Brane Bulk Couplings and Condensation from REA Fusion

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ABSTRACT: The physical meaning of the Reflection Equation Algebras of \cite{1} is elucidated in the context of Wess–Zumino–Witten D-brane geometry, as determined by couplings of closed-string modes to the D-brane. Particular emphasis is laid on the rôle of algebraic fusion of the matrix generators of the Reflection Equation Algebras. The fusion is shown to induce transitions among D-brane configurations admitting an interpretation in terms of RG-driven condensation phenomena.

KEYWORDS: (twisted) D-branes, WZW models, D-brane condensation, quantum groups, (twisted) Reflection Equation Algebras.

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1. Introduction

Physics of D-branes has long been a subject of intense study. Of particular interest are branes on compact WZW manifolds. They provide interesting examples of brane behaviour in non-trivial backgrounds with fluxes. But even for these, highly symmetric, cases the full BCFT analysis is rather complicated \(^2\) and stands in shocking contrast with the very simple matrix model of brane condensation advocated in \(^3\). There is a price to pay for the simplicity of the latter - it cannot describe all branes on the group manifold. Some years ago, a matrix model based on quantum-group symmetries was advanced \(^1\). It seems to work properly for all branes but many of its features are still mysterious. The model uses the celebrated Reflection Equation (RE) \(^4\) and its representation theory to derive brane properties.

In this short paper, we shall follow this track and analyse fusing properties of the matrices entering the RE. It appears that there are two types of fusions. We shall show how both of them lead to some known brane properties. As we shall see, the first type of fusion (which we dub Bound-State Fusion – BSF) can be interpreted as describing a process of formation of extended branes out of D0-branes. It also chooses a particular set of representations of the RE as the relevant ones. The second type (to be called Bulk-Weight Fusion – BWF) is just the standard representation-theoretic fusion of the function algebra on a given brane. Mastering this last kind of fusion is necessary to keep control of functions on a given brane and yields couplings of gravity to the brane.

To begin with, we recall the Reflection Equation (RE) \(^4\):

\[
R_{21}^{\Lambda_1,\Lambda_2} M_1^{\Lambda_1} R_{12}^{\Lambda_1,\Lambda_2} M_2^{\Lambda_2} = M_2^{\Lambda_2} R_{21}^{\Lambda_1,\Lambda_2} M_1^{\Lambda_1} R_{12}^{\Lambda_1,\Lambda_2},
\]

written for matrices \(M^{\Lambda_1,\Lambda_2}\) in the two respective representations of \(\mathcal{U}_q(\mathfrak{g}^L \times \mathfrak{g}^R)_{\mathcal{R}}\) of the highest weights \(\Lambda_1, \Lambda_2\). Here, the deformation parameter is \(q = e^{\frac{\pi i g}{\dim(G)}}\) and \(R_{12}^{\Lambda_1,\Lambda_2} = (\pi_{\Lambda_1} \otimes \pi_{\Lambda_2})(\mathcal{R}_{12})\).
is the suitably represented $R$-matrix of the Drinfel’d–Jimbo quantum group $\mathcal{U}_q(\mathfrak{g})$. Also, $M^\Lambda \in \text{End}(V_{\Lambda}) \otimes \mathcal{M}$, with $\mathcal{M}$ the (abstract) Reflection Equation Algebra (REA), and we take $\Pi$ to denote the fundamental representation $^2$ of $\mathcal{U}_q(\mathfrak{g})$. The RE (1.1) induces a $\mathcal{U}_q(\mathfrak{g}^L \times \mathfrak{g}^R)$-module structure on $\mathcal{M}$, that is the RE guarantees that $\mathcal{M}$ are tensors under quantum-group action. Indeed, 

\[(u_L \otimes u_R) \triangleright (M^\Lambda)^{ij}_k = \pi_\Lambda(Su_L)^i_k (M^\Lambda)^{jk}_i \pi_\Lambda(u_R)^l_j, \quad (1.2)\]

with $S$ the antipode of $\mathcal{U}_q(\mathfrak{g})$, preserves (1.1).

