SPECTRA AND MAXIMAL ORTHOGONAL SETS OF CANTOR MEASURES

XIN-RONG DAI

ABSTRACT. Let $\mu_{q,b}$ be the Cantor measure associated with the iterated function system $f_i(x) = x/b + i/q, 0 \leq i \leq q-1$, where $2 \leq q, b/q \in \mathbb{Z}$. In this paper, we consider spectra and maximal orthogonal sets of the Cantor measure $\mu_{q,b}$ and their rational rescaling. We introduce a quantity to measure level difference between a branch and its subbranch for the labeling tree corresponding to a maximal orthogonal set of the Cantor measure $\mu_{q,b}$, and use certain boundedness property of that quantity as sufficient and necessary conditions for a maximal orthogonal set of the Cantor measure $\mu_{q,b}$ to be its spectrum. We show that the integrally rescaled set $K\Lambda$ is still a spectrum if it is a maximal orthogonal set, and we provide a simple characterization for the integrally rescaled set to be a maximal orthogonal set. As an application of the above characterization, we find all integers $K$ such that $K\Lambda_4$ are spectra of the Cantor measure $\mu_{2,4}$, where $\Lambda_4 := \{\sum_{n=0}^{\infty} d_n 4^n : d_n \in \{0,1\}\} \subset \mathbb{Z}$ is the first known spectrum for the Cantor measure $\mu_{2,4}$. Finally we discuss rescaling spectra rationally and construct a spectrum $\Lambda$ for the Cantor measure $\mu_{q,b}$ such that $\Lambda/(b-1)$ is a maximal orthogonal set but not a spectrum.

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1. Introduction and main theorems

A probability measure $\mu$ on the real line $\mathbb{R}$ is said to be a spectral measure if there exists a countable set $\Lambda$ of real numbers such that $\{e_\lambda := \exp(-2\pi i \lambda \cdot) : \lambda \in \Lambda\}$ forms an orthonormal basis for the Hilbert space $L^2(\mu)$ of all square-integrable functions with respect to the measure $\mu$. The set $\Lambda$ in the above definition is known as a spectrum of the measure $\mu$. Spectral problem for probability measures is one of fundamental problems in Fourier analysis, and it has close connection to tiling, as formulated in Fuglede’s spectral

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A measurable set is a spectral set if and only if it tiles the whole Euclidean space by translation. The spectral set conjecture has been proved to be false by Tao and others in dimension 3 or higher \[18, 29\], while it remains open in dimensions one and two, see for instance \[15, 16, 20, 21, 28\] and references therein. In this paper, we consider spectral problem for the Cantor measure \(\mu_{q,b}\), one of the most celebrated fractal measures, \[
\mu_{q,b} = \frac{1}{q} \sum_{i=0}^{q-1} \mu_{q,b}(f_i^{-1}(\cdot)),
\]
associated with the iterated function system (IFS), \[
f_i(x) = x/b + i/q, \quad i = 0, 1, \ldots, q - 1,
\]
where \(q \geq 2\) is an integer and \(b > q\) is a real number. For \(q = 2\), the Cantor measure \(\mu_{2,b}\) is also known as a Bernoulli convolution with contraction ratio \(1/b\).

The first known singular, non-atomic, spectral measures with compact support are Bernoulli convolutions \(\mu_{2,2n}\) with contraction ratio being the reciprocal of an even integer \(2n \geq 4\) \[17\]. In fact, a Bernoulli convolution \(\mu_{2,b}\) with contraction ratio \(1/b\) is a spectral measure if and only if the contraction ratio is the reciprocal of an even integer \[2\]. The above equivalence was recently extended to Cantor measures \(\mu_{q,b}\) with arbitrary \(q \geq 2\): \(\mu_{q,b}\) is a spectral measure if and only if \(b/q \in \mathbb{Z}\) \[3, 4\]. So in this paper, we restrict ourselves to consider Cantor measures \(\mu_{q,b}\) with \[
2 \leq q, b/q \in \mathbb{Z}.
\]

A spectral measure may admit various spectra and the Fourier expansion associated with distinct spectra could have completely different convergence property \[6, 27\]. Define the upper Beurling dimension \(\dim^+(\Lambda)\) of a discrete set \(\Lambda\) of real numbers by \[
\dim^+(\Lambda) := \inf \{r > 0 : \limsup_{h \to \infty} \sup_{x \in \mathbb{R}} \frac{\#(\Lambda \cap [x-h, x+h])}{(2h)^r} < \infty\},
\]
where \(\#(E)\) is the cardinality of a finite set \(E\) \[11\]. Spectra of the Cantor measure \(\mu_{b,q}\) have their upper Beurling dimension less than \(\log q / \log b\) \[7\], but unlike Fourier frames on the unit interval \[23\], they could be arbitrarily sparse and have zero upper Beurling dimension \[3\]. We refer the reader to \[2-8, 10, 13-15, 17, 19, 20, 22-27, 30\] and references therein for additional information on self-similar/self-affine spectral measures and Fourier expansions.

A countable set \(\Lambda\) of real number is said to be an orthogonal set (resp. a maximal orthogonal set) of a probability measure \(\mu\) if \(\{e_\lambda : \lambda \in \Lambda\}\) is an orthogonal set (resp. a maximal orthogonal set) of \(L^2(\mu)\). Clearly a spectrum of a probability measure is its maximal orthogonal set and also its orthogonal set. As orthogonal sets (maximal orthogonal sets, spectra) \(\Lambda\) of a probability measure \(\mu\) are invariant under translations, in this paper we always assume that they contain zero, i.e., \[
0 \in \Lambda.
\]
Define the Fourier transform \( \hat{\mu} \) of a probability measure \( \mu \) by

\[
\hat{\mu}(\xi) := \int_{\mathbb{R}} e^{-2\pi i \xi x} d\mu(x).
\]

For a probability measure \( \mu \), the inner product \( \langle e_A, e_{A'} \rangle_{L^2(\mu)} \) between exponentials \( e_A \) and \( e_{A'} \) in the Hilbert space \( L^2(\mu) \) is given by the evaluation \( \hat{\mu}(\lambda - \lambda') \) of the Fourier transform of the measure \( \mu \). Thus the zero set of the Fourier transform \( \hat{\mu} \) could be used to characterize an orthogonal set of the measure \( \mu \). Specifically for the Cantor measure \( \mu_{q,b} \),

(1.4) a discrete set \( \Lambda \) is its orthogonal set if and only if \( \Lambda - \Lambda \subset Z_{q,b} \cup \{0\} \), where \( Z_{q,b} = \{b^j a : a \in \mathbb{Z} \setminus q \mathbb{Z}, 0 \leq j \in \mathbb{Z} \} \) [3,4].

Let \( \Sigma_q^n := \Sigma_q \times \cdots \times \Sigma_q \), 1 \( \leq n \leq \infty \), be the \( n \) copies of \( \Sigma_q := \{0, \ldots, q - 1\} \), and \( \Sigma_q^* = \bigcup_{1 \leq n < \infty} \Sigma_q^n \) contain all tree branches of finitely many levels. Define the growing \( E : \Sigma_q^* \to \Sigma_q \) by

(1.5) \[
E(\delta) := \delta 0^{\infty};
\]

and the heading \( R_n : \Sigma_q^\infty \cup \Sigma_q^* \to \Sigma_q^n, n \geq 1 \), by

(1.6) \[
R_n(\delta) := \begin{cases} 
\delta_1 \delta_2 \cdots \delta_n & \text{if } \delta = \delta_1 \delta_2 \cdots \in \Sigma_q^\infty \\
R_n(E(\delta)) & \text{if } \delta \in \Sigma_q^*,
\end{cases}
\]

where \( 0^{\infty} = 000 \cdots \in \Sigma_q^\infty \) and \( \delta \delta' \) is the concatenation of \( \delta \in \Sigma_q^* \) and \( \delta' \in \Sigma_q^\infty \cup \Sigma_q^* \).

**Definition 1.1.** Let \( 2 \leq q, b/q \in \mathbb{Z} \), and define \( 0^n := R_n(0^{\infty}), n \geq 1 \). A mapping \( \tau : \Sigma_q^* \to \{-1, 0, \ldots, b - 2\} \) is said to be a tree mapping if

(i) \( \tau(0^n) = 0 \) for all positive integers \( n \), and

(ii) \( \tau(\sigma) \in \sigma_n + q\mathbb{Z} \) if \( \sigma = \sigma_1 \cdots \sigma_n \in \Sigma_q^n, n \geq 1 \).

A tree mapping \( \tau \) is said to be a maximal tree mapping if

(iii) for any \( \delta \in \Sigma_q^* \) there exists \( \delta' \in \Sigma_q^* \) such that \( \tau(R_n(\delta \delta')) = 0 \) for sufficiently large integers \( n \).

For the Cantor measure \( \mu_{q,b} \), its maximal orthogonal sets (and spectra) \( \Lambda \) with \( 0 \in \Lambda \) have integral elements, i.e.,

(1.7) \[
\Lambda \subset \mathbb{Z},
\]

by (1.2) and (1.4), but they cannot be described simply. The first attempt to classify those maximal orthogonal sets and spectra was made in [5], where certain tree structure of a maximal orthogonal set of the Cantor measure \( \mu_{2,4} \) was discovered. The latest conclusion in [3] states that a maximal orthogonal set of the Cantor measure \( \mu_{q,b} \) could be characterized via some maximal tree mapping.

**Theorem 1.2.** ([3]) Let \( 2 \leq q, b/q \in \mathbb{Z} \), and \( \Lambda \subset \mathbb{Z} \) contain zero. Then \( \Lambda \) is a maximal orthogonal set of the Cantor measure \( \mu_{q,b} \) if and only if there exists a maximal tree mapping \( \tau \) such that \( \Lambda = \Lambda(\tau) \), where

(1.8) \[
\Lambda(\tau) := \left\{ \sum_{n=1}^{\infty} \tau(R_n(\delta))b^{n-1} : \delta \in \Sigma_q \text{ such that } \tau(R_m(\delta)) = 0 \text{ for sufficiently large } m \right\}.
\]
Let $\tau : \Sigma_q^* \to \{-1, 0, \ldots, b - 2\}$ be a maximal tree mapping. We say that $\delta \in \Sigma_q^*, n \geq 1$, is a $\tau$-regular branch if $\tau(R_m(\delta)) = 0$ for sufficiently large $m$, and a $\tau$-main branch if $\tau(R_m(\delta)) = 0$ for all $m > n$. Clearly $\delta \in \Sigma_q^*$ is a $\tau$-regular branch if and only if either $\delta$ is a $\tau$-main branch or $\delta 0^k$ is for some $k \geq 1$; and for any $\delta \in \Sigma_q^*$ there exists a $\tau$-main subbranch $\delta' \in \Sigma_q^*$.

