Quantum Systems Related to Root Systems, and Radial Parts of Laplace Operators

M.A. Olshanetsky and A.M. Perelomov

Institute for Theoretical and Experimental Physics, 117259 Moscow, USSR

1 Introduction

Laplace operators on semisimple Lie groups and symmetric spaces were studied by Gel'fand [1,2]. The explicit form of the radial parts of Laplace operators (RPLO) was calculated only for symmetric spaces with a complex group of motions [3] and for Cartan domains of the first type [4].

In this paper we shall present the explicit form of generating algebras of RPLO for symmetric spaces that have a bounded system of roots of type $A_n$, and for all symmetric spaces of rank 2, with the exception of $G_2/(SU(2) \otimes SU(2))$. These results can be obtained by studying a quantum system constructed on the basis of root systems by a method proposed by the authors. The corresponding classical systems were studied previously [5].

The connection between dynamical systems and symmetric spaces is useful in studying both of them.

In particular, it can be proved that for certain values of the constants occurring in the Hamiltonian, the quantum systems and the classical systems are completely integrable for all the systems of roots.

Let us note that by constructing completely integrable quantum systems, we obtain an explicit description of a commutative algebra of differential operators of several variables.

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2 Quantum Systems Related to Root Systems

Let $R$ be a root system in the space $\mathcal{H}$ ($\dim \mathcal{H} = n$), and let $R_+$ be a subsystem of roots that are positive under a certain order (the definitions and terms used here can be found in [6]).

Let $q \in \mathcal{H}$ and let $q_\alpha = (q, \alpha)$ be the scalar product of $q$ and of the root $\alpha$. As usual, we shall introduce the operators of the coordinate $q_j$ and of the momentum $p_j = -i \partial/\partial q_j$, and let $g_\alpha^2$ be positive constants on roots of equal length. As in [5], we shall consider systems with a Hamiltonian of the form

$$ H = \frac{p^2}{2} + U(q), \quad (1) $$

where

$$ U(q) = \sum_{\alpha \in R_+} g_\alpha^2 V(q_\alpha), \quad (2) $$

and the function $V(\xi)$ can have one of the following forms:

I. $\xi^{-2}$,

II. $a^2 \sin^{-2} a \xi$,

III. $a^2 \gamma(a \xi)$,

IV. $a^2 \gamma'(a \xi)$,

V. $\xi^{-2} + \omega^2 \xi^2$.

(3)

For a root system of type $A_{n-1}$ we have $q_\alpha = q_k - q_l$ ($k \neq l$). In this case,

$$ U(q) = g^2 \sum_{k<l} V(q_k - q_l). \quad (4) $$

Such quantum systems describe $n$ particles on a straight line; they were studied in [7-13]. A quantum system of type $G_2$ has been considered in [14], [15].

Let us note that although the Hamiltonian (1) is defined in the space $\mathcal{H}$, it is in fact impossible for a particle to move in the entire space $\mathcal{H}$.

Let $\Lambda$ be a Weyl chamber defined on the basis of a subsystem of positive roots,

$$ \Lambda = \{ q \in \mathcal{H} | q_\alpha = (q, \alpha) > 0, \alpha \in R_+ \}, \quad (5) $$

and $\Lambda_a$ be the Weyl alcove

$$ \Lambda_a = \{ q \in \mathcal{H} | 0 < a q_\alpha < \pi, \alpha \in R_+ \}. \quad (6) $$
Then the potential $U(q)$ in (2) becomes infinite on the walls of the Weyl chamber $\Lambda$ (5) for systems of type I, II and V, and on the walls of the alcove $\Lambda_a$ (6) for systems of type III. It is possible to indicate an alcove form also for the functions $a^2 \gamma(a\xi)$ (IV (3), see [5]). Thus we have found that for the systems I, II and V the particle moves in the Weyl chamber $\Lambda$, whereas for the systems III and IV it moves in the Weyl alcove $\Lambda_a$.

Now let us present two definitions.

**Definition 1.** A differential operator that commutes with the Hamiltonian $H$ is called an integral of motion (IM).

It is evident that the IM form an algebra.

