Constant mean curvature surfaces of Delaunay type along a closed geodesic

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Abstract

In this paper, we construct Delaunay type constant mean curvature surfaces along a non degenerate closed geodesic in a 3-dim Riemannian manifold.

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1 Introduction

Delaunay surface is a kind of important non-compact constant mean curvature (CMC) surface in Euclidean space $\mathbb{R}^3$. In this paper, we construct a kind of CMC surface along a non-degenerate geodesic in Riemannian 3-manifold. This CMC surface is compact and it resembles Delaunay surface in $\mathbb{R}^3$. It can be regarded as a parallel work as that of Mazzeo and Pacard\cite{2} where they constructed a foliation by tubes along a non-degenerate geodesic. The original example in Euclidean space of the tube is column. Notice that the column embedded in $\mathbb{R}^3$ can be thought of as the limit case of Delaunay undoloid. However the Delaunay surface is much more complex than column.

Let’s state the main theorem of this paper roughly:

**Theorem 1.1.** Suppose $(M^3, g)$ is a Riemannian manifold of 3 dimensional and $\Gamma$ is a simple closed embedded geodesic with nondegenerate Jacobi operator. Then for any Delaunay parameter $\tau \in (0, \frac{1}{4})$ we can find $\varepsilon_0 > 0$ which depends on $\tau$ such that there is a monotone sequence $\varepsilon_n \to 0$ with $\varepsilon_0 > \varepsilon_1 > \cdots > \varepsilon_n > \cdots$ such that there are at least two constant mean curvature surfaces of Delaunay type, with mean curvature $2/\varepsilon_n$, along the geodesic which is of size $\varepsilon_n$.

The terms “Delaunay type” and “size $\varepsilon_n$” will be made clear in the next section.

This paper is organized as follows: In Section 2, we revise some basic facts of Delaunay unduloid (which we call Delaunay surface) and we make a discription of how we arrange an initial surface along the closed geodesic and how we perturb the initial surface. Then we calculate the first fundamental form and second fundamental of the perturbed initial surface. In Section 2, after long calculation we get the mean curvature of the perturbed initial surface. In Section 3 we analyze the Jacobi operator of the perturbed initial surface. We divide the function space and Jacobi operator into 3 parts, i.e. high mode, 1st mode and 0th mode. For high mode, it is easy to prove that the operator is invertible and the inverse has good bound. For 1st mode, after careful examination, we find the Jacobi operator converges in some sense to the Jacobi operator of the geodesic, which is invertible by the assumption that the geodesic is non-degenerate. Here we prove an “average 1” lemma which verified this convergence. For 0th mode, it is not easy to analyze the linearized equation directly. First we can solve the 0th mode ODE in nonlinear way through the first integral $\tau$. Then we have some basic estimates for the 0th mode ODE. Then we can start to analyze the linearized ODE of 0th mode. Another thing which needs to be considered is that the 0th mode linearized ODE has kernel. So one can only solves this mode up to the kernel. In section 4, we use fixed point theorem to solve the 3 modes together. In the expression of mean curvature, there are some big terms which prevent the application of fixed point theorem. It is fortunate that we can find an “average 0” lemma which helps to deal with this issue. At last, to kill the kernel of 0th mode we analyze the energy functional. One last thing we can choose is the starting point of the initial surface. Then for the maximal point and the minimal point of the Energy, one can at least get two constant mean curvature surfaces of Delaunay type.
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2 Geometry of Delaunay surfaces

2.1 Delaunay surfaces in Euclidean space $\mathbb{R}^3$

First we give a brief revise of the definition of Delaunay surfaces in Euclidean space $\mathbb{R}^3$. There are two kinds of Delaunay surfaces in $\mathbb{R}^3$, Delaunay nodoids and Delaunay unduloids. The first kind can be immersed into $\mathbb{R}^3$, and the second type can be embedded into $\mathbb{R}^3$. In this paper, by Delaunay surface, we always mean Delaunay unduloids. The Delaunay unduloid $D_\tau$ can be parameterized by

$$X_\tau(s, \theta) := (\phi_\tau(s) \cos \theta, \phi_\tau(s) \sin \theta, \psi_\tau(s)),$$

where $(s, \theta) \in \mathbb{R} \times S^1$. Here $\tau$ is a real parameter and $0 < \tau \leq \frac{1}{4}$. $(\phi, \psi)$ is the solution to the following system

$$\begin{cases} 
\dot{\phi}^2 + (\phi^2 + \tau)^2 = \phi^2, & \phi(0) = \frac{1 - \sqrt{1 - 4\tau}}{2} \\
\dot{\psi} = \phi^2 + \tau, & \psi(0) = 0,
\end{cases}$$

where the derivative “$\cdot$” is taken with respect to parameter $s$. This solution is periodic.

**Remark.** This solution satisfies

$$\frac{1 - \sqrt{1 - 4\tau}}{2} \leq \phi(s) \leq \frac{1 + \sqrt{1 - 4\tau}}{2}.$$

So the initial value is the minimal value of $\phi$.

**Remark.** From $\dot{\psi} = \phi^2 + \tau$ we know $\psi$ is always increasing as $s$ increases. So the surface $X_\tau(s, \theta)$ is embedded. We can also regard $\phi$ and a function of $\psi$.

If we regard $\phi$ as a function of $\psi$. Then $\phi(\psi)$ has to satisfies some second order ODE. This ODE can also be regard as another definition of Delaunay surfaces. From

$$\dot{\phi}^2 + (\phi^2 + \tau)^2 = \phi^2,$$

we have

$$\ddot{\phi} = \phi - 2\phi(\phi^2 + \tau).$$

And $\dot{\phi} = \phi'(\psi)\dot{\psi}$ and $\ddot{\phi} = \phi''(\psi)(\phi^2 + \tau)^2 + 2\phi\phi_\psi(\phi^2 + \tau)$, where “$'$” or “$\phi_\psi$” denote the derivative of $\phi$ with respect to $\psi$. At last, we have

$$\phi_\psi^2 = \frac{\phi'^2}{\psi^2} = \frac{\phi^2}{(\phi^2 + \tau)^2} - 1.$$
Combining all these, we get the ODE satisfied by $\phi(\psi)$,

$$
\begin{cases}
\phi_{\psi\psi} - \phi^{-1}(1 + \phi^2) + 2(1 + \phi^2)^{\frac{3}{2}} = 0, \\
\phi(0) = \frac{1 - \sqrt{1 - 4\tau}}{2}, \\
\phi_{\psi}(0) = 0.
\end{cases}
$$

(1)

We can also find this from [1]. By direct calculation, we know

$$X_{\tau}(s, \theta) = (\phi \cos \theta, \phi \sin \theta, \psi) \subset (\mathbb{R}^3, d\psi^2 + d\phi^2 + \phi^2 d\theta^2)$$

has mean curvature

$$-\phi_{\psi\psi}(1 + \phi^2)^{-3/2} + \phi^{-1}(1 + \phi^2)^{-1/2}.$$

From this and the ODE (1), we get the mean curvature of Delaunay surfaces is 2 no matter how we choose $\tau$. Actually, when $\tau = \frac{1}{4}$, we get a limit solution to ODE (1), which is a column of radius $\frac{1}{2}$. If $\tau \to 0$, the limit solution of ODE (1) is $\phi = \sqrt{1 - (\psi - 2i - 1)^2}, i \in \mathbb{Z}$. This is a singular limit of Delaunay surfaces which is in fact infinite many spheres of radius 1 with each sphere meeting the two neighbors at the two poles.

An important property of ODE (1) is that $\tau$ can be regarded as the first integral

$$\tau = -\phi^2 + \frac{\phi}{\sqrt{1 + \phi^2}}.$$

Along the ODE, $\tau$ is conservative.

2.2 Fermi coordinates and Taylor expansion of the metric near the geodesic

From now on we discuss the geometry of Delaunay surface along a geodesic in Riemannian 3-manifold. Fix an arc length parametrization $\gamma(x_0)$ of the geodesic $\Gamma$, $x_0 \in [a, b]$. We denote the normal bundle of $\Gamma$ by $NT$. Choose a parallel orthonormal $E_1, E_2$ for $NT$ along [a, b]. This determines a coordinate system

$$x : (x_0, x_1, x_2) \mapsto \exp_{\gamma(x_0)}(x_1 E_1 + x_2 E_2) := F(x),$$

and we denote the corresponding coordinate vector fields by $X_{\alpha} := F_*(\partial_{x_{\alpha}})$. We adopt the convention that indices $i, j, k, \cdots \in \{1, 2\}$ while $\alpha, \beta, \cdots \in \{0, 1, 2\}$. Let $r = \sqrt{x_1^2 + x_2^2}$. By Gauss’ Lemma $r$ is the geodesic distance from $x$ to $\Gamma$ and the vector $\partial_r = \frac{1}{r}(x_1 X_1 + x_2 X_2)$ is perpendicular to $X_0$ and $\partial_0 = -x_2 X_1 + x_1 X_2$.

By the setting above we have in fact arranged the metric coefficients $g_{\alpha\beta} = <x_{\alpha}, x_{\beta}> \text{ equal } \delta_{\alpha\beta} \text{ along } \Gamma$. Now we are going to calculate higher terms in the Taylor expansions of $g_{\alpha\beta}$. By the notation $O(r^m)$, we mean a function $f$ such that it and its partial derivatives of any order, with respect to the vector fields $X_0$ and $x_i X_i$, are bounded by $Cr^m$ in some fixed $T_{\rho}(\Gamma)$.

First for the covariant derivative, we have
Lemma 2.1. For $\alpha, \beta = 0, 1, 2$,
\[
\nabla X_\alpha X_\beta = \sum_{\gamma=0}^{2} O(r)X_\gamma,
\]
and more precisely for $\alpha = \beta = 0$, we have
\[
\nabla X_0 X_0 = -2\sum_{i,j=1} R(X_j, X_0, X_i, X_0) p x_i x_j + \sum_{\gamma=0}^{2} O(r^2) X_\gamma,
\]
where $R(X_i, X_j, X_k, X_l) = <\nabla X_i \nabla X_j X_k - \nabla X_j \nabla X_i X_k - \nabla [X_i, X_j] X_k, X_l>$.

See [2] Lemma 2.1 for the proof.

The next lemma gives the expansion of the metric coefficients in Fermi coordinates.

Lemma 2.2. In the same notation as before, we have
\[
g_{ij}(q) = \delta_{ij} + \frac{1}{3} R(X_k, X_i, X_j) p x_k x_l + O(r^3)
\]
\[
g_{0i}(q) = \frac{2}{3} R(X_k, X_0, X_i) p x_k x_l + O(r^3)
\]
\[
g_{00}(q) = 1 + R(X_k, X_0, X_0) p x_k x_l + O(r^3)
\]
where $R(X_i, X_j, X_k, X_l) = <\nabla X_i \nabla X_j X_k - \nabla X_j \nabla X_i X_k - \nabla [X_i, X_j] X_k, X_l>$.

See [2] Proposition 2.1 for the proof of the first one and third one. The second one can be proved in the same way.

2.3 Initial Delaunay surfaces and the perturbation

First we arrange an initial Delaunay surface of size $\varepsilon$ along the geodesic. Choose a starting point $p_0$ on the closed geodesic $\Gamma$. Suppose the length of the geodesic is $L$. We fixed a parameter $\tau_0 \in (0, 1/4)$. We assume $\phi_{p_0, \tau_0}(\psi)$ is the solution to ODE[1] with $\tau = \tau_0$. Suppose the period of $\phi(\psi)$ is $\text{Per}(\tau_0)$. Suppose $\varepsilon$ is very small and such that $L = \varepsilon \text{Per}(\tau_0) N$, where $N \in \mathbb{N}^+$ and is very big. It is obvious that one can only choose a sequence of such $\varepsilon$ which tends to 0. We call this sequence the proper size for $\tau_0$ and $L$. With such $\varepsilon$, we can arrange a Delaunay type initial surface around the geodesic. To be precise, if we have found an orthonormal basis $\{E_1, E_2\}$ for the normal bundle of the geodesic in $[p_0, p_1]$, in local coordinates $\{x_0, x_1, x_2\}$, the surface can be expressed as
\[
\begin{aligned}
x_0 &= \varepsilon \psi, \\
r &= \sqrt{x_1^2 + x_2^2} = \varepsilon \phi,
\end{aligned}
\]
\[
\psi(p_0) = 0, \quad \phi(p_0) = \frac{1 - \sqrt{1 - 4\tau_0}}{2}.
\]
We can extend this definition along the geodesic. This definition does not depend on the choice of $\{E_1, E_2\}$. So from the condition $L = \varepsilon \text{Per}(\tau_0) N$, we can make the left side and right side of $p_0$ coincide. We have got a well defined smooth initial surface of Delaunay type. We denote this initial surface by $\partial_{\phi_{p_0, \tau_0}, p_0, \tau}$. We know
that when $\epsilon$ is very small and proper for $\tau_0$ and $L$, the mean curvature of this surface is nearly $\frac{2}{3}$. Our aim is to perturb this initial surface such that it will have exactly constant mean curvature $\frac{2}{3}$.

Consider the following perturbation of $\mathcal{D}_{\phi_{\tau_0},\gamma_0}(w,\eta)$, denoted by $\mathcal{D}_{\phi_{\tau_0},\gamma_0}(w,\eta)$, where $w$ is a function on unit circle bundle $\mathcal{N}G$ and $\eta$ is a section of $\mathcal{N}G$.

The unit circle bundle is locally trivialized by the map

$$[a,b] \times S^1 \ni (x_0, \Upsilon) \mapsto (\gamma(x_0), \sum_{j=1}^{2} \Upsilon_j E_j) \in \mathcal{N}G.$$ 

Fix $\epsilon > 0$, and define

$$G(x_0, \Upsilon) := F(x_0, \epsilon(\phi(x_0) + w(x_0, \theta)) \Upsilon + \eta(x_0));$$

the image of this map is denoted by $\mathcal{D}_{\phi_{\tau_0},\gamma_0}(w,\eta)$. $\mathcal{D}_{\phi_{\tau_0},\gamma_0}(w,\eta)$ is obtained by first taking the vertical graph of the function $\epsilon w$ over the initial Delaunay surface $\mathcal{D}_{\phi_{\tau_0}}$ and then translating by $\eta$.

**Remark.** We distinguish $\Upsilon$ from $\theta$. $\Upsilon$ denotes the unit vector from origin to some point of $S^1$ while $\theta$ is the standard parameter of $S^1$. Later $\Upsilon_\theta$ will denote the derivative of $\Upsilon$ with respect to $\theta$, which is the tangential vector of $S^1$.

The spaces we will work in is the $H^1_\epsilon$ Hölder spaces. $C^{m,\alpha}_{\epsilon_0}$ will represents $C^{m,\alpha}(S\mathcal{N}G)$ or $C^{m,\alpha}_{\epsilon_0}(\mathcal{N}G)$. Also we will use modified $H^1_\epsilon$ Hölder spaces $C^{m,\alpha}_{\epsilon}(S\mathcal{N}G), C^{m,\alpha}_{\epsilon}(\mathcal{N}G)$ which are based on differentiations with respect to the vector fields $\epsilon \partial x_0 = \frac{\partial}{\partial \psi}$ and $\partial \theta$ where $\theta$ is the coordinates on $S^1$.

For $p \in \Gamma$, let $S^1_p$ denote the circle fibre of $\mathcal{N}G$ over $p$. Any function $w$ on $\mathcal{N}G$ decomposes into a sum of three terms

$$w = w_0 + w_1 + \tilde{w},$$

where the restriction to any $S^1_p$ of each of these terms lies in the span of the eigenfunctions $\xi_j$ on $S^1$ with $j = 0, j = 1, 2$, and $j > 2$, respectively. $w_0$ is a function on $\Gamma$. Next, the eigenfunctions $\xi_1 = \cos \theta, \xi_2 = \sin \theta$. So any linear combination of $\xi_1$ and $\xi_2$ can be identified with a translation in $\mathbb{R}^2$ ($\xi_1$ and $\xi_2$ correspond to the translations in $x$ and $y$ direction). Correspondingly, $w_1$ is canonically associated to a section $\eta$ of the normal bundle $\mathcal{N}G$.

We shall assume that the functions $w$ has “linear component” $w_1 = 0$, and shall regard the linear part of the perturbation as a section of $\mathcal{N}G$, as described above.

Now we can state Theorem 1.1 rigorously.

**Theorem 2.3.** For any $\tau_0 \in (0, 1/4)$ there is $\epsilon_0 > 0$ such that when $0 < \epsilon < \epsilon_0$ and $\epsilon$ is proper size for $\tau_0$ and $L$, we can choose at least two points $p_1, p_2$ on the geodesic and $w_1, \eta_1, w_2, \eta_2$ such that

$$H(\mathcal{D}_{\phi_{\tau_i},\gamma_0}(w_i,\eta_i)) = \frac{2}{\epsilon}, i = 1, 2.$$
and for uniform constant $C$ which

$$
\|w_i\|_{C^2_{\varepsilon}} \leq C\varepsilon
$$
$$
\|\eta_i\|_{C^1_{\varepsilon}} \leq C\varepsilon
$$
$$
\|\eta_i\|_{C^2_{\varepsilon}} \leq C\varepsilon
$$
$$
\|\eta_i\|_{C^2_{\varepsilon}} \leq C\varepsilon^{1-\alpha}.
$$

Moreover $\mathcal{D}_{\phi_{p_1,\tau_0,\varepsilon}}(w_1,\eta_1)$ is different from $\mathcal{D}_{\phi_{p_2,\tau_0,\varepsilon}}(w_2,\eta_2)$.

### 2.4 The first fundamental form

Now we calculate the first fundamental form of $\mathcal{D}_{\phi_{p_0,\tau_0,\varepsilon}}(w,\eta)$ with respect to the coordinate $(s,\theta)$. At the point

$$
q = F(\varepsilon\psi(s), \varepsilon(\phi(s) + w(s, \theta))\Upsilon(\theta) + \eta(\varepsilon\psi(s))).
$$

Suppose $p = F(\varepsilon\psi(s), 0)$. First we have

$$
\partial_s = \varepsilon(\dot{\psi}X_0 + (\dot{\phi} + \frac{\partial w}{\partial s})\Upsilon + \dot{\psi} \frac{\partial \eta}{\partial x_0})
$$
$$
\partial_\theta = \varepsilon((\phi + w)\Upsilon_\theta + \frac{\partial w}{\partial \theta} \Upsilon),
$$

and $x_k(q) = \varepsilon(\phi(s) + w(s, \theta))\Upsilon^k + \eta^k, k = 1, 2$.

From $[2]$ we know

**Lemma 2.4.**

$$
<X_0, X_0 >_q = 1 + \varepsilon^2 \phi^2 R(Y, X_0, Y, X_0)_p + 2\varepsilon\phi R(Y, X_0, \eta, X_0)_p + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3),
$$
$$
<X_i, X_j >_q = \delta_{ij} + \frac{1}{3}\varepsilon^2 \phi^2 R(Y, X_i, Y, X_j)_p + \frac{1}{3}\varepsilon\phi(R(Y, X_i, \eta, X_j)_p + R(\eta, X_i, Y, X_j)_p) + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3),
$$
$$
<X_0, X_i >_q = \frac{2}{3}\varepsilon^2 \phi^2 R(Y, X_0, Y, X_i)_p + \frac{2}{3}\varepsilon\phi(R(Y, X_0, \eta, X_i)_p + R(\eta, X_0, Y, X_i)_p) + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3).
$$

We use these expansions to obtain the expansion of the first fundamental form,
Lemma 2.5.

\[ \varepsilon^{-2} < \partial_s, \partial_s > = \phi^2 + \varepsilon^2 \phi^2 \psi^2 \mathcal{R}(Y, X_0, Y, X_0) + 2 \varepsilon \phi \dot{\psi} \mathcal{R}(Y, X_0, \eta, X_0) + 2 \phi \frac{\partial w}{\partial s} + 2 \phi \dot{\psi} < \eta, \frac{\partial \eta}{\partial x_0} > + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3), \]

\[ \varepsilon^{-2} < \partial_s, \partial_s > = \frac{2}{3} \varepsilon^2 \phi^2 \psi^2 \mathcal{R}(Y, X_0, \eta, \dot{\eta}) \]

\[ + \frac{1}{3} \varepsilon^2 \phi^2 \dot{\psi}^2 \mathcal{R}(\eta, \dot{\eta}, \dot{\eta}) + \frac{1}{3} \varepsilon^2 \phi^2 \dot{\psi} \mathcal{R}(\eta, \dot{\eta}, \dot{\eta}) + \phi \frac{\partial w}{\partial \theta} \]

\[ + \phi \dot{\psi} < \frac{\partial \eta}{\partial x_0}, \dot{\eta} > + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3), \]

\[ \varepsilon^{-2} < \partial_s, \partial_s > = \phi^2 + 2 \phi \dot{\psi} + \varepsilon^2 \phi \dot{\psi}^2 \mathcal{R}(\eta, \dot{\eta}, \dot{\eta}) + \frac{2}{3} \varepsilon^2 \phi^2 \mathcal{R}(\eta, \dot{\eta}, \dot{\eta}) \]

\[ + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3). \]

Note that all the curvatures here and after are taken on \( p \in \Gamma. \)

**Proof.** The proof is long and direct calculation, by using Lemma 2.4. For example for the first one \( \varepsilon^{-2} < \partial_s, \partial_s >, \) first we have

\[ \varepsilon^{-2} < \partial_s, \partial_s > = < \dot{\psi} X_0 + (\dot{\phi} + \frac{\partial w}{\partial s}) Y + \dot{\psi}, \frac{\partial \eta}{\partial x_0}, \dot{\eta} >. \]

And we get 6 different terms on the right hand side. For each one we can use Lemma 2.4. Finally we can get the results.

\[ \square \]

### 2.5 Normal vector

To calculate the mean curvature, one need to know the normal vector. Now we are going to find expansions of the unit normal vector of \( \mathcal{D}_{\phi_{p_0, \tau_0}, p_0, \varepsilon}(w, \eta). \) First we take

\[ N_0 = \frac{1}{\phi}(\phi X_0 - \dot{\psi} Y), \]

which is the unit normal vector of \( \mathcal{D}_{\phi_{p_0, \tau_0}, p_0, \varepsilon} \) when curvature vanishes. We expect the unit normal vector \( N \) of \( \mathcal{D}_{\phi_{p_0, \tau_0}, p_0, \varepsilon}(w, \eta) \) is small perturbation of \( N_0. \) Namely, we assume

\[ N = \frac{1}{k}(N_0 + a_1 \partial_s + a_2 \partial_{\theta}) \quad (4) \]

where \( k \) is the norm of \( N_0 + a_1 \partial_s + a_2 \partial_{\theta}. \)

\[ 0 = < k N, \partial_s > = < N_0, \partial_s > + a_1 < \partial_s, \partial_s > + a_2 < \partial_s, \partial_{\theta} >, \]

\[ 0 = < k N, \partial_{\theta} > = < N_0, \partial_{\theta} > + a_1 < \partial_s, \partial_{\theta} > + a_2 < \partial_{\theta}, \partial_{\theta} >. \quad (5) \]
Lemma 2.6.

\[ \varepsilon^{-1} < N_0, \partial_s > = -\frac{\psi}{\phi} \frac{\partial w}{\partial s} - \frac{\psi^2}{\phi} < \frac{\partial \eta}{\partial x_0}, \Upsilon >_\varepsilon + \varepsilon^2 \phi \dot{\psi} R(\Upsilon, X_0, \Upsilon, X_0) \]
\[ + 2 \varepsilon \dot{\psi} R(\Upsilon, X_0, \eta, X_0) + \frac{2}{3} (\dot{\phi}^2 - \dot{\psi}^2) R(\Upsilon, X_0, \eta, \Upsilon) \]
\[ + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3) \]

\[ \varepsilon^{-1} < N_0, \partial_\theta > = \frac{2}{3} \varepsilon^2 \phi^2 \dot{\phi} R(\Upsilon, X_0, \Upsilon, \Upsilon_\theta) \]
\[ + \frac{2}{3} \varepsilon \phi (R(\Upsilon, X_0, \eta, \Upsilon_\theta) + R(\eta, X_0, \Upsilon, \Upsilon_\theta)) \]
\[ + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3) \]

Proof. The proof is again direct calculation.

We denote \( g_{ss} = < \partial_s, \partial_s >, g_{s\theta} = < \partial_s, \partial_\theta >, g_{\theta\theta} = < \partial_\theta, \partial_\theta > \). From Lemma 2.5 we have

\[ \begin{pmatrix} g_{ss} & g_{s\theta} \\ g_{s\theta} & g_{\theta\theta} \end{pmatrix} = \varepsilon^2 \phi^2 \begin{pmatrix} 1 + \sigma_1 & \sigma_2 \\ \sigma_2 & 1 + \sigma_3 \end{pmatrix}, \]

where

\[ \sigma_1 = \varepsilon^2 \dot{\psi}^2 R(\Upsilon, X_0, \Upsilon, X_0) + 2 \varepsilon \dot{\phi}^{-1} \dot{\psi}^2 R(\Upsilon, X_0, \eta, X_0) \]
\[ + \frac{4}{3} \varepsilon \dot{\phi}^{-1} \dot{\psi} R(\Upsilon, X_0, \eta, \Upsilon) + 2 \phi^{-2} \phi \frac{\partial w}{\partial s} \]
\[ + 2 \phi^{-2} \dot{\phi} < \Upsilon, \frac{\partial \eta}{\partial x_0} >_\varepsilon + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3) \]

\[ \sigma_2 = \frac{2}{3} \varepsilon^2 \dot{\psi} R(\Upsilon, X_0, \Upsilon, \Upsilon_\theta) + \frac{2}{3} \varepsilon \dot{\psi} (R(\Upsilon, X_0, \eta, \Upsilon_\theta) + R(\eta, X_0, \Upsilon, \Upsilon_\theta)) \]
\[ + \frac{1}{3} \varepsilon \phi R(\eta, \Upsilon, \Upsilon, \Upsilon_\theta) + \phi^{-2} \phi \frac{\partial w}{\partial \theta} + \phi^{-1} \dot{\psi} < \frac{\partial \eta}{\partial x_0}, \Upsilon_\theta >_\varepsilon \]
\[ + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3) \]

\[ \sigma_3 = 2 \phi^{-1} w + \frac{1}{3} \varepsilon^2 \phi^2 R(\Upsilon, \Upsilon_\theta, \Upsilon, \Upsilon_\theta) + \frac{2}{3} \varepsilon \phi R(\Upsilon, \Upsilon_\theta, \eta, \Upsilon_\theta) \]
\[ + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3) \]

(6)

Notice that \( \sigma_1 \sigma_3 - \sigma_2^2 = \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3) \). We have

\[ \det \begin{pmatrix} g_{ss} & g_{s\theta} \\ g_{s\theta} & g_{\theta\theta} \end{pmatrix} \cong \varepsilon^4 \phi^4 (1 + \sigma_1 + \sigma_3) \]

and

\[ \det \begin{pmatrix} g_{ss} & g_{s\theta} \\ g_{s\theta} & g_{\theta\theta} \end{pmatrix}^{-1} \cong \varepsilon^{-4} \phi^{-4} (1 - \sigma_1 - \sigma_3). \]
So we get the inverse matrix
\[
\begin{pmatrix}
g_{ss} & g_{s\theta} \\
g_{s\theta} & g_{\theta\theta}
\end{pmatrix}^{-1} = \det \begin{pmatrix}
g_{ss} & g_{s\theta} \\
g_{s\theta} & g_{\theta\theta}
\end{pmatrix}^{-1} \begin{pmatrix}
g_{\theta\theta} & -g_{s\theta} \\
-g_{s\theta} & g_{ss}
\end{pmatrix}
\approx \varepsilon^{-2} \phi^{-2} \begin{pmatrix}
1 - \sigma_1 & -\sigma_2 \\
-\sigma_2 & 1 - \sigma_3
\end{pmatrix}.
\tag{7}
\]

From this and Lemma 2.6, we can get

**Lemma 2.7.**
\[
\begin{pmatrix}
a_1 \\
a_2
\end{pmatrix} \approx -\varepsilon^{-2} \phi^{-2} \begin{pmatrix}
< N_0, \partial_s > \\
< N_0, \partial_\theta >
\end{pmatrix} = \begin{pmatrix}
O(\varepsilon) + \varepsilon^{-1} L(w, \eta) + O(\varepsilon^3) \\
O(\varepsilon) + \varepsilon L(w, \eta) + O(\varepsilon^3)
\end{pmatrix}
\tag{8}
\]

**Proof.** From (7) we have
\[
\begin{pmatrix}
a_1 \\
a_2
\end{pmatrix} = \varepsilon^{-2} \phi^{-2} \begin{pmatrix}
1 - \sigma_1 & -\sigma_2 \\
-\sigma_2 & 1 - \sigma_3
\end{pmatrix} \begin{pmatrix}
- < N_0, \partial_s > \\
- < N_0, \partial_\theta >
\end{pmatrix}.
\]

By direct calculation, we have \( \sigma_i < N_0, \partial_s > \) and \( \sigma_i < N_0, \partial_\theta > \) are in fact \( \varepsilon^3 L(w, \eta) + \varepsilon Q(w, \eta) + O(\varepsilon^4) \). So we get the result.

