In this study, for COVID-19, we divide people into four categories: susceptible $S(t)$, closely contacted $C(t)$, infective $I(t)$, and removed $R(t)$ according to the current epidemic situation and then investigate two models: the SCIR models with immigration (Model (2)) and without immigration (Model (1)). For the former, Model 1, we obtain the condition for global stability of its disease-free equilibrium. For the latter, Model 2, we establish the local asymptotic stability of its endemic equilibrium by constructing Lyapunov function. Afterwards, by the bifurcation theory, we qualitatively analyze the properties of its Hopf bifurcations of the latter. Finally, numerical simulations are given to illustrate the obtained results of two models. The results imply the importance of finding closely contacted and overseas imports on epidemic control. It indicates that not only the incubation delay $\tau$ is crucial for the containment of the COVID-19 but also the scientific and rigorous containment measures are the key factors of the success of the containment.

1. Introduction

The 2019 novel coronavirus disease (COVID-19) is running rampantly in the world, which has caused a great concern on the global public health. Research on infectious diseases has never stopped [1–10]. In [11], the authors investigated global stability for the SEIR model in epidemiology. The dynamical behavior of an epidemic model with nonlinear incidence was studied in Ruan and Wang [12]. Peng [13] considered asymptotic curve of the positive steady state of the SIS epidemic reaction diffusion system. From 2020 to 2021, to defeat the epidemic, scientists in different fields cooperated to investigate COVID-19 from different points of view. Sex ratio, media reports, super-spreaders, cytokine storm, mask wearing, healthcare, vaccination, age structure, impact on energy demand and consumption, etc., are covered [14–26].

In order to prevent the spread of the epidemic, the suspected cases and occasional local cases should be checked, including nasopharyngeal and oropharyngeal swabs, CT scans, serum testing, bronchoalveolar lavage, and other different methods of inspection [27]. In [28], their model is based on the usual SIR model, dividing the total proportion of infected individuals into two parts: patients who have not yet been detected and patients who have been detected by tests. In addition, because a large number of people travel from one country to another, the disease is mainly spread between different countries through air travel. For countries with better epidemic control, imported cases have become the main source of cases. Therefore, it is important to consider foreign inputs.

Based on the global pandemic COVID-19, the epidemic lasts for such a long time; personnel exchanges between countries are inevitable. In this study, we discuss the impact of immigration between countries on the prevention and control of epidemic. The total population of each group is divided into four compartments, that is, susceptible, closely contacted, infective, and removed compartments. We will investigate two models: the SCIR models with immigration (Model (2)) and without immigration (Model (1)), respectively.

The arrangement of this study is as follows. In Section 2, we establish the global stability of disease-free equilibrium for Model 1. In Section 3, we obtain the local asymptotic stability of endemic equilibrium and analyze the Hopf bifurcations for Model 2. In Section 4, this study is concluded.
2. Model 1

2.1. Construction of Model 1. Take one country as a whole, not taking into account the number of imported people, the birth rate, or the normal death rate in the countries.

The model (Model 1) reads

\[
\begin{align*}
S'(t) &= aC(t) - \frac{bSI(t)I(t)}{N}, \\
C'(t) &= \frac{bSI(t)I(t)}{N} - aC(t) - \beta C(t - \tau), \\
I'(t) &= \beta C(t - \tau) - rI(t), \\
R'(t) &= rI(t),
\end{align*}
\]

where \(S(t)\) means the number of susceptible individuals, \(C(t)\) means the closely contacted individuals, that is the exposed, \(I(t)\) means the infected at time \(t\), \(R(t)\) is removed from the infected system including recovered and dead, and \(\tau\) indicates the incubation delay, i.e., the duration between the time when the people who is exposed to the infected cases, but not yet infectious, and later the time when they become infectious. \(a\) means transition rate of closely contacted individuals to susceptible by testing and quarantine, \(b\) means contact rate of transmission per contact from infected class, \(\beta\) means transition rate of closely contacted individuals to the infected class, and \(r\) means removal rate including mortality of infectious disease and recovery rate of infected individuals. Furthermore, \(N\) is nearly a constant, the total population of a country or region.

Since \(R(t)\) does not intervene in the dynamics of \(S(t), C(t),\) and \(I(t)\), the following system can be separated from system (1):

\[
\begin{align*}
S'(t) &= aC(t) - \frac{bSI(t)I(t)}{N}, \\
C'(t) &= \frac{bSI(t)I(t)}{N} - aC(t) - \beta C(t - \tau), \\
I'(t) &= \beta C(t - \tau) - rI(t).
\end{align*}
\]

The initial conditions of the delayed system (2) are

\[
\begin{align*}
S_0(\theta) &= \varphi_1(\theta), \\
C_0(\theta) &= \varphi_2(\theta), \\
I_0(\theta) &= \varphi_3(\theta)(-\tau \leq \theta \leq 0); \quad S(\theta) > 0, C(\theta) \geq 0, I(\theta) \geq 0,
\end{align*}
\]

where \(\varphi = (\varphi_1, \varphi_2, \varphi_3)^T \in C([-\tau, 0], R^3)\) is the initial function, \(C([-\tau, 0], R^3)\) is the Banach space of continuous functions mapping the interval \((-\tau, 0]\) into \(R^3\), \(R^3 = [(S, C, I), S \geq 0, C \geq 0, I \geq 0]\) and \(\|\varphi\| = \max \{|\varphi(\theta)|: \theta \in (-\tau, 0]\}\) and \(|\varphi(\theta)|\) is any norm in \(R^3\). Usually, we use regular symbols \(C_\theta(\theta) = C(t + \theta)\) for \(\theta \in (-\tau, 0]\).

2.2. Global Asymptotical Stability for Model 1. System (2) does not have the endemic equilibrium and has only one equilibrium, i.e., the disease-free equilibrium \(E_0(S_0, 0, 0)\), where \(S_0\) is the number of the initial susceptible individuals.

**Theorem 1.** If \(b < r/2\), then \(E_0(S_0, 0, 0)\) of system (2) has global asymptotical stability.

**Proof.** Assume \(V(t) = |S(t) - S_0| + C(t) + I(t)\). If \(S(t) > S_0\), \(V(t) = S(t) - S_0 + C(t) + I(t)\), we have

\[
V'(t) = S'(t) + C'(t) + I'(t) = -rI < 0.
\]

If \(S(t) < S_0\), \(V(t) = S_0 - S(t) + C(t) + I(t)\), we have

\[
V'(t) = -2aC(t) + \left[\frac{2bSI(t)I(t)}{N} - r\right] I(t).
\]

Because of \(S(t)/N \leq 1\), if \(2b < r\), we can get \(V'(t) \leq -2aC(t) < 0\).

If \(S(t) = S_0\), \(V(t) = C(t) + I(t)\), we have

\[
V'(t) = -aC(t) + \left[\frac{bSI(t)I(t)}{N} - r\right] I(t).
\]

Because of \(S(t)/N \leq 1\), if \(b < r\), we can get \(V'(t) \leq -aC(t) < 0\).

Therefore, when \(b < r/2\), then the unique equilibrium, \(E_0\) has global asymptotical stability.

**Example 1.** Let \(a = 0.1, b = 0.01, \beta = 0.058, r = 0.025\), and \(N = 1.0 \times 10^7\); then, system (2) reads

\[
\begin{align*}
S'(t) &= 0.1C(t) - \frac{0.01SI(t)I(t)}{1.0 \times 10^7}, \\
C'(t) &= \frac{0.01SI(t)I(t)}{1.0 \times 10^7} - 0.1C(t) - 0.058C(t - \tau), \\
I'(t) &= 0.058C(t - \tau) - 0.025I(t).
\end{align*}
\]

For system (7), apparently \(b - r/2 < 0\), by Theorem 1, the equilibrium \(E_0(S_0, 0, 0)\) has global asymptotical stability. We have Figure 1 by simulations. Note that, in Figure 1, the time histories of \(S(t), C(t),\) and \(I(t)\) in system (7), for \(r = 4\), with initial functions \((S(-\tau), C(-\tau), I(-\tau)) = (10^5, 30, 5)\) and \((S(0), C(0), I(0)) = (10^6, 2000, 100)\) and for \(t: 0 \rightarrow 4000(a, c, e)\) and \(t: 7000 \rightarrow 8000(b, d, f)\) are presented, respectively, and all converge to \((10^5, 0, 0)\) as \(t\) tends to \(+\infty\).