**Definition 2.** A quantum system with a Hamiltonian $H$ is said to be completely integrable if the algebra IM contains a commutative subalgebra that has at least $n$ functionally independent operators ($n = \dim H$).

### 3 The Mapping $j$

In this section we shall study a mapping $j$ in the algebra of differential operators that carries the Laplace–Beltrami operator into the Hamiltonian operator of a quantum system.

Let us note that this mapping has been used in [3] and [4] for the calculation of the RPLO.

At first we shall consider symmetric spaces of negative curvature $X^-$. If $G$ is a semisimple Lie group with a finite center, and $K$ is its maximal compact subgroup, then $X^- = G/K$. The Laplace operators are $G$-invariant differential operators on $X^-$ [16].

Let $\mathcal{G}$ and $\mathcal{K}$ be Lie algebras of the groups $G$ and $K$, and let $\mathcal{L}$ be an orthogonal complement of $\mathcal{K}$ in $\mathcal{G}$ in the sense of Cartan’s scalar product. By $\mathcal{H}$ we shall denote the Cartan subalgebra in $\mathcal{L}$. (The subalgebra $\mathcal{H}$ is the vector space introduced in Sec. 2).

Let $R = \{\alpha\}$ be a bounded system of roots of the space $X^-$. [18]. The roots $\alpha \in R$ are linear forms on $\mathcal{H}$. If $\Lambda$ is a fixed Weyl chamber of the root system $R$, and $x_0k = x_0$ for $k \in K$, then for any point $x \in X^-$ there exists an unique element $q(x) \in \Lambda$ such that

$$x = x_0 \exp\{q(x)\} k, \quad k \in K$$

(7)

(the Cartan decomposition).

Functions that are invariant under rotations ($f(xk) = f(x)$) depend only on $q(x)$. On these functions the Laplace operators induce their radial part.
Let us denote the RPLO algebra by $D$. The RPLO is $W$-invariant ($W$ being Weyl’s group).

Let $m_\alpha$ be the multiplicity of the root $\alpha$, let $R_+$ be a subsystem of roots in $R$ that is positive with respect to $\Lambda$, and let $x(q_\alpha) = a \text{sh}^{-1} a q_\alpha$ ($V(q_\alpha) = x^2(q_\alpha)$). \[\]

On $\Lambda$ let us define a function $\xi(q)$:

$$\xi(q) = \prod_{\alpha \in R_+} x(q_\alpha)^{-m_\alpha/2}. \quad (8)$$

Let us consider a mapping $j$ of the algebra $T$ of all differential operators on $\mathcal{H}$:

$$j: t \rightarrow \xi(q) t \xi^{-1}(q), \quad t \in T. \quad (9)$$

Let us define the operator $B \in D$ as

$$B = -\xi^{-2}(q) \sum_{l=1}^{n} \hat{p}_l \xi^2(q) \hat{p}_l. \quad (10)$$

This operator is the radial part of the Laplace–Beltrami operator on the space $X^-$.

**Lemma 1.** If $H$ is the Hamiltonian (1) with a function $V(q)$ of type II (3), then

$$H = -j \left[ \frac{1}{2} (B + \rho^2) \right], \quad \rho = \frac{1}{2} \sum_{\alpha \in R_+} m_\alpha \alpha, \quad (11)$$

the constants $g_\alpha$ in (2) being related to the root multiplicities $m_\alpha$ as follows:

$$g_\alpha^2 = \frac{1}{8} m_\alpha (m_\alpha + 2 m_{2\alpha} - 2) |\alpha|^2. \quad (12)$$

**Proof.** According to (10) and (11), we have to show that

$$H = -\frac{1}{2} \xi(q) (B + \rho^2) \xi^{-1}(q). \quad (13)$$

\[1\] For simplicity, in intermediate calculations the parameter $a$ will be set equal to unity.

