Path-Independent Quantum Gates with Noisy Ancilla

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Ancilla systems are often indispensable to universal control of a nearly isolated quantum system. However, ancilla systems are typically more vulnerable to environmental noise, which limits the performance of such ancilla-assisted quantum control. To address this challenge of ancilla-induced decoherence, we propose a general framework that integrates quantum control and quantum error correction, so that we can achieve robust quantum gates resilient to ancilla noise. We introduce the path independence criterion for fault-tolerant quantum gates against ancilla errors. As an example, a path-independent gate is provided for superconducting circuits with a hardware-efficient design.

An outstanding challenge of quantum computing is building quantum devices with both excellent coherence and reliable universal control [1–3]. For good coherence, we may choose physical systems with low dissipation (e.g., superconducting cavities [4–6] and nuclear spins [7–10]) or further boost the coherence with active quantum error correction [11, 12]. As we improve the coherence by better isolating the central system from the noisy environment, it becomes more difficult to process information stored in the central system. To control the nearly isolated central system, we often introduce an ancilla system (e.g., transmon qubits [13–15] and electron spins [8, 9]) that is relatively easy to control, but the ancilla system typically suffers more decoherence than the central system, which often limits the fidelity of the ancilla-assisted quantum operations. Therefore, it is crucial to develop quantum control protocols that are fault-tolerant against ancilla errors. This can boost the performance of ancilla-assisted quantum operations by suppressing ancilla errors to higher orders.

For noise with temporal or spatial correlations, we can use techniques of dynamical decoupling [16–18] or decoherence-free encoding [19, 20] to suppress errors to higher order and achieve noise-resilient control of the central system. When the noise has no correlations (e.g., Markovian noise), we need active quantum error correction (QEC) to extract the entropy. For qubit systems, a common strategy to suppress ancilla errors is to use the transversal approach [1, 21–26], which may have a significant hardware overhead and cannot provide universal control [1], and it is desirable to have a hardware-efficient approach to fault-tolerant operations against ancilla errors [27, 28]. Different from qubit systems, each bosonic mode has a large Hilbert space that can encode quantum information using various bosonic quantum codes, which have been successfully demonstrated in recent experiments [11, 29–31]. However, there is no simple way to divide the bosonic mode into separate subsystems, which prevents us from extending the transversal approach to the bosonic central system. Ancilla errors may propagate to the bosonic mode and compromise the encoded quantum information [32]. Nevertheless, a recent experiment with a hardware-efficient three-level ancilla demonstrated fault-tolerant readout of an error syndrome of the central system against the decay of the ancilla [34], while it remains unknown whether there exists a similar approach to fault-tolerant gates against ancilla errors. There is an urgent need of a general theoretical framework that integrates quantum control and quantum error correction, to guide the design of hardware-efficient robust quantum operations against ancilla errors.

In this letter, we provide a general criterion for fault-tolerant quantum gates on the central system robust against ancilla errors [Fig. 1(a)]. Our general criterion
of path-independence (PI) requires that for given initial and final ancilla states, the central system undergoes a unitary operation independent of the specific ancilla trajectory induced by control drives and ancilla error events. For a subset of final ancilla states, the desired quantum gate on the central system is successfully implemented [Fig. 1(b)], while other other final ancilla states herald a failure of the attempted operation, but the central system still undergoes a deterministic unitary evolution without loss of coherence. Thus we may repeat our attempts of PI gates on the central system until the gate succeeds. As an application of our general criterion, we will provide a PI design of the photon-number selective phase (SNAP) gates [14, 15] for universal control and quantum error correction of superconducting circuits.

Ancilla-assisted quantum control.—— Suppose we intend to implement some unitary gate on the central system assisted by a d-level ancilla system. The total Hamiltonian is

$$H_{\text{tot}} = H_0 + H_c,$$

where $H_c$ is the control Hamiltonian and $H_0 = H_{\text{as}} + H_{\text{cs}} + H_{\text{int}}$ is the static part of the total Hamiltonian with contributions from the ancilla system, the central system and their interaction. We assume that $\{ H_{\text{as}}, H_{\text{int}} \} = 0$, so that $H_{\text{int}}$ preserves the eigenbasis $\{|m\rangle \}_{m=0}^{d-1}$ of the ancilla ($H_{\text{as}}|m\rangle = \varepsilon_m|m\rangle$), which naturally provides the projection basis of the initial and final ancilla states. The static Hamiltonian $H_0$ can also be diagonalized in the eigenbasis $\{|m\rangle \}_{m=0}^{d-1}$ of the ancilla ($H_{\text{as}}|m\rangle = \varepsilon_m|m\rangle$),

$$H_0 = \sum_{m=0}^{d-1} |m\rangle \langle m| \otimes (\varepsilon_m + H_{\text{cs}} + H_{\text{int},m}),$$

with $H_{\text{int},m} = \langle m|H_{\text{int}}|m\rangle$. The propagator for total system can be expanded in the ancilla eigenbasis as

$$U(t_2, t_1) = \mathcal{T} \exp \left( -i \int_{t_1}^{t_2} H_{\text{tot}}(t) \, dt \right) = \sum_{m,n} \eta_{mn}(t_2, t_1) |m\rangle \langle n| \otimes V_{mn}(t_2, t_1),$$

where $\mathcal{T}$ is the time-ordering operator, $\eta_{mn}(t_2, t_1)$ is a complex function and and $V_{mn}(t_2, t_1)$ is an operator on the central system. For pre-selection of the ancilla state on $|i\rangle$ at time $t_1$ and post-selection on state $|f\rangle$ at $t_2$, the central system undergoes an quantum operation $V_{mn}(t_2, t_1)$. Below we will show that for specific sets of unitary $\{V_{mn}(t_2, t_1)\}$, the quantum gates on the central system can be PI of Markovian ancilla errors.

Markovian ancilla noise.—— We assume that the central system suffers much weaker noise than the ancilla and therefore can be regarded as noise-free within the ancilla coherence time [Fig. 1(a)]. Suppose the ancilla suffers from Markovian noise and the dynamics of the total system is

$$\frac{d\rho}{dt} = i[\rho, H_{\text{tot}}] + \left( \sum_i D[\sqrt{\kappa_i} L_i] + \sum_j D[\sqrt{\gamma_j} J_j] \right) \rho,$$

where $D[A] \rho = A \rho A^\dagger - \{ A^\dagger A, \rho \}/2$ is the Lindbladian dissipator, $\{L_i\}/\{J_j\}$ are the Lindblad operators describing the ancilla relaxation/dephasing errors ($L_i = |m\rangle \langle n|$, $J_j = \sum_{m=0}^{d-1} \Delta^{(m)}_j |m\rangle \langle m|$ with $\Delta^{(m)}_j$ being a complex number), and $\kappa_i/\gamma_j$ is the relaxation/dephasing rate. The ancilla dephasing and relaxation errors can be unified into a general class of ancilla errors [33].

