A Curvature identity on a 4-dimensional Riemannian manifold

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Abstract

We give a curvature identity derived from the generalized Gauss-Bonnet formula for 4-dimensional compact oriented Riemannian manifolds. We prove that the curvature identity holds on any 4-dimensional Riemannian manifold which is not necessarily compact. We also provide some applications of the identity.

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1 Introduction

We recall that the Gauss-Bonnet formula for a compact oriented surface \( M = (M, g) \)

\[
2\pi \chi(M) = \int_M K dv_g,
\]

(1.1)

where \( \chi(M) \) is the Euler number of \( M \), \( K \) is the Gaussian curvature of \( M \) and \( dv_g \) is the volume element of \( M \). The Gaussian curvature \( K \) of \( M \) is also expressed by using the scalar curvature \( \tau \) of \( M \) as \( K = \frac{\tau}{2} \).

Now we consider any one-parameter smooth deformation \( g(t) \) of \( g \). Since the Euler number \( \chi(M) \) is a topological invariant of \( M \), from (1.1), we have

\[
0 = \left. \frac{d}{dt} \right|_{t=0} \int_M \tau(t) dv_{g(t)} = \int_M \left( - \rho^{ij} + \frac{\tau}{2} g^{ij} \right) h_{ij} dv_g
\]

(1.2)

for symmetric (0,2)-tensor field \( h_{ij} = \left. \frac{d}{dt} \right|_{t=0} g(t)_{ij} \) (for more details, refer to the next section). Thus we have

\[
\rho_{ij} = \frac{\tau}{2} g_{ij}.
\]

(1.3)

It is well-known that the equality (1.3) holds for any 2-dimensional Riemannian manifold without the compactness assumption.

Motivated by the above observation, the following question will naturally arise.
**Question.** As mentioned above, does the above phenomenon also occur for any $2n(n \geq 2)$-dimensional Riemannian manifold?

Concerning the Question, Berger [2] discussed the Gauss-Bonnet formula for the 4-dimensional compact Riemannian manifold $(M, g)$ from the variational theoretic viewpoint. From one of his results ([2], pp. 292), and taking account of the well-known fact that the Euler number $\chi(M)$ is a topological invariant of $M$, we can see that the following curvature identity holds on any 4-dimensional compact Riemannian manifold $(M, g)$:

$$\hat{R} - 2\hat{\rho} - L\rho + \tau \rho - \frac{1}{4}(|R|^2 - 4|\rho|^2 + \tau^2)g = 0.$$ (1.4)

Here,

$$\hat{R} : \hat{R}_{ij} = \sum_{a,b,c} R_{abci} R^{abcj}, \quad \hat{\rho} : \hat{\rho}_{ij} = \sum_a \rho_{ai} \rho^a_j,$$

$$L : (L\rho)_{ij} = 2 \sum_{a,b} R_{iabj} \rho^{ab},$$

where $R$ is the curvature tensor of $M$ and $\rho$ is the Ricci tensor of $M$.

It is a remarkable fact that the identity is a quadratic equation of the curvature tensor which does not involve the covariant derivatives of the curvature tensor. Recently, Labbi [8] extended the above curvature identity to the higher dimensional cases by using an elegant method. He considered the only compact case in his paper, however his equality (10) on p.178 allows us to deduce that his equation also applies to the non-compact case. One just have to use a purely algebraic computations in the ring of double forms. As a final result, we shall prove the following the Main Theorem.

**Main Theorem.** Equation (1.4) holds on any 4-dimensional Riemannian manifold.

**Remark.** We can check that equation (1.4) is valid for any 4-dimensional pseudo-Riemannian manifold with the aid of “Mathematica”.

In the present paper, first we shall review Berger’s arguments in [2] and also give a direct proof of the Main Theorem in view of its applications to the related topics.

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2 Preliminaries

In this section, we prepare some fundamental formulas derived from one-parameter deformations of Riemannian metrics and further introduce a curvature identity on a 4-dimensional compact oriented Riemannian manifold derived from the generalized Gauss-Bonnet formula by making use of those fundamental formulas.

Let $M = (M, g)$ be an $n$-dimensional Riemannian manifold and $\mathfrak{X}(M)$ the Lie algebra of all smooth vector fields on $M$. We denote the Levi-Civita connection, the curvature tensor, the Ricci tensor and the scalar curvature of $M$ by $\nabla$, $R$, $\rho$ and $\tau$, respectively. The curvature tensor is defined by $R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$ for $X$, $Y$, $Z \in \mathfrak{X}(M)$.

