Remarks on the nonlocal Dirichlet problem

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Abstract

We study translation-invariant integrodifferential operators that generate Lévy processes. First, we investigate different notions of what a solution to a nonlocal Dirichlet problem is and we provide the classical representation formula for distributional solutions. Second, we study the question under which assumptions distributional solutions are twice differentiable in the classical sense. Sufficient conditions and counterexamples are provided.

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1 Introduction

The aim of this article is to provide two results on translation-invariant integrodifferential operators, which are not surprising but have not been systematically covered in the literature. Let us briefly explain these results in case of the classical Laplace operator.

The classical result of Weyl says the following. Assume $D \subset \mathbb{R}^d$ is an open set, $f \in C^\infty(D)$, and $u \in \mathcal{D}'(D)$ is a Schwartz distribution satisfying $\Delta u = f$ in the distributional sense, i.e. $\langle u, \Delta \psi \rangle = \langle \psi, f \rangle$ for every $\psi \in C^\infty_c(D)$. Then $u \in C^\infty(D)$ and $\Delta u = f$ in $D$. This is the starting point for the study of distributional solutions to boundary value problems. Our first aim is to study distributional solutions to nonlocal boundary value problems of the form

\[ \mathcal{L} u = f \quad \text{in } D, \]
\[ u = g \quad \text{in } D^c, \]

\[ \mathcal{L} = \int_{\mathbb{R}^d} (t_1 \xi_1 + \cdots + t_d \xi_d) \, \tau(d\xi), \]

where $\tau$ is a nonnegative finite Borel measure on $\mathbb{R}^d$. The second aim is to study distributional solutions to nonlocal boundary value problems of the form

\[ \mathcal{L} u = f \quad \text{in } D, \]
\[ u = g \quad \text{in } D^c, \]

where $\mathcal{L}$ is a differential operator with constant coefficients.
where $\mathcal{L}$ is an integrodifferential operator generating a unimodal Lévy process. Our second aim is to provide sufficient conditions such that distributional solutions $u$ to the nonlocal Dirichlet problem are twice differentiable in the classical sense. In case of the Laplace operator, it is well known that Dini continuity of $f : D \to \mathbb{R}$, i.e. finiteness of the integral $\int_0^1 \omega_f(r)/r \, dr$ for the modulus of continuity $\omega_f$, implies that the distributional solution $u$ to the classical Dirichlet problem satisfies $u \in C^2_{\text{loc}}(D)$. On the other hand, one can construct a continuous function $f : B_1 \to \mathbb{R}$ and a distribution $u \in D'(B_1)$ such that $\Delta u = f$ in the distributional sense, but $u \notin C^2_{\text{loc}}(B_1)$. These observation have been made long time ago [24]. They have been extended to non-translation-invariant operators by several authors [11, 30] and to nonlinear problems [28, 14]. Note that there are many more related contributions including treatments of partial differential equations on non-smooth domains. In the present work we treat the simple linear case for a general class of nonlocal operators generating unimodal Lévy processes.

Let us introduce the objects of our study and formulate our main results. Let $\nu : \mathbb{R}^d \setminus \{0\} \to [0, \infty)$ be a function satisfying

$$
\int (1 \wedge |h|^2)\nu(h) \, dh < \infty.
$$

The function $\nu$ induces a measure $\nu(dh) = \nu(h) \, dh$, which is called the Lévy measure. We note that we use the same symbol for the measure as well as for the density. We study operators of the form

$$
\mathcal{L}u(x) = \lim_{\epsilon \to 0} \int_{|y| > \epsilon} (u(x + y) - u(x))\nu(y) \, dy.
$$

This expression is well defined if $u$ is sufficiently regular in the neighbourhood of $x \in \mathbb{R}^d$ and satisfies some integrability condition at infinity. We recall that for $\alpha \in (0,2)$ and $\nu(dh) = c_\alpha |h|^{-d-\alpha} \, dh$ with some appropriate constant $c_\alpha$, the operator $\mathcal{L}$ equals the fractional Laplace operator $-(-\Delta)^{\alpha/2}$ on $C^2_b(\mathbb{R}^d)$. The regularity theory of such operators has been intensively studied recently. For instance, it is well known [3, 33, 18, 34, 32] that the solution of $-(-\Delta)^{\alpha/2}u = f$ with $f \in C^\beta$ belongs to $C^{\alpha+\beta}$ provided that neither $\beta$ nor $\alpha + \beta$ is an integer. The same result in more general setting is derived in [2].

Our standing assumption is that $h \to \nu(h)$ is a non-increasing radial function and that there exists a Lévy measure $\nu^*$ resp. a density $\nu^*$ such that $\nu \leq \nu^*$ and

$$
\nu^*(r) \leq C\nu^*(r + 1), \quad r \geq r_0
$$

for some $r_0, C \geq 1$. Given an open set $D \subset \mathbb{R}^d$, denote by $\mathcal{L}^1(D)$ the vector space of all Borel functions $u \in L^1_{\text{loc}}$ satisfying

$$
\int_D |u(x)|(1 \wedge \nu^*(x)) \, dx < \infty.
$$

The condition $u \in \mathcal{L}^1(D)$ is the integrability condition needed to ensure well-posedness in the definition of $\mathcal{L}u$ in distributional sense. Given an open set, we denote by $G_D$ resp. $P_D$ the usual Green resp. the Poisson operator, cf. Section 2. For a definition of the Kato class $\mathcal{K}$ and $\mathcal{K}(D)$ see Definition 2.4 below. Here is our first result.
Theorem 1.1. Let $D$ be a bounded open set. Suppose $f \in L^1(D)$ and $g \in L^1(D^c)$. Let $u \in L^1(\mathbb{R}^d)$ be a distributional solution of the Dirichlet problem

$$\begin{align*}
\mathcal{L} u &= f \quad \text{in } D, \\
u &= g \quad \text{in } D^c.
\end{align*}$$

Then $u(x) + G_D[f](x)$ satisfies the mean-value property inside $D$. Furthermore, if $D$ is a Lipschitz domain and there exists $V \subset\subset D$ such that $f$ and $g * \nu$ belongs to the Kato class $K(D \setminus V)$, then there is a unique solution which is bounded close to the boundary of $D$

$$u(x) = -G_D[f](x) + P_D[g](x).$$

The theorem above says that the distributional solution of (1.4) is unique up to a harmonic function. If, additionally, $D$ is a Lipschitz domain and we impose some regularity, then the solution is unique. Boundedness of $u, f, g$ would suffice, of course. It is obvious that one has to impose some regularity condition on $f$ in order to prove uniqueness of solutions. Note that, in the case where $\mathcal{L}$ equals the fractional Laplace operator, similar results like Theorem 1.1 are proved in [6]. A result similar to Theorem 1.1 has recently been proved in [26]. The authors consider a smaller class of operators and concentrate on viscosity solutions instead of distributional solutions.

Variational solutions to nonlocal operators have been studied by several authors, e.g., in [17, 35]. The problem to determine appropriate function spaces for the data $g$ leads to the notion of nonlocal traces spaces introduced in [15]. It is interesting that the study of Dirichlet problems for nonlocal operators leads to new questions regarding the theory of function spaces.

The formulation of our second main result requires some further preparation. They are rather technical because we cover a large class of translation-invariant operators. The similar condition to the following appears in [7].

(A) $\nu$ is twice continuously differentiable and there is a positive constant $C$ such that

$$|\nu'(r)|, |\nu''(r)| \leq C \nu^*(r) \quad \text{for } r \geq r_0.$$  

(A) and (1.2) are essential for proving that functions with the mean-value property are twice continuously differentiable, see Lemma 2.3. We emphasize that in general this is not the case and usually harmonic functions lack sufficient regularity if no additional assumptions are imposed. The reader is referred to [29, Example 7.5], where a function $f$ with the mean-value property is constructed for which $f'(0)$ does not exist.

Let $G$ be a fundamental solution of $\mathcal{L}$ on $\mathbb{R}^d$ (see (2.2) for definition). Note that in the case of the fractional Laplace operator $G(x) = c_{d,\alpha}|x|^\alpha - d$ for $d \neq \alpha$ and some constant $c_{d,\alpha}$. In what follows we will assume the kernel $G$ to satisfy the following growth condition:

(G) $G \in C^2(\mathbb{R}^d \setminus \{0\})$ and there exists a non-increasing function $S : (0, \infty) \mapsto [0, \infty)$ and $r_0 > 0$ such that
Section 3 is a new result even in this case. We also study the Theorem 1.2, if Theorem 1.2 Section 2 we provide the main definitions and

Theorem 1.2. Let \( D \) be an open bounded set. Assume that the measure \( \nu \) satisfies (A) and (1.2) and the fundamental solution \( G \) satisfies (G). Let \( g \in \mathcal{L}^1(D^c) \) and \( f : D \rightarrow \mathbb{R} \).

If \( \int_0^1 |G'(t)| t^{d-1} \, dt < \infty \) \( G(r), |G'(r)|, r|G''(r)| \leq S(r), \quad r < r_0, \)

(ii) if \( \int_0^1 |G'(t)| t^{d-1} \, dt < \infty \), then additionally \( G \in C^3(\mathbb{R}^d \setminus \{0\}) \) and \( G(r), |G'(r)|, |G''(r)|, r|G'''(r)| \leq S(r), \quad r < r_0. \)

Then the solution \( u \in \mathcal{L}^1(\mathbb{R}^d) \) of the problem

\[
\begin{align*}
\mathcal{L} u &= f \quad \text{in } D, \\
u u &= g \quad \text{in } D^c. 
\end{align*}
\]

belongs to \( C^2_{\text{lo}}(D) \) and is unique up to a harmonic function (with respect to \( \mathcal{L} \)).

Remark 1.3. (1.5) or (1.6) imply \( f \in K(D) \), so by Theorem 1.1, if \( D \) is a Lipschitz domain and \( g \ast \nu \in K(D) \) then the solution is unique.

The result uses quite involved conditions because the measure \( \nu \) interacts with the Dirichlet assumptions for the right-hand side function \( f \). Looking at examples, we see that the two cases described in the theorem appear naturally. In the fractional Laplacian case \( (G(x) = c_{d, \alpha} |x|^{\alpha-d}) \), finiteness of the expression \( \int_0^{1/2} |G'(t)| t^{d-1} \, dt \) depends on the value of \( \alpha \in (0, 2) \). We show in Section 6 that the conditions hold true when \( \mathcal{L} \) is the generator of a rotationally symmetric \( \alpha \)-stable process, i.e., when \( \mathcal{L} \) equals the fractional Laplace operator. Note that Theorem 1.2 is a new result even in this case. We also study the more general class, e.g. operators of the form \( -\varphi(-\Delta) \), where \( \varphi \) is a Bernstein function. Note that in the theorem above we do not assume that \( g \) is bounded.

Remark 1.4. We emphasize that in the case of \( \mathcal{L} \) being the fractional Laplace operator of order \( \alpha \in (0, 2) \) and \( f \in C^2_{\text{loc}}(D^c) \), it is not true that every solution of \( \mathcal{L} u = f \) belongs to \( C^2_{\text{loc}}(D) \) as is stated in [1, Theorem 3.7]. A similar phenomenon has been mentioned in [3] and is visible here as well. Observe that in such case the integrals (1.5) and (1.6) are clearly divergent and consequently, Theorem 1.2 cannot be applied. We devote Section 5 to the construction of counterexamples for any \( \alpha \in (0, 2) \).

The article is organized as follows: in Section 2 we provide the main definitions and some preliminary results. The proof of Theorem 1.1 is provided in Section 3. Section 4 contains several rather technical computations and the proof of Theorem 1.2. We discuss
the necessity of the assumptions of Theorem 1.2 through examples in Section 5. Finally, in Section 6 we provide examples that show that the assumptions of Theorem 1.2 are natural.

