Noncommutivity and Scalar Field Cosmology

W. Guzmán\(^1\)
M. Sabido\(^1\)
and J. Socorro\(^1,2\)

\(^1\) Instituto de Física de la Universidad de Guanajuato, A.P. E-143, C.P. 37150, León, Guanajuato, México

\(^2\) Facultad de Ciencias de la Universidad Autónoma del Estado de México, Instituto Literario No. 100, Toluca, C.P. 50000, Edo Mex, México.

(Dated: February 2, 2008)

PACS numbers: 02.40.Gh, 04.60.Kz, 98.80.Jk, 98.80.Qc

It is a common issue in Cosmology to make use of scalar fields \(\phi\) as the responsible agents of some of the most intriguing aspects of our universe (see [1] and references therein). Just to mention a few, we find that scalar fields are used as the inflaton which seeds the primordial perturbations for structure formation during an early inflationary epoch; as the cold dark matter candidate responsible for the formation of the actual cosmological structure and as the dark energy component which seems to be driving the current accelerated expansion of the universe. The key feature for such flexibility of scalar fields (spin-0 bosons) is the freedom one has to propose a scalar potential \(V(\phi)\), which encodes in itself the (non gravitational) self-interactions among the scalar particles.

To study the very early universe one of the simplest approaches is Quantum Cosmology (QC). In this framework gravitational and matter variables have been reduced to a finite number of degrees of freedom. For homogenous cosmological models the metric depends only on time and gives a model with a finite dimensional configuration space, minisuperspace [2]. This simplification is used because a full quantum theory of gravity does not exist (although two main candidates have been constructed: String Theory and Loop Quantum Gravity) and these approximate models can be canonically quantized.

Since the end of the XX century the old idea of noncommutative space-time [3] has been rekindled, the renewed interest is a consequence of the developments in M-Theory and String Theory [4]. Although most of the work of noncommutative field theory has been done in connection with Yang-Mills theory [5], several noncommutative models of gravity have been constructed [6]. All of these formulations of gravity in noncommutative space-time share a common issue; they are highly non-linear and solutions to the noncommutative equations of motion are incredibly difficult. One may expect that at very early times non trivial effects of noncommutativity could affect the evolution of the universe, making the study of noncommutative cosmological models an interesting testing ground for noncommutativity. Unfortunately, as already mentioned, the complexity of the field equations for noncommutative gravity makes a direct calculation of cosmological models a very complex task. On this line of reasoning, in the last few years there have been several attempts to study the possible effects of noncommutativity in the cosmological and inflationary scenario [7, 8, 9]. In [7] the authors in a cunning way avoid the difficult technicalities of analyzing noncommutative cosmological models when these are derived from the full noncommutative theory of gravity. They achieved this by introducing the effects of noncommutativity in quantum cosmology by deforming the minisuperspace variables and introducing the Moyal product of functions in the Wheeler-DeWitt equation (WDW), similar to noncommutative quantum mechanics [10]. Other authors have analyzed some phenomenological effects of noncommutativity in cosmology, [8] but in their analysis they neglect the noncommutative effects on the gravitational sector. The aim of this paper is to construct a noncommutative formulation of a simple scalar field cosmological model in which the matter and gravitational sector are affected by the noncommutative deformation.

Let us start with the line element for a homogeneous, flat and isotropic universe, the Friedmann-Robertson-Walker (FRW) metric

\[
 ds^2 = -N^2(t)dt^2 + e^{2a(t)}[dr^2 + r^2d\Omega^2],
\]

where \(a(t) = e^{\alpha(t)}\) is the scale factor and \(N(t)\) is the lapse function. The classical evolution is obtained by solving Einstein’s field equations. Using the energy momentum tensor of a scalar field we get the Einstein-Klein-Gordon equations

\[
 H^2 = \frac{8\pi G}{3} \left( \frac{1}{2} \dot{\phi}^2 + V(\phi) \right), \quad \ddot{\phi} + 3H\dot{\phi} = -\frac{\partial V(\phi)}{\partial \phi},
\]
where $H = \dot{a}/a$, $\rho = \frac{1}{2} \dot{\phi}^2 + V(\phi)$ and $p = \frac{1}{2} \dot{\phi}^2 - V(\phi)$. As stated at the beginning, by choosing appropriate potentials, different aspects of the universe can be modeled.

