Hypergeometric functions and the Tricomi operator

J. Barros-Neto*
Rutgers University, Hill Center
110 Frelinghuysen Rd, Piscataway, NJ 08854-8019
e-mail: jbn@math.rutgers.edu

Fernando Cardoso†
Departamento de Matemática, Universidade Federal de Pernambuco
50540-740 Recife, Pe, Brazil
e-mail: fernando@dmat.ufpe.br

Abstract
In this paper we show how certain hypergeometric functions play an important role in finding fundamental solutions for a generalized Tricomi operator.

1 Introduction
In this article we consider the operator

$$T = y \Delta_x + \frac{\partial^2}{\partial y^2},$$

(1.1)
in $\mathbb{R}^{n+1}$, where $\Delta_x = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$, $n \geq 1$. This is a natural generalization of the classical Tricomi operator in $\mathbb{R}^2$ already considered by us in the article [2] where it was called generalized Tricomi operator.

*Partially supported by NSF, Grant # INT 0124940
†Partially supported by CNPq (Brazil)
In that article we obtained, by the method of partial Fourier transformation, explicit expressions for fundamental solutions to $\mathcal{T}$, relative to points on the hyperplane $y = 0$. That lead us to calculate inverse Fourier transforms of Bessel functions which, in turn, revealed the importance of certain hypergeometric functions (depending on the “space dimension” $n$) that are intimately related to the operator $\mathcal{T}$.

In the present article we look for fundamental solutions of $\mathcal{T}$ relative to an arbitrary point $(x_0, y_0)$, located in the hyperbolic region ($y < 0$) of the operator, and which are supported by the “forward” characteristic conoid of $\mathcal{T}$ with vertex at $(x_0, y_0)$. We follow the method of S. Delache and J. Leray in [5] where they introduced hypergeometric distributions, a notion also considered by I. M. Gelfand and G. E. Shilov in [7].

The plan of this article is the following. In Section 2 we deal with preliminary material that is needed throughout the paper. Hypergeometric distributions are introduced in Section 3 where we obtain the basic formula (3.21) which is used in Sections 4, 5, and 6 to obtain fundamental solutions respectively in the cases $n = 1$ (the classical Tricomi operator), $n$ even, and $n$ odd. The case $n$ odd $\geq 3$ differs from the other two cases by the fact that the fundamental solution is then a sum of two terms one supported by the “forward” conoid (as in the cases $n = 1$ and $n$ even) and another supported by the boundary of the conoid. In Section 4 we also show how to derive from the method used in this paper the results obtained previously by Barros-Neto and Gelfand in [4]. Finally, in the Appendix we prove or indicate the proof of results mentioned and utilized in Section 4.

2 Preliminaries

Let $\mathcal{T}$ be the operator given by (1.1) at the beginning of Section 1. Defining the modified gradient

$$\hat{\nabla} u = (yu_{x_1}, \ldots, yu_{x_n}, u_y),$$

one verifies that $\mathcal{T}u = \text{div}(\hat{\nabla} u)$ and that

$$\int\int_D (u\mathcal{T}v - v\mathcal{T}u) \, dV = \int_{\partial D} (u\hat{\nabla}v - v\hat{\nabla}u) \cdot \vec{n} \, dS$$

(2.2)

for all smooth $u$ and $v$ on the closure of an open bounded domain $D$ with smooth boundary $\partial D$. 

2
If \( y < 0 \) and we set \( t = 2(-y)^{3/2}/3 > 0 \), then the change of variables

\[
x = x, \quad t = 2(-y)^{3/2}/3 \quad \iff \quad x = x, \quad y = -(2/3)^{-2/3}t^{2/3},
\]

(2.3)

whose Jacobian is

\[
\frac{\partial(x, y)}{\partial(x, t)} = -(2/3)^{1/3}t^{-1/3},
\]

(2.4)

transforms \( T \) into the operator

\[
2\left(\frac{3t}{2}\right)^{2/3} T_h,
\]

(2.5)

with

\[
T_h = \frac{1}{2} \left( \frac{\partial^2}{\partial t^2} - \Delta_x \right) + \frac{1}{6t} \frac{\partial}{\partial t}.
\]

(2.6)

We call \( T_h \) the reduced hyperbolic Tricomi operator. Its formal adjoint is

\[
T_h^* = \frac{1}{2} \left( \frac{\partial^2}{\partial t^2} - \Delta_x \right) - \frac{1}{6t} \frac{\partial}{\partial t} + \frac{1}{6t^2}.
\]

(2.7)

It is a matter of verification that

\[
T_h^* (t^{1/3} u) = t^{1/3} T_h (u).
\]

Thus, if \( u \) is a solution of \( T_h (u) = 0 \), then \( v = t^{1/3} u \) is a solution of \( T_h^* v = 0 \), and conversely. Moreover, suppose that \( E(x, t; 0, t_0) \), with \( t_0 \neq 0 \), is a fundamental solution of \( T_h \) relative to the point \( (0, t_0) \), that is,

\[
T_h E = \delta(x, t - t_0),
\]

then \( (t/t_0)^{1/3} E \) is a fundamental solution of \( T_h^* \) relative to the same point.

We now recall the definition of the distribution \( \chi_q(s) \) (see \[5, 7, 8\]). Let \( q \in \mathbb{C} \) be such that \( \text{Re} \, q > -1 \). The locally integrable function

\[
\chi_q(s) = \frac{s^q}{\Gamma(q+1)} \text{ if } s > 0, \quad \chi_q(s) = 0 \text{ if } s \leq 0
\]

(2.8)

defines a distribution in \( \mathbb{R} \) that depends analytically on \( q \) and that extends by analytic continuation to an entire function of \( q \). We have

\[
\chi_q(s) = \frac{d}{ds} \chi_{q+1}(s).
\]
Moreover, $\chi_q(s)$ is positive homogeneous of degree $q$ and Euler’s formula holds
\[ s\chi_{q-1}(s) = q\chi_q(s). \] (2.9)
We also have that $\chi_q(s) = \delta^{(-q-1)}(s)$ if $q$ is an integer $< 0$ (see [7]).

Consider the function
\[ k(x, t - t_0) = \begin{cases} 
(t - t_0)^2 - |x|^2 & \text{if } t - t_0 > |x| \\
0 & \text{if } t - t_0 \leq |x|,
\end{cases} \] (2.10)
defined in the whole of $\mathbb{R}^{n+1}$. Since $k(x, t - t_0)$ is positive in the semi-cone $C = \{(x, t) \in \mathbb{R}^{n+1} : t - t_0 > |x|\}$, and identically zero outside of $C$, it follows that $\chi_q(k(x, t - t_0))$ (which, for simplicity and when no confusion is possible, we denote by $\chi_q(k(\cdot))$) is a distribution in $\mathbb{R}^{n+1}$ which is an entire analytic function of $q \in \mathbb{C}$. In particular, if $q$ is an integer $< 0$, then
\[ \chi_q(k(\cdot)) = \delta^{(-q-1)}((k(\cdot))) \]
is a distribution concentrated on the boundary of $C$ (see [4]).