2 Preliminaries

In this section we explain our use of notation, define several objects and collect some basic facts. We write \( f \asymp g \) when \( f \) and \( g \) are comparable, that is the quotient \( f/g \) stays between two positive constants. To simplify the notation, for a radial function \( f \) we use the same symbol to denote its radial profile. In the whole paper \( c \) and \( C \) denote constants which may vary from line to line. We write \( c(a) \) when the constant \( c \) depends only on \( a \). By \( B(x,r) \) we denote the ball of radius \( r \) centered at \( x \), that is \( B(x,r) = \{ y \in \mathbb{R}^d : |y-x| < r \} \). For convenience we set \( B_r = B(0,r) \). For an open set \( D \) and \( x \in D \) we define \( \delta_D(x) = \text{dist}(x,\partial D) \) and \( \text{diam}(D) = \sup_{x,y \in D}|x-y| \). The modulus of continuity of a continuous function \( f : D \to \mathbb{R} \) is defined by
\[
\omega_f(t, D) = \sup\{|f(x) - f(y)| : x,y \in D, |x-y| < t\} \quad (t > 0).
\]
For a differentiable function \( f : D \to \mathbb{R} \) we set
\[
\omega f(t, D) = \max_{i \in \{1, \ldots, d\}} \sup_{x,y} |\partial_{x_i} f(x) - \partial_{x_i} f(y)| \quad (t > 0).
\]
We say that a Borel measure is isotropic unimodal if it is absolutely continuous on \( \mathbb{R}^d \setminus \{0\} \) with respect to the Lebesgue measure and has a radial, non-increasing density. Given an isotropic unimodal Lévy measure \( \nu(dx) = \nu(|x|)dx \), we define a Lévy-Khinchine exponent
\[
\psi(\xi) = \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot x)) \nu(dx), \quad \xi \in \mathbb{R}^d.
\]
\( \psi \) is usually called the characteristic exponent. It is well known (e.g. [29, Lemma 2.5]) that if \( \nu(\mathbb{R}^d) = \infty \), there exist a continuous function \( p_t \geq 0 \) in \( \mathbb{R}^d \setminus \{0\} \) such that
\[
\tilde{p}_t(\xi) = \int_{\mathbb{R}^d} e^{-i\xi \cdot x} p_t(x) dx = e^{-t\psi(\xi)}, \quad \xi \in \mathbb{R}^d.
\]
The family \( \{p_t\}_{t \geq 0} \) induces a strongly continuous contraction semigroup on \( C_0(\mathbb{R}^d) \) and \( L^2(\mathbb{R}^d) \)
\[
P_tf(x) = \int_{\mathbb{R}^d} f(y)p_t(y-x)dy, \quad x \in \mathbb{R}^d,
\]
whose generator \( \mathcal{A} \) has the Fourier symbol \(-\psi\). Using the Kolmogorov theorem one can construct a stochastic process \( X_t \) with transition densities \( p_t(x,y) = p_t(y-x) \), namely \( \mathbb{P}^x(X_t \in A) = \int_A p_t(x,y)dy \). Here \( \mathbb{P}^x \) is the probability corresponding to a process \( X_t \) starting from \( x \), that is \( \mathbb{P}^x(X_0 = x) = 1 \). By \( \mathbb{E}^x \) we denote the corresponding expectation. In fact, \( X_t \) is a pure-jump isotropic unimodal Lévy process in \( \mathbb{R}^d \), that is a stochastic process with stationary and independent increments and càdlàg paths (see for instance [36]).
One of the objects of significant importance in this paper is the potential kernel defined as follows:

$$ U(x, y) = \int_0^\infty p_t(x, y) \, dy. $$

Clearly $U(x, y) = U(y - x)$. The potential kernel can be defined in our setting if $\int_{B_1} \frac{1}{\psi(\xi)} \, d\xi < \infty$. In particular, for $d \geq 3$ the potential kernel always exists (see [36, Theorem 37.8]). If this is not the case, one can consider the compensated potential kernel

$$ W_{x_0}(x - y) = \int_0^\infty (p_t(x - y) - p_t(x_0)) \, dt $$

for some fixed $x_0 \in \mathbb{R}^d$. If $d = 1$ and $\int_{B_1} \frac{1}{\psi(\xi)} \, d\xi < \infty$, we can set $x_0 = 0$. In other cases the compensation must be taken with $x_0 \in \mathbb{R}^d \setminus \{0\}$. For details we refer the reader to [21] and to the Appendix A.

Slightly abusing the notation, we let $W_1$ be (2.1) for $x_0 = (0, ..., 0, 1) \in \mathbb{R}^d$. Thus, we have arrived with three potential kernels: $U$, $W_0$ and $W_1$. Each one corresponds to a different type of process $X_t$ and an operator associated with it. In order to merge these cases in one object, we let

$$ G(x) = \begin{cases} 
    U(x), & \text{if } \int_{B_1} \frac{1}{\psi(\xi)} \, d\xi < \infty, \\
    W_0(x), & \text{if } d = 1, \int_{B_1} \frac{1}{\psi(\xi)} \, d\xi = \infty \text{ and } \int_0^\infty \frac{1}{1 + \psi(\xi)} \, d\xi < \infty, \\
    W_1(x), & \text{otherwise}.
\end{cases} $$

For instance, in the case of $\mathcal{L} = \Delta$ we have

$$ G(x) = \begin{cases} 
    c_d |x|^{2-d}, & d \geq 3, \\
    \frac{1}{\pi} \ln \frac{1}{|x|}, & d = 2, \\
    |x|, & d = 1.
\end{cases} $$

The basic object in the theory of stochastic processes is the first exit time of $X$ from $D$, $\tau_D = \inf\{t > 0 : X_t \notin D\}$. Using $\tau_D$ we define an analogue of the generator of $X_t$, namely, the characteristic operator or Dynkin operator. We say a Borel function $f$ is in a domain $\mathcal{D}_U$ of Dynkin operator $\mathcal{U}$ if there exists a limit

$$ \mathcal{U}f(x) = \lim_{B \to \{x\}} \frac{\mathbb{E}^x(X_{\tau_B}) - f(x)}{\mathbb{E}^x\tau_B}. $$

Here $B \to \{x\}$ is understood as a limit over all sequences of open sets $B_n$ whose intersection is $\{x\}$ and whose diameters tend to 0 as $n \to \infty$. The characteristic operator is an extension of $\mathcal{A}$, that is $\mathcal{D}_A \subset \mathcal{D}_U$ and $\mathcal{U}|_{\mathcal{D}_A} = \mathcal{A}$. For a wide description of characteristic operator and its relation with the generator of $X_t$ we refer the reader to [16, Chapter V].

Instead of the whole $\mathbb{R}^d$, one can consider a process $X$ killed after exiting $D$. By $p_t^D(x, y)$ we denote its transition density (or, in other words, the fundamental solution of $\partial_t - \mathcal{L}$)
in $D$). We have
\[ p_t^D(x, y) = p_t(x, y) - \mathbb{E}^x[\tau_D < t; p_t(X_{\tau_D}, y)], \quad x, y \in \mathbb{R}^d. \]

It follows that $0 \leq p_t^D \leq p_t$. By $P_D(x, dz)$ we denote the distribution of $X_{\tau_D}$ with respect to $\mathbb{P}^x$, that is $P_D(x, A) = \mathbb{P}^x(X_{\tau_D} \in A)$. We call $P_D(x, dz)$ a harmonic measure and its density $P_D(x, z)$ on $\mathbb{R}^d \setminus \overline{D}$ with respect to the Lebesgue measure — a Poisson kernel. For $g : D^c \mapsto \mathbb{R}$ we let
\[ P_D[g](x) = \int_{D^c} g(z) P_D(x, dz), \quad x \in D, \]
if the integral exists. For $x \in D^c$ we set $P_D[g](x) = g(x)$.

**Remark 2.1.** If $D$ is an open bounded set and $g \in L^1(D^c)$ then $P_D[g] \in L^1$. Indeed, since $P_D[g] \equiv g$ on $D^c$, it is enough to prove that $P_D[g] \in L^1(D)$. By the mean-value property, for any $B \subset D$ we have $P_D[P_D[g](x) = P_D[g](x)]$ for $x \in B$. It follows, by the Ikeda-Watanabe formula, that for any fixed $x \in B$
\[ \infty > \int_{B^c} P_B(x, z) P_D[g](z) dz \geq c \int_{A \setminus D} P_D[g](z) dz, \]
where $A = B^c \cap (B + \text{supp}(\nu)/2)$. Arbitrary choice of $B$ yields the claim.

We define a Green function for the set $D$
\[ G_D(x, y) = \int_0^{\infty} p_t^D(x, y) dt, \quad x, y \in D, \]
and the Green operator
\[ G_D[f](x) = \int_D G_D(x, y) f(y) dy. \]

We note that $G_D(x, y)$ can be interpreted as the occupation time density up to the exit time $\tau_D$, $G_D[f]$ — as a mean value of $f(X_t)$. Using that we obtain $G_D[1] = \mathbb{E}^x \tau_D$. For bounded sets $D$ we have $\sup_{x \in \mathbb{R}^d} \mathbb{E}^x \tau_D < \infty$ ([31], [8]). By the strong Markov property for any open $\Omega \subset D$ we have
\[ G_D(x, y) = G_{\Omega}(x, y) + \mathbb{E}^x G_D(X_{\tau_\Omega}, y), \quad x, y \in \Omega. \quad (2.3) \]

Obviously we have $G_{\mathbb{R}^d} = U$. If $U$ is well-defined (finite) a.s., the well-known Hunt formula holds:
\[ G_D(x, y) = U(y - x) - \mathbb{E}^x U(y - X_{\tau_D}), \quad x, y \in D. \]

In case of compensated potential kernels, a similar formula is valid, namely,
\[ G_D(x, y) = G(y - x) - \mathbb{E}^x G(y - X_{\tau_D}), \quad x, y \in D. \quad (2.4) \]

See Theorem A.4.

**Definition 2.2.** We say that a function $g : \mathbb{R}^d \mapsto \mathbb{R}$ satisfies the mean-value property in an open set $D \subset \mathbb{R}^d$ if $g(x) = P_D[g](x)$ for all $x \in D$. Here we assume that the integral is absolutely convergent. If $g$ has the mean-value property in every bounded open set whose closure is contained in $D$ then $u$ is said to have the mean-value property inside $D$. 

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Clearly if $f$ has the mean-value property inside $D$, then $U f = 0$ in $D$.

In general, functions with the mean-value property lack sufficient regularity if no additional assumptions are imposed. In our setting, however, we can show that they are, in fact, twice continuously differentiable in $D$.

**Lemma 2.3.** Let $g \in L^1$ and $D$ be an open set. Suppose that (A) and (1.2) hold. If $g$ has the mean-value property inside $D$, then $g \in C^2_{\text{loc}}(D)$.

The proof is similar to the proof of [7, Theorem 4.6] and is omitted.

**Definition 2.4 ([38], [23]).** We say that a Borel function $f$ belongs to the Kato class $\mathcal{K}$ if it satisfies the following condition

$$\lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_0^r P_t |f|(x) \, dt = 0.$$  \hspace{1cm} (2.5)

We say that $f \in \mathcal{K}(D)$, where $D$ is an open set, if $f \mathbf{1}_D \in \mathcal{K}$.

This is one of three conditions discussed by Zhao in [38]. A detailed description of different notions of the Kato class and related conditions can be found in [23].

**Lemma 2.5.** Let $V \subset \subset D$ and $\rho := \text{dist}(V, \partial D)$. Suppose $f \in \mathcal{K}(D \setminus V)$. Then $G_D[f]$ is bounded in $D_1 := \{ x \in D \setminus V : \delta_D(x) < \rho/2 \}$.

**Proof.** Let $x \in V_1$ and define $V_2 := \{ x \in D \setminus V : \delta_D(x) < 3\rho/4 \}$. We have

$$(G_D[f \mathbf{1}_{V_2}](x)) \leq \int_{V_2} G_D(x,y) |f(y)| \, dy.$$  \hspace{1cm} (2.3)

Let $r = 2 \sup_{x \in D} |x|$. Then $D \subset B_r$ and by [20, Theorem 1.3]

$$\int_{V_2} G_D(x,y) |f(y)| \, dy \leq \int_{V_2} G_{B_r}(x,y) |f(y)| \, dy \leq c(\rho) \|f\|_1.$$  

Moreover, by (2.3)

$$G_D[f \mathbf{1}_{V_2}](x) = G_{D \setminus V}[f \mathbf{1}_{V_2}](x) + \mathbb{E}^x G_D[f \mathbf{1}_{V_2}] \left( X_{T_D,V} \right).$$

Observe that

$$|\mathbb{E}^x G_D[f \mathbf{1}_{V_2}](X_{T_D,V})| \leq \mathbb{E}^x \int_{V_2} G_D(x_{T_D,V},y) |f(y)| \, dy \leq c(\rho/4) \|f\|_1$$

again by [20, Theorem 1.3]. Finally, we have

$$|G_{D \setminus V}[f \mathbf{1}_{V_2}](x)| \leq G_{D \setminus V}[|f| \mathbf{1}_{D \setminus V}](x).$$

A straightforward application of the proof of [12, Theorem 4.3] to the last term gives the claim. \hfill \Box

**Proposition 2.6.** If $f$ satisfies (1.5) then it is uniformly continuous in $D$. If (1.6) holds then $\frac{\partial}{\partial x_i} f$, $i = 1, \ldots, d$, is uniformly continuous in $D$. 

