Black hole initial data without elliptic equations

István Rácz§♭ and Jeffrey Winicour♭♮

§ Wigner RCP, 
H-1121 Budapest, Konkoly Thege Miklós út 29-33., Hungary
♭ Department of Physics and Astronomy, 
University of Pittsburgh, Pittsburgh, PA, 15260, USA

Abstract

We explore whether a new method to solve the constraints of Einstein’s equations, which does not involve elliptic equations, can be applied to provide initial data for black holes. We show that this method can be successfully applied to a nonlinear perturbation of a Schwarzschild black hole by establishing the well-posedness of the resulting constraint problem. We discuss its possible generalization to the boosted, spinning multiple black hole problem.

1 Introduction

The prescription of physically realistic initial data for black holes is a crucial ingredient to the simulation of the inspiral and merger of binary black holes and the computation or the radiated gravitational waveform. Initialization of the simulation is a challenging problem due to the nonlinear constraint equations that the data must satisfy. The traditional solution expresses the constraints in the form of elliptic equations. Here we consider a radically new method of solving the constraints which does not require elliptic solvers §. We show, at least for small perturbations of Schwarzschild black hole data, that the constraints can be formulated as a well-posed problem consisting of one algebraic equation and a system of three strongly hyperbolic equations. This possibility of extending this approach to binary black holes offers a simpler way to provide boundary conditions for the initialization problem that might prove to be more physically realistic.

The inspiral and merger of a binary black hole is expected to be the strongest possible source of gravitational radiation for the emerging field of gravitational wave
astronomy. The details of the gravitational waveform supplied by numerical simulation is a key tool to enhance detection of the gravitational signal and interpret its scientific content. It is thus important that the initial data does not introduce spurious effects in the waveform. Such “junk radiation” is common to all current methods to supply initial data. Although this junk radiation appears to dissipate after some early time in the simulation, it is a troublesome feature with regard to matching the waveform in the nonlinear regime spanned by the simulation to the post-Newtonian chirp waveform provided by perturbation theory. All initialization methods presently in use reduce the constraint problem to a system of elliptic equations, which require boundary conditions at inner boundaries surrounding the singularities inside the black holes, as well as at an outer boundary surrounding the entire system. The new method we consider here only requires data on the outer boundary and in this way offers the possibility of suppressing junk radiation.

The initial data for solving Einstein’s equations consist of a pair of symmetric tensor fields \((h_{ij}, K_{ij})\) on a smooth three-dimensional manifold \(\Sigma\), where \(h_{ij}\) is a Riemannian metric and, \(K_{ij}\) is interpreted as the extrinsic curvature of \(\Sigma\) after its embedding in a 4-dimensional space-time. The constraints on a vacuum solution (see, e.g. Refs. [2, 3]) consist of

\[
\begin{align*}
(3)^R + (K_{ij}^i)^2 - K_{ij}K_{ij} & = 0, \\
D_jK_{ij}^i - D_iK_{ij}^j & = 0,
\end{align*}
\]

where \((3)^R\) and \(D_i\) denote the scalar curvature and the covariant derivative operator associated with \(h_{ij}\), respectively.

The standard approach to solving the constraints is based upon the conformal method, introduced by Lichnerowicz [4] to recast the Hamiltonian constraint (1.1) as an elliptic equation and later extended by York [5, 6] to reduce the momentum constraint (1.2) also to an elliptic system. For a review of the historic implementation of this method in numerical relativity see [7].

A major obstacle in prescribing black hole initial data is the presence of a singularity inside the black hole. The initial strategy for handling the singularity was the excision of the singular region inside the black hole [8]. Other strategies have since been proposed. One is the puncture method in which the initial hypersurface extends though a wormhole to an internal asymptotically flat spatial infinity, which is then treated by conformal compactification [9]. Another, the trumpet method, extends the initial slice to an internal timelike infinity with asymptotically finite surface area [10, 11].

Coupled to these techniques for avoiding singularities is the choice of initial time slice. For example, there are many ways to prescribe Schwarzschild initial data depending upon whether, say, the initial Cauchy hypersurface is time symmetric or horizon penetrating. Here we will focus on initial data in Kerr-Schild form [12, 13],
which for the Schwarzschild case corresponds to ingoing Eddington-Finklestein coordinates, which extend from spatial infinity to the singularity and penetrate the horizon. The approach to solving the constraints that we consider here does not in fact work for a time symmetric initial slice, whose extrinsic curvature vanishes. However, time symmetric space times contain as much ingoing as outgoing gravitational waves, so they are not the appropriate physical models for studying binary waveforms. Although our focus here is on data in Kerr-Schild form, we do not wish to imply that this approach would not work for puncture or trumpet data.

