Analysis of Stochastic Switched Systems with Application to Networked Control Under Jamming Attacks
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Abstract—We investigate the stability problem for discrete-time stochastic switched linear systems under the specific scenarios where information about the switching patterns and the probability of switches are not available. Our analysis focuses on the average number of times each mode becomes active in the long run and, in particular, utilizes their lower- and upper-bounds. This setup is motivated by cyber security issues for networked control systems in the presence of packet losses due to malicious jamming attacks where the attacker’s strategy is not known a priori. We derive a sufficient condition for almost sure asymptotic stability of the switched systems which can be examined by solving a linear programming problem. Our approach exploits the dynamics of an equivalent system that describes the evolution of the switched system’s state at every few steps; the stability analysis may become less conservative by increasing the step size. The computational efficiency is further enhanced by exploiting the structure in the stability analysis problem, and we introduce an alternative linear programming problem that has fewer variables. We demonstrate the efficacy of our results by analyzing networked control problems where communication channels face random packet losses as well as jamming attacks.

Index Terms—Stochastic switched systems, stability analysis, linear programming, networked control systems, jamming, packet losses.

I. INTRODUCTION

In the recent studies of hybrid systems, switched systems represent an important and fundamental class due to their simple structures. Switched systems are composed of a number of subsystems that possess different continuous dynamics and a discrete-valued switching mode signal which determines the active subsystem. Complicated behaviors in the state evolutions can be demonstrated, depending on the nature of the switching as well as the dynamics of the subsystems. For this reason, analyses of switched systems concerning their stability and performance have posed various challenges and led researchers to interesting results [1], [2].

Recently, the importance of this class of hybrid systems is rising from the application side. In particular, in large-scale networked control systems, switchings in the system dynamics frequently take place whenever the status in the communication networks changes. In such systems, network channels connect the plant having many sensors and actuators with remote controllers. Thus, transmission times and patterns of control-related signals affect the dynamics of the plant as well as the overall closed-loop system. Furthermore, the communication is often unreliable in that the transmitted data may become lost, not reaching the destination depending on the condition of the channels. Since such channel behaviors are commonly modeled in a probabilistic manner, the study of networked control systems often call for the framework of stochastic switched systems.

Stability of switched systems has been studied for different types of mode signals. The case of (deterministic) arbitrary switching has been explored in a number of works including [3], [4], establishing conditions under which a switched system remains stable for all possible mode switching scenarios. The switching frequency in the mode signal can be restricted by utilizing dwell-time and average dwell-time notions. The works [5], [6] dealt with systems composed of only stable subsystems while [7], [8] made extensions to systems also consisting of unstable subsystems. In addition, [9] investigated the problem of designing state-dependent switching rules to guarantee stability.

For stochastic mode signals, stability issues of switched systems have attracted considerable attention as well (see [10], [11] and the references therein). In most cases, stability results for such systems rely on statistical information on the mode signal such as the probability of mode switches and the stationary distributions associated with the modes. An important class there is that of Markov jump systems, for which the mode signal is dominated by underlying Markov chains [12]–[16]. Moreover, some works characterized both stochastic and deterministic effects in the switching of the mode signal, referred to as “dual switching” [17], [18].

In this paper, we investigate a stability problem for discrete-time stochastic switched linear systems with a special emphasis on the case where information about the mode switching probabilities or the stationary distributions are not available for analysis. Our interest stems from the current research activities on cyber security of networked control systems [19]–[21]. Today, more control systems are connected to the Internet and wireless networks for their remote operation and monitoring. Such communication settings significantly increase the risk to be targeted by malicious cyber attackers. Clearly, the system dynamics can change, for example, if attackers interfere with the communication of the control related signals. Under such conditions, the networked system may be represented as a stochastic switched system, but the a priori knowledge on the switching, whether it is deterministic or probabilistic, would be extremely limited. In the networked control literature, recent works dealt with denial-of-service (DoS) and jamming attacks [22]–[24] or packet drops by compromised routers [25], [26].

Our problem formulation is based on the approach in our
In this section we first describe the dynamics of a stochastic switched system. We then discuss the stability problem and provide sufficient almost sure asymptotic stability conditions.
A. Switched System Dynamics

Consider the discrete-time switched linear system with \( M \in \mathbb{N} \) modes described by
\[
x(t+1) = A_r(t)x(t), \quad x(0) = x_0, r(0) = r_0, t \in \mathbb{N}_0,
\]
where \( x(t) \in \mathbb{R}^n \) denotes the state vector, \( \{r(t) \in \{1, \ldots, M\}\} \in \mathbb{N}_0 \) is the mode signal, and \( A_r \in \mathbb{R}^{n \times n} \), \( s \in \{1, \ldots, M\} \), represent the system matrices for each mode. We use \( \mathcal{M} \triangleq \{1, \ldots, M\} \) to denote the set of modes.

The mode signal \( \{r(t) \in \mathcal{M}\} \in \mathbb{N}_0 \) is assumed to be a stochastic process that satisfies the following assumption.

Assumption 2.1: There exist scalars \( \underline{\rho}_s, \overline{\rho}_s \in [0, 1], s \in \mathcal{M} \), \( \liminf_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} 1[r(t) = s] \geq \underline{\rho}_s \), \( \limsup_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} 1[r(t) = s] \leq \overline{\rho}_s \), almost surely.

In Assumption 2.1, the scalars \( \underline{\rho}_s \) and \( \overline{\rho}_s \) respectively represent lower- and upper-bounds on the long-run average number of times mode \( s \) is active. If no information is available on the long-run average for mode \( s \), the scalars \( \underline{\rho}_s \) and \( \overline{\rho}_s \) can be selected as \( \underline{\rho}_s = 0 \) and \( \overline{\rho}_s = 1 \), since \( \underline{\rho}_s \) and \( \overline{\rho}_s \) are trivially satisfied with those values for all \( s \in \mathcal{M} \).

Our main motivation for considering Assumption 2.1 is to analyze switched systems for which precise information about the mode switching rules is not available for analysis. In Section II-B we show that the scalars \( \underline{\rho}_s, \overline{\rho}_s \in \mathcal{M} \), can be used for analysis, even if we do not know how the mode may switch at each time \( t \).

Assumption 2.1 allows the mode signal \( \{r(t) \in \mathcal{M}\} \in \mathbb{N}_0 \) to be generated in many different ways either randomly according to a probability distribution or in a deterministic fashion. For instance, \( \{r(t) \in \mathcal{M}\} \in \mathbb{N}_0 \) may be a stationary and ergodic stochastic process with stationary distribution \( \pi \in [0, 1]^\mathcal{M} \). In that case \( \underline{\rho}_s \) and \( \overline{\rho}_s \) would be scalars that satisfy \( \underline{\rho}_s \leq \pi_s \leq \overline{\rho}_s \), \( s \in \mathcal{M} \). On the other hand, \( \{r(t) \in \mathcal{M}\} \in \mathbb{N}_0 \) may also represent a deterministically generated switching sequence.

Remark 2.1: In the literature of stochastic switched systems the mode signal is typically characterized as a Markov chain [13]. In this paper, \( \{r(t) \in \mathcal{M}\} \in \mathbb{N}_0 \) is not necessarily a Markov chain. In fact in certain cases, we may have
\[
\mathbb{P}[r(t+1) = q|r(0), r(1), \ldots, r(t)] \neq \mathbb{P}[r(t+1) = q|r(t)],
\]
which indicates that \( r(\cdot) \) fails to satisfy the Markov property [39]. Furthermore, \( r(\cdot) \) may also depend on other processes.

B. Stability Analysis

In this section, we explore the stability of the switched system (1), where the mode signal satisfies Assumption 2.1. We use the stochastic stability notion of almost sure asymptotic stability in our analysis.

Definition 2.2: The zero solution \( x(t) \equiv 0 \) of the stochastic system (1) is almost surely stable if for each \( \epsilon > 0 \) and \( \overline{\rho} > 0 \), there exists \( \delta = \delta(\epsilon, \overline{\rho}) > 0 \) such that \( \|x_0\|_2 < \delta \), then
\[
\mathbb{P}[\sup_{t \in \mathbb{N}_0} \|x(t)\|_2 > \epsilon] < \overline{\rho},
\]
where \( \| \cdot \|_2 \) denotes the Euclidean norm. Moreover, the zero solution \( x(t) \equiv 0 \) is asymptotically stable almost surely if it is almost surely stable and
\[
\mathbb{P}[\lim_{t \to \infty} \|x(t)\|_2 = 0] = 1.
\]

Stability of discrete-time switched systems have been investigated in many studies under different assumptions on the mode signal. For instance, in several works (see [2] and the references therein) researchers explore stability of systems with a form similar to (1) under arbitrary switching. A necessary condition is stability of each mode (i.e., \( A_1, A_2, \ldots, A_M \) need to be Schur matrices). In our problem setting we allow some of the modes to be unstable, and hence this approach is not applicable.

On the other hand, researchers also studied stability of systems similar to (1) for the case where \( r(\cdot) \) is a Markov process (see, e.g., [12], [13], [40]). The stability analysis in those studies relies on transition probabilities and stationary distributions associated with the Markov process that characterize the switching sequence. Note again that in our case, \( r(\cdot) \) need not be a Markov process. Furthermore, to account for the uncertainty in generation of the mode signal, we assume that statistical information concerning transition probabilities and stationary distributions is not available. Hence, the stability results reported in the above-mentioned literature are not applicable to the present problem.
In our stability analysis of the switched system, we follow the approach in [40, 42] and investigate the evolution of the system’s state at every \( h \in \mathbb{N} \) steps. First, let \( \mathcal{M}^h \) denote the set of sequences of length \( h \) with entries in \( \mathcal{M} \), that is,

\[
\mathcal{M}^h \triangleq \{(q_1, q_2, \ldots, q_h) : q_j \in \mathcal{M}, j \in \{1, \ldots, h\}\}.
\]

With this definition, \( q_i \) (ith entry of a sequence \( q \)) represents a mode in the set of modes \( \mathcal{M} \). Now, let \( \{\bar{r}(i) \in \mathcal{M}^h\}_{i \in \mathbb{N}_0} \) be a sequence-valued process defined by

\[
\bar{r}(i) \triangleq (r(ih), r(ih + 1), \ldots, r((i + 1)h - 1)), \ i \in \mathbb{N}_0. \quad (6)
\]

It then follows that the state evaluated at every \( h \) steps is described by

\[
x((i + 1)h) = \Gamma_q(i)x(ih), \quad i \in \mathbb{N}_0, \quad (7)
\]

where

\[
\Gamma_q \triangleq A_{q_1}A_{q_{h-1}} \cdots A_{q_1}, \quad q \in \mathcal{M}^h. \quad (8)
\]

The dynamical system (7) is a “lifted” switched system with \( \mathcal{M}^h \) number of modes. Each mode of this system is identified by a sequence of \( h \) numbers from \( \mathcal{M} \) representing the modes of the original switched system (1).

Now, let \( c_s : \mathcal{M}^h \to \{0, \ldots, h\} \) be defined by \( c_s(q) \triangleq \sum_{j=1}^{h} [q_j = s], q \in \mathcal{M}^h, s \in \mathcal{M} \). With this definition, the number of entries with value \( s \) in the sequence \( q \in \mathcal{M}^h \) is represented with \( c_s(q) \). Note that \( c_s \) satisfies

\[
\sum_{i=0}^{k-1} c_s(q) \mathbb{1}[\bar{r}(i) = q] = \sum_{i=0}^{k-1} \mathbb{1}[r(i) = s], \quad k \in \mathbb{N}, \quad (9)
\]

which establishes a key relation between the mode signal \( r(\cdot) \) and the sequence-valued process \( \bar{r}(\cdot) \).

In Lemma 2.3 below, we use (9) to obtain a relation between \( \rho_s, \overline{\rho}_s \) in Assumption 2.1 and the long-run average numbers of the occurrences of all sequences in \( \mathcal{M}^h \). The long run average for a sequence \( q \in \mathcal{M}^h \) is given by

\[
\lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q],
\]

whenever this limit exists, that is, \( \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q] \) converges almost surely to a random variable as \( k \to \infty \).

