EXACT BOUNDS ON THE INVERSE MILLS RATIO AND ITS DERIVATIVES

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Abstract. The inverse Mills ratio is \( R := \frac{\varphi}{\Psi} \), where \( \varphi \) and \( \Psi \) are, respectively, the probability density function and the tail function of the standard normal distribution. Exact bounds on \( R(z) \) for complex \( z \) with \( \Re z \geq 0 \) are obtained, which then yield logarithmically exact bounds on high-order derivatives of \( R \). The main idea of the proof is a non-asymptotic version of the so-called stationary-phase method.

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1. Introduction, summary, and discussion

The inverse Mills ratio \( R \) is defined by the formula

\[
R := \frac{\varphi}{\Psi},
\]

where \( \varphi \) and \( \Psi \) are, respectively, the probability density function and the tail function of the standard normal distribution, so that \( \varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \) and \( \Psi(x) = \int_x^\infty \varphi(u) \, du \) for all real \( x \). These expressions for \( \varphi \) and \( \Psi \) in fact define entire functions on the complex plane \( \mathbb{C} \), if the upper limit of the integral is still understood as the point \( \infty = \infty + 0i \) on the extended real axis. One may note that \( \Psi \) is a rescaled version of the complementary error function: \( \Psi(z) = \text{erfc}(z/\sqrt{2})/2 \) for all \( z \in \mathbb{C} \).

Theorem 1.1.

(I) The function \( \Psi \) has no zeros on the right half-plane

\[
H_+ := \{ z \in \mathbb{C} : \Re z \geq 0 \}.
\]

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So, the function $S: H_+ \to \mathbb{C}$ defined by the formula

$$S(z) := \frac{R(z)}{z + \sqrt{2/\pi}}$$

for $z \in H_+$ is holomorphic on the interior of $H_+$.

(II) One has

$$1 > |S(z)| > |S(iy_*)| = 0.686 \ldots \quad \text{for all } z \in H_+ \setminus \{0, iy_*, -iy_*\},$$

where $y_* = 1.6267 \ldots$ is the only minimizer of $|S(iy)|$ in $y \in (0, \infty)$.

(III) It obviously follows that the lower bound $|S(iy_*)|$ on $|S(z)|$ in (1.4) is exact. The upper bound $1$ on $|S(z)|$ in (1.4) is also exact, in following rather strong sense:

$$S(0) = 1 = \lim_{H^+ \ni z \to \infty} S(z).$$

In view of the maximum modulus principle (see e.g. Section 3.4 of [1]) applied to the function $S$ and its reciprocal $1/S$, Theorem 1.1 is an immediate corollary of Lemma 1.2.

(i) For all $z \in H_+$ one has $\varphi(\overline{z}) = \overline{\varphi(z)}$ and $\Psi(\overline{z}) = \overline{\Psi(z)}$, where, as usual, the bar denotes the complex conjugation.

(ii) The function $\Psi$ has no zeros on $H_+$. Moreover, $\Re R(z) > 0$ and $\Re R(z)$ equals $\Im z$ in sign, for any $z \in H_+$.

(iii) Equalities (1.5) hold.

(iv) There is a (necessarily unique) point $y_* \in (0, \infty)$ such that $|S(iy_*)|$ (strictly) decreases from $S(0) = 1$ to $|S(iy_*)|$ as $y$ increases from 0 to $y_*$, and $|S(iy_*)|$ (strictly) increases from $|S(iy_*)|$ to 1 as $y$ increases from $y_*$ to $\infty$. In fact, $y_* = 1.6267 \ldots$ and $|S(iy_*)| = 0.686 \ldots$.

All necessary proofs are deferred to Section 2.

The main idea of the proof is a non-asymptotic version of the so-called stationary-phase method, which latter is described and used for asymptotics e.g. in [2]. The mentioned version of the method is given by formulas (1.4) and (1.2), which provide comparatively easy to analyze integral expressions for the real and imaginary parts of the Mills ratio $\Psi(z)/\varphi(z)$ for $z$ with $\Re z > 0$.

