Euclidean Travelling Salesman Problem with Location-Dependent and Power-Weighted Edges

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Abstract
Consider \( n \) nodes \( \{X_i\}_{1 \leq i \leq n} \) independently distributed in the unit square \( S \), each according to a density \( f \), and let \( K_n \) be the complete graph formed by joining each pair of nodes by a straight line segment. For every edge \( e \) in \( K_n \), we associate a weight \( w(e) \) that may depend on the individual locations of the endvertices of \( e \) and is not necessarily a power of the Euclidean length of \( e \). Denoting \( TSP_n \) to be the minimum weight of a spanning cycle of \( K_n \) corresponding to the travelling salesman problem (TSP) and assuming an equivalence condition on the weight function \( w(\cdot) \), we prove that \( TSP_n \) appropriately scaled and centred converges to zero almost surely and in mean as \( n \to \infty \). We also obtain upper and lower bound deviation estimates for \( TSP_n \).

Keywords Travelling salesman problem · Location-dependent edge weights · Deviation estimates

Mathematics Subject Classification (2020) 60D05

1 Introduction
The travelling salesman problem (TSP) is the study of finding the minimum weight cycle containing all the nodes of a graph where each edge is assigned a certain weight. Often, the location of the nodes by itself is random and the edge weights are taken to be the Euclidean distance between the nodes. Thus, the edge weight is a metric, and for such models, there is extensive literature dealing with various properties of the TSP including the variance, deviation and almost sure convergence. Beardwood et al. [1] use subadditive techniques to determine a.s. convergence of the TSP length appropriately scaled to a constant \( \beta_{TSP} \) and recently [7] has obtained improved bounds...
for $\beta_{\text{TSP}}$. We refer to the books by [4,5] and [9] and papers by [6,8] and references therein for various other aspects of TSP like the variance and complete convergence.

In this paper, we assume that the edge weight function is not necessarily a metric and in fact might depend on the location of the endvertices of the edge. This arises, for example, in broadcasting problems of wireless ad hoc networks where each node is constrained to broadcast any received message at most once. The cost of transmitting a message from a node might depend on the geographical conditions surrounding and the node and the goal is therefore to minimize the total cost incurred in sending packets to all nodes of the network.

In the rest of this section, we briefly describe the model under consideration and state our result Theorem 1 regarding the deviation estimates for the TSP length.

### 1.1 Model Description

Let $f$ be any density on the unit square $S$ satisfying such that

$$
\epsilon_1 \leq \inf_{x \in S} f(x) \leq \sup_{x \in S} f(x) \leq \epsilon_2 \tag{1.1}
$$

for some constants $0 < \epsilon_1 \leq \epsilon_2 < \infty$. Throughout, all constants are independent of $n$.

Let $\{X_i\}_{i \geq 1}$ be independently and identically distributed (i.i.d.) with the distribution $f(\cdot)$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For $n \geq 1$, let $K_n = K(X_1, \ldots, X_n)$ be the complete graph whose edges are obtained by connecting each pair of nodes $X_i$ and $X_j$ by the straight line segment $(X_i, X_j)$ with $X_i$ and $X_j$ as endvertices.

A path $P = (Y_1, \ldots, Y_t)$ is a subgraph of $K_n$ with vertex set $\{Y_j\}_{1 \leq j \leq t}$ and edge set $\{(Y_j, Y_{j+1})\}_{1 \leq j \leq t-1}$. The nodes $Y_1$ and $Y_t$ are said to be connected by edges of the path $P$.

Let $Y_1, \ldots, Y_t \subset \{X_k\}_{1 \leq k \leq n}$ be $t$ distinct nodes. The subgraph $C = (Y_1, Y_2, \ldots, Y_t, Y_1)$ with vertex set $\{Y_j\}_{1 \leq j \leq t}$ and edge set $\{(Y_j, Y_{j+1})\}_{1 \leq j \leq t-1} \cup \{(Y_t, Y_1)\}$ is said to be a cycle. If $C$ contains all the nodes $\{X_i\}_{1 \leq i \leq n}$, then $C$ is said to be a spanning cycle of $K_n$.

In what follows, we assign weights to edges of the graph $K_n$ and study minimum weight spanning cycles.

### 1.2 Travelling Salesman Problem

For points $x, y \in S$, we let $d(x, y)$ denote the Euclidean distance between $x$ and $y$ and let $h : S \times S \to (0, \infty)$ be a deterministic measurable function satisfying

$$
c_1 d(x, y) \leq h(x, y) = h(y, x) \leq c_2 d(x, y) \tag{1.2}
$$

for some positive constants $c_1, c_2$. For $\alpha > 0$ a constant and for $1 \leq i < j \leq n$ we let $h^{\alpha}(X_i, X_j) = h^{\alpha}(e)$ denote the weight of the edge $e = (X_i, X_j)$, with exponent $\alpha$. We remark here that unlike the Euclidean distance function $d$, the edge
weight function $h$ is not necessarily a metric. Also throughout, the quantity $\alpha$ appears in the superscript of any term only as an exponent.

The weight of a cycle $C$ in the graph $K_n$ is defined to be the sum of the weights of the edges in $C$; i.e.

$$W(C) := \sum_{e \in C} h^{\alpha}(e). \quad (1.3)$$

Let $C_n$ be a spanning cycle of the graph $K_n$ satisfying

$$\text{TSP}_n = W(C_n) := \min_C W(C), \quad (1.4)$$

where the minimum is taken over all spanning cycles $C$. We refer to $C_n$ as the travelling salesman problem (TSP) cycle with corresponding weight $\text{TSP}_n$. If there is more than one choice for $C_n$, we choose one according to a deterministic rule.

Let $\epsilon_1, \epsilon_2$ be as in (1.1) and set $\delta = \delta(\alpha) = \epsilon_1$ if the edge weight exponent is $\alpha \leq 1$ and $\delta = \epsilon_2$ if $\alpha > 1$. Recalling that $c_1$ and $c_2$ are the bounds for the edge weight function $h$ as in (1.2), we define for $A > 0$ the terms $C_1(A) = C_1(A, \epsilon_1, \epsilon_2, \alpha) \text{ and } C_2(A) = C_2(A, \epsilon_1, \epsilon_2, \alpha)$ as

$$C_1(A) := \frac{(c_1 A)^{\alpha}}{A^2} (1 - e^{-\epsilon_1 A^2}) e^{-8\epsilon_2 A^2} \text{ and }$$

$$C_2(A) := (2c_2 A)^{\alpha} \left( 1 + \frac{\mathbb{E} T^\alpha}{A^2} \right), \quad (1.5)$$

where $T$ is a geometric random variable with success parameter $p = 1 - e^{-\delta A^2}$, independent of the node locations $\{X_i\}$; i.e. $\mathbb{P}(T = k) = (1 - p)^{k-1} p$ for all integers $k \geq 1$. We have the following result.

**Theorem 1**  Let $0 < \alpha < 2$ be the edge weight exponent. For every $A > 0$ and every integer $k \geq 1$ and all $n \geq n_0(A, k, \epsilon_1, \epsilon_2, \alpha, c_1, c_2)$ large,

$$\mathbb{P} \left( \text{TSP}_n \geq C_1(A)n^{1-\frac{\alpha}{2}} \left( 1 - \frac{4\sqrt{A}}{n^{1/4}} \right) \right) \geq 1 - e^{-n^{1/3}}, \quad (1.6)$$

$$\mathbb{P} \left( \text{TSP}_n \leq C_2(A)n^{1-\frac{\alpha}{2}} \left( 1 + \frac{2}{n^{1/16}} \right) \right) \geq 1 - \frac{1}{n^{2k}}, \quad (1.7)$$

and so

$$C_1^k(A) \left( 1 - \frac{37k\sqrt{A}}{n^{1/4}} \right) \leq \mathbb{E} \left( \frac{\text{TSP}_n^k}{n^{k(1-\frac{\alpha}{2})}} \right) \leq C_2^k(A) \left( 1 + \frac{3k}{n^{1/16}} \right).$$
For the lower deviation estimates, we determine the probability of favourable configurations that ensure a large enough number of relatively long edges in the TSP. To prove the upper deviation estimate, we tile the unit square into small subsquares and join nodes within these subsquares to form an overall spanning cycle. We then estimate the length of this cycle via stochastic domination by homogenous processes (see Sect. 2).

1.3 Remarks on Theorem 1

From Theorem 1, we see that the weights of the TSP in the location-dependent case, is of the same order $n^{1-\frac{\alpha}{2}}$ as in the location independent case (see Steele 1988). Strictly speaking, in the proof of Theorem 1 below, we show that the bounds in Theorem 1 hold for some $A_n$ lying between $A$ and $A + \frac{1}{\log n}$. In any case, by standard truncation arguments [3] we have that $\lim_n C_i(A_n) = C_i(A)$ for $i = 1, 2$ and so using (1.8), we get that the normalized TSP weight $\frac{\mathbb{E} T_{SP}}{n^{1-\frac{\alpha}{2}}}$ in fact satisfies

$$c_1^\alpha \cdot \beta_{low}(\alpha) \leq \lim inf_n \frac{\mathbb{E} T_{SP}}{n^{1-\frac{\alpha}{2}}} \leq \lim sup_n \frac{\mathbb{E} T_{SP}}{n^{1-\frac{\alpha}{2}}} \leq c_2^\alpha \cdot \beta_{up}(\alpha),$$

where

$$\beta_{low}(\alpha) = \beta_{low}(\alpha, \epsilon_1, \epsilon_2) := \sup_{A > 0} \frac{A^\alpha}{A^2} (1 - e^{-\epsilon_1 A^2}) e^{-8 \epsilon_2 A^2},$$

$$\beta_{up}(\alpha) = \beta_{up}(\alpha, \epsilon_1, \epsilon_2) := \inf_{A > 0} (2A)^\alpha \left(1 + \frac{\mathbb{E} T^\alpha}{A^2}\right),$$

and $T$ is a geometric random variable as defined in (1.5). In particular, for the case of homogenous distribution $\epsilon_1 = \epsilon_2 = 1$ we get

$$\beta_{down}(\alpha) = \sup_{A > 0} A^{\alpha-2} (1 - e^{-A^2}) e^{-8A^2} > 0$$

and

$$\beta_{up}(\alpha) = \inf_{A > 0} (2A)^\alpha \left(1 + \frac{\mathbb{E} T^\alpha}{A^2}\right) < \infty.$$ 

For illustration, we plot $\beta_{down}(\alpha)$ and $\beta_{up}(\alpha)$ as a function of $\alpha$ in Figs. 1 and 2, respectively. As we see from the figures $\beta_{up}(\alpha)$ increases with $\alpha$ and $\beta_{down}(\alpha)$ decreases with $\alpha$.

As a final remark, we provide a simple upper bound for $\mathbb{E} T^\alpha$ for a geometric random variable $T$ with success parameter $p$, in order to obtain quick evaluations of $\beta_{up}(\alpha)$ in (1.10). Using $\mathbb{P}(T \geq k) = (1 - p)^{k-1} \leq e^{-p(k-1)}$, we see that relation (7.6) in Appendix is satisfied and so letting $r$ be the smallest integer greater than or equal $\alpha$,
we have from (7.7) that

$$\mathbb{E}T^\alpha \leq \mathbb{E}T^r \leq \frac{r!}{(1-e^{-p})^r}.$$  

Plugging this estimate into (1.10) provides an upper bound for $\beta_{up}(\alpha)$. 

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**Fig. 1** Plot of $\beta_{up}(\alpha)$ as a function of $\alpha$ for the homogenous case $\epsilon_1 = \epsilon_2 = 1$.

**Fig. 2** Plot of $\beta_{down}(\alpha)$ as a function of $\alpha$ for the homogenous case $\epsilon_1 = \epsilon_2 = 1$. 
The paper is organized as follows. In Sect. 2, we prove the deviation estimates in Theorem 1. In Sect. 3, we obtain the variance upper bounds for TSP\(_n\) and in Sect. 4, we obtain the variance lower bounds for TSP\(_n\). Next in Sect. 5, we prove the a.s. convergence for TSP\(_n\) (Theorem 7), appropriately scaled and centred and finally in Sect. 6, we obtain bounds for the scaled TSP weight when the nodes are uniformly distributed in the unit square (Theorem 9).

### 2 Proof of Theorem 1

To prove the deviation estimates, we use Poissonization and let \(\mathcal{P}\) be a Poisson process in the unit square \(S\) with intensity \(f(\cdot)\). We join each pair of nodes by a straight line segment and denote the resulting complete graph as \(K_{\mathcal{P}}(n)\). We let \(\text{TSP}_{\mathcal{P}}(n)\) be the TSP of \(K_{\mathcal{P}}(n)\) by an analogous definition as in (1.4). We first find deviation estimates for \(\text{TSP}_{\mathcal{P}}(n)\) and then use dePoissonization to obtain corresponding estimates for \(\text{TSP}_n\), the TSP for the Binomial process as defined in (1.4).

For a real number \(A > 0\), we tile the unit square \(S\) into small \(\frac{A(n)}{\sqrt{n}} \times \frac{A(n)}{\sqrt{n}}\) squares \(\{R_i\}_{1 \leq i \leq \frac{n}{A^2}}\) where \(A(n) \in \left[A, A + \frac{1}{\log n}\right]\) is chosen such that \(\frac{\sqrt{n}}{A(n)}\) is an integer. This is possible since

\[
\frac{\sqrt{n}}{A} - \frac{\sqrt{n}}{A + (\log n)^{-1}} = \frac{\sqrt{n}}{A} \cdot \frac{1}{A(A + (\log n)^{-1})} \geq \frac{\sqrt{n}}{2A^2 \log n} \quad (2.1)
\]

for all \(n\) large. For notational simplicity, we denote \(A(n)\) as \(A\) henceforth and label the squares as in Fig. 3 so that \(R_i\) and \(R_{i+1}\) share an edge for \(1 \leq i \leq \frac{n}{A^2} - 1\).

#### 2.1 Lower Deviation Bounds

For \(1 \leq i \leq \frac{n}{A^2}\), let \(E(R_i)\) denote the event that the \(\frac{A}{\sqrt{n}} \times \frac{A}{\sqrt{n}}\) square \(R_i\) is occupied, i.e. contains at least one node of \(\mathcal{P}\), and all squares sharing a corner with \(R_i\) are
empty. If $E(R_i)$ occurs, then there are at least two edges in the TSP of $K_n^{(P)}$ with one endvertex in $R_i$ and other endvertex in a square not sharing a corner with $R_i$. Each such edge has a Euclidean length of at least $\frac{A}{\sqrt{n}}$ and so a weight of at least $(\frac{c_1 A}{\sqrt{n}})^\alpha$ (see (1.2)). Consequently,

$$\text{TSP}_n^{(P)} \geq \frac{1}{2} \cdot \sum_{i=1}^{\frac{n^2}{\sqrt{n}}} 2 \left( \frac{c_1 A}{\sqrt{n}} \right)^\alpha \mathbb{I}(E(R_i)) = \left( \frac{c_1 A}{\sqrt{n}} \right)^\alpha \cdot G_\alpha,$$  \hspace{1cm} (2.2)

where $G_\alpha := \sum_{i=1}^{\frac{n^2}{\sqrt{n}}} \mathbb{I}(E(R_i))$ and the factor $\frac{1}{2}$ occurs, since each edge is counted twice in the summation.

To estimate $G_\alpha$, we would like to split it into sums of independent r.v.s using the following construction. For a square $R_i$, let $\mathcal{N}(R_i)$ be the set of all squares sharing a corner with $R_i$, including $R_i$. If $R_i$ does not intersect the sides of the unit square $S$, then there are 9 squares in $\mathcal{N}(R_i)$ and if $R_j$ is another square such that $\mathcal{N}(R_i) \cap \mathcal{N}(R_j) = \emptyset$, then the corresponding events $E(R_i)$ and $E(R_j)$ are independent, by Poisson property. We therefore extract nine disjoint subsets $\{U_l\}_{1 \leq l \leq 9}$ of $\{R_i\}$ with the following properties:

(A) If $R_i, R_j \in U_l$, then $\#\mathcal{N}(R_i) = \#\mathcal{N}(R_j) = 9$ and $\mathcal{N}(R_i) \cap \mathcal{N}(R_j) = \emptyset$. This is possible since there are at most $\frac{4\sqrt{n}}{A} - 4 < \frac{4\sqrt{n}}{A}$ squares in $\{R_k\}$ intersecting the sides of the unit square $S$ and the total number of squares in $\{R_k\}$ is $\frac{n}{A^2}$.

We now write $G_\alpha = \sum_{i=1}^{\frac{n^2}{\sqrt{n}}} \mathbb{I}(E(R_i)) \geq \sum_{i=1}^{9} \sum_{R_i \in U_l} \mathbb{I}(E(R_i))$, where each inner summation on the right side is a sum of independent Bernoulli random variables, which we bound via standard deviation estimates. Indeed for $1 \leq l \leq 9$ and $R_i \in U_l$, the number of nodes $N(R_i)$ is Poisson distributed with mean $n \int_{R_i} f(x) dx \in [\epsilon_1 A^2, \epsilon_2 A^2]$ (see (1.1)) and so $R_i$ is occupied with probability at least $1 - e^{-\epsilon_1 A^2}$. Also each of the eight squares sharing a corner with $R_i$ is empty with probability at least $e^{-\epsilon_2 A^2}$, implying that $\mathbb{P}(E(R_i)) \geq (1 - e^{-\epsilon_1 A^2})e^{-8\epsilon_2 A^2}$. Using the deviation estimate (7.2) in Appendix with $\mu_1 = (1 - e^{-\epsilon_1 A^2})e^{-8\epsilon_2 A^2}$, $m = \frac{n}{9A^2} - \frac{4\sqrt{n}}{A}$ and $\epsilon = \frac{1}{m^{1/4}}$ we then get that

$$\mathbb{P}_0 \left( \sum_{R_i \in U_l} \mathbb{I}(E(R_i)) \geq (1 - \epsilon) \left( \frac{n}{9A^2} - \frac{4\sqrt{n}}{A} \right) (1 - e^{-\epsilon_1 A^2})e^{-8\epsilon_2 A^2} \right) \geq 1 - e^{-D_1 \epsilon^2 n},$$  \hspace{1cm} (2.3)

for some constant $D_1 > 0$ not depending on $l$. Since $m^{1/4} < \left( \frac{n}{9A^2} \right)^{1/4}$, we get that $D_1 \epsilon^2 n \geq 2 D_2 \sqrt{n}$ for some constant $D_2 > 0$ and since $m^{1/4} > \left( \frac{n}{10A^2} \right)^{1/4}$ for
all $n$ large, we have
\[
(1 - \epsilon) \left( \frac{n}{9A^2} - \frac{4\sqrt{n}}{A^2} \right) \geq \frac{n}{9A^2} - \frac{4\sqrt{n}}{A^2} \frac{n}{A^2 m^{1/4}} \geq \frac{n}{9A^2} \left( 1 - \frac{36 \sqrt{A}}{n^{1/4}} \right)
\]
for all $n$ large.

