On the extended $W$-algebra of type $\mathfrak{sl}_2$ at positive rational level

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Abstract

The extended $W$-algebra of type $\mathfrak{sl}_2$ at positive rational level, denoted by $\mathcal{M}_{p_+,p_-}$, is a vertex operator algebra that was originally proposed in [FGST06]. This vertex operator algebra is an extension of the minimal model vertex operator algebra and plays the role of symmetry algebra for certain logarithmic conformal field theories. We give a construction of $\mathcal{M}_{p_+,p_-}$ in terms of screening operators and use this construction to prove that $\mathcal{M}_{p_+,p_-}$ satisfies Zhu’s $c_2$-cofiniteness condition, calculate the structure of the zero mode algebra (also known as Zhu’s algebra) and classify all simple $\mathcal{M}_{p_+,p_-}$-modules.

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1 Introduction

The theory of vertex operator algebras (VOA), which was developed by Borcherds [Bor86], is an algebraic counterpart to conformal field theory and gives an algebraic meaning to the notions of locality and operator product expansions. For general facts about VOAs we refer to [FBZ01, NT05, MNT10].

Examples of conformal field theories on general Riemann surfaces for which vertex operator algebraic descriptions are known, are given by lattice VOAs, VOAs associated to integrable representations of affine Lie algebras and VOAs associated to minimal representations of the Virasoro algebra. The abelian categories associated to the representation theory of all these examples are semi-simple and the number of irreducible representations is finite.

In order to define conformal field theories over a Riemann surface associated to a VOA, the VOA needs to satisfy certain finiteness conditions. Zhu found such a finiteness condition [Zhu96], which is now called Zhu’s $c_2$-cofiniteness condition. For a VOA satisfying Zhu’s $c_2$-cofiniteness condition it is known that the abelian category of its representations is both Noetherian and Artinian. Additionally the number of simple objects is finite. In general this abelian category is not semisimple [Zhu96, FZ92], though so far the semi-simple case is much better understood. Conformal field theories associated to VOAs with a non-semisimple representation theory are called logarithmic or non-semisimple.

Examples of VOAs which satisfy Zhu’s $c_2$-cofiniteness condition but not semisimplicity are given by so called $W_p$ theories for which the representation category is by now well established [GK96, GR08, AM08, NT11, TW12].

In this paper we analyse a different example which generalises the $W_p$ theories. This example was originally defined in [FGST06] and was called $W_{p_+,p_-}$. Unfortunately the letter “W” is rather overused in this context, so we will denote these VOAs by $\mathcal{M}_{p_+,p_-}$ and call them “extended $W$-algebras of type $\mathfrak{sl}_2$ at rational level”.

The $\mathcal{M}_{p_+,p_-}$ are a family of VOAs parametrised by two coprime integers $p_+, p_- \geq 2$. They are defined by means of a lattice VOA $V_{p_+,p_-}$ and two screening operators $S_+, S_-$. The $\mathcal{M}_{p_+,p_-}$ have the same central charge

$$c_{p_+,p_-} = 1 - \frac{(p_+ - p_-)^2}{p_+ p_-},$$

as the minimal models. However, the Virasoro subtheory of $\mathcal{M}_{p_+,p_-}$ is not isomorphic to the minimal model VOA $\text{Vir}_{p_+,p_-}$; rather, the minimal model VOA $\text{Vir}_{p_+,p_-}$ is obtained from $\mathcal{M}_{p_+,p_-}$ by taking a quotient.

A number of results in this paper have already been described in [FGST06]. In [FGST06] the construction of integration cycles, over which products of screening operators are integrated, are described by using a Kazhdan-Lusztig type correspondence between the homology groups on configuration spaces of points the projective line with local coefficients and quantum groups at roots of unity. However, this correspondence is not yet well understood in this case [KL93a, KL93b, KL94a, KL94b, FGST07]. The motivation for this paper was to reformulate the setup of [FGST06] in a way intrinsic to VOAs without any reference to quantum groups. By developing VOA techniques, we have succeeded in not only proving the results of [FGST06] but also going beyond that paper.
We introduce free field VOAs over the discrete valuation ring \( \mathcal{O} = \mathbb{C}[[\varepsilon]] \) of formal complex power series. We discuss their representation theory, screening operators and the construction of cycles on which products of screening operators can be integrated. These integrated products of screening operators define intertwining operators of the Virasoro action on free field modules – called Fock modules – over \( \mathcal{O} \), which allow us to explicitly compute all the data required for analysing the VOA structure of \( \mathcal{M}_{p_+,p_-} \). Thus we are able to analyse the zero mode algebra of \( \mathcal{M}_{p_+,p_-} \) and prove that \( \mathcal{M}_{p_+,p_-} \) satisfies Zhu’s c2-cofiniteness condition. For \( p_+ = 2 \) and \( p_- \) odd, the c2-cofiniteness of \( \mathcal{M}_{p_+,p_-} \) has already been shown [AMI1].

The results of this paper form a necessary starting point for studying problems such as the representation theory; the Kazhdan-Lusztig correspondence between \( \mathcal{M}_{p_+,p_-} \) and quantum groups; and conformal field theories with \( \mathcal{M}_{p_+,p_-} \) symmetry on the Riemann sphere and elliptic curves – as was done in [BPZ84, TK88] for the semi-simple case – and more generally on moduli spaces of \( N \) point genus \( g \) stable curves.

This paper is organised as follows: In Section 2 we introduce some basic notation and definitions. We define VOAs with an emphasis on the Heisenberg and lattice VOAs as well as their screening operators. We also briefly explain how to construct the Poisson and zero mode algebra associated to a VOA as well as their implications for the representation theory of a VOA.

In Section 3 we develop the techniques required for analysing the extended \( W \)-algebra \( \mathcal{M}_{p_+,p_-} \). We construct cycles over which products of screening operators can be integrated. These integration cycles are elements of the homology groups of configuration spaces of \( N \) points on the projective line, with local coefficients defined by the monodromy of products of screening operators. Due to so called resonance problems homology and cohomology groups with these local coefficients exhibit very complicated behaviour [OT01, Var03]. To overcome these complications we deform the Heisenberg VOA, including its energy momentum tensor and screening operators, and construct the theory over the ring \( \mathcal{O} \) and its field of fractions \( \mathcal{K} = \mathbb{C}((\varepsilon)) \). The problem is thus translated into constructing well behaved cycles such that all matrix elements of integrals of products of screening operators lie in \( \mathcal{O} \) rather than \( \mathcal{K} \). We show this by using the theory of Jack polynomials [Mac93]. By setting \( \varepsilon = 0 \) we then obtain integration cycles over \( \mathbb{C} \) for products of screening operators. By integrating these products of screening operators over the constructed cycles we obtain local primary fields \( S^r_+ (z), S^s_- (z), r, s \in \mathbb{N} \) of conformal weight 1. The zero mode operators \( S^r_+, S^s_- \), \( r, s \in \mathbb{N} \) of these primary fields define Virasoro intertwining operators.

In Section 4 we review the decomposition of Fock modules as Virasoro modules due to Feigin and Fuchs [FF84, FF88, FF90]. By using the intertwining operators \( S^r_+ \) and \( S^s_- \), we construct all Virasoro singular vectors of these Fock modules at central charge \( c_{p_+,p_-} \). We define operators \( E,F \) which we call Frobenius homomorphisms. The Frobenius homomorphisms are Virasoro homomorphisms between the kernels of \( S^r_+ \) and \( S^s_- \) respectively and define derivations of the extended \( W \)-algebra \( \mathcal{M}_{p_+,p_-} \).

In Section 5 the extended \( W \)-algebra \( \mathcal{M}_{p_+,p_-} \) is introduced. We give a decomposition of \( \mathcal{M}_{p_+,p_-} \) as a Virasoro module, determine a generating set of fields, analyse its zero mode and Poisson algebra and classify and construct all simple \( \mathcal{M}_{p_+,p_-} \) modules. The screening operators \( S^r_+, S^s_- \) and the Frobenius homomorphisms \( E,F \) are crucial to all these calculations.

In Section 6 we give our conclusions and state a list of future problems and conjectures associated to conformal field theories with \( \mathcal{M}_{p_+,p_-} \) symmetry.
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2 Basic definitions and notation

In this section we review basic definitions and notation for VOAs, in particular Heisenberg and lattice VOAs.

2.1 Vertex operator algebras

For a detailed discussion of vertex operator algebras see [FZ92, NT05, MNT10].

Definition 2.1. A tuple \((V, |0\rangle, T, Y)\) is called a vertex operator algebra (VOA for short) where

1. \(V\) is a complex non-negative integer graded vector space

\[ V = \bigoplus_{n=0}^{\infty} V[n], \]

called the vacuum space of states.
2. \(|0\rangle \in V[0]\) is called the vacuum state.
3. \(T \in V[2]\) is called the conformal vector.
4. \(Y\) is a \(\mathbb{C}\)-linear map

\[ Y : V \to \text{End}_\mathbb{C}(V)[[z, z^{-1}]] \]

called the vertex operator map.

These data are subject to the axioms:

1. Each homogeneous subspace \(V[n]\) of the space of states is finite dimensional and in particular \(V[0]\) is spanned by the vacuum state.

2. For each \(A \in V[h]\) there exists a Laurent expansion

\[ Y(A; z) = A(z) = \sum_{n \in \mathbb{Z}} A[n] z^{-n-h}, \]
where $Y(A; z)$ is called a field and the $A[n]$ are called field modes. Each field satisfies the state field correspondence

$$Y(A; z)|0\rangle = A \in V[[z]]z.$$

The field corresponding to the vacuum is the identity field

$$Y(|0\rangle; z) = \text{id}_V.$$

3. The field modes of the field corresponding to the conformal vector

$$Y(T; z) = T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2},$$

satisfy the commutation relations of the Virasoro algebra with fixed central charge $c = c_V$

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c_V}{12}(m^3 - m)\delta_{m+n,0}.$$

The field $T(z)$ is called the Virasoro field.

4. The zero mode of the Virasoro algebra $L_0$ acts semi-simply on $V$ and the eigenvalues of $L_0$ define the grading of $V$, that is,

$$V[h] = \{ A \in V | L_0 A = h A \}.$$

5. The Virasoro generator $L_{-1}$ acts as the derivative with respect to $z$

$$\frac{d}{dz} Y(A; z) = Y(L_{-1} A; z),$$

for all $A \in V$.

6. For any two elements $A, B \in V$ the fields $Y(A; z)$ and $Y(B; w)$ are local, i.e. there exists a sufficiently large $N \in \mathbb{Z}$ such that

$$(z - w)^N [Y(A; z), Y(B; w)] = 0,$$

as elements of $\text{End}(V)[[z, z^{-1}, w, w^{-1}]]$.

7. For a homogeneous element $A \in V[h]$ and an element $B \in V$ the fields $Y(A; z)$ and $Y(B; w)$ satisfy the operator product expansion

$$Y(A; z)Y(B; w) = Y(Y(A; z - w)B; w)$$

$$= \sum_{n \in \mathbb{Z}} Y(A[n]B; w)(z - w)^{-n-h}.$$

When there is no chance of confusion we will refer to a VOA just by its graded vector space $V$. 

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Remark 2.2. 1. For $A \in V[h]$ the Virasoro generators $L_0$ and $L_{-1}$ satisfy
\[
[L_{-1}, A[n]] = -(n + h - 1)A[n - 1],
\]
\[
[L_0, A[n]] = -nA[n].
\]
2. The operator product expansion of the Virasoro field with itself is
\[
T(z)T(w) = \frac{cV/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w) + \cdots.
\]
3. For any homogeneous element $A \in V[h]$ such that $L_nA = 0$ for $n \geq 1$, the operator product expansion of the Virasoro field with $Y(A; z)$ is
\[
T(z)Y(A; w) = \frac{h}{(z-w)^2}Y(A; w) + \frac{1}{z-w}\partial Y(A; w) + \cdots.
\]
Such fields $Y(A; w)$ are called primary fields.

Definition 2.3. A VOA module $M$ is a vector space that carries a representation $Y_M$
\[
Y_M : V \to \text{End}_\mathbb{C}(M)[[z, z^{-1}]],
\]
of the vertex operator $Y$, such that for all $A, B \in V$ and $C \in M$
1. $Y_M(A, z)C \in M((z))$,
2. $Y_M(|0\rangle; z) = \text{id}_M$, is the identity on $M$,
3. $Y_M(A; z)Y_M(B; w) = Y_M(Y(A; z-w)B; w)$.

Let $O(V)$ be the vector subspace of $V$ spanned by vectors
\[
A \circ B = \text{Res}_{z=0} Y(A; z)B \frac{(1+z)^{h_A}}{z^2} \, dz = \sum_{n=0}^{h_A} \binom{h_A}{n} A[-n-1]B,
\]
for $A \in V[h_A], B \in V$. Furthermore, let $A \ast B$ be the binary operation
\[
A \ast B = \text{Res}_{z=0} Y(A; z)B \frac{(1+z)^{h_A}}{z} \, dz = \sum_{n=0}^{h_A} \binom{h_A}{n} A[-n]B.
\]

Proposition 2.4. The space $A_Z(V) = V/O(V)$ is called Zhu’s algebra and carries the structure of an associative $\mathbb{C}$ algebra. Let $[A], [B] \in A_Z(V)$ denote the classes represented by $A, B \in V$, then the multiplication in $A_Z(V)$ is given by
\[
[A] \cdot [B] = [A \ast B].
\]
1. The unit of $A_Z(V)$ is the class of the vacuum state $1 = [|0\rangle]$. 
2. The class of the conformal vector $[T]$ lies in the centre of $A_Z(V)$.

3. $(L_0 + L_{-1})A \in O(V)$ for all $A \in V$.

4. There is a 1 to 1 correspondence between finite dimensional simple $A_Z(V)$-modules and simple $V$-modules. The simple $A_Z(V)$-modules are isomorphic to the homogeneous space of least conformal weight of the corresponding simple $V$-module.

5. For $A \in V[h_A]$, $B \in V$ and $m \geq n \geq 0$

$$\text{Res}_{z=0} Y(A; z) B \frac{(1 + z)^{h_A+n}}{z^{2+m}} \, dz \in O(V).$$

There is an equivalent definition of Zhu’s algebra as a quotient of the algebra of zero modes. Let

$$U(V) = \bigoplus_{d \in \mathbb{Z}} U(V)[d]$$

$$U(V)[d] = \{ P \in U(V) | [L_0, P] = -d \}$$

be the graded associative algebra of modes of all the fields in $V$. This algebra is called the current algebra [NT05]. Furthermore, let

$$F_p(V) = \bigoplus_{d \leq p} U(V)[d]$$

be a descending filtration of $U(V)$ and let

$$\mathcal{I} = U(V) \cdot F_{-1}(V)$$

be the closure of the $U(V)$ left ideal generated by $F_{-1}(V)$, then

$$I = \mathcal{I} \cap F_0(V)$$

is a closed two sided $F_0(V)$ ideal.

**Definition 2.5.** The zero mode algebra is the quotient algebra

$$A_0(V) = F_0(V)/I.$$ 

**Proposition 2.6.**

1. There exists a canonical surjective $\mathbb{C}$ algebra homomorphism

$$U(V)[0] \rightarrow A_0(V)$$

which maps an element $P \in U(V)[0] \subset F_0(V)$ to its class in the zero mode algebra $A_0(V) = F_0(V)/I$. 

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2. There exists a well defined canonical isomorphism of \( \mathbb{C} \) algebras from Zhu’s algebra \( A_Z(V) \) to the zero mode algebra \( A_0(V) \), such that for \( A \in V \)

\[
A_Z(V) \rightarrow A_0(V) \\
[A] \mapsto [A[0]].
\]

**Remark 2.7.** Since the zero mode algebra and Zhu’s algebra are canonically isomorphic, we identify these two algebras and denote both by \( A_0(V) \). For \( A \in V \) we denote the corresponding class in \( A_0(V) \) by \( [A] = [A[0]] \).

Let \( c_2(V) \) be the subspace of \( V \) given by

\[
c_2(V) = \text{span}\{A[-(h_A + n)]B | n \geq 1\}.
\]

**Definition 2.8.** The VOA \( V \) is said to satisfy Zhu’s \( c_2 \)-cofiniteness condition if the quotient

\[
p(V) = V/c_2(V)
\]

is finite dimensional.

**Proposition 2.9.** The quotient space \( p(V) = V/c_2(V) \) carries the structure of a commutative Poisson algebra. Let \([A]_p, [B]_p\) be the classes of \( A \in V[h_A] \) and \( B \in V \), then the multiplication and bracket are given by

\[
[A]_p \cdot [B]_p = [A[-h_A]B]_p, \\
\{[A]_p, [B]_p\} = [A[-h_A + 1]B]_p.
\]

**Remark 2.10.** The space \( O(V) \) is not spanned by homogeneous vectors; therefore Zhu’s algebra is not graded, it is merely filtered by conformal weight. The space \( c_2(V) \), on the other hand, is spanned by homogeneous vectors, so the Poisson algebra \( p(V) \) is graded by conformal weight.

Let \( F_0(A_0(V)) \subset F_1(A_0(V)) \subset \cdots \) be the filtration of \( A_0(V) \) by conformal weight, that is

\[
F_p(A_0(V)) = \bigoplus_{h=0}^{p} V[h].
\]

Then the gradification of \( A_0(V) \) is the graded algebra

\[
\text{Gr}(A_0(V)) = \bigoplus_{p \geq 0} G_p(A_0(V)),
\]

where

\[
G_0(A_0(V)) = F_0(A_0(V)), \\
G_p(A_0(V)) = F_p(A_0(V))/F_{p-1}(A_0(V)), p \geq 1.
\]

**Proposition 2.11.** There exists a surjection of graded \( \mathbb{C} \) algebras

\[
p(V) \rightarrow \text{Gr}(A_0(V)).
\]

For proofs of the properties of Zhu’s algebra see [Zhu96, FZ92] and for a proof of the existence of the canonical isomorphism between \( A_Z(V) \) and \( A_0(V) \) see [NT05].
2.2 The Heisenberg vertex operator algebra

The Heisenberg VOA is a central building block for all the VOAs considered in this paper. Before we can define the Heisenberg VOA, we must first define the Heisenberg algebra and its highest weight modules, called Fock modules.

Definition 2.12.

1. Let $U(b_{\pm})$ and $U(b_0)$ be $\mathbb{Z}$-graded polynomial algebras over $\mathbb{C}$ given by

$$U(b_{\pm}) = \mathbb{C}[b_{\pm 1}, b_{\pm 2}, \ldots], \quad U(b_0) = \mathbb{C}[b_0],$$

where the degree of $b_n$ is $\deg(b_n) = -n$.

2. The associative $\mathbb{Z}$-graded degree wise completed $\mathbb{C}$-algebra $U(\overline{b})$ is given by

$$U(\overline{b}) = U(b_-) \hat{\otimes} U(b_+)$$

as a vector space, where $\hat{\otimes}$ denotes the degree wise completed tensor product. The algebra structure on $U(\overline{b})$ is defined by the Heisenberg commutation relations

$$[b_m, b_n] = m\delta_{m,-n} \cdot \text{id}, \quad m, n \in \mathbb{Z} \setminus \{0\}.$$

3. The Heisenberg algebra is the $\mathbb{Z}$-graded associative algebra $U(b)$ given by

$$U(b) = U(\overline{b}) \otimes U(b_0)$$

and satisfies the commutation relations

$$[b_m, b_n] = m\delta_{m,-n} \cdot \text{id}, \quad m, n \in \mathbb{Z}.$$

Definition 2.13. Let $\beta \in \mathbb{C}$.

1. We define the left $U(b)$-module $F_\beta$ – called a Fock module. It is generated by the state $|\beta\rangle$, which satisfies

$$b_0|\beta\rangle = \beta|\beta\rangle, \quad b_n|\beta\rangle = 0, \quad n \geq 1,$$

such that

$$U(b_-) \to F_{\beta},$$

$$P \mapsto P|\beta\rangle$$

is an isomorphism of complex vector spaces.
2. We define the right $U(b)$-module $F_\beta^\vee$ — called a dual Fock module. It is generated by the state $\langle \beta \rangle$, which satisfies

$$\langle \beta \rangle b_0 = \langle \beta \rangle \beta, \quad \langle \beta \rangle b_n = 0, \quad n \leq -1,$$

such that

$$U(b_+) \rightarrow F_\beta^\vee$$

$$P \mapsto \langle \beta \rangle P$$

is an isomorphism of complex vector spaces.

3. The two Fock modules $F_\beta, F_\beta^\vee$ are equipped with an inner product $F_\beta^\vee \times F_\beta \rightarrow \mathbb{C}$, characterised by $\langle \beta \rangle \langle \beta \rangle = 1$.

The parameter $\beta$ is called the Heisenberg weight.

**Remark 2.14.** Let $\hat{b}$ be the conjugate of $b_0$, that is, it satisfies the commutation relations

$$[b_m, \hat{b}] = \delta_{m,0}1.$$

For each $\gamma \in \mathbb{C}$, $e^{\gamma \hat{b}}$ defines a Heisenberg weight shifting map

$$e^{\gamma \hat{b}} : F_\beta \rightarrow F_{\beta+\gamma},$$

that satisfies

1. $e^{\gamma \hat{b}} \langle \beta \rangle = \langle \beta + \gamma \rangle$,

2. $e^{\gamma \hat{b}}$ commutes with $U(b_-)$ and $U(b_+)$.

For $\alpha_0 \in \mathbb{C}$ let

$$T = \frac{1}{2}(b_{-1}^2 + \alpha_0 b_{-2})|0\rangle \in F_0.$$

**Proposition 2.15.** The Fock space $F_0$ carries the structure of a VOA, with

$$Y(|0\rangle; z) = \text{id}, \quad Y(b_{-1}|0\rangle; z) = b(z) = \sum_{n \in \mathbb{Z}} b_n z^{-n-1},$$

$$Y(T; z) = T(z) = \frac{1}{2} \big( : b(z)^2 : + \alpha_0 \partial b(z) \big).$$

We denote this VOA by $\mathcal{F}_{\alpha_0} = (F_0, |0\rangle, T, Y)$ and call it the Heisenberg VOA.
Remark 2.16. The operator product expansion of the field \( b(z) \) with itself is given by

\[
b(z)b(w) = \frac{1}{(z-w)^2} + \cdots
\]

and the central charge of \( \mathcal{F}_{\alpha_0} \) is given by

\[
c_{\alpha_0} = 1 - 3\alpha_0^2.
\]

Proposition 2.17.

1. The Fock module \( F_\beta \) is a simple \( \mathcal{F}_{\alpha_0} \)-module for all \( \beta \in \mathbb{C} \).

2. The abelian category of \( \mathcal{F}_{\alpha_0} \)-modules, \( \mathcal{F}_{\alpha_0} \text{-mod} \), is semisimple and the set of simple objects is given by \( \{F_\beta\}_{\beta \in \mathbb{C}} \).