**Lemma 2.3:** Suppose \( \{r(i) \in \mathcal{M}\}_{i \in \mathbb{N}_0} \) satisfies Assumption 2.1 with \( \rho_s, \overline{\rho}_s \in [0, 1], s \in \mathcal{M} \). If \( \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q] \) exists for each \( q \in \mathcal{M}^h \), then we have

\[
\sum_{q \in \mathcal{M}^h} \frac{c_s(q)}{h} \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q] \geq \rho_s, \quad (10)
\]

\[
\sum_{q \in \mathcal{M}^h} \frac{c_s(q)}{h} \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q] \leq \overline{\rho}_s, \quad (11)
\]

for \( s \in \mathcal{M} \), almost surely.

**Proof:** We first show (11). By (9),

\[
\sum_{q \in \mathcal{M}^h} \frac{c_s(q)}{h} \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q] = \lim_{k \to \infty} \frac{1}{kh} \sum_{i=0}^{k-1} c_s(q) \mathbb{1}[\bar{r}(i) = q] = \lim_{k \to \infty} \frac{1}{kh} \sum_{i=0}^{k-1} \mathbb{1}[r(i) = s]. \quad (12)
\]

Here, we have

\[
\left\{ \frac{1}{kh} \sum_{i=0}^{k-1} \mathbb{1}[r(i) = s] : \bar{k} \geq k \right\} 
\subset \left\{ \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[r(i) = s] : \bar{k} \geq k \right\}, \quad k \in \mathbb{N}, \quad (13)
\]

and hence

\[
\sup_{\bar{k} \geq k} \frac{1}{kh} \sum_{i=0}^{k-1} \mathbb{1}[r(i) = s] \leq \sup_{\bar{k} \geq k} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[r(i) = s], \quad k \in \mathbb{N}. \quad (14)
\]

Therefore,

\[
\limsup_{k \to \infty} \frac{1}{kh} \sum_{i=0}^{k-1} \mathbb{1}[r(i) = s] \leq \limsup_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[r(i) = s]. \quad (15)
\]

As a result, (11) follows from (13), (12), and (15).

The inequality (10) can be shown using a similar approach.

Next, we employ Lemma 2.3 to establish sufficient conditions for almost sure asymptotic stability. To this end, first, for a given matrix \( N \in \mathbb{R}^{n \times n} \), let \( ||N|| \) denote the induced matrix norm defined by

\[
||N|| \triangleq \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{||Nx||}{||x||}, \quad (16)
\]

where \( || \cdot \| \) on the right-hand side denotes a vector norm in \( \mathbb{R}^n \).

In the proof of the next result, we use the submultiplicativity property of induced matrix norms, i.e., \( ||N_1N_2|| \leq ||N_1|| ||N_2|| \) for \( N_1, N_2 \in \mathbb{R}^{n \times n} \) (see Section 5.6 in [50]).

**Theorem 2.4:** Consider the switched system (1). Suppose that the mode signal \( \{r(t) \in \{1, \ldots, M\}\}_{t \in \mathbb{N}_0} \) satisfies Assumption 2.1 with \( \overline{\rho}_s, \rho_s \in [0, 1], s \in \mathcal{M} \), and \( \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q] \) exists for each \( q \in \mathcal{M}^h \) for a given \( h \in \mathbb{N} \). If there exist an induced matrix norm \( || \cdot \| \) and a scalar \( \varepsilon \in (0, 1) \) such that the inequality

\[
\sum_{q \in \mathcal{M}^h} \gamma_q \rho_q < 0, \quad (17)
\]

holds with

\[
\gamma_q \triangleq \begin{cases} \ln ||\Gamma_q||, & \Gamma_q \neq 0, \\ \ln \varepsilon, & \Gamma_q = 0, \end{cases} \quad q \in \mathcal{M}^h, \quad (18)
\]
for all $\rho_q \in [0, 1], q \in \mathcal{M}^h$, that satisfy
\[
\sum_{q \in \mathcal{M}^h} \rho_q = 1, \tag{19}
\]
\[
P_s \leq \sum_{q \in \mathcal{M}^h} \frac{c_s(q)}{h} \rho_q \leq \mathcal{P}_s, \quad s \in \mathcal{M}, \tag{20}
\]
then the zero solution $x(t) \equiv 0$ of the dynamical system is asymptotically stable almost surely.

**Proof:** First, it follows from (1) that
\[
\|x((k+1)h)| = \|\Gamma_{\bar{r}}(x(kh)) \leq \|\Gamma_{\bar{r}}(x(h))\|
\]
and hence by the submultiplicativity property of the induced matrix norm $\|\cdot\|$, we have
\[
\|x(kh)\| \leq \eta(k)\|x_0\|, \quad k \in \mathbb{N}_0, \tag{21}
\]
where $\eta(k) \triangleq \prod_{i=0}^{k-1} \|\Gamma_{\bar{r}}(i)\|$. Now we define $\mu(k) \triangleq \sum_{i=0}^{k-1} \gamma(i), k \in \mathbb{N}_0$, where $\gamma(i), \rho \in \mathcal{M}^h$, are given by (18). It follows from (18) together with the definitions of $\eta(k)$ and $\mu(k)$ that $\eta(k) \leq e^{\mu(k)}, k \in \mathbb{N}_0$. Furthermore, since $\gamma(i) = \sum_{q \in \mathcal{M}^h} \gamma_q \mathbb{I}[\bar{r}(i) = q]$, we have
\[
\mu(k) = \sum_{q \in \mathcal{M}^h} \sum_{i=0}^{k-1} \gamma_q \mathbb{I}[\bar{r}(i) = q] = \sum_{q \in \mathcal{M}^h} \gamma_q \sum_{i=0}^{k-1} \mathbb{I}[\bar{r}(i) = q],
\]
for $k \in \mathbb{N}$, and as a result,
\[
\lim_{k \to \infty} \frac{1}{k} \mu(k) = \sum_{q \in \mathcal{M}^h} \gamma_q \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{I}[\bar{r}(i) = q], \tag{22}
\]
almost surely. Here, note that $\lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{I}[\bar{r}(i) = q] \in [0, 1]$. Furthermore,
\[
\sum_{q \in \mathcal{M}^h} \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{I}[\bar{r}(i) = q] = \lim_{k \to \infty} \frac{1}{k} \sum_{q \in \mathcal{M}^h} \sum_{i=0}^{k-1} \mathbb{I}[\bar{r}(i) = q] = \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} 1 = 1.
\]
\[
\tag{23}
\]
Let $\rho_q^* \triangleq \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{I}[\bar{r}(i) = q], q \in \mathcal{M}^h$, and
\[
\vartheta \triangleq \max \left\{ \sum_{q \in \mathcal{M}^h} \gamma_q \rho_q^* : \rho_q \in [0, 1], q \in \mathcal{M}^h, \text{s.t. (19), (20)} \right\}.
\]
Furthermore, let $E, F \subset \mathcal{F}$ be the events defined by
\[
E \triangleq \{ \omega \in \Omega : \sum_{q \in \mathcal{M}^h} \rho_q^* = 1, \quad \rho_s \leq \sum_{q \in \mathcal{M}^h} \frac{c_s(q)}{h} \rho_q^* \leq \mathcal{P}_s, \quad s \in \mathcal{M} \},
\]
\[
F \triangleq \{ \omega \in \Omega : \sum_{q \in \mathcal{M}^h} \gamma_q \rho_q^* \leq \vartheta \}.
\]
We first show that $\mathbb{P}[E] = 1$. Observe that (23) implies (19) with $\rho_q$ replaced by $\rho_q^*$. Moreover, Lemma 2.3 implies that (20) with $\rho_q$ replaced by $\rho_q^*$ holds almost surely. Hence, we have $\mathbb{P}[E] = 1$. Now, since $E \subseteq F$, we also have $\mathbb{P}[F] = 1$.

Furthermore, (17) implies that $\vartheta < 0$. As a result, by noting that $\mathbb{P}[F] = 1$, we obtain $\sum_{q \in \mathcal{M}^h} \gamma_q \rho_q^* \leq \vartheta < 0$, almost surely. We use this fact together with (22) to obtain
\[
\lim_{k \to \infty} \frac{1}{k} \mu(k) = \sum_{q \in \mathcal{M}^h} \gamma_q \rho_q^* \leq \vartheta < 0,
\]
almost surely. Here, (23) implies $\lim_{k \to \infty} \mu(k) = -\infty$, almost surely. As a result, by noting that $\eta(k) \leq e^{\mu(k)}$, we obtain $\mathbb{P}[\lim_{k \to \infty} \eta(k) = 0] = 1$. Thus, for any $\epsilon > 0$, $\lim_{j \to \infty} \mathbb{P}[\sup_{k \geq j} \eta(k) > \epsilon] = 0$ (see Proposition 5.6 of [51]). Therefore, for any $\epsilon > 0$ and $\bar{p} > 0$, there exists a positive integer $N(\epsilon, \bar{p})$ such that
\[
\mathbb{P}[\sup_{k \geq j} \eta(k) > \epsilon] < \bar{p}, \quad j \geq N(\epsilon, \bar{p}). \tag{25}
\]
In what follows, we show almost sure stability of the switched system by using (21) and (25). First, we define
\[
\phi \triangleq \max \{ 1, \max_{s \in \mathcal{M}} \|A_s\| \} \tag{26}
\]
and $T_k \triangleq \{kh, \ldots, (k+1)h - 1\}, \quad k \in \mathbb{N}_0$. Using these definitions, we obtain
\[
\|x(t+1)\| = \|A_{\bar{r}}(t)x(t)\| \leq \|A_{\bar{r}}(t)\| \|x(t)\| \leq \phi \|x(t)\|, \quad t \in T_k.
\]
It then follows from (27) that $\|x(t)\| \leq \phi^{t-kh}\|x(kh)\|$, $t \in T_k$. Since $T_k$ has $h$ time instants and $\phi \geq 1$, we have $\phi^{t-kh} \leq \phi^{h-1} \leq \phi^h$ and hence $\|x(t)\| \leq \phi^h \|x(kh)\|$ for all $t \in T_k$.

Consequently,
\[
\max_{t \in T_k} \|x(t)\| \leq \phi^h \|x(kh)\|, \quad k \in \mathbb{N}_0. \tag{28}
\]
Now by (21) and (28),
\[
\eta(k) \geq \|x(kh)\| \|x_0\|^{-1} \geq \max_{t \in T_k} \|x(t)\| \phi^{-h} \|x_0\|^{-1}, \quad k \in \mathbb{N}_0.
\]
Then it follows from (25) that for all $\epsilon > 0$ and $\bar{p} > 0$,
\[
\mathbb{P}[\sup_{k \geq j} \max_{t \in T_k} \|x(t)\| > \epsilon \phi^h \|x_0\|^{-1}] = \mathbb{P}[\sup_{k \geq j} \max_{t \in T_k} \|x(t)\| \phi^{-h} \|x_0\|^{-1}] > \epsilon] \leq \mathbb{P}[\sup_{k \geq j} (\eta(k) > \epsilon)] < \bar{p}, \quad j \geq N(\epsilon, \bar{p}). \tag{29}
\]
Now let $\delta_1 \triangleq \phi^{-h}$. Notice that if $\|x_0\| \leq \delta_1$, then $\phi^h \|x_0\| \leq 1$, and therefore, for all $j \geq N(\epsilon, \bar{p})$, we have
\[
\mathbb{P}[\sup_{k \geq j} \max_{t \in T_k} \|x(t)\| > \epsilon] \leq \mathbb{P}[\sup_{k \geq j} \max_{t \in T_k} \|x(t)\| > \epsilon \phi^h \|x_0\|^{-1}] < \bar{p}. \tag{29}
\]
Furthermore, observe that for all $k \in \{0, 1, \ldots, N(\epsilon, \bar{p}) - 1\}$, we have $\|x(kh)\| \leq \phi^k \|x_0\| \leq \phi^{N(\epsilon, \bar{p}) - 1} \|x_0\|$. Hence, as a result of (28),
\[
\max_{t \in T_k} \|x(t)\| \leq \phi^h \|x(kh)\| \leq \phi^h \phi^{N(\epsilon, \bar{p}) - 1} \|x_0\|, \tag{30}
\]
for all $k \in \{0, 1, \ldots, N(\epsilon, \bar{p}) - 1\}$. Let $\delta_2(\epsilon, \bar{p}) \triangleq \epsilon \phi^{3-h-N(\epsilon, \bar{p})}$. Now, if $\|x_0\| \leq \delta_2(\epsilon, \bar{p})$, then by (30),
max_{t \in T_k} \| x(t) \| \leq \epsilon, k \in \{0, 1, \ldots, N(\epsilon, \bar{p}) - 1\}. Thus, if 
\| x_0 \| \leq \delta_2(\epsilon, \bar{p})$, then

\[
P\left[ \max_{k \in \{0, 1, \ldots, N(\epsilon, \bar{p})\}} \max_{t \in T_k} \| x(t) \| > \epsilon \right] = 0. \quad (31)
\]

