MULTIGRADED FUJITA APPROXIMATION

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Abstract. The original Fujita approximation theorem states that the volume of a big divisor $D$ on a projective variety $X$ can always be approximated arbitrarily closely by the self-intersection number of an ample divisor on a birational modification of $X$. One can also formulate it in terms of graded linear series as follows: let $W_\bullet = \{W_k\}$ be the complete graded linear series associated to a big divisor $D$:

$$W_k = H^0(X, \mathcal{O}_X(kD)).$$

For each fixed positive integer $p$, define $W_\bullet(p)$ to be the graded linear subseries of $W_\bullet$ generated by $W_p$:

$$W_m^{(p)} = \begin{cases} 0, & \text{if } p \nmid m; \\ \text{Image}(S^kW_p \to W_{kp}), & \text{if } m = kp. \end{cases}$$

Then the volume of $W_\bullet(p)$ approaches the volume of $W_\bullet$ as $p \to \infty$. We will show that, under this formulation, the Fujita approximation theorem can be generalized to the case of multigraded linear series.

1. Introduction

Let $X$ be an irreducible variety of dimension $d$ over an algebraically closed field $K$, and let $D$ be a (Cartier) divisor on $X$. When $X$ is projective, the following limit, which measures how fast the dimension of the section space $H^0(X, \mathcal{O}_X(mD))$ grows, is called the volume of $D$:

$$\text{vol}(D) = \text{vol}_X(D) = \lim_{m \to \infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^d/d!}.$$  

One says that $D$ is big if $\text{vol}(D) > 0$. It turns out that the volume is an interesting numerical invariant of a big divisor ([Laz04 §2.2.C]), and it plays a key role in several recent works in birational geometry ([BDPP04], [Tsu00], [HM06], [Tak06]).

When $D$ is ample, one can show that $\text{vol}(D) = D^d$, the self-intersection number of $D$. This is no longer true for a general big divisor $D$, since $D^d$ may even be negative. However, it was shown by Fujita [Fuj94] that the volume of a big divisor can always be approximated arbitrarily closely by the self-intersection number of an ample divisor on a birational modification of $X$. This theorem, known as Fujita approximation, has

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several implications on the properties of volumes, and is also a crucial ingredient in [BDPP04] (see [Laz04, §11.4] for more details).

In their recent paper [LM08], Lazarsfeld and Mustaţă obtained, among other things, a generalization of Fujita approximation to graded linear series. Recall that a graded linear series \( W_\bullet = \{ W_k \} \) on a (not necessarily projective) variety \( X \) associated to a divisor \( D \) consists of finite dimensional vector subspaces

\[
W_k \subseteq H^0(X, \mathcal{O}_X(kD))
\]

for each \( k \geq 0 \), with \( W_0 = K \), such that

\[
W_k \cdot W_\ell \subseteq W_{k+\ell}
\]

for all \( k, \ell \geq 0 \). Here the product on the left denotes the image of \( W_k \otimes W_\ell \) under the multiplication map \( H^0(X, \mathcal{O}_X(kD)) \otimes H^0(X, \mathcal{O}_X(\ell D)) \to H^0(X, \mathcal{O}_X((k+\ell)D)) \).

In order to state the Fujita approximation for \( W_\bullet \), they defined, for each fixed positive integer \( p \), a graded linear series \( W^{(p)}_\bullet \) which is the sub graded linear series of \( W_\bullet \) generated by \( W_p \):

\[
W^{(p)}_m = \begin{cases} 
0, & \text{if } p \nmid m; \\
\text{Im} \left( S^k W_p \to W_{kp} \right), & \text{if } m = kp.
\end{cases}
\]

Then under mild hypotheses, they showed that the volume of \( W^{(p)}_\bullet \) approaches the volume of \( W_\bullet \) as \( p \to \infty \). See [LM08, Theorem 3.5] for the precise statement, as well as [LM08, Remark 3.4] for how this is equivalent to the original statement of Fujita when \( X \) is projective and \( W_\bullet \) is the complete graded linear series associated to a big divisor \( D \) (i.e. \( W_k = H^0(X, \mathcal{O}_X(kD)) \) for all \( k \geq 0 \)).

The goal of this note is to generalize the Fujita approximation theorem to multigraded linear series. We will adopt the following notations from [LM08, §4.3]: let \( D_1, \ldots, D_r \) be divisors on \( X \). For \( \vec{m} = (m_1, \ldots, m_r) \in \mathbb{N}^r \) we write \( \vec{m}D = \sum m_i D_i \), and we put \( |\vec{m}| = \sum |m_i| \).

**Definition 1.** A multigraded linear series \( W_\bullet \) on \( X \) associated to the \( D_i \)'s consists of finite-dimensional vector subspaces

\[
W_{\vec{k}} \subseteq H^0(X, \mathcal{O}_X(\vec{k}D))
\]

for each \( \vec{k} \in \mathbb{N}^r \), with \( W_0 = K \), such that

\[
W_{\vec{k}} \cdot W_{\vec{m}} \subseteq W_{\vec{k}+\vec{m}},
\]

where the multiplication on the left denotes the image of \( W_{\vec{k}} \otimes W_{\vec{m}} \) under the natural map \( H^0(X, \mathcal{O}_X(\vec{k}D)) \otimes H^0(X, \mathcal{O}_X(\vec{m}D)) \to H^0(X, \mathcal{O}_X((\vec{k}+\vec{m})D)) \).

