Weighted zero-sum problems over $C^r_3$

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Abstract. Let $C_n$ be the cyclic group of order $n$ and set $s_A(C^r_n)$ as the smallest integer $\ell$ such that every sequence $S$ in $C^r_n$ of length at least $\ell$ has an $A$-zero-sum subsequence of length equal to $\exp(C^r_n)$, for $A = \{-1, 1\}$. In this paper, among other things, we give estimates for $s_A(C^r_3)$, and prove that $s_A(C^r_3) = 9$, $s_A(C^r_3) = 21$ and $41 \leq s_A(C^r_3) \leq 45$.

Introduction

Let $G$ be a finite abelian group (written additively), and $S$ be a finite sequence of elements of $G$ and of length $m$. For simplicity we are going to write $S$ in a multiplicative form

$$S = \prod_{i=1}^{\ell} g_i^{v_i},$$

where $v_i$ represents the number of times the element $g_i$ appears in this sequence. Hence $\sum_{i=1}^{\ell} v_i = m$.

Let $A = \{-1, 1\}$. We say that a subsequence $a_1 \cdots a_s$ of $S$ is an $A$-zero-sum subsequence, if we can find $\epsilon_1, \ldots, \epsilon_s \in A$ such that

$$\epsilon_1 a_1 + \cdots + \epsilon_s a_s = 0 \text{ in } G.$$
Here we are particularly interested in studying the behavior of \( s_A(G) \) defined as the smallest integer \( \ell \) such that every sequence \( S \) of length greater than or equal to \( \ell \), satisfies the condition \( (s_A) \), which states that there must exist an \( A \)-zero-sum subsequence of \( S \) of length \( \exp(G) \) (the exponent of \( G \)).

For this purpose, two other invariants will be defined to help us in this study. Thus, define \( \eta_A(G) \) as the smallest integer \( \ell \) such that every sequence \( S \) of length greater than or equal to \( \ell \), satisfies the condition \( (\eta_A) \), which says that there exists an \( A \)-zero-sum subsequence of \( S \) of length at most \( \exp(G) \). Define also \( g_A(G) \) as the smallest integer \( \ell \) such that every sequence \( S \) of distinct elements and of length greater than or equal to \( \ell \), satisfies the condition \( (g_A) \), which says that there must exist an \( A \)-zero-sum subsequence of \( S \) of length \( \exp(G) \).

The study of zero-sums is a classical area of additive number theory and goes back to the works of Erdős, Ginzburg and Ziv [6] and Harborth [9]. A very thorough survey up to 2006 can be found on Gao-Geroldinger [7], where applications of this theory are also given.

In [8], Grynkiewicz established a weighted version of Erdős-Ginzburg-Ziv theorem, which introduced the idea of considering certain weighted subsequence sums, and Thangadurai [13] presented many results on a weighted Davenport’s constant and its relation to \( s_A \).

For the particular weight \( A = \{-1,1\} \), the best results are due to Adhikari et al [1], where it is proved that \( s_A(C_n) = n + \lfloor \log_2 n \rfloor \) (here \( C_n \) is a cyclic group of order \( n \)) and Adhikari et al [2], where it is proved that \( s_A(C_n \times C_n) = 2n - 1 \), when \( n \) is odd. Recently, Adhikari et al proved that \( s_A(G) = \exp(G) + \log_2 |G| + O(\log_2 \log_2 |G|) \) when \( \exp(G) \) is even and \( \exp(G) \rightarrow +\infty \) (see [3]).

The aim of this paper is to give estimates for \( s_A(C_n^r) \), where as usual \( C_n^r = C_n \times \cdots \times C_n \) (\( r \) times), and here are our results.

**Theorem 1.** Let \( A = \{-1,1\}, n > 1 \) odd and \( r \geq 1 \). If \( n = 3 \) and \( r \geq 2 \), or \( n \geq 5 \) then

\[
2^{r-1}(n-1) + 1 \leq s_A(C_n^r) \leq (n^r - 1) \left( \frac{n-1}{2} \right) + 1.
\]

For the case of \( n = 3 \) we present a more detailed study and prove

**Theorem 2.** Let \( A = \{-1,1\} \) and \( r \geq 5 \).

(i) If \( r \) is odd then

\[
s_A(C_3^r) \geq 2^r + 2 \left( \frac{r-1}{2} \right) - 1.
\]
(ii) If $r$ is even, with $m = \left\lfloor \frac{3r-4}{4} \right\rfloor$, then

(a) If $r \equiv 2 \pmod{4}$, then $s_A(C_r^n) \geq 2 \sum_{1 \leq j \leq m} \binom{r}{j} + 2\left(\frac{r-2}{2}\right) + 1$, where $j$ takes odd values.

