Abstract. Let $L/K$ be a Galois extension of fields with Galois group $G \cong (\mathbb{F}_p^n, +)$, an elementary abelian $p$-group of rank $n$ for $p$ an odd prime. It is known that nilpotent $\mathbb{F}_p$-algebra structures $A$ on $G$ yield regular subgroups of the holomorph $\text{Hol}(G)$, and in turn Hopf Galois structures on $L/K$. In this paper we describe these structures in two cases. If $A^3 = 0$ the corresponding regular subgroup yields Hopf Galois structures directly, without translating from $\text{Hol}(G)$ to $\text{Perm}(G)$. We determine the number of Hopf Galois structures that arise when $\dim(A^2) = 1$, which yields a new lower bound on the number of Hopf Galois structures on $L/K$ when $n = 4$. We also look at the case when $\dim(A/A^2) = 1$.

1. Introduction

In 1969, Chase and Sweedler defined the notion of a Hopf Galois extension of fields by abstracting the formal properties of a classical Galois extension of fields. In 1987, Greither and Pareigis discovered that a classical Galois extension $L/K$ of fields with Galois group $G$ could also be a Hopf Galois extension for a $K$-Hopf algebra $H$ other than $H = KG$, the group ring of the Galois group, acting in the obvious way on $L$. [GP87] showed that determining the number of Hopf Galois structures on $L/K$ depends solely on the Galois group $G$: more precisely, they correspond to regular subgroups of the permutation group of $G$ that are normalized by the image $\lambda(G)$ of the left regular representation of $G$ in $\text{Perm}(G)$.

Since the appearance of [GP87] there has been a fairly steady sequence of papers studying the number of Hopf Galois structures on a Galois extension of fields $L/K$ with Galois group $G$. These range from Byott’s uniqueness paper [By96] and his theorem [By04] that if $G$ is a non-abelian simple group then $L/K$ has exactly two Hopf Galois structures, to papers that describe large numbers of Hopf Galois structures on $L/K$ for suitable Galois groups $G$. A recent example of the latter is [Ch15], which showed that the number of abelian Hopf Galois...
structures on $L/K$ where $G$ is elementary abelian of order $p^n$, $p$ an odd prime, is a number asymptotic to $p^b$ where $b = (2/27)n^3$, as $n$ goes to infinity.

But there has been relatively little work that says much about what the Hopf Galois structures look like. The case where $G$ has order $p^2$ was handled by Byott [By02], the Hopf algebras corresponding to the Hopf Galois structures on a cyclic Kummer extension of degree $p^n$ were studied in [Ch11], and in [Ko07], [Ko13], and [Ko16] Kohl works entirely within the Greither-Pareigis setting. Other examples are rare. Counting Hopf Galois structures on a field extension with Galois group $G$ is made easier by translating the problem of finding regular subgroups $J$ of $\text{Perm}(G)$ that are isomorphic to a given group $N$ and are normalized by $\lambda(G)$ to a problem of finding regular subgroups $T$ of $\text{Hol}(N)$. This translation from $\text{Perm}(G)$ to $\text{Hol}(N)$ was first codified in [By96], and has been the approach of choice for most papers devoted to counting Hopf Galois structures.

Let $L/K$ be a Galois extension of fields with Galois group $G \cong (\mathbb{F}_p^n, +)$. This is the third paper to apply the result of [CDVS06] that relates commutative nilpotent $\mathbb{F}_p$-algebra structures $A$ on $G$ to regular subgroups of the holomorph of $G \cong (\mathbb{F}_p^n, +)$, and hence to Hopf Galois structures on $L/K$. The first was [FCC12], which proved that if $N$ is a regular subgroup of $\text{Hol}(G)$ of $p$-rank $m$ where $m + 1 < p$, then $N \cong G$, thereby limiting the type of abelian Hopf Galois structures on $L/K$. In [Ch15], we used results on the number of isomorphism types of nilpotent algebras of dimension $n$ for large $n$ to obtain asymptotic results on the number of Hopf Galois structures on $L/K$ for large $n$, in part by specializing to algebras $A$ with $A^3 = 0$. The number of isomorphism types of commutative nilpotent $\mathbb{F}_p$-algebras $A$ of dimension $n$ with $A^3 = 0$, the number of Hopf Galois structures on a Galois extension with Galois group $G \cong (\mathbb{F}_p^n, +)$, and the number of isomorphism types of all commutative nilpotent dimension $n \mathbb{F}_p$-algebras are all asymptotic to $p^{27n^3}$ as $n$ goes to infinity. A fourth, [Ch16b], will study the Galois correspondence for Hopf Galois extensions that correspond to nilpotent algebra structures.

In this paper we begin studying the Hopf Galois structures on $L/K$ themselves. We observe that if $(G, +)$ is a finite abelian $p$-group, $p \geq 5$ and $A$ is a commutative nilpotent ring structure on $(G, +)$ with $A^3 = 0$, then all regular subgroups of $\text{Hol}(G) \subset \text{Perm}(G)$ arising from $A$ are normalized by $\lambda(G)$ and isomorphic to $G$. Thus if $L/K$ is a Galois extension of fields with Galois group $G$, then the nilpotent ring structure $A$ on $G$ yields a Hopf Galois structure on $L/K$ directly by descent, as
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in [GP87], avoiding the translation from the holomorph to the permutation group. The process of going from the commutative nilpotent algebra to the descent data needed to describe the Hopf Galois structure is straightforward.

In particular, we determine the isomorphism types of commutative nilpotent \( \mathbb{F}_p \)-algebras \( A \) of dimension \( n \) with \( A^3 = 0 \) and \( \dim(A^2) = 1 \), and determine the number of regular subgroups of \( \text{Hol}(G) \) associated to each isomorphism type. For \( n = 4 \) this approach yields more than \( p^9 \) regular subgroups. We describe the Hopf Galois structure on \( L/K \) corresponding to each regular subgroup arising from a given isomorphism type of algebra.

In a final section we obtain the data needed to construct, by descent, a Hopf Galois structure on a Galois extension \( L/K \) with Galois group \( G \cong (\mathbb{F}_p^n, +) \) where the corresponding nilpotent algebra \( A \) has basis \( \{x, \ldots, x^n\} \) with \( x^{n+1} = 0 \). These examples include the only degree 3 commutative nilpotent algebra \( A \) which does not have \( A^3 = 0 \), hence is the only algebra for which the corresponding regular subgroups are not normalized by \( \lambda(G) \), and so the data for constructing the Hopf Galois structure must be obtained by translating from the holomorph to \( \text{Perm}(G) \). This section complements results in [Ch07], which determined the number of Hopf Galois structures corresponding to such algebras.

Throughout, let \( L/K \) be a Galois extension of fields with Galois group \( \Gamma \), an elementary abelian \( p \)-group of order \( p^n \), \( p \) an odd prime. For a finite abelian \( p \)-group \( G \) with operation \(+\), a ring structure \( A = (G, +, \cdot) \) on \( G \) will be called nilpotent if \( A \) is commutative, associative and nilpotent: \( A^m = 0 \) for some \( m > 1 \).

This note was inspired by a remark by Tim Kohl that for \( G \) abelian many regular subgroups of \( \text{Perm}(G) \) normalized by \( \lambda(G) \) can be found inside \( \text{Hol}(G) \). Many thanks to him for sharing his enthusiasm with me.

2. HOPF GALOIS STRUCTURES WHEN \( A^3 = 0 \)

Let \( G \) be a finite abelian \( p \)-group, written additively. We use the idea, due to Caranti, Dalla Volta and Sala [CDVS06], c. f. [FCC12], that regular subgroups of \( \text{Hol}(G) \), the holomorph of \( G \), correspond to commutative, associative nilpotent ring structures on the additive group \( (G, +) \).