3. Model 2

3.1. Construction of Model 2. With the joint efforts of the whole country, the epidemic in some countries has been brought under control. However, due to the immigration of imported cases and the virus on imported cold-chain food, the epidemic has occurred many times. For the convenience of discussion, we will collectively refer to imported
personnel and imported cold-chain food as imported from abroad.

The corresponding model (Model 2) is

\[
\begin{align*}
S'(t) & = \Lambda_1 + aC(t) - \frac{bS(t)I(t)}{N} - d_SS(t), \\
C'(t) & = \Lambda_2 + \frac{bS(t)I(t)}{N} - aC(t) - \beta C(t - \tau) - d_CC(t), \\
I'(t) & = \Lambda_3 + \beta C(t - \tau) - rI(t) - d_I I(t), \\
R'(t) & = rI(t) - d_R R(t),
\end{align*}
\]

Figure 1: The time histories of \(S(t), C(t),\) and \(I(t)\) in system (7), for \(\tau = 4,\) with initial functions \((S(-\tau), C(-\tau), I(-\tau)) = (10^7, 30, 5)\) and \((S(0), C(0), I(0)) = (10^7, 2000, 100)\), and \(t: 0 \rightarrow 4000\) (a, c, e) and \(t: 7000 \rightarrow 8000\) (b, d, f), respectively.

where \(\Lambda_1, \Lambda_2,\) and \(\Lambda_3\) present the immigration rates of susceptible class, closely contacted class, and infected class from the foreign countries, respectively. \(d_S, d_C, d_I,\) and \(d_R\) present the death rates of susceptible individuals, closely contacted individuals, infected, and removed, respectively. Moreover, \(N = S + C + I + R\) is a variable, which keeps increasing with the input of \(\Lambda_1, \Lambda_2,\) and \(\Lambda_3\) and holds decreasing with output of the population which we do not consider. Therefore, combining the two cases into consideration, we can assume that the population is nearly a constant, the total population of a country or region.

System (8) has no disease-free equilibrium \((0, 0, 0, 0)\) or other boundary equilibrium; there exists only the endemic equilibrium \(E^* \left(S^*, C^*, I^*, R^*\right).\) From (8), it follows that
\[ C^* = \frac{\Lambda_1 + \Lambda_2 - d_s S^*}{\beta + d_C}. \]

\[ I^* = \frac{\Lambda_1 + \beta(\Lambda_1 + \Lambda_2 - d_s S^*)}{r + d_I} \left( \frac{r + d_I}{\beta + d_C} \right). \]

\[ R^* = \frac{r I^*}{d_R}. \]

where \( S^* \) satisfies

\[ m_1 (S^*)^2 - m_2 S^* + m_3 = 0, \quad (10) \]

\[ m_1 = b \beta d_C > 0, m_2 = b \beta (\Lambda_1 + \Lambda_2) + b \Lambda_3 (\beta + d_C) + N d_s (r + d_I) (\beta + d_C + a) > 0, m_3 = N (r + d_I) [\Lambda_1 (\beta + d_C) + a (\Lambda_1 + \Lambda_2)] > 0. \]

We analyze the root of (10), by calculation:

\[ \Delta = m_1^2 - 4m_1m_3 = \left[ a N d_s (r + d_I) - b \beta (\Lambda_1 + \Lambda_2) \right]^2 + \left[ N d_s (r + d_I) (\beta + d_C) \right]^2 + (b \beta d_C \Lambda_3)^2 \]

\[ + 2b^2 \beta d_C \Lambda_3 (\Lambda_1 + \Lambda_2 + \Lambda_3) + 2ab \beta N d_s \Lambda_3 (r + d_I) + 2ab N d_s d_C \Lambda_3 (r + d_I) \]

\[ + 2N d_s (r + d_I) (\beta + d_C) [a + b \beta (\Lambda_2 + \Lambda_3) + b d_c \Lambda_3 - b \beta \Lambda_1]. \quad (11) \]

When \( \Delta > 0 \), equation (10) has two positive roots \( S_1 \) and \( S_2 \). Furthermore, if \( \Lambda_1 + \Lambda_2 - d_s S_i > 0 \), \( i = 1, 2 \), then system (8) has two positive equilibria \( E_1^*(S_1^*, C_1^*, I_1^*, R_1^*) \) and \( E_2^*(S_2^*, C_2^*, I_2^*, R_2^*) \).

When \( \Delta > 0 \), equation (10) has two positive roots \( S_1 \) and \( S_2 \). However, if \( \Lambda_1 + \Lambda_2 - d_s S_1 < 0 \), or \( \Lambda_1 + \Lambda_2 - d_s S_2 < 0 \), then system (8) has the unique equilibrium, i.e., the positive equilibrium \( E^*(S^*, C^*, I^*, R^*) \).

When \( \Delta = 0 \), equation (10) has one positive root \( S^* \). Furthermore, if \( \Lambda_1 + \Lambda_2 - d_s S^* > 0 \), then system (8) has the unique equilibrium, i.e., the positive equilibrium \( E^*(S^*, C^*, I^*, R^*) \).

3.2. Locally Asymptotically Stability for Model 2. The Jacobian matrix [29] on \( E^* \) of system (8) is

\[ (\lambda + d_R) \left[ \begin{array}{ccc} b I^* - d_s & a & b S^* \\ -d_s & \frac{b I^*}{N} & \frac{b S^*}{N} \\ -a + d_C - \beta e^{-\lambda r} & \frac{b S^*}{N} & 0 \end{array} \right] (\lambda + r + d_R) + \beta e^{-\lambda r} b I^* = 0. \quad (13) \]

So, we have the characteristic root \( \lambda = -d_R \), and the other characteristic roots satisfying

\[ \lambda^3 + a_1 \lambda^2 e^{-\lambda r} + a_2 \lambda e^{-\lambda r} + a_3 \lambda + a_4 e^{-\lambda r} + a_5 = 0, \quad (14) \]

with
Complexity

\[ a_1 = \beta, \]
\[ a_2 = a + d_c + r + d_l + \frac{bl^*}{N} + d_s, \]
\[ a_3 = \beta \left( r + d_l + \frac{bl^*}{N} + d_s - \frac{bS^*}{N} \right), \]
\[ a_4 = \frac{bl^*}{N} (d_c + r + d_l) + d_s (a + d_c + r + d_l) \]
\[ + (r + d_l) (a + d_c), \]
\[ a_5 = \beta \left[ (r + d_l) \left( \frac{bl^*}{N} + d_s \right) - \frac{bS^*}{N} \right], \]
\[ a_6 = d_s (r + d_l) (a + d_c) + d_c \frac{bl^*}{N} (r + d_l). \]

By (14), when \( \tau = 0 \), we have
\[ \lambda^3 + (a_1 + a_2) \lambda^2 + (a_3 + a_4) \lambda + a_5 + a_6 = 0, \] (16)

with

\[ a_1 + a_2 = \beta + a + d_c + r + d_l + \frac{bl^*}{N} + d_s > 0, \]
\[ a_3 + a_4 = \beta \left( r + d_l + \frac{bl^*}{N} + d_s - \frac{bS^*}{N} \right) + \frac{bl^*}{N} (d_c + r + d_l) + d_s (a + d_c + r + d_l) + (r + d_l) (a + d_c), \] (17)
\[ a_5 + a_6 = \beta \left[ (r + d_l) \left( \frac{bl^*}{N} + d_s \right) - \frac{bS^*}{N} \right] + d_s (r + d_l) (a + d_c) + d_c \frac{bl^*}{N} (r + d_l). \]

Assume
\[ \Delta_1 = a_1 + a_2, \]
\[ \Delta_2 = \begin{vmatrix} a_1 + a_2 & 1 \\ a_3 + a_6 & a_3 + a_4 \end{vmatrix}, \]
\[ \Delta_3 = \begin{vmatrix} a_1 + a_2 & 1 & 0 \\ a_5 + a_6 & a_3 + a_4 & a_1 + a_2 \end{vmatrix} \]
Using Routh–Hurwitz criterion, when
\[ \Delta_1 > 0, \]
\[ \Delta_2 > 0, \] (18)
\[ \Delta_3 > 0. \]

Then, all roots of (16) have negative real parts. So, we can obtain the following result.