(b) If $r \equiv 0 \pmod{4}$, then $s_A(C_r^n) \geq 2 \sum_{1 \leq j \leq m} \binom{r}{j} + \binom{r}{2} + 1$, where $j$ takes odd values.

It is simple to check that $s_A(C_3) = 4$, and it follows from Theorem 3 in [2] that $s_A(C_3^2) = 5$. Our next result presents both exact values of $s_A(C_r^n)$, and $r = 3, 4$ as well as estimates for $s_A(C_r^n)$, $r = 3, 4, 5$, for all $a \geq 1$.

**Theorem 3.** Let $A = \{-1, 1\}$. Then

(i) $s_A(C_3^3) = 9$, $s_A(C_3^4) = 21$, $41 \leq s_A(C_3^5) \leq 45$

(ii) $s_A(C_3^{3a}) = 4 \times 3^a - 3$, for all $a \geq 1$

(iii) $8 \times 3^a - 7 \leq s_A(C_3^{4a}) \leq 10 \times 3^a - 9$, for all $a \geq 1$

(iv) $16 \times 3^a - 15 \leq s_A(C_3^{5a}) \leq 22 \times 3^a - 21$, for all $a \geq 1$

1. Relations between the invariants $\eta_A$, $g_A$ and $s_A$

We start by proving the following result.

**Lemma 1.** For $A = \{-1, 1\}$, we have

(i) $\eta_A(C_3) = 2$, $g_A(C_3) = 3$ and $s_A(C_3) = 4$, and

(ii) $\eta_A(C_3^n) \geq r + 1$ for any $r \in \mathbb{N}$.

**Proof.** The proof of item (i) is very simple and will be omitted. For (ii), the proof follows from the fact that the sequence $e_1 e_2 \cdots e_r$ with $e_j = (0, \ldots, 1, \ldots, 0)$, has no $A$-zero-sum subsequence.

**Proposition 1.** For $A = \{-1, 1\}$, we have $g_A(C_3^n) = 2\eta_A(C_3^n) - 1$.

**Proof.** The case $r = 1$ follows from Lemma 1. Let $S = \prod_{i=1}^{m} g_i$ of length $m = \eta_A(C_3^n) - 1$ which does not satisfy the condition ($\eta_A$). In particular $S$ has no $A$-zero-sum subsequences of length 1 and 2, that is, all elements of $S$ are nonzero and distinct. Now, let $S^*$ be the sequence $\prod_{i=1}^{m} g_i \prod_{i=1}^{m} (-g_i)$. Observe that $S^*$ has only distinct elements, since $S$ has no $A$-zero-sum subsequences of length 2. It is easy to see that any $A$-zero-sum of $S^*$ of length 3 is also an $A$-zero-sum of $S$, for $A = \{-1, 1\}$. Hence $g_A(C_3^n) \geq 2\eta_A(C_3^n) - 1$. 
Let $S$ be a sequence of distinct elements and of length $m = 2\eta_A(C^r_3) - 1$, and write

$$S = \prod_{i=1}^{t} g_i \prod_{i=1}^{t} (-g_i) \prod_{i=2t+1}^{m} g_i$$

where $g_r \neq -g_s$ for $2t+1 \leq r < s \leq m$. If $t = 0$, then $S$ has no $A$-zero-sum of length 2, and 0 can appear at most once in $S$. Let $S^*$ be the subsequence of all nonzero elements of $S$, hence $|S^*| = 2\eta_A(C^r_3) - 2 > \eta_A(C^r_3)$, for $r \geq 2$ (see Lemma 1(ii)), hence it must contain an $A$-zero-sum of length 3.