Due to (29) and (31), for all $\epsilon > 0, \bar{p} > 0$, we have

\[
P[\sup_{t \in T_0} \| x(t) \| > \epsilon] = P[\sup_{k \in \{0, 1, \ldots, N(\epsilon, \bar{p})-1\}} \max_{t \in T_k} \| x(t) \| > \epsilon] 
= P\left\{ \max_{k \in \{0, 1, \ldots, N(\epsilon, \bar{p})-1\}} \max_{t \in T_k} \| x(t) \| > \epsilon \right\} 
\cup \left\{ \sup_{k \geq N(\epsilon, \bar{p})} \max_{t \in T_k} \| x(t) \| > \epsilon \right\} 
\leq P\left[ \max_{k \in \{0, 1, \ldots, N(\epsilon, \bar{p})-1\}} \max_{t \in T_k} \| x(t) \| > \epsilon \right] 
+ P\left[ \sup_{k \geq N(\epsilon, \bar{p})} \max_{t \in T_k} \| x(t) \| > \epsilon \right] < \epsilon < \bar{p}, \quad (32)
\]

whenever $\| x_0 \| < \min(\delta_1, \delta_2(\epsilon, \bar{p}))$.

Now, by Corollary 5.4.5 of [50], there exist $c_1, c_2 > 0$ such that

\[
c_1 \| x \| \leq \| x \|_2 \leq c_2 \| x \|, \quad x \in \mathbb{R}^n. \quad (33)
\]

By (32) and (33), we obtain that for all $\epsilon > 0, \bar{p} > 0$, 

\[
P[\sup_{t \in T_0} \| x(t) \|_2 > \epsilon] \leq P[\sup_{t \in T_0} \| x(t) \| > \epsilon \| \| x \|_2 < \epsilon] < \epsilon \leq \frac{c_1}{c_2} < \bar{p},
\]

whenever $\| x_0 \| < \min(\delta_1, \delta_2(\epsilon, \bar{p}))$. Now, since $\| x_0 \| \leq \| x_0 \|_2$, we have that for all $\epsilon > 0, \bar{p} > 0$, the inequality (41) holds whenever $\| x_0 \|_2 < \epsilon(\bar{p}, \bar{p}) \leq c_1 \min(\delta_1, \delta_2(\epsilon, \bar{p}))$, which implies almost sure stability.

Next, we show (5) to establish almost sure asymptotic stability of the zero solution. In this regard, first notice that $P[\lim_{k \to \infty} \eta(k) = 0] = 1$. By using (21), we obtain $P[\lim_{k \to \infty} \| x(kh) \| = 0] = 1$, which implies $P[\lim_{k \to \infty} \| x(t) \| = 0] = 1$. Now as a consequence of (33), we have (5). Hence the zero solution of the switched system (1) is asymptotically stable almost surely.

Theorem 2.4 provides an almost sure asymptotic stability condition for the switched system (1). This result indicates that the stability can be assessed by checking the inequality (17) for all scalars $\rho_q \in [0, 1], q \in M^h$, such that (19), (20) hold. In (17), the scalar $\gamma_q \in \mathbb{R}$ represents the effect of mode sequence $q$. Specifically, for mode sequences with $\| \Gamma_q \| < 1$, we have $\gamma_q < 0$. A negative value for $\gamma_q$ implies that the norm of the system’s state gets smaller after $\rho_q$ steps, if the mode within those $\rho_q$ steps follows the sequence $q$. Note that for the case $\Gamma_q = 0, \epsilon \in (0, 1)$ in (18) ensures that $\gamma_q$ is well defined and negative. In practice, $\epsilon$ can be selected as a very small positive number. On the other hand, for mode sequences with $\| \Gamma_q \| > 1$, we have $\gamma_q > 0$. A positive value for $\gamma_q$ indicates that the mode sequence $q$ may cause the norm of the system’s state to increase. Hence, the term $\sum_{q \in M^h} \gamma_q \rho_q \bar{p}$ in (17) with $\rho_q = \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} 1[\tilde{r}(i) = q]$ would correspond to the average of the effects of all $h$-length sequences in $M^h$. However, this average cannot be computed directly, since in this paper, we consider the case where the specific values of $\lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} 1[\tilde{r}(i) = q]$ are not available.

On the other hand, we show by using Lemma 2.5 that if the long-run average activity of modes is known to be bounded as in (2) and (3), then $\rho_q = \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} 1[\tilde{r}(i) = q]$ would satisfy (19), (20). Hence, for stability analysis, one can check the sign of $\sum_{q \in M^h} \gamma_q \rho_q$ in (17) for all $\rho_q \in [0, 1], q \in M^h$, that satisfy (19), (20). This is equivalent to checking the stability for all possible mode sequence scenarios that satisfy (2) and (3), since different values of the limits $\rho_q = \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} 1[\tilde{r}(i) = q], q \in M^h$, represent different scenarios. We will show in Section 3 that rather than checking the condition in (17) for all possible scenarios, we can utilize linear programming methods to identify the worst scenario in terms of stability, and check the condition only for that scenario.

Note that in Theorem 2.4 we require the existence of $\lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} 1[\tilde{r}(i) = q]$ for all $q \in M^h$, even though the particular values of these limits are not needed for stability analysis. The following result identifies a class of mode signals $\{r(t) \in \{0, 1\}\}_{t \in \mathbb{N}_0}$ for which these limits exist. Using this result, we will show that our analysis technique is applicable in a variety of scenarios.

**Proposition 2.5:** Let $\{g(t) \in S\}_{t \in \mathbb{N}_0}$ with $g(0) = g_0 \in S$ be a finite-state irreducible Markov chain. Assume $\{r(t) \in M\}_{t \in \mathbb{N}_0}$ is given by

\[
r(t) \triangleq \begin{cases} 
1, & g(t) \in S_1; \\
\vdots & t \in \mathbb{N}_0, \\
M, & g(t) \in S_M,
\end{cases}
\]

where $S_1, S_2, \ldots, S_M$ form a partition of the set $S$, i.e., $\bigcup_{M \in \mathbb{N}} S_i = S$ and $S_i \cap S_j = \emptyset, i \neq j$. Then for all $h \in \mathbb{N}$, the limits $\lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} 1[\tilde{r}(i) = q], q \in M^h$, exist.

The proof of this result is composed of several key steps. The first step is based on the observation that $\tilde{r}(\cdot)$ is generated from sequences of values that the process $r(\cdot)$ takes between every $h$ time steps. By exploiting this observation, we construct a new process $\bar{g}(\cdot)$ representing the sequences of values that $g(\cdot)$ takes between every $d$ steps, where $d$ is a carefully chosen period length that is an integer multiple of $h$. In the second step, we establish the relation between the processes $\tilde{r}(\cdot)$ and $\bar{g}(\cdot)$ by using (34). Then, in the final step, we show that $\lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} 1[\tilde{r}(i) = q]$ can be obtained by utilizing invariant distributions of $\bar{g}(\cdot)$.

**Proof:** In the proof, we use the notion of period for Markov chains [52]. Specifically, the period $\tau_\sigma \in \mathbb{N}$ of a state $\sigma \in S$ is defined by

\[
\tau_\sigma \triangleq \gcd\{t \in \mathbb{N} : P[g(t) = \sigma | g(0) = \sigma] > 0\}, \quad \sigma \in S,
\]

where $\gcd(T)$ denotes the greatest common denominator of the elements of the set $T$. By this definition, the random time intervals between revisits to state $\sigma$ are guaranteed to be integer multiples of $\tau_\sigma$. Since $\{g(t) \in S\}_{t \in \mathbb{N}_0}$ is an irreducible finite-state Markov chain, it follows from Corollary 8.3.7 of [2] that the period is the same for all states. We use $\tau \in \mathbb{N}$ to denote this period, i.e., $\tau = \tau_1 = \tau_2 = \cdots = \tau_\sigma$, where $|S|$ denotes the number of elements in the set $S$. Now let $d \triangleq \tau h$.

Next, we define a sequence-valued process to characterize the evolution of $g(\cdot)$ in every $d$ steps. To this end, first, for each $\sigma \in S$, let $I_{\sigma,k} \triangleq \{s \in S : P[g(kd) = s | g(0) = \sigma] > 0\}$,
Now we define the sequence-valued process \( \{ \bar{g}(i) \} \) by
\[
\bar{g}(i) \triangleq (g(id), g(id + 1), \ldots, g((i + 1)d - 1)).
\]
(35)

Notice that \( \bar{g}(i) \in \tilde{S}_{\sigma_0}, i \in \mathbb{N}_0 \).

Our next goal is to show that the sequence-valued Markov chain \( \{ \bar{g}(i) \} \) is an irreducible Markov chain. Specifically, we prove that for every \( \sigma, \bar{s} \in \tilde{S}_{\sigma_0} \), there exists \( \bar{k} \in \mathbb{N} \) such that
\[
\mathbb{P}[\bar{g}(i + \bar{k}) = \bar{s} | \bar{g}(i) = \sigma] > 0.
\]
(36)

To this end, first note that \( \bar{g}_1(i) \in \mathcal{I}_{\sigma_0} \), \( i \in \mathbb{N}_0 \), that is, the first elements of the sequence-values that \( \bar{g}(\cdot) \) takes are elements of the set \( \mathcal{I}_{\sigma_0} \). It follows from the definition of \( \mathcal{I}_{\sigma_0} \), that for all \( \sigma, \bar{s} \in \mathcal{I}_{\sigma_0} \),
\[
\{ k \in \mathbb{N} : \mathbb{P}[g(kd) = s | g(0) = \sigma] > 0 \} \neq \emptyset.
\]

Now, define \( k : \mathcal{I}_{\sigma_0} \times \mathcal{I}_{\sigma_0} \to \mathbb{N} \) by
\[
k(\sigma, \bar{s}) \triangleq \min\{ k \in \mathbb{N} : \mathbb{P}[g(kd) = s | g(0) = \sigma] > 0 \}.
\]
(37)

Moreover, note that for any given \( \bar{\sigma}, \bar{s} \in \tilde{S}_{\sigma_0} \), we can always pick a state \( c \in \mathcal{I}_{\sigma_0} \) and let \( \bar{k} \triangleq k(c, \bar{s}_1) + 1 \) so that
\[
\mathbb{P}[g(1) = c | g(0) = \bar{s}_d] > 0.
\]
By (37), we have \( \mathbb{P}[g(k(c, \bar{s}_1)d) = \bar{s}_1 | g(0) = c] > 0 \), and consequently
\[
\mathbb{P}[\bar{g}(i + \bar{k}) = \bar{s} | \bar{g}(i) = \bar{\sigma}] = \mathbb{P}[\bar{g}(i + \bar{k}) = \bar{s} | g((i + 1)d - 1) = \bar{\sigma}_d] = \mathbb{P}[g((i + \bar{k})d + 1) = \bar{s}_2, \ldots, g((i + \bar{k})d + 1) = \bar{s}_d, \ldots, g((i + \bar{k})d + 1) = \bar{s}_1] = \mathbb{P}[g((i + \bar{k})d + 1) = \bar{s}_d, g((i + \bar{k})d + 1) = \bar{s}_1]
\]
\[
\geq \mathbb{P}[g((i + \bar{k})d + 1) = \bar{s}_2, \ldots, g((i + \bar{k})d + 1) = \bar{s}_d, \ldots, g((i + \bar{k})d + 1) = \bar{s}_1 | g((i + 1)d - 1) = \bar{\sigma}_d] = \mathbb{P}[g((i + \bar{k})d + 1) = \bar{s}_2, \ldots, g((i + \bar{k})d + 1) = \bar{s}_d, \ldots, g((i + \bar{k})d + 1) = \bar{s}_1 | g((i + 1)d - 1) = \bar{\sigma}_d] = \mathbb{P}[g((i + \bar{k})d + 1) = \bar{s}_2, \ldots, g((i + \bar{k})d + 1) = \bar{s}_d, \ldots, g((i + \bar{k})d + 1) = \bar{s}_1 | g((i + 1)d - 1) = \bar{\sigma}_d] = \mathbb{P}[g((i + \bar{k})d + 1) = \bar{s}_2, \ldots, g((i + \bar{k})d + 1) = \bar{s}_d, \ldots, g((i + \bar{k})d + 1) = \bar{s}_1] = 1.
\]