2. Bound-State Fusion

In this section, we establish a non-trivial relation between D-brane physics in boundary WZW models and the REA’s defined by (1.1). We want to propose an algebraic cousin of the D-brane condensation effect $^3$, discussed at great length in $^3$ $^5$ with reference to the seminal papers by Affleck and Ludwig $^7$. In the case at hand, the fusion shows that an arbitrary brane can be built out of trivial representations of the RE, describing D0-branes. The algebraic fusion algorithm has been devised in direct reference to the techniques of the principal chiral model presented in $^9$, in which there is additional structure (dependence on a dynamical parameter) justifying its interpretation. In the present setup, lacking this extra structure, some elementary tests of its validity are performed explicitly below, as well as in the Appendix.

One can show that $M^\Lambda(\Lambda B)$ given by either side of the RE (1.1),

\[M_1^\Lambda(\Lambda B) = M_3^\Lambda(\Lambda B) R_{31}^{\Lambda \Lambda B} M_1^\Lambda R_{13}^{\Lambda \Lambda B}, \quad (2.1)\]

also satisfies an appropriate RE:

\[R_{21}^{\Lambda_1 \Lambda_2} M_1^\Lambda(\Lambda B) R_{12}^{\Lambda_1 \Lambda_2} M_2^\Lambda(\Lambda B) = M_2^\Lambda(\Lambda B) R_{21}^{\Lambda_1 \Lambda_2} M_1^\Lambda(\Lambda B) R_{12}^{\Lambda_1 \Lambda_2}, \quad (2.2)\]

for arbitrary $\Lambda B$. The latter follows straightforwardly from the RE’s and the QYBE satisfied by the $M$-matrices fused and the $R$-matrix, respectively.

The Bound-State Fusion (BSF) thus defined, (2.1), singles out a set of REA representations of special relevance to the study of WZW branes. Take $c$-number matrices $M^\Lambda$, $M^{\Lambda \Lambda B}$ respecting the RE, that is a trivial realisation of the RE (considered, e.g., in $^8$). Then, the right-hand side of (2.1) belongs to $(\pi_\Lambda \otimes \pi_{\Lambda B})(\mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{U}_q(\mathfrak{g}))$. For the $c$-number solutions $M^\Lambda = \mathbb{I}$ (with $\Lambda$ arbitrary), realisations of this type shall be denoted by $M^{\Lambda \Lambda B}$, i.e.

\[M^{\Lambda \Lambda B} \equiv R_{21}^{\Lambda \Lambda B} R_{12}^{\Lambda \Lambda B} = (\pi_\Lambda \otimes \pi_{\Lambda B})(R_{21} R_{12}). \quad (2.3)\]

$^1$ is the irreducible representation of $\mathcal{U}_q(\mathfrak{g})$ of the highest weight $\Lambda$, a dominant integral affine weight of $\mathfrak{g}$. We denote the set of all such weights (the fundamental affine alcove) by $P^0_{\kappa}$.

$^2$ Displaying the indices explicitly, the RE reads (here $M^i_j \equiv (M^{|\Lambda\Lambda B}|)$,

\[(\text{RE})^k_i^{\; \; \; \; \; \; \; \; k} : \quad R^k_i a_i b M^a_j c M^d_j a = M^k_d a R^a_i b M^d_j R^j_i b . \quad (2.4)\]

The indices $\{i, j\}$ and $\{k, l\}$ correspond to the first (1) and the second (2) vector space in (1.1), respectively.
This, however, is none other but the Faddeev–Reshetikhin–Takhtajan embedding \( \text{REA}_q(\mathfrak{g}) \leftarrow \mathcal{U}_q(\mathfrak{g}) \), chosen in [1] to provide a realisation of the REA for the simple reason: it induces a representation theory of \( \text{REA}_q(\mathfrak{g}) \) whose elements, irreducible highest-weight representations of \( \mathcal{U}_q(\mathfrak{g}) \) of a non-vanishing quantum dimension, are in a straightforward one-to-one correspondence with all the candidate algebraic branes associated, in [10], with untwisted maximally symmetric WZW boundary conditions on the compact Lie group \( G \). Thus, we can postulate a principle saying that untwisted maximally symmetric WZW branes on \( G \) correspond to those irreducible representations of \( \text{REA}_q(\mathfrak{g}) \) which are generated through the BSF (2.1).