For the case $t \geq 1$, we may assume $g_j \neq 0$, for every $j = 2t+1, \ldots , m$ since otherwise, $g_l + (-g_l) + g_{j_0}$ is $A$-zero-sum subsequence of length 3. But now, either $t \geq \eta_A(C^r_3)$, so that $\prod_{i=1}^{t} g_i$ has an $A$-zero-sum of length 3, or $m - t \geq \eta_A(C^r_3)$, so that $\prod_{i=1}^{t} (-g_i) \prod_{i=2t+1}^{m} g_i$ has an $A$-zero-sum subsequence of length 3. \hfill \Box

Here we note that by the definition of these invariants and the proposition above, we have

$$s_A(C^r_3) \geq g_A(C^r_3) = 2\eta_A(C^r_3) - 1. \quad (1)$$

**Proposition 2.** For $A = \{-1, 1\}$, we have $s_A(C^r_3) = g_A(C^r_3)$, for $r \geq 2$.

*Proof.* From Theorem 3 in [2] we have $s_A(C^2_3) = 5$ and, on the other hand, the sequence $(1, 0)(0, 1)(2, 0)(0, 2)$ does not satisfy the condition $(g_A)$, hence $s_A(C^2_3) = g_A(C^2_3)$ (see (1)). From now on, let us consider $r \geq 3$.

Let $S$ be a sequence of length $m = s_A(C^r_3) - 1$ which does not satisfy the condition $(s_A)$. In particular $S$ does not contain three equal elements, since $3g = 0$. If $S$ contains only distinct elements, then it does not satisfy also the condition $(g_A)$, and then $m \leq g_A(C^r_3) - 1$, which implies $s_A(C^r_3) = g_A(C^r_3)$ (see (1)). Hence, let us assume that $S$ has repeated elements and write

$$S = \ell^2 F = \prod_{i=1}^{t} g_i^2 \prod_{j=2t+1}^{m} g_j \quad (2)$$

where $g_1, \ldots , g_t, g_{2t+1}, \ldots , g_m$ are distinct. If for some $1 \leq j \leq m$ we have $g_j = 0$, then the subsequence of all nonzero elements of $S$ has length at least equal to $s_A(C^r_3) - 3 \geq 2\eta_A(C^r_3) - 4 \geq \eta_A(C^r_3)$ for $r \geq 3$ (see Lemma 1 (ii)). Then it must have an $A$-zero-sum of length 2 or 3. And if the $A$-zero-sum is of length 2, together with $g_j = 0$ we would have an $A$-zero-sum of length 3 in $S$, contradicting the assumption that it does not satisfy the condition $(s_A)$. 

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Hence let us assume that all elements of $S$ are nonzero. Observe that we can not have $g$ in $E$ and $h$ in $F$ (see (2)) such that $h = -g$, for $g + g - h = 3g = 0$, an $A$-zero-sum of length 3. Therefore the new sequence

$$R = \prod_{i=1}^{t} g_i \prod_{i=1}^{t} (-g_i) \prod_{i=2t+1}^{m} g_i$$

has only distinct elements, length $m = s_A(C_3^n) - 1$, and does not satisfy the condition $(g_A)$. Hence $m \leq g_A(C_3^n) - 1$, and this concludes the proof according to (1).

\[\square\]

2. Proof of Theorem 1

2.1. The lower bound for $s_A(C_n^r)$

Let $e_1, \ldots, e_r$ be the elements of $C_n^r$ defined as $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$, and for every subset $I \subset \{1, \ldots, r\}$, of odd cardinality, define $e_I = \sum_{i \in I} e_i$ (e.g., taking $I = \{1, 3, r\}$, we have $e_I = (1, 0, 1, 0, \ldots, 0, 1)$), and let $\mathcal{I}_m$ be the collection of all subsets of $\{1, \ldots, r\}$ of cardinality odd and at most equal to $m$.