Now, we denote by $\mathfrak{M}(M)$ the space of all Riemannian metrics on $M$. Let $g(t) \in \mathfrak{M}(M)$ be a smooth curve through $g(0) = g$. We shall also call it a one-parameter deformation of $g$. Let $(U; x^1, \cdots, x^n)$ be a local coordinate system on a coordinate neighborhood $U$ of $M$. With respect to the natural frame $\{\partial_i = \frac{\partial}{\partial x^i}\}_{i=1,2,\cdots,n}$, we set $g(t)(\partial_i, \partial_j) = g(t)_{ij}$, $R(t)(\partial_i, \partial_j)\partial_k = R(t)_{ijk}^l \partial_l$, $\rho(t)(\partial_i, \partial_j) = \rho(t)_{ij}$, $\tau(t) = g(t)^{ij} \rho(t)_{ij}$ and $(g(t)^{ij}) = (g(t)_{ij})^{-1}$.

In particular, we have $g(0)_{ij} = g_{ij}$, $R(0)_{ijk}^l = R_{ijk}^l$, $\rho(0)_{ij} = \rho_{ij}$ and $\tau(0) = \tau$.

We set
\[
\frac{d}{dt}\bigg|_{t=0} g(t)_{ij} = h_{ij}. \tag{2.1}
\]
Then we see that the $(0, 2)$-tensor field $h = (h_{ij})$ is symmetric on $M$ and we also have
\[
\frac{d}{dt}\bigg|_{t=0} g(t)^{ij} = -h^{ij}, \tag{2.2}
\]
where we adopt the standard notational convention of tensor analysis; for example $h^{ij} = g^{ia}g^{jb}h_{ab}$ and so on. We denote by $dv_{g(t)}$ the volume element of $(M, g(t))$. Then, we have
\[
\frac{d}{dt}\bigg|_{t=0} dv_{g(t)} = \frac{1}{2} g^{ij} h_{ij} dv_g. \tag{2.3}
\]

From (2.1) and (2.2), we see that the coefficient $\Gamma(t)^k_{ij}$ of $\nabla^{(t)}$, where $\nabla^{(t)}$ denotes the Levi-Civita connection with respect to the metric $g(t)$ satisfies
\[
\frac{d}{dt}\bigg|_{t=0} \Gamma(t)^k_{ij} = \frac{1}{2} g^{ka}(\nabla_i h_{aj} + \nabla_j h_{ia} - \nabla_a h_{ij}). \tag{2.4}
\]
Thus, from (2.4), the derivatives of $R(t)^l_{ijk}$, $\rho(t)_{ij}$ and $\tau(t)$ at $t = 0$ are given respectively by
\[
\frac{d}{dt}\bigg|_{t=0} R(t)^l_{ijk} = \frac{1}{2} \left( - R_{ijk}^a h_{al}^i + R_{i ja}^l h_{k}^a 
\right.
\left. + \nabla_i \nabla_k h_{jl} - \nabla_j \nabla_k h_{il} - \nabla_i \nabla^l h_{jk} + \nabla_j \nabla^l h_{ik}\right). \tag{2.5}
\]
\[
\frac{d}{dt} |t=0| \rho(t)_{ij} = \frac{1}{2} \left( - R_{aij} h_b^a + \rho_a h_j^a + \nabla_a \nabla_j h_i^a - \nabla_i \nabla_j h_a^a - \nabla^a \nabla_a h_{ij} + \nabla_i \nabla_a h_j^a \right), \tag{2.6}
\]

\[
\frac{d}{dt} |t=0| \tau(t) = - \rho_i h^i_j + \nabla^i \nabla_j h_i^j - \nabla^i \nabla_i h^i_j. \tag{2.7}
\]

Equation (1.2) is derived from (2.3) and (2.7). We refer to [9] in detail.