\[2\] In the particular case $m_\alpha = 1$ and root systems of type $A_n$, we can find in [17] a remark that corresponds to this lemma.
Let us calculate the operator in the right-hand side. According to (10), we obtain

\[-\frac{1}{2} \xi(q) (B + \rho^2) \xi^{-1}(q) = \frac{1}{2} \xi^{-1}(q) \sum_{i=1}^{n} p_i \xi^2(q) p_i \xi^{-1}(q) - \frac{\rho^2}{2} = \frac{1}{2} p^2 + U(q),\]

where

\[U(q) = \frac{1}{2} \left( \sum_{j=1}^{n} \xi^{-1}(q) \xi_{jj}(q) - \rho^2 \right)\]

and \(\xi_{jj}\) is the second derivative with respect to \(q_j\). It can be asserted that this expression coincides with the potential (2). By substituting into (15) the explicit form of the function \(\xi(q)\) (8), we obtain

\[U(q) = \sum_{\alpha \in R_+} \frac{m_\alpha}{4} \left( \frac{m_\alpha}{2} - 1 \right) |\alpha|^2 \xi(q_\alpha) + F(q),\]

where

\[F(q) = \frac{1}{8} \sum_{\alpha, \beta \in R_+, \alpha \neq \beta} m_\alpha m_\beta (\alpha, \beta) [\cosh q_\alpha \cosh q_\beta - 1].\]

(19)

Let \(R\) be a reduced system of roots. This signifies that \(m_\alpha m_{2\alpha} = 0\). Let us show that in this case,

\[F(q) \equiv 0.\]

(18)

If (18) holds, then the assertion of the lemma follows from (16) and (13), (14).

From (7) it follows that (18) is equivalent to the vanishing of the function \(\Phi(q) = F(q) \xi^{(0)}(q)\), where \(\xi^{(0)}(q) = \prod_{\alpha \in R_+} \sinh q_\alpha\). The function \(\Phi(q)\) has the following form:

\[\Phi(q) = c \sum_{\alpha, \beta \in R_+, \alpha \neq \beta} m_\alpha m_\beta (\alpha, \beta) \left( e^{q_\alpha - q_\beta} + e^{q_\beta - q_\alpha} \right) \prod_{\gamma \in R_+, \gamma \neq \alpha, \gamma \neq \beta} (e^{q_\gamma} - e^{-q_\gamma}).\]

(19)

Let \(W\) be Weyl’s group of the root system \(R\). Let us note that the function \(F(q)\) is \(W\)-invariant by virtue of its construction (see (16)), whereas the function \(\xi^{(0)}(q)\) is anti-invariant. Hence, \(\Phi(q)\) is also anti-invariant.

Let us consider the group algebra \(A(P)\) over the ring \(R\) of the additive weight group \(P\) of the root system \(R\) (see [6], Chap.VI, Sec.3). Its elements are linear combinations of formal exponents \(e^p\ (p \in P)\), and \(e^p e^{p'} = e^{p+p'}\).
In particular, the element
\[
\varphi = \sum_{\alpha, \beta \in R_+} m_{\alpha} m_{\beta} (\alpha, \beta) \left( e^{\alpha - \beta} + e^{\beta - \alpha} \right) \prod_{\gamma \in R_+, \gamma \neq \alpha, \gamma \neq \beta} \left( e^\gamma - e^{-\gamma} \right)
\] (20)
belongs to \( A(P) \). Since \( \Phi(q) = \varphi(q) \) (see (19)), it follows that \( \varphi \) is antiinvariant. From (20) it follows that the maximal term of the element \( \varphi \) (see [6], Chap.VI, Sec.3.2) is \( e^{\rho - \alpha} \), where \( \alpha \in R_+ \). An antiinvariant element with such a maximal term vanishes identically. Therefore, the function \( \Phi(q) \), and hence also \( F(q) = \Phi(q) \xi^{(0)}(q) \) is equal to zero.

