NONCOMMUTATIVITY AS A MAPPING OF PATHS

José Manuel Carmona, José Luis Cortés, Javier Induráin and
Diego Mazón

Departamento de Física Teórica, Facultad de Ciencias,
Universidad de Zaragoza, 50009 Zaragoza, Spain

Abstract. A reinterpretation of noncommutativity as a mapping of paths is proposed at the level of quantum mechanics.

Keywords: Noncommutativity, path integral.

PACS classification: 11.10.Nx; 03.65.-w; 02.40.Gh.

1. Introduction

In memory of Julio Abad, with whom one of the authors (J.L.C.) shared more than thirty years of professional experience, another one (J.M.C.) fifteen years and the other two the beginning of their research career. During all these years we have been lucky enough to experience Julio’s kindness, his availability to speak with him at any time, his joy for physics and love for good books. We are proud to contribute to a volume in his honor with an article that, we like to think, he might well have enjoyed.

Noncommutative geometry was considered and developed as a mathematical generalization of commutative geometry, with an application to physics, during the 1980s, mainly from Alain Connes approach to gauge theories [1].

Noncommutativity had entered physics, however, much earlier, with the advent of quantum mechanics. In ordinary quantum mechanics, position and momentum are described by noncommutative self-adjoint operators, but the geometry of space is the usual one. There are however arguments suggesting that in a quantum theory including gravity, position measurements will be problematic at the Planck length and the geometry of space will have to be changed at these small scales [2].

The arising of noncommutative spaces in string theory [3] has in fact given support to this idea and led to a stronger interest in the study of physical systems on a noncommutative geometry, in particular of the quantum mechanics of particles on such spaces [4]. More recently, it has been shown that the effective low-energy limit of 3-d quantum gravity coupled to quantum matter (non gravitational) fields (i. e., when the gravitational degrees of freedom are integrated out) is equivalent to a quantum field theory on a 3-d noncommutative spacetime [5].
In this paper we treat a very simple problem, the harmonic oscillator on the noncommutative plane, and try to give a new perspective of its analogies and differences with respect to the standard quantum harmonic oscillator on the commutative plane, by means of a path integral approach to both systems.

2. Noncommutative Quantum Mechanics

The spectrum of a harmonic oscillator on the noncommutative plane was analyzed in Refs. [6]. Let us review the main results. The Hamiltonian of the system is

\[ H = \frac{1}{2m}(\tilde{p}_1^2 + \tilde{p}_2^2) + \frac{1}{2}m\omega^2(\tilde{q}_1^2 + \tilde{q}_2^2), \]  

(1)

with \([\tilde{q}_i, \tilde{p}_j] = i\hbar\delta_{ij}, [\tilde{q}_1, \tilde{q}_2] = i\tilde{\theta}\), where \(\tilde{\theta}\) is the noncommutativity parameter. It is convenient to rescale phase space coordinates so that the Hamiltonian is written in terms of dimensionless variables:

\[ q_i = \sqrt{\frac{m}{\hbar\omega}}\tilde{q}_i, \quad p_i = \frac{1}{\sqrt{m\omega\hbar}}\tilde{p}_i, \]  

(2)

so that \([q_i, p_j] = i\delta_{ij}, [q_1, q_2] = i\theta\), where

\[ \theta = \frac{m\omega}{\hbar}\tilde{\theta} \]  

(3)

is the dimensionless rescaled noncommutativity parameter, and the Hamiltonian is re-expressed as

\[ H = \hbar\omega (\frac{1}{2}(p_1^2 + p_2^2) + q_1^2 + q_2^2). \]  

(4)

In the following we will omit the \(\hbar\) factors. A simple way to solve this system is to make a Darboux transformation \((q_i, p_i) \mapsto (Q_i, P_i)\) such that the modes of vibration are decoupled. Defining the quantities

\[ \lambda_\pm = \sqrt{1 + \frac{\theta^2}{4} \pm \frac{\theta}{2}} = (\lambda_\pm)^{-1}, \]  

(5)

then the change of variables is given (up to rotations in the \(\{Q_1, P_1\}\) and \(\{Q_2, P_2\}\) planes) by

\[ \xi = \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} = \sqrt{\frac{\lambda_+}{1 + \lambda_+^2}} \begin{pmatrix} \lambda_+ & 0 & \lambda_0 & 0 \\ 0 & 0 & \lambda_+ & -\lambda_- \\ 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \\ P_1 \\ P_2 \end{pmatrix}. \]  

