A linear bound on the k-rendezvous time for primitive sets of NZ matrices

Umer Azfar\textsuperscript{1}, Costanza Catalano\textsuperscript{2}, Ludovic Charlier\textsuperscript{1}, and Raphaël M. Jungers\textsuperscript{1}\textsuperscript{*}

\textsuperscript{1} ICTEAM, UCLouvain, Av. Georges Lemaîtres 4-6, Louvain-la-Neuve, Belgium.
umer.azfar,ludovic.charlier@student.uclouvain.be, raphael.jungers@uclouvain.be

\textsuperscript{2} Gran Sasso Science Institute, Viale Francesco Crispi 7, L’Aquila, Italy.
costanza.catalano@gssi.it

Abstract. A set of nonnegative matrices is called primitive if there exists a product of these matrices that is entrywise positive. Motivated by recent results relating synchronizing automata and primitive sets, we study the length of the shortest product of a primitive set having a column or a row with \( k \) positive entries (the \( k \)-RT). We prove that this value is at most linear w.r.t. the matrix size \( n \) for small \( k \), while the problem is still open for synchronizing automata. We then report numerical results comparing our upper bound on the \( k \)-RT with heuristic approximation methods.

Keywords: Primitive set of matrices, synchronizing automaton, Černý conjecture.

1 Introduction

Primitive sets of matrices. The notion of primitive matrix\textsuperscript{3}, introduced by Perron and Frobenius at the beginning of the 20th century in the theory that carries their names, can be extended to sets of matrices: a set of nonnegative matrices \( \mathcal{M} = \{M_1, \ldots, M_m\} \) is called primitive if there exists some indices \( i_1, \ldots, i_r \in \{1, \ldots, m\} \) such that the product \( M_{i_1} \cdots M_{i_r} \) is entrywise positive. A product of this kind is called a positive product and the length of the shortest positive product of a primitive set \( \mathcal{M} \) is called its exponent and it is denoted by \( \exp(\mathcal{M}) \). The concept of primitive set was just recently formalized by Protasov and Vojnov\textsuperscript{28}, but has been appearing before in different fields as in stochastic switching systems\textsuperscript{18,27} and time-inhomogeneous Markov chains\textsuperscript{17,31}. It has lately gained more importance due to its applications in consensus of discrete-time multi-agent systems\textsuperscript{8}, cryptography\textsuperscript{11} and automata theory\textsuperscript{3,13,5}.

\textsuperscript{*} R. M. Jungers is a FNRS Research Associate. He is supported by the French Community of Belgium, the Walloon Region and the Innoviris Foundation.

\textsuperscript{3} A nonnegative matrix \( M \) is primitive if there exists \( s \in \mathbb{N} \) such that \( M^s > 0 \) entrywise.
Deciding whether a set is primitive is an NP-hard problem if the set contains at least three matrices (while it is still of unknown complexity for sets of two matrices) [19], and so is computing its exponent. For sets of matrices having at least a positive entry in every row and every column (called NZ [18] or allowable matrices [10,18]), the primitivity problem becomes decidable in polynomial-time [28], although computing the exponent remains NP-hard [13]. Methods for approximating the exponent have been proposed [6] as well as upper bounds that depend just on the matrix size; in particular, if we denote with $exp_{NZ}(n)$ the maximal exponent among all the primitive sets of $n \times n$ NZ matrices, it is known that $exp_{NZ}(n) \leq (15617n^3 + 7500n^2 + 56250n - 78125)/46875$ [33]. Better upper bounds have been found for some classes of primitive sets [13,17]. The NZ condition is often met in applications and in particular in the connection with synchronizing automata.

**Synchronizing automata.** A (complete deterministic finite state) automaton is a 3-tuple $\mathcal{A} = (Q, \Sigma, \delta)$ where $Q = \{q_1, \ldots , q_n\}$ is a finite set of states, $\Sigma = \{a_1, \ldots , a_m\}$ is a finite set of input symbols (the letters of the automaton) and $\delta : Q \times \Sigma \to Q$ is the transition function. Let $i_1, i_2, \ldots , i_l \in \{1, \ldots , m\}$ be indices. Then $w = a_{i_1}a_{i_2}\ldots a_{i_l}$ is called a word and we define $\delta(q, w) = \delta(q, a_{i_1}a_{i_2}\ldots a_{i_{l-1}}, a_{i_l})$. An automaton is synchronizing if it admits a word $w$, called a synchronizing or a reset word, and a state $q$ such that $\delta(q', w) = q$ for any state $q' \in Q$. In other words, the reset word $w$ brings the automaton from every state to the same fixed state. The automaton $\mathcal{A}$ can be equivalently represented by the set of matrices $\{A_1, \ldots , A_m\}$ where, for all $i = 1, \ldots , m$ and $l, k = 1, \ldots , n$, $(A_i)_{lk} = 1$ if $\delta(q_i, a_l) = q_k$, $(A_i)_{lk} = 0$ otherwise. The action of a letter $a_l$ on a state $q_k$ is represented by the product $c_j^T A_i$, where $c_j$ is the $j$-th element of the canonical basis. Notice that the matrices $A_1, \ldots , A_m$ are binary and row-stochastic, i.e. each of them has exactly one entry equal to 1 in every row and zero everywhere else. In this representation, the automaton $\mathcal{A}$ is synchronizing if and only if there exists a product of its matrices with a column whose entries are all equal to 1 (also called an all-ones column).