There is a natural isomorphism between the cyclic groups $C_n^r \cong (\mathbb{Z}/n\mathbb{Z})^r$, and this result here will be proved for $(\mathbb{Z}/n\mathbb{Z})^r$. Let $\phi : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ be the canonical group epimorphism, and define $\varphi : \mathbb{Z}^r \to (\mathbb{Z}/n\mathbb{Z})^r$ as $\varphi(a_1, \ldots, a_r) = (\phi(a_1), \ldots, \phi(a_r))$. If $S = g_1 \cdots g_m$ is a sequence over the group $\mathbb{Z}^r$, let us denote by $\varphi(S)$ the sequence $\varphi(S) = \varphi(g_1) \cdots \varphi(g_m)$ of same length over the group $(\mathbb{Z}/n\mathbb{Z})^r$.

Let $e_1^*, \ldots, e_r^*$ be the canonical basis (i.e., $e_j^* = (0, \ldots, 0, 1, 0, \ldots, 0)$) of the group $\mathbb{Z}^r$, and define, as above

$$e_I^* = \sum_{i \in I} e_i^*$$

Now consider the sequence

$$S = \prod_{I \in \mathcal{I}_r} (e_I^*)^{n-1},$$

of length $2^{r-1}(n - 1)$. We will prove that the corresponding sequence

$$\varphi(S) = \prod_{I \in \mathcal{I}_r} e_I^{n-1},$$

has no A-zero-sum subsequences of length $n$, which is equivalent to prove that given $A = \{-1, 1\}$ and any subsequence $R = g_1 \cdots g_n$ of $S$, it is not
possible to find \( \epsilon_1, \ldots, \epsilon_s \in A \) such that (with an abuse of notation)
\[
\epsilon_1 g_1 + \cdots + \epsilon_n g_n \equiv (0, \ldots, 0) \pmod{n}.
\]

Let us consider the sequence \( S = \phi(S) = \prod_{I \in \mathcal{J}} \phi I \), for some \( \phi \in \mathcal{J} \), for \( n = 3 \), we see that this does not satisfy the condition \((\eta_A)\). So \( \eta_A(C^r_n) \geq 2^{r-1} + 1 \) for any \( r \in \mathbb{N} \), which is an improvement of the item (ii) of the Lemma 1.

2.2. The upper bound for \( s_A(C^r_n) \)

Let us consider the set of elements of the group \( C^r_n \) as the union \( \{0\} \cup G^+ \cup G^- \), where if \( g \in G^+ \) then \(-g \in G^- \). And write the sequence \( S \) as
\[
S = 0^n \prod_{g \in G^+} (g^{v_g(S)}(-g)^{v_g(S)}).
\]
First observe that if for some $g$, $v_g(S) + v_{-g}(S) \geq n$, then we can find a subsequence $R = c_1 \cdots c_n$ of $S$, which is an $A$-zero-sum, for $A = \{-1, 1\}$, and any sum of $n$ equal elements is equal to zero in $C_n$. Now consider $m \geq 1$ and $m + v_{g}(S) + v_{-g}(S) > n$, then we can find a subsequence $R = h_1 \cdots h_l$ of $S$ of even length $t \geq n - m$ with $h_j \in \{-g, g\}$. Since $A = \{-1, 1\}$, this is an $A$-zero-sum. Hence, the subsequence $T = 0^{m^*}R$ ($m^* \leq m$) of $S$ is an $A$-zero-sum of length $n$.

Thus assume that, for every $g$ in $S$ we have $v_g(S) + v_{-g}(S) \leq n - m$, which gives

$$|S| \leq \begin{cases} \frac{m + n^{r-1}}{2}(n - m) & \text{if } m > 0 \text{ even} \\ m - 1 + \frac{n^{r-1}}{2}(n - m) & \text{if } m > 0 \text{ odd} \\ \frac{n^{r-1}}{(n - 1)} & \text{if } m = 0, \end{cases}$$

for $|G^+| = \frac{n^{r-1}}{2}$. We observe than in the case $m$ even $m + \frac{n^{r-1}}{2}(n - m) \leq 2 + \frac{n^{r-1}}{2}(n - 2) \leq 2 + \frac{n^{r-1}}{2}(n - 2) + \frac{n^{r-1}}{2} - 1$ and the equality only happens when $n = 3$ and $r = 1$. In any case, if $|S| \geq \frac{n^{r-1}}{2}(n - 1) + 1$, it has a subsequence of length $n$ which is an $A$-zero-sum.

