CHARACTER AND DIMENSION FORMULAE FOR
GENERAL LINEAR SUPERALGEBRA

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Abstract. The generalized Kazhdan-Lusztig polynomials for the finite dimensional irreducible representations of the general linear superalgebra are computed explicitly. Using the result we establish a one to one correspondence between the set of composition factors of an arbitrary \( r \)-fold atypical \( \mathfrak{gl}_{m|n} \)-Kac-module and the set of composition factors of some \( r \)-fold atypical \( \mathfrak{gl}_{r|r} \)-Kac-module. The result of Kazhdan-Lusztig polynomials is also applied to prove a conjectural character formula put forward by van der Jeugt et al in the late 80s. We simplify this character formula to cast it into the Kac-Weyl form, and derive from it a closed formula for the dimension of any finite dimensional irreducible representation of the general linear superalgebra.

1. Introduction

Formal characters of finite dimensional irreducible representations of complex simple Lie superalgebras encapsulate rich information on the structure of the representations themselves. In his foundational papers \([6, 7, 8, 9]\) on Lie superalgebras, Kac raised the problem of determining the formal characters of finite dimensional irreducible representations of Lie superalgebras, and developed a character formula for the so-called typical irreducible representations. However, the problem turned out to be quite hard for the so-called atypical irreducible representations. In the early 80s Bernstein and Leites \([1]\) gave a formula for the general linear superalgebra, which produces the correct formal characters for the singly atypical irreducible representations \([15]\), but fails for the multiply atypical irreducibles (e.g., the trivial representation is multiply atypical). Since then much further research was done on the problem. For the orthosymplectic superalgebra \( \mathfrak{osp}_{2|2n} \), van der Jeugt \([13]\) constructed a character formula for all finite dimensional irreducible representations (which are necessarily singly atypical). There were also partial results and conjectures in other cases. Particularly noteworthy is the conjectural character formula for arbitrary finite dimensional irreducible representations of \( \mathfrak{gl}_{m|n} \) put forward by van der Jeugt, Hughes, King and Thierry-Mieg \([16]\), which was the result of extensive research carried out by the authors over several years period. Their formula withstood the tests of large scale computer calculations for a wide range of irreducible representations. However, the full problem of determining the formal characters of finite dimensional irreducible representations of Lie superalgebras remained open until 1995 when Serganova \([11, 12]\) used a combination of geometric and algebraic techniques to obtain a general solution.

Serganova’s approach was based on ideas from Kazhdan-Lusztig theory. She introduced some generalized Kazhdan-Lusztig polynomials, the values of which at \( q = -1 \) determine the formal characters of finite dimensional irreducible representations of Lie superalgebras. Serganova’s work was further developed in the papers \([18, 17, 2, 3, 4]\). Particularly important is the work of Brundan, developed a very practicable algorithm for computing the generalized Kazhdan-Lusztig polynomials, by using quantum
group techniques. This enables him to gain sufficient knowledge on the generalized Kazhdan-Lusztig polynomials to prove the conjecture of [17] on the composition factors of Kac-modules.

In this paper, we shall further investigate Brundan’s algorithm and implement it to compute the generalized Kazhdan-Lusztig polynomials for the finite dimensional irreducible representations of the general linear superalgebra. A closed formula is obtained for the generalized Kazhdan-Lusztig polynomials, which is essentially given in terms of the permutation group of the atypical roots (see Theorem 3.24 for details).

The formula for the generalized Kazhdan-Lusztig polynomials is quite explicit and easy to apply. It leads to a relatively explicit character formula for all the finite dimensional irreducible representations (see Theorem 4.1). By analysing this formula we prove that the conjecture of van der Jeugt et al [16] holds true for all finite dimensional irreducible $\mathfrak{gl}_{m|n}$-modules.

A general fact in the context of Lie superalgebras is that the character formula constructed by using Kazhdan-Lusztig theory is always in the form of an infinite sum. This makes the character formula rather unwieldy to use for, e.g., determining dimensions of finite dimensional irreducible representations. Therefore, it is highly desirable to sum up the infinite series to cast the character formula into the Kac-Weyl form. This is done in Theorem 4.9.

Equipped with Theorem 4.9 we are able to work out the dimension of any finite dimensional irreducible representation of $\mathfrak{gl}_{m|n}$, and the result is given by the closed formula of Theorem 4.14. In the special case of singly atypical irreducible representations, our dimension formula reproduces what one obtains from the Bernstein-Leites character formula [14].

In proving Theorem 3.24 on the Kazhdan-Lusztig polynomials we have introduced the notion of heights of a weight with respect to its atypical roots. This notion proves to be extremely useful. All the results in this paper can be presented using this concept. In particular, the Kazhdan-Lusztig polynomial $K_{\lambda\mu}(q)$ depends only on the heights of $\lambda$ and $\mu$ with respect to their atypical roots. This latter fact enables one to reduce the study of $r$-fold atypical $\mathfrak{gl}_{m|n}$-Kac-modules to the study of $r$-fold atypical $\mathfrak{gl}_{r|r}$-Kac-modules. We state this precisely in Theorem 3.29 which establishes a one to one correspondence between the set of composition factors of an arbitrary $r$-fold atypical $\mathfrak{gl}_{m|n}$-Kac-module and the set of composition factors of some $r$-fold atypical $\mathfrak{gl}_{r|r}$-Kac-module.

The organization of the paper is as follows. In Section 2 we present some background material on $\mathfrak{gl}_{m|n}$, which will be used throughout the paper. In Section 3 we investigate the generalized Kazhdan-Lusztig polynomials for finite dimensional irreducible $\mathfrak{gl}_{m|n}$-modules. This section contains two main results, Theorem 3.24 and Theorem 3.29. While Theorem 3.24 gives an explicit formula for the Kazhdan-Lusztig polynomials, Theorem 3.29 establishes a one to one correspondence between the set of composition factors of an arbitrary $r$-fold atypical Kac-module over $\mathfrak{gl}_{m|n}$ and the set of composition factors of some $r$-fold atypical Kac-module $\mathfrak{gl}_{r|r}$. In Section 4 we first use Theorem 3.24 to prove the conjectural character formula of van der Jeugt et al [16] (see Theorem 4.2), then we re-write the formula into the Kac-Weyl form (see Theorem 4.9). Finally we derive from Theorem 4.9 a closed formula for the dimension of any finite dimensional irreducible representation of the general linear superalgebra (see Theorem 4.14).
2. Preliminaries

We explain some basic notions of Lie superalgebras here and refer to \([7, 5, 16]\) for more details. We shall work over the complex number field \(\mathbb{C}\) throughout the paper. Given a \(\mathbb{Z}_2\)-graded vector space \(W = W_0 \oplus W_1\), we call \(W_0\) and \(W_1\) the even and odd subspaces, respectively. Define a map \([\ ] : W_0 \cup W_1 \to Z_2\) by \([w] = \hat{i}\) if \(w \in W_i\). For any two \(\mathbb{Z}_2\)-graded vector spaces \(V\) and \(W\), the space of morphisms \(\text{Hom}_\mathbb{C}(V, W)\) (in the category of \(\mathbb{Z}_2\)-graded vector spaces) is also \(\mathbb{Z}_2\)-graded with \(\text{Hom}_\mathbb{C}(V, W)_{\hat{k}} = \sum_{\hat{i} + \hat{j} = \hat{k}} \text{Hom}_\mathbb{C}(V_\hat{i}, W_\hat{j})\).

We write \(\text{End}_\mathbb{C}(V)\) for \(\text{Hom}_\mathbb{C}(V, V)\).

Let \(\mathcal{C}^{m|n}\) be the \(\mathbb{Z}_2\)-graded vector space with even subspace \(\mathcal{C}^m\) and odd subspace \(\mathcal{C}^n\). Then \(\text{End}_\mathbb{C}(\mathcal{C}^{m|n})\) with the \(\mathbb{Z}_2\)-graded commutator forms the general linear superalgebra. To describe its structure, we choose a homogeneous basis \(\{v_a \mid a \in \mathcal{I}\}\), for \(\mathcal{C}^{m|n}\), where \(\mathcal{I} = \{1, 2, \ldots, m + n\}\), and \(v_a\) is even if \(a \leq m\), and odd otherwise. The general linear superalgebra relative to this basis of \(\mathcal{C}^{m|n}\) will be denoted by \(\mathfrak{gl}_{m|n}\), which shall be further simplified to \(\mathfrak{g}\) throughout the paper. Let \(E_{ab}\) be the matrix unit, namely, the \((m + n)\times(m + n)\)-matrix with all entries being zero except that at the \((a, b)\) position which is 1. Then \(\{E_{ab} \mid a, b \in \mathcal{I}\}\) forms a homogeneous basis of \(\mathfrak{g}\), with \(E_{ab}\) being even if \(a, b \leq m\), or \(a, b > m\), and odd otherwise. For convenience, we define the map \([\ ] : \mathcal{I} \to \mathbb{Z}_2\) by \([a] = \{0, 1\} \text{ if } a \leq m\), Then the commutation relations of the Lie superalgebra can be written as

\[
[E_{ab}, E_{cd}] = E_{ad}b_{ec} - (-1)^{([a]-[b])([c]-[d])}E_{ab}d_{ad}.
\]

The upper triangular matrices form a Borel subalgebra \(\mathfrak{b}\) of \(\mathfrak{g}\), which contains the Cartan subalgebra \(\mathfrak{h}\) of diagonal matrices. Let \(\{\epsilon_a \mid a \in \mathcal{I}\}\) be the basis of \(\mathfrak{h}^*\) such that \(\epsilon_a(E_{bb}) = \delta_{ab}\). The supertrace induces a bilinear form \((\ , \) : \(\mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}\) on \(\mathfrak{h}^*\) such that

\[
(\epsilon_a, \epsilon_b) = (-1)^{[a]d_{ab}}.
\]

Relative to the Borel subalgebra \(\mathfrak{b}\), the roots of \(\mathfrak{g}\) can be expressed as \(\epsilon_a - \epsilon_b\), \(a \neq b\), where \(\epsilon_a - \epsilon_b\) is even if \([a] + [b] = 0\) and odd otherwise. The set of the positive roots is \(\Delta^+ = \{\epsilon_a - \epsilon_b \mid a < b\}\), and the set of simple roots is \(\{\epsilon_a - \epsilon_{a+1} \mid a < m + n\}\).

We denote \(\mathcal{I}^1 = \{1, 2, \ldots, m\}\) and \(\mathcal{I}^2 = \{1, 2, \ldots, n\}\). We also set \(\delta_\zeta = \epsilon_\zeta\) for \(\zeta \in \mathcal{I}^2\), where we use the notation

\[
\dot{\zeta} = \zeta + m.
\]

Then the sets of positive even roots and odd roots are respectively

\[
\Delta^+_0 = \{\epsilon_i - \epsilon_j, \delta_\zeta - \delta_\eta \mid 1 \leq i < j \leq m, 1 \leq \zeta < \eta \leq n\},
\]

\[
\Delta^+_1 = \{\epsilon_i - \delta_\zeta \mid i \in \mathcal{I}^1, \zeta \in \mathcal{I}^2\}.
\]

The Lie algebra \(\mathfrak{g}\) admits a \(\mathbb{Z}_2\)-consistent \(\mathbb{Z}\)-grading

\[
\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}, \quad \text{where} \quad \mathfrak{g}_0 = \mathfrak{g}_0 \cong \mathfrak{gl}(m) \oplus \mathfrak{gl}(n) \quad \text{and} \quad \mathfrak{g}_{\pm 1} \subset \mathfrak{g}_1,
\]

with \(\mathfrak{g}_{+1}\) (resp. \(\mathfrak{g}_{-1}\)) being the nilpotent subalgebra spanned by the odd positive (resp. negative) root spaces. We define a total order on \(\Delta^+_1\) by

\[
\epsilon_i - \delta_\zeta < \epsilon_j - \delta_\eta \iff \zeta - i < \eta - j \text{ or } \zeta - i = \eta - j \text{ but } i > j. \tag{2.1}
\]

An element in \(\mathfrak{h}^*\) is called a weight. A weight \(\lambda \in \mathfrak{h}^*\) will be written in terms of the \(\epsilon\delta\)-basis as

\[
\lambda = (\lambda_1, \ldots, \lambda_m | \lambda_1, \ldots, \lambda_n) = \sum_{i \in \mathcal{I}^1} \lambda_i \epsilon_i - \sum_{\zeta \in \mathcal{I}^2} \lambda_\zeta \delta_\zeta, \tag{2.2}
\]
where we have adopted an unusual (but convenient) sign convention for \( \lambda_i \)'s. Thus \( \lambda_i = (\lambda, \epsilon_i) \), called the \( i \)-th entry of \( \lambda \) for \( i \in I^1 \), and \( \lambda_\zeta = (\lambda, \delta_\zeta) \), called the \( \zeta \)-th entry of \( \lambda \) for \( \zeta \in I^2 \). A weight \( \lambda \) is called

\[
\begin{align*}
\text{integral} & \iff \lambda_i, \lambda_\zeta \in \mathbb{Z} \quad \text{for} \quad i \in I^1, \zeta \in I^2; \\
\text{dominant} & \iff 2(\Lambda, \alpha)/(\alpha, \alpha) \geq 0 \quad \text{for all positive even roots} \ \alpha \ \text{of} \ \mathfrak{g}, \ \text{namely}, \\
& \quad \lambda_1 \geq \ldots \geq \lambda_m, \quad \lambda_1 \leq \ldots \leq \lambda_n.
\end{align*}
\]

(2.3)

Denote by \( P \) (resp. \( P_+ \)) the set of integral (resp. dominant integral) weights. Using notation (2.2), \( P \) coincides with the set of the \( m|n \)-tuples of integers, thus \( P \) is also denoted by \( \mathbb{Z}^{m|n} \), and \( P_+ \) by \( \mathbb{Z}_+^{m|n} \).