**Theorem 2.** If condition (19) holds, all roots of (16) have negative real parts. The endemic equilibrium \( E^* \) of system (8) is asymptotically stable for \( \tau = 0 \) if condition (19) holds.

Next, we suppose condition (19) is established. In order to determine if the real parts of some roots of (9) reaches to zero and ultimately become positive as \( \tau \) varies continuously, let \( \lambda = i \omega (\omega \neq 0) \) be the root; we have
\[ (a_5 - a_1 \omega^2) \cos \omega \tau + a_4 \omega \sin \omega \tau = a_2 \omega^2 - a_6, \] (20)
\[ a_3 \omega \cos \omega \tau + \left( a_1 \omega^2 - a_5 \right) \sin \omega \tau = \omega^3 - a_4 \omega. \]

Thus,
\[ (a_5 - a_1 \omega^2)^2 + (a_4 \omega)^2 = (a_2 \omega^2 - a_6)^2 + \omega^2 (\omega^2 - a_4^2), \]

in other words,
\[ \omega^6 + \left( a_2^2 - a_1^2 - 2a_4 \right) \omega^4 + \left( a_4^2 - a_2^2 + 2a_4 a_5 - 2a_2 a_6 \right) \omega^2 + a_6^2 - a_5^2 = 0. \] (21)

Assume \( z = \omega^2 \), equation (21) can be converted to
\[ z^3 + p z^2 + q z + g = 0, \] (22)

with \( p = a_2^2 - a_1^2 - 2a_4, q = a_4^2 - a_2^2 + 2a_4 a_5 - 2a_2 a_6, \) and \( g = a_6^2 - a_5^2 \).

Assume \( h(z) = z^3 + p z^2 + q z + g \) and \( \Gamma = p^2 - 3q < 0 \). We have the following result.

**Lemma 1** (see [30]). (i) When \( g < 0 \), equation (22) has at least one positive root.

(ii) When \( g \geq 0 \) and \( \Gamma = p^2 - 3q < 0 \), equation (22) has no positive root.

\]
(ii) When $g \geq 0$ and $I = p^2 - 3q \geq 0$, equation (22) has positive root if and only if $z_1 = -p + \sqrt{T/3} > 0$ and $h(z_1) \leq 0$

Therefore, the following theorem is obtained by Lemma 1.

**Theorem 3.** When

$$g < 0. \quad (23)$$

Equation (22) has at least one positive root.

Moreover, we suppose condition (23) is established. Equation (14) has at least one positive root, i.e., equation (22) has at least one positive root. And equation (22) is a cubic, so it may have one, two, or three positive roots. Without loss of generality, we suppose that it has three positive roots, expressed as $z_1$, $z_2$, and $z_3$. Then, equation (21) has three positive roots, that is, $\omega_1 = \sqrt{z_1}$, $\omega_2 = \sqrt{z_2}$, and $\omega_3 = \sqrt{z_3}$.

Assume

$$\tau \in (0, 2\pi],$$

meeting

$$\sin \theta_k = \frac{a_k \left( a_k^2 - a_k \right) \left( \omega_k^2 - \omega_k \right)}{\left( a_k \omega_k \right)^2 + \left( 1 \omega_k^2 - \omega_k \right)^2},$$

$$\cos \theta_k = \frac{a_k \left( a_k^2 - a_k \right) \left( \omega_k - a_k \omega_k \right) + a_k \omega_k (\omega_k^2 - \omega_k)}{\left( a_k \omega_k \right)^2 + \left( 1 \omega_k^2 - \omega_k \right)^2},$$

so $\pm i\omega_k$ are a pair of purely imaginary roots of equation (14) with $\tau = \tau^0_k, k = 1, 2, 3, j = 0, 1, 2, \ldots$; obviously,

$$\lim_{j \to \infty} \tau^0_k = \infty,$$

$$k = 1, 2, 3.$$

So, we can assume

$$\tau_0 = \tau^0_k = \min \{ \tau^0_k \},$$

$$\omega_0 = \omega_k.$$

When condition (23) is established, owing to Theorem 2, equation (14) always has positive root and has roots having zero real part for $\tau = \tau^0_k$, and $\tau_0$ is the minimum of $\tau$. Therefore, for all $\tau \in (0, \tau_0)$, the roots of equation (14) have negative real parts. We can have the following theorem.

**Theorem 4.** When $\tau \in (0, \tau_0)$, all roots of equation (14) have negative real parts. That is, when $\tau \in (0, \tau_0)$, the endemic equilibrium $E^*$ of system (8) is locally asymptotically stable.

Assume $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ is the root of equation (14) satisfying $\alpha(\tau_0) = 0$ and $\omega(\tau_0) = \omega_0$. We suppose $h'(\tau_0) \neq 0$ in order to ensure that $\pm i\omega_0$ are simple pure imaginary roots of equation (14) for $\tau = \tau_0$ and $\lambda(\tau)$ satisfies the transversality condition.

The two sides of Equation (14) are differentiated with respect to $\tau$; we can obtain

$$\frac{d\lambda(\tau)}{d\tau} = \frac{(\alpha \lambda^2 + \alpha \lambda + \alpha_3) \lambda e^{-\lambda \tau}}{3\lambda^2 + 2a \lambda^2 + a_4 - e^{-\lambda \tau} (\alpha \lambda^2 \rho + \alpha \lambda + \alpha_3 \rho - 2a \lambda + \alpha_5 \rho - \alpha_3 \rho)},$$

with

$$A(\tau) = e^{-\alpha \tau} \left[ a_4 (\alpha^2 - \omega^2) + a_4 \alpha + a_4 \omega \right] (a \cos \omega \tau + \omega \sin \omega \tau) - e^{-\alpha \tau} (2a_4 \omega + a_4 \omega) (a \cos \omega \tau - \omega \sin \omega \tau),$$

$$B(\tau) = e^{-\alpha \tau} \left[ a_4 (\alpha^2 - \omega^2) + a_4 \alpha + a_4 \omega \right] (a \cos \omega \tau - \omega \sin \omega \tau) + e^{-\alpha \tau} (2a_4 \omega + a_4 \omega) (a \cos \omega \tau + \omega \sin \omega \tau),$$

$$D(\tau) = 3(\alpha^2 - \omega^2) + 2a_4 \alpha + a_4 - e^{-\alpha \tau} \cos \omega \left[ a_4 (\alpha^2 - \omega^2) + a_4 \alpha - 2a_4 \alpha + a_4 \alpha + a_4 \alpha - a_4 \right] + e^{-\alpha \tau} \sin \omega \left[ a_4 \alpha \omega + a_4 \alpha \omega - a_4 \alpha \omega \right],$$

$$E(\tau) = 6a_4 + 2a_4 \omega - e^{-\alpha \tau} \cos \omega \left[ a_4 \alpha \omega + a_4 \alpha \omega - a_4 \alpha \omega \right] - e^{-\alpha \tau} \sin \omega \left[ a_4 \alpha \omega + a_4 \alpha \omega - a_4 \alpha \omega \right].$$

Putting $\tau = \tau_0$ into the above equation, separating the real and imaginary parts, and by Equation (20) and the definition of $h(z)$, we obtain

Complexity
When \( d\Re \lambda(\tau)/d\tau < 0 \) for \( \tau < \tau_0 \) and close to \( \tau_0 \), we obtain equation (14) has a root \( \lambda(\tau) = \alpha(\lambda) + i\omega(\lambda) \) with \( \alpha(\lambda) \). This contradicts with Theorem 4, so we have the following conclusion.

**Theorem 5.** When \( \tau = \tau_0 \), a pair of simple purely imaginary roots of equation (14) are \( \pm i\omega_0 \) and all other roots have negative real parts. In addition, \( d\Re \lambda(\tau_0)/d\tau > 0 \).

Hence, \( E^* \) of system (8) is asymptotically stable (unstable) when \( \tau < \tau_0 \) (\( \tau > \tau_0 \)), respectively, and \( \tau_0 \) satisfying Equation (27).