A very attractive feature of Kerr-Schild initial data is that it provides a preferred Minkowski background to construct boosted black holes by means of a Lorentz transformation. Two independent ways of prescribing Kerr-Schild initial data have been proposed. In one version, the 4-dimensional aspect of the Kerr-Schild ansatz is preserved as much as possible \cite{14}. This leads to a workable scheme for superimposing non-spinning black holes but the generalization to the spinning case remains problematic. In the other case, the Kerr-Schild ansatz is loosened to a 3-dimensional version that allows superposition of multiple spinning black holes \cite{15}. This has been implemented to provide data for boosted, spinning binary black holes and plays an important role in current simulations \cite{16}.

There are several variants to the new method of solving the constraints proposed in \cite{1, 17, 18, 19}, depending upon which components of the initial data are assigned freely. They all avoid elliptic equations. Here we apply the simplest of these variants to the initial data problem for black holes. In this variant, the Hamiltonian constraint reduces to an algebraic equation and the momentum constraint reduces to a system of strongly hyperbolic equations, which only require data on a 2-surface surrounding the black holes.

In Sec. \ref{sec:2}, we review this new approach. In Sec. \ref{sec:3} we show that the underlying requirements for well-posedness of the resulting algebraic-hyperbolic constraint problem are satisfied by a Schwarzschild black hole described in Kerr-Schild form. In Sec \ref{sec:4} we present an explicit proof that well-posedness extends to nonlinear perturbations of Schwarzschild black hole data in Kerr-Schild form. In Sec \ref{sec:5}, we conclude with a discussion of the possibility of extending this approach to general data for a system of boosted, spinning multiple black holes.

## 2 A new approach to the constraints

We assume that the topology of $\Sigma$ allows a smooth foliation by a one-parameter family of homologous two-surfaces. In the application to black hole initial data, we assume for simplicity a foliation $\mathcal{F}_\rho$ by topological spheres described by the level surfaces $\rho = \text{const}$ of a smooth function.

Choose now a vector field $\rho^i$ on $\Sigma$ such that $\rho^i \partial_i \rho = 1$. Then the unit normal $\hat{n}^i$...
to $\mathcal{S}_\rho$ has the decomposition

$$\hat{n}^i = \hat{N}^{-1} [\rho^i - \hat{N}^i],$$  \hspace{1cm} (2.1)

where the ‘lapse’ $\hat{N}$ and ‘shift’ $\hat{N}^i$ of the vector field $\rho^i$ are determined by $\hat{n}_i = \hat{N} \partial_i \rho$ and $\hat{N}^i = \hat{\gamma}_{ij} \rho^j$, with $\hat{\gamma}_{ij} = \delta_{ij} - \hat{n}^i \hat{n}_j$.

The 3-metric $h_{ij}$ on $\Sigma$ then has the $2+1$ decomposition

$$h_{ij} = \hat{\gamma}_{ij} + \hat{n}_i \hat{n}_j,$$  \hspace{1cm} (2.2)

where $\hat{\gamma}_{ij}$ is the metric induced on the surfaces $\mathcal{S}_\rho$. The extrinsic curvature $\hat{K}_{ij}$ of $\mathcal{S}_\rho$ is given by

$$\hat{K}_{ij} = \hat{\gamma}^l_{ij} D_l \hat{n}_j = \frac{1}{2} \mathcal{L}_{\hat{n}} \hat{\gamma}_{ij},$$  \hspace{1cm} (2.3)

where the ‘lapse’ $\hat{N}$ and ‘shift’ $\hat{N}^i$ of the vector field $\rho^i$ are determined by $\hat{n}_i = \hat{N} \partial_i \rho$ and $\hat{N}^i = \hat{\gamma}_{ij} \rho^j$, with $\hat{\gamma}_{ij} = \delta_{ij} - \hat{n}^i \hat{n}_j$.

