A note on the convergence of renewal and regenerative processes to a Brownian bridge

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Abstract

The standard functional central limit theorem for a renewal process with finite mean and variance, results in a Brownian motion limit. This note shows how to obtain a Brownian bridge process by a direct procedure that does not involve conditioning. Several examples are also considered.

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1 The basic theorem

In proving convergence results for a stochastic ordered graph on the integers [2], we noticed that one can obtain a Donsker-like theorem for Brownian bridge in a somewhat non-standard manner. The result appears to be new. As it may be of potential interest in some related areas (statistics, large deviations), we summarise it in this short note.

Consider a (possibly delayed) renewal process on [0, ∞) with renewal epochs

\[ 0 < R_1 < R_2 < \cdots \]

We assume that \( \{R_{n+1} - R_n\}_{n \geq 1} \) are i.i.d. with mean \( \mu \) and variance \( \sigma^2 \), both finite. Let

\[ A_t := \#\{n \geq 1 : R_n \leq t\} \]

be the associated counting process. The standard functional central limit theorem for a renewal process, see, e.g., [1], states that the sequence of processes \( \xi_1, \xi_2, \ldots \), where

\[ \xi_n(t) := \frac{A_{nt} - \mu^{-1}nt}{\sqrt{n}}, \quad t \geq 0, \]

converges weakly, as \( n \to \infty \), to \( \mu^{-3/2}\sigma W \), where \( W \) is a standard Brownian motion on [0, ∞). Weak convergence (denoted by \( \Rightarrow \) below) means weak convergence of probability
measures on the space $D[0, \infty)$ of functions which are right continuous with left limits, equipped with the usual Skorokhod topology (see, e.g., [3], [7]).

A standard Brownian bridge \[3\] p. 84] $W^0$ is defined, in distribution, as a standard Brownian motion $W$ on $[0, 1]$, conditional on $W_1 = 0$, i.e. as the weak limit of the sequence of probability measures

$$P(W \in \cdot \mid 0 \leq W_1 \leq 1/n), \quad n \in \mathbb{N},$$

as $n \to \infty$. Often, when Brownian bridge is obtained as a limit by a functional central limit theorem, there is an explicit underlying conditioning that takes place. One first proves convergence to a Brownian motion and uses conditioning to prove convergence to a Brownian bridge. Brownian bridges appear in limits of urn processes, and also in limits of empirical distributions [3 Thm. 13.1].

In this note we remark that it is possible to obtain a Brownian bridge from a renewal process, without the use of conditioning.

**Theorem 1.** Define, for $u > 0$,

$$\eta_u(t) := \frac{R_{[tA_u]} - tu}{\sqrt{u}}, \quad 0 \leq t \leq 1.$$

Considering $\eta_u$ as a random element of $D[0, 1]$ (equipped with the topology of uniform convergence on compacta), we have

$$\eta_u \Rightarrow \mu^{-1/2} \sigma W^0, \quad \text{as } u \to \infty,$$

where $W^0$ is a standard Brownian bridge.

Here, $[x]$ denotes the largest integer not exceeding the real number $x$. We remark that $R_{A_u}$ is “close” to $u$, in the sense that $R_{A_u} \leq u < R_{1+A_u}$. In fact, the difference $u - R_{A_u}$ (known as the age of the renewal process) is a tight family (over $u \geq 0$) of random variables. In the above theorem, we just introduce another parameter, $t$, and measure the difference between $tu$ and $R_{[tA_u]}$. When $t = 0$ or 1, this difference is “negligible” with respect to any power of $u$. When $t$ is between 0 and 1, then the difference is of the “order of $\sqrt{u}$” in the sense that when divided by $\sqrt{u}$ it converges to a normal random variable. Jointly, over all $t \in [0, 1]$, we have convergence to a Brownian bridge, and this is what we show next.

