AUTOMORPHIC LOOPS AND METABELIAN GROUPS

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ABSTRACT. Given a uniquely 2-divisible group $G$, we study a commutative loop $(G, \circ)$ which arises as a result of a construction in \cite{1}. We investigate some general properties and applications of $\circ$ and determine a necessary and sufficient condition on $G$ in order for $(G, \circ)$ to be Moufang. In \cite{6}, it is conjectured that $G$ is metabelian if and only if $(G, \circ)$ is an automorphic loop. We answer a portion of this conjecture in the affirmative: in particular, we show that if $G$ is a split metabelian group of odd order, then $(G, \circ)$ is automorphic.

1. INTRODUCTION

A loop $(Q, \cdot)$ consists of a set $Q$ with a binary operation $\cdot : Q \times Q \to Q$ such that (i) for all $a, b \in Q$, the equations $ax = b$ and $ya = b$ have unique solutions $x, y \in Q$, and (ii) there exists $1 \in Q$ such that $1x = x1 = x$ for all $x \in Q$. Standard references for loop theory are \cite{3, 14}.

Let $G$ be a uniquely 2-divisible group, that is, a group in which the map $x \mapsto x^2$ is a bijection. On $G$ we define a new binary operation as follows:

$$x \circ y = x(y[y, x]^{1/2}).$$  

Here $a^{1/2}$ denotes the unique $b \in G$ satisfying $b^2 = a$ and $[y, x] = y^{-1}x^{-1}yx$. Though it is not obvious, $(G, \circ)$ is a commutative loop with neutral element 1. Moreover, this loop is power-associative, which informally means that integer powers of elements can be defined unambiguously, and powers in $G$ and powers in $(G, \circ)$ coincide. It turns out that $(G, \circ)$ lives in a variety of loops called $\Gamma$-loops (defined in §22), which include commutative RIF loops \cite{9} and commutative automorphic loops \cite{8}.

If $G$ is nilpotent of class at most 2, then $(G, \circ)$ is an abelian group. In this case, the passage from $G$ to $(G, \circ)$ is called the “Baer trick” \cite{7}. This construction seems to first appear in \cite{11}. It was utilized by Bender in \cite{2} to provide an alternative proof of the following result due to Thompson in \cite{15}.

**Theorem 1.1.** Let $p$ be an odd prime and let $A$ be the semidirect product of a $p$-subgroup $P$ with a normal $p'$-subgroup $Q$. Suppose that $A$ acts on a $p$-group $G$ such that

$$C_G(P) \leq C_G(Q).$$

Then $Q$ acts trivially on $G$.

Our goal is to study $(G, \circ)$ with different restrictions on $G$. We show that $(G, \circ)$ is a commutative Moufang loop if and only if $G$ is uniquely 2-divisible 2-Engel (Theorem 2.9) and give an alternative proof to Baer that if $(G, \circ)$ is an abelian group then $G$ has nilpotency class at most 2 (Corollary 3.11). Our main result is that if $G$ is uniquely 2-divisible

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split-metabelian then \((G, \circ)\) is a commutative automorphic loop (Theorem 3.3). Finally, we end with some general facts about \((G, \circ)\) when \(G\) is metabelian and open problems.

2. Preliminaries

To avoid excessive parentheses, we use the following convention:

- multiplication \(\cdot\) will be less binding than divisions \(\backslash, /\).
- divisions are less binding than juxtaposition

For example \(xy/z \cdot y\backslash xy\) reads as \(((xy)/z)(y\backslash(xy))\). To avoid confusion when both \(\cdot\) and \(\circ\) are in a calculation, we denote divisions by \(\backslash\) and \(\circ\) respectively.