(6)

The new coordinates in the configuration space are now commutative still verifying \([Q_i, P_j] = i\delta_{ij}\), and the Hamiltonian gets the following form in terms of them:

\[ H = \frac{\omega_+}{2}(P_1^2 + Q_1^2) + \frac{\omega_-}{2}(P_2^2 + Q_2^2), \]  

(7)

where \(\omega_\pm = \omega\lambda_\pm\). In this formulation (which we will refer to as canonical formulation in contrast to the path integral) it is easy to see therefore that the isotropic quantum oscillator of
frequency $\omega$ on the noncommutative plane has the same spectrum as an anisotropic oscillator on the commutative plane with frequencies $\omega_1, \omega_2$.

Defining the matrix $(\Omega^{-1})_{ij} = -i[\xi_i, \xi_j]$ is also useful to pass to the first order noncommutative Lagrangian which is written as

$$L_{nc} = \frac{1}{2} \Omega_{ij} \dot{\xi}_i \dot{\xi}_j - H(\xi) = \dot{q}_1 p_1 + \dot{q}_2 p_2 + \theta \dot{p}_2 p_1 - H(q_i, p_i),$$

up to total derivatives.

3. Noncommutativity in the path integral formulation

We are going now to reformulate the problem of the quantum harmonic oscillator on the noncommutative plane, which was discussed in the canonical formulation in the previous section, by using the path integral formalism. One has a sum over all paths in a four-dimensional phase space, each one weighted by a phase factor (action) which depends on the noncommutativity parameter $\theta$. This will allow us to reinterpret the effect of the space noncommutativity as a mapping of paths.

The starting point is the expression for the action of the two dimensional isotropic harmonic oscillator in the commutative plane

$$S_c = \int d\tau L_c(\tau) = \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi \Omega} L_c(\epsilon),$$

with

$$L_c(\epsilon) = -\frac{1}{2} \xi_c^\dagger(\epsilon) A_c(\epsilon) \xi_c(\epsilon),$$

$$A_c(\epsilon) = I + i \frac{\epsilon}{\omega} \Omega(\theta = 0),$$

$\xi_c$ is a matrix notation for the four phase space coordinates in the commutative plane and

$$f(\epsilon) = \omega \int d\tau e^{i\epsilon \tau} f(\tau),$$

with time parameter $\tau$.

The action in the noncommutative case can be expressed in a similar way

$$S_{nc} = \int dt L_{nc}(t) = \int_{-\infty}^{\infty} \frac{dE}{2\pi \omega} L_{nc}(E),$$

$$L_{nc}(E) = -\frac{1}{2} \xi_{nc}^\dagger(E) A_{nc}(E) \xi_{nc}(E),$$

$$A_{nc}(E) = I + i \frac{E}{\omega} \Omega(\theta).$$

---

8In the former equation and from now on we will use the same notation for the phase space coordinates in the classical action as we have used for the quantum operators.

9We are going to assume that all the integrals appearing throughout this work are sufficiently well defined.
The subscript in $\xi_{nc}$ only emphasizes that it corresponds to the column vector of phase space coordinates $\{\xi_i\}$ of the noncommutative plane as defined in (6). For completeness we give the explicit form of the matrix $\Omega(\theta)$

$$
\Omega(\theta) = \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & \theta \\
0 & 1 & -\theta & 0
\end{pmatrix}.
$$

(16)

The notation for the time parameter is now $t$ and

$$
f(E) = \omega \int dte^{itE}f(t).
$$

(17)

All the effect of the noncommutativity at this level is concentrated in the phase space matrix $A_{nc}$ to be compared with $A_c$ in the commutative case. The matrix $A_c$ has two (doubly) degenerate eigenvalues $1 + \epsilon/\omega$, $1 - \epsilon/\omega$, and the corresponding normalized eigenvectors are

$$
v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} q_- \end{pmatrix},
v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} i q_- \end{pmatrix},
v_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} q_- \end{pmatrix},
v_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} q_+ \end{pmatrix},
$$

(18)

where $q_-, q_+$ are normalized two component column vectors satisfying

$$
\sigma_2 q_- = -q_- \quad \quad \sigma_2 q_+ = q_+,
$$

(19)

where

$$
\sigma_2 = \begin{pmatrix} 0 & -i \\
i & 0
\end{pmatrix}
$$

(20)

is the second Pauli matrix.