The idea of synchronization is quite simple: we want to restore control over a device whose current state is unknown. For this reason, synchronizing automata are often used as models of error-resistant systems [10,7], but they also find application in other fields as in symbolic dynamics [22], in robotics [22] or in resilience of data compression [30]. For a recent survey on synchronizing automata we refer the reader to [33]. We are usually interested in the length of the shortest reset word of a synchronizing automaton $\mathcal{A}$, called its reset threshold and denoted by $rt(\mathcal{A})$. Despite determining whether an automaton is synchronizing can be done polynomial time (see e.g. [33]), computing its reset threshold is an NP-hard problem [10]. One of the most longstanding open questions in automata theory concerns the maximal reset threshold of a synchronizing automaton, presented by Černý in 1964 in his pioneering paper:

**Conjecture 1 (The Černý conjecture [33]).** Any synchronizing automaton on $n$ states has a synchronizing word of length at most $(n - 1)^2$. 
Černý also presented in [35] a family of automata having reset threshold of exactly \((n - 1)^2\), thus demonstrating that the bound in his conjecture (if true) cannot be improved. Exhaustive search confirmed the Černý conjecture for small values of \(n\) [4,34] and within certain classes of automata (see e.g. [20,32,37]), but despite a great effort has been made to prove (or disprove) it in the last decades, its validity still remains unclear. Indeed on the one hand, the best upper bound known on the reset threshold of any synchronizing \(n\)-state automaton is cubic in \(n\) [25,12,33], while on the other hand automata having quadratic reset threshold, called extremal automata, are very difficult to find and few of them are known (see e.g. [14,9,29,21]). Some of these families have been found by Ananichev et. al. [2] by coloring the digraph of primitive matrices having large exponent; this has been probably the first time where primitivity has been successfully used to shed light on synchronization.

Connecting primitivity and synchronization. The following definition and theorem establish the connection between primitive sets of binary \(n\) NZ matrices and synchronizing automata.

**Definition 1.** Let \(\mathcal{M}\) be a set of binary NZ matrices. The automaton associated to the set \(\mathcal{M}\) is the automaton \(\text{Aut}(\mathcal{M})\) such that \(A \in \text{Aut}(\mathcal{M})\) if and only if \(A\) is a binary and row-stochastic matrix and there exists \(M \in \mathcal{M}\) such that \(A \leq M\) (entrywise). We denote with \(\text{Aut}(\mathcal{M}^T)\) the automaton associated to the set \(\mathcal{M}^T = \{M^1, \ldots, M^m\}\).

**Theorem 1 ([3], Theorems 16-17).** Let \(\mathcal{M} = \{M_1, \ldots, M_m\}\) be a primitive set of binary NZ matrices. Then \(\text{Aut}(\mathcal{M})\) and \(\text{Aut}(\mathcal{M}^T)\) are synchronizing and it holds that:

\[
rt(\text{Aut}(\mathcal{M})) \leq \exp(\mathcal{M}) \leq rt(\text{Aut}(\mathcal{M})) + rt(\text{Aut}(\mathcal{M}^T)) + n - 1. \tag{1}
\]

Notice that the requirement in Theorem 1 that the set \(\mathcal{M}\) has to be made of binary matrices is not restrictive, as the primitivity property does not depend on the magnitude of the positive entries of the matrices of the set. We can thus restrict ourselves to the set of binary matrices by using the boolean product between them, that is setting for any \(A\) and \(B\) binary matrices, \((AB)_{ij} = 1\) any time that \(\sum_s A_{is}B_{sj} > 0\). In this framework, primitivity can be also rephrased as a membership problem (see e.g. [24,26]), where we ask whether the all-ones matrix belongs to the semigroup generated by the matrix set. The following example reports a primitive set \(\mathcal{M}\) of NZ matrices and the synchronizing automata \(\text{Aut}(\mathcal{M})\) and \(\text{Aut}(\mathcal{M}^T)\).

**Example 1.** Here we present a primitive set and its associated automata:

\[\mathcal{M} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} , \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right\}, \quad \text{Aut}(\mathcal{M}) = \left\{ a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} , b_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}, \quad \text{Aut}(\mathcal{M}^T) = \left\{ a' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} , b_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} , b_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\} \]

It holds that \(\exp(\mathcal{M}) = 7\), \(rt(\text{Aut}(\mathcal{M})) = 2\) and \(rt(\text{Aut}(\mathcal{M}^T)) = 3\). See also Fig. 1.

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4 A binary matrix is a matrix having entries in \{0,1\}. 
Equation (11) shows that the behavior of the exponent of a primitive set of NZ matrices is tightly connected to the behavior of the reset threshold of its associated automaton. A primitive set $\mathcal{M}$ with quadratic exponent implies that one of the automata $\text{Aut}(\mathcal{M})$ or $\text{Aut}(\mathcal{M}^T)$ has quadratic reset threshold; in particular, a primitive set with exponent greater than $2(n-1)^2 + n - 1$ would disprove the Černý conjecture. This property has been used by the authors in [5] to construct a randomized procedure for finding extremal synchronizing automata.