**Proof.** Consider, for $u > 0$,

$$y_u(t) := \frac{R_{[tu]} - \mu tu}{\sqrt{u}}, \quad t \geq 0.$$

From Donsker’s theorem [3] for the random walk $\{R_n\}$ we have that $y_u \Rightarrow \sigma W$, where $W$ is a standard Brownian motion. Define also, for $u > 0$,

$$\varphi_u(t) := \frac{tA_u}{u}.$$

From the law of large numbers for the renewal process, $A_u/u \to \mu^{-1}$, a.s., as $u \to \infty$. Hence, $\varphi_u$ converges a.s. (and weakly) to the deterministic process $\{\mu^{-1}t\}$. Since composition is a continuous function [3], we have that

$$\{(y_u \circ \varphi_u)(t)\} \Rightarrow \{\sigma W_{\mu^{-1}t}\} \stackrel{d}{=} \{\mu^{-1/2}\sigma W_t\}. \quad (1)$$
We also have
\[(y_u \circ \varphi_u)(t) = \frac{R_{[tA_u]} - \mu t A_u}{\sqrt{u}},\]
and so
\[\eta_u(t) = (y_u \circ \varphi_u)(t) + \mu \frac{A_u - \mu^{-1} u}{\sqrt{u}}\]
\[= (y_u \circ \varphi_u)(t) - t(y_u \circ \varphi_u)(1) - t \frac{u - R_{A_u}}{\sqrt{u}}. \tag{2}\]

Observe now that \(\{u - R_{A_u}, u \geq 0\}\) is a tight family. Indeed, from standard renewal theory (see, e.g., [1]), if \(R_1\) has a non-lattice distribution, then \(u - R_{A_u}\) converges weakly as \(u \to \infty\). And if \(R_1\) has a lattice distribution with span \(h\), then a similar convergence takes places for \(nh - R_{A_nh}\) as \(n \to \infty\). Since, for all \(u \geq 0\), \(0 \leq u - R_{A_u} \leq (\lfloor u/h \rfloor + 1)h - R_{A_u}\lfloor u/h \rfloor\), the family \(\{u - R_{A_u}, u \geq 0\}\) is tight even in the lattice case. Tightness implies that the last term of (2) converges to 0 in probability. From the convergence stated in (1) and the decomposition (2), we have that
\[\{\eta_u(t)\}_{0 \leq t \leq 1} \Rightarrow \mu^{-1/2} \sigma \{W_t - tW_1\}_{0 \leq t \leq 1}.\]

It is well known [4] that a standard Brownian bridge \(W^0\) can be represented as \(W_t^0 = W_t - tW_1\), and so the process above is the limit we were looking for.

\[\square\]

2 Extensions, discussion, and examples

Here is a different version that, perhaps, makes Theorem 1 clearer: Suppose that \(M\) is a regenerative random measure on \([0, \infty)\). That is, there is some renewal process with points \(T_0 < T_1 < T_2 < \cdots\) such that the random measures obtained by restricting \(M\) onto \([T_n, T_{n+1})\), \(n = 0, 1, 2, \ldots\), are i.i.d. Suppose that
\[\mu := E(T_2 - T_1), \quad \var(T_2 - T_1) < \infty,\]
\[\alpha := EM([T_1, T_2)), \quad 0 < \var(M([T_1, T_2))) < \infty.\]

Define the random distribution function of \(M\) by
\[S(t) = M((0, t]), \quad u \geq 0.\]

By the law of large numbers, \(S(t)/t \to \mu^{-1} \alpha\), a.s. as \(t \to \infty\). Consider the generalised inverse
\[S^{-1}(u) := \inf\{t \geq 0 : S(t) > u\}, \quad u \geq 0.\]

Then, in some naive sense, \(S^{-1}\) composed with \(S\) is “approximately” the identity function, but what can we say about the composition of \(S^{-1}\) with a fraction \(tS\) of \(S\) where \(0 < t < 1\)\? The law of large numbers tells us that, almost surely,
\[\frac{S(tS^{-1}(u))}{u} \xrightarrow{u \to \infty} t.\]

An extension of the previous theorem quantifies the deviation:
Theorem 2. As $u \to \infty$, the sequence of processes $\eta_u$ where
\[ \eta_u(t) := \frac{S(tS^{-1}(u)) - tu}{\sqrt{u}}, \quad 0 \leq t \leq 1, \]
converges weakly to a Brownian bridge.

