1. Introduction

The main purpose of this article is to lay the groundwork for [AT13], whose main result is as follows:

**Theorem 1.0.1 ([AT13, Theorem 1.3.2]).** Let \( \phi : X_1 \to X_2 \) be the blowing up of a coherent ideal sheaf \( I \) on a qe (quasi-excellent) scheme \( X_2 \) and let \( U \subset X_2 \) be the complement of the support of \( I \). Then \( \phi \) can be factored, functorially for regular surjective morphisms on \( X_2 \), into

\[
X_1 = V_0 \xrightarrow{\varphi_1} V_1 \xrightarrow{\varphi_2} \ldots \xrightarrow{\varphi_{l-1}} V_{l-1} \xrightarrow{\varphi_l} V_l = X_2,
\]

where

1. \( \phi = \varphi_l \circ \varphi_{l-1} \circ \ldots \circ \varphi_1 \circ \varphi_1 \).
2. \( V_i \) are nonsingular qe schemes.
3. \( \varphi_i \) are birational and induce isomorphisms over \( U \).
4. For every \( i = 1, \ldots, l \) either \( \varphi_i : V_{i-1} \twoheadrightarrow V_i \) or \( \varphi_{i}^{-1} : V_i \twoheadrightarrow V_{i-1} \) is a morphism given as the blowing up of a \( Z_i \), which is respectively a subscheme of \( V_i \) or \( V_{i-1} \) disjoint from \( U \).
5. For every \( i = 1, \ldots, l \), the map \( V_i \twoheadrightarrow X_2 \) is a morphism given as the blowing up of the corresponding coherent ideal sheaf \( J_i \) on \( X_2 \).

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This result generalizes the main theorems of [AKMW02] and [Wlo03], to the appropriate generality of qe schemes, and enables one to further prove factorization results in other geometric categories of interest.

Our method of proof in [AT13] follows [AKMW02] closely, and requires generalizing the foundations of group actions, quotient, and toroidal theory used in [AKMW02] from varieties to schemes.

The two main results proven here are:

1. *Luna’s Fundamental Lemma for diagonalizable group actions* 5.6.4: a regular morphism \( f : X' \to X \) of locally noetherian schemes, equivariant and \emph{inert} under a relatively affine action of a diagonalizable group is \emph{strongly equivariant}.

2. *Torification of actions on toroidal schemes* 7.5.1: if a toroidal scheme \((X, U)\) is provided with a relatively affine action of a diagonalizable group \(G\) then there exists a sequence of \(G\)-equivariant normalized blowings up with nowhere dense centers \(X_n \to \cdots \to X_0 = X\) making the action toroidal. This construction is functorial with respect to surjective regular morphisms which are compatible with the action and the toroidal structure in a strong sense, see 7.5.1(ii).

For (1), we review in Section 2 the notions of regularity and formal smoothness, and prove Theorems 2.2.9 and 2.2.12 which may be of independent interest. After reviewing groups, actions and diagonalizable groups in Section 3 we study diagonalizable group actions on affine schemes in Section 4 and continue to relatively affine actions in Section 5.

For (2), we introduce actions on toroidal schemes in Section 6 and the torification process in Section 7.

While [AT13] only requires actions of \(G_m\), we find it both convenient and fruitful to work with an arbitrary diagonalizable group.

1.1. **Mixed characteristics and further directions.** It is our goal in [AT13] to show that weak factorization works also for schemes in positive and mixed characteristics, assuming that the requisite equivariant resolution results are proven in such contexts. The present manuscript does not yet provide full foundations for such a result: a chart for a logaritmically regular scheme is given by a certain hypersurface inside a toric scheme, see [Kat94, Theorem 3.2 (2)]. This necessitates modifying our treatment in Sections 6 and 7, and as a result even Luna’s Fundamental Lemma 5.6.4 needs to be generalized. We have provided foundations through Section 4 and aim to address the remaining issues in a later version of this manuscript.

It would be interesting, perhaps in future work, to further extend Luna’s fundamental lemma and the torification procedure to other tame actions, i.e. actions having linearly reductive (or even reductive) stabilizers and affine orbits. Moreover, one may hope to extend this to tame groupoids and their quotients, that one may call tame stacks (if the stabilizers are of dimension zero then those are the tame stacks of [AOV08]).
2. Regularity and formal smoothness

In toroidal geometry of varieties one uses étale charts. Since we work with general schemes, both formally smooth and regular charts will be used as needed. We are going to recall in §2 basic facts about formal smoothness in the local case, which is the only case we will use later. The cited results are due to Grothendieck, but we will cite [Sta] in addition to [Gro67]. In addition, we will prove in Theorems 2.2.9 and 2.2.12 a splitting result for local formally smooth homomorphisms, which seems to be new, although it is close in spirit to what was known.

2.1. Definitions.

2.1.1. Regular morphisms. Regular morphisms are a generalization of smooth morphisms in situations of morphisms which are not necessarily of finite type. Following [Gro67, IV, 6.8.1] a morphism of schemes \( f: Y \to X \) is said to be regular if

- the morphism \( f \) is flat and
- all geometric fibers of \( f: X \to Y \) are regular.

2.1.2. Qe schemes. The class of quasi-excellent schemes was introduced by Grothendieck as the natural class where problems related to resolution of singularities behave well. The name “quasi-excellent” is perhaps not very elegant (it was not introduced by Grothendieck), and we feel it harmless to refer to them as qe schemes.

A locally noetherian scheme \( X \) is a qe scheme if the following two conditions hold:

- for any scheme \( Y \) of finite type over \( X \), the regular locus \( Y_{\text{reg}} \) is open; and
- for any point \( x \in X \), the completion morphism \( \text{Spec} \, \hat{\mathcal{O}}_{X,x} \to \text{Spec} \, \mathcal{O}_{X,x} \) is regular.

It is a known, but nontrivial fact, that a scheme \( Y \) of finite type over a qe scheme is also a qe scheme, see, for example, [Mat80, 34.A]. A ring \( A \) is a qe ring if \( \text{Spec} \, A \) is a qe scheme.

2.1.3. Formal smoothness of local homomorphisms. Let \( f: (A, m_A) \to (B, m_B) \) be a local homomorphism of local rings. By saying that \( f \) is formally smooth we mean that it is formally smooth with respect to the \( m_A \)-adic and \( m_B \)-adic topologies. This means that for any ring \( C \) with a square zero ideal \( I \) and compatible homomorphisms \( A \to C \) and \( B \to C/I \) that vanish on large powers of \( m_A \) and \( m_B \), respectively,

\[
\begin{array}{ccc}
A & \longrightarrow & C \\
\downarrow^f & & \downarrow \ \\
B & \longrightarrow & C/I
\end{array}
\]

there exists a lifting \( B \to C \) making the diagram commutative. Using a slightly non-standard terminology, we say that a morphism of schemes \( f: Y \to X \) is formally smooth at \( y \in Y \) if the local homomorphism \( \mathcal{O}_{X, f(y)} \to \mathcal{O}_{Y,y} \) is formally smooth.

Remark 2.1.4. Unlike regularity, formal smoothness is not a local property. Consider an example of a non-qe DVR: a DVR \( R \) such that \( \bar{K} = \text{Frac}(\bar{R}) \) is not separable over \( K = \text{Frac}(R) \). For example, set \( \bar{R} = k[[x]] \), take an element \( y \in k[[x]] \) which is
transcendental over \(k(x)\), and set \(R = \hat{R} \cap k(x, y^p)\). Then \(\hat{R}\) is, indeed, the completion of \(R\), and the homomorphism \(R \to \hat{R}\) is formally smooth but its generic fiber \(K \to \hat{K}\) is not.

2.2. Properties of formal smoothness.

2.2.1. Criteria. Here are the main criteria for formal smoothness of local homomorphisms.

**Theorem 2.2.2.** For a local homomorphism \(f : (A, m) \to (B, n)\) of noetherian local rings the following conditions are equivalent:

(i) \(f\) is formally smooth,

(ii) the completion \(\hat{f} : \hat{A} \to \hat{B}\) is formally smooth,

(ii)' the completion \(\hat{f} : \hat{A} \to \hat{B}\) is a regular homomorphism,

(iii) \(f\) is flat and its closed fiber \(f : k = A/m \to B/mB\) is formally smooth,

(iii)' \(f\) is flat and its closed fiber \(f : k = A/m \to B = B/mB\) is a regular homomorphism (i.e. \(B\) is geometrically regular over \(k\)).

**Proof.** The equivalence of (i) and (ii) is almost obvious, see [Sta, tag/07ED]. See [Sta, tag/07NQ] for the equivalence of (ii), (iii), and (iii)' and see [Sta, tag/07PM] for the equivalence of (ii) and (ii)'.

**Corollary 2.2.3.** A morphism of noetherian schemes \(f : Y \to X\) is regular if and only if it is formally smooth at all points of \(Y\).

**Proof.** Both directions are proved by the same argument. If \(f\) is formally smooth (resp. regular) at all points of \(Y\) then it is flat by part (iii) of the theorem (resp. by definition). Fix a point \(x \in X\). The fiber \(f_x : Y_x \to \text{Spec } k(x)\) is formally smooth (resp. regular) at any point \(y \in Y_x\), hence the homomorphism \(k(x) \to \mathcal{O}_{Y_x,y}\) is regular (resp. formally smooth) by the theorem. Therefore, the fibers of \(f\) are regular (resp. formally smooth at all points) and hence \(f\) itself is regular (resp. formally smooth at all points).

As another corollary, we obtain that formal smoothness behaves nicely for qe schemes. In particular, it becomes a local property (this is due to André, see [And74]).

**Corollary 2.2.4.** Assume that \(\phi : A \to B\) is a local homomorphism of noetherian rings and \(A\) is a qe ring. Then \(\phi\) is formally smooth if and only if it is regular. In particular, if \(f : Y \to X\) is a morphism of noetherian schemes with \(X\) a qe scheme, and if \(f\) is formally smooth at all closed points of \(Y\), then \(f\) is regular.

**Proof.** By Theorem 2.2.2, the completion \(\hat{\phi} : \hat{A} \to \hat{B}\) is regular. Since \(A\) is quasi-excellent, the composition \(A \to \hat{A} \to \hat{B}\) is regular. Since the completion homomorphism \(B \to \hat{B}\) is flat, \(\phi\) is regular by [Sta, tag/07NT].

2.2.5. Adic lifting property. It is shown in [Sta, tag/07NJ] that a formally smooth \(f\) satisfies a strong lifting property with respect to continuous homomorphisms to adic rings. We will need the following particular case:
Lemma 2.2.6. Let $f: A \to B$ be a formally smooth local homomorphism, let $C$ be a complete noetherian local ring, and let $I \subset C$ be any ideal. Then any pair of compatible homomorphisms of topological rings $A \to C$ and $B \to C/I$ admits a lifting $B \to C$.

Proof. Since $C$ is noetherian $I$ is a closed ideal, see [Mat89, Theorem 8.14]. Also $I$ is contained in the maximal ideal of $C$, which is an ideal of definition. Therefore the claim follows from [Sta, tag/07NJ].

Corollary 2.2.7. Let $A$ be a local ring with residue field $k$, let $B$ and $C$ be complete noetherian local $A$-algebras, and assume that $B$ is formally smooth over $A$. Then an $A$-homomorphism $g: C \to B$ is an isomorphism if and only if its closed fiber $\overline{g} = g \otimes_A k$ is an isomorphism.

Proof. Only the inverse implication needs a proof. Since $B$ and $C$ are complete with respect to their maximal ideals, they are also complete with respect to $m_A B$ and $m_A C$, respectively. The closed fiber $\overline{g}$ is surjective, hence the $n$-th fibers $C/m_A^n C \to B/m_A^n B$ are surjective by Nakayama’s lemma for nilpotent ideals, and we obtain that $g$ is surjective too. Setting $I = \text{Ker}(g)$ and using Lemma 2.2.6, we obtain that there exists a homomorphism $h: B \to C$ making the following diagram commutative:

\[
\begin{array}{ccc}
A & \to & C \\
\downarrow h & & \downarrow g \\
B & \to & C/I \\
\end{array}
\]

In other words, we have found a section $h$ of $g$. It remains to note that the closed fiber of $h$ is surjective (in fact, $\overline{h} = \overline{g}^{-1}$), so $h$ is surjective by the same argument as above.

2.2.8. Splitting of formally smooth homomorphisms. Now we are ready to prove the main result of section 2 in the equal characteristic case: any formally smooth homomorphisms between complete noetherian local rings is a base change of its closed fiber. For convenience of the exposition we will deal with the mixed characteristic case separately.

Recall that if $A$ is a complete noetherian local ring with residue field $k$, and if $A$ contains a field, then $A$ contains $k$ as a coefficient field \( i: k \hookrightarrow A \) by Cohen’s Structure Theorem, [Sta, tag/032A].

Theorem 2.2.9. Let $f: A \to B$ be a homomorphism of complete noetherian local rings with closed fiber $\overline{f}: k = A/m_A \to \overline{B} = B/m_A B$. Assume that $A$ contains a field.

(i) Assume $\overline{f}$ is formally smooth. Then there is a field of definition $i: k \to A$ and a section $j: \overline{B} \to B$ of the surjection $B \to \overline{B}$ which extends the composition $f \circ i: k \hookrightarrow A \to B$.

(ii) Assume $f$ is formally smooth. Then for any choice of such $i$ and $j$, the homomorphism $A \otimes_k \overline{B} \to B$ is an isomorphism. In particular, $f$ is (non-canonically) isomorphic to the formal base change of its closed fiber $\overline{f}$ with respect to $i$. 
Proof. To establish existence of $j$ we apply Lemma 2.2.6 to the following diagram:

\[
\begin{array}{c}
  k_f \\
  \downarrow \\
  f \circ i \\
  B \\
  \downarrow \\
  B \\
\end{array}
\]

Once $j$ is fixed, we obtain a homomorphism $g: \hat{A} \otimes_k B \to B$ of complete noetherian local $A$-algebras whose closed fiber $\mathfrak{f}: k \otimes_k B \to B$ is an isomorphism. Hence $g$ is an isomorphism by Corollary 2.2.7.

Remark 2.2.10. In a sense, Theorem 2.2.9 reduces the classification of formally smooth homomorphisms $f: A \to B$ of complete local rings to the case when the source is a field. By Theorem 2.2.2, $g: k \to C$ is formally smooth if and only if $C$ is geometrically regular over $k$. Perhaps, this is the “best” characterization of formally smooth $k$-algebras one can give in general. On the other hand, if we further assume that $K = C/m_C$ is separable over $k$ (e.g., $k$ is perfect) then a better characterization is possible: $g$ is formally smooth if and only if $C$ is of the form $K[[t_1, \ldots, t_n]]$. Indeed, since $k \to K$ is formally smooth we can extend $g$ to a field of coefficients $K \hookrightarrow C$. Then we choose $t_1, \ldots, t_n$ to be any family of regular parameters.

2.2.11. The mixed characteristic case. Recall that given a field $k$ with char($k$) = $p > 0$, a Cohen ring $C(k)$ is a complete DVR with residue field $k$ and maximal ideal $(p)$. Since $C(k)$ is formally smooth over $\mathbb{Z}_p$ (see Theorem 2.2.2), given a complete local ring $A$ with $A/m_A = k$, the homomorphism $C(k) \to A$ lifts to $f: C(k) \to A$. In fact, this argument is used in the proof of Cohen’s Structure Theorem, [Sta, tag/032A]: in the mixed characteristic case, $C(k) \to A$ is a ring of coefficients of $A$, and in the equal characteristic case $Im(f) = k$ is a field of coefficients of $A$.

Assume that $A$ is a complete local ring and $k = A/m_A$ is of characteristic $p > 0$. By [Sta, tag/07NR], any formally smooth homomorphism $g: k \to D$ admits a formally smooth lifting $f: C(k) \to E$ in the sense that $g = f \otimes_A k$. Moreover, if $f': C(k) \to E'$ is another such lifting then by formal smoothness of $f$ the homomorphism $E \to D$ lifts to a homomorphism $E \to E'$, which is necessarily an isomorphism. For this reason, we will use the notation $C(D) = E$ and $C(g) = f$.

In order to unify the notation, if $R$ is a ring containing $\mathbb{Q}$, we set $C(R) = R$, and for any homomorphism $f: R \to S$ we set $C(f) = f$. Here is the analogue of Theorem 2.2.9.

Theorem 2.2.12. Let $f: A \to B$ be a homomorphism of complete noetherian local rings with closed fiber $\mathfrak{f}: k = A/m_A \to \mathfrak{f} = B/m_AB$.

(i) Assume $\mathfrak{f}$ is formally smooth. Then there exist homomorphisms $i: C(k) \to A$ and $j: C(B) \to B$ making the following diagram commutative.
(ii) Assume $f$ is formally smooth. Then for any choice of $i$ and $j$ the homomorphism $A \hat{\otimes}_{C(k)} C(B) \to B$ is an isomorphism. In particular, $f$ is (non-canonically) isomorphic to the formal base change of a Cohen lift $C(T)$ of its closed fiber $T$.

Proof. As we saw in the beginning of Section 2.2.11, the homomorphism $C(k) \to k$ lifts to $i: C(k) \to A$. In particular, $B$ becomes a $C(k)$-algebra. Since $C(T)$ is formally smooth by Theorem 2.2.2, the $C(k)$-homomorphism $C(B) \to B$ lifts to a $C(k)$-homomorphism $j: C(B) \to B$.

Given $i$ and $j$, we obtain a homomorphism $g: A \hat{\otimes}_{C(k)} C(B) \to B$ of complete noetherian local $A$-algebras whose closed fiber $\overline{g}: k \otimes_k B \to B$ is an isomorphism. Hence $g$ is an isomorphism by Corollary 2.2.7.

3. Generalities on group scheme actions

In this section we fix some basic terminology, including group schemes, orbits, stabilizers, etc.

3.1. General groups.

3.1.1. Group schemes and actions. A $\mathbb{Z}$-flat group scheme (resp. flat $S$-group scheme) $G$ will be simply referred to as a group (resp. $S$-group). An action of a group $G$ on a scheme $X$ is given by an action morphism $\mu: G \times X \to X$ satisfying the usual compatibilities: associativity and the triviality of the unit action. Similarly an action of an $S$-group $G$ on an $S$-scheme $X$ is given by a morphism $\mu: G \times_X X \to X$ satisfying the analogous requirements. If $X$ is an $S$-scheme and $G$ is a group then an action of $G$ on $X$ is called an $S$-action if $\mu$ is an $S$-morphism. Giving such an action is equivalent to providing $X$ with an action of the $S$-group $G_S = G \times S$.

Remark 3.1.2. The projection and the action morphisms give rise to an fppf groupoid $G \times X \rightrightarrows X$, see [Sta, tag/0234], which often shows up in constructions related to the action.

3.1.3. Stabilizers. The stabilizer or inertia group of an action of $G$ on $X$ is the $X$-group scheme

$$I_X = G_X \times_{X \times X} X,$$

where the maps $G_X \to X \times X$ are the action and projection maps and $X \to X \times X$ on the right is the diagonal. This is a subgroup of the $X$-group $G_X$, which is often not flat. For any point $x \in X$, we define its stabilizer as $G_x = I_X \times_X \text{Spec}(k(x))$. 

\[\begin{array}{c}
A \xrightarrow{f} B \\
\downarrow \quad \quad \downarrow \\
C(k) \quad \quad C(B) \\
\downarrow \quad \quad \downarrow \\
k \xrightarrow{T} \overline{B}
\end{array}\]
3.1.4. Orbits. Working with varieties one usually considers only “classical” orbits of group actions, which can be characterized as orbits of closed points or locally closed orbits. When $G$ acts on a more general scheme $X$ it is more natural to take into account orbits of all points. A set-theoretic orbit of $x \in X$ is the image of the map $G \times \text{Spec}(k(x)) \to X$. This definition ignores the non-reduced structure which becomes essential when $G$ is non-reduced. For example, free and non-free actions of $\mu_p$ are distinguished by the nilpotent structure of the orbits.

In order to define scheme-theoretic orbits one should use the scheme-theoretic image, see [Har77, II, Exercise 3.11 (d)], [Sta, tag/01R6]. Let $O_x$ be the scheme-theoretic image of $G \times \text{Spec}(k(x)) \to X$, and let $O_x$ be obtained from $O_x$ by removing all proper closed subsets of the form $O_y$. We provide $O_x$ with the structure sheaf $O_{O_x}|_{O_x}$ and call it the $G$-orbit of $x$. We do not know general criteria for $O_x$ to be a scheme, but the orbits we will use below are in fact schemes. Note that in this case, $O_x$ is a limit of open subschemes of $O_x$.

3.2. Diagonalizable groups. Starting from this point, we consider only diagonalizable groups $G$. Probably, many results can be extended to the case of arbitrary linearly reductive or even reductive groups, but we do not pursue that direction.

3.2.1. The definition. By a diagonalizable group we mean a finite type diagonalizable group $G$ over $\mathbb{Z}$, see [SGA70b, VIII.1.1]. In other words, $G = D_L = \text{Spec}(\mathbb{Z}[L])$ for a finitely generated abelian group $L$. Note that for any subgroup $L' \subseteq L$ with the quotient $L'' = L/L'$, we have a natural embedding $D_{L''} \hookrightarrow D_L$ and $D_L/D_{L''} \to D_{L'}$. Moreover, this construction exhausts all subgroups and quotient groups of $D_L$.

3.2.2. Action of a diagonalizable group. Any element $d \in L$ induces a character $\chi_d: G \to \mathbb{G}_m = D_\mathbb{Z}$, and this construction identifies $L$ with the group of all characters of $G$. An action of a diagonalizable group $G = D_L$ on a scheme $X$ can also be described in the dual language by giving a comultiplication homomorphism of $O_X$-algebras $\mu^#: O_X \to O_X[L]$.

3.2.3. The affine case. If $X = \text{Spec}(A)$ is affine then the action is described by the homomorphism $\mu^#: A \to A[L]$, and it is easy to see that such a homomorphism $\mu^#$ is, indeed, a comultiplication if and only if it corresponds to an $L$-grading $A = \oplus_{n \in L} A_n$ on $A$, see [SGA70a, I.4.7.3]; if $a = \sum a_n$ with $a_n \in A_n$ then $\mu^#(a) = \sum a_n L$. In the sequel, we will identify $G$-actions on $X = \text{Spec}(A)$ with the corresponding $L$-gradings of $A$.