Letting
\[
E_{low} := \left\{ G_\alpha \geq (1 - e^{-\epsilon_1 A^2}) e^{-8 \epsilon_2 A^2} \frac{n}{A^2} \left( 1 - \frac{36 \sqrt{A}}{n^{1/4}} \right) \right\},
\]
we get from (2.3) that $\mathbb{P}_0(E_{low}) \geq 1 - 9e^{-2D_2 \sqrt{n}}$ and moreover, from (2.2) we also get that
\[
\text{TSP}_n^P \mathbb{1}(E_{low}) \geq \Delta_n := C_1(A) n^{1 - \frac{9}{2}} \left( 1 - \frac{36 \sqrt{A}}{n^{1/4}} \right)
\]
where $C_1(A)$ is as defined in (1.5). From the estimate for the probability of the event $E_{low}$ above we therefore get $\mathbb{P}_0 \left( \text{TSP}_n^P \geq \Delta_n \right) \geq 1 - 9e^{-2D_2 \sqrt{n}}$ for all $n$ large. To convert the estimate from Poisson to the Binomial process, we let $E := \left\{ \text{TSP}_n \geq \Delta_n \right\}$ and use the dePoissonization formula ([2], also proved below)
\[
\mathbb{P}(E) \geq 1 - D \sqrt{n} \mathbb{P}(E^c)
\]
for some constant $D > 0$ to get that $\mathbb{P}(E) \geq 1 - D \sqrt{n} e^{-2D_2 \sqrt{n}} \geq 1 - e^{-D_2 \sqrt{n}}$ for all $n$ large. This proves (1.6) and so using
\[
\mathbb{E} \text{TSP}_n^k \geq \mathbb{E} \text{TSP}_n^k \mathbb{1}(\text{MST}_n \geq \Delta_n) \geq \Delta_n^k \left( 1 - e^{-D_2 \sqrt{n}} \right)
\]
and
\[
\left( 1 - \frac{36 \sqrt{A}}{n^{1/4}} \right)^k \left( 1 - e^{-D_2 \sqrt{n}} \right) \geq \left( 1 - \frac{36k \sqrt{A}}{n^{1/4}} \right) \left( 1 - e^{-D_2 \sqrt{n}} \right) \geq 1 - \frac{37k \sqrt{A}}{n^{1/4}}
\]
for all $n$ large, we also obtain the lower bound on the expectation in (1.8).

To see that (2.4) is true, we let $N_P$ denote the random number of nodes of $\mathcal{P}$ in all the squares $\{S_j\}$ so that $\mathbb{E}_0 N_P = n$ and $\mathbb{P}_0(N_P = n) = e^{-n n^n / n!} \geq \frac{D_1}{\sqrt{n}}$ for some

\[\mathbb{E}_0 \text{MT}_n^k \geq \mathbb{E}_0 \text{MT}_n^k \mathbb{1}(\text{MST}_n \geq \Delta_n) \geq \Delta_n^k \left( 1 - e^{-D_2 \sqrt{n}} \right) \geq \Delta_n^k \left( 1 - \frac{37k \sqrt{A}}{n^{1/4}} \right).
\]

\[\mathbb{E}_0 \text{MT}_n^k \geq \mathbb{E}_0 \text{MT}_n^k \mathbb{1}(\text{MST}_n \geq \Delta_n) \geq \Delta_n^k \left( 1 - e^{-D_2 \sqrt{n}} \right) \geq 1 - \frac{37k \sqrt{A}}{n^{1/4}}.
\]

\[\mathbb{E}_0 \text{MT}_n^k \geq \mathbb{E}_0 \text{MT}_n^k \mathbb{1}(\text{MST}_n \geq \Delta_n) \geq \Delta_n^k \left( 1 - e^{-D_2 \sqrt{n}} \right) \geq 1 - \frac{37k \sqrt{A}}{n^{1/4}}.
\]
constant $D_1 > 0$, using the Stirling formula. Given $N_P = n$, the nodes of $P$ are i.i.d. with distribution $f(\cdot)$ as defined in (1.1), i.e. $\mathbb{P}_0(E_P^c|N_P = n) = \mathbb{P}(E^c)$ and so

$$\mathbb{P}_0(E_P^c) \geq \mathbb{P}_0(E_P^c|N_P = n)\mathbb{P}_0(N_P = n) = \mathbb{P}(E^c)\mathbb{P}_0(N_P = n) \geq \mathbb{P}(E^c)\frac{D_1}{\sqrt{n}},$$

proving (2.4).

2.2 Upper Deviation Bounds (1.7)

As before, we first obtain upper bounds for the Poissonized TSP $\text{TSP}_n^P$. The idea is to first connect all the nodes within each square $R_j$ to get a collection of spanning subpaths and then join all these subpaths together to get an overall spanning path. Joining the endvertices of this path by an edge (whose length is at most $\sqrt{2}$ and weight therefore is at most $(c_2\sqrt{2})^\alpha$ by (1.5)), then gives us the desired spanning cycle. Suppose there is at least one node of the Poisson process $P$ in the unit square $S$ and let $R_{i_1}, R_{i_2}, \ldots, R_{i_Q}, 1 \leq i_1 < i_2 < \ldots < i_Q \leq \frac{n}{\lambda}$, $Q \leq \frac{n}{\lambda^2}$ be the $\frac{A}{\sqrt{n}} \times \frac{A}{\sqrt{n}}$ squares containing all the nodes of $P$. For $1 \leq j \leq Q$, let $T_{i_j}$ be any spanning path containing all the nodes of $R_{i_j}$ and for $1 \leq j \leq Q-1$ let $e_{j+1}$ be any edge joining some node in $R_{i_j}$ and some node in $R_{i_{j+1}}$ so that the union $T_{uni} := \bigcup_{1 \leq j \leq Q} T_{i_j} \bigcup \bigcup_{2 \leq l \leq Q} \{e_l\}$ is a spanning path of the complete graph $K_n^P$. As argued before, the weight

$$W(T_{uni}) \geq \text{TSP}_n^P - (c_2\sqrt{2})^\alpha \quad (2.5)$$

and so it suffices to upper bound $W(T_{uni})$.

For $1 \leq j \leq Q$, there are $N(R_{i_j})$ nodes of the Poisson process in the square $R_{i_j}$ and any two such nodes are connected by an edge of length at most $\frac{A\sqrt{2}}{\sqrt{n}}$. Therefore, the spanning path $T_{i_j}$ has a total weight of at most $N(R_{i_j}) \cdot \left(\frac{c_2A\sqrt{2}}{\sqrt{n}}\right)^\alpha$, using the bounds for the function $h$ in (1.2). The edge $e_{j+1}$ that connects some node in $R_{i_j}$ with some node of $R_{i_{j+1}}$ has a Euclidean length of at most $\frac{2T_{j+1}A}{\sqrt{n}}$ where $T_{j+1} := i_{j+1} - i_j$ and therefore has a weight of at most $\left(\frac{2c_2T_{j+1}A}{\sqrt{n}}\right)^\alpha$, by (1.2). In effect,

$$W(T_{uni}) \leq \sum_{j=1}^{Q} N(R_{i_j}) \cdot \left(\frac{c_2A\sqrt{2}}{\sqrt{n}}\right)^\alpha + \sum_{j=1}^{Q-1} \left(\frac{2c_2T_{j+1}A}{\sqrt{n}}\right)^\alpha$$

$$= \sum_{i=1}^{n/\lambda^2} N(R_i) \cdot \left(\frac{c_2A\sqrt{2}}{\sqrt{n}}\right)^\alpha + \sum_{j=2}^{Q} \left(\frac{2c_2T_jA}{\sqrt{n}}\right)^\alpha.$$
Setting $T_1 := i_1 - 1$, $T_{Q+1} := \frac{n}{A^2} - i_Q$ and $S_\alpha := \sum_{j=1}^{Q+1} T_j^\alpha$ and using (2.5), we then get

$$\text{TSP}^{(P)}_n \leq \left( \frac{2c_2 A}{\sqrt{n}} \right)^\alpha \left( \sum_{i=1}^{n} N(R_i) + S_\alpha \right) + (c_2 \sqrt{2})^\alpha. \quad (2.6)$$

The first sum $\sum_{i=1}^{n} N(R_i)$ in the right side of (2.6) is a Poisson random variable with mean $n$ since this denotes the total number of nodes of the Poisson process in the unit square. Using the deviation estimate (7.1) in Appendix with $m = 1$, $\mu_2 = n$ and $\epsilon = \log n / \sqrt{n}$, we have that

$$\mathbb{P}_0 \left( \sum_{i=1}^{n} N(R_i) \leq n \left( 1 + \frac{\log n}{\sqrt{n}} \right) \right) \geq 1 - e^{-D_1 \left( \log n \right)^2} \quad (2.7)$$

for some constant $D_1 > 0$. Setting $E_{\text{node}} := \left\{ \sum_{i=1}^{n} N(R_i) \leq n \left( 1 + \frac{\log n}{\sqrt{n}} \right) \right\}$ and absorbing the final term in (2.6) (this is possible since $\alpha < 2$) we then get that

$$\text{TSP}^{(P)}_n \mathbb{P}(E_{\text{node}}) \leq \left( \frac{2c_2 A}{\sqrt{n}} \right)^\alpha \left( n \left( 1 + \frac{\log n}{\sqrt{n}} \right) + S_\alpha \right). \quad (2.8)$$

The second term $S_\alpha$ in (2.8) is well defined for any configuration $\omega$ of the Poisson process provided we set $S_\alpha(\omega_0) = \left( \frac{n}{A^2} - 1 \right)^\alpha$ where $\omega_0$ is the configuration containing no node of the Poisson process in the unit square $S$. The following lemma obtains an estimate on $S_\alpha$.

**Lemma 2** Let $T$ be a geometric random variable with success parameter $p = 1 - e^{-A^2 \delta}$ independent of the node locations $\{X_i\}$. For every even integer $m \geq 1$ and every $A > 0$ and for all $n \geq n_0(m, A)$ large, we have

$$\mathbb{P}_0 \left( S_\alpha \leq \left( 1 + \frac{1}{n^{1/16}} \right) \frac{n}{A^2} \mathbb{E} T^\alpha \right) \geq 1 - \frac{D}{n^{2m/16}}, \quad (2.9)$$

where $D > 0$ is a constant.

Since $S_\alpha$ is not an i.i.d. sum, we use coupling techniques to estimate $S_\alpha$ and prove Lemma 2 at the end of this section.

We continue with the proof of the deviation upper bound. Let $m \geq 2$ be an even integer to be determined later. Letting $E_S$ be the event on the left side of (2.9), we have from the previously computed upper bound on $\text{TSP}^{(P)}_n$ (see (2.8)) that

$$\text{TSP}^{(P)}_n \mathbb{P}(E_{\text{node}} \cap E_S) \leq \left( \frac{2c_2 A}{\sqrt{n}} \right)^\alpha$$
we use the estimate $MST_n$ from (2.7) and (2.9), respectively, we have

\[
\left( n \left( 1 + \frac{\log n}{\sqrt{n}} \right) + \left( 1 + \frac{1}{n^{1/16}} \right) \frac{n}{A^2 \mathbb{E} T^\alpha} \right) \leq C_2(A)n^{1-\frac{\alpha}{2}} \left( 1 + \frac{1}{n^{1/16}} \right)
\]

(2.10)

where $C_2(A)$ is as in (1.5) and the final estimate in (2.10) is obtained using $\frac{2\log n}{\sqrt{n}} \leq \frac{1}{n^{1/16}}$ for all $n$ large. From (2.10) and the estimates for the events $E_{node}$ and $E_S$ from (2.7) and (2.9), respectively, we have

\[
P_0 \left( TSP_n \leq C_2(A)n^{1-\frac{\alpha}{2}} \left( 1 + \frac{1}{n^{1/16}} \right) \right) \geq 1 - e^{-D(\log n)^2} - \frac{D}{n^{7m/16}}
\]

\[
\geq 1 - \frac{2D}{n^{7m/16}}
\]

(2.11)

for all $n$ large and some constant $D > 0$.

From (2.11) and the dePoissonization formula (2.4), we obtain

\[
P \left( TSP_n \leq C_2(A)n^{1-\frac{\alpha}{2}} \left( 1 + \frac{1}{n^{1/16}} \right) \right) \geq 1 - \frac{2D\sqrt{n}}{n^{7m/16}}.
\]

(2.12)

Choosing $m$ large enough such that $\frac{7m}{16} - \frac{1}{2} \geq k$, we obtain the estimate in (1.7).

For bounding the expectation of $TSP_n^k$, we let $\Delta_n = C_2(A)n^{1-\frac{\alpha}{2}} \left( 1 + \frac{1}{n^{1/16}} \right)$ and write

\[
\mathbb{E} TSP_n^k = \mathbb{E} TSP_n^k \mathbb{I}(TSP_n \leq \Delta_n) + \mathbb{E} TSP_n^k \mathbb{I}(TSP_n > \Delta_n)
\]

\[
\leq \Delta_n^k + \mathbb{E} TSP_n^k \mathbb{I}(TSP_n > \Delta_n).
\]

(2.13)

To estimate $\Delta_n^k$, we use $\left( 1 + \frac{1}{n^{1/16}} \right)^k \leq e^{k/n^{1/16}} \leq 1 + \frac{2k}{n^{1/16}}$ for all $n$ large and get that $\Delta_n^k \leq C_2^k(A)n^{k(1-\frac{\alpha}{2})} \left( 1 + \frac{2k}{n^{1/16}} \right)$ for all $n$ large. For the second term in (2.13), we use the estimate $MST_n \leq n \cdot \left( c_2 \sqrt{2} \right)^{\alpha}$ since there are $n$ edges in the spanning tree, each such edge has an Euclidean length of at most $\sqrt{2}$ and so the weight of any edge is at most $(c_2 \sqrt{2})^\alpha$, using (1.2). Letting $\theta_m = \frac{7m}{16} - \frac{1}{2} - \frac{ak}{2}$ and using the probability estimate (2.12), we then get that

\[
\mathbb{E} MST_n^k \leq \Delta_n^k + \frac{3D_3n^k\sqrt{n}}{n^{7m/16}} \leq C_2^k(A)n^{k(1-\frac{\alpha}{2})} \left( 1 + \frac{2k}{n^{1/16}} + \frac{D_3}{n^{\theta_m}} \right)
\]

for all $n$ large and some constant $D_3 > 0$. Choosing $m$ larger if necessary so that $\theta_m \geq 1 > \frac{1}{16}$, we obtain the expectation upper bound in (1.8).

\[\square\]

**Proof of Lemma 2** We show that $S_\alpha(\omega)$ is monotonic in $\omega$ in the sense that adding more nodes increases $S_\alpha$ if $\alpha < 1$ and decreases $S_\alpha$ if $\alpha > 1$. This then allows us to use coupling and upper bound $S_\alpha$ by simply considering homogenous Poisson processes.
Monotonicity of $S_\alpha$: We recall that $\omega_0$ is the configuration containing no node of the Poisson process in the unit square. For a configuration $\omega \neq \omega_0$ let $1 \leq i_1(\omega) < \ldots < i_Q(\omega) \leq n/\lambda^2$ be the indices of the squares in $\{R_j\}$ containing at least one node of the Poisson process $\mathcal{P}$. Letting $i_0(\omega) = 1$ and $i_{Q+1}(\omega) = n/\lambda^2$, we have $S_\alpha(\omega) = \sum_{j=0}^Q (i_{j+1}(\omega) - i_j(\omega))^\alpha$. Suppose $\omega' = \omega \cup \{x\}$ is obtained by adding a single extra node at $x \in R_{j_0}$ for some $1 \leq j_0 \leq n/\lambda^2$. If $j_0 \in \{i_k(\omega)\}_{0 \leq k \leq Q+1}$, then $S_\alpha(\omega') = S_\alpha(\omega)$. Else there exists $0 \leq a \leq Q$ such that $i_a(\omega) < j_0 < i_{a+1}(\omega)$ and so

$$S_\alpha(\omega') = S_\alpha(\omega) + (i_{a+1}(\omega) - j_0)^\alpha + (j_0 - i_a(\omega))^\alpha - (i_{a+1}(\omega) - i_a(\omega))^\alpha.$$  

If $\alpha \leq 1$ then using $a^\alpha + b^\alpha \geq (a + b)^\alpha$ for positive numbers $a, b$ we get that $S_\alpha(\omega') \geq S_\alpha(\omega)$. If $\alpha > 1$ then $a^\alpha + b^\alpha \leq (a + b)^\alpha$ and so $S_\alpha(\omega') \leq S_\alpha(\omega)$. This monotonicity property together with coupling allows us to upper bound $P$ of the Poisson process in the unit square. For a configuration $\{\tilde{T}_j\}$...
and $T^{(Q)}_{Q+1} \leq \tilde{T}_{Q+1}$. Consequently,

$$S^{(Q)}_\alpha \| (F_\delta) \leq \sum_{i=1}^{Q+1} \tilde{T}_i^{\alpha} \| (F_\delta) \leq \sum_{i=1}^n \tilde{T}_i^{\alpha} \| (F_\delta) \leq \sum_{i=1}^n \tilde{T}_i^{\alpha}, \quad (2.15)$$

since $Q_\delta \leq \frac{n}{A^2}$.