The Liouville superoperator $\mathcal{L}(t)$ generating the Markovian dynamics in Eq. (4) can be divided into two parts [35],

$$\frac{d\rho}{dt} = \mathcal{L}(t) \rho(t) = (\mathcal{L}_{\text{eff}}(t) + \mathcal{S}) \rho(t),$$

where $\mathcal{L}_{\text{eff}} \rho = i(\rho H_{\text{eff}}^\dagger - H_{\text{eff}} \rho)$ represents the no-jump evolution with $H_{\text{eff}}(t) = H_{\text{tot}}(t) - \sum_i (\sum_j \kappa_j L_j^\dagger L_j + \sum_j \gamma_j J_j^\dagger J_j)$, and $\mathcal{S} \rho = \sum_i \kappa_i L_i \rho L_i^\dagger + \sum_j \gamma_j J_j \rho J_j^\dagger$ represents the quantum jumps associated with the no-jump evolution. The propagator for the whole system can be represented by the generalized Dyson expansion as

$$\rho(t) = \sum_{p=0}^{\infty} \mathcal{G}_p(t, 0) \rho(0),$$

with

$$\mathcal{G}_0(t, 0) = \mathcal{W}(t, 0),$$

$$\mathcal{G}_p(t, 0) = \int_0^t dt_1 \cdots \int_0^{t_p} dt_2 \int_0^{t_1} dt_1 \mathcal{W}(t, t_p) \cdots \mathcal{W}(t_2, t_1) \mathcal{W}(t_1, 0), \quad p \geq 1,$$

where $\rho(0) = |m\rangle \langle m| \otimes \rho_{cs}$ is a product state with $m \in [0, d - 1]$ and $\rho_{cs}$ being the initial density matrix of the central system, and $\mathcal{W}(t_2, t_1) \rho = W(t_2, t_1) \rho W^\dagger(t_2, t_1)$ with $W(t_2, t_1) = \mathcal{T} \exp(-i \int_{t_1}^{t_2} H_{\text{eff}}(t) \, dt)$ being the no-jump propagator. The $p$th-order Dyson expansion $\mathcal{G}_p(t, 0)$ contains all the possible trajectories with any sequence of $p$ ancilla jump events, therefore describing the $p$th-order ancilla errors. When $\kappa t, \gamma t \ll 1$, the Liouville superoperator is well approximated by a finite-order Dyson expansion. Below we will show how quantum gate on the central system can be PI of the ancilla relaxation and dephasing errors with a specific no-jump propagator.

Path independence for ancilla errors.—— The PI gates in this letter can be understood as follows. With an initial ancilla eigenstate $|i\rangle$ of $H_{\text{as}}$, some control Hamiltonian acting during $[0, t]$ and a final projective measurement on the ancilla with result $|r\rangle$, the central system undergoes a deterministic unitary evolution up to finite-order or
infinite-order Dyson expansion in Eq. (6) (a sequence of finite or infinite ancilla errors). For a specific final ancilla state, the desired unitary gate is implemented; for all other final ancilla states, the gate fails but the central system still undergoes a deterministic unitary evolution so that conditional operation may be applied until the gate succeeds. Now we provide a formal definition of path independence.

**Definition 1.**— (Path independence) Let the ancilla start from $|i\rangle$ and end in $|r\rangle$, with $|i\rangle, |r\rangle \in \{|m\}_{\ell=0}^{d-1}$. Suppose

$$
|r\rangle \left[ \sum_{p=0}^{k} \mathcal{G}_p(t, 0) (|i\rangle \langle i| \otimes \rho_{\text{cs}}) \right] |r\rangle \propto \mathcal{U}_{r_i}(t, 0) \rho_{\text{cs}},
$$

(9)

applies for $k \leq n$ but does not hold for $k > n$, where $\mathcal{U}_{r_i}(t, 0) \rho_{\text{cs}} = U_{r_{t_1}}(t, 0) \rho_{\text{cs}} U_{r_{t_1}}^\dagger(t, 0)$ is a unitary channel on the central system. Then we say the quantum gate on the central system is PI of the ancilla error up to the $n$th-order from $|i\rangle$ to $|r\rangle$.

The path independence for ancilla errors is possible if the no-jump propagator is in a PI form below.

**Lemma 1.**— Let $\{U_{mn}(t_{2_1}, t_{1})\}_{m,n=0}^{d-1}$ be a set of unitaries on the central system that are differentiable with respect to $t_2$ and $t_1$ and also satisfy the PI condition

$$
U_{me}(t_3, t_2) U_{en}(t_2, t_1) = U_{mn}(t_3, t_1),
$$

(10)

with $m, e, n \in [0, d - 1]$, there exists a PI no-jump propagator

$$
W(t_2, t_1) = \sum_{m,n} \xi_{mn}(t_2, t_1) |m\rangle \langle n| \otimes U_{mn}(t_2, t_1),
$$

(11)

where $\{\xi_{mn}(t_2, t_1)\}$ are a set of complex functions of $t_2$ and $t_1$ satisfying $\xi_{mn}(t_3, t_1) = \sum_{e=0}^{d-1} \xi_{me}(t_3, t_2) \xi_{en}(t_2, t_1)$ and $\xi_{mn}(t, t) = \delta_{mn}$.

**Lemma 2.**— The PI condition for $\{U_{mn}(t_2, t_1)\}$ in Eq. (10) is satisfied if and only if

$$
U_{mn}(t_2, t_1) = R_m(t_2) U_{mn} R_n^\dagger(t_1),
$$

(12)

where $R_m(t) = T \{ e^{-i \int_0^t H_m(t') dt'} \}$ with $H_m(t)$ being an arbitrary time-dependent Hamiltonian on the central system and $U_{mn} = U_{mn}(0, 0)$ satisfy the cocycle condition

$$
U_{me} U_{en} = U_{mn}.
$$

(13)

From the cocycle condition, we have $U_{me} \cdots U_{ia} U_{ba} U_{am} = \mathbb{1}$ with $a, b, c, \ldots, n \in [0, d - 1]$. From the viewpoint of non-Abelian path integration (Table I) [36], the set of discrete ancilla states $\{|m\}$ defines a manifold, $U_{mn}$ is the parallel-transport operator from $|n\rangle$ to $|m\rangle$, and then the cocycle condition means that the holonomy for any loop path $|m\rangle \rightarrow |a\rangle \rightarrow |b\rangle \rightarrow |c\rangle \cdots \rightarrow |e\rangle \rightarrow |m\rangle$ is always the identity.

**Definition 2.**— (Noiseless ancilla subspace) In the interaction picture associated with $H'_n(t) = \sum_{m=0}^{d-1} |m\rangle \langle m| \otimes H_m(t)$ [note that $H_0$ in Eq. (2) and $H'_1(t)$ are similar but can be different], $\{U_{mn}(t_2, t_1)\}$ become constant unitaries $\{U_{mn}\}$. Denote the ancilla subspace $\{|k\}$ with $H_k(t)$ for all the ancilla states therein at most differing by some real constants as the noiseless ancilla subspace.

A special case of PI gates is the previously proposed error-transparent gates [37, 38] for a QEC code, with the error syndromes corresponding to the ancilla states here. The error transparency requires the physical Hamiltonian implementing the gate commutes with the errors when acting on the QEC code space, corresponding to the case where all the identity loop paths are around single ancilla states in the noiseless ancilla subspace ($\xi_{mn} = 0$ if $m \neq n$), and thus fulfill the PI criterion. However, the PI gates contains a larger set of operations, because it is possible to fulfill the PI criterion with the physical Hamiltonian not commuting with the errors.

We now classify the ancilla errors into three types: dephasing errors (type-I), relaxation errors (type-II), and both dephasing and relaxation errors (type-III). Then we can decompose Eq. (8) as $\mathcal{G}_p = \sum_{\alpha=1}^3 \mathcal{G}^\alpha_p$ with $\mathcal{G}^\alpha_p$ representing the part of $\mathcal{G}_p$ containing only type-$\alpha$ errors, and the path independence for type-$\alpha$ of ancilla error can be analysed by replacing $\mathcal{G}_p$ with $\mathcal{G}^\alpha_p$. Below we gave the main result of the letter.