Let \( \rho_0 \) (resp. \( \rho_1 \)) be half the sum of positive even (resp. odd) roots, and let \( \rho = \rho_0 - \rho_1 \). Then

\[
\begin{align*}
\rho_0 &= \frac{1}{2} \left( \sum_{i=1}^{m} (m - 2i + 1) \epsilon_i + \sum_{\zeta=1}^{n} (n - 2\zeta + 1) \delta_\zeta \right) \\
&= \frac{1}{2} (m - 1, m - 3, \ldots, 1 - m \mid 1 - n, 3 - n, \ldots, n - 1), \\
\rho_1 &= \frac{1}{2} \left( \sum_{i=1}^{m} \epsilon_i - \sum_{\zeta=1}^{n} \delta_\zeta \right) = \frac{1}{2} (n, \ldots, n \mid m, \ldots, m), \\
\rho &= \rho' - \frac{m+n+1}{2}, \quad \text{where} \quad \rho' = (m, \ldots, 2, 1 \mid 1, 2, \ldots, n), \quad 1 = (1, \ldots, 1 \mid 1, \ldots, 1).
\end{align*}
\]

(2.5)

For all purposes, we can replace \( \rho \) by \( \rho' \). Therefore, from here on we shall denote

\[
\rho = (m, \ldots, 2, 1 \mid 1, 2, \ldots, n).
\]

(2.6)

Let \( W = \text{Sym}_m \times \text{Sym}_n \) be the Weyl group of \( \mathfrak{g} \), where \( \text{Sym}_m \) is the symmetric group of degree \( m \). We define the dot action of \( W \) on \( P \) by

\[
w \cdot \mu = w(\mu + \rho) - \rho \quad \text{for} \quad w \in W, \mu \in P.
\]

(2.7)

An integral weight \( \lambda \) is called

\begin{itemize}
  \item regular or non-vanishing (in sense of [5, 16]) if it is \( W \)-conjugate under the dot action to a dominant weight (which is denoted by \( \lambda^+ \) throughout the paper);
  \item vanishing otherwise (since the right-hand side of (2.13) is vanishing in this case, cf. [4, 8]).
\end{itemize}

Obviously,

\[
\lambda \text{ is regular } \iff \begin{cases} 
\lambda_1 + m, \lambda_2 + m - 1, \ldots, \lambda_n + 1 \text{ are all distinct, and} \\
\lambda_1 + 1, \lambda_2 + 2, \ldots, \lambda_n + n \text{ are all distinct.}
\end{cases}
\]

(2.8)

Let \( \lambda \) in (2.2) be a regular weight. A positive odd root \( \epsilon_i - \delta_\zeta \) is an atypical root of \( \lambda \) if

\[
(\lambda + \rho, \epsilon_i - \delta_\zeta) = (\lambda_i + m + 1 - i) - (\lambda_\zeta + \zeta) = 0.
\]

(2.9)

Denote by \( \Gamma_\lambda \) the set of atypical roots of \( \lambda \) (cf. (3.1) and (3.2)):

\[
\Gamma_\lambda = \{ \epsilon_i - \delta_\zeta \mid (\lambda + \rho, \epsilon_i - \delta_\zeta) = 0 \}.
\]

(2.10)

Set \( r = \# \Gamma_\lambda \). We also denote \( \# \lambda = r \), called the degree of atypicality of \( \lambda \). A weight \( \lambda \) is called

\begin{itemize}
  \item typical if \( r = 0 \);
  \item atypical if \( r > 0 \) (in this case \( \lambda \) is also called an \( r \)-fold atypical weight).
\end{itemize}
Let \( V = \oplus_{\mu \in \mathfrak{h}^*} V_\mu \) be a weight module over \( \mathfrak{g} \), where
\[
V_\mu = \{ v \in V \mid hv = \mu(h)v, \forall h \in \mathfrak{h} \}
\]
is the weight space of weight \( \mu \). The character \( \text{ch} V \) is defined to be
\[
\text{ch} V = \sum_{\mu \in \mathfrak{h}^*} (\text{dim} V_\mu) e^\mu,
\]
where \( e^\mu \) is the formal exponential, which will be regarded as an element of an additive group isomorphic to \( \mathfrak{h}^* \) under \( \mu \mapsto e^\mu \). Then \( \text{ch} V \) is an element of the completed group algebra
\[
\varepsilon = \left\{ \sum_{\mu \in \mathfrak{h}^*} a_\mu e^\mu \mid a_\mu \in \mathbb{C}, a_\mu = 0 \text{ except } \mu \text{ is in a finite union of } Q_\lambda \right\},
\]
where for \( \lambda \in \mathfrak{h}^* \),
\[
Q_\lambda = \left\{ \lambda - \sum_{\alpha \in \Delta^+} i_\alpha \alpha \in \mathfrak{h}^* \mid i_\alpha \in \mathbb{Z}_+ \right\}.
\]

For every integral dominant weight \( \lambda \), we denote by \( V^{(0)}(\lambda) \) the finite-dimensional irreducible \( \mathfrak{g}_0 \)-module with highest weight \( \lambda \). Extend it to a \( \mathfrak{g}_0 \oplus \mathfrak{g}_{+1} \)-module by putting
\[
\mathfrak{g}_{+1} V^{(0)}(\lambda) = 0.
\]
Then the Kac-module \( \nabla(\lambda) \) is the induced module
\[
\nabla(\lambda) = \text{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_{+1}}^{\mathfrak{g}} V^{(0)}(\lambda) \cong U(\mathfrak{g}_{-1}) \otimes_{\mathbb{C}} V^{(0)}(\lambda).
\]
Denote by \( V(\lambda) \) the irreducible module with highest weight \( \lambda \) (which is the unique irreducible quotient module of \( \nabla(\lambda) \)).

The following result is due to Kac [8, 9]:

**Theorem 2.1.** If \( \lambda \) is a dominant integral typical weight, then \( V(\lambda) = \nabla(\lambda) \), and
\[
\text{ch} V(\lambda) = \text{ch} \nabla(\lambda) = \frac{L_1}{L_0} \sum_{w \in W} \varepsilon(w) e^{w(\lambda + \rho)},
\]
where \( \varepsilon(w) \) is the signature of \( w \in W \), and
\[
L_0 = \prod_{\alpha \in \Delta^+_0} (e^{\alpha/2} - e^{-\alpha/2}), \quad L_1 = \prod_{\beta \in \Delta^+_1} (e^{\beta/2} + e^{-\beta/2}).
\]

Since any finite-dimensional irreducible \( \mathfrak{g} \)-module is either a typical module or is a tensor module of \( V(\lambda) \) with one-dimensional module for some \( \lambda \in P_+ \), in the rest of the paper there is no loss of generality in restricting our attention to integral weights \( \lambda \).

3. Kazhdan-Lusztig polynomials

3.1. **The height vectors and the \( c \)-relationship.** Let \( \lambda \in \mathfrak{h}^* \) be an \( r \)-fold atypical dominant integral weight:
\[
\lambda = (\lambda_1, \ldots, \lambda_{m_r}, \ldots, \lambda_i, \ldots, \lambda_{m_1}, \ldots, \lambda_m) \mid \gamma_1 \ldots, \gamma_r
\]
with the set of atypical roots (cf. (2.9), (2.10) and (2.11))
\[
\Gamma_\lambda = \{ \gamma_1, \ldots, \gamma_r \}, \quad \text{where } \gamma_1 = \epsilon_{m_1} - \delta_{n_1} < \ldots < \gamma_r = \epsilon_{m_r} - \delta_{n_r},
\]
and \( m_r < \ldots < m_1, \ n_1 < \ldots < n_r \). We call \( \gamma_s \) the \( s \)-th atypical root of \( \lambda \) for \( s = 1, \ldots, r \).

For convenience, we introduce the notation \( \lambda^\rho \) for the \( \rho \)-translation of \( \lambda \):
\[
\lambda^\rho = \lambda + \rho.
\]
Thus \( \{(m_s, n_s) \mid s = 1, \ldots, r\} \) is the maximal set of pairs \((i, \zeta)\) satisfying \( \lambda_i^\rho = \lambda_\zeta^\rho \) by (2.9). We define the atypical tuple of \( \lambda \)
\[
\text{aty}_\lambda = (\lambda_\\eta_1^\rho, \ldots, \lambda_\\eta_s^\rho) = (\lambda_\\eta_1 + n_1, \ldots, \lambda_\\eta_r + n_r) \in \mathbb{Z}^r,
\] (3.4)
and call the \( s \)-th entry of \( \text{aty}_\lambda \) the \( s \)-th atypical entry of \( \lambda \) for \( s = 1, \ldots, r \). We also define the typical tuple of \( \lambda \)
\[
\text{typ}_\lambda \in \mathbb{Z}^{m-r|n-r}
\] (3.5)
to be the element obtained from \( \lambda^\rho \) by deleting all entries \( \lambda_\\eta_m^\rho, \lambda_\\eta_n^\rho \) for \( s = 1, \ldots, r \). Thus all entries of \( \text{typ}_\lambda \), called the typical entries of \( \lambda \), are distinct by (2.8).

**Definition 3.1.** Corresponding to each atypical root \( \gamma_s \) of \( \lambda \), we define the \( \gamma_s \)-height of \( \lambda \) (the height of \( \lambda \) with respect to the \( s \)-th atypical root)
\[
h_s(\lambda) = \lambda_\\eta_m - n_s + s \quad \text{for} \quad s = 1, \ldots, r.
\] (3.6)
We also introduce the height vector of \( \lambda \) and the height of \( \lambda \) respectively:
\[
h(\lambda) = (h_1(\lambda), \ldots, h_r(\lambda)), \quad |h(\lambda)| = \sum_{s=1}^r h_s(\lambda).
\] (3.7)

**Remark 3.2.** As we shall see later, the concept of heights of a weight with respect to its atypical roots is extremely useful. In fact, the Kazhdan-Lusztig polynomials are completely determined by the height vectors of the weights involved (see Theorem 3.24 and Theorem 3.29).

**Example 3.3.** Suppose \( \lambda \) is the weight
\[
\lambda = (7, 6, 5, 5, 3, 2, 0 \mid 1, 2, 3, 4, 5, 7, 7) \in \mathbb{Z}_+^{9|8},
\] (3.8)
where we put a label \( t \) over a pair of entries to indicate that they are the entries associated with the \( t \)-th atypical root. Then
\[
\lambda^\rho = (16, 14, 12, 11, 8, 7, 5, 4, 1 \mid 2, 4, 6, 8, 9, 11, 14, 15),
\] (3.9)
\[
\text{aty}_\lambda = (4, 8, 11, 14),
\] (3.10)
\[
\text{typ}_\lambda = (16, 12, 7, 5, 1 \mid 2, 6, 9, 15),
\] (3.11)
\[
h(\lambda) = (1, 1, 2, 3),
\] (3.12)
where the underlined integers in \( \lambda^\rho \) are entries of atypical tuple \( \text{aty}_\lambda \).

We define the partial order \( \preccurlyeq \) on \( P_+ \) for \( \lambda, \mu \in P_+ \) by
\[
\mu \preccurlyeq \lambda \iff \#\mu = \#\lambda, \; \text{aty}_\mu \leq \text{aty}_\lambda \; \text{and} \; \text{typ}_\mu = \text{typ}_\lambda,
\] (3.13)
where the partial order \( \preceq \) on \( \mathbb{Z}^r \) is defined for \( a = (a_1, \ldots, a_r), \, b = (b_1, \ldots, b_r) \in \mathbb{Z}^r \) by
\[
a \preceq b \iff a_i \leq b_i, \quad \forall i \in [1, r].
\] (3.14)

Now suppose \( \mu \) is another \( r \)-fold atypical dominant integral weight with atypical roots:
\[
\gamma'_1 = \epsilon_{m'_1} - \delta_{n'_1} < \ldots < \gamma'_r = \epsilon_{m'_r} - \delta_{n'_r}.
\] (3.15)
We define (cf. [8,22] below)
\[
\ell_s(\lambda, \mu) = h_s(\lambda) - h_s(\mu) = \lambda_\\eta_m - \mu_\\eta_m + n'_s - n_s \quad \text{for} \quad s \in [1, r],
\] (3.16)
and define the length between \( \lambda \) and \( \mu \) to be
\[
\ell(\lambda, \mu) = \sum_{s=1}^r \ell_s(\lambda, \mu) = |h(\lambda)| - |h(\mu)|.
\] (3.17)
(In general $\ell(\lambda, \mu)$ is not necessarily non-negative, but when $\mu \preceq \lambda$, it is indeed non-negative.)

**Remark 3.4.** The height $|h(\lambda)|$ of $\lambda$ turns out to be the absolute length $\ell(\lambda)$ defined by Brundan in [2] §3-g, and the length $\ell(\lambda, \mu)$ coincides with the length $\ell(\mu, \lambda)$ defined in [2] §3-g.

For $1 \leq s \leq t \leq r$, we denote
\[
d_{s,t}(\lambda) = h_t(\lambda) - h_s(\lambda) = \lambda_m - \lambda_{m_s} - n_t + n_s + t - s. \tag{3.18}
\]
The fact that $d_{s,t}(\lambda)$ is non-negative is not obvious, but one can observe that it is the number of integers between the $s$-th atypical entry $\lambda^p_{n_s}$ and the $t$-th atypical entry $\lambda^p_{n_t}$ which are not entries of $\lambda^p$, namely (cf. (3.20), (3.21) and (3.22))
\[
d_{s,t}(\lambda) = \# \left( [\lambda^p_{n_s}, \lambda^p_{n_t}] \backslash \text{Set}(\lambda^p) \right). \tag{3.19}
\]
Because of this fact, we call $d_{s,t}(\lambda)$ the *distance between two atypical roots* $\gamma_s$ and $\gamma_t$ of $\lambda$. Here and below we use the notation
\[
\text{Set}(\mu) = \text{the set of the entries of a weight } \mu, \tag{3.20}
\]
and the notation
\[
[i, j] = \begin{cases} \{k \in \mathbb{Z} | i \leq k \leq j\} & \text{if } i \leq j, \\ \emptyset & \text{otherwise,} \end{cases} \quad \text{for } i, j \in \mathbb{Z}. \tag{3.21}
\]
(There will be no danger of confusing this notation with the Lie bracket as the later will not be used in the remainder of the paper.)

One can generalize (3.13) to obtain the following proposition, which will be used in the proof of Theorem 3.29.

**Proposition 3.5.** Let $\lambda, \mu$ be $r$-fold atypical weights with $\mu \preceq \lambda$. For any $s, t \in [1, r]$ with $\mu^p_{n_s} \leq \lambda^p_{n_t}$, we have
\[
\# \left( [\mu^p_{n_t} + 1, \lambda^p_{n_t}] \backslash \text{Set}(\text{typ}_\lambda) \right) = h_t(\lambda) - h_s(\mu) + t - s. \tag{3.22}
\]
**Proof.** By (3.6) and the fact that $\lambda^p_{n_s} = \lambda^p_{n_t}$, the right-hand side of (3.22) is equal to
\[
\lambda_{n_t} + m_t - (\mu^p_{n_t} + m^p_t) + 2(t - s) = \lambda^p_{n_t} - \mu^p_{n_t} - (m^p_t - m_t + n_t - n^p_t) - 2(s - t).
\]
Thus (3.22) is equivalent to
\[
\# \left( [\mu^p_{n_t} + 1, \lambda^p_{n_t}] \cap \text{Set}(\text{typ}_\lambda) \right) = m^p_t - m_t + n_t - n^p_t + 2(s - t). \tag{3.23}
\]
Note from (3.13) that typ$_\lambda = \text{typ}_\mu$. By (2.4), (2.8), and (3.1), in $\lambda^p$ (resp., $\mu^p$), the number of typical entries to the left of entry $\lambda^p_{n_t}$ (resp., $\mu^p_{m_t}$) is $m_t - 1 - r + t$ (resp., $m^p_t - 1 - r + s$). Thus the number of typical entries to the left of the entry $\mu^p_{m_t}$ which are in $[\mu^p_{m_t} + 1, \lambda^p_{m_t}]$ is $m^p_t - m_t + s - t$. Similarly, the number of typical entries to the right of the entry $\mu^p_{m_t}$ which are in $[\mu^p_{m_t} + 1, \lambda^p_{m_t}]$ is $n_t - n^p_t + s - t$. Hence we obtain (3.22). \(\square\)

For any two distinct atypical roots $\gamma_s$ and $\gamma_t$, there exist unique positive even roots $\alpha_{st}$ and $\beta_{st}$ such that $(\alpha_{st}, \beta_{st}) = 0$, and the composite of the actions of the Weyl group elements $\sigma_{\alpha_{st}}, \sigma_{\beta_{st}}$, respectively corresponding to reflections with respect to these even roots, send $\gamma_s$ to $\gamma_t$, namely, $\gamma_t = \sigma_{\alpha_{st}} \sigma_{\beta_{st}}(\gamma_s)$. In terms of explicit formulæ, we have
\[
\alpha_{st} = \epsilon_{m_t} - \epsilon_{m_s}, \quad \beta_{st} = \delta_{n_s} - \delta_{n_t} \in \Delta_0^+ \quad \text{for} \quad 1 \leq s < t \leq r. \tag{3.24}
\]
The concept of $c$-relationship defined below was first introduced in [5] from a different point of view.

**Definition 3.6.** For $s \leq t$, two atypical roots $\gamma_s, \gamma_t$ of $\lambda$ are called $c$-related (in the sense of [5]) or connected (in the sense of [17]) if $s = t$ or $d_{s,t}(\lambda) < t - s$. The later relation has the nice interpretation that

the distance (i.e., $d_{s,t}(\lambda)$) between two atypical roots is smaller than the number (i.e., $t - s$) of atypical roots between them.

Relation $d_{s,t}(\lambda) < t - s$ is also equivalent to

$$\lambda_{m_t} - \lambda_{m_s} < n_t - n_s,$$

or in terms of weights,

$$\frac{2(\lambda, \alpha_{st})}{(\alpha_{st}, \alpha_{st})} + \frac{2(\lambda, \beta_{st})}{(\beta_{st}, \beta_{st})} < \frac{2(\rho, \alpha_{st})}{(\alpha_{st}, \alpha_{st})} + \frac{2(\rho, \beta_{st})}{(\beta_{st}, \beta_{st})}.$$  (3.25')

We define

$$c_{s,t}(\lambda) = \begin{cases} 1 & \text{if the atypical roots } \gamma_s, \gamma_t \text{ of } \lambda \text{ are } c\text{-related}, \\ 0 & \text{otherwise}. \end{cases}$$  (3.26)

We also define

$$\tilde{c}_{s,t}(\lambda) = \begin{cases} 1 & \text{if } c_{s,s}(\lambda) = c_{s,s+1}(\lambda) = \ldots = c_{s,t}(\lambda) = 1, \\ 0 & \text{otherwise}, \end{cases}$$  (3.27)

and we say that $\gamma_s, \gamma_t$ are strongly $c$-related if $\tilde{c}_{s,t}(\lambda) = 1$.

