ \text{Aut} \left( X_{\Omega_{et}}/X \right) \cong \pi_1^{et} (X) \]
where $(X_{\Omega_{et}}, p_X)$ is a universal cover for the group $\pi_1^{et}(X)$.

Proof. As $k(X_{\Omega_{et}}) = k(X)^{un}$ is a Galois extension over $k(X)$, by Lemma 3.2 we have $\text{Aut} \left( X_{\Omega_{et}}/X \right) \cong \text{Gal} \left( k(X_{\Omega_{et}})/k(X) \right)$. Then it is immediate from Lemma 4.1 above. $\Box$

Now let $X$ and $Y$ be two arithmetic varieties such that $k(Y)$ is contained in an algebraic closure $\Omega$ of the function field $k(X)$.

Fixed a group homomorphism
\[ \sigma : \pi_1^{et} (X) \rightarrow \pi_1^{et} (Y). \]

By Lemma 4.2 we have a group homomorphism, namely
\[ \sigma : \text{Aut} \left( X_{\Omega_{et}}/X \right) \rightarrow \text{Aut} \left( Y_{\Omega_{et}}/Y \right). \]

Lemma 4.3. (Monodromy action) Assume that $(X_{\Omega_{et}}, p_X)$ and $(Y_{\Omega_{et}}, p_Y)$ are the universal covers for the groups $\pi_1^{et}(X)$ and $\pi_1^{et}(Y)$, respectively.

Fixed a group homomorphism
\[ \sigma : \text{Aut} \left( X_{\Omega_{et}}/X \right) \rightarrow \text{Aut} \left( Y_{\Omega_{et}}/Y \right). \]

Then there is a bijection
\[ \tau : \text{Hom} (X, Y) \rightarrow \text{Hom} (X_{\Omega_{et}}, Y_{\Omega_{et}}), f \mapsto f_{et} \]
between sets given in a canonical manner:

- Let $f \in \text{Hom} (X, Y)$. Then the map
  \[ g (x_0) \mapsto \sigma (g) (h (x_0)) \]
defines a morphism
\[ f_{et} : X_{\Omega_{et}} \rightarrow Y_{\Omega_{et}} \]
for any $x_0 \in X$ and any $g \in \text{Aut} \left( X_{\Omega_{et}}/X \right)$.

- Let $f_{et} \in \text{Hom} (X_{\Omega_{et}}, Y_{\Omega_{et}})$. Then the map
  \[ p_X (x) \mapsto p_Y (f_{et} (x)) \]
defines a morphism
\[ f : X \rightarrow Y \]
for any $x \in X_{\Omega_{et}}$.

In particular, we have
\[ f \circ p_X = p_Y \circ f_{et}. \]

Proof. It is immediate from Lemmas 3.2-4.2.
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5. sp-Completion

In this section we will use Weil’s theory of specializations (see [1] for detail) to give the completion of rational maps between schemes.

5.1. Definition for specializations. Let $E$ be a topological space $E$ and $x, y \in E$. If $y$ is in the closure $\{x\}$, $y$ is said to be a specialization of $x$ (or, $x$ is said to be a generalization of $y$) in $E$, denoted by $x \rightarrow y$. Put $Sp(x) = \{y \in E \mid x \rightarrow y\}$. It is evident that $Sp(x) = \{x\}$ is an irreducible closed subset in $E$.

If $x \rightarrow y$ and $y \rightarrow x$ both hold in $E$, $y$ is said to be a generic specialization of $x$ in $E$, denoted by $x \leftrightarrow y$. The point $x$ is said to be generic (or initial) in $E$ if we have $x \leftrightarrow z$ for any $z \in E$ such that $z \rightarrow x$. And $x$ is said to be closed (or final) if we have $x \leftrightarrow z$ for any $z \in E$ such that $x \rightarrow z$. We say that $y$ is a closest specialization of $x$ in $X$ if either $z = x$ or $z = y$ holds for any $z \in X$ such that $x \rightarrow z$ and $z \rightarrow y$.

