Linewidth of the laser optical frequency comb with arbitrary temporal profile

Cite as: Appl. Phys. Lett. 113, 131104 (2018); https://doi.org/10.1063/1.5049583
Submitted: 23 July 2018. Accepted: 11 September 2018. Published Online: 26 September 2018

Jacob B. Khurgin, Nathan Henry, David Burghoff, and Qing Hu

COLLECTIONS

Paper published as part of the special topic on On-Chip Mid-Infrared and THz Frequency Combs for Spectroscopy

ARTICLES YOU MAY BE INTERESTED IN

On-chip mid-infrared and THz frequency combs for spectroscopy
Applied Physics Letters 114, 150401 (2019); https://doi.org/10.1063/1.5097933

Shortwave quantum cascade laser frequency comb for multi-heterodyne spectroscopy
Applied Physics Letters 112, 141104 (2018); https://doi.org/10.1063/1.5020747

Widely tunable harmonic frequency comb in a quantum cascade laser
Applied Physics Letters 113, 031104 (2018); https://doi.org/10.1063/1.5039611
Linewidth of the laser optical frequency comb with arbitrary temporal profile

Jacob B. Khurgin,1,a) Nathan Henry,1 David Burghoff,2 and Qing Hu2

1Whiting School of Engineering, Johns Hopkins University, Baltimore, Maryland 21218, USA
2Department of Electrical Engineering and Computer Science, Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

(Received 23 July 2018; accepted 11 September 2018; published online 26 September 2018)

For many applications, optical frequency combs (OFCs) require a high degree of temporal coherence (narrow linewidth). Commonly, OFCs are generated in nonlinear media from a monochromatic narrow linewidth laser source or from a mode-locked laser pulse, but in all the important mid-infrared (MIR) and terahertz (THz) regions of the spectrum, OFCs can be generated intrinsically by free-running quantum cascade lasers (QCLs) with high efficiency. These combs do not look like conventional OFCs as the phases of each mode are different, and in the temporal domain, OFCs are a seemingly random combination of amplitude- and phase-modulated signals rather than a short pulse. Despite this “pseudo-randomness,” the experimental evidence suggests that the linewidth of a QCL OFC is just as narrow as that of a QCL operating in a single mode.

While universally acknowledged, this observation is seemingly not fully understood. In this work, we explicate this fact by deriving the expression for the Schawlow-Townes linewidth of QCL OFCs and offer a transparent physical interpretation based on the orthogonality of laser modes, indicating that despite their very different temporal profiles, MIR and THz QCL OFCs are just as good for most applications as any other OFCs. Published by AIP Publishing.

https://doi.org/10.1063/1.5049583

Optical frequency combs (OFCs) have been enjoying a healthy increase in interest in the last decade due to their applications in metrology, frequency standards, and spectroscopy.1,2 The major difference between an OFC and an arbitrary signal with a discrete periodic spectrum (such as produced by a multi-mode laser) is that in OFCs, the phases of all spectral components are locked, and it assures a narrow linewidth and stability (in the sense that all the frequencies remain equidistant in a long term) which is essential for all the existing and potential OFC applications. Following the original work,1,2 the OFCs are routinely generated by mode-locked lasers,3–5 which are subsequently spectrally broadened in nonlinear fiber. Nonlinear processes (self-phase modulation in the temporal domain or four-wave mixing (FWM) in the spectral domain) require dispersion compensation, and once, it is attained, a short soliton-like pulse is formed in the time domain. More recently, another strategy has been advancing—using a continuous-wave narrow linewidth laser source or from a mode-locked laser pulse, but in all the important mid-infrared (MIR) and terahertz (THz) regions of the spectrum, OFCs can be generated intrinsically by free-running quantum cascade lasers (QCLs) with high efficiency. These combs do not look like conventional OFCs as the phases of each mode are different, and in the temporal domain, OFCs are a seemingly random combination of amplitude- and phase-modulated signals rather than a short pulse. Despite this “pseudo-randomness,” the experimental evidence suggests that the linewidth of a QCL OFC is just as narrow as that of a QCL operating in a single mode.

While OFCs have been most successful in the near-IR region of the spectrum, from the point of view of spectroscopy, it is mid-infrared (MIR) and terahertz (THz) regions of the spectra which are more interesting because they contain information about vibrational and rotational structures of many organic and inorganic substances. In the absence of a direct laser pump source, one must revert to using optical parametric oscillators9,10,12 or to down converting of near infrared OFCs13,14 which greatly increases the complexity, dimensions, and cost of MIR OFC sources.