The extrinsic curvature $K_{ij}$ of $\Sigma$, which forms part of the data, has the decomposition

$$K_{ij} = \kappa \hat{n}_i \hat{n}_j + \hat{n}_i k_j + \hat{n}_j k_i + K_{ij},$$  \hspace{1cm} (2.4)

where $k_i = \hat{n}^k \hat{n}_l K_{kl}$, $k_i = \hat{\gamma}^k \hat{n}_l K_{kl}$ and $K_{ij} = \hat{\gamma}^k \hat{\gamma}^l K_{kl}$. Here we use boldfaced symbols to indicate tensor fields tangent to $\mathcal{S}_\rho$. In addition, we shall denote the trace and trace free parts of $K_{ij}$ and $K_{ij}$ by $K^l = \hat{\gamma}^l K_{kl}$, $K^l = \hat{\gamma}^l K_{kl}$, $K_{ij} = \hat{K}_{ij} - \frac{1}{2} \hat{\gamma}_{ij} \hat{K}^l l$ and $\dot{K}_{ij} = k_{ij} - \frac{1}{2} \hat{\gamma}_{ij} K^l l$, respectively.

By replacing the initial data set $(h_{ij}, K_{ij})$ by the seven fields $(\hat{N}, \hat{N}^i, \hat{\gamma}_{ij}, \hat{K}_{ij}, k_i, K^l l)$, the Hamiltonian and momentum constraints (1.1) and (1.2) can be expressed as (1) (see also [17, 18, 19])

$$\mathcal{L}_\partial(K^l l) - \hat{D}^l k_l + 2 \hat{n}^l k_l - [\kappa - \frac{1}{2} \hat{K}_{kl} \hat{K}^k l] = 0,$$  \hspace{1cm} (2.5)

where $k_i = \hat{n}^k \hat{n}_l K_{kl}$, $k_i = \hat{\gamma}^k \hat{n}_l K_{kl}$ and $K_{ij} = \hat{\gamma}^k \hat{\gamma}^l K_{kl}$. Here we shall denote the covariant derivative operator and scalar curvature associated with $\hat{\gamma}_{ij}$, respectively, and $\hat{n}_k = \hat{n}^l \hat{D}_l \hat{n}_k = -\hat{D}_k (\ln \hat{N})$. Here (2.7) provides an algebraic solution to the Hamiltonian constraint (1.1) (for more details see [1]).

Given the free data $(\hat{N}, \hat{N}^i, \hat{\gamma}_{ij}, \hat{K}_{ij})$, the equations (2.6)-(2.7) were shown to comprise a first order strongly hyperbolic system for the vector valued variable $(K^l l, k_i)$ provided $\kappa$ and $K^l l$ are of opposite sign,

$$\kappa K^l l = -C^2, \hspace{1cm} C \neq 0.$$  \hspace{1cm} (2.8)
It was verified in [1] that, given the values of \((k_i, K^i)\) on some “initial” surface \(S_0\), solutions to the nonlinear system (2.5)-(2.7) exist (at least locally) in a neighborhood of \(S_0\), and they are guaranteed to satisfy the full constraint system (1.1)-(1.2).

### 3 The free and constrained Schwarzschild data

The successful application of this new approach to the constraint problem depends upon a judicious choice of gauge, determined by the lapse of the initial Cauchy hypersurface \(\Sigma\), and a judicious choice of foliation \(S_{\rho}\). We begin by considering data in Kerr-Schild form, in which the space-time metric has the form

\[
g_{ab} = \eta_{ab} + 2H\ell_a\ell_b, \quad g^{ab} = \eta^{ab} - 2H\ell^a\ell^b,
\]

where \(H\) is a smooth function (except at singularities) on \(\mathbb{R}^4\) and \(\ell_a\) is null with respect to both \(g_{ab}\) and an implicit background Minkowski metric \(\eta_{ab}\). In inertial coordinates \((t, x^i)\) adapted to \(\eta_{ab}\),

\[
g_{ab}dx^a dx^b = (-1 + 2H\ell_t^2)dt^2 + (\delta_{ij} + 2H\ell_i\ell_j)dx^i dx^j,
\]

where \(\ell^a = g^{ab}\ell_b = \eta^{ab}\ell_b\) and \(g^{ab}\ell_a\ell_b = \eta^{ab}\ell_a\ell_b = -(\ell_t)^2 + \ell_t\ell_i = 0\). The Kerr-Schild metrics also satisfy the geodesic condition

\[
\ell^i \partial_i \ell_t + \ell^k \partial_k \ell_i = 0.
\]