5.2. Any specialization is contained in an affine open set. Let $E = \text{Spec}(A)$ be an affine scheme. For any point $z \in \text{Spec}(A)$, denote by $j_z$ the corresponding prime ideal in $A$. Then we have a specialization $x \rightarrow y$ in $\text{Spec}(A)$ if and only if $j_x \subseteq j_y$ holds in $A$. Hence, there is a generic specialization $x \leftrightarrow y$ in $\text{Spec}(A)$ if and only if $x = y$ holds.

Now consider a scheme $X$.

Lemma 5.1. ([1]) For any points $x, y \in X$, we have $x \leftrightarrow y$ in $X$ if and only if $x = y$.

Proof. $\Leftarrow$. Trivial. Prove $\Rightarrow$. Assume $x \leftrightarrow y$ in $X$. Let $U$ be an affine open set of $X$ containing $x$. From $x \leftrightarrow y$ in $X$, we have $Sp(x) = Sp(y)$; then $x \in Sp(x) \cap U = Sp(y) \cap U \ni y$; hence, $x \leftrightarrow y$ in $U$. It follows that $x = y$ holds in $U$ (and of course in $X$).

Lemma 5.2. ([1]) Fixed any specialization $x \rightarrow y$ in $X$. Then there is an affine open subset $U$ of $X$ such that the two points $x$ and $y$ are both contained in $U$. In particular, any affine open set in $X$ containing the specialization $y$ must contain the generalization $x$.

Proof. Assume $x \neq y$. Then $y$ is a limit point of the one-point set $\{x\}$ since $y$ is contained in the topological closure $Sp(x)$ of $\{x\}$. Let $U \subseteq X$ be an open set containing $y$. We have $U \cap (\{x\} \setminus \{y\}) \neq \emptyset$ by the definition for a limit point of a set (see any standard textbook for general topology). We choose $U$ to be an affine open set of $X$. $\square$
5.3. Any morphism preserves specializations. Let $f : E \to F$ be a map of spaces. The map $f$ is said to be specialization-preserving if there is a specialization $f(x) \to f(y)$ in $F$ for any specialization $x \to y$ in $E$.

**Lemma 5.3.** ([1]) Any morphism between schemes is specialization-preserving.

*Proof.* It is immediate from *Lemma 5.2.*

5.4. The graph functor $\Gamma$ from schemes to graphs. We have such a covariant functor from the category of schemes to the category of (combinatorial) graphs. See [18] for preliminaries on graph theory.

**Lemma 5.4.** (Graph functor [1]) There exists a covariant functor $\Gamma$ from the category $\text{Sch}$ of schemes to the category $\text{Grph}$ of graphs given in such a natural manner.

- To any scheme $X$, assign the graph $\Gamma(X)$ in which the vertex set is the set of points in the underlying space $X$ and the edge set is the set of specializations in $X$.
  
  Here, for any points $x, y \in X$, we say that there is an edge from $x$ to $y$ if and only if there is a specialization $x \to y$ in $X$.

- To any scheme morphism $f : X \to Y$, assign the graph homomorphism $\Gamma(f) : \Gamma(X) \to \Gamma(Y)$.
  
  Here, any specialization $x \to y$ in the scheme $X$ as an edge in $\Gamma(X)$, is mapped by $\Gamma(f)$ into the specialization $f(x) \to f(y)$ as an edge in $\Gamma(Y)$.

*Proof.* It is immediate from *Lemmas 5.2-3.*

The above functor $\Gamma : \text{Sch} \to \text{Grph}$ is said to be the graph functor.

**Remark 5.5.** There are many beautiful graphs $\Gamma(X)$ associated with schemes $X$. For example, it is easily seen that

- $\Gamma(\text{Spec}(\mathbb{Z}))$ is a star-shaped graph;
- $\Gamma(\text{Spec}(\mathbb{Z}[t]))$ is a graph of infinitely many loops.

**Remark 5.6.** By the graph functor $\Gamma$, many invariants that are defined on graphs can be introduced into schemes in a natural manner, for example, the discrete Morse theory, the Kontsevich’s graph homology theory, etc.

5.5. $sp$-completion. In virtue of the graph functor $\Gamma$, we can give the completion of a rational maps between schemes, which will be applied to the proofs of the main theorems of the paper.
Let’s recall basic definitions for graphs (see [18]). Fixed a graph $X$. Let $V(X)$ be the set of vertices in $X$ and $E(X)$ the set of edges in $X$.