Most recently, though, an entirely new method of generating MIR15 and THz16 OFCs based on free running quantum cascade lasers (QCLs)17 has been developed, and dual comb spectroscopic measurements have been performed using these unconventional OFCs. What makes these OFCs “unconventional” is the fact that the OFC regime in QCLs does not require any additional intra-cavity phase locking mechanism but is achieved by means of four wave-mixing in the fast (picosecond) saturable gain medium. This fast response time is a salient feature of intersubband transitions in semiconductor quantum wells and is unique to QCLs, and it is the reason that in the time domain, QCL frequency combs15,16 look nothing like a short pulse but is a predominately frequency-modulated (FM) signal. As explained in Refs. 18 and 19, the QCL active medium is a fast saturable gain (i.e., negative loss), which is the exact opposite of the fast saturable absorbing medium that enables production of short pulses in a mode-locked laser. Therefore, the effect of the fast saturable gain is just the opposite—it favors a constant intensity output which can be either a single mode laser or a FM signal. The single mode regime has higher threshold due to burning of spatial and spectral holes, while the multimode FM regime mitigates the spatial and spectral hole burning and is therefore a regular operational regime of free running QCLs in which group velocity dispersion (GVD) has been compensated.19,20 More recent measurements have shown that the actual operating regime of THz and MIR QCL OFCs is more complicated than a simple FM21,22 and has a significant intensity modulation on it as explained in Refs. 23 and 24. While the signal is obviously periodic with a period of cavity round trip, within this

a)Author to whom correspondence should be addressed: jakek@jhu.edu
interval, it appears entirely “random” which is the best way to mitigate spatial hole burning. Despite this, QCL OFCs have been very successfully used in spectroscopy, which indicates that all the relevant parameters of QCL OFCs are comparable to those in other OFCs.

One of the most important parameters describing OFCs is the linewidth of each spectral line in the comb. According to the original work of Schawlow and Townes refined by Lax, the linewidth of a single mode laser is inversely proportional to the power \( P_{\text{out}} \) or to the total number of photons in the cavity \( N_p \). For a multimode laser operating in \( N \) modes, one would then expect that the linewidth of each line would be inversely proportional to the number of photons in that mode or \( P_{\text{out}}/N \), i.e., \( N \) times wider. However, it is experimentally well known that the linewidth of the mode locked lasers is comparable to the linewidth a single mode laser of equal average power since all the modes are locked into the same phase. The linewidth and phase noise of mode locked lasers have been theoretically explored in a number of works where it was assumed that all the modes are locked into the same phase since until recently that was the only practical way to lock all the phases using a saturable absorber or an active intensity of a phase modulator. However, as QCL OFCs mostly with FM character appeared, it became important to understand what kind of linewidth can be achieved in them. Measurements have shown that the linewidth is indeed very narrow, comparable to the linewidth of a single-mode QCL operating at the same power. This narrowness has been explained by introducing a concept of “supermode” but without a detailed derivation of the amount of noise going into that mode and just assuming a Langenvin spontaneous emission term. As shown below, this approach is correct but would benefit from a stronger theoretical foundation.

This foundation is provided in this work with a simple derivation of the Schawlow-Townes (ST) linewidth for arbitrary laser OFCs based on the orthogonality of modes and the fact that the spontaneous noise amplitudes in each mode are uncorrelated. We also provide a simple intuitive and physically transparent picture of why only a small fraction of spontaneous noise actually affects the linewidth of the arbitrary OFC. Our conclusion is that despite their seemingly “random” temporary profiles, OFCs produced by the lasers with short gain recovery times (such as QCLs) are every bit as good as OFCs produced by single mode lasers and microresonators.

We consider the situation in which the steady phase locking of the modes has already taken place, and as explained in Ref. 31, following the standard methods in calculating oscillator linewidths, we assume that the noise sources are not strong enough to disrupt the phase relationship between modes, and hence, the total field can be written as

\[
E(t,z) = \sum_{n=1}^{N} E_n(t)e^{-j\omega_n t}a_n(z) = A(t) \sum_{n=1}^{N} f_n e^{-j\omega_n t}a_n(z), \tag{1}
\]

where \( \omega_n \) is a frequency of the \( n \)-th mode, \( a_n(z) \sim \sin(\omega_n z/\nu) \) is the normalized shape of the \( n \)-th orthogonal mode, \( \nu \) is the phase velocity of light, \( \int a_m a_n dz = \delta_{mn} \), and the Fourier amplitudes \( f_n \) have been normalized as \( \sum_{n=1}^{N} |f_n|^2 = 1 \). The average power can then be defined as \( P = |A|^2 \). The normalized distribution of normalized instant electric field \( E(z) \) inside the cavity and one of the modes \( a_m(z) \) are shown in Fig. 1(a).