Thus, the sequence-valued Markov chain \( \{ \bar{g}(i) \} \) is irreducible. Now, define the function \( \alpha : \tilde{S}_{\sigma_0} \times \mathcal{M}^h \to \mathbb{N}_0 \) by
\[
\alpha(\bar{s}, q) \triangleq \sum_{j=0}^{\tau - 1} 1[\bar{s}_{j+1} \in S_{q_1}, \ldots, \bar{s}_{j+h} \in S_{q_h}],
\]
(38)

for \( \bar{s} \in \tilde{S}_{\sigma_0}, q \in \mathcal{M}^h \). Note that \( \alpha(\bar{s}, q) \in \mathbb{N}_0 \) is the number of times the sequence \( q \) appears in the process \( r(\cdot) \), when the process \( g(\cdot) \) takes the values \( \bar{s}_1, \bar{s}_2, \ldots, \bar{s}_d \). This number is computed by dividing the \( d \)-length sequence \( \bar{s} \) into \( \tau \) number of \( h \)-length sequences and counting the number of \( h \)-length sequences whose elements are from sets \( S_{q_1}, S_{q_2}, \ldots, S_{q_h} \).
We would like to highlight that the stability results on Markov ordinary Markov jump systems, neither transition probabilities nor stationary distributions of the process \( g(\cdot) \) are available for analysis. In fact even the size of the set \( S \) may not be known. In particular, when we consider the application to networked control under jamming attacks (see Sections \[IV\] and \[V\]), the process \( \{ g(t) \in S \}_{t \in \mathbb{N}_0} \) may characterize a jamming attacker’s strategy and its properties are not available for analysis.

Furthermore, we remark that \((34)\) is only one of the possible characterizations of the mode signal for our analysis to be applicable. The mode signal may also be generated in other ways following different random or deterministic characterizations. Our stability analysis in Theorem \([2.4]\) relies on the bounds on the average number of times each mode is active in the long run (see Assumption \([2.1]\)) instead of the properties of particular mode signal characterizations.

The characterization through \((34)\) is general enough to model various types of mode signals. We present two examples in this regard.

**Example 1:** Periodic mode switchings can be described with an irreducible and periodic Markov chain \( \{ g(t) \in S \}_{t \in \mathbb{N}_0} \). Consider for example a switched system with 2 modes. The mode sequence is assumed to repeat itself in every 4 time steps. Specifically, in every 4 time steps, mode 1 is active for 1 time step then mode 2 becomes active for the next 3 time steps. This periodic switching scenario can be characterized by setting \( S_1 \triangleq \{ 1, 2, 3, 4 \} \), \( S_2 \triangleq \{ 2, 3, 4 \} \), and \( \{ g(t) \in S \}_{t \in \mathbb{N}_0} \) as a Markov chain with transition probabilities shown on the edges of the transition graph in Fig. 1. In this situation \( g(\cdot) \) repeatedly takes the values \( 1, 2, 3, 4, 1, 2, 3, 4, \ldots \). As a result, by the definition in \((34)\), the mode signal \( r(\cdot) \) takes the values \( 1, 2, 2, 1, 2, 2, \ldots \) indicating the periodic change in the mode.

In this example, the switched system can be used for modeling a networked control system under periodic attacks. In particular, the first mode corresponds to a successful packet exchange between the plant and the controller, and the second mode represents the dynamics when there is a transmission failure due to attacks. The characterization of the mode signal through the setup in Fig. 1 represents an example of the discrete-time version of the periodic attacks discussed in \([19]\). Here the attacker periodically repeats sleeping for 1 time step and emitting a jamming signal to block network transmissions for 3 consecutive time steps. It is important to note that when the networked control system is periodically attacked, the specific failure sequence and the period itself are part of attacker’s strategy and in general they are not available to the system operator. We consider a networked control problem that covers this case in Section \[IV\]. There, we show that to check networked control system’s stability through Theorem \([2.4]\) only the knowledge of the upper-bound on the average attack ratio is needed. For the periodic attack in Fig. 1 the upper-bound on this ratio is given by \( \overline{p}_2 \), since the second mode corresponds to the attacks. Notice that in this case we can select \( \overline{p}_1 = 1 - \overline{p}_2, \overline{p}_1 = 1, \) and \( \overline{p}_2 = 0 \).

**Example 2:** The characterization in \((34)\) can also be used to describe random packet transmission failures. For example, communication channels following the Markov model can be described simply by setting \( S_1 \triangleq \{ 1 \}, S_2 \triangleq \{ 2 \}, \) and \( \{ g(t) \in S = S_1 \cup S_2 \}_{t \in \mathbb{N}_0} \) as a Markov chain with certain transition probabilities. In addition, the Gilbert-Elliott model and other more advanced models based on Markov chains (see \([44, 45]\)) can also be described within the framework.

Note that when \( \{ r(t) \in M \}_{t \in \mathbb{N}_0} \) is characterized through \((34)\), the limits \( \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} 1 \{ r(i) = q \}, q \in M^h, \) exist for all \( h \in \mathbb{N} \). Hence, in such cases, the stability analysis in Theorem \([2.4]\) can be conducted with any \( h \in \mathbb{N} \). On the other hand, for other characterizations of \( \{ r(t) \in M \}_{t \in \mathbb{N}_0} \), it may be the case that the limits exist for \( h \in \{ 1, 2, \ldots, \tilde{h} \} \) but not for \( h > \tilde{h} \), where \( \tilde{h} \in \mathbb{N} \). In those situations, Theorem \([2.4]\) is applicable only for \( h \in \{ 1, 2, \ldots, \tilde{h} \} \).
Assumption 2.1 implies

III. LINEAR PROGRAMMING METHODS FOR STABILITY ASSESSMENT

In this section, we investigate two closely-related linear programming problems and present a method for checking the almost sure asymptotic stability condition given in Theorem 2.4 through their optimal solutions.

A. Linear Programming Problem 1

Theorem 2.4 states that the switched system (1) is stable if there exists an induced matrix norm \( \| \cdot \| \) and a scalar \( \varepsilon \in (0, 1) \) such that the inequality (17) holds for all \( \rho_q \in [0, 1], q \in M_h \), that satisfy (19), (20). In what follows, we provide a linear programming problem to check this condition for a given induced matrix norm \( \| \cdot \| \) and scalar \( \varepsilon \in (0, 1) \).

Now define \( \gamma_q \in \mathbb{R}, q \in M_h \), as in (13) and consider the linear programming problem

\[
\text{maximize} \quad \sum_{q \in M_h} \gamma_q \rho_q \\
\text{subject to} \quad (19), (20).
\]

For the stability analysis, different values of \( \rho_q, q \in M_h \), that satisfy (19), (20) represent possible mode activity scenarios such that the long run average conditions (2) and (3) hold. The linear programming problem (46) allows us to identify scenarios that maximize \( \sum_{q \in M_h} \gamma_q \rho_q \). We can then check the stability condition (17) with the maximum value of \( \sum_{q \in M_h} \gamma_q \rho_q \) instead of checking it for all possible scenarios.

In the following lemma, we show that the linear programming problem (46) is feasible, that is, there always exist \( \rho_q \in [0, 1], q \in M_h \), that satisfy (19), (20). Furthermore, we show that the problem is bounded (i.e., the objective function \( \sum_{q \in M_h} \gamma_q \rho_q \) in (46) is bounded).

**Lemma 3.1:** The linear programming problem (46) is feasible and bounded.

**Proof:** First, we show that the feasible region of the linear programming problem is not empty. To this end, first observe that \( \sum_{q \in M_h} \rho_q \leq 1 \leq \sum_{q \in M_h} \gamma_q \). This is because Assumption 2.1 implies

\[
1 = \lim \inf_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} \sum_{s=1}^{M} I[r(t) = s] \\
\leq \sum_{s=1}^{M} \lim \sup_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} I[r(t) = s] \leq \sum_{s=1}^{M} \gamma_q.
\]

and similarly,

\[
1 = \lim \inf_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} \sum_{s=1}^{M} I[r(t) = s] \\
\geq \sum_{s=1}^{M} \lim \sup_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} I[r(t) = s] \geq \sum_{s=1}^{M} \rho_q.
\]

Now let \( \rho_q = \sum_{s=1}^{M} \rho_q, \gamma_q = \sum_{s=1}^{M} \gamma_q, \text{ and } \beta_q = (\gamma_q - \rho_q) \frac{1 - \rho}{\beta - \rho} \). We can then define

\[
\rho_q = \begin{cases} \rho_q + \beta_q, & \text{if } c_q(q) = h, s \in M, \\ 0, & \text{otherwise}. \end{cases}
\]

To establish that the feasible region contains \( \rho_q, q \in M_h \), given by (47), we show that (19) and (20) hold. First, (19) holds because

\[
\sum_{q \in M_h} \rho_q = \sum_{s=1}^{M} \left( \rho_q + \beta_q \right) = \rho + \frac{1 - \rho}{\beta - \rho} \sum_{s=1}^{M} (\gamma_q - \rho_q) = \rho \left( 1 - \frac{1 - \rho}{\beta - \rho} \right) + \frac{1 - \rho}{\beta - \rho} = 1.
\]

Now we show (20). Since \( \bar{\gamma} \leq 1 \leq \bar{\rho} \) we have \( \beta_s \in [0, \bar{\gamma} - \rho_q] \). It then follows that

\[
\sum_{q \in M_h} c_s(q) \rho_q = \rho_q + \beta_s \in [0, \gamma_q], \quad s \in M.
\]

It now remains to show that the solutions to the linear programming problem are bounded. Note that since \( \rho_q \leq 1, q \in M_h \), we have \( \sum_{q \in M_h} \gamma_q \rho_q \leq (\max_{q \in M_h} \gamma_q) \sum_{q \in M_h} \rho_q \leq (\max_{q \in M_h} \gamma_q) M_h < \infty \), which completes the proof.

**Lemma 3.1** implies that there exists an optimal solution to the linear programming problem (46) (see Proposition 3.1 of [64]). Even though there may be multiple optimal solutions, we can always compute the optimal value of the objective function using any one of those solutions. Let \( J_h \) denote the optimal value of the objective function when \( h \)-length sequences are considered, that is,

\[
J_h \equiv \max \left\{ \sum_{q \in M_h} \gamma_q \rho_q : \rho_q \in [0, 1], q \in M, \text{ s.t. (19), (20)} \right\}.
\]

The stability of the switched system (1) can be assessed by checking the sign of the optimal value \( J_h \). Specifically, the zero solution \( x(t) \equiv 0 \) of the switched system (1) is asymptotically stable almost surely if

\[
J_h < 0.
\]

This is because (50) implies that (17) in Theorem 2.4 holds for all \( \rho_q \in [0, 1], q \in M_h \), that satisfy (19), (20).