Now we introduce a quantity to measure the level difference between branch $\delta$ and its subbranch $\delta'$. For $\delta \in \Sigma_q^*, n \geq 1$, and $\delta' \in \Sigma_q^* \cup \Sigma_q^*$, define

$$D_{\tau, \delta}(\delta') := \begin{cases} 0 & \text{if } \tau(R_{n+m}(\delta')) = 0 \text{ for all } m \geq 1 \\ L(\eta) & \text{if } \eta \neq 0^\infty \text{ and } \delta' \text{ is a } \tau\text{-regular branch} \\ +\infty & \text{if } \delta\delta' \text{ is not a } \tau\text{-regular branch}, \end{cases}$$

where $\eta = \eta_1 \eta_2 \cdots \in \{-1, 0, \ldots, b - 2\}^\infty$ (the infinite copies of $\{-1, 0, \ldots, b - 2\}$) is given by $\eta_m = \tau(R_{n+m}(\delta'))$, $m \geq 1$, and $L(\eta)$ is the length of a trimmed tree branch $\tilde{\eta}$ obtained from $\eta$ by squeezing out all consecutive zeros in the front of $\eta_m \in q\mathbb{Z}\setminus\{0\}$ and then cutting consecutive zeros $0^\infty$ at the end of $\eta$. For instance, for $q = 3$ and $b = 6$, the trimmed tree branch of $0301003(-1)0400 \cdots$ is $3013(-1)0404$, which has length 7. One may verify that

$$D_{\tau, \delta}(\delta') := \begin{cases} \#A_\delta(\delta') + \sum_{n_j \in B_\delta(\delta')} (n_j - n_{j-1} - 1) & \text{if } A_\delta(\delta') \neq 0 \\ 0 & \text{if } A_\delta(\delta') = 0, \end{cases}$$

where $A_\delta(\delta') := \{m \geq 1 : \tau(R_m(\delta')) = 0\}$, $B_\delta(\delta') := \{m \geq 1 : \tau(R_m(\delta')) \notin q\mathbb{Z}\}$, $n_0 = 0$, and $\{n_j : j \geq 1\}$ is a strictly increasing sequence of positive integers given by $\{n_j : j \geq 1\} = A_\delta(\delta')$.

**Definition 1.3.** Let $2 \leq q, b/q \in \mathbb{Z}$ and $\tau : \Sigma_q^* \to \{-1, 0, \ldots, b - 2\}$ be a maximal tree mapping. Define the distance $D_{\tau, \delta}$ of $\delta \in \Sigma_q^*$ to its $\tau$-main subbranches by

$$D_{\tau, \delta} := \inf \{D_{\tau, \delta}(\delta') : \delta\delta' \text{ are } \tau\text{-main branches}\} < \infty.$$

A challenging problem in spectral theory for the Cantor measure $\mu_{q,b}$ is when a maximal orthogonal set becomes a spectrum. Various sufficient and necessary conditions have been proposed in [3-6, 9, 17]. In this paper, we provide a sufficient condition for a maximal orthogonal set of the Cantor measure $\mu_{q,b}$ to be its spectrum via boundedness of the distances $D_{\tau, \delta}, \delta \in \Sigma_q^*$.

**Theorem 1.4.** Let $2 \leq q, b/q \in \mathbb{Z}$. If $\tau : \Sigma_q^* \to \{-1, 0, \ldots, b - 2\}$ is a maximal tree mapping such that

$$D_\tau := \sup \{D_{\tau, \delta} : \delta \in \Sigma_q^*\} < \infty,$$

then $\Lambda(\tau)$ in (1.8) is a spectrum of the Cantor measure $\mu_{q,b}$.

For $\delta \in \Sigma_q^*$ define the distance $N_{\tau, \delta}$ to its natural $\tau$-main subbranches by

$$N_{\tau, \delta} := D_{\tau, \delta}(0^\infty).$$

Clearly

$$D_{\tau, \delta} \leq N_{\tau, \delta} \text{ for all } \delta \in \Sigma_q^*$$

with the equality holds when $D_{\tau, \delta} = 0$. Thus by Theorem 1.4 $\Lambda(\tau)$ in (1.8) is a spectrum of the Cantor set $\mu_{q,b}$ if $\tau$ is a maximal tree mapping with $N_{\tau, \delta}$ in (1.13) being bounded on
\( \Sigma_q \). As shown in the next theorem, the above boundedness condition on the distance \( N_{r,\delta} \) is close to be necessary for a maximal orthogonal set of the Cantor measure \( \mu_{q,b} \) to be a spectrum.

**Theorem 1.5.** Let \( 2 \leq q, b/q \in \mathbb{Z} \) and \( \tau : \Sigma_q^* \to \{-1,0,\ldots,b-2\} \) be a maximal tree mapping. If there exists a positive number \( \varepsilon_0 \) such that
\[
(1.15) \quad N_{r,\delta} \geq \varepsilon_0 n
\]
for all \( \delta \in \Sigma_q^n\backslash \Sigma_q^{n-1} := \{\delta^j : \delta^j \in \Sigma_q^{j-1}, j \in \{1,2,\ldots,q-1\}\} \) and \( n \geq 2 \), then \( \Lambda(\tau) \) in (1.3) is not a spectrum of the Cantor measure \( \mu_{q,b} \).

The proofs of Theorems 1.4 and 1.5 are given in Sections 2 and 3 respectively. We believe that the boundedness assumption (1.12) about the distance to \( \tau \)-main subbranches is a very weak sufficient condition for a maximal orthogonal set to be a spectrum.

A difficult problem in the study of spectral theory for the Cantor measure \( \mu_{q,b} \) is the explicit construction. For the Cantor measure \( \mu_{2,4} \), the rescaled sets \( K\Lambda_4 := \{K\lambda : \lambda \in \Lambda_4\} \) of its first known spectrum \( \Lambda_4 := \{\Sigma_{n=0}^{\infty} d_n 4^n : d_n \in \{0,1\}\} \) have been shown to be spectra for \( K = 5^k, k \geq 1 \), while they are not spectra for \( K = 3,9 \) as both \( 3\Lambda_4 \) and \( 9\Lambda_4 \) are not maximal orthogonal sets [6,9,17]. This leads to the rescaling-invariant problem when the integrally rescaled set \( K\Lambda \) is a spectrum of the Cantor measure \( \mu_{q,b} \) if \( \Lambda \) is. In the next theorem, we show that the integrally rescaled set \( K\Lambda \) is a spectrum of the Cantor measure \( \mu_{q,b} \) if it is a maximal orthogonal set.

**Theorem 1.6.** Let \( 2 \leq q, b/q \in \mathbb{Z}, \tau : \Sigma_q^* \to \{-1,0,\ldots,b-2\} \) be a maximal tree mapping satisfying (1.12), and \( \Lambda(\tau) \) be as in (1.3). Then for any integer \( K \) being prime with \( b \), \( K\Lambda(\tau) \) is a spectrum of the Cantor measure \( \mu_{q,b} \) if and only if it is a maximal orthogonal set.

The next natural question about rescaled spectra is how to verify the maximal orthogonality of the rescaled set \( K\Lambda \). In the following theorem, we show that the integrally rescaled set \( K\Lambda \) is not a maximal orthogonal set of the Cantor measure \( \mu_{q,b} \) if and only if the labeling tree \( \tau(\Sigma_q^*) \) has certain periodic properties (1.16) and (1.18).

**Theorem 1.7.** Let \( 2 \leq q, b/q \in \mathbb{Z}, \tau : \Sigma_q^* \to \{-1,0,\ldots,b-2\} \) be a maximal tree mapping, \( \Lambda := \Lambda(\tau) \) be as in (1.3), and let \( K > 1 \) be an integer prime with \( b \). Then \( K\Lambda \) is not a maximal orthogonal set of the Cantor measure \( \mu_{q,b} \) if and only if there exist \( \delta \in \Sigma_q^{\infty} \) and a nonnegative integer \( M \) such that \( \{\tau(R_n(\delta))\}_{n=M+1}^{\infty} \) is a periodic sequence with positive period \( N \), i.e.,
\[
(1.16) \quad \tau(R_n(\delta)) = \tau(R_{n+N}(\delta)), \ n \geq M + 1,
\]
and that the word \( W = \omega_1\omega_2\cdots\omega_N \) defined by
\[
(1.17) \quad \omega_j = \tau(R_{M+j}(\delta)), \ 1 \leq j \leq N,
\]
is a repetend of the recurring \( b \)-band decimal expression of \( i/K \) for some \( i \in \mathbb{Z}\backslash\{0\} \), i.e.,
\[
(1.18) \quad \frac{i}{K} = 0.\omega_N\cdots\omega_2\omega_1\omega_N\cdots\omega_2\omega_1\cdots = \sum_{n=1}^{\infty} \sum_{j=1}^{N} \omega_j b^{j-Nn-1} = \frac{\sum_{j=1}^{N} \omega_j b^{j-1}}{b^N - 1}.
\]
We prove Theorems 1.6 and 1.7 in Sections 4 and 5 respectively. Applying Theorems 1.6 and 1.7, we immediately obtain the following characterization for the integrally rescaled spectrum $K\Lambda_4 := \{K \sum_{j=0}^{\infty} d_j 4^j : d_j \in \{0, 1\}\}$ for the Cantor measure $\mu_{2,4}$.

**Corollary 1.8.** Let $K \geq 3$ be an odd integer. Then $K\Lambda_4$ is a spectrum of the Cantor measure $\mu_{2,4}$ if and only if there does not exist a positive integer $N$ such that

$$K \sum_{j=1}^{N} d_j 4^{j-1} \in (4^N - 1)\mathbb{Z}\setminus{0}$$

for some $d_j \in \{0, 1\}$, $1 \leq j \leq N$.

For a spectral set $\Lambda$ of the Cantor measure $\mu_{q,b}$, its irrational rescaling set $r\Lambda$ (i.e., $r \notin \mathbb{Q}$) is not an orthogonal set (and hence not a spectrum) by (1.7). The next question then is when a rational rescaling set $r\Lambda$ is an orthogonal set, or a maximal orthogonal set, or a spectrum. A necessary condition is that $r\Lambda \subset \mathbb{Z}$ by (1.7), but unlike integral rescaling discussed in Theorems 1.6 and 1.7 there are lots of interesting problems unsolved yet. In Section 6 (the last section of this paper), we construct a spectrum $\Lambda$ of the Cantor measure $\mu_{q,b}$ such that the rescaled set $\Lambda/(b-1)$ is its maximal orthogonal set but not its spectrum.

