A PATHOLOGICAL CONSTRUCTION FOR REAL FUNCTIONS WITH LARGE COLLECTIONS OF LEVEL SETS

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Abstract. Consider all the level sets of a real function. We can group these level sets according to their Hausdorff dimensions. We show that the Hausdorff dimension of the collection of all level sets of a given Hausdorff dimension can be arbitrarily close to 1, even if the function is differentiable to some level. By definition of Hausdorff dimension it is clear, for any real function \( f(x) \) and any \( \alpha \in [0,1] \), that
\[
\dim_H \{ y : \dim_H(f^{-1}(y)) \geq \alpha \} \leq 1.
\]
What is surprising, and what we show, is that this is actually a sharp bound. That is,
\[
\sup \{ \dim_H \{ y : \dim_H(f^{-1}(y)) = 1 \} : f \in C^k \} = 1,
\]
for any \( k \in \mathbb{Z}_{\geq 0} \).

1. Preliminaries

For the purposes of this paper it will be sufficient to consider functions of the form
\[
f : [0, 1] \to [0, 1].
\]

Let \( y \in [0, 1] \) and consider the level set \( f^{-1}(y) \subseteq [0, 1] \).

For any \( d \in [0, \infty) \), this level set has a \( d \)-dimensional Hausdorff content given by
\[
C_H^d \left( f^{-1}(y) \right) = \inf \left\{ \sum_i r_i^d : \text{there is a cover of } f^{-1}(y) \text{ by balls of radii } r_i > 0 \right\}.
\]

Further, \( f^{-1}(y) \) has a Hausdorff dimension given by
\[
\dim_H \left( f^{-1}(y) \right) = \inf \left\{ d \geq 0 : C_H^d \left( f^{-1}(y) \right) = 0 \right\}.
\]

We are interested in all those \( y \) whose pre-images have positive Hausdorff dimension:
\[
\{ y \in [0, 1] : \dim_H \left( f^{-1}(y) \right) > 0 \}.
\]

More specifically though we are interested in the sets
\[
\{ y \in [0, 1] : \dim_H \left( f^{-1}(y) \right) \geq \alpha \},
\]
where \( 0 \leq \alpha \leq 1 \).

We wish to find functions, \( f(x) \), that maximize the Hausdorff dimension of this set.

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**Definition 1.1.** Let $0 \leq \alpha \leq 1$. Define

$$I_\alpha(f) = \{ y \in [0, 1] : \dim_H (f^{-1}(y)) \geq \alpha \}.$$  

Note: Trivially, for any function $f(x)$, we have $I_0(f) = Range(f)$.

### 2. Examples

**Example 2.1 (Trivial Example).**
Consider the graph of the function

$$f : [0, 1] \rightarrow [0, 1], \quad x \mapsto -x(x - 1).$$

As expected, in this case $I_0(f) = [0, 0.25]$.

Note that the pre-image of each point in the range of $f(x)$ is at most finite. Thus the pre-image of each point has trivial Hausdorff dimension. Hence

$$I_\alpha(f) = \emptyset \quad \text{and} \quad \dim_H I_\alpha(f) = 0,$$

for all $\alpha > 0$. 
Example 2.2 (Another Trivial Example).
Consider any constant function. For example:
\[ f : [0, 1] \rightarrow [0, 1], \quad x \mapsto 0.5. \]

\[ I_0(f) \quad \quad f(x) = 0.5 \]

In this case the only non-trivial pre-image is \( f^{-1}(0.5) = [0, 1] \).

The unit interval has Hausdorff dimension 1, and so
\[ I_\alpha(f) = \{0.5\} \quad \text{and} \quad \dim_H I_\alpha(f) = 0, \]
for all \( 0 \leq \alpha \leq 1 \).

The next question is: How large can we make \( I_\alpha(f) \), for \( \alpha > 0 \), while preserving continuity or even differentiability?