Let $Y$ be a graph. Then $Y$ is said to be a subgraph of $X$ if the following conditions are satisfied:

- $V(X) \supseteq V(Y)$;
- $E(X) \supseteq E(Y)$;
- Every $L \in E(Y)$ has the same ends in $Y$ as in $X$.

Recall that an isomorphism $t$ from $X$ onto $Y$ is a ordered pair $(t_V, t_E)$ satisfying the conditions:

- $t_V$ is a bijection from $V(X)$ onto $V(Y)$;
- $t_E$ is a bijection from $E(X)$ onto $E(Y)$;
- Let $x \in V(X)$ and $L \in E(X)$. Then $x$ is incident with $L$ if and only if $t_V(x) \in V(Y)$ is incident with $t_E(L) \in E(Y)$.

Now we consider the graphs of integral schemes.

**Definition 5.7.** An integral scheme $X$ is said to be sp-complete if $X$ and $Y$ must be essentially equal for any integral scheme $Y$ such that

- $\Gamma(X)$ is isomorphic to a subgraph of $\Gamma(Y)$;
- $k(Y)$ is contained in an algebraic closure of $k(X)$.

**Remark 5.8.** Let $X$ be an sp-complete integral variety. It is easily seen that the function field $k(X)$ must be algebraically closed. In such a case, the graph $\Gamma(X)$ is maximal (by set-inclusion).

For example, let $\mathcal{O}$ be the set of all algebraic numbers over $\mathbb{Q}$. Then $\text{Spec}(\mathbb{Z}[\mathcal{O}])$ is sp-complete.

For the sp-complete, we have the following theorem.

**Theorem 5.9.** (sp-completion) For any integral variety $X$, there exists an integral variety $X_{sp}$ and a surjective morphism $\lambda_X : X_{sp} \rightarrow X$ such that

- $\lambda_X$ is affine;
- $X_{sp}$ is sp-complete;
- $k(X_{sp})$ is an algebraic closure of $k(X)$;
- $X_{sp}$ is quasi-galois closed over $X$ by $\lambda_X$.

Such an integral scheme $X_{sp}$, is said to be an sp-completion of $X$. We will denote this by $(X_{sp}, \lambda_X)$.

**Proof.** Let $K = k(X)$ and $L = K^{al}$. Fixed a set $\Delta$ of generators of the field $L$ over $K$. Put $G = \text{Gal}(L/K)$. By Lemma 3.1, without loss of generality, assume that $X$ has a reduced affine covering $\mathcal{C}_X$ with values in $\Omega$. We choose $\mathcal{C}_X$ to be maximal (in the sense of set inclusion).
We will proceed in several steps to give the construction:

- For any local chart \((V, \psi_V, B_V) \in C_X\), define \(A_V = B_V [\Delta_V]\), where \(\Delta_V = \{\sigma(x) \in L : \sigma \in G, x \in \Delta\}\). Set \(i_V : B_V \to A_V\) to be the inclusion.
- Define
  \[
  \Sigma = \bigsqcup_{(V, \psi_V, B_V) \in C_X} \Spec(A_V)
  \]
  to be the disjoint union. Let \(\pi_X : \Sigma \to X\) be the projection induced by the inclusions \(i_V\).
- Define an equivalence relation \(R_{\Sigma}\) in \(\Sigma\) in such a manner:
  For any \(x_1, x_2 \in \Sigma\), we say \(x_1 \sim x_2\) if and only if \(j_{x_1} = j_{x_2}\) holds in \(L\).
  Here \(j_x\) denotes the corresponding prime ideal of \(A_V\) to a point \(x \in \Spec(A_V)\).
- Define \(X_{sp}\) be the quotient space \(\Sigma / \sim\) and let \(\pi_{sp} : \Sigma \to X_{sp}\) be the projection of spaces.
- Define a map \(\lambda_X : X_{sp} \to X\) of spaces by \(\pi_{sp}(z) \mapsto \pi_X(z)\) for each \(z \in \Sigma\).
- Define
  \[
  C_{X_{sp}} = \{(U_V, \varphi_V, A_V) \mid (V, \psi_V, B_V) \in C_X\}
  \]
  where \(U_V = \pi_X^{-1}(V)\) and \(\varphi_V : U_V \to \Spec(A_V)\) is the identity map for each \((V, \psi_V, B_V) \in C_X\).
  Hence, we obtain a scheme, namely \(X_{sp}\), by gluing the affine schemes \(\Spec(A_V)\) for all \((U_V, \varphi_V, A_V) \in C_X\) with respect to the equivalence relation \(R_{\Sigma}\). Naturally, \(\lambda_X\) becomes a morphism of schemes.