2. Maximal orthogonal sets and spectra: a sufficient condition

In this section, we prove Theorem 1.4.

For the Cantor measure $\mu_{q,b}$ in (1.1) and an orthogonal set $\Lambda$ of $L^2(\mu_{q,b})$ containing zero, define

$$Q_\Lambda(\xi) := \sum_{\lambda \in \Lambda} |\hat{\mu}_{q,b}(\xi + \lambda)|^2.$$  

Then $Q_\Lambda$ is a real analytic function on the real line with $Q_\Lambda(0) = 1$, and

$$Q_\Lambda(\xi) = \sum_{\lambda \in \Lambda} |\langle e_\lambda, e_{-\xi} \rangle_{L^2(\mu_{q,b})}|^2 \leq \|e_{-\xi}\|_{L^2(\mu_{q,b})}^2 = 1, \xi \in \mathbb{R},$$

where the equality holds if $\Lambda$ is a spectrum. The converse is shown to be true in [3,17], which provides a characterization for a maximal orthogonal set of the Cantor measure $\mu_{q,b}$ to be its spectrum.

**Lemma 2.1.** ([3,17]) Let $\mu$ be a probability measure with compact support. Then an orthogonal set $\Lambda$ of the measure $\mu$ is its spectrum if and only if the function $Q_\Lambda(\xi) := \sum_{\lambda \in \Lambda} |\hat{\mu}(\xi + \lambda)|^2$ satisfies

$$Q_\Lambda(\xi) = 1 \text{ for all } \xi \in \mathbb{R}.$$  

Taking Fourier transform at both sides of the equation (1.1), we obtain the following refinement equation in the Fourier domain:

$$\hat{\mu}_{q,b}(\xi) = H_{q,b}(\xi/b) \cdot \hat{\mu}_{q,b}(\xi/b),$$

where

$$H_{q,b}(\xi) := \frac{1}{q} \sum_{l=0}^{q-1} e^{-2\pi ib l / q}$$

(2.3)
is a periodic function with $H_{q,b}(0) = 1$. Set

\begin{equation}
H_m(\xi) := \prod_{j=1}^{m} H_{q,b}(\xi / b^j), \ m \geq 1.
\end{equation}

Then

\begin{equation}
\hat{\mu}_{q,b}(\xi) = H_m(\xi) \cdot \hat{\mu}_{q,b}(\xi / b^m), \ m \geq 1
\end{equation}

by applying (2.2) repeatedly, and

\begin{equation}
\hat{\mu}_{q,b}(\xi) = \prod_{j=1}^{\infty} H_{q,b}(\xi / b^j)
\end{equation}

by taking limit $m \to \infty$ in (2.5). We remark that the characterization (1.4) for orthogonal sets of the Cantor measure $\mu_{q,b}$ follows from (2.6) and the fact that

\begin{equation}
H_{q,b}(\xi) = 0 \text{ if and only if } b\xi \in \mathbb{Z} \setminus q\mathbb{Z}.
\end{equation}

Let

\begin{equation}
r_0 := \inf_{|\xi| \leq (b-2)/(b-1)} |\hat{\mu}_{q,b}(\xi)| \text{ and } r_1 := \inf_{1 \leq j \leq q-1} \inf_{|\xi| \leq (b-2)/(b-1)} |\xi|^{-1} |H_{q,b}(\xi / b + j / b)|.
\end{equation}

Observe that

\begin{equation}
H'_{q,b}(j / b) \neq 0 \text{ for all } j \in \mathbb{Z}.
\end{equation}

This, together with (2.6) and (2.7), imply that both $r_0$ and $r_1$ are well-defined and positive,

\begin{equation}
r_0 > 0 \text{ and } r_1 > 0.
\end{equation}

Set

\begin{equation}
T_b = \left( -\frac{1}{b-1}, \frac{b-2}{b-1} \right) \cup \left( -\frac{1}{b(b-1)}, \frac{b-2}{b(b-1)} \right).
\end{equation}

For any $m \geq 1$ and $d_j \in \{-1, 0, \ldots, b-2\}$, $1 \leq j \leq m$, with $d_m \neq 0$, one may verify that

\begin{equation}
(\xi + \sum_{j=1}^{m} d_j b^{j-1}) b^{-m} \in T_b \text{ for all } \xi \in \left( -\frac{1}{b-1}, \frac{b-2}{b-1} \right).
\end{equation}

To prove Theorem [1.4], we need a lower bound estimate of $|\hat{\mu}_{q,b}(\xi + \lambda)|$ for $\xi \in T_b$ and $\lambda \in \mathbb{Z}$, which is crucial for our proof of Theorem [1.4].

**Lemma 2.2.** Let $2 \leq q, b / q \in \mathbb{Z}$, $\mu_{q,b}$ be the Cantor measure in (1.1), and let $\lambda = \sum_{j=1}^{K} d_j b^{j-1}$ for some positive integers $n_j$, $1 \leq j \leq K$, satisfying $1 := n_0 \leq n_1 < \ldots < n_K$, and for some $d_{n_j}$, $1 \leq j \leq K$, belonging to the set $\{-1, 1, 2, \ldots, b-2\}$. Then

\begin{equation}
|\hat{\mu}_{q,b}(\xi + \lambda)| \geq r_0^{K+1} \left( \frac{r_1}{b(b-1)} \right)^B b^{-\sum_{j=n_{n_j-1}}^{n_{n_j-1}}}, \ \xi \in T_b,
\end{equation}

where $B = \{1 \leq j \leq K : \ d_{n_j} \notin q\mathbb{Z}\}$ and $r_0, r_1$ are given in (2.8).
Proof. For $0 \leq i \leq K$, define $\xi_0 = \xi$ and $\xi_i = (\xi + \sum_{j=1}^{i} d_n b^n / b^n)$ for $1 \leq i \leq K$. Then
\begin{equation}
(2.14) \quad \xi_i \in T_b \text{ for all } 0 \leq i \leq K
\end{equation}
by (2.12). Observe that
\begin{equation}
(2.15) \quad |H_{q,b}(\eta)| \leq 1 \text{ for all } \eta \in \mathbb{R} \text{ and } \sup_{b \in T_b} |H_{q,b}(\eta)| < 1.
\end{equation}
The above observation, together with (2.5), (2.14) and the fact that $H_{q,b}$ has period $q/b$, implies
\begin{align*}
&\prod_{\ell=n_{i-1}+1}^{n_i} |H_{q,b}(\xi + \lambda / b^\ell)| = \prod_{\ell=n_{i-1}+1}^{n_i} |H_{q,b}(\xi + \sum_{j=1}^{i-1} d_n b^n + d_n b^n)| \\
&= \prod_{\ell'=1}^{n_i-n_{i-1}} |H_{q,b}(\xi_{i-1} / b^{\ell'})| \geq |\mu_{q,b}(\xi_{i-1})| \geq r_0
\end{align*}
if $d_n \in q \mathbb{Z}$; and
\begin{align*}
&\prod_{\ell=n_{i-1}+1}^{n_i} |H_{q,b}((\xi + \lambda) / b^\ell)| \\
&= \left( \prod_{\ell=n_{i-1}+1}^{n_i} |H_{q,b}(\xi + \sum_{j=1}^{i-1} d_n b^n) / b^\ell)| \cdot |H_{q,b}(\xi + \sum_{j=1}^{i-1} d_n b^n)| \right) \\
&\geq |\mu_{q,b}(\xi_{i-1})| \cdot |H_{q,b}(\xi_{i-1} / b^{n_{i-1}} + d_n / b)| \geq r_0 r_1 |\xi_{i-1}| / b^{n_{i-1}} \\
&\geq r_0 r_1 b^{-n_{i-1}} / (b - 1)
\end{align*}
if $d_n \notin q \mathbb{Z}$. Combining the above two lower bound estimates with
\begin{equation}
(2.16) \quad \mu_{q,b}(\xi + \lambda) = \left( \prod_{i=1}^{K} \prod_{\ell=n_{i-1}+1}^{n_i} H_{q,b}(\xi + \lambda / b^\ell) \right) \cdot \mu_{q,b}(\xi + \lambda / b^{n_1})
\end{equation}
proves (2.13). \hfill \Box

Let $\tau : \Sigma_q^* \to \mathbb{R}$ be a tree mapping. Define $\Pi_{r,m} : \Sigma_q^\infty \cup \Sigma_q^* \to \mathbb{R}, m \geq 1$, by
\begin{equation}
(2.17) \quad \Pi_{r,m}(\delta) = \sum_{k=1}^{m} \tau(R_k(\delta)) b^{k-1}.
\end{equation}
One may verify that
the restriction of $\Pi_{r,m}$ onto $\Sigma_q^m$ is one-to-one,
as $\mathbb{Z}/(b\mathbb{Z}) = \{-1, 0, \cdots, b-2\}$. Observe that the filter $H_{q,b}(\xi)$ satisfies
\begin{equation}
(2.18) \quad \sum_{j=0}^{q-1} |H_{q,b}(\xi + j / b)|^2 = 1.
\end{equation}
To prove Theorem 1.4, we need a similar identity for filters $H_m(\xi) := \prod_{j=1}^m H_{q,b}(\xi / b^j), m \geq 1$, in (2.4) with shifts in $\Pi_{r,m}(\Sigma_q^m)$.
Lemma 2.3. Let \( 2 \leq q, b/q \in \mathbb{Z} \), filters \( H_m(\xi), m \geq 1 \), be as in (2.4), and let \( \tau : \Sigma_q^* \to \mathbb{R} \) be a tree mapping. Then

\[
(2.19) \quad \sum_{\delta \in \Sigma_q^*} |H_m(\xi + \Pi_{\tau,m}(\delta))|^2 = 1, \quad \xi \in \mathbb{R}.
\]

Proof. For \( m = 1 \),

\[
\sum_{\delta \in \Sigma_q^*} |H_m(\xi + \Pi_{\tau,m}(\delta))|^2 = \sum_{j=0}^{q-1} |H_{q,b}(\xi + j/b)|^2 = \sum_{j=0}^{q-1} |H_{q,b}(\xi + j/b)|^2 = 1,
\]

where the last equality follows from (2.18), and the second one holds as \( H_{q,b} \) has period \( q/b \) and \( \tau(j) - j \in q\mathbb{Z}, 0 \leq j \leq q - 1 \), by the tree mapping property for \( \tau \). This proves (2.19) for \( m = 1 \).