The next example shows that we can construct a continuous function \( f(x) \) such that \( I_1(f) \) is infinite.
Example 2.3. (Non-Trivial $I_\alpha(f)$)

Consider the function

$$f_1 : [0, 0.5] \rightarrow [0, 1], \quad x \mapsto \begin{cases} \frac{3}{2}x & \text{if } x \in [0, \frac{1}{6}] \\ \frac{1}{4} & \text{if } x \in \left[\frac{1}{6}, \frac{1}{3}\right] \\ \frac{3}{2}x - \frac{1}{4} & \text{if } x \in \left[\frac{1}{3}, \frac{1}{2}\right] \end{cases}$$

Take this function and make scaled copies of it with dimensions $\frac{1}{2^k} \times \frac{1}{2^k}$. Then graph these scaled functions end-to-end so that the bottom left coordinate of the $k$-th graph coincides with the point $\left(1 - \frac{1}{2^k}, 1 - \frac{1}{2^k} \right)$.

This gives us a continuous (although not differentiable) function $f : [0, 1] \rightarrow [0, 1]$ such that

$$I_\alpha(f) = \left\{ \frac{1}{4}, \frac{5}{8}, \ldots, \frac{2^i - \frac{3}{2}}{2^i}, \ldots \right\} \quad \text{and} \quad \dim_H I_\alpha(f) = 0,$$

for all $0 < \alpha \leq 1$. 

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3. **Main Theorem**

In this paper we show the following very counterintuitive result: We can make \( \dim H I_\alpha(f) \leq 1 \) arbitrarily close to 1, for all \( 0 \leq \alpha \leq 1 \), while still maintaining the continuity and even differentiability of \( f(x) \).

**Theorem 3.1.** For any \( k \in \mathbb{Z}_{\geq 0} \) and any \( 0 \leq \alpha \leq 1 \) we have

\[
\sup \{ \dim H (I_\alpha(f)) : f \in C^k \} = 1.
\]

**Example 3.2.** *(Main Function)* Consider the following iteratively defined function.

Let \( k \in \mathbb{Z}_{\geq 0} \) and \( \beta < 1 \).

Let \( n \) refer to the level of iteration we are considering at a given time.

Let \( b \) be the number of boxes in the initial iteration level \((n = 1)\), and let \( m \) be the total number of solid curves and boxes in the initial iteration. We shall choose \( b = \frac{m+1}{2} \).

Note: This forces \( m \) to be an odd natural number.

**Construction at iteration level** \( n = 1 \). We begin with \( b \) boxes of dimension \( \frac{1}{m} \times \frac{\beta}{m} \) arranged in the unit square so that the first \( b - 1 \) boxes form a diagonal with bottom left corners having coordinates \( \left( \frac{2i}{m}, \frac{i}{m} \right) \), for \( 0 \leq i \leq b - 2 \). The remaining box then has its bottom left corner placed at \( \left( \frac{2b-2}{m}, 0 \right) \).
To connect the first $b - 1$ boxes we use smooth curves beginning at the bottom right-hand corner of one box and ending at the bottom left-hand corner of the next box. We choose these curves, $g_k(x)$, to be translations of the solution to
\[
\frac{dg_k}{dx} = C(mx)^k(1 - mx)^k, \quad g_k(0) = 0, \quad g_k \left( \frac{1}{m} \right) = \frac{\beta}{m},
\]
on the interval $[0, \frac{1}{m}]$, for some constant $C$. This constant is given in [2].
Note: Any suitable flat function would work here, all we require is a $C^k$ function on a closed interval with trivial first $k$-derivatives at both ends.
Solving the above ODE gives us the following connecting curves
\[
g_k(x) = \frac{\beta}{m} \sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^i}{(k+1+i)!} (mx)^{k+1+i} \sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^i}{(k+1+i)!} (mx)^{k+1+i},
\]
To join the penultimate box to the final box we use a translation of the previous curve combined with a reflection and scaling:
\[
h_k(x) = \frac{\beta}{m} (b - 2) \frac{(2k+1)!}{(k!)^2} \sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^i}{(k+1+i)!} \frac{1}{(2mx)^{k+1+i}},
\]
for $0 \leq x \leq \frac{1}{m}$, and $h_k(x) = 0$ for $\frac{1}{2m} \leq x \leq \frac{1}{m}$.