Note that $c$-relationship is reflexive and transitive but not symmetric ($c_{t,s}(\lambda)$ is not defined when $t > s$).

**Example 3.7.** Let $\lambda$ be the weight in (3.3). By (3.25),

$$c_{1,2}(\lambda) = 1 \text{ since } \lambda_{m_2} - \lambda_{m_1} = 3 - 2 < 4 - 3 = n_2 - n_1;$$
$$c_{1,3}(\lambda) = 1 \text{ since } \lambda_{m_3} - \lambda_{m_1} = 5 - 2 < 6 - 2 = n_3 - n_1;$$
$$c_{1,4}(\lambda) = 1 \text{ since } \lambda_{m_4} - \lambda_{m_1} = 6 - 2 < 7 - 2 = n_4 - n_1;$$
and
$$c_{s,t}(\lambda) = 0 \text{ for any other pair } (s,t).$$

Thus

$$\tilde{c}_{1,2}(\lambda) = \tilde{c}_{1,3}(\lambda) = \tilde{c}_{1,4}(\lambda) = 1 \text{ and } \tilde{c}_{s,t}(\lambda) = 0 \text{ for any other pair } (s,t).$$

**Remark 3.8.** If $\lambda$ is regular but not necessarily dominant, we shall generalize the notions $c, \tilde{c}, d, h$ to $\lambda$ by defining them with respect to the dominant weight $\lambda^+$. For instance, $c_{s,t}(\lambda) = c_{s,t}(\lambda^+)$. Sometimes even if $\lambda$ is not regular but lexical (in the sense of Definition 3.9), one can still define the $c$-relationship by using the distance (3.18).

### 3.2. Lexical weights

Let $\lambda$ be an $r$-fold atypical regular weight (not necessarily dominant) with the set $\Gamma_\lambda = \{\gamma_1, \ldots, \gamma_r\}$ of atypical roots ordered according to (3.2). We call $\lambda$ lexical if its atypical tuple aty$_\lambda$ is lexical in the following sense:

**Definition 3.9.** An element $a = (a_1, \ldots, a_r) \in \mathbb{Z}^r$ is called lexical if

$$a_1 \leq \ldots \leq a_r.$$  (3.28)

The two sets $P_r$ and $P_r^{\text{Lex}}$ to be defined below will be frequently used throughout this section.
Definition 3.10. We denote by $P_r$ the set of $r$-fold atypical regular weights $\lambda$ of the form \( (3.1) \) such that the atypical roots of $\lambda$ can be ordered as in \( (3.2) \) and that the typical tuple $\text{typ}_\lambda \in \mathbb{Z}_{+}^{m-r|n-r}$ is dominant as a weight for $\mathfrak{gl}_{m-r|n-r}$.

We denote by $P_r^{\text{lex}}$ the subset of $P_r$ consisting of the lexical weights of $P_r$.

3.3. The $r$-tuple of positive integers associated with $\lambda$. The $r$-tuple associated with $\lambda \in P_r^{\text{lex}}$ defined below was first introduced in [17].

Definition 3.11. Define the $r$-tuple $(k_1, ..., k_r)$ of positive integers associated with $\lambda \in P_r^{\text{lex}}$ in the following way: each $k_s$ is the smallest positive integer such that

\[(\lambda + \theta, k_1) + k_s \gamma_s \quad \text{is regular for all } t = s + 1, s + 2, ..., r \text{ and } \theta_t \in \{0,1\} . \quad (3.29)\]

The following lemma gives a way to compute $k_s$’s. For $\lambda \in P_r^{\text{lex}}$ and $s \in [1, r]$, we set

\[
\max_s^\lambda = \max \{ p \in [s, r] | c_{s,p}(\lambda) = 1 \} \quad (3.30)
\]

to be the maximal number $p \in [s, r]$ satisfying the condition that the $p$-th atypical root $\gamma_p$ of $\lambda$ is strongly $c$-related to $\gamma_s$. One immediately sees that

\[
\max_1^\lambda \leq \max_s^\lambda \leq \max_r^\lambda \quad \text{for any } t \text{ with } s \leq t \leq \max_s^\lambda. \quad (3.31)
\]

Lemma 3.12. Let $\lambda \in P_r^{\text{lex}}$.

1. For $s \in [1, r]$, $k_s$ is the integer such that $\lambda_{n_s}^\rho + k$ is the $(\max_s^\lambda + 1 - s)$-th smallest integer bigger than $\lambda_{n_s}^\rho$ and not in the entry set $\text{Set}(\lambda^\rho)$, i.e.,

\[
 k_s = \min \{ k > 0 | \# (\lambda_{n_s}^\rho, \lambda_{n_s}^\rho + k) \setminus \text{Set}(\lambda^\rho) \} = \max_s^\lambda + 1 - s . \quad (3.32)
\]

2. The tuple $(k_r, ..., k_1)$ is the lexicographically smallest tuple of positive integers such that for all $\theta = (\theta_1, ..., \theta_r) \in \{0, 1\}^r$, $\lambda + \sum_{s=1}^r \theta_s k_s \gamma_s$ is regular. Thus $(k_r, ..., k_1)$ is the tuple satisfying [2] Main Theorem.

Proof. Denote by $k'_r$ the right-hand side of \( (3.32) \). Obviously, $k'_r$ is the smallest positive integer such that $\lambda + k'_r \gamma_s$ is regular because for any $0 < k < k'_r$, by definition the integer $\lambda_{n_s}^\rho + k$ which is equal to $\lambda_{n_s}^\rho + k$ already appears in the entry set $\text{Set}(\lambda^\rho)$ and thus $\lambda + k'_r \gamma_s$ is not regular by \( (2.9) \).

If $s < r$, by induction on $\max_s^\lambda$, it is straightforward to see that $\lambda_{n_s}^\rho + k'_s$ is the smallest integer (bigger than $\lambda_{n_s}^\rho$) which is not in $\text{Set}(\lambda^\rho + \theta_s k'_s \gamma_s)$ for $\theta_s \in \{0, 1\}$ and $s < t \leq r$. Thus $k'_s$ is the smallest positive integer satisfying \( (3.29) \).

Similarly, $\lambda_{n_s}^\rho + k'_r$ is also the smallest integer (bigger than $\lambda_{n_s}^\rho$) which is not in $\text{Set}(\lambda^\rho + \sum_{s=1}^r \theta_s k'_s \gamma_s)$ for $\theta = (\theta_1, ..., \theta_r) \in \{0, 1\}^{r+1-s}$, i.e., $(k'_r, ..., k'_1)$ is the lexicographically smallest tuple of positive integers such that for all $\theta = (\theta_s, ..., \theta_r) \in \{0, 1\}^{r+1-s}$, the weight $\lambda + \sum_{s=1}^r \theta_s k'_s \gamma_s$ is regular.

Lemma 3.12 allows us to compute $k_s$ by the following procedure.

Procedure 3.13. First set $S = \text{Set}(\lambda^\rho)$. Suppose we have computed $k_r, ..., k_{s+1}$. To compute $k_s$, we count the numbers in the set $S$ starting with $\lambda_{n_s}^\rho$ until we find a number; say $k$, not in $S$. Then $k_s = k - \lambda_{n_s}^\rho$. Now add $k$ into the set $S$, and continue.

Example 3.14. Let $\lambda$ be given in \( (3.8) \). Using the above procedure we obtain $(k_1, k_2, k_3) = (3, 2, 2, 14)$ (cf. \( (3.29) \)).

Remark 3.15. If $\lambda \in P_r$ (not necessarily in $P_r^{\text{lex}}$), we can still compute $k_s$ by the above procedure, but the difference lies in that the $k_s$’s are computed not in the order $s = r, ..., 1$, but in the order that each time we compute $k_s$ with $\lambda_{n_s}^\rho$ being the largest among all those $\lambda_{n_s}^\rho$’s, the corresponding $k_s$’s of which are not yet computed.
3.4. Raising operators. Following [2], we define the raising operator $R_{m,s,\hat{n}}$ on $P_r$ by
\[ R_{m,s,\hat{n}}(\lambda) = \lambda + k_s \gamma_s \quad \text{for} \quad \lambda \in P_r \quad \text{and} \quad s \in [1, r], \]  
where $\gamma_s, k_s$ are defined in \((3.2)\) and Definition \((3.11)\) respectively. Obviously, for $\lambda, \mu \in P_r^{\text{Lex}}$,
\[ \mu = R_{m,s,\hat{n}}(\lambda) \implies \text{typ}_\lambda = \text{typ}_\mu \quad \text{(cf. \((3.5)\))}. \]  
Denote $\mathbb{N} = \{0, 1, \ldots\}$, and let $\theta = (\theta_1, \ldots, \theta_r) \in \mathbb{N}^r$. We define
\[ R^\theta_\theta(\lambda) = (R^\theta_{m_1,\hat{n_1}} \circ \cdots \circ R^\theta_{m_r,\hat{n_r}}(\lambda))^+, \]  
where in general $\mu^+$ denotes the unique dominant element which is $W$-conjugate under the dot action to $\mu$ (cf. \((2.7)\)).

Let $\mu \in P_r$ be another weight with atypical roots $\gamma'_s = \epsilon_m - \delta_n$, $1 \leq s \leq r$, being as in \((3.15)\). Then we have (cf. [2], §3-\f)
\[ \mu \triangleleft \lambda \iff \#\mu = \#\lambda =: r, \quad \text{and} \quad \exists \theta \in \mathbb{N}^r \text{ with } R^\theta_\theta(\mu) = \lambda. \]  
For convenience, we denote
\[ \mu \triangleleft \lambda \quad \text{if} \quad \text{typ}_\mu = \text{typ}_\lambda \quad \text{and} \quad \max\{\mu^s_\hat{n} | s \in [1, r]\} \leq \min\{\lambda^s_\hat{n} | s \in [1, r]\}. \]  

3.5. Definitions of $S^\lambda$ and $S^{\lambda,\mu}$. The symmetric group $\text{Sym}_r$ of degree $r$ acts on $\mathbb{Z}^r$ by permuting entries. This action induces an action on $P_r$ given by
\[ \sigma(\lambda) = (\lambda_1, \ldots, \lambda_{m_1}, \ldots, \lambda_1, \ldots, \lambda_{m_2}, \ldots, \lambda_1, \ldots, \lambda_{m_r}, \ldots, \lambda_1, \ldots, \lambda_{\hat{n}}), \]  
for $\sigma \in \text{Sym}_r$ and $\lambda \in P_r$. With this action on $P_r$, the group $\text{Sym}_r$ can be regarded as a subgroup of $W$, such that every element is of even parity. Thus we also have the dot action
\[ \sigma \cdot \lambda = \sigma(\lambda + \rho) - \rho \quad \text{for} \quad \sigma \in \text{Sym}_r. \]  

**Definition 3.16.** Let $\lambda, \mu \in P_r^{\text{Lex}}$. Define $S^\lambda$ to be the subset of the symmetric group $\text{Sym}_r$ consisting of permutations $\sigma$ which do not change the order of $s < t$ when the atypical roots $\gamma_s$ and $\gamma_t$ of $\lambda$ are strongly $e$-related. That is,
\[ S^\lambda = \{ \sigma \in \text{Sym}_r | \sigma^{-1}(s) < \sigma^{-1}(t) \quad \text{for all} \quad s < t \quad \text{with} \quad \tilde{c}_{s,t}(\lambda) = 1 \}, \]  
where $\tilde{c}_{s,t}(\lambda)$ is defined in \((3.27)\). We also define $S^{\lambda,\mu}$ to be subset of $S^\lambda$ consisting of permutations $\sigma$ such that $\mu \ll \sigma \cdot \lambda$, namely
\[ S^{\lambda,\mu} = \{ \sigma \in S^\lambda | \mu \ll \sigma \cdot \lambda \}. \]  
For convenience, we denote
\[ \tilde{S}^{\lambda,\mu} = \{ \sigma \in \text{Sym}_r | \mu \ll \sigma \cdot \lambda \}. \]  
Thus $S^{\lambda,\mu} = S^\lambda \cap \tilde{S}^{\lambda,\mu}$.

**Example 3.17.** If $\lambda$ is the weight in \((3.8)\), then $S^\lambda = \{ \sigma \in \text{Sym}_4 | \sigma(1) = 1 \} \cong \text{Sym}_3$, is a subgroup of $\text{Sym}_4$ (however in general $S^\lambda$ is not a subgroup).

Let $\ell(\sigma)$ denote the normal length function on $\text{Sym}_r$, namely
\[ \ell(\sigma) = \sum_{s=1}^r \ell(\sigma, s), \quad \text{where} \]  
\[ \ell(\sigma, s) = \# \{ t > s | \sigma(t) < \sigma(s) \} \quad \text{for} \quad s = 1, \ldots, r. \]
For any subset $B \subset Sym_r$, we define the $q$-length function of $B$ by:

$$B(q) = \sum_{\sigma \in B} q^\ell(\sigma).$$

(3.45)

**Proposition 3.18.** Let $\lambda, \mu \in P^{\text{Lex}}_r$ with $\mu \not\ll \lambda$. We have

$$Sym_r(q) = (1 + q)(1 + q + q^2) \cdots (1 + q + \cdots + q^{r-1}) = \prod_{s=1}^{r} \frac{q^s - 1}{q - 1},$$

(3.46)

$$S^\lambda(q) = \prod_{s=1}^{r} \frac{(q^s - 1)}{\prod_{s=1}^{n} (1 + q^{s+1/2})},$$

(3.47)

$$\hat{S}^{\lambda,\mu}(q) = (1 + q + \cdots + q^{r-1})(1 + q + \cdots + q^{r-1-i_r}) \cdots$$

$$= \prod_{s=1}^{r} \frac{q^{s+1-i_s} - 1}{q - 1},$$

(3.48)

where

$$i_s = \min\{i \in [1, r] \mid \mu^\rho_{n_s'} \leq \lambda^\rho_{n_s} \} \, \text{for} \ s \in [1, r].$$

(3.49)

Proof. First we compute $\hat{S}^{\lambda,\mu}(q)$. Elements $\sigma \in \hat{S}^{\lambda,\mu}$ can be easily described as follows (cf. Example 3.20): for each $s = r, \ldots, 1$, suppose for all $t > s$, $\sigma(t)$ have been chosen, then $\sigma(s)$ can be any of $s - i_s + 1$ integers $r, r - 1, \ldots, i_s$ which have not yet been taken by the $\sigma(t)$’s for $t > s$. We order the elements of $\{r, \ldots, i_s\} \setminus \{\sigma(r), \ldots, \sigma(s + 1)\}$ in descending order, and denote by

$$\{r, \ldots, i_s\} \setminus \{\sigma(r), \ldots, \sigma(s + 1)\} = \{x_1 > \ldots > x_{s-i_s+1}\}. \quad (3.50)$$

Then each choice of $\sigma(s) = x_k$ for $k = 1, \ldots, s - i_s + 1$ contributes $k - 1$ to the length $\ell(\sigma)$. Thus we have (3.48). Since $Sym_r = \hat{S}^{\lambda,\mu}$ for any $\lambda, \mu \in P^{\text{Lex}}_r$ with $\mu \not\ll \lambda$ (cf. (3.42)), and when $\mu \not\ll \lambda$, all $i_s$’s are equal to 1, we obtain (3.46), which is a well-known formula.