Given \( M^{\Lambda B, \lambda} \) and \( M^{\Lambda, \lambda} \), the BSF (2.1) takes the form:

\[
M^{\Lambda, \lambda} \times \lambda = M_2^{\Lambda B, \lambda} R_{21}^{\Lambda \lambda} M_1^{\Lambda, \lambda} R_{12}^{\Lambda \lambda}.
\]  

Here, the left-hand side belongs to \([\pi_{\Lambda} \otimes (\pi_{\Lambda B} \otimes \pi_{\lambda})](\mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{U}_q(\mathfrak{g}))\) and hence - as a tensor operator\(^3\) - it can be decomposed as

\[
M^{\Lambda, \lambda} \times \lambda = \bigoplus_{\mu \in P^+_{\Lambda}} N_{\Lambda B, \lambda}^{\mu} M^{\Lambda, \mu},
\]

where \( N_{\Lambda B, \lambda}^{\mu} \) are the standard fusion rules of the WZW model for \( \hat{\mathfrak{g}}_\kappa \) and the usual restriction to irreducible representations of \( \mathcal{U}_q(\mathfrak{g}) \) of a non-vanishing quantum dimension has been imposed. It ought to be noted that the BSF (2.4) admits a very natural interpretation in terms of a perturbation of a set of D-branes by gauge fields. This resembles CFT perturbations inducing transitions through condensation between these original (stacked) D-branes and a final (metastable) state [6]. In order to give some flesh to this claim, we should go back to [11] and identify nontrivial gauge degrees of freedom on a stack of quantum D-branes. Furthermore, we decompose some of the terms in (2.4), \( X \in \{M_2^{\Lambda B, \lambda}, R_{21}^{\Lambda \lambda}, R_{12}^{\Lambda \lambda}\} \), as follows

\[
X = I + x,
\]

where \( x \) is \((q-)\)traceless and of the order of \( \mathcal{O}(1/\kappa) \). With this decomposition, the leading term in (2.4) reads \( \tau_1,2(\dim V_{\Lambda B} \otimes M^{\Lambda, \lambda}) \) (\( \tau_1,2 \) interchanges the first and second tensor components) and shall be denoted as \( M_0^{\Lambda, \lambda} \). Thus, the right-hand side of (2.4) can be rewritten as

\[
M^{\Lambda, \lambda} \times \lambda = M_0^{\Lambda, \lambda} + A^{\Lambda, \lambda} \times \lambda,
\]

where \( A^{\Lambda, \lambda} \times \lambda \sim \mathcal{O}(1/\kappa) \) can be interpreted as a gauge field\(^4\). Very much in the spirit of [3], we then see that the stack of \( \dim V_{\Lambda B} \) quantum D-branes thus perturbed by the gauge field condenses to a D-brane configuration described by (2.5), again in perfect agreement with the BCFT. This simple picture lends support to our interpretation of the BSF.

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\(^3\)Cp [1] and the Appendix.

\(^4\)Using the BWF of Sec [3], the gauge field is readily shown to be a “function” of the background geometry \( M^{\Lambda, \lambda} \).
3. Bulk-Weight Fusion and brane-gravity couplings

Let us begin by recalling that the quantised algebra of functions on untwisted branes, \( \mathcal{M} \), is generated by the elements \((M^\Pi)^{i}_{j}\). There is a convenient basis of the algebra, regarded here as a vector space. We claim that

\[
\mathcal{M} = \oplus_{\Lambda, i,j} \text{span}((M^\Lambda)^{i}_{j})
\]

is the basis sought. Above, \(M^\Lambda\) denote matrices respecting the RE \([1, 1]\), the latter being written in the representation \(\pi_{\Lambda} \otimes \pi_{\Lambda}\). Thus, \(M^{\Lambda, \lambda}\) (see the previous section) are quantum-group tensors with transformation properties appropriate for functions on the standard set of branes,

\[
(1 \otimes \pi_{\Lambda}(u_1))M^{\Lambda, \lambda}(1 \otimes \pi_{\Lambda}(u_2)) = (\pi_{\Lambda}(Su_1) \otimes 1)M^{\Lambda, \lambda}(\pi_{\Lambda}(u_2) \otimes 1).
\]

The road to (3.1) goes through the definition of the Bulk-Weight Fusion:

\[
M^{\Lambda_1 \times \Lambda_2}_{12} = \left( R^{\Lambda_1, \Lambda_2}_{21} \right)^{-1} M^{\Lambda_1}_{12} R^{\Lambda_1, \Lambda_2}_{12} M^{\Lambda_2}_{21},
\]