Given \( \vec{a} \in \mathbb{N}^r \), denote by \( W_{\vec{a}, \bullet} \) the singly graded linear series associated to the divisor \( \vec{a}D \) given by the subspaces \( W_{k\vec{a}} \subseteq H^0(X, \mathcal{O}_X(k\vec{a}D)) \). Then put

\[
\text{vol}_{W_\bullet}(\vec{a}) = \text{vol}(W_{\vec{a}, \bullet})
\]
(assuming that this quantity is finite). It will also be convenient for us to consider $W_{\tilde{a}}$ when $\tilde{a} \in \mathbb{Q}^r_{\geq 0}$, given by

$$W_{\tilde{a}} = \begin{cases} W_{k\tilde{a}}, & \text{if } k\tilde{a} \in \mathbb{N}^r; \\ 0, & \text{otherwise.} \end{cases}$$

Our multigraded Fujita approximation, similar to the singly-graded version, is going to state that (under suitable conditions) the volume of $W_{\tilde{a}}$ can be approximated by the volume of the following finitely generated sub multigraded linear series of $W_{\tilde{a}}$:

**Definition 2.** Given a multigraded linear series $W_{\tilde{a}}$ and a positive integer $p$, define $W_{\tilde{a}}^{(p)}$ to be the sub multigraded linear series of $W_{\tilde{a}}$ generated by all $W_{\tilde{m}}$ with $|\tilde{m}| = p$, or concretely

$$W_{\tilde{m}}^{(p)} = \begin{cases} 0, & \text{if } p \nmid |\tilde{m}|; \\ \sum_{\tilde{m} \mid \tilde{m}_1 + \cdots + \tilde{m}_k = \tilde{m}} W_{\tilde{m}_1} \cdots W_{\tilde{m}_k}, & \text{if } |\tilde{m}| = kp. \end{cases}$$

We now state our multigraded Fujita approximation when $W_{\tilde{a}}$ is a complete multigraded linear series, since this is the case of most interest and allows for a more streamlined statement. We will point out in Remark 4 afterward what assumptions on $W_{\tilde{a}}$ are actually needed in the proof.

**Theorem 3.** Let $X$ be an irreducible projective variety of dimension $d$, and let $D_1, \ldots, D_r$ be big divisors on $X$. Let $W_{\tilde{a}}$ be the complete multigraded linear series associated to the $D_i$’s, namely

$$W_{\tilde{m}} = H^0 \left( X, \mathcal{O}_X(\tilde{m}D) \right)$$

for each $\tilde{m} \in \mathbb{N}^r$. Then given any $\varepsilon > 0$, there exists an integer $p_0 = p_0(\varepsilon)$ having the property that if $p \geq p_0$, then

$$\left| 1 - \frac{\text{vol}_{W_{\tilde{a}}}^{(p)}(\tilde{a})}{\text{vol}_{W_{\tilde{a}}}(\tilde{a})} \right| < \varepsilon$$

for all $\tilde{a} \in \mathbb{N}^r$.

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## 2. Proof of Theorem 3

The main tool in our proof is the theory of Okounkov bodies developed systematically in [LM08]. Given a graded linear series $W_{\tilde{a}}$ on a $d$-dimensional variety $X$, its Okounkov body $\Delta(W_{\tilde{a}})$ is a convex body in $\mathbb{R}^d$ that encodes many asymptotic invariants of $W_{\tilde{a}}$, the most prominent one being the volume of $W_{\tilde{a}}$, which is precisely
Proof of Theorem 3. Let $T = \{(a_1, \ldots, a_r) \in \mathbb{R}_{\geq 0}^r \mid a_1 + \cdots + a_r = 1\}$, and let $T_Q$ be the set of all points in $T$ with rational coordinates. The fraction inside (11) is invariant under scaling of $\vec{a}$ due to homogeneity, hence it is enough to prove (11) for $\vec{a} \in T_Q$.