3 Hypergeometric distributions

Our aim is to find fundamental solutions for the Tricomi operator $T$ relative to an arbitrary point $(0, b)$, $b < 0$, in $\mathbb{R}^{n+1}$, that is, a distribution $E(x, y; 0, b)$ defined in $\mathbb{R}^{n+1}$ so that $TE = \delta(x, y - b)$. In guise of motivation, suppose that $E(x, y; 0, b)$ is locally integrable function. Then in view of formulas (2.3), (2.4), (2.5), and (2.6), we may write
\[ \phi(0, b) = \langle TE, \phi \rangle = \int_{\mathbb{R}^{n+1}} E(x, y; 0, b)T\phi \, dx \, dy \] (3.1)
\[ = 2\left(\frac{3}{2}\right)^{1/3} \int_{\mathbb{R}^{n+1}} t^{1/3}E^\sharp(x, t; 0, t_0)T_h\psi(x, t) \, dx \, dt, \]
where in the last formula $E^\sharp(x, t; 0, t_0)$ and $\psi(x, t)$ denote, respectively, $E(x, y; 0, b)$ and $\phi$ in the variables $x$ and $t$, and we have set $t_0 = 2(-b)^{3/2}/3$. Thus, our problem reduces to finding fundamental solutions relative to $(0, t_0)$ for the adjoint operator $T^*_h$, which according to our remark in Section 1 is equivalent to finding fundamental solutions for $T^*_h$ relative to the same point.
The operator \( T_h \) belongs to a class of operators, called Euler–Poisson–Darboux operators, studied by Delache and Leray in [5], where they obtained explicit formulas for fundamental solutions of those operators. For sake of completeness, we outline Delache and Leray’s method in [5] relative to the reduced hyperbolic Tricomi operator \( T_h \), or more generally, the operator

\[
\mathcal{P}_\alpha = \frac{1}{2} \left( \frac{\partial^2}{\partial t^2} - \Delta_x \right) + \frac{\alpha}{t} \frac{\partial}{\partial t},
\]

(3.2)

where \( \alpha \in \mathbb{C} \). Note that \( \mathcal{P}_\alpha \) remains invariant under the action of the group that leaves \( t \) unchanged and transforms \((x_1, \ldots, x_n)\) by translations. Since \( \mathcal{P}_\alpha \) is homogeneous of degree \(-2\) and \( \delta(x, t - t_0) \) is homogeneous of degree \(-(n + 1)\), a fundamental solution \( E_\alpha \) of \( \mathcal{P}_\alpha \) should be homogeneous of degree \( 1 - n \). Monomials of the type

\[
t_0^{-j} t^{-\alpha - j} x_{j+1/2-n/2}(k(x, t - t_0)),
\]

have the desired homogeneity degree. On the other hand, if \( \Box_{(x,t)} = (\partial^2 / \partial t^2 - \Delta_x) \) denotes the wave operator in \( \mathbb{R}^{n+1} \), it is shown in [5] that

\[
\Box_{(x,t)} \left( \frac{1}{2} \pi^{1/2-n/2} x_{1/2-n/2}(k(x, t)) \right) = \delta(x, t),
\]

(3.3)

in other words, the distribution \( \pi^{1/2-n/2} x_{1/2-n/2}(k(x, t))/2 \) is a fundamental solution of the wave operator.

As a consequence, it is natural to look for a fundamental solution to \( \mathcal{P}_\alpha \) as a formal series

\[
E_\alpha(x, t; 0, t_0) = \pi^{1/2-n/2} t_0^\alpha \sum_{j=0}^{\infty} c_j(t_0 t)^{-j} x_{j+1/2-n/2}(k(\cdot)),
\]

(3.4)

with a suitable choice of the coefficients \( c_j \). By applying \( \mathcal{P}_\alpha \) to both sides of (3.4) one obtains, after routine calculations where the two identities

\[
\Box_{(x,t)} x_{j+1/2-n/2}(k(\cdot)) = 4 j x_{j-1/2-n/2}(k(\cdot))
\]

and

\[
\frac{\partial}{\partial t} [x_{j+1/2-n/2}(k(\cdot))] = 2(t - t_0) x_{j-1/2-n/2}(k(\cdot))
\]

are used, the following result:

\[
\mathcal{P}_\alpha E_\alpha = c_0 \delta(x, t - t_0) +
\]

(3.5)
\[ + \pi^{1/2-n/2} \sum_{j=1}^{\infty} \left\{ \frac{1}{2} (j - 1 + \alpha)(j - \alpha)c_{j-1} + 2jc_j \right\} t_0^{-j+1} t^{\alpha-j-1} \chi_{j-1/2-n/2}(k(\cdot)). \]

If we choose the coefficients \( c_j \) so that

\[ c_0 = 1 \quad \text{and} \quad \frac{1}{2} (j - 1 + \alpha)(j - \alpha)c_{j-1} + 2jc_j = 0, \quad j \geq 1, \quad (3.6) \]

then (3.5) reduces to

\[ \mathcal{P}_\alpha E_\alpha = \delta(x, t - t_0), \quad (3.7) \]

that is \( E_\alpha \) is a fundamental solution of \( \mathcal{P}_\alpha \). Now recalling notations

\[ (a)_0 = 1, \quad (a)_j = a(a + 1) \cdots (a + j - 1) = \frac{\Gamma(a + j)}{\Gamma(a)}, \quad j \geq 1, \quad (3.8) \]

it follows from (3.6) that

\[ c_j = (-\frac{1}{4})^j (\alpha)_j(1 - \alpha)_j \frac{1}{j!}, \quad j \geq 0. \quad (3.9) \]

Hence we may rewrite (3.4) as

\[ E_\alpha(x, t; 0, t_0) = \pi^{1/2-n/2} (\frac{t_0}{t})^\alpha \Phi_\alpha(x, t), \quad (3.10) \]

where

\[ \Phi_\alpha(x, t) = \sum_{j=0}^{\infty} \frac{(\alpha)_j(1 - \alpha)_j}{j!} (-\frac{1}{4t_0t})^j \chi_{j+1/2-n/2}(k(\cdot)). \quad (3.11) \]

This series converges for \(|k(\cdot)/4t_0t| < 1\). \( \Phi_\alpha(x, t) \) is the hypergeometric distribution introduced by Delache and Leray in [5]. Hypergeometric distributions were also considered by Gelfand and Shilov in [7].

The expression of \( \Phi_\alpha \) depends on the space dimension \( n \). To see this consider three cases.