In the remainder of this section, we review Berger's result [2]. Let \( M = (M, g) \) be a 4-dimensional compact oriented Riemannian manifold. Then it is well-known that the Euler number \( \chi(M) \) of \( M \) is given by the following integral formula

\[
\chi(M) = \frac{1}{32\pi^2} \int_M \{|R|^2 - 4|\rho|^2 + \tau^2\} dv_g, \tag{2.8}
\]

where \(|R|^2\) and \(|\rho|^2\) are the square norms of the curvature tensor and the Ricci tensor, respectively (namely, \(|R|^2 = - g^{ai} g^{bj} R_{aij} R_{abl}^k \) and \(|\rho|^2 = g^{ai} g^{bj} \rho_{ab} \rho_{ij} \)). It is well-known as the generalized Gauss-Bonnet formula. Since \( \chi(M) \) is a topological invariant of \( M \), it follows that

\[
0 = \frac{d}{dt} \bigg|_{t=0} \int_M \{|R(t)|^2 - 4|\rho(t)|^2 + \tau(t)^2\} dv_{g(t)}. \tag{2.9}
\]

Here, from (2.2), (2.3), (2.5) and taking account of Green’s theorem and Ricci and Bianchi identities, we get

\[
\frac{d}{dt} \bigg|_{t=0} \int_M |R(t)|^2 dv_{g(t)} = \int_M \left\{ h^{ai} g^{bj} R_{ijk} R_{abl}^k + g^{ai} h^{bj} R_{ijk} R_{abl}^k - \frac{1}{2} g^{ai} g^{bij} ( - R_{ijkl} c h^l_c + R_{ijkl} ( - R_{ab} c h^c_k + R_{abc} h^c_i + \nabla_a \nabla_i h_b^k - \nabla_b \nabla_i h_a^k \nabla_a \nabla^k_i h_k^l - \nabla_j \nabla^l_i h_k^l ) R_{abl}^k - \nabla_a \nabla^k h_{bi} + \nabla_b \nabla^k h_{al} ) + \frac{1}{2} |R|^2 g^{ij} h_{ij} \right\} dv_g. \tag{2.10}
\]

Similarly, from (2.2), (2.3) and (2.6), we get (by taking account of Green’s theorem and...
the Ricci identity)

\[
\frac{d}{dt} \int_M |\rho(t)|^2 dv_g(t) = \int_M \left\{ -h^{ai}g^{bj}\rho_{ab\rho_{ij}} - g^{ai}h^{bj}\rho_{ab\rho_{ij}} + \frac{1}{2}g^{ai}g^{bj}\left(-R_{uab}^{\vphantom{\rho}} h_{iv}^{\vphantom{a}} + \rho_{au}h_{ib}^{\vphantom{a}}
\right.
\right.
\left. + \nabla_u\nabla_v h_{ia}^{\vphantom{a}} - \nabla_a\nabla_b h_{iv}^{\vphantom{a}} - \nabla^u\nabla_a h_{ab} + \nabla_a\nabla_u h_{ib}^{\vphantom{a}}\right)_{\rho_{ij}}
\right.
\left. + \frac{1}{2}g^{ai}g^{bj}\rho_{ab}\left(-R_{uij}^{\vphantom{\rho}} h_{iv}^{\vphantom{a}} + \rho_{iu}h_{jv}^{\vphantom{a}} + \nabla_u\nabla_j h_{iv}^{\vphantom{a}} - \nabla_i\nabla_j h_{iv}^{\vphantom{a}}
\right.
\right.
\left. - \nabla^u\nabla_i h_{ij} + \nabla_i\nabla_u h_{iv}^{\vphantom{a}}\right) + \frac{1}{2}|\rho|^2g_{ij}h_{ij}\right\} dv_g
\]

(2.11)

where $\Delta$ is the Laplace-Beltrami operator acting on differentiable functions on $M$. Further, from (2.2), (2.3) and (2.7), we get

\[
\frac{d}{dt} \int_M \tau(t)^2 dv_g(t) = \int_M \left\{ 2\tau\left(-\rho_{ij}h^{ij} + \nabla^j\nabla^i h_{ij} - \nabla^i\nabla_i h_{ij}^{\vphantom{a}}\right) + \frac{1}{2}\tau^2g_{ij}h_{ij}\right\} dv_g
\]

(2.12)

From (2.9) $\sim$ (2.12), we see that the following integral formula

\[
\int_M \left\{ R_{abc}^{\vphantom{R}} R_{abc}^{\vphantom{R}} - 2\rho^{ia}\rho_{a}^{\vphantom{a} j} - 2\rho^{ab}R_{iab}^{\vphantom{R}} j
\right.
\left. + \tau\rho^{ij} - \frac{1}{4}|R|^2g_{ij} + |\rho|^2g_{ij} - \frac{\tau^2}{4}g_{ij}\right\} dv_g = 0
\]