Inductively we assume that (2.19) hold for all \( m \leq k \). Then for \( m = k + 1 \),

\[
\sum_{\delta \in \Sigma_q^*} |H_m(\xi + \Pi_{\tau,m}(\delta))|^2 = \sum_{\delta' \in \Sigma_q^*} \sum_{j=0}^{q-1} \left| H_k(\xi + \Pi_{\tau,k+1}(\delta') \cdot j) \cdot H_{q,b}(\xi + \Pi_{\tau,k+1}(\delta') \cdot j/b^{k+1}) \right|^2 = \sum_{\delta' \in \Sigma_q^*} \sum_{j=0}^{q-1} \left| H_k(\xi + \Pi_{\tau,k}(\delta')) \cdot H_{q,b}(\xi + \Pi_{\tau,k}(\delta') \cdot j/b^{k+1} + j/b) \right|^2 = 1,
\]

where the first equality holds as \( H_{k+1}(\xi) = H_k(\xi)H_{q,b}(\xi/b^{k+1}) \), the second one follows from the observations that \( H_k \) and \( H_{q,b} \) are periodic functions with period \( b^{k+1}q \) and \( q/b \) respectively and that

\[ \Pi_{\tau,k+1}(\delta') = \Pi_{\tau,k}(\delta') + \tau(\delta') b^k \in \Pi_{\tau,k}(\delta') + j b^k + q b^k \mathbb{Z}, 0 \leq j \leq q - 1, \]

by the tree mapping property for \( \tau \), and the last one is true by (2.18) and the inductive hypothesis. This completes the inductive proof. \( \square \)

Before we start our proof of Theorem 1.4, let us extend the definition \( \Pi_{\tau,n}(\delta), n \geq 1 \), in (2.17) to \( n = \infty \) by taking limit in (2.17),

\[
(2.20) \quad \Pi_{\tau,\infty}(\delta) := \sum_{k=1}^{\infty} \tau(R_k(\delta)) b^{k-1},
\]

provided that \( \tau \) is regular on \( \delta \in \Sigma_q^* \). The infinite series in (2.20) converges as it is, in fact, a finite sum. Applying the above \( b \)-nary expression, we can rewrite the set \( \Lambda(\tau) \) in (1.3) as follows:

\[
\Lambda(\tau) = \{ \Pi_{\tau,\infty}(\delta) : \delta \in \Sigma_q^* \text{ are } \tau \text{-regular branches} \} = \{ \Pi_{\tau,\infty}(\delta) : \delta \in \Sigma_q^* \text{ are } \tau \text{-main branches} \}.
\]

Proof of Theorem 1.4. Let \( Q(\xi) := Q_\Lambda(\xi) \) be the function in (2.1) associated with the maximal orthogonal set \( \Lambda := \Lambda(\tau) \) of \( L^2(\mu_{q,b}) \). As \( Q \) is an analytic function on the real
line, the spectral property for the maximal orthogonal set \( \Lambda \) reduces to proving \( Q(\xi) \equiv 1 \) for all \( \xi \in T_b \) by Lemma 2.1. Suppose, on the contrary, there exists \( \xi_0 \in T_b \) such that

\[
Q(\xi_0) < 1.
\]

For \( n \geq 1 \), set

\[
\Lambda_n := \{ \Pi_{\tau,\infty}(\delta) : \delta \in \Sigma_q^n \text{ such that } \tau \text{ is regular on } \delta \},
\]

and define

\[
Q_n(\xi) := \sum_{\lambda \in \Lambda_n} |\hat{\mu}_{q,b}(\xi + \lambda)|^2, \quad \xi \in \mathbb{R}.
\]

Then

\[
\lim_{n \to \infty} \Lambda_n = \Lambda \quad \text{and} \quad \Lambda_n \subset \Lambda_{n+1} \quad \text{for all } n \geq 1,
\]

since \( \Lambda = \Lambda(\tau) \) and \( \Sigma_q^q = \bigcup_{n=1}^{\infty} \Sigma_q^n \). This implies that \( Q_n(\xi), n \geq 1 \), is an increasing sequence that converges to \( Q(\xi) \), i.e.,

\[
\lim_{n \to \infty} Q_n(\xi) = Q(\xi), \quad \xi \in \mathbb{R}.
\]

Thus for sufficiently small \( \epsilon > 0 \) chosen later, there exists an integer \( N \) such that

\[
Q(\xi_0) - \epsilon \leq Q_N(\xi_0) \leq Q_n(\xi_0) \leq Q(\xi_0) < 1 \quad \text{for all } n \geq N.
\]

For any \( \delta \in \Sigma_q^n \) being \( \tau \)-regular,

\[
\lim_{m \to \infty} H_m(\xi + \Pi_{\tau,m}(\delta)) = \lim_{m \to \infty} H_m(\xi + \Pi_{\tau,\infty}(\delta)) = \hat{\mu}_{q,b}(\xi + \Pi_{\tau,\infty}(\delta)), \quad \xi \in \mathbb{R}.
\]

For any \( \delta \in \Sigma_q^n \) such that \( \delta \) is not \( \tau \)-regular, the set \( \{ m \geq n + 1 : \tau(R_m(\delta)) \neq 0 \} \) contains infinite many integers. Denote that set by \( \{ m_j, j \geq 1 \} \) for some strictly increasing sequence \( \{ m_j \}_{j=1}^{\infty} \). Recall that

\[
\tau(R_{m_j}(\delta)) \in q\mathbb{Z} \cap \{-1, 1, 2, \ldots, b - 2\} \quad \text{for all } j \geq 1
\]

by the tree mapping property for \( \tau \). Therefore for \( m_j \leq m < m_{j+1} \) with \( j \geq 1 \),

\[
|H_m(\xi + \Pi_{\tau,m}(\delta))| \leq |H_{m_j}(\xi + \Pi_{\tau,m_j}(\delta))| = |H_{m_j}(\xi + \Pi_{\tau,m_j}(\delta))| \leq \prod_{k=1}^{j-1} |H_{q,b}(\xi + \Pi_{\tau,m_k}(\delta))/b^{m_k+1})| = \prod_{k=1}^{j-1} |H_{q,b}(\xi + \Pi_{\tau,m_k}(\delta))/b^{m_k+1})| = \left( \sup_{b \in T_b} |H_{q,b}(\eta)| \right)^{j-1}, \quad \xi \in T_b,
\]

where the three inequalities follow from (2.12), (2.15) and (2.27), and the two equalities hold by the tree mapping property \( \tau \) and the \( q/b \) periodicity of the filter \( H_{q,b} \). Combining (2.15) and (2.28) proves that

\[
\lim_{m \to \infty} |H_m(\xi + \Pi_{\tau,m}(\delta))| = 0, \quad \xi \in T_b
\]

if \( \delta \in \Sigma_q^n \) is not \( \tau \)-regular.
Applying \((2.26)\) and \((2.29)\) with \(n\) and \(\xi\) replaced by \(N\) and \(\xi_0\) respectively, we can find a sufficient large integer \(M \geq N + 1\) such that

\[
\sum_{\delta \in \Sigma^N_q} |H_M(\xi_0 + \Pi_{r,M}(\delta))|^2 \leq \sum_{\lambda \in \Lambda_N} |\tilde{\mu}_{q,b}(\xi_0 + \lambda)|^2 + \varepsilon \leq Q(\xi_0) + \varepsilon.
\]

This together with Lemma 2.3 implies that

\[
\sum_{\delta \in \Sigma^M_q \setminus \Sigma^N_q} |H_M(\xi_0 + \Pi_{r,M}(\delta))|^2 > 1 - Q(\xi_0) - \varepsilon > 0,
\]

where

\[
\Sigma^M_q \setminus \Sigma^N_q = \{ \delta \in \Sigma^M_q : E(R_N(\delta)) \neq E(\delta) \}.
\]

Take \(\delta \in \Sigma^M_q \setminus \Sigma^N_q\) and let \(r_0, r_1\) be as in \((2.3)\). Define

\[
\lambda(\delta) := \Pi_{r,0^m}(\delta)
\]

if \(\delta\) is a \(\tau\)-main branch. In this case,

\[
|\tilde{\mu}_{q,b}(\xi + \lambda(\delta))| = |\tilde{\mu}_{q,b}(\xi + \Pi_{r,M}(\delta))| = |H_M(\xi + \Pi_{r,M}(\delta))| \cdot |\tilde{\mu}_{q,b}(\xi + \Pi_{r,M}(\delta))/b^M| \\
\geq \inf_{\eta \in (\xi + (b-1)/b, (b-2)/(b-1))} |H_M(\xi + \Pi_{r,M}(\delta))| \cdot r_0|\tilde{\mu}_{q,b}(\xi + \Pi_{r,M}(\delta))|, \xi \in T_b,
\]

where the first and second equalities follows from \((2.32)\) and \((2.25)\) respectively, while the first inequality holds as \(b^{-M}(\xi + \Pi_{r,M}(\delta)) \in (-1/(b-1), (b-2)/(b-1))\) for all \(\xi \in T_b\).