From (2.15), it suffices to estimate the sum on the right side to find an upper bound for $S^{(Q)}_\alpha$. The advantage of (2.15) is that $\{\tilde{T}_i\}$ are i.i.d. geometric random variables with success parameter $p = 1 - e^{-A^2 \delta}$ and so all moments of $\tilde{T}_i^{\alpha}$ exist. Letting $\beta_i = \left( \tilde{T}_i^{\alpha} - E_\delta \tilde{T}_i^{\alpha} \right)$ and $\beta_{tot} = \sum_{i=1}^n \beta_i$, we obtain for an even integer constant $m$ that

$$E_\delta \beta_{tot}^m = E_\delta \sum_{(i_1, \ldots, i_m)} \beta_{i_1} \ldots \beta_{i_m}. \quad (2.16)$$

For a tuple $(i_1, \ldots, i_m)$ let $\{j_1, \ldots, j_w\}$ be the distinct integers in $\{i_1, \ldots, i_m\}$ with corresponding multiplicities $l_1, \ldots, l_w$ so that

$$E_\delta \beta_{i_1} \ldots \beta_{i_m} = \left[ E_\delta \beta_{j_1} \ldots \beta_{j_w} = \prod_{k=1}^w E_\delta \beta_{j_k} \right].$$

If $l_k = 1$ for some $1 \leq k \leq w$, then $E_\delta \beta_{i_1} \ldots \beta_{i_m} = 0$ and so for any nonzero term in the summation in (2.16), there are at most $\frac{m}{2}$ distinct terms in $\{i_1, \ldots, i_m\}$. This implies that

$$E_\delta \beta_{tot}^m \leq D(m) \left( \frac{n}{m/2} \right) \leq D(m)n^{m/2}$$

for some constant $D(m) > 0$. For $\epsilon > 0$, we therefore get from Chebyshev's inequality that

$$P_\delta \left( |\beta_{tot}| > \epsilon \left( \frac{n}{A^2} + 1 \right) E_\delta \tilde{T}_1^{\alpha} \right) \leq D_1 \frac{E_\delta (\beta_{tot})}{n^m \epsilon^m} \leq \frac{D_2}{n^{m/2} \epsilon^m}$$

for some constants $D_1, D_2 > 0$. Setting $\epsilon = \frac{1}{n^{1/16}}$ and using $\epsilon \left( \frac{n}{A^2} + 1 \right) \leq \left( 1 + \frac{1}{n^{1/16}} \right) \frac{n}{A^2}$ for all $n$ large, we then get

$$P_\delta \left( \sum_{i=1}^n \tilde{T}_i^{\alpha} \leq \left( 1 + \frac{1}{n^{1/16}} \right) \frac{n}{A^2} \bar{E}_\delta \tilde{T}_1^{\alpha} \right) \geq 1 - \frac{D_2}{n^{7m/16}}, \quad (2.17)$$

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From the upper bound for \( S_\alpha^{(\delta)} \) in (2.15) and the fact that there is at least one node of \( P_\delta \) in the unit square \( S \) with probability \( 1 - e^{-\delta n} \), we further get
\[
P_\delta \left( S_\alpha^{(\delta)} \leq \left( 1 + \frac{1}{n^{1/16}} \right) \frac{n}{A^2} \widetilde{T}_1^\alpha \right) \geq 1 - \frac{D_2}{n^{7m/16}} - e^{-\delta n} \geq 1 - 2D_2 \]
for all \( n \) large. Using the coupling relation (2.14), we finally get (2.9) proving Lemma 2.

\[ \square \]

3 Variance Upper Bound for TSP

In this section, we obtain the variance upper bound for the overall TSP length \( TSP_n \) as defined in (1.4). We have the following result.

**Theorem 3** Let \( \epsilon_1 \) and \( \epsilon_2 \) be as in (1.1) and suppose the edge weight exponent is \( 0 < \alpha < 2 \). For every \( \epsilon > \frac{(2-\alpha)(1+\alpha)}{2+\alpha} \), there is a constant \( D_1 = D_1(\epsilon, \alpha, \epsilon_1, \epsilon_2) > 0 \) such that
\[
\text{var}(TSP_n) \leq D_1 n^\epsilon
\]
for all \( n \) large.

The desired bound is obtained via the martingale difference method that estimates the change in TSP lengths after adding or removing a single node.

We perform some preliminary computations.

3.1 One Node Difference Estimates

As in (1.4), let \( TSP_{n+1} = W(C_{n+1}) \) be the weight of the TSP \( C_{n+1} \) formed by the nodes \( \{X_k\}_{1 \leq k \leq n+1} \) and for \( 1 \leq j \leq n+1 \), let \( TSP_n(j) = W(C_n(j)) \) be the weight of the TSP \( C_n(j) \) formed by the nodes \( \{X_k\}_{1 \leq k \neq j \leq n+1} \). In this subsection, we find estimates for \( |TSP_{n+1} - TSP_n(j)| \), the change in the length of the TSP upon adding or removing a single node.

Suppose \( v_j \) and \( v_{j+1} \) are the neighbours of the node \( X_j \) in the TSP \( C_{n+1} \) so that the edges \( (v_j, X_j) \) and \( (X_j, v_{j+1}) \) both belong to \( C_{n+1} \). Removing the edges \( (v_j, X_j) \) and \( (X_j, v_{j+1}) \) and adding the edge \( (v_j, v_{j+1}) \), we get a spanning cycle containing the nodes \( \{X_k\}_{1 \leq k \neq j \leq n+1} \) and so
\[
TSP_n(j) \leq TSP_{n+1} + h^\alpha(v_j, v_{j+1}) \leq TSP_{n+1} + c_2^\alpha d^\alpha(v_j, v_{j+1}), \quad (3.1)
\]
using (1.2).

By the triangle inequality, we have that \( d(v_j, v_{j+1}) \leq d(X_j, v_j) + d(X_j, v_{j+1}) \) and so using \( (a + b)^\alpha \leq 2^\alpha (a^\alpha + b^\alpha) \) for all \( a, b, \alpha > 0 \) we get that
\[
d^\alpha(v_j, v_{j+1}) \leq 2^\alpha \left( d^\alpha(X_j, v_j) + d^\alpha(X_j, v_{j+1}) \right)
\]
\[= 2^\alpha \sum_{v \in \mathcal{N}(X_j, C_{n+1})} d^\alpha(X_j, v),\]

where \(\mathcal{N}(X_j, C_{n+1})\) denotes the set of neighbours of \(X_j\) in the spanning cycle \(C_{n+1}\). Thus,

\[TSP_n(j) \leq TSP_{n+1} + g_1(X_j), \quad (3.2)\]

where \(g_1(X_j) \coloneqq (2c_2)^\alpha \sum_{v \in \mathcal{N}(X_j, C_n)} d^\alpha(X_j, v).\)

To get an estimate in the reverse direction, we would like to identify edges of the TSP \(C_n(j)\) with small length that are close to the (yet to be added) node \(X_j\). Recalling that \(\epsilon > \frac{(2-\alpha)(1+\alpha)}{2+\alpha}\), we let \(\theta > 0\) such that \(0 < \theta < \frac{\alpha}{2(2+\alpha)}\) and \(1 - \epsilon < 2\theta\alpha < \frac{\alpha}{2}\). (3.3)

To see such a \(\theta\) exists, we use \(\epsilon > \frac{(2-\alpha)(1+\alpha)}{2+\alpha}\) and choose \(\theta > 0\) sufficiently close to but less than \(\frac{\alpha}{2(2+\alpha)}\), so that \(\epsilon > 1 - 2\theta\alpha > 1 - 2\left(\frac{\alpha}{2(2+\alpha)}\right)\alpha = \frac{(2-\alpha)(1+\alpha)}{2+\alpha}\). Fixing such a \(\theta\) we use \(\alpha < 2\) to get that \(\theta < \frac{\alpha}{2(2+\alpha)} < \frac{1}{4}\) and therefore that \(2\theta\alpha < \frac{\alpha}{2}\).

Let \(E_{\text{good}}(\theta)\) be the event that there is an edge \((u, v) \in C_n(j)\) such that each edge in the triangle formed by the nodes \(X_j, u, v\) has Euclidean length at most \(\frac{1}{n^\theta}\). If \(E_{\text{good}}(\theta)\) occurs, then removing the edge \((u, v)\) and adding the edges \((X_j, u)\) and \((X_j, v)\), we obtain a spanning cycle containing all the nodes \(\{X_k\}_{1 \leq k \leq n+1}\). The Euclidean length of each added edge is at most \(\frac{1}{n^\theta}\) and so the weight of each added edge is at most \(\frac{c_2}{n^\theta}\alpha\), by (1.2).

If \(E_{\text{good}}(\theta)\) does not occur, we then remove the shortest edge \((u_1, v_1) \in C_n(j)\) in terms of Euclidean length and add the edges \((X_j, u_1)\) and \((X_j, v_1)\). The Euclidean length of each of these newly added edges is at most \(\sqrt{2}\) and so the corresponding weights are at most \(c_2\sqrt{2})\alpha\), each. Thus, we get from the above discussion that \(TSP_{n+1} \leq TSP_n(j) + g_2(X_j)\), where

\[g_2(X_j) \coloneqq 2\left(\frac{c_2}{n^\theta}\right)^\alpha \mathbb{1}(E_{\text{good}}(\theta)) + 2(c_2\sqrt{2})^\alpha \mathbb{1}(E_{\text{good}}(\theta)). \quad (3.4)\]

Combining with (3.2), we get

\[|TSP_{n+1} - TSP_n(j)| \leq g_1(X_j) + g_2(X_j). \quad (3.5)\]

The following lemma collects properties regarding \(g_1\) and \(g_2\) used later.

**Lemma 4** There is a constant \(D > 0\) such that for all \(n\) large,

\[\sum_{j=1}^{n+1} \mathbb{E}g_1(X_j) = (n+1)\mathbb{E}g_1(X_{n+1}) \leq Dn^{1-\frac{q}{2}}. \quad (3.6)\]
\[
g_1^2(X_j) \leq D \sum_{v \in N(X_j, C_{n+1})} d^\alpha(X_j, v),
\]

\[
(g_1^2(X_j))^2 \leq (n+1)g_1^2(X_1) \leq 2Dn^{1-\frac{\alpha}{2}},
\tag{3.7}
\]

\[
\Pr(E_{good}(\theta)) \geq 1 - \frac{1}{n^{2\theta \alpha}} \text{ and } \max_{1 \leq j \leq n+1} \mathbb{E}g_2^2(X_j) \leq \frac{D}{n^{2\theta \alpha}}.
\tag{3.8}
\]

Moreover,

\[
\mathbb{E}|TSP_{n+1} - TSP_n| \leq \frac{2D}{n^{1-\epsilon}}.
\tag{3.9}
\]

The proof of (3.9) follows from (3.5) since

\[
\mathbb{E}|TSP_{n+1} - TSP_n| \leq g_1(X_j) + \mathbb{E}g_2(X_j) \leq \frac{D}{n^{\frac{\alpha}{2}}} + \frac{D}{n^{2\theta \alpha}} \leq \frac{2D}{n^{1-\epsilon}},
\]

since both $\frac{\alpha}{2}$ and $2\theta \alpha$ are at least $1 - \epsilon$, by our choice of $\theta$ in (3.3).

**Proof of (3.6) in Lemma 4** Using (1.2), we have that

\[
g_1(X_j) = (2c_2)^\alpha \sum_{v \in N(X_j, C_{n+1})} d^\alpha(X_j, v) \leq \left(\frac{2c_2}{c_1}\right)^\alpha \sum_{v \in N(X_j, C_{n+1})} h^\alpha(X_j, v)
\]

and so

\[
\sum_{j=1}^{n+1} \mathbb{E}g_1(X_j) \leq \left(\frac{2c_2}{c_1}\right)^\alpha \left(\sum_{j=1}^{n+1} \sum_{v \in N(X_j, C_{n+1})} h^\alpha(X_j, v)\right). \tag{3.10}
\]

The final double summation in (3.10) is simply twice the weight of the TSP cycle containing all the $n+1$ nodes and so using the expectation upper bound (1.8), we get the desired estimate in (3.6).

**Proof of (3.7) in Lemma 4** There are two edges containing $X_j$ as an endvertex in the cycle $C_{n+1}$ and so using $(a + b)^2 \leq 2(a^2 + b^2)$ we first get that

\[
g_1^2(X_j) \leq 2(2c_2)^{2\alpha} \sum_{v \in N(X_j, C_{n+1})} d^{2\alpha}(X_j, v).
\]

Since the Euclidean length of any edge is at most $\sqrt{2}$, we have that $d^{2\alpha}(X_j, v) \leq (\sqrt{2})^{\alpha} d^\alpha(X_j, v)$ and so

\[
g_1^2(X_j) \leq 2(4c_2^2 \sqrt{2})^\alpha \sum_{v \in N(X_j, C_{n+1})} d^\alpha(X_j, v),
\]
proving the first relation in (3.7). Consequently,
\[
\sum_{j=1}^{n+1} \mathbb{E} g_j^2(X_j) \leq 2(4c_2^2\sqrt{2})^\alpha \mathbb{E} \sum_{j=1}^{n+1} \sum_{v \in \mathcal{N}(X_j, \mathcal{C}_{n+1})} d^\alpha(X_j, v)
\]
\[
\leq 2 \left( \frac{4c_2^2\sqrt{2}}{c_1} \right)^\alpha \mathbb{E} \sum_{j=1}^{n+1} \sum_{v \in \mathcal{N}(X_j, \mathcal{C}_{n+1})} h^\alpha(X_j, v),
\]
using (1.2). As before, the double summation in the right side of (3.11) is simply twice the weighted TSP length of \( \mathcal{C}_{n+1} \) and so using expectation upper bound (1.8), we also get the second estimate in (3.7).

**Proof of (3.8) in Lemma 4** Let \( a_0 < \frac{1}{16\sqrt{2}} \) be a positive constant and tile the unit square \( S \) into \( \frac{8A_0}{n^2} \times \frac{8A_0}{n^2} \) squares \( Q_i \), \( 1 \leq i \leq \frac{n^{2\theta}}{64A_0} \) where \( A_0 = A_0(n) \in [a_0, 2a_0) \) is such that \( \frac{n^{2\theta}}{8A_0} \) is an integer for all \( n \) large. If \( Q_i^0 \) is the \( \frac{2A_0}{n^\theta} \times \frac{2A_0}{n^\theta} \) square with the same centre as \( Q_i \), then the number of nodes \( N_{i}^{0} \) of \( \{ X_i \}_{1 \leq i < j \leq n+1} \) within the square \( Q_i^0 \) is Binomially distributed with mean
\[
n \int_{Q_i^0} f(x) \, dx \geq \epsilon_1 n \cdot \frac{4A_0^2}{n^{2\theta}} \geq 4\epsilon_1 a_0^2 n^{1-2\theta},
\]
using the bounds for \( f(\cdot) \) in (1.1). From the deviation estimate (7.2) in Appendix, we then get that \( N_i^{0} \) is at least \( 2\epsilon_1 a_0^2 n^{1-2\theta} \) with probability at least \( 1 - e^{-2D_1 n^{1-2\theta}} \), for some constant \( D_1 > 0 \). Setting
\[
E_{dense} := \bigcap_{1 \leq i \leq \frac{n^{2\theta}}{64A_0}} \{ N_i^{0} \geq 2\epsilon_1 a_0^2 n^{1-2\theta} \},
\]
we therefore have
\[
\mathbb{P}(E_{dense}) \geq 1 - \frac{n^{2\theta}}{64A_0} e^{-2D_1 n^{1-2\theta}} \geq 1 - e^{-D_1 n^{1-2\theta}}
\]
(3.12) for all \( n \) large.

The event \( E_{dense} \) defined above is useful in the following way. Say that a \( \frac{8A_0}{n^\theta} \times \frac{8A_0}{n^\theta} \) square \( Q_i \) is a bad square if every edge of the TSP \( C_n \) with one endvertex within \( Q_i^0 \), has its other endvertex outside \( Q_i \) and let \( E_{no\_bad} \) be the event that no square in \( \{ Q_k \} \) is bad. Suppose the event \( E_{dense} \cap E_{no\_bad} \) occurs and the (new) node \( X_j \) is added to the square \( Q_i \). There exists an edge \( e = (u, v) \) of the TSP \( C_n \) with both endvertices in the square \( Q_i \) and so the Euclidean lengths \( d(u, v), d(X_j, u) \) and \( d(X_j, v) \) are each no more than \( \frac{8A_0 \sqrt{2}}{n^\theta} \), the length of the diagonal of \( Q_i \). Using \( A_0 \leq 2a_0 < \frac{1}{8\sqrt{2}} \).
(see the first sentence of this proof) we then that \( \frac{8A_0 \sqrt{2}}{n^\theta} \leq \frac{16A_0 \sqrt{2}}{n^\theta} < \frac{1}{n^\theta} \) and so the event \( E_{\text{good}}(\theta) \) occurs.