### 3.3. Local Asymptotic Stability of Model 2

Assume

\[
\begin{align*}
S(t) &= S^* + \bar{S} \\
C(t) &= C^* + \bar{C} \\
I(t) &= I^* + \bar{I} \\
R(t) &= R^* + \bar{R}
\end{align*}
\]

then, system (8) is linearized as

\[
\begin{align*}
\dot{S}(t) &= \left( \frac{bI^*}{N} + d_S \right) S(t) + aC(t) - \frac{bS^*}{N} I(t), \\
\dot{C}(t) &= \frac{bI^*}{N} S(t) - (a + d_C) C(t) - \beta C(t - \tau) + \frac{bS^*}{N} I(t), \\
\dot{I}(t) &= \beta C(t - \tau) - (r + d_I) I(t), \\
\dot{R}(t) &= rI(t) - d_R R(t).
\end{align*}
\]

For the simplicity of the symbol, we omit the tildes; system (31) can be rewritten as

\[
\begin{align*}
S'(t) &= \left( \frac{bI^*}{N} + d_S \right) S(t) + aC(t) - \frac{bS^*}{N} I(t), \\
C'(t) &= \frac{bI^*}{N} S(t) - (a + d_C) C(t) - \beta C(t - \tau) + \frac{bS^*}{N} I(t), \\
I'(t) &= \beta C(t - \tau) - (r + d_I) I(t), \\
R'(t) &= rI(t) - d_R R(t)
\end{align*}
\]

(32)

We rewrite the second equation of system (32) as

\[
C'(t) = \frac{bI^*}{N} S(t) - (a + d_C + \beta) C(t) + \frac{bS^*}{N} I(t) + \beta \int_{t-\tau}^{t} C'(x)dx
\]

(33)

By the same method, from the third equation of the linearized system (32), we have

\[
C'(t) = \beta C(t - \tau) - (r + d_I) I(t) - \beta \int_{t-\tau}^{t} \left[ \frac{bI^*}{N} S(x) - (a + d_C) C(x) - \beta C(x - \tau) + \frac{bS^*}{N} I(x) \right] dx.
\]

(34)
In order to prove the local asymptotic stability of the equilibrium \((0, 0, 0, 0)\) of system (32), it is sufficient to consider the existence of a strict Lyapunov functional. Let \(V_1(t) = S^2(t)\); the full-time derivative of \(V_1(t)\) is

\[
V_1'(t) = -2 \left( \frac{bI^*}{N} + d_s \right) S^2 + 2aSC - \frac{2bS^*}{N} SI. \tag{35}
\]

Considering \(V_2(t) = \beta C^2(t)\), from (33) and the inequality \(ab \leq a^2 + b^2/2\), we obtain

\[
V_2'(t) = \frac{2bI^*}{N} S(t)C(t) - 2(a + d_c + \beta)C^2(t) + \frac{2bS^*}{N} C(t)I(t) + 2\beta C(t) \int_{t-\tau}^t \left[ \frac{bI^*}{N} S(x) - (a + d_c)C(x) - \beta C(x - \tau) + \frac{bS^*}{N} I(x) \right] dx
\]

\[
\leq \frac{2bI^*}{N} S(t)C(t) - 2(a + d_c + \beta)C^2(t) + \frac{2bS^*}{N} C(t)I(t) + \beta \left[ \frac{bI^*}{N} + (a + d_c) + \beta + \frac{bS^*}{N} \right] \tau C^2(t)
\]

\[
+ \beta \int_{t-\tau}^t \left[ \frac{bI^*}{N} S(x) + (a + d_c)C^2(x) + \beta C^2(x - \tau) + \frac{bS^*}{N} I^2(x) \right] dx. \tag{36}
\]

Considering \(V_3(t) = \beta I^2(t)\) from (33) and the inequality \(ab \leq a^2 + b^2/2\), we have

\[
V_3'(t) = \beta^2 \tau \int_{t-\tau}^t \left[ \frac{bI^*}{N} S^2(t) + (a + d_c)C^2(t) + \beta C^2(t - \tau) + \frac{bS^*}{N} I^2(t) \right]
\]

\[
- \beta \int_{t-\tau}^t \left[ \frac{bI^*}{N} S^2(x) + (a + d_c)C^2(x) + \beta C^2(x - \tau) + \frac{bS^*}{N} I^2(x) \right] dx. \tag{37}
\]

In addition, considering \(V_3(t) = \beta I^2(t) \int_{t-\tau}^t C^2(x)dx\) and \(V_2(t) = V_2(t) + V_2(t) + V_3(t)\), we obtain

\[
V_2'(t) \leq \frac{2bI^*}{N} S(t)C(t) - 2(a + d_c + \beta)C^2(t) + \frac{2bS^*}{N} C(t)I(t) + \beta \left[ \frac{bI^*}{N} + (a + d_c) + \beta + \frac{bS^*}{N} \right] \tau C^2(t)
\]

\[
+ \beta \tau \left[ \frac{bI^*}{N} S^2(t) + (a + d_c)C^2(t) + \beta C^2(t) + \frac{bS^*}{N} I^2(t) \right]. \tag{38}
\]

Similarly, from (34) and \(ab \leq a^2 + b^2/2\), let \(V_3(t) = I^2(t)\), \(V_3(t) = V_3(t)\), \(V_3(t) = V_3(t)\), and \(V_3(t) = V_3(t) + V_3(t)\); we obtain

\[
V_3'(t) \leq 2\beta C(t)I(t) - 2(r + d_i)I^2(t) + \beta \left[ \frac{bI^*}{N} + (a + d_c + \beta) + \frac{bS^*}{N} \right] \tau I^2(t)
\]

\[
+ \beta \tau \left[ \frac{bI^*}{N} S^2(t) + (a + d_c + \beta)C^2(t) + \frac{bS^*}{N} I^2(t) \right]. \tag{39}
\]

Assume \(V_4(t) = R^2(t)\) and \(V(t) = V_1(t) + V_2(t)
\]

\(+ V_3(t) + V_4(t)\). Obviously, \(V(t) \geq 0\) and \(V(t) = 0\) if and
only if $S(t) = C(t) = R(t) = I(t) = 0$. Additionally, if $S(t) = C(t) = R(t) = I(t) = 0$, then $V'(t) = 0$. Furthermore, using $ab \leq a^2 + b^2/2$, we have

$$
V'(t) \leq \left[ \frac{b l^*}{N} + 2d_s - 2\beta \tau \frac{b l^*}{N} - a - \frac{b S^*}{N} \right] S^2(t)
- \left[ a + 2d_c + \beta - 3\beta \tau (a + d_c + \beta) - (1 + \beta \tau) \left( \frac{b l^*}{N} + \frac{b S^*}{N} \right) \right] C^2(t)
- \left[ r + 2d_I - 3\beta \tau \frac{b S^*}{N} - \frac{2b S^*}{N} - \beta - \beta \tau \frac{b l^*}{N} - \beta \tau (a + d_c + \beta) \right] I^2(t) - (2d_R - r)R^2(t).
$$

(40)

Therefore, when

$$
\begin{align*}
\frac{b l^*}{N} + 2d_s - 2\beta \tau \frac{b l^*}{N} - a - \frac{b S^*}{N} > 0, \\
a + 2d_c + \beta - 3\beta \tau (a + d_c + \beta) - (1 + \beta \tau) \left( \frac{b l^*}{N} + \frac{b S^*}{N} \right) > 0, \\
r + 2d_I - 3\beta \tau \frac{b S^*}{N} - \frac{2b S^*}{N} - \beta - \beta \tau \frac{b l^*}{N} - \beta \tau (a + d_c + \beta) > 0,
\end{align*}
$$

(41)

then $V'$ is negative definite. Hence, we have the following theorem.

**Theorem 6.** When the following conditions,

$$
\tau \in (0, \tau_1)
$$

$$
\tau_1 = \min \left\{ \frac{b l^*}{N} + 2d_s - a - \frac{b S^*}{N}, \frac{a + 2d_c + \beta - b l^* / N - b S^* / N}{\beta \left[ 3(a + d_c + \beta) + b l^* / N + b S^* / N \right]}, \frac{r + 2d_I - 2b S^* / N - \beta}{\beta \left[ 3b S^* / N + b l^* / N + a + d_c + \beta \right]} \right\}
$$

(42)

$$
\frac{b S^*}{N} < \min \left\{ \frac{b l^*}{N} + 2d_s - a, a + 2d_c + \beta - \frac{b l^* + r + 2d_I - \beta}{2} \right\}, 2d_R - r > 0,
$$

hold, then equilibrium $(0, 0, 0, 0)$ is locally asymptotically stable for system (32), that is, the positive equilibrium $E^*(S^*, C^*, I^*, R^*)$ is also locally asymptotically stable for system (8).