We can relate the Kerr-Schild metric to the 3 + 1 decomposition of the space-time metric

\[
g_{ab} = h_{ab} - n_a n_b,
\]

where \(n^a\) is the future directed unit normal to the \(t = \text{const}\) hypersurfaces. Choose a time evolution field \(t^a\) satisfying \(t^a \partial_a t = 1\). Then \(n^a\) has the decomposition

\[
n^a = N^{-1}(t^a - N^a),
\]

where \(N\) and \(N^a\) denote the spacetime lapse and shift, determined by

\[
N = -(t^e n_e), \quad n_a = -N \partial_a t \quad \text{and} \quad N^a = h^a{}_{e} t^e,
\]

respectively.

In the Kerr-Schild spacetime coordinates \((t, x^i)\), the metric has components

\[
g_{\alpha\beta} = \begin{pmatrix} -N^2 + N_i N^i & N_i \\ N_j & h_{ij} \end{pmatrix}.
\]
It follows that
\[ h_{ij} = \delta_{ij} + 2H \ell_i \ell_j, \quad h^{ij} = \delta^{ij} - \frac{2H}{1 + 2H} \ell^i \ell^j, \] (3.16)
\[ N = \frac{1}{\sqrt{1 + 2H \ell_i^2}}, \] (3.17)
\[ N_i = 2H \ell_i \ell_i, \quad N^i = 2HN^2 \ell_t \ell^i. \] (3.18)

A direct calculation of the extrinsic curvature
\[ K_{ij} = \frac{1}{2} \mathcal{L}_n h_{ij} = (2N)^{-1} [\partial_t h_{ij} - (D_i N_j + D_j N_i)] \] (3.19)
gives
\[ N^{-1} K_{ij} = -\ell_t [\partial_i (H \ell_j) + \partial_j (H \ell_i)] + N^{-2} \partial_t (H \ell_i \ell_j) + 2H \ell^k \partial_k (H \ell_i \ell_j) - H (\ell_i \partial_j \ell_t + \ell_j \partial_i \ell_t). \] (3.20)

For a Kerr spacetime
\[ H = \frac{rM}{r^2 + a^2 \cos^2 \theta}, \] (3.21)
where the Boyer-Lindquist radial coordinate \( r \) is related to the Cartesian inertial spatial coordinates \( x^i = (x_1, x_2, x_3) \) according to
\[ r^2 = \frac{1}{2} \left[ (\rho^2 - a^2) + \sqrt{(\rho^2 - a^2)^2 + 4a^2 x_3^2} \right] \] (3.22)
with
\[ \rho^2 = x_1^2 + x_2^2 + x_3^2 \] (3.23)
and
\[ \ell_a = \left( 1, \frac{r x_1 + a x_2}{r^2 + a^2}, \frac{r x_2 - a x_1}{r^2 + a^2}, \frac{x_3}{r} \right). \] (3.24)

As \( H \) and \( \ell_a \) are \( t \)-independent and \( \ell_t = 1 \), the extrinsic curvature (3.20) simplifies to
\[ K_{ij} = -\ell_t N \left[ \partial_i (H \ell_j) + \partial_j (H \ell_i) + 2H \ell_i \ell_j \ell^k \partial_k H \right]. \]

For the purpose of applying the approach in Sec. 2 to a generic inspiral and merger, it would be necessary to show that the required sign condition (2.8) holds for a boosted Kerr black hole. Here we restrict our investigation to the Schwarzschild case, where the choice of foliation \( \mathcal{S}_\rho \) is guided by the spherical symmetry and the algebraic simplicity allows a clear exposition of the approach.