Also we need to get the expansion of \( k \) which is the norm of \( N_0 + a_1 \partial_s + a_2 \partial_\theta \).
\[
k^2 = < N_0, N_0 > + a_1 < N_0, \partial_s > + a_2 < N_0, \partial_\theta >
\]

From (5) we can simplify this to
\[
k^2 = < N_0, N_0 > + a_1 < N_0, \partial_s > + a_2 < N_0, \partial_\theta >.
\]

From
\[
< N_0, N_0 > = \frac{1}{\phi} (\dot{\phi} X_0 - \dot{\psi} \Upsilon) + \frac{1}{\phi} (\dot{\phi} X_0 - \dot{\psi} \Upsilon)
\]

\[
= \frac{\dot{\phi}^2}{\phi} < X_0, X_0 > + \frac{\dot{\psi}^2}{\phi^2} < \Upsilon, \Upsilon > - 2 \frac{\dot{\phi} \dot{\psi}}{\phi^2} < X_0, \Upsilon >
\]

\[
= 1 + \varepsilon^2 \frac{\dot{\phi}^2}{\phi} R(\Upsilon, X_0, \Upsilon, X_0) + 2 \varepsilon \frac{\dot{\phi} \dot{\psi}}{\phi} R(\Upsilon, X_0, X_0) + \frac{4}{3} \varepsilon^3 \frac{\dot{\phi} \dot{\psi} R(\Upsilon, X_0, \Upsilon, \Upsilon)}{\phi} + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3),
\]

\[
a_1 < N_0, \partial_s > = (O(\varepsilon) + \varepsilon^{-1} L(w, \eta) + \varepsilon^{-1} Q(w, \eta)) \varepsilon (\frac{\dot{\psi}}{\phi} \frac{\partial w}{\partial \phi} - \frac{\dot{\psi}^2}{\phi^2} \frac{\partial \eta}{\partial \phi}) \Upsilon > e
\]

\[
+ \varepsilon^2 \phi \dot{\psi} R(\Upsilon, X_0, \Upsilon, X_0) + 2 \varepsilon \dot{\phi} \dot{\psi} R(\Upsilon, X_0, \Upsilon, \Upsilon)
\]

\[
+ \frac{2}{3} (\dot{\psi}^2 - \dot{\psi}) R(\Upsilon, X_0, \Upsilon, \Upsilon) + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3)
\]

\[
= \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^4),
\]

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\[ a_2 < N_0, \partial_\theta > = \left( O(\varepsilon) + \varepsilon L(w, \eta) + \varepsilon^{-1} Q(w, \eta) \right) \varepsilon \left( \frac{2}{3} \varepsilon^2 \phi^2 \phi R(\Upsilon, X_0, \Upsilon, \Upsilon) \right) + 2 \varepsilon \phi \psi (R(\Upsilon, X_0, \eta, \Upsilon) + R(\eta, X_0, \Upsilon, \Upsilon)) + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3) \]

we have

\[ k^2 = 1 + \varepsilon^2 \phi^2 R(\Upsilon, X_0, \Upsilon, X_0) + 2 \varepsilon \phi^2 R(\Upsilon, X_0, \eta, X_0) - \frac{4}{3} \varepsilon \phi R(\Upsilon, X_0, \eta, \Upsilon) \]

So

\[ k = 1 + \frac{\varepsilon^2}{2} \phi^2 R(\Upsilon, X_0, \Upsilon, X_0) + \varepsilon \phi R(\Upsilon, X_0, \eta, X_0) - \frac{2}{3} \varepsilon \phi^2 R(\Upsilon, X_0, \eta, \Upsilon) \]

\[ + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3). \]

### 2.6 The second fundamental form and the mean curvature

of \( D_{\phi_{p_0, \tau_0, p_0, \varepsilon}}(w, \eta) \).

In the end of this subsection we will get the mean curvature of \( D_{\phi_{p_0, \tau_0, p_0, \varepsilon}}(w, \eta) \)

\[ H(D_{\phi_{p_0, \tau_0, p_0, \varepsilon}}(w, \eta)) = \frac{2}{\varepsilon} + \mathcal{L}_{\text{SN}} w + < \mathcal{J} \eta, \Gamma > \]

\[ + \varepsilon (F_1(\phi, \phi) \ast R_1 + F_2(\phi, \phi) \ast R_2) + E \]

\[ + F_3(\phi, \phi) \ast R_3(\eta) + T(w, \eta) \]
Remark. The reason why $F_1, F_2, F_3$ only depends on $\phi, \phi_\psi$ is

\[
\begin{align*}
\phi(s) &= \phi(\psi) \\
\dot{\psi}(s) &= \phi(\psi)^2 + \tau(\psi) \\
\dot{\phi}(s) &= \frac{\partial \phi(\psi)}{\partial \psi} \left( \phi(\psi)^2 + \tau(\psi) \right) \\
\dot{\psi}(s) &= \frac{\partial \phi(\psi)}{\partial \psi} \left( \phi(\psi)^2 + \tau(\psi) \right) = 2\phi \frac{\partial \phi(\psi)}{\partial \psi} (\phi^2 + \tau(\psi)) \\
\tau(s) &= -\phi^2 + \frac{\phi}{\sqrt{1 + \phi^2}}
\end{align*}
\]

Before proving it we make some analysis for the terms appearing in this expression. The biggest term on the right hand side is $\frac{2}{3}$ which is the mean curvature of the standard Delaunay surface embedded in flat manifold. $\mathcal{L}_{SN}^w$ is the Jacobi operator for $w$ with $\Pi_1(w) = 0$. $\mathcal{J} \eta$ is the Jacobi operator for $\eta, \eta$ can be regarded as the $\Pi_1$ part of the variation (so it is reasonable to assume $\Pi_1(w) = 0$).

$$
\epsilon(F_1(\phi, \phi_\psi) \ast R_1 + F_2(\phi, \phi_\psi) \ast R_2) + E + F_3(\phi, \phi_\psi) \ast R_3(\eta) + T(w, \eta)
$$

are tail terms we have to deal with. We have to perturb the initial surface in certain way to cancel all these terms. We can use $\Pi_0, \Pi, \tilde{\Pi}$ to project the tail.
terms into three different subspaces and deal with them in three different modes. We can see that these three modes are involved in each other. Also we notice

\[ \Pi_0(\varepsilon F_1(\phi, \phi \psi) \ast R_1 + F_2(\phi, \phi \psi) \ast R_2) + F_3(\phi, \phi \psi) \ast R_3(\eta)) \]

\[ = \varepsilon F_1(\phi, \phi \psi) \ast \Pi_0(R_1) + F_2(\phi, \phi \psi) \ast \Pi_0(R_3(\eta)), \]

\[ \Pi_1(\varepsilon F_1(\phi, \phi \psi) \ast R_1 + F_2(\phi, \phi \psi) \ast R_2) + F_3(\phi, \phi \psi) \ast R_3(\eta)) \]

\[ = \varepsilon F_2(\phi, \phi \psi) \ast \Pi_1(R_2). \]  \hspace{1cm} (11)

Now we calculate the expression of the mean curvature. First we have

\[ H(\mathcal{D}_{\phi_0, \tau_0, p_0}(w, \eta)) = g^{ss} < N, \nabla_{\partial_s} \partial_s > + 2g^{s\theta} < N, \nabla_{\partial_s} \partial_s > + g^{\theta\theta} < N, \nabla_{\partial_\theta} \partial_\theta > \]

\[ = \frac{1}{k}(g^{ss} < kN, \nabla_{\partial_s} \partial_s > + 2g^{s\theta} < kN, \nabla_{\partial_s} \partial_s > + g^{\theta\theta} < kN, \nabla_{\partial_\theta} \partial_\theta >). \]

For this we have to calculate the second fundamental form. For

\[ g^{ss} < kN, \nabla_{\partial_\theta} \partial_s > = g^{ss} < N_0 + a_1 \partial_s + a_2 \partial_\theta, \nabla_{\partial_s} \partial_s > \]

\[ < \partial_s, \nabla_{\partial_\theta} \partial_s > = \frac{1}{2} \partial_s < \partial_s, \partial_s > \]

\[ = \frac{\varepsilon^2}{2} \partial_s (\phi^2 + \varepsilon^2 \phi^2 \dot{\psi}^2 R(\Upsilon, X_0, \Upsilon, X_0) + 2\varepsilon \phi \dot{\psi}^2 R(\Upsilon, X_0, \eta, X_0) \]

\[ + \frac{4}{3} \varepsilon^2 \phi \dot{\psi}^2 R(\Upsilon, X_0, \eta, \Upsilon) + 2\phi \frac{\partial w}{\partial s} + 2\phi \dot{\psi} < \Upsilon, \frac{\partial \eta}{\partial x^0} > \epsilon \]

\[ + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3)), \]

\[ = \frac{\varepsilon^2}{2} (2\phi \dot{\phi} + L(w, \eta) + \varepsilon L(\frac{\partial^2 \eta}{(\partial x^0)^2}) + Q(w, \eta) + O(\varepsilon^2)), \]  \hspace{1cm} (12)
\[
< \partial_\theta, \nabla_\theta, \partial_s > = \partial_s < \partial_\theta, \partial_s > - < \nabla_\theta, \partial_\theta, \partial_s > \\
= \partial_s < \partial_\theta, \partial_s > - \frac{1}{2} \partial_\theta < \partial_s, \partial_s > \\
= \varepsilon^2 \partial_s (\frac{2}{3} \varepsilon^2 \partial^3 \dot{\psi} \varepsilon \dot{R}(Y, X_0, Y, Y_\theta) + \frac{2}{3} \varepsilon \partial^2 \dot{\psi} \varepsilon \dot{R}(X_0, \eta, Y_\theta) \\
+ R(\eta, X_0, Y, Y_\theta) + \frac{1}{3} \varepsilon \partial^2 \dot{\psi} \varepsilon \dot{R}(Y, Y, Y_\theta) + \partial \frac{\partial \psi}{\partial \theta} \\
+ \partial \dot{\psi} < \partial \psi, \partial \theta > + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3)) \\
- \varepsilon^2 \partial_\theta (\dot{\phi}^2 + \varepsilon^2 \partial^2 \dot{\psi} \varepsilon \dot{R}(Y, X_0, Y, X_0) + 2 \varepsilon \partial \dot{\psi} \varepsilon \dot{R}(X_0, \eta, X_0) \\
+ \frac{4}{3} \varepsilon \phi \dot{\psi} \varepsilon \dot{R}(Y, X_0, \eta, Y) + 2 \phi \partial \psi \varepsilon \dot{R}(Y, X_0, \eta, \partial_x) > e \\
+ \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3)), \\
= \varepsilon^2 (O(\varepsilon^2) + L(w, \eta) + \varepsilon L(\frac{\partial^2 \psi}{\partial x^2}) + Q(w, \eta)) \\
\tag{13}
\]

\[
\nabla_\theta, \partial_s = \varepsilon \nabla_\theta, (\dot{\psi} X_0 + (\phi + \frac{\partial \psi}{\partial x}) \partial_x + \psi \frac{\partial \eta}{\partial x_0} X_i) \\
= \varepsilon (\dot{\psi} X_0 + (\phi + \frac{\partial \psi}{\partial x}) \partial_x + \psi \frac{\partial \eta}{\partial x_0} X_i) \\
+ \varepsilon^2 \dot{\psi} \nabla_\theta X_0 + (\phi + \frac{\partial \psi}{\partial x}) \partial_x \partial_x + \dot{\psi} \psi \frac{\partial \eta}{\partial x_0} \partial_x \partial_x + \dot{\psi} \psi \frac{\partial \eta}{\partial x_0} \partial_x \partial_x \\
+ 2 \dot{\psi} \dot{\psi} \frac{\partial \eta}{\partial x_0} \partial_x \partial_x + 2 \dot{\psi} \psi \frac{\partial \eta}{\partial x_0} \partial_x \partial_x + 2 (\phi + \frac{\partial \psi}{\partial x}) \partial_x \partial_x \partial_x \partial_x \\
\]

We calculate

\[
< N_0, \nabla_\theta, \partial_s > = < \frac{1}{\phi} (\dot{\phi} X_0 - \dot{\psi} Y), \nabla_\theta, \partial_s > \\
term by term. There would be 20 terms totally.
\]

\[
< \frac{\dot{\phi}}{\phi} X_0, \varepsilon \dot{\psi} X_0 > = \varepsilon \frac{\dot{\phi}}{\phi} < X_0, X_0 > \\
= \varepsilon \frac{\dot{\phi}}{\phi} (1 + \varepsilon^2 \partial^2 \dot{\psi} \varepsilon \dot{R}(Y, X_0, Y, X_0) + 2 \varepsilon \phi \dot{R}(Y, X_0, \eta, X_0) \\
+ \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3)) \\
= \varepsilon \frac{\dot{\phi}}{\phi} + \varepsilon^3 \phi \dot{\psi} \varepsilon \dot{R}(Y, X_0, Y, X_0) + 2 \varepsilon^2 \phi \dot{\psi} \varepsilon \dot{R}(Y, X_0, \eta, X_0) \\
+ \varepsilon^3 L(w, \eta) + \varepsilon Q(w, \eta) + O(\varepsilon^4),
\]

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The method to calculate this will be used frequently, so we write it in detail. Although \( Y \) actually depends on \( \theta \), here we are only interested in \( \nabla_Y Y \), we can assume \( Y \) doesn’t depend on \( \theta \) and is constant vector in the coordinate \( \{x_0, x_1, x_2\} \). We may make such assumption when it is convenient.

\[
< \frac{\dot{\phi}}{\phi} X_0, \varepsilon(\ddot{\phi} + \frac{\partial^2 w}{\partial s^2}) Y > = \varepsilon \frac{\dot{\phi}^2}{\phi} < \phi + \frac{\partial^2 w}{\partial s^2} X_0, Y >
\]

\[
= \varepsilon \frac{\dot{\phi}^2}{\phi} (\phi + \frac{\partial^2 w}{\partial s^2})(\frac{2}{3} \varepsilon \phi R(\dot{Y}, X_0, \eta, \dot{Y}) + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3))
\]

\[
= \frac{2}{3} \varepsilon^2 \frac{\dot{\phi}^2}{\phi} R(\dot{Y}, X_0, \eta, \dot{Y}) + \varepsilon^3 L(w, \eta) + \varepsilon^3 Q(w, \eta) + O(\varepsilon^4),
\]

\[
< \frac{\dot{\phi}}{\phi} X_0, \varepsilon \psi \frac{\partial \eta^i}{\partial x^0} X_i > = \varepsilon \frac{\dot{\phi} \psi}{\phi} \frac{\partial \eta^i}{\partial x^0} < X_0, X_i >
\]

\[
= \varepsilon \frac{\dot{\phi} \psi}{\phi} \frac{\partial \eta^i}{\partial x^0} (O(\varepsilon^2) + \varepsilon L(w, \eta) + Q(w, \eta) + O(\varepsilon))
\]

\[
= \varepsilon^3 L(w, \eta) + \varepsilon Q(w, \eta),
\]

\[
< \frac{\dot{\phi}}{\phi} X_0, \varepsilon^2 \psi^2 \frac{\partial^2 \eta^i}{\partial x_0^2} X_i > = \varepsilon^2 \frac{\dot{\phi} \psi^2}{\phi} \frac{\partial^2 \eta^i}{\partial x_0^2} (O(\varepsilon^2) + \varepsilon L(w, \eta)
\]

\[
+ Q(w, \eta) + O(\varepsilon))
\]

\[
= \varepsilon^4 L(\partial_{x_0}^2 \eta) + \varepsilon^2 Q(w, \partial_{x_0}^2 \eta),
\]

\[
< \frac{\dot{\phi}}{\phi} X_0, \varepsilon^2 \psi^2 \nabla X_0 X_0 > = \varepsilon^2 \frac{\dot{\phi} \psi^2}{\phi} < X_0, \nabla X_0 X_0 >
\]

\[
= \varepsilon^2 \frac{\dot{\phi} \psi^2}{\phi} (O(\varepsilon^2) + \varepsilon L(w, \eta) + Q(w, \eta))
\]

\[
= O(\varepsilon^4) + \varepsilon^3 L(w, \eta) + \varepsilon^2 Q(w, \eta),
\]

\[
< \frac{\dot{\phi}}{\phi} X_0, \varepsilon^2 (\phi + \frac{\partial w}{\partial s})^2 \nabla_Y Y > = \varepsilon^2 \frac{\dot{\phi}}{\phi} (\phi + \frac{\partial w}{\partial s})^2 < X_0, \nabla_Y Y >
\]

\[
= \varepsilon^2 \frac{\dot{\phi}}{\phi} (\phi + \frac{\partial w}{\partial s})^2 (Y < X_0, Y > - \nabla_{X_0} Y, Y >).
\]
\[ < \nabla_{X_0} Y, Y >_p = 0 \]

for all \( p \in \Gamma \). We consider

\[
X_j < \nabla_{X_0} Y, Y >_p = < \nabla_{X_j} \nabla_{X_0} Y, Y >_p + < \nabla_{X_0} Y, \nabla_{X_j} Y >_p .
\]

From Lemma 2.1 we know \( (\nabla_{X_\alpha} X_\beta)_p = 0 \). So

\[
X_j < \nabla_{X_0} Y, Y >_p = < \nabla_{X_0} \nabla_{X_j} Y, Y >_p + R(X_j, X_0, Y, Y)_p.
\]

We know \( R(X_j, X_0, Y, Y)_p = 0 \). And we know \( \nabla_{X_j} Y = 0 \) always holds on the geodesic. So \( (\nabla_{X_0} \nabla_{X_j} Y)_p = 0 \). So we know

\[
< \nabla_{X_0} Y, Y > (x_0, x_1, x_2) = O(r^2) = O(\varepsilon^2) + \varepsilon L(w, \eta) + Q(w, \eta).
\]

Now we get

\[
\varepsilon^2 < \frac{\dot{\phi}}{\phi} X_0, 2\dot{\psi}(\phi + \frac{\partial w}{\partial s})^2 \nabla_{\eta} Y > = 2\frac{\varepsilon^2}{3} \frac{\dot{\phi}}{\phi} R(Y, X_0, \eta, Y) + O(\varepsilon^4) + \varepsilon^3 L(w, \eta) + \varepsilon Q(w, \eta),
\]

By using the skill above we can calculate all the remaining terms. We state the result directly.

\[
\varepsilon^2 < \frac{\dot{\phi}}{\phi} X_0, 2\dot{\psi} \frac{\partial \eta^i}{\partial x_0} (\frac{\partial \eta^j}{\partial x_0}) \nabla_{X_0} X_j > = \varepsilon^3 Q(w, \eta),
\]

\[
\varepsilon^2 < \frac{\dot{\phi}}{\phi} X_0, 2\dot{\psi} \frac{\partial \eta^i}{\partial x_0} \nabla_{X_0} X_i > = \varepsilon^3 L(w, \eta),
\]

\[
\varepsilon^2 < \frac{\dot{\phi}}{\phi} X_0, 2\dot{\psi} \frac{\partial \eta^i}{\partial x_0} \nabla_{\eta} X_i > = \varepsilon^3 L(w, \eta) + \varepsilon^2 Q(w, \eta),
\]

\[
< -\frac{\psi}{\phi} Y, \varepsilon \frac{\dot{\phi}}{\phi} X_0 > = -\varepsilon \frac{\psi}{\phi} < Y, X_0 > = -\frac{2}{3} \varepsilon^2 \dot{\phi} \psi R(Y, X_0, \eta, Y) + \varepsilon^3 L(w, \eta) + \varepsilon Q(w, \eta) + O(\varepsilon^4),
\]

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\[
< - \frac{\dot{\psi}}{\phi} \nabla, \epsilon (\dot{\phi} + \frac{\partial^2 w}{\partial s^2}) Y > = - \epsilon \frac{\dot{\phi} \dot{\psi}}{\phi} - \epsilon \frac{\dot{\psi} \partial^2 w}{\phi \partial s} + \epsilon^3 L(w, \eta) + \epsilon Q(w, \eta) + O(\epsilon^4),
\]

\[
< - \frac{\dot{\psi}}{\phi} \nabla, \epsilon^2 \psi \frac{\partial \eta^i}{\partial x_0} X_i > = - \epsilon \frac{\dot{\psi} \dot{\psi}}{\phi} < \frac{\partial \eta}{\partial x_0}, Y >,
\]

\[
< - \frac{\dot{\psi}}{\phi} \nabla, \epsilon^2 \psi \frac{\partial^2 \eta^i}{\partial x_0^2} X_i > = - \epsilon^2 \frac{\dot{\psi} \dot{\psi}}{\phi} < \frac{\partial^2 \eta}{\partial x_0^2}, Y >,
\]

\[
< - \frac{\dot{\psi}}{\phi} \nabla, \epsilon^2 \psi^2 \nabla X_0 Y > = \epsilon^3 \psi^3 R(Y, X_0, Y, X_0) + \epsilon^2 \psi^3 R(Y, X_0, \eta, X_0) + \epsilon^3 L(w, \eta) + \epsilon^2 Q(w, \eta) + O(\epsilon^4),
\]

\[
< - \frac{\dot{\psi}}{\phi} \nabla, \epsilon^2 \psi \frac{\partial w}{\partial s} \nabla Y > = \epsilon^3 L(w, \eta) + \epsilon^2 Q(w, \eta) + O(\epsilon^4),
\]

\[
< - \frac{\dot{\psi}}{\phi} \nabla, \epsilon^2 \psi^2 (\frac{\partial \eta^i}{\partial x_0})(\frac{\partial \eta^j}{\partial x_0}) \nabla X_i X_j > = \epsilon^3 Q(w, \eta),
\]

\[
< - \frac{\dot{\psi}}{\phi} \nabla, \epsilon^2 \psi^2 (\frac{\partial w}{\partial s}) \nabla X_i Y > = \epsilon^3 L(w, \eta) + \epsilon^2 Q(w, \eta) + O(\epsilon^4),
\]

\[
< - \frac{\dot{\psi}}{\phi} \nabla, \epsilon^2 \psi^2 \frac{\partial \eta^i}{\partial x_0} \nabla X_i Y > = \epsilon^3 L(w, \eta),
\]

\[
< - \frac{\dot{\psi}}{\phi} \nabla, \epsilon^2 \psi^2 \frac{\partial \eta^i}{\partial x_0} \nabla Y X_i > = \epsilon^3 L(w, \eta) + \epsilon^2 Q(w, \eta).
\]

Collecting all the terms and notice that

\[
\frac{\dot{\phi} \dot{\psi}}{\phi} - \frac{\dot{\phi} \dot{\psi}}{\phi} = \phi^2 - \tau_0
\]

we get

\[
< N_0, \nabla \phi, \partial_s > = \epsilon (\phi^2 - \tau_0) - \epsilon \frac{\psi \partial^2 w}{\phi} - \epsilon^2 \frac{\psi \dot{\psi}}{\phi} < \frac{\partial^2 \eta}{\partial x_0^2}, Y > - \epsilon \frac{\dot{\psi} \dot{\psi}}{\phi} < \frac{\partial \eta}{\partial x_0}, Y >
\]

\[
+ \epsilon^3 (\phi \dot{\phi} \dot{\psi} + 2 \phi \dot{\psi}^2 + \psi^3) R(Y, X_0, Y, X_0)
\]

\[
+ \epsilon^2 (2 \phi \dot{\phi} \dot{\psi} + 2 \phi \dot{\psi}^2 + \psi^3) R(Y, X_0, \eta, X_0)
\]

\[
+ \epsilon^2 \frac{2}{3} \phi \dot{\phi} \dot{\psi} + \frac{2}{3} \phi \dot{\psi} \dot{\psi} \dot{\psi} R(Y, X_0, \eta, Y)
\]

\[
+ \epsilon^3 L(w, \eta) + \epsilon Q(w, \eta) + \epsilon^4 L(\partial^2 \eta) + \epsilon^2 Q(w, \partial^2 \eta) + O(\epsilon^4),
\]

(15)
\[ g^{\theta \theta} < kN, \nabla \partial_\theta \partial_\theta > = g^{\theta \theta} < N_0 + a_1 \partial_s + a_2 \partial_\theta, \nabla \partial_\theta \partial_\theta >. \]

\[ < \partial_s, \nabla \partial_\theta \partial_\theta > = \partial_\theta < \partial_s, \partial_\theta > - \frac{1}{2} \partial_s < \partial_\theta, \partial_\theta > \]

\[ = \varepsilon^2 \partial_\theta \left( \frac{2}{3} \varepsilon^2 \dot{\psi} R(Y, X_0, \nabla \theta) + \frac{2}{3} \varepsilon \phi \dot{\psi} (R(Y, X_0, \eta) + R(\eta, X_0, Y, Y_\theta)) \right) + \frac{1}{3} \varepsilon \phi \dot{\psi} R(Y, \dot{Y}, Y_\theta) \]

\[ + \phi \frac{\partial w}{\partial \theta} + \phi \dot{\psi} < \frac{\partial \eta}{\partial x_0}, Y_\theta > + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3) \]

\[ = \varepsilon^2 \phi \dot{\phi} + O(\varepsilon^4) + \varepsilon^2 L(w, \eta) + \varepsilon^2 Q(w, \eta), \quad (16) \]

\[ < \partial_\theta, \nabla \partial_\theta \partial_\theta > = \frac{1}{2} \partial_\theta < \partial_\theta, \partial_\theta > \]

\[ = \varepsilon^2 \partial_\theta \left( \phi^2 + 2 \phi w + \frac{1}{3} \varepsilon^2 \phi^4 R(Y, Y_\theta, Y, Y_\theta) \right) \]

\[ + \frac{2}{3} \varepsilon \phi \dot{\psi} R(Y, \eta, \dot{Y}_\theta + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3)) \]

\[ = O(\varepsilon^4) + \varepsilon^2 L(w, \eta) + \varepsilon^2 Q(w, \eta), \quad (17) \]

\[ \nabla \partial_\theta \partial_\theta = \frac{\partial w}{\partial \theta} Y_\theta + \varepsilon \frac{\partial^2 w}{\partial \theta^2} Y + \varepsilon^2 (\phi + w)^2 \nabla\nabla \theta \]

\[ + \varepsilon^2 \left( \frac{\partial w}{\partial \theta} \right)^2 \nabla \dot{Y} + \varepsilon^2 (\phi + w) \frac{\partial w}{\partial \theta} \left( \nabla \nabla\nabla \theta + \nabla \nabla \theta \right) \]

\[ < \frac{\dot{\phi}}{\phi} X_0, \varepsilon \frac{\partial w}{\partial \theta} Y_\theta > = \varepsilon^3 L(w, \eta) + \varepsilon Q(w, \eta), \]

\[ < \frac{\dot{\phi}}{\phi} X_0, \varepsilon \frac{\partial^2 w}{\partial \theta^2} Y > = \varepsilon^4 L(w, \eta) + \varepsilon Q(w, \eta), \]

\[ < \frac{\dot{\phi}}{\phi} X_0, \varepsilon^2 (\phi + w)^2 \nabla\nabla \theta \nabla \theta > = \varepsilon^2 \frac{\dot{\phi}}{\phi} (\phi + w)^2 < X_0, \nabla\nabla \theta \nabla \theta >, \]

notice that

\[ < X_0, \nabla\nabla \theta \nabla \theta > = \nabla \partial_\theta Y_\theta - < X_0, \nabla\nabla \theta >, \quad (18) \]
We recall \( Y(x_0, x_1, x_2) = \frac{(x_2 - \eta_2, x_1 - \eta_1)}{\sqrt{(x_1 - \eta_1)^2 + (x_2 - \eta_2)^2}} = (\cos \theta, \sin \theta) \) and \( Y_{\theta}(x_0, x_1, x_2) = \frac{(-x_2 + \eta_2, x_1 - \eta_1)}{\sqrt{(x_1 - \eta_1)^2 + (x_2 - \eta_2)^2}} = (-\sin \theta, \cos \theta) \). We denote \((-x_2 + \eta_2, x_1 - \eta_1)\) by \( \tilde{\phi}_\theta \), and \( \sqrt{(x_1 - \eta_1)^2 + (x_2 - \eta_2)^2} \) by \( \tilde{r} \).

\[ Y_\theta < X_0, Y_\theta > = \frac{1}{\tilde{r}} \tilde{\phi}_\theta < X_0, -\sin \theta X_1 + \cos \theta X_2 > \]
\[ = \frac{1}{\tilde{r}} < X_0, -\bar{Y} > + (\sin \theta) Y_\theta < X_0, X_1 > + \cos \theta Y_\theta < X_0, X_1 > \]
\[ = \frac{2}{3} \varepsilon \phi R(Y_\theta, X_0, \bar{Y}, Y_\theta) + \frac{2}{3} R(Y_\theta, X_0, \eta, Y_\theta) - \frac{2}{3} R(Y, X_0, \eta, Y) \]
\[ + \varepsilon L(w, \eta) + \varepsilon^{-1} Q(w, \eta) + O(\varepsilon^2). \]

To calculate \( < \nabla_{(\bar{Y})} X_0, Y_\theta > \) we can regard \( Y_\theta \) as constant in \( (x_0, x_1, x_2) \) coordinate.

\[ < \nabla_{(\bar{Y})} X_0, Y_\theta > = \frac{1}{2} X_0 < Y_\theta, Y_\theta > \]
\[ = O(\varepsilon^2) + \varepsilon L(w, \eta) + Q(w, \eta). \]

We have

\[ < \frac{\dot{\phi}}{\phi} X_0, \varepsilon^2(\phi + w)^2 \nabla_{(\bar{Y})} Y > = \frac{2}{3} \varepsilon^2 \phi \phi R(Y_\theta, X_0, \bar{Y}, Y_\theta) \]
\[ + \frac{2}{3} \varepsilon \phi (R(Y_\theta, X_0, \eta, Y_\theta) - R(Y, X_0, \eta, Y)) \]
\[ + \varepsilon^4 L(w, \eta) + \varepsilon Q(w, \eta) + O(\varepsilon^4). \]

\[ < \frac{\dot{\phi}}{\phi} X_0, \varepsilon^2 (\frac{\partial w}{\partial \theta})^2 \nabla_{(\bar{Y})} Y > = \varepsilon^3 Q(w, \eta). \]

\[ < \frac{\dot{\phi}}{\phi} X_0, \varepsilon^2 (\phi + w) \frac{\partial w}{\partial \theta} \nabla_{(\bar{Y})} Y > = \varepsilon^3 L(w, \eta) + \varepsilon^2 Q(w, \eta). \]

\[ < \frac{\dot{\phi}}{\phi} X_0, \varepsilon^2 (\phi + w) \frac{\partial w}{\partial \theta} \nabla_{(\bar{Y})} Y_\theta > = \varepsilon^3 L(w, \eta) + \varepsilon^2 Q(w, \eta). \]

Note that \( \nabla_{(\bar{Y})} Y \) and \( \nabla_{(\bar{Y})} Y_\theta \) are not the same.

\[ < -\frac{\dot{\psi}}{\phi} Y, \varepsilon \frac{\partial w}{\partial \theta} Y_\theta > = \varepsilon^3 L(w, \eta) + \varepsilon Q(w, \eta). \]

\[ < -\frac{\dot{\psi}}{\phi} Y, \varepsilon \frac{\partial^2 w}{\partial \theta^2} Y > = -\varepsilon \frac{\dot{\psi}}{\phi} \frac{\partial^2 w}{\partial \theta^2} + \varepsilon^3 L(w, \eta) + \varepsilon^3 Q(w, \eta). \]
\[
< -\frac{\dot{\psi}}{\phi} \varepsilon^2 (\phi + w)^2 \nabla_\theta \Psi > = -\varepsilon^2 \frac{\dot{\psi}}{\phi} (\phi + w)^2 < \Psi, \nabla_\theta \Psi > \\
= -\varepsilon^2 \frac{\dot{\psi}}{\phi} (\phi + w)^2 (\Psi \Psi < \Psi, \nabla_\theta \Psi > - < \nabla_\theta \Psi, \Psi >).
\]