4. $L$-graded rings

In this section we study how a diagonalizable group $G = D_L$ acts on affine schemes $X = \text{Spec}(A)$. On the algebraic side, this corresponds to studying $L$-graded rings $A = \oplus_{n \in L} A_n$.

4.1. Coinvariants and the scheme of fixed points.
4.1. Coinvariants. Given an $L$-graded ring $A$ consider the ideal $I$ generated by all modules $A_n$ with $n \neq 0$. It is a graded ideal, and one has $I_n = A_n$ for $n \neq 0$ and $I_0 = \sum_{n \geq 0} A_n A_{-n}$. Note that $A/I = A_0/I_0$ is the maximal graded quotient of $A$ with trivial $L$-grading. We call it the ring of coinvariants of $A$ and denote $A_G$ or $A_L$.

4.1.1. The scheme of fixed points. If $X = \text{Spec}(A)$ then $X^G := \text{Spec}(A_G)$ is the maximal closed subscheme of $X$ on which the action is trivial. Obviously, $X \mapsto X^G$ is a functor on the category of affine $G$ schemes. We call it the fixed points functor, see also §5.1.10 below.

4.2. Invariants and the quotient.

4.2.1. The definition. If $A$ is an $L$-graded ring then $G = D_L$ acts on $A$, and $A_d$ is the set of all elements on which $G$ acts through $\chi_d$. In particular, the ring of invariants $A^G$ coincides with $A_0$. The quotient of $X = \text{Spec}(A)$ by the action is the scheme $X / G := \text{Spec}(A_0)$. Obviously, $X \mapsto X / G$ is a functor on the category of affine $G$ schemes. We call it the quotient functor.

4.2.2. Some properties preserved by the quotient functor. It is classical, that the properties of being reduced, integral, and normal are preserved by quotients.

**Lemma 4.2.3.** Assume that an $L$-graded ring $A = \oplus_{n \in L} A_n$ satisfies one of the following properties: reduced, an integral domain, a normal domain. Then $A_0$ satisfies the same property.

**Proof.** Only the last case needs a justification. Assume that $A$ is a normal domain and so $A_0$ is a domain. If $a, b \in A$ are such that $b \neq 0$ and $c = \frac{a}{b}$ is integral over $A_0$ then $c \in A$. Since $bc = a$ in the domain $A$, and $a, b \in A_0$, we obtain that $c$ is also homogeneous of degree zero. $\blacklozenge$

Another property preserved by quotients is being of finite type.

**Lemma 4.2.4.** Assume that an $L$-graded ring $A = \oplus_{n \in L} A_n$ is finitely generated over a subring $C \subseteq A_0$. Then $A_0$ is finitely generated over $C$.

**Proof.** We can choose homogeneous $C$-generators of $A$, and this gives a presentation of $A$ as a quotient of an $L$-graded polynomial algebra $B = C[t_1, \ldots, t_l]$ by a homogeneous ideal $I$. Then $A_n = B_n/I_n$ for any $n \in L$, and so $A_0$ is a quotient of $B_0$. Note that $B_0 = C[M]$, where $M$ is the kernel of $\phi: \mathbb{N}^l \to L$ given by $\phi(a_1, \ldots, a_l) = \sum_{i=1}^l a_i \deg(t_i)$. Since the monoid $M$ is finitely generated, $B_0$ is finitely generated over $C$, and the lemma follows. $\blacklozenge$

4.2.5. Universality of quotients. As opposed to the case of general reductive groups, the quotient is universal in the sense of [MFK94, Ch. 0, §1], regardless of the characteristic.

**Lemma 4.2.6.** Assume that $G = D_L$ acts on $X = \text{Spec}(A)$, $Y = X / G$, $Y' \to Y$ is an affine morphism, and $X' = Y' \times_Y X$. Then $X' / G = Y'$ and $(X')^G = X^G \times_X X'$. 

**Proof.** Let $Y' = \text{Spec}(B_0)$. Since $Y = \text{Spec}(A_0)$, we obtain that $X' = \text{Spec}(B)$, where $B = B_0 \otimes_{A_0} A$ with the grading $B = \oplus_{n \in L} B_0 \otimes_{A_0} A_n$. So, $B_0$ is the degree zero component of $B$, and hence $X' / G = Y'$. In addition, $A_nB = B_nB$ for any $n \in L$, hence $B_G = B \otimes_A A_G$ and we obtain the last assertion of the lemma. $\blacklozenge$
Remark 4.2.7. A much more general result is proved by Alper in [Alp08, Prop. 4.7(i)].

4.2.8. Fibers. The quotient morphism $X \to Y = X \sslash G$ is $G$-equivariant with respect to the trivial action on $Y$, hence it contracts the orbits of the action on $X$. It is well known that two orbits of closed points are mapped to the same point if and only if their closures intersect, so on the set-theoretical level one can view $Y$ as the “separated” quotient of $X$. For example, if $X$ is defined over a field then the same fact is proved for any reductive group action in [MFK94, Cor. 1.2]. In the case of a diagonalizable group, one can obtain a more precise description as follows.

**Lemma 4.2.9.** Assume that $G = D_L$ acts on $X = \text{Spec}(A)$ and let $y$ be a point of $Y = X \sslash G$. Then the fiber $X_y$ contains a single orbit $O$ which is closed in $X_y$, and this orbit belongs to the closure of any other orbit contained in $X_y$.

**Proof.** By Lemma 4.2.6 we can replace $Y$ with $y = \text{Spec}(k(y))$ and $X$ with $X_y = X \times_Y y$. So, we can assume that $Y = \text{Spec}(k)$ for a field $k$ and $X = \text{Spec}(A)$ for an $L$-graded $k$-algebra $A = \oplus_{n \in L} A_n$ with $A_0 = k$. The set $L'$ of elements $n \in L$, such that $A_n$ contains a unit of $A$, is a subgroup of $L$. Furthermore, if $n \notin L'$ then any non-zero element of $A_n$ is a unit because $A_0$ is a field. So, $I = \oplus_{n \in L \setminus L'} A_n$ is the maximal homogeneous ideal of $A$. The closure of any orbit is given by a homogeneous ideal, hence contains $O = \text{Spec}(A/I)$. It remains to observe that $O = \text{Spec}(k[L'])$ is a single orbit with stabilizer $D_{L/L'}$.

**Example 4.2.10.** If $G = \mathbb{G}_m$ then there are two types of fibers: either $X_y$ contains a single $G$-invariant closed point, or $X_y$ is a single orbit with stabilizer $\mu_n$.

Recall that a morphism $X \to Y$ is submersive if $U \subset Y$ is open if and only if its preimage is open [Sta, tag/0406]. Lemmas 4.2.6 and 4.2.9, and [MFK94, §0.2, Remarks 5 and 6] imply the following result.

**Corollary 4.2.11.** If $G = D_L$ acts on $X = \text{Spec}(A)$ then $X \sslash G$ is a universal categorical quotient and the quotient morphism $X \to X \sslash G$ is submersive.

**Example 4.2.12.** For completeness, we note that the above theory completely breaks down for non-reductive groups. The classical example is obtained when $X = \text{Spec}(A)$ is $\text{GL}_2(k)$ and $G$ is a Borel subgroup acting on $X$ on the left. Then $A^G = k$, but the categorical quotient is $\mathbb{P}^1_k$. In particular, $\text{Spec}(A^G)$ is just the affine hull of the categorical quotient.

### 4.3. Noetherian $L$-graded rings

A theorem of S. Goto and K. Yamagishi states that any noetherian $L$-graded ring is finitely generated over the subring of invariants, see [GY83]. In the case of a finite group action (not necessarily commutative) an analogous claim was recently proved by Gabber, see [IL012, Exp. IV, Prop. 2.2.3]. It seems that the work [GY83] is not widely known in algebraic geometry; at least, we have reproved the theorem (in a more complicated way!) before finding the reference. For the sake of completeness, we outline the proof of [GY83, Theorem 1.1] below.

**Proposition 4.3.1 (Goto-Yamagishi).** Assume that $A = \oplus_{n \in L} A_n$ is a noetherian $L$-graded ring. Then $A_0$ is noetherian, each $A_n$ is a finitely generated $A_0$-module, and $A$ is a finitely generated $A_0$-algebra.
Proof: If \( n \in L \) then for any \( A_0 \)-submodule \( M \subseteq A_n \) the ideal \( MA \) of \( A \) satisfies \( MA \cap A_n = M \). It follows that each \( A_n \) is a noetherian \( A_0 \)-module, so \( A_0 \) is a noetherian ring and each \( A_0 \)-module \( A_n \) is finitely generated. It remains to prove that \( A \) is finitely generated over \( A_0 \), and the case of a finite \( L \) is, now, obvious.

In general, \( L \) is a direct sum of cyclic groups, and using that \( A^{L' \oplus L''} = (A^{L'})^{L''} \), we reduce the claim to the case when \( L \) is cyclic. Thus, we can assume that \( L = \mathbb{Z} \). It suffices to prove that \( A_{\geq 0} = \oplus_{n \geq 0} A_n \) is finitely generated over \( A_0 \). Indeed, the same is then true for \( A_{<0} = \oplus_{n \leq 0} A_n \) by the symmetry, and we win.

Set \( A_+ = \oplus_{n > 0} A_n \). The homogeneous ideal \( A_+ A \) is finitely generated, so we can choose its homogeneous generators \( f_1, \ldots, f_l \in A_+ \). Note that \( n_i = \deg(f_i) > 0 \).

Set \( n = \max_i n_i \) and let \( C \subseteq A_{\geq 0} \) be the \( A_0 \)-subalgebra generated by \( A_0, \ldots, A_n \). We claim that \( C = A_{\geq 0} \), so the latter is finitely generated over \( A_0 \). Indeed, by induction on \( m > n \) we can assume that \( A_0, \ldots, A_{m-1} \) lie in \( C \). Any \( g \in A_m \) can be represented as \( \sum g_i f_i \), and the equality is preserved when we remove from each \( g_i \) the components of degree different from \( m - n_i \). Then but \( g_i \in A_{m-n_i} \subseteq C \) by the induction assumption, and so \( g \in C \). ♣

4.4. \( L \)-local rings. In this section we study \( L \)-local rings that play the role of local rings among \( L \)-graded rings.

Remark 4.4.1. In fact, one can develop an \( L \)-graded analogue of commutative algebra (and algebraic geometry) which goes rather far. See [Tem04, §1], where graded versions of fields, local rings, fields of fractions, prime ideals, spectra, valuation rings, etc., were introduced. Many formulations and arguments are extended to the graded case just by replacing “elements” (of a ring or a module) with “homogeneous elements”.

4.4.2. Maximal homogeneous ideals. By a maximal homogeneous ideal of an \( L \)-graded ring \( A \) we mean any homogeneous ideal \( m \subsetneq A \) such that no homogeneous ideal \( n \) satisfies \( m \subsetneq n \subsetneq A \). Note that \( m \) does not have to be a maximal ideal of \( A \).

4.4.3. Graded fields. An \( L \)-graded ring \( k \) is called an \( L \)-graded field if 0 is the only proper homogeneous ideal of \( A \). Equivalently, any non-zero homogeneous element of \( A \) is invertible. Note that \( m \subsetneq A \) is a maximal homogeneous ideal if and only if \( A/m \) is a graded field. Graded fields are analogues of fields in the category of graded rings. In particular, it is easy to see that any graded module over \( k \) is a free \( k \)-module, see [Tem04, Lemma 1.2].

4.4.4. \( L \)-local rings. An \( L \)-graded ring \( A \) that possesses a single maximal homogeneous ideal \( m \) will be called \( L \)-local, and we will often use the notation \((A,m)\). A homogeneous homomorphism \( \phi \colon A \to B \) of \( L \)-local rings is called \( L \)-local if it takes the maximal homogeneous ideal of \( A \) to that of \( B \). Here are a few other ways to characterize \( L \)-local rings.

Lemma 4.4.5. For an \( L \)-graded ring \( A \) the following conditions are equivalent:

(i) \( A \) is \( L \)-local.

(ii) The ring \( A_0 \) is local.

(iii) The action of \( \mathbf{D}_L \) on \( \text{Spec}(A) \) possesses a single closed orbit.
Proof. Homogeneous ideals of $A$ correspond to closed $D_L$-equivariant subsets of $\Spec(A)$, hence (i) is equivalent to (iii). On the other hand, each fiber of the quotient map $\Spec(A) \to \Spec(A_0)$ contains a single closed orbit hence (ii) is equivalent to (iii).

Example 4.4.6. A homogeneous ideal $p \subseteq A$ is called $L$-prime in [Tem04, §1] if for any two homogeneous elements $x, y$ with $xy \in p$ at least one of them lies in $p$. Inverting all homogeneous elements in $A \setminus p$ one obtains an $L$-local ring $A_{p,L}$ that we call the homogeneous localization of $A$ at $p$. Even if $p$ is prime in the usual sense, $A_{p,L}$ is usually smaller than the usual localization $A_p$.

4.4.7. Graded Nakayama’s lemma.

Proposition 4.4.8. Let $(A,m)$ be an $L$-local ring and let $M$ be a finitely generated $L$-graded $A$-module. Then $mM = M$ if and only if $M = 0$.

Proof. Note that $M = 0$ if and only if its support is empty. By Nakayama’s lemma, the latter consists of all points $x \in \Spec(A)$ such that $M(x) = M \otimes_A k(x) \neq 0$. Since the support is $D_L$-equivariant, it is given by a homogeneous ideal, and hence it is either empty or contains $V(m)$. It follows that $M = 0$ if and only if $M(x) = 0$ for any $x \in V(m)$. The latter is equivalent to vanishing of $M/mM$.

Corollary 4.4.9. Let $(A,m)$ be an $L$-local ring with residue graded field $k = A/m$, and let $M$ a finitely generated $L$-graded $A$-module. Then,

(i) A homogeneous homomorphism of $L$-graded $A$-modules $\phi: N \to M$ is surjective if and only if $\phi \otimes_A k$ is surjective.

(ii) Homogeneous elements $m_1, \ldots, m_l$ generate $M$ if and only if their images generate $M/mM$.

(iii) The minimal cardinality of a set of homogeneous generators of $M$ equals to the rank of the free $k$-module $M/mM$.

4.4.10. Equivariant Cartier divisors. As a corollary of the graded Nakayama’s lemma we can give the following characterization of equivariant divisors that will be used in §6. (The finite presentation assumption is only essential in the non-noetherian case.)

Proposition 4.4.11. Assume that $(A,m)$ is an $L$-local integral domain and $D \subset \Spec(A)$ is an equivariant finitely presented closed subscheme. Let $x$ be an arbitrary point of $V(m)$ and let $X_x = \Spec(O_{X,x})$ be the localization at $x$. Then the following conditions are equivalent:

(i) $D = V(f)$ for a homogeneous element $f \in A$,

(ii) $D$ is a Cartier divisor in $X$,

(iii) $D_x = D \times_X X_x$ is a Cartier divisor in $X_x$.

Proof. The only implication that requires a proof is (iii) $\implies$ (i). By our assumptions, $D = V(I)$ for a finitely generated homogeneous ideal $I$, so $I/mI$ is a free $A/m$-module of a finite rank $d$. Since $D_x$ is Cartier, the $k(x)$-vector space $I \otimes_A k(x) = (I/mI) \otimes_A k(x)$ is one-dimensional, and we obtain that $d = 1$. Then $I$ is generated by a single homogeneous element $f$ by Corollary 4.4.9.
4.4.12. Regular parameters. Let \((A, m)\) be an \(L\)-local ring. Then \(O = \text{Spec}(A/m)\) is the closed orbit of \(\text{Spec}(A)\) and hence \(A/m = k[L']\) for a subgroup \(L' \subseteq L\).

**Lemma 4.4.13.** Keep the above notation and assume that \(A\) is a regular ring and the torsion degree of \(L'\) is invertible in \(k\). Let \(n\) be the codimension of \(O\) in \(\text{Spec}(A)\), then there exist homogeneous elements \(t_1, \ldots, t_n \in A\) that generate \(m\).

**Proof.** By Nakayama’s lemma 4.4.9(iii), we should only check that the rank of the free \(A/m\)-module \(m/m^2\) is \(n\). Note that \(m/m^2\) defines the conormal sheaf to \(O\). But \(O\) is regular by our assumption on the torsion of \(L'\), hence the rank of the conormal sheaf is \(n\). \(\blacklozenge\)

4.5. **Strictly \(L\)-local rings.**

4.5.1. The definition. An \(L\)-local ring \((A, m)\) is called strictly \(L\)-local if \(m\) contains any \(A_n\) with \(n \neq 0\). Here are a few natural ways to reformulate this:

**Lemma 4.5.2.** Let \((A, m)\) be an \(L\)-local ring and let \(m_0 = m \cap A_0\). Then the following conditions are equivalent:

(i) \(A\) is strictly \(L\)-local.

(ii) the closed orbit of \(D_L\) on \(\text{Spec}(A)\) is a point.

(iii) \(A/m = A_0/m_0\).

(iv) \(m = m_0 \oplus (\oplus_{0 \neq n \in L} A_n)\).

4.5.3. Regularity and coinvariants.

**Lemma 4.5.4 ([Fog73, Corollary of Theorem 5.4]).** If a strictly \(L\)-local ring \((A, m)\) is regular then the ring of coinvariants \(A_L\) is regular.

**Proof.** Choose \(t_1, \ldots, t_l\) as in Lemma 4.4.13. Since \(A/m\) is a field, they form a regular family of homogeneous parameters. First, consider the case when all \(t_i\) are of degree zero. We claim that \(A = A_0\), and so \(A_L = A\) is regular. Indeed, we have that \(m = m_0 A\), where \(m_0 = m \cap A_0\). Choose \(0 \neq n \in L\). Since \(A_n \subset m\), we obtain that \(m_0 A_n = A_n\). But \(A_n\) is a finitely generated \(A_0\)-module by Proposition 4.3.1, and so \(A_n = 0\) by Nakayama’s lemma.

Assume now that the degrees are arbitrary, and reorder \(t_i\) so that \(t_1, \ldots, t_q\) are the only elements of degree 0. Then \(A' = A/(t_{q+1}, \ldots, t_l)\) is regular and we have that \(A'_L = A_L\). It remains to observe that the images of \(t_1, \ldots, t_q\) form a regular family of parameters of \(A'\), and so \(A'_L = A'\) by the above case. \(\blacklozenge\)

4.5.5. **Completion.** The completion of a noetherian strictly \(L\)-local ring is called an \(L\)-complete local ring. It admits a nice description as follows:

**Proposition 4.5.6.** Assume that \(L\) is a finitely generated abelian group and \((A, m)\) is a noetherian strictly \(L\)-local ring. Set \(m_0 = m \cap A_0\) and for each \(n \in L\) let \(\hat{A}_n\) denote the \(m_0\)-adic completion of the \(A_0\)-module \(A_n\). Then, the \(m\)-adic completion of \(A\) is isomorphic to \(\prod_{n \in L} \hat{A}_n\).

**Proof.** Write \(m_0^A = m_0 A\). We need to relate the \(m\)-adic and \(m_0^A\)-adic topologies on \(A\). We have the following:
Lemma 4.5.7.  
(1) There is a constant $b > 0$ and, for every $n \in L$ a constant
$r(n) > 0$, such that the following holds. Let $a \in A_n \cap m^N$. Then if $N \geq \sum bq + r(n)$ then $a \in (m_0^N)^q$.

(2) For each $N$ the set $\{n \in L | A_n \nsubseteq m^N\}$ is finite.

Proof of the lemma. (1) By Proposition 4.3.1, $A$ is finitely generated over $A_0$, hence we can choose homogeneous $A_0$-generators $f_1, \ldots, f_k \in A \setminus A_0$. Denote their degrees by $n_1, \ldots, n_k \in L \setminus \{0\}$ and consider the monoid homomorphism $\phi : \mathbb{N}^k \to L$ sending the $i$-th generator to $n_i$. An element of $A_n$ is a polynomial in the $f_i$ whose monomials have exponents in $\phi^{-1}(n)$. Any monomial in $\phi^{-1}(0)$ lies in $m_0$, and the claim is proven if we find $b > 0$ and show that each monomial appearing in $a \in A_n \cap m^N$ with $N \geq bq$ monomials in $m_0$. Writing $|(l_1, \ldots, l_k)| = \sum l_i$, we need to find $b > 0$ and show that each monomial $f_1^{l_1} \cdots f_k^{l_k}$ of degree $|(l_1, \ldots, l_k)| = N \geq bq$ in $f_i$ such that $\phi(l_1, \ldots, l_k) = n$ factors at least $q$ monomials lying in $m_0$. This is a combinatorial question on lattices.

Consider first the case $n = 0$. The monoid $\phi^{-1}(0)$ is finitely generated, $\phi^{-1}(0) = \langle g_1, \ldots, g_k \rangle$. Write $b = \max(|g_i|)$. If $(l_1, \ldots, l_k) \in \phi^{-1}(0)$ then it has an integer expression $(l_1, \ldots, l_k) = \sum q_i g_i$, and $q = \sum q_i \geq N/\max(|g_i|)$. So, $f_1^{l_1} \cdots f_k^{l_k}$ is the product of at least $q$ elements of the form $f^q$ in $m_0$. This gives the result in this case.

Consider the general case. Since the monoid $\mathbb{N}^k$ is noetherian, the ideal generated by $\phi^{-1}(n)$ has finitely many generators $h_1, \ldots, h_t \in \phi^{-1}(n)$. It follows that $\phi^{-1}(n) = \bigcup_i \langle h_i + \phi^{-1}(0) \rangle$. Write $r = \max(|h_i|)$. Then any $(l_1, \ldots, l_k) \in \phi^{-1}(n)$ of degree $N \geq bq + r$ can be written as $h_i + t$, with $t \in \phi^{-1}(0)$ of degree $\geq bq$, hence $t$ is the sum of at least $q$ elements of $\phi^{-1}(0)$, which gives the general case.

(2) Let $a \in A_n \subset A$ with $a \notin m^N$. Then any expression $a = \sum c_j f_1^{i_1} \cdots f_k^{i_k}$ has at least one monomial of degree $|(l_1, \ldots, l_k)| < N$. But the set of $(l_1, \ldots, l_k)$ of degree $< N$ is finite, therefore the set of $n = \phi(l_1, \ldots, l_k)$ with $|(l_1, \ldots, l_k)| < N$ is finite, as required.

