Spaceability of the sets of surjective and injective operators between sequence spaces

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Abstract
We investigate algebraic structures within sets of surjective and injective linear operators between sequence spaces, completing results of Aron et al.

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1 Introduction

If \( V \) is a vector space and \( \alpha \) is a cardinal number, a subset \( A \) of \( V \) is called \( \alpha \)-lineable in \( V \) if \( A \cup \{0\} \) contains an \( \alpha \)-dimensional linear subspace \( W \) of \( V \). When \( V \) has a topology and the subspace \( W \) can be chosen to be closed, we say that \( A \) is spaceable. This line of research has its starting point with the seminal paper [2] by Aron, Gurariy, and Seoane-Sepúlveda. Nowadays, this theme has profound inroads in Set Theory and Real Analysis, with investi-
gations involving themes such as undecidability and consistency (see, for instance [11] and the references therein). It has been also successfully explored in several research branches, with increasingly relevant applications in areas such as norm-attaining operators, multilinear forms, homogeneous polynomials, sequence spaces, holomorphic mappings, absolutely summing operators, Peano curves, fractals, topological dynamical systems and many others (see, for instance, [1,4–10,12,14] and the references therein).

From now on all vector spaces are considered over a fixed scalar field \( \mathbb{K} \) which can be either \( \mathbb{R} \) or \( \mathbb{C} \). For any set \( X \) we shall denote by \( \text{card} (X) \) the cardinality of \( X \); in particular, we denote \( c = \text{card} (\mathbb{R}) \) and \( \aleph_0 = \text{card} (\mathbb{N}) \).

In this paper we are interested in lineability and spaceability properties of sets of injective and surjective continuous linear operators between sequence spaces. The following results were recently proved in [3]:

**Theorem 1.1** [3, Theorem 4.1 and Corollary 3.4] The set

\[
S = \{ T : \ell_p \to \ell_p : T \text{ is linear, continuous and surjective} \}
\]

is spaceable in \( \mathcal{L} (\ell_p; \ell_p) \) for all \( p \in [1, \infty) \) and the set

\[
I = \{ T : c_0 \to c_0 : T \text{ is linear, continuous and injective} \}
\]

is spaceable in \( \mathcal{L} (c_0, c_0) \).

For surjective operators, the proof has some matrix arguments split for different choices of \( p \) and duality. In the case of injective operators, the argument used in the proof is strongly connected with the sup norm of \( c_0 \) having no immediate adaptation to \( \ell_p \) spaces (see [3, Theorem 3.3]). The main results of the present paper extend, with different techniques, the above results to a wide class of sequence spaces. For instance, (1.1) is extended to a class of sequence spaces encompassing the spaces \( \ell^u_p (X) \) of unconditionally \( p \)-summable sequences; and (1.2) is extended to a very general class of sequence spaces containing \( \ell^u_p (X) \), the spaces of weakly \( p \)-summable sequences \( \ell^w_p (X) \), among others. These classes of sequence spaces will be formally defined in the beginning of Section 2. Our main results read as follows:

**Theorem 1.2** Let \( E \) be a \( c_{00} \)-dense standard Banach sequence space. The set

\[
S = \{ T : E \to E : T \text{ is linear, continuous and surjective} \}
\]

is spaceable in \( \mathcal{L} (E; E) \).

**Theorem 1.3** Let \( V \) be an infinite-dimensional Banach space and let \( E \) be a standard Banach sequence space. The set

\[
I = \{ T : V \to E : T \text{ is linear, continuous and injective} \}
\]

is either empty or spaceable in \( \mathcal{L} (V; E) \).

As a matter of fact, in Theorem 1.3 we prove an even stronger result: we show that \( I \) is \((1, c)\)-spaceable, according to the notion recently introduced in [13] which shall be recalled later.

The paper is organized as follows: in Sect. 2 we introduce the definition of standard Banach sequence spaces and prove Theorem 1.2 and some corollaries. In Sect. 3 we prove Theorem 1.3 and present some consequences.
2 Spaceability of continuous surjective linear operators

Let $X \neq \{0\}$ be a Banach space. By a standard Banach sequence space over $X$ we mean an infinite-dimensional Banach space $E$ of $X$-valued sequences enjoying the following conditions:

(i) There is $C > 0$ such that

$$\|x_j\|_X \leq C \|x\|_E$$

for every $x = (x_j)_{j=1}^\infty \in E$ and all $j \in \mathbb{N}$.

