The ground state of a spin-1/2 neutral particle with anomalous magnetic moment in a Aharonov-Casher configuration

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(March 1999)

We determine the (bound) ground state of a spin 1/2 chargless particle with anomalous magnetic moment in certain Aharonov-Casher configurations. We recast the description of the system in a supersymmetric form. Then the basic physical requirements for unbroken supersymmetry are established. We comment on the possibility of neutron trapping in these systems.

PACS number(s): 03.65Ge, 03.65.Bz, 12.60.J, 11.30.P.

Supersymmetric quantum mechanics (SUSYQM) was discussed by Witten as a laboratory for understanding (super)symmetry breakdown in one-dimensional field theories [1,2]. It constitutes a simplified arena where new ideas are generated, tested, and subsequently generalized. As an example of a one-dimensional SUSYQM system, let us consider the problem of a chargless spin 1/2 particle with anomalous magnetic moment (v.gr. a neutron) confined to move on the real line under the influence of an axially symmetric electrostatic field. Aharonov and Casher pointed out the existence of a quantum mechanical process [3–5] wherein the behavior of an uncharged magnetic dipole is affected by the presence of an electric field. Let us imagine an electrically charged object with axial symmetry centered around the z axis. The neutrons are completely polarized along, say, the positive y direction. It is straightforward to see that this system can be recast in a supersymmetric form. In order to study supersymmetry breaking, we solve the corresponding eigenvalue problem for the ground state. We conclude that, although in this circumstance there is apparently no force on the particles, slow neutrons (s states) will tend to move toward regions where the gradient of the field increases. The next step considers the standard (two-dimensional) AC configuration. We treat this problem in the same framework of supersymmetry. Then we perform the corresponding calculations to determine the physical conditions for unbroken supersymmetry. Finally we briefly compare our results with those obtained in the one-dimensional case.

To be specific, let us consider a spin 1/2 chargless particle with an anomalous magnetic moment $\kappa_n$. The Dirac equation can be written [3,6,7] in a covariant form as

$$\left(\gamma_\mu p^\mu - \frac{e\kappa_n}{2M_n} F^{\mu\nu} \sigma_{\mu\nu} - M_n\right) \Psi(r,t) = 0,$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the electromagnetic field tensor. In 1+1 dimensions we can set

$$\Psi(z,t) = \begin{pmatrix} \psi_u(z,t) \\ \psi_l(z,t) \end{pmatrix}.$$

Thus from (1) and (2) we can write the differential equations for the upper and lower components, $\psi_u$ and $\psi_l$, as follows

$$\frac{1}{2M_n} \left( p_z + \frac{e\kappa_n}{2M_n} E(z) \right) \left( p_z - i \frac{e\kappa_n}{2M_n} E(z) \right) \psi_u(z,t) = i \frac{\partial \psi_u(z,t)}{\partial t},$$

$$\frac{1}{2M_n} \left( p_z - i \frac{e\kappa_n}{2M_n} E(z) \right) \left( p_z + i \frac{e\kappa_n}{2M_n} E(z) \right) \psi_l(z,t) = i \frac{\partial \psi_l(z,t)}{\partial t}.$$

Assuming that $A^\mu$ is time independent, we let the time dependence of $\Psi$ be given by

$$\Psi_E(z,t) = \Psi_E(z) e^{-iEt} = \begin{pmatrix} \psi_u(z) e^{-iEt} \\ \psi_l(z) e^{-iEt} \end{pmatrix}.$$

The resulting set of uncoupled differential equations can be straightforwardly rewritten in the (non-relativistic) supersymmetric form

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\[
H_{SS} \Psi_E(z,t) = \varepsilon \Psi_E(z,t),
\]
with
\[
H_{SS} = \{Q, Q^\dagger\}, \quad [H_{SS}, Q] = [H_{SS}, Q^\dagger] = 0,
\]
where \(\varepsilon \equiv (E^2 - M_n^2) / 2M_n \geq 0\). Here
\[
Q \equiv \tau^- \otimes \left( p_z - i \frac{e\kappa_n}{2M_n} E(z) \right)
\]
is the supersymmetric charge with \(\tau^- = (1/2)(\tau_1 - i\tau_2)\), where the \(\tau_1, \tau_2\) are Pauli matrices. The alternative eigenvalue equation (5) is indistinguishable from (1) (for stationary states) and emerges naturally from it.

