RATIONAL HOMOTOPY AND INTERSECTION COHOMOLOGY

DAVID CHATAUR, MARTINTXO SARALEGI-ARANGUREN, AND DANIEL TANRÉ

Abstract. In this text, we extend Sullivan’s presentation of rational homotopy type to Goresky and MacPherson’s intersection cohomology.

We choose the context of face sets, also called simplicial sets without degeneracies, in the sense of Rourke and Sanderson. We define filtered face sets, perverse local systems over them and intersection cohomology with coefficients in a perverse local system. In particular, we get a perverse local system of cochains quasi-isomorphic to the intersection cochains of Goresky and MacPherson, over a field. We show also that these two complexes are quasi-isomorphic to a filtered version of Sullivan’s differential forms over the field \( \mathbb{Q} \).

We construct a functor from the category of filtered face sets to a category of perverse commutative differential graded algebras (CDGA’s) due to Hovey. We establish also the existence and uniqueness of a positively graded, minimal model of some perverse CDGA’s, including the perverse forms over a filtered face set and their intersection cohomology. Finally, we prove the topological invariance of the minimal model of a PL-pseudomanifold whose regular part is connected, and this theory creates new topological invariants. This brings a definition of formality in the intersection setting and examples are given. We show that the Calabi Yau quintic, and more generally, any nodal hypersurface in \( \mathbb{CP}(4) \), are intersection-formal.

Intersection cohomology of pseudomanifolds has been introduced by Goresky and MacPherson in [24] and [25]. The main feature of their theory concerns the transversality of a simplex relatively to a stratum, the notion of general position being replaced by a less restrictive one depending on a parameter called perversity. We supply the necessary definitions all along the text but, for a more complete background, the reader can consult the original papers or one of the books that have appeared in the last years as, for instance, [4], [33], [26], [2], [1], [19] or the historical development presented in [34].

Our work is motivated by a presentation of the intersection cohomology, similar to Sullivan’s presentation of the rational homotopy type, including the notion of minimal model, with its topological invariance for PL-pseudomanifolds whose regular part is connected. This rational intersection theory answers a question raised by Goresky ([27, Introduction]) in view of a treatment of formality. In fact, we offer more than a rational point of view, by providing a homotopical framework for intersection cohomology and many sections of this paper are concerned with homotopy type without localization over the field of rational numbers. This brings a new approach for the study of Steenrod operations in intersection cohomology, as quoted by MacPherson in [27, Question 4.3]. We come back to this aspect in [10].

Date: April 16, 2014.

2000 Mathematics Subject Classification. 55N33, 55P62, 57N80.

The third author is partially supported by the MICINN grant MTM2010-18089, ANR-11-BS01-002-01 “HOGT” and ANR-11-LABX-0007-01 “CEMPI”.

1
Our work is developed along the lines of Sullivan’s theory. (The reader can consult [18], [37], [13], [11], [29] or [49] for a presentation of rational homotopy.) This theory begins with a functor from simplicial sets to the category of commutative graded differential algebras (henceforth CDGA’s), denoted by $A_{PL}$, such that the cohomology of $A_{PL}(K)$ is isomorphic to the cohomology algebra of the simplicial set, $K$, with rational coefficients. One important step is thus the determination of a simplicial setting providing a homotopical context to intersection cohomology.

This work is organized as follows. The topological setting is concentrated in the three appendices. The original work of Goresky and MacPherson is designed for PL-pseudomanifolds filtered by PL-subspaces, $(X_i)_{0 \leq i \leq n}$, of respective dimension $i$. As in King’s paper ([32]), we are concerned with general topological spaces, $X$, together with a filtration by closed subspaces, $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X$. We consider singular simplices, $\sigma: \Delta = \Delta^0 \ast \cdots \ast \Delta^k \to X$, whose domain is decomposed as a join product such that $\sigma^{-1}(X_k) = \Delta^0 \ast \cdots \ast \Delta^k$. To any such simplex, called filtered simplex, and to any number $i \in \{1, \ldots, n\}$, we associate the perverse degree $|\sigma|_i = \dim(\Delta^0 \ast \cdots \ast \Delta^{n-1})$. Now comes the fundamental notion of perversity. The lack of relation between the filtration degree and a geometrical dimension leads us to consider a notion of perversity with less constraints than in the Goresky and MacPherson original work. We call loose perversity (see [32]) any map $\bar{p}: N \to Z$ such that $\bar{p}(0) = 0$. A filtered simplex as above is said $\bar{p}$-admissible if the inequality $|\sigma|_i \leq \dim \Delta - \bar{p}(i) + i$ is satisfied for all $i \in \{1, \ldots, n\}$ and a chain $c$ is $\bar{p}$-admissible if there exist $\bar{p}$-admissible simplices, $\sigma_j$, so that $c = \sum_j \lambda_j \sigma_j$, $\lambda_j \in \mathbb{Z}$. We denote by $C^\bar{p}_\bullet(X)$ the complex of $\bar{p}$-admissible chains, $c$, whose boundary $\partial c$ is $\bar{p}$-admissible also. There come three results.

- Proposition A.14 The homology theory associated to $C^\bar{p}_\bullet(X)$ owns a Mayer-Vietoris exact sequence, for any loose perversity $\bar{p}$.
- Theorem E For any loose perversity $\bar{p}$, a stratified map (see Definition B.2) $f: X \to Y$, induces a chain map $f_*: C^\bar{p}_\bullet(X) \to C^\bar{p}_\bullet(Y)$.
- Proposition C.4 In the case of a pseudomanifold (see Definition C.2) our complex gives the same intersection homology than the Goresky and MacPherson original one.

The first property is the main result of Appendix A. The second result is a key point in the proof of the topological invariance of the minimal model, stated in Section 10. In the proof of the third statement, we rely on King’s paper where a chain complex, similar to ours, is connected to the Goresky and MacPherson complex.

The description of a simplicial setting for the study of intersection cohomology is done in Sections 1 to 6. We choose the context of face sets, also called simplicial sets without degeneracies, see ([11]). We denote by $\Delta^n_f$ the category whose objects are the decomposed euclidian simplices, $\Delta^0 \ast \cdots \ast \Delta^n$, and maps are joins of face operators, i.e., they are compositions of maps $f = \ast^n_{i=0} \partial_i$, with $\partial_i$ a face operator. The filtered face sets are defined as functors, $K$, from $\Delta^n_f$ to the category of sets. For instance, the filtered simplices of a filtered topological space constitute a filtered face set, see Example 1.5.

Sections 2, 3 and 4 are concerned with various cohomologies defined on a filtered face set $K$. Adapting the topological situation described above, we associate, to $K$ and to a
loose perversity \( \overline{p} \), a cochain complex \( C^\ast_{GM,\overline{p}}(K; R) \), with coefficients in a commutative ring \( R \), corresponding to the original Goresky and MacPherson one. We look also for other cochain complexes built on \( K \) and quasi-isomorphic to the previous one. To proceed with, we introduce a simplicial version of a blow-up, already defined in [2] for singular manifolds and for some particular simplicial complexes in [9]. To any filtered simplex, \( \Delta = \Delta^j_0 \ast \cdots \ast \Delta^j_n \), we associate \( \Delta = c\Delta^j_0 \times \cdots \times c\Delta^j_{n-1} \times \Delta^j_n \), called the blow-up of \( \Delta \). If \( A_{PL}(\Delta^i) \) is the CDGA of rational polynomial forms on \( \Delta^i \), we define \( \tilde{A}_{PL}(\Delta) = A_{PL}(c\Delta^j_0) \otimes \cdots \otimes A_{PL}(c\Delta^j_{n-1}) \otimes A_{PL}(\Delta^j_n) \). We do not want to go into technical points in this introduction but the reader should be aware that this definition makes sense only if \( \Delta^j_n \neq \emptyset \). Thus, the compatibility condition for the definition of the CDGA, \( A_{PL}(K) \), of global forms on \( K \) concerns only the face operators which satisfy this condition. For any global form \( \omega \in A_{PL}(K) \), we define a perverse degree, \( ||\omega|| \), which is a sort of degree along a fiber. For any loose perversity \( q \), a global form \( \omega \in A_{PL}(K) \) is said \( \overline{q} \)-admissible if \( ||\omega||_i \leq \overline{q}(i) \) for all \( i \). The complex \( \tilde{A}_{PL,\overline{q}}(K) \) is generated by the \( \overline{q} \)-admissible forms, \( \omega \), whose boundary \( d\omega \) is \( \overline{q} \)-admissible also. We prove that the integration map induces a quasi-isomorphism,

\[
\int : \tilde{A}_{PL,\overline{q}}(K) \rightarrow C^\ast_{GM,\overline{p}}(K; \mathbb{Q}),
\]

if \( \overline{p} \) and \( \overline{q} \) are perversities verifying \( \overline{q} \geq 0 \) and \( \overline{p}(j) + \overline{q}(j) = j - 2 \).

We may consider again this construction, replacing \( A_{PL}(\Delta^i) \) by the cochain algebra \( C^\ast(\Delta^i; R) \). This gives a cochain complex \( \tilde{C}^\ast(K; R) \), called the Thom-Whitney complex. When \( R \) is a field, it is connected to the Goresky and MacPherson cochain complex by a quasi-isomorphism,

\[
\tilde{C}^\ast_{\overline{q}}(K; R) \simeq C^\ast_{GM,\overline{p}}(K; R),
\]

if \( \overline{p} \) and \( \overline{q} \) are perversities verifying \( \overline{q} \geq 0 \) and \( \overline{p}(j) + \overline{q}(j) = j - 2 \). (This opens a treatment of intersection cohomology with coefficients in \( \mathbb{Z}_2 \) from local cochains.) The existence of the previous quasi-isomorphisms are contained in Theorem A (Section 2) and Theorem B (Section 1).

In Section 3, we study particular cases and examples. For instance, we prove that, in the case \( \overline{p}(i) = \infty \), the intersection cohomology is the cohomology of the regular part, \( K^{(0)} \), which consists of simplices \( \Delta^j_0 \ast \cdots \ast \Delta^j_n \) such that \( \Delta^j_i = \emptyset \) if \( i \leq n - 1 \). Also, the \( \overline{0} \)-intersection cohomology is the cohomology of \( K \), when \( K \) is a normal filtered face set (see Definition 5.2). As in [24], we prove that any filtered face set, \( K \), admits a unique normalization, \( N(K) \), which has the same intersection cohomology than \( K \), see Proposition 5.7. Examples of cone and suspension of a face set are detailed. We emphasize the fact that we can handle intersection cohomologies associated to a filtration, independently of any geometrical dimension. This can be done with the original Goresky and MacPherson perversities if we work with the blow-up of differential forms or cochains but we must consider negative perversities if we use the original Goresky and MacPherson cochains. Example 5.9 is a concrete illustration of this situation.

In Section 6, we supply a cylinder object, define the product of a filtered face set with a face set and a notion of homotopy between maps of filtered face sets.
In Sections 7 to 11, we come back to our rational homotopy project and recall the algebraic category, CDGA\textsubscript{F}, of perverse CDGA’s defined by Hovey in [31], see Definition 7.1 and Definition 7.5. Our previous construction of the blow-up of Sullivan’s forms gives a functor \( A: \Delta[n] \rightarrow \text{Sets} \rightarrow \text{CDGA}_{F} \), \( K \mapsto A(K)_{\bullet} = \tilde{A}_{PL,\bullet}(K) \). We define also Sullivan minimal perverse models and prove they are unique up to isomorphism.

In Section 8, we introduce the notion of balanced perverse CDGA, based on an explicit description of the predecessors of a GM-perversity. The main examples of balanced perverse CDGA are the blow-ups, \( \tilde{F}(K) \), and their cohomology, \( H_{\bullet}(K; F) \), for any filtered face set, \( K \), and universal system of CDGA’s, \( F \). In Section 9, we prove that any balanced, cohomologically connected, perverse CDGA admits a Sullivan minimal perverse model, see Theorem C.

In Section 10, we consider a first series of geometric properties of the minimal model of a filtered face set, \( K \). If \( K \) is connected and normal, we prove that the CDGA of elements of perverse degree 0 in the minimal model of \( A(K) \) is the minimal model of the face set associated to \( K \). We prove also that the two perverse algebras of cohomology, built from the PL-forms on one side and from the Thom-Whitney cochains on the other side, are isomorphic. If \( K \) is the filtered face set associated to a pseudomanifold, \( X \), these algebras are also isomorphic to the algebra of cohomology defined by G. Friedman and J. McClure in [20]. Moreover, if \( X \) is a PL-pseudomanifold whose regular part is connected, we establish the topological invariance of its minimal model, see Theorem D. This uses the existence of an intrinsic filtration of a CS set, due to Sullivan, and detailed in [32], see also [19]. If \( X \) is also normal, the CDGA of elements of perverse degree 0 in the minimal model of \( A(K) \) is the minimal model of \( X \).

Section 11 is devoted to a notion of perverse formality and to the study of some concrete examples. A connected filtered face set, \( K \), is said intersection-formal if there is an isomorphism between the minimal models of \( A(K)_{\bullet} \) and \( H_{\bullet}(A(K)) \). The analogue of triple Massey products is also introduced and the relationship with intersection-formality is detailed. We continue with a series of examples. The cone on a PL-pseudomanifold furnishes an example of a formal PL-pseudomanifold which is not intersection-formal. We present also (non cofibrant) models of PL-pseudomanifolds with isolated singularities: Thom spaces of vector bundles on a formal space, projective cones of a smooth projective variety, the Calabi Yau quintic, and more generally, any nodal hypersurface in \( \mathbb{C}P(4) \), are intersection-formal.

Some of our results are established in the general case of loose perversities but the construction of the minimal models relies strongly on the structure of the lattice of Goresky and MacPherson perversities. This was first observed by Stienne ([47]) who determined the properties of the predecessors of a GM-perversity and a part of the construction of our perverse minimal model is inspired from his work.

In all this text, for a graded perverse object, \( a \), we denote by \( |a| \) its degree and by \( ||a|| \) its perverse degree. A chain map is called a quasi-isomorphism (or a weak equivalence) if it induces an isomorphism in homology.

We would like to thank Greg Friedman for reading the first version and making valuable suggestions which have contributed to improve the writing. We thank also the referee for many helpful comments.
Part 1. Simplicial setting

1. Filtered face sets

Singular homology and cohomology of a topological space, $X$, can be obtained simplicially, thanks to the simplicial set of continuous simplices. In the case of a filtered space, there is a notion of filtered simplices (Definition A.3) whose set, denoted by $\text{ISing}_F^*(X)$, generates a chain complex (Definition A.10) giving the intersection homology of a pseudomanifold (Proposition C.4). In this section, we abstract the properties of $\text{ISing}_F^*(X)$ in the concept of filtered face set, giving a simplicial setting to intersection homology and cohomology. We present also the simplicial notions of link and expanded link which are crucial in the proofs of Sections 2, 3 and 4.

Let us recall the basics on face sets, introduced by Rourke and Sanderson ([41]). For any integer $n$, we set $[n] = \{0, 1, \ldots, n\}$. Denote by $\Delta$ the category whose morphisms are the injective order-preserving maps $f: [n] \to [m]$. A morphism of $\Delta$ is a composition
of face operators \( \delta_i : [n] \to [n+1] \) defined by
\[
\delta_i(j) = \begin{cases} 
  j & \text{if } j < i, \\
  j+1 & \text{if } j \geq i,
\end{cases}
\]
which satisfy \( \delta_j \delta_i = \delta_i \delta_{j-1} \), if \( i < j \).

**Definition 1.1.** A face set is a contravariant functor, \( K \), from \( \Delta \) to the category of sets, \([n] \to K_n\). An element \( \sigma \in K_n \) is called a simplex of dimension \( n \), that we denote \( |\sigma| = n \). The \( n \)-skeleton of \( K \) is \( K^{(n)} = \bigcup_{p \leq n} K_p \). The images by \( K \) of face operators are \( \partial_i : K_{n+1} \to K_n \) such that \( \partial_i \partial_j = \partial_j \partial_{i-1} \), if \( i < j \).

If \( K \) and \( K' \) are face sets, a face map, \( f : K \to K' \), is a natural transformation between the two functors \( K \) and \( K' \). We denote by \( \Delta - \text{Sets} \) the category of face maps between face sets.

We emphasize that morphisms between face sets are called face maps and that we keep the expression “face operators” for the maps \( \partial_i \) or \( \delta_i \). For basic properties of face sets and their relation with the more classical simplicial setting, we refer to the foundational paper [41], observing that the homotopy theory of face sets is the same as the homotopy theory of simplicial sets.

**Example 1.2.** The face set \( \Delta^n \) is defined as follows: \( \Delta^n_p \) is the set of injective order-preserving maps from \([p]\) to \([n]\). It has only one simplex of dimension \( n \), denoted by \([\Delta^n]\). If \( K \) is a face set, there is a bijection between the set of elements \( \sigma \in K_n \) and the set of face maps, \( \sigma : \Delta^n \to K \), defined by \( \sigma(f) = K(f)(\sigma) \). In the sequel, we do not make any distinction between \( \sigma : \Delta^n \to K \) and \( \sigma \in K_n \).

Recall from [41], that the realization of a face set, \( K \), is the CW-complex, \( \|K\| \), defined by
\[
\|K\| = \bigcup_{n=0}^\infty (K_n \times \Delta^n)/(\partial_i x, t) \sim (x, \delta_i t),
\]
where
- \( \Delta^n \) is the standard \( n \)-simplex of \( \mathbb{R}^{n+1} \), whose vertices \( v_0, \ldots, v_n \) verify \( v_i = (t_0, \ldots, t_n) \), \( t_j = 0 \) if \( i \neq j \) and \( t_i = 1 \),
- the map \( \delta_i : \Delta^{n-1} \to \Delta^n \) is the linear application defined by \( \delta_i(v_j) = v_{k(j)} \).

For instance, the realization of the face set \( \Delta \) described in Example 1.2 is the standard simplex \( \Delta^n \) and, in the sequel, we identify \([n] \), \( \Delta^n \) and \( \Delta^n \). The CW-complex \( \|K\| \) has one \( n \)-cell for each \( \sigma \in K_n \), of characteristic map \( \tilde{\sigma} : \Delta^n \to \|K\| \) defined by \( \tilde{\sigma}(t_0, \ldots, t_n) = [\sigma, (t_0, \ldots, t_n)] \), where \([\cdot]\) denotes the equivalence classes in the realization \( \|K\| \).

We adapt now these objects to the context of intersection theory. First, we fix an integer \( n \) which corresponds to the formal dimension of filtered spaces; that means, we fix the number of elements in the filtration to \( n \), see Remark 1.7 for the topological meaning of this process.

Let \( \Delta^n_{\bar{j}} \) be the category whose
- objects are the join \( \Delta = \Delta^{j_0} \star \Delta^{j_1} \star \cdots \star \Delta^{j_n} \), where \( \Delta^{j_i} \) is the simplex of dimension \( j_i \), possibly empty, with the conventions \( \Delta^{-1} = \emptyset \) and \( \emptyset \star X = X \).
• maps are the $\sigma: \Delta = \Delta^j_0 \ast \Delta^j_1 \ast \cdots \ast \Delta^j_n \to \Delta' = \Delta^{k_0} \ast \Delta^{k_1} \ast \cdots \ast \Delta^{k_n}$, of the shape $\sigma = \ast_{i=0}^n \sigma_i$, with $\sigma_i: \Delta^j_i \to \Delta^{k_i}$ an injective order-preserving map for each $i$ or the map $\emptyset \to \Delta^{k_i}$.

For all $0 \leq i \leq n$, any face operator, $\delta_i: \Delta^j_i \to \Delta^j_{i+1} \in \Delta$, gives rise to a face operator in $\Delta^j_n$, obtained from the join with the identity map, and still denoted $\delta_i: \Delta^{j_0} \ast \cdots \ast \Delta^{j_i} \ast \cdots \ast \Delta^{j_n} \to \Delta^{j_0} \ast \cdots \ast \Delta^{j_i+1} \ast \cdots \ast \Delta^{j_n}$.

The category $\Delta^j_0^\ast$ is the full subcategory of $\Delta^j_n$ whose objects are the $\Delta^{j_0} \ast \cdots \ast \Delta^{j_n}$ with $\Delta^{j_n} \neq \emptyset$, i.e., $j_n \geq 0$.

**Definition 1.3.** A filtered face set is a contravariant functor, $K$, from the category $\Delta^j_0$, to the category of sets, i.e., $(\delta_j, \cdots, j_n) \mapsto K_{j_0, \cdots, j_n}$. Any face operator, $\delta_i: \Delta^{j_i} \to \Delta^{j_{i+1}}$, induces a face operator, defined by $\delta_i: K_{j_0, \cdots, j_{i+1}, \cdots, j_n} \to K_{j_0, \cdots, j_{i+1}}$, $\delta_i(\sigma) = \sigma \circ \delta_i$.

The restriction of a filtered face set, $K$, to $\Delta^j_0^\ast$ is denoted $K_\ast$. Morphisms between filtered face sets are natural transformations; we call them filtered face maps.

**Face sets** are filtered face sets whose simplices, $\sigma: \Delta^{j_0} \ast \cdots \ast \Delta^{j_n} \to K$, are such that $j_i = -1$ for all $i < n$.

**Definition 1.4.** The perverse degree of a simplex, $\sigma: \Delta^{j_0} \ast \cdots \ast \Delta^{j_n} \to K$, of a filtered face set, $K$, is the $(n + 1)$-uple,

$$\|\sigma\| = (\|\sigma\|_0, \cdots, \|\sigma\|_n),$$

where $\|\sigma\|_\ell = \dim(\Delta^{j_0} \ast \cdots \ast \Delta^{j_{n-\ell}})$ if $\Delta^{j_0} \ast \cdots \ast \Delta^{j_{n-\ell}} \neq \emptyset$ and $\|\sigma\|_\ell = -\infty$ otherwise.

Observe that any filtered face map preserves the perverse degree. The next example is the main motivation for the introduction of filtered face sets.

**Example 1.5.** In Appendix A for any filtered space, $X_0 \subset X_1 \subset \cdots \subset X_n = X$, we define a filtered singular simplex as a continuous map, $\sigma: \Delta^m \to X$, such that each $\sigma^{-1}X_i$ is a face of $\Delta^m$ or is the empty set. Such a map induces a decomposition $\Delta^m = \Delta^{j_0} \ast \cdots \ast \Delta^{j_n} \to X$, with $\sigma^{-1}X_i = \Delta^{j_0} \ast \cdots \ast \Delta^{j_i}$. (Example A.6 provides an illustration of this situation.) All together, these filtered singular simplices define a filtered face set, ISing$^\sigma(X)$, by

$$\text{ISing}^\sigma(X)_{j_0, \cdots, j_n} = \{ \sigma: \Delta^{j_0} \ast \cdots \ast \Delta^{j_n} \to X, \ \sigma \ \text{continuous and filtered} \}.$$ 

Moreover, any stratified preserving stratified map, $X \to Y$, induces a filtered face map, ISing$^\sigma(X) \to \text{ISing}^\sigma(Y)$, see Theorem E and Corollary B.9.

**Example 1.6.** The second barycentric decomposition of a triangulated pseudomanifold gives rise also to a filtered face set, see [9, Proposition 1.1.4].

**Remark 1.7.** Imposing a common formal dimension, $n$, to filtered face sets, is not a restriction for the study of intersection homology. Observe, from Definition A.7, that the admissibility of a simplex of a filtered space, $X$, depends only on the codimension in $X$. Therefore, as noticed by Friedman in [16], one can add a fixed integer to each index of the filtration of a space without modifying its intersection homology. A practical consequence is that we can always suppose that the spaces we are considering have the same formal dimension without losing generality.
The restriction to the subcategory \( \Delta_j^{[n]} \) is due to the fact that we do not consider simplices entirely included in the singular part, see Remark 1.6. One may observe also (see Definition 2.13) that the blow-ups are not defined if \( \Delta^j_n = \emptyset \).

In [48] Theorem 7.1, D. Sullivan gives a short proof of de Rham’s theorem for PL-forms, based on the fact that the CDGA of forms on the relative skeleton \((X^{(p)}, X^{(p-1)})\) “breaks into a product over \(p\) cells”. For the filtered face sets, we need the analogue of the previous relative skeleton, satisfying a breaking decomposition, as PL-forms do in the case of a simplicial set. Proposition 1.19 reaches this objective and we introduce first the necessary tools, as the filtered notions of skeleta and links. They coincide with usual notions when \( n = 0 \). At the end of this section, we detail Example 1.20 which illustrates these definitions and may help to their understanding.

For face sets, a skeleton is characterized by an integer, the dimension of simplices. Here, for the filtered face sets, we need two integers. One is still the dimension of \( K \), filtered face set, ISing \((\sigma)\) that the blow-ups are not defined if \( \Delta^j_n = \emptyset \).

For face sets, a skeleton is characterized by an integer, the dimension of simplices. For face sets, a skeleton is characterized by an integer, the dimension of simplices.
We set $\xi \triangleleft \sigma$ if $\xi$ is a face (of any codimension) of a simplex $\sigma$.

**Definition 1.11.** Let $K$ be a filtered face set and $\tau \in \mathcal{J}(K,[r],k)$. 

(a) The star of $\tau$ in $K$ is the filtered face subset, $K(\tau)$, of $K$ defined by

$$K(\tau) = \{ \xi \in K \mid \exists \sigma \in K \text{ with } R_1(\sigma) = \tau \text{ and } \xi \triangleleft \sigma \}.$$ 

(b) The link of $\tau$ in $K$ is the filtered face subset, $\mathcal{L}(K,\tau)$, of $K$ defined by

$$\mathcal{L}(K,\tau) = \{ \xi \in K \mid \exists \sigma \in K \text{ with } R_1(\sigma) = \tau \text{ and } \xi \triangleleft R_2(\sigma) \}.$$ 

The filtered face set $K(\tau)$ is never empty and the link $\mathcal{L}(K,\tau)$ can be empty. We observe also that

$$K^{[r],k} = K^{[r],k-1} \cup_{\tau \in \mathcal{J}(K,[r],k)} K(\tau).$$

Let $\tau \in \mathcal{J}(K,[r],k)$, $r \geq 0$ and $k \geq 1$. As $K(\tau)$ is a filtered face set, the filtered face set $K(\tau)(\partial_i \tau)$ is already defined for any face operator $\partial_i : \Delta^k \to \Delta^{k-1}$, i.e.,

$$K(\tau)(\partial_i \tau) = \{ \xi \in K(\tau) \mid \exists \sigma \in K(\tau) \text{ with } R_1(\sigma) = \partial_i \tau \text{ and } \xi \triangleleft \sigma \}.$$ 

The fact that $\xi \in K(\tau)$ can be detailed as: $\xi \in K$ with the existence of $\gamma \in K$ such that $R_1(\gamma) = \tau$ and $\xi \triangleleft \gamma$. If, moreover, $\xi \in K(\tau)(\partial_i \tau)$, there exists also $\sigma \in K(\tau)$ with $R_1(\sigma) = \partial_i \tau$ and $\xi \triangleleft \sigma$, from which we deduce $\xi \triangleleft \partial_i \gamma$. Thus, we can write,

$$K(\tau)(\partial_i \tau) = \{ \xi \in K \mid \exists \gamma \in K \text{ with } R_1(\gamma) = \tau \text{ and } \xi \triangleleft \partial_i \gamma \},$$ 

since $R_1(\gamma) = \tau$ implies $R_1(\partial_i \gamma) = \partial_i \tau$.

**Definition 1.12.** Let $K$ be a filtered face set, $r \geq 0$, $k \geq 1$ and $\tau \in \mathcal{J}(K,[r],k)$. We define a filtered face set $K(\tau,\partial \tau)$ by

$$K(\tau,\partial \tau) = \bigcup_{i=0}^k K(\tau)(\partial_i \tau).$$ 

Observe that $K(\tau) = K^{[r],k}$ and $K(\tau,\partial \tau) = K(\tau,\partial \tau)^{[r],k-1}$. The next result is the determination of the intersection of the two filtered face sets in the right-hand side of equation (1).

**Proposition 1.13.** If $K$ is a filtered face set, $r \geq 1$, $k \geq 0$ and $\tau \in \mathcal{J}(K,[r],k)$, we have

$$K^{[r],k-1} \cap K(\tau) = \begin{cases} K(\tau,\partial \tau), & \text{if } k \geq 1, \\ \mathcal{L}(K,\tau), & \text{if } k = 0. \end{cases}$$

In the sequel, we abuse the notation by using $\mathcal{L}(K,\tau) = K(\tau,\partial \tau)$ when $k = |r| = 0$.

**Proof.** Suppose first $k \geq 1$. An element $\xi \in K^{[r],k-1} \cap K(\tau)$ is a face of a $\sigma \in K(\tau)$ with $R_1(\sigma) = \tau$ and we have two possibilities:

- $v(\xi) = r$ and $j_n-r \leq k-1$. In this case, $R_1(\xi)$ is a proper face of $\tau$. Let $\tau' = \partial_i \tau$ such that $R_1(\xi) \subset \partial_i \tau$ and set $\sigma' = \partial_i \sigma$. Then we have $R_1(\sigma') = \tau'$, $\sigma' \in K(\tau)$ and $\xi$ is a face of $\sigma'$. This implies $\xi \in K(\tau,\partial \tau)$.

- or $v(\xi) \leq r-1$. In this case, $\xi$ is a face of $R_2(\sigma)$. Set $\tau' = \partial_i \tau$ and $\sigma' = \partial_i \sigma$ for a face operator $\partial_i$. Then, we have $\sigma' \in K(\tau)$, $R_1(\sigma') = \tau'$ and $R_2(\sigma)$ is a face of $\sigma'$. Thus $\xi$ is also a face of $\sigma'$ and we conclude $\xi \in K(\tau,\partial \tau)$.

The reverse inclusion comes directly from $K(\tau,\partial \tau) = K(\tau,\partial \tau)^{[r],k-1}$. The case $k = 0$ is proved in a similar way, by using $\mathcal{L}(K,\tau) = \mathcal{L}(K,\tau)^{[r-1]}$. \qed
This result can also be stated as follows.

**Corollary 1.14.** Let \( K \) be a filtered face set, \( r \geq 1, k \geq 0 \). Then we have a push-out,

\[
\begin{array}{c}
\bigsqcup_{\tau \in \delta(K^{[r]}, k)} K(\tau, \partial \tau) \\
\downarrow \\
\bigsqcup_{\tau \in \delta(K^{[r]}, k)} K(\tau, \partial \tau) \\
\downarrow \\
K^{[r], k-1} \rightarrow K^{[r], k}.
\end{array}
\]

The expanded link, defined below, allows “the break” of \((K^{[r], k}, K^{[r], k-1})\) in Proposition 1.19.

**Definition 1.15.** Let \( K \) be a filtered face set, \( r \geq 1, k \geq 0 \) and \( \tau \in \delta(K, [r], k) \). The expanded link of \( \tau \) in \( K \) is the filtered face set \( L^{\text{exp}}(K, \tau) \) defined by

\[ L^{\text{exp}}(K, \tau) = \{(R_2 \sigma, \sigma) \mid \sigma \in K, R_1(\sigma) = \tau \text{ and } \sigma \neq \tau\}, \]

with face operators,

\[ \partial_i(R_2(\sigma), \sigma) = (\partial_i R_2(\sigma), \partial_i+1 + k \sigma) = (R_2(\partial_i+1 + k \sigma), \partial_i+1 + k \sigma), \]

for all \( i \in \{0, \ldots, \dim(R_2(\sigma), \sigma) = \dim R_2(\sigma)\} \), and perverse degrees,

\[ ||(R_2(\sigma), \sigma)||_\ell = ||R_2(\sigma)||_\ell, \]

for all \( \ell \in \{0, \ldots, n\} \).

First, we note that the previous definition of \( \partial_i \) implies that the expanded link is stable with face operators and thus is a well defined filtered face set. By definition, a simplex \((\alpha, \sigma) \in L^{\text{exp}}(K, \tau)\) if, and only if, \( R_1(\sigma) = \tau \) and \( R_2(\sigma) = \alpha \). The difference between the link and the expanded link lies in the fact that two distinct simplices, \( \sigma \) and \( \sigma' \), of \( K(\tau) \), verifying \( R_2(\sigma) = R_2(\sigma') \), give the same element in \( L(K, \tau) \) and two distinct elements in \( L^{\text{exp}}(K, \tau) \). In the expanded link, we keep track of \( \sigma \) and \( \sigma' \) as parameters. Also, the condition \( \sigma \neq \tau \) means \( R_2(\sigma) \neq \emptyset \), for any \((\alpha, \sigma) \in L^{\text{exp}}(K, \tau)\). Example 1.20 illustrates the difference between the two links. Observe also that, by definition, we have

\[ L^{\text{exp}}(K, \tau) = L^{\text{exp}}(K, \tau)^{[r-1]} . \]

**Definition 1.16.** A morphism of pairs of filtered face sets, \( f: (K_1, L_1) \rightarrow (K_2, L_2) \), is a filtered face map, \( f: K_1 \rightarrow K_2 \), such that \( f(L_1) \subseteq L_2 \). A relative isomorphism, \( f: (K_1, L_1) \rightarrow (K_2, L_2) \), is a morphism of pairs of filtered face sets, such that the restriction of \( f \) to the complementary subsets is a bijection respecting the perverse degree, \( f: K_1 \setminus L_1 \cong K_2 \setminus L_2 \).

**Example 1.17.** From Proposition 1.13, we deduce that the map

\[
(l_{\tau \in \delta([K, [r], k])}) K(\tau, \partial \tau) \rightarrow (K^{[r], k}, K^{[r], k-1})
\]

is a relative isomorphism.

**Example 1.18.** Let \( K \) be a filtered face set of depth \( v(K) \leq r - 1 \) with \( r \in \{1, \ldots, n\} \). Let \( k \in \mathbb{N} \). We denote by \( \Delta^k * K \) the filtered face set, of depth \( r \), generated by the simplices,

\[ \Delta^k * \Delta^j_{n-r-1} * \cdots * \Delta^j_n \xrightarrow{id_{\Delta^k}} \Delta^k * K, \]
with \( \sigma \in K \). Set \( j_{n-r} = k \). The simplices of this filtered face set \( \Delta^k \ast K \) are of three types: \( \partial_0 \Delta^k \), \( (\partial_0 \Delta^k) \ast \sigma \) with \( I \varsubsetneq \{0, \ldots, k\} \), and those of \( K \). The perverse degree of the simplices of \( K \) is keeping unchanged. For the other ones, we set:

\[
\| \partial_1 \Delta^k \|_{r'} = \begin{cases} 
\pm \infty & \text{if } r < r', \\
-k - |I| & \text{if } r \ge r', 
\end{cases}
\]

\[
\| \partial_1 \Delta^k \ast \sigma \|_{r'} = \begin{cases} 
\pm \infty & \text{if } r < r', \\
k - |I| & \text{if } r' = r, \\
k - |I| + \| \sigma \|_{r'} + 1 & \text{if } r > r' \text{ and } \| \sigma \|_{r'} \neq -\infty.
\end{cases}
\]

By construction, we have \( f(\partial_1 \Delta^k) = \partial_1 \tau_f \), \( f(\alpha, \sigma) = \alpha \) and \( f(\partial_1 \Delta^k \ast (\alpha, \sigma)) = \partial_1 \sigma \).

Proposition 1.19. Let \( K \) be a filtered face set, \( r \ge 1 \), \( k \ge 0 \) and \( \tau \in \mathcal{J}(K, [r], k) \). There exists a relative isomorphism,

\[
f : (\Delta^k \ast \mathcal{L}^{\exp}(K, \tau), \partial \Delta^k \ast \mathcal{L}^{\exp}(K, \tau)) \to (K(\tau), K(\tau, \partial \tau)),
\]

preserving the perverse degree.

Proof. Let \((\alpha, \sigma) \in \mathcal{L}^{\exp}(K, \tau)\) and \( I \varsubsetneq \{0, \ldots, k\} \). The application \( f \) is defined by

\[
f(\partial_1 \Delta^k) = \partial_1 \tau_f, \quad f(\alpha, \sigma) = \alpha \quad \text{and} \quad f(\partial_1 \Delta^k \ast (\alpha, \sigma)) = \partial_1 \sigma.
\]

By construction, we have \( f(\partial \Delta^k \ast \mathcal{L}^{\exp}(K, \tau)) = K(\tau, \partial \tau) \). Decompose \( \partial \Delta^k \ast (\alpha, \sigma) \) in \( \Delta^j_0 \ast \cdots \ast \Delta^j_{n-r'} \). If \( r < r' \), then the product \( \Delta^j_0 \ast \cdots \ast \Delta^j_{n-r'} \) is empty and we have \( \| \partial_1 \Delta^k \ast (\alpha, \sigma) \|_{r'} = -\infty \). As \( \tau \in \mathcal{J}(K, [r], k) \) and \( R_1(\sigma) = \tau \), we know that \( \| \partial_1 \Delta^k \ast (\alpha, \sigma) \|_{r'} = k - |I| \), if \( r' = r \) or \((r > r' \text{ and } \| \alpha \|_{r'} = -\infty) \). Finally, in perverse degree \( r' \), \( r' < r \) and \( \| \alpha \|_{r'} \neq -\infty \), we have to take in account \( \alpha = R_2(\sigma) \), and we get \( \| \partial_1 \Delta^k \ast (\alpha, \sigma) \|_{r'} = k - |I| + \| \alpha \|_{r'} + 1 \), the element +1 coming from the fact that a join \( \beta_1 \ast \beta_2 \) has for dimension \( \dim \beta_1 + \dim \beta_2 + 1 \). From these observations, the definition of \( f \) and Example 1.18, we may check first that \( f \) keeps the perverse degree.

\[
\| f(\partial_1 \Delta^k \ast (\alpha, \sigma)) \|_{r'} = \| \partial_1 \sigma \|_{r'} = -\infty = \| \partial_1 \Delta^k \ast (\alpha, \sigma) \|_{r'}, \quad \text{if } r < r',
\]

\[
\| f(\partial_1 \Delta^k \ast (\alpha, \sigma)) \|_{r'} = \| \partial_1 \sigma \|_{r'} = k - |I| = \| \partial_1 \Delta^k \ast (\alpha, \sigma) \|_{r'}, \quad \text{if } r' = r \text{ or } (r > r' \text{ and } \| \alpha \|_{r'} = -\infty),
\]

\[
\| f(\partial_1 \Delta^k \ast (\alpha, \sigma)) \|_{r'} = \| \partial_1 \sigma \|_{r'} = k - |I| + \| \alpha \|_{r'} + 1 = \| \partial_1 \Delta^k \ast (\alpha, \sigma) \|_{r'}, \quad \text{if } r > r' \text{ and } \| \alpha \|_{r'} \neq -\infty.
\]

A similar computation is done for the other cases. We leave also to the reader the verifications of the compatibility with face operators.

We study now the restriction to the complementary subsets. Let \( \gamma \in K(\tau) \setminus K(\tau, \partial \tau) \). By definition, we have \( R_1(\gamma) = \tau \). If \( R_2(\gamma) = 0 \), then \( \gamma = \tau = f(\Delta^k) \). If \( R_2(\gamma) \neq 0 \), then \( \gamma = f(\Delta^k \ast (R_2(\gamma), \tau)) \). This gives the surjectivity and the injectivity is obvious from inspection of the definition of \( f \). \( \square \)
Example 1.20. We determine $K_{[r,k]}$ and the links for the face set, $K_i$, defined by the following picture

\[ \begin{array}{c}
\begin{array}{c}
\alpha \\
\downarrow \\
\tau \\
\downarrow \\
b \\
\end{array}
\quad
\begin{array}{c}
\beta' \\
\downarrow \\
\tau' \\
\downarrow \\
b' \\
\end{array}
\quad
\begin{array}{c}
\alpha'' \\
\downarrow \\
\tau'' \\
\downarrow \\
a \\
\end{array}
\end{array} \]

and $K_{(-1,0)} = \{a, b''\}$, $K_{(1,1)} = \{\beta'\}$, $K_{(0,-1)} = \{b, b'\}$, $K_{(0,0)} = \{\alpha, \alpha', \alpha'', \beta\}$, $K_{(0,1)} = \{F''\}$, $K_{(1,-1)} = \{\tau\}$, $K_{(1,0)} = \{F, F'\}$.

Denoting by $(-)$ the face set generated by $-$, we have:

\[
\begin{align*}
K_{[r,0]} &= K_{(-1,0)} = \langle a, b'' \rangle, \\
K_{[r,1]} &= K_{[r,0]} \cup K_{(-1,1)} = \langle \beta' \rangle, \\
K_{[1,0]} &= K_{[0,1]} \cup K_{(0,0)} \cup K_{(0,1)} = \langle \alpha, \alpha'', F'' \rangle, \\
K_{[1,1]} &= K_{[1,0]} \cup K_{(1,1)} \cup K_{(1,0)} = K, \\
\overline{K}(\tau) &= \langle F, F' \rangle, \\
\overline{K}(\tau, \partial \tau) &= \langle \alpha, \alpha', \alpha'', \beta' \rangle, \\
\overline{\beta}(K, [0], 0) &= \langle a, b'' \rangle, \\
\overline{\beta}(K, [0], 1) &= \langle \beta' \rangle, \\
\overline{\beta}(K, [1], 0) &= \langle b, b' \rangle, \\
\overline{\beta}(K, [1], 1) &= \{\tau\}, \\
\overline{L}(K, \tau) &= \langle \alpha \rangle, \\
\overline{L}^{\exp}(K, \tau) &= \langle \langle a, F \rangle, \langle a, F' \rangle \rangle.
\end{align*}
\]

2. Perverse local systems on filtered face sets

We define perverse local systems of coefficients, $\mathcal{M}$, over a filtered face set $K$ and the intersection cohomology of $K$ with coefficients in $M$, for a loose perversity $\overline{q}$. The blow-up $\tilde{F}$ of certain local systems over face sets, $F$, are examples of this situation. We detail the particular case of universal systems, as cochains and Sullivan forms (over $\mathbb{Q}$). The main result (Theorem A) gives sufficient conditions on the local systems, $F$ and $G$, for having an isomorphism between the intersection cohomologies of a filtered face set, $K$, with coefficients in the blow-ups $\tilde{F}$ and $\tilde{G}$.

In this section, all cochain complexes are over a commutative ring $R$.

The particular case $R = \mathbb{Q}$ is explicitly quoted if necessary.

We introduce the notion of perversity which is the fundamental tool of intersection theory. We follow the convention of [32].

Definition 2.1. A loose perversity is a map $\overline{q} : \mathbb{N} \to \mathbb{Z}$, $i \mapsto \overline{q}(i)$, such that $\overline{q}(0) = 0$. A loose perversity is positive if $\overline{q}(i) \geq 0$, for any $i \in \mathbb{N}$.

A perversity is a loose perversity such that $\overline{q}(i) \leq \overline{q}(i + 1) \leq \overline{q}(i) + 1$, for all $i \geq 1$.

A Goresky-MacPherson perversity (or GM-perversity) is a perversity such that $\overline{q}(1) = \overline{q}(2) = 0$.

If $\overline{q}_1$ and $\overline{q}_2$ are two loose perversities, we set $\overline{q}_1 \leq \overline{q}_2$ if we have $\overline{q}_1(i) \leq \overline{q}_2(i)$, for all $i \in \mathbb{N}$. The lattice of GM-perversities, denoted $\mathfrak{P}^n$, admits a maximal element, $\overline{t}$, called the top perversity and defined by $\overline{t}(i) = i - 2$, if $i \geq 2$. 

To these perversities, we add an element, $\infty$, which is the constant map on $\infty$. We call it the infinite perversity despite the fact that it is not a perversity in the sense of the previous definition.

Homology and cohomology theories on a face set, $K$, can be expressed with local systems, see [29] for instance in the simplicial case. We adapt these local systems to the filtered situation as follows.

**Definition 2.2.** A strict perverse cochain complex is a cochain complex, $(C, d)$, in which each element, $\omega$, has a perverse degree,

$$\|\omega\| = (\|\omega\|_1, \ldots, \|\omega\|_n),$$

with $\|\omega\|_i \in \mathbb{N} \cup \{\infty, \infty\}$, such that $\|\omega + \omega'\|_i \leq \max(\|\omega\|_i, \|\omega'\|_i)$, for any $i \in \{1, \ldots, n\}$.

If $\overline{q}$ is a loose perversity, a cochain $\omega \in C$ is $\overline{q}$-admissible if $\|\omega\|_i \leq \overline{q}(i)$, for any $i \in \{1, \ldots, n\}$. A cochain $\omega \in C$ is of intersection for $\overline{q}$ (or of $\overline{q}$-intersection) if $\omega$ and $d\omega$ are $\overline{q}$-admissible. We denote by

$$C_{\overline{q}} = \{ \omega \in C \mid \|\omega\|_i \leq \overline{q}(i) \text{ and } \|d\omega\|_i \leq \overline{q}(i) \text{ for any } i \in \{1, \ldots, n\} \},$$

the complex of $\overline{q}$-intersection cochains and by $H_{\overline{q}}(C)$ its homology.

A morphism of strict perverse cochain complexes, $f : (C, d) \to (C', d')$, is a morphism of cochain complexes which decreases the perverse degree, i.e., $\|f(\omega)\| \leq \|\omega\|$, for any $\omega$.

A strict perverse differential graded algebra (henceforth strict perverse DGA) is a strict perverse cochain complex and a DGA such that $\|\omega \cdot \omega'\|_i \leq \|\omega\|_i + \|\omega'\|_i$, for any $i \in \{1, \ldots, n\}$. A morphism of strict perverse cochain complexes, compatible with the products, is called a morphism of strict perverse DGA’s.

**Definition 2.3.** A strict perverse local system of cochains over a filtered face set, $K$, is a family of strict perverse cochain complexes, $M_\sigma$, indexed by the simplices $\sigma$ of $K^+$, and a family of cochain maps, $\partial_i : M_\sigma \to M_{\partial_i \sigma}$, decreasing the perverse degree (i.e., $\|\partial_i \omega\|_j \leq \|\omega\|_j$, for any $j \in \{1, \ldots, n\}$) and such that $\partial_i \partial_j = \partial_j \partial_{i-1} \partial_i$, if $i < j$ and $\partial_i \partial_j$ corresponds to $\delta_j \delta_i \in \Delta[n]_\sigma^+$. The space of global sections of $M$ over $K$ is the strict perverse cochain complex $M(K)$ defined as follows: an element $\omega \in M^j(K)$ is a function which assigns to each simplex $\sigma \in K^+$ an element $\omega_\sigma \in M_\sigma^j$ such that $\omega_{\partial_i \sigma} = \partial_i(\omega_\sigma)$ for all $\sigma$ and all face operators $\delta_i \in \Delta[n]_\sigma^+$. The perverse degree of $\omega$ in $M(K)$ is the supremum (possibly infinite) of the perverse degrees of the $\omega_\sigma$, for all $\sigma \in K^+$, i.e., $\|\omega\| = (\sup_\sigma \|\omega_\sigma\|_1, \ldots, \sup_\sigma \|\omega_\sigma\|_n)$. The laws and differential on $M(K)$ are defined by $((\lambda \omega + \mu \omega')_\sigma = \lambda \omega_\sigma + \mu \omega'_{\sigma}, (d\omega)_\sigma = d\omega_\sigma$.

If $M$ is a local system of strict perverse DGA’s, the space of global sections is a strict perverse DGA, with the law $(\omega \cdot \omega')_\sigma = \omega_\sigma \cdot \omega'_{\sigma}$.

**Definition 2.4.** Let $\overline{q}$ be a loose perversity and $M$ be a strict perverse local system of cochains over a filtered face set $K$. We denote by $M_{\overline{q}}(K)$ the complex of global sections which are of intersection for $\overline{q}$ and by $H^i_{\overline{q}}(K, M)$ its homology, called the intersection cohomology of $K$ with coefficients in $M$, for the loose perversity $\overline{q}$.

In the case of a filtered face set of formal dimension 0 (i.e., $n = 0$) the strict perverse local systems are the usual local systems.
Let $M$ be a strict perverse local system on a filtered face set $K$. Observe that any filtered face map, $f: L \rightarrow K$, induces a strict perverse local system, $f^*M$, on $L$, defined by

$$(f^*M)_\sigma = M_{f_0\sigma}.$$ 

Therefore, a strict perverse local system, $M$, on $K$ induces a strict perverse local system on any filtered face subset of $K$; we still denote by $M$ these induced systems.

**Definition 2.5.** A universal system (of cochains) is a contravariant functor $F$ from the category $\Delta$ to the category of cochain complexes, free as $R$-modules. If $F$ takes its values in the category of DGA’s, free as $R$-modules, we say that $F$ is a universal system of DGA’s.

A universal system of cochains defines a local system of cochains on any face set, $K$, by setting $F_\sigma = F(\Delta^{[\sigma]})$ and $\partial_\sigma = F(\delta_\sigma): F_\sigma \rightarrow F_{\partial_\sigma}$. The cochain complex of global sections is denoted $F(K)$. If $f: L \rightarrow K$ is a filtered face map, then $f^*F = F$ and we obtain a morphism of cochain complexes, $F(f): F(K) \rightarrow F(L)$.

The PL-forms of Sullivan and the cochains are examples of universal systems of DGA’s. We recall their construction.

**Example 2.6** (Cochain algebra). Let $R$ be a commutative ring. We define a universal system of DGA’s from the simplicial cochain complex, by setting $C^*(\{n\}) = C^*(\Delta^n; R)$. If $K$ is a face set, the associated CDGA of global sections, $C^*(K; R)$, is defined by:

- $C^j(K; R)$, is the free $R$-module of all set maps $f: K_j \rightarrow R$,
- the differential $\delta$ is given by

$$\delta f(\sigma) = \sum_{i=0}^{j+1} (-1)^i f(\partial_i \sigma), \quad f \in C^j(K; R), \quad \sigma \in K_{j+1},$$

- the product of $f \in C^j(K; R)$ and $g \in C^k(K; R)$ is defined by

$$(f \cdot g)(\sigma) = f(\partial^F_j \sigma) g(\partial^B_k \sigma), \quad \sigma \in K_{j+k},$$

where $\partial^F_j: K_{j+k} \rightarrow K_j$ and $\partial^B_k: K_{j+k} \rightarrow K_k$ are given by $\partial^F_j \sigma = (\partial_{j+1} \circ \cdots \circ \partial_k) \sigma$ and $\partial^B_k \sigma = (\partial_0 \circ \cdots \circ \partial_0) \sigma$.

**Example 2.7** (Sullivan’s polynomial forms). Let $R = \mathbb{Q}$. We define a universal system of CDGA’s by setting $A_{PL}(\{n\}) = \wedge(t_0, \ldots, t_n, dt_0, \ldots, dt_n)/(t_0 + \cdots + t_n = 1, dt_0 + \cdots + dt_n = 0)$, with $|t_i| = 0$ and $|dt_i| = 1$. Geometrically, the elements of $A_{PL}(\{n\})$ are the polynomial differential forms, with rational coefficients, on the simplex $\Delta^n$. The face operator, $\delta_i: \{n\} \rightarrow \{n + 1\}$, induces $A_{PL}(\delta_i): A_{PL}(\{n + 1\}) \rightarrow A_{PL}(\{n\})$, defined by: $A_{PL}(\delta_i)(t_k)$ is equal to $t_k$ if $k < i$, $0$ if $k = i$, and $t_{k-1}$ if $k > i$. The associated CDGA of global sections on a face set, $K$, is denoted $A_{PL}(K)$.

We mention the existence of local systems which are not universal. For instance, if $f: E \rightarrow K$ is a Kan fibration between face sets, for any $\sigma: \Delta \rightarrow K$, we denote by $E_\sigma$ the pullback of $f$ and $\sigma$. The association $\sigma \mapsto A_{PL}(E_\sigma)$ is a local system over $K$ which is the key tool in [29] and [23]. These more general local systems are not considered here.
We introduce now the fundamental notion of blow-up of a universal system, based on the blow-up of a simplex which first appears in \cite{7}, see also \cite{44}.

**Definition 2.8.** The blow-up of a filtered simplex, $\Delta = \Delta^{j_0} \ast \ldots \ast \Delta^{j_n}$, with $j_n \geq 0$, is the map,

$$
\mu: \widetilde{\Delta} = c\Delta^{j_0} \times \ldots \times c\Delta^{j_{n-1}} \times \Delta^{j_n} \to \Delta = \Delta^{j_0} \ast \ldots \ast \Delta^{j_n},
$$

defined by

$$
\mu([y_0,s_0],\ldots,[y_{n-1},s_{n-1}],y_n) = s_0y_0 + (1-s_0)s_1y_1 + \cdots + (1-s_0)\cdots(1-s_{n-2})(1-s_{n-1})y_n,
$$

where $c\Delta^k = \Delta^k \times [0,1]/\Delta^k \times \{0\}$ is the cone on $\Delta^k$, $[y_i,s_i] \in c\Delta^{j_i}$ and $y_n \in \Delta^{j_n}$.

The prism $\widetilde{\Delta}$ is sometimes also called the blow-up of $\Delta = \Delta^{j_0} \ast \ldots \ast \Delta^{j_n}$. The map $\mu$ induces a diffeomorphism between the interior of the blow-up $\widetilde{\Delta}$ and the interior of the simplex $\Delta$. In the previous representation of the elements of $\widetilde{\Delta}$, the face $\Delta^{j_i} \times \{1\}$ of the simplex $c\Delta^{j_i}$ corresponds to $s_i = 1$.