First we have
\[
\Psi < \Psi, \Psi > = \frac{1}{r} (< \Psi, \Psi > - < \Psi >) + \begin{cases} (- \sin \theta \cos \theta \Psi < X_1, X_1 > + \sin \theta \cos \theta \Psi < X_2, X_2 > \\ + (\cos^2 \theta - \sin^2 \theta) \Psi < X_1, X_2 >) \end{cases}
= \frac{1}{3} \Psi \Psi < \Psi, \Psi > + \varepsilon L(w, \eta) + \varepsilon^{-1} Q(w, \eta) + O(\varepsilon^2),
\]
\[
< \nabla_\theta \Psi, \Psi > = \frac{1}{r} < \nabla_\theta \Psi, \Psi > \\
= \frac{1}{r} < \nabla_\theta \Psi, \Psi > \\
= \frac{1}{r} < \Psi, \Psi > + < \nabla_\theta \Psi, \Psi >.
\]

From
\[
\frac{1}{r} < \Psi, \Psi > = \frac{1}{r} + \frac{1}{r} \varepsilon \phi R(\Psi, \Psi, \Psi, \Psi) + \frac{2}{3} \Psi \Psi < \Psi, \Psi > + \varepsilon L(w, \eta) + \varepsilon^{-1} Q(w, \eta) + O(\varepsilon^2),
\]
\[
< \nabla_\theta \Psi, \Psi > = \frac{1}{3} \Psi < \Psi, \Psi > \\
= \frac{1}{3} \varepsilon \phi R(\Psi, \Psi, \Psi, \Psi) + \frac{1}{3} \Psi \Psi < \Psi, \Psi > + \varepsilon L(w, \eta) + \varepsilon^{-1} Q(w, \eta) + O(\varepsilon^2).
\]

We get
\[
< \nabla_\theta \Psi, \Psi > = \frac{1}{r} + \frac{2}{3} \varepsilon \phi R(\Psi, \Psi, \Psi, \Psi) + \frac{2}{3} \Psi \Psi < \Psi, \Psi > + \varepsilon L(w, \eta) + \varepsilon^{-1} Q(w, \eta) + O(\varepsilon^2),
\]
\[
< -\frac{\dot{\psi}}{\phi} \varepsilon^2 (\phi + w)^2 \nabla_\theta \Psi > = \varepsilon (\phi^2 + \tau_0) + \varepsilon \frac{\dot{\psi}}{\phi} w + \frac{2}{3} \varepsilon^2 \phi^2 \Psi R(\Psi, \Psi, \Psi, \Psi) \\
+ \frac{2}{3} \Psi \Psi < \Psi, \Psi > + \varepsilon L(w, \eta) + \varepsilon Q(w, \eta) + O(\varepsilon^4).
\]
\[
< -\frac{\dot{\psi}}{\phi} \varepsilon^2 (\frac{\partial w}{\partial \theta})^2 \nabla_\theta \Psi > = \varepsilon^3 Q(w, \eta).
\]
\[-\frac{\dot{\psi}}{\phi} \psi, \epsilon^2 (\phi + w) \frac{\partial w}{\partial \theta} \nabla Y, \psi > = \epsilon^3 L(w, \eta) + \epsilon^2 Q(w, \eta).\]

\[-\frac{\dot{\psi}}{\phi} \psi, \epsilon^2 (\phi + w) \frac{\partial w}{\partial \theta} \nabla Y, \theta > = \epsilon^3 L(w, \eta) + \epsilon^2 Q(w, \eta).\]

We collect all the terms and get

\[
\begin{align*}
<N_0, \nabla & \partial_s \partial_\theta > = \epsilon(\phi^2 + \tau_0) - \frac{\epsilon}{\phi} \frac{\partial^2 w}{\partial \theta^2} + \frac{\epsilon}{\phi} \psi + \\
&+ \frac{2}{3} \epsilon^3 \phi^2 \psi R(Y, Y_\theta, Y, Y_\theta) + \frac{2}{3} \epsilon^3 \phi^2 R(Y_\theta, X_0, Y, Y_\theta) \\
&+ \frac{2}{3} \epsilon^2 \phi \phi R(Y_\theta, X_0, \eta, Y_\theta) - \frac{2}{3} \epsilon^2 \phi \phi R(Y, X_0, \eta, Y) \\
&+ 2 \frac{2}{3} \phi \phi R(Y, Y_\theta, \eta, Y_\theta) + \epsilon^3 L(w, \eta) + \epsilon Q(w, \eta) + O(\epsilon^4).\]
\end{align*}
\]

For

\[
<N, \nabla \partial_s \partial_\theta > = < N_0 + a_1 \partial_s + a_2 \partial_\theta, \nabla \partial_s \partial_\theta >
\]

we don’t need very precise expansion because $g^{s\theta}$ is small relatively.

\[
\begin{align*}
<N, \nabla \partial_\theta \partial_s > = \epsilon(\phi^2 + \tau_0) + \epsilon^2 L(w, \eta) + \epsilon^2 Q(w, \eta), & \quad (19) \\
<N, \nabla \partial_\theta \partial_s > = \epsilon^2 \phi \phi + O(\epsilon^4) + \epsilon^2 L(w, \eta) + \epsilon^2 Q(w, \eta). & \quad (20) \\
<N_0, \nabla \partial_\theta \partial_s > = \epsilon(\phi^2 + \tau_0) + \epsilon^2 L(w, \eta) + \epsilon Q(w, \eta). & \quad (21)
\end{align*}
\]

Now we can calculate the mean curvature

\[
H(\mathcal{D}_{\tau_0, \eta_0}, \epsilon(w, \eta)) = g^{ss} < N, \nabla \partial_s \partial_s > + 2 g^{s\theta} < N, \nabla \partial_s \partial_\theta > + g^{\theta \theta} < N, \nabla \partial_\theta \partial_\theta >
\]

\[
= \frac{1}{k}(g^{ss} < kN, \nabla \partial_s \partial_s > + 2 g^{s\theta} < kN, \nabla \partial_s \partial_\theta > + g^{\theta \theta} < kN, \nabla \partial_\theta \partial_\theta >)
\]

\[
\text{21}
\]
From (6), (7), (8), (12), (13) and (15) we know

\[ g^{ss} < kN, \nabla_{\partial_s} \partial_s > = \varepsilon^{-2} \phi^{-2}(1 - (\varepsilon^2 \psi^2 R(Y, X_0, Y, X_0) + 2\varepsilon \phi^{-1} \psi^2 R(Y, X_0, \eta, X_0) + 4\varepsilon \frac{\phi}{\psi^2} R(Y, X_0, \eta, Y) + 2\phi^{-2} \psi \frac{\partial \psi}{\partial s} + 2\phi^{-2} \dot{\phi} \psi < Y, \frac{\partial \eta}{\partial x^0} > + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3)) \] \\
= + \varepsilon^3 (\phi \dot{\psi} + 2\psi \dot{\phi} + \psi^3)R(Y, X_0, \eta, X_0) + \varepsilon^2 (2\dot{\phi} \psi + 2\frac{\partial \psi}{\partial \phi} + \frac{\psi^3}{\phi})R(Y, X_0, \eta, X_0) \\
= + \varepsilon^3 \left( \frac{2\dot{\phi} \psi + 2\frac{\partial \psi}{\partial \phi} + \frac{\psi^3}{\phi}}{2} \right) \right) R(Y, X_0, \eta, X_0) \\
= \varepsilon^2 (\phi \dot{\psi} + 2\psi \dot{\phi} + \psi^3)R(Y, X_0, \eta, X_0) \\
= + \varepsilon^3 L(w, \eta) + \varepsilon Q(w, \eta) + \varepsilon^4 L(\partial^2 \eta) + \varepsilon^2 Q(w, \partial^2 \eta) + O(\varepsilon^4)) \]
From (6), (7), (8), (16), (17) and (18) we know

\[ g^{\theta\theta} < kN, \nabla_{\partial_\theta} s > = \varepsilon^{-2}\phi^{-2}(1 - (2\phi^{-1}w + \frac{1}{3}\varepsilon^{2})R(\Upsilon, \Upsilon_\theta, \Upsilon, \Upsilon_\theta)
\]
\[ + \frac{2}{3}\varepsilon \phi R(\Upsilon, \Upsilon_\theta, \eta, \Upsilon_\theta) + \varepsilon^{2}L(w, \eta) + Q(w, \eta) + O(\varepsilon^{3}))
\]
\[ (a_1(-\varepsilon^{2}\phi + O(\varepsilon^{4}) + \varepsilon^{2}L(w, \eta) + \varepsilon^{2}Q(w, \eta))
\]
\[ + a_2(O(\varepsilon^{4}) + \varepsilon^{2}L(w, \eta) + \varepsilon^{2}Q(w, \eta))
\]
\[ + \varepsilon(\phi^{2} + \tau_0) - \varepsilon \frac{\psi}{\phi} \frac{\partial^{2}w}{\partial \theta^{2}} + \varepsilon \frac{\psi}{\partial \theta} w + \frac{2}{3}\varepsilon^{3}\phi^{2}\phi^{2}R(\Upsilon, \Upsilon_\theta, \Upsilon, \Upsilon_\theta)
\]
\[ + \frac{2}{3}\varepsilon^{2}\phi^{2}R(\Upsilon_\theta, X_0, \Upsilon, \Upsilon_\theta) + \frac{2}{3}\varepsilon^{2}\phi^{2}R(\Upsilon_\theta, X_0, \eta, \Upsilon_\theta)
\]
\[ + \frac{2}{3}\varepsilon^{2}\phi^{2}R(\Upsilon, X_0, \eta, \Upsilon) + \frac{2}{3}\varepsilon^{2}\phi^{2}R(\Upsilon, \Upsilon_\theta, \eta, \Upsilon_\theta)
\]
\[ + \frac{2}{3}\varepsilon L(w, \eta) + \varepsilon Q(w, \eta) + O(\varepsilon^{4})
\]
\[ = \varepsilon^{-2}\phi^{-2}(-a_1\phi + \varepsilon(\phi^{2} + \tau_0) - \varepsilon \frac{\psi}{\phi} \frac{\partial^{2}w}{\partial \theta^{2}} - \varepsilon \frac{\psi}{\phi} w
\]
\[ + \frac{1}{3}\varepsilon^{3}\phi^{2}R(\Upsilon, \Upsilon_\theta, \Upsilon, \Upsilon_\theta) + \frac{2}{3}\varepsilon^{3}\phi^{2}R(\Upsilon_\theta, X_0, \Upsilon, \Upsilon_\theta)
\]
\[ + \frac{2}{3}\varepsilon^{2}\phi^{2}R(\Upsilon_\theta, X_0, \eta, \Upsilon_\theta) - \frac{2}{3}\varepsilon^{2}\phi^{2}R(\Upsilon, X_0, \eta, \Upsilon)
\]
\[ + \varepsilon^{3}L(w, \eta) + \varepsilon Q(w, \eta) + O(\varepsilon^{4})
\]

From (6), (7), (8), (19), (20) and (21) we know

\[ g^{s\theta} < kN, \nabla_{\partial_\theta} s > = \varepsilon^{-2}\phi^{-2}(\frac{2}{3}\varepsilon^{3})\phi^{2}R(\Upsilon, X_0, \Upsilon, \Upsilon_\theta) + \frac{2}{3}\varepsilon^{3}\phi^{2}R(\Upsilon, X_0, \eta, \Upsilon_\theta)
\]
\[ + R(\eta, X_0, \Upsilon, \Upsilon_\theta) + \frac{1}{3}\varepsilon^{3}\phi^{2}R(\eta, \Upsilon, \Upsilon, \Upsilon_\theta)
\]
\[ + \phi^{-2}\phi \frac{\partial w}{\partial \theta} + \phi^{-1}\psi < \frac{\partial \eta}{\partial x_0}, \Upsilon_\theta > e + \varepsilon^{2}L(w, \eta) + Q(w, \eta) + O(\varepsilon^{3}))
\]
\[ (a_1(O(\varepsilon^{4}) + \varepsilon^{2}L(w, \eta) + \varepsilon^{2}Q(w, \eta))
\]
\[ + a_2(O(\varepsilon^{4}) + \varepsilon^{2}L(w, \eta) + \varepsilon^{2}Q(w, \eta))
\]
\[ + O(\varepsilon^{3}) + \varepsilon^{2}L(w, \eta) + \varepsilon Q(w, \eta)
\]
\[ = \varepsilon^{-2}\phi^{-2}(\varepsilon^{3}L(w, \eta) + \varepsilon Q(w, \eta) + O(\varepsilon^{4})).
\]
And we know

\[ H = \frac{1}{k} e^{-2\phi - 2(2\varepsilon \phi^2 - \varepsilon \dot{\phi} \frac{\partial^2 w}{\partial s^2} + \frac{\partial^2 w}{\partial \theta^2} - 2\varepsilon (\phi^2 - \tau_0) \frac{\dot{\phi}}{\phi^2} \frac{\partial w}{\partial s} - \varepsilon \frac{\dot{\phi}}{\phi} w} \]

\[ -\varepsilon^2 \frac{\dot{\psi}}{\phi} < \frac{\partial^2 \eta}{\partial x^2}, \mathbf{Y} > - \varepsilon (\frac{\dot{\psi} \dot{\psi}}{\phi} + 2(\phi^2 - \tau_0) \frac{\dot{\phi}}{\phi^2}) < \frac{\partial \eta}{\partial x^2}, \mathbf{Y} > \]

\[ + \varepsilon^3 (\dot{\phi} \dot{\psi} + 2 \dot{\phi}^2 \dot{\psi} + \dot{\psi}^3 - (\phi^2 - \tau_0) \dot{\psi}^2) R(\mathbf{Y}, X_0, Y, X_0) \]

\[ + \frac{1}{3} \varepsilon^3 \phi^2 \psi R(\mathbf{Y}, T, T, T) + \frac{2}{3} \varepsilon^3 \phi^2 \phi R(\mathbf{Y}, T, X_0, Y, T) \]

\[ + \varepsilon^2 (2 \dot{\phi} \dot{\psi} + 2 \dot{\phi}^2 \dot{\psi} + \dot{\psi}^3 - \phi^2 - \tau_0 \frac{\dot{\phi}}{\phi^2} (\phi^2 - \tau_0)) R(\mathbf{Y}, X_0, \eta, X_0) \]

\[ + \varepsilon^2 \frac{2}{3} \frac{\dot{\phi} \dot{\psi}}{\phi} + \frac{2}{3} \frac{\dot{\phi}^3}{\phi} - \frac{2}{3} \frac{\dot{\psi} \dot{\psi}}{\phi} - \frac{2}{3} \frac{\dot{\phi} \dot{\phi}}{\phi} \]

\[ + \frac{2}{3} \varepsilon^3 \phi^2 \phi R(\mathbf{Y}, X_0, \eta, T) \]

\[ + \varepsilon^2 L(w, \eta) + \varepsilon Q(w, \eta) + \varepsilon^4 L(\partial^2 \eta) + \varepsilon^4 Q(\partial w, \partial^2 \eta) + O(\varepsilon^4) \]

From (9) we know

\[ \frac{1}{k} = 1 - \varepsilon^2 \frac{\dot{\psi}^2}{\phi^2} R(\mathbf{Y}, X_0, T, X_0) - \varepsilon \frac{\dot{\phi} \dot{\psi}}{\phi} R(\mathbf{Y}, T, X_0, \eta, X_0) \]

\[ + \frac{2}{3} \varepsilon \frac{\dot{\phi} \dot{\psi}}{\phi} R(\mathbf{Y}, X_0, \eta, Y) + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3). \]

So at last we get result mentioned at the beginning of this subsection.

3 Jacobi operator

In this section we study the linear operators which appear in the expression of \( H(\mathcal{D}_{\phi_0, \tau_0, \phi_0, \varepsilon}(w, \eta)) \).

3.1 Definitions

The two linear operators appearing in (10) are

\[ w \mapsto \mathcal{L}_{SN} w := -\frac{\dot{\psi}^3}{\phi^3} (\frac{\partial^2 w}{\partial x^2} - \frac{\partial w}{\partial x^2} - 2(\phi^2 - \tau_0) \frac{\dot{\phi}}{\phi^4} \frac{\partial w}{\partial s} - \frac{\dot{\phi}}{\phi^3} w \]

\[ \eta \mapsto \mathcal{J} \eta := -\frac{\dot{\psi}^3}{\phi^3} (\frac{\partial \eta}{\partial x^0} - \frac{1}{\varepsilon} (\frac{\dot{\psi} \dot{\psi}}{\phi^3} + 2(\phi^2 - \tau_0) \frac{\dot{\phi}}{\phi^4}) \frac{\partial \eta}{\partial x^0} \frac{\partial \eta}{\partial x^0} \]

\[ -\phi^{-2} (2 \dot{\phi} \dot{\psi} + 2 \frac{\dot{\phi}^2 \dot{\psi}}{\phi} + \frac{\dot{\psi}^3}{\phi} - 2 \frac{\dot{\psi}^2}{\phi} (\phi^2 - \tau_0) - 2 \phi \dot{\phi}^2) R(\eta, X_0) X_0. \]
When $\tau_0 = \frac{1}{4}$, i.e. the Delaunay surface is a column, the second operator $J$ reduces to the Jacobi operator $J_A$ of the geodesic. The geodesic is called non-degenerate when $J_A$ is invertible. And the first operator is conjugate to the Jacobi operator which corresponds to the second variation of the area functional.

It is easy to see that in terms of the coordinate $(s, \theta)$

$$L_{SN} : \mathcal{C}^2_{\epsilon} (SN) \rightarrow \mathcal{C}^0_{\epsilon} (SN)$$

is bounded uniformly in $\epsilon$. We can analyze this operator using the eigendecomposition for $\partial_s^2$ on $S^1$. As in (3), if the eigenfunction decomposition of $w$ is given by

$$w(s, \theta) = w_0 + w_1 \cos \theta + w_2 \sin \theta + \sum_{j>2} \tilde{w}_j \xi_j$$

then

$$w_1 = w_1 \cos \theta + w_2 \sin \theta$$

and

$$\tilde{w} = \sum_{j>2} \tilde{w}_j \xi_j.$$ 

We denote by $\Pi_0$, $\Pi_1$ and $\tilde{\Pi}$ the projections onto these three components, respectively. From now on, we assume that we are working with functions $w$ such that $\Pi_1 w = 0$, and thus we only need to be concerned with the operators $(L_{SN})_0$ and $\tilde{L}_{SN}$.

We are going to study the mapping properties of the operator in 3 different modes.

### 3.2 High mode

In this mode, we are going to prove that

$$L_{SN} : \tilde{\Pi} \mathcal{C}^2_{\epsilon} (SN) \rightarrow \tilde{\Pi} \mathcal{C}^0_{\epsilon} (SN)$$

is an isomorphism whose inverse is bounded independent of $\epsilon$.

First it is clear that

$$L_{SN}(\tilde{\Pi} \mathcal{C}^2_{\epsilon} (SN)) \subseteq \tilde{\Pi} \mathcal{C}^2_{\epsilon} (SN).$$

From $\frac{\partial}{\partial s} = \epsilon \dot{\psi} \frac{\partial}{\partial \psi}$, we have, for $w, v \in \tilde{\Pi}W^1_{\epsilon}(SN)$

$$L_{SN} w = -\frac{\dot{\psi}}{\phi^3} \left( \frac{\partial^2 w}{\partial s^2} + \frac{\partial^2 w}{\partial \theta^2} \right) - 2(\phi^2 - \tau_0) \frac{\dot{\phi}}{\phi^3} \frac{\partial w}{\partial s} - \frac{\dot{\psi}}{\phi^3} w$$

$$= -\frac{\dot{\phi}}{\phi^3} \frac{\dot{\psi}^3}{\phi^2} \frac{\partial}{\partial \psi} \left( \frac{\phi^3}{\dot{\phi}^2} \frac{\partial}{\partial \psi} w \right) + \frac{\dot{\psi}^4}{\phi^4} w + \frac{\dot{\psi}^4}{\phi^4} \frac{\partial^2 w}{\partial \theta^2}.$$ 

Consider the bounded bilinear functional

$$L(v, w) = \int_{SN} (v L_{SN} w) d\theta d\psi.$$ 

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We have, for some positive constant $C$ which does not depend on $\varepsilon$,

$$L(w, w) = \int_{\mathcal{S}\mathcal{N}\Gamma} \left( \frac{\dot{\psi}^3}{\phi^2} \left| \frac{\partial}{\partial \psi} w \right|^2 - \frac{\dot{\psi}}{\phi^2} w^2 + \frac{\dot{\psi}}{\phi^2} \left| \frac{\partial w}{\partial \theta} \right|^2 \right) d\theta d\psi$$

$$\geq C \int_{\mathcal{S}\mathcal{N}\Gamma} \left( \left| \frac{\partial w}{\partial \psi} \right|^2 + \left| \frac{\partial w}{\partial \theta} \right|^2 \right) d\theta d\psi. \quad (22)$$

The inequality holds because for $w \in \bar{\Pi}^{1,2}_{\varepsilon}(\mathcal{S}\mathcal{N}\Gamma)$, we have

$$\int_{\mathcal{S}\mathcal{N}\Gamma \cap \{ \psi = \psi_0 \}} \left| \frac{\partial w}{\partial \theta} \right|^2 d\theta \geq 4 \int_{\mathcal{S}\mathcal{N}\Gamma \cap \{ \psi = \psi_0 \}} |w|^2 d\theta,$$

for every $\psi_0$.

From (22) we know $L_{\mathcal{S}\mathcal{N}\Gamma}$ is invertible and

$$\|w\|_{W^{1,2}_{\varepsilon}} \leq C \|L_{\mathcal{S}\mathcal{N}\Gamma} w\|_{W^{-1,2}_{\varepsilon}}$$

with $C$ does not depend on $\varepsilon$. And from standard regularity theory of elliptic PDE we can get

$$\|w\|_{C^{2,\alpha}_{\varepsilon}} \leq C \|L_{\mathcal{S}\mathcal{N}\Gamma} w\|_{C^{0,\alpha}_{\varepsilon}}.$$

### 3.3 1st-mode

In this mode, we are going to prove that

$$\mathcal{J} \eta : C^{2,\alpha}(\mathcal{N}\Gamma) \rightarrow C^{0,\alpha}(\mathcal{N}\Gamma)$$

is invertible and has certain estimates.

To prove this we need to find the relationship of the operator $\mathcal{J}$ and the Jacobi operator of the geodesic $\mathcal{J}_A$. Notice that

$$\frac{\dot{\psi}^3}{\phi} \mathcal{J} \eta = -\frac{\dot{\psi}^3}{\phi^2} \frac{\partial}{\partial x_0} \left( \frac{\dot{\psi}^3}{\phi^2} \frac{\partial \eta}{\partial x_0} \right)_{x_0}$$

$$-\frac{\dot{\psi}^3}{\phi^3} \left( 2\ddot{\phi} + 2\dot{\phi}^2 \dot{\psi} \right) \phi + \frac{\dot{\psi}^3}{\phi} - 2\frac{\ddot{\phi}^2}{\phi} (\phi^2 - \tau_0) - 2\phi\dot{\phi}^2 \mathcal{R}(\eta, X_0)X_0|_{x_0}.$$ 

Let $\frac{\ddot{\phi}}{\phi} \frac{\partial}{\partial x_0} \frac{\partial}{\partial y_0} = \frac{\partial}{\partial y_0}$, then $dy_0 = \frac{\phi^2}{\psi^3} dx_0$. We can solve

$$dy_0 = \frac{\phi^2}{\psi^3} dx_0$$

$$y_0(0) = 0 \quad (23)$$

and get $y_0(x_0)$. We know $x_0 = \varepsilon \psi$, so the period of $\phi$ and $\psi$ or the derivatives of them have period of order $\varepsilon$. So coefficients such as $\frac{\phi^2}{\psi^3}$ and $\frac{\ddot{\phi}}{\phi} \frac{\partial}{\partial x_0} \frac{\partial}{\partial y_0}$
\[ \frac{\dot{\psi}^3}{\phi} - 2\frac{\ddot{\psi}^2}{\phi}(\phi^2 - \tau_0) - 2\dot{\phi}\ddot{\psi}^2 \] are highly oscillating. To understand the mean value of the coefficients in the right way is the key to under the operator \( J \).

Suppose \( \psi \in [a_1, b_1] \) is one period of \( \phi \). Suppose

\[ \frac{\int_{a_1}^{b_1} \frac{\phi^2}{\psi^3} d\psi}{\int_{a_1}^{b_1} d\psi} = I_1. \]

Actually \( I_1 \) indicates the ratio of the length of \( y_0 \) and that of \( x_0 \). In some sense \( dy_0 \equiv I_1 dx_0 \).

And then from

\[ \frac{\dot{\psi}^3}{\phi} J \eta = \frac{\partial^2 \eta}{\partial y_0^2} \bigg|_{y_0(x_0)} - \frac{\dot{\psi}^3}{\phi^3} (2\dot{\phi} \ddot{\psi} + 2\dot{\phi}^2 \ddot{\psi} \phi + \dot{\psi}^3 \phi + 2\dot{\psi}^2 \ddot{\phi} (\phi^2 - \tau_0) - 2\phi \ddot{\phi}^2) R(\eta, X_0) X_0 |_{x_0}. \]

Now we need the average of \( \frac{\dot{\psi}^3}{\phi^3} (2\dot{\phi} \ddot{\psi} + 2\dot{\phi}^2 \ddot{\psi} \phi + \dot{\psi}^3 \phi + 2\dot{\psi}^2 \ddot{\phi} (\phi^2 - \tau_0) - 2\phi \ddot{\phi}^2) \) in the coordinate \( y_0 \). Note that \( dy_0 = \frac{\phi^2}{\psi^3} dx_0 = \frac{\phi^2}{\psi^3} d\psi \). If we assume \( y_0(a_1) = y_1, y_0(b_1) = y_2 \), then we have

\[
\frac{\int_{a_1}^{b_1} \frac{\phi^2}{\psi^3} \left(2\dot{\phi} \ddot{\psi} + 2\dot{\phi}^2 \ddot{\psi} \phi + \dot{\psi}^3 \phi + 2\dot{\psi}^2 \ddot{\phi} (\phi^2 - \tau_0) - 2\phi \ddot{\phi}^2\right) d\psi}{\int_{a_1}^{b_1} \frac{\phi^2}{\psi^3} d\psi} = I_2.
\]

\( I_2 \) indicates that in some sense

\[
-\frac{\partial^2 \eta}{\partial y_0^2} \bigg|_{y_0} = -I_2 R(\eta, X_0) X_0 |_{x_0}.
\]

But \( \frac{\partial}{\partial y_0} \cong I_1^{-1} \frac{\partial}{\partial x_0} \). So

\[
-\frac{\partial^2 \eta}{\partial y_0^2} \bigg|_{y_0} - I_2 R(\eta, X_0) X_0 |_{x_0} \cong -I_1^{-2} \left( \frac{\partial^2 \eta}{\partial x_0^2} |_{x_0} + I_1^2 I_2 R(\eta, X_0) X_0 \right) |_{x_0}.
\]

So if

\[ I_1^2 I_2 = 1, \]

we will have the chance to get the Jacobi operator of the geodesic \( J_A \). Fortunately, it is true. Due to easy calculation, it is equivalent to the following lemma

**Lemma 3.1.** ("average 1" lemma)

\[
\int_{a_1}^{b_1} \frac{1}{\phi} \left(2\dot{\phi} \ddot{\psi} + 2\dot{\phi}^2 \ddot{\psi} \phi + \dot{\psi}^3 \phi + 2\dot{\psi}^2 \ddot{\phi} (\phi^2 - \tau_0) - 2\phi \ddot{\phi}^2\right) d\psi \cdot \int_{a_1}^{b_1} \frac{\phi^2}{\psi^3} d\psi = (b_1 - a_1)^2.
\]
The proof of this lemma is long and technical calculation. It depends on some special properties of Delaunay surface. We write it in the Appendix A.

Now we can discuss the operator $\mathcal{J}$. We have the following lemma.

**Lemma 3.2.** If $\varepsilon$ is sufficiently small, for every $f \in C^{0,\alpha}(N\Gamma)$, there exists a unique $\eta \in C^{2,\alpha}(N\Gamma)$

$$\mathcal{J}\eta = f$$

such that

$$\|\eta\|_{C^0} \leq C\|f\|_{C^0}$$

and

$$\|\eta\|_{C^{\alpha}_y} \leq C\|f\|_{C^0}, \quad \|\eta\|_{C^{0,\alpha}_y} \leq C(\|f\|_{C^0} + \varepsilon^{-\alpha}\|f\|_{C^0})$$

for some constant $C$ which does not depend on $\varepsilon$.

**Proof.** We do this in several steps

**Step 1. Uniqueness of the solution** Note that $\mathcal{J}$ depends on $\varepsilon$. We use $\mathcal{J}_\varepsilon$ to indicate this dependence. We first consider

$$\mathcal{J}_\varepsilon\eta = 0.$$ 

We are going to prove that for $\varepsilon$ very small, this equation only has trivial solution. Suppose it were false, then, for a sequence $\varepsilon_n \to 0$,

there exist non trivial solutions $\eta_n$ such that

$$\mathcal{J}_{\varepsilon_n}\eta_n = 0.$$ 

One can rescale $\eta_n$ such that $\|\eta_n\|_{C^0} = 1$. So we assume $\|\eta_n\|_{C^0} = 1$. So in the coordinate $y_0$, we have

$$\frac{\partial^2 \eta}{\partial y_0^2}(y_0(x_0)) - \frac{\psi^3}{\phi^4}(2\phi \dot{\psi} + 2\dot{\phi}^2) + \frac{\psi^3}{\phi} - 2\frac{\psi^2}{\phi} (\phi^2 - \tau_0) - 2\phi\dot{\phi}^2 R(\eta, X_0)X_0|_{x_0} = 0.$$ 

d$y_0 = \frac{\psi^3}{\phi^3} dx_0$, so $y_0$ also depends on $\varepsilon_n$. We also denote $y_0$ as $y_{0,\varepsilon_n}(x_0)$. We have

$$y_{0,\varepsilon_n}(\bar{x}_0) = \int_{y_0(0)}^{\bar{x}_0} \frac{\phi^2}{\psi^3} dx_0, \bar{x}_0 \in [0, L_\Gamma]$$

where $L_\Gamma$ is the length of the geodesic. Notice that

$$y_{0,\varepsilon_n}(L_\Gamma) = \int_0^{L_\Gamma} \frac{\phi^2}{\psi^3} dx_0 = I_1 L_\Gamma$$

which does not depends on $\varepsilon_n$. And we have, for every $\bar{x}_0$ fixed,

$$\lim_{\varepsilon_n \to 0} y_{0,\varepsilon_n}(\bar{x}_0) = I_1 \bar{x}_0. \quad (24)$$

Note that all the functions like $\phi, \psi$ and their derivatives are bounded. The curvature $R$ depends only on the manifold and geodesic. So there exists some
constant $C$ that only depends on the manifold $M$ and geodesic $\Gamma$ and the Delaunay parameter $\tau_0$ such that
\[ \| \eta_n \|_{C^2_{\gamma_0}} \leq C \| \eta_n \|_{C^0} = C. \]
So for some $0 < \alpha < 1$, we have
\[ \eta_n \rightarrow \eta_\infty \]
in $C^{1,\alpha}_{\gamma_0}$.