3. The generating state \( |\beta\rangle \) of \( F_\beta \) satisfies

\[
L_0|\beta\rangle = h_\beta|\beta\rangle, \quad L_n|\beta\rangle = 0, \quad n \geq 1
\]

where

\[
h_\beta = \frac{1}{2}\beta(\beta - \alpha_0).
\]

We introduce an auxiliary field \( \varphi(z) \), which is a formal primitive of \( b(z) \)

\[
\varphi(z) = \hat{b} + b_0 \log z - \sum_{n \neq 0} \frac{b_n}{n} z^{-n}
\]

and which satisfies the operator product expansion

\[
\varphi(z)\varphi(w) = \log(z-w) + \cdots
\]

Definition 2.18. For all \( \beta \in \mathbb{C} \), let \( V_\beta(z) \) denote the field

\[
V_\beta(z) = e^{\beta \varphi(z)} := e^{\beta \hat{b} + \beta b_0 \log(z)}
\]

\[
\overline{V}_\beta(z) = e^{\beta \sum_{n \geq 1} \frac{b_n}{n} z^n} e^{-\beta \sum_{n \geq 1} \frac{b_n}{n} z^{-n}} \in U(\mathfrak{g}) \otimes \mathbb{C}[z,z^{-1}],
\]

where \( z^{b_0} = \exp(\beta b_0 \log(z)) \).

Problems arising from the multivaluedness of \( z^{b_0} \) will be resolved in Section 3.

Proposition 2.19. For all \( \beta \in \mathbb{C} \) the fields \( V_\beta(z) \) satisfy:

1. For \( \gamma \in \mathbb{C} \) the field \( V_\beta(z) \) defines a map

\[
V_\beta(z) : F_\gamma \to F_{\beta+\gamma}[z,z^{-1}] z^{\beta\gamma}.
\]
2. The field \( V_\beta(z) \) corresponds to the state \( |\beta\rangle \), that is,
\[
V_\beta(z)|0\rangle - |\beta\rangle \in F_\beta[[z]]z.
\]

3. The field \( V_\beta(z) \) is primary and satisfies the operator product expansion
\[
T(z)V_\beta(w) = \frac{h_\beta}{(z-w)^2}V_\beta(w) + \frac{1}{z-w}\partial V_\beta(w) + \cdots.
\]

4. For \( \beta_1, \ldots, \beta_k \in \mathbb{C} \) the product of the \( k \) fields \( V_{\beta_i}(z_i) \) satisfies the operator product expansion
\[
\prod_{i=1}^{k} V_{\beta_i}(w_i) = e^{\sum_{i=1}^{k} \beta_i \hat{b}_i} \prod_{i=1}^{k} w_i^{\beta_i b_0} \prod_{1\leq i<j\leq k} (w_j - w_i)^{\beta_i \beta_j} : \prod_{i=1}^{k} \overline{V}_{\beta_i}(w_i) :,
\]
where \( : \prod_{i=1}^{k} \overline{V}_{\beta_i}(w_i) : \) is an element of \( U(\mathfrak{b}) \otimes \mathbb{C}[z_1, z_1^{-1}, \ldots, z_k, z_k^{-1}] \)
\[
: \prod_{i=1}^{k} \overline{V}_{\beta_i}(w_i) : = e^{\sum_{i=1}^{k} \beta_i \sum_{n \geq 1} \frac{b_n}{n} w_i^n} e^{-\sum_{i=1}^{k} \beta_i \sum_{n \geq 1} \frac{b_n}{n} w_i^{-n}}.
\]

**Remark 2.20.** The Virasoro generator \( L_0 \) is diagonalisable on the Fock spaces \( F_\beta \). The eigenvalues of \( L_0 \) define a grading of \( F_\beta \)
\[
F_\beta = \bigoplus_{n \geq 0} F_\beta[h_\beta + n],
\]
where
\[
F_\beta[h] = \{ u \in F_\beta | L_0 u = h u \}.
\]
The dimension of these homogeneous subspaces is
\[
\dim F_\beta[h_\beta + n] = p(n),
\]
where \( p(n) \) is the number of partitions of the integer \( n \).

The Heisenberg algebra \( U(\mathfrak{b}) \) admits an anti-involution
\[
\sigma : U(\mathfrak{b}) \to U(\mathfrak{b})
\]
\[
b_n \mapsto \delta_{n,0} a_0 - b_{-n},
\]
such that
\[
\sigma(L_n) = L_{-n}.
\]
Definition 2.21. The dual Fock module $F^{\vee}_{\beta}$ is isomorphic to the graded dual space of $F_{\beta}$

$$F^{\vee}_{\beta} = \bigoplus_{n \geq 0} \text{Hom}(F_{\beta}[h_{\beta} + n], \mathbb{C}) .$$

The anti-involution $\sigma$ induces the structure of a left $U(b)$-module on $F^{\vee}_{\beta}$ by

$$\langle b_n \varphi, u \rangle = \langle \varphi, \sigma(b_n) u \rangle$$

for all $n \in \mathbb{Z}$, $\varphi \in F_{\beta}^{\ast}$, $u \in F_{\beta}$. We denote this left module by $F_{\beta}^{\ast}$ and call it the contragredient dual of $F_{\beta}$.

Proposition 2.22. Taking the contragredient defines a contravariant functor

$$* : \mathcal{F}_{\alpha_0} - \text{mod} \rightarrow \mathcal{F}_{\alpha_0} - \text{mod},$$

satisfying

$$F_{\beta}^{\ast} = F_{\alpha_0 - \beta} ,$$

such that $(F_{\beta}^{\ast})^{\ast} = F_{\beta}$.

2.3 The lattice vertex operator algebra $\mathcal{V}_{p_+,p_-}$

The lattice VOA $\mathcal{V}_{p_+,p_-}$ is defined for special values of the parameter $\alpha_0$ and by restricting the weights of the Fock spaces to a certain lattice. Let $p_+, p_- \geq 2$ be two coprime integers, such that

$$\alpha_+ = \sqrt{\frac{2p_-}{p_+}} \quad \alpha_- = -\sqrt{\frac{2p_+}{p_-}} \quad \alpha_0 = \alpha_+ + \alpha_-$$

$$\kappa_+ = \frac{\alpha_+^2}{2} = \frac{p_-}{p_+} \quad \kappa_- = \frac{\alpha_-^2}{2} = \frac{p_+}{p_-} \quad \alpha = p_+ \alpha_+ = -p_- \alpha_- .$$

The parameters $\kappa_+ = \kappa_-^{\ast}$ are called the level of $\mathcal{V}_{p_+,p_-}$. This is the positive rational level mentioned in the title of this paper. Next we define the rank 1 lattices

$$Y = \mathbb{Z} \sqrt{2p_+p_-} \quad X = \text{Hom}_\mathbb{Z}(Y, \mathbb{Z}) = \mathbb{Z} \frac{1}{\sqrt{2p_+p_-}} .$$

Both $\alpha_+$ and $\alpha_-$ lie in $X$ and give rise to the parametrisation

$$\beta_{r,s} = \frac{1 - r}{2} \alpha_+ + \frac{1 - s}{2} \alpha_- , \quad r, s \in \mathbb{Z} .$$

Note that $\beta_{r,s} = \beta_{r+p_+s+p_-}$ and we use the shorthand

$$\beta_{r,s;n} = \beta_{r-np_+s} = \beta_{r,s+np_-} .$$

When denoting the weights of Fock spaces, we will only write the indices and drop the “$\beta$” from $\beta_{r,s;n}$ or $\beta_{r,s}$, that is

$$F_{\beta_{r,s;n}} = F_{r,s;n} , \quad F_{\beta_{r,s}} = F_{r,s} .$$
Remark 2.23.

1. For $\alpha_0 = \alpha_+ + \alpha_-$ we denote the Heisenberg VOA by $F_{p_+ p_-}$ instead of $F_{\alpha_0}$.

2. Taking the contragredient of a Fock space $F_{r,s;n}$ reverses the sign of the indices:

$$F_{r,s;n}^* = F_{-r,-s;-n}.$$

Definition 2.24. The lattice VOA $V_{p_+ p_-}$ is the tuple $(V_0, |0\rangle, \frac{1}{2}(\beta_1^2 - \alpha_0 b_2)|0\rangle, Y)$, where the underlying vector space of $V_{p_+ p_-}$ is given by

$$V_0 = \bigoplus_{\beta \in Y} F_\beta = \bigoplus_{n \in \mathbb{Z}} F_{n\alpha}.$$

The fields corresponding to $|0\rangle, b_- |0\rangle$ and $T$ are those of $F_{p_+ p_-}$ and

$$Y(|\beta\rangle; z) = V_\beta(z), \quad \beta \in Y.$$

Remark 2.25. The relations for the vertex operator map in the definition above uniquely define the VOA structure of $V_{p_+ p_-}$. The central charge of the Virasoro field $T(z)$ is

$$c_{p_+ p_-} = 1 - 6 \frac{(p_+ - p_-)^2}{p_+ p_-},$$

and the conformal weight of the generating state $|\beta\rangle$ of a Fock module $F_\beta$ is $h_\beta = \frac{1}{2} \beta (\beta - \alpha_0)$. We define $h_{r,s} = h_{\beta_{r,s}}$, $r, s \in \mathbb{Z}$, then we have

$$h_{r,s} = \frac{r^2 - 1}{4} \kappa_+ - \frac{rs - 1}{2} + \frac{s^2 - 1}{4} \kappa_-.$$

Proposition 2.26. The abelian category $V_{p_+ p_-}$-mod of $V_{p_+ p_-}$-modules is semi-simple with $2p_+ p_-$ simple objects. These simple objects are parametrised by the classes of $X/Y$

$$V_{\beta} = \bigoplus_{\gamma \in \beta + Y} F_\gamma, \quad \beta \in X.$$

Remark 2.27. Using the $\beta_{r,s;n}$ we parametrise the simple $V_{p_+ p_-}$-modules as

$$V^+_{r,s} = \bigoplus_{n \in \mathbb{Z}} F_{r,s;2n},$$

$$V^-_{r,s} = \bigoplus_{n \in \mathbb{Z}} F_{r,s;2n+1},$$

for $1 \leq r \leq p_+$, $1 \leq s \leq p_-$. In this notation $V_0 = V^+_{1,1}$. 

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By the formula for conformal weights in Proposition 2.17, the two Heisenberg weights $\alpha_+, \alpha_-$ have conformal weight $h_{\alpha_\pm} = 1$. These are the only Heisenberg weights with conformal weight 1. We call the fields corresponding to $|\alpha_\pm\rangle$ screening operators and denote them by

$$S_\pm(z) = : e^{\alpha_\pm \phi(z)} : .$$

Since $h_{\alpha_\pm} = 1$, these fields define intertwining operators, that is, the map

$$S_\pm = \oint S_\pm(z) \, dz : V[0] \to V[\alpha_\pm]$$

is $\mathbb{C}$-linear and commutes with the Virasoro algebra. Note that since $\alpha_\pm \notin Y$ the fields $S_\pm(z)$ do not belong to $V_{p_+, p_-}$. We will later define the extended $W$-algebra $M_{p_+, p_-}$ as the subVOA of $V_{p_+, p_-}$ given by the intersection of the kernels of $S_+$ and $S_-$.  

### 3 Deformation of screening operators

For the purposes of this paper it is necessary to consider integrals of products of screening operators and not just the residues of individual screening operators. In order to perform these integrals one needs to consider homology groups of configuration spaces of $N$ points on the projective line with local coefficients. It is necessary to use local coefficients because these products of screening operators are not single valued, but have non-trivial monodromies that are roots of unity. The homology groups with such local coefficients exhibit very complicated behaviour and in order to make them tractable we deform the Heisenberg VOA, that is, we deform its conformal structure and its screening operators. The associated local systems then no longer exhibits monodromy at roots of unity and the homology groups of these deformed local systems are very simple. After analysing the deformed case in detail, we will show that one can take a meaningful limit to the undeformed case.

#### 3.1 Deformation of the Heisenberg vertex operator algebra

Let $\mathcal{O} = \mathbb{C}[[\varepsilon]]$ be the ring of formal power series with coefficients in $\mathbb{C}$ and let $\mathcal{K} = \mathbb{C}((\varepsilon))$ be the fraction field of $\mathcal{O}$. To any module over $\mathcal{O}$ we can associate a $\mathbb{C}$ vector space by taking the tensor product $- \otimes_{\mathcal{O}} \mathbb{C}$, that is, by setting $\varepsilon$ to zero. We enlarge the ground field $\mathbb{C}$ of the Heisenberg algebra $U(b)$ introduced in Section 2.2 to the rings $\mathcal{O}$ and $\mathcal{K}$.

**Definition 3.1.** Let $\mathcal{K} U(b_\pm)$ and $\mathcal{O} U(b_\pm)$ be the Heisenberg algebra over $\mathcal{K}$ and $\mathcal{O}$ respectively, that is

$$\mathcal{K} U(b_\pm) = \mathcal{K}[b_{\pm 1}, b_{\pm 2}, \ldots] \quad \mathcal{O} U(b_\pm) = \mathcal{O}[b_{\pm 1}, b_{\pm 2}, \ldots]$$

$$\mathcal{K} U(b_0) = \mathcal{K}[b_0] \quad \mathcal{O} U(b_0) = \mathcal{O}[b_0]$$

$$\mathcal{K} U(\bar{b}) = \mathcal{K} U(b_-) \otimes_{\mathcal{K}} \mathcal{K} U(b_+)$$

$$\mathcal{O} U(\bar{b}) = \mathcal{O} U(b_-) \otimes_{\mathcal{O}} \mathcal{O} U(b_+)$$

$$\mathcal{K} U(b) = \bigoplus_{d \in \mathbb{Z}} \mathcal{K} U(\bar{b})[d]$$

$$\mathcal{O} U(b) = \bigoplus_{d \in \mathbb{Z}} \mathcal{O} U(\bar{b})[d]$$

The Heisenberg algebra over $\mathcal{K}$ contains the $\mathcal{O}$ subalgebra $\mathcal{O} U(b)$ as an $\mathcal{O}$ lattice.
We deform the parameters $\alpha_\pm$ and $\kappa_\pm$ as follows. Let

$$\alpha_\pm(\varepsilon) = \alpha_\pm^{(0)} + \alpha_\pm^{(1)} \varepsilon + \alpha_\pm^{(2)} \varepsilon^2 + \cdots \in \mathcal{O}$$

such that $\alpha_\pm^{(0)} = \alpha_\pm$ and that $\alpha_\pm^{(1)} \neq 0$ as well as $\alpha_+(\varepsilon) \alpha_-(\varepsilon) = -2$. Furthermore, let

$$\kappa_\pm(\varepsilon) = \frac{1}{2} \alpha_\pm(\varepsilon)^2 \in \mathcal{O},$$

$$\alpha_0(\varepsilon) = \alpha_+(\varepsilon) + \alpha_-(\varepsilon) \in \mathcal{O}.$$ 

We define the rank 2 abelian group

$$\sigma X = \mathbb{Z} \alpha_+^{(\varepsilon)} \oplus \mathbb{Z} \alpha_-^{(\varepsilon)} \subset \mathcal{O}$$

and for $r, s \in \mathbb{Z}$

$$\beta_{r,s}(\varepsilon) = \frac{1-r}{2} \alpha_+(\varepsilon) + \frac{1-s}{2} \alpha_-(\varepsilon) \in \sigma X.$$

**Definition 3.2.**

1. For each $\beta \in \sigma X$ we define the left $\mathcal{K} U(b)$-module $\mathcal{K} F_\beta$ generated by $|\beta\rangle$

$$b_0|\beta\rangle = \beta |\beta\rangle, \quad b_n|\beta\rangle = 0, \quad n \geq 1,$$

such that

$$\mathcal{K} U(b-) \rightarrow \mathcal{K} F_\beta, \quad P \mapsto P |\beta\rangle$$

is an isomorphism of $\mathcal{K}$-vector spaces.

2. Let $\sigma F_\beta$ be the subspace of $\mathcal{K} F_\beta$ given by

$$\sigma F_\beta = \sigma U(b)|\beta\rangle.$$

The energy momentum tensor and other fields are defined in the same way as in Section 2

$$T(z) = \frac{1}{2} :b(z)^2: + \frac{\alpha_0(\varepsilon)}{2} \partial b(z),$$

$$V_\beta(z) = :e^{\beta \varphi(z)}:,$$

with $\beta$ now in $\mathcal{O}$ instead of $\mathbb{C}$. Also as in Section 2, we drop $\beta$ from the index of Fock spaces $\mathcal{K} F_{\beta_{r,s}} = \mathcal{K} F_{r,s}, \sigma F_{\beta_{r,s}} = \sigma F_{r,s}$. By evaluating operator product expansions it follows that the central charge and conformal weights are given by the same formulae as before

$$c_{p_+, p_-}(\varepsilon) = 1 - 3\alpha_0(\varepsilon)^2 \in \mathcal{O},$$

$$h_\beta(\varepsilon) = \frac{1}{2} \beta(\beta - \alpha_0(\varepsilon)) \in \mathcal{O}.$$ 

Set $h_{r,s}(\varepsilon) = h_{\beta_{r,s}}(\varepsilon)$, $r, s \in \mathbb{Z}$, then

$$h_{r,s}(\varepsilon) = \frac{r^2 - 1}{4} \kappa_+^{(\varepsilon)} - \frac{rs - 1}{2} + \frac{s^2 - 1}{4} \kappa_-^{(\varepsilon)}.$$
Proposition 3.3. Let $\mathcal{K}_0 = (\mathcal{K}_F, |0\rangle, \frac{1}{2}(b^2_{-1} + \alpha_0(\varepsilon)b_{-2})|0\rangle, Y)$, then $\mathcal{K}_0$ has the structure of a VOA over the field $\mathcal{K}$.

Proposition 3.4. For each $A \in \mathcal{O}_0$, the field $Y(A; z)$ preserves the $\mathcal{O}$ lattice $\mathcal{O}_0$ of $\mathcal{K}_0$, that is

$$Y(A; z) \in \text{End}_{\mathcal{O}}(\mathcal{O}_0)[[z, z^{-1}]].$$

The two Heisenberg weights $\alpha_{\pm}(\varepsilon)$ have conformal weight $h_{\alpha_{\pm}}(\varepsilon) = 1$. Therefore the fields

$$S_+(z) = e^{\alpha_+(\varepsilon)}\varphi(z);$$

$$S_-(z) = e^{\alpha_-(\varepsilon)}\varphi(z);$$

define screening operators for $\mathcal{K}_0$.

3.2 The construction of renormalisable cycles

In this section we construct cycles over which we can integrate products of the screening operators $S_+(z)$ and $S_-(z)$. In order to construct these cycles, we make extensive use of local systems as well as de Rham theory twisted by these local systems. We refer readers unfamiliar with these topics to Aomoto and Kita’s book [AK].

For $m \geq 1$ let $Y_m$ be the complex manifold

$$Y_m = \{(y_1, \ldots, y_m) \in \mathbb{C}^m | y_i \neq y_j, y_i \neq 0, 1\}$$

Let $\rho, \sigma, \tau \in \mathcal{O}$ then

$$G_m(\rho, \sigma, \tau; y) = \prod_{i=1}^{m} y_i^\rho (1 - y_i)^\sigma \prod_{1 \leq i \neq j \leq m} (y_i - y_j)^\tau$$

is a multivalued function on $Y_m$. Denote by $\mathcal{O}\mathcal{L}_m(\rho, \sigma, \tau)$ and $\mathcal{K}\mathcal{L}_m(\rho, \sigma, \tau)$ the local systems defined by the multivaluedness of $G_m(\rho, \sigma, \tau; y)$ over $\mathcal{O}$ and $\mathcal{K}$ respectively and denote their duals by $\mathcal{O}\mathcal{L}_m(\rho, \sigma, \tau)$ and $\mathcal{K}\mathcal{L}_m(\rho, \sigma, \tau)$. Furthermore the twisted homology groups with coefficients in $\mathcal{O}\mathcal{L}_m(\rho, \sigma, \tau)$ and $\mathcal{K}\mathcal{L}_m(\rho, \sigma, \tau)$ are denoted by $H_p(Y_m, \mathcal{O}\mathcal{L}_m(\rho, \sigma, \tau))$ and $H_p(Y_m, \mathcal{K}\mathcal{L}_m(\rho, \sigma, \tau))$ and the twisted cohomology groups by

$$H^p(Y_m, \mathcal{O}\mathcal{L}_m(\rho, \sigma, \tau)) = \text{Hom}_{\mathcal{O}}(H^p(Y_m, \mathcal{O}\mathcal{L}_m(\rho, \sigma, \tau)), \mathcal{O}),$$

$$H^p(Y_m, \mathcal{K}\mathcal{L}_m(\rho, \sigma, \tau)) = \text{Hom}_{\mathcal{O}}(H^p(Y_m, \mathcal{K}\mathcal{L}_m(\rho, \sigma, \tau)), \mathcal{O}).$$

By the theorem of twisted de Rham theory these cohomology groups are isomorphic to twisted de Rham cohomology groups

$$H^p(Y_m, \mathcal{O}\mathcal{L}_m(\rho, \sigma, \tau)) \cong H^p(Y_m, \nabla_{\omega_m(\rho, \sigma, \tau)}), \quad H^p(Y_m, \mathcal{K}\mathcal{L}_m(\rho, \sigma, \tau)) \cong H^p(Y_m, \nabla_{\omega_m(\rho, \sigma, \tau)}),$$

where the differential $\nabla_{\omega_m(\rho, \sigma, \tau)}$ is given by

$$\nabla_{\omega_m(\rho, \sigma, \tau)} = d + \omega_m(\rho, \sigma, \tau) \wedge,$$

$$\omega_m(\rho, \sigma, \tau) = d \log G_m(\rho, \sigma, \tau).$$
It is known that \[ \text{[AK11 Chapter 2]} \]
\[
\dim H^p(Y_m, \mathcal{O}'_m(\rho, \sigma, \tau)) = 0, p > m.
\]

The symmetric group $S_m$ acts in a compatible fashion on both $Y_m$ and $\mathcal{L}_m(\rho, \sigma, \tau)$, therefore the cohomology group $H^p(X_N, \mathcal{L}_m(\rho, \sigma, \tau))$ carries the structure of a finite dimensional representation of $S_m$. We can therefore decompose $H^p(X_N, \mathcal{L}_m(\rho, \sigma, \tau))$ into a direct sum of irreducible $S_m$ modules. For any $S_m$ module $M$, let $M^{S_m-}$ be the skew symmetric part of $M$. For purposes of this paper we are only interested in the skew symmetric parts of the $m$th cohomology groups $H^m(Y_m, \mathcal{L}_m(\rho, \sigma, \tau))^{S_m-}$.