Consider $S^\lambda$ defined in (3.40), which can be re-written as

$$S^\lambda = \{\sigma \in Sym_r \mid \sigma^{-1}(s) < \sigma^{-1}(t) \text{ for all } s, t \text{ with } s < t \leq \max^\lambda\}\{r\}.$$ 

Since for each $s = 1, \ldots, r$, we cannot change the order of $s$ and $t$ for $s < t \leq \max^\lambda$, we shall remove the factor $1 + q + \cdots + q^{\max^\lambda - s}$ from $Sym_r(q)$. Thus we obtain (3.47). $\square$

Similar to (3.48), we also have

$$\hat{S}^{\lambda,\mu}(q) = \prod_{s=1}^{r} \frac{q^{s+1-j_s} - 1}{q - 1}, \text{ where } j_s = \max\{j \in [1, r] \mid \lambda^\rho_{n_s} \geq \mu^\rho_{n_s'} \} \text{ for } s \in [1, r].$$

(3.48)

3.6. The $q$-length function of $S^{\lambda,\mu}$. Elements $\sigma \in S^{\lambda,\mu}$ can be described in the following way (cf. the proof of Proposition 3.18 and Example 3.20):

**Description 3.19.** For $s = r, \ldots, 1$, each $\sigma(s)$ can be any one of the numbers $r, r - 1, \ldots, i_s$ which has not yet been occupied by $\sigma(t)$ for some $t > s$ (cf. (3.50)), with an additional condition that if $\tilde{c}_{a,b}(\lambda) = 1$ for some $a < b$ such that $b$ has not yet been chosen, then $\sigma(s) \neq a$.

We can associate each $\sigma \in S^{\lambda,\mu}$ with a graph defined as follows: Put $r$ weighted points at the bottom such that the $s$-th point (which will be referred to as point $s^-$) has weight $\mu^\rho_{n_s}$. Similarly we put $r$ weighted points on the top such that the $s$-th point (which will be referred to as point $s^+$) has weight $\lambda^\rho_{n_s}$. Two points $s^+$ and $t^+$ on the top are connected by a line if and only if $\tilde{c}_{s,t}(\lambda) = 1$, in this case we say that the two points are linked. Note that if $s^+$ is linked to $t^+$, then $s^+$ is linked to $p^+$ for all $p$ with
for any lexical (3.26) and (3.27) by

\[ b = (3, 4, 4, 4, 2, 1, 1, 1, 0, 1, 1, 1, 2, 4, 4, 4, 7), \] and so

\[ \mu^p = (16, 12, 11, 10, 7, 5, 4, 3, 1, 2, 3, 4, 6, 9, 10, 11, 15) \] (cf. (3.3)).

Thus \( \mu \not\leq \lambda \). The elements of \( \tilde{S}^{\lambda, \mu} \) correspond to the following graphs:

\[
\begin{array}{cccc|cccc|cccc|cccc}
1^+ & 2^+ & 3^+ & 4^+ & 1^+ & 2^+ & 3^+ & 4^+ & 1^+ & 2^+ & 3^+ & 4^+ \\
4 & 8 & 11 & 14 & 4 & 8 & 11 & 14 & 4 & 8 & 11 & 14 \\
\hline \\
3 & 4 & 10 & 11 & 3 & 4 & 10 & 11 & 3 & 4 & 10 & 11 \\
1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \\
\end{array}
\]

We see that \( i_4 = i_3 = 3, i_2 = i_1 = 1 \) (where \( i_s \)'s are defined in (3.3)), thus

\( \tilde{S}^{\lambda, \mu}(q) = (1 + q^{1-3})1(1 + q^{2-1})1(1 + q)^2 \), which agrees with the above graphs.

The elements of \( S^{\lambda, \mu} \) are represented by the first two of the above graphs. Thus

\[ S^{\lambda, \mu}(q) = 1 + q. \]

Now let us compute the \( q \)-length function \( S^{\lambda, \mu}(q) \). First we introduce a family of \( q \)-functions \( Z_q(x; b) \) defined on the set of pairs \((x, b)\) of lexical \( r \)-tuples \( x = (x_1, \ldots, x_r) \), \( b = (b_1, \ldots, b_r) \in \mathbb{Z}^r \) satisfying \( 1 \leq b_s \leq s \) for \( s \in [1, r] \), i.e.,

\[ x_1 \leq x_2 \leq \ldots \leq x_r, \quad b_1 \leq b_2 \leq \cdots \leq b_r \quad \text{and} \quad 1 \leq b_s \leq s, \quad \forall s \in [1, r]. \] (3.53)

**Definition 3.21.** Define the \( q \)-function \( Z_q(x; b) \) as follows: Set \( Z_q(x; b) = 0 \) if (3.3) is not satisfied, and define \( Z_q(x; b) \) inductively on \( r \) by:

\[
\begin{align*}
Z_q(x_1; 1) & = 1, \\
Z_q(x_1, x_2; 1, b_2) & = 1 + \theta(x_2 - x_1 - 1)\theta(1 - b_2)q. \\
Z_q(x; b) & = Z_q(x_1, \ldots, x_{r-1}; b^{(r-1)}) + \sum_{i=b_r}^{r-1} \theta(x_{i+1} - x_i - 1)Z_q(x_1, \ldots, x_{i-1}, x_{i+1} - 1, \ldots, x_r - 1; b^{(r-1)})q^{r-i},
\end{align*}
\] (3.54)

where \( b^{(r-1)} = (b_1, \ldots, b_{r-1}) \), and \( \theta(x) \) is the *step function* defined by

\[
\theta(x) = \begin{cases} 
1 & \text{if} \ x \geq 0, \\
0 & \text{otherwise}.
\end{cases}
\] (3.56)

Note that there are only finite many functions \( Z_q(x; b) \) for each fixed \( r \). To see this, for any lexical \( x \in \mathbb{Z}^r \) and for \( 1 \leq s \leq t \leq r \), we define \( c_{s,t}(x) \) and \( \tilde{c}_{s,t}(x) \) analogous to (3.26) and (3.27) by

\[
\begin{align*}
c_{s,t}(x) & = \begin{cases} 
1 & \text{if} \ t = s \text{ or } x_t - x_s < t - s, \\
0 & \text{otherwise},
\end{cases} \\
\tilde{c}_{s,t}(x) & = \begin{cases} 
1 & \text{if} \ c_{s,s}(x) = c_{s,s+1}(x) = \ldots = c_{s,t}(x) = 1, \\
0 & \text{otherwise},
\end{cases}
\end{align*}
\] (3.57)
and define $\widehat{x} = (\widehat{x}_1, ..., \widehat{x}_r)$ with
\begin{equation}
\widehat{x}_s = \# \{ p \in [1, s-1] \mid \hat{c}_{p,s}(x) = 0 \} \quad \text{for} \ s \in [1, r].
\end{equation}
Then one can prove that $\widehat{x}$ is lexical and $0 \leq \widehat{x}_s < s$ for $s \in [1, r]$, and
\begin{equation}
Z_q(x; b) = Z_q(\widehat{x}; b).
\end{equation}
Thus there are only finite many functions $Z_q(x, b)$ for each fixed $r$.

**Proposition 3.22.** Let $\lambda, \mu \in P_{r, lex}^\lambda$ with $\mu \preceq \lambda$. Denote
\begin{equation}
b_{\lambda, \mu} = (i_1, ..., i_r) \in \mathbb{Z}^r, \ \text{where} \ i_s = \min \{ i \in [1, r] \mid \lambda_n^s \leq \mu_n^s \} \ \text{for} \ s \in [1, r],
\end{equation}

namely $i_s$ is defined in (3.49). Let $h(\lambda)$ be the height vector of $\lambda$ defined in (3.7). Then the $q$-length function of $S^{\lambda, \mu}$ is given by
\begin{equation}
S^{\lambda, \mu}(q) = Z_q(h(\lambda); b_{\lambda, \mu}).
\end{equation}

**Proof.** The proposition follows from Description 3.19 and Definition 3.6. □

### 3.7. Generalized Kazhdan-Lusztig polynomials

Let $K_{\lambda, \mu}(q)$ be the Kazhdan-Lusztig polynomial defined in [12] (which was denoted as $K_{\mu, \lambda}(q)$ in [2], but we prefer its original notation in [12]), and $\ell_{\lambda, \mu}(q)$ the polynomial defined in [2] (which was denoted as $\ell_{\mu, \lambda}(q)$ in [2]). Then [2, Corollary 3.39] states that
\begin{equation}
K_{\lambda, \mu}(q) = \ell_{\lambda, \mu}(-q^{-1}) = \sum_{\theta \in \mathbb{N}^r : \rho_s(\mu) = \lambda} q^{||\theta||}, \ \text{where} \ ||\theta|| = \sum_{s=1}^{r} \theta_s,
\end{equation}
for dominant weights $\lambda, \mu$. Let $\ell(\lambda, \mu)$ be the length defined in (3.7). First let us see an example of computing $\ell(\lambda, \mu)$.

**Example 3.23.** Suppose $\lambda$ is given in (3.8) and $\mu$ given in (3.51). By (3.16),
\begin{align*}
\ell_1(\lambda, \mu) &= 2 - 1 + 2 - 2 = 1, \quad \ell_2(\lambda, \mu) = 3 - 1 + 3 - 4 = 1, \\
\ell_3(\lambda, \mu) &= 5 - 4 + 6 - 6 = 1, \quad \ell_4(\lambda, \mu) = 6 - 4 + 7 - 7 = 2.
\end{align*}

Thus $\ell(\lambda, \mu) = 7$.

One of the main results of this paper is the following.

**Theorem 3.24.** Suppose $\lambda, \mu$ are dominant weights. Then $K_{\lambda, \mu}(q) \neq 0$ if and only if $\mu \preceq \lambda$ and in this case
\begin{equation}
K_{\lambda, \mu}(q) = q^{\ell(\lambda, \mu)} \sum_{\sigma \in S^{\lambda, \mu}} q^{-2\ell(\sigma)} = q^{\ell(\lambda, \mu)} S^{\lambda, \mu}(q^{-2}),
\end{equation}
where $S^{\lambda, \mu}$ is defined by (3.44), and $S^{\lambda, \mu}(q)$ is the $q$-length function of $S^{\lambda, \mu}$ defined in (3.47) and can be determined by (3.62).

A weight $\lambda$ is said to be

- totally disconnected if $c_{s,t}(\lambda) = 0$ for all pairs $(s, t)$ with $s < t$;
- totally connected if $c_{s,t}(\lambda) = 1$ for all pairs $(s, t)$ with $s \leq t$.

From Theorem 3.24 one immediately obtains

**Corollary 3.25.** Let $\lambda, \mu \in P_{r, lex}^\lambda$ with $\mu \preceq \lambda$, and let $S^\lambda(q)$, $\tilde{S}^{\lambda, \mu}(q)$ be as in Proposition 3.18. We have
\begin{enumerate}
\item If $\lambda$ is totally connected, then $K_{\lambda, \mu}(q) = q^{\ell(\lambda, \mu)}$.
\item If $\lambda$ is totally disconnected, then $K_{\lambda, \mu}(q) = q^{\ell(\lambda, \mu)} \tilde{S}^{\lambda, \mu}(q^{-2})$.
\item If $\mu \preceq \lambda$ (recall (3.37)), then $K_{\lambda, \mu}(q) = q^{\ell(\lambda, \mu)} S^\lambda(q^{-2})$.
\end{enumerate}
3.8. **Proof of Theorem 3.24**. Using notations as above, we set (cf. (3.63))
\[ \Theta^\lambda_\mu = \{ \theta \in \mathbb{N}^r \mid R_\theta^r(\mu) = \lambda \}, \quad \text{where } r = \# \lambda. \]

In the three lemmas below, we shall establish a bijection between \( \Theta^\lambda_\mu \) and \( S^\lambda^\mu \). Theorem 3.24 is then a simple consequence of this bijection.

First let us define a map
\[ \Theta^\lambda_\mu \rightarrow \widetilde{S}^\lambda^\mu : \theta \mapsto \sigma_\theta \]  
(3.65)
in the following way. Suppose \( \theta \in \Theta^\lambda_\mu \), i.e. (cf. (3.15)) for notations \( m'_s, n'_s \),
\[ \lambda = R_\theta^r(\mu) = (R_{m'_1, n'_1}^{\theta_1} \circ \cdots \circ R_{m'_r, n'_r}^{\theta_r}(\mu))^+. \]

We denote
\[ \mu^{(s)} = \mu^{(s-1)} = R_{m'_s, n'_s}^{\theta_s}(\mu^{(s)}) \quad \text{for } s = r, r-1, \ldots, 1. \]  
(3.67)
Then each \( \mu^{(s-1)} \) is obtained from \( \mu^{(s)} \) by adding some number, say, \( K_s \), to its \( m'_s \)-th, \( n'_s \)-th entries. In fact (we use the notation \( k^{(s,i)}_s \) to denote the integer \( k_s \) defined in (3.32) with \( \lambda \) replaced by \( \mu^{(s,i)} \))
\[ K_s = \sum_{i=1}^{\theta_s} k^{(s,i)}_s, \quad \text{where } \mu^{(s,0)} = \mu^{(s)}, \ \mu^{(s,i)} = R_{m'_s, n'_s}^{\theta_s}(\mu^{(s,i-1)}). \]  
(3.68)
Thus
\[ \mu^{(0)} = \mu + \sum_{s=1}^r K_s \gamma_s. \]  
(3.69)
Since \( \lambda = (\mu^{(0)})^+ \) and both \( \lambda \) and \( \mu^{(0)} \) are regular, there exists a \( \sigma \in \text{Sym}_r \) uniquely determined by \( \theta \), such that (cf. (3.38) and (3.39))
\[ \text{aty}_{\mu^{(0)}} = \sigma(\text{aty}_\lambda) = \text{aty}_{\sigma, \lambda}, \quad \text{i.e.}, \quad (\mu^{(0)})^\sigma = \lambda^{\sigma}_{\dot{s}(\sigma)} \quad \text{for } s \in [1, r] \quad \text{(cf. (3.14))}. \]  
(3.70)
By (3.69), \( \text{aty}_\mu \leq \text{aty}_{\mu^{(0)}} = \text{aty}_{\sigma, \lambda} \), and by (3.13) and (3.36), \( \text{typ}_\mu = \text{typ}_\lambda \). But \( \text{typ}_\lambda = \text{typ}_{\sigma, \lambda} \) (typical tuples are invariant under the dot action of \( \text{Sym}_r \), cf. (3.3), (3.38) and (3.39)). Thus \( \mu \preceq \sigma \cdot \lambda \) by (3.13), i.e., \( \sigma \in \widetilde{S}^\lambda^\mu \) by definition (4.1). We denote \( \sigma \) by \( \sigma_{\theta} \). Thus we obtain the map (3.65).

**Lemma 3.26.** The map (3.65) is an injection. More precisely, suppose \( \sigma \in \text{Sym}_r \) such that \( \sigma = \sigma_{\theta} \) for some \( \theta = (\theta_1, \ldots, \theta_r) \in \Theta^\lambda_\mu \), then such \( \theta \) is unique and is given by
\[ \theta_s = \theta'_s - 2\ell(\sigma, s) \quad \text{(cf. (3.44))}, \quad \text{where} \]
\[ \theta'_s = \# Q_s, \quad Q_s = [\mu^{\sigma}_{\dot{n}_s} + 1, \lambda^{\sigma}_{\dot{n}_s} + 1] \setminus \text{Set}(\text{typ}_\lambda) \quad \text{for } s = r, r-1, \ldots, 1. \]  
(3.72)
Thus by (3.22), \( \theta'_s \) is in fact \( \theta'_s = h_{\sigma(s)}(\lambda) - h_{s}(\mu) + \sigma(s) - s \).

**Proof.** The proof of the lemma is divided into the following cases.

**Case 1:** \( s = r \).
We want to prove \( \theta_r = \theta'_r \). Note that each time when we apply \( R_{m'_r, n'_r}^{\theta_r} \) to \( \mu \), the \( r \)-th entry \( \mu^{\sigma}_{\dot{n}_r} \) of the atypical tuple \( \text{aty}_\mu \) reaches an integer in the set \( Q_r \) (cf. (3.32) and (3.33)), and no integer in this set \( Q_r \) can be skipped. Thus after applying \( \theta'_r \) times, this entry reaches the integer \( \lambda^{\sigma}_{\dot{n}_{\sigma(r)}} \) (it is an entry of \( \text{aty}_\lambda \), thus not in the typical entry set \( \text{Set}(\text{typ}_\lambda) \)), and \( \mu \) becomes \( \mu^{(r-1)} \) (cf. (3.67)). Thus \( \theta_r = \theta'_r \).