**Example 2.** Assume $a = 0.01, b = 0.025, \beta = 0.058, r = 0.015, \tau = 4, N = 1.0 \times 10^7, \lambda_1 = 1000, \lambda_2 = 200, \lambda_3 = 1, d_s = 0.0095, d_c = 0.015, d_I = 0.055, \text{ and } d_R = 0.012$; then, system (8) reads
Through calculation, \((S^*, C^*, I^*, R^*) = (107749, 2416, 2016, 2520), bI^*/N + 2d_S - a - bS^*/N/2\beta bI^*/N = 14939, a + 2d_C + \beta - bI^*/N - bS^*/N/\beta [3(a + d_C + \beta) + bI^*/N + bS^*/N] = 6.7593, r + 2d_t - 2bS^*/N - \beta [3bS^*/N + bI^*/N + a + d_C + \beta] = 13.6719, 2d_R - r = 0.009 > 0, \) and \(\tau_1 = \min \{14939, 6.7593, 13.6719\} = 6.7593.\) Because condition (42) for Theorem 6 is satisfied, \((S^*, C^*, I^*, R^*) = (107749, 2416, 2016, 2520)\) is locally asymptotically stable, which is also obtained from Figure 2. Note that, in Figure 2, the time histories of \(S(t), C(t), I(t), \) and \(R(t)\) in system (43), for \(\tau = 4,\) with initial functions \((S(-\tau), C(-\tau), I(-\tau)) = (10^2, 30, 5)\) and \((S(0), C(0), I(0)) = (10^2, 2000, 100),\) and \(t: 0 - 4000,\) respectively, all converge to \((107749, 2416, 2016, 2520)\) as \(t\) tends to \(+\infty.\) The time histories are presented as follows.

3.4. Hopf Bifurcation for Model 2. Simply, system (8) was nondimensionalized using the scaling \(S(t) - S^* \rightarrow S(t),\) \(C(t) - C^* \rightarrow C(t), I(t) - I^* \rightarrow I(t), R(t) - R^* \rightarrow R(t),\) and \(t \rightarrow t\tau;\) then, we have

\[
\begin{align*}
S' (t) &= 1000 + 0.01C(t) - \frac{0.025S(t)I(t)}{1.0 \times 10^7} - 0.0095S(t), \\
C' (t) &= 200 + \frac{0.025S(t)I(t)}{1.0 \times 10^7} - 0.01C(t) - 0.058C(t - \tau) - 0.015C(t), \\
I' (t) &= 1 + 0.058C(t - \tau) - 0.015I(t) - 0.055I(t), \\
R' (t) &= 0.015I(t) - 0.012R(t).
\end{align*}
\]

(43)

Assume \(\tau = \tau_0 + \mu,\) then the Hopf bifurcation value of (44) is \(\mu = 0.\) When \(\phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T \in C([-1, 0], R^4),\) let

\[
L_\mu (\phi) = (\tau_0 + \mu) [B\phi(0) + C\phi(-1)].
\]

(46)

Using the Riesz representation theorem, there is a \(4 \times 4\) matrix \(\eta(\theta, \mu), \theta \in [-1, 0],\) whose elements are functions of bounded variation such that

\[
L_\mu (\phi) = \int_{-1}^{0} [d\eta(\theta, \mu)]\phi(\theta), \phi \in C([-1, 0], R^4).
\]

(47)

Actually, we choose

\[
\eta(\theta, \mu) = \begin{cases} 
(\tau_0 + \mu)B, \theta = 0, \\
0, \theta \in (-1, 0), \\
-(\tau_0 + \mu)C, \theta = -1,
\end{cases}
\]

(48)

which satisfies (47).

When \(\phi \in C([-1, 0], R^4),\) we let the operator \(A(\mu)\) be

\[
A(\mu)\phi(\theta) = \begin{pmatrix}
\frac{d\phi(\theta)}{d\theta}, \theta \in [-1, 0), \\
\int_{-1}^{0} [d\eta(\xi, \mu)]\phi(\xi), \theta = 0,
\end{pmatrix}
\]

(49)

and

\[
R(\mu)\phi(\theta) = \begin{pmatrix}
0, \theta \in [-1, 0), \\
F(\mu, \phi), \theta = 0,
\end{pmatrix}
\]

(50)

where
Figure 2: The time histories of $S(t), C(t), I(t),$ and $R(t)$ in system (43), for $\tau = 4$, with initial functions $(S(\tau), C(\tau), I(\tau), R(\tau)) = (10^7, 30, 5, 0),$ $(S(0), C(0), I(0), R(0)) = (10^7, 2000, 100, 0),$ and $t: 0 - 4000,$ respectively.

\[ F(\mu, \phi) = (\tau_0 + \mu) \begin{pmatrix} \frac{1}{2} f_{20} \phi_1^2(0) + f_{11} \phi_1(0) \phi_2(0) + \frac{1}{2} f_{02} \phi_2^2(0) + \frac{1}{6} f_{30} \phi_1^3(0) \\ \frac{1}{2} f_{20} \phi_1^2(0) + l_{11} \phi_1(0) \phi_2(0) + \frac{1}{2} f_{02} \phi_2^2(0) + \frac{1}{6} f_{30} \phi_1^3(0) \\ + \frac{1}{2} f_{12} \phi_1(0) \phi_2^2(0) + \frac{1}{2} f_{21} \phi_2(0) \phi_2(0) + \frac{1}{6} f_{03} \phi_2^3(0) + \cdots \\ + \frac{1}{2} f_{12} \phi_1(0) \phi_2^2(0) + \frac{1}{2} f_{21} \phi_2(0) \phi_2(0) + \frac{1}{6} f_{03} \phi_2^3(0) + \cdots \end{pmatrix} \]

Therefore, system (8) is equal to the following operator equation:

\[ u_t = A(\mu)u + R(\mu)u, \tag{52} \]

with $u(t) = (S(t), C(t), I(t), R(t))^T$ and $u_0 = u(t + \theta)$ for $\theta \in [-1, 0].$

When $\psi \in C^1([0, 1], (R^5)^*),$ let

\[ A^* \psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0, 1], \\ \phi(-\xi)d\eta(\xi, 0), & s = 0. \end{cases} \tag{53} \]

And a bilinear form is

\[ \langle \psi(s), \phi(\theta) \rangle = \overline{\psi}(0)\phi(0) - \int_{-1}^{0} \int_{\xi=0}^{\theta} \overline{\psi}(\xi - \theta)d\eta(\theta)\phi(\xi)d\xi, \]

with $\eta(\theta) = \eta(\theta, 0).$ Hence, $A(0)$ and $A^*$ are adjoint operators.