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Proof. Suppose \( \frac{\partial}{\partial x_i} f \) for some \( i = 1, \ldots, d \) is not uniformly continuous, i.e. \( \omega \chi(t, D) \geq c > 0 \) for \( t \leq 1 \). If (1.6) holds then in particular

\[
\infty > \int_{0}^{1/2} S(t) \omega \chi(t, D) t^{d-1} dt \geq c \int_{0}^{1/2} |G'(t)| t^{d-1} dt,
\]

which is a contradiction. Now let \( \omega(t, D) \geq c \) for \( t \leq 1 \), and suppose (1.5). For \( d \geq 3 \) we have

\[
\infty > \int_{0}^{1/2} S(t) \omega(t, D) t^{d-1} dt \geq c \int_{0}^{1/2} |G''(t)| t^{d-1} dt.
\]

By integration by parts

\[
\int_{0}^{1/2} G'''(t) t^{d-1} dt = G'(t) t^{d-1} \bigg|_{0}^{1/2} - (d-1) \int_{0}^{1/2} G'(t) t^{d-2} dt.
\]

Observe that \( G' \) is of constant sign. Hence, both \( \lim_{t \to 0^+} G'(t) t^{d-1} \) and the integral are finite. In particular, integration by parts once again yields

\[
\int_{0}^{1/2} G'(t) t^{d-2} dt = G(t) t^{d-2} \bigg|_{0}^{1/2} - (d-2) \int_{0}^{1/2} G(t) t^{d-3} dt.
\]

Both \( \lim_{t \to 0^+} G(t) t^{d-2} \) and the integral are positive. Hence, both must be finite. By [19, Proposition 1 and 2] we have \( \int_{0}^{1} G(t) t^{d-1} dt \geq c \psi(1/r)^{-1} \). It follows that

\[
\int_{0}^{1} G(t) t^{d-3} dt = \int_{B_1} G(|x|) \frac{dx}{|x|^2} = \int_{B_1} \frac{1}{s^d} \int_{|x|}^{\infty} G(|x|) d|x| \geq \int_{1}^{\infty} \frac{1}{s^d} \int_{B_{1/s}} G(|x|) d|x|
\]

\[
\geq \int_{0}^{1} \frac{1}{\psi(1/s)^2} ds + \psi(1) \geq \int_{1}^{\infty} \frac{u^2}{\psi(u)} du \geq \int_{1}^{\infty} \frac{du}{u} = \infty,
\]

which is a contradiction. Now let \( d = 2 \). By the same argument

\[
\int_{0}^{1/2} G''(t) t dt = G'(t) t \bigg|_{0}^{1/2} - \int_{0}^{1/2} G'(t) dt
\]

and we conclude that the integral is finite. Hence, \( \lim_{t \to 0^+} G(t) \leq \infty \). By [36, Theorems 41.5 and 41.9] we get the contradiction. Finally, for \( d = 1 \) we get that \( \lim_{t \to 0^+} G'(t) < \infty \). It follows that \( \limsup_{t \to 0^+} G(t)/t < \infty \). Due to [4, Theorem 16] and [21, Lemma 2.14] we obtain that

\[
\liminf_{x \to \infty} \psi(x)/x^2 > 0,
\]

which is a contradiction, since \( \limsup_{x \to \infty} \psi(x)/x^2 = 0 \).

\[\square\]

Lemma 2.7. Let \( D \) be bounded open and \( k \in \mathbb{N} \). If \( g \in C^k_{\text{loc}}(\mathbb{R}^d \setminus \{0\}) \cap L^1_{\text{loc}}(D) \) and \( f \in C^k(D) \) then \( g * f \in C^k_{\text{loc}}(D) \).

Proof. Fix \( x_0 \in D \). Let \( l = \delta_D(x_0) \). Let \( \chi_1, \chi_2 \in C^\infty(\mathbb{R}^d) \) be such that \( \mathbf{1}_{B(x_0, l/4)} \leq \chi_1 \leq \mathbf{1}_{B(x_0, l/2)} \) and \( \mathbf{1}_{B(x_0, l/8)} \leq \chi_2 \leq \mathbf{1}_{B(x_0, l/16)} \). Observe that \( g * f = g *(f \chi_1) + (g \chi_2) * (f(1 - \chi_1)) \) on \( B(x_0, l/8) \). Since \( f \chi_1, g \chi_2 \in C^k(\mathbb{R}^d) \), it follows that \( g * f \in C^k(B(x_0, l/8)) \). Since \( x_0 \) was arbitrary, the claim follows by induction.

\[\square\]
A consequence of Lemma 2.7 is the following corollary.

**Corollary 2.8.** Let $D$ be open and bounded and $k \in \mathbb{N}$. If $g \in C^k_{\text{loc}}(\mathbb{R}^d \setminus \{0\}) \cap L^1_{\text{loc}}$ then $g \ast 1_D \in C^k_{\text{loc}}(D)$.

The following lemma is crucial in one of the proofs.

**Lemma 2.9 ([20, Proposition 3.2]).** Let $X_t$ be an isotropic unimodal Lévy process in $\mathbb{R}^d$. For every $r > 0$ there is a radial kernel function $\mathcal{P}_r(z)$ and a constant $C(r) > 0$ such that $\mathcal{P}_r(z) = C(r)$ for $x \in B_r$, $0 \leq \mathcal{P}_r(z) \leq C(r)$ for $z \in \mathbb{R}^d$ and the profile function of $\mathcal{P}_r$ is non-increasing. Furthermore, if $f$ has the mean-value property in $B(x,r)$

$$f(x) = \int_{\mathbb{R}^d} f(z) \mathcal{P}_r(x - z) \, dz = f \ast \mathcal{P}_r(x).$$

### 3 Weak solutions

The aim of this section is to prove Theorem 1.1. For the fractional Laplacian related results are known, cf. [6, Section 3]. A similar result has recently been obtained in [10] using purely analytic methods instead of probabilistic ones exploited in [6]. When the generalization of these results to more general nonlocal operators is immediate, we omit the proof.

**Lemma 3.1.** Suppose $u \in L^1(\mathbb{R}^d)$ has the mean-value property inside $D$ with respect to $X_t$. Then $\mathcal{L}u = 0$ in $D$ in distributional sense.

**Proof.** Let $\varphi \in C_c^\infty(D)$ and $\phi_\epsilon$ be a standard mollifier (i.e. $\phi_\epsilon \in C_c^\infty(\mathbb{R}^d)$ and $\text{supp} \phi_\epsilon = \overline{B_\epsilon}$). Using (1.2) it is easy to check that $\phi_\epsilon \ast u \in L^1(\mathbb{R}^d) \cap C^\infty(D)$. Hence, $\mathcal{L}(\phi_\epsilon \ast u)$ can be calculated pointwise for $x \in D$ and we have

$$(\phi_\epsilon \ast u, \mathcal{L} \varphi) = (\mathcal{L}(\phi_\epsilon \ast u), \varphi).$$

We consider Dynkin characteristic operator $\mathcal{U}$. Since it is an extension of $\mathcal{L}$ and is translation-invariant, we obtain

$$\mathcal{L}(\phi_\epsilon \ast u) = \mathcal{U}(\phi_\epsilon \ast u) = \phi_\epsilon \ast \mathcal{U}u.$$

We have $\mathcal{U}u(x) = 0$ for $x \in D$, hence

$$0 = (\mathcal{L}(\phi_\epsilon \ast u), \varphi) = (\phi_\epsilon \ast u, \mathcal{L} \varphi), \quad \varphi \in C_c^\infty(D_\epsilon),$$

where $D_\epsilon = \{x \in D : \delta_D(x) > \epsilon\}$. Passing $\epsilon \to 0$ we get the claim.

The following lemma is a generalization of [6, Theorem 3.9 and Corollary 3.10], where the fractional Laplace operator is considered.

**Lemma 3.2.** Let $u \in L^1(\mathbb{R}^d) \cap C^2_{\text{loc}}(D)$ be a solution of $\mathcal{L}u = 0$ in $D$ in distributional sense. Then $u$ has the mean-value property inside $D$. 


We have

Lemma 2.9
it is continuous in

\[ D \]

\[ n \]

\[ D \]

\[ h \]

which implies that

\[ h \]

a function of

The first integral is clearly absolutely convergent. We claim that it is also continuous as

\[ x \in \mathbb{R}^d. \]

by [8, Lemma 2.1 and 2.9], by arbitrary

choice of \( \epsilon \) we get the claim.

Furthermore, from monotonicity of \( 1 \wedge \nu^*(h) \) we obtain

\[ P_D(x, z) \leq (1 \wedge \nu^*(\text{dist}(z, D_1))) E^{\tau_{D_1}}_z, \quad x \in D_1, \ z \in D_2. \]

Since \( u \in L^1(\mathbb{R}^d) \), (1.3) implies the absolute convergence of the second integral. Since

by [8, Lemma 2.9 and Remark 2] \( E^{\tau_{D_1}} \in C_0(D_1) \), it is continuous as well. Hence \( u \) is

continuous and has the mean-value property in \( D_1 \). Note that \( u = u \) on \( D_1 \), since \( D_1 \) is

a Lipschitz domain.

Let \( h = \tilde{u} - u \). We now verify that \( h \geq 0 \) so that \( u = u \) has the mean-value property in

\( D_1 \). Since \( \mathcal{L}u = 0 \) in \( D_1 \), from Lemma 3.1 we have \( \mathcal{L}h(x) = 0 \) for \( x \in D_1 \). Observe \( h \) is

continuous and compactly supported. Suppose it has a positive maximum at \( x_0 \in D_1 \), then

\[ 0 = \mathcal{L}h(x_0) = \int_{\mathbb{R}^d} (h(y) - h(x_0)) \nu(x_0 - y) \, dy, \]

which implies that \( h \) is constant on \( \text{supp}(\nu) + x_0 \). If \( D_1 \subset \text{supp}(\nu) + x_0 \) we get that

\( h \leq 0 \). If not we can use the chain rule to get for any \( n \in \mathbb{N} \) that \( h \) is constant on

\( \text{nsupp}(\nu) + x_0 \) and consequently \( h \leq 0 \). Similarly, \( h \) must be non-negative. \( \square \)

Lemma 3.3. Let \( u \in L^1(\mathbb{R}^d) \) be a solution of \( \mathcal{L}u = 0 \) in \( D \) in distributional sense.

Then \( u \) has the mean-value property inside \( D \).

Proof. Let \( \Omega \subset D \) be a bounded Lipschitz domain. By [37] and the Ikeda-Watanabe

formula we have that the harmonic measure \( P_{\Omega}(x, d\nu) \) is absolutely continuous with

respect to the Lebesgue measure. Define \( \rho = (1 \wedge \text{dist}(\Omega, D^c))/2 \) and let

\( V = \Omega + B_\rho \).

For \( \epsilon < \rho/2 \) we consider standard mollifiers \( \phi_\epsilon \) (i.e. \( \phi_\epsilon \in C^\infty(\mathbb{R}^d) \) and \( \text{supp} \phi_\epsilon = \overline{B_\epsilon} \)).
Since $\mathcal{L}$ is translation-invariant we have that $\mathcal{L}(\phi_\epsilon * u) = \mathcal{L} u * \phi_\epsilon = 0$ in $V_\epsilon = \{ x \in D : \text{dist}(x, V^c) > \epsilon \}$ in distributional sense. By Lemma 3.2 we obtain

$$\phi_\epsilon * u(x) = P_\Omega[\phi_\epsilon * u](x), \quad x \in \Omega.$$ 

Note $u \in L^1_{\text{loc}}$ implies $\phi_\epsilon * u \to u$ in $L^1_{\text{loc}}$. Hence, up to the subsequence

$$\lim_{\epsilon \to 0} \phi_\epsilon * u(x) = u(x) \quad \text{a. e.}$$

Moreover, since $\phi_\epsilon * u$ has the mean-value property in $\overline{P}_{\rho/2}$, by Lemma 2.9

$$\phi_\epsilon * u(z) = \phi_\epsilon * u * \overline{P}_r(z)$$

for a fixed $0 < r < \rho/4$. Hence, for any $E \subset \Omega^c$

$$P_U[|\phi_\epsilon * u| ; V_{\rho/2} \cap E](x) \leq \int_{V_{\rho/2} \cap E} |\phi_\epsilon * u(z)| P_\Omega(x, z) \, dz$$

$$= \int_{V_{\rho/2} \cap E} |\phi_\epsilon * u * \overline{P}_r(z)| P_\Omega(x, z) \, dz$$

$$\leq \int_{B_{\rho/2}} \phi_\epsilon(s) \int_{\mathbb{R}^d} |u(y)| \int_{V_{\rho/2} \cap E} \overline{P}_r(z - y - s) P_\Omega(x, z) \, dz \, dy \, ds.$$ 

Let $c = 2 \sup_{x \in V} |x|$. Then from boundedness of $\overline{P}_r$ and local integrability of $u$ we get

$$\int_{|y| \leq c} |u(y)| \int_{V_{\rho/2} \cap E} \overline{P}_r(z - y - s) P_\Omega(x, z) \, dz \, dy \leq C \int_{|y| \leq c} |u(y)| \, dy \int_{E} P_\Omega(x, z) \, dz$$

$$\leq C \|u\|_{L^1} \int_{E} P_\Omega(x, z) \, dz.$$ 

Furthermore, for $|y| > c$ we have $|z - y - s| > r$, hence $\overline{P}_r(z - y - s) \leq P_{B_{r}}(0, z - y - s)$. From (1.2) and monotonicity of the Lévy measure we get

$$P_{B_{r}}(0, y + s - z) \leq 1 \wedge \nu^r(|y - s - z| - r) \mathbb{E}^x \tau_{B_{r}} \leq C(1 \wedge \nu^r(|y|)).$$ 

Thus,

$$\int_{|y| > c} |u(y)| \int_{V_{\rho/2} \cap E} \overline{P}_r(z - y - s) P_\Omega(x, z) \, dz \, dy \leq C \|u\|_{L^1} \int_{E} P_\Omega(x, z) \, dz.$$ 

It follows that $\phi_\epsilon * u$ are uniformly integrable with respect to the measure $P_\Omega(x, z) \, dz$ in $V_{\rho/2}$. By the Vitali convergence theorem

$$\lim_{\epsilon \to 0} P_\Omega[\phi_\epsilon * u ; V_{\rho/2}](x) = P_\Omega[u ; V_{\rho/2}](x).$$

It remains to show that $\lim_{\epsilon \to 0} P_\Omega[\phi_\epsilon * u ; V_{\rho/2}^c] = P_\Omega[u ; V_{\rho/2}^c]$. Since $\text{dist}(\Omega, V_{\rho/2}^c) = \rho/2$, by the Ikeda-Watanabe formula

$$P_\Omega[\phi_\epsilon * u ; V_{\rho/2}^c](x) = \int_{V_{\rho/2}} \phi_\epsilon * u(z) \int_{\Omega} G_\Omega(x, y) \nu(z - y) \, dz \, dy$$

$$= \int_{B_{\rho/2}} \nu(z) \, dz \int_{\Omega} \phi_\epsilon * u(z + y) 1_{V_{\rho/2}^c}(z + y) G_\Omega(x, y) \, dy.$$ 

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Using the fact that \( \int_{\Omega} G(\Omega(x,y)) \, dy = \mathbb{E}^{\pi} \tau_{\Omega} < \infty, \nu(B_{\rho/2}^c) < \infty \) and \( \lim_{\delta \to 0} \phi_{\delta} \ast u = u \) in \( L^1(\mathbb{R}^d) \) we obtain

\[
\lim_{\delta \to 0} P_{\Omega}[\phi_{\delta} \ast u; V_{\rho/2}^c] = P_{\Omega}[u; V_{\rho/2}^c].
\]

Thus \( u(x) = P_{\Omega}[u](x) \) for a.e. \( x \in \Omega \).

