ON THE DEPTH $r$ BERNSTEIN PROJECTOR

ROMAN BEZRUKAVNIKOV, DAVID KAZHDAN, AND YAKOV VARSHAVSKY

Abstract. In this paper we prove an explicit formula for the Bernstein projector to representations of depth $\leq r$. As a consequence, we show that the depth zero Bernstein projector is supported on topologically unipotent elements and equals the restriction of the character of the Steinberg representation. As another application, we deduce that the depth $r$ Bernstein projector is stable. Moreover, for integral depths, our proof is purely local.

To Joseph Bernstein on his 70th birthday

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INTRODUCTION

Let $G$ be a reductive $p$-adic group. Recall that the Bernstein center $Z_G$ of $G$ is a commutative ring which plays a role in representation theory of $G$ similar to the role played by the center of the group ring of a finite group in representation theory of the group. Elements of $Z(G)$ can be thought of as invariant distributions on $G$. While
Bernstein center is an important tool in the structural theory of representations of $G$, known explicit formulas for its elements are rather rare. In this paper we provide explicit descriptions for some natural elements in $Z_G$.

Recall also that $Z_G$ admits a natural injective homomorphism to the ring of functions on the set $\text{Irr}(G)$ of irreducible smooth representations.

Fix a number $r \geq 0$ and consider the function $f_r$ on $\text{Irr}(G)$ such that $\phi_r(V) = 1$, if the depth of $V$ is $\leq r$, and $f_r(V) = 0$, otherwise. The main results of the paper describe the element $E_r \in Z_G$ for which the corresponding function on $\text{Irr}(G)$ equals $f_r$. (We call it the depth $r$ projector).

The first result (available only for $r = 0$) is an equality between $E_0$ and the restriction of the character of the Steinberg representation to the locus of topologically unipotent elements of $G$. This can be thought of as a $p$-adic group analogue of the well known fact that the character of the Steinberg representation of a finite Chevalley group restricted to the set of unipotent elements is proportional to the delta function of the unit element.

Let $\mathfrak{g}^*$ be the linear dual of the Lie algebra $\mathfrak{g}$ of $G$. Another description of $E_r$ links it to Fourier transform of the characteristic function of a certain subset of $\mathfrak{g}^*$. This formula fits naturally into the standard analogy between harmonic analysis on the group $G$ and on its Lie algebra $\mathfrak{g}$ (notice that under this analogy elements of $Z_G$ correspond to invariant distributions on $\mathfrak{g}$ whose Fourier transform is locally constant).

As a corollary of our description we show that $E_r$ is a stable distribution. This property of $E_r$ is suggested by the conjectural theory of $L$-packets and its relation to endoscopy for invariant distributions. The set $\text{Irr}(G)$ is conjectured to be partitioned into finite subsets called $L$-packets; among many expected properties of $L$-packets we mention the following: an element $E \in Z_G$ is a stable distribution if and only if the corresponding function on $\text{Irr}(G)$ is constant on $L$-packets. It is also expected that the set of irreducible representations of a given depth is a union of $L$-packets, thus the above conjectures imply that $E_r$ is a stable distribution; we prove this fact unconditionally.

This result also provides evidence for another conjecture which has the advantage of being a self-contained formal statement. The so called stable center conjecture asserts that the subspace of stable distributions in $Z_G$ is a subring, while it follows from our results that the space of stable distributions in $Z_G$ does contain a large subring: the linear span of the projectors $E_r$ $(r \geq 0)$.

This work is an outgrowth of a project described in [BKV] whose goal is to construct elements in $Z_G$ and more general invariant distributions of interest using $l$-adic sheaves on loop groups. Such a construction for $E_0$ (for split groups in positive characteristic) was presented in [BKV].
Though $l$-adic sheaves are not used in the present paper, our main technical result Theorem 1.6 was suggested by [BKV]. Namely, the $l$-adic sheaf counterpart of $E_0$ can be constructed by taking derived invariants of the affine Weyl group acting on the loop group version of the Springer sheaf. Moreover, using a standard resolution for the trivial representation of $W_{\mathrm{aff}}$ whose terms are indexed by standard parabolic subgroups therein, we get an explicit resolution for this sheaf. This leads to the formula for the corresponding function appearing in Theorem 1.6.

Acknowledgements. We thank Akshay Venkatesh whose question motivated us to rewrite a geometric formula from [BKV] in elementary terms. We also thank Gopal Prasad for stimulating conversations and Ju-Lee Kim and Allen Moy for useful discussions.

1. Statement of results

1.1. Notation. (a) Let $F$ be a local non-archimedean field of residual characteristic $p$, $F_{nr}$ be the maximal unramified extension, $\overline{F}$ be the algebraic closure of $F$, and $\text{val}_F$ be the valuation on $F$ such that $\text{val}_F(F^\times) = \mathbb{Z}$.

(b) Let $G$ be a connected reductive group over $F$, $G := G(F)$, and $\mathcal{X} = \mathcal{X}(G)$ be the reduced Bruhat–Tits building of $G$, viewed as a metric space, equipped with extra structure (see 2.1). To every pair $(x, r) \in \mathcal{X} \times \mathbb{R}_{\geq 0}$, Moy–Prasad [MP1, MP2] associate an open-compact subgroup $G_{x,r} \subset G$ (see 2.7).

1.2. Depth of a representation. (a) Let $R(G)$ be a category of smooth complex representations of $G$, and let $\text{Irr}(G)$ be the set of equivalence classes of irreducible objects of $R(G)$. To each $V \in R(G)$, Moy–Prasad associate a depth $r \in \mathbb{Q}_{\geq 0}$, which is defined to be the smallest $r \in \mathbb{R}$ such that $V_{G_{x,r}} \neq 0$ for some $x \in \mathcal{X}$. Actually, for our purposes slightly weaker results of DeBacker ([DB]) are sufficient.

(b) For every $r \in \mathbb{Q}_{\geq 0}$, we denote by $\text{Irr}(G)_{\leq r}$ (resp. $\text{Irr}(G)_{> r}$) the set of $V \in \text{Irr}(G)$ of depth $\leq r$ (resp. $> r$), and denote by $R(G)_{\leq r}$ (resp. $R(G)_{> r}$) the full subcategory of $R(G)$, consisting of representations $V$, all of whose irreducible subquotients belong to $\text{Irr}(G)_{\leq r}$ (resp. $\text{Irr}(G)_{> r}$).

(c) It follows from a combination of results of Bernstein [Be] and Moy–Prasad (or DeBacker) that for every $r \in \mathbb{Q}_{> 0}$ and $V \in R(G)$ there exists a unique direct sum decomposition $V = V_{\leq r} \oplus V_{> r}$ such $V_{\leq r} \in R(G)_{\leq r}$ and $V_{> r} \in R(G)_{> r}$. (We provide an alternative proof of this fact in 5.2).

1.3. The Bernstein center. (a) Let $Z_G$ be the algebra of endomorphisms of the identity functor $\text{End} \text{Id}_{R(G)}$, called the Bernstein center of $G$. In particular, for every $z \in Z_G$ and $V \in R(G)$, we are given an endomorphism $z|_V \in \text{End} V$.

(b) Let $\mathcal{H}(G)$ be the algebra of smooth measures with compact support on $G$. Then $\mathcal{H}(G)$ is a smooth representation of $G$ with respect to the left action. Then
the map $z \mapsto z|_{\mathcal{H}(G)}$ identifies $Z_G$ with the algebra $\text{End}_{\mathcal{H}(G) \otimes \mathcal{H}(G)^{op}} \mathcal{H}(G)$ of endomorphisms of $\mathcal{H}(G)$, commuting with the left and the right convolution.

(c) Note that for every $V \in \mathcal{R}(G)$ and $v \in V$ the map $h \mapsto h(v)$ defines a $G$-equivariant map $\mathcal{H}(G) \to V$. Therefore for every $h \in \mathcal{H}(G)$ and $z \in Z_G$ we have $z(h(v)) = z(h)(v)$.

(d) For every $z \in Z_G$ there exists a unique invariant distribution $E_z \in D^G(G)$ on $G$ such that $z(h) = E_z \ast h$ for every $h \in \mathcal{H}(G)$. Moreover, the map $z \mapsto E_z$ identifies $Z_G$ with the space of $E \in D^G(G)$ such that the convolution $E \ast h$ has compact support for every $h \in \mathcal{H}(G)$.

1.4. The Bernstein projector. (a) By 1.2(c), there exists an idempotent $\Pi_r \in Z_G$ such that for every object $V \in \mathcal{R}(G)$ the endomorphism $\Pi_r|_V$ is the projection $V \to V_{\leq r} \hookrightarrow V$. We call $\Pi_r$ the Bernstein projector to the depth $\leq r$ spectrum.

(b) Let $E_r$ be the invariant distribution on $G$, corresponding to $\Pi_r$ (see 1.3(d)).

A particular case of the stable center conjecture (see [BK]) asserts that the distribution $E_r$ is stable. The goal of this work is to give an explicit formula for the Bernstein projector $\Pi_r$, and to use this description to show the stability of $E_r$.

From now on we fix $m \in \mathbb{N}$ and $r \in \frac{1}{m} \mathbb{Z}_{\geq 0}$.

1.5. Notation. (a) Denote by $[\mathcal{X}]$ the set of open polysimplexes of $\mathcal{X}$ (see 2.12(a)), and by $[\mathcal{X}_m]$ the set of open polysimplexes of $\mathcal{X}$, obtained by "subdividing of each polysimplex $\sigma \in [\mathcal{X}]$ into $m^{\dim \sigma}$ smaller polysimplices" (see 2.12(c)).

(b) For each $\sigma \in [\mathcal{X}_m]$, we choose $x \in \sigma$ and define $G_{\sigma,r^+} := G_{x,r^+}$. Since $r \in \frac{1}{m} \mathbb{Z}$, the subgroup $G_{\sigma,r^+}$ does not depend on a choice of $x$ (see 2.12(d)).

(c) To every finite subset $\Sigma \subset [\mathcal{X}_m]$, we can associate an element

$$E^\Sigma_r = \sum_{\sigma \in \Sigma} (-1)^{\dim \sigma} \delta_{G_{\sigma,r^+}} \in \mathcal{H}(G),$$

where $\delta_{G_{\sigma,r^+}}$ is the Haar measure on $G_{\sigma,r^+}$ with total measure one.

(d) Denote by $\Theta_m$ the set of all non-empty finite convex subcomplexes $\Sigma \subset [\mathcal{X}_m]$ (see 3.1).

The following result provides an explicit formula for the projector $\Pi_r$.

**Theorem 1.6.** For every $V \in \mathcal{R}(G)$ and $v \in V$, the sequence $\{E^\Sigma_r(v)\}_{\Sigma \in \Theta_m} \subset V$ stabilizes, and $\Pi_r(v)$ equals the limiting value of $E^\Sigma_r(v)$, that is, $\Pi_r(v) = \lim_{\Sigma \in \Theta_m} E^\Sigma_r(v)$.

1.7. Strategy of the proof. Analysing the combinatorics of the Bruhat–Tits building, we show that for every $x \in \mathcal{X}$ and $s \in \mathbb{R}_{\geq 0}$ the sequence $\{E^\Sigma_r \ast \delta_{G_{x,s^+}}\}_{\Sigma \in \Theta_m}$ stabilizes. This implies that the sequence $\{E^\Sigma_r \ast h\}_{\Sigma \in \Theta_m}$ stabilizes for all $h \in \mathcal{H}(G)$, and that there exists a unique element of the Bernstein center $z \in Z_G$ such that $z(h) = \lim_{\Sigma \in \Theta_m} E^\Sigma_r \ast h$. 
Next, using 1.3 (c), for every $V \in \text{Irr}(G_{\leq r})$ and $v \in V$, the sequence $\{E^\Sigma_r(v)\}_{\Sigma \in \Theta_m}$ stabilizes, and we have $z(v) = \lim_{\Sigma \in \Theta_m} E^\Sigma_r(v)$. In particular, $z|_V = 0$ for every $V \in \text{Irr}(G)_{>r}$.

It remains to show that $z = \Pi_r$. By a theorem of Bernstein, we have to check that $z|_V = \text{Id}$ for every $V \in \text{Irr}(G)_{\leq r}$. Using 1.3 (c) again, it remains to show that $z(\delta_{G_{x,r^+}}) = \delta_{G_{x,r^+}}$ for every $x \in \mathcal{X}$. To prove this, we show a stronger assertion that $E^\Sigma_r \ast \delta_{G_{x,r^+}} = \delta_{G_{x,r^+}}$ for all $\Sigma \in \Theta_m$ such that $x \in \Sigma$.

1.8. Remark. Our argument also provides an alternative proof of the decomposition $V = V_{\leq r} \oplus V_{>r}$ from 1.2 (b), hence an alternative proof of an existence of the projector $\Pi_r$ (see 5.2).

Consider an open and closed subset $G_{r^+} := \bigcup_{x \in \mathcal{X}} G_{x,r^+} \subset G$ (see Lemma 7.4 or [ADB, Cor 3.7.21]). Notice that $G_{0^+}$ is what is sometimes called the set of topologically unipotent elements. Theorem 1.6 has the following consequence.

**Corollary 1.9.** (a) For every $f \in C^\infty_c(G)$ the sequence $\{E^\Sigma_r(f)\}_{\Sigma \in \Theta_m}$ stabilizes, and we have $E_r(f) = \lim_{\Sigma \in \Theta_m} E^\Sigma_r(f)$.

(b) The invariant distribution $E_r$ is supported on $G_{r^+}$.

As a further consequence, we get the following variant of the character formula of Meyer–Solleveld [MS].

**Corollary 1.10.** For every admissible $V \in \text{R}(G)_{\leq r}$ and every $h \in \mathcal{H}(G)$ we have

$$\text{Tr}(h, V) = \lim_{\Sigma \in \Theta_m} \left[ \sum_{\sigma \in \Sigma} (-1)^{\dim \sigma} \text{Tr}(\delta_{G_{x,r^+}} \ast h \ast \delta_{G_{x,r^+}}, V_{G_{x,r^+}}) \right].$$

1.11. Notation. Let $I^+$ be the pro-unipotent radical of an Iwahori subgroup $I$ of $G$, and let $\mu^{I^+}$ be the Haar measure on $G$ normalized by the condition that $\int_{I^+} \mu^{I^+} = 1$.

Since by Corollary 1.9 (b), the invariant distribution $E_0$ is supported on $G_{0^+}$, the following result describes $E_0$ in terms of the character $\chi_{St_G}$ of the Steinberg representation $St_G$ of $G$.

**Theorem 1.12.** We have the equality $E_0|_{G_{0^+}} = (\chi_{St_G}|_{G_{0^+}}) \cdot \mu^{I^+}$.

To prove this result, we compare the explicit formula Corollary 1.9 for $E_0$ with the corresponding formula of Meyer–Solleveld [MS] for $\chi_{St_G}$, generalizing formula of Schneider–Stuhler [SS].

Since the character of the Steinberg representation is known to be stable (see 7.3 (b)), we deduce from Theorem 1.12 the following corollary.
Corollary 1.13. The invariant distribution $E_0$ is stable.

1.14. The Moy–Prasad filtration for the Lie algebras. (a) Let $\mathfrak{g}$ be the Lie algebra of $G$, and let $\mathfrak{g}^*$ be the dual vector space. For every $(x,r) \in X \times \mathbb{R}_{\geq 0}$, Moy–Prasad define $\mathcal{O}$-lattices $\mathfrak{g}_{x,r} \subset \mathfrak{g}$ and $\mathfrak{g}^*_{x,-r} \subset \mathfrak{g}^*$ (see Corollary 3.4.3).

(b) As in the group case, for every $\sigma \in \mathcal{X}_m$ we define $\mathfrak{g}_{x,r+} := \mathfrak{g}_{x,r}$ for $x \in \sigma$ and for every $\Sigma \subset \Theta_m$, we associate an element

$$E_\Sigma^r = \sum_{\sigma \in \Sigma} (-1)^{\dim \sigma} \mu_{\mathfrak{g}_{x,r+}} \in \mathcal{H}(\mathfrak{g}).$$

Here $\mathcal{H}(\mathfrak{g})$ denotes the space of smooth measures with compact support on $\mathfrak{g}$, and $\mu_{\mathfrak{g}_{x,r+}}$ is the Haar measure on $\mathfrak{g}_{x,r+}$ with total measure one.

(c) Consider open-closed subsets $\mathfrak{g}_{r+} := \bigcup_{x \in \mathcal{X}} \mathfrak{g}_{x,r+} \subset \mathfrak{g}$ and $\mathfrak{g}^*_{-r} := \bigcup_{x \in \mathcal{X}} \mathfrak{g}^*_{x,-r} \subset \mathfrak{g}^*$ (see Lemma 7.4 or [ADB, Cor 3.4.3]), and denote by $1_{\mathfrak{g}^*_{-r}}$ the characteristic function of $\mathfrak{g}^*_{-r}$.

1.15. The Fourier transform. (a) Let $\mathcal{O} \subset \mathcal{C}$ be the ring of integers, let $\varpi \in \mathcal{O}$ be an uniformizer, and let $\psi : \mathcal{C} \to \mathbb{C}$ be an additive character, trivial on $(\varpi)$ but nontrivial on $\mathcal{O}$. Then $\psi$ gives rise to a Fourier transform $\mathcal{F} : \mathcal{H}(\mathfrak{g}^*) \to C^\infty_c(\mathfrak{g})$, where $\mathcal{H}(\mathfrak{g}^*)$ denotes the space of smooth measures with compact support on $\mathfrak{g}^*$. Explicitly, $\mathcal{F}(h)(a) = \int_{b \in \mathfrak{g}^*} \psi((b,a))h$ for every $h \in \mathcal{H}(\mathfrak{g}^*)$ and $a \in \mathfrak{g}$.