Evidently, it needs only to verify that \(X_{sp}\) is \(sp\)–complete.

In deed, take any integral scheme \(Y\) such that

- \(\Gamma(X)\) is isomorphic to a subgraph of \(\Gamma(Y)\);
- \(k(Y)\) is contained in an algebraic closure of \(k(X)\).

Hypothesize that there is some \(x_0 \in Y \setminus X_{sp}\). Let \(z_0 \in Y\) be final. By Lemma 5.2 it is seen that \(z_0\) is not contained in \(X_{sp}\). Evidently, there is an affine open set \(W = \Spec(A_0)\) in \(Y\) such that \(z_0 \in W\).

On the other hand, there must be an affine open set \(W_{sp} = \Spec(B_0)\) in \(X_{sp}\) such that the ring \(A_0\) is contained in \(B_0\) from the construction above. It follows that \(z_0\) must be contained in \(W_{sp}\) and hence in \(X_{sp}\), which will be in contradiction. \(\Box\)

The \(sp\)–completions have the following property.
Lemma 5.10. (Uniqueness up to isomorphisms) Let $X$ and $Y$ be integral varieties such that $k(X) = k(Y)$. Then the $sp$–completions $X_{sp}$ and $Y_{sp}$ are essentially equal. In particular, $X_{sp}$ and $Y_{sp}$ are isomorphic schemes.

Proof. The essential equality is from Definition 5.7 and Theorem 5.9. The isomorphism can be proved in a manner similar to the proof of Lemma 2.15 in [5].

6. Monodromy Action, II

Let $X$ and $Y$ be two integral varieties. Put

$$G_X = \text{Aut}(X_{sp}/X);$$

$$G_Y = \text{Aut}(Y_{sp}/Y).$$

Here $(X_{sp}, \lambda_X)$ and $(Y_{sp}, \lambda_Y)$ are $sp$–completions of $X$ and $Y$, respectively.

Now we can give the monodromy actions on $sp$–completions.

Lemma 6.1. (Monodromy action) Suppose that there is a group homomorphism $\sigma : G_X \to G_Y$. Then there is a bijection

$$\tau : \text{Hom}(X, Y) \to \text{Hom}(X_{sp}, Y_{sp}), f \mapsto f_{sp}$$

between sets given in a canonical manner:

- Let $f \in \text{Hom}(X, Y)$. Then the map
  $$g(x_0) \mapsto \sigma(g)(h(x_0))$$
  defines a morphism
  $$f_{sp} : X_{sp} \to Y_{sp}$$
  for any $x_0 \in X$ and any $g \in G_X$.
- Let $f_{sp} \in \text{Hom}(X_{sp}, Y_{sp})$. Then the map
  $$\lambda_X(x) \mapsto \lambda_Y(f_{sp}(x))$$
  defines a morphism
  $$f : X \to Y$$
  for any $x \in X_{sp}$.

In particular, we have

$$f \circ \lambda_X = \lambda_Y \circ f_{sp}.$$ 

Proof. It is immediate from Theorem 5.9 and Lemma 3.3.
Lemma 6.2. \textit{(sp—completion of rational maps, I)} Let \(k(X) = k(Y)\). Then there is a bijection \(\tau\) from \(\text{Hom}(X,Y)\) onto \(\text{Hom}(X_{sp},Y_{sp})\) given in a canonical manner.

In particular, \(\text{Hom}(X,Y)\) must be a non-void set.