**Proposition 3.5.** Let $\rho = \rho_0 + \rho_1 \varepsilon + \cdots, \sigma = \sigma_0 + \sigma_1 \varepsilon + \cdots, \tau = \tau_0 + \tau_1 \varepsilon + \cdots \in \mathcal{O}$ such that the constant terms of $\sigma, \tau$ lie in $\mathbb{C} \setminus \mathbb{Q}_{\leq 0}$ and
\[
d(d+1)\tau \notin \mathbb{Z}, \ d(d-1)\tau + d\rho \notin \mathbb{Z}, \ d(d-1)\tau + d\sigma \notin \mathbb{Z},
\]
then there exists a construction of a cycle $[\Delta_m(\rho, \sigma, \tau)] \in H_m(Y_m, \mathcal{L}_m(\rho, \sigma, \tau))$ such that

1. for $f(y) \in \mathcal{K}[y_1^+, \ldots, y_m^+]$
\[
\int_{[\Delta_m(\rho, \sigma, \tau)]} G_m(\rho, \sigma, \tau; y) f(y) \frac{dy_1 \cdots dy_m}{y_1^+ \cdots y_m} = \int_{[\Delta_m]} G_m(\rho, \sigma, \tau; y) f(y) \frac{dy_1 \cdots dy_m}{y_1^+ \cdots y_m},
\]
where $[\Delta_m] = \{1 > y_1 > \cdots > 0\}$ is the $m$-simplex.

2. The integration of $G_m(\rho, \sigma, \tau)$ over $[\Delta_m]$ is given by the Selberg integral
\[
S_m(\rho, \sigma, \tau) = \int_{[\Delta_m]} G_m(\rho, \sigma, \tau; y) \frac{dy_1 \cdots dy_m}{y_1^+ \cdots y_m}
= \frac{1}{m!} \prod_{i=1}^m \frac{\Gamma(1 + i\tau)\Gamma(\rho + (i-1)\tau)\Gamma(1 + \sigma + (i-1)\tau)}{\Gamma(1 + \tau)\Gamma(1 + \rho + \sigma + (m+i-2)\tau)}.
\]

**Proof.**

1. The cycle $[\Delta_m(\rho, \sigma, \tau)]$ was constructed in \[ \text{[TK86]} \] by means of the isomorphism of homology groups $H_m(Y_m, \mathcal{L}_m(\rho, \sigma, \tau)) \cong H^1_m(Y_m, \mathcal{L}_m(\rho, \sigma, \tau))$ between the twisted homology group and the locally finite homology group. If $\varphi$ is the principal branch of $G_m(\rho, \sigma, \tau; y)$ on $\Delta_m$, then $[\Delta_m; \varphi] \in H^1_m(Y_m, \mathcal{L}_m(\rho, \sigma, \tau))$. The corresponding cycle in $H_m(Y_m, \mathcal{L}_m(\rho, \sigma, \tau))$ was then constructed by means of a blow up $\hat{Y}_m \to Y_m$.

2. This integral is due to Selberg \[ \text{[Sel44]} \].

We will now use these cycles $[\Delta_m(\rho, \sigma, \tau)]$ in order to integrate $N$-fold products of screening operators. Consider the $N$-fold product of $S_\pm(z)$
\[
\prod_{i=1}^N S_\pm(z_i) = e^{N\alpha_\pm(\varepsilon)k} \prod_{i=1}^N z_i^{\alpha_\pm(\varepsilon)\theta_0} \prod_{1 \leq i \neq j \leq N} (z_i - z_j)^{\alpha_\pm(\varepsilon)^2/2} \prod_{i=1}^N S_\pm(z_i).
\]
where: \( \prod_{i=1}^{N} S_{\pm}(z_i) \in \mathcal{O}(U(\mathbb{D})) \hat{\otimes} \mathcal{O}([z_1^+, \ldots, z_n^+]) \)

\[
\prod_{i=1}^{N} S_{\pm}(z_i) := \prod_{k \geq 0} \prod_{i=1}^{N} e^{\alpha_{\pm}(\varepsilon) \sum_{k=1}^{N} \frac{z_i^k}{k^b_k}} e^{-\alpha_{\pm}(\varepsilon) \sum_{k=1}^{N} \frac{z_i^{-k}}{k^b_k}} .
\]

Note that: \( \prod_{i=1}^{N} S_{\pm}(z_i) \) is symmetric with respect to permuting the variables \( z_i \).

Let \( r \geq 1, s \in \mathbb{Z} \), then if we evaluate the operator \( S_{+}(z_1) \cdots S_{+}(z_r) \) on \( \mathcal{K} F_{r,s} \) we have

\[
\prod_{i=1}^{r} S_{+}(z_i) = e^{r\alpha_{+}(\varepsilon)b} U_r(z_1, \ldots, z_r; \kappa_{+}(\varepsilon)) \prod_{i=1}^{r} \left( z_i^{s-1} \right) \prod_{i=1}^{r} S_{+}(z_i) : .
\]

where

\[
U_r(z_1, \ldots, z_r; \kappa_{+}(\varepsilon)) = \prod_{1 \leq i \neq j \leq r} (z_i - z_j)^{\kappa_{+}(\varepsilon)} \prod_{i=1}^{r} z_i^{(1-r)\kappa_{+}(\varepsilon)} .
\]

Similarly one can also evaluate \( S_{-}(z_1) \cdots S_{-}(z_s) \) on \( \mathcal{K} F_{r,s} \) for \( s \geq 1, r \in \mathbb{Z} \).

We can use the twisted de Rham theory, developed above, to integrate the multivalued function \( U_N(z_1, \ldots, z_N; \kappa) \). Consider the \( N \) dimensional complex manifold

\[
X_N = \{(z_1, \ldots, z_N) \in \mathbb{C}^N | z_i \neq z_j, z_i \neq 0 \} .
\]

and fix

\[
\kappa = \kappa_0 + \kappa_1 \varepsilon + \cdots \in \mathcal{O}
\]

such that \( \kappa_0 \in \mathbb{C} \setminus \mathbb{Q}_{\leq 0} \) and \( \kappa_1 \neq 0 \). Then

\[
U_N(z; \kappa) = \prod_{1 \leq i \neq j \leq N} (1 - \frac{z_i}{z_j})^{\kappa}
\]

defines a multivalued holomorphic function on \( X_N \). Denote by \( \mathcal{K} \mathcal{L}_N(\kappa) \) the local system defined by the multivaluedness of \( U_N(z; \kappa) \) over \( \mathcal{K} \) and its dual by \( \mathcal{K} \mathcal{L}_N^*(\kappa) \). We introduce new variables \( z, y_1, \ldots, y_N \) such that

\[
z_i = z, \ z_i = z y_{i-1}, i = 2, \ldots, N .
\]

Then it is clear that \( X_N \cong \mathbb{C}^* \times Y_{N-1} \). If we write \( U_N(z; \kappa) \) in terms of the new variables, we see that the \( z \) dependence drops out

\[
U_N(z, z y_1, \ldots; \kappa) = \prod_{i=1}^{N-1} y_i^{(1-N)\kappa} (1 - y_i)^{2\kappa} \prod_{1 \leq i \neq j \leq N-1} (y_i - y_j)^{\kappa} = G_{N-1}((1-N)\kappa, 2\kappa, \kappa) .
\]
By the Künneth formula we therefore have

\[ H_N(X_N, \kappa L_N^\vee(\kappa)) = H_1(C^*, \mathcal{K}) \otimes H_{N-1}(Y_{N-1}, \kappa L_{m_0}^\vee((1 - N)\kappa, 2\kappa, \kappa)) . \]

It is known that [AK11]

\[ \dim_{\kappa} H_N(X_N, \kappa L_N^\vee(\kappa)) = (N - 1)! , \]

\[ \dim_{\kappa} H_N(X_N, \kappa L_N^\vee(\kappa))^{SN-1} = 1 . \]

Let \( 2\pi i[\gamma] \in H_1(C^*, \mathcal{K}) \) be the class of a circle about the origin, then we can use Proposition 3.5.

**Definition 3.6.** Let \( [\Gamma_N(\kappa)] \) and \( [\bar{\Gamma}_N(\kappa)] \) be the renormalised cycles

\[ [\Gamma_N(\kappa)] = \frac{1}{S_{N-1}((1 - N)\kappa, 2\kappa, \kappa)}[\Delta_{N-1}((1 - N)\kappa, 2\kappa, \kappa)] , \quad [\bar{\Gamma}_N(\kappa)] = [\gamma] \times [\bar{\Gamma}_N(\kappa)] \]

such that

\[ \int_{[\Gamma_N(\kappa)]} U_N(z; \kappa) \prod_{i=1}^N \frac{dz_i}{z_i} = 1 , \quad \int_{[\bar{\Gamma}_N(\kappa)]} U_N(z, zy_1, \ldots, zy_{N-1}; \kappa) \prod_{i=1}^{N-1} \frac{dy_i}{y_i} = 1 . \]

**Definition 3.7.** For \( f \in \mathcal{K}[^{\pm}_{1}, \ldots, ^{\pm}_{N}]^{SN} \) the cycle \( [\Gamma_N(\kappa)] \) defines a \( \mathcal{K} \)-linear map

\[ \langle f(z) \rangle_N^\kappa = \int_{[\Gamma_N(\kappa)]} U_N(z; \kappa) f(z) \prod_{i=1}^N \frac{dz_i}{z_i} . \]

**Proposition 3.8.**

1. \( \langle 1 \rangle_N^\kappa = 1 \)

2. \( \langle f \rangle_N^\kappa = 0 \) if \( \deg f \neq 0 \)

3. \( \langle f \rangle_N^\kappa = (\overline{f})_N^\kappa \) where \( \overline{f}(z_1, \ldots, z_N) = f(z_1^{-1}, \ldots, z_N^{-1}) \).

**Definition 3.9.** Let \( (\ , \ )_N^\kappa \) be the bilinear \( \mathcal{K} \)-form

\[ (\ , \ )_N^\kappa : \mathcal{K}[^{\pm}_{1}, \ldots, ^{\pm}_{N}]^{SN} \otimes_\mathcal{K} \mathcal{K}[^{\pm}_{1}, \ldots, ^{\pm}_{N}]^{SN} \rightarrow \mathcal{K} \]

defined by

\[ (f, g)_N^\kappa = \langle \overline{f} g \rangle_{N-1}^\kappa . \]

Note that the inner product above is defined by means of \( \kappa^{-1} \) in order to be closer to the notation of Macdonald’s book [Mac95]. Following Macdonald’s book [Mac95], we will evaluate this bilinear form in the next section by using the theory of Jack polynomials. We will also be able to show that it defines an inner product of symmetric polynomials over \( \mathcal{O} \), that is

\[ (\ , \ )_N^\kappa : \mathcal{O}[^{\pm}_{1}, \ldots, ^{\pm}_{N}]^{SN} \otimes_\mathcal{O} \mathcal{O}[^{\pm}_{1}, \ldots, ^{\pm}_{N}]^{SN} \rightarrow \mathcal{O} . \]
3.3 The theory of Jack polynomials

We introduce the theory of Jack polynomials following Macdonald’s book [Mac95, Chapter 6]. Fix the parameter \( \kappa \) to be

\[
\kappa = \kappa_0 + \kappa_1 \varepsilon + \cdots \in \mathcal{O},
\]

such that \( \kappa_0 \in \mathbb{C} \setminus \mathbb{Q}_{\leq 0} \) and \( \kappa_1 \neq 0 \).

**Definition 3.10.** The rings of symmetric polynomials over \( \mathcal{O} \) and \( \mathcal{K} \) in \( N \) variables \( x_i \) are given by

\[
\mathcal{K} \Lambda_N = \mathcal{K}[x_1, \ldots, x_N]^{S_N} \quad \mathcal{O} \Lambda_N = \mathcal{O}[x_1, \ldots, x_N]^{S_N}
\]

where the

\[
p_n = \sum_{i=1}^{N} x_i^n,
\]

are called power sums. The ring of symmetric polynomials \( \mathcal{K} \Lambda_N \) forms a graded commutative algebra with \( \deg p_n = n \). Rings of symmetric polynomials in different numbers of variables are related by the homomorphisms

\[
\rho_{N,M} : \mathcal{K} \Lambda_N \to \mathcal{K} \Lambda_M
\]

\[
x_i \mapsto x_i \quad i \leq M
\]

\[
x_i \mapsto 0 \quad i > M,
\]

where \( N > M \). The ring of symmetric polynomials in a countably infinite number variables is given by the projective limit

\[
\mathcal{K} \Lambda = \lim_{\leftarrow N} \mathcal{K} \Lambda_N,
\]

relative to the homomorphisms \( \rho_{N,M} \). One can return to the finite variable case by the projection

\[
\rho_N : \mathcal{K} \Lambda \to \mathcal{K} \Lambda_N
\]

\[
x_i \mapsto x_i \quad i \leq N
\]

\[
x_i \mapsto 0 \quad i > N.
\]

A convenient way of parametrising symmetric polynomials is by partitions of integers.

**Definition 3.11.** A partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \) is a weekly descending sequence of non-negative integers. We refer to \( |\lambda| = \sum \lambda_i \) as the degree of \( \lambda \) and to \( \ell(\lambda) = \# \{ \lambda_i \neq 0 \} \) as the length of \( \lambda \).

To each partition \( \lambda \) we associate a Young diagram, that is a collection of left aligned rows of boxes where the \( i \)-th row consists of \( \lambda_i \) boxes. The boxes of a diagram are labelled by two integers \((i,j)\), where \( i \) labels the row and \( j \) the column. For every partition \( \lambda \) there is also conjugate
partition $\lambda'$ which is obtained by exchanging rows and columns in the Young diagram. For example the conjugate of the partition $(4,2)$ is $(2,2,1,1)$. For a box $s = (i,j)$ in a Young diagram let

\[
\begin{align*}
a_\lambda(s) &= \lambda_i - j \\
a'_\lambda(s) &= j - 1 \\
\ell_\lambda(s) &= \lambda'_j - i \\
\ell'_\lambda(s) &= i - 1.
\end{align*}
\]

Partitions admit a partial ordering $\geq$ called the dominance ordering. For two partitions $\lambda, \mu$ of equal degree, $\lambda \geq \mu$ if and only if

\[
\sum_{i=1}^{n} \lambda_i \geq \sum_{i=1}^{n} \mu_i, \quad n \geq 1.
\]

Two examples of bases of $\mathcal{K}\Lambda$ parametrised by partitions are the power sums

\[p_\lambda(x) = p_{\lambda_1}(x)p_{\lambda_2}(x) \cdots\]

and the symmetric monomials

\[m_\lambda(x) = \sum_{\sigma} \sum_{i \geq 1} x_{\sigma(i)}^{\lambda_i},\]

where the first sum runs over all distinct permutations of the entries of the partition $\lambda$. Note that in the case of symmetric polynomials in $N$ variables one must restrict oneself to partitions of length at most $N$, since

\[\rho_N(m_\lambda(x)) = 0, \quad \text{for } \ell(\lambda) > N.\]

**Definition 3.12.** Let $(\ , \ )_\kappa$ be the inner product of symmetric polynomials defined by

\[
(\ , \ )_\kappa : \mathcal{K}\Lambda \otimes \mathcal{K}\Lambda \to \mathcal{K}
\]

\[
(p_\lambda(x), p_\mu(x))_\kappa \mapsto \delta_{\lambda,\mu} z_\lambda \kappa^{\ell(\lambda)},
\]

where

\[z_\lambda = \prod_{i \geq 1} i^{m_i(\lambda)} m_i(\lambda)!,\]

and $m_i(\lambda)$ is the multiplicity of $i$ in $\lambda$.

**Proposition 3.13.** For $\kappa = \kappa_0 + \kappa_1 \varepsilon + \cdots \in \mathcal{O}$ such that $\kappa_0 \in \mathcal{O} \setminus \mathbb{Q}\leq_0$ and $\kappa_1 \neq 0$, the symmetric polynomials $\mathcal{K}\Lambda$ admit a basis of polynomials called Jack polynomials $P_\lambda(x; \kappa)$ that satisfy

1. $(P_\lambda(x; \kappa), P_\mu(x; \kappa))_\kappa = 0$ if $\lambda \neq \mu$
2. $P_\lambda(x; \kappa) = \sum_{\mu \geq \lambda} u_{\lambda,\mu}(\kappa) m_\mu(x)$, where $u_{\lambda,\lambda}(\kappa) = 1$, $u_{\lambda,\mu}(\kappa) \in \mathcal{O}$ and the partial ordering $\geq$ is the dominance ordering.
Definition 3.14. Let
\[ b_\lambda(\kappa) = ((P_\lambda(x; \kappa), P_\lambda(x; \kappa))^1, \]
then the polynomials
\[ Q_\lambda(x; \kappa) = b_\lambda(\kappa)P_\lambda(x; \kappa), \]
form a basis dual to \( P_\lambda(x; \kappa) \), such that,
\[ (P_\lambda(x; \kappa), Q_\mu(x; \kappa))^\kappa = \delta_{\lambda,\mu}. \]

Jack polynomials in an infinite number of variables satisfy a number of remarkable properties which we summarise in the following proposition.

Proposition 3.15.
\begin{enumerate}
\item For \( \kappa = \kappa_0 + \kappa_1 \varepsilon + \cdots \in \mathcal{O} \) such that \( \kappa_0 \in \mathbb{C} \setminus \mathbb{Q}_{\geq 0} \) and \( \kappa_1 \neq 0 \), the coefficients \( b_\lambda(\kappa) \) are given by
\[ b_\lambda(\kappa) = \left( (P_\lambda(x; \kappa), P_\lambda(x; \kappa))^\kappa \right)^{1} = \prod_{s \in \lambda} \frac{\kappa a_\lambda(s) + \ell_\lambda(s) + 1}{\kappa a_\lambda(s) + \ell_\lambda(s) + \kappa} \]
and are units of \( \mathcal{O} \), that is, the coefficient of \( \varepsilon^0 \) is non-zero.
\item \[ \prod_{i,j \geq 1} (1 - x_i y_j)^{-1/\kappa} = \prod_{k \geq 1} e^{\frac{1}{k} \sum_{p_r(x)p_k(y)}} = \sum_{\lambda} P_\lambda(x; \kappa)Q_\lambda(y; \kappa) \]
\item Let \( \omega_\beta, \beta \in \mathcal{O} \) be the \( \mathcal{K} \) algebra endomorphism of \( \mathcal{K} \Lambda \) given by
\[ \omega_\beta(p_r) = (-1)^{r-1} \beta p_r, \]
then the Jack polynomials satisfy
\[ \omega_\kappa(P_\lambda(x; \kappa)) = Q_\lambda'(x; \kappa^{1}). \]
\item Let \( \Xi_X \) be the \( \mathcal{O} \) algebra homomorphism defined by
\[ \Xi_X : \mathcal{K} \Lambda \rightarrow \mathcal{K} \]
\[ p_r \mapsto X \]
for any \( X \in \mathcal{O} \), then the Jack polynomials satisfy
\[ \Xi_X(Q_\lambda(x; \kappa)) = \prod_{s \in \lambda} \frac{X^s + \kappa a_\lambda(s) - \ell_\lambda'(s)}{\kappa a_\lambda(s) + \ell_\lambda(s) + \kappa}. \]
\end{enumerate}
The map \( \Xi_X \) allows us to decompose certain products in terms of Jack polynomials.
\[ \Xi_X\left( \prod_{i,j \geq 1} (1 - x_i y_j)^{-1/\kappa} \right) = \prod_{k \geq 1} e^{\frac{1}{k} \sum_{p_r(x)p_k(y)}} = \prod_{i \geq 1} (1 - y_i)^{-X} \]
\[ = \sum_{\lambda} P_\lambda(y; \kappa)\Xi_X(Q_\lambda(x; \kappa)) \]
We return to symmetric polynomials in a finite number of variables.

**Proposition 3.16.** 1. A partition $\lambda$ defines a non-zero Jack polynomial $P_\lambda(x; \kappa) \in \mathcal{K}\Lambda_N$ if and only if $\ell(\lambda) \leq N$.

2. For the partition $\lambda = (m, \ldots, m)$, that is a partition consisting of $N$ copies of an integer $m$, the Jack polynomial $P_\lambda(x; \kappa) \in \mathcal{K}\Lambda_N$ is given by

$$P_\lambda(x; \kappa) = m_\lambda(x) = \prod_{i=1}^{N} x_i^m.$$  

3. For a partition $\lambda = (\lambda_1, \ldots, \lambda_N)$ and $m \geq 0$, let $\lambda + m = (\lambda_1 + m, \ldots, \lambda_N + m)$, then the Jack polynomial $P_\lambda(x; \kappa) \in \mathcal{K}\Lambda_N$ satisfies

$$P_\lambda(x; \kappa) \cdot \prod_{i=1}^{N} x_i^m = P_{\lambda+m}(x; \kappa).$$

**Definition 3.17.** For the symmetric polynomials in a finite number of variables, let $(\ , \ )_\kappa^N : \mathcal{K}\Lambda_N \otimes \mathcal{K}\Lambda_N \to \mathcal{K}$ be the inner product

$$(f, g)_\kappa^N : \mathcal{K}\Lambda_N \otimes \mathcal{K}\Lambda_N \to \mathcal{K}$$

such that

$$(f, g)_\kappa^N = \int_{\Gamma_N(\kappa)} U_N(x; \kappa^{-1}) \overline{f} \prod_{i=1}^{N} \frac{dx_i}{x_i},$$

where $\overline{f} = f(x_1^{-1}, \ldots, x_N^{-1})$.

**Proposition 3.18.** For $\kappa = \kappa_0 + \kappa_1 \varepsilon + \cdots \in \mathcal{O}$ such that $\kappa_0 \in \mathbb{C} \setminus \mathbb{Q}_{\leq 0}$ and $\kappa_1 \neq 0$, the two inner products $(\ , \ )_\kappa^N$ and $(\ , \ )_\kappa$ satisfy:

1. For two partitions $\lambda, \mu$

$$(P_\lambda(x; \kappa), P_\mu(x; \kappa))_\kappa = \delta_{\lambda, \mu} b_\lambda(\kappa)^{-1} \in \mathcal{O}$$

$$b_\lambda(\kappa) = \prod_{s \in \lambda} \frac{\kappa a_\lambda(s) + \ell_\lambda(s) + 1}{\kappa a_\lambda(s) + \ell_\lambda(s) + \kappa}.$$  

2. For two partitions $\lambda, \mu$ with $\ell(\lambda), \ell(\mu) \leq N$

$$(P_\lambda(x; \kappa), P_\mu(x; \kappa))_\kappa^N = \delta_{\lambda, \mu} \frac{b_\lambda^N(\kappa)}{b_\lambda(\kappa)} \in \mathcal{O}$$

$$b_\lambda^N(\kappa) = \prod_{s \in \lambda} \frac{N + \kappa a_\lambda(s) - \ell_\lambda(s)}{N + (a_\lambda(s) + 1)\kappa - \ell_\lambda(s) - 1}.$$
3. For positive integers $M, N$, let $\lambda_{N,M} = (M, \ldots, M)$ denote the length $N$ partition consisting of $N$ copies of $M$. The coefficients $b_{\lambda_{N,M}}^N(\kappa), b_{\lambda_{N,M}}(\kappa)$ satisfy the remarkable identity

$$b_{\lambda_{N,M}}^N(\kappa) = b_{\lambda_{N,M}}(\kappa).$$

As a direct consequence of the above proposition we have the following proposition.