(ii) If $x = (x_j)_{j=1}^\infty \in E$ and $(x_{n_k})_{k=1}^\infty$ is a subsequence of $x$ then $(x_{n_k})_{k=1}^\infty \in E$ and

$$\|(x_{n_k})_{k=1}^\infty\|_E \leq \|x\|_E .$$

(iii) If $(x_j)_{j=1}^\infty \in E$ and $\{n_1 < n_2 < n_3 < \cdots \}$ is an infinite subset of $\mathbb{N}$, then the $X$-valued sequence $(y_j)_{j=1}^\infty$ defined as

$$y_j = \begin{cases} x_i, & \text{if } j = n_i, \\ 0, & \text{otherwise} \end{cases}$$

belongs to $E$ and

$$\|(y_j)_{j=1}^\infty\|_E \leq \|(x_j)_{j=1}^\infty\|_E .$$

From now on we shall call standard Banach sequence space for a standard Banach sequence space over some Banach space $X$ and, when $c_{00}(X)$ is dense in $E$, we say that $E$ is a $c_{00}$-dense standard Banach sequence space.

Notice that (i) ensures that the $m$-th projection over $X$

$$\pi_m : E \to X$$

$$(x_j)_{j=1}^\infty \mapsto x_m$$

is a continuous linear operator for every $m$. Therefore, convergence implies coordinatewise convergence. Also, (ii) yields that if $x \in E$ then each subsequence of $x$ belongs to $E$. Moreover, if $\mathbb{N}'$ is an infinite subset of positive integers, then the linear operator

$$T : E \to E$$

$$(x_j)_{j=1}^\infty \mapsto (x_k)_{k \in \mathbb{N}'}$$

is well-defined and continuous.

Finally, from (iii) we have that if $\mathbb{N}' = \{n_1 < n_2 < n_3 < \cdots \}$ is an infinite subset of positive integers then the linear operator

$$S : E \to E$$

$$(x_j)_{j=1}^\infty \mapsto (y_j)_{j=1}^\infty$$

where

$$y_j = \begin{cases} x_i, & \text{if } j = n_i \in \mathbb{N}', \\ 0, & \text{otherwise} \end{cases}$$
is continuous and well-defined. In particular, if

$$F^n : E \to E$$

$$(x_j)_{j=1}^{\infty} \mapsto (0, \ldots, 0, x_1, x_2, x_3, \ldots)$$

is the forward $n$-shift then $F^n$ is continuous and well-defined.

Now we are ready to prove Theorem 1.2. Splitting the natural numbers in disjoint infinite subsets $\mathbb{N}_1, \mathbb{N}_2, \ldots$ and denoting the elements of $\mathbb{N}_k$ as

$$\mathbb{N}_k = \{n_k, 1, n_k, 2, n_k, 3, \ldots\}$$

we define, for all $k$, the operators

$$S_k : E \to E$$

$$(a_j)_{j=1}^{\infty} \mapsto (a_j)_{j \in \mathbb{N}_k}.$$

By (ii), for all $k$, we have

$$\|S_k\| = \sup_{\|a_j\|_{j=1}^{\infty} \leq 1} \|S_k(a_j)\|_{j=1}^{\infty} \leq 1.$$

It is obvious that $S_k$ is surjective for all $k$. In fact, given $c = (c_j)_{j=1}^{\infty} \in E$, note that by (iii) we have that $a = (a_j)_{j=1}^{\infty}$ defined as

$$a_j = \begin{cases} c_j, & \text{if } j = n_k, i \in \mathbb{N}_k, \\ 0, & \text{otherwise,} \end{cases}$$

belongs to $E$ and it is obvious that $S_k(a) = c$. It is also simple to verify that non trivial linear combinations of $S_k$ are also surjective. Let

$$S = \sum_{i=1}^{n} b_i S_i$$

be a non trivial linear combination of $S_1, \ldots, S_n$ and let $k$ be an index such that $b_k \neq 0$. Note that, given $c = (c_j)_{j=1}^{\infty} \in E$, if we consider the sequence $a = (a_j)_{j=1}^{\infty}$ defined as