Let us now write the expression for the temporal development of the slow-varying amplitude of the \( n \)-th mode

\[
dE_n(t)/dt = \frac{1}{2}g_n(\bar{P})E_n - \frac{1}{2}E_n/\tau_c + iD_nE_n + \sum_{l \neq n, m \neq l}^{N} g_{nlm}(\bar{P})\kappa_{nlm}f^*_mE_{n+m-l} + \frac{S_n(t)}{\tau_c}, \tag{2}
\]

where \( 1/2g_n(\bar{P}) \) is the gain that includes both self-saturation and cross-saturation, \( g_{nlm}(\bar{P})\kappa_{nlm}f^*_m \) is the four-wave mixing (FWM) term (but in the case of active mode locking, it may have a different shape corresponding to the sideband generation), \( \tau_c \) is the cavity lifetime, \( D_n = 2n^2\pi^2\tau_c^{-2}\beta_2\nu_g \) is the dispersive term, \( \nu_g \) is the velocity, \( \beta_2 \) is the GVD, and \( S_n(t) \) is the noise complex amplitude in the \( n \)-th mode. The gain and FWM term dependence on average power becomes important further on [see (8)] because in order to estimate the linewidth, one must take into account the fact that the population inversion is not pinned at the threshold but is slightly less than that. The noise can be caused by the spontaneous emission of photons in the \( n \)-th mode, but it can also be due to a far more mundane causes such as cavity length oscillations due to random vibrations. The noise amplitudes of different modes are not correlated, \( \langle S'_n(t)S_m(t') \rangle = S^2\delta_{nm}\delta(t-t') \).

Upon substituting (1) into (2), we obtain

\[
dA(t)/dt = \frac{1}{2}g_n(\bar{P})A(t)f_n + iD_nA(t)f_n - \frac{A(t)f_n}{2\tau_c} + A(t)|A(t)|^2 \sum_{l \neq n, m \neq l}^{N} g_{nlm}(\bar{P})\kappa_{nlm}f^*_mE_{n+m-l} + \frac{S_n(t)}{\tau_c}. \tag{3}
\]

The FWM terms under double summation have “amplitude” or in-phase parts and “phase” or quadrature parts. The quadrature terms cause frequency chirp, and once the stable regime has been reached, they all sum up to zero, i.e.,

\[
|A(t)|^2 \text{Im} \sum_{l \neq n, m \neq l}^{N} g_{nlm}(\bar{P})\kappa_{nlm}f^*_mE_{n+m-l} + iD_nA(t)f_n = 0. \tag{4}
\]

Multiply now both sides of (3) by \( f^*_n \) and perform summation. The amplitude terms only cause the energy redistribution between the modes, and hence, once the stable operational regime has been reached, they all sum up to zero, i.e.,

\[
\text{Re} \sum_n^{N} \sum_{l \neq n, m \neq l}^{N} g_{nlm}(\bar{P})\kappa_{nlm}f^*_mE_{n+m-l} = 0. \tag{5}
\]

The new differential equation then becomes...
Siegman by introducing the photon number in the cavity as the framework in original ST work as elucidated by the only noise source is the spontaneous emission following obtained instantly. 

that \( (7) \) becomes

\[
\frac{dA(t)}{dt} = \frac{1}{2} g(\mathcal{P})A(t) - \frac{A(t)}{2\tau_c} + \sum_{n=1}^{N} S_n(t)f_n^2.
\]  

Obviously, any small deviations from (4) and (5) can be treated as additional sources of noise and simply added to (6). The deviation from (4) caused by the stochastic nature of power \( |A(t)|^2 \) is nothing but additional phase noise due to gain/index coupling which can be taken care of by introducing the linewidth enhancement factor later on and which we are not considering here for the sake of simplicity as it is known to be small in the QCLs. Let us now examine the noise term. Since the phases of complex noise amplitudes in each mode are random, one must sum up the noise powers, i.e., the total noise power is \( \langle |S_{\text{out}}|^2 \rangle = \sum_{m=1}^{N} \langle |S_m|^2 \rangle \). 