Let \( A \) be a nilpotent ring structure \((G, +, \cdot)\) on \( G \). Define a new group structure on the set \( G \), call it \((G, \circ)\), by

\[ x \circ y = x + y + x \cdot y. \]
Then it is obvious that $\circ$ is commutative, and easy to see that $\circ$ is associative. The element 0 is the additive identity, and the inverse of $x$ is

$$-x + x^2 - x^3 + \ldots,$$

a finite sum since $A$ is nilpotent. Then $N = (G, \circ)$ is the group associated to $A$. Define an embedding

$$\tau : N \to \text{Perm}(G)$$

by

$$\tau(x)(z) = x \circ z = x + z + x \cdot z$$

for all $z$ in $G$. Then $\tau$ is a homomorphism, because

$$\tau(x)\tau(y)(z) = \tau(x)(y \circ z) = x \circ (y \circ z) = (x \circ y) \circ z = \tau(x \circ y)(z).$$

So $T = \tau(N)$ is a regular subgroup of $\text{Perm}(G)$ because $\tau(x)(0) = x \circ 0 = x$ for all $x$ in $G$. Moreover, $\tau(N) \subseteq \text{Hol}(G)$ the normalizer of $\lambda(G)$ in $\text{Perm}(G)$, because $\lambda(z)(w) = z + w$ for $y, w$ in $(G, +)$, and so

$$\tau(x)\lambda(z)(w) = \lambda(z + x \cdot z)\tau(x)(w)$$

for all $w$ in $G$.

A key observation of this paper is:

**Proposition 2.1.** Let $(G, +)$ be an abelian $p$-group of finite order $p^n$, and let $A = (G, +, \cdot)$ be a nilpotent ring structure on $(G, +)$. Let $N = (G, \circ)$ and let $\tau$ be the associated regular embedding of $N$ in $\text{Hol}(G)$ with $T = \tau(N) \subseteq \text{Perm}(G)$. Then $T$ is normalized by $\lambda(G)$ if and only if $A^3 = 0$.

**Proof.** We have that for all $z$ in $G$,

$$\tau(x)(z) = x + z + x \cdot z$$

while

$$\lambda(y)(z) = y + z.$$

Suppose for each $x, y$ in $A$ there is some $w$ in $A$ so that

$$\lambda(y)\tau(x)\lambda(-y) = \tau(w).$$

Applied to an element $z$ of $G$, the left side is

$$\lambda(y)\tau(x)\lambda(-y)(z) = y + \tau(x)(z - y) = y + x + (z - y) + x \cdot (z - y) = x + z + x \cdot z - x \cdot y.$$

The right side is

$$w + z + wz.$$
So for all \( z \) in \( A \), we must have
\[
x + z + x \cdot z - x \cdot y = w + z + w \cdot z.
\]
In particular, for \( z = 0 \), we have
\[
x - x \cdot y = w.
\]
Then for all \( z \) in \( A \), we must have
\[
x - x \cdot y + x \cdot z = x - x \cdot y + x \cdot z - x \cdot y \cdot z.
\]
This holds if and only if \( x \cdot y \cdot z = 0 \). Thus if \( A^3 = 0 \), then for all \( x, y \) in \( A \),
\[
\lambda(y)\tau(x)\lambda(-y) = \tau(x - x \cdot y)
\]
so \( T \) is normalized by \( \lambda(G) \). But if \( A^3 \neq 0 \), there exist \( x, y \) in \( A \) so that \( \lambda(y)\tau(x)\lambda(-y) \) is not \( \tau(w) \) for any \( w \) in \( A \), and so \( T = \tau(N) \) is not normalized by \( \lambda(G) \) in \( \text{Perm}(G) \). \( \square \)

We identify the corresponding Hopf Galois structures:

Corollary 2.2. Let \( L/K \) be a Galois extension with Galois group \( G \), an abelian \( p \)-group of order \( p^n \). Let \( A = \langle G, +, \cdot \rangle \) be a commutative nilpotent ring structure on \( (G, +) \) and suppose \( A^3 = 0 \). Let \( T \cong \langle G, \circ \rangle \) be the regular subgroup of \( \text{Hol}(G) \subset \text{Perm}(G) \) corresponding to \( A \). Then \( T \) yields a Hopf Galois structure on \( L/K \) by a \( K \)-Hopf algebra \( H \), where

i) \( H \) is the fixed ring of \( LT \) under the action of \( G \):
\[
H = LT^G = \{ \sum_{x \in G} b_x \tau(x) : b_{x-x \cdot z} = b_x^z \text{ for all } z \text{ in } G \};
\]

ii) \( H \) acts on \( L \) by
\[
(\sum_{x \in G} b_x \tau(x))(a) = \sum_{x \in G} b_x a^{-x+x^2}
\]
for \( b, a \) in \( L \).

Proof. Let \( \{e_z : z \in G\} \) be the dual basis to the basis \( G \) of the group ring \( L[G] \). The action of \( T \) on \( GL = \sum_{z \in G} Le_z \) is by
\[
\tau(x)(e_z) = e_{xoz}
\]
for \( x \) in \( G \). This yields an action of the group ring \( LT \) on \( GL \) making \( GL \) an \( LT \)-Hopf Galois extension of \( L \). Since \( \lambda(G) \) acts on \( T \) by
\[
\lambda(z)\tau(x)\lambda(-z) = \tau(x - x \cdot z),
\]
the corresponding \( K \)-Hopf algebra is
\[
H = LT^G = \{ \sum_{x \in G} b_x \tau(x) : b_{x-x \cdot z} = b_x^z \text{ for all } z \text{ in } G \}
where for $a$ in $L$ and $y$ in $G$, $a^y$ is the image of $a$ under the Galois action of $y$ on $L$, and $H$ acts on $GL$ by

$$\left(\sum_{x \in G} b_x \tau(x)\right)\left(\sum_{y \in G} a_y e_y\right) = \sum_{x, y \in G} b_x a_y e_{x \circ y}.$$ 

Now $L$ embeds in $GL$ by

$$a \mapsto \sum_{y \in G} a^y e_y.$$ 

So the action of $H$ on $L$ is by

$$\left(\sum_{x \in G} b_x \tau(x)\right)(a) = \sum_{x, y \in G, x \circ y = 0} b_x a^y.$$ 

Since $x^3 = 0$, $x \circ (-x + x^2) = 0$, and so the action of $H$ on $L$ can be written

$$\left(\sum_{x \in G} b_x \tau(x)\right)(a) = \sum_{x \in G} b_x a^{-x + x^2}.$$

□

There is no a priori reason why $(G, \circ)$ and hence $T$ should be isomorphic to $G$, so that $H$ has type $G$. But it is true: we note the following variant of Theorem 1 of [FCC12]:

**Proposition 2.3.** Let $p > 3$ be an odd prime and $G = (G, +)$ be a finite abelian $p$-group of order $p^n$. Let $A = (G, +, \cdot)$ be a commutative nilpotent ring structure on $(G, +)$ and suppose $A^3 = 0$. Then the regular subgroup $N = (G, \circ)$ of $\text{Hol}(G) \subset \text{Perm}(G)$ is isomorphic to $(G, +)$.

The statement of Theorem 1 of [FCC12] replaces the condition $A^3 = 0$ in Proposition 2.3 by the condition that the $p$-rank $m$ of $G$ should satisfy $m + 1 < p$. The proof of Proposition 2.3 is essentially the same as that of Theorem 1 of [FCC12]. The only change is that the condition $A^3 = 0$ implies that $a^p = 0$ for all $a$ in $A$, which slightly simplifies the proof in [FCC12] by eliminating the need to apply a condition on the $p$-rank of $G$ to insure that $a^p$ does not interfere with the induction argument.