**Case I: \( n = 1 \).** We have

\[ \Phi_\alpha(x, t) = \chi_0(k(\cdot)) + \sum_{j=1}^{\infty} \frac{(\alpha)_j(1 - \alpha)_j}{j!} (-\frac{1}{4t_0t})^j \chi_j(k(\cdot)). \quad (3.12) \]
From Euler’s formula (2.9) it follows that \( \chi_j(s) = s^j \chi_0(s)/j! \), \( j \geq 0 \). By recalling the expression of \( \chi_j(k(\cdot)) \) we rewrite (3.12) as follows

\[
\Phi_\alpha(x,t) = \chi_0(k(\cdot)) \sum_{j=0}^{\infty} \frac{(\alpha)_{j}(1-\alpha)_{j}}{j!j!} \frac{(t-t_0)^2 - x^2}{-4t_0t} j^j
\]

(3.13)

\[
= \chi_0(k(\cdot)) F(\alpha, 1-\alpha, 1; \frac{(t-t_0)^2 - x^2}{-4t_0t}).
\]

**Case II:** \( n \) even. We have

\[
\Phi_\alpha(x,t) = \chi_{1/2-n/2}(k(\cdot)) + \sum_{j=1}^{\infty} \frac{(\alpha)_{j}(1-\alpha)_{j}}{j!} \frac{\alpha_j}{4t_0t} \chi_{j+1/2-n/2}(k(\cdot)).
\]

(3.14)

From Euler’s formula (2.9) it follows by induction that

\[
s^j \chi_q(s) = (q+1) j \chi_{q+j}(s),
\]

for all integer \( j \geq 0 \). Inserting the corresponding formula with \( q = 1/2 - n/2 \) into (3.14) we obtain

\[
\Phi_\alpha(x,t) = \chi_{1/2-n/2}(k(\cdot)) \sum_{j=0}^{\infty} \frac{(\alpha)_{j}(1-\alpha)_{j}}{(3/2-n/2)_{j!} j!} \frac{(t-t_0)^2 - |x|^2}{-4t_0t} j^j
\]

(3.15)

\[
= \chi_{1/2-n/2}(k(\cdot)) F(\alpha, 1-\alpha, 3/2 - n/2; \frac{(t-t_0)^2 - |x|^2}{-4t_0t}).
\]

**Case III:** \( n \) odd > 1. Let \( n = 2m + 1, m \geq 1 \). Note that in this case \( 1/2 - n/2 = -m \), a negative integer. We split \( \Phi_\alpha \) into two terms:

\[
\Phi_\alpha(x,t) = \sum_{j=0}^{m-1} \frac{(\alpha)_{j}(1-\alpha)_{j}}{j!} \frac{\alpha_j}{4t_0t} \chi_{j-m}(k(\cdot)) + \sum_{j=m}^{\infty} \frac{(\alpha)_{j}(1-\alpha)_{j}}{j!} \frac{\alpha_j}{4t_0t} \chi_{j-m}(k(\cdot)).
\]

(3.16)

Whenever \( j - m < 0 \), \( \chi_{j-m}(k(\cdot)) = \delta^{(m-j-1)}(k(\cdot)) \) is a distribution concentrated on the surface of the semi-cone \( C \). Thus the first term in (3.16) corresponds to a finite sum of distributions supported by the boundary of \( C \).
Recalling that $\chi_j(s) = s^j \chi_0(s)/j!$ and setting $j' = j - m$, rewrite the second term in (3.16) as

$$S = \chi_0(k(\cdot)) \left( \frac{1}{-4t_0t} \right)^m \sum_{j'=0}^{\infty} \frac{(\alpha)_{j'+m}(1-\alpha)_{j'+m}}{(j'+m)!} \left( \frac{k(\cdot)}{-4t_0t} \right)^{j'}$$

Now $(\alpha)_{j'+m} = (\alpha)_m(\alpha + m)_{j'}$, $(1-\alpha)_{j'+m} = (1-\alpha)_m(1-\alpha + m)_{j'}$, and $(j'+m)! = m!(m+1)_{j'}$. Therefore

$$S = \chi_0(k(\cdot)) c_m \left( \frac{1}{-4t_0t} \right)^m F(\alpha + m, 1 - \alpha + m, m + 1, \frac{(t - t_0)^2 - |x|^2}{-4t_0t}),$$

where $c_m = (\alpha)_m(1-\alpha)_m/m!$. Thus the expression (3.16) for $\Phi_\alpha$ becomes

$$\Phi_\alpha(x, t) = \sum_{j=0}^{m-1} \frac{(\alpha)_j(1-\alpha)_j}{j!} \left( -\frac{1}{4t_0t} \right)^j \delta^{(m-j-1)}(k(\cdot)) + \chi_0(k(\cdot)) c_m \left( \frac{1}{-4t_0t} \right)^m F(\alpha + m, 1 - \alpha + m, m + 1, \frac{(t - t_0)^2 - |x|^2}{-4t_0t}).$$

**Remarks**

1) The support of all fundamental solutions above described is the closure of the semi-cone $C$ defined at the end of Section 2. In the case $n$ odd integer $> 1$, besides the term that contains the hypergeometric function whose support is the closure of $C$ there are a finite number of terms whose support is the boundary of $C$.

2) Formula (3.17) can be viewed as a derivative with respect to $k(\cdot)$ of a certain hypergeometric distribution. More precisely, consider the hypergeometric distribution $\chi_0(s) F(a, b, 1; rs)$ where $r$ is a real or complex parameter. The following formula holds

$$\frac{d^m}{ds^m}[\chi_0(s) F(a, b, 1; rs)] = \sum_{j=0}^{m-1} \frac{(a)_j(b)_j}{j!} r^j \delta^{(m-j-1)}(s) + \chi_0(s)c_mr^m F(a + m, b + m, m + 1; rs).$$

Indeed, just note that if $f(s)$ is a smooth function defined near $s = 0$, then $f(s) \delta(s) = f(0) \delta(s)$, and whenever $c \neq 0, -1, -2, \cdots$ one has

$$\frac{d}{dz} F(a, b, c; z) = \frac{ab}{c} F(a + 1, b + 1, c + 1; z).$$
Thus we may rewrite (3.17) as a derivative:

\[ \Phi_\alpha(x,t) = \frac{d^n}{d(k(\cdot))^n}[\chi_0(k(\cdot))F(\alpha, 1 - \alpha, 1, \frac{k(\cdot)}{-4t_0})]. \]  

Formulas (3.18) and (3.19) are analogous to formulas considered by Gelfand and Shilov in [7] and involving complex order derivatives of hypergeometric distributions of the type \( \chi_0(s)F(a, b, c; s) \).

Returning to the operator \( \mathcal{T}_h \) formula (3.10) with \( \alpha = 1/6 \) gives us a fundamental solution relative to the point \((0, t_0)\):

\[ E_{1/6}(x, t; 0, t_0) = \pi^{1/2-n/2}(\frac{t_0^0}{t})^{1/6}\Phi_{1/6}(x, t). \]  

In view of our remarks at the end of Section 1, the distribution

\[ (t/t_0)^{1/3}E_{1/6}(x, t; 0, t_0) = \pi^{1/2-n/2}(\frac{t}{t_0})^{1/6}\Phi_{1/6}(x, t) \]

is then a fundamental solution of \( \mathcal{T}_h^* \) relative to the same point. Motivated by formula (3.1) we define the distribution \( E^\sharp \) by

\[ 2(\frac{3}{2})^{1/3}t^{1/3}E^\sharp(x, t; 0, t_0) = \pi^{1/2-n/2}(\frac{t}{t_0})^{1/6}\Phi_{1/6}(x, t), \]

or,

\[ E^\sharp(x, t; 0, t_0) = \frac{\pi^{1/2-n/2}}{2^{1/3}3^{1/3}4^{1/6}t_0^0}\Phi_{1/6}(x, t). \]  

In the next sections, we derive from this formula fundamental solutions to the Tricomi operator (1.1) and relative to a point \((0, b)\), \( b < 0 \). We must distinguish three cases: (I) \( n = 1 \) which corresponds to the classical Tricomi operator, (II) \( n \) an even, and (III) \( n \) odd > 1. In order to simplify notations we write, in what follows, \( E(x, t; 0, t_0) \) instead of \( E^\sharp(x, t; 0, t_0) \).