On the other hand $\eta_n$ satisfies
\[
\int \frac{\partial}{\partial y_0} \eta_n \frac{\partial}{\partial y_0} \varphi - \frac{\dot{\psi}^3}{\phi^3} (2 \dot{\phi} \ddot{\psi} + 2 \dot{\phi}^2 \dot{\psi} + \ddot{\psi}^3 - 2 \dot{\phi}^2 (\phi^2 - \tau_0) - 2 \phi \ddot{\phi})
\]
\[
< R(\eta_n, X_0) |_{x_0(y_0)} \varphi > |_{x_0(y_0)} dy_0
\]
\[= 0 \]
for any $\varphi \in C^\infty(N \Gamma)$. Suppose for different $\varepsilon_n$ we have chosen the same $\varphi(y_0)$, i.e. $\varphi_n(x_0) = \varphi(y_0, \varepsilon_n(x_0))$. Let $\varepsilon_n \rightarrow 0$, we have
\[
\lim_{\varepsilon_n \rightarrow 0} \int \frac{\partial}{\partial y_0} \eta_n \frac{\partial}{\partial y_0} \varphi dy_0 = \int \frac{\partial}{\partial y_0} \eta_\infty \frac{\partial}{\partial y_0} \varphi
\]
and
\[
\lim_{\varepsilon_n \rightarrow 0} \int \frac{\dot{\psi}^3}{\phi^3} (2 \dot{\phi} \ddot{\psi} + 2 \dot{\phi}^2 \dot{\psi} + \ddot{\psi}^3 - 2 \dot{\phi}^2 (\phi^2 - \tau_0) - 2 \phi \ddot{\phi})
\]
\[
< R(\eta_n, X_0) |_{x_0(y_0)} \varphi(y_0, \varepsilon_n(x_0)) > dy_0
\]
\[= I_1 \int < R(\eta_\infty, X_0) |_{x_0(y_0)} \varphi(I_1 x_0) > dy_0. \]
So we have, for every smooth $\varphi$,
\[
\int \frac{\partial}{\partial y_0} \eta_\infty \frac{\partial}{\partial y_0} \varphi dy_0 = I_1 \int < R(\eta_\infty, X_0) |_{x_0(y_0)} \varphi(I_1 x_0) > dy_0. \]
From the regularity theory of differential equation, we know $\eta_\infty \in C^\infty_{\gamma_0}$ if the curvature is in $C^\infty$. So
\[
\frac{\partial^2 \eta_\infty}{\partial y_0^2} |_{y_0(x_0)} + I_2 R(\eta_\infty, X_0) |_{x_0} = 0.
\]
However in this limit case, $y_0 = I_1 x_0$ from (24). So we have
\[
\frac{\partial^2 \eta_\infty}{\partial x_0^2} |_{x_0} + I_1^2 I_2 R(\eta_\infty, X_0) |_{x_0} = 0.
\]
From the Lemma 3.1 we know $I_1^2 I_2 = 1$. So $\eta_\infty$ solves the Jacobi Equation of the geodesic,
\[
\frac{\partial^2 \eta_\infty}{\partial x_0^2} + R(\eta_\infty, X_0) X_0 = 0.
\]
However, as $\eta_n \to \eta_\infty$ in $C_{y_0,0}^{1,\alpha}$, $\|\eta_\infty\|_{C^0} = 1$, which contradicts the Jacobi operator $J_A$ of the geodesic is non degenerate. So we proved that, for $\varepsilon$ sufficient small, the equation

$$J_\varepsilon \eta = 0$$

only has trivial solution.

**Step 2. Existence of the solution** Consider

$$J_\varepsilon \eta = f$$

when $\varepsilon > 0$ is very small and $f \in C_{x_0}^{0,\alpha}(N\Gamma)$. Because $J_\varepsilon$ is self-adjoint, the existence of the solution follows from step 1.

**Step 3. Estimates of the solution** If $\eta$ satisfies

$$J_\varepsilon \eta = f,$$

first in $y_0$ coordinate we have

$$\frac{\partial^2 \eta}{\partial y_0^2} |_{y_0,\varepsilon(x_0)} - \frac{\dot{\psi}^3}{\phi^3} (2\dot{\phi}\dot{\psi} + 2\frac{\phi^2}{\dot{\phi}} \dot{\psi}^2) + \dot{\psi}^3 - 2\frac{\dot{\psi}^2}{\dot{\phi}} (\phi^2 - \tau_0) - 2\phi\dot{\phi}^2) R(\eta, X_0)X_0 |_{x_0} = f.$$  

Using the same method as in Step 1, we can prove

$$\|\eta\|_{C^0} \leq C\|f\|_{C^0}.$$  

So we have

$$\|\eta\|_{C_{y_0}^2} \leq C\|f\|_{C^0}, \|\eta\|_{C_{y_0}^{2,\alpha}} \leq C(\|f\|_{C_{y_0}^{0,\alpha}} + \varepsilon^{-\alpha}\|f\|_{C^0}) \quad (25)$$

However we are fond of $f$ with special properties for our needs later. In the expression of the mean curvature $[10]$, we see the two important terms that will appear on the right hand side of the first mode are

$$\frac{2}{3} \varepsilon \dot{\phi} \Pi_1 R(Y_\theta, X_0, Y, Y_\theta) + \Pi_1(E),$$

where $\|E\|_{C^0} = O(\varepsilon^2)$ and $\|E\|_{C_{y_0}^{0,\alpha}} = O(\varepsilon^{2-\alpha})$. In fact $E$ comes from the part $O(\varepsilon^2)$ in the expression $[10]$. We can also write the expression of this $O(\varepsilon^2)$ in detail and will find that the terms may look like some good functions (like the curvatures of the manifold along the geodesic) times some oscillating functions (like $\phi$ and $\psi$). So we can get the estimate of $E$ as above.

First look at

$$J_\varepsilon \eta = \Pi_1(E).$$
From (25) and 
\[
\begin{align*}
\frac{\partial \eta}{\partial y_0} &= \psi^3 \frac{\partial \eta}{\phi^2 \partial x_0}, \\
\frac{\partial^2 \eta}{\partial y_0^2} &= \frac{\psi^6}{\phi^4 (\partial x_0^2)} + 1 \frac{\psi^3}{\phi^2} \left( \frac{\dot{\psi} \ddot{\psi}}{\phi^2} + 2 (\dot{\phi}^2 - \tau_0) \frac{\dot{\psi}}{\phi^3} \right) \frac{\partial \eta}{\partial x_0},
\end{align*}
\]
we know
\[
\|\eta\|_{C^1_{x_0}} \leq C \|E\|_{C^0} \leq C \varepsilon^2
\]
and
\[
\|\eta\|_{C^2_{x_0}} \leq C \varepsilon; \|\eta\|_{C^{2,\alpha}_{x_0}} \leq C \varepsilon^{1-\alpha}.
\]

Then look at
\[
\mathcal{J}_\varepsilon \hat{\eta} = \frac{2}{3} \varepsilon \dot{\phi} \hat{\Pi}(\Upsilon_\theta, X_0, \Upsilon, \Upsilon_\theta).
\]

To get good estimates for the solution of this equation, we use some special property of the right hand side. In \(\frac{2}{3} \varepsilon \dot{\phi} \hat{\Pi}(\Upsilon_\theta, X_0, \Upsilon, \Upsilon_\theta)\), we find that \(\hat{\Pi}(\Upsilon_\theta, X_0, \Upsilon, \Upsilon_\theta)\) is a good function, all of whose derivatives (with respect to \(x_0\)) have uniform bounds. \(\dot{\phi}\) is highly oscillating but it has average 0 in one period. This property is important for us.

From (25) we have
\[
\|\hat{\eta}\|_{C^2_{x_0}} \leq C \varepsilon; \|\hat{\eta}\|_{C^{2,\alpha}_{x_0}} \leq C \varepsilon^{1-\alpha}.
\]

Then we can get
\[
\|\hat{\eta}\|_{C^2_{x_0}} \leq C \varepsilon^2; \|\hat{\eta}\|_{C^0} \leq C \varepsilon^2.
\]
We will prove it in Appendix 3, namely,

**Lemma 3.3.**

\[
\begin{align*}
\|\hat{\eta}\|_{C^2_{x_0}} &\leq C \varepsilon^2, \\
\|\hat{\eta}\|_{C^1_{x_0}} &\leq C \varepsilon^2, \\
\|\hat{\eta}\|_{C^{2,\alpha}_{x_0}} &\leq C \varepsilon, \\
\|\hat{\eta}\|_{C^{2,\alpha}_{x_0}} &\leq C \varepsilon^{1-\alpha}.
\end{align*}
\]

### 3.4 0th-mode

To solve this mode, we need both consider a nonlinear ODE and its linearization.

Our aim is to perturb the initial surface in some way such that the surface after perturbation has trivial 0th projection. See (10) for the expression of \(H(\varphi_{(0)}, \gamma, \phi_0, \varepsilon(w, \eta))\). Suppose we have some method to perturb the initial surface such that the right hand side has trivial 0th projection (except for \(\frac{2}{\varepsilon}\)). However after we perturb the surface in high mode and 1st-mode, 0th projection...
may appear again. We expect the new 0th projection is much smaller than the original one. Then a fixed point argument will work. To accomplish this we solve
\[
\frac{\partial^2 \phi}{\partial \psi^2} - \phi^{-1} \left( 1 + \left( \frac{\partial \phi}{\partial \psi} \right)^2 \right) + (2 + \rho) \left( 1 + \left( \frac{\partial \phi}{\partial \psi} \right)^2 \right)^2 = 0
\]
(26)
\[
\frac{\partial \phi}{\partial \psi}(0) = \frac{\partial \phi}{\partial \psi}(\frac{L \Gamma}{\varepsilon}) = 0
\]
(27)
\[
\phi(0) = \phi(\frac{L \Gamma}{\varepsilon}) = 1 - \sqrt{1 - 4\tau(0)}
\]
(28)
\[
\rho = -\varepsilon^2 F_1(\phi, \phi_\psi) * \Pi_0(R_1) + \varepsilon F_4(\phi, \phi_\psi) \xi(x_0)
\]
\[
+ \varepsilon^3 \mu(\psi) + \varepsilon^3 \omega \phi_\psi
\]
(29)
where
\[
\|\xi\|_{C^1} \leq C_1 \varepsilon^2 , \|\mu\|_{C^2} \leq C_2, |\omega| \leq C_3, |\phi(0) - \frac{1 - \sqrt{1 - 4\tau_0}}{2}| \leq \varepsilon^2 C_4.
\]
(30)

In this chapter, we use $\phi$ to represent a general function and $\phi_0$ to represent the defining function of standard Delaunay surface.

Note that in $\rho$ we prescribe two functions $\xi(x_0)$, $\mu(\psi)$ and two constants $\omega$ and $\tau(0)$.

First we explain the reason for $\omega \phi_\psi$ and $\tau(0)$. To solve the ODE system above, we expect to have a global smooth function $\phi$ on the closed geodesic. This means
\[
\phi(0) = \phi(\frac{L \Gamma}{\varepsilon})
\]
\[
\phi_\psi(0) = \phi_\psi(\frac{L \Gamma}{\varepsilon}) = 0.
\]
We can adjust $\omega$ and $\tau(0)$ (or $\phi(0)$) to achieve this.

The two functions $\mu$ and $\xi$ are used to do fixed point theorem to get CMC surfaces. The aim of 0th mode is to use $\mu$ and $\xi$ to cancel the 0th projection of tail terms. Recall (11),
\[
\Pi_0(\varepsilon F_1(\phi, \phi_\psi) * R_1 + F_2(\phi, \phi_\psi) * R_2) + F_3(\phi, \phi_\psi) * R_3(\eta) + E + T(w, \eta)
\]
\[
= \varepsilon F_1(\phi, \phi_\psi) * \Pi_0(R_1) + F_3(\phi, \phi_\psi) * \Pi_0(R_3(\eta)) + \Pi_0(E) + \Pi_0(T(w, \eta))
\]
\[
\varepsilon F_1(\phi, \phi_\psi) * \Pi_0(R_1)
\]
is a big term, however, we can solve it in the ODE directly. $\mu$ is used to deal with the terms
\[
\Pi_0(E) + \Pi_0(T(w, \eta)).
\]
One thing we have to remember is that $\mu$ and $\xi$ will influence $\phi$, hence $\eta$ and $w$. When $\varepsilon^2 \mu$ is prescribed as large as $\Pi_0(E) + \Pi_0(T(w, \eta))$, this term will vary much smaller than $\varepsilon^2 \mu$. So the fixed point argument works. However, after the
analysis of 1st mode, we know \( \eta \) will vary nearly as much as \( \varepsilon^2 \mu \). So the fixed point argument does not work for these terms. However, we notice a non trivial cancellation property of these terms. First

\[
F_3(\phi, \phi_\psi) \ast \Pi_0(R_3(\eta)) = F_4(\phi, \phi_\psi)\Pi_0(R(\Upsilon, X_0, \eta, \Upsilon))
\]

because

\[
\Pi_0(R(\Upsilon, X_0, \eta, \Upsilon)) - R(\Upsilon_0, X_0, \eta, \Upsilon_0)) = 0.
\]

Then if we look at

\[
\frac{d\tau}{d\psi} = -\phi_\psi F_4(\phi, \phi_\psi)\Pi_0(R(\Upsilon, X_0, \eta, \Upsilon))
\]

we notice that \( R(\Upsilon, X_0, \eta, \Upsilon) \) is a good function whose derivative with respect to \( \psi \) is of \( \varepsilon \) order. What is more important is the following lemma.

**Lemma 3.4.** ("average 0" lemma) Suppose \( \phi_0 \) is the defining function for standard Delaunay surfaces of parameter \( \tau_0 \) and \( [a_1, b_1] \) is one period of \( \phi_0 \). Then we have

\[
\int_{a_1}^{b_1} \phi_0(\frac{\partial \phi_0}{\partial \psi})F_4(\phi_0, \frac{\partial \phi_0}{\partial \psi})d\psi = 0
\]

We will prove this lemma in Appendix A. Note that \( \phi_0(\frac{\partial \phi_0}{\partial \psi})F_4(\phi_0, \frac{\partial \phi_0}{\partial \psi}) \) is not trivially average 0 as the functions \( \phi \), so it is not a trivial lemma. With this lemma we know \(-\phi_\psi F_4(\phi, \phi_\psi)\Pi_0(R(\Upsilon, X_0, \eta, \Upsilon))\) is nearly average 0 in one period. So although \( \eta \) varies nearly as much as \( \varepsilon^2 \mu \), it will influence the 0th mode much less. This is the key observation to solve the 0th mode. We use \( \xi \) to cancel \( \Pi_0(R(\Upsilon, X_0, \eta, \Upsilon)) \).

Now we state the main theorem of 0th mode.

**Theorem 3.5.** For fixed \( C_3, C_2 > 0 \), we can choose \( C_3, C_4 > 0 \) and \( \varepsilon > 0 \) sufficiently small, such that for every \( \|\xi\|_{C_{4_0}} \leq C_1\varepsilon^2, \|\mu\|_{C_{2}} \leq C_2 \), we can find unique

\[
\|\omega_{\xi,\mu}\| \leq C_3, |\phi(0)_{\xi,\mu} - \frac{1 - \sqrt{1 - 4\eta_0}}{2}| \leq C_4\varepsilon^2
\]

and \( \phi_{\xi,\mu} \) which solves the ODE system from 26 to 29 with \( \omega = \omega_{\xi,\mu}, \phi(0) = \phi(0)_{\xi,\mu} \).

Moreover the following estimates hold

\[
\begin{align*}
|\omega_{\xi_2,\mu_2} - \omega_{\xi_1,\mu_1}| & \leq C_5(\varepsilon^2\|\mu_2 - \mu_1\|_{C_{2}} + \varepsilon\|\xi_2 - \xi_1\|_{C_{4_0}}) \\
|\phi(0)_{\xi_2,\mu_2} - \phi(0)_{\xi_1,\mu_1}| & \leq C(\varepsilon^2\|\mu_2 - \mu_1\|_{C_{2}} + \varepsilon\|\xi_2 - \xi_1\|_{C_{4_0}})
\end{align*}
\]

\[
\|\phi_{\xi_2,\mu_2}\omega_{\xi_2,\mu_2}\tau(0)_{\xi_2,\mu_2}(\psi) - \phi_{\xi_1,\mu_1}\omega_{\xi_1,\mu_1}\tau(0)_{\xi_1,\mu_1}(\psi)\|_{C_{2}}\leq \frac{C}{\varepsilon}\varepsilon^2\|\mu_2 - \mu_1\|_{C_{2}} + \varepsilon\|\xi_2 - \xi_1\|_{C_{4_0}}
\]

\[
\leq \frac{C}{\varepsilon}\varepsilon^2\|\mu_2 - \mu_1\|_{C_{2}} + \varepsilon\|\xi_2 - \xi_1\|_{C_{4_0}}
\] (31)
For given $C_1, C_2, C_3, C_4$ and $\varepsilon$ sufficiently small we intend to solve this ODE system in the period $\psi \in [0, \frac{1}{2} L_T]$. First we can prove the local existence of this ODE. Then we can use the Hamiltonian function

$$\tau = -\phi^2 + \frac{\phi}{\sqrt{1 + \phi_\psi^2}}$$

to extend the solution to the whole interval $[0, \frac{L_T}{\varepsilon}]$. Then we can adjust $\omega$ and $\tau(0)$ to satisfy these boundary conditions (27) and (28).

We prove Theorem 3.5 in steps. We solve the ODE system in Step 1 to Step 5 and get the estimates in Step 6.

**Step 1. Local existence and uniqueness of ODE (26)** Choose $A_1, A_2, B_1, B_2, K_1, K_2$ that only depend on $\tau_0$ such that

$$0 < A_1 < A_2 < \frac{1 - \sqrt{1 - 4\tau(0)}}{2} < \frac{1 + \sqrt{1 - 4\tau(0)}}{2} < B_2 < B_1 < 1,$$

and $0 < K_2 < K_1$ to be specified later. Define $C^1_{\lambda_1, r_1, K_1}([0, T]) = \{ \phi(\psi) \in C^1([0, T]) | A_i \leq \phi(\psi) \leq B_i, |\phi_\psi| \leq K_i \}, i = 1, 2$. We have

**Lemma 3.6.** If $\phi(\psi) \in C^1_{A_2, B_2, K_2}([0, T])$ solves the ODE system for $T \geq 0$. Then for some $\delta_1 > 0$, this solution can be uniquely extended to $\phi(\psi) \in C^1_{A_1, B_1, K_1}([0, T + \delta_1])$.

**Proof.** Denote $\frac{\partial \phi}{\partial \psi}$ by $\zeta$. Then $(\phi, \zeta)$ satisfies the following equivalent system

$$\begin{cases}
\frac{\partial \phi}{\partial \psi} = \zeta, \\
\frac{\partial \zeta}{\partial \psi} = \phi^{-1}(1 + \zeta^2) - (2 + \rho)(1 + \zeta^2)\frac{\tau}{2}, \\
\phi(0) = 1 - \sqrt{1 - 4\tau(0)}
\end{cases}$$

The right hand side is continuous in $(\phi, \zeta, \psi)$ and is a Lipschitz function with respect to $(\phi, \zeta)$ in the domain

$$\begin{cases}
|\zeta| < K_1 \\
A_1 < \phi < B_1 \\
0 < \psi < T + \delta_3
\end{cases}$$

for some $\delta_3 > 0$. So from standard theory of ODE we get the conclusion.

**Step 2. Global existence** To get the global existence, we need do apriori estimates for the solution. The key to do this is the first integral $\tau$. Recall

$$\tau = -\phi^2 + \frac{\phi}{\sqrt{1 + \phi_\psi^2}}.$$
We look at the orbit space of this dynamical system. In the graph below the horizontal direction represents $\phi$ and the vertical direction represents $\zeta$. The central circle represents the orbit of standard Delaunay surface with $\tau = \tau_0$. We choose $\delta_1(\tau_0) > 0$ sufficiently small. The outside circle represents Delaunay surface with $\tau = \tau_0 - \delta_1$ while the inside one represents $\tau = \tau_0 + \delta_1$. We can choose $K_1 > K_2$ and $K_2$ bigger than the maximal value of $\zeta$ of the outside circle. We denote the domain between the outside and inside circle by $Dom(\tau_0, \delta_1)$. Apparently $Dom(\tau_0, \delta_1) \subset \{(\zeta, \phi) | A_i \leq \phi \leq B_i, |\zeta| \leq K_i \}, i = 1, 2$. And we denote $C^1_{Dom(\tau_0, \delta_1)}([0, T_0]) = \{ \phi(\psi) \in C^1([0, T_0]) | (\phi, \phi(\psi)) \in Dom(\tau_0, \delta_1) \}$.

We can prove the following lemma.

**Lemma 3.7.** (Global existence) For fixed $\xi, \mu, \omega, \tau(0)$, when $\varepsilon$ is very small, there is a unique solution $\phi(\psi) \in C^1_{Dom(\tau_0, \delta_1)}([0, \frac{1}{\varepsilon} L_\Gamma])$ to ODE (26).

**Proof.** By using the same method as in Lemma [3.6], the solution will at least exist in $C^1_{Dom(\tau_0, \delta_1)}([0, T_0])$ for some positive $T_0$ because Lemma [3.6] holds for $T = 0$. If this lemma were not true, we assume $T_0$ is the maximal value such that $(\phi, \zeta) \in Dom(\tau_0, \delta_1)$ when $\psi \in [0, T_0)$. We know $0 < T_0 \leq \frac{1}{\varepsilon} L_\Gamma$ and $\tau(T_0) = \tau_0 \pm \delta_1$.

However, on the other hand, for $\tau$ we have

$$\frac{\partial \tau}{\partial \psi} = \rho(\psi) \frac{\partial \phi}{\partial \psi}.$$
where \( \| \rho(\psi) \|_{C^0} \leq C \varepsilon^2 (1 + \| \mu \|_{C^0}) \). So

\[
|\tau(T_0) - \tau(0)| = | \int_0^{T_0} \rho(\psi) \frac{\partial \phi}{\partial \psi} d\psi |
\leq CT_0 \varepsilon^2 + \varepsilon \| \xi \|_{C^0} + \varepsilon^3 \| \mu \|_{C^0} + \varepsilon^3 |\omega|
\leq C(\varepsilon + \| \xi \|_{C^0} + \varepsilon^2 \| \mu \|_{C^0} + \varepsilon^2 |\omega|)
\leq C(\varepsilon + (C_1 + C_2 + C_3) \varepsilon^2)
\] (32)

So if we choose \( \varepsilon \) sufficient small, there is no chance that \( \tau_\rho(T_0) = \tau_0 \pm \delta_1 \), which is a contradiction. So we proved the lemma.

Now we got a solution for ODE system but the solution may not satisfy the boundary conditions \( \text{[27][28]} \). We denote this solution as \( \phi_{\xi,\mu,\omega,\tau(0)}(\psi) \).

**Step 3. Estimates of the solution** \( \phi_{\xi,\mu,\omega,\tau(0)}(\psi) \) From the last step we know, when \( \varepsilon \) is small, if we choose \( \tau(0) \) as \( |\tau(0) - \tau_0| \leq C_4 \varepsilon^2 \), the orbit of the ODE will keep in a very small neighborhood (so far we know it is \( \varepsilon \) neighborhood) of the circle \( \tau \equiv \tau_0 \). Actually we can improve this estimate to \( \varepsilon^2 \) neighborhood. This result comes from a simple observation. Note that

\[
|\tau(T_0) - \tau_0| \\
\leq |\tau(T_0) - \tau(0)| + |\tau(0) - \tau_0| \\
= | \int_0^{T_0} \rho(\psi) \frac{\partial \phi}{\partial \psi} d\psi | + |\tau(0) - \tau_0| \\
= | \int_0^{T_0} \frac{\partial \phi}{\partial \psi} (-\varepsilon \Pi_0(\frac{1}{3} \varepsilon \dot{\psi} R(Y, Y_\theta, Y, Y_\theta)) \\
+ \varepsilon \phi^{-2} (\phi \ddot{\psi} + 2 \frac{\dot{\phi}}{\phi} \dot{\psi} + \psi^3 - (\phi^2 - \tau_0) \psi^2 - \phi^2 \phi^{-2}) R(Y, X_\theta, Y, X_\theta)) \\
+ \varepsilon \phi^{-2} \frac{2 \dot{\phi} \dot{\psi}}{3 \phi} + \frac{2 \dot{\phi}^2}{3} - \frac{2}{3} \dot{\psi} \dot{\psi} - \frac{4}{3} (\phi^2 - \tau_0) \frac{\phi \dot{\psi}}{\dot{\phi}} + \frac{4}{3} \phi \dot{\phi} \psi) \xi \\
+ \varepsilon \phi^{-2} + \varepsilon^2 \mu + \varepsilon^3 \omega \phi \psi \phi \dot{\psi} \phi \dot{\psi}) d\psi | + |\tau(0) - \tau_0|.
\]

On the right hand side \( \varepsilon \xi = O(\varepsilon^3) \), \( \varepsilon^2 \mu = O(\varepsilon^3) \), \( \varepsilon^3 \omega \phi \psi = O(\varepsilon^3) \). The integral of these three terms is \( O(\varepsilon^2) \) because \( T_0 \) can be as large as \( \frac{L_\rho}{\varepsilon} \) and \( |\tau(0) - \tau_0| \leq C_4 \varepsilon^2 \). The other two integrals are of order \( O(\varepsilon) \). However there is a cancellation property of these two terms. For example look at

\[
\varepsilon^2 \int_0^{T_0} \phi \frac{\partial \phi}{\partial \psi} \dot{\psi} R(Y, Y_\theta, Y, Y_\theta).
\]

In one period of \( \phi \), the function \( \phi \frac{\partial \phi}{\partial \psi} \dot{\psi} \) is nearly average 0 (up to order \( \varepsilon \)) from estimates \( \text{[34]} \) and \( \nabla_\phi R(Y, Y_\theta, Y, Y_\theta) = O(\varepsilon) \). So in one period the integral is only of size \( \varepsilon^2 \). When \( T_0 \) is as large as \( \frac{L_\rho}{\varepsilon} \), the integral is of size \( \varepsilon^2 \). The second
term on the right hand side also has the same property. This is the key to the
new estimate for \( \tau(\psi) \). However to prove this in detail is a little long.

To be precise, we have

**Lemma 3.8.** There exists \( C \) which doesn't depend on \( \varepsilon \) such that for \( \psi \in [0, \frac{L}{2\pi}] \),
\[(\phi(\psi), \zeta(\psi)) \in \text{Dom}(\tau_0, C\varepsilon^2).\]

**Proof.** We know that for each \( \psi \in [0, \frac{L}{2\pi}] \), \(|\tau(\phi(\psi), \zeta(\psi)) - \tau_0| \leq C(\varepsilon + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|) + \|\tau(0) - \tau_0\| \) from (32). Suppose \( \phi_0(\psi) \) defines the standard Delaunay surface. We assume the arc length of the curve \((\phi_0(\psi), \zeta_0(\psi))\) in the orbit space is \( s_0 \) (\( s_0 \) is a multi-valued function on the orbit \((\phi_0, \zeta_0))\) i.e.

\[ ds_0 = \sqrt{d\phi_0^2 + d\zeta_0^2}. \]

We can extend \( s_0 \) to some small neighborhood of \((\phi_0(\psi), \zeta_0(\psi))\) such that \( \frac{\partial}{\partial \tau} \perp \frac{\partial}{\partial \phi} \)
everywhere in this neighborhood. Note that \( ds_0 \) is not the arc length on the orbits nearby. We know that

\[ d\phi^2 + d\zeta^2 = <\partial_{s_0}, \partial_{s_0}> ds_0^2 + <\partial_{\tau}, \partial_{\tau}> d\tau^2 \]

where \(<,>\) denotes the inner product of the metric \( d\phi^2 + d\zeta^2 \). Also from the definition of \( s_0 \) we know that \(< \partial_{s_0}, \partial_{s_0}> = 1 \) on \((\phi_0(\psi), \zeta_0(\psi))\).