One can also use the following arguments to prove \( \theta_r = \theta'_r \): The integer \( \lambda^{\sigma}_{\dot{n}_{\sigma(r)}} \) is the \( r \)-th entry of the atypical tuple \( \text{aty}_{\mu^{(r-1)}} \), which by definitions (3.33) and (3.35) and by (3.32), is equal to the \( \theta_r \)-th smallest integer bigger than \( \mu^{\sigma}_{\dot{n}_r} \) and not in \( \text{Set}(\text{typ}_\lambda) \). But
by the definition of $\theta_r'$ in (3.72), $\lambda^\rho_{\hat{\eta}_{\sigma(r)}}$ is the $\theta_r'$-th smallest integer bigger than $\mu^\rho_{\hat{\eta}_r}$ and not in Set(typ$_\lambda$). Thus $\theta_r = \theta_r'$.

**Case 2:** $s = r - 1$ and $\sigma(r - 1) < \sigma(r)$.

We want to prove $\theta_{r-1} = \theta'_{r-1}$. There are two possibilities to consider.

**Subcase 2.i:** Suppose $\mu^\rho_{\hat{\eta}'_{r-1}} = \lambda^\rho_{\hat{\eta}_{\sigma(r-1)}}$. Then $\mu^{(r-2)} = \mu^{(r-1)}$ and we obviously have $\theta_{r-1} = 0 = \theta'_{r-1}$. We are done.

**Subcase 2.ii:** Suppose $\mu^\rho_{\hat{\eta}'_{r-1}} < \lambda^\rho_{\hat{\eta}_{\sigma(r-1)}}$. Note that $\lambda^\rho_{\hat{\eta}_{\sigma(r-1)}} < \lambda^\rho_{\hat{\eta}_{\sigma(r)}}$. Also note that

$$\text{Set}(\mu^{(r-1)}) = (\text{Set}(\mu^\rho) \setminus \{\mu^\rho_{\hat{\eta}_r}\}) \cup \{\lambda^\rho_{\hat{\eta}_{\sigma(r)}}\};$$

(recall the $\rho$-translated notation in (3.3) and note that the only difference between $\mu^{(r-1)}$ and $\mu$ is their $r$-th atypical entries). We observe that $\lambda^\rho_{\hat{\eta}_{\sigma(r-1)}}$ is not an entry of the typical tuple typ$_n = \text{typ}_\lambda$ because of the regularity of $\lambda$, it is not the $t$-th entry of the atypical tuple atyp$_n$ either for any $t \leq r - 1$ because of the assumption that $\mu^\rho_{\hat{\eta}'_{r-1}} < \lambda^\rho_{\hat{\eta}_{\sigma(r-1)}}$. Thus $\lambda^\rho_{\hat{\eta}_{\sigma(r-1)}}$ is not in (3.73). But it is in the set

$$[(\mu^{(r-1)})^\rho_{\hat{\eta}'_{r-1}}, (\mu^{(r-1)})^\rho_{\hat{\eta}'_{r-1}}] = [\mu^\rho_{\hat{\eta}'_{r-1}}, \lambda^\rho_{\hat{\eta}_{\sigma(r)}}].$$

Hence there is at least an integer in (3.74) but not in (3.73). By Definition 3.6, the $(r - 1)$-th and $r$-th atypical roots of $\mu^{(r-1)}$ are not $c$-related.

Similarly, the $(r - 1)$-th and $r$-th atypical roots of $\mu^{(r-1,i)}$ (cf. (3.68)) are not $c$-related for all $i$ with $1 \leq i < \theta'_{r-1}$. Thus by the arguments in Case 1, we need to apply the raising operator $R_{m'_{r-1},\hat{\eta}'_{r-1}}$ to $\mu^{(r-1)}$ exactly $\theta'_{r-1}$ times in order to obtain $\mu^{(r-2)}$. So $\theta'_{r-1} = \theta_{r-1}$.

**Case 3:** $s = r - 1$ and $\sigma(r - 1) > \sigma(r)$.

We want to prove $\theta_{r-1} = \theta'_{r-1} - 2$. Note that

$$(\mu^{(r-1)})^\rho_{\hat{\eta}'_{r-1}} = \mu^\rho_{\hat{\eta}'_{r-1}} < \mu^\rho_{\hat{\eta}_r} \leq (\mu^{(0)})^\rho_{\hat{\eta}_r} = \lambda^\rho_{\hat{\eta}_{\sigma(r)}} < \lambda^\rho_{\hat{\eta}_{\sigma(r-1)}} \quad \text{ (cf. (3.70))},$$

(recall that $\lambda, \mu$ are dominant). Thus we need to apply $R_{m'_{r-1},\hat{\eta}'_{r-1}}$ to $\mu^{(r-1)}$ at least once.

Suppose after applying $i$ times of $R_{m'_{r-1},\hat{\eta}'_{r-1}}$ to $\mu^{(r-1)}$ ($i$ can be zero), the $(r - 1)$-th entry $\mu^{(r-1)}_{\hat{\eta}'_{r-1}}$ of the atypical tuple atyp$_{\mu^{(r-1)}}$ reaches an integer, say $p$, such that

$$p < \lambda^\rho_{\hat{\eta}_{\sigma(r)}} \quad \text{ but } [p, \lambda^\rho_{\hat{\eta}_{\sigma(r)}}] \subset \text{Set}(\lambda^\rho).$$

By (3.75), such $i$ must exist since the $(r - 1)$-th entry will finally reach the integer $\lambda^\rho_{\hat{\eta}_{\sigma(r-1)}}$.

Note that $p$ and $\lambda^\rho_{\hat{\eta}_{\sigma(r)}}$ are respectively the $(r - 1)$-th and $r$-th entries of atyp$_{\mu^{(r-1,i)}}$ (cf. (3.68)). Thus by (3.70) and Definition 3.6 the $(r - 1)$-th and $r$-th atypical roots of $\mu^{(r-1,i)}$ are $c$-related. Then by (3.32) and (3.33), when we apply $R_{m'_{r-1},\hat{\eta}'_{r-1}}$ to $\mu^{(r-1,i)}$, the $(r - 1)$-th entry of atyp$_{\mu^{(r-1,i)}}$ reaches an integer in the set

$$[\lambda^\rho_{\hat{\eta}_{\sigma(r)}}, \lambda^\rho_{\hat{\eta}_{\sigma(r-1)}}] \setminus \text{Set}(\mu^{(r-1)}) = [\lambda^\rho_{\hat{\eta}_{\sigma(r)}}, \lambda^\rho_{\hat{\eta}_{\sigma(r-1)}}] \setminus \text{Set}(\mu^{(r-1,i)}),$$

(recall that Set(typ$_{\mu^{(r-1,i)}}$) = Set(typ$_\lambda$)), such that an integer in this set is skipped. So

$$\#\{[\lambda^\rho_{\hat{\eta}_{\sigma(r)}}, \lambda^\rho_{\hat{\eta}_{\sigma(r-1)}}] \setminus \text{Set}(\mu^{(r-1)})\} \geq 2.$$
In particular there is at least an element in the set \((3.77)\), which means
\[
\hat{c}_{\sigma(r),\sigma(r-1)}(\lambda) = 0 \quad \text{if} \quad \sigma(r-1) - \sigma(r) = 1 \quad \text{(cf. (3.26))}.
\] (3.79)
Note that \(\lambda^p_{\sigma(r)} \in \Q_{r-1} = [\mu^p_{\hat{n}_{r-1}} + 1, \lambda^p_{\sigma(r-1)}] \setminus \Set(\typ_{\lambda})\) by \([3.77]\), but not in the set \((3.77)\). Thus we have in fact skipped two integers in the set \(\Q_{r-1}\). Therefore
\[
\theta_{r-1} = \theta'_{r-1} - 2 \quad \text{if} \quad \sigma(r-1) > \sigma(r).
\]

**Case 4:** The general case.

In general, when we apply \(R_{m',\tilde{n}',u'}\) to \(\mu^{(s)}\) in order to obtain \(\mu^{(s-1)}\), for each \(t > s\) with \(\sigma(t) < \sigma(s)\) the above arguments show that we have to skip two integers in the set \(\Q_s\). Therefore we have \([3.71]\) and the lemma.

**Lemma 3.27.** The image of the map \([3.65]\) is contained in \(S^{\lambda,\mu}\), i.e., if \(\sigma = \sigma_\theta\) for some \(\theta \in \Theta^\lambda_{\mu}\), then
\[
\hat{c}_{\sigma(t),\sigma(s)}(\lambda) = 0 \quad \text{if} \quad \sigma(t) < \sigma(s) \quad \text{for any} \quad t > s.
\] (3.80)

**Proof.** Let \(p\) be the number such that \(\sigma(p)\) is minimal among those \(\sigma(u)\) with \(u < t\) and \(\sigma(u) > \sigma(t)\). Then \(p = s\) or \(\sigma(p) < \sigma(s)\). The definition \([3.27]\) means that the relation \(\hat{c}_{\sigma(t),\sigma(p)}(\lambda) = 0\) implies the relation \([3.80]\). Thus it suffices to prove \(\hat{c}_{\sigma(t),\sigma(p)}(\lambda) = 0\).

For any \(p'\) with \(\sigma(t) \leq \sigma(p') < \sigma(p)\), by the choice of \(p\), we have \(p' \geq t\) and so \(\sigma(p') > p\). Thus the arguments in the proof of Lemma \([3.26]\) show that when we apply \(R_{m'_p,\tilde{n}'_p,u'_p}\) to \(\mu^{(p)}\) in order to obtain \(\mu^{(p-1)}\), we need to pass over the integer \(\lambda^p_{\sigma(p')}\) and skip another integer for all such \(p'\). Hence there are at least \(2(\sigma(p) - \sigma(t))\) integers in the set \([\lambda^p_{\sigma(t)}, \lambda^p_{\sigma(p)}] \setminus \Set(\typ_{\lambda})\) (cf. \([3.78]\)). This means that there are at least \(\sigma(p) - \sigma(t)\) integers in the set \([\lambda^p_{\sigma(t)}, \lambda^p_{\sigma(p)}] \setminus \Set(\lambda^p)\) (since there are exactly \(\sigma(p) - \sigma(t)\) integers in \([\lambda^p_{\sigma(t)}, \lambda^p_{\sigma(p)}]\) which are entries of atypical tuple \(\typ_{\lambda}\)), which implies that \(c_{\sigma(t),\sigma(p)}(\lambda) = 0\) by \([3.26]\) and so \(\hat{c}_{\sigma(t),\sigma(p)}(\lambda) = 0\) (cf. \([3.79]\)).

**Lemma 3.28.** The map \([3.65]\) is a bijection between \(\Theta^\lambda_{\mu}\) and \(S^{\lambda,\mu}\).

**Proof.** For any \(\sigma \in S^{\lambda,\mu}\), we define \(\theta_s\) as in \([3.71]\). We want to prove
\[
\theta_s \geq 0 \quad \text{for} \quad s = 1, \ldots, r.
\] (3.81)
Suppose \(t > s\) such that \(\sigma(t) < \sigma(s)\). So
\[
\hat{c}_{\sigma(t),\sigma(s)}(\lambda) = 0.
\] (3.82)
Denote
\[
X_{s,t} = \{p > s \mid \sigma(t) \leq \sigma(p) < \sigma(s)\} \quad \text{and} \quad x_{s,t} = \#X_{s,t}.
\] (3.83)
First we prove by induction on \(\sigma(s) - \sigma(t)\) that
\[
\# \left( [\lambda^p_{\sigma(t)}, \lambda^p_{\sigma(s)}] \setminus \Set(\lambda^p) \right) \geq x_{s,t}.
\] (3.84)
If \(\sigma(s) - \sigma(t) = 1\), then \([3.84]\) follows from \([3.82]\) and Definition \([3.6]\) (note that obviously \(\sigma(s) - \sigma(t) \geq x_{s,t}\)).

In general set \(p' = x_{s,t}\) and we write the set \(\{\sigma(p) \mid p \in X_{s,t}\}\) in ascending order:
\[
\{\sigma(p) \mid p \in X_{s,t}\} = \{\sigma(t_1) < \sigma(t_2) < \cdots < \sigma(t_{p'})\},
\] (3.85)
where \(t_1 = t\) and \(\sigma(t_{p'}) < \sigma(s)\). Since \(\hat{c}_{\sigma(t),\sigma(s)}(\lambda) = 0\), by Definition \([3.6]\) there exists some number, denoted by \(\sigma(i)\), lies in between \(\sigma(t)\) and \(\sigma(s)\), i.e.,
\[
\sigma(t) < \sigma(i) \leq \sigma(s),
\] (3.86)
such that $c_{\sigma(t),\sigma(i)}(\lambda) = 0$, that is,
\[
\# \left( \{\lambda_{\sigma(t)}^0, \lambda_{\sigma(i)}^0\} \setminus \text{Set}(\lambda^0) \right) \geq \sigma(i) - \sigma(t) \quad (\text{cf. } \text{(3.19)}) \tag{3.87}
\]

Now we prove (3.84) by induction on $\sigma(s) - \sigma(t)$ in two cases.

**Case 1:** Suppose $i \in X_{s,t}$, say $i = t_{p''}$ for some $1 < p'' \leq p'$. Then
\[
\# \left( \{\lambda_{\sigma(t)}^0, \lambda_{\sigma(i)}^0\} \setminus \text{Set}(\lambda^0) \right) \geq x_{s,t_{p''}} \geq p' - p'' + 1, \tag{3.88}
\]
where the first inequality follows from the inductive assumption that (3.84) holds for $t_{p''}$ since $\sigma(s) - \sigma(t_{p''}) < \sigma(s) - \sigma(t)$, and the second from the fact that $t_{p''}, t_{p''+1}, \ldots, t_p' \in X_{s,t_p'}$. Then (3.84) follows from (3.88) and (3.87) (with $i$ replaced by $t_{p''}$) by noting that $\sigma(t_{p''}) - \sigma(t) \geq p'' - 1$ (cf. (3.86)) and that $p' = x_{s,t}$.

**Case 2:** Suppose $i \notin X_{s,t}$. This means that $i < s$. Let $p''$ be the minimal integer with $1 \leq p'' \leq p'$ such that $\sigma(t_{p''}) > \sigma(i)$. Then $\sigma(s) - \sigma(t_{p''}) < \sigma(s) - \sigma(t)$ and so (3.88) holds again in this case by the inductive assumption. Furthermore since $\sigma(i) - \sigma(t) < \sigma(s) - \sigma(t)$ (cf. (3.86)), the inductive assumption also gives
\[
\# \left( \{\lambda_{\sigma(t)}^0, \lambda_{\sigma(i)}^0\} \setminus \text{Set}(\lambda^0) \right) \geq x_{i,t} \geq p'' - 1, \tag{3.89}
\]
where the last inequality follows from the fact that
\[
t_1, \ldots, t_{p''-1} \in \{p > i \mid \sigma(s) \leq \sigma(p) \leq \sigma(i)\} = X_{i,t},
\]
(recall that $i < s$). Now (3.84) follows from (3.89) and (3.88). This completes the proof of (3.84).

Now set $p = \ell(\sigma, s)$ and write
\[
\{t > s \mid \sigma(t) < \sigma(s)\} = \{s_1, s_2, \ldots, s_p \mid \sigma(s_1) < \sigma(s_2) < \cdots < \sigma(s_p) < \sigma(s)\}. \tag{3.90}
\]
(Then the left-hand side of (3.90) is in fact the set $X_{s,s_1}$, cf. (3.35).) Thus (3.84) means that the set
\[
\{\lambda_{\sigma(s_1)}^0, \lambda_{\sigma(s_1)}^0\} \setminus \text{Set}(\lambda^0) \tag{3.91}
\]
has cardinality $\geq x_{s,s_1} = p$, and so the set
\[
\{\mu_{\sigma(s_1)}^0 + 1, \lambda_{\sigma(s_1)}^0\} \setminus \text{Set}(\lambda^0) \tag{3.92}
\]
has cardinality $\geq 2p$, because $\mu_{\sigma(s_1)}^0 < \mu_{\sigma(s_1)}^0 \leq \lambda_{\sigma(s_1)}^0 < \lambda_{\sigma(s_1)}^0$; i.e., we have $p$ more elements $\lambda_{\sigma(s_1)}^0$ with $i \in [1, p]$ which are not in the first set (3.91) but in the second set (3.92).