**Proposition 3.19.** For $\kappa = \kappa_0 + \kappa_1 \varepsilon + \cdots \in \mathcal{O}$ such that $\kappa_0 \in \mathcal{O} \setminus \mathbb{Q}_{\leq 0}$ and $\kappa_1 \neq 0$

1. the inner product $(\ , \ )_\kappa^N$ defines an inner product on $\mathcal{O} \Lambda^N$

$$\mathcal{O} \Lambda^N \otimes \mathcal{O} \mathcal{O} \Lambda^N \to \mathcal{O},$$

2.

$$\int_{[\Gamma_N(\kappa)]} : H^N(X_N; \mathcal{O} \mathcal{L}(\kappa))^{S_N} \to \mathcal{O},$$

where $S_N$ denotes the elements of the cohomology groups that are skew-symmetric with respect to the action of the symmetric group $S_N$.

**Proposition 3.20.** Let $\rho = \rho_0 + \rho_1 \varepsilon + \cdots, \sigma = \sigma_0 + \sigma_1 \varepsilon + \cdots, \tau = \tau_0 + \tau_1 \varepsilon + \cdots \in \mathcal{O}$ such that the constant terms of $\sigma, \tau$ lie in $\mathbb{C} \setminus \mathbb{Q}_{\leq 0}$ and

$$d(d+1) \tau \notin \mathbb{Z}, \ d(d-1) \tau + d \rho \notin \mathbb{Z}, \ d(d-1) \tau + d \sigma \notin \mathbb{Z}.$$

Let $\lambda$ be a partition and $Q_\lambda(x; \kappa) \in \mathcal{O} \Lambda^N$ a dual Jack polynomial. Then the Kadell integral is given by

$$I_{\lambda,N}(\rho, \sigma, 1/\kappa) = \int_{[\Delta_N(\rho, \sigma, 1/\kappa)]} \prod_{i=1}^N x_i^\rho (1 - x_i)^\sigma \prod_{1 \leq i < j \leq N} (x_i - x_j)^{1/\kappa} Q_\lambda(x; \kappa) \frac{dx_1 \cdots dx_N}{x_1 \cdots x_N}$$

$$= S_N(\rho, \sigma, 1/\kappa) \Xi_{\rho+(N-1)/\kappa}(Q_\lambda(x; \kappa)) \Xi_{1+\rho+\sigma+2(N-1)/\kappa}(Q_\lambda(x; \kappa)) \Xi_N(Q_\lambda(x; \kappa)).$$

The evaluation of the above integral is a conjecture due to Macdonald [Mac87] and was first proven by Kadell [Kad97].

### 3.4 Integrating on the $\mathcal{O}$-lattice

In this section we consider the action of the screening operators on the Fock modules $\mathcal{O} F_{r,s}$ over $\mathcal{O}$.

**Definition 3.21.**
1. For each \( r \geq 1, s \in \mathbb{Z} \) we define the field \( S_+^{[r]}(z) \in \text{Hom}_K(\mathcal{O}_rF_{s,r}, \mathcal{O}F_{r-s})[[z, z^{-1}]] \) by

\[
S_+^{[r]}(z) = \int_{[F_r(z)z^{-1}]} S_+(z)z^{-1} \prod_{i=1}^{r-1} S_+(zy_i) \prod_{i=1}^{r-1} d y_i.
\]

The field \( S_+^{[r]}(z) \) admits the the mode expansion

\[
S_+^{[r]}(z) = \sum_{n \in \mathbb{Z}} S_+^{[r]}[n]z^{-n-1}
\]

\[
S_+^{[r]}[n] = \text{Res}_{z=0} z^n S_+^{[r]}(z) \text{ d } z \in \text{Hom}_K(\mathcal{O}_rF_{s,r}, \mathcal{O}F_{r-s}).
\]

2. For each \( r \in \mathbb{Z}, s \geq 1 \) we define the field \( S_-^{[s]}(z) \in \text{Hom}_K(\mathcal{O}_rF_{s,r}, \mathcal{O}F_{r-s})[[z, z^{-1}]] \) by

\[
S_-^{[s]}(z) = \int_{[F_r(z)z^{-1}]} S_-(z)z^{-s} \prod_{i=1}^{s-1} S_-(zy_i) \prod_{i=1}^{s-1} d y_i.
\]

The field \( S_-^{[s]}(z) \) admits the the mode expansion

\[
S_-^{[s]}(z) = \sum_{n \in \mathbb{Z}} S_-^{[s]}[n]z^{-n-1}
\]

\[
S_-^{[s]}[n] = \text{Res}_{z=0} z^n S_-^{[s]}(z) \text{ d } z \in \text{Hom}_K(\mathcal{O}_rF_{s,r}, \mathcal{O}F_{r-s}).
\]

The fields \( S_+^{[r]}(z) \) and \( S_-^{[s]}(z) \) are primary fields of conformal weight 1, that is, they satisfies

\[
T(z)S_+^{[k]}(w) = \frac{1}{(z-w)^2}S_+^{[k]}(z) + \frac{1}{(z-w)}\partial S_+^{[k]}(z) + \cdots
\]

\[
[L_n, S_+^{[k]}(z)] = z^n(z \frac{d}{dz} + (n+1))S_+^{[k]}(z)
\]

for \( n \in \mathbb{Z} \) and \( k = r, s \).

Proposition 3.22.

1. For \( n \in \mathbb{Z} \), the modes \( S_+^{[s]}[n], S_-^{[s]}[n] \) satisfy

\[
S_+^{[s]}[n](\mathcal{O}_rF_{s,r}) \subseteq \mathcal{O}F_{r-s}, \quad r \geq 1, s \in \mathbb{Z},
\]

\[
S_-^{[s]}[n](\mathcal{O}_rF_{s,r}) \subseteq \mathcal{O}F_{r-s}, \quad r \in \mathbb{Z}, s \geq 1.
\]

2. The zero modes \( S_+^{[r]} = S_+^{[r]}[0] \) and \( S_-^{[s]} = S_-^{[s]}[0] \) are Virasoro homomorphisms, that is, they are Virasoro intertwining operators of Fock modules.
Proof. This proposition follows from Proposition 3.20. We only prove the \( S_+^{[r]}[n] \) case, since the \( S_-^{[s]}[n] \) case follows by the same arguments.

1. The mode \( S_+^{[r]}[n] \) is given by

\[
S_+^{[r]}[n] = \int_{[\gamma \times [\kappa_+ (\varepsilon)]) \times [\kappa_+ (\varepsilon)])} z^{n+1} S_+ (z_1) \cdots S_+ (z_r) \, dz_1 \cdots dz_r
\]

\[
= \int_{[\gamma \times [\kappa_+ (\varepsilon)]) \times [\kappa_+ (\varepsilon)])} z^{n+1+k} G_{r-1} ((1-r)\kappa_+ (\varepsilon), 2\kappa_+ (\varepsilon), \kappa_+ (\varepsilon); y)
\]

\[
\times \prod_{i=1}^{r-1} y_i^s : \mathcal{F}_+ (z) \prod_{i=1}^{r-1} \mathcal{F}_+ (zy_i) : \frac{dz_1 \cdots dz_{r-1}}{yz_1 \cdots y_{r-1}}
\]

The product \( \mathcal{F}_+ (z) \prod_{i=1}^{r-1} \mathcal{F}_+ (zy_i) \) admits a decomposition into Jack polynomials of the form

\[
\mathcal{F}_+ (z) \prod_{i=1}^{r-1} \mathcal{F}_+ (zy_i) := \sum_{k \in \mathbb{Z}} \sum_{\lambda} \sum_{r,k} \prod_{i=1}^{r-1} y_i^k Q_\lambda (z, zy_1, \ldots, zy_{r-1}; \kappa_- (\varepsilon)) A_{\lambda,k},
\]

where \( \lambda \) is a partition of length at most \( r-1 \) and \( A_{\lambda,k} \) is an element of \( U(b) [\lambda + k] \). Therefore the action \( S_+^{[r]}[n] \) reduces to evaluating integrals of the form

\[
\int_{[\kappa_+ (\varepsilon)]) \times [\kappa_+ (\varepsilon)])} G_{r-1} ((1-r)\kappa_+ (\varepsilon) + k, 2\kappa_+ (\varepsilon), \kappa_+ (\varepsilon); y)Q_\lambda (y; \kappa_- (\varepsilon)) \frac{dy_1 \cdots dy_{r-1}}{y_1 \cdots y_{r-1}},
\]

where \( k \in \mathbb{Z} \) and \( \lambda \) is a partition of length at most \( r-1 \). By the using the Kadell integral formulae of Proposition 3.20 one sees that all these integrals lie in \( \mathcal{O} \) and thus \( S_+^{[r]}[n] (\mathcal{O} F_{r,s}) \subseteq \mathcal{O} F_{r,s} \).

2. follows from \( S_+^{[r]}(z) \) being a primary field. \(\square\)

Definition 3.23. By setting \( \varepsilon = 0 \) in Definition 3.21 we define the primary fields:

\[
S_+^{[r]} (z) \in \text{Hom}_C (F_{r,s}, F_{r,s}) [\mathbb{Z}, z^{-1}], \quad r \geq 1, s \in \mathbb{Z},
\]

\[
S_+^{[s]} (z) \in \text{Hom}_C (F_{r,s}, F_{r,s}) [\mathbb{Z}, z^{-1}], \quad r \in \mathbb{Z}, s \geq 1.
\]

In the same way we also define the Virasoro homomorphisms:

\[
S_+^{[r]} := \text{Res}_{z=0} S_+^{[r]} (z) \, dz,
\]

\[
S_+^{[s]} := \text{Res}_{z=0} S_+^{[s]} (z) \, dz.
\]

For each \( \gamma \in \mathbb{C} \setminus \{0\} \), let \( \rho_\gamma \) be the \( \mathbb{C} \) algebra isomorphism

\[
\rho_\gamma : \Lambda \to U(b_-)
\]

\[
p_\gamma (x) \to \gamma b_{-n}, \quad n = 1, 2, \ldots
\]

Proposition 3.24. The action of the screening operators on \( |\beta_{r,s} \rangle \) is given by

1. For \( r \geq 1 \) and \( s \in \mathbb{Z} \)

\[
S_+^{[r]} : F_{r,s} \to F_{r,s},
\]

such that

\[
S_+^{[r]}|\beta_{r,s} \rangle = \begin{cases} 0 & s \geq 1 \\ \rho_\frac{2}{a_+} (Q_{\lambda_{r-s}} (x; \kappa_-)) |\beta_{r,s} \rangle & s \leq 0 \end{cases}
\]

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2. For \( r \in \mathbb{Z} \) and \( s \geq 1 \)

\[
S_+^{[s]} : F_{r,s} \to F_{r,-s},
\]
such that

\[
S_+^{[s]} |\beta_{r,s}\rangle = \begin{cases} 
0 & r \geq 1 \\
p \frac{e_{-}}{\alpha_{-}} (Q_{\lambda_{s,-}}(\kappa_{+})) |\beta_{r,-s}\rangle & r \leq 0 
\end{cases}.
\]

3. For \( r, s \geq 1 \)

\[
S_+^{[r]} |\beta_{r,-s}\rangle = (-1)^{rs} b_{\kappa_{s}}(\kappa_{-}) S_+^{[s]} |\beta_{r,s}\rangle.
\]

**Proof.** This proposition is proved by using Proposition 3.18 to evaluate the action of the screening operators on \(|\beta_{r,s}\rangle\).

**Remark 3.25.** The results of Proposition 3.24 are truly remarkable. The screening operators on the left-hand side of points 1 and 2 are defined in terms of integrals over renormalised cycles and Jack polynomials in a finite number of variables, while the right-hand side is written in terms of Jack polynomials in the infinite variable case with Heisenberg generators inserted.

For generic values of the central charge \( c \notin \mathbb{Q} \), the singular vectors of Fock modules were identified with Jack polynomials by direct calculation in [MY95].

4 **Virasoro representation theory**

The way in which Fock spaces decompose into Virasoro modules was determined by Feigin and Fuchs in [FF84, FF88, FF90]. For a more modern and detailed account see [IK10]. In this section we will see how Fock-modules decompose as Virasoro modules, calculate the kernels and images of screening operators mapping between Fock-modules and introduce infinite sums of kernels and images that will later turn out to be \( M_{p_{+},p_{-}} \)-modules.

Let \( \mathcal{L} \) be the Virasoro algebra at fixed central charge

\[
c_{p_{+},p_{-}} = 1 - \frac{6(p_{+} - p_{-})^2}{p_{+}p_{-}}
\]

and let \( U(\mathcal{L}) \) be the universal enveloping algebra of \( \mathcal{L} \). The Virasoro vertex operator algebra \( \text{Vir}_{p_{+},p_{-}} \) is given by the restriction of \( \mathcal{F}_{p_{+},p_{-}} \) to the subVOA \( \text{Vir}_{p_{+},p_{-}} = (U(\mathcal{L})|0\rangle,|0\rangle,\frac{1}{2}(b_{-1}^2 - \alpha_0 b_2)|0\rangle,Y) \). Furthermore let \( \mathcal{K} U(\mathcal{L}) \) and \( \mathcal{O} U(\mathcal{L}) \) be the universal enveloping algebras of the Virasoro algebra at central charge \( c_{p_{+},p_{-}}(\varepsilon) \) over \( \mathcal{K} \) and \( \mathcal{O} \) respectively. By \( U(\mathcal{L}_{-}), \mathcal{K} U(\mathcal{L}_{-}) \) and \( \mathcal{O} U(\mathcal{L}_{-}) \) we denote the associative algebras of Virasoro generators with negative mode numbers over \( \mathbb{C}, \mathcal{K} \) and \( \mathcal{O} \) respectively.
4.1 Virasoro representation theory over $\mathcal{K}$ and $\mathcal{O}$

Let

$$\mathcal{K}K_{1,1} = \ker(S \colon \mathcal{K}F_{1,1} \to \mathcal{K}F_{-1,1}) \cap \ker(S \colon \mathcal{K}F_{1,1} \to \mathcal{K}F_{1,-1})$$
$$\mathcal{O}K_{1,1} = \ker(S \colon \mathcal{O}F_{1,1} \to \mathcal{O}F_{-1,1}) \cap \ker(S \colon \mathcal{O}F_{1,1} \to \mathcal{O}F_{1,-1}) ,$$

then $(\mathcal{O}K_{1,1}, |0\rangle, \frac{1}{2}(b_{-1}^2 + \alpha_0(\varepsilon)b_{-2})|0\rangle, Y)$ carries the structure of a VOA over $\mathcal{O}$ and is an $\mathcal{O}$-lattice of the VOA $(\mathcal{K}K_{1,1}, |0\rangle, \frac{1}{2}(b_{-1}^2 + \alpha_0(\varepsilon)b_{-2})|0\rangle, Y)$. 

**Remark 4.1.** It is known that $(\mathcal{K}K_{1,1}, |0\rangle, \frac{1}{2}(b_{-1}^2 + \alpha_0(\varepsilon)b_{-2})|0\rangle, Y)$ is isomorphic to the Virasoro VOA over $\mathcal{K}$ at central charge $c_{p_+,p_-}(\varepsilon)$. However, as we will see in Section 3, the VOA $(\mathcal{O}K_{1,1} \otimes \mathcal{O} \mathbb{C}, |0\rangle, \frac{1}{2}(b_{-1}^2 + \alpha_0(\varepsilon)b_{-2})|0\rangle, Y)$ is larger than just the Virasoro VOA at central charge $c_{p_+,p_-}$.

For each $h \in \mathcal{K}$, let $\mathcal{K}M(h)$ be the $\mathcal{K}U(\mathcal{L})$-Verma module with highest weight $h$. Then the following is well known due to Feigin and Fuchs [FF84, FF88, FF90].

**Proposition 4.2.**

1. The Verma module $\mathcal{K}M(h)$ is not simple as a $\mathcal{K}U(\mathcal{L})$ module if and only if $h = h_{r,s}(\varepsilon)$ for some $r \geq 1, s \geq 1$.

2. For each $\beta \in \mathcal{K}$, consider the left $\mathcal{K}U(\mathcal{L})$-module $\mathcal{K}F_{\beta}$, then there is a canonical $\mathcal{K}U(\mathcal{L})$-module map

$$\mathcal{K}M_{h_{\beta}} \rightarrow \mathcal{K}F_{\beta}$$
$$u_{h_{\beta}} \mapsto |\beta\rangle ,$$

where $u_{h_{\beta}}$ is the highest weight state that generates $\mathcal{K}M_{h_{\beta}}$. This map is not an isomorphism if and only if $\beta = \beta_{r,s}(\varepsilon)$ or $\beta = \beta_{-r,-s}(\varepsilon)$ for some $r \geq 1, s \geq 1$.

3. Let $r \geq 1, s \geq 1$. The sequences

$$0 \rightarrow \mathcal{K}L(h_{r,s}(\varepsilon)) \rightarrow \mathcal{K}F_{r,s} \xrightarrow{S_{[r]}^{[r]}} \mathcal{K}F_{-r,s} \rightarrow 0$$
$$0 \rightarrow \mathcal{K}L(h_{r,s}(\varepsilon)) \rightarrow \mathcal{K}F_{r,s} \xrightarrow{S_{[s]}^{[s]}} \mathcal{K}F_{r,-s} \rightarrow 0$$
$$0 \rightarrow \mathcal{K}F_{r,-s} \xrightarrow{S_{[r]}^{[r]}} \mathcal{K}F_{-r,-s} \rightarrow \mathcal{K}L(h_{r,-s}(\varepsilon)) \rightarrow 0$$
$$0 \rightarrow \mathcal{K}F_{r,-s} \xrightarrow{S_{[s]}^{[s]}} \mathcal{K}F_{r,-s} \rightarrow \mathcal{K}L(h_{r,-s}(\varepsilon)) \rightarrow 0$$

are exact as $\mathcal{K}U(L)$-modules and the screening operators $S_{[r]}^{[r]}$ and $S_{[s]}^{[s]}$ commute.

4. The maps

$$\mathcal{K}M(h_{r,s}(\varepsilon)) \rightarrow \mathcal{K}F_{r,-s} ,$$
$$\mathcal{K}F_{r,s} = \mathcal{K}F_{r,-s}^{*} \rightarrow \mathcal{K}M(h_{r,s}(\varepsilon))^{*}$$
are isomorphisms of left $\mathcal{K}U(\mathcal{L})$-modules. These $\mathcal{K}U(\mathcal{L})$-module maps preserve the $\mathcal{O}$-lattices
\[
\mathcal{O}M(h_{r,s}(\varepsilon)) \rightarrow \mathcal{O}F_{-r,-s},
\]
\[
\mathcal{O}F_{r,s} ightarrow \mathcal{O}M(h_{r,s}(\varepsilon))^*,
\]
where
\[
\mathcal{O}M(h_{r,s}(\varepsilon)) = \mathcal{O}U(\mathcal{L})u_{h_{r,s}(\varepsilon)} \subset \mathcal{K}M(h_{r,s}(\varepsilon))
\]
\[
\mathcal{O}M(h_{r,s}(\varepsilon))^* = \bigoplus_{d \geq 0} \text{Hom}_\mathcal{O}(\mathcal{O}M(h_{r,s}(\varepsilon))[h_{r,s}(\varepsilon) + d], \mathcal{O}).
\]

5. For each $r \geq 1, s \geq 1$ there exists a unique element $S_{r,s}(\kappa) \in \mathcal{O}U(\mathcal{L}-)$ such that
\[
S_{r,s}(\kappa) = (L_{-1})^{rs} + \cdots
\]
which satisfies
\[
L_n S_{r,s}(\kappa)u_{h_{r,s}(\varepsilon)} = 0, \ n \geq 1
\]
\[
L_0 S_{r,s}(\kappa)u_{h_{r,s}(\varepsilon)} = (h_{r,s}(\varepsilon) + rs)S_{r,s}(\kappa)u_{h_{r,s}(\varepsilon)}
\]
\[
h_{r,s}(\varepsilon) + rs = h_{-r,-s}(\varepsilon) = h_{-r,s}(\varepsilon).
\]

6. The conformal weight $h_{r,s}(\varepsilon) + rs = h_{r,-s}(\varepsilon) = h_{-r,s}(\varepsilon)$ is precisely the conformal weight of $S_{r,s}(\kappa)u_{h_{r,s}(\varepsilon)}$, $S^{[r]}_{\beta} |\beta_{r,-s}(\varepsilon))$ and $S^{[s]}_{\beta} |\beta_{-r,s}(\varepsilon))$. Under the identification
\[
\mathcal{K}M(h_{r,s}) \cong \mathcal{K}F_{-r,-s}
\]
these three singular vectors are proportional to each other.

4.2 The category $\mathcal{U}(\mathcal{L})$-mod of Virasoro modules at central charge $c_{p+,p-}$

The purpose of this subsection is to give a description of the category $\mathcal{U}(\mathcal{L})$-mod and to give socle sequence decompositions of Fock-modules in terms of simple Virasoro modules.

Definition 4.3. Let
\[
H = \{h_\beta | \beta \in X\}
\]
be the set of highest conformal weights. Two non-equal Heisenberg weights $\beta, \beta' \in X$ correspond to the same conformal weight if and only if $\beta' = \alpha_0 - \beta$.

Definition 4.4. 1. For $1 \leq r < p_+, 1 \leq s < p_-$, let
\[
\Delta_{r,s} = \Delta_{p_+-r,p_--s} = h_{r,s}(0).
\]
2. The Kac table $\mathcal{T}$ is the quotient set
\[ \mathcal{T} = \{(r, s) | 1 \leq r < p_+, 1 \leq s < p_- \}/ \sim, \]
where $(r, s) \sim (r', s')$ if and only if $r' = p_+ - r, s' = p_- - s$. The Kac table is the set of all classes $[(r, s)]$ such that the conformal weights $\Delta_{r,s}$ are distinct.