Regard \((s_0, \tau)\) as local coordinates of \(\text{Dom}(\tau_0, C\varepsilon)\). For \( \psi \in [0, \frac{L}{2\pi}] \), the new coordinates of \((\phi, \zeta)\) is \((s_0(\phi, \zeta), \tau(\phi, \zeta))\). We define a function \( \tilde{\Phi}_{\xi, \mu, \omega, \tau(0)} : [0, \frac{L}{2\pi}] \to \mathbb{R} \) such that

\[ s_0(\phi, \zeta) = s_0(\phi_0(\tilde{\Phi}_{\xi, \mu, \omega, \tau(0)}(\psi)), \zeta_0(\tilde{\Phi}_{\xi, \mu, \omega, \tau(0)}(\psi))). \]

So we have

\[
\begin{aligned}
&\left| \phi(\psi) - \phi_0(\tilde{\Phi}_{\xi, \mu, \omega, \tau(0)}(\psi)) \right| \leq C(\varepsilon + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|) + |\tau(0) - \tau_0|, \\
&\left| \zeta(\psi) - \zeta_0(\tilde{\Phi}_{\xi, \mu, \omega, \tau(0)}(\psi)) \right| \leq C(\varepsilon + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|) + |\tau(0) - \tau_0|.
\end{aligned}
\]

(34)

For \( \tilde{\Phi}_{\xi, \mu, \omega, \tau(0)} \) we have estimates

\[
\begin{aligned}
&\tilde{\Phi}_{\xi, \mu, \omega, \tau(0)}(0) = 0, \\
&\left| \tilde{\Phi}_{\xi, \mu, \omega, \tau(0)}(\psi) - \psi \right| \leq C(\varepsilon + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|) + |\tau(0) - \tau_0|, \\
&\left| \tilde{\Phi}_{\xi, \mu, \omega, \tau(0)}'(\psi) - 1 \right| \leq C(\varepsilon + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|) + |\tau(0) - \tau_0|.
\end{aligned}
\]

(35)

Now we prove the estimates (35). First we prove

\[ \left| \tilde{\Phi}_{\xi, \mu, \omega, \tau(0)}'(\psi) - 1 \right| \leq C(\varepsilon + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|) + |\tau(0) - \tau_0| \]

Note that

\[
\tilde{\Phi}_{\xi, \mu, \omega, \tau(0)}'(\psi) = \frac{d\tilde{\Phi}_{\xi, \mu, \omega, \tau(0)}(\psi)}{d\psi} = \frac{d\tilde{\Phi}_{\xi, \mu, \omega, \tau(0)}(\psi)}{ds_0} \frac{ds_0}{d\psi} + \frac{dl}{d\psi}.
\]
where
\[
 dl = \sqrt{<\partial_{s_0}, \partial_{s_0}> ds_0^2 + <\partial_{\tau}, \partial_{\tau}> d\tau^2}
\]
is the arc length of \((\phi(\psi), \zeta(\psi))\). We have
\[
1 = \sqrt{<\partial_{s_0}, \partial_{s_0}> (\frac{ds_0}{dl})^2 + <\partial_{\tau}, \partial_{\tau}> (\frac{d\tau}{dl})^2}.
\]
In \(C(\varepsilon + ||\xi||_{C^0} + \varepsilon^2 ||\mu||_{C^0} + \varepsilon^2 |\omega| + |\tau(0) - \tau_0|)\) neighborhood of \((\phi_0(\psi), \zeta_0(\psi))\),
\[
|<\partial_{s_0}, \partial_{s_0}> - 1| \leq C(\varepsilon + ||\xi||_{C^0} + \varepsilon^2 ||\mu||_{C^0} + \varepsilon^2 |\omega| + |\tau(0) - \tau_0|).
\]
And
\[
|\frac{d\tau}{d\psi}| \leq C\varepsilon(\varepsilon + ||\xi||_{C^0} + \varepsilon^2 ||\mu||_{C^0} + \varepsilon^2 |\omega|).
\]
So we know
\[
|\frac{ds_0}{dl} - 1| \leq C(\varepsilon + ||\xi||_{C^0} + \varepsilon^2 ||\mu||_{C^0} + \varepsilon^2 |\omega|).
\]
Note that
\[
\frac{dl}{d\psi} = \sqrt{(\frac{d\phi_0}{d\psi})^2 + (\frac{d\zeta_0}{d\psi})^2} = \sqrt{\zeta_0^2 + (\phi_0^{-1}(1 + \phi_0^2) - (2 + \rho)(1 + \zeta_0^2)\frac{2}{3})^2}
\]
\[
\frac{ds_0}{d\Phi(\psi)} = \sqrt{(\frac{d\phi_0}{d\Phi(\psi)})^2 + (\frac{d\zeta_0}{d\Phi(\psi)})^2} = \sqrt{\zeta_0^2 + (\phi_0^{-1}(1 + \phi_0^2) - 2(1 + \zeta_0^2)\frac{2}{3})^2} |\Phi(\psi)|
\]
where \(\Phi(\psi) = \Phi_{\xi, \mu, \omega, \tau(0)}(\psi)\). So
\[
|\Phi'(\psi) - 1| = \frac{|\sqrt{\zeta_0^2 + (\phi_0^{-1}(1 + \phi_0^2) - (2 + \rho)(1 + \zeta_0^2)\frac{2}{3})^2} |\Phi(\psi)| ds_0}{\sqrt{\zeta_0^2 + (\phi_0^{-1}(1 + \phi_0^2) - 2(1 + \zeta_0^2)\frac{2}{3})^2} |\Phi(\psi)| ds_0 - 1}
\]
\[
|\Phi(\psi) - \psi| \leq C(\varepsilon + ||\xi||_{C^0} + \varepsilon^2 ||\mu||_{C^0} + \varepsilon^2 |\omega| + |\tau(0) - \tau_0|).
\]
We know the range of \(\psi\) is \([0, \frac{L}{\varepsilon}]\), so by integration we know that
\[
|\Phi(\psi) - \psi| \leq \frac{C}{\varepsilon}(\varepsilon + ||\xi||_{C^0} + \varepsilon^2 ||\mu||_{C^0} + \varepsilon^2 |\omega| + |\tau(0) - \tau_0|).
\]
There is a special property that they are nearly average locally (upto order \( \varepsilon \)). And for the curvature we have

\[
|\tau(T_0) - \tau_0| 
\leq |\int_0^{T_0} \rho(\psi) \frac{\partial \phi}{\partial \psi} d\psi| + |\tau_0 - \tau(0)|
\]

\[
= |\int_0^{T_0} \phi \frac{\partial \phi}{\partial \psi} \left( -\varepsilon \Pi_0 \left( \frac{1}{3} \right) \partial \psi \right) R(\Upsilon, \Upsilon_0, \Upsilon, \Upsilon_0) + \varepsilon \phi^{-2} \left( \phi \dot{\psi} + 2 \ddot{\phi} \dot{\psi} + 3 \phi^3 - (\phi^2 - \tau_0) \dot{\psi}^2 - \phi^2 \ddot{\phi} \right) R(\Upsilon, X_0, \Upsilon, X_0)) + \varepsilon \phi^{-2} \left( 2 \frac{\dot{\psi}}{\phi} + 2 \frac{\dot{\phi}}{\phi} \right) \left( 2 \frac{\dot{\phi}}{\phi} - 2 \frac{\phi^3}{\phi} \right) \left( \phi \dot{\psi} - 4 \left( \phi^2 - \tau_0 \right) \right) \frac{\partial \phi}{\phi} + 4 \frac{\phi \dot{\phi} \dot{\psi}}{\phi} \xi + \varepsilon^3 \mu + \varepsilon^3 \omega \phi \dot{\phi}) d\psi| + |\tau_0 - \tau(0)|.
\]

Consider the coefficients \( \phi \frac{\partial \phi}{\partial \psi} \) and \( \phi \frac{\partial \phi}{\partial \psi} \phi^{-2} \left( \phi \dot{\psi} + 2 \ddot{\phi} \dot{\psi} + 3 \phi^3 - (\phi^2 - \tau_0) \dot{\psi}^2 - \phi^2 \ddot{\phi} \right) \phi \dot{\phi} \). There is a special property that they are nearly average \( 0 \) locally (upto order \( \varepsilon \)). And for the curvature we have

\[
|\frac{\partial}{\partial \psi} R(\Upsilon, \Upsilon_0, \Upsilon, \Upsilon_0)| + |\frac{\partial}{\partial \psi} R(\Upsilon, X_0, \Upsilon, X_0)| \leq C \varepsilon.
\]

Suppose \( \psi^1 > 0 \) is the first \( 0 \) point of \( \frac{\partial \phi}{\partial \psi} \) on the geodesic. We have

\[
\left| \int_0^{\psi^1} \phi \frac{\partial \phi}{\partial \psi} \Pi_0(R(\Upsilon, \Upsilon_0, \Upsilon, \Upsilon_0)) d\psi \right|
\leq C(\varepsilon + ||\xi||_{C^0} + \varepsilon^2 ||\mu||_{C^0} + \varepsilon^2 ||\omega|| + |\tau(0) - \tau_0|)
\]

and

\[
\int_0^{\psi^1} \phi \frac{\partial \phi}{\partial \psi} \Pi_0(R(\Upsilon, \Upsilon_0, \Upsilon, \Upsilon_0)) \frac{\partial \Pi_0}{\partial \Phi} \left( \Phi(\psi) \right) \left( \phi \frac{\partial \phi}{\partial \psi} \right) d\psi
\leq C(\varepsilon + ||\xi||_{C^0} + \varepsilon^2 ||\mu||_{C^0} + \varepsilon^2 ||\omega|| + |\tau(0) - \tau_0|)
\]

where \( \psi^1 > 0 \) is the first \( 0 \) point of \( \phi^1(\psi_0) \). So we have

\[
\int_0^{\psi^1} \phi \frac{\partial \phi}{\partial \psi} \Pi_0(R(\Upsilon, \Upsilon_0, \Upsilon, \Upsilon_0)) d\psi \leq C(\varepsilon + ||\xi||_{C^0} + \varepsilon^2 ||\mu||_{C^0} + \varepsilon^2 ||\omega|| + |\tau(0) - \tau_0|).
\]

\( T_0 \) is of order \( \frac{1}{\varepsilon} \). So

\[
\varepsilon^2 \int_0^{T_0} \phi \frac{\partial \phi}{\partial \psi} \Pi_0(R(\Upsilon, \Upsilon_0, \Upsilon, \Upsilon_0)) d\psi \leq C(\varepsilon + ||\xi||_{C^0} + \varepsilon^2 ||\mu||_{C^0} + \varepsilon^2 ||\omega|| + |\tau(0) - \tau_0|).
\]
The term \( \phi \frac{\partial \zeta}{\partial \xi} \phi^{-2}(\phi \dot{\psi} + 2\dot{\phi}^2 \dot{\psi} + \psi^3 - (\phi^2 - \tau) \dot{\psi}^2 - \phi^2 \dot{\phi}^2) \) has the same cancellation property as \( \phi \frac{\partial \zeta}{\partial \xi} \).

So we can get better estimate for \( \tau \),

\[
|\tau(T_0) - \tau_0| \leq C(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega| + |\tau(0) - \tau_0|)
\]

for \( T_0 \in [0, \frac{1}{\varepsilon} \mathcal{L}_1] \).

Now we can get better estimates for \( \dot{\Phi}(\xi, \mu, \omega, \tau, 0) \) and \( (\phi_{\xi, \mu, \omega, \tau}(0), \zeta_{\xi, \mu, \omega, \tau}(0)) \), i.e.

\[
\begin{cases}
|\phi(\psi) - \phi_0(\dot{\Phi}(\psi))| \leq C(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega| + |\tau(0) - \tau_0|), \\
|\zeta(\psi) - \zeta_0(\dot{\Phi}(\psi))| \leq C(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega| + |\tau(0) - \tau_0|).
\end{cases}
\]

and

\[
\begin{cases}
\dot{\Phi}(0) = 0, \\
|\dot{\Phi}(\psi) - \dot{\psi}| \leq C_1^1(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega| + |\tau(0) - \tau_0|), \\
|\dot{\Phi}'(\psi) - 1| \leq C(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega| + |\tau(0) - \tau_0|).
\end{cases}
\]

So we have proved that the solution can be extended to the whole geodesic.

**Corollary 3.9.**

\[
|\phi(\psi) - \phi_0(\psi)| + |\zeta(\psi) - \zeta_0(\psi)| + \left| \frac{\partial^2 \phi}{\partial \psi^2}(\psi) - \frac{\partial^2 \phi_0}{\partial \psi^2}(\psi) \right| 
\leq C \frac{1}{\varepsilon}(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega| + |\tau(0) - \tau_0|)
\]

**Proof.** From (36) and (37) and \( \zeta_0 \) and \( \frac{\partial^2 \phi_0}{\partial \psi^2} \) is uniformly bounded, we have

\[
|\phi(\psi) - \phi_0(\psi)| \leq |\phi(\psi) - \phi_0(\dot{\Phi}(\psi))| + |\phi_0(\dot{\Phi}(\psi)) - \phi_0(\psi)| 
\leq C(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega| + |\tau(0) - \tau_0|) 
+ \sup \{\zeta_0 : |\dot{\Phi}(\psi) - \dot{\psi}| 
\leq C \frac{1}{\varepsilon}(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega| + |\tau(0) - \tau_0|)
\]

and

\[
|\zeta(\psi) - \zeta_0(\psi)| \leq C \frac{1}{\varepsilon}(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega| + |\tau(0) - \tau_0|).
\]

Finally from the ODE satisfied by \( \phi \) and \( \phi_0 \) and the two estimates above we have

\[
|\frac{\partial^2 \phi}{\partial \psi^2}(\psi) - \frac{\partial^2 \phi_0}{\partial \psi^2}(\psi)| \leq C \frac{1}{\varepsilon}(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega| + |\tau(0) - \tau_0|).
\]

\[\square\]
Step 4. The linearized equation at \( \phi_{\xi,\mu,\omega,\tau(0)}(\psi) \) Now we want to analyze the linearized equation of

\[
\begin{aligned}
\frac{\partial^2 \phi}{\partial \psi^2} - \phi^{-1}(1 + (\frac{\partial \phi}{\partial \psi})^2) + (2 + \rho)(1 + (\frac{\partial \phi}{\partial \psi})^2)^{\frac{3}{2}} &= 0 \\
\phi(0) &= \frac{1 - \sqrt{1 - 4\tau(0)}}{2} \\
\phi'(0) &= 0
\end{aligned}
\]

where

\[
\rho = -\varepsilon^2 F_1(\phi, \phi_\psi) \ast \Pi_0(R_1) + \varepsilon F_4(\phi, \phi_\psi) \xi(x_0) \\
+ \varepsilon^3 \mu(\psi) + \varepsilon^3 \omega \phi_\psi
\]

where

\[
\|\xi\|_{C_1} \leq C_1\varepsilon^2, \|\mu\|_{C_2} \leq C_2, |\omega| \leq C_3, |\phi(0) - \frac{1 - \sqrt{1 - 4\tau(0)}}{2}| \leq C(\tau_0)\varepsilon
\]

where \( C(\tau_0) \) is a small constant which depends only on \( \tau_0 \). First we can also get global solution for such parameters from Step 1 and Step 2. We can linearize the equation in four methods, i.e. to perturb \( \xi, \mu, \omega \) and \( \phi(0) \) (or \( \tau(0) \)). Suppose \( \mathcal{L}_{\xi,\mu,\omega,\tau(0)} \) is the linearized operator, i.e.

\[
\mathcal{L}_{\xi,\mu,\omega,\tau(0)} \beta(\psi) = \frac{\partial^2 \beta}{\partial \psi^2} + (6(1 + (\frac{\partial \phi_{\tau(0)}}{\partial \psi})^2)^{\frac{3}{2}} \phi_{\tau(0)} - 2\phi_{\tau(0)}^{-1} \phi_\psi + F_1 \frac{\partial \beta}{\partial \psi} + (\phi_{\tau(0)}^{-2}(1 + \phi_{\tau(0)}^2) + F_2)\beta
\]

where \( \|F_1\|_{C_2} + \|F_2\|_{C_2} \leq \varepsilon^2(C + C_1 + C_2 + C_3) \). First we analyze the fundamental solution of this equation. Suppose \( \beta_1(\psi), \beta_2(\psi) \) satisfy

\[
\mathcal{L}_{\xi,\mu,\omega,\tau(0)} \beta_i(\psi) = 0, \psi \in [0, \frac{L_\Gamma}{\varepsilon}]
\]

and

\[
\begin{pmatrix}
\frac{\beta_1(0)}{\phi_{\tau(0)}(0)} & \frac{\beta_2(0)}{\phi_{\tau(0)}(0)}
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

Consider \( \phi_{\tau(0)}(\psi_0) \) as the solution to

\[
\begin{aligned}
\frac{\partial^2 \phi}{\partial \psi^2} - \phi^{-1}(1 + (\frac{\partial \phi}{\partial \psi})^2) + 2(1 + (\frac{\partial \phi}{\partial \psi})^2)^{\frac{3}{2}} &= 0 \\
\phi(0) &= \frac{1 - \sqrt{1 - 4\tau(0)}}{2} \\
\phi'(0) &= 0
\end{aligned}
\]

i.e. \( \phi_{\tau(0)}(\psi_0) \) defines the standard Delaunay surface of initial parameter \( \tau(0) \). Compare \( \mathcal{L}_{\xi,\mu,\omega,\tau(0)} \) with the linearized operator at \( \phi_{\tau(0)}(0) \),

\[
\mathcal{L}_{\tau(0)} \beta(\psi_0) = \frac{\partial^2 \beta}{\partial \psi^2_0} + (6(1 + (\frac{\partial \phi_{\tau(0)}}{\partial \psi_0})^2)^{\frac{3}{2}} \phi_{\tau(0)} - 2\phi_{\tau(0)}^{-1} \phi_{\tau(0)} \frac{\partial \phi_{\tau(0)}}{\partial \psi_0} + (\phi_{\tau(0)}^{-2}(1 + \phi_{\tau(0)}^2) + F_2)\beta
\]

+ \phi_{\tau(0)}^{-2}(1 + (\frac{\partial \phi_{\tau(0)}}{\partial \psi_0})^2)\beta.
Remember in the last Step we got \([36]\) and \([37]\). We use the method of Step 3 and get a map \(\tilde{\Phi}_{r(0)} : \Gamma \to \mathbb{R}\) such that

\[
\begin{align*}
|\phi(\psi) - \phi_{r(0)}(\tilde{\Phi}_{r(0)}(\psi))| & \leq C(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^{0}} + \varepsilon^2|\omega|), \\
|\zeta(\psi) - \zeta_{r(0)}(\tilde{\Phi}_{r(0)}(\psi))| & \leq C(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^{0}} + \varepsilon^2|\omega|).
\end{align*}
\]

(38)

and

\[
\begin{align*}
|\tilde{\Phi}_{r(0)}(0)| = 0, \\
|\tilde{\Phi}_{r(0)}(\psi) - \psi| & \leq C(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^{0}} + \varepsilon^2|\omega|), \\
|\tilde{\Phi}'_{r(0)}(\psi) - 1| & \leq C(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^{0}} + \varepsilon^2|\omega|).
\end{align*}
\]

(39)

We denote \(\psi_i, i = 0, 1, 2, \ldots\) as the zero points of \(\phi_0\) in \([0, L]\) and \(\psi^0 = 0\) and \(|\psi^1 - \tilde{\Phi}_{r(0)}(\psi^1)| \leq C(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^{0}} + \varepsilon^2|\omega|)\). From the definition of \(\tilde{\Phi}_{r(0)}\) we have \(\frac{\partial \phi_{r(0)}}{\partial \psi_0}(\tilde{\Phi}_{r(0)}(\psi^1)) = 0\). When \(\psi \in [0, \psi^1]\)

\[
|\phi(\psi) - \phi_{r(0)}(\tilde{\Phi}_{r(0)}(\psi^1))| \leq |\phi(\psi) - \phi_{r(0)}(\tilde{\Phi}_{r(0)}(\psi))| + |\phi_{r(0)}(\tilde{\Phi}_{r(0)}(\psi)) - \phi_{r(0)}(\frac{\tilde{\Phi}_{r(0)}(0)}{\psi^1} \psi)| \\
\leq C(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^{0}} + \varepsilon^2|\omega|), \\
|\zeta(\psi) - \zeta_{r(0)}(\tilde{\Phi}_{r(0)}(\psi^1))| \leq C(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^{0}} + \varepsilon^2|\omega|).
\]

(40)

Note that the two fundamental solutions of \(L_{r(0)}\) is

\[
W_1(\psi_0) = h(\tau(0))(\psi_0 - \phi_{r(0)}(0) + v_{r(0)}(\psi_0)) \\
W_2(\psi_0) = h(\tau(0))\phi_{r(0)}(0)
\]

where \(h(\tau(0))\) is a normalized constant with \(h(\tau(0))\frac{\partial^2 \phi_{r(0)}}{\partial \psi^2}(0) = 1\) and \(v_{r(0)}(\psi_0)\) is a periodic function and

\[
\begin{pmatrix}
W_1(\psi_0) & W_2(\psi_0) \\
\frac{\partial W_1}{\partial \psi^0}(0) & \frac{\partial W_2}{\partial \psi^0}(0)
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

So we have

\[
\begin{pmatrix}
W_1(\tilde{\Phi}_{r(0)}(\psi^1)) & W_2(\tilde{\Phi}_{r(0)}(\psi^1)) \\
\frac{\partial W_1}{\partial \psi^0}(\tilde{\Phi}_{r(0)}(\psi^1)) & \frac{\partial W_2}{\partial \psi^0}(\tilde{\Phi}_{r(0)}(\psi^1))
\end{pmatrix} = \begin{pmatrix} \tilde{\Phi}_{r(0)}(\psi^1) & 0 \\ 0 & 1 \end{pmatrix}.
\]

Comparing the operator \(L_{\xi,\mu,\omega,\tau(0)}\) with \(L_{r(0)}\) we can get

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Lemma 3.10.

\[ ||\beta_1(\psi) - h(\tau(0))(\psi)\left(\frac{\partial\phi}{\partial\psi} + v_1(\psi)\right)||_{C^1([0,\psi])} \leq C(\varepsilon^2 + ||\xi||_{C^0} + \varepsilon^2||\mu||_{C^0} + \varepsilon^2|\omega|) \]

\[ ||\beta_2(\psi) - h(\tau(0))(\psi)\left(\frac{\partial\phi}{\partial\psi}\right)||_{C^1([0,\psi])} \leq C(\varepsilon^2 + ||\xi||_{C^0} + \varepsilon^2||\mu||_{C^0} + \varepsilon^2|\omega|) \]

where \( v_1(\psi) = v_{\tau(0)}(\frac{\phi(\psi)}{\psi}) \).

Proof. Note that from (40), when \( \psi \in [0, \psi_1] \), the coefficients of \( L_{\xi,\mu,\omega,\tau(0)} \) at \( \psi \) and the coefficients of \( L_{\tau(0)} \) at \( \frac{\phi(\psi)}{\psi} \) have a difference of order

\[ C(\varepsilon^2 + ||\xi||_{C^0} + \varepsilon^2||\mu||_{C^0} + \varepsilon^2|\omega|). \]

Also we know

\[ |\right(\frac{\phi(\psi)}{\psi}\right) - \psi| \leq C(\varepsilon^2 + ||\xi||_{C^0} + \varepsilon^2||\mu||_{C^0} + \varepsilon^2|\omega|). \]

So from basic ODE theory we know

\[ ||\beta_1(\psi) - W_1(\right(\frac{\phi(\psi)}{\psi}\right)||_{C^1([0,\psi])} \leq C(\varepsilon^2 + ||\xi||_{C^0} + \varepsilon^2||\mu||_{C^0} + \varepsilon^2|\omega|) \]

\[ ||\beta_2(\psi) - W_2(\right(\frac{\phi(\psi)}{\psi}\right)||_{C^1([0,\psi])} \leq C(\varepsilon^2 + ||\xi||_{C^0} + \varepsilon^2||\mu||_{C^0} + \varepsilon^2|\omega|) \]

So the lemma follows easily.

As a consequence we have

\[ \left( \begin{array}{c} \beta_1(\psi) \\ \beta_2(\psi) \\ \end{array} \right) = \left( \begin{array}{cc} 1 + e_{11} & e_{12} \\ e_{11} & 1 + e_{12} \\ \end{array} \right) \left( \begin{array}{cc} \psi \\ e_{21} \\ \end{array} \right) \]

where \( |e_{ij}| \leq C(\varepsilon^2 + ||\xi||_{C^0} + \varepsilon^2||\mu||_{C^0} + \varepsilon^2|\omega|). \)

Suppose \( \psi^i \) is the \( i \)th 0 point of \( \phi_\psi \) on the geodesic. From the theory of linear ODE, we know

\[ \left( \begin{array}{c} \beta_1(\psi^i) \\ \beta_2(\psi^i) \\ \end{array} \right) = \left( \begin{array}{cc} 1 + e_{11} & e_{12} \\ e_{11} & 1 + e_{12} \\ \end{array} \right) \cdots \left( \begin{array}{cc} 1 + e_{11} & e_{12} \\ e_{11} & 1 + e_{12} \\ \end{array} \right) \left( \begin{array}{c} \psi \\ e_{21} \\ \end{array} \right) \]

By analysis of the matrix, we can prove

**Lemma 3.11.**

\[ \exp(- (C + C_1 + C_2 + C_3)) \leq A_{11} \leq \exp(C + C_1 + C_2 + C_3), \]

\[ \exp(- (C + C_1 + C_2 + C_3)) \leq A_{22} \leq \exp(C + C_1 + C_2 + C_3), \]

\[ \exp(- (C + C_1 + C_2 + C_3)) \varepsilon^{-1} \leq A_{21} \leq \exp(C + C_1 + C_2 + C_3) \varepsilon^{-1}, \]

\[ |A_{12}| \leq C(C_1, C_2, C_3) \varepsilon \exp(C + C_1 + C_2 + C_3). \]

(41)
We will prove this result in Appendix C. So we have

**Lemma 3.12.** Suppose $|\tau(0) - \tau_0| < C(\tau_0)\varepsilon$, there is $\delta > 0$ such that when $0 < \varepsilon < \delta$, in the interval $[\psi^i, \psi^{i+1}]$, we have

$$
\|\beta_1(\psi) - (A_{11}^i h(\tau(0))((\psi - \psi^i) \frac{\partial \phi}{\partial \psi} + v_i(\psi)) + A_{21}^i h(\tau(0)) \frac{\partial \phi}{\partial \psi})\|_{C^1} \\
\leq \frac{C}{\varepsilon}(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|)
$$

$$
\|\beta_2(\psi) - (A_{12}^i h(\tau(0))((\psi - \psi^i) \frac{\partial \phi}{\partial \psi} + v_i(\psi)) + A_{22}^i h(\tau(0)) \frac{\partial \phi}{\partial \psi})\|_{C^1} \\
\leq C(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|)
$$

where $C$ is a uniform constant which does not depend on $\varepsilon$ and $v_i((1 - \lambda)\psi_i + \lambda\psi_{i+1}) = v(\tau(0)((1 - \lambda)\Phi(\psi_i) + \lambda\Phi(\psi_{i+1}))$ for $\lambda \in [0, 1]$ and $A_{kl}^i$ satisfy (41).

In particular, $\beta_1$ has linear growth and $\beta_2$ is bounded.

We perturb $\mu$ to $\mu + t\Delta \mu$, if we denote

$$
\frac{d}{dt}\phi_{\xi, \mu + t\Delta \mu, \tau(0)}(\psi) = \beta_\mu(\psi)
$$

then

$$
\begin{cases}
\mathcal{L}_{\xi, \mu, \omega, \tau(0)}\beta_\mu(\psi) = -\varepsilon^3(1 + \phi_\psi^3)\frac{d}{dt}\Delta \mu, \\
\beta_\mu(0) = 0, \\
\beta_\mu'(0) = 0
\end{cases}
$$

(42)

$$
|\beta_\mu(\psi)| = \int_0^\psi (\beta_2(\psi)\beta_1(t) - \beta_1(\psi)\beta_2(t))(-\varepsilon^3(1 + \phi_\psi^3(t))^\frac{3}{2}\Delta \mu(dt)| \\
\leq C(C_1, C_2, C_3)\varepsilon\|\Delta \mu\|_{C^0}
$$

(43)

By differentiation and (42), we have

$$
\|\beta_\mu(\psi)\|_{C^2} \leq C(C_1, C_2, C_3)\varepsilon\|\Delta \mu\|_{C^0}
$$

We perturb $\xi$ to $\xi + t\Delta \xi$, if we denote

$$
\frac{d}{dt}\phi_{\xi + t\Delta \xi, \mu, \tau(0)}(\psi) = \beta_\xi(\psi)
$$

$$
\begin{cases}
\mathcal{L}_{\xi, \mu, \omega, \tau(0)}\beta_\xi(\psi) = -\varepsilon F_4(\phi, \phi_\psi)(1 + \phi_\psi^3)^\frac{3}{2}\Delta \xi, \\
\beta_\xi(0) = 0, \\
\beta_\xi'(0) = 0
\end{cases}
$$

(44)

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at first we can get

\[ \| \beta_\xi(\psi) \|_{C^2} \leq \frac{C(C_1, C_2, C_3)}{\varepsilon} \| \Delta \xi \|_{C^0}. \]

We are going to prove

\[ \| \beta_\xi(\psi) \|_{C^2} \leq C(C_1, C_2, C_3) \| \Delta \xi \|_{C^1}. \quad (45) \]

This comes from Lemma 3.4, the “average 0” lemma.