(2.13)

holds for any symmetric $(0, 2)$-tensor field $h = (h_{ij})$. Therefore, we see that finally the curvature identity

\[
R_{abc}^{\vphantom{R}} R_{abc}^{\vphantom{R}} j - 2\rho^{ia}\rho_{a}^{\vphantom{a} j} - 2\rho^{ab}R_{iab}^{\vphantom{R}} j + \tau\rho^{ij} - \frac{1}{4}(|R|^2 - 4|\rho|^2 + \tau^2)g_{ij} = 0
\]

(2.14)

holds on $M$. Contracting (2.14) with $g_{iu}g_{ju}$ we can confirm that Main Theorem is valid for the compact case. Now, let $\{e_i\}$ be an orthonormal basis of $T_pM$ at any point $p \in M$. Then, we may rewrite (2.14) as follows:

\[
\sum_{a,b,c} R_{abc}^{\vphantom{R}} R_{abc} - 2\sum_a \rho_{ia}\rho_{ja} - 2\sum_{a,b} \rho_{ab}R_{iab}
\]

\[
+ \tau\rho_{ij} - \frac{1}{4}(|R|^2 - 4|\rho|^2 + \tau^2)\delta_{ij} = 0,
\]

(2.15)
where \( R_{ijkl} = g(R(e_i, e_j)e_k, e_l) \), \( \rho_{ij} = \rho(e_i, e_j) \). In this paper, we shall adopt the notational conventions with respect to a natural basis and an orthonormal basis alternatively for Main Theorem components of tensors.

From (2.10), (2.11) and (2.12), we see that each term of (2.9) contains the covariant derivatives of the Ricci tensor and scalar curvature but equation (2.9) no longer involves the covariant derivatives as equation (2.13). Based on this observation Berger inquired whether the similar phenomenon would hold true for the higher dimension. In [8], Labbi recently gave the positive answer to this question.

### 3 Proof of Main Theorem

In this section, we shall give a proof of Main Theorem, namely, the curvature equality (2.15) holds for any (not necessarily compact) 4-dimensional Riemannian manifold.

Let \( M = (M, g) \) be a 4-dimensional Riemannian manifold and \( \{e_i\} \) a Chern basis of \( T_pM \) at any point \( p \in M \), namely, an orthonormal basis of \( T_pM \) satisfying

\[
R_{1213} = R_{1214} = R_{1323} = R_{1314} = R_{1323} = 0
\] (3.1)

We note that a Singer-Thorpe basis for a 4-dimensional Einstein manifold is a special kind of a Chern basis [10]. From (3.1), we get

\[
\sum_{a,b,c} R_{abc1} = 2\{R_{1212} + R_{1313} + R_{1414} + R_{1234} + R_{1324} + R_{1423} + \rho_{12} + \rho_{13} + \rho_{14} \}.
\] (3.2)

Similarly, we get

\[
\sum_{a,b,c} R_{abc2} = 2\{R_{1212} + R_{2323} + R_{2424} + R_{1234} + R_{1324} + R_{1423} + \rho_{12} + \rho_{23} + \rho_{24} + 2\rho_{34} \},
\]

\[
\sum_{a,b,c} R_{abc3} = 2\{R_{1313} + R_{2323} + R_{3434} + R_{1234} + R_{1324} + R_{1423} + \rho_{13} + 2\rho_{14} + \rho_{23} + 2\rho_{24} + \rho_{34} \},
\] (3.3)

\[
\sum_{a,b,c} R_{abc4} = 2\{R_{1414} + R_{2424} + R_{3434} + R_{1234} + R_{1324} + R_{1423} + 2\rho_{12} + 2\rho_{13} + 2\rho_{14} + 2\rho_{23} + 2\rho_{24} + \rho_{34} \}.
\]

