Wildly Compatible Systems and Six Operations

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September 18, 2018

Abstract

Let $S$ be an excellent regular scheme and let $X$ be a scheme separated and of finite type over $S$. Let $K_c(X,\mathbb{F}_\lambda)$ be the Grothendieck ring of $\mathbb{F}_\lambda$-constructible sheaves on $X$, where $\mathbb{F}_\lambda$ is the finite field with $\lambda$ elements. Given an index set $I$ and for certain $\mathbb{Q}$-vector subspaces $V \subseteq \prod_{i \in I} \mathbb{Q}_{\lambda_i}$, we define wildly compatible systems of virtual constructible sheaves on $X$. The main result is that for dim $S \leq 1$, wildly compatible systems are preserved by Grothendieck’s six operations and Verdier’s duality, with further assumption that $V$ is a sub-algebra for derived Hom and tensor product. Finally, when $X$ is a curve over a finite field we prove all $\ell$-adic compatible systems will give wildly compatible systems.

1 Introduction

Let $k$ be an algebraically closed field with characteristic $p$ and let $Z$ be a proper normal connected scheme over $k$. Let $X \subset Z$ be an open dense subscheme. Let $\ell$ be a prime distinct with $p$ and let $\lambda$ be a power of $\ell$. Deligne-Illusie ([7]) have defined the notion of “same wild ramification” and proved that if two locally constant sheaves $\mathcal{F}_1, \mathcal{F}_2$ of $\mathbb{F}_\lambda$-modules on $X$ have the same rank and the same wild ramification along $Z \setminus X$, then they have the same Euler-Poincaré characteristic $\chi_{\ell}(X, \mathcal{F}_1) = \chi_{\ell}(X, \mathcal{F}_2)$ with $\chi_{\ell}(X, -) = \sum (-1)^i \dim H^i_\ell(X, -)$.

Let $S$ be the spectrum of an excellent henselian discrete valuation ring with residue characteristic exponent $p \geq 1$ or Spec$(k)$, where $k$ is a field of characteristic exponent $p \geq 1$. Let $X, Y$ be two schemes separated and of finite type over $S$. Vidal ([14], [15]) has defined a subgroup $K_{\ell}(X, \mathbb{F}_\lambda)_t^{0} \subset K_{\ell}(X, \mathbb{F}_\lambda)$, where $K_{\ell}(X, \mathbb{F}_\lambda)$ is the Grothendieck group of constructible sheaves of $\mathbb{F}_\lambda$-modules on $X$. This subgroup is called the Grothendieck ring of constructible sheaves of virtual wild ramification 0. Vidal has proved this subgroup is preserved by Grothendieck’s six operations, which generalizes the result of Deligne-Illusie.

Let $S$ and $X$ be as in Vidal’s case. For $\mathcal{F}_i \in K_{\ell}(X, \mathbb{F}_\lambda), (i = 1, 2)$ with $(\lambda_i, p) = 1$, Yatagawa ([16]) has defined the subgroup of having the same wild ramification and proved it is preserved by four operations and Verdier’s duality.

In this paper, we introduce and study a general notion of wildly compatible systems of virtual $\mathbb{F}_\lambda$-constructible sheaves on $X$ for $i \in I$, where $X$ is a scheme separated and of finite type over $S$ and $S$ is an excellent scheme. We assume that $\lambda_i$ are all invertible over $S$. All wildly compatible systems associated to index set $I$ and a fixed subspace $V \subseteq \prod_{i \in I} K_{\lambda_i}$ form a subgroup $K_{\ell}(X, I) \subset K_{\ell}(X, I) := \prod_{i \in I} K_{\ell}(X, \mathbb{F}_\lambda)$, where $K_{\lambda_i} := \mathbb{Q}_{\lambda_i} \cap \mathbb{Q}(\wp)$. Virtual constructible sheaves in a wildly compatible system can be viewed as having “compatible wild ramifications”: for a wildly compatible system Brauer traces defined by virtual locally constant sheaves, which reflect wild ramifications of these sheaves, form vectors in $V$ as components. The main result of this paper is that when $S$ is excellent regular with dim $S \leq 1$, wildly compatible systems are preserved by Grothendieck’s four operations $f^*, f^!, f_!, f_!$ and Verdier’s duality $D_X$. With further assumption that $V$ is a sub-algebra, $R\text{Hom}$ and $- \otimes -$ also preserve wildly compatible systems. In fact, this result generalizes those of Vidal and Yatagawa. Besides, as a new special example of our notion, we will define wildly compatible systems associated to $L$ and $\{\iota_i\}$, where $L$ is a fixed field of characteristic zero and $\iota_i : L \hookrightarrow \overline{\mathbb{Q}}_{\ell_i}$ are fixed embeddings of fields. Finally, this example is related to $\ell$-adic compatible systems of virtual constructible sheaves when $X$ is a curve over a finite field.
In Section 2, we define the wild part of $\pi_1$, which generalizes Vidal’s definition in [14] and prove a formula connecting global and local wild parts of $\pi_1$ (Theorem 2.4). Then we explore some functorial properties of wild part of $\pi_1$ (e.g., Proposition 2.8, Corollaries 2.10, 2.11).

In Section 3, in order to compare wild ramifications we define wildly compatible system of virtual constructible (resp. locally constant) $F_{\lambda_i}$-sheaves and prove these definitions have some nice properties (e.g., Corollary 3.7). Then we give three special examples of wildly compatible systems: Vidal’s notion, Yatagawa’s notion and wildly compatible systems associated to $L$ a field of characteristic zero (Remark 3.8).

In Section 4, we show that wildly compatible systems are preserved by Grothendieck’s six operations and Verdier’s duality (Theorem 4.1).

In Section 5, with the assumption that $X$ is a curve over a finite field, compatible $\ell$-adic virtual constructible sheaves are related to our wildly compatible virtual constructible sheaves. In fact, we prove that the decomposition map $d : K(X,E) \to K(X,F)$ sends compatible $\lambda$-adic virtual constructible sheaves to wildly compatible virtual constructible $F_{\lambda}$-sheaves (Theorem 5.1).

Acknowledgement

Thanks to my advisor, Weizhe Zheng. With his forward-looking instruction and guidance, this paper was written smoothly. He also have helped me to correct mistakes and defects with infinite patience.

2 The wild part of $\pi_1$

The wild part of $\pi_1$ is used to describe wild ramifications of sheaves. To define it, we recall compactifications of schemes.

Nagata compactifications Let $S$ be a quasi-compact and quasi-separated scheme and let $X$ be a scheme over $S$. We say $f : X \to S$ is compactifiable or has a compactification, if there is an $S$-scheme $\overline{X}$ satisfying the following commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{i} & \overline{X} \\
\downarrow{f} & & \downarrow{\overline{f}} \\
S & & \overline{S}
\end{array}
\]

where $i$ is an open immersion and $\overline{f}$ is a proper morphism. Nagata’s compactification theorem says that with the assumptions of $X$ and $S$ as above, $X \to S$ is compactifiable if and only if $X \to S$ is of finite type and separated ([1], Theorem 4.1).

If $X \to S$ is compactifiable, all compactifications of it form a category $\mathcal{C}$. The objects of $\mathcal{C}$ are triplets $(\overline{X}, i, \overline{f})$ as in the above diagram. The morphisms in $\mathcal{C}$ for compactifications $(Z_1, i_1, f_1), (Z_2, i_2, f_2)$ are such $g : Z_1 \to Z_2$ satisfying the following commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{i_1} & Z_1 & \xrightarrow{g} & Z_2 \\
\downarrow{f_1} & & \downarrow{f} & & \downarrow{f_2} \\
S & & & & \overline{S}
\end{array}
\]

so that $g \circ i_1$ is the open immersion $i_2 : X \hookrightarrow Z_2$.

In fact, $\mathcal{C}$ is a cofiltered category ([3], 3.2.6): 1) Given two objects $Z_1, Z_2$ in $\mathcal{C}$, there exists an object $Z_3$ and morphisms $g_1 : Z_3 \to Z_1$ and $g_2 : Z_3 \to Z_1$ in $\mathcal{C}$. 
2) Given two morphisms $Z_1 \xrightarrow{r} Z_2$ in $C$, there exists an object $Z_3$ and a morphism $t : Z_3 \to Z_1$ in $C$ such that $s \circ t = r \circ t$.

For 1), it suffices to take $Z' := Z_1 \times_S Z_2$ and $Z_3$ a compactification of $(i_1, i_2) : X \to Z'$. For 2), the equalizer exists in the category of $S$-schemes and is denoted by $K$. By universal property, $i_1$ is factorized as $i_1 = k \circ i$ and $k : K \to Z_1$ is proper. Note that $K \to S$ is proper but $i$ is not necessarily an open immersion. So we take a compactification of $i : X \to K$ as $X \hookrightarrow Y \to K$ then $k \circ p : Y \to Z_1$ is what we want.

In the sequel we assume that $S$ is an excellent scheme.