In a loop \(Q\), the left and right translations by \(x \in Q\) are defined by \(yL_x = xy\) and \(yR_x = yx\) respectively. We thus have \(\backslash, /\) as \(x\backslash y = yL_x^{-1}\) and \(y/x = yR_x^{-1}\). We define the left section of \(Q\), \(L_Q = \{L_x \mid x \in Q\}\), left multiplication group of \(Q\), \(\text{Mlt}_L(Q) = \langle L_x \mid x \in Q\rangle\) and multiplication group of \(Q\), \(\text{Mlt}(Q) = \langle R_x, L_x \mid x \in Q\rangle\). We define the inner mapping group of \(Q\), \(\text{Inn}(Q) = \langle \theta \mid \theta \in \text{Mlt}(Q) \mid 1\theta = 1\rangle\). It is well known that \(\text{Inn}(Q)\) has the standard generators \(L_{x,y}, R_{x,y}\), and \(T_x\) (see 3) where

\[
L_{x,y} = L_x L_y L_{y^{-1}} x, R_{x,y} = R_x R_y R_{y^{-1}} y, T_x = R_x L_x^{-1} y.
\]

A loop \(Q\) is an automorphic loop if every inner mapping of \(Q\) is an automorphism of \(Q\), \(\text{Inn}(Q) \leq \text{Aut}(Q)\). A loop is Moufang if it satisfies \(xy \cdot zx = (xy)z\) and (xy)\(^{-1}\) = \(x^{-1}y^{-1}\) where \(x^{-1}\) is the unique two-sided inverse of \(x\).

**Definition 2.1.** A loop \((Q, \cdot)\) is a \(\Gamma\)-loop if the following hold

- \((\Gamma_1)\) \(Q\) is commutative.
- \((\Gamma_2)\) \(Q\) has the automorphic inverse property (AIP): \(\forall x, y \in Q, (xy)^{-1} = x^{-1}y^{-1}\).
- \((\Gamma_3)\) \(\forall x \in Q, L_x L_{x^{-1}} = L_{x^{-1}} L_x\).
- \((\Gamma_4)\) \(\forall x, y \in Q, P_x P_y P_x = P_y P_x\) where \(P_x = R_x L_{x^{-1}} = L_x L_{x^{-1}}\).

We recall some definitions and notation, which is standard in most group theory books. We define \([x_0, x_1, \ldots, x_n] = \left\lfloor x_0, x_1, \ldots, x_n \right\rfloor\). Hence, \([x, y, z] = [x, y, z]\). The following identities are well-known:

**Lemma 2.2.** Let \(x, y, z \in G\) for a group \(G\).

- \([xy, z] = [x, z]^y[y, z] = [x, z][x, y][y, z]\)
- \([x, yz] = [x, z][x, y]^z = [x, z][x, y][x, y, z]\)
- \([x, y^{-1}] = [y, x]^y^{-1}\) and \([x^{-1}, y] = [y, x]^x^{-1}\)
- \([x, y^{-1}, z]^y[y, z^{-1}, x]^z[x, x^{-1}, y]^z = [x, y, z][x, y, z][y, z, x]^y = 1\)

Recall that the lower central series of a group is \(G = \gamma_1(G) \geq \gamma_2(G) \geq \ldots\), with \(\gamma_i(G)\) defined inductively by

\[
\gamma_1(G) = G, \quad \gamma_{i+1}(G) = [\gamma_i(G), G]
\]

and the upper central series of a group \(G\) is \(1 = \zeta^0(G) \leq \zeta^1(G) \leq \ldots\), with \(\zeta^i(G)\) defined inductively by

\[
\zeta^0(G) = 1, \quad \zeta^{i+1}(G)/\zeta^i(G) = Z(G/\zeta^i(G))
\]
Lastly recall the proof of associativity. An immediate question is what properties does a group $G$ have if its upper central series has finite length? Therefore, we have $G$ is nilpotency of class $n$ if and only if $[x_0, x_1, \ldots, x_n] = 1 \ \forall x_i \in G$. A group $G$ is 2-Engel if $[x, y, y] = 1$, alternatively $x^y = x^y x$, for all $x, y \in G$.

Theorem 2.3. ([1]) Let $G$ be a uniquely 2-divisible group. For all $x, y \in G$, define $x \circ y = xy[y, x]^{-\frac{1}{2}}$. Then $(G, \circ)$ is an abelian group if and only if $G$ is has nilpotency class 2. Moreover, powers in $G$ coincide in $(G, \circ)$.

Note that in the proof of the above theorem the restriction to class 2 only appears in the proof of associativity. An immediate question is what properties does $(G, \circ)$ have without the restriction that $G$ be nilpotent of class 2?