One can easily show that the above yields a \((\pi_{\Lambda_1} \otimes \pi_{\Lambda_2})/2\text{nd}\) tensor product decomposition of \(\mathcal{M}\). The latter is the (symmetric) quadratic tensor generated by the elements \((M^\Lambda)^{i}_{j}\) up to a \(\Lambda\)-dependent factor only. Let us calculate \(f(\Lambda, \lambda)\) of (3.6) for the brane labeled by the weight \(\lambda\). Accordingly, we calculate the \((q)\)-trace of \(M^\Lambda\) over the module \(V_{\lambda}\). It is straightforward to demonstrate \([1]\) that the \((q)\)-trace is proportional to the unit matrix, that is

\[
\text{tr}_q^{(\lambda)}((M^\Lambda)^{i}_{j}) = \text{tr}_{V_{\lambda}}((M^\Lambda)^{i}_{j} \cdot q^{2H_{\rho}}) = f(\Lambda, \lambda) \delta^i_j,
\]

where \(\rho\) is the Weyl vector of \(\mathfrak{g}\). It was shown in \([1]\) that (2.3) encode certain properties of the brane \(\lambda\) in an algebraic manner. Since we have not normalised \(M\) so far, we can specify the function \(f(\Lambda, \lambda)\) up to a \(\lambda\)-dependent factor only. Let us calculate \(f(\Lambda, \lambda)\). We use

\[
\mathcal{R}_{12} = q^{H_{i}F_{ij} \otimes H_{j}} \left( I \otimes I + \sum_{U^\pm} U^+ \otimes U^- \right),
\]

while for \(\mathcal{R}_{21}\) we transpose \(U^+ \leftrightarrow U^-\) in the expression above. Here, \(F\) is the (symmetric) quadratic matrix of \(\mathfrak{g}\), and \(U^+, U^-\) stand for terms in the Borel subalgebras of rising and lowering operators, respectively. As the left-hand side of (3.4) does not depend on the vector from the module \(V_{\Lambda}\) that it acts upon, we can evaluate it on the highest-weight vector (annihilated by \(U^+\)), \(|\Lambda\rangle\). Then, only the generators of the Cartan subalgebra in \((\mathcal{R})\) contribute,

\[
q^{2H_{i}F_{ij} \otimes H_{j}}|_{|\Lambda\rangle} = q^{2\Lambda_i F_{ij} \otimes H_{j}} = \mathbb{I}_{\dim \pi_{\Lambda}} \otimes q^{2H_{\lambda}},
\]
and (3.4) becomes\footnote{The explicit formula relating entries of the modular S-matrix to Lie-algebra characters can be found, e.g., in \cite{12}.}

\[
\text{tr}_q^{(\lambda)}((M^\lambda)^i_j) = \delta^i_j \text{tr}_V(\psi^{2(H^\lambda + H^\rho)}) = \delta^i_j \chi_\lambda \left( \frac{2\pi i (\Lambda + \rho)}{\kappa + g^{V}(\mathfrak{g})} \right) = \delta^i_j \frac{S_{\Lambda+\lambda}}{S_{\Lambda+0}},
\]

(3.7)

where \(\chi_\lambda\) and \(S_{\Lambda+\lambda}\) are the standard character over the \(g\)-module of the highest weight \(\lambda\) and the modular matrix of the WZW model associated to \(\widehat{g}_\kappa\), respectively, whereas \(\Lambda^+\) is the unique charge conjugate of the weight \(\Lambda\). In particular, for \(g = su_2\), we obtain

\[
f_{su_2}(\Lambda, \lambda) = \text{tr}_V(q^{(\Lambda+1)H}) = \frac{\sin \frac{\pi (\Lambda+1)(\lambda+1)}{k+2}}{\sin \frac{\pi (\lambda+1)}{k+2}},
\]

(3.8)

which agrees with \(S_{su_2}^{\lambda\lambda} = \sqrt{\frac{2}{k+2}} \sin \frac{\pi (\lambda+1)(\lambda+1)}{k+2}\) and \(\Lambda^+ \equiv \Lambda\) for all \(\Lambda\).