For a Schwarzschild black hole, the spin parameter \( a = 0 \) and the Kerr-Schild form of the metric simplifies to
\[ H = \frac{M}{r}, \quad \ell_i = \frac{x_i}{r} = \partial_i r, \quad \ell^2 = \delta^{ij} x_i x_j, \] (3.25)
with lapse
\[ N = (1 + 2H)^{-1/2} \] (3.26)
and 3-metric
\[ h_{ij} = \delta_{ij} + 2H\ell_i\ell_j, \] (3.27)
(Here \(-\ell^a\) is a future directed ingoing null vector, which corresponds to the convention for ingoing Eddington-Finklestein coordinates.) Thus
\[ \partial_i H = -\frac{M}{r^3} x_i, \quad \partial_j (H\ell_i) = \frac{M}{r^4} [r^2\delta_{ij} - 2x_i x_j] \] (3.28)
and (3.25) reduces to
\[ K_{ij} = -\frac{2M}{r^2 \sqrt{1 + 2H}} (\delta_{ij} - [2 + H] \ell_i \ell_j). \] (3.29)

We choose the foliation \( S_\rho \) by setting \( \rho = r \), with \( \rho^i = \ell^i \), corresponding to the “spatial” lapse and shift
\[ \hat{N} = \sqrt{1 + 2H}, \quad \hat{N}^i = 0, \] (3.30)
unit normal
\[ \hat{n}_i = \sqrt{1 + 2H} \ell_i, \quad \hat{n}^i = h^{ij} \hat{n}_j = \frac{1}{\sqrt{1 + 2H}} \ell^i, \] (3.31)
and intrinsic 2-metric
\[ \hat{\gamma}_{ij} = h_{ij} - \hat{n}_i \hat{n}_j = \delta_{ij} - \ell_i \ell_j, \quad \hat{\gamma}^{ij} = \delta^{ij} - \ell^i \ell^j. \] (3.32)
A straightforward calculation gives the extrinsic curvature components of \( \Sigma \),
\[ \kappa = \hat{n}^k \hat{n}^l K_{kl} = \frac{2M}{r^2 \sqrt{1 + 2H}} (1 + H), \] (3.33)
\[ k_i = \hat{\gamma}^k \hat{n}^l K_{kl} = 0, \] (3.34)
\[ K_{ij} = \hat{\gamma}^k \hat{\gamma}^l \hat{K}_{kl} = -\frac{2M}{r^2 \sqrt{1 + 2H}} \hat{\gamma}_{ij}, \] (3.35)
\[ K^l_l = \hat{\gamma}^{kl} \hat{K}_{kl} = -\frac{4M}{r^2 \sqrt{1 + 2H}}, \] (3.36)
and
\[ \hat{\kappa}_{ij} = K_{ij} - \frac{1}{2} \hat{\gamma}_{ij} K^l_l = 0. \] (3.37)
Note that \( \kappa \) and \( K^l_l \) are globally non-vanishing and have opposite sign, in agreement with the condition (2.8) for strong hyperbolicity.
The extrinsic curvature of $\mathcal{S}_\rho$ is given by

$$\hat{K}_{ij} = \gamma^l_i D_l \hat{n}_j = \frac{1}{2} \mathcal{L}_{\hat{n}} \hat{\gamma}_{ij}, \quad (3.38)$$

where

$$\mathcal{L}_{\hat{n}} \hat{\gamma}_{ij} = \hat{n}^k \partial_k \hat{\gamma}_{ij} + \hat{\gamma}_{kj} (\partial_i \hat{n}^k) + \hat{\gamma}_{ik} (\partial_j \hat{n}^k) = \frac{1}{\sqrt{1 + 2H}} \left[ \frac{1}{x^k} \partial_k \left[ \frac{x^j}{r^2} \right] \right] + \left( \delta_{ik} - \frac{x^i x^k}{r^2} \right) \partial_j \left[ \frac{1}{\sqrt{1 + 2H}} \frac{x^k}{r} \right] = \frac{2}{r \sqrt{1 + 2M}} \left( \delta_{kj} - \frac{x_k x_j}{r^2} \right). \quad (3.39)$$

It follows that

$$\hat{K}_{ij} = \frac{1}{r \sqrt{1 + 2H}} \hat{\gamma}_{ij}, \quad (3.40)$$

$$\hat{K}^l_l = \hat{\gamma}^{kl} \hat{K}_{kl} = \frac{2}{r \sqrt{1 + 2H}}, \quad (3.41)$$

and

$$\hat{\kappa} = \hat{K}_{ij} - \frac{1}{2} \hat{\gamma}_{ij} \hat{K}^l_l = 0. \quad (3.42)$$

4 Nonlinear perturbations of a Schwarzschild black hole

Here we investigate nonlinear perturbations of the Kerr-Schild initial data for a Schwarzschild black hole. In doing so, we simplify the discussion by assigning Schwarzschild values to the freely specifiable variables $(\hat{N}, \hat{N}^i, \hat{\gamma}_{ij}, \hat{K}_{ij})$. As a result, the initial metric $h_{ij}$ retains its Schwarzschild value and, in particular, $\hat{K}_{ij} = 0$ and $\hat{N}$ and $^{(3)}R$ have no angular dependence. For a more general perturbation, $(\hat{N}, \hat{N}^i, \hat{\gamma}_{ij}, \hat{K}_{ij})$ would enter as explicit terms in the resulting system for $(\kappa, K^l_l, k, \cdots)$.