**Nagata normal compactifications** Let $X$ be a normal connected scheme separated and of finite type over $S$. For any Nagata compactification $\overline{X}$ of $f : X \to S$, one can consider its normalization $\overline{\overline{X}}$. By the universal property of normalization there is a morphism $i : X \to \overline{\overline{X}}$. We claim that $X \xrightarrow{i} \overline{\overline{X}} \xrightarrow{\nu \circ f} S$ is another compactification of $f : X \to S$.

The base change $Y := \overline{\overline{X}} \times_{\overline{X}} X \to X$ of $\overline{\overline{X}} \to \overline{X}$ by $X \to \overline{X}$ is a finite birational morphism from integral scheme to normal connected schemes, hence is an isomorphism. Then the section morphism $s$ is an isomorphism and the composition $j' \circ s$ is an open immersion. By excellence assumption of schemes over $S$, $\nu$ is finite hence the composition $\nu \circ f$ is proper. This completes the proof.

All normal compactifications of $X \to S$ form a full subcategory $C'$ of $C$ defined above. Further $C'$ is a co-cofinal subcategory of $C$:

1) For every $x \in \text{Obj}(C)$, there is an object $y \in \text{Obj}(C')$ and a morphism $y' \to x$ in $\text{Morph}(C)$.

2) Given $x \in \text{Obj}(C)$ and $y_1, y_2 \in \text{Obj}(C')$ with $y_1 \to x$ and $y_2 \to x$, there exists $y_3 \in \text{Obj}(C')$ and morphisms

$$y_1 \leftarrow y_3 \rightarrow y_2$$

which form a commutative diagram.

It is clear that 1) and 2) follows directly from the fact that given any two normal compactifications in $C'$ there is always a normal compactification dominating them.

**Nagata normal dense compactifications** Further we consider normal compactifications $\overline{X}$ of $X \to S$ such that $X \to \overline{X}$ are dominant open immersions. We assume that $X$ is a normal connected scheme separated and of finite type over an excellent scheme $S$. In this case for an arbitrary normal compactification $\overline{X}$ we can always take the connected component $\overline{X}'$ of $\overline{X}$ that contains $X$. Hence $X$ is open dense in $\overline{X}'$, which is also a normal compactification of $X \to S$ dominating $\overline{X}$. Thus all normal compactifications of $X \to S$ such that $X$ is an open subscheme form a cofiltered co-cofinal subcategory of $C'$ and $C$.

By the results about co-cofinality above, from now on for convenience we always assume $X$ is an open dense subscheme in every normal compactification of $X \to S$.
Wild part of $\pi_1$

**Definition 2.1.** Let $\mathcal{X}$ be a normal connected scheme and let $X \hookrightarrow \mathcal{X}$ be an open immersion, where $X$ is an arbitrary scheme. Let $\eta$ be a geometric generic point of $X$. We define some subsets of $\pi_1(X, \eta)$ as follows.

(i) Let $\overline{\pi}$ be a geometric point of $\mathcal{X}$ and let $\overline{\nu}$ be a geometric point of $\mathcal{X}(\overline{\pi}) \times_{\mathcal{X}} X$ lying above $\overline{\eta}$. We take $p \geq 1$ to be the characteristic exponent of $\overline{\pi}$.

We define a subset $E_{\overline{\pi}, \overline{\nu}}$ of $\pi_1(\mathcal{X}(\overline{\pi}) \times_{\mathcal{X}} X, \overline{\nu})$ to be the union of images of $p$-Sylow subgroups of $\pi_1(\mathcal{X}(\overline{\pi}) \times_{\mathcal{X}} X, \overline{\nu})$, where $\mathcal{X}(\overline{\pi})$ denotes the strict localization of $\mathcal{X}$ at $\overline{\pi}$.

(ii) We define $E_{\overline{\pi}, \overline{\nu}}$ as $\bigcup_{g \in \pi_1(X, \nu)} g^{-1}E_{\overline{\pi}, \overline{\nu}}g$, where $g$ goes through all elements in $\pi_1(X, \nu)$ and $\overline{\nu}$ is a fixed geometric point of $\mathcal{X}(\overline{\pi}) \times_{\mathcal{X}} X$ lying over $\overline{\eta}$.

(iii) We define $E_{\overline{\pi}, \overline{\nu}}$ as $\bigcup_{\overline{\pi} \to \mathcal{X}} E_{\overline{\pi}, \overline{\nu}}$, where $\overline{\pi}$ goes through all geometric points of $\mathcal{X}$.

(iv) Let $S$ be an excellent scheme and let $X$ be a normal connected scheme separated and of finite type over $S$. We define $E_{X/S}$ as

$$E_{X/S} := \bigcap_{\overline{\nu}} E_{\overline{\pi}, \overline{\nu}}$$

where $\overline{\nu}$ goes through every normal compactification $\overline{\mathcal{X}}$ of $X \to S$.

(v) With the same assumption as (iv) let $\tau$ be the generic point of $X$. We define $E_{\tau/S}(X)$ as

$$E_{\tau/S}(X) := \bigcap_{U} \text{Image}(E_{U/S} \to E_{X/S})$$

where $U$ goes through every open dense subscheme of $X$.

In fact, the definition (ii) above is independent of choices of $\overline{\nu}$. Let $\overline{\nu}'$ be another geometric point above $\overline{\eta}$ then there is an isomorphism of fibre functors $\gamma : F_{\overline{\pi}} \cong F_{\overline{\nu}'}$ which induces $\phi : \pi_1(\mathcal{X}(\overline{\pi}) \times_{\mathcal{X}} X, \overline{\nu}) \cong \pi_1(\mathcal{X}(\overline{\nu}') \times_{\mathcal{X}} X, \overline{\nu}')$. The image of $\gamma$ in $\pi_1(X, \eta)$ is denoted by $\overline{\gamma}$. We have the following commutative diagram

$$\begin{array}{ccc}
\pi_1(\mathcal{X}(\overline{\pi}) \times_{\mathcal{X}} X, \overline{\nu}) & \xrightarrow{\phi} & \pi_1(\mathcal{X}(\overline{\nu}') \times_{\mathcal{X}} X, \overline{\nu}') \\
\phi \downarrow & & \downarrow \gamma \\
\pi_1(\mathcal{X}(\overline{\pi}) \times_{\mathcal{X}} X, \overline{\nu}') & \xrightarrow{\overline{\gamma}} & \pi_1(\mathcal{X}, \overline{\eta})
\end{array}$$

where $-\overline{\gamma}$ is the conjugation by $\overline{\gamma}$. Thus $E_{\mathcal{X}, \overline{\pi}}$ is independent of choices of $\overline{\nu}$.

**Remark 2.2.** Here we compare our Definition 2.1 with Vidal’s notion([14], Subsection 2.1, [15], Section 1).

- Let $S$ be the spectrum of an excellent henselian discrete valuation ring of mixed characteristic $(0, p)$ (i.e., the characteristic of residue field at generic point $\eta$ is 0 and for closed point $s$ is $p$). Let $\overline{\pi}$ be the geometric point above $\eta$, then $E_{\mathcal{X}, \overline{\pi}} = \{\text{id}\}$. On the other hand $E_{X, \overline{\pi}}^{\text{Vidal}}$ is not trivial in general. Hence $E_{\mathcal{X}, \overline{\pi}} \neq E_{X, \overline{\pi}}^{\text{Vidal}}$ in this case.
We further consider $E_{X,X}$ and $E^{\text{Vidal}}_{X,X}$ for a normal compactification $\overline{X}$ of $X \to S$. We consider a finite quotient $Q$ of $\pi_1(X)$ which corresponds to a Galois cover $Y \to X$. Let $\overline{Y}$ be the normalization of $\overline{X}$ in $Y$. From [15], Proof 6.1 we know that $g \in Q$ belongs to $E_{X,X}$ if and only if $g$ fixes a point of $(\overline{Y})_s$. Similarly, $g \in Q$ belongs to $E^{\text{Vidal}}_{X,X}$ if and only if $g$ fixes a point of $\overline{Y}$. But the fixed point locus $\overline{Y}^g$ of $\overline{Y}$ by $g$ is proper over $S$, since proper morphism is stable under composition. Therefore $\overline{Y}^g \cap \overline{Y}_s \neq \emptyset$, we find that two conditions are equivalent and $E_{X,X} = E^{\text{Vidal}}_{X,X}$.

Remark 2.3. With Definition 2.1 we can also define tameness for Galois covers, which is a classical notion. Let $X$ be a normal connected scheme separated and of finite type over $\text{Spec}(\mathbb{Z})$. A Galois cover $Y \to X$ is called tame if $E_{X/\text{Spec}(\mathbb{Z})}(Q) = \{1\}$, where $Q$ is the finite quotient of $\pi_1(X)$ associated to the Galois cover $Y \to X$. When $Y$ is regular, this is the notion of numerical tameness in [10].