3. For $1 \leq r \leq p_+, 1 \leq s \leq p_-, n \geq 0$ let
\[ \Delta^+_{r,s;n} = \begin{cases} h_{p_+,p_-;2n} & r = p_+, s = p_- \\ h_{p_+-r,p_-;2n-1} & r \neq p_+, s = p_- \\ h_{p_+,p_-;2n+1} & r = p_+, s \neq p_- \\ h_{p_+-r,p_-;2n-1} & r \neq p_+, s \neq p_- \end{cases}, \quad \Delta^-_{r,s;n} = \begin{cases} h_{p_+,p_-;2n-1} & r = p_+, s = p_- \\ h_{p_+-r,p_-;2n-2} & r \neq p_+, s = p_- \\ h_{p_+,p_-;2n+2} & r = p_+, s \neq p_- \\ h_{p_+-r,p_-;2n-2} & r \neq p_+, s \neq p_- \end{cases}. \]

Definition 4.5. Let $U(\mathcal{L})$-$\text{Mod}$ be the abelian category whose morphisms are Virasoro-homomorphisms and whose objects are left $U(\mathcal{L})$ modules that satisfy the following:

1. Every object $M$ decomposes into a direct sum of generalised $L_0$ eigenspaces
\[ M = \bigoplus_{h \in \mathbb{C}} M[h], \]
\[ M[h] = \{ u \in M | \exists n \geq 1, \text{ s.t. } (L_0 - h)^n u = 0 \} \]
where $\dim M[h] < \infty$. For all $h \in \mathbb{C}$ and there are only a countable number of $h$ for which $M[h]$ is non-trivial.

2. For every object $M \in U(\mathcal{L})$-$\text{mod}$, there exists the contragredient object $M^*$
\[ M^* = \bigoplus_{h \in \mathbb{C}} \text{Hom}(M[h], \mathbb{C}), \]
on which the anti-involution $\sigma(L_n) = L_{-n}$ induces the structure of a left $U(\mathcal{L})$-module by
\[ \langle L_n \varphi, u \rangle = \langle \varphi, \sigma(L_n) u \rangle, \quad \varphi \in M^*, u \in M. \]

Note that $(M^*)^* \cong M$.

Definition 4.6. A socle series of a Virasoro module $M$, is a certain series of semi-simple modules $S_i(M)$. Let
\[ 0 = M_0 \subset M_1 \subset M_2 \subset \cdots \]
be an ascending series of submodules of $M$, such that $S_i(M) = M_i/M_{i-1}$, $i \geq 1$ is the maximal semi-simple submodule of $M_{i+1}/M_{i-1}$, that is, $M_i$ is the largest submodule of $M$ such that $M_i/M_{i-1}$ is semi-simple. The $S_i(M)$ are called the components of $M$. If there exists an element $M_n$ of the filtration, such that, $M_n = M$ and $M_{n-1} \neq M$, then we say that the socle series has length $n$ or that it has finite length.

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**Definition 4.7.** We define $U(\mathcal{L})$-mod to be the full subcategory of $U(\mathcal{L})$-Mod such that all objects $M$ in $U(\mathcal{L})$-mod satisfy the following two conditions:

1. The socle series of $M$ has finite length,

2. The conformal weights $h$ of the simple modules $L(h)$, appearing in the components of $M$, are elements of $H$.

Note that the Verma modules $M(h_r,s)$ and their duals $M(h_r,s)^*$ are not objects of $U(\mathcal{L})$-mod, however, the Fock modules $F_\beta, \beta \in X$ and the simple modules $L(h), h \in H$ are objects of $U(\mathcal{L})$-mod.

**Proposition 4.8.**

1. For each $\beta \in X$ the Fock module $F_\beta$ is an object of $U(\mathcal{L})$-mod.

2. There are four cases of socle series for the Fock modules $F_{r,s;n}$:

   (I) For $1 \leq r < p_+, 1 \leq s < p_-, n \in \mathbb{Z}$,

   \[ S_1(F_{r,s;n}) = \bigoplus_{k \geq 0} \mathcal{L}(h_{r-p_--s;n+2k+1}), \]

   \[ S_2(F_{r,s;n}) = \bigoplus_{k \geq a} \mathcal{L}(h_{r,s;n+2k}) \]

   \[ \quad \oplus \bigoplus_{k \geq 1-a} \mathcal{L}(h_{p_+--r-p_--s;n+2k+1}), \]

   \[ S_3(F_{r,s;n}) = \bigoplus_{k \geq 0} \mathcal{L}(h_{p_+--r,s;n+2k+1}), \]

   where $a = 0$ if $n \geq 0$ and $a = 1$ if $n < 0$.

   (II+) For $1 \leq s < p_-, n \in \mathbb{Z}$,

   \[ S_1(F_{p_+,s;n}) = \bigoplus_{k \geq 0} \mathcal{L}(h_{p_+,p_--s;n+2k+1}), \]

   \[ S_2(F_{p_+,s;n}) = \bigoplus_{k \geq a} \mathcal{L}(h_{p_+,s;n+2k}), \]

   where $a = 0$ if $n \geq 1$ and $a = 1$ if $n < 1$.

   (II-) For $1 \leq r < p_+, n \in \mathbb{Z}$,

   \[ S_1(F_{r,p_-;n}) = \bigoplus_{k \geq 0} \mathcal{L}(h_{r,p_-;n+2k}), \]

   \[ S_2(F_{r,p_-;n}) = \bigoplus_{k \geq a} \mathcal{L}(h_{p_+--r,p_-;n+2k-1}), \]

   where $a = 1$ if $n \geq 1$ and $a = 0$ if $n < 1$. 

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Proposition 4.10. For $n \in \mathbb{Z}$, the Fock space $F_{p_+, p_-; n}$ is semi-simple as a Virasoro module

$$S_1(F_{p_+, p_-; n}) = F_{p_+, p_-; n} = \bigoplus_{k \geq 0} \mathcal{L}(h_{p_+, p_-; n} + 2k).$$

Figure 1 visualises the socle sequence decomposition of Fock modules.

4.3 Kernels and images of screening operators

In this subsection we give the socle sequences of the kernels and images of the screening operators $S_+^{[r]}$, $S_-^{[s]}$.

Definition 4.9. We denote the kernels and images of screening operators $S_+^{[r]}$, $S_-^{[s]}$ by

1. For $1 \leq r < p_+$, $1 \leq s \leq p_-$, $n \in \mathbb{Z}$

$$K_{r,s;n;+} = \ker S_+^{[r]} : F_{r,s;n} \to F_{p_+, r,s;n+1}$$
$$X_{p_+, r,s;n+1;+} = \im S_+^{[r]} : F_{r,s;n} \to F_{p_+, r,s;n+1}$$

2. For $1 \leq r \leq p_+$, $1 \leq s < p_-$, $n \in \mathbb{Z}$

$$K_{r,s;n;-} = \ker S_-^{[s]} : F_{r,s;n} \to F_{r,p_-, s;n-1}$$
$$X_{r,p_-, s;n-1;-} = \im S_-^{[s]} : F_{r,s;n} \to F_{r,p_-, s;n-1}$$

3. For $1 \leq r < p_+$, $1 \leq s < p_-$, $n \in \mathbb{Z}$

$$K_{r,s;n} = K_{r,s;n;+} \cap K_{r,s;n;-}$$
$$X_{r,s;n} = X_{r,s;n;+} \cap X_{r,s;n;-}$$
$$K_{p_+, s;n} = K_{p_+, s;n;+}$$
$$K_{p_+, p_-; n} = X_{p_+, p_-; n} = F_{p_+, p_-; n}.$$
Figure 1: The socle sequences of Fock modules: The arrows indicate which states one can reach by acting with the Virasoro algebra, that is, an arrow $v \rightarrow w$ means $w \in U(L)v$.  

(1) $F_{r,s;n}$, $1 \leq r < p_+$, $1 \leq s < p_-$

$n \geq 0$

$\square = h_{p_+ - r, s; n + k}, \quad \Box = h_{r, s; n + k}, \quad \framebox{1} = h_{p_+ - r, p_- - s; n + k}, \quad \framebox{2} = h_{r, p_- - s; n + k}$

$n \leq 0$

$\square = h_{p_+ - r, s; - n + k}, \quad \Box = h_{p_+ - p_-, s; - n + k}, \quad \framebox{1} = h_{r, s; - n + k}, \quad \framebox{2} = h_{r, p_- - s; - n + k}$

(II$_-$) $F_{p_+, s;n}$, $1 \leq s < p_-$

$n \geq 1$

$\triangle = h_{p_+, s; n + k}, \quad \square = h_{p_+, p_-, s; n + k}$

$n < 1$

$\triangle = h_{p_+, p_-, s; - n + k}, \quad \square = h_{p_+, s; - n + k}$

(II$_+$) $F_{r, p_-; n}$, $1 \leq r < p_+$

$n \geq 0$

$\triangle = h_{p_+ - r, p_-, n + k}, \quad \square = h_{r, p_-; n + k}$

$n < 0$

$\triangle = h_{r, p_-; - n + k}, \quad \square = h_{p_+ - r, p_-, - n + k}$
and those of the screening operator $S_-$ by

$$n \geq 1 \quad S_1(K_{r,s;n+1-}) = \bigoplus_{k \geq 1} L(h_{r,p_- - s,n+2k-1}), \quad S_1(K_{r,s;n-1}) = \bigoplus_{k \geq 1} L(h_{r,p_- - s,n+2k-1}),$$

$$n \leq 0 \quad S_2(K_{r,s;n+1-}) = \bigoplus_{k \geq 1} L(h_{p_+ - r,p_- - s,n+2k}), \quad S_2(K_{r,s;n-1}) = \bigoplus_{k \geq 1} L(h_{p_+ - r,p_- - s,n+2k(k-1)}),$$

$$S_1(X_{r,s;n+1-}) = \bigoplus_{k \geq 1} L(h_{r,p_- - s,n+2k-1}), \quad S_1(X_{r,s;n+1-}) = \bigoplus_{k \geq 1} L(h_{p_+ - r,p_- - s,n+2k}),$$

$$S_2(X_{r,s;n+1-}) = \bigoplus_{k \geq 1} L(h_{p_+ - r,p_- - s,n+2k-1}), \quad S_2(X_{r,s;n+1-}) = \bigoplus_{k \geq 1} L(h_{p_+ - r,p_- - s,n+2k-1}).$$

For $r = p_+$ or $s = p_-$ the kernels and images are semisimple and we have

$$K_{r,p_-; n+1} = X_{r,p_-; n+1} = S_1(F_{r,p_-; n}), \quad K_{p_+, s;n; n-1} = X_{p_+, s;n; n-1} = S_1(F_{p_+, s;n}).$$

The following proposition is due to Felder [Fel89].

**Proposition 4.11.** 1. For $1 \leq r < p_+$, $1 \leq s \leq p_-$ and $n \in \mathbb{Z}$ the screening operator $S_+$ defines the Felder complex

$$\cdots \rightarrow s_t^{[r]} \rightarrow F_{r,p_- - s,n+1} \rightarrow s_t^{[p_+ - r]} \rightarrow F_{r,p_- - s,n} \rightarrow s_t^{[r]} \rightarrow F_{r,p_- - s,n+1} \rightarrow \cdots.$$  

This complex is exact for $s = p_-$. For $1 \leq s < p_-$ it is exact everywhere except in $F_{r,s;0}$, where the cohomology is given by

$$K_{r,s;0; n+1}/X_{r,s;0; n} \cong L(h_{r,s; 0}).$$

2. For $1 \leq r \leq p_+$, $1 \leq s < p_-$ and $n \in \mathbb{Z}$ the screening operator $S_-$ defines the Felder complex

$$\cdots \rightarrow s_t^{[r]} \rightarrow F_{r,p_- - s,n+1} \rightarrow s_t^{[p_+ - r]} \rightarrow F_{r,p_- - s,n} \rightarrow s_t^{[r]} \rightarrow F_{r,p_- - s,n+1} \rightarrow \cdots.$$  

This complex is exact for $r = p_+$. For $1 \leq r < p_+$ it is exact everywhere except in $F_{r,s;0}$, where the cohomology is given by

$$K_{r,s;0; n}/X_{r,s;0; n} \cong L(h_{r,s; 0}).$$

**Theorem A.** Let $1 \leq r \leq p_+$, $1 \leq s \leq p_-$, all singular vectors of the Fock modules $F_{r,s; n}$ can be expressed as the images of the screening operators $S_+$ and $S_-^\dagger$.

1. The singular vectors at conformal weights $h_{r,p_- - s;n|n+2k-1}$, $k \geq 0$ are given by

$$S_{+ (k+n+1)p_+ - r} | \beta_{p_+ - r,s; 1 - 2k} \rangle \in F_{r,s; n}$$

for $n \geq 0$ and

$$S_{+ (k+1)p_+ - r} | \beta_{p_+ - r,s; 1 - 2k+n} \rangle \in F_{r,s; n}$$

for $n \leq 0$. 

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2. The singular vectors at conformal weights \( h_{r,p_-;n+2k-1} \), \( k \geq 0 \) are given by
\[
S_{\pm}^{[(k+1)p_-;n]} \beta_{r,p_-;2k+n+1} \in F_{r,s;n}
\]
for \( n \geq 0 \) and
\[
S_{\pm}^{[(k-n+1)p_-;n]} \beta_{r,p_-;2k-n+1} \in F_{r,s;n}
\]
for \( n \leq 0 \).

Proof. The formulae for the singular vectors follow from Proposition 3.24. The fact that these lists exhaust all singular vectors follows from the socle sequence decompositions in Proposition 4.8.

4.4 From \( U(\mathcal{L})\)-mod to \( \mathcal{M}_{p_+,p_-}\)-mod

The purpose of this subsection is to define certain infinite direct sums of kernels and images of \( S_+^{[s]} \), \( S_-^{[s]} \) which will later turn out to be modules of \( \mathcal{M}_{p_+,p_-} \).

Definition 4.12.

1. For \( 1 \leq r < p_+, 1 \leq s < p_- \) and \( n \geq 0 \), let \( K_{r,s}^\pm \) and \( \mathcal{X}_{r,s}^\pm \) be the direct sums
\[
K_{r,s}^+ = \ker S_+^{[r]} |_{V_+^{r,s}} \cap \ker S_-^{[s]} |_{V_-^{r,s}} = \bigoplus_{n \in \mathbb{Z}} K_{r,s;2n},
\]
\[
\mathcal{X}_{r,s}^+ = \operatorname{im} S_+^{[p_+-r]} |_{V_+^{p_+,r,s}} \cap \operatorname{im} S_-^{[p_-;s]} |_{V_-^{r,p_-;s}} = \bigoplus_{n \in \mathbb{Z}} \mathcal{X}_{r,s;2n}
\]
\[
K_{r,s}^- = \ker S_+^{[r]} |_{V_+^{r,s}} \cap \ker S_-^{[s]} |_{V_-^{r,s}} = \bigoplus_{n \in \mathbb{Z}} K_{r,s;2n+1},
\]
\[
\mathcal{X}_{r,s}^- = \operatorname{im} S_+^{[p_+-r]} |_{V_+^{p_+,r,s}} \cap \operatorname{im} S_-^{[p_-;s]} |_{V_-^{r,p_-;s}} = \bigoplus_{n \in \mathbb{Z}} \mathcal{X}_{r,s;2n+1}
\]
\[
K_{r,p_-}^+ = \mathcal{X}_{r,p_-}^+ = \ker S_+^{[r]} |_{V_+^{r,p_-}} = \bigoplus_{n \in \mathbb{Z}} K_{r,p_-;2n}, \quad K_{r,p_-}^- = \mathcal{X}_{r,p_-}^- = \ker S_+^{[r]} |_{V_+^{r,p_-}} = \bigoplus_{n \in \mathbb{Z}} K_{r,p_-;2n+1},
\]
\[
K_{p_+,s}^+ = \mathcal{X}_{p_+,s}^+ = \ker S_-^{[s]} |_{V_+^{p_+,s}} = \bigoplus_{n \in \mathbb{Z}} K_{p_+,s;2n}, \quad K_{p_+,s}^- = \mathcal{X}_{p_+,s}^- = \ker S_-^{[s]} |_{V_+^{p_+,s}} = \bigoplus_{n \in \mathbb{Z}} K_{p_+,s;2n+1},
\]
\[
K_{p_+,p_-}^+ = \mathcal{X}_{p_+,p_-}^+ = \bigoplus_{n \in \mathbb{Z}} F_{p_+,p_-;2n}, \quad K_{p_+,p_-}^- = \mathcal{X}_{p_+,p_-}^- = \bigoplus_{n \in \mathbb{Z}} F_{p_+,p_-;2n+1}.
\]

2. For \( 1 \leq r \leq p_+, 1 \leq s \leq p_- \) and \( n \geq 0 \), let \( V_{r,s;n}^\pm \) be the spaces of singular vectors of conformal weight \( \Delta_{r,s;n}^\pm \)
\[
V_{r,s;n}^\pm = \{ u \in \mathcal{X}_{r,s}^\pm | L_0 u = \Delta_{r,s;n}^\pm u \}.
\]

We call these spaces soliton sectors. The elements of \( V_{r,s;n}^\pm \) are called soliton vectors.
Note that by Theorem A it is easy to give explicit bases of the soliton sector $V_{r,s;n}^\pm$ in terms of $S_+^+$ and $S_-^-$.

**Proposition 4.13.** As Virasoro modules the spaces $K_{r,s}^\pm$ and $X_{r,s}^\pm$ decompose as

1. For $1 \leq r < p_+$ and $1 \leq s < p_-$
   
   $$K_{r,s}^+ = U(\mathcal{L})|_{\beta_{r,s;0}} \oplus \bigoplus_{n \geq 1} L(\Delta_{r,s;\pm n}^+) \otimes V_{r,s;n}^+$$
   
   $$X_{r,s}^+ = \bigoplus_{n \geq 0} L(\Delta_{r,s;+n}^+) \otimes V_{r,s;n}^+$$
   
   $$K_{r,s}^- = X_{r,s}^- = \bigoplus_{n \geq 0} L(\Delta_{r,s;-n}^-) \otimes V_{r,s;n}^-$$

2. For $r = p_+$ and $1 \leq s < p_-$
   
   $$K_{p+,s}^+ = X_{p+,s}^+ = \bigoplus_{n \geq 0} L(\Delta_{p+,s;+n}^+) \otimes V_{p+,s;n}^+$$
   
   $$K_{p+,s}^- = X_{p+,s}^- = \bigoplus_{n \geq 0} L(\Delta_{p+,s;-n}^-) \otimes V_{p+,s;n}^-$$

3. For $1 \leq r < p_+$ and $s = p_-$
   
   $$K_{r,p-}^+ = X_{r,p-}^+ = \bigoplus_{n \geq 0} L(\Delta_{r,p-;+n}^+) \otimes V_{r,p-;n}^+$$
   
   $$K_{r,p-}^- = X_{r,p-}^- = \bigoplus_{n \geq 0} L(\Delta_{r,p-;-n}^-) \otimes V_{r,p-;n}^-$$

4. For $r = p_+$ and $s = p_-$
   
   $$K_{p+,p-}^+ = X_{p+,p-}^+ = \bigoplus_{n \geq 0} L(\Delta_{p+,p-;+n}^+) \otimes V_{p+,p-;n}^+$$
   
   $$K_{p+,p-}^- = X_{p+,p-}^- = \bigoplus_{n \geq 0} L(\Delta_{p+,p-;-n}^-) \otimes V_{p+,p-;n}^-$$

**Proposition 4.14.** For $1 \leq r < p_+$, $1 \leq s < p_-$, the $K_{r,s}^+$ satisfy the following exact sequence as Virasoro modules

$$0 \rightarrow X_{r,s}^+ \rightarrow K_{r,s}^+ \rightarrow L(h_{r,s;0}) \rightarrow 0.$$

### 4.5 Frobenius homomorphisms

In this section we introduce a class of Virasoro homomorphisms

$$E, F \in \text{End}_C(K_{r,s}^\pm), \ 1 \leq r < p_+, 1 \leq s < p_-$$
such that $E$ and $F$ define derivations of the universal enveloping algebra $U(\mathcal{M}_{p_{+},p_{-}})$. We call $E$ and $F$ Frobenius homomorphisms. These Frobenius homomorphisms will prove to be essential tools for analysing the VOA structure of $\mathcal{M}_{p_{+},p_{-}}$ and for analysing the action of $\mathcal{M}_{p_{+},p_{-}}$ on $K_{r,s}^{\pm}$.

Recall that by the Felder complexes the maps

$$S_{+}^{[p_{+}]} \circ S_{+}^{[r]} : F_{r,s;n} \to F_{r,s;n+2}, \quad S_{-}^{[p_{-}]} \circ S_{-}^{[s]} : F_{r,s;n} \to F_{r,s;n-2}$$

are zero. The Frobenius homomorphisms are regularisations of these maps, such that they become non-trivial.