Theorem 2.4. ([6]) Let $G$ be a uniquely 2-divisible group. Then $(G, \circ)$ is a $\Gamma$-loop. Moreover, powers coincide in $G$ and $(G, \circ)$.

The main goal of [6] was to establish a connection to Bruck loops and $\Gamma$-loops of odd order.

Theorem 2.5. ([6]) There is a one-to-one correspondence between left Bruck loops of odd order $n$ and $\Gamma$-loops of odd order $n$. That is

(i) If $(Q, \cdot)$ is a left Bruck loop of odd order $n$ with $1 \in Q$ identity element, then $(Q, \circ)$ is a $\Gamma$-loop of order $n$ where $x \circ y = (1)L_x L_y [L_y, L_x]^{1/2}$.

(ii) If $(Q, \cdot)$ is a $\Gamma$-loop of odd order $n$, then $(Q, \oplus)$ is a left Bruck loop of order $n$ where $x \oplus y = (x^{-1}(y^2 x))^{1/2}$.

(iii) The mappings in (i) and (ii) are mutual inverses.

In general, not much can be said about $(G, \circ)$ without any restrictions on $G$. However, we do have the following.

Lemma 2.6. Let $G$ be a uniquely 2-divisible group. Then $Z(G) \leq Z(G, \circ)$.

Proof. Let $g \in Z(G)$. Then we have

$$g \circ (x \circ y) = gxy[y, x]^{\frac{1}{2}}[xy[y, x]]^{\frac{1}{2}} g = gx y[y, x]^{\frac{1}{2}} g = gxy[y, g]^{\frac{1}{2}} = (g \circ x) \circ y,$$

$$x \circ (g \circ y) = xgy[gy, x]^{\frac{1}{2}} = xgy[y, x]^{\frac{1}{2}} g = (x \circ g) \circ y,$$

$$x \circ (y \circ g) = xyg[yg, x]^{\frac{1}{2}} = xyg[y, x]^{\frac{1}{2}} g = (x \circ y) \circ g.$$ 

Thus $g \in Z(G, \circ)$. 

It turns out that $(G, \circ)$ has a lot of structure if $G$ is 2-Engel.

Lemma 2.7. Let $G$ be uniquely 2-divisible. Then $xy[y, x]^{1/2} = (xy^2 x)^{1/2}$ if and only if $G$ is 2-Engel.

Proof. Before beginning the proof, we first note that if $G$ is uniquely 2-divisible and $a, b \in G$ commute, then $a$ commutes with $b^{1/2}$. Indeed, since $(a^{-1}b^{1/2} a)^2 = a^{-1}b a$, it follows that $(a^{-1} b a)^{1/2} = a^{-1} b^{1/2} a$. Thus, since $a$ and $b$ commute, we have that $b^{1/2} = a^{-1} b^{1/2} a$, as
desired. 
Suppose $G$ is 2−Engel. Hence, both $x$ and $y$ commute with $[y, x]$. Then by the note above,
\[(xy[y, x]^{1/2})^2 = xy[y, x]^{1/2}xy[y, x]^{1/2} = (xy)^2[y, x] = xy^2x.
\]
Taking square roots of both sides gives the desired results.
For the reverse direction, set $u = [y, x]^{1/2}$. By hypothesis, $xyxyu = yxyx$ and canceling gives $uxyu = yx$. Multiplying both sides on the right by $u$ gives $yxu = uxu^2 = uxu^{−1}x^{−1}yx = uyx$. Since $yx$ commutes with $u$ (Theorem 2.4) it commutes with any power of $u$. Thus $yx[y, x] = [y, x]yx$. Replacing $x$ with $y^{−1}x$ to get $x[y, y^{−1}x] = [y, y^{−1}x]x$. But $[y, y^{−1}x] = y^{−1}x^{−1}yy^{−1}x = [y, x]$. Therefore $x[y, x] = [y, x]x$, that is, $[y, x, x] = 1$. Thus, $G$ is 2−Engel.

Defining multiplication with $x \oplus y = (xy^2x)^{1/2}$ has been well studied by Bruck, Glaubermann, and others.