(b) By duality, $\mathcal{F}$ induces an isomorphism $\mathcal{F} : D^G(\mathfrak{g}) \cong \hat{C}^G(\mathfrak{g}^*)$ between the space of invariant distributions on $\mathfrak{g}$ and invariant generalized functions on $\mathfrak{g}^*$. Explicitly, $\mathcal{F}(E)(h) = E(\mathcal{F}(h))$ for every $E \in D^G(\mathfrak{g})$ and $h \in \mathcal{H}(\mathfrak{g}^*)$.

1.16. The Lie algebra analog of the center. (a) We denote by $Z_{\mathfrak{g}} \subset D^G(\mathfrak{g})$ the subspace of all $E$ such that the convolution $E \ast h$ has compact support for every $h \in \mathcal{H}(\mathfrak{g})$. Equivalently, $E \in D^G(\mathfrak{g})$ belongs to $Z_{\mathfrak{g}}$ if and only if the Fourier transform $\mathcal{F}(E) \in \hat{C}^G(\mathfrak{g})$ is locally constant.

(b) We set $\mathcal{E}_r := \mathcal{F}^{-1}(1_{\mathfrak{g}^*_{-r}}) \in D^G(\mathfrak{g})$. This distribution is called the Lie algebra analog of the depth $\leq r$ projector. Since $\mathfrak{g}^*_{-r} \subset \mathfrak{g}^*$ is open and closed, the function $1_{\mathfrak{g}^*_{-r}}$ is locally constant, thus $\mathcal{E}_r \in Z_{\mathfrak{g}}$.

The following result is the Lie algebra analog of Corollary 3.9

Proposition 1.17. For every $f \in C^\infty_c(\mathfrak{g})$ the sequence $\{\mathcal{E}_r^\Sigma(f)\}_{\Sigma \in \Theta_m}$ stabilizes, and $\mathcal{E}_r(f) = \lim_{\Sigma \in \Theta_m} \mathcal{E}_r^\Sigma(f)$. In particular, $\mathcal{E}_r$ is supported on $\mathfrak{g}_{r+}$.

1.18. An $r$-logarithm. By an $r$-logarithm, we mean an $\text{Ad}G$-equivariant homeomorphism $L : G_{r+} \to \mathfrak{g}_{r+}$, which induces a homeomorphism $L_x : G_{x,r+} \to \mathfrak{g}_{x,r+}$ for all $x \in X$. 

Corollary 1.19. Let $\mathcal{L} : G_{r^+} \rightarrow g_{r^+}$ be an $r$-logarithm. Then the pushforward $\mathcal{L}^*(E_r|_{G_{r^+}})$ equals $E_r|_{g_{r^+}}$.

By a theorem of Waldspurger, the Fourier transform preserves stability (see [Wa] or [KP]), therefore Corollary 1.19 easily implies that $E_r$ is stable if $G$ admits an $r$-logarithm (see Corollary 7.7). Moreover, extending theory of quasi-logarithms, introduced in [KV1], we show the following result.

Theorem 1.20. Assume that $p$ is very good for $G$ (see 7.9). Then the invariant distribution $E_r$ is stable.

1.21. Remarks. (a) If $F$ is of characteristic zero, one can show that $E_r$ is stable if $p$ is good (see 7.9 and compare 7.14).
(b) Notice that since the proof of a theorem of Waldspurger is global, our proof of the stability of $E_r$ is global in general. On the other hand, when $r \in \mathbb{N}$, we can deduce the stability of $E_r$ from that of $E_0$ (see 7.8 (c)), thus providing a purely local proof in this case.
(c) Allen Moy has informed us that he has independently conjectured Corollary 1.19 (for integral $r$ and fields of characteristic zero), found a proof for $G = \text{SL}_2$ and discovered its relation to stability of Bernstein projectors.

1.22. Plan of the paper. The paper is organized as follows. In Section 2 we review basic properties of the Bruhat–Tits building and the Moy–Prasad filtrations, needed for this work. In order to make our presentation more elementary, we present all the constructions, without using Neron models.

In Sections 3-4 we carry out all the preliminaries, needed for the stabilization assertion of Theorem 1.6. Then, in Section 5, we complete the proof of Theorem 1.6, deduce corollaries 1.9 and 1.10, and prove Lie algebra analogs (Proposition 1.17 and Corollary 1.19).

In Section 6 we compare the projector to depth zero with the character of the Steinberg representation (Theorem 1.12). Finally, in Section 7, we show the stability of the projector (Corollary 1.13 and Theorem 1.20).

In the appendices we prove several assertions, stated in the main part of the paper. Namely, in Appendix A we provide details to various properties of the Moy–Prasad filtrations, stated in Section 2 without proofs. Then, in Appendix B, we study congruence subsets, used in Section 7.

Finally, in Appendix C we review theory of quasi-logarithms introduced in [KV1, KV2] and deduce existence of $r$-logarithms, used for the proof of Theorem 1.20, and has other applications as well.
2. Moy–Prasad filtrations

In this Section we review basic properties of the Bruhat–Tits buildings (see [BT1, BT2]), and the Moy–Prasad filtrations (see [MP1, MP2]).

2.1. The reduced Bruhat–Tits building. Let $G^{\text{ad}}$ be the adjoint group of $G$.

(a) For every maximal split torus $S \subset G$, we denote by $S_{G^{\text{ad}}}$ the corresponding split maximal torus of $G^{\text{ad}}$ and consider the $\mathbb{R}$-vector space $V_{G,S} := X_*(S_{G^{\text{ad}}}) \otimes_{\mathbb{Z}} \mathbb{R}$. We denote by $A_S = A_{G,S}$ the "canonical" affine space under $V_{G,S}$ (see [Ti, 1.2] or [La, 1.9]), called the apartment, corresponding to $S$. Non-canonically, $A_S$ can be identified with $V_{G,S}$. Moreover, $A_S$ is equipped with a Euclidean metric, unique up to a scalar.

(b) The reduced Bruhat-Tits building $\mathcal{X} = \mathcal{X}(G)$ of $G$ is a metric space $\mathcal{X} = \mathcal{X}(G)$, equipped with an action of $G$, and a decomposition $\mathcal{X} = \bigcup A_S$ into a union of apartments. Note that $\mathcal{X}(G)$ only depends on $G^{\text{ad}}$.

2.2. The affine root subgroups. Let $S \subset G$ be a maximal split torus.

(a) For every root $\alpha \in \Phi(G,S)$, we denote by $g_{\alpha} \subset g$ the corresponding root subspace. We also denote by $U_{\alpha} \subset G$ the corresponding root subgroup (see [Bo2, 21.9]), and set $U_\alpha := U_{\alpha}(F)$. By definition, $U_{\alpha}$ is a connected unipotent group, whose Lie algebra is $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$. One can show that there exists a canonical isomorphism $U_{\alpha}/U_{2\alpha} \cong \mathfrak{g}_\alpha$, hence a canonical surjection $\iota_{\alpha} : U_{\alpha} \to \mathfrak{g}_\alpha$.

(b) Let $A := A_S$ be the apartment, corresponding to $S$. We denote by $\Psi(A)$ the set of affine roots (see [Ti, 1.6]). Each $\psi \in \Psi(A)$ is an affine function of $A$, whose vector part $\alpha = \alpha_\psi \in (V_{G,S})^*$ belongs to $\Phi(A) := \Phi(G,S)$.

(c) We denote by $U_{\psi} \subset U_{\alpha}$ the corresponding affine root subgroup (see [Ti, 1.4]), and we set $u_{\psi} := \iota_{\alpha}(U_{\psi}) \subset u_{\alpha}$. Then $u_{\psi} \subset u_{\alpha}$ is an $O$-submodule (see [AS (a)]).

2.3. Properties of buildings. The following standard properties of the Bruhat–Tits building $\mathcal{X}$ will be used later.

(a) For every root $\alpha \in \Phi(G,S)$, we denote by $g_{\alpha} \subset g$ the corresponding root subspace. We also denote by $U_{\alpha} \subset G$ the corresponding root subgroup (see [Bo2, 21.9]), and set $U_{\alpha} := U_{\alpha}(F)$. By definition, $U_{\alpha}$ is a connected unipotent group, whose Lie algebra is $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$. One can show that there exists a canonical isomorphism $U_{\alpha}/U_{2\alpha} \cong \mathfrak{g}_\alpha$, hence a canonical surjection $\iota_{\alpha} : U_{\alpha} \to \mathfrak{g}_\alpha$.

(b) For every two apartments $A, A' \subset \mathcal{X}$ there exists a distance preserving isomorphism of affine spaces $A \to A'$, which is the identity on $A \cap A'$ and induces an isomorphism of affine roots $\Psi(A') \cong \Psi(A)$ (see [La, Prop 13.6]).

(c) For every two points $x, y \in \mathcal{X}$ there exists a unique geodesics $[x,y] \subset \mathcal{X}$. Moreover, $[x,y]$ is a geodesics in $A$ for every apartment $A \ni x, y$ (by (b)).

(d) For every finite extension $K/F$, the building $\mathcal{X}(G_K)$ is equipped with an action of the Galois group $\text{Gal}(K/F)$. Furthermore, we have a natural $G$-equivariant embedding $\mathcal{X}(G) \hookrightarrow \mathcal{X}(G_K)^{\text{Gal}(K/F)}$, which is an isomorphism, if $K/F$ is unramified. We denote the image of $x \in \mathcal{X}(G)$ in $\mathcal{X}(G_K)$ simply by $x$. 

2.4. The Kottwitz homomorphism. (a) Following Borovoi, to a connected reductive group \( H \) over an algebraically closed field \( K \), one can associate its algebraic fundamental group \( \pi_1(H) \). It is defined to be the quotient \( X_*(T_{H^\text{sc}})/X_*(T_H) \), where \( H^\text{sc} \) is the simply connected covering of the derived group of \( H \), \( T_G \) denotes the abstract Cartan subgroup of \( H \), and similarly \( T_{H^\text{sc}} \). In particular, \( \pi_1(H) = 1 \), if \( H = H^\text{sc} \). Moreover, the construction \( H \mapsto \pi_1(H) \) is functorial. 

(b) For a connected reductive group \( H \) over a field \( K \), the fundamental group \( \pi_1(H_{\overline{K}}) \) is equipped with an action of the Galois group \( \Gamma_K := \text{Gal}(\overline{K}/K) \), and we denote by \( \Lambda(H) \) the group of coinvariants \( \pi_1(H_{\overline{K}})^{\Gamma_K} \).

(c) For a connected reductive group \( H \) over \( K := F^\text{nr} \), Kottwitz ([Ko, Section 7]) defined a canonical continuous surjective homomorphism \( \kappa_H : H(K) \to \Lambda(H) \), functorial in \( H \), and we set \( H(K)^0 := \ker \kappa_H \).

(d) For a connected reductive group \( G \) over \( F \), we set \( G^0 := G \cap G_K(K)^0 \). In particular, we have \( G = G^0 \), if \( G = G^\text{sc} \).

2.5. Remark. The group \( H(K)^0 \) is often denoted by \( H(K)_1 \). However, our terminology is better adjusted to the one of Moy–Prasad.

2.6. Filtrations for tori. Let \( T \) be a torus over \( F \), \( t := \text{Lie} T \), and \( r \in \mathbb{R}_{\geq 0} \).

(a) Assume that \( T \) is split. Then we denote by \( T_r \subset T \) the subgroup of all \( t \in T \) such that \( \text{val}_F(\lambda(t) - 1) \geq r \) for every character \( \lambda \) of \( T \). Similarly, we denote by \( t_r \subset t \) the \( O \)-module of all \( a \in t \) such that \( \text{val}_F(d\lambda(a)) \geq r \) for every \( \lambda \).

(b) For an arbitrary \( T \), let \( F'/F \) be the splitting field and \( e \) be the ramification degree of \( F'/F \). We set \( T' := T_{F'}, t' := \text{Lie} T', T_r := T_r \cap T^0 \) and \( t_r := t_{re} \cap t \).

2.7. The Moy–Prasad filtrations. Fix \( x \in \mathcal{X} \) and \( r \geq 0 \).

(a) Assume that \( G \) is quasi-split. Choose an apartment \( \mathcal{A} \subset \mathcal{X} \) containing \( x \), and let \( S \subset G \) be the corresponding maximal split torus. Then \( T := Z_G(S) \) is a maximal torus of \( G \), and the Moy–Prasad subgroup \( G_{x,r} \subset G \) is defined to be the subgroup, generated by \( T_r \) and affine root subgroups \( U_{\psi} \), where \( \psi \) runs over all elements of \( \Psi(\mathcal{A}) \) such that \( \psi(x) \geq r \).

Similarly, we denote by \( g_{x,r} \subset g \) the \( O \)-submodule, spanned by \( t_r \) and \( u_{\psi} \) for all \( \psi \in \Psi(\mathcal{A}) \) with \( \psi(x) \geq r \). Then both \( G_{x,r} \) and \( g_{x,r} \) do not depend on the choice \( \mathcal{A} \).

(b) For an arbitrary \( G \), let \( F'/F \) be a finite unramified extension of minimal degree such that \( G' := G_{F'} \) is quasi-split (see Lemma [A.2]). In this case, we set \( G_{x,r} := G'_{x,r} \cap G \) and \( g_{x,r} := g'_{x,r} \cap g \).

(c) We set \( G_{x,r^+} := \cup_{s>r} G_{x,s} \), and \( g_{x,r^+} := \cup_{s>r} g_{x,s} \). Clearly, \( G_{x,r^+} = G_{x,r'} \) and \( g_{x,r^+} = g_{x,r'} \) for some \( r' > r \). We also denote by \( g'_{x,-r} \subset g' \) the \( O \)-submodule, consisting of all \( b \in g' \) such that \( \langle b, a \rangle \in (\varnothing) \) for every \( a \in g_{x,r^+} \).


Let \( S \subset G \) be a maximal split torus. Then \( M := Z_G(S) \) is a minimal Levi subgroup of \( G \), thus \( M^{ad} \) is anisotropic, hence the building \( \mathcal{X}(M) \) is a single point \( \{ x_M \} \). We set \( m := \text{Lie} M \) and define \( M_r := M_{x_M,r} \) and \( m_r := m_{x_M,r} \).

**2.12. The polysimplicial decomposition.**

(a) Recall that the Bruhat–Tits building \( X \) equipped with a decomposition into a disjoint union of (open) polysimplices, and that each apartment \( A \subset X \) is a union of polysimplices. We denote by \([X] \) (resp. \([A] \)) the set of polysimplices in \( X \) (resp. \( A \)).

(b) Let \( F^0/F \) be a finite unramified extension, set \( \Gamma^0 := \text{Gal}(F^0/F) \) and \( G^0 := G_{F^0} \). Then we have the equality \( G_{x,r} = (G^0_{x,r})^0 \). Indeed, for \( r = 0 \), this follows from (a), while for \( r > 0 \) this follows from Lemma \([A.9] \) (b) and observation \([A.7] \) (e).

(c) Formally speaking, we define \( G_{x,r} \) and \( g_{x,r} \) slightly differently from Moy–Prasad. However, our definitions are equivalent. For example, the assertion for \( G_{x,r} \) follows from (a) and (b), while the assertion for \( g_{x,r} \) can be shown by the same argument as in \([A.8] \) (a).

(d) It can be shown that every \( g_{x,r} \subset g \) is a Lie subalgebra over \( O \), but we are not going to use this fact.

The following two results will be proven in the Appendix \([A] \)

**Proposition 2.9.** Fix a maximal split torus \( S \subset G \), set \( M := Z_G(S) \) and \( A := A_S \), and choose \( x \in A \), and \( r \in \mathbb{R}_{\geq 0} \).

1. The subgroup \( G_{x,r} \) (resp. \( G_{x,r}^+ \)) of \( G \) is generated by \( M_r \) and the affine root subgroups \( U_\psi \), where \( \psi \) runs over all elements of \( \Psi(A) \) such that \( \psi(x) \geq r \) (resp. \( \psi(x) > r \)).

2. The \( O \)-module \( g_{x,r} \) (resp. \( g_{x,r}^+ \)) of \( g \) is spanned by \( m_r \) and \( O \)-submodules \( u_\psi \), where \( \psi \) runs over all elements of \( \Psi(A) \) such that \( \psi(x) \geq r \) (resp. \( \psi(x) > r \)).

**Corollary 2.10.** For every \( g \in G \) (resp. \( a \in g, \) resp. \( b \in g^* \)) and \( r \in \mathbb{R}_{\geq 0} \) the subset \( \mathcal{X}(g,r) \) (resp. \( \mathcal{X}(a,r) \), resp. \( \mathcal{X}(b,r) \)) of \( \mathcal{X} \) consisting of \( x \in \mathcal{X} \) such that \( g \in G_{x,r} \) (resp. \( a \in G_{x,r} \), resp. \( b \in g_{x,r}^* \)) is convex.

**2.11. Refined affine roots.** Let \( A \subset \mathcal{X} \) be an apartment and \( m \in \mathbb{N} \).

1. For every \( \psi \in \Psi(A) \) there exists \( n_\psi \in \mathbb{Z}_{>0} \) such that the set of \( \psi' \in \Psi(A) \) with \( \alpha_{\psi'} = \alpha_\psi \) equals \( \psi + \frac{1}{n_\psi} \mathbb{Z} \) (see \([A.8] \) (c)). In particular, we have \( \psi + \mathbb{Z} \subset \Psi(A) \) for every \( \psi \in \Psi(A) \).

2. We denote by \( \Psi_m(A) \) the set of affine functions on \( A \) of the form \( \psi + \frac{k}{mn_\psi} \), where \( \psi \in \Psi(A) \) and \( k \in \mathbb{Z} \). In particular, \( \Psi_m(A) \supset \Psi(A) \), and for every \( \psi \in \Psi_m(A) \), we have \( \psi + \frac{1}{m} \mathbb{Z} \subset \Psi_m(A) \).