### 3. Working in the Affine Group

For the remainder of the paper we restrict $G$ to be an elementary abelian $p$-group of $p$-rank $n > 1$. For such groups $G$ there are non-trivial commutative nilpotent ring structures $A$ on $G$ with $A^3 = 0$. In fact, as shown in [Ch15], the number of isomorphism types of such structures is bounded from below by $p^b$ where $b = O(n^3)$. 
Each such ring is an $\mathbb{F}_p$-algebra. Let $\dim_{\mathbb{F}_p} A/A^2 = r$ and $\dim_{\mathbb{F}_p} A^2 = n - r$. Given an $\mathbb{F}_p$-basis $(x_1, \ldots, x_r)$ of $A/A^2$ and a basis $(y_1, \ldots, y_{n-r})$ of $A^2$, the multiplicative structure of $A$ with those bases is given by a set $\Phi^{(k)} = (\phi_{ij}^{(k)})$ of $r \times r$ matrices with coefficients in $\mathbb{F}_p$, by the equations

$$x_i x_j = \sum_{k=1}^{n-r} \phi_{ij}^{(k)} y_k.$$  

The group structure $(G, \circ)$ on $G$ arising from $(A, +, \cdot)$ depends on the matrices $\{\Phi^{(k)}\}$, and hence so does the regular subgroup $T$ of $\text{Perm}(G)$.

It is convenient to view $\text{Hol}(G)$ as the affine group $\text{Aff}_n(\mathbb{F}_p)$ and realize the regular subgroup $T$ inside $\text{Aff}_n(\mathbb{F}_p)$.

Let $\text{Aff}_n(\mathbb{F}_p)$ be the subset of $\text{GL}_{n+1}(\mathbb{F}_p)$ consisting of matrices of the form

$$\begin{pmatrix} B & \overline{v} \\ 0 & 1 \end{pmatrix},$$

where $B$ is an $n \times n$ matrix, $\overline{v}$ is a column vector in $\mathbb{F}_p^n$, $0$ is a $n$-row vector of zeros and $1$ is a $1 \times 1$ identity matrix. Then $\text{Aff}_n(\mathbb{F}_p)$ may be identified as the holomorph $\text{Hol}(\mathbb{F}_p^n) = \lambda(\mathbb{F}_p^n) \cdot \text{Aut}(\mathbb{F}_p^n)$ of the additive group $\mathbb{F}_p^n$, where the matrices

$$\begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}$$

with $P$ in $\text{GL}_n(\mathbb{F}_p)$ form the subgroup $\text{Aut}(\mathbb{F}_p^n)$ of $\text{Hol}(\mathbb{F}_p^n)$, and matrices

$$\begin{pmatrix} I & \overline{x} \\ 0 & 1 \end{pmatrix}$$

for $\overline{x}$ in $\mathbb{F}_p^n$ form the subgroup $\lambda(\mathbb{F}_p^n)$.

Let $A = (\mathbb{F}_p^n, +, \cdot)$ be a commutative nilpotent $\mathbb{F}_p$-algebra of dimension $n$. The circle product on $\mathbb{F}_p^n$ by

$$\overline{x} \circ \overline{y} = \overline{x} + \overline{y} + \overline{x} \cdot \overline{y}$$

yields a group $N$ which embeds as a regular subgroup of $\text{Aff}_n(\mathbb{F}_p)$ as follows:

Let $\tau(\overline{x})$ be the function from $\mathbb{F}_p^n$ to $\mathbb{F}_p^n$ given by $\tau(\overline{x})(\overline{y}) = \overline{x} \circ \overline{y} = \overline{x} + \overline{y} + \overline{x} \cdot \overline{y}$. Write $\tau(\overline{x})(\overline{y}) = \overline{x} + \overline{y} + L_{\overline{x}}(\overline{y})$, where $L_{\overline{x}}(\overline{y}) = \overline{x} \cdot \overline{y}$. Then $L_{\overline{x}}$ is a linear function from $\mathbb{F}_p^n$ to $\mathbb{F}_p^n$, so has a matrix relative to the standard basis of $\mathbb{F}_p^n$ that we also call $L_{\overline{x}}$.

Then $\tau(\overline{x})$ in $\text{Aff}_n(\mathbb{F}_p)$ becomes the $n + 1 \times n + 1$ matrix

$$T_{\overline{x}} = \begin{pmatrix} I + L_{\overline{x}} & \overline{x} \\ 0 & 1 \end{pmatrix},$$
because for any \( \overline{y} \) in \( \mathbb{F}_p^n \), we have

\[
T \begin{pmatrix} \overline{y} \\ 1 \end{pmatrix} = \begin{pmatrix} \overline{y} + L_T(\overline{y}) + \overline{x} \\ 1 \end{pmatrix} = \begin{pmatrix} \overline{y} + \overline{x} \cdot \overline{y} + \overline{x} \\ 1 \end{pmatrix} = \begin{pmatrix} \tau_T(\overline{y}) \\ 1 \end{pmatrix}.
\]

4. The case \( r = n - 1 \)

We consider the class of examples where \( \dim(A) = n, \dim(A^2) = 1, A^3 = 0 \). This class includes all examples of nilpotent \( \mathbb{F}_p \)-algebra structures on \( (\mathbb{F}_p^n, +) \) for \( n = 2 \) and all but one for \( n = 3 \). Then \( A \) has the \( \mathbb{F}_p \)-basis \((\overline{z}_1, \ldots, \overline{z}_{n-1}, \overline{z}_n)\) with \( \overline{z}_i \overline{z}_j = \phi_{i,j} \overline{z}_n \), so \( A \) is determined by that basis and the single \( n \times n \) structure matrix \( \Phi = (\phi_{ij}) \) satisfying

\[
\overline{z} \overline{z}^T = \Phi \overline{z}_n
\]

where \( \overline{z}^T = (\overline{z}_1, \ldots, \overline{z}_n) \). Then \( \phi_{ni} = \phi_{in} = 0 \) for all \( i \). In this section we determine the regular subgroups of \( \text{Aff}_n(\mathbb{F}_p) \) associated to \( A \).

Since \( A \) is commutative, the structure matrix \( \Phi \) is symmetric, hence defines a quadratic form

\[
\overline{x}^T \Phi \overline{x} = \sum_{i,j} \phi_{ij} x_i x_j
\]

which is diagonalizable. More precisely (c.f. [BM53], IX, 8), there is an \( n - 1 \times n - 1 \) invertible matrix \( P_{n-1} \) (not necessarily orthogonal) so that setting

\[
P = \begin{pmatrix} P_{n-1} \\ 0 \\ 1 \end{pmatrix},
\]

\[
P \Phi P^T = \text{diag}(d_1, d_2, \ldots, d_{n-1}, 0) = D.
\]

If \( \overline{x}^T \Phi \overline{x} = \Phi \overline{x}_n \), then

\[
P \overline{x} \overline{x}^T P^T = P \Phi P^T \overline{y} = D \overline{y}.
\]

So set \( \overline{z} = P \overline{x} \). Then \( z_n = x_n \) and with respect to the basis \((z_1, \ldots, z_{n-1}, z_n)\), \( A \) has the structure matrix \( D \):

\[
\overline{z} \overline{z}^T = D \overline{z}_n.
\]