Combining Lemma 3.1 and Lemma 3.3 we obtain a following result.

**Theorem 3.4.** Let \( D \) be an open set and \( u \in L^1 \). Then \( u \) has the mean-value property inside \( D \) if and only if \( L u = 0 \) in distributional sense.

**Lemma 3.5.** Let \( D \) be a bounded open set and \( f \in L^1(D) \). Then \( -G_D[f] \) is a distributional solution of (1.4) with \( g \equiv 0 \).

**Proof.** First assume \( f \) is continuous. Then by [16, Chapter V] we have

\[
U G_D[f](x) = -f(x), \quad x \in D.
\]

Let \( \phi_\epsilon, \epsilon > 0 \), be a standard mollifier. Since \( U \) is an extension of \( L \) and is translation-invariant we get

\[
L(\phi_\epsilon \ast G_D[f]) = U(\phi_\epsilon \ast G_D[f]) = \phi_\epsilon \ast U G_D[f] = -\phi_\epsilon \ast f.
\]

Thus

\[
(-\phi_\epsilon \ast G_D[f], L \phi) = (\phi_\epsilon \ast f, \phi).
\]

Passing \( \epsilon \to 0 \) we obtain

\[
(-G_D[f], L \phi) = (f, \varphi), \quad \varphi \in C_c^\infty(\mathbb{R}^d).
\]

In general case, since \( D \) is bounded, we have \( \|G_D[f]\|_{L^1} \leq \|G_D[1]\|_\infty \|f1_D\|_{L^1} \) and

\[
\|G_D[1]\|_\infty = \sup_{x \in \mathbb{R}^d} G_D[1](x) = \sup_{x \in \mathbb{R}^d} \mathbb{E}^{\pi} \tau_D \leq \mathbb{E}^{\pi} \tau_{B(0,diam(D))} < \infty.
\]

Using mollification of \( f \) we get the claim. \( \square \)

**Proof of Theorem 1.1.** Let \( h = u + G_D[f] \). By Lemma 3.5 \( h \) is a harmonic function in distributional sense. Hence, by Lemma 3.3 \( h \) has the mean-value property, which finishes the first claim.

Now let \( f, g \ast \nu \in K(D \setminus \overline{\nu}) \) and \( D \) be a Lipschitz domain. Then it follows that

\[
\tilde{u}(x) = -G_D[f](x) + P_D[g](x).
\]

is a solution of (1.4), which is bounded near to the boundary. Let \( U_n \nearrow D \) be a sequence of Lipschitz domains approaching \( D \). We have

\[
P_{U_n}[h](x) = P_{U_n}[h; D^c](x) + P_{U_n}[h; D \setminus U_n](x).
\]
By the dominated convergence theorem \( P_{U_n}[h;D^c](x) \xrightarrow{n \to \infty} P_D[h;D^c](x) = P_D[g](x) \).

Note that by our additional assumptions on \( g \) and \( \nu \) we have that \( P_D[g] \) is well-defined. Furthermore, since \( f \in K(D \setminus \overline{V}) \), there exists \( n_0 \in \mathbb{N} \) such that for \( n \geq n_0 \) we have \( V \subset U_n \). From boundedness of \( u \) and Lemma 2.5 we get that \( h \) is bounded in \( D \setminus U_n \) for \( n > n_0 \) and

\[
P_{U_n}[h;D \setminus U_n](x) \leq CP_{U_n} \left( x, \overline{D} \setminus U_n \right).
\]

By [37, Theorem 1] we have

\[
P_{U_n}(x, \overline{D} \setminus U_n) \xrightarrow{n \to \infty} P_D(x, \partial D) = 0
\]

Hence, \( u = \bar{u} \).

\[\square\]

4  The sufficient condition for twice differentiability

In this section, we provide auxiliary technical results and the proof of Theorem 1.2. Throughout this section we assume \( D \subset \mathbb{R}^d \) be an open bounded set. The following lemmas are modifications of Lemma 2.2 and Lemma 2.3 in [11].

**Lemma 4.1.** Suppose \( f \) is a uniformly continuous function on \( D \) and \( H(x,y) \) is a continuous function for \( x,y \in D, x \neq y \) satisfying

\[
|H(x,y)| \leq F(|x-y|), \quad \left| \frac{\partial H(x,y)}{\partial x_i} \right| \leq \frac{F(|x-y|)}{|x-y|}, \quad i = 1,\ldots,d
\]

for some non-increasing function \( F : (0, \infty) \mapsto [0, \infty) \). If the following holds

\[
\int_{0}^{1/2} F(t) \omega_f(t,D) t^{d-1} dt < \infty,
\]

then the function \( g(x) = \int_{D} H(x,y) (f(y) - f(x)) dy \) is uniformly continuous in \( D \).

**Remark 4.2.** The integral condition (4.1) and boundedness of the integrand for \( 1/2 \leq t \leq \text{diam}(D) \) imply that

\[
\int_{0}^{\text{diam}(D)} F(t) \omega_f(t,D) t^{d-1} dt < \infty.
\]

Moreover,

\[
\lim_{h \to 0} \int_{h}^{\text{diam}(D)} F(t) \omega_f(t,D) t^{d-2} dt = 0.
\]

Indeed, clearly we have

\[
h \int_{h}^{\text{diam}(D)} F(t) \omega_f(t,D) t^{d-2} dt = \int_{0}^{\text{diam}(D)} 1_{[h,\infty)}(t) F(t) \omega_f(t,D) t^{d-1} \frac{h}{t} dt.
\]

Since \( 1_{[h,\infty)}(t) h/t \leq 1 \), the claim follows by the dominated convergence theorem.
\textbf{Proof.} First note that by integration in polar coordinates one can check that the integral defining \( g \) actually exists. Set \( \epsilon > 0 \). Let \( 0 < h < \delta(D) \) and \( x, z \) be arbitrary fixed points in \( D \) such that \( |x - z| = h \). Denote \( j(x, y) := H(x, y) \left( f(y) - f(x) \right) \). Observe that 
\[ |g(x) - g(z)| \] is bounded by the sum of two integrals \( I_1 \) and \( I_2 \) of \( j(x, \cdot) - j(z, \cdot) \) over the sets \( D \cap B(x, 2h) \) and \( D \setminus B(x, 2h) \) respectively. On \( D \cap B(x, 2h) \) we have

\[ I_1 = \left| \int_{D \cap B(x, 2h)} H(x, y) \left( f(y) - f(x) \right) \, dy - \int_{D \cap B(x, 2h)} H(z, y) \left( f(y) - f(z) \right) \, dy \right| \]
\[ \leq \int_{D \cap B(x, 3h)} |H(x, y)||f(y) - f(x)| \, dy + \int_{D \setminus B(z, 3h)} |H(z, y)||f(y) - f(z)| \, dy \]
\[ \leq 2 \int_0^{3h} F(t)\omega_f(t, D)t^{d-1} \, dt < \frac{\epsilon}{3} \]
for sufficiently small \( h \). Obviously \( I_2 \leq I_3 + I_4 \), where

\[ I_3 := \left| \int_{D \setminus B(x, 2h)} (f(y) - f(z)) \left( H(x, y) - H(z, y) \right) \, dy \right| , \]
\[ I_4 := |f(z) - f(x)| \int_{D \setminus B(x, 2h)} H(x, y) \, dy . \]

By the mean value theorem

\[ I_3 \leq |x - z| \sum_{i=1}^{d} \int_{D \setminus B(x, 2h)} |H_x(i, y)| |f(y) - f(z)| \, dy \]
for some \( \tilde{x} = \theta x + (1 - \theta)z, \theta \in (0, 1) \). Note that for \( y \in D \setminus B(x, 2h) \) we have 
\[ |x - y| \geq 2|x - z| = 2h > 0 . \] It follows that \( |\tilde{x} - y| \geq h \) and consequently \( |z - y| \leq |z - \tilde{x}| + |\tilde{x} - y| \leq 2|\tilde{x} - y| . \) Thus,

\[ I_3 \leq Ch \int_{D \setminus B(x, 2h)} \frac{F(|\tilde{x} - y|)}{|\tilde{x} - y|} |f(y) - f(z)| \, dy \]
\[ \leq Ch \int_{D \setminus B(x, 2h)} \frac{F(|z - y|/2)}{|z - y|} |f(y) - f(z)| \, dy \]
\[ \leq h \int_{D \setminus B(x, h)} \frac{F(|z - y|/2)}{|z - y|} |f(y) - f(z)| \, dy \leq h \int_h^{\text{diam}(D)} F(t/2)\omega_f(t, D)t^{d-2} \, dt \]
\[ \leq h \int_{h/2}^{\text{diam}(D)/3} F(t)\omega_f(2t, D)t^{d-2} \, dt . \]

Thus, by Remark 4.2 we see that \( I_3 < \epsilon/3 \) for sufficiently small \( h \). Finally, (4.1) implies

\[ I_4 \leq \omega_f(h, D) \int_{D \setminus B(x, 2h)} F(|x - y|) \, dy = \int_0^{\text{diam}(D)} 1_{[2h, \infty)}(t)F(t)\omega_f(h, D)\omega_f(t, D)t^{d-1} \, dt . \]

Observe that \( 1_{[2h, \infty)}(t)\omega_f(h, D) \omega_f(t, D) \leq 1 \) by monotonicity of \( \omega_f(\cdot, D) \). Thus, (4.1) justifies the application of the dominated convergence theorem and we obtain
\[
\lim_{h \to 0} \omega_f(h, D) \int_{D \setminus B(x, 2h)} F(|x - y|) \, dy = 0.
\]

In particular, \( I_4 \leq \epsilon/\sqrt{3} \) for sufficiently small \( h \). It follows that \(|g(x) - g(z)| < \epsilon\), if \( h \) is sufficiently small. Thus, \( g \) is uniformly continuous. □

**Lemma 4.3.** Suppose \( f \) is a uniformly continuous function on \( D \) and \( H(x, y) \) is a continuous function for \( x, y \in D, \ x \neq y \) such that \( \int_D H(x, y) \, dy \) is continuously differentiable with respect to \( x \). Assume there exists a non-increasing function \( F: [0, \infty) \to [0, \infty) \) such that for \( i, j = 1, \ldots, d \)

\[
|H(x, y)|, \left| \frac{\partial H(x, y)}{\partial x_i} \right| \leq F(|x - y|), \quad \left| \frac{\partial^2 H(x, y)}{\partial x_i \partial x_j} \right| \leq \frac{F(|x - y|)}{|x - y|}. \tag{4.2}
\]

If the following holds

\[
\int_0^{1/2} F(t) \omega_f(t, D) t^{d-1} \, dt < \infty, \tag{4.3}
\]

then \( u(x) = \int_D H(x, y) f(y) \, dy \) is continuously differentiable with respect to \( x \in D \) and

\[
\frac{\partial u(x)}{\partial x_i} = \int_D \frac{\partial H(x, y)}{\partial x_i} (f(y) - f(x)) \, dy + f(x) \frac{\partial}{\partial x_i} \int_D H(x, y) \, dy, \quad x \in D, \quad i = 1, \ldots, d. \tag{4.4}
\]

**Proof.** Fix \( s > 0 \). Let \( V_s = \{x \in D : \text{dist}(x, \partial D) \geq s\} \). We will show that (4.4) holds for \( x \in B(\bar{x}, r) \), where \( r > 0 \) is such that \( B(\bar{x}, 4r) \subset V_s \). For \( \epsilon < r \) we consider standard mollifiers \( \phi_\epsilon(x) \) and set \( f_\epsilon(x) = \phi_\epsilon \ast f \). Note that \( \omega_f(h, B(\bar{x}, 2r)) \leq \omega_f(h, D) \).