**2.12. The polysimplicial decomposition.** (a) Recall that the Bruhat–Tits building \( \mathcal{X} \) equipped with a decomposition into a disjoint union of (open) polysimplices, and that each apartment \( A \subset \mathcal{X} \) is a union of polysimplices. We denote by \([X] \) (resp. \([A] \)) the set of polysimplices in \( \mathcal{X} \) (resp. \( A \)).
(b) More precisely, two points \( x, y \in A \) belong to a polysimplex if and only if for every \( \psi \in \Psi(A) \) we have \( \psi(x) \geq 0 \) if and only if \( \psi(y) \geq 0 \), while two points \( x, y \in X \) belong to a polysimplex if and only if they belong to a polysimplex in \( A \) for every apartment \( A \ni x, y \).

(c) For \( m \in \mathbb{N} \), we denote by \( [\mathcal{X}_m] \) (resp. \( [\mathcal{A}_m] \)) the set of polysimplices in \( \mathcal{X} \) (resp. \( \mathcal{A} \)), obtained by the same procedure as in (b), but replacing \( \Psi(A) \) by \( \Psi_m(A) \). Alternatively, polysimplices in \( [\mathcal{X}_m] \) (resp. \( [\mathcal{A}_m] \)) are obtained by a subdivision of each polysimplex \( \sigma \in [\mathcal{X}] \) (resp. \( \sigma \in [\mathcal{A}] \)) into \( m^{\dim \sigma} \) smaller polysimplices.

(d) For every \( \sigma \in [\mathcal{X}_m] \), \( x, y \in \sigma \) and \( r \in \frac{1}{m} \mathbb{Z} \), we have equalities \( G_{x,r} = G_{y,r} \), \( G_{x,r+} = G_{y,r+} \), \( \mathfrak{g}_{x,r} = \mathfrak{g}_{y,r} \) and \( \mathfrak{g}_{x,r+} = \mathfrak{g}_{y,r+} \). Indeed, this follows from the fact that for every \( \mathcal{A} \supset \sigma \) and \( \psi \in \Psi(A) \), we have \( \psi - r \in \Psi_m(A) \) (by \( \ref{2.11} \) (b)), thus \( (\psi - r)(x) \geq 0 \) (resp. \( (\psi - r)(x) > 0 \)) if and only if \( (\psi - r)(y) \geq 0 \) (resp. \( (\psi - r)(y) > 0 \)).

2.13. "Bad" groups. We say that \( G \) is "bad", if \( p = 2 \), and \( G^0 \) has a factor \( R_{K/F}^\text{ad} \text{PSU}_{2n+1} \), where \( R \) denotes the restriction of scalars, and \( \text{PSU}_{2n+1} \) denotes the adjoint unitary group.

The following result is a (slightly corrected) version of \([\text{Ad}]\) Prop 1.4.1. It asserts that in almost all cases, questions about Moy–Prasad filtrations can be reduced to the split case.

**Lemma 2.14.** Assume that \( G \) is not "bad". Let \( F^e/F \) be a finite Galois extension of ramification degree \( e \), and set \( G^e := G_{F^e} \), and \( \mathfrak{g}^e := \text{Lie} G^e \). Then for every \( x \in \mathcal{X} \) and \( r \in \mathbb{R}_{\geq 0} \) we have an equalities \( G_{x,r} = G^0 \cap G_{x,re}^0 \) and \( \mathfrak{g}_{x,r} = \mathfrak{g} \cap \mathfrak{g}_{x,re}^0 \).

3. Combinatorics of the building

3.1. Notation. (a) We define a partial ordering \( < \) on \( [\mathcal{X}_m] \) by requiring that \( \sigma' < \sigma \), if \( \sigma' \) is contained in the closure \( \text{cl}(\sigma) \) of \( \sigma \). In this case, we say that \( \sigma' \) is a face of \( \sigma \).

(b) We say that \( \Sigma \subset [\mathcal{X}_m] \) is a subcomplex, if the union \( |\Sigma| := \cup_{\sigma \in \Sigma} \sigma \subset \mathcal{X} \) is closed. Furthermore, we say that \( \Sigma \) is convex, if \( |\Sigma| \) is convex, that is, for every \( x, y \in |\Sigma| \), the geodesics \( [x, y] \) in \( \mathcal{X} \) is also contained in \( |\Sigma| \).

(c) By a chamber, we mean a polysimplex \( \sigma \in [\mathcal{X}_m] \) of maximal dimension. We also denote by \( [\mathcal{X}_m]_0 \) the set of \( x \in [\mathcal{X}_m] \) of dimension zero. We will not distinguish between \( x \in [\mathcal{X}_m]_0 \) and the corresponding point of \( \mathcal{X} \).

(d) Let \( \mathcal{A} \subset \mathcal{X} \) be an apartment, \( \psi \in \Psi_m(A) \) and \( \sigma \in [\mathcal{A}_m] \). We say that \( \psi(\alpha) > 0 \), if \( \psi(y) > 0 \) for every \( y \in \sigma \). Similarly, we define \( \psi(\alpha) = 0 \), \( \psi(\alpha) \geq 0 \), etc.

(e) Let \( \mathcal{A} \subset \mathcal{X} \) be an apartment, and \( \sigma \in [\mathcal{A}_m] \) be a chamber. Denote by \( \Delta_{\mathcal{A}}(\sigma) \) the set of \( \psi \in \Psi_m(A) \) such that \( \psi(\sigma) > 0 \), and \( \psi(\sigma') = 0 \) for some face \( \sigma' < \sigma \) of codimension one. We call elements of \( \Delta_{\mathcal{A}}(\sigma) \) simple affine roots, relative to \( \sigma \).
3.2. The basic subcomplex. Fix \( \sigma \in [\mathcal{X}_m], x \in [\mathcal{X}_m]_0 \), and choose an apartment \( \mathcal{A} \subset X \) such that \( \sigma, x \in [\mathcal{A}_m] \).

(a) Denote by \( \Gamma(\sigma, x) \subset [\mathcal{X}_m] \) the smallest convex subcomplex, containing \( \sigma, x \). One can show that \( \Gamma(\sigma, x) \) can be characterized as the set of all \( \sigma' \in [\mathcal{A}_m] \) such that for every \( \psi \in \Psi_m(\mathcal{A}) \) satisfying \( \psi(\sigma) \leq 0 \) and \( \psi(\sigma') > 0 \), we have \( \psi(x) > 0 \).

(b) For every \( s \in \mathbb{R}_{>0} \), we denote by \( \Gamma_s(\sigma, x) \subset \Gamma(\sigma, x) \) the subcomplex consisting of \( \sigma' \in [\mathcal{A}_m] \) such that for every \( \psi \in \Psi_m(\mathcal{A}) \), satisfying \( \psi(\sigma) \leq 0 \) and \( \psi(\sigma') > 0 \), we have \( \psi(x) > s \).

3.3. Remarks. (a) The subcomplex \( \Gamma_s(\sigma, x) \) does not depend on the choice of \( \mathcal{A} \). Indeed, for every other apartment \( \mathcal{A}' \subset X \), there exists an isomorphism of affine spaces \( \mathcal{A} \sim \mathcal{A}' \), which is the identity on \( \mathcal{A} \cap \mathcal{A}' \), and induces a bijection \( \Psi(\mathcal{A}') \sim \Psi(\mathcal{A}) \), hence a bijection \( \Psi_m(\mathcal{A}') \sim \Psi_m(\mathcal{A}) \).

(b) Note that \( \sigma' \in \Gamma_s(\sigma, x) \) if and only if for every \( \psi \in \Psi_m(\mathcal{A}) \) satisfying \( \psi(\sigma') \leq 0 \) and \( \psi(x) \leq s \) we have \( \psi(\sigma') \leq 0 \). In particular, the complex \( \Gamma_s(\sigma, x) \) is convex.

(c) Let \( x, \sigma \) and \( \mathcal{A} \) be as in 3.2. Then there exists a chamber \( \tilde{\sigma} \in [\mathcal{A}_m] \) such that \( \sigma < \tilde{\sigma} \) and for every \( \psi \in \Delta_A(\tilde{\sigma}) \) with \( \psi(\sigma) = 0 \), we have \( \psi(x) \geq 0 \).

Indeed, choose a point \( \sigma \in \mathcal{A} \), and a chamber \( \sigma \in [\mathcal{A}_m] \) such that \( \sigma < \tilde{\sigma} \) and \( \text{cl}(\tilde{\sigma}) \cap (y, x] \neq \emptyset \). We claim that this chamber satisfies the required property. Indeed, let \( \psi \in \Psi_m(\mathcal{A}) \) such that \( \psi(\sigma) = 0 \) and \( \psi(x) < 0 \). Then \( \psi(y) = 0 \), and for \( \psi|_{[y, x]} < 0 \). Hence \( \psi(\tilde{\sigma}) < 0 \), therefore \( \psi \notin \Delta_A(\tilde{\sigma}) \).

The following result is an analog of [MS, Lem. 2.8].

**Lemma 3.4.** Let \( x \in [\mathcal{X}_m]_0, s \in \mathbb{R}_{>0} \) and \( \sigma \in [\mathcal{X}_m] \).

(a) There exists a unique minimal face \( \sigma' = m_{x,s}(\sigma) \) of \( \sigma \) such that \( \sigma \in \Gamma_s(\sigma', x) \).

(b) Assume that \( \sigma \neq x \), and the root system of \( G \) is irreducible. Let \( \mathcal{A} \) and \( \tilde{\sigma} \) be as in 3.3 (c). Then the face \( \sigma' < \sigma \) from (a) is characterized by the condition that for \( \psi \in \Delta_A(\tilde{\sigma}) \) we have \( \psi(\sigma') = 0 \) if and only if \( \psi(\sigma) = 0 \) or \( \psi(x) > s \).

**Proof.** If \( \sigma = x \), then \( \sigma' := x \) satisfies the property, so we can assume that \( \sigma \neq x \).

If the root system of \( G \) is reducible, then \( [\mathcal{X}_m] = [\mathcal{X}_m(G)] \) decomposes as a product \( [\mathcal{X}_m(G)] = \prod_i [\mathcal{X}_m(G_i)] \), so \( \sigma \) and \( x \) decompose as products \( \sigma = \prod \sigma_i \) and \( x = \prod x_i \). Moreover, every face \( \sigma' < \sigma \) decomposes as \( \sigma' = \prod \sigma_i' \), and we have \( \sigma \in \Gamma_s(\sigma', x) \) if and only if \( \sigma_i \in \Gamma_s(\sigma_i', x_i) \) for all \( i \). Thus we can assume that the root system of \( G \) is irreducible.

Fix \( \mathcal{A} \) and \( \tilde{\sigma} \) as in 3.3 (c). Assume that for every \( \psi \in \Delta_A(\tilde{\sigma}) \) we have \( \psi(\sigma) = 0 \) or \( \psi(x) > s \). Then, by our choice of \( \tilde{\sigma} \), for every \( \psi \in \Delta_A(\tilde{\sigma}) \) we have \( \psi(x) \geq 0 \). Thus \( x \leq \tilde{\sigma} \). Since \( \sigma \neq x \), there exists \( \psi_0 \in \Delta_A(\tilde{\sigma}) \) such that \( \psi_0(x) = 0 \) and \( \psi_0(\sigma) > 0 \). Then \( \psi_0(x) \leq s \), contradicting our assumption.

By the previous paragraph, there exists a unique face \( \sigma' < \sigma \) be such that for every \( \psi \in \Delta_A(\tilde{\sigma}) \) we have \( \psi(\sigma') = 0 \) if and only if \( \psi(\sigma) = 0 \) or \( \psi(x) > s \).
that $\sigma \in \Gamma_s(\sigma', x)$. Choose any $\xi \in \Psi_m(\mathcal{A})$ satisfying $\xi(\sigma') \leq 0$ and $\xi(\sigma) > 0$. We want to show that $\xi(x) > s$.

Since $\xi(\sigma) > 0$, thus $\xi(\tilde{\sigma}) > 0$, the affine root $\xi$ has the form $\sum_{\psi \in \Delta(\tilde{\sigma})} n_\psi \psi$, where $n_\psi \in \mathbb{Z}_{\geq 0}$ for all $\psi$, and there exists $\psi_0$ with $\psi_0(\sigma') > 0$ and $n_{\psi_0} > 0$. Since $\xi(\sigma') \leq 0$, we get $n_\psi = 0$, when $\psi(\sigma') > 0$. Thus every $\psi \in \Delta(\tilde{\sigma})$ with $n_\psi > 0$ satisfies $\psi(\sigma') = 0$, that is, $\psi(\sigma) = 0$ or $\psi(x) > s$. In particular, there exists $\psi_0$ with $\psi_0(x) > s$ and $n_{\psi_0} > 0$. By definition of $\tilde{\sigma}$, the assumption $\psi(\sigma) = 0$ implies that $\psi(x) \geq 0$, therefore $\xi(x) \geq n_{\psi_0} \psi_0(x) > s$, as claimed.

It remains to show that for every $\sigma'' < \sigma$ such that $\sigma \in \Gamma_s(\sigma'', x)$ we have $\sigma' < \sigma''$. Choose $\psi \in \Delta(\tilde{\sigma})$ such that $\psi(\sigma'') = 0$. We want to show that $\psi(\sigma') = 0$, that is, $\psi(\sigma) = 0$ or $\psi(x) > s$.

Equivalently, assuming that $\psi(x) \leq s$, we want to conclude that $\psi(\sigma) = 0$, that is, $\psi(\sigma) \leq 0$ and $\psi(\sigma) \geq 0$. Since $\sigma \in \Gamma_s(\sigma'', x)$, we have $\psi(\sigma) \leq 0$ (see [3.3] (b)). On the other hand, since $\psi \in \Delta(\tilde{\sigma})$ and $\sigma < \tilde{\sigma}$, we have $\psi(\tilde{\sigma}) > 0$, thus $\psi(\sigma) \geq 0$. $\square$

The following result is an analog of [MS, Lem. 2.9].

**Lemma 3.5.** For every $\sigma \in [\mathcal{X}_m]$ there exists a unique maximal polysimplex $\sigma'' \in \Gamma_s(\sigma, x)$ such that $\sigma < \sigma''$.

**Proof.** Consider two maximal polysimplices $\sigma''_1, \sigma''_2 \in \Gamma_s(\sigma, x) \subset [\mathcal{A}_m]$ such that $\sigma < \sigma''_1, \sigma''_2$. First we claim that $\sigma''_1$ and $\sigma''_2$ are faces of the same chamber. For this we have to show that there is no $\psi \in \Psi_m(\mathcal{A})$ such that $\psi(\sigma''_1) > 0$ and $\psi(\sigma''_2) < 0$. Indeed, assume that such $\psi \in \Psi_m(\mathcal{A})$ exists. Since $\sigma < \sigma''_2$ and $\sigma''_1 \in \Gamma_s(\sigma, x)$, this would imply that $\psi(\sigma) \leq 0$, thus $\psi(x) > s \geq 0$. On the other hand, since $\sigma < \sigma''_1$ and $\sigma''_2 \in \Gamma_s(\sigma, x)$, this would imply that $\psi(x) < 0$, a contradiction.

Since $\sigma''_1$ and $\sigma''_2$ are faces of the same chamber, they generate a polysimplex $\sigma''_3$ such that $\sigma''_2, \sigma''_2 < \sigma''_3$. Moreover, since $\Gamma_s(\sigma, x)$ is convex, we conclude that $\sigma''_3 \in \Gamma_s(\sigma, x)$. Since $\sigma''_1$ and $\sigma''_2$ are assumed to be maximal, we thus conclude that $\sigma''_1 = \sigma''_2 = \sigma''_3$. $\square$

**3.6. Notation.** For $x \in [\mathcal{X}_m]_0$ and $s \in \mathbb{R}_{\geq 0}$, we denote by $\Upsilon_{x,s}$ the set of all chambers $\sigma \in [\mathcal{X}_m]$ such that for every apartment $\mathcal{A} \subset \mathcal{X}$ such that $\sigma, x \in [\mathcal{A}_m]$ and every $\psi \in \Delta(\sigma)$ we have $\psi(x) \leq s$.

**Lemma 3.7.** For every $s$ and $x$, the set $\Upsilon_{x,s}$ is finite, and $\Upsilon_{x,0} = \emptyset$.

**Proof.** To show that $\Upsilon_{x,0} = \emptyset$, we fix a chamber $\sigma \in [\mathcal{X}_m]$ and an apartment $\mathcal{A} \supset \sigma$. Decomposing $\mathcal{X}(G)$ into a product, if necessary, we may assume that the root system of $\mathcal{G}$ is irreducible. Then there exist positive numbers $\{n_\psi\}_{\psi \in \Delta(\sigma)}$ such that the affine function $\sum_{\psi} n_\psi \psi$ is 1. Indeed, this is standard for $m = 1$, and the general case follows from it. Since in the linear combination $\sum_{\psi \in \Delta(\sigma)} n_\psi \psi(x) = 1 > 0$, we have $n_\psi > 0$ for all $\psi$, there exists $\psi \in \Delta(\sigma)$ such that $\psi(x) > 0$. Hence $\sigma \notin \Upsilon_{x,0}$. "$\square$
Next, we notice that the group $G$ has transitively on the set of chambers in $[X]$, hence it acts with finitely many orbits on the set of chambers in $[X_m]$. Therefore in order to show that the set $Y_{x,s}$ is finite, it suffices to show that for every chamber $\sigma \in [X_m]$, the set of $x \in [X_m]_0$ such that $\sigma \in Y_{x,s}$ is finite.

Choose an apartment $A = A_\Sigma \supset \sigma$. Note that every $x \in [X_m]_0$ belongs to an apartment $A' \supset \sigma$, the group $G_{\sigma}$ acts transitively on the set of such apartments, and the building is locally finite. Therefore it remains to show that the set of $x \in [A_m]_0$ such that $\sigma \in Y_{x,s}$ is finite.