**Remark 2.** For $n = 3$, the upper bound for $s_A(C_3^n)$ can be improved using the result of Meshulam[12] as follows. According to Proposition 2, $s_A(C_3^n) = g_A(C_3^n)$ for $r \geq 2$, and it follows from the definition that $g_A(C_3^n) \leq g(C_3^n)$, where $g(C_3^n)$ is the invariant $g_A(C_3^n)$ with $A = \{1\}$. Now we use the Theorem 1.2 of [12] to obtain $s_A(C_3^n) = g_A(C_3^n) \leq g(C_3^n) \leq 2 \times 3^{r}/r$.

### 3. Proof of Theorem 2

Now we turn our attention to prove the following proposition.

**Proposition 3.** If $r > 3$ is odd and $A = \{-1, 1\}$ then $\eta_A(C_3^n) \geq 2^{r-1} + \binom{r-1}{\delta}$, where

$$\delta = \delta(r) = \begin{cases} \frac{(r-3)}{2} & \text{if } r \equiv 1 \pmod{4} \\ \frac{(r-5)}{2} & \text{if } r \equiv 3 \pmod{4}. \end{cases} \tag{6}$$

**Proof.** We will prove this proposition by presenting an example of a sequence of length $2^{r-1} + \binom{r-1}{\delta} - 1$ with no $A$-zero-sum subsequences of length smaller or equal to 3. Let $\ell = \binom{r-1}{\delta}$, and consider the sequence

$$S = \mathcal{E} \cdot \mathcal{G} = \left( \prod_{i \in \mathcal{I}_{r-2}} e_i \right) \cdot g_1 \cdots g_\ell,$$
with

\[ g_1 = (-1, -1, \ldots, -1, 1, 1, \ldots, 1) \]

\[ \vdots \]

\[ g_\ell = (-1, 1, \ldots, 1, -1, \ldots, -1) \],

where \( \epsilon_I \) and \( \mathcal{J}_{r-2} \) are defined in the beginning of section 2. Clearly \( S \) has no \( A \)-zero-sum subsequences of length 1 or 2 and also sum or difference of two elements of \( G \) will never give another element of \( G \), for no element of \( G \) has zero as one of its coordinates. Now we will consider \( \epsilon_s - \epsilon_t \), where \( \epsilon_s \) and \( \epsilon_t \) represent the \( \epsilon_I \)’s for which \( s \) coordinates are equal to 1 and \( t \) coordinates are equal to 1 respectively. Thus, we see that \( \epsilon_s - \epsilon_t \) will never be an element of \( G \) since it necessarily has either a zero coordinate or it has an odd number of 1’s and -1’s (and \( \delta + 1 \) is even).

Now, if for some \( s, t \) we would have

\[ \epsilon_s + \epsilon_t = g_i, \]

Then \( \epsilon_t, \epsilon_s \) would have \( \delta + 1 \) nonzero coordinates at the same positions (to obtain \( \delta + 1 \) coordinates -1’s). Hence we would need to have

\[ r + (\delta + 1) = s + t \]

Which is impossible since \( s + t \) is even and \( r + (\delta + 1) \) is odd, for \( \delta \) is odd in any of the two cases.

Thus, the only possible \( A \)-zero-sum subsequence of length 3 would necessarily include one element of \( E \) and two elements of \( G \).

Let \( v, w \) be elements of \( G \). Now it simple to verify that (the calculations are modulo 3) either \( v + w \) or \( v - w \) have two of their entries with opposite signs (for \( \delta(r) < (r - 1)/2 \) and hence either of them can not be added to an \( \pm \epsilon_I \) to obtain an \( A \)-zero-sum, since all its nonzero entries have the same sign.

