Rastall’s Cosmology and its Observational Constraints

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The Rastall’s theory is a modification of General Relativity touching one of the cornerstone of gravity theory: the conservation laws. In Rastall’s theory, the energy-momentum tensor is not conserved anymore, depending now on the gradient of the Ricci curvature. In this sense, this theory can be seen as a classical implementation of quantum effects in a curved background space-time. We exploit this structure in order to reproduce some results of an effective theory of quantum loop cosmology. Later, we propose a model for the dark sector of the universe. In this case, the corresponding ΛCDM model appears as the only model consistent with observational data.

I. RASTALL’S THEORY

The Rastall’s proposal is based on the following observation: The conservation laws are tested effectively only on flat spacetime [1]. Hence, in a non-flat spacetime, there is room for generalizations of the usual conservation laws.

The choice made by Rastall to introduce a modification of the usual conservation law is:

\[ T_{\mu\nu;\mu} = 0 \Rightarrow T_{\mu\nu;\mu} = \frac{(\lambda - 1)}{16\pi G} R_{\mu\nu}. \]  

(1)

with \( \lambda \) a constant. When \( \lambda = 1 \), General Relativity, with its usual conservation law, is recovered.

Today, a new motivation to Rastall’s theory can be given. Since the violation of the usual conservation law is related to the gradient of the curvature scalar, such violation can be associated to quantum effects in curved spacetime. In fact, the expression (1), in a particular form, appears in the so-called gravitational anomaly, a anomaly in the usual gravitational equations due to quantum effects [2]. According to this remark, the Rastall’s theory can be considered as a classical, effective implementation of quantum effects due to fundamental fields living in a curved spacetime.

The field equations of the Rastall’s theory can be written alternatively as follows:

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \{ T_{\mu\nu} - \frac{\gamma - 1}{2} g_{\mu\nu} T \}, \]  

(2)

\[ T_{\mu\nu;\mu} = \frac{\gamma - 1}{2} T_{\mu\nu}. \]  

(3)

In these equations \( \gamma \) is a dimensionless parameter related to the previous one, \( \lambda \), by the relation,

\[ \gamma = \frac{2 - 3\lambda}{1 - 2\lambda}. \]  

(4)

When \( \gamma = 1 \), General Relativity, with the usual conservation law, is recovered.

Is this theory well justified theoretically? First of all, there is in principle no Lagrangian formulation leading to the Rastall’s field equation. But this is strictly true only at the context of the Riemannian geometry. Possible exterions (e.g., Weyl geometry) may cure this possible drawback. Moreover an introduction of an “external field” may lead to a Lagrangian formulation even in the context of the Riemannian geometry.

In fact, consider the following Lagrangian:

\[ L = \sqrt{-g} \left( R + \phi (R - \Lambda) \right) + L_m, \]  

(5)

where \( L_m \) is the usual matter Lagrangian, \( \Lambda \) is an external constant field, and \( \phi \) is a Lagrange multiplier. Such modification of the usual Hilbert Lagrangian of General Relativity in order to obtain the Rastall’s field equation. The parameter \( \gamma \) is connected with \( \Lambda \).

Moreover, besides the initial motivations for the Rastall’s theory, there are other possible connections with effective quantum models. As discussed in references [2, 3], quantum effects in curved spacetime lead to violations of the usual conservation law of the energy-momentum tensor. This phenomena is usually known as gravitational anomalies. Such gravitational anomaly may be very important, for example, in the computation of the Hawking radiation of a black hole [4].

For example, consider the effective two-dimensional metric of a black hole:

\[ ds^2 = f(r) dt^2 - f(r)^{-1} dr^2. \]  

(6)
The absence of ingoing models leads to

$$\nabla_\mu T^\mu_\nu (r_H) \equiv \Xi_\nu (r) \equiv \frac{1}{\sqrt{-g}} \partial_\mu N^\mu_\nu ,$$

where,

$$N^\mu_\nu = \frac{1}{96\pi} \epsilon^{\beta\mu} \partial_\alpha \Gamma^\alpha_{\nu\beta} \rightarrow N^r_t = \frac{1}{192\pi} \left( ff'' + f^2 \right) ,$$

with $\epsilon^{01} = \epsilon^{rr} = 1$ and $'$ is a derivative with respect to $r$.