Fix $y \in \sigma$. Then for every $\psi \in \Delta_A(\sigma)$ and every $x \in [A_m]_0$ such that $\sigma \in Y_{x,s}$ we have $\psi(x) \leq s$ and $\psi(y) > 0$. Then the difference $x - y \in V_{G,s}$ satisfies $\alpha_w(x - y) < s$ (see 2.2 (b)) for all $\psi \in \Delta_A(\sigma)$. From this we conclude that $x - y$ lies in a bounded set, thus the set of such $x$ is finite. □

3.8. Notation. Fix $x \in [X_m]_0$ and $s \in \mathbb{R}_{\geq 0}$, and let $m_{x,s} : [X_m] \to [X_m]$ be the map, defined in Lemma 3.4. For $\sigma \in [X_m]$, we set $\sigma' := m_{x,s}(\sigma)$. By Lemma 3.3 there exists a unique maximal $\sigma'' \in \Gamma_s(\sigma', x)$ such that $\sigma' < \sigma''$.

Claim 3.9. In the situation of 3.8, let $\Sigma, \Sigma' \in \Theta_m$ satisfy $x \in \Sigma' \subset \Sigma$ and $\sigma \in \Sigma \setminus \Sigma'$.

(a) For $\tau \in [X_m]$, we have $\sigma' < \tau < \sigma''$ if and only if $\tau \in \Sigma \setminus \Sigma'$ and $m_{x,s}(\tau) = \sigma'$.

(b) If $Y_{x,s} \subset \Sigma'$, then $\sigma'' \neq \sigma'$.

Proof. (a) Assume that $m_{x,s}(\tau) = \sigma'$. Then $\sigma' < \tau$ and $\tau \in \Gamma_s(\sigma', x)$. Hence $\tau < \sigma''$ by the definition of $\sigma''$.

Conversely, assume that $\sigma' < \tau < \sigma''$. First we claim that $\tau \in \Sigma$ and $\tau \notin \Sigma'$. Since $\Sigma'$ and $\Sigma$ are subcomplexes, it suffices to show that $\sigma' \notin \Sigma'$ and $\sigma'' \in \Sigma$.

Assume that $\sigma' \in \Sigma'$. Since $x \in \Sigma'$ and $\Sigma'$ is convex, we conclude that $\Gamma(\sigma', x) \subset \Sigma'$, thus $m_{x,s}(\sigma', x) \subset \Gamma(\sigma', x)$ is contained in $\Sigma'$. But this contradicts to the assumptions $\sigma \in m_{x,s}(\sigma')$ and $\sigma \notin \Sigma'$.

Next, since $\sigma \in \Sigma$, $\sigma' < \sigma$ and $\Sigma$ is a subcomplex, we conclude that $\sigma' \in \Sigma$. Thus, arguing as above we conclude that $\Gamma_s(\sigma', x) \subset \Sigma$, thus $\sigma'' \in \Sigma$.

It remains to show that $\tau' := m_{x,s}(\tau)$ equals $\sigma' = m_{x,s}(\sigma)$. Since $\sigma'' \in \Gamma_s(\sigma', x)$ and $\tau < \sigma''$, we conclude that $\tau \in \Gamma_s(\sigma', x)$, thus $\tau' < \sigma'$. On the other hand, since $\tau \in \Gamma_s(\tau', x)$ and $\sigma' < \tau$, we have $\sigma' \in \Gamma_s(\tau', x)$, thus $m_{x,s}(\sigma') < \tau'$. Finally, since $\sigma' = m_{x,s}(\sigma)$, we conclude that $m_{x,s}(\sigma') = \sigma'$, thus $\tau' = \sigma'$.

(b) We have to show that there exists $\tau \neq \sigma'$ such that $\sigma' = m_{x,s}(\tau)$. Decomposing the building into a product, if necessary, we may assume that the root system of $G$ is irreducible. Since $x \in \Sigma'$ and $\sigma \notin \Sigma'$, we conclude that $\sigma \neq x$.

Let $A$ and $\tilde{\sigma}$ be as in 3.3 (c). Since $\sigma < \tilde{\sigma}$ and $\sigma \notin \Sigma'$, we conclude that $\tilde{\sigma} \notin \Sigma'$. Using the assumption $Y_{x,s} \subset \Sigma'$, we conclude that $\tilde{\sigma} \notin Y_{x,s}$. Thus there exists $\psi_0 \in \Delta_A(\tilde{\sigma})$ such that $\psi_0(x) > s$. 


By Lemma 3.4 (b), we conclude that \( \psi_0(\sigma') = 0 \). Hence there exists a unique \( \tau < \tilde{\sigma} \) such that \( \tau \) is a face of \( \tilde{\sigma} \) of codimension one, and \( \psi_0(\tau) > 0 \). By assumption, \( \tilde{\sigma} \) satisfies \( \psi(x) \geq 0 \) for every \( \psi \in \Delta_\mathcal{A}(\tilde{\sigma}) \) such that \( \psi(\sigma') = 0 \). Hence it satisfies this property for each \( \psi \) with \( \psi(\tau) = 0 \). Since \( \psi_0(x) > s \), we conclude from Lemma 3.4 (b) that \( m_{x,s}(\tau) < \sigma' \). Then \( m_{x,s}(\tau) = \sigma' \), because \( \sigma' = m_{x,s}(\sigma') < m_{x,s}(\tau) \). □

4. Stabilization

The complex \( \Gamma_s(\sigma,x) \) is important to us because of the following fact.

**Lemma 4.1.** Let \( \sigma, \sigma' \in [\mathcal{X}_m], x \in [\mathcal{X}_m]_0 \) and \( r, s \in \frac{1}{m}\mathbb{Z}_{\geq 0} \) such that \( \sigma' < \sigma \) and \( \sigma \in \Gamma_s(\sigma',x) \). Then we have \( \delta_{G_{\sigma,r+}} * \delta_{G_{x,(r+s)+}} = \delta_{G_{\sigma',r+}} * \delta_{G_{x,(r+s)+}} \).

**Proof.** Notice that \( \delta_{G_{\sigma',r+}} * \delta_{G_{x,(r+s)+}} \) is the image of \( \delta_{G_{\sigma,r+}} \times G_{x,(r+s)+} \) under the multiplication map \( G_{\sigma,r+} \times G_{x,(r+s)+} \to G_{\sigma,r+} \cdot G_{x,(r+s)+} \), and similarly, for \( \sigma' \). Thus we have to check that \( G_{\sigma,r+} \cdot G_{x,(r+s)+} = G_{\sigma',r+} \cdot G_{x,(s+r)+} \). Since \( \sigma' < \sigma \), we conclude that \( G_{\sigma',r+} \subset G_{\sigma,r+} \) is a normal subgroup. Hence we have to show that

\[
G_{\sigma,r+} \subset G_{\sigma',r+} \cdot (G_{\sigma,r+} \cap G_{x,(r+s)+}).
\]

Choose an apartment \( \mathcal{A} \in \mathcal{X} \) such that \( \sigma, x \in [\mathcal{A}_m] \). Then, by Proposition 2.9 the subgroup \( G_{\sigma,r+} \) is generated by \( M_{r+} \) and affine root subgroups \( U_\psi \), where \( \psi \) runs over elements of \( \Psi(\mathcal{A}) \) such that \( \psi(\sigma) > r \), and similarly for \( G_{\sigma',r+} \) and \( G_{x,(r+s)+} \).

Thus, it suffices to show that for every \( \psi \in \Psi(\mathcal{A}) \) satisfying \( \psi(\sigma) > r \), we have \( \psi(\sigma') > r \) or \( \psi(x) > r + s \). Recall that for every \( \psi \in \Psi(\mathcal{A}) \), we have \( \psi - r \in \Psi_m(\mathcal{A}) \) (see 2.11 (b)). Replacing \( \psi \) by \( \psi - r \), we reduce to the assertion that for each \( \psi \in \Psi_m(\mathcal{A}) \) satisfying \( \psi(\sigma) > 0 \) and \( \psi(\sigma') \leq 0 \), we have \( \psi(\tau') > s \), which is precisely the assumption \( \sigma \in \Gamma_s(\sigma',x) \). □

**Proposition 4.2.** Let \( x \in [\mathcal{X}_m]_0 \) and \( r, s \in \frac{1}{m}\mathbb{Z}_{\geq 0} \).

(a) For every \( \Sigma, \Sigma' \in \Theta_m \) satisfying \( x \in \Sigma', \Sigma' \subset \Sigma \) and \( \Upsilon_{x,s} \subset \Sigma' \), we have

\[
E_{r, \Sigma' +} * \delta_{G_{x,(r+s)+}} = E_{r, \Sigma' +} * \delta_{G_{x,(r+s)+}}.
\]

(b) For every \( \Sigma \in \Theta_m \) such that \( x \in \Sigma \), we have \( E_{r, \Sigma +} * \delta_{G_{x,r+}} = \delta_{G_{x,r+}} \).

(c) For every \( \Sigma \in \Theta_m \) and \( \sigma \in \Sigma \), we have \( E_{r, \Sigma +} * \delta_{G_{\sigma,r+}} = \delta_{G_{\sigma,r+}} \).

**Proof.** (a) Setting \( \Sigma'' := \Sigma \setminus \Sigma' \), we have to show that \( E_{r, \Sigma'' +} * \delta_{G_{x,(r+s)+}} = 0 \). Let \( m_{x,s} \) be as in 3.8, and define an equivalence relation on \( \Sigma'' \) by the requiring that \( \sigma_1 \sim \sigma_2 \) if and only if \( m_{x,s}(\sigma_1) = m_{x,s}(\sigma_2) \). For every \( \sigma \in \Sigma'' \), we denote by \( \Sigma'' \subset \Sigma'' \) the equivalence class of \( \sigma \). Then \( \Sigma'' \) decomposes as a disjoint union of the \( \Sigma'' \)'s, so it remains to show that \( E_{r, \Sigma'' +} * \delta_{G_{x,(r+s)+}} = 0 \) for every \( \sigma \in \Sigma'' \).

By Lemma 4.1 for every \( \tau \in [\mathcal{X}_m]_0 \), we have

\[
\delta_{G_{x,r+}} * \delta_{G_{x,(r+s)+}} = \delta_{G_{m_{x,s}(r+r),r+}} * \delta_{G_{x,(r+s)+}}.
\]
Since every $\tau \in \Sigma''$ satisfies $m_{x,s}(\tau) = m_{x,s}(\sigma)$, we have
\[
E^{\Sigma}_{r} \ast \delta_{G_{x,(r+s)^+}} = (\sum_{\tau \in \Sigma''} (-1)^{\dim \tau})(\delta_{G_{m_{x,s}(\sigma),r^+}} \ast \delta_{G_{x,(r+s)^+}}).
\]
Thus it remains to show that $\sum_{\tau \in \Sigma''} (-1)^{\dim \tau} = 0$.

Let $\sigma', \sigma'' \in [\mathcal{X}_m]$ be as in 3.3. By Claim 3.9 (a), $\Sigma''_\sigma \subset \Sigma''$ consists of all $\tau$ such that $\sigma' < \tau < \sigma''$. Thus the sum $\sum_{\tau \in \Sigma''} (-1)^{\dim \tau}$ equals $\sum_{\tau, \sigma' < \tau < \sigma''} (-1)^{\dim \tau}$, and the latter expression vanishes, because $\sigma'' \neq \sigma'$.

(b) Since $\Upsilon_{x,0}$ is empty (by Lemma 3.7), the subcomplex $\Sigma' := \{x\}$ satisfies the assumptions of (a) for $s = 0$. Thus we have
\[
E^{\Sigma}_{r} \ast \delta_{G_{x,r^+}} = E^{\{x\}}_{r} \ast \delta_{G_{x,r^+}} = \delta_{G_{x,r^+}} + \delta_{G_{x,r^+}} = \delta_{G_{x,r^+}}.
\]

(c) Choose $x \in [\mathcal{X}_m]_0$ such that $x < \sigma$. Then $G_{x,r^+} \subset G_{\sigma,r^+}$, hence $\delta_{G_{x,r^+}} + \delta_{G_{\sigma,r^+}} = \delta_{G_{\sigma,r^+}}$, and the assertion follows from (b).

\[\Box\]

5. Formula for the projector

In this section we prove Theorem 1.6, Corollaries 1.9 and 1.10, Proposition 1.17 and Corollary 1.19.

5.1. Proof of Theorem 1.6. We divide the proof into six steps.

\textbf{Step 1.} For every $h \in \mathcal{H}(G)$, the sequence $\{E^{\Sigma}_{r} \ast h\}_{\Sigma \in \Theta_m}$ stabilizes.

\textit{Proof.} Fix $x \in [\mathcal{X}_m]_0$ and $n \in \mathbb{N}$ such that $\delta_{G_{x,n^+}} \ast h = h$. It is enough to show that the sequence $\{E^{\Sigma}_{r} \ast \delta_{G_{x,n^+}}\}_{\Sigma \in \Theta_m}$ stabilizes, so the assertion follows from Proposition 4.2 (a). \[\Box\]

\textbf{Step 2.} There exists a unique element $z \in Z_G$ such that $z(h) = E^{\Sigma}_{r} \ast h$ for every $h \in \mathcal{H}(G)$ and every sufficiently large $\Sigma \in \Theta_m$, that is, $z(h) = \lim_{\Sigma \in \Theta_m}(E^{\Sigma}_{r} \ast h)$.

\textit{Proof.} By Step 1, there exists a unique endomorphism $z \in \text{End}_C \mathcal{H}(G)$ such that $z(h) = \lim_{\Sigma \in \Theta_m} E^{\Sigma}_{r} \ast h$ for every $h \in \mathcal{H}(G)$. We claim that $z \in Z_G$.

Since $z$ commutes with the right convolutions, it suffices to show that $z$ is $\text{Ad} G$-equivariant (use 1.3 (b)). First we claim that $z$ is $\text{Ad} K$-invariant for every compact subgroup $K \subset G^\text{ad}$. Indeed, the $\Sigma \in \Theta_m$'s in the equality $z(h) = \lim_{\Sigma \in \Theta_m}(E^{\Sigma}_{r} \ast h)$ can be chosen to be $\text{Ad} K$-invariant, thus $z$ is $\text{Ad} K$-equivariant.

It remains to show that the group $G^\text{ad}$ is generated by compact subgroups. Since the corresponding simply connected group $G^\text{sc}$ is known to be generated by compact subgroups, and $G^\text{sc}$ acts transitively on the set of chambers in $[\mathcal{X}(G)]$, the assertion follows from the fact that a stabilizer $\text{Stab}_{G^\text{ad}}(\sigma)$ of every chamber is compact. \[\Box\]
**Step 3.** For every \( V \in R(G) \) and \( v \in V \), the sequence \( \{E_r^{\Sigma}(v)\}_{\Sigma \in \Theta_m} \) stabilizes, and \( z(v) = \lim_{\Sigma \in \Theta_m} E_r^{\Sigma}(v) \).

*Proof.* Choose \( h \in \mathcal{H}(G) \) such that \( h(v) = v \). Then \( E_r^{\Sigma}(v) = E_r^{\Sigma}(h(v)) = (E_r^{\Sigma} \ast h)(v) \) stabilizes (by Step 1), and the limiting value equals \( z(h)(v) = z(h(v)) = z(v) \) (see 1.3 (c)). \( \Box 

**Step 4.** For every \( V \in \text{Irr}(G)_{\leq r} \), we have \( z|_V = \text{Id}_V \).

*Proof.* By definition, there exists \( x \in \mathcal{X} \) such that \( V^{G_{x,+}} \neq 0 \). Thus, by the Schur lemma, it remains to show that \( z(v) = v \) for all \( v \in V^{G_{x,+}} \). By Proposition 4.2 (c), we conclude that \( z(\delta_{G_{x,+}}) = \delta_{G_{x,+}} \). Note that for each \( v \in V^{G_{x,+}} \) we have \( \delta_{G_{x,+}}(v) = v \). Therefore, by 1.3 (c), we conclude that \( z(v) = z(\delta_{G_{x,+}}(v)) = (z(\delta_{G_{x,+}}))(v) = \delta_{G_{x,+}}(v) = v \). \( \Box 

**Step 5.** For every \( V \in R(G)_{>r} \), we have \( z|_V = 0 \).

*Proof.* For every \( V \in R(G)_{>r} \) and \( x \in \mathcal{X} \), we have \( V^{G_{x,+}} = 0 \). Thus \( \delta_{G_{x,+}}(v) = 0 \) for all \( v \in V \). Therefore we have \( E_r^{\Sigma}(v) = 0 \) for all \( \Sigma \in \Theta_m \) and \( v \in V \), hence \( z(v) = 0 \) by Step 3. \( \Box 

**Step 6.** Since an element of \( z \in Z_G \) is determined by its action on irreducible representations, it follows from Steps 4 and 5 that \( z = \Pi_r \) (see 5.2 for a more direct argument). \( \Box 

5.2. An alternative proof. Using arguments, described above, we can give both an alternative proof of the existence of the decomposition \( R(G) = R(G)_{\leq r} \oplus R(G)_{>r} \) and a more direct proof of the equality \( z = \Pi_r \). We do it in two steps.

(I) The element \( z \in Z_G \), constructed in the Step 2 of 5.1, is idempotent.

*Proof.* We have to show that \( z \circ z = z \). By 1.3 (b), it suffices to show that for every \( h \in \mathcal{H}(G) \) we have \( z(h) = z(h) \). By the definition of \( z \), we have to show that \( z(E_r^{\Sigma} \ast h) = E_r^{\Sigma} \ast h \) for all sufficiently large \( \Sigma \in \Theta_m \). By construction, we have \( z(E_r^{\Sigma} \ast h) = z(E_r^{\Sigma}) \ast h \), so it suffices to show that \( z(E_r^{\Sigma}) = E_r^{\Sigma} \) for every \( \Sigma \in \Theta_m \), or equivalently that \( z(\delta_{\sigma,+}) = \delta_{\sigma,+} \) for every \( \sigma \in [\mathcal{X}_m] \). But this follows from Proposition 4.2 (c). \( \Box 

(II) For every \( V \in R(G) \), set \( V_{\leq r} := \text{Im}(z|_V) \subset V \) and \( V_{>r} := \text{Ker}(z|_V) \subset V \).