4 The classical Tricomi operator

If \( n = 1 \), then (1.1) is the classical Tricomi operator in two variables. For this operator we will obtain two distinct fundamental solutions: one with support in a region entirely contained in the hyperbolic half-plane and the
other with support in the complement of that region. From formula (3.21) we get

\[ E(x,t;0,t_0) = \frac{1}{2^{1/3}3^{1/3}} \left( \frac{1}{4t_0} \right)^{1/6} \Phi_{1/6}(x,t). \]  

(4.1)

On the other hand, since \( F(a,b,c;z) = F(b,a,c;z) \), we get from (3.13) that

\[ \Phi_{1/6}(x,t) = \chi_0(k(\cdot))F\left(\frac{5}{6}, \frac{1}{6}; 1; -\frac{k(\cdot)}{4t_0}\right). \]  

(4.2)

Note that \( \chi_0(k(\cdot)) \) is the characteristic function of the semi-cone \( C \). Recall that

\[ F(a,b,c;z) = (1 - z)^{-b}F(c-a,b,c; \frac{z}{z-1}). \]  

(4.3)

If we set \( z = (t-t_0)^2 - x^2 / -4t_0t \), then

\[ 1 - z = \frac{(t+t_0)^2 - x^2}{4t_0t} \quad \text{and} \quad \frac{z}{z-1} = \frac{(t-t_0)^2 - x^2}{(t+t_0)^2 - x^2}, \]  

(4.4)

hence

\[ F\left(\frac{5}{6}, \frac{1}{6}; 1; -\frac{k(\cdot)}{4t_0}\right) = \left( \frac{(t+t_0)^2 - x^2}{4t_0t} \right)^{-1/6} F\left(\frac{1}{6}, \frac{1}{6}; 1; \frac{(t-t_0)^2 - x^2}{(t+t_0)^2 - x^2}\right), \]  

(4.5)

and we rewrite (4.1) as follows

\[ E(x,t;0,t_0) = \chi_0(k(\cdot)) \left( \frac{(t+t_0)^2 - x^2}{2^{1/3}3^{1/3}} \right)^{-1/6} F\left(\frac{1}{6}, \frac{1}{6}; 1; \frac{(t-t_0)^2 - x^2}{(t+t_0)^2 - x^2}\right), \]  

(4.6)

which is, as we pointed out at the end of Section 3, a fundamental solution to \( T_b^* \). Since \( \chi_0(k(\cdot)) \) is the characteristic function of the semi-cone \( C \), it follows that \( E(x,t;0,t_0) \) is supported by the closure of \( C \). Moreover, the last factor in formula (4.6) represents the hypergeometric series, because the absolute value of its argument (denoted by \( z/z-1 \) in formula (4.4)) is less than 1.

If one translates formula (4.6) in terms of the variables \( x \) and \( y \), one obtains a fundamental solution of the classical Tricomi operator, relative to the point \((0,b), b < 0\), and supported by the closure of the region in \( \mathbb{R}^2 \) that corresponds to the semi-cone \( C \). More specifically, consider the change of variables (4.1) and let \( a > 0 \) be such that \( t_0 = 2(-b)^{3/2}/3 = a \). Then, we have

\[ (t-t_0)^2 - x^2 = -\frac{1}{9} \left[ 9(x^2 - a^2) + 12a(-y)^{3/2} + 4y^3 \right] \]  

(4.7)
and
\[(t + t_0)^2 - x^2 = -\frac{1}{9}[9(x^2 - a^2) - 12a(-y)^{3/2} + 4y^3]. \tag{4.8}\]

In what follows and in order to simplify notations, we set
\[u = 9(x^2 - a^2) + 12a(-y)^{3/2} + 4y^3, \quad v = 9(x^2 - a^2) - 12a(-y)^{3/2} + 4y^3. \tag{4.9}\]

One can see that
\[u = [3(x - a) + 2(-y)^{3/2}][3(x + a) - 2(-y)^{3/2}]
\]
where the curve \(3(x - a) + 2(-y)^{3/2} = 0\) is one of the characteristics of \(T\) through \((0, b)\) and \(3(x + a) - 2(-y)^{3/2} = 0\), the other. Similarly,
\[v = [3(x - a) - 2(-y)^{3/2}][3(x + a) + 2(-y)^{3/2}].\]

The curve \(r_a\) of equation \(3(x - a) - 2(-y)^{3/2} = 0\) corresponds to one of the branches of the characteristic curve originating from \((a, 0)\) while \(r_{-a}\), the curve of equation \(3(x + a) + 2(-y)^{3/2} = 0\), corresponds to one of the branches of the characteristic originating from \((-a, 0)\).

It is a matter of verification that the semi-cone \(C\) corresponds in \(\mathbb{R}^2\) to the region
\[D_{b,-} = \{(x, y) \in \mathbb{R}_-^2 : 9(x^2 - a^2) + 12a(-y)^{3/2} + 4y^3 < 0, y < b\}, \tag{4.10}\]
denoted by \(D_I\) in the article \[4\]. One may now represent \(E(x, t; 0, t_0)\) in terms of \(x\) and \(y\), via the expressions \(u\) and \(v\), by
\[E_-(x, y; 0, b) = \chi_{D_{b,-}}(x, y) \cdot \frac{(-v)^{-1/6}}{2^{1/3}} F\left(\frac{1}{6}, \frac{1}{6}, 1; \frac{u}{v}\right) \tag{4.11}\]
where \(\chi_{D_{b,-}}\) is the characteristic function of \(D_{b,-}\).

In order to get another fundamental solution supported by the closure of the complement of \(D_{b,-}\) we introduce, as explained in the Appendix, \(\tilde{F}(1/6, 1/6, 1; \zeta)\), the principal branch of the analytic continuation of the corresponding hypergeometric series, and define in the whole of \(\mathbb{R}^2\) the function
\[\tilde{E}(x, y; 0, b) = \frac{(-v)^{-1/6}}{2^{1/3}} \tilde{F}\left(\frac{1}{6}, \frac{1}{6}, 1; \frac{u}{v}\right). \tag{4.12}\]

We will see in the Appendix that \(\tilde{E}(x, y; 0, b)\) is locally integrable in \(\mathbb{R}^2\), singular when \(v = 0\), real analytic in \(\mathbb{R}^2 \setminus (r_a \cup r_{-a})\), and a solution \(Tu = 0\) in the sense of distributions. We have the following result:
Theorem 4.1. The distribution \( E_\cdot \) defined by

\[
E_\cdot(x, y; 0, b) = \begin{cases} 
\tilde{E}(x, y; 0, b) & \text{in } D_{b,-} \\
0 & \text{elsewhere}
\end{cases}
\] (4.13)

is a fundamental solution of the Tricomi operator \( T \) relative to the point \((0, b)\). Its support is the closure of \( D_{b,-} \).

Proof. \( E_\cdot \) is just another way of writing the expression (4.11). \( \square \)

Remarks. 1. Since in \( D_{b,-} \) both \( u \) and \( v \) are \(< 0\), it follows that \( E_\cdot(x, y; 0, b) \) is real valued.

2. In [2] this fundamental solution was obtained by a method different than the one here described and based upon the existence of the Riemann function for the reduced hyperbolic operator \( T_h \).