**Remark 3.2:** The optimal solution \( J_h \) for the linear programming problem (46) may be positive when \( h \) is small and negative for sufficiently large \( h \). The reason is that with large \( h \), stability/instability properties of more mode activity patterns are taken into account. For instance, consider a switched system with two modes. When \( h = 2 \), the effects of the dynamics associated with packet failure sequences in \( M^2 \) \( \{1, 1\}, \{1, 2\}, \{2, 1\}, \{2, 2\} \) are represented by \( \gamma_q, q \in M^2 \).
However, $\gamma_q, q \in \mathcal{M}^2$, cannot be used to distinguish the difference between stabilizing (destabilizing) effects of longer mode activity sequences $(1, 2, 2, 1)$ and $(2, 1, 1, 2)$, since both of them are composed of the same smaller sequences $(1, 2)$, $(2, 1)$, $\text{etc}$. To avoid showing stability, we may need to take into account longer mode sequences and obtain $J_h$ for larger values of $h \in \mathbb{N}$. Let us note that if $\gamma_q$ associated with a mode sequence $q = (1, 2, 2, 1)$ may be negative, even though $\gamma_{(1,2)}$ and $\gamma_{(2,1)}$ associated with the smaller sequences $(1, 2), (2, 1)$ are positive. This is similar to the observation that a switched system with individually unstable modes may be stable if the switching is constrained in a certain way (see Chapter 2 of [3]).

Even though there are efficient algorithms for solving linear programming problems, it is difficult to solve (46) and obtain $J_h$ when $h \in \mathbb{N}$ is large. This is because the number of variables $q_h, q \in \mathcal{M}^h$, of the problem (46) grows exponentially in $h$. Specifically, the number of elements of the set $\mathcal{M}^h$, and hence the number of variables of the linear programming problem (46) is given by $h!$. In the following section, we show that an alternative linear programming problem with fewer variables shares the same optimal objective function value as that of (46). In particular, the number of variables in this alternative problem grows only polynomially in $h$.

### B. Linear Programming Problem 2

We observe in the linear programming problem (46) that due to the particular structure of our problem setting, some of the variables have the same coefficients in the constraints. In particular, if two (or more) mode sequences are reordered versions of each other, then the variables associated with those mode sequences have the same coefficients in the constraints. For instance, in the case with $M = 2, h = 2$, the constraints in (20) are $\rho_1, \leq \rho_{(1,1)} + 0.5 \rho_{(1,2)} + 0.5 \rho_{(2,1)} + 0.5 \rho_{(1,2)} \leq \mathcal{P}_1$, and $\rho_2, \leq \rho_{(1,1)} + 0.5 \rho_{(1,2)} + 0.5 \rho_{(2,1)} + 1 \rho_{(2,2)} \leq \mathcal{P}_2$, where the variables $\rho_{(1,1)}$ and $\rho_{(2,1)}$ have the same coefficients. For this example case, if $\max \{ \gamma_{(1,2)}, \gamma_{(2,1)} \} = \gamma_{(2,1)} > 0$, then it means that we can maximize the objective $\gamma_{(1,1)} \rho_{(1,1)} + \gamma_{(1,2)} \rho_{(1,2)} + \gamma_{(2,1)} \rho_{(2,1)} + \gamma_{(2,2)} \rho_{(2,2)}$ by choosing $\rho_{(2,2)}$ as large as we can and setting $\rho_{(1,2)}$ to 0. Then the variable $\rho_{(1,2)}$ can be removed, since the terms that involve $\rho_{(1,2)}$ are all 0. Similar techniques can be used for all $M \in \mathbb{N}$ and $h \in \mathbb{N}$ leading us to more variable reductions as $M$ and $h$ increase.

In particular, we obtain the following linear programming problem:

Let $Z_h \triangleq \{(z_1, z_2, \ldots, z_M) : z_s \in \{0, 1, \ldots, h\}, s \in \mathcal{M}, \sum_{s=1}^M z_s = h\}$, and consider the linear programming problem

\[
\begin{align*}
\text{maximize} & \quad \sum_{z \in Z_h} \gamma_z \rho_z' \\
\text{subject to} & \quad \sum_{z \in Z_h} \rho_z = 1, \\
& \quad \rho_z, \leq \frac{z_s \rho_z,}{h} \leq \mathcal{P}_s, s \in \mathcal{M}, (a) \quad (51)
\end{align*}
\]

where

\[
\gamma_z' \triangleq \max_{q \in \mathcal{M}^{h,z}} \gamma_q, \quad z \in Z_h,
\]

with $\mathcal{M}^{h,z} \triangleq \{ q \in \mathcal{M}^h : c_1(q) = z_1, c_2(q) = z_2, c_3(q) = z_3, \ldots, c_M(q) = z_M\}, z \in Z_h$.

In what follows, we first show that the objective functions of the linear programming problems (46) and (51) have the same optimal values. After that we discuss the advantage of the linear programming problem (51) over the problem (46). Specifically, we show that it is easier to solve the linear programming problem (51) because it involves fewer variables.

**Lemma 3.3:** The linear programming problem (51) is feasible and bounded.

**Proof:** The proof is similar to that of Lemma 3.1. First, we show that the feasible region of the linear programming problem (51) is not empty. To this end, consider $\rho_z', z \in Z_h$, given by

\[
\rho_z' = \sum_{q \in \mathcal{M}^{h,z}} \rho_q,
\]

with $\rho_q$ given in (47). Notice that $\rho_z' \in [0, 1], z \in Z_h$. This is because $\rho_q \geq 0, q \in \mathcal{M}^h$, implies $\rho_z' \geq 0$, and moreover, $\rho_z' = \sum_{q \in \mathcal{M}^{h,z}} \rho_q \leq \sum_{q \in \mathcal{M}^h} \rho_q = 1$ due to (48). To establish that the feasible region contains $\rho_z', z \in Z_h$, given by (53), we show that (51b) hold. Now, (51b) holds, because it follows from (48) that

\[
\sum_{z \in Z_h} \rho_z' = \sum_{z \in Z_h} \sum_{q \in \mathcal{M}^{h,z}} \rho_q = \sum_{q \in \mathcal{M}^h} \rho_q = 1,
\]

where we also used the fact that $\mathcal{M}^{h,z} \cap \mathcal{M}^{h,z} = \emptyset$ for $z \neq \hat{z}, z, \hat{z} \in Z_h$, and $\bigcup_{z \in Z_h} \mathcal{M}^{h,z} = \mathcal{M}^h$.

Next, to show (51a), note that $\sum_{z \in Z_h} \frac{z_s \rho_z'}{h} = \sum_{z \in Z_h} \sum_{q \in \mathcal{M}^{h,z}} \frac{z_s \rho_q}{h} = \sum_{q \in \mathcal{M}^h} \frac{c_q(q)}{h} \rho_q$. Hence, (49) implies (51a).

It now remains to show that the solutions to the linear programming problem are bounded. Note that since $\rho_z' \leq 1, z \in Z_h$, it follows from (52) that $\sum_{z \in Z_h} \gamma_z' \rho_z' \leq \max_{\gamma_q \in \mathcal{M}^h} \gamma_q \rho_q \leq (\max_{\gamma_q \in \mathcal{M}^h} \gamma_q) (h+M-1) < \infty$, which completes the proof. □

By Lemma 3.3 there exists an optimal solution to the linear programming problem (51). Let $J_h'$ denote the optimal value of the objective function for a given $h \in \mathbb{N}$, that is,

\[
J_h' \triangleq \max \left\{ \sum_{z \in Z_h} \gamma_z' \rho_z' : \rho_z', z \in Z_h, \text{s.t. } (51a), (51b) \right\}.
\]

**Lemma 3.4:** The objective functions of the linear programming problems (46) and (51) have the same optimal values, that is, $J_h = J_h'$.

**Proof:** We prove this result by showing $J_h \leq J_h'$ and $J_h \geq J_h'$ separately.

To establish $J_h \leq J_h'$, we show that for all $\rho_q \in [0, 1], q \in \mathcal{M}^h$, such that (19), (20) hold, we have $\sum_{q \in \mathcal{M}^h} \gamma_q \rho_q \leq J_h'$. Now, notice that for all $\rho_q \in [0, 1], q \in \mathcal{M}^h$, such that (19), (20) hold, the constraints in (51a) and (51b) hold with $\rho_z' \triangleq \gamma_z' \rho_z' \leq \mathcal{P}_s$. Hence, by Lemma 3.3, we have $J_h \leq J_h'$. □
$\sum_{q \in M^{h,z}} \rho_q z \in Z_h$, and as a result, \( \sum_{q \in Z_h} \gamma_q \rho_q z \leq J'_h \).

Hence, for all \( \rho_q, q \in M^h \), such that (19), (20) hold, we have

\[
\sum_{q \in M^h} \gamma_q \rho_q z = \sum_{z \in Z_h, q \in M^{h,z}} \gamma_q \rho_q \leq \sum_{z \in Z_h, q \in M^{h,z}} \gamma_q \rho_q = \sum_{z \in Z_h} \gamma_z \rho_z^* = J'_h,
\]

which implies \( J_h \leq J'_h \).

To prove \( J_h \geq J'_h \), we now show that there exists \( \rho_q \in [0, 1], q \in M^h \), such that \( \sum_{q \in M^h} \gamma_q \rho_q z = J'_h \) and (19), (20) hold. Here, let \( \rho^*_z, z \in Z_h \), denote an optimal solution to the linear programming problem \( J'_h \), that is,

\[
\sum_{z \in Z_h} \gamma_z \rho_z^* = J'_h.
\]

Now, for each \( z \in Z_h \), let \( q^{(z)} \in \arg\max_{q \in M^{h,z}} \gamma_q z \) and set \( \rho_{q^{(z)}} = \rho_z^* \), \( \rho_q = 0, q \notin M^{h,z} \). It follows that \( \rho_q, q \in M^h \), satisfy (19), (20); furthermore,

\[
\sum_{q \in M^h} \gamma_q \rho_q z = \sum_{z \in Z_h} \gamma_z \rho_z^* = J'_h.
\]

This establishes that there exist \( \rho_q \in [0, 1], q \in M^h \), such that \( \sum_{q \in M^h} \gamma_q \rho_q z = J'_h \) and (19), (20) hold, which implies that \( J_h \geq J'_h \).

A direct consequence of Lemma 3.4 is that if \( J'_h < 0 \), then the zero solution \( x(t) \equiv 0 \) of the switched system (1) is asymptotically stable almost surely.

So far we established that almost sure asymptotic stability of the switched system (1) can be assessed by checking signs of the optimal objective function values \( (J_h, J'_h) \) of linear programming problems \( \text{(46)} \) and \( \text{(51)} \). The switched system (1) is stable if the value of \( J_h = J'_h \) is negative.

We observe that solving the linear programming problem \( \text{(51)} \) can be computationally more advantageous in comparison to the problem \( \text{(46)} \). This is because \( \text{(51)} \) involves fewer variables. Specifically, the number of elements of the set \( Z_h \), and hence the number of variables of the linear programming problem \( \text{(51)} \), is \( f'(h, M) = (h^M - 1)/(M - 1) \). For \( h \) and \( M \) larger than 1, the number of variables in the problem \( \text{(51)} \) is strictly smaller than that of the problem \( \text{(46)} \), that is,

\[
f'(h, M) < f(h, M), \quad h > 1, \quad M > 1.
\]

We note again that \( f \) grows exponentially in \( h \). On the other hand \( f' \), the number of variables in the problem \( \text{(51)} \), grows only polynomially in \( h \). Specifically, we have

\[
f'(\alpha h, M) \leq \alpha^{M-1} f'(h, M), \quad \alpha, h, M \in \mathbb{N}.
\]

As a result, when \( h \) is large, obtaining \( J_h \) is much faster than obtaining \( J_h \). We also remark that the number of variables grows polynomially in the number of modes \( M \) as well. In particular, we have

\[
f'(h, \alpha M) \leq \alpha^h f'(h, M), \quad \alpha, h, M \in \mathbb{N}.
\]

Fig. 3 shows graphs of \( \ln f(h, M) \) and \( \ln f'(h, M) \) indicating how the numbers of variables in problems \( \text{(46)} \) and \( \text{(51)} \) grow with respect to \( h \) and \( M \). The difference becomes larger as \( M \) and \( h \) get larger. For instance, in the case where \( h = 15 \) and \( M = 3 \), there are \( f(15, 3) = 14,348,907 \) variables in the problem \( \text{(46)} \), whereas \( f'(15, 3) = 136 \) variables.