For our purposes the Hamiltonian formulation is more appropriate and is constructed from the effective action

$$S = \int dx^4 \sqrt{-g} \left[ R + \frac{1}{2} g^{\mu \nu} \partial_{\mu} \phi \partial_{\nu} \phi + V(\phi) \right],$$  \hspace{1cm} (3)

where we have used the units $8 \pi G = 1$. Instead of solving Einstein’s equations for the FRW metric with the scalar field energy momentum tensor, we will be using the Hamiltonian dynamics derived from Eq. (3) as well as one of the most used and better understood scalar field potentials, the exponential potential $V(\phi) = V_0 e^{-\lambda \phi}$. This potential has the advantage that analytical solutions can be found and for appropriate values of the parameters inflation can be obtained. Unfortunately, when used in connection with inflation there is no mechanism to end the inflationary epoch. We can write the canonical Hamiltonian by means of the Legendre transformation and arrive to the FRW Hamiltonian

$$\mathcal{H} = \frac{N}{12} e^{-3\alpha} \left[ \Pi^2_\phi - 6 \Pi_\phi^2 - 12 e^{6\alpha} V(\phi) \right].$$  \hspace{1cm} (4)

The Poisson algebra for the minisuperspace variables is $\{\alpha, \phi\} = 0$, $\{\Pi_\alpha, \Pi_\phi\} = 0$, $\{\alpha, \Pi_\phi\} = 1$, $\{\phi, \Pi_\alpha\} = 1$, if we set $N = 1$ we recover the equations derived from GR. To simplify the calculations we will be working in the gauge $N = 12 e^{3\alpha}$, the advantage is that in this gauge we can find exact analytical solutions to the noncommutative quantum models. Because of the reparametrization invariance of the theory the physical implications are independent of the gauge. For $N = 12 e^{3\alpha}$ the equations of motion take the simple form

$$\dot{\alpha} = 2 \Pi_\alpha, \quad \Pi_\alpha = 72 e^{6\alpha} V(\phi),$$
$$\dot{\phi} = -12 \Pi_\phi, \quad \Pi_\phi = 12 e^{6\alpha} \frac{dV(\phi)}{d\phi}.$$  \hspace{1cm} (5)

As already pointed, we will be using the exponential potential $V(\phi) = e^{-\lambda \phi}$, Eq. (5) is simplified by making the canonical transformation

$$x = -6 \alpha + \lambda \phi, \quad \Pi_x = \frac{1}{\lambda^2 - 6} (\Pi_\alpha + \lambda \Pi_\phi),$$
$$y = -\sqrt{6} \alpha - \sqrt{6} \phi, \quad \Pi_y = \frac{1}{\sqrt{6}(6 - \lambda^2)} (\lambda \Pi_\alpha + 6 \Pi_\phi).$$  \hspace{1cm} (6)

In this new set of minisuperspace variables we will be making our analysis, the classical Hamiltonian Eq. (4) in the new variables has the simple form

$$\mathcal{H} = -\beta \Pi_x^2 + \beta \Pi_y^2 - 12 V_0 e^{-x} \approx 0,$$  \hspace{1cm} (7)

where $\beta$ is defined as $\beta \equiv 6(\lambda^2 - 6)$. In these variables the equations of motion are considerably simplified, in particular $\Pi_x = 0$ gives a constant of motion, which helps to solve the rest of the equations, actually the canonical transformation was used in order to have a simplified Hamiltonian that has at least one constant of motion. The solutions in the minisuperspace variables $(x, y)$ are

$$x(t) = \ln \left( \cosh^2 \sqrt{12 \beta V_0} t \right), \quad y(t) = 2 \beta \Pi_{y_0} (t - t_0).$$  \hspace{1cm} (8)