Remark 1.3. It also follows from Lemma 1.2 by the maximum modulus principle (or, slightly more immediately, by the minimum modulus principle) that for each $x \in [0, \infty)$ the minimum $\min \{|S(x + iy)|: y \in \mathbb{R}\}$ is attained and is (strictly) increasing in $x \in [0, \infty)$, from 0.686 $\ldots$ to 1. Of course, instead of the family of vertical straight lines $(\{x + iy: y \in \mathbb{R}\})_{x \in [0, \infty)}$, one can take here any other appropriate family of curves, e.g. the family $(\{x + \varphi(y) + iy: y \in \mathbb{R}\})_{x \in [0, \infty)}$.

In particular, Theorem 1.1 implies that $\max_{x \geq 0} S(x) = 1$. One can also find $\min_{x \geq 0} S(x)$. More specifically, one has the following proposition, complementing Theorem 1.1.

Proposition 1.4. There is a (necessarily unique) point $x_* \in (0, \infty)$ such that $S(x)$ decreases from $S(0) = 1$ to $S(x_*)$ as $x$ increases from 0 to $x_*$, and $S(x)$ increases back to 1 as $x$ increases from $x_*$ to $\infty$. In fact, $x_* = (\pi - 1)\sqrt{2/\pi} = 1.7087 \ldots$ and $S(x_*) = 0.844 \ldots$. 
The Mills ratio of the real argument, including related bounds and monotonicity patterns, has been studied very extensively; see e.g. [3, 4] and further references therein.

Theorem 1.1, Lemma 1.2, Remark 1.3, and Proposition 1.4 are illustrated in Figure 1.

An immediate corollary of Theorem 1.1 is

**Corollary 1.5.** For all $z \in H_+ \setminus \{0\}$,

$$|R(z)| < \sqrt{\frac{2}{\pi}}|z + \sqrt{\frac{2}{\pi}}|,$$

whereas $R(0) = \sqrt{\frac{2}{\pi}}$. Moreover,

$$R(z) \sim z \quad \text{as} \quad H_+ \ni z \to \infty.$$

Here and in the sequel we use the following standard notation: $F \sim G$ meaning that $F/G \to 1$; $F \leq G$ or, equivalently, $G \geq F$ meaning that $\limsup F/G \leq 1$; $F \ll G$ or, equivalently, $G \gg F$ meaning that $F/G \to 0$; $F \leq G$ or, equivalently, $G \geq F$ meaning that $\limsup |F/G| < \infty$, and $F \asymp G$ meaning that $F \leq G \leq F$.

In turn, Corollary 1.5 yields the following bound on the $n$th derivative $R^{(n)}$ of the inverse Mills ratio $R$.

**Corollary 1.6.** For any natural $n$ and any $z \in H_+$ with $x := \Re z > 0$,

$$|R^{(n)}(z)| \leq R^{(n)}_{\max}(z) := \frac{n!}{x^n} \sqrt{|z + \sqrt{\frac{2}{\pi}}|^2 + x^2}.$$

Moreover, for each natural $n$,

$$|R^{(n)}(z) - I\{n = 1\}| \ll \frac{|z|}{x^n} \quad \text{as} \quad z \in H_+ \text{ and } x \to \infty,$$