Such interpretation of the Rastall’s theory as an classical, effective implementation of quantum effects leads to many interesting applications, as we will see in what follows. First, we will use the Rastall’s theory to implement a classical version of the Loop quantum cosmology effective equation, showing that singularity-free solutions emerge in that classical context. Second, we will apply Rastall’s theory to the description of the present universe. It will be shown that the corresponding $\Lambda$CDM model is unique when a confrontation with observations is made.

II. APPLICATION I: LQC

Among the many approaches to construct a Quantum Theory of Gravity, there is the Loop quantum gravity, based on the use of the connections as the fundamental quantities. A quantum cosmological model can be constructed from LQG, called Loop quantum cosmology, LQC. One of the predictions of LQC is the existence of an upper critical density $\rho_c$ leading to the avoidance of the cosmological singularity. This prediction is encoded in a modification of the Friedmann’s equation such that

$$H^2 = \frac{8\pi G}{3} \left( \rho - \frac{\rho^2}{\rho_c} \right) .$$

Let us introduce in the Rastall’s theory a fluid with an equation of state of the type,

$$p = \alpha \rho + \omega \rho^2 .$$

For our purposes, such equation of state is purely phenomenological. But, it is in principle possible to evoke a Bose-Einstein Condensation (BEC) origin, even if some details must be worked out in order to have a consistent scenario. With this equation of state, the equations of motion are:

$$3 \frac{\dot{a}^2}{a^2} = 4\pi G \left\{ [(3 - \gamma) + 3\alpha(\gamma - 1)]\rho + 3\omega(\gamma - 1)\rho^2 \right\} ,$$

$$\dot{\rho} = -6 \frac{\dot{a}}{a} \rho \frac{(1 + \alpha + \omega \rho)}{[(3 - \gamma) + 3\alpha(\gamma - 1) + 6\omega(\gamma - 1)\rho] .}$$

Let us consider $\alpha \neq -1$. The conservation equation takes then the form,

$$\rho^A (1 + \alpha + \omega \rho) \frac{\dot{a}}{a} = \lambda a^{-6} ,$$

with the definition,

$$A \equiv \frac{(3 - \gamma) + 3\alpha(\gamma - 1)}{1 + \alpha} , \quad B \equiv \frac{(7\gamma - 9) + 3\alpha(\gamma - 1)}{1 + \alpha} ,$$

where $\lambda$ is a positive integration constant.

The existence of the pressure term, with the equation of state, in the modified Friedmann’s equation allows to have a term similar to that predicted by LQC. This modified Friedmann’s equation obtained from the Rastall’s theory leads to many new type of solutions, describing interesting cosmological scenarios. Two of them, of particular interest, are:

1. Soft singular solutions.
2. Non-singular, bounce solution.
One example of a soft singular solution is given when we choose $\alpha \neq -1$, $\gamma = 3/2$. The solution reads:

$$\rho = \frac{-1 + \alpha}{2\omega} \left( 1 \pm \sqrt{1 + \frac{4\omega\lambda}{(1 + \alpha)^2} a^{-4}} \right), \quad a \propto t^4, \quad H \propto t^{-1}, \quad (15)$$

Remark that this solution gives a behaviour for the scale factor identical to the radiative phase of the standard model. However, $a$ cannot go to zero if $\omega$ is negative, otherwise the density becomes imaginary. The cosmic evolution begins necessarily at a $t = t_0 \neq 0$. This curious solution may contain a geodesic singularity. In some sense it remembers the sudden singularity solution [9] but for the primordial universe. It may also have some connections with the singularities found in the DBI context in reference [10]: singularities that can be crossed by a pointlike object. This may lead also to some possible similarities with the cosmic genesis project developed in reference [11].