**Theorem 2.8.** ([5]) Let $G$ be uniquely 2−divisible group. For all $x, y \in G$, define $x \oplus y = (xy^2x)^{1/2}$. Then $(G, \oplus)$ is a Bruck loop. Moreover, powers in $G$ coincide with powers in $(G, \circ)$.

Finally, it is well known that commutative Bruck loops are Moufang [3].

**Theorem 2.9.** Let $G$ be uniquely 2−divisible. Then $G$ is 2−Engel if and only if $(G, \circ)$ is a commutative Moufang loop.

**Proof.** If $G$ is 2−Engel then $(G, \circ) = (G, \oplus)$, and hence a commutative Bruck loop, so Moufang.

Alternatively, set $u = [y, x]^{1/2}$. Using the inverse property,
\[y = x^{−1} \circ (x \circ y) = x^{−1}xyu^{−1}[xyu^{−1}x^{−1}]^{1/2}.
\]Cancel and multiply on the left by $u$ to get $u = [xyu^{−1}x^{−1}]^{1/2}$. Squaring both sides gives $u^2 = [xyu^{−1}x^{−1}] = [yu^{−1}x^{−1}] = uy^{−1}xyu^{−1}x^{−1}$. Hence $u = y^{−1}xyu^{−1}xy$ after canceling. Multiplying on the left by $x^{−1}$ to get $x^{−1}u = [x, y]u^{−1}x^{−1} = u^2u^{−1}x^{−1} = ux^{−1}$. Since $x^{−1}$ commutes with $u$ it commutes with $u^2 = [x, y]$. Similarly, since $[x, y]$ commutes with $x^{−1}$, it commutes with $x$. Hence, $G$ is 2-Engel. 

3. **Split Metabelian Groups**

Let $G$ be the semidirect product of a normal abelian subgroup $H$ of odd order acted on (as a group of automorphisms) by an abelian group $F$ of odd order. Products in $H$ and in $F$ are written multiplicatively. We use exponential notation for the action of $\mathrm{Aut}(H)$ on $H$: given $\theta \in \mathrm{Aut}(H)$, $h \in H$, define $h^\theta = \theta(h)$.

Further, given $m, n \in \mathbb{Z}$ with $m$ and $n$ relatively prime to $|H|$, we make special use of the notation $h^{\frac{m}{n} \theta} = (h^\frac{m}{n})^\theta = (h^\theta)^\frac{m}{n}$. Note that since $H$ is abelian, this convention is consistent with an additional notation: given commuting automorphisms $\theta, \psi \in \mathrm{Aut}(H)$, $h^{\theta + \psi} = h^\theta h^\psi$. Then $G = H \rtimes F = HF$, where
\[h_1f_1h_2f_2 = h_1f_1 \cdot h_2f_2 = h_1h_2^{f_1}f_1f_2
\]for all $h_1, h_2 \in H, f_1, f_2 \in F$. Note that $G$ is metabelian (we refer to such groups as *split metabelian*). To proceed, we need a proposition.
Proposition 3.1. Let $H$ be an abelian group of odd order. Suppose $\alpha$ and $\beta$ are commuting automorphisms of $H$ with odd order in $\text{Aut}(H)$. Then the map $h \mapsto h^{\alpha+\beta}$ is an automorphism of $H$.

Proof. Define $\phi : H \to H$ by $\phi(h) = h^{\alpha+\beta}$. Clearly, $\phi$ is a homomorphism. We will show that $\phi$ is injective. Suppose $h_0 \in H$ such that $\phi(h_0) = 1$. It follows that $h_0^\alpha = h_0^{-\beta}$, and thus $h_0^{2\alpha} = (h_0^\alpha)^\alpha = (h_0^{-\beta})^\alpha = (h_0^{-\beta})^{-\beta} = h_0^{2\beta}$.