We give the details of the construction for $S_{+}$, the $S_{-}$ case follows in the same way. At first we fix $1 \leq r \leq p_{+}$ and $s \in \mathbb{Z}$ and consider the action of $S_{+}(z_{1}) \cdots S_{+}(z_{p_{+}})$ on $K F_{r,s}$

$$S_{+}(z_{1}) \cdots S_{+}(z_{p_{+}}) = e^{p_{+}\alpha(z)} U_{p_{+}}(z;\kappa_{+}(z),r) \prod_{i=1}^{p_{+}} z_{i}^{s-1} : \prod_{i=1}^{p_{+}} S_{+}(z_{i}) :$$

$U_{p_{+}}(z;\kappa_{+}(z),r) = \prod_{1 \leq i \neq j \leq p_{+}} (z_{i} - z_{j})^{\kappa_{+}(z)} \prod_{i=1}^{p_{+}} z_{i}^{(1-r)\kappa_{+}(z)}.$

Let $K_{p_{+}}(\kappa_{+}(z),r)$ be the local system on $X_{p_{+}}$ over $K$ defined by the multivaluedness of $U_{p_{+}}(z;\kappa_{+}(z),r)$ and let $K_{p_{+}}^{\vee}(\kappa_{+}(z),r)$ be its dual. For $r = p_{+}$ we can integrate the above screening operators over the cycle $[\Gamma_{p_{+}}(\kappa_{+})]$ to obtain $S_{+}^{[p_{+}]}$, however, for $r < p_{+}$ the homology group $H_{\kappa_{+}}(X_{p_{+}},K_{p_{+}}^{\vee}(\kappa_{+}(z),r))$ is trivial and no such cycle exists. We remedy this problem by changing the domain of the $z_{i}$ to

$$Y_{p_{+}} = \{(z_{1},\ldots,z_{p_{+}}) \in \mathbb{C}^{p_{+}} | z_{i} \neq z_{j}, z_{i} \neq 0, 1 \} \subset X_{p_{+}},$$

then it is known that $[\text{AKP}]$

$$\dim_{K} H_{p_{+}}^{p_{+}}(Y_{p_{+}},K_{p_{+}}(\kappa_{+}(z),r)) = p_{+}!, \quad \dim_{K} H_{p_{+}}^{p_{+}}(Y_{p_{+}},K_{p_{+}}^{\vee}(\kappa_{+}(z),r))^{S_{+}^{r}} = 1, \quad \dim_{K} H_{p_{+}}^{p_{+}}(Y_{p_{+}},K_{p_{+}}^{\vee}(\kappa_{+}(z),r))^{p_{+}} = p_{+}!.$$

We introduce new variable names $w_{i} = z_{i+p_{+}-r}, \ i = 1,\ldots,r$ and consider the open domain $U_{1}^{p_{+}-r} \times U_{2}^{r} \subset Y_{p_{+}}$

$$U_{1}^{p_{+}-r} = \{(z_{1},\ldots,z_{p_{+}-r}) \in \mathbb{C}^{p_{+}-r} | z_{i} \neq z_{j}, z_{i} \neq 1, |z_{i}| > \delta \}, \quad U_{2}^{r} = \{(w_{1},\ldots,w_{r}) \in \mathbb{C}^{r} | w_{i} \neq w_{j}, w_{i} \neq 0, \delta > |w_{i}| \} ,$$

where $1 > \delta > 0$ is a fixed real constant, and define the multivalued holomorphic functions

$$G_{p_{+}-r}(z;\kappa_{+}(z)) = \prod_{1 \leq i \neq j \leq p_{+}-r} (z_{i} - z_{j})^{\kappa_{+}(z)} \prod_{i=1}^{p_{+}-r} z_{i}^{(1+r)\kappa_{+}(z)},$$

$$U_{r}(w;\kappa_{+}(z)) = \prod_{1 \leq i \neq j \leq r} (w_{i} - w_{j})^{\kappa_{+}(z)} \prod_{i=1}^{r} w_{i}^{(1-r)\kappa_{+}(z)}.$$
on $U_1^{p_+-r} \times U_2^r$ respectively. Then on $U_1^{p_+-r} \times U_2^r$ the product $S_+(z_1) \cdots S_+(z_{p_+-r})S_+(w_1) \cdots S_+(w_r)$ factorises as

$$\prod_{i=1}^{p_+-r} S_+(z_i) \prod_{i=1}^r S_+(w_i) = e^{p_+ \alpha_+(\epsilon)} G_{p_+-r}(z; \kappa_+(\epsilon)) \prod_{i=1}^{p_+-r} z_i^{s_i-1} : \prod_{i=1}^r S_+(w_i) : .$$

Next we consider the local system $\mathcal{K}G_{p_+-r}(\kappa_+(\epsilon))$ on $U_1^{p_+-r}$ defined by $G_{p_+-r}(z; \kappa_+(\epsilon))$ and the local system $\mathcal{K}L_r(\kappa_+(\epsilon))$ on $U_2^r$ defined by $U_r(w; \kappa_+(\epsilon))$. We define regularised cycles

$$[\Gamma_{p_+-r}^\infty(\kappa_+(\epsilon))] \in H_{p_+-r}(U_1^{p_+-r}; \mathcal{K}G_{p_+-r}^\vee(\kappa_+(\epsilon)))$$
$$[\Gamma_r(\kappa_+(\epsilon))] \in H_r(U_2^r; \mathcal{K}L_r^\vee(\kappa_+(\epsilon))),$$

where the regularised cycles $[\Gamma_r(\kappa_+(\epsilon))]$ are those of Definition 3.6. We define $[\Gamma_{p_+-r}^\infty(\kappa_+(\epsilon))]$ as

$$[\Gamma_{p_+-r}^\infty(\kappa_+(\epsilon))] = \frac{1}{c_r^\infty(\kappa_+(\epsilon))}\Delta_{p_+-r}^\infty; \varphi,$$

where $\Delta_{p_+-r}^\infty = \{ \infty > z_1 > \cdots > z_{p_+-r} > 1 \}$, $\varphi$ is the principal branch of $G_{p_+-r}(z; \kappa_+(\epsilon))$ and

$$c_r^\infty(\kappa_+(\epsilon)) = \int_{[\Gamma_r(\kappa_+(\epsilon))] \setminus \{ 1 \}} \prod_{1 \leq i \neq j \leq p_+-r} (z_i - z_j)^{\kappa_+(\epsilon)} \prod_{i=1}^{p_+-r} z_i^{2\kappa_+(\epsilon)} \frac{d z_1 \cdots d z_{p_+-r}}{z_1 \cdots z_{p_+-r}} \prod_{i=1}^{r-1} \frac{y_i \cdots d y_{r-1}}{y_1 \cdots y_{r-1}},$$

where $[\Gamma_r(\kappa_+(\epsilon))]$ is the regularised cycle of Definition 3.6. By the change of variables $u_i = 1/z_i$ one can evaluate $c_r^\infty(\kappa_+(\epsilon))$ by means of Selberg integral

$$c_r^\infty(\kappa_+(\epsilon)) = \frac{S_{p_+-r}((1-2p_+)\kappa_+(\epsilon), 0, \kappa_+(\epsilon))S_r(2\kappa_+(\epsilon), 0, \kappa_+(\epsilon))}{S_{r-1}((1-r)\kappa_+(\epsilon), 2\kappa_+(\epsilon), \kappa_+(\epsilon))} .$$

We define the cycle $[\Gamma_{p_+-r}(\kappa_+(\epsilon))] \in H_{p_+-r}(Y_{p_+}; \mathcal{K}L_{p_+}^\vee(\kappa_+(\epsilon), r))$ as the image of

$$[\Gamma_{p_+-r}^\infty(\kappa_+(\epsilon))] \otimes [\Gamma_r(\kappa_+(\epsilon))] \in H_{p_+-r}(U_1^{p_+-r}; \mathcal{K}G_{p_+-r}^\vee(\kappa_+(\epsilon))) \otimes H_r(U_2^r; \mathcal{K}L_r^\vee(\kappa_+(\epsilon)))$$

under the map

$$H_{p_+-r}(U_1^{p_+-r}; \mathcal{K}G_{p_+-r}^\vee(\kappa_+(\epsilon))) \otimes H_r(U_2^r; \mathcal{K}L_r^\vee(\kappa_+(\epsilon))) \rightarrow H_{p_+}(Y_{p_+}; \mathcal{K}L_{p_+}(\kappa_+(\epsilon), r)) .$$

Definition 4.15.

1. For $1 \leq r \leq p_+$, $s \in \mathbb{Z}$, the Frobenius operator $E$ associated to $S_+$ is the map $E : \mathcal{K}F_{r,s} \rightarrow \mathcal{K}F_{r-2p_+,s}$, where

$$E = \int_{[\Gamma_{p_+-r}(\kappa_+(\epsilon))] \setminus \{ 1 \}} \prod_{i=1}^{p_+-r} S_+(z_i) \prod_{i=1}^r S_+(w_i) \prod_{i=1}^{p_+-r} d z_i \prod_{i=1}^r d w_i .$$

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2. For \(1 \leq s \leq p_-, r \in \mathbb{Z}\), the Frobenius operator \(F\) associated to \(S_-\) is the map \(F : \kappa F_{r,s} \to \kappa F_{r,s-2p_-}\), where

\[
F = \int_{[\Gamma_{p_-,s}(\kappa_-(\varepsilon))]} \prod_{i=1}^{p_- - s} S_-(z_i) \prod_{i=1}^s S_-(w_i) \prod_{i=1}^{p_- - s} dz_i \prod_{i=1}^s dw_i.
\]

**Theorem B.** For \(1 \leq r \leq p_+, 1 \leq s \leq p_-\) and \(n \in \mathbb{Z}\) the Frobenius homomorphisms \(E\) and \(F\) induce well defined maps over \(\mathbb{C}\)

\[
E : K_{r,s;n+} \to K_{r,s;n+2+} \quad \quad F : K_{r,s;n-} \to K_{r,s;n-2-}
\]

that satisfy the following properties:

1. The maps \(E\) and \(F\) are Virasoro homomorphisms.
2. The maps \(E\) and \(F\) act transitively on the soliton sectors, that is, for \(m \geq 0, l \geq 1\)

\[
ES_+^{(m+1)p_+,r} |_{\beta_{p_-,r,s;-m-1-l}} = \text{const} \ S_+^{(m+2)p_+,r} |_{\beta_{p_-,r,s;-m-1-l}}
\]

\[
FS_+^{(m+1)p_+,s} |_{\beta_{r,-p_-;s+m+1+l}} = \text{const} \ S_+^{(m+2)p_+,s} |_{\beta_{r,-p_-;s+m+1+l}}
\]

where the constants are non-zero complex numbers.
3. The maps \(E\) and \(F\) are derivations, that is, for \(A \in K_{1,1;2n+}, B \in K_{r,s;m+}\) and \(m, n \in \mathbb{Z}\)

\[
EY(A;u)B = Y(EA;u)B + Y(A;u)EB\; ,
\]

while for \(C \in K_{1,1;2n-}, D \in K_{r,s;m-}\)

\[
FY(C;u)D = Y(FC;u)D + Y(C;u)FD\; .
\]

**Proof.** We prove the Theorem for \(E\), since \(F\) follows in the same way. We already know from our previous calculations regarding the screening operator \(S_+^{(\varepsilon)}\), that integrating the \(w_i\) coordinates over the cycle \([\Gamma_{r}(\kappa_+^{(\varepsilon)})]\) is well defined over \(\mathbb{C}\) after setting \(\varepsilon = 0\). In order to show that \(E\) is well defined over \(\mathbb{C}\), we must consider integrals of the form

\[
\int_{[\Gamma_{p_-,r}(\kappa_+^{(\varepsilon)})]} \prod_{1 \leq i \neq j \leq p_+ - r} (z_i - z_j)^{\kappa_+^{(\varepsilon)}(i)} \prod_i z_i^{(1+r)\kappa_+^{(\varepsilon)} + k} f(z) \prod_{i=1}^{p_+ - r} \frac{dz_1 \cdots dz_{p_+ - r}}{z_1 \cdots z_{r}}
\]

for \(k \in \mathbb{Z}\) and \(f(z) \in O[z_1^\pm, \ldots, z_{p_+ - r}^\pm] S_{p+ - r}^{(\varepsilon)}\). By appropriately shifting \(k\) we can assume without loss of generality that \(f(z) = Q_{\lambda}(z^{-1}; \kappa_-(\varepsilon))\) is a dual Jack polynomial in \(z_i^{-1}\) and \(\lambda\) is a partition of length at most \(p_+ - r - 1\). If we perform a change of variables \(u_i = 1/z_i\), then we can evaluate this expression using the Kaudell integral formula in Proposition 3.20

\[
\int_{[\Gamma_{p_-,r}(\kappa_+^{(\varepsilon)})]} \prod_{1 \leq i \neq j \leq p_+ - r} (z_i - z_j)^{\kappa_+^{(\varepsilon)}(i)} \prod_i z_i^{(1+r)\kappa_+^{(\varepsilon)} + k} Q_{\lambda}(u; \kappa_-(\varepsilon)) \prod_{i=1}^{p_+ - r} \frac{du_1 \cdots du_{p_+ - r}}{u_1 \cdots u_{r}}
\]

\[
= I_{p_+ - r}((1 - 2p_+ + r)\kappa_+^{(\varepsilon)} - k, 0, \kappa_+^{(\varepsilon)}) \text{ } \epsilon^{-1}O_{X}^{(\kappa_+^{(\varepsilon)})} \begin{cases} \varepsilon & k = -p_+ \text{ and } \lambda = 0 \\ 0 & \text{else} \end{cases}
\]
It thus follows that $E : \mathcal{O} F_{r,s} \to \varepsilon^{-1} \mathcal{O} F_{r-2p_+,s}$. Furthermore, if $A \in K_{r,s;n;\pm}$ then for any lift $\tilde{A} \in \mathcal{O} F_{r,s}$, we have
\[
\int_{[\Gamma_r(\kappa_+(\varepsilon))]} \prod_{i=1}^r S_+(w_i) \tilde{A} \prod_{i=1}^r d w_i \in \varepsilon \mathcal{O} F_{-r,s}
\]
Therefore it follows that $E$ induces a well defined map $E : K_{r,s;n;\pm} \to F_{r,s;n;\mp}$ over $\mathbb{C}$.

Next we show that $E$ commutes with the Virasoro algebra. Let $A \in K_{r,s;n;\pm}$ and let $\tilde{A} \in \mathcal{O} F_{r,s}$ be a lift of $A$. Recall that
\[
[L_m, S_+(z_i)] = z_i^m (z_i \partial z_i + n + 1) S_+(z_i).
\]
Then by the formulae for Kadell integrals it follows that when $m \neq 0$
\[
\int_{[\Gamma_r(\kappa_+(\varepsilon))]} \prod_{i=1}^r S_+(w_i) \tilde{A} \prod_{i=1}^r d w_i = \frac{1}{\mathcal{O}} \left( \sum_{i=1}^r z_i^m (z_i \partial z_i + m + 1) \prod_{1 \leq i \neq j \leq r} (z_i - z_j)^{\kappa_+(\varepsilon)} \prod_{i=1}^{p_+ - r} z_i^{(1+r)\kappa_+(\varepsilon) + k} \quad \times Q_\lambda(z^{-1}; \kappa_-(\varepsilon)) \frac{d z_1 \cdots d z_{p_+ - r}}{z_1 \cdots z_r} \right) \in \mathcal{O}.
\]
for all partitions $\lambda$ and $k \in \mathbb{Z}$. Thus it follows that $[L_m, E] \tilde{A} \in \varepsilon \mathcal{O} F_{-2p_+,s}$ for $m \neq 0$. The case for $L_0$ follows by using the Jaccobi identity and $[L_1, L_{-1}] = 2L_0$. Since $E$ is a Virasoro homomorphism, it follows that $im E(K_{r,s;n;\pm}) \subset K_{r,s;n;\pm}$ from the socle sequence decompositions of Proposition 4.10.

Next we prove that $E$ acts transitively on soliton vectors. In order to show that
\[
ES_+^{(m+1)p_+ - r} |\beta_{p_+-r,s;\pm;m-1-l} = \text{const } S_+^{(m+2)p_+ - r} |\beta_{p_+-r,s;\pm;m-1-l}
\]
it is sufficient to show that $ES_+^{(m+1)p_+ - r} |\beta_{p_+-r,s;\pm;m-1-l} \neq 0$, since singular vectors of a given conformal weight are unique up to normalisation and, due to $E$ being a Virasoro homomorphism, $E$ maps singular vectors to singular vectors. We consider the the matrix element
\[
\langle \beta_{r,s+k_1-k_2;m+l+2} | \prod_{i=1}^{k_1} V_{-2p}(v_i) V_{-k_2 \alpha_-/2} \prod_{i=1}^{k_2} S_+(z_i) S_+(w_i) \rangle E S_+^{(m+1)p_+ - r} |\beta_{p_+-r,s;\pm;m-1-l} \rangle,
\]
k_1 = (m + 1)p_-, k_2 = (l + 1)p_--s and we assume that $v_i, u$ lie on the positive imaginary axis, satisfying $|v_{k_1}| > \cdots > |v_1| > |u| > 1$. A lift of this matrix element to $\mathcal{O}$ is given by
\[
M(u, v) = \int_{[\Gamma_{p_+}(\kappa_+(\varepsilon))]} \langle \beta_{r-2p_+,s+k_1-k_2+m-l}(\varepsilon) | \prod_{i=1}^{k_1} V_{-2p}(v_i) V_{-k_2 \alpha_-/2} \prod_{i=1}^{k_2} S_+(z_i) S_+(w_i) \rangle
\]
\[
\times \rho_2(\alpha_+(\varepsilon)) Q_\lambda(m+1)p_+ - r, (l+1)p_+ - s (x; \kappa_-(\varepsilon)) \langle \beta_{r,s+(m-1)p_+}(\varepsilon) | \prod_{i=1}^{k_1} d z_i \prod_{i=1}^{k_2} d w_i \rangle,
\]

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where \( \rho \) is the map introduced for Proposition 3.34. Multiplying this matrix element by

\[
\prod_{i=1}^{k_1} \frac{1}{v_i} \frac{1}{u} \alpha_{-}(e) \beta_{r,+}(m-t)_{-}(e) \frac{2}{k_2} \alpha_{-}(e) \beta_{r,+}(m-t)_{-}(e) + ((m+1)p_+ - r)((\ell+1)p_- - s) \\
\times \prod_{1 \leq i \neq j \leq k_1} (v_i - v_j)^{-\frac{\kappa_{-}(e)}{4}} \prod_{i=1}^{k_1} (v_i - u)^{k_2 \frac{\kappa_{-}(e)}{4}} ,
\]

and then taking the limit \( u \to 0 \) along the positive imaginary axis the integrand becomes

\[
\tilde{M}(v) = (-1)^{p_+ k_2} \prod_{i=1}^{k_1} \prod_{j=1}^{p_+ - r} (v_i - z_j)^{-1} \prod_{i=1}^{k_1} \prod_{j=1}^{r} (v_i - w_j)^{-1} \\
\Xi_{-k_1 \kappa_{-}(e)} \left(Q_{\lambda(m+1)p_+ - r, (\ell+1)p_- - s} (x; \kappa_{-}(e))\right) \\
\times U_{p_+ - r}(z; \kappa_{+}(e)) \prod_{i=1}^{p_+ - r} (z_i + 2)_{\kappa_{+}(e)} \prod_{i=1}^{p_+ - r} \frac{d z_i}{z_i} \prod_{i=1}^{p_+ - r} \frac{d w_i}{w_i} ,
\]

where \( \Xi \) is the map introduced in Proposition 3.15. Next we take the limit \( v_1 \to 0 \) along the positive imaginary axis and then \( v_2 \to 0 \) and so on. As the \( v_i \) become sufficiently small the integrand \( \tilde{M}(v) \) picks up degree 1 poles at \( w_i = v_j \). Upon evaluating the residues at these poles, one sees that they do not contribute in the limit \( v_i \to 0 \) and it therefore follows that

\[
\lim_{v_{k_1} \to 0} \cdots \lim_{v_1 \to 0} \int_{[\Gamma_{p_+ - r}(\kappa_{+}(e))]} \tilde{M}(v) = \\
(-1)^{p_+(k_2 - k_1)} \Xi_{-k_1 \kappa_{-}(e)} \left(Q_{\lambda(m+1)p_+ - r, (\ell+1)p_- - s} (x; \kappa_{-}(e))\right) \frac{S_{p_+ - r}((1 - 2p_+ + r)_{\kappa_{+}(e)}, 0, \kappa_{+}(e))}{e^{\infty}_{\kappa_{+}(e)}} .
\]

This lies in \( V^\times \) and thus \( E_{\infty}^{\ell + (m+1)p_+ - r}(\beta_{p_+ - r,s;-(m+1+\ell)}) \) is non-zero.

Finally we prove the derivation property of \( E \). Let \( A \in K_{1,1+2n:+}; B \in K_{r,s;2m:+}, n, m \in \mathbb{Z} \) and let \( \bar{A} \in \mathcal{O}K_{1,1+2n:+}; \bar{B} \in \mathcal{O}K_{r,s+2m:+} \) be lifts of \( A \) and \( B \). Then

\[
EY(\bar{A}; u)\bar{B} = [E, Y(\bar{A}; u)]\bar{B} + Y(\bar{A}; u)EB .
\]

Let \( [\Gamma_{r}(\kappa_{+}(e))] \) be the renormalised cycles given in Definition 3.6. The commutator above is given by

\[
[E, Y(\bar{A}; u)] \\
= \int_{[\Gamma_{p_+ - r}(\kappa_{+}(e))]} Y \left( \prod_{i=1}^{p_+ - r} S_{+}(z_i) \text{ Res}_{u=w} \int_{[\Gamma_{r}(\kappa_{+}(e))]} \prod_{i=1}^{r} S_{+}(wy_{i-1} - u)\bar{A}; u \right) w^{r-1} \prod_{i=1}^{p_+ - r} d z_i d w \prod_{i=1}^{r-1} d y_i \\
= \int_{[\Gamma_{p_+ - r}(\kappa_{+}(e))]} \int_{[\Gamma_{r}(\kappa_{+}(e))]} Y \left( \prod_{i=1}^{p_+ - r} S_{+}(z_i - u) \prod_{i=1}^{r} S_{+}(u(y_{i-1} - 1))(S_{+}\bar{A}); u \right) u^{r-1} \prod_{i=1}^{p_+ - r} d z_i d w \prod_{i=1}^{r-1} d y_i .
\]
Note that we have suppressed total derivative terms in the $y_i$ coordinates. Setting $u = -1$, then implies
\[
[E, Y(\hat{A}; -1)]
\]
\[
= \int_{[\Gamma_{p_+^{\infty}-r}(\kappa_+^{(r)})]} \int_{[\Gamma_{r}(\kappa_+^{(r)})]} (-1)^{r-1} Y \left( \prod_{i=1}^{p_+ - r} S_+(z_i + 1) \prod_{i=1}^{r} S_+(1 - y_i) \right) (S_+ \hat{A}; -1) \prod_{i=1}^{p_+ - r} d z_i d y_i \prod_{i=1}^{r-1} d y_i.
\]

Therefore we have an orientation reversal of the usual integration of the $y_i$ coordinates and a shift in the lower bound of the cycle for the $z_i$ coordinates. Since changing the lower bound of the cycle for the $z_i$ coordinates only leads to corrections of order $\varepsilon$ and greater, we can shift the lower bound of the cycle for the $z_i$ to 1 and obtain
\[
[E, Y(\hat{A}; -1)]
\]
\[
= \int_{[\Gamma_{p_+^{\infty}-r}(\kappa_+^{(r)})]} \int_{[\Gamma_{r}(\kappa_+^{(r)})]} Y \left( \prod_{i=1}^{p_+ - r} S_+(z_i) \prod_{i=1}^{r-1} S_+(y_i) \right) (S_+ \hat{A}; -1) \prod_{i=1}^{p_+ - r} d z_i d y_i \mod \varepsilon.
\]

By using the Kadell integral formulae in Proposition 3.20 one can see that the normalisation of the cycles $[\Gamma_{p_+^{\infty}-r}(\kappa_+^{(r)})], [\Gamma_{r}(\kappa_+^{(r)})]$ is such that
\[
\int_{[\Gamma_{p_+^{\infty}-r}(\kappa_+^{(r)})]} \int_{[\Gamma_{r}(\kappa_+^{(r)})]} Y \left( \prod_{i=1}^{p_+ - r} S_+(z_i) \prod_{i=1}^{r-1} S_+(y_i) \right) (S_+ \hat{A}; -1) \prod_{i=1}^{p_+ - r} d z_i d y_i
\]
\[
= \int_{[\Gamma_{p_+^{\infty}-r}(\kappa_+^{(r)})]} Y \left( \prod_{i=1}^{p_+ - r} S_+(z_i) \right) (S_+ \hat{A}; -1) \prod_{i=1}^{p_+ - r} d z_i \mod \varepsilon.
\]

Setting $\varepsilon = 0$, it therefore follows that
\[
[E, Y(A; -1)] = Y(\varepsilon A; -1).
\]

Since $E$ is a Virasoro homomorphism it then follows that
\[
[E, Y(A; u)] = e^{(1-u)L_{-1}} [E, Y(A; -1)] e^{(u-1)L_{-1}} = Y(\varepsilon A; u).
\]

**Definition 4.16.** For $1 \leq r \leq p_+, 1 \leq s \leq p_-$ and $n \geq 0$, bases of the sectors $V_{r,s;n}^\pm \in K_{r,s;n}^\pm$ in terms of the Frobenius homomorphisms $E$ and $F$ are given by:

1. $E$ basis

\[
V_{r,s;n}^+ = \begin{cases} 
\bigoplus_{m=-n}^{n} \mathbb{C} E^{n+m} |_{\beta_{p_+ p_-; -2n}} & r = p_+, s = p_- \\
\bigoplus_{m=-n}^{n} \mathbb{C} E^{n+m} |_{\beta_{p_+ s_-; -2n}} & r = p_+, s \neq p_- \\
\bigoplus_{m=-n}^{n} \mathbb{C} E^{n+m} S_+^{p_+ - r} |_{\beta_{p_+ - p_-, -2n-1}} & r \neq p_+, s = p_- \\
\bigoplus_{m=-n}^{n} \mathbb{C} E^{n+m} S_+^{p_+ - r} |_{\beta_{p_+ - s_-, -2n-1}} & r \neq p_+, s \neq p_- 
\end{cases}
\]

\[
V_{r,s;n}^- = \begin{cases} 
\bigoplus_{m=-n}^{n+1} \mathbb{C} E^{n+m} |_{\beta_{p_+ p_-; -2n-1}} & r = p_+, s = p_- \\
\bigoplus_{m=-n}^{n+1} \mathbb{C} E^{n+m} |_{\beta_{p_+ s_-; -2n-1}} & r = p_+, s \neq p_- \\
\bigoplus_{m=-n}^{n+1} \mathbb{C} E^{n+m} S_+^{p_+ - r} |_{\beta_{p_+ - p_-, -2n-2}} & r \neq p_+, s = p_- \\
\bigoplus_{m=-n}^{n+1} \mathbb{C} E^{n+m} S_+^{p_+ - r} |_{\beta_{p_+ - s_-, -2n-2}} & r \neq p_+, s \neq p_- 
\end{cases}
\]

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5 The extended $W$-algebra $\mathcal{M}_{p^+, p^-}$

In this section we apply the theory developed in the previous sections to analysing the extended $W$-algebra $\mathcal{M}_{p^+, p^-}$.