We can realize the group \( T \) conveniently in

\[
\text{Hol}(G) \cong \text{Aff}_n(\mathbb{F}_p) = \begin{pmatrix} \text{GL}_n(\mathbb{F}_p) \\ \mathbb{F}_p^n \\ 0 \end{pmatrix}.
\]

by picking the basis \((z_1, \ldots, z_n)\) for \( G \) so that

\[
\Phi = D = \text{diag}(d_1, \ldots, d_n)
\]
with \( d_n = 0 \). Let \( \{ \overline{e}_1, \ldots, \overline{e}_n \} \) be the standard basis of \( \mathbb{F}_p^n \) corresponding to the basis \( \{ z_1, \ldots, z_n \} \) of \( A = (G, +, \cdot) \). Then \( \tau(z_i) = T_i \) is the element

\[
T_i = \begin{pmatrix} L_i & \overline{e}_i \\ 0 & 1 \end{pmatrix}
\]

which acts on \( G = \{ \bar{r} = \sum_{i=1}^{n} r_i \overline{e}_i : \bar{r} \in \mathbb{F}_p^n \} \) embedded as elements \( (\bar{r}) \) in \( \mathbb{F}_{p^{n+1}} \) by

\[
\begin{pmatrix} L_i & \overline{e}_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \overline{e}_j \\ 1 \end{pmatrix} = \begin{pmatrix} e_i \circ e_j \\ 1 \end{pmatrix} = \begin{pmatrix} \overline{e}_i + \overline{e}_j \\ 1 \end{pmatrix} \text{ for } i \neq j
\]

\[
\begin{pmatrix} L_i & \overline{e}_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \overline{e}_i \\ 1 \end{pmatrix} = \begin{pmatrix} e_i \circ e_i \\ 1 \end{pmatrix} = \begin{pmatrix} \overline{e}_i + \overline{e}_i + d_i \overline{e}_n \\ 1 \end{pmatrix}.
\]

So the columns of the matrix \( L_i \) are

\[
L_i(\overline{e}_j) = \overline{e}_j \text{ for } j \neq i
\]

\[
L_i(\overline{e}_i) = \overline{e}_i + d_i \overline{e}_n
\]

and \( L_i = I + d_i E_{n,i} \), where \( E_{n,i} \) is the zero matrix except for a single 1 in the \((n, i)\)-component. So for \( \bar{r} = \sum_{i=1}^{n} r_i \overline{e}_i \) in \( \mathbb{F}_p^n \),

\[
\tau(\bar{r}) = \begin{pmatrix} I + \sum_{k=1}^{n-1} E_{n,k} d_k r_k \\ 0 \\ 1 \end{pmatrix}
\]

in \( \text{Aff}_n(\mathbb{F}_p) \) for all \( \bar{r} \) in \( G \). (Note that \( \tau \) is a homomorphism from \((G, +)\) to \( \text{Hol}(G) \) only when \( \tau = \lambda \).)

5. Determining the Number of Hopf Galois Structures Associated to \( A \)

In this section we determine the number of Hopf Galois structures on a Galois extension \( L/K \) of fields with Galois group \( G = (\mathbb{F}_p^n, +) \) that correspond to certain isomorphism types of nilpotent algebra structures on \( G \).

To do so, we have

**Proposition 5.1.** Let \( A \) be a nilpotent \( \mathbb{F}_p \)-algebra structure on \( A = (\mathbb{F}_p^n, +) \). Then the number of Hopf Galois structures on \( L/K \) corresponding to the isomorphism type of \( A \) is equal to

\[
|\text{GL}_n(\mathbb{F}_p)|/|\text{Sta}(\Theta)|
\]

where \( T \) is the regular subgroup of \( \text{Aff}_n(\mathbb{F}_p) \) corresponding to \( A \) and \( \text{Sta}(T) = \{ P \in \text{GL}_n(\mathbb{F}_p) : \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} T = T \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} \}. \)
This follows by [CDVS06], which showed that two nilpotent $F_p$-algebras on $(F_n^p, +)$ are isomorphic if and only if the corresponding regular subgroups of $\text{Aff}_n(F_p)$ are conjugate by an element of $\text{Aut}(G) = \left( \begin{array}{cc}
\text{GL}_n(F_p) & 0 \\
0 & 1 \end{array} \right)$ in $\text{Aff}_n(F_p)$.

We note that given a regular subgroup $T$ of $\text{Aff}_n(F_p)$ normalized by $\lambda(G)$, corresponding to $A$ with $A^3 = 0$, then all of the regular subgroups in the orbit of $T$ under conjugation by $\text{Aut}(G)$ correspond to algebras $A_1$ isomorphic to $A$, hence have $A_1^3 = 0$. Thus all are normalized by $\lambda(G)$. Hence by Galois descent, all of those regular subgroups give rise directly to Hopf Galois structures on a Galois extension $L/K$ with Galois group $G$.

The commutative nilpotent $F_p$-algebra $A$ with $A^2 = 0$ yields the classical Galois structure on a Galois extension with Galois group $G$. For then $\Phi = D = 0$, and the corresponding regular subgroup $T$ of $\text{Aff}_n(F_p)$ is $\lambda(G)$, which is stable under conjugation by every element of $\text{Aut}(G)$, hence yields only the classical Galois structure on $L/K$.

In this section we determine the number of Hopf Galois structures when $\text{dim}(A/A^2) = 1$, $\text{dim}(A^2) = 1$, $A^3 = 0$ and $\Phi = D \neq 0$ by determining the isomorphism types of nilpotent algebras. As always, $G \cong (F_n^p, +)$ and $p$ is odd.

The theorem that quadratic forms over a field of characteristic not $= 2$ can be diagonalized is stronger for a finite field of odd characteristic. Namely (c.f. [Wi09, Section 3.4.6], there is an invertible matrix $P$ so that $P\Phi P^T = D = \text{diag}(D_s, 0)$, where $D_s = \text{diag}(1, \ldots, 1, s)$ is $k \times k$ for some $k \leq n$, with $s = 1$ if $k$ is odd, and $s$ is either 1 or any fixed non-square in $F_p$ when $k$ is even.

In the last section we described the regular subgroup $T$ of $\text{Aff}_n(F_p)$ corresponding to $\Phi$, where $\Phi$ is diagonal. We specialize to the case where $\Phi = \text{diag}(D_s, 0)$. Let $\overline{v}_s = (r_1, r_2, \ldots, r_{k-1}, sr_k)^T$, $\overline{v} = (r_1, r_2, \ldots, r_{k-1}, r_k)^T$, and $\overline{w} = (r_{k+1}, \ldots, r_{n-1})^T$ (column vectors of elements of $F_p$). Then it is convenient to write elements of $T$ as block matrices of the form

$$T = \{ \tau(\overline{v}) = \left( \begin{array}{ccc}
I & 0 & 0 \\
0 & I & 0 \\
\overline{v}_{\overline{s}}^T & 0 & 1 \\
0 & 0 & 1 \end{array} \right) : \overline{v} \in F_p^n \}$$

where the diagonal entries are identity matrices of size $k \times k$, $(n - 1 - k) \times (n - 1 - k)$, $1 \times 1$ and $1 \times 1$, respectively.

To determine the number of Hopf Galois structures corresponding to regular subgroups in the orbit of $T$, we need to find the stabilizer of $T$ under conjugation by the elements of $\text{Aut}(G) = \text{GL}_n(F_p)$. 
To determine the stabilizer of $T$, we seek the set of $(n + 1) \times (n + 1)$ matrices

$$Q = \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}$$

in $\text{Aut}(G) \subset \text{Aff}_n(G)$ so that $QTQ^{-1} = T$, where $P$ in $\text{GL}_n(\mathbb{F}_p)$ has the form

$$P = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix}$$

with blocks of the same size as $\tau(\tau')$. Let $\tau(\tau')$ be another element of $T$. We compute $P\tau(\tau')$ and $\tau(\tau')P$ and set $P\tau(\tau') = \tau(\tau')P$.