We continue to prove Proposition 4.5.6. The module homomorphism $A \to A_n$ induces $A/m^N \to A_n/(m^N \cap A_n)$. Writing $q(N) = \lceil (N - r(n))/b \rceil$ we obtain by part (1) of the lemma a homomorphism $A/m^N \to A_n/m_0^q(N)A_n$, giving a homomorphism $\hat{A} \to \hat{A}_n$. Taking the product we obtain a homomorphism $\Phi : \hat{A} \to \prod \hat{A}_n$.

Note also that $\hat{A}_n = \varprojlim A_n/(m^N \cap A_n)$ since $m_0^N A_n \subset (m^N \cap A_n) \subset m_0^q(N) A_n$. An element in the kernel of $\Phi$ is represented by a sequence of elements in $A/m^N$ which necessarily map to 0 in $\prod A_n/(m^N \cap A_n)$. But $A/m^N \to \prod A_n/(m^N \cap A_n)$ is injective. Therefore $\Phi$ is injective. On the other hand by part (2) of the lemma $\prod A_n/(m^N \cap A_n)$ is a finite product, so $A/m^N \to \prod A_n/(m^N \cap A_n)$ is surjective, and so $\Phi$ is surjective.

4.5.8. Formal lci homomorphisms. Recall that a local homomorphism $\phi : A \to B$ is formally smooth if and only if so is its completion $\hat{\phi} : \hat{A} \to \hat{B}$. Analogously, we say that $\phi$ is formally lci if the completion $\hat{\phi}$ factors into a composition of a formally smooth homomorphism $\hat{A} \to C$ and a quotient $C \to C/(x_1, \ldots, x_n) = \hat{B}$, where $x_1, \ldots, x_n$ is a regular sequence of elements of $C$. 

Given an $L$-local homomorphism $\phi: (A, m_A) \to (B, m_B)$ of strictly $L$-local rings, we say that it is \textit{formally invariantly lci} if the completion $\hat{\phi}: \hat{A} \to \hat{B}$ admits an $L$-homogeneous factorization $\hat{A} \xrightarrow{\psi} \hat{C} \to \hat{B}$ with $\hat{C}$ an $L$-complete local ring and such that $\psi$ is formally smooth and $\hat{B} = C/(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ lie in $C_0$ and form a regular sequence in $C$.

4.5.9. \textit{Descent of formal smoothness.} The following lemma will be a main ingredient in the proof of Luna’s fundamental lemma.

\textbf{Lemma 4.5.10.} Let $\phi: (A, m_A) \to (B, m_B)$ be an $L$-local homomorphism of noetherian strictly $L$-local rings, and assume that the grading on $\Lambda = B/m_AB$ is trivial (i.e. $m_AB$ contains any $B_n$ with $n \neq 0$).

(i) If $\phi$ is formally smooth at $m_B$ then the homomorphism $\phi_0: A_0 \to B_0$ is formally smooth and $\phi$ is its base change.

(ii) If $\phi$ is formally invariantly lci then the homomorphism $\phi_0: A_0 \to B_0$ is formally lci and $\phi$ is its base change.

\textit{Proof.} Both cases will be dealt with simultaneously. We will first prove the formal part of the lemma: if $\Lambda$ is trivially graded and $\hat{\phi}$ is formally smooth (resp. formally invariantly lci) then $\hat{\phi}: \hat{A}_0 \to \hat{B}_0$ is formally smooth (resp. formally lci) and $\psi: \hat{A} \otimes_{\hat{A}_0} \hat{B}_0 \to \hat{B}$.

Note that since $\Lambda = B/m_AB$ is trivially graded, the same is true for $\hat{\Lambda} = \hat{B}/m_\Lambda \hat{B}$. In case (ii), fix a factorization $\hat{A} \to C \to \hat{B} = C/(x_1, \ldots, x_n)$ of $\hat{\phi}$ as in Section 4.5.8. Then $C_0 \to \hat{B}_0 = C_0/(x_1, \ldots, x_n)$ is formally lci and $\hat{C} \otimes_{\hat{C}_0} \hat{B}_0 \xrightarrow{\sim} \hat{B}$.

In addition, we claim that $\Sigma = C/m_AC$ is trivially graded. Indeed, $\Sigma/(x_1, \ldots, x_n) = \hat{\Lambda}$ is trivially graded, hence each $\Sigma_l$ with $l \neq 0$ satisfies $x_1\Sigma_l + \cdots + x_n\Sigma_l = \Sigma_l$, and by Nakayama’s lemma $\Sigma_l = 0$. Thus, we are reduced to proving the claim for $\Sigma_l$. Since $\Lambda$ is formally smooth morphism $\psi: \hat{A} \to C$, and replacing $\hat{\phi}$ with $\psi$ we can assume in the sequel that it is formally smooth.

Since $A$ is strictly $L$-local, the residue field $k = A/m_A$ is also the residue field of $A_0$. Since $\hat{\Lambda} = \hat{\Lambda}_0$, the isomorphism $\hat{B} \otimes_{\hat{A}} k \xrightarrow{\sim} \hat{\Lambda}$ induces the isomorphism $\hat{B}_0 \otimes_{\hat{A}_0} k \xrightarrow{\sim} \hat{\Lambda}$ of the weight zero components, i.e. $\lambda: k \to \hat{\Lambda}$ is the fiber of both $\hat{\phi}$ and $\hat{\phi}_0$, in particular, it is formally smooth. Now, Theorem 2.2.12(i) provides compatible morphisms $i: C(k) \to \hat{A}_0$ and $j: C(\hat{\Lambda}) \to \hat{B}_0$ making diagram (1) of that theorem commutative.

We provide $C(k)$ and $C(\hat{\Lambda})$ with the trivial gradings so that the compositions $C(k) \to \hat{A}$ and $C(\hat{\Lambda}) \to \hat{B}$ are homogeneous. By part (ii) of Theorem 2.2.12, the homomorphism $\hat{\phi}$ is the base change of $C(\Lambda): C(k) \to C(\hat{\Lambda})$, i.e. $\psi: \hat{A} \otimes_{C(k)} C(\hat{\Lambda}) \to \hat{B}$ is an isomorphism. Since $C(\hat{\Lambda})$ is trivially graded, and $\hat{A} = \prod_{n \in \Lambda} \hat{A}_n$ and $\hat{B} = \prod_{n \in \Lambda} \hat{B}_n$, by Proposition 4.5.6, the zeroth component of $\psi$ provides an isomorphism $\hat{A}_0 \otimes_{C(k)} C(\hat{\Lambda}) \xrightarrow{\sim} \hat{B}_0$. Thus, $\hat{\phi}_0: \hat{A}_0 \to \hat{B}_0$ is a formal base change of $C(\lambda)$, and we obtain that $\hat{\phi}_0$ is formally smooth and $\hat{\phi}$ is its formal base change.

Now, let us deduce the lemma. First, $\phi_0$ is formally lci (resp. formally smooth) because its completion $\hat{\phi}_0$ is. It remains to show that the homogeneous homomorphism $\psi: D = A \otimes_{A_0} B_0 \to B$ is an isomorphism. We showed that the completed homomorphism $\hat{\psi}$ is an isomorphism. From Proposition 4.5.6 we then obtain the
isomorphisms $\hat{\psi}_n: \hat{D}_n \to \hat{B}_n$ for any $n$. By Proposition 4.3.1, each homogeneous component $A_n$ (resp. $B_n$) is a finitely generated $A_0$-module (resp $B_0$ module). It follows that $D_n$ is finitely generated over $B_0$ and therefore $D_n = B_n \otimes_{B_0} \hat{B}_0$, $\hat{B}_n = B_n \otimes_{B_0} \hat{B}_0$. Thus, the base change of $\psi_n: D_n \to B_n$ with respect to the faithfully flat homomorphism $B_0 \to \hat{B}_0$ is the isomorphism $\hat{\psi}_n$, and we obtain that $\psi_n$ is an isomorphism. So, $\psi$ is an isomorphism, as claimed. ♣

5. **Luna’s fundamental lemma**

In Section 5 we study relatively affine actions of diagonalizable groups on general noetherian schemes and extend the classical Luna’s fundamental lemma to this case. To simplify the exposition we work with split groups, and indicate in Section 5.7 how the non-split case can be deduced.

5.1. **Relatively affine actions.**

5.1.1. *The definition.* An action of $G = D_L$ on a scheme $X$ is called *relatively affine* if there exists a scheme $Z$ provided with the trivial $G$-action and an affine $G$-equivariant morphism $f: X \to Z$. In this case we define the *quotient* $X \sslash G = \text{Spec}_Z(f^*(O_X)^G)$. We omit $Z$ in the notation because the quotient is categorical by the following theorem, and hence it is independent of the scheme $Z$. By definition, if $Y = X \sslash G$ is covered by affine open subschemes $Y_i$ then $X_i = Y_i \times_Y X$ form an open affine equivariant covering of $X$ and $Y_i = X_i \sslash G$. Therefore, Lemma 4.2.9 and Corollary 4.2.11 extend to the relative situation:

**Theorem 5.1.2.** Assume that a scheme $X$ is provided with a relatively affine action of a diagonalizable group $G$. Then,

(i) The morphism $X \to Y = X \sslash G$ is submersive and $Y$ is the universal categorical quotient of the action.

(ii) For each $y \in Y$, the fiber $X_y$ contains a single orbit $O$ which is closed in $X_y$, and this orbit belongs to the closure of any other orbit contained in $X_y$.

In the sequel, we will only consider relatively affine actions.

**Remark 5.1.3.** The notion of a relatively affine action is not as meaningful for non-reductive groups because it does depend on $Z$. For instance, in the situation of Example 4.2.12 we can take $Z$ to be either $\text{Spec}(k)$ or $\mathbb{P}^1_k$. For both choices, the relative quotient coincides with $Z$.

5.1.4. *Geometric quotients.* If the action of $G$ on $X$ is relatively affine and each fiber of the quotient morphism $p: X \to X \sslash G$ consists of a single orbit then we say that the quotient is *geometric* and use the notation $X/G$ instead of $X \sslash G$. This matches the terminology, but not the notation, of GIT, see [MFK94, Definition 0.6].

5.1.5. *Special orbits.* If an orbit of a $G$-action on $X$ is closed in the fiber of $X \to X \sslash G$ then we say that the orbit is *special*. Obviously, such an orbit is a scheme.
5.1.6. Local actions. We say that a relatively affine action of $G$ on a scheme $X$ is local if $X$ is quasi-compact and contains a single closed orbit.

Lemma 5.1.7. Assume we have a relatively affine action of $G = D_L$ on a scheme $X$. Then the following conditions are equivalent:

(i) The action is local.

(ii) $X$ is affine, say $X = \text{Spec}(A)$, and the $L$-graded ring $A$ is $L$-local.

(iii) The quotient $Y = X \sslash G$ is local.

Proof. A scheme is local if and only if it is quasi-compact and contains a single closed point. Therefore (i) and (iii) are equivalent. Equivalence of (ii) and (iii) was proved in Lemma 4.4.5.

5.1.8. Localization along a special orbit. Assume that $O$ is a special orbit of a relatively affine action on $X$, and $y$ is its image in $Y$. Consider the localization $Y_y = \text{Spec}(\mathcal{O}_{Y,y})$ and set $X_O = X \times_Y Y_y$. We call $X_O$ the equivariant localization of $X$ along $O$. Note that $X_O \sslash G = Y_y$ by universality of the quotient, in particular, $G$ acts locally on $X_O$.

Remark 5.1.9. (i) Set-theoretically, $X_O$ consists of all orbits whose closure contains $O$. So, even if $O = \{x\}$ is a closed point, it typically happens that $X_O$ is larger than the localization of $X$ at $x$. Equivariant localization of a scheme $X = \text{Spec}(A)$ corresponds to homogeneous localization of $A$ in the sense of Example 4.4.6.

(ii) Even if $O$ is only locally closed in the fiber of $X \to X \sslash G$, one can define an equivariant localization $X_O \hookrightarrow X$ whose only closed orbit is $O$. We will not use this construction. If $O$ is not special then the localization morphism $X_O \to X$ is not inert (see §5.5 below), and the morphism $X_O \sslash G \to Y$ can be bad (e.g. not flat).

5.1.10. The schemes of fixed points. If $X$ is acted on by $G$ then the scheme of fixed points of $X$ is the maximal closed subscheme $X^G$ of $X$ such that $X^G$ is equivariant and the action of $G$ on it is trivial. In other words, it is the maximal closed subscheme over which the inclusion $I_X \hookrightarrow G \times X$ becomes an isomorphism. For diagonalizable groups, existence and functoriality of fixed points schemes is guaranteed by [SGA70b, VIII.6.5(a)], where one sets $Y = G$, $Z = X \times X$ and $Z' \subset Z$ the diagonal. If $X = \text{Spec}(A)$ is affine then, as we noted in §4.1.2, $X^G = \text{Spec}(A_G)$.

5.1.11. Inertia stratification. If $G$ is diagonalizable and $G' \subseteq G$ is a subgroup then

$$X(G') := X^{G'} \setminus \bigcup_{G'' \supseteq G'} X^{G''}$$

is the maximal $G$-equivariant subscheme $Y$ with constant inertia equal to $G'$ (i.e. such that $I_Y = G' \times Y$). The family of subschemes $\{X(G')\}_{G' \subseteq G}$ provides a $G$-equivariant stratification of $X$ that we call the inertia stratification. Set-theoretically, this is the stratification of $X$ by the stabilizers of points.

5.1.12. Regularity. The fixed points functor preserves regularity, see [Fog73, Corollary of Theorem 5.4]. We provide a simple proof in our situation.

Proposition 5.1.13. Assume that a diagonalizable group $G$ acts on a regular scheme $X$. Then the scheme of fixed points $X^G$ is regular. In particular, the strata of the inertia stratification of $X$ are regular.
Proof. The claim is local at a point \( x \in X^G \hookrightarrow X \). Since \( x \) is \( G \)-invariant, we can replace \( X \) with the equivariant localization along \( x \). Then \( X \) is the spectrum of a strictly \( L \)-local ring, and it remains to use Lemma 4.5.4. \( \diamondsuit \)

5.1.14. The case of \( \mathbb{G}_m \). We discuss a construction specific to \( G = \mathbb{G}_m = \mathbb{D}_L \); we indicate the general case in Remark 5.1.15 below. Assume that \( X \) is provided with a relatively affine action of \( G \). Following [Wlo00] we define \( X_+ \) (resp. \( X_- \)) to be the open subscheme obtained by removing all orbits that have a limit at \(+\infty\) (resp. \(-\infty\)). The construction of \( X_\pm \) is local on \( X/\sslash G \) and if the latter is affine, say \( X = \text{Spec} \ A \) and \( X/\sslash G = \text{Spec} \ A_0 \), then \( X_+ = X \setminus V(A_-) \) with \( A_- = \oplus_{n<0} A_n \).

Similarly, \( X_- = X \setminus V(A_+) \) for \( A_+ = \oplus_{n>0} A_n \).

Remark 5.1.15 (see [Tha96]). For arbitrary \( G = \mathbb{D}_L \) and affine \( X = \text{Spec} \ A \) one defines \( X \sslash_m G = \text{Proj} \ A[z]^G \) with \( G \) acting on \( z \) via \( -m \in L \). Then there is a locally finite decomposition \( \Sigma = \bigsqcup \sigma_i^G \) of the monoid \( \Sigma \subset L \) of characters figuring in \( A \) into relative interiors of polyhedral cones, and \( X \sslash_m G = X \sslash_{\sigma_i^G} G \) is constant on such interiors of cones. The replacement of \( X_\pm \) is \( X_{\text{ss}}(m) \), the semistable locus of \( X \) with respect to the linearization by \( m \).

5.2. Basic properties of the quotient functor. In this section, we study the quotient map \( X \to X/\sslash G \) and properties of schemes and morphisms preserved by the quotient functor. The most subtle result is that noetherianity is preserved by quotients.

5.2.1. Quotients of schemes.

Theorem 5.2.2. Assume that a diagonalizable group \( G = \mathbb{D}_L \) acts trivially on a scheme \( S \) and an \( S \)-scheme \( X \) is provided with a relatively affine action of \( G \), then:

(i) Assume that \( X \) satisfies one of the following properties: (a) reduced, (b) integral, (c) normal with finitely many connected components, (d) locally of finite type over \( S \), (e) of finite type over \( S \), (f) quasi-compact over \( S \), (g) locally noetherian, (h) noetherian. Then \( X/\sslash G \) satisfies the same property.

(ii) If \( X \) is locally noetherian then the quotient morphism \( X \to X/\sslash G \) is of finite type.

Proof. Note that \( X \to S \) factors through \( Y = X/\sslash G \) because the latter is the categorical quotient by Theorem 5.1.2. Claim (f) is obvious since \( X \to Y \) is onto. Furthermore, a morphism is of finite type if and only if it is quasi-compact and locally of finite type, hence (e) follows from (d) and (f). It remains to prove all assertions except (e) and (f). As shown in [MFK94, Section 2 of Chapter 0], claims (a), (b) and (c) hold for any categorical quotient, hence they are implied by Theorem 5.1.2(i). Note that the assertion of (d) is local on \( S \) hence we can assume that \( S \) is affine. Furthermore, all assertions of the theorem we are dealing with are local on \( Y \), so we can assume that \( Y \) is affine. In this case, (d) was proven in Lemma 4.2.4, and (g), (h) and (ii) were proven in Proposition 4.3.1. \( \diamondsuit \)

5.2.3. Quotients of morphisms. We start with the following corollary of the above theorem.
Corollary 5.2.4. Assume that locally noetherian schemes $X$ and $X'$ are provided with relatively affine actions of a diagonalizable group $G$, and $f : X' \to X$ is a $G$-equivariant morphism. If $f$ is of finite type then the quotient morphism $f // G$ is of finite type.

Proof. The morphism $X \to X // G$ is of finite type by Theorem 5.2.2(ii). Hence the composition $X' \to X // G$ is of finite type, and then $X' // G \to X // G$ is also of finite type by Theorem 5.2.2(i)(d).

Proposition 5.2.5. Let $G$ be a diagonalizable group and $f : X' \to X$ a $G$-equivariant morphism such that the actions on $X$ and $X'$ are relatively affine. Then,

(i) If $f$ satisfies one of the following properties: (a) affine, (b) integral, (c) a closed embedding, then $f // G$ satisfies the same property.

(ii) If $X$ is locally noetherian and $f$ is finite then $f // G$ is finite.

Proof. The claim is local on $Y = X // G$, hence we can assume that $X = \text{Spec}(A)$ and $Y = \text{Spec}(A_0)$ are affine. Since $f$ is affine in each case, $X' = \text{Spec}(A')$ is affine, and so $Y' = \text{Spec}(A'_0)$ is affine. This proves (a). If $A \to A'$ is onto then $A_0 \to A'_0$ is onto and we obtain (c).

If $A \to A'$ is integral then any $x \in A'$ satisfies an integral equation $x^n = a_1 x^1 + \cdots + a_n x^n$ with coefficients in $A$. If $x \in A'_0$ then replacing each $a_i$ with its component of degree zero we obtain an integral equation for $x$ with coefficients in $A_0$. Thus, $A'_0$ is integral over $A_0$ and we obtain (b). Finally, (ii) follows from (b) and Corollary 5.2.4.

5.2.6. Some bad examples. Basic properties of $G$-morphisms, including, smoothness and separatedness, are not preserved by the quotient functor. Here are some classical bad examples.

Example 5.2.7. (i) Let $k$ be a field with char $k \neq 2$ and let $X = \text{Spec}(k[x, y])$ be an affine plane with $\mu_2$ acting by changing the sign of $x$ and $y$. Then $Y = X // \mu_2$ is the quotient singularity $Y = \text{Spec}(k[x^2, xy, y^2])$ and the quotient morphism $X \to Y$ is not even flat. On the other hand, set $X' = X \times \mu_2$ with $\mu_2$ acting diagonally. The projection $X' \to X$ is a split étale covering which is not invert above the origin $O \in X$ because $O$ is fixed by $\mu_2$ and the action of $\mu_2$ on $X'$ is free. Moreover, $Y' = X' // \mu_2 \to X$, so the quotient map $f // \mu_2$ is the projection $X' \to Y$. Although $f$ is split étale, its quotient is not even flat.

(ii) Similarly, it is easy to construct a $\mathbb{G}_m$-action on $X = \mathbb{A}^3_k$ such that the quotient $Y = X // \mathbb{G}_m$ is not smooth (e.g., if $X = \text{Spec}(A)$ for $A = k[x, y, z]$ with $x, y \in A_1$ and $z \in A_{-2}$, then $Y = \text{Spec}(k[x^2z, xyz, y^2z])$ is an orbifold). Then automatically the morphism $X \to Y$ is not flat, and setting $X' = X \times \mathbb{G}_m$ we obtain another bad example, where $X' \to X$ is smooth but the morphism between $X' // \mathbb{G}_m = X$ and $X // \mathbb{G}_m$ is not flat.

(ii) Let $k$ be a field and $G = \mathbb{G}_m = \mathbb{D}_z$. Consider the $\mathbb{Z}$-graded $k$-algebra $A = k[x, y, z]$ with $x \in A_1$ and $y, z \in A_{-1}$, and set $X = \text{Spec}(A) = \mathbb{A}^3_k$. Then $Y = X // G$ equals to $\text{Spec}(k[xz, xy]) = \mathbb{A}^2_k$ and the orbits over the origin are: the origin, the punctured $(z)$-axis, all punctured lines through the origin in the $(yz)$-plane. Consider the equivariant subspace $X' = \text{Spec}(k[x, y, z^\pm 1])$. The open embedding $X' \to X$ preserves stabilizers but takes some closed orbits to non-closed orbits. For example, the punctured $(z)$-axis is closed in $X'$. The quotient $Y' = X' // G = \text{Spec}(k[xz, \frac{y}{z}])$ is not flat over $Y$. In fact, it is an affine chart of the
blowing up of $Y$ at the origin, and the quotient map $X' \to Y'$ separates all orbits of $X$ sitting over the origin of $Y$ and contained in $X'$.