The synchronization problem for automata is about finding the length of the shortest word mapping the whole set of states onto one single state. We can weaken this request by asking what is the length of the shortest word mapping a set of $k \geq 2$ states onto one single state. In the matrix framework, we are asking what is the length of the shortest product having a column with $k$ positive entries. The case $k = 2$ is trivial, as any synchronizing automaton has a letter mapping two states onto one; for $k = 3$ Gonze and Jungers [15] presented a quadratic upper bound in the number of the states of the automaton while, to the best of our knowledge, the cases $k \geq 4$ are still open. Clearly, the case $k = n$ is the problem of computing the reset threshold.

In view of the connection between synchronizing automata and primitive sets, we extend the above described problem to primitive sets by introducing the $k$-rendezvous time ($k$-RT): the $k$-RT of a primitive set $\mathcal{M}$ is the length of the shortest product having a row or a column with $k$ positive entries. The following proposition shows how the $k$-RT of a primitive set $\mathcal{M}$ of NZ matrices (denoted by $rt_k(\mathcal{M})$) is linked to the length of the shortest word mapping a set of $k$ states onto one single state in the automata $\text{Aut}(\mathcal{M})$ and $\text{Aut}(\mathcal{M}^T)$ (denoted respectively by $rt_k(\text{Aut}(\mathcal{M}))$ and $rt_k(\text{Aut}(\mathcal{M}^T))$).

**Proposition 1.** Let $\mathcal{M}$ be a primitive set of $n \times n$ binary NZ matrices and let $\text{Aut}(\mathcal{M})$ and $\text{Aut}(\mathcal{M}^T)$ be the automata defined in Definition 1. Then for every $2 \leq k \leq n$, it holds that $rt_k(\mathcal{M}) = \min \{rt_k(\text{Aut}(\mathcal{M})), rt_k(\text{Aut}(\mathcal{M}^T))\}$.

**Proof.** See Appendix. $\square$

**Our contribution.** In this paper we prove that for any primitive set $\mathcal{M}$ of $n \times n$ NZ matrices, the $k$-rendezvous time $rt_k(\mathcal{M})$ is upper bounded by a linear function in $n$ for any fixed $k \leq \sqrt{n}$, problem that is still open for synchronizing automata. Our result also implies that $\min \{rt_k(\text{Aut}(\mathcal{M})), rt_k(\text{Aut}(\mathcal{M}^T))\}$ is upper bounded by a linear function in $n$ for any fixed $k \leq \sqrt{n}$, in view of Proposition 1. We then show that our technique for upper bounding $rt_k(\mathcal{M})$ cannot be much improved as it is, and so new strategies have to be implemented in order to possibly achieve better upper bounds. Finally, we report some numerical...
Definition 2. Let $\mathcal{M}$ be a finite set of $n \times n$ NZ matrices. We define the square graph of the set $\mathcal{M}$ as the labeled directed graph $SG(\mathcal{M}) = (V, E)$ where $V = \{1, \ldots, n\}$ is the set of all the products of\footnote{Since $\mathcal{M}$ is irreducible, it follows that $SG(\mathcal{M})$ is strongly connected and so, since $E$ has cardinality $n$, any pair of vertices in $SG(\mathcal{M})$ are connected by a path of length at most $n - 1$. Consider a path connecting vertex $i$ to vertex $j$ whose edges are sequentially labeled by the matrices $A_{s_i}, \ldots, A_{s_j}$ from $\mathcal{M}$ and let $B = A_{s_i} \cdots A_{s_j}$. Clearly $B \in \mathcal{M}^{n-1}$; furthermore it holds that $B_{i,j} > 0$ and so $supp(A_{s_i}) \subseteq supp((AB)_{i,j})$.} at most $d$ matrices from $\mathcal{M}$ and with $v \rightarrow w$ the directed edge leaving $v$ and entering $w$. Notice that a directed graph $SG(\mathcal{M})$ is strongly connected if and only if there exists a directed path from any vertex to any other vertex.
Definition 3. Let $\mathcal{M}$ be a primitive set of $n \times n$ NZ matrices and $2 \leq k \leq n$. We define the $k$-rendezvous time ($k$-RT) to be the length of the shortest product of matrices from $\mathcal{M}$ having a column or a row with $k$ positive entries and we denote it by $rt_k(\mathcal{M})$. We indicate with $rt_k(n)$ the maximal value of $rt_k(\mathcal{M})$ among all the primitive sets $\mathcal{M}$ of $n \times n$ NZ matrices.

Our goal is to find, for any $n \geq 2$ and $2 \leq k \leq n$, a function $B_k(n)$ such that $rt_k(n) \leq B_k(n)$.

Definition 4. Let $n$, $k$ integers such that $n \geq 2$ and $2 \leq k \leq n - 1$. We indicate with $S_k^n$ the set of all the $n \times n$ NZ matrices having every row and
column of weight \( k \) at most \( k \) and at least one column of weight exactly \( k \). For any \( A \in S^k_n \), let \( C_A \) be the set of the indices of the columns of \( A \) having weight equal to \( k \). We define \( a^k(A) = \min_{c \in C_A} \{ i : \text{supp}(A_{ci}) \not\subseteq \text{supp}(A_{ei}) \} \) and \( a^k = \min_{A \in S^k_n} a^k(A) \).