5.1 The algebra structure of $\mathcal{M}_{p^+, p^-}$

Definition 5.1.

1. The extended $W$-algebra $\mathcal{M}_{p^+, p^-} = (\mathcal{K}^+_{1,1}, |0\rangle, T, Y)$ is a subVOA of $\mathcal{V}_{p^+, p^-}$, where the vacuum vector, conformal vector and vertex operator map are those of $\mathcal{V}_{p^+, p^-}$, and

$$\mathcal{K}^+_{1,1} = \ker S_+ \cap \ker S_- \subset \mathcal{V}^+_1.$$

2. Let $\mathcal{M}^{(0)}_{p^+, p^-} = \mathcal{M}_{p^+, p^-} \cap F_{1,1,0} = (K_{1,1,0}, |0\rangle, T, Y)$ be the restriction of $\mathcal{M}_{p^+, p^-}$ to its Heisenberg weight 0 subspace. Then $\mathcal{M}^{(0)}_{p^+, p^-}$ contains the vacuum and Virasoro states $|0\rangle$, $T$ and $Y$ is a subVOA of $\mathcal{M}_{p^+, p^-}$.

We specialise some of the notation of Section 4.

Definition 5.2.

1. For $n \geq 0$ let

$$\alpha^+_n = \beta_{p^+ - 1,1; -2n-1} = \alpha_+ - (n+1)\alpha, \quad \alpha^-_n = \beta_{1, p^- - 1; 2n+1} = \alpha_- + (n+1)\alpha,$$

such that $\alpha^-_n = \alpha^+_n \vee$. The conformal weight corresponding to these Heisenberg weights is

$$\Delta^+_{1,1,n} = h_{\alpha^+_n} = ((n+1)p_+ - 1)((n+1)p_- - 1).$$

We abbreviate $\Delta^+_{1,1,n}$ as $\Delta_n$. 

2. $F$ basis

$$V^+_{r,s:n} = \begin{cases} \bigoplus_{m=-n}^{n} \mathbb{C}[n-m][\beta_{p^+, p^-; 2n}] & r = p^+, s = p^- \\ \bigoplus_{m=-n}^{n} \mathbb{C}[n-m]S[p^-; s][\beta_{p^+, p^-; 2n+1}] & r = p^+, s \neq p^- \\ \bigoplus_{m=-n}^{n} \mathbb{C}[n-m][\beta_{r, p^-; 2n}] & r \neq p^+, s = p^- \\ \bigoplus_{m=-n}^{n} \mathbb{C}[n-m]S[p^-; s][\beta_{r, p^-; 2n+1}] & r \neq p^+, s \neq p^- \end{cases}$$

$$V^-_{r,s:n} = \begin{cases} \bigoplus_{m=-n}^{n+1} \mathbb{C}[n-m+1][\beta_{p^+, p^-; 2n+1}] & r = p^+, s = p^- \\ \bigoplus_{m=-n}^{n+1} \mathbb{C}[n-m+1]S[p^-; s][\beta_{p^+, p^-; 2n+2}] & r = p^+, s \neq p^- \\ \bigoplus_{m=-n}^{n+1} \mathbb{C}[n-m+1][\beta_{r, p^-; 2n+1}] & r \neq p^+, s = p^- \\ \bigoplus_{m=-n}^{n+1} \mathbb{C}[n-m+1]S[p^-; s][\beta_{r, p^-; 2n+2}] & r \neq p^+, s \neq p^- \end{cases}$$
Proof. We show the

\[ \lambda_{n,m}^+ = \lambda_{(n+m+1)p_+ - 1, (n-m+1)p_- - 1} \]

\[ \lambda_{n,m}^- = \lambda_{(n-m+1)p_- - 1, (n+m+1)p_+ - 1} \cdot \]

3. For \( n \geq 0 \) and \(-n \leq m \leq n\) let \( W_{n,m} \) be the basis of the soliton sector \( V_{1;1;n}^+ \), given by

\[ W_{n,m} = E_{n+m}^p S_{\alpha_n^+}^p | \alpha_n^+ \rangle. \]

**Proposition 5.3.** For \( 1 \leq r < p_+ \) and \( 1 \leq s < p_- \) the screening operators \( S_{\alpha_r^+}^p \) and \( S_{\alpha_s^+}^p \) are \( \mathcal{M}_{p_+, p_-} \) homomorphisms, that is, for \( A \in \mathcal{K}_{1,1}^+ \)

\[ [S_{\alpha_r^+}^p, Y(A; w)] = 0, \quad [S_{\alpha_s^+}^p, Y(A; w)] = 0. \]

**Proof.** We show the \( S_+ \) case.

\[ [S_{\alpha_r^+}^p, Y(A; w)] = \text{Res}_{z=w} S_{\alpha_r^+}^p (z) Y(A; w) \, dz \]

\[ = \text{Res}_{z=w} \int_{[r(\alpha_r^+)]} z^{r-1} Y \left( \prod_{i=1}^{r-1} S_+((z-w)y_i)S_+(z-w)A; w \right) \, dz \, dy_1 \cdots dy_{r-1} \]

\[ = \int_{[r(\alpha_r^+)]} z^{r-1} Y \left( \prod_{i=1}^{r-1} S_+((z-w)y_i)(S_+ A); w \right) \, dy_1 \cdots dy_{r-1} = 0. \]

Note that we have suppressed total derivative terms in the \( y_i \) coordinates.

**Theorem C.** The extended \( W \)-algebra \( \mathcal{M}_{p_+, p_-} \) is generated by the fields \( T(z), Y(W_{1;1}; z) \) and \( Y(W_{1,-1}; z) \), while \( \mathcal{M}_{p_+, p_-}^{(0)} \) is generated by the fields \( T(z) \) and \( Y(W_{1,0}; z) \). Furthermore the VOAs \( \mathcal{M}_{p_+, p_-} \) and \( \mathcal{M}_{p_+, p_-}^{(0)} \) satisfy:

1. The Frobenius homomorphisms \( E \) and \( F \) are derivations of the extended \( W \)-algebra \( \mathcal{M}_{p_+, p_-} \).
2. The module \( X_{1,1}^+ \subset \mathcal{K}_{1,1}^+ \) is simple as an \( \mathcal{M}_{p_+, p_-} \)-module and is generated by the soliton vector \( W_{0,0} \).
3. The quotient \( \mathcal{K}_{1,1}^+ / X_{1,1}^+ \cong L(0) \) is a simple \( \mathcal{M}_{p_+, p_-} \)-module on which \( Y(A; z) = 0 \) for \( A \in X_{1,1}^+ \).
4. The sequence

\[ 0 \longrightarrow X_{1,1}^+ \longrightarrow \mathcal{K}_{1,1}^+ \longrightarrow L(0) \longrightarrow 0, \]

is an exact sequence of \( \mathcal{M}_{p_+, p_-} \)-modules.
5. The module \( X_{1,1;0} \) is simple as an \( \mathcal{M}_{p_+, p_-}^{(0)} \)-module and is generated by the soliton vector \( W_{0,0} \).
6. The quotient \( K_{1,1;0} / X_{1,1;0} \cong L(0) \) is a simple \( \mathcal{M}_{p_+, p_-}^{(0)} \)-module on which \( Y(A; z) = 0 \) for \( A \in X_{1,1;0} \).

45
Proposition 5.4.

1. is proven by showing that the matrix elements are non-zero. We detail how to evaluate one of these matrix elements. The rest follow in the same way.

2. follows from the fact that \( W_{1,1}[\Delta+1-\Delta_0] W_{n,n} = a_n W_{n+1,n+1} \), \( W_{1,1}[\Delta-\Delta_0-1] W_{n,n-1} = b_n W_{n+1,n} \), \( W_{1,1}[\Delta+1-\Delta] W_{n+1,n} = c_n W_{n,n} \), \( W_{1,-1}[\Delta+1-\Delta] W_{n+1,n+1} = d_n W_{n,n} \),

where \( a_n, b_n, c_n \) and \( d_n \) are non-zero constants.

2. follows from the fact that \( S^{p-1} W_{n+1,n} \) are given by \( S^{p-1} W_{n+1,n} \).

Proof. Since up to multiplication by non-zero constants the \( W_{n,\pm} \) are given by \( S^{p-1} W_{n,\pm} \),

1. is proven by showing that the matrix elements

\[
\int_{[\Gamma_{p-1}(\kappa_+)]} \langle \alpha_{n+1}^\pm | S_-(z_1 + w) \cdots S_-(z_{p-1} + w) V_{\alpha_1^\pm}(w) | \alpha_n^\pm \rangle \prod_{i=1}^{p-1} d z_i,
\]

\[
\int_{[\Gamma_{p-1}(\kappa_-)]} \langle \alpha_{n-1}^\pm | S_-(z_1 + w) \cdots S_-(z_{p-1} + w) V_{\alpha_1^\pm}(w) | \alpha_n^\pm \rangle \prod_{i=1}^{p-1} d z_i,
\]

are non-zero. We detail how to evaluate one of these matrix elements. The rest follow in the same way.

\[
\int_{[\Gamma_{p-1}(\kappa_-)]} \langle \alpha_{n+1}^- | S_-(z_1 + w) \cdots S_-(z_{p-1} + w) V_{\alpha_1^-}(w) | \alpha_n^- \rangle \prod_{i=1}^{p-1} d z_i,
\]

\[
= \prod_{1 \leq i < j \leq p-1} (z_i - z_j)^{\kappa_-} \prod_{i=1}^{p-1} z_i^{(2-p_-)\kappa_- + 1 - 3p_+}
\]

\[
\times \prod_{i=1}^{p-1} \left(1 + \frac{z_i}{w}\right)^{2\kappa_- - 2(n+1)p_+ w^{2(n+1)p_+ - 2p_+}} \prod_{i=1}^{p-1} \frac{d z_i}{z_i}
\]

\[
= w^{\Delta_1+1-\Delta_n-\Delta_0} \Xi_{2(n+1)p_+ - 1}(Q_{\lambda_2^+} (x; \kappa_+)) \neq 0,
\]

where \( \Xi \) is the map described in Proposition 3.15.

2. follows from the fact that \( W_{1,-1}[m] | \alpha_1^+ \rangle = 0 \), \( W_{1,1}[m] | \alpha_1^- \rangle = 0 \) for \( m > \Delta_1 - \Delta_0 \).
Proof of Theorem C. 1. follows directly from Theorem B. We prove the points 2. - 4. pertaining to $\mathcal{M}_{p_+p_-}$, points 5. - 7. follow by the same methods.

2. Since $W_{0,0}$ is a Virasoro descendant of the vacuum $|0\rangle$, proving that $\mathcal{M}_{p_+p_-}$ is generated by $T(z), Y(W_{1,-1}; z)$ and $Y(W_{1,1}; z)$, reduces to showing that all soliton vectors $W_{n,m}$ can be reached by acting on $W_{0,0}$ with the modes of $Y(W_{1,-1}; z)$ and $Y(W_{1,1}; z)$. We prove this by showing that $W_{1,-1}[\Delta_n - \Delta_{n+1}]W_{n,m}$ and $W_{1,1}[\Delta_n - \Delta_{n+1}]W_{n,m}$ have non-zero contributions from $W_{n+1,m-1}$ and $W_{n+1,m+1}$ respectively.

We show the $W_{1,-1}$ case. Since $FW_{1,-1} = 0$, the derivation property of the Frobenius homomorphisms implies that $Y(W_{1,-1}; z)$ commutes with $F$. Therefore

$$F^{m+n}(W_{1,-1}[\Delta_n - \Delta_{n+1}]W_{n,m}) = \text{const } W_{n+1,-(n+1)},$$

which implies that $W_{1,-1}[\Delta_n - \Delta_{n+1}]W_{n,m}$ has non-zero contributions from $W_{n+1,m-1}$, similarly

$$W_{1,-1}[\Delta_n - \Delta_{n-1}]W_{n,m} = \text{const } F^{n-m}W_{n-1,n-1},$$

which implies that $W_{1,-1}[\Delta_n - \Delta_{n-1}]W_{n,m}$ has non-zero contributions from $W_{n-1,m-1}$.

3. and 4. Since $\mathcal{K}_{1,1}^+$ and $\mathcal{X}_{1,1}^+$ are $\mathcal{M}_{p_+p_-}$-modules their quotient $\mathcal{K}_{1,1}^+/\mathcal{X}_{1,1}^+ \cong L(0)$ is an $\mathcal{M}_{p_+p_-}$-module on which all fields $Y(A; z), A \in \mathcal{X}_{1,1}^+$ act trivially. Thus the sequence

$$0 \longrightarrow \mathcal{X}_{1,1}^+ \longrightarrow \mathcal{K}_{1,1}^+ \longrightarrow L(0) \longrightarrow 0,$$

is an exact sequence of $\mathcal{M}_{p_+p_-}$-modules. \hfill \Box

Proposition 5.5. Let $\omega_n(\beta)$ be the eigenvalues of the zero modes $W_{n,0}|0\rangle$ acting on the generating state $|\beta\rangle \in F_\beta$, for $\beta \in \mathbb{C}$ and $n \geq 0$.

1. For $n \geq 0$ the eigenvalue $\omega_n(\beta)$ is a degree $\Delta_n$ polynomial in $\beta$

$$\omega_n(\beta) = \langle \beta|W_{n,0}|0\rangle|\beta\rangle = \text{const } \prod_{i=1}^{(n+1)p_- - 1} \prod_{j=1}^{(n+1)p_+ - 1} (\beta - \beta_{i,j}).$$

2. For $n \geq 0$ and $h_\beta = \frac{1}{2}\beta(\beta - \alpha_0)$ there exist polynomials $g_n(h) \in \mathbb{C}[h]$ such that

$$g_n(h_\beta) = \begin{cases} \omega_n(\beta) & \text{n even} \\ \omega_n(\beta)^2 & \text{n odd} \end{cases}.$$
For $n = 0, 1, 2$ these polynomials are

\[
g_0(h_\beta) = \omega_0(\beta) = \text{const} \cdot \prod_{(i,j) \in \mathcal{T}} (h_\beta - \Delta_{i,j}),
\]

\[
g_1(h_\beta) = \omega_1(\beta)^2 = \text{const} \cdot \prod_{(i,j) \in \mathcal{T}} (h_\beta - \Delta_{i,j}) \prod_{i=1}^{p-1} \prod_{j=1}^{p-1} (h_\beta - \Delta_{i,j,0})^2 (h_\beta - \Delta_{i,j,0})^2
\]

\[
g_2(h_\beta) = \omega_2(\beta) = \text{const} \cdot \prod_{(i,j) \in \mathcal{T}} (h_\beta - \Delta_{i,j})^3
\]

\[
\prod_{i=1}^{p-1} \prod_{j=1}^{p-1} (h_\beta - \Delta_{i,j,0})^2 \prod_{i=1}^{p-1} \prod_{j=1}^{p-1} (h_\beta - \Delta_{i,j,0})^2 (h_\beta - \Delta_{i,j,0})
\]

\[
\prod_{j=1}^{p-1} (h_\beta - \Delta_{i,j,0})^2 \prod_{j=1}^{p-1} (h_\beta - \Delta_{i,j,0})^2 (h_\beta - \Delta_{i,j,0}),
\]

\[
(h_\beta - \Delta_{p+,p-}) (h_\beta - \Delta_{p+,p-})
\]

Where $\mathcal{T}$ is the Kac table and $\Delta_{r,s}$ and $\Delta_{r,s,0}$ are the conformal weights of Definition 4.12.

Proof. 1. can be expressed in terms of Jack polynomials and evaluated accordingly.

\[
\int_{[\tau_{(n+1)^p+1}]} w^{\Delta_n} \langle \beta | S_+(z_1 + w) \cdots S_+(z_{(n+1)p+1} + w) V_{\alpha^+}(w) | \beta \rangle \prod_{i=1}^{(n+1)p+1} dz_i = \Xi_{\alpha+\beta} (Q_{\lambda^+}^{\kappa^+}(x; \kappa^-)) = \prod_{i=1}^{(n+1)p+1} \prod_{j=1}^{(n+1)p+1} \frac{\beta - \beta_{i,j}}{\beta_{(n+1)p+1+1,j-(n+1)p+1}},
\]

where $\varepsilon$ is the map described in Proposition 3.15.

2. follows from 1. by using the identity

\[
(\beta - \beta_{i,j})(\beta - \beta_{-i,-j}) = \beta^2 - \beta(\beta_{i,j} + \beta_{-i,-j}) + \beta_{i,j} \beta_{-i,-j} = 2(h_\beta - h_{i,j}).
\]

5.2 The zero mode algebra and the $c_2$-cofiniteness condition

In this section we will determine the structure of the zero mode algebra $A_0(\mathcal{M}_{p+,p-})$ and the Poisson algebra $\mathfrak{p} (\mathcal{M}_{p+,p-})$, as well as prove that $\mathcal{M}_{p+,p-}$ satisfies Zhu’s $c_2$-cofiniteness condition.
Let $A_0 = A_0(\mathcal{M}_{p^+,p^-})$. We will first compute relations for the zero mode algebra $A_0$ and then show that they imply that the Poisson algebra is finite dimensional and that therefore $\mathcal{M}_{p^+,p^-}$ satisfies Zhu’s $c_2$-cofiniteness condition.

**Proposition 5.6.**

1. The $\mathcal{M}_{p^+,p^-}$ derivations $E$ and $F$ are also derivations of $A_0$.

2. The $\mathcal{M}_{p^+,p^-}$ anti-involution $\sigma$ defines an anti-involution of $A_0$ and $A_0(\mathcal{M}_{p^+,p^-}^{(0)})$. On $A_0$ it satisfies

   \[ \sigma([T]) = [T] \quad \sigma([W_{1,m}]) = -[W_{1,m}] \]

   for $m = -1, 0, 1$ and on $A_0(\mathcal{M}_{p^+,p^-}^{(0)})$

   \[ \sigma([T]) = [T] \quad \sigma([W_{n,0}]) = (-1)^n[W_{n,0}] \]

   for $n \geq 0$.

3. We have the following surjection onto the zero mode algebra $A_0$ from a subspace of $K_{1,1}^+$

   \[ \mathbb{C}[T] \oplus \bigoplus_{m=-1}^1 \mathbb{C}[T] * W_{1,m} \rightarrow A_0, \]

   where $\mathbb{C}[T]$ denotes polynomials in $T$ with multiplication $*$.

4. We have the following surjection onto the zero mode algebra $A_0(\mathcal{M}_{p^+,p^-}^{(0)})$ from a subspace of $K_{1,1;0}$

   \[ \mathbb{C}[T] \oplus \bigoplus_{n \geq 1} \mathbb{C}[T] * W_{n,0} \rightarrow A_0(\mathcal{M}_{p^+,p^-}^{(0)}), \]

   where $\mathbb{C}[T]$ denotes polynomials in $T$ with multiplication $*$.

**Proof.** 1. and 2. follow because they hold in $U(\mathcal{M}_{p^+,p^-})[0]$ and descend to $A_0$ and $A_0(\mathcal{M}_{p^+,p^-}^{(0)})$.

We prove 3. by using Zhu’s formulation of the zero mode algebra. It follows from Lemma 5.4 and Proposition 2.4 that for $n \geq 2$

\[ W_{n,n} = \text{const} \cdot W_{1,1}[\Delta_{n-1} - \Delta_n]W_{n-1,n-1} \in O(\mathcal{M}_{p^+,p^-}). \]

By applying $F$ multiple times to $W_{n,n}$ we see that $W_{n,m} \in O(\mathcal{M}_{p^+,p^-})$ for $n \geq 2$ and $-n \leq m \leq n$.

From Proposition 2.4 it also follows that the image of $L_{-n}$ for $n \geq 3$ satisfies

\[ L_{-n}A \in \mathbb{C}[L_{-2}, L_{-1}, L_0]A \mod O(\mathcal{M}_{p^+,p^-}). \]

Since $(L_0 + L_{-1})A \in O(\mathcal{M}_{p^+,p^-})$ we therefore have the following surjection of vector spaces

\[ \mathbb{C}[L_{-2}]^0 \oplus \bigoplus_{m=-1}^1 \mathbb{C}[L_{-2}]W_{1,m} \rightarrow A_0. \]

Thus $A_0$ is spanned by polynomials in $T$ and polynomials in $T$ times $W_{1,m}$.

4. follows from the same arguments used to prove 3. and the decomposition of $K_{1,1;0}$ as a $U(\mathcal{L})$ module.  \[ \square \]
Theorem D.