For \( x \in D \) we define \( u_\epsilon(x) = \int_D H(x, y) f_\epsilon(y) \, dy \). From boundedness of \( f_\epsilon \) we see that the integral defining \( u_\epsilon \) is well defined and by the dominated convergence theorem \( u_\epsilon(x) \to u(x) \) for \( x \in V_s \), as \( \epsilon \to 0 \). By Lemma 4.1 applied to \( \frac{\partial H(x, y)}{\partial x_i} \) we have that the function

\[
\int_D \frac{\partial H(x, y)}{\partial x_i} (f_\epsilon(y) - f_\epsilon(x)) \, dy + f_\epsilon(x) \frac{\partial}{\partial x_i} \int_D H(x, y) \, dy \tag{4.6}
\]

is continuous on \( V_s \). Let \( x \in B(\bar{x}, r) \). Integrating (4.6) with respect to \( x_i \) from \( \bar{x}_i \) to \( x_i \) we obtain a continuously differentiable function \( \Psi_\epsilon(x) \) with respect to \( x_i \) with (4.6) being its derivative. Denote \( x = (\bar{x}, x_d) \) and \( \bar{x} = (\bar{x}_1, \ldots, \bar{x}_{d-1}) \) and \( x_d \) is fixed. The Fubini theorem and interchanging the order of integration yields

\[
\Psi_\epsilon(x) = \int_{x_d}^{\infty} \left( \int_{D} \left( \frac{\partial H(\bar{x}, s, y)}{\partial s} (f_\epsilon(y) - f_\epsilon(\bar{x}, s)) \right) \, dy \right) ds + f_\epsilon(\bar{x}, s) \frac{\partial}{\partial s} \int_{D} H(\bar{x}, s, y) \, dy \bigg|_{x_d}^{\infty} ds
\]

\[
= \int_{D} H(\bar{x}, s, y) (f_\epsilon(y) - f_\epsilon(\bar{x}, s)) \bigg|_{x_d}^{\infty} dy - \int_{D} \int_{x_d}^{\infty} H(\bar{x}, s, y) \frac{\partial}{\partial s} (f_\epsilon(y) - f_\epsilon(\bar{x}, s)) \bigg|_{x_d}^{\infty} ds dy
\]

\[
+ f_\epsilon(\bar{x}, s) \int_{D} H(\bar{x}, s, y) \bigg|_{x_d}^{\infty} dy - \int_{D} \int_{x_d}^{\infty} \frac{\partial f_\epsilon(\bar{x}, s)}{\partial s} \bigg|_{x_d}^{\infty} ds dy = u_\epsilon(x) - u_\epsilon(\bar{x}).
\]
Thus, for $x \in B(\mathbf{T}, r)$ the partial derivative $\frac{\partial u(x)}{\partial x_i}$ exists and is equal to (4.6). The same argument applies to any $i = 1, \ldots, d$. It remains to prove that (4.6) converges uniformly to (4.4), as $\epsilon \to 0$. Since $f_\epsilon \to f$ uniformly, as $\epsilon \to 0$, it is enough to prove the convergence of first integral in (4.6). Fix $\delta > 0$. Since $\int_0^{\text{diam}(D)} F(t) \omega_f(t, D) t^{d-1} dt < \infty$, there is $\gamma > 0$ such that $\int_0^\gamma F(t) \omega_f(t, D) t^{d-1} dt < \delta/4$. (4.5) implies
\[
\left| \int_{B(x, \gamma)} \frac{\partial H(x, y)}{\partial x_i} (f_\epsilon(y) - f_\epsilon(x)) \, dy - \int_{B(x, \gamma)} \frac{\partial H(x, y)}{\partial x_i} (f(y) - f(x)) \, dy \right| 
\leq 2 \left| \int_{B(x, \gamma)} \frac{\partial H(x, y)}{\partial x_i} \omega_f(|x - y|, D) \, dy \right| \leq 2 \int_0^\gamma S(t) \omega_f(t, D) t^{d-1} dt < \frac{\delta}{2}.
\]
On the complement of $B(x, \gamma)$ the function $\left| \frac{\partial H(x, y)}{\partial x_i} \right|$ is bounded by some constant $C > 0$. Choose $\epsilon_0 > 0$ such that $\|f_\epsilon - f\|_\infty \leq \delta/(4C|D|)$ for $\epsilon < \epsilon_0$. Then
\[
\left| \int_{D \setminus B(x, \gamma)} \frac{\partial (x, y)}{\partial x_i} (f_\epsilon(y) - f_\epsilon(x) - f(y) + f(x)) \, dy \right| \leq 2 \frac{\delta}{4C|D|}|D|C = \frac{\delta}{2},
\]
which combined with (4.7) and arbitrary choice of $\delta$ ends the proof. \hfill \square

Now we are ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** Let $u$ be of the form
\[
u(x) = -G_D[f](x) + P_D[g](x)
= -\int_D G(x, y)f(y) \, dy + \int_D \mathbb{E}^x G(X_{\tau_D}, y) f(y) \, dy + P_D[g](x)
=: I_1(x) + I_2(x) + I_3(x).
\]
Observe $I_3$ has the mean-value property in $D$, thus, by Remark 2.1 and Lemma 2.3 it belongs to $C^{2,\text{loc}}_{\text{loc}}(D)$. Moreover, for $x \in D$ from symmetry of $G$ and (G) we obtain that both $G$ and its first and second derivative are bounded either by $S(\delta_D(x))$ or $S(\delta_D(x))/\delta_D(x)$, depending on the finiteness of $\int_0^{1/2} |G'(t)| t^{d-1} dt$, and we are allowed to differentiate under the integral sign. Hence, it is enough to prove that $g(x) := \int_D G(x, y)f(y) \, dy$ is in $C^{2,\text{loc}}_{\text{loc}}(D)$. Fix $i, j \in \{1, \ldots, d\}$. Consider two cases.

1. Let $\int_0^1 |G'(t)| t^{d-1} dt = \infty$. Fix $x \in D$. From Lemma 2.7 we get
\[
\frac{\partial}{\partial x_i} g(x) = \int_{\mathbb{R}^d} G(x - y) \frac{\partial}{\partial x_i} (f_\chi_1)(y) \, dy + \int_{\mathbb{R}^d} \frac{\partial}{\partial x_i} (G_\chi_2)(x - y) (f1_D)(y) (1 - \chi_1)(y) \, dy
=: w_1(x) + w_2(x),
\]
where the localization functions $\chi_1$ and $\chi_2$ are chosen in dependence of $x$. Note that in the integral defining $w_2$, due to the function $\chi_2$ and (G), integration w.r.t.
y takes place in a region where $G$ and its derivative are bounded. Hence, from (G) we see that differentiating under the integral sign is justified. We obtain

$$\frac{\partial}{\partial x_j} w_2(x) = \int_{\mathbb{R}^d} \frac{\partial^2}{\partial x_i \partial x_j} (G\chi_2)(x-y) \left( f 1_D \right)(y) [1 - \chi_1](y) \, dy.$$ 

If we split $w_1$ into two integrals

$$w_1(x) = \int_{D_1} G(x-y) \frac{\partial}{\partial y_i} (f \chi_1)(y) \, dy + \int_{D \setminus D_1} G(x-y) \frac{\partial}{\partial y_i} (f \chi_1)(y) \, dy
=: w_3(x) + w_4(x),$$

where $D_1 \subset D$ is such that $\chi_1|_{D_1} \equiv 1$ then the same argument can be applied to $w_1$. Thus

$$\frac{\partial}{\partial x_j} w_4(x) = \int_{D \setminus D_1} \frac{\partial}{\partial x_j} G(x-y) \frac{\partial}{\partial y_i} (f \chi_1)(y) \, dy.$$ 

Next, observe that

$$\int_0^{\text{diam}(D_1)} S(t) \omega_4 f(t, D_1)t^{d-1} \, dt \leq \int_0^{\text{diam}(D)} S(t) \omega_4 f(t, D)t^{d-1} \, dt < \infty.$$ 

Moreover, by Corollary 2.8 the function $x \mapsto \int_D G(x,y) \, dy$ is continuously differentiable and from (G) we see that (4.2) of Lemma 4.3 is satisfied for $H(x,y) = G(|x-y|)$ and $F = S$. Hence, for $h(x) = \frac{\partial}{\partial x_i} f(x)$ we obtain

$$\frac{\partial}{\partial x_j} w_3(x) = \int_D \frac{\partial G(x,y)}{\partial x_j} (h(y) - h(x)) \, dy + h(x) \frac{\partial}{\partial x_i} \int_D G(x,y) \, dy.$$ 

2. Now let $\int_0^1 |G'(t)|t^{d-1} \, dt < \infty$. In this case, by the Fubini theorem and the fundamental theorem of calculus we get

$$\frac{\partial}{\partial x_i} \int_D G(x,y) f(y) \, dy = \int_D \frac{\partial G(x,y)}{\partial x_i} f(y) \, dy.$$ 

A similar argument applied to $H(x,y) = \frac{\partial G(x,y)}{\partial x_i}$ shows that the assumptions of Lemma 4.3 are satisfied with $F = S$. Note that here we use the additional assumption on $G''$. Thus,

$$\frac{\partial^2}{\partial x_i \partial x_j} \int_D G(x,y) f(y) \, dy = \int_D \frac{\partial^2 G(x,y)}{\partial x_i \partial x_j} (f(y) - f(x)) \, dy + f(x) \frac{\partial}{\partial x_j} \int_D \frac{\partial G(x,y)}{\partial x_i} \, dy
+ \frac{\partial}{\partial x_j} \int_D \frac{\partial G(x,y)}{\partial x_i} f(y) \, dy.$$ 

We have proved that $u \in C^2_{\text{loc}}(D)$. Then by [7, Lemma 4.7] the Dynkin characteristic operator $\mathcal{U}$ coincides with $\mathcal{L}$. Hence $u$ indeed is a solution of the problem (1.7).

Now suppose $\tilde{u}$ is another solution of (1.7). By Theorem 1.1 we find that it is of the form

$$\tilde{u}(x) = -G_D[f](x) + P_U[h](x), \quad x \in U,$$
where \( h(x) = u + G_D[f](x) \) and \( U \) is any Lipschitz domain such that \( U \subset D \). Fix \( x_0 \in D \). Then \( U_0 = B(x_0, r) \subset D \) for any \( r < \operatorname{dist}(x_0, D^c) \) and obviously \( U_0 \) is also Lipschitz. Hence,
\[
\tilde{u}(x) - u(x) = P_{U_0} [h](x) - P_D [g](x), \quad x \in U_0,
\]
is harmonic in \( U_0 \), so it belongs to \( C^2_{\text{loc}}(U_0) \). The proof yields \( -G_D[f] \in C^2_{\text{loc}}(D) \), thus \( \tilde{u} \) is twice continuously differentiable in the neighbourhood \( x_0 \). Since \( x_0 \) was arbitrary, it follows that every solution of (1.7) is \( C^2_{\text{loc}}(D) \).

\[\square\]

5 Counterexamples for the case „\( \alpha + \beta = 2 \)“

In this section we provide several counterexamples for Theorem 1.2. These examples are of the nature „\( \alpha + \beta = 2 \)“, i.e., for \( \alpha \in (0, 2) \) we give a function \( f \in C^{2-\alpha}(D) \) for which the solution of the Dirichlet problem (5.1) is not twice continuously differentiable inside of \( D \). In Section 6 we explain how the counterexamples can be modified in order to match the assumptions of Theorem 1.2.

Let \( D = B_1 \). Consider a Dirichlet problem
\[
\begin{cases}
\Delta^{\alpha/2} u = f & \text{in } D, \\
u = 0 & \text{in } D^c,
\end{cases}
\tag{5.1}
\]
where \( \alpha \in (0, 2) \). It is known (see [6] or Theorem 1.1) that \( u(x) = \int_D G_D(x, y)f(y) \, dy \), where \( G_D(x, y) \) is Green function for the operator \( \Delta^{\alpha/2} \) and domain \( D \) solves (5.1). By the Hunt formula
\[
G_D(x, y) = G(x, y) - \mathbb{E}^x G(X_{\tau_D}, y),
\]
where \( G \) is the (compensated) potential for process \( X_t \) whose generator is \( \Delta^{\alpha/2} \). Note that since \( \mathbb{E}^x G(X_{\tau_D}, y) \) is \( C^\infty \), the regularity problem is reduced to the regularity of the function \( x \mapsto g(x) = \int_{B(0, 1)} G(x, y)f(y) \, dy = G \ast f(x) \).

5.1 Case \( \alpha \in (0, 1) \)

We follow closely the idea from the proof of Theorem 1.2 apart from the fact that at the end we will show that the last function \( w_3 \) is not continuously differentiable. From Lemma 2.7 we get
\[
\begin{align*}
\frac{\partial}{\partial x_d} g(x) &= \int_{\mathbb{R}^d} G(x - y) \frac{\partial}{\partial y_d} (f \chi_1) (y) \, dy + \int_{\mathbb{R}^d} \frac{\partial}{\partial x_d} (G \chi_2) (x - y) (f 1_{B_1}) (y) (1 - \chi_1) (y) \, dy \\
&=: w_1(x) + w_2(x),
\end{align*}
\tag{5.2}
\]
if only \( f \in C^1_b(B_1) \). \( \chi_1 \) and \( \chi_2 \) in (5.2) are chosen for \( x_0 = 0 \). Put \( f(y) = ((y_d)_x)^{2-\alpha} \) and calculate \( \frac{\partial^2}{\partial x_d^2} g(x) \) \( w = 0 \). Since in \( w_2 \) we are separated from the origin, it follows that
\[
\frac{\partial}{\partial x_d} w_2(x) = \int_{\mathbb{R}^d} \frac{\partial^2}{\partial x_d^2} (G \chi_2) (x - y) \left(f 1_{B_1}\right) (y) (1 - \chi_1) (y) \, dy.
\]

If we split \( w_1 \) into
\[
w_1(x) = \int_{B_{1/4}} G(x - y) \frac{\partial}{\partial y_d} (f \chi_1) (y) \, dy + \int_{B_{1/4}} G(x - y) \frac{\partial}{\partial y_d} (f \chi_1) (y) \, dy =: w_3(x) + w_4(x),
\]
then the same argument applies for \( w_3 \). Therefore, it remains to calculate the derivative of \( w_3 \). Observe that on \( B_{1/4} \) we have \( f \chi_1 \equiv f \). To simplify the notation we accept a mild ambiguity and by \( h \) we denote, depending on the context, either a real number or a vector in \( \mathbb{R}^d \) of the form \((0, \ldots, 0, h)\). Let \( h > 0 \).

\[
\frac{1}{-h} (w_3(-h) - w_3(0)) = \frac{2 - \alpha}{-h} \int_{B_{1/4}} \left(1 - h - y|\alpha-d| - |y|\alpha-d\right) (|y_d|)_{1-\alpha} \, dy = (2 - \alpha) \int_A |y|\alpha-d - |y + h|\alpha-d \, y_d^{1-\alpha} \, dy =: (2 - \alpha) I(h),
\]

where \( A = B_{1/4} \cap \{y_d > 0\} \).