In other words, \( a^k(A) \) is the minimum over all the indices \( c \in C_A \) of the number of columns of \( A \) whose support is not contained in the support of the \( c \)-th column of \( A \). Since the matrices are NZ, it clearly holds that for any \( A \in S^k_n \), \( 1 \leq a^k \leq a^k(A) \). The following theorem shows that for every \( n \geq 2 \), we can recursively define a function \( B_k(n) \geq rt_k(n) \) on \( k \) by using the term \( a^k \).

**Theorem 2.** Let \( n \geq 2 \) integer. The following recursive function \( B_k(n) \) is such that for all \( 2 \leq k \leq n \), \( rt_k(n) \leq B_k(n) \).

\[
\begin{cases}
B_2(n) = 1 \\
B_{k+1}(n) = B_k(n) + n(1 + n - a^k)/2 \quad \text{for } 2 \leq k \leq n - 1. 
\end{cases}
\]

**Proof.** We prove the theorem by induction.

Let \( k = 2 \). Any primitive set of NZ matrices must have a matrix with a row or a column with two positive entries, as otherwise it would be made of just permutation matrices and hence it would not be primitive. This trivially implies that \( rt_2(n) = 1 \leq B_2(n) \).

Suppose now that \( rt_k(n) \leq B_k(n) \), we show that \( rt_{k+1}(n) \leq B_{k+1}(n) \). We remind that we denote with \( M^d \) the set of all the products of matrices from \( M \) having length smaller or equal than \( d \). If in \( M^{rt_k(M)+n-1} \) there exists a product having a column or a row with \( k+1 \) positive entries then \( rt_{k+1}(M) \leq rt_k(M) + n - 1 \leq B_{k+1}(n) \). Suppose now that this is not the case. This means that in \( M^{rt_k(M)+n-1} \) every matrix has all the rows and columns of weight at most \( k \). Let \( A \in M^{rt_k(M)} \) be a matrix having a row or a column of weight \( k \), and suppose it is a column. The case when \( A \) has a row of weight \( k \) will be studied later. By Lemma \( \ref{lem:primitive} \) applied on the matrix \( A \), for every \( i \in [n] \) there exists a matrix \( W_i \in M^{rt_k(M)+n-1} \) having the \( i \)-th column of weight \( k \) (and all the other columns and rows of weight \( \leq k \)). Every \( W_i \) has at least \( a^k \) (see Definition \( \ref{def:weight} \)) columns whose support is not contained in the support of the \( i \)-th column of \( W_i \); let \( c_{1i}, c_{2i}, \ldots, c_{ni} \) be the indices of these columns. Notice that any product \( B \) of matrices from \( M \) of length \( l \) such that \( B_{is} > 0 \) and \( B_{js} > 0 \) for some \( s \in [n] \) and \( j \in [a^k] \) would imply that \( W_iB \) has the \( s \)-th column of weight at least \( k+1 \) and so \( rt_{k+1}(M) \leq rt_k(M) + n - 1 + l \). We now want to minimize this length \( l \) over all \( i, s \in [n] \) and \( j \in [a^k] \); we will prove that there exists \( i, s \in [n] \) and \( j \in [a^k] \) such that \( l \leq n(n - 1 - a^k)/2 + 1 \). To do this, we consider the square graph \( SG(M) = (V, \mathcal{E}) \) (see Definition \( \ref{def:graph} \)) and the vertices

\[
(1, c_1^1), (1, c_1^2), \ldots, (1, c_{a^k}^1), (2, c_2^1), \ldots, (2, c_{a^k}^2), \ldots, (n, c_n^1), \ldots, (n, c_n^{a^k}).
\]

\(^5\) We remind that the **weight** of a nonnegative vector is the number of its positive entries.

\(^6\) We remind that the **support** of a nonnegative vector is the set of the indices of its positive entries.
By Lemma 2, for each vertex in Eq. (4) there exists a path in $SG(M)$ connecting it to a singleton. By the same lemma, a path of length $l$ from $(i, c_i^j)$ to a singleton $(s, s)$ would result in a product $B_j$ of matrices from $M$ of length $l$ such that $W_i B_j$ has the $s$-th column of weight at least $k + 1$. We hence want to estimate the minimal length among the paths connecting the vertices in Eq. (4) to a singleton. Notice that Eq. (4) contains at least $\lceil na_k^2/2 \rceil$ different elements, since each element occurs at most twice. It is clear that the shortest path from a vertex in the list (4) to a singleton does not contain any other element from that list. The vertex set $V$ of $SG(M)$ has cardinality $n(n+1)/2$ and it contains $n$ vertices of type $(s, s)$. It follows that the length of the shortest path connecting some vertex from the list (4) to some singleton is at most of $n(n+1)/2 - n - \lceil na_k^2/2 \rceil + 1 \leq n(n-1-a_k^2)/2 + 1$.