In this setting, $(2.5)-(2.6)$ reduce to

$$\mathcal{L}_{\hat{n}} (K^l_l) - \hat{D}^l k_l - [\kappa - \frac{1}{2} (K^l_l)] (\hat{K}^l_l) = 0, \quad (4.43)$$

$$\mathcal{L}_{\hat{n}} k_l + (K^l_l)^{-1} [\kappa \hat{D}_l (K^l_l) - 2 k^l \hat{D}_l k_l] + (\hat{K}^l_l) k_l = 0, \quad (4.44)$$

where $\kappa$, determined by $(2.7)$, reduces to

$$\kappa = (2 K^l_l)^{-1} [2 k^l k_l - \frac{1}{2} (K^l_l)^2 - ^{(3)}R]. \quad (4.45)$$

It is easy to check that these equations hold for a Schwarzschild solution, for which $^{(3)}R = \frac{8M^2}{r^2 (1 + 2H)}$, $\hat{n}^i \partial_i = \frac{1}{\sqrt{1 + 2H}} \partial_r$, $k_l = 0$ and neither $K^l_l$ nor $\kappa$ have angular dependence.
In spherical coordinates $x^i = (r, x^A)$, $x^A = (\theta, \phi)$,

$$\gamma_{ij}dx^idx^j = r^2q_{AB}dx^Adx^B$$ (4.46)

where $q_{AB}$ is the unit sphere metric. Then (4.43)-(4.44) become

$$\frac{1}{\sqrt{1+2H}} \partial_\tau K^l_l - \hat{D}^Bk_B - [\kappa - \frac{1}{2} (K^l_l) (\hat{K}^l_l)] = 0,$$ (4.47)

$$\frac{1}{\sqrt{1+2H}} \partial_\tau k_A + (K^l_l)^{-1} [\kappa \partial_A (K^l_l) - 2 k^B \hat{D}_A k_B] + (\hat{K}^l_l) k_A = 0.$$ (4.48)

Now consider nonlinear perturbations of Schwarzschild. We denote by $\delta V = V - V_S$ the deviation of a variable $V$ from its Schwarzschild value $V_S$. Then (4.47)-(4.48) take the form

$$\frac{1}{\sqrt{1+2H}} \partial_\tau (\delta K^l_l) - \frac{q^{BC}}{r^2} \partial_C \delta k_B = F_1,$$ (4.49)

$$\frac{1}{\sqrt{1+2H}} \partial_\tau \delta k_A + \kappa (K^l_l)^{-1} \partial_A (K^l_l) - 2 k^B \hat{D}_A k_B] + (\hat{K}^l_l) k_A = F_A,$$ (4.50)

where $F_1$ and $F_A$ represent lower differential order terms. This is a coupled quasilinear system for the vector valued variable $U_\alpha = (u^1, u^A) = (\delta K^l_l, \delta k_A)$. The system (4.49)-(4.50) has matrix form

$$\partial_\tau U_\alpha = \mathcal{L}_\alpha^\beta C \partial_C U_\beta + F_\alpha,$$ (4.51)

where $\partial_\tau = (1+2H)^{-1/2} \partial_\tau$, $F_\alpha = (F_1, F_A)$ and

$$\mathcal{L}_1^{1C} = 0, \quad \mathcal{L}_1^{BC} = \frac{1}{r^2} q^{BC},$$ (4.52)

$$\mathcal{L}_A^{1C} = -\frac{\kappa}{K^l_l} \delta_C^A, \quad \mathcal{L}_A^{BC} = \frac{2}{r^2 K^l_l} q^{BP} k_D^C.$$ (4.53)