Let $\Delta(W_{\bullet}) \subseteq \mathbb{R}^d \times \mathbb{R}^r$ be the global Okounkov cone of $W_{\bullet}$ as in [LM08, Theorem 4.19], and let $\pi: \Delta(W_{\bullet}) \rightarrow \mathbb{R}^r$ be the projection map. For each $\vec{a} \in T$ we write $\Delta(W_{\bullet})_{\vec{a}}$ for the fiber $\pi^{-1}(\vec{a})$. We also define in a similar fashion the convex cone $\Delta(W_{\bullet}(p))$ and the convex bodies $\Delta(W_{\bullet}(p))_{\vec{a}}$. By [LM08, Theorem 4.19],

$$\Delta(W_{\bullet})_{\vec{a}} = \Delta(W_{\bullet, \vec{a}}) \quad \text{for all } \vec{a} \in T_Q.$$

Note that although [LM08] Theorem 4.19 requires $\vec{a}$ to be in the relative interior of $T$, here we know that (2) holds even for those $\vec{a}$ in the boundary of $T$ because the big cone of $X$ is open and $W_{\bullet}$ was assumed to be the complete multigraded linear series. By the singly-graded Fujita approximation, $\text{vol}(W_{\bullet})_{\vec{a}}$ can be approximated arbitrarily closely by $\text{vol}(W_{\bullet}(p))_{\vec{a}}$ if $p$ is sufficiently large. (Here by $W_{\bullet}(p)$ we mean $W_{\bullet}$ restricted to the $\vec{a}$ direction, which certainly contains $(W_{\bullet, \vec{a}}(p))$. Hence given any finite subset $S \subset T_Q$ and any $\varepsilon' > 0$, we have

$$\text{vol}(\Delta(W_{\bullet}(p))_{\vec{a}}) \geq \text{vol}(\Delta(W_{\bullet})_{\vec{a}}) - \varepsilon' \quad \text{for all } \vec{a} \in S$$

as soon as $p$ is sufficiently large.

Because the function $\vec{a} \mapsto \text{vol}(\Delta(W_{\bullet})_{\vec{a}})$ is uniformly continuous on $T$, given any $\varepsilon' > 0$, we can partition $T$ into a union of polytopes with disjoint interiors $T = \bigcup T_i$, in such a way that the vertices of each $T_i$ all have rational coordinates, and on each $T_i$ we have a constant $M_i$ such that

$$M_i \leq \text{vol}(\Delta(W_{\bullet})_{\vec{a}}) \leq M_i + \varepsilon' \quad \text{for all } \vec{a} \in T_i.$$

Let $S$ be the set of vertices of all the $T_i$’s. Then as we saw in the end of the previous paragraph, as soon as $p$ is sufficiently large we have

$$\text{vol}(\Delta(W_{\bullet}(p))_{\vec{a}}) \geq \text{vol}(\Delta(W_{\bullet})_{\vec{a}}) - \varepsilon' \quad \text{for all } \vec{a} \in S.$$

We claim that this implies

$$\text{vol}(\Delta(W_{\bullet}(p))_{\vec{a}}) \geq \text{vol}(\Delta(W_{\bullet})_{\vec{a}}) - 2\varepsilon' \quad \text{for all } \vec{a} \in T_Q.$$

To show this, it suffices to verify it on each of the $T_i$’s. Let $\vec{v}_1, \ldots, \vec{v}_k$ be the vertices of $T_i$. Then each $\vec{a} \in T_i$ can be written as a convex combination of the vertices:
\( \vec{a} = \sum t_j \vec{v}_j \) where each \( t_j \geq 0 \) and \( \sum t_j = 1 \). Since \( \Delta(W^{(p)}) \) is convex, we have
\[
\Delta(W^{(p)})_{\vec{a}} \supseteq \sum t_j \Delta(W^{(p)})_{\vec{v}_j},
\]
where the sum on the right means the Minkowski sum. By (3) and (4), the volume of each \( \Delta(W^{(p)})_{\vec{v}_j} \) is at least \( M_i - \varepsilon' \), hence by the Brunn-Minkowski inequality [KK08, Theorem 5.4], we have
\[
\text{vol}(\Delta(W^{(p)})_{\vec{a}}) \geq M_i - \varepsilon' \quad \text{for all } \vec{a} \in T_i \cap T_Q.
\]
This combined with (3) shows that (5) is true on \( T_i \cap T_Q \), hence it is true on \( T_Q \) since the \( T_i \)'s cover \( T \).

Since (1) follows from (5) by choosing a suitable \( \varepsilon' \), the proof is thus complete. \( \square \)

**Remark 4.** In the statement of Theorem 3 we assume that \( W \vec{\cdot} \) is the complete multigraded linear series associated to big divisors. But in fact since the main tool we used in the proof is the theory of Okounkov bodies established in [LM08], in particular [LM08, Theorem 4.19], the really indispensable assumptions on \( W \vec{\cdot} \) are the same as those in [LM08] (which they called Conditions (A') and (B'), or (C')). The only place in the proof where we invoke that we are working with a complete multigraded linear series is the sentence right after (2), where we want to say that (2) holds not only in the relative interior of \( T \) but also in its boundary. Hence if \( W \vec{\cdot} \) is only assumed to satisfy Conditions (A') and (B'), or (C'), then given any \( \varepsilon > 0 \) and any compact set \( C \) contained in \( T \cap \text{int}(\text{supp}(W \vec{\cdot})) \), there exists an integer \( p_0 = p_0(C, \varepsilon) \) such that if \( p \geq p_0 \) then
\[
\text{vol}_{W^{(p)}}(\vec{a}) > \text{vol}_{W^{(p)}}(\vec{a}) - \varepsilon
\]
for all \( \vec{a} \in C \cap T_Q \).

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