**Theorem 1.**— Let the ancilla start from $|i\rangle$, evolve with the no-jump propagator in Eq. (11), and finally be projected to $|r\rangle$. Then

(i) if $\xi_{r_i} \neq 0$, the quantum gate on the central system is PI of any ancilla dephasing (type-I) errors up to infinite-order;
(ii) if $\xi_{r_i} = 0$ but the ancilla can go from $|i\rangle$ to $|r\rangle$ with a unique sequence of at most $n$ ancilla relaxation errors in the noiseless ancilla subspace, the central system gate is PI of the ancilla relaxation (type-II) errors up to the $n$th-order;
(iii) with the same conditions as those in (ii), the central system gate is PI of the combination of up to the $n$th-order ancilla relaxation and up to infinite-order ancilla dephasing (type-III) errors.

| Non-Abelian path integration | PI propagator |
|-----------------------------|---------------|
| **Parallel-transport operator** | $\exp(i \int \mathbf{A})$ | $U_{mn}$ |
| **Holonomy** | $\exp(i \oint A)$ | $U_{me} \cdots U_{ia} U_{ba} U_{am}$ |

**Table I:** Comparison between the non-Abelian path integration and the PI propagator. Here $\mathbf{A}$ is the connection one-form.
FIG. 2: Schematic of ancilla evolution paths with different kinds of ancilla errors. (a) In the path with blue line, the ancilla goes from $|i\rangle$ to $|f\rangle$ without any ancilla error and the unitary gate on the central system is $U_{fi}$; In the paths with green lines, the ancilla suffers two dephasing errors $|a\rangle\langle a|$ and at $|c\rangle\langle c|$, while the unitary gate on the central system is still $U_{fi}U_{ea}U_{ai}=U_{fi}$ that is independent of the dephasing error times. (b) In the path with red lines, the ancilla suffers a relaxation error $|b\rangle\langle a|$ (dashed red arrow lines) with the unitary gate as $U_{eb}U_{ai}$; In the path with yellow line, the ancilla suffers two additional dephasing errors $|a\rangle\langle a|$ and $|c\rangle\langle c|$ but with the same unitary gate as that for a single relaxation error. The solid (hollow) circles represent the initial and final (intermediate) ancilla states, and the red dashed arrows represent the ancilla relaxation errors. Here we adopt the interaction picture associated with $H_0(t)$.

The proof of Lemma 1, Lemma 2, Theorem 1 and construction of PI control Hamiltonian and jump operators can be found in [33]. To provide an intuitive picture for Theorem 1, we show the ancilla evolution paths with different kinds of ancilla errors in Fig. 2. Without ancilla errors [blue trajectory in Fig. 2(a)], the ancilla goes directly from the initial state $|i\rangle$ to the final state $|f\rangle$ with the target unitary gate $U_{fi}$ on the central system. With ancilla dephasing errors [green trajectories in Fig. 2(a)], the ancilla takes a different continuous path from $|i\rangle$ to $|f\rangle$, but the final unitary gate on the central system remains unchanged since it depends only on the initial and final ancilla states [Eq. (13)]. With only ancilla relaxation errors [red trajectories in Fig. 2(b)], the ancilla path is composed of discontinuous segments connected by the relaxation error operators, and the final unitary gate on the central system is often different from that without ancilla errors, but if the ancilla ends in other states, the central system still undergoes a deterministic unitary evolution. With both ancilla relaxation and dephasing errors [orange trajectories in Fig. 2(b)], for each path segment caused by the relaxation errors, the ancilla goes another continuous way with the same initial and final states, so the final unitary gate on the central system is the same as that with only relaxation errors.

Example: PI gates in superconducting circuits.— We consider the implementation of the photon-number selective arbitrary phase (SNAP) gates in superconducting circuits [14, 15]. The superconducting cavity (central system) dispersively couples to a nonlinear transmon device (ancilla system) with Hamiltonian $H_0 = \omega_a |e\rangle\langle e| + \omega_f |f\rangle\langle f| + \omega_s a^\dagger a - \chi |e\rangle\langle e| + |f\rangle\langle f| |a\rangle\langle a|$, where $\omega_a$ ($\omega_s$) are the transmon (cavity) frequency, $a$ ($a^\dagger$) is the annihilation (creation) operator of the cavity mode, $\chi$ is the dispersive coupling strength, and $|e\rangle$ ($|g\rangle$) denotes the excited (ground) state of the ancilla transmon. The SNAP gate on the cavity, $S(\varphi) = \sum_{n=0}^{\infty} e^{i\varphi n} |n\rangle\langle n|$ imparts arbitrary phases $\varphi = \{\varphi_n\}_{n=0}^\infty$ to the different Fock states of the cavity.

In the interaction picture associated with $H_0$, the PI operation can be obtained from $H^{(I)} = \Omega |g\rangle\langle e| \otimes S(\varphi) + |e\rangle\langle g| \otimes S(-\varphi) + \delta |e\rangle\langle e| = \Omega \sum_{n=0}^{\infty} e^{i\varphi n} |g, n\rangle\langle e, n| + e^{i\varphi n} |e, n\rangle\langle g, n| + \delta |e\rangle\langle e|$. One can show that the propagator produced by such a PI operation is in the form of Eq. (11) [33]. Returning to the Schrödinger’s picture, $H_c = \Omega \sum_{n=0}^{\infty} e^{i(\omega_a-\omega_s)n t + \varphi_n - \delta} |g, n\rangle\langle e, n| + \text{h.c.}$. When $\Omega \ll \chi$, the control Hamiltonian acting on both the transmon and cavity can be simplified as the driving acting on the transmon alone but with multiple frequency components to distinguish the different cavity Fock states, $H_c \approx \epsilon_{ge}(t) e^{i (\omega_a-\delta) t} |g\rangle\langle e| + \text{h.c.}$ with $\epsilon_{ge}(t) = \sum_n \Omega e^{i(\varphi_n - \varphi_n - \delta)}$, which is a simplified and generalized control Hamiltonian compared to the original proposal for SNAP gates [14] (Note that the equal driving amplitude for all the driving frequencies is required for the PI control Hamiltonian). According to Theorem 1, the SNAP gates are PI of any transmon dephasing error with $D[\sqrt{\chi} (c|e\rangle\langle e| + c^\dagger|g\rangle\langle g|)]$ with $c_{ge}$ being any complex number, but not PI of the transmon relaxation error with $D[\sqrt{\chi} |g\rangle\langle e|]$ since the $|e\rangle \rightarrow |g\rangle$ transition can be induced by either the driving or relaxation with different unitary operations on the cavity.

To make the SNAP gates PI of the dominant transmon relaxation error, we may use a 3-level transmon with $H_0 = \omega_a |e\rangle\langle e| + \omega_f |f\rangle\langle f| + \omega_s a^\dagger a - \chi |e\rangle\langle e| + |f\rangle\langle f| |a\rangle\langle a|$, where the dispersive coupling strength is engineered to the same for the first-excited transmon state $|e\rangle$ and the second-excited state $|f\rangle$ [34] ($\chi$-matching condition). The SNAP gate is implemented by driving the $|g\rangle \leftrightarrow |f\rangle$ transition instead of the $|g\rangle \leftrightarrow |e\rangle$ transition above [39]. Since $|e\rangle, |f\rangle$ forms a noiseless ancilla subspace, according to Theorem 1, the SNAP gate is PI of the dominant transmon relaxation error with $D[\sqrt{\chi} (c|e\rangle\langle f|)]$ and also to any transmon dephasing error with $D[\sqrt{\chi} (c|g\rangle\langle g| + c^\dagger|e\rangle\langle e| + c_{fj}|f\rangle\langle f|)]$. Note that the PI SNAP gates are not error-transparent (with $H_c$ not commuting with the transmon relaxation or dephasing errors), but it is still PI to enable robustness against transmon errors.

