An undecidable property of context-free languages

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Abstract

We prove that there exists no algorithm to decide whether the language generated by a context-free grammar is dense with respect to the lexicographic ordering. As a corollary to this result, we show that it is undecidable whether the lexicographic orderings of the languages generated by two context-free grammars have the same order type.

1 Introduction

Suppose that Σ is an alphabet equipped with a (strict) linear order relation <. We may extend < to a lexicographic ordering <ℓ of Σ∗ by defining, for all words u, v ∈ Σ∗, u <ℓ v if either u is a proper prefix of v, or u = xay and v = xbz for some a, b ∈ Σ and x, y, z ∈ Σ∗ with a < b. Thus, when L ⊆ Σ∗, then (L, <ℓ) is a linear ordering. It is known (see e.g. [BE07, Cour78a]) that if the size of Σ is two or more, then every countable linear ordering is isomorphic to a linear ordering (L, <ℓ) for some language L ⊆ Σ∗. Let us call a linear ordering regular, context-free, or deterministic context-free if it is isomorphic to the linear ordering of a language of the appropriate type.

It follows by the characterization of regular and algebraic trees by their branch languages [Cour78a, Cour78b] that the regular (deterministic context-free) linear orderings are exactly those that can be defined by recursion schemes of order 0 (order 1, respectively). See also [BE07]. Moreover, a well-ordering is regular if and only if its order type is less than ωω, and deterministic context-free if and only if its order type is less than ωωω, cf. [BE10]. (These well-orderings have other characterizations using operations on well-orderings or automata, cf. [Del04, KRS03].) Moreover, it follows from results proved in [Heil80] that the

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Hausdorff rank \cite{Ros82} of every scattered regular linear ordering is finite. As shown in \cite{BE09}, the Hausdorff rank of every scattered deterministic context-free linear ordering is less than \(\omega\). Ordinals and scattered linear orderings defined by higher order recursion schemes are studied in \cite{BC10}.

It was shown in \cite{Thom86} that it is decidable for regular linear orderings (given as lexicographic orderings of regular languages) whether they are isomorphic. The decidability status of the isomorphism problem for deterministic context-free linear orderings is open. Here, we show that it is undecidable for context-free linear orderings (given by context-free grammars) whether they are isomorphic. Moreover, we show that it is undecidable whether a context-free language defines a dense linear ordering.

2 Linear orderings and context-free grammars

A linear ordering \cite{Ros82} is a set \(S\) equipped with a strict linear order relation \(<\). In this paper, we restrict ourselves to linear orderings \((S, <)\), where \(S\) is a countable set. A linear ordering \((S, <)\) is dense if it has at least two elements and for any \(x, y \in S\) with \(x < y\) there is some \(z\) with \(x < z < y\). Two linear orderings \((S, <)\) and \((S', <)\) are isomorphic if there is a bijection \(h : S \to S'\) such that \(xh < yh\) for all \(x, y \in S\) with \(x < y\). Isomorphic linear orderings have the same order type. It is known that up to isomorphism there are 4 dense (countable) linear orderings, the ordering \(\mathbb{Q}\) of the rationals possibly equipped with a least or greatest element (or both). The order type of \(\mathbb{Q}\) is denoted \(\eta\).

A context-free grammar \(G\) over a (terminal) alphabet \(\Sigma\) consists of a finite nonempty set \(N\) of nonterminals and a finite set of productions \(A \to u\), where \(A \in N\) and \(u \in (N \cup \Sigma)^*\). It is assumed that \(N\) and \(\Sigma\) are disjoint. A nonterminal \(A_0\), called the start symbol, is distinguished. The derivation relation \(\Rightarrow^*\) is defined as usual. For each nonterminal \(A\), we let \(L(G, A) = \{u \in \Sigma^* : A \Rightarrow^* u\}\) denote the language generated from \(A\). The context-free language \(L(G) \subseteq \Sigma^*\) generated by \(G\) is \(L(G, A_0)\). We call \(G\) a prefix grammar if the languages \(L(G, A)\) are all prefix (or prefix-free) languages. A right linear grammar is a context-free grammar such that, except possibly for the last letter, each letter occurring in the word on the right side of a production is a terminal letter. It is well-known that a language is regular if and only if it can be generated by a right-linear grammar. For all unexplained notions on context-free grammars and languages refer to any standard book on formal languages.