From (3), (4) and (5) we find
\[
\frac{1}{2M_n} \left\{ p_z^2 - \tau_3 \frac{e\kappa_n}{2M_n} \frac{dE(z)}{dz} + \left( \frac{e\kappa_n}{2M_n} \right)^2 E^2(z) \right\} \Psi_E(z) = \varepsilon \Psi_E(z).
\]
Thus the superpotential is proportional to the electric field.

Supersymmetry is unbroken if
\[
Q \Psi^{(0)}_{E=M_n}(z) = 0, \quad Q^\dagger \Psi^{(0)}_{E=M_n}(z) = 0,
\]
where \(\Psi^{(0)}_{E=M_n}(z)\) is the normalizable (non-degenerate) ground state. Without lack of generality we can set
\[
\Psi^{(0)}_{E=M_n}(z) = \begin{pmatrix} \phi(z) \\ 0 \end{pmatrix}, \quad (s \text{ states}).
\]
The second equation of (9) is satisfied identically. From the first one we get
\[
\left( p_z - i \frac{e\kappa_n}{2M_n} E(z) \right) \phi(z) = 0.
\]

As a first instance, let us consider the field \(E(z)\) created by a uniformly charged ring of radius \(r_0\), centered around the origin of the \(z\) axis. From (12) we obtain
\[
\phi(z) = C \exp \left( \frac{|\beta|}{(z^2 + r_0^2)^{1/2}} \right),
\]
with \(\beta \equiv -eQ\kappa_n/4M_n\), \(Q\) the total charge of the distribution and \(C\) a complex constant. Note that when \(|z| \to \infty\), \(\phi(z) \to C\). Thus \(\phi(z)\) is not normalizable on the real axis. Therefore \(\Psi^{(0)}_{E=M_n}\) does not belong to the Hilbert space, i.e., supersymmetry is broken in this case.

An analogous result is obtained for the case of a disk of radius \(r_0\), with uniform charge per unit area and total charge \(Q\). Here \(\phi(z)\) turns out to be
\[
\phi(z) = C' \exp \left( -\frac{|\beta|}{\pi r_0} \left( |z| - \left( z^2 + r_0^2 \right)^{1/2} \right) \right).
\]
Notice that again \(\phi(z)\) is not normalizable.

Figure 1 shows the non-normalizable ground state probability densities \(|\phi|^2\) for a neutron in the field of a uniformly charged ring, and in the field of a uniformly charged disk. Both configurations have the same total electric charge \(Q\).

The third instance regards an infinite plane with charge density per unit area \(\sigma\). In this example we get
\[
\phi(z) = \sqrt{\frac{\alpha}{2}} \exp \left( -\frac{|\alpha|}{2} |z| \right),
\]
where \(\alpha = -e\sigma\kappa_n/2M_n\). Here \(\phi(z)\) has finite norm and thus supersymmetry is unbroken. Note that \(\phi(z)\) in (13) and (14) is not differentiable at \(z = 0\) since we have taken an (idealized) thickless charged surface as a source of the field.
Finally, let us consider an infinitely large uniform charge distribution with density per unit volume \( \rho \), where a symmetric infinite plane of thickness \( L \) has been removed. This situation resembles a potential well in one-dimensional quantum mechanics. In this case we have

\[
\phi(z) = C'' \exp \left( -\frac{1}{2} |\alpha| (z^2 - L |z|) \right), \quad L/2 \leq |z|; \quad C'' \exp \left( -\frac{1}{2} |\alpha| \left( \frac{1}{4} - \frac{1}{2} L \right) \right), \quad |z| < L/2,
\]

where now \( \alpha = -e \rho \kappa_n / 4M_n \). Here \( \phi(z) \) is also normalizable and supersymmetry is then unbroken.