Discussion: PI gates for both ancilla errors and central system errors. The PI gates for ancilla errors can also be made PI of the central system errors. We assume that the central system also suffers Markovian noise with the Lindbladian dissipators $\sum_{\zeta=0}^{n-1} D[\sqrt{\zeta} E_{\zeta}]$. Suppose we can find a quantum error correction (QEC) code for the central system [1, 40, 41], which means that the error set $\{E_{\zeta}\}$ satisfy the Knill-Laflamme condition.
$P_0 E_j^t E_j P_0 = A_{ij} P_0$ with $E_0 = P_0$ being the projection to the code subspace and $A$ a Hermitian matrix. We may diagonalize $A$ as $B = u^\dagger A u$ to obtain another set of correctable errors $\{F_k\}$ with $F_k = \sum_{ik} u_{ik} E_i$, satisfying $P_0 F_n^t F_l P_0 = r_k \delta_{kl} P_0$ with $P_0 = P_0$, $B_{00} = 1$ and $r_k = B_{kk}$. Then the condition for path independence against the central system errors is

$$[H'_0(t), F_k] = \sum_{m=0}^{d-1} c_{m,k}(t) |m\rangle \langle m| \otimes F_k,$$

where $m \in [0, d-1]$, $k \in [0, q-1]$ and $c_{m,k} \in \mathbb{R}$. This condition ensures that $\mathcal{T} e^{\int_{t_0}^{t_1} H'_0(t') dt'} F_k (\mathcal{T} e^{\int_{t_0}^{t_1} H'_0(t') dt'})^\dagger = \sum_m e^{i \int_{t_0}^{t_1} c_{m,k}(t') dt'} |m\rangle \langle m| \otimes F_k$ is a tensor product of a ancilla dephasing operator and the same error operator $F_k$. Then in the interaction picture associated with $H'_0(t)$, the PI no-jump propagator for both ancilla and central system errors $[33, 37, 38]$ can be constructed as in Eq. (11) with

$$U_{mn} = \sum_k e^{i \phi_{mn,k}} F_k U_{mn,0} F_k^\dagger / r_k,$$

where $U_{mn,0}$ is the target unitary in the code subspace satisfying $U_{mn,0} U_{mn,0}^\dagger = P_0$ and $\phi_{mn,k} \in \mathbb{R}$. After such a PI gate, we can make a joint measurement on both the ancilla state and the error syndromes of the central system, then the path independence of ancilla errors is ensured and the first-order central system errors during the gate can also be corrected at the end of the gate.

**Summary.** To address the challenge of ancilla-induced decoherence, we provide a general criterion of path independence, so that we can achieve robust quantum control over the central system even with noisy ancilla. For quantum information processing with bosonic encoding, such PI design will be crucial in protecting the encoded information from ancilla errors, while the previous transversal approach does not apply. Moreover, different from the traditional approach with separated quantum control and error correction tasks, our approach integrates quantum control and error correction. This can help achieve a hardware-efficient design with both suppression of the ancilla errors and robust quantum operations on the central system.

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*Note added.*—Recently the PI SNAP gates have been experimentally implemented in a superconducting circuits [42], with the SNAP gate fidelity significantly improved by the PI design.

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Supplementary Information for “Path-Independent Quantum Gates with Noisy Ancilla”

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I. PROOF OF LEMMA 1

Lemma 1.— Let \{U_{mn}(t_2, t_1)\}_{m,n=0}^{d-1} be a set of unitaries on the central system that are differentiable with respect to \(t_2\) and \(t_1\) and also satisfy the PI condition

\[
U_{mc}(t_3, t_2)U_{en}(t_2, t_1) = U_{mn}(t_3, t_1),
\]

(S1)

with \(m, e, n \in [0, d - 1]\), there exists a PI no-jump propagator

\[
W(t_2, t_1) = \sum_{m,n} \xi_{mn}(t_2, t_1)|m\rangle\langle n| \otimes U_{mn}(t_2, t_1),
\]

(S2)

where \(\{\xi_{mn}(t_2, t_1)\}\) are a set of complex functions of \(t_2\) and \(t_1\) satisfying \(\xi_{mn}(t_3, t_1) = \sum_{e=0}^{d-1} \xi_{me}(t_3, t_2)\xi_{en}(t_2, t_1)\) and \(\xi_{mn}(t, t) = \delta_{mn}\).

Proof. — Note that the no-jump propagator should satisfy

\[
W(t_2, t_1) = W(t_3, t_2)W(t_2, t_1),
\]

(S3)

implying that any matrix element between the ancilla states for both sides of the equation should be the same,

\[
\langle m|W(t_3, t_1)|n\rangle = \langle m|W(t_3, t_2)W(t_2, t_1)|n\rangle = \sum_{e=0}^{d-1} \xi_{me}(t_3, t_2)\xi_{en}(t_2, t_1)U_{me}(t_3, t_2)U_{en}(t_2, t_1) = \sum_{e=0}^{d-1} \xi_{me}(t_3, t_2)\xi_{en}(t_2, t_1)U_{mn}(t_3, t_1),
\]

(S4)

where we have used \(U_{me}(t_3, t_2)U_{en}(t_2, t_1) = U_{mn}(t_3, t_1)\) from the second line to the third line. So \(W(t_3, t_1)\) is still in the same form as Eq. (S2) with \(\{U_{mn}(t_2, t_1)\}\) satisfying the PI condition. If we require \(\xi_{mn}(t_2, t_1)\) to be a well-defined function of \(t_2\) and \(t_1\), then \(\xi_{mn}(t_2, t_1)\) should satisfy

\[
\xi_{mn}(t_3, t_1) = \sum_{e=0}^{d-1} \xi_{me}(t_3, t_2)\xi_{en}(t_2, t_1).
\]

(S5)

The no-jump propagator should also satisfy that \(W(t, t) = I\) with \(I\) being the identity operator for the whole system, which requires that \(\xi_{mn}(t, t) = \delta_{mn}\). Moreover, we have \(W(t_2, t_2) = W(t_2, t_1)W(t_1, t_2) = I\), so \(W(t_2, t_1)^{-1} = W(t_1, t_2)\). □
II. PROOF OF LEMMA 2

Lemma 2.— The PI condition for \( \{U_{mn}(t_2, t_1)\} \) in Eq. (S1) is satisfied if and only if
\[
U_{mn}(t_2, t_1) = R_m(t_2)U_{mn}R_n^\dagger(t_1),
\]
where \( R_m(t) = T\{e^{-i\int_0^t H_m(t')dt'}\} \) with \( H_m(t) \) being an arbitrary time-dependent Hamiltonian on the central system and \( U_{mn} = U_{mn}(0, 0) \) satisfy the cocycle condition
\[
U_{me}U_{en} = U_{mn}.
\]