Let us consider now $\alpha \neq -1$, $\gamma = \frac{3(\alpha + 3)}{3\alpha + 7}$. The solution reads:

$$a = \left[ \frac{2\pi G\lambda(1 + \alpha)}{\gamma - 1} t^2 - \frac{\omega\lambda}{2(1 + \alpha)} \right]^\gamma^{-1}. \quad (16)$$

If $\alpha > -1$, then $\gamma - 1$ and $1 + \alpha$ is positive. If $\omega < 0$, the scale factor varies from infinity ($t \to -\infty$) to infinity ($t \to +\infty$) with a minimum value $a = \left[ \frac{\omega\lambda}{2(1 + \alpha)} \right]^\gamma^{-1}$ and the energy density varies from zero to zero with the maximum value $\rho = 2(1 + \alpha)/|\omega|$. Hence, there is no singularity in the cosmic evolution for this case. One important advantage of such approach to the LQC effective Friedmann is that we have a full set of covariant equations, what may allow to inspect the model at perturbation level. Of course, we can not assure that all the structure of the underlying quantum model is present in this classical approach; but, some hints on the behaviour of the corresponding quantum model at linear perturbation level may be obtained.

Remark that, in order to have a singularity free solution, it is necessary to have a connection of the equation of state parameter $\alpha$ and the Rastall’s parameter $\gamma$. But, since $\gamma$ quantifies the violation of the usual conservation law, it is not impossible that it could depend on the kind of fluid considered.

III. APPLICATION II: THE PRESENT UNIVERSE

The present universe is very well described by the ΛCDM model. However, this description has some important drawbacks. Among then, there is the excess of power in the matter spectrum at small scales. Can the Rastall’s cosmology give some new insight on these problems preserving the success of the ΛCDM model?

Let us consider a two fluid model, when representing dark matter (and also baryons) and the other the dark energy component under the form of a cosmological term [12]:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \left\{ T^m_{\mu\nu} + T^x_{\mu\nu} - \frac{\gamma - 1}{2} g_{\mu\nu} (T^m + T^x) \right\}, \quad (17)$$

$$T^x_{\mu\nu,:\mu} = \frac{\gamma - 1}{2} (T_m + T_x)^{,\nu}, \quad (18)$$

$$T^m_{\mu\nu,:\mu} = 0. \quad (19)$$

The structure displayed above takes into account the fact that it is necessary to have the usual conservation law for one of the fluids (the pressureless one representing at this stage baryons and dark matter) in order to allow structure formation through gravitational collapse. The second fluid, represented by the subscript $x$, obeys the vacuum equation of state, $p_x = -\rho_x$.

The equations of motion are

$$H^2 = \frac{8\pi G}{3} \left\{ (3 - 2\gamma)\rho_x + \frac{\gamma + 3}{2}\rho_m \right\}, \quad (20)$$

$$\dot{\rho}_m + 3H\rho_m = 0, \quad (21)$$

$$\frac{3 - 2\gamma}{2}\rho_x = \frac{\gamma - 1}{2}\dot{\rho}_m, \quad (22)$$

$$\rho_m = \frac{\rho_0}{a^3}, \quad (23)$$

$$\rho_x = \frac{\rho_0}{3 - 2\gamma} + \frac{\gamma - 1}{2(3 - 2\gamma)}\rho_m. \quad (24)$$
Combining these expressions we obtain:

\[ H^2 = \frac{8\pi G}{3} (\rho_{x0} + \rho_m), \]  
(25)

\[ 2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = 8\pi G \rho_{x0}. \]  
(26)

It is the same background dynamics of the ΛCDM model. Hence, for this model, the so-called kinematical observational tests, based on the dynamics of the background, are satisfied in exactly the same way as in the ΛCDM model.

At linear perturbation level, using the standard calculations in the synchronous coordinate condition, we find the following equation for the evolution of the density contrast of the pressureless component [12].

\[ \ddot{\delta}_m + 2\frac{\dot{a}}{a}\delta_m - 4\pi G \rho_m \delta_m = 0. \]  
(27)

Again, it is the same perturbed equation as in the standard ΛCDM model. Hence, the perturbation tests based on the linear approximation, like the matter power spectrum, are identical to the ΛCDM model.