where $I\{\cdot\}$ stands for the indicator function.
In applications of Corollary 1.6 to the calculation of sums of the form \( s_{x_0, \delta, N} := \sum_{i=0}^{N-1} R(x_0 + i\delta) \) for natural \( N \) and positive \( x_0 \) and \( \delta \) (see e.g. [4]), of special interest is the case when \( x >> n >> 1 \), which makes the bound \( R_{n}^{(n)}(x) \) small (here \( x \) is real, as before). In such a case, the bound \( R_{\max}^{(n)}(x) \) is optimal at least in the logarithmic sense – which is the appropriate sense as far as the desired number of digits in the calculation of the sums \( s_{x_0, \delta, N} \) is concerned. Indeed, let us compare the bound \( R_{\max}^{(n)}(x) \) on the \( n \)th derivative of the function \( R \) with the \( n \)th derivative of the function \( f(x) = x + 1/x \), which is asymptotic to \( R(x) \) as \( x \to \infty \). We see that \( \log R_{\max}^{(n)}(x) \sim -n \log \frac{x}{n} \sim \log |f^{(n)}(x)| \) if \( x >> n >> 1 \).

2. Proofs

Proof of Lemma 1.2

(i) The conjugation symmetry property of the function \( \varphi \) is obvious. That of \( \Psi \) follows because \( \overline{\Psi(z)} = \int_{z}^{\infty} \varphi(w) \, dw = \int_{z}^{\infty} \varphi(w) \, dw = \psi(\overline{z}) \) for all \( z \in \mathbb{C} \). for all real \( x \).

(ii) Take any \( z = x + iy \in H_{+} \) with \( x := \Re z \) and \( y := \Im z \). By part (i) of Lemma 1.2 without loss of generality \( y \geq 0 \). If \( x = 0 \), then \( \overline{\Psi(z)} = \overline{\Psi(iy)} = -\int_{y}^{0} \varphi(iv) \, dv + \Psi(0) = -i \int_{0}^{y} \frac{1}{\sqrt{2\pi}} e^{v^2/2} \, dv + \frac{1}{2} \), so that \( \Re \Psi(z) > 0 \) and \( \Im \Psi(z) \) equals \( -3z \) in sign. So, since \( \varphi(iy) \) is real for real \( y \), the statements made in part (i) follow, in the case \( x = 0 \). Suppose now that \( x > 0 \). Then, integrating from \( z = x + iy \) to \( \infty + 0i \) along a curve with the image set \( C := \{ w \in H_{+} : uv = xy, u \geq x \} \), where \( u := \Re w \) and \( v := \Im w \), one has

\[
(2.1) \quad \frac{\Psi(z)}{\varphi(z)} = A - iB,
\]

where

\[
(2.2) \quad A := A(z) := \int_{x}^{\infty} e^{a(u) \varphi(x) - a(x)} \, du \quad \text{and} \quad B := B(z) := \int_{0}^{y} e^{a(y) \varphi(x) - a(v)} \, dv,
\]

where

\[
(2.3) \quad a(u) := a_{xy}(u) := \frac{x^2y^2}{2u^2} - \frac{u^2}{2}.
\]

Now the statements made in part (ii) follow from (2.1), because \( A(z) > 0 \) and \( B(z) \geq 0 \) (for \( y \geq 0 \)), with \( B(z) = 0 \) only if \( \Im z = 0 \).

(iii) The first equality in (1.3) is trivial. Let us prove the second equality there. Let \( H_{+} \ni z = x + iy \to \infty \), where \( x := \Re z \) and \( y := \Im z \); that is, \( x > 0 \), \( y \in \mathbb{R} \), and \( x^2 + y^2 \to \infty \). In view of the continuity of the function \( S \) and its conjugation symmetry property, without loss of generality \( x > 0 \) and \( y > 0 \).

Let \( r \) denote the Mills ratio, so that \( r = 1/R = \psi/\varphi \). It is well known and easy to prove using l’Hospital’s rule that

\[
(2.4) \quad r(x) \sim 1/x \quad \text{as} \quad x \to \infty.
\]