**Remark 2.9.** The blow-up map, $\mu$, can also be defined as follows. We express a point of $c\Delta^{j_i}$ by $(x_i,t_i)$, with $x_i = (x_{i,0},\ldots,x_{i,j_i}) \in \mathbb{R}^{j_i+1}$, $t_i \in \mathbb{R}$, $t_i + \sum_{k=0}^{j_i} x_{i,k} = 1$, and set

$$
\mu((x_0,t_0),\ldots,(x_{n-1},t_{n-1}),x_n) = x_0 + t_0x_1 + t_0t_1x_2 + \cdots + t_0\cdots t_{n-1}x_n.
$$

In this context, the face $\Delta^{j_i} \times \{1\}$ of the simplex $c\Delta^{j_i}$ corresponds to the set of elements satisfying the equation $t_i = 0$ and the cone point corresponds to $(0,1)$. In the case $\Delta^{j_i} = \emptyset$ with $i < n$, we have $c\Delta^{j_i} = \{(0,1)\}$.

**Example 2.10.** We draw two explicit examples of the blow-up of a simplex.
The faces containing $\Delta^i \times \{1\}$ as a factor, which play a fundamental role in the next definition, have been shadowed in the previous drawings.

As already observed in [7] in the case of differential forms, any universal system of cochains, $F$, gives rise to a strict perverse local system of cochains, $\tilde{F}$, on any filtered face set, $K$, defined as follows. For any simplex, $\sigma: \Delta^j_0 \ast \cdots \ast \Delta^j_n \to K_+$, we set

$$\tilde{F}_\sigma = F(c\Delta^j_0) \otimes \cdots \otimes F(c\Delta^j_{n-1}) \otimes F(\Delta^j_n).$$

With a method similar to a construction of Brylinski ([8]), any $\omega_{\sigma} \in \tilde{F}_\sigma$ can be provided with an extra degree, called the perverse degree, that we describe now. Let $\ell \in \{1, \ldots, n\}$ such that $\Delta^{j_{n-\ell}}_n \neq \emptyset$, the restriction of $\omega_{\sigma}$,

$$\omega_{\sigma, n-\ell} \in F(c\Delta^j_0) \otimes \cdots \otimes F(\Delta^j_{n-\ell} \times \{1\}) \otimes \cdots \otimes F(c\Delta^j_{n-1}) \otimes F(\Delta^j_n),$$

can be written

$$\omega_{\sigma, n-\ell} = \sum_k \omega'_{\sigma, n-\ell}(k) \otimes \omega''_{\sigma, n-\ell}(k),$$

with

- $\omega'_{\sigma, n-\ell}(k) \in F(c\Delta^j_0) \otimes \cdots \otimes F(c\Delta^j_{n-\ell-1}) \otimes F(\Delta^j_{n-\ell} \times \{1\})$ and

- $\omega''_{\sigma, n-\ell}(k) \in F(c\Delta^j_{n-\ell+1}) \otimes \cdots \otimes F(\Delta^j_n)$.

**Definition 2.11.** If $\omega_{\sigma, n-\ell} \neq 0$, the $\ell$-perverse degree, $||\omega_{\sigma}||_\ell$, of $\omega_{\sigma}$ is equal to

$$||\omega_{\sigma}||_\ell = \sup_k \{|\omega''_{\sigma, n-\ell}(k)| \text{ such that } \omega'_{\sigma, n-\ell}(k) \neq 0\},$$

where $|\omega''_{\sigma, n-\ell}(k)|$ is the degree of $\omega''_{\sigma, n-\ell}(k)$, as an element of the graded module $F(c\Delta^j_{n-\ell+1}) \otimes \cdots \otimes F(\Delta^j_n)$. If $\omega_{\sigma, n-\ell} = 0$ or $\Delta^{j_{n-\ell}} = \emptyset$, we set $||\omega_{\sigma}||_\ell = -\infty$.

Observe now that any face operator, $\delta_i: \Delta^j_0 \ast \cdots \ast \Delta^j_n \to \Delta^j_{0+} \ast \cdots \ast \Delta^j_{n+} \in \Delta^{[n]}_g$, induces a chain map

$$\delta_i^*: F(c\Delta^j_0) \otimes \cdots \otimes F(c\Delta^j_{n-1}) \otimes F(\Delta^j_n) \to F(c\Delta^j_0) \otimes \cdots \otimes F(c\Delta^j_{n-1}) \otimes F(\Delta^j_n),$$

which decreases the perverse degree. These chain maps are the operators, $\hat{\partial}_i: F_\sigma \to F_{\partial_0,\sigma}$, required in Definition 2.3 and we have proved that $\tilde{F}$ is a strict perverse local system of cochains over $K$.

Observe that we have two notions of perverse degree, one for the simplices of a filtered face set (Definition 1.4) and the previous one for global sections associated to a universal system. As they qualify different objects, they are different in essence but we will specify the one in use when there is some possible ambiguity. They are also relied in some sense, as we show now in the following remark.
Remark 2.12. Let \( \sigma : \Delta^0 \ast \cdots \ast \Delta^{j_n} \to \mathbb{K}_+ \). In the case of differential forms, the \( \ell \)-perverse degree of \( \omega_\sigma \) is the degree along the fibers of \( \omega_{\sigma,n-\ell} \) relatively to the projection (see [28, Page 122])
\[
c\Delta^0 \times \cdots \times (\Delta^{j_{n-\ell}} \times \{1\}) \times \cdots \times c\Delta^{j_n} \times \Delta^{j_n} \to c\Delta^0 \times \cdots \times (\Delta^{j_{n-\ell}} \times \{1\}).
\]
Because of the dimension of the fiber, the perverse degree, \( \|\omega_\sigma\| \), of the form \( \omega_\sigma \) and the perverse degree, \( \|\sigma\| \), of the simplex \( \sigma \) satisfy the inequality,
\[
\|\omega_\sigma\| \leq (\dim \sigma) - 1 - \|\sigma\|.
\]

Definition 2.13. Let \( F \) be a universal system and \( \mathbb{K} \) be a filtered face set. The strict perverse local system, \( \tilde{F} \), described above, is called the blow-up of \( F \) over \( \mathbb{K} \).

Example 2.14. The algebra of Sullivan’s polynomial forms and the cochain algebra of Examples [2.7 and 2.6] give the main examples of blow-ups used in this work, denoted respectively by \( \tilde{A}_{PL} \) and \( \tilde{C}^* \). The cochain complexes of \( \tilde{q} \)-intersection are denoted by \( \tilde{A}_{PL,q} \) and \( \tilde{C}^*_q \). The differential forms corresponding to \( \tilde{q} = 0 \) are the forms introduced by A. Verona in [50].

Definition 2.15. Let \( R \) be a commutative ring and \( \mathbb{K} \) be a filtered face set. The complex \( \tilde{C}^*(\mathbb{K}) \) over \( R \) is called the Thom-Whitney complex with coefficients in \( R \). If \( \tilde{q} \) is a loose perversity, the homology of \( \tilde{C}^*_q(\mathbb{K}) \) is denoted \( H_{\tilde{TW},q}(\mathbb{K} ; R) \).

We compare it with the Goresky-MacPherson cohomology in Section 4.

Proposition 2.16. Let \( F \) be a universal system. Any filtered face map, \( f : \mathbb{L} \to \mathbb{K} \), induces a cochain map, \( f^* : \tilde{F}(\mathbb{K}) \to \tilde{F}(\mathbb{L}) \), defined by \( (f^*\omega)_\sigma = \omega_{f(\sigma)} \). Moreover, the map \( f^* \) decreases the perverse degree and induces a cochain map \( f^* : \tilde{F}_q(\mathbb{K}) \to \tilde{F}_q(\mathbb{L}) \), for any loose perversity \( \tilde{q} \). In the case of a universal system of DGA’s, the map \( f^* : \tilde{F}(\mathbb{K}) \to \tilde{F}(\mathbb{L}) \) is a morphism of strict perverse DGA’s.

Proof. Let \( \omega \in \tilde{F}(\mathbb{K}) \) and \( \sigma \in \mathbb{L} \). Any map \( f \) which is compatible with face operators satisfies
\[
(f^*\omega)_{\partial_{\sigma}} = \omega_{f(\partial_{\sigma})} = \omega_{f_{\partial(\sigma)}} = \tilde{\partial}_{f_{\partial(\sigma)}} = \tilde{\partial}_{(f^*\omega)_{\sigma}}.
\]
Thus \( f^*\omega \) is an element of \( \tilde{F}(\mathbb{L}) \). The compatibility with the perverse degree (or the laws of algebras) is direct.

The next result is the key point in the existence of quasi-isomorphisms. It is based on the following definitions introduced in [29].

Definition 2.17. A universal system, \( F \), is extendable if the restriction map, \( F(\Delta^n) \to F(\partial\Delta^n) \), is surjective, for any \( n \geq 1 \).

Definition 2.18. A universal system, \( F \), is of differential coefficients if, for any \( n \) and any \( i \), the face operator \( \delta_i : \Delta^{n-1} \to \Delta^n \) induces an isomorphism \( H(F(\Delta^n)) \cong H(F(\Delta^{n-1})) \).

Theorem A. Let \( R \) be a commutative ring and \( \tilde{q} \) be a loose perversity. Let \( (F,G) \) be a pair of extendable universal systems of differential coefficients, with a natural transformation, \( \psi : F \to G \), given by \( \psi_{\Delta^i} : F(\Delta^i) \to G(\Delta^i) \). Then, there exists a morphism
\[ \Psi : \tilde{F}_q(K) \rightarrow \tilde{G}_t(K), \] defined as follows: if \( \omega \in \tilde{F}(K) \), \( \sigma : \Delta^0 \ast \cdots \ast \Delta^n \rightarrow K_+ \) and \( \omega_\sigma = \omega_0 \otimes \cdots \otimes \psi_\Delta^0(\omega_0) \otimes \cdots \otimes \psi_\Delta^n(\omega_n) \). If \( \psi \) is also a natural transformation of DGA’s, then \( \Psi \) induces a morphism, \( F(K) \rightarrow G(K) \), of strict perverse DGA’s.

Moreover, if \( \psi_\Delta^0 \) induces an isomorphism, \( H^0(\psi_\Delta^0) : H^0(F(\Delta^0)) \cong H^0(G(\Delta^0)) \cong R \), then the map, induced by \( \Phi \) in homology, is an isomorphism,

\[ \Phi^* : H^*_q(K; F) \cong H^*_q(K; G). \]

If \( R = \mathbb{Q} \), S. Halperin proves ([29, Chapters 13 and 14]) that the two local systems of Example 2.6 and Example 2.7 are extendable universal systems of differential coefficients. Furthermore, the classical integration map, \( \int : A_{PL}(\Delta^n) \rightarrow C^*(\Delta^n) \), defines a morphism between the blow-ups of Sullivan’s forms and the Thom-Whitney cochains,

\[ \int : \tilde{A}_{PL}(K) \rightarrow C^*_q(K). \]

In the case of face sets, Sullivan proves that this integration is a quasi-isomorphism, see [43, Theorem 7.1]. This theorem of Sullivan and Theorem A imply the next result.

**Corollary 2.19.** Let \( R = \mathbb{Q} \), \( K \) be a filtered face set and \( \bar{q} \) be a loose perversity. The integration map \( \int : A_{PL}(\bar{q}, K) \rightarrow C^*_q(K) \) induces an isomorphism in homology.

The compatibility with products, of the map induced by \( \int \) in cohomology, is detailed in Proposition 10.4.

### 3. Proof of Theorem A

This section is devoted to the proof of Theorem A. For that, we introduce the notion of filtered theory of cochains that will be used also for the comparison between Goresky-MacPherson and Thom-Whitney cochains in Section 4. All cochain complexes are over a commutative ring \( R \).

**Definition 3.1.** A functor \( C^* \) from the category of filtered face sets to the category of cochain complexes, is a filtered theory of cochains if the following properties are satisfied.

(a) **Extension Axiom.** For any filtered face set, \( K_r \), any \( r \geq 0 \) and any \( k \geq 0 \), the restriction map, \( C^*(K_r^k) \rightarrow C^*(K_r^{k-1}) \), is surjective. The kernel is denoted by \( C^*(K_r^k, K_r^{k-1}) \).

(b) **Relative Isomorphism Axiom.** Each relative isomorphism (see Definition 1.16), of the shape \( f : (K_r^k, K_r^{k-1}) \rightarrow (L_r^k, L_r^{k-1}) \), induces an isomorphism

\[ C^*(K_r^k, K_r^{k-1}) \cong C^*(L_r^k, L_r^{k-1}). \]

(c) **Wedge Axiom.** Let \( (K_i) \) be a family of filtered face sets such that the complementary subsets, \( K_i^k, K_i^{k-1} \), are disjoint. Then, there is an isomorphism

\[ C^*(\cup_i K_i^k, K_i^{k-1}) \cong \prod_i C^*(K_i^k, K_i^{k-1}). \]

(d) **Filtered Dimension Axiom.** For any filtered face set, \( K_r \), and any \( r \), we have \( H^m(C^*(K_r^k, K_r^{k-1})) = 0 \) if \( m < k \).
Recall the filtered face set $\Delta^k \ast K$ defined in Example 1.18

**Definition 3.2.** Let $R$ be a commutative ring and $\Psi: C^* \to D^*$ be a natural transformation between two filtered theories of cochains, such that $\Psi(K): C^*(K) \to D^*(K)$ is a quasi-isomorphism for a filtered face set $K$, of depth $v(K) < n$. The natural transformation $\Psi$ is *cone-compatible* for $K$ if the map

$$\Psi(\Delta^k \ast K): C^*(\Delta^k \ast K) \to D^*(\Delta^k \ast K)$$

is a quasi-isomorphism, for any $k \geq 0$.

**Proposition 3.3.** Let $R$ be a commutative ring. Let $\Psi: C^* \to D^*$ be a natural transformation between two filtered theories of cochains, cone-compatible for any filtered face set of depth strictly less than $n$, and inducing an isomorphism in homology, $H^*(\Psi): C^*(\{\emptyset\}) \to D^*(\{\emptyset\})$, for any singleton $\emptyset$, of any depth $\leq n$. Then the cochain map $\Psi(K): C^*(K) \to D^*(K)$ is a quasi-isomorphism for any filtered face set $K$.

**Proof.** Suppose the result is true for any filtered face set $L$ such that $L = L[r,k]$. If $r = 0, k = 1$, the hypothesis of induction is satisfied by hypothesis and the wedge axiom. The extension axiom gives a morphism of exact sequences

$$
\begin{array}{cccccc}
0 & \to & C^*(K[r,k],K[r,k]_{k-1}) & \to & C^*(K[r,k]) & \to & C^*(K[r,k]_{k-1}) & \to & 0 \\
& & f_1 & & f_2 & & f_3 & & \\
0 & \to & D^*(K[r,k],K[r,k]_{k-1}) & \to & D^*(K[r,k]) & \to & D^*(K[r,k]_{k-1}) & \to & 0,
\end{array}
$$

where the morphisms $f_i$ are induced by $\Psi$, $i = 1, 2, 3$. The map $f_3$ is a quasi-isomorphism by induction. To prove that $f_2$ is one also, we are reduced to study the map $f_1$ induced between the kernels. Recall that $\mathcal{J}(K[r],k)$ is the set of $\tau = R_1(\sigma)$, when $\sigma \in K[r,k]/K[r,k]_{k-1}$. From Example 1.17, the wedge and the relative isomorphism axioms, we get

$$C^*(K[r,k],K[r,k]_{k-1}) = \prod_{\tau \in \mathcal{J}(K[r],k)} C^*(K(\tau),K(\tau,\partial\tau))$$

and a similar result for $D^*$.

- Suppose first $k \neq 0$. From Proposition 1.19 and the relative isomorphism axiom, there exist isomorphisms,

$$C^*(K(\tau),K(\tau,\partial\tau)) \cong C^*(\Delta^k \ast \mathcal{L}^{\exp}(K,\tau),\partial\Delta^k \ast \mathcal{L}^{\exp}(K,\tau))$$

and

$$D^*(K(\tau),K(\tau,\partial\tau)) \cong D^*(\Delta^k \ast \mathcal{L}^{\exp}(K,\tau),\partial\Delta^k \ast \mathcal{L}^{\exp}(K,\tau)).$$

As we have already observed, we have

$$\mathcal{L}^{\exp}(K,\tau) = (\mathcal{L}^{\exp}(K,\tau))^{r-1} \text{ and } \partial\Delta^k \ast \mathcal{L}^{\exp}(K,\tau) = (\partial\Delta^k \ast \mathcal{L}^{\exp}(K,\tau))^{r,k-1}.$$  

Therefore, from the induction hypothesis and the cone compatibility condition, we obtain quasi-isomorphisms, induced by $\Psi$, $C^*(\Delta^k \ast \mathcal{L}^{\exp}(K,\tau)) \to D^*(\Delta^k \ast \mathcal{L}^{\exp}(K,\tau))$ and $C^*(\partial\Delta^k \ast \mathcal{L}^{\exp}(K,\tau)) \to D^*(\partial\Delta^k \ast \mathcal{L}^{\exp}(K,\tau))$, which bring out a quasi-isomorphism, also induced by $\Psi$,

$$C^*(\Delta^k \ast \mathcal{L}^{\exp}(K,\tau),\partial\Delta^k \ast \mathcal{L}^{\exp}(K,\tau)) \to D^*(\Delta^k \ast \mathcal{L}^{\exp}(K,\tau),\partial\Delta^k \ast \mathcal{L}^{\exp}(K,\tau)).$$
If $k = 0$, we have $K(\tau, \partial \tau) = \mathcal{L}(K, \tau)$ and $K^{[r]-1} = K^{[r-1]}$. If $\mathcal{L}^{\exp}(K, \tau) \neq \emptyset$, the argument above is still valid. If $\mathcal{L}^{\exp}(K, \tau) = \emptyset$, then $K(\tau) = \tau = \{ \emptyset \}$ and we apply the hypothesis on the singletons.

Finally, we have proved that $f_1$ is a quasi-isomorphism and the result is established in the case $K^{[r]} = K^{[r],k}$ for some $k$.

The axiom of filtered dimension implies the nullity of the relative cohomologies, $H^m(C(K^{[r]}, K^{[r],k}))$ and $H^m(D(K^{[r]}, K^{[r],k}))$, in any degree $m$ for $k$ great enough. Therefore, we get the isomorphism $H^m(C(K^{[r]})) \cong H^m(D(K^{[r]}))$, for any $m$. As $K = K^{[n]}$, the proof is done.

Proposition 3.3 admits a variation on the connectivity of the involved filtered face sets (used in Proposition 5.4) whose proof follows from an inspection of the previous arguments.

**Proposition 3.4.** Let $R$ be a commutative ring. Let $\Psi : C^* \to D^*$ be a natural transformation between two filtered theories of cochains, cone-compatible for any connected filtered face set of depth strictly less than $n$, and inducing an isomorphism in homology, $H^*(\Psi) : C^*(\{\emptyset\}) \to D^*(\{\emptyset\})$, for any singleton $\{\emptyset\}$, of any depth $\leq n$. Then $\Psi(K) : C^*(K) \to D^*(K)$ is a quasi-isomorphism for any filtered face set $K$ whose expanded links, $\mathcal{L}^{\exp}(K, \tau)$, are connected.

We determine a class of universal local systems generating filtered theory of cochains by blow-ups.

**Proposition 3.5.** Let $R$ be a commutative ring. Let $F$ be an extendable universal system of differential coefficients, such that $H^0(F(\{\emptyset\})) = R$, and $\overline{q}$ be a loose perversity. Then, the cochain complex $\widetilde{F}_\overline{q}$ is a filtered theory of cochains.

**Proof.** We follow the properties of Definition 3.1.

- **Extension Axiom.** Let $K$ be a filtered face set and $\tau \in \mathcal{J}(K, [r], k)$. We first prove that $\widetilde{F}_\overline{q}(K(\tau)) \to \widetilde{F}_\overline{q}(K(\tau, \partial \tau))$ is surjective.

Begin with $k > 0$. Let $\sigma \in K(\tau) \setminus K(\tau, \partial \tau)$ and $\omega \in \widetilde{F}(K(\tau, \partial \tau))$. If $\sigma : \Delta^k \ast \Delta^{j_{n-r+1}} \ast \ldots \ast \Delta^j \to K_+$, we denote by $\omega_{\overline{q}\sigma}$ the restriction of $\omega$ to $\partial \sigma = \partial \Delta^k \ast \Delta^{j_{n-r+1}} \ast \ldots \ast \Delta^j$. This restriction can be decomposed as $\omega_{\overline{q}\sigma} = \sum_m \omega'_m \otimes \omega''_m$ with $\omega'_m \in F(c\partial \Delta^k)$ and $\omega''_m \in F(c\Delta^{j_{n-r+1}}) \otimes \ldots \otimes F(\Delta^j)$. By hypothesis and [29] Proposition 12.21, the map $F(c\Delta^k) \to F(c\partial \Delta^k)$ admits a section, $\rho$, and we may define $\mu(\omega)_\sigma = \sum_m \rho(\omega'_m) \otimes \omega''_m$. By construction, the correspondence $\sigma \to \mu(\omega)_\sigma$ is compatible with face operators and we get a global section $\mu(\omega) \in \widetilde{F}(K(\tau))$.

As in the expression of $\sigma : \Delta^k \ast \Delta^{j_{n-r+1}} \ast \ldots \ast \Delta^j \to K_+$, the euclidian simplices, $\Delta^j$, are empty for $i < n - r$, we have $\|\mu(\omega)\|_r = -\infty$ if $r < \ell$. If $\ell = r$, by definition of the perverse degree of a global section, we consider the restriction, $\beta = \sum_m \beta'_m \otimes \omega''_m$ of $\mu(\omega)_\sigma$ to $F(\Delta^k \times \{1\}) \otimes (c\Delta^{j_{n-r+1}}) \otimes \ldots \otimes F(\Delta^j)$. The $r$-perverse degree of $\mu(\omega)_\sigma$ is the maximum of the (cohomological) degrees of the $\omega''_m$’s, with $\beta'_m \neq 0$. As $k > 0$, we have $\partial \Delta^k \neq \emptyset$ and $\|\mu(\omega)_\sigma\|_r \leq \|\omega_{\overline{q}\sigma}\|_r$. A similar argument works for $\ell < r$ and we have proved $\|\mu(\omega)_\sigma\| \leq \|\omega_{\overline{q}\sigma}\|$. As $\omega$ is of $\overline{q}$-intersection, we have

$$\|\mu(\omega)\|_r \leq \overline{q}(r) \text{ and } \|\mu(d\omega)\|_r \leq \overline{q}(r).$$
This proves the \( q \)-admissibility of \( \mu(\omega) \) and gives also

\[
\|d\mu(\omega)\|_r \leq (1) \max(\|d\mu(\omega) - \mu(d\omega)\|_r, \|\mu(d\omega)\|_r)
\leq (2) \max(\|\sum_{m} (d\rho(\omega'_m) - \rho(d\omega'_m)) \otimes \omega''_m\|_r, q(r))
\leq (3) \max(q(r), q(r)) = q(r),
\]

where

- \( \leq (1) \) comes from the compatibility condition of the perverse degree with sums, see Definition \( 2.2 \).
- \( \leq (2) \) is the replacement of \( \mu(\omega) \) by its value,
- \( \leq (3) \) is a computation similar at the previous one used in the proof of the displayed formula \( 3 \).

Therefore, we get an induced surjective map, \( \tilde{F}_q(K(\tau)) \to \tilde{F}_q(K(\tau, \partial \tau)) \).

— We consider now the case \( k = 0 \). Here, in perverse degree \( r \), we have \( \|\omega|_{\partial_{\tau}}\|_r = -\infty \). Therefore, for having the inequality \( \|\mu(\omega)\|_r \leq \|\omega|_{\partial_{\tau}}\|_r \), we need a section, \( \mu(\omega) \), with a vanishing restriction to \( F(\{\vartheta\} \times \{1\}) \otimes F(c\Delta^{j_{n+1}}) \otimes \cdots \otimes F(\Delta^{j_n}) \). We proceed now to the construction of this section.

As \( H^0(F(\{\vartheta\}) = R \), there are two cocycles in \( F^0(\{\vartheta\}) \) which correspond to \( 0 \in R \) and \( 1 \in R \), respectively. We denote them by 0 and 1. The surjectivity of \( F(\Delta^1) = F(c\vartheta) \to F(\partial \Delta^1) = F((\{\vartheta\} \times \{1\}) \cup (\{\vartheta\} \times \{0\})) \) implies the existence of \( \alpha \in F^0(c\vartheta) \) such that

\[
\alpha|_{\{\vartheta\} \times \{1\}} = 0 \text{ and } \alpha|_{\{\vartheta\} \times \{0\}} = 1.
\]

Let \( \sigma : \{\vartheta\} \star \Delta^{j_n} \to K_{+} \) be a simplex of \( K(\vartheta) \). We identify the product \( c\Delta^{j_{n-1}} \otimes \cdots \otimes \Delta^{j_n} \) to the subset \( \{\vartheta\} \times \{0\} \times c\Delta^{j_{n-1}} \times \cdots \times \Delta^{j_n} \) of \( c\vartheta \times c\Delta^{j_{n-1}} \times \cdots \times \Delta^{j_n} \). Let \( \omega \in \tilde{F}(\mathcal{L}(K, \{\vartheta\})) \). We set \( \mu(\omega) = \alpha \otimes \omega_{\partial_{\tau}}. \) This construction is compatible with the face operators and define an element \( \mu(\omega) \in \tilde{F}(K(\{\vartheta\})) \) whose restriction to \( \tilde{F}(\mathcal{L}(K, \{\vartheta\})) \) coincides with \( \omega \). As the restriction of \( \alpha \) and \( d\alpha \) to \( \{\vartheta\} \times \{1\} \) are equal to 0, we have, for any \( \omega' \in \tilde{F}(\mathcal{L}(K, \{\vartheta\})) \),

\[
\|d\alpha \otimes \omega'\|_{\ell} \leq \|\alpha \otimes \omega'\|_{\ell} = \left\{ \begin{array}{ll} -\infty & \text{if } \ell \geq r, \\ \|\omega'\|_{\ell} & \text{if } \ell < r. \end{array} \right.
\]

By applying these inequalities alternatively to \( \omega' = \omega_{\partial_{\tau}} \) and \( \omega' = d\omega_{\partial_{\tau}} \), we obtain

\[
\|\mu(\omega)\| = \|\alpha \otimes \omega_{\partial_{\tau}}\| \leq \|\omega_{\partial_{\tau}}\|
\leq \max(\|d\alpha \otimes \omega_{\partial_{\tau}}\|, \|\alpha \otimes d\omega_{\partial_{\tau}}\|)
\leq \max(\|\omega_{\partial_{\tau}}\|, \|d\omega_{\partial_{\tau}}\|).
\]

The surjectivity of \( \tilde{F}_q(K(\{\vartheta\})) \to \tilde{F}_q(\mathcal{L}(K, \{\vartheta\})) \) follows.

— We prove now the surjectivity of \( \tilde{F}_q(K^{[r],k}) \to \tilde{F}_q(K^{[r],k-1}) \), by detailing only the case \( k > 0 \), the case \( k = 0 \) being similar. Observe, from Corollary \( 1.14 \) that \( K^{[r],k} \) can
be obtained as a push-out, built on disjoint unions,

\[ \bigcup_{\tau \in \mathcal{I}} (K_{[r],k} \mathcal{K}(\tau, \partial \tau)) \rightarrow \bigcup_{\tau \in \mathcal{I}} (K_{[r],k} \mathcal{K}(\tau)) \]

\[ K_{[r],k-1} \rightarrow K_{[r],k}. \]

By definition, the blow-up \( \tilde{F}_\mathcal{K} \) is compatible with colimits and we get a pullback

\[ \tilde{F}_\mathcal{K}(\bigcup_{\tau \in \mathcal{I}} (K_{[r],k} \mathcal{K}(\tau, \partial \tau))) \rightarrow \tilde{F}_\mathcal{K}(\bigcup_{\tau \in \mathcal{I}} (K_{[r],k} \mathcal{K}(\tau))) \]

\[ \tilde{F}_\mathcal{K}(K_{[r],k-1}) \rightarrow \tilde{F}_\mathcal{K}(K_{[r],k}). \]

By definition also, we have \( \tilde{F}_\mathcal{K}(\bigcup_{\tau \in \mathcal{I}} (K_{[r],k} \mathcal{K}(\tau, \partial \tau))) = \prod_{\tau \in \mathcal{I}} (K_{[r],k} \mathcal{K}(\tau, \partial \tau)) \) and \( \tilde{F}_\mathcal{K}(\bigcup_{\tau \in \mathcal{I}} (K_{[r],k} \mathcal{K}(\tau))) = \prod_{\tau \in \mathcal{I}} (K_{[r],k} \mathcal{K}(\tau)) \). The surjectivity of the map \( \tilde{F}_\mathcal{K}(\mathcal{K}(\tau)) \rightarrow \tilde{F}_\mathcal{K}(K_{[r],k} \mathcal{K}(\tau), \partial \tau)) \) implies the surjectivity of \( \tilde{F}_\mathcal{K}(K_{[r],k}) \rightarrow \tilde{F}_\mathcal{K}(K_{[r],k-1}). \)

Moreover, observe that, if \( L' \) and \( L'' \) are filtered face sets, connected by a push-out of filtered face maps,

\[ \bigcup_{\tau \in \mathcal{I}, \mathcal{L}'} \mathcal{K}(\tau, \partial \tau) \rightarrow \bigcup_{\tau \in \mathcal{I}, \mathcal{L}''} \mathcal{K}(\tau) \]

\[ L' \rightarrow L. \]

as in (4), the previous argument implies the surjectivity of \( \tilde{F}_\mathcal{K}(L) \rightarrow \tilde{F}_\mathcal{K}(L'') \).

**Relative isomorphism axiom.**

A relative isomorphism \( f: (K_{[r],k} , K_{[r],k-1}) \rightarrow (L_{[r],k} , L_{[r],k-1}) \) induces a morphism of complexes

\[ f^* : \tilde{F}_\mathcal{K}(L_{[r],k} , L_{[r],k-1}) \rightarrow \tilde{F}_\mathcal{K}(K_{[r],k} , K_{[r],k-1}). \]

If \( \eta \in \tilde{F}_\mathcal{K}(K_{[r],k} , K_{[r],k-1}) \) and \( \sigma \in L_{[r],k} \), we set

\[ \omega_\sigma = \begin{cases} \eta_j^{-1}(\sigma) & \text{if } \sigma \in L_{[r],k} \backslash L_{[r],k-1}, \\ 0 & \text{if } \sigma \in L_{[r],k-1}. \end{cases} \]

As \( f^{-1}(\sigma) \) exists and is uniquely defined, the element \( \omega \) is well defined. We have to check the compatibility with face operators. Let \( \partial_j \) be a face operator and \( \sigma \in L_{[r],k} \).

- if \( \partial_j \sigma \in L_{[r],k} \backslash L_{[r],k-1} \), then we have
  \[ \omega_{\partial_j \sigma} = \eta_j f^{-1}(\partial_j \sigma) = \eta_j \partial_j f^{-1}(\sigma) = \partial_j \omega_\sigma. \]

  (We used \( \partial_j f^{-1}(\sigma) = f^{-1}(\partial_j \sigma) \) which can be verified by composing with \( f \).)

- if \( \partial_j \sigma \in L_{[r],k-1} \), then we have \( \partial_j \omega_\sigma = \partial_j \eta_j f^{-1}(\sigma) = \eta_j \partial_j f^{-1}(\sigma) = 0 \), because \( \partial_j f^{-1}(\sigma) \in K_{[r],k-1} \). (If \( \partial_j f^{-1}(\sigma) \in K_{[r],k} \backslash K_{[r],k-1} \) then we should have \( \partial_j \sigma \in L_{[r],k} \backslash L_{[r],k-1} \). By construction, we have also \( \omega_{\partial_j \sigma} = 0 \).
• **Wedge Axiom.** The restriction maps give a morphism of complexes

$$\tilde{F}_\tau(\bigcup_i K^{|i|,k}, \cup_i K^{|i|,k-1}) \to \prod_i \tilde{F}_\tau(K^{|i|,k}, K^{|i|,k-1}).$$

This morphism admits an inverse: for any family $(\omega_i)_i \in \prod_i \tilde{F}_\tau(K^{|i|,k}, K^{|i|,k-1})$, we define a global section $\omega \in \tilde{F}_\tau(\bigcup_i K^{|i|,k}, \cup_i K^{|i|,k-1})$ by

$$\omega_\sigma = \begin{cases} (\omega_i)_\sigma & \text{if } \sigma \in K^{|i|,k}, \chi^{|i|,k-1}, \\ 0 & \text{otherwise}. \end{cases}$$

The index $i$ such that $\sigma \in K^{|i|,k}\backslash K^{|i|,k-1}$ being unique, the element $\omega_\sigma$ is well defined. We have to check its compatibility with face operators. Let $\sigma \in K^{|i|,k}$.

- If $\partial_j \sigma \in K^{|i|,k}\backslash K^{|i|,k-1}$, we have $\omega_{\partial_j \sigma} = (\omega_i)_{\partial_j \sigma} = \tilde{\partial}_j(\omega_i)_\sigma = \partial_j \omega_\sigma$.
- If $\partial_j \sigma \in K^{|i|,k-1}$, we have $\omega_{\partial_j \sigma} = 0$ and $\tilde{\partial}_j \omega_\sigma = \partial_j(\omega_i)_\sigma = (\omega_i)_{\partial_j \sigma} = 0$, because $\omega_i \in \tilde{F}_\tau(K^{|i|,k}, K^{|i|,k-1})$.

• **Filtered dimension Axiom.** We have to prove the nullity of the relative cohomology with coefficients in $\tilde{F}_\tau$, i.e., $H_q^m(K^{|i|,k}, K^{|i|,k}; \tilde{F}) = 0$ for $m < k$. We establish it in three steps.

  (i) If $\tau \in \mathcal{J}(K^{|i|,k})$ and $m < k$, we prove

$$H_q^m(K^{|i|,k}, K^{|i|,k}; \tilde{F}) = 0.$$

We denote by $\Lambda^{m,k}$ the subcomplex of $\Delta^k$ generated by the simplices containing the vertex $v_0$ and of dimension less than, or equal, to $m$ and by $\Lambda^{m,\tau}$ the restriction of $\tau$ to $\Lambda^{m,k}$. (Observe that $\Lambda^{k,k} = \Delta^k$ and $\Lambda^{k,\tau} = \tau$.) Let $f_i: \Delta^m \to \Lambda^{m,k}$, $i \in I_m$, the $m$-dimensional simplices of $\Lambda^{m,k}$. We set $\tau_i = \Lambda^{m,\tau} \circ f_i$ and define $K(\Lambda^{m,\tau}) = \cup i \in I \Lambda^m(\tau_i)$, where (see Definition 1.11)

$$K(\tau_i) = \{ \xi \in K^{|i|,k} | \exists \sigma \in K^{|i|,k} \text{ with } \xi < \sigma \text{ and } R_1(\sigma) = \tau_i \}.$$

We first prove that

(6) \hspace{1cm} $H_q^m(K^{|i|,k}, K(\Lambda^{m,\tau}); \tilde{F}) = 0$, for any $m$, $0 \leq m \leq k$.

If $\tau \in \mathcal{J}(K^{|i|,k})$, then $\Lambda^{0,\tau} = \tau$ and we have $H_q^m(K^{|i|,k}, K(\Lambda^{0,\tau}); \tilde{F}) = 0$ as required. We use a first induction on the dimension $|\tau|$ of $\tau$, by supposing that (6) is true for any $\tau'$ of dimension strictly less than $|\tau|$.

Secondly, we do a decreasing induction assuming $H_q^m(K^{|i|,k}, K(\Lambda^{j,\tau}); \tilde{F}) = 0$ for some $j$, $j \leq k$. As $\Lambda^{k,\tau} = \tau$, we have $H_q^m(K^{|i|,k}, K(\Lambda^{k,\tau}); \tilde{F}) = 0$ and the inductive property is fulfilled for $j = k$. The proof of (6) is reduced to

(7) \hspace{1cm} $H_q^m(K^{|i|,k}, K(\Lambda^{j-1,\tau}); \tilde{F}) = 0$.

Using the (already established) wedge axiom, we have an isomorphism

$$\tilde{F}_\tau(K(\Lambda^{j,\tau}), K(\Lambda^{j-1,\tau})) \cong \prod_{i \in I_j} \tilde{F}_\tau(K(\tau_i), K(\Lambda^{j-1,\tau})).$$
The first induction hypothesis (on the dimension of $\tau$) gives
\[ H^r_q(K(\tau_i), K(A^{j-1,\tau_i}); \tilde{F}) = 0, \]
from which we deduce
\[ H^r_q(K(A^{j,\tau}), K(A^{j-1,\tau}); \tilde{F}) = 0. \]
By combining the short exact sequence of pairs with the snake lemma, the extension axiom generates a short exact sequence
\[ 0 \to \tilde{F}_q(K(\tau), K(A^{j,\tau})) \to \tilde{F}_q(K(\tau), K(A^{j-1,\tau})) \to \tilde{F}_q(K(A^{j,\tau}), K(A^{j-1,\tau})) \to 0. \]
This sequence and the equality (8) imply
\[ H^r_q(K(A^{j,\tau}), K(A^{j-1,\tau}); \tilde{F}) \cong H^r_q(K(\tau), K(A^{j,\tau}); \tilde{F}), \]
which is trivial with the second (decreasing) induction. Finally, we have proved the equality (6).

Consider now $\iota: \Delta^{k-1} \to \Delta^k$, the face opposite to $v_0$ and set $\tau' = \tau \circ \iota$. With the extension axiom, we have an exact sequence,
\[ 0 \to \tilde{F}_q(K(\tau), K(\tau, \partial \tau)) \to \tilde{F}_q(K(\tau), K(A^{k-1,\tau})) \to \tilde{F}_q(K(\tau'), K(\tau', \partial \tau')) \to 0. \]
The equality (6) implies the existence of an isomorphism of degree +1 between the relative cohomologies involving $\tau$ and $\tau'$. Starting from $\tau' = \{\emptyset\}$ and $\tilde{F}_q(K(\tau'), K(\tau', \partial \tau')) = \tilde{F}_p(K(\{\emptyset\}), L(K, \{\emptyset\}))$, whose cohomology is zero in negative degree, we deduce
\[ H^m_q(K(\tau), K(\tau, \partial \tau); \tilde{F}) = 0, \text{ if } m < |\tau| = k, \]
and the result follows.

(ii) As the map $(\cup_{\tau \in \mathcal{D}(K_{r|}[r], k)} K(\tau), \cup_{\tau \in \mathcal{D}(K_{r|}[r], k)} K(\tau, \partial \tau)) \to (K_{r|}[r], K_{r|}[r], k-1))$ is a relative isomorphism, the previous step, the wedge and the relative isomorphism axioms imply
\[ H^m_q(K_{r|}[r], K_{r|}[r], k-1); \tilde{F}) = \prod_{\tau \in \mathcal{D}(K_{r|}[r], k)} H^m_q(K(\tau), K(\tau, \partial \tau), \tilde{F}) = 0, \]
for any $m$, $m < k$.

(iii) Finally, we prove $H^m_q(K_{r|}[r], K_{r|}[r], k); \tilde{F}) = 0$, if $m < k$.
Let $\omega \in \tilde{F}_q(K_{r|}[r], K_{r|}[r], k)$, of degree $m$, with $d\omega = 0$. The restriction $\omega|_{K_{r|}[r], k+1}$ is a coboundary (with step (ii)) and there exists $\psi_0 \in \tilde{F}_q(K_{r|}[r], k+1, K_{r|}[r], k)$ such that $\omega|_{K_{r|}[r], k+1} = d\psi_0$. By iterating the extension axiom, the global section $\psi_0$ can be extended in a global section defined on $K_{r|}[r]$ that we still denote $\psi_0$, i.e., we have $\psi_0 \in \tilde{F}_q(K_{r|}[r], K_{r|}[r], k)$ and $\omega - d\psi_0 \in \tilde{F}_q(K_{r|}[r], K_{r|}[r], k+1)$.

Suppose that, for any $i$, $0 \leq i \leq j$, we have built global sections $\psi_i \in \tilde{F}_q(K_{r|}[r], K_{r|}[r], k+i)$ such that $\omega - d(\sum_{i=0}^j \psi_i) \in \tilde{F}_q(K_{r|}[r], K_{r|}[r], k+j+1)$. The restriction of $\omega - d(\sum_{i=0}^j \psi_i)$ to $K_{r|}[r], k+j+2$ being a coboundary (by step (ii)), there exists $\psi_{j+1} \in \tilde{F}_q(K_{r|}[r], k+j+2, K_{r|}[r], k+j+1)$
such that \( d\psi_{j+1} = (\omega - d(\sum_{i=0}^{j} \psi_i))|_{K^{[r]}}. \) We extend \( \psi_{j+1} \) in a global section defined on \( K^{[r]} \), still denoted \( \psi_{j+1} \). This section verifies

\[ \psi_{j+1} \in \tilde{F}_Q(K^{[r]}, K^{[r], k+j+1}) \text{ and } \omega - \sum_{i=0}^{j+1} d\psi_i \in \tilde{F}_Q(K^{[r]}, K^{[r], k+j+2}). \]

We set \( \psi = \sum_{i=0}^{\infty} \psi_i \). When we apply \( \psi \) to a simplex \( \sigma \in K^{[r]}_+ \), the sum \( \psi_\sigma \) is finite and we have \( \psi \in \tilde{F}_Q(K^{[r]}, K^{[r], k}) \) with \( \omega = d\psi \). \( \square \)

**Definition 3.6.** For any positive integer \( s \) and any cochain complex, \( C^* \), over the commutative ring \( R \), the \( s \)-truncation of \( C^* \) is the cochain complex, \( \tau_{\leq s} C^* \), defined by

\[(\tau_{\leq s} C)^r = \begin{cases} \ C^r & \text{if } r < s, \\ \mathcal{Z}C^s & \text{if } r = s, \\ 0 & \text{if } r > s, \end{cases} \]

where \( \mathcal{Z}C^s \) denote the \( R \)-module of cocycles of \( C \) in degree \( s \).

**Proposition 3.7.** Let \( R \) be a commutative ring and \( \ell \) be an integer such that \( 1 \leq \ell \leq n \). Let \( F \) be an extendable universal system of differential coefficients and \( \tilde{Q} \) be a loose perversity. Let \( K \) be a filtered face set of depth \( v(K) \leq \ell - 1 \) and \( \Delta^k \ast K \) be a join of depth \( \ell \) with \( k \geq 0 \). Then we have

\[ H^q_Q(\Delta^k \ast K; \tilde{F}) = H^q(F(\{0\}) \otimes \tau_{\leq 0} \tilde{F}(K)). \]

**Proof.** An element of \( F(c \Delta^k) \) is determined by its value on the maximal simplex \([c \Delta^k]\). As the universal system, \( F \), and \( \Delta^k \) are given, \( F(c \Delta^k)|_{[c \Delta^k]} \) is already determined and an element \( \eta \in F(c \Delta^k) \otimes \tilde{F}(K) \) is given by the values \( \eta_\alpha \in \tilde{F}(K) \) for \( \alpha \in K^+_1 \). Also, for the same reason, any element of \( F(\Delta^k \ast K) \) is determined by the values \( \omega_\alpha \in \tilde{F}(K) \) for \( \alpha \in K^+_1 \). This remark implies the existence of an isomorphism,

\[ \tilde{F}(\Delta^k \ast K) \cong F(c \Delta^k) \otimes \tilde{F}(K). \]

Consider now the short exact sequence

\[ 0 \longrightarrow F(c \Delta^k; \Delta^k \times \{1\}) \longrightarrow F(c \Delta^k) \longrightarrow F(\Delta^k \times \{1\}) \longrightarrow 0. \]

As \( F(-) \) is free as \( R \)-module, this sequence splits and we have an isomorphism of cochain complexes,

\[ F(c \Delta^k) \cong F(c \Delta^k, \Delta^k \times \{1\}) \oplus F(\Delta^k \times \{1\}). \]

Let \( \alpha : \Delta^{j_{n-\ell+1}} \ast \cdots \ast \Delta^{j_n} \to K^+_1 \), \( \eta \in F(c \Delta^k) \otimes \tilde{F}(K) \) and \( \tilde{Q} \) be a loose perversity. We decompose

\[ \eta_{\Delta^k \ast \alpha} = \sum_i (\eta_{i, \alpha} + \eta''_{i, \alpha}) \otimes \eta'''_{i, \alpha} \in (F(c \Delta^k, \Delta^k \times \{1\}) \oplus F(\Delta^k \times \{1\})) \otimes \tilde{F}(K). \]

By definition, the perverse degree of \( \eta_{\Delta^k \ast \alpha} \) is determined as follows:

- if \( \ell < \ell' \), then \( \| \eta_{\Delta^k \ast \alpha} \|_{\ell'} = -\infty \),
- if \( \ell' = \ell \), then, the \( \ell \)-perverse degree of \( \eta_{i, \alpha} \otimes \eta'''_{i, \alpha} \) is equal to \( -\infty \). Therefore, if \( \eta_{i, \alpha}|_{\Delta^k \times \{1\}} = 0 \) for all \( i \), we have \( \| \eta_{\Delta^k \ast \alpha} \|_{\ell} = -\infty \),
have where the last equality comes from Proposition 3.7. By definition of the truncation, we obtain:

$$\|\eta_{\Delta^k \ast \alpha}\|_{\ell} = \max_i \{\|\eta_{i,\alpha}''\|_{\ell} \text{ such that } \eta_{i,\alpha}''|_{\Delta^k \times \{1\}} \neq 0\},$$

- if not, we have $$\|\eta_{\Delta^k \ast \alpha}\|_{\ell} = \max_i \{\|\eta_{i,\alpha}''\|_{\ell} \text{ such that } \eta_{i,\alpha}'' \neq 0\},$$

and the perverse degree of $$\eta$$ coincide with the perverse degree of its component in $$\tilde{F}(K)$$.

As a consequence, if we consider the $$\ell'$$-perverse degree for $$\ell' < \ell$$, the fact of being of $$\mathfrak{q}$$-intersection involves only the $$\mathfrak{q}$$-intersection condition of the component in $$\tilde{F}(K)$$. For the $$\ell$$-perverse degree, the situation is different: there is no restriction for the component which involves $$F(c\Delta^k, \Delta^k \times \{1\})$$ and there is a restriction on the (cohomological) degree in $$\tilde{F}(K)$$ for the component having $$F(\Delta^k \times \{1\})$$ as first factor. In summary, we have a decomposition of $$\tilde{F}_{\mathfrak{q}}(\Delta^k \ast K)$$ as

$$(F(c\Delta^k, \Delta^k \times \{1\}) \otimes \tilde{F}_{\mathfrak{q}}(K)) \oplus (F(\Delta^k \times \{1\}) \otimes \tau_{\leq \mathfrak{q}(\ell)} \tilde{F}_{\mathfrak{q}}(K)).$$

As $$F$$ is a system of differential coefficients, there is a quasi-isomorphism between $$F(\Delta^k \times \{1\})$$ and $$F(\{0\})$$ and we get the quasi-isomorphisms:

$$\tilde{F}_{\mathfrak{q}}(\Delta^k \ast K) \simeq F(\Delta^k \times \{1\}) \otimes \tau_{\leq \mathfrak{q}(\ell)} \tilde{F}_{\mathfrak{q}}(K)$$

$$\simeq F(\{0\}) \otimes \tau_{\leq \mathfrak{q}(\ell)} \tilde{F}_{\mathfrak{q}}(K).$$

**Corollary 3.8.** If we add $$H^\ast(F(\{\vartheta\})) = R$$ to the hypotheses of Proposition 3.7, we obtain:

$$H_{\mathfrak{q}}^i(\Delta^k \ast K; \tilde{F}) = \begin{cases} H_{\mathfrak{q}}^i(K; \tilde{F}) & \text{if } i \leq \mathfrak{q}(\ell), \\ 0 & \text{if } i > \mathfrak{q}(\ell). \end{cases}$$

**Proof.** By Definition 2.3, the module $$F(\{\vartheta\})$$ is $$R$$-free. Thus, there is a Künneth spectral sequence, converging to $$H^\ast(F(\{\vartheta\}) \otimes \tau_{\leq \mathfrak{q}(\ell)} \tilde{F}_{\mathfrak{q}}(K))$$, and whose second page is

$$E_2^{ij} = \oplus_{u+v=j} \text{Tor}^i(H^u(F(\{0\})), H^v(\tau_{\leq \mathfrak{q}(\ell)} \tilde{F}_{\mathfrak{q}}(K))).$$

As $$H^\ast(F(\{0\}) = R$$, all the terms $$\text{Tor}^i(-, -)$$ are zero if $$i > 0$$, and this spectral sequence collapses in

$$E_2^{ij} = H^j(\tau_{\leq \mathfrak{q}(\ell)} \tilde{F}_{\mathfrak{q}}(K)) \cong E_{\infty}^{0j} \cong H^j(F(\{\vartheta\}) \otimes \tau_{\leq \mathfrak{q}(\ell)} \tilde{F}_{\mathfrak{q}}(K)) = H_{\mathfrak{q}}^j(\Delta^k \ast K; \tilde{F}),$$

where the last equality comes from Proposition 3.7. By definition of the truncation, we have

$$H^j(\tau_{\leq \mathfrak{q}(\ell)} \tilde{F}_{\mathfrak{q}}(K)) = \begin{cases} H_{\mathfrak{q}}^j(K; \tilde{F}) & \text{if } j \leq \mathfrak{q}(\ell), \\ 0 & \text{if } j > \mathfrak{q}(\ell). \end{cases}$$

as announced. □
Proof of Theorem A. Let $\sigma: \Delta^{j_0} \ast \cdots \ast \Delta^{j_n} \rightarrow K$, We define
\[
\psi_\sigma: \tilde{F}_\sigma = F(c\Delta^{j_0}) \cdots \otimes F(\Delta^{j_n}) \rightarrow \tilde{G}_\sigma = G(c\Delta^{j_0}) \cdots \otimes G(\Delta^{j_n})
\]
by $\psi_\sigma = \psi_{c\Delta^{j_0}} \otimes \cdots \otimes \psi_{\Delta^{j_n}}$. By construction, the map $\psi_\sigma$ decreases the perverse degree and induces a cochain map $\Psi(K): \tilde{F}_\sigma(K) \rightarrow \tilde{G}_\sigma(K)$. Also, in the case of a natural transformation between universal systems of DGA's, the induced map $\tilde{F}(K) \rightarrow \tilde{G}(K)$ is compatible with the laws of algebras, as follows directly from their construction in Definition 2.3.

The hypothesis of Proposition 3.3 are satisfied: the blow-ups $\tilde{F}_\sigma$ and $\tilde{G}_\sigma$ are filtered theories of cochains thanks to Proposition 3.5. Moreover, the natural transformation $\tilde{F}_\sigma(-) \rightarrow \tilde{G}_\sigma(-)$ is cone-compatible for any filtered face set, thanks to Corollary 3.8. \qed

4. COCHAINS ON FILTERED FACE SETS

We compare the Thom-Whitney complex (cf. Definition 2.15) to a cochain complex which is the linear dual of the analogous for filtered face sets of the classical Goresky and MacPherson intersection chain complex. The main result (Theorem B) is the existence of a quasi-isomorphism between these two complexes, over any field of coefficients, for complementary per-versities.

Let $K$ be a filtered face set and $R$ be a commutative ring. The chain complex $C^\ast_{GM}(K; R)$ is defined as follows,

- as module, $C^\ast_{GM}(K; R)$ is the free $R$-module generated by the simplices $\sigma \in K$,
- any simplex $\sigma: \Delta^{j_0} \ast \cdots \ast \Delta^{j_n} \rightarrow K$ has a perverse degree defined by $\|\sigma\|_\ell = \text{dim}(\Delta^{j_0} \ast \cdots \ast \Delta^{j_{n-\ell}})$, if $\Delta^{j_0} \ast \cdots \ast \Delta^{j_{n-\ell}} \neq \emptyset$ and $\|\sigma\|_\ell = -\infty$ otherwise,
- the differential of a simplex $\sigma: \Delta^k = \Delta^{j_0} \ast \cdots \ast \Delta^{j_n} \rightarrow K$ is given as usual by $\partial\sigma = \sum_{i=0}^k (-1)^i \sigma \circ \delta_i$, with $\delta_i \in \Delta_{[n]}$.

Let $p$ be a loose perversity. A simplex $\sigma: \Delta = \Delta^{j_0} \ast \cdots \ast \Delta^{j_n} \rightarrow K$ is $p$-admissible if
\[
\text{dim}(\Delta^{j_0} \ast \cdots \ast \Delta^{j_{n-\ell}}) = \|\sigma\|_\ell \leq \text{dim} \Delta - \ell + p(\ell),
\]
for any $\ell$, such that $\ell \in \{1, \ldots, n\}$. (See Remark 3.8 for the case $\ell = 0$.) A chain $c$ is $p$-admissible if $c$ can be written as a linear combination of $p$-admissible simplices. A chain $c$ is of $p$-intersection if $c$ and $\partial c$ are $p$-admissible. We denote by $C^\ast_{GM,p}(K; R)$ the complex of $p$-intersection chains. The GM-cochain complexes are the duals of these complexes, i.e., $C^\ast_{GM}(K; R) = \text{hom}(C^\ast_{GM,p}(K; R), R)$ and $C^\ast_{GM,p}(K; R) = \text{hom}(C^\ast_{GM,p}(K; R), R)$.

We denote by $H^\ast_{GM,p}(K; R)$ the homology of the complex $C^\ast_{GM,p}(K; R)$, by $H^\ast_{GM,p}(K; R)$ the homology of the complex $C^\ast_{GM,p}(K; R)$ and called them, respectively, Goresky-MacPherson homology and cohomology of $K$.

If $X$ is a filtered space and $L = ISing^p(X)$, as in Example 1.5, we have $C^\ast_{GM,p}(L; R) = \text{hom}(C^\ast_p(X; R), R)$, where $C^\ast_{GM}(X; R)$ is introduced in Definition 2.3.