**A global-local formula** Given a geometric point $\overline{s}$ of $S$, the strict henselization $S(\overline{s}) \to S$ at $\overline{s}$ induces base changes of $X$ and $\overline{X}$.

\[
\begin{array}{ccc}
\overline{X}(\overline{s}) & \rightarrow & \overline{X} \\
\downarrow & & \downarrow \\
X(\overline{s}) & \rightarrow & X \\
\downarrow & & \downarrow \\
S(\overline{s}) & \rightarrow & S
\end{array}
\]

So we can get a normal compactification of $X(\overline{s})$ over $S(\overline{s})$. Conversely, we take an arbitrary normal compactification $\overline{X}(\overline{s})$ of $X(\overline{s}) \to S(\overline{s})$ and prove it can be dominated by the base change of a normal compactification of $X \to S$. This leads to a formula relating global and local wild parts of $\pi_1$.

**Theorem 2.4.** Let $S$ be an excellent regular scheme with $\dim S < 3$. Let $X$ be a normal connected scheme separated and of finite type over $S$. Then

$$E_{X/S} = \bigcup_{\overline{s} \to S} \bigcup_{g \in \pi_1(X,\overline{s})} g^{-1}E'_{X(\overline{s})/S(\overline{s})}g.$$  

Here $E'_{X(\overline{s})/S(\overline{s})}$ are images of $E_{X(\overline{s})/S(\overline{s})}$ in $\pi_1(X,\overline{\eta})$, where $\overline{\eta}$ is the geometric generic point of $X$, $\overline{s}$ goes through all geometric points of $S$, $S(\overline{s})$ is the strict henselization $S$ at $\overline{s}$ and $X(\overline{s})$ is the base change of $X$ to $S(\overline{s})$.

**Proof.** By Definition 2.1 the wild part of $\pi_1$ for an arbitrary normal compactification $\overline{X}$ can be written as

$$E_{X,X} = \bigcup_{\overline{s} \to X} \bigcup_{g \in \pi_1(X,\overline{s})} g^{-1}E_{\overline{s},\overline{g}}$$

$$= \bigcup_{\overline{s} \to S} \bigcup_{\overline{s} \to \overline{X}(\overline{s})} \bigcup_{g \in \pi_1(X,\overline{s})} g^{-1}E_{\overline{s},\overline{g}}$$

$$= \bigcup_{\overline{s} \to S} \bigcup_{g \in \pi_1(X,\overline{s})} g^{-1}E'_{X(\overline{s}),X(\overline{s})}g,$$

where $\overline{s}$ are geometric points of $S$. For a given $\overline{s}$ we have $\overline{X}(\overline{s})$ as the base change of $\overline{X}$. The geometric point $\overline{s}'$ of $\overline{X}(\overline{s})$ is given by a geometric point $\overline{x}$ of $\overline{X}$ such that its composition with $\overline{X} \to S$ is $\overline{s}$. We use such a
triplet \((\pi, \overline{x}, \pi')\) to define \(E_{\pi, \overline{x}, \pi'}\) as \(E_{\pi, \overline{x}, S}\) where \(\overline{x}\) is the image of \(x'\). The images of \(E_{X(\pi), X(\overline{x})}^\pi\) in \(\pi_1(X, \overline{x})\) are denoted by \(E_{X(\pi), X(\overline{x})}^\pi\). For a fixed \(\overline{x} \to S\), by Proposition 2.5 below, the base changes of \(X\) are co-cofinal in the category of normal compactifications of \(X(\overline{x}) \to S(\overline{x})\). Take intersections on both sides,

\[
E_{X/S} = \bigcap_{X \to \overline{X} \to S} E_{X, \overline{X}} = \bigcap_{X \to \overline{X} \to S} \left( \bigcup_{\overline{x} \to S \in \pi_1(X, \overline{x})} \left( \bigcup_{\pi \to S \in \pi_1(X, \overline{x})} g^{-1}E_{X(\pi), X(\overline{x})}^\pi \right) \right)
\]

\[
= \lim_{Q \to \overline{X} \to S} \left( \bigcup_{\pi \to S \in \pi_1(X, \overline{x})} \left( \bigcup_{\overline{x} \to S \in \pi_1(X, \overline{x})} g^{-1}E_{X(\pi), X(\overline{x})}^\pi \right) \right) (Q)
\]

\[
= \lim_{Q \to \overline{X} \to S} \left( \bigcup_{\pi \to S \in \pi_1(X, \overline{x})} \left( \bigcup_{\overline{x} \to S \in \pi_1(X, \overline{x})} g^{-1}E_{X(\pi), X(\overline{x})}^\pi \right) \right) (Q)
\]

\[
= \lim_{Q \to \overline{X} \to S} \left( \bigcup_{\pi \to S \in \pi_1(X, \overline{x})} \left( \bigcup_{\overline{x} \to S \in \pi_1(X, \overline{x})} g^{-1}E_{X(\pi), X(\overline{x})}^\pi \right) \right) (Q)
\]

\[
= \bigcup_{\pi \to S \in \pi_1(X, \overline{x})} \left( \bigcup_{\overline{x} \to S \in \pi_1(X, \overline{x})} g^{-1}E_{X(\pi), X(\overline{x})}^\pi \right).
\]

Here \(Q\) goes through all finite quotients of \(\pi_1(X, \overline{x})\). The third and the sixth equalities come from that a closed subset in a profinite group is the inverse limit of all its finite quotients. The fourth equality is by distribution law for intersection and union and the fact that the category of normal compactifications is cofiltered. □

**Proposition 2.5.** Let \(S\) and \(X\) be as before Theorem 2.4. Let \(\overline{x}\) be a geometric point of \(S\) and let \(S(\overline{x}) \to S\) be the henselization of \(S\) at \(\overline{x}\). Let \(S(\overline{x})\) be the base change of \(X\) to \(S(\overline{x})\). Then given an arbitrary normal compactification \(\overline{X}(\overline{x})\) of \(X(\overline{x}) \to S(\overline{x})\), there exists a normal compactification \(\overline{X}\) of \(X \to S\), such that its base change by \(S(\overline{x}) \to S\) dominates \(\overline{X}(\overline{x})\) as a normal compactification of \(X(\overline{x}) \to S(\overline{x})\).

**Proof.** If \(\overline{X}\) is a normal compactification of \(X \to S\), its base change to \(S(\overline{x})\) is also a normal compactification of \(X(\overline{x}) \to S(\overline{x})\). It remains to show the existence of such \(\overline{X}\) that satisfies the condition above. The proof proceeds in the following steps.

(i) We can find a compactification of \(X \times_S S'\) over \(S'\) such that its pullback over \(S(\overline{x})\) is \(\overline{X}(\overline{x})\), where \(S'\) is a scheme such that \(S(\overline{x}) \to S' \to S\) and \(S' \to S\) is étale. This descent is by [5](EGA IV §8), since a strict henselization is a filtered limit of pointed étale objects. The category of pointed étale objects over \(S\) has a co-cofinal subcategory which consists of all affine étale objects over \(S\). We may assume that \(S' \to S\) is affine. The base change of \(X\) along \(S' \to S\) is denoted by \(X_{S'}\) and its normal compactification obtained by descent of \(\overline{X}(\overline{x})\) is denoted by \(\overline{X}_{S'}\).

(ii) Since \(S' \to S\) is étale and affine, it is quasi-finite and separated. By Zariski’s main theorem, it can be decomposed into an open immersion followed by a finite morphism as \(S' \to T \to S\). Further we may replace \(T\) by its normalization if necessary.

(iii) Now we extend \(\overline{X}_{S'}\) to a scheme over \(T\). We have the following commutative diagram

\[
\begin{array}{ccc}
\overline{X}_{S'} & \longrightarrow & X_T \\
\downarrow & & \downarrow \\
X_{S'} & \longrightarrow & X_T \\
\downarrow & & \downarrow \\
S' & \longrightarrow & T
\end{array}
\]
with pullback square. We glue \( \overline{X_{S'}} \) and \( X_T \) along \( X_{S'} \) to obtain \( Z = \overline{X_{S'}} \cup X_{S'} X_T \). The morphism \( Z \to T \) is also obtained by glueing morphisms. Then we have the following diagram

\[
\begin{array}{c}
\overline{X_{S'}} & \overset{\beta}{\rightarrow} & Z \\
\downarrow & & \downarrow \\
X_{S'} & \overset{\alpha}{\rightarrow} & X_T \\
\downarrow & & \downarrow \\
S' & \overset{\gamma}{\rightarrow} & T
\end{array}
\]

Note that \( Z \) is not necessarily a compactification of \( X_T \to T \). By Nagata’s compactification theorem if \( Z \to T \) is of finite type and separated, there is a compactification of \( Z \to T \). Since \( Z \to T \) is of finite type, it suffices to show that \( Z \) is separated, i.e., the diagonal morphism \( \Delta_{Z/T} : Z \to Z \times_T Z \) is a closed immersion. Since closed immersion can be tested locally on the target, it suffices to test on four open subschemes of \( Z \times_T Z \)

\[
Z \times_T Z = (\overline{X_{S'}} \times_T \overline{X_{S'}}) \cup (\overline{X_{S'}} \times_T X_T) \cup (X_T \times_T \overline{X_{S'}}) \cup (X_T \times_T X_T).
\]