3. \( E_\cdot \) is the unique fundamental solution of \( T \), relative to \((0, b)\) whose support is \( \bar{D}_{b,-} \). Indeed, any other such fundamental solution is of the form \( E_\cdot + f \), with \( Tf = 0 \) and \( y \leq b \) on \( \text{supp} f \). Since the convolution \( E_\cdot * f \) is well defined because the map

\[
\text{supp } E_\cdot \times \text{supp } f \ni ((x, y), (x', y')) \rightarrow (x + x', y + y')
\]

is proper, we have

\[
f = TE_\cdot * f = E_\cdot * Tf = 0.
\]

As a consequence of Theorem 4.1 we obtain one of the fundamental solutions described in [3].

Corollary 4.1. As \((0, b) \rightarrow (0, 0)\), the fundamental solution (4.13) converges, in the sense of distributions, to the fundamental solution

\[
F_\cdot(x, y) = \begin{cases} 
\frac{1}{2^{1/3}} F(\frac{1}{6}, \frac{1}{6}, 1; 1)|9x^2 + 4y^3|^{-1/6} & \text{in } D_- \\
0 & \text{elsewhere}
\end{cases}
\] (4.14)

where \( D_- = \{(x, y) \in \mathbb{R}^2 : 9x^2 + 4y^3 < 0\} \).
Since $\mathcal{T} \tilde{E} = 0$ in the sense of distributions, it follows that the distribution $E_0 - \tilde{E}$, identically zero in the region $D_{b_0}$, is also a fundamental solution of $\mathcal{T}$. Denote by $D_{b_+}$ the complement in $\mathbb{R}^2$ of $D_{b_0}$ and define the distribution

$$E_+(x, y; 0, b) = \begin{cases} -\tilde{E}(x, y; 0, b) & \text{in } D_{b_+} \\ 0 & \text{elsewhere.} \end{cases}$$

(4.15)

We clearly have

**Theorem 4.2.** $E_+$ is a fundamental solution of $\mathcal{T}$ relative to $(0, b)$ whose support is the closure of the region $D_{b_+}$.

This fundamental solution is not unique. If we replace the exponential factor in (4.12) by $e^{-i\pi/6}$ we obtain another fundamental solution. Moreover, it does not follow as in the case of $E_+$, that $E_+$ converges, as $b \to 0$, to the fundamental solution $F_+(x, y)$ described in [3]. In order to obtain such a result, one needs to consider a suitable linear combination of these two fundamental solutions before taking limits (see [4]).

We thus have

**Corollary 4.2.** As $(0, b) \to (0, 0)$, a suitable linear combination of fundamental solutions of the type $E_+$ converges, in the sense of distributions, to the fundamental solution

$$F_+(x, y) = \begin{cases} -\frac{1}{2^{1/3}3^{1/2}}F(\frac{1}{6}, \frac{1}{6}, 1; 1)(9x^2 + 4y^3)^{-1/6} & \text{in } D_+ \\ 0 & \text{elsewhere,} \end{cases}$$

(4.16)

where $D_+ = \{(x, y) \in \mathbb{R}^2 : 9x^2 + 4y^3 > 0\}$.

5 The Tricomi operator, $n$ even

We begin with formula (3.21)

$$E(x, t; 0, t_0) = \frac{\pi^{1/2-n/2}}{2^{1/3}3^{1/3}} \left(\frac{1}{4t_0 t}\right)^{1/6} \Phi_{1/6}(x, t),$$

where $\Phi_{1/6}$, given by (3.15), is

$$\Phi_{1/6}(x, t) = \chi_{1/2-n/2}(k(\cdot)) F\left(\frac{5}{6}, \frac{1}{6}, \frac{3}{2} - \frac{n}{2}; -\frac{k(\cdot)}{4t_0 t}\right).$$

(5.1)
Recalling formulas (2.8), (4.3) and (4.4) we obtain

\[
E(x, t; 0, t_0) = \frac{\pi^{1/2-n/2}}{2^{1/3}3^{1/3}} \chi_{1/2-n/2}([k(\cdot)]) [(t + t_0)^2 - |x|^2]^{-1/6} \times \\
\times F(\frac{2}{3} - \frac{n}{2}, \frac{3}{2} - \frac{n}{2}, (t + t_0)^2 - |x|^2). \tag{5.2}
\]

To obtain a fundamental solution to \( T \) we represent (5.2) in terms of \( x \) and \( y \). If we set

\[
u = 9(|x|^2 - a^2) + 12a(-y)^{3/2} + 4y^3, \quad v = 9(|x|^2 - a^2) - 12a(-y)^{3/2} + 4y^3, \tag{5.3}
\]

then

\[(t - t_0)^2 - |x|^2 = -\frac{1}{9} u \quad \text{and} \quad (t + t_0)^2 - |x|^2 = -\frac{1}{9} v. \tag{5.4}
\]

These two formulas are the counterpart to (4.7) and (4.8) in the case \( n = 1 \).

Define, as we did in Section 4 case \( n = 1 \), the region

\[D_{b_{-}}^{n} = \{(x, y) \in \mathbb{R}^{n+1} : 9(|x|^2 - a^2) + 12a(-y)^{3/2} + 4y^3 < 0, y < b\} \tag{5.5}
\]

which corresponds to the semi-cone \( C \), and let \( \chi_{D_{b_{-}}^{n}} \) be its characteristic function.

In terms of \( x \) and \( y \) the distribution (5.2) becomes

\[
E_{-}(x, y; 0, b) = c(n)\chi_{D_{b_{-}}^{n}}(x, y)(-u)^{1/2-n/2}(-v)^{-1/6}F(\frac{2}{3} - \frac{n}{2}, \frac{3}{2} - \frac{n}{2}, \frac{u}{v}), \tag{5.6}
\]

where

\[
c(n) = \frac{\pi^{1/2-n/2}}{2^{1/3}3^{1/3} \Gamma(\frac{3}{2} - \frac{n}{2})}. \tag{5.7}
\]

Thus we obtain the following result:

**Theorem 5.1.** \( E_{-}(x, y; 0, b) \) is a fundamental solution of \( T \) relative to \( (0, b) \) whose support is the closure of the region \( D_{b_{-}}^{n} \).

If we let \( b \to 0 \), we obtain a fundamental solution of \( T \) relative to the origin, namely
Corollary 5.1. The limit, in the sense of distributions, of \( E_-(x,y;0,b) \) as \((0,b) \to (0,0)\) is

\[
F_-(x,y) = \begin{cases} 
\frac{\pi^{1/2-n/2}}{2^{1/3}3^{1-n}\Gamma\left(\frac{3}{2} - \frac{n}{2}\right)} F\left(\frac{2}{3} - \frac{n}{2}, \frac{1}{2}; \frac{3}{2} - \frac{n}{2}, 1\right) |9|x|^2 + 4y^3|^{\frac{1}{3} - \frac{n}{2}} & \text{in } D^n \\
0 & \text{elsewhere,}
\end{cases}
\] (5.8)

a fundamental solution of \( T \) relative to the origin whose support is the closure of the region \( D^n = \{(x,y) \in \mathbb{R}^{n+1} : 9|x|^2 + 4y^3 < 0\} \).