Consider

\[ \| (\beta_\xi, \beta'_\xi) \cdot (\tau_\phi, \tau_\phi) \|_{\bar{\psi}} \]

\[ = \left| \frac{d}{dt} \tau_\phi \right|_{\bar{\psi}} \]

\[ = \left| \frac{d}{dt} \int_0^\psi \phi \phi_\psi \rho d\psi \right| \]

\[ = \left| \int_0^\psi \varepsilon^2 \left( \frac{\partial \hat{F}_1}{\partial \phi} \beta_\xi + \frac{\partial \hat{F}_1}{\partial \phi_\psi} \beta'_\xi \right) + \varepsilon^3 (\beta_\xi \phi_\psi + \phi \beta'_\xi) \mu + \varepsilon (\frac{\partial \hat{F}_2}{\partial \phi} \beta_\xi + \frac{\partial \hat{F}_2}{\partial \phi_\psi} \beta'_\xi) + \varepsilon^2 \hat{F}_2 \Delta \xi + \varepsilon^3 \omega (\frac{\partial \hat{F}_3}{\partial \phi} \beta_\xi + \frac{\partial \hat{F}_3}{\partial \phi_\psi} \beta'_\xi) d\psi \right| \]

\[ \leq C(C_1, C_2, C_3) \| \Delta \xi \|_{C^0} \]

where

\[ \hat{F}_1 = \phi \frac{\partial \phi}{\partial \psi} F_1(\phi, \phi_\psi) \ast R_1 \]

\[ \hat{F}_2 = \phi \frac{\partial \phi}{\partial \psi} F_4(\phi, \phi_\psi) \]

\[ \hat{F}_3 = \phi \phi_\psi^2 \]

On the points \( \psi_i \) where \( \phi_\psi = 0 \), we have \( \tau_\phi = 0 \) and \( |\tau_\phi| \) has uniform positive lower bound. So

\[ |\beta_\xi(\psi_i)| \leq C(C_1, C_2, C_3) \| \Delta \xi \|_{C^0} \]

\[ |\beta'_\xi(\psi_i)| \leq \frac{C(C_1, C_2, C_3)}{\varepsilon} \| \Delta \xi \|_{C^0}. \]

So in the interval \( [\psi_i, \psi_{i+1}] \)

\[ \| \beta_\xi(\psi) - (\beta_\xi(\psi_i) h(\tau(0))((\psi - \psi_i) \frac{\partial \phi}{\partial \psi} + \nu_i(\psi)) + \beta'_\xi(\psi_i) h(\tau(0)) \frac{\partial \phi}{\partial \psi})\|_{C^1} \]

\[ \leq C(C_1, C_2, C_3) \varepsilon \| \Delta \xi \|_{C^0}. \]
So in $\beta_\xi(\psi), \beta'_\xi(\psi) h(\tau) \frac{\partial \phi}{\partial \psi}$ is the big term and

$$\|\beta_\xi(\psi)\|_{C^1} \leq \frac{C(C_1, C_2, C_3)}{\varepsilon} \|\Delta \xi\|_{C^0}.$$ 

Then we have, for $\psi \in [0, \frac{4\varepsilon}{5}]$

$$\left| \int_0^\psi \varepsilon^2 \left( \frac{\partial \hat{F}_1}{\partial \phi} \beta_\xi + \frac{\partial \hat{F}_1}{\partial \phi \psi} \beta'_\xi \right) d\psi \right|$$

$$\leq \sum_{i=0}^k \int_{\psi_i}^{\psi_{i+1}} \varepsilon^2 \left( \frac{\partial \hat{F}_1}{\partial \phi} \beta_\xi + \frac{\partial \hat{F}_1}{\partial \phi \psi} \beta'_\xi \right) d\psi$$

$$+ \int_{\psi_{k+1}}^\psi \varepsilon^2 \left( \frac{\partial \hat{F}_1}{\partial \phi} \beta_\xi + \frac{\partial \hat{F}_1}{\partial \phi \psi} \beta'_\xi \right) d\psi$$

$$\leq \frac{C(C_1, C_2, C_3)}{\varepsilon} \|\Delta \xi\|_{C^0} \left| \sum_{i=0}^k \int_{\psi_i}^{\psi_{i+1}} \varepsilon^2 \left( \frac{\partial \hat{F}_1}{\partial \phi} \frac{\partial \phi}{\partial \psi} + \frac{\partial \hat{F}_1}{\partial \phi \psi} \frac{\partial^2 \phi}{\partial \psi^2} \right) d\psi \right|$$

$$+ C(C_1, C_2, C_3) \varepsilon \|\Delta \xi\|_{C^0}$$

$$\leq C(C_1, C_2, C_3) \varepsilon \|\Delta \xi\|_{C^0} \left| \sum_{i=0}^k \left( \hat{F}_1(\psi_{i+1}) - \hat{F}_1(\psi_i) \right) \right|$$

$$+ C(C_1, C_2, C_3) \varepsilon \|\Delta \xi\|_{C^0}$$

$$\leq C(C_1, C_2, C_3) \varepsilon \|\Delta \xi\|_{C^0}.$$ 

and

$$\left| \int_0^\psi \left( \varepsilon^3 (\beta_\xi \phi + \phi \beta'_\xi) \mu + \varepsilon (\frac{\partial \hat{F}_2}{\partial \phi} \beta_\xi + \frac{\partial \hat{F}_2}{\partial \phi \psi} \beta'_\xi) \right) d\psi \right|$$

$$+ C(C_1, C_2, C_3) \varepsilon \|\Delta \xi\|_{C^0}$$

$$\leq C(C_1, C_2, C_3) \varepsilon \|\Delta \xi\|_{C^0}$$

The last term is

$$\int_0^\psi \varepsilon \hat{F}_2 \Delta \xi d\psi$$

$$= \varepsilon \int_0^\psi \phi \left( \frac{\partial \phi}{\partial \psi} \right) F_4(\phi, \phi_\psi) \Delta \xi d\psi.$$
Note that

\[ \varepsilon \int_{\psi_i}^{\psi_{i+1}} \phi \left( \frac{\partial \phi}{\partial \psi} \right) F_4(\phi, \phi) \Delta \xi d\psi \]

\[ \leq \varepsilon \int_{\Phi_{i+1}^\tau(\psi)}^{\Phi_{i+1}^\tau(\psi_i)} \phi_{\tau}(0) \left( \frac{\partial \phi_{\tau}(0)}{\partial \psi_{\tau}(0)} \right) F_4(\phi_{\tau}(0), \frac{\partial \phi_{\tau}(0)}{\partial \psi_{\tau}(0)}) \Delta \xi d\Phi(\psi) \]

\[ + C(C_1, C_2, C_3) \varepsilon^2 \| \Delta \xi \|_{C^0} \]

\[ \leq \varepsilon \int_{\Phi_{i+1}^\tau(\psi)}^{\Phi_{i+1}^\tau(\psi_i)} \Delta \xi d\chi + C(C_1, C_2, C_3) \varepsilon^2 \| \Delta \xi \|_{C^0} \]

\[ \leq \varepsilon \| \Delta \xi \|_{C^0} \left( \chi(\Phi_{i+1}^\tau(\psi)) - \chi(\Phi_{i+1}^\tau(\psi_i)) \right) \]

\[ - \int_{\Phi_{i+1}^\tau(\psi_i)}^{\Phi_{i+1}^\tau(\psi_{i+1})} \chi \left( \frac{\partial \Delta \xi}{\partial \psi} \right) d\psi \right) + C(C_1, C_2, C_3) \varepsilon^2 \| \Delta \xi \|_{C^0} \]

where \( \chi \) is the integral of \( \phi_{\tau}(0) \left( \frac{\partial \phi_{\tau}(0)}{\partial \psi_{\tau}(0)} \right) F_4(\phi_{\tau}(0), \frac{\partial \phi_{\tau}(0)}{\partial \psi_{\tau}(0)}) \).

From Lemma 3.4 we have

\[ \chi(\Phi_{i+1}^\tau(\psi)) - \chi(\Phi_{i+1}^\tau(\psi_i)) = 0 \]

and

\[ \left| \frac{\partial \Delta \xi}{\partial \psi} \right| \leq C \varepsilon \| \Delta \xi \|_{C^1} \]

So we have

\[ \varepsilon \left| \int_{\psi_i}^{\psi_{i+1}} \phi \left( \frac{\partial \phi}{\partial \psi} \right) F_4(\phi, \phi) \Delta \xi d\psi \right| \]

\[ \leq C(C_1, C_2, C_3) \varepsilon^2 \| \Delta \xi \|_{C^1} \]

So

\[ \int_0^\psi \varepsilon F_2 \Delta \xi d\psi \leq C(C_1, C_2, C_3) \varepsilon^2 \| \Delta \xi \|_{C^1} \]

So we have

\[ |\beta_2(\psi)| \leq C(C_1, C_2, C_3) \varepsilon \| \Delta \xi \|_{C^1} \]

Note that

\[ |\beta_2'(\psi_{i+1}) - (\beta_1(\psi_{i+1}) + \beta_2(\psi_{i+1}))| \]

\[ = \left| \int_{\psi_i}^{\psi_{i+1}} (\beta_2(\psi) - \beta_1(\psi)) \beta_2(\psi) d\psi \right| \]

where

\[ \beta_{1,i}(\psi) = \begin{pmatrix} \beta_1(\psi) & \beta_1'(\psi) \\ \beta_2(\psi) & \beta_2'(\psi) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \]

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We have
\[ |\beta_{1,i}^\prime(\psi) - h(\tau_0)(\phi_{\psi} + (\psi - \psi_i)\phi_{\psi\psi} + \frac{\partial v_i}{\partial \psi})| \leq C(C_1, C_2, C_3)\varepsilon^2 \]
\[ |\beta_{2,i}^\prime(\psi) - h(\tau_0)\phi_{\psi\psi}| \leq C(C_1, C_2, C_3)\varepsilon^2. \]

Then
\[ |\beta_{1,i}^\prime(\psi^{i+1}) - \beta_{1,i}^\prime(\psi^i)| \leq C(C_1, C_2, C_3)\varepsilon|\Delta \xi|_{C^0} + C(C_1, C_2, C_3)|\beta_{1,i}^\prime(\psi_i)| \]
\[ \leq C(C_1, C_2, C_3)\varepsilon|\Delta \xi|_{C^1_0}. \]

From $\beta_{1,i}^\prime(0) = 0$, by an induction argument, we get
\[ |\beta_{1,i}^\prime(\psi^i)| \leq C(C_1, C_2, C_3)|\Delta \xi|_{C^1_0}. \]

At last from the equation (44) we get
\[ \|\beta_{\xi}(\psi)\|_{C^2_0} \leq C(C_1, C_2, C_3)|\Delta \xi|_{C^1_0}. \]

If we perturb $\omega$ to $\omega + t$, then we denote
\[ \frac{d}{dt}\phi_{\xi,\mu,\omega+t,\tau(0)}(\psi) = \beta_{\omega}(\psi). \]

In the same way as before we can get
\[ \|\beta_{\omega}(\psi)\|_{C^2_0} \leq C(C_1, C_2, C_3)\varepsilon. \] (46)

However one special thing in this case is that
\[ \frac{d}{dt}\tau(\bar{\psi}) = \frac{d}{dt}\int_0^\bar{\psi} \phi_{\psi\psi} p d\psi \]
\[ = \frac{d}{dt}\int_0^\bar{\psi} \phi_{\psi\psi} p d\psi \]
\[ = \int_0^{\bar{\psi}} \varepsilon^2(\frac{\partial \hat{F}_1}{\partial \phi} \beta_{\omega} + \frac{\partial \hat{F}_1}{\partial \phi_{\psi}} \beta'_{\omega}) + \varepsilon^3(\beta_{\omega}\phi_{\psi} + \phi \beta'_{\omega})\mu \]
\[ + \varepsilon(\frac{\partial \hat{F}_2}{\partial \phi} \beta_{\omega} + \frac{\partial \hat{F}_2}{\partial \phi_{\psi}} \beta'_{\omega} \phi) R(\xi) + \varepsilon^3\omega(\frac{\partial \hat{F}_3}{\partial \phi} \beta_{\omega} + \frac{\partial \hat{F}_3}{\partial \phi_{\psi}} \beta'_{\omega}) + \varepsilon^3\phi \delta^2 d\psi. \]

In the similar way as we did for $\xi$, we can get
\[ |\int_0^\bar{\psi} \varepsilon^2(\frac{\partial \hat{F}_1}{\partial \phi} \beta_{\omega} + \frac{\partial \hat{F}_1}{\partial \phi_{\psi}} \beta'_{\omega}) + \varepsilon^3(\beta_{\omega}\phi_{\psi} + \phi \beta'_{\omega})\mu \]
\[ + \varepsilon(\frac{\partial \hat{F}_2}{\partial \phi} \beta_{\omega} + \frac{\partial \hat{F}_2}{\partial \phi_{\psi}} \beta'_{\omega} \phi) R(\xi) + \varepsilon^3\omega(\frac{\partial \hat{F}_3}{\partial \phi} \beta_{\omega} + \frac{\partial \hat{F}_3}{\partial \phi_{\psi}} \beta'_{\omega})| \]
\[ \leq C(C_1, C_2, C_3)\varepsilon^3 \]
The biggest term turns out to be

\[ \int_0^\infty \varepsilon^3 \phi \sigma_\nu^2 d\psi. \]

And \( \phi \sigma_\nu^2 > 0 \). So for \( |\phi(0) - \frac{1-\sqrt{1-4\tau_{0}}}{2}| \leq C(\tau_0)\varepsilon \) there is a uniform constant \( C_5 > 0 \), which does not depend on \( \varepsilon \) and \( C_1, C_2, C_3 \) such that

\[ \frac{d}{dt} \tau \left( \frac{L_{\Gamma}}{\varepsilon} \right) \geq C_5\varepsilon^2. \]  

(47)

At last if we perturb \( \phi(0) = \frac{1-\sqrt{1-4\tau(0)}}{2} \) to \( \frac{1-\sqrt{1-4\tau(0)}}{2} + t \). The linearized function is just \( \beta_1(\psi) \).

Now we consider

\[ \frac{\partial(\Delta \tau, \zeta(L_{\Gamma} \varepsilon))}{\partial(\omega, \phi(0))} \]

where \( \Delta \tau = \tau(L_{\Gamma} \varepsilon) - \tau(0) \) and \( \zeta = \phi_0 \). First from \( (47) \) we have

\[
\begin{cases}
\frac{\partial \Delta \tau}{\partial \omega} \geq C_5\varepsilon^2 \\
|\frac{\partial \Delta \tau}{\partial \phi(0)}| = |\int_0^{L_{\Gamma} \varepsilon} \varepsilon^2 \left( \frac{\partial F_{\epsilon}}{\partial \phi_0} \beta_1 + \frac{\partial F_{\epsilon}}{\partial \phi_0} \beta_1' \right) + \varepsilon \left( \frac{\partial F_{\epsilon}}{\partial \phi_0} \beta_1 + \frac{\partial F_{\epsilon}}{\partial \phi_0} \beta_1' \right) d\psi| \\
\leq K_1(C_1, C_2, C_3)\varepsilon \\
|\frac{\partial \beta'(L_{\Gamma} \varepsilon)}{\partial \phi(0)}| = |\beta'_2(L_{\Gamma} \varepsilon)| \leq K_2(C_1, C_2, C_3)\varepsilon \\
|\frac{\partial \beta'(L_{\Gamma} \varepsilon)}{\partial \phi(0)}| = |\beta'_1(L_{\Gamma} \varepsilon)| \geq \exp(-C_0(1 + C_1 + C_2 + C_3)) \frac{1}{\varepsilon} 
\end{cases}
\]

(48)

where \( C_0 \) is a uniform constant.

Step 5. Match the boundary value

**Lemma 3.13.** For \( C_1, C_2 > 0 \) we can choose \( C_3, C_4 \) and \( \varepsilon \) such that if \( ||\xi||C_{\phi_{0}} \leq C_1\varepsilon^2 \) and \( ||\mu||C_{\phi_{0}} \leq C_2 \) there is only one \( \omega_{\xi, \mu}, \phi(0)_{\xi, \mu} \) (or \( \tau(0)_{\xi, \mu} \)) such that

\[
\begin{align*}
\phi_{\xi, \mu, \omega_{\xi, \mu}}(0) &= \phi_{\xi, \mu, \omega_{\xi, \mu}}(0)_{\xi, \mu} \left( \frac{L_{\Gamma} \varepsilon}{\varepsilon} \right) \\
\phi'_{\xi, \mu, \omega_{\xi, \mu}}(0) &= \phi'_{\xi, \mu, \omega_{\xi, \mu}}(0)_{\xi, \mu} \left( \frac{L_{\Gamma} \varepsilon}{\varepsilon} \right)
\end{align*}
\]

with

\[
\begin{align*}
|\omega_{\xi, \mu}| &\leq C_3 \\
|\phi(0)_{\xi, \mu} - \frac{1-\sqrt{1-4\tau_{0}}}{2}| &\leq C_4\varepsilon^2.
\end{align*}
\]

**Proof.** We begin with \( \omega = 0, \tau(0) = \tau_0 \).

|\Delta \tau| = |\tau(L_{\Gamma} \varepsilon) - \tau(0)| \leq \tilde{C} \varepsilon^2 (1 + C_1 + C_2)

\[
|\zeta(L_{\Gamma} \varepsilon)| \leq \tilde{C} \varepsilon (1 + C_1 + C_2)
\]

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where $\tilde{C}$ is a uniform constant. We choose
\[
C_3 = \frac{4\tilde{C}(1 + C_1 + C_2)}{C_5}
\]
and
\[
C_4 = 4\tilde{K}_2(C_1, C_2, C_3, C_5)C_5^{-1}\tilde{C}(1 + C_1 + C_2)\exp(C_0(1 + C_1 + C_2 + C_3))
\]
where $\tilde{K}_2(C_1, C_2, C_3, C_5) = K_2(C_1, C_2, C_3) + C_5$. We choose
\[
0 < \varepsilon \leq \min\left\{ \frac{C(\tau_0)}{C_4}, \frac{C_5 \exp(-C_0(1 + C_1 + C_2 + C_3))}{2|K_1(C_1, C_2, C_3)\tilde{K}_2(C_1, C_2, C_3, C_5)| + 1}\right\}
\]
Note that first $C_4\varepsilon^2 \leq C(\tau_0)\varepsilon$, so one can work with a uniform constant $C_5$ such that
\[
\frac{\partial \Delta \tau}{\partial \omega} \geq C_5\varepsilon^2.
\]
From (48), we know
\[
\det \frac{\partial(\Delta \tau, \zeta(\frac{\varepsilon}{C}))}{\partial(\omega, \phi(0))} \geq C_5 \exp(-C_0(1 + C_1 + C_2 + C_3))\varepsilon - K_1(C_1, C_2, C_3)K_2(C_1, C_2, C_3)\varepsilon^2.
\]
When
\[
\varepsilon \leq \frac{C_5 \exp(-C_0(1 + C_1 + C_2 + C_3))}{2|K_1(C_1, C_2, C_3)\tilde{K}_2(C_1, C_2, C_3, C_5)| + 1},
\]
we have
\[
\det \frac{\partial(\Delta \tau, \zeta(\frac{\varepsilon}{C}))}{\partial(\omega, \phi(0))} \geq \frac{C_5}{2} \exp(-C_0(1 + C_1 + C_2 + C_3))\varepsilon.
\]
So if there is some $|\omega| \leq C_5, |\tau(0) - \tau_0| \leq C_4\varepsilon^2$ such that $\Delta \tau = 0$ and $\zeta(\frac{\varepsilon}{C}) = 0$, the solution has to be unique.

For the existence first we perturb $\omega$ within $\tilde{C}(1 + C_1 + C_2)\varepsilon$ such that $\Delta \tau = 0$. Then $\zeta(\frac{\varepsilon}{C})$ will change no more than $K_2(C_1, C_2, C_3)\tilde{C}(1 + C_1 + C_2)\exp(C_0(1 + C_1 + C_2 + C_3))\varepsilon^2$. Then we perturb $\phi(0)$ within
\[
\tilde{K}_2(C_1, C_2, C_3, C_5)C_5^{-1}\tilde{C}(1 + C_1 + C_2)\exp(C_0(1 + C_1 + C_2 + C_3))\varepsilon^2
\]
such that $\zeta(\frac{\varepsilon}{C}) = 0$. Then $\Delta \tau$ will change no more than
\[
K_1(C_1, C_2, C_3)\tilde{K}_2(C_1, C_2, C_3, C_5)C_5^{-1}
\]
\[
\tilde{C}(1 + C_1 + C_2)\exp(C_0(1 + C_1 + C_2 + C_3))\varepsilon^3.
\]
From
\[
\varepsilon \leq \frac{C_5 \exp(-C_0(1 + C_1 + C_2 + C_3))}{2|K_1(C_1, C_2, C_3)\tilde{K}_2(C_1, C_2, C_3, C_5)| + 1}
\]

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we know

\[ K_1(C_1, C_2, C_3)K_2(C_1, C_2, C_3, C_3)C_5^{-1} \]
\[ \hat{C}(1 + C_1 + C_2) \exp(C_0(1 + C_1 + C_2 + C_3))\varepsilon^3 \]
\[ \leq \frac{1}{2} \hat{C}(1 + C_1 + C_2)\varepsilon^2. \]

Then by an iteration argument, we have, there exists \(|\omega_{\xi,\mu}| \leq C_3\) and \(|\phi(0)\xi_{\mu} - \frac{1 - \sqrt{1 - 4\varepsilon^2}}{2}| \leq C_4\varepsilon^2\) such that \(\Delta\tau = 0\) and \(\zeta(L_{\varepsilon'})\). From \(|\phi_{\xi,\mu,\omega_{\xi,\mu},\tau(0)\xi_{\mu}}(\frac{L_{\varepsilon'}}{\varepsilon}) - \phi_{\xi,\mu,\omega_{\xi,\mu},\tau(0)\xi_{\mu},(0)}| \leq C\varepsilon\), we know \(\phi_{\xi,\mu,\omega_{\xi,\mu},\tau(0)\xi_{\mu},(0)}(\varepsilon) = \phi_{\xi,\mu,\omega_{\xi,\mu},\tau(0)\xi_{\mu},(0)}(0)\).

**Step 6. Main estimates of 0th mode** For

\[ \|\xi_1\|_{C^1_{t_0}}, \|\xi_2\|_{C^2_{t_0}} \leq C_1\varepsilon^2, \|\mu_1\|_{C^3_{t_0}}, \|\mu_2\|_{C^3_{t_0}} \leq C_2, \]

we can get unique \(\phi_{\xi_1,\mu_1,\omega_{\xi_1,\mu_1},\tau(0)\xi_1,\mu_1}(\psi)\) and \(\phi_{\xi_2,\mu_2,\omega_{\xi_2,\mu_2},\tau(0)\xi_2,\mu_2}(\psi)\) such that

\[ |\omega_{\xi,\mu_1}|, |\omega_{\xi,\mu_2}| \leq C_3 \]

and

\[ |\phi(0)\xi_2| \leq C_3 \]
and

\[ |\phi(0)\xi_2| - \frac{1 - \sqrt{1 - 4\varepsilon^2}}{2}\leq C_4\varepsilon^2 \]
and

\[ \left\{ \begin{array}{l}
\phi_{\xi_1,\mu_1,\omega_{\xi_1,\mu_1},\tau(0)\xi_1,\mu_1}(0) = \phi_{\xi_1,\mu_1,\omega_{\xi_1,\mu_1},\tau(0)\xi_1,\mu_1}(L_{\varepsilon'}) \\
\zeta_{\xi,\mu_1,\omega_{\xi_1,\mu_1},\tau(0)\xi_1,\mu_1}(0) = \zeta_{\xi,\mu_1,\omega_{\xi_1,\mu_1},\tau(0)\xi_1,\mu_1}(L_{\varepsilon'})
\end{array} \right. \]

for \(i = 1, 2\).

\[ |\phi_{\xi_1,\mu_1,\omega_{\xi_1,\mu_1},\tau(0)\xi_1,\mu_1}(\psi) - \phi_{\xi,\mu,\omega_{\xi,\mu},\tau(0)\xi,\mu}(\psi)| \leq \int_0^1 \left| \frac{d}{dt}\phi_{\xi_1,\mu_1,\omega_{\xi_1,\mu_1},\tau(0)\xi_1,\mu_1}(\psi) \right| d\psi \]
\[ + \int_0^1 \left| \frac{d}{dt}\phi_{\xi_2,\mu_2,\omega_{\xi_2,\mu_2},\tau(0)\xi_1,\mu_1}(\psi) \right| d\psi \]
\[ \leq C\varepsilon \|\mu_2 - \mu_1\|_{C^3_{t_0}} + C\|\xi_2 - \xi_1\|_{C^3_{t_0}}, \]

and

\[ \sup_{\psi \in [0, L_{\varepsilon'}]} |\zeta_{\xi_1,\mu_1,\omega_{\xi_1,\mu_1},\tau(0)\xi_1,\mu_1}(\psi) - \zeta_{\xi_2,\mu_2,\omega_{\xi_2,\mu_2},\tau(0)\xi_1,\mu_1}(\psi)| \leq C\varepsilon \|\mu_2 - \mu_1\|_{C^3_{t_0}} + C\|\xi_2 - \xi_1\|_{C^3_{t_0}}. \]
And we know
\[ |\Delta \phi_{\xi_2, \mu_2, \omega, \xi_1, \mu_1, \tau(0)\xi_1, \mu_1} \tau| = |\tau(L_{\xi}) - \tau(0)| \]
\[ \leq C \varepsilon^2 \|\mu_2 - \mu_1\| c_\gamma + C \varepsilon \|\xi_2 - \xi_1\| c_{i_0}. \]

If we denote
\[ \left( \begin{array}{cc} B_{11} & B_{12} \\ B_{21} & B_{22} \end{array} \right) = \frac{\partial(\Delta \tau, \zeta(\xi_2))}{\partial(\omega, \phi(0))} \]
then we can get
\[ |B_{11}| \leq 2 C^{-1} \varepsilon^{-2} \]
\[ |B_{22}| \leq 2 \exp(C_0(1 + C_1 + C_2 + C_3)) \varepsilon \]
\[ |B_{12}| \leq 2 K_1(C_1, C_2, C_3) C^{-1} \exp(C_0(1 + C_1 + C_2 + C_3)) \]
\[ |B_{21}| \leq 2 K_2(C_1, C_2, C_3) C^{-1} \exp(C_0(1 + C_1 + C_2 + C_3)). \]

So we know easily
\[ \left\{ \begin{array}{l} |\omega_{\xi_2, \mu_2} - \omega_{\xi_1, \mu_1}| \\ |\phi(0)_{\xi_2, \mu_2} - \phi(0)_{\xi_1, \mu_1}| \end{array} \right\} \leq C_\alpha \varepsilon \left( \varepsilon^2 \|\mu_2 - \mu_1\| c_\gamma + \varepsilon \|\xi_2 - \xi_1\| c_{i_0} \right) \]

So from (46) and Lemma 3.12 we have
\[ \left| \phi_{\xi_2, \mu_2, \omega_{\xi_2, \mu_2, \tau(0)\xi_2, \mu_2} (\psi) - \phi_{\xi_1, \mu_1, \omega_{\xi_1, \mu_1, \tau(0)\xi_1, \mu_1} (\psi)} \right| \]
\[ \leq C \varepsilon^2 \|\mu_2 - \mu_1\| c_\gamma + \varepsilon \|\xi_2 - \xi_1\| c_{i_0} \]
\[ \left| \zeta_{\xi_2, \mu_2, \omega_{\xi_2, \mu_2, \tau(0)\xi_2, \mu_2} (\psi) - \zeta_{\xi_1, \mu_1, \omega_{\xi_1, \mu_1, \tau(0)\xi_1, \mu_1} (\psi)} \right| \]
\[ \leq C \varepsilon^2 \|\mu_2 - \mu_1\| c_\gamma + \varepsilon \|\xi_2 - \xi_1\| c_{i_0} \]

And from the ODE satisfied by \( \phi \), we can get high order estimates, i.e.
\[ \|\phi_{\xi_2, \mu_2, \omega_{\xi_2, \mu_2, \tau(0)\xi_2, \mu_2} (\psi) - \phi_{\xi_1, \mu_1, \omega_{\xi_1, \mu_1, \tau(0)\xi_1, \mu_1} (\psi)} \| C^\alpha \]
\[ \leq C \varepsilon^2 \|\mu_2 - \mu_1\| c_\gamma + \varepsilon \|\xi_2 - \xi_1\| c_{i_0} \]

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4 The existence of CMC surfaces

4.1 A fixed point argument

If we prescribe $\xi \in C^1$ and $\mu \in C^\alpha$ in 0th mode and get $\phi_{\xi,\eta}$, first we have the mean curvature of $\mathcal{D}_{\phi_{\xi,\mu}}(w, \eta)$ is

$$H(\mathcal{D}_{\phi_{\xi,\mu}}(w, \eta)) = \frac{2}{\varepsilon} + \frac{\rho}{\varepsilon} + \frac{1}{\varepsilon} L_{\xi,\mu} w + \langle \mathcal{F}_{\xi,\mu} \eta, \mathcal{Y} \rangle$$

$$+ \varepsilon (F_1(\phi_{\xi,\mu}, \frac{\partial \phi_{\xi,\mu}}{\partial \psi}) \ast R_1 + F_2(\phi_{\xi,\mu}, \frac{\partial \phi_{\xi,\mu}}{\partial \psi}) \ast R_2) + E_{\xi,\mu}$$

$$+ F_3(\phi_{\xi,\mu}, \frac{\partial \phi_{\xi,\mu}}{\partial \psi}) \ast R_3(\eta) + T_{\xi,\mu}(w, \eta)$$

$$+ \varepsilon^{-1} \rho L_\mu^3(w, \eta) + \varepsilon^{-1} \rho Q_\mu^3(w, \eta) + \varepsilon L_\mu^4(\rho) + \varepsilon L_\mu^4(\rho)$$

where

$$L_{\xi,\mu} w$$

$$= -\psi_{\xi,\mu}(\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial \theta^2}) w - (\phi_{\xi,\mu}^2 - \tau_{\xi,\mu}) \frac{\partial w}{\partial \phi_{\xi,\mu}^2} - \frac{\partial}{\partial \psi} - \psi_{\xi,\mu} w$$

$$\mathcal{F}_{\xi,\mu} \eta$$

$$= -\frac{\psi_{\xi,\mu}^3}{\phi_{\xi,\mu}^3} \frac{\partial^2 \eta}{\partial x_0^2} - \frac{1}{\varepsilon} (\psi_{\xi,\mu} \psi_{\xi,\mu}^2 + 2(\phi_{\xi,\mu}^2 - \tau_{\xi,\mu}) \frac{\partial \psi_{\xi,\mu}}{\partial \phi_{\xi,\mu}^2} \frac{\partial \eta}{\partial x_0})$$

$$- \phi_{\xi,\mu}^2 (2\phi_{\xi,\mu} \psi_{\xi,\mu} + 2\phi_{\xi,\mu} \psi_{\xi,\mu} + \phi_{\xi,\mu} \phi_{\xi,\mu})$$

$$- 2 \frac{\psi_{\xi,\mu}^2}{\phi_{\xi,\mu}^2} (\phi_{\xi,\mu}^2 - \tau_{\xi,\mu}) - 2\phi_{\xi,\mu} \psi_{\xi,\mu} R(\eta, X_0)X_0$$

$$T_{\xi,\mu}(w, \eta)$$

$$= \varepsilon L_{\xi,\mu}^1(w, \eta) + \varepsilon^{-1} Q_{\xi,\mu}^1(w, \eta) + \varepsilon^2 L_{\xi,\mu}^2(\partial^2 \eta) + Q_{\xi,\mu}^2(w, \partial^2 \eta)$$

Now we solve

$$\frac{1}{\varepsilon} L_{\xi,\mu}(w_{\xi,\mu}) = -\mathcal{F}(\phi_{\xi,\mu}, \frac{\partial \phi_{\xi,\mu}}{\partial \psi}) \ast R_1 + F_2(\phi_{\xi,\mu}, \frac{\partial \phi_{\xi,\mu}}{\partial \psi}) \ast R_2) + E_{\xi,\mu}$$

$$+ F_3(\phi_{\xi,\mu}, \frac{\partial \phi_{\xi,\mu}}{\partial \psi}) \ast R_3(\eta_{\xi,\mu}) + T_{\xi,\mu}(w_{\xi,\mu}, \eta_{\xi,\mu})$$

$$+ \varepsilon^{-1} \rho L_\mu^3(w_{\xi,\mu}, \eta_{\xi,\mu}) + \varepsilon^{-1} \rho Q_\mu^3(w_{\xi,\mu}, \eta_{\xi,\mu}) + \varepsilon L_\mu^4(\rho)$$
\[ J_{\xi,\mu}(\eta_{\xi,\mu}) = -\Pi_1(\varepsilon (F_1(\phi_{\xi,\mu}, \frac{\partial \phi_{\xi,\mu}}{\partial \eta})) \ast R_1 + F_2(\phi_{\xi,\mu}, \frac{\partial \phi_{\xi,\mu}}{\partial \eta}) \ast R_2) + E_{\xi,\mu} + \varepsilon F_3(\phi_{\xi,\mu}, \frac{\partial \phi_{\xi,\mu}}{\partial \eta}) \ast R_3(\eta_{\xi,\mu}) + T_{\xi,\mu}(w_{\xi,\mu}, \eta_{\xi,\mu}) \]

where

\[ \varepsilon^{-1} \rho L^3(\eta_{\xi,\mu}) + \varepsilon^{-1} \rho Q^3(\eta_{\xi,\mu}) + \varepsilon L^4(\rho) \]

is some function depending only on the curvature of the manifold. 

Remark. Note that the terms in \( T(w, \eta) \) also depend on \( \phi \) (and its derivatives) because each term, for example, \( Q^1(\eta, \eta) \) has Taylor expansion and each term of the expansion looks like

\[
F_{a_1,a_2,b_1,b_2,c_1,c_2}(\phi, w) R_{a_1,a_2,b_1,b_2,c_1,c_2} \cdot w^{a_1}(\partial_{a_1}w)^{a_2} \]

\[
(\partial_2^2 w)^{a_1}(\partial_{a_1}w)^{b_1}(\partial_2^2 w)^{b_2}(\partial_{a_2}w)^{c_2},
\]

where \( R_{a_1,a_2,b_1,b_2,c_1,c_2} \) is some function depending only on the curvature of the manifold. \( E \) also has similar Taylor expansion so it also depends on \( \phi \).