(ii) Analogous examples related to cobordisms will play a crucial role in the proof of the factorization theorem: $(\mathcal{B}_a)_+ \to \mathcal{B}_a$ is an open embedding preserving the stabilizers (see [AT13, §3.4]), but $(\mathcal{B}_a)_+ / \mathcal{G} \to \mathcal{B}_a / \mathcal{G}$ is usually a non-trivial modification.

(iii) Let $\mathcal{G} = \mathbb{G}_m$ act on $T = \text{Spec}(k[x,y])$ as $x \mapsto tx$ and $y \mapsto t^{-1}y$, and let $X'$ be obtained from $T$ by removing the origin. Then $X' / \mathcal{G}$ is an affine line with doubled origin. In particular, the morphism $f : X' \to \text{Spec}(k)$ is separated but its quotient is not.

5.3. Strongly equivariant morphisms. The situation improves drastically if one considers quotients of a more restrictive class of strongly equivariant morphisms.

5.3.1. The definition. We say that a $\mathcal{G}$-morphism $f : X' \to X$ is strongly equivariant if the actions are relatively affine and $f$ is base change of its quotient $f / \mathcal{G}$, i.e. the morphism $\phi : X' \to X \times_Y Y'$ is an isomorphism, where $Y = X / \mathcal{G}$ and $Y' = X' / \mathcal{G}$. Furthermore, we say that $f$ is strongly equivariant over a point $y' \in Y'$ if $\phi$ is an isomorphism over $y'$ in the sense that

$$\phi_{y'} : X' \times_{Y'} \text{Spec}(\mathcal{O}_{Y',y'}) \to X \times_Y \text{Spec}(\mathcal{O}_{Y',y'})$$

is an isomorphism. Note that $f$ is strongly equivariant if and only if it is strongly equivariant over all points of $Y'$, and if $Y'$ is quasi-compact then it suffices to consider only closed points.

5.3.2. Basic properties. The proof of the following result is a simple diagram chase of the relevant cartesian squares, so we omit it.

**Lemma 5.3.3.** Let $\mathcal{G}$ be a diagonalizable group.

(i) The composition of strongly $\mathcal{G}$-equivariant morphisms is strongly $\mathcal{G}$-equivariant.

(ii) If $Y \to X$ is a strongly $\mathcal{G}$-equivariant morphism and $g : Z \to Y$ is a $\mathcal{G}$-equivariant morphism such that the composition is strongly $\mathcal{G}$-equivariant, then $g$ is strongly $\mathcal{G}$-equivariant.

(iii) If $Y \to X$ and $Z \to X$ are strongly $\mathcal{G}$-equivariant then the projections of $Y \times_X Z$ onto $Y$ and $Z$ are strongly $\mathcal{G}$-equivariant.

(iv) If $f : Y \to X$ is strongly equivariant then the diagonal $\Delta_f : Y \to Y \times_X Y$ is strongly equivariant and $\Delta_f / \mathcal{G}$ is the diagonal of $f / \mathcal{G}$.

5.3.4. Strongly satisfied properties. Let $P$ be a property of morphisms preserved by base changes. We say that an equivariant morphism $f : X' \to X$ strongly satisfies $P$ if it is strongly equivariant and the quotient $f / \mathcal{G}$ satisfies $P$. In particular, $f$ itself satisfies $P$.

**Remark 5.3.5.** (i) In the case of strongly étale morphisms we recover the definition from [MFK94, Appendix 1.D]. Such morphisms played important role in [AKMW02], and we will use strongly regular morphisms in [AT13] for analogous purposes.

(ii) For various properties $P$ it is true that a morphism strongly satisfies $P$ if and only if it satisfies $P$ and is strongly equivariant. In fact, one should only check that if $f$ satisfies $P$ and is strongly equivariant then $f / \mathcal{G}$ satisfies $P$. However, such descent results may be difficult to prove because the morphisms $X \to X / \mathcal{G}$
are not always flat; in particular, flat descent is unapplicable. For the strong étale property such descent claim is a part of Luna’s fundamental lemma.

5.3.6. Some descent results. Here is a (rather incomplete) list of properties for which the descent is easy.

**Proposition 5.3.7.** Let $G$ be a diagonalizable group and $f : X' \rightarrow X$ a $G$-equivariant morphism of schemes with relatively affine $G$-actions. Then,

(i) Let $P$ be any of the following properties: (a) has finite fibers, (b) a monomorphism, (c) separated, (d) universally closed. Then $f$ strongly satisfies $P$ if and only if it satisfies $P$ and is strongly equivariant.

(iii) Assume that $X$ is locally noetherian and let $P$ be one of the following properties: (e) quasi-finite, (f) proper. Then $f$ strongly satisfies $P$ if and only if it satisfies $P$ and is strongly equivariant.

**Proof.** In all claims we assume that $f$ is strongly equivariant and satisfies $P$ and we should prove that $g = f \sslash G$ satisfies $P$. The assertion of (a) follows from the surjectivity of $X \rightarrow X/\sslash G$. Note that a morphism is a monomorphism if and only if its diagonal is an isomorphism. Hence (b) follows from Lemma 5.3.3(iv). Similarly, (c) follows from Lemma 5.3.3(iv) and Proposition 5.2.5(i)(c).

(d) First, assume only that $f$ is closed. Then for any closed set $T \subset X/\sslash G$ the preimage of $g(T)$ in $X$ is closed. Since the quotient morphism $X \rightarrow X/\sslash G$ is submersive, $g(T)$ is closed. Thus $g$ is closed. The assertion of (ii) follows from this and the fact that the base change of $f$ by a strongly equivariant morphism is strongly equivariant by Lemma 5.3.3(iii).

Finally, (e) follows from (a) and Corollary 5.2.4, and (f) follows from (d) and Corollary 5.2.4. ♣

5.3.8. The case of $G_m$. Assume that $X$ and $Y$ are provided with a relatively affine action of $G = G_m$. We refer to Section 5.1.14 for the definitions of $X_\pm$ and $Y_\pm$.

**Lemma 5.3.9.** Keeping the above notation, if $f : X \rightarrow Y$ is strongly equivariant then $X_\pm = Y_\pm \times_Y X$.

**Proof.** The claim is local on $Y/\sslash G$ and $X/\sslash G$, so we can assume that $Y = \text{Spec } B$, $Y/\sslash G = \text{Spec } B_0$, $X = \text{Spec } A$ and $X/\sslash G = \text{Spec } A_0$. By strong equivariance, $A = B \otimes_{B_0} A_0$. Hence $A_n = B_n \otimes_{B_0} A_0$ for any $n$ and we obtain equalities of ideals $B_- A = A_-$ and $B_+ A = A_+$. ♣

**Remark 5.3.10.** For general $G = D_L$, we similarly have that $X^{ss}(m) = (Y^{ss}(m)) \times_Y X$.

5.4. Free actions. In §4.4 we studied the quotients when the stabilizers are maximal. This section is devoted to the other extreme case when the stabilizers are trivial. We will show in Corollary 5.4.5 that any quotient of a relatively affine action of $G = D_L$ can be described in terms of these two cases.

5.4.1. Definitions. We say that an action of $G$ on $X$ is regular or split free if there is an equivariant isomorphism $X \rightarrow G \times Y$, where $Y = X/\sslash G$. If such an isomorphism only exists locally on $Y$ for a topology $\tau$ (e.g. flat, étale, or Zariski) then we say that the action is $\tau$-split free. Finally, if there is a $\tau$-open morphism $g : Y' \rightarrow Y$
and an equivariant isomorphism $X \times_Y Y' \to G \times Y'$ then for any point $y \in g(Y')$ we say that the action is $\tau$-split free over $y$.

**Remark 5.4.2.** Recall that an action of $G$ on $X$ is semi-regular or free if the morphism $G \times X \to X \times X$ is a closed embedding, and this condition is equivalent to the condition that $\psi: G \times X \to X \times_Y X$ is an isomorphism; in other words, $X$ is a pseudo $G$-torsor over $Y$, see [Sta, tag/0498]. This pseudo-torsor is a $\tau$-torsor of $G$ if and only if the action is $\tau$-split free. In fact, any semi-regular action has free orbits hence it follows from Lemma 5.4.4 below that it is flat-split free over the quotient $Y$. In particular, we will not distinguish free and flat-split free actions.

5.4.3. A criterion for splitting. Flat-split freeness of an action can be tested very easily: the stabilizers of the fibers should be trivial.

**Lemma 5.4.4.** Given a relatively affine action of a diagonalizable group $G = D_L$ on a scheme $X$, let $f: X \to Y = X \sslash G$ be the quotient morphism and $y \in Y$ a point. Then the following conditions are equivalent:

- (i) the action is flat-split free over $y$,
- (ii) the scheme-theoretic fiber $X_y = y \times_Y X$ coincides with a single free orbit $O = \text{Spec}(k[L])$,
- (iii) the set-theoretic fiber $f^{-1}(y)$ is a single free orbit,
- (iv) all points of the fiber $f^{-1}(y)$ have trivial stabilizer.

Moreover, if the degree of the torsion of $L$ is invertible on $Y$ (respectively, $L$ is torsion free) then one may replace the flat topology in (i) with the étale (respectively, Zariski) topology.

**Proof.** The implications (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (iv) are obvious. In the opposite direction, if (iv) holds then all orbits in $f^{-1}(y)$ are free and closed, hence the fiber is a single orbit by Theorem 5.1.2(ii), and we obtain (iii).

If (iii) holds then $X_y = \text{Spec}(A)$ such that $A_0 = k$ and $k[L]$ is the reduction of $A$. This implies that each $A_n$ contains a unit, and hence is an invertible $k$-module. Therefore, the reduction $A \to k[L]$ is an isomorphism, and we obtain (ii).

Finally, assume that $X_y = \text{Spec}(k[L])$. Let $L = \bigoplus_{i=1}^n L_i$ be a decomposition of $L$ into cyclic groups, and choose a generator $m_i$ of $L_i$. Shrinking $Y$ around $y$ we can assume that $X = \text{Spec}(A)$, $Y = \text{Spec}(A_0)$, and each $m_i \in k[L]$ lifts to a unit $u_i \in A_{m_i}$ with respect to the homomorphism $A \to k[L]$. For each $m_i$, of a finite order $d_i$ we have that $u_i^{d_i} = a_i \in A_0$. Obviously, $X$ is a principal $G$-torsor over $Y$ which is trivialized by adjoining the roots $a_i^{1/d_i}$ to $A_0$. The latter defines a flat covering of $Y$, which is étale (respectively, an isomorphism) if $d_i$ are invertible on $Y$ (respectively, $L$ is torsion free).

The following corollary shows that locally one can construct quotients in two steps: first by dividing by the stabilizer and then by dividing by a locally free action.

**Corollary 5.4.5.** Assume that a scheme $X$ is provided with a relatively affine action of a diagonalizable group $G = D_L$, $y$ is a point of $Y = X \sslash G$, $G'$ is the stabilizer of the closed orbit of the fiber $X_{y'}$, $Z = X \sslash G'$, and $G'' = G/G'$. Then $Z \sslash G'' = Y$ and the $G''$-action on $Z$ is flat-split free over $y$. Moreover, if the torsion of $L$ is invertible on $Y$ (resp. $L$ is torsion free) then the action is étale-split (resp. Zariski-split) free over $y$.\[\]
5.4.4. \textbf{Proof.} By universality, quotients are compatible with taking fibers. So, in view of Lemma 5.4.4 it suffices to show that $X_y//G'$ is a single free $G''$-orbit. Note that $X_y = \text{Spec}(A)$, where $(A, m)$ is an $L$-local ring with $A_0 = k$. Let $0 \to L'' \to L \to L' \to 0$ be the exact sequence corresponding to $1 \to G' \to G \to G'' \to 1$. Since $A/m = k[L'']$, each $A_n$ with $n \in L''$ contains a unit and we obtain that $A^{L'} = \oplus_{n \in L''} k = k[L'']$. Thus, $X_y//G'' = \text{Spec}(k[L''])$, and we are done. ♣

5.4.6. \textbf{Descend of regularity.} We show below that dividing by a free action is a nice functor that preserves various properties of equivariant morphisms. In fact, the following lemma reduces this to the usual flat descent.

\textbf{Lemma 5.4.7.} Assume that a diagonalizable group $G$ acts on a scheme $X$ and the action is flat-split free over a point $y \in Y = X//G$. Then the quotient morphism $f: X \to Y$ is flat along the fiber $f^{-1}(y)$. In particular, if $X$ is regular at a point $x \in f^{-1}(y)$ then $Y$ is regular at $y$.

\textbf{Proof.} By definition, there exists a flat morphism $Y' \to Y$, whose image contains $y$, and an isomorphism $X \times_Y Y' \to G \times Y'$. Then the flat base change $X \times_Y Y' \to Y'$ of $f$ is flat, and hence $f$ is flat over $y$. If $X$ is regular at a point $x$ then $Y$ is regular at $y$ by [Sta, tag/00EF]. ♣

\textbf{Corollary 5.4.8.} Assume that schemes $X$ and $X'$ are provided with relatively affine actions of a diagonalizable group $G = D_L$, and $f: X' \to X$ is a $G$-equivariant morphism with quotient $g = f//G: Y' \to Y$. Suppose $y' \in Y'$ is a point such that the actions on $X'$ and $X$ are flat-split free over $y'$ and $g(y')$, respectively. Then $f$ is strongly equivariant over $y'$. In particular, if $f$ is regular at a point $x' \in X'_y$, then $g$ is regular at $y'$.

\textbf{Proof.} The statement is local at $y'$ and $y$. So, we can assume that $Y$ and $Y'$ are local, and the claim reduces to showing that $\phi: X' \to X \times_Y Y'$ is an isomorphism. By flat descent, this can be checked flat-locally on $Y$, so by Lemma 5.4.7 we can assume that $X \to G \times Y$. By the same lemma, there exists a flat covering $Y'' \to Y'$ such that $X' \times_Y Y'' \to G \times Y''$. The base change of $\phi$ with respect to the morphism $Y'' \to Y'$ is the isomorphism $X' \times_Y Y'' \to G \times Y'' \to X \times_Y Y''$. Thus, $\phi$ is an isomorphism by flat descent.

Suppose $f$ is regular. To check that $g: Y' \to Y$ is regular it suffices to check that its pullback by a flat surjective morphism is regular; but $X \to Y$ is flat and surjective by Lemma 5.4.7, and the pullback is $f: X' \to X$, as we have shown. ♣

5.5. \textbf{Inert morphisms.} The main disadvantage in the definition of strong equivariance (Section 5.3.1) is that it involves the quotient morphism. It is desirable to have an explicit criterion of strong equivariance in terms of the morphism itself. This leads to the notion of inert morphisms.

5.5.1. \textbf{The definition.} A $G$-equivariant morphism $f: X' \to X$ is called \textit{inert} if it satisfies the following two conditions:

(i) $f$ preserves inertia groups, i.e. $I_X \times_X X' = I_{X'}$.

(ii) $f$ takes special orbits (5.1.5) to special orbits.

\textbf{Remark 5.5.2.} (i) Condition (i) implies that for any $x' \in X'$ and $x = f(x')$ the inclusion of the stabilizers $G_{x'} \hookrightarrow G_x$ is an equality; a morphism satisfying the
latter property is usually called a fixed point reflecting morphism (e.g. [Knu71, IV.1.8]). In some situations the latter property is easier to verify. Note that if the fibers of $f$ are reduced, e.g. if $f$ is regular, then (i) is equivalent to being fixed point reflecting. In fact, we will be mostly interested in situations where $f$ is regular.

(ii) It follows directly from the definition (5.3.1) that any strongly equivariant morphism is inert.

(iii) If either of the conditions (i) and (ii) is violated then the quotient functor might behave rather badly. For example, condition (i) is violated in Example 5.2.7(i) and (i)', and condition (ii) is violated in Example 5.2.7(ii) and (ii)'.

(iv) If $f: X' \to X$ is inert and $x \in X$ is a point, then the stabilizers along the fiber $f^{-1}(x)$ coincide with $G_x$, and hence the action of $G_x$ on the fiber is trivial.

(v) Inert regular morphisms play important role in this paper. We will prove an analogue of Luna’s fundamental lemma for them, and many our constructions will be functorial with respect to such morphisms.

5.5.3. **Comparison with the literature.** In the case of a reductive group $G$ and a $G$-equivariant étale morphism of varieties, the inertness condition is essentially due to Luna. Although not formulated explicitly, it can be extracted from [Lun73, Lemme fondamental].

For finite groups, fixed point reflecting morphisms were called inert in [ILO12, Section VIII.5.3.6]. Note that condition (ii) did not appear in [ILO12] because it is automatic for finite groups. An analogue of our version of inert morphism was called “schematically inert” in [ILO12, Remark VIII.5.6.2]. The motivation in [ILO12] was similar to ours – (schematical) inertness is the “right” condition to impose on morphisms in order to obtain functorial constructions.

An analogue of inertness is used by Alper in a much more general context of good and adequate moduli spaces, see for example [Alp08]. It is needed to guarantee that the moduli space functor (which generalizes the quotient functor) behaves nicely. In fact, Alper calls condition (i) “stabilizer preserving” and replaces (ii) with the following condition: (ii)' closed orbits go to closed orbits (closed point of stacks in the context of [Alp08]). For general schemes, closed orbits can go to non-closed special orbits, hence condition (ii)' is not sharp.

5.6. **Main results about the quotient functor.** In this section we will prove that inert regular morphisms are strongly regular. In the case of diagonalizable groups, this extends the classical Luna’s fundamental lemma, which deals with inert étale morphisms of varieties, to regular morphisms of schemes.

5.6.1. **Local case.** We start with the following local statement.

**Lemma 5.6.2.** Assume that locally noetherian schemes $X$ and $X'$ are provided with relatively affine actions of a diagonalizable group $G = D_L$, and $f: X' \to X$ is a $G$-equivariant morphism with quotient $g = f \sslash G: Y' \to Y$. Let $x' \in X'$ be a point with images $x = f(x') \in X$, $y' \in Y'$, and $y \in Y$. Assume that

(a) $f$ is formally smooth at $x'$,
(b) $x$ lies in a special orbit,
(c) the inclusion of the stabilizers $G_{x'} \subseteq G_x$ is an equality,
(d) $G_x$ acts trivially on the fiber $X'_{x'}$.

Then $f$ is strongly equivariant over $y'$ and $g$ is formally smooth at $y'$.
5.1.6. The claim is local at $y'$ and $y$, so we can assume that $Y' = \text{Spec} \mathcal{O}_{Y',y'}$ and $Y = \text{Spec} \mathcal{O}_{Y,y}$. Then the actions on $X'$ and $X$ are local (5.1.6), and $x$ lies in the closed orbit by assumption (b). We need to prove that $f$ is strongly equivariant and $g$ is formally smooth. Let us recall two cases of the lemma that were already established earlier.

Case 1. The lemma holds true when $G_x = G$. Indeed, in this case the closed orbits are $x$ and $x'$, so $X = \text{Spec}(A)$ and $X' = \text{Spec}(A')$, where $A$ and $A'$ are strictly $L$-local rings. By assumption (a), the homomorphism of localizations is formally smooth, and $A'/m_AA'$ is trivially graded because the action on $X'_z$ is trivial by assumption (d). So, Case 1 is covered by Lemma 4.5.10.

Case 2. The lemma holds true when $G_x = \{1\}$. Indeed, in this case the actions are free by Lemma 5.4.4, hence Case 2 is covered by Corollary 5.4.8.

Assume, now that $G_x$ is arbitrary. Recall that the actions of $G' := G/G_x$ on $Z = X/G_x$ and $Z' = X'/G_x$ are free by Corollary 5.4.5. Consider the $G'$-equivariant morphism $h = f/G_x$. By Case 1 above, $h$ is formally smooth at the image $z' \in Z'$ of $x'$, and $f$ is strongly equivariant over $z'$. Since $Z_y$ consists of a single orbit by local freeness of the action, $G'$, $h$, and $z'$ satisfy all assumptions of the lemma. Therefore, Case 2 applies and we obtain that $g = h \parallel G'$ is formally smooth at $y'$, and $h$ is strongly equivariant over $y'$. The lemma follows.

5.6.3. Luna’s fundamental lemma.

**Theorem 5.6.4.** Let $X$ and $X'$ be locally noetherian schemes provided with relatively affine actions of a diagonalizable group $G$, and let $f: X' \to X$ be a $G$-equivariant morphism. Then,

(i) The following conditions are equivalent: (a) $f$ is regular and inert, (b) $f$ is regular and strongly equivariant, (c) $f$ is strongly regular.

(ii) The assertion of (i) holds true if one replaces regularity with any of the following properties: (a) smooth, (b) étale, (c) an open embedding.

**Proof.** (i) The implications $(c) \implies (b) \implies (a)$ are obvious, so let us prove that $(a) \implies (c)$. By Corollary 2.2.3, $f$ is formally smooth at all points of $Y$. Also, for any point $x \in X$ the stabilizer $G_x$ acts trivially on the fiber $X'_x$ by Remark 5.5.2. Thus, Lemma 5.6.2 applies to all points of the special orbits of $X'$, and we obtain that $f$ is strongly equivariant, and $g$ is formally smooth at all points of $X' \sslash G$. Using Corollary 2.2.3 once again we obtain that $g$ is regular.

(ii) A morphism is smooth if and only if it is regular and of finite type. Hence (a) follows from (i) and Corollary 5.2.4. A smooth morphism is étale or an open embedding if and only if it has finite fibers or is a monomorphism, respectively. Hence (b) and (c) follow from (a) and assertions (a) and (b) of Proposition 5.3.7.

5.7. Groups of multiplicative type. Assume that $S$ is a scheme and $G_S$ is an $S$-group. We say that $G_S$ is of multiplicative type if it is of finite type and $G_S = G_S \times_S S'$ is isomorphic to a diagonalizable $S'$-group $D_L \times S'$ for a surjective flat finitely presented morphism $h: S' \to S$. Our definition is more restrictive than in [SGA70b, IX.1.1] because we use the fppf topology instead of the fqc topology and consider only groups of finite type. Recall that by [SGA70b, X.4.5] any such group $G_S$ is quasi-isotrivial, i.e. it can be split by an étale surjective morphism $h: S' \to S$. 