In view of what said before, we have that there exists a product $B$ of matrices from $M$ of length $n(n-1-a_k^2)/2 + 1$ and $i \in [n]$ such that $W_iB_j$ has a column of weight at least $k + 1$. Since $W_iB_j$ belongs to $M^{rt_n(M)} + n-1 + n(n-1-a_k^2)/2 + 1$, it follows that $r_{t_{k+1}}(M) \leq r_k(M) + n(n+1-a_k^2)/2 \leq B_{k+1}(n)$.

Suppose now $A \in M^{rt_n(M)}$ has a row of weight $k$. We can use the same argument as above on the matrix set $M^T$ made of the transpose of all the matrices in $M$.

Notice that the above argument stays true if we replace $a_k^2$ by a function $b(n,k)$ such that for all $n \geq 2$ and $2 \leq k \leq n-1$, $1 \leq b(n,k) \leq a_k^2$. It follows that Eq. (3) still holds true if we replace $a_k^2$ by $b(n,k)$.

## 4 Solving the recurrence

We now find an analytic expression for a lower bound on $a_k^2$ and we then solve the recurrence $a_k^2$ in Theorem 2 by using this lower bound. We then show that this is the best estimate on $a_k^2$ we can hope for.

**Lemma 3.** Let $n$, $k$ integers such that $n \geq 2$ and $2 \leq k \leq n-1$, and let $a_k^2$ as in Definition 4. It holds that $a_k^2 \geq \max\{n-k(k-1)-1, [(n-k)/k], 1\}$. 

**Proof.** We have that $a_k^2 \geq 1$ since $k \leq n-1$ and the matrices are NZ.

Let now $A \in S_k^k$ (see Definition 4) and let $a$ be one of its columns of weight $k$. Let $S = \text{supp}(a)$; by assumption, the rows of $A$ have at most $k$ positive entries, so there can be at most $(k-1)k$ columns of $A$ different from $a$ whose support is contained in $S$. Therefore, since $A$ is NZ, there must exist at least $n - k(k-1) - 1$ columns of $A$ whose support is not contained in $\text{supp}(a)$ and so $a_k^2 \geq n - k(k-1) - 1$.

Let again $A \in S_k^k$ and let $a$ be one of its columns of weight $k$. Let $S = [n]\text{supp}(a)$; $S$ has cardinality $n-k$ and since $A$ is NZ, for every $s \in S$ there exists $s' \in [n]$ such that $A_{ss'} > 0$. By assumption each column of $A$ has weight of at most $k$, so there must exist at least $\lceil (n-k)/k \rceil$ columns of $A$ different from $a$ whose support is not contained in $\text{supp}(a)$. It follows that $a_k^2 \geq \lceil (n-k)/k \rceil$. 

Since $\lceil (n-k)/k \rceil \geq (n-k)/k$, $n-k(k-1)-1 \geq (n-k)/k$ for $k \leq \sqrt{n}$ and $(n-k)/k \geq 1$ for $k \leq \lceil n/2 \rceil$, the recursion $a_k^2$ with $a_k^2$ replaced by $\max(n-k(k-1)-1, (n-k)/k, 1)$ now reads as:
Imposing the initial condition

\[ B_k(n) + n(1 + n(k - 1)/2) \]

Proposition 2. Equation (5) is fulfilled by the following function:

\[ B_k(n) + n^2/2 \]

The following proposition shows the solution of the recursion (5):

\[ \text{Proof.} \quad n \text{ is at most linear in } k, \text{ therefore, for any constant } \alpha, \beta, \gamma, \delta, \text{ we have that } C_k(n) = k^3/6 - k^2/2 + 4k/3 + \delta. \]

Imposing the initial condition \( B_2(n) = 1 \) gives finally the desired result \( B_k(n) = n(k^3 - 3k^2 + 8k - 12)/6 + 1 \).

Proposition 2. Equation (6) is fulfilled by the following function:

\[ B_k(n) = \begin{cases} \frac{n(k^3 - 3k^2 + 8k - 12) + 1}{6} & \text{if } 2 \leq k \leq \lfloor \sqrt{n} \rfloor \\ B_{[\sqrt{n}]}(n) + \frac{n(n + 2)(k - \lfloor \sqrt{n} \rfloor)}{2} - \frac{n^2}{2} \sum_{i=\lfloor \sqrt{n} \rfloor}^{k-1} \frac{1}{i} & \text{if } \lfloor \sqrt{n} \rfloor + 1 \leq k \leq \lfloor \frac{n}{2} \rfloor \\ B_{[\frac{n}{2}]}(n) + \frac{(k - \lfloor \frac{n}{2} \rfloor)n^2}{2} & \text{if } \lfloor \frac{n}{2} \rfloor + 1 \leq k \leq n \end{cases} \]

Therefore, for any constant \( k \) such that \( k \leq \sqrt{n} \), the \( k \)-rendezvous time \( r_t_k(n) \) is at most linear in \( n \).

Proof. If \( 2 \leq k \leq \lfloor \sqrt{n} \rfloor \), let \( C_k(n) = B_k(n)/n \). By Eq. (6), it holds that \( C_{k+1}(n) - C_k(n) = 1 + n(k - 1)/2 \). By setting \( C_k(n) = \alpha k^2 + \beta k + \gamma + \delta \), it follows that \( 3\alpha k^2 + (3\alpha + 2\beta)k + \alpha + \beta + \gamma = k^2/2 - k/2 + 1 \). Since this must be true for all \( k \), by equating the coefficients we have that \( C_k(n) = k^3/6 - k^2/2 + 4k/3 + \delta \). Imposing the initial condition \( B_2(n) = 1 \) gives finally the desired result \( B_k(n) = n(k^3 - 3k^2 + 8k - 12)/6 + 1 \).