Then (3.72), (3.44) and the fact that (3.92) has cardinality $\geq 2p$ show that $\theta_s' \geq 2p$ and so $\theta_s \geq 0$. (In fact if $p > 0$ then $\theta_s \geq 1$ since in this case there exists at least one more integer $\lambda_{\sigma(s_1)}^0$ which is in $[\mu_{\sigma(s_1)}^0 + 1, \lambda_{\sigma(s_1)}^0] \setminus \text{Set}(\lambda^0)$.) This proves (3.81).

Now we define $\mu^{(0)}$ as in (3.67). Then the arguments in the proof of Lemma 3.26 show that (3.70) holds, i.e., $(\mu^{(0)})^+ = \lambda$. Thus $0 \in \Theta^\lambda_{\mu}$ and $0 = \sigma_0$. Therefore $\theta \mapsto \sigma_0$ is a bijection between $\Theta^\lambda_{\mu}$ and $S^\lambda_{\mu}$.

**Proof of Theorem 3.24.** Finally we return to the proof of Theorem 3.24. By (3.17), (3.23), (3.43) and (3.72), we have
\[
|\theta| = \sum_{s=1}^r \theta_s' - \sum_{s=1}^r \ell(s, s) = \ell(\lambda, \mu) - \ell(\sigma).
\]
Now (3.64) follows from (3.63).
3.9. A correspondence between \( r \)-fold atypical modules over \( \mathfrak{gl}_{m|n} \) and \( \mathfrak{gl}_{r|r} \).

A subset \( B \subset \mathbb{Z}^{m|n}_+ \) of dominant integral weights is called a block of \( \mathbb{Z}^{m|n}_+ \) for \( \mathfrak{gl}_{m|n} \) if it is a maximal subset such that for any two weights \( \lambda, \mu \in B \), there exist weights \( \mu^1, \mu^2, \ldots, \mu^k \in B \) with \( \lambda = \mu^1, \mu^k = \mu \) such that the extension group 
\[
\text{Ext}^1(V(\mu), V(\mu^{i+1})) \neq 0 \quad \text{for} \quad i = 1, \ldots, k - 1.
\]
Then \( \mathbb{Z}^{m|n}_+ \) is divided into a disjoint union of blocks. Lemma 1.12 in [1] says that for any \( \lambda, \mu \in B \), one has \( \#\lambda = \#\mu \), which is called the degree of atypical type of \( B \), and denoted by \( \#B \). Let \( \mathcal{E}^{m|n} \) be the category of finite dimensional \( \mathfrak{gl}_{m|n} \)-modules. A dominant weight \( \mu \) is called a primitive weight of a module \( V \) if it is the highest weight of a composition factor of \( V \). Denote by \( \text{Prim}(V) \) the set of primitive weights of \( V \). For \( \lambda, \mu \in \mathbb{Z}^{m|n}_+ \), we denote by \( a_{\lambda,\mu} = |\mathcal{V}(\mu) : V(\mu)| \) the multiplicity of the composition factor \( V(\mu) \) in the Kac-module \( \mathcal{V}(\lambda) \). It was proved in [2] that
\[
a_{\lambda,\mu} \leq 1, \quad \text{and the matrix } (a_{\lambda,\mu}) \text{ is the inverse of the matrix } (K_{\lambda,\mu}(-1)), \quad (3.93)
\]

where the matrices were defined with respect to some total order of weights compatible with the partial order “\( \leq \)”. An application of Theorem 3.24 is the following.

**Theorem 3.29.** Let \( B \) be a block of \( \mathbb{Z}^{m|n}_+ \) for \( \mathfrak{gl}_{m|n} \) with \( \#B = r \). Let \( B' \) be the unique block of \( \mathbb{Z}^{r|r}_+ \) for \( \mathfrak{gl}_{r|r} \) with \( \#B' = r \).

1. There exists a bijection
\[
\phi : B \to B', \quad \phi(\lambda) = (h'(\lambda) | h(\lambda)), \quad (3.94)
\]
where \( h(\lambda) \) is the height vector of \( \lambda \) defined by (3.7) and \( h'(\lambda) = (h_r(\lambda), \ldots, h_1(\lambda)) \).

2. The Kazhdan-Lusztig polynomials \( K_{\lambda,\mu}(q) \) of \( \mathfrak{gl}_{m|n} \) and \( K_{\phi(\lambda),\phi(\mu)}(q) \) of \( \mathfrak{gl}_{r|r} \) coincide, that is,
\[
K_{\lambda,\mu}(q) = K_{\phi(\lambda),\phi(\mu)}(q) \quad (3.95)
\]
for \( \lambda, \mu \in B \).

3. Under the mapping (3.94), the set \( \text{Prim}(\mathcal{V}(\lambda)) \) of primitive weights of the \( \mathfrak{gl}_{m|n} \)-Kac-module \( \mathcal{V}(\lambda) \) for \( \lambda \in B \) is in one to one correspondence to the set \( \text{Prim}(\mathcal{V}(\phi(\lambda))) \) of primitive weights of the \( \mathfrak{gl}_{r|r} \)-Kac-module \( \mathcal{V}(\phi(\lambda)) \), namely
\[
\text{Prim}(\mathcal{V}(\phi(\lambda))) = \{\phi(\mu) \mid \mu \in \text{Prim}(\mathcal{V}(\lambda))\}. \quad (3.96)
\]

**Proof.** [1] From Theorem 3.24 (cf. (3.31)) we can deduce that any two dominant weights are in the same block if and only if their typical tuples are equal (thus blocks of \( \mathbb{Z}^{m|n}_+ \) are in one to one correspondence with the typical tuples). Therefore for any block \( B \) of \( \mathbb{Z}^{m|n}_+ \) with \( \#B = r \), we can denote the typical tuple of any weight in \( B \) by
\[
\text{tt}_B = (tt_1, \ldots, tt_{m-r} | tt_{r+1}, \ldots, tt_{n}), \quad (3.97)
\]
where \( tt_1 > \ldots > tt_{m-r}, tt_{r+1} < \ldots < tt_n \) are all distinct.

First we prove that the map \( \phi \) is injective, which is equivalent to proving that for any two dominant weights \( \lambda, \mu \in B \) with \( h(\lambda) = h(\mu) \), one has the equality of the atypical tuples \( \text{aty}_\lambda = \text{aty}_\mu \). Thus assume that \( \lambda_0^0 \neq \mu_0^0 \) for some \( s \). Say, \( \lambda_0^0 > \mu_0^0 \). Then the right-hand side of (3.22) with \( t = s \) is zero, but the left-hand side is not since \( \lambda_0^0 \in [\mu_{n'_s}^0 + 1, \lambda_{n'_s}^0] \setminus \text{Set(\text{tt}_B)} \).
Next we prove that the map $\phi$ is surjective, which is equivalent to proving that for any lexical $r$-tuple $a = (a_1, ..., a_r) \in \mathbb{Z}^r$, there exists $\lambda \in \mathcal{B}$ such that $h(\lambda) = a$. Fix a lexical $r$-tuple $b = (b_1, ..., b_r) \in \mathbb{Z}^r$ such that $b_r < \min\{tt_{m-r}, tt_{r+1}, a_1\} - r$. Let
\[
\mu = -\rho + (tt_{1}, ..., tt_{m-r}, b_r + r, ..., b_1 + 1 | b_1 + 1, ..., b_r + r, tt_{r+1}, ..., tt_{n}) \in \mathbb{Z}^{|m|n}. \tag{3.98}
\]
Then $\mu$ is dominant such that $\text{typ}_\mu = tt_\mathcal{B}$ (thus $\mu \in \mathcal{B}$) and $h(\mu) = b$. Take $t = s$ and $h_s(\lambda) = a_s$ in (3.22), then (3.22) uniquely determines a number $\lambda^\rho_s \in \text{Set}(tt_\mathcal{B})$. This uniquely determines a weight $\lambda \in \mathcal{B}$. To see $h(\lambda) = a$, use (3.22) again. Thus $\phi$ is a bijection.

In fact the atypical tuple $\text{aty}_\lambda$ determines the height vector $h(\lambda)$ through the following formula:
\[
h_s(\lambda) + s = \lambda^\rho_s - \#\{\text{typical entries which are smaller than } \lambda^\rho_s\}. \tag{3.99}
\]
Conversely the height vector $h(\lambda)$ determines the atypical tuple $\text{aty}_\lambda$ as follows: First set $\text{aty}_\lambda = h(\lambda)$, and denote $\text{aty}_\lambda$ as $\text{aty}_\lambda = (a_1, ..., a_r)$. Label the typical entries of $\text{typ}_\lambda$ in ascending order: $x_1 < x_2 < ... < x_{m+n-r}$. For each $i = 1, 2, ..., m + n - r$, if $a_t \geq x_i$ then replace $a_t$ by $a_t + 1$ for all $t = 1, 2, ..., r$.

Note that in $\mathfrak{g}_l^{ir}$, we have $\text{aty}_{\phi(\lambda)} = (\phi(\lambda))^{-\rho_r | r}$ (cf. notation (3.3)), where $\rho_r | r = (r, ..., 1 | 1, ..., r)$, and the typical tuple $\text{typ}_{\phi(\lambda)}$ is empty. Also we have the equality of the height vectors: $h(\lambda) = h(\phi(\lambda))$. Thus by Definition 3.6 we have $c_{s,t}(\lambda) = c_{s,t}(\phi(\lambda))$ for all $s, t \in [1, r]$. Let $\mu$ be another dominant weight with $\mu \leq \lambda$. By (3.22) and (3.61), we have $b^{\lambda, \mu} = b^{\phi(\lambda), \phi(\mu)}$. This gives (3.95) by (3.60) and (3.64).

Using (3.93) and (3.95), we have
\[
\text{Prim}(\mathcal{V}(\phi(\lambda))) = \{\mu' | a_{\phi(\lambda), \mu'} = 1\}
= \{\phi(\mu) | a_{\phi(\lambda), \phi(\mu)} = 1\} = \{\phi(\mu) | a_{\lambda, \mu} = 1\} = \phi(\text{Prim}(\mathcal{V}(\lambda))). \tag{3.96}
\]

4. Character formulae

As mentioned in the introduction, this section contains three main results: the proof of the conjecture due to van der Jeugt et al, the construction of a Kac-Weyl type character formula, and the derivation of a dimension formula.

We shall continue to use notations in the previous sections. Moreover, we define
\[
m(\Lambda)_\lambda = \#S^{\Lambda, \lambda} \tag{4.1}
\]
to be the cardinality of the set $S^{\Lambda, \lambda}$ (cf. (3.30)). Then $m(\Lambda)_\lambda = S^{\Lambda, \lambda}(1)$ (cf. (3.35)).

For any weight $\lambda \in \mathcal{P}$, we define what is called the Kac-character of $\lambda$:
\[
\chi^{\text{Kac}}(\lambda) = \frac{L_1}{L_0} \sum_{w \in \mathcal{W}} \epsilon(w)e^{w(\lambda + \rho)}. \tag{4.2}
\]
Namely, $\chi^{\text{Kac}}(\lambda)$ is defined by the right-hand side of (2.13). Thus it is the character of the Kac-module $\mathcal{V}(\lambda)$ when $\lambda$ is dominant.

4.1. Proof of the conjecture of van der Jeugt et al. As an immediate consequence of Theorem 3.24, we have

**Theorem 4.1.** The formal character $\text{ch}V(\Lambda)$ of the finite dimensional irreducible $\mathfrak{g}$-module $V(\Lambda)$ is given by
\[
\text{ch}V(\Lambda) = \sum_{\lambda \in \mathcal{P}_+, \text{dominant}} (-1)^{\ell(\Lambda, \lambda)} m(\Lambda)_\lambda \chi^{\text{Kac}}(\lambda), \tag{4.3}
\]
where $P_+$ is the set of dominant integral weights, the length $\ell(\Lambda, \lambda)$ is defined in (3.14), and the partial order “$\preceq$” is defined in (3.13).

Proof. This follows from (3.4) and (1.1) and [12, Lemma 3.4], which states
\[
\text{ch}\, V(\Lambda) = \sum_{\lambda \in P_+} K_{\lambda, \lambda}(-1)\chi^{\text{Kac}}(\lambda).
\]

One can re-write (4.3) to obtain the conjecture of van der Jeugt et al. To state it, we need to introduce the following notations.

Let $\lambda$ as in (3.1) be an $r$-fold atypical weight (not necessarily dominant) with atypical roots ordered as in (3.2): $\gamma_1 < \ldots < \gamma_r$. We define the normal cone with vertex $\lambda$:
\[
\mathcal{C}_\lambda^\text{Nor} = \{\lambda - \sum_{s=1}^r i_s \gamma_s | i_s \geq 0\}.
\]

We also define $\mathcal{C}_\lambda^\text{Trun}$, which was referred to as the truncated cone with vertex $\lambda$ in [16], to be the subset of $\mathcal{C}_\lambda^\text{Nor}$ consisting of weights $\mu$ such that the $s$-th entry of the atypical tuple $\text{aty}_\mu$ (cf. (2.9)) is smaller than or equal to the $t$-th entry of $\text{aty}_\mu$ when the atypical roots $\gamma_s, \gamma_t$ of $\lambda$ (not $\mu$) are strongly $c$-related for $s < t$. Namely (recall the $\rho$-translated notation in (3.3))
\[
\mathcal{C}_\lambda^\text{Trun} = \{\mu \in \mathcal{C}_\lambda^\text{Nor} | \mu^\rho_{\gamma_s} \leq \mu^\rho_{\gamma_t} \text{ if } \widehat{c}_{s,t}(\lambda) = 1 \text{ for } s < t\}.
\]

For $\lambda = \Lambda - \sum_{s=1}^r i_s \hat{\gamma}_s \in \mathcal{C}_\lambda^\text{Nor}$, we denote
\[
|\Lambda - \lambda| = \sum_{s=1}^r i_s \quad \text{(called the relative level of } \lambda\text{)}.
\]

As an application of Theorem 4.1 we prove the following character formula which was a conjecture put forward by van der Jeugt, Hughes, King and Thierry-Mieg in [16] as the result of in depth research carried out by the authors over several years time.

Theorem 4.2.
\[
\text{ch}\, V(\Lambda) = \sum_{\lambda \in \mathcal{C}_\lambda^\text{Trun}} (-1)^{|\Lambda - \lambda|}\chi^{\text{Kac}}(\lambda).
\]

Proof. First we remark that for any weight $\lambda \in P$, we have (cf. (1.2))
\[
\chi^{\text{Kac}}(\lambda) = 0 \quad \text{if } \lambda \text{ is vanishing (i.e., not regular).}
\]

Note that for any weight $\lambda$ in the truncated cone $\mathcal{C}_\lambda^\text{Trun}$ or in the set
\[
R_\Lambda := \{\mu \in P_+ | \mu \preceq \Lambda\} \quad \text{(cf. the right-hand side of (1.3)),}
\]
we have the equality of the typical tuples:
\[
\text{typ}_\lambda = \text{typ}_\Lambda \quad \text{(cf. (3.5)).}
\]

Thus $\lambda$ is uniquely determined by the atypical tuple $\text{aty}_\lambda$ (cf. (2.9)). Clearly for any $\mu \in \mathcal{C}_\lambda^\text{Trun}$, it is either vanishing (so $\chi^{\text{Kac}}(\mu) = 0$) or is $W$-conjugate under the dot action to some unique $\lambda \in R_\Lambda$, where the later fact is equivalent to the existence of a unique $\sigma \in \text{Sym}_r$ satisfying $\text{aty}_{\sigma, \mu} = \text{aty}_\lambda$ (cf. (3.38) and (3.39)). If $\sigma(t) < \sigma(s)$ for some $1 \leq s < t \leq r$. Then
\[
\mu^\rho_{\hat{\gamma}_s(t)} = \lambda^\rho_{\hat{\gamma}_t} > \lambda^\rho_{\hat{\gamma}_s} = \mu^\rho_{\hat{\gamma}_s(s)}.
\]

Thus $\widehat{c}_{\sigma(t), \sigma(s)}(\lambda) = 0$ by (4.5). Therefore $\sigma \in S^\Lambda$ by (3.30). Obviously $\text{aty}_\lambda = \text{aty}_{\sigma, \mu} \preceq \text{aty}_{\sigma, \Lambda}$ (cf. (3.14)), i.e., $\sigma \in S^{\Lambda, \lambda}$ by definitions (3.41) and (3.13) and the fact that...
Thus we have the equality of typical tuples: \( \text{typ}_\sigma \lambda = \text{typ}_\lambda (\text{cf. } (4.10)) \). Conversely, for any \( \lambda \in R_\Lambda \) and \( \sigma \in S_\Lambda, \lambda \), there corresponds to a unique \( \mu \in C^\text{Trun}_\Lambda \) such that \( \text{typ}_\sigma \mu = \text{typ}_\lambda \).