$$a_j = \begin{cases} b_k^{-1} c_i, & \text{if } j = n_k, i \in \mathbb{N}_k, \\ 0, & \text{if } j \notin \mathbb{N}_k, \end{cases}$$

then $S_i(a) = 0$ if $i \neq k$ and

$$S(a) = b_k S_k(a) = c.$$

As a consequence, $\{S_k : k \in \mathbb{N}\}$ is a linearly independent subset of $\mathcal{L}(E, E)$. In fact, any non trivial linear combination of elements of $\{S_k : k \in \mathbb{N}\}$ is surjective and, in particular, different from 0. Now consider

$$\Psi : \ell_1 \to \mathcal{L}(E; E)$$

$$(b_k)_{k=1}^{\infty} \mapsto \sum_{k=1}^{\infty} b_k S_k.$$
Note that $\Psi$ is well-defined. In fact, since $\|S_k\| \leq 1$, we have
\[
\sum_{k=1}^{\infty} \|b_k S_k\| \leq \sum_{k=1}^{\infty} |b_k| < \infty
\]
and since $\mathcal{L}(E; E)$ is complete, it follows that $\sum_{k=1}^{\infty} b_k S_k$ belongs to $\mathcal{L}(E; E)$.

Also, the same argument used before for finite sums is straightforwardly adapted to prove that $\sum_{k=1}^{\infty} b_k S_k$ is always surjective whenever $(b_k)_{k=1}^{\infty} \neq 0$ and, in particular, $\Psi$ is injective.

It remains to prove the spaceability of the set $S$ defined in Theorem 1.2. Let us denote by $\text{Im}(\Psi)$ the image of $\Psi$ and by $\overline{\text{Im}(\Psi)}$ its closure in $\mathcal{L}(E; E)$. Let $0 \neq S \in \overline{\text{Im}(\Psi)}$; we only need to prove that $S$ is surjective. Consider a sequence of elements $g_n = \sum_{k=1}^{\infty} b_k^{(n)} S_k \in \text{Im}(\Psi)$ converging to $S$. Hence, for each $a = (a_j)_{j=1}^{\infty} \in E$ we have
\[
S(a) = \lim_{n \to \infty} \sum_{k=1}^{\infty} b_k^{(n)} S_k(a)
\]
and, since $\pi_m$ is continuous for all $m \in \mathbb{N}$,
\[
\lim_{n \to \infty} \pi_m \left( \sum_{k=1}^{\infty} b_k^{(n)} S_k(a) \right) = \pi_m \left( S(a) \right).
\]
So, we have
\[
S(a) = \left( \lim_{n \to \infty} \pi_1 \left( \sum_{k=1}^{\infty} b_k^{(n)} S_k(a) \right), \lim_{n \to \infty} \pi_2 \left( \sum_{k=1}^{\infty} b_k^{(n)} S_k(a) \right), \ldots \right), \quad (2.2)
\]
for all $a = (a_j)_{j=1}^{\infty} \in E$. Since
\[
\pi_1 \left( \sum_{k=1}^{\infty} b_k^{(n)} S_k(a) \right) = b_1^{(n)} a_{n,1} + b_2^{(n)} a_{n,2} + b_3^{(n)} a_{n,3} + \cdots = \sum_{k=1}^{\infty} b_k^{(n)} a_{nk,1}
\]
\[
\pi_2 \left( \sum_{k=1}^{\infty} b_k^{(n)} S_k(a) \right) = b_1^{(n)} a_{n,2} + b_2^{(n)} a_{n,2} + b_3^{(n)} a_{n,3} + \cdots = \sum_{k=1}^{\infty} b_k^{(n)} a_{nk,2}
\]
\[
\vdots
\]
by (2.2) we conclude that
\[
S(a) = \left( \lim_{n \to \infty} \sum_{k=1}^{\infty} b_k^{(n)} a_{nk,1}, \lim_{n \to \infty} \sum_{k=1}^{\infty} b_k^{(n)} a_{nk,2}, \lim_{n \to \infty} \sum_{k=1}^{\infty} b_k^{(n)} a_{nk,3}, \ldots \right)
\]
for all $a = (a_j)_{j=1}^{\infty} \in E$. Denote by $xe_i$ the sequence having $x$ in the $i$-th entry and zero elsewhere. Thus, for all $i \in \mathbb{N}$, there are $k, m \in \mathbb{N}$ such that $i = n_{k,m} \in \mathbb{N}_k$ and hence
\[
S(xe_i) = (0, \ldots, 0, x \lim_{n \to \infty} b_k^{(n)}, 0, 0, \ldots),
\]
for all \( x \in X \). This shows that the all the limits \( \lim_{n \to \infty} b_j^{(n)} \) exist, for all \( j \in \mathbb{N} \). Since \( S \neq 0 \) is continuous and \( c_{00}(X) \) is dense in \( E \), we conclude that
\[
\lim_{n \to \infty} b_j^{(n)} \neq 0
\]
for some \( j_0 \). There is no loss of generality in supposing \( j_0 = 1 \) and \( \mathbb{N}_1 = \{1, 2, 3, 5, 7, 11, \ldots\} \).