If \( \lfloor \sqrt{n} \rfloor + 1 \leq k \leq \lfloor n/2 \rfloor \), let again \( C_k(n) = B_k(n)/n \). By Eq. (6), it holds that \( C_{k+1}(n) - C_k(n) = 1 + n(k - 1)/2k \) and so \( C_k(n) = C_{[\sqrt{n}]}(n) + (k - 2)(1 + n/2) - (n/2) \sum_{i=\lfloor \sqrt{n} \rfloor}^{k-1} i^{-1} \). Since \( C_{[\sqrt{n}]}(n) = B_{[\sqrt{n}]}(n)/n \), it follows that \( B_k(n) = B_{[\sqrt{n}]}(n) + (k - \lfloor \sqrt{n} \rfloor)n(n + 2)/2 - (n^2/2) \sum_{i=\lfloor \sqrt{n} \rfloor}^{k-1} i^{-1} \).

If \( \lfloor n/2 \rfloor + 1 \leq k \leq n - 1 \), by Eq. (6) it is easy to see that \( B_k(n) = B_{[n/2]}(n) + (k - \lfloor n/2 \rfloor)n^2/2 \), which concludes the proof. \( \square \)

We now show that \( a_k^n = \max\{n - k(k - 1) - 1, [(n - k)/k], 1\} \), and so we cannot improve the upper bound \( B_k(n) \) on \( r_t_k(n) \) by improving our estimate of \( a_k^n \).

Lemma 4. Let \( n, k \) integers such that \( n \geq 2 \) and \( 2 \leq k \leq n - 1 \). It holds that:

\[ 1 \leq a_k^n \leq u(n, k) := \begin{cases} n - k(k - 1) - 1 & \text{if } n - k(k - 1) - 1 \geq [(n - k)/k] \\ [(n - k)/k] & \text{otherwise} \end{cases} \]

Proof. We need to show that for every \( n \geq 2 \) and \( 2 \leq k \leq n - 1 \), there exists a matrix \( A \in S_k \) such that \( a_k^n(A) = u(n, k) \) (see Definition 4). We define the matrix \( C_{m_1 \times m_2} \) as the \( m_1 \times m_2 \) matrix having all the entries of the \( i \)-th column
Suppose that \( v^i_n \) is such that \( -\hat{v}^i_n \) is equal to 1 and all the other entries equal to 0, and the matrix \( R^m_{m_1 \times m_2} \) as the \( m_1 \times m_2 \) matrix having all the entries of the \( i \)-th row equal to 1 and all the other entries equal to 0. We indicate with \( 0_{m_1 \times m_2} \) the \( m_1 \times m_2 \) matrix having all its entries equal to zero and with \( I^m_{m \times m} \) the \( m \times m \) identity matrix. Let \( v^0_n = \lfloor (n-k)/k \rfloor + 1 \) and \( q = n \mod k \).

Suppose that \( n-k(k-1)-1 \geq \lfloor (n-k)/k \rfloor \) and set \( \alpha = n-k(k-1)-1 - \lfloor (n-k)/k \rfloor \). Then the following matrix \( \hat{A} \) is such that \( a^v_n(\hat{A}) = n-k(k-1)-1 = u(n,k) \):

\[
\hat{A} = \begin{bmatrix}
C_{1}^{k \times v^0_n} & R_1^{k \times (k-1)} & \cdots & R_{k}^{k \times (k-1)} \\
C_{2}^{k \times v^0_n} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
C_{v^0_n}^{k \times v^0_n} & 0 & \cdots & 0
\end{bmatrix}
\]

Indeed by construction, the first column of \( \hat{A} \) has exactly \( k \) positive entries. The columns of \( \hat{A} \) whose support is not contained in \( A_{i} \), for \( i = 1, \ldots, v^0_n \) and all the columns of \( D \). In total we have \( [(n-k)/k] + \alpha = n-k(k-1)-1 \) columns, so it holds that \( a^v_n(\hat{A}) = n-k(k-1)-1 \).

Suppose that \( n-k(k-1)-1 \leq \lfloor (n-k)/k \rfloor \). Then the following matrix \( \hat{A} \) is such that \( a^v_n(\hat{A}) = \lfloor (n-k)/k \rfloor = u(n,k) \):

\[
\hat{A} = \begin{bmatrix}
C_{1}^{k \times v^0_n} & R_1^{k \times (k-1)} & \cdots & R_{k}^{k \times (k-1)} & R_{k+1}^{k \times (n-v^0_n(k-2))} \\
C_{2}^{k \times v^0_n} & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
C_{v^0_n}^{k \times v^0_n} & 0 & \cdots & 0 & 0
\end{bmatrix}
\]

Indeed by construction, the first column of \( \hat{A} \) has exactly \( k \) positive entries and the columns of \( \hat{A} \) whose support is not contained in \( A_{i} \), for \( i = 1, \ldots, v^0_n \). Therefore it holds that \( a^v_n(\hat{A}) = v^0_n - 1 = \lfloor (n-k)/k \rfloor \).