We know the eigenvalues of $A(0)$ are $\pm i\omega_0 \tau_0$ and $A^*$ has the same eigenvalues. Furthermore, we can verify that the eigenvectors of $A(0)$ and $A^*$ corresponding to the eigenvalues $i\omega_0 \tau_0$ and $-i\omega_0 \tau_0$ are vectors $q(\theta) = (q_1, q_2)^T e^{i\omega_0 \tau_0 \theta} (\theta \in [-1, 0])$ and $q^*(s) = D(q_1^*, q_2^*) e^{i\omega_0 \tau_0 s} (s \in [0, 1]),$ respectively, and
we obtain Equation (57) has not trivial solution. Assume $q_4 = 1$, we obtain

\[
\begin{pmatrix}
(i \omega_0 \tau_y I - A(0))
\begin{pmatrix}
q_1 \\
q_2 \\
q_3 \\
q_4
\end{pmatrix} e^{i \omega_0 \tau_y \theta} =
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix},
\end{pmatrix}
\]

(55)

\[
\begin{pmatrix}
(i \omega_0 \tau_y I - \tau_y B - \tau_y e^{-i \omega_0 \tau_y C})
\begin{pmatrix}
q_1 \\
q_2 \\
q_3 \\
q_4
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}.
\end{pmatrix}
\]

(56)

\[
\begin{pmatrix}
(-i \omega_0 \tau_y I - A^*)
\begin{pmatrix}
q_1^* \\
q_2^* \\
q_3^* \\
q_4^*
\end{pmatrix} e^{i \omega_0 \tau_y \theta} =
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix},
\end{pmatrix}
\]

Namely,

\[
\begin{pmatrix}
\begin{pmatrix}
i \omega_0 + \frac{bl^*}{N} + d_s & -a & \frac{bS^*}{N} & 0 \\
-\frac{bl^*}{N} & i \omega_0 + a + d_C + \beta e^{-i \omega_0 \tau_y} & \frac{bS^*}{N} & 0 \\
0 & -\beta e^{-i \omega_0 \tau_y} & i \omega_0 + d_I & 0 \\
0 & 0 & 0 & -r
\end{pmatrix}
\end{pmatrix}
\begin{pmatrix}
q_1 \\
q_2 \\
q_3 \\
q_4
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.
\]

(57)

It means that $q_1, q_2, q_3$, and $q_4$ follow as

\[
q_3 = \frac{d_r}{r},
\]

\[
q_2 = \frac{[i \omega_0 + r + d_I]q_3}{\beta e^{-i \omega_0 \tau_y}},
\]

(59)

\[
q_1 = \frac{[i \omega_0 + a + d_C + \beta e^{-i \omega_0 \tau_y}]q_2 - bS*/Nq_3}{bl^*/N}.
\]

By the same method, we have

\[
\begin{pmatrix}
\begin{pmatrix}
-i \omega_0 + \frac{bl^*}{N} + d_s & \frac{bl^*}{N} & 0 \\
-a & -i \omega_0 + a + d_C + \beta e^{-i \omega_0 \tau_y} & -\beta e^{-i \omega_0 \tau_y} \\
0 & \frac{bS^*}{N} & \frac{bS^*}{N} \\
0 & -i \omega_0 + r + d_I
\end{pmatrix}
\end{pmatrix}
\begin{pmatrix}
q_1^* \\
q_2^* \\
q_3^* \\
q_4^*
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.
\]

(60)

$q_1^*, q_2^*, q_3^*$, and $q_4^*$ have no trivial solution. Assume $q_1^* = 1$; we obtain
coordinates describing the center manifold direction of
Complexity 13
and for center manifold which can be rewritten as
\[ q_2^* = \frac{-i\omega_0 + bi^* / N + d_S}{bi^* / N}, \]
\[ q_3^* = \frac{-a + [-i\omega_0 + a + d_c + \beta e^{-i\omega_0 t_\xi}] q_2^*}{\beta e^{-i\omega_0 t_\xi}}, \]
\[ q_4^* = 0. \]

In order to satisfy the normalized condition
\[ \langle q^* (s), q(\theta) \rangle = 1 \]
and the orthogonal condition
\[ \langle q^* (s), \overline{q}(\theta) \rangle = 0, \]
we need to determine the value \( D \). From (54), we know

\[ \langle q^* (s), q(\theta) \rangle = \overline{q}^T (0) q(0) - \int_{-1}^{0} \int_{\xi=0}^{\theta} \overline{q} (\xi - \theta) d\eta(\theta) q(\xi) d\xi \]
\[ = \overline{D} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} - \overline{D} \int_{-1}^{\theta} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} e^{-i\omega_0 \tau_1 (\xi - \theta)} d\eta(\theta) \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} e^{i\omega_0 \tau_1 \xi} d\xi \]
\[ = \overline{D} \begin{pmatrix} q_1 + q_2 q_2 + q_3 q_3 + q_4 q_4 \end{pmatrix} - \overline{D} \int_{-1}^{\theta} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} e^{-i\omega_0 \tau_1 \theta} d\eta(\theta) \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} + \overline{D} r_0 e^{-i\omega_0 \tau_1 \beta q_3 (q_3 - q_2)} = 1. \]

So, we choose \( D \) satisfying
\[ \frac{1}{D} = \overline{q_1} q_1 + \overline{q_2} q_2 + \overline{q_3} q_3 + \overline{q_4} q_4 + r_0 e^{-i\omega_0 \tau_1 \beta q_3 (q_3 - q_2)}. \]

With the same notations in [31], we first calculate the coordinates describing the center manifold \( C_0 \) at \( \mu = 0 \). Assume \( u_t \) is the solution of system (8) for \( \mu = 0 \).

Let \( z(t) = \langle q^* , u_t \rangle \) and
\[ W(t, \theta) = u_t (\theta) - 2 Re [ z(t) q(\theta) ]. \]

On the center manifold \( C_0 \), we obtain
\[ W(t, \theta) = W(z(t), \Xi(t), \theta). \]

With
\[ W(z(t), \Xi(t), \theta) = W_{20} (\theta) \frac{z^2}{2} + W_{11} (\theta) z \Xi + W_{02} (\theta) \frac{\Xi^2}{2} + W_{30} (\theta) \frac{z^3}{6} + \cdots \]

and for center manifold \( C_0 \), the local coordinates in the direction of \( q^* \) and \( \overline{q}^* \) are \( z \) and \( \Xi \), respectively. We know if \( u_t \) is real, then \( W \) is real. Here, we only consider real solutions.

To the solution \( u_t \in C_0 \) of system (8), owing to \( \mu = 0 \),

\[ z'(t) = i\omega_0 r_0 z(t) + \langle q^* (\theta), r_0 F(W(z(t), \Xi(t), \theta) + 2 Re [ z(t) q(\theta) ] \rangle \]
\[ = i\omega_0 r_0 z + \overline{q}^* (0) r_0 F(W(z(t), \Xi(t), 0) + 2 Re [ z(t) q(0) ] = i\omega_0 r_0 z + q^* (0) r_0 F_0 (z, \Xi), \]

which can be rewritten as
\[ z'(t) = i\omega_0 r_0 z(t) + g(z, \Xi). \]
With
\[ g(z, \bar{z}) = q^*(0)\tau_0 F(W(z(t), \bar{z}(t), 0) + 2\text{Re}(z(t)q(0))) \]
\[ = g_{20} \frac{z^2}{2} + g_{11} z\bar{z} + g_{02} \frac{\bar{z}^2 z}{2} + g_{21} \frac{z^2 \bar{z}}{2} + \cdots. \]  

(69)

From (52), (68), and (64), we can obtain
\[ W' = u'_t - z' q - \bar{z} q' = \begin{cases} AW - 2\text{Re}(q^*(0)\tau_0 F_0 q(\theta)), \theta \in [-1, 0), \\ AW - 2\text{Re}(q^*(0)\tau_0 F_0 q(0)) + \tau_0 F_0, \theta = 0, \end{cases} \]

(70)

with
\[ H(z, \bar{z}, \theta) = H_{20}(\theta) \frac{z^2}{2} + H_{11}(\theta) z\bar{z} + H_{02}(\theta) \frac{\bar{z}^2 z}{2} + \cdots. \]  

(71)

If we expand the above series and compare the corresponding coefficients, then we can obtain
\[ \begin{align*}
-H_{20}(\theta) &= (A - 2i\omega_0 \tau_0) W_{20}(\theta), \\
-H_{11}(\theta) &= AW_{11}(\theta), \\
-H_{02}(\theta) &= (A + 2i\omega_0 \tau_0) W_{02}(\theta), \\
&\quad \vdots
\end{align*} \]

(72)

By (67), assume \( F_0(z, \bar{z}) = F_{20} z^2/2 + F_{11} z\bar{z} + F_{02} \bar{z}^2 z/2 + F_{21} z^2 \bar{z}/2 + \cdots \) and (68); we obtain
\[ \begin{align*}
g_{20} &= q^*(0)\tau_0 F_{20}, \\
g_{11} &= q^*(0)\tau_0 F_{11}, \\
g_{02} &= q^*(0)\tau_0 F_{02}, \\
g_{21} &= q^*(0)\tau_0 F_{21}. \end{align*} \]

(73)