Proposition 4.1. Let $R$ be a commutative ring and $p$ be a loose perversity. The GM-cochain complex $C^\ast_{GM,p}(-; R)$ is a filtered theory of cochains.
Proof. As all the chain and cochain complexes of this proof are over $R$, we do not mention explicitly the coefficients. We establish the properties of Definition 3.1 for a filtered face set, $K$.

- Extension axiom. We denote by $\mathcal{R}$ the map induced by the canonical inclusion,
  $\mathcal{R}: \hom(C_s^{GM, p}(K^{[r], k}), R) \to \hom(C_s^{GM, p}(K^{[r], k-1}, R)$.

A chain $c \in C_s(K^{[r], k})$ can be decomposed as $c = c_1 + c_2$, with $c_1 \in C_s(K^{[r], k-1})$ and $c_2 \in C_s(K^{[r], k} \setminus K^{[r], k-1})$. If $c$ is of $p$-intersection, then $c_1, c_2$ are $p$-admissible and the boundary, $\partial c$, can be written as a linear combination of $p$-admissible simplices. We decompose $\partial c_2$ in $\partial c_2 = \partial' c_2 + \partial'' c_2$, with $\partial' c_2 \in C_s(K^{[r], k-1})$ and $\partial'' c_2 \in C_s(K^{[r], k} \setminus K^{[r], k-1})$. The elements of the linear combination $\partial'' c_2$ cannot be canceled with elements coming from $\partial' c_2$ or $\partial c_1$, therefore $\partial'' c_2$ is $p$-admissible.

Let $\partial_i$ be any face operator and $\sigma$ be a $p$-admissible simplex. If we have strict inequalities, $\|\partial_i \sigma\| < \|\sigma\|$, for any $\ell$, then, directly from the definition, we observe that $\partial_i \sigma$ is $p$-admissible. As these strict inequalities occur when $\partial_i$ is a face operator acting on the first factor $\Delta^{n-r}$ of $\Delta^{j_{n-r}} \cdots \Delta^{j_{l}}$, we get the $p$-admissibility of $\partial' c_2$. We have proved that $c_2$ is of $p$-intersection; therefore $c = c_1 - c_2$ is of $p$-intersection also.

This property allows the construction of a section, $s$, to $\mathcal{R}$, as follows: Let $\Phi \in \hom(C_s^{GM, p}(K^{[r], k-1}, R)$, we define $s(\Phi) \in \hom(C_s^{GM, p}(K^{[r], k}, R)$ by $s(\Phi)(c_1 + c_2) = \Phi(c_1)$.

- Relative isomorphism axiom. Let
  $f: (K^{[r], k}, K^{[r], k-1}) \to (L^{[r], k}, L^{[r], k-1})$

be a relative isomorphism preserving the perverse degree. The relative isomorphism axiom is equivalent to the fact that $f$ induces an isomorphism between the quotients,

$$\overline{f}_*: \frac{C_s^{GM, p}(K^{[r], k})}{C_s^{GM, p}(K^{[r], k-1})} \cong \frac{C_s^{GM, p}(L^{[r], k})}{C_s^{GM, p}(L^{[r], k-1})}.$$ $$

Let $c \in C_s(K^{[r], k})$. As in the previous item, we have a unique decomposition of $c$ as $c = c_1 + c_2$, with $c_1 \in C_s(K^{[r], k-1})$ and $c_2$ a linear combination of simplices of $K^{[r], k} \setminus K^{[r], k-1}$. Moreover, if $c \in C_s^{GM, p}(K^{[r], k})$ then $c_1 \in C_s^{GM, p}(K^{[r], k-1})$ and $c_2$ is a linear combination of simplices of $K^{[r], k} \setminus K^{[r], k-1}$ that is of $p$-intersection. A similar decomposition exists for the elements of $C_s^{GM, p}(L^{[r], k})$.

We denote by $[c]$ the class of $c \in C_s^{GM, p}(K^{[r], k})$ in the quotient above. As $f$ is a bijection between $K^{[r], k} \setminus K^{[r], k-1}$ and $L^{[r], k} \setminus L^{[r], k-1}$, we have $\overline{f}_*([c]) = \overline{f}_*([c_2])$, which is a linear combination of simplices in $L^{[r], k} \setminus L^{[r], k-1}$, that is of $p$-intersection, by the hypothesis on the conservation of the perverse degree. This implies that $\overline{f}_*$ is an isomorphism.

- Wedge axiom. The dual of a direct sum being a product, we have to prove that the canonical injections induce an isomorphism,

$$\psi: \bigoplus_i \frac{C_s^{GM, p}(K_i^{[r], k})}{C_s^{GM, p}(K_i^{[r], k-1})} \cong \frac{C_s^{GM, p}(\bigcup_i K_i^{[r], k})}{C_s^{GM, p}(\bigcup_i K_i^{[r], k-1})}.$$
Let \( ([c_i])_i \) be an element of the left-hand term. The map \( \psi \) is defined by \( \psi(([c_i])_i) = \sum_i [c_i] \), where \([ \cdot ]\) denotes the equivalence classes of the quotients.

As we already did, we decompose a chain \( c \in C^\text{GM,}\mathcal{P}_* (\bigcup_i K^{[r],k}_i) \) in \( c = c' + c'' \), with \( c' \in C^\text{GM,}\mathcal{P}_* (\bigcup_i K^{[r],k-1}_i) \) and \( c'' \in C^\text{GM,}\mathcal{P}_* (\bigcup_i K^{[r],k}_i \setminus \bigcup_i K^{[r],k-1}_i) \). As the intersection of the sets, \( K^{[r],k}_i \setminus K^{[r],k-1}_i \), is empty, the element \( c'' \) can be written \( c'' = \sum_i c''_i \), in a unique way, with \( c''_i \in C^\text{GM,}\mathcal{P}_* (K^{[r],k}_i \setminus K^{[r],k-1}_i) \). An inverse to \( \psi \) is defined by \( \psi^{-1}(\{c\}) = ([c''_i])_i \).

- **FILTERED DIMENSION AXIOM.** Let \( c \in C^\text{GM,}\mathcal{P}_* (K^{[r]}_i) \) of homological degree \(|c|\) such that \(|c| < k\). Denote by \([c]\) the class of \( c \) in the quotient

\[
C^\text{GM,}\mathcal{P}_* (K^{[r]}_i) / C^\text{GM,}\mathcal{P}_* (K^{[r],k}_i).
\]

For degree reason, we have \([c] = 0\) and the filtered dimension axiom is proved. \( \square \)

The next result generalizes Proposition \ref{A.14} (v), which corresponds to the case \( k = 0 \) and \( K = \text{ISing}^\mathcal{P}(X) \).

**Proposition 4.2.** Let \( R \) be a commutative ring and \( \ell \) be an integer such that \( 1 \leq \ell \leq n \). Let \( K \) be a filtered face set of depth \( v(K) \leq \ell - 1 \) and \( \Delta^k \ast K \) be a join of depth \( \ell \) with \( k \geq 0 \). For any perversity \( p \) and any \( k \in \mathbb{N} \), we have

\[
H^\text{GM,}\mathcal{P}_*(\Delta^k \ast K; R) = \begin{cases} 
H^\text{GM,}\mathcal{P}_*(K; R) & \text{if } i \leq \ell - 2 - p(\ell), \\
R & \text{if } i > \ell - 2 - p(\ell) \text{ and } i = 0, \\
0 & \text{if } i > \ell - 2 - p(\ell) \text{ and } i \neq 0.
\end{cases}
\]

**Proof.** As the elements of degree zero are playing an important role, we suppose that each chain complex \( C_*(-) \) is augmented. We denote by \( \varepsilon \) the various augmentations and by \( \overline{C}_*(-) = C_*(-) \oplus R \) the associated augmented complexes, with the component \( R \) in degree -1. We define a linear map,

\[
\psi : \overline{C}_*(-) \otimes \overline{C}_*(K) \to \overline{C}_{*+1}(\Delta^k \ast K),
\]

by \( \psi(\gamma \otimes \sigma) = \gamma \ast \sigma, \) for any \( \gamma \in \overline{C}_*(-) \) and any \( \sigma \in \overline{C}_*(K) \), with the convention \( \gamma \ast 1 = \gamma, \ 1 \ast \sigma = \sigma \) and \( 1 \ast 1 = 1 \). We check easily that this map, of degree +1, is compatible with the differentials. Moreover, \( \psi \) is injective on the basis of simplices, thus injective. The surjectivity comes from the definition of \( \Delta^k \ast K \) in Example \ref{1.18}.

Recall, from Example \ref{1.18}, the determination of the perverse degrees of the simplices of \( \Delta^k \ast K \). Let \( \ell' \geq 1, \sigma; \Delta^N = \Delta^{1+1} \ast \cdots \ast \Delta^N \to K \) and \( \gamma; \Delta^{k'} \to \Delta^k \). The perverse degree of \( \sigma \) is the same, as simplex of \( K \) or as simplex of \( \Delta^k \ast K \). For \( \gamma \in \Delta^k \subset \Delta^k \ast K \), we have

\[
\|\gamma\|_{\ell'} = \begin{cases} 
-\infty & \text{if } \ell < \ell', \\
k' & \text{if } \ell' \leq \ell.
\end{cases}
\]

The perverse degree of \( \gamma \ast \sigma \in \Delta^k \ast K \) is determined by

\[
\|\gamma \ast \sigma\|_{\ell'} = \begin{cases} 
-\infty & \text{if } \ell < \ell', \\
k' & \text{if } \ell' = \ell \text{ or } (\ell > \ell' \text{ and } \|\sigma\|_{\ell'} = -\infty), \\
k' + \|\sigma\|_{\ell'} + 1 & \text{if } \ell > \ell' \text{ and } \|\sigma\|_{\ell'} \neq -\infty.
\end{cases}
\]
From these determinations, we deduce the $p$-admissibility conditions of the various types of simplices of $\Delta^k \ast K$.

- The simplex $\sigma$ is $p$-admissible as simplex of $K$ if, and only if, it is admissible as simplex of $\Delta^k \ast K$.
- The $p$-admissibility condition of $\gamma: \Delta^{k'} \rightarrow \Delta^k$ as simplex of $\Delta^k \ast K$ is, by definition,
  \begin{equation}
  \|\gamma\|_v \leq k' - \ell' + p(\ell'), \text{ for any } \ell' \geq 1.
  \end{equation}

  - If $\ell < \ell'$, this condition is always satisfied.
  - If $\ell' \leq \ell$, this condition is satisfied if, and only if, $0 \leq -\ell' + p(\ell')$. As $p$ is a perversity, all these conditions are equivalent to $\ell \leq p(\ell)$.
- The $p$-admissibility condition of $\gamma \ast \sigma \in \Delta^k \ast K$ is, by definition,
  \begin{equation}
  \|\gamma \ast \sigma\|_v \leq k' + N + 1 - \ell' + p(\ell'), \text{ for any } \ell' \geq 1.
  \end{equation}

  - If $\ell < \ell'$, this condition is always satisfied.
  - If $\ell' < \ell$ and $\|\sigma\|_v \neq -\infty$, this condition is satisfied if, and only if, $\|\sigma\|_v \leq N - \ell' + p(\ell')$. This last inequality is exactly the $p$-admissibility of $\sigma$ at $\ell'$.
  - If $\ell' = \ell$ or $\ell' > \ell$ and $\|\sigma\|_v = -\infty$, the condition (10) is equivalent to $N \geq \ell' - 1 - p(\ell')$. As $p$ is a perversity, all these inequalities are equivalent to $N \geq \ell - 1 - p(\ell)$.

We determine now $H^*_{GM,p}(\Delta^k \ast K; R)$ by considering two cases.

**First case:** $p(\ell) \geq \ell$. The previous determination of perverse degrees implies the bijectivity of

$$
\psi: \overline{C}_*(\Delta^k) \otimes \overline{C}_{*-1}^{GM,p}(K) \rightarrow \overline{C}_{*-1}^{GM,p}(\Delta^k \ast K),
$$

from which we deduce $H^*_{GM,p}(\Delta^k \ast K; R) = 0$.

**Second case:** $p(\ell) \leq \ell - 1$. By comparing to the previous case, we observe that the chains of $p$-intersection are sums of chains of the shape $\nu_1 \ast \nu_2$ with $\nu_2 \in \overline{C}_*^{GM,p}(K)$, such that $|\nu_2| > \ell - 1 - p(\ell)$ or, with $|\nu_2| = \ell - 1 - p(\ell)$ and $\partial \nu_2 = 0$. By setting,

$$
\tau_\ell \overline{C}_*(K) = \begin{cases} 
0 & \text{if } * < \ell - 1 - p(\ell), \\
\overline{C}_{*-1}^{GM,p}(K) & \text{if } * = \ell - 1 - p(\ell), \\
\overline{C}_*^{GM,p}(K) & \text{if } * > \ell - 1 - p(\ell).
\end{cases}
$$

the previous calculations of perverse degree give an isomorphism

$$
\psi: D_* = \left( C_* (\Delta^k) \otimes \tau_\ell \overline{C}_*(K) \right) \oplus \left( R \otimes \overline{C}_{*-1}^{GM,p}(K) \right) \rightarrow \overline{C}_{*-1}^{GM,p}(\Delta^k \ast K).
$$

For computing the homology of $D_*$, we decompose it in the next short exact sequence,

1. $\begin{array}{c}
0 \rightarrow R \otimes \overline{C}_{*-1}^{GM,p}(K) \rightarrow D_* \rightarrow C_* (\Delta^k) \otimes \tau_\ell \overline{C}_*(K) \rightarrow 0
\end{array}$

In the first term, on the left, the factor $R$ is concentrated in degree -1 and the exact sequence can be rewritten as

$$
\begin{array}{c}
0 \rightarrow \overline{C}_{*-1}^{GM,p}(K) \rightarrow D_* \rightarrow C_* (\Delta^k) \otimes \tau_\ell \overline{C}_*(K) \rightarrow 0.
\end{array}
$$
The associated long exact sequence with its connecting map, $\delta$, appears as
\[
\ldots \to H_{i+1}(C^{\text{GM},\tau}(K)) \to H_i(D) \to H_i(C^{\text{GM},\tau}(K)) \xrightarrow{\delta} H_i(C^{\text{GM},\tau}_\ell(K)) \to H_{i-1}(D) \to \ldots
\]
The connecting map, $\delta$, is induced by the canonical inclusion $\tau^{\text{GM},\tau}_\ell(K) \hookrightarrow C^{\text{GM},\tau}(K)$.
By definition of $\tau^{\text{GM},\tau}_\ell(K)$ this inclusion induces an injection in homology. Thus, with the isomorphism (11), we have
\[
H^i_{\text{GM},\tau}(\Delta^k \ast K; R) = \text{Coker} \left( H_i(\tau^{\text{GM},\tau}_\ell(K)) \to H^i_{\text{GM},\tau}(K; R) \right)
\]
\[
= \begin{cases} 
H^i_{\text{GM},\tau}(K; R) & \text{if } i \leq \ell - 2 - \overline{p}(\ell), \\
\text{Ext}(H^i_{\text{GM},\tau}(K; R), R) & \text{if } i = \ell - 1 - \overline{p}(\ell) \\
H^i_{\text{GM},\tau}(K; R) & \text{if } i \geq \ell - 2 - \overline{p}(\ell).
\end{cases}
\]
This argument determines the reduced cohomology of $\Delta^k \ast K$. The non-reduced one has an extra term $R$ in degree $i = 0$. In the case $i \leq \ell - 2 - \overline{p}(\ell)$, this component appears in $H^i_{\text{GM},\tau}(K; R)$ but for $0 \geq \ell - 1 - \overline{p}(\ell)$, we have to add $H^0_{\text{GM},\tau}(\Delta^k \ast K; R) = R$, as stated.

We may note that the isomorphism $H^i_{\text{GM},\tau}(\Delta^k \ast K; R) \cong H^i_{\text{GM},\tau}(K; R)$, obtained for low degrees in Proposition 4.2, is induced by the inclusion $K \hookrightarrow \Delta^k \ast K$.

In the particular case of the cone on a smooth manifold, the next result already appears in [42].

**Corollary 4.3.** Let $R$ be a principal ideal domain. Let $K$ be a filtered face set of depth $v(K) \leq \ell - 1$, with $\ell \in \{1, \ldots, n\}$, and $\Delta^k \ast K$ be a join of depth $\ell$ with $k \geq 0$. For any perversity $\overline{p}$ and any $k \in \mathbb{N}$, we have
\[
H^i_{\text{GM},\overline{p}}(\Delta^k \ast K; R) = \begin{cases} 
H^i_{\text{GM},\overline{p}}(K; R) & \text{if } i \leq \ell - 2 - \overline{p}(\ell), \\
\text{Ext}(H^i_{\text{GM},\overline{p}}(K; R), R) & \text{if } i \geq \ell - 1 - \overline{p}(\ell) \text{ and } i = 0, \\
H^i_{\text{GM},\overline{p}}(K; R) & \text{if } i = \ell - 1 - \overline{p}(\ell) \text{ and } i \neq 0, \\
0 & \text{if } i \geq \ell - 2 - \overline{p}(\ell) \text{ and } i \neq 0.
\end{cases}
\]

**Proof.** As $R$ is a principal ideal domain, the complex $C^p_{\text{GM},\overline{p}}(K; R)$ is free as $R$-module and we may use the universal coefficients formula. The announced results for $i = \ell - 1 - \overline{p}(\ell)$ and $i \geq \ell - 2 - \overline{p}(\ell)$ are direct consequences of this formula and Proposition 4.2.

For the last case, $i \leq \ell - 2 - \overline{p}(\ell)$, we consider the morphism of short exact sequences induced by the filtered face set map $K \to \Delta^k \ast K$:
\[
0 \to \text{Ext}(H^i_{\text{GM},\overline{p}}(\Delta^k \ast K), R) \to H^i_{\text{GM},\overline{p}}(\Delta^k \ast K) \to \text{hom}(H^i_{\text{GM},\overline{p}}(\Delta^k \ast K), R) \to 0
\]
\[
0 \to \text{Ext}(H^i_{\text{GM},\overline{p}}(K), R) \to H^i_{\text{GM},\overline{p}}(K) \to \text{hom}(H^i_{\text{GM},\overline{p}}(K), R) \to 0
\]
As the left-hand side and right-hand side arrows are isomorphisms, the middle one is an isomorphism also.

**Theorem B.** Let $R$ be a field. Let $K$ be a filtered face set, $\overline{p}$ and $\overline{q}$ be two perversities such that $\overline{q} \geq 0$ and $\overline{p}(i) + \overline{q}(i) = i - 2$ for any $i \in \{1, \ldots, n\}$. Then, the GM-cochain complex, $C^p_{\text{GM},\overline{p}}(K)$, and the Thom-Whitney complex, $\tilde{C}^p_{\overline{q}}(K)$, are related by a quasi-isomorphism, i.e., $H^i_{\text{GM},\overline{p}}(K; R) \cong H^{i+1}_{\text{TW},\overline{q}}(K; R)$. 

\[\]
Proof. First, we define \( \tilde{\chi}: \tilde{C}^s(K) \to \text{hom}(C^*(GM(K), R)) \). If \( \sigma: \Delta = \Delta_0 \ldots \Delta_n \to K_+ \), recall that \( \tilde{C}^s(K)_\sigma = C^*(c\Delta_0) \otimes \cdots \otimes C^*(\Delta_n) \). If \( \Phi \in \tilde{C}^s(K) \), we may write \( \Phi_\sigma = \sum_i \Phi_{0,\sigma,i} \otimes \cdots \otimes \Phi_{n,\sigma,i} \in C^*(c\Delta_0) \otimes \cdots \otimes C^*(\Delta_n) \). The cochains \( \Phi_{k,\sigma,i} \) can be evaluated on the maximal simplex in each factor and we set
\[
\chi(\Phi)(\sigma) = \sum_i \Phi_{0,\sigma,i}([c\Delta_0]) \cdots \Phi_{n,\sigma,i}([\Delta_n]).
\]
Denote by \( \Phi_i = \Phi_{0,\sigma,i} \otimes \cdots \otimes \Phi_{n,\sigma,i} \). The compositions of \( \chi \) with the differentials verify
\[
\chi(d\Phi_i)(\sigma) = \sum_{k=0}^n \pm \Phi_{0,\sigma,i}([c\Delta_0]) \cdots d\Phi_{k,\sigma,i}([c\Delta_k]) \cdots \Phi_{n,\sigma,i}([\Delta_n]),
\]
d\( \chi(\Phi_i)(\sigma) = \chi(\Phi_i)(\partial\sigma)
\]
\[
= \sum_{k=0}^n \pm \Phi_{0,\sigma,i}([c\Delta_0]) \cdots \Phi_{k,\sigma,i}([c\partial\Delta_k]) \cdots \Phi_{n,\sigma,i}([\Delta_n]).
\]
From the definition of the differential of a cochain, we have \( d\Phi_{k,\sigma,i}(\partial\Delta_k) = \Phi_{k,\sigma,i}(\partial[c\Delta_k]) \). The sign convention of the boundary operator of a cone is given by \( \partial[c\Delta_k] = c\partial[\Delta_k] + (-1)^{j+1}[\Delta_k \times \{1\}] \). Therefore, to get the equality \( d\chi(\Phi) = \chi(d\Phi) \), we have to prove the nullity of the products
\[
\Pi_i^k = \Phi_{0,\sigma,i}([c\Delta_0]) \cdots \Phi_{k,\sigma,i}([\Delta_k \times \{1\}]) \cdots \Phi_{n,\sigma,i}([\Delta_n]),
\]
for any \( k \in \{0, \ldots, n-1\} \). Suppose \( \Delta_k \neq \emptyset \) and the restriction of \( \Phi_{k,\sigma,i} \) to \( \Delta_k \times \{1\} \) not equal to 0. For having \( \Pi_i^k \neq 0 \), the perverse degrees of \( \Phi_i \) must verify
\[
\|\Phi_i\|_{n-k} = \dim(c\Delta_{k+1} \times \cdots \times c\Delta_{n-1} \times \Delta_j),
\]
which is equivalent to
\[
\|\Phi_i\|_{n-k} + \|\sigma\|_{n-k} = \dim \Delta - 1.
\]
If \( \Phi_i \in \tilde{C}^s_\sigma(K) \) and \( \sigma \in C^s_{GM}(K) \), their perverse degrees verify
\[
\|\Phi_i\|_{n-k} \leq \sigma(n-k) \ 	ext{and} \ \|\sigma\|_{n-k} \leq \dim \Delta - (n-k) + \sigma(n-k)
\]
which imply
\[
\|\Phi_i\|_{n-k} + \|\sigma\|_{n-k} \leq \dim \Delta - (n-k) + \sigma(n-k) + \sigma(n-k) \leq \dim \Delta - 2.
\]
Thus the condition (12) cannot be satisfied and the products \( \Pi_i^k \) are equal to 0 for any \( k \in \{0, \ldots, n-1\} \). We have proved the compatibility of the map
\[
\chi: \tilde{C}^s_\sigma(K) \to \text{hom}(C^*_{GM,\sigma}(K), R)
\]
with the differentials. The conclusion is a recollection of previous results.

- From Proposition 3.5 and the fact that \( C^* \) is an extendable universal system of coefficients \( (29, \text{Chapter 14}) \), \( \tilde{C}^s_\sigma(K) \) is a filtered theory of cochains.
- Proposition 4.1 says that \( C^*_{GM,\sigma} \) is a filtered theory of cochains too.
As we are working over a field and with a perversity, \( \overline{p} \), the conclusion of Corollary 4.3 simplifies in
\[
H^i_{\text{GM,F}}(\Delta^k * K; R) = \begin{cases} 
H^i_{\text{GM,F}}(K; R) & \text{if } i \leq \ell - 2 - \overline{p}(\ell), \\
0 & \text{if } i \geq \ell - 1 - \overline{p}(\ell).
\end{cases}
\]
(Observe that, with the hypotheses on \( \overline{p} \) and \( \overline{q} \), we cannot have \( 0 = \ell - 1 - \overline{p}(\ell) \).) This computation coincides with Corollary 3.8.

As noted before, the equality in low degrees between the cohomology of \( K \) and the cohomology of \( \Delta^k * K \) is induced by the inclusion \( K \hookrightarrow \Delta^k * K \). Therefore, a quasi-isomorphism, \( \tilde{C}^*(K) \to \text{hom}(C^\text{GM}_*(K), R) \), induced by a natural map, gives also a quasi-isomorphism \( \tilde{C}^*(\Delta^k * K) \to \text{hom}(C^\text{GM}_*(\Delta^k * K), R) \).

Finally, Proposition 3.3 implies that the map \( \chi \) is a quasi-isomorphism.

Remark 4.4. The hypothesis \( R \) is a field is used for having an isomorphism between \( H^*_{\text{GM,F}}(\Delta^k * K; R) \) and \( H^*_{\text{TW,F}}(\Delta^k * K; R) \), by killing the Ext-term. We may observe also that this Ext-term may also be avoided in a particular case on a principal ideal domain.

More precisely, let \( \overline{t} \) be the perversity defined by \( \overline{t}(i) = i - 2 \), for any \( i \in \{1, \ldots, n\} \). This perversity coincides with the top perversity \( \overline{t} \) (Definition 2.1) for any \( i \in \{2, \ldots, n\} \) and the difference between \( \overline{t} \) and \( \overline{t} \) does not matter if we work with filtered spaces without strata of codimension 1, as it is the case for pseudomanifolds. Directly from the previous proof, for any filtered face set, \( K \), and any principal ideal domain, \( R \), we have an isomorphism
\[
H^*_{\text{TW,F}}(K; R) \cong H^*_{\text{GM,F}}(K; R),
\]
since, in Corollary 4.3, the Ext-term appears in homological degree 0 and we have \( \text{Ext}(H^0_{\text{GM,F}}(K; R), R) = 0 \).

The first part of the next corollary is a direct consequence of Theorem 3 and Corollary 2.19. The second part follows from Proposition C.4 and [32].

Corollary 4.5. Let \( R = \mathbb{Q} \). Let \( K \) be a filtered face set, \( \overline{p} \) and \( \overline{q} \) be perversities such that \( \overline{q} \geq 0 \) and \( \overline{p}(i) + \overline{q}(i) = i - 2 \) for any \( i \in \{1, \ldots, n\} \). Then the complexes \( A\text{PL,}{\overline{q}}(K) \), \( \tilde{C}^*_\overline{q}(K) \) and \( C^\text{GM}_*(K) \) are related by quasi-isomorphisms.

In the particular case that \( \overline{p} \) and \( \overline{q} \) are GM-perversities and \( K \) is the filtered face set associated to a pseudomanifold, \( X \), the homology of these complexes coincides with the original Goresky-MacPherson intersection cohomology of \( X \).

Remark 4.6. The simplices in \( K \backslash K_+ \) (i.e., \( j_n = -1 \)) are not considered in \( \tilde{C}^*_\overline{q}(K) \), by definition, and they do not appear in \( C^\text{GM}_*(K) \) if \( \overline{q}(1) \geq 0 \), or equivalently if \( \overline{p}(1) \leq -1 \). Indeed, any \( \overline{p} \)-admissible simplex \( \sigma : \Delta = \Delta^j_0 * \cdots * \Delta^j_n \to K \) verifies
\[
\|\sigma\|_1 = \dim(\Delta^j_0 * \cdots * \Delta^j_{n-1}) \leq \dim \Delta - 1 + \overline{p}(1) \leq \dim \Delta - 2.
\]
If \( \Delta^j_0 * \cdots * \Delta^j_{n-1} \neq \emptyset \), we get \( 2 \leq \dim \Delta - \dim(\Delta^j_0 * \cdots * \Delta^j_{n-1}) = j_n + 1 \) and \( j_n \geq 1 \). This implies also that all the \( \overline{p} \)-admissible 0-simplices and 1-simplices belong to the regular part of \( K \).
5. PARTICULAR CASES: $q = \emptyset$, $q = \infty$, CONE AND SUSPENSION

We characterize the Thom-Whitney intersection cohomology in the cases $q = \infty$ and $q = \emptyset$. For $q = \emptyset$, we need to introduce the normalization of a filtered face set, as Goresky and MacPherson do (see [24]) in the framework of pseudomanifolds. Here, the connectivity of the link is not sufficient, we need to use the expanded link.

Recall from Definition 1.9 that the face set $K^{[0]}$ is the regular part of a filtered face set, $K$.

**Proposition 5.1.** Let $R$ be a commutative ring. For any filtered face set, $K$, the restriction map, $\tilde{C}^*_\infty(K) \to C^*(K^{[0]})$, induces an isomorphism

$$H^*_{TW,\infty}(K; R) \cong H^*(K^{[0]}; R),$$

where the last term is the ordinary cohomology of the face set $K^{[0]}$.

**Proof.** If $\omega \in \tilde{C}^*_\infty(K)$, we have an element $\omega_\sigma \in C^*(c\Delta^{j_0}) \otimes \cdots \otimes C^*(c\Delta^{j_{n-1}}) \otimes C^*(\Delta^1)$, for each $\sigma: \Delta^{j_0} \ast \cdots \ast \Delta^{j_n} \to K_\sigma$. The simplices $\sigma$ of $K^{[0]}$ correspond to the particular case $j_i = -1$ for all $i \leq n - 1$. A global section, still denoted $\omega \in C^*(K^{[0]})$, is thus defined by restriction from $K_\sigma$ to $K^{[0]}$.

We check easily that $K \mapsto \tilde{C}^*_\infty(K)$ and $K \mapsto C^*(K^{[0]})$ are filtered theory of cochains, see Proposition 3.5. Let $K$ be a filtered face set of depth $v(K) = \ell - 1$, $\ell \in \{1, \ldots, n\}$, and such that $\tilde{C}^*_\infty(K) \to C^*(K^{[0]})$ is a quasi-isomorphism. Let $\Delta^k \ast K$ be the join of depth $\ell$. Using Proposition 3.3, we are reduced to prove that $\tilde{C}^*_\infty(\Delta^k \ast K) \to C^*((\Delta^k \ast K)^{[0]})$ is a quasi-isomorphism. On the right-hand side, we have $(\Delta^k \ast K)^{[0]} = K^{[0]}$. On the left-hand side, with Corollary 3.8, we know that

$$H^*_{TW,\infty}(\Delta^k \ast K; R) \cong H^*_\infty(K^{[0]}; R).$$

This isomorphism being induced by the canonical inclusion $K \hookrightarrow \Delta^k \ast K$, the hypotheses of Proposition 3.3 are fulfilled and the statement is proved.

For the study of the $\emptyset$-cohomology, we introduce a concept similar to the normalization of Goresky and MacPherson in [24, Page 151]. We denote by $\sigma \triangleleft \phi$ the relation "$\sigma$ is a face of $\phi$".

**Definition 5.2.** A filtered face set, $K$, is called **normal** if the two following conditions are satisfied.

(a) Any simplex $\sigma \in K \setminus K_+$ is a face of a simplex $\phi \in K_+$, i.e., $\sigma \triangleleft \phi$.

(b) Moreover, the simplex $\phi \in K_+$ is unique in the following sense.

Let $\sigma \in K \setminus K_+$. For any pair $(\phi, \phi')$ of simplices of $K_+$ such that $\sigma \triangleleft \phi$ and $\sigma \triangleleft \phi'$, there exists a family $\phi_1, \ldots, \phi_m$ of simplices of $K_+$ such that $\sigma \triangleleft \phi_i$ for $i \in \{1, \ldots, m\}$ and $\phi \triangleleft \phi_1 \ast \cdots \ast \phi_m \ast \phi'$.

Recall the expanded link, $L^{exp}(K, \tau)$, introduced in Definition 1.15.

**Proposition 5.3.** Let $K$ be a normal filtered face set. For any $r > 0$, $k \geq 0$ and $\tau \in \mathcal{J}(K, [r], k)$, the expanded link, $L^{exp}(K, \tau)$, is a connected, non empty and normal filtered face set.
Proof. As $r > 0$, we have $\tau \in K \setminus K_-$ and there exists $\sigma \in K_+$ such that $\tau \prec \sigma$. We may suppose $R_1(\sigma) = \tau$. As $\sigma \neq \tau$, we have $(R_2(\sigma), \sigma) \in L^{\exp}(K, \tau)$ and $L^{\exp}(K, \tau) \neq \emptyset$.

We prove now the connectivity of $L^{\exp}(K, \tau)$ by connecting any element $(R_2(\sigma'), \sigma') \in L^{\exp}(K, \tau)$ to $(R_2(\sigma), \sigma)$. We consider two cases.

- If $\sigma' \in K_+$, as $R_1(\sigma') = R_1(\sigma) = \tau$, we may apply the uniqueness axiom of normal filtered face set. There exists a family $\phi_1, \ldots, \phi_m$ of simplices of $K_+$ such that $\tau \prec \phi_i$ for $i \in \{1, \ldots, m\}$ and $\sigma \prec \phi_1 \cdots \prec \phi_m \prec \sigma'$. We may suppose $R_1(\phi_i) = \tau$ and, as $\phi_i \neq \tau$, we have $(R_2(\phi_i), \phi_i) \in L^{\exp}(K, \tau)_+$ for $i \in \{1, \ldots, m\}$. This implies the next relations in $L^{\exp}(K, \tau)$,

  

  $$(R_2(\sigma), \sigma) \prec (R_2(\phi_1), \phi_1) \cdots \prec (R_2(\phi_m), \phi_m) \prec (R_2(\sigma'), \sigma').$$

- If $\sigma' \not\in K_+$, there exists $\phi \in K_+$ such that $\sigma' \prec \phi$. As $R_1(\sigma') = \tau$, we may suppose $R_1(\phi) = \tau$ and we get $(R_2(\phi), \phi) \prec (R_2(\phi), \phi)$ with $(R_2(\phi), \phi) \in L^{\exp}(K, \tau)_+$. This gives us the first property of a normal filtered face set. For the second one, consider $(R_2(\sigma), \sigma) \in L^{\exp}(K, \tau) \setminus L^{\exp}(K, \tau)_+$, $(R_2(\phi), \phi) \in L^{\exp}(K, \tau)_+$ and $(R_2(\phi'), \phi') \in L^{\exp}(K, \tau)_+$ such that

  

  $$(R_2(\phi), \phi) \succ (R_2(\sigma), \sigma) \prec (R_2(\phi'), \phi').$$

We deduce $\phi$ and $\phi'$ in $K_+$, $\sigma \in K \setminus K_+$, $R_1(\phi) = R_1(\phi') = R_1(\sigma) = \tau$ and $\phi \succ \sigma \prec \phi'$. From the uniqueness condition in a normal filtered face set, we deduce the existence of a family $\phi_1, \ldots, \phi_m$ of simplices of $K_+$ such that $\sigma \prec \phi_i$ for $i \in \{1, \ldots, m\}$ and $\phi \prec \phi_1 \cdots \prec \phi_m \prec \phi'$. We may suppose $R_1(\phi_i) = \tau$ which gives $(R_2(\sigma), \sigma) \prec (R_2(\phi_i), \phi_i)$ and $(R_2(\phi_i), \phi_i) \in L^{\exp}(K, \tau)_+$ for all $i \in \{1, \ldots, m\}$, and

  

  $$(R_2(\phi), \phi) \prec (R_2(\phi_i), \phi_i) \cdots \prec (R_2(\phi_m), \phi_m) \prec (R_2(\phi'), \phi').$$

$\square$

Proposition 5.4. Let $R$ be a principal ideal domain and $\overline{t}$ be the perversity defined by $\overline{t}(i) = i - 2$. For any normal filtered face set, $K$, we have isomorphisms,

$$H^*_\text{TW,0}(K; R) \cong H^*_\text{GM,0}(K; R) \cong H^*(K; R),$$

where the last term is the ordinary cohomology of the face set underlying $K$.

Proof. The first isomorphism comes directly from Theorem 3.3 and Remark 4.4.

For the second one, we use the canonical inclusion, $C^*_\text{GM,0}(K) \rightarrow C_*(K)$, to define, by duality, a cochain map, $C^*(K) \rightarrow C^*_\text{GM,0}(K)$. Using the classical theory of ordinary cochains on a face set, we know that the association $K \mapsto C^*(K)$ is a filtered theory of cochains. (The proof of Proposition 4.1 in which all perverse degrees are taking out, gives an explicit confirmation of this fact.) Therefore, $C^*(K) \rightarrow C^*_\text{GM,0}(K)$ is a cochain map between two filtered theories of cochains. As $K$ is normal, its expanded links are connected (cf. Proposition 3.3) and an application of Proposition 3.4 gives the second isomorphism. $\square$
Remark 5.5. The cochain map, $C^*(-) \to C^*_{GM,F}(-)$, is not cone-compatible in general. Indeed, if $L$ is a filtered face set, non connected as face set, such that $C^*(L) \to C^*_{GM,F}(L)$ is a quasi-isomorphism, we have $H^0(\Delta^k \ast L; R) = R$, because $\Delta^k \ast L$ is connected as face set, and $H^0_{GM,F}(\Delta^k \ast L; R) \cong H^0_{GM,F}(L; R) \cong H^0(L; R) \neq R$. Thus, we must restrict to connected expanded links on this point. That justifies the use of Proposition [5.4] in the proof of Proposition [5.4]. (Example 5.8 comes back on this point.)

We prove now that we can associate to any filtered face set a unique normal filtered face set, with the same intersection cohomology.

**Definition 5.6.** The normalization of a filtered face set, $K$, is a map of filtered face sets, $N: N(K) \to K$, such that $N(K)$ is normal and the restriction map, $N: N(K)_+ \to K_+$, is an isomorphism.

**Proposition 5.7.** Any filtered face set, $K$, admits a unique normalization defined by $N(K) = K_+ \cup \{ (\sigma, \phi) \mid \sigma \in K \setminus K_+, \phi \in K_+ \text{ and } \sigma \triangleleft \phi \}$, where

- $\langle -, - \rangle$ denotes the equivalence class for the equivalence relation generated by \( (\sigma, \phi) \sim (\sigma, \phi') \iff \phi \triangleleft \phi' \),
- the perverse degree of elements of $K_+$ is kept and \( \| (\sigma, \phi) \|_i = \| \sigma \|_i \).

Moreover, if $F$ is an extendable universal system of differential coefficients, over a commutative ring $R$, of blow-up $\tilde{F}$, the map $N$ induces

- an isomorphism $H^*_\mathbb{P}(N(K); \tilde{F}) \cong H^*_{\mathbb{P}}(K; \tilde{F})$, for any loose perversity $\overline{\mathbb{P}}$ or $\overline{\mathbb{P}} = \infty$,
- an isomorphism $H^*_{GM,F}(N(K); R) \cong H^*_{GM,F}(K; R)$, for the Goresky-MacPherson homology if $\mathbb{P}$ is a loose perversity such that $\mathbb{P}(1) < 0$.

**Proof.** Let $\sigma: \Delta^j_0 \ast \cdots \ast \Delta^j_k \to K$ be a simplex with $k < n$ and $\phi \in K_+$ with $\sigma \triangleleft \phi$. The simplex $(\sigma, \phi)$ of $N(K)$ is considered also as $(\sigma, \phi): \Delta^j_0 \ast \cdots \ast \Delta^j_k \to N(K)$. A simplex of this type cannot be in $N(K)_+$ and we have $N(K)_+ = K_+$. We specify now the definition of a face operator, $\partial_i$, by considering three cases:

- if $\alpha \in K_+$ is a simplex whose $\partial_i$-face in $K$ belongs to $K_+$, we keep the same face operator than in $K$, 
- if $\alpha \in K_+$ is a simplex whose $\partial_i$-face in $K$ belongs to $K \setminus K_+$, we set $\partial_i \alpha = \langle \partial_i \alpha, \alpha \rangle$ (in this case, $\partial_i$ is the last face operator of $\alpha$), 
- we set $\partial_i \langle \sigma, \phi \rangle = \langle \partial_i \sigma, \phi \rangle$ otherwise.

Let $\alpha: \Delta = \Delta^j_0 \ast \cdots \ast \Delta^j_m \to K$ with $j = \text{dim } \Delta$. For the commutation rule of face operators, we can reduce the verification to the next two cases:

- if $j_n = 0$ and $i < j$, we have $\partial_i \partial_j \alpha = \langle \partial_i \partial_j \alpha, \alpha \rangle = \langle \partial_j \partial_i \alpha, \alpha \rangle = \langle \partial_j \partial_{i-1} \alpha, \partial_i \alpha \rangle = \langle \partial_{j-1} \partial_i \alpha, \partial_i \alpha \rangle = \partial_{j-1} \partial_i \alpha$,
- if $j_n = 1$ and $i = j - 1$, we have $\partial_i \partial_j \alpha = \langle \partial_i \partial_j \alpha, \partial_j \alpha \rangle = \langle \partial_i \partial_j \alpha, \alpha \rangle = \langle \partial_j \partial_i \alpha, \partial_i \alpha \rangle = \langle \partial_{j-1} \partial_i \alpha, \partial_i \alpha \rangle = \partial_{j-1} \partial_i \alpha$. 


We first have to prove that \( \langle \sigma, \phi \rangle \in N(K) \setminus N(K)_+ \) with \( \phi : \Delta^n * \cdots * \Delta^n \to K_+ \). Let \( J \) be the smallest subset of indices such that \( \partial J \phi \notin K_+ \) and choose \( I \) such that \( \partial I \partial J \phi = \sigma \). By definition of face operators, we have
\[
\partial I \partial J \phi = \partial I \langle \partial J \phi, \phi \rangle = \langle \partial I \partial J \phi, \phi \rangle = \langle \sigma, \phi \rangle
\]
and \( \langle \sigma, \phi \rangle \ll \phi \), which means that \( \langle \sigma, \phi \rangle \) is a face of an element of \( N(K)_+ \). We study now the uniqueness property of the definition of normal filtered face set. Let \( \langle \sigma, \phi \rangle \in N(K) \setminus N(K)_+ \) and \( \beta, \beta' \in N(K)_+ \) such that \( \langle \sigma, \phi \rangle \ll \beta \) and \( \langle \sigma, \phi \rangle \ll \beta' \). Let \( J \) be the smallest subset such that \( \partial J \beta \notin K_+ \) and \( I \) such that \( \langle \sigma, \phi \rangle = \partial I \partial J \beta \). By definition of the face operator, we have
\[
\langle \sigma, \phi \rangle = \partial I \partial J \beta = \partial I \langle J \beta, \beta \rangle = \langle \partial I \partial J \beta, \beta \rangle,
\]
which implies \( \sigma = \partial I \partial J \beta \) (in \( K_! \)) and \( \langle \sigma, \phi \rangle = \langle \sigma, \beta \rangle \). Similarly, we prove \( \langle \sigma, \beta' \rangle = \langle \sigma, \phi \rangle \) and deduce
\[
\langle \sigma, \phi \rangle = \langle \sigma, \beta \rangle = \langle \sigma, \beta' \rangle.
\]
By definition of the equivalence relation, there exists a family of elements of \( K_+ = N(K)_+, (\beta_1, \ldots, \beta_p, \ldots, \beta_m) \), with
\[
\beta \ll \beta_1 \gg \cdots \gg \beta_p \gg \phi \ll \beta_{p+1} \gg \cdots \gg \beta_m \gg \beta' \quad \text{and} \quad \sigma \ll \beta_i,
\]
for all \( i \in \{1, \ldots, m\} \). We are reduced to prove that \( \langle \sigma, \phi \rangle \ll \beta_i \), for all \( i \in \{1, \ldots, m\} \). Observe that the relations (13) imply \( \langle \sigma, \phi \rangle = \langle \sigma, \beta_i \rangle \), for all \( i \in \{1, \ldots, m\} \). Let \( J \) be the smallest subset of indices such that \( \partial I \beta_i \notin K_+ \) and \( I \) such that \( \partial I \partial J \beta_i = \sigma \). By definition of the face operators, we get
\[
\partial I \partial J \beta_i = \partial I \langle J \beta_i, \beta_i \rangle = \langle \partial I \partial J \beta_i, \beta_i \rangle = \langle \sigma, \beta_i \rangle,
\]
which implies \( \langle \sigma, \beta_i \rangle \ll \beta_i \). We have obtained \( \langle \sigma, \phi \rangle = \langle \sigma, \beta_i \rangle \ll \beta_i \).

A map of filtered face sets, \( N : N(K) \to K_+ \), is defined by \( N(\alpha) = \alpha \), if \( \alpha \in K_+ \) and \( N(\langle \sigma, \phi \rangle) = \sigma \). The verification of the compatibility with face operators is direct from the definitions and the restriction of \( N \) is the identity map from \( N(K)_+ \) to \( K_+ \).

Suppose now that \( N : N(K) \to K_+ \) and \( N' : N'(K) \to K_+ \) are two normalizations of \( K_+ \). We construct a map of filtered face sets, \( \mathcal{M} : N(K) \to N'(K) \), such that \( N' \circ \mathcal{M} = N \), by:
\[
\mathcal{M}(\alpha) = N'^{-1}(N(\alpha)), \quad \text{if} \ \alpha \in N(K)_+,
\]
\[
\mathcal{M}(\alpha) = \partial I(N'^{-1}(N(\phi))), \quad \text{if} \ \alpha = \partial I \phi \in N(K) \setminus N(K)_+ \quad \text{and} \ \phi \in N(K)_+.
\]
We first have to prove that \( \mathcal{M} \) is well defined in the case \( \alpha \notin N(K)_+ \). Let \( \phi' \in N(K)_+ \) such that \( \alpha = \partial I \phi' \). We may suppose that \( \alpha \ll \phi \ll \phi' \) and set \( \phi = \partial I \phi' \). We have
\[
\alpha = \partial I \partial J \phi' \quad \text{and} \quad \partial I(N'^{-1}(N(\phi))) = \partial I(N'^{-1}(N(\partial J \phi'))) = \partial I \partial J(N'^{-1}(N(\phi'))),
\]
which proves that \( \mathcal{M} \) is well defined.

We establish now the bijectivity of the map \( \mathcal{M} \). A map \( \mathcal{M}' : N'(K) \to N(K) \) is defined in a similar manner and, on \( N(K)_+ \), they are obviously inverse. If \( \alpha = \partial I \phi \), we have
\[
\mathcal{M}(\mathcal{M}'(\partial I(N'^{-1}(N(\phi)))) = \partial I(N^{-1}(N'(N'^{-1}(N(\phi)))))) = \partial I \phi = \alpha.
\]
The map $\mathcal{R}$ preserves the perverse degree. That is obvious on $N(K)_+$. Let $\alpha \in N(K)_-$. Then there exists $\phi \in N(K)_-$ and we may suppose $\alpha = \partial_\alpha \phi$, with $m = |\phi|$. This implies

$$||\mathcal{R}(\alpha)||_\ell = ||\partial_\alpha N^{\ell-1}(N(\phi))||_\ell = ||N^{\ell-1}(N(\phi))||_\ell - 1 = ||\phi||_\ell - 1 = ||\alpha||_\ell.$$ 

For proving the compatibility of $\mathcal{R}$ with face operators, we consider three cases.

- It is direct if $\alpha$ and $\partial_\alpha$ are in $N(K)_-$. 
- Let $\alpha \in N(K)_+$ with $\partial_\alpha \phi \notin N(K)_+$. Then, we have
  $$\partial_\alpha \mathcal{R}(\alpha) = \partial_\alpha (N^{\ell-1}(N(\phi))) = \mathcal{N}(\partial_\alpha \phi).$$
- Let $\alpha \notin N(K)_+$. There exists $\phi \in N(K)_+$ with $\alpha = \partial_1 \phi$ and we have
  $$\partial_1 \mathcal{R}(\alpha) = \partial_1 \mathcal{R}(N^{\ell-1}(N(\phi))) = \mathcal{N}(\partial_1 \phi) = N(\alpha).$$

The equality $N' \circ \mathcal{R} = N$ is obvious on $N(K)_+$. If $\alpha = \partial_1 \phi$, $\alpha \notin N(K)_+$ and $\phi \in N(K)_+$, we have

$$N'(\mathcal{R}(\alpha)) = N'(\partial_1 (N^{\ell-1}(N(\phi)))) = \partial_1 N(\phi) = N(\partial_1 \phi) = N(\alpha).$$

We have established the existence and uniqueness of the normalization. We show now the existence of isomorphisms in cohomology and homology.

The restriction map of $N$ to $N(K)_+$ being an isomorphism, $N(K)_+ \cong K_+$, the map $\mathcal{N}$ induces an isomorphism for the cohomology with coefficients in $F_q$, for any loose perversity $\underline{q}$ or $\underline{q} = \infty$.

Let $\sigma : \Delta = \Delta^0 \star \cdots \star \Delta^m \to N(K)$ be a $\overline{p}$-admissible simplex for a loose perversity $\overline{p}$ such that $\overline{p}(1) < 0$. From $||\sigma||_1 = \dim(\Delta^0 \star \cdots \star \Delta^m-1)$ and

$$||\sigma||_1 \leq \dim \Delta - 1 + \overline{p}(1) \leq \dim \Delta - 2,$$

we deduce $j_n \geq 1$ and $\sigma \in K_+ \cong N(K)_+$. A similar argument works for $\partial \sigma$ and we get an isomorphism of chain complexes, $C_{GM,\overline{p}}^*(N(K)) \cong C_{GM,\overline{p}}^*(K)$.

\[ \Box \]

\textbf{Example 5.8.} This example shows that the connectivity of $L(K, \tau)$ is not sufficient for having a quasi-isomorphism between $C(K)$ and $C_{\overline{p}}(K)$. Consider a face set, $K_+$, which looks like a pinch ribbon. It is formed of two triangles, $\Delta^2_1$ and $\Delta^2_2$, with a common vertex and the opposite edge in common also. We can represent it as

```
  a
 / \
|   |
|   |
 b
```

with $|ad| = |be|$. We decompose the two triangles as $\{c\} * [ad]$ and $\{c\} * [be]$ creating a filtered face set still denoted $K_+$. The blow-up of these triangles are $\Delta^2_1 = c \Delta^0 \times \Delta^1 = \Delta^1_1 \times \Delta^0$ and $\Delta^2_2 = c \Delta^0 \times \Delta^1 = \Delta^1_2 \times \Delta^0$, with $\Delta^1_1 \neq \Delta^1_2$ and the second factor in common. The cochains on these blow-ups (compatible with the restriction to the common faces) have the behaviour of cochains on a rectangle. Thus, there is no cohomology of degree 1 in perverse degree $\overline{0}$, i.e., $H^1_{TW,\overline{0}}(K; \mathbb{Z}) = 0$. The cohomology of the face set $K_+$ has a generator in degree 1 and, in this case, we have $H^1_{TW,\overline{5}}(K; \mathbb{Z}) \neq 0$. The cochains on these blow-ups (compatible with the restriction to the common faces) have the behaviour of cochains on a rectangle.
$H^1(K; \mathbb{Z})$. It is easy to check that the link of \{c\} is connected but its expanded link is not connected.

The normalization $N(K)$ of $K$ is formed of $K_+$ and of two 0-simplices $c_1 = \langle \{c\}, \Delta^2_{(1)} \rangle$ and $c_2 = \langle \{c\}, \Delta^2_{(2)} \rangle$. The determination of the other boundaries shows that in $N(K)$ the two triangles still have a common edge but the two cone points are now different. Thus $N(K)$ can be represented as

$$
c_1 \quad \bullet \quad a = b \quad \bullet \quad c = d \quad c_2
$$

with $N(K)_+ \cong K_+ \cong N(K) \setminus \{c_1, c_2\} \cong K \setminus \{c\}$.

**Example 5.9.** Here, we work over $\mathbb{Q}$. The cone on the face set $\mathcal{S}$ is the filtered face set $c\mathcal{S} = \{\vartheta\} \ast \emptyset \ast \cdots \ast \emptyset \ast \mathcal{S} \in \Delta^{|\mathcal{S}|}_0$. The simplices of $c\mathcal{S}$ are of three kinds, $\{\vartheta\} \ast \vartheta : \Delta^0 \ast \emptyset \ast \cdots \ast \emptyset \ast \Delta^{|\mathcal{S}|} \to c\mathcal{S}$, $\{\vartheta\} : \emptyset \ast \cdots \ast \emptyset \ast \emptyset \ast \Delta^{|\mathcal{S}|} \to c\mathcal{S}$, or $\vartheta : \emptyset \ast \cdots \ast \emptyset \ast \vartheta : \emptyset \ast c\mathcal{S}$, where $\vartheta \in \Delta^{|\mathcal{S}|}$. Observe that the $\{\vartheta\} \ast \vartheta$ and $\vartheta$ are the only elements of $c\mathcal{S}_+$. The blow-up of Sullivan’s forms on these simplices is defined by

$$
\tilde{A}_{PL}(c\mathcal{S})_{(t)\vartheta} = A_{PL}(c\Delta^0) \otimes A_{PL}(\Delta^{|\mathcal{S}|}) \text{ and } \tilde{A}_{PL}(c\mathcal{S})_{\vartheta} = \mathbb{Q} \otimes A_{PL}(\Delta^{|\mathcal{S}|}).
$$

As the compatibility to face operators involves only the face operators on $\{\vartheta\} \ast \emptyset \ast \cdots \ast \emptyset \ast \mathcal{S}$, we are reduced to the compatibility on the factors $A_{PL}(\Delta^{|\mathcal{S}|})$ and we get

$$
\tilde{A}_{PL}(c\mathcal{S}) = A_{PL}(\Delta^1) \otimes A_{PL}(\mathcal{S}) = \wedge(t, dt) \otimes A_{PL}(\mathcal{S}).
$$

Recall from Remark 2.9 that in this system of coordinates, the face $\{\vartheta\} \times \{1\}$ of $c\Delta^0$ corresponds to $t = 0$. Thus the perverse degrees are determined by $||t \otimes \omega|| = ||dt \otimes \omega|| = -\infty$, $||1 \otimes \omega|| = ||\omega||$, if $\omega \in A_{PL}(\mathcal{S})$ with $\omega \neq 0$.