In what follows, we estimate the probability of the event \( E_{\text{dense}} \cap E_{\text{no bad}}^c \). Suppose that the event \( E_{\text{dense}} \cap E_{\text{no bad}}^c \) occurs so that there is at least one bad square and let \( Q_l \) be a bad square so that each node within the smaller subsquare \( Q_l^0 \) belongs to a (long) edge of the cycle \( C_n(j) \) whose other endvertex lies outside the bigger square \( Q_l \). The Euclidean length of each such long edge is at least \( 3A_0 \), the width of the annulus \( Q_l \setminus Q_l^0 \) and so using \( A_0 \geq a_0 \), the weight of the long edge is at least \( (3a_0 c_1) \alpha n^{1-(2+\alpha)\theta} \), using the bounds for the weight function \( h \) in (1.2). Additionally, since the event \( E_{\text{dense}} \) occurs, there are at least \( 2\epsilon_1 a_0^2 n^{1-2\theta} \) nodes in \( Q_l^0 \) and so

\[
\text{TSP}_n(j) \mathbb{I}(E_{\text{dense}} \cap E_{\text{no bad}}^c) \geq 2\epsilon_1 a_0^2 n^{1-2\theta} \cdot \left( \frac{3a_0 c_1}{n^\theta} \right) \alpha \mathbb{I}(E_{\text{dense}} \cap E_{\text{no bad}}^c) = 2\epsilon_1 a_0^2 (3a_0 c_1)^\alpha n^{1-(2+\alpha)\theta} \mathbb{I}(E_{\text{dense}} \cap E_{\text{no bad}}^c).
\]

For any integer constant \( k \geq 1 \), we therefore get that

\[
\mathbb{E}\text{TSP}_n^k(j) \mathbb{I}(E_{\text{dense}} \cap E_{\text{no bad}}^c) \geq \left( 2\epsilon_1 a_0^2 (3a_0 c_1)^\alpha \right)^k n^{k(1-(2+\alpha)\theta)} \mathbb{P}(E_{\text{dense}} \cap E_{\text{no bad}}^c).
\]

But using the expectation upper bound in (1.8), we have

\[
\mathbb{E}\text{TSP}_n^k(j) \mathbb{I}(E_{\text{dense}} \cap E_{\text{no bad}}^c) \leq \mathbb{E}\text{TSP}_n^k(j) \leq D_1 n^{k\left(1-\frac{\theta}{2}\right)}
\]

for some constant \( D_1 > 0 \) and so \( \mathbb{P}(E_{\text{dense}} \cap E_{\text{no bad}}^c) \leq \frac{D_2}{n^{k\theta_1}} \), where \( D_2 > 0 \) is a constant and \( \theta_1 := \frac{\alpha}{2} - (2 + \alpha)\theta \). By choice \( \theta < \frac{\alpha}{2(2+\alpha)} \) (see (3.3)) and so choosing \( k \) sufficiently large satisfying \( k\theta_1 > 2\theta\alpha \), we get

\[
\mathbb{P}(E_{\text{dense}} \cap E_{\text{no bad}}^c) \leq \frac{D_2}{n^{k\theta_1}} \leq \frac{1}{n^{2\theta\alpha}}
\]

for all \( n \) large. Using the bounds for the probability of the event \( E_{\text{dense}} \) from (3.12), we then get that

\[
\mathbb{P}(E_{\text{dense}} \cap E_{\text{no bad}}) \geq 1 - e^{-D_1 n^{1-2\theta}} - \frac{1}{n^{2\theta\alpha}} \geq 1 - \frac{2}{n^{2\theta\alpha}}
\]

for all \( n \) large. This implies that \( \mathbb{P}(E_{\text{good}}(\theta)) \geq 1 - \frac{2}{n^{2\theta\alpha}} \). Using this along with the expression for \( g_2 \) in (3.4), we then get (3.8). \( \square \)
3.2 Variance Upper Bound for TSP

We use one node difference estimate (3.5) together with the martingale difference method to obtain a bound for the variance. For $1 \leq j \leq n + 1$, let $\mathcal{F}_{j} = \sigma \left( \{X_{k}\}_{1 \leq k \leq j} \right)$ denote the sigma field generated by the node positions $\{X_{k}\}_{1 \leq k \leq j}$. Defining the martingale difference

$$H_{j} = \mathbb{E}(TSP_{n+1}|\mathcal{F}_{j}) - \mathbb{E}(TSP_{n+1}|\mathcal{F}_{j-1}),$$  

we then have that $TSP_{n+1} - \mathbb{E}TSP_{n+1} = \sum_{j=1}^{n+1} H_{j}$ and so by the martingale property

$$\text{var}(TSP_{n+1}) = \mathbb{E}\left( \sum_{j=1}^{n+1} H_{j} \right)^{2} = \sum_{j=1}^{n+1} \mathbb{E}H_{j}^{2}. \quad (3.14)$$

To evaluate $\mathbb{E}H_{j}^{2}$, we rewrite the martingale difference $H_{j}$ in a more convenient form. Letting $X'_{j}$ be an independent copy of $X_{j}$ which is also independent of $\{X_{k}\}_{1 \leq k \neq j \leq n+1}$, we rewrite

$$H_{j} = \mathbb{E}(TSP_{n+1}(X_{j}) - TSP_{n+1}(X'_{j})|\mathcal{F}_{j}), \quad (3.15)$$

where $TSP_{n+1}(X_{j})$ is the weight of the MST formed by the nodes $\{X_{i}\}_{1 \leq i \leq n+1}$ and $TSP_{n+1}(X'_{j})$ is the weight of the TSP formed by the nodes $\{X_{i}\}_{1 \leq i \neq j \leq n+1} \cup \{X'_{j}\}$. Using the triangle inequality and the one node difference estimate (3.5), we have that $|TSP_{n+1}(X_{j}) - TSP_{n+1}(X'_{j})|$ is bounded above as

$$|TSP_{n+1}(X_{j}) - TSP_{n}(j)| + |TSP_{n+1}(X'_{j}) - TSP_{n}(j)| \leq g_{1}(X_{j}) + g_{2}(X_{j}) + g_{1}(X'_{j}) + g_{2}(X'_{j}),$$

where $g_{1}$ and $g_{2}$ are as in (3.5) and we recall that $TSP_{n}(j)$ is the weight of the MST formed by the nodes $\{X_{k}\}_{1 \leq k \neq j \leq n+1}$. Thus,

$$|H_{j}| \leq \mathbb{E}(|TSP_{n+1}(X_{j}) - TSP_{n+1}(X'_{j})||\mathcal{F}_{j})$$

$$\leq \mathbb{E}(g_{1}(X_{j})|\mathcal{F}_{j}) + \mathbb{E}(g_{2}(X_{j})|\mathcal{F}_{j}) + \mathbb{E}(g_{1}(X'_{j})|\mathcal{F}_{j}) + \mathbb{E}(g_{2}(X'_{j})|\mathcal{F}_{j})$$

$$= \mathbb{E}(g_{1}|\mathcal{F}_{j}) + \mathbb{E}(g_{2}|\mathcal{F}_{j}) + \mathbb{E}(g_{1}|\mathcal{F}_{j-1}) + \mathbb{E}(g_{2}|\mathcal{F}_{j-1}).$$

Using $(a_{1} + a_{2} + a_{3} + a_{4})^{2} \leq 4(a_{1}^{2} + a_{2}^{2} + a_{3}^{2} + a_{4}^{2})$, we then get

$$H_{j}^{2} \leq 4 \left( (\mathbb{E}(g_{1}|\mathcal{F}_{j})^{2} + (\mathbb{E}(g_{2}|\mathcal{F}_{j})^{2} + (\mathbb{E}(g_{1}|\mathcal{F}_{j-1})^{2} + (\mathbb{E}(g_{2}|\mathcal{F}_{j-1})^{2} \right)$$

$$\leq 4 \left( (g_{1}^{2}|\mathcal{F}_{j}) + (g_{2}^{2}|\mathcal{F}_{j}) + (g_{1}^{2}|\mathcal{F}_{j-1}) + (g_{2}^{2}|\mathcal{F}_{j-1}) \right)$$
since \((E(X|F))^2 \leq E(X^2|F)\). Thus, \(E H_j^2 \leq 8 \left( E g_1^2(X_j) + E g_2^2(X_j) \right)\) and plugging this in (3.14), we have

\[
var(TSP_{n+1}) = \sum_{j=1}^{n+1} E H_j^2 \leq 8 \left( \sum_{j=1}^{n+1} E g_1^2(X_j) + \sum_{j=1}^{n+1} E g_2^2(X_j) \right).
\]

Plugging the estimates for \(E g_1^2(X_j)\) and \(E g_2^2(X_j)\) from (3.7) and (3.8), respectively, we get

\[
var(TSP_{n+1}) \leq D \left( n^{1-\frac{\alpha}{2}} + n^{1-2\alpha} \right) \leq 2Dn^\epsilon \text{ for some constant } D > 0, \text{ by our choice of } \theta \text{ in (3.3)}.
\]

\[\square\]

4 Variance Lower Bound for TSP

**Theorem 5** Suppose there is a square \(S_0\) with a constant side length \(s_0\) such that \(h(u, v) = d(u, v)\) if either \(u\) or \(v\) is in \(S_0\). For every \(\alpha < 1\), there is positive constant \(D_2\) such that

\[
var(TSP_n) \geq D_2 n^{1-\alpha}
\]

for all \(n\) large.

We indirectly compute the variance lower bound for the TSP by computing a related spanning path \(P_{app}(n + 1)\) which has approximately the same length as the TSP of the nodes \(\{X_i\}_{1 \leq i \leq n+1}\) and estimate the variance of the weight of \(P_{app}(n + 1)\). Using this estimate, we obtain the desired variance lower bound for \(TSP_{n+1}\).

We tile the unit square \(S\) into disjoint \(\frac{400A}{\sqrt{n}} \times \frac{400A}{\sqrt{n}}\) squares \(\{R_i\}\) where \(1 \leq A \leq 1 + \frac{1}{\log n}\) is such that \(\sqrt{n}\frac{400A}{\sqrt{n}}\) is an integer for all \(n\) large. This is possible by an argument following (2.1). Let \(R(1) \in \{R_i\}\) be the (random) square containing the node \(X_1\) and let \(R^{small}(1)\) be the \(\frac{200A}{\sqrt{n}} \times \frac{200A}{\sqrt{n}}\) square with the same centre as \(R(1)\) and contained in \(R(1)\).

Let \(P_{in}(n + 1)\) be the minimum weight spanning path formed by the nodes of \(\{X_i\}_{1 \leq i \leq n+1}\) present within the square \(R^{small}(1)\) and let \(u_{in}\) and \(v_{in}\) be the endvertices of \(P_{in}(n + 1)\). We define \(P_{in}(n + 1)\) to be the minimum weight *in*-spanning path. Let \(v_{close}\) be the node closest (in terms of Euclidean length) to but outside \(R^{small}(1)\) and suppose \(v_{in}\) is closer in terms of Euclidean length to \(v_{close}\) than \(u_{in}\). We define \(e_{min} = (v_{in}, v_{close})\) to be the *cross-edge*. Finally, let \(P_{out}(n + 1)\) be the minimum weight spanning path formed by the nodes of \(\{X_i\}_{1 \leq i \leq n+1}\) present outside \(R^{small}(1)\) and having \(v_{close}\) as an endvertex. We denote \(P_{out}(n + 1)\) to be the minimum weight *out*-spanning path. By construction, the union

\[
P_{app}(n + 1) = P_{app}(n + 1, \{X_i\}_{1 \leq i \leq n+1}) := P_{in}(n + 1) \cup \{e_{min}\} \cup P_{out}(n + 1)
\]

(4.1)
is an overall spanning path containing all the nodes \( \{X_i\}_{1 \leq i \leq n + 1} \) and we define \( P_{\text{app}}(n + 1) \) to be the approximate overall TSP path.

As in (1.3), we define 
\[
W_{n+1} := W(P_{\text{app}}(n + 1)) = \sum_{e \in P_{\text{app}}(n+1)} h^\alpha(e)
\]
to be the weight of the approximate overall TSP path \( P_{\text{app}}(n + 1) \) and recall that \( \text{TSP}_{n+1} \) is the minimum weight of the overall spanning cycle containing all the \( n + 1 \) nodes \( \{X_i\}_{1 \leq i \leq n+1} \). The following result shows that the approximate overall TSP path has nearly the same weight as the optimal overall TSP cycle and obtains a lower bound on the variance.

**Lemma 6** There exists a constant \( D > 0 \) such that
\[
\mathbb{E}|\text{TSP}_{n+1} - W_{n+1}|^2 \leq D(\log n)^2 
\]
and
\[
\text{var}(W_{n+1}) \geq D \cdot n^{1-\alpha}
\]
for all \( n \) large.

From the bounds in Lemma 6, we obtain a lower bound for the variance of \( \text{TSP}_{n+1} \) as follows. Using \( (a + b)^2 \leq 2(a^2 + b^2) \), we have for any two random variables \( X \) and \( Y \) that
\[
\text{var}(X + Y) \leq 2(\text{var}(X) + \text{var}(Y)) \leq 2\text{var}(X) + 2\mathbb{E}Y^2.
\]
Setting \( X = \text{TSP}_{n+1}, Y = W_{n+1} - \text{TSP}_{n+1} \) and using the bounds in Lemma 6, we therefore get
\[
\text{var}(\text{TSP}_{n+1}) \geq \frac{1}{2} \cdot \text{var}(W_{n+1}) - \mathbb{E}|\text{TSP}_{n+1} - W_{n+1}|^2 \\
\geq \frac{1}{2} \cdot D \cdot n^{1-\alpha} - D(\log n)^2,
\]
obtaining the desired bound on the variance of \( \text{TSP}_{n+1} \).

**Proof of (4.2) in Lemma 6** Joining the endvertices of the approximate overall TSP path \( P_{\text{app}}(n + 1) \) by an edge, we get an overall spanning cycle containing all the nodes \( \{X_i\}_{1 \leq i \leq n+1} \). The added edge has a Euclidean length of at most \( \sqrt{2} \) and so a weight of at most \( (c_2 \sqrt{2})^\alpha \), by (1.2). Thus,
\[
\text{TSP}_{n+1} \leq W_{n+1} + (c_2 \sqrt{2})^\alpha. 
\]

To get an estimate in the other direction, we first get from (4.1) that
\[
W_{n+1} = W(P_{\text{in}}(n + 1)) + h^\alpha(e_{\text{min}}) + W(P_{\text{out}}(n + 1))
\]
and estimate the lengths of \( P_{\text{in}}(n + 1), e_{\text{min}} \) and \( P_{\text{out}}(n + 1) \) in that order. If \( N_{\text{in}} \) is the number nodes of \( \{X_i\}_{1 \leq i \leq n+1} \) present in the square \( R_{\text{small}}(1) \), then the number of
edges in the in-spanning path $P_{in}(n + 1)$ is $N_{in} - 1 < N_{in}$. As argued before, each such edge has a weight of at most $(c_2 \sqrt{2})^\alpha$ and so $W(P_{in}(n + 1)) \leq N_{in} \cdot (c_2 \sqrt{2})^\alpha$. Similarly, $h^\alpha(e_{min}) \leq (c_2 \sqrt{2})^\alpha$.

We now find bounds for $W(P_{out}(n + 1))$ in terms of TSP$_{n+1}$ as follows. We recall that $P_{out}(n + 1)$ is an out-spanning path with vertex set being the nodes of $\{X_i\}_{1 \leq i \leq n+1}$ outside $R_{small}(1)$. Also $P_{out}(n + 1)$ has $v_{close}$ as an endvertex, where $v_{close}$ is the node closest to but outside $R_{small}(1)$. We therefore obtain an upper bound for $W(P_{out}(n + 1))$ by constructing an out-spanning path $P_{fin}$ with $v_{close}$ as an endvertex, starting from the overall spanning cycle $C_{n+1} = (u_1, \ldots, u_{n+1}, u_1)$ with weight TSP$_{n+1}$.

Let $(u_{j_1}, \ldots, u_{j_2})$, $(u_{j_3}, \ldots, u_{j_4})$, $(u_{j_5}, \ldots, u_{j_6})$ be the subpaths of $C_{n+1}$ that are contained within the square $R_{small}(1)$ in the following sense: For example, in the path $(u_{j_1}, \ldots, u_{j_2})$, the nodes $u_{j_1}$ and $u_{j_2}$ are present outside $R_{small}(1)$ and the rest of the nodes are present inside $R_{small}(1)$. Removing all these subpaths and adding the edges $(u_{j_1}, u_{j_2})$, $(u_{j_3}, u_{j_4})$, $(u_{j_5}, \ldots, u_{j_6})$, we therefore get an out-spanning cycle $C_{temp}$. Further removing an edge containing $v_{close}$ from $C_{temp}$, we get an out-spanning path $P_{fin}$ having $v_{close}$ as an endvertex. Thus, $W(P_{out}(n + 1)) \leq W(P_{fin})$ and it suffices to upper bound $W(P_{fin})$.

Recalling that $N_{in}$ is the number nodes of $\{X_i\}_{1 \leq i \leq n+1}$ present in $R_{small}(1)$, we get that the total number of edges added in the above procedure is at most $s \leq N_{in}$. As argued before, the weight of any edge is at most $(c_2 \sqrt{2})^\alpha$ by (1.2) and so

$$W(P_{out}(n + 1)) \leq W(P_{fin}) \leq \text{TSP}_{n+1} + N_{in} \cdot (c_2 \sqrt{2})^\alpha.$$ 

Combining this with the estimates for $W(P_{in}(n + 1))$ and the weight of the edge $e_{min}$ obtained before, we get

$$W_{n+1} \leq \text{TSP}_{n+1} + (2N_{in} + 1) \cdot (c_2 \sqrt{2})^\alpha.$$ 

and so from (4.4) we get

$$|W_{n+1} - \text{TSP}_{n+1}| \leq (2N_{in} + 2) \cdot (c_2 \sqrt{2})^\alpha.$$ 

Taking expectations, we get

$$\mathbb{E}|W_{n+1} - \text{TSP}_{n+1}|^2 \leq 4 \cdot (c_2 \sqrt{2})^{2\alpha} \cdot \mathbb{E}(N_{in} + 1)^2.$$ 

and we prove below that $\mathbb{E}(N_{in} + 1)^2 \leq 2(\log n)^2$, thus obtaining (4.2).