In fig. 2, we plot the normalized ground-state probability density of a neutron in the field of a uniformly charged plane. In this picture we also show the corresponding probability density of an uniformly charged infinite volume configuration, where a symmetric plane section of thickness \( L \) has been removed.

Notice that, in the first two examples, confinement mainly comes (although not completely achieved for the ground state) from the gradient of the electric field along the \( z \) axis. Furthermore, we have found confinement in one direction assuming confinement in the other spatial degrees of freedom.

Next let us examine the standard 1+2 AC configuration \([3,7]\). Here again we are concerned with the conditions for finding the ground state of a system with unbroken supersymmetry. To this end we have to assume connectivity in the configuration space in order to be able to define a normalizable ground state. The problem turns out to have exact supersymmetry only under the fulfillment of a condition for the magnitude of the charge distribution which generates the electric field. We also discuss the possibility of breaking supersymmetry by examining the requirements for the existence of lower energy bound states.

To start with, let us consider an infinite cylinder with uniform charge per unit volume \( \rho \) centered along the \( z \) axis, so that there exists an electric field

\[
E_z(r) = \rho r/2, \quad 0 \leq r \leq r_0; \quad E_z(r) = \rho \sigma_2 r/2, \quad r_0 < r < \infty,
\]

where \( r_0 \) is the radius of the cylinder and for simplicity we have chosen \( \hat{\mathbf{r}} \hat{\mathbf{z}} = 0 \). Here \( \hat{\mathbf{r}} \) and \( \hat{\mathbf{z}} \) are unit vectors in the \( r \) and \( z \) directions respectively. The neutrons are completely polarized along the positive \( z \) direction. They move on a plane in the the presence of \( \mathbf{E} \). In this circumstance there is no direct force on the neutrons but there exists a kind of Aharonov-Bohm effect \([3,4]\). Nevertheless, if the singularity on the \( z \) axis is removed, as is implied in \([10]\), the neutrons are allowed to penetrate the charged line. Therefore a new question is to be considered: It regards the problem of the possible bound states of the neutron in this new AC configuration.

The Aharonov-Casher effective wave equation is obtained by making \( A_0 \neq 0, B = 0 \), with \( \nabla \cdot \mathbf{E} = \rho \). For stationary states of energy \( E \) we write

\[
\Psi_E(r,t) = \begin{pmatrix} \phi(r) \\ \chi(r) \end{pmatrix} e^{-iEt}.
\]

Thus from \([3]\) and \([17]\) we get

\[
\frac{1}{2M_n} \sigma \cdot \left( \mathbf{p} + i\kappa_n \frac{\mathbf{E}(r)}{2M_n} \right) \sigma \cdot \left( \mathbf{p} + i\kappa_n \frac{\mathbf{E}(r)}{2M_n} \right) \phi(r) = \varepsilon \phi(r),
\]

\[
\frac{1}{2M_n} \sigma \cdot \left( \mathbf{p} + i\kappa_n \frac{\mathbf{E}(r)}{2M_n} \right) \sigma \cdot \left( \mathbf{p} + i\kappa_n \frac{\mathbf{E}(r)}{2M_n} \right) \chi(r) = \varepsilon \chi(r),
\]

where \( \sigma = (\sigma_1, \sigma_2) \) and \( \varepsilon \equiv (E^2 - M_n^2) / 2M_n \geq 0 \). As before, this set of differential equations can be rewritten in the supersymmetric form

\[
H_{SS} = \{Q, Q^\dagger\}, \quad [H_{SS}, Q] = [H_{SS}, Q^\dagger] = 0,
\]

with

\[
H_{SS} \Psi_E(r,t) = \varepsilon \Psi_E(r,t),
\]

where now

\[
Q \equiv \frac{1}{\sqrt{2M_n}} \tau^- \otimes \sigma \cdot \left[ \mathbf{p} - i \left( \kappa_n / 2M_n \right) \mathbf{E}(r) \right]
\]