Let \( S_1 \) be a \( d \)-dimensional cube contained in \( A \), that is
\[
S_1 = \{y \in \mathbb{R}^d : |y_i| < a, 0 < y_d < a, \ i = 1, \ldots, d - 1\}, \quad (5.3)
\]
where \( a = (4\sqrt{d})^{-1} \). Define \( S_2 \subset S_1 \)
\[
S_2 = \{y \in S_1 : |y_i| < y_d, \ i = 1, \ldots, d - 1\}. \quad (5.4)
\]

By the Fatou lemma and the Fubini theorem
\[
\liminf_{h \to 0} I(h) \geq \int_A \liminf_{h \to 0} \frac{|y|^{\alpha-d} - |y_d + h|^{\alpha-d}}{h} y_d^{1-\alpha} \, dy = \int_A \frac{y_d}{|y|^{\alpha+2-\alpha}} y_d^{1-\alpha} \, dy \geq \int_{S_2} \frac{y_d}{|y|^{\alpha+2-\alpha}} y_d^{1-\alpha} \, dy \geq \frac{1}{\sqrt{d}} \int_{S_2} \frac{y_d}{y_d^{1-\alpha}} y_d^{1-\alpha} \, dy = C \int_0^a \frac{dy}{y}.
\]

Hence \( \frac{\partial^2}{\partial x_d} g_\alpha (0) = \infty \).

### 5.2 Case \( \alpha = 1 \)

Let \( d = 1 \). The compensated kernel is of the form \( G(x, y) = \frac{1}{\pi} \ln \frac{1}{|x-y|} \). Note that we cannot apply \([11, \text{Lemma 2.3}]\) because (ii) does not hold. Instead write
\[
\frac{g(x+h) - g(x)}{h} = \int_{-1}^{1} \frac{G(x + h - y) - G(x - y)}{h} (f(y) - f(x)) \, dy
\]
\[
+ f(x) \int_{-1}^{1} \frac{G(x + h - y) - G(x - y)}{h} \, dy =: I_1(h) + I_2(h).
\]

Let \( f \) be a Lipschitz function. By the mean value theorem
\[
\lim_{h \to 0} \int_{-1}^{1} \frac{G(x + h - y) - G(x - y)}{h} (f(y) - f(x)) \, dy = \int_{-1}^{1} G'(x - y) (f(y) - f(x)) \, dy.
\]

Furthermore, denote
\[
F(x) := \int_{-1}^{1} G(x - y) \, dy = -\int_{-1}^{1} \ln |y - x| \, dy = \int_{-1}^{1-x} \ln |s| \, ds.
\]

It follows that
\[
\lim_{h \to 0} \int_{-1}^{1} \frac{G(x + h - y) - G(x - y)}{h} \, dy = F'(x) = \ln \frac{1 - x}{1 + x}.
\]

Hence,
\[
g'(x) = \int_{-1}^{1} G'(x - y) (f(y) - f(x)) \, dy + f(x)F'(x). \quad (5.5)
\]

Put \( f(y) = y_+ \ln^{-\beta} \left( 1 + (y^{-1})_+ \right), \beta \in (0, 1). \) It is easy to check that \( f \) is a Lipschitz function. Let \( h < 0 \). Since \( f(y) = 0 \) for \( y \leq 0 \), from (5.5) we obtain
\[
\frac{1}{h} (g'(h) - g'(0)) = \frac{1}{h} \int_{0}^{1} \left( \frac{1}{|h - y|} - \frac{1}{|y|} \right) y_+ \ln^{-\beta} \left( 1 + \frac{1}{y} \right) \, dy
\]
\[
= \frac{1}{h} \int_{0}^{1} \left( \frac{1}{y - h} - \frac{1}{y} \right) y \ln^{-\beta} \left( 1 + \frac{1}{y} \right) \, dy
\]
\[
= \frac{1}{h} \int_{0}^{1} \frac{h}{y(y - h)} y \ln^{-\beta} \left( 1 + \frac{1}{y} \right) \, dy = \int_{0}^{1} \frac{1}{y - h} \ln^{-\beta} \left( 1 + \frac{1}{y} \right) \, dy.
\]

By the Monotone Convergence Theorem
\[
\int_{0}^{1} \frac{1}{y - h} \ln^{-\beta} \left( 1 + \frac{1}{y} \right) \, dy \xrightarrow{h \to 0^+} \int_{0}^{1} \frac{1}{y} \ln^{-\beta} \left( 1 + \frac{1}{y} \right) \, dy. \quad (5.6)
\]

Since
\[
\lim_{y \to 0^+} \frac{\ln \left( 1 + \frac{1}{y} \right)}{\ln \frac{1}{y}} = 1,
\]

we obtain \( g''(0) = \infty. \) For \( d > 1 \) and \( \beta \in (0, 1) \) we apply [11, Lemma 2.3] to the function \( f(y) = (y_d)_+ \ln^{-\beta} \left( 1 + \left( y_d^{-1} \right)_+ \right), \) and \( G(x, y) = |x - y|^{-d+1} \) in order to obtain
\[
\frac{\partial}{\partial x_d} g(x) = \int_{B_1} \frac{\partial G(x, y)}{\partial x_d} [f(y) - f(x)] \, dy + f(x) \frac{\partial}{\partial x_d} \int_{B_1} G(x, y) \, dy. \quad (5.7)
\]

By Corollary 2.8 the condition (iii) of [11, Lemma 2.3] holds. Denote
\[
H(x, y) := \frac{\partial G(x, y)}{\partial x_d} = (1 - d) \frac{(x - y)_d}{|x - y|^{d+1}} = -C \frac{(x - y)_d}{|x - y|^{d+1}},
\]

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$C > 0$. Let $h > 0$. We calculate the left-sided second partial derivative $\frac{\partial^2}{\partial x^2} g(x)$ in $x = 0$.

Note that some of terms vanish and the remaining limit is

$$\lim_{h \to 0^+} \frac{1}{h} \int_{B(0,1)} (H(y + h) - H(y)) f(y) dy.$$ 

Let $f_1(s) = f((0, ..., 0, s))$. We have

$$\int_{B_1} (H(y + h) - H(y)) f(y) dy = \int_{B_1} (H(y + h) - H(y)) f_1(y_d) dy\]

$$= \int_{B_1} (H(y + h) - H(y)) \int_0^{y_d} f_1'(s) ds dy = \int_0^1 ds f_1'(s) \int_{B_1 \cap \mathbb{H}_s} (H(y + h) - H(y)) dy,$$

where $\mathbb{H}_s = \{y : y_d > s\}$. Denote $\tilde{y} = (y_1, ..., y_{d-1})$. Then

$$\int_0^1 ds f_1'(s) \int_{B_1 \cap \mathbb{H}_s} (H(y + h) - H(y)) dy\]

$$= \int_0^1 ds f_1'(s) \int_{|\tilde{y}| < 1} d\tilde{y} \int_s^{1-|\tilde{y}|^2} [H(y + h) - H(y)] dy_d\]

$$= \int_0^1 ds f_1'(s) \int_{|\tilde{y}| < 1} d\tilde{y} \left[ \int_s^{1-|\tilde{y}|^2 + h} H(y) dy_d - \int_s^{1-|\tilde{y}|^2} H(y) dy_d \right]\]

$$= \int_0^1 ds f_1'(s) \int_{|\tilde{y}| < 1} d\tilde{y} \left[ \int_s^{1-|\tilde{y}|^2 + h} H(y) dy_d - \int_s^{1-|\tilde{y}|^2 + h} H(y) dy_d \right]\]

$$= \int_0^1 ds f_1'(s) \int_{|\tilde{y}| < 1} \left[ (G(\tilde{y}, s + h) - G(\tilde{y}, s)) - (G(\tilde{y}, \sqrt{1-|\tilde{y}|^2 + h}) - G(\tilde{y}, \sqrt{1-|\tilde{y}|^2})) \right] d\tilde{y}\]

$$= : I_1(h) - I_2(h).$$

The Dominated Convergence Theorem implies

$$\lim_{h \to 0^+} \int_0^1 ds f_1'(s) \int_{|\tilde{y}| < 1} \frac{G(\tilde{y}, \sqrt{1-|\tilde{y}|^2 + h}) - G(\tilde{y}, \sqrt{1-|\tilde{y}|^2})}{-h} d\tilde{y}\]

$$= - \int_0^1 ds f_1'(s) \int_{|\tilde{y}| < 1} \lim_{h \to 0^+} \frac{G(\tilde{y}, \sqrt{1-|\tilde{y}|^2 + h}) - G(\tilde{y}, \sqrt{1-|\tilde{y}|^2})}{h} d\tilde{y}\]

$$= - \int_0^1 ds f_1'(s) \int_{|\tilde{y}| < 1} H(\tilde{y}, \sqrt{1-|\tilde{y}|^2}) d\tilde{y} = - \int_{B_1} H(\tilde{y}, \sqrt{1-|\tilde{y}|^2}) f_1'(y_d) dy. $$

Note that the function $H$ under the integral sign is bounded on $B_1$. It follows that

$$\lim_{h \to 0^+} \frac{I_2(h)}{h} \leq C \int_{B_1} f_1'(y_d) dy < \infty.$$

By the Fatou lemma
We have
\[ f_1'(s) = \ln^{-\beta} \left( 1 + s^{-1} \right) + \frac{\beta}{s+1} \ln^{-\beta-1} \left( 1 + s^{-1} \right), \quad s > 0. \]

Thus,
\[ \int_{B_1} \frac{y_d}{|y|^{d+1}} \ln^{-\beta} \left( 1 + \left( \frac{y_d^{-1}}{y} \right) \right) dy \geq \int_{S_2} \frac{y_d}{|y|^{d+1}} \ln^{-\beta} \left( 1 + y_d^{-1} \right) dy \]
\[ \geq \int_{S_2} \frac{y_d}{y_d^{d+1}} \ln^{-\beta} \left( 1 + y_d^{-1} \right) dy \geq C \int_0^1 \frac{1}{y} \ln^{-\beta} \left( 1 + y^{-1} \right) dy. \]

Hence \( \frac{\partial^2}{\partial x_d} g_- (0) = \infty. \)

5.3 Case \( \alpha \in (1, 2) \)

Let \( d = 1 \). The compensated potential kernel is of the form \( G(x, y) = c_\alpha |x - y|^{\alpha - 1}. \)

From [11, Lemma 2.1] we have
\[ g'(x) = \int_{-1}^1 G'(y - x) f(y) dy. \]

We count the second derivative \( g(x) \) for \( |x| < 1 \). Observe that
\[ I_1(x) := \frac{d}{dx} \int_{|y|<1, |y-x|>\frac{1}{2|x|}} G'(y - x) f(y) dy = \int_{|y|<1, |y-x|>\frac{1}{2|x|}} G''(y - x) f(y) dy. \]

Hence \( g''(x) = I_1(x) + I_2(x) \), where
\[ I_2(x) := \lim_{h \to 0} \int_{|y-x|<\frac{1}{2|x|}} G'(y - x - h) - G'(y - x) \frac{h}{f(y) dy}. \]

Put \( f(y) = (y_1)^{2-\alpha} \). Then
\[ I_2(0) = \lim_{h \to 0} \int_0^{1/2} \frac{G'(y - h) - G'(y)}{h} y^{2-\alpha} dy. \]

We count the left-sided limit. Let \( h > 0 \).
\[ \int_0^{1/2} \frac{G'(y + h) - G'(y)}{y^{2-\alpha}} dy = C \int_0^{1/2} \frac{y^{\alpha-2} - (y + h)^{\alpha-2}}{h} y^{2-\alpha} dy \]
\[ = C \int_0^{1/2} \frac{1 - (1 + h/y)^{\alpha-2}}{h} dy \]
\[ = C \int_0^{1/(2h)} \left( 1 - (1 + y^{-1})^{\alpha-2} \right) dy \]
\[ = C \int_0^{\infty} \left( 1 - (1 + s)^{\alpha-2} \right) \frac{ds}{s^2}. \]
Thus $g''(0-) = \infty$. Now let $d > 1$. Then $G(x, y) = |x - y|^{-d-\alpha}$. Denote

$$g(x) = \int_{B_1} G(x, y) f(y) \, dy,$$

where $f(y) = ((y_d)_+)^{2-\alpha}$. [11, Lemma 2.1] implies

$$\frac{\partial g(x)}{\partial x_d} = \int_{B_1} \frac{\partial G(x, y)}{\partial x_d} f(y) \, dy.$$

We follow closely the argumentation from the case $\alpha = 1, d > 1$. We introduce the same notation

$$H(x, y) := \frac{\partial G(x, y)}{\partial x_d} = -(d - \alpha) \frac{(x - y)_d}{|x - y|^{d+2-\alpha}} = -C \frac{(x - y)_d}{|x - y|^{d+\beta}},$$

$C > 0, \beta := 2 - \alpha \in (0, 1)$. Let $h > 0$. By repeating (5.8) — (5.11) we conclude that it remains to calculate

$$\int_{B_1} -H(y) f_1(y_d) \, dy,$$

where $f_1$ is the same as for $\alpha = 1$. Here the derivative has simpler form. Note that the argumentation (5.8) — (5.11) is correct even though $f_1$ does not belong to $C^1(B_1)$ for $\alpha > 1$. We obtain

$$\int_{B_1} \frac{y_d}{|y|^{d+\beta}} (y_d)_+^{1-\alpha} \, dy = \int_A \frac{y_d}{|y|^{d+\beta}} y_d^{1-\alpha} \, dy \geq \int_{S_2} \frac{y_d}{|y|^{d+\beta}} y_d^{1-\alpha} \, dy \geq \int_{S_2} \frac{y_d}{y_d^{d+\beta}} y_d^{1-\alpha} \, dy \geq C \int_0^a \frac{dy}{y} = \infty.$$

Hence $\frac{\partial^2}{\partial x_d^2} g_-(0) = \infty$.