1. The zero mode algebra $A_0$ is finite dimensional and is generated by $[T], [W_{1,m}], m = -1, 0, 1$.

2. In the zero mode algebra $A_0(\mathcal{M}^{(0)}_{p,+p-})$ the Virasoro element satisfies the polynomial relations

\[ g_n([T]) = \begin{cases} [W_{n,0}] & n \text{ even} \\ [W_{n,0}]^2 & n \text{ odd} \end{cases} \]

where $n \geq 0$ and the $g_i$ are the polynomials of Proposition 5.5.

3. In the zero mode algebra $A_0$ the Virasoro element satisfies the polynomials relations

\[ [W_{0,0}] = g_0([T]) \]
\[ [W_{2,0}] = g_2([T]) = 0 \]

where the $g_i$ are the polynomials of Proposition 5.5

4. The products of the generators $[W_{1,m}], m = -1, 0, 1$ are given by

\[
\begin{array}{|c|c|c|c|}
\hline
 & [W_{1,-1}] & [W_{1,0}] & [W_{2,1}] \\
\hline
[W_{1,-1}] & 0 & -f([T])[W_{1,-1}] & -g_1([T]) - f([T])[W_{1,0}] \\
[W_{1,0}] & f([T])[W_{1,-1}] & g_1([T]) & -f([T])[W_{1,1}] \\
[W_{1,1}] & -g_1([T]) + f([T])[W_{1,0}] & f([T])[W_{1,1}] & 0 \\
\hline
\end{array}
\]

where $f$ is a degree less than $\Delta_1/2$ polynomial and $g_1$ is that of Proposition 5.5

5. The commutators of generators $[W_{1,m}], m = -1, 0, 1$ are given by

\[ [[W_{1,0}, [W_{1,1}]] = -2f([T])[W_{1,1}], \quad [[W_{1,0}, [W_{1,-1}]] = 2f([T])[W_{1,-1}], \]
\[ [[W_{1,1}, [W_{1,-1}]] = 2f([T])[W_{1,0}], \]

Proof. We first prove the polynomial relations of 2. Let $k \geq 0$. Since

\[ W_{2k+1,0} \ast W_{2k+1,0} \in U(\mathcal{L})[0] \oplus \bigoplus_{n=1}^{m-1} U(\mathcal{L})W_{n,0}, \]

where $m$ is the smallest integer such that $\Delta_m > 2\Delta_{2k+1}$, it follows that in $A_0(\mathcal{M}^{(0)}_{p,+p-})$

\[ [W_{2k+1,0}]^2 = f_0^{(2k+1)}([T]) + \sum_{n=1}^{m-1} f_n^{(2k+1)}([T])[W_{n,0}], \]

for some polynomials $f_n^{(2k+1)}$. Furthermore, since

\[ W_{1,0}([\Delta_{2k+1} - \Delta_{2k+2}] W_{2k+1,0} = \text{const} \ W_{2k+2,0} + \bigoplus_{n=0}^{2k+1} U(\mathcal{L})W_{n,0} \]
lies in $O(\mathcal{M}_{p+,p-}^{(0)})$, it follows that in $A_0(\mathcal{M}_{p+,p-}^{(0)})$
\[ [W_{2k+2},0] = f_0^{(2k+2)}([T]) + \sum_{n=1}^{2k+1} f_n^{(2k+2)}([T])[W_{n,0}], \]
for some polynomials $f_n^{(2k+2)}$. Additionally, since $\sigma([W_{2k+1,0}])^2 = [W_{2k+1,0}]^2$ and $\sigma([W_{2k+2,0}]) = [W_{2k+2,0}]$, the polynomials $f_n^{2k+1}, f_n^{2k+2}$ must vanish for $n$ odd. Finally, it follows by induction in $k$ that in $A_0(\mathcal{M}_{p+,p-}^{(0)})$ the elements $[W_{2k+1,0}]^2$ and $[W_{2k+2,0}]$ for $k \geq 0$ are polynomials in $[T]$, since $W_{0,0}$ is a Virasoro descendant of the vacuum. The formulae for these polynomials follow from Proposition 5.5.

Next we prove 3. The above calculations for $A_0(\mathcal{M}_{p+,p-}^{(0)})$ imply that in $A_0$, $[W_{0,0}] = g_0([T]), [W_{1,0}] = g_1([T]), [W_{2,0}] = g_2([T])$ and since $[W_{2,0}]$ vanishes in $A_0$, it follows that $g_2([T]) = 0$. The finite dimensionality of $A_0$ then follows from Proposition 5.6 and $g_2([T]) = 0$, thus proving 1.

Finally we prove 4. and 5. The relation $[W_{1,\pm 1}]^2 = 0$ follows from the fact that $W_{1,\pm 1}W_{1,\pm 1} = 0$ in $\mathcal{K}_{1,1}$. To show the relation for $[W_{1,0}] \cdot [W_{1,-1}]$ recall that
\[ W_{1,0} \cdot W_{1,-1} \in U(\mathcal{L})W_{1,-1}. \]
This implies that in $A_0$
\[ [W_{1,0}] \cdot [W_{1,-1}] = f([T])[W_{1,-1}], \]
for some degree less than $\Delta_1/2$ polynomial $f$. We take this to be the polynomial $f$ in the theorem. The remaining relations follow by the application of $E$ and $\sigma$ to the above relations.

Theorem E.

1. The Poisson algebra $\mathfrak{p}(\mathcal{M}_{p+,p-})$ is finite dimensional and $\mathcal{M}_{p+,p-}$ therefore satisfies Zhu’s $c_2$-cofiniteness condition.

2. As a commutative algebra $\mathfrak{p}(\mathcal{M}_{p+,p-})$ is generated by $[T], [W_{1,-1}], [W_{1,0}]$ and $[W_{1,1}]$.

3. The maps $E$ and $F$ act as derivations on $\mathfrak{p}(\mathcal{M}_{p+,p-})$.

4. $[W_{1,\pm 1}]^2 = 0$
\[ [W_{1,1}]_p \cdot [W_{1,-1}]_p = -[W_{1,0}]_p^2 \]
\[ [W_{1,0}]_p^2 = \text{const.} \cdot [T]^{\Delta_1} \]

Proof. 2. and 3. follow directly from the properties of $\mathcal{M}_{p+,p-}$.

The first relation of 4. follows from
\[ W_{1,\pm 1}[-\Delta_1]W_{1,\pm 1} = 0. \]
The second relation follows by applying $E^2$ to $[W_{1,-1}]^2 = 0$. The third relation is a consequence of Proposition 2.11 which implies that the relation $[W_{1,0}]^2 = g_1([T])$ in $A_0$ becomes
\[ [W_{1,0}]^2 = \text{const.} \cdot [T]^{\Delta_1} \]
in $\text{Gr}_{\Delta_1}(A_0)$ and therefore the same relation follows in $\mathfrak{p}(\mathcal{M}_{p+,p-})$.

The Poisson algebra $\mathfrak{p}(\mathcal{M}_{p+,p-})$ is finite dimensional because it is finitely generated and all of its generators are nilpotent, thus proving 1.
5.3 Classification of simple $\mathcal{M}_{p_+,p_-}$ modules

In this section we classify all simple modules of both the zero mode algebra and of $\mathcal{M}_{p_+,p_-}$. We will see that all simple $\mathcal{M}_{p_+,p_-}$-modules are either isomorphic to minimal model modules $L_{r,s}$ or submodules of the lattice modules $V_{r,s}^\pm$.

**Proposition 5.7.** Let $\mathcal{K}_{r,s}^\pm$ and $\mathcal{X}_{r,s}^\pm$ be the Virasoro modules of Proposition 4.13 where $1 \leq r \leq p_+$, $1 \leq s \leq p_-$.

1. The $\mathcal{K}_{r,s}^\pm$ and $\mathcal{X}_{r,s}^\pm$ are $\mathcal{M}_{p_+,p_-}$-modules.
2. The $\mathcal{X}_{r,s}^\pm$ are simple $\mathcal{M}_{p_+,p_-}$-modules for $1 \leq r \leq p_+$, $1 \leq s \leq p_-$.
3. For $1 \leq r < p_+$, $1 \leq s < p_-$ the modules $\mathcal{K}_{r,s}^+$ satisfy the exact sequences

$$0 \rightarrow \mathcal{X}_{r,s}^+ \rightarrow \mathcal{K}_{r,s}^+ \rightarrow L_{[r,s]} \rightarrow 0,$$

where the arrows are $\mathcal{M}_{p_+,p_-}$-homomorphisms and

$$L_{[r,s]} = L_{[p_+ - r, p_- - s]} = \mathcal{K}_{r,s}^+ / \mathcal{X}_{r,s}^+ = L(\Delta_{r,s})$$

are the simple modules of the minimal model VOA MinVir$_{p_+,p_-}$.

4. For $1 \leq r \leq p_+$, $1 \leq s \leq p_-$ the spaces of least conformal weight $\mathcal{K}_{r,s}^+[\Delta_{r,s}]$ and $\mathcal{X}_{r,s}^+[\Delta_{r,s,0}]$ of $\mathcal{K}_{r,s}^+$ and $\mathcal{X}_{r,s}^+$ are one dimensional and the zero modes $W_{1,m}[0], m = -1, 0, 1$ act trivially. The spaces of least conformal weight $\mathcal{K}_{r,s}^-[\Delta_{r,s,0}] = \mathcal{X}_{r,s}^-[\Delta_{r,s,0}]$ of $\mathcal{K}_{r,s}^-$ and $\mathcal{X}_{r,s}^-$ are two dimensional and the zero modes $W_{1,m}[0], m = -1, 0, 1$ act non-trivially.

**Proof.** The fact that $\mathcal{K}_{r,s}^\pm$ and $\mathcal{X}_{r,s}^\pm$ are $\mathcal{M}_{p_+,p_-}$-modules follows analogously to the proof of Theorem C by showing that $Y(W_{1,\pm 1}; z)$ act transitively on the soliton vectors. This then also implies that the $\mathcal{X}_{r,s}^\pm$ are simple.

The $L_{[r,s]}$ are $\mathcal{M}_{p_+,p_-}$-modules, because they are quotients of $\mathcal{M}_{p_+,p_-}$-modules. They are simple, because they are already simple as Virasoro modules.

The spaces of least conformal weight of $\mathcal{K}_{r,s}^+$ are just $\mathbb{C}[\beta_{r,s,0}]$ while for $\mathcal{X}_{r,s}^+$ they are the soliton sectors $V_{r,s,0}^\pm$.

For a simple $\mathcal{M}_{p_+,p_-}$-module $M$ let $\overline{M}$ be its corresponding simple $A_0$-module.

**Proposition 5.8.** The simple modules of the zero mode algebra corresponding to the simple $\mathcal{M}_{p_+,p_-}$-modules of Proposition 5.7 have the following structure:

1. For $1 \leq r < p_+$, $1 \leq s < p_-$ the simple $A_0$-modules $\overline{L}_{[r,s]} = L_{[p_+ - r, p_- - s]}$ are 1-dimensional. The Virasoro $[T]$ element acts as $\Delta_{r,s} \cdot \text{id}$, while the $[W_{1,m}]$ act trivially.

2. For $1 \leq r < p_+$, $1 \leq s < p_-$, the simple $A_0$-modules $\overline{\mathcal{X}}_{r,s}^+$ are 1-dimensional. The $[W_{1,m}]$ act trivially, while the Virasoro element acts as

$$[T] = \Delta_{r,s,0}^+ \cdot \text{id}.$$
3. For $1 \leq r < p_+$, $1 \leq s < p_-$, the simple $A_0$-modules $X_{r,s}^-$ are 2-dimensional. Let $v_\pm$ be a basis such that

$$v_- = [W_{1,-1}]v_+,$$

then

$$[T]v_\pm = \Delta_{r,s;0}^- v_\pm$$
$$[W_{1,0}]v_\pm = \mp f([T])v_\pm$$

and $\Delta_{r,s;0}^-$ is not a root of $f$.

Proof. Recall the structure of the $\mathcal{K}_{r,s}^\pm$ and $X_{r,s}^\pm$ as laid out in Proposition 4.13.

1. For $1 \leq r < p_+$, $1 \leq s < p_-$ the factorisation of $\omega_1(\beta)$ in Proposition 5.5 contains a factor $h_\beta - \Delta_{r,s}^-$. Therefore the $A_0$ generator $[W_{1,0}]$ acts trivially on $\overline{L}_{[r,s]}$. By applying $E$ and $F$ one sees that $[W_{1,-1}]$ must also act trivially. Thus $[T]$ is the only generator of $A_0$ that acts non-trivially on $\overline{L}_{[r,s]}$ and it acts by multiplying by the conformal weight $\Delta_{r,s}^-$. 

2. The case for $X_{r,s}^+$ follows in the same way as for $\overline{L}_{[r,s]}$.

3. For $1 \leq r \leq p_+$, $1 \leq s \leq p_-$, the zero modes $W_{1,m}[0], m = -1, 0, 1$ act non-trivially on the two dimensional soliton sector $V_{r,s;0}^- \subset X_{r,s}^-$. Therefore there exits a basis $v_\pm$ of $X_{r,s}^-$ such that

$$v_- = [W_{1,-1}]v_+ \quad \quad v_+ = \text{const}[W_{1,1}]v_-.$$

It then follows from the relations of $A_0$ and the conformal weight of $V_{r,s;0}^-$ that

$$[T]v_\pm = \Delta_{r,s;0}^- v_\pm$$
$$[W_{1,0}]v_\pm = \mp f([T])v_\pm.$$

Since $[W_{1,0}]$ acts non-trivially on $\overline{X}_{r,s}^-$, the conformal weight $\Delta_{r,s;0}^-$ cannot be a root of $f(h_\beta)$.

\[ \square \]

**Theorem F.** The list of $2p_-/(p_+ - 1) (p_- - 1)/2$ simple $\mathcal{M}_{p_+,p_-}$-modules and simple $A_0$-modules of Propositions 5.4 and 5.5 are a full classification of all simple $\mathcal{M}_{p_+,p_-}$ and $A_0$-modules.

Proof. From Theorem D we know that $g_2([T]) = 0$ in $A_0$. Therefore the minimal polynomial of $[T]$ must divide $g_2([T])$, that is the conformal weight of any simple $A_0$-module must be a zero of $g_2(h_\beta)$. The list of simple $A_0$-modules in Proposition 5.8 exhausts all these possibilities, therefore there can be no other inequivalent modules other than those listed. \[ \square \]

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5.4 The Whittaker category $\mathcal{M}_{p^+,p^-}$-Whitt-mod

Let $\mathcal{M}_{p^+,p^-}$-mod be the category of left $\mathcal{M}_{p^+,p^-}$-modules. Then in the sense of abelian categories any object $M$ of $\mathcal{M}_{p^+,p^-}$-mod has a finite Jordan-Hölder composition series and the simple objects of $\mathcal{M}_{p^+,p^-}$-mod are given by

$$\{L_{[r,s]} | 1 \leq r < p^+, 1 \leq s < p^-\} \cup \{X_{r,s}^\pm | 1 \leq r \leq p^+, 1 \leq s \leq p^-\}.$$

**Definition 5.9.** We define the full subcategory $\mathcal{M}_{p^+,p^-}$-Whitt-mod of $\mathcal{M}_{p^+,p^-}$-mod as the category of all objects $M \in \mathcal{M}_{p^+,p^-}$-mod such that the composition series of $M$ only contains simple objects in

$$\{X_{r,s}^\pm | 1 \leq r \leq p^+, 1 \leq s \leq p^-\}.$$

We call the category $\mathcal{M}_{p^+,p^-}$-Whitt-mod the Whittaker category of $\mathcal{M}_{p^+,p^-}$-mod and the objects of $\mathcal{M}_{p^+,p^-}$-Whitt-mod Whittaker $\mathcal{M}_{p^+,p^-}$-modules.

The full subcategory $\mathcal{M}_{p^+,p^-}$-Whitt-mod has properties crucial to developing the conformal field theory associated to $\mathcal{M}_{p^+,p^-}$.

6 Concluding remarks and further problems

Since the VOA $\mathcal{M}_{p^+,p^-}$ satisfies the $c_2$-cofiniteness condition, one can develop the conformal field theory over general Riemann surfaces associated to $\mathcal{M}_{p^+,p^-}$. A paper on this subject is being prepared by the first author together with Y. Hashimoto [HT].

We have some conjectures regarding the conformal field theory over general Riemann surfaces associated to $\mathcal{M}_{p^+,p^-}$. By considering the conformal field theory on the Riemann sphere associated to $\mathcal{M}_{p^+,p^-}$, the fusion tensor product $\otimes$ induces the structure of a braided monoidal category on $\mathcal{M}_{p^+,p^-}$-mod. However, as was noted in [GRW09], this fusion tensor product is not exact on $\mathcal{M}_{p^+,p^-}$-mod. We conjecture the following:

1. The full abelian subcategory $\text{Vir}_{\text{min}} \subset \mathcal{M}_{p^+,p^-}$-mod, generated by the simple modules

$$\{L_{[r,s]} | 1 \leq r < p^+, 1 \leq s < p^-\}$$

forms a tensor ideal in $\mathcal{M}_{p^+,p^-}$-mod. Thus $\mathcal{M}_{p^+,p^-}$-Whitt-mod is endowed with a quotient braided monoidal structure ($\mathcal{M}_{p^+,p^-}$-Whitt-mod, $\otimes$).

2. The category ($\mathcal{M}_{p^+,p^-}$-Whitt-mod, $\otimes$) is rigid as a monoidal category.

3. In their seminal paper [FGST07] Feigin, Gainutdinov, Semikhatov and Tipunin defined the quantum group $\mathfrak{g}_{p^+,p^-}$ – a finite dimensional complex Hopf algebra. They determined all simple modules of $\mathfrak{g}_{p^+,p^-}$ and their projective covers. There are exactly $2p^+p^-$ simple modules, which is also the number of simple objects in $\mathcal{M}_{p^+,p^-}$-Whitt-mod. The Hopf algebra $\mathfrak{g}_{p^+,p^-}$ is not quasi-triangular, nevertheless, we expect the monoidal category ($\mathfrak{g}_{p^+,p^-}$-mod, $\otimes$)
to be braided. Since $\mathfrak{g}_{p_+, p_-}$ is a Hopf algebra, $(\mathfrak{g}_{p_+, p_-}\text{-mod}, \otimes)$ is a rigid monoidal category. We conjecture that

$$(\mathcal{M}_{p_+, p_-}\text{-Whitt-mod}, \otimes) \cong (\mathfrak{g}_{p_+, p_-}\text{-mod}, \otimes)$$

as braided monoidal categories.\footnote{The first author attributes this conjecture to exciting discussions with Semikhatov and Tipunin during his stay in Moscow in January 2013 and is convinced of its validity.}

4. In [FGST07] it was also shown that the centre $Z(\mathfrak{g}_{p_+, p_-})$ of $\mathfrak{g}_{p_+, p_-}$ is $\frac{1}{2}(3p_+ - 1)(3p_- - 1)$ dimensional and carries the structure of a $\text{SL}(2, \mathbb{Z})$-module. The space of vacuum amplitudes of $\mathcal{M}_{p_+, p_-}\text{-mod}$ on the torus also carries an $\text{SL}(2, \mathbb{Z})$ action. Explaining the relation between these two $\text{SL}(2, \mathbb{Z})$ modules will be an important future problem.

5. For any rank $\ell$ simple Lie algebra $\mathfrak{g}$ of ADE type, we fix coprime positive integers $p_+, p_- \geq h = h^\vee$, where $h$ is the Coxeter number of $\mathfrak{g}$. By using free bosons and $2\ell$ screening operators, we can define an extended $W$ algebra of type $\mathfrak{g}$, which we call $\mathcal{M}_{p_+, p_-}(\mathfrak{g})$. When $\mathfrak{g} = \mathfrak{sl}_2$, then $\mathcal{M}_{p_+, p_-}(\mathfrak{sl}_2)$ is just $\mathcal{M}_{p_+, p_-}$, the VOA considered in this paper. We conjecture that $\mathcal{M}_{p_+, p_-}(\mathfrak{g})$ satisfies Zhu’s $c_2$-cofiniteness condition and that one can construct Frobenius homomorphisms $E_i, F_i$, $i = 1, \ldots, \ell$ subject to relations that are similar to those encountered in the case $\mathfrak{g} = \mathfrak{sl}_2$.

References

[AK11] Kazuhiko Aomoto and Michitake Kita. Theory of Hypergeometric Functions. Springer, 2011.

[AM08] Dražen Adamović and Antun Milas. On the triplet vertex algebra $W(p)$. Adv. Math., 217:2664–2699, 2008. arXiv:0707.1857

[AM11] Dražen Adamović and Antun Milas. On $W$-algebra extensions of $(2, p)$ minimal models: $p > 3$. J. Alg, 344:313–332, 2011. arXiv:1101.0803

[Bor86] Richard E. Borcherds. Vertex algebras, Kac-Moody algebras, and the monster. Proc. Nat. Acad. Sci., 83:3068–3071, 1986.

[BPZ84] Alexander A. Belavin, Alexander M. Polyakov, and Alexander B. Zamolodchikov. Infinite conformal symmetry in two-dimensional quantum field theory. Nucl. Phys. B, 241:333–380, 1984.

[FBZ01] Edward Frenkel and David Ben-Zvi. Vertex Algebras and Algebraic Curves. Mathematical Surveys and Monographs, Amer. Math. Soc., 88, 2001.

[Fel89] Giovanni Felder. BRST approach to minimal models. Nuc. Phy. B, 317:215–236, 1989.

[FF84] Boris L. Feigin and Dimitry B. Fuchs. Verma modules over the Virasoro algebra. Lect. Notes in Math., 1060:230–245, 1984.

[FF88] Boris L. Feigin and Dimitry B. Fuchs. Cohomologies of some nilpotent subalgebras of the Virasoro and Kac-Moody Lie algebras. Jour. Geom. Phys., 5:209–235, 1988.

[FF90] Boris L. Feigin and Dmitry B. Fuchs. Representations of the Virasoro algebra. Representations of Lie Groups and Related Topics, Adv. Stud. Contemp. Math., Gordon and Breach, New York, 7:465–554, 1990.
Akihiro Tsuchiya and Yukihiro Kanie. Vertex operators in the conformal field theory on \( \mathbb{P}1 \) and monodromy representations of the braid group. *Adv. Stud. Pure Math., Amer. Math. Soc.*, 16:297–373, 1988.

Akihiro Tsuchiya and Simon Wood. The tensor structure on the representation category of the \( W_p \) triplet algebra. 2012. [arXiv:1201.0419](http://arxiv.org/abs/1201.0419).

Alexander Varchenko. *Special Functions, KZ Type Equations, and Representation Theory*. Amer. Math. Soc., 2003.

Yongchang Zhu. Modular invariance of characters of vertex operator algebras. *J. Amer. Math. Soc.*, 9:237–302, 1996.