It is important to note that although solving the linear programming problem \( \text{(51)} \) is easier due to fewer variables, it requires precomputation of coefficients \( \gamma_q z \) by \( \text{(52)} \). Notice that, for each \( z \), the complexity of computing \( \gamma_q z \) is linear in the number of variables of the set \( M^{h,z} \), which is given by \( h^M - 1 \). It turns out that this computation can be carried out for all coefficients \( \gamma_z \) at the same time using parallel computing techniques. For instance, in the case of \( h = 15 \) and \( M = 3 \), the set \( Z_h \) of \( f'(15, 3) = 136 \) variables can be partitioned into 8 subsets with size 17. We can then utilize a computer with 8 central processing units to carry out the computation for each of these subsets. Specifically, the \( i \)th processing unit computes \( \gamma_q z \) for all \( z \) in the \( i \)th subset. Both linear programming problems \( \text{(46)} \) and \( \text{(51)} \) require calculation of norms of matrix products in the computation of \( \gamma_q z, q \in M^h \), given in (18). This calculation can also be conducted in parallel for different \( q \) values. We also note that using different matrix norms in the definition of \( \gamma_q z, q \in M^h \), can be useful to check stability in the case of limited computational resources. This is because \( J_h \) and \( J'_h \) may be positive for a particular matrix norm and negative for another.

We remark that for stability analysis, the sign of \( J_h \) and \( J'_h \) can also be assessed by solving linear feasibility problems without computing the actual values of those scalars. In particular, we have \( J_h \geq 0 \) if and only if there exist \( \rho_q \in [0, 1], q \in M^h \), such that (19), (20), and \( \sum_{q \in M^h} \gamma_q \rho_q z \geq 0 \) hold. Similarly, we have \( J'_h \geq 0 \) if and only if there exist \( \rho_z^* \in [0, 1], z \in Z_h \), such that (19), (51), and \( \sum_{z \in Z_h} \gamma_z^* \rho_z^* \geq 0 \) hold. Note that solving these feasibility problems is not necessarily faster in comparison to solving the associated linear programming problems, since the numbers of variables in these two feasibility problems are equal to those in the associated linear programming problems.

IV. APPLICATION TO NETWORKED CONTROL UNDER JAMMING ATTACKS

In this section we consider two problem settings where we model the networked control system as a switched system.
A. Control over Delay-Free Communication Links

First, we explore the networked control problem where at each time instant, the plant and the controller attempt to exchange state and control input packets over a communication channel. In this problem setting, network transmissions do not face delay, but packet exchange attempts between the plant and the controller may be subject to packet losses due to malicious jamming attacks or nonmalicious communication errors. In a successful packet exchange attempt, the plant transmits the state information to the controller; the controller uses the received state information to compute the control input through a linear control law and sends back the control input to the plant. The transmitted control input is then applied at the plant side. Packet exchange attempt failures happen when either the measured state packets or the control input packets are lost. In the case of a failed exchange attempt, the control input at the plant side is given by

\[
\tilde{x}(t+1) = A\tilde{x}(t) + Bu(t), \quad \tilde{x}(0) = \tilde{x}_0, \quad t \in \mathbb{N}_0, \tag{54}
\]

where \(\tilde{x}(t) \in \mathbb{R}^n\) and \(u(t) \in \mathbb{R}^m\) denote the state and the control input, respectively; furthermore, \(A \in \mathbb{R}^{n \times n}\) and \(B \in \mathbb{R}^{n \times m}\) are the state and input matrices, respectively.

We use the binary-valued process \(\{l(t) \in \{0, 1\}\}_{t \in \mathbb{N}_0}\) to describe success or failure states of packet exchange attempts. Specifically, the state \(l(t) = 0\) indicates that the packet exchange attempt at time \(t\) is successful, whereas \(l(t) = 1\) indicates failure. In this case, the control input \(u(t)\) applied at the plant side is given by

\[
u(t) \triangleq (1-l(t))K\tilde{x}(t), \quad t \in \mathbb{N}_0, \tag{55}\]

where \(K \in \mathbb{R}^{m \times n}\) denotes the feedback gain.

In [21], we proposed a characterization for \(\{l(t) \in \{0, 1\}\}_{t \in \mathbb{N}_0}\) that allows us to model the effects of random packet losses and jamming attacks in a unified manner. This characterization relies on the following assumption.

Assumption 4.1: There exists a scalar \(\rho \in [0, 1]\) such that

\[
\sum_{k=1}^{\infty} \mathbb{P} \left[ \sum_{i=0}^{k-1} l(i) > \rho k \right] < \infty. \tag{56}\]

Here, the inequality (56) can be considered as a condition on the evolution of tail probability \(\mathbb{P} \left[ \sum_{i=0}^{k-1} l(i) > \rho k \right]\). The scalar \(\rho \in [0, 1]\) in (56) plays a key role in characterizing a probabilistic bound on the average ratio of packet exchange failures. Observe that the case where all packet transmission attempts result in failure can be described by setting \(\rho = 1\). However, in such cases, the plant cannot receive control inputs and if the uncontrolled \((u(t) \equiv 0)\) dynamics are unstable, the state of the control system diverges. It is shown in [21] that \(\rho\) in (56) can be obtained to be strictly smaller than 1 for certain random and malicious packet loss models. These models include time-inhomogeneous Markov chains for describing random packet losses, as well as a discrete-time version of the malicious attack model in [20], where the number of packet exchange attempts that face attacks is upper bounded by a certain ratio of the total number of packet exchange attempts.

We showed in [21] that the closed-loop networked control system is stable, when \(\rho\) in (56) takes a sufficiently small value. In what follows we provide an alternative stability analysis method for the networked control system by utilizing Theorem 2.4. This new method turns out to be less conservative than the results in [21] in certain scenarios. To utilize Theorem 2.4 we first describe the closed-loop system as a discrete-time switched system.

The networked control system (54), (55) is equivalently described as a discrete-time switched system (1) with the state \(x(t) = \tilde{x}(t)\) and the mode signal given by \(r(t) = l(t) + 1\). Furthermore, the subsystem matrices are given by \(A_1 = A + BK\), \(A_2 = A\). Now, under Assumption 4.1 the inequalities (2) and (3) in Assumption 2.1 hold with \(\rho_1 = 1 - \rho, \rho_2 = 1, \rho_0 = 0\), and \(\rho_2 = \rho\). This is because (56) implies

\[
\limsup_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} l(t) \leq \rho, \tag{57}\]

and hence \(\limsup_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} 1[r(t) = 2] \leq \rho\). Through this switched system characterization, stability of the networked control system (54), (55) can be analyzed by using Theorem 2.4. Moreover, the linear programming problems developed in Section III can also be employed.

In [21] (see also [22]), an event-triggering controller is used for stabilization, and the packet exchange attempt times are decided by utilizing a set of triggering conditions. These conditions can be adjusted to consider the problem setting where the plant and the controller attempt packet exchanges at each time instant. For this problem setting, Theorem 2.4 is less conservative and can be considered as an enhancement of the stability result presented in [21]. In fact, the stability result in [21] is obtained by analyzing the evolution of a Lyapunov-like function \(V(x) = x^T P x\) at each time step. This analysis idea can be recovered by our approach presented in this paper through setting \(h = 1\), and defining the norm in (13) as the matrix norm induced by the vector norm \(\|x\|_P = \sqrt{x^T P x}\). In particular, for the case where \(\lim_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} l(t)\) exists, for all scenarios in which the stability condition in Theorem 3.5 in [21] holds, the stability condition in Theorem 2.4 also holds. Furthermore, as we illustrate in Section V there are cases where the condition in Theorem 3.5 in [21] does not hold but the condition in Theorem 2.4 is satisfied.

The following result shows that when \(\lim_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} l(t)\) exists, for all scenarios in which the stability condition in Theorem 3.5 in [21] holds, the stability condition in Theorem 2.4 also holds.
Proposition 4.1: Assume \( \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} l(i) \) exists. Suppose the stability condition in Theorem 3.5 in [21] holds, that is, there exist a positive-definite matrix \( P \in \mathbb{R}^{n \times n} \) and scalars \( \beta \in (0,1), \varphi \in [1,\infty) \) such that
\[
\beta P - (A + BK)^T P (A + BK) \geq 0, \\
\varphi P - A^T PA \geq 0, \\
(1 - \rho) \ln \beta + \rho \ln \varphi < 0,
\]
hold. Then with \( h \equiv 1 \) the stability condition in Theorem 2.4 also holds, i.e., \( \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} l(i) \) implies that \( \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \tilde{r}(i) \), \( q \in M^h \), exist; moreover, there is a matrix norm \( \| \cdot \| \) and a scalar \( \varepsilon \in (0,1) \) such that \( \tilde{r} \) holds for all \( \rho_q \in [0,1], q \in M^h \), that satisfy (19) and (20).

Proof: First, note that when \( h = 1 \), we have \( M^h = \{(1),(2)\} \). As a result, existence of \( \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} l(i) \) implies that \( \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \tilde{r}(i) = q \), \( q \in M^h \), also exist.

Next, let the norm in (13) be the matrix norm induced by the vector norm \( \| x \|_p \equiv \sqrt{\sum_{x=1}^n p_x} \), that is, \( \| M \| \equiv \sup_{x \in \mathbb{R}^{n \times n}} \frac{\| M x \|_p}{\| x \|_p} \), \( M \in \mathbb{R}^{n \times n} \). Under this norm, the inequalities (58) and (59) imply \( \| A \| = \| A + BK \| \leq \sqrt{\beta} \) and \( \| A \| \leq \sqrt{\beta} \). Now let \( \varepsilon \equiv \sqrt{\beta} \). Since \( \varepsilon = \sqrt{\beta} < \sqrt{\varepsilon} \), by using (60) and \( \rho_{(2)} \leq \rho_2 \), we get
\[
\sum_{q \in M^h} \gamma_q \rho_q = \gamma(1) \rho(1) + \gamma(2) \rho(2) = \gamma(1) (1 - \rho(2)) + \gamma(2) \rho(2)
\]
(1 - \rho(2)) \ln \max \{ \| A + BK \|, \varepsilon \} + \rho(2) \ln \max \{ \| A \|, \varepsilon \}
\leq (1 - \rho(2)) \ln \sqrt{\beta} + \rho(2) \ln \frac{1}{\sqrt{\varepsilon}}
\leq (1 - \rho) \ln \beta + \rho \ln \frac{1}{\sqrt{\varepsilon}} = \frac{1}{2} ((1 - \rho) \ln \beta + \rho \ln \varphi) < 0,

for all \( \rho_q \in [0,1], q \in M^h = \{(1),(2)\} \), such that (19) and (20) hold, which completes the proof. □

B. Control over Delay-Free and One-Step Delayed Communication Links

The switched system framework generalizes [21] to the multiple mode case with \( M \geq 2 \). This aspect is now illustrated through the networked control system depicted in Fig. 5 where the control actions are transmitted to the plant over two separate communication channels. We assume that one of these channels faces no delay and the other one faces 1-step delay in transmissions. Investigation of a networked control setup involving multiple channels with different delays is useful for analyzing systems that incorporate multiple actuators placed at different locations. The nodes that relay the information coming from the controller to certain actuators may induce delays due to different security measures in transmission powers, encryptions, and so on.