Under a reasonable assumption that all the noise powers \( \langle |S_m|^2 \rangle \) are identical, we obtain \( \langle |S_{\text{out}}|^2 \rangle = \langle |S_m|^2 \rangle \), i.e., the total noise added to a given stable phase-locked signal is equal to the noise in any given mode, and we have

\[
\frac{dA(t)}{dt} = \frac{1}{2} g(\mathcal{P})A(t) - \frac{A(t)}{2\tau_c} + \frac{S_{\text{out}}(t)}{\tau_c},
\]  

where \( S_{\text{out}}(t) \) is the noise source whose power is equal to the power of noise in any given mode.

In other words, this equation looks exactly like the equation for a single mode laser and is therefore bound to yield the linewidth that is determined by the total average power \( \mathcal{P} = |A|^2 \) rather than by a power in a given mode. One easy way to interpret this situation is to simply assume that one can take any linear combination of modes in the cavity and just call it an eigenmode of the system. Of course, as different modes do have different frequencies, their linear combination does not necessarily amount to an eigenmode, which after all is supposed to have a well-defined frequency, but this is still a useful analogy which allows Eq. (7) to be obtained instantly.

One can now obtain the ST linewidth for the case when the only noise source is the spontaneous emission following the framework in original ST work as elucidated by Siegman by introducing the photon number in the cavity as

\[ N_p = |A|^2 \tau_c/\hbar \omega, \]

where \( \tau_{rt} \) is the cavity round trip time, so that (7) becomes

\[
\frac{dN_p}{dt} = g(N_p) - \frac{1}{\tau_c} N_p + n_{sp} g(N_p),
\]  

where \( n_{sp} \geq 1 \) is the excess noise factor caused by the finite population of the lower laser level (subband in the case of QCL). In the steady state, the saturated gain is \( g(N_p) = N_p \tau_c/(N_p + n_{sp}) \) and the linewidth can be introduced as the effective photon decay rate

\[
\Delta \nu_{ST} = \frac{1}{\tau_c} - \frac{g(N_p)}{2\pi} = \frac{n_{sp}}{N_p + n_{sp}} \tau_c^{-1} \Delta \nu_0 \eta_{out} N_p^{-1},
\]  

where \( \Delta \nu_0 = \tau_c^{-1}/2\pi \) is the cold cavity linewidth. Relating the photon density inside the cavity to the output power as \( P_{out} = \hbar \omega \eta_{out} \tau_c^{-1} N_p \), where \( \eta_{out} \) is the outcoupling efficiency, the ST linewidth assumes a more familiar shape

\[
\Delta \nu_{ST} = 2\pi \eta_{out} \Delta \nu_0 \hbar \omega / P_{out}.
\]

Let us now consider the physical interpretation of the result—how come that despite the fact that each of N modes contributes to the spontaneous emission, only 1/Nth of that radiation contributes to both the phase and amplitude noise of the laser comb? For a short (mode-locked with all phases being equal) pulse, the interpretation in the time domain is obvious: the pulse sequence containing N cavity modes has the duty cycle of exactly 1/N—hence, only 1 of each N spontaneously emitted photons will affect the phase (and amplitude) of the pulse—the rest of them will appear as additive noise which can always be filtered out. For a FM frequency comb, a similar time domain interpretation can be made: assume for simplicity that the laser simply jumps from one mode to another—i.e., instant frequency is always equal to one of the cavity frequencies. Obviously, the spontaneous emission into other N-1 modes does not contribute to the multiplicative phase noise but simply presents an additive noise.

To understand the narrow linewidth of the arbitrary comb rather than using the temporal domain, it is helpful to turn instead to the spatial dependence of the signal (1) and consider the change in the phase imposed by the emission of a single photon into the m-th mode. According to (8), each photon is emitted every \( \delta t_m = \tau_c/n_{sp} \), and the power of that photon \( \delta P_m = |\delta E_m|^2 = \hbar \omega \tau_{rt}^{-1} \). Therefore, the phase change imposed by this photon is [see Fig. 1(a)]
\[ \delta \varphi_m = \int E(t, z) \delta E_m a_m(z) dz \int E^2(t, z) dz = f_m \delta E_m Q / |A|, \]

where \( \delta E_m Q \) is the quadrature component of the spontaneous emission, and we have used the orthogonality of the laser modes in space. Now, in the time interval \( \Delta t \), there will be \( \Delta N_m = \Delta t / \delta t_m = n_{sp} \tau_m^{-1} \Delta t \) photons emitted into the m-th mode, and according to random walk process theory, the variance of the phase will be

\[ \langle \delta \varphi^2 \rangle = \frac{\Delta N_m}{|A|^2} = \frac{n_{sp} \tau_m^{-1} |f_m|^2}{N_p} \Delta t. \]