The proof of this is analogous to the previous one, so it is omitted. Observe that the “tying down” of the Brownian motion occurs naturally at $t = 0$ and $t = 1$.

The Brownian bridge has a scaling constant depending on the parameters of the process $S$.

Note that the regenerative assumption is not crucial. All we need is to have a process for which a Donsker theorem with a Brownian limit holds. This is then translatable to a Brownian bridge limit.

If we interchange the roles of $S$ and $S^{-1}$ we still get a Brownian bridge but with different constant. For instance, interchanging the roles of $\{R_n\}$ and $\{A_u\}$ in Theorem 1 we obtain that
\[ \eta_n'(t) := \frac{A(tR_n) - tn}{\sqrt{n}}, \quad 0 \leq t \leq 1, \]
converges weakly, as $n \to \infty$, to $\kappa W^0$, where $W^0$ is a standard Brownian bridge and $\kappa = \sigma \mu^{-1}$.

2.1 An interpretation

To better understand the phenomenon, we cast the limit theorem as follows: We have a random function $S$, composed with scaling functions
\[ \rho_t : x \mapsto tx \]
and composed again with the inverse function $S^{-1}$ and we look at the asymptotic behaviour of the family of random functions
\[ S \circ \rho_t \circ S^{-1} - \rho_t, \quad 0 \leq t \leq 1, \quad (3) \]
(or of $S^{-1} \circ \rho_t \circ S$), as a function of the parameter $t$. Thus, the time parameter of the Brownian bridge obtained in the limit plays the role of a scaling factor. When $t$ is 0 or 1, $S \circ \rho_t \circ S^{-1} - \rho_t$ is approximately zero (with respect to the normalising factor). This raises the following three questions:
(i) How much “one-dimensional” is this phenomenon?
(ii) Can we replace the family $\rho_t$ by a more general homotopy?
(iii) Are different kind of bridges possible to obtain?

With respect to the latter question, we could start with a regenerative process with finite mean but infinite variance, one that belongs to the domain of attraction of, say, a self-similar Lévy process.
2.2 Four examples

**EXAMPLE 1** The first is a simple example involving a standard Brownian motion $W$. Let $X$ denote the (strong) Markov process

$$X_t = (W_t - t) - \min_{0 \leq s \leq t} (W_s - s), \quad t \geq 0,$$

which is the reflection of the drifted Brownian motion $\{W_t - t\}$. This process in natural in many areas of applied probability, e.g. in the diffusion approximation of a queue. We have $X_0 = 0, X_t \geq 0$. The Brownian area process

$$S(t) = \int_0^t X_r \, dr$$

is non-decreasing. Fix some $u \geq 0$ and $t \in [0,1]$. By continuity, there is a unique point between 0 and $u$ that splits the area $S(u)$ into two parts with ratio $t : (1 - t)$. Call this point $H_u(t)$. Specifically,

$$H_u(t) := \min \left\{ v \geq 0 : t \int_0^v X_r \, dr = (1 - t) \int_v^u X_r \, dr \right\}, \quad 0 \leq t \leq 1.$$

We then claim that

$$\eta_u(t) := \frac{H_u(t) - tu}{\sqrt{u}}, \quad 0 \leq t \leq 1,$$

converges weakly to a Brownian bridge as $u \to \infty$. To see this, observe that

$$S^{-1}(x) = \min \{ v \geq 0 : S(v) = x \},$$

and hence

$$S^{-1}(tS(u)) = \min \{ v \geq 0 : S(v) = tS(u) \}
= \min \{ v \geq 0 : S(v) = t(S(v) + S(u) - S(v)) \}
= \min \{ v \geq 0 : (1 - t)S(v) = t(S(u) - S(v)) \} = H_u(1 - t).$$

Apply Theorem 2 to get the result. (Notice that $\eta_u(1 - t)$ also converges to a Brownian bridge.)