This gives us the first iteration: $n = 1$.
For the next iteration, $n = 2$, we take the $b$ boxes of dimension $\frac{1}{m} \times \frac{\beta}{m}$, and into each of these boxes we identically construct a new collection of boxes and curves similar to those in iteration $n = 1$, with the exception that the new boxes have dimension $\frac{1}{m^2} \times \frac{\beta^2}{m^2}$ and the new curves are all appropriately scaled so that they are all translations of
\[
g_{k2}(x) = \frac{\beta}{m} g_k(mx) \quad \text{and} \quad h_{k2}(x) = \frac{\beta}{m} h_k(mx).
\]
We then repeat this process ad infinitum, for each iteration $n$.
This gives us our function $f(x) : [0, 1] \rightarrow [0, 1]$.  

Claim 3.3. $f(x) \in C^k([0,1])$.

Proof. The domain of $f(x)$ can be broken into two groups: interior points on which the solid curves are defined and boundary points at the left and right endpoints of some box.

It is clear that $f \in C^k$ for any interior point on which a solid curve is defined. It remains to establish that $f \in C^k$ at the endpoints of the boxes. More specifically, it remains to establish that $f(x)$ is $k$-times differentiable from the left for right-hand endpoints, and from the right for left-hand endpoints. We prove this by induction on order of differentiation $1 \leq j \leq k$.

Let $x = e$ be any left endpoint of some box from our construction process.

Case: $j = 1$.

Let $(e_i)_{i \in \mathbb{N}}$ be any sequence of points, for which we have defined right-hand derivatives, that converge from the right to $e$.

By construction, for any given $i \in \mathbb{N}$ there exists $N(i) \in \mathbb{N}$ that tells us the level of the iterative process at which $f(e_i)$ was defined. Since $\lim_{i \to \infty} e_i = e$ it follows that $\lim_{i \to \infty} N(i) = \infty$.

If $f(e_i)$ is defined in the $N(i)$-th level of the iterative process then

$$\left(\frac{1}{m}\right)^{N(i)} \leq |e - e_i| < \left(\frac{1}{m}\right)^{N(i)-1} \quad \text{and} \quad |f(e) - f(e_i)| < \left(\frac{\beta}{m}\right)^{N(i)-1}.$$

Hence

$$\frac{|f(e) - f(e_i)|}{|e - e_i|} < \beta^{N(i)-1}m.$$

By definition, $\beta < 1$, and thus

$$\partial_+ f(e) = \lim_{i \to \infty} \frac{|f(e) - f(e_i)|}{|e - e_i|} \leq \lim_{i \to \infty} \beta^{N(i)-1}m = 0 = \partial_- f(e),$$

the last equality comes from our choice of the solid curves.

This argument is virtually identical for right endpoints. Therefore $f(x) \in C^1$ and $f^{(1)}(e) = 0$.

Case $j = l \leq k$.

Assume that $f^{(1)}(e) = \cdots = f^{(l-1)}(e) = 0$ for some left endpoint, $e$, of a box. Again, let $(e_i)_{i \in \mathbb{N}}$ be any sequence of points, for which we have defined right-hand derivatives, that converge from the right to $e$.

As above, there exists $N(i) \in \mathbb{N}$ telling us the level of the iterative process at which $f(e_i)$ is defined.

Consider $|f^{(l-1)}(e) - f^{(l-1)}(e_i)| = |f^{(l-1)}(e_i)|$. When defining the solid curve on $x = e_i$ we used a translation of one of the polynomials $g_{kN(i)}(x)$ or $h_{kN(i)}(x)$. Thus

$$|f^{(l-1)}(e) - f^{(l-1)}(e_i)| = |f^{(l-1)}(e_i)| = \left|g_{kN(i)}^{(l-1)}(x)\right| \text{ or } \left|h_{kN(i)}^{(l-1)}(x)\right|.$$
In our construction we chose that
\[ g_{kN(i)}^{(1)}(x) = \frac{\beta^{N(i)}}{m^{N(i)}} \frac{(2k + 1)!}{(k!)^2} (m^{N(i)} x)^k (1 - m^{N(i)} x)^k \]
on \([0, \frac{1}{m^{N(i)}}]\) and
\[ h_{kN(i)}^{(1)}(x) = \frac{d}{dx} \left( g_{kN(i)} \left( \frac{1}{m^{N(i)}} - 2x \right) \right) = -2g_{kN(i)}^{(1)} \left( \frac{1}{m^{N(i)}} - 2x \right), \]
on \([0, \frac{1}{2m^{N(i)}}]\) and \( h_{kN(i)}^{(1)}(x) = 0 \) on \([\frac{1}{2m^{N(i)}}, \frac{1}{m^{N(i)}}] \).