Proof. — The ‘if’ part is easy to prove. Suppose \( U_{mn}(t_2, t_1) = R_m(t_2)U_{mn}R_n^\dagger(t_1) \) with \( \{U_{mn}\} \) satisfying the cocycle condition, then
\[
U_{me}(t_3, t_2)U_{en}(t_2, t_1) = R_m(t_3)U_{me}R_n^\dagger(t_2)U_{en}R_n(t_1)
\]

\[
= R_m(t_3)U_{me}U_{en}R_n(t_1) = R_m(t_3)U_{mn}R_n(t_1) = U_{mn}(t_3, t_1).
\]

Conversely, suppose \( U_{me}(t_3, t_2)U_{en}(t_2, t_1) = U_{mn}(t_3, t_1) \), then
\[
U_{me}(t_3, t_2 + \Delta t)U_{en}(t_2 + \Delta t, t_1) = U_{mn}(t_3, t_1).
\]

Since we assume that \( \{U_{mn}(t_2, t_1)\} \) are differentiable with respect to both \( t_1 \) and \( t_2 \), so we can define
\[
\frac{\partial U_{mn}(t_2, t_1)}{\partial t_2} = -iH_{mn}^{(l)}(t_2, t_1)U_{mn}(t_2, t_1),
\]
\[
\frac{\partial U_{mn}(t_2, t_1)}{\partial t_1} = iU_{en}(t_2, t_1)H_{en}^{(r)}(t_2, t_1),
\]
so we have
\[
U_{me}(t_3, t_2 + \Delta t)U_{en}(t_2 + \Delta t, t_1) = \left[U_{me}(t_3, t_2) - i\Delta tU_{me}(t_3, t_2)H_{me}^{(r)}(t_3, t_2)\right][U_{en}(t_2, t_1) + i\Delta tH_{en}^{(l)}(t_2, t_1)U_{en}(t_2, t_1)]
\]
\[
= U_{me}(t_3, t_2)U_{en}(t_2, t_1) - i\Delta tU_{me}(t_3, t_2)[H_{me}^{(r)}(t_3, t_2) - H_{en}^{(l)}(t_2, t_1)]U_{en}(t_2, t_1) + O(\Delta t^2).
\]

Eq. (S9) requires that \( H_{me}^{(r)}(t_3, t_2) = H_{en}^{(l)}(t_2, t_1) \) for any \( t_1, t_2, t_3 \), and \( m, e, n \in [0, d - 1] \), implying that \( H_{me}^{(r)}(t_3, t_2) \) is independent of \( t_3 \) and \( m \) and \( H_{me}^{(l)}(t_3, t_2) \) is independent of \( t_1 \) and \( n \), so \( H_{me}^{(r)}(t_3, t_2) = H_{en}^{(l)}(t_2, t_1) = H_{e}(t) \). Then \( U_{mn}(t_2, t_1) \) can be obtained from \( U_{mn} = U_{mn}(0, 0) \) by first integrating the first variable from 0 to \( t_2 \) and then the second variable from 0 to \( t_1 \), so \( U_{mn}(t_2, t_1) = R_m(t_2)U_{mn}R_n^\dagger(t_1) \) with \( R_m(t) = T\{e^{-i\int_0^t H_m(t')dt'}\} \). Inserting this expression of \( U_{mn}(t_2, t_1) \) into \( U_{me}(t_3, t_2)U_{en}(t_2, t_1) = U_{mn}(t_3, t_1) \), we have the cocycle condition \( U_{me}U_{en} = U_{mn} \). From the cocycle condition, we can derive that (i) \( U_{mn} = \mathbb{1} \), (ii) \( U_{mn} = U_{nm}^\dagger \); (iii) \( U_{ma} = U_{me}\cdots U_{ma}U_{ba} \), or equivalently \( U_{me}\cdots U_{ca}U_{ba}U_{am} = \mathbb{1} \) with \( m, a, b, \cdots, e \in [0, d - 1] \).

III. PROOF OF THEOREM 1

Theorem 1.— Let the ancilla start from \( |i\rangle \), evolve with the no-jump propagator in Eq. (S2) and finally be projected to \( |r\rangle \). Then
(i) if \( \xi_{ri} \neq 0 \), the quantum gate on the central system is PI of any ancilla dephasing (type-I) errors up to infinite-order; (ii) if \( \xi_{ri} = 0 \) but the ancilla can go from \( |i\rangle \) to \( |r\rangle \) with a unique sequence of at most \( n \) ancilla relaxation errors in the noiseless ancilla subspace, the central system gate is PI of the ancilla relaxation (type-II) errors up to the \( nth \)-order; (iii) with the same conditions as those in (ii), the central system gate is PI of the combination of up to the \( nth \)-order ancilla relaxation and up to infinite-order ancilla dephasing (type-III) errors.

Proof. — Consider a sequence of \( n \) dephasing errors \( J_j(t_1) \rightarrow J_k(t_2) \cdots \rightarrow J_q(t_n) \) with \( 0 \leq t_1 < t_2 \cdots < t_n \leq t \), then the propagator for the central system can be expressed as the matrix element of the propagator between the initial
state $|i\rangle$ and final state $|l\rangle$ of the ancilla,

$$
\langle r | W(t, t_n) J_q \cdots J_b W(t_2, t_1) J_j W(t_1, 0) |i\rangle
$$

$$
= \sum_{a=0}^{d-1} \sum_{b=0}^{d-1} \sum_{c=0}^{d-1} \Delta_q^{(a)} \cdots \Delta_b^{(b)} \Delta_j^{(c)} \xi_{ra}(t, t_n) \cdots \xi_{bc}(t_2, t_1)

\times \xi_{ci}(t_1, 0) U_{ra}(t, t_n) \cdots U_{bc}(t_2, t_1) U_{ci}(t_1, 0)

\propto U_{rl}(t, 0),
$$

(S13)

where we have used Eq. (S1) in Lemma 1. Since the final unitary gate on the central system is independent of the dephasing jump sequence \{J_q, J_b, J_j\} and the jump times \{t_1, t_2, \cdots, t_n\}, so the gate on the central system remains unchanged even after the path integral in Eq. (8) in the main text. Here $\xi_{ri} \neq 0$ ensures that the ancilla $|i\rangle \rightarrow |r\rangle$ transition is permitted by the PI propagator.

Suppose from $|i\rangle$ to $|r\rangle$, the only path is through a single relaxation operator $L_i$, then the propagator for the central system is

$$
\langle r | W(t, t_1) L_i W(t_1, 0) |i\rangle \propto U_{r,m_i}(t, t_1) U_{n_i,l}(t_1, 0)
$$

$$
= R_r(t) U_{r,m_i} R_{n_i}(t) R_{n_i}(t_1) U_{n_i,l},
$$

(S14)

where we have used Eq. (S6) in Lemma 2. This implies that the central system undergoes the same final unitary evolution $R_r(t) U_{r,m_i} U_{n_i,l}$ independent of the jump time $t_1$ if $|m_i\rangle$ and $|n_i\rangle$ are in the noiseless ancilla subspace with $H_{m_i}(t)$ and $H_{n_i}(t)$ differing only by some real constant. If the only path consists of a finite sequence of $n$ relaxation jumps $L_j(t_1) \rightarrow L_k(t_2) \cdots \rightarrow L_q(t_n)$ in the noiseless ancilla subspace, the final unitary operation on the central system becomes $R_r(t) U_{r,m_n} U_{n_n-1} U_{n_n-1,m_n} U_{n_n,l}$. Here $\xi_{ri} = 0$ ensures that the ancilla relaxation path does not interfere with the path without errors or with only dephasing errors. In practice, there may be more than one relaxation paths (consisting of no more than $n$ relaxation jumps) from $|i\rangle$ to $|r\rangle$, which produces the same unitary operation on the central system, then the central system gate is also PI of the relaxation errors up to the $n$th-order. Any infinite sequence of dephasing errors in any time interval (say $[t_1, t_2]$) leave the unitary operation for each time interval $(U_{n_n,m_n})$ unchanged, therefore the central system undergoes the same final unitary operation as that with only relaxation errors.