From (3.2) and (3.3), we have also

\[
|R|^2 = 4\{R_{1212} + R_{1313} + R_{1414} + R_{2323} + R_{2424} + R_{3434} + 2R_{1234} + 2R_{1324} + 2(\rho_{12} + \rho_{13} + \rho_{14} + \rho_{23} + \rho_{24} + \rho_{34}) \}.
\] (3.4)
Further, we get the following

$$\sum_a \rho_{1a}^2 = R_{1112}^2 + R_{1133}^2 + R_{1414}^2 + 2R_{1212}R_{1133} + 2R_{1133}R_{1414}$$

$$+ 2R_{1212}R_{1414} + \rho_{12}^2 + \rho_{13}^2 + \rho_{14}^2,$$

$$\sum_{a,b} \rho_{ab}R_{1ab} = R_{1112}^2 + R_{1133}^2 + R_{1414}^2 + R_{1112}R_{2323} + R_{1212}R_{2424}$$

$$+ R_{1313}R_{2323} + R_{1313}R_{3434} + R_{1414}R_{2424} + R_{1414}R_{3434}.$$  

$$\tau_{11} = 2\{R_{1112}^2 + R_{1133}^2 + R_{1414}^2 + 2R_{1212}R_{1133} + 2R_{1133}R_{1414}$$

$$+ 2R_{1212}R_{1414} + R_{1112}R_{2323} + R_{1212}R_{2424} + R_{1133}R_{3434} + R_{1313}R_{2323}$$

$$+ R_{1313}R_{2424} + R_{1313}R_{3434} + R_{1414}R_{2323} + R_{1414}R_{2424} + R_{1414}R_{3434}\},$$

$$|\rho|^2 = \rho_{11}^2 + \rho_{22}^2 + \rho_{33}^2 + \rho_{44}^2 + 2(\rho_{12}^2 + \rho_{13}^2 + \rho_{14}^2 + \rho_{23}^2 + \rho_{24}^2 + \rho_{34}^2),$$

$$= 2\{R_{1112}^2 + R_{1133}^2 + R_{1414}^2 + R_{2323}^2 + R_{2424}^2 + R_{3434}^2 + R_{1212}R_{1313}$$

$$+ R_{1313}R_{1414} + R_{1212}R_{1414} + R_{1112}R_{2323} + R_{1212}R_{2424} + R_{1313}R_{3434} + R_{1414}R_{2323}$$

$$+ R_{1414}R_{2424} + R_{1313}R_{3434} + R_{1414}R_{2424} + R_{1414}R_{3434}$$

$$+ R_{1414}R_{3434} + \rho_{12}^2 + \rho_{13}^2 + \rho_{14}^2 + \rho_{23}^2 + \rho_{24}^2 + \rho_{34}^2\},$$

$$\tau^2 = 4\{R_{1112}^2 + R_{1133}^2 + R_{1414}^2 + R_{2323}^2 + R_{2424}^2 + R_{3434}^2 + 2R_{1212}R_{1313}$$

$$+ 2R_{1212}R_{1414} + 2R_{1112}R_{2323} + 2R_{1212}R_{2424} + 2R_{1133}R_{3434}$$

$$+ 2R_{1313}R_{1414} + 2R_{1313}R_{2323} + 2R_{1313}R_{2424} + 2R_{1313}R_{3434}$$

$$+ 2R_{2424}R_{3434} + 2R_{1414}R_{2323} + 2R_{1414}R_{2424} + 2R_{1414}R_{3434}$$

$$+ 2R_{2323}R_{2424} + 2R_{2323}R_{3434}\}.$$  

From (3.2), (3.4) and (3.5), we obtain the following equality

$$\sum_{a,b,c} R_{abc}^2 - 2\sum_a \rho_{1a}^2 - 2\sum_{a,b} \rho_{ab}R_{1ab} + \tau\rho_{11} - \frac{1}{4}|R|^2 + |\rho|^2 - \frac{\tau^2}{4} = 0.$$  

(3.6)

Similarly, we obtain

$$\sum_{a,b,c} R_{abcd}^2 - 2\sum_a \rho_{da}^2 - 2\sum_{a,b} \rho_{ab}R_{dab} + \tau\rho_{dd} - \frac{1}{4}|R|^2 + |\rho|^2 - \frac{\tau^2}{4} = 0.$$  

(3.7)
for \( d = 2, 3, 4 \). Further, we get the following

\[
\sum_{a,b,c} R_{abc} R_{abc} - 2 \sum_a \rho_{1a} \rho_{2a} - 2 \sum_{a,b} \rho_{ab} R_{1ab2} + \tau \rho_{12} = 0. \tag{3.9}
\]

Thus, from (3.8), we have

\[
\sum_{a,b,c} R_{abc} R_{abc} - 2 \sum_a \rho_{1a} \rho_{2a} - 2 \sum_{a,b} \rho_{ab} R_{1ab2} + \tau \rho_{12} = 0. \tag{3.9}
\]

Similarly, we see that

\[
\sum_{a,b,c} R_{abc} R_{abc} - 2 \sum_a \rho_{1a} \rho_{2a} - 2 \sum_{a,b} \rho_{ab} R_{1ab2} + \tau \rho_{12} = 0. \tag{3.10}
\]

holds for \((i, j) = (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\). From (3.6), (3.7), (3.9) and (3.10), we have (2.15) with respect to a Chern basis \(\{\varepsilon_i\}\). Since (2.15) is a tensor equation, (2.15) is valid for any orthonormal basis. This completes the proof of Main Theorem.