Equating the (11) terms yields that $P_{13} = 0$.
Equating the (21) terms yields that $P_{23} = 0$.
Equating the (32) terms yields that $P_{12} = 0$.
Then equating the (31) terms yields

$$\tau_s^TP_{11} = P_{33}\tau_s^T.$$ 

Equating the (14) terms yields

$$P_{11}v = v'.$$

Equating the (24) terms yields

$$\overline{w} = P_{21}v + P_{22}w.$$ 

Equating the (34) terms yields

$$t' = P_{31}v + P_{32}w + P_{33}t.$$ 

The (24) and (34) equations define $\overline{w}$ and $t'$. Setting $P_{33} = q$, a non-zero element of $\mathbb{F}_p$, then from (14) and (31) we have

$$P_{11}^T\tau_s' = q\tau_s' \text{ and } P_{11}\overline{v} = \overline{v}'.$$ 

Recalling that $D_s = \text{diag}(1, \ldots, 1, s)$, a $k \times k$ matrix, then $D_s\overline{v} = \overline{v}_s$, $D_s\overline{v}' = \overline{v}_s'$. So

$$P_{11}^TD_s\overline{v}' = qD_s\overline{v},$$

hence

$$P_{11}^TD_sP_{11}\overline{v} = qD_s\overline{v}.$$ 

Thus $P$ is in the stabilizer of $T$ if

$$P = \begin{pmatrix} P_{11} & 0 & 0 \\ P_{21} & P_{22} & 0 \\ P_{31} & P_{32} & P_{33} \end{pmatrix}$$

where

- $P_{33} = q$ is in $\text{GL}_1(\mathbb{F}_p)$;
- $P_{32}$ is $1 \times (n - 1 - k)$ and arbitrary;
$P_{31}$ is $1 \times k$ and arbitrary;  
$P_{22}$ is in $\text{GL}_{n-1-k}(\mathbb{F}_p)$;  
$P_{21}$ is $(n-1-k) \times k$ and is arbitrary; and  
$P_{11}$ is in $\text{GL}_k(\mathbb{F}_p)$ and satisfies $P_{11}^T D_s P_{11} = qD_s$.

As noted above, by an appropriate change of basis for $A$, we may assume that $A$ has a basis for which $A$ has one of the following structure matrices

1. $\Phi = \text{diag}(I_k,0)$ (with $I_k$ the $k \times k$ identity matrix) and $k$ is odd;  
2. $\Phi = \text{diag}(I_k,0)$ and $k$ is even;  
3. $\Phi = \text{diag}(D_s,0)$ where $D_s$ is the $k \times k$ matrix $\text{diag}(1, \ldots, 1, s)$, $k$ is even and $s$ is a non-square in $\mathbb{F}_p$.

We may then determine the possible $P_{11}$ in each of the three cases. The notation for the orthogonal groups over $\mathbb{F}_p$ is from [Wi09], Section 3.7.

**Proposition 5.2.** For Case 1), let $k = 2m + 1$ and $s = 1$ (so $D_s = I$). There exists a $k \times k$ matrix $C$ so that $C^T C = qI$ if and only if $q$ is a square. For each non-zero square number $q$ in $\mathbb{F}_p$, $P_{11}^T P_{11} = qI$ if and only if $P_{11} = CU$ for $U$ in $\text{GO}_{2m+1}^+(\mathbb{F}_p)$.

For Case 2), let $k = 2m$ and $s = 1$. For all $q \neq 0$ in $\mathbb{F}_p$, there exists a $k \times k$ matrix $C$ so that $C^T C = qI$. Then $P_{11}^T P_{11} = qI$ if and only if $P_{11} = CU$ for $U$ in $\text{GO}_{2m}^+(\mathbb{F}_p)$.

For Case 3), let $k = 2m$ and $s$ be a non-square in $\mathbb{F}_p$. For all $q \neq 0$ in $\mathbb{F}_p$, there exists a $k \times k$ matrix $C$ so that $C^T D_s C = qD_s$. Then $P_{11}^T D_s P_{11} = qD_s$ if and only if $P_{11} = CU$ for $U$ in $\text{GO}_{2m}^+ (\mathbb{F}_p)$.

**Proof.** In Case 1) with $k$ odd, if there exists $C$ so that $C^T C = qI$, then taking determinants gives $\det(C)^2 = q^k$, hence $q$ must be a square.

For the rest, it suffices to find the matrix $C$ in each case.

For Case 1), let $q = t^2$, then $C = tI$ satisfies $C^T C = qI$.

For Case 2), let $q = f^2 + g^2$, let $Q = \begin{pmatrix} f & g \\ -g & f \end{pmatrix}$ and let $C = \text{diag}(Q, Q, \ldots, Q)$. Then $C^T C = qI$.

For Case 3), let $q = f^2 + g^2$ and $Q$ as in Case 2). For $s$ a non-square in $\mathbb{F}_p$, find $w$ and $x$ in $\mathbb{F}_p$ so that $w^2 = sx^2 = q$. (If $q$ is a square, let $x = 0$, $w^2 = q$; if $q$ is a non-square, let $w = 0$ and find $x$ so that $sx^2 = w$, possible because the squares have index 2 in $\mathbb{F}_p^\times$.) Then $R = \begin{pmatrix} w & sx \\ x & -w \end{pmatrix}$ satisfies $R^T \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix} R = \begin{pmatrix} q & 0 \\ 0 & sq \end{pmatrix}$. Let $C = \text{diag}(Q, Q, \ldots, Q, R)$. Then $C^T D_s C = qD_s$. \qed
Corollary 5.3. Let $A$ be a commutative nilpotent $\mathbb{F}_p$-algebra of dimension $n$ with $A^3 = 0$ and $\dim(A^2) = 1$. Suppose the structure matrix of $A$ is $\Phi = \text{diag}(D_s, 0)$ where $D_s$ is $k \times k$ and

1) $k = 2m + 1, s = 1$
2) $k = 2m, s = 1$
3) $k = 2m, s$ is a non-square in $\mathbb{F}_p$.

Then the number of distinct regular subgroups of $\text{Aff}_n(\mathbb{F}_p)$ associated to $A$, and hence the number of Hopf Galois structures on $L/K$ associated to the isomorphism type of $A$, is

1) \[
\frac{|\text{GL}_n(\mathbb{F}_p)|}{(\frac{p-1}{2}) \cdot |\text{GO}_{2m+1}| \cdot |\text{GL}_{n-1-k}| \cdot p^{k(n-1-k)+(n-1)}}
\]

2) \[
\frac{|\text{GL}_n(\mathbb{F}_p)|}{(p-1) \cdot |\text{GO}_{2m}| \cdot |\text{GL}_{n-1-k}| \cdot p^{k(n-1-k)+(n-1)}}
\]

3) \[
\frac{|\text{GL}_n(\mathbb{F}_p)|}{(p-1) \cdot |\text{GO}_{2m}^-| \cdot |\text{GL}_{n-1-k}| \cdot p^{k(n-1-k)+(n-1)}}
\]

The orders of the orthogonal groups are polynomials in $p$ of degree $(k^2 - 2k + 3)/2$ when $k$ is odd and $(k^2 - k)/2$ when $k$ is even (c.f. [Wi09], p. 72).

6. For $n = 2, 3, 4$

We compare the counts of Hopf Galois structures in the last section to the number of Hopf Galois structures found by formal group methods in [Ch05] for $n = 2, 3$.