As there is only one singular stratum, a loose perversity, $\bar{\vartheta}$, is given by a non-negative number, $\bar{\vartheta}(n)$. We want to determine the forms of $\bar{\vartheta}$-intersection. First, if this form has a component in $t$ or $dt$, there is no restriction on the second factor and all the elements of $\wedge^+(t, dt) \otimes A_{PL}(\mathcal{S})$ are $\bar{\vartheta}$-admissible. (Here, $\wedge^+(t, dt)$ is the sum of elements of the shape $t^n, t^\beta dt$, with $\alpha \geq 1$ and $\beta \geq 0$.) On the other hand, the element $1 \otimes \omega$ is $\bar{\vartheta}$-admissible if, and only if, $||\omega|| = \bar{\vartheta}(n)$. From that, we deduce that the set of forms of $\bar{\vartheta}$-intersection is

$$
(A_{PL}(\Delta^1) \otimes A_{PL}(\mathcal{S}))_{\bar{\vartheta}} = (\wedge^+(t, dt) \otimes A_{PL}(\mathcal{S})) \oplus A_{PL}^{\bar{\vartheta}(n)}(\mathcal{S}) \oplus \mathbb{Z}A_{PL}^{\bar{\vartheta}(n)}(\mathcal{S}).
$$

The CDGA $\wedge^+(t, dt)$ being contractible, there is a quasi-isomorphism,

$$
(\tilde{A}_{PL}(c\mathcal{S}))_{\bar{\vartheta}} \simeq A_{PL}^{\bar{\vartheta}(n)}(\mathcal{S}) \oplus \mathbb{Z}A_{PL}^{\bar{\vartheta}(n)}(\mathcal{S}) = \tau_{\leq \bar{\vartheta}(n)}A_{PL}(\mathcal{S}),
$$

where the truncation $\tau_{\leq \bar{\vartheta}(n)}$ is introduced in Definition 3.6. This last cochain complex involves only the CDGA $A_{PL}(\mathcal{S})$ and inherits a structure of CDGA. We may define a structure of strict perverse algebra on $A_{PL}(\mathcal{S})$ (see Definition 2.2) by setting $||\omega|| = \max(||\omega||, ||d\omega||)$, if $\omega \neq 0$. For this structure, we observe that $A_{PL}(\mathcal{S})_{\bar{\vartheta}} = A_{PL}^{\bar{\vartheta}(n)}(\mathcal{S}) \oplus \mathbb{Z}A_{PL}^{\bar{\vartheta}(n)}(\mathcal{S}) = \tau_{\leq \bar{\vartheta}(n)}A_{PL}(\mathcal{S})$. In conclusion, the blow-up of Sullivan’s forms on a cone, $c\mathcal{S}$, is related by a quasi-isomorphism to a strict perverse CDGA, defined on the PL-forms, $\tilde{A}_{PL}(\mathcal{S})$. 

We end this example with an explicit computation. In the case of the filtered face set associated to $cCP(2)$, only three values of $\overline{q}(n)$ suffice for the description of all the possible intersection cohomologies; they are $\overline{q}(n) = 0$, 2 and 4. If we denote by $\overline{\ell}$ the perversity such that $\overline{\ell}(n) = \ell$, we have

$$H^{i}_\ell(cCP(2); \mathbb{Q}) = \mathbb{Q}, \quad \text{if } i = 0 \text{ and } 0 \text{ otherwise},$$

$$H^{i}_\ell(cCP(2); \mathbb{Q}) = \mathbb{Q}, \quad \text{if } i = 0, 2 \text{ and } 0 \text{ otherwise},$$

$$H^{i}_\ell(cCP(2); \mathbb{Q}) = \mathbb{Q}, \quad \text{if } i = 0, 2, 4 \text{ and } 0 \text{ otherwise}.$$

If we choose $n = 1$, we did a computation of perverse cohomologies with filtrations that do not correspond to any geometrical notion of dimension but, as expected, the results coincide with the classical determination of the perverse cohomology of a cone. If we want to obtain these cohomologies from $C^*_{\text{GM}}(\overline{K}; \mathbb{Q})$ with the same filtration, we need to choose perversities $\overline{p}$ related to the previous $\overline{q}$’s by $\overline{p} + \overline{q} = -1$, which corresponds to the loose perversities, $\overline{p}(n) = -1$, $\overline{p}(n) = -3$ and $\overline{p}(n) = -5$. Evidently, if we choose a geometric filtration of length $n = 5$, we recover usual GM-perversities, $\overline{p}$.

Example 5.10. We work over $\mathbb{Q}$. The suspension of the face set $\Sigma S$ is the filtered face set $\Sigma S = \{\emptyset, \emptyset, \emptyset \ast \cdots \ast \emptyset \ast S \in \Delta^{[n]} \}$. By definition, the blow-up of Sullivan’s forms is

$$\tilde{A}_{PL}(\Sigma S) = (A_{PL}(\Delta^1) \oplus A_{PL}(\Delta^1))_{t_1=t_2=0} \otimes A_{PL}(S),$$

where $(A_{PL}(\Delta^1) \oplus A_{PL}(\Delta^1))_{t_1=t_2=0}$ is the subspace of the direct sum generated by polynomials, $P(t_1 dt_1) + Q(t_2 dt_2)$, such that $P(0,0) = Q(0,0)$. These polynomials coincide on one side of the intervals; thus, the situation corresponds to two intervals attached together by one vertex and this can be assimilated to one interval, $I$. We may therefore replace the previous complex by $C = A_{PL}(\Delta^1) \otimes A_{PL}(S) = \wedge(t, dt) \otimes A_{PL}(S)$. In this replacement, the boundary of the first component, is given by the values $t = 0$ and $t = 1$. Thus any polynomial $f$ in $t$ such that $f(0) = f(1) = 0$ has a restriction to these faces equal to zero. This observation allows the determination of the perverse degrees as,

- if $f \in \wedge t$, then $\|f \otimes \omega\| = -\infty$ if $(f(0) = f(1) = 0 \text{ or } \omega = 0)$ and $\|f \otimes \omega\| = |\omega|$ otherwise,

- if $\alpha = f dt$ with $f \in \wedge t$, then $\|\alpha \otimes \omega\| = -\infty$.

In conclusion, the blow-up of Sullivan’s forms on the suspension, $\Sigma S$, is related by a quasi-isomorphism to a strict perverse CDGA, defined on $\wedge(t, dt) \otimes A_{PL}(S)$.

We introduce now a second cochain complex, $E_{\overline{q}}$, related to $C_{\overline{q}}$ by a quasi-isomorphism. Let $\overline{q}$ be a loose perversity, given by a non-negative number, $\overline{q}(n)$. We define a cochain complex by

$$(\overline{\tau}^{\geq \overline{q}(n)} A_{PL}(S))^t = \begin{cases} 0 & \text{if } i < \overline{q}(n), \\ (\mathbb{Z}cA_{PL}(S))^t & \text{if } i = \overline{q}(n), \\ A_{PL}(S)^t & \text{if } i > \overline{q}(n), \end{cases}$$

where $\mathbb{Z}cA_{PL}(S)$ denotes a supplementary subspace of the vector space of cocycles. This complex has the same cohomology than the truncation denoted by $\overline{\tau}^{\geq \overline{q}(n)+1} A_{PL}(S)$ in [4] Page 52, i.e., $H^i(\overline{\tau}^{\geq \overline{q}(n)} A_{PL}(S)) = H^i(A_{PL}(S))$, if $i \geq \overline{q}(n) + 1$ and 0 otherwise. We note that our truncation, $\overline{\tau}$, depends on the choice of a supplementary subspace and
We construct a morphism \( \varphi: E_\eta = \tau_{\le \eta(n)}A_{PL}(S) \oplus s(\tau_{> \eta(n)}A_{PL}(S)) \to C_\eta \),
defined by \( \varphi(\eta) = 1 \otimes \eta \) if \( |\eta| \le \eta(n) \) and \( \varphi(s\eta) = dt \otimes \eta \), if \( \eta \in \tau_{> \eta(n)}A_{PL}(S) \). We have to prove that \( \varphi \) is a quasi-isomorphism. It is easy to see that \( C_\eta^{< \eta(n)} = (\wedge(t, dt) \otimes A_{PL}(S))^{< \eta(n)} \) and \( \mathcal{Z}_\eta C_\eta^{\eta(n)} = \mathcal{Z}(\wedge(t, dt) \otimes A_{PL}(S))^{\eta(n)} \), which imply that \( H^j(\varphi) \) is an isomorphism for \( j \le \eta(n) \).

In degree \( \eta(n) + 1 \), if \( \omega = \sum_{i \ge 0} t^i \alpha_i + \sum_{i \ge 0} t^i dt \beta_i \in \wedge(t, dt) \otimes A_{PL}(S) \) is such that \( d\omega = 0 \), we know that

\[
\omega - \alpha_0 = d \left( \sum_{i \ge 0} \frac{t^{i+1}}{i+1} \beta_i \right).
\]

If \( \omega \) is an element of \( C_\eta \), we have \( \alpha_0 = 0 \) and we observe that the forms \( \frac{t^{i+1}}{i+1} \beta_i \) are elements of \( C_\eta^{\eta(n)} \). Thus any cocycle of degree \( \eta(n) + 1 \) is a coboundary and we obtain \( H^{\eta(n)+1}(C_\eta) = 0 \).

In degrees strictly greater than \( \eta(n) + 1 \), from the previous determination of the perverse degree, there is no restriction for the polynomials in \( t, f(t) \), in the expressions \( f(t)dt \beta_i \in C_\eta^{> \eta(n)} \) but the polynomials \( f(t) \) in the expressions \( f(t)\alpha_i \in C_\eta^{> \eta(n)} \) have to verify \( f(0) = f(1) = 0 \). In short, an element \( \omega \in C_\eta^{> \eta(n)} \) is of the shape

\[
\omega = \sum_{i \ge 0} (t^{i+2} - t^{i+1})\alpha_i + \sum_{i \ge 0} t^i dt \beta_i.
\]

We construct a morphism \( \psi: C_\eta^{> \eta(n)} \to s(A_{PL}^{> \eta(n)}(S)) \) and a homotopy \( K \) by

\[
\psi(\omega) = s \sum_{i \ge 0} \frac{\beta_i}{i+1} \text{ and } K(\omega) = \sum_{i \ge 0} \frac{t^{i+1}}{i+1} \beta_i - t \left( \sum_{i \ge 0} \frac{\beta_i}{i+1} \right).
\]

We check \( \psi \circ d = d \circ \psi \), \( K \circ d + d \circ K = \text{id} - \varphi \circ \psi \), which imply that \( H^j(\varphi) \) is an isomorphism for \( j > \eta(n) \).

In the case of \( \Sigma \mathbb{C}P(2) \), only three values of \( \eta(n) \) are sufficient for the description of all the possible intersection cohomologies. If we denote by \( \bar{\ell} \) the perversity such that \( \bar{\ell}(n) = \ell \), we have

\[
H^i_0(\Sigma \mathbb{C}P(2); \mathbb{Q}) = \mathbb{Q} \quad \text{if } i = 0, 3, 5 \text{ and } 0 \text{ otherwise,}
\]

\[
H^i_2(\Sigma \mathbb{C}P(2); \mathbb{Q}) = \mathbb{Q} \quad \text{if } i = 0, 2, 5 \text{ and } 0 \text{ otherwise,}
\]

\[
H^i_4(\Sigma \mathbb{C}P(2); \mathbb{Q}) = \mathbb{Q} \quad \text{if } i = 0, 2, 4 \text{ and } 0 \text{ otherwise.}
\]

6. Homotopy of filtered face sets

In this section, we define the product of a filtered face set with a face set and use it for a definition of homotopy between filtered face maps. Kan
fibrations are also defined in the category of filtered face sets and linked to their analogue in the category of face sets, see Proposition 6.3.

Let $\Delta^N = \Delta^{j_0} \times \cdots \times \Delta^{j_n}$ be a filtered simplex and $\Delta^k$ be a simplex. We describe a filtered face set $(\Delta^{j_0} \times \cdots \times \Delta^{j_n}) \otimes \Delta^k$ by its vertices, as follows. Denote by $z_0 < \cdots < z_k$ the vertices of $\Delta^k$ and by $a^j_0 < \cdots < a^j_i$ the vertices of $\Delta^{j_i}$, for any $i \in \{0, \ldots, n\}$. We extend the order between the vertices of the $\Delta^{j_i}$’s to an order on the vertices of $\Delta^N$ by setting $a^j_i \leq a^{j'}_i$ if $\ell < \ell'$ or ($\ell = \ell'$ and $s \leq s'$). The vertices of $(\Delta^{j_0} \times \cdots \times \Delta^{j_n}) \otimes \Delta^k$ are the couples $(a^j_i, z_i)$ of vertices of $\Delta^N$ and $\Delta^k$.

Consider a sequence of distinct vertices, $\left((a^j_i, z_i)\right)_{0 \leq i \leq \nu}$, with $\ell_i \in \{0, \ldots, n\}$, $s_i \in \{0, \ldots, j_i\}$, and $w_i \in \{0, \ldots, k\}$. Such a sequence spans a simplex of $(\Delta^{j_0} \times \cdots \times \Delta^{j_n}) \otimes \Delta^k$ if, and only if, we have $a^j_i \leq a^{j_{i+1}}_i$ and $z_{w_i} \leq z_{w_{i+1}}$, for all $i \in \{0, \ldots, \nu - 1\}$. For instance, the maximal simplices are sequences of distinct $(k + N + 1)$ couples. The filtration degree is given by the index $\ell$, i.e., if $\nabla$ is a simplex of $(\Delta^{j_0} \times \cdots \times \Delta^{j_n}) \otimes \Delta^k$, the number $1 + ||\nabla||_\ell$ is equal to the cardinal of the set of vertices whose first component belongs to $\Delta^{j_0} \times \cdots \times \Delta^{j_n} \otimes \Delta^k$.

This product is still of formal dimension $n$, as filtered face set, and this construction is compatible with face operators, see [11, Section 3]. This justifies the next definition.

**Definition 6.1.** Let $K \in \Delta^{|n|}-Sets$ be a filtered face set and $T$ be a face set. The product of $K$ and $T$ is the filtered face set defined by

$$K \otimes T = \text{colim} \left( \Delta^{j_0} \times \cdots \times \Delta^{j_n} \rightarrow \text{colim} \left( (\Delta^{j_0} \times \cdots \times \Delta^{j_n}) \otimes \Delta^k \right) \right).$$

This product is clearly associative, i.e.,

$$K \otimes (T \otimes S) = (K \otimes T) \otimes S,$$

extends naturally in a bifunctor,

$\otimes : \Delta^{|n|}-Sets \times \Delta-\text{Sets} \rightarrow \Delta^{|n|}-\text{Sets},$

and commutes with colimits by definition.

We define now some structures on the sets of morphisms. We keep the previous notation: $K, L$ are filtered face sets and $T$ is a face set. We define a filtered face set, $K^T$, by

$$(K^T)_{j_0 \cdots j_n} = \text{hom}_{\Delta^{|n|}-Sets}((\Delta^{j_0} \times \cdots \times \Delta^{j_n}) \otimes T, K),$$

and a face set, $\text{hom}_\Delta(K, L)$, by

$$\text{hom}_\Delta(K, L)_k = \text{hom}_{\Delta^{|n|}-Sets}(K \otimes \Delta^k, L).$$

The following formulae of adjunctions derive directly from the definitions and the usual preservation of colimits by a hom-functor.

**Proposition 6.2.** If $K, L$ are two filtered face sets and $T$ is a face set, we have natural bijections,

$$\text{hom}_{\Delta^{|n|}-Sets}(K \otimes T, L) \cong \text{hom}_{\Delta^{|n|}-Sets}(K, L^T) \cong \text{hom}_{\Delta-\text{Sets}}(T, \text{hom}_\Delta(K, L)).$$
The two canonical maps, $\iota_0, \iota_1: \Delta^0 \to \Delta^1$, give a notion of homotopy in the category $\Delta_\mathbb{F}^{[n]} - \text{Sets}$.

**Definition 6.3.** Let $f, g: K \to L$ be two filtered face maps. They are *homotopic* if there exists a filtered face map, $F: K \otimes \Delta^1 \to L$, such that $F \circ (\text{id} \otimes \iota_0) = f$ and $F \circ (\text{id} \otimes \iota_1) = g$. We denote this relation by $f \simeq g$.

In particular, the two injections, $\text{id} \otimes \iota_i: K \to K \otimes \Delta^1$, for $i = 0, 1$, are homotopic.

As in the case of face sets, for getting an equivalence relation, we have to impose some restrictions. Let $m \geq 1$. We denote by $\Delta_{m,k}$ the face subset of $\partial \Delta^m$ obtained by removing the $k$-th $(m - 1)$-face.

**Definition 6.4.** A filtered face map, $f: K \to K'$, is a *Kan fibration* if, for each filtered face set, $L$, and each commutative diagram of filtered face maps,

$$
\begin{array}{ccc}
L \otimes \Delta_{m,k} & \xrightarrow{\varphi} & K \\
\downarrow g & & \downarrow f \\
L \otimes \Delta^m & \xrightarrow{\psi} & K',
\end{array}
$$

there exists a filtered face map, $g: L \otimes \Delta^m \to K$, extending $\varphi$ and lifting $\psi$. In the case $K' = K_{\mathbb{F}}^* = \emptyset \ast \cdots \ast \emptyset \ast \Delta^0$, the filtered face set $K$ is called a *Kan filtered face set*.

This definition is connected to the notion of Kan fibration between face sets, see [41, Section 5].

**Proposition 6.5.** A filtered face map, $f: K \to K'$, is Kan if, and only if, the corresponding filtered face map, $\hat{f}: \text{hom}(\Delta^\mathbb{F}(L, K)) \to \text{hom}(\Delta^\mathbb{F}(L, K'))$, is a Kan fibration for any filtered face set, $L$.

**Proof.** With Proposition 6.2, the existence of a filtered face map $g$, making commutative the following diagram,

$$
\begin{array}{ccc}
L \otimes \Delta_{m,k} & \xrightarrow{\varphi} & K \\
\downarrow g & & \downarrow f \\
L \otimes \Delta^m & \xrightarrow{\psi} & K',
\end{array}
$$

is equivalent to the existence of a filtered face map, $\hat{g}$, making commutative the following diagram,

$$
\begin{array}{ccc}
\Delta_{m,k} & \xrightarrow{\hat{\varphi}} & \text{hom}(\Delta^\mathbb{F}(L, K)) \\
\downarrow \hat{g} & & \downarrow \hat{f} \\
\Delta^m & \xrightarrow{\hat{\psi}} & \text{hom}(\Delta^\mathbb{F}(L, K')),
\end{array}
$$

where $\hat{\varphi}, \hat{\psi}$ correspond to $\varphi, \psi$ in the adjunction correspondence, respectively. This last property is the definition of a Kan fibration in $\Delta^\mathbb{F}-\text{Sets}$, see [41, Section 5].

**Corollary 6.6.** Let $K$ be a Kan filtered face set and $(T, S)$ be a pair of face sets. Then the map $K^T \to K^S$ is a Kan fibration in $\Delta_\mathbb{F}^{[n]} - \text{Sets}$.
Proof. Let \( L \) be a filtered face set. With Proposition 6.5, we have to show the existence of a dotted arrow making commutative the following diagram in \( \Delta^{-Sets} \),

\[
\begin{array}{c}
\Delta^{m,k} \\
\downarrow \\
\Delta^m \\
\downarrow \\
\hom^\Delta(L, K^T) \\
\downarrow \\
\hom^\Delta(L, K^S).
\end{array}
\]

This problem of existence can also be rewritten as the following lifting problem in \( \Delta^{-Sets} \),

\[
\begin{array}{c}
\Delta^m \otimes S \cup \Delta^{m,k} \otimes S \\
\downarrow \\
\Delta^m \otimes T \\
\downarrow \\
\hom^\Delta(L, K).
\end{array}
\]

As \( K \to \emptyset \to \Delta^0 \) is a Kan fibration, by Proposition 6.5, the induced map \( \hom^\Delta(L, K) \to \hom^\Delta(L, \emptyset \to \Delta^0) = \Delta^0 \) is a Kan fibration (i.e., \( \hom^\Delta(L, K) \) is a Kan face set) and this last dotted arrow exists. \( \square \)

Two maps, \( f, g : K \to L \) in \( \Delta^{[n]}^{-Sets} \), can be viewed as elements of \( \hom^\Delta(K, L)_{0} \). From Definition 6.3, they are homotopic in \( \Delta^{[n]}^{-Sets} \) if, and only if, there exists \( F \in \hom^\Delta(K, L)_1 \) such that \( \partial_0 F = f \) and \( \partial_1 F = g \), where \( \partial_0, \partial_1 \) are face operators of the face set \( \hom^\Delta(K, L) \). This observation implies the next result, by definition of homotopy groups of Kan face sets, see [41, Section 6].

**Corollary 6.7.** If \( K \) is a Kan filtered face set, the relation of homotopy between filtered face maps, of codomain \( K \), is an equivalence relation and the set of equivalence classes verifies

\[
[L, K]_{\Delta^{[n]}^{-Sets}} = \pi_0 \hom^\Delta(L, K).
\]

Proof. We have only to check that \( \hom^\Delta(L, K) \) is a Kan face set and this is already done at the end of the proof of Corollary 6.6. \( \square \)

We study now the behavior of the product of a filtered face set and a face set with intersection cohomology.

**Proposition 6.8.** Let \( K \) be a filtered face set, \( f : X \to Y \) be a face map between face sets and \( p \) be a loose perversity. Then, the map \( \id \otimes f : K \otimes X \to K \otimes Y \) induces a chain map

\[
(id \otimes f)_* : C_*(K \otimes X) \to C_*(K \otimes Y).
\]

Moreover, if \( f \) is homotopic to \( g \), then we have \( H_*^{GM,F}(id \otimes f) = H_*^{GM,F}(id \otimes g) \). In particular, if \( f \) is a homotopy equivalence, the chain map \( (id \otimes f)_* \) is a quasi-isomorphism.

Proof. The map \( \id \otimes f \) induces a chain map \( C_*(K \otimes X) \to C_*(K \otimes Y) \) and we have only to determine its behavior with the perverse degree. As it is a local computation, we suppose \( f : \Delta^k \to \Delta^{k'} \) and consider the map \( \id \otimes f : (\Delta^{j_0} \star \cdots \star \Delta^{j_n}) \otimes \Delta^k \to (\Delta^{j_0} \star \cdots \star \Delta^{j_n}) \otimes \Delta^{k'} \).

Let \( \sigma : \nabla \to (\Delta^{j_0} \star \cdots \star \Delta^{j_n}) \otimes \Delta^k \) be a simplex of the product defined at the beginning of this section; it has a perverse degree induced by the perverse degree of a product. Thus,
has a decomposition such that \( \|\sigma\|_\epsilon = \dim (\nabla^{k_0} \ast \cdots \ast \nabla^{k_{n-\epsilon}}) \). Recall also that \( \nabla \) can be described by its vertices and these last ones are couples of a vertex of \( \Delta^{j_0} \ast \cdots \ast \Delta^{j_n} \) and a vertex of \( \Delta^k \). In this context, the integer \( 1 + \|\sigma\|_\epsilon \) is the cardinal of the set of vertices whose first component belongs to \( \Delta^{j_0} \ast \cdots \ast \Delta^{j_{n-\epsilon}} \). The image \( ((\text{id} \otimes f) \circ \sigma)(\nabla) \) is a simplex whose vertices are the images of the vertices of \( \nabla \) by \( \text{id} \otimes f \). Thus the number of vertices whose first components are in \( \Delta^{j_0} \ast \cdots \ast \Delta^{j_{n-\epsilon}} \) cannot increase and we get an induced map \( C^*_s \hat{\Gamma}_M((\Delta^{j_0} \ast \cdots \ast \Delta^{j_n}) \otimes \Delta^k) \rightarrow C^*_s \hat{\Gamma}_M((\Delta^{j_0} \ast \cdots \ast \Delta^{j_n}) \otimes \Delta^k) \).

For the second part of the statement, it is sufficient to prove that the two injections \( \iota_0, \iota_1: \mathbb{K} \rightarrow \mathbb{K} \otimes \Delta^1 \) induce the same map in homology. The argument used in the proof of Lemma \( A.15 \) can be, word for word, adapted to this situation.

\[ \square \]

**Part 2. Rational algebraic models**

### 7. Perverse differential graded algebras

We collect here some results of Hovey [31] concerning the closed model structure on the category of perverse CDGA’s and specify the notion of homotopy. A definition of minimal model and its uniqueness up to isomorphisms are provided. In Sections \( 7 \) to \( 11 \) we consider algebraic structures defined on the field of rational numbers. Thus, in these sections, any vector space is supposed to be a rational vector space.

The notion of strict perverse CDGA, introduced in Section \( 2 \) is extended in the one of perverse CDGA defined by M. Hovey, see [31]. We enlarge the lattice \( \hat{\mathcal{P}}^n \) of GM-perversities used in [31] by considering the lattice \( \hat{\mathcal{P}}^n = \mathcal{P}^n \cup \{ \infty \} \). If \( \hat{p} \) and \( \hat{q} \) are elements of \( \hat{\mathcal{P}}^n \), we define \( \hat{p} \oplus \hat{q} \) as the smallest element, \( \hat{r} \), of \( \hat{\mathcal{P}}^n \) such that \( \hat{p} + \hat{q} \leq \hat{r} \), see [23]. (In particular, if \( \hat{p}(i) + \hat{q}(i) > i - 2 \) for some \( i > 2 \), then \( \hat{p} \oplus \hat{q} = \infty \).) We may construct \( \hat{p} \oplus \hat{q} \) by induction (see [18]), starting from \( (\hat{p} \oplus \hat{q})(n) = \hat{p}(n) + \hat{q}(n) \). The inductive step is given by:

- if \( \hat{p}(k) + \hat{q}(k) < (\hat{p} \oplus \hat{q})(k+1) \), one sets \( (\hat{p} \oplus \hat{q})(k) = (\hat{p} \oplus \hat{q})(k+1) - 1 \),
- if \( \hat{p}(k) + \hat{q}(k) = (\hat{p} \oplus \hat{q})(k+1) \), one sets \( (\hat{p} \oplus \hat{q})(k) = (\hat{p} \oplus \hat{q})(k+1) \).

This law is commutative, associative and has the null perversity as neutral element.

**Definition 7.1.** A *perverse vector space* is a functor from \( \hat{\mathcal{P}}^n \) to the category of vector spaces. If this functor takes value in the category of graded vector spaces, we use the expression *perverse graded vector space*. *Perverse linear maps* are natural transformations between perverse vector spaces.

If \( M_* \) is a perverse vector space, we denote by \( \phi_{\hat{p}}: M_{\hat{p}} \rightarrow M_{\hat{q}} \) the morphism associated to \( (\hat{p} \leq \hat{q}) \). As in [31] Section 2, we define a tensor product of perverse vector spaces by:

\[
(M_* \otimes N_*)_{\hat{r}} = \colim_{\hat{p} + \hat{q} \leq \hat{r}} M_{\hat{p}} \otimes N_{\hat{q}}.
\]

For any perversity \( \hat{p} \in \hat{\mathcal{P}}^n \), there is an evaluation functor, \( \text{ev}_{\hat{p}} \), which associates to a perverse vector space, \( M_* \), the vector space \( M_{\hat{p}} \). This functor admits a left adjoint, \( \mathcal{V}_{\hat{p}} \), that generates free objects.
Definition 7.2. Let $V$ be a $\mathbb{Q}$-vector space and $p \in \hat{P}^n$. The $p$-free perverse vector space generated by $V$ is defined by,

- $\mathcal{V}_p(V)_r = V$ if $r \geq p$ and $0$ otherwise;
- the map $\varphi^p_{r_2}_{r_1}$ is the identity map $V \to V$ if $p \leq r_1 \leq r_2$, and the canonical injection $0 \to V$ or the identity map $0 \to 0$ otherwise.

The projective perverse vector spaces are direct factors of $\bigoplus_p \mathcal{V}_p(V(p))$, where the $V(p)$'s are vector spaces. More precisely, ([31 Proposition 2.1]), a perverse vector space, $M_\bullet$, is projective if the map $\text{colim}_{q \leq p} M_p \to M_q$ is a split monomorphism for all $q$.

Definition 7.3. A perverse algebra is a monoid in the category of perverse vector spaces. In the case of a commutative monoid, we have a perverse commutative algebra (henceforth perverse CGA). We denote by $Q_\bullet$ the commutative perverse algebra defined by $\varphi^q_p = \text{id}: Q_p = \mathbb{Q} \to Q_q = \mathbb{Q}$, for any $p \leq q$.

A perverse chain complex is a functor from $\hat{P}^n$ in the category of chain complexes. A perverse chain map is a perverse linear map compatible with the differentials. We denote by $\text{Ch}(\hat{P}^n)$ the associated category.

The projective objects of $\text{Ch}(\hat{P}^n)$ are the perverse chain complexes, $A_\bullet$, formed of perverse projective vector spaces and such that each perverse chain map into an acyclic complex, $A_\bullet \to B_\bullet$, is chain homotopic to zero, ([31 Section 3]).

Theorem 7.4 (cf. [11], [31]). The category $\text{Ch}(\hat{P}^n)$ is a closed model category for the following classes of objects.

- The weak equivalences are the perverse chain maps $f_\bullet: M_\bullet \to N_\bullet$ such that $H_\bullet(f_p): H_\bullet(M_p) \to H_\bullet(N_p)$ is an isomorphism for any perversity $p \in \hat{P}^n$.
- The fibrations are the surjections.
- The cofibrations are the injective maps with a projective cokernel.

This structure is monoidal and each object of $\text{Ch}(\hat{P}^n)$ is fibrant.

Definition 7.5. A perverse differential graded algebra (henceforth perverse DGA) is a monoid in the category $\text{Ch}(\hat{P}^n)$. In the case of a commutative monoid, we have a perverse commutative differential graded algebra (henceforth perverse CDGA). We denote by $\text{CDGA}_F$ the category of perverse CDGA’s.

Theorem 7.6 (cf. [31]). The category $\text{CDGA}_F$ is a closed model category for the following classes of objects.

- The weak equivalences are the morphisms which are weak equivalences in $\text{Ch}(\hat{P}^n)$.
- The fibrations are the surjections.
- The cofibrations are defined by the lifting property relatively to trivial fibrations.

To any strict perverse cochain complex, $A$, we associate a perverse cochain complex, $A_\bullet$, defined by

$A_p = \{ \omega \mid \|\omega\|_i \leq \overline{p}(i) \text{ and } \|d\omega\|_i \leq \overline{p}(i) \text{ for all } i \}.$

Moreover, if $A$ is a strict perverse DGA, then $A_\bullet$ is a perverse DGA and $H(A_\bullet)$ is a perverse algebra.
For instance, if $K$ is a filtered face set and $F$ is a universal system, the blow-up, $\tilde{F}(K)$, generates a perverse cochain complex $\tilde{F}(K)_\bullet$. In particular, the blow-up of Sullivan’s forms gives a perverse CDGA, $A(K)_\bullet$, defined by $A(K)_{\overrightarrow{p}} = \hat{A}_{PL,\overrightarrow{p}}(K)$.

In the perverse cohomology of the suspension of $CP(2)$ (see Example 5.10 on an element of $H^3_2(\Sigma CP(2))$ disappears in $H^3_2(\Sigma CP(2))$ and we have no inclusion of $H^3_2(\Sigma CP(2))$ into $H^3_2(\Sigma CP(2))$). As in the case where the perverse cochain complex is coming from a strict perverse cochain complex, the morphisms $\varphi_{\overrightarrow{p}}: A_{\overrightarrow{p}} \to A_{\overrightarrow{p}}$ are injective, we deduce that the cohomology of a strict perverse CDGA is a perverse algebra which, in general, is not coming from a structure of strict perverse DGA. For the study of formality, it is important that the involved cochain complexes and their cohomology are objects of the same category. Thus, the introduction of the perverse CDGA’s defined by M. Hovey (31) is essential in this context.

We introduce now free objects in the category of perverse CDGA’s. The perverse tensor algebra on a perverse graded vector space, $M_\bullet$, is defined by $T(M_\bullet) = \oplus_{k \geq 0} M^{\otimes k}_\bullet$, where the tensor products $(M^{\otimes k})_\bullet$ are defined in (14). The perverse commutative tensor algebra is defined as $\Lambda M_\bullet = \oplus_{k \geq 0} M^{\otimes k}/\Sigma_k$, where the action of the symmetric group $\Sigma_k$ is characterized by the next action of the transposition $(i, i + 1)$,

$$x_0 \otimes \cdots \otimes x_k \mapsto (-1)^{x_i |x_{i+1}|} x_0 \otimes \cdots \otimes x_{i-1} \otimes x_{i+1} \otimes x_i \otimes x_{i+2} \otimes \cdots \otimes x_n.$$  

If there is no ambiguity, the element $x \otimes x'$ of $\Lambda M_\bullet$ is also denoted $xx'$.

**Definition 7.7.** A perverse CGA, $B_\bullet$, is called free if there exists a family of graded vector spaces, $(V_{[\overrightarrow{p}]})_{\overrightarrow{p} \in \overrightarrow{\mathbb{Z}}^n}$, such that $B_\bullet = \Lambda \left( \oplus_{\overrightarrow{p} \in \overrightarrow{\mathbb{Z}}^n} V_{[\overrightarrow{p}]}(B_\bullet) \right)$.

From the definition, we deduce the existence of a canonical injection $Q_\bullet \to B_\bullet$, for any free perverse CGA, $B_\bullet$. Observe also that a free perverse CGA, $B_\bullet$, comes from a strict perverse CGA: the elements $v \in V_{[\overrightarrow{p}]}$ have a degree denoted by $|v| = i$, and a perverse degree denoted by $\|v\| = \overrightarrow{p}$. This perverse degree is extended to the free CGA by $\|v_1 v_2\| = \|v_1\| + \|v_2\|$. We use this structure to simplify the notation in

$$B = \Lambda \left( \oplus_{\overrightarrow{p} \in \overrightarrow{\mathbb{Z}}^n} V_{[\overrightarrow{p}]} \right),$$  

with $B_{\overrightarrow{p}}$ the vector space generated by the products, $v_1 \ldots v_k$, with $v_i \in V_{[\overrightarrow{p}]}$ and $\overrightarrow{p}_1 + \cdots + \overrightarrow{p}_k \leq \overrightarrow{p}$, i.e.,

$$B_{\overrightarrow{p}} = \left( v_1 \ldots v_k \mid v_i \in V_{[\overrightarrow{p}]} \text{ and } \overrightarrow{p}_1 + \cdots + \overrightarrow{p}_k \leq \overrightarrow{p} \right).$$  

(16)

For a perverse CGA, $B_\bullet$, we denote by $B^d_\overrightarrow{p}$ the set of elements of degree $k$ and perverse degree $\overrightarrow{p}$. We describe some particular cases of free perverse CGA’s.

- If $V_{[\overrightarrow{p}]} = \mathbb{Q}x$, with $x$ of odd degree, and $V_{[\overrightarrow{r}]} = 0$ for all $[\overrightarrow{r}] \neq [\overrightarrow{p}]$, then $B_{\overrightarrow{q}} = \mathbb{Q}x$ if $\overrightarrow{q} \geq \overrightarrow{p}$ and $B_{\overrightarrow{q}} = \mathbb{Q}$ otherwise.
- If $V_{[\overrightarrow{p}]} = \mathbb{Q}x$, with $x$ of even degree $k$, and $V_{[\overrightarrow{r}]} = 0$ for all $[\overrightarrow{r}] \neq [\overrightarrow{p}]$, then $B^d_{\overrightarrow{q}} = \mathbb{Q}x^k$ if $\overrightarrow{q} \geq \overrightarrow{p}$ and $B_{\overrightarrow{q}} = \mathbb{Q}$ otherwise.

As $\Lambda (V \otimes W) = \Lambda V \otimes \Lambda W$, the general situation can be expressed as tensor products of these two cases.
We introduce now the concept of Sullivan minimal model of perverse CDGA’s. Recall that, if \( V \) is a graded vector space, \((\wedge V)^{+}\) is the ideal of \( \wedge V \) formed of the elements of strictly positive degree and \( \wedge^{\geq 2} V \) the ideal formed of decomposable elements in \( \wedge V \).

**Definition 7.8.** A Sullivan minimal perverse CDGA is a strict perverse CDGA, \((B,d)\), free as perverse CGA, i.e., \( B = \wedge(\oplus_{\overline{p} \in \mathcal{P}^{n}} V_{[\overline{p}]}) \), and such that, for any \( \overline{p} \in \mathcal{P}^{n} \) and any \( m \geq 1 \), we have \( V_{\overline{p}} = \oplus_{k \geq 1} V_{\overline{p}}^{k} \), and \( V_{\overline{p}}^{m} \) can be written as, \( V_{\overline{p}}^{m} = \bigcup_{i=0}^{\infty} V_{[\overline{p}]}(i)^{m} \), with \( V_{[\overline{p}]}(i)^{m} = V_{[\overline{p}]}(i-1)^{m} \oplus V_{[\overline{p}]}[i]^{m} \) and

\[
dV_{[\overline{p}]}^{m}[i] \subset \left( (\wedge \oplus_{r < \overline{p}} V_{[\overline{r}]}) \otimes (\wedge V_{[\overline{p}]}) \otimes (\wedge V_{[\overline{p}]}(i-1)) \right)_{\overline{p}}.
\]

Observe that the condition (17) implies that \((\wedge V_{[\overline{p}]}, d)\) is a classical minimal CDGA, and that the canonical inclusion, \((\wedge(\oplus_{\overline{p} \in \mathcal{P}^{n}} V_{[\overline{p}]}, d) \rightarrow (\wedge(\oplus_{\overline{p} \in \mathcal{P}^{n}} V_{[\overline{p}]}, d)\), is a relative minimal CDGA.

**Definition 7.9.** If \( A_{*} \) is a Sullivan minimal perverse CDGA, a Sullivan minimal model of \( A_{*} \) is a quasi-isomorphism of CDGA’s, \( \rho_{*}: (\wedge(\oplus_{\overline{p} \in \mathcal{P}^{n}} V_{[\overline{p}]}, d)_{*} \rightarrow A_{*} \), whose domain is a Sullivan minimal perverse CDGA.

**Proposition 7.10.** Sullivan minimal perverse CDGA’s are cofibrant in CDGA

**Proof.** This is a classical argument. If \( k \geq 1 \) and \( \overline{p} \in \mathcal{P}^{n} \), we denote by \( Q(k, \overline{p}) \) the free perverse cochain complex generated by \( z, |z| = k, \|z\| = \overline{p}, dz = 0 \). If \( k \geq 2 \), we define \( Q(k-1, k, \overline{p}) \) as the free perverse cochain complex generated by \( x \) and \( y, |x| = k-1, |x| = \overline{p}, |y| = k, \|y\| = \overline{p} \) and \( dx = y \). The natural perverse cochain maps, \( Q \rightarrow Q(k, \overline{p}) \) and \( Q(k, \overline{p}) \rightarrow Q(k-1, k, \overline{p}), z \mapsto y, \) are cofibrations in Ch(\( \mathcal{P}^{n} \)), see [21] Section 3. Thus the perverse CDGA’s maps, \( Q \rightarrow \wedge Q(k, \overline{p}) \) and \( \wedge Q(k, \overline{p}) \rightarrow \wedge Q(k-1, k, \overline{p}) \), are cofibrations in CDGA’s.

As Sullivan perverse CDGA’s can be obtained as successive push-outs built from the previous cofibrations, we get the result, see [5] Proposition 7.5 for a similar treatment.

As in the classical case ([38] Theorem 5.1), minimal models are unique up to isomorphisms.

**Proposition 7.11.** Any weak equivalence between Sullivan minimal perverse CDGA’s is an isomorphism.

**Proof.** Let \( \varphi: B = (\wedge(\oplus_{\overline{r} \in \mathcal{P}^{n}} V_{[\overline{r}]}, d) \rightarrow C = (\wedge(\oplus_{\overline{r} \in \mathcal{P}^{n}} W_{[\overline{r}]}, d) \) be a weak equivalence between two Sullivan minimal perverse CDGA’s. We adapt to the perverse setting a proof made by Antonio Gómez-Tato in [22], by building a perverse CDGA’s map, \( \psi: C \rightarrow B \), such that \( \varphi \circ \psi = id \).

First, we observe that the restriction \( \varphi: (\wedge V_{[\overline{p}]}, d) \rightarrow (\wedge W_{[\overline{p}]}, d) \) is a weak equivalence between (classical) minimal models therefore it is an isomorphism and the map \( \psi \) is defined on \( \wedge W_{[\overline{p}]}. \)

Let \( \overline{p} \) be a perversity. By definition, (see [17]) there exists a filtration of \( W_{[\overline{p}]}^{k} \) as \( W_{[\overline{p}]}^{k}(i) = W_{[\overline{p}]}^{k}(i-1) \oplus W_{[\overline{p}]}^{k}[i] \) such that

\[
dW_{[\overline{p}]}^{k}[i] \subset (\wedge \oplus_{r \not\in \overline{p}} W_{[\overline{r}]}) \otimes (\wedge W_{[\overline{p}]}) \otimes (\wedge W_{[\overline{p}]}(i-1)).
\]
Set \( M = (\land \oplus_{i < p} W_{[i]}^k) \otimes (\land W^k_{[p]}(i - 1)) \). We suppose that \( \psi \), such that \( \varphi \circ \psi = \text{id} \), is defined on \( M \). By induction, the construction of \( \psi \) on \( C \) is therefore reduced to its extension to \( W^k_{[p]}[i] \). With the 5-lema, we get a quasi-isomorphism
\[
\varphi: (B/\psi(M))_p \to (C/M)_p.
\]
Let \( (w_i) \) be a basis of \( W^k_{[p]}[i] \). Any \( w_i \) is a cocycle in \( (C/M)_p \) which cannot be a coboundary. We deduce the existence of \( v_i \in B_p, m_i \in M_p, m'_i \in M_p \) such that
\[
\varphi(v_i) = w_i + m_i \quad \text{and} \quad dw_i = \psi(m'_i).
\]
We set \( \psi(w_i) = v_i - \psi(m_i) \) and we check:
\[
\begin{align*}
\quad \quad dw_i &= d\varphi(v_i) - dm_i = \varphi(\psi(m'_i)) - dm_i = m'_i - dm_i, \\
\quad \psi(dw_i) &= \psi(\varphi(m'_i)) - \psi(dm_i) = dw_i - \psi(m_i) = d\psi(w_i), \\
\quad \varphi(\psi(w_i)) &= \varphi(v_i) - \varphi(\psi(m_i)) = \varphi(v_i) - m_i = w_i.
\end{align*}
\]
From the equality \( \varphi \circ \psi = \text{id} \), we deduce that \( \psi \) is a quasi-isomorphism as well and the same construction applied to \( \psi \) brings a morphism \( \varphi' \) such that \( \psi \circ \varphi' = \text{id} \). Therefore \( \psi \) is an isomorphism and \( \varphi = \psi^{-1} \) is one also.

**Definition 7.12.** A path object of an element \( B_\bullet \) of CDGA\(_\mathcal{F}\) is given by
\[
(B \land (t, dt))_\bullet \xrightarrow{\pi_0} B_\bullet
\]
where \(|t| = 0, |dt| = 1\), \((B \land (t, dt))_p = B_p \land (t, dt)\) and the maps \( \pi_i \) are defined by the identity on \( B_\bullet \) and the evaluations \( \pi_i(t) = i \), for \( i = 0, 1 \). Let \( A_\bullet \) be cofibrant. Two morphisms, \( f_0, f_1: A_\bullet \to B_\bullet \) of CDGA\(_\mathcal{F}\), are homotopic if there exists a diagram in CDGA\(_\mathcal{F}\),
\[
\begin{array}{ccc}
A_\bullet & \xrightarrow{f_0} & B_\bullet \\
\downarrow{f_1} & & \downarrow{\pi_1} \\
(B \land (t, dt))_\bullet & \xrightarrow{\pi_0} & B_\bullet
\end{array}
\]
such that \( \pi_0 \circ F = f_0 \) and \( \pi_1 \circ F = f_1 \).

**Remark 7.13.** The sequence \( B_\bullet \xrightarrow{\Delta} (B \land (t, dt))_\bullet \xrightarrow{(\pi_0, \pi_1)} B_\bullet \oplus B_\bullet \) is a decomposition of the diagonal \( B_\bullet \to B_\bullet \oplus B_\bullet \) in a weak-equivalence followed by a fibration. This corresponds to the classical definition of a path object in a closed model category.

As CDGA\(_\mathcal{F}\) is a closed model category, the following properties are standard, see [36].

**Proposition 7.14.** Let \( A'_\bullet \) and \( A_\bullet \) be cofibrant objects of CDGA\(_\mathcal{F}\).

1. If
\[
A'_\bullet \xrightarrow{g} A_\bullet \xrightarrow{f_0} B_\bullet \xrightarrow{f_1} B'_\bullet
\]
is a diagram of morphisms of CDGA\(_\mathcal{F}\), the next properties are satisfied.

(a) If \( f_0 \) is homotopic to \( f_1 \), then \( h \circ f_0 \circ g \) is homotopic to \( h \circ f_1 \circ g \).
(b) Homotopy is an equivalence relation on the set of morphisms from $A_\bullet$ to $B_\bullet$. We denote this relation by $\simeq$.

(2) Any weak equivalence, $h: B_\bullet \to B'_\bullet$ in CDGA$_F$, induces an isomorphism between the homotopy classes,

$$h_\sharp: [A_\bullet, B_\bullet] \to [A_\bullet, B'_\bullet].$$

(3) Any weak equivalence, $g: A'_\bullet \to A_\bullet$ in CDGA$_F$, induces an isomorphism between the homotopy classes,

$$g_\sharp: [A_\bullet, B_\bullet] \to [A'_\bullet, B_\bullet].$$

The notion of homotopy can be extended to any pair of morphisms, not necessarily with a cofibrant domain. With Proposition 7.14 (3), the next definition does not depend on the choice of a model of $B_\bullet$.

**Definition 7.15.** Let $f_0, f_1: B'_\bullet \to B_\bullet$ be two morphisms in CDGA$_F$ and $\varphi: A_\bullet \to B'_\bullet$ be a cofibrant model. The morphisms $f_0$ and $f_1$ are homotopic if $f_0 \circ \varphi \simeq f_1 \circ \varphi$.

### 8. Balanced perverse cochain complex

In this section, we define and study a notion of balanced perverse CDGA's, which are the basis of the construction of a minimal model in the next section. If $K$ is a connected filtered face set and $F$ is a universal system of CDGA's, the blow-up, $\tilde{F}(K)_\bullet$, and its cohomology, $H(\tilde{F}(K)_\bullet)$, are examples of balanced perverse CDGA's.

We begin with a description of the predecessors of a given GM-perversity $\p$.

**Definition 8.1.** Let $\p \in \mathbb{P}^n$ be a GM-perversity. A peak of $\p$ is an integer $j \in \{3, \ldots, n-1\}$ such that $\p(j+1) = \p(j) > \p(j-1)$. The integer $n$ is a peak if $\p(n) > \p(n-1)$. We order the peaks $(n_1, \ldots, n_s)$ of $\p$ by $3 \leq n_1 < n_2 < \ldots < n_s \leq n$. For each peak $n_i$ of $\p$, we define a perversity $\p_i$ by

$$\begin{cases} 
\p_i(k) = \p(k), & \text{if } k \neq n_i, \\
\p_i(n_i) = \p(n_i) - 1.
\end{cases}$$

We check easily the following properties.

(i) The previous perversities $(\p_1, \ldots, \p_s)$ are in $\mathbb{P}^n$.

(ii) The perversities $(\p_1, \ldots, \p_s)$ are the predecessors of $\p$. We order them by the corresponding peaks.

(iii) We have $\p_i(n_i) \geq i$ and $\p_i(n_i) \geq i - 1$.

(iv) If $(\p_1, \ldots, \p_s)$ are the ordered predecessors of $\p$, we denote by $\p_{i,\ell}$ the GM-perversity $\inf(\p_i, \p_\ell)$. The GM-perversities

$$(\p_{1,2}, \p_{1,3}, \ldots, \p_{1,s})$$

are ordered predecessors of $\p_1$ and the corresponding ordered peaks are $(n_2, \ldots, n_s)$.

We define now the main objects of this section.

**Definition 8.2.** A perverse cochain complex, $A_\bullet$, defined by the morphisms, $\varphi_{\p}: A_\p \to A_{\p'}$, for any $\p \leq \p$, is called balanced if it satisfies the next properties, for any GM-perversity $\p$, and any sequence of ordered predecessors of $\p$, $(\p_{n_1}, \ldots, \p_{n_s})$. 


(a) Recall \( p_{\nu_j,\nu} = \inf(p_{\nu_j}, p_{\nu}) \). For any \( j \geq 1 \) and any \( k \leq p(n_{\nu_j}) - j + 1 \), the sequence,
\[
\bigoplus_{1 \leq i < \ell \leq j} A^k_{\beta_{\nu_i,\nu_j}} \xrightarrow{\varphi} \bigoplus_{i=1}^j A^k_{\beta_{\nu_i,\nu_j}} \xrightarrow{\psi} A^k_p,
\]
is exact, where

- the morphism \( \varphi \) is defined on \( \beta_{i,\ell} \in A_{\beta_{\nu_i,\nu_j}} \) by \( \varphi(\beta_{i,\ell}) = (\alpha_m)_{1 \leq m \leq j} \), with \( \alpha_1 = \varphi_{\beta_{\nu_i,\nu_j}}(\beta_{i,\ell}) \), \( \alpha_\ell = -\varphi_{\beta_{\nu_i,\nu_j}}(\beta_{i,\ell}) \) and \( \alpha_m = 0 \) otherwise,
- the morphism \( \psi \) by \( \psi(\alpha_1, \ldots, \alpha_j) = \varphi_{\beta_{\nu_j,\nu_j}}(\alpha_1) + \cdots + \varphi_{\beta_{\nu_j,\nu_j}}(\alpha_j) \).

(b) The map \( \varphi^p_{\beta_{\nu_j,\nu_j}} : A^0_{\beta_{\nu_j,\nu_j}} \to A^0_p \) is surjective if \( p(n_{\nu_j}) \geq 2 \).

(c) For any \( \nu \) with \( n \leq p(n) \), the induced morphisms,
\[
H^1(\varphi^p_{\beta_{\nu_j,\nu_j}}) : H^1(A_{\beta_{\nu_j,\nu_j}}) \to H^1(A_p) \quad \text{and} \quad H^1(\varphi^p_{\beta_{\nu_j,\nu_j}}) : H^1(A_{\beta_{\nu_j,\nu_j}}) \to H^1(A_{\beta_{\nu_j,\nu_j}}),
\]
are injective.

A balanced perverse CDGA is a perverse CDGA which is balanced as perverse cochain complex.

**Remark 8.3.** In the case of a sequence of predecessors of length \( j = 1 \), Property (a) of Definition 8.2 can also be stated as follows:

for any \( \nu \) with \( p(n) \leq p(n_j) \), the map \( \varphi^k_{\beta_{\nu_j,\nu_j}} : A^k_{\beta_{\nu_j,\nu_j}} \to A^k_p \) is an injection, for any \( k \leq p(n) \).

**Definition 8.4.** Let \( A_{\nu} \) be a perverse cochain complex and \( (\beta_{\nu_j,\nu})_{1 \leq i \leq j} \) be an ordered sequence of predecessors of a GM-perversity, \( \nu \). The dotted sum, \( \bigoplus_{i=1}^j A^k_{\beta_{\nu_i,\nu_j}} \), is the cochain complex, quotient of \( \bigoplus_{i=1}^j A^k_{\beta_{\nu_i,\nu_j}} \) by the image of \( \varphi \), i.e.,
\[
\bigoplus_{i=1}^j A^k_{\beta_{\nu_i,\nu_j}} = \bigoplus_{i=1}^j A^k_{\beta_{\nu_i,\nu_j}} / \varphi(\bigoplus_{1 \leq i \leq j} A^k_{\beta_{\nu_i,\nu_j}}).
\]
We denote by \( \langle \alpha_1, \ldots, \alpha_j \rangle \in \bigoplus_{i=1}^j A^k_{\beta_{\nu_i,\nu_j}} \) the class of \( (\alpha_1, \ldots, \alpha_j) \in \bigoplus_{i=1}^j A^k_{\beta_{\nu_i,\nu_j}} \) and by
\[
\bigoplus_{i=1}^j A^k_{\beta_{\nu_i,\nu_j}} \xrightarrow{\psi} \bigoplus_{i=1}^j A^k_{\beta_{\nu_i,\nu_j}} \xrightarrow{\psi} A^k_p
\]
the decomposition of \( \psi \).

The dotted sum, \( \bigoplus \), can also be defined inductively, along the number of predecessors, as shows the second property of the next statement.

**Proposition 8.5.** Let \( A_{\nu} \) be a perverse cochain complex and \( (\beta_{\nu_j,\nu})_{1 \leq i \leq j} \) be an ordered sequence of predecessors of a GM-perversity, \( \nu \). The following properties are satisfied.

(i) Property (a) of Definition 8.2 is equivalent to the injectivity, in degrees \( k \leq p(n_{\nu_j}) - j + 1 \), of the map, \( \bigoplus_{i=1}^j A^k_{\beta_{\nu_i,\nu_j}} \to A^k_p \).

(ii) Let \( (\overline{\psi}_1, \overline{\varphi}) : \bigoplus_{i=2}^j A^k_{\beta_{\nu_i,\nu_j}} \to A^k_{\beta_{\nu_j,\nu_j}} \to \bigoplus_{i=1}^j A^k_{\beta_{\nu_i,\nu_j}} \), where \( \overline{\psi}_1 \) is the map \( \overline{\psi} \) associated to \( p_{\nu} \) and \( \overline{\varphi} \) is induced by \( \varphi \). (See the proof for an explicit description of these maps.)