\[ \square \]

## 5 Numerical results

We report here some numerical results that compare the theoretical bound \( B_k(n) \) on \( r_k(n) \), Eq. (6) with either the exact \( k \)-RT or with an heuristic approximation of the \( k \)-RT when the computation of the exact value is not computationally feasible. In Fig. 2 we compare our bound with the real \( k \)-RT of the primitive sets \( \mathcal{M}_{CPR} \) and \( \mathcal{M}_K \) reported here below:

\[
\mathcal{M}_{CPR} = \left\{ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \right\} 
\]

\[
\mathcal{M}_K = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} 
\]
The sets $\mathcal{M}_K$ and $\mathcal{M}_{CPR}$ are primitive sets of matrices that are based on the Kari automaton [19] and the Černý-Piricka-Rozenaurova automaton [36] respectively. We can see that for small values of $k$, the upper bound is fairly close to the actual value of $rt_k(\mathcal{M})$.

When $n$ is large, computing the $k$-RT for every $2 \leq k \leq n$ becomes hard, so we compare our upper bound on the $k$-RT with a method for approximating it. The Eppstein heuristic is a greedy algorithm developed by Eppstein in [10] for approximating the reset threshold of a synchronizing automaton. Given a primitive set $\mathcal{M}$ of binary NZ matrices, we can apply a slightly modified Eppstein heuristic to obtain, for any $k$, an upper bound on $rt_k(\mathcal{M})$. This modified heuristic is explained in the Appendix.

In Fig. 3 we compare our upper bound with the results of the Eppstein heuristic on the $k$-RT of the primitive sets with quadratic exponent presented by Catalano and Jungers in [5], Section 4; here we denote these sets by $\mathcal{M}_{Cn}$ where $n$ is the matrix dimension. Finally, Fig. 4 compares the evolution of our bound with the results of the Eppstein heuristic on the $k$-RT of the family $\mathcal{M}_{Cn}$ for fixed $k = 4$ and as $n$ varies. It can be noticed that the bound $B_k(n)$ does not increase very rapidly as compared to the Eppstein approximation.

6 Conclusions

In this paper we have shown that we can upper bound the length of the shortest product of a primitive NZ set $\mathcal{M}$ having a column or a row with $k$ positive entries by a linear function of the matrix size $n$, for any constant $k \leq \sqrt{n}$. We have called this length the $k$-rendezvous time ($k$-RT) of the set $\mathcal{M}$, and we have shown that the same linear upper bound holds for $\min \{ rt_k(\text{Aut}(\mathcal{M})), rt_k(\text{Aut}(\mathcal{M}^T)) \}$, where $\text{Aut}(\mathcal{M})$ and $\text{Aut}(\mathcal{M}^T)$ are the synchronizing automata defined in Definition.4 We have also showed that our technique cannot be improved as it already takes into account the worst cases, so new strategies have to be implemented in order to possibly obtain a better upper bound on $rt_k(n)$. The notion of $k$-RT
Fig. 3: Comparison between $B_k(n)$ and the Eppstein approx. of $rt_k(M)$, for $M = M_{C_{10}}$ (left) and $M = M_{C_{25}}$ (right). We recall that $B_k(n)$ is a generic bound valid for all primitive NZ sets, while the Eppstein bound is computed on each particular set.

Fig. 4: Comparison between $B_k(n)$ and the Eppstein approx. of $rt_k(M_{C_n})$ for $k = 4$. We recall that $B_k(n)$ is a generic bound valid for all primitive NZ sets, while the Eppstein bound is computed on each particular set.

for primitive sets comes as an extension to primitive sets of the one introduced for synchronizing automata. For automata, the problem whether there exists a linear upper bound on the $k$-RT for small $k$ is still open, as the only nontrivial result on the $k$-RT that appears in the literature, to the best of our knowledge, proves a quadratic upper bound on the $3$-RT [13]. We believe that our result could help in shedding light to this problem and possibly to the Černý conjecture, in view of the connection between synchronizing automata and primitive NZ sets established by Theorem 1.
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Proof of Proposition 1

We remind that the support of a nonnegative vector $v$ is the set of the indices of its positive entries and its weight is the cardinality of its support. By Definition, each matrix of $Aut(M)$ and $Aut(M^T)$ is entrywise smaller than a matrix of $M$. It follows that $rt_k(M) \leq \min\{rt_k(Aut(M)), rt_k(Aut(M^T))\}$. Let now $M = M_1 \cdots M_u$ be the product that attains the $k$-RT, that is a product of length $rt_k(M)$ having a column or a row with $k$ positive entries. Suppose that $M$ has column with $k$ positive entries: we show that $rt_k(M) \geq rt_k(Aut(M))$. Let $j$ be the index of this column and $S$ be its support. We claim that for every $r \in [u]$ we can safely set to zero some entries of $M_r$ in order to make its rows be stochastic while making sure that the final product still has the $j$-th column
with support $S$. In other words, we claim that for every $r \in [u]$ we can select a binary row-stochastic matrix $A_r \leq M_{u_r}$ (entrywise) such that the $j$-th column of the product $A_1 \cdots A_u$ has support $S$. If this is true, since by hypothesis the matrices $A_1, \ldots, A_u$ belong to $Aut(M)$, it holds that $rt_k(Aut(M)) \leq rt_k(M)$. We now prove the claim: let $D_r$ be the digraph on $n$ vertices and edge set $E_r$ such that $p \rightarrow q \in E_r$ if and only if $(M_{u_r})_{pq} > 0$. The fact that the $j$-th column of $M_{u_1} \cdots M_{u_u}$ has support $S$ means that for every $s \in S$ there exists a sequence of vertices $v^s_1, \ldots, v^s_{u+1} \in [n]$ such that:

$$v^s_1 = s,$$

$$v^s_{u+1} = j,$$

$$v^s_r \rightarrow v^s_{r+1} \in E_r \quad \forall r = 1, \ldots, u.$$

We can impose an additional property on these sequences: if at step $t$ two sequences share the same vertex, then they have to coincide for all the steps $t' > t$. More formally, if for some $t \in [u]$ we have that $v^s_t = v^{s'}_t$ for $s \neq s'$, then we set $v^s_{t'} = v^{s'}_{t'}$ for all $t' > t$ as the new sequence $v^s_1, \ldots, v^s_t, v^s_{t+1}, \ldots, v^s_{u+1}$ for vertex $s'$ fulfills all the requirements (7), (8) and (9). For every $r \in [u]$, we now remove from $E_r$ all the edges that are not of type (9). Furthermore, for every $r \in [u]$ and vertex $w \notin \{v^s_r\}_{s \in S}$, we remove from $E_r$ all the outgoing edges of $w$ but one. We call this new edge set $\tilde{E}_r$ and let $\tilde{D}_r$ be the subgraph of $D_r$ with edge set $\tilde{E}_r$. Then, for every $r \in [u]$, we set $A_r$ to be the adjacency matrix of $\tilde{D}_r$. Since $M_{u_r}$ is NZ, if $A_r$ has some zero-rows we can always add a one in each of them while preserving the property $A_r \leq M_{u_r}$. We do so in order to make $A_r$ row stochastic. By construction, for all $r \in [u]$, $A_r$ has exactly one positive entry in each row and it is entrywise smaller than $M_{u_r}$, so $A_r \in Aut(M)$. Finally, the $j$-th column of $A_1 \cdots A_u$ has support $S$ by construction.

The case when $M$ has a row with $k$ positive entries can be proved via a similar reasoning by observing that the product $M^T = M^T_{u_1} \cdots M^T_{u_u}$ has a column with $k$ positive entries, and so for every $r \in [u]$ we can select a binary matrix $B_r \leq M^T_r$ (entrywise) such that $B_r \in Aut(M^T)$ and $B_1 \cdots B_u$ has a column with $k$ positive entries. This implies that $rt_k(Aut(M^T)) \leq rt_k(M)$. □

**Description of the modified Eppstein heuristic:**

For any nodes $(i, j)$ and $(k, l)$ in $G(M)$ (see Definition 2), let us denote by $d_{G(M)}(i, j), (k, l)$ the length of the shortest path from $(i, j)$ to $(k, l)$ in $G(M)$. The modified Eppstein heuristic for primitive sets is formalized by Algorithm 1.

Since this algorithm increases the weight of the column $i$ at each iteration of the while loop, it allows us to give an upper bound on the the $k$-RT for all $k$ by producing a (reasonably short) product whose $i$-th column is of weight greater or equal than $k$. At the same time, we also check the weights of the rows of $A$ in case the algorithm happens to produce a larger row weight than the maximal column weight, thus improving the bound on the $k$-RT.

**Remark 1.** In our implementation we look for the shortest path to a specific singleton $(i, i)$, whereas it would, in general, be better to find the shortest path to
Algorithm 1: Pseudo-code algorithm for the modified Eppstein heuristic

**Input:** A primitive matrix set $\mathcal{M}$

**Output:** A matrix $A$ of elements from $\mathcal{M}$ with a positive column

1. $A \leftarrow \text{arg max}_{X \in \mathcal{M}} \max_{i \in [n]} |\text{supp}(X_{*i})|$;
2. $i \leftarrow \text{arg max}_{j \in [n]} |\text{supp}(A_{*j})|$;
3. $S \leftarrow \text{supp}(A_{*i})$

while $S \neq [n]$ do

4. $C \leftarrow \{j \in [n] : \text{supp}(A_{*j}) \not\subseteq S\}$;
5. $j^* \leftarrow \text{arg min}_{j \in C} d_{SG(\mathcal{M})}((i, j), (i, i))$; /* See Remark 1. below */
6. $A_{p_1}, \ldots, A_{p_l} \leftarrow$ labels of shortest path from $(i, j)$ to $(i, i)$;
7. $A \leftarrow A A_{p_1} \cdots A_{p_l}$;
8. $S \leftarrow \text{supp}(A_{*i})$

end

return $A$;

any singleton. However, one can show that in the case where among the columns of the matrices of the set $\mathcal{M}$ there is only one column with two positive entries, both implementations are equivalent. Since this is the case for all the matrix sets used in the numerical experiments of this paper, the choice of implementation was based on considerations of simplicity.