The requirement that (4.51) is a strongly hyperbolic system [20, 21] is that there exists a positive bilinear form $H_{\beta\gamma}$ such that $\mathcal{L}(\omega)_{\beta\alpha} = H_{\beta\gamma} \omega^\gamma C$ is symmetric for each choice of $\omega_C$. It is straightforward to check that such a symmetrizer is given by

$$H_{11} = -\frac{K^l_l}{\kappa}, \quad H_{1A} = 0,$$ (4.54)

$$H_{A1} = \frac{2k_A}{\kappa}, \quad H_{AB} = r^2 q_{AB}.$$ (4.55)

The positivity of the symmetrizer for perturbations of Schwarzschild,

$$H_{\alpha\beta} v^\alpha v^\beta = -\frac{K^l_l}{\kappa} (v^1)^2 + \frac{2}{\kappa} k_A v^1 v^A + r^2 q_{AB} v^A v^B > 0, \quad v^\alpha \neq 0,$$ (4.56)
follows from the near Schwarzschild approximations

\[
\begin{align*}
- \frac{K_l^i}{\kappa} & \approx \frac{2(1 + 2H)}{1 + H}, & k_A^i & \approx 0.
\end{align*}
\] (4.57)

Furthermore, the \( \omega_A \) independence of \( H_{\alpha\beta} \) implies that the system is symmetric hyperbolic as well as strongly hyperbolic.

Given near Schwarzschild data for \((K_l^i, k_A^i)\) on a surface \( \mathcal{S}_R \) surrounding a Schwarzschild black hole, strong hyperbolicity is a sufficient condition for the system (4.47) - (4.48) to produce a local solution of the constraint problem in some neighborhood of \( \mathcal{S}_R \). Furthermore, the problem is well-posed so that the solution depends continuously on the data. For linearized perturbations the solution extends globally to \( r = 0 \).

5 Future prospects

We have shown that the new treatment of the constraints proposed in [1] leads to a well-posed constraint problem for nonlinear perturbations of a Schwarzschild black hole in Kerr-Schild form. As is generally the case for nonlinear problems, the solution is only guaranteed locally in a neighborhood of the surface \( \mathcal{S}_R \) on which the data is prescribed. The issue of a global solution seems best explored by numerical techniques.

Our result immediately extends to perturbations representing a Kerr black hole with small spin and boost. The full question whether it extends further to a Kerr black hole with maximal spin and arbitrary boost is much more complicated. Its resolution would depend, among other things, on a judicious choice of foliation \( \mathcal{S}_\rho \).

The ultimate utility of this new approach rests upon its extension to multiple black holes. Formally, it can be applied to the multiple black hole problem using a modification of the superimposed Kerr-Schild data proposed in [15, 16], which is based upon the ansatz that the initial three metric for a binary black hole is given by

\[
\begin{align*}
h_{ij} = & \delta_{ij} + 2H^{[1]} \ell_i^{[1]} \ell_j^{[1]} + 2H^{[2]} \ell_i^{[2]} \ell_j^{[2]},
\end{align*}
\] (5.58)

where \( H^{[n]} \) and \( \ell_i^{[n]} \) correspond to the Kerr-Schild data for individual boosted, spinning black holes. In [15, 16], the actual 3-metric data is only conformal to (5.58), with the conformal factor chosen to satisfy the Hamiltonian constraint.

In our modified approach to the constraints, we propose to retain the superimposed Kerr-Schild ansatz in its strict 4-dimensional form

\[
\begin{align*}
g_{ab} = & \eta_{ab} + 2H^{[1]} \ell_a^{[1]} \ell_b^{[1]} + 2H^{[2]} \ell_a^{[2]} \ell_b^{[2]},
\end{align*}
\] (5.59)

where \( \ell_a^{[n]} \) are null with respect to the background Minkowski metric. This determines the the initial lapse and shift as well as the initial 3-metric (5.58). Then we
use the Hamiltonian constraint to express the extrinsic curvature component $\kappa$ algebraically in terms of $K^l_l$ and explicitly known terms via (2.7). The extrinsic curvature components $\dot{K}_{ij} = K_{ij} - \frac{1}{2} \dot{\gamma}_{ij} K^l_l$, can be freely prescribed, say, by superposition of their individual Kerr-Schild values. Given a foliation of the initial hypersurface $\mathscr{I}_\rho$ and vector field $\rho^i$, perhaps determined by the radial coordinate (3.23) intrinsic to the Minkowski background, the remaining components of the extrinsic curvature data, $K^l_l$ and $k_i$, could then be determined from the hyperbolic system (2.5)-(2.7). The only necessary data are the values of $K^l_l$ and $k_i$ on a large surface $\mathscr{I}_{\rho_0}$ surrounding the system. The surface data for $K^l_l$ and $k_i$ could again tentatively be prescribed by the superposition of their individual Kerr-Schild values.