To estimate $N_{in}$, we write $N_{in} \leq \max_i N(R_i)$, where $N(R_i)$ is the number of nodes of $\{X_i\}_{1 \leq i \leq n+1}$ in the square $R_i$. Any square in $\{R_i\}$ has an area of $\frac{10^4 A^2}{n}$ and so using (1.1) and $A \leq 1 + \frac{1}{\log n} \leq 2$, we get that $N(R_i)$ is Binomially distributed with a mean and a variance of at most $\epsilon_2(n + 1) \frac{10^4 A^2}{n} \leq D_1$ for some constant $D_1 \geq 1$. Therefore, from the proof of the deviation estimate (7.1) in Appendix, we get for $s > 0$ that $\mathbb{E}^s N(R_i) \leq e^{D_1(e^s - 1)}$ and so by Chernoff bound,

$$\mathbb{P}(N(R_i) \geq \log n) \leq e^{-s \log n} e^{D_1(e^s - 1)} \leq \frac{D_2}{n^s}.$$
by setting \( s = 4 \) and \( D_2 = e^{D_1(e^4 - 1)} \). Since there are at most \( \frac{n}{10^4 A^2} \) squares in \( \{ R_i \} \) we further get

\[
P(\max_i N(R_i) \geq \log n) \leq \frac{n}{10^4 A^2} \cdot \frac{D_2}{n^4} \leq \frac{D_3}{n^3}
\]

(4.5)

for some constant \( D_3 > 0 \) and so

\[
\mathbb{E}(N_{in} + 1)^2 = \mathbb{E}(N_{in} + 1)^2 \mathbb{I}(\max_i N(R_i) \leq \log n)
+ \mathbb{E}(N_{in} + 1)^2 \mathbb{I}(\max_i N(R_i) > \log n)
\leq (\log n + 1)^2 + \mathbb{E}(N_{in} + 1)^2 \mathbb{I}(\max_i N(R_i) > \log n).
\]

Using \( N_{in} \leq n + 1 \) and (4.5), we get

\[
\mathbb{E}(N_{in} + 1)^2 \mathbb{I}(\max_i N(R_i) > \log n) \leq (n + 1)^2 \mathbb{P}(\max_i N(R_i) > \log n)
\leq (n + 1)^2 \cdot \frac{D_3}{n^3}
\leq 1
\]

for all \( n \) large. Thus, \( \mathbb{E}(N_{in} + 1)^2 \leq 2(\log n)^2 \), proving the desired estimate. \( \square \)

**Proof of (4.3) in Lemma 6** We perform some preliminary computations. As in the proof of the variance upper bound for \( 1 \leq j \leq n + 1 \), we let \( F_j \) denote the sigma field generated by the nodes \( \{ X_i \}_{1 \leq i \leq j} \) and define the martingale difference \( G_j = \mathbb{E}(W_{n+1}|F_j) - \mathbb{E}(W_{n+1}|F_{j-1}) \) so that

\[
var(W_{n+1}) = \left( \sum_{j=1}^{n+1} \mathbb{E}G_j \right)^2 = \sum_{j=1}^{n+1} \mathbb{E}G_j^2,
\]

(4.6)

by the martingale property. To evaluate \( \mathbb{E}G_j^2 \), we rewrite the martingale difference \( G_j \) in a more convenient form as

\[
G_j = \mathbb{E}(W_{n+1}(X_j) - W_{n+1}(X'_j)|F_j),
\]

(4.7)

where \( X'_j \) is an independent copy of \( X_j \), also independent of \( \{ X_i \}_{1 \leq i \neq j \leq n+1} \) and \( W_{n+1}(X_j) \) and \( W_{n+1}(X'_j) \) are the weights of the approximate TSP paths formed by the nodes \( \{ X_i \}_{1 \leq i \leq n+1} \) and \( \{ X_i \}_{1 \leq i \neq n+1} \cup \{ X'_i \} \). In words, \( W_{n+1}(X_j) - W_{n+1}(X'_j) \) represents the change in the length of the approximate overall TSP path after “replacing” the node \( X_j \) by \( X'_j \).

Below, we define an event \( E_{good}(j) \) with the following properties:
(p1) There exists a constant $D > 0$ not depending on $j$ such that

$$
(W_{n+1}(X_{j}) - W_{n+1}(X'_{j}))\mathbb{I}(E_{good}(j)) \geq D \cdot n^{-\frac{a}{2}} \cdot \mathbb{I}(E_{good}(j)).
$$

(p2) The minimum probability $\min_{2 \leq j \leq n+1} \mathbb{P}(E_{good}(j)) \geq c_0$ for some constant $c_0 > 0$.

Assuming (p1) − (p2), we get from the expression for $G_j$ in (4.7) that

$$
\mathbb{E}|G_j| \geq \mathbb{E}|G_j|\mathbb{I}(E_{good}(j)) = \mathbb{E}G_j\mathbb{I}(E_{good}(j)) \geq Dn^{-\frac{a}{2}}\mathbb{P}(E_{good}(j)) \geq Dn^{-\frac{a}{2}}c_0.
$$

Thus, from (4.6), we have

$$
\text{var}(W_{n+1}) \geq \sum_{j=2}^{n+1} \mathbb{E}G_j^2 \geq \sum_{j=2}^{n+1} (\mathbb{E}|G_j|)^2 \geq D^2c_0^2n^{1-a},
$$

proving (4.3).

Proof of (p1) − (p2): We recall the tiling of the unit square $S$ into small $\frac{400A}{\sqrt{n}} \times \frac{400A}{\sqrt{n}}$ squares $\{R_i\}$. The side length of the smaller square $S_0 \subseteq S$ is $s_0$, a constant and there are $W \geq \frac{1}{2} \cdot \left(\frac{s_0}{\sqrt{n}}A\right)^2$ squares of $\{R_i\}$ completely contained in $S_0$, which we label as $R_1, \ldots, R_W$. For each square $R_i$, we let $R_i^B, R_i^C, R_i^D$ and $R_i^E$ be $\frac{A}{\sqrt{n}} \times \frac{A}{\sqrt{n}}$ subsquares contained in $R_i$, as illustrated in Fig. 4, equidistant from the left and right sides of $R_i$. Say that $R_i$ is a $j$-good square if the $\frac{A}{\sqrt{n}} \times \frac{A}{\sqrt{n}}$ square $R_i^B$ contains a single node of $\{X_k\}_{2 \leq k \leq n+1}$ and the rest of $R_i$ does not contain any node of $\{X_k\}_{2 \leq k \neq j \leq n+1}$. Recalling that $R(1) \in \{R_i\}$ is the random square containing the node $X_1$, we define the event

$$
E_{good}(j) := \{R(1) \in \{R_i\}_{1 \leq i \leq W} \cap \{R(1) \text{ is } j \text{ - good}\} \cap \{X_1 \in R^C(1)\} \cap \{X' \in R^D(1)\} \cap \{X_j \in R^E(1)\}.
$$

Assuming the event $E_{good}(j)$ occurs, we now determine the change in the weight of the approximate overall TSP path upon “replacing” the node $X_j$ with the node $X'$. The square $R(1)$ containing the node $X_1$ is contained within the square $S_0$ and so the weight of any edge with an endvertex in $R(1)$ is simply the Euclidean length of the edge (see statement of the Theorem). Since there are only two nodes $\{X_j, X_1\}$ present in the square $R^{small}(1) \subset R(1)$, the in-spanning path $\mathcal{P}_{in}(n+1)$ is the single edge $(X_j, X_1)$.

Next, the node $v$ present in the subsquare $R^B(1) \subset R(1)$ is the node closest to but outside $R^{small}(1)$ and $v$ is closer to $X_1 \in R^C(1)$ than $X_j \in R^D(1)$. Therefore, $v_{close} = v$ and the minimum length cross-edge $e_{min} = (X_1, v)$. Combining, we get

$$
\mathcal{P}_{app}(n+1) = \mathcal{P}_{app}(n+1, \{X_i\}_{1 \leq i \leq n+1})
$$
By an analogous analysis, we have
\[
\mathcal{P}_{app}(n + 1, \{X_i\}_{1 \leq i \leq n+1} \cup X_j') = \mathcal{P}_{out}(n + 1) \cup \{(X_1, X_j)\} \cup \{(X_1, v)\}. \tag{4.10}
\]

Denoting \( W_{n+1}(X_j) \) and \( W_{n+1}(X_j') \) to be the respective weights of the approximate paths formed by \( \{X_i\}_{1 \leq i \leq n+1} \) and \( \{X_i\}_{1 \leq i \neq j \leq n+1} \cup \{X_j'\} \), we get from (4.9) and (4.10) that
\[
W_{n+1}(X_j)\mathbb{1}(E_{good}(j)) = W_{n+1}(X_j')\mathbb{1}(E_{good}(j))
+ \left( d^\alpha(X_1, X_j) - d^\alpha(X_1, X_j') \right) \mathbb{1}(E_{good}(j)) \tag{4.11}
\]
since \( X_1, X_j \) and \( X_j' \) belong to \( S_0 \) and so as mentioned before, the weight of the edges is simply the corresponding Euclidean lengths raised to the power \( \alpha \).

From Fig. 4, we have that the squares \( R^C(1) \), \( R^D(1) \) and \( R^E(1) \) are spaced 10\( a \) apart. Since the nodes \( X_1 \in R^C(1), X_j' \in R^D(1) \) and \( X_j \in R^E(1) \), we have that \( d(X_1, X_j) \geq d(X_1, X_j') + 10a \) and \( 10a \leq d(X_1, X_j') \leq 14a \). Thus,
\[
d^\alpha(X_1, X_j) = (d(X_1, X_j') + 10a)^\alpha \geq (20a)^\alpha
\]
and so we get from (4.11) that
\[
W_{n+1}(X_j)\mathbb{1}(E_{good}(j)) \geq W_{n+1}(X_j')\mathbb{1}(E_{good}(j))
+ ((20a)^\alpha - (14a)^\alpha)\mathbb{1}(E_{good}(j)) \tag{4.12}
\]

Setting \( a = \frac{A}{\sqrt{n}} \) and using \( A \geq 1 \), we finally get
\[
W_{n+1}(X_j)\mathbb{1}(E_{good}(j)) \geq W_{n+1}(X_j')\mathbb{1}(E_{good}(j))
+ ((20a)^\alpha - (14a)^\alpha) \left( \frac{1}{\sqrt{n}} \right)^\alpha \mathbb{1}(E_{good}(j)) \tag{4.13}
\]

It remains to estimate the probability of the event \( E_{good}(j) \) and we use Poissonization. None of the constants appearing below depend on the index \( j \). As before, we let \( \mathcal{P} \) be a Poisson process with intensity \( nf(\cdot) \) in the unit square with corresponding probability measure \( \mathbb{P}_0 \). A \( \frac{400A}{\sqrt{n}} \times \frac{400A}{\sqrt{n}} \) square in \( R_i \in \{R_k\} \) is good if the \( \frac{A}{\sqrt{n}} \times \frac{A}{\sqrt{n}} \) subsquare \( R_i^B \) contains exactly one node of \( \mathcal{P} \) and the rest of \( R_i \) contains no node of \( \mathcal{P} \). The mean number of nodes of \( \mathcal{P} \) in \( R_i \) is \( n \int_{R_i} f(x)dx \) and \( \mathbb{P}_0(R_i \text{ contains exactly one node of } \mathcal{P}) \) equals
\[
n \int_{R_i} f(x)dx \exp \left( -n \int_{R_i} f(x)dx \right) \geq A^2 \varepsilon_1 e^{-A^2\varepsilon_2} \geq \varepsilon_1 e^{-4\varepsilon_2} \tag{4.14}
\]
where the first inequality in (4.14) follows from the bounds for \( f(\cdot) \) in (1.1) and the second inequality in (4.14) is true since \( 1 \leq A \leq 1 + \frac{1}{\log n} \leq 2 \). Given that \( R_i \) contains a single node of \( \mathcal{P} \), the node is distributed in \( R_i \) with density \( f(\cdot) \) and so is present in the subsquare \( R^i \) with conditional probability \( \frac{\int_{R^i} f(x) \, dx}{\int_{R_i} f(x) \, dx} \geq \frac{\epsilon_1}{(400)^2 \epsilon_2} \), again using the bounds for \( f(\cdot) \) in (1.1), i.e.

\[
\mathbb{P}_0 \left( R_i \text{ is good} \mid R_i \text{ contains exactly one node of } \mathcal{P} \right) = \int_{R^i} f(x) \, dx \geq \frac{\epsilon_1}{(400)^2 \epsilon_2}. \tag{4.15}
\]

Combining (4.15) and (4.14), we get that \( \mathbb{P}_0(R_i \text{ is good}) \geq p_0 \) for some constant \( p_0 > 0 \) not depending on \( i \). The Poisson process is independent on disjoint sets and so \( N_{\text{good}} = \sum_{i=1}^W \mathbb{I}(R_i \text{ is good}) \) is a sum of independent Bernoulli random variables with mean at least \( p_0 \), where we recall that \( \{R_i\}_{1 \leq i \leq W} \) are all the squares contained completely inside the smaller square \( S_0 \). Using the standard deviation estimate (7.2) in Appendix with \( m = W, \mu_1 = p_0 \) and \( \epsilon = \frac{1}{2} \), we therefore have that \( \mathbb{P}_0 \left( N_{\text{good}} \geq \frac{p_0 W}{2} \right) \geq 1 - e^{-2 C_2 n} \) for some positive constants \( C_1 \) and \( C_2 \), since \( W \geq \frac{1}{2} \cdot (\frac{s_0 \sqrt{n}}{400 A})^2 \) (see discussion prior to (4.8)).

Using the dePoissonization formula (2.4), we therefore get that

\[
\mathbb{P} \left( N_{\text{good}} \geq \frac{p_0 W}{2} \right) \geq 1 - C_3 \sqrt{n} e^{-2 C_2 n} \geq 1 - e^{-C_2 n}
\]

where \( N_{\text{good}} \) is the number of \( j \)-good squares (containing exactly one node of \( \{X_k\}_{2 \leq k \neq j \leq n+1} \) contained within \( S_0 \). Given that \( N_{\text{good}} \geq \frac{p_0 W}{2} \geq C_4 n \), where \( C_4 > 0 \) is a constant (see previous paragraph), the nodes \( X_1, X'_j \) and \( X'_j \) are all present in the same \( \frac{400A}{\sqrt{n}} \times \frac{400A}{\sqrt{n}} \) good square \( R_i \subset S_0 \) in the corresponding \( \frac{A}{\sqrt{n}} \times \frac{A}{\sqrt{n}} \) subsquares \( R^C_i, R^D_i \) and \( R^E_i \) with probability at least \( \left( \frac{p_0 W}{2} \epsilon_1 \frac{A^2}{n} \right)^3 \geq p_1 \) for some constant \( p_1 > 0 \), using the bounds for \( f(\cdot) \) in (1.1). From (4.8), we therefore get that \( \mathbb{P}(E_{\text{good}}(j)) \geq p_1 (1 - e^{-C_2 n}) \).

\[ \square \]

5 Convergence of TSP\(_n\)

In this section, we prove the following convergence results for TSP\(_n\) appropriately scaled and centred.

**Theorem 7** (i) If \( 0 < \alpha < 2(\sqrt{2} - 1) \), then

\[
\frac{1}{n^{1-\frac{\alpha}{2}}} (\text{TSP}_n - \mathbb{E}\text{TSP}_n) \longrightarrow 0 \text{ a.s.}
\]
as \( n \to \infty \).

(ii) Suppose the edge weight function \( h \) is a metric and the edge weight exponent is \( 0 < \alpha < 1 \). We have that
\[
\frac{1}{n^{1-\frac{\alpha}{2}}} (\text{TSP}_n - \mathbb{E}\text{TSP}_n) \longrightarrow 0 \text{ a.s.}
\]
as \( n \to \infty \).

The advantage of allowing \( h \) to be a metric is that the TSP length satisfies the monotonicity property if the edge weight exponent is \( \alpha \leq 1 \); in other words, adding more nodes increases the length of the TSP. We then obtain a.s. convergence via a subsequence argument by simply considering an upper bound on the TSP formed by the nodes with indices present outside the subsequence (see Sect. 5).

To construct an example of \( h \) which is a metric, we recall that \( S = [-\frac{1}{2}, \frac{1}{2}]^2 \) is the unit square and define \( h_1 : S \to (0, \infty) \) as \( h_1(u, v) = |(1 + u)^2 - (1 + v)^2| \) for \( u, v \in [-\frac{1}{2}, \frac{1}{2}] \) so that \( h_1 \) is a metric. Also,
\[
|u - v| \leq h_1(u, v) = |u - v|(2 + u + v) \leq 3|u - v|.
\] (5.1)

For \( x = (x_1, x_2) \), \( y = (y_1, y_2) \), define
\[
h(x, y) = h_1(x_1, y_1) + h_1(x_2, y_2)
\]
so that \( h \) is a metric in \( S \) and using (5.1) we also get
\[
d(x, y) \leq |x_1 - y_1| + |x_2 - y_2| \leq h(x, y) \\
\leq 3(|x_1 - y_1| + |x_2 - y_2|) \leq 3\sqrt{2}d(x, y).
\]

To prove Theorem 7, we have the following preliminary lemma. We recall that \( \text{TSP}_k \) denotes the length of the TSP cycle formed by the nodes \( \{X_i\}_{1 \leq i \leq k} \) as defined in (1.4).
For a constant $M > 1$, we let

$$J_n = J_n(M) := \max_{nM \leq k < (n+1)M} |TSP_k - TSP_{nM}|$$

(5.2)

and have the following result.

**Lemma 8** There is a constant $D > 0$ such that

$$\mathbb{E} \left( \frac{J_n}{nM^{(1-\frac{\alpha}{2})}} \right) \leq \frac{D}{n^{1-\frac{\alpha}{2}}} \quad \text{and} \quad \mathbb{E} \left( \frac{J_n}{nM^{(1-\frac{\alpha}{2})}} \right)^2 \leq \frac{D}{n^{2-\alpha}},$$

(5.3)

where $y = \max \left( \frac{M\alpha}{2}, \alpha \right)$. If the edge weight function $h$ is a metric and $0 < \alpha \leq 1$, then

$$\mathbb{E} \left( \frac{J_n}{nM^{(1-\frac{\alpha}{2})}} \right) \leq \frac{D}{n^{1-\frac{\alpha}{2}}} \quad \text{and} \quad \mathbb{E} \left( \frac{J_n}{nM^{(1-\frac{\alpha}{2})}} \right)^2 \leq \frac{D}{n^{2-\alpha}}.$$  

(5.4)

**Proof of (5.3) in Lemma 8** From the partial one node estimate (3.1), we have that $TSP_l \leq TSP_{l+1} + g_{1,l}$, where $g_{1,l} \geq 0$ is such that

$$\max \left( \mathbb{E}g_{1,l}, \mathbb{E}g_{1,l}^2 \right) \leq \frac{D}{l^{\frac{1}{2}}}$$

(5.5)

for some constant $D_1 > 0$, using estimates (3.6) and (3.7). For $nM + 1 \leq k < (n+1)M$, we therefore have

$$TSP_{nM} \leq TSP_k + \sum_{l=nM}^{k} g_{1,l}.$$  

To get an estimate in the reverse direction, we use relation (7.4) in Appendix and get that

$$TSP_k \leq TSP_{nM} + TSP(X_{nM+1}, \ldots, X_k) + (c_2\sqrt{2})^\alpha.$$  

Combining, we get $J_n \leq J_{n,1} + J_{n,2} + (c_2\sqrt{2})^\alpha$ where

$$J_{n,1} := \sum_{l=nM}^{(n+1)M} g_{1,l} \quad \text{and} \quad J_{n,2} := \max_{nM + 1 \leq k < (n+1)M} TSP(X_{nM+1}, \ldots, X_k).$$

We evaluate $J_{n,1}$ and $J_{n,2}$ separately. Using (5.5), we have that

$$\mathbb{E}J_{n,1} \leq \sum_{l=nM}^{(n+1)M} \frac{D}{l^{\frac{1}{2}}} \leq ((n+1)^M - n^M) \cdot \frac{D}{n^{\frac{M}{2}}} \leq \frac{2MD}{n} \cdot n^{M(1-\frac{\alpha}{2})}$$

(5.6)
since
\[(n + 1)^M - n^M \leq M(n + 1)^{M-1} \leq 2Mn^{M-1} \tag{5.7}\]
for all \(n \) large. Similarly, using \((\sum_{i=1}^{t} a_i)^2 \leq t \sum_{i=1}^{t} a_i^2\), we get
\[
\mathbb{E}J_{n,1}^2 \leq ((n + 1)^M - n^M) \sum_{l=n^M}^{(n+1)^M} \mathbb{E}g_{1,l}^2 \leq ((n + 1)^M - n^M) \sum_{l=n^M}^{(n+1)^M} \frac{D}{l^{\alpha^2}},
\]
using (5.5). Again using (5.7), we then get that

\[
\mathbb{E}J_{n,1}^2 \leq D_1 n^{2M-2-M\frac{\alpha}{2}} \tag{5.8}
\]
for some constant \(D_1 > 0\).