Now, since $\alpha$, $\beta$ are commuting odd-ordered automorphisms of $H$, there exists some positive odd integer $k$ such that $\alpha^k = \beta^k$. In particular,

$$h_0^{\alpha k} = h_0^{\beta k}$$

$$(h_0^{2\alpha})^{ak-2} = (h_0^{2\beta})^{ak-2}$$

$$(h_0^{2\beta})^{ak-2} = (h_0^{2\alpha})^{ak-2}$$

$$(h_0^{ak-2})^{2\beta} = (h_0^{ak-2})^{2\alpha}.$$

Since $\beta^2 \in \text{Aut}(H)$, it follows that $h_0^{ak-2} = h_0^{ak-2}$. Continuing in this manner, we have that $h_0^\alpha = h_0^\beta$, and hence $h_0^\beta = h_0^{-\beta}$. Since $|H|$ is odd, this implies that $h_0 = 1$. Therefore, $\phi$ is an injective homomorphism $H \to H$ and is thus an automorphism of $H$. \qed

Since $F$ is abelian, Proposition 3.1 implies that if $\theta$ is a $\mathbb{Q}$-linear combination of elements of $F$ (where the numerators and denominators of the coefficients are relatively prime to $|H|$), the map $H \to H$, $h \mapsto h^\theta$ is an automorphism of $H$ which commutes with any other such linear combination $\psi$. In particular, note that the aforementioned automorphism has an inverse in $\text{Aut}(H)$; we denote this inverse by $h \mapsto h^{\theta^{-1}}$, and this map also commutes with $\psi$. We will use this fact throughout the following calculations without specific reference.

Lemma 3.2. Let $u = hf$, $x = h_1f_1$, $y = h_2f_2 \in G$. Then

- $u^{-1} = h^{-f^{-1}}f^{-1}$
- $u^{1/2} = h^{(1+f)(1/2)^{-1}}f^{1/2}$
- $[x, y] = h_1^{f_1^{-1}(-1)f_2^{-1}}h_2^{f_2^{-1}(-1)f_1^{-1}+1} \in H$
- $x \circ y = h_1^{1+f_2^{-1}}h_2^{1+f_1^{-1}}f_1f_2$
- $x \backslash y = x \backslash_\phi y = \left( h_1^{-1-f_1}h_2^{(1+f_1)^{-1}}f_1^{-1}f_2 \right)^{1+f_1^{-1}}f_2$
- $uL_{x,y} = \left( h_2^{(1+f_1)(1+f_2)}h_2^{1+f_1-f_1} \right)^{(1+f_1f_2)^{-1}}f$

Proof. First, we compute

$$u \cdot h^{-f^{-1}}f^{-1} = hf \cdot h^{-f^{-1}}f^{-1}$$

$$= hh^{-f^{-1}}ff^{-1}$$

$$= hh^{-1}ff^{-1}$$

$$= 1,$$
and first item is proved. Next, we compute

\[
\left( h^{(1+f^{1/2})^{-1} f^{1/2}} \right)^2 = h^{(1+f^{1/2})^{-1} f^{1/2} \cdot h^{(1+f^{1/2})^{-1} f^{1/2}}} = h^{(1+f^{1/2})^{-1} h^{(1+f^{1/2})^{-1} f^{1/2} f^{1/2}}}. 
\]

Setting \( k = h^{(1+f^{1/2})^{-1}} \in H \) gives

\[
\left( h^{(1+f^{1/2})^{-1} f^{1/2}} \right)^2 = kk^{f^{1/2}} f
= k^{1+f^{1/2}} f
= hf
= u,
\]

and thus \( u^{1/2} = h^{(1+f^{1/2})^{-1} f^{1/2}}. \)

Now,

\[
[x, y] = x^{-1} y^{-1} xy
= (h_1^{-1} f_1^{-1} h_2^{-1} f_2^{-1}) (h_1 f_1 \cdot h_2 f_2)
= (h_1^{-1} f_1^{-1} h_2^{-1} f_2^{-1} f_1^{-1} f_2^{-1}) (h_1 f_1 f_2)
= h_1^{-1} f_1^{-1} h_2^{-1} f_2^{-1} f_1^{-1} f_2^{-1} f_1 f_2
= h_1^{-1} f_1^{-1} (f_1 f_2)^{-1} h_2^{-1} (f_1 f_2)^{-1} + f_2^{-1} \cdot 1
= h_1^{-1} f_1^{-1} (1+f_2^{-1}) h_2^{-1} (-f_1^{-1} + 1).
\]