\[\square\]

IV. CONSTRUCTION OF PI CONTROL HAMILTONIAN AND JUMP OPERATORS

The general effective Hamiltonian generating the PI no-jump propagator can be derived as

$$
H_{\text{eff}}(t) = i \frac{\partial W(t, t_0)}{\partial t} W^{-1}(t, t_0) = i \frac{\partial W(t, t_0)}{\partial t} W(t_0, t)
$$

$$
= i \sum_{m=0}^{d-1} \xi_{mp}(t_0, t_0) |m\rangle \langle p| \otimes \frac{\partial U_{mp}(t, t_0)}{\partial t} |m\rangle \langle p| \otimes U_{mp}(t, t_0)

\Bigg[ \sum_{q=0}^{d-1} \xi_{qn}(t_0, t_0) |q\rangle \langle n| \otimes U_{qn}(t_0, t) \Bigg],
$$

$$
= \sum_{m=0}^{d-1} |m\rangle \langle m| \otimes H_m(t) + i \sum_{m,n} \sum_{p=0}^{d-1} \frac{\partial \xi_{mp}(t_0, t_0)}{\partial t} \xi_{pn}(t_0, t_0) \Bigg[ |m\rangle \langle n| \otimes U_{mn}(t, t) \Bigg],
$$

(S15)

where the first part corresponds to $H_0$ or $H_0^\dagger(t)$ in the main text and the second part corresponds to $H_c - iH_{\text{jump}}/2$ with $H_{\text{jump}} = \sum_i \alpha_i K_i^\dagger K_i$. In the derivation, we have used $\xi_{mn}(t_3, t_1) = \sum_{q=0}^{d-1} \xi_{mq}(t_3, t_2) \xi_{qn}(t_2, t_1)\delta_{mn}$, $\partial U_{mp}(t_0, t)/\partial t = -iH_{mn}(t) U_{mp}(t, t_0) - U_{mp}(t, t_0) U_{pm}(t_0, t) = U_{mn}(t, t))$.

So the general form of the PI control Hamiltonian $H_c$ should be

$$
H_c(t) = \sum_{m,n}^{d-1} \varepsilon_{mn}(t) |m\rangle \langle n| \otimes U_{mn}(t, t) = \sum_{m,n}^{d-1} \varepsilon_{mn}(t) |m\rangle \langle n| \otimes R_m(t) U_{mn} R_n^\dagger(t),
$$

(S16)

where $\varepsilon_{mn}(t) = \varepsilon_{nm}^*(t)$.

The jump Hamiltonian $H_{\text{jump}} = \sum_i \alpha_i K_i^\dagger K_i$ can be in the PI form if each positive operator $K_i^\dagger K_i$ is in the PI form,
i.e. $K_i^d K_i = \sum_{m=0}^{d-1} \omega_i^{(m)} |m\rangle \langle m|$ with $\omega_i^{(m)} \geq 0$ for any $m$. By polar decomposition, the general form of $K_i$ is

$$K_i = S_i \sqrt{K_i^d K_i} = \sum_{m=0}^{d-1} \sqrt{\omega_i^{(m)}} S_i |m\rangle \langle m|,$$

(S17)

with $S_i$ being an unitary matrix for the ancilla. To be PI of the ancilla error $K_i$, $S_i$ can only non-zero off-diagonal elements only in the noiseless ancilla subspace. To see this, consider

$$\langle r | W(t, t_1) K_i W(t_1, 0) | i \rangle = \sum_{n=0}^{d-1} \sum_{m=0}^{d-1} \sqrt{\omega_i^{(m)}} \langle r | W(t, t_1) | n \rangle \langle n | S_i | m \rangle \langle m | W(t_1, 0) | i \rangle = \sum_{n=0}^{d-1} \sum_{m=0}^{d-1} \sqrt{\omega_i^{(m)}} \xi_{fn}(t, t_1) \xi_{mi}(t_1, 0) \langle n | S_i | m \rangle R_i(t_1) U_{rn} R_i^\dagger(t_1) R_m(t_1) U_{mi},$$

(S18)

For $\langle n | S_i | m \rangle \neq 0$, when $|m\rangle$ and $|n\rangle$ are in the noiseless ancilla subspace with $H_m(t)$ and $H_n(t)$ differing by a constant, then $R_i^\dagger(t_1) R_m(t_1)$ is a trivial phase factor. An additional requirement for being PI of $K_i$ is that there is only one error path from $|i\rangle$ to $|f\rangle$.

The dephasing and relaxation Lindblad operators in the main text are specific examples of the general Lindblad operators $\{K_i\}$. The dephasing operator $J_j = \sum_{m=0}^{d-1} \Delta_j^{(m)} |m\rangle \langle m|$ = $\sum_{m=0}^{d-1} |\Delta_j^{(m)} S_j |m\rangle \langle m|$ with $S_j = \sum_{m=0}^{d-1} |\Delta_j^{(m)} S_j |m\rangle \langle m|$ while the relaxation operator $L_i = |m_i\rangle \langle n_i| = U_{il} |n_i\rangle \langle m_i|$ with $S_t = |m_i\rangle \langle n_i| + |n_i\rangle \langle m_i| + \sum_{m\neq m_i, n_i} |m\rangle \langle m|.$

V. EXACT EXPRESSIONS FOR THE PI NO-JUMP PROPAGATOR

We now construct a specific PI control Hamiltonian. For simplicity, we move to the interaction picture associated with $H_0(t) = \sum_{m=0}^{d-1} |m\rangle \langle m| \otimes H_m(t)$, and the PI no-jump propagator in Eq. (S2) becomes

$$W^{(I)}(t_2, t_1) = R^I(t_2) W(t_2, t_1) R(t_1) = \sum_{m,n} \xi_{mn}(t_2, t_1) |m\rangle \langle n| \otimes U_{mn},$$

(S19)

where $R(t) = T\{e^{-i \int_0^t H_0^{(t')}dt'}\} = \sum_{m=0}^{d-1} |m\rangle \langle m| \otimes R_m(t)$ with $R_m(t) = T\{e^{-i \int_0^t H_m^{(t')}dt'}\}$. The effective Hamiltonian in the interaction picture

$$H_{\text{eff}}^{(I)}(t) = R^I(t) H_{\text{eff}}(t) R(t) - i R^I(t) \frac{\partial R(t)}{\partial t} = H_c^{(I)} + H_0^{(I)} - H_0^{(t)} - i H_{\text{jump}}^{(I)},$$

(S20)

with $H_c^{(I)} = R^I(t) H_c(t) R(t)$, $H_0^{(I)} = R^I(t) H_0(t) R(t) = H_0$ due to $[H_0, H_0(t)] = 0$, $H_{\text{jump}}^{(I)} = \sum_j \kappa_j \left( L^{(I)}_j \right)^\dagger L^{(I)}_j + \sum_k \gamma_k \left( J^{(I)}_k \right)^\dagger J^{(I)}_k$ with $L^{(I)}_j / J^{(I)}_k = R^I(t) (L_j / J_k) R(t)$. Setting $H_0^{(t)} = H_0$ below, we have $H_{\text{eff}}^{(I)}(t) = H_c^{(I)} - i H_{\text{jump}}^{(I)}$.