In our problem setting, the plant is as given in (54). The controller receives the system state \( \hat{x}(t) \) at each time \( t \), and computes two control inputs \( K_N \hat{x}(t) \) and \( K_D \hat{x}(t) \) that are attempted to be transmitted on the delay-free and 1-step-delayed channels, respectively. We respectively use \( \{ l_N(t) \in \{0,1\} \}_{t \in \mathbb{N}_0} \) and \( \{ l_D(t) \in \{0,1\} \}_{t \in \mathbb{N}_0} \) to indicate failures on the delay-free and the one-step-delayed channels. At time \( t \), the plant receives the control data \( K_N \hat{x}(t) \) if the transmission on the delay-free channel is successful \( (l_N(t) = 0) \) and \( K_D \hat{x}(t - 1) \) if the transmission on the delayed channel is successful \( (l_D(t) = 0) \). If both channels fail \( (l_N(t) = 1, l_D(t) = 1) \), the control input at the plant side is set to 0. Furthermore, the control data \( K_D \hat{x}(t - 1) \) received from the delayed channel is used only if the transmission on the delay-free channel fails \( (l_N(t) = 1, l_D(t) = 0) \). Otherwise \( (l_N(t) = 0, l_D(t) = 1) \) or \( (l_N(t) = 0, l_D(t) = 0) \), the control input at the plant side is set to \( K_N \hat{x}(t) \) received from the delay-free channel. Hence, the control input applied at the plant is given by
\[
u(t) = (1 - l_N(t)) K_N \hat{x}(t) + l_N(t)(1 - l_D(t)) K_D \hat{x}(t - 1),
\]
for \( t \geq 1 \). Assuming \( u(0) = 0 \), the closed-loop dynamics (54), (61) can be given by
\[
\begin{bmatrix}
\hat{x}(t + 2) \\
\hat{x}(t + 1)
\end{bmatrix} =
\begin{bmatrix}
A(1 - l_N(t)) BK_N & l_N(t)(1 - l_D(t)) BK_D \\
l_N(t) & 0
\end{bmatrix}
\begin{bmatrix}
\hat{x}(t + 1) \\
\hat{x}(t)
\end{bmatrix}, \quad t \in \mathbb{N}_0.
\]
By setting
\[
x(t) \triangleq \begin{bmatrix}
\hat{x}(t + 1) \\
\hat{x}(t)
\end{bmatrix}, \quad r(t) \triangleq \begin{cases}
1, & l_N(t) = 0, \\
2, & l_N(t) = 1, l_D(t) = 0, \\
3, & l_N(t) = 1, l_D(t) = 1,
\end{cases}
\]
for \( t \in \mathbb{N}_0 \), the closed-loop dynamics (62) forms a switched system (1) with 3 modes represented by
\[
A_1 \triangleq \begin{bmatrix}
A + BK_N & 0 \\
I_N & 0
\end{bmatrix},
A_2 \triangleq \begin{bmatrix}
A & BK_D \\
I_N & 0
\end{bmatrix},
A_3 \triangleq \begin{bmatrix}
A & 0 \\
I_N & 0
\end{bmatrix}.
\]
Concerning the communication channels in the networked control system, we assume the following.

Assumption 4.2: There exist scalars \( \sigma_N, \sigma_D, \rho_N, \rho_D \in [0,1] \) such that
\[
\lim \inf_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} l_N(t) \geq \sigma_N, \quad \lim \sup_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} l_N(t) \leq \rho_N,
\]
\[
\lim \inf_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} l_D(t) \geq \sigma_D, \quad \lim \sup_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} l_D(t) \leq \rho_D,
\]
hold almost surely.
Note that the scalars $\sigma_N, \sigma_D, \rho_N, \rho_D$ in this assumption provide lower- and upper-bounds on the long-run average numbers of transmission failures on the delay-free and 1-step-delayed channels.

In the following result, we show that under Assumption 4.2, the mode signal of the switched system representing the networked control system satisfies Assumption 2.1.

Proposition 4.2: Suppose (66) and (67) in Assumption 4.2 are satisfied. Then (2) and (3) in Assumption 2.1 hold with
\begin{align*}
\varrho_1 &= 1 - \rho_N, \varrho_2 = 1 - \sigma_N, \quad (68) \\
\varrho_2 &= \max\{0, \sigma_N - \rho_D\}, \varrho_3 = \min\{\rho_N, 1 - \sigma_D\}, \quad (69) \\
\varrho_3 &= \max\{0, \sigma_N + \sigma_D - 1\}, \varrho_4 = \min\{\rho_N, \rho_D\}. \quad (70)
\end{align*}

Proof: We use Lemma A.1 to show the result. To this end, first note that
\begin{align*}
1[r(t) = 1] &= 1 - l_N(t), \\
1[r(t) = 2] &= l_N(t)(1 - l_D(t)), \\
1[r(t) = 3] &= l_N(t)l_D(t), \quad t \in \mathbb{N}_0.
\end{align*}

First, we show (2) and (3) hold for the case $s = 1$ with $\varrho_1, \varrho_2$ given in (68). By (66) and (67), the inequalities (75) and (76) in Lemma A.1 hold with $\xi_1(\cdot), \xi_2(\cdot)$ given by $\xi_1(t) = 1 - l_N(t), \xi_2(t) = 1$, $t \in \mathbb{N}_0$, and $\varsigma_1 = 1 - \rho_N, \varsigma_1 = 1 - \sigma_N, \varsigma_2 = 1, \varsigma_2 = 1$. By the lemma, we obtain (2) from (77) and (3) from (78) with $\varrho_1, \varrho_2$ given in (68).

Next, we show (2) and (3) hold for the case $s = 2$ with $\varrho_3, \varrho_2$ given in (69). By (66) and (67), the inequalities (75) and (76) hold with $\xi_1(\cdot), \xi_2(\cdot)$ given by $\xi_1(t) = l_N(t), \xi_2(t) = 1 - l_D(t), t \in \mathbb{N}_0$, and $\varsigma_1 = \sigma_N, \varsigma_1 = \sigma_N, \varsigma_2 = 1, \varsigma_2 = 1$. By applying Lemma A.1, we obtain (2) from (77) and (3) from (78) with $\varrho_3, \varrho_2$ given in (69). The result for the other case ($s = 3$) is obtained similarly by using Lemma A.1 together with (75).

Proposition 4.2 shows that the networked control system (54), (61) with communication channels satisfying Assumption 4.2 can be represented by a switched system with a mode signal that satisfies Assumption 2.1. As a result, Theorem 2.4 and the linear programming problems developed in Section III can be used for the stability analysis.

V. Numerical Examples

In this section, we illustrate the efficacy of our results by investigating stability properties of networked control systems discussed in Sections IV-A and IV-B.

A) Example 1: Consider the system (54) with
\begin{align*}
A &= \begin{bmatrix} 1 & 0.1 \\ -0.5 & 1.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.1 \\ 1.2 \end{bmatrix}. \quad (74)
\end{align*}

In [21], we explored stabilization of this system over a network that faces random and malicious packet losses. To guarantee stabilization, we proposed a linear state feedback controller with feedback gain $K = \begin{bmatrix} -2.9012 & -0.9411 \end{bmatrix}$. By utilizing Theorem 3.5 of [21], we see that the closed-loop system is almost surely asymptotically stable whenever the scalar $\rho$ identified in Assumption 4.1 is inside the range $[0, 0.411]$. In the following we show that even for strictly larger values of $\rho$, the closed-loop system (54), (55) with above-mentioned parameters remains almost surely asymptotically stable.

For investigating stability of the closed-loop system (54), (55), we first characterize it as a switched system (1) with two modes represented with $A_1 = A + BK$ and $A_2 = A$. Notice that for this switched system, Assumption 4.1 implies (57), and as a result, Assumption 2.1 holds with $\varrho_1 = 1 - \rho, \varrho_2 = 0, \varrho_3 = \rho$.

We numerically solve the linear programming problems (46) and (51) to obtain $J_h$ and $J_h'$ for different values of $\rho$ and $h$. For finding the coefficients $\gamma_q$ and $\gamma''_q$ of the objective functions, we use (13) (with the matrix norm induced by the Euclidean norm and $\varepsilon = 10^{-24}$) and (52). We numerically confirm that $J_h = J_h'$ for $h = \{1, \ldots, 11\}$. For $h \geq 12$, we utilize only the linear programming problem (51) and obtain $J_h'$, as solving the problem (46) takes excessively long times. We see in Fig. 6 that for smaller values of $\rho$, $J_h'$ takes a negative value indicating almost sure asymptotic stability. In particular, we see in Fig. 7 that when $\rho = 0.5$, we obtain $J_{22}' < 0$, which implies that (17) holds for all $\rho_q \in [0, 1], q \in \mathcal{M}$, that satisfy (19), (20). It follows from Theorem 2.4 that if Assumption 4.1 holds with $\rho = 0.5$, and $\lim_{k \to \infty} \frac{1}{l} \sum_{l=0}^{l-1} I[\ell(t) = q]$ exists for each $q \in \mathcal{M}$, then the zero solution of the closed-loop system is almost surely asymptotically stable. We note again that for $\rho = 0.5$, the stability conditions of Theorem 3.5 in [21] do not hold. This indicates that the stability conditions obtained in this paper are less conservative than those in [21].

Note that for a given $\rho$, obtaining a nonnegative value for $J_h = J_h'$ does not necessarily imply that the system is unstable. For the same $\rho$, the value of $J_h'$ may be positive for small $h$ and negative for sufficiently large $h$ (see Remark 3.2). For instance, in this example, $J_{10}'$ takes a negative value for $\rho = 0.3$, but not for $\rho = 0.5$ or $\rho = 0.7$ (Fig. 7). If one can only compute $J_h'$ up to $h = 10$ due to limited computational power, then stability for $\rho = 0.5$ cannot be concluded.

We also remark that using different matrix norms in the definition of $\gamma_q, q \in \mathcal{M}$, given in (13) results in different trajectories for $J_h'$. Checking $J_h'$ for different matrix norms can be useful to analyze stability in the case of limited...
computational resources. For illustration, we compute $J'_h$ with matrix norms $\| \cdot \|_1$, $\| \cdot \|_2$, $\| \cdot \|_\infty$ (induced respectively by the 1-norm, the Euclidean norm, and the infinity norm of vectors), as well as the matrix norm $\| \cdot \|_P$ induced by the vector norm $\|x\|_P \triangleq \sqrt{x^T P x}$. Here, we use the positive-definite matrix $P$ that we previously utilized in [21] for the Lyapunov-based stability analysis of this system. Fig. 8 shows the optimal solution value $J'_h$ of the linear programming problem (51) for different values of $h$ when $\rho = 0.6$. For this example, the values of $J'_h$ obtained with the matrix norm $\| \cdot \|_P$ is clearly lower than others. We also observe that $J'_{30}$ obtained with $\| \cdot \|_P$ is negative. Thus, we can conclude that (17) holds for all $\rho_q \in [0, 1], q \in \mathcal{M}^{30}$, that satisfy (19), (20). It then follows from Theorem 2.4 if Assumption 4.1 holds with $\rho = 0.6$, and $\lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{I}[l(t) = q]$ exists for each $q \in \mathcal{M}^{30}$, then the zero solution of the closed-loop system is almost surely asymptotically stable.

By calculating $J'_{30}$ with the matrix norm $\| \cdot \|_P$ for different values of $\rho$, we confirm that stability is guaranteed for all $\rho \in [0, 0.6]$. Hence, for instance, the system is stable under all periodic attack scenarios with $\rho \leq 0.6$, since Proposition 2.5 implies that the limit $\lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{I}[l(t) = q], q \in \mathcal{M}^{30}$, exist in the periodic case. In fact this stability region $[0, 0.6]$ is quite tight, as there exists a destabilizing attack strategy for $\rho = 0.64$. This attack strategy is periodic with a period of 150 time steps. It was identified through the solution $\rho_{12}, z \in \mathcal{Z}_{30}$, to the linear programming problem (51) solved with $h = 30$ by using the matrix norm $\| \cdot \|_2$ and the bounds $\rho_1 = 1 - \rho$, $\rho_2 = 0, \rho_2 = \rho$, where $\rho = 0.64$. Under this attack strategy, the mode signal of the associated switched system (1) repeats the same pattern in every 150 time steps. This pattern is composed of a particular sequence $\bar{q} \in \mathcal{M}^{30}$ appearing once and then another sequence $\bar{q} \in \mathcal{M}^{30}$ appearing 4 times. Note that under this attack strategy, Assumption 4.1 holds with $\rho = 0.64$, and furthermore, the monodromy matrix associated with the closed-loop periodic networked control system possesses an eigenvalue that is outside the unit circle of the complex plane indicating divergence of the state.