First we note that for \( \varepsilon \) very small, \( L_{\xi,\mu} \) and \( J_{\xi,\mu} \) is invertible.

**Proof.** Take \( J_{\xi,\mu} \) for example.

\[
\frac{\psi_{\xi,\mu}^3}{\phi_{\xi,\mu}} J_{\xi,\mu} \eta = -\psi_{\xi,\mu}^3 \frac{\partial}{\partial x_0} \left( \frac{\psi_{\xi,\mu}^3}{\phi_{\xi,\mu}} \frac{\partial \eta}{\partial x_0} \right) + O(\varepsilon) \frac{\partial \eta}{\partial x_0}
\]

\[
-\phi_{\xi,\mu}^2 \left( 2 \phi_{\xi,\mu} \frac{\psi^2_{\xi,\mu}}{\phi_{\xi,\mu}} + 2 \phi_{\xi,\mu} \frac{\psi^2_{\xi,\mu}}{\phi_{\xi,\mu}} + \frac{\psi_{\xi,\mu}^3}{\phi_{\xi,\mu}} \right)
\]

\[
-2 \psi^2_{\xi,\mu} \left( \phi_{\xi,\mu}^3 - \tau_{\xi,\mu} \right) - 2 \phi_{\xi,\mu} \frac{\psi_{\xi,\mu}^2}{\phi_{\xi,\mu}} R(\eta, X_0) X_0.
\]

Suppose \( \frac{\partial}{\partial \eta_0} = \frac{\psi^2_{\xi,\mu}}{\phi_{\xi,\mu}} \frac{\partial}{\partial x_0} \). By contradiction, if there were \( \varepsilon_n \to 0 \), \( ||\xi_n||_{C^2} \leq C_1 \varepsilon_n^2, ||\mu_n||_{C^4} \leq C_2 \), such that

\[ J_{\xi_n,\mu_n} \eta_n = 0 \]

always has non trivial solution \( \eta_n \), then by passing to subsequence, \( \eta_n(y_0) \) will converge to non trivial \( \eta_{\infty}(y_0) \) in \( C^1,\alpha \) sense. One can get a contradiction because

\[ \frac{\psi_{\xi_n,\mu_n}^3}{\phi_{\xi_n,\mu_n}} J_{\xi_n,\mu_n} \eta_n \to J A \eta_{\infty} \]

from estimates (36) and (37). So if \( f \in C^\alpha \), there is one unique \( \eta_f \) such that

\[ J_{\xi,\mu} \eta_f = f \]

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and
\[ \|
\eta
\|_{C^1_{x_0}} \leq C \|
\|f\|_{C^0} \\
\|
\eta
\|_{C^2_{x_0}} \leq \frac{C}{\varepsilon} \|
\|f\|_{C^0} \\
\|
\eta
\|_{C^2_{x_0}, \alpha} \leq \frac{C}{\varepsilon^{1+\alpha}} \|f\|_{C^2} \]

Also by similar method as in Appendix B and the estimates 36 and 37 we can get if \( \eta \) solves
\[ \mathcal{J}_{\xi, \mu} \eta = \frac{2}{3} \varepsilon \hat{\phi}_{\xi, \mu} R(X_0, Y_0) Y_0 \]
we have
\[ \|
\eta
\|_{C^1_{x_0}} \leq C \varepsilon^2 \\
\|
\eta
\|_{C^2_{x_0}} \leq C \varepsilon \\
\|
\eta
\|_{C^2_{x_0}, \alpha} \leq C \varepsilon^{1-\alpha}. \]

So \( \mathcal{J}_{\xi, \mu} \) is invertible. Also \( \mathcal{L}_{\xi, \mu} \) is invertible and if \( w_f \) satisfies
\[ \mathcal{L}_{\xi, \mu} w_f = f \]
we have
\[ \|w_f\|_{C^2_{x_0}, \alpha} \leq C \|f\|_{C^2}. \]

Now by an iteration argument, we can solve the system (49) (50). And we have
\[ \|w_{\xi, \mu}\|_{C^2_{x_0}, \alpha} \leq C \varepsilon^2 \\
\|\eta_{\xi, \mu}\|_{C^1_{x_0}} \leq C \varepsilon^2 \\
\|\eta_{\xi, \mu}\|_{C^2_{x_0}} \leq C \varepsilon \\
\|\eta_{\xi, \mu}\|_{C^2_{x_0}, \alpha} \leq C \varepsilon^{1-\alpha}. \]

We are going to see how \( w_{\xi, \mu} \) and \( \eta_{\xi, \mu} \) vary when \( \xi \) or \( \mu \) varies. For example, if we change \( \mu \) into \( \mu + \Delta \mu \), then we linearize (49) (50) at \( \mu + t \Delta \mu \) for \( t \in [0, 1] \).

\[
\frac{1}{\varepsilon} \mathcal{L}_{\xi, \mu + t \Delta \mu} \left( \frac{\partial w_{\xi, \mu + t \Delta \mu}}{\partial t} \right) = -\frac{1}{\varepsilon} \left( \frac{\partial}{\partial t} \mathcal{L}_{\xi, \mu + t \Delta \mu} (w_{\xi, \mu + t \Delta \mu}) \right) + \frac{\partial}{\partial t} \left( \frac{1}{\varepsilon} \mathcal{L}_{\xi, \mu + t \Delta \mu} (w_{\xi, \mu + t \Delta \mu}) \right) \\
\mathcal{J}_{\xi, \mu + t \Delta \mu} \left( \frac{\partial \eta_{\xi, \mu + t \Delta \mu}}{\partial t} \right) = -(\frac{\partial}{\partial t} \mathcal{J}_{\xi, \mu + t \Delta \mu} (\eta_{\xi, \mu + t \Delta \mu}) \right) + \frac{\partial}{\partial t} (\mathcal{J}_{\xi, \mu + t \Delta \mu} (\eta_{\xi, \mu + t \Delta \mu}))
\]
Note that

\[ \frac{\partial \phi_{\xi,\mu + \Delta \mu}}{\partial t} \leq C \varepsilon \| \Delta \mu \|_{C^0} \]
\[ \frac{\partial \xi_{\mu + \Delta \mu}}{\partial t} \leq C \varepsilon \| \Delta \mu \|_{C^0} \]
\[ \frac{d \omega_{\xi,\mu + \Delta \mu}}{dt} \leq C \| \Delta \mu \|_{C^0} \]

So we know

\[ \frac{\partial}{\partial t} w_{\xi,\mu + \Delta \mu} \leq C \varepsilon^2 \| \Delta \mu \|_{C^2} \]
\[ \frac{\partial}{\partial t} \eta_{\xi,\mu + \Delta \mu} \leq C \varepsilon \| \Delta \mu \|_{C^0} \]
\[ \frac{\partial}{\partial t} (\eta_{\xi,\mu + \Delta \mu} - \varepsilon L \rho^{3 \xi,\mu} (w_{\xi,\mu}, \eta_{\xi,\mu})) \leq C \varepsilon \| \Delta \mu \|_{C^0} \]

So we have

\[ \| w_{\xi,\mu + \Delta \mu} - w_{\xi,\mu} \|_{C^2} \leq C \varepsilon^3 \| \Delta \mu \|_{C^2} \]
\[ \| \eta_{\xi,\mu + \Delta \mu} - \eta_{\xi,\mu} \|_{C^1} \leq C \varepsilon^2 \| \Delta \mu \|_{C^0} \]
\[ \| \eta_{\xi,\mu + \Delta \mu} - \eta_{\xi,\mu} \|_{C^2} \leq C \varepsilon \| \Delta \mu \|_{C^0} \]
\[ \| \eta_{\xi,\mu + \Delta \mu} - \eta_{\xi,\mu} \|_{C^2} \leq C \varepsilon \| \Delta \mu \|_{C^0} \]

In the same way

\[ \| w_{\xi + \Delta \xi,\mu} - w_{\xi,\mu} \|_{C^2} \leq C \varepsilon^2 \| \Delta \xi \|_{C^1} \]
\[ \| \eta_{\xi + \Delta \xi,\mu} - \eta_{\xi,\mu} \|_{C^1} \leq C \varepsilon \| \Delta \xi \|_{C^0} \]

With such \( w_{\xi,\mu}, \eta_{\xi,\mu} \) and \( \rho \), one can calculate

\[
H(D_{\phi_{\xi,\mu}} (w_{\xi,\mu}, \eta_{\xi,\mu})) = \frac{2}{\varepsilon} + F_i(\phi_{\xi,\mu}, \frac{\partial \phi_{\xi,\mu}}{\partial \psi}) \Pi_0 (R(\Upsilon, X_0, \eta_{\xi,\mu}, \Upsilon) + \xi(x_0)) + \varepsilon^2 \mu + \Pi_0 E_{\xi,\mu} + T_{\xi,\mu} (w_{\xi,\mu}, \eta_{\xi,\mu}) + \varepsilon^{-1} \rho I_{\xi,\mu}^3 (w_{\xi,\mu}, \eta_{\xi,\mu}) + \varepsilon^{-1} \rho Q_{\xi,\mu}^3 (w_{\xi,\mu}, \eta_{\xi,\mu}) + \varepsilon L^4_{\xi,\mu}(\mu) + \varepsilon^2 \omega \frac{\partial \phi_{\xi,\mu}}{\partial \psi}
\]

Now we define a map

\[ \Omega : (\xi, \mu) \rightarrow (\Omega^1(\xi, \mu), \Omega^2(\xi, \mu)) \]
where
\[ \Omega^1(\xi, \mu) = -R(\eta, X_0, \eta, \xi, \mu, \nu) \]
\[ \Omega^2(\xi, \mu) = -\varepsilon^{-2}(T_{\xi, \mu}(w_{\xi, \mu}, \eta, \mu) + \varepsilon^{-1} \rho L^4(\eta, \mu) + E_{\xi, \mu} + \varepsilon L^4(\mu)) \]

We define a norm for \((\xi, \mu)\),
\[ \| (\xi, \mu) \|_{\varepsilon, \alpha} = \| \xi \|_{c_{x_0}} + \varepsilon \| \mu \|_{c_{x_2}}^2 \]

We are going to prove that

**Lemma 4.1.** For fixed \(C_1, C_2\) if \(\| \xi \|_{c_{x_0}} \leq C_1\xi^2, \| \mu \|_{c_{x_2}} \leq C_2\), then
\[ \| (\Omega^1(\xi_1, \mu_1), \Omega^2(\xi_1, \mu_1)) - (\Omega^1(\xi_2, \mu_2), \Omega^2(\xi_2, \mu_2)) \|_{\varepsilon, \alpha} \leq C(C_1, C_2) \| (\xi_1 - \xi_2, \mu_1 - \mu_2) \|_{\varepsilon, \alpha} \]

**Proof.** Note that
\[
\| L^1_{\xi, \mu}(w_1, \eta_1) - L^1_{\xi, \mu}(w_2, \eta_2) \|_{c_{x_2}}^2 \leq C\| w_1 - w_2 \|_{c_{x_2}}^2 + C\| \eta_1 - \eta_2 \|_{c_{x_2}}^2 \]  
(53)
\[
\| L^2_{\xi, \mu}(w_1, \partial^2_{x_0} \eta_1) - L^2_{\xi, \mu}(w_2, \partial^2_{x_0} \eta_2) \|_{c_{x_2}}^2 \leq C\| w_1 - w_2 \|_{c_{x_2}}^2 + C\| \eta_1 - \eta_2 \|_{c_{x_2}}^2 \]  
(54)
\[
\| L^3_{\xi, \mu}(w_1, \eta_1) - L^3_{\xi, \mu}(w_2, \eta_2) \|_{c_{x_2}}^2 \leq C\| w_1 - w_2 \|_{c_{x_2}}^2 + C\| \eta_1 - \eta_2 \|_{c_{x_2}}^2 \]  
(55)
\[
\| Q^1_{\xi, \mu}(w_1, \eta_1) - Q^1_{\xi, \mu}(w_2, \eta_2) \|_{c_{x_0}}^2 \leq C(\| \eta_1 \|_{c_{x_0}}^2 + \| \eta_2 \|_{c_{x_0}}^2 + \| w_1 \|_{c_{x_2}}^2 + \| w_2 \|_{c_{x_2}}^2 \) \]
(56)
\[
\| Q^2_{\xi, \mu}(w_1, \partial^2_{x_0} \eta_1) - Q^2_{\xi, \mu}(w_2, \partial^2_{x_0} \eta_2) \|_{c_{x_0}}^2 \leq C(\| \eta_1 \|_{c_{x_0}}^2 + \| \eta_2 \|_{c_{x_0}}^2 + \| w_1 \|_{c_{x_2}}^2 + \| w_2 \|_{c_{x_2}}^2 \) \]
(57)
\[
\| Q^3_{\xi, \mu}(w_1, \partial^2_{x_0} \eta_1) - Q^3_{\xi, \mu}(w_2, \partial^2_{x_0} \eta_2) \|_{c_{x_0}}^2 \leq C(\| \eta_1 \|_{c_{x_0}}^2 + \| \eta_2 \|_{c_{x_0}}^2 + \| w_1 \|_{c_{x_2}}^2 + \| w_2 \|_{c_{x_2}}^2 \) \]
(58)
and if we assume norms of the curvatures along the geodesic, such that curvature terms along the geodesic. So there is lemma.

\[ \xi \]

From (51) and the remark after (50) we can prove the lemma.

Note that \(Q^2\) is linear in \(\partial^2_{\alpha_0} \eta_1\). From (31) we know

\[ \frac{\partial}{\partial \psi} F_i(\phi_{\xi_1, \mu_1}, \frac{\partial \phi_{\xi_1, \mu_1}}{\partial \psi} - F_i(\phi_{\xi_2, \mu_2}, \frac{\partial \phi_{\xi_2, \mu_2}}{\partial \psi}) \leq C \frac{\varepsilon^2}{\varepsilon^2} \mu_1 \leq C_7 \varepsilon^2 \]

From (51) and the remark after (50) to (61) we can prove the lemma.

If \(\xi = 0, \mu = 0\), we can get \(w_{0,0}, \eta_{0,0}\), whose norms only depend on the curvature terms along the geodesic. So there is \(C_7\) which only depends on the norms of the curvatures along the geodesic, such that

\[ \|\Omega^1(0,0)\|_{C_{x_0}^1} \leq C_7 \varepsilon^2 \]
\[ \varepsilon \|\Omega^2(0,0)\|_{C_\gamma^1} \leq C_7 \varepsilon \]

and if we assume

\[ \xi^1 = \Omega^1(0,0) \]
\[ \mu^1 = \Omega^2(0,0) \]

\[ \varepsilon \|\Omega^1(\xi^1, \mu^1) - \xi^1\|_{C_{x_0}^1} \leq C_7 \varepsilon^2 \]
\[ \varepsilon \|\Omega^2(\xi^1, \mu^1) - \mu^1\|_{C_\gamma^1} \leq C_7 \varepsilon^2 \]

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where we can use the same constant $C_7$. Then

$$\|\Omega(\xi^1, \mu^1) - (\xi^1, \mu^1)\|_{\epsilon, \alpha} \leq 2C_7\epsilon^2.$$ 

Let

$$E(5C_7) = \{ (\xi, \mu) : \| (\xi, \mu) - (\xi^1, \mu^1) \|_{\epsilon, \alpha} \leq 5C_7\epsilon^2 \}$$

We assume $C_1 = C_2 = 10C_7$. Let

$$E(C_1, C_2) = \{ (\xi, \mu) : \| \xi \|_{C_1}, \| \mu \|_{C_2} \leq C_1 \epsilon^2, \| \mu \|_{C_2} \leq C_2 \}.$$ 

It is obvious that

$$E(5C_7) \subset E(C_1, C_2).$$

So if $(\xi_1, \mu_1), (\xi_2, \mu_2) \in E(5C_7)$, then

$$\| \Omega(\xi_1, \mu_1) - \Omega(\xi_2, \mu_2) \|_{\epsilon, \alpha} \leq C(C_1, C_2)\epsilon \| (\xi_1 - \xi_2, \mu_1 - \mu_2) \|_{\epsilon, \alpha}.$$ 

If we choose $\epsilon$ such that

$$C(C_1, C_2)\epsilon \leq \frac{1}{100},$$

then $\Omega_2$ maps $E(5C_7)$ into itself. Note that $E(5C_7)$ is a complete metric space. From fixed point theorem, there is a unique

$$(\hat{\xi}, \hat{\mu}) \in E(5C_7)$$

such that

$$\Omega_2(\hat{\xi}, \hat{\mu}) = (\hat{\xi}, \hat{\mu}).$$

For this $(\hat{\xi}, \hat{\mu})$, we have

$$H(D_{\phi_{\hat{\xi}, \hat{\mu}}} (w_{\hat{\xi}, \hat{\mu}}, \eta_{\hat{\xi}, \hat{\mu}})) = \frac{2}{\epsilon} + \omega_{\hat{\xi}, \hat{\mu}}\epsilon^2 \frac{\partial \phi_{\hat{\xi}, \hat{\mu}}}{\partial \psi}.$$ 

4.2 The energy of the surface

So far what we’ve got is a global smooth surface $D_{\phi_{\hat{\xi}, \hat{\mu}}} (w_{\hat{\xi}, \hat{\mu}}, \eta_{\hat{\xi}, \hat{\mu}})$ whose mean curvature is

$$\frac{2}{\epsilon} + \omega_{\hat{\xi}, \hat{\mu}}\epsilon^2 \frac{\partial \phi_{\hat{\xi}, \hat{\mu}}}{\partial \psi}.$$ 

Remember that in the ODE system (26) to (29), the starting point $p$ where $\phi_{\xi, \mu}$ attains its minimum can be every point on the geodesic. The last thing we can do is to choose a particular point $p$ to make

$$\omega_{\hat{\xi}, \hat{\mu}} = 0.$$ 

Note that in the ODE system above we have the initial value condition

$$\frac{\partial \phi_{\hat{\xi}, \hat{\mu}}}{\partial \psi} (0) = 0$$

$$\phi_{\hat{\xi}, \hat{\mu}} (0) = \frac{1 - \sqrt{1 - 4\tau(0)}}{2}.$$
Now for a small $\delta$ and $\psi \in [\delta, \frac{L}{\varepsilon} + \delta]$, we solve

\[
\begin{cases}
\frac{\partial^2 \phi}{\partial \psi^2} & - \phi^{-1} (1 + (\frac{\partial \phi}{\partial \psi})^2) + (2 + \rho)(1 + (\frac{\partial \phi}{\partial \psi})^2)^\frac{3}{2} = 0 \\
\frac{\partial \phi}{\partial \psi} (\delta) & = 0 \\
\phi (\delta) & = \frac{1 - \sqrt{1 - 4\tau(\delta)}}{2} \\
\rho & = -\varepsilon^2 F_1(\phi, \phi_\psi) \star \Pi_0(R_1) + \varepsilon F_4(\phi, \phi_\psi) \xi(x_0) + \varepsilon^3 \mu(\psi) + \varepsilon^3 \omega \phi_\psi
\end{cases}
\]

For each $\xi$ and $\mu$ with

\[
\|\xi\|_{C^1_0} \leq C_1 \varepsilon^2 \\
\|\mu\|_{C^2} \leq C_2
\]

we can solve the above system. And we can choose unique $|w_{\xi,\mu}| \leq C_3$ and $|\phi(\delta)_{\xi,\mu} - \frac{1 - \sqrt{1 - 4\tau(\delta)}}{2}| \leq C_4$ as before such that

\[
\frac{\partial \phi}{\partial \psi} (\delta) = \frac{\partial \phi}{\partial \psi} (\delta + \frac{L}{\varepsilon}) = 0
\]

\[
\phi (\delta) = \phi (\delta + \frac{L}{\varepsilon}) = \frac{1 - \sqrt{1 - 4\tau(\delta)}}{2}.
\]

Also as before we can solve 1st mode and high mode and at last by a fixed point argument we get $\hat{\xi}_\delta, \hat{\mu}_\delta$. Now all we get are $\hat{\phi}_{\xi,\mu,\delta}$, $\hat{w}_{\xi,\mu,\delta}$, $\hat{\eta}_{\xi,\mu,\delta}$ with

\[
H(\hat{\xi}_{\delta,\mu,\delta}) = \frac{2}{\varepsilon} + \omega_{\hat{\xi}_{\delta,\mu,\delta}} \varepsilon^2 \frac{\partial \phi_{\hat{\xi}_{\delta,\mu,\delta}}}{\partial \psi}.
\]

We make the following notations

\[
\frac{\partial f_\delta}{\partial \delta} |_{\delta=0} = \lim_{\delta \to 0} \frac{f_\delta (\psi) - f_0 (\psi)}{\delta} \\
\frac{\partial' f_\delta}{\partial \delta} |_{\delta=0} = \lim_{\delta \to 0} \frac{f_\delta (\psi) - f_0 (\psi - \delta)}{\delta}
\]

So we have

\[
\frac{\partial f_\delta (\psi)}{\partial \delta} |_{\delta=0} = \lim_{\delta \to 0} \frac{f_\delta (\psi) - f_0 (\psi)}{\delta}
\]

\[
= \lim_{\delta \to 0} \frac{f_\delta (\psi) - f_0 (\psi - \delta)}{\delta} + \lim_{\delta \to 0} \frac{f_0 (\psi - \delta) - f_0 (\psi)}{\delta}
\]

\[
= \frac{\partial' f_\delta (\psi)}{\partial \delta} |_{\delta=0} - \frac{\partial f_0 (\psi)}{\partial \psi}
\]

Note that
The ODE system (62) together with (63) is a linear system of

\[
\begin{align*}
\sum_{i=1}^{n} a_{ij} \phi_j \bigg|_{\delta=0} &= b_i, \quad i = 1, 2, \ldots, n, \\
\sum_{j=1}^{n} b_{ij} \delta_j \bigg|_{\delta=0} &= c_i, \quad i = 1, 2, \ldots, n.
\end{align*}
\]

where

\[
\begin{align*}
E_1 &= \left( -\varepsilon^2 F_1(\phi_{\xi_0, \mu_0, \omega}) + \frac{\partial}{\partial \delta} \mathcal{P} \right) \bigg|_{\delta=0} + \varepsilon F_1(\phi_{\xi_0, \mu_0, \omega}) \frac{\partial}{\partial \delta} \mathcal{P} \bigg|_{\delta=0} \\
E_2 &= -\varepsilon^2 \frac{\partial}{\partial \delta} \mathcal{P} \bigg|_{\delta=0} + \varepsilon^2 \frac{\partial}{\partial \delta} \mathcal{P} \bigg|_{\delta=0} \\
E_3 &= \Omega_{2, \delta}(\xi_\delta, \mu_\delta) \\
\end{align*}
\]

We may denote the symbols like $\phi_{\xi_\delta, \mu_\delta}$ as $\phi_{\delta}$ for short. From

\[
\begin{align*}
0 &= \delta \bigg( \frac{\partial}{\partial \delta} \mathcal{P} \bigg|_{\delta=0} \\
0 &= \frac{\partial}{\partial \delta} \mathcal{P} \bigg|_{\delta=0} \\
0 &= \frac{\partial}{\partial \delta} \mathcal{P} \bigg|_{\delta=0} \\
\end{align*}
\]

we can get

\[
\begin{align*}
E_1 &= \left( -\varepsilon^2 F_1(\phi_{\xi_0, \mu_0, \omega}) + \frac{\partial}{\partial \delta} \mathcal{P} \right) \bigg|_{\delta=0} + \varepsilon F_1(\phi_{\xi_0, \mu_0, \omega}) \frac{\partial}{\partial \delta} \mathcal{P} \bigg|_{\delta=0} \\
E_2 &= -\varepsilon^2 \frac{\partial}{\partial \delta} \mathcal{P} \bigg|_{\delta=0} + \varepsilon^2 \frac{\partial}{\partial \delta} \mathcal{P} \bigg|_{\delta=0} \\
E_3 &= \Omega_{2, \delta}(\xi_\delta, \mu_\delta) \\
\end{align*}
\]

The ODE system (62) together with (63) is a linear system of

\[
\begin{align*}
\frac{\partial}{\partial \delta} \phi_{\delta}, \frac{\partial}{\partial \delta} w_{\delta}, \frac{\partial}{\partial \delta} \eta_{\delta}, \frac{\partial}{\partial \delta} \hat{\xi}_{\delta}, \frac{\partial}{\partial \delta} \hat{\mu}_{\delta}, \frac{\partial}{\partial \delta} \phi_{\delta}(0), \frac{\partial}{\partial \delta} \omega_{\delta} \bigg|_{\delta=0}.
\end{align*}
\]

Note that for curvature terms like $R(\mathbf{Y}, X_0, \mathbf{Y}, X_0)$ we have

\[
\frac{\partial}{\partial x_0} R(\mathbf{Y}, X_0, \mathbf{Y}, X_0) = \varepsilon \frac{\partial R}{\partial x_0}
\]
where \( \frac{\partial}{\partial \tau_0} R \) is also a good function with \( \frac{\partial}{\partial \tau_0}(\frac{\partial}{\partial \tau_0} R) = O(\varepsilon) \). Paying attention to every cancellation property, we can solve the system and can get

\[
\| \frac{\partial}{\partial \delta} \phi_{\delta}(\psi) \|_{\delta=0} \|_{C^1} = O(\varepsilon^2)
\]

\[
\| \frac{\partial}{\partial \delta} w_\delta(\psi) \|_{\delta=0} \|_{C^2,0} = O(\varepsilon^3)
\]

\[
\| \frac{\partial}{\partial \delta} \eta_\delta(\psi) \|_{\delta=0} \|_{C^3} = O(\varepsilon^3)
\]

by an iteration argument.

**Remark.** This iteration argument is a little complex. First look at

\[
\mathcal{L}_{\xi_0,\mu_0,\omega_{\xi_0,0},\tau(0)}(\frac{\partial}{\partial \delta} \phi_{\xi_0,\mu_0,\delta}(\psi))|_{\delta=0} = E_1
\]

together with (63). At first assume \( \frac{\partial}{\partial \tau_0} w_\delta, \frac{\partial}{\partial \tau_0} \eta_\delta, \frac{\partial}{\partial \tau_0} \xi_\delta, \frac{\partial}{\partial \tau_0} \mu_\delta \) do not exist and solve the three equations by an iteration argument. Then one get a global smooth \( \frac{\partial}{\partial \tau_0} \phi_{\delta} \) and one can solve the equations

\[
\frac{1}{\varepsilon} \mathcal{L}_{\xi_0,\mu_0,\omega_{\xi_0,0},\tau(0)}(\frac{\partial}{\partial \delta} \phi_{\xi_0,\mu_0,\delta}(\psi))|_{\delta=0} = E_2
\]

\[
\mathcal{J}_{\xi_0,\mu_0,\omega_{\xi_0,0},\tau(0)}(\frac{\partial}{\partial \delta} \eta_{\xi_0,\mu_0,\delta}(\psi))|_{\delta=0} = E_3
\]

Then we can get \( \frac{\partial}{\partial \tau_0} \xi_\delta, \frac{\partial}{\partial \tau_0} \mu_\delta \) by the equation

\[
\Omega_{2,\delta}(\xi_\delta, \mu_\delta) = \left( \Omega_{1,\delta}^2(\xi_\delta, \mu_\delta), \Omega_{2,\delta}^2(\xi_\delta, \mu_\delta) \right)
\]

\[
= \left( R(T, X_0, \eta_\delta, T), \varepsilon^{-2} (T_{\xi_\delta, \mu_\delta, 3}(w_\delta, \eta_\delta) + \varepsilon^{-1} \rho \xi_{\delta, \mu_\delta, 3}(w_\delta, \eta_\delta)) \right)
\]

\[
+ \varepsilon^{-1} \rho \xi_{\delta, \mu_\delta, 3}(w_\delta, \eta_\delta) + E_{\delta, \mu_\delta, 3}.
\]

Then one can solve

\[
\mathcal{L}_{\xi_0,\mu_0,\omega_{\xi_0,0},\tau(0)}(\frac{\partial}{\partial \delta} \phi_{\xi_0,\mu_0,\delta}(\psi))|_{\delta=0} = E_1
\]

together with (63) for the second time.

Note that any time when the solution of

\[
\mathcal{L}_{\xi_0,\mu_0,\omega_{\xi_0,0},\tau(0)}(\frac{\partial}{\partial \delta} \phi_{\xi_0,\mu_0,\delta}(\psi))|_{\delta=0} = E_1
\]

has a perturbation of order \( O(\varepsilon^k) \), this perturbation behaves like

\[
O(\varepsilon^k) \phi_\psi + O(\varepsilon^{k+1}).
\]

It is an important property which makes the iteration work.
So we have
\[ \frac{\partial}{\partial \delta} (\varepsilon \phi_\delta \psi + \eta_\delta, \Upsilon >) \big|_{\delta=0} = -\varepsilon \frac{\partial \phi_{\delta=\hat}{\psi}, \delta}{\partial \psi} + O(\varepsilon^3) \]

Consider the energy functional of the surface
\[ E(\mathcal{D}_\delta(w_\delta, \eta_\delta)) = \text{Area}(\mathcal{D}_\delta(w_\delta, \eta_\delta)) - \frac{2}{\varepsilon} \text{Vol}(\mathcal{D}_\delta(w_\delta, \eta_\delta)). \]

We have
\[ \frac{d}{d\delta} E(\mathcal{D}_\delta(w_\delta, \eta_\delta)) \big|_{\delta=0} \]
\[ = \frac{d}{d\delta} \left( \text{Area}(\mathcal{D}_\delta(w_\delta, \eta_\delta)) - \frac{2}{\varepsilon} \text{Vol}(\mathcal{D}_\delta(w_\delta, \eta_\delta)) \right) \]
\[ = \int H \frac{\partial}{\partial \delta} (\varepsilon \phi_\delta \psi + \eta_\delta, \Upsilon >) \big|_{\delta=0} < N, \Upsilon > dS \]
\[ - \frac{2}{\varepsilon} \int \frac{\partial}{\partial \delta} (\varepsilon \phi_\delta \psi + \eta_\delta, \Upsilon >) \big|_{\delta=0} < N, \Upsilon > dS \]
\[ = -\varepsilon^3 \omega_\delta \left( \int (\partial \phi_\delta \psi, \partial \psi)^2 < N, \Upsilon > dS + O(\varepsilon^2) \right). \]

In the same way we can prove that
\[ \frac{d}{d\delta} E(\mathcal{D}_\delta(w_\delta, \eta_\delta)) \]
\[ = -\varepsilon^3 \omega_\delta \left( \int (\partial \phi_\delta \psi, \partial \psi)^2 < N, \Upsilon > dS + O(\varepsilon^2) \right). \]

Note that \( < N, \Upsilon > \) has uniform positive lower bound. So
\[ \int (\partial \phi_\delta \psi, \partial \psi)^2 < N, \Upsilon > dS + O(\varepsilon^2) > 0. \]

It is easy to see that \( E \) is a smooth function with respect to \( \delta \) and \( E \) takes the same value at 0 and \( \frac{L}{\varepsilon} \). So if we let \( \delta \) goes from 0 to \( \frac{L}{\varepsilon} \), if \( E \) is always constant, then \( \omega_\delta \equiv 0 \), so we can get infinitely many different constant mean curvature surfaces of Delaunay type. If \( E \) is not always constant, we will at least get two 0 points of
\[ \frac{d}{d\delta} E(\mathcal{D}_\delta(w_\delta, \eta_\delta)), \]

where \( \omega_\delta = 0 \). The two surfaces are not the same, because they corresponds to the maximal value and minimal value of \( E \). So we proved the main theorem.