However, according to the theory for the random walk process with the variance \( \langle \delta \varphi^2 \rangle = 2 n C \Delta t \), the power spectral density of the phase noise can be found as \( S_\varphi(\omega) = C / \omega^2 \) with the linewidth being \( \Delta \nu = C \). Therefore, the linewidth can be found as

\[ \Delta \nu = \langle \delta \varphi^2 \rangle / 2 \pi \Delta t = \frac{n_{sp} \Delta \nu_0}{N_p} |f_m|^2. \]

Comparing (12) with (9), one can state that the linewidth’s contribution from spontaneous emission in the m-th mode is proportional to its relative weight \( |f_m|^2 \), i.e., roughly 1/N. Thus, physical interpretation in the spatial domain is simple—since the spontaneous emission into a given resonator mode only weakly overlaps in space with the actual distribution of the laser field inside the cavity, most of this emission ends up as an additive noise and does not contribute to the phase/frequency noise and linewidth. This situation is schematically explained in Fig. 1(b).

The key conclusion reached in this work is that we have confirmed the previously made argument that the linewidth of a given multi-mode phase locked laser emission does not depend on exactly what is the phase relationship between the individual modes as long as this relation exists, i.e., as long as the locking mechanism (which can be a fast saturable absorption as in conventional mode locked lasers or a fast saturable gain in QCLs) is strong enough to overcome the dispersion of group velocity and gain. The OFC signal may be found using nonlinear processes in microresonators given comparable signal to noise ratios combined the simplicity, smaller size, and higher efficiency of the free running laser OFCs.

The authors acknowledge the generous support provided by the DARPA SCOUT Program. Additionally, they would like to thank Professor Jérôme Faist and Matt Singleton at ETH Zurich for many stimulating discussions.