Similarly, for the rescaled version \( \tilde{r} \) of the Dawson function defined by the formula

\[
(2.5) \quad \tilde{r}(y) := e^{-y^2/2} \int_{0}^{y} e^{u^2/2} \, du
\]

for \( y > 0 \), one has

\[
(2.6) \quad \tilde{r}(y) \sim 1/y \quad \text{as} \quad y \to \infty.
\]
Recalling (2.3), note that \( a'(u) = -x^2y^2/u^3 - u \geq -(u+y^2/x) \) for \( u \geq x \), whence \( a(u) - a(x) \geq \frac{1}{2}(x + y^2/x)^2 - \frac{1}{2}(u + y^2/x)^2 \) and, by (2.2),

\[
A \geq \int_{x}^{\infty} \exp \left\{ \frac{1}{2} \left( x + y^2/x \right)^2 - \frac{1}{2} \left( u + y^2/x \right)^2 \right\} \, du \\
\geq \int_{x+y^2/x}^{\infty} \exp \left\{ \frac{1}{2} \left( x + y^2/x \right)^2 - \frac{1}{2} \left( \sqrt{x^2 + y^2} \right)^2 \right\} \, ds \\
= r(x+y^2/x) \sim \frac{x}{x^2 + y^2}.
\]

Here we used the observation that \( x + y^2/x = (x^2 + y^2)/x \geq \sqrt{x^2 + y^2} \to \infty \) and (2.4).

Similarly, \(-a'(v) = x^2y^2/v^3 + v \geq v + x^2/y \) for \( v \in (0, y] \), whence \( a(y) - a(v) \leq \frac{1}{2} \left( v + x^2/y \right)^2 - \frac{1}{2} \left( y + x^2/y \right)^2 \) and, by (2.2),

\[
B \leq \int_{0}^{y} \exp \left\{ \frac{1}{2} \left( v + x^2/y \right)^2 - \frac{1}{2} \left( x + x^2/y \right)^2 \right\} \, dv \\
\leq \int_{0}^{y+x^2/y} \exp \left\{ \frac{1}{2} v^2 - \frac{1}{2} \left( y + x^2/y \right)^2 \right\} \, dt \\
= r(y+x^2/y) \sim \frac{y}{x^2 + y^2},
\]

in view of (2.4).

Next, fix any \( c \in (0, 1) \). Then for all \( v \in [cy, y] \) one has \(-a'(v) = x^2y^2/v^3 + v \leq v + x^2/(c^3y) \), whence \( a(y) - a(v) \geq \frac{1}{2} \left( v + x^2/(c^3y) \right)^2 - \frac{1}{2} \left( y + x^2/(c^3y) \right)^2 \). So,

\[
B \geq \int_{cy}^{y} \exp \left\{ \frac{1}{2} \left( v + \frac{x^2}{c^3y} \right)^2 - \frac{1}{2} \left( y + \frac{x^2}{c^3y} \right)^2 \right\} \, dv \\
= r \left( y + \frac{x^2}{c^3y} \right) \exp \left\{ - \frac{1}{2} \left( y + \frac{x^2}{c^3y} \right)^2 \right\} \exp \left\{ \frac{1}{2} \left( cy + \frac{x^2}{c^3y} \right)^2 \right\} \tilde{r} \left( cy + \frac{x^2}{c^3y} \right) \\
\sim \frac{c^3y^3}{x^2 + c^3y^2};
\]

here we again used (2.4) and the condition \( x^2 + y^2 \to \infty \). Letting now \( c \uparrow 1 \) and recalling (2.8), we have

\[
B \sim \frac{y}{x^2 + y^2} \quad \text{as } x > 0, \ y > 0, \ x^2 + y^2 \to \infty.
\]

Further, fix any real \( k > 1 \). Then for all \( u \in [x, kx] \) one has \( a'(u) = -x^2y^2/u^3 - u \leq -(u+y^2/(k^3x)) \), whence \( a(u) - a(x) \leq -\frac{1}{2} \left( u + y^2/(k^3x) \right)^2 + \frac{1}{2} \left( x + y^2/(k^3x) \right)^2 \). So,

\[
A \leq A_1(k) + A_2(k),
\]
where

\[ A_1(k) := \int_0^k \exp \left\{ -\frac{1}{2} \left( u + \frac{y^2}{k^3 x} \right)^2 + \frac{1}{2} \left( x + \frac{y^2}{k^3 x} \right)^2 \right\} \, du \]