Consider a simple case of the PI propagator,

$$W^{(I)}(t_2, t_1) = \sum_{m=0}^{d-1} \xi_{mm} |m\rangle \otimes \mathbb{I} + \sum_{\mu} \xi_{\mu} |m_{\mu}\rangle \otimes U_{\mu} + |n_{\mu}\rangle \langle m_{\mu}| \otimes U_{\mu}^\dagger,$$

(S21)

where $\{|m_{\mu}\rangle, |n_{\mu}\rangle\}_{\mu=1}^{\mu_{\text{max}}} \quad (\mu_{\text{max}} < \lfloor d/2 \rfloor)$ are unoverlapping pairs of ancilla states. For any pair $\{|m_{\mu}\rangle, |n_{\mu}\rangle\}$, the central system undergoes the unitary evolution $U_{\mu}$ ($U_{\mu}^\dagger$) from $|n_{\mu}\rangle$ to $|m_{\mu}\rangle$ (from $|m_{\mu}\rangle$ to $|n_{\mu}\rangle$), while for the remaining unpaired ancilla state, the state of the central system remain unchanged. One possible class of control Hamiltonian producing the FT propagator in Eq. (S21) is

$$H_c^{(I)} = \sum_{\mu} \Omega_\mu |m_{\mu}\rangle \langle n_{\mu}| \otimes U_{\mu} + |n_{\mu}\rangle \langle m_{\mu}| \otimes U_{\mu}^\dagger + (\delta_{m_{\mu}} |m_{\mu}\rangle \langle m_{\mu}| + \delta_{n_{\mu}} |n_{\mu}\rangle \langle n_{\mu}|),$$

(S22)
FIG. S1: Schematic of the PI-driving protocol for a $d$-level ancilla. The $d$ levels are grouped into non-overlapping pairs and singlets. The dephasing errors make the ancilla undergo additional 1-site or 2-site loops, while ancilla relaxation errors (represented by the red dashed arrows) in the noiseless ancilla subspace (represented by blue hollow circles) connect different pairs or unpaired states.

with the terms in the first (second) line representing the drive (detuning). One can verify that even with the addition of the non-Hermitian Hamiltonian, the above the control Hamiltonian produces the PI propagator in Eq. (S21). If there is a single relaxation path from one pair of ancilla states to another pair or to the unpaired ancilla states, with all the ancilla states in the relaxation path in the noiseless ancilla subspace, as shown in Fig. S1, then the quantum gate on central system is PI of any possible ancilla dephasing and relaxation errors. In practice, not all ancilla relaxation errors are within the noiseless ancilla subspace, so the gate may be PI of only finite-order relaxation errors.

With the PI control Hamiltonian in Eq. (S22), the effective non-Hermitian Hamiltonian is

$$H_{\text{eff}}^{(I)} = H_{c}^{(I)} - \frac{i}{2} \left( \sum_{j} \kappa_{j} (L_{j}^{(I)})^{\dagger} L_{j}^{(I)} + \sum_{k} \gamma_{k} (J_{k}^{(I)})^{\dagger} J_{k}^{(I)} \right)$$

$$= \sum_{\mu} \Omega_{\mu} (|m_{\mu}\rangle \langle n_{\mu}| \otimes U_{\mu} + |n_{\mu}\rangle \langle m_{\mu}| \otimes U_{\mu}^{\dagger}) + (\delta_{m_{\mu}} + i\lambda_{m_{\mu}})|m_{\mu}\rangle \langle m_{\mu}| + (\delta_{n_{\mu}} + i\lambda_{n_{\mu}})|n_{\mu}\rangle \langle n_{\mu}|$$

$$+ \sum_{k \in \{\text{unpaired}\}} i\lambda_{k} |k\rangle \langle k|,$$  \hspace{1cm} (S23)

where one can see the non-Hermitian terms modify the diagonal elements. Since the ancilla states are grouped into unoverlapped pairs and single unpaired states, so the total no-jump propagator is the direct sum of the propagators for the subspaces of the paired states and unpaired states. The propagator for the unpaired states is trivial, so here we try to derive the no-jump propagator for the ancilla subspace spanned by a single pair of ancilla states.

The effective non-Hermitian Hamiltonian for the subspace $\{ |m\rangle, |n\rangle \}$ (the subscripts omitted for simplicity) is

$$H_{\text{eff,mn}}^{(I)} = \Omega_{mn} (|m\rangle \langle n| \otimes U_{mn} + |n\rangle \langle m| \otimes U_{mn}^{\dagger}) + (\delta_{m} + i\lambda_{m})|m\rangle \langle m| + (\delta_{n} + i\lambda_{n})|n\rangle \langle n|,$$  \hspace{1cm} (S24)

Define the identity operator and Pauli operators for the ancilla subspace $\{ |m\rangle, |n\rangle \}$ as

$$I_{mn} = |m\rangle \langle m| + |n\rangle \langle n|,$$  \hspace{1cm} (S25a)

$$\sigma_{mn}^{x} = (|m\rangle \langle n| - |n\rangle \langle m|), \hspace{1cm} (S25b)$$

then $H_{\text{eff,mn}}^{(I)}$ can be recast as

$$H_{\text{eff,mn}}^{(I)} = \omega_{mn}^{0} I_{mn} + \omega_{mn}^{x} \sigma_{mn}^{x} \otimes \text{Re}(U_{mn}) + \omega_{mn}^{y} \sigma_{mn}^{y} \otimes \text{Im}(U_{mn}) + \omega_{mn}^{z} \sigma_{mn}^{z} \otimes \mathbb{I},$$  \hspace{1cm} (S26)

where $\omega_{mn}^{0} = [\delta_{m} + \delta_{n} + i(\lambda_{m} + \lambda_{n})]/2$, $\omega_{mn}^{x} = [\delta_{m} - \delta_{n} + i(\lambda_{m} - \lambda_{n})]/2$, $\omega_{mn}^{y} = \Omega_{mn}/2$, and $\text{Re}(U_{mn})/\text{Im}(U_{mn})$ denotes the real/imaginary part of $U_{mn}$ $[\text{Re}(U_{mn})^{2} + \text{Im}(U_{mn})^{2} = 1]$ due to $U_{mn} U_{mn}^{\dagger} = U_{mn}^{\dagger} U_{mn} = \mathbb{I}$. Then the
no-jump propagator generated by $H_{	ext{eff},mn}$ is

$$W_{mn}(t_2,t_1) = \exp \left[ -iH_{\text{eff},mn}(t_2 - t_1) \right]$$

$$= e^{-i\omega_{mn}(t_2 - t_1)} \left\{ \cosh[\omega_{mn}(t_2 - t_1)] - \sinh[\omega_{mn}(t_2 - t_1)] \right\} n_{xy} \sigma_{mn}^x \otimes \Re(U_{mn}) + n_{xy} \sigma_{mn}^y \otimes \Im(U_{mn}) + n_z \sigma_{mn}^z \otimes \mathbb{I} \right\}$$

$$= \xi_{mn} \langle m | \otimes \mathbb{I} + \xi_{mn} \langle n | \otimes \mathbb{I} + \xi_{mn} \langle m | \otimes U_{mn} + \xi_{mn} \langle n | \otimes U^\dagger_{mn},$$

(S27)

with

$$\xi_{mn} = e^{-i\omega_{mn}(t_2 - t_1)} \left\{ \cosh[\omega_{mn}(t_2 - t_1)] + n_z \sinh[\omega_{mn}(t_2 - t_1)] \right\},$$