References:
1. I. Galli, F. Cappelli, P. Cancio, G. Giusfredi, D. Mazzotti, S. Bartalini, and M. Lipson, “Silicon-chip mid-infrared frequency comb generation,” Nature Photonics 6, 480–487 (2012).
2. A. G. Griffith, R. K. W. Lau, J. Cardenas, Y. Okawachi, A. Mohanty, R. Fain, Y. H. D. Lee, M. Yu, C. T. Phare, C. A. Poitras, A. L. Gaeta, and M. Lipson, “Silicon-chip mid-infrared frequency comb generation,” Nature Commun. 4, 1345 (2013).
3. T. Herr, K. Hartinger, J. Riemensberger, C. Y. Wang, E. Gavartin, R. Holzwarth, M. L. Gorodetsky, and T. J. Kirchenberg, “Universal formation dynamics and noise of Kerr-frequency combs in microresonators,” Nat. Photonics 6, 6209 (2015).
4. V. Brisch, M. Geiselmann, T. Herr, G. Lihachev, M. H. P. Pfeiffer, M. L. Gorodetsky, and T. J. Kirchenberg, “Photonic chip-based optical frequency comb using soliton Cherenkov radiation,” Science 351, 357–360 (2016).
5. K. L. Vodopyanov, E. Sorokin, I. T. Sorokina, and P. G. Schunemann, “Mid-IR frequency comb source spanning 4.4–5.4 μm based on subharmonic GaAs optical parametric oscillator,” Opt. Lett. 36, 2275–2277 (2011).
6. I. Galli, F. Cappelli, P. Cancio, G. Giusfredi, D. Mazzotti, S. Bartalini, and P. De Natale, “High-coherence mid-infrared frequency comb,” Opt. Express 21, 28877–28885 (2013).
7. A. Ruehl, A. Gambetta, I. Hartl, M. E. Ferrmann, K. S. E. Eikema, and M. Marangoni, “Widely-tunable mid-infrared frequency comb source based on difference frequency generation,” Opt. Lett. 37, 2232–2234 (2012).
8. A. Hagi, G. Villares, S. Blaser, H. C. Liu, and J. Faist, “Mid-infrared frequency comb based on a quantum cascade laser,” Nature 492, 229–233 (2012).
9. D. Burghoff, T.-Y. Kao, N. Han, C. W. I. Chan, X. Cai, Y. Yang, D. J. Hayton, J.-R. Gao, J. L. Reno, and Q. Hu, “Terahertz laser frequency combs,” Nat. Photonics 8, 462–467 (2014).
10. J. Faist, F. Capasso, D. L. Sivco, C. Sirtori, A. L. Hutchinson, and A. Y. Cho, “Quantum cascade laser,” Science 264, 553–556 (1994).
16. J. B. Khurgin, Y. Dikmelik, A. Hugi, and J. Faist, “Coherent frequency combs produced by self frequency modulation in quantum cascade lasers,” *Appl. Phys. Lett.* **104**, 081118 (2014).
17. G. Villares and J. Faist, “Quantum cascade laser combs: Effects of modulation and dispersion,” *Opt. Express* **23**, 1651–1669 (2015).
18. G. Villares, S. Riedi, J. Wolf, D. Kazakov, M. Süess, P. Jouy, M. Beck, and J. Faist, “Dispersion engineering of quantum cascade laser frequency combs,” *Optica* **3**, 252–258 (2016).
19. D. Burghoff, Y. Yang, D. Hayton, J. Gao, J. Reno, and Q. Hu, “Evaluating the coherence and time-domain profile of quantum cascade laser frequency combs,” *Opt. Express* **23**, 1190–1202 (2015).
20. M. Singleton, P. Jouy, M. Beck, and J. Faist, “Evidence of linear chirp in mid-infrared quantum cascade lasers,” *Optica* **5**, 948–953 (2018).
21. N. Henry, D. Burghoff, Q. Hu, and J. B. Khurgin, “Temporal characteristics of quantum cascade laser frequency modulated combs in long wave infrared and THz regions,” *Opt. Express* **26**, 14201–14212 (2018).
22. P. Tzenov, D. Burghoff, Q. Hu, and C. Jirauschek, “Time domain modeling of terahertz quantum cascade lasers for frequency comb generation,” *Opt. Express* **24**, 23232–23247 (2016).
23. G. Villares, A. Hugi, S. Blaser, and J. Faist, “Dual-comb spectroscopy based on quantum-cascade-laser frequency combs,” *Nat. Commun.* **5**, 5192 (2014).
24. Y. Yang, D. Burghoff, D. J. Hayton, J.-R. Gao, J. L. Reno, and Q. Hu, “Terahertz multiheterodyne spectroscopy using laser frequency combs,” *Optica* **3**, 499–502 (2016).
25. A. L. Schawlow and C. H. Townes, “Infrared and optical masers,” *Phys. Rev.* **112**(6), 1940 (1958).
26. M. Lax, “Classical noise. V. Noise in self-sustained oscillators,” *Phys. Rev.* **160**, 290 (1967).
27. D. W. Rush, P.-T. Ho, and G. L. Burdge, “The coherence time of a mode-locked pulse train,” *Opt. Commun.* **52**, 41–45 (1984).
28. D. W. Rush, G. L. Burdge, and P.-T. Ho, “The linewidth of a mode-locked semiconductor laser. Caused by spontaneous emission. Experimental comparison to single-mode operation,” *IEEE J. Quantum Electron.* **22**, 2088–2091 (1986).
29. P.-T. Ho, “Phase and amplitude fluctuation a mode-locked laser,” *IEEE J. Quantum Electron.* **21**, 1806–1813 (1985).
30. H. A. Haus and A. Mecozzi, “Noise of mode-locked lasers,” *IEEE J. Quantum Electron.* **29**, 983–995 (1993).
31. A. L. Jiang, M. E. Grein, H. A. Hermann, and E. P. Ippen, “Noise of mode-locked semiconductor lasers,” *IEEE J. Sel. Top. Quantum Electron.* **7**(2), 159–166 (2001).
32. F. Cappelli, G. Villares, S. Riedi, and J. Faist, “Intrinsic linewidth of quantum cascade laser frequency combs,” *Optica* **2**, 836–840 (2015).
33. K. Karokawa, *An Introduction to the Theory of Microwave Circuits* (New York, Academic, 1969), Chap. 9.
34. J. A. Mullen, “Background noise in nonlinear oscillators,” *Proc. IRE* **48**, 1467 (1960).
35. C. H. Henry, “Theory of the linewidth of semiconductor lasers,” *IEEE J. Quantum Electron.* **18**, 259–264 (1982).
36. A. E. Siegman, *Lasers* (University Science Books Sausalito, 1986), Chap. 11.