\textit{Proof.} By Theorem 5.9 we have
\[
\text{Aut}(X_{sp}/X) \cong \text{Gal}(k(X)^{\text{al}}/k(X)) \cong \text{Aut}(Y_{sp}/Y).
\]
It follows that there is a group isomorphism \(\sigma : G_X \cong G_Y\). Then it is immediate from Lemma 6.1. \hfill \square

Lemma 6.3. \textit{(sp—completion of rational maps, II)} Suppose \(k(X) \supsetneq k(Y)\). Then there is a bijection \(\tau\) from \(\text{Hom}(X,Y)\) onto \(\text{Hom}(X_{sp},Y_{sp})\) given in a canonical manner.

In particular, \(\text{Hom}(X,Y)\) must be a non-void set and there is a homomorphism \(\sigma : G_X \rightarrow G_Y\).

\textit{Proof.} By Theorem 5.9 we have
\[
\text{Aut}(X_{sp}/X) \cong \text{Gal}(k(X)^{\text{al}}/k(X));
\]
\[
\text{Aut}(Y_{sp}/Y) \cong \text{Gal}(k(Y)^{\text{al}}/k(Y)).
\]
As \(k(X) \supsetneq k(Y)\), we have \(k(X)^{\text{al}} \supsetneq k(Y)^{\text{al}}\). In particular, \(k(X)^{\text{al}}\) is a Galois extension over \(k(Y)^{\text{al}}\). There is a homomorphism from \(\text{Gal}(k(X)^{\text{al}}/k(X))\) onto \(\text{Gal}(k(Y)^{\text{al}}/k(Y))\), which is the composite of the maps
\[
\text{Gal}(k(X)^{\text{al}}/k(X)) \rightarrow \text{Gal}(k(X)^{\text{al}}/k(Y))
\]
and
\[
\frac{\text{Gal}(k(X)^{\text{al}}/k(Y))}{\text{Gal}(k(X)^{\text{al}}/k(Y)^{\text{al}})} \cong \text{Gal}(k(Y)^{\text{al}}/k(Y)).
\]
It follows that there is a homomorphism \(\sigma : G_X \rightarrow G_Y\). Then it is immediate from Lemma 6.1. \hfill \square

Remark 6.4. The \(sp\)—completions of rational maps between integral schemes, as stated above, can be regarded as a generalization of the correspondences between dominant rational maps of algebraic varieties and homomorphisms of algebras in the classical algebraic geometry.

7. \textit{qc Fundamental Groups}

To prove the main theorems of the paper, we also need some results on the \textit{qc} fundamental group of an arithmetic scheme. See [6] for details.
7.1. Definition for qc fundamental groups. Let $X$ be an arithmetic variety. Fixed an algebraically closed field $\Omega$ that contains $k(X)$. Here, $\Omega$ is not necessarily algebraic over $k(X)$.

Define $X_{qc}[\Omega]$ to be the set of arithmetic varieties $Z$ satisfying the two conditions:

- $Z$ has a reduced affine covering with values in $\Omega$;
- There is a surjective morphism $f : Z \to X$ of finite type such that $Z$ is quasi-galois closed over $X$.

Naturally there is a partial order $\leq$ in the set $X_{qc}[\Omega]$ given in such a manner:

- For any $Z_1, Z_2 \in X_{qc}[\Omega]$, we say $Z_1 \leq Z_2$ if there is a surjective morphism $\varphi : Z_2 \to Z_1$ of finite type such that $Z_2$ is quasi-galois closed over $Z_1$.

By Lemmas 3.6,3.8-10 in [6], it is seen that $X_{qc}[\Omega]$ is a directed set and

$\{\text{Aut}(Z/X) : Z \in X_{qc}[\Omega]\}$

is an inverse system of groups.

The inverse limit

$$\pi_1^{qc}(X; \Omega) \triangleq \lim_{\longleftarrow \ Z \in X_{qc}[\Omega]} \text{Aut}(Z/X)$$

of the inverse system $\{\text{Aut}(Z/X) : Z \in X_{qc}[\Omega]\}$ of groups is said to be the qc fundamental group of the scheme $X$ with coefficient in $\Omega$.