We claim that all results of Section 5 extend to the case when an $S$-scheme $X$ is provided with a relatively affine action of an $S$-group $G_S$ of multiplicative type. The only exception is part (ii) of Lemma 5.1.7, which makes no sense for non-diagonalizable group. The proofs we provided of equivalence of (i) and (iii) in Lemma 5.1.7 and parts (a), (b) and (c) of Theorem 5.2.2(i) apply to groups of multiplicative type without change. All remaining results, including Luna’s fundamental lemma, can be extended to $G_S$ of multiplicative type with $L$ denoting the abelian group Cartier dual to $G_S'$ by use of étale descent with respect to $h$. Namely, $X_{S'} / G_S = (X / G_S) \times_S S'$ so the result for $X_{S'} / G_S$ extends to $X / G$ by descent. In fact, already flat descent suffices in almost all cases, with the main exception being Proposition 5.1.13.

6. Group actions on toroidal schemes

Toroidal schemes generalize the classical toroidal embeddings of varieties. Although one can describe them by formal charts, it is more convenient to use an equivalent approach via logarithmically regular logarithmic schemes. For simplicity, we will consider Zariski logarithmic schemes and toroidal schemes without self intersections. We remark, however, that analogously to [ILO12, Exp. VI, §1] almost everything can be done for general logarithmic schemes at the cost of replacing points and localizations with geometric points and strict henselizations. We do not pursue this further here.

6.1. Monoids.

6.1.1. Conventiones. We will use the following notation for commutative monoids: we denote by $\overline{M} = M/M^\times$ the sharpening of $M$, by $M^{gp}$ the Grothendieck group of $M$, and $M^+ = M \setminus M^\times$ denotes the maximal ideal of $M$. The rank of a monoid $M$ is $\text{rk}(M) = \dim_\mathbb{Q}(M^{gp} \otimes_\mathbb{Z} \mathbb{Q})$.

6.1.2. Toric monoids. A toric monoid is an fs (i.e. fine and saturated) monoid $M$ without torsion.

Remark 6.1.3. Usually, one also requires toric monoids to be sharp but we prefer to modify the terminology in this paper.

6.1.4. Prime ideals. Any subset $S \subseteq M$ generates an ideal $(S) = \cup_{f \in S} (f + M)$, and an ideal $(f) = f + M$ with $f \in M$ is called principal. Prime ideals and their height are naturally defined, analogously to the case of rings, see [Kat94, §5]. In toric monoids, prime ideals are of the form $p = M \setminus F$, where $F$ is a face, and such a $p$ is of height one if and only if $F$ is a facet.

6.1.5. Inner elements. An element $v \in M$ will be called inner if it lies in the interior of $M_R$. In toric monoids the following conditions are easily seen to be equivalent: (a) $v$ is inner, (b) $v$ is not contained in any facet of $M$, (c) $v$ lies in all prime ideals of $M$.

6.1.6. Divisorial prime ideals. Prime ideals of height one are analogous to divisorial ideals. If $p$ is prime of height one then $F = M \setminus p$ is a facet and the image of $M$ in $M^{gp}/p^{gp}$ is isomorphic to $\mathbb{N}$. So, to any $f \in M$ we can associate a number $\nu_p(f) \in \mathbb{N}$, which is an analogue of the order of $f$ with respect to $p$. 
Lemma 6.1.7. Assume that $M$ is a sharp toric monoid, $H$ is the set of prime ideals of $M$ of height one, and $f \in M$. Then $(f) = \bigcap_{\mathfrak{p} \in H} \nu_{\mathfrak{p}}(f)\mathfrak{p}$.

Proof. This follows from [GR13, Proposition 3.4.33].

6.1.8. Splitting faces and facets off. We say that a face $N \subseteq M$ splits off if there exists a face $K$ such that $M = N \oplus K$. If $M$ is sharp and toric then $K$ is the face spanned by all edges not contained in $N$.

Lemma 6.1.9. Let $M$ be a sharp toric monoid with a face $N$. Then $N$ splits off if and only if the following two conditions hold:

(a) All edges not contained in $N$ span a face $K$ and $K^{\text{gp}} \cap N^{\text{gp}} = 0$ in $M^{\text{gp}}$.

(b) The ideal $(N^+) \subset M$ generated by the maximal ideal $N^+$ of the monoid $N$ is prime in $M$.

Proof. The direct implication is clear: if $N$ splits off then $M = N \oplus K$ and $(N^+) = M \setminus K$ is prime. Conversely, assume that (a) and (b) hold. The edges of $K$ are precisely the edges of $M$ not contained in the prime ideal $(N^+)$. Since any prime ideal is a complement of a face we obtain that $(N^+) = M \setminus K$. Therefore $N \oplus K \to M$ is surjective. The condition $K^{\text{gp}} \cap N^{\text{gp}} = 0$ in $M^{\text{gp}}$ implies that, in fact, $N \oplus K = M$.

Corollary 6.1.10. If $M$ is a sharp toric monoid and $\mathfrak{p} \subset M$ is a prime ideal of height one then the following conditions are equivalent: (a) $\mathfrak{p}$ is principal, (b) the facet $F = M \setminus \mathfrak{p}$ splits off, (c) $M$ contains a single edge not lying in $F$ and the ideal $(F^+) \subset M$ is prime in $M$. If these conditions are satisfied then $M = F \oplus \mathbb{N}e$, where $e$ is the generator of $\mathfrak{p}$ and $\mathbb{N}e$ is the only edge not contained in $F$.

Proof. Assertions (b) and (c) are equivalent by Lemma 6.1.9. Assume (a) holds, say $\mathfrak{p} = e + M$. For any element $x \in M$ we have that $x - \nu_{\mathfrak{p}}(x)e \in F$, hence $M = F + \mathbb{N}e$ and using that $M$ is toric we obtain that $M = F \oplus \mathbb{N}e$. In particular, (b) holds. In addition, it is clear that $\mathbb{N}e$ is the only edge not in $F$, hence (a) implies all the claims in the end of the corollary. Assume (b) holds, say $M = F \oplus K$. Since $F$ is a facet, $K^{\text{gp}}$ is of dimension 1, and so $K = \mathbb{N}e$ for an element $e \in \mathfrak{p}$. In particular, $\mathfrak{p} = (e)$.

6.2. Logarithmic schemes.

6.2.1. Conventiones. We refer to [Kat89] for the general definition of logarithmic schemes. A Zariski logarithmic scheme is defined similarly but using the Zariski topology: $(X, \mathcal{M}_X, \alpha)$, where $\alpha : \mathcal{M}_X \to (\mathcal{O}_X, \cdot)$ is a logarithmic structure in the Zariski topology. See also [Kat94]. Unless said otherwise, all logarithmic schemes are assumed to be fine and Zariski, and we will use the shorter notation $(X, \mathcal{M}_X)$.

Remark 6.2.2. In fact, one can view Zariski logarithmic schemes as usual logarithmic schemes $(X, \mathcal{M}_X)$ such that $\varepsilon^*\varepsilon_*\mathcal{M}_X = \mathcal{M}_X$, for $\varepsilon : X_{\text{et}} \to X_{\text{Zar}}$. In this case, $\mathcal{M}_X$ is determined by its restriction onto the Zariski site.

6.2.3. Ranks. Given a fine logarithmic scheme $(X, \mathcal{M}_X)$ we will use the notation $r(x) = \text{rk}(\mathcal{M}_{X,x})$ for $x \in X$, and $\text{rk}(\mathcal{M}_X) = \max_{x \in X} r(x)$. 
6.2.4. Charts. By a monoidal chart of a fine logarithmic scheme \((X, \mathcal{M}_X)\) we mean an open subscheme \(V \hookrightarrow X\), a fine monoid \(M\), and a homomorphism \(\phi : M \to (\mathcal{O}_X(V), \cdot)\) such that \(\mathcal{M}_X\) is the logarithmic structure associated with the pre-logarithmic structure induced by \(\phi\). To give this data is the same as to give a strict morphism \(f : (V, \mathcal{M}_X|_V) \to (\mathbf{A}_M, \mathcal{M}_{\mathbf{A}_M})\), where \(\mathbf{A}_M = \text{Spec}(\mathbb{Z}[M])\) and \(\mathcal{M}_{\mathbf{A}_M}\) is the logarithmic structure associated with \(M\). So, we will also call \(f\) a monoidal chart of \((X, \mathcal{M}_X)\). Recall that any fine logarithmic scheme is coherent by definition, hence it can be covered by monoidal charts.

6.2.5. Sharp charts. We say that a monoidal chart is sharp or fs, if the defining monoid \(M\) is sharp or fs, respectively.

**Lemma 6.2.6.** If an fs logarithmic scheme \((X, \mathcal{M}_X)\) admits an fs monoidal chart \(M \to \mathcal{O}_X(V)\) then it also admits a sharp monoidal chart \(\overline{M} \to \mathcal{O}_X(V)\).

**Proof.** Any fs monoid (non-canonically) splits as \(M = \overline{M} \oplus M^\times\), see [GR13, Lemma 3.2.10]. The composition \(\overline{M} \hookrightarrow M \to \mathcal{O}_X(V)\) is a sharp chart. \(\Box\)

6.2.7. The center. The center of a monoidal chart \(\phi : M \to \mathcal{O}_X(V)\) is the closed subscheme \(V(\phi(M^+))\) of \(V\). If it is non-empty then we say that the chart is central. Obviously, the following conditions are equivalent: (i) \(x\) lies in the center, (ii) \(r(x) = \text{rk}(\overline{M})\), (iii) the induced homomorphism \(\phi_x : M \to \mathcal{O}_{X,x}\) is local, i.e. satisfies \(\phi_x^{-1}(\mathcal{O}_{X,x}^\times) = M^\times\). In particular, a chart is central if and only if \(\text{rk}(\overline{M}) = \text{rk}(\mathcal{M}_X)\).

**Remark 6.2.8.** The definition of the center is of global nature and does not make sense for sheaves because \(\text{rk}(\overline{M})\) can jump. For example, the ideal \(\alpha(M^+_X)\mathcal{O}_X\) is not coherent already for the affine plane with its toric logarithmic structure.

**Lemma 6.2.9.** Assume that a fine logarithmic scheme \((X, \mathcal{M}_X)\) admits a global central monoidal chart \(\phi : M \to \Gamma(\mathcal{O}_X)\). Then the center \(C\) of the chart depends only on \((X, \mathcal{M}_X)\).

**Proof.** Set-theoretically, \(C\) is the set of points where \(r(x)\) is maximal. In particular, \(|C|\) is independent of the chart. For any \(x \in C\) the homomorphism \(\phi_x : M \to \mathcal{O}_{X,x}\) is local, hence the homomorphism \(\overline{M} \to \overline{M}_{X,x}\) is an isomorphism, and we obtain that the ideal \(\phi_x(M^+)_X\) coincides with \(\alpha_x(M^+_X)\mathcal{O}_{X,x}\). In particular, \(C \times_X \text{Spec}(\mathcal{O}_{X,x}) = V(\phi_x(M^+)\mathcal{O}_{X,x})\) is independent of the chart, and hence the same is true for \(C\). \(\Box\)

6.2.10. Logarithmic stratification. Assume that \((X, \mathcal{M}_X)\) is a fine logarithmic scheme. For any point \(x \in X\) there exists a monoidal chart \(\phi : M \to \mathcal{O}_X(V)\) such that \(x \in V\) and \(\overline{M} = \overline{M}_{X,x}\), in particular, the chart is central. Cover \(X\) with central charts \(\phi_i : M_i \to \mathcal{O}_X(V_i)\) so that any point \(x \in X\) lies in the center of some chart, and let \(C_i\) denote the center of \(\phi_i\) and \(r_i = \text{rk}(\overline{M}_i)\). If \(r_i = r_j\) then the restrictions of \(C_i\) and \(C_j\) to \(V_i \cap V_j\) coincide by Lemma 6.2.9. It follows that for any \(n \in \mathbb{N}\), all \(C_i\) with \(r_i = n\) glue to a locally closed subscheme \(X(n) \hookrightarrow X\). Furthermore, set-theoretically \(X(n)\) is the set of all points \(x \in X\) with \(r(x) = n\), hence we obtain a stratification of \(X\), that will be called the logarithmic stratification.

**Remark 6.2.11.** (i) The reduction of the logarithmic stratification was considered in [ILO12, Exp. VI, §1.5] under the name “canonical stratification” or “stratification by rank of \(\overline{M}\).”
(ii) $X(0)$ is the triviality locus of $\mathcal{M}_X$, i.e. the open subscheme on which the logarithmic structure is the trivial one.

(iii) It follows from the definition that the logarithmic stratification is compatible with strict morphisms: if $(Y, \mathcal{M}_Y) \to (X, \mathcal{M}_X)$ is strict then $Y(n) = X(n) \times_X Y$ for any $n$.

6.2.12. Center of a logarithmic scheme. Given a fine logarithmic scheme $(X, \mathcal{M}_X)$ let $X_i$ be the connected components of the logarithmic strata of $X$, and let $C(X, \mathcal{M}_X)$ be the union of those $X_i$ that are closed in $X$. We call $C(X, \mathcal{M}_X)$ the center of $(X, \mathcal{M}_X)$.

Lemma 6.2.13. Let $(X, \mathcal{M}_X)$ be an fs logarithmic scheme. Then the sheaf $\mathcal{M}_X = \mathcal{M}_X/\mathcal{O}_X^*$ is locally constant along each logarithmic stratum. In particular, it is locally constant along the center of $(X, \mathcal{M}_X)$.

**Proof.** If $x$ is a specialization of $y$ in $X$ then a surjective cospecialization map $h: \mathcal{M}_{X,x} \to \mathcal{M}_{X,y}$ arises, see [Niz06, Lemma. 2.12(1)] and its proof. If $x$ and $y$ lie in the same logarithmic stratum then the monoids have the same rank and hence $h$ is an isomorphism. \(\blacklozenge\)

6.3. Group actions.

6.3.1. The definition. An action of $G$ on a logarithmic scheme $(X, \mathcal{M}_X)$ consists of an action $m: G \times X \to X$ on $X$ and an isomorphism $\phi: p^{-1}{\mathcal{M}_X} \cong m^{-1}{\mathcal{M}_X}$, where $p: G \times X \to X$ is the projection and the pullbacks of $\phi$ to $G \times G \times X$ satisfy the usual cocycle condition. In fact, one can view $G$ with the trivial logarithmic structure as a group object in the category of logarithmic schemes, and then this data reduces to a usual categorical action.

6.3.2. Equivariant charts. Assume that $G = D_L$ acts on $(X, \mathcal{M}_X)$. By a (strongly) equivariant monoidal chart we mean a $G$-equivariant open subscheme $V \hookrightarrow X$ and a (strongly) $G$-equivariant strict morphism $f: (V, \mathcal{M}_X|_V) \to (\mathbb{A}_M, \mathcal{M}_{A_M})$ (see Section 5.3.1), where $G$ acts on the target via a homomorphism $h: M \to L$. A chart $f$ is equivariant if and only if the corresponding homomorphism $\phi: M \to \mathcal{O}_X(V)$ is homogeneous, i.e. takes each $h^{-1}(l)$ to the $l$-homogeneous component of $\mathcal{O}_X(V)$.

One may wonder when an action possesses a (strongly) equivariant chart. This naturally leads to the definitions of $G$-simple and toroidal actions below.

6.3.3. $G$-simple actions. Assume that a diagonalizable group $G$ acts on a logarithmic scheme $(X, \mathcal{M}_X)$. We say that the action is $G$-simple at a point $x \in X$ if $G_x$ acts trivially on $\mathcal{M}_{X,x}$. The action is $G$-simple if it is $G$-simple at all points of $X$. When necessary we will emphasize $G$, referring to a $G$-simple action. In particular, this will be useful to avoid confusion with strict morphisms of logarithmic schemes.

Remark 6.3.4. (i) If $G$ is connected then any action is $G$-simple. In general, $G_x$ acts on $\mathcal{M}_{X,x}$ through the quotient by its connected component, i.e. through a finite group.

(ii) Our definition is taken from [ILO12, Exp. VI, 3.1(ii)], where it appears without name. The terminology differs slightly from $G$-strict actions of [dJ97, 7.1], which is concerned with regular schemes with simple normal crossings divisors acted on by a finite group. Since the word “strict” conflicts with logarithmic strict morphisms, we prefer ”$G$-simple”, in analogy with simple normal crossings divisors.
Lemma 6.3.5. Let $G = D_L$ be a diagonalizable group. Then,

(i) Assume that $f : (Y, M_Y) \to (X, M_X)$ is a strict $G$-equivariant morphism between fine logarithmic schemes. If the action on $(X, M_X)$ is $G$-simple then the action on $(Y, M_Y)$ is $G$-simple, and the converse is true whenever $f$ is surjective.

(ii) Assume that $M$ is a toric monoid and $h : M \to L$ is a homomorphism. Then the induced action on $(A_M, M_{A_M})$ is $G$-simple.

(iii) Assume that $G$ acts on an fs logarithmic scheme so that there exists a global equivariant monoidal chart. Then the action is $G$-simple.

Proof. To prove (i) it suffices to note that $f$ induces isomorphism of stalks $M_{X,f(y)} = M_{Y,g}$. The action of $G$ on $A_M$ is through the torus $A_M^{gp}$. The latter is connected hence the action is $G$-simple. Finally, (iii) follows from (i) and (ii).

6.3.6. Toroidal actions. Consider the logarithmic stratum $Z = X(r(x))$ through $x$, and let $Z_x$ be its localization at $x$. We say that the action is toroidal at $x$ if it is $G$-simple and $G_x$ acts trivially on $Z_x$. The action is toroidal if it is so at all points. This is compatible with the situation when $X$ is toroidal in Section 6.5.5.

Remark 6.3.7. (i) A similar notion was introduced by Gabber in [ILO12, Exp. VI, 3.1] under the name “very tame action”. This terminology seems not ideal as the tameness condition [ILO12, Exp. VI, 3.1(i)] simply means that the stabilizers are diagonalizable. We prefer to replace “very tame” with “toroidal”.

(ii) Our definition slightly differs from that in [ILO12] because we have defined the logarithmic stratification in a scheme-theoretic way, and the logarithmic strata may be non-reduced. However, in the logarithmically regular case both definitions agree, and this is the only case we will work with.

Lemma 6.3.8. Let $G = D_L$ be a diagonalizable group. Then,

(i) A $G$-simple action of $G$ on an fs logarithmic scheme $(X, M_X)$ is toroidal if and only if the logarithmic stratification refines the inertia stratification, i.e. each connected component of a logarithmic stratum $X(r)$ is contained in an inertia stratum $X(G')$ for some $G' \subseteq G$.

(ii) Assume that $f : (Y, M_Y) \to (X, M_X)$ is a strict inert $G$-equivariant morphism between fine logarithmic schemes. If the action on $(X, M_X)$ is toroidal then the action on $(Y, M_Y)$ is toroidal, and the converse is true whenever $f$ is surjective.

(iii) Assume that $M$ is a toric monoid and $h : M \to L$ is a homomorphism. Then the induced action of $G$ on $(A_M, M_{A_M})$ is toroidal.

(iv) Assume that $G$ acts on an fs logarithmic scheme so that there exists a global strongly equivariant monoidal chart. Then the action is toroidal.

Proof. Part (i) is obvious. Part (ii) follows from (i) and Lemma 6.3.5(i) because $f$ is strict and inert, and hence is compatible with both the logarithmic stratification and the inertia stratification. The action in (iii) is $G$-simple by Lemma 6.3.5(ii), hence we should only check that the stabilizers are constant along the connected components of the logarithmic strata of $A_M$. The closed logarithmic stratum of $A_M$ is its center $V(M^+) = \text{Spec} \mathbb{Z}[M^+]$, and all orbits in $V(M^+)$ have the same stabilizer $D_{L/\phi(M^+)}$. The general case reduces to this one because $A_M \setminus V(M^+)$ is the union of the logarithmic schemes $A_M[m-1]$ for $m \in M^+$. Finally, (iv) follows from (ii) and (iii) because any strongly equivariant morphism is inert.
6.4. Toroidal schemes. We now focus on toroidal schemes, which form the main case we are interested in.

6.4.1. Logarithmic regularity. An fs logarithmic noetherian scheme \((X, \mathcal{M}_X)\) is called \textit{logarithmically regular} if each of its non-empty logarithmic strata \(X(n)\) is regular, and for any \(x \in X(n)\) the equality \(\dim(\mathcal{O}_{X,x}) = n + \dim(\mathcal{O}_{X(n),x})\) holds.

Remark 6.4.2. (i) Logarithmic regularity of Zariski logarithmic schemes was introduced by Kato in [Kat94]. To the general (étale) case it was extended by Nizioł in [Niz06].

(ii) A logarithmically regular scheme \(X\) is Cohen-Macaulay by [Kat94, Theorem 4.1]. Hence \(X\) is catenary, and therefore each non-empty \(X(n)\) is of pure codimension \(n\).

We refer to [Kat94, Theorem 11.6] for the proof of the following result.

Proposition 6.4.3. Assume that \((X, \mathcal{M}_X, \alpha)\) is a logarithmically regular logarithmic scheme. Let \(U = X(0)\) denote the triviality locus and \(j: U \hookrightarrow X\) the open embedding. Then \(\alpha: \mathcal{M}_X \rightarrow \mathcal{O}_X\) is injective, \(D = X \setminus U\) is a divisor, and \(\mathcal{M}_X = j_* \mathcal{O}_U^X \cap \mathcal{O}_X\).

6.4.4. Toroidal schemes. Given a scheme \(X\) with an open subscheme \(U\), we say that the pair \((X, U)\) is a \textit{toroidal scheme} if the logarithmic structure \(\mathcal{M}_X := j_* \mathcal{O}_U^X \cap \mathcal{O}_X \hookrightarrow \mathcal{O}_X\) makes \(X\) into a logarithmically regular logarithmic scheme. Sometimes we will use the divisor \(D = X \setminus U\) instead of \(U\) in the notation of toroidal schemes.

Remark 6.4.5. (i) The correspondences \((X, \mathcal{M}_X) \mapsto (X, X(0))\) and \((X, U) \mapsto (X, j_* \mathcal{O}_U^X \cap \mathcal{O}_X)\) establish a bijection between logarithmically regular logarithmic schemes and toroidal schemes.

(ii) In the case of varieties over an algebraically closed field, toroidal schemes are the classical toroidal embeddings without self intersections of [KKMSD73]. Furthermore, étale logarithmically regular varieties correspond to general toroidal embeddings.