B) Example 2: In this example we demonstrate the results for the networked control setup discussed in Section 4. V-B. Specifically, we consider the plant with $A$ and $B$ given by (74) in the previous subsection. The control packets are transmitted to the plant over the delay-free and the 1-step-delayed channels depicted in Fig. 5. The feedback gains associated with these channels are given by $K_N = [ -2.9012 -0.9411 ]$ and $K_D = [ -0.04 -0.3 ]$. We note that $K_N$ is the feedback gain considered in the previous subsection, and $K_D$ ensures that $A_2$ is a Schur matrix. The closed-loop system (54), (61) can be represented as a switched system (1) with 3 modes described by matrices $A_1, A_2, A_3$ given in (64), (65) and the mode signal given by (63).

We consider the case where the channels are subject to coordinated periodic jamming attacks. In this case, $\{l_N(t) \in \{0, 1\}_t \in \mathbb{N}_0\}$ and $\{l_D(t) \in \{0, 1\}_t \in \mathbb{N}_0\}$, the failure indicators of the delay-free and the one-step-delayed channels, can be given by

$$l_N(t) = \begin{cases} 1, & g(t) \in \mathcal{S}_N, \\ 0, & \text{otherwise}, \end{cases}$$

$$l_D(t) = \begin{cases} 1, & g(t) \in \mathcal{S}_D, \\ 0, & \text{otherwise}, \end{cases}$$

where $\{g(t) \in \mathcal{S}\}_t \in \mathbb{N}_0$ is a finite-state irreducible and periodic Markov chain with transition probabilities either 0 or 1, and moreover, $\mathcal{S}_N$ and $\mathcal{S}_D$ are subsets of $\mathcal{S}$. Fig. 9 shows the transition diagram of $\{g(t) \in \mathcal{S}\}_t \in \mathbb{N}_0$ with an 8-periodic pattern. At time $t$, the transmission on the delay-free channel is attacked if $g(t) \in \mathcal{S}_N = \{3, 4, 7, 8\}$; moreover, the transmission on the 1-step-delayed channel is attacked if $g(t) \in \mathcal{S}_D = \{2, 3\}$, and both channels are attacked when $g(t) = 3$. We also remark that $\mathcal{S}_N$, $\mathcal{S}_D$, and $\mathcal{S}_{30}$ describe the attackers’ strategy for the timings of the attacks but are unknown to the system operator. However, note that for any attack strategy represented with an irreducible $\{g(t) \in \mathcal{S}\}_t \in \mathbb{N}_0$, the mode signal given by (63) satisfies (44) with $\mathcal{S}_1 = \mathcal{S} \setminus \mathcal{S}_N$, $\mathcal{S}_2 = \mathcal{S} \setminus (\mathcal{S} \setminus \mathcal{S}_D)$, and $\mathcal{S}_3 = \mathcal{S} \setminus \mathcal{S}_D$. Hence, it follows from Proposition 2.5 that for all $h \in \mathbb{N}$, the limits $\lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{I}[r(i) = q], q \in \mathcal{M}^h$, exist.
Hence, the stability can be concluded by Theorem 2.4 for all faces more failures, the transmission on the channel becomes more important. In particular, the long-run average number of failures on both channels to set the lower-bounds on the long-run average number of failures on delay-free and transmission failures on both channels satisfy Assumption 4.2.

We first consider the case where the information about the average number of failures on delay-free and 1-step-delayed channels is limited. In particular, we set the lower-bounds on the long-run average number of failures on both channels to 0, that is, \( \sigma_N = \sigma_D = 0 \). Our goal is to identify upper-bounds for the values \( \rho_N, \rho_D \) for which the closed-loop networked control system is stable. To this end, we solve the linear programming problem (51) for different values of \( \rho_N \) and \( \rho_D \). For finding the coefficients \( \gamma_q^s = \max_{q \in \mathcal{M}^s} \gamma_q, z \in \mathcal{Z}_h \), of the objective function, we use (18) (with the matrix norm induced by the Euclidean norm and \( \varepsilon = 10^{-24} \)).

Fig. 10 shows how the optimal objective function \( J_{20}^{\rho} \) of the linear programming problem (51) changes with respect to \( \rho_D \) for the values \( \rho_N = 0.4, \rho_N = 0.5, \) and \( \rho_N = 0.6 \). Observe that when the long-run average number of failures on the delay-free communication channel is sufficiently small, stability can be achieved regardless of the amount of transmission failures on the 1-step-delayed channel. This is seen in Fig. 10 for the case \( \rho_N = 0.4 \). Specifically, we have \( J_{20}^{\rho} < 0 \), which implies that (17) holds for all \( \rho_q \in [0, 1] \) satisfying (19), (20). Hence, the stability can be concluded by Theorem 2.4 for all \( \rho_D \) in the interval [0, 1]. On the other hand, when the delay-free channel faces more failures, the transmissions on the 1-step-delayed channel become more important. In particular for \( \rho_N = 0.5 \) and \( \rho_N = 0.6 \), the value of \( J_{20}^{\rho} \) is negative and the stability can be concluded only for sufficiently small values of \( \rho_D \).

Next, we consider the situation where the delay-free channel is known to be completely blocked due to jamming attacks at each time instant. To explore this case, we set \( \sigma_N = \rho_N = 1 \). In this setup, all control packets are to be transmitted over the 1-step-delayed channel. We would like to find out the long-run average number of failures that can be tolerated on this channel. For this purpose, we solve the linear programming problem (51) to obtain \( J_h^{\rho} \) with respect to \( h \) for three different cases: \( \rho_D = 0.1, \rho_D = 0.2, \) and \( \rho_D = 0.3 \). In all cases we set \( \sigma_D = 0 \). Fig. 11 shows \( J_h^{\rho} \) for different values of \( \rho_D \).

We observe that \( J_{14}^{\rho} < 0 \) when \( \rho_D = 0.1 \), implying that (17) holds for all \( \rho_q \in [0, 1] \) satisfying (19), (20). Hence, by Theorem 2.4 the zero solution of the closed-loop system is almost surely asymptotically stable if the ratio of the transmission failures on the 1-step-delayed channel is bounded by 0.1 in the long-run. On the other hand, for \( \rho_D = 0.2 \) and \( \rho_D = 0.3 \), we cannot guarantee stability, since we have \( J_h^{\rho} > 0, h \in \{1, \ldots, 20\} \). We remark that the scalars \( \rho_q, q \in \mathcal{M}^s, \) and \( \rho_z, z \in \mathcal{Z}_h \), that are associated with the optimal values \( J_h \) and \( J_h^{\rho} \) provide useful information about damaging attack patterns and packet loss scenarios, even when stability cannot be concluded.

VI. CONCLUSION

We explored almost sure asymptotic stability of a stochastic switched linear system. Our proposed stability analysis approach relies on studying the switched system’s state at every \( h \in \mathbb{N} \) steps. We obtained sufficient stability conditions and showed that the stability can be checked by solving a linear programming problem. The number of variables in this problem grows polynomially in the number of subsystems, but exponentially in \( h \), which makes the computation difficult when \( h \) is large. To overcome this issue, we also constructed an alternative linear programming problem, where the number of variables grows polynomially in both the number of subsystems and \( h \). Even though the calculation of the coefficients in the alternative problem takes additional time, the solution is obtained faster compared to the original problem.

Our linear programming-based analysis approach allows us to check stability without relying on statistical information on the mode signal. In particular, the probability of mode switches and the stationary distributions associated with the modes are not needed for stability analysis. We applied our approach in exploring networked control systems under malicious jamming attacks. The technical challenge there is that the attackers’ specific strategies are not available for analysis. By using our approach, we showed that stability can be guaranteed under all possible attack strategies when the long-run average number of network transmission failures satisfies certain conditions. In practice, our approach can be used by system operators to assess the safety of industrial processes. Specifically, our stability results can be utilized in identifying the level of...
jamming attacks that can be tolerated on the communication channels used for the measurement and the control of a plant.

Investigations of the case with noisy dynamics and the stabilization problem are part of our future extensions. In the stabilization problem, the controller may not have access to precise information of the active mode. In those cases, the system modes are divided into several groups, and the controller only knows which group contains the currently active mode. This problem was considered for discrete- and continuous-time Markov jump systems by [55, 56]. For this problem setting, our approaches may be extended for the case where the mode signal is not necessarily a Markov process.

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APPENDIX

The following result provides lower- and upper-bounds for the long-run average of the product of two binary-valued processes.

Lemma A.1: For all binary-valued processes \( \{\xi_1(t) \in \{0, 1\}\}_{t \in \mathbb{N}_0} \) and \( \{\xi_2(t) \in \{0, 1\}\}_{t \in \mathbb{N}_0} \) that satisfy
\[
\begin{align*}
\liminf_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} \xi_1(t) &\geq \varsigma_1, \\
\limsup_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} \xi_1(t) &\leq \varrho_1, \quad i \in \{1, 2\},
\end{align*}
\]
almost surely with \( \varsigma_1, \varrho_1 \in [0, 1], \) \( i \in \{1, 2\}, \) we have
\[
\begin{align*}
\liminf_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} \xi_1(t) \xi_2(t) &\geq \max\{0, \varsigma_1 + \varsigma_2 - 1\}, \\
\limsup_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} \xi_1(t) \xi_2(t) &\leq \min\{\varrho_1, \varrho_2\},
\end{align*}
\]
almost surely.

Proof: To show (77), first note that
\[
\begin{align*}
\liminf_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} \xi_1(t) \xi_2(t) &= 1 - \limsup_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} (1 - \xi_1(t)) \xi_2(t) \cdot
\end{align*}
\]
For all \( i, j \in \{1, 2\}, i \neq j, \) we have
\[
\begin{align*}
\limsup_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} (1 - \xi_1(t)) \xi_2(t)) \\
&= \limsup_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} (\xi_1(t)(1 - \xi_j(t)) + (1 - \xi_i(t))) \\
&\leq \limsup_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} (\xi_1(t)(1 - \xi_j(t))) \\
&+ \limsup_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} (1 - \xi_i(t)).
\end{align*}
\]
Since \( \xi_i(t)(1 - \xi_j(t)) \leq \xi_i(t) \) and \( \xi_j(t)(1 - \xi_j(t)) \leq 1 - \xi_j(t) \),
by (76), we have
\[
\limsup_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} \xi_1(t) \leq \varrho_1,
\]
and by (75), we have
\[
\begin{align*}
\limsup_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} \xi_i(t) &\leq \min\{\varrho_i, 1 - \varsigma_j\}. \quad (81)
\end{align*}
\]
Furthermore, since \( \limsup_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} (1 - \xi_i(t)) \leq 1 - \varsigma_i \),
it follows from (79)–(81) that
\[
\begin{align*}
\liminf_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} \xi_1(t) \xi_2(t) &\geq 1 - \min\{\varrho_i, 1 - \varsigma_j\} + \varsigma_i \\
&= \varsigma_i - \min\{\varrho_i, 1 - \varsigma_j\} = \max\{\varsigma_i - \varrho_i, \varsigma_i + \varsigma_j - 1\}. \quad (82)
\end{align*}
\]
Now, noting that \( \varsigma_i - \varrho_i \leq 0, \) \( i \in \{1, 2\}, \) and
\[
\begin{align*}
\liminf_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} \xi_1(t) \xi_2(t) &\geq 0, \quad \text{almost surely, we have (83)}
\end{align*}
\]
Next, we prove (78). Since \( \xi_1(t) \xi_2(t) \leq \xi_1(t) \) and \( \xi_1(t) \xi_2(t) \leq \xi_2(t), \) we obtain
\[
\begin{align*}
\limsup_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} \xi_1(t) \xi_2(t) &\leq \limsup_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} \xi_1(t),
\end{align*}
\]
for \( i \in \{1, 2\}. \) It then follows from (76) that
\[
\begin{align*}
\limsup_{k \to \infty} \frac{1}{k} \sum_{t=0}^{k-1} \xi_1(t) \xi_2(t) &\leq \varrho_i, \quad i \in \{1, 2\}, \quad \text{almost surely, which then implies (78).}
\end{align*}
\]