As an application of our Main Theorem, we shall provide a new proof of the following classical Theorem [1].

**Corollary 3.1.** Let \(M' = (M', g')\) be any 3-dimensional Riemannian manifold, and denote by \(R'\), \(\rho'\) and \(\tau'\) the curvature tensor, the Ricci tensor and the scalar curvature of \(M'\), respectively. Then from (2.15), we can see that the following equality

\[
R'_{abcd} = \rho'_{ad} \delta_{bc} - \rho'_{ac} \delta_{bd} + \delta_{ad} \rho'_{bc} - \delta_{ac} \rho'_{bd} - \frac{\tau}{2} (\delta_{ad} \delta_{bc} - \delta_{ac} \delta_{bd}) \tag{3.11}
\]

\((1 \leq a, b, c, d \leq 3)\) holds with respect to any orthonormal basis \(\{e'_a\}\) of \(T_{p'} M'\) at any point \(p' \in M'\).
Proof. Let \( M \) be the Riemannian product of \( M' \) and a real line \( \mathbb{R} \) and \( \{e_i\} = \{e_a = e'_a, e_4 = \frac{d}{dt}\} \) be any orthonormal basis of \( p = (p', t) \in M' \times \mathbb{R} \), where \( \{e'_a\} = \{e'_1, e'_2, e'_3\} \) is an orthonormal basis of \( T_{p'}M' \). Now, setting \( i = j = 4 \) in (2.15), we can easily get the following equality
\[
\frac{1}{4} |R'|^2 - |\rho'|^2 + \frac{\tau'^2}{4} = 0. \tag{3.12}
\]
Then, we can see that the equality (3.12) is equivalent to the following
\[
\sum_{a,b,c,d} \left\{ R'_{abcd} - \rho'_ad\delta_{bc} + \rho'_ac\delta_{bd} - \delta_{ad}\rho'_{bc} + \delta_{ac}\rho'_{bd} + \frac{\tau'}{2} (\delta_{ad}\delta_{bc} - \delta_{ac}\delta_{bd}) \right\}^2 = 0. \tag{3.13}
\]
Thus, from (3.13), we have the equality (3.11). \(\square\)

An \( n \)-dimensional Einstein manifold \( M = (M, g) \) is called super-Einstein if \( M \) satisfies
\[
R^a_{bcd}R_{aef} = \frac{1}{n} |R|^2 g_{ef} \tag{3.14}
\]
[6]. We here remark that the constancy of \( |R|^2 \) follows from the condition for an \( n(\neq 4) \)-dimensional super-Einstein manifold ([4], Lemma 3.3). For a 4-dimensional super-Einstein manifold, the constancy of \( |R|^2 \) is usually required [6]. Then from Main Theorem, we have immediately the following.

Corollary 3.2. A 4-dimensional Einstein manifold satisfies the condition (3.14).

We note that Corollary 3.2 can be also proved by making use of a Singer-Thorpe basis in the 4-dimensional Einstein manifold. We shall call a Riemannian manifold \( M = (M, g) \) satisfying the condition (3.14) a weakly super-Einstein manifold. Especially, from the Corollary 3.2 we shall call a 4-dimensional Riemannian manifold \( M = (M, g) \) satisfying the condition (3.14) (with \( |R|^2 \) not necessarily constant) a weakly Einstein manifold in short. The following example shows that a weakly Einstein manifold is not necessarily Einstein.

Example. Let \( M \) be a Riemannian product manifold of 2-dimensional Riemannian manifolds \( M_1^2(c) \) and \( M_2^2(-c) \) of constant Gaussian curvatures \( c \) and \( -c \) \((c \neq 0)\), respectively. Then we can easily check that \( M \) is not Einstein. We also can easily check that \( M \) satisfies (3.14), thus \( M \) is weakly Einstein.

The detailed study of the weakly Einstein spaces is in procedure and it will be published elsewhere.
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