Given
\[
c = (c_j)_{j=1}^{\infty} \in E,
\]
by (iii), the sequence \( d = (d_j)_{j=1}^{\infty} \) defined as
\[
d_j = \begin{cases} c_i \left( \lim_{n \to \infty} b_j^{(n)} \right)^{-1}, & \text{if } j = n_1, i, \\ 0, & \text{if } j \not\in \mathbb{N}_1 \end{cases}
\]
belongs to \( E \) and recalling that
\[
S(a) = \left( \lim_{n \to \infty} \sum_{k=1}^{\infty} b_j^{(n)} a_k, 1 \right), \quad \lim_{n \to \infty} \sum_{k=1}^{\infty} b_j^{(n)} a_k, 2, \quad \lim_{n \to \infty} \sum_{k=1}^{\infty} b_j^{(n)} a_k, 3, \ldots
\]
for all \( a = (a_j)_{j=1}^{\infty} \in E \), we have
\[
S(d) = \left( \lim_{n \to \infty} b_1^{(n)} d_1, 1, \lim_{n \to \infty} b_1^{(n)} d_2, 2, \lim_{n \to \infty} b_1^{(n)} d_3, 3, \lim_{n \to \infty} b_1^{(n)} d_5, 5, \lim_{n \to \infty} b_1^{(n)} d_7, \ldots \right) = c.
\]
This proves Theorem 1.2.

**Corollary 2.1** Let \( E_1, \ldots, E_m \) be infinite-dimensional Banach spaces and let \( E \) be a \( c_{00} \)-dense standard Banach sequence space. If there is a surjective multilinear operator from \( E_1 \times \cdots \times E_m \) to \( E \), then the set of all surjective multilinear forms from \( E_1 \times \cdots \times E_m \) to \( E \) is \( c \)-lineable.

**Proof** Let \( S_m \) be the set of all surjective multilinear operators from \( E_1 \times \cdots \times E_m \) to \( E \). Let us fix \( T_0 \in S_m \) and consider the set
\[
W_m = \{ u \circ T_0 : u \in W \},
\]
where \( W \) is the \( c \)-dimensional subspace of \( S \cup \{0\} \) in the proof of Theorem 1.2. It is plain that \( W_m \) is a \( c \)-dimensional subspace contained in \( S_m \cup \{0\} \).

\[\square\]

A similar argument proves that the same holds for polynomials:

**Corollary 2.2** Let \( E \) be an infinite-dimensional Banach space and \( F \) be a \( c_{00} \)-dense standard Banach sequence space. If there is a surjective \( m \)-homogeneous polynomial from \( E \) to \( F \), then the set of all surjective \( m \)-homogeneous polynomials from \( E \) to \( F \) is \( c \)-lineable.