The fundamental solution given by formula (5.8) coincides with the fundamental solution given by formula (4.2) in Theorem 4.1 of [2]. The only apparent discrepancy between these two formulas is the multiplying constants. In (5.8), the multiplicative constant is

\[
A = \frac{\pi^{1/2-n/2}}{2^{1/3}3^{1-n}\Gamma\left(\frac{3}{2} - \frac{n}{2}\right)} F\left(\frac{2}{3} - \frac{n}{2}, \frac{1}{2}; \frac{3}{2} - \frac{n}{2}, 1\right) \] (5.9)

while in [2], page 490, the multiplicative constant for \( F_-(x,y) \) is

\[
C_- = \frac{3^n\Gamma(4/3)}{2^{2/3}\pi^{n/2}\Gamma\left(\frac{4}{3} - \frac{n}{2}\right)}.
\] (5.10)

Since

\[
F(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}
\]
a formula that holds whenever \( \text{Re} \, c > \text{Re} \, b + \text{Re} \, a \), we may rewrite \( A \) as

\[
A = \frac{\pi^{1/2-n/2}\Gamma(2/3)}{2^{1/3}3^{1-n}\Gamma(5/6)\Gamma\left(\frac{4}{3} - \frac{n}{2}\right)}.
\]

In order for \( A = C_- \) one must have the identity

\[
\frac{2^{1/3}\pi^{1/2}\Gamma(2/3)}{3\Gamma(5/6)\Gamma(4/3)} = 1.
\]

But this is a consequence of the following relations for the Gamma function: \( \Gamma(z + 1) = z\Gamma(z) \), \( \Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(z + 1/2) \), and \( \Gamma(z)\Gamma(1 - z) = \pi \csc(\pi z) \).
In [2] we showed that the distribution

\[ F_+(x, y) = \begin{cases} 
C_+(9|x|^2 + 4y^3)^{1/3-n/2} & \text{in } D_+^n \\
0 & \text{elsewhere,} 
\end{cases} \]  

(5.11)

where

\[ C_+ = -\frac{3^{n-2}\Gamma\left(\frac{n}{2} - \frac{1}{3}\right)}{2^{2/3}\pi^{n/2}\Gamma(2/3)} \]  

(5.12)

and

\[ D_+^n = \{(x, y) \in \mathbb{R}^{n+1} : 9|x|^2 + 4y^3 > 0\}, \]  

(5.13)

is a fundamental solution of \( \mathcal{T} \) supported by the closure of \( D_+^n \). It is a matter of verification that the ratio between the constants \( C_+ \) and \( C_- \) is

\[ \frac{C_+}{C_-} = -\frac{1}{2\sqrt{3}\sin\left(\frac{n}{2} - \frac{1}{3}\right)}. \]  

(5.14)

Therefore the constant \( C_+ \) in [2] can also be represented in terms of the above hypergeometric function.

6 The Tricomi operator, \( n \text{ odd} \geq 1 \)

Let \( n = 2m + 1 \) with \( m \geq 1 \). Again from formula (3.21) we have

\[ E(x, t; 0, t_0) = A_m\left(\frac{1}{4t_0t}\right)^{1/6}\Phi_{1/6}(x, t), \]  

(6.1)

where \( A_m = 1/2^{1/3}3^{1/3}\pi^m \). From formula (3.17) \( \Phi_{1/6} \) is given by

\[ \Phi_{1/6}(x, t) = \sum_{j=0}^{m-1} c_j\left(-\frac{1}{4t_0t}\right)^{5/6}(m-j-1)(k(\cdot)) + \]  

(6.2)

\[ + \chi_0(k(\cdot))c_m\left(-\frac{1}{4t_0t}\right)^{5/6}F\left(m + \frac{5}{6}, m + \frac{1}{6}, m + 1, \frac{k(\cdot)}{-4t_0t}\right). \]

with

\[ c_j = \frac{\Gamma(j + 5/6)\Gamma(j + 1/6)}{\Gamma(5/6)\Gamma(1/6)\Gamma(j + 1)}, \quad 0 \leq j \leq m. \]  

(6.3)
In view of (4.3) and (4.4), the hypergeometric function in (6.2) is equal to
\[
\frac{(t + t_0)^2 - |x|^2}{4t_0 t} - m^{-1/6} \left( \frac{1}{6}, m + \frac{1}{6}, m + 1; \frac{(t - t_0)^2 - |x|^2}{(t + t_0)^2 - |x|^2} \right) - \frac{1}{m - 1/6} F \left( 1, \frac{m + 1}{6}, \frac{m + 1}{6}, \frac{(t - t_0)^2 - |x|^2}{(t + t_0)^2 - |x|^2} \right)
\]
and we may rewrite \( E(x, t; 0, t_0) \) as
\[
E(x, t; 0, t_0) = \sum_{j=0}^{m-1} (-1)^j c_j \left( \frac{4t_0 t}{j} \right)^{-j - 1/6} \delta^{(m-j-1)}(k(\cdot)) + \]
\[
+ (-1)^m A m \sum_{j=0}^{m-1} \left( \frac{4t_0 t}{j} \right)^{-j - 1/6} \delta^{(m-j-1)}(k(\cdot)) \chi_0(k(\cdot)).
\]
Note that all terms in the sum contain distributions of the form \( \delta^{(q)}(k(\cdot)) \) which are supported by the surface of the semi-cone \( C \). However, the support of the last term in (6.4) is the closure of \( C \).

In [7] Gelfand and Shilov introduced the distribution \( \delta(P) \) supported by the surface \( S \) given by \( P = 0 \), where \( P \) is a smooth function such that \( \nabla P \neq 0 \) on \( S \). In particular, they proved that if \( a(\cdot) \) is a nonvanishing function, then
\[
\delta^{(q)}(aP) = a^{-(q+1)} \delta^{(q)}(P).
\]
These results extend to our case, where \( P = k(\cdot) \) has a singular point at \((0, t_0)\). We have the following

Lemma 6.1. For all \( 0 \leq j \leq m - 1 \),
\[
(4t_0 t)^{-j - 1/6} \delta^{(m-j-1)}(k(\cdot)) = (4t_0 t)^{5/6} \delta^{(m-j-1)}((4t_0 t)^{j+1/m-j} k(\cdot)).
\]

Proof. Indeed we have
\[
(4t_0 t)^{-j - 1/6} \delta^{(m-j-1)}(k(\cdot)) = (4t_0 t)^{5/6} (4t_0 t)^{-j - 1/6} \delta^{(m-j-1)}(k(\cdot)) =
\]
\[
= (4t_0 t)^{5/6} \left( (4t_0 t)^{j+1/m-j} \right)^{-j - 1/6} \delta^{(m-j-1)}((4t_0 t)^{j+1/m-j} k(\cdot)),
\]
by virtue of (6.5) and the fact that \( 4t_0 t \neq 0 \) in the region \( t - t_0 > |x| \).

As a consequence of this lemma, all terms that contain derivatives of \( \delta \) in (6.4) tend to zero, as \( t_0 \to 0 \). By taking limits, it follows that the distribution
\[
E(x, t; 0, 0) = \]
\[
(6.6)
\]
is a fundamental solution of \( T_b \) supported by the closure of the semi-cone \( \{(x, t) : t > |x|\} \).