**EXAMPLE 2** Same as Example 1, but with $W$ being a zero-mean Lévy process. The Brownian bridge in Example 1 was obtained not from the fact that $W$ was Brownian, but from the regenerative structure of $S$. It is this that allows us to replace $W$ by a more general, say a Lévy process, as long as we maintain the finite variance assumptions. The latter hold once we add a strictly negative drift to a zero-mean Lévy process $W$, reflect it, precisely as in (4), and integrate just as in (5). Whereas $W$ may be discontinuous, $S$ is continuous and the conclusion remains the same.
EXAMPLE 3 The third example is an application of the above in proving a limit theorem for a random digraph. We consider a random directed graph $G_n = (V_n, E_n)$ on the set of vertices $V_n := \{1, \ldots, n\}$ by letting the set of edges $E_n$ contain the pair $(i, j)$, $i < j$, with probability $p$, independently from pair to pair. This is a directed version of the (nowadays) so-called Erdős-Rényi graph.

A path starting in $i$ and ending in $j$ is a sequence of vertices $i_0 = i, i_1, \ldots, i_n = j$ such that $(i, i_1), \ldots, (i_{n-1}, j)$ are edges. Amongst all paths in $G_n$ there is one with maximum length; this length is denoted by $L_n$. Amongst all paths in $G_n$ that end at a vertex $j \in V_n$ there is one with maximum length; this length is called weight of vertex $j$. We keep track of vertices with a specific weight and let $S_n(\ell)$ be the number of vertices with weights at least $\ell$. (Here $\ell$ ranges between 0 and $L_n$.) So, for example, $S_n(0)$ is the number of vertices in $V_n$ that are endpoints of no edge in $E_n$, and $S_n(L_n)$ is the number of paths of maximal length in $G_n$.

**Theorem 3.**

$$S_n(\lfloor tL_n \rfloor) - tn \sqrt{n}, \quad 0 \leq t \leq 1,$$

converges, as $n \to \infty$, weakly to a Brownian bridge.

The proof of this theorem can be found in [2, p. 453].

EXAMPLE 4 Here is an illustration, of the kind of phenomenon described around (3), in Stochastic Geometry. We consider a Poisson point process $N$ in $\mathbb{R}^d$ with intensity, say, 1; that is, $N$ is a random discrete subset of $\mathbb{R}^d$ such that the cardinalities of $N \cap B_1, \ldots, N \cap B_n$ are independent random variables whenever $B_1, \ldots, B_n$ are disjoint Borel sets, for any $n \in \mathbb{N}$, and the expectation of the cardinality of $N \cap B$ equals the Lebesgue measure of $B$. For each $x$ in $\mathbb{R}^d$ we let $\pi(x)$ be the point of $N$ closest to $x$ (there is a.s. a unique such point). For each point $z$ of $N$, we let $\sigma(z)$ be the Voronoi cell associated to $z$:

$$\sigma(z) := \{x \in \mathbb{R}^d : ||x - z|| \leq ||x - z'|| \text{ for all points } z' \text{ of } N\},$$

where $|| \cdot ||$ is the Euclidean norm on $\mathbb{R}^d$. The Voronoi tessellation of $\mathbb{R}^d$ is the tiling of $\mathbb{R}^d$ by the Voronoi cells. If $z$ is not a point of $N$ we define $\sigma(z)$ to be the Voronoi cell containing $z$ (again this cell is a.s. unique). The distance of a closed set $A \subset \mathbb{R}^d$ from a point $x \in \mathbb{R}^d$ is

$$\text{dist}(A, x) = \inf\{||x - y|| : y \in A\}.$$ 

Consider now the process

$$D(t, x) := \text{dist}(\sigma(t\pi(x)), tx),$$

where $t \in [0, 1]$ and $x \in \mathbb{R}^d$. The claim is that

$$||x||^{-1/2} D(\cdot, x) \Rightarrow |W^0|,$$

$|W^0|$ being the absolute value of a Brownian bridge.

\footnote{More general point processes can be allowed here.}
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