\[ = \int_0^\infty \exp \left\{ -\frac{1}{2} \left( u + \frac{y^2}{k^3 x} \right)^2 + \frac{1}{2} \left( x + \frac{y^2}{k^3 x} \right)^2 \right\} \, du \]

\[ \sim k^3 x \sim \frac{k^3 x}{x^2 + y^2}, \]

in view of (2.4) and the condition \( x^2 + y^2 \to \infty \), and

\[ A_2(k) := \int_{kx}^\infty e^{a(u) - a(x)} \, du = \int_{kx}^\infty \exp \left\{ \frac{x^2 y^2}{2u^2} - \frac{u^2}{2} - a(x) \right\} \, du \]

\[ \leq \exp \left\{ \frac{y^2}{2k^2} - a(x) \right\} \int_{kx}^\infty e^{-u^2/2} \, du \]

\[ < \exp \left\{ \frac{y^2}{2k^2} - a(x) \right\} e^{-k^2 x^2/2} \frac{1}{kx} \]

\[ \leq \frac{k^2 - 1}{2k^2} (x^2 + y^2) \frac{1}{kx} \]

\[ \ll \frac{x}{x^2 + y^2} \quad \text{if} \quad x^2 (x^2 + y^2) \gtrsim 1, \]

because then \( \frac{1}{kx} \leq x(x^2 + y^2) \), while \( \exp \left\{ -\frac{k^2 - 1}{2k^2} (x^2 + y^2) \right\} \ll 1/(x^2 + y^2)^2 \).

Letting now \( k \downarrow 1 \) and recalling (2.7), we have

\[ A \sim \frac{x}{x^2 + y^2} \quad \text{as} \quad x > 0, \quad y > 0, \quad x^2 + y^2 \to \infty, \quad x^2 (x^2 + y^2) \gtrsim 1. \]

In view of (1.1), (2.1), (2.9), and (2.11),

\[ (2.12) \quad R(z) = \frac{1}{A - iB} \sim \frac{x^2 + y^2}{x - iy} = z \]

as \( x > 0, \ y > 0, \ x^2 + y^2 \to \infty, \) and \( x^2 (x^2 + y^2) \gtrsim 1. \)

Suppose now the condition \( x^2 (x^2 + y^2) \gtrsim 1 \) does not hold, so that without loss of generality \( x^2 (x^2 + y^2) \to 0 \) (while still \( x > 0, \ y > 0, \ x^2 + y^2 \to \infty \)). Then \( x \to 0, \ xy \to 0, \ y \to \infty, \ z \sim iy, \) and \( \varphi(x + iy) = \varphi(iy) \exp \{-x^2/2 - ixy\} \sim \varphi(iy) \to \infty. \)

So,

\[ (2.13) \quad \Psi(z) = -I_1 - I_2 + \Psi(0) = -I_1 - I_2 + 1/2, \]

where

\[ (2.14) \quad I_1 := \int_0^x \varphi(u + iy) \, du \quad \text{and} \quad I_2 := i \int_0^u \varphi(iv) \, dv. \]

At that,

\[ (2.15) \quad |I_1| \leq x\varphi(iy) \ll \frac{\varphi(iy)}{\sqrt{x^2 + y^2}} \sim |\varphi(z)/z| \]

and, by (2.6),

\[ (2.16) \quad I_2 \sim i\varphi(iy)/y \sim -\varphi(z)/z. \]

It also follows that \( |I_2| \to \infty. \)
So, in view of (1.1) and (2.13), asymptotic equivalence (2.12) holds whenever $x > 0$, $y > 0$, and $x^2 + y^2 \to \infty$. This completes the proof of part (iii) of Lemma 1.2.