On the first and the last open subschemes, restrictions of \( \Delta_{Z/T} \) are closed immersions by separatedness of \( X_T \to T \) and \( \overline{X_{S'}} \to T \). For \( \overline{X_{S'}} \times_T X_T \), it is just \( \overline{X_{S'}} \times_T X_T \) by the diagram above. Note that \( X_{S'} \to X_{S'} \times_T X_{S'} \) as the restriction of \( \Delta_{Z/T} \) on \( \overline{X_{S'}} \times_T X_{S'} \) is a closed immersion, since \( \overline{X_{S'}} \to T \) is separated and \( \overline{X_{S'}} \to \overline{X_{S'}} \times_T \overline{X_{S'}} \) is a closed immersion and closed immersions are stable under base change. Symmetrically, \( X_{S'} \to X_T \times_T \overline{X_{S'}} \) is a closed immersion. Hence \( Z \to T \) is separated. The base change \( (\overline{X_T})_{S'} \) of \( \overline{X_T} \) along \( S' \to T \) is a normal compactification of \( X_{S'} \to S' \). In fact, it has \( X_{S'} \) or \( \overline{X_{S'}} \) as an open dense subscheme. Thus the image of open immersion \( \overline{X_{S'}} \to (\overline{X_T})_{S'} \) is \( (\overline{X_T})_{S'} \) itself, which means \( \overline{X_{S'}} \cong (\overline{X_T})_{S'} \). We have the following commutative diagram

\[
\begin{array}{c}
\overline{X_{S'}} & \overset{\beta}{\rightarrow} & X_T \\
\downarrow & & \downarrow \\
X_{S'} & \overset{\alpha}{\rightarrow} & X_T \\
\downarrow & & \downarrow \\
S' & \overset{\gamma}{\rightarrow} & T
\end{array}
\]

such that the base change of the right triangle along \( S' \to T \) is the left triangle.

(iv) Since \( T \to S \) is finite, it follows that \( \overline{X_T} \to S \) is proper. On the other hand, \( X_T \to S \) is separated and of finite type. Therefore \( X_T \to \overline{X_T} \) viewed as a morphism of \( S \)-schemes is also a compactification of \( X_T \to S \). Now we have \( X_T \to X \), a finite \( S \)-morphism. Because \( T \) is normal and of dimension smaller than 3, it is Cohen-Macaulay. So \( T \to S \) is dominant implies it is also flat and so does \( X_T \to X \). According to the Lemma 2.6 below, there is a normal compactification \( \overline{X} \) of \( X \to S \) such that its base change along \( T \to S \) dominates \( \overline{X_T} \) as a compactification of \( X_T \to T \).

\[\square\]

Lemma 2.6. Let \( f : X \to Y \) be a finite flat morphism from a connected scheme to a normal connected scheme separated and of finite type over \( S \). Then every compactification of \( X \) can be dominated by a normal compactification of \( Y \).
Proof. This proof comes from O. Gabber’s idea and is similar to Vidal’s proof in [14], Proposition 2.1.1 (ii). Given a compactification $X$ of $X \to S$ and a normal compactification $Y$ of $Y \to S$, we take the closure of the graph of $f$ as $\Gamma \subset X \times_S Y$. Then $p_2 : \Gamma \to Y$ is flat over $Y \to Y$, since $X \to Y$ is proper and $p_2^{-1}(Y) = X \subset \Gamma$. By the flattening theorem ([13], Théorème 5.2.2), there is a $Y$-admissible blowing-up $\overline{Y'} \to \overline{Y}$ such that the strict transform $\Gamma' \to \overline{Y'}$ is flat, hence also finite. Replace $\overline{Y'}$ by its normalization if necessary, the normalization $\overline{X}$ of $\Gamma'$ dominates $X$.

Remark 2.7. If we consider normal compactifications which are algebraic spaces, the assumption in Theorem 2.4 for $S$ can be more general: $S$ can be an arbitrary excellent scheme. The idea is from Chenyang Xu, who suggests to consider equivariant normal compactifications. As in the proof of Theorem 2.4 (ii) let $K(S')$ and $K(S)$ be fraction fields of $S'$ and $S$ respectively. Then we take a Galois extension $K/K(S)$ with Galois group $G$ containing $K(S')$ and the normalization $R$ of $S$ in $K$. The normalization of $S$ in $K(S')$ is denoted by $T$. With the similar technique in (iii) before, one can extend $X_S$ to a normal compactification $\overline{X_T}$ of $X_T \to T$ ($X_T$ can be replaced with its normalization if necessary). Then the base change of $\overline{X_T}/T$ to $R$ can be dominated by a $G$-equivariant normal compactification of $X_R \to R$ by the construction in [18], 3.7. The normalization of the quotient $\overline{X_R}/G$ is a normal compactification and its base change along $S' \to S$ dominates $\overline{X_S'}$.

Functoriality of wild parts of $\pi_1$ With the wild part of $\pi_1$, we have some functorial results. The following lemma is similar to Vidal’s results ([14], Proposition 2.1.1).

Proposition 2.8. Let $S, R$ be excellent schemes. Let $X \to \overline{X}$ and $Y \to \overline{Y}$ be two open immersions of normal connected schemes. Let $f : X \to \overline{Y}$ be a morphism of schemes. Let $a$ be a geometric generic point of $X$. Let $b : \overline{Y} \to Y$ be a geometric generic point such that $b = f(a)$. We also use $f : \pi_1(X, a) \to \pi_1(Y, b)$ to denote the morphism of étale fundamental groups induced by $f$.

(i) If there is a commutative square

$$
\begin{array}{ccc}
\overline{X} & \longrightarrow & \overline{Y} \\
\uparrow & & \uparrow \\
X & \longrightarrow & Y \\
\end{array}
$$

then $f(E_{X, \overline{X}}) \subset E_{Y, \overline{Y}}$.

(ii) Assume that $X$ and $Y$ are separated and of finite type over $S$ and $R$ respectively. Let $\overline{X}$ and $\overline{Y}$ be normal compactifications of $X \to S$ and $Y \to R$ respectively. If there is a commutative square

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
S & \longrightarrow & R \\
\end{array}
$$

then $f(E_{X/S}) \subset E_{Y/R}$.

(iii) When $f$ is a finite étale cover, we have $E_{X, \overline{X}} = E_{Y, \overline{Y}} \cap \pi_1(X, a)$. Finally, for finite étale cover $f$ and $X, Y$ separated of finite type over $S$, we have $E_{X/S} = E_{Y/S} \cap \pi_1(X, a)$.

(iv) With the condition of (ii) we further assume that $X = Y$ with $f = \text{id}$, then $E_{X/S} \subset E_{X/R}$. When $S \to R$ is proper and the diagram is commutative we have $E_{X/S} = E_{X/R}$.

Proof. (i) By definition $E_{X, \overline{X}} = \bigcup_{\pi \in \overline{X}} \bigcup_{g \in \pi_1(X, a)} g^{-1}E_{\overline{Y}, \overline{Y}}$, where $\pi$ and $\overline{\pi}$ are geometric points of $\overline{X}$ and $X \times \overline{X}\overline{X}(\overline{\pi})$ respectively. It suffices to show that $f(\bigcup_{\pi \in \overline{X}} \bigcup_{g \in \pi_1(X, a)} g^{-1}E_{\overline{Y}, \overline{Y}}) \subset E_{Y, \overline{Y}}$, since $E_{Y, \overline{Y}}$ is
(iii) We first prove for the case when $\gamma_1 \in \pi_1(X, a)$ and $a$ is a geometric point of $X$ and $\hat{a}$ is a geometric point above $a$. The geometric point of $\overline{X}$ coming from $\gamma$ is denoted by $\overline{\gamma}$ and its image in $\overline{Y}$ is denoted by $\overline{y}$. Let $b$ and $\hat{b}$ be images of $a$ and $\hat{a}$ respectively. Let $\phi_2$ be the image of $\phi_1$ in $\pi_1(Y \times_\overline{\gamma} \overline{y}, \hat{b})$. Let $\gamma_2$ be the image of $\gamma_1$ in $\pi_1(Y, b)$. Then we have $f(\sigma) = \gamma_2^{-1} \phi_2 \gamma_2 \in E_{Y, Y}$. In conclusion we have $f(E_{X, \overline{X}}) \subset E_{Y, Y}$.

(ii) For any normal compactification $\overline{Y}$ of $Y \to R$, we take a normal compactification of $\overline{Y} \times_R S$ to obtain a normal compactification of $X \to S$. By the result of (i) the assertion follows.