There is an isomorphism,
\[
\bigoplus_{1 \leq i \leq j} A^k_{\beta_{\nu_i,\nu_j}} \cong (A^k_{\beta_{\nu_j,\nu_j}} \oplus (\bigoplus_{i=2}^j A^k_{\beta_{\nu_i,\nu_j}})) / (\overline{\psi}_1, \overline{\varphi})(\bigoplus_{i=2}^j A^k_{\beta_{\nu_i,\nu_j}}).
\]

In the sequel, we identify these two expressions.

**Proof.** The first property is obvious. As for the second one, let \( (\alpha_1, \ldots, \alpha_j) \in \bigoplus_{i=1}^j A^k_{\beta_{\nu_i,\nu_j}} \) and \( (\beta_{i,\ell})_{1 \leq i \leq j} \in \bigoplus_{1 \leq i \leq j} A^k_{\beta_{\nu_i,\nu_j}} \). Using the notation of Definition 8.2 we define:
$f((\beta_1, \ldots, \beta_k)) = (\beta_1, \ldots, \beta_k) \in \bigoplus_{k=2}^j A_{p_{1\cdots k}},$

$f'(\alpha_1, \ldots, \alpha_j) = (\alpha_1, (\alpha_2, \ldots, \alpha_j)) \in A_{p_{1\cdots k}} \oplus (\bigoplus_{k=2}^j A_{p_{1\cdots k}}),$

$\overline{\psi}_1((\beta_1, \ldots, \beta_{j})) = \overline{\psi}_{p_1}(\beta_1, 2) + \cdots + \overline{\psi}_{p_{1\cdots j}}(\beta_{j}) \in A_{p_{1\cdots j}},$

$\overline{\varphi}((\beta_1, \ldots, \beta_{j})) = \varphi(\beta_1, 2), \ldots, \varphi(\beta_{j}) \in \bigoplus_{\ell=2}^j A_{p_{1\cdots \ell}}.$

The equality $f' \circ \varphi = (\overline{\psi}_1, \overline{\varphi}) \circ f$ follows from the definitions of these maps. In the next commutative diagram, the maps $\Psi$ and $\overline{\Psi}$ are defined as cokernels of $\varphi$ and $(\overline{\psi}_1, \overline{\varphi})$, respectively.

\[
\begin{array}{ccc}
\bigoplus_{1\leq i\leq j} A_{p_{1\cdots i}} & \xrightarrow{\varphi} & \bigoplus_{i=1}^j A_{p_{1\cdots i}} \\
f \downarrow & & \downarrow f' \\
(\bigoplus_{\ell=2}^j A_{p_{1\cdots \ell}}) A_{p_{1\cdots i}} & \xrightarrow{\overline{\psi}_1, \overline{\varphi}} & A_{p_{1\cdots i}} \oplus (\bigoplus_{\ell=2}^j A_{p_{1\cdots \ell}}) \\
& \xrightarrow{f''} & C
\end{array}
\]

As $\overline{\Psi}$ and $f'$ are surjective, the map $f''$ is surjective also. Let $\overline{\varphi} \in \bigoplus_{i=1}^j A_{p_{1\cdots i}}$ such that $f''(x) = 0$. Then there exists $x \in \bigoplus_{1\leq i\leq j} A_{p_{1\cdots i}}$ such that $\Psi(x) = \overline{\varphi}$. By exactitude of the bottom line, there is $z \in \bigoplus_{i=1}^j A_{p_{1\cdots i}}$ such that $(\overline{\psi}_1, \overline{\varphi})(z) = f'(x)$ and $u \in \bigoplus_{1\leq i\leq \ell} A_{p_{1\cdots \ell}}$ such that $f(u) = z$. The element $x - \varphi(u)$ is in the kernel of $f'$, which is the image by $\varphi$ of $\bigoplus_{2\leq i\leq \ell} A_{p_{1\cdots \ell}}$, by definition of $\bigoplus$. This implies, the existence of $v \in \bigoplus_{2\leq \ell\leq \ell} A_{p_{1\cdots \ell}}$ such that $x - \varphi(u) = \varphi(v)$. Finally, $\overline{\varphi} = \Psi(x) = \Psi(\varphi(u + v)) = 0$ and the map $f''$ is injective.

The next statement is a major point in the construction of a minimal model of balanced perverse CDGA’s.

**Corollary 8.6.** Let $(p_{1\cdots j})_{1\leq j \leq \ell}$ be an ordered sequence of predecessors of a GM-perversity, $p$. For any balanced perverse cochain complex, $A_\bullet$, the following properties are satisfied.

(i) The sequence

\[
0 \to \bigoplus_{\ell=2}^j A_{p_{1\cdots \ell}} \overline{\psi}_{p_{1\cdots \ell}} A_{p_{1\cdots \ell}}^k \oplus (\bigoplus_{\ell=2}^j A_{p_{1\cdots \ell}})^k \overline{\varphi} \to \bigoplus_{i=1}^j A_{p_{1\cdots i}}^k \to 0,
\]

is exact, for any $k \leq \overline{p}(n_{\ell}) - j + 2$.

(ii) The homomorphism $H^1(\overline{\psi}) : H^1(\bigoplus_{\ell=1}^j A_{p_{1\cdots \ell}}) \to H^1(A_\overline{\psi})$ is injective.

**Proof.** (i) With property (ii) of Proposition 8.5, the proof is reduced to the fact that the map $(\overline{\psi}_1, \overline{\varphi})$ is injective in the specified degrees, for $j \geq 2$. From the first property of Proposition 8.5, we know that the map $\overline{\psi}_{p_{1\cdots \ell}} : \bigoplus_{\ell=2}^j A_{p_{1\cdots \ell}} \to A_{p_{1\cdots \ell}}$ is injective in degree $k \leq \overline{p}(n_{\ell}) - (j - 1) + 1 = \overline{p}(n_{\ell}) - j + 2$, and the result is established.

(ii) This property is true for $j = 1$ by (a) of Definition 8.2, see Remark 8.3. Let $j \geq 2$ and consider a cocycle, $(\alpha_1, \ldots, \alpha_j) \in A_{p_{1\cdots 1}} + \cdots + A_{p_{1\cdots j}}$, of degree 1 such that there exists $f \in A_{p_{1\cdots 1}}^0$ with $df = \varphi_{p_{1\cdots 1}}(\alpha_1) + \cdots + \varphi_{p_{1\cdots j}}(\alpha_j)$. By (b) of Definition 8.2, there exists $g \in A_{p_{1\cdots j}}^0$ such that $f = \varphi_{p_{1\cdots j}}(g)$ and $(\alpha_1, \ldots, \alpha_j - dg)$ is in the kernel of $\psi: A_{p_{1\cdots 1}}^1 + \cdots + A_{p_{1\cdots j}}^1 \to A_{p_{1\cdots j}}^1$. 

As \( p(n_v) - j + 1 \geq 1 \), we obtain \( \langle \alpha_1, \ldots, \alpha_j - dg \rangle = 0 \) (cf. Proposition 8.5(i)) and \( \langle \alpha_1, \ldots, \alpha_j \rangle = d(0, \ldots, 0, g) \).

If, for any \( p \), the cochain complex \( A_p \) is a subcomplex of \( A_{\infty} \), the ordinary sum of sub vector spaces is defined and can be different from the dotted sum. (We consider the case of \( A(K) \) in Remark 9.4.) But these two sums, + and \( +, \) coincide for free perverse CGA’s.

**Proposition 8.7.** Let \( B = \wedge \oplus_{p} V_p \) be a free perverse CGA. The following properties are satisfied.

(i) For any perversities, \( \bar{p}, \bar{q}_1, \ldots, \bar{q}_j \), we have

\[
B_{\bar{q}_1} \cap B_{\bar{q}_2} = B_{\inf(\bar{q}_1, \bar{q}_2)},
\]

\[
B_{\bar{p}} \cap (B_{\bar{q}_1} + \cdots + B_{\bar{q}_j}) = (B_{\bar{p}} \cap B_{\bar{q}_1}) + \cdots + (B_{\bar{p}} \cap B_{\bar{q}_j}).
\]

(ii) For any ordered sequence, \( (\bar{p}_v)_1 \leq j \), of predecessors of a GM-perversity, \( \bar{p} \), one has

\[
B_{\bar{p}_1} + \cdots + B_{\bar{p}_j} = B_{\bar{p}_1} + \cdots + B_{\bar{p}_j}.
\]

**Proof.** (i) Define \( B_{\bar{p}} = \{ v_1 \ldots v_k \mid v_i \in V_{\bar{q}_i} \} \) and \( \bar{q}_1 + \cdots + \bar{q}_k = \bar{p} \). From Definition 7.7, the equality (16) can also be written as \( B_{\bar{p}} = \oplus_{\tau \leq \bar{p}} B_{\tau} \). The inclusion \( (B_{\bar{p}} \cap B_{\bar{q}_1}) + \cdots + (B_{\bar{p}} \cap B_{\bar{q}_j}) \subset B_{\bar{p}} \cap (B_{\bar{q}_1} + \cdots + B_{\bar{q}_j}) \) being obvious, let \( \omega \in B_{\bar{p}} \cap (B_{\bar{q}_1} + \cdots + B_{\bar{q}_j}) \). Denote by

\[
\mathcal{J}_i = \{ \tau \mid \tau \leq \bar{q}_i \} \text{ and } \mathcal{J} = \{ \tau \mid \tau \leq \bar{p} \}.
\]

We write \( \omega \) as a linear combination of elements of \( B_{(\tau)} \), \( \tau \in \cup_{i=1}^j \mathcal{J}_i \),

\[
\omega = \sum_{i=1}^j \sum_{\tau \in \mathcal{J}_i} v_i,_{(\tau)} \text{ with } v_i,_{(\tau)} \in B_{(\tau)}.
\]

Decompose now each \( \mathcal{J}_i \) in two disjoint subsets

\[
\mathcal{J}_{i,1} = \mathcal{J}_i \cap \mathcal{J} = \{ \tau \mid \tau \leq \bar{q}_i \text{ and } \tau \leq \bar{p} \}, \mathcal{J}_{i,2} = \mathcal{J}_i \setminus \mathcal{J}_{i,1} = \{ \tau \mid \tau \leq \bar{q}_i \text{ and } \tau \not\leq \bar{p} \}.
\]

Observe that

\[
\sum_{i=1}^j \sum_{\tau \in \mathcal{J}_{i,2}} v_i,_{(\tau)} = \omega - \sum_{i=1}^j \sum_{\tau \in \mathcal{J}_{i,1}} v_i,_{(\tau)} \in B_{\bar{p}}.
\]

Any element of \( B_{\bar{p}} \) can be written as a sum of elements of \( B_{(\tau)} \) with \( \tau \in \mathcal{J} \). The intersections \( \mathcal{J} \cap \mathcal{J}_{i,2} \), \( B_{(\tau)} \cap B_{(\sigma)} \) being the empty set if \( \tau \neq \sigma \), we get \( \sum_{i=1}^j \sum_{\tau \in \mathcal{J}_{i,2}} v_i,_{(\tau)} = 0 \) and

\[
\omega = \sum_{i=1}^j \sum_{\tau \in \mathcal{J}_{i,1}} v_i,_{(\tau)} \in (B_{\bar{p}} \cap B_{\bar{q}_1}) + \cdots + (B_{\bar{p}} \cap B_{\bar{q}_j}).
\]

(ii) Let \( j = 2 \). By definition of \( + \) and Property (i), we have \( B_{\bar{p}_{v_1}} + B_{\bar{p}_{v_2}} = B_{\bar{p}_{v_1} + B_{\bar{p}_{v_2}}} \).

From (i), we deduce that \( +_{i \geq 2} B_{\bar{p}_{v_1}} \) is the kernel of \( B_{\bar{p}_{v_1}} \).
\( +_{i \geq 2}B_{\mathfrak{p}} \rightarrow +_{i \geq 1}B_{\mathfrak{p}} \uparrow \). Thus, by induction, we have a commutative diagram,

\[
\begin{array}{cccc}
0 & \rightarrow & +_{i \geq 2}B_{\mathfrak{p},1} & \rightarrow & B_{\mathfrak{p},1} \oplus +_{i \geq 2}B_{\mathfrak{p}} & \rightarrow & +_{i \geq 1}B_{\mathfrak{p}} & \rightarrow & 0 \\
\uparrow \cong && \uparrow \cong && \uparrow \cong && \uparrow \cong && \uparrow \cong
\end{array}
\]

in which the two vertical isomorphisms induce a dotted isomorphism between \( +_{i \geq 1}B_{\mathfrak{p},1} \# \) and \( +_{i \geq 1}B_{\mathfrak{p}} \).

We exhibit now examples of balanced perverse cochain complexes.

**Proposition 8.8.** Let \( F \) be a universal system of coefficients and \( K \) be a filtered face set. Then, the perverse cochain complex of global sections, \( \tilde{F}(K)_\bullet \), and its cohomology, \( H(\tilde{F}(K)_\bullet) \), are balanced.

The proof uses interesting properties of the perverse cochain complex of global sections, that we state independently.

**Lemma 8.9.** Let \( F \) be a universal system of coefficients, \( K \) be a filtered face set, \( \mathfrak{p} \) be a GM-perversity and \( \overline{q} \) be a predecessor of \( \mathfrak{p} \). We denote by \( m \) the associated peak, i.e., \( \mathfrak{p}(i) = \overline{q}(i) \) if \( i \neq m \) and \( \mathfrak{p}(m) = \overline{q}(m) + 1 \). Then the following properties are satisfied.

(i) For any \( k \leq \mathfrak{p}(m) - 2 \), we have \( \tilde{F}(K)^k_\mathfrak{p} = \tilde{F}(K)^k_{\overline{q}} \).

(ii) The morphism \( f^k : H^k(\tilde{F}(K)_{\overline{q}}) \rightarrow H^k(\tilde{F}(K)_\mathfrak{p}) \), induced by the inclusion, \( \tilde{F}(K)_{\overline{q}} \subseteq \tilde{F}(K)_\mathfrak{p} \), is surjective if \( k \leq \mathfrak{p}(m) - 1 \) and injective if \( k \leq \mathfrak{p}(m) \).

**Proof.** (i) Let \( \omega \in \tilde{F}(K)^k_\mathfrak{p} \). We consider two cases:

- if \( \ell \neq m \), then \( \max(\|\omega\|_{\ell}, \|d\omega\|_{\ell}) \leq \mathfrak{p}(\ell) = \overline{q}(\ell) \),
- if \( \ell = m \), then \( \max(\|\omega\|_m, \|d\omega\|_m) \leq k + 1 \leq \mathfrak{p}(m) - 1 = \overline{q}(m) \).

Therefore \( \omega \in \tilde{F}(K)^k_{\overline{q}} \). The reverse inclusion is obvious.

(ii) (a) Let \( \omega \in \tilde{F}(K)^k_\mathfrak{p} \), \( d\omega = 0 \). We consider two cases:

- if \( \ell \neq m \), then \( \|\omega\|_{\ell} \leq \mathfrak{p}(\ell) = \overline{q}(\ell) \),
- if \( \ell = m \), then \( \|\omega\|_m \leq k \leq \mathfrak{p}(m) - 1 = \overline{q}(m) \).

This implies \( \omega \in \tilde{F}(K)^k_{\overline{q}} \) and the surjectivity of \( f^k \).

(b) Let \( \omega \in \tilde{F}(K)^k_{\overline{q}} \) and \( \alpha \in \tilde{F}(K)^{k-1}_{\overline{q}} \) with \( d\alpha = \omega \). We consider two cases:

- if \( \ell \neq m \), then \( \max(\|\alpha\|_{\ell}, \|\omega\|_{\ell}) \leq \overline{q}(\ell) \),
- if \( \ell = m \), then \( \max(\|\alpha\|_m, \|\omega\|_m) \leq \max(k - 1, \overline{q}(m)) = \overline{q}(m) \).

This implies \( \alpha \in \tilde{F}(K)^k_{\overline{q}} \) and the injectivity of \( f^k \).

(Observe that the key point of this proof is the inequality \( \|\omega\| \leq |\omega| \).)

Proof of Proposition 8.3 (i) First, we verify the properties of Definition 8.2 for the perverse cochain complex \( \tilde{F}(K)_\bullet \).

We begin with Property (a). It is trivially verified for \( j = 1 \). Suppose \( j \geq 2 \). Let \( k \leq \mathfrak{p}(n_{r_j}) - j + 1 \) and \( (\omega_i)_{1 \leq i \leq j} \in \oplus_{i=1}^j \tilde{F}(K)_{\mathfrak{p},i} \) such that \( \omega_1 + \cdots + \omega_j = 0 \). If \( j = 2 \),
we obtain $\omega_1 = -\omega_2 \in \tilde{F}(K)^2_{p_{n_1}q_2}$ and $(\omega_1, \omega_2) = \varphi(\omega_1)$, as expected. If $j \geq 3$, from (i) of Lemma 8.9, we have $\omega_i \in \tilde{F}(K)^{p_{n_i}q_j}_{p_{n_j}q_j}$, for any $i \neq j$. As $\omega_j = -\omega_1 - \cdots - \omega_{j-1}$, we can write $(\omega_1, \ldots, \omega_j) = \varphi(\omega_{i,\ell})$, where $(\omega_{i,\ell}) \in \bigoplus_{1 \leq i < \ell \leq j} \tilde{F}(K)^{k}_{p_{n_i}q_j}$ is defined by $\omega_{i,\ell} = 0$ if $\ell \neq j$ and $\omega_{i,j} = \omega_i$, (cf. Definition 8.2 for the construction of $\varphi$).

For proving (b), let $\omega \in \tilde{F}(K)^{0}_{p_{n_j}}$ and $p_{n_j}$ be a predecessor of $p$ with $p(n_{v_j}) \geq 2$. If $\ell \neq n_{v_j}$, then max$(||\omega||_{\ell}, ||d\omega||_{\ell}) \leq p(\ell) = p_{n_j}(\ell)$ and max$(||\omega||_{n_{v_j}}, ||d\omega||_{n_{v_j}}) \leq 1 \leq p(n_{v_j}) - 1 = p_{n_j}(n_{v_j})$. We get $\omega \in \tilde{F}(K)^{0}_{p_{v_j}}$.

As for (c), let $\omega \in \tilde{F}(K)^{0}_{\eta}$ such that $\omega = df$ with $f \in \tilde{F}(K)^{0}_{\eta}$ and $\eta \leq p$. For any $j$, we have, max$(||f||_j, ||df||_j) \leq \max(0, \eta(j)) \leq \eta(j)$, which implies $f \in \tilde{F}(K)^{0}_{\eta}$ and the injectivity of $H^1(\tilde{F}(K)_{\eta}) \to H^1(\tilde{F}(K)_{\eta})$. The same argument works if $\eta = \infty$.

(ii) We study now the perverse cohomology $H(\tilde{F}(K)_{\eta})$. Property (c) comes from the first case and Property (b) is obvious from the previous result on $\tilde{F}(K)^{0}$. Thus, we are reduced to Property (a), which is a consequence of Lemma 8.9(ii) for $j = 1$. Let $j \geq 2$. From Proposition 8.5 we have to prove that the morphism,

$$f^k: H^k(\tilde{F}(K)_{p_{n_j}}) \to H^k(\tilde{F}(K)_{p_{n_j}}),$$

which sends $(\omega_1, \ldots, \omega_j)$ on $f_1(\omega_1) + \cdots + f_j(\omega_j)$, is injective for any $k \leq p(n_{v_j}) - j + 1$. Let

$$\Upsilon = (\omega_1, \ldots, \omega_j) \in H^k(\tilde{F}(K)_{p_{n_j}}),$$

such that $f^k(\Upsilon) = 0$. The map $f_{i,j}: H^k(\tilde{F}(K)_{p_{n_i}q_j}) \to H^k(\tilde{F}(K)_{p_{n_j}})$ being surjective for any $i \in \{1, \ldots, j-1\}$ (see (ii) of Lemma 8.9), we have

$$\Upsilon = (f_{1,j}(\omega'_1), \ldots, f_{j-1,j}(\omega'_{j-1}), \omega_j) = (0, \ldots, 0, [\omega_j]).$$

This implies $0 = f^k(\Upsilon) = f_j([\omega_j])$ and $\Upsilon = 0$ because $f_j$ is injective (see (ii) of Lemma 8.9).

\textbf{Example 8.10.} Let $K$ be a filtered face set. The statement of Lemma 8.9 contains the fact that $H^1(A(K)_{\eta}) = 0$ implies $H^1(A(K)_{p}) = 0$, for any GM-perversity $p$. This example shows that the reverse way is not true.

We choose $n = 3$ and the filtered face set, $K$, associated to the cone on the torus, $c(S^1 \times S^1)$, stratified by the cone point. As a cone is a contractible space, from Proposition 5.4 we have $H^0(A(K)_{\eta}) = H^0(K; \mathbb{Q}) = \mathbb{Q}$ and $H^1(A(K)_{\eta}) = H^1(K; \mathbb{Q}) = 0$. The filtered face set $K$ has one singular stratum and a perversity consists of the integer $\eta(3)$. There are only two possibilities, $\eta(3) = 0$ or 1, the value 1 corresponding to $\tilde{\eta}$. From Example 5.9 we have,

$$H^i(K; \mathbb{Q}) = \begin{cases} H^i(S^1 \times S^1; \mathbb{Q}) & \text{if } i \leq 1, \\ 0 & \text{if } i > 1. \end{cases}$$

Thus we have got $H^1(A(K)_{\eta}) = 0$ and $H^1(A(K)_{\eta}) = \mathbb{Q} \oplus \mathbb{Q}$.
9. Minimal models of balanced perverse CDGA’s

In this section, we construct a Sullivan minimal model, in the sense of Definition 9.1, of any cohomologically connected, balanced perverse CDGA’s. Geometrical applications are given in the next section.

As in the classical case, we need some connectivity hypothesis for the construction of a model.

**Definition 9.1.** A perverse CDGA, $A_\bullet$, is cohomologically connected if $H^0(A_\bullet) = \mathbb{Q}_\bullet$.

**Theorem C** (Construction of a minimal model). Let $A_\bullet$ be a cohomologically connected, balanced perverse CDGA. Then, there exists a Sullivan minimal model of $A_\bullet$.

\[
\rho_\bullet : B_\bullet = (\wedge_{\pi} V_{[\pi]}, d) \to A_\bullet.
\]

i.e., $\rho_\pi(B_\pi) \subset A_\pi$, the restriction $\rho_\pi : B_\pi \to A_\pi$ is a quasi-isomorphism for any $\pi \in 2^n$ and the elements of $V_{[\pi]}$ have a strictly positive degree. This model is unique up to isomorphism.

The construction is done by induction on the degree and on the perverse degree. We first establish some properties of an inductive step, described in the next statement.

**Lemma 9.2.** Let $\pi$ be a GM-perversity of ordered set of predecessors, $(\pi_1, \ldots, \pi_s)$. Let $A_\bullet$ be a cohomologically connected, balanced perverse CDGA and

\[
\rho_\bullet : B_\bullet = (\wedge_{\tau < \pi} V_{[\tau]}, d) \to A_\bullet.
\]

be a morphism of perverse CDGA’s, such that the restriction $\rho_{\tau} : B_{\tau} \to A_{\tau}$ is a quasi-isomorphism and $V^0_{[\tau]} = 0$, for any $\tau < \pi$. Then, the following properties are satisfied.

(i) $B^1_{\pi} = B^1_{\pi_1} + \cdots + B^1_{\pi_s}$.

(ii) Let $(\tau_{v_1}, \ldots, \tau_{v_j})$ be any ordered set of $j$ predecessors of a GM-perversity $\tau$, with $\tau \leq \pi$. We denote by $\nu_{v_i}$ the peak associated to $\tau_{v_i}$. Then, the map $\bar{\rho} : B_{\tau v_1} + \cdots + B_{\tau v_j} \to A_{\nu_{v_1}} + \cdots + A_{\nu_{v_j}}$, generated by $\rho$, is such that $H^k(\bar{\rho})$ is an isomorphism for $k \leq \tau(n_{v_1}) - j + 1$. As a direct consequence, the map $H^1(\bar{\rho}) : H^1(B_{\tau v_1} + \cdots + B_{\tau v_j}) \to H^1(A_{\nu_{v_1}} + \cdots + A_{\nu_{v_j}})$ is an isomorphism for any $j$.

(iii) The map $H^1(\rho_{\tau}) : H^1(B_{\tau}) \to H^1(A_{\tau})$ is injective.

**Proof.** (i) The inclusion $B^1_{\tau v_1} + \cdots + B^1_{\tau v_j} \subset B^1_{\pi}$ being obvious, we consider $\omega \in B^1_{\pi}$. This element can be written as a sum $\omega = \sum \lambda_i \omega_i$, with $\lambda_i$ of degree 0 and $\omega_i \in \wedge_{\tau < \pi} V^1_{[\tau]}$. As $V^0_{[\tau]} = 0$ and $V^1_{[\pi]} = 0$, we have $\lambda_i \in \mathbb{Q}$ and $\omega_i \in B_{\pi_{v_i}}$ for a certain predecessor, $\pi_{v_i}$, of $\pi$. Regrouping these elements, we write $\omega$ as a sum of elements of $(B_{\pi_{v_i}})_{1 \leq i \leq s}$ and the property (i) is proved.

(ii) Observe that this property is true if $j = 1$, by hypothesis. Suppose now that (ii) is satisfied for any ordered sequence of $(j - 1)$ predecessors of any GM-perversity less than or equal to $\pi$. We consider now an ordered family of $j$ predecessors, $(\tau_{v_1}, \ldots, \tau_{v_j})$, of a GM-perversity $\tau$, with $\tau \leq \pi$. With Corollary 8.6, applied to the bottom line, and Proposition 8.7, applied to the upper line, we have a morphism of short exact sequences,
whose vertical maps are induced by $\rho$:

$$
\begin{align*}
0 \to B^*_{r_{v_1,v_2}} + \cdots + B^*_{r_{v_1,v_j}} \to B^*_{r_{v_1}} \oplus (B^*_{r_{v_2}} + \cdots + B^*_{r_{v_j}}) \to B^*_{r_{v_1}} + \cdots + B^*_{r_{v_j}} \to 0 \\
0 \to A^*_{r_{v_1,v_2}} + \cdots + A^*_{r_{v_1,v_j}} \to A^*_{r_{v_1}} \oplus (A^*_{r_{v_2}} + \cdots + A^*_{r_{v_j}}) \to A^*_{r_{v_1}} + \cdots + A^*_{r_{v_j}} \to 0
\end{align*}
$$

for $* \leq \tau(n_{u_j}) - j + 2$. This morphism induces a morphism between long exact sequences and the induction hypothesis, associated to the five lemma, gives the result.

(iii) For getting the injectivity of $H^1(\rho_\bar{\tau}) : H^1(B_\bar{\tau}) \to H^1(A_{\bar{\tau}})$, we decompose it in two maps,

$$
H^1(B_\bar{\tau}) = H^1(B_{\bar{\tau}_1} + \cdots + B_{\bar{\tau}_j}) \to H^1(A_{\bar{\tau}_1} + \cdots + A_{\bar{\tau}_j}) \overset{H^1(\rho_\bar{\tau})}{\to} H^1(A_\bar{\tau}),
$$

where the left-hand equality comes from (i). The injectivity of $H^1(\rho_\bar{\tau})$ comes from Property (ii) and the injectivity of $H^1(\rho_\bar{\tau})$ has been proved in Corollary 8.6.

**Proof of Theorem C** The uniqueness up to isomorphism is a direct consequence of Proposition 7.11 and (2) of Proposition 7.14. The construction of this model begins with a classical minimal model $(\wedge V_{[\bar{\tau}],d})$ of the CDGA $A_{\bar{\tau}}$.

Let $\bar{\tau} \in \check{\mathcal{P}}^n$ be a fixed perversity and suppose we have a morphism of perverse CDGA’s,

$$
\rho_\bullet : B_\bullet = (\wedge \oplus_{\tau < \bar{\tau}} V_{[\tau],d}, d)_\bullet \to A_\bullet,
$$

satisfying the next properties for any GM-perversity $\tau$, with $\tau < \bar{\tau}$,

(i) $V^{\leq 0} = 0$,

(ii) $\rho(B_\tau) \subset A_\tau$ and the restriction $\rho_\tau : B_\tau \to A_\tau$ is a quasi-isomorphism.

For having a model for the perversity $\bar{\tau}$, we have to construct

$$
\rho_\bullet : B_\bullet = (\wedge \oplus_{\tau \leq \bar{\tau}} V_{[\tau],d}, d)_\bullet \to A_\bullet,
$$

verifying (i) and (ii) for $\tau \leq \bar{\tau}$. Suppose that we have extended $\rho_\bullet$ in a morphism of perverse CDGA’s, still denoted $\rho_\bullet$,

$$
\rho_\bullet : B'_\bullet = (B \otimes \wedge V_{[\bar{\tau}],d})_{\bullet} = (\wedge (\oplus_{\tau \leq \bar{\tau}} V_{[\tau],d}) + \wedge V_{[m]})_{\bullet} \to A_\bullet,
$$

such that

(iii) $V^{\leq 0} = 0$,

(iv) $\rho(V^{\leq m}) \subset A_\tau$ and the restriction $\rho_\tau : B'_\tau \to A_\tau$ verifies,

a) $H^i(\rho_\tau)$ is an isomorphism for any $i \leq m - 1$,

b) $H^i(\rho_\tau)$ is injective if $i = m$.

If $\tau < \bar{\tau}$, as $B_\tau = B'_\tau$, the map $\rho_\tau : B'_\tau \to A_\tau$ is a quasi-isomorphism. Moreover, when this construction is performed for any $m$, by denoting $V_{[\bar{\tau}]} = \oplus_m V_{[m]}$, we get a morphism,

$$
\rho_\bullet : (\wedge \oplus_{\tau \leq \bar{\tau}} V_{[\tau],d}, d)_\bullet \to A_\bullet,
$$

satisfying (i) and (ii) for any $\tau \leq \bar{\tau}$. Thus, we are reduced to properties (iii) and (iv).

In degree $m = 0$, as $V^{\leq 0} = 0$ for any $\tau \leq \bar{\tau}$, we have $(\wedge \oplus_{\tau \leq \bar{\tau}} V_{[\tau],0})_\bullet = Q_\bullet$ and the map $\rho_\bullet : Q_\bullet \to A_\bullet$ is the canonical inclusion. The perverse CDGA, $A_\bullet$, being cohomologically connected by hypothesis, $H^0(\rho_\bullet)_\tau$ is an isomorphism, for any $\tau \leq \bar{\tau}$. 


Observe that Lemma 9.2 implies properties (iii) and (iv) for \( m = 1 \). We show now that the morphism \( \rho_\bullet \), defined on \( B_\bullet \), can be extended in a morphism of perversive CDGA’s, \( \rho_\bullet : (B' \otimes \Lambda V_{[p]}^m, d) \rightarrow A_\bullet \), so that properties (iii) and (iv) are satisfied for \( m + 1 \). We set

\[
\begin{align*}
Y_{[p]}^m[0] & = \text{Coker} \, H^m(\rho_\bullet) : H^m(B_\bullet) \rightarrow H^m(A_\bullet), \\
Z_{[p]}^m[0] & = \text{Ker} \, H^{m+1}(\rho_\bullet) : H^{m+1}(B_\bullet) \rightarrow H^{m+1}(A_\bullet), \\
V_{[p]}^m[0] & = Y_{[p]}^m[0] \oplus Z_{[p]}^m[0].
\end{align*}
\]

We extend the differential \( d \) and the morphism \( \rho_\bullet \) by:

- if \( y \in Y_{[p]}^m[0] \), then \( dy = 0 \) and \( \rho(y) \in A_{[p]}^m \) is a cocycle representing the class corresponding to \( y \in \text{Coker} \, H^m(\rho_\bullet) \),
- if \( z \in Z_{[p]}^m[0] \), then \( dz \in B_{[p]}' \) is a cocycle representing the class corresponding to \( z \in \text{Ker} \, H^{m+1}(\rho_\bullet) \). The element \( \rho(z) \in A_{[p]}^m \) is defined by the equality \( \rho \circ d = d \circ \rho \).

By construction, we have

\[
d(V_{[p]}^m[0]) \subset B_{[p]}' = (B \otimes \Lambda V_{[p]}^m)_{[p]} \quad \text{and} \quad \rho(V_{[p]}^m[0]) \subset A_{[p]}.\]

Let \( j \) be a fixed integer. Suppose we have defined, for all \( i \in \{0, \ldots, j\} \), \( V_{[p]}^m[i] \), \( \rho \) and \( d \) on \( B' \otimes \Lambda_{i \leq j} V_{[p]}^m[i] \) so that, for all \( k \leq j \),

\[
\begin{align*}
& d(V_{[p]}^m[i]) \subseteq (B' \otimes \Lambda_{i \leq k-1} V_{[p]}^m[i])_{[p]}, \\
& \rho \circ d = d \circ \rho \quad \text{and} \quad \rho(V_{[p]}^m[k]) \subseteq A_{[p]}.
\end{align*}
\]

We set

\[ V_{[p]}^m[j+1] = \text{Ker} \, H^{m+1}(\rho_\bullet) : H^{m+1}(B' \otimes \Lambda_{i \leq j} V_{[p]}^m[i])_{[p]} \rightarrow H^{m+1}(A_{[p]}). \]

If \( z \in V_{[p]}^m[j+1] \), then \( dz \in (B' \otimes \Lambda_{i \leq j} V_{[p]}^m[i])_{[p]} \) is a cocycle representing to \( z \) in \( \text{Ker} \, H^{m+1}(\rho_\bullet) \). The element \( \rho(z) \in A_{[p]}^m \) is defined by the equality \( \rho \circ d = d \circ \rho \). Finally, we set

\[ V_{[p]}^m = \otimes_{i \geq 0} V_{[p]}^m[i]. \]

We verify now properties (iii), (iv) for \( \rho_\bullet : (B' \otimes \Lambda V_{[p]}^m, d) \rightarrow A_\bullet \). Property (iii) is satisfied by construction and we are reduced to Property (iv).

- If \( i \leq m - 1 \), then \( (B' \otimes \Lambda V_{[p]}^m)_{[p]} = B'_{[p]} \) and \( H^i(\rho_\bullet) \) is an isomorphism.
- The morphism \( H^m(\rho_\bullet) \) is surjective by construction of \( Y_{[p]}^m[0] \).
- The morphism \( H^m(\rho_\bullet) \) is injective. To prove that, we consider \( \omega \in (B' \otimes \Lambda V_{[p]}^m)_{[p]} \) such that \( d\omega = 0 \) and \( \rho(\omega) \) is a boundary in \( A_{[p]}^m \). There exists \( j \geq -1 \) with \( \omega \in B' \otimes \Lambda_{i \leq j+1} V_{[p]}^m[i] \). We set \( \tilde{B} = B' \otimes \Lambda_{i \leq j} V_{[p]}^m[i] \) and decompose \( \omega \) in

\[ \omega = \omega_1 + \omega_2 \in (B'_{[p]} \otimes V_{[p]}^m[j+1]) \oplus B'^m. \]

As, by induction, \( B'_{[p]} = \mathbb{Q} \), we have \( \omega_1 \in V_{[p]}^m[j+1] \). The equality \( d\omega = 0 \) implies \( d\omega_1 \in d(\tilde{B}^m) \). By construction of \( V_{[p]}^m[j+1] \), see (19), this implies \( \omega_1 = 0 \) and \( \omega \in B' \otimes \Lambda_{i \leq j} V_{[p]}^m[i] \). We iterate this process and we obtain \( \omega \in B' \otimes \Lambda V_{[p]}^m[0] \).

As \( \rho(\omega) = 0 \), we have \( \omega \in B' \otimes Z_{[p]}^m[0] \) that we decompose, as above, in

\[ \omega = \omega_1 + \omega_2 \in Z_{[p]}^m[0] \oplus \tilde{B}^m. \]

The equality \( d\omega = 0 \) implies \( d\omega_1 \in d(\tilde{B}^m) \) and,
by construction of $Z^m_n[0]$, we get $\omega_1 = 0$ and $\omega \in B'_{\mathcal{P}}$. Now hypothesis (iv) b) implies the nullity in $B$ of the cohomology class associated to $\omega$.

- The morphism $H^{m+1}(\rho_{\mathcal{P}})$ is injective. Let $\omega \in (B' \otimes \Lambda^m_{[\mathcal{P}]})^{m+1}$ such that $d\omega = 0$ and $\rho(\omega)$ is a boundary in $A^m_{\mathcal{P}}$. There exists an integer $j \geq -1$ with $\omega \in (B' \otimes \Lambda_{i \leq j+1} V^m_{[\mathcal{P}]}[i])^{m+1}$ and, by construction of $V^m_{[\mathcal{P}]}[j+2]$, there exists also an element $v \in V^m_{[\mathcal{P}]}[j+2]$ such that $\omega - d(1 \otimes v)$ is a boundary in $B' \otimes \Lambda_{i \leq j+1} V^m_{[\mathcal{P}]}[i]$. Thus $\omega$ is a boundary in $B' \otimes \Lambda^m_{[\mathcal{P}]}$.

Set $V^m_{[\mathcal{P}]}(i) = \oplus_{j \leq i} V^m_{[\mathcal{P}]}[i]$. By construction, the differential, $d$, verifies, for all $i \geq 0$,

$$dV^m_{[\mathcal{P}]}[i] \subset \left( \Lambda_{\mathcal{P}} \otimes (\Lambda^\leq V^m_{[\mathcal{P}]}) \otimes (\Lambda^\leq V^m_{[\mathcal{P}])(i-1)} \right)_{[\mathcal{P}]}.$$ 

This is Property [17] and we have constructed a Sullivan minimal model in the sense of Definition [7,8].

**Example 9.3.** Consider a lattice of perversities containing

![Diagram of perversities]

Suppose that $A_{\bullet}$ is a perverse CDGA such that $A_{\mathcal{P}_6} = \mathbb{Q} e$ with $|e| = 2$ and $H^+(A_{\mathcal{P}_6}) = 0$ if $i < 8$. Then the minimal model will contain elements,

- of degree 1, $x$ of perversity $\mathcal{P}_5$, $y$ of perversity $\mathcal{P}_6$ and $z$ of perversity $\mathcal{P}_7$ such that $dx = dy = dz = e$,
- of degree 0, $f$ of perversity $\mathcal{P}_2$, $g$ of perversity $\mathcal{P}_3$, $h$ of perversity $\mathcal{P}_4$ such that $df = x - y$, $dg = x - z$, $dh = y - z$,
- of degree -1, $\alpha$ of perversity $\mathcal{P}_1$ such that $d\alpha = f - g + h$.

This example illustrates that we cannot have a model with generators of positive degree for an arbitrary perverse CDGA. As $H^1(A_{\mathcal{P}_1} + A_{\mathcal{P}_2}) \to H^1(A_{\mathcal{P}_1})$ is not injective, this perverse CDGA is not balanced, see Corollary [8,6](ii). Moreover, it cannot be the perverse CDGA of forms on a filtered face set, as shows Proposition [10,1].

**Remark 9.4.** In general, the operations $+$ and $\dot{+}$, applied to the perverse CDGA of forms on a filtered face set, $K$, are different. More precisely, we have $A(K)_{\mathcal{P}_1} + A(K)_{\mathcal{P}_2} = A(K)_{\mathcal{P}_1} \dot{+} A(K)_{\mathcal{P}_2}$ but for a sum of three terms or more these operations do not coincide. For instance, in the case of three terms, they correspond to the next short exact sequences,

$$0 \rightarrow A_{\mathcal{P}_1} \cap (A_{\mathcal{P}_2} + A_{\mathcal{P}_3}) \rightarrow A_{\mathcal{P}_1} \oplus (A_{\mathcal{P}_2} + A_{\mathcal{P}_3}) \rightarrow A_{\mathcal{P}_1} + A_{\mathcal{P}_2} + A_{\mathcal{P}_3} \rightarrow 0$$
and
\[ 0 \to (A_{\mathcal{F}_1} \cap A_{\mathcal{F}_2}) + (A_{\mathcal{F}_1} \cap A_{\mathcal{F}_3}) + (A_{\mathcal{F}_1} \cap A_{\mathcal{F}_2} + A_{\mathcal{F}_3}) + A_{\mathcal{F}_1} + A_{\mathcal{F}_2} + A_{\mathcal{F}_3} \to 0, \]
where we have denoted \( A(K) \) by \( A \). But the equality \( A_{\mathcal{F}_1} \cap (A_{\mathcal{F}_2} + A_{\mathcal{F}_3}) = (A_{\mathcal{F}_1} \cap A_{\mathcal{F}_2}) + (A_{\mathcal{F}_1} \cap A_{\mathcal{F}_3}) \) is true only in degree less than, or equal to, \( p(n_3) - 1 \). This can be proved by arguments similar to those in the proof of Lemma 8.9; we do not need this result in the sequel.

### 10. Minimal models of filtered spaces

The construction of a Sullivan minimal model can be done in the cases of the algebra of perverse forms on a connected filtered face set, \( K \), and of its cohomology. Moreover, in the case \( K \) is coming from a pseudomanifold, we prove that the model we have built is a topological invariant.

Recall, from Definition 1.10, that a filtered face set, \( K \), is connected if the face set \( K^{[0]} \) is connected. For a \( \Gamma \)-perversity \( p \), it is easy to see (cf. Remark 4.6) that the only \( p \)-admissible 0-simplices and 1-simplices are elements of \( K^{[0]} \), thus, if \( K \) is connected, the perverse CDGA, \( \mathcal{A}(K)_* = \tilde{A}_{PL,*}(K) \), is cohomologically connected. The next result is a direct consequence of Proposition 8.8.

**Proposition 10.1.** The perverse CDGA of forms on a filtered face set, \( K \) and their cohomology are balanced. Therefore, if \( K \) is connected, the perverse CDGA’s, \( \mathcal{A}(K)_* \), and \( H(\mathcal{A}(K)_*) \) with the trivial differential, admit Sullivan minimal models.

The next definition arises naturally from Proposition 10.1.

**Definition 10.2.** If \( K \) is a connected filtered face set, the Sullivan minimal model of \( \mathcal{A}(K)_* \) is called the minimal model of \( K \).

In perverse degree \( \eta = 0 \), the Sullivan minimal model of a normal filtered face set, \( K \), is the minimal model of the associated face set.

**Proposition 10.3.** Let \( K \) be a connected normal filtered face set, of minimal model \( (\wedge \oplus_{\eta \in \mathbb{Z}^n} V_{[\eta]}, d)_* \). Then \( (\wedge V_{[\eta]}, d) \) is a minimal model of the face set associated to \( K \).

**Proof.** For that, we need a version of Proposition 5.4 taking in account the structure of algebras. Let \( \Delta = \Delta^{j_0} \times \cdots \times \Delta^{j_n} \) be a filtered simplex and \( \mu: \Delta = c\Delta^{j_0} \times \cdots \times c\Delta^{j_{n-1}} \times \Delta^{j_n} \to \Delta \) its blow-up. We work with the coordinates introduced in Remark 2.9, where \( \mu \) sends the element \( ((a_{j_0}^0, \ldots, a_{j_0}^0, t_0), (a_{j_0}^1, \ldots, a_{j_1}^1, t_1), \ldots, (a_{j_n}^n, \ldots, a_{j_n}^n)) \) on \( (a_{j_0}^0, t_0 a_{j_0}^1, \ldots, t_0 a_{j_0}^n, t_0 t_1 a_{j_1}^2, \ldots, t_0 t_1 t_2 a_{j_2}^2, \ldots, t_0 \cdots t_{n-1} a_{j_{n-1}}^n, t_0 \cdots t_{n-1} a_{j_n}^n) \).

Denote by \( (u_i)_{0 \leq i \leq j_0 + \cdots + j_n + n} \) the barycentric coordinates of \( \Delta \) and by \( (v_i^k)_{0 \leq i \leq j_k} \) the barycentric coordinates of \( \Delta^{j_k} \). The map \( \mu \) induces a map \( \mu^*: A_{PL}(\Delta) \to \tilde{A}_{PL}(\Delta) = A_{PL}(c\Delta^{j_0} \times \cdots \times A_{PL}(\Delta^{j_n})) \), defined by \( \mu^*(u_i) = \)

\[
\begin{align*}
v^0_1 \otimes 1 \otimes \cdots \otimes 1, & \quad \text{if } 0 \leq i \leq j_0 \\
t_0 \otimes v^1_{j_0-1} \otimes 1 \otimes \cdots \otimes 1, & \quad \text{if } j_0 + 1 \leq i \leq j_0 + j_1 + 1 \\
t_0 \otimes t_1 \otimes v^2_{j_0-2-j_1} \otimes 1 \otimes \cdots \otimes 1, & \quad \text{if } j_0 + j_1 + 2 \leq i \leq j_0 + j_1 + j_2 + 2 \\
& \quad \vdots \\
t_0 \otimes t_1 \otimes \cdots \otimes t_{n-1} \otimes v^n_{j_0-\cdots-j_{n-1}-n} & \quad \text{if } j_0 + \cdots + j_{n-1} + n \leq i \leq j_0 + j_1 + \cdots + j_n + n.
\end{align*}
\]
The faces in $\tilde{\Delta}$ having a factor $\Delta^{n-\ell} \times \{1\}$, for $\ell \in \{1, \ldots, n\}$, are characterized by $t_\ell = 0$. Thus, the forms $\mu^*(u_i)$ are of perverse degree 0 and we have $\mu^*(u_i) \in \tilde{A}_{PL,0}(\Delta)$.

This local construction generates a morphism of CDGA’s, $A_{PL}(K) \to \tilde{A}_{PL,0}(K)$, which satisfies the hypothesis of Theorem A for any connected normal filtered face set, $K$.

Thus, in the next diagram,

$$
\begin{array}{ccc}
A_{PL}(K) & \xrightarrow{\mu^*} & \tilde{A}_{PL,0}(K) \\
\downarrow & & \downarrow \rho_0 \\
(\wedge V_{[0]}, d) & & (\wedge V_{[0]}, d),
\end{array}
$$

the morphism $\mu^*$ is a quasi-isomorphism (Theorem A). Let $\rho_0$ be the minimal model of $\tilde{A}_{PL,0}(K)$. As $(\wedge V_{[0]}, d)$ is a cofibrant CDGA, there exists a morphism of CDGA’s, $\psi$, making the diagram commutative up to homotopy, and $\psi$ is a quasi-isomorphism. □

We may also compare the structure of products induced, in cohomology, on the blow-ups of Sullivan’s forms and on the Thom-Whitney cochains, on one side, with the product defined by G. Friedman and J. McClure (20) in the topological case.

**Proposition 10.4.** Let $K$ be a filtered face set. Then, the integration map, $\int: \tilde{A}_{PL}(K) \to \tilde{C}^*(K)$, induces an isomorphism of perverse algebras,

$$
H^*_\bullet(K; \tilde{A}_{PL}) \cong H^*_\bullet(K; \tilde{C}).
$$

If $X$ is an $n$-dimensional pseudomanifold and $K = \text{ISing}^T(X)$, the cup-product of the previous perverse cohomology algebras coincides with the cup-product defined in [20].

**Proof.** We use an argument due to C. Watkiss (unpublished) in the classical case of Sullivan’s theory. The two inclusions of universal systems of DGA’s,

$$
A_{PL}(\Delta^n) \longrightarrow (A_{PL} \otimes C^*)(\Delta^n) \longrightarrow C^*(\Delta^n),
$$

satisfy the hypotheses of Theorem A and induce an isomorphism of perverse CDGA’s,

$$
H^*_\bullet(K; \tilde{A}_{PL}) \cong H^*_\bullet(K; A_{PL} \otimes C) \cong H^*_\bullet(K; \tilde{C}).
$$

This establishes the first assertion. The last part of the statement is proved in [10, End of Section 4]. □

To any filtered space $X$ (Definition A.1) whose regular part is connected, we associate a filtered face set, $L = \text{ISing}^T(X)$, such that $A(L)_\bullet$ is a cohomologically connected, balanced perverse CDGA, see Proposition 8.8. Thus, by Theorem C, a perverse minimal model of $A(L)_\bullet$ exists. We call it the perverse minimal model of $X$. In the case $X$ is normal, the 0-part of this model is the minimal model of $X$, as shows Proposition 10.3.

The next result is the topological invariance of the minimal model of $\text{ISing}^T(X)$.

**Theorem D.** Let $X$ be a PL-pseudomanifold whose regular part is connected. Then, the minimal model of $X$ does not depend on the stratification of $X$, in GM-perversity degrees strictly less than $\infty$. 
As the \( \infty \)-intersection cohomology of a filtered face set, \( K \), is the ordinary cohomology of its regular part (Proposition 5.1), we cannot expect any topological invariance for the perversity \( \infty \). The simple example of a sphere \( S^2 \) with strata of codimension 0 shows that the regular part can have different homotopy types. Nevertheless, the knowledge of a model of the regular part of a PL-pseudomanifold reveals crucial for some computation, see Proposition 11.15.

**Proof of Theorem A** In [32, Page 150], King associates to any CS set, \( X \), a CS set, \( X^* \), which is an intrinsic coarsest stratification of \( X \). In particular, the identity map induces a stratified map, \( X \to X^* \), in the sense of Definition 3.2. In [19, Chapter 2], G. Friedman proves that the CS set, \( X^* \), associated to a PL-pseudomanifold is also a PL-pseudomanifold. We begin by proving that the regular part of \( X^* \) is connected.

Let \( R = X \setminus X_{n-1} \) and \( R^* = X^* \setminus X_{n-1}^* \) be the regular parts of \( X \) and \( X^* \), respectively. By construction, we have \( R \subset R^* \). If \( R^* \) is the union of two disjoint open sets, \( R^* = U_1 \cup U_2 \), then we have \( R = (R \cap U_1) \cup (R \cap U_2) \). As \( R \) is connected, by hypothesis, and open, we must have \( U_1 \cap R = \emptyset \) or \( U_2 \cap R = \emptyset \). The open set \( R \) being dense in \( X \), we obtain \( U_1 = \emptyset \) or \( U_2 = \emptyset \), which implies the connectivity of \( R^* \).

We denote by \( L \) and \( L^* \) the filtered face sets associated to \( X \) and \( X^* \), respectively, and consider the following diagram whose elements are detailed below,

\[
\begin{array}{ccc}
\tilde{A}_{PL,pt}(L^*) & \xrightarrow{\varphi_1} & \tilde{A}_{PL,pt}(L) \\
\downarrow f_L^* & & \downarrow f_L \\
\tilde{C}_q^r(L^*) & \xrightarrow{\varphi_2} & \tilde{C}_q^r(L) \\
\downarrow g_L^* & & \downarrow g_L \\
C_{GM,pt}(L^*) = \text{hom}(C^r_q(X^*), \mathbb{Q}) & \xrightarrow{\varphi_3} & C_{GM,pt}(L) = \text{hom}(C^r_q(X), \mathbb{Q}) \\
\downarrow h_L^* & & \downarrow h_L \\
\text{hom}(K^r_q(X^*), \mathbb{Q}) & \xrightarrow{\varphi_4} & \text{hom}(K^r_q(X), \mathbb{Q})
\end{array}
\]

The perversities \( p \) and \( q \) are elements of \( \mathcal{P} \) such that \( p(k) + q(k) = k - 2 \), if \( k \geq 2 \). The bottom square commutes by construction and the vertical maps are quasi-isomorphisms, as this is proved in

- Corollary 4.5 for \( f_L, f_L^*, g_L^* \), and \( g_L \).
- and, as \( X^* \) is a PL-pseudomanifold, we can use Proposition 4.4 for \( h_L^* \) and \( h_L \).

The proof that \( \varphi_3 \) is a quasi-isomorphism being done in [32, Theorem 9], the proof of Theorem 3 is reduced to the construction of \( \varphi_1 \) making the diagram commutative.

From Theorem 3 (see also Corollary 3.9), we know that the stratified map, \( X \to X^* \), induces an amalgamation of the singular simplices, generated by elementary amalgamations of the type \( \Delta^p \star \Delta^q \to \emptyset \star \Delta^{p+q+1} \). To prove that these amalgamations induce a morphism of perverse CDGA’s, \( \varphi_1 : \tilde{A}_{PL,pt}(L^*) \to \tilde{A}_{PL,pt}(L) \), making commutative the previous diagram, it is sufficient to show it locally. This is done in Lemma 10.5 \( \square \)
Lemma 10.5. Let \( i \in \{0, \ldots, n-1\} \). We denote by \( \Phi_i \) the identity map on a simplex \( \Delta \), corresponding to the elementary amalgamation

\[
\Delta^{j_0} \ast \cdots \ast \Delta^{j_n} \mapsto \Delta^{k_0} \ast \cdots \ast \Delta^{k_n},
\]

with

\[
k_a = j_a, \quad \text{if} \quad a < i \text{ or } a > i + 1, \\
k_i = -1, \quad \text{and} \quad k_{i+1} = j_i + j_{i+1} + 1.
\]

Let \( \tilde{\Delta}_1 = c\Delta^{j_0} \ast \cdots \ast \Delta^{j_n} \rightarrow \Delta \) and \( \mu' : \tilde{\Delta}_2 = c\Delta^{k_0} \ast \cdots \ast \Delta^{k_n} \rightarrow \Delta \) be the two blow-ups. Then, for all \( i \in \{0, \ldots, n-1\} \), the map \( \Phi_i \) lifts in a map \( \tilde{\Phi}_i \), between the two blow-ups, compatible with the face operators of \( \Delta \) and such that \( \mu' \circ \tilde{\Phi}_i = \mu \). The induced cochain map, \( \tilde{\Phi}_i^* \), is such that, the following diagram commutes,

\[
\begin{array}{ccc}
A_{PL}(c\Delta^{j_0}) \otimes \cdots \otimes A_{PL}(\Delta^{j_n}) & \xrightarrow{f} & \mathbb{Q} \\
\Phi_i^* \downarrow & & \downarrow f \\
A_{PL}(c\Delta^{k_0}) \otimes \cdots \otimes A_{PL}(\Delta^{k_n}) & & \\
\end{array}
\]

where \( \int \omega = \int_{\Delta} (\mu^{-1})^* \omega \) if \( \omega \in A_{PL}(c\Delta^{j_0}) \otimes \cdots \otimes A_{PL}(\Delta^{j_n}) \) and a similar formula for the second integral map. Moreover, the map \( \tilde{\Phi}_i^* \) verifies

\[
\tilde{\Phi}_i^*(\tilde{A}_{PL,\bar{\sigma}}(\Delta^{k_0} \ast \cdots \ast \Delta^{k_n})) \subset \tilde{A}_{PL,\bar{\sigma}}(\Delta^{j_0} \ast \cdots \ast \Delta^{j_n}),
\]

for all positive loose perversity \( \bar{\sigma} \).