Next,

\[
x \circ y = h_1 f_1 \circ h_2 f_2
= (h_1 f_1)(h_2 f_2) \cdot [h_2 f_2, h_1 f_1]^{1/2}
= (h_1 h_2 f_1 f_2) \left( h_2^{-1} f_2^{-1} (1+f_1^{-1}) h_1^{-1} (-f_2^{-1} + 1) \right)^{1/2}
= h_1 h_2 f_1 \left( h_2^{-1} f_2^{-1} (1+f_1^{-1}) h_1^{-1} (-f_2^{-1} + 1) \right)^{1/2} f_1 f_2
= h_1^{1+f_2^{-1}} f_1 f_2.
\]

To compute \( x \backslash y \), observe that

\[
x \circ (h_1^{-1} f_1^{-1} f_2 h_2^{-1})^{(1+f_1^{-1})^{-1}} f_1^{-1} f_2 = h_1 f_1 \circ (h_1^{-1} f_1^{-1} f_2 h_2^{-1})^{(1+f_1^{-1})^{-1}} f_1^{-1} f_2
= h_1^{1+f_2^{-1}} f_1^{1+f_1^{-1}} (h_1^{-1} f_1^{-1} f_2 h_2^{-1})^{(1+f_1^{-1})^{-1}} (1+f_1^{-1}) f_1 f_2
\]
Proof. Since 
\[ uL_{x,y} = (x \circ y)/(u \circ x) \circ y \]
\[ = \left( h_1^{x/f_2} h_2^{x/f_1} f_1 f_2 \right) \circ \left( \left( h_1^{x/f_2} h_2^{x/f_1} f_1 f_2 \right) \circ h_2 f_2 \right) \]
\[ = \left( h_1^{x/f_2} h_2^{x/f_1} f_1 f_2 \right) \circ \left( \left( h_1^{x/f_2} h_2^{x/f_1} f_1 f_2 \right) \circ h_2 f_2 \right) \]
\[ = \left( h_1^{x/f_2} h_2^{x/f_1} f_1 f_2 \right) \circ \left( \left( h_1^{x/f_2} h_2^{x/f_1} f_1 f_2 \right) \circ h_2 f_2 \right) \]
\[ = \left( h_1^{x/f_2} h_2^{x/f_1} f_1 f_2 \right) \circ \left( \left( h_1^{x/f_2} h_2^{x/f_1} f_1 f_2 \right) \circ h_2 f_2 \right) \]
\[ = \left( \left( h_1^{x/f_2} h_2^{x/f_1} f_1 f_2 \right) \circ h_2 f_2 \right) \circ \left( h_1^{x/f_2} h_2^{x/f_1} f_1 f_2 \right) \circ h_2 f_2 \]

and thus \[ x \setminus y = \left( h_1^{x/f_1} f_2 h_2^2 \right) \left( h_1^{x/f_2} f_1 f_2 \right) \]
Finally,
\[ uL_{x,y} = (x \circ y)/(u \circ x) \circ y \]
\[ = (x \circ y)/(u \circ x) \circ y \]
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\[ \square \]

Theorem 3.3. Let \( G \) be a split metabelian group of odd order. Then \( (G, \circ) \) is an automorphic loop.

Proof. Since \( (G, \circ) \) is commutative, for any \( x, y \in G \), \( L_{x,y} = R_{x,y} \) and \( T_x = id_G \). Thus, to prove that \( (G, \circ) \) is automorphic, it suffices to show that \( L_{x,y} \) is a loop homomorphism. We must show that \( uL_{x,y} \circ vL_{x,y} = (u \circ v)L_{x,y} \) for all \( u, v, x, w \in G \). Thus, let \( u = h_1 f_1, v = k_1, x = h_1 f_1, y = h_2 f_2 \in G \). We first compute, by Lemma 3.2

\[ uL_{x,y} \circ vL_{x,y} \]
\[ = \left( h_1^{(1+f_1)(1+f_2)} h_2^{1+f_1-f_1} \right) \left( h_1^{(1+f_1)(1+f_2)} h_2^{1+g_1-g_1} \right) \]
\[ = \left( h_1^{(1+f_1)(1+f_2)} h_2^{1+f_1-f_1} \right) \left( h_1^{(1+f_1)(1+f_2)} h_2^{1+g_1-g_1} \right) \]
\[ = \left( h_1^{(1+f_1)(1+f_2)} h_2^{1+f_1-f_1} \right) \left( h_1^{(1+f_1)(1+f_2)} h_2^{1+g_1-g_1} \right) \]
\[ = \left( h_1^{(1+f_1)(1+f_2)} h_2^{1+f_1-f_1} \right) \left( h_1^{(1+f_1)(1+f_2)} h_2^{1+g_1-g_1} \right) \]