In the noncommutative case one has the eigenvalues and normalized eigenvectors of the matrix $A_{nc}$

$$
\left(1 + \lambda_+ \frac{E}{\omega}\right) \quad \quad u_1 = \frac{1}{\sqrt{1 + \lambda_+^2}} \begin{pmatrix} q_- \\
i\lambda_+ q_-
\end{pmatrix}
$$

(21)

$$
\left(1 + \lambda_- \frac{E}{\omega}\right) \quad \quad u_2 = \frac{1}{\sqrt{1 + \lambda_-^2}} \begin{pmatrix} q_+ \\
i\lambda_- q_+
\end{pmatrix}
$$

(22)

$$
\left(1 - \lambda_+ \frac{E}{\omega}\right) \quad \quad u_3 = \frac{1}{\sqrt{1 + \lambda_+^2}} \begin{pmatrix} q_- \\
-i\lambda_+ q_-
\end{pmatrix}
$$

(23)

$$
\left(1 - \lambda_- \frac{E}{\omega}\right) \quad \quad u_4 = \frac{1}{\sqrt{1 + \lambda_-^2}} \begin{pmatrix} q_+ \\
-i\lambda_- q_+
\end{pmatrix}
$$

(24)

According to the signs of the eigenvalues of the matrix $A_c(\epsilon)$ one has a decomposition of the commutative action

$$
S_c = \int_{-\omega}^{\omega} \frac{d\epsilon}{2\pi \omega} L_c(\epsilon) + \int_{-\omega}^{\omega} \frac{d\epsilon}{2\pi \omega} L_c(\epsilon) + \int_{\omega}^{\infty} \frac{d\epsilon}{2\pi \omega} L_c(\epsilon).
$$

(25)
In the first term one has signs \((-, +)\) for the two doubly degenerate eigenvalues of \(A_c(\epsilon)\), in the second term both eigenvalues are positive, and in the third term one has signs \((+, -)\).

A similar decomposition for the noncommutative action leads us to

\[
S_{nc} = S_{nc}^\theta + \bar{S}_{nc}^\theta
\]  

(26)

with

\[
S_{nc}^\theta = \int_{-\infty}^{\omega/A_-} \frac{dE}{2\pi\omega} L_{nc}(E) + \int_{\omega/A_-}^{\omega/A_+} \frac{dE}{2\pi\omega} L_{nc}(E) + \int_{\omega/A_+}^{\infty} \frac{dE}{2\pi\omega} L_{nc}(E)
\]  

(27)

\[
\bar{S}_{nc}^\theta = \int_{-\omega/A_-}^{\omega/A_-} \frac{dE}{2\pi\omega} L_{nc}(E) + \int_{\omega/A_-}^{\omega/A_+} \frac{dE}{2\pi\omega} L_{nc}(E)
\]  

(28)

The three terms in \(S_{nc}^\theta\) correspond to the combinations of signs \((-,-,+,+), (+,+,+,+)\) and \((+,+,-,-)\) respectively for the four eigenvalues of \(A_{nc}(E)\), and the two terms in \(\bar{S}_{nc}^\theta\) to \((-,-,+,+)\) and \((+,+,+,-)\).

The three terms in the commutative action can be mapped to the three terms in \(S_{nc}^\theta\) with the identifications \(E = \epsilon/A_-\), \(E = \epsilon/A_+\), and \(E = \epsilon/A_+\). To make explicit the mapping we introduce new phase space coordinates \(a_i\) by

\[
\xi_{nc}(E) = \sum_{i=1}^{4} a_i(E) u_i
\]  

(29)

where \(u_i\) are the four eigenvectors of \(A_{nc}(E)\). In a similar way one can use the four eigenvectors \(v_i\) of the matrix \(A_c(\epsilon)\) to introduce in the commutative case phase space coordinates \(b_i\) through the expansion

\[
\xi_c(\epsilon) = \sum_{i=1}^{4} b_i(\epsilon) v_i
\]  

(30)

Now we can define a mapping between paths in the phase spaces of the commutative and noncommutative systems

\[
a_1(\epsilon/A_-) = \sqrt{\omega + \epsilon/\lambda_-} b_1(\epsilon) \quad a_2(\epsilon/A_-) = \sqrt{\lambda_+} b_2(\epsilon) \\
a_3(\epsilon/A_-) = \sqrt{\lambda_+} b_3(\epsilon) \quad a_4(\epsilon/A_-) = \sqrt{\omega - \epsilon/\lambda_-} b_4(\epsilon)
\]  