After applying the inverse canonical transformation we get the solutions in the original variables

$$\phi(t) = \frac{1}{6 - \lambda^2} \left( 2 \sqrt{6} \beta \Pi_{y_0} t - \lambda \ln \left( \cosh^2 \sqrt{12 \beta V_0} t \right) \right),$$
$$\alpha(t) = \ln \left( \cosh \frac{2}{6 - \lambda^2} \left[ \sqrt{12 \beta V_0} t \right] \right) - 2 \lambda \sqrt{6} \beta \Pi_{y_0} t,$$  \hspace{1cm} (9)

and can easily calculate the expression for the scale factor $a(t)$. In these solutions the parameter $t$ is not the usual time that is used in cosmology. In order to have exactly the same solutions, we need to work in the gauge $N = 1$ and to find the inflationary epoch we apply the slow roll condition $\epsilon = \left( V''(\phi) / V(\phi) \right)^2 \ll 1$, this yields the condition $\lambda < \sqrt{2}$. Carrying out the same analysis in the gauge and variables we have chosen, we obtain the same result, this is to be expected as the theory is invariant under time reparametrization.

The original proposal for noncommutative cosmology, was done at the quantum level \[7\]. The simplest approach for quantum cosmology is canonical quantization and is achieved by applying the well-known result $\mathcal{H} = 0$. The WDW equation for this model is achieved by the usual identifications, $\Pi_x = -i \partial_x$ and $\Pi_y = -i \partial_y$ in Eq. (5) and gives a Klein-Gordon type equation

$$\frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 \Psi}{\partial y^2} + \frac{2V_0}{6 - \lambda^2} e^{-x} \Psi = 0.$$  \hspace{1cm} (10)

In this formalism, $\Psi$ is called the wave function of the universe. To extract a normalizable wave function we need to construct wave packets to form a Gaussian state from which some physical information can be obtained. The proposal to introduce the noncommutative minisuperspace deformation is achieved by introducing the following commutation relation between the minisuperspace variables

$$[x, y] = i \theta,$$  \hspace{1cm} (11)

this can be seen as an effective noncommutativity that could arise from a fundamental noncommutative theory of gravity. For example, if we start with the Lagrangian derived in \[1\] (this is a higher order Lagrangian and is expanded in the usual noncommutative parameter), the noncommutative fields are a consequence of noncommutativity among the coordinates and then the minisuperspace variables would inherit some effective noncommutativity. This we assume to be encoded in \[11\], otherwise we would have a very complicated Hamiltonian for the higher order Lagrangian.

This effective noncommutativity can be formulated in terms of product of functions of the minisuperspace variables, with the Moyal star product of functions. The
The noncommutative WDW (NCWDW) equation is obtained by replacing the products of functions by star products. We can show that the effects of the Moyal star product are reflected only in a shift in the potential $V(x, y) \ast \Psi(x, y) = V(x + \frac{\theta}{2} \Pi y, y - \frac{\theta}{2} \Pi x) \Psi(x, y)$. Applying this to Eq. (10) we can write

$\left(-\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \gamma e^{-\left(x - \frac{\theta}{2} \Pi y\right)}\right) \Psi = 0,$

(12)

and the solutions are given by

$\Psi_{\theta}^{nc}(x, y) = e^{\pm i\theta y} \left[J_{\pm i2\eta} \left(\sqrt{\gamma} e^{-\frac{x}{2} - \frac{\theta}{2} \Pi y}\right)\right].$

(13)

These solutions give the quantum description of the noncommutative universe. The wave packet can be constructed as in the commutative case, but unfortunately is difficult to obtain physical information. In [2] the authors arrive to a similar equation, but they are interested only in the quantum epoch of the Kantowski-Sachs universe. By solving the NCWDW equation and plotting the probability density after constructing a wave packet. In order to extract physical information we may try to find the classical evolution for this model, this can be achieved by several equivalent ways (i.e. WKB type analysis, Hamiltonian dynamics, etc.).