By (64), we can get \( u_t(\theta) = W(t, \theta) + z(t) q(\theta) + \bar{z}(t) \bar{q}(\theta) \), so
\[ u(t) = W(t, 0) + z(t) q(0) + \bar{z}(t) \bar{q}(0) \]
\[ = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} z + \begin{pmatrix} \bar{q}_1 \\ \bar{q}_2 \\ \bar{q}_3 \\ \bar{q}_4 \end{pmatrix} \bar{z} + W_{20}(0) \frac{z^2}{2} + W_{11}(0) z\bar{z} + W_{02}(0) \frac{\bar{z}^2 z}{2} + \cdots, \]

(74)

with
\[
S(t) = q_1 z + \overline{q_1} \overline{z} + W^{(1)}(t, 0),
\]
\[
C(t) = q_1 z + \overline{q_1} \overline{z} + W^{(2)}(t, 0),
\]
\[
I(t) = q_1 z + \overline{q_1} \overline{z} + W^{(3)}(t, 0),
\]
\[
R(t) = q_1 z + \overline{q_1} \overline{z} + W^{(4)}(t, 0),
\]

From (70), we can directly compute, but we need to compute \( W_20(\theta) \) and \( W_{11}(\theta) \) for \( \theta \in [-1, 0] \) in the following order to compute \( g_{21} \).

\[
g_{20} = \frac{2br_0 \mathcal{D}}{N} (\overline{q_2} - \overline{q_1}) q_1 q_3, \quad g_{11} = \frac{br_0 \mathcal{D}}{N} (\overline{q_2} - \overline{q_1})(q_1 \overline{q_3} + \overline{q_1} q_3),
\]
\[
g_{21} = \frac{2br_0 \mathcal{D}}{N} (\overline{q_2} - \overline{q_1}) \left[ q_1 W_{11}^{(3)}(0) + q_3 W_{11}^{(1)}(0) + \frac{1}{2} \right] + \frac{1}{2} \frac{W_{20}^{(1)}(0)}{2} + \frac{1}{2} \frac{W_{20}^{(1)}(0)}{2}.
\]

In addition, we have

\[
\begin{align*}
S(t-1) &= e^{-i\omega t} q_1 z + \overline{e^{-i\omega t} q_1} \overline{z} + W^{(1)}(t-1), \\
C(t-1) &= e^{-i\omega t} q_1 z + \overline{e^{-i\omega t} q_1} \overline{z} + W^{(2)}(t-1), \\
I(t-1) &= e^{-i\omega t} q_1 z + \overline{e^{-i\omega t} q_1} \overline{z} + W^{(3)}(t-1), \\
R(t-1) &= e^{-i\omega t} q_1 z + \overline{e^{-i\omega t} q_1} \overline{z} + W^{(4)}(t-1),
\end{align*}
\]

and (76), we compare the coefficients with (69) and have

\[
\begin{pmatrix}
-b N S I \\
-b N S I \\
0 \\
0
\end{pmatrix} = \frac{br_0 \mathcal{D}}{N} (\overline{q_2} - \overline{q_1}) S I.
\]
\[ W' = AW - 2\text{Re}[\bar{q}^* (0) \tau_0 F_0 q(\theta)] = AW - \bar{q}^* (0) \tau_0 F_0 q(\theta) - q^* (0) \tau_0 F_0 \bar{q}(\theta) \]

and

\[ W' = AW + H(z, \bar{z}, \theta). \]

Hence,

\[ H(z, \bar{z}, \theta) = -g(z, \bar{z})q(\theta) - \bar{g}(z, \bar{z})\bar{q}(\theta) = -(A - 2i\omega_0 \tau_0)W_{20}(\theta), \]

(82)

and

\[ H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta) = -AW_{11}(\theta). \]

(83)

By (82), we obtain

\[ AW_{20}(\theta) = 2i\omega_0 \tau_0 W_{20}(\theta) + g_{20}q(\theta) + \frac{\bar{g}_{02}}{2} q(\theta). \]

(84)

From (76), hence,

\[ W_{20}(\theta) = 2i\omega_0 \tau_0 W_{20}(\theta) + g_{20}q(0)e^{i\omega_0 \tau_0} + \frac{\bar{g}_{02}}{2} q(0)e^{-i\omega_0 \tau_0}. \]

(85)

If we solve the above equation for \( W_{20}(\theta) \), then we have

\[ W_{20}(\theta) = K_1 e^{2i\omega_0 \tau_0} - \frac{g_{20}q(0)}{i\omega_0 \tau_0} + \frac{\bar{g}_{02}}{3i\omega_0 \tau_0} q(0). \]

(86)

By the same method, from (83), we can obtain

\[ AW_{11}(\theta) = g_{11}q(\theta) + \bar{g}_{11}\bar{q}(\theta). \]

(87)

From (76), hence,

\[ W_{11}(\theta) = g_{11}q(0)e^{i\omega_0 \tau_0} + \bar{g}_{11}\bar{q}(0)e^{-i\omega_0 \tau_0}. \]

(88)

If we solve the above equation for \( W_{11}(\theta) \), then we have

\[ W_{11}(\theta) = K_2 + \frac{g_{11}q(0)}{i\omega_0 \tau_0} e^{i\omega_0 \tau_0} - \frac{\bar{g}_{11}}{i\omega_0 \tau_0} \bar{q}(0). \]

(89)

Secondly, to obtain \( K_1 \) and \( K_2 \), we study the situation of \( \theta = 0 \) of (70). By (70), we obtain

\[ W' = AW + \tau_0 F_0 - 2\text{Re}[\bar{q}^* (0) \tau_0 F_0 q(\theta)] = AW + H(z, \bar{z}, 0). \]

(90)

So,

\[ H(z, \bar{z}, 0) = -2\text{Re}[\bar{q}^* (0) \tau_0 F_0 q(0)] + \tau_0 F_0 \]

\[ = \left[ g_{20} \frac{\bar{z}^2}{2} + g_{11}z\bar{z} + g_{02} \frac{\bar{z}^2}{2} \right] q(0) - \left[ \frac{\bar{g}_{20}}{2} z\bar{z} + \bar{g}_{11} z\bar{z} + \bar{g}_{02} \frac{\bar{z}^2}{2} \right] \bar{q}(0) + \begin{pmatrix} \frac{b}{N} SI \\ 0 \\ 0 \end{pmatrix} \]

\[ = H_{20}(0) \frac{\bar{z}^2}{2} + H_{11}(0) z\bar{z} + H_{02}(0) \frac{\bar{z}^2}{2} \]
If we put (76) into the above equation and compare the coefficient of $z^2/2$ and $z\overline{z}$, then we obtain

$$H_{20}(0) = -g_{20}q(0) - \overline{a_0}\overline{q}(0) + \begin{pmatrix} \frac{2b}{N}q_1q_3 \\ \frac{2b}{N}q_1q_3 \\ 0 \\ 0 \end{pmatrix},$$

$$H_{11}(0) = -g_{11}q(0) - \overline{a}_1\overline{q}(0) + \begin{pmatrix} \frac{-b}{N}(q_1\overline{q}_3 + q_3\overline{q}_1) \\ \frac{b}{N}(q_1\overline{q}_3 + q_3\overline{q}_1) \\ 0 \\ 0 \end{pmatrix}.$$  \hfill (92)

By $A W_{20}(0) = 2i\omega_0\tau_0\overline{W}_{20}(0) - H_{20}(0)$, we obtain

$$\tau_0 B W_{20}(0) + \tau_0 C W_{20}(-1) = 2i\omega_0\tau_0\overline{W}_{20}(0) - H_{20}(0).$$  \hfill (93)

If (86) and (92) are put into (93), then we can obtain

$$\tau_0 B K_1 = \frac{g_{20}}{i\omega_0\tau_0}q(0) - \frac{\overline{a}_0}{3i\omega_0\tau_0}\overline{q}(0) + e^{-2i\omega_0\tau_0}\tau_0 C K_1 - \frac{g_{20}e^{-i\omega_0\tau_0}}{i\omega_0\tau_0}\tau_0 B q(0) - \frac{\overline{a}_0 e^{i\omega_0\tau_0}}{3i\omega_0\tau_0}\tau_0 B \overline{q}(0)$$

$$-2i\omega_0\tau_0\left(K_1 - \frac{g_{20}}{i\omega_0\tau_0}q(0) - \frac{\overline{a}_0}{3i\omega_0\tau_0}\overline{q}(0)\right).$$  \hfill (94)