To estimate \(J_{n,2}\), we let \(L_k := \text{TSP}(X_{n^M+1}, \ldots, X_k)\) so that

\[L_k \leq (k - n^M - 1) \cdot (c_2 \sqrt{2})^\alpha \text{ by (1.2)}\] since there are \(k - n^M - 1\) edges each of length at most \(\sqrt{2}\).

Thus,

\[J_{n,2} \leq ((n + 1)^M - n^M) \cdot (c_2 \sqrt{2})^\alpha \leq Dn^{M-1}\tag{5.9}\]
for some constant \(D > 0\) using (5.7) and if \(x = (M - 1) \left(1 - \frac{\alpha}{2}\right)\), then

\[
\max_{n^M+1 \leq k < n^M+1+n^x} L_k \leq D_1 n^x = D_1 n^{(M-1)(1-\frac{\alpha}{2})}, \tag{5.10}
\]
for some constant \(D_1 > 0\). To bound \(L_k\) for the remaining range of \(k\), we use the deviation estimate in Theorem 1. For \(\Delta > 0\) and \(n^M + 1 + n^x \leq k \leq (n + 1)^M\), we have from (1.7) that

\[
\mathbb{P}(L_k \leq D_1(k - n^M - 1)^{1-\frac{\alpha}{2}}) \geq 1 - \frac{1}{(k - n^M - 1)^\Delta} \geq 1 - \frac{1}{n^x \Delta}
\]
for some constant \(D_1 > 0\). In other words, we have that

\[L_k \leq D_1((n + 1)^M - n^M)^{(1-\frac{\alpha}{2})} \leq D_2 n^{(M-1)(1-\frac{\alpha}{2})} \tag{5.11}\]
for some constant \(D_2 > 0\), with probability at least \(1 - \frac{1}{n^x \Delta}\), where the final estimate in (5.11) follows from (5.7). Setting \(D_3 = \max(D_1, D_2)\), where \(D_1\) and \(D_2\) are as in (5.10) and (5.11), respectively, we therefore get that

\[
\mathbb{P}(J_{n,2} \leq D_3 n^{(M-1)(1-\frac{\alpha}{2})}) \geq 1 - \frac{1}{n^x \Delta}((n + 1)^M - n^M)
\geq 1 - \frac{2Mn^{M-1}}{n^x \Delta}. \tag{5.12}
\]
By (5.7).

From (5.12) and (5.9), we get

$$
\mathbb{E} J_{n, 2} \leq Dn^{(M-1)(1-\frac{\alpha}{2})} + Dn^{M-1} \cdot \frac{Dn^{M-1}}{n^\Delta} \leq 2 Dn^{(M-1)(1-\frac{\alpha}{2})},
$$

(5.13)

provided \( \Delta > 0 \) large. Choosing \( \Delta > 0 \) larger if necessary and arguing as above, we also get that

$$
\mathbb{E} J_{n, 2}^2 \leq 4 D^2 n^{(M-1)(2-\alpha)}. \quad (5.14)
$$

From (5.6) and (5.13), we get the desired bound for \( \mathbb{E} J_n \) in (5.3). Also, using \((a+b)^2 \leq 2(a^2 + b^2)\) and (5.8) and (5.14), we get the desired bound for \( \mathbb{E} J_n^2 \) in (5.3). \( \square \)

**Proof of (5.4) in Lemma 8** From (7.4) and the monotonicity relation (7.5), we get that

$$
J_n \leq TSP(X_{n \alpha+1}, \ldots, X_{(n+1)\alpha})
$$

and so

$$
\mathbb{E} J_n^2 \leq \mathbb{E} TSP^2(X_{n \alpha+1}, \ldots, X_{(n+1)\alpha}) \leq D_1 ((n+1)M - n^M)^{2-\alpha} \leq D_2 n^{(M-1)(2-\alpha)}
$$

(5.15)

for some constants \( D_1, D_2 > 0 \), where the second inequality in (5.15) follows from the expectation upper bound for \( TSP^k(\cdot) \) in (1.8) with \( k = 2 \) and the final inequality in (5.15) is true by (5.7). Thus,

$$
\left( \frac{\mathbb{E} J_n}{n^{M(1-\frac{\alpha}{2})}} \right)^2 \leq \frac{\mathbb{E} J_n^2}{n^{2M-M \alpha}} \leq \frac{D_2}{n^{2-\alpha}},
$$

proving (5.4). \( \square \)

**Proof** (Proof of Theorem 7(i)) We prove a.s. convergence via a subsequence argument using the variance upper bound for \( TSP_n \) obtained in Theorem 3. Letting \( \epsilon_0 := \frac{(2-\alpha)(1+\alpha)}{2+\alpha} \), we use \( \alpha < 2(\sqrt{2} - 1) < 1 \) to get that \( \frac{2}{\alpha} > \frac{2+\alpha}{2-\alpha} = \frac{1}{2-\alpha-\epsilon_0} \). We therefore choose \( M > 1 \) such that \( \frac{2}{\alpha} > M > \frac{1}{2-\alpha-\epsilon_0} \) and let \( \epsilon > \epsilon_0 \) be sufficiently close to \( \epsilon_0 \) so that \( \frac{2}{\alpha} > M > \frac{1}{2-\alpha-\epsilon} \). Fixing such \( M \) and \( \epsilon \), we also have that \( y = \max(M \frac{\alpha}{2}, \alpha) < 1 \).

For simplicity, we treat \( n^M \) as an integer for all \( n \) and work the subsequence \( \{n^M\} \).

From the variance estimate in Theorem 3 (i), we have that \( \text{var} \left( \frac{TSP_n}{n^{\frac{\alpha}{2}}} \right) \leq \frac{D}{n^{2-\alpha-\epsilon}} \)
for some constant \( D > 0 \) and all \( n \) large. Since \( M(2-\alpha-\epsilon) > 1 \), we get by an application of the Borel–Cantelli lemma that

$$
\frac{1}{n^M(1-\frac{\alpha}{2})} (TSP_n^M - \mathbb{E} TSP_n^M) \longrightarrow 0 \quad a.s.
$$

\( \square \) Springer
as \( n \to \infty \). To prove convergence along the subsequence \( a_n = n \), we let \( J_n \) be as in (5.2) and obtain from (5.3) that

\[
\left( \frac{\mathbb{E} J_n}{n^{M(1-\frac{\alpha}{2})}} \right)^2 \leq \frac{\mathbb{E} J_n^2}{n^{M(2-\alpha) \frac{y}{2}}} \leq \frac{D_1}{n^{2-\gamma}} \to 0
\]

as \( n \to \infty \) and since \( y < 1 \) (see the first paragraph of this proof), we also get from Borel–Cantelli lemma that \( \frac{J_n}{n^{(1-\frac{\alpha}{2})}} \to 0 \) a.s. as \( n \to \infty \). For \( n^M \leq k < (n+1)^M \), we then write

\[
|TSP_k - \mathbb{E}TSP_k| \leq \frac{|TSP_k - TSP_{nM}|}{k^{1-\frac{\alpha}{2}}} + \frac{\mathbb{E}|TSP_k - TSP_{nM}|}{k^{1-\frac{\alpha}{2}}}
\]

\[
\leq \frac{J_n}{k^{1-\frac{\alpha}{2}}} + \frac{\mathbb{E}J_n}{k^{1-\frac{\alpha}{2}}}
\]

\[
\leq \frac{J_n}{n^{M(1-\frac{\alpha}{2})}} + \frac{\mathbb{E}J_n}{n^{M(1-\frac{\alpha}{2})}}
\]

and get that \( \frac{TSP_k - \mathbb{E}TSP_k}{k^{1-\frac{\alpha}{2}}} \to 0 \) a.s. as \( n \to \infty \). \( \square \)

**Proof** (Proof of Theorem 7 (iii)) Here, we prove a.s. convergence for \( 0 < \alpha < 1 \) assuming that the edge weight function \( h \) is a metric. The proof is analogous as before with some modifications. Let \( 2-\alpha > \epsilon > (2-\alpha) \cdot \frac{1+\alpha}{2+\alpha} \). From the variance estimate in Theorem 3 (i), we have that

\[
\text{var} \left( \frac{TSP_n}{n^{1-\frac{\alpha}{2}}} \right) \leq \frac{D}{n^{2-\alpha-\epsilon}}
\]

for some constant \( D > 0 \) and all \( n \) large. Letting \( M > 1 \) be large so that \( M(2-\alpha-\epsilon) > 1 \), we get by an application of the Borel–Cantelli lemma that

\[
\frac{1}{n^{M(1-\frac{\alpha}{2})}} (TSP_{nM} - \mathbb{E}TSP_{nM}) \to 0 \text{ a.s.}
\]

as \( n \to \infty \).

To prove a.s. convergence along the sequence \( a_n = n \), we let \( J_n \) be as in (5.2) and get from (5.4) in property (p2) above that

\[
\left( \frac{\mathbb{E} J_n}{n^{M(1-\frac{\alpha}{2})}} \right)^2 \leq \frac{\mathbb{E} J_n^2}{n^{2M-M\alpha}} \leq \frac{D}{n^{2-\alpha}} \to 0
\]

as \( n \to \infty \), for some constant \( D > 0 \). Thus, \( \frac{\mathbb{E} J_n}{n^{M(1-\frac{\alpha}{2})}} \to 0 \) as \( n \to \infty \) and since \( \alpha < 1 \) we get from Borel–Cantelli lemma and (5.17) that

\[
\frac{J_n}{n^{M(1-\frac{\alpha}{2})}} \to 0 \text{ a.s.}
\]
as \( n \to \infty \). Arguing as in (5.16), we then get the desired a.s. convergence. \( \square \)

6 Uniform TSPs

In this section, we assume that the nodes \( \{X_i\}_{1 \leq i \leq n} \) are uniformly distributed in the unit square and obtain bounds on the asymptotic values of the expected weight, appropriately scaled and centred. We assume that the positive edge weight function \( h : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \) satisfies (1.2) along with the following two properties:

(b1) For every \( a > 0 \), we have

\[
h(au, av) = a \cdot h(u, v) \quad \text{for all } u, v \in \mathbb{R}^2.
\]

(b2) There exists a constant \( h_0 > 0 \) such that for all \( b \in \mathbb{R}^2 \) we have

\[
h(b + u, b + v) \leq h_0 \cdot h(u, v).
\]

(6.1)

For example, recalling that \( d(u, v) \) denotes the Euclidean distance between \( u \) and \( v \), we have that the function

\[
h(u, v) = d(u, v) + \frac{1}{2} |d(u, 0) - d(v, 0)|
\]

is a metric since

\[
d(u, 0) \leq d(0, v) + d(v, u) \quad \text{and} \quad d(v, 0) \leq d(0, u) + d(u, v)
\]

by triangle inequality and so

\[
d(u, v) \leq h(u, v) \leq d(u, v) + \frac{1}{2} d(u, v).
\]

This implies that \( h(u, v) \) satisfies (1.2) and by definition \( h \) also satisfies (b1). Moreover, using the triangle inequality we have

\[
h(b + u, b + v) = d(b + u, b + v) + \frac{1}{2} |d(u + b, 0) - d(v + b, 0)|
\]

\[
= d(u, v) + \frac{1}{2} |d(u + b, 0) - d(v + b, 0)|
\]

\[
\leq d(u, v) + \frac{1}{2} d(u + b, v + b)
\]

\[
= d(u, v) + \frac{1}{2} d(u, v)
\]

\[
\leq \frac{3}{2} h(u, v)
\]

by definition of \( h \) in (6.2). Thus, (b2) is also satisfied with \( h_0 = \frac{3}{2} \).

\( \square \) Springer
We have the following result.

**Theorem 9** Suppose the distribution function \( f(\cdot) \) is uniform, i.e. \( \epsilon_1 = \epsilon_2 = 1 \) in (1.1) and the edge weight function \( h(u,v) \) satisfies (1.2) and properties (b1) – (b2) above. If the edge weight exponent is \( 0 < \alpha < 2 \), then

\[
0 < \liminf_n \frac{\mathbb{E}TSP_n}{n^{1-\frac{\alpha}{2}}} \leq \limsup_n \frac{\mathbb{E}TSP_n}{n^{1-\frac{\alpha}{2}}} \leq h_0^\alpha \cdot \liminf_n \frac{\mathbb{E}TSP_n}{n^{1-\frac{\alpha}{2}}} < \infty. \tag{6.3}
\]

Thus, the scaled weight of the TSP remains bounded within a factor of \( h_0^\alpha \).

Letting

\[
J_n = \max_{n^2 \leq k < (n+1)^2} |TSP_k - TSP_{n^2}|
\]

be as in (5.2) with \( M = 2 \), we get for \( n^2 \leq k < (n+1)^2 \) that

\[
\frac{\mathbb{E}TSP_k}{k^{1-\frac{\alpha}{2}}} \leq \frac{\mathbb{E}TSP_k}{n^{2-\alpha}} \leq \frac{\mathbb{E}TSP_{n^2}}{n^{2-\alpha}} + \frac{\mathbb{E}J_n}{n^{2-\alpha}}
\]

and

\[
\frac{\mathbb{E}TSP_k}{k^{1-\frac{\alpha}{2}}} \geq \frac{\mathbb{E}TSP_k}{(n+1)^{2-\alpha}} \geq \frac{\mathbb{E}TSP_{n^{2M}}}{(n+1)^{2-\alpha}} - \frac{\mathbb{E}J_n}{(n+1)^{2-\alpha}}.
\]

Using the estimate (5.3) with \( M = 2 \), we have \( \frac{\mathbb{E}J_n}{n^{2-\alpha}} \to 0 \) as \( n \to \infty \) and so we get

\[
\limsup_n \frac{\mathbb{E}TSP_n}{n^{1-\frac{\alpha}{2}}} = \limsup_n \frac{\mathbb{E}TSP_{n^2}}{n^{2-\alpha}}
\]

and

\[
\liminf_n \frac{\mathbb{E}TSP_n}{n^{1-\frac{\alpha}{2}}} = \liminf_n \frac{\mathbb{E}TSP_{n^2}}{n^{2-\alpha}}.
\]

In what follows therefore, we work with the sequence \( a_n = n^2 \).

We begin with the following preliminary Lemma.

**Lemma 10** There is a constant \( D > 0 \) such that for any positive integers \( n_1, n_2 \geq 1 \) we have

\[
\mathbb{E}TSP_{n_1+n_2} \leq \mathbb{E}TSP_{n_1} + Dn_2^{1-\frac{\alpha}{2}} + 2(\sqrt{2})^\alpha. \tag{6.4}
\]

Also for any fixed integer \( m \geq 1 \) we have

\[
\limsup_n \frac{\mathbb{E}TSP_{n^2}}{n^{2-\alpha}} \leq \limsup_k \frac{\mathbb{E}TSP_{k^2m^2}}{(km)^{2-\alpha}}. \tag{6.5}
\]
Proof of (6.4) in Lemma 10 From the additive relation (7.4) in Appendix, we have
\[ \text{TSP}_{n_1+n_2} \leq \text{TSP}_{n_1} + \text{TSP}(X_{n_1+1}, \ldots, X_{n_1+n_2}) + 2(\sqrt{2})^{\alpha} \]
and using the TSP expectation upper bound in (1.8), the middle term is bounded above by \( Dn_2^{1-\alpha/2} \) for some constant \( D > 0 \).

Proof of (6.5) in Lemma 10 Fix an integer \( m \geq 1 \) and write \( n =qm + s \) where \( q = q(n) \geq 1 \) and \( 0 \leq s = s(n) \leq m - 1 \) are integers. As \( n \to \infty \),
\[ q(n) \to \infty \quad \text{and} \quad \frac{n}{q(n)} \to m. \quad (6.6) \]
Also using \( s < m \), we have \( qm + s < (q + 1)m \) and so \( q_1 := (qm + s)^2 - (qm)^2 \) is bounded above by
\[ q_1 \leq m^2((q + 1)^2 - q^2) \leq 4m^2 q. \quad (6.7) \]
Using property \((b1)\), we therefore have that \( \mathbb{E}\text{TSP}_n = \mathbb{E}\text{TSP}_{(qm+s)^2} \) equals
\[ \mathbb{E}\text{TSP}_{(qm)^2+q_1} \leq \mathbb{E}\text{TSP}_{(qm)^2} + Dq_1^{1-\alpha/2} + 2(\sqrt{2})^{\alpha} \]
for some constant \( D > 0 \). From (6.7), we have \( Dq_1^{1-\alpha/2} \leq \frac{D_1}{q^{1-\alpha/2}} \cdot (qm)^{2-\alpha} \) for some constant \( D_1 > 0 \) and so
\[ \limsup_n \frac{\mathbb{E}\text{TSP}_n}{n^{2-\alpha}} \leq \limsup_n \left( \frac{q_m}{n} \right)^{2-\alpha} \frac{\mathbb{E}\text{TSP}_{(qm)^2}}{(qm)^{2-\alpha}} + \limsup_n \left( \frac{q_m}{n} \right)^{2-\alpha} \left( \frac{D_1}{q^{1-\alpha/2}} + \frac{2(\sqrt{2})^{\alpha}}{(qm)^{2-\alpha}} \right). \quad (6.8) \]
Since \( q(n) \to \infty \) and \( \frac{n}{q(n)} \to m \) (see (6.6)), \( m \) is fixed and \( 0 < \alpha < 2 \), the second term in (6.8) is zero and the first term in (6.8) equals \( \limsup_n \frac{\mathbb{E}\text{TSP}_{(qm)^2}}{(qm)^{2-\alpha}} \).