Since \( z \in Z_G \) is an idempotent, we have a direct sum decomposition \( V = V_{\leq r} \oplus V_{>r} \), and we also have \( z|_W = \text{Id}_W \) (resp. \( z|_W = 0 \)) for every irreducible subquotient \( W \) of \( V_{\leq r} \) (resp. \( V_{>r} \)). Then the result of Step 4 (resp. Step 5) of 5.1 implies that \( V_{\leq r} \in R(G)_{\leq r} \) (resp. \( V_{>r} \in R(G)_{>r} \)). This implies both the desired decomposition \( R(G) = R(G)_{\leq r} \oplus R(G)_{>r} \) and the desired equality \( z = \Pi_r \).
5.3. **Proof of Corollary [1.10]**  (a) For \( f \in C_c^\infty(G) \) and \( E \in D(G) \), we define the convolution \( E \ast f \in C_c^\infty(G) \) by the rule \( (E \ast f)dg := E \ast (fdg) \) for a Haar measure \( dg \) on \( G \). Then \( E(f) = (E \ast \iota(f))(1) \), where \( \iota : G \to G \) is the map \( g \mapsto g^{-1} \).

By Theorem [1.6] for every \( h \in \mathcal{H}(G) \), we have \( E_r \ast h = \lim_{\Sigma \in \Theta_m} (E_r^\Sigma \ast h) \). Therefore, for every \( f \in C_c^\infty(G) \), we have \( E_r \ast f = \lim_{\Sigma \in \Theta_m} (E_r^\Sigma \ast f) \), hence \( E_r(f) = \lim_{\Sigma} E_r^\Sigma(f) \).

(b) Since each \( E_r^\Sigma \) is supported on \( G_{r+} \), we conclude by (a). \( \square \)

5.4. **Generalized functions.**  (a) Since the space of generalized functions \( \hat{C}(G) \) is the linear dual of \( \mathcal{H}(G) \), the Bernstein center \( Z_G \) acts on \( \hat{C}(G) \) by the formula \( z(\chi)(h) := \chi(z(h)) \) for every \( z \in Z_G \), \( \chi \in \hat{C}(G) \) and \( h \in \mathcal{H}(G) \). We say that \( \chi \in \hat{C}(G) \) is of depth \( \leq r \) if \( \Pi_r(\chi) = \chi \).

(b) By the Schur lemma, for every \( z \in Z_G \) and \( V \in \text{Irr}(G) \), there exists \( f_z(V) \in \mathbb{C} \) such that \( z|_{V} = f_z(V) \text{Id}_V \). Moreover, the character \( \chi_V \) of \( V \) satisfies \( z(\chi_V) = f_z(V)\chi_V \). In particular, for an admissible representation \( V \in R(G)_{\leq r} \), its character \( \chi_V \) is of depth \( \leq r \). Thus the following result is a generalization of Corollary [1.10]

**Corollary 5.5.**  For every invariant generalized function \( \chi \in \hat{C}^G(G) \) of depth \( \leq r \) and every \( h \in \mathcal{H}(G) \) we have

\[
\chi(h) = \lim_{\Sigma \in \Theta_m} \left[ \sum_{\sigma \in \Sigma} (-1)^{\dim \sigma} \chi(\delta_{G_{\sigma,r+}} \ast h \ast \delta_{G_{\sigma,r+}}) \right].
\]

**Proof.** Since \( \chi \) is of depth \( \leq r \), we have the equality \( \chi(h) = \Pi_r(\chi)(h) = \chi(\Pi_r(h)) \). Then, by Theorem [1.6], \( \chi(h) \) equals

\[
\lim_{\Sigma \in \Theta_m} \chi(E_r^\Sigma \ast h) = \lim_{\Sigma \in \Theta_m} \left[ \sum_{\sigma \in \Sigma} (-1)^{\dim \sigma} \chi(\delta_{G_{\sigma,r+}} \ast h) \right].
\]

Finally, since \( \chi \) is \( \text{Ad}G \)-invariant, we have

\[
\chi(\delta_{G_{\sigma,r+}} \ast h \ast \delta_{G_{\sigma,r+}}) = \chi(\delta_{G_{\sigma,r+}} \ast \delta_{G_{\sigma,r+}} \ast h) = \chi(\delta_{G_{\sigma,r+}} \ast h),
\]

and the assertion follows. \( \square \)

5.6. **Proof of Proposition [1.17]**  For every \( \Sigma \in \Theta_m \), we set \( g^*_\Sigma_{\sigma,-r} := \cup_{\sigma \in \Sigma} g^*_\sigma_{\sigma,-r} \). Then \( g^*_\Sigma_{\sigma,-r} \subset g^* \) is an open and compact subset, and \( g^*_r = \cup_{\Sigma \in \Theta_m} g^*_\Sigma \). Thus we have \( 1_{g^*_r} = \lim_{\Sigma \in \Theta_m} 1_{g^*_\Sigma} \), hence \( \mathcal{E}_r = \lim_{\Sigma \in \Theta_m} \mathcal{F}^{-1}(1_{g^*_\Sigma}) \). It therefore suffices to show that \( \mathcal{F}^{-1}(1_{g^*_\Sigma}) = \mathcal{E}^*_r \), that is, \( \mathcal{F}(\mathcal{E}^*_r) = 1_{g^*_\Sigma} \).

Notice that the restriction of the Fourier transform \( \mathcal{F} : D(g) \to \hat{C}(g^*) \) to \( \mathcal{H}(g) \) is the Fourier transform \( \mathcal{H}(g) \to C_c^\infty(g^*) \).

Since \( \psi \) is trivial on \( \langle \pi \rangle \) but nontrivial on \( \mathcal{O} \), for every \( \sigma \in [\mathcal{X}_m] \), the lattice \( g^*_{\sigma,-r} \subset g^* \) is the orthogonal complement of \( g_{\sigma,r+} \subset g \) with respect to the pairing \( g \times
representation, the space of invariants \( St_{G,I} \) of \( G(I) \), and for every element \( w \) of the affine Weyl group \( W_{aff} \) of \( G \), the element \( 1_{IwI} \delta_I \in \mathcal{H}(G,I) \) acts on \( St_G \) as \( sgn(w) \) Id.

6. The Steinberg representations of \( p \)-adic groups (see [Bo1] and compare [Ca] Section 8 and [BW] p. 199-205).

(a) Let \( St_G \) be the Steinberg (or special) representation of \( G = G(F) \). Then \( St_G \) is irreducible, the space of Iwahori invariants \( St_G \) is a one-dimensional module of the Hecke algebra \( \mathcal{H}(G,I) \), and for every element \( w \) of the affine Weyl group \( W_{aff} \) of \( G \), the element \( 1_{IwI} \delta_I \in \mathcal{H}(G,I) \) acts on \( St_G \) as \( sgn(w) \) Id.
(b) As a virtual representation, $\text{St}_G$ equals the alternative sum of the non-normalized induced representations $\text{Ind}_Q^G(1_Q)$, where $Q = Q(F)$, and $Q$ runs over a set of standard parabolic subgroups $Q \subset G$.

**6.3. Parahoric subgroups.** (a) Fix a parahoric subgroup $P \subset G$, and an Iwahori subgroup $I \subset P$. Then the quotient $P/P^+$ is naturally isomorphic to $L = L(\mathbb{F}_q)$ for some connected reductive group $L = L_P$ over $\mathbb{F}_q$, and under this isomorphism $I/P^+ \subset P/P^+$ corresponds to $B = B(\mathbb{F}_q)$ for some Borel subgroup $B = B_P \subset L$.

(b) Note that for every representation $V \in R(G)$, the space of invariants $V^{P^+}$ is a representation of $P/P^+ = L$.

**Claim 6.4.** The $L$-representation $\text{St}_G^{P^+}$ is isomorphic to the Steinberg representation $\text{St}_L$.

**Proof.** Denote the $L$-representation $\text{St}_G^{P^+}$ by $\text{St}'$. Then we have equality $(\text{St}')^B = \text{St}_G^I$ and a natural identifications $W_L \cong W_P \subset W$ and $H(L,B) \cong H(P,I) \subset H(G,I)$ under which $h_w$ (from 6.1 (a)) corresponds to $1_{IwI}$. Therefore, by 6.2 (b), $(\text{St}')^B$ is a one-dimensional representation of the Hecke algebra $H(L,B)$ such that $h_w$ acts by $\text{sgn}(w)$ for every $w \in W_L$. Hence, by 6.1 (b), $\text{St}'$ is isomorphic to a direct sum $\text{St}_L \oplus V$, there $V^B = 0$. It remains to show that $\text{St}'$ is generated by its $B$-invariants. But this follows from Lemma [6.5] below. □

**Lemma 6.5.** For every smooth representation $V$ of $G$, which is generated by its $I$-invariants, the $L$-representation $V^{P^+}$ is generated by $B$-invariants.

**Proof.** Since $V$ is generated by $V^I$, it is a quotient of a direct sum of the $C_\infty(I\backslash G)$'s. Thus, it is enough to prove the claim in the case $V = C_\infty(I \backslash G)$. Notice that the space $V$, considered as a $P$-representation, decomposes as a sum $V = \sum_{g \in G} V_g$, where $V_g := \mathbb{C}[I \backslash IgP]$. Thus it remains to show that each $V_g^{P^+}$ is generated by its $B$-invariants. It suffices to show that $V_g^{P^+} \cong \mathbb{C}[B' \backslash L]$, where $B' = B'(\mathbb{F}_q)$ for some Borel subgroup $B' \subset L$.

Notice that we have a natural isomorphism of $P$-representations $V_g \cong \mathbb{C}[P \cap I' \backslash P]$, where $I' := g^{-1}Ig$. Therefore $V_g^{P^+} \cong \mathbb{C}[P^+(P \cap I') \backslash P]$, so it suffices to show that $J := P^+(P \cap I') \subset P$ is an Iwahori subgroup (compare 6.3).

Let $\sigma \in [\mathcal{A}]$ (resp. $\tau \in [\mathcal{A}]$) be the polysimplex such that $P = G_\sigma$ (resp. $I' = G_\tau$). Choose an apartment $\mathcal{A}$ of $\mathcal{X}$ such that $\sigma, \tau \in [\mathcal{A}]$ and points $x \in \sigma$ and $y \in \tau$. Since $I'$ is an Iwahori subgroup, $\tau$ is a chamber. Hence we have $\psi(y) \neq 0$ for every $\psi \in \Psi(\mathcal{A})$. Therefore every point $z \in [x, y]$, close to $x$, lies in $\bar{\sigma}$ for some chamber $\bar{\sigma} \in [\mathcal{A}]$ such that $\sigma < \bar{\sigma}$. We claim that $J = G_{\bar{\sigma}}$, that is, $G_{\bar{\sigma}} = G_{\sigma,0^+}(G_\sigma \cap G_\tau)$.

By Proposition 2.9, the subgroup $G_{\bar{\sigma}}$ is generated by $M_\bar{\sigma}$ and affine root subgroups $U_\psi$ for $\psi \in \Psi(\mathcal{A})$ satisfying $\psi(\bar{\sigma}) > 0$. Since $\sigma < \bar{\sigma}$, we have $\psi(\bar{\sigma}) > 0$ if and only
if we have either \( \psi(\sigma) > 0 \) or \( \psi(\sigma) = 0 \) and \( \psi(\bar{\sigma}) > 0 \). Thus, to show the inclusion \( G_{\bar{\sigma}} \subset G_{\sigma,0^+} (G_{\sigma} \cap G_{\tau}) \), we have to check that for every \( \psi \in \Psi(A) \), satisfying \( \psi(\sigma) = 0 \) and \( \psi(\bar{\sigma}) > 0 \) we have \( \psi(\tau) > 0 \). Equivalently, we have to check that for every \( \psi \in \Psi(A) \), satisfying \( \psi(x) = 0 \) and \( \psi(z) > 0 \) we have \( \psi(y) > 0 \), which follows from the assumption \( z \in \{x,y\} \).

The converse inclusion is easier. Namely, inclusion \( G_{\sigma} \cap G_{\tau} \subset G_{\bar{\sigma}} \) or, equivalently, \( G_{x} \cap G_{y} \subset G_{z} \) follows from Corollary 2.10 while the inclusion \( G_{\sigma,0^+} \subset G_{\bar{\sigma}} \) follows from the fact that \( \sigma < \bar{\sigma} \). \(\square\)

6.6. Proof of Theorem 1.12. We have to show the equality

\[
E_0(f) = \chi_{St_G}(f \mu^{I^+}),
\]

valid for every \( f \in C^\infty_c(G_{0^+}) \). Moreover, since \( E_0 \) and \( \chi_{St_G} \) are Ad\( G \)-invariant and \( G_{0^+} = (\text{Ad} \ G)(I^+) \), it is enough to prove (6.1) for \( f \in C^\infty_c(I^+) \).

To calculate the right hand side, we apply result of Meyer-Solleveld [MS, Thm 4.1] for \( K = I^+ \) and \( \pi = St_G \).

For every \( \sigma \in [X] \), we denote by \( G_\sigma^+ \subset G \) be the stabilizer of \( \sigma \) (compare [MS, Section 4]), and set \( G_{\sigma} := G_{\sigma,0}, G_{\sigma}^+ := G_{\sigma,0^+}, L_{\sigma} := L_{G_{\sigma}} \), and let \( U_{\sigma} \subset L_{\sigma} \) be a maximal unipotent subgroup.

Since \( G_{\sigma}^+ \) normalizes \( G_{\sigma}^+ \), it acts on the space of invariants \( St_{G_{\sigma}^+} \). We denote by \( \text{sgn}_\sigma : G_{\sigma}^+ \rightarrow \{ \pm 1 \} \) the orientation character, that is, \( \text{sgn}_\sigma(g) = 1 \) if and only if \( g \in G_{\sigma}^+ \) preserves an orientation of \( \sigma \).

Since the Steinberg representation \( St_G \) is generated by \( St_{G_{\sigma}^+} \), [MS, Thm 4.1] asserts that for every sufficiently large \( I^+ \)-invariant \( \Sigma \in \Theta_1 \), we have

\[
\chi_{St_G}(f \mu^{I^+}) = \int_{g \in I^+} f(g) \left( \sum_{\sigma \in \Sigma \mid g \in G_{\sigma}^+} (-1)^{\dim \sigma} \text{Tr}(g, St_{G_{\sigma}^+} \text{sgn}_\sigma) \right) \mu^{I^+}.
\]

Notice that for every \( g \in I^+ \cap G_{\sigma}^+ \), we have \( g \in G_{\sigma} \), and the image \( [g] \in L_{\sigma} \) is unipotent. In particular, \( \text{sgn}_\sigma(g) = 1 \). Since the space of invariants \( St_{G_{\sigma}^+} \) is the Steinberg representation of \( L_{\sigma} \) by Claim 6.4, we conclude from 6.1 (c) that for every \( g \in I^+ \cap G_{\sigma}^+ \), the trace \( \text{Tr}(g, St_{G_{\sigma}^+}) \) equals \( |U_{\sigma}| 1_{G_{\sigma}^+}(g) \).

Hence the right hand side of (6.2) equals

\[
\int_{g \in I^+} f(g) \left( \sum_{\sigma \in \Sigma \mid g \in G_{\sigma}^+} (-1)^{\dim \sigma} |U_{\sigma}| 1_{G_{\sigma}^+}(g) \right) \mu^{I^+} = \int_{g \in I^+} f(g) \left( \sum_{\sigma \in \Sigma} (-1)^{\dim \sigma} |U_{\sigma}| 1_{G_{\sigma}^+}(g) \right) \mu^{I^+}.
\]
Using the equality $|U_\sigma|1_{G^+_+}\mu^{I^+} = \delta_{G^+_+}$, the latter expression equals $\int_{g \in I^+} f(g) E_0^\Sigma = E_0^\Sigma(f)$. This shows that $\chi_{\text{St}_G}(f\mu^{I^+}) = E_0^\Sigma(f)$ for every sufficiently large $I^+$-invariant $\Sigma \in \Theta_1$. Now the assertion follows from Corollary 1.9 (a).

\section{Stability}

In this section we prove Corollary 1.13 and Theorem 1.20.

\subsection{Set up}

(a) We fix a non-zero translation invariant differential form $\omega_G$ on $G$ and a similar form $\omega_T$ on $T$ for each maximal torus $T \subset G$. Then $\omega_G/\omega_T$ defines a translation invariant differential form on $G/T$, hence a $G$-invariant measure $|\omega_G/\omega_T|$ on $(G/T)(F)$. Also $\omega_G$ defines a Haar measure $|\omega_G|$ on $G$.

(b) Let $X$ be either $G$, or $g$, or $g^*$, where $g$ denotes the Lie algebra $g$, viewed as an algebraic variety, and similarly $g^*$. Then $X$ is equipped with an adjoint action of $G$, and we denote by $X^{sr} \subset X$ the open subset consisting of all $x \in X$ such that the stabilizer $G_x := \text{Stab}_G(x) \subset G$ is a maximal torus. We set $X := X(F)$ and $X^{sr} := X^{sr}(F)$. We assume that $X^{sr} \neq \emptyset$, which is always satisfied, if $X = G$ or the characteristic of $F$ is not two.

\subsection{Stability}

Suppose that we are in the situation of $A.1$ (c).

(a) For every $x \in X^{sr}$ we have a natural map $a_x : G/G_x \to X : [g] \mapsto g(x)$, hence a map $(G/G_x)(F) \to X^{sr}$, whose image we call the \textit{stable orbit}.

(b) Notice that each stable orbit is closed in $X$, hence we can define an invariant distribution $O_x^{st} \in D_G^\infty(X)$ by the formula $O_x^{st}(f) := \int_{(G/G_x)(F)} a_x^*(f)|\omega_G/\omega_G_x|$ for every smooth function with compact support $f \in C^\infty_c(X)$. The distribution $O_x^{st}$ is called the \textit{stable orbital integral}. It is defined uniquely up to a constant.