(iii) We first prove for the case when $\overline{X}$ is dominated by the normalization of a normal compactification $\overline{Y}$ for $Y \to S$. Let $\overline{\gamma} : \overline{\pi} \to X$ be a geometric point and let $\hat{a} : \overline{\pi} \to X_{\overline{\gamma}} := \overline{X}_{\overline{\gamma}} \times \overline{X} X$ be a geometric point above $a$. Let the image of $\hat{a}$ be $\hat{b} : \overline{\pi} \to Y_{\overline{\gamma}} = \overline{Y}_{\overline{\gamma}} X \times Y$. Then $X_{\overline{\gamma}} \subset Y_{\overline{\gamma}} X$ is a pointed connected component. We now prove that $E_{X, \overline{X}} = E_{Y, Y} \cap \pi_1(X, a)$. It suffices to show that the following diagram is cartesian

\[
\begin{array}{ccc}
\pi_1(X_{\overline{\gamma}}, \hat{a}) & \xrightarrow{\phi} & \pi_1(X, a) \\
\downarrow & & \downarrow \\
\pi_1(Y_{\overline{\gamma}}, \hat{b}) & \xrightarrow{\phi} & \pi_1(Y, b)
\end{array}
\]

In this diagram vertical arrows are all injective. Let $U = \phi^{-1}(\pi_1(X, a)) \subset \pi_1(Y_{\overline{\gamma}}, \hat{b})$. This is an open subgroup associated to an intermediate finite pointed étale covering $X_{\overline{\gamma}} \to V_{\overline{\gamma}} \to Y_{\overline{\gamma}}$. The correspondent point is denoted by $c : \overline{\pi} \to V_{\overline{\gamma}}$. As the map $\pi_1(V_{\overline{\gamma}}, c) \to \pi_1(Y_{\overline{\gamma}}, \hat{b})$ is factorized by $\pi_1(X, a)$, then by [6], V.6.4, the natural projection $V_{\overline{\gamma}} \times_Y X \to V_{\overline{\gamma}}$ admits a pointed section $s$. Composite the section $s$ with the natural pointed morphism $V_{\overline{\gamma}} \times_Y X \to V_{\overline{\gamma}} \times_Y X$, we find that the pointed morphism $V_{\overline{\gamma}} \to V_{\overline{\gamma}} \times_Y X$ must factors through $X_{\overline{\gamma}}$. That means $V_{\overline{\gamma}} \cong X_{\overline{\gamma}}$, hence that diagram is cartesian.

Let $q \in \pi_1(Y, b)$ and let $(X', a')$ be a pointed connected finite étale cover of $Y$ corresponding to the open subgroup $\pi_1(X', a) = q \pi_1(X, a)q^{-1}$. We denote by $\phi : \pi_1(X, a) \to \pi_1(X', a')$ the isomorphism induced by conjugation by $q$ in $\pi_1(Y, b)$ and $\psi : (X', a') \to (X, a)$ is denoted for its inverse. Then $q^{-1}E_{x, \overline{\gamma}} \cap \pi_1(X, a) = \psi(E_{x, \overline{\gamma}} \cap \pi_1(X', a')) = \psi(E_{x, \overline{\gamma}})$ (for $X'$ it is the same with the above argument). By the result of (i) we find that $\psi(E_{x, \overline{\gamma}}) \subset \psi(E_{X', \overline{X}}) \subset E_{X, \overline{X}}$. As this can be applied to all $q$ and all geometric point $\overline{\gamma} : \overline{\pi} \to \overline{X}$, we then use the result of (i) to find that $E_{Y, Y} \cap \pi_1(X, a) \subset E_{X, \overline{X}}$.

By Lemma 2.6 above we finish the proof.

(iv) Note that every normal compactification of $X \to S$ is also a normal compactification of $X \to R$. Further given any normal compactification $W$ of $X \to R$, we take a normal compactification $Z$ of
Now we consider a family of virtual constructible (resp., locally constant) sheaves \( F \) of all excellent schemes \( S \) separated and of finite type over \( S \). Let Definition 2.9.

(i) Let \( B \) be the (pointed) category whose objects consist of all open immersions \((X \hookrightarrow X)\) of normal connected schemes with fixed geometric generic points. Morphisms in \( B \) consists of all commutative squares as in Proposition 2.8 (i) such that induced morphisms of geometric generic points of \( Y, Y', X, X' \) are compatible, where \((X \hookrightarrow X)\) and \((Y \hookrightarrow Y)\) are objects in \( B \).

(ii) Let \( D \) be the (pointed) category whose objects consist of all \((X \to S)\), where \( S \) is an excellent scheme \( X \) is normal connected separated finite type over \( S \) with fixed geometric generic points. Morphisms in \( D \) consists of all commutative squares as in Proposition 2.8 (ii) such that induced morphisms of geometric generic points of \( X, S, Y, R \) are compatible, where \((X \to S)\) and \((Y \to R)\) are objects in \( D \).

With Definition 2.9 and Proposition 2.8 (i)& (ii) we obtain the following corollary.

**Corollary 2.10.** Here we use the same notations as Definition 2.9.

(i) There is a functor \( b : B \to \text{Sets} \) defined as follows. For each \((X \to X) \in \text{Obj}(B)\), \( b \) sends it to \( E_{X, X} \).

(ii) There is a functor \( E_{-/-} : D \to \text{Sets} \) defined as follows. For each \((X \to S) \in \text{Obj}(D)\), \( E_{-/-} \) sends it to \( E_{X/S} \).

(iii) Let \((Y \to R)\) be an object in \( D \). Let \( f : X \to Y \) be a morphism separated and of finite type. Then \( f \) gives a morphism in \( D \) and hence we have \( E_{X/R} \subseteq E_{Y/R} \).

**Corollary 2.11.** Let \( S \) be an excellent scheme. Let \( D_S \) be the category of all normal connected schemes separated and of finite type over \( S \). Let \( X \) be a normal connected scheme. Let \( \mathcal{E}^X \) be the category consisting of all excellent schemes \( S \) with \( X \to S \) separated and of finite type.

(i) There is a covariant functor \( E_{-/-} : D_S \to \text{Sets} \) defined as follows. For each object \( X \in D_S \), \( E_{-/-} \) sends it to \( E_{-/-}(X) = E_{X/S} \) an object in \( \text{Sets} \).

(ii) There is a covariant functor \( E_{X/-} : \mathcal{E}^X \to \text{Sets} \) defined as follows. For each object \( S \in \mathcal{E}^X \), \( E_{X/-} \) sends it to \( E_{X/-}(S) = E_{X/S} \). For each morphism \( h : S \to R \) in \( \mathcal{E}^X \), \( E_{X/-} \) sends it to \( E_{X/-}(h) : E_{X/S} \to E_{X/R} \) a inclusion morphism in \( \text{Sets} \).

**Proof.** By Corollary 2.10 (ii) the assertion of (i)&(ii) follows.

### 3 Wildly compatible systems of virtual constructible sheaves

**Brauer traces** Let \( S \) be an excellent scheme and let \( X \) be a scheme separated and of finite type over \( S \). Now we consider a family of virtual constructible (resp., locally constant) sheaves \( \{ F_i \}_{i \in I} \) on \( X \). Here \( F_i \) are elements in Grothendieck ring \( K_0(X, \mathbb{F}_\lambda) \) (resp., \( K_{coh}(X, \mathbb{F}_\lambda) \)) of constructible (resp., locally constant) sheaves of \( \mathbb{F}_\lambda \)-modules on \( X \), and each \( \lambda_i \) is a power of prime \( \ell_i \). In order to compare the wild ramifications of \( F_i, i \in I \) we use Brauer trace.

Let \( G \) be a profinite group. For a finite field \( \mathbb{F}_\lambda \), where \( \lambda \) is a power of a prime \( \ell \), the subset consisting of \( \ell \)-regular elements (i.e., elements of orders prime to \( \ell \)) is denoted by \( G_{\ell-\text{reg}} \). For \( M \in K_0(\mathbb{F}_\lambda[G]) \) an element of Grothendieck ring of finite dimensional \( \mathbb{F}_\lambda \)-vector spaces with continuous \( G \)-actions, the Brauer trace is a central function \( \text{Tr}_{M}^{B_{\ell}} : G_{\ell-\text{reg}} \to W(\mathbb{F}_\lambda) \), where \( W(\mathbb{F}_\lambda) \) is the Witt ring of \( \mathbb{F}_\lambda \). Concretely, each eigenvalue \( \zeta \) of action of \( g \in G_{\ell-\text{reg}} \) on \( M \) is in \( \mathbb{F}_\lambda \) and has a unique lift \( [\zeta] \) as a root of unity of order prime to \( \ell \), and
Brauer trace is given by $\text{Tr}^{Br}_F(g) = \sum [\zeta]$. Brauer trace is an additive function and is multiplicative with respect to the element of $K((\mathbb{F}_\lambda(G))).$

Note that the Brauer trace, when restricted to $E_{X/S}$, takes value in a CM field $K\lambda := \mathbb{Q}_\lambda \cap \mathbb{Q}(\zeta_{<\infty})$ considered as a subfield of $\mathbb{Q}_\lambda$. Let $V$ be a $\mathbb{Q}$-vector subspace of $\prod_{i \in I} K\lambda_i$. In the sequel we use this notation of $K\lambda$ for convenience and $\lambda_i$ are all assumed to be invertible on $S$.

Wildly compatible locally constant sheaves

**Definition 3.1.** Let $S$ be an excellent scheme and let $X$ be a normal connected scheme separated and of finite type over $S$. Let $V$ be a fixed subspace of $\prod_{i \in I} K\lambda_i$. We say a system $\{F_i\}_{i \in I}$ of virtual locally constant $\mathbb{F}_\lambda_i$-sheaves on $X$ is a wildly compatible system for $V$ if $(\text{Tr}^{Br}_F(g))_{i \in I} \in V$ for all $g \in E_{X/S}$. We define a subgroup $K_{coh}(X/S, I, V)$ of $K_{coh}(X, I) := \prod_{i \in I} K_{coh}(X, \mathbb{F}_\lambda_i)$ to be the subring consisting of $(F_i)_{i \in I}$ in wildly compatible systems.