(31)

when \(|\epsilon| > \omega\) and

\[
a_1(\epsilon/A_+) = \sqrt{\lambda_-} b_1(\epsilon) \quad a_2(\epsilon/A_+) = \sqrt{\omega + \epsilon/\lambda_+} b_2(\epsilon) \\
a_3(\epsilon/A_+) = \sqrt{\omega - \epsilon/\lambda_-} b_3(\epsilon) \quad a_4(\epsilon/A_+) = \sqrt{\lambda_-} b_4(\epsilon)
\]  

(32)

when \(|\epsilon| < \omega\). With this mapping one has

\[
S_{nc}^\theta[a_i] = S_c[b_i]
\]  

(33)
The decomposition of the noncommutative action as a sum of two contributions (26) can be translated at the level of phase space paths

\[ \xi_{nc}(t) = \xi_{nc}^\theta(t) + \bar{\xi}_{nc}(t) \]  \hspace{1cm} (34)

with

\[ \xi_{nc}^\theta(t) = \int_{-\omega/\lambda}^{\omega/\lambda} dE \frac{2\pi \omega}{2\pi \omega} e^{-iEt} \xi_{nc}(E) + \int_{\omega/\lambda}^{\infty} dE \frac{2\pi \omega}{2\pi \omega} e^{-iEt} \xi_{nc}(E), \]  \hspace{1cm} (35)

\[ \bar{\xi}_{nc}(t) = \int_{-\omega/\lambda}^{\omega/\lambda} dE \frac{2\pi \omega}{2\pi \omega} e^{-iEt} \xi_{nc}(E) + \int_{-\omega/\lambda}^{-\omega/\lambda} dE \frac{2\pi \omega}{2\pi \omega} e^{-iEt} \xi_{nc}(E). \]  \hspace{1cm} (36)

Since every path in phase space \( \xi_{nc} \) is decomposed into a direct sum of two paths \( \xi_{nc}^\theta \) and \( \bar{\xi}_{nc} \), this amounts to a decomposition of the space of paths.

The mapping between paths can then be re-expressed in the form

\[ \xi_{nc}^\theta(t) = \omega \int d\tau K_\xi(t, \tau) \xi_c(\tau). \]  \hspace{1cm} (37)

The explicit form of the kernel matrix \( K_\xi(t, \tau) \) can be obtained by combining (35) with the expansions of \( \xi_{nc}(E) \) and \( \xi_c(\epsilon) \) in eigenvectors of the matrices \( A_{nc}(E) \) and \( A_c(\epsilon) \) (29-30) and the mapping from \( a_i \) to \( b_i \) (31-32).

To summarize one has a noncommutative action

\[ S_{nc}[\xi_{nc}] = S_{nc}^\theta[\xi_{nc}^\theta] + \bar{S}_{nc}[\bar{\xi}_{nc}] = S_c[\xi_c] + \bar{S}_{nc}[\bar{\xi}_{nc}]. \]  \hspace{1cm} (38)

The effect of the noncommutativity in the path integral formulation is twofold: a mapping of paths (37) and the addition of a term in the action (\( \bar{S}_{nc} \)).

A similar analysis of the effect of noncommutativity can be made at the level of paths in configuration space. One can repeat step by step the discussion of the path integral formulation in phase space. We give directly the result. One has

\[ S_{nc}[q_{nc}] = S_{nc}^\theta[q_{nc}^\theta] + \bar{S}_{nc}[\bar{q}_{nc}] \]  \hspace{1cm} (39)

with

\[ q_{nc}(t) = q_{nc}^\theta(t) + \bar{q}_{nc}(t) \]  \hspace{1cm} (40)

\[ q_{nc}^\theta(t) = \int_{-\omega/\theta}^{\omega/\theta} dE \frac{2\pi \theta}{2\pi \omega} e^{-iEt} q_{nc}(E) \]  \hspace{1cm} (41)

\[ \bar{q}_{nc}(t) = \int_{\omega/\theta}^{\infty} dE \frac{2\pi \theta}{2\pi \omega} e^{-iEt} q_{nc}(E). \]  \hspace{1cm}