Now let \( R \) be an unreduced system of roots. There exists only one such system \( BC_n \). This root system has subsystems of type \( B_n \) and \( C_n \). Let us denote by \( \bar{R} \) a root subsystem of \( R \) of the form \( \bar{R} = \left\{ \alpha \in R_+ | 2\alpha \in R \right\} \). Then the function \( F(q) \) (see (17)) can be expressed in the form

\[
F(q) = F_{B_n}(q) + F_{C_n}(q) + \frac{1}{4} \sum_{\alpha, \beta \in \bar{R}} (m_{\alpha} m_{2\beta \beta}) (c \text{th} q_{2\alpha} c \text{th} 2q_{2\beta} - 1),
\] (21)

where \( F_{B_n} \) (\( F_{C_n} \)) is a function \( F(q) \) that corresponds to the root system \( B_n \) (\( C_n \)). Since \( B_n \) and \( C_n \) are reduced root systems, it follows that \( F_{B_n} = F_{C_n} = 0 \). Let us also note that if \( \alpha, \beta \in \bar{R} \), then \( (\alpha, \beta) = \delta_{\alpha \beta} \). Therefore we can rewrite (21) in the form

\[
F(q) = \frac{1}{2} \sum_{\alpha \in \bar{R}} m_{\alpha} m_{2\alpha} (c \text{th} q_{2\alpha} c \text{th} 2q_{2\alpha} - 1).
\]

By substituting this expression into (16) and performing very simple trigonometric transformations, we obtain the required equation

\[
U(q) = \frac{1}{2} \sum_{\alpha \in R_+} \frac{m_{\alpha}}{2} \left( \frac{m_{\alpha}}{2} + m_{2\alpha} - 1 \right) |\alpha|^2 V(q_{\alpha}).
\] (22)

From this lemma and from the form of \( j \) we obtain the following

**Proposition 1.**

1. \( \text{Ker } j = 0 \), \hspace{1cm} (23)
2. \( j \) preserves leading terms, \hspace{1cm} (24)
3. \( j \) preserves \( W \)-invariance, \hspace{1cm} (25)
4. \( j(D) \subset S \) (an algebra of IM). \hspace{1cm} (26)
It is evident that (26) makes sense only for group values of $g_\alpha$ (12). The root multiplicities $m_\alpha$ that determine $g_\alpha$ can be found in [18] for all symmetric spaces.

**Remark.** The analysis of this section is completely applicable also to systems of type I and III (3). For this purpose let us introduce the parameter $a$ into the function $\xi(q)$ (8)

\[ \xi_a(q) = \prod_{\alpha \in R^+} (a^{-1} \sin a q_\alpha)^{\mu_\alpha}, \quad \mu_\alpha = \frac{1}{2} m_\alpha. \]

Let $a$ tends to zero. We obtain a mapping of the RPLO on symmetric spaces of zero curvature $X^0$ in an IM algebra of systems of type I. But if $a$ is imaginary, then we obtain systems of type III and spaces of positive curvature $X^+$ that are dual in the sense of Cartan to $X^-$. Let us note that the curvature of a symmetric space is proportional to $a^2$.

**Theorem 1.** Quantum systems of type I, II, III, and V are completely integrable for group values of $g_\alpha$.

**Proof.** Since the algebra $\tilde{S} = \text{Im} j(D)$ is isomorphic to the algebra $D$ (23) and $\tilde{S} \subset S$ (26), the assertion of the theorem for the systems I, II and III will follow from the existence of $n$ generators of the commutative algebra $D$ on the symmetric spaces $X^0$, $X^-$, and $X^+$ [16]. For systems of type V the assertion follows from the complete integrability of systems of type I when we change the variables $p_j \rightarrow p_j \pm i \omega q_j$ (see [19]).

**Corollary 1.** For group values of $g_\alpha$, the classical systems of type I, II, III, and V are completely integrable.

**Proof.** Let us introduce the Planck constant into the momentum operators $p_j = -i \hbar \partial / \partial q_j$, and let it tends to zero. Then the commutators of the operators will go over into the Poisson brackets of classical variables. Let us note that in the classical case, complete integrability has been proved earlier (in [5]) for classical root systems only.

### 4 Algebra of Integrals of Motion

In this section we shall describe a subalgebra $\tilde{S}$ of the IM algebra $S$ which for systems of type I - III and group values of $g_\alpha$ (12) is an image of the RPLO algebra $D$ under the mapping $j$.