**Remark 3.2 ([14], Remark 2.2.1).** Let $S$ and $X$ be the same as in Definition 3.1. Assume $\{F_i\}_{i \in I}$ are finitely many locally constant virtual sheaves over $\mathbb{F}_\lambda$. Let $V$ be a subspace of $\prod_{i \in I} K\lambda_i$. Then $\{F_i\}$ form a wildly compatible system for $V$ if and only if there exists a normal compactification $X_0$ of $X \to S$ such that $\{F_i\}$ satisfy the following condition: for every \(g \in E_{X/S,0}\) we have $(\text{Tr}^{Br}_F(g))_{i \in I} \in V$. One side is easy to prove since $E_{X/S,0} \subset E_{X,\mathbb{F}_0}$. For the other side, we notice that $\pi_1(X)$ acts on $F_i$ by a finite quotient $Q$. Since $Q$ is finite and the category $\mathcal{C}$ of normal compactifications of $X \to S$ is cofiltered, for the projection $p_Q : \pi_1(X, \pi) \to Q$ the finite quotient parts $E_{X/S}(Q) := p_Q(E_{X/S})$ of $E_{X/S}$ and $E_{X,\mathbb{F}}(Q) := p_Q(E_{X,\mathbb{F}})$ satisfy $E_{X/S}(Q) = \bigcap_{\mathbb{F} \in Q} E_{X,\mathbb{F}}(Q) = E_{X,\mathbb{F}}(Q)$ for a normal compactification $X_0$ of $X \to S$ then the assertion follows.

**Wildly compatible constructible sheaves**

Given a virtual constructible sheaf $F_i$ on $X$, there is a stratification $X = \bigcup_{i \in I} U_i$ denoted by $P_i$, where $U_i$ are normal connected and locally closed subschemes of $X$ such that restrictions of $\mathcal{F}_i$ on $U_i$ are locally constant. Here by “stratification” we mean a “nice stratification”, i.e., for each stratum $U_i$, its closure is a disjoint union of strata $U_i = \bigcup_{j \in J} U_i^j$, where $J \subset I$. But $X$ can have another stratification $Q_i$ for $F_i$: $X = \bigcup_{j \in J} V_j$. We say a stratification $P_i$ is finer than $Q_i$ or $P_i \geq Q_i$, if $V_j = \bigcup_{i \in I} U_i^j$ for each $j$, where $i(j)$ are functions of index sets $I$ and $J$. Now we consider finitely many virtual constructible sheaves $F_i$ on $X$ with stratifications $P_i$. Then there is a common refinement $P$ so that we can compare their wild ramifications in terms of Brauer traces.

**Assumption 3.3.** In the sequel when considering virtual constructible sheaves, we further assume that $V = \bigcap_{j} p_{j-1}^j p_j(V)$, where $J$ goes through all finite subsets of $I$ and $p_j$ are projections to subspaces.

**Definition 4.4.** Let $S$ be an excellent scheme and let $X$ be a scheme separated and of finite type over $S$. Let $V$ be a fixed $\mathbb{Q}$-vector subspace of $\prod_{i \in I} K\lambda_i$. We say a system $\{F_i\}_{i \in I}$ of virtual constructible $\mathbb{F}_\lambda$-sheaves on $X$ is a wildly compatible system for $V$, if for any finite subset $J \subset I$ there is a common stratification $X = \bigcup_{\sigma \in \Sigma_J} X_\sigma$ such that $\{F_i|_{X_\sigma}\}_{\sigma \in \Sigma_J}$ is a wildly compatible system of virtual locally constant sheaves on $X_\sigma$ for $p_j(V)$ and for each $\sigma \in \Sigma_J$. We define a subgroup $K_c(X/S, I, V)$ of $K_c(X, I) := \prod_{i \in I} K_c(X, \mathbb{F}_\lambda_i)$ to be the subring consisting of such $(F_i)_{i \in I}$.

**Remark 3.5.** From Definition 3.4, it is clear that a system $\{F_i\}_{i \in I}$ of virtual constructible sheaves is a wildly compatible system for $V$ if and only if its restriction to $J$ is a wildly compatible system for $p_j(V)$ for every finite subset $J \subset I$.

**Proposition 3.6.** Let $S$ be an excellent scheme with $\dim S \leq 1$. Let $X$ be an $S$-scheme separated of finite type over $S$. Let $\{F_i\}_{i \in I}$ be finitely many virtual locally constant sheaves on an $S$-scheme $X$. We fix a $\mathbb{Q}$-vector subspace $V$ of $\prod_{i \in I} K\lambda_i$. If there is a common stratification $X = \bigcup_{\sigma \in \Sigma_J} X_\sigma$ for $\{F_i\}_{i \in I}$ such that for each $\sigma$ we have $(\text{Tr}^{Br}_F|_{X_\sigma}(g))_{i \in I} \in V$ for all $g \in E_{X_\sigma/S}$, then $(\text{Tr}^{Br}_F(g))_{i \in I} \in V$ for all $g \in E_{X/S}$.

**Proof.** This proposition and its proof is similar to Vidal's in [15, 6.2(ii)]. By the valuative criterion of O. Gabber ([15], 6.1), $E_{X/S} = \bigcup_{\sigma} \text{Im} E_{X_\sigma/S}$. Hence for every $g \in E_{X/S}$, we have $(\text{Tr}^{Br}_F(g))_{i \in I} \in V$. 

\[
\]
Corollary 3.7. Assumption of $X$ being as in Proposition 3.6, for any index set $I$ we have $K_{\text{coh}}(X/S,I,V) = K_c(X/S,I,V) \cap K_{\text{coh}}(X,I)$ for arbitrary $V$.

Remark 3.8. Here we give three special examples of wildly compatible systems of virtual locally constant sheaves and constructible sheaves.

(i) When $\#I = 1$ and $V = 0$, a virtual constructible sheaf $F$ in $K_c(X/S,I,V)$ can be viewed as it “has zero wild ramification”.

(ii) When $\#I = 2$ and let $V$ be the kernel of the map

$$K_{\lambda_1} \times K_{\lambda_2} \to \mathbb{Q}, \quad a \mapsto \frac{1}{[K_{\lambda_1} : \mathbb{Q}]} \text{Tr}_{K_{\lambda_1}/\mathbb{Q}}(a) - \frac{1}{[K_{\lambda_2} : \mathbb{Q}]} \text{Tr}_{K_{\lambda_2}/\mathbb{Q}}(a)$$

For $(F_1,F_2) \in K_{\text{coh}}(X/S,I,V)$ they “have the same wild ramification” in a weaker sense.

(iii) Let $L$ be a field of characteristic zero with embeddings $\iota = (\iota_i)_{i \in I} : L \to \prod_{i \in I} \mathbb{Q}_p$ and let $V$ be $(\prod_{i \in I} K_{\lambda_i}) \cap \text{Image}((\iota_i)_{i \in I})$. If $(\text{Tr}_{F_i}^B(g))_{i \in I} \in V$ for all $g \in E_{X/S}$, then $(F_i)_{i \in I} \in K_{\text{coh}}(X/S,I,V)$. In this case, for any fixed index set $I$, $(F_i)_{i \in I} \in K_{\text{coh}}(X/S,I,V)$ if and only if for all subset $J \subset I$ with $\#J \leq 2$, $(F_j)_{j \in J}$ is a wildly compatible system for $p_j(V)$.

(a) For (i), when $S$ is the spectrum of an excellent henselian discrete valuation ring with residue characteristic exponent $p \geq 1$ or an algebraically closed field $k$, and $X$ is a normal connected scheme separated and of finite type over $S$, by 2.2 we obtain $K_c(X/S,I,V) = K_c(X,F_{\lambda})^0$ which is Vidal’s notion.

(b) For (ii), when $S$ is the spectrum of an excellent henselian discrete valuation ring of residue characteristic exponent $p \geq 1$ and $X$ is a scheme separated and of finite type over $S$, we obtain $K_{\text{coh}}(X/S,I,V) = \Delta_{\text{coh}}(X,F_{\lambda_1},F_{\lambda_2})$ which is Yatagawa’s notion. Further with Assumption 3 for $V$, we have $K_c(X/S,I,V) = \Delta_c(X,F_{\lambda_1},F_{\lambda_2})$. In fact, our generalization is the same with Kato’s, see [9], Definition 2.2.

(c) Let $I$, $V$, and $X$ be the same with (iii) with extra Assumption 3 for $V$. If for any $J \subset I$ a finite subset, there is a stratification $X = \bigsqcup_{\sigma \in \Sigma_j} X_{\sigma}$ such that for each $g \in E_{X/S}$ there exists $a \in L$ and $\text{Tr}_{F_{\lambda_1}(X)}^B(g) = \iota(a)$, then $(F_i)_{i \in I} \in K_c(X/S,I,V)$. In this case, for any fixed index set $I$, $(F_i)_{i \in I} \in K_c(X/S,I,V)$ if and only if for all subset $J \subset I$ with $\#J \leq 2$, $(F_j)_{j \in J}$ is a wildly compatible system for $V$.