Hence
\[ g_{kN(i)}^{(l-1)}(x) = \frac{\beta^{N(i)}}{m^{N(i)}} (m^{N(i)} x)^{k+2-l} (1 - m^{N(i)} x)^{k+2-l} p_{k,l-1,i}(m^{N(i)} x), \]
for some polynomial \( p_{k,l-1,i}(m^{N(i)} x) \) of order \( l - 2 \) defined on \([0, \frac{1}{m^{N(i)}}]\). Also
\[ h_{kN(i)}^{(l-1)}(x) = (-2)^{(l-1)} g_{kN(i)}^{(l-1)} \left( \frac{1}{m^{N(i)}} - 2x \right), \]
on \([0, \frac{1}{2m^{N(i)}}]\) and \( h_{kN(i)}^{(l-1)}(x) = 0 \) on \([\frac{1}{2m^{N(i)}}, \frac{1}{m^{N(i)}}] \).

This tells us three things:
1. The first \( k \) right-derivatives of the solid curves at their left end-points are equally 0,
2. The first \( k \) left-derivatives of the solid curves at their right end-points are equally 0,
3. Since \( p_{k,l-1,i}(m^{N(i)} x) \) is a polynomial defined on \([0, \frac{1}{m^{N(i)}}]\) it must be bounded by some constant \( c(k, l) \) only depending on \( k \) and \( l \). Therefore
\[ |f^{(l-1)}(e) - f^{(l-1)}(e_i)| \leq \max \left\{ \left| g_{kN(i)}^{(l-1)}(x) \right|, \left| h_{kN(i)}^{(l-1)}(x) \right| \right\} \]
\[ \leq \frac{\beta^{N(i)}}{m^{N(i)}} C(k, l), \]
where \( C(k, l) \) is some constant depending on \( k \) and \( l \).

Now, as in the initial case, we have that if \( f(e_i) \) is defined in the \( N(i) \)-th level of the iterative process then
\[ \left( \frac{1}{m} \right)^{N(i)} \leq |e - e_i| < \left( \frac{1}{m} \right)^{N(i)-1} \text{ and } \left| f^{(l-1)}(e) - f^{(l-1)}(e_i) \right| < \left( \frac{\beta}{m} \right)^{N(i)} C(k, l). \]

Hence
\[ \left| \frac{f^{(l-1)}(e) - f^{(l-1)}(e_i)}{e - e_i} \right| \leq \beta^{N(i)} C(k, l). \]
Taking the limit as \( i \to \infty \):
\[ \partial_+ f^{(l-1)}(e) = \lim_{i \to \infty} \left| \frac{f(e) - f(e_i)}{e - e_i} \right| \leq \lim_{i \to \infty} \beta^{N(i)} C(k, l) = 0 = \partial_- f^{(l-1)}(e). \]
The argument is virtually identical for right endpoints. Thus \( f(x) \in C^l \). This gives us the inductive step.

Therefore, by strong induction, \( f(x) \in C^k ([0, 1]) \). \( \square \)

**Claim 3.4.** \( \text{dim}_H I_1 = \frac{\log (b - 1)}{\log (2b + 1) - \log (\beta)} \).

**Proof.** In each level of the iteration we added flat sections of curves. These flat sections mean that \( f(x) \) has points in its range whose pre-images have Hausdorff dimension 1. We want to calculate the Hausdorff dimension of the collection of all these points in the range of \( f(x) \), which is equivalent to calculating the Hausdorff dimension of the intersection of all the boxes in the range. Let us denote this set by \( S \).