For the classical evolution, we start by proposing that the noncommutative nature of the minisuperspace variables can be encoded in a deformation of the Poisson structure this has the advantage that in this approach the Hamiltonian is not modified. For our model the new algebra for the minisuperspace variables is, \{x, y\} = \theta, \{\Pi x, \Pi y\} = 0, \{x, \Pi x\} = 1, \{y, \Pi y\} = 1. From this algebra we find the noncommutative equations of motion

$x = -2\beta \Pi x, \quad \Pi x = -12V_0 e^{-x},$

$y = 2\beta \Pi y - 12\theta V_0 e^{-x}, \quad \Pi y = 0,$

(14)

as in the commutative model the equation associated to the momenta \Pi y is a constant of motion, this simplifies the calculations considerably. Solving this set of equations we arrive to the classical evolution for noncommutative cosmology

$x(t) = \ln \left[\cosh^2 \left(\sqrt{12\beta V_0} t\right)\right],$

$y(t) = 2\beta y_0 t + \theta \left(\frac{12V_0}{\beta}\right)^{\frac{1}{2}} \tanh \left(\sqrt{12\beta V_0} t\right).$

(15)

From the canonical transformation, Eq. (11), we reconstruct the noncommutative scale factor and the time dependence of scalar field. As in the commutative case, the parameter $t$ does not correspond to the usual cosmic time and as already stressed, the physical implications are independent of the chosen gauge.

We can see in Eq. (15) that the noncommutative deformations only modify the variable $y$. Furthermore, the hyperbolic tangent asymptotically goes to \pm 1, then the effects of noncommutativity in the evolution of the universe can only be felt for a short time, after which the behavior is exactly the same as in the commutative case, non the less there is the possibility that the effects of this minisuperspace noncommutativity may change the dynamics during this short period of time. In order to study this possibility we will analyze the inflationary scenario that can be constructed from this noncommutative model. For the commutative case, the potential that we are using gives inflation if we fix the parameter in the potential to \(\lambda > \sqrt{2}\). One of the advantages of this inflation potential, is that exact analytical solutions can be found, but the lack of a mechanism to end inflation is its biggest drawback. This can be seen from the constant value we get for the slow roll parameter $\epsilon = \lambda^2/2$, eliminating any possibility to violate the slow roll condition $\epsilon < 1$. To study some consequences of the noncommutative model we start by calculating $\epsilon$, this can easily be done from the definition of the slow roll parameter and the noncommutative solutions. To first order in $\theta$, we find

$\epsilon = \frac{\lambda^2}{2} \left[1 - \theta \mu \left(1 - \frac{\sqrt{6}}{\lambda} \tanh \left(3\mu(\lambda^2 - 6)t\right)\right)\right],$

(16)

where $\mu = \sqrt{\frac{3V_0}{\lambda}}$. We can write the slow roll parameter as the sum of to contributions $\epsilon = \epsilon_c + \epsilon_{nc}$, the first term corresponds to the usual slow roll parameter and the second one to the noncommutative contribution. As expected in the limit $\theta \to 0$ we get the usual condition for inflation $\lambda < \sqrt{2}$, also for $t \to \infty$, the slow roll parameter tends to a constant value, meaning that the effects of the noncommutative deformation can not be seen. If we start in an inflationary epoch for the commutative case ($\lambda < \sqrt{2}$), in the noncommutative case the slow roll parameter also starts at a value smaller than 1, so in both cases we have inflation, but as the universe evolves the value of $\epsilon$ in the noncommutative model increases and by choosing appropriately the parameters of the potential, as well as the value of $\theta$, we can violate the slow roll condition and end inflation; this is in contrast with the commutative case. Then we can argue that the introduction of the noncommutative minisuperspace gives a new mechanism to end inflation. Also the time that takes to end inflation is regulated by the value of $3\mu(\lambda^2 - 6)$, due to the freedom that we have on the parameters of the model this time can be fixed to give the correct number of e-folds. In Fig. (11), we have plotted the slow roll parameter $\epsilon$ for the commutative and noncommutative models. The values we have taken are $\lambda = 6/7$, $\theta = 1$ and $\mu = 0.53$, we can see that before $t = 0.6$, $\epsilon < 1$, after that time the slow roll condition isviolated and inflation ends, after which the value of $\epsilon$ is asymptotically constant.

The other interesting possibility is in connection with the current acceleration of the universe. If we analyze the same model but start with a value of $\lambda > \sqrt{2}$, again we can fix the values of the parameters in order to have a transition from a non accelerating universe to an ac-
FIG. 1: The dotted line corresponds to the commutative slow roll parameter $\epsilon_c$, the thin line is the value of $\epsilon$ that breaks inflation, the thick line corresponds to the new slow roll parameter.