(c) A function $f \in C^\infty_c(X)$ is called \textit{unstable}, if $O_x^{st}(f) = 0$ for every $x \in X^{sr}$. An invariant distribution $F \in D_G^\infty(X)$ is called \textit{stable}, if $F(f) = 0$ for every unstable $f \in C^\infty_c(X)$. An invariant generalized function $\chi \in \hat{C}_G^\infty(X)$ is called \textit{stable}, if $\chi dx \in D_G^\infty(X)$ is stable for a Haar measure $dx$ on $X$.

(d) We call a $G$-equivariant open and closed subset $Y \subset X$ \textit{stable}, if $Y \cap X^{sr}$ is a union of stable orbits (see (a)).

\subsection{Examples}

(a) If $Y \subset X$ is a stable subset (see 7.2 (d)), then the characteristic function $1_Y \in \hat{C}_G^\infty(X)$ is stable. Indeed, we want to show that for every unstable function $f \in C^\infty_c(G)$, we have $\int_G (f \cdot 1_Y)dx = 0$. Since $Y$ is stable, the function $f \cdot 1_Y \in C^\infty_c(X)$ is unstable. Thus it remains to check that for every unstable function $f \in C^\infty_c(X)$, we have $\int_G f dx = 0$. This follows from the fact $X^{sr} \subset X$ is dense.

(b) The character $\chi_{\text{St}_G}$ of the Steinberg representation is stable. Indeed, by 6.2 (b), it remains to show that each character $\chi_{\text{Ind}_G^G(1_Q)}$ is stable. This follows from the
fact that the constant function is stable (by (a)) and that the parabolic induction preserves stability (see [KV3]).

The following lemma will be proven in Appendix B (see B.2 (b)).

**Lemma 7.4.** For every \( r \in \mathbb{R}_{\geq 0} \), the open \( G \)-invariant subsets \( G_r+ \subset G \), \( g_r+ \subset g \) and \( g^*_{-r} \subset g \) are closed and stable.

**7.5. Remark.** The fact that \( G_r+ \subset G \) and \( g^+_r \subset g \) are closed was also proven by Adler and DeBacker (see [ADB, Cor 3.4.3 and Cor 3.7.21]). On the other hand, our proof is completely different.

**7.6. Proof of Corollary 1.13.** We have to show that for every unstable \( f \in C_c^\infty(G) \), we have \( E_0(f) = 0 \).

Since \( G_{0^+} \subset G \) is open and closed (by Lemma 7.4), \( f \) decomposes as \( f = f' + f'' \), where \( f' := f \cdot 1_{G_{0^+}} \) and \( f'' := f \cdot 1_{G \setminus G_{0^+}}. \) Since \( f \) is unstable, while \( G_{0^+} \subset G \) is stable (by Lemma 7.4), we conclude that \( f' \) is unstable.

Since \( E_0 \) is supported on \( G_{0^+} \) (by Corollary 1.12), and \( f'' \) is supported on \( G \setminus G_{0^+} \), we conclude that \( E_0(f'') = 0 \). Therefore \( E_0(f) = E_0(f') \) equals \( \chi_{\text{St} G}(f' \mu^+_1) \) (by Theorem 1.12), hence it vanishes, because \( \chi_{\text{St} G} \) is stable (see 7.3 (b)), while \( f' \) is unstable. \( \square \)

**Corollary 7.7.** Assume that the characteristic of \( F \) is different from two, and that \( G \) admits an \( r \)-logarithm. Then the invariant distribution \( E_r \) is stable.

**Proof.** By observation 7.3 (a) and Lemma 7.4, the invariant generalized function \( 1_{g^*_{-r}} \in \hat{C}^G_G(g^*) \) is stable. Hence, by a generalization [KP] of a theorem of Waldspurger [Wa], the distribution \( \mathcal{E}_r = F^{-1}(1_{g^*_{-r}}) \) is stable.

The rest of the argument is similar to 7.6. For every unstable function \( f \in C_c^\infty(G) \), functions \( f' := f \cdot 1_{G_{r^+}} \) and \( \mathcal{L}_1(f') \in C_c^\infty(g) \) are unstable. On the other hand, we have \( E_r(f) = E_r(f') \), because \( E_r \) supported on \( G_{r^+} \), and \( E_r(f') = \mathcal{E}_r(L_1(f')) \) by Corollary 1.19. Hence \( E_r(f) = \mathcal{E}_r(L_1(f')) = 0 \), because \( \mathcal{E}_r \) is stable. \( \square \)

**7.8. Remarks.** (a) Formally speaking, the theorem of Waldspurger and its generalization in [KP] are only proved when \( F \) is of characteristic zero. On the other hand, the arguments can be extended to local fields of positive odd characteristic.

(b) In all known cases when \( G \) admits an \( r \)-logarithm, the Lie algebra admits a non-degenerate quadratic form. In this case, we can identify the dual \( g^* \) with \( g \), thus the original theorem of Waldspurger suffices.

(c) When \( r \in \mathbb{N} \), we can prove Corollary 1.17 without a theorem of Waldspurger. Namely, arguing as in the second paragraph of the proof of Corollary 1.17, we see that \( E_r \) is stable if and only if \( \mathcal{E}_r \) is stable. Hence, by Corollary 1.13 it suffices to show that \( \mathcal{E}_r \) is stable if and only if \( \mathcal{E}_0 \) is stable.
Let \( \mu_r : g \to g \) be the homothety map \( a \mapsto \sigma^r a \). Since \( r \in \mathbb{N} \), for every \( x \in X \), we have the equality \( g \sigma_{x,r} = \sigma^r g_{x,0} \) (see [A.3](b)). Then the pullback \( \mu_r^* : D(g) \to D(g) \) satisfies \( \mu_r^*(\delta_{g_{x,r,\pm}}) = \delta_{g_{x,0}} \) for all \( x \in X \), hence \( \mu_r^*(\mathcal{E}_r^\Sigma) = \mathcal{E}_r^{\Sigma_0} \) for all \( \Sigma \in \Theta_m \), thus \( \mu_r^*(\mathcal{E}_r) = \mathcal{E}_0^s \) (by Proposition [I.17]). Since \( \mu_r^* \) maps stable distributions to stable, the assertion follows.

### 7.9. (Very) good primes

(a) Let \( G_{\text{sc}} \) be the simply connected covering of the derived group of \( G \). Then \( G_{\text{sc}} \) decomposes a product \( G_{\text{sc}} = \prod_i R_{F_i/F} H_i \), such that each \( F_i/F \) is a finite separable extension, \( H_i \) is an absolutely simple algebraic group over \( F_i \), and \( R_{F_i/F} \) denotes the Weil restriction of scalars. We denote by \( H_i^s \) the quasi-split inner form of \( H_i \) and by \( F_i[H_i^s] \) the splitting field of \( H_i^s \).

(b) We say that \( p \) is **good** for \( G \), if either \( p > 5 \), or \( p = 5 \) and neither of the \( H_i \)'s is of type \( E_8 \), or \( p = 3 \), all the \( H_i \)'s are of type \( A - D \), and satisfy \( [F_i[H_i^s]] : F_i \leq 2 \).

(c) We say that \( p \) is **very good** for \( G \), if \( p \) is good, and \( p \) does not divide \( n \), if some of the \( H_i \)'s is of type \( A_n \).

The following assertion is an immediate consequence of Lemma [C.3](C) and Lemma [C.4](C).

**Corollary 7.10.** If \( p \) is very good for \( G \), then \( G_{\text{sc}} \) admits an \( r \)-logarithm for every \( r \in \mathbb{R}_{\geq 0} \).

The proof of following assertion is given in Appendix [B](B) (see [B.4](B)).

**Lemma 7.11.** Let \( \pi : G' \to G \) be an isogeny of degree prime to \( p \). Then \( \pi \) induces homeomorphisms \( G'_{x,r} \to G_{x,r} \) and \( G'_{x,r} \to G_{x,r} \) for all \( r \) and \( x \in X(G') = X(G) \).

**Corollary 7.12.** In the situation of Lemma 7.11 the distribution \( E_r \) on \( G \) is stable if and only if \( E_r \) on \( G' \) is stable.

**Proof.** Since \( E_r \) is supported on \( G_{r,+} \) (by Corollary [I.19](b)), to show that it is stable, we have to check that \( E_r(f) = 0 \) for every unstable \( f \) supported on \( G_{r,+} \), and similarly for \( G' \). Thus, the assertion follows from Lemma 7.11 and Corollary [I.19](a).

#### 7.13. Proof of Theorem 1.20.

Consider the natural isogeny \( \pi : G_{\text{sc}} \times \mathbb{Z}(G)^0 \to G \). Since the degree of \( \pi \) divides \( |\mathbb{Z}(G_{\text{sc}})| \), and \( p \) is very good, the degree of \( \pi \) is prime to \( p \). Hence (by Corollary 7.12), to show the stability of \( E_r \) on \( G \), it is enough to show the stability of \( E_r \) on \( G_{\text{sc}} \). Since \( G_{\text{sc}} \) admits an \( r \)-logarithm by Corollary 7.10, the assertion follows from Corollary 7.7.

#### 7.14. Remark

If \( F \) is of characteristic zero and \( p \) is good, then \( E_r \) is stable. Indeed, arguing similarly to [7.13], we reduce to the assertion that \( E_r \) is stable, if each \( H_i \) if of type \( A_p \) and \( p > 2 \). Then, using classification, we deduce to the case when \( G \) is either \( \text{GL}_n \) or \( \text{GU}_n \) (in this step we use the assumption that the characteristic is zero). In both cases, \( G \) admits an \( r \)-logarithm, so the assertion follows from Corollary 7.7.
APPENDIX A. Properties of the Moy–Prasad filtrations

In this section we provide proofs of some of the results, formulated in Section 2. We are going to follow a standard strategy, first to pass to an unramified extension, thus reducing to a quasi-split case, then to pass to a Levi subgroup, thus reducing to a rank one case, and to finish by direct calculations. Though most of the results in this sections are well-known to specialists (see, for example, [Vi, Section 1]), we include details for completeness.

A.1. Set-up. Let $S \subset G$ be a maximal split torus, $M := Z_G(S)$ be the corresponding minimal Levi subgroup of $G$, set $A := A_S$, and let $\Phi(A)_{nd} \subset \Phi(A)$ be the set of non-divisible roots, that is, those $\alpha \in \Phi(A)$ such that $a/2 \notin \Phi(A)$.

Lemma A.2. There exists a finite unramified extension $F'/F$ such that $G' := G_{F'}$ is quasi-split. Moreover, for every such extension, there exists a subtorus $S' \supset S$ of $G$ defined over $F$ such that $S'_{F'} \subset G'$ is a maximal split torus.

Proof. Assume first that $G = GL_1(D)$ for some finite-dimensional central division algebra $D$ over $F$. In this case, both assertions are easy. Indeed, let $\dim_F D = d^2$, and let $F'/F$ is an unramified extension. Then $G_{F'}$ is quasi-split if and only if $F'$ splits $D$. Moreover, this happens if and only if $F' \supset F^{(d)}$, where $F^{(d)}/F$ is an unramified extension of degree $d$. Furthermore, there exists an embedding $F^{(d)} \hookrightarrow D$ of $F$-algebras, whose image corresponds to a torus $S'$ we are looking for.

Assume next that $G = GL_1(D)$ for some finite-dimensional division algebra $D$ over $F$. This case reduces to the first one, and is left to the reader.

Finally, the general case follows from the previous one. Indeed, $G_{F'}$ is quasi-split if and only if $M_{F'}$ is quasi-split, and if and only if the simply connected covering $M_{F'}^{sc}$ of $M_{F'}$ is quasi-split. Thus we may replace $G$ by $M^{sc}$, thus assuming that $G$ is semisimple, simply-connected, and anisotropic. Next, decomposing $G$ into simple factors, we may further assume that $G$ is simple. Then $G = SL_1(D)$ for some finite-dimensional division algebra over $F$, and $SL_1$ denotes the kernel of the reduced norm (see [PfR], Thm 6.5. p. 285]). Since the assertion for $SL_1(D)$ follows from the assertion for $GL_1(D)$, the proof is now complete. $\square$

A.3. Roots subgroups. (a) We choose a set of positive roots $\Phi(A)_{nd}^{+} \subset \Phi(A)_{nd}$, and a total order on $\Phi(A)_{nd} \cup \{0\}$ such that $\alpha > 0$ if and only if $\alpha \in \Phi(A)_{nd}^{+}$. Set $U_0 := M$. Then the product map $\prod_{\alpha \in \Phi(A)_{nd} \cup \{0\}} U_\alpha \to G$ is an open embedding.

(b) For every $\alpha \in \Phi(A)$, $x \in A$ and $r \in \mathbb{R}_{\geq 0}$, we denote by $\psi_{\alpha,x,r}$ the smallest affine root $\psi \in \Psi(A)$ such that $\alpha_{\psi} = \alpha$ and $\psi(x) \geq r$. Set $\bar{U}_{\alpha,x,r} := U_{\psi_{\alpha,x,r}} \subset U_\alpha$ and $u_{\bar{U}_{\alpha,x,r}} := u_{\psi_{\alpha,x,r}} \subset u_\alpha$.

(c) We also set $U_{\alpha,x,r} := U_{\alpha,x,r} \cdot U_{2\alpha,x,r} \subset U_\alpha$, if $2\alpha \in \Phi(A)$; $U_{\alpha,x,r} := U_{\alpha,x,r}$, if $2\alpha \notin \Phi(A)$, and $U_{0,x,r} := M_r$. 

ON THE DEPTH $r$ BERNSTEIN PROJECTOR
A.4. The SL₂ case. Let \( G = \text{SL}_2 \), and let \( S \subset G \) be the group of diagonal matrices. In this case, we have a natural identification of \( A \) with \( V_S \), and we have \( \Phi(A) = \pm \alpha \) and \( \Psi(A) = \pm \alpha + \mathbb{Z} \). Moreover, if the root subgroup \( U_\alpha \) consists of matrices \( g_a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \) with \( a \in F \), then the affine root subgroup \( U_{\alpha + n} \subset U_\alpha \) consists of \( g_a \in U_\alpha \) with \( \text{val}_F(a) \geq n \).

A.5. The SU₃ case (compare [Tits Ex. 1.15]). (a) Let \( K/F \) be a separable totally ramified quadratic extension, and let \( \tau \in \text{Gal}(K/F) \) be a non-trivial element. Let \( G = \text{SU}_3 \), be the special unitary group over \( F \), split over \( K \) and corresponding to the quadratic form \( (x, y) \mapsto \sum_i x_i y_{n-i} \). Let \( S \subset G \) the maximal torus, corresponding to diagonal matrices, and let \( \alpha \in \Phi(G, S) \) the non-divisible root such that \( U_\alpha \) consists of upper triangular matrices. Then \( U_\alpha \) consists of all elements of the form

\[
g_{a,b} = \begin{pmatrix} 1 & -a & -b \\ 0 & 1 & \tau a \\ 0 & 0 & 1 \end{pmatrix}, \quad a, b \in K \text{ such that } a\tau a + b + b^\tau = 0, \text{ while } U_{2\alpha} \text{ consists of all } g_{0,0} \in U_\alpha.
\]

(b) Set \( \delta := \max\{\text{val}_K(b)|b + \tau b + 1 = 0\} \). Then \( \delta \leq 0 \), and \( \delta = 0 \) if and only if \( p \neq 2 \). Then for every \( g_{a,b} \in U_\alpha \), we have \( \text{val}_K(b) \leq 2 \text{val}(a) + \delta \), and for every \( a \in K^\times \) there exists \( g_{a,b} \in U_\alpha \) with \( \text{val}_K(b) = 2 \text{val}(a) + \delta \). On the other hand, as it was explained by Tits, for every \( g_{0,0} \in U_{2\alpha} \), we have \( \text{val}_K(b) \in 2\mathbb{Z} + \delta + 1 \).

(c) Identifying \( A \) with \( V_S \), we identify the set of affine roots \( \Psi(A) \) with the set \( (\pm \alpha + \frac{1}{2}(2\mathbb{Z} + \delta)) \cup (\pm \alpha + \frac{1}{2}(2\mathbb{Z} + \delta + 1)) \), where we divide by extra factor of 2, because our normalization uses valuation \( \text{val}_F = \frac{1}{2} \text{val}_K \).

(d) In the notation of (c), for \( \psi := \alpha + \frac{1}{2}(2n + \delta) \), the subgroup \( U_\psi \) consists of all \( g_{a,b} \in U_\alpha \) such that \( \text{val}_K(b) \geq 2n + \delta \), while for \( \psi := 2\alpha + \frac{1}{2}(2n + \delta + 1) \) the subgroup \( U_\psi \) consists of \( g_{0,0} \in U_\alpha \) such that \( \text{val}_K(b) \geq 2n + \delta + 1 \).

(e) Using (d), for every \( x \in A \) and \( r \in \mathbb{R}_{\geq 0} \), the subgroup \( U_{\alpha, x, r} \) consists \( g_{a,b} \in U_\alpha \) such that \( \text{val}_K(b) \geq 2r - 2\alpha(x) \), while subgroup \( U_{2\alpha, x, r} \) consists \( g_{0,0} \in U_\alpha \) such that \( \text{val}_K(b) \geq r - 2\alpha(x) \). In particular, we have \( U_{\alpha, x, r} \cap U_{2\alpha, x, r} = U_{2\alpha, x, 2r} \).

(f) We claim that an element \( g_{a,b} \in U_\alpha \) belongs to \( U_{(\alpha), x, r} \) if and only if \( \text{val}_K(a) \geq r - \alpha(x) - \frac{1}{2}\delta \) and \( \text{val}_K(b) \geq r - 2\alpha(x) \).

By definition, \( U_{(\alpha), x, r} \) consists of elements of the form \( g_{a,b} = g_{a,b} \cdot g_{0,b'} \) such that \( g_{a,b} \in U_{\alpha, x, r} \) and \( g_{0,b'} \in U_{2\alpha, x, r} \). In particular, we have \( \text{val}_K(b') \geq r - 2\alpha(x) \), and \( \text{val}_K(b') \geq 2r - 2\alpha(x) \) (by (c)), hence \( 2 \text{val}_K(a) + \text{val}_K(b') - \delta \geq 2r - 2\alpha(x) - \delta \) (by (b)) and \( \text{val}_K(b) \geq \min\{\text{val}_K(b'), \text{val}_K(b'')\} \geq r - 2\alpha(x) \).