### 3 Spaceability of continuous injective linear operators

We begin by recalling a more restrictive and somewhat geometric approach to lineability and spaceability, recently introduced in [13]. Namely, let \( \alpha, \beta \) and \( \lambda \) be cardinal numbers and \( V \) be a vector space, with \( \dim V = \lambda \) and \( \alpha < \beta \leq \lambda \). A set \( A \subset V \) is \((\alpha, \beta)\)-lineable if it is \( \alpha \)-lineable and for every subspace \( W_\alpha \subset V \) with \( W_\alpha \subset A \cup \{0\} \) and \( \dim W_\alpha = \alpha \), there is a
subspace \( W_\beta \subset V \) with \( \dim W_\beta = \beta \) and \( W_\alpha \subset W_\beta \subset A \cup \{0\} \). Furthermore, if \( W_\beta \) can be chosen to be a closed subspace, we say that \( A \) is \((\alpha, \beta)\)-spaceable. Observe that the ordinary notions of lineability and spaceability are recovered when \( \alpha = 0 \).

Now we are able to begin the proof of Theorem 1.3. Let us assume that \( I \) is non-empty, and let us fix \( T \in I \). For all \( n \in \mathbb{N} \), let \( F^n: E \to E \) be the forward \( n \)-shift defined in (2.1). Defining \( T_1 = T \) and \( T_{n+1} = F^n \circ T \), since \( F^n \) is a continuous linear operator, it is immediate that

- \( T_n \in I \), for all \( n \in \mathbb{N} \);
- \( \|T_{n+1}\| \leq \|F^n \| \|T\| \), for all \( n \in \mathbb{N} \).

For the sake of simplicity, we will write \( T(\alpha, \beta) = \left((T(\alpha, \beta))_n\right)_{n=1}^{\infty} \). Notice that, at this point, it was established that, whenever \( \alpha, \beta \) are non-null scalars, \( j_1 < \cdots < j_k \) be natural numbers and we let consider the continuous linear operator

\[
A = \alpha_1 T_{j_1} + \alpha_2 T_{j_2} + \cdots + \alpha_k T_{j_k}
\]

Let us show that \( A \) is injective. Let \( x \in V \setminus \{0\} \).

In this case, \( T(x) \neq 0 \) and, consequently, if \( n_0 \) is the smallest positive integer such that \( (T(x))_n \neq 0 \), then \( (T_{j_1}(x))_{j_1+n_0-1} = (T(x))_{n_0} \) is the first non-null coordinate of \( T_{j_1}(x) \). Since \( j_1 < j_i, i = 2, \ldots, k \), we conclude that \( (T_{j_i}(x))_{j_i+n_0-1} = 0 \) for each \( i = 2, \ldots, k \) and, therefore, summing coordinate by coordinate, we can infer that

\[
\alpha_1 (T_{j_i}(x))_{j_i+n_0-1} = \alpha_1 (T(x))_{n_0}
\]

is the first non-null coordinate of \( A(x) \). In particular, \( A(x) \neq 0 \); therefore \( A \) is injective. Hence, every finite non trivial linear combination of \( T_n \) originating an injective linear operator and, in particular, also originates a non-null linear operator and, consequently, \( \{T_n: n \in \mathbb{N}\} \) is linearly independent. Notice that, at this point, it was established that, whenever \( E \) is a Banach sequence space in which the forward \( n \)-shift \( F^n \) is well-defined and continuous, then \( I \) is \((1, N_0)\)-lineable.

As we know, \( E \) is a standard Banach sequence space over a Banach space \( X \) and, by condition (iii) of the definition of standard Banach sequence space, we have \( \|F^n\| \leq 1 \) for all \( n \). Let \( x \in V \) and \( (\lambda_k)_{k=1}^{\infty} \in \ell_1 \). Since

\[
\left\| \sum_{n=1}^{\infty} \lambda_n T_n(x) \right\|_E \leq \sum_{n=1}^{\infty} |\lambda_n| \left\| T_n(x) \right\|_E \\
\leq \sum_{n=1}^{\infty} |\lambda_n| \left\| T_n \right\| \left\| x \right\|_V \\
\leq \|T\| \left\| x \right\|_V \sum_{n=1}^{\infty} |\lambda_n| ,
\]

it follows that the linear operator

\[
\Phi: \ell_1 \to \mathcal{L}(V, E) \\
(\lambda_k)_{k=1}^{\infty} \mapsto \sum_{n=1}^{\infty} \lambda_n T_n
\]