As we did in the previous sections, we rewrite \( E(x, t; 0, t_0) \) in terms of the variables \( x \) and \( y \). From formulas (5.3) and (5.4) we derive that 

\[
4t_0 t = \frac{v - u}{9}
\]

and, following Gelfand and Shilov’s notations, we replace \( \delta^{(q)}(k(\cdot)) \) by \( \delta^{(q)}(u(\cdot)) \), with the understanding that \( u(\cdot) \) now means \( u(x, y) \), with \( y \leq b \). Thus (6.4) becomes

\[
E_-(x, y; 0, b) = \sum_{j=0}^{m-1} (-1)^{j} c_j \left( \frac{v - u}{9} \right)^{-j-1/6} \delta^{(m-j-1)}(u(\cdot)) + (-1)^m A_m c_m \left( \frac{v}{9} \right)^{-m-1/6} F \left( \frac{1}{6}, m + \frac{1}{6}, m + 1; \frac{u}{v} \right) \chi_{D_{b,-}}(x, y)
\]

where \( \chi_{D_{b,-}} \) is the characteristic function of the set (5.5). Then the following result holds:

**Theorem 6.1.** The distribution \( E_-(x, y; 0, b) \) is a fundamental solution of \( T \) relative to \( (0, b) \) supported by the closure of the set \( D_{b,-} \).

Note that in (6.7) all terms inside the summation are supported by the boundary of \( D_{b,-} \) while the last term is supported by the closure of \( D_{b,-} \). If we let \( b \to 0 \), we obtain at the limit, the fundamental solution \( F_-(x, y) \) described in our previous paper \([2]\), namely

**Theorem 6.2.** The distribution

\[
F_-(x, y) = \begin{cases} 
3^m \Gamma \left( \frac{4}{3} \right) 2^{3/2} \pi^{n/2} \Gamma \left( \frac{4}{3} - \frac{n}{2} \right) \frac{9}{2} |x|^2 + 4y^3 \frac{1}{3} \frac{1}{2} & \text{in } D_- \\
0 & \text{elsewhere,}
\end{cases}
\]

supported by the closure of the region \( D_- = \{(x, y) \in \mathbb{R}^{n+1} : 9|x|^2 + 4y^3 < 0\} \), is a fundamental solution of \( T \).

**Proof.** Recall that \( t = 2(-y)^{3/2}/3 \) and that \( t^2 - |x|^2 = \frac{1}{9} (-9|x|^2 - 4y^3) \). Hence, the right hand-side of (6.6) equals

\[
AF \left( \frac{1}{6}, m + \frac{1}{6}, m + 1 + 1; 1 \right) 9 |x|^2 + 4y^3 \frac{1}{2} \frac{1}{2} - m - 1/6
\]

18
where
\[ A = \frac{(-1)^m 3^m}{2^{n/3} \pi n} \frac{\Gamma(m + 5/6) \Gamma(m + 1/6)}{\Gamma(5/6) \Gamma(1/6) \Gamma(m + 1)}. \] (6.9)

Now the exponent \(-m - 1/6\) equals \(1/3 - n/2\) because \(n = 2m + 1\). On the other hand, it is a matter of verification that the constant
\[ \frac{3^n \Gamma(4/3)}{2^{2n/3} \pi n/2 \Gamma(4/3 - n/2)} \] (6.10)
(denoted by \(C_\cdot\) in [2]) which appears in (6.8) is the same as \(A\). \(\square\)

7 Appendix

We are going to prove that the function \(\tilde{E}(x, y; 0, b)\) defined by formula (4.12) in Section 4 is locally integrable in \(\mathbb{R}^2\), singular when \(v = 0\), real analytic in \(\mathbb{R}^2 \setminus (r_a \cup r_{-a})\), and a solution of \(T w = 0\) in the sense of distributions. Recall that \(r_a\) is the characteristic curve \(3(x - a) - 2(-y)^{3/2} = 0\) originating from \((a, 0)\), and \(r_{-a}\), the characteristic curve \(3(x + a) + 2(-y)^{3/2} = 0\), originating from \((-a, 0)\).

Following Whittaker and Watson [10], let \(\alpha, \beta, \text{ and } \gamma\) be complex numbers, \(\gamma \neq 0, -1, -2, \cdots\), and let
\[ (\alpha)_0 = 1, (\alpha)_n = \alpha(\alpha + 1) \cdots (\alpha + n - 1) = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)}. \] (7.1)

The power series
\[ F(\alpha, \beta, \gamma; \zeta) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} \zeta^n \] (7.2)
is called the hypergeometric series. The ratio test guarantees absolute convergence for \(|\zeta| < 1\). If \(\Re(\gamma - \alpha - \beta) > 0\), then the series converges for \(|\zeta| \leq 1\) and
\[ F(\alpha, \beta, \gamma; 1) = \frac{\Gamma(\gamma) \Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha) \Gamma(\gamma - \beta)}. \] (7.3)

Barnes’ contour integral defines a single-valued analytic function of \(\zeta\) in the region \(|\arg(-\zeta)| < \pi\), that is, \(\mathbb{C}\) minus the positive real axis, which gives the principal branch of the analytic continuation of the hypergeometric series \(F(\alpha, \beta, \gamma; \zeta)\). More precisely we quote the following theorem whose proof is found in [10].
Theorem 7.1. (Barnes) The integral
\[ \frac{1}{2\pi i} \int_{-\infty}^{i\infty} \frac{\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(-s)}{\Gamma(\gamma+s)} (-\zeta)^s \, ds, \] (7.4)
where the contour of integration is curved (if necessary) to ensure that the poles of \( \Gamma(\alpha+s)\Gamma(\beta+s) \), i.e., \( s = -\alpha - n, -\beta - n, n = 0, 1, 2, \cdots \), lie on the left of the contour and the poles of \( \Gamma(-s) \), i.e., lie on the right of the contour, define a single-valued analytic function in the region \(|\arg(-\zeta)| < \pi\). Moreover, in the unit disk \(|\zeta| < 1\), it coincides with the hypergeometric series
\[ \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)} F(\alpha, \beta, \gamma; \zeta). \]

Following traditional practice we use the notation \( F(\alpha, \beta, \gamma; \zeta) \) to denote either the hypergeometric series or the principal branch of its analytic continuation, and call it the hypergeometric function.

Barnes’ integral may also be used to obtain a representation of the hypergeometric function in the form of a power series in \( \zeta^{-1} \), convergent when \(|\zeta| > 1\). By choosing a suitable contour of integration one can prove (see [10]) that if \( \alpha - \beta \) is not an integer or zero, then
\[ F(\alpha, \beta, \gamma; \zeta) = A(-\zeta)^{-\alpha} F(\alpha, 1 - \gamma + \alpha, 1 - \beta + \alpha; \zeta^{-1}) \] (7.5)
where \( A \) and \( B \) are suitable constants and \(|\arg(-\zeta)| < \pi\). This formula also describes the asymptotic behaviour of the function \( F(\alpha, \beta, \gamma; \zeta) \) near \(|\zeta| = \infty\). If \( \alpha - \beta \) is an integer or zero, formula (7.5) must be modified because some of the poles of \( \Gamma(\alpha+s)\Gamma(\beta+s) \) are double poles. The reader should find the expression for \( F(\alpha, \beta, \gamma; \zeta) \) in [3], chapter on hypergeometric functions.