Proof. Recall, from Remark 2.9, that the two blow-ups can be described by

\[
\mu((x_0, t_0), \ldots, (x_{n-1}, t_{n-1}), x_n) = x_0 + t_0 x_1 + t_0 t_1 x_2 + \cdots + t_0 \cdots t_{n-1} x_n
\]

with \( x_i = (x_{i,0}, \ldots, x_{i,j_i}) \in \mathbb{R}^{j_i+1} \), \( t_i \in \mathbb{R} \), \( t_i + \sum_{k=0}^{j_i} x_{i,k} = 1 \), for all \( i \), and a similar formula for \( \mu' \). (With this setting, in the particular case \( \Delta^{j_n} = \emptyset \), we have \( c\Delta^{j_n} = \{(0, 1)\} \).) For the study of \( \Phi_i \), we have two cases, depending if \( i + 1 = n \) or not.

1) We begin with \( i + 1 \neq n \). We construct a map \( \tilde{\Phi}_i : c\Delta^{j_0} \ast \cdots \ast \Delta^{j_n} \rightarrow c\Delta^{k_0} \ast \cdots \ast \Delta^{k_n} \) by \( \tilde{\Phi}_i = \text{id} \times f_i \times \text{id} \), where

\[
f_i : c\Delta^{j_i} \times c\Delta^{j_{i+1}} \rightarrow c\emptyset \times c(\Delta^{j_i} \ast \Delta^{j_{i+1}})
\]

is defined by

\[
f_i((x_i, t_i), (x_{i+1}, t_{i+1})) = ((0, 1), ((x_i, t_i x_{i+1}), t_i t_{i+1})).
\]

We check easily from the definition that the map \( \tilde{\Phi}_i \) verifies \( \mu' \circ \tilde{\Phi}_i = \mu \) and is compatible with face operators. Thus it induces a CDGA’s map \( \tilde{\Phi}_i^* : A_{PL}(c\Delta^{k_0}) \otimes \cdots \otimes A_{PL}(\Delta^{k_n}) \rightarrow A_{PL}(c\Delta^{j_0}) \otimes \cdots \otimes A_{PL}(\Delta^{j_n}) \).

Let \( \omega = \omega_0 \otimes \cdots \otimes \omega_n \in A_{PL}(c\Delta^{k_0}) \otimes \cdots \otimes A_{PL}(\Delta^{k_n}) \). In the next equalities, we use the fact that \( \mu \) and \( \mu' \) are diffeomorphisms on the interior of the integration domains,

\[
\int_{\tilde{\Delta}_2} \omega = \int_{\Delta} (\mu'^{-1})^* \omega = \int_{\Delta} ((\mu^{-1})^* \circ \tilde{\Phi}_i^*) (\omega) = \int_{\tilde{\Delta}_1} \tilde{\Phi}_i^* (\omega).
\]
This proves the equality \( f \circ \tilde{\Phi}_i = f \). We show now the compatibility with perversities, i.e.,

\[
\|\omega\|_\ell \leq \overline{q}(\ell) \Rightarrow \|\tilde{\Phi}_i^* \omega\|_\ell \leq \overline{q}(\ell),
\]

for all \( \ell \in \{1, \ldots, n\} \). If \( \Delta_{j_{n-\ell}} = \emptyset \), the previous implication follows from \( \|\tilde{\Phi}_i^* \omega\|_\ell = -\infty \).

Suppose now that \( \Delta_{j_{n-\ell}} \neq \emptyset \).

— Consider first \( n - \ell = 0 \), with \( i \neq 0 \). (The case \( i = 0, \ell = n \) is done below.) The commutativity of the next diagram,

\[
\begin{array}{c}
\Delta_j \times \{1\} \times \cdots \times c\Delta_{\bar{j}_i} \times c\Delta_{\bar{j}_i+1} \times \cdots \times \Delta_j^n \\
\Phi_i \\
\end{array}
\]

\[
\begin{array}{c}
\Delta_j \times \{1\} \times \cdots \times c\emptyset \times c(\Delta_{\bar{j}_i} \times \Delta_{\bar{j}_i+1}) \times \cdots \times \Delta_j^n,
\end{array}
\]

implies

\[
\|\tilde{\Phi}_i^*(\omega)\|_\ell \leq \|\omega\|_\ell,
\]

since the projections \( \text{pr} \) are used for the determination of \( \| - \|_\ell \), see Remark 2.12. A similar argument works for all the perverse degrees, except in the cases \( n - \ell = i \) and \( n - \ell = i + 1 \).

— We continue with \( n - \ell = i + 1 \). In this case, the restriction of the map \( \tilde{\Phi}_i = \text{id} \times f_i \times \text{id} \),

\[
c\Delta_j \times \cdots \times c \Delta_{\bar{j}_i} \times (\Delta_{\bar{j}_i+1} \times \{1\}) \times \cdots \times \Delta_j^n
\]

\[\tilde{\Phi}_i = \text{id} \times f_i \times \text{id} \]

\[c\Delta_j \times \cdots \times c \emptyset \times ((\Delta_{\bar{j}_i} \times \Delta_{\bar{j}_i+1}) \times \{1\}) \times \cdots \times \Delta_j^n,
\]

is defined by \( f_i : c\Delta_j \times (\Delta_{\bar{j}_i+1} \times \{1\}) \rightarrow c\emptyset \times ((\Delta_{\bar{j}_i} \times \Delta_{\bar{j}_i+1}) \times \{1\}) \), which sends the element \((x, t), (y, 0)\) on \((0, 1), ((x, ty), 0)\). (Observe that the face \( \Delta_{\bar{j}_i} \times \{1\} \) of \( c\Delta_{\bar{j}_i} \), used in the definition of perverse degree, corresponds to \( t = 0 \), see Remark 2.9.) We denote by \( \text{pr}_1 \) the projection of the domain of \( \tilde{\Phi}_i \) on \( c\Delta_j \times \cdots \times c \Delta_{\bar{j}_i} \times (\Delta_{\bar{j}_i+1} \times \{1\}) \), which is used for the determination of \( \|\tilde{\Phi}_i^*(\omega)\|_\ell \), and by \( \text{pr}_2 \) the projection of the codomain of \( \tilde{\Phi}_i \) on \( c\Delta_j \times \cdots \times c \emptyset \times ((\Delta_{\bar{j}_i} \times \Delta_{\bar{j}_i+1}) \times \{1\}) \), which is used for the determination of \( \|\omega\|_\ell \).

The equality \( \text{pr}_2 \circ \tilde{\Phi}_i = (\text{id} \times f_i) \circ \text{pr}_1 \) implies

\[
\|\tilde{\Phi}_i^*(\omega)\|_\ell \leq \|\omega\|_\ell.
\]

— The last perversity we have to study corresponds to \( n - \ell = i \). The restriction of \( \tilde{\Phi}_i \) to \( \tilde{S}_i = c\Delta_j \times \cdots \times (\Delta_{\bar{j}_i} \times \{1\}) \times c\Delta_{\bar{j}_i+1} \times \cdots \times \Delta_j^n \) is defined by \( f_i((x, 0), (y, s)) = ((0, 1), ((x, 0), 0)) \). Thus we have

\[
\tilde{\Phi}_i(\tilde{S}_i) = c\Delta_j \times \cdots \times c \emptyset \times (\Delta_{\bar{j}_i} \times \{1\}) \times c\Delta_{\bar{j}_i+2} \times \cdots \times \Delta_j^n \subseteq c\Delta_j \times \cdots \times c \emptyset \times (\Delta_{\bar{j}_i} \times \Delta_{\bar{j}_i+1} \times \{1\}) \times c\Delta_{\bar{j}_i+2} \times \cdots \times \Delta_j^n.
\]

We denote by \( \text{pr}_1 \) the projection of \( \tilde{S}_i \) on \( c\Delta_j \times \cdots \times c \Delta_{\bar{j}_i-1} \times (\Delta_{\bar{j}_i} \times \{1\}) \) and by \( \text{pr}_2 \) the projection of \( c\Delta_j \times \cdots \times c \emptyset \times (\Delta_{\bar{j}_i} \times \Delta_{\bar{j}_i+1} \times \{1\}) \times c\Delta_{\bar{j}_i+2} \times \cdots \times \Delta_j^n \) on \( c\Delta_j \times \cdots \times c \emptyset \times (\Delta_{\bar{j}_i} \times \Delta_{\bar{j}_i+1} \times \{1\}) \). The projection \( \text{pr}_1 \) is used for the determination of \( \|\tilde{\Phi}_i^*(\omega)\|_\ell \) and the projection \( \text{pr}_2 \) for the determination of \( \|\omega\|_{\ell-1} \). The equality \((\text{id} \times
The commutativity of this diagram implies, for $\ell \geq 1$, \[ \|\Phi^*_{\ell}(\omega)\| \leq \|\omega\|_{\ell-1} \leq \bar{q}(\ell - 1) \leq \bar{q}(\ell). \]

2) We study now the case $i+1 = n$. The map $\Phi_{n-1} = \text{id} \times f_{n-1}$, with $f_{n-1} : c\Delta^{j_{n-1}} \times \Delta^n \to c\emptyset \times (\Delta^{j_{n-1} \times j_n})$, is defined by $f_{n-1}(x, t, y) = ((0, 1), (x, t, y))$. The proof goes like in the previous case, except for $\ell = 1$. The restriction of $\Phi_{n-1}$ to $\tilde{S}_{n-1} = c\Delta^{j_0} \times \cdots \times (\Delta^{j_{n-1} \times \{1\}} \times \Delta^n$ is defined by $f_{n-1}(0, y) = ((0, 1), (x, 0))$ and we have $\tilde{S}_{n-1} \subset c\Delta^{j_0} \times \cdots \times c\emptyset \times (\Delta^{j_{n-1} \times j_n})$. In the next diagram, the map $h_{n-1}$ is defined by $h_{n-1}(x, 0) = ((0, 1), (x, 0))$,

\[
\begin{array}{ccc}
c\Delta^{j_0} \times \cdots \times (\Delta^{j_{n-1} \times \{1\}}) \times \Delta^n & \xrightarrow{\text{pr}} & c\Delta^{j_0} \times \cdots \times (\Delta^{j_{n-1} \times \{1\}}) \\
\Phi_{n-1} = \text{id} \times f_{n-1} & & \text{id} \times h_{n-1}
\end{array}
\]

The commutativity of this diagram implies, for $\ell = 1$, \[ \|\Phi_{n-1}^* (\omega)\|_{\ell} = \|\text{pr}^* \circ (\text{id} \times h_{n-1})^* \omega\|_{\ell} \leq 0, \]

since the projection $\text{pr}$ is used for the determination of $\| - \|_{\ell}$. \qed

Remark 10.6. (In this remark, we keep the notations of the proof of Theorem 11) In this proof, the hypothesis “$X$ PL-pseudomanifold” is used only for proving that the maps $h_1$ and $h_{L^*}$ are quasi-isomorphisms. We believe that the map $\varphi_2$ is a quasi-isomorphism under the weaker hypothesis that $X$ is a recursive CS set in the sense of G. Friedman, [19]. This should need a direct proof, similar to the proof made by Friedman in [19]. We do not go further in this direction.

11. Formality and Examples

A notion of formality is defined and examples are given. This opens a framework for a study of the formality of singular projective algebraic varieties, as asked by M. Goresky in the introduction of [27]. In this section, we use Example 5.9 and Example 5.10 to construct models, not necessarily cofibrant, for the cone and suspension of a face set. They provide an explicit example of a non intersection-formal PL-pseudomanifold which is formal, as space.

More generally, we show how to construct models of PL-pseudomanifolds with isolated singularities. We apply it first to the case of the Thom space associated to a vector bundle and specialize the result to projective cones; we deduce that all singular quadrics are intersection-formal. We continue with the study of nodal hypersurfaces in $\mathbb{CP}(4)$ and prove their intersection-formality.
11.1. Intersection-formality. When $K$ is a filtered face set, a notion of perverse formality comes naturally from the fact that the perverse CDGA of forms, $A(K)_\bullet$, and its perverse cohomology algebra are belonging to the same category.

**Definition 11.1.** A connected filtered face set, $K$, is intersection-formal if there is an isomorphism between the minimal models of $A(K)_\bullet$ and $H(A(K)_\bullet)$, for $\bullet < \infty$.

An equivalent formulation is the existence of a sequence of quasi-isomorphisms in CDGA$_\infty$, between $A(K)_\bullet$, and its cohomology.

**Definition 11.2.** A PL-pseudomanifold, $X$, whose regular part is connected, is intersection-formal if it associated filtered face set, $\text{ISing}_\bullet^\phi(X)$, is intersection-formal.

**Theorem** [1] and [22] imply immediately the next result.

**Proposition 11.3.** The intersection-formality of a PL-pseudomanifold, whose regular part is connected, is independent of the stratification.

In the literature, Massey products are sometimes introduced for the detection of non formal spaces. We emphasize that the existence of non trivial Massey products guaranties the non-formality of a space but the reverse is harder to express. Mention, for instance from [12], that the vanishing of each Massey product is not sufficient for getting the formality; for that, this vanishing has to be done in an uniform way. Before the introduction of examples, we adjust the definition of Massey products and their basic properties (see [14] Definition 2.89 and Proposition 2.90) to the perverse frame.

**Definition 11.4.** Let $(A, d)_\bullet$ be a perverse CDGA, $x \in A_{[1]}$, $y \in A_{[2]}$, $z \in A_{[3]}$ be cocycles, of associated cohomology classes, $[x]$, $[y]$, $[z]$, and such that there exist $\alpha \in A_{[1]\oplus [2]}$, $\beta \in A_{[2]\oplus [3]}$ with $d\alpha = xy$ and $d\beta = yz$. The element $\alpha z - (-1)^{|x|} x \beta$ is a cocycle.

The *triple Massey product* is the set $\langle [x], [y], [z] \rangle \subseteq H_{[1]\oplus [2]\oplus [3]}(A, d)$ formed by all the cohomology classes $[\alpha z - (-1)^{|x|} x \beta]$, constructed using all the possible choices of the elements $\alpha$ and $\beta$. This Massey product is said trivial if $0 \in \langle [x], [y], [z] \rangle$.

If we quotient $H_\bullet(A, d)$ by the ideal generated by $[x]$ and $[z]$, the set $\langle [x], [y], [z] \rangle$ projects to a single element. Also, this element is zero if, and only if, $\langle [x], [y], [z] \rangle$ is trivial.

**Proposition 11.5.** Let $K$ be a connected filtered face set, of minimal model, $\rho_\bullet : (\wedge \oplus_{\mathbb{P} \leq 1} V_{[\mathbb{P}]})_\bullet \to A(K)_\bullet$. If there exists a non trivial triple Massey product in $(\wedge \oplus_{\mathbb{P} \leq 1} V_{[\mathbb{P}]})_\bullet$, then $K$ is not intersection-formal.

**Proof.** Suppose that $(\wedge \oplus_{\mathbb{P} \leq 1} V_{[\mathbb{P}]})_\bullet$ is also a perverse minimal model of $(\oplus_{\mathbb{P} \leq 1} H_\mathbb{P}(A(K)))_\bullet$, i.e., there is a quasi-isomorphism $\varphi_\bullet : (\wedge \oplus_{\mathbb{P} \leq 1} V_{[\mathbb{P}]})_\bullet \to (\oplus_{\mathbb{P} \leq 1} H_\mathbb{P}(A(K)))_\bullet$. Let $x \in (\wedge \oplus_{\mathbb{P} \leq 1} V_{[\mathbb{P}]})_{[1]}$, $y \in (\wedge \oplus_{\mathbb{P} \leq 1} V_{[\mathbb{P}]})_{[2]}$, $z \in (\wedge \oplus_{\mathbb{P} \leq 1} V_{[\mathbb{P}]})_{[3]}$ be cocycles such that there exist $\alpha \in (\wedge \oplus_{\mathbb{P} \leq 1} V_{[\mathbb{P}]})_{[1]\oplus [2]}$, $\beta \in (\wedge \oplus_{\mathbb{P} \leq 1} V_{[\mathbb{P}]})_{[2]\oplus [3]}$ with $d\alpha = xy$ and $d\beta = yz$. The cocycle $\alpha z - (-1)^{|x|} x \beta$ is sent by $\varphi$ to an element of the ideal generated by $\varphi(x)$ and $\varphi(z)$. The morphism $\varphi$ being a quasi-isomorphism of algebras, the element $[\alpha z - (-1)^{|x|} x \beta]$ is in the ideal generated by $[x]$ and $[z]$. Thus, the Massey product $\langle [x], [y], [z] \rangle$ is trivial.

We give now some easy constructions of models, beginning with the cone of a face set, see Example [5.9].
Proposition 11.6. Let $S$ be a connected face set of Sullivan minimal model $(\wedge V, d)$. Then a perverse model of the cone, $cS$, is $(\wedge V, d)_\bullet$, where the perverse degree of $\omega \in \wedge V$ is defined by $\|\omega\| = \max(|\omega|, |d\omega|)$.

In the case of the cone, $cS \in \Delta[5]$, a perversity is determined by the value $\overline{q}(n)$ and, with the described perverse degree, we have $(\wedge V, d)_{\overline{q}} = \tau_{\leq \overline{q}(n)}(\wedge V, d)$, see Definition 3.6.

Proof. From Example 5.9, we know that the perverse CDGA, $A_{PL}(cS)$, is quasi-isomorphic to $A_{PL}(S)_\bullet$, with $A_{PL}(S)_\bullet = \{ \omega \in A_{PL}(S) \mid \|\omega\| = \max(|\omega|, |d\omega|) \leq \overline{q}(n) \}$. Let $\varphi: (\wedge V, d) \to A_{PL}(S)_\bullet$ be the minimal model of the CDGA, $A_{PL}(S)$. As above, we define a perverse CDGA, $(\wedge V, d)_{\overline{q}}$, by $(\wedge V, d)_{\overline{q}} = \{ \omega \in \wedge V \mid \|\omega\| = \max(|\omega|, |d\omega|) \leq \overline{q}(n) \}$.

As $\varphi$ keeps the degree, i.e., $|\varphi(\omega)| = |\omega|$ if $\varphi(\omega) \neq 0$, we have an induced morphism of perverse CDGA’s, $\varphi_{\overline{q}}: (\wedge V, d)_{\overline{q}} \to A_{PL}(cS)_\bullet$, which induces a quasi-isomorphism, $\varphi_{\overline{q}}: (\wedge V, d)_{\overline{q}} \to A_{PL}(cS)_\partial$, for any GM-perversity, $\overline{q}$, entirely determined by $\overline{q}(n)$. □

Observe that $(\wedge V, d)_{\overline{q}} = Q$ is a model of the contractible face set, $cS$, as expected.

As the perverse model of a PL-pseudomanifold “contains” the model of the underlying space in perverse degree 0, we see that any intersection-formal PL-pseudomanifold is formal as space. The next result shows that these two notions, formal and intersection-formal, are distinct.

Proposition 11.7. There exist PL-pseudomanifolds that are formal as spaces but not intersection-formal.

The next example is an illustration of this statement.

Example 11.8. Denote by $\psi: S^2 \times S^2 \to S^4$ the map obtained by collapsing the two 2-dimensional spheres. We denote by $E$ the pullback of the Hopf fibration $S^3 \to S^7 \to S^4$ along $\psi$. The minimal model of the PL-pseudomanifold $E$ is (see Example 2.91) $(\wedge(x, \alpha, y, \beta, a), d)$, with $|x| = |y| = 2$, $|\alpha| = |\beta| = |a| = 3$, $d\alpha = x^2$, $d\beta = y^2$, $da = xy$.

Set $u = ay - xa$ and $v = x\beta - ay$.

The non-zero groups of rational cohomology are $H^0(E; Q) = Q$, $H^2(E; Q) = Q[x] \oplus Q[y]$, $H^4(E; Q) = Q[u] \oplus Q[v]$, $H^6(E; Q) = Q[uy]$. We observe that $\{[u]\} = \{[x], [x], [y]\}$ and $\{[v]\} = \{[x], [y], [y]\}$. As $ux - uy = d(\alpha \beta)$, we have also $[uy] = [xv]$. The cone, $cE$, being of dimension 8, a perversity is determined by one integer and we denote by $\overline{t}$ the perversity such that $\overline{t}(8) = t$, the top GM-perversity being $\overline{5}$. From Proposition 11.6 we compute the perverse minimal model of $cE$ and find $\rho_{\bullet}: (\wedge_{p \leq 1} V|_{\overline{p}}, d)_{\bullet} \to (\wedge(x, \alpha, y, \beta, a), d)_{\bullet}$, with

- $V|_2 = Qx \oplus Qy$, $dx = dy = 0$, $\rho(x) = x$, $\rho(y) = y$.
- $V|_3 = Q\alpha \oplus Q\beta \oplus Q\hat{a}$, $d\alpha = \hat{x}^2$, $d\beta = \hat{y}^2$, $d\hat{a} = \hat{x}\hat{y}$, $\rho(\alpha) = \alpha$, $\rho(\beta) = \beta$, $\rho(\hat{a}) = a$.
- $V|_5 = Q\hat{u} \oplus Q\hat{v}$, $d\hat{u} = d\hat{v} = 0$, $\rho(\hat{u}) = u$, $\rho(\hat{v}) = v$.
- $V|_7 = Q\hat{\xi}_1 \oplus Q\hat{\xi}_2$, $d\hat{\xi}_1 = \hat{\alpha}\hat{y} - \hat{x}\hat{a} - \hat{u}$, $d\hat{\xi}_2 = \hat{x}\hat{\beta} - \hat{a}\hat{y} - \hat{v}$.

From this determination, we note the existence of two non trivial triple Massey products, $\hat{u} \in \{[x], [\hat{x}], [y]\}$ and $\hat{v} \in \{[\hat{x}], [y], [y]\}$. Proposition 11.5 implies the non-intersection-formality of $cE$ despite the formality of $cE$ as space.

We continue with perverse models of the suspension of a face set.
Proposition 11.9. Let $\mathcal{S}$ be a connected face set of Sullivan minimal model $(\Lambda V, d)$. Then, the perverse minimal model of the suspension, $\Sigma \mathcal{S} \in \Delta^{|\mathcal{S}|}$, is the perverse minimal model of the perverse CDGA, $(\Lambda(t, dt) \otimes (\Lambda V, d))_{\bullet}$, with $(\Lambda(t, dt) \otimes (\Lambda V, d))_{\mathcal{S}} = dt \otimes \Lambda(t) \wedge \Lambda V \oplus \{f(t) \wedge w \in \Lambda(t) \wedge \Lambda V \mid f(0) = f(1) = 0 \text{ or } |\omega| \leq \mathcal{q}(n)\}$, for any GM-perversity, $\mathcal{q}$.

Proof. We proceed as in the proof of Proposition [11.6] by using the results obtained in Example [5.10] and replacing the CDGA, $A_{PL}(\mathcal{S})$, by the minimal model, $(\Lambda V, d)$, of $\mathcal{S}$. □

We note that $(\Lambda(t, dt) \otimes (\Lambda V, d))_{\bullet}$ contains elements of degree 0 and is not a model as in Definition [7.8]. On the other side, an explicit description of the perverse minimal model of $\Sigma \mathcal{S}$ is awkward because it is built by starting with the minimal model of the face set $\Sigma \mathcal{S}$. This last one has the (suspended) dual of a free Lie algebra as vector spaces of indecomposables. For instance, if $\mathcal{S}$ corresponds to the space $\mathbb{CP}(2)$, we have $\Sigma(\mathbb{CP}(2)) = S^3 \vee S^5$ whose minimal model is $(\Lambda S^2 L(x, y), d)$, with $|x| = 2$, $|y| = 4$ and the differential $d$ is the suspension of the transposition of the bracket of the free Lie algebra, $L(x, y)$, $a \otimes b \mapsto [a, b]$. Below, we built the minimal perverse model, $\varphi : B_{\bullet} \rightarrow A_{PL}(\Sigma(\mathbb{CP}(2)))$, in low degrees, such that $H^i(\varphi)$ is an isomorphism for $i \leq 6$. (This gives the minimal perverse model in the range of the cohomology.) Taking in account the dimension of the space, the perversities are determined by the value of $\mathcal{q}(5)$, the top perversity being given by $\mathcal{q}(5) = 3$. As previously, we denote by $\mathcal{q}$ the perversity such that $\mathcal{q}(5) = \ell$. Observe also that, as $\mathbb{CP}(2)$ is a formal space, we may replace the model of $\mathbb{CP}(2)$ by its cohomology algebra, $(H, 0)$. In this case, there is no ambiguity in the definition of $\mathcal{q} \geq \ell H$ (see Example [5.10]) and we are reduced to the determination of the perverse minimal model of $E_\ell = H^{\geq \ell} \oplus s(H^{>\ell})$, with a trivial differential and $H = (\Lambda u)/u^3$, $|u| = 2$. The product on $E_{\bullet}$ comes from the product induced by the product of $H$ on the quotient $H^{\leq \ell}$ and the formulæ, $s\eta_1 \cdot s\eta_2 = 0$, $\eta_1 \cdot s\eta_2 = 0$. Let us write $E_{\ell} = \mathbb{Q} \oplus sH$ and $B_{\ell}^{\leq \ell} = (\wedge(\alpha_3, \alpha_5), d)^{\leq \ell}$, with $d\alpha_3 = d\alpha_5 = 0$, $|\alpha_4| = 3$ and $||\alpha_4|| = 0$. The map $\varphi$ sends $\alpha_3$ to $s\alpha_3$ and $\alpha_5$ to $s(u^2)$.

- If $\ell = 0$ or 1, we have $E_{\ell} = \mathbb{Q} \oplus sH$ and $B_{\ell}^{\leq \ell} = (\wedge(\alpha_3, \alpha_5), d)^{\leq \ell}$, with $d\alpha_3 = d\alpha_5 = 0$, $|\alpha_4| = 3$ and $||\alpha_4|| = 0$. The map $\varphi$ sends $\alpha_3$ to $s\alpha_3$ and $\alpha_5$ to $s(u^2)$.

- If $\ell = 2$ or 3, we have $E_{\ell} = \mathbb{Q} \oplus \mathbb{Q} u \oplus s(\mathbb{Q} u^2)$ and, in the model, we have to kill the class associated to $\alpha_3$ and add a new cocycle which reaches $u$. Thus, we introduce $\beta_2$, $\beta'_2$, with $d\beta_2 = 0$, $d\beta'_2 = \alpha_3$. Doing that, we have also introduced a cocycle, $\alpha_3 \beta'_2$, which has to be killed. Also, we observe that the product $\alpha_3 \beta'_2$ has to be identified to $\alpha_5$. Finally, we set $B_{\ell}^{\leq \ell} = (\wedge(\alpha_3, \alpha_5, \beta_2, \beta'_2, \beta_4, \beta'_4), d)^{\leq \ell}$, with $|\beta_2| = |\beta'_2| = 2$, $|\beta_4| = |\beta'_4| = 4$, $d\beta_2 = 0$, $d\beta'_2 = \alpha_3$, $d\beta_4 = \alpha_3 \beta - \alpha_5$ and $d\beta'_4 = \alpha_3 \beta'_2$. The map $\varphi$ sends $\beta_2$ to $u$ and $\beta'_2$, $\beta_4$, $\beta'_4$ to 0. All the $\beta_i$ and $\beta'_i$ are of perverse degree 2. (Observe that we do not have to consider $\beta'_2$ and $\beta'_2 \beta_2$ which are not of this perversity.)

11.2. A model in CDGA for isolated singularities. We extend now these two situations, the cone and the suspension, to the case of PL-pseudomanifolds with isolated singularities. (They are also considered in [10] Proposition 5.1 where the structure of their Steenrod squares is determined.) They can be expressed under the following shape.
Let $M$ be a PL-pseudomanifold, obtained from a manifold with boundary, $(W, \partial W)$, by attaching cones on the connected components of the boundary, i.e., $M$ is the push out

$$
\begin{array}{ccc}
\bigcup_{u \in I} \partial_u W & \overset{\iota}{\longrightarrow} & W \\
\downarrow & & \\
\bigcup_{u \in I} c(\partial_u W) & \longrightarrow & M,
\end{array}
$$

where the $\partial_u W$'s are the connected components of $\partial W$, and $c(\partial_u W)$ is the cone on a component. We filter the pseudomanifold, $M$, by the cone points of $c(\partial_u W)$. As the singularities are points, a perversity $\eta$ is determined by one number, $\eta(n)$, with $n = \dim M$.

**Proposition 11.10.** Let $\eta$ be a GM-perversity and $M$ be an $n$-dimensional PL-pseudomanifold, as above. Let $\varphi: (A_1, d_1) \rightarrow (A_2, d_2) = \oplus_{u \in I} (A_2(u), d_2)$ be a surjective model of the inclusion $\iota: \bigcup_{u \in I} \partial_u W \rightarrow W$. Then a (non cofibrant) perverse model of $M$ is given by:

$$
M(M)_\eta = (A_1, d_1) \oplus A_2 \left( \oplus_{u \in I} \tau_{\leq \eta(n)}(A_2(u), d_2) \right),
$$

where the truncation $\tau_{\leq \eta(n)}$ is defined in Definition 3.6.

**Proof of Proposition 11.10.** We start with a pushout of spaces

$$
\begin{array}{ccc}
\bigcup_{u \in I} \partial_u W & \overset{\iota}{\longrightarrow} & W \\
\downarrow & & \\
\bigcup_{u \in I} c(\partial_u W) & \longrightarrow & M.
\end{array}
$$

We may suppose that $M$, $W$, $\partial_u W$ and $c(\partial_u W)$ are triangulated in such a way that any simplex is filtered, for the filtration by the cone point.

Let $X$ be one of the spaces above, of associated simplicial complex $X^\tau$ and of associated filtered face set, $X^\tau$. G. Friedman proves, in particular, that if the triangulation is full (which is the case in our situation) then the cochains $C^*_{GM,\eta}(X)$ and $C^*_{GM,\eta}(X^\tau)$ are quasi-isomorphic for any GM-perversity $\eta$ (see [19, Chapter 3 and Chapter 5]). Thus, the isomorphism

$$
C^*_{GM,\eta}(M^\tau) \cong C^*(W^\tau) \oplus (\oplus_{u \in I} C^*(\partial_u W^\tau)) \left( \oplus_{u \in I} C^*_{GM,\eta}(c(\partial_u W)^\tau) \right)
$$

gives a quasi-isomorphism

$$
C^*_{GM,\eta}(M) \cong C^*(W) \oplus (\oplus_{u \in I} C^*(\partial_u W)) \left( \oplus_{u \in I} C^*_{GM,\eta}(c(\partial_u W)) \right) .
$$

Moreover, we know, from Corollary 4.5, that $\tilde{A}_{PL,\eta}(X)$ and $C^*_{GM,\eta}(X; \mathbb{Q})$ are quasi-isomorphic if $\eta + \tilde{\eta} = \tilde{\eta}$. This implies that the canonical CDGA map,

$$
\tilde{A}_{PL,\eta}(M) \rightarrow A_{PL}(W) \oplus (\oplus_{u \in I} \tilde{A}_{PL,\eta}(c(\partial_u W)) \left( \oplus_{u \in I} \tilde{A}_{PL,\eta}(c(\partial_u W)) \right),
$$

is a quasi-isomorphism. The CDGA of forms on the cone is quasi-isomorphic to a truncation, i.e., there is a quasi-isomorphism $\tilde{A}_{PL,\eta}(c(\partial_u W)) \cong \tau_{\leq \eta(n)} A_{PL}(c(\partial_u W))$ which induces a quasi-isomorphism

$$
\tilde{A}_{PL,\eta}(M) \cong A_{PL}(W) \oplus (\oplus_{u \in I} \tilde{A}_{PL,\eta}(c(\partial_u W)) \left( \oplus_{u \in I} \tau_{\leq \eta(n)} A_{PL}(c(\partial_u W)) \right).
$$
With the surjective model \( \varphi : (A_1, d_1) \to \bigoplus_{u \in I} (A_2(u), d_2) \) of the statement, we get a morphism of short exact sequences,

\[
0 \to \text{Ker} \varphi \to (A_1, d_1) \oplus \bigoplus_{u \in I} (A_2(u), d_2) \to \bigoplus_{u \in I} \tau \leq \varphi(n)(A_2(u), d_2) \to 0
\]

The result follows with an application of the five lemma to the associated long exact sequences.

In Proposition 11.10 the elements of \( M(M)_\varphi \) are couples, \((\omega, \varphi(\omega))\) such that \( \omega \in A_1 \) and

\[
\begin{align*}
\varphi(\omega) &= 0 & \text{if } |\omega| > \varphi(n), \\
\varphi(\omega) &\text{ is a cocycle} & \text{if } |\omega| = \varphi(n), \\
\text{no condition} &\text{ if } |\omega| < \varphi(n).
\end{align*}
\]

This implies immediately:

\[
H^k(\varphi(M, \mathbb{Q}) = \begin{cases} 
H^k(W; \mathbb{Q}) & \text{if } k \leq \varphi(n), \\
\ker (H^k(W; \mathbb{Q}) \to H^k(\partial W; \mathbb{Q})) & \text{if } k = \varphi(n) + 1, \\
H^k(W, \partial W; \mathbb{Q}) & \text{if } k > \varphi(n) + 1.
\end{cases}
\]

In the case of simply connected spaces, the model arising in Proposition 11.10 can be simplified as follows.

**Corollary 11.11.** Let \( \varphi \) be a GM-perversity and \( M \) be an \( n \)-dimensional PL-pseudo-manifold, as above, and such that \( W \) and the \( \partial W \)'s are simply connected. Let \( \varphi : (A_1, d_1) \to (A_2, d_2) = \bigoplus_{u \in I} (A_2(u), d_2) \) be a model of the inclusion \( \iota : \bigcup_{u \in I} \partial_u W \to W \), such that \( A_1^0 = A_2(u)^0 = \mathbb{Q}, A_1^1 = A_2(u)^1 = 0 \) and \( \varphi \) surjective in strictly positive degrees. Then a (non cofibrant) perverse model of \( M \) is given by:

\[
M(M)_\varphi = (A_1, d_1) \oplus A_2 \left( \bigoplus_{u \in I} \tau \leq \varphi(n)(A_2(u), d_2) \right),
\]

where the truncation \( \tau \leq \varphi(n) \) is defined in Definition 3.6.

**Proof.** Proof of Proposition 11.10 is still valid with the following modifications.

- The arguments with long exact sequences are starting in degree 1 instead of degree 0.
- The result is still true in degree 0 because

\[
\left( (A_1, d_1) \oplus \bigoplus_{u \in I} (A_2(u), d_2) \right)^0 = \mathbb{Q}.
\]

**11.3. Thom spaces.** Let \( \mathbb{R}^m \to E \to B \) be a vector bundle. We denote by \( D_E \to B \) the associated disk-bundle and by \( S_E \to B \) the associated sphere-bundle. The *Thom space*, \( \text{Th}(E) \), is the quotient of the disk-bundle by the sphere-bundle. We filter \( \text{Th}(E) \) by the point of compactification and a perversity is determined by the number \( \varphi(n) \) where \( n = \dim \text{Th}(E) \). The next result suffices for the study of projective cones.

**Proposition 11.12.** Let \( f : E \to B \) be a vector bundle of rank \( 2r \), with \( B \) a formal manifold. The associated Thom space, \( \text{Th}(E) \), filtered by the compactification point, is an intersection-formal space.
The manifold as follows.

\[ M(\text{Th}(E)) = (M(B) \otimes \Lambda(x, y), D) \oplus_{M(B) \otimes \Lambda x} \tau_{\leq q(n)}(M(B) \otimes \Lambda x, d). \]

The manifold \( B \) being formal, we may choose \( M(B) = (H^*(B), 0) \) and obtain,

\[ M(\text{Th}(E)) = (H^*(B) \otimes \Lambda(x, y), D) \oplus_{H^*(B) \otimes \Lambda x} \tau_{\leq q(n)}(H^*(B) \otimes \Lambda x, d). \]

This pullback can be described by:

\[ M(\text{Th}(E))_{\mathcal{P}} = \begin{cases} (H(B) \otimes \Lambda(x, y))^k & \text{if } k < \mathcal{P}(n), \\ \{ \omega \in (H(B) \otimes \Lambda(x, y))^k \mid D(\omega)|_{y=0} = 0 \} & \text{if } k = \mathcal{P}(n), \\ J(y) & \text{if } k > \mathcal{P}(n). \end{cases} \]

where \( J(y) = H(B) \otimes \Lambda x \otimes y \) is the differential ideal generated by \( y \). If \( \mathcal{P} \leq \mathcal{Q} \), the morphisms \( \psi_{\mathcal{P}}^{\mathcal{Q}} : M(\text{Th}(E))_{\mathcal{P}} \to M(\text{Th}(E))_{\mathcal{Q}} \) are the canonical inclusions. From this presentation, we recover (see Page 77) the intersection cohomology vector space of the Thom space,

\[ H^k_{\mathcal{P}}(\text{Th}(E)) = \begin{cases} H^k(B) & \text{if } k \leq \mathcal{Q}(n), \\ \text{Im}(- \cup c : H^{k-2r}(B) \to H^k(B)) & \text{if } k = \mathcal{Q}(n) + 1, \\ H^{k-2r}(B) & \text{if } k > \mathcal{Q}(n) + 1. \end{cases} \]

If \( \mathcal{P} \leq \mathcal{Q} \), the morphisms, \( \psi_{\mathcal{P}}^{\mathcal{Q}} : H^k_{\mathcal{P}}(\text{Th}(E)) \to H^k_{\mathcal{Q}}(\text{Th}(E)) \) are the canonical inclusions, except for \( \mathcal{P}(n) + 1 < k \leq \mathcal{Q}(n) \) where \( \psi_{\mathcal{P}}^{\mathcal{Q}}(\gamma) = \gamma \cup c \). Let \( \mathcal{Q}_1, \mathcal{Q}_2 \) be two GM-perversities and \( a_1 \in H^*_\mathcal{Q}_1(\text{Th}(E)) \), \( a_2 \in H^*_\mathcal{Q}_2(\text{Th}(E)) \). We specify the product \( a_1 \cdot a_2 \in H^*_\mathcal{Q}_1 \cup \mathcal{Q}_2(\text{Th}(E)) \) as follows.

- If \( |a_1| \leq \mathcal{Q}_1(n) + 1 \) and \( |a_2| < \mathcal{Q}_2(n) + 1 \), we have \( a_1 \cdot a_2 = a_1 \cup a_2 \).
- If \( |a_1| = \mathcal{Q}_1(n) + 1 \) and \( |a_2| = \mathcal{Q}_2(n) + 1 \), then \( a_1 = a_1' \cup c \) and \( a_2 = a_2' \cup c \) and we have \( a_1 \cdot a_2 = a_1' \cup a_2' \cup c \).
- If \( |a_1| \leq \mathcal{Q}_1(n) + 1 \) and \( |a_2| > \mathcal{Q}_2(n) + 1 \), with \( |a_1| + |a_2| < (\mathcal{Q}_1 \cup \mathcal{Q}_2)(n) + 1 \), we have \( a_1 \cdot a_2 = a_1 \cup a_2 \).
- If \( |a_1| \leq \mathcal{Q}_1(n) + 1 \) and \( |a_2| > \mathcal{Q}_2(n) + 1 \), with \( |a_1| + |a_2| > (\mathcal{Q}_1 \cup \mathcal{Q}_2)(n) + 1 \), we have \( a_1 \cdot a_2 = a_1 \cup a_2 \).
- If \( |a_1| \geq \mathcal{Q}_1(n) + 1 \) and \( |a_2| \geq \mathcal{Q}_2(n) + 1 \), we have \( a_1 \cdot a_2 = a_1 \cup a_2 \).

We construct now an explicit CDGA’s map, \( \Phi_{\mathcal{Q}} : M(\text{Th}(E))_{\mathcal{Q}} \to (H^*_\mathcal{Q}(\text{Th}(E)), 0) \). An element \( \omega \in H^*(B) \otimes \Lambda(x, y) \) is a sum of terms, \( a \otimes y^i \) and \( b \otimes xy^i \), with \( i \geq 0 \) and \( a, b \in H^*(B) \). We define \( \Phi_{\mathcal{Q}}(\omega) \in H^*_\mathcal{Q}(\text{Th}(E)) \) as follows.

- If \( |\omega| \leq \mathcal{Q}(n) + 1 \), we set \( \Phi_{\mathcal{Q}}(a \otimes y^i) = a \cup c^i \) and \( \Phi_{\mathcal{Q}}(b \otimes xy^i) = 0 \).
- If \( |\omega| > \mathcal{Q}(n) + 1 \), we have \( i \geq 1 \) and set \( \Phi_{\mathcal{Q}}(a \otimes y^i) = a \cup c^{i-1} \) and \( \Phi_{\mathcal{Q}}(b \otimes xy^i) = 0 \).
By definition, the map \(\Phi_{\mathcal{T}}\) takes value in \(H^k_{\mathbb{Q}}(\text{Th}(\mathcal{E}))\). We prove that \(\Phi_{\mathcal{T}} \circ \varphi^\mathcal{T}_p = \psi_p^\mathcal{T} \circ \Phi_{\mathcal{T}}\), if \(p \leq \mathcal{T}\). This is clear except for \(p(n) + 1 < k \leq \mathcal{Q}(n)\), where we have,

\[
(\psi_p^\mathcal{T} \circ \Phi_{\mathcal{T}})(a \otimes y^i) = \psi_p^\mathcal{T}(a \cup c^{i-1}) = a \cup c^i = \Phi_{\mathcal{T}}(a \otimes y^i) = (\Phi_{\mathcal{T}} \circ \varphi^\mathcal{T}_p)(a \otimes y^i).
\]

We check now the compatibility of \(\Phi_{\mathcal{T}}\) with differentials, which reduces, in this case, to \(\Phi_{\mathcal{T}}(D\omega) = 0\). The differential of \(\omega\) is determined by

\[
D(a \otimes y^i) = 0 \quad \text{and} \quad D(b \otimes x y^i) = (-1)^{|b|}(b \cup c \otimes y^i - b \otimes y^{i+1}).
\]

Thus, we are only concerned with the terms \(\omega = b \otimes x y^i\).

- If \(|\omega| < \mathcal{Q}(n)\), then there is no restriction on \(\omega\) and we have,
  \[
  \Phi_{\mathcal{T}}(D(b \otimes x y^i)) = (-1)^{|b|}(b \cup c \otimes c^i - b \cup c^{i+1}) = 0.
  \]

- If \(|\omega| = \mathcal{Q}(n)\), then \(D(\omega)_{|y = 0} = 0\) implies \(b \cup c = 0\) if \(i = 0\) and we have,
  \[
  \Phi_{\mathcal{T}}(D(b \otimes x)) = -(1)^{|b|}\Phi_{\mathcal{T}}(b \otimes y) = -(1)^{|b|}b \cup c = 0,
  \]
  \[
  \Phi_{\mathcal{T}}(D(b \otimes x y^i)) = (-1)^{|b|}(b \cup c \otimes c^i - b \cup c^{i+1}) = 0, \quad \text{if} \ i > 0.
  \]

- If \(|\omega| > \mathcal{Q}(n)\), then \(b = 0\) if \(i = 0\), and we have,
  \[
  \Phi_{\mathcal{T}}(D(b \otimes x y^i)) = (-1)^{|b|}(b \cup c \otimes c^{i-1} - b \cup c^i) = 0.
  \]

Let \(\mathcal{Q}_1\) and \(\mathcal{Q}_2\) be two GM-perversities. For the compatibility with products, it is sufficient to establish \(\Phi_{\mathcal{Q}_1 \oplus \mathcal{Q}_2}((a_1 \otimes y^i)(a_2 \otimes y^j)) = \Phi_{\mathcal{Q}_1}(a_1 \otimes y^i) \cdot \Phi_{\mathcal{Q}_2}(a_2 \otimes y^j)\), i.e.,

\[
\Phi_{\mathcal{Q}_1 \oplus \mathcal{Q}_2}(a_1 \otimes a_2 \otimes y^{i+j}) = \Phi_{\mathcal{Q}_1}(a_1 \otimes y^i) \cdot \Phi_{\mathcal{Q}_2}(a_2 \otimes y^j).
\]

We specify the different cases and set \(\omega_1 = a_1 \otimes y^i, \omega_2 = a_2 \otimes y^j\).

- Suppose \(|\omega_1| \leq \mathcal{Q}_1(n) + 1\) and \(|\omega_2| < \mathcal{Q}_2(n) + 1\). Then, we have,
  \[
  \Phi_{\mathcal{Q}_1}(\omega_1) \cdot \Phi_{\mathcal{Q}_2}(\omega_2) = (a_1 \cup c^i) \cdot (a_2 \cup c^j) = a_1 \cup a_2 \cup c^{i+j} = \Phi_{\mathcal{Q}_1 \oplus \mathcal{Q}_2}(\omega_1 \omega_2).
  \]

- Suppose \(|\omega_1| = \mathcal{Q}_1(n) + 1\) and \(|\omega_2| = \mathcal{Q}_2(n) + 1\). Then, we have,
  \[
  \Phi_{\mathcal{Q}_1}(\omega_1) \cdot \Phi_{\mathcal{Q}_2}(\omega_2) = (a_1 \cup c^i) \cdot (a_2 \cup c^j) = a_1 \cup a_2 \cup c^{i+j+1} = \Phi_{\mathcal{Q}_1 \oplus \mathcal{Q}_2}(\omega_1 \omega_2).
  \]

- Suppose \(|\omega_1| \leq \mathcal{Q}_1(n) + 1\) and \(|\omega_2| > \mathcal{Q}_2(n) + 1\), with \(|\omega_1| + |\omega_2| \leq (\mathcal{Q}_1 + \mathcal{Q}_2)(n) + 1\). Then we have,
  \[
  \Phi_{\mathcal{Q}_1}(\omega_1) \cdot \Phi_{\mathcal{Q}_2}(\omega_2) = (a_1 \cup c^i) \cdot (a_2 \cup c^j) = (a_1 \cup c^i) \cup (a_2 \cup c^{j-1}) \cup c
  = a_1 \cup a_2 \cup c^{i+j} = \Phi_{\mathcal{Q}_1 \oplus \mathcal{Q}_2}(\omega_1 \omega_2).
  \]

- Suppose \(|\omega_1| > \mathcal{Q}_1(n) + 1\) and \(|\omega_2| > \mathcal{Q}_2(n) + 1\). Then, we have,
  \[
  \Phi_{\mathcal{Q}_1}(\omega_1) \cdot \Phi_{\mathcal{Q}_2}(\omega_2) = (a_1 \cup c^{i-1}) \cdot (a_2 \cup c^{j-1}) = (a_1 \cup c^{i-1}) \cup (a_2 \cup c^{j-1})
  = a_1 \cup a_2 \cup c^{i+j-1} = \Phi_{\mathcal{Q}_1 \oplus \mathcal{Q}_2}(\omega_1 \omega_2).
  \]
Recall the notion of projective cone of a smooth projective variety. Let $S$ be a smooth algebraic subvariety of $\mathbb{C}P^n$, we embedd it in $\iota: S \to \mathbb{C}P^{n+1}$ through the canonical map, $\mathbb{C}P^n \to \mathbb{C}P^{n+1}$, sending $[x_0 : \ldots : x_n]$ to $[x_0 : \ldots : x_n : 0]$. By definition, the projective cone of $S$ is the union of all the lines that intersect $S$ and contain the point $[0 : \ldots : 0 : 1]$. It can also be expressed as the Thom space of the restriction of the tautological line bundle over $\mathbb{C}P^{n+1}$.

The characteristic class of this vector bundle is $c_1(E) = \iota^*(c_1) \in H^2(S)$, with $c_1$ the Kähler class of $\mathbb{C}P^{n+1}$. From [12] and Proposition 11.12, we deduce directly the next result.

**Proposition 11.13.** The projective cones of a smooth projective variety are intersection-formal. Thus, any singular quadric is intersection-formal.

We use the model of Proposition 11.12 for the complete determination of the intersection cohomology algebra of the Segre embedding.

**Example 11.14.** The Segre embedding (see [6]) is defined by $\mathbb{C}P(1) \times \mathbb{C}P(1) \to \mathbb{C}P(3)$, $([x_0 : x_1], [y_0 : y_1]) \mapsto [x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1]$. Let $u$ and $v$ be the fundamental classes of the $\mathbb{C}P(1)$’s. The corresponding line bundle, $\mathbb{C} \to E \to \mathbb{C}P(1) \times \mathbb{C}P(1)$, has for Chern class $c = c_1(E) = u + v$. The regular component is $\mathbb{C}P(1) \times \mathbb{C}P(1)$ and the link is the associated sphere-bundle.

We denote by $\ell$ the perversity such that $\ell(6) = \ell$. For the description of the intersection cohomology of the Thom space, $\text{Th}(E)$, we have only to consider the GM-perversities, $0, 2, 4$. A perverse model of the associated Thom space is given by

\[(A, \delta)_{\ell} = (H \wedge (x, y), D) \oplus_{H \wedge x} \tau_{\leq \ell}(H \wedge x, d),\]

where $H$ is the quotient $H = \wedge(u, v)/(u^2, v^2)$, $|x| = 1$, $|y| = |u| = |v| = 2$, $dx = u + v$, $Dx = u + v - y$, $Dy = 0$. From this model, we deduce the perverse cohomology algebra, whose elements are specified below.

| $H^k_{\ell}(\text{Th}(E); \mathbb{Q})$ | $\ell = 0$ or 1 | $\ell = 2$ | $\ell = 3$ or 4 |
|---|---|---|---|
| $k = 0$ | $\mathbb{Q}$ | $\mathbb{Q}$ | $\mathbb{Q}$ |
| $k = 1$ | 0 | 0 | 0 |
| $k = 2$ | $\mathbb{Q}[y]$ | $\mathbb{Q}[u] \oplus \mathbb{Q}[v]$ | $\mathbb{Q}[u] \oplus \mathbb{Q}[v]$ |
| $k = 3$ | 0 | 0 | 0 |
| $k = 4$ | $\mathbb{Q}[\alpha] \oplus \mathbb{Q}[\beta]$ | $\mathbb{Q}[\alpha] \oplus \mathbb{Q}[\beta]$ | $\mathbb{Q}[uv]$ |
| $k = 5$ | 0 | 0 | 0 |
| $k = 6$ | $\mathbb{Q}[\gamma]$ | $\mathbb{Q}[\gamma]$ | $\mathbb{Q}[\gamma]$ |
We specify now the maps \( \psi_\eta^p : H^2_{\eta}(\text{Th}(E); \mathbb{Q}) \to H^2_{\eta}(\text{Th}(E); \mathbb{Q}) \), with \( p \leq \eta \).

- \( \psi_\eta^p \) is determined by \([y] \mapsto [u] + [v]\) in degree 2 and the identity otherwise.
- \( \psi_\eta^p \) is determined by \([\alpha] \mapsto [uv], \ [\beta] \mapsto [uv]\) in degree 4 and the identity otherwise.

The structure of perverse algebra is given by the following equalities.

- The classes \([\alpha]\) and \([\beta]\) correspond to the cocycles \( yu \) and \( yv \), respectively. The differential \( D(xy) = uy + vy - y^2 \) implies \([y]^2 = [\alpha] + [\beta]\). Using similar arguments, we obtain the cohomology algebra of \( H^*_\eta(\text{Th}(E); \mathbb{Q}) = H^*(\text{Th}(E); \mathbb{Q}) \), with \( 2[y] \alpha = 2[y] \beta = [y]^3 = [\gamma] \).
- As \([\alpha]\) and \([\beta]\) are represented by \( yu \) and \( yv \), respectively, the product \( H^*_\eta \otimes H^*_\eta \to H^*_\eta \) is completed by \([y][u] = [\alpha], [y][v] = [\beta], [u][\alpha] = [v][\beta] = 0, [u][\beta] = [v][\alpha] = [\gamma] \).
- Finally, the product \( H^*_\eta \otimes H^*_\eta \to H^*_\eta \) reduces to \([u][v] = [uv]\).

The other products are zero for reason of dimension or belong to the cohomology algebra of the regular component, \( H^*_\text{reg}(\text{Th}(E); \mathbb{Q}) = H \).

### 11.4. Nodal Hypersurfaces in \( \mathbb{CP}(4) \)

In this paragraph, we work over the field \( \mathbb{Q} \) and study the intersection-formality of nodal hypersurfaces in \( \mathbb{CP}(4) \); i.e., of hypersurfaces in \( \mathbb{CP}(4) \) whose singularities are ordinary double points, called nodes. For instance, mention the Calabi-Yau quintic, \( X_0 \), with 125 nodes, defined by the polynomial

\[
P(z) = z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5z_0z_1z_2z_3z_4.
\]

**Proposition 11.15.** Any nodal hypersurface in \( \mathbb{CP}(4) \) is intersection-formal.

Come back to the general case and let \( \overline{V} \) be a nodal hypersurface in \( \mathbb{CP}(4) \), with \( N \) isolated singularities. We denote by \( \overline{V}_\text{reg} \) the regular component and by \( V \) a small resolution of \( \overline{V} \). Our strategy of proof is the construction of a model of \( \overline{V} \) in the spirit of Corollary 11.11. First, we determine a (classical) Sullivan model of the regular component, \( \overline{V}_\text{reg} \), from a geometrical local analysis. The link around each singular point is a product \( \mathbb{CP}(1)_{(i)} \times S^3_{(i)} \) that can be obtained as follows.