Let us define the subalgebra $\tilde{S}$ in a more general situation. Let us consider systems of type I - IV for any values of $g_\alpha$. The subalgebra $\tilde{S}$ is specified by the following conditions:
1. \[ I(sp, sq) = I(p, q) \quad (W - \text{invariance}), \quad (27) \]

2. For systems of type I, the algebra \( \tilde{S}^I \) is graded,
\[ \tilde{S}^I = \oplus_{k=0}^{\infty} \tilde{S}^I_k, \quad \tilde{S}^I_k = \{I^I_k(p, q)\}, \quad I^I_k(\lambda^{-1}p, \lambda q) = \lambda^{-k} I^I_k(p, q), \quad (28) \]
where \( I^I_k(p, q) \) is a polynomial of degree not higher than \( k \) in \( p \).

3. For systems of type II - IV,
\[ I(p, q) = I^I(p, q) (1 + O(|q|)) \quad \text{for} \quad q \to 0, \quad (29) \]
where \( I^I(p, q) \) is an IM of systems of type I. This relation makes it possible to introduce the gradation in \( \tilde{S} \) for systems II - IV.

Let \( I^I_k(p, q) \in \tilde{S}_k \) and suppose that it has the form
\[ I^I_k(p, q) = \sum_{m=0}^{s} \sum_{l=1}^{i_1} c^i_1 \cdots c^i_m(q) p_{i_1} \cdots p_{i_m} \quad (s \leq k). \quad (30) \]

**Lemma 2.** If the leading coefficients \( c^{i_1 \cdots i_m}(q) \) in (30) are not constant, then \( I^I_k(p, q) = 0 \).

**Proof.** Since \( I^I_k \in \tilde{S}_k \), it follows that \([H, I^I_k] = 0\). Let the coefficients of the leading powers \( p_{i_1} \cdots p_{i_{s+1}} \) of this commutator are equal to zero. By virtue of the explicit form of \( H \) (see (1) and (2)), we obtain
\[ \sum_{\sigma} \partial_{q_l} c^{i_1 \cdots i_{s+1}}(q) = 0, \quad (31) \]
where the sum is taken over all the permutations of indices \( \sigma(l, j_1, \ldots, j_s) = (i_1, \ldots, i_{s+1}) \). It is proved ([3], Lemma 2.5, p.407) that the system (31) has only polynomial solutions. For systems of type I, from the gradation conditions (28) it follows that \( c^{i_1 \cdots i_s}(\lambda q) = \lambda^{s-k} c^{i_1 \cdots i_s}(q) \). Since \( k > s \), the only polynomial to satisfy this condition is the polynomial being equal to zero identically. Thus, \( c^{i_1 \cdots i_s}(q) \equiv 0 \). The same result can be obtained by considering the asymptotic behavior, for \(|q| \to 0\) of systems of type II - IV in (29).

**Corollary 2.** The integrals \( I^I_k(p, q) \in \tilde{S}_k \) are uniquely determined by their leading terms.
Suppose that $I'_k$ and $I''_k$ belong to $\tilde{S}_k$, and that their leading terms coincide. Then $\Delta I_k = I'_k - I''_k$ will belong to $S_k$ and it does not contain constant leading terms. Hence, $\Delta I_k = 0$.

Corollary 3. The algebra $\tilde{S}$ is commutative.

Proof. Let $I_k \in \tilde{S}_k$ and $I_m \in \tilde{S}_m$. It is evident that $[I_k, I_m]$ belongs to $\tilde{S}_{k+m}$ and that it does not contain constant leading terms. Therefore, the commutator will be equal to zero.

Now let us confine ourselves to systems of type I - III and to group values of the constants $g_\alpha$ (12).

Theorem 2. The mapping $j$ is an isomorphism of an RPLO algebra on symmetric spaces of zero, negative, and positive curvature into the algebra $\tilde{S}$ of systems of type I, II and III, respectively.

Proof. By virtue of Proposition 1, the mapping $j$ is a monomorphism of the algebra $D$ into an algebra of $W$-invariant IM.

