BOUNDDEDNESS OF MULTIDIMENSIONAL HAUSDORFF OPERATORS ON $L^1$ AND $H^1$ SPACES

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ABSTRACT. For a wide family of multivariate Hausdorff operators, a new stronger condition for the boundedness of an operator from this family on the real Hardy space $H^1$ by means of atomic decomposition.

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1. Introduction

In the one-dimensional case Hausdorff operators on the real line were introduced in [3] and studied on the Hardy space in [7]. As in [1], we define a multidimensional Hausdorff type operator by

$$ (Hf)(x) = (H_\Phi f)(x) = (H_{\Phi,A} f)(x) = \int_{\mathbb{R}^n} \Phi(u) f(xA(u)) \, du, $$

where $A = A(u) = (a_{ij})^n_{i,j=1} = (a_{ij}(u))^n_{i,j=1}$ is the $n \times n$ matrix with the entries $a_{ij}(u)$ being measurable functions of $u$. This matrix may be singular at most on a set of measure zero; $xA(u)$ is the row $n$-vector obtained by multiplying the row $n$-vector $x$ by the matrix $A$. Of course, $xA$ can be written as $A^T x^T$, where both matrix and vector are transposed, the latter to a column vector.

We are going to prove sufficient conditions, in terms of $\Phi$ and $A$, for the boundedness of the whole range of Hausdorff type operators (1) in $H^1(\mathbb{R}^n)$.

Before proving the result we give natural assumptions on $\Phi$ and $A$, which provide the boundedness of the Hausdorff operator in $L^1(\mathbb{R}^n)$.

Let the following condition be satisfied:

$$ \|\Phi\|_{L_A} = \int_{\mathbb{R}^n} |\Phi(u)| |\det A^{-1}(u)| \, du < \infty, $$

or $\varphi(u) = \Phi(u) \det A^{-1}(u) \in L^1(\mathbb{R}^n)$.

Among the other basic properties of Hausdorff operators, one may find in [1], in slightly different terms, that the operator $Hf$ is bounded taking $L^1$ into $L^1$, with

$$ \|Hf\|_{L^1} \leq \|\Phi\|_{L_A} \|f\|_{L^1}. $$
It was proved in [8] that the same condition provides the boundedness of Hausdorff type operators on $H^1(\mathbb{R}^n)$ for diagonal matrices $A$ with all diagonal entries equal to one another.

We denote

$$\|B\| = \|B(u)\| = \max_j (|b_{1j}(u)| + \ldots + |b_{nj}(u)|),$$

where $b_{nj}$ are the entries of the matrix $B$, to be the operator $\ell$-norm. We will say that $\Phi \in L^*_{\Lambda}$ if

$$\|\Phi\|_{L^*_{\Lambda}} = \int_{\mathbb{R}^n} |\Phi(u)| \|B(u)\|^n du < \infty.$$

The following result was proved by Lerner and Liflyand [5] for the boundedness of Hausdorff type operators in $H^1(\mathbb{R}^n)$ for general matrices $A$. The proof used duality argument.

**Theorem 1.** The Hausdorff operator $Hf$ is bounded on the real Hardy space $H^1(\mathbb{R}^n)$ provided $\Phi \in L^*_{\Lambda}$, and

$$\|Hf\|_{H^1(\mathbb{R}^n)} \leq \|\Phi\|_{L^*_{\Lambda}} \|f\|_{H^1(\mathbb{R}^n)}.$$  

(3)

The difference in conditions $\Phi \in L_{A^{-1}}$ and $\Phi \in L^*_{\Lambda}$ seemed to be quite natural. In [5] and then in [6] the problem of the sharpness of Theorem 1 was posed. We will prove that a weaker condition provides the boundedness of Hausdorff type operators on $H^1(\mathbb{R}^n)$. The proof will be based on atomic decomposition of $H^1(\mathbb{R}^n)$.

In what follows $a \ll b$ means that $a \leq Cb$ for some absolute constant $C$ but we are not interested in explicit indication of this constant.

2. **Main result and proof**

Let $\|B\|_2 = \max_{|x|=1} |Bx^T|$, where $| \cdot |$ denotes the Euclidean norm. It is known (see, e.g., [4, Ch.5, 5.6.35]) that this norm does not exceed any other matrix norm. We will say that $\Phi \in L^2_B$ if

$$\|\Phi\|_{L^2_B} = \int_{\mathbb{R}^n} |\Phi(u)| \|B(u)\|^2 du < \infty.$$

The following result is true.

**Theorem 2.** The Hausdorff operator $Hf$ is bounded on the real Hardy space $H^1(\mathbb{R}^n)$ provided $\Phi \in L^2_{\Lambda^{-1}}$, and

$$\|Hf\|_{H^1(\mathbb{R}^n)} \ll \|\Phi\|_{L^2_{\Lambda^{-1}}} \|f\|_{H^1(\mathbb{R}^n)}.$$  

(4)

Proof. Let $a(x)$ denote an atom (a $(1, \infty, 0)$-atom), a function satisfying the following conditions:

$$\text{supp } a \subset B(x_0, r);$$  

(5)
\( ||a||_\infty \leq \frac{1}{|B(x_0, r)|}; \)

\( \int_{\mathbb{R}^n} a(x) \, dx = 0. \)

It is well known that

\( ||f||_{H^1} \sim \inf \{ \sum_k |c_k| : f(x) = \sum_k c_k a_k(x) \}, \)

where \( a_k \) are atoms.