By

$$\left(\tau_0 B + e^{-i\omega_0\tau_0}C\right)q(0) = i\omega_0\tau_0 q(0),$$

$$\left(\tau_0 B + e^{i\omega_0\tau_0}C\right)\overline{q}(0) = -i\omega_0\tau_0 \overline{q}(0).$$  \hfill (95)

We obtain

$$\left(2i\omega_0\tau_0 I - \tau_0 B - e^{-2i\omega_0\tau_0}C\right)K_1 = \begin{pmatrix} \frac{2b}{N}q_1q_3 \\ \frac{2b}{N}q_1q_3 \\ 0 \\ 0 \end{pmatrix}.$$  \hfill (96)

So,

$$K_1 = \tau_0 \left(2i\omega_0 I - B - e^{-2i\omega_0\tau_0}C\right)^{-1} \begin{pmatrix} \frac{2b}{N}q_1q_3 \\ \frac{2b}{N}q_1q_3 \\ 0 \\ 0 \end{pmatrix}. \hfill (97)

By $A W_{11}(0) = -H_{11}(0)$, we obtain

$$\tau_0 B W_{11}(0) + \tau_0 C W_{11}(-1) = -H_{11}(0).$$  \hfill (98)

If (89) and (92) are put into (98), then we can obtain
By (95), we obtain
\[
(\tau_0 B + \tau_0 C)K_2 = -\begin{pmatrix}
\frac{b}{N} (q_1 \bar{q}_3 + \bar{q}_1 q_3) \\
0 \\
0
\end{pmatrix}.
\] (100)

So,
\[
K_2 = -\tau_0 (B + C)^{-1} \begin{pmatrix}
\frac{b}{N} (q_1 \bar{q}_3 + \bar{q}_1 q_3) \\
0 \\
0
\end{pmatrix}.
\] (101)

From \( K_1 \) and \( K_2 \), we have \( W_{11}(\theta) \) and \( W_{20}(\theta) \). Hence, we can determine \( g_{21} \). On the basis of the above analysis, we know each \( g_{ij} \) in (78) is decided by the parameters and delays in system (8). Therefore, we can have the following quantities:
\[
C_1(0) = \frac{i}{2\omega_0 \tau_0} \left( g_{20} g_{11} - 2 |g_{11}|^2 - \frac{1}{3} |g_{02}|^2 \right) + \frac{g_{21}}{2},
\]
\[
\mu_2 = \frac{\text{Re}[C_1(0)]}{\text{Re}[\lambda'(\tau_0)]},
\]
\[
T_2 = -\frac{\text{Im}[C_1(0)] + \mu_2 \text{Im}[\lambda'(\tau_0)]}{\omega_0 \tau_0},
\]
\[
\beta_2 = 2 \text{Re}[C_1(0)].
\]

It determines the properties of bifurcating periodic solutions near the critical value \( \tau_0 \), and we get the following theorem.

**Theorem 7.** The following results hold for the expression of (102).

(i) The direction of the Hopf bifurcation is determined by the sign of \( \mu_2 \), that is, Hopf bifurcations are forward (backward) for \( \mu_2 > 0 \) \( (< 0) \), respectively

(ii) The stability of bifurcating periodic solutions is determined by the sign of \( \beta_2 \), that is, the bifurcation periodic solutions on the center manifold are stable (unstable) for \( \beta_2 < 0 \) \( (> 0) \), respectively
(iii) The periods of the bifurcating periodic solutions is determined by the sign of $T_2$, that is, the periods increase (decrease) for $T_2 > 0$ ($< 0$), respectively.

Example 3. Assume $a = 0.1, b = 0.025, \beta = 0.058, r = 0.015, N = 1.0 \times 10^4, \Lambda_1 = 5000, \Lambda_2 = 300, \Lambda_3 = 2, \Delta_s = 1.81 \times 10^{-5}, d_C = 2.33 \times 10^{-5}, d_I = 5.48 \times 10^{-5}$, and $d_R = 1.95 \times 10^{-5}$, then system (8) reads

$$
\begin{align*}
S'(t) &= 5000 + 0.1C(t) - \frac{0.025S(t)I(t)}{1.0 \times 10^7} - 1.81 \times 10^{-5}S(t), \\
C'(t) &= 300 + \frac{0.025S(t)I(t)}{1.0 \times 10^7} - 0.1C(t) - 0.058C(t-r) - 2.33 \times 10^{-5}C(t), \\
I'(t) &= 2 + 0.058C(t-r) - 0.015I(t) - 5.48 \times 10^{-5}I(t), \\
R'(t) &= 0.015I(t) - 1.95 \times 10^{-5}R(t).
\end{align*}
$$

(103)

Through calculation, system (103) has a positive equilibrium $(S^*, C^*, I^*, R^*) = (16040222, 86339, 332760, 2.5597 \times 10^8)$. $a_1 = 0.058, a_2 = 0.1159, a_3 = -0.0014, a_4 = 0.0015, a_5 = 7.0011 \times 10^{-7}, a_6 = 2.7547 \times 10^{-8}, \Delta_1 = 0.1739, \Delta_2 = 1.9640 \times 10^{-5},$ and $\Delta_3 = 1.4291 \times 10^{-11}$, satisfying condition (19). Furthermore, $p = 0.0070, q = 4.1721 \times 10^{-7}$, and $g = -4.8939 \times 10^{-13}$, so the Hopf critical value $\tau_c = 311$ by the calculation process in Section 3.2.

From Theorems 3 and 4, this equilibrium $(16040222, 86339, 332760, 2.5597 \times 10^8)$ of system (103) is asymptotically stable for $\tau \in [0, 311)$ and unstable for $\tau > 311$, which can also be found from Figures 3–6 for the
Figure 4: The time histories of $S(t), C(t), I(t),$ and $R(t)$ in system (103), for $\tau = 10$, with initial functions $(S(-\tau), C(-\tau), I(-\tau), R(-\tau)) = (S(0), C(0), I(0), R(0)) = (1.6 \times 10^7, 8.6 \times 10^4, 3.3 \times 10^5, 2.5 \times 10^8)$ and $t: 68000 - 70000$, respectively.

Figure 5: The time histories of $S(t), C(t), I(t),$ and $R(t)$ in system (103), for $\tau = 15$, with initial functions $(S(-\tau), C(-\tau), I(-\tau), R(-\tau)) = (S(0), C(0), I(0), R(0)) = (1.6 \times 10^7, 8.6 \times 10^4, 3.3 \times 10^5, 2.5 \times 10^8)$ and $t: 68000 - 70000$, respectively.
values of interest, $\tau = 4, 10, 15, 25$, respectively, since to the best knowledge of the authors, the most incubation delay is 24 days. Note that, in Figures 3–6, four orbits best knowledge of the authors, the most incubation delay is simulate it any more.

4. Conclusions

In this study, we divided people into four categories: $S(t)$ susceptible, $C(t)$ closely contacted, $I(t)$ infective, and $R(t)$ removed according to the current situation in the middle and late stages of COVID-19, investigating the SCIR models with immigration and without immigration, and obtained the following main results:

1. If $b < r/2$, then the disease-free equilibrium $E_0(S_0, 0, 0)$ of Model 1 is globally asymptotically stable
2. If $\Delta_1 > 0, \Delta_2 > 0, \Delta_3 > 0$, the endemic equilibrium $E^*$ of Model 2 is asymptotically stable for $r \in [0, r_0)$
3. If condition (42) holds, then the positive equilibrium $E^*(S^*, C^*, I^*, R^*)$ of Model 2 is locally asymptotically stable by constructing Lyapunov function
4. By the bifurcation formulae, in [31], we qualitatively analyze the properties for the occurring Hopf bifurcations of Model 2

Moreover, considering neither the immigration nor the death rate during the outbreak, Model 1 has only the disease-free equilibrium. However, considering both the immigration and the death rate during late stage, Model 2 has only one or two endemic equilibria. Both indicate the importance of finding closely contacted and overseas imports on epidemic control, respectively.

Data Availability

The numerical simulation data used to support the findings of the study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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