Conversely, assume that element \( g_{a,b} \in U_\alpha \) satisfies \( \text{val}_K(a) \geq r - \alpha(x) - \frac{1}{2}\delta \) and \( \text{val}_K(b) \geq r - 2\alpha(x) \). Choose \( g_{a,b'} \in U_\alpha \) with \( \text{val}_K(b') = 2 \text{val}_K(a) + \delta \), and set
Let $b'' := b - b'$. Then \( \val(b') \geq 2r - 2\alpha(x) \) and \( \val(b'') \geq \min\{\val(b), \val(b')\} \geq r - 2\alpha(x) \). Thus \( g_{a,b} \in U_{a,x,r} \) and \( g_{0,b'} \in U_{2\alpha,x,r} \), hence \( g_{a,b} \in U_{(a),x,r} \).

**A.6. Levi subgroups.** Let \( L \supset S \) be a Levi subgroup of \( G \), and \( A_L := A_{L,S} \).

(a) We have a natural projection \( \pr_L : A \to A_L \) of affine spaces, compatible with the projection \( V_{G,S} \to V_{L,S} \) of vector spaces (see [La, 1.10 and 1.11]).

(b) We have an inclusion \( \Phi(A_L) \subset \Phi(A) \), and every affine root \( \psi \in \Psi(A) \) such that \( \alpha_\psi \in \Phi(A_L) \) induces an affine function \( \psi_L \) on \( A_L \), which belongs to \( \Psi(A_L) \). Moreover, the correspondence \( \psi \mapsto \psi_L \) induces a bijection between the set of \( \psi \in \Psi(A) \) such that \( \alpha_\psi \in \Phi(A_L) \) and the set \( \Psi(A_L) \).

(c) By definition, for every \( \psi \in \Psi(A) \) such that \( \alpha_\psi \in \Phi(A_L) \subset \Phi(A) \), the affine root subgroup \( U_\psi \subset U_{a_\psi} \) equals \( U_{\psi_L} \).

(d) By (b) and (c), for every \( \alpha \in \Phi(A_L) \subset \Phi(A) \), \( x \in A \) and \( r \in \mathbb{R}_{\geq 0} \), the affine root subgroup \( U_{a,x,r} \subset U_a \subset G \) equals \( U_{a,pr(x),r} \subset U_a \subset L \).

(e) For every \( \alpha \in \Phi(A) \subset X^*(S) \), let \( S_\alpha \) be the connected component \( (\ker \alpha)^0 \) and set \( L_\alpha := Z_G(S_\alpha) \). Then \( L_\alpha \) is a Levi subgroup of semisimple rank 1, and \( \Phi(L_\alpha, S) \) is either \( \pm \alpha \) or \( \{ \pm \alpha, \pm 2\alpha \} \).

**A.7. Descent.** (a) Let \( F'/F \), \( G' := G_{F'} \), and \( S' \subset S \) be as in Lemma A.2, where we assume in addition that the splitting field of \( G_{F'} \) is totally ramified. We set \( \Gamma' := \Gal(F'/F) \), and let \( A' := A_{S,F'} \subset \mathcal{A}(G') \) be the corresponding apartment. Then \( \mathcal{A}' \) is equipped with an action of \( \Gamma' \), and we have a natural identification \( A \cong A'' \).

(b) Note that for \( \alpha \in \Phi(A) \), the root group \( U_\alpha' := (U_{\alpha})_{F'} \) equals the product \( \prod_{\alpha'} U_{\alpha'} \), where \( \alpha' \) runs over the union of all \( \alpha' \in \Phi(A') \) such that \( \alpha'|_A = \alpha \) and all \( \alpha' \in \Phi(A')_{nd} \) such that \( \alpha'|_A = 2\alpha \) (compare [Bo2, 21.9]).

(c) Moreover, for every \( x \in A \) and \( r \in \mathbb{R}_{\geq 0} \), the affine root subgroup \( U_{a,x,r} \subset U_a \) equals \( U_{a,x,r} = (U_{a,x,r})'' \), where \( U_{a,x,r}'' \subset U_a'' \) is the product

\[
\left( \prod_{\alpha' \in \Phi(A'), \alpha'|_A = \alpha} U_{\alpha',x,r} \right) \times \left( \prod_{\alpha' \in \Phi(A')_{nd}, \alpha'|_A = 2\alpha} U_{\alpha',x,2r} \right),
\]

taken in order (see, for example, [La, 10.19 and 11.5]).

(d) For every triple \( (\alpha, x, r) \) as in (b), (c) such that \( 2\alpha \in \Phi(A) \), we have the equality \( U_{a,x,r} \cap U_{2\alpha,x,r} = U_{2\alpha,x,2r} \). Indeed, by (c), it suffices to show that \( U_{a,x,r}'' \cap U_{2\alpha,x,r}'' = U_{2\alpha,x,2r}'' \), which reduces to the equality \( U_{\alpha',x,r}'' \cap U_{2\alpha',x,r}'' = U_{2\alpha',x,2r}'' \) for every \( \alpha' \in \Phi(A') \) such that \( 2\alpha' \in \Phi(A') \). By A.6 (d), (e), we reduce to the case \( G = SU_3 \), in which case the assertion was shown in A.3 (e).

(e) We set \( U_{(a),x,r} := U_{(a),x,r}'' \cdot U_{(a),x,2r}'' \subset U'' \), if \( 2\alpha \in \Phi(A) \), and \( U_{(a),x,r} := U_{a,x,r}'' \), otherwise. We claim that \( U_{(a),x,r} = (U_{(a),x,r})'' \). If \( 2\alpha \notin \Phi(A) \), this follows from (c).
Let $2\alpha \in \Phi(\mathcal{A})$, and we have to show that $(U'_{\alpha,x,r} \cdot U_{2\alpha,x,r})^{\Gamma'} = (U'_{\alpha,x,r})^{\Gamma'} \cdot (U_{2\alpha,x,r})^{\Gamma'}$. By (d), it remains to show that $H^1(\Gamma', U'_{2\alpha,x,2r}) = 0$. Using the Shapiro lemma, the assertion reduces to the vanishing of $H^1(\Gamma', \mathcal{O}_{F'})$, which follows from the additive Hilbert 90 theorem.

(f) For every two triples $(\alpha, x, r)$ and $(\alpha, y, s)$ as in (b),(c) such that $2\alpha \in \Phi(\mathcal{A})$ we have $U_{(\alpha),x,r} \cap U_{(\alpha),y,s} = (U_{\alpha,x,r} \cap U_{\alpha,y,s}) \cdot (U_{2\alpha,x,r} \cap U_{2\alpha,y,s})$.

Indeed, using (d) and arguing as in (e), we reduce the assertion to the corresponding equality of the $U'$s. Then by A.9 (d),(e), we reduce to the case $G = SU_3$, in which case we finish by precisely the same arguments as A.5 (f).

A.8. Applications. (a) Each $u_\psi \subset u_\alpha$ is an $\mathcal{O}$-lattice (2.2 (c)). Indeed, by A.7 (c), we reduce to the case when $G$ is quasi-split and split over a totally ramified extension. Then by A.6 (d),(e), we reduce to the rank one case, in which case, the assertion follows from formulas of A.4 and A.5. Again, this can be shown by the same strategy as in (a).

(b) For every $x \in \mathcal{A}, r \in \mathbb{R}_{0}$ and $n \in \mathbb{N}$, we have the equality $\pi^n g_{x,r} = g_{x,r+n}$. Notice that the extra factor of 2 in A.5 (c) is essential.

Lemma A.9. (a) In the situation of Proposition 2.9, the subalgebra $g_{x,r}$ decomposes as a direct sum $g_{x,r} = m_r \oplus \prod_{\alpha \in \Phi(\mathcal{A})} u_{\alpha,x,r}$.

(b) Assume in addition that either $r > 0$ or $x$ lies in a chamber in $[\mathcal{X}]$. For every order of $\Phi(\mathcal{A})_{nd} \cup \{0\}$ as in A.3 (a), the product map $\prod_{\alpha \in \Phi(\mathcal{A})_{nd} \cup \{0\}} U_{(\alpha),x,r} \rightarrow G_{x,r}$ is bijective.

A.10. Remark. Actually, the map in (b) is bijective for every order of $\Phi(\mathcal{A})_{nd} \cup \{0\}$.

Proof. We show only (b), while the proof of (a) is similar but much easier.

Since $U_{(\alpha),x,r} \subset U_\alpha$ for all $\alpha$, the injectivity follows from A.3 (a). To show the surjectivity, assume first that $G$ is quasi-split. In this case, the argument is standard (compare [PRK 2.9]), and can be carried out as follows.

Let $Y \subset G_{x,r}$ be the image of the product map. Since $Y$ is closed, and $\{G_{x,s}\}_{s \geq 0}$ form a basis of open neighbourhoods, it remains to show that $G_{x,r} \subset Y \cdot G_{x,s}$ for every $s \geq r$. Since $G_{x,s}$ is generated by subgroups $U_{(\alpha),x,s}$, it remains to show that $Y \cdot U_{(\alpha),x,s} \subset Y \cdot G_{x,s}$. If $s > 0$, this follows from the inclusion $(G_{x,r}, G_{x,s}) \subset G_{x,r+s} \subset G_{x,s}$ (use [PRK 2.4 and 2.7]). If $s = r = 0$, and $\alpha = 0$, this follows from the fact that $M_r$ normalizes each $U_{(\alpha),x,r}$. If $\alpha \neq 0$, then $U_{(\alpha),x,0} = U_{(\alpha),x,s}$ for some $s > 0$, because $x$ belongs to a chamber, and the assertion is immediate.
For an arbitrary $G$, let $F'/F$ and $G'$ be as in A.7. Note that the embedding $\mathcal{X}(G) \hookrightarrow \mathcal{X}(G')$ maps chambers into chambers. Set $U'_{(0),x,r} := M'_{x,M,r}$. As it was already shown, the assertion holds for $G'_{x,r}$ and $M'_{x,M,r}$. This implies that the product $\prod_{\alpha \in \Phi(\mathcal{A})_{nd} \cup \{0\}} U'_{(\alpha),x,r} \to G'_{x,r}$ is bijective. Now the assertion follows from equalities $G_{x,r} = (G'_{x,r})^{F'}$, $M_r = (M'_{x,M,r})^{F'}$, which were our definition, and $U_{(\alpha),x,r} = (U'_{(\alpha),x,r})^{F'}$ for all $\alpha \in \Phi(\mathcal{A})_{nd}$ (see A.7 (d)).

Corollary A.11. Let $(x, r)$ be as in Lemma A.9 (b), $y \in \mathcal{A}$ and $s \in \mathbb{R}_{\geq 0}$. Then

(a) For every order of $\Phi(\mathcal{A})_{nd} \cup \{0\}$ as in A.3 (a), the product map

$$\prod_{\alpha \in \Phi(\mathcal{A})_{nd} \cup \{0\}} (U_{(\alpha),x,r} \cap U_{(\alpha),y,s}) \to G_{x,r} \cap G_{y,s}$$

is bijective.

(b) The subgroup $G_{x,r} \cap G_{y,s}$ is generated by $M_{\max\{r,s\}}$ and affine root subgroups $U_\sigma$, where $\sigma$ runs over all elements of $\Phi(\mathcal{A})$ such that $\psi(x) \geq r$ and $\psi(y) \geq s$.

Proof. (a) It follows from Lemma A.9 that the product map is injective and that every $g \in G_{x,r} \cap G_{y,s}$ uniquely decomposes as $g = \prod_{\alpha} g_\alpha$ such that $g_\alpha \in U_{(\alpha),x,r}$. It remains to show that $g_\alpha \in U_{(\alpha),y,s}$ for all $\alpha$.

If $(y, s)$ also satisfies the assumption of Lemma A.9 (b), the assertion follows from Lemma A.9 together with the observation that the product map $\prod_{\alpha} U_\alpha \to G$ is injective. Thus we may assume that $s = 0$.

If $r = 0$, then, by our assumption, $x$ lies in a chamber of $[\mathcal{A}]$. Then every $y' \in [x, y)$, close enough to $y$, lies in a chamber $\sigma$ such that $y \in \text{cl}(\sigma)$. Then $g \in G_x \cap G_y \subset G_{y'}$ (see the second paragraph of A.13), thus $g \in G_x \cap G_{y'}$. Thus, by the previous case, $g_\alpha \in U_{(\alpha),y',0} \subset U_{(\alpha),y,0}$.

Finally, if $r > 0$, then there exists a point $x' \in \mathcal{A}$, lying in a chamber in $[\mathcal{A}]$ such that $G_{x',r} \subset G_{x',0}$. Thus, $g \in G_{x',0} \cap G_{y,s}$, hence $g_\alpha \in U_{(\alpha),y,s}$ by the $r = 0$ case.

(b) The assertion (b) follows from (a) and A.7 (f). \hfill \Box

A.12. Proof of Proposition 2.9. Lemma A.9 implies all the cases, except the one for $G_x$, which is not Iwahori. To show the remaining case (which is not used in this work), note that $G_x$ is generated by its Iwahori subgroups $G_y$, where $y$ lies in a chamber $\sigma \subset \mathcal{A}$ such that $x \in \text{cl}(\sigma)$. Since each $G_y$ is generated by $T_0$ and $U_\psi$ with $\psi(y) \geq 0$ by Lemma A.9 (b), the assertion for $G_x$ follows as well. \hfill \Box

A.13. Proof of Corollary 2.10. We have to show that for every $x, y \in \mathcal{X}$, $z \in [x, y]$ and $r \in \mathbb{R}_{\geq 0}$ we have inclusions $G_{x,r} \cap G_{y,r} \subset G_{z,r}$, $g_{x,r} \cap g_{y,r} \subset g_{z,r}$ and $g_{x,-r}^* \cap g_{y,-r}^* \subset g_{z,-r}^*$. Choose an apartment $\mathcal{A} = \mathcal{A}_S \in \mathcal{X}$ such that $x, y \in \mathcal{A}$, and use notation of A.1.
First we show the inclusion \( G_x \cap G_y \subset G_z \). By Lemma 2.14 (a), it remains to show that for every \( g \in G \), the set of fixed points \( x_g^r \) is convex. But this follows from the fact that the action of \( G \) on \( X \) is distance preserving and the uniqueness of geodesics.

Next we show that \( G_{x,r} \cap G_{y,r} \subset G_{z,r} \) for \( r > 0 \). By Corollary B.11 (b), it remains to show that for every \( \psi \in \Psi(\mathcal{A}) \) such that \( \alpha(x) \geq r \) and \( \alpha(y) \geq r \) we have \( \alpha(z) \geq r \). But this follows from the assumption \( z \in [x, y] \). The proof of the inclusion \( g_{x,r} \cap g_{y,r} \subset g_{z,r} \) is similar, but easier.

Finally, to show the inclusion \( g_{x,r}^+ \cap g_{y,r}^+ \subset g_{z,r}^+ \), it remains to show the inclusion \( g_{x,r}^+ \subset g_{x,r}^+ + g_{y,r}^+ \). By Lemma A.9, \( g_{x,r}^+ \) is spanned by \( m_r \) and \( u_\psi \), where \( \psi \) runs over all elements of \( \Psi(\mathcal{A}) \) such that \( \psi(z) > r \) and similarly for \( x \) and \( y \). Thus we have to show that for every \( \psi \in \Psi(\mathcal{A}) \) satisfying \( \psi(z) > r \), we have \( \psi(x) > r \) or \( \psi(y) > r \). But this follows again from the assumption \( z \in [x, y] \). \( \square \)

**A.14. Proof of Lemma 2.14**. For \( r = 0 \), the assertion follows immediately from the observation 2.8 (a). Assume now that \( r > 0 \). Replacing \( F \) by \( F' \) as in A.7, we may assume that \( G \) is quasi-split and split over a totally ramified extension. Then, by Lemma A.9 it remains to show that \( T_r = T_0 \cap T_{r_0} \), which was our definition, and that \( U(\alpha)_{x,r} = (U(\alpha)_{x,r})' \) for every \( \alpha \in \Phi(\mathcal{A})_{nd} \). By A.6 (d), (e), we reduce to the case of \( SL_2 \) and \( SU_3 \), which follow from formulas in A.4 and A.5 (f), respectively. \( \square \)

**A.15. Remark.** The formula of A.5 (f) also implies that the conclusion of Lemma 2.14 is false, if \( G \) is \( SU_3 \), split over a wildly ramified quadratic extension. Therefore, by A.6 (d), (e), the result is also false, if \( G \) is "bad".

**Appendix B. Congruence subsets**

**B.1. Notation.** For every \( r \in \mathbb{R}_{\geq 0} \), we set \( G_r := \bigcup_{x \in X} G_{x,r} \subset G \) and \( g_r := \bigcup_{x \in X} g_{x,r} \subset g \). By construction, both \( G_r \subset G \) and \( g_r \subset g \) are open and \( \text{Ad} G \)-invariant. Moreover, we have \( G_{r+} = \bigcup_{s \geq r} G_s \subset G_r \) and \( g_{r+} = \bigcup_{s > r} g_s \subset g_r \).

**B.2. Remark.** (a) The set of \( r \in \mathbb{R}_{\geq 0} \) such that \( G_{r+} \neq G_r \) (resp. \( g_{r+} \neq g_r \)) is discrete. For example, this follows from the fact that any such \( r \) is optimal in the sense of [ADB 2.3]. Alternatively, this can be seen as follows.

Choose any \( r \) such that \( G_{r+} \neq G_r \), and choose a chamber \( \sigma \in [X] \). Since all chambers are \( G \)-conjugate, there exists \( x \in \text{cl}(\sigma) \) such that \( G_{x,r} \notin G_{r+} \). Choose \( k \in \mathbb{Z} \) such that \( r \in [k, k+1] \). It remains to show that the set of subgroups \( \{ G_{x,s} \}_{x \in \text{cl}(\sigma), s \in [k, k+1]} \) is finite.