Let

\begin{equation}
(2.17) \quad s(y) := |S(iy)|^2 = \frac{f(y)}{g(y)},
\end{equation}

where

\begin{equation}
(2.18) \quad f(y) := \frac{\varphi(iy)^2}{y^2 + 2/\pi}, \quad g(y) := \frac{1}{4} + E(y)^2, \quad E(y) := \int_0^y \varphi(iv) \, dv.
\end{equation}

Here and in the rest of the proof of Lemma 1.2, $y$ stands for an arbitrary nonnegative real number. Consider first two “derivative ratios” for the ratio $s = f/g$:

\begin{equation}
(2.19) \quad s_1 := \frac{f'}{g'} = \frac{f_1}{g_1} \quad \text{and} \quad s_2 := \frac{f_1}{g_1},
\end{equation}

where $g_1 := E$ and $f_1 := s_1 E$. Then $s_2$ is a rational function, that is, the ratio of two polynomials (each of degree 6). So, it is straightforward to find that, for some algebraic numbers $y_{21}$ and $y_{22}$ such that $0 < y_{21} < y_{22}$, the function $s_2$ is (strictly) increasing on the interval $[0, y_{21}]$, decreasing on $[y_{21}, y_{22}]$, and increasing on $[y_{22}, \infty)$; in fact, $y_{21} = 0.685 \ldots$ and $y_{22} = 1.407 \ldots$.

Using the l’Hospital rule for limits, one easily finds that $\lim_{y \to 0} s_1'(y)/y = \frac{1}{\pi}(2 - 4\pi + 3\pi^2) > 0$. So, by the general l’Hospital-type rule for monotonicity, given e.g. by line 1 of Table 1.1 in [6], the function $s_1$ is increasing on the interval $[0, y_{21}]$.

Next, $s_1'(y_{22}) = 0.054 \ldots > 0$; so, by lines 2 and 1 of Table 1.1 in [6], $s_1$ is increasing on the intervals $[y_{21}, y_{22}]$ and $[y_{22}, \infty)$ as well. Thus, the first “derivative ratio” $s_1$ for the ratio $s$ is increasing on the entire interval $[0, \infty)$.

The values of the function $s$ at the points 0, 1, and 3 are 1, 0.553\ldots, and 0.670\ldots, respectively, so that $s(0) > s(1) < s(3)$. Using again line 1 of Table 1.1 in [6], we conclude that $s$ is decreasing\-increasing on $[0, \infty)$; that is, there is a uniquely determined number $y_∗ ∈ (0, \infty)$ such that $s$ decreases on $[0, y_∗]$ and increases on $[y_∗, \infty)$. In other words, $|S(iy)| = \sqrt{s(y)}$ decreases in $y ∈ [0, y_∗]$ and increases in $y ∈ [y_∗, \infty)$. Let $y_{01} := 16267/10000$ and $y_{02} := 16268/10000$. Then $s'(y_{01}) < 0 < s'(y_{02})$, whence $0 < y_{01} < y_∗ < y_{02}$ and $y_∗ = 1.6267 \ldots$.

So, in view of (2.17) and (2.18),

\begin{equation}
(2.20) \quad \min_{y>0} |S(iy)| = |S(iy_*)| > \frac{\varphi(iy_{01})}{\sqrt{(y_{02}^2 + 2/\pi)(1/4 + E(y_{02})^2)}} = 0.6861 \ldots;
\end{equation}

here we used the obvious fact that the expressions $\varphi(iy)$, $y^2 + 2/\pi$, and $1/4 + E(y)^2$ are each increasing in $y \geq 0$. On the other hand, $\min_{y>0} |S(iy)| = |S(iy_*)| \leq |S(iy_{02})| = 0.6862 \ldots$. Thus, $|S(iy_*)| = 0.686 \ldots$.

This concludes the proof of part (iv) of Lemma 1.2 and thereby that of the entire lemma.