**Proposition 4.** Let \( r > 4 \) be even, \( m = \left\lfloor \frac{3r-4}{4} \right\rfloor \) and \( A = \{-1, 1\} \). Then

\[ \eta_A(C_r^3) \geq \sum_{\substack{j=1 \atop j \text{ odd}}}^{m} \binom{r}{j} + \ell(r) + 1, \]

where

\[ \ell(r) = \begin{cases} \binom{r-2}{2} & \text{if } r \equiv 2 \pmod{4}, \\ \binom{r}{2}/2 & \text{if } r \equiv 0 \pmod{4}. \end{cases} \]
Proof. Consider the sequence \( K = g_1 \cdots g_\tau \) with
\[
g_1 = (-1, \ldots, -1, 1, 1, \ldots, 1)
\]
\[
\vdots
\]
\[
g_\tau = (1, 1, \ldots, 1, -1, \ldots, -1)
\]
where
\[
\tau = \begin{cases} \ell(r) & \text{if } r \equiv 2 \pmod{4} \\ 2\ell(r) & \text{if } r \equiv 0 \pmod{4} \end{cases}
\]
and \( \delta = \begin{cases} \frac{r-2}{2} & \text{if } r \equiv 2 \pmod{4} \\ \frac{r}{2} & \text{if } r \equiv 0 \pmod{4} \end{cases} \)
and rearrange the elements of the sequence \( K \), and write it as
\[
K = \prod_{i=1}^{\tau/2} g_i \prod_{i=1}^{\tau/2} (-g_i) = K^+K^-.
\]
It is simple to observe that if \( r \equiv 2 \pmod{4} \), then \( \tau = \ell \) and \( K^- = \emptyset \).

Now define the sequence
\[
S = \left( \prod_{I \in \mathcal{I}_m} \epsilon_I \right) G,
\]
where \( G = K \) if \( r \equiv 2 \pmod{4} \) or \( G = K^+ \) if \( r \equiv 0 \pmod{4} \), and \( m = \left\lfloor \frac{3r-4}{4} \right\rfloor \), a sequence of length \( |S| = \sum_{j=1}^{m} \binom{r}{j} + \ell(r) + 1 \).

The first important observation is that \( S \) has no \( A \)-zero-sum subsequences of length 1 or 2. And also sum or difference of two elements of \( G \) will never be another element of \( G \), for it necessarily will have a zero as coordinate. Also \( \epsilon_I - \epsilon_J \) will never be an element of \( G \) since it necessarily has either a zero coordinate or it has an odd number of 1's and -1's (and \( \delta \) is even). Now, if for some \( s, t \) (both defined as in the proof of the Proposition 3) we would have
\[
\epsilon_s + \epsilon_t = \pm g_j, \quad \text{for some } j
\]
then \( \epsilon_t, \epsilon_s \) would necessarily have \( \delta \) nonzero coordinates at the same positions (to obtain \( \delta \) coordinates -1's). But then
\[
s + t = r + \delta \geq \frac{3r-2}{2}, \quad \text{for any value of } \delta
\]
which is impossible since
\[ s + t \leq 2m \leq \frac{3r - 4}{2}. \]

Thus the only \( A \)-zero-sum subsequence of length 3 possible necessarily includes an element \( e_t \) and two elements of \( G \).

Let \( v, w \) elements of \( G \). First, observe that if they do not have \(-1\)'s in common positions, then \( v + w \) has an even amount of zeros and an even amount of \(-1\)'s (since \( r \) and \( \delta \) are both even), i.e., \( v + w \neq \pm e_I \). If we make \( v - w \) also have an even amount of nonzero coordinates, i.e., we haven’t \( \pm e_I \). Now, assuming that \( v, w \) have at least a \(-1\) in same position, it simple to verify that (the calculations are modulo 3) either \( v + w \) or \( v - w \) have two or more of their entries with opposite signs and hence either of them can not be added to an \( \pm e_I \) to obtain an \( A \)-zero-sum, since all its nonzero entries have the same sign. \( \square \)

Theorem 2 now follows from propositions 1, 2, 3 and 4.

4. Proof of Theorem 3

We start by proving the following proposition.

**Proposition 5.** For \( A = \{-1, 1\} \), we have

(i) \( \eta_A(C_3^2) = 3 \);
(ii) \( \eta_A(C_3^3) = 5 \);
(iii) \( \eta_A(C_3^4) = 11 \);
(iv) \( 21 \leq \eta_A(C_3^5) \leq 23 \).