The simplicity of such a scheme for multiple black hole data makes it extremely attractive. Whether it can be successfully implemented is again a matter for numerical study. If this is indeed successful then it leads to the question of the most physical importance: Does the resulting multiple black hole initial data suppress junk radiation? The sole need for data on a single large surface surrounding the system distinguishes this approach from other solutions to the constraint problem which rely on elliptic equations. Whether this feature suppresses junk radiation would again be a matter for numerical investigation.

Acknowledgments

The authors are grateful for the kind hospitality of the Albert Einstein Institute in Golm, Germany, where this work was initiated. IR was supported in part by the Die Aktion Österreich-Ungarn, Wissenschafts- und Erziehungskooperation grant 90öu1. JW was supported by NSF grant PHY-1201276 to the University of Pittsburgh.

References

[1] Rácz I: Constrains in new dress, in preparation

[2] Choquet-Bruhat Y: General relativity and Einstein’s equations, Oxford University Press Inc., New York (2009)

[3] Wald R M: General relativity, University of Chicago Press, Chicago (1984)

[4] Lichnerowicz A: L’integration des Equations de la Gravitation Relativiste et le Probleme des n Corps, J. Math. Pures Appl., 23, 39-63 (1944)

[5] York J W: Role of conformal three-geometry in the dynamics of gravitation, Phys. Rev. Letters 28, 1082-1085 (1972)
[6] York J W: Covariant decompositions of symmetric tensors in the theory of gravitation, Ann. Inst. Henri Poincaré A 21 319-332 (1974)

[7] Cook G B: Initial data for numerical relativity, Living Rev. Relativity 3 5 (2000)

[8] Thornburg J: Coordinates and boundary conditions for the general relativistic initial data problem, Class. Quantum Grav. 4 1119 (1987)

[9] Brandt S and Brügmann B: A simple construction of initial data for multiple black holes, Phys. Rev. Letters 78 3006 (1997)

[10] Hannam M, Husa S, Ohme F, Brügmann B and Murchadha N Ó: Wormholes and trumpets: Schwarzschild space-time for the moving-puncture generation, Phys. Rev. D 78 064020 (2008)

[11] Baumgarte T W and Shapiro S L: Numerical Relativity: Solving Einstein’s Equations on the computer Cambridge University Press , Cambridge, England (2010)

[12] Kerr R P and Schild A: Some algebraically degenerate solutions of Einstein’s gravitational field equations, Proc. Symp. Appl. Math. 17 199 (1965)

[13] Kerr R P and Schild A: A new class of vacuum solutions of the Einstein field equations, Atti degli Convegno Sulla Relativita generale p. 222 (Firenze, 1966).

[14] Bishop N T, Isaacson R, Maharaj M and Winicour J: Black hole data via a Kerr-Schild approach, Phys. Rev. D 57 6113 (1998)

[15] Matzner R A, Huq M F and Shoemaker D: Initial data and coordinates for multiple black hole systems, Phys. Rev. D 59 024015 (1999)

[16] Bonning E, Marronetti P, Neilson D and Matzner R A: Physics and initial data for multiple black hole spacetimes, Phys. Rev. D. 68 044019 (2003)

[17] Rácz I: Is the Bianchi identity always hyperbolic?, Class. Quant. Grav. 31 155004 (2014)

[18] Rácz I: Cauchy problem as a two-surface based ‘geometrodynamics’, Class. Quant. Grav. 32 015006 (2015)

[19] Rácz I: Dynamical determination of the gravitational degrees of freedom, submitted to Class. Quant. Grav.; arXiv:gr-qc/1412.0667

[20] Kreiss H-O and Lorenz J: Initial-boundary value problems and the Navier-Stokes equations, (Academic Press, Boston, 1989), reprinted as SIAM Classic (2004)

[21] Reula O A: Strongly hyperbolic systems in general relativity J. Hyper. Differential Equations 01 251 (2004)