Choose an apartment \( \mathcal{A} \subset X \) such that \( \sigma \subset \mathcal{A} \) and fix \( x \in \text{cl}(\sigma) \) and \( s \in [k, k+1] \). Then the set \( \{ \psi \in \Psi(\mathcal{A}) \mid \psi(x) \geq s \} \) contains the set \( \{ \psi \in \Psi(\mathcal{A}) \mid \psi(x) > k + 1 \} \) and is contained in the set \( \{ \psi \in \Psi(\mathcal{A}) \mid \psi(x) > k \} \). This implies the assertion.

(b) It can be shown that every \( r \) from (a) is rational. But even without this fact it follows from (a) that for every \( r \in \mathbb{R}_{\geq 0} \), there exist \( r', r'' \in \mathbb{Q}_{\geq 0} \) such that \( G_r = G_{r'} \).
and \( G_{r^+} = G_{r''} \), and similarly for \( g \). Thus Lemma 7.4 follows from the following assertion.

**Lemma B.3.** For every \( r \in \mathbb{R}_{\geq 0} \), the subsets \( G_r \subset G \), \( g_r \subset g \) and \( g^*_{r^+} \subset g^* \) are open, closed and stable.

**Proof.** First we show that \( G_0 \subset G \) is closed. Note that \( G^0 \subset G \) is closed (see 2.4 (c),(d)). By the Bruhat-Tits fixed point theorem and observation 2.8 (a), \( G_0 \) coincides with the set of all compact elements of \( G \). But the set of all compact elements of \( G \) is closed. Indeed, choose a faithful representation \( \rho : G \twoheadrightarrow \text{GL}_n \), and notice that \( g \in G \) is compact if and only if \( \det(\rho(g)) \in \mathcal{O}^\times \) and the characteristic polynomial of \( \rho(g) \) has coefficients in \( \mathcal{O} \).

Next we show that \( G_r \subset G \) is closed for \( r > 0 \). Since \( G_0 := \bigcup_{x \in [X]_0} G_x \subset G \) is closed, and each \( G_x \) is open and compact, it remains to show that for every \( x \in [X]_0 \), the intersection \( G_x \cap G_r \) is compact. By 3.2 (b), we may assume that \( r \in \mathbb{Q} \), hence \( r = \frac{1}{m} \mathbb{Z}_{\geq 0} \) for some \( m \in \mathbb{N} \). As in 5.6, for every \( \Sigma \in \Theta_m \), we set \( G_{\Sigma,r} := \bigcup_{\sigma \in \Sigma} G_{\sigma,r} \). Then each \( G_x \cap G_{\Sigma,r} \) is compact, and it suffices to show that \( G_x \cap G_r = G_x \cap G_{\Sigma,r} \) for every \( \Sigma \supset \Upsilon_{x,r} \). Equivalently, it suffices to show the equality of functions \( 1_{G_x} \cdot 1_{G_{\Sigma,r}} = 1_{G_x} \cdot 1_{G_{\Sigma',r}} \) for every \( \Sigma, \Sigma' \in \Theta_m \) such that \( \Upsilon_{x,r} \subset \Sigma' \subset \Sigma \).

As in Lemma 5.7, we deduce from Corollary 2.10 that for every \( \Sigma \in \Theta_m \) we have \( 1_{G_{\Sigma,r}} = \sum_{\sigma \in \Sigma} (-1)^{\dim \sigma} 1_{G_{\sigma,r}} \). Thus we have to show that for every \( \Sigma' \subset \Sigma \) as above, we have \( \sum_{\sigma \in \Sigma' \setminus \Sigma} (-1)^{\dim \sigma} (1_{G_x} \cdot 1_{G_{\sigma,r}}) = 0 \). Arguing as in Proposition 4.2 (a), it remains to show that for every \( \sigma, \sigma' \in [X]_m \) with \( \sigma' < \sigma \) and \( \sigma \in \Gamma_r(\sigma', x) \) we have the equality \( 1_{G_x} \cdot 1_{G_{\sigma,r}} = 1_{G_x} \cdot 1_{G_{\sigma',r}} \). Equivalently, we have to show that \( G_x \cap G_{\sigma,r} = G_x \cap G_{\sigma',r} \), that is, \( G_x \cap G_{\sigma',r} \subset G_{\sigma,r} \).

Choose an apartment \( A \subset X \), containing \( \sigma, x \). By Corollary A.11 (b), the intersection \( G_x \cap G_{\sigma',r} \) is generated by \( M_r \) and affine root subgroups \( U_\psi \), where \( \psi \in \Psi(A) \) satisfies \( \psi(x) \geq 0 \) and \( \psi(\sigma') \geq r \). Thus we have to show that for every \( \psi \in \Psi(A) \) such that \( \psi(x) \geq 0 \) and \( \psi(\sigma') \geq r \) we have \( \psi(\sigma) \geq r \). Replacing \( \psi \) with \( r - \psi \) it suffices to show that for every \( \psi \in \Psi_m(A) \) with \( \psi(x) \leq r \) and \( \psi(\sigma') \leq 0 \), we have \( \psi(\sigma) \leq 0 \). But this is precisely the assumption \( \sigma \in \Gamma_r(\sigma', x) \).

This shows that every \( G_r \) is closed. To show that \( G_r \) is stable, we need to show that for every \( \mathbf{G}^{(F)} \)-conjugate \( g, g' \in G^{ur} \) such that \( g \in G_r \), we have \( g' \in G_r \). In other words, we have to show that the subset \( X(\sigma', x) \subset X \) consisting of all \( x \in X \) such that \( g' \in G_{x,r} \), is non-empty.

Since \( g \) and \( g' \) are \( \mathbf{G}^{(F)} \)-conjugate, and \( F^{ur} \) is of cohomological dimension one, we conclude that \( g, g' \) are \( \mathbf{G}(F^{ur}) \)-conjugate, thus \( \mathbf{G}(F^p) \)-conjugate for some finite unramified extension \( F^p/F \). Set \( \mathbf{G}^p := \mathbf{G}_{F^p}, X^p := X(\mathbf{G}^p) \) and \( \Gamma^p := \text{Gal}(F^p/F) \). Then \( g \in G_r \subset \mathbf{G}^p \), hence \( g' \in G \cap G^p \), because \( G^p \) is \( \text{Ad} G^p \)-invariant. Thus the subset \( X^p(\sigma', x) \subset X^p \) is non-empty. On the other hand, \( X^p(\sigma', x) \) is \( \Gamma^p \)-invariant,
because $g' \in G$, and convex, by Corollary 2.10. Thus, by the Bruhat–Tits fixed point theorem, the set of fixed points $X^g(g', r)^{\Gamma_0}$ is non-empty. Since $X^g(g', r)^{\Gamma_0}$ equals $X^g(g', r) \cap X = X(g', r)$ (by 2.3 (d) and 2.8 (b)), we are done.

The proof for $g_r$ is similar. Namely, for every $x \in [X]_0$, we have $g = \bigcup_n x^{-n}g_x$. Thus to show that $g_r$ is closed, it remains to show that every $g_r \cap x^{-n}g_x$ is compact. Since $x^{-n}g_x = g_{r+n}$ (see A.8 (b)), it remains to show that the intersection $g_{r+n} \cap g_x$ is compact. This case be shown, as in the group case.

Finally, the prove the result for $g^{*r}$ we can either mimic the proof for $g_r$, using the decomposition for $g_{x-r}$, obtained from Lemma A.9 (a) by duality, or to deduce it from a Lie algebra version of Proposition 4.2 (a) by the Fourier transform.

\begin{proof}[B.4. Proof of Lemma 7.11] It suffices to show that $\pi$ induces bijections $\pi_x : G_{x,r+}^{\Gamma_0} \to G_{x,r+}$ and $\pi_{x,y} : G_{x,r+}^{\Gamma_0} \cap G_{y,r+}^{\Gamma_0} \to G_{x,r+} \cap G_{y,r+}$ for every $x, y \in X(G) = X(G')$. Indeed, the surjectivity of $G_{r+}^{\Gamma_0} \to G_{r+}$ follows from the surjectivity of the $\pi_x$’s, while injectivity follows from the injectivity of the $\pi_{x,y}$’s.

Replacing $F$ by $F'$ as in A.7, we may assume that $G$ and $G'$ are quasi-split over $F$ and split over a totally ramified extension. Choose an apartment $A \ni x, y$, corresponding to a maximal split torus $S \subset G$, and set $T := Z_G(S)$, and $T' := \pi^{-1}(T) \subset G'$. Then $T \subset G$ is a maximal torus, and both $G_{x,r+}$ and $G_{x,r+} \cap G_{y,r+}$ decompose as $T_{x+} \times \prod U_{\alpha, x,r+}$ and $T_{r+} \times \prod U_{\alpha, x,r+} \cap U_{\alpha, y,r+}$, respectively (by Lemma A.9 and Corollary A.11), and similarly for $G'$.

Since $\pi$ induces an isomorphisms between the $U_{\alpha}$’s, it remains to show that the induced map $T'_r \to T_r$ is an isomorphism. If $T$ and $T'$ are split, the assertion is easy. Namely, $\pi$ induces a morphism of $F$-vector spaces $\pi_n : T'_n/T_{n+1} \to T_n/T_{n+1}$ for every $n > 0$. Hence each $\pi_n$ is an isomorphism, because the degree of $\pi$ is prime to $p$, hence $T'_r \to T_r$ is an isomorphism as well.

In general, let $F^0$ be the splitting field of $T$ (and $T'$), $e$ be the ramification degree of $F^0/F$, and set $r^0 := er$, and $\Gamma^0 := \text{Gal}(F^0/F)$. Then $T_{r^0} = \text{Ker} \kappa_T \cap T(F^0)^{\Gamma^0}_{(r^0)^+}$, where $\kappa_T$ is the Kottwitz homomorphism $T(F^0) \to X_*(T)_{\Gamma^0}$ (see 2.4), and similarly for $T'$. By the split case, $\pi$ induces an isomorphism $T'(F^0)^{\Gamma^0}_{(r^0)^+} \to T(F^0)^{\Gamma^0}_{(r^0)^+}$ of pro-$p$-groups. So it remains to check that every element in the kernel of the homomorphism $X_*(T')_{\Gamma^0} \to X_*(T)_{\Gamma^0}$ is torsion of prime to $p$ order. Since this kernel is killed by $\deg \pi$, the proof is complete.
\end{proof}

**Appendix C. Quasi-logarithms**

**C.1. Quasi-logarithms.** Let $G$ be a reductive group over a field $F$. 
(a) Following [KV1, 1.8], we call an $\text{Ad} \ G$-equivariant algebraic morphism $L : G \to g$ a \textit{quasi-logarithm}, if $L(1) = 0$, and the induced map on tangent spaces $dL_1 : g = T_1(G) \to T_0(g) = g$ is the identity map.

(b) Let $F^0/F$ be a field extension. Then a quasi-logarithm $L : G \to g$ induces a quasi-logarithm $L_{F^0} : G_{F^0} \to g_{F^0}$. Conversely, a quasi-logarithm $L^0 : G_{F^0} \to g_{F^0}$ induces a quasi-logarithm $R_{F^0/F}(L^0) : R_{F^0/F}(G_{F^0}) \to R_{F^0/F}(g_{F^0}) = g \otimes_F F^0$.

c) Since $L$ is $\text{Ad} \ G$-equivariant, it induces a morphism $[L] : c_G \to c_g$ of the corresponding Chevalley spaces (compare [KV2, 5.2]).

C.2. Quasi-logarithms defined over $\mathcal{O}$. Let $F$ be a local non-archimedean field of residual characteristic $p$.

(a) Assume that $G$ is split over $F$. Then the Chevalley spaces $c_G$ and $c_g$ have natural structures over $\mathcal{O}$. In this case, we say that a quasi-logarithm $L : G \to g$ is \textit{defined over $\mathcal{O}$}, if the corresponding map $[L]$ is defined over $\mathcal{O}$ (compare [KV2, 5.2]). Note that by [KV2, Lem 5.2.1] this notion is equivalent to the corresponding notion of [KV1, 1.8.8].

(b) For an arbitrary $G$, we say that $L : G \to g$ is defined over $\mathcal{O}$, if $L_{F^0}$ is defined over $\mathcal{O}_{F^0}$ for some or, equivalently, every splitting field $F^0$ of $G$.

(c) Let $F^0/F$ be a finite Galois extension, and let $L^0 : G_{F^0} \to g_{F^0}$ be a quasi-logarithm defined over $\mathcal{O}_{F^0}$. Then the quasi-logarithm $R_{F^0/F}(L^0)$ (see C.1 (b)) is also defined over $\mathcal{O}$.

(d) In the situation of (c), assume that $[F^0 : F]$ is prime to $p$. Then the composition

$$L : G \hookrightarrow R_{F^0/F}G_{F^0} \xrightarrow{R_{F^0/F}(L^0)} g \otimes_F F^0 \xrightarrow{[F^0 : F]} \text{Tr}_{F^0/F} g$$

is a quasi-logarithm defined over $\mathcal{O}$.

Lemma C.3. Assume that $G$ is semisimple and simply connected and $p$ is very good for $G$ (see 7.9). Then $G$ admits a quasi-logarithm defined over $\mathcal{O}$.

Proof. (compare [KV1 Lem 1.8.12]). Assume that $G = \prod_i R_{F_i/F}H_i$ as in 7.9. By C.2 (c), we can replace $G$ by $H_i$, thus assuming that $G$ is absolutely simple. Using [KV1 Lem 1.8.9], we can replace $G$ by its quasi-split inner form. Since $p$ is good, $G$ splits over a tamely ramified extension. Hence, using C.2 (d), we may extend scalars to the splitting field of $G$, thus assuming that $G$ is split. In this case, the assertion was shown in [KV1 Lem 1.8.12], using the fact that $G$ has a faithful representation, whose Killing form is non-degenerate over $\mathcal{O}$. Namely, one uses the standard representation, if $G$ is classical, and the adjoint representation, if $G$ is exceptional.
Lemma C.4. Let $G$ be semi-simple and simply connected, $p \neq 2$, and let $\mathcal{L} : G \to g$ be a quasi-logarithm defined over $O$. Then for every $x \in X$ and $r \in \mathbb{R}_{\geq 0}$, $\mathcal{L}$ induces analytic isomorphisms $\mathcal{L}_r : G_{x,r} \to g_{x,r}$ and $\mathcal{L}_{x,r} : G_{x,r} \to g_{x,r}$.

Proof. Assume first that $G$ is split. The assertion for $r = 0$ was shown in [KV1 Prop 1.8.16]. Next we want to show that $\mathcal{L}$ induces an analytic isomorphism $\mathcal{L}_{x,r} : G_{x,r} \to g_{x,r}$ when $x \in X'$ is a hyperspecial vertex and $r = n \in \mathbb{Z}$. In this case, $G_{x,r} = G_{x,n+1}$ and $g_{x,r} = g_{x,n+1}$, so we have to show that $\mathcal{L}$ induces an analytic isomorphism $G_{n+1} \iso g_{n+1}$. But is easy and it was shown in the course of the proof of [KV1 Prop 1.8.16]. We are going to deduce the general case from the particular case, shown above.

Let $F^0/F$ be a finite Galois extension of ramification degree $e$, and set $r^0 := er$, $\Gamma^0 := \text{Gal}(F^0/F)$ and $G^0 := G_{F^0}$. Then $\mathcal{L}$ induces a quasi-logarithm $\mathcal{L}^0 := \mathcal{L}_{F^0} : G^0 \to g^0$, which is $\Gamma^0$-equivariant and defined over $O_{F^0}$. Moreover, since $G$ is semi-simple and simply connected, we have $G^0 = G$ (see 2.4 (d)), thus $G_{x,r} = (G^0_{x,(r^0)})^{F^0}$ and $g_{x,r} = (g^0_{x,(r^0)})^{F^0}$ (see Lemma 2.11).

Note that the assertion for $\mathcal{L}^0$ and $r^0$ implies that for $\mathcal{L}$ and $r$. Indeed, if $\mathcal{L}^0$ induces an isomorphism $\mathcal{L}_{x,r}^0 := (\mathcal{L}_{x,r})^\Gamma$ of Galois invariants. Therefore $\mathcal{L}$ induces a morphism $\mathcal{L}_r : G_{r} \to g_{r}$, which is surjective, because each $\mathcal{L}_{x,r}$ is surjective, and injective, because $\mathcal{L}_{r}^0$ is injective. Thus we can replace $F$ by $F^0$, $G$ by $G^0$, and $r$ by $r^0$.

Now the assertion is easy. Indeed, choosing $F^0$ to be a splitting field of $G$, we can assume that $G$ is split. Since $\mathcal{L}_0$ is injective, it is enough to show that $\mathcal{L}$ induces an isomorphism $\mathcal{L}_{x,r}$. Observe that both $G_{x,r}$ and $g_{x,r}$ do not change if we replace pair $(x,r)$ by a close pair $(x',r')$. Thus we may assume that $r \in \frac{1}{m}\mathbb{Z}_{\geq 0}$ and $x$ is a hyperspecial vertex of $[X_m(G)]$ for some $m$.

Choose a finite extension $F^0$ of $F$ of ramification degree $m$. Thus, $r^0 = mr \in \mathbb{N}$ and $x$ is a hyperspecial vertex of $[X_m(G)] \subset [X(G^0)]$. Hence the assertion for $\mathcal{L}_{x,r}$, shown in the first paragraph of the proof, implies the assertion for $\mathcal{L}_{x,r}$. \qed

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Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139, USA
E-mail address: bezrukav@math.mit.edu

Institute of Mathematics, The Hebrew University of Jerusalem, Givat-Ram, Jerusalem, 91904, Israel
E-mail address: kazhdan@math.huji.ac.il

Institute of Mathematics, The Hebrew University of Jerusalem, Givat-Ram, Jerusalem, 91904, Israel
E-mail address: vyakov@math.huji.ac.il