The reverse of a word \(u\) will be denoted \(u^{-1}\).

**Remark 2.1** It was pointed out by Luc Boasson that there is no algorithm to decide for a context-free grammar \(G\) whether it is a prefix grammar. Moreover, there is no algorithm to decide whether a given context-free grammar generates a prefix language.
3 Some undecidability results

In this section our aim is to prove that it is undecidable for a context-free (prefix) grammar $G$ over a 2-letter alphabet whether or not $(L(G), <_\ell)$ is a dense ordering, or a linear ordering isomorphic to the ordering $\mathbb{Q}$ of the rationals. It follows from this result that it is undecidable whether or not the lexicographic orderings of two context-free languages, given by context-free (prefix) grammars, are isomorphic.

In our proofs, we will use reduction from the Post Correspondence Problem (PCP).

Let $(\alpha, \beta)$ be an instance of PCP, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ are nonempty sequences of nonempty words over the two-letter alphabet $\{a, b\}$. Then consider the alphabet

$$\Gamma = \{1, \ldots, n, a, b, \$, \}$,$

ordered as indicated. For convenience, we will also refer to the elements of $\Gamma$ by the letters $c_1, c_2, \ldots, c_{n+4}$ with $c_1$ denoting 1, $c_2$ denoting 2, etc. For $j = 1, \ldots, n+2$, define $\Delta_j$ as the 3-letter alphabet $\{d_{j0}, d_{j1}, d_{j2}\}$ and extend the linear order on $\Gamma$ to a linear ordering of the set

$$\Delta = \Gamma \cup \bigcup_{j=1}^{n+2} \Delta_j$$

so that

$$c_j < d_{j0} < d_{j1} < d_{j2} < c_{j+1}$$

for all $j = 1, \ldots, n+2$. Note that $\Delta$ contains $4n + 10$ letters and there is no “extra letter” between $\$ and $\$.

We will construct a (prefix) grammar $G = G_{\alpha, \beta}$ over the alphabet $\Delta$ such that $(L(G), <_\ell)$ is dense if and only if $(\alpha, \beta)$ has no solution. The grammar $G$ will be designed so that it will generate the language

$$L = L_\alpha \cup L_\beta \cup L_1 \cup \ldots \cup L_{n+2}$$

where

1. $L_\alpha = \{i_1 \ldots i_m (\alpha_{i_1} \ldots \alpha_{i_m})^{-1} \$ : $1 \leq i_k \leq n$, $m \geq 1\}$

2. $L_\beta = \{i_1 \ldots i_m (\beta_{i_1} \ldots \beta_{i_m})^{-1} \$ : $1 \leq i_k \leq n$, $m \geq 1\}$

3. $L_j = \{1, \ldots, n, a, b\}^* Q_j$, where $Q_j = \{d_{j0}, d_{j2}\}^* d_{j1}$, $j = 1, \ldots, n+2$.

Note that each $Q_j$ and each $L_j$ is a dense regular language whose order type is $\eta$, the order type of the rationals. The same fact holds for the languages $Q = \bigcup_{j=1}^{n+2} Q_j$ and $L' = \bigcup_{j=1}^{n+2} L_j$, since the order type of any finite nonempty sum $\sum_{i \in I} P_i$ of linear orderings $P_i$ of order type $\eta$ is also $\eta$. 

3
The grammar $G$ has start symbol $S$ and contains the following productions in BNF:

$$
S \rightarrow A¢ \mid BS \mid C \\
A \rightarrow iAα_{i-1} \mid iα_i^{-1} \\
B \rightarrow iBβ_i^{-1} \mid iβ_i^{-1} \\
C \rightarrow iC \mid aC \mid bC \\
C \rightarrow D_1 \mid \ldots \mid D_{n+2} \\
D_j \rightarrow d_j0D_j \mid d_j2D_j \mid d_j1
$$

It is clear that $G$ is a prefix grammar.