(S28a)

$$\xi_{nn} = e^{-i\omega_{mn}(t_2 - t_1)} \left\{ \cosh[\omega_{mn}(t_2 - t_1)] - n_z \sinh[\omega_{mn}(t_2 - t_1)] \right\},$$

(S28b)

$$\xi_{mn} = \xi_{nm} = e^{-i\omega_{mn}(t_2 - t_1)} n_{xy} \sinh[\omega_{mn}(t_2 - t_1)],$$

(S28c)

where $\omega_{mn} = \sqrt{(\omega_{mn,x}^y)^2 + (\omega_{mn,z}^y)^2}$, $n_z = \omega_{mn,z}/\omega_{mn}$ and $n_{xy} = \omega_{mn,x}/\omega_{mn}$ (Note that $\omega_{mn}, n_z, n_{xy}$ are all complex numbers). In the derivation of Eq. (S27), we have used $[\omega_{mn}^x \sigma_{mn}^x \otimes \mathbb{I} + \omega_{mn}^y \sigma_{mn}^y \otimes \Re(U_{mn}) + \omega_{mn}^y \sigma_{mn}^y \otimes \Im(U_{mn})]^2 = \omega_{mn}^2 U_{mn} \otimes \mathbb{I}$. For hyperbolic function with complex numbers, we have the following useful formula

$$\cosh(ix) = \cos(x),$$

(S29a)

$$\sinh(ix) = i \sin(x),$$

(S29b)

$$\cosh(x + iy) = \cosh(x) \cos(y) + i \sinh(x) \sin(y),$$

(S29c)

$$\sinh(x + iy) = \sinh(x) \cos(y) + i \cosh(x) \sin(y).$$

(S29d)

If there are no ancilla errors ($\lambda_m = \lambda_n = 0$) and no detuning ($\delta_m = \delta_n = 0$), then $W_{mn}(t_2,t_1)$ becomes a simple unitary propagator

$$W_{mn}(t_2,t_1) = \cos \theta I_{mn} \otimes \mathbb{I} - i \sin \theta (|m\rangle \langle n| \otimes U_{mn} + |n\rangle \langle m| \otimes U^\dagger_{mn}),$$

(S30)

with $\theta = \Omega_{mn}(t_2 - t_1)$. When $\theta = \pi/2$ or $t_2 - t_1 = \pi/(2\Omega_{mn})$, $W_{mn}(t_2,t_1) = |m\rangle \langle n| \otimes U_{mn} + |n\rangle \langle m| \otimes U^\dagger_{mn}$ (with the trivial phase factor neglected), implying that with ancilla transition $|n\rangle \rightarrow |m\rangle (|m\rangle \rightarrow |n\rangle)$ a quantum gate $U_{mn}$ ($U^\dagger_{mn}$) is implemented on the central system.

VI. PI GATES FOR BOTH ANCILLA ERRORS AND CENTRAL SYSTEM ERRORS

Now we demonstrate that the PI propagator

$$W(t_2,t_1) = \sum_{m=0}^{d-1} \xi_{mn} |m\rangle \langle m| \otimes \mathbb{I} + \sum_{m < n} (\xi_{mn} |m\rangle \langle n| \otimes U_{mn} + \xi_{nm} |n\rangle \langle m| \otimes U^\dagger_{mn}),$$

(S31)

with

$$U_{mn} = \sum_k e^{i\phi_{mn,k}} F_k U_{mn,0} F_k^\dagger / r_k,$$

(S32)

is PI of both ancilla errors (as in Theorem 1 in the main text) and first-order central system errors.

First note that $U_{mn}$ is actually a block-diagonal unitary matrix as follows

$$U_{mn} = \\
\begin{bmatrix}
e^{i\phi_{mn,0}} U_{mn,0} & e^{i\phi_{mn,1}} U_{mn,0} \\
e^{i\phi_{mn,1}} U_{mn,0} & \ddots \\
& \ddots \\
e^{i\phi_{mn,k}} U_{mn,0}
\end{bmatrix}
$$

(S33)

where the different blocks represent the code subspace (with the projection operator $P_0$) and different errors subspaces.
\[
(P_k)_{k=1}^{q-1} \text{ with } P_k = F_k P_0 F_k^\dagger / r_k \text{ of the central system, correspondingly. Then we have}
\]
\[
U_{mn} F_k |\psi_0\rangle = \sum_j e^{i\phi_j} F_j P_0 U_{mn,0} P_0 F_j^\dagger F_k P_0 |\psi_0\rangle = e^{i\phi_k} F_k U_{mn} |\psi_0\rangle.
\]  
(S34)

where we have used that \( P_0 F_k^\dagger F_k P_0 = r_k \delta_{kk} P_0 \). Also the product of any possible sequence of the elements in \( \{U_{mn}\} \) is in the same form as that of \( U_{mn} \),
\[
U_{ab} U_{cd} \cdots U_{em} = \sum_k e^{i(\phi_{ab,k} + \phi_{cd,k} + \ldots + \phi_{em,k})} F_k (U_{ah,0} U_{cd,0} \cdots U_{em,0}) F_k^\dagger / r_k,
\]  
(S35)

since the product of any block-diagonal matrix is still a block-diagonal matrix. So we also have
\[
U_{ab} U_{bc} \cdots U_{em} F_k |\psi_0\rangle = e^{i(\phi_{ah,k} + \phi_{cd,k} + \ldots + \phi_{em,k})} F_k U_{ab} U_{bc} \cdots U_{em} |\psi_0\rangle.
\]  
(S36)

Suppose that there is no ancilla errors but a central system error \( F_k^{(I)}(t) = e^{iH_0 t} F_k e^{-iH_0 t} = \sum_m e^{i\phi_{em,k} t} |m\rangle \otimes F_k \) at time \( t_1 \) during the PI gate \([0, t]\), then the wavefunction of the central system after the gate is
\[
\langle l | W^{(I)}(t, t_1) F_k^{(I)}(t_1) W^{(I)}(t_1, 0) |\phi\rangle |\psi_0\rangle
\]
\[
= \sum_{a,b} \sum_{c,d} \sum_m \sum_{l,m} \xi_{ab}(t-t_1) \xi_{cd}(t_1) e^{i\phi_{em,k}} |l\rangle |a\rangle |b\rangle |c\rangle |d\rangle U_{ab} |\psi_0\rangle,
\]
\[
\langle l | U_{mn} F_k |\psi_0\rangle
\]
\[
= \sum_m \xi_{lm}(t-t_1) \xi_{mi}(t_1) e^{i\phi_{em,k}} U_{lm} F_k |\psi_0\rangle,
\]
\[
\langle l | F_k U_{lm} |\psi_0\rangle
\]
\[
= \sum_m \xi_{lm}(t-t_1) \xi_{mi}(t_1) e^{i\phi_{em,k}} F_k U_{lm} |\psi_0\rangle,
\]
\[
= \left[ \sum_m \xi_{lm}(t-t_1) \xi_{mi}(t_1) e^{i\phi_{em,k}} \right] F_k U_{lk} |\psi_0\rangle,
\]  
(S37)

which implies that a single central system error happening during the gate is equivalent to the error happening after the gate and therefore can be corrected at the end of the gate. One can verify that the same conclusion still holds if the ancilla suffers any errors (permitted by Theorem 1 in the main text) by using Eq. (S36). Thus the gate is PI of both ancilla errors and first-order central system errors.