FIG. 2: The dotted line corresponds to the commutative slow roll parameter $\epsilon_c > 1$ so there is no acceleration, the thin line corresponds to $\epsilon = 1$. In the noncommutative model (thick line), there is a transition from a non-accelerating universe to an accelerating one. This can be seen from the behavior of the slow roll parameter which goes from $\epsilon > 1$ to $\epsilon < 1$ and again the parameters can be tuned to have the correct value for the current observed acceleration. In Fig. 2, we have taken for the parameters the values $\lambda = \sqrt{7} + 1$, $\theta = .23$ and $\mu = 1.6$. By analyzing the time evolution of the noncommutative slow roll parameter we can see that we have a transition from a non-accelerating universe to an accelerating one, this behavior is completely different from the commutative case.

Summarizing, we have introduced a noncommutative deformation to the minisuperspace of scalar field cosmology in the gravitational and matter sectors of the model, this has been done at the quantum and classical levels. The corrections of the noncommutative deformation have been analyzed at the classical level and the most striking effect, is the presence of a mechanism that ends inflation. This mechanism also permits that a non-accelerating universe after a period of time can start an acceleration. This might give some evidence on a possible relationship between dark energy and noncommutativity, although this is only an speculative statement. Further research in this direction might establish or discard this possibility. As a final remark, we have shown in this simple toy model that noncommutativity may have very important implications in the evolution of the universe.

Acknowledgments

This work was partially supported by CONACYT grants 47641, 51306-F and PROMEP grants UGTO-CA-3, UGTO-PTC-085, M.S. is also supported by CONCYTEG grant 0716K662062.

[1] E. J. Copeland, M. Sami and S. Tsujikawa, Int. J. Mod. Phys. D 15, 1753 (2006); B. Ratra and P. J. E. Peebles, Phys. Rev. D 37, 3406 (1988); L. P. Chimento and A. S. Jakubi, Int. J. Mod. Phys. D 5, 71 (1996); A. R. Liddle and L. A. Urena-Lopez, Phys. Rev. Lett. 97, 161301 (2006).
[2] M.P. Ryan, in: Hamiltonian Cosmology (Springer, Berlin, 1972).
[3] H. Snyder, Phys. Rev. 71, 38 (1947).
[4] A. Connes, M. R. Douglas, and A. Schwarz, JHEP 9802:003 (1998); N. Seiberg and E. Witten, JHEP 9909:032 (1999).
[5] M. R. Douglas and N. A. Nekrasov, Rev. Mod. Phys. 73, 977 (2001).
[6] H. Garcia-Compean, O. Obregon, C. Ramirez and M. Sabido, Phys. Rev. D 68, 044015 (2003); H. Garcia-Compean, O. Obregon, C. Ramirez and M. Sabido, Phys. Rev. D 68, 045010 (2003); A.H. Chamseddine, J. Math. Phys. 44, 2534 (2003); J.W. Moffat, Phys. Lett. B 491, 345 (2000); P. Aschieri, C. Blohmann, M. Dimitrijevic, F. Meyer, P. Schupp and J. Wess, Class. Quant. Grav. 22, 3511 (2005); L. Alvarez-Gaume, F. Meyer and M. A. Vazquez-Mozo, Nucl. Phys. B 753, 92 (2006).
[7] H. Garcia-Compean, O. Obregon and C. Ramirez, Phys. Rev. Lett. 88, 161301 (2002).
[8] G. D. Barbosa, Phys. Rev. D 71, 063511 (2005); G. D. Barbosa and N. Pinto-Neto, Phys. Rev. D 70, 103512 (2004); L. O. Pimentel and C. Mora, Gen. Rel. Grav. 37, 817 (2005).
[9] R. Brandenberger and P. M. Ho, Phys. Rev. D 66, 023517 (2002); Q. G. Huang and M. Li, Nucl. Phys. B 713, 219 (2005); Q. G. Huang and M. Li, Mod. Phys. Lett. A 20, 271 (2005).
[10] M. Chaichian, M. M. Sheikh-Jabbari, and A. Tureanu, Phys. Rev. Lett. 86, 2716 (2001); L. Mezirosce, “Star Operation in Quantum Mechanics,” arXiv:hep-th/0007046.