**Proof.** By Propositions 1 and 2, we have that \( s_A(C_3^r) = g_A(C_3^r) = 2\eta_A(C_3^r) - 1 \), for \( r > 1 \), and by definition, we have \( g_A(C_3^r) \leq g(C_3^r) \) resulting in \( \eta_A(C_3^r) \leq \frac{g(C_3^r)+1}{2} \), for \( r > 1 \). It follows from

\[ g(C_3^2) = 5 \ (|10|), g(C_3^3) = 10, g(C_3^4) = 21 \ (|11|), g(C_3^5) = 46 \ (|5|), \]

that \( \eta_A(C_3^2) \leq 3, \eta_A(C_3^3) \leq 5, \eta_A(C_3^4) \leq 11 \) and \( \eta_A(C_3^5) \leq 23 \). It is easy to see that the sequences \((1, 0)(0, 1)\) and \((1, 0, 0)(0, 1, 0)(0, 0, 1)\) (1, 1, 1) has no \( A \)-zero-sum of length at most three, so \( \eta_A(C_3^2) = 3 \) and \( \eta_A(C_3^3) = 5 \). It is also simple to check that following sequences of lengths 10 and 20 respectively do not satisfy the condition (\( \eta_A \)):

\[
\begin{align*}
(1, 1, 0, 0) & \cdots (0, 0, 1, 1)(1, 1, 1, 0) \cdots (0, 1, 1, 1) \\
(1, 1, 0, 0, 0) & \cdots (0, 0, 0, 1, 1)(1, 1, 1, 0, 0) \cdots (0, 0, 1, 1, 1),
\end{align*}
\]

(7)
hence $\eta_A(C_3^4) = 11$ and $\eta_A(C_3^5) \geq 21$.  

Proposition 5 together with propositions 1 and 2 gives the proof of item (i) of Theorem 3. The proof of the remaining three items is given in Proposition 7 below.

Before going further, we need a slight modification of a result due to Gao et al for $A = \{1\}$ in [4]. Here we shall use it in the case $A = \{-1, 1\}$. The proof in this case is analogous to the original one, and shall be omit it.

**Proposition 6.** Let $G$ be a finite abelian group, $A = \{-1, 1\}$ and $H \leq G$. Let $S$ be a sequence in $G$ of length

$$m \geq (s_A(H) - 1) \exp(G/H) + s_A(G/H).$$

Then $S$ has an $A$-zero-sum subsequence of length $\exp(H) \exp(G/H)$. In particular, if $\exp(G) = \exp(H) \exp(G/H)$, then

$$s_A(G) \leq (s_A(H) - 1) \exp(G/H) + s_A(G/H).$$

**Proposition 7.** For $A = \{-1, 1\}$, we have

(i) $s_A(C_{3^a}^3) = 4 \times 3^a - 3$, for all $a \geq 1$;

(ii) $8 \times 3^a - 7 \leq s_A(C_{3^a}^4) \leq 10 \times 3^a - 9$, for all $a \geq 1$;

(iii) $16 \times 3^a - 15 \leq s_A(C_{3^a}^5) \leq 22 \times 3^a - 21$, for all $a \geq 1$.

**Proof.** It follows of (i) from Theorem 3 that $s_A(C_3^3) = 4 \times 3 - 3 = 9$. Now assume that $s_A(C_{3^a-1}^3) = 4 \cdot 3^{a-1} - 3$. Thus, Proposition 6 yields

$$s_A(C_{3^a}^3) \leq 3 \times (s_A(C_{3^a-1}^3) - 1) + s_A(C_3^3) \leq 4 \times 3^a - 3.$$  

On the other hand, Theorem 1 gives $s_A(C_{3^a}^3) \geq 4 \times 3^a - 3$, concluding the proof of (i).

Again by (i) from Theorem 3, we have that $s_A(C_3^4) = 10 \times 3 - 9 = 21$. Now, assume that $s_A(C_{3^a-1}^4) \leq 10 \cdot 3^{a-1} - 9$. It follows from Proposition 6 that

$$s_A(C_{3^a}^4) \leq 3 \times (s_A(C_{3^a-1}^4) - 1) + s_A(C_3^4) \leq 10 \times 3^a - 9.$$  

On the other hand, Theorem 1 gives the lower bound $s_A(C_{3^a}^4) \geq 8 \times 3^a - 7$, concluding the proof of (ii). The proof of item (iii) is analogous to the proof of item (ii), again using (i) of the Theorem 3 and Theorem 1.  

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