\[ \square \]
\[
= \left( h^{\frac{(1+f_1)(1+f_2)}{2}} k^{\frac{(1+f_1)(1+f_2)}{2}} h_2^{1-f_g+f_gf_1-f_1} \right)^{\frac{(1+f_1f_2)}{2}-1} f_g.
\]

On the other hand,
\[
(u \circ v)L_{x,y} = \left( h^{\frac{1+g}{2}} k^{\frac{1+f_1}{2}} f_g \right) L_{x,y} = \left( \left( h^{\frac{1+g}{2}} k^{\frac{1+f_1}{2}} \right) \left( h_2^{1+f_gf_1-f_g-f_1} \right) \right)^{\frac{(1+f_1f_2)}{2}-1} f_g = \left( h^{\frac{(1+g)(1+f_1)(1+f_2)}{2}} k^{\frac{(1+f_1)(1+f_2)}{2}} h_2^{1+f_gf_1-f_g-f_1} \right)^{\frac{(1+f_1f_2)}{2}-1} f_g = uL_{x,y} \circ vL_{x,y}.
\]

As an immediate corollary, we see that if \( G \) is any group such that all groups of order \(|G|\) are split metabelian, then \((G, \circ)\) is an automorphic loop. In particular, disregarding the cases where \( G \) is abelian, we obtain the following.

**Corollary 3.4.** If \(|G|\) is any one of the following (for distinct odd primes \( p \) and \( q \)), then \((G, \circ)\) is automorphic.

- \( pq \) (where \( p \) divides \( q - 1 \))
- \( p^2q \)
- \( p^2q^2 \)

**Corollary 3.5.** Let \( p \) and \( q \) be distinct odd primes with \( p \) dividing \( q - 1 \). Then there is exactly one nonassociative, commutative, automorphic loop of order \( pq \).

**Proof.** Let \( G \) be a group of order \( pq \). Then \((G, \circ)\) is automorphic (Theorem 3.3). Suppose \( Q \) is a \( \Gamma \)-loop of order \( pq \). Then \((Q, \oplus)\) is a Bruck loop. The only two options are (i) \((Q, \oplus)\) is abelian or (ii) \((Q, \oplus)\) is the unique nonassociative Bruck loop of order \( pq \) \([10]\). For (i), \( Q = (Q, \oplus) \) and hence an abelian group (so automorphic). For (ii), \((G, \oplus_\circ) = (Q, \oplus) \) must be the same nonassociative Bruck loop, and hence, \( Q = (G, \circ) \). \( \square \)

The only known examples where \((G, \circ)\) is not automorphic occur when \( G \) is not metabelian.

**Conjecture 3.6.** Let \( G \) be a uniquely 2-divisible group. Then \((G, \circ)\) is automorphic if and only if \( G \) is metabelian.

For a general metabelian group \( G \), we have the following results.

**Lemma 3.7.** Let \( G \) be a uniquely 2-divisible, metabelian group. Then for all \( x, y, z \in G \).

- \([[x, y]^\frac{1}{2}, z] = [[x, y], z]^\frac{1}{2} \)
- \([x, y, z][z, x, y][y, z, x] = 1 \).