The case $n = 2$. Let $n = 2, k = 1$. Then $\Phi = (1)$. For $P$ to stabilize $T$,

\[
P = \begin{pmatrix}
P_{11} & 0 \\
P_{21} & P_{22}
\end{pmatrix},
\]

and the number of choices for each submatrix in $P$ is

\[
\begin{pmatrix}
|\text{GO}_1| & 1 \\
p & p-1
\end{pmatrix}.
\]

Since $\text{GO}_1 = \{(1), (-1)\}$, the size of the stabilizer of the regular subgroup is

\[
2 \cdot p \cdot \frac{p-1}{2} = p(p-1).
\]
The order of $GL_2(\mathbb{F}_p)$ is $(p^2 - 1)(p^2 - p)$. So there are $p^2 - 1$ distinct regular subgroups in the orbit of the regular subgroup corresponding to $\Phi$.

Since every nilpotent algebra structure $A$ on $(\mathbb{F}_p^2, +)$ has $A^3 = 0$, we have counted all Hopf Galois structures on a Galois extension with Galois group $C_p^2$.

**The case $n = 3$.**

Subcase: $k = 1$: The matrix $P$ is in the stabilizer of the regular subgroup $T$ corresponding to $\Phi = \text{diag}(1, 0)$ if

$$P = \begin{pmatrix} P_{11} & 0 & 0 \\ P_{21} & P_{22} & 0 \\ P_{31} & P_{32} & P_{33} \end{pmatrix},$$

all submatrices being $1 \times 1$. So the number of choices for each entry is

$$\begin{pmatrix} |\text{GO}_1| & 1 & 1 \\ p & |\text{GL}_1| & 1 \\ p & p & \frac{p-1}{2} \end{pmatrix}.$$  

Then $|\text{GO}_1| = 2$ and $|\text{GL}_1| = p - 1$, so the size of the stabilizer $\text{Sta}(T)$ is

$$p^3(p - 1)^2,$$

and the orbit has cardinality

$$|\text{GL}_3(\mathbb{F}_p)|/|\text{Sta}(T)| = (p - 1)^3(p + 1).$$

Subcase: $k = 2$, $s = 1$, $\Phi = \text{diag}(1, 1)$: The matrix $P$ is in the stabilizer if

$$P = \begin{pmatrix} P_{11} & 0 \\ P_{21} & P_{22} \end{pmatrix},$$

where $P_{11}$ is in $\text{GO}_2^+$. The number of choices for each submatrix is

$$\begin{pmatrix} |\text{GO}_2^+| & 1 \\ p^2 & p - 1 \end{pmatrix},$$

and $|\text{GO}_2^+| = 2(p - 1)$, so the size of the stabilizer is

$$2(p - 1)^2p^2.$$  

Subcase: $k = 2$, $s$ a non-square, $\Phi = \text{diag}(1, s)$: The matrix $P$ is in the stabilizer if

$$P = \begin{pmatrix} P_{11} & 0 \\ P_{21} & P_{22} \end{pmatrix},$$

where $P_{11}$ is in $\text{GO}_2^+$. The number of choices for each submatrix is

$$\begin{pmatrix} |\text{GO}_2^+| & 1 \\ p^2 & p - 1 \end{pmatrix},$$

and $|\text{GO}_2^+| = 2(p - 1)$, so the size of the stabilizer is

$$2(p - 1)^2p^2.$$
where $P_{11}$ is in $\text{GO}_2^-$. The number of choices for each submatrix is
\[
\begin{pmatrix}
|\text{GO}_2^-| & 1 \\
p^2 & p - 1
\end{pmatrix},
\]
and $|\text{GO}_2^-| = 2(p + 1)$, so the size of the stabilizer is
\[
2(p^2 - 1)p^2.
\]
The number of regular subgroups corresponding to each case is $|\text{GL}_3|$ divided by the order of the stabilizer:
- For $k = 1$, the number of regular subgroups is $(p^3 - 1)(p + 1)$.
- For $k = 2$, $s = 1$, the number of regular subgroups is $(p^3 - 1)p(p + 1)/2$.
- For $k = 2$, $s$ a non-square, the number of regular subgroups is $(p^3 - 1)p(p - 1)/2$.

These agree with the counts found in [Ch05].
The only isomorphism type of nilpotent algebras $A = (\mathbb{F}_{p^3}^3, +, \cdot)$ with $A^3 \neq 0$ is the algebra with $\dim(A/A^2) = 1$. We’ll look at that case in Section 7, below.

The case $n = 4$. This case has not previously been looked at.
For $n = 4$ there are four subcases:

$k = 1$. Here
\[
P = \begin{pmatrix}
P_{11} & 0 & 0 \\
P_{21} & P_{22} & 0 \\
P_{31} & P_{32} & P_{33}
\end{pmatrix},
\]
where $P_{22}$ is $2 \times 2$. The number of choices for each submatrix is
\[
\begin{pmatrix}
|\text{GO}_1| & 1 & 1 \\
p^2 & |\text{GL}_2| & 1 \\
p & p^2 & p - 1
\end{pmatrix}.
\]
So the size of the stabilizer is
\[
2p^5(p^2 - 1)(p^2 - p)(\frac{p - 1}{2}).
\]

$k = 2, s = 1$: Here $P_{11}$ is $2 \times 2$. The number of choices for each matrix is
\[
\begin{pmatrix}
|\text{GO}_2^+| & 1 & 1 \\
p^2 & |\text{GL}_1| & 1 \\
p^2 & p & p - 1
\end{pmatrix}.
\]
So the size of the stabilizer is 
\[ 2p^5(p - 1)^3. \]

\( k = 2, \) \( s \) a non-square. It is the same as the last case except \( P_{11} \) is in \( GO_2 \), so the size of the stabilizer is 
\[ 2p^5(p - 1)^2(p + 1). \]

\( k = 3. \) Here

\[ P = \begin{pmatrix} P_{11} & 0 \\ P_{21} & P_{22} \end{pmatrix} \]

where \( P_{11} \) is in \( GO_3 \), which has order \( 2(p^2 - 1) \), and \( P_{22} = (q) \) where \( q \) is a square. So the order of the stabilizer is 
\[ 2p(p^2 - 1)p^3 \frac{(p - 1)}{2}. \]

The number of regular subgroups in each case is the order of \( GL_4(F_p) \) divided by the orders of the respective stabilizers:

| Case | number of regular subgroups |
|------|-----------------------------|
| \( k = 1 \) | \( (p^2 + 1)(p + 1)(p^3 - 1) \) |
| \( k = 2, s = 1 \) | \( p(p^2 + 1)(p^3 - 1)(p + 1)^2/2 \) |
| \( k = 2, s \) a non-square | \( p(p^4 - 1)(p^3 - 1)/2 \) |
| \( k = 3 \) | \( p^2(p^4 - 1)(p^3 - 1) \) |

The total number of Hopf Galois structure exceeds \( p^9 \).

**The Hopf Galois structure.** Given a Galois extension \( L/K \) with Galois group \( G \cong \text{F}_p^n \), if the commutative nilpotent algebra \( A \) with \( \dim(A^2) = 1, A^3 = 0 \) has diagonal structure matrix \( \Phi = \text{diag}(d_1,\ldots,d_k,0,\ldots,) \), then the regular subgroup \( T \) corresponding to \( D \) acts on \( GL \) by

\[ \tau(\bar{r})e_\bar{t} = e_{\tau\bar{r}\bar{t}} = e_{\bar{w}} \]

where

\[ \bar{w} = \bar{r} + \bar{t} + \left( \sum_{i=1}^{k} r_i t_i d_i \right)x_n, \]

and \( \lambda(G) \) conjugates \( T \) by

\[ \lambda(\bar{t})\tau(\bar{r})\lambda(-\bar{t}) = \tau(\bar{r} - \bar{r} \cdot \bar{t}) \]

\[ = \tau(\bar{r} - \sum_{i=1}^{k} r_i t_i d_i x_n). \]
7. The case \( \dim(A/A^2) = 1 \)

Let \( L/K \) be a Galois extension of fields with Galois group \( \Gamma \) an elementary abelian \( p \)-group of order \( p^n \), and let \( G \) also be an elementary abelian \( p \)-group of order \( p^n \). The opposite extreme to Hopf Galois structures on \( L/K \) that arise from nilpotent \( \mathbb{F}_p \)-algebra structures \( A = (G, +, \cdot) \) with \( \dim(A/A^2) = n - 1 \) are structures that correspond to the case where \( \dim(A/A^2) = 1 \), that is, where \( A = \langle x \rangle \) with \( x^{n+1} = 0 \). For \( n \geq 3 \), to describe these Hopf Galois structures we need to translate from the holomorph of \( G \) to \( \text{Perm}(\Gamma) \).