For any \( \lambda \in R_\Lambda \), let \( \tilde{\lambda} \in C^\text{Trun}_\Lambda \cap W \cdot \lambda \) be the (unique) lexical weight in the sense of Definition 3.9. Then

\[
\chi^Kac(\mu) = \chi^Kac(\tilde{\lambda}) \quad \text{for } \mu \in C^\text{Trun}_\Lambda \cap W \cdot \lambda,
\]

since elements of \( Sym_r \) correspond to elements of \( W \) with even parity (cf. (3.38) and (3.39)).

The above arguments have in fact shown that the right-hand side of (4.7) is equal to

\[
\sum_{\lambda \in P_\Lambda : \lambda \leq \Lambda} (-1)^{|A - \lambda|} \sum_{\sigma \in S_\Lambda, \lambda} \chi^Kac(\tilde{\lambda}) = \sum_{\lambda \in P_\Lambda : \lambda \leq \Lambda} (-1)^{|A - \lambda|} m(\lambda) \chi^Kac(\tilde{\lambda}).
\]

By (4.3), what remains to prove is the following: if \( w(\tilde{\lambda}^\rho) = \lambda^\rho \) for \( w \in W, \lambda \in R_\Lambda \), then

\[
(-1)^{|A - \lambda|} = (-1)^{\ell(A, \lambda)} \epsilon(w).
\]

Note that in order to obtain \( \lambda^\rho \) from \( \tilde{\lambda}^\rho \) by moving all entries \( \tilde{\lambda}^\rho_{m,s}, \tilde{\lambda}^\rho_{n,s} \) for \( s = 1, \ldots, r \) to suitable positions step by step, each time exchanging nearest neighbor entries only, the total number of movements is \( N = \sum_{s=1}^r N_s \), where

\[
N_s = \#([\tilde{\lambda}^\rho_{n_s} + 1, \Lambda^\rho_{n_s}] \cap \text{Set}(\text{typ}_\Lambda)) \quad (\text{cf. notation } (3.20)).
\]

To see this, note that \( \tilde{\lambda} \) is obtained from \( \Lambda \) by subtracting some atypical roots (cf. (4.11)). Thus the entries \( \tilde{\lambda}^\rho_{m,s}, \tilde{\lambda}^\rho_{n,s} \) of \( \tilde{\lambda}^\rho \), which seat at the positions \( m_s, n_s \) of \( \Lambda \) should be moved to appropriate positions in order to make the resultant weight \( \lambda^\rho \) dominant. The number of steps needed for these two entries is obviously \( N_s \).

From (3.17) and (3.16), we have \( |A - \lambda| = \sum_{s=1}^r (\Lambda_{n_s} - \lambda_{n_s}) = \ell(A, \lambda) + N \). But \( \epsilon(w) = (-1)^N \), we obtain (4.13). \( \square \)

### 4.2. Definitions of \( \lambda_t \) and \( C_r \)

Our purpose is to re-write (4.7) into a finite sum. Since the sum over the truncated cone \( C^\text{Trun}_\Lambda \) (cf. 4.7) is difficult to compute, we want to change this sum into several sums over some normal cones \( C^\text{Norm}_\mu \) by making use of the fact that the Kac-character \( \chi^Kac(\lambda) \) is \( Sym_r \)-invariant under the dot action (cf. (3.38) and (3.39)), as a sum over a cone \( C^\text{Norm}_\mu \) is easy to compute.

This will be done in two steps.

First we need to introduce more notations. Define another partial order "\( \leq \)" on \( C^\text{Norm}_\Lambda \) such that for \( \lambda, \mu \in C^\text{Norm}_\Lambda \),

\[
\mu \leq \lambda \iff \text{every entry of } \mu \leq \text{the corresponding entry of } \lambda.
\]

**Definition 4.3.** For \( \lambda \in C^\text{Norm}_\Lambda \), denote by \( \lambda_t \in C^\text{Norm}_\Lambda \) the maximal lexical weight (cf. (3.28)) which is \( \leq \lambda \), namely,

\[
\lambda_t = \max \{ \mu \in C^\text{Norm}_\Lambda \mid \mu \leq \lambda, \text{ and } \mu \text{ is lexical} \}.
\]

Thus we have the equality of typical tuples: \( \text{typ}_{\lambda_t} = \text{typ}_\lambda \) (cf. (3.34)) and the entries of atypical tuple \( \text{aty}_{\lambda_t} \) (cf. (3.41)) are defined by

\[
(\lambda_t)_{n_t}^s = \min \{ \lambda_{n_t}^s \mid s \leq t \leq r \} \quad \text{for } s = 1, \ldots, r.
\]
Denote by $C^{\text{Lexi}}_\Lambda$, called the \textit{lexical cone with vertex} $\lambda$, the subset of the truncated cone $C^{\text{Trun}}_{\Lambda}$ (cf. \eqref{eq:4.28}) consisting of lexical weights (cf. \eqref{eq:3.29}), namely
\begin{equation}
C^{\text{Lexi}}_\Lambda = \{ \mu \in C^{\text{Trun}}_{\Lambda} \mid \mu \text{ is lexical} \}. \tag{4.16}
\end{equation}

Our first step is to change the sum over $C^{\text{Trun}}_{\Lambda}$ in \eqref{eq:4.17} to several sums over some lexical cones $C^{\text{Lexi}}_{(\sigma \cdot \Lambda)\uparrow}$ (see Proposition 4.4).

The proof of Theorem 3.24 (cf. the arguments of proving \eqref{eq:4.12}) and \eqref{eq:4.8} show that \eqref{eq:4.17} can be re-written as
\begin{equation}
\text{ch} V(\Lambda) = \sum_{\sigma \in S_\Lambda} \sum_{\lambda \in C^{\text{Lexi}}_{(\sigma \cdot \Lambda)\uparrow}} (-1)^{|\Lambda - \lambda|} \chi^{\text{Kac}}(\lambda). \tag{4.17}
\end{equation}

The definition of $\lambda_\ast$ in \eqref{eq:4.15} shows that we have the equality of the following two sets of regular weights:
\begin{equation}
\{ \lambda \in C^{\text{Lexi}}_\Lambda \mid \lambda \preceq \sigma \cdot \Lambda, \lambda \text{ is regular} \} = \{ \lambda \in C^{\text{Lexi}}_{(\sigma \cdot \Lambda)\uparrow} \mid \lambda \text{ is regular} \}. \tag{4.18}
\end{equation}

Thus \eqref{eq:4.17} leads to the following.

\begin{proposition} \tag{4.19}
\begin{equation}
\text{ch} V(\Lambda) = \sum_{\sigma \in S_\Lambda} \sum_{\lambda \in C^{\text{Lexi}}_{(\sigma \cdot \Lambda)\uparrow}} (-1)^{|\Lambda - \lambda|} \chi^{\text{Kac}}(\lambda). \tag{4.19}
\end{equation}
\end{proposition}

The second step is to change the sum over the lexical cone $C^{\text{Lexi}}_{(\sigma \cdot \Lambda)\uparrow}$ in \eqref{eq:4.19} to several sums over the normal cones $C^{\text{Norm}}_{(\pi \cdot (\sigma \cdot \Lambda)\uparrow)}$. This will be achieved by Lemma 4.6 below. To state lemma, we need some further notations.

\begin{definition} \tag{4.4}
Denote by $C_r$ the subset of $\text{Sym}_r$ consisting of permutations $\pi$ which can be written as a product of cyclic permutations of the form
\begin{equation}
\pi = (1, 2, \ldots, i_1)(i_1+1, i_1+2, \ldots, i_1+i_2) \cdots (i_1+\ldots+i_{t-1}+1, i_1+\ldots+i_{t-1}+2, \ldots, r), \tag{4.20}
\end{equation}
where $i_1, \ldots, i_t$ are positive integers such that $\sum_{s=1}^t i_s = r$ (namely, $(i_1, \ldots, i_t)$ is a \textit{composition} of $r$). Associated to $\pi$, there is the multi-nomial coefficient
\begin{equation}
\binom{r}{\pi} = \frac{r!}{i_1! \cdots i_t!}. \tag{4.21}
\end{equation}
\end{definition}

\subsection{A technical lemma.} \tag{4.3}
The following technical lemma is crucial in obtaining our character formula in Theorem 4.9.

\begin{lemma} \tag{4.6}
Let $\lambda \in C^{\text{Lexi}}_\Lambda$. We have
\begin{equation}
\sum_{\mu \in C^{\text{Lexi}}_\Lambda} (-1)^{|\Lambda - \mu|} \chi^{\text{Kac}}(\mu) = \frac{1}{r!} \sum_{\pi \in C_r} \binom{r}{\pi} (-1)^{\ell(\pi)} \sum_{\mu \in C^{\text{Norm}}_{(\sigma \cdot \Lambda)\uparrow}} (-1)^{|\Lambda - \mu|} \chi^{\text{Kac}}(\mu), \tag{4.22}
\end{equation}
where $\ell(\pi)$ is the length of $\pi$ (cf. \eqref{eq:3.43}), namely $\ell(\pi) = \sum_{s=1}^t (i_s - 1)$ for $\pi$ in \eqref{eq:4.20}.
\end{lemma}

\begin{proof}
We denote by $C'_r$, the set of compositions of $r$, and denote
\begin{equation}
\pi_\downarrow = \pi \text{ which is defined by } \eqref{eq:4.20} \text{ for } \pi = (i_1, \ldots, i_t) \in C'_r. \tag{4.23}
\end{equation}
We also denote \eqref{eq:4.21} by \eqref{eq:4.21}. For any subset $S$ of $C^{\text{Norm}}_{\Lambda}$, denote by $S^{\text{reg}}$ the regular weights of $S$. We define
\begin{equation}
\chi^{\text{sum}}(S) := \sum_{\mu \in S} (-1)^{|\Lambda - \mu|} \chi^{\text{Kac}}(\mu). \tag{4.24}
\end{equation}
\end{proof}
Thus
\[ \chi^{\text{sum}}(S) = \chi^{\text{sum}}(S^{\text{reg}}). \] (4.25)

Note that any element \( \mu \in C_{\Lambda}^{\text{Norm}} \) is uniquely determined by the atypical tuple \( \text{aty}_{\mu} \) (cf. (3.4)), thus there is a bijection
\[ \varphi : \Lambda + \sum_{s=1}^{r} \mathbb{Z} \gamma_s \to \mathbb{Z}^r, \quad \varphi(\mu) = \text{aty}_{\mu} = (\mu_{\ell_1}, ..., \mu_{\ell_r}). \] (4.26)

For simplicity, we use \( g = \varphi(\lambda) = (g_1, ..., g_r) \) to represent \( \lambda \) (and transfer all terminologies to \( g \)). Then the normal cone \( C_{\lambda}^{\text{Norm}} \) defined in (4.16) and the lexical cone \( C_{\lambda}^{\text{Lexi}} \) defined in (4.16) correspond to the following sets respectively:
\[ C_g^{\text{Norm}} = \{ x = (x_1, ..., x_r) \in \mathbb{Z}^r \mid x_s \leq g_s, s = 1, ..., r \}, \]
\[ C_g^{\text{Lexi}} = \{ x \in \mathbb{Z}^r \mid x_1 \leq x_2 \leq \cdots \leq x_s \text{ and } x_s \leq g_s, s = 1, ..., r \}. \] (4.27)

We define the half-lexical cone with vertex \( \lambda \):
\[ H_{g_1}(g_2, ..., g_r) = \chi^{\text{sum}}(C_g^{\text{Lexi}}), \quad \mathcal{L}(g_1, ..., g_r) = \chi^{\text{sum}}(C_g^{\text{Half}}), \] (4.28)
which are the sign sums of Kac-characters over the normal cone \( C_g^{\text{Norm}} \), the lexical cone \( C_g^{\text{Lexi}} \) and the half-lexical cone \( C_g^{\text{Half}} \) respectively (cf. (4.24)). Then (4.22) is equivalent to
\[ \mathcal{L}(g_1, ..., g_r) = \frac{1}{r!} \sum_{\underline{i} \in \mathbb{I}_r} \left( \begin{array}{c} r \\ \underline{i} \end{array} \right) (-1)^{\ell(\underline{i})} \mathcal{N}(g_{\underline{i}}), \] (4.29)
where \( g_{\underline{i}} \) is defined by
\[ g_{\underline{i}} = (g_1, ..., g_1, g_{i_1+1}, ..., g_{i_1+1}, ..., g_{\sum_{s=1}^{r-i_1} i_s+1}, ..., g_{\sum_{s=1}^{r-i_1} i_s+1}) \] (4.30)
for \( \underline{i} = (i_1, ..., i_t) \). This is because from the definition (4.13), one can easily check
\[ g_2 = \varphi((\pi_2 \cdot \lambda)_t) \] (4.31)
Denote
\[ g^{(s)} = (g_1, ..., g_1, g_{s+1}, ..., g_r) = g|_{g_2=...=g_s=g_1} \in \mathbb{Z}^r, \]
\[ g^{(s)} = (g_1, ..., g_1, g_{s+1}, ..., g_r) \in \mathbb{Z}^{r-1}, \]
\( (g^{(s)}) \) is obtained from \( g^{(s)} \) by deleting the first entry) for \( s = 1, ..., r \). Note that
\[ H_{g_1}(g^{(1)}) = \sum_{s=1}^{r} \mathcal{L}(g^{(s)}). \] (4.32)
To see this, observe that for any regular weight \( x = (x_1, ..., x_r) \in C_g^{\text{Half}} \) (cf. (4.25)), there is a unique \( s \in [1, r] \) such that
\[ x_2 < ... < x_s < x_1 < x_{s+1} < ... < x_r. \] (4.33)
So \( x \) is \( Sym_r \)-conjugate (thus under the inverse mapping \( \varphi^{-1} \), it is \( Sym_r \)-conjugate under the dot action, cf. (3.38), (3.39)) to
\[ (x_2, ..., x_s, x_1, x_{s+1}, ..., x_r) \in C_{g^{(s)}}^{\text{Lexi}} \] since \( x_1 < g_1 \).
Conversely every regular weight of \( C_{g^{(s)}}^{\text{Lexi}} \) is \( Sym_r \)-conjugate under the dot action to a unique weight \( x \) of \( C_g^{\text{Half}} \) satisfying (4.33). This proves (4.32).
By (4.32), we obtain
\[ \mathcal{L}(g) = \mathcal{L}(g^{(1)}) = \mathcal{H}_{g_1}(g^{(1)}) - \sum_{s=2}^{r} \frac{(-1)^{s-1}}{s(s-1)} \mathcal{H}_{g_1}(g^{(s)}). \] (4.34)

Since the first variable \( x_1 \) in \( C^\text{Half}_1 \) does not relate to any other variable, when \( g_1 \) is fixed, \( \mathcal{H}_{g_1}(g^{(s)}) \) is in fact \( \mathcal{L}(g^{(s)}) \) with respect to the \( r-1 \) variables \( g_2, \ldots, g_r \). By the inductive assumption on \( r \) that (4.29) holds for \( r-1 \), we have
\[ \mathcal{H}_{g_1}(g^{(s)}) = \frac{1}{(r-1)!} \sum_{j \in C_{r-1}} \binom{r-1}{j} (-1)^{j} \mathcal{N}(g^{(s)}_{(1,j)}), \quad s = 1, \ldots, r, \] (4.35)

where
\[
(1, j) = \begin{cases} 
(1, j_1, j_2, \ldots) & \text{if } s = 1, \\
(j_1 + 1, j_2, \ldots) & \text{otherwise, for } j = (j_1, j_2, \ldots) \in C_{r-1}.
\end{cases}
\]