7.2. Main result for qc fundamental groups. There are the following result for qc fundamental groups.

Lemma 7.1. (6) Let $X$ be an arithmetic variety. Suppose that $\Omega$ is an algebraically closed field containing $k(X)$. There are the following statements.

- There is a group isomorphism

$$\pi_1^{qc}(X; \Omega) \cong \text{Gal}(\Omega/k(X)).$$

- Take any geometric point $s$ of $X$ over $\Omega$. Then there is a group isomorphism

$$\pi_1^{et}(X; s) \cong \pi_1^{qc}(X; \Omega)_{et}$$

where $\pi_1^{qc}(X; \Omega)_{et}$ is a subgroup of $\pi_1^{qc}(X; \Omega)$. In particular, $\pi_1^{qc}(X; \Omega)_{et}$ is a normal subgroup of $\pi_1^{qc}(X; \Omega)$.
8. Proofs of the Main Theorems

8.1. Preparatory lemmas. Let’s first prove the below result on the surjection of the sets that are considered.

Let $X$ and $Y$ be arithmetic varieties.

**Lemma 8.1. (sp–completion of rational maps)** Assume $k(X) \supseteq k(Y)$. Then $\text{Hom}(X, Y)$ must be a non-void set.

**Proof.** Using the $sp$-completions $(X_{sp}, \lambda_X)$ and $(Y_{sp}, \lambda_Y)$ of $X$ and $Y$, respectively. Then $k(X_{sp})$ (resp. $k(Y_{sp})$)is the algebraic closure of $k(X)$ (resp. $k(Y)$).

As $k(X) \supseteq k(Y)$, we have $\overline{k(X)} \supseteq \overline{k(Y)}$. Then it is seen that the ring $B$ of an affine open set $V$ in $Y_{sp}$ must be embedded into the ring $A$ of some certain affine open set $U$ in $X_{sp}$; conversely, each $A$ must contain some $B$. It follows that there is a homomorphism $f_U : U = \text{Spec}(A) \to V = \text{Spec}(B)$ defined by the inclusion. This gives us a scheme homomorphism $f_{sp} : X_{sp} \to Y_{sp}$.

By the projections $\lambda_X : X_{sp} \to X$ and $\lambda_Y : Y_{sp} \to Y$ we have a unique homomorphism $f : X \to Y$ satisfying the condition $\lambda_{sp} \circ f_{sp} = f \circ \lambda_{sp}$.

This completes the proof. \qed

**Lemma 8.2.** Suppose $k(Y) \subseteq k(X)$. Then each element of the set $\text{Hom}(\pi_1^{et}(X), \pi_1^{et}(Y))$ and of the set $\text{Hom}(\pi_1^{et}(k(X)), \pi_1^{et}(k(Y)))$ gives an element of the set $\text{Hom}(X, Y)$ in a canonical manner, respectively.

**Proof.** Let $\delta$ be a homomorphism from $\pi_1^{et}(X)$ into $\pi_1^{et}(Y)$.

As $k(X) \supseteq k(Y)$, by Lemmas 6.2-3 it is seen that there is a group homomorphism $\sigma$ from $G_X = \text{Aut}(X_{sp}/X)$ into $G_Y = \text{Aut}(Y_{sp}/Y)$.

From Lemma 7.1 it is seen that $k(X)^{un}/k(X)$ and $k(Y)^{un}/k(Y)$ are both Galois extensions.

Then we have

$$\pi_1^{et}(X) \cong \frac{\text{Gal}(k(X)^{al}/k(X))}{\text{Gal}(k(X)^{al}/k(X)^{un})};$$

$$\pi_1^{et}(Y) \cong \frac{\text{Gal}(k(Y)^{al}/k(Y))}{\text{Gal}(k(Y)^{al}/k(Y)^{un})}.$$

It is easily seen that the homomorphisms $\delta$ and $\sigma$ are compatible in a canonical manner. From Lemmas 6.2-3 it is seen that for the homomorphism $\delta$ there is a corresponding morphism $f : X \to Y$ which is given in a canonical manner. \qed
Lemma 8.3. Every morphism \( f : X \to Y \) arise from a morphism \( f_{qc} : X_{qc} \to Y_{qc} \) of integral schemes given in such a manner:

\[ f \circ \phi_X = \phi_Y \circ f_{qc}. \]

In particular, for the function fields, we have

\[ k(X) \subseteq k(X_{qc}); k(Y) \subseteq k(Y_{qc}). \]

Here, \( X_{qc} \) is quasi-galois closed over \( X \) by a surjective morphism \( \phi_X \); \( Y_{qc} \) is quasi-galois closed over \( Y \) by a surjective morphism \( \phi_Y \) (see §3).