is the supersymmetric charge and \( \tau^- = (1/2) (\tau_1 - i\tau_2) \), where the \( \tau_1, \tau_2 \) are Pauli matrices. Thus \( H_{SS} \) is invariant under \( N = 1 \) supersymmetry. Note that the AC effect has also been discussed in the framework of \( N = 2 \) nonrelativistic supersymmetry \([3]\).
From (18) we find \([\lambda]\) that

\[
\frac{1}{2M_n} \left\{ \mathbf{p}^2 + \frac{e \kappa_n}{2M_n} r_3 \otimes (\nabla \cdot \mathbf{E}(r) + 2\sigma_3 \mathbf{E}(r) \times \mathbf{p})_3 + \left( \frac{e \kappa_n}{2M_n} \right)^2 \mathbf{E}^2(r) \right\} \Psi(r) = \varepsilon \Psi(r).
\]  

(23)

Supersymmetry is unbroken if

\[
Q \phi^{(0)}(r) = 0, \quad Q^I \phi^{(0)}(r) = 0,
\]

(24)

where \(\phi^{(0)}(r) = \phi^{(0)}(r) \quad (r \equiv |r| = \sqrt{r_1^2 + r_2^2})\) is the ground state of the system. In other words, the generators of supersymmetry annihilate the vacuum state in order to have an exact symmetry. We also have the constraint

\[
(E(r) \times \mathbf{p})_3 \phi^{(0)}(r) = \frac{|E(r)|}{r} L_3 \phi^{(0)}(r) = 0 \quad \text{(s states)},
\]

(25)

with \(L_3 = (r \times \mathbf{p})_3\) the \(z\) component of the orbital angular momentum operator. Here we are concerned with states for which \(E^2 = M_n^2\), i.e., \(\varepsilon = 0\).

The second equation of (24) is satisfied identically since in the nonrelativistic limit the lower components \(\Psi_{E=M_n}\) vanish. From the first one we get

\[
\sigma \cdot (\mathbf{p} - i (e \kappa_n / 2M_n) \mathbf{E}(r)) \phi^{(0)}(r) = 0.
\]

(26)

Without lack of generality we can set

\[
\phi^{(0)}(r) = \binom{\phi(r)}{0}, \quad \chi^{(0)}(r) = \binom{0}{0}.
\]

(27)

Then from (26) we find the differential equations

\[
\left( \frac{d}{dr} - \beta r \right) \phi_<(r) = 0, \quad 0 \leq r \leq r_0; \quad \left( \frac{d}{dr} - \frac{\beta r_0^2}{r} \right) \phi_>(r) = 0, \quad r_0 \leq r < \infty,
\]

(28)

where \(\beta \equiv -e \rho \kappa_n / 4M_n\). Thus

\[
\phi_<(r) = Ae^{\frac{1}{2} \beta r^2}, \quad 0 \leq r \leq r_0; \quad \phi_>(r) = Br^{\beta r_0^2}, \quad r_0 \leq r < \infty,
\]

(29)

with \(A, B\) complex constants.

Next we demand continuity of the wavefunction and its derivative at \(r = r_0\). Both conditions give the same information: \(A \exp \left( (1/2) \beta r_0^2 \right) = Br_0^{\beta r_0^2}\). Furthermore, if \(\Psi_{E=M_n}\) belongs to the Hilbert space, \(\phi\) must be normalizable on the plane \([0, 2\pi] \times [0, \infty]::\)

\[
2\pi \int_0^\infty |\phi(r)|^2 r dr = 2\pi \left\{ |A|^2 \int_0^{r_0} dr e^{\beta r^2} + |B|^2 \int_{r_0}^{\infty} dr r^{2\beta r_0^2+1} \right\} = 1,
\]

(30)

from where we get

\[
|A|^2 = \frac{|B|}{\pi} \frac{(\beta |r_0^2 - 1| e^{\frac{1}{2} \beta |r_0^2}}}{2 (\beta |r_0^2 - 1| \sinh (\frac{1}{2} \beta |r_0^2) + |\beta| r_0^2 e^{-\frac{1}{2} \beta |r_0^2})}.
\]

(31)

Notice that in (30) we must require that

\[
\beta r_0^2 < -1.
\]

(32)