The map

\[ q_{nc}(E) = \sqrt{\frac{dE}{d\epsilon}} q_c(\epsilon) \]  \hspace{1cm} (42)

with

\[ \frac{E}{\sqrt{1 + \theta \frac{E}{\omega}}} = \epsilon \]  \hspace{1cm} (43)
for $E > -\omega/\theta$ leads to
\[
S^\theta_{mc}[q^\theta_{mc}] = S_c[q_c] \tag{44}
\]
where $S_c$ is the action in the commutative case that corresponds to the path
\[
q_c(\tau) = \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi\omega} e^{-i\epsilon\tau} q_c(\epsilon). \tag{45}
\]
Then once more one has that the effect of the noncommutativity at the level of the path integral formulation in configuration space is a mapping of paths
\[
q^\theta_{mc}(t) = \omega \int d\tau \ K_\theta(t,\tau) \ q_c(\tau) \tag{46}
\]
and the addition of a term in the action ($\overline{S}^\theta_{mc}$).

The characterization of noncommutativity as a decomposition of the space of paths \[\xi_c\], together with a mapping of the space of paths $\xi_c$ into the subspace of paths $\xi^\theta_{mc}$ corresponding to one of the components of the decomposition, is a consequence of the path integral formulation of the quantum system. In the canonical formulation of the previous section one had just a (noncanonical) change of variables in phase space associated to the noncommutativity of space. In fact this change of variables could have been introduced also at the level of the action as a functional defined in the space of paths to show that it can be written as a sum of two actions, each one corresponding to a one dimensional harmonic oscillator of different frequencies.

### 4. Summary

We have shown that for a quantum linear system (quadratic action) it is possible to identify a mapping of paths that allows to go partially from the quantum system defined in a commutative space to the system in a noncommutative space. We say partially because the noncommutative action can be written as a sum of two independent contributions (in the sense that they involve different decoupled degrees of freedom) and it is only one of them that can be obtained from the commutative action through a mapping of paths.

The identification of this mapping associated to noncommutativity opens a new way to introduce nonlinear effects. Instead of including directly non-quadratic terms at the level of the noncommutative action, one can apply the mapping identified in the linear system to a commutative action including non-quadratic terms, and then add the quadratic additional action $S^\theta_{mc}$ of the linear system to the resulting action. In this way one generalizes the correspondence of the commutative and noncommutative actions to nonlinear systems. It seems interesting to investigate the consistency and properties of a quantum system defined in this way.

The possibility to use an alternative characterization of space noncommutativity to introduce nonlinear effects in a different way while keeping a simple relation with the commutative case can be just a curiosity at the level of quantum mechanics but the extension of the discussion presented in this work to quantum field theory (\[29\]) may be essential to formulate a theory with interactions which is consistent in the presence of noncommutativity at the level of fields.
Acknowledgements

This work has been partially supported by CICYT (grant FPA2006-02315) and DGIID-DGA (grant2008-E24/2). J.I. acknowledges a FPU grant and D.M. a FPI grant from MICINN.

References

[1] A. Connes, *Noncommutative Geometry* (Academic Press, 1994).
[2] F. Lizzi, arXiv:0811.0268; L.J. Garay, Int. J. Mod. Phys. A10 (1995) 145; S. Doplicher, K. Fredenhagen and J. E. Roberts, Commun. Math. Phys. 172 (1995) 187; M. R. Douglas and N. A. Nekrasov, Rev. Mod. Phys. 73 (2001) 97.
[3] A. Connes, M. Douglas and A. S. Schwarz, JHEP 02 (1998) 003; N. Seiberg and E. Witten, JHEP 09 (1999) 032; T. Yoneya, Prog. Theor. Phys. 103 (2000) 1081.
[4] L. Mezincescu, arXiv:hep-th/0007046; C. Duval and P.A. Horvathy, Phys. Lett. B 479 (2000) 284; J. Gamboa, M. Loewe and J.C. Rojas, Phys. Rev. D64 (2001) 067901.
[5] L. Freidel and E. R. Livine, Phys. Rev. Lett. 96 (2006) 221301.
[6] J. Lukierski, P. Stichel and W. Zakrzewski, Ann. Phys. 260 (1997) 224; V.P. Nair and A.P. Polychronakos, Phys. Lett. B505 (2001) 267.
[7] J.M. Carmona, J.L. Cortés, J. Gamboa and F. Méndez, JHEP 03 (2003) 058.
[8] G. Mangano, J. Math. Phys. 39 (1998) 2584; C. Acatrinei, JHEP 09 (2001) 007.
[9] J.M. Carmona, J.L. Cortés, J. Induráin and D. Mazón, in preparation.