**Proposition 3.1** $(L(G_{α,β}), <_ℓ)$ is dense if and only if $(α, β)$ has no solution.

**Proof.** Assume that $i_1 \ldots i_m$ is a solution of $(α, β)$. Let $u = (α_{i_1} \ldots α_{i_m})^{-1} = (β_{i_1} \ldots β_{i_m})^{-1}$. Then

$$
u_α = i_1 \ldots i_m u¢ \quad \text{and} \quad v_β = i_1 \ldots i_m u$$(

are in $L$. However, there is no word $v$ in $L$ with

$$u_α <_ℓ v <_ℓ u_β,$$

showing that $L$ is not dense.

Suppose now that $(α, β)$ has no solution. We show that $L$ is dense. To this end, suppose that $u, v \in L$ with $u <_ℓ v$. Since $L$ is a prefix language, $u$ and $v$ can be decomposed as

$$u = wcu', \quad v = wdv'$$

where $c$ and $d$ are letters with $c < d$. It is not possible that $c = ¢$ and $d = $, since otherwise we would have $u' = v' = ¢$ and the maximal prefix of $w$ that is in $\{1, \ldots, n\}^*$ would give a solution of $(α, β)$.

Thus, either $c \in Δ_i$ or $c = c_i$ for some $i = 1, \ldots, n + 2$. There are three cases to consider.

1. $c \in Δ_i$ for some $i = 1, \ldots, n + 2$, so that $cu' \in Q_i$. If $d$ is also in $Δ_i$, then $dv' \in Q_i$, and since $cu' <_ℓ dv'$, there exists some $x \in Q_i$ with $cu' <_ℓ x <_ℓ dv'$ and thus $u = wcu' <_ℓ wx <_ℓ wdv'$, where $wx$ is in $L$. If $d \notin Δ_i$, then choose any word $x \in Q_i$ with $cu' <_ℓ x$ and $wx <_ℓ wdv'$, where $wx \in L$. We again have $u = wcu' <_ℓ wx <_ℓ wdv'$ and $wx \in L$.

2. $d \in Δ_i$ for some $i = 1, \ldots, n + 2$. This case is symmetrical to the previous case.

3. Thus the only remaining case is when $c = c_i$ for some $i = 1, \ldots, n + 2$ and $d = c_j$ for some $j = 1, \ldots, n + 4$ with $i < j$. In this case let $x$ be any word in $Q_i$. We have that $u = wcu' <_ℓ wx <_ℓ wdv'$ and $wx \in L$. 

4
Thus, we have shown that if \((\alpha, \beta)\) has no solution, then between any two words of \(L\) there is a third word of \(L\), completing the proof of the fact that \(L\) is dense.

\[\square\]

**Remark 3.2** The language \(L = L(G_{\alpha, \beta})\) generated by the above grammar \(G_{\alpha, \beta}\) has no least or greatest element with respect to the lexicographic order. Indeed, if \(v \in L'\), then there exist words \(u, w \in L'\) with \(u <_\ell v <_\ell w\) since the order type of \(L'\) is \(\eta\). Now consider a word \(v = i_1 \ldots i_m(\alpha_{i_1} \ldots \alpha_{i_m})^{-1}c\) in \(L_\alpha\). Then let \(u = i_1 \ldots i_m 1(\alpha_{i_1} \ldots \alpha_{i_m} \alpha_1)^{-1}c\) and let \(w = d_{i_11}\) or any other word in \(Q_{i_1}\). We have that \(u <_\ell v <_\ell w\) for the words \(u = i_1 \ldots i_m 1(\alpha_{i_1} \ldots \alpha_{i_m} \alpha_1)^{-1}c\) and \(w = d_{i_11}\) in \(L\).

We order the binary alphabet \(\{0, 1\}\) by \(0 < 1\).