Thus \((1, j)\) is a composition of \( r \). For \( i \in C_r \), \( g^{(s)}_i \) is defined by (4.30) with \( g_2, \ldots, g_s \) being set to \( g_1 \). Thus each \( g^{(s)}_{(1,i)} \) has the form \( g_1 \) for some \( i \in C_r \), and
\[
g^{(s)}_{(1,i)} = g_1 \iff \{ i_1 = 1, j = (i_2, i_3, \ldots) \text{ if } s = 1, \text{ or } i_1 \geq s, 1 + j_1 + \ldots + j_s = i_1 \text{ and } (j_{s+1}, j_{s+2}, \ldots) = (i_2, i_3, \ldots) \text{ otherwise.}
\]

From this one can prove by induction on \( r \) and \( s \) that the coefficient of \( \mathcal{N}(g_1) \) in (4.35) is \( \frac{1}{i_1! \cdots i_t!} b_{s,i_1} \), where \( b_{0,i_1} = 0 \) and
\[
b_{s,i_1} = \frac{1}{i_1!} \left( \frac{1}{i_1!(i_1-1)!} + \frac{1}{2!(i_1-2)!} + \cdots + (-1)^{i_1-s} \frac{1}{s!(i_1-s)!} \right) \quad \text{for } 1 \leq s \leq i_1.
\]

Using this in (4.34), we obtain that the coefficient of \( \mathcal{N}(g_1) \) in \( \mathcal{L}(g) \) is \( \frac{1}{i_1! \cdots i_t!} \). This proves (4.29) and the lemma. \( \square \)

**Remark 4.7.** Note that a special case of (4.29) is when \( g_1 = \ldots = g_r \). In this case we have
\[
\mathcal{L}(g_1, \ldots, g_1) = \frac{1}{r!} \mathcal{N}(g_1, \ldots, g_1) \quad \text{(cf. definition (4.28))}. \] (4.36)

**Remark 4.8.** Using notation (4.24), formula (4.30) can be written as
\[
\text{ch} V(\Lambda) = \sum_{\sigma \in S_r} \chi^\text{sum}(C_{\text{Lexi}}^\Lambda) (\sigma, \Lambda). \] (4.37)

We also have
\[
\chi^\text{sum}(C_{\text{Norm}}^\Lambda) = \sum_{\sigma \in S_r} \chi^\text{sum}(C_{\text{Lexi}}^\Lambda) (\sigma, \Lambda) \quad \text{for } \lambda \in C_{\text{Norm}}^\Lambda. \] (4.38)

To prove (4.38), note that the derivation of (4.37) from formula (4.17) does not depend on how \( \mathcal{C}_{s,t}(\lambda) \)'s are defined. Thus if we simply regard \( \mathcal{C}_{s,t}(\lambda) \) as zero for all \( s, t \), then \( C_{\text{Trun}}^\lambda \) coincides with \( C_{\text{Norm}}^\lambda \), and \( S^\lambda \) becomes \( S_{\text{Sym}}^r \). Hence (4.38) can be regarded as a special case of (4.37). Formula (4.22) can be re-written as
\[
\chi(C_{\text{Lexi}}^\Lambda) = \sum_{\pi \in C_r} \frac{1}{r!} \left( \frac{r}{\pi} \right) (-1)^{\ell(\pi)} \chi(C_{\text{Norm}}^\Lambda) (\pi, \Lambda). \] (4.39)

Thus formula (4.39) is the inverse formula of (4.38).
4.4. Kac-Weyl type formula. While the character formula $(4.13)$ or $(4.17)$ is extremely useful for understanding structural features of irreducible $\mathfrak{g}$-modules, such as their resolutions in terms of Kac-modules, it is not easy to use for purposes like determining the dimensions of irreducibles. For such purposes, a Kac-Weyl type formula is more desirable. We now derive such a formula.

For any weight $\lambda \in P$ and any subset $\Gamma \subset \Delta^+_l$, we define

$$\chi^\text{BL}_\Gamma (\lambda) = L_0^{-1} \sum_{\omega \in W} \epsilon(w) e^{\omega(\lambda + \rho_0)} \prod_{\beta \in \Delta^+_l \setminus \Gamma} (1 + e^{-w(\beta)}), \quad (4.40)$$

(which was referred to as the Bernstein-Leites type character in [16]). Here and below $L_0, L_1$ are defined in $(2.14)$. For any $\lambda \in C^\text{Norm}_\Lambda$ (not necessarily regular), by $(4.24)$ and the definition of the Kac-character $\chi^{\text{Kac}}(\Lambda)$ in $(4.2)$, we have

$$\chi^\text{sum}(C^\text{Norm}_\Lambda) = (-1)^{|\Lambda - \lambda|} L_1 L_0^{-1} \sum_{\omega \in W} \sum_{0 \leq i_1, ..., i_r < \infty} (-1)^{\sum s s i_s} e^{\lambda + \rho - \sum s s i_s \gamma_s} \prod_{s = 1}^r (1 + e^{-\gamma_r}) \quad (4.41)$$

where $\Gamma_\Lambda = \{ \gamma_1, ..., \gamma_r \}$ defined in $(2.10)$ is the set of atypical roots of $\Lambda$ (cf. (3.2)). In deriving the last equality one has made use of the expression of $L_1$ in $(2.14)$.

This together with $(4.37)$ and $(4.22)$ (or $(4.39)$) proves

$$\text{ch} V(\Lambda) = \sum_{\sigma \in S^\Lambda, \pi \in C_r} \frac{1}{r!} \left( \frac{r}{\pi} \right) (-1)^{|\Lambda - (\pi \cdot (\sigma \cdot \Lambda))_\tau| + \ell(\pi)} \chi^\text{BL}_\Gamma (\pi \cdot (\sigma \cdot \Lambda)_\tau). \quad (4.42)$$

Thus we obtain the following theorem.

**Theorem 4.9.** The formal character $\text{ch} V(\Lambda)$ of the finite dimensional irreducible $\mathfrak{g}$-module $V(\Lambda)$ is given by

$$\text{ch} V(\Lambda) = \sum_{\sigma \in S^\Lambda, \pi \in C_r} \frac{1}{r!} \left( \frac{r}{\pi} \right) (-1)^{|\Lambda - (\pi \cdot (\sigma \cdot \Lambda))_\tau| + \ell(\pi)} \frac{1}{L_0} \sum_{\omega \in W} \epsilon(w) w(\pi) e^{(\pi \cdot (\sigma \cdot \Lambda))_\tau + \rho_0} \prod_{\beta \in \Delta^+_l \setminus \Gamma_\Lambda} (1 + e^{-w(\beta)}). \quad (4.43)$$

where notations $S^\Lambda$, $\lambda$, $C_r$ are defined by $(4.40)$, $(4.13)$ and Definition 4.5 (see also $(2.10)$, $(2.14)$, $(4.6)$, $(3.43)$ and $(4.21)$ for other notations).

**Remark 4.10.** Let $\lambda = \lambda + \sum_{s = 1}^r s \gamma_s$. Denote the set of regular lexical weights which are $\leq \lambda$ by:

$$\text{Reg}_{\lambda} = \{ \mu \in C^\text{Norm}_\Lambda \mid \mu \leq \lambda, \text{ and } \mu \text{ is regular and lexical} \}. \quad (4.44)$$

Define

$$\lambda^\dagger = \max \{ \mu \in \Lambda + \sum_{s = 1}^r s \gamma_s \mid \text{Reg}_{\mu} = \text{Reg}_{\lambda} \text{ and } \mu \text{ is lexical} \}. \quad (4.45)$$

Thus $\lambda^\dagger$ is obtained from $\lambda$ by replacing the atypical entries by (cf. $(4.15)$)

$$(\lambda^\dagger)^\rho_\sigma = (\lambda^\rho_\sigma)^\rho_\sigma, \quad \text{where } t(s) = \max \{ t \geq s \mid \tilde{c}_{p,t}(\lambda^\rho_\sigma) = 1 \text{ for } s \leq p \leq t \}, \quad (4.45)$$

for $s = 1, ..., r$ (note that although $\lambda^\rho_\sigma$ is not necessarily regular, one can still define $\tilde{c}_{p,t}(\lambda^\rho_\sigma)$ as stated in Remark 3.8). Then in formula $(4.13)$ (hence also in formula $(4.43)$), for each $\sigma \in S^\Lambda$, the $\sigma \cdot \Lambda$ is replaced by any lexical weight $\eta$ with
\[(\sigma \cdot \Lambda)^{\dagger} \leq \eta \leq (\sigma \cdot \Lambda)^{\vee}.\] In particular, we have another character formula:

\[
\text{ch} \, V(\Lambda) = \sum_{\sigma \in S^{\Lambda}, \pi \in C_r} \frac{1}{r!} \frac{(-1)^{|\Lambda - (\pi \cdot (\sigma \cdot \Lambda)^{\dagger})_0| + \ell(\pi)}}{r} \sum_{w \in W} \epsilon(w) w \left( e^{\pi \cdot (\sigma \cdot \Lambda)^{\dagger}} + \prod_{\beta \in \Delta_+^{\dagger} \setminus \Gamma_0} (1 + e^{-\beta}) \right). \tag{4.43'}
\]

**Remark 4.11.** The main difference between (4.43) and (4.43)’ lies in that formula (4.43)’ keeps the number of the distinct weights \((\pi \cdot (\sigma \cdot \Lambda)^{\dagger})_0\) to be minimal. For example when \(\Lambda = 0\), then \((\pi \cdot (\sigma \cdot \Lambda)^{\dagger})_0 = 0^{\dagger}\) for all \(\pi \in C_r\) and \(\sigma \in S^{\Lambda} = \{1\}\) (See Corollary (1.12) below). But all \((\pi \cdot (\sigma \cdot \Lambda),)_0\) for \(\sigma = 1, \pi \in C_r\) are distinct (in this case \((\sigma \cdot \Lambda)^{\dagger} = 0\)).

**Corollary 4.12.** If \(\Lambda\) is totally connected (cf. definition after Theorem 3.24), then

\[
\text{ch} \, (\Lambda) = \frac{1}{r! L_0} \sum_{w \in W} \epsilon(w) w \left( e^{\Lambda^{\dagger} + \rho_0} \prod_{\beta \in \Delta_+^{\dagger} \setminus \Gamma_0} (1 + e^{-\beta}) \right) = \frac{1}{r!} \chi_{\Gamma_0}^B(\Lambda^{\dagger}), \tag{4.46}
\]

where \(\Lambda^{\dagger}\) is defined by the way that \(\Lambda^{\dagger} + \rho\) is obtained from \(\Lambda + \rho\) by replacing all atypical entries (cf. (3.4)) by the largest one. In particular, by taking \(\Lambda = 0\) we obtain the following denominator formula

\[
L_0 = \frac{1}{r_0!} \sum_{w \in W} \epsilon(w) w \left( e^{\Lambda^{\dagger} + \rho_0} \prod_{\beta \in \Delta_+^{\dagger} \setminus \Gamma_0} (1 + e^{-\beta}) \right), \tag{4.47}
\]

where \(r_0 = \min\{m, n\}\), \(\Gamma_0 = \{s_0^n = \epsilon_{m - \delta_1}, \ldots, s_0^n = \epsilon_{m + 1} - \delta_0\}\) and

\[
0^{\dagger} = \sum_{s = 1}^{r_0} (r_0 - s) s_0^n = (0, \ldots, 0, 1, \ldots, 0^{\dagger}) = (0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0) \tag{cf. notation (2.2)}.
\]

**Proof.** If \(\Lambda\) is totally connected, then \(S^{\Lambda} = \{1\}\) and (4.46) follows from (4.36) and (1.19) with \((\sigma \cdot \Lambda)_0\) replaced by \(\Lambda^{\dagger}\). Observe that when \(\Lambda = 0\), we have \(\text{ch} \, V(\Lambda) = 1\) and \(\Gamma_0 = \Gamma_0\). Thus we obtain (4.47).

**Corollary 4.13.** If \(\Lambda\) is totally disconnected (cf. definition after Theorem 3.24), then

\[
\text{ch} \, (\Lambda) = \frac{1}{L_0} \sum_{w \in W} \epsilon(w) w \left( e^{\Lambda^\dagger + \rho_0} \prod_{\beta \in \Delta_+^{\dagger} \setminus \Gamma_0} (1 + e^{-\beta}) \right) = \chi_{\Gamma_0}^B(\Lambda). \tag{4.48}
\]

**Proof.** If \(\Lambda\) is totally disconnected, then \(S^{\Lambda} = \text{Sym}_r\) and the result follows from the fact that (1.22) is the inverse formula of (1.38) (cf. (1.41)).

### 4.5. Dimension formula
An important application of Theorem 4.9 is the derivation of a dimension formula.

**Theorem 4.14.** The dimension \(\dim V(\Lambda)\) of the finite dimensional irreducible \(g\)-module \(V(\Lambda)\) is given by

\[
\dim V(\Lambda) = \sum_{\sigma \in S^{\Lambda}, \pi \in C_r} \frac{1}{r!} \frac{(-1)^{|\Lambda - (\pi \cdot (\sigma \cdot \Lambda)^{\dagger})_0| + \ell(\pi)}}{r} \sum_{\alpha \in \Delta_+^{\dagger}} \frac{\prod_{\alpha \in \Delta_+^{\dagger}} (a_{\alpha, \rho_0} + (\pi \cdot (\sigma \cdot \Lambda)^{\dagger})_0 - \sum_{\beta \in \Gamma_0}}{(a_{\alpha, \rho_0})}. \tag{4.48}
\]
Proof. Regard an element of $\varepsilon$ (cf. (2.12)) as a function on $\mathfrak{h}^*$ such that the evaluation of $e^\varepsilon$ on $\mu$ is $e^\varepsilon(\mu) = e^{(\lambda,\mu)}$ for $\mu \in \mathfrak{h}^*$. Then $\text{ch} V(\Lambda) \in \varepsilon$ is a function on $\mathfrak{h}^*$, and

$$\dim V(\Lambda) = \lim_{x \to 0} (\text{ch} V(\Lambda))(x\rho_0). \tag{4.49}$$

First we calculate $\lim_{x \to 0} \chi^{BL}_\Lambda(\lambda)(x\rho_0)$ (cf. (4.40)). Using

$$\prod_{\beta \in \Delta_1^+ \backslash \Gamma_\lambda} (1 + e^{-\beta}) = \sum_{B \subset \Delta_1^+ \backslash \Gamma_\lambda} e^{-\sum_{\beta \in B} \beta},$$

and the well-known denominator formula

$$L_0 = \sum_{w \in W} \varepsilon(w) e^{w(\rho_0)} \quad (\text{cf. (2.14)}), \tag{4.50}$$

we have

$$\lim_{x \to 0} \chi^{BL}_\Lambda(\lambda)(x\rho_0) = \lim_{x \to 0} \sum_{B \subset \Delta_1^+ \backslash \Gamma_\lambda, w \in W} \varepsilon(w) \frac{e^{(\lambda + \rho_0 - \sum_{\beta \in B} \beta, x\rho_0)}}{L_0(x\rho_0)}$$

$$= \lim_{x \to 0} \sum_{B \subset \Delta_1^+ \backslash \Gamma_\lambda} \frac{L_0(x(\lambda + \rho_0 - \sum_{\beta \in B} \beta))}{L_0(x\rho_0)}$$

$$= \lim_{x \to 0} \sum_{B \subset \Delta_1^+ \backslash \Gamma_\lambda} \prod_{\alpha \in \Delta_0^+} \frac{e^{(\alpha/2, x(\lambda + \rho_0 - \sum_{\beta \in B} \beta))} - e^{(-\alpha/2, x(\lambda + \rho_0 - \sum_{\beta \in B} \beta))}}{e^{(\alpha/2, \rho_0)} - e^{(-\alpha/2, \rho_0)}}$$

$$= \sum_{B \subset \Delta_1^+ \backslash \Gamma_\lambda} \prod_{\alpha \in \Delta_0^+} \frac{(\alpha, \lambda + \rho_0 - \sum_{\beta \in B} \beta)}{(\alpha, \rho_0)},$$

where the second equality follows from (1.50) and the fact that $(w(\lambda), \mu) = (\lambda, w(\mu))$ for $w \in W, \lambda, \mu \in \mathfrak{h}^*$. This together with (4.49) and (4.42) gives the result. \qed

Remark 4.15. As far as we are aware, there exists no dimension formula for the finite dimensional irreducible $\mathfrak{g}$-modules in the literature except the one for singly atypical modules given by van der Jeugt [13].

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