The Lie algebra $\mathcal{G}$ of the group of motions of a space of zero curvature has a decomposition $\mathcal{G} = K + L$, where $K$ is a maximal compact subalgebra and $L$ is a subalgebra of parallel translations of the space $X^0$. The Lie operators belonging to $K$ have degree 0 under similarity transformations, whereas the Lie operators belonging to $L$ have degree 1. Therefore, an universal enveloping algebra has a natural gradation.

This gradation can be transferred to the algebra of Laplace operators on the space $X^0$. This follows from the fact that in canonical coordinates the Laplace operators have constant coefficients, and also from the homogeneity of Lie operators. Therefore, any homogeneous component of the Laplace operator is also Laplace operator. This gradation can evidently be transferred to the RPLO algebra $D$. From the explicit form of $j$ (see (9)) it follows that $j$ preserves this gradation.

Now let us consider symmetric spaces of nonzero curvature.

Let the point $x$ tends to $x_0$ (see (7)). This corresponds to $q \to 0$. The space which is tangent at the point $x_0$ to $X$ is a symmetric space of zero curvature $X^0$. Hence, for $x \to x_0$ the Lie operators on $X$ will go over into Lie operators on $X^0$. It hence follows that the RPLO on $X$ has an asymptotic behavior (29). Let us note that the mapping $j$ preserves this asymptotic behavior. Thus, $j$ is a monomorphism of $D$ into $\tilde{S}$.

Let $I_k \in \tilde{S}_k$. Let us consider the operator $j^{-1} I_k$. By virtue of Proposition 1 we have $[j^{-1} I_k, B] = 0$, and its leading terms are $W$-invariant. Moreover, there exists a unique $\Delta_k \in D$ with the same leading terms [16]. By virtue of the uniqueness, from Corollary 2 it follows that $\Delta_k = j^{-1} I_k$. 

9
5 Explicit Form of Radial Parts of Laplace Operators

In [19] we have obtained explicit formulas for certain integrals in $\tilde{S}$. With the aid of the theorem presented in Sec. 4 it is easy to obtain formulas for the radial parts of Laplace operators: $\Delta_k = \xi^{-1} I_k \xi$.

Below we denote

\[ x(\xi) = \]

I. $\xi^{-1} (X^0)$,
II. $\sh^{-1} \xi \ (X^-)$,
III. $\sin^{-1} \xi \ (X^+)$.

Let us note that the IM depends only on the function $x^2(\eta) = V(\eta)$ and its derivatives, whereas the constants $g$, $g_1$, and $g_2$ are related to the root multiplicities $m$, $m_1$, and $m_2$ by the formula (12).

a) Spaces with root systems of type $A_{n-1}$. Here generators are $\Delta_2 = B$, $\Delta_3, \ldots, \Delta_n$, $\Delta_k = \xi^{-1} J_k \xi$.

\[ J_n = \exp \left\{ -\frac{g^2}{2} \sum_{k \neq l} x^2(q_k - q_l) \frac{\partial}{\partial p_k} \frac{\partial}{\partial p_l} \right\} p_1 p_2 \cdots p_n. \] (32)

Let $Q = \sum_{j=1}^n q_j$. Then the other $J_k$ can be defined recursively $^3$

\[ J_{k-1} = i \frac{k - n - 1}{Q} [Q, J_k]. \] (33)

In lower dimensions, it is possible to present an explicit form of the operators $I_k$ with other leading terms. Let us denote by the brackets $\langle A_{kl}^r \rangle$ the trace of the matrix $A^r$, and let $p = \text{diag} (p_1, \ldots, p_n)$. Then

\[ I_3 = \sum_{k=1}^n p_k^3 + 3g^2 \sum_{k \neq l} x^2(q_k - q_l) p_l, \]
\[ I_4 = \sum_{k=1}^n p_k^4 + 2g^2 \sum_{k \neq l} x^2(q_k - q_l)(2p_l^2 + p_k p_l) + g^2 \langle x^4(q_k - q_l) \rangle \]