6.4.6. The center.

Lemma 6.4.7. Let \((X, U)\) be a toroidal scheme and let \(C\) be its center, then

(i) \(C\) is regular.

(ii) If \(G\) acts on \((X, U)\) locally then any connected component of \(C\) intersects with the closed orbit \(O\). In particular, if the action is strictly local then \(C\) is integral.

Proof. Claim (i) follows from unraveling the definitions. For claim (ii), assume to the contrary that \(V\) is a connected component of \(C\) which is disjoint from \(O\). Then the union \(\overline{V}\) of all \(G\)-translates of \(V\) is disjoint from \(O\). On the other hand, \(\overline{V}\) is a closed \(G\)-equivariant subscheme and hence contains \(O\). The contradiction concludes the proof. ♣

6.4.8. Toroidal divisors. By a \textit{toroidal divisor} in \((X, U)\) we mean any divisor \(E\) of \(X\) with support in \(X \setminus U\).

Lemma 6.4.9. Assume that \((X, U)\) is a toroidal scheme and \(X\) is local with the closed point \(x\). Let \(\mathcal{M}_X\) be the logarithmic structure of \((X, U)\) and \(M = \overline{M}_{X,x}\). Then,
(i) The map \( m \mapsto V(m) \) establishes an isomorphism of \( M \) and the monoid of effective toroidal Cartier divisors of \( X \).

(ii) The map \( p \mapsto V(p\mathcal{O}_X) \) establishes a bijection between prime ideals of height one in \( M \) and integral toroidal Weil divisors.

Proof. The first claim follows from the definition of \( \mathcal{M}_X \). Let \( H \) be the set of prime ideals of \( M \) of height one. By [Kat94, Cor. 7.3], \( V(p\mathcal{O}_X) \) is an integral Weil divisor for any \( p \in H \). It remains to show that any integral toroidal Weil divisor is of this form.

Fix an inner element \( f \in M (\mathsection 6.1.5) \). Note that \( X \setminus U \) is the support of \( V(f) \). By Lemma 6.1.7 and [Kat94, Prop. 6.4], \( f\mathcal{O}_{X,x} = \bigcap_{p \in H} (p\mathcal{O}_{X,x})^{\nu(f)} \), hence any irreducible component of \( X \setminus U \) is one of the divisors \( V(p\mathcal{O}_{X,x}) \).

Corollary 6.4.10. Keep the assumptions of Lemma 6.4.9. Let \( Z \) be an irreducible component of \( D = X \setminus U \) with corresponding prime ideal \( p \subset M (6.4.9(ii)) \) and \( D' = D - Z \). Then,

(i) \((X, D')\) is a toroidal scheme if and only if \( Z \) is Cartier.

(ii) Assume \((X, D')\) is a toroidal scheme, and let \( \mathcal{M}_X \) be the logarithmic structure associated with \((X, D')\) and \( M' = \mathcal{M}_{X,x} \). Then \((a) M' = M \setminus p \), \((b) Z = V(e)\), and \((c) M = M' \oplus \mathbb{N}e\).

Proof. We start with the “if” direction of (i). Assume that \( Z \) is Cartier. Let \( X(r) \) and \( X'(r) \) denote the logarithmic strata of \((X, D)\) and \((X, D')\). By definition, \( X(r) \) is regular of codimension \( r \) and we should prove that the same is true for \( X'(r) \). Fix \( r \) and a point \( z \in X'(r) \). If \( z \notin Z \) then \( X'(r) \) coincides with \( X(r) \) locally at \( z \), hence it is regular at \( z \). If \( z \in Z \) then \( X'(r) = X(r + 1) \) locally at \( z \). Since \( X'(r) \) is of codimension at most \( r \) and \( X(r + 1) \) is regular and of codimension \( r + 1 \), it follows that \( X'(r) \) is regular of codimension \( r \) at \( z \), and thus \((X, D')\) is toroidal.

Assume that \((X, D')\) is toroidal. We prove \( Z \) is Cartier as well as (ii). Define \( M' \) as in (ii). Consider the facet \( F = M \setminus p \) and the ideal \( I = (F^+) \subset M \) generated by \( F^+ \). Note that \( V(I\mathcal{O}_{X,x}) \) is the center \( C^+ \) of \((X, D')\). Since \( C^+ \) is regular by Lemma 6.4.7, \( I \) is a prime ideal of \( M \). In addition, \( M' = F \) because both consist of the elements of \( M \) not vanishing on \( D \), proving (ii)(a). Thus, the facets of \( F \) are parameterized by the irreducible components of \( D' \) by Lemma 6.4.9(ii). Therefore, if \( G \neq F \) is a facet of \( M \) then \( G \cap F \) is a facet of \( F \). Note that \((G \cap F)^{\mathbb{P}} = G^{\mathbb{P}} \cap F^{\mathbb{P}} \). Since \( \cap_{G}(G \cap F)^{\mathbb{P}} = \{0\} \) we have that \( \text{rk} (\cap_{G}G^{\mathbb{P}}) \leq 1 \). It follows that \( M \) contains a single edge not lying in \( F \). Thus, \( p = (e) \) is principal by the equivalence of (a) and (c) in Corollary 6.1.10, and so \( Z = V(e) \) is Cartier, proving (ii)(b) and (i). By Corollary 6.1.10 we have that \( M = M' \oplus \mathbb{N}e \), proving (ii)(c).

6.4.11. Morphisms of toroidal schemes and toroidal morphisms. By a morphism \( f: (X', U') \to (X, U) \) of toroidal schemes we mean any morphism \( f: X' \to X \) with \( f(U') \subseteq U \). Note that \( f \) induces a morphism \( h: (X', \mathcal{M}_X) \to (X, \mathcal{M}_X) \) of the corresponding logarithmic schemes. If \( h \) is logarithmically smooth then we say that \( f \) is a toroidal morphism. This generalizes [AK00, Definition 1.2].

6.4.12. Toroidal charts. If \((X, U)\) is a toroidal scheme and \( M \to \mathcal{O}_X(X) \) gives rise to a global monoidal chart then we obtain a strict morphism of toroidal schemes \((X, U) \to (A_M, A_{M^\mathbb{P}}) \) that will be called a global toroidal chart of \((X, U)\). If
X is defined over a field $\mathbb{F}$ then we also consider $\mathbb{F}$-charts with target $\mathbb{A}_{M,\mathbb{F}} = \text{Spec}(\mathbb{F}[M])$.

**Lemma 6.4.13.** Assume that $M$ is a toric monoid, $f : X \to \mathbb{A}_M$ (resp. $f : X \to \mathbb{A}_{M,\mathbb{F}}$) is a regular morphism and $U = f^{-1}(\mathbb{A}_{M^0})$ (resp. $U = f^{-1}(\mathbb{A}_{M^0,\mathbb{F}})$). Then $(X, U)$ is a toroidal scheme and $f$ gives rise to a global toroidal chart.

**Proof.** Set $Y = \mathbb{A}_M$ (resp. $Y = \mathbb{A}_{M,\mathbb{F}}$). We provide $X$ and $Y$ with the logarithmic structures induced by $M$. Since $\mathcal{M}_X = f^{-1}\mathcal{M}_Y$, the logarithmic strata are compatible with $f$, i.e. $X(n) = Y(n) \times_Y X$. Since $Y(n)$ and $f$ are regular, $X(n)$ is regular. In the same way, $\dim(O_{X, x}) = n + \dim(O_{X(n), x})$ for any $x \in X(n)$. Thus, $(X, \mathcal{M}_X)$ is logarithmically regular and it remains to note that $X(0) = f^{-1}(Y(0)) = U$. $\blacklozenge$

As a partial converse to the above lemma we have the following result.

**Lemma 6.4.14.** If $(X, U)$ is a toroidal scheme defined over a perfect field $\mathbb{F}$ and $(X, U) \to (\mathbb{A}_{M,\mathbb{F}}, \mathbb{A}_{M^0,\mathbb{F}})$ is a global central sharp toroidal chart over $\mathbb{F}$, then the morphism $X \to \mathbb{A}_{M,\mathbb{F}}$ is formally smooth at any point of the center $C = V(M^0 + O_X)$ of the chart.

**Proof.** Let $x \in C$ be a point and $A = \hat{O}_{X, x}$. Note that $\dim(A) \geq \text{rk}(M) + \dim O_{C, x}$. Choose a coefficient field $K \subseteq A$ and a regular family of parameters $x_1, \ldots, x_n$ of $O_{C, x}$. Then the homomorphism

$$K[[x_1, \ldots, x_n]] \to A$$

is onto, and hence an isomorphism by comparing the dimensions. It follows that $A$ is formally smooth over $\mathbb{F}[M]$, but the latter is the completed local ring of the center of $\mathbb{A}_M$.

$\blacklozenge$

### 6.5. Existence of equivariant charts.

**6.5.1. G-simple actions on toroidal schemes.** By an $L$-grading on a monoid $M$ we simply mean a homomorphism $\chi : M \to L$. By the following lemma $G$-simple actions induce gradings on the sharpened stalks of $\mathcal{M}_X$.

**Lemma 6.5.2.** Assume that a diagonalizable group $G$ acts on a toroidal scheme $(X, U)$. Let $x \in X$ be a point with stabilizer $G_x = D_{l_x}$ and $X_{x, G_x} = \text{Spec}(A)$ the $G_x$-equivariant localization at $x$. Then,

(i) The action is $G$-simple at $x$ if and only if any integral toroidal Weil divisor of $X_{x, G_x}$ is $G_x$-equivariant.

(ii) Assume that the action is $G$-simple at $x$. Then any element $\overline{m} \in \overline{\mathcal{M}}_{X, x}$ admits a lifting $f \in \mathcal{M}_X(X_{x, G_x}) \subset A$ which is $l$-homogeneous for some $l \in L_x$. Moreover, $l = \chi_x(\overline{m})$ is uniquely determined by $\overline{m}$ and the obtained map $\chi_x : \overline{\mathcal{M}}_{X, x} \to L_x$ is a homomorphism.

**Proof.** Both claims are $G_x$-local at $x$, so we can replace $X$, $U$, and $G$ with $X_{x, G_x}$, $U \times_X X_{x, G_x}$, and $G_x$, respectively, so that the action on $X = \text{Spec}(A)$ is strictly local.

(i) By Lemma 6.4.9(i), the action is $G$-simple if and only if all toroidal Cartier divisors are equivariant. This implies the inverse implication. Conversely, assume that a toroidal integral Weil divisor $E$ is not equivariant. Being closed and equivariant the union of all $G$-translates of $E$ contains $x$, hence $x \in E$. Applying
Lemma 6.4.9(ii) we obtain that $G$ acts non-trivially on the set of prime ideals of $\overline{M}_{X,x}$, hence it acts non-trivially on $\overline{M}_{X,x}$ and the action is not $G$-simple.

(ii) We can lift $\overline{m}$ to a toroidal Weil divisor $E$ which is Cartier at $x$. Since $E$ is equivariant by (i), Proposition 4.4.11 applies and we obtain that $E = V(f)$ for a homogeneous element $f \in A_l$. Let us check that $l$ depends only on $\overline{m}$. If $f' \in A_l$ is another lifting then $fA_m = f'A_m$, where $m \subset A$ is the maximal ideal corresponding to $x$. It then follows from the graded Nakayama’s lemma that $fA = fA + f'A = f'A$, see Corollary 4.4.9(ii). So, $f = uf'$ for a unit $u \in A^\times$. Since $A$ is strictly local, all graded units are of degree zero, and we obtain that $l = l'$. Thus, the map $\chi_x: \overline{M}_{X,x} \to L$ is well defined, and then it is obviously a homomorphism.

6.5.3. A local characterization of $G$-simple actions. We proved in Lemma 6.3.5(iii) that any action admitting a toroidal chart is $G$-simple. Here is a result in the opposite direction. In fact, it can be extended to any local action, but we only consider the strictly local case that will be used later.

**Proposition 6.5.4.** Assume that a toroidal scheme $(X,U)$ is provided with a $G$-simple action of a diagonalizable group $G = D_L$. If the action on $X$ is strictly local then $(X,U)$ admits a global equivariant sharp central toroidal chart.

**Proof.** Let $x$ be the closed $G$-invariant point and $M = \overline{M}_{X,x}$. Recall from Lemma 5.1.7(ii) that $X = \text{Spec}(A)$ for an $L$-local ring $(A,m)$. Moreover, $(A,m)$ is strictly local since $A/m = k(x)$. By Lemma 6.5.2(ii), any $t \in M$ admits a lifting $h(t) \in A_l$ so we obtain a homogeneous map $h: M \to A$. Since $h(t)$ is unique up to a unit of degree zero, for any $a,b \in M$ there exists a unit $u(a,b) \in A^\times_0$ such that $h(a)h(b) = h(a + b)u(a,b)$. By [Kat94, Th. 4.1], $X$ is normal. Since $x$ lies in any irreducible component of $X$, the ring $A$ is an integral domain. In particular, the monoid $Q = A \smallsetminus \{0\}$ is integral, i.e. $Q \subseteq Q^{\text{gp}}$. Note that $M^{\text{gp}} = Z^r$ and there exists a basis $t_1,\ldots,t_r \in M$. Let $\phi: M^{\text{gp}} \to Q^{\text{gp}}$ be the homomorphism with $\phi(t_i) = h(t_i)$, then for any $t \in M$ the element $\phi(t)$ can be obtained from $h(t)$ by multiplying it with units of the form $u(a,b)^\pm 1$. In particular, $\phi(t) \in Q \subset A$ is another homogeneous lifting of $t$, and the map $\phi: M \to A$ is a homomorphism.

We claim that $\phi$ is a homogeneous monoidal chart. By construction, the logarithmic structure induced by $\phi$ embeds into $M_X$, hence it suffices to show that the homomorphism $\phi_y: M \to \overline{M}_{X,y}$ is onto for any $y \in X$. Furthermore, by Lemma 6.2.13, we can replace $y$ with a generic point of its logarithmic stratum. Then $y$ specializes to a point $z$ in the center of $(X,U)$. By Lemma 6.4.7, $x$ is the only closed point of the center, hence $y$ specializes to $x$. It remains to note that $\phi_x$ is an isomorphism by construction, and the cospecialization map $\overline{M}_{X,x} \to \overline{M}_{X,y}$ is surjective.

We proved that $\phi$ is a chart. It is sharp because $M = \overline{M}_{X,x}$ is sharp and it is central because its center contains $x$.

6.5.5. Toroidal actions on toroidal schemes. The notion of a toroidal action of a diagonalizable group $G$ on a toroidal scheme $(X,U)$ generalizes the definition of [AKMW02]. A basic example of a toroidal action is provided by Lemma 6.3.8(iii).
6.5.6. A local characterization of toroidal actions. The following result shows that the converse of Lemma 6.3.8(iv) holds strictly locally for toroidal schemes. We only consider the equicharacteristic case.

**Proposition 6.5.7.** Let \((X, U)\) be a toroidal scheme provided with a toroidal action of a diagonalizable group \(G = D_L\). Assume that \(X\) is defined over a perfect field \(F\) and the action on \(X\) is strictly local. Then \((X, U)\) possesses a sharp central strongly equivariant chart, and any sharp central equivariant chart \(f : (X, U) \to (A_M, \mathbb{P}, A_M^p, \mathbb{P})\) is strongly regular.

**Proof.** By Proposition 6.5.4, \((X, U)\) possesses a sharp central equivariant chart, hence we should only prove the second claim. The center \(C\) of \((X, U)\) is irreducible by Lemma 6.4.7 and contains the closed orbit \(O = \{x\}\). Since the action is toroidal, the stabilizer is constant on \(C\), and we obtain that \(G\) acts trivially on \(C\).

Let \(z\) be the center of \(A_M\). Then the center of the chart \(f^{-1}(z)\) is a component of \(C\) hence coincides with \(C\). Thus \(G\) acts trivially on the fiber and \(f\) is formally smooth at \(x\) by Lemma 6.4.14. Therefore, Lemma 5.6.2 applies and we obtain that \(f\) is strongly regular. ☐

6.6. Toroidal quotients. In this section we define quotients of toroidal schemes and prove that they always exist for relatively affine toroidal actions. This is the main property of toroidal actions used in applications.

6.6.1. The definition. Assume that a toroidal scheme \((X, U)\) is provided with a relatively affine action of \(G = D_L\). Set \(Y = X / \! / G\) and let \(V\) be the image of \(U\) in \(Y\). If \(V\) is open and \((Y, V)\) is a toroidal scheme then we say that \((Y, V)\) is the quotient of \((X, U)\) by \(G\) and denote it \((X, U) / \! / G\). It satisfies the universal property analogous to categorical quotients: any \(G\)-equivariant morphism of toroidal schemes \((X, U) \to (Z, W)\) with the trivial action on the target factors uniquely through \((Y, V)\).

**Lemma 6.6.2.** Assume that \((Y, V) = (X, U) / \! / G\) is a toroidal quotient as above. Then \(U\) is the preimage of \(V\) if and only if the open embedding \(U \hookrightarrow X\) is inert. Furthermore, if these conditions are satisfied then \(V = U / \! / G\).

**Proof.** The direct implication is obvious. Conversely, if \(U \hookrightarrow X\) is inert then by Theorem 5.6.4, \(U / \! / G \to Y\) is an open embedding and \(U\) is the preimage of \(V = U / \! / G\). In particular, \(V = U / \! / G\). ☐

6.6.3. Taut quotients. Motivated by the lemma, we say that the toroidal quotient \((Y, V) = (X, U) / \! / G\) is taut if \(U\) is the preimage of \(V\). More generally, the quotient will be called loose if \(V = U / \! / G\).

**Remark 6.6.4.** Let \(f : X \to Y\) be the quotient morphism and consider the toroidal divisors \(D = X \setminus U\) and \(E = Y \setminus V\). The quotient is taut if and only if the inclusion \(f^{-1}(E) \subseteq D\) is an equality, and this happens if and only if \(D\) has no horizontal components, i.e. \(f(D)\) contains no generic points of \(Y\). One can show that for taut quotients the induced logarithmic morphism \((X, M_X) \to (Y, M_Y)\) satisfies the universal property of quotients in the category of logarithmic schemes. This last observation will not be used, so we omit its verification.
6.6.5. The toric case. In the toroidal case we characterize when the quotient is loose or taut in terms of the action itself. Let us consider the toric case first.

**Lemma 6.6.6.** Assume that $G = D_L$ acts on $(A_M, A_{M^\sigma})$ via a homomorphism $\phi: M \to L$. Let $M_0 = \phi^{-1}(0)$ and let $K$ be the kernel of $\phi^{gp}: M^{gp} \to L$. Then,

(i) The quotient is taut if and only if $M_0$ contains an inner element $v \in M$ (§6.1.2).

(ii) The quotient is loose if and only if the inclusion $M_0^{gp} \subseteq K$ is an equality.

Furthermore, if $\phi^{gp}$ is surjective then these properties can be characterized in terms of the dual monoid $\sigma = M^\vee$ and the dual map $(\phi^{gp})^\vee: L^\vee \to N = (M^{gp})^\vee$ as follows:

(i') The quotient is taut if and only if $L^\vee \cap \sigma = \{0\}$. Equivalently, $U \subset X^s(0)$, the Geometric Invariant Theory stable locus with respect to the trivial linearization $0 \in L$.

(ii') The quotient is loose if and only if $L^\vee \cap \sigma \subset \sigma$ is a whole face of $\sigma$. Equivalently, $\dim X / G = \dim X - \dim G$.

**Proof.** We start with (ii) and (ii'). By definition, $Y = X / G = A_{M_0}$ and $U / G = A_K$, which is a geometric quotient. This implies (ii). If $\phi^{gp}$ is surjective then the lattice $L^\vee$ is the annihilator of $K$ in $(M^{gp})^\vee$. So $K = M_0^{gp}$ if and only if the dual cone of $M_0$ in $N / L^\vee$ is strictly convex, which is equivalent to $L^\vee \cap \sigma \subset \sigma$ being a whole face of $\sigma$, proving (ii'). The second claim of (ii') is immediate.

Now, let us prove (i) and (i'). The quotient is taut if and only if $D = X \setminus U$ is the preimage of $Y \setminus V$. This happens if and only if there exists a Cartier divisor $D'$ on $X$ such that (a) $D'$ is a toroidal divisor induced from $Y$, and (b) $|D'| = |D|$. Note that a Cartier divisor on $X$ satisfies (a) if and only if it is of the form $V(v)$ for $v \in M_0$, and such divisor satisfies (b) if and only if $v$ is inner, giving (i). Assume, now, that $\phi^{gp}$ is surjective. The first claim of (i') is immediate. An inner element $m \in M_0$ is an invariant function vanishing on $X \setminus U$, implying $U \subset X^s(0)$; conversely, if $U \subset X^s(0)$ there is a homogeneous invariant element separating it from any other torus orbit. The product of finitely many of these corresponds to an inner element in $M_0$.

6.6.7. Taut toroidal actions. Lemma 6.6.6 motivates the following general definition. Assume that $G = D_L$ acts toroidally on a toroidal scheme $(X, U)$. For any point $x \in X$ let $L_x$ be the quotient of $L$ such that $G_x = D_{L_x}$. Then the action induces a homomorphism $\phi_x: \overline{M}_{X,x} \to L_x$. We say that the action is taut at $x$ if $\text{Ker}(\phi_x)$ contains an inner element. We say that the action is loose at $x$ if $(\text{Ker}(\phi_x))^{gp} = \text{Ker}(\phi_x)^{gp}$. The action is taut or loose if it is so at all points of $X$.

**Example 6.6.8.** (i) If $G$ is finite then any $G$-simple action on a toroidal scheme $(X, U)$ is taut.

(ii) If $G = \mathbb{G}_m = D_{\mathbb{Z}}$ then the action can fail to be taut only at $G$-invariant points. Assume that $x \in X$ is $G$-invariant. Then the action is taut if and only if the image of $\phi_x: \overline{M}_{X,x} \to \mathbb{Z}$ is unbounded on both sides. The action is loose but not taut if and only if $\text{Ker}(\phi_x)$ contains a facet of $\overline{M}_{X,x}$.

For the sake of reference we record the following immediate result.

**Lemma 6.6.9.** Assume that $f: (X', D') \to (X, D)$ is a strict $G$-equivariant morphism of toroidal schemes. If the action on $(X, D)$ is taut (resp. loose) then the
action on \((X', D')\) is taut (resp. loose). The converse is true whenever \(f\) is surjective.