![Diagram of Hausdorff dimension calculation](image)

Set \( d = \frac{\log (b - 1)}{\log (2b + 1) - \log (\beta)} \). We first prove that \( \text{dim}_H (S) \leq d \). Suppose \( \gamma > d \). The iterative process used to construct \( f(x) \) gives us a sequence of coverings of \( S \). At level \( n = 1 \) we can cover \( S \) by \( b - 1 \) intervals of length \( \frac{\beta}{m} \). At level \( n = 2 \) we can cover \( S \) by \( (b - 1)^2 \) intervals of length \( \left( \frac{\beta}{m} \right)^2 \). After \( n \) iterations we can cover \( S \) by \( (b - 1)^n \) intervals of length \( \left( \frac{\beta}{m} \right)^n \). The \( \gamma \)-total length of the \( n \)-th cover of \( S \) is then \( (b - 1)^n \left( \frac{\beta}{m} \right)^\gamma \).

If we take the limit of this as \( n \to \infty \) we get

\[
\lim_{n \to \infty} (b - 1)^n \left( \frac{\beta}{m} \right)^\gamma = \lim_{n \to \infty} \exp \left[ n \left( \log (b - 1) - \gamma \left( \log (m) - \log (\beta) \right) \right) \right] = 0.
\]

Therefore \( C_H^\gamma (S) = 0 \) and \( \text{dim}_H (S) \leq d = \frac{\log (b - 1)}{\log (2b + 1) - \log (\beta)} \).
For the other direction we will show that $C_H^d(S) > 0$.

Let $(S_i)_{i \in \mathbb{N}}$ be a countable cover of $S$.

By compactness [3], given any $\varepsilon > 0$, there exist a finite collection of open intervals $D_1, \ldots, D_l$ such that $\cup_{i=1}^\infty S_i \subseteq \cup_{j=1}^l D_j$ and

\[
\sum_{j=1}^l |D_j|^\alpha < \sum_{i=1}^\infty |S_i|^\alpha + \varepsilon.
\]

Let us choose $n$ such that

\[
\left( \frac{\beta}{m} \right)^n \leq \min \{|D_j| : j = 1, \ldots, l\}.
\]

For $i = 1, \ldots, n$ define

\[
M_i = \# \left\{ D_j : \left( \frac{\beta}{m} \right)^i \leq |D_j| < \left( \frac{\beta}{m} \right)^{i-1}\right\}.
\]

It follows that

\[
\sum_{j=1}^l |D_j|^\alpha \geq \sum_{j=1}^n M_j \left( \frac{\beta}{m} \right)^{j\alpha} = \sum_{j=1}^n M_j \left( \frac{1}{b-1} \right)^j.
\]

Consider any $D_j$. There must exist some $i$ such that $\left( \frac{\beta}{m} \right)^i \leq |D_j| < \left( \frac{\beta}{m} \right)^{i-1}$. Thus $D_j$ can intersect at most 2 of the $(b-1)^i$ intervals obtained in the $i$-th level of the iterative process. Each of these intervals produces $(b-1)^{n-i}$ sub-intervals at the $n$-th level of the iterative process, hence $D_j$ contains at most $2(b-1)^{n-i}$ intervals from the $n$-th level of the construction process. In total, the $n$-th step of the construction process has $(b-1)^n$ intervals. Therefore

\[
(b-1)^n \leq \sum_{i=1}^l 2M_i(b-1)^{n-i} \quad \Rightarrow \quad \frac{1}{2} \leq \sum_{i=1}^l \frac{M_i}{(b-1)^i}.
\]

Combining this with the above equation gives:

\[
\frac{1}{2} \leq \sum_{j=1}^l |D_j|^d < \sum_{i=1}^\infty |S_i|^d + \varepsilon.
\]

Let $\varepsilon = \frac{1}{4}$. Then

\[
\frac{1}{4} \leq \sum_{i=1}^\infty |S_i|^d.
\]

Therefore $\sum_{i=1}^\infty |S_i|^d$ is bounded below and hence

\[
\dim_H(S) \geq d = \frac{\log(b-1)}{\log(2b+1) - \log(\beta)}.
\]

\[\square\]
Using the previous claim and letting $b \to \infty$, L'Hôpital's Rule tells us that:

$$\lim_{b \to \infty} \frac{\log(b - 1)}{\log(2b + 1) - \log(\beta)} = \lim_{b \to \infty} \left[ \frac{1}{2b+1} \right] = \lim_{b \to \infty} \left[ 1 + \frac{3}{2b - 2} \right] = 1.$$

Therefore 1 is indeed a sharp bound for $\dim_H I_1(f) \leq 1$.

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