**Proof of Proposition 1.4.** This proof is similar to, and even significantly simpler than, the proof of part (iv) of Lemma 1.2. Indeed, let here $s$ be the restriction of the function $S$ to $[0, \infty)$, so that $s = f/g$, where $f(x) := \varphi(x)/(x + \sqrt{2/\pi})$ and $g(x) := \Psi(x)$ for $x \geq 0$. Then the “derivative ratio” $f'/g'$ is a (rather simple) rational function, which decreases on the interval $[0, x_\ast]$ (with $x_\ast = (\pi - 1)\sqrt{2/\pi} = 1.7087 \ldots$, as in the statement of Proposition 1.4) and increases on the interval
\( [x_*, \infty) \). Also, clearly \( f(∞) = g(∞) = 0 \). Moreover, \( S(x_*) = \frac{0.844}{1} = S(0) = S(∞) \). Now Proposition 1.4 follows immediately by the derived special \( \text{H} \)ospital-type rule for monotonicity given in the last line of Table 4.1 in [6]. \( \square \)

**Proof of Corollary 1.7.** Take indeed any natural \( n \) and any \( z \in H_+ \) with \( x = Rz > 0 \). For any real \( \varepsilon > 0 \), let \( C_{\varepsilon} \) denote the circle of radius \( \varepsilon \) centered at the point \( z \), traced out counterclockwise. By the Cauchy integral formula,

\[
R^{(n)}(z) = \frac{n!}{2\pi i} \int_{C_{\varepsilon}} \frac{R(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta = \frac{n!}{2\pi x^n} \int_0^{2\pi} R(z + x e^{it}) e^{-int} \, dt.
\]

So, by (1.6),

\[
\frac{2\pi x^n}{n!} |R^{(n)}(z)| \leq \int_0^{2\pi} |z + \sqrt{\frac{2}{\pi}} x e^{it}| \, dt = \int_0^{2\pi} \sqrt{a + b \cos(t - \theta)} \, dt = \int_0^{2\pi} \sqrt{a + b \cos t} \, dt =: J(a, b),
\]

where

\[
(a) := |z + \sqrt{\frac{2}{\pi}} x| + x^2, \quad b := 2x |z + \sqrt{\frac{2}{\pi}}|, \quad \theta := \arctan \frac{x}{\sqrt{\frac{2}{\pi}}}. \]

The integral \( J(a, b) \) is an elliptic one. It admits a simple upper bound, \( J(a, 0) = 2\pi \sqrt{a} \), which is rather accurate (not exceeding \( \frac{\pi}{4} J(a, b) \)) for any \( a > 0 \) and \( b \) in \([0, a]\). This follows because \( J(a, b) \) is obviously concave in \( b \) at \( b = 0 \) equal 0, so that \( J(a, b) \) is nonincreasing in \( b \) in \([0, a]\), from \( J(a, 0) = 2\pi \sqrt{a} \) to \( J(a, a) = 4\sqrt{2a} > \frac{\pi}{4} J(a, 0) \). Now (1.8) follows immediately from (2.22) and (2.23).

Clearly, the identities in (2.21) hold with \( x/2 \) in place of \( x \). Let now \( t \in [0, 2\pi] \), \( z \in H_+ \), and \( x = Rz \rightarrow \infty \). Then \( 2|z| \geq |z + \frac{x}{2} e^{it}| \geq |z|/2 \rightarrow \infty \). So, in view of (1.7),

\[
2\pi \left( \frac{x}{2} \right)^n R^{(n)}(z) = \int_0^{2\pi} R \left( z + \frac{x}{2} e^{it} \right) e^{-int} \, dt = \int_0^{2\pi} \left( z + \frac{x}{2} e^{it} + o(|z|) \right) e^{-int} \, dt
\]

\[
= 2\pi \frac{x}{2} 1\{n = 1\} + \int_0^{2\pi} o(|z|) e^{-int} \, dt = 2\pi \frac{x}{2} 1\{n = 1\} + o(|z|),
\]

so that (1.9) follows as well. \( \square \)

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