Now consider \(\delta \in \Sigma^M_q \setminus \Sigma^N_q\) with \(\delta\) being not a \(\tau\)-main branch. In this case, define

\[
\lambda(\delta) := \Pi_{r,0^m}(\delta^0 \delta'),
\]

where \(m \geq 1\) is the smallest integer such that \(\tau(R_{m+M}(\delta)) \neq 0\), and \(\delta' \in \Sigma^*_q\) is so chosen that the quantities \(D_{\tau,0^m}(\delta')\) in \((1.10)\) and \(D_{\tau,0^m}\) in \((1.1)\) are the same,

\[
D_{\tau,0^m}(\delta') = D_{\tau,0^m}.
\]

Let \(\eta_1 = (\xi + \Pi_{r,M+m}(\delta^0 \delta'))/b^{M+m}\) and \(\eta_2 = (\xi + \Pi_{r,M}(\delta))/b^M\) for \(\xi \in T_b\). Then

\[
\eta_1 \in T_b \quad \text{and} \quad \eta_2 \in \left(-\frac{1}{b-1}, \frac{b-2}{b-1}\right)
\]

by \((2.12)\) and \(\tau(\delta^0 \delta') = \tau(R_{m+M}(\delta)) \neq 0\). Set

\[
r = \min \left(r_0, \frac{1}{b}, \frac{r_1}{b(b-1)}\right) \in (0, 1)
\]

and write

\[
(P_{\tau,0^m}(\delta^0 \delta') - \Pi_{r,M+m}(\delta^0 \delta'))/b^{M+m} = \sum_{j=1}^{K} d_{n_j} b^{n_j-1}
\]
for some integers $n_j, 1 \leq j \leq K$, satisfying $1 \leq n_1 < n_2 < \ldots < n_K$ and some $d_{n_j} \in \{-1, 1, 2, \ldots, b - 2\}, 1 \leq j \leq K$. Therefore

$$
|\mu_{q,b}(\xi + \lambda(\delta))| = |H_M(\xi + \lambda(\delta))| \cdot \prod_{l=M+1}^{M+m} H_{q,b}(\xi + \lambda(\delta))/b^l \bigg| \bigg| \frac{\mu_{q,b}(\xi + \lambda(\delta))/b^{M+m}}{\mu_{q,b}(\xi + \lambda(\delta))/b^l} \bigg|
$$

$$
\geq r_0^{2D_{r,\lambda}(\delta')}|\mu_{q,b}(\eta_2)| \cdot |H_M(\xi + \Pi_{\tau,M}(\delta))| \geq r_0^2 r^{2D_{r,\lambda}(\delta')}|H_M(\xi + \Pi_{\tau,M}(\delta))|,
$$

where the first inequality follows from (2.6), (2.15) and Lemma 2.2. Combining (2.33) and (2.37), we obtain the following important low bound estimate for $|\mu_{q,b}(\xi + \lambda(\delta))|$

$$
(2.38) \quad |\mu_{q,b}(\xi + \lambda(\delta))| \geq r_0^2 r^{2D_{r,\lambda}(\delta')}|H_M(\xi + \Pi_{\tau,M}(\delta))|, \quad \xi \in T_b,
$$

where $\lambda(\delta) \in \Lambda, \delta \in \Sigma_{q} M_1 \Sigma_{q}^N$, are defined by (2.32) and (2.34).

Observe that $\lambda(\delta) - \Pi_{\tau,M}(\delta) \in b^N \mathbb{Z}$ for all $\delta \in \Sigma_{q} M_1 \Sigma_{q}^N$. This implies that $\lambda(\delta_1) \neq \lambda(\delta_2)$ for two distinct $\lambda_1, \lambda_2 \in \Sigma_{q} M_1 \Sigma_{q}^N$. Therefore

$$
Q(\xi_0) = \sum_{\lambda \in \Lambda} |\mu_{q,b}(\xi_0 + \lambda)|^2 \geq \sum_{\lambda \in \Lambda_N} |\mu_{q,b}(\xi_0 + \lambda)|^2 + \sum_{\delta \in \Sigma_{q} M_1 \Sigma_{q}^N} |\mu_{q,b}(\xi_0 + \lambda(\delta))|^2
$$

$$
\geq Q(\xi_0) - \varepsilon + r_0^4 r^{4D_{r}} \sum_{\delta \in \Sigma_{q} M_1 \Sigma_{q}^N} |H_M(\xi + \Pi_{\tau,M}(\delta))|^2
$$

$$
\geq Q(\xi_0) - \varepsilon + r_0^4 r^{4D_{r}} (1 - Q(\xi_0) - \varepsilon),
$$

where the second inequality follows from (2.25) and (2.38), and the last holds by (2.31). This contradicts to (2.21) by letting $\varepsilon$ chosen sufficiently small. \qed

3. Maximal orthogonal sets and spectra: a necessary condition

Let $r_2 = \max \{|H_{q,b}(\xi)| : 1/b \leq b(b - 1)|\xi| \leq b - 2\}$, and define

$$
N_n = \inf_{\delta \in \Sigma_{q} M_1 \Sigma_{q}^N} N_{\tau,\delta}, \quad n \geq 2.
$$

In this section, we prove the following strong version of Theorem 1.5.

**Theorem 3.1.** Let $2 \leq q, b/q \in \mathbb{Z}$, and let $\tau : \Sigma_{q}^n \to \{-1, 0, \ldots, b - 2\}$ be a maximal tree mapping. If $\sum_{n=2}^{\infty} N_n r_2^n < \infty$, then $\Lambda(\tau)$ in (1.8) is not a spectrum of $L^2(\mu_{q,b})$.

**Proof.** Let $n \geq 2$ and take $\delta \in \Sigma_{q}^n \Sigma_{q}^{n-1}$ being $\tau$-regular. Write

$$
\{m \geq n + 1 : \tau(R_m(\delta)) \neq 0\} = \{n_k : 1 \leq k \leq K\}
$$
for some integers \( n < n_1 < n_2 < \ldots < n_K \), where \( K \geq N_n \). Therefore for \( \xi \in T_b \),

\[
|\overline{\mu}_{q,b}(\xi + \Pi_{\tau,\omega}(\delta))| = |H_n(\xi + \Pi_{\tau,\omega}(\delta))| \cdot |\overline{\mu}_{q,b}((\xi + \Pi_{\tau,\omega}(\delta))/b^n)|
\]

\[
\leq |H_n(\xi + \Pi_{\tau,\omega}(\delta))| \cdot \prod_{k=1}^{K} |H_{q,b}((\xi + \Pi_{\tau,\omega}(\delta))/b^{-n_k})|
\]

\[
\leq \left( \sup_{\eta \in T_b} |H_{q,b}(\eta/b)| \right)^K |H_n(\xi + \Pi_{\tau,\omega}(\delta))|
\]

(3.1)

where the first equality holds by (2.5); the first inequality follows from (2.6), (2.15) and \( \tau(R_{n_k}\delta) \in q\mathbb{Z}, 1 \leq k \leq K \), by the tree mapping property for \( \tau \); the second inequality is true since \( (\xi + \Pi_{\tau,\omega}(\delta))/b^{-n_k} \in T_b \) by (2.12); and the last inequality follows from the definition of the quality \( N_n \).

Let \( \Lambda_n \) and \( Q_n, n \geq 1 \), be as in (2.22) and (2.23) respectively, and set \( \Lambda_0 = \{0\} \) and \( Q_0(\xi) = |\overline{\mu}_{q,b}(\xi)|^2 \). Then for \( n \geq 1 \) and \( \xi \in T_b \),

\[
1 - Q_n(\xi) = 1 - Q_{n-1}(\xi) - \sum_{\delta \in \Lambda_n \setminus \Lambda_{n-1}} |\overline{\mu}_{q,b}(\xi + \Pi_{\tau,\omega}(\delta))|^2
\]

\[
\geq 1 - Q_{n-1}(\xi) - r_2^{2N_n} \sum_{\delta \in \Lambda_n \setminus \Lambda_{n-1}} |H_n(\xi + \Pi_{\tau,\omega}(\delta))|^2
\]

\[
\geq 1 - Q_{n-1}(\xi) - r_2^{2N_n} \left( 1 - \sum_{\lambda \in \Lambda_{n+1}} |\overline{\mu}_{q,b}(\xi + \lambda)|^2 \right)
\]

\[
= (1 - r_2^{2N_n}) \cdot (1 - Q_{n-1}(\xi)),
\]

(3.2)

where the first equality holds because

\[
\Lambda_n \setminus \Lambda_{n-1} = \{\Pi_{\tau,\omega}(\delta) : \delta \in \Sigma_q \setminus \Sigma_{q-1} \text{ is } \tau\text{-regular};
\]

the first inequality is true by (3.1); and the second inequality follows from Lemma 2.3 and

\[
\sum_{\lambda \in \Lambda_{n+1}} |\overline{\mu}_{q,b}(\xi + \lambda)|^2 \leq \sum_{\delta \in \Sigma_q \setminus \Sigma_{q-1}} |H_n(\xi + \Pi_{\tau,\omega}(\delta))|^2, \ \xi \in \mathbb{R},
\]

by (2.6) and (2.15). Recall that \( \lim_{n \to \infty} Q_{n}(\xi) = Q(\xi), \xi \in \mathbb{R} \), by (2.24). Applying (3.2) repeatedly and using the convergence of \( \sum_{n=1}^{\infty} r_2^{-2N_n} \) gives

\[
1 - Q(\xi) \geq \left( \prod_{n=1}^{\infty} (1 - r_2^{2N_n}) \right) \cdot (1 - Q_0(\xi)), \ \xi \in T_b.
\]

(3.3)

On the other hand,

\[
Q_0(\xi) = |\overline{\mu}_{q,b}(\xi)|^2 < 1, \ \xi \in T_b
\]

by (2.6) and (2.15). This together with (3.3) proves that \( Q(\xi) < 1 \) for all \( \xi \in T_b \), and hence \( \Lambda = \Lambda(\tau) \) is not a spectrum for \( L^2(\mu_{q,b}) \) by Lemma 2.1.
4. **Integrally rescaled spectra**

In this section, we prove Theorem 1.6.

As the necessity is obvious, it suffices to prove the sufficiency. Take \( \delta \in \Sigma_q^n, n \geq 1 \), and let \( \delta_1 \in \Sigma_q^* \) be so chosen that \( \delta \delta_1 \) is \( \kappa \)-regular, where \( \kappa \) is the maximal tree mapping associated with the maximal orthogonal set \( K \Lambda \). Thus \( \Pi_{K,\infty}(\delta \delta_1) \in K \Lambda \), which implies the existence of \( \zeta \in \Sigma_q^\infty \) such that

\[
(4.1) \quad K \Pi_{\tau,n}(\zeta) - \Pi_{K,n}(\delta) \in b^n \mathbb{Z}.
\]

Let \( \zeta' \in \Sigma_q^* \) be so chosen that \( \zeta \zeta' \) is a \( \tau \)-main subbranch of \( \zeta \) and

\[
(4.2) \quad D_{\tau,\zeta}(\zeta') = D_{\tau,\zeta},
\]

where \( D_{\tau,\zeta} \) is given in (1.11). Denote the integral part of a real number \( x \) by \( \lfloor x \rfloor \). By Theorem 1.4 and the assumption that \( D_{\tau} < \infty \), it suffices to verify that

\[
(4.3) \quad K \Pi_{\tau,\infty}(\zeta'') = \Pi_{K,\infty}(\delta \delta')
\]

for some \( \kappa \)-main subbranch \( \delta \delta' \) of \( \delta \), and

\[
(4.4) \quad D_{\kappa,\delta}(\delta') \leq (\lfloor \log_b K \rfloor + 2)(D_{\tau,\zeta} + 1).
\]

By Theorem 1.2 there exists a \( \kappa \)-regular branch \( \delta_2 \in \Sigma_q^* \) such that \( \Pi_{K,\infty}(\delta_2) = K \Pi_{\tau,\infty}(\zeta'') \).