63
A The proof of Lemma 3.1 and Lemma 3.4

The proof of Lemma 3.1. The goal is to prove that
\[
\int_{a_1}^{b_1} \frac{1}{\phi} (2\dot{\phi} \dot{\psi} + 2\dddot{\phi} \dot{\psi} + \psi^3 - 2\dddot{\phi} (\phi^2 - \tau_0) - 2\phi \dot{\phi}^2) d\psi \cdot \int_{a_1}^{b_1} \frac{\phi^2}{\psi^3} d\psi
\]
\[
= (b_1 - a_1)^2,
\]
where \([a_1, b_1]\) is one period for \(\phi(\psi)\).

Proof.
\[
\int_{a_1}^{b_1} \frac{\phi^2}{\psi^3} d\psi = \int_{a_1}^{b_1} \frac{1}{\phi}(1 + \phi_0^2) \frac{2}{\phi} d\psi.
\]
From
\[
\phi \dot{\phi} - \phi^{-1}(1 + \phi_0^2) + 2(1 + \phi_0^2) \frac{d}{d\psi} = 0
\]
we have
\[
\int_{a_1}^{b_1} \frac{\phi^2}{\psi^3} d\psi = \int_{a_1}^{b_1} \left(\frac{1 + \phi_0^2}{2\phi^2} - \frac{\phi \dot{\phi} \dot{\psi}}{2\phi} \right) d\psi.
\]
Note that
\[
\int_{a_1}^{b_1} - \frac{\phi \dot{\phi} \dot{\psi}}{2\phi} d\psi = \frac{1}{2} \left(\int_{a_1}^{b_1} \phi \frac{-\phi \dot{\phi} \dot{\psi}}{\phi^2} d\psi\right)
\]
\[
= -\frac{1}{2} \int_{a_1}^{b_1} \phi \frac{-\phi \dot{\phi} \dot{\psi}}{\phi^2} d\psi.
\]
So we have
\[
\int_{a_1}^{b_1} \frac{\phi^2}{\psi^3} d\psi = \int_{a_1}^{b_1} \frac{1}{2\phi^2} d\psi.
\]
Also from direct computation one can get
\[
\int_{a_1}^{b_1} \frac{1}{\phi} (2\dot{\phi} \dot{\psi} + 2\dddot{\phi} \dot{\psi} + \psi^3 - 2\dddot{\phi} (\phi^2 - \tau_0) - 2\phi \dot{\phi}^2) d\psi
\]
\[
= \int_{a_1}^{b_1} \phi^2 (2\phi \dot{\phi} \dot{\psi} (1 + \phi_0^2)^{-\frac{1}{2}} - (1 + \phi_0^2)^{-\frac{3}{2}} \phi \dot{\phi} \dot{\psi} \phi_0^2)
\]
\[
+ \phi (3(1 + \phi_0^2)^{-\frac{3}{2}} \phi_0^2 + (1 + \phi_0^2)^{-\frac{5}{2}})) d\psi
\]
\[
= \int_{a_1}^{b_1} (2\phi^2 + \frac{2\phi^2 \phi \dot{\phi} \dot{\psi}}{(1 + \phi_0^2)^{\frac{3}{2}}}) d\psi.
\]
We assume that
\[
\phi(s) = \sqrt{T_0} \exp(\sigma(s))
\]
then we have
\[
\begin{align*}
\dot{\psi} &= \sqrt{\tau_0} \exp(\sigma(s)) \sqrt{1 - \sigma_s^2} \\
\phi \dot{\psi} &= \frac{\sigma_s}{\sqrt{1 - \sigma_s^2}} \\
\phi \psi \dot{\psi} &= \frac{\sigma_{ss}}{\sqrt{\tau_0} \exp(\sigma)(1 - \sigma_s^2)}
\end{align*}
\]

Also we have
\[
\begin{align*}
1 - \sigma_s^2 &= 4\tau_0 \cosh^2 \sigma \\
\sigma_{ss} &= -2\tau_0 \sinh 2\sigma.
\end{align*}
\]

For \( \phi \) there are two particular points in one period such that \( \sigma = 0 \). Suppose these two points are \( s = s_1, s = s_2 \) and suppose \([s_1, s_3]\) is one period for \( \phi(s) \). Note that \( s_2 \in (s_1, s_3) \).

Here we use one particular property of Delaunay surface
\[
\sigma(s_2 - t) = -\sigma(s_2 + t).
\]

Then from direct computation we know
\[
\begin{align*}
\int_{a_1}^{b_1} \frac{1}{2\phi^2} d\phi &= \int_{s_1}^{s_3} \cosh 2\sigma ds \\
\int_{a_1}^{b_1} \left(2\phi^2 + \frac{2\phi^2 \phi \dot{\psi}}{(1 + \phi^2)^2}\right) d\psi &= 4\tau_0 \int_{s_1}^{s_3} \cosh 2\sigma ds
\end{align*}
\]
and
\[
\begin{align*}
b_1 - a_1 &= 2\tau_0 \int_{s_1}^{s_3} \cosh 2\sigma ds.
\end{align*}
\]

So we know
\[
\begin{align*}
\int_{a_1}^{b_1} \frac{1}{\phi} \left(2\phi \ddot{\psi} + \frac{2\phi^2 \ddot{\phi}}{\phi} + \frac{3\ddot{\phi}}{\phi} - 2\dddot{\psi}(\phi^2 - \tau_0) - 2\phi \dddot{\phi}\right) d\psi &= (b_1 - a_1)^2.
\end{align*}
\]

Proof of Lemma 3.4. The goal is to prove
\[
\int_{a_1}^{b_1} \phi \left(\frac{\partial \phi}{\partial \psi}\right) \phi^{-2} \left(\frac{2}{3} \phi \ddot{\phi} + \frac{2}{3} \phi^3 - \frac{2}{3} \psi \dddot{\phi} - \frac{4}{3}(\phi^2 - \tau) \dddot{\phi} \phi + \frac{4}{3} \phi \dddot{\phi} \right) d\psi = 0
\]
which is equivalent to
\[
\int_{a_1}^{b_1} \phi^2 \phi^{-2} \psi (\phi \dddot{\phi} + \phi^2 - 2(\phi^2 + \tau)(\phi^2 - \tau)) d\psi = 0.
\]
Proof. Note that

\[
\int_{a_1}^{b_1} \phi^2 \phi^{-2} \dot{\psi} (\phi \ddot{\phi} + \dot{\phi}^2 - 2(\phi^2 + \tau)(\phi^2 - \tau)) d\psi = 0
\]

\[
= \int_{a_1}^{b_1} \frac{\sigma^2_s}{1 - \sigma^2_s \tau \exp(2\sigma)} \tau \exp(2\sigma)(1 - \sigma^2_s)(\phi \ddot{\phi} + \dot{\phi}^2 - 2(\phi^2 + \tau)(\phi^2 - \tau)) d\psi
\]

\[
= \int_{a_1}^{b_1} \sigma^2_s (\phi \ddot{\phi} + \dot{\phi}^2 - 2(\phi^2 + \tau)(\phi^2 - \tau)) d\psi.
\]

\[
\int_{a_1}^{b_1} \sigma^2_s (\phi \ddot{\phi} + \dot{\phi}^2) d\psi
\]

\[
= \int_{a_1}^{b_1} \sigma^2_s (\phi \ddot{\phi})_s ds
\]

\[
= - \int_{a_1}^{b_1} \phi \dot{\phi} 2 \sigma_s \sigma_{ss} ds
\]

\[
= - 2 \int_{a_1}^{b_1} \exp(\sigma) \exp(\sigma) \sigma^2_s \sigma_{ss} ds
\]

\[
= 4\tau^2 \int_{a_1}^{b_1} \exp(2\sigma)(1 - 4\tau \cosh^2(\sigma)) \sinh(2\sigma) ds
\]

\[
= 2\tau^2 \int_{a_1}^{b_1} (1 - 4\tau \cosh^2(\sigma)) \sinh^2(2\sigma) ds.
\]

\[
-2 \int_{a_1}^{b_1} \sigma^2_s (\phi^2 + \tau)(\phi^2 - \tau) ds
\]

\[
= -2 \int_{a_1}^{b_1} \sigma^2_s (\tau \exp(2\sigma) + \tau)(\tau \exp(2\sigma) - \tau) ds
\]

\[
= -2\tau^2 \int_{a_1}^{b_1} \sigma^2_s 4 \sinh(\sigma) \cosh(\sigma) \exp(2\sigma) ds
\]

\[
= -2\tau^2 \int_{a_1}^{b_1} \sigma^2_s 2 \sinh(2\sigma) \exp(2\sigma) ds
\]

\[
= -2\tau^2 \int_{a_1}^{b_1} \sigma^2_s \sinh^2(2\sigma) ds
\]

\[
= -2\tau^2 \int_{a_1}^{b_1} (1 - 4\tau \cosh^2(\sigma)) \sinh^2(2\sigma) ds.
\]

So we proved that

\[
\int_{a_1}^{b_1} \phi^2 \phi^{-2} \dot{\psi} (\phi \ddot{\phi} + \dot{\phi}^2 - 2(\phi^2 + \tau)(\phi^2 - \tau)) d\psi = 0.
\]

\[\square\]
B The proof of Lemma 3.3

Consider

\[
\frac{\dot{\psi}^3}{\phi} \int_x \eta = \frac{\dot{\psi}^3}{\phi} \left( -\frac{\dot{\psi}^3}{\phi} \frac{\partial^2 \eta}{\partial \phi^2} - \frac{1}{\varepsilon} \left( \frac{\dot{\psi}^3}{\phi^3} + 2(\phi^2 - \tau_0) \frac{\dot{\phi}}{\phi^2} \right) \frac{\partial \eta}{\partial \phi} \right)
\]

\[
- \phi^{-2}(2\dot{\phi} + 2\ddot{\phi} \frac{\phi^2}{\phi} + \frac{\dot{\psi}^3}{\phi} \frac{\partial \phi}{\partial \phi}) - 2\dot{\phi}^2 - \phi^2(\phi^2 - \tau_0) - 2\phi \dot{\phi}^2)R(\eta, X_0)X_0
\]

\[
= - \frac{\partial^2 \eta}{\partial y^0} \mid_{y_0(x_0)} - \frac{\dot{\psi}^3}{\phi^3} \left( 2\dot{\phi} + 2\ddot{\phi} \frac{\phi^2}{\phi} + \frac{\dot{\psi}^3}{\phi} \frac{\partial \phi}{\partial \phi} \right) - 2\dot{\phi}^2 - \phi^2(\phi^2 - \tau_0) - 2\phi \dot{\phi}^2)R(\eta, X_0)X_0\mid_{x_0}
\]

\[
= \frac{2}{3} \varepsilon \dot{\phi} \hat{R}(Y_0, X_0, Y, Y_0)\mid_{x_0}
\]

where \( \frac{\partial}{\partial y^0} = \frac{\dot{\psi}^3}{\phi^3} \frac{\partial}{\partial x^0} \). We denote \( x_0 = x_{0,\varepsilon}(y_{0,\varepsilon}(x_0)) \). The aim is to prove

\[
\| \eta \|_{C_{y^0}^{\varepsilon}} \leq C \varepsilon^2
\]

\[
\| \eta \|_{C_{y^0}^{1,\varepsilon}} \leq C \varepsilon^2
\]

\[
\| \eta \|_{C_{y^0}^{1,\alpha}} \leq C \varepsilon
\]

\[
\| \eta \|_{C_{y^0}^{2,\alpha}} \leq C \varepsilon^{1-\alpha}
\]

Proof. We know the average of \( \Psi_1(\psi) \) in \( y_0 \) coordinate is

\[
\int_{y_0}^b \frac{\dot{\psi}^3}{\phi^3} (2\dot{\phi} + 2\ddot{\phi} \frac{\phi^2}{\phi} + \frac{\dot{\psi}^3}{\phi} \frac{\partial \phi}{\partial \phi} - 2\dot{\phi}^2)dy_0
\]

\[
= \int_{y_1}^{b_1} \frac{\dot{\psi}^3}{\phi^3} (2\dot{\phi} + 2\ddot{\phi} \frac{\phi^2}{\phi} + \frac{\dot{\psi}^3}{\phi} \frac{\partial \phi}{\partial \phi} - 2\dot{\phi}^2)dy
\]

\[
= I_2.
\]
Lemma B.1. For fixed $y_0$, $G(y_0, z_0)$ have the following properties

1. $G(y_0, z_0) \in C^{0,1}_{\Gamma \setminus z_0}$, $\|G(y_0, z_0)\|_{C^{0,1}_{\Gamma \setminus z_0}} \leq C$. $C$ does not depends on $y_0$.

2. $G(y_0, z_0) \in C^{\infty}(\Gamma \setminus z_0)$.

Proof. For simplicity the reader can regard this ODE system as a single ODE. For example regard $\hat{R}(\Upsilon_\theta, X_0, Y, Y_\theta)$ as a function and $G(y_0, z_0)$ as the a function for fixed $y_0$. The argument for the ODE system is essentially the same thing.

From $f_A$ is invertible, we can find a solution $\eta_1$ for

$$-\frac{\partial^2 \eta_1}{\partial y_0^2} - I_2 R(\eta_1, X_0)X_0 = \frac{2}{3} \varepsilon \phi \hat{R}(\Upsilon_\theta, X_0, Y, Y_\theta).$$

Suppose $G(y_0, z_0)$ is the Green function for the operator $I_1^{-2} f_A(I_1^{-2} f_A$ acts on $z_0)$, i.e.

$$I_1^{-2} f_A G(y_0, z_0) = \delta_{y_0}^I.$$
we have, apart from \( y_0 \), \( G(y_0, z_0) \) is a \( C^\infty \) function of \( z_0 \). Note that in dimension 1, we have \( \delta_{y_0} \in W^{-\frac{2}{3}, q} \) for fixed \( q_0 < 2 \). So

\[
\| G(y_0, z_0) \|_{C^0} \leq C \| G(y_0, z_0) \|_{W^{\frac{1}{4}, q}} \leq C \| \delta_{y_0} \|_{W^{-\frac{1}{2}, q_0}} \leq C(q_0).
\]

From the equation

\[
\mathcal{J}_A G(y_0, z_0) = I_1^2 \delta_{y_0}^I
\]

we know, apart from \( y_0 \), \( \| G(y_0, z_0) \|_{C^2(\Gamma \setminus y_0)} \leq C \| G(y_0, z_0) \|_{C^0(\Gamma)} \leq C(q_0) \). So \( G_{z_0}(y_0, y_0 - 0), G_{z_0}(y_0, y_0 + 0) \) exist. But they are not equal because at \( y_0 \), the right hand side of the ODE is Dirac function \( I_1^2 \delta_{y_0}^I \).

Consider

\[
\eta_1(y_0) = \int_{\Gamma} G(y_0, z_0) \frac{2}{3} \varepsilon \tilde{\Phi} R(\mathbf{Y}_\theta, X_0, \mathbf{Y}, \mathbf{Y}_\theta) dz_0,
\]

(64)

Regard \( \phi \) as a function of \( z_0 \). Locally it is average 0. So we have

\[
\eta_1(y_0) = \int_{\Gamma} G(y_0, z_0) \frac{2}{3} \varepsilon \tilde{\Phi} R(\mathbf{Y}_\theta, X_0, \mathbf{Y}, \mathbf{Y}_\theta) d\chi(z_0),
\]

(65)

where \( d\chi(z_0) = \phi dz_0 \). We know \( \chi(z_0) \) is also periodic because \( \phi \) is locally average 0. \( \chi(z_0) \) is decided up to a constant. So we can choose this constant properly so that \( \chi(z_0) \) is also average 0 locally. This special \( \chi(z_0) \) satisfies

\[
\| \chi(z_0) \|_{C^0} \leq C \varepsilon.
\]

So

\[
\eta_1(y_0) = -\frac{2}{3} \varepsilon \int_{\Gamma} \chi(z_0)(G_{z_0}(y_0, z_0)\tilde{\Phi} R(\mathbf{Y}_\theta, X_0, \mathbf{Y}, \mathbf{Y}_\theta)
+ G(y_0, z_0)(\tilde{\Phi} R(\mathbf{Y}_\theta, X_0, \mathbf{Y}, \mathbf{Y}_\theta))_{z_0}) dz_0,
\]

\[
= -\frac{2}{3} \varepsilon \int_{\Gamma} (G_{z_0}(y_0, z_0)\tilde{\Phi} R(\mathbf{Y}_\theta, X_0, \mathbf{Y}, \mathbf{Y}_\theta)
+ G(y_0, z_0)(\tilde{\Phi} R(\mathbf{Y}_\theta, X_0, \mathbf{Y}, \mathbf{Y}_\theta))_{z_0}) d\chi_1(z_0)
\]

\[
= \frac{2}{3} \varepsilon \int_{\Gamma} \chi_1(z_0)(G_{z_0 z_0}(y_0, z_0)\tilde{\Phi} R(\mathbf{Y}_\theta, X_0, \mathbf{Y}, \mathbf{Y}_\theta)
+ 2G(y_0, z_0)(\tilde{\Phi} R(\mathbf{Y}_\theta, X_0, \mathbf{Y}, \mathbf{Y}_\theta))_{z_0}
+ G(y_0, z_0)(\tilde{\Phi} R(\mathbf{Y}_\theta, X_0, \mathbf{Y}, \mathbf{Y}_\theta))_{z_0 z_0}) dz_0
\]

where \( d\chi_1(z_0) = \chi(z_0)dz_0 \) and as before \( \chi_1(z_0) \) is locally average 0 and

\[
\| \chi_1(z_0) \|_{C^0} \leq C \varepsilon^2.
\]
Note that $G_{z_0 z_0}(y_0, z_0) = -I_1 \delta_{y_0} - R(G, X_0)X_0$, so we have
\[
\frac{2}{3} \varepsilon \left| \int \chi_1(z_0)G_{z_0 z_0}(y_0, z_0)\Pi R(\Upsilon_\theta, X_0, \Upsilon, \Upsilon_\theta) dz_0 \right|
\leq \frac{2}{3} \varepsilon (C\varepsilon^2) = \frac{2}{3} C\varepsilon^3.
\]
And because $G(y_0, z_0), G(y_0, z_0)z_0$ are bounded, so we have
\[
\frac{2}{3} \varepsilon \left| \int \chi_1(z_0)G(y_0, z_0)(\Pi R(\Upsilon_\theta, X_0, \Upsilon, \Upsilon_\theta))z_0 dz_0 \right|
\leq \frac{2}{3} \varepsilon (C\varepsilon^2) = \frac{2}{3} C\varepsilon^3.
\]
So
\[
|\eta_1(y_0)| \leq C\varepsilon^2
\]
where $C$ does not depends on $y_0$. So
\[
\|\eta_1\|_{C^0} \leq C\varepsilon^2
\]
Take derivative with respect to $y_0$ on both sides of (65) and we can get
\[
\|\eta_1\|_{C^1_{\psi_0}} \leq C\varepsilon^2, \|\eta_1\|_{C^2_{\psi_0}} \leq C\varepsilon.
\]
Consider
\[
-\frac{\partial^2 (\eta_2 - \eta_1)}{\partial y_0^2} - I_2 R(\eta_2 - \eta_1, X_0)X_0
= (\Psi_1(\psi) - I_2)R(\eta_1, X_0)X_0 + O(\varepsilon^2)L(\eta_1).
\]
Note that $(\Psi_1(\psi) - I_2)$ is locally average 0 as $\dot{\phi}$. From the estimate of $\eta_1$, by using similar argument we can get
\[
\|\eta_2 - \eta_1\|_{C^0} \leq C\varepsilon^3
\]
\[
\|\eta_2 - \eta_1\|_{C^1_{\psi_0}} \leq C\varepsilon^3
\]
\[
\|\eta_2 - \eta_1\|_{C^2_{\psi_0}} \leq C\varepsilon^2.
\]
Generally if we consider
\[
-\frac{\partial^2 (\eta_{k+1} - \eta_k)}{\partial y_0^2} - I_2 R(\eta_{k+1} - \eta_k, X_0)X_0 = (\Psi_1(\psi) - I_2)R(\eta_k - \eta_{k-1}, X_0)X_0 + O(\varepsilon^2)L(\eta_k - \eta_{k-1})
\]

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we can get
\[
\|\eta_{k+1} - \eta_k\|_{C^0} \leq C\varepsilon^{k+2}
\]
\[
\|\eta_{k+1} - \eta_k\|_{C^1_{\nu_0}} \leq C\varepsilon^{k+2}
\]
\[
\|\eta_{k+1} - \eta_k\|_{C^2_{\nu_0}} \leq C\varepsilon^{k+1}.
\]

Note that \(\hat{\eta} = \lim_{k \to \infty} \eta_k\) satisfies
\[
-\frac{\partial^2 \hat{\eta}}{\partial y_0^2} - \Psi_1(\psi) R(\hat{\eta}, X_0)X_0 = \frac{2}{3} \varepsilon \dot{\phi} \Pi R(Y_0, X_0, Y_0) + O(\varepsilon^2)L(\eta).
\]

So \(\hat{\eta}\) satisfies
\[
\|\hat{\eta}\|_{C^0} \leq C\varepsilon^2
\]
\[
\|\hat{\eta}\|_{C^1_{\nu_0}} \leq C\varepsilon^2
\]
\[
\|\hat{\eta}\|_{C^2_{\nu_0}} \leq C\varepsilon^1.
\]

So we have
\[
\|\hat{\eta}\|_{C^1_{\nu_0}} \leq C\varepsilon^2.
\]

From the equation in \(x_0\) coordinate we have
\[
\|\hat{\eta}\|_{C^2_{\nu_0}} \leq C\varepsilon
\]
\[
\|\hat{\eta}\|_{C^2_{\nu_0}} \leq C\varepsilon^{1-\alpha}.
\]

\[\square\]

C  The proof of Lemma 3.11

The proof of Lemma 3.11

\[
\begin{pmatrix}
\frac{\beta_1(\psi)}{\partial \psi} & \frac{\beta_2(\psi)}{\partial \psi} \\
\frac{\beta_1(\psi)}{\partial \psi} & \frac{\beta_2(\psi)}{\partial \psi}
\end{pmatrix}
\]
\[
= \begin{pmatrix}
1 + e_{11}^1 & e_{12}^1 \\
\psi + e_{21} & 1 + e_{22}^1
\end{pmatrix} \cdots \begin{pmatrix}
1 + e_{11}^2 & e_{12}^2 \\
\psi^1 + e_{21} & 1 + e_{22}^2
\end{pmatrix}
\]
\[
= \begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix}
\]

where \(|e_{kl}^i| \leq C(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|)\). Note that
\[
\begin{pmatrix}
1 & f_{12}^1 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
1 + e_{11}^1 & e_{12}^1 \\
\psi^1 + e_{21} & 1 + e_{22}^2
\end{pmatrix} = \begin{pmatrix}
1 + \tilde{e}_{11}^1 & 0 \\
\psi^1 + \tilde{e}_{21} & 1 + \tilde{e}_{22}^2
\end{pmatrix}
\]

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where $f^i_{12} = -\frac{e_{11}^i}{1 + e_{22}^i}$. We have $|\tilde{e}_{kl}^i| \leq C(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2 \|\mu\|_{C^0} + \varepsilon^2 |\omega|)$ and

$$
\begin{pmatrix}
1 + e_{11}^i & e_{12}^i \\
\psi^1 + e_{21}^i & 1 + e_{22}^i
\end{pmatrix} = 
\begin{pmatrix}
1 & -f^i_{12} \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 + e_{11}^i & 0 \\
\psi^1 + e_{21}^i & 1 + e_{22}^i
\end{pmatrix}
$$

And note that

$$
\begin{pmatrix}
1 + e_{11}^i & e_{12}^i \\
\psi^1 + e_{21}^i & 1 + e_{22}^i
\end{pmatrix} = 
\begin{pmatrix}
1 & -f^i_{12} \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 + e_{11}^i & 0 \\
\psi^1 + e_{21}^i & 1 + e_{22}^i
\end{pmatrix}
$$

where $|e_{ij}^i| \leq 2C(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2 \|\mu\|_{C^0} + \varepsilon^2 |\omega|)$. And

$$
\begin{pmatrix}
1 + e_{11}^i & e_{12}^i \\
\psi^1 + e_{21}^i & 1 + e_{22}^i
\end{pmatrix} = 
\begin{pmatrix}
1 & -f^i_{12} \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 + e_{11}^i & 0 \\
\psi^1 + e_{21}^i & 1 + e_{22}^i
\end{pmatrix}
$$

where $|f^i_{12}|, |\tilde{e}_{kl}^i| \leq 2C(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2 \|\mu\|_{C^0} + \varepsilon^2 |\omega|)$. We go on like this. At last we have

$$
\begin{pmatrix}
A_{11}^i & A_{12}^i \\
A_{21}^i & A_{22}^i
\end{pmatrix} = 
\begin{pmatrix}
1 & -f^i_{12} \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 + e_{11}^i & 0 \\
\psi^1 + e_{21}^i & 1 + e_{22}^i
\end{pmatrix} \cdots 
\begin{pmatrix}
1 + e_{11}^i & 0 \\
\psi^1 + e_{21}^i & 1 + e_{22}^i
\end{pmatrix}
$$

where

$$ |-f^i_{12}|, |\tilde{e}_{kl}^i| \leq \frac{C}{\varepsilon}(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2 \|\mu\|_{C^0} + \varepsilon^2 |\omega|).$$

If we change $-f^i_{12}$ and $\tilde{e}_{kl}^i$ to $E_r = \frac{C}{\varepsilon}(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2 \|\mu\|_{C^0} + \varepsilon^2 |\omega|)$ then all $A_{kl}^i$ will be bigger. Note that

$$
\begin{pmatrix}
1 + E_r & 0 \\
\psi^1 + E_r & 1 + E_r
\end{pmatrix} \cdots 
\begin{pmatrix}
1 + E_r & 0 \\
\psi^1 + E_r & 1 + E_r
\end{pmatrix} = 
\begin{pmatrix}
1 + E_r & 0 \\
\psi^1 + E_r & 1 + E_r
\end{pmatrix}
\begin{pmatrix}
(1 + E_r)^i & 0 \\
i(\psi^1 + E_r)(1 + E_r)^{i-1} & (1 + E_r)^i
\end{pmatrix}
$$

Note that $i \leq \frac{C}{\varepsilon}$ and $E_r = \frac{C}{\varepsilon}(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2 \|\mu\|_{C^0} + \varepsilon^2 |\omega|) \leq \varepsilon(C + C_1 + C_2 + C_3)$ so

$$(1 + E_r)^i \leq \exp(C + C_1 + C_2 + C_3)$$

$$i(\psi^1 + E)r(1 + E_r)^{i-1} \leq \frac{1}{\varepsilon} \exp(C + C_1 + C_2 + C_3)$$

In the same way if we change $-f^i_{12}$ all $\tilde{e}_{kl}^i$ to $-E_r$ then all $A_{kl}^i$ will be smaller. Note that

$$
\begin{pmatrix}
1 - E_r & 0 \\
\psi^1 - E_r & 1 + E_r
\end{pmatrix} \cdots 
\begin{pmatrix}
1 - E_r & 0 \\
\psi^1 - E_r & 1 - E_r
\end{pmatrix} = 
\begin{pmatrix}
1 - E_r & 0 \\
\psi^1 - E_r & 1 - E_r
\end{pmatrix}
\begin{pmatrix}
(1 - E_r)^i & 0 \\
i(\psi^1 - E_r)(1 - E_r)^{i-1} & (1 - E_r)^i
\end{pmatrix}
$$

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and

$$(1 - Er)^i \geq \exp(-(C + C_1 + C_2 + C_3))$$

$$i(\psi^1 - Er)(1 - E_r)^{i-1} \geq \frac{1}{\varepsilon} \exp(-(C + C_1 + C_2 + C_3))$$

So we know

$$\exp(-(C + C_1 + C_2 + C_3)) \leq A_{i_{22}} \leq \exp(C + C_1 + C_2 + C_3),$$

$$\exp(-(C + C_1 + C_2 + C_3)\varepsilon^{-1}) \leq A_{i_{21}} \leq \exp(C + C_1 + C_2 + C_3)\varepsilon^{-1},$$

$$|A_{i_{12}}| \leq C(C_1, C_2, C_3)\varepsilon \exp(C + C_1 + C_2 + C_3).$$

There is another way we can do this

$$\begin{pmatrix}
A_{i_{11}} & A_{i_{12}} \\
A_{i_{21}} & A_{i_{22}}
\end{pmatrix}
= \begin{pmatrix}
1 + e_{11}^1 & e_{12}^1 \\
\psi^1 + e_{21}^1 & 1 + e_{22}^1
\end{pmatrix} \cdots \begin{pmatrix}
1 + e_{11}^1 & e_{12}^1 \\
\psi^1 + e_{21}^1 & 1 + e_{22}^1
\end{pmatrix}
= \begin{pmatrix}
1 + e_{11}^1 & e_{12}^1 \\
\psi^1 + e_{21}^1 & 1 + e_{22}^1
\end{pmatrix}
= \begin{pmatrix}
1 + e_{11}^1 & e_{12}^1 \\
\psi^1 + e_{21}^1 & 1 + e_{22}^1
\end{pmatrix}
\cdots \begin{pmatrix}
1 + e_{11}^1 & e_{12}^1 \\
\psi^1 + e_{21}^1 & 1 + e_{22}^1
\end{pmatrix}
\begin{pmatrix}
1 + e_{11}^1 & e_{12}^1 \\
\psi^1 + e_{21}^1 & 1 + e_{22}^1
\end{pmatrix}
= \cdots$$

And we can prove that

$$\exp(-(C + C_1 + C_2 + C_3)) \leq A_{i_{11}} \leq \exp(C + C_1 + C_2 + C_3).$$

References

[1] R. Mazzeo and F. Pacard. Constant mean curvature surfaces with delauay ends. Commun. Anal. Geom., 9(1):169–237, 2001.

[2] R. Mazzeo and F. Pacard. Foliations by constant mean curvature tubes. Commun. Anal. Geom., 13(4):633–670, 2005.