is well-defined and continuous, then \( I \) is \((0, N_0)\)-lineable.
is well-defined and continuous. Note that the same argument we have used to prove that the operator $A$ above is injective shows that if $(\lambda_k)_{k=1}^\infty \in \ell_1 \setminus \{0\}$, then $\sum_{n=1}^{\infty} \lambda_n T_n$ is injective and, therefore, $\Phi$ is also injective. Thus,

$$\text{Im} (\Phi) = \left\{ \sum_{n=1}^{\infty} \lambda_n T_n : (\lambda_k)_{k=1}^\infty \in \ell_1 \right\} \subset \mathcal{I} \cup \{0\}$$

and

$$\dim (\text{Im} (\Phi)) = \dim (\ell_1) = c.$$

Finally, from the arbitrariness of $T \in \mathcal{I}$, we finally conclude the proof that $\mathcal{I}$ is $(1, c)$-spaceable if we show that $\text{Im} (\Phi) = \left\{ \sum_{n=1}^{\infty} \lambda_n T_n : (\lambda_k)_{k=1}^\infty \in \ell_1 \right\} \subset \mathcal{I} \cup \{0\}$.

If $S \in \text{Im} (\Phi)$, then there are sequences $(\lambda_n^{(k)})_{n=1}^{\infty} \in \ell_1$ such that

$$S = \lim_{k \to \infty} \sum_{n=1}^{\infty} \lambda_n^{(k)} T_n.$$

Since $E$ is a standard Banach sequence space and $S$ is continuous, we have

$$S (x) = \lim_{k \to \infty} \sum_{n=1}^{\infty} \lambda_n^{(k)} T_n (x)$$

$$= \lim_{k \to \infty} \sum_{n=1}^{\infty} \lambda_n^{(k)} (0, \ldots, 0, \pi_1 (T (x)), \pi_2 (T (x)), \pi_3 (T (x)), \ldots)$$

$$= \lim_{k \to \infty} \left[ \lambda_n^{(k)} (0, \pi_1 (T (x)), \pi_2 (T (x)), \pi_3 (T (x)) \ldots) + \lambda_n^{(k)} (0, \pi_1 (T (x)), \pi_2 (T (x)), \pi_3 (T (x)) \ldots) + \cdots \right]$$

$$= \lim_{k \to \infty} \left( \lambda_n^{(k)} \pi_1 (T (x)), \lambda_n^{(k)} \pi_2 (T (x)) + \lambda_n^{(k)} \pi_1 (T (x)), \ldots \right)$$

$$= \left( \lim_{k \to \infty} \lambda_1^{(k)} \pi_1 (T (x)), \lim_{k \to \infty} \left[ \lambda_1^{(k)} \pi_2 (T (x)) + \lambda_2^{(k)} \pi_1 (T (x)) \right], \ldots \right).$$

Let us show that $S$ is either identically zero or injective. Assume that $S$ is not injective, i.e. that there exists $w_0 \in V \setminus \{0\}$ such that $S (w_0) = 0$. Since $T$ is injective, then $T (w_0) \neq 0$; let $n_0$ be the smallest positive integer such that $\pi_{n_0} (T (w_0)) \neq 0$. Notice that

$$0 = \pi_n (S (w_0)) = \lim_{k \to \infty} \sum_{j=1}^{n} \lambda_n^{(k)} \pi_{n-j+1} (T (w_0)),$$
for all $n \in \mathbb{N}$. Now, we shall proceed by induction on $m$ to show that $\lim_{k \to \infty} \lambda^m = 0$ for all $m \in \mathbb{N}$. We can see that

$$0 = \pi_{n_0} (S (w_0)) = \lim_{k \to \infty} \sum_{j=1}^{n_0} \lambda_{n_0-j+1} (T (w_0))$$

$$= \lim_{k \to \infty} \lambda^k_1 \pi_{n_0} (T (w_0)) = \left( \lim_{k \to \infty} \lambda^k_1 \right) \pi_{n_0} (T (w_0))$$

and, so,

$$\lim_{k \to \infty} \lambda^k_1 = 0.$$

Assuming by induction hypothesis that

$$\lim_{k \to \infty} \lambda^k_1 = \lim_{k \to \infty} \lambda^k_2 = \cdots = \lim_{k \to \infty} \lambda^k_{m-1} = 0,$$