The other value to which \( ||f||_{H^1} \) is equivalent is

\( \sum_{p=0}^n \int_{\mathbb{R}^n} |R_p f(x)| \, dx, \)

where \( R_0 f \equiv f \) and \( R_p \) are \( n \) Riesz transforms (see, e.g., [9]).

We now have

\[ ||\mathcal{H} f||_{H^1} = ||\int_{\mathbb{R}^n} \Phi(u) f(\cdot A(u)) \, du||_{H^1} \]
\[ \ll \sum_{p=0}^n \int_{\mathbb{R}^n} |R_p \mathcal{H} f(x)| \, dx \leq \int_{\mathbb{R}^n} |\Phi(u)| \sum_{p=0}^n ||R_p f(\cdot A(u))||_{L^1} \, du \]
\[ \ll \int_{\mathbb{R}^n} |\Phi(u)| ||f(\cdot A(u))||_{H^1} \, du. \]

We wish to estimate the right-hand side from above by using (8). Let

\[ f(xA(u)) = \sum_k c_k a_k(xA(u)). \]

We will show that multiplying \( a_k(xA(u)) \) by a constant depending on \( u \) (actually on \( A(u) \)) we get an atomic decomposition of \( f \) itself, with no composition in the argument. Since we analyze all such decompositions for \( f \), the upper bound will be \( ||f||_{H^1} \) times the mentioned constant, which completes the proof.

Thus, let us figure out when, or under which transformation \( a_k(xA(u)) \) becomes an atom. We have

\[ \int_{\mathbb{R}^n} a_k(xA(u)) \, dx = \int_{a_k(xA(u)) \neq 0} a_k(xA(u)) \, dx, \]

and under substitution \( xA(u) = v \) the integral becomes \( \int_{\mathbb{R}^n} a_k(v) \, dv \) times a Jacobian depending only on \( u \). This integral vanishes because of (7).

The support of \( a_k(xA(u)) \) is \( < xA, xA > \leq r^2 \), an ellipsoid. To use known results, let us represent it in the transposed form \( < A^T x^T, A^T x^T > \leq r^2 \).

Let us solve the following extremal problem. We are looking for the min of the quadratic form \( < Bx^T, Bx^T > \), where \( B \) is a non-singular \( n \times n \) real-valued matrix - we denote the linear transformation and its matrix with the same symbol - on the unit sphere \( < x^T, x^T > = 1 \).
Denoting by $B^*$ the adjoint to $B$, we arrive to the equivalent problem for $<B^*Bx, x^T>$. Since $(B^*)^* = B$, the operator $B^*B$ is self-adjoint:

$$<B^*Bx, y> = <Bx, By> = <x, B^*By>,$$

and thus positive definite. Since $B$ is non-singular, the same $B^*B$ is.

If a transformation is positive definite, all its eigenvalues are non-negative; if it is also non-singular all the eigenvalues are strictly positive. Define the minimal eigenvalue of $B^*B$ by $l_1$.

By Theorem 1 from Ch.II, §17 of [2], for self-adjoint $C$ the form $<Cx, x>$ on the unit sphere attains its minimum equal to the least eigenvalue of $C$.

Hence the solution of the initial problem is just $l_1$. We mention also that the matrix of the adjoint real transformation is the transposed initial. Therefore, the desired minimum is the least eigenvalue $l_1$ of $B^TB$.

It follows from this by taking $B = A^T$

$$l_1 < x^T, x^T> = <A^Tx^T, A^Tx^T> \leq r^2,$$

and every point of the ellipsoid $<A^Tx^T, A^Tx^T> \leq r^2$ lies in the ball $<x^T, x^T> \leq r^2/l_1$, where $l_1$ is the minimal eigenvalue of the matrix $A^TA$.

It remains to check the $\infty$ norm. Instead of the measure of the ball in (6) we must have the measure of the ball of radius $r/\sqrt{l_1}$. This is achieved by multiplying $a_k(xA(u))$ by $l_1^{n/2}$ and hence $l_1^{n/2}a_k(xA(u))$ is an atom. Correspondingly,

$$||f(A(u))||_{H^1} \ll l_1^{-n/2} ||f||_{H^1},$$

and finally $Hf$ belongs to $H^1$ provided

$$\int_{\mathbb{R}^n} |\Phi(u)| l_1^{-n/2}(u) \, du < \infty.$$

Further, $l_1^{-n/2} = L_n^{n/2}$, where $L_n$ is the maximal eigenvalue of the matrix $(A^TA)^{-1} = A^{-1}(A^T)^{-1}$. But it is known that such $L_n$ is equal to the spectral radius of the corresponding matrix $A^{-1}(A^T)^{-1}$ and, in turn, to $||A^{-1}||_2^2$. Replacing $l_1^{-n/2}(u)$ in (11) with the obtained bound completes the proof.

As is mentioned above, the obtained condition (11) is weaker that (3) but of course still more restrictive than (2).

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References

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