Let \( D^2 = E_D(i) \xrightarrow{\pi} \mathbb{CP}(1)_{(i)} \times \mathbb{CP}(1)_{(i)} \) be the normal bundle associated to the Segre embedding \( \mathbb{CP}(1)_{(i)} \times \mathbb{CP}(1)_{(i)} \hookrightarrow \mathbb{CP}(3) \). We set \( E_D = \bigsqcup_{i=1}^N E_D(i) \) and denote by \( \partial E_D = \bigsqcup_{i=1}^N \mathbb{CP}(1)_{(i)} \times S^3_{(i)} \xrightarrow{\pi} \bigsqcup_{i=1}^N \mathbb{CP}(1)_{(i)} \times \mathbb{CP}(1)_{(i)} \) the associated circle-bundles.

We consider also a second resolution, \( W \), obtained from the minimal resolution, \( V \), with a blow-up, \( \mathcal{B}l \), of the \( \mathbb{CP}(1)_{(i)} \)'s, i.e., we have a commutative diagram, where \( f \) and \( g \) are embeddings,

\[
\begin{array}{ccc}
\bigsqcup_{i=1}^N \mathbb{CP}(1)_{(i)} \times \mathbb{CP}(1)_{(i)} & \xrightarrow{g} & \bigsqcup_{i=1}^N \mathbb{CP}(1)_{(i)} \\
W & \xrightarrow{\mathcal{B}l} & V \\
\end{array}
\]

The regular part, \( \overline{V}_\text{reg} \), is the complement of the divisor \( \mathcal{D} = \bigsqcup_{i=1}^N \mathbb{CP}(1)_{(i)} \times \mathbb{CP}(1)_{(i)} \) in the non-singular manifold, \( W \), and we know from J. Morgan (see [35]) how to build a rational (non-free) Sullivan model of it. The two resolutions can also be described as...
pushout’s,
\[ W = \overline{V}_{\text{reg}} \cup \bigcup_{i=1}^{N} \mathbb{C}P(1)_{(i)} \times S^3_{(i)} \ E_D \]
in contrast with
\[ V = \overline{V}_{\text{reg}} \cup \bigcup_{i=1}^{N} \mathbb{C}P(1)_{(i)} \times S^3_{(i)} \ (\bigcup_{i=1}^{N} \mathbb{C}P(1)_{(i)} \times D^4_{(i)}). \]

In terms of normal bundle, the construction of \( W \) is done from a replacement of the fibration \( \pi : \partial E \to \bigcup_{i=1}^{N} \mathbb{C}P(1)_{(i)} \times S^3_{(i)} \) by its composition with the canonical projection \( \text{pr}_1 : \mathbb{C}P(1)_{(i)} \times \mathbb{C}P(1)_{(i)} \to \mathbb{C}P(1)_{(i)} \). This gives a trivial sphere-bundle, of basis \( \mathbb{C}P(1)_{(i)} \) and fiber \( F_{(i)} = S^3_{(i)} \), as shows the next diagram, restricted to one component,

\[ \begin{array}{ccc}
F & \to & \partial E_D \\
\downarrow b & & \downarrow \pi_S & \downarrow c \\
\mathbb{C}P(1) & \to & \mathbb{C}P(1) \times \mathbb{C}P(1) & \to & \mathbb{C}P(3) \\
\downarrow a & & \downarrow \text{pr}_1 & & \\
\ast & \to & \mathbb{C}P(1).
\end{array} \]

The square \([a]\) and the rectangle \([a] \cup [b]\) being pullbacks, the square \([b]\) is a pullback. As \([c]\) is a pullback also, the rectangle \([b] \cup [c]\) is a pullback. This implies that the left-hand vertical map is the Hopf fibration, \( S^1 \to F = S^3 \to \mathbb{C}P(1) \). The next cube provides a map, \( W \to V \), compatible with the decomposition in pushouts of (22) and (23),

\[ \begin{array}{ccc}
W & \to & E_D \\
\downarrow \overline{V}_{\text{reg}} & & \downarrow \partial E_D \\
V & \to & \bigcup_{i=1}^{N} \mathbb{C}P(1)_{(i)} \times D^4_{(i)} \\
\downarrow \psi & & \downarrow \overline{V}_{\text{reg}} \\
\overline{V}_{\text{reg}} & \to & \bigcup_{i=1}^{N} \partial (\mathbb{C}P(1)_{(i)} \times D^4_{(i)})
\end{array} \]

We use the front face for a construction of rational models, starting from \( H^*(\mathbb{C}P(1)_{(i)}) \otimes H^*(S^3_{(i)}) \) as model of \( \partial (\mathbb{C}P(1)_{(i)} \times D^4_{(i)}) \) and \( \bigoplus_{i=1}^{N} (H^*(\mathbb{C}P(1)_{(i)} \times \mathbb{C}P(1)_{(i)}) \otimes \wedge \theta_i, d) \) as model of \( \partial E_D \), with \( d\theta_i = a_i + b_i \), where \( a_i \) and \( b_i \) are generators of \( H^2(\mathbb{C}P(1)_{(i)} \times \mathbb{C}P(1)_{(i)}) \). The model of \( \partial E_D \) arises as a model of the total spaces of the fibrations, \( S^1 \to \partial E_D(i) \to \mathbb{C}P(1)_{(i)} \times \mathbb{C}P(1)_{(i)}, \) whose Euler class is \( a_i + b_i \), see the definition of the Segre embedding in Example 11.14. In summary, we have a commutative square of
CDGA’s,

\[
\begin{array}{c}
\xymatrix{
\mathcal{N}' \ar[r]^{\oplus_{i=1}^N (H^*(CP(1)_{(i)}) \otimes H^*(CP(1)_{(i)}) \otimes \wedge \theta_i, d)} \ar[d]_{\sim} & \mathcal{N} \ar[d]_{\sim} \\
(\oplus_{i=1}^N H^*(CP(1)_{(i)}) \otimes H^*(S^3_{(i)}), 0)}
\end{array}
\]

where

- the top horizontal map is a surjective model of the inclusion \( \partial E_D \to \overline{V}_{reg} \),
- the right-hand vertical map is a model of \( \psi \),
- the CDGA, \( N \), is the fibered product of these two maps.

By construction, the two vertical maps are quasi-isomorphisms. For the determination of the model \( M_\psi \) of \( \psi \), we start from the commutative diagram

\[
\begin{array}{c}
\xymatrix{
\sqcup_{i=1}^N \partial(CP(1)_{(i)} \times D^1_{(i)}) \ar[r]^\psi \ar[d] & \partial E_D \ar[d]_{\pi_S} \\
\sqcup_{i=1}^N CP(1)_{(i)} \ar[r]^{pr_1} & \sqcup_{i=1}^N CP(1)_{(i)} \times CP(1)_{(i)} \times CP(1)_{(i)}
\end{array}
\]

We denote by \((1_i, \alpha_i)\) a basis of \( H^*(CP(1)_{(i)}) \) and \((\overline{T}_i, \overline{\pi}_i)\) a basis of \( H^*(S^3_{(i)}) \). The left-hand vertical map being a trivial bundle, we have \( M_\psi(\alpha_i) = a_i \). For degree reasons, the image of \( \overline{\pi}_i \in H^3(S^3_{(i)}) \) can be written as \( M(\overline{\pi}_i) = \lambda_i a_i \theta_i + \mu_i b_i \theta_i \), with \( \lambda_i, \mu_i \in \mathbb{Q} \). This image being a cocycle, we have

\[
\lambda_i a_i (a_i + b_i) + \mu_i b_i (a_i + b_i) = (\lambda_i + \mu_i) a_i b_i = 0,
\]

which implies \( \lambda_i = -\mu_i \). As we are working over the field \( \mathbb{Q} \), we may suppose \( \mu_i = 1 \) and \( M(\overline{\pi}_i) = (b_i - a_i) \theta_i \). In diagram (25), we filter \( N' \) by the set of elements sent in the ideal generated by the \( \theta_i \)'s and \( N \) by the set of elements sent in the ideal generated by the \( \overline{\pi}_i \)'s. These filtrations are of length 1 and, thanks to the commutativity of (25), the map \( N \to N' \) is compatible with the filtrations and induces a morphism of spectral sequences, \( E_j^{*,*} : E_j^{*,*} N' \to E_j^{*,*} N' \), \( j = 0, 1 \).

The CDGA \( (E_1^{*,*} N', d_1) \) is determined by Morgan in [35, Theorem 2.3]. In this reference the previous decreasing filtration, indexed by \( \{0, 1\} \), is replaced by the increasing filtration indexed by \( \{0, -1\} \). We use here the framework of decreasing filtration and, in this context, Theorem 2.3 of [35] states as:

\[
E_1^{0,0} N' = H^*(W), \quad E_1^{*,*} N' = \oplus_{i=1}^N (H^*(CP(1)_{(i)} \times CP(1)_{(i)})),
\]

the differential \( d_1 \) coming from the Gysin sequence of the sphere-bundle associated to the normal bundle. Moreover, with [35, Theorem 10.1], we know also that the CDGA \( (E_1^{*,*} N', d_1) \) is a rational model of the complement \( \overline{V}_{reg} \). If we use the convention \( (s^r A)^k = A^{r+k} \) for any \( r \in \mathbb{Z} \) and any graded vector space, \( A \), this model is the CDGA

\[
M_{\overline{V}_{reg}} = (H^*(W) \oplus_{i=1}^N s^{-1}(H^*(CP(1)_{(i)} \times CP(1)_{(i)}), d).
\]

The law of algebra is defined by:

- \( \alpha \cdot \beta = \alpha \cup \beta \), if \( \alpha, \beta \in H^*(W) \).
We conclude that, as CDGA, we have

\[ \mathcal{V} \Rightarrow \mathcal{V}_{\text{reg}} \]

\[ (E_{1}N, d_{1}) \Rightarrow \oplus_{i=1}^{N} H^{*}(\mathbb{C}P(1)_{(i)}), \]

This square is a model of the (homotopy) pushout

\[ \mathcal{V}_{\text{reg}} \leftarrow \bigcup_{i=1}^{N} \mathbb{C}P(1)_{(i)} \times S^{3}_{(i)} \]

Therefore \((E_{0}N, d_{0})\) is a model of \(V\) and \(E_{1}N = H(E_{0}N, d_{0}) = H^{*}(V)\). The CDGA, \(N\), being the set of elements of filtration degree 0, the previous observation gives a short exact sequence \((E_{0}N, d_{0}) \hookrightarrow (N, d_{0}) \rightarrow (\mathcal{V}_{1}N, d_{0})\). The square (26) being a pullback, the cokernels of the two vertical maps coincide and we have,

\[ \oplus_{i=1}^{N} H^{*}(\mathbb{C}P(1)_{(i)}) \leftarrow \oplus_{i=1}^{N} H^{*}(\mathbb{C}P(1)_{(i)}) \otimes H^{*}(S^{3}_{(i)}) \leftarrow \oplus_{i=1}^{N} s^{-3} H^{*}(\mathbb{C}P(1)_{(i)}) \]

We conclude that, as CDGA, we have

\[ (E_{1}N, d_{1}) = (H^{*}(V) \oplus \bigoplus_{i=1}^{N} s^{-3} H^{*}(\mathbb{C}P(1)_{(i)}), d_{1}) \]

with a product given by

- \(s^{-1}\alpha \cdot s^{-1}\beta = 0\), if \(s^{-1}\alpha, s^{-1}\beta \in \bigoplus_{i=1}^{N} s^{-1}(H^{*}(\mathbb{C}P(1) \times \mathbb{C}P(1))_{i})\),
- \(\alpha \cdot s^{-1}\beta = s^{-1}(g^{*}(\alpha) \cup \beta)\), if \(\alpha \in H^{*}(W)\) and \(s^{-1}\beta \in \bigoplus_{i=1}^{N} s^{-1}(H^{*}(\mathbb{C}P(1) \times \mathbb{C}P(1))_{i})\),
- \(s^{-1}\alpha \cdot s^{-1}\beta = s^{-1}(f^{*}(\alpha) \cup \beta)\), if \(\alpha \in H^{*}(V)\), \(s^{-1}\beta \in \bigoplus_{i=1}^{N} s^{-1}(H^{*}(\mathbb{C}P(1))_{i})\),

and a differential \(d_{1}\) which is the Gysin differential. This differential can be seen as the transfer map of the embedding \(f: \bigcup_{i=1}^{N} \mathbb{C}P(1)_{(i)} \to V\) and \(d_{1}(s^{-3}\alpha_{i})\) is the fundamental class of \(H^{*}(V)\), through the isomorphisms,

\[ H^{2}(\mathbb{C}P(1)_{i}) \xrightarrow{\cong} H_{0}(\mathbb{C}P(1)_{i}) \xrightarrow{\cong} H_{0}(V) \xrightarrow{\cong} H^{0}(V). \]

The map of CDGA’s, \(\Phi: (E_{1}N, d_{1}) \to (E_{1}N', d_{1})\), is defined by \(H^{*}(\mathbb{C}P(1)): H^{*}(V) \to H^{*}(W)\) and \(\Phi(s^{-3}\alpha_{i}) = s^{-1}(b_{i} - a_{i})\), \(\Phi(s^{-3}\alpha_{i}) = s^{-1}\theta_{i}(b_{i} - a_{i})\). As this map is a quasi-isomorphism, we choose \((E_{1}N, d_{1})\) as CDGA model of \(\mathcal{V}_{\text{reg}}\). With the description of \(H^{*}(V)\) in [30, Theorem 3.2], we have proved the next statement. (We notice also
that the cohomology of $V$ coincides with the intersection cohomology for the middle perversity of $V$, see [25].

Lemma 11.16. A model, $\mathcal{M}_{\text{reg}} = \oplus_{k=0}^{N} \mathcal{M}_{\text{reg}}^k$, of the complement, $V_{\text{reg}} = V \setminus \bigcup_{i=1}^{N} \mathbb{CP}(1)_{(i)}$, is described in the next array, the differential being specified below.

| $k$  | $\mathbb{Q}$ | 0          |
|------|--------------|------------|
| 0    | $\mathbb{Q}$ | 0          |
| 1    | 0            | 0          |
| 2    | $\mathbb{Q}[\omega, \mathcal{E}]$ | 0          |
| 3    | $\mathbb{Q}[\mathcal{A}, \mathcal{A}']$ | $\mathbb{Q}[s^{-3}1_i | 1 \leq i \leq N]$ |
| 4    | $\mathbb{Q}[\omega^2, \mathcal{E}']$ | 0          |
| 5    | 0            | $\mathbb{Q}[s^{-3}1_i | 1 \leq i \leq N]$ |
| 6    | $\mathbb{Q}[\omega^3]$ | 0          |

Most of the law of algebra of $H^*(V)$ can be deduced from Poincaré duality and the Hard Lefschetz theorem ([3]). (For instance, we choose Poincaré dual bases, $\mathcal{A} = (a_1, \ldots, a_v)$ and $\mathcal{A}' = (a'_1, \ldots, a'_v).)$ A part stays in the shadow, as the product of two elements of $\mathcal{E}$, but this information plays no role for the proof of the intersection-formality. The structure of $H^*(V)$-module on the generators $s^{-3}1_i$, $s^{-3}a_i$, has been described before the statement.

As for the differential, we already know that $d(s^{-3}a_i) = \omega^3$ for any $i = 1, \ldots, N$. We specify now the differential $d^3: \mathbb{Q}[s^{-3}1_i | 1 \leq i \leq N] \rightarrow \mathbb{Q}[\omega^2, \mathcal{E}']$. With $\text{Im} d^3 = \mathbb{Q}[\mathcal{E}']$, we choose a basis of $\mathbb{Q}[\mathcal{E}']$, extracted from $(d^3(s^{-3}1_i))_{1 \leq i \leq N}$, and suppose that is $(d^3(s^{-3}1_j))_{1 \leq j \leq p}$. We set $e'_j = d^3(s^{-3}1_j)$, for $j = 1, \ldots, p$. As basis of $\text{Ker} d^3$, we may choose a family $(s^{-3}1_k = s^{-3}1_k + \sum_{j=1}^{p} \nu_{kj} s^{-3}1_j)_{p+1 \leq k \leq N}$, $\nu_{kj} \in \mathbb{Q}$. In summary, we have

$$d^3(s^{-3}1_i) = \left\{ \begin{array}{ll}
  e'_i & \text{if } 1 \leq i \leq p, \\
  -\sum_{j \leq p} \nu_{ij} e'_j & \text{if } p + 1 \leq i \leq N.
\end{array} \right.$$  

We denote by $\mathcal{E} = (e_1, \ldots, e_p)$ the orthogonal basis of $\mathcal{E}'$, for the Poincaré duality. The equalities (29) are connected to $f^*(e_j) = \sum_{i=1}^{N} \lambda_{ji} a_i$, $\lambda_{ji} \in \mathbb{Q}$, as follows. By definition of the algebra structure, we have $e_j \cdot (s^{-3}1_i) = s^{-3}(f^*(e_j) \cup 1_i) = \lambda_{ji} s^{-3}a_i$. Applying the differential on each side of this equality, we get

$$\lambda_{ji} \omega^3 = \left\{ \begin{array}{ll}
  e_j \cdot e'_i & \text{if } 1 \leq i \leq p, \\
  -e_j \cdot \sum_{j \leq p} \nu_{ij} e'_j & \text{if } p + 1 \leq i \leq N.
\end{array} \right.$$  

This implies

$$f^*(e_j) = \alpha_j - \sum_{k=p+1}^{N} \nu_{kj} a_k.$$  

(In the case of the Calabi-Yau quintic, we have $N = 125$, $p = 24$ and $v = 1$.)

The injection of the link in the regular space, $L = \bigcup_{i=1}^{N} \partial(\mathbb{CP}(1)_{(i)} \times D^4_{(i)}) \rightarrow V_{\text{reg}}$, is the bottom map of [24] and its model, given by (28), is described in the next lemma.
Lemma 11.17. A model of the inclusion of the link, $L$, in $V_{\text{reg}}$,
$$
\mu: (M_{V_{\text{reg}}}, d_f) \rightarrow (M_L, 0) = \oplus_{i=1}^{N}(H^*(\text{CP}(1)_i)) \oplus s^{-3}H^*(\text{CP}(1)_i), 0)
$$
with
- $\mu = \text{id}$ on $s^{-3}1_i$ and $s^{-3}\alpha_i$,
- $\mu(e_j) = f^*(e_j)$, $j = 1, \ldots, p$,
- $\mu(1) = +\sum_{i=1}^{N}1_i$ and $\mu = 0$ on the other elements.

Concerning the product on $M_L$, the elements $1_i$’s are neutral elements and we have
$(s^{-3}1_i) \cdot \alpha_i = s^{-3}\alpha_i$. In summary, $M_L$ is the cohomology algebra of $\sqcup_{i=1}^{N}\text{CP}(1)_i \times S^3$.

Proof of Proposition 11.13. We transform $\mu$ in a surjective map, in strictly positive degrees, by
$$
\mu' = (M_{V_{\text{reg}}}', d_f) = (M_{V_{\text{reg}}}' \otimes \wedge(x_{p+1}, \ldots, x_N, y_{p+1}, \ldots, y_N), d_f) \rightarrow (M_L, 0),
$$
with $|x_k| = 2$, $|y_k| = 3$, $d_f x_k = y_k$, $\mu'(x_k) = \alpha_k$, $\mu'(y_k) = 0$. Now, Corollary 11.11 gives an intersection model of $V$
$$
(M_{\text{reg}}) = M_{V_{\text{reg}}}' \otimes M_L \tau \leq \mathbb{P}(n)M_L,
$$

with the canonical inclusions, $\varphi_{\mathbb{P}}: (M_{\mathbb{P}})_{\mathbb{P}} \rightarrow (M_{\mathbb{P}})_{\mathbb{P}}$, if $\mathbb{P} \leq \mathbb{P}$. Denote by $\overline{\ell}$ the GM-perversity such that $\overline{\ell}(6) = \ell$. In the next array, we collect the various models and their cohomology.

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|-----|---|---|---|---|---|---|---|
| $(M_{\text{reg}})^{k}_{\mathbb{P}}$ | $\mathbb{Q}$ | 0 | $\mathbb{Q}[\omega]$ | $\mathbb{Q}[A, A', B]$ | $(M_{V_{\text{reg}}}')^4$ | $(\text{Ker } \mu')^5$ | $(M_{V_{\text{reg}}}')^6$ |
| $H^k_{\mathbb{P}}(V)$ | $\mathbb{Q}$ | 0 | $\mathbb{Q}[\omega]$ | $\mathbb{Q}[A, A', B]$ | $\mathbb{Q}[\omega^2, \mathcal{E}']$ | 0 | $\mathbb{Q}[\omega^3]$ |
| $(M_{\text{reg}})^{k}_{\mathbb{P}}$ | $\mathbb{Q}$ | 0 | $(M_{V_{\text{reg}}}')^2$ | $\mathbb{Q}[A, A', B]$ | $(M_{V_{\text{reg}}}')^4$ | $(\text{Ker } \mu')^5$ | $(M_{V_{\text{reg}}}')^6$ |
| $H^k_{\mathbb{P}}(V)$ | $\mathbb{Q}$ | 0 | $\mathbb{Q}[\omega, \mathcal{E}]$ | $\mathbb{Q}[A, A', \mathcal{E}']$ | $\mathbb{Q}[\omega^2, \mathcal{E}']$ | 0 | $\mathbb{Q}[\omega^3]$ |
| $(M_{\text{reg}})^{k}_{\mathbb{P}}$ | $\mathbb{Q}$ | 0 | $(M_{V_{\text{reg}}}')^2$ | $(M_{V_{\text{reg}}}')^3$ | $(M_{V_{\text{reg}}}')^4$ | $(\text{Ker } \mu')^5$ | $(M_{V_{\text{reg}}}')^6$ |
| $H^k_{\mathbb{P}}(V)$ | $\mathbb{Q}$ | 0 | $\mathbb{Q}[\omega, \mathcal{E}]$ | $\mathbb{Q}[A, A', \mathcal{E}']$ | $\mathbb{Q}[\omega^2]$ | 0 | $\mathbb{Q}[\omega^3]$ |

with $\mathcal{B} = (y_{p+1}, \ldots, y_N)$, and $\mathcal{B}' = (s^{-3}1_{p+1}, \ldots, s^{-3}1_N)$. Let $\overline{\mathbb{P}} \leq \mathbb{P}$. The morphisms
$\psi_{\mathbb{P}}: H_{\mathbb{P}}(V) \rightarrow H_{\overline{\mathbb{P}}}(V)$ are defined by
- $\psi_{\mathbb{P}} = \text{id}$ and $\psi_{\mathbb{P}}$ is the identity except on $\mathcal{B}$ where it takes the value 0,
- $\psi_{\overline{\mathbb{P}}}$ is the identity except on $\mathcal{E}'$ where it takes the value 0, and $\psi_{\overline{\mathbb{P}}} = \text{id}$.

We construct now a perverse CDGA’s map, $\chi_{\bullet}$, such that the following diagram commutes.

$$
\begin{array}{cccc}
(M_{\mathbb{P}})_{\mathbb{P}} & \overset{\varphi_{\mathbb{P}}}{\longrightarrow} & (M_{\mathbb{P}})_{\mathbb{P}} & \overset{\varphi_{\mathbb{P}}}{\longrightarrow} & (M_{\mathbb{P}})_{\mathbb{P}} \\
\chi_{\mathbb{P}} & & & & \chi_{\mathbb{P}} \\
H_{\mathbb{P}}(V) & \overset{\psi_{\mathbb{P}}}{\longrightarrow} & H_{\mathbb{P}}(V) & \overset{\psi_{\mathbb{P}}}{\longrightarrow} & H_{\mathbb{P}}(V)
\end{array}
$$

(30)
In each degree, we choose a supplementary subspace, \( Z^c \), of the subspace of cocycles, \( Z \), and vector space bases \( B_{2c}, B_2 \), for each of them, with \( \nu \in \{1, \ldots, v\} \), \( j \in \{1, \ldots, p\} \), \( k, k', k'' \in \{p + 1, \ldots, N\} \).

- **Degree 2**: \( B_{2c} = (\omega, \xi) \), \( B_2 = (x_k) \).
- **Degree 3**: \( B_{2c} = (A, A', y_k, s^{-1}1^1_k), B_2 = (s^{-1}j_1) \).
- **Degree 4**: \( B_{2c} = (\omega^2, \xi'), B_2 = (\omega x_k, e_j x_k, x_k x_k') \) with \( k \leq k' \).
- **Degree 6**: \( B_{2c} = (\omega^3, \omega y_k, a y_k, A', y_k, s^{-1}1^1_k y_k, y_k y_k', e_j x_k - s^{-1}1_j y_k), \) with \( k'' < k'' \) and \( B_{2c} = (\omega^2 x_k, e_j x_k x_k', \omega x_k x_k', x_k x_k' x_k', s^{-1}1_j y_k), \) with \( k \leq k' \leq k'' \).

The description of the elements of degree 5 is not necessary because all of them are sent to 0 by \( \chi \). For any element, \( a \), in one of these bases, we set \( \chi(a) = [a] \), if \( a \in B_{2c} \), and \( \chi(a) = 0 \), if \( a \in B_2 \). (In this definition, we mean that \( \chi_\Sigma(a) \) is defined if the element \( a \) is of perverse degree \( q \).) The compatibility of \( \chi \) with the differentials and the commutativity of \((31)\) are direct. We need only to verify,

\[
\chi(a)(b) = \chi(ab).
\]

This is immediate if \( a, b \in B_{2c} \). Consider now \( a \in B_{2c} \). If the product \( ab \in B_{2c} \), the equality \((31)\) is satisfied. An inspection of the previous list of bases shows that the only case where \( ab \notin B_{2c} \) is \( e_j x_k \). We have \( \chi(a) = 0 \), by definition, and

\[
\chi(e_j x_k) = \chi(e_j x_k - s^{-1}1_j y_k) + \chi(s^{-1}1_j y_k) = \chi(e_j x_k - s^{-1}1_j y_k).
\]

For the determination of the cohomology class associated to \( e_j x_k - s^{-1}1_j y_k \), we first observe that this element is in the kernel of \( \mu' \) and thus has perverse degree \( \overline{0} \). The triviality of this class comes from \( d_f(s^{-1}1_j x_k) = e_j x_k - s^{-1}1_j y_k \) and \( s^{-1}1_j x_k \in \text{Ker} \mu' \).

\[\square\]

**Part 3. Topological setting**

**Appendix A. Filtered spaces**

A filtered space is a topological space together with a filtration by closed subsets. This simple notion is sufficient to define an intersection homology associated to a loose perversity, with classical properties, as the existence of a Mayer-Vietoris sequence and some particular case of Künneth formula.

**Definition A.1.** A *filtered space* is a topological space, \( X \), together with a filtration by closed subspaces,

\[X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n = X,\]

such that \( X_n \setminus X_{n-1} \) is not empty. The *formal dimension* of \( X \) is \( n \), denoted by \( d(X) = n \). The subspace \( X_i \) is called the *\( i \)-skeleton of \( X \)* but the index \( i \) and the formal dimension, \( d(X) \), are not necessarily related to a notion of geometrical dimension.

The connected components, \( S \), of \( X_i \setminus X_{i-1} \) are called *strata* of \( X \) and we set \( d(S) = i \). The strata in \( X_n \setminus X_{n-1} \) are called the *regular strata* of \( X \). We denote by \( S_X \) the family of non-empty strata of \( X \). The subspace \( X_{n-1} \) is the *singular set* and is also denoted by \( \Sigma \).
By convention, we set $X_j = \emptyset$ if $j < 0$ and $X_j = X$ if $j > n$. The filtration of a non-empty topological space $X$ by $X_{-1} = \emptyset$ and $X_n = X$ is called trivial of formal dimension $n$. If $(X, (X_i)_{0 \leq i \leq n})$ and $(Y, (Y_i)_{0 \leq i \leq m})$ are filtered spaces, we define a canonical filtration on the product $X \times Y$ by

$$(X \times Y)_i = \bigcup_{k+j=i} X_k \times Y_j, \text{ for } i \in \{0, \ldots, n + m\}.$$ 

**Definition A.2.** Let $(X, (X_i)_{0 \leq i \leq n})$ be a filtered space, the open cone on $X$ is the quotient $\mathcal{c}X = X \times [0, 1[\times X \times \{\emptyset\}$. We denote by $\emptyset$ the cone point. The conic filtration, $(\mathcal{c}X)_{0 \leq i \leq n+1}$, on the open cone is defined by $(\mathcal{c}X)_i = \mathcal{c}X_{i-1}$, with the convention $\mathcal{c}\emptyset = \emptyset$. Observe that $\mathcal{c}X \setminus \{\emptyset\}$ is the product $X \times [0, 1[$, where $[0, 1[$ owns the trivial filtration of formal dimension 1.

This conic filtration generates a canonical filtration on the suspension $\Sigma X$. The join $X * Y$ of two filtered spaces is also canonically filtered by observing that $X * Y$ is the union of three open sets, $X \times Y \times [0, 1[\times X \times \mathcal{c}Y \times \mathcal{c}X \times Y$, and the filtrations on the respective intersections coincide.

**Definition A.3.** Let $(X, (X_i)_{0 \leq i \leq n})$ be a filtered space and $\Delta$ be an euclidian simplex. A filtered simplex is a continuous map, $\sigma: \Delta \rightarrow X$, such that one of the following equivalent properties is satisfied.

(a) For all $i \in \{0, \ldots, n\}$, $\sigma^{-1}X_i$ is a face, $[0, 1[, m_i]$, of $\Delta$ or $\sigma^{-1}X_i = \emptyset$.

(b) There exists a decomposition $\Delta = \Delta^j_0 * \Delta^j_1 * \cdots * \Delta^j_n$ such that

$$\sigma^{-1}X_i = \Delta^j_0 * \Delta^j_1 * \cdots * \Delta^j_i,$$

for all $i \in \{0, \ldots, n\}$. We call it the $\sigma$-decomposition of $\Delta$.

Let $R$ be a commutative ring. We denote by $C_\ast(X)$ the usual singular chain complex of $X$, with coefficients in $R$, and by $C^s_\ast(X)$ the subcomplex generated by the filtered simplices.

The simplices $\Delta^j_i$, which arise in Definition A.3, can be empty. (An empty set appears automatically if the filtration of $X$ is stationary for an index $i$.) We use the convention $\emptyset * Y = Y$, for any space $Y$.

**Definition A.4.** Let $X$ be a filtered space. The solid $\sigma$-decomposition of a filtered simplex, $\sigma: \Delta \rightarrow X$, is obtained by keeping only the non-empty elements, $\Delta^j_i$, of the $\sigma$-decomposition, written in the same order, i.e., $\Delta = \Delta^j_1 * \cdots * \Delta^j_p$.

**Definition A.5.** Let $(X, (X_i)_{0 \leq i \leq n})$ be a filtered space and let $\sigma: \Delta \rightarrow X$ be a filtered simplex. The perverse degree of $\sigma$ is the $(n + 1)$-uple,

$$\|\sigma\| = (\|\sigma\|_0, \ldots, \|\sigma\|_n),$$

where $\|\sigma\|_i$ is the dimension of the smallest skeleton containing $\sigma^{-1}X_{n-\ell}$, with the convention $\|\sigma\|_{-\ell} = -\infty$ if $\sigma^{-1}X_{n-\ell} = \emptyset$.

If $\sigma: \Delta = \Delta^j_0 * \cdots * \Delta^j_n \rightarrow X$ is such that $\Delta^j_0 * \cdots * \Delta^j_{n-\ell} \neq \emptyset$, then $\|\sigma\|_{\ell} = \dim \sigma^{-1}X_{n-\ell} = \dim (\Delta^j_0 * \cdots * \Delta^j_{n-\ell})$. 

Example A.6. The next picture corresponds to a filtered space of dimension 2 with a curve as subspace $X_1$ and a singleton as subspace $X_0$. We have represented several situations of filtered simplices together with their decomposition and their perverse degree.

Definition A.7. \((\mathbb{Z}, \mathbb{Z})\) Let \((X, (X_i)_{0 \leq i \leq n})\) be a filtered space and let $p$ be a loose perversity. A \(p\)-admissible simplex is a filtered simplex, $\sigma: \Delta \to X$, such that
$$\|\sigma\|_{\ell} \leq \dim \Delta - \ell + p(\ell),$$
for all $\ell \in \{0, \ldots, n\}$. A chain $c$ is \(p\)-admissible if there exist \(p\)-admissible simplices, $\sigma_j$, so that $c = \sum_j \lambda_j \sigma_j$, $\lambda_j \in \mathbb{R}$.

Remark A.8. A \(p\)-admissible simplex, $\sigma: \Delta \to X$, verifies
$$\dim \Delta = \|\sigma\|_0 \leq \dim \Delta - 0 + p(0).$$
We impose $p(0) = 0$ in the definition of loose perversity and this restriction makes vacuous the condition on $\|\sigma\|_0$. (On the other side, we may observe also that, with a negative value for $p(0)$, we should not have any \(p\)-admissible simplex.)

Let $\overline{p}$ be a GM-perversity and $(X, (X_i)_{0 \leq i \leq n})$ be a filtered space. We check easily (see Remark 4.6) that a filtered 0-simplex, $\sigma: \Delta^0 \to X$, is $\overline{p}$-admissible if, and only if, $\sigma(\Delta^0) \subset X \backslash X_{n-1}$. This leads naturally to the next definition.

Definition A.9. A filtered space, $(X, (X_i)_{0 \leq i \leq n})$, is connected, if the regular part is it.

As usual in this theory, the submodule generated by the admissible $\overline{p}$-chains is not a subcomplex of the singular chains and we need to set the following definition.

Definition A.10. Let $(X, (X_i)_{0 \leq i \leq n})$ be a filtered space and let $\overline{p}$ be a loose perversity. A chain $c$ is an intersection chain for $\overline{p}$ if $c$ and its boundary $\partial c$ are $\overline{p}$-admissible chains. We denote by $C^p_\overline{p}(X)$ the complex of intersection chains, with coefficients in a commutative ring, $R$. The associated homology and cohomology are denoted by $H^p_\overline{p}(X)$ and $H_\overline{p}^p(X)$ and called intersection homology and cohomology for $\overline{p}$.

First, we characterize the $\overline{p}$-admissible simplices which are of $\overline{p}$-intersection.

Definition A.11. Let $X$ be a filtered space, $\overline{p}$ be a loose perversity and $\sigma: \Delta = \Delta^{j_0} \ast \cdots \ast \Delta^{j_n} \to X$ be a $\overline{p}$-admissible simplex. We denote by $\ell(\sigma) \in \{0, \ldots, n\}$ the integer such that
• \( \|\sigma\|_{\ell(\sigma)} = \dim \Delta - \ell(\sigma) + p(\ell(\sigma)) \), and

• \( \|\sigma\|_m < \dim \Delta - m + p(m) \), for all \( m > \ell(\sigma) \).

As \( \|\sigma\|_0 = \dim \Delta - 0 + p(0) \), the integer \( \ell(\sigma) \) exists. When \( \dim(\Delta^{j_0} \cdots \Delta^{j_{n-\ell(\sigma)}}) \neq \dim \Delta \), the restriction, \( \tau_\sigma: \Delta^{j_0} \cdots \Delta^{j_{n-\ell(\sigma)}} \to X \), is called the bad face of \( \sigma \).

**Example A.12.** Example A.6 provides an illustration of the previous definition. We choose the perversity \( \overline{0} \); a similar argument works for any loose perversity. Here the dimension is equal to 2. For the simplex \( \emptyset \ast \Delta^1 \ast \Delta^0 \), a computation gives:

\[
\dim \Delta - m + \overline{0}(m) = \begin{cases} 
2 & \text{if } m = 0, \\
1 & \text{if } m = 1, \\
0 & \text{if } m = 2,
\end{cases}
\]

and \( \|\sigma\|_m = \begin{cases} 
2 & \text{if } m = 0, \\
1 & \text{if } m = 1, \\
-\infty & \text{if } m = 2.
\end{cases} \)

Thus, by definition, its bad face is \( \emptyset \ast \Delta^1 \ast \emptyset \). To be complete, mention that \( \emptyset \ast \Delta^0 \ast \Delta^1 \) and \( \emptyset \ast \emptyset \ast \Delta^2 \) have no bad face. Also, the bad face of \( \Delta^0 \ast \Delta^0 \ast \Delta^0 \) is \( \emptyset \ast \emptyset \ast \emptyset \).

**Proposition A.13.** Let \( X \) be a filtered space and \( \sigma: \Delta \to X \) be a \( \overline{p} \)-admissible simplex.

(a) The simplex \( \sigma \) is of \( \overline{p} \)-intersection if, and only if, it does not contain bad faces.

(b) A codimension 1 face of \( \sigma \) is not \( \overline{p} \)-admissible if, and only if, it contains the bad face of \( \sigma \).

(c) Let \( \sigma': \Delta \to X \) be a \( \overline{p} \)-admissible simplex of \( X \), having a not \( \overline{p} \)-admissible face \( \sigma'' \) (of codimension 1) in common with \( \sigma \). Then \( \sigma'' \) contains the bad face of \( \sigma \), i.e., \( \sigma \) and \( \sigma' \) have the same bad face.

**Proof.** Recall that, by definition, \( \sigma: \Delta = \Delta^{j_0} \ast \cdots \ast \Delta^{j_n} \to X \) is \( \overline{p} \)-admissible if we have

\[
\dim \Delta^{j_0} \ast \cdots \ast \Delta^{j_{n-m}} = \|\sigma\|_m \leq \dim \Delta - m + \overline{p}(m),
\]

for all \( m \in \{1, \ldots, n\} \). Let \( F_i \) be a face of codimension 1 of \( \Delta^{j_i} \), with \( \Delta^{j_i} \neq \emptyset \). The restriction \( \sigma_i: \Delta^{j_0} \ast \cdots \ast \Delta^{j_{i-1}} \ast F_i \ast \Delta^{j_{i+1}} \ast \cdots \ast \Delta^{j_n} \to X \) of \( \sigma \) is \( \overline{p} \)-admissible if, and only if,

\[
\|\sigma_i\|_m \leq \dim \Delta - 1 - m + \overline{p}(m),
\]

for all \( m \in \{1, \ldots, n\} \). This perverse degree verifies

\[
\|\sigma_i\|_m = \begin{cases} 
\|\sigma\|_m - 1 & \text{if } m \leq n - i, \\
\|\sigma\|_m & \text{if } m > n - i.
\end{cases}
\]

As \( \|\sigma\|_m \leq \dim \Delta - 1 - m + \overline{p}(m) \) if \( m > \ell(\sigma) \), the simplex \( \sigma_i \) is \( \overline{p} \)-admissible if, and only if,

\[
(32) \quad \|\sigma\|_m < \dim \Delta - m + \overline{p}(m),
\]

for any \( m \) and \( i \) such that \( n - i < m \leq \ell(\sigma) \) and \( \Delta^{j_i} \neq \emptyset \).

(a) Inequality (32) cannot be verified for \( m = \ell(\sigma) \), because of the next contradiction,

\[
(33) \quad \dim \Delta - \ell(\sigma) + p(\ell(\sigma)) = \|\sigma\|_{\ell(\sigma)} < \dim \Delta - \ell(\sigma) + p(\ell(\sigma)).
\]
Recall that $\tau_\sigma$ denotes the bad face of $\sigma$, relatively to $\mathcal{P}$, and that $\sigma_i$ is a face of codimension 1 of $\sigma$. The previous observation implies:

$$\sigma \text{ is of } \mathcal{P}\text{-intersection } \iff \sigma_i \text{ is } \mathcal{P}\text{-admissible for any } i \in \{0, \ldots, n\}$$
$$\text{such that } \Delta_i \neq \emptyset$$
$$\iff \Delta_i = \emptyset \text{ if } n - \ell(\sigma) < i$$
$$\iff \dim \sigma = \dim \tau_\sigma.$$

But, by definition of the bad face, the equality $\dim \sigma = \dim \tau_\sigma$ is a contradiction.

(b) Thanks to the conditions (32) and (33), we know that the restriction $\sigma_i$ is $\mathcal{P}$-admissible if, and only if, $i \leq n - \ell(\sigma)$ and $\Delta_i \neq \emptyset$. In other words, $\sigma_i$ is not $\mathcal{P}$-admissible if, and only if, the bad face $\tau_\sigma$ of $\sigma$ is a face of $\sigma_i$. (Observe that $\dim \tau_\sigma \neq \dim \sigma$ as required in the definition of a bad face.)

(c) Let $\sigma' : \Delta = \Delta^{k_0} \ast \cdots \ast \Delta^{k_n} \to X$ and $\sigma'' : \Delta^{j_0} \ast \cdots \ast \Delta^{j_i-1} \ast F_i \ast \Delta^{j_i} \ast \cdots \ast \Delta^{j_n} \to X$ be the not $\mathcal{P}$-admissible face of $\sigma'$, in common with $\sigma$. As proved in (b), we have $i > n - \ell(\sigma)$. For $* \leq n - \ell(\sigma)$, if the integers $k_i$ and $j_i$ are not equal, they differ only for one index, $m \in \{0, \ldots, n - \ell(\sigma)\}$, for which we have $k_m = j_m + 1$. By definition, the $\mathcal{P}$-admissibility of $\sigma''$ implies $\|\cdot\|_{\ell(\sigma)} \leq \dim \Delta - \ell(\sigma) + \mathcal{P}(\ell(\sigma))$ and we have:

$$\dim \Delta - \ell(\sigma) + \mathcal{P}(\ell(\sigma)) \geq \|\sigma''\|_{\ell(\sigma)} = \dim(\Delta^{k_0} \ast \cdots \ast \Delta^{k_{n-\ell(\sigma)}})$$
$$= \dim(\Delta^{j_0} \ast \cdots \ast \Delta^{j_{n-\ell(\sigma)}}) + 1 = \|\sigma\|_{\ell(\sigma)} + 1$$
$$= \dim \Delta - \ell(\sigma) + \mathcal{P}(\ell(\sigma)) + 1.$$  

Thus, the integers $k_\ast$ and $j_\ast$ are equal for $* \leq n - \ell(\sigma)$ and the bad face of $\sigma$ is a face of $\sigma''$. \qed

In this context of filtered spaces, we prove the existence of a Mayer-Vietoris sequence and recall the intersection homology of a cone.

**Proposition A.14.** Let $X$ be a filtered space of formal dimension $d(X) = n$ and let $\mathcal{P}$ be a loose perversity. Then the following properties are satisfied, for intersection homology with coefficients in a commutative ring, $R$.

(i) If $\mathcal{U} = \{U, V\}$ is an open cover of $X$, there exists a long exact sequence

$$\cdots \to H_\mathcal{P}(U \cap V) \to H_\mathcal{P}(U) \oplus H_\mathcal{P}(V) \to H_\mathcal{P}(X) \to H_{\mathcal{P}-1}(U \cap V) \to \cdots$$

(ii) The canonical projection $\text{pr} : \mathbb{R} \times X \to X$ induces an isomorphism

$$H_\mathcal{P}(\text{pr}) : H_\mathcal{P}(\mathbb{R} \times X) \to H_\mathcal{P}(X),$$

whose inverse is induced by the inclusion $j_0 : X \to \mathbb{R} \times X$, $x \mapsto (0, x)$.

(iii) If $U_0 \subset U_1 \subset \cdots$ are open sets of $X$ so that $X = \bigcup_i U_i$, then the natural map

$$\lim_{\to} H_\mathcal{P}(U_i) \to H_\mathcal{P}(X)$$

is an isomorphism.

(iv) If $M$ is a topological manifold, then the following sequence is split exact for each $i$,

$$0 \to (H_\ast(M) \otimes H_\mathcal{P}(X))_i \to H_i^\mathcal{P}(M \times X) \to (H_\ast(M) \ast H_\mathcal{P}(X))_{i-1} \to 0.$$
Suppose $X$ is compact and $\mathcal{p}$ is a perversity. If the cone $\mathcal{c}X$ is endowed with the conic filtration, we have

$$\mathcal{H}^{\mathcal{p}}_k(\mathcal{c}X) = \begin{cases} 
\mathcal{H}^{\mathcal{p}}_k(X) & \text{if } k \leq n - \mathcal{p}(n + 1) - 1, \\
0 & \text{if } k \geq n - \mathcal{p}(n + 1) \text{ and } k \neq 0, \\
R & \text{if } k \geq n - \mathcal{p}(n + 1) \text{ and } k = 0.
\end{cases}$$

**Proof.** We will use the fact that any map, $\varphi: X \to Y$, between filtered spaces of the same formal dimension, such that $\varphi^{-1}Y_i \subset X_i$ for any $i$, induces a chain map $C^\mathcal{p}_*(X) \to C^\mathcal{p}_*(Y)$.

(i) This is a consequence of Lemma A.16, as in the classical case.

(ii) The map $f: \mathbb{R} \times X \times [0, 1] \to \mathbb{R} \times Y$, $(u, x, t) \mapsto (ut, x)$ induces a chain map $C^\mathcal{p}_*(\mathbb{R} \times X \times [0, 1]) \to C^\mathcal{p}_*(\mathbb{R} \times Y)$ and thus a homomorphism $\mathcal{H}^{\mathcal{p}}_k(\mathbb{R} \times X \times [0, 1]) \to \mathcal{H}^{\mathcal{p}}_k(\mathbb{R} \times Y)$. With the notation and the result of Lemma A.15, as $f \circ i_0 = j_0 \circ \varphi$ and $f \circ i_1 = \text{id}$, we deduce $\mathcal{H}^{\mathcal{p}}_k(j_0) \circ \mathcal{H}^{\mathcal{p}}_k(\varphi) = \text{id}$. This equality and $\varphi \circ j_0 = \text{id}_X$ imply that $\mathcal{H}^{\mathcal{p}}_k(j_0)$ and $\mathcal{H}^{\mathcal{p}}_k(\varphi)$ are inverse of each other.

(iii) This is obvious.

(iv) A careful reading of the proof of [32, Theorem 4] shows that this particular K"unneth formula exists if properties (i), (ii) and (iii) are satisfied.

(v) Let $X$ be a compact filtered space and $\mathcal{c}X$ be the associated open cone, endowed with the conic filtration. The next computation is already done in previous works.

Let $\sigma: \Delta^k \to \mathcal{c}X$ be a $\mathcal{p}$-admissible filtered simplex. We consider two cases.

- Let $k = \dim \Delta \leq n - \mathcal{p}(n + 1)$. As $\|\sigma\|_{n+1} \leq k - (n + 1) + \mathcal{p}(n + 1) < 0$, we have $\sigma^{-1}(\mathcal{c}X)_0 = \emptyset$ and the simplex $\sigma$ does not meet the cone point, $\vartheta$, of $\mathcal{c}X$. This implies $C^{\mathcal{p}}_{\leq n-\mathcal{p}(n+1)}(\mathcal{c}X) = C^{\mathcal{p}}_{\leq n-\mathcal{p}(n+1)}(X \times [0, 1])$ and property (ii) implies $\mathcal{H}^{\mathcal{p}}_k(\mathcal{c}X) = \mathcal{H}^{\mathcal{p}}_k(X \times [0, 1]) = \mathcal{H}^{\mathcal{p}}_k(X)$, for all $k \leq n - \mathcal{p}(n + 1) - 1$.

- Let $k = \dim \Delta \geq n - \mathcal{p}(n + 1)$. In this case, we show that the classical homotopy operator, used to prove the contractibility of the singular chain complex of a cone, is compatible with the intersection setting. If $s \in [0, 1]$ and $(x, t) \in \mathcal{c}X$, we set $s \cdot (x, t) = (x, st)$. If $\sigma: \Delta \to \mathcal{c}X$ is a simplex, we define $cs: \{\vartheta\} * \Delta \to \mathcal{c}X$ by $cs(sa + (1 - s)\vartheta) = s \cdot \sigma(a)$.

As $X$ is compact, this map is continuous and we extend it in a linear map $cs: C_*(\mathcal{c}X) \to C_*(\mathcal{c}X)$. Let $i \in \{1, \ldots, n + 1\}$ and $\sigma: \Delta \to \mathcal{c}X$ be a $\mathcal{p}$-admissible simplex. We have

$$\sigma^{-1}((\mathcal{c}X)_{n+1-i}) = \{sa + (1 - s)\vartheta \mid s \cdot \sigma(a) \in (\mathcal{c}X)_{n+1-i} = \mathcal{c}X_{n-i}\} = \{\vartheta\} \ast \sigma^{-1}(\mathcal{c}X_{n-i}),$$

which is a face of $\Delta$. If $\sigma^{-1}(\mathcal{c}X_{n-i}) \neq \emptyset$, we have

$$\dim(\sigma^{-1}((\mathcal{c}X)_{n+1-i})) = 1 + \dim(\sigma^{-1}(\mathcal{c}X_{n-i})) \leq 1 + \dim \Delta - i + \mathcal{p}(i) = \dim(\{\vartheta\} \ast \Delta) - i + \mathcal{p}(i).$$
Therefore, the simplex $\sigma \iota$ is $p$-admissible.

If $\sigma^{-1}(\check{c}X_{n-i}) = \emptyset$, then $\dim(\check{c}X_{n+1-i}) = 0$ and, by hypothesis, we have

$$0 \leq k - n + p(n + 1) = k + 1 - (n + 1) + p(n + 1) \leq k + 1 - i + p(i),$$

because $p$ is a perversity. The last expression is equal to $\dim(\{\vartheta\} \star \Delta) - i + p(i)$ and the simplex $\sigma \iota$ is $p$-admissible.

Now we compute $H^p_k(\check{c}X)$ by distinguishing two cases.

— $k \neq 0$. The result is clear for $k < 0$ and we may suppose $k > 0$. For any $\eta \in C^p_0(\check{c}X)$, we have $\partial \eta + c \partial \eta = \eta$, which implies $H^p_k(\check{c}X) = 0$.

— $k = 0$. For any $\eta \in C^0_0(\check{c}X)$, we have the formula $\partial \eta = \eta - \vartheta$. This implies that all the generators of $C^0_0(\check{c}X)$ are identified in homology and, therefore, $H^0_0(\check{c}X) = R$, because $X_n \setminus X_{n-1} \neq \emptyset$.

\[\square\]

**Lemma A.15.** Let $p$ be a loose perversity. For any filtered space, $X$, the canonical injections, $\iota_0, \iota_1: X \to X \times I$, defined by $\iota_k(x) = (x, k)$ for $k = 0, 1$, induce the same homomorphism in intersection homology, $H^p(\iota_0) = H^p(\iota_1): H^p(\check{c}X) \to H^p(X \times [0, 1])$.

**Proof.** In the non-stratified setting, the proof (see [15, Théorème 5.33] for instance) uses a chain homotopy between $C_\iota$ and $C_{\iota_1}$, defined as follows. If $\Delta^m = \{e_0, \ldots, e_m\}$, we denote by $a_j = (e_j, 0)$ and $b_j = (e_j, 1)$ the points of $\Delta^m \times I$ corresponding to vertices. A singular $(m+1)$-chain of $\Delta^m \times I$ is defined by

$$P = \sum_{j=0}^{m} (-1)^j (a_0, \ldots, a_j, b_j, \ldots, b_m).$$

This chain generates a homotopy $h: C_\iota(X) \to C_{\iota_1}(X \times I)$ defined by $h(\sigma) = (\sigma \times \text{id})_\iota(P)$ and which satisfies the expected equality $(dh + hd)(\sigma) = C_{\iota_1}(\sigma) - C_\iota(\sigma_0)(\sigma)$.

Let $\sigma$ be a $p$-admissible simplex of $X$, we are reduced to prove that $h(\sigma)$ is a $p$-admissible simplex of $X \times [0, 1]$. Let $j \in \{0, \ldots, m\}$. We denote by $\tau_j: \Delta^{m+1} = \{v_0, \ldots, v_{m+1}\} \to \Delta^m \times I$ the $(m+1)$-simplex defined by $(v_0, \ldots, v_j, v_{j+1}, \ldots, v_{m+1}) \mapsto (a_0, \ldots, a_j, b_j, \ldots, b_m)$. Let $F$ be any face of the simplicial decomposition

$$\Delta^m \times I = \bigcup_{j=0}^{m} \tau_j(\Delta^{m+1}).$$

As $\tau_j^{-1}(F)$ is identified to $F \cap \tau_j(\Delta^{m+1})$, we have $\dim \tau_j^{-1}(F) \leq \dim F$, for any $0 \leq j \leq m$. From this inequality, we deduce

$$\dim \tau_j^{-1}((\sigma \times \text{id})^{-1}(X_{n-i} \times I)) \leq \dim \tau_j^{-1}(\sigma^{-1}(X_{n-i}) \times I) \leq \dim \sigma^{-1}(X_{n-i}) \times I \leq m - i + p(i) + 1.$$
and \( h(\sigma) \) is \( \mathcal{P} \)-admissible.

**Lemma A.16.** Let \( \mathcal{U} = (U, V) \) be an open cover of the filtered space, \( X \), and let \( \mathcal{P} \) be a loose perversity. We define a homomorphism, \( \varphi: \mathcal{C}^p_U(U) \oplus \mathcal{C}^p_V(V) \to \mathcal{C}^p_X(X) \), by \( \varphi([\xi], [\xi]) = [\xi_1 + \xi_2] \). Then the following properties are satisfied.

(i) For any \( \xi \in \mathcal{C}^p_X(X) \), there exists an integer \( k \geq 0 \), such that the iterated barycentric subdivision of \( \xi \) verifies \( (sd)^k \xi \in \text{Im} \varphi \).

(ii) The injection \( \text{Im} \varphi \hookrightarrow \mathcal{C}^p_X(X) \) induces an isomorphism in homology.