This inequality constitutes a necessary condition on the possible values of \(\rho\) and \(r_0\) (or equivalently on \(\lambda \equiv \rho \tau \rho_0\)) if we want to preserve unbroken supersymmetry. Inserting \(c^2\) in (12), we can estimate the minimum value of \(\lambda\) to be able to obtain a normalizable ground state: \(|\lambda|_\text{min} \simeq 4\pi M_n c^2 / |e\kappa_n| \simeq 4.697 \times 10^{-3} \text{ C/cm}\). As \(\lambda\) depends linearly on \(r_0\), one can in principle set up a configuration with the required \(\lambda\) \([\text{3}]\).
Figure 3 shows the neutron density of probability $|\phi|^2$ as a function of the dimensionless parameter $r/r_0$ for different values of $\beta < -1$, in natural units. Notice that when $\beta$ approaches $-1$, $|\phi|^2$ becomes flatter, i.e., there exists a larger probability that the neutron be outside the charged distribution than within it.

To treat the general eigenvalue problem, we observe that the eigenvalue problem stated by (23) has two kinds of solutions: a) non-normalizable scattering-like states for $\varepsilon > 0$ ($E^2 > M_n^2$); b) (normalizable) bound states for $\varepsilon < 0$ ($E^2 < M_n^2$).

The energy levels are obtained by requiring that the radial solutions and their derivatives be continuous at $r = r_0$, i.e., this is the quantization condition for the remaining energy levels. This involves non-trivial numerical calculation and is now being studied. Notice that the existence of further bound states would break exact supersymmetry, as expressed by (21) and (24), since $(E^2 - M_n^2)_{\text{min}} < 0$.

From the above we can draw at least two main conclusions: First, in the one-dimensional systems the electric charge distribution has to be sufficiently spread out in space in order to preserve unbroken supersymmetry. If this be the case, $\phi(z)$ is normalizable and thus $\Psi^{(0)}_{E=M_n}$ constitutes a (unique ground) bound state of the system. Furthermore, in the standard two-dimensional system, the magnitude of the electric charge distribution has to be sufficiently large ($\lambda \gtrsim 4\pi M_nc^2/|e\kappa_n|$) in order to generate a bound (ground) state. Second, in both the one and two-dimensional systems, we are not asserting that the neutron directly "feels" a force due to the electric field generated by the charge density. Rather, from the second terms on the left hand sides of (8) and (23), we state that the neutron tends to move toward regions where the gradient of the electric field increases. The third term in the same equations corresponds to the appearance of an induced electric dipole moment on the particle [3].

Note that, in the standard AC configuration, the fulfillment of the condition $E^2 \leq M_n^2$ would allow cold neutron trapping by an electrostatic field as a physical consequence of a purely quantum mechanical effect. Confinement is usually achieved by means of diverse magnetic trap systems [10]. Cold neutrons are extensively used: in tests of fundamental quantum theory [11], and in applied physics [12].

Acknowledgments

This work was supported by Dirección de Investigación, Universidad de Concepción, through grants P.I. 96.11.19-1.0 and Fondecyt #1970995.

One of us (SB) is grateful to the School of Physics, University of Melbourne, Australia, for its warm hospitality. We are very thankful to Professors A. G. Klein and G. I. Opat for their valuable criticisms and helpful suggestions on the experimental and theoretical aspects of this paper.
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Figure Captions

FIG. 1. The non-normalizable ground state probability densities for a neutron in the field of a uniformly charged ring (blue colour), and of a uniformly charged disk (green colour); both charge configurations have total charge $Q$.

FIG. 2. The normalizable ground state probability density for a neutron in the field of a uniformly charged plane (blue colour). In green colour we show the corresponding probability density for a uniformly charged infinite volume configuration, where a symmetric plane section of thickness $L$ has been removed.

FIG. 3. The neutron ground state probability density $|\phi|^2$ as a function of the dimensionless parameter $r/r_0$ for different values of $\beta < -1$. The units used are $\hbar = c = 1$. 
$\beta = -3$

$\beta = -2$

$\beta = -1.1$