6 Examples

In the last section we present some examples of operators $L$ resp. corresponding Dirichlet problems that allow for an application of Theorem 1.2. In Example 6.1 we modify the considerations from Section 5 in order to match the assumptions of Theorem 1.2. In Example 6.2 we generalize to subordinated Brownian motion. Finally, in Example 6.4 we extend the above class and discuss the process which is assumed only to have the lower scaling property on the characteristic exponent.

**Example 6.1** (fractional Laplace operator). Let $X_t$ be strictly stable process whose generator is the fractional Laplace operator $-(-\Delta)^{\alpha/2}$. Let $D$ be a bounded open set.

1. Let $\alpha \in (0, 1)$. The potential kernel is of the form $G(y) = c_{d,\alpha} |y|^{\alpha-d}$ and satisfies

$$\int_0^{1/2} |G'(t)| t^{d-1} \, dt = \infty. \quad (6.1)$$

Here $S(r) = |G'(r)|$. According to Theorem 1.2, there is a $C^2_{\text{loc}}(D)$ solution of (1.7) if the following holds:

$$\int_0^{1/2} |G'(t)| \omega_{\nabla f}(t, D) t^{d-1} \, dt = \int_0^{1/2} t^{\alpha-2} \omega_{\nabla f}(t, D) \, dt < \infty. \quad (6.2)$$
Theorem 1.2

Section 5

The solution of (6.1), there will be

\[ \alpha \]

\[ \beta > \]

Calculations in the cases below are very similar and therefore will be omitted.

2. Let \( \alpha = d = 1 \). The compensated potential kernel is of the form \( G(y) = \frac{1}{\pi} \ln \frac{1}{|y|} \) and (6.1) holds for \( S(r) = |G'(r)| \). Note that in this case \( |G'(r)| \neq G(r) \). By Theorem 1.2 the solution of (1.7) will be in \( C^2_{\text{loc}}(D) \) if

\[
\int_0^{1/2} |G'(t)|\omega_{\nabla f}(t, D) t^{d-1} \, dt = \int_0^{1/2} t^{\alpha-2} t^{1-\alpha} \ln^{-\beta} \left( 1 + t^{-1} \right) \, dt \\
\leq C \int_0^{1/2} t^{-1} \ln^{-\beta} \left( t^{-1} \right) \, dt \\
= C \int_{\ln 2}^{\infty} \frac{dt}{t^\beta} < \infty.
\]

Hence, it suffices that \( \omega_{\nabla f}(t, D) \leq C \ln^{-\beta} (1 + t^{-1}), \beta > 1 \).

3. Let \( \alpha = 1, d > 1 \). The potential kernel has a form \( G(y) = c_{d,\alpha} |y|^{1-d} \) and (6.1) holds for \( S(r) = |G'(r)| \). Analogous to the case \( \alpha \in (0, 1) \) it suffices that \( \omega_{\nabla f}(t, D) \leq C \ln^{-\beta} (1 + t^{-1}), \beta > 1 \).

4. \( \alpha \in (1, 2), d = 1 \). The compensated potential kernel is of the form \( G(y) = c_\alpha |y|^{\alpha-1} \), \( S(r) = |G''(r)| \), and we have \( \int_0^{1/2} |G'(t)| \, dt < \infty \), thus by Theorem 1.2, there will be a \( C^2_{\text{loc}}(D) \) solution if

\[
\int_0^{1/2} |G''(t)|\omega_{\nabla f}(t, D) t^{d-1} \, dt = \int_0^{1/2} t^{\alpha-3} \omega_{f}(t, D) \, dt < \infty.
\]  

(6.3)

Clearly the function \( f(y) = (y_+)^{2-\alpha} \) from Section 5 does not satisfy (6.3). In order to correct it we must either take a function from \( C^{2-\alpha+\varepsilon}(D) \), \( \varepsilon > 0 \) (i.e. \( f(y) = (y_+)^{2-\alpha+\varepsilon} \)) or a function whose modulus of continuity is of the form \( \omega_{f}(t, D) = t^{2-\alpha} \ln^{-\beta} (1 + t^{-1}), \beta > 1 \).

5. \( \alpha \in (1, 2), d \geq 2 \). The potential kernel has the form \( G(y) = c_{d,\alpha} |y|^{\alpha-d} \) and \( S(r) = |G''(r)| \). We have

\[
\int_0^{1/2} |G'(t)| t^{d-1} \, dt < \infty.
\]
By Theorem 1.2 we have to take a function $\tilde{f}$ from $C^{2-\alpha+\epsilon}(D)$ or such that its modulus of continuity has the form $\omega_{f}(t, D) = t^{2-\alpha}\ln(1 + t^{-1}), \beta > 1$.

**Example 6.2** (Subordinate Brownian motion). Let $(B_{t}, t \geq 0)$ be a Brownian motion in $\mathbb{R}^{d}$ and $(S_{t}, t \geq 0)$ — a subordinator independent from $B_{t}$, i.e., a Lévy process in $\mathbb{R}$ which starts from 0 and has non-negative trajectories. Process $(X_{t}, t \geq 0)$ defined by $X_{t} = B_{S_{t}}$ is called a subordinated Brownian motion. Denote by $\phi$ the Laplace exponent of $S_{t}$:

$$
\mathbb{E}\exp\{-\lambda S_{t}\} = \exp\{-t\phi(\lambda)\}.
$$

It is well known that $\phi$ is of the form

$$
\phi(\lambda) = \gamma t + \int_{0}^{\infty} \left(1 - e^{-\lambda t}\right) \mu(dt)
$$

where $\mu$ is the Lévy measure of $S_{t}$ satisfying $\int_{0}^{\infty}(1 + t)\mu(dt) < \infty$. The corresponding operator is of the form $\mathcal{L} = -\phi(-\Delta)$ and we have $\psi(\xi) = \phi(|\xi|^{2})$. An example of subordinated Brownian motion is the process from Example 6.1 with $\phi(\lambda) = \lambda^{\alpha/2}$, $\alpha \in (0, 2)$. Another example is geometric stable process with $\phi(\lambda) = \ln \left(1 + \lambda^{\alpha/2}\right), \alpha \in (0, 2)$. Denote by $G_{d}(r)$ the potential of $d$-dimensional subordinated Brownian motion $X_{t}$. From [13, Theorem 5.17] we have

$$
G_{d}(r) \asymp r^{-d-2}\frac{\phi'(r^{-2})}{\phi^{2}(r^{-2})}, \quad r \to 0^{+}, \quad (6.4)
$$

if $d \geq 3$ and there exist $\beta \in [0, d/2 + 1)$ and $\alpha > 0$ such that $\phi^{-2}\phi'$ satisfies weak lower and upper scaling condition at infinity with exponents $-\beta$ and $-\alpha$, respectively (see [13]). The same result under slightly stronger assumptions is derived in [27, Proposition 3.5]. For $d$-dimensional subordinated Brownian motion $X_{t}, d \geq 3$, we have

$$
G_{d}(r) = \int_{0}^{\infty} (4\pi t)^{-d/2} \exp\left(-\frac{r^{2}}{4t}\right) u(dt).
$$

It follows that

$$
G'_{d}(r) = G_{d}(r) = -\int_{0}^{\infty} (4\pi t)^{-d/2} \exp\left(-\frac{r^{2}}{4t}\right) \frac{2r}{4t} u(dt)
$$

$$
= -2\pi \int_{0}^{\infty} (4\pi t)^{-(d+2)/2} \exp\left(-\frac{r^{2}}{4t}\right) u(dt) = -2\pi G_{d+2}(r). \quad (6.5)
$$

That and (6.4) imply

$$
\left|G'_{d}(r)\right| \leq Cr \cdot r^{-(d+1)-2}\frac{\phi'(r^{-2})}{\phi^{2}(r^{-2})} = C r^{-d-2}\frac{\phi'(r^{-2})}{\phi^{2}(r^{-2})} \leq C\frac{G_{d}(r)}{r}.
$$

By induction

$$
\left|G^{(k)}_{d}(r)\right| \leq C\frac{G(r)}{r^{k}}, \quad k \in \mathbb{N}.
$$
Thus, the necessary conditions involving \( G \) and its derivatives hold true for \( S(r) = G_d(r)/r^2 \). Note that the density of Lévy measure of \( X_t \)

\[
\nu(r) = \int_0^\infty (4\pi t)^{-d/2} \exp \left( -\frac{r^2}{4t} \right) \mu(dt)
\]

belongs to \( C^\infty \). By [7, Lemma 7.4] the assumptions of Theorem 1.2 are satisfied with \( \nu^* \equiv \nu \) if \( \phi \) is a complete Bernstein function.

Take geometric stable process with \( \phi(\lambda) = \ln \left( 1 + \frac{\lambda \alpha}{2} \right) \). Then by (6.4) and (6.5)

\[
\int_0^{1/2} |G_d'(t)| t^{d-1} dt \geq C \int_0^{1/2} t^{-d-3} \phi'(t^{-2}) \frac{\phi(t^{-2})}{\phi'^2(t^{-2})} \frac{1}{t^{1+t-\alpha}} \frac{1}{t^{\frac{d}{2}}} dt = C \int_0^{1/2} \frac{1}{t^{2}} \ln^2 (1 + t^{-\alpha}) dt \geq \int_0^{1/2} \frac{1}{t^{2}} \ln^2 t dt = \infty,
\]

hence, for the solution of (1.7) to be in \( C^\infty_{loc}(D) \), it suffices that the modulus of continuity of gradient of function \( f \) is of the form \( \omega_{\nabla f}(t, D) = t \ln^2 (1 + t^{-\alpha}) \), \( \alpha \in (0, 1) \).

Before moving to the last example, let us define concentration functions \( K \) and \( h \) by setting

\[
K(r) = \frac{1}{r^2} \int_{|x| < r} |x|^2 \nu(dx), \quad r > 0,
\]

\[
h(r) = \int_{\mathbb{R}^d} \left( 1 \wedge \frac{|x|^2}{r^2} \right) \nu(dx), \quad r > 0.
\]

**Proposition 6.3.** Let \( d \geq 3 \). Suppose there exist \( c > 0 \) and \( \alpha \geq 3/2 \) such that

\[
h(r) \leq c\lambda^\alpha h(\lambda r), \quad \lambda \leq 1, r > 0.
\]

(6.6)

Then there exists \( c > 0 \) such that \( |U'(r)| \leq cU(r)/r, \ |U''(r)| \leq cU(r)/r^2, \ |U'''(r)| \leq cU(r)/r^3 \) for \( r > 0 \).

**Proof.** Observe that for \( d \geq 3 \) the potential \( U \) always exists. By [19, Theorem 3] there exists \( c > 0 \) such that

\[
U(x) \geq \frac{c}{|x|^{d-2}h(1/r)}, \quad r > 0.
\]

Our aim is to prove (G). By definition and isotropy of \( p_t \)

\[
U(r) = \int_0^\infty p_t(\tilde{r}) dt,
\]

where by \( \tilde{r} = (0, ..., 0, r) \in \mathbb{R}^d \). Since \( p_t \) is radially decreasing, by the Tonelli theorem
\[ U(r) - U(1) = \int_0^\infty \int_1^r \partial_x p_t(y) \, dy \, dt = \int_1^r \int_0^\infty \partial_x p_t(\tilde{y}) \, dt \, dy, \]

where \( \tilde{y} = (0, \ldots, 0, y) \in \mathbb{R}^d \). Hence,

\[ U'(r) = \int_0^\infty \partial_x p_t(\tilde{r}) \, dt, \quad r > 0. \]

By [22, Theorem 5.6 and Corollary 6.8]

\[ |\partial_x^\beta p_t(x)| \leq c \left(h^{-1}(1/t)^{1/2}\right)^{-|\beta|} \varphi_t(x), \quad t > 0, x \in \mathbb{R}^d, \]

where

\[ \varphi_t(x) = \begin{cases} (h^{-1}(1/t))^{-d}, & |x| \leq h^{-1}(1/t), \\
\lambda K(|x|) |x|^{-d}, & |x| > h^{-1}(1/t). \end{cases} \]

Let us estimate \(|U'(r)|\). We have

\[ |U'(r)| \leq \frac{K(|x|)}{|x|^d} \int_0^{1/h(|x|)} \frac{t}{h^{-1}(1/t)} \, dt + \int_1^{\infty} \frac{dt}{t^{d+1}}. \]

The scaling property of \( h \) for \(|x| > h^{-1}(1/t)\) yields

\[ h(|x|) \leq c \left(\frac{h^{-1}(1/t)}{|x|}\right)^\alpha h(1/t). \]

It follows that

\[ \frac{K(|x|)}{|x|^d} \int_0^{1/h(|x|)} \frac{t}{h^{-1}(1/t)} \, dt \leq c \frac{K(|x|)}{|x|^{d+1}} \int_0^{1/h(|x|)} \frac{1}{h(|x|)} \, dt \leq c \frac{K(|x|)}{|x|^{d+1}} h(|x|)^{-1/\alpha} \int_0^{1/h(|x|)} t^{1-1/\alpha} \, dt. \]