On the BCFT side, recall that untwisted maximally symmetric D-branes are represented by Cardy states \footnote{Actually, on the level of bulk-boundary couplings, the only piece of data that cannot be retrieved from the algebra is \(S_{00}\). Indeed, we have \cite{13}.}:

\[
|\lambda\rangle_C = \sum_{\Lambda \in P^+_\kappa} \frac{S_{\Lambda\lambda}}{\sqrt{S_{\Lambda0}}} |\Lambda\rangle_I.
\]

(3.9)

Above, \(|\Lambda\rangle_I\) are Ishibashi (character) states \cite{15}. The data encoded in (3.9) turns out to be sufficient to determine, to the leading order in \(1/\kappa\), the coupling of \emph{graviton} modes:

\[
[a, b, \gamma^i_j] = J^{(a)}_{-1} J^{(b)}_{-1} |\gamma^i \rangle \otimes |\gamma^+_j\rangle, \quad |\gamma^i \rangle \otimes |\gamma^+_j\rangle \in V_\gamma \otimes V_\gamma
\]

(3.10)

to the brane defined by (3.9). Indeed, one verifies that \(\mathcal{N}\) is an irrelevant normalisation constant

\[
\langle a, b, \gamma^i_j |\lambda\rangle_C = \mathcal{N} \delta^{ab} \delta^i_j \frac{S_{\gamma\lambda}}{\sqrt{S_{\gamma0}}}.
\]

(3.11)

In the present context, we are dealing with a matrix model whose elementary degrees of freedom are D0-branes (the D0-brane enters the quantum-algebraic construction as the trivial representation of \(\text{REA}_q(\mathfrak{g})\), on which \(M^{\lambda,0} = \mathbb{I}_{\dim \pi_\lambda}\)), hence it seems only natural to consider couplings normalised relative to the reference D0-brane,

\[
\frac{\langle a, a, \gamma^i_j |\lambda\rangle_C}{\langle a, a, \gamma^i_j |0\rangle_C} = \delta^i_j \frac{S_{\gamma\lambda}}{S_{\gamma0}}.
\]

(3.12)

From direct comparison between (3.7) and (3.12), we conclude that \(M^{\lambda,\lambda}\) encode the full gravitational data on the untwisted D-brane labeled by \(\lambda \in P^+_\kappa(\mathfrak{g})\), relative to an elementary D0-brane.

We emphasise that it is not just the numerical values of the couplings but also their structure, diagonal in the bulk representation indices, that can be read off from (3.7). The D0-brane data (e.g. the D0-brane tension), on the other hand, have to be supplemented independently of the algebra\footnote{where we have first used the symmetries of the modular S-matrix: \(S_{\lambda+\mu} = S_{\lambda\mu+}\) and \(S_{\lambda\mu} = S_{\mu\lambda}\), and later reiterated the first equality.}.
4. Conclusions

In the present paper, we have discussed several application of the algebraic RE fusion to the description of physics of untwisted WZW branes. The Bound-State Fusion has been shown to lead to the appropriate choice of realisations of the REA and to give a nice picture of higher-dimensional quantum branes as condensates of the elementary quantum D0-branes. It also seems to offer some insight into the structure of gauge fluctuations of the non-commutative geometry defined by the quantised function algebra $\mathcal{M}$, all highly reminiscent of the familiar construction of Affleck and Ludwig. The Bulk-Weight Fusion, on the other hand, has been demonstrated to encode a fairly complete information on the gravitational brane couplings. Both are amazingly simple and follow straightforwardly from the structure of the RE.

In spite of the progress, signified by our results, in formulating a compact description of quantum WZW geometry and elucidating the quantum-group structure of the associated BCFT, a lot more still needs to be understood in this context. We hope to return to these riddles soon.