for a certain $m \in \mathbb{N}$, it is obvious that for all $n = 1, \ldots, m - 1$, we have

$$0 = \lim_{k \to \infty} \lambda^k_n \pi_j (T (w_0))$$

for all $j \in \mathbb{N}$. So, since

$$0 = \pi_{n_0+m-1} (S (w_0)) = \lim_{k \to \infty} \sum_{n=1}^{m} \lambda^k_n \pi_{n_0+m-n} (T (w_0)),$$

we obtain

$$\lim_{k \to \infty} \lambda^k_m \pi_{n_0} (T (w_0)) = \lim_{k \to \infty} \left[ \sum_{n=1}^{m} \lambda^k_n \pi_{n_0+m-n} (T (w_0)) - \sum_{n=1}^{m-1} \lambda^k_n \pi_{n_0+m-n} (T (w_0)) \right]$$

$$= \lim_{k \to \infty} \sum_{n=1}^{m} \lambda^k_n \pi_{n_0+m-n} (T (w_0)) - \lim_{k \to \infty} \sum_{n=1}^{m-1} \lambda^k_n \pi_{n_0+m-n} (T (w_0))$$

$$= \pi_{n_0+m-1} (S (w_0)) - \sum_{n=1}^{m-1} \lim_{k \to \infty} \lambda^k_n \pi_{n_0+m-n} (T (w_0))$$

$$= 0,$$

and thus

$$\lim_{k \to \infty} \lambda^k_m = 0.$$

Hence, $\lim_{k \to \infty} \lambda^k_m = 0$ for all $m \in \mathbb{N}$. Consequently, for all $j \in \mathbb{N}$ and all $x \in V$, the limit $\lim_{k \to \infty} \lambda^k_m \pi_j (T (x))$ exists and it is equal to zero. Thus, by (3.1), we have

$$S (x) = \left( \lim_{k \to \infty} \lambda^k_1 \pi_1 (T (x)) , \lim_{k \to \infty} \lambda^k_1 \pi_2 (T (x)) + \lambda^k_2 \pi_1 (T (x)) \right) , \cdots$$

$$= \left( \lim_{k \to \infty} \lambda^k_1 \pi_1 (T (x)) , \lim_{k \to \infty} \lambda^k_1 \pi_2 (T (x)) + \lim_{k \to \infty} \lambda^k_2 \pi_1 (T (x)) , \cdots \right)$$

$$= 0$$

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for all \( x \in V \). Therefore,
\[
\text{Im} (\Phi) \subset \mathcal{I} \cup \{0\}
\]
and the proof of Theorem 1.3 is completed.

The following result is a straightforward consequence of the above proof:

**Corollary 3.1** Let \( V \) be an infinite-dimensional Banach space and let \( E \) be an infinite-dimensional Banach sequence space in which the forward shift \( F = F_1 : E \rightarrow E \) is well-defined and continuous. The set
\[
\mathcal{I} = \{ T : V \rightarrow E : T \text{ is linear, continuous and injective} \}
\]
is either empty or \((1, \aleph_0)\)-lineable.

As in the case of surjective polynomials, we have now a similar result for injective polynomials.

**Corollary 3.2** Let \( V \) be an infinite-dimensional Banach space, let \( E \) be a standard Banach sequence space and consider the set
\[
\mathcal{I}_m = \{ P : V \rightarrow E : P \text{ is an injective } m\text{-homogeneous polynomial} \}.
\]
If \( \mathcal{I}_m \neq \emptyset \), then \( \mathcal{I}_m \) is \((1, \epsilon)\)-lineable.

**Proof** Let \( P_0 \in \mathcal{I}_m \) and consider the set
\[
W_m = \{ u \circ P_0 : u \in W \},
\]
where \( W \) is a closed \( \epsilon \)-dimensional subspace within \( \mathcal{I} \cup \{0\} \) that we can choose containing the identity operator \( \text{id} : E \rightarrow E \in W \). Notice that \( P_0 \in W_m \) (we just have to take \( u = \text{id} \)). Since \( W_m \) is \( \epsilon \)-dimensional and \( W_m \subset \mathcal{I}_m \cup \{0\} \), the proof is done. \( \square \)

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