**Theorem 3.8.** Let \( G \) be uniquely 2–divisible and metabelian. Then \( \zeta^2(G) \leq Z(G, \circ) \).
Proof. If \( g \in \zeta^2(G) \), then it is clear that \( gT_x = x \). We show \( gL_{x,y} = g \). First, it is clear that \([g, x, y] = 1 \iff [x, g, y] = 1 \iff [x, y, g] = 1\). Thus, we have \([g, x]y = y[g, x]\) and \([x, y]g = g[x, y]\). Now,

\[
y \circ (x \circ g) = yxyg[g, x]^{x/2}[xg, y]^{x/2}[[x, g]^{x/2}, y]^{x/2}
= yxyg[g, x]^{x/2}[xg, y]^{x/2}
= yxyg[g, x]^{x/2}[x, y]^{x/2}[g, y]^{x/2}
= yxyg[x, y]^{x/2}[g, y]^{x/2}[g, x]^{x/2}
= yxyx[g, y]^{x/2}[g, y]^{x/2}[g, x]^{x/2}
= yxyx[g, y]^{x/2}[g, y]^{x/2}[g, x]^{x/2}
= (y \circ x) \circ g
\]

Hence, \( gL_{x,y} = g \). \( \square \)

**Theorem 3.9.** Let \( G \) be uniquely 2-divisible and of nilpotency class 3. Then \( Z(G, \circ) = \zeta^2(G) \).

**Proof.** By the previous theorem, we have \( \zeta^2(G) \leq Z(G, \circ) \). From Lemma 3.7, we have \([y, x, z][z, y, x] = [y, [x, z]]\) by interchanging \( x \) and \( y \). Thus,

\[
[[[y, x]^{x/2}, z]^{x/2}[[z, y]^{x/2}, x]^{x/2} = [y, [x, z]^{x/2}]^{x/2} \quad (\ast)
\]

Let \( g \in Z(G, \circ) \). We show \([g, x, y] = 1\) for all \( x, y \in G \) and therefore, \( g \in \zeta^2(G) \). Since \( g \in Z(G, \circ) \), we have \( g \circ (x \circ y) = x \circ (y \circ g) \). Hence, we have

\[
gxy[y, x]^{x/2}[xy, g]^{x/2}[[y, x]^{x/2}, g]^{x/2} = xyg[y, y]^{x/2}[yg, x]^{x/2}[[g, y]^{x/2}, x]^{x/2}
\]

\[
xyg[g, xy][y, x]^{x/2}[xy, g]^{x/2}[[y, x]^{x/2}, g]^{x/2} = xyg[y, y]^{x/2}[yg, x]^{x/2}[[g, y]^{x/2}, x]^{x/2}
\]

\[
[g, xy][y, x]^{x/2}[[y, x]^{x/2}, g]^{x/2} = [g, y]^{x/2}[yg, x]^{x/2}[[g, y]^{x/2}, x]^{x/2}
\]

\[
[g, y]^{x/2}[g, x, y]^{x/2}[[y, x]^{x/2}, g]^{x/2} = [g, y]^{x/2}[y, x, g]^{x/2}[[g, y]^{x/2}, g]^{x/2}
\]

\[
[g, y]^{x/2}[[y, x]^{x/2}, g]^{x/2} = [g, y]^{x/2}[[g, y]^{x/2}, x]^{x/2}
\]

\[
[g, y]^{x/2}[[g, y]^{x/2}, x]^{x/2} = 1
\]

\[
[g, x]^{x/2}[[g, x]^{x/2}, y]^{x/2} = 1
\]

\[
[g, x, y] = 1.
\]

\( \square \)

**Corollary 3.10.** Let \( G \) be uniquely 2-divisible and of nilpotency class 3. Then \((G, \circ)\) is a commutative loop of nilpotency class 2.

**Proof.** We have as sets, \( G/\zeta^2(G) = (G, \circ)/Z(G, \circ) \) by Theorem 3.9. Now, since \( G/\zeta^2(G) \) is an abelian group, the two sets have the same operation and thus, \((G, \circ)/Z(G, \circ)\) is an abelian group. \( \square \)
Finally, we give an alternative proof of Baer’s result that if \( (G, \circ) \) is an abelian group, then \( G \) is of nilpotency class at most 2.

**Corollary 3.11.** Let \( G \) be uniquely 2–divisible. If \( (G, \circ) \) is an abelian group, then \( G \) is of class at most 2.

**Proof.** Since \( (G, \circ) \) is an abelian group, \( (G, \circ) \) is a commutative Moufang loop. Thus, \( G \) is 2–Engel, which implies \( G \) is of class at most 3. Thus, by Theorem 3.9, \( G = \zeta^2(G) \), and hence \( G \) has class at most 2.

\[\square\]

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