In order that the group \( (G, \circ) \) arising from \( A \) be isomorphic to \( \Gamma \), we require, and will assume, that \( p > n \) (c. f. [Ch07], Corollary 2).

To establish notation, let \( G = (\mathbb{F}_p^n, +) \) and define the primitive nilpotent \( \mathbb{F}_p \)-algebra structure \( A = (\mathbb{F}_p^n, +, \cdot) \) on \( G \) by picking the standard basis \( (e_1, \ldots, e_n) \) for \( \mathbb{F}_p^n \), let \( z = e_1 \) and let \( e_k = z^k \) for \( k \geq 1 \), and \( z^{n+1} = 0 \).

The corresponding group structure \( (A, \circ) \) is defined by

\[
z^i \circ z^j = z^i + z^j + z^{i+j},
\]

and in general by

\[
(\sum_i r_i z^i) \circ (\sum_j s_j z^j) = (\sum_i r_i z^i) + (\sum_j s_j z^j) + (\sum_i r_i z^i) \cdot (\sum_j s_j z^j).
\]

Let \( \mathbb{F}_p[x] \) be the polynomial ring in \( x \) with coefficients in \( \mathbb{F}_p \).

**Proposition 7.1.** The map \( y \mapsto 1 + y \) defines an isomorphism from \( (A, \circ) \) to the group \( (1+x)\mathbb{F}_p[x]/x^{n+1}\mathbb{F}_p[x] \) of principal units of \( \mathbb{F}_p[x]/x^{n+1}\mathbb{F}_p[x] \).

This follows immediately from the definition of the \( \circ \) operation. For

\[
a \circ b \mapsto 1 + a \circ b = 1 + a + b + a \cdot b = (1 + a) \cdot (1 + b).
\]

Then an easy induction argument yields

**Proposition 7.2.** For all \( a_1, \ldots, a_n \) in \( (A, \circ) \),

\[
a_1 \circ a_2 \circ \ldots \circ a_m \mapsto (1 + a_1) \cdot (1 + a_2) \cdots \cdot (1 + a_n).
\]

Given \( (A, \circ) \) where \( A = (\mathbb{F}_p^n, +, \cdot) \), define a map \( \tau : (A, \circ) \to \text{Perm}(\mathbb{F}_p^n) \) by

\[
\tau(a)(x) = a \circ x
\]

for \( a, x \) in \( \mathbb{F}_p^n \). Then \( \tau(a)(0) = a \), so the image \( T \subset \text{Perm}(\mathbb{F}_p^n) \) is a regular subgroup of \( \text{Perm}(G) \). Also \( \tau \) is a homomorphism from \( (A, \circ) \) to \( \text{Perm}(\mathbb{F}_p^n) \), because

\[
\tau(a)\tau(b)(x) = \tau(a)(b \circ x) = a \circ (b \circ x) = (a \circ b) \circ x = \tau(a \circ b)(x).
\]

Also, \( \tau(a)\lambda(b) = \lambda(b + a \cdot b)\tau(a) \), so \( T \subset \text{Hol}(\mathbb{F}_p^n) \).
Since $p > n$, $(G, \circ)$ is an elementary abelian $p$-group with $p$-basis $z, z^2, \ldots, z^n$ by Proposition 1 of [Ch07]. So let $\Gamma = (\mathbb{F}_p^n, +)$ have basis $x_1, \ldots, x_n$ and define $\xi : \Gamma \to (\mathbb{F}_p, \circ)$ by $\xi(x_i) = z^i$ for $i = 1, \ldots, n$, and

$$\xi(\sum_i r_i x_i) = (r_1 \circ z) \circ (r_2 \circ z^2) \circ \ldots \circ (r_n \circ z^n)$$

where for $s$ in $\mathbb{F}_p$ and $w$ in $\Gamma$,

$$s \circ w = w \circ w \circ \ldots \circ w \quad (s \text{ factors}).$$

Then $\xi$ is an isomorphism.

Define $\beta = \tau \xi : \Gamma \to T \subset \text{Hol}(G)$. Then $\beta$ is a regular embedding of $\Gamma$ in $\text{Hol}(G)$.

To obtain the corresponding Hopf Galois structure on $L/K$, we construct the embedding $\alpha : G \to \text{Perm}(\Gamma)$ corresponding to $\beta$. To do so, we first define

$$b : \Gamma \to G$$

by $b(\gamma) = \beta(\gamma)(0) = \tau(\xi(\gamma))(0) = \xi(\gamma)$. So $b = \xi$ is the isomorphism from $\Gamma$ to $(G, \circ)$.

Using Proposition 7.2 we can write

$$b(\sum_i r_i x_i) = (1 + z)^{r_1} \cdot (1 + z^2)^{r_2} \cdot \ldots \cdot (1 + z^n)^{r_n} - 1 = \sum_{j=1}^n s_j z^j$$

in $(\mathbb{F}_p^n, +)$ for some $s_1, \ldots, s_n$ in $\mathbb{F}_p$.

From $b : \Gamma \to (G, \circ)$, we define $\alpha : \Gamma \to \text{Perm}(G)$ by

$$\alpha(g)(\gamma) = b^{-1}(\lambda(g)b(\gamma)).$$

Then $\alpha(G)$ is a regular subgroup of $\text{Perm}(\Gamma)$ normalized by $\lambda(\Gamma)$. So the $L$-Hopf algebra $L[\alpha(G)]$ descends to the $K$-Hopf algebra $H = L[\alpha(G)]^G$, which acts on $L$ as follows: if $h = \sum_{g \in G} s_g g$ is in $H$ and $a$ is in $L$, then

$$h(a) = \sum_{g \in G} s_g b^{-1}(g^{-1})(a).$$

So to understand the Hopf algebra $H$ and its action on $L$, we need $b^{-1}$, where

$$b(\sum_i r_i x_i) = (1 + z)^{r_1} \cdot (1 + z^2)^{r_2} \cdot \ldots \cdot (1 + z^n)^{r_n} = \sum_{j=1}^n s_j z^j.$$

To find $b^{-1}$, that is, to find $r_1, \ldots, r_n$ in terms of $s_1, \ldots, s_n$ we can use the logarithm function

$$\log_z(1 + w) = \sum_{i=1}^n (-1)^{i+1} \frac{w^i}{i}.$$
for \( w \) in \( z\mathbb{F}_p[z]/z^{n+1}\mathbb{F}_p[z] \). Then for \( w_1, w_2 \) multiples of \( z \),

\[
\log_z((1 + w_1)(1 + w_2)) = \log_z(1 + w_1) + \log_z(1 + w_2).
\]

Applying \( \log_z \) to the equation

\[
(1 + z)^{r_1}(1 + z^2)^{r_2} \cdots (1 + z^n)^{r_n} = \sum_{j=1}^{n} s_j z^j
\]

yields

\[
\sum_{i=1}^{n} r_i \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} z^{ik} = \sum_{j=1}^{n} \frac{(-1)^{i+1}}{i} (s_1 z + \ldots + s_n z^n)^j).
\]

Looking at this equation modulo \( z^2, z^3, \ldots \), one can see that \( r_1 = s_1 \) and there are polynomials \( f_{i+1}(x_1, \ldots, x_n), g_{i+1}(x_1, \ldots, x_i) \) for \( i = 1, 2, \ldots, n - 1 \) so that

\[
r_{i+1} + f_{i+1}(r_1, \ldots, r_i) = s_{i+1} + g_{i+1}(s_1, \ldots, s_i).
\]

By successive substitutions, we find that for \( i = i, \ldots, n, \)

\[
s_i = r_i + ( \text{polynomial function of } r_1, \ldots, r_{i-1}) \\
r_i = s_i + ( \text{polynomial function of } s_1, \ldots, s_{i-1}).
\]

Thus for any particular \( n \) the equations defining \( b \) and \( b^{-1} \) can be determined.