Then

\[
(4.5) \quad \Pi_{K,n}(\delta_2) - \Pi_{K,n}(\delta) \in K \Pi_{\tau,n}(\zeta) - \Pi_{K,n}(\delta) + b^n \mathbb{Z} = b^n \mathbb{Z}
\]

by (4.1). This together with one-to-one correspondence of the mapping \( \Pi_{K,n} : \Sigma_q^n \rightarrow \mathbb{Z} \) proves (4.3).

Now we prove (4.4). Without loss of generality, we assume that \( \Pi_{K,\infty}(\delta \delta') \neq \Pi_{K,n}(\delta) \), because otherwise \( D_{\kappa,\delta}(\delta') = 0 \) and hence (4.4) follows immediately. Thus we may write

\[
(4.6) \quad \Pi_{K,\infty}(\delta \delta') = \Pi_{K,n}(\delta) + \sum_{l=1}^L d_l b^{n+m_{l-1}}
\]

for a strictly increasing sequence \( \{m_l\}_{l=1}^L \) of integers and some \( d_l \in \{-1, 1, \ldots, b - 2\}, 1 \leq l \leq L \).

Consider the case that \( \Pi_{\tau,\infty}(\zeta'') = \Pi_{\tau,n}(\zeta) \). In this case,

\[
K \Pi_{\tau,\infty}(\zeta'') = K \Pi_{\tau,n}(\zeta) \in K(-b^n/(b-1), (b-2)b^n/(b-1))
\]

and

\[
\Pi_{K,\infty}(\delta \delta') \notin (-b^{n+m_{L-1}}/(b-1), (b-2)b^{n+m_{L-1}}/(b-1))
\]

by (2.12). This together with (4.3) implies that \( b^{m_{L-1}} \leq K \). Hence

\[
D_{\kappa,\delta}(\delta') \leq m_L \leq \lfloor \log_b K \rfloor + 1
\]

and (4.4) is proved.

Next consider the case that \( \Pi_{\tau,\infty}(\zeta'') \neq \Pi_{\tau,n}(\zeta) \). In this case,

\[
(4.7) \quad \Pi_{\tau,\infty}(\zeta'') = \Pi_{\tau,n}(\zeta) + \sum_{j=1}^N c_j b^{n+j_{j-1}}
\]
where \(c_j \in \{-1, 1, \ldots, b-2\}, 1 \leq j \leq N\), and \(\{n_j\}_{j=1}^N\) is a strictly increasing sequence of integers. To prove (4.4) for the case that \(\Pi_{r,\infty}(\zeta') \neq \Pi_{r,n}(\zeta)\), we need the following claim:

**Claim 1:** \(\{m_l, 1 \leq l \leq L\} \subset \bigcup_{l=0}^{N} [n_{j_l}, n_{j_l} + \lfloor \log_b K \rfloor + 1]\).

Suppose, on the contrary, that Claim 1 does not hold. Then there exists \(1 \leq l \leq L\) such that \(n_{j_{l-1}} + \lfloor \log_b K \rfloor + 1 < m_l < n_{j_{l+1}}\) for some \(0 \leq j_0 \leq N\), where we set \(n_0 = 0\) and \(n_{N+1} = +\infty\). Observe that

\[
\Pi_{\kappa,n+m_l}(\delta\delta') - K\Pi_{r,n+n_{j_0}}(\zeta') \in b^{n+m_l}\mathbb{Z}
\]

by (4.3) and the assumption \(m_l < n_{j_{l+1}}\), and

\[
\Pi_{\kappa,n+m_l}(\delta\delta') - K\Pi_{r,n+n_{j_0}}(\zeta') \subset d_l b^{n+m_l-1} + \frac{b^{n+m_l-1} - 1}{b-1} [-1, b-2] - K \frac{b^{n+n_{j_0}} - 1}{b-1} [-1, b-2]
\]

by the definitions of \(\Pi_{\kappa,n+m_l}\) and \(\Pi_{r,n+n_{j_0}}\) and the assumption \(n_{j_l} + \lfloor \log_b K \rfloor + 1 > m_l\). Combining (4.8) and (4.9) leads to the contradiction that \(d_l \in \{-1, 1, \ldots, b-2\}\). This completes the proof of Claim 1.

To prove (4.4) for the case that \(\Pi_{r,\infty}(\zeta') \neq \Pi_{r,n}(\zeta)\), we need another claim:

**Claim 2:** If \(n_j + \lfloor \log_b K \rfloor + 1 < n_{j+1}\), then there exists \(l_0\) such that \(m_{l_0} = n_{j+1}, m_{l_0-1} \in [n_j, n_j + \lfloor \log_b K \rfloor + 1]\) and \(d_{l_0} \in q\mathbb{Z}\) if and only if \(c_{j+1} \in q\mathbb{Z}\).

Let \(l_0\) be the smallest integer \(l\) with \(m_l \geq n_{j+1}\). By Claim 1, \(m_{l_0-1} \leq n_j + \lfloor \log_b K \rfloor + 1 \leq n_{j+1} - 1\). Observe that \(\Pi_{\kappa,n+m_{l_0}}(\delta\delta') - K\Pi_{r,n+n_{j+1}}(\zeta') \in b^{n+n_{j+1}}\mathbb{Z}\) by (4.3); and

\[
\Pi_{\kappa,n+m_{l_0}}(\delta\delta') - K\Pi_{r,n+n_{j+1}}(\zeta') \subset d_{l_0} b^{n+n_{l_0}-1} - K c_{j+1} b^{n+n_{j+1}-1} + \frac{b^{n+m_{l_0}-1} - 1}{b-1} [-1, b-2] - K \frac{b^{n+n_{j+1}} - 1}{b-1} [-1, b-2]
\]

(4.10)

Thus \(d_{l_0} b^{m_{l_0}-n_{j+1}} - K c_{j+1} b^{n+n_{j+1}-1} \in b\mathbb{Z}\). This together, with the assumptions that \(c_{j+1} \in \{-1, 1, \ldots, b-2\}\) and that \(K\) and \(b\) are coprime, implies that \(m_{l_0} = n_{j+1}\) and \(d_{l_0} \in q\mathbb{Z}\) if and only if \(c_{j+1} \in q\mathbb{Z}\). From the argument in (4.10), we see that

\[
\Pi_{\kappa,m_{l_0}}(\delta\delta') = K\Pi_{r,n}(\zeta').
\]

Thus \(m_{l_0-1} \geq n_j\), as \(\Pi_{\kappa,m_{l_0}}(\delta\delta') \in b^{m_{l_0}-1}(-1/(b-1), (b-2)/(b-1))\) and \(\Pi_{r,n}(\zeta') \neq Kb^{n-1}(-1/(b-1), (b-2)/(b-1))\) by (2.12). This completes the proof of Claim 2.

Having established the above two claims, let us return to the proof of the inequality (4.4). Write

\[
\bigcup_{j=0}^{N} [n_j, n_j + \lfloor \log_b K \rfloor + 1] = \bigcup_{l=0}^{L} [n_{p_l}, n_{p_{l+1}} - [\log_b K] + 1]
\]

where \(p_{l+1} = N + 1\) and \(0 \leq p_1 < p_2 < \cdots < p_L \leq N\) satisfies \(n_{p_{l+1}} - [\log_b K] + 1 < n_{p_{l+1}}\). It follows from Claim 2 that in order to obtain the trimmed tree branch \(\tilde{\eta}\), for any \(0 \leq l \leq L - 1\) with \(c_{p_{l+1}} \in q\mathbb{Z}\), at least \(n_{p_{l+1}} - n_{p_{l+1}} - [\log_b K] - 2\) conservative zeros before \(n_{p_{l+1}}\) in \(q\mathbb{Z}\) should be squeezed out. Therefore the length \(D_{\kappa,\delta}(\delta')\) of the trimmed tree branch \(\tilde{\eta}\) is at most the...
numbers of integers in \( \bigcup_{i=0}^{\lambda} [n_{p_1}, n_{p_{i+1}} + \lfloor \log_b K \rfloor + 1] \cup \bigcup_{\ell \notin q \mathbb{Z}} (n_{p_{i+1}} + \lfloor \log_b K \rfloor + 1, n_{p_{i+1}}) \). Thus the inequality (4.4) follows from

\[
\mathcal{D}_{\kappa, \delta}(\delta') \leq (\lfloor \log_b K \rfloor + 2)(N + 1) + \sum_{c_i \notin q \mathbb{Z}} (n_j - n_{j-1} - 1) \leq (\lfloor \log_b K \rfloor + 2)(\mathcal{D}_{\tau, \zeta}(\zeta') + 1).
\]

This completes the proof of Theorem 1.6. \( \Box \)

5. Integrially rescaled maximal orthogonal sets

In this section, we prove Theorem 1.7.

First the sufficiency. Take \( \lambda \in \Lambda \). By the maximality of the tree mapping \( \tau \), there exists a \( \tau \)-main branch \( \zeta \in \Sigma^{n} \) for some \( m \geq 1 \) by Theorem 1.2 such that

\[
(5.1) \quad \lambda = \Pi_{\tau, \zeta}(\zeta).
\]

Let

\[
(5.2) \quad \lambda_0 = K\Pi_{\tau, M}(\delta) - ib^M,
\]

where \( i \in \mathbb{Z} \) is given in (1.18). Inductively applying (1.18) proves that

\[
(5.3) \quad \lambda_0 = K\Pi_{\tau, M+N}(\delta) - ib^{M+N} = \cdots = K\Pi_{\tau, M+nN}(\delta) - ib^{M+nN}, \quad n \geq 1.
\]

Also for sufficiently large \( n \geq 1 \), there exists \( \lambda \neq \lambda_n \in \Lambda \) by the maximality of the tree mapping \( \tau \) such that

\[
(5.4) \quad \lambda_n = \Pi_{\tau, M+nN}(\delta) \in b^{M+nN}\mathbb{Z}
\]

The reason for \( \lambda_n \neq \lambda \) is that \( \Pi_{\tau, M+nN}(\delta) \neq \Pi_{\tau, M+nN}(\zeta) \) for sufficiently large \( n \) by \( W = \omega_1 \cdots \omega_N \neq 0^N \) by (1.18).

