On Critical Threshold Value for Simple Games

Kanstantsin Pashkovich

Department of Combinatorics and Optimization,
University of Waterloo,
200 University Avenue West Waterloo, ON, Canada N2L 3G1
kpashkov@uwaterloo.ca

Abstract

In this note, we show that for every simple game with \( n \) players the critical threshold value is at most \( n/4 \). This verifies the conjecture of Freixas and Kurz.

1 Introduction

Let \( N \) be a finite set of players. We call a function \( v : 2^N \to \{0,1\} \) monotone if \( v(C) \leq v(S) \) for all \( C, S \subseteq N \) such that \( C \subseteq S \). The pair \( (N,v) \) is called a simple game if \( v(\emptyset) = 0 \), \( v(N) = 1 \) and \( v \) is a monotone 0/1 function. We refer the reader to [7], [6] for further reading on simple games. A simple game \( (N,v) \) partitions the collection of all possible player coalitions into two collections: the collection of winning coalitions \( W := \{C \subseteq N : v(C) = 1\} \) and the collection of losing coalitions \( L := \{C \subseteq N : v(C) = 0\} \).

Weighted voting games are a natural family of simple games. A weighted voting game is defined by a finite set of players \( N \) and a vector \( p \in \mathbb{R}^N, p \geq 0 \), \( p(N) \geq 1 \), where

\[
v(C) := \begin{cases} 1 & \text{if } p(C) \geq 1 \\ 0 & \text{otherwise.} \end{cases}
\]

Here, we use the notation \( q(C) := \sum_{i \in C} q_i \) for a vector \( q \in \mathbb{R}^N \) and \( C \subseteq N \).

Clearly, every weighted voting game is a simple game. To show that the reverse is not true let us consider the following example from [3].

Example 1. Let \( N = \{1,2,\ldots,n\} \) for some even \( n \) and the value function \( v : 2^N \to \{0,1\} \) be as follows

\[
v(C) := \begin{cases} 1 & \text{if } \{2i - 1, 2i\} \subseteq C \text{ for some } i \in \{1,\ldots,n/2\} \\ 0 & \text{otherwise.} \end{cases}
\]

Obviously, coalitions \( \{2i - 1, 2i\}, i \in \{1,\ldots,n/2\} \) are winning while the two coalitions \( \{1,3,\ldots,n-1\}, \{2,4,\ldots,n\} \) are losing.
If the desired vector \( p \in \mathbb{R}^N \) exists for Example 1, then on one side \( p(N) \geq n/2 \) and on the other side \( p(N) < 2 \), showing that for \( n \geq 4 \) this game is not a weighted voting game.

To understand whether a simple game is a weighted voting game, we could use the critical threshold value introduced in [4]. Before we define the critical threshold value of a simple game, let us define the following polyhedron

\[
Q(W) := \{ x \in \mathbb{R}^N : x(C) \geq 1 \text{ for } C \in W, \quad x \geq 0 \}.
\]

The critical threshold value can be defined as

\[
\alpha = \alpha(N, v) := \min_{p \in Q(W)} \max_{C \in \mathcal{L}} p(C).
\]

Observe, that \( \alpha < 1 \) if and only if the simple game \((N, v)\) is a weighted voting game.

The example in [3] shows that \( \alpha \) can be as large as \( n/4 \), because \( \frac{2}{n} \mathbf{1} \) lies in the convex hull of the characteristic vectors of winning coalitions while \( \frac{1}{2} \mathbf{1} \) lies in the convex hull of the characteristic vectors of losing coalitions. Freixas and Kurz [3] conjectured that there is no simple game with a larger value of \( \alpha \). Here, we state the variant of the conjecture of Freixas and Kurz from [5].

**Conjecture 2** (Conjecture of Freixas and Kurz). For a simple game with \( n \) players, the collection of winning coalitions \( W \) and the collection of losing coalitions \( \mathcal{L} \), we have

\[
\alpha = \min_{p \in Q(W)} \max_{C \in \mathcal{L}} p(C) \leq n/4.
\]

In [5] the conjecture of Freixas and Kurz was verified for simple games with all minimal winning coalitions of size 3 and for simple games with no minimal winning coalitions of size 3. In [5] it was shown that \( \alpha \leq 2n/7 \) for general simple games.