For \( n = 3 \) we obtain the only commutative nilpotent \( \mathbb{F}_p \)-algebra \( A \) of dimension 3, up to isomorphism, that does not have \( \dim(A^2) = 1 \) and \( A^3 = 0 \). If we write

\[
\overline{r} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}, \quad \overline{s} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix},
\]

and view \( b : \mathbb{F}_p^3 \rightarrow \mathbb{F}_p^3 \) as

\[
b(\overline{r}) = \overline{s},
\]

then

\[
\overline{s} = b(\overline{r}) = \begin{pmatrix} r_1 \\ r_2 + \binom{r_1}{2} \\ r_3 + r_1 r_2 + \binom{r_1}{3} \end{pmatrix}.
\]
Then $b^{-1}$ satisfies

$$
\begin{pmatrix}
  s_1 \\
  s_2 - \binom{s_1}{2} \\
  s_3 - s_1 s_2 + s_1 \binom{s_1}{2} - \binom{s_1}{3}
\end{pmatrix}
$$

$$
= \begin{pmatrix}
  s_1 \\
  s_2 - \binom{s_1}{2} \\
  s_3 - s_1 s_2 + 2 \binom{s_1}{3}
\end{pmatrix}.
$$

To obtain a regular subgroup of $\text{Perm}(\Gamma)$ from $\beta$, we define the embedding $\alpha : G \to \text{Perm}(\Gamma)$ by

$$
\alpha(\tau) = b^{-1} \lambda(\tau) b : G \to \text{Perm}(\Gamma).
$$

Thus

$$
\alpha(\tau)(x) = b^{-1} \lambda(\tau) b(\tau)
$$

$$
= b^{-1} \lambda(\tau) \begin{pmatrix}
  x_1 \\
  x_2 + \binom{x_1}{2} \\
  x_3 + x_1 x_2 + \binom{x_1}{3}
\end{pmatrix}
$$

$$
= b^{-1} \begin{pmatrix}
  r_1 + x_1 \\
  r_2 + x_2 + \binom{x_1}{2} \\
  r_3 + x_3 + x_1 x_2 + \binom{x_1}{3}
\end{pmatrix}
$$

$$
= \begin{pmatrix}
  r_1 + x_1 \\
  r_2 + x_2 - \frac{r^2}{f} - r_1 x_1 + \frac{r_1}{2}
\end{pmatrix}
$$

where

$$
f = r_3 + x_3 - r_1 x_2 - r_1 x_2 - r_2 x_1
$$

$$
- \frac{r_1}{3} + \frac{r_1 x_1}{2} + \frac{r^2}{3} + \frac{r_1^2 x_1}{3} + \frac{r_1 x_1^2}{2}.
$$

This example for $n = 3$ illustrates that when $A^3 = 0$, so that the nilpotent algebra $A$ yields a regular subgroup $T$ of $\text{Aff}_n(\mathbb{F}_p)$ that is normalized by $\lambda(G)$, the resulting Hopf Galois structure is a bit more transparent than when we need to translate from the holomorph of $G$ to the permutation group of $\Gamma$.

If $T$ is a regular subgroup of $\text{Aff}_n(\mathbb{F}_p)$ corresponding to the nilpotent algebra $A = (\mathbb{F}_p^n, +, \cdot)$ with $\dim(A/A^2) = 1$, then $T = (\mathbb{F}_p^n, \circ)$ is isomorphic to the group of principal units of $\mathbb{F}_p[x]/x^{n+1}\mathbb{F}_p[x]$. Suppose $L/K$ is a Galois extension with Galois group $\Gamma \cong (\mathbb{F}_p^n, \circ)$. Then [Ch07] determines the isomorphism type of $\Gamma$ as an abelian $p$-group and counts the number of Hopf Galois structures on $L/K$ of type $(G, +)$. If $p > n$ then $\Gamma \cong (G, +)$ and from [Ch05] the number of Hopf Galois structures arising from $A$ with $\dim(A/A^2) = 1, A^{n+1} = 0$ is $|GL_n(\mathbb{F}_p)|/(p^n - p^{n-1})$. 

References

[BM53] G. Birkhoff, S. MacLane, A Survey of Modern Algebra, revised edition, MacMillan, 1953.

[By96] N. P. Byott, Uniqueness of Hopf Galois structure of separable field extensions, Comm. Algebra 24 (1996), 3217–3228.

[By02] N. P. Byott, Integral Hopf-Galois structures on degree $p^2$ extensions of $p$-adic fields, J. Algebra, 248, (2002), 334–365.

[By04] N. P. Byott, Hopf-Galois structures on field extensions with simple Galois groups, Bull. London Math. Soc. 36 (2004), 23–29.

[CDVS06] A. Caranti, F. Dalla Volta, M. Sala, Abelian regular subgroups of the affine group and radical rings, Publ. Math. Debrecen 69 (2006), 297–308.

[CS69] S. U. Chase, M. E. Sweedler, Hopf Algebras and Galois Theory, Lecture Notes in Mathematics, vol 97, Springer-Verlag, Berlin-New York, 1969.

[Ch05] L. N. Childs, Elementary abelian Hopf Galois structures and polynomial formal groups, J. Algebra 283 (2005), 292–316.

[Ch07] L. N. Childs, Some Hopf Galois structures arising from elementary abelian $p$-groups, Proc. Amer. Math. Soc. 135 (2007), 3453–3460.

[Ch11] L. N. Childs, Hopf Galois structures on Kummer extensions of prime power degree, New York J. Math. 17 (2011). 5174.

[Ch15] L. N. Childs, On Abelian Hopf Galois structures and finite commutative nilpotent rings, New York J. Math., 21 (2015), 205–229.

[Ch16b] L. N. Childs, On the Galois correspondence for Hopf Galois structures, in preparation.

[FCC12] S. C. Featherstonhaugh, A. Caranti, L. N. Childs, Abelian Hopf Galois structures on prime-power Galois field extensions, Trans. Amer. Math. Soc. 364 (2012), 3675–3684.

[GP87] C. Greither, B. Pareigis, Hopf Galois theory for separable field extensions, J. Algebra 106 (1987), 239–258.

[Ko07] T. Kohl, Groups of order $4p$, twisted wreath products and Hopf-Galois theory. J. Algebra 314 (2007), 42–74.

[Ko13] T. Kohl, Regular permutation groups of order $mp$ and Hopf Galois structures, Algebra & Number Theory 7 (2013), 22032240.

[Ko16] T. Kohl, Hopf-Galois structures arising from groups with unique subgroup of order $p$, Algebra & Number Theory 10 (2016), 37–59.

[Wi09] R. A. Wilson, The Finite Simple Groups, London, Springer-Verlag, 2009

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