As both \( \lambda, \lambda_n \in \Lambda \), there exists a nonnegative integer \( l \) and an integer \( a \in \mathbb{Z} \setminus q\mathbb{Z} \) by (1.4) such that

\[
(5.5) \quad \lambda - \lambda_n = ab^l.
\]

Now we show that

\[
(5.6) \quad l < M + Nn
\]

when \( n \) is sufficiently large. Suppose, on the contrary, that \( l \geq M + Nn \). Then

\[
(5.7) \quad \lambda - \Pi_{\tau, M+nN}(\delta) \in b^{M+nN}\mathbb{Z}.
\]

On the other hand,

\[
\Pi_{\tau, M+nN}(\delta) \in b^{M+nN}[{-1}/(b-1), (b-2)/(b-1)]
\]

by the tree mapping property for \( \tau \). Therefore \( \lambda = \Pi_{\tau, M+nN}(\delta) \) for sufficiently large \( n \), which is a contradiction as

\[
\Pi_{\tau, M+nN}(\delta) \notin b^{M+(N-1)n}(-1/(b-1), (b-2)/(b-1))
\]

by \( W = \omega_1 \cdots \omega_N \neq 0^N \) and the tree mapping property for \( \tau \).

Combining (5.4), (5.5) and (5.6) and recalling that \( K \) and \( b \) are co-prime, we obtain that

\[
(5.8) \quad K\lambda - K\Pi_{\tau, M+nN}(\delta) = \tilde{a}b^l
\]
for some integers \(0 \leq l < M + Nn\) and \(a \in \mathbb{Z}\backslash q\mathbb{Z}\). Thus the inner product between \(e_{k_0}\)
and \(e_{K,l}\) is equal to zero by (1.4), (5.3) and (5.8). This proves that \(K\Lambda\) is not a maximal orthogonal set as \(\lambda \in \Lambda\) is chosen arbitrarily.

Next the necessity. By (1.4) and the assumption on the rescaled set \(K\Lambda\), there exists a maximal orthogonal set \(\Theta\) of the Cantor measure \(\mu_{q,b}\) such that

\[
K\Lambda \subset \Theta \subset \mathbb{Z}.
\]

Take \(\vartheta_0 \in \Theta \backslash (K\Lambda)\). Then

\[
\vartheta_0 = \Pi_{\kappa,\infty}(\vartheta_0) = \Pi_{\kappa,m}(\vartheta_0)
\]

for some \(\kappa\)-main branch \(\vartheta_0 \in \Sigma^n_q, m \geq 1\), where \(\kappa\) is the maximal tree mapping given in Theorem 1.2.

Let \(\tau\) be the maximal tree mapping in Theorem 1.2 such that \(\Lambda = \Lambda(\tau)\). To establish the necessity, we need the following claim:

**Claim 1:** Let \(n \geq 1\). For any \(\zeta \in \Sigma_q^n\) there exists a unique \(\delta \in \Sigma^n_q\) such that \(\Pi_{\kappa,n}(\zeta) - K\Pi_{\tau,n}(\delta) \in b^n\mathbb{Z}\).

Observe that

\[
K\Pi_{\tau,n}(\delta_1) - K\Pi_{\tau,n}(\delta_2) \notin b^n\mathbb{Z} \quad \text{for all distinct} \quad \delta_1, \delta_2 \in \Sigma^n_q,
\]

because \(b/q \in \mathbb{Z}, K\) and \(b\) are coprime, and \(K\Pi_{\tau,n}(\delta_1) - K\Pi_{\tau,n}(\delta_2) = ab^l\) for some \(0 \leq l \leq n - 1\) and \(a \notin q\mathbb{Z}\). On the other hand,

\[
\{K\Pi_{\tau,n}(\delta) : \delta \in \Sigma^n_q\} + b^n\mathbb{Z} = K\Lambda + b^n\mathbb{Z} \subset \Theta + b^n\mathbb{Z} = \{\Pi_{\kappa,n}(\zeta) : \zeta \in \Sigma^n_q\} + b^n\mathbb{Z}
\]

by (5.9). Combining (5.11) and (5.12) leads to

\[
\{K\Pi_{\tau,n}(\delta) : \delta \in \Sigma^n_q\} + b^n\mathbb{Z} = \{\Pi_{\kappa,n}(\zeta) : \zeta \in \Sigma^n_q\} + b^n\mathbb{Z}.
\]

Then Claim 1 follows from (5.13) and (5.11).

To establish the necessity, we need another claim:

**Claim 2:** \(\vartheta_0 \notin K\mathbb{Z}\).

Suppose, on the contrary, that \(\vartheta_0 \in K\mathbb{Z}\). Then for any \(\lambda \in \Lambda\), there exist \(a \in \mathbb{Z}\backslash q\mathbb{Z}\) and \(0 \leq l \in \mathbb{Z}\) by (1.4) and (5.9) such that \(\vartheta_0 - K\lambda = ab^l\). This together with the co-prime assumption between \(K\) and \(b\) implies that \(a/K \in \mathbb{Z}\) and \(0 \neq \vartheta_0/K - \lambda \in (a/K)b^l\). Thus \(\Lambda \cup \{\vartheta_0/K\}\) is an orthogonal set for the measure \(\mu_{q,b}\) by (1.4), which contradicts to the maximality of the set \(\Lambda\).

Now we continue our proof of the necessity. Let \(N\) be the smallest positive integer such that \((b^N - 1)\vartheta_0/K \in \mathbb{Z}\), where the existence follows from the co-prime property between \(K\) and \(b\). By Claim 2, there exists \(\omega_j \in \{-1, 0, \ldots, b - 2\}, 1 \leq j \leq N\), such that the word \(W := \omega_1\omega_2 \cdots \omega_N \neq 0\) and

\[
\frac{\vartheta_0}{K} = \frac{c.\omega_N \cdots \omega_2\omega_1 \omega_N \cdots \omega_2\omega_1}{b^N - 1} = c + \frac{\sum_{j=1}^{N} \omega_j b^{j-1}}{b^N - 1}
\]
for some integer $c \in \mathbb{Z}$. Let $W' = \omega'_1 \omega'_2 \cdots \omega'_N$ be so chosen that $\omega'_j \in \{-1, 0, \ldots, b-2\}, 1 \leq j \leq N$, and

\begin{equation}
\sum_{j=1}^{N} (\omega'_j + \omega_j)b^{j-1} = \begin{cases} 0 & \text{if } \sum_{j=1}^{N} \omega_j b^{j-1} \in \frac{b^{N-1}-1}{b-1}(-1, 1) \\
b^{N} - 1 & \text{if } \sum_{j=1}^{N} \omega_j b^{j-1} \in \frac{b^{N-1}-1}{b-1}[1, b-2]. \end{cases}
\end{equation}

The existence of such a word $W'$ follows from the observation that

\[ \left\{ \sum_{j=1}^{N} \omega_j b^{j-1}, \omega_j \in \{-1, 0, \ldots, b-2\} \right\} = \left( \frac{b^{N}-1}{b-1} [1, b-2] \right) \cap \mathbb{Z}. \]

Let $n > m/N$ and set $\zeta_{nN} = \zeta_0 0^{nN-m} \in \Sigma^n_{q}$. By Claim 1 and the $\kappa$-main branch assumption for $\zeta_0$, there exists $\delta_{nN} \in \Sigma^n_{q}$ such that

\begin{equation}
K\Pi_{\tau,nN}(\delta_{nN}) - \theta_0 \in b^{nN}\mathbb{Z}.
\end{equation}

Combining (5.14), (5.15) and (5.16) and recalling that $K$ and $b$ are coprime, we obtain

\[(b^{N}-1)(\Pi_{\tau,nN}(\delta_{nN}) - \bar{c}) + \sum_{j=1}^{N} \omega_j b^{j-1} \in b^{nN}\mathbb{Z},\]

where

\[\bar{c} = \begin{cases} c & \text{if } \sum_{j=1}^{N} \omega_j b^{j-1} \in \frac{b^{N}-1}{b-1}(-1, 1) \\
c - 1 & \text{if } \sum_{j=1}^{N} \omega_j b^{j-1} \in \frac{b^{N}-1}{b-1}[1, b-2]. \end{cases}\]

Therefore

\begin{equation}
\Pi_{\tau,nN}(\delta_{nN}) - \bar{c} - \left( \sum_{j=1}^{N} \omega'_j b^{j-1} \right)(1 + b^{N} + \cdots + b^{(N-1)N}) \in b^{nN}\mathbb{Z}.
\end{equation}

By the construction of $\omega'_j, 1 \leq j \leq N$, $\sum_{j=1}^{N} \omega'_j b^{j-1} \in \frac{b^{N}-1}{b-1}(-1, b-2)$. If either $\sum_{j=1}^{N} \omega'_j b^{j-1} \in \frac{b^{N}-1}{b-1}(-1, b-2)$ or $\sum_{j=1}^{N} \omega'_j b^{j-1} = \frac{b^{N}-1}{b-1}(b-2)$ and $\bar{c} \leq 0$, then for sufficiently large $k$,

\[\bar{c} + \left( \sum_{j=1}^{N} \omega'_j b^{j-1} \right)(1 + b^{N} + \cdots + b^{(N-1)N}) = \sum_{j=1}^{kN} \theta_j b^{j-1}\]

for some $\theta_j \in \{-1, 0, \ldots, b-2\}, 1 \leq j \leq kN$, as it is contained in $[-(b^{kN} - 1)/(b-1), (b^{kN} - 1)(b-2)/(b-1)]$. This together with (5.17) implies that

\[\Pi_{\tau,nN}(\delta_{nN}) = \sum_{j=1}^{kN} \theta_j b^{j-1} + \sum_{j=1}^{N} \omega'_j b^{j-1}(b^{kN} + \cdots + b^{(N-1)N})\]

for $n \geq k$. Thus there exists $\delta \in \Sigma^\infty_{q}$ such that $R_{nN}(\delta) = \delta_{nN}$ and

\[\tau(R_{nN+j}(\delta)) = \omega'_j, 1 \leq j \leq N\]

for $n \geq k$, which proves the desired conclusion.
Now consider the case that \( \sum_{j=1}^{N} \omega_j b_j^{j-1} = b^{N-1}(b-2) \) and \( \tilde{c} > 0 \). In this case, \( \omega_j = b - 2 \) for all \( 1 \leq j \leq N \) and \( N = 1 \) by the selection of the integer \( N \). Further we obtain from (5.17) that

\[
\Pi_{\tau_n}(\delta_n) - \tilde{c} + 1 + \sum_{j=1}^{n} b^{j-1} \in b^n \mathbb{Z},
\]

which implies that there exists \( \delta \in \Sigma_q^\infty \) such that \( R_n(\delta) = \delta_n \) and \( \tau(\delta_n(\delta)) = -1 \) for sufficiently large \( n \), which proves the desired conclusion. \( \square \)

6. Rationally rescaled spectra

In this section, we construct a spectrum \( \Lambda \) of the Cantor measure \( \mu_{q,b} \) such that \( \Lambda / (b-1) \) is its maximal orthogonal set but not its spectrum.