Before going to the proof, we would like to say that our approach is inspired by the work of Ahmad Abdi, Gérard Cornuéjols and Dabeen Lee on identically self-blocking clutters [1] (Section 3).

## 2 Proof

To prove the conjecture we reformulate, strengthen and only then verify it. A coalition \( C, C \subseteq N \) is called a cover of \( W \) if \( C \) has at least one common player with every coalition in \( W \). We call the collection of covers of \( W \) the blocker of \( W \) and denote it by \( b(W) \) [2]. Due to the definition of simple games, we have

\[
\mathcal{L} = \{ N \setminus C : C \in b(W) \}.
\]

\footnote{Usually, blocker is defined as the collection of minimal covers. Here, for simplicity of exposition we define blocker as the collection of all covers.}
Hence, the critical threshold value can be reformulated as follows
\[
\alpha = \min_{p \in Q(W)} \max_{L \in \mathcal{L}} p(L) = \min_{p \in Q(W)} \max_{C \in \mathcal{C}(W)} p(N \setminus C) = \min_{p \in Q(W)} \max_{q \in \{0, 1\}^N} \langle p, 1 - q \rangle.
\]

Here, \(\langle p, q \rangle\) stands for the scalar product of two vectors \(p\) and \(q\).

**Conjecture 3** (Reformulation of Conjecture of Freixas and Kurz). For a simple game with \(n\) players and the collection of winning coalitions \(W\), we have
\[
\min_{p \in Q(W)} \max_{q \in \{0, 1\}^N} \langle p, 1 - q \rangle \leq n/4.
\]

Next, we prove Theorem 5, which is a strengthening of Conjecture 2. For the proof we need the following straightforward remark, which we leave as an exercise.

**Remark 4.** Let \(P\) be a polyhedron and let \(p^*\) be the optimal solution of the program \(\min\{\|p\|_2 : p \in P\}\). Then \(p^*\) is an optimal solution of the linear program \(\min\{\langle p^*, q \rangle : q \in P\}\).

**Theorem 5** (Strengthening of Conjecture of Freixas and Kurz). For a simple game with \(n\) players and the collection of winning coalitions \(W\), we have
\[
\min_{p \in Q(W)} \max_{q \in \{0, 1\}^N} \langle p, 1 - q \rangle \leq n/4.
\]

In particular, if \(p^*\) is the optimal solution for the program
\[
\min\{\|p\|_2 : p \in Q(W)\},
\]
then
\[
\max_{q \in Q(W)} \langle p^*, 1 - q \rangle \leq n/4.
\]

**Proof.** Let us consider the unique optimal solution \(p^*\) for the program \(\min\{\|p\|_2 : p \in Q(W)\}\). By Remark 4, \(p^*\) is an optimal solution for the program \(\min\{\langle p^*, q \rangle : q \in Q(W)\}\). Thus, \(p^*\) is an optimal solution for the program \(\max_{q \in Q(W)} \langle p^*, 1 - q \rangle\). Thus, we have
\[
\max_{q \in Q(W)} \langle p^*, 1 - q \rangle = \langle p^*, 1 - p^* \rangle = \frac{n}{4} - (\frac{1}{2} - p^*, \frac{1}{2} - p^*) \leq \frac{n}{4},
\]
finishing the proof.

To finish the note, let us discuss when Conjecture 2 provides a tight upper bound for the critical threshold value. The next theorem shows that if the upper bound in Conjecture 2 is tight, then this fact can be certified in the same way as in Example 1.
Theorem 6. For a simple game with \( n \) players and the collection of winning coalitions \( W \) and the collection of losing coalitions \( L \), we have

\[
\alpha = \min_{p \in Q(W)} \max_{L \in L} p(L) = n/4
\]

if and only if \( \begin{cases} 2 \end{cases} \mathbf{1} \) lies in the convex hull of the characteristic vectors of winning coalitions and \( \begin{cases} 1/2 \end{cases} \mathbf{1} \) lies in the convex hull of the characteristic vectors of losing coalitions.