Such an integral scheme \( X_{qc} \) is said to be a quasi-galois closed cover of \( X \), denoted by \( (X_{qc}, \phi_X) \).

Proof. Just repeat the procedure for a universal cover in [4] or as in the previous section §3.2 of the present paper.

Lemma 8.4. The \( sp \)-completions \( X_{sp} \) and \( Y_{sp} \) are large enough for the morphisms from \( X \) into \( Y \). That is, there is a surjection from \( \text{Hom}(X_{sp}, Y_{sp}) \) onto \( \text{Hom}(X, Y) \).

Proof. Let \( f \in \text{Hom}(X, Y) \). Hypothesize that \( f \) does not arise from any morphism \( f_{sp} : X_{sp} \to Y_{sp} \) in a canonical manner.

Suppose that \( f \) arises from a morphism

\[ h_{qc} : W_{qc} \to Z_{qc} \]

between quasi-galois closed covers, where \( h_{qc} \) is given in a canonical manner by Lemma 8.3.

Consider the graphs of schemes (see §5). It is seen that \( \Gamma(X_{sp}) \) must be contained in \( \Gamma(W_{qc}) \).

There are two cases.

Case (i): Assume \( \dim X_{sp} = \dim W_{qc} \).

Assume \( W_{qc} = (W_{qc})_{sp} \) without loss of generality.

The function fields \( k(X_{sp}) \) and \( k(W_{qc}) \) are two algebraic closure of the field \( k(X) \) and hence are isomorphic over \( k(X) \).

It is seen that \( W_{qc} \) and \( X_{sp} \) are isomorphic schemes over \( X \) by the construction for \( sp \)-completion. Hence, \( f \) arises from \( h : X_{sp} \to Y_{sp} \), where there will be in contradiction.

Case (ii): Suppose \( \dim X_{sp} < \dim W_{qc} \).

First consider the commutative diagrams which are all given in a canonical manner:

\[ h_{qc} \circ \lambda_{W_{qc}} = \lambda_{Z_{qc}} \circ (h_{qc})_{sp} : (W_{qc})_{sp} \to Z_{qc}; \]

\[ f \circ \phi_X = \phi_Y \circ h_{qc} : W_{qc} \to Y. \]
Then consider the commutative diagrams which are all given in a canonical manner:

\[ h \circ \lambda_X = \phi_Y \circ (h_{qc})_{sp} : (W_{qc})_{sp} \to Y_{sp}; \]
\[ f \circ \lambda_X = \lambda_Y \circ h : X_{sp} \to Y, \]
where \( h : X_{sp} \to Y_{sp} \) is uniquely defined in a canonical manner.

It follows that \( f \) arises from \( h \), where there will be in contradiction.

This completes the proof. \( \square \)

8.2. **Proofs of the main theorems.** Now we can give the proofs of the main theorems in the paper.

**Proof.** (Proof of Theorem 1.1) By Lemmas 8.1-4 it is seen that there is a surjection

\[ t : \text{Hom}(\pi_1^{et}(k(X)), \pi_1^{et}(k(Y))) \to \text{Hom}(X, Y). \]

Let \( \pi \) be the projection from the set

\[ \text{Hom}(\pi_1^{et}(k(X)), \pi_1^{et}(k(Y))) \]

onto the sets

\[ \text{Hom}^{out}_{\pi_1^{et}(k(X)), \pi_1^{et}(k(Y))} \left( \pi_1^{et}(X), \pi_1^{et}(Y) \right) \]

given by \( f \mapsto [f] \).