**Proof.** We adapt the classical proof to the intersection setting (see also \[17\] and \[43\]). For that, we refer to the presentation of \[46\] Section 4 of Chapter 4 or \[30\] Section 2.1]. The key is the construction of chain maps, \( sd: \mathcal{C}^p_X(X) \to \mathcal{C}^p_X(X) \) and \( T: \mathcal{C}^p_X(X) \to \mathcal{C}^p_{X+1}(X) \), such that \( \partial T + T \partial = \text{id} - sd \). We proceed in several steps.

- First, we define \( sd \) in the intersection setting. Denote by \( \Delta^{p\text{er}}(\Delta) \) the subcomplex generated by the linear simplices, \( \xi \), of \( \Delta \) such that

\[
\dim \xi^{-1}(F) \leq \dim F,
\]

for any face \( F \) of \( \Delta \). (Recall from \[46\] Page 176] that a simplex \( \sigma: \Delta' \to \Delta \) is said linear if \( \sigma(\sum t_i e_i) = \sum t_i \sigma(e_i) \) for \( t_i \in [0,1] \) and \( \sum t_i = 1 \). A linear simplex is therefore entirely determined by its values on the vertices.) Observe that if \( \sigma: \Delta \to X \) is filtered and \( \zeta \in \Delta^{p\text{er}}(\Delta) \), then

\[
- \sigma_*(\zeta)^{-1}(X_i) = \zeta^{-1}\sigma^{-1}(X_i) \text{ is a face of } \Delta \text{ because } \sigma \text{ is filtered and } \zeta \text{ linear,}
- \|\sigma_*(\zeta)\| \leq \dim \zeta^{-1}\sigma^{-1}(X_{n-i}) \leq \dim \sigma^{-1}(X_{n-i}) = \|\sigma\|.
\]

Therefore, if \( \sigma \) is \( p \)-admissible and \( \zeta \in \Delta^{p\text{er}}_{\dim \Delta}(\Delta) \) then we have

\[
\|\sigma_*(\zeta)\| \leq \|\sigma\| \leq \dim \Delta - i + p(i),
\]

and \( \sigma_*(\zeta) \) is \( p \)-admissible. We define \( sd \) on a linear simplex, \( \xi: \Delta^\ell \to \Delta \), by

\[
\text{sd}(\xi) = \left\{ \begin{array}{ll}
\xi & \text{if } \ell = 0, \\
(\xi_b(\xi) \cdot \text{sd}(\partial \xi)) & \text{if } \ell > 0,
\end{array} \right.
\]

where \( b(\xi) \) is the barycenter of \( \xi(\Delta^\ell) \) and the definition of \( b(\xi) \cdot \text{sd}(\partial \xi) \) is recalled below.

If \( \sigma: \Delta \to X \) is a singular simplex, we set \( \text{sd}(\sigma) = \sigma_*(\text{sd}([\Delta])) \), as in \[46\] or \[30\].

We want to prove that the image by \( sd \) of a chain of \( p \)-intersection is of \( p \)-intersection. As \( sd \) commutes with the differential, it is sufficient to prove that the image by \( sd \) of a \( p \)-admissible simplex is \( p \)-admissible. Therefore, we are reduced to the verification of the inclusion,

\[
\text{sd}(\Delta^{p\text{er}}(\Delta)) \subset \Delta^{p\text{er}}(\Delta).
\]

(Observe that \([\Delta]\) \( \in \Delta^{p\text{er}}_{\dim \Delta}(\Delta) \).) The inclusion \( \text{sd}(\Delta^{p\text{er}}(\Delta)) \subset \Delta^{p\text{er}}(\Delta) \) is clear for \( \ell = 0 \). Let \( \xi \in \Delta^{p\text{er}}(\Delta) \). By induction, we know that \( \text{sd}(\partial(\xi)) = \sum n_i \tau_i \), with \( \tau_i \in \Delta^{p\text{er}}_{\ell-1}(\Delta) \). Therefore, we have \( b(\xi) \cdot \text{sd}(\partial(\xi)) = \sum n_i b(\xi) \cdot \tau_i \), where \( b(\xi) \cdot \tau_i \) is defined on \( \Delta^\ell = \Delta^{\ell-1} \ast \{\vartheta\} \) by

\[
(b(\xi) \cdot \tau_i)(t \vartheta + (1-t)a) = t b(\xi) + (1-t) \tau_i(a).
\]

We have to prove that \( b(\xi) \cdot \tau_i \in \Delta^{p\text{er}}(\Delta) \). If \( F \) is a face of \( \Delta \), we claim that \( \dim (b(\xi) \cdot \tau_i)^{-1}(F) \leq \dim F \). We consider two cases:
We extend this definition to the chain $\xi$ and $\ell$ induction, we know that $\xi^{-1}(F) = (b(\xi) \cdot \tau_i)^{-1}(F)$ which gives $\dim(b(\xi) \cdot \tau_i)^{-1}(F) = \dim \xi^{-1}(F) \leq \dim F$.

- $b(\xi) \notin F$. Then $(b(\xi) \cdot \tau_i)^{-1}(F) = \tau_i^{-1}(F)$ and we already know that $\dim \tau_i^{-1}(F) \leq \dim F$.
- $b(\xi) \in F$. In this case, $\Delta^\ell = \xi^{-1}(F) = (b(\xi) \cdot \tau_i)^{-1}(F)$ which gives $\dim(b(\xi) \cdot \tau_i)^{-1}(F) = \dim \xi^{-1}(F) \leq \dim F$.

\begin{itemize}
\item We extend now the definition of $T$ to the intersection setting. Denote by $\Delta_{per}^{\ell+1}(\Delta)$ the subspace generated by the linear simplices, $\xi$, of $\Delta$ such that
\[
\dim \xi^{-1}(F) \leq 1 + \dim F,
\]
for any face $F$ of $\Delta$. If $\sigma : \Delta \to X$ is filtered and $\zeta \in \Delta_{per}^{\ell+1}(\Delta)$, then
\begin{align*}
- \sigma_*(\zeta) &= \zeta^{-1}\sigma_*(X_i) \text{ is a face of } \Delta \text{ because } \sigma \text{ is filtered and } \zeta \text{ linear,}
- \|\sigma_*(\zeta)\|_i &= \dim \zeta^{-1}\sigma^{-1}(X_{n-i}) \leq \dim \sigma^{-1}(X_{n-i}) + 1 = \|\sigma\|_i + 1.
\end{align*}

Therefore, if $\sigma$ is $p$-admissible and $\zeta \in \Delta_{per}^{\ell+1}(\Delta)$, then we have
\[
\|\sigma_*(\zeta)\|_i \leq \|\sigma\|_i + 1 \leq \dim \Delta + 1 - i + p(i),
\]
and $\sigma_*(\zeta)$ is $p$-admissible. We define $T$ on a linear simplex, $\xi : \Delta^\ell \to \Delta$, by
\[
T(\xi) = \begin{cases}
0 & \text{if } \ell = -1, \\
b(\xi) \cdot (\xi - T(\partial(\xi))) & \text{if } \ell \geq 0,
\end{cases}
\]
where $b(\xi)$ is the barycenter of $\xi(\Delta^\ell)$. If $\sigma \in C_p^\ell(X)$, we set $T(\sigma) = \sigma_*(T(\Delta^\ell))$. As in the case of $sd$, since $T\partial + \partial T = \id - sd$, for proving that the image by $T$ of a $p$-intersection simplex is of $p$-intersection, it is sufficient to prove that
\[
T(\Delta_{per}^\ell(\Delta)) \subseteq \Delta_{per}^{\ell+1}(\Delta).
\]
The inclusion $T(\Delta_{per}^\ell(\Delta)) \subseteq \Delta_{per}^{\ell+1}(\Delta)$ is obvious if $\ell = -1$. Let $\xi \in \Delta_{per}^\ell(\Delta)$. By induction, we know that $\xi - T(\partial(\xi)) = \sum_i n_i \tau_i$, with $\tau_i \in \Delta_{per}^{\ell+1}(\Delta)$. Therefore, we have $b(\xi) \cdot (\xi - T(\partial(\xi))) = \sum_i n_i b(\xi) \cdot \tau_i$ and we have to prove that $b(\xi) \cdot \tau_i \in \Delta_{per}^{\ell+1}(\Delta)$. If $F$ is a face of $\Delta^\ell$, we claim that $\dim (b(\xi) \cdot \tau_i)^{-1}(F) \leq \dim F + 1$. We consider two cases:

- $b(\xi) \notin F$. Then $(b(\xi) \cdot \tau_i)^{-1}(F) = \tau_i^{-1}(F)$ and we already know that $\dim \tau_i^{-1}(F) \leq \dim F + 1$.
- $b(\xi) \in F$. In this case, $\Delta^\ell = \xi^{-1}(F)$ and $\Delta^{\ell+1} = (b(\xi) \cdot \tau_i)^{-1}(F)$ which gives $\dim (b(\xi) \cdot \tau_i)^{-1}(F) = \dim \xi^{-1}(F) + 1 \leq \dim F + 1$.

\begin{itemize}
\item Proof of (i). Let $\xi = \sum_{j\in J} n_j \sigma_j$ be a chain of $p$-intersection, written as a sum of filtered simplices. Recall from Definition [11] the integer $\ell(\sigma_j)$, associated to a simplex $\sigma$. We extend this definition to the chain $\xi$ by $\ell(\xi) = \max\{\ell(\sigma_j) \mid j \in J \text{ such that } n_j \neq 0\}$ and $\ell(0) = -\infty$.

We have to show the existence of an integer $k$ such that the iterated barycentric subdivision of $\xi$ verifies
\[
(34) \quad sd^k \xi \in C^p_\star(U) + C^p_\star(V).
\]
From the classical theory, we can find $r \geq 0$, with $sd^r \xi \in C_\star(U) + C_\star(V)$. Thus, we can suppose
\[
(35) \quad \xi \in C^p_\star(X) \cap (C_\star(U) + C_\star(V)).
\]
On the set of chains of $\mathcal{P}$-intersection, we define an equivalence relation by
\[ \xi \sim \zeta \text{ if there exists } k \geq 0, \text{ such that } sd^k \xi - sd^k \zeta \in C_\mathcal{P}^*(U) + C_\mathcal{P}^*(V). \]
This equivalence relation is clearly compatible with sums, i.e., $\xi \sim \zeta$ and $\xi' \sim \zeta'$ imply $\xi + \xi' \sim \zeta + \zeta'$. The interest of this relation relies in the fact that Property (i) is equivalent to $\xi \sim 0$. We prove it by induction on $\ell(\xi)$. This is obviously true if $\ell(\xi) = 0$. We suppose it is true for any chain of $\mathcal{P}$-intersection, $\zeta$, such that $\ell(\zeta) < m$ and we consider a chain of $\mathcal{P}$-intersection, $\xi$, with $\ell(\xi) = m$. We decompose $\xi$ in $\xi = \xi_0 + \cdots + \xi_m$, with the next characterization of the components $\xi_i$.

- The chain $\xi_0$ is formed of simplices which are of $\mathcal{P}$-intersection, i.e., $\xi_0$ is composed of simplices $\sigma$ such that $\ell(\sigma) = 0$.
- The chain $\xi_i$, for $i > 0$, is composed of simplices $\sigma$ having a bad face and such that $\ell(\sigma) = i$.

By induction and the compatibility of the equivalence relation with sums, we may suppose $\xi = \xi_m$. Denote by $\{\tau_i\}_{i \in I}$ the bad faces that appear for the simplices of $\xi$. We decompose $\xi$ in $\xi = \sum_{i \in J} \xi(i)$, where the simplices of $\xi(i)$ have $\tau_i$ as bad face. Let $\sigma$ be a simplex of $\xi(i)$. As the sum $\xi$ is of $\mathcal{P}$-intersection, there exists at least a simplex $\sigma'$ which has $\tau_i$ as face. From (c) of Proposition A.13 we know that $\sigma$ and $\sigma'$ have the same bad face. Therefore $\sigma'$ is a simplex of $\xi(i)$ and $\xi(i)$ is of $\mathcal{P}$-intersection. With the compatibility of the equivalence relation with sums, we may suppose that each simplex of $\xi$ has the same bad face, $\tau: \Delta^{j_0} \ast \cdots \ast \Delta^{j_n-m} \to X$.

Grants to (35), we may suppose also $\text{Im } \tau \subset U$. There exists a subdivision such that
\[ sd^k \xi = \sum_{\text{Im } \beta_a \cap \text{Im } \tau = \emptyset} n_{\alpha} \beta_a + \sum_{\text{Im } \beta_b \cap \text{Im } \tau \neq \emptyset} n_b \beta_b \in C_\mathcal{P}^*(X), \]
with $\text{Im } \beta_b \subset U$, for all $b$. Each element $\beta$ of these sums is obtained from a simplex, $\sigma: \Delta = \Delta^{j_0} \ast \cdots \ast \Delta^{j_n} \to X$, of $\xi$, verifying $\ell(\sigma) = m$ and $\dim \Delta^{j_0} \ast \cdots \ast \Delta^{j_n-m} \neq \text{dim } \Delta$, and from an element $F$ of the iterated barycentric subdivision of $\Delta$, of the same dimension than $\Delta$, i.e.,
\[ \beta: F \cap (\Delta^{j_0} \ast \cdots \ast \Delta^{j_n}) \to X. \]

Observe:
\[
\|\beta\|_n = \dim(F \cap \Delta^{j_0}) \leq \dim \Delta^{j_0} < \dim \Delta - n + p(n), \\
\|\beta\|_{m+1} = \dim(F \cap (\Delta^{j_0} \ast \cdots \ast \Delta^{j_{n-m-1}})) \leq \dim(\Delta^{j_0} \ast \cdots \ast \Delta^{j_{n-m-1}}) \\
< \dim \Delta - (m+1) + p(m+1) = \dim F - (m+1) + p(m+1), \\
\|\beta\|_m = \dim(F \cap (\Delta^{j_0} \ast \cdots \ast \Delta^{j_{n-m}})) = \dim(\Delta^{j_0} \ast \cdots \ast \Delta^{j_{n-m}}) \\
\leq \dim \Delta - m + p(m) = \dim F - m + p(m),
\]
where the strict inequalities appear for $\|\beta\|_i$ with $i > m = \ell(\sigma)$. From the previous inequalities, we deduce $\ell(\beta) \leq m$. Thus, we have proved that
\[ sd^k \xi = (sd^k \xi)_0 + \cdots + (sd^k \xi)_{m-1} + (sd^k \xi)_m. \]
From the induction and the compatibility of the equivalence relation with sums, it is sufficient to prove that $(sd^k \xi)_m \sim 0$. For that, we observe that we have $\ell(\beta) = m$
if, and only if, \( \dim(F \cap (\Delta^{\ast} \cdots \ast \Delta^{\ast} \cdots \ast \Delta^{n-m})) = \dim(\Delta^{\ast} \cdots \ast \Delta^{\ast} \cdots \ast \Delta^{n-m}) \), which implies \( F \cap (\Delta^{\ast} \cdots \ast \Delta^{\ast} \cdots \ast \Delta^{n-m}) \neq \emptyset \). In particular, if \( \ell(\beta) = m \), we have \( \operatorname{Im} \beta \cap \operatorname{Im} \tau \neq \emptyset \) and thus \( \operatorname{Im} \beta \subset U \).

As \( \dim(F \cap (\Delta^{\ast} \cdots \ast \Delta^{n-m})) = \dim(\Delta^{\ast} \cdots \ast \Delta^{n-m}) \neq \dim \Delta = \dim F \), we have proved that the chain of \( \mathcal{P} \)-intersection, \( (sd^k \xi)_m \), verifies

\[
(sd^k \xi)_m = \sum_{\operatorname{Im} \beta \cap \operatorname{Im} \tau \neq \emptyset, \ell(\beta_b) = m} n_b \beta_b \in C_p^\mathcal{P}(U)
\]

and this gives the inductive step.

- Proof of (ii). We prove that the inclusion \( i: \operatorname{Im} \varphi \hookrightarrow C_p^\mathcal{P}(X) \) induces an isomorphism in homology. Let \( [\xi] \in H^*_p(X) \). With the already established properties of \( sd \) and \( T \), we have \( [\xi] = [sd^k \xi] \), for all \( i \geq 0 \).
  - For the surjectivity, let \( [\xi] \in H^*_p(X) \). Thanks to the property (i), there exists \( k \geq 0 \) with \( sd^k \xi \in \operatorname{Im} \varphi \). We get
    \[
    [\xi] = i_*[sd^k \xi] \in \operatorname{Im} i_*.
    \]
  - Concerning the injectivity, let \( [\alpha] \in H^*_p(\operatorname{Im} \varphi) \) and \( \xi \in C_p^{\mathcal{P} + 1}(X) \) such that \( \alpha = \partial \xi \). With the property (i), there exists \( k \geq 0 \) with \( sd^k \xi \in \operatorname{Im} \varphi \). This implies
    \[
    [\alpha] = [sd^k \alpha] = [sd^k(\partial \xi)] = [\partial(sd^k \xi)]
    \]
    and \( [\alpha] = 0 \) in \( H^*_p(\operatorname{Im} \varphi) \).

\[\square\]

Appendix B. Stratified spaces

Filtered spaces are sufficient for a definition of intersection homology, but this generality does not bring a good behavior for maps. For having induced morphisms between intersection homology groups, we introduce stratified spaces and maps, the key point (Theorem \ref{thm:main}) being the fact that stratified maps decrease the perverse degree of Definition \ref{def:perverse}. We define also a notion of \( s \)-homotopy between stratified maps and prove that two \( s \)-homotopic stratified maps induce the same homomorphism in intersection homology.

**Definition B.1.** A stratified space is a filtered space such that any pair of strata, \( S \) and \( S' \) with \( S \cap S' \neq \emptyset \), verifies \( S \subset S' \).

For instance, the CS sets of King \cite{1989} are stratified spaces, see Theorem \ref{thm:CS} and Definition \ref{def:CS}. Also, the previous constructions of cone, suspension, product and join of stratified spaces are stratified.

**Definition B.2.** A stratified map, \( f: (X, (X_j)_{0 \leq j \leq n}) \to (Y, (Y_j)_{0 \leq j \leq m}) \), is a continuous map between stratified spaces, such that, for any stratum \( S \in \mathcal{S}_X \), there exists a stratum \( S' \in \mathcal{S}_Y \) with \( f(S) \subset S' \) and \( m - d(S') \leq n - d(S) \). A stratified map is stratum preserving if \( n = m \) and \( f^{-1} Y_{n-\ell} = X_{n-\ell} \), for any \( \ell \geq 0 \).
The previous conditions on stratum preserving map correspond to definitions of Friedman \cite{10} and Quinn, \cite{39}, \cite{38}. Observe that a continuous map is stratified if, and only if, for any stratum $S' \in S_Y$, $f^{-1}(S')$ is the empty set or there exists a family of strata, \{S_i \mid i \in I\} \subset S_X$, such that

$$f^{-1}(S') = \bigcup_{i \in I} S_i, \text{ with } m - d(S') \leq n - d(S_i).$$

In the particular case of two stratified spaces on the same space, $(X, (X_i)_{1 \leq i \leq n})$ and $(X, (X'_i)_{1 \leq i \leq n})$, the identity map, id: $(X, (X_i)_{1 \leq i \leq n}) \to (X, (X'_i)_{1 \leq i \leq n})$, is stratified if, and only if, the strata coming from the filtration $(X'_i)_{1 \leq i \leq n}$ are union of strata associated to the filtration $(X_i)_{1 \leq i \leq n}$.

**Theorem E.** A stratified map, $f: X \to Y$, induces a chain map $f_*: C^\pi_\ell(X) \to C^\pi_\ell(Y)$, defined by $\sigma \mapsto f \circ \sigma$. Moreover, for all $\ell \geq 0$, we have

$$\|f_*(\sigma)\|_\ell \leq \|\sigma\|_\ell.$$  

The next result is a direct consequence of Definitions A.7, A.10 and Theorem E.

**Corollary B.3.** A stratified map, $f: X \to Y$, induces a morphism between the complexes of intersection chains, $f_*: C^\pi(X) \to C^\pi(Y)$, for any loose perversity $\pi$.

In the next definition, $[0, 1]$ is trivially filtered of formal dimension 1 and $X \times [0, 1]$ endows the product filtration.

**Definition B.4.** Two stratified maps, $f_0, f_1: X \to Y$, are $s$-homotopic if there exists a stratified map, $f: X \times [0, 1] \to Y$, such that $f(x, 0) = f_0(x)$ and $f(x, 1) = f_1(x)$.

**Proposition B.5.** If $f_0, f_1: X \to Y$ are two $s$-homotopic stratified maps, they induced the same homomorphism in intersection homology,

$$H^\pi_\ell(f_0) = H^\pi_\ell(f_1): H^\pi_\ell(X) \to H^\pi_\ell(Y),$$

for any loose perversity $\pi$.

**Proof.** Let $f: X \times I \to Y$ be an $s$-homotopy from $f_0$ to $f_1$ and $\iota_k: X \to X \times I, x \mapsto (x, k)$, be the canonical inclusions for $k = 0, 1$. The proof follows directly from Corollary B.3 and Lemma A.15.

Before proving Theorem E, we need results on the behavior of strata.

**Proposition B.6.** If $(X, (X_i)_{0 \leq i \leq n})$ is a stratified space, the following properties are satisfied.

(i) The relation $S \preceq S'$, defined on the set of strata by $S \subset \overline{S'}$, is an order relation.

(ii) The formal dimension map, $d: S_X \to \mathbb{N}$, is strictly increasing.

(iii) The closure of a stratum is a union of lower strata, i.e., $\overline{S} = \cup_{S' \preceq S} S'$.

**Proof.** (i) Let $S$ and $S'$ be strata such that $S \subset \overline{S'}$ and $S' \subset \overline{S}$. Suppose $d(S) \leq d(S')$.

Since each stratum, $S$, is closed in $X \setminus X_{d(S)-1}$ and $X_{d(S)}$ is closed in $X$, the stratum $S$ is closed in $X \setminus X_{d(S)-1}$, which implies

$$S = \overline{S} \cap (X \setminus X_{d(S)-1}).$$
Then, the inclusion $S' \subset \mathcal{S}$ implies $S' \setminus X_{d(S)-1} \subset S$. The inclusion $S' \cap X_{d(S)-1} \subset S' \cap X_{d(S')-1}$ and the equality $S' \cap X_{d(S')-1} = \emptyset$, by definition of $d(S')$, imply $S' \subset S$ and the equality $S' = S$. This establishes the antisymmetry property; the reflexivity and transitivity are obvious.

(ii) Suppose $S \preceq S'$. As $X_{d(S')}$ is closed, we have $S \subset \mathcal{S}' \subset X_{d(S')}$ which implies $d(S) \leq d(S')$. Since $S \subset X \setminus X_{d(S)-1}$, we have

$$S = S \cap (X \setminus X_{d(S)-1}) \subset \mathcal{S}' \cap (X \setminus X_{d(S')-1}).$$

If $d(S) = d(S')$, we know that $S'$ is closed in $X \setminus X_{d(S)-1}$, which implies

$$S' = \mathcal{S}' \cap (X \setminus X_{d(S)-1}).$$

We get $S \subset S'$ and $S = S'$.

(iii) The inclusion $\cup_{S' \preceq S} S' \subset \mathcal{S}$ is obvious. Consider $x \in \mathcal{S}$. Since the strata form a partition, there exists a unique stratum $S'$ such that $x \in S'$. From $\mathcal{S}' \cap S' \neq \emptyset$ and from Definition B.1, we deduce $S' \subset \mathcal{S}$. We get $x \in S'$ and $S' \preceq S$, which proves the remaining part of the equality.

As strata give a partition, the stratum $S'$ of Definition B.2 is uniquely determined by $f$ and $S$.

**Proposition B.7.** If $f : X \to Y$ is a stratified map, the induced map, $S_f : (\mathcal{S}_X, \preceq) \to (\mathcal{S}_Y, \preceq)$, defined by $S_f(S) = S'$, is increasing.

**Proof.** Let $S_1 \preceq S_2$ in $\mathcal{S}_X$. Since $f$ is continuous and $S_1 \subset \overline{S_2}$, we have $f(S_1) \subset \overline{f(S_2)}$. This implies $S_1' \cap \overline{S_2} \neq \emptyset$, which gives $S_f(S_1) \preceq S_f(S_2)$. □

**Lemma B.8.** Let $(X, (X_i)_{0 \leq i \leq n})$ be a stratified space and let $\sigma : \Delta \to X$ be a filtered simplex. Then the following properties are satisfied.

(i) There is a stratum $S \in \mathcal{S}_X$ such that $\sigma(\Delta) \subset S$.

(ii) The family of strata that have a non-empty intersection with the image of $\sigma$ is totally ordered. If $S_1 \prec S_2 \prec \cdots \prec S_p$ is this family, then we have

$$\sigma^{-1} X_i = \begin{cases} \emptyset & \text{if } i < d(S_1), \\ \sigma^{-1}(S_1 \sqcup \cdots \sqcup S_k) & \text{if } d(S_k) \leq i < d(S_{k+1}), 1 \leq k < p, \\ \sigma^{-1}(S_1 \sqcup \cdots \sqcup S_p) & \text{if } i \geq d(S_p). \end{cases}$$

**Proof.** (i) Let $k$ be the integer such that $\sigma(\Delta) \subset X_k$ and $\sigma(\Delta) \not\subset X_{k-1}$. Thus there exists $t \in \Delta$ with $\sigma(t) \not\in X_{k-1}$ and we consider the stratum, $S$, characterized by $\sigma(t) \in S \subset X_k \setminus X_{k-1}$. By hypothesis, the pullback $\Delta' = \sigma^{-1}(X_k)$ is a face (perhaps empty) of $\Delta$. We have $\Delta' \neq \Delta$ because $t \not\in \Delta'$. From $\sigma(\Delta \setminus \Delta') \subset X_k \setminus X_{k-1}$ and $\sigma(t) \in \sigma(\Delta \setminus \Delta') \cap S$, we deduce $\sigma(\Delta \setminus \Delta') \subset S$ and

$$\sigma(\Delta) \subset \sigma(\Delta \setminus \Delta') \subset S.$$
(ii) Let \( i \in \{0, \ldots, n\} \). From the definition of filtered simplex, we know that \( \sigma^{-1}X_i \cap \sigma^{-1}X_{i-1} \) is connected. Moreover, this complement can be decomposed in a disjoint union of closed subsets of \( \sigma^{-1}X_i \cap \sigma^{-1}X_{i-1} \),

\[
\sigma^{-1}X_i \cap \sigma^{-1}X_{i-1} = \bigcup_{d(S)=i} \sigma^{-1}S.
\]

Suppose \( \sigma^{-1}X_i \cap \sigma^{-1}X_{i-1} \neq \emptyset \). As it is connected, it is the pullback of one stratum \( S_i \) of \( X_i \setminus X_{i-1} \). Therefore, there exists a family of integers, \( 0 \leq d_1 < d_2 < \cdots < d_p \leq n \), such that

\[
\sigma^{-1}X_i \cap \sigma^{-1}X_{i-1} = \begin{cases} 
\emptyset & \text{if } i \notin \{d_1, \ldots, d_p\}, \\
\sigma^{-1}(S_i) & \text{if } i = d_k \text{ for some } k \in \{1, \ldots, p\}.
\end{cases}
\]

Setting \( S_{d_k} = S_k \), this gives the equalities \( \text{(37)} \).

Let \( k \in \{2, \ldots, p\} \), we are reduced to prove \( S_{k-1} \prec S_k \). Recall the equality \( \sigma^{-1}X_{d_k} = \sigma^{-1}(S_{d_k} \cup \cdots \cup S_k) \). We have proved the equalities \( \sigma^{-1}X_{d_k} \setminus \sigma^{-1}X_{d_{k-2}} = \sigma^{-1}(S_{k-1} \cup S_k) \) and

\[
\sigma^{-1}S_k = \bigcup_{S \preceq S_k} \sigma^{-1}S,
\]

see (iii) of Proposition \( \text{B.6} \). In particular, \( \sigma^{-1}(S_k \cap (X_{d_{k-1}} \setminus X_{d_{k-2}})) = \sigma^{-1}S_{k-1} \cup \sigma^{-1}S_k \), which imply \( 0 \leq d_1 < d_2 < \cdots < d_p \leq n \), such that

\[
\sigma^{-1}X_i \cap \sigma^{-1}X_{i-1} = \begin{cases} 
\emptyset & \text{if } i \notin \{d_1, \ldots, d_p\}, \\
\sigma^{-1}(S_i) & \text{if } i = d_k \text{ for some } k \in \{1, \ldots, p\}.
\end{cases}
\]

Setting \( S_{d_k} = S_k \), this gives the equalities \( \text{(37)} \).

The subset \( \sigma^{-1}S_{k-1} \) (resp. \( \sigma^{-1}(S_k \cap (X_{d_k} \setminus X_{d_{k-2}})) \)) is closed in \( \sigma^{-1}X_{d_{k-1}} \setminus \sigma^{-1}X_{d_{k-2}} \) (resp. \( \sigma^{-1}X_{d_k} \setminus \sigma^{-1}X_{d_{k-2}} \)), which is closed in the connected space \( \sigma^{-1}X_{d_k} \setminus \sigma^{-1}X_{d_{k-2}} \).

From the formula above and the connectivity of \( \sigma^{-1}X_{d_k} \setminus \sigma^{-1}X_{d_{k-2}} \), we deduce that

\[
\sigma^{-1}(S_{k-1}) \cap \sigma^{-1}(S_k \cap (X_{d_k} \setminus X_{d_{k-2}})) \neq \emptyset \text{ which implies } S_{k-1} \cap S_k \neq \emptyset \text{ and } S_{k-1} \preceq S_k,
\]

by definition. Finally, as \( d(S_{k-1}) = d_{k-1} \neq d_k = d(S_k) \), we have \( S_{k-1} \prec S_k \). \( \square \)

**Proof of Theorem \( \text{E} \)**. Let \( f : (X, (X_i)_{0 \leq i \leq n}) \to (Y, (Y_i)_{0 \leq i \leq m}) \) be a stratified map and \( \sigma : \Delta \to X \) be a filtered simplex.

If \( (f \circ \sigma)^{-1}(Y_{m-\ell}) = \emptyset \) then \( \|f \circ \sigma\|_{\ell} = -\infty \).

If \( \sigma^{-1}X_{n-\ell} = \emptyset \), then from \( (f \circ \sigma)^{-1}(Y_{m-\ell}) \subset \sigma^{-1}X_{n-\ell} = \emptyset \), we deduce \( \|f \circ \sigma\|_{\ell} = \|\sigma\|_{\ell} = -\infty \).

We may thus suppose \( (f \circ \sigma)^{-1}(Y_{m-\ell}) \neq \emptyset \) and \( \sigma^{-1}X_{n-\ell} \neq \emptyset \), which imply \( 0 \leq \ell \leq m \).

We have to prove that \( (f \circ \sigma)^{-1}(Y_{m-\ell}) \) is a face of \( \Delta \) and that

\[
\dim(f \circ \sigma)^{-1}(Y_{m-\ell}) \leq \dim \sigma^{-1}(X_{n-\ell}).
\]

Recall that the family of strata meeting the image of \( \sigma \) can be ordered as

\[
S_1 \prec S_2 \prec \cdots \prec S_p,
\]

which implies, with Proposition \( \text{B.7} \)

\[
S^f_1 \preceq S^f_2 \preceq \cdots \preceq S^f_p.
\]

Let \( T \) be a stratum of \( Y \) such that \( T \cap (f \circ \sigma)(\Delta) \neq \emptyset \), i.e., \( f^{-1}(T) \cap \sigma(\Delta) \neq \emptyset \). As the map \( f \) is stratified, there exists \( k \in \{1, \ldots, p\} \), such that \( T = S^f_k \) and

\[
m - d(T) \leq n - d(S_k).
\]
We denote by \( 0 \leq k_0 < \cdots < k_q = p \) the integers such that
\[
\begin{align*}
T_1 &= S^f_1 = \cdots = S^f_{k_1}, \text{ with } S^f_{k_1} \neq S^f_{k_1+1}, \\
T_2 &= S^f_{k_1+1} = \cdots = S^f_{k_2}, \text{ with } S^f_{k_2} \neq S^f_{k_2+1}, \\
& \hphantom{= S^f_{k_1+1} = \cdots = S^f_{k_2}, \text{ with }} \cdots \\
T_q &= S^f_{k_{q-1}+1} = \cdots = S^f_{k_q} = S^f_p.
\end{align*}
\]
From that, we deduce:

1. \( \{ T \in S_Y \mid T \cap (f \circ \sigma)(\Delta) \neq \emptyset \} = \{ T_1 < T_2 < \cdots < T_q \} \),
2. \( m - d(T_j) \leq n - d(S(k_j)) \),
3. \( f^{-1}(T_j) = \bigcup_{k=k_{j-1}+1}^{k_j} S_k \text{, for any } j \in \{1, \ldots, q\} \).

From (ii) of Proposition B.6 we deduce:
\[ 0 \leq d(T_1) < d(T_2) < \cdots < d(T_q) \leq m. \]

With the convention \( d(T_{q+1}) = m \) and \( d(T_0) = 0 \), we denote by \( a \in \{0, \ldots, q\} \) the integer such that
\[ d(T_a) \leq m - \ell < d(T_{a+1}). \]

As \( (f \circ \sigma)^{-1} Y_{m-\ell} \neq \emptyset \), the integer \( a \) is different of 0 and we have obtained
\[
(f \circ \sigma)^{-1}(Y_{m-\ell}) = \bigcup_{j=0}^{m-\ell} \sigma^{-1} f^{-1}(Y_j \setminus Y_{j-1}) = \bigcup_{i=0}^{a} \sigma^{-1} f^{-1}(T_i) = \bigcup_{k=1}^{k_a} \sigma^{-1}(S_k) = \sigma^{-1}(X_{d(S_{k_a})}),
\]
which is a face of the simplex \( \Delta \). The simplex \( f \circ \sigma \) is thus filtered. From Property (2) above, we deduce
\[ d(S_{k_a}) \leq n - m + d(T_a) \leq n - m + (m - \ell) \leq n - \ell, \]
which implies \( \sigma^{-1}(X_{d(S_{k_a})}) \subseteq \sigma^{-1}(X_{n-\ell}) \) and the expected inequality between the dimensions,
\[ \dim(f \circ \sigma)^{-1}(Y_{m-\ell}) = \dim \sigma^{-1}(X_{d(S_{k_a})}) \leq \dim \sigma^{-1}(X_{n-\ell}). \]

\[ \square \]

**Corollary B.9.** Let \( f : X \to Y \) be a stratified map and \( \sigma : \Delta \to X \) be a filtered simplex of solid \( \sigma \)-decomposition \( \Delta = \Delta_1 \cdots \Delta_p \). Then, the solid \( (f \circ \sigma) \)-decomposition is of the shape \( \Delta = \Delta'_1 \cdots \Delta'_q \), where the \( \Delta'_k \)'s are obtained by amalgamations of the type \( \Delta^i \star \Delta^j \mapsto \emptyset \star \Delta^i \cdots \Delta^j \). In the case of a stratum preserving stratified map, the \( \sigma \)- and \( f \circ \sigma \)-decompositions of \( \Delta \) are identical, as well as their solid decompositions.

**Proof.** In the previous proof, with the same notation, we have shown that the solid \( (f \circ \sigma) \)-decomposition, \( \Delta = \Delta'_1 \cdots \Delta'_q \), verifies \( \Delta'_a = \Delta_{k_a-1+1} \cdots \Delta_{k_a} \), for any \( a \in \{1, \ldots, q\} \) and
\[ (f \circ \sigma)^{-1}(T_1 \sqcup \cdots \sqcup T_a) = \Delta'_1 \cdots \Delta'_a, \text{ for any } a \in \{1, \ldots, q\}. \]
This establishes the first part of the statement.

In the case of a stratum preserving stratified map, with property (ii) of Proposition B.6, the relation \( S_1 \prec S_2 \prec \cdots \prec S_p \) and the stratum preserving condition imply \( S_f^1 \prec S_f^2 \prec \cdots \prec S_f^p \). Therefore the equation (3) above reduces to \( f^{-1}(T_j) = S_{k_j} \) and we have \((f \circ \sigma)^{-1}(T_j) = \sigma^{-1}(S_{k_j})\), as announced. \( \square \)

**Appendix C. CS Sets and pseudomanifolds**

Independence of the stratification is one important feature of the theory of Goresky and MacPherson. Proved in [25] for pseudomanifolds by using sheaves, this result is also established for CS sets, by King, with a singular point of view. Here, we prove the existence of a quasi-isomorphism between King’s intersection complex and the complex \( C^p_*(X) \) introduced in Appendix A, when \( X \) is a pseudomanifold. From [32, Theorem 9], we deduce the independence of the stratification for our intersection homology.

**Definition C.1.** A CS set of dimension \( n \) is a filtered space,

\[
\emptyset \subset X_0 \subseteq X_1 \subseteq \cdots \subseteq X_{n-2} \subseteq X_{n-1} \subseteq X_n = X,
\]

such that, for all \( i \), \( X_i \setminus X_{i-1} \) is an \( i \)-dimensional metrizable topological manifold or the empty set. Moreover, for each point \( x \in X_i \setminus X_{i-1}, i \neq n \), there exist

- (a) an open neighborhood, \( V \), of \( x \) in \( X \), endowed with the induced filtration,
- (b) an open neighborhood, \( U \), of \( x \) in \( X_i \setminus X_{i-1} \),
- (c) a compact, filtered space, \( L \), of formal dimension \( n-i-1 \), whose cone, \( \hat{c}L \) is endowed with the conic filtration,
- (d) a homeomorphism, \( \varphi: U \times \hat{c}L \to V \), such that
  - (i) \( \varphi(u, \vartheta) = u \), for any \( u \in U \), with \( \vartheta \) the cone point,
  - (ii) \( \varphi(U \times L_j \times ]0,1[) = V \cap X_{i+j+1} \), for any \( j \in \{0, \ldots, n-i-1\} \).

The couple \((V, \varphi)\) is called a conic chart of \( x \) and the filtered space, \( L \), the link of \( x \).

Observe that property (d) (i) implies \( V \cap X_{i-1} = \emptyset \) and both properties (d) (i) and (ii) imply \( V \cap X_i = \varphi(U \times \{\vartheta\}) = U \). As \( X_i \setminus X_{i-1} \) is locally connected, we may choose \( U \) connected. Observe also, from Definition A.1 that the link \( L \) is not empty.

The CS sets were introduced by Siebenmann ([45]), without a restriction of finite formal dimension on the filtration. They include metrizable topological manifolds, differentiable stratified sets \( X \) in the sense of Thom,...

Pseudomanifolds are particular cases of CS sets which allow proofs by induction on the dimension.

**Definition C.2.** An \( n \)-dimensional topological pseudomanifold (or pseudomanifold in short) is a CS set of dimension \( n \), in which \( X_{n-2} = X_{n-1} \) and all the links, \( L \), are \((n-i-1)\)-dimensional topological pseudomanifolds.

Observe that this definition makes sense with an induction on the dimension, starting from pseudomanifolds of dimension 0 which are discrete topological spaces, by definition. Also, one can prove that, in a pseudomanifold, \( X \), the subspace \( X_n \setminus X_{n-2} \) is dense.

**Theorem F.** A CS set is a stratified space.
Proof. Let $S$ and $S'$ be strata such that $S \cap S' \neq \emptyset$, we have to prove $S \subset S'$. Set $i = d(S)$ and $j = d(S')$. The space $X_i \setminus X_{i-1}$ being locally connected, we may cover $S$ by connected open sets $U$ of $S$ satisfying the local condition of Definition C.1 for some chart $(V, \varphi)$. Denote by $U$ this cover of $S$ and let $U \in U$. We first prove the

**Claim:** if $U \cap S' \neq \emptyset$, then $U \subset S'$.

Suppose that $U \cap S' \neq \emptyset$ and recall $U = V \cap S = \varphi(U \times \{\vartheta\})$ and $V \cap X_{i-1} = \emptyset$. The property $U \cap S' \neq \emptyset$ implies $V \cap S' \neq \emptyset$. As $V$ is open, we deduce $V \cap S' \neq \emptyset$ and $j \geq i$, because $V \cap X_{i-1} = \emptyset$.

- If $i = j$, we have
  \[
  \emptyset \neq S' \cap V \subset X_i \cap V = U = S \cap V,
  \]
  which implies $S \cap S' \neq \emptyset$ and $S = S'$. This gives $U \subset S = S' \subset S'$.

- If $i < j$, we have
  \[
  \emptyset \neq S' \cap V \subset (X_j \setminus X_{j-1}) \cap V = \varphi(U \times (L_{j-1} \setminus L_{j-2}) \times ]0, 1[)
  = \varphi(\bigcup_{T} U \times T \times ]0, 1[),
  \]
  where $T$ runs over the connected components of $L_{j-1} \setminus L_{j-2}$. We may write
  \[
  (X_j \setminus X_{j-1}) \cap V = \bigcup_{S''} \bigcup_{C''} C'',
  \]
  where $S''$ runs over the strata of $X_j \setminus X_{j-1}$ and $C''$ over the connected components of $S'' \cap V$. By local connectivity of $X_j \setminus X_{j-1}$, the $C''$ are open in $(X_j \setminus X_{j-1}) \cap V$. As $S' \cap V \neq \emptyset$, there exists a non empty connected component, $C'$, of $S' \cap V$ and a non empty connected component, $T$, of $L_{j-1} \setminus L_{j-2}$ with
  \[
  C' = \varphi(U \times T \times ]0, 1[).
  \]
  Observe that $U \times \{\vartheta\}$ is included in the closure of $U \times T \times ]0, 1[$ in $U \times \partial L$. This implies that $U$ is included in the closure of $C'$ in $V$ and gives $U \subset S'$.

As the claim is proved, we know that, for any $U \in U$, we have $U \subset S'$ or $U \cap S' = \emptyset$. The condition $S \cap S' \neq \emptyset$ and the connectivity of $S$ imply $U \subset S'$, and therefore $S \subset S'$.

In [32], King defines a subcomplex of the singular chain complex of a filtered space as follows.

**Definition C.3.** Let $(X, (X_i)_{0 \leq i \leq n})$ be a filtered space and let $\mathfrak{p}$ be a loose perversity. A singular simplex, $\sigma : \Delta \to X$, is King $\mathfrak{p}$-admissible if $\sigma^{-1}(X_{n-j} \setminus X_{n-j-1})$ is contained in the $(\dim \Delta - j + \mathfrak{p}(j))$-skeleton of $\Delta$, for all $j \in \{0, 1, \ldots, n\}$. A chain $c$ is King $\mathfrak{p}$-admissible if there exist $\mathfrak{p}$-admissible simplices, $\sigma_j$, so that $c = \sum \lambda_j \sigma_j$, $\lambda_j \in R$.

Let $K_{\mathfrak{p}}^T(X)$ be the chain complex which consists of the singular chains $c$, such that $c$ and its boundary $\partial c$ are King $\mathfrak{p}$-admissibles.

**Proposition C.4.** Let $X$ be a pseudomanifold and let $\mathfrak{p}$ be a perversity. Then the canonical inclusion $C_\mathfrak{p}^T(X) \to K_{\mathfrak{p}}^T(X)$ induces an isomorphism in homology.
Proof. Let $\sigma : \Delta \to X$ be a $p$-admissible simplex. As the set $\sigma^{-1}X_{n-i}$ is a face of $\Delta$ of dimension less than, or equal to, $(\dim \Delta - i + p(i))$, its subset $\sigma^{-1}(X_{n-i}\setminus X_{n-i-1})$ is also included in this skeleton. This shows that any $\sigma$ in $C^p_*(X)$ belongs also to $K^p_*(X)$ and we have a canonical inclusion $C^p_*(X) \subset K^p_*(X)$. We have now only to check the five conditions of [32, Theorem 10]. They are direct consequences of Proposition A.14. □

Theorem 9 of [32] implies immediately:

Corollary C.5. Let $X$ be a pseudomanifold and let $p$ be a perversity. Then the homology of $C^p_*(X)$ is independent of the stratification of $X$.

This result is not true if $p$ is not a perversity, as shows an example of King, [32, Page 155]. Also, we mention that, in [19], G. Friedman establishes this result in the more general setting of recursive CS sets.

Remark C.6. If $\sigma : \Delta \to X$ is a simplex of a filtered space $X$ of formal dimension $n$, we consider the two following conditions,

(a) $\sigma^{-1}X_{n-j}$ is included in the $(\dim \Delta - j + p(j))$-skeleton of $\Delta$, for all $j$,
(b) $\sigma^{-1}(X_{n-j}\setminus X_{n-j-1})$ is included in the $(\dim \Delta - j + p(j))$-skeleton of $\Delta$, for all $j$.

In the proof of Proposition C.4, we used that condition (a) implies condition (b). As already quoted by King ([32]), in the case of a perversity $\overline{p}$, condition (b) implies also condition (a). For proving that, we observe that $\sigma^{-1}X_{n-j}$ is a union of $\sigma^{-1}(X_{n-j-k}\setminus X_{n-j-k-1})$ and that

$$\dim \Delta - (j + k) + p(j + k) \leq \dim \Delta - (j + k) + p(j) + k \leq \dim \Delta - j + p(j),$$

for any $k, k \geq 0$.

References

1. Markus Banagl, Topological invariants of stratified spaces, Springer Monographs in Mathematics, Springer, Berlin, 2007. MR 2286904 (2007j:55007)

2. ———, Intersection spaces, spatial homology truncation, and string theory, Lecture Notes in Mathematics, vol. 1997, Springer-Verlag, Berlin, 2010. MR 2662593

3. A. A. BeUılinson, J. Bernstein, and P. Deligne, Faisceaux pervers, Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque, vol. 100, Soc. Math. France, Paris, 1982, pp. 5–171. MR 751966 (86g:32015)

4. A. Borel and et al., Intersection cohomology, Modern Birkhäuser Classics, Birkhäuser Boston Inc., Boston, MA, 2008, Notes on the seminar held at the University of Bern, Bern, 1983, Reprint of the 1994 edition. MR 2401086 (2009k:14046)

5. A. K. Bousfield and V. K. A. M. Gugenheim, On PL de Rham theory and rational homotopy type, Mem. Amer. Math. Soc. 8 (1976), no. 179, ix+94. MR 0425956 (54 #13906)

6. J.-P. Brasselet and G. Gonzalez-Sprinberg, Espaces de Thom et contre-exemples de J.-L. Verdier et M. Goresky, Bol. Soc. Brasil. Mat. 17 (1986), no. 2, 23–50. MR 901594 (89h:32031)

7. Jean-Paul Brasselet, Gilbert Hector, and Martin Saralegi, Théorème de de Rham pour les variétés stratifiées, Ann. Global Anal. Geom. 9 (1991), no. 3, 211–243. MR 1143404 (93g:55009)

8. Jean-Luc Brylinski, Equivariant intersection cohomology, Kazhdan-Lusztig theory and related topics (Chicago, IL, 1989), Contemp. Math., vol. 139, Amer. Math. Soc., Providence, RI, 1992, pp. 5–32. MR 1197827 (94c:55010)

9. Bohumil Cenkl, Gilbert Hector, and Martin Saralegi, Cohomologie d’intersection modérée. Un théorème de de Rham, Pacific J. Math. 169 (1995), no. 2, 235–289. MR 1346255 (96k:55007)
10. David Chataur, Martintxo Saralegi-Aranguren, and Daniel Tanré, Steenrod squares on Intersection cohomology and a conjecture of M. Goresky and W. Pardon, ArXiv e-prints (2013).

12. J. Daniel Christensen and Mark Hovey, Quillen model structures for relative homological algebra, Math. Proc. Cambridge Philos. Soc. 133 (2002), no. 2, 261–293. MR 1912401 (2003f:18012)

14. Yves Félix, Stephen Halperin, and Jean-Claude Thomas, Rational homotopy theory, Graduate Texts in Mathematics, vol. 205, Springer-Verlag, New York, 2001. MR 1802847 (2002d:55014)

17. Yves Félix and Daniel Tanré, Topologie algébrique, Sciences SUP. Dunod, Paris, 2010.

19. Yves Félix and Daniel Tanré, Stratified fibrations and the intersection homology of the regular neighborhoods of bottom strata, Topology Appl. 134 (2003), no. 2, 69–109. MR 2009092 (2004g:55008)

20. Greg Friedman and James E. McClure, Cup and cap products in intersection (co)homology, Adv. Math. 240 (2013), 383–426. MR 3046315

21. Rudolf Fritsch and Marek Golasiński, Topological, simplicial and categorical joins, Arch. Math. (Basel) 82 (2004), no. 5, 468–480. MR 2061452 (2005a:55012)

22. Antonio Gómez-Tato, Modèles minimaux résolubles, J. Pure Appl. Algebra 85 (1993), no. 1, 43–56. MR 1207067 (94c:55017)

23. Antonio Gómez-Tato, Stephen Halperin, and Daniel Tanré, Rational homotopy theory for non-simply connected spaces, Trans. Amer. Math. Soc. 352 (2000), no. 4, 1493–1525. MR 1653355 (2000i:55035)

28. Werner Greub, Stephen Halperin, and Ray Vanstone, Connections, curvature, and cohomology. Vol. I: De Rham cohomology of manifolds and vector bundles, Academic Press, New York, 1972. MR 0336650 (49 #1423)

29. S. Halperin, Lectures on minimal models, Mém. Soc. Math. France (N.S.) (1983), no. 9-10, 261. MR 736299 (85j:55009)

30. Allen Hatcher, Algebraic topology, Cambridge University Press, Cambridge, 2002. MR 1867354 (2002k:55001)

31. Mark Hovey, Intersection homological algebra, New topological contexts for Galois theory and algebraic geometry (BIRS 2008), Geom. Topol. Monogr., vol. 16, Geom. Topol. Publ., Coventry, 2009, pp. 133–150. MR 2544388 (2010g:55009)

32. Steven L. Kleiman, The development of intersection homology theory, Pure Appl. Math. Q. 3 (2007), no. 1, Special Issue: In honor of Robert D. MacPherson, Part 3, 225–282. MR 2330160 (2008e:55006)

35. John W. Morgan, The algebraic topology of smooth algebraic varieties, Inst. Hautes Études Sci. Publ. Math. (1978), no. 48, 137–204. MR 516917 (80e:55020)

36. Daniel G. Quillen, Homotopical algebra, Lecture Notes in Mathematics, No. 43, Springer-Verlag, Berlin, 1967. MR 0223432 (36 #6480)
37. _______, *Rational homotopy theory*, Ann. of Math. (2) **90** (1969), 205–295. MR 0258031 (41 #2678)
38. Frank Quinn, *Intrinsic skeleta and intersection homology of weakly stratified sets*, Geometry and topology (Athens, Ga., 1985), Lecture Notes in Pure and Appl. Math., vol. 105, Dekker, New York, 1987, pp. 233–249. MR 873296 (88g:57022)
39. _______, *Homotopically stratified sets*, J. Amer. Math. Soc. **1** (1988), no. 2, 441–499. MR 928266 (89g:57050)
40. Michele Rossi, *Geometric transitions*, J. Geom. Phys. **56** (2006), no. 9, 1940–1983. MR 2240431 (2007e:14061)
41. C. P. Rourke and B. J. Sanderson, *Δ*-sets. I. Homotopy theory*, Quart. J. Math. Oxford Ser. (2) **22** (1971), 321–338. MR 0300281 (45 #9327)
42. Ghada Salem, *Homologie d’intersection géométrique pour les singularités coniques isolées*, Ph.D. thesis, Toulouse, 2011.
43. Martin Saralegi, *Homological properties of stratified spaces*, Illinois J. Math. **38** (1994), no. 1, 47–70. MR 1245833 (95a:55011)
44. Martintxo Saralegi-Aranguren, *de Rham intersection cohomology for general perversities*, Illinois J. Math. **49** (2005), no. 3, 737–758 (electronic). MR 2210257 (2006k:55013)
45. L. C. Siebenmann, *Deformation of homeomorphisms on stratified sets. I, II*, Comment. Math. Helv. **47** (1972), 123–136; ibid. 47 (1972), 137–163. MR 0319207 (47 #7752)
46. Edwin H. Spanier, *Algebraic topology*, McGraw-Hill Book Co., New York, 1966. MR 0210112 (35 #1007)
47. Paul Stienne, *Modèles minimaux et cohomologie d’intersection*, Lille, 2002.
48. Dennis Sullivan, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. (1977), no. 47, 269–331 (1978). MR 0646078 (58 #31119)
49. Daniel Tanré, *Homotopie rationnelle: modèles de Chen, Quillen, Sullivan*, Lecture Notes in Mathematics, vol. 1025, Springer-Verlag, Berlin, 1983. MR 764769 (86b:55010)
50. Andrei Verona, *Stratified mappings—structure and triangulability*, Lecture Notes in Mathematics, vol. 1102, Springer-Verlag, Berlin, 1984. MR 771120 (86k:58010)

DÉPARTEMENT DE MATHEMATIQUES, UMR 8524 ET FÉDÉRATION CNRS NORD-PAS-DE-CALAIS FR 2956, UNIVERSITÉ DE LILLE 1, 59655 VILLENEUVE D’ASCQ CEDEX, FRANCE
E-mail address: David.Chataur@math.univ-lille1.fr

LAbORATOIRE DE MathéMATIQUES DE Lens, EA 2462 ET FÉDÉRATION CNRS NORD-PAS-DE-CALAIS FR 2956, Université d’Artois, SP18, rue jean Souvraz, 62307 Lens Cedex, France
E-mail address: saralegi@euler.univ-artois.fr

DÉPARTEMENT DE MATHEMATIQUES, UMR 8524 ET FÉDÉRATION CNRS NORD-PAS-DE-CALAIS FR 2956, UNIVERSITÉ DE LILLE 1, 59655 VILLENEUVE D’ASCQ CEDEX, FRANCE
E-mail address: Daniel.Tanre@univ-lille1.fr
URL: http://www.tanre.org/Pro