Proposition 3.9. Let $S$ be an excellent scheme. Let $X$ be a scheme separated and of finite type over $S$.

(i) Let $h : T \to S$ be a morphism of excellent schemes. Let $X_T$ be the base change of $X$ over $T$. We also denote $X_T \to X$ by $h$. If $(F_i)_{i \in I}$ is in $K_c(X/S,I,V)$ then $(h^*F_i)_{i \in I}$ is in $K_c(X_T/T,V,I)$.

(ii) Suppose $S$ is an excellent regular scheme of dimension smaller than 3. Let $(F_i)_{i \in I}$ be a system of virtual constructible sheaves on $X$. Then $(F_i)_{i \in I}$ is in $K_c(X,I,V)$ if and only if for every geometric point $\bar{s} \to S$ and the strict henselization $s : S(\bar{s}) \to S$, $s^*(F_i)_{i \in I}$ is in $K_c(X(\bar{s})/S(\bar{s}),I,V)$.

(iii) Let $S \to R$ be a finite type and separated morphism between excellent schemes. Let $(F_i)_{i \in I}$ be a wildly compatible system of virtual constructible sheaves on $X$ with respect to $E_{X/R}$. Then $(F_i)_{i \in I}$ is also a wildly compatible system of virtual constructible sheaves on $X$ with respect to $E_{X/S}$.

Proof. (i) It suffices to prove when $F_i$ are locally constant. The assertion follows directly from Corollary 2.8,(ii).

(ii) Assume $(F_i)_{i \in I}$ is in $K_c(X/S,I,V)$. By result of (i), we conclude that $(s^*(F_i))_{i \in I}$ is a wildly compatible system.

For the other side, as above we may assume $\#I$ is finite and $F_i$ are locally constant. Then there exists a finite quotient $E_{X/S}(Q)$ such that for all $g \in E_{X/S}$, $\text{Tr}_{F_i}^B(g) = \text{Tr}_{F_i}^B(\bar{g})$ for all $i \in I$, where $\bar{g}$ is the
(i) The proof is directly from the result of Corollary 2.11, (ii). 

\[ \text{(iii) } \]

4 Preservation by Six Operations

**Theorem 4.1.** Let \( S \) be an excellent regular scheme with \( \dim S \leq 1 \). Let \( f : X \to Y \) be an \( S \)-morphism of schemes over \( S \) separated and of finite type. Then wildly compatible systems are preserved under Grothendieck’s six operations and Verdier’s duality in the following sense,

(i) \( f^* : K_c(Y, I) \to K_c(X, I) \) induces \( f^* : K_c(Y/S, I, V) \to K_c(X/S, I, V) \).

(ii) \( f_* : K_c(X, I) \to K_c(Y, I) \) induces \( f_* : K_c(X/S, I, V) \to K_c(Y/S, I, V) \).

(iii) \( f_! : K_c(X, I) \to K_c(Y, I) \) induces \( f_! : K_c(X/S, I, V) \to K_c(Y/S, I, V) \).

(iv) \( f^! : K_c(X, I) \to K_c(Y, I) \) induces \( f^! : K_c(X/S, I, V) \to K_c(Y/S, I, V) \).

(v) \( R\text{Hom} : K_c(X, I) \times K_c(X, I) \to K_c(X, I) \) induces \( R\text{Hom} : K_c(X/S, I, V) \times K_c(X/S, I, V) \to K_c(X/S, I, V) \) when \( V \) is a sub-algebra.

(vi) \( \otimes - : K_c(X, I) \times K_c(X, I) \to K_c(X, I) \) induces \( \otimes - : K_c(X/S, I, V) \times K_c(X/S, I, V) \to K_c(X/S, I, V) \) when \( V \) is a sub-algebra.

(vii) \( D_X : K_c(X, I) \to K_c(Y, I) \) induces \( D_X : K_c(X/S, I, V) \to K_c(X/S, I, V) \).

This theorem also implies results for the three cases in Remark 3.8, including Vidal’s ([15], Corollaire 0.2) and Yatagawa’s ([16] Corollary 4.1), except that the last one of Vidal’s. But by the same proof we also have \( R\text{Hom} \) induces \( K_c(X, \mathcal{F}_\lambda)^0_\sigma \times K_c(X, \mathcal{F}_\lambda) \) or \( K_c(X, \mathcal{F}_\lambda) \times K_c(X, \mathcal{F}_\lambda)^0_\sigma \) into \( K_c(X, \mathcal{F}_\lambda)^0_\sigma \). Note that \( V \) are sub-algebras for the first and the third examples but not the second.

**Proof.** By the Lemma 3.9, it suffices to prove in the case when \( S \) is strict local. Further \( f^* = f^! \) by the Corollary 9.5 in [17], \( f_* = f_! \) by the Theorem 3.14 in [8] generalizing a theorem of Laumon ([11]). So in order to prove (i) to (iv) we only need to prove for \( f^* \) and \( f_! \). By 3.5 we may assume \( I \) is a finite set.

(i)\&( iv) This proof is similar to Vidal ([14], Proposition 2.3.3) and Yatagawa’s proof ([16], Corollary 4.1). For \( (\mathcal{F}_i)_{i \in I} \in K_c(Y, I) \), we may assume they are constructible sheaves. There is a common stratification for \( \mathcal{F}_i : Y = \bigsqcup Y_\sigma \) such that \( (\mathcal{F}_i)_{i \in I} \in K_{coh}(Y_\sigma/S, I, V) \). Take preimages of \( Y_\sigma \) and take their stratifications we have \( X = \bigsqcup X_{\sigma, \mu} \) such that \( f^* \mathcal{F}_i |_{X_{\sigma, \mu}} \) are locally constant sheaves. By Proposition 2.8 the assertion follows.
(ii) & (iii) Let \((a_i)_{i \in I}\) be an element of \(K_c(X/S, I, V)\). First we can prove the assertion is true when \(f\) is an immersion. If \(f\) is an open immersion then \(Y = X \setminus \{(Y\setminus X)\) and \(f|_{a_i} = a_i\), \(f|_{a_i}^{-1}X = 0\). If \(f\) is a closed immersion, we have \(f|_{a_i}X = a_i\), \(f|_{a_i}^{-1}X = 0\). Take a stratification of \(Y\setminus X\) then we find that \(\{a_i\}_{i \in I}\) form a wildly compatible system.

For general case we decompose \(X = X_n \cup X_s\), \(Y = Y_n \cup Y_s\) as closed fibre parts and generic fibre parts. Write \(f_n : X_n \rightarrow Y_n\) and \(f_s : X_s \rightarrow Y_s\). Let \(i_{Y_n} : Y_n \rightarrow Y\) and \(i_{Y_s} : Y_s \rightarrow Y\) be immersions. By \(f|_{a_i} = i_{Y_n}f_n((a_i)|_{X_n}) + i_{Y_s}f_s((a_i)|_{X_s})\) for \(i \in I\) and the proof above for immersions, it suffices to prove for \(f_n\) or \(f_s\). The assertion is local so it suffices to prove when \(Y = \text{Spec} A\) is affine. Further we take \(X = \text{Spec} B\) affine and view \(Y\) as \(\text{Spec} B[x_1, \ldots, x_r]/I\). Decompose \(X \rightarrow Y\) into a closed immersion followed by a series of morphisms of relative dimension one.

When \(f\) is a closed immersion the proof has been finished above. Now let \(f\) be relative dimension one morphism. Since \(a_i\) are constructible, we may assume \(Y\) is normal connected and \((a_i)_{i \in I} \in K_{coh}(X, I, V)\). Further we may assume \(X\) is normal connected and \((a_i)_{i \in I} \in K_{coh}(X/S, I, V)\). For any pair of components \((a_1, a_2)\) of an element \((a_i)_{i \in I} \in K_{coh}(X/S, I, V)\) its associated galois étale covering \(p : V \rightarrow X\) trivializing them with galois group \(H\). By shrinking \(Y\) to an open dense sub scheme if necessary, we may assume images of this pair of components under \((f \circ p)_i\) are in Grothendieck rings of locally constant sheaves. Let \(Q\) be a finite quotient of \(\pi_1(Y)\) acting on \(f|_{a_1}, f|_{a_2}, (f \circ p)_i\) \(\mathbb{F}_{\lambda_i}\) and \((f \circ p)_i\) \(\mathbb{F}_{\lambda_2}\).