**Theorem 3.3** There exists no algorithm to decide for a context-free (prefix) grammar \(G\) over \(\{0, 1\}\) whether \((L(G), <_\ell)\) is dense. Moreover, there exists no algorithm to decide for a context-free (prefix) grammar \(G\) over \(\{0, 1\}\) whether the order type of \((L(G), <_\ell)\) is \(\eta\).

**Proof.** This follows from Proposition 3.1 and Remark 3.2 by an appropriate order preserving coding of the letters of the alphabet \(\Delta\) by words over \(\{0, 1\}^*\) of length \(\lceil \log(4n + 10) \rceil\).

\[\square\]

**Theorem 3.4** There exists no algorithm to decide for a context-free (prefix) grammar \(G\) and a right linear (prefix) grammar \(G'\) over \(\{0, 1\}\) whether \((L(G), <_\ell)\) and \((L(G'), <_\ell)\) are isomorphic.

**Proof.** Consider an instance \((\alpha, \beta)\) of PCP and the grammar \(G = G_{\alpha, \beta}\) constructed above. As before, let us code terminal letters by words of length \(\lceil \log(4n + 10) \rceil\) by an order preserving coding. Thus, \(L(G)\) is a language over the alphabet \(\{0, 1\}^*\) such that the order type of \((L(G), <_\ell)\) is \(\eta\) if and only if \((\alpha, \beta)\) has no solution. Then let \(G'\) be the right linear (prefix) grammar with productions

\[
S \to 00S \mid 11S \mid 01
\]

generating the language \(\{00, 11\}^*01\) of order type \(\eta\). Then \((L(G), <_\ell)\) and \((L(G'), <_\ell)\) are isomorphic if and only if \((\alpha, \beta)\) has no solution.

\[\square\]

**4 Conclusion**

We have proved that there is no algorithm to decide whether a context-free grammar (even prefix grammar) generates a dense language with respect to the lexicographic ordering. As a corollary to this result, we have shown that it is
undecidable whether two prefix grammars generate languages of the same order type.

We can prove that it is decidable in polynomial time whether the lexicographic ordering of the language generated by a prefix grammar is scattered, or a well-ordering. Moreover, we can extend the decidability part of this result to arbitrary context-free grammars. It is likely that a PTIME algorithm can be obtained for all context-free grammars.

References

[BE07] S.L. Bloom and Z. Ésik. Regular and algebraic words and ordinals. In: CALCO 2007, Bergen, LNCS 4624, Springer, 2007, 1–15.

[BE09] S.L. Bloom and Z. Ésik. Scattered algebraic linear orderings. In: 6th Workshop on Fixed Points in Computer Science, Coimbra, 2009, Edited by Ralph Matthes and Tarmo Uustalu, Institute of Cybernetics at Tallinn University of Technology, 2009, 25–29.

[BE10] S.L. Bloom and Z. Ésik. Algebraic ordinals. Fundamenta Informaticae, to appear in 2010.

[BC10] L. Braud and A. Carayol. Linear orders in the pushdown hierarchy. ICALP 2010, to appear.

[Cour78a] B. Courcelle. Frontiers of infinite trees. RAIRO Theoretical Informatics and Applications, 12(1978), 319–337.

[Cour78b] B. Courcelle. A representation of trees by languages, Parts I and II, Theoretical Computer Science, 6 (1978), 255–279 and 7(1978), 25–55.

[Del04] Ch. Delhommé. Automaticity of ordinals and of homogeneous graphs. C. R. Math. Acad. Sci. Paris 339(2004), no. 1, 5–10. (in French)

[Heil80] S. Heilbrunner. An algorithm for the solution of fixed-point equations for infinite words. RAIRO Theoretical Informatics and Applications, 14(1980), 131–141.

[KRS03] B. Khoussainov, S. Rubin and F. Stephan. On automatic partial orders. Proceedings of Eighteenth IEEE Symposium on Logic in Computer Science, LICS, 168-177, 2003.

[Ros82] J.B. Rosenstein. Linear Orderings. Academic Press, New York, 1982.

[Thom86] W. Thomas. On frontiers of regular trees. RAIRO Theoretical Informatics and Applications, 20(1986), 371–381.