For \( \alpha > 1/2 \) the integral is finite and we get

\[ \frac{K(|x|)}{|x|^d} \int_0^{1/h(|x|)} \frac{t}{h^{-1}(1/t)} \, dt \leq c \frac{K(|x|)}{|x|^{d+1} h(|x|)^2}. \]

The comparability \( K \) and \( h \) (22, Lemma 2.3) implies

\[ \frac{K(|x|)}{|x|^d} \int_0^{1/h(|x|)} \frac{t}{h^{-1}(1/t)} \, dt \leq c \frac{1}{|x|^{d+1} h(|x|)} \leq \frac{U(r)}{r}. \]

Furthermore, we always have \( h(r) \geq \lambda^2 h(\lambda r) \) for \( \lambda \leq 1 \) and \( r > 0 \). Thus,
yields that the assumptions of Theorem 1.2 imposed on function $f$ on the context, either a real number or the vector $(0,\ldots,0)$. In the following part we introduce a mild ambiguity by denoting by $1$, depending

**Proof.** Let $\text{Lemma A.1.}$

$\int_{1/h(|x|)}^{\infty} \frac{dt}{(h^{-1}(1/t))^{d+1}} = \frac{1}{|x|^{d+1}} \int_{1/h(|x|)}^{\infty} \frac{|x|^{d+1}}{(h^{-1}(1/t))^{d+1}} dt \leq \frac{1}{|x|^{d+1}} \int_{1/h(|x|)}^{\infty} \left( \frac{1}{th(|x|)} \right)^{(d+1)/2} dt.$

Since $d > 1$, the integral is finite and we get

$$\int_{1/h(|x|)}^{\infty} \frac{dt}{(h^{-1}(1/t))^{d+1}} \leq c \frac{1}{|x|^{d+1}h(|x|)} \leq \frac{cU(r)}{r}.$$  

Hence, for $\alpha > 1/2$ we obtain $|U'(r)| \leq cU(r)/r$, $r > 0$. By similar argument one may conclude that $|U''(r)| \leq cU(r)/r^2$ if $\alpha > 1$ and $|U'''(r)| \leq cU(r)/r^3$ for $\alpha > 3/2$.  

**Example 6.4.** Let $d \geq 3$, $\alpha > 3/2$, and $X_t$ be a truncated $\alpha$-stable Lévy process in $\mathbb{R}^d$, i.e. with Lévy measure $\nu(dx) = |x|^{-d-\alpha} \varphi(x)$, where $\varphi$ is a cut-off function, i.e. $\varphi \in C^\infty(\mathbb{R}^d)$ and $\mathbf{1}_{B_{1/2}} \leq \varphi \leq \mathbf{1}_{B_1}$. One can easily check that $h(r) \asymp r^{-\alpha} \wedge r^{-2}$.

Proposition 6.3 yields that the assumptions of Theorem 1.2 imposed on function $G$ are satisfied. Observe that (A) and (1.2) is satisfied for $\nu^\alpha \equiv 0$. In that case the appropriate $\mathcal{L}^1$ space is simply $L^1_{loc}$. 

## A Potential theory for recurrent unimodal Lévy process

In this appendix we establish a formula for the Green function for a bounded open set $D$ in case of recurrent unimodal Lévy process $X_t$. Contrary to the transient case, here the potential kernel $U(x) = \int_0^\infty p_t(x) dt$ is infinite, so the classical Hunt formula has no application. Instead, one can define the $\lambda$-potential kernel $U^\lambda$ by setting

$$U^\lambda(x) = \int_0^\infty e^{-\lambda t} p_t(x) dt.$$  

Similarly, we define the $\lambda$-Green function for an open set $D$

$$G_D^\lambda(x,y) = \int_0^\infty e^{-\lambda t} p_t^D(x-y) dt.$$  

Note that both $U^\lambda$ and $G_D^\lambda$ exist. An analogue of the Hunt formula for $G_D^\lambda$ holds, namely, for $x,y \in D$

$$G_D^\lambda(x,y) = U^\lambda(y-x) - \mathbb{E}^x \left[ e^{-\lambda \tau_D} U^\lambda(y-X_{\tau_D}) \right].$$

**Lemma A.1.** Let $d \geq 1$. For any fixed $x_0 \in \mathbb{R}^d \setminus \{0\}$ we have $\lambda U^\lambda(x_0) \to 0$ as $\lambda \to 0$.

**Proof.** In the following part we introduce a mild ambiguity by denoting by 1, depending on the context, either a real number or the vector $(0,\ldots,0,1) \in \mathbb{R}^d$. Set $x_0 = 1$. Let $f_\lambda(r) = \int_{|x| < r} dx \int_0^\infty e^{-\lambda u} p_u(x) du$. We have
By [19, Lemma 6]
\[
\int_{\mathbb{R}^d} e^{-s|x|^2} p_u(x) \, dx = c_d \int_{\mathbb{R}^d} e^{-u\psi(\sqrt{s}x)} e^{-|x|^2/4} \, dx.
\]
Hence, we have for \( \lambda > 0 \)
\[
sL_\lambda f(s) = c_d \int_0^\infty e^{-\lambda u} du \int_{\mathbb{R}^d} e^{-u\psi(\sqrt{s}x)} e^{-|x|^2/4} \, dx = c_d \int_{\mathbb{R}^d} \frac{1}{\lambda + \psi(\sqrt{s}x)} e^{-|x|^2/4} \, dx.
\]
By monotonicity of \( f \)
\[
f_\lambda(r) = \frac{e}{r} \int_0^\infty e^{-u/r} f(r) \, du \leq \frac{e}{r} \int_0^\infty e^{-u/r} f_\lambda(u) \, du = \frac{e}{r} L_\lambda f(1/r)
\]
\[
= c' \int_{\mathbb{R}^d} \frac{1}{\lambda + \psi(\sqrt{1/r}x)} e^{-|x|^2/4} \, dx.
\]
Since by [19, Lemma 1 and Proposition 1]
\[
\sup_{|x| \leq 1} \psi(x) \leq 4 \sup_{|\xi| \leq 1} \psi(x) \leq \frac{\psi(\xi)}{|\xi|^2},
\]
we obtain
\[
\lambda G^\lambda(1) \leq \frac{f_\lambda(1)}{|B_1|} \leq c_d \int_{\mathbb{R}^d} \frac{\lambda}{\lambda + \psi(\xi)} e^{-|\xi|^2/4} \, d\xi \leq \frac{\lambda}{\psi(1)} \int_{B_1^c} e^{-|\xi|^2/4} \, d\xi + \int_{B_1} \frac{\lambda}{\lambda + |\xi|^2} \, d\xi.
\]
Hence, \( \lambda U^\lambda(1) \to 0 \) as \( \lambda \to 0 \). The extension to arbitrary \( x_0 \) is immediate. \( \square \)

**Lemma A.2.** Let \( x_0 \in \mathbb{R}^d \setminus \{0\} \) be an arbitrary fixed point. For all \( x \in \mathbb{R}^d \setminus \{0\} \) we have \( \int_0^\infty |p_t(x) - p_t(x_0)| \, dt < \infty \).

**Proof.** Let \( f \in C_c^\infty(\mathbb{R}^d) \) be such that \( 1_{B_\epsilon} \leq f \leq 1_{B_\delta}, \) where \( 0 < 4\epsilon < 1 \). Denote
\[
W_{x_0}^\lambda(x) = \int_0^\infty e^{-\lambda t} (p_t(x) - p_t(x_0)) \, dt, \quad x \neq 0,
\]
\[
W_{x_0}^\lambda(x) = \int_0^\infty (p_t(x) - p_t(x_0)) \, dt, \quad x \neq 0.
\]

Let \( x_0 = 1 \). Observe that
\[
W_1^\lambda * f(0) = \int_0^\infty e^{-\lambda t} (p_t * f(0) - p_t(1) \|f\|_1) \, dt.
\]
Note that the integrand has a positive sign. Indeed,
\[
p_t * f(0) - p_t(1) \|f\|_1 = \int_{B_\delta} (p_t(y) f(y) - p_t(1) f(y)) \, dy > 0,
\]

since $4\epsilon < 1$. Furthermore,

$$p_t(1)||f||_1 = \int_{B_{4\epsilon}} p_t(1)f(y) \, dy \geq \int_{B_{4\epsilon}} p_t(1 + 4\epsilon - y)f(y) \, dy = p_t * f(1 + 4\epsilon).$$

Hence, by the Fourier inversion theorem

$$\int_0^\infty e^{-\lambda t} (p_t * f(0) - p_t(1)||f||_1) \, dt \leq \int_0^\infty e^{-\lambda t} \int_{\mathbb{R}^d} (1 - \cos((1 + 4\epsilon)x))\tilde{p}_t(x) \, d\xi \, dx \, dt \leq \int_{\mathbb{R}^d} (1 - \cos((1 + 4\epsilon)x)) \frac{\hat{f}(\xi)}{\psi(\xi)} \, d\xi.$$

By the monotone convergence theorem and the fact that $|\hat{f}(\xi)|$ decays faster than any polynomial

$$W_1 * f(0) = \lim_{\lambda \to 0} W_1^\lambda * f(0) \leq \int_{\mathbb{R}^d} (1 - \cos((1 + 4\epsilon)x)) \frac{\hat{f}(\xi)}{\psi(\xi)} \, d\xi < \infty.$$ 

Hence,

$$\int_{B_1} W_1(x) \, dx \leq W_1 * f(0) < \infty. \quad (A.1)$$

Since $W_1$ is radially decreasing and positive for $|x| < 1$, (A.1) implies that it may be infinite only for $x = 0$. It follows that $W_1$ is well defined for $0 < |x| \leq 1$. Similarly $0 \leq W_{x_0} < \infty$ for $0 < |x| \leq |x_0|$.

It remains to notice that for $|x| > |x_0|$ we have $0 \leq |W_{x_0}(x)| = -W_{x_0}(x) = W_x(x_0) < \infty$ by the first part of the proof.

Lemma A.2 allows us to introduce, following [5], [25], [9], a compensated potential kernel by setting for $x \in \mathbb{R}^d \setminus \{0\}$

$$W_{x_0}(x) := \int_0^\infty (p_t(x) - p_t(x_0)) \, dt, \quad (A.2)$$

where $x_0 \in \mathbb{R}^d \setminus \{0\}$ is an arbitrary but fixed point. From the proof of Lemma A.2 we immediately obtain the following corollary.

**Corollary A.3.** $W$ is locally integrable in $\mathbb{R}^d$.

**Theorem A.4.** Let $x_0 \in D^c$, $d \leq 2$ and $D$ be bounded. Then for $x, y \in D$

$$G_D(x, y) = W_{x_0}(y - x) - \mathbb{E}^x W_{x_0}(y - X_{\tau_D}). \quad (A.3)$$

**Proof.** Let $x, y \in D$. Fix $x_0 \in D^c$ and observe that

$$G_D^\lambda(x, y) = U^\lambda(y - x) - \mathbb{E}^x \left[e^{-\lambda \tau_D} U^\lambda(y - X_{\tau_D})\right]$$

$$= U^\lambda(x - y) - U^\lambda(x_0) - \mathbb{E}^x \left[e^{-\lambda \tau_D} \left(U^\lambda(y - X_{\tau_D}) - U^\lambda(x_0)\right)\right] + U^\lambda(x_0) \mathbb{E}^x \left[1 - e^{-\lambda \tau_D}\right]. \quad (A.4)$$
We want to pass with $\lambda$ to 0. The limit of left-hand side is well defined and is equal to $G_D(x,y)$. From Lemma A.1 we get

$$U^\lambda(x_0)E^x [1 - e^{-\lambda \tau_D}] \leq \lambda U^\lambda(x_0) \sup_{x \in \mathbb{R}^d} E^x \tau_D \xrightarrow{\lambda \to 0} 0.$$ 

Moreover, from Lemma A.2 we obtain that

$$\lim_{\lambda \to 0} \left( U^\lambda(y-x) - U^\lambda(x_0) \right) = W_{x_0}(y-x). \tag{A.5}$$

It remains to show the convergence of the middle term of (A.4). Since $U^\lambda$ is radially decreasing, $U^\lambda(y-X_{\tau_D}) - U^\lambda(x_0)$ is positive on the set $\{y \in \mathbb{R}^d : |y-X_{\tau_D}| \leq |x_0|\}$ and non-positive on its complement. By Lemma A.2 and the Monotone Convergence Theorem

$$\lim_{\lambda \to 0} E^x \left[ e^{-\lambda \tau_D} \left( U^\lambda(y-X_{\tau_D}) - U^\lambda(x_0) \right) ; |y-X_{\tau_D}| < |x_0| \right] = E^x [W_{x_0}(y-X_{\tau_D}) ; |y-X_{\tau_D}| < |x_0|] \leq W_{x_0}(\delta_D(y)) < \infty.$$ 

Observe that the left-hand side of (A.4) converges to $G_D$ so it is finite. The remaining integral on the right-hand side converges as well by the monotone convergence theorem, but since all the other terms are finite, it follows that the integral is also finite and we obtain

$$\lim_{\lambda \to 0} E^x \left[ e^{-\lambda \tau_D} \left( U^\lambda(y-X_{\tau_D}) - U^\lambda(x_0) \right) \right] = E^x W_{x_0}(y-X_{\tau_D}),$$

which ends the proof.

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