By shrinking \(Y\) to an open dense sub scheme, we may assume that \(E_{Y/S}(Q) = (E_{\tau/S}(Y))(Q)\), where \(\tau\) is the generic point of \(Y\). Let \(g \in E_{Y/S}\), then by [15](Corollary 3.0.5 and the proof of Théorème 0.1), we express the Brauer traces of \(a_i\) for \(i = 1, 2\)

\[
\text{Tr}_{f|_{a_i}}^{Br}(g) = \frac{1}{|H|} \sum_{h' \in H, p_H(h') = g'} \text{Tr}_{H \times \mathbb{Q}_l}^{Br}(V_{\tau, E_{\lambda_i}})(h') \times \text{Tr}_{f|_{a_i}}^{Br}(p_H(h')), 
\]

where \(p'\) is a pro-p-subgroup of \(Gal(\bar{\tau}/\tau)\) with image \(P\) under \(Gal(\bar{\tau}/\tau) \rightarrow \pi_1(Y)\) and \(H' = P' \times H\), \(E_{\lambda_i}\) is a finite extension of \(\mathbb{Q}_l\), such that its integer ring has the residue field \(\mathbb{F}_{\lambda_i}\). For all \(g \in E_{X/S}\) we have \((\text{Tr}_{f|_{a_i}}^{Br}(g))_{i \in I} \in V\). We need to show that for all \(g' \in E_{Y/S}\) we have \((\text{Tr}_{f|_{a_i}}^{Br}(g'))_{i \in I} \in V\).

By Proposition 2.2.1 of [15] and Proposition 2.3.1 of [15], the traces \(\text{Tr}_{f|_{a_i}}^{Br}(g')_{i \in I}\) are integers independent of \(i\). Then by [15](Corollary 3.0.7 and the proof of Théorème 0.1), if \(h' \in H'\) such that \(\text{Tr}_{f|_{a_i}}^{Br}(g')(h') \neq 0\) then \(p_H(h')\) is in the image of \(E_{X/S}\) by \(\pi_1(X) \rightarrow H\). Thus \((\text{Tr}_{f|_{a_i}}^{Br}(g))_{i \in I} \in V\) for every \(g \in E_{Y/S}\). So we finish the proof.

(v) Since \(R\text{Hom}_X(\mathcal{F}_i, \mathcal{G}_i) \cong D_X(\mathcal{F}_i \otimes D_X(\mathcal{G}_i))\), the assertion follows from (vi) and (vii).

(vi) The assertion follows since Brauer trace of a tensor product is a product of Brauer traces.

(vii) Let \(\{\mathcal{F}_i\}_{i \in I}\) be an element of \(K_c(X, I, V)\). Here we use Noether’s induction. Let \(j : U \rightarrow X\) be an open immersion such that \(\mathcal{F}_i\) are lisse and \(U \rightarrow S\) is smooth. Let \(i : Y = X \setminus U \rightarrow X\) be the closed immersion. Then \(D_X \mathcal{F}_i = D_X j^* \mathcal{F}_i + D_X i^* \mathcal{F}_i\). It suffices to prove that \(D_X j^* \mathcal{F}_i\) form a wildly compatible system. By definition we have \(D_X j^* \mathcal{F}_i = j_! D_U j^* \mathcal{F}_i = j_! R\text{Hom}_U(j^* \mathcal{F}_i, \mathcal{F}_\lambda)(d)[2d]\). By the result of (ii), it suffices to consider \(\text{RHom}_U(j^* \mathcal{F}_i, \mathcal{F}_\lambda)(-d)[2d]\). In fact, for a dual module \(M^\vee\) of \(M\), the Brauer trace satisfies \(\text{Tr}_{M^\vee}(g) = \text{Tr}_M(g^{-1})\). Note that as a result of definition \(E_{X/S}\) is symmetric: if \(g \in E_{X/S}\) then \(g^{-1} \in E_{X/S}\). By (i), \(j^* \mathcal{F}_i\) form a compatible system. So \((j^* \mathcal{F}_i)^{\vee}(-d)[2d]\) also form a compatible system. Thus \(D_U j^* \mathcal{F}_i\) form a compatible system and by Noether’s induction we finish the proof.
5 Relation with compatible \( \ell \)-adic virtual constructible sheaves

Let \( X \) be a smooth curve over \( \mathbb{F}_p \). Let \( \ell_i \) be primes distinct with \( p \), where \( i \in I \). Let \( E_i/\mathbb{Q}_{\ell_i} \) be finite extensions of fields with residue fields \( \mathbb{F}_{\lambda_i} \) and \( \lambda_i \) are integer powers of \( \ell_i \). We fix a field \( L \) of characteristic zero and embeddings \( \{ \iota_i : L \to E_i \to \mathbb{Q}_{\ell_i} \} \). Let \( K \) be the function field of \( X \). We recall the definition of (strictly) compatible system of \( E_i \)-virtual sheaves on \( X \) ([4], 1.2). A family of virtual constructible sheaves \( \{ \mathcal{F}_i \}_{i \in I} \) where \( \mathcal{F}_i \) are virtual sheaves over \( E_i \), is called an \( L \)-compatible system if for all \( x \in |X| \), all geometric points \( \pi \to x \) and all \( g \in \text{Gal}(\kappa(\pi)/\kappa(x)) \) such that \( g \) are integer powers of geometric Frobenius as \( F_0^g \), there exist \( a_g \in L \) depending on \( g \) such that \( \text{Tr}(g, (\mathcal{F}_i)_{\pi}) = \iota_i(a_g) \). Now we find a connection between \( E_i \)-compatible systems and wildly compatible systems.

Let \( \mathcal{O} \) be a complete DVR and let its fraction field be \( E \) of characteristic zero and its residue field be \( F \) of characteristic \( \ell > 0 \). Let \( X \) be a Noetherian and connected scheme. Then the category of lisse \( \ell \)-adic sheaves on \( X \) is equivalent to the category of continuous \( \ell \)-adic representations of \( \pi_1(X, \pi) \), where \( \pi \) is a geometric point of \( X \). Let \( K(X, \mathcal{O}), K(X, E), K(X, F) \) be the Grothendieck groups of constructible sheaves of \( \mathcal{O}, E, F \)-modules over \( X \) respectively. Let \( j^* : K(X, \mathcal{O}) \to K(X, E) \) be the ring homomorphism induced by the exact functor

\[
\text{Mod}_c(X, \mathcal{O}) \to \text{Mod}_c(X, E), \quad M \mapsto E \otimes_{\mathcal{O}} M
\]

Let \( i^* : K(X, \mathcal{O}) \to K(X, F) \) be the ring homomorphism given by the triangulated functor

\[
D^b_c(X, \mathcal{O}) \to D^b_c(X, F), \quad M \mapsto F \otimes_{\mathcal{O}} M
\]

By Proposition 9.4 of [17], \( j^* \) is an isomorphism. Thus we obtain the decomposition morphism \( d := i^* \circ j^{*-1} : K(X, E) \to K(X, F) \).

**Theorem 5.1.** Let \( X \) be a smooth curve over \( \mathbb{F}_p \). Then given any \( L \)-compatible system \( \{ \mathcal{F}_i \}_{i \in I} \) of virtual constructible sheaves on \( X \), the image under decomposition map \( \{ d(\mathcal{F}_i) \}_{i \in I} \) is a wildly compatible system of virtual constructible \( \mathbb{F}_{\lambda_i} \)-sheaves.

**Proof.** It suffices to prove for \( \#I = 2 \) and locally constant sheaves \( \{ \mathcal{F}_i \}_{i \in I} \) on \( X \). If \( \mathcal{F}_i \) are \( L \)-compatible, then by [18], Proposition 3.4 or [2], Théorème 9.8, for any normal compactification \( j : X \to \overline{X} \), all closed point \( x \in |\overline{X}| \) and all \( \pi \to x \), there are \( a \in L \) such that \( \text{Tr}(g, (j^*(\mathcal{F}_i))_{\pi}) = \iota_i(a) \), where \( g \in \text{Gal}(\kappa(\pi)/\kappa(x)) \) is an integer power of \( F_0^g \). Note that there is a surjective homomorphism \( \pi_1(X \times \overline{X}(\pi), \overline{b}) \to \pi_1(\overline{X}(\pi), \overline{b}) \cong \text{Gal}(\kappa(\pi)/\kappa(x)) \), where \( \overline{b} \) is a geometric point of \( \overline{X}(\pi) \). For any \( h \in \pi_1(X \times \overline{X}(\pi), \overline{b}) \) such that its image is \( g \in \pi_1(\overline{X}(\pi), \overline{b}) \), we have \( \text{Tr}(g, (j^*(\mathcal{F}_i))_{\pi}) = \text{Tr}(h, ((\mathcal{F}_i)|_{X \times \overline{X}(\pi)} \overline{b})) = \iota_i(a) \) for \( a \in L \). Further if we restrict the fundamental group \( \pi_1(X \times \overline{X}(\pi), \overline{b}) \) to its wild inertia subgroup, it means that \( \text{Tr}_{(\mathcal{F}_i)|_{X \times \overline{X}(\pi)}}(h) = \text{Tr}(h, (\mathcal{F}_i)|_{X \times \overline{X}(\pi)}) = \iota_i(a_h) \) for \( a_h \in L \). Therefore \( \{ \mathcal{F}_i \}_{i \in I} \) is a wildly compatible system (Remark 3.8, (c)).

**Remark 5.2.** Qing Lu and Weizhe Zheng have proved that Theorem 5.1 can be extended to higher dimensions, see [12], Theorem 1.2, Corollary 3.7.

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