6.6.10. Existence of toroidal quotients. Here is our main result about toroidal quotients.

**Theorem 6.6.11.** Assume that a toroidal scheme \((X, U)\) defined over a field \(\mathbb{F}\) is provided with a toroidal action of a diagonalizable group \(G = \mathbb{D}_L\) such that the torsion degree of \(L\) is invertible in \(\mathbb{F}\). Then the toroidal quotient \((X, U) / G\) exists and the quotient morphism \((X, U) \to (X, U) / G\) is toroidal. In addition, the quotient is taut or loose if and only if the action is taut or loose, respectively.

**Proof.** Replacing \(\mathbb{F}\) with the prime subfield we can make it perfect. The claim is local over a point \(y \in Y = X / G\). So, we can assume that the action is local and \(y\) is the image of the closed orbit \(O\). By Corollary 5.4.5 it suffices to establish two cases: (a) the action is strictly local, (b) the action is free. Let \(V\) denote the image of \(U\) in \(Y\).

Case (a). Let \(x\) be the closed \(G\)-invariant point and \(M = \mathcal{M}_{X,x}\). By Proposition 6.5.7 there exists a strongly equivariant toroidal chart \(f : (X, U) \to (X_0, U_0) = (\mathbb{A}_{M, \mathbb{F}}, \mathbb{A}_{M_{\mathbb{F}}, \mathbb{F}})\), so we obtain a Cartesian square

\[
\begin{array}{ccc}
X & \xrightarrow{\phi} & Y \\
\downarrow f & & \downarrow g \\
X_0 & \xrightarrow{\phi_0} & Y_0
\end{array}
\]

where \(Y_0 = X_0 / G\). Set \(V = \phi(U)\) and \(V_0 = \phi_0(U_0)\) and observe that \(V = g^{-1}(V_0)\) because \(U = f^{-1}(U_0)\) and the square is Cartesian. By Lemma 6.4.14, \(f\) is formally smooth at \(x\) and hence \(g = f / G\) is regular by Lemma 5.6.2. Applying Lemma 6.4.13 we obtain that \((Y, V) \to (Y_0, U_0)\) is a toroidal chart and so \((Y, V)\) is toroidal.

By definition, the action is taut or loose at \(x\) if and only if the action on \((X_0, U_0)\) is taut or loose, respectively. By Lemma 6.6.6 this happens if and only if the quotient \((Y_0, V_0)\) is taut or loose, and by base change the latter is equivalent to the quotient \((Y, V)\) being taut or loose, respectively. The results we proved so far apply to any \(L\). Finally, by our assumption on the torsion in \(L\), the morphism \((X_0, U_0) \to (Y_0, U_0)\) is toroidal, hence its base change \((X, U) \to (Y, V)\) is toroidal.

Case (b). Since \(U\) is \(G\)-equivariant and \(X \to Y\) is flat, \(U\) is the preimage of an open subscheme \(V \hookrightarrow Y\) by flat descent. In particular, the quotient is automatically taut. Let \(\mathcal{M}_X\) and \(\mathcal{M}_Y\) be the logarithmic structures associated with \(U\) and \(V\), respectively. By [GR13, Proposition 7.5.46(i)], logarithmic regularity descends with respect to flat strict morphism, hence it suffices to prove that the morphism \(h : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)\) is strict. By [Kat94, Theorem 4.1], \(X\) is normal, hence \(Y\) is normal by Theorem 5.2.2(i)(c). Note that \(\mathbb{D}_{L, \mathbb{F}}\) is smooth over \(\mathbb{F}\) by our assumption on \(L\), hence \(X \to Y\) is a smooth morphism and \(h\) is strict by [Tem13, Lemma 5.2.3(i)]. We note that in [Tem13] one formulates a result when \(X\) and \(Y\) are varieties, but the proof applies to normal noetherian schemes.

6.6.12. The case of \(\mathbb{G}_m\). If a scheme \(X\) is provided with a relatively affine action of \(G = \mathbb{G}_m\) then we defined in Section 5.1.14 the open subschemes \(X_+\) and \(X_-\) of
If $X$ underlies a toroidal $G$-equivariant scheme $(X,D)$ then $D_+ = X_+ \cap D$ and $D_- = X_- \cap D$ define equivariant toroidal subschemes of $(X,D)$.

**Proposition 6.6.13.** Keeping the above notation, the morphisms $(X_\pm, D_\pm) \sslash G \to (X,D) \sslash G$ are toroidal.

**Proof.** By symmetry it suffices to consider $X_+$. The claim is local on $X \sslash G$, hence we can assume that $X \sslash G$ is local and the action on $X$ is strictly local, in particular, $X = \text{Spec}(A)$. By Proposition 6.5.7, $(X,D)$ possesses a strongly equivariant chart $\phi: X \to Y = \mathbb{A}_M$, where the action on $Y$ is via a homomorphism $M \to \mathbb{Z}$. Note that the morphism $Y_- \sslash G \to Y \sslash G$ is toroidal, hence we should only prove that the bottom square in the following commutative diagram is cartesian

$$
\begin{array}{ccc}
X_+ & \xrightarrow{\phi} & X \\
\downarrow & & \downarrow \\
X_+ \sslash G & \xrightarrow{\phi_+ \sslash G} & X \sslash G \\
\downarrow & & \downarrow \\
Y_+ & \xleftarrow{\phi_+} & Y
\end{array}
$$

By strong equivariance, $\phi$ is the base change of $\phi \sslash G$. Since $\phi_+$ is the base change of $\phi$ by Lemma 5.3.9, it is also a base change of $\phi \sslash G$. The bottom square is cartesian by Lemma 6.6.14 below.

**Lemma 6.6.14.** Assume that $X,Y,X',Y'$ are acted on by a diagonalizable group $G$ so that the actions on $X$ and $Y$ are relatively affine and the actions on $X'$ and $Y_0$ are trivial. If a $G$-equivariant morphism $f: X \to Y$ is a base change of a morphism $f': X' \to Y'$, then also $f \sslash G$ is a base change of $f'$.

**Proof.** Since the quotients are categorical, the morphisms $X \to X'$ and $Y \to Y'$ factor through $X \sslash G$ and $Y \sslash G$, respectively. The problem is local on $X'$ and $Y'$, so we can assume that $X' = \text{Spec}(A')$ and $Y' = \text{Spec}(B')$. So, the morphism $f$ is affine. In addition, the problem is local on $Y' \sslash G$, so we can assume that it is affine and then $Y = \text{Spec}(B)$, $Y \sslash G = \text{Spec}(B_0)$, $X = \text{Spec}(A)$ and $X \sslash G = \text{Spec}(A_0)$. By our assumption, $A = B \otimes_{B'} A'$. Since the gradings on $A'$ and $B'$ are trivial, this implies that $A_0 = B_0 \otimes_{B'} A'$, as required.

6.7. Making actions toroidal by increasing the toroidal structure. The fact that general $G$-simple actions are not toroidal may be compensated by the following result asserting that strictly locally any $G$-simple action can be made toroidal simply by increasing the toroidal divisor.

**Proposition 6.7.1.** Let $(X,U)$ be a toroidal scheme provided with a $G$-simple action of a diagonalizable group $G = D_L$. Assume that the action on $X$ is strictly local and $X$ is defined over a field. Then there exists an equivariant open subset $U' \subseteq U$ such that $(X,U')$ is a toroidal scheme and the action of $G$ on $(X,U')$ is toroidal.

**Proof.** As usual, $X = \text{Spec}(A)$ for a strictly $L$-local ring $(A,m)$, and by $F$ we denote the prime field of definition. Let $x = \text{Spec}(k) = \text{Spec}(A/m)$ denote the closed $G$-invariant point. By Proposition 6.5.4, $(X,U)$ has an equivariant central
sharp toroidal chart \( f: X \to \mathbb{A}_M \); denote by \( \phi: M \to A \) the corresponding homogeneous homomorphism. The center of the chart is \( C = \text{Spec}(B) \), where \((B, n)\) is the \( L\)-local ring \( A/\phi(M^+)A \). The ring \( B \) is regular since \((X, U)\) is toroidal, hence Lemma 4.4.13 implies that \( n \) is generated by homogeneous elements \( s_1, \ldots, s_l \in B \), where \( l = \dim(C) \). Choose homogeneous \( t_i \in A \) lifting \( s_i \). Then the homomorphism \( k[M][t_1, \ldots, t_n] \to \tilde{O}_{X,x} \) is onto, and by dimension counting it is an isomorphism.

Consider the homogeneous homomorphism \( \phi': M' = M \oplus \mathbb{N}^l \to A \) extending \( \phi \) and sending the basis of \( \mathbb{N}^l \) to \( t_1, \ldots, t_l \). We claim that the logarithmic structure induced by \( \phi' \) as required. Let \( f': X \to \mathbb{A}_{M', \mathbb{N}} \) be the equivariant morphism induced by \( \phi' \) and \( U' = f'^{-1}(\mathbb{A}_{M^1, \mathbb{N}}) \). The fiber of \( f' \) over the center \( z \) of the target is \( \{x\} \), hence \( \tilde{O}_{X,x} \), as computed above, is formally smooth over \( \tilde{O}_z = \mathcal{F}[M][t_1, \ldots, t_n] \). Thus, Lemma 5.6.2 applies and hence \( f' \) is regular. It follows that \((X, U')\) is a toroidal scheme. The action of \( G \) on \((X, U')\) is toroidal by Lemma 6.3.8.

\[\text{Remark 6.7.2.} \quad (i) \text{ Let us say that an action of } G \text{ on a toroidal scheme } (X, D) \text{ is } \text{pretoroidal} \text{ if for any } x \in X \text{ there exists a larger divisor } D' \supset D \text{ such that } (X, D') \text{ is still toroidal, } D' \text{ is equivariant and the action on } (X, D') \text{ is toroidal at } x. \text{ In fact, Proposition 6.7.1 proves that a pretoroidal action is nothing else but a } G\text{-simple action. Pretoroidal actions were introduced in [AdJ97] for finite groups, and are related to the locally toric actions of } \mathbb{G}_m \text{ in [AKMW02], where one did not have a given toroidal structure } (X, D). \text{ Note that the definition is local and for a } G\text{-simple action it may happen that there is no larger global equivariant toroidal structure such that the action is toroidal everywhere.}

(ii) \text{Proposition 6.7.1 and Theorem 6.6.11 imply that for any toroidal scheme } (X, D) \text{ with a } G\text{-simple action of } G \text{ the singularities of } X / G \text{ are locally isomorphic to the singularities of toroidal schemes; such schemes were called } \text{locally toric} \text{ in [AKMW02]. However, there is no canonical way to find such an isomorphism; e.g. the cone } C = \text{Spec}(k[x, y, z]/(xy - z^2)) \text{ with the empty divisor is not a toroidal scheme, and there are many different ways to choose a divisor that makes it into a toroidal scheme. This makes locally toric schemes difficult to work with. For example, one can locally resolve their singularities in a combinatorial way, but Wlodarczyk [Wlo03, Theorem 8.3.2] had to develop a theory of stratified toroidal varieties to resolve them canonically, and hence globally.}

6.8. Decreasing the toroidal structure. We will also need to know when a given toroidal structure can be decreased without loosing good properties of the action. In the following lemma we use divisors in the notation of toroidal schemes.

\[\text{Proposition 6.8.1.} \quad \text{Let } (X, D) \text{ be a toroidal scheme provided with a toroidal action of a diagonalizable group } G = D_L. \text{ Assume that the action on } X \text{ is strictly local with closed } G\text{-invariant point } x \text{ and } M = \overline{M}_{X,x}. \]

(i) \text{If } Z \text{ is an irreducible component of } D \text{ and } D' = D - Z, \text{ then the following conditions are equivalent: (a) } (X, D') \text{ is a toroidal scheme and the action of } G \text{ on } (X, D') \text{ is toroidal, (b) } Z \text{ is Cartier and the corresponding element } e \in M \text{ is of degree zero with respect to the grading } \chi: \mathbb{M} \to \mathbb{L} \text{ induced by the action.} \]
(ii) Let $E$ be obtained from $D$ by removing all irreducible Cartier subdivisors $Z \subseteq D$ such that the corresponding character in $L$ is trivial. Then $E \subseteq D$ is the minimal subdivisor such that the action on $(X, E)$ is toroidal.

Proof. It suffices to prove (i) as (ii) is an immediate corollary. We have that $X = \text{Spec}(A)$ for a strictly $L$-local ring $A$. If $(X, D')$ is toroidal then $Z$ is Cartier at $x$ by Corollary 6.4.10(i), and hence $Z = V(f)$ for a homogeneous element $f \in A_l$ by Proposition 4.4.11. Recall that $Z$ does not contain the center $C' = \text{Spec}(B)$ of $(X, D')$ (e.g. because $\text{rk}(M) = \text{rk}(\mathcal{M}_{X,x}) + 1$ by Corollary 6.4.10(ii)). Therefore, the image of $f$ in $B$ is not zero. If the action on $(X, D')$ is toroidal then the action on $C'$ is trivial. So, $B$ is trivially graded and $l = 0$.

Conversely, assume that (b) is satisfied. By Proposition 4.4.11, $Z = (f)$ for $f \in A_l$, and $l = 0$ by our assumption on its image in $M$. We should check that (a1) for any point $y \in X$ the localization of $(X, D')$ at $y$ is toroidal and (a2) the action of $G_y$ on the logarithmic stratum through $y$ is trivial. Claim (a1) holds by Corollary 6.4.10(i). To prove claim (a2) we can replace $X$ with the $G_y$-equivariant localization at $y$, reducing to the case $y = x$. Then the stratum is the center $C' = \text{Spec}(B)$ of $(X, D')$. Since $C = \text{Spec}(B/fB)$ is the center of $(X, D)$, the ring $B/fB$ is trivially graded and so $B$ itself is trivially graded. Thus the action on $C'$ is trivial. $\hfill \Box$

6.8.2. Potentially taut actions. Assume that the action of $G = D_L$ on a toroidal scheme $(X, D)$ is $G$-simple. If the action on $X$ is strictly local then by Proposition 6.7.1 there exists a larger equivariant divisor $E$ that makes the action of $G$ toroidal. Let $\mathcal{N}_X$ be the associate logarithmic structure. By Proposition 6.8.1(ii) the monoid $N = \mathcal{N}_{X,x}$ with the grading $N \to L$ is uniquely determined by $(X, D)$ up to trivially graded direct summands $N$. It follows that if the action on $(X, E)$ is taut or loose then the same is true for any other equivariant divisor $E' \supseteq D$ making the action toroidal. Thus, we say that the action on $(X, D)$ is potentially taut or loose if the action on $(X, E)$ is taut or loose.

In general, a $G$-simple action is potentially taut or loose at $x \in X$ if the action of $G_x$ on the $G_x$-equivariant localization at $x$ is taut or loose.

7. Torification

Although a $G$-simple action on a toroidal scheme $(X, U)$ can be locally “improved” to a toroidal action by increasing the toroidal divisor, this procedure is neither global nor canonical. Some drawbacks of this were discussed in Remark 6.7.2. The goal of this section is to establish a better way, called torification, to make actions toroidal. Torification, introduced in [AdJ97] and developed in [AKMW02], will be achieved by a functorial blowing up of $X$ and will only increase the toroidal divisor by adding the exceptional divisor. So, the exceptional divisor plays a role analogous to its role in desingularization theory – providing new parameters in a canonical way.

We will mostly follow the methods of [AKMW02, §3], the main modification being the use of strongly regular charts instead of strongly étale charts.
7.1. Making an action $G$-simple. The restriction on an action to be $G$-simple is very mild; it can always be achieved by a simple combinatorial construction recalled below.

7.1.1. Kato Fans. To any toroidal scheme $(X, U)$ Kato associates in [Kat94, Section 10.1] a combinatorial structure we call a Kato fan $F = F(X, U)$, to distinguish it from the fans of toric geometry. It is defined as follows: points of $F$ are the maximal points of the logarithmic stratification of $(X, \mathcal{M}_X)$ and the structure sheaf of monoids $\mathcal{M}_F$ is the pullback of $\mathcal{M}_X$. Since logarithmic strata of $(X, \mathcal{M}_X)$ are irreducible, one obtains a natural retraction map $c: X \to F$, that can be viewed as a “combinatorial chart” of $(X, U)$. The polyhedral cone complex with integral structure used in [KKMSD73] for toroidal varieties can be recovered as $F(\mathbb{R}_{\geq 0})$, see [Uli13]. Any subdivision $F' \to F$ can be pulled back to $X$: Kato defines a “base change” modification $f: X' = X \times_F F' \to X$ such that $(X', f^{-1}(U))$ is a toroidal scheme with Kato fan $F'$. In particular, a sequence of subdivisions $F_n \to \cdots \to F_1 \to F_0 = F$ induces a sequence of toroidal modifications $(X_n, U_n) \to \cdots \to (X, U)$. Moreover, if the subdivisions are given by order functions (see [AW97, Proposition 2.3]) then the modifications are toroidal blowings up.

7.1.2. Barycentric subdivision. For any Kato fan $F$ its barycentric subdivision is defined as a composition of subdivisions $B(F) = F_n \to \cdots \to F_0 = F$, where $F_1 \to F$ performs simultaneous subdivisions of all cones of maximal dimension at their barycenters, $F_2 \to F_1$ subdivides the preimages of the original cones of the next dimension at their barycenters, and so on.

For any toroidal scheme $(X, U)$ the barycentric subdivision $B(F) \to F$ of its Kato fan induces via a ”base change” a sequence of toroidal blowings up $(X', U') := (X_n, U_n) \to \cdots \to (X, U)$ and we say that $(X', U') \to (X, U)$ is the barycentric modification.

Remark 7.1.3. (i) In fact, one can even realize $(X', U') \to (X, U)$ as a single toroidal blowing up along a toroidal ideal $J$, see [Niz06, Theorem 5.6] for details.

(ii) The barycentric subdivision provides a standard procedure to subdivide any Kato fan to a simplicial one. There also is a much more delicate canonical desingularization procedure that subdivides $F$ into a non-singular one, i.e. a Kato fan whose cones are of the form $\mathbb{N}^n$. The latter can be used for a canonical desingularization of toroidal schemes, see [ILO12, Theorem 3.4.9].

Proposition 7.1.4 ([AW97, Proposition 2.3]). Assume that a diagonalizable group $G$ acts on a toroidal scheme $(X, U)$. Then the barycentric modification $(X', U') \to (X, U)$ is $G$-equivariant and the action on $(X', U')$ is $G$-simple.

Proof. An action of a group $G$ on $(X, U)$ induces by functoriality an action of $G$ on $F$. Since the set of barycenters of cones of a given dimension is stable under $G$, the action lifts to $B(F)$ and by pullback the action on $(X, U)$ lifts to $(X', U')$. An irreducible component of $D' = X' \setminus U'$ is old if it is a strict transform of a component of $D = X \setminus U$. All new components of $D'$ are equivariant and all old components are disjoint. Thus, the action on $(X', U')$ is $G$-simple by Lemma 6.5.2(i)\(\clubsuit\).

7.2. Torific ideals.
7.2.1. The definition. Assume that a diagonalizable group $G = D_l$ acts on an affine scheme $X = \text{Spec}(A)$. For any element $l \in L$ let $I^A_l$ denote the ideal $A_lA$ generated by all $l$-homogeneous elements. The formation of such ideals is compatible with localization by elements of $A_0$ hence the definition globalizes to any scheme $X$ with a relatively affine $G$-action. We call the corresponding ideal $l$-torific and denote it $I^X_l \subseteq O_X$. For any finite subset $S \subseteq L$ we define the $S$-torific ideal $I^S_X = \prod_{l \in S} I^X_l$.

Remark 7.2.2. The construction of torific ideals is local on $X \sslash G$ but not on $X$. One can easily give examples where $I^S_X$ is not compatible with a non-inert open embedding.

7.2.3. Functoriality of torific ideals. Compatibility of torific ideals with inert open embeddings is a particular case of the following result.

Lemma 7.2.4. Torific ideals are compatible with strongly $G$-equivariant morphisms $f: Y \to X$ in the sense that $I^Y_S = I^S_X \mathcal{O}_Y$ for any finite subset $S \subseteq L$.

Proof. The claim is local on the quotients, hence we can assume that the quotients are affine, and so $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$ are affine. The morphism $f$ is strongly equivariant precisely when $A \otimes_{A_0} B_0 \to B$ is an isomorphism. Then $I^B_l = I^A_l B$ for any $l \in L$, and hence $I^B_S = I^S_X B$.

7.2.5. Torific blowings up. By the $S$-torific blowing up $b_{X,S}: X_S \to X$ we mean the normalized blowing up $\text{NorBl}_{I^S_X} (X) \to X$ along $I^S_X$. If needed, we will also indicate $G$ by writing $b_{X,S,G}$. The $S$-torific ideal $I^X_S$ is, by definition, $G$-equivariant. Therefore, the action of $G$ naturally lifts to $X_S$ making $b_{X,S}$ a $G$-equivariant morphism.

Remark 7.2.6. A normalized blowing up of a noetherian scheme $X$ does not have to be of finite type and can even be non-noetherian. For this reason, when working with normalized blowings up of $X$ we will always assume that $X$ is qe. In fact, an unpublished argument of Gabber implies that torific blowings up are always of finite type, but it is rather involved and we prefer not to work it out here.

7.2.7. Functoriality of torific blowings up. Recall that a morphism is normal if it is flat and has normal fibers. Torific blowings up are compatible with normal strongly equivariant morphisms.

Lemma 7.2.8. Let $G = D_l$ be a diagonalizable group, $f: Y \to X$ a normal strongly $G$-equivariant morphism, and $S \subseteq L$ a finite subset. Then $b_{Y,S}$ is the base change of $b_{X,S}$.

Proof. By Lemma 7.2.4, $I^Y_S = I^S_X \mathcal{O}_Y$. Since blowings up are compatible with flat morphisms, and normalizations are compatible with normal morphisms, we obtain that $b_{Y,S} = b_{X,S} \times_X Y$.