But since \( q(n) \geq \frac{n-m}{m} \geq \frac{l-m}{m} \) for \( n \geq l \), we have that
\[ \sup_{n \geq l} \frac{\mathbb{E}\text{TSP}_{q^2m^2}}{(qm)^{2-\alpha}} = \sup_{n \geq l} \frac{\mathbb{E}\text{TSP}_{q^2(n)m^2}}{(q(n)m)^{2-\alpha}} \leq \sup_{k \geq \frac{l-m}{m}} \frac{\mathbb{E}\text{TSP}_{k^2m^2}}{(km)^{2-\alpha}} \quad (6.9) \]
and as \( l \uparrow \infty \), the final term in (6.9) converges to the second term in (6.5). \( \square \)

If \( \lambda := \liminf_n \frac{\mathbb{E}\text{TSP}_n}{n^{2-\alpha}} \), then from (1.8), we have that \( \lambda > 0 \). We show below that for every \( \epsilon > 0 \), there exists \( m \) sufficiently large such that
\[ \limsup_k \frac{\mathbb{E}\text{TSP}_{k^2m^2}}{(km)^{2-\alpha}} \leq h_0^\alpha \cdot \lambda + \epsilon. \quad (6.10) \]
Since $\epsilon > 0$ is arbitrary, the desired bound in Theorem 9 follows from (6.5). In the rest of this section, we prove (6.10) via a series of Lemmas.

Let $k$ and $m$ be positive integers and distribute $k^2m^2$ nodes $\{X_i\}_{1 \leq i \leq k^2m^2}$ independently and uniformly in the unit square $S$. Also divide $S$ into $k^2$ disjoint squares $\{W_j\}_{1 \leq j \leq k^2}$ each of size $\frac{1}{k} \times \frac{1}{k}$ as in Fig. 3 so that the top left most square is labelled $W_1$, the square below $W_1$ is $W_2$ and so on until we reach the square $W_k$ intersecting the bottom edge of the unit square $S$. The square to the right of $W_k$ is then labelled $W_{k+1}$ and the square above $W_{k+1}$ is $W_{k+2}$ and so on.

The next result obtains an upper bound for $TSP_{k^2m^2}$ in terms of the local TSPs of the squares $W_j$. For $1 \leq j \leq k^2$, let $N(j)$ be the number of nodes of $\{X_i\}$ present in the square $W_j$ and let $TSP_j(N(j))$ be the TSP length of the nodes present in $W_j$. We have the following lemma.

**Lemma 11** There exists a constant $D > 0$ not depending on $k$ or $m$ such that

$$\mathbb{E}TSP_{k^2m^2} \leq \sum_{j=1}^{k^2} \mathbb{E}TSP_j(N(j)) + D \cdot k^{2-\alpha} + (c_2\sqrt{2})^\alpha,$$  \hspace{1cm} (6.11)

where $c_2$ is as in (1.2).

**Proof of Lemma 11** We first connect all nodes within each square $W_j$ by a path of minimum weight to get local minimum weight paths. We then join all these paths together to get an overall spanning path containing all the $k^2m^2$ nodes. Joining the endvertices of this new path by an edge, we get a spanning cycle whose weight is at least $TSP_{k^2m^2}$.

For $1 \leq j \leq k^2$, let $P(j)$ and $C(j)$, respectively, denote the minimum spanning path and minimum spanning cycle, containing all the $N(j)$ nodes. If $N(j) = 0$ we set $P(j) = \emptyset$ and if $N(j) \leq 2$, we set $P(j) = C(j)$. For any $1 \leq j \leq k^2m^2$ we first show that

$$W(P(j)) \leq W(C(j)) \leq W(P(j)) + \left(\frac{c_2\sqrt{2}}{k}\right)^\alpha.$$  \hspace{1cm} (6.12)

To see that (6.12) is true, let $C$ be any spanning cycle containing all the nodes in the square $W_j$. Removing an edge of $C$ we get a spanning path $P$ and so $W(P(j)) \leq W(P) \leq W(C)$. Taking minimum over all spanning cycles we obtain the first relation in (6.12). Taking the minimum spanning path $P(j)$ with endvertices $u$, $v$ and adding the edge $(u, v)$, we obtain a spanning cycle $C_{new}$ and since the length of the edge $(u, v)$ is at most $\frac{\sqrt{2}}{k}$ we use (1.2) to get that

$$W(C(j)) \leq W(C_{new}) \leq W(P(j)) + \left(\frac{c_2\sqrt{2}}{k}\right)^\alpha,$$

proving (6.12).
For $1 \leq j \leq k^2 - 1$, let $j + T_{j}^{\text{next}} := \min\{i \geq j + 1 : W_i \text{ is not empty}\}$ be the next nonempty square after $W_j$, containing at least one node of $\{X_i\}_{1 \leq i \leq k^2 m^2}$. If no such square exists, set $j + T_{j}^{\text{next}} = k^2$. If both $W_j$ and $W_{j + T_{j}^{\text{next}}}$ are nonempty, let $e_j$ be the edge with one endvertex in $W_j$ and the other endvertex in $W_{j + T_{j}^{\text{next}}}$, having the smallest Euclidean length $d(e_j)$ and set $e_j = \emptyset$ and $d(e_j) = 0$, otherwise. The union

$$
\mathcal{P}_{ov} := \bigcup_{1 \leq j \leq k^2} \mathcal{P}(j) \cup \bigcup_{1 \leq j \leq k^2 - 1} \{e_j\}
$$

is a spanning path containing all the $k^2 m^2$ nodes and joining the endvertices of $\mathcal{P}_{ov}$ (by an edge of length at most $\sqrt{2}$), we again use (1.2) to get

$$
\text{TSP}_{k^2 m^2} \leq W(\mathcal{P}_{ov}) + (c_2 \sqrt{2})^\alpha
$$

$$
= \sum_{j=1}^{k^2} W(\mathcal{P}(j)) + \sum_{j=1}^{k^2-1} h^\alpha(e_j) + (c_2 \sqrt{2})^\alpha
$$

$$
\leq \sum_{j=1}^{k^2} W(\mathcal{P}(j)) + c_2^\alpha \sum_{j=1}^{k^2-1} d^\alpha(e_j) + (c_2 \sqrt{2})^\alpha
$$

$$
\leq \sum_{j=1}^{k^2} W(\mathcal{C}(j)) + c_2^\alpha \sum_{j=1}^{k^2-1} d^\alpha(e_j) + (\sqrt{2})^\alpha,
$$

(6.13)

using (6.12).

Taking expectations in (6.13) and recalling that $\text{TSP}_j(N(j)) := W(\mathcal{C}(j))$, we get

$$
\mathbb{E}\text{TSP}_{k^2 m^2} \leq \sum_{j=1}^{k^2} \mathbb{E}\text{TSP}_j(N(j)) + c_2^\alpha \cdot k^2 \max_{1 \leq j \leq k^2 - 1} \mathbb{E}d^\alpha(e_j) + (c_2 \sqrt{2})^\alpha. \quad (6.14)
$$

We show that there exists a constant $D > 0$ not depending on $k$ or $m$ such that

$$
\max_{1 \leq j \leq k^2 - 1} \mathbb{E}d^\alpha(e_j) \leq \frac{D}{k^\alpha}
$$

(6.15)

and this proves Lemma 11. Indeed, the Euclidean length of the edge $e_j$ is at most $\frac{2T^{\text{next}}_{j} \sqrt{2}}{k^2}$ and so $\mathbb{E}d^\alpha(e_j) \leq \frac{(2 \sqrt{2} \alpha)}{k^\alpha} \mathbb{E}(T^{\text{next}}_{j})^\alpha$. Next, for any $l \geq 1$ we have $T_{j}^{\text{next}} > l$ if and only if $W_{j+1}, \ldots, W_{j+l}$ are empty, which happens with probability $\left(1 - l \cdot \frac{1}{k^2}\right)^{k^2 m^2} \leq e^{-lm^2} \leq e^{-l}$, since $m \geq 1$. Thus,

$$
\mathbb{E}(T^{\text{next}}_{j})^\alpha \leq \mathbb{E}(T^{\text{next}}_{j})^{2} \leq \sum_{l \geq 1} l \mathbb{P}\left(T^{\text{next}}_{j} \geq l\right) \leq \sum_{l \geq 1} le^{-(l-1)} \leq \frac{1}{(1 - e^{-1})^2},
$$

proving (6.15). \qed
Lemma 11 is useful in the following manner. For $1 \leq j \leq k^2$ let TSP$(N(j))$ be the shifted TSP of the nodes present in $W_j$ defined as follows: If $v_1, \ldots, v_t$ are the nodes of $\{X_i\}$ present in the square $W_j$ with centre $s_j$, then TSP$(N(j))$ is the TSP formed by the nodes $v_1 - s_j, \ldots, v_t - s_j$. From the translation relation (7.11) in Appendix, we have TSP$_j(N(j)) \leq h_0^2 \cdot$ TSP$(N(j))$ and so we get from Lemma 11 that

\[
\mathbb{E}\text{TSP}_{k^2m^2} \leq h_0^2 \cdot \sum_{j=1}^{k^2} \mathbb{E}\text{TSP}(N(j)) + D \cdot k^{2-\alpha} + (c_2\sqrt{2})^\alpha
\]

To evaluate TSP$(N(1))$, we let $0 < \gamma < \frac{1}{2}$ and $(\frac{2-\alpha}{2+\alpha}) < \epsilon_1 < 2 - \alpha$ be constants (not depending on $k$ or $m$) such that

\[2 - \alpha + 2\gamma - 2\epsilon_1 > 0 \quad (6.16)\]

This is possible since $2 (\frac{2-\alpha}{2+\alpha}) - (2 - \alpha) = \frac{\alpha(2-\alpha)}{2+\alpha} < 1$ for all $0 < \alpha < 2$ and so we choose $\epsilon_1$ greater than but sufficiently close to $(\frac{2-\alpha}{2+\alpha})$ and $2\gamma$ less than but sufficiently close to one so that (6.16) holds.

We have the following lemma.

**Lemma 12** There is a constant $D > 0$ not depending on $k$ or $m$ such that

\[
\mathbb{E}\text{TSP}(N(1)) \leq \frac{1}{k^\alpha} \left( \mathbb{E}\text{TSP}_{m^2} + Dm^2(\epsilon_1 - \gamma) + D \cdot \frac{m^{2-\alpha}}{m^{\alpha(1-2\gamma)}} \right). \quad (6.17)
\]

**Proof Lemma 12** We write $\mathbb{E}\text{TSP}(N(1)) = I_1 + I_2$, where

\[
I_1 = \mathbb{E}\text{TSP}(N(1))\mathbb{I}(F_1), \quad I_2 = \mathbb{E}\text{TSP}(N(1))\mathbb{I}(F_1^c)
\]

and $F_1 := \{m^2 \left( 1 - \frac{1}{m^{2\gamma}} \right) \leq N(1) \leq m^2 \left( 1 + \frac{1}{m^{2\gamma}} \right) \}$. Each node $X_i$, $1 \leq i \leq k^2m^2$ has a probability $\frac{1}{k^2}$ of being present in the $\frac{1}{k} \times \frac{1}{k}$ square $W_1$. Therefore, the number of nodes $N(1)$ in the square $W_1$ is binomially distributed with mean $\mathbb{E}N(1) = m^2$ and $\text{var}(N_1) \leq m^2$. We therefore get from Chebyshev’s inequality that

\[
\mathbb{P}(F_1^c) \leq \frac{1}{m^{2(1-2\gamma)}}, \quad (6.18)
\]

**Evaluation of $I_1$:** We write

\[
I_1 = \sum_{j=\text{low}}^{j_{\text{up}}} \mathbb{E}\text{TSP}(N(1))\mathbb{I}(N(1) = j).
\]
where \( j_{\text{low}} \coloneqq m^2 \left(1 - \frac{1}{m^2} \right) \leq m^2 \left(1 + \frac{1}{m^2} \right) =: j_{\text{up}} \). Given \( N_1 = j \), the nodes in \( W_1 \) are independently and uniformly distributed in \( W_1 \) and we let \( \mathbb{E}_{\text{TSP}} \left( j; \frac{1}{k} \right) \) be the expected length of the TSP containing \( j \) nodes independently and uniformly distributed in the \( \frac{1}{k} \times \frac{1}{k} \) square centred at origin. From the scaling relation (7.10) in Appendix, we have

\[
I_1 = \sum_{j=j_{\text{low}}}^{j_{\text{up}}} \mathbb{E}_{\text{TSP}} \left( j; \frac{1}{k} \right) \mathbb{P}(N(1) = j)
\]

by (7.10).

Letting \( 2 - \alpha > \epsilon_1 > \frac{(2-\alpha)(1+\alpha)}{2+\alpha} \) be as in (6.16), we have from the one node difference estimate (3.9) that for any integers \( j_1 \) and \( j_2 \) lying between \( j_{\text{low}} \) and \( j_{\text{up}} \), the term \( \mathbb{E} |\text{TSP}_{j_2} - \text{TSP}_{j_1}| \) is bounded above by

\[
\sum_{u=j_{\text{low}}}^{j_{\text{up}}-1} \mathbb{E} |\text{TSP}_{u+1} - \text{TSP}_{u}| \leq \sum_{u=j_{\text{low}}}^{j_{\text{up}}-1} \frac{D_1}{u^{1-\epsilon_1}} \leq \frac{D_2(j_{\text{up}} - j_{\text{low}})}{m^{2(1-\epsilon_1)}} = 2D_2 \frac{m^{2(1-\gamma)}}{m^{2(1-\epsilon_1)}},
\]

where \( D_1, D_2 > 0 \) are constants not depending on \( j_1 \) or \( j_2 \). Setting \( j_1 = m^2 \) and \( j_2 = j \) and using (6.20) we get \( \mathbb{E}_{\text{TSP}} j \leq \mathbb{E}_{\text{TSP}} m^2 + D_2 m^{2(\epsilon_1-\gamma)} \) for all \( j_{\text{low}} \leq j \leq j_{\text{up}} \).

From the expression for \( I_1 \) in (6.19), we therefore have that

\[
I_1 \leq \frac{1}{k^\alpha} \left( \mathbb{E}_{\text{TSP}} m^2 + D_2 m^{2(\epsilon_1-\gamma)} \right).
\]

**Evaluation of \( I_2 \):** There are \( N(1) \) nodes in the \( \frac{1}{k} \times \frac{1}{k} \) square \( W_1 \) and given \( N(1) = l \), the \( l \) nodes are uniformly distributed in \( W_1 \) and so

\[
\mathbb{E}(\text{TSP}(N(1))|N(1) = l) = \mathbb{E}_{\text{TSP}} \left( l; \frac{1}{k} \right) = \frac{1}{k^\alpha} \mathbb{E}_{\text{TSP}} l \leq D_1 \frac{l^{1-\frac{\alpha}{2}}}{k^\alpha}
\]

for some constant \( D_1 > 0 \) using the expectation upper bound in (1.8). Thus,

\[
\mathbb{E}_{\text{TSP}}(N(1))\mathbb{I}(N(1) = l) \leq \mathbb{E}(\text{TSP}(N(1))|N(1) = l)\mathbb{P}(N(1) = l) \leq D_1 \frac{l^{1-\frac{\alpha}{2}}}{k^\alpha} \mathbb{P}(N(1) = l)
\]
and consequently \( I_2 = \mathbb{E}_{TSP}(N(1)) \mathbb{I}(F_1^c) \) equals

\[
\sum_{l \leq \text{low}} + \sum_{l \geq \text{up}} \mathbb{E}_{TSP}(N(1)) \mathbb{I}(N(1) = l) \\
\leq \sum_{l \leq \text{low}} + \sum_{l \geq \text{up}} D_1 \frac{l^{1-\frac{q}{2}}}{k^{\alpha}} \mathbb{P}(N(1) = l) \\
= \frac{D_1}{k^{\alpha}} \mathbb{E}(N(1))^{1-\frac{q}{2}} \mathbb{I}(F_1^c).
\]

(6.22)

Using Holder’s inequality

\[
\mathbb{E}_{XY} \leq \left( \mathbb{E}_{X^p} \right)^{\frac{1}{p}} \left( \mathbb{E}_{Y^q} \right)^{\frac{1}{q}}
\]

with \( X = (N(1))^{1-\frac{q}{2}}, Y = \mathbb{I}(F_1^c), p = \frac{2}{2-\alpha} > 1 \) and \( q = \frac{2}{\alpha} > 1 \), we get

\[
\mathbb{E}(N(1))^{1-\frac{q}{2}} \mathbb{I}(F_1^c) \leq \left( \mathbb{E}N(1) \right)^{1-\frac{q}{2}} \left( \mathbb{P}(F_1^c) \right)^{\frac{q}{2}} \\
= m^{2-\alpha} \left( \mathbb{P}(F_1^c) \right)^{\frac{q}{2}} \\
\leq m^{2-\alpha} \frac{1}{m^{\alpha(1-2\gamma)}},
\]

(6.23)

by (6.18). From (6.22), we therefore get that

\[
I_2 \leq \frac{D_2 m^{2-\alpha}}{k^{\alpha}} \frac{1}{m^{\alpha(1-2\gamma)}}.
\]

(6.24)

From the estimate for \( I_1 \) in (6.21), we therefore get that

\[
\mathbb{E}_{TSP}(N(1)) = I_1 + I_2 \leq \frac{1}{k^{\alpha}} \left( \mathbb{E}_{TSP} m^2 + D_2 m^{2(\epsilon_1-\gamma)} + D_2 \frac{m^{2-\alpha}}{m^{\alpha(1-2\gamma)}} \right),
\]

proving Lemma 11. \( \square \)

Substituting (6.17) of Lemma 11 into (6.11), we get

\[
\mathbb{E}_{TSP} k^2 m^2 \leq k^{2-\alpha} \left( h_0^a \cdot \mathbb{E}_{TSP} m^2 + D m^{2(\epsilon_1-\gamma)} + D \frac{m^{2-\alpha}}{m^{\alpha(1-2\gamma)}} + D \right)
\]

(6.25)

for some constant \( D > 0 \). Thus,

\[
\limsup_k \frac{\mathbb{E}_{TSP} k^2 m^2}{(km)^{2-\alpha}} \leq h_0^a \cdot \frac{\mathbb{E}_{TSP} m^2}{m^{2-\alpha}} + D \frac{m^{2-\alpha}}{m^{\alpha(1-2\gamma)}} + D \frac{m^{2-\alpha}}{m^{2-\alpha}}
\]

for all \( m \) large. By choice of \( \gamma, \epsilon_1 > 0 \) in (6.16) we have \( 2\gamma < 1 \) and \( 2 - \alpha + 2\gamma - 2\epsilon_1 > 0 \) and so

\[
\limsup_k \frac{\mathbb{E}_{TSP} k^2 m^2}{(km)^{2-\alpha}} \leq h_0^a \cdot \frac{\mathbb{E}_{TSP} m^2}{m^{2-\alpha}} + \epsilon,
\]

(6.26)
for all $m$ large by definition of $\lambda$ in (6.5). Letting $\{m_j\}$ be any sequence such that $\frac{\mathbb{E}TSP_{m_j}^2}{m_j^{2-a}} \to \lambda = \liminf_n \frac{\mathbb{E}TSP_n^2}{n^{2-a}}$ and allowing $m \to \infty$ through the sequence $\{m_j\}$ in (6.26), we get that $\limsup_n \frac{\mathbb{E}TSP_n^2}{n^{2-a}} \leq h^2_0 \cdot \lambda + \epsilon$. Since $\epsilon > 0$ is arbitrary, this obtains (6.10). \qed

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7 Appendix: Miscellaneous Results

7.1 Standard Deviation Estimates

We use the following standard deviation estimates for sums of independent Poisson and Bernoulli random variables.