**Theorem 6.1.** Consider \( 2 \leq q, b/q \in \mathbb{Z} \) and \( b > 4 \). Define a tree mapping \( \kappa : \Sigma_q^* \to \{-1, 0, 1, \ldots, b-2\} \) by

\[
\kappa(R_{k+1}(\delta)) = \begin{cases} 
0 & \text{if } \delta = 0 \text{ and } k \geq 0 \\
\delta & \text{if } 1 \leq \delta \leq q-1 \text{ and } k = 0 \\
q & \text{if } 1 \leq \delta \leq q-1 \text{ and } k \in \{1, 2, \ldots, K_\delta, 2b\} \\
0 & \text{if } 1 \leq \delta \leq q-1 \text{ and } K_\delta < k \neq 2b
\end{cases}
\]

where \( 0 \leq K_\delta \leq b-2 \) is the unique integer such that \( q(K_\delta + 1) + \delta \in (b-1)\mathbb{Z} \), and inductively

\[
\kappa(R_{k+n}(\delta)) = \begin{cases} 
0 & \text{if } k = 0 \\
q & \text{if } k \in \{1, 2, \ldots, K_\delta, n+2b-1\} \\
0 & \text{if } k > K_\delta \text{ and } k \neq n+2b-1
\end{cases}
\]

if \( \delta = \delta' \) for some \( \delta' \in \Sigma_q^{n-1}, n \geq 2 \) and \( j \in \{1, \ldots, q-1\} \), where \( K_\delta \in \{0, 1, \ldots, b-2\} \) is the unique integer such that

\[
\left( \sum_{i=1}^{n-1} \kappa(R_i(\delta)) + q(K_\delta + 1) + j \right) \in (b-1)\mathbb{Z}.
\]

Then

\[
\Lambda_{q,b} := \{ \Pi_{\kappa,\omega}(\delta) : \delta \in \Sigma_q^* \}
\]

is a spectrum of the Cantor measure \( \mu_{q,b} \) and the rationally rescaled set \( \Lambda_{q,b} / (b-1) \) is its maximal orthogonal set but not its spectrum.

**Proof.** First we show that \( \Lambda_{q,b} \) is a spectrum of the Cantor measure \( \mu_{q,b} \). Observe that \( \kappa \) is a maximal tree mapping, every \( \delta \in \Sigma_q^* \) is \( \kappa \)-regular, and \( \Lambda_{q,b} = \Lambda(\kappa) \). We then obtain from Theorem 1.2 that

\[
\Lambda_{q,b} \text{ is a maximal orthogonal set of the Cantor measure } \mu_{q,b}.
\]

From the definition of the maximal tree mapping \( \kappa \) it follows that

\[
D_{\kappa,\delta} \leq D_{\kappa,\delta}(0^\omega) \leq K_\delta + 1 \leq b - 1 \text{ for all } \delta \in \Sigma_q^*.
\]

where \( K_\delta \) is given in (6.3). Therefore the spectral property for \( \Lambda_{q,b} \) holds by (6.5), (6.6) and Theorem 1.4.
Next we prove that $\Lambda_{q,b}/(b-1)$ is a maximal orthogonal set for the Cantor measure $\mu_{q,b}$. By (1.4), $\Lambda$ is an orthogonal set of the Cantor measure $\mu_{q,b}$ if and only if
\begin{equation}
\Lambda - \Lambda \subset \{b^j a : 0 \leq j \in \mathbb{Z}, a \in \mathbb{Z} \setminus q\mathbb{Z} \} \cup \{0\}.
\end{equation}
Thus we obtain from the spectral property for the set $\Lambda$, that
\begin{equation}
\Lambda_{q,b} - \Lambda_{q,b} \subset \{b^j a : 0 \leq j \in \mathbb{Z}, a \in \mathbb{Z} \setminus q\mathbb{Z} \} \cup \{0\}.
\end{equation}
On the other hand,
\begin{equation}
0 \in \Lambda_{q,b} \subset \mathbb{Z}
\end{equation}
and for any $\delta \in \Sigma_q^*$,
\begin{equation}
\Pi_{\kappa,\infty}(\delta) = \sum_{j=1}^{\infty} \tau(R_j(\tau)) b^{j-1} \in \sum_{j=1}^{\infty} \tau(R_j(\tau)) + (b - 1)\mathbb{Z} = (b - 1)\mathbb{Z}
\end{equation}
by (6.1)–(6.3). Combining (6.8) and (6.9) leads to
\begin{equation}
(\Lambda_{q,b} - \Lambda_{q,b})/(b - 1) \subset \{b^j a : 0 \leq j \in \mathbb{Z}, a \in \mathbb{Z} \setminus q\mathbb{Z} \} \cup \{0\},
\end{equation}
and hence $\Lambda_{q,b}/(b - 1)$ is an orthogonal set for the Cantor measure $\mu_{q,b}$. Now we establish the maximality of the rescaled set $\Lambda_{q,b}/(b - 1)$. Suppose, on the contrary, that there exists $\lambda_0 \notin \Lambda_{q,b}/(b - 1)$ such that $\tilde{\Lambda}_{q,b} := \Lambda_{q,b}/(b - 1) \cup \{\lambda_0\}$ is an orthogonal set for the Cantor measure $\mu_{q,b}$. Then $(b - 1)\tilde{\Lambda}_{q,b}$ is an orthogonal set for the Cantor measure $\mu_{q,b}$ by (6.7), because
\begin{equation}
(b - 1)\tilde{\Lambda}_{q,b} - (b - 1)\tilde{\Lambda}_{q,b} \subset (b - 1)((\{b^j a : 0 \leq j \in \mathbb{Z}, a \in \mathbb{Z} \setminus q\mathbb{Z} \} \cup \{0\})
\end{equation}
This contradicts the spectral property for $\Lambda_{q,b}$, and hence it completes the proof that $\Lambda_{q,b}/(b - 1)$ is a maximal orthogonal set of the Cantor measure $\mu_{q,b}$.

Finally we prove that $\Lambda_{q,b}/(b - 1)$ is not a spectrum of the Cantor measure $\mu_{q,b}$. Let $\tau_{q,b} : \Sigma_q^* \rightarrow \{-1, 0, \ldots, b - 2\}$ be the maximal tree mapping such that $\Lambda_{q,b}/(b - 1) = \Lambda(\tau_{q,b})$. The existence of such a maximal tree mapping follows from Theorem 1.2 and the maximality of the orthogonal set $\Lambda_{q,b}/(b - 1)$. By Theorem 1.5 the non-spectral property for the set $\Lambda_{q,b}/(b - 1)$ reduces to showing that
\begin{equation}
N(\tau_{q,b,\delta} \geq n)
\end{equation}
for all $\delta \in \Sigma_q^\prime \setminus \Sigma_q^{n-1}, n \geq 2$, being $\tau_{q,b}$-regular. Recall that $\Lambda_{q,b} = \Lambda(\kappa)$. This together with (6.1) and (6.2) implies the existence of $\eta \in \Sigma_q^\prime, m \geq 1$, such that
\begin{equation}
(b - 1)\Pi_{\tau_{q,b,\infty}}(\delta) = \Pi_{\kappa,\infty}(\eta) = \sum_{j=1}^{m+b-2} d_j b^{j-1} + q \cdot b^{2m+2b-2},
\end{equation}
where $d_j \in \{0, 1, \ldots, q\}$ for all $1 \leq j \leq m + b - 2$ and $d_m \in \{1, \ldots, q - 1\}$. Write
\begin{equation}
\Pi_{\tau_{q,b,\infty}}(\delta) = \sum_{j=1}^{\infty} c_j b^{j-1} = \sum_{j=1}^{M} c_j b^{j-1}
\end{equation}
where $c_j := \tau_{q,b}(R_j(\delta)) \in \{-1, 0, \ldots, b - 2\}$ and $M \geq n$ is so chosen that $c_M \neq 0$. The existence of such an integer follows from $\tau_{q,b}(R_n(\delta)) \in \mathbb{Z} \setminus q\mathbb{Z}$ and $\tau_{q,b}(R_j(\delta)) = 0$ for sufficiently large $j$. Combining (6.11) and (6.12) leads to

\begin{equation}
(b - 1) \sum_{j=1}^{M} c_j b^{j-1} = \sum_{j=1}^{M} d_j b^{j-1} + q \cdot b^{2m+2b-2}.
\end{equation}

Thus

\[
\sum_{j=1}^{M} c_j b^{j-1} = \frac{1}{b-1} \left( \sum_{j=1}^{M} d_j b^{j-1} + q \cdot b^{m+b-2} \right) + q \sum_{j=m+b-2}^{2m+2b-3} b^j
\]

\[
\leq q \sum_{j=m+b-2}^{2m+2b-3} b^j + (0, \frac{b-2}{b-1})b^{m+b-2},
\]

where the last inequality follows as $q \leq b - 3$ and

\[
\sum_{j=1}^{M} d_j b^{j-1} + q \cdot b^{m+b-2} \leq \frac{q(b^{m+b-2} - 1)}{b-1} + q \cdot b^{m+b-2} < (b - 2)b^{m+b-2}.
\]

This, together with $c_j \in \{-1, 0, \ldots, b - 2\}, 1 \leq j \leq M$, implies that

\begin{equation}
M = 2m + 2b - 2 \quad \text{and} \quad c_j = q, m + b - 2 < j \leq M.
\end{equation}

On the other hand, for $\delta \in \Sigma_q \setminus \Sigma_q^{n-1}$ it follows from the tree mapping property for $\tau_{q,b}$ that $c_n \notin q\mathbb{Z}$. Thus $n \leq m + b - 2$ according to (6.14). Therefore

\[
N_{\tau_{q,b}, \delta} \geq 2m + 2b - 2 - n \geq n.
\]

This proves (6.10) and then the conclusion that $\Lambda_{q,b}$ is not a spectrum of the Cantor set $\mu_{q,b}$ by Theorem [1.5].

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School of Mathematics and Computational Science, Sun Yat-sen University, Guangzhou, 510275, P. R. China

E-mail address: daixr@mail.sysu.edu.cn