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Appendix

Below, we consider $\mathcal{U}_q(\mathfrak{g})$-covariance properties of the various generalised reflection matrices introduced in the main text. In particular, we give a simple proof of (3.2) and (3.4), essentially repeating the original one from [10]. First, we show, for $M_{univ} := \mathcal{R}_{21}\mathcal{R}_{12}$,

$$\Delta(u)M_{univ} = \Delta(u)\mathcal{R}_{21}\mathcal{R}_{12} = \mathcal{R}_{21}\Delta^{cop}(u)\mathcal{R}_{12} = \mathcal{R}_{21}\mathcal{R}_{12}\Delta(u) = M_{univ}\Delta(u), \quad (1)$$

where we have invoked the twisting property of $\mathcal{R}$ [16, 17]:

$$\Delta^{cop}(u) = \mathcal{R}\Delta(u)\mathcal{R}^{-1}, \quad \Delta^{cop}(u) := u_2 \otimes u_1. \quad (2)$$

Using Hopf-algebra identities for the coproduct and the antipode of $\mathcal{U}_q(\mathfrak{g})$ (i.e. multiplying (1) from the left with $(Su_0 \otimes 1)$ and from the right with $(1 \otimes Su_3)$, and subsequently representing both sides on $V_\Lambda \otimes V_\Lambda$), we turn (1) into (3.2), or

$$\pi_{\Lambda}(u_1)M^{\Lambda,\Lambda}\pi_{\Lambda}(Su_2) = \pi_{\Lambda}(Su_1)M^{\Lambda,\Lambda}\pi_{\Lambda}(u_2), \quad (3)$$

for any $u \in \mathcal{U}_q(\mathfrak{g})$.

In order to prove (3.4), we recall the definition of the quantum trace: $tr_q(x) := \text{tr}(xv)$, where $v := (S \otimes \text{id})(\mathcal{R}_{21})$ is the distinguished invertible ($v^{-1} \equiv Sv$) element of $\mathcal{U}_q(\mathfrak{g})$ satisfying $S^2u = vuv^{-1}$ for any $u \in \mathcal{U}_q(\mathfrak{g})$ [16]. This, together with (3.2), immediately implies

$$\pi_{\Lambda}(Su_1)tr_q^{(\Lambda)}(M^\Lambda)\pi_{\Lambda}(u_2) = tr_q^{(\Lambda)}(u_1M^\Lambda Su_2) = tr_q^{(\Lambda)}(M^\Lambda Su_2u_1) = \varepsilon(u)tr_q^{(\Lambda)}(M^\Lambda), \quad (4)$$

or, equivalently,

$$[\pi_{\Lambda}(u), tr_q^{(\Lambda)}(M^\Lambda)] = 0. \quad (5)$$

Last, we may verify the tensorial character of (2.1), on which our physical interpretation of the BSF has been based. Our proof is in fact a slight variation of the trick used above. We begin by defining the operator $M^\chi_{univ} = \mathcal{R}_{32}\mathcal{R}_{23}\mathcal{R}_{21}\mathcal{R}_{31}\mathcal{R}_{13}\mathcal{R}_{12}$ such that $M_{1, 2 \otimes 3}^{\Lambda, \Lambda_B \times \lambda} \equiv (\pi_{\Lambda} \otimes \pi_{\Lambda_B} \otimes \pi_{\lambda})(M^{\chi}_{univ})$. Using (3) again, we then obtain

$$[(\Delta \otimes \text{id}) \otimes \Delta](u)M^\chi_{univ} = M^\chi_{univ}[(\Delta \otimes \text{id}) \otimes \Delta](u) \quad (6)$$

and hence

$$(Su_1 \otimes \mathbb{I} \otimes \mathbb{I})M^\chi_{univ}(u_2 \otimes \mathbb{I} \otimes \mathbb{I}) = (\mathbb{I} \otimes u_1 \otimes u_2)M^\chi_{univ}(\mathbb{I} \otimes Su_4 \otimes Su_3). \quad (7)$$

The latter formula ultimately turns into an appropriate analogon of (3),

$$\pi_{\Lambda}(Su_1)M^{\Lambda, \Lambda_B \times \lambda}\pi_{\Lambda}(u_2) = \pi_{\Lambda_B \otimes \lambda}(u_1)M^{\Lambda, \Lambda_B \times \lambda}\pi_{\Lambda_B \otimes \lambda}(Su_2), \quad (8)$$

once we invoke one of the fundamental properties of a Hopf algebra [16]: $\Delta \circ S = (S \otimes S) \circ \Delta^{cop}$, and use the standard definition of a tensor-product representation of a coalgebra, $\pi_{\Lambda_1 \otimes \Lambda_2} := (\pi_{\Lambda_1} \otimes \pi_{\Lambda_2}) \circ \Delta$. 
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