From the maps \( t \) and \( \pi \), we have a map

\[ \xi : \text{Hom}^{out}_{\pi_1^{et}(k(X)), \pi_1^{et}(k(Y))} \left( \pi_1^{et}(X), \pi_1^{et}(Y) \right) \to \text{Hom}(X, Y) \]

given by

\[ [f] \mapsto t(f). \]

It is clear that \( \xi \) is a surjection of sets.

In the following we prove that \( \xi \) is an injection.

In fact, according to the properties of quasi-galois closed schemes (see Lemmas 3.2-4), we have

\[ \text{Gal}(k(X_{sp})/k(X)) \cong \text{Aut}(X_{sp}/X); \]
\[ \text{Gal}(k(Y_{sp})/k(Y)) \cong \text{Aut}(Y_{sp}/Y); \]
\[ \pi_1^{et}(X) \cong \text{Gal}(k(X_{\Omega_{et}})/k(X)) \cong \text{Aut}(X_{\Omega_{et}}/X); \]
\[ \pi_1^{et}(Y) \cong \text{Gal}(k(Y_{\Omega_{et}})/k(Y)) \cong \text{Aut}(Y_{\Omega_{et}}/Y). \]

By Lemma 4.3 and Lemma 6.1, it is seen that there are the following monodromy actions:

For the scheme \( X \), we have

- the monodromy action of \( \text{Aut}(X_{\Omega_{et}}/X) \) on the universal cover \( X_{\Omega_{et}} \);
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• the monodromy action of $\text{Aut}(X_{sp}/X)$ on the $sp$-completion $X_{sp}$.

For the scheme $Y$, we have

• the monodromy action of $\text{Aut}(Y_{\Omega_{et}}/Y)$ on the universal cover $Y_{\Omega_{et}}$;

• the monodromy action of $\text{Aut}(Y_{sp}/Y)$ on the $sp$-completion $Y_{sp}$.

It is seen that there is a bijection

$$\text{Hom}(\frac{\text{Aut}(X_{sp}/X)}{\text{Aut}(X_{\Omega_{et}}/X)}, \frac{\text{Aut}(Y_{sp}/Y)}{\text{Aut}(Y_{\Omega_{et}}/Y)}) \rightarrow \text{Hom}(X, Y)$$

between sets.

In deed, take any $f \in \text{Hom}(X, Y)$. There is an $f_{sp} \in \text{Hom}(X_{sp}, Y_{sp})$ which produces $f$ in a canonical manner. Then $f_{sp}$ produces canonically an $f_{et} \in \text{Hom}(X_{\Omega_{et}}, Y_{\Omega_{et}})$. It follows that all elements of $\text{Hom}(X, Y)$ arise from the elements of $f_{et} \in \text{Hom}(X_{\Omega_{et}}, Y_{\Omega_{et}})$.

On the other hand, different elements of the set

$$\text{Hom}(\frac{\text{Aut}(X_{sp}/X)}{\text{Aut}(X_{\Omega_{et}}/X)}, \frac{\text{Aut}(Y_{sp}/Y)}{\text{Aut}(Y_{\Omega_{et}}/Y)})$$

produce different elements of the set

$$\text{Hom}(X_{\Omega_{et}}, Y_{\Omega_{et}})$$

and then different elements of

$$\text{Hom}(X, Y)$$

in a canonical manner, respectively, by the monodromy actions.

Hence, $\xi$ is a bijection. This completes the proof. \hfill \Box

**Proof. (Proof of Theorem 1.2)** It is immediate from Theorem 1.1. \hfill \Box

**Proof. (Proofs of Theorem 1.3-4)** It is immediate from the following fact that

$$G(k(X))^{un} \cong \pi_{1}^{et}(X)$$

holds for any arithmetic variety $X$ (see Lemma 4.1). \hfill \Box

**Remark 8.5.** In the above we indeed have proved that there is a bijection

$$\text{Hom}(X, Y) \cong \text{Hom}(\frac{\text{Aut}(X_{sp}/X)}{\text{Aut}(X_{\Omega_{et}}/X)}, \frac{\text{Aut}(Y_{sp}/Y)}{\text{Aut}(Y_{\Omega_{et}}/Y)})$$

between sets. This is the key point of the section conjecture.
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