**Lemma 13** Suppose $W_i, 1 \leq i \leq m$ are independent Bernoulli random variables satisfying $\mu_1 \leq \mathbb{P}(W_1 = 1) = 1 - \mathbb{P}(W_1 = 0) \leq \mu_2$. For any $0 < \epsilon < \frac{1}{2},$

$$
\mathbb{P}\left( \sum_{i=1}^{m} W_i > m\mu_2 (1 + \epsilon) \right) \leq \exp\left( -\frac{\epsilon^2}{4} m\mu_2 \right) \quad (7.1)
$$

and

$$
\mathbb{P}\left( \sum_{i=1}^{m} W_i < m\mu_1 (1 - \epsilon) \right) \leq \exp\left( -\frac{\epsilon^2}{4} m\mu_1 \right) \quad (7.2)
$$

Estimates (7.1) and (7.2) also hold if $\{W_i\}$ are independent Poisson random variables with $\mu_1 \leq \mathbb{E}W_1 \leq \mu_2$.

For completeness, we give a quick proof.

**Proof of Lemma 13** First suppose that $\{W_i\}$ are independent Poisson with $\mu_1 \leq \mathbb{E}W_i \leq \mu_2$ so that $\mathbb{E}e^{sW_i} = \exp(\mathbb{E}W_i(e^s - 1)) \leq \exp(\mu_2(e^s - 1))$ for $s > 0$. By Chernoff bound, we then have

$$
\mathbb{P}\left( \sum_{i=1}^{m} W_i > m\mu_2 (1 + \epsilon) \right) \leq e^{-sm\mu_2(1+\epsilon)} \exp(\mu_2(e^s - 1)) = e^{m\mu_2\Delta_1},
$$

where $\Delta_1 = e^s - 1 - s - s\epsilon$. For $s \leq 1$, we have the bound

$$
e^s - 1 - s = \sum_{k \geq 2} \frac{s^k}{k!} \leq s^2 \sum_{k \geq 2} \frac{1}{k!} = s^2(e - 2) \leq s^2
$$

and so we set $s = \frac{\epsilon}{2}$ to get that $\Delta_1 \leq s^2 - s\epsilon = -\frac{\epsilon^2}{4}$. 

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Similarly, for $s > 0$, we have
\[ \mathbb{E} e^{-s W_i} = \exp(\mathbb{E} W_i(e^{-s} - 1)) \leq \exp(\mu_1(e^{-s} - 1)) \]
and so
\[ P\left( \sum_{i=1}^{m} W_i < m \mu_1(1 - \epsilon) \right) \leq e^{sm\mu_1(1-\epsilon)} \exp(m \mu_1(e^{-s} - 1)) = e^{-m \mu_1 \Delta_2}, \]
where $\Delta_2 = 1 - s - e^{-s} + s \epsilon$. For $s \leq 1$, the term $e^{-s} \leq 1 - s + \frac{s^2}{2}$ and so we get $\Delta_2 \geq -\frac{s^2}{2} + s \epsilon = \frac{s}{2}$ for $s = \epsilon$.

The proof for the Binomial distribution follows from the fact that if $\{W_i\}_{1 \leq i \leq m}$ are independent Bernoulli distributed with $\mu_1 \leq \mathbb{E} W_i \leq \mu_2$, then for $s > 0$ we have $\mathbb{E} e^{s W_i} = 1 - \mathbb{E} W_i + e^s \mathbb{E} W_i \leq \exp(\mathbb{E} W_i(e^s - 1)) \leq \exp(\mu_2(e^s - 1))$ and similarly $\mathbb{E} e^{-s W_i} \leq \exp(\mu_1(e^{-s} - 1))$. The rest of the proof is then as above. \( \square \)

### 7.2 Proof of the Monotonicity Property (2.14)

For $\alpha \leq 1$, we couple the original Poisson process $\mathcal{P}$ and the homogenous process $\mathcal{P}_\delta$ in the following way. Let $V_i, i \geq 1$ be i.i.d. random variables each with density $f(\cdot)$ and let $N_V$ be a Poisson random variable with mean $n$, independent of $\{V_i\}$. The nodes $\{V_i\}_{1 \leq i \leq N_V}$ form a Poisson process with intensity $nf(\cdot)$ which we denote as $\mathcal{P}$ and colour green.

Let $U_i, i \geq 1$ be i.i.d. random variables each with density $\epsilon_2 - f(\cdot)$ where $\epsilon_2 \geq 1$ is as in (1.1) and let $N_U$ be a Poisson random variable with mean $n(\epsilon_2 - 1)$. The random variables $\{U_i\}, N_U$ are independent of $\{V_i\}, N_V$ and the nodes $\{U_i\}_{1 \leq i \leq N_U}$ form a Poisson process with intensity $n(\epsilon_2 - f(\cdot))$ which we denote as $\mathcal{P}_{ext}$ and colour red. The nodes of $\mathcal{P}$ and $\mathcal{P}_{ext}$ together form a homogenous Poisson process with intensity $n\epsilon_2$, which we denote as $\mathcal{P}_\delta$ and define it on the probability space $(\Omega_\delta, \mathcal{F}_\delta, \mathbb{P}_\delta)$.

Let $\omega_\delta \in \Omega_\delta$ be any configuration and as above let $\{i_{j(\delta)}\}_{1 \leq j \leq Q_\delta}$ be the indices of the squares in $\{R_j\}$ containing at least one node of $\mathcal{P}_\delta$ and let $\{i_{j}\}_{1 \leq j \leq Q}$ be the indices of the squares in $\{R_j\}$ containing at least one node of $\mathcal{P}$. The indices in $\{i_{j(\delta)}\}$ and $\{i_{j}\}$ depend on $\omega_\delta$. Defining $S_{\alpha} = S_\alpha(\omega_\delta)$ and $S_{\alpha(\delta)} = S_{\alpha(\delta)}(\omega_\delta)$ as before, we have that $S_{\alpha}$ is determined only by the green nodes of $\omega_\delta$ while $S_{\alpha(\delta)}$ is determined by both green and red nodes of $\omega_\delta$.

From the monotonicity property, we therefore have that $S_{\alpha}(\omega_\delta) \leq S_{\alpha(\delta)}(\omega_\delta)$ and so for any $x > 0$ we have
\[ \mathbb{P}_\delta(S_{\alpha(\delta)} < x) \leq \mathbb{P}_\delta(S_\alpha < x) = \mathbb{P}_0(S_\alpha < x), \tag{7.3} \]
proving (2.14).

If $\alpha > 1$, we perform a slightly different analysis. Letting $\epsilon_1 \leq 1$ be as in (1.1), we construct a Poisson process $\mathcal{P}_{ext}$ with intensity $n(f(\cdot) - \epsilon_1)$ and colour nodes of $\mathcal{P}_{ext}$ red. Letting $\mathcal{P}_\delta$ be another independent Poisson process with intensity $n\epsilon_1$,.
we colour nodes of $\mathcal{P}_\delta$ green. The superposition of $\mathcal{P}_{\text{ext}}$ and $\mathcal{P}_\delta$ is a Poisson process with intensity $nf(\cdot)$, which we define on the probability space $(\Omega_\delta, \mathcal{F}_\delta, \mathbb{P}_\delta)$. In this case, the sum $S_{\alpha}$ is determined by both green and red nodes while $S_{\alpha}^{(\delta)}$ is determined only by the green nodes. Again using the monotonicity property of $S_{\alpha}$, we get (7.3).

7.3 Additive Relations

If $\text{TSP}(x_1, \ldots, x_j), j \geq 1$ denotes the length of the TSP cycle with vertex set $\{x_1, \ldots, x_j\}$, then for any $k \geq 1$, we have

$$\text{TSP}(x_1, \ldots, x_{j+k}) \leq \text{TSP}(x_1, \ldots, x_j) + \text{TSP}(x_{j+1}, \ldots, x_{j+k}) + (c_2\sqrt{2})^\alpha$$

(7.4)

and if $\alpha \leq 1$ and the edge weight function $h$ is a metric, then

$$\text{TSP}(x_1, \ldots, x_j) \leq \text{TSP}(x_1, \ldots, x_{j+k}).$$

(7.5)

Proof of (7.4) and (7.5) To prove (7.4), suppose $C_1$ is the minimum weight spanning cycle formed by the nodes $\{x_l\}_{1 \leq l \leq j}$ and $C_2$ is the minimum weight spanning cycle formed by $\{x_l\}_{j+1 \leq l \leq j+k}$. Let $e_1 = (u_1, v_1) \in C_1$ and $e_2 = (u_2, v_2) \in C_2$ be any two edges. The cycle

$$C_{\text{tot}} = (C_1 \setminus \{e_1\}) \cup (C_2 \setminus \{e_2\}) \cup \{(u_1, u_2), (v_1, v_2)\}$$

obtained by removing the edges $e_1, e_2$ and adding the “cross”-edges $(u_1, u_2)$ and $(v_1, v_2)$ is a spanning cycle containing all the nodes $\{x_l\}_{1 \leq l \leq j+k}$. The edges $(u_1, u_2)$ and $(v_1, v_2)$ have a Euclidean length of at most $\sqrt{2}$ and so a weight of at most $(c_2\sqrt{2})^\alpha$ using the bounds for the metric $h$ in (1.2). This proves (7.4).

It suffices to prove (7.5) for $k = 1$. Let $C = (y_1, \ldots, y_{j+1}, y_1)$ be any cycle with vertex set $\{y_i\}_{1 \leq i \leq j+1} = \{x_i\}_{1 \leq i \leq j+1}$ and without loss of generality suppose that $y_{j+1} = x_{j+1}$. Removing the edges $(y_j, y_{j+1})$ and $(y_{j+1}, y_1)$, and adding the edge $(y_1, y_j)$ we get a new cycle $C'$ with vertex set $\{x_i\}_{1 \leq i \leq j}$.

Since the edge weight function $h$ is a metric, we have by triangle inequality that $h(y_1, y_j) \leq h(y_j, y_{j+1}) + h(y_{j+1}, y_1)$. Using $(a + b)^\alpha \leq a^\alpha + b^\alpha$ for $a, b > 0$ and $0 < \alpha \leq 1$, we get that $h^\alpha(y_1, y_j) \leq h^\alpha(y_j, y_{j+1}) + h^\alpha(y_{j+1}, y_1)$. Therefore, the weight $W(C')$ of $C'$

$$W(C') = \sum_{i=1}^{j-1} h^\alpha(y_i, y_{i+1}) + h^\alpha(y_j, y_1) \leq \sum_{i=1}^{j} h^\alpha(y_i, y_{i+1}) + h^\alpha(y_{j+1}, y_1) = W(C).$$

Therefore, $\text{TSP}(x_1, \ldots, x_j) \leq W(C') \leq W(C)$. Taking minimum over all cycles $C$ with vertex set $\{x_i\}_{1 \leq i \leq j+1}$, we get (7.5) for $k = 1$. \qed
7.4 Moments of Random Variables

Let $X \geq 1$ be any integer valued random variable such that

$$P(X \geq l) \leq e^{-\theta(l-1)} \tag{7.6}$$

for all integers $l \geq 1$ and some constant $\theta > 0$ not depending on $l$. For every integer $r \geq 1$,

$$E X^r \leq r \sum_{l \geq 1} l r^{-1} \frac{P(X \geq l)}{e^{-\theta(l-1)}} \leq \frac{r!}{(1 - e^{-\theta})^r} \tag{7.7}$$

**Proof of (7.7)** For $r \geq 1$, we have

$$E X^r = \sum_{l \geq 1} l r^{-1} P(X = l) = \sum_{l \geq 1} l r^{-1} P(X \geq l) - l r^{-1} P(X \geq l + 1) \tag{7.8}$$

and substituting the $l r^{-1}$ in the final term of (7.8) with $(l + 1)^r - ((l + 1)^r - l^r)$, we get

$$E X^r = \sum_{l \geq 1} \left( l r^{-1} P(X \geq l) - (l + 1)^r P(X \geq l + 1) \right)$$

$$+ \sum_{l \geq 1} ((l + 1)^r - l^r) P(X \geq l + 1)$$

$$= 1 + \sum_{l \geq 1} ((l + 1)^r - l^r) P(X \geq l + 1)$$

$$= \sum_{l \geq 0} ((l + 1)^r - l^r) P(X \geq l + 1) \tag{7.9}$$

where the second equality is true since $l r^{-1} P(X \geq l) \leq l^r e^{-\theta(l-1)} \rightarrow 0$ as $l \rightarrow \infty$. Using $(l + 1)^r - l^r \leq r \cdot (l + 1)^{r-1}$ in (7.9), we get the first relation in (7.7).

We prove the second relation in (7.7) by induction as follows. Let $\gamma = e^{-\theta} < 1$ and $J_r := \sum_{l \geq 1} l r^{-1} \gamma^{l-1}$ so that

$$J_{r+1}(1 - \gamma) = \sum_{l \geq 1} l r^{-1} \gamma^{l-1} - \sum_{l \geq 1} l r^{-1} \gamma^l = \sum_{l \geq 1} (l^r - (l - 1)^r) \gamma^{l-1}.$$ 

Using $l^r - (l - 1)^r \leq r \cdot l^{r-1}$ for $l \geq 1$ we therefore get that

$$J_{r+1}(1 - \gamma) \leq r \sum_{l \geq 1} l^{r-1} \gamma^{l-1} = r J_r$$

and so the second relation in (7.7) follows from induction. \qed
7.5 Scaling and Translation Relations

For a set of nodes \( \{ x_1, \ldots, x_n \} \) in the unit square \( S \), recall from Sect. 1 that \( K_n(x_1, \ldots, x_n) \) is the complete graph formed by joining all the nodes by straight line segments and the edge \((x_i, x_j)\) is assigned a weight of \( d^\alpha(x_i, x_j) \), where \( d(x_i, x_j) \) is the Euclidean length of the edge \((x_i, x_j)\). We denote TSP\((x_1, \ldots, x_n)\) to be the length of the minimum spanning cycle of \( K_n(x_1, \ldots, x_n) \) with edge weights obtained as in (1.4).

**Scaling:** For any \( a > 0 \), consider the graph \( K_n(ax_1, \ldots, ax_n) \) where the length of the edge between the vertices \( ax_1 \) and \( ax_2 \) is simply \( a \) times the length of the edge between \( x_1 \) and \( x_2 \) in the graph \( K_n(x_1, \ldots, x_n) \) Using the definition of TSP in (1.4), we then have \( \text{TSP}(ax_1, \ldots, ax_n) = a^\alpha \text{TSP}(x_1, \ldots, x_n) \) and so if \( Y_1, \ldots, Y_n \) are \( n \) nodes uniformly distributed in the square \( aS \) of side length \( a \), then

\[
\text{TSP}(n; a) := \text{TSP}(Y_1, \ldots, Y_n) = a^\alpha \text{TSP}(X_1, \ldots, X_n),
\]

where \( X_i = \frac{Y_i}{a} \), \( 1 \leq i \leq n \) are i.i.d. uniformly distributed in \( S \). Recalling the notation \( \text{TSP}_n = \text{TSP}(X_1, \ldots, X_n) \) from (1.4), we therefore get

\[
\mathbb{E} \text{TSP}(n; a) = a^\alpha \mathbb{E} \text{TSP}_n. \tag{7.10}
\]

**Translation:** For \( b \in \mathbb{R}^2 \), consider the graph \( K_n(x_1 + b, \ldots, x_n + b) \). Using the translation property (b2), the weight \( h(x_1 + b, x_2 + b) \leq h_0 \cdot h(x_1, x_2) \), the weight of the edge between \( x_1 \) and \( x_2 \). Using the definition of TSP in (1.4), we therefore have

\[
\text{TSP}(x_1 + b, \ldots, x_n + b) \leq h_0^\alpha \cdot \text{TSP}(x_1, \ldots, x_n), \tag{7.11}
\]
obtaining the desired bound. \( \square \)

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