Proof. Clearly, if \( \begin{cases} 2 \end{cases} \mathbf{1} \) lies in the convex hull of the characteristic vectors of winning coalitions and \( \begin{cases} 1/2 \end{cases} \mathbf{1} \) lies in the convex hull of the characteristic vectors of losing coalitions, then for every \( p \in Q(W) \) we have

\[
\max_{L \in L} p(L) \geq \langle p, \begin{cases} 1/2 \end{cases} \mathbf{1} \rangle = \frac{n}{4} \langle p, \begin{cases} 2 \end{cases} \mathbf{1} \rangle = \frac{n}{4},
\]

showing that \( \alpha \geq n/4 \) and hence \( \alpha = n/4 \) by Theorem 5.

On the other hand, from the proof of Theorem 5 we know that if \( \alpha = n/4 \) then \( p^* = \begin{cases} 1/2 \end{cases} \mathbf{1} \) is an optimal solution for \( \min \{ \langle p^*, q \rangle : q \in Q(W) \} \) with value \( n/4 \). Let us show that \( \begin{cases} 2 \end{cases} \mathbf{1} \) lies in the convex hull of the characteristic vectors of winning coalitions. To do that consider an optimal dual solution \( y^* \) for the program \( \min \{ \langle p^*, q \rangle : q \in Q(W) \} \). Using complementary slackness it is straightforward to show that \( \begin{cases} 4 \end{cases} y^* \) provides coefficients of a convex combination of characteristic vectors of winning coalitions, where the convex combination equals \( \begin{cases} 2 \end{cases} \mathbf{1} \).

In the same way as the proof of Theorem 5 we could show that

\[
\alpha \leq \max_{q \in \{0,1\}^N} \langle q^*, \mathbf{1} - q \rangle = \langle q^*, \mathbf{1} - q^* \rangle = \frac{n}{4} - \langle \begin{cases} 1 \end{cases} \mathbf{1} - q^*, \begin{cases} 1 \end{cases} \mathbf{1} - q^* \rangle \leq \frac{n}{4},
\]

where \( q^* \) is the optimal solution for the program

\[
\min \{ \|q\|_2 : q \in \text{conv} \{ r \in \{0,1\}^N : r \in Q(W) \} \}.
\]

Thus, if \( \alpha \) equals \( n/4 \), then \( q^* = \begin{cases} 1/2 \end{cases} \mathbf{1} \) and \( \begin{cases} 1/2 \end{cases} \mathbf{1} \) lies in \( \text{conv} \{ r \in \{0,1\}^N : r \in Q(W) \} \). Hence, if \( \alpha \) equals \( n/4 \), then \( \mathbf{1} - q^* = \begin{cases} 1 \end{cases} \mathbf{1} \) lies in the convex hull of the characteristic vectors of losing coalitions, finishing the proof.

3 Open Questions

The question about asymptotic behaviour of the critical threshold value of complete simple games remains open. These are the games with a total order of players by "winning power". Freixas and Kurz [3] conjectured that the critical threshold value of a complete simple game with \( n \) players equals \( O(\sqrt{n}) \). Recently, in [5] it was shown that the critical threshold value of such games is \( O((\ln n)\sqrt{n}) \).
Acknowledgements.

We would like to thank Ahmad Abdi for helpful comments on the first version of this note.

References

[1] Ahmad Abdi, *Ideal clutters*, Ph.D. thesis, University of Waterloo, 2018.

[2] Jack Edmonds and D.R. Fulkerson, *Bottleneck extrema*, Journal of Combinatorial Theory 8 (1970), no. 3, 299 – 306.

[3] Josep Freixas and Sascha Kurz, *On α-roughly weighted games*, International Journal of Game Theory 43 (2014), no. 3, 659–692.

[4] Tatiana Gvozdeva, Lane A. Hemaspaandra, and Slinko Arkadii, *Three hierarchies of simple games parameterized by “resource” parameters*, International Journal of Game Theory 42 (2013), no. 1, 1–17.

[5] Frits Hof, Walter Kern, Sascha Kurz, and Daniël Paulusma, *Simple Games versus Weighted Voting Games*, ArXiv e-prints (2018).

[6] Shapley L. S., *Simple games: An outline of the descriptive theory*, Behavioral Science 7 (1962), no. 1, 59–66.

[7] John von Neumann, Oskar Morgenstern, Harold W. Kuhn, and Ariel Rubinstein, *Theory of games and economic behavior (60th anniversary commemorative edition)*, Princeton University Press, 1944.