But now, there is the new relation relating the fluctuations of dark matter to the fluctuations of dark energy,

\[ \delta \rho_x = \frac{\gamma - 1}{2(3 - 2\gamma)} \delta \rho_m. \]  
(28)

Dark energy agglomerates, even if it is in the form of a vacuum energy term. This may modify the ΛCDM scenario at non-linear level, and may lead to some new scenarios at this regime where the standard model faces some difficulties.

In the previous model, it has been introduced a cosmological constant and a dark matter component. It is possible to have realistic variations around this scenario as in the usual standard model?

Let us consider the same structure as before but with a more general equation of state for the dark energy component [13]:

\[ p_x = w_x \rho_x. \]  
(29)

The previous case implies \( w_x = -1 \).

The equations of motion are now,

\[ \left(\frac{\dot{a}}{a}\right)^2 = \frac{4\pi G}{3} \left\{ \rho_x [3 - \gamma - 3(1 - \gamma)w_x] + (3 - \gamma)\rho_m \right\}, \]  
(30)

\[ \frac{\dot{a}}{a} = \frac{4\pi G}{3} \left\{ [3(\gamma - 2)w_x - \gamma] \rho_x - \gamma \rho_m \right\}, \]  
(31)

\[ \dot{\rho}_x + 3w_x \frac{\dot{a}}{a} \rho_x = \frac{\gamma - 1}{2} [(1 - 3w_x)\dot{\rho}_x + \dot{\rho}_m], \]  
(32)

\[ \dot{\rho}_m + \frac{\dot{a}}{a} \rho_m = 0. \]  
(33)

The density parameter are now given by:

\[ \rho_x = \rho_{de0} a^{-\frac{6(1+w_x)}{(3-\gamma)(1-3w_x)}} + \frac{(1 - \gamma)\rho_{m0}}{2w_x + (\gamma - 1)(1 - 3w_x)} a^{-3}, \]  
(34)

\[ \rho_m = \frac{\rho_{m0}}{a^3}. \]  
(35)

The situation now changes drastically. In principle, the background tests can be satisfactorily satisfied for some range of values of \( \gamma \) and \( w_x \), as shown in figure 1. But, at perturbation level the constraints are much more strict: a small deviation even of the order of \( 10^{-4} \) from \( w_x = -1 \) is catastrophic at perturbation level. Essentially, only the case \( w_x = -1 \) survives. Hence, in the Rastall’s cosmology the comparison to observation seems to single out the ΛCDM model, mainly due to the perturbation behaviour even at linear level.
FIG. 1. Probability distribution function for the parameters of the Rastall’s cosmology (at the left panel $\gamma$, in the middle panel the matter density parameter and at the right panel the dark energy equation of state parameter) obtained by using supernova type Ia data.

FIG. 2. Typical behaviour of the time evolution of the matter (left panel) and dark energy (right panel) density contrasts in the Rastall’s cosmology with $\omega_x \neq -1$ (red curve) and with $\omega_x = -1$ (black curve).

IV. CONCLUSIONS

Rastall’s theory is a possible mechanism to introduce in a gravity theory quantum effects in a classical, effective approach. Rastall’s theory touches one of the cornerstone of the General Relativity theory: the conservation of the energy-momentum tensor. Now, the divergence of the energy-momentum tensor is proportional to the gradient of the Ricci curvature scalar.

One natural application of the Rastall’s theory, viewed as an effective theory implementing classically the general features of quantum effects, is the primordial universe. With a suitable equation of state, it is possible to reproduce in the context of the Rastall’s cosmology, the general behaviour predicted by Loop quantum cosmology. Singularity-free cosmological models are obtained. Some types of soft singularities are also displayed.

In applying the Rastall’s cosmology to the present universe, and using a model containing pressureless matter and a cosmological term, it is possible to keep the success of the $\Lambda$CDM model, but introducing new features at non-linear perturbation level. This modified $\Lambda$CDM model is, in some sense, unique, since any deviation of this configuration, by generalizing the equation of the state of the dark energy component, leads to a complete disagreement with observations, mainly with respect to the power spectrum data.

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