In the case that interests us, that is, \( \alpha = \beta \), that expression is
\[ F(\alpha, \alpha, \gamma; \zeta) = (-\zeta)^{-\alpha} [\log(-\zeta) U(\zeta) + V(\zeta)], \] (7.6)
where \(|\arg(-\zeta)| < \pi\), and both \( U(\zeta) \) and \( V(\zeta) \) are power series in \( \zeta^{-1} \) convergent for \(|\zeta| > 1\). The reader expressions are found in [3] or [4]. We also mention that if \( \Re(\gamma - \alpha - \beta) > 0 \), we have convergence for \(|\zeta| \geq 1\).

From the above results and in particular from (7.6) it follows that
\[ \tilde{E}(x, y; 0, b) = \frac{(\log b)^{-1/6}}{2^{1/3}} F\left(\frac{1}{6}, \frac{1}{6}, 1; \frac{u}{\log b}\right), \]
with $u$ and $v$ defined by (4.9), is locally integrable in $\mathbb{R}^2$, singular when $v = 0$, and real analytic in $\mathbb{R}^2 \setminus (r_a \cup r_{-a})$.

It remains to prove that $\tilde{E}(x, y; 0, b)$ is a solution to $Tw = 0$ in the sense of distributions. For this we need several results proved in the paper [3]. In that paper we showed that the function

$$E(\ell, m; \ell_0, m_0) = (\ell - m)^{-1/6}(\ell_0 - m)^{-1/6}F\left(\frac{1}{6}, \frac{1}{6}, 1; \frac{(\ell - \ell_0)(m - m_0)}{(\ell_0 - m_0)(m - \ell_0)}\right)$$

is a classical solution of

$$Thw = \frac{\partial^2 w}{\partial \ell \partial m} - \frac{1/6}{\ell - m} \left( \frac{\partial w}{\partial \ell} - \frac{\partial w}{\partial m} \right) = 0,$$

the reduced hyperbolic Tricomi equation. Here

$$\ell = x + \frac{2}{3}(-y)^{3/2}, \quad m = x - \frac{2}{3}(-y)^{3/2}$$

are the characteristic coordinates. Now, except for the constant $1/2^{1/3}$, $\tilde{E}(x, y; 0, b)$ is obtained from $E(\ell, m; \ell_0, m_0)$ after replacement of $\ell$ and $m$ by their expressions above and by setting $\ell_0 = -m_0 = 2(b)^{3/2}/3$. Thus, away from the set $\{v = 0\} = r_a \cup r_{-a}$, $\tilde{E}(x, y; 0, b)$ is a classical solution of $Tw = 0$.

To show that $T\tilde{E} = 0$, in the sense of distributions, we have to contend with the fact that $\tilde{E}(x, y; 0, b)$ has logarithmic singularities along the two characteristics $r_{-a}$ and $r_a$, or, equivalently, that $E(\ell, m; \ell_0, m_0)$ has logarithmic singularities along the lines $\ell = -\ell_0$ and $m = \ell_0$. Since, as we have remarked, $T\tilde{E} = 0$ away from the characteristics $r_{-a}$ and $r_a$, in order to prove that $T\tilde{E} = 0$ in the sense of distributions, it suffices to prove that

$$\langle \tilde{E}, T\phi \rangle = \int \int_{\mathbb{R}^2} \tilde{E}T\phi dx \, dy = 0 \quad (7.7)$$

for all $\phi \in C_c^\infty(\mathbb{R}^2)$ whose support intersects at least one of the characteristics $r_{-a}$ or $r_a$. If $\text{supp} \phi$ does not intersect either of these characteristics, then (7.7) is automatically satisfied.

Suppose that $\text{supp} \phi$ is contained in an open disk $D$ centered, say at $(a, 0)$, and with radius $R$. Let $0 < r < R$ and denote by $D_r$ the set of points of $D$.
at a distance $> \epsilon$ from the characteristic $r_\alpha$. Then, from Green’s formula for $T$ (see [4], formula (4.5)) one gets

$$
\int \int_D \tilde{E} T \phi \, dx \, dy = \lim_{\epsilon \to 0} \int \int_{D_\epsilon} \tilde{E} T \phi \, dx \, dy
$$

(7.8)

$$
= \lim_{\epsilon \to 0} \int \int_{\Gamma_\epsilon \cup \gamma_\epsilon \cup \Gamma'_\epsilon} \tilde{E} (y \phi_x \, dy - \phi_y \, dx) - \phi (y \tilde{E}_x \, dy - \tilde{E}_y \, dx),
$$

where $\Gamma_\epsilon$ is the characteristic $3(x-\alpha+\epsilon)-2(-y)^{3/2}=0$, $\gamma_\epsilon$ the circumference center at $\alpha$ with radius $\epsilon$, and $\Gamma'_\epsilon$ the characteristic $3(x-\alpha-\epsilon)-2(-y)^{3/2}=0$. In order to prove (7.7) we must prove that the last limit in (7.8) is zero.

Most details of the proof are to be found in Section 4 of the paper [4]. We just point out that the integrand in (7.8) remains bounded along $\gamma_\epsilon$ thus, along this contour, the integral tends to zero with $\epsilon$. Along both $\Gamma_\epsilon$ and $\Gamma'_\epsilon$ we must take into account the asymptotic behaviour of $F(1/6, 1/6, 1; \zeta)$ and its derivative $F(7/6, 7/6, 2; \zeta)$, at $\zeta = \infty$, according with (7.6). It turns out that at the limit, the values of these integrals cancel each other and this completes the proof.

References

[1] J. Barros-Neto, *On Fundamental Solutions for the Tricomi Operator*, Atas do 49$^{o}$ Seminário Brasileiro de Análise, 1999, 69–88.

[2] J. Barros-Neto and F. Cardoso, *Bessel integrals and fundamental solutions for a generalized Tricomi operator*, Jour. of Funct. Analysis, 183 (2001), 472–497.

[3] J. Barros-Neto and I. M. Gelfand, *Fundamental solutions for the Tricomi operator*, Duke Math. J. 98 (1999), 465–483.

[4] J. Barros-Neto and I. M. Gelfand, *Fundamental solutions for the Tricomi operator, II*, Duke Math. J. 111 (2002), 561–584.

[5] S. Delache and J. Leray, *Calcul de la solution élémentaire de l’opérateur d’Euler-Poisson-Darboux et de l’opérateur de Tricomi-Clairaut, hyperbolique d’ordre 2*, Bull. Soc. Math. France, 99 (1971), 313–336.

[6] A. Erdély, *Higher Transcendental Functions*, Vols. I, II, III, McGraw-Hill, New York (1953).
[7] I. M. Gelfand and G. E. Shilov, *Generalized Functions, Vol. I: Properties and Operations*, Academic Press, New York (1964).

[8] L. Schwartz, *Théorie des Distributions*, Hermann, Paris, (1950–51).

[9] F. Tricomi, *Sulle equazioni lineari alle derivate parziali di secondo ordine, di tipo misto*, Rendiconti, Atti dell’Accademia Nazionali dei Lincei, Serie 5, 14, (1923), 134-247.

[10] E. T. Whittaker and G. N. Watson, *A course of Modern Analysis*, 4th ed., Cambridge Press, New York, 1962.