$^3$ In different form, the formulas for $J_k$ for systems of type $A_{n-1}$ have appeared for the first time in [13]. The proof that $J_k \in \tilde{S}_k$ can be found in [19].
\[ + g^2 \sum_{k \neq l} \left\{ 2 \left[ x^2(q_k - q_l) \right]' ip_l - \left[ x^2(q_k - q_l) \right]'' \right\}, \]

\[ I_5 = \sum_{k=1}^{n} p_k^5 + 5g^2 \sum_{k \neq l} x^2(q_k - q_l) (p_l^3 + p_k^2 p_l) + 5g^4 \langle x^4(q_k - q_l) p_l \rangle \]

\[ + 5g^2 \sum_{k \neq l} \left\{ \left[ x^2(q_k - q_l) \right]' ip_l^2 - \left[ x^2(q_k - q_l) \right]'' p_l \right\}. \]

There exist two infinite series and one exceptional symmetric space with a root system of type \( A_{n-1} \). Here we list the spaces of noncompact type (the notation has been adopted from [16]):

|     | \(X^-\)                      | \(g^2\) | Rank |
|-----|-----------------------------|-------|------|
| \(AI\) | \(SL(n, R)/SO(n)\) | \(-\frac{1}{4}\) | \(n - 1\) |
| \(AII\) | \(SU^*(2n)/Sp(n)\) | 2 | \(n - 1\) |
| \(EIV\) | \(E_6/F_4\) | 12 | 2 |

b) Spaces with root systems of type \( B_n, C_n, BC_n, \) and \( D_n \). Here the generators are \( \Delta_2 = B, \Delta_4, \ldots, \Delta_{2n} \) (or \( \Delta'_n \) for \( D_n \)). Only the operator \( \Delta_4 = \xi^{-1}I_4 \xi \) is known.

\[ I_4 = 2 \sum_{k=1}^{n} p_k^4 + 8g^2 \sum_{k \neq l} \left[ x^2(q_k - q_l) + x^2(q_k + q_l) \right] p_l^2 + 8g^2 \sum_{k \neq l} x^2(q_l) p_l^2 \]

\[ + 8g^2 \sum_{l=1}^{n} x^2(2q_l)p_l^2 + 4g^2 \sum_{k \neq l} \left[ x^2(q_k - q_l) - x^2(q_k + q_l) \right] p_k p_l \]

\[ + 4g^2 \sum_{k \neq l} \left[ x^2(q_k - q_l) + x^2(q_k + q_l) \right]' ip_l + g^4 \langle x(q_k - q_l) \rangle^4 \]

\[ ^4 \text{Here and below we shall not consider symmetric spaces with a complex group of motion for which } U(q) = 0 \text{ [3].} \]
Thus, for symmetric spaces of rank 2, with the exception of spaces with a group of motions $G_2$, we have a complete system of generators. Here we list the nonisomorphic spaces of noncompact type [16], [18]:

|      | $X^-$                        | $g^2$             | $g_1^2$            | $g_2^2$            |
|------|------------------------------|-------------------|-------------------|-------------------|
| 1    | BDI $SO_0(n,2)/(SO(n) \otimes SO(2))$, $n > 2$ | $-\frac{1}{4}$  | $\frac{1}{8}(n-2)(n-4)$ | 0                   |
| 2    | CHI $Sp(2,2)/(Sp(2) \otimes Sp(2))$    | $\frac{3}{4}$  | 1                 | 0                   |
| 3    | AIII $SU(n,2)/(SU(n) \otimes U(2))$, $n > 2$ | 0                 | $\frac{1}{2}(n-2)^2$ | $-\frac{1}{2}$    |
| 4    | CHI $Sp(n,2)/(Sp(n) \otimes Sp(2))$, $n > 2$ | 2                 | $2(n-1)(n-2)$     | $\frac{3}{2}$     |
| 5    | DIII $SO^*(10)/U(5)$              | 2                 | 2                 | $-\frac{1}{2}$    |
| 6    | EIII $E_6/(SO(10) \otimes SO(2))$  | 6                 | 8                 | $-\frac{1}{2}$    |

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