7.2.9. Charts of torific blowings up. Note that $b_{X,S}$ can be covered by charts of the form $Y' \to X'$ where $X' = \text{Spec}(A)$ is an open affine strongly $G$-equivariant subscheme of $X$ and $Y'$ is a chart of the blowing up along $I^A_X$. Moreover, $I^A_X$ is generated by elements $t_1, \ldots, t_n \in A_d$ where $d = \sum_{s \in S} l$, hence it suffices to consider charts $Y' = \text{Spec}(B)$ with $B = \text{Spec}(A[t_1, \ldots, t_n])$. All fractions lie in $B_0$ hence $A_lB_0 = B_l$ and therefore $I^B_{Y'} = I^A_l B$ for any finite set of characters $T \subseteq L$. Globalizing, we obtain the following lemma that extends [AKMW02, Lemma 3.2.6].
Lemma 7.2.10. Given a relatively affine action of $G = \mathcal{D}_L$ on $X$ and finite sets $S, T \subseteq L$ of characters, let $Y = X_S$. Then $I_Y^Y = I_Y^X \mathcal{O}_Y$.

7.2.11. Relative affineness. Using charts it is also easy to see that the action remains relatively affine after a torific blowing up.

Lemma 7.2.12. If $G$ acts relatively affine on $X$ and $b_{X,S} : X' \rightarrow X$ is a torific blowing up then the induced action of $G$ on $X'$ is relatively affine.

Proof. The question is local on $X$, so we can assume that $X = \text{Spec}(A)$. Then $Y$ is covered by $G$-equivariant charts $Y_i = \text{Spec}(A[\frac{t_1}{t_i}, \ldots, \frac{t_n}{t_i}])$. The intersection $Y_{ij} = Y_i \cap Y_j$ is the localization of $Y_i$ along $\frac{t_j}{t_i}$. Since the latter is of degree 0, the open embedding $Y_{ij} \hookrightarrow Y_i$ is inert.

7.2.13. Composition of torific blowings up. Torific blowings up are compatible with compositions in the following sense.

Lemma 7.2.14. Given a relatively affine action of $G = \mathcal{D}_L$ on $X$ and finite sets $S, T \subseteq L$ of characters with $R = S \cup T$, the $R$-torific blowing up $b_{X,R}$ factors into the composition of torific blowings up $b_{X,S} : Y = X_S \rightarrow X$ and $b_{Y,T} : X_R \rightarrow Y$.

Proof. Set $Z = X_R$ for simplicity. By Lemma 7.2.10, $I_Y^Y$ is the inverse image of $I_R^X$. So, $I_Y^Y$ is invertible, and hence all ideals $I_l^Y$ with $l \in R$ are invertible. In particular, $I_S^Y$ is invertible. Recall that a normalized blowing up along $I$ is the universal morphism such that the source is normal and the inverse image of $I$ is invertible. Therefore $b_S^X$ factors through $Y$ and $Z \rightarrow Y$ is the normalized blowing up along the inverse image of $I_R^X$. The latter coincides with $I_Y^Y$ by Lemma 7.2.10, and so $Z = Y_R$. The same argument as above shows that $I_Y^Y$ is invertible for any $l \in S$. In particular, the factors $I_l^Y$ in the formula $I_Y^Y = I_T^Y \cdot \prod_{l \in S \setminus T} I_l^Y$ are invertible, and so $Y_T = Y_R = Z$, as required.

7.2.15. Ample sets of characters. We say that a subset $S \subseteq L$ is ample with respect to the action of $G$ on a scheme $X$ if for any point $x \in X$, the set $S$ contains all characters of $G$ through which it acts on the cotangent space $m_x/m_x^2$.

Lemma 7.2.16. If a noetherian scheme $X$ is provided with an action of a diagonalizable group $G = \mathcal{D}_L$, then there exists a finite subset $S \subseteq L$ ample with respect to $X$.

Proof. Since $X$ is covered by finitely many $G$-equivariant affine open subschemes we can assume that $X = \text{Spec}(A)$. Then $A$ is $L$-graded and finitely generated over $A_0$. Therefore, $A = A_0[f_1, \ldots, f_n]$, where $f_i \in A_{t_i}$, and we can take $S = \{l_1, \ldots, l_n\}$. ♣

7.2.17. Functoriality of ampleness.

Lemma 7.2.18. Let $G = \mathcal{D}_L$ be a diagonalizable group, $f : Y \rightarrow X$ a strongly $G$-equivariant morphism, and $S \subseteq L$ a finite subset. If $S$ is ample with respect to the action on $X$ then it is ample with respect to the action on $Y$, and the converse is true whenever $f$ is regular and surjective.

Proof. For any point $y \in Y$ and $x = f(y)$ we have a map of the cotangent spaces $h : m_x/m_x^2 \rightarrow m_y/m_y^2$, and the cokernel is the cotangent space to the fiber $f^{-1}(x)$ at $y$. The action on $f^{-1}(x)$ is trivial by the strong equivariance, hence $G$ acts
trivially on the cokernel of $h$. Thus, if a character $\chi$ of $G$ appears in the action on $m_{x}/m_{y}^{2}$ then it also appears in the action on $m_{x}/m_{y}$, and we obtain the direct implication.

Conversely, if $f$ is regular then $h$ is an embedding hence any character appearing in the action on $m_{x}/m_{y}^{2}$ appears also in the action on $m_{y}/m_{y}^{2}$. ♣

7.3. Toric blowings up of toroidal schemes. In the sequel we denote toroidal schemes using divisors, i.e., instead of $(X, U)$ we will write $(X, D)$, where $D = X \setminus U$.

7.3.1. Blowings up of toroidal schemes. Let $(X, D)$ be a toroidal scheme and $Z \subseteq X$ a closed subscheme. Consider the (normalized) blowing up $X' \to X$ along $Z$ and set $D' = f^{-1}(D \cup Z)$. In other words, $D'$ is the union of the preimage of $D$ and the exceptional divisor $f^{-1}(Z)$. If $(X', D')$ is a toroidal scheme then we say that a (normalized) blowing up of $(X, D)$ along $Z$ is permissible, and refer both to $(X', D')$ and $f : (X', D') \to (X, D)$ as the (normalized) blowing up along $Z$. If $Z$ is a toroidal subscheme, i.e. $Z = V(I\mathcal{O}_X)$, where $I$ is an ideal in $\mathcal{M}_X$, then $f$ is called toroidal blowing up.

Remark 7.3.2. Typically a blowing up of $(X, D)$ along a subscheme $Z$ is not permissible, and even if it is permissible then the blowing up morphism $f$ is usually not toroidal. However, toroidal blowings up are always permissible and are toroidal morphisms, as can be easily seen using charts.

7.3.3. Toric blowings up. If $G$ acts on $(X, D)$ and $X' \to X$ is a toric blowing up then we say that the normalized blowing up $f : (X', D') \to (X, D)$ is a toric blowing up. Obviously, the morphism $f$ is $G$-equivariant.

Lemma 7.3.4. Assume that $G = \mathbb{D}_L$ acts toroidally on a toroidal scheme $(X, D)$ and $X$ is defined over a field. Then,

(i) Any toric ideal $I_X^{\Sigma}$ is toroidal, in particular, the toric blowing up $b_{X,S} : (X', D') \to (X, D)$ is permissible and toroidal.

(ii) The action on $(X', D')$ is toroidal. Furthermore, if the action on $(X, D)$ is taut or loose, then the same holds for the action on $(X', D')$.

Proof. Assume first that $X = \mathbb{A}_{M,\Sigma}$ with toric divisor $D$ and $G$ acts via $\chi : M \to L$. Then $I_X^{\Sigma}$ is generated by elements $m_1, \ldots, m_n \in M$ so $b_{X,S}$ is toric, hence toroidal, and (i) holds. Furthermore, $X'$ is glued from blow up charts $X'_i$ of the form $\mathbb{A}_{M'_i}$ where $M'_i = M[m_1 - m_i, \ldots, m_n - m_i]$ and $G$ acts through the homomorphism $M'_i \to L$ extending $\chi$. By Lemma 6.3.8(iii) the action is toroidal. If the action on $(X, D)$ is taut then $\text{Ker}(\chi)$ contains an inner element $v \in M$. Since $v$ is also inner in the larger monoids $M'_i$, the action on $(X', D')$ is taut. The claim about looseness is also simple, so we omit the justification.

Now let us consider the general case. The claim is local at $x \in X$, hence replacing $G$ and $X$ with $G_x$ and a $G_x$-equivariant localization of $X$ at $x$ we can assume that the action is strictly local. By Proposition 6.5.7, $(X, U)$ admits a strongly regular atlas $h : (X, U) \to (\mathbb{A}_{M,\Sigma}, \mathbb{A}_{M_{\Theta},\Sigma})$. The statement has been proven above for the target, so it remains to recall that all ingredients are compatible with strongly regular strict morphisms: Lemma 7.2.8 deals with toric blowings up. Lemma 6.3.8(ii) deals with the action being toroidal, and compatibility of tautness and looseness follows from Lemma 6.6.9. ♣
7.4. Torification of G-simple actions. In this section we prove the following torification theorem.

**Theorem 7.4.1.** Assume that a toroidal scheme \((X, D)\) is provided with a \(G\)-simple relatively affine action of a diagonalizable group \(G = \mathbb{D}_L\) and \(S \subseteq L\) is a finite set of characters, which is ample with respect to the actions of \(G\) on the logarithmic strata of \((X, D)\). Assume also that \(X\) is defined over a field. Then the torific blowing up \(b_{X,S}: (X', D') \rightarrow (X, D)\) is permissible and \(G\) acts on \((X', D')\) toroidally. Moreover, if the action on \((X, D)\) is potentially taut or loose then the action on \((X', D')\) is taut or loose, respectively.

Note that it may happen that \(I_S^X = 0\). In this case the assertion of the theorem holds true but becomes vacuous. See Section 7.4.8 for a way around this issue.

7.4.2. **Proof of Theorem 7.4.1: the plan.** The proof will be in three steps. First, one establishes the model case of toric varieties over \(\mathbb{Z}\). Then, we will deduce the case when the action is strictly local, and after that will deal with the general case.

7.4.3. **The model case.** The following special model case models a \(G\)-simple action which becomes toroidal after adjoining a few regular parameters. Let \(P\) be a toric monoid and \(M = P \oplus \mathbb{N}^r\), and set \(X = \mathbb{A}_M\), \(D = \bigcup_{f \in P^*} V(f)\), and \(\overline{D} = \bigcup_{f \in M^*} V(f)\). By \(e_1, \ldots, e_r\), we denote the basis of \(\mathbb{N}^r\). Both \((X, D)\) and \((X, \overline{D})\) are toroidal schemes, and we have a chart \(h: (X, D) \rightarrow (\mathbb{A}_P, \mathbb{A}_P^*)\). Assume that \(G = \mathbb{D}_L\) acts on \(X\) via a homomorphism \(\chi: M \rightarrow L\), then both \((X, \overline{D})\) and \(h\) acquire an action of \(G\). The actions on \((X, \overline{D})\) and \((\mathbb{A}_P, \mathbb{A}_P^*)\) are toroidal by Proposition 6.3.8(iii). Recall that by Proposition 6.8.1(ii), the action on \((X, D)\) is toroidal if and only if \(\chi(\mathbb{N}^r) = 0\).

**Lemma 7.4.4.** Keeping the above notation, let \(S \subseteq L\) be a finite subset containing \(\chi(e_1), \ldots, \chi(e_r)\). Consider the \(S\)-torific blowing up \(b_{X,S}: X' \rightarrow X\) and let \(D' \subset X'\) be the union of the preimage of \(D\) and the exceptional divisor. Then the action of \(G\) on \((X', D')\) is toroidal. In addition, if the action on \((X, \overline{D})\) is taut or loose then the same is true for \((X', D')\).

The following special monoidal case is established in [AKMW02, Proposition 3.2.5(2)]: \(L = \mathbb{Z}\) and \(X = \mathbb{A}_{M,K}\), where \(K\) is a field of characteristic zero. However, the arguments are combinatorial and apply to our more general situation verbatim. For the sake of completeness we briefly outline the proof.

**Proof.** We can assume that \(I_S^X \neq 0\) as otherwise \(X'\) is empty and there is nothing to prove. Let \(z^m\) denote the image of \(m \in M\) in \(\mathbb{Z}[M]\) and \(t_i = z^{e_i}\). Note that \(D\) is obtained from \(\overline{D}\) by removing \(r\) divisors \(D_1 := V(t_i)\). Hence \(D'\) is obtained from the preimage \(\overline{D}\) of \(\overline{D}\) by removing the strict transforms \(D'_1, \ldots, D'_r\) of \(D_1, \ldots, D_r\).

The action on \((X', \overline{D'})\) is toroidal by Lemma 7.3.4(ii), and by Lemma 6.8.1(ii) it suffices to prove that it remains toroidal if we remove from \(\overline{D'}\) a single component \(D'_i\).

Set \(e = e_i\) and \(l = \chi(e)\), and split \(M\) as \(Q \oplus Ne\) (i.e. \(e\) is the generator of the second summand). First we consider the particular case when \(S = \{l\}\). We can choose generators \(z^e, z^g, \ldots, z^a\) of \(\mathbb{Z}[M]\) such that \(q_j \in Q\). The strict transform of \(D_i\) is non-empty on the charts \(\mathbb{A}_{M_j}\) where \(M_j = M[e - q_j, q_i - q_j, \ldots, q_n - q_j]\). Clearly, \(M_j = Q[q_1 - q_j, \ldots, q_n - q_j] \oplus \mathbb{N}(e - q_j)\) and the generator of the second summand is...
of degree zero. Hence after removing the strict transform \( D'_i = V(e - q_j) \) from \( \overline{D'} \), the action remains toroidal by Lemma 6.8.1(i). In addition, if \( \ker(\chi) \) contains an inner vector of \( M \) then the same vector is inner in \( M_j \) and lies in \( \chi_j : \ker(M_j \to L) \), and if \( M \cap \ker(\chi) \) spans \( \ker(\chi^{\text{sp}}) \) then the same is true for \( \ker(\chi_j) \supseteq \ker(\chi) \). It follows that tautness and looseness are preserved in this case.

In general, Lemma 7.2.14 tells us that \( b_{X,S} \) is the composition of the toric blowings up \( b_{X,(1)} : X'' \to X \) and \( b_{X'',S} : X' \to X''. \) Let \( D'' \subset X'' \) be the union of the preimage of \( D \) and the exceptional divisor. By the above case, \((X'',D'')\) is a toroidal scheme acted on toroidally by \( G \). Since \( D' \) is the union of the preimage of \( D'' \) and the exceptional divisor, Lemma 7.3.4 tells us that \( b_{X'',S} : (X',D') \to (X'',D'') \) is a toroidal blowing up and the action on the source is toroidal. The same lemma also says that tautness and looseness are preserved by \( b_{X'',S} \).

7.4.5. The strictly local case. Back to the proof of Theorem 7.4.1. Assume that the action on \( X \) is strictly local, and let \( x \) be the closed \( G \)-invariant point. By Proposition 6.7.1 there exists a larger divisor \( 
abla \supseteq D \) such that the action on \((X,\nabla)\) is toroidal. By definition, if the action on \((X,D)\) is potentially taut (resp. loose) then the action on \((X,\nabla)\) is taut (resp. loose). Let \( \mathcal{M}_X \) and \( N_X \) be the logarithmic structures associated with \( D \) and \( \nabla \), respectively, then \( M = N_X \mid_X \) and \( P = \mathcal{M}_X \mid_X \) are related by \( M = P \oplus \mathbb{N}^- \).

By Proposition 6.5.7, there exists a strongly equivariant sharp central chart \( f : (X,D) \to (X_0,\nabla_0) \), where \( X_0 = A_{M,\mathbb{F}} \) with \( \mathbb{F} \) the prime field of definition of \( X \), and \( \nabla_0 \) is the toric divisor. Set \( D_0 = \cup_{f \in P} V(f) \subset \nabla_0 \), then \((X_0,D_0)\) is a toroidal scheme, and locally at \( x \), the preimage of \( D_0 \) coincides with \( D \). Each irreducible component of \( \nabla \) is equivariant, and hence passes through \( x \). It follows that \( D \) is the preimage of \( D_0 \), and we obtain a strict inert equivariant chart \( f : (X,D) \to (X_0,D_0) \).

The toric blowing up \( b_{X,S} \) is the pullback of the toric blowing up \( b_{X_0,S} : X'_0 \to X_0 \) by Lemma 7.2.8, so we obtain an equivariant chart \((X',D') \to (X'_0,D'_0) \). By Lemma 7.4.4, the action on \((X'_0,D'_0)\) is toroidal, hence the action on \((X',D')\) is toroidal by Lemma 6.3.8(ii).

7.4.6. The general case. Now, let us prove Theorem 7.4.1 in general. By Lemma 7.2.8, toric ideals and blowings up are compatible with equivariant localizations along special orbits, since such localizations are strongly equivariant. Since \( X \) is covered by such localizations, it suffices to prove the theorem for them. So, we can assume that the action is local. The strictly local case was proven above, so assume that the stabilizer \( G' = D_{L'} \) of the closed orbit is strictly smaller than \( G \). Then by noetherian induction on the quotients of \( L \) we can assume that the theorem holds for \( G' \). The stabilizer of any point of \( X \) and \( X' \) is contained in \( G' \), hence the action of \( G \) on \((X',D')\) is toroidal if and only if the action of \( G' \) is toroidal. Since the image \( S' \subseteq L' \) of \( S \) is an ample set for the action of \( G' \) on the logarithmic strata of \((X,D)\), it suffices to prove that \( b_{X,S,G} \) coincides with \( b_{X,S',G'} \). This is done in the following lemma.

Lemma 7.4.7. Let \( \phi : L \to L' \) be a surjective homomorphism of finitely generated abelian groups, \( S \subseteq L \) a finite subset, and \( S' = \phi(S) \). Assume that \( A \) is an \( L \)-graded ring such that for any \( n \in \ker(\phi) \) there exists a unit \( u_n \in A_n \), and let \( A' \) denote the
ring $A$ with the induced $L'$-grading. Then there is an equality of the torific ideals $I^A_S = I^Y_S$.

**Proof.** By our assumption on the units, $I^A_l = I^A_{f \circ l}$ for any $l \in L$ and $n \in \operatorname{Ker}(\phi)$. Therefore, $I^A_l = I^{X'}_{\phi(l)}$ and the lemma follows. ✷

7.4.8. Adjusted torific ideals. The $l$-torific ideal might vanish identically on open subsets of $X$. Assume $X$ is noetherian and normal. We devise the following ad-hoc variant, the adjusted $l$-torific ideal $\tilde{I}^X_l$ defined on the connected components $X_i$ of $X$: we define

$$(\tilde{I}^X_l)_{|X_i} = \begin{cases} I^{X'}_{l_i} & \text{if } I^{X'}_{l_i} \text{ does not vanish identically, and} \\ \mathcal{O}_{X_i} & \text{otherwise.} \end{cases}$$

For any finite subset $S \subseteq L$ we define the adjusted $S$-torific ideal $\tilde{I}^X_S = \prod_{l \in S} \tilde{I}^X_l$. We define the adjusted torific blowing up $\tilde{b}_{X,S} : (\tilde{X}', D') \to (X, D)$ to be the normalized blowing up of $\tilde{I}^X_S$. All the results proven so far about torific ideals and blowings up extend immediately to the adjusted case, simply working component by component.

7.5. Main torification theorem. We can summarize all results about modification of toroidal schemes in the following theorem.

**Theorem 7.5.1.** Let $G$ be a diagonalizable group. For any toroidal scheme $(X, D)$ defined over a field and provided with a relatively affine action of $G$ there exists a sequence $f_{(X,D)} : X = X_n \to \cdots \to X_0 = X$ of normalized blowings up with nowhere dense centers such that the following conditions are satisfied:

(i) Let $D'$ be the union of the preimage of $D$ and the exceptional divisor of $f_{(X,D)}$. Then the pair $(X', D')$ is toroidal and the natural $G$-action on $(X', D')$ is toroidal.

(ii) The sequences $f_{\bullet}$ are functorial with respect to surjective strict inert regular equivariant morphisms $h : (Y, E) \to (X, D)$ of toroidal schemes in the sense that $f_{(Y,E)} : Y' \to Y$ is the base change $f_{(X,D)} \times_X Y$ of $f_{(X,D)}$.

**Proof.** Note that the barycentric subdivision $(\tilde{X}, \tilde{D}) \to (X, D)$ is a sequence of normalized blowings up, $\tilde{D}$ is the preimage of $D$ and $\tilde{D}$ contains the exceptional divisor.

Let $S = S(\tilde{X}, \tilde{D}) \subseteq L$ be the minimal subset which is ample with respect to the action of $G$ on the logarithmic strata of $(\tilde{X}, \tilde{D})$. In particular, each element of $S$ appears as a character of the action of $G$ on the cotangent space $m_x/m_x^2$ to some point $x \in X$. Since $\tilde{X}$ is integral, its support is nowhere dense. We take the adjusted torific blowing up $\tilde{b}_{\tilde{X}, S} : X' \to \tilde{X}$ to be the last blowing up in the sequence, so let $D'$ be the union of the preimage of $\tilde{D}$ and the exceptional divisor. We claim that the sequence $f_{(X,D)} : X' \to \tilde{X} \to \cdots \to X$ is as required. Indeed, the action on $(\tilde{X}, \tilde{D})$ is strict by Proposition 7.1.4, hence the action on $(X', D')$ is toroidal by Theorem 7.4.1, and our construction of $f_{(X,D)}$ satisfies (i).

It remains to check that this construction is compatible with $h$, so consider the analogous sequence $f_{(Y,E)} : (Y', E') \to (\tilde{Y}, \tilde{E}) \to \cdots \to (Y, E)$. Barycentric subdivisions are compatible with strict morphisms hence the morphism $\tilde{h} : (\tilde{Y}, \tilde{E}) \to (\tilde{X}, \tilde{D})$ is the base change of $h$. In particular, $\tilde{h}$ is regular and inert, and hence
strongly equivariant by Theorem 5.6.4. Being a base change of $h$, the morphism $\tilde{h}$ is strict and hence respects the logarithmic strata. Since $h$ is surjective, $\tilde{h}$ is surjective and by Lemma 7.2.18, $S := S(\tilde{X}, \tilde{D}) = S(\tilde{Y}, \tilde{E})$. Finally, $b_{\tilde{Y},S}$ is the base change of $b_{\tilde{X},S}$ by Lemma 7.2.8, hence the construction of $f_*$ satisfies (ii).  

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