Independent linear forms on the group $\Omega_p$

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Abstract

Let $\Omega_p$ be the group of $p$-adic numbers, $\xi_1, \xi_2, \xi_3$ be independent random variables with values in $\Omega_p$ and distributions $\mu_1, \mu_2, \mu_3$. Let $\alpha_j, \beta_j, \gamma_j$ be topological automorphisms of $\Omega_p$. We consider linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2 + \alpha_3\xi_3$, $L_2 = \beta_1\xi_1 + \beta_2\xi_2 + \beta_3\xi_3$ and $L_3 = \gamma_1\xi_1 + \gamma_2\xi_2 + \gamma_3\xi_3$. Assuming that the linear forms $L_1$, $L_2$ and $L_3$ are independent, we describe possible distributions $\mu_1$, $\mu_2$, $\mu_3$. This theorem is an analogue of the well-known Skitovich-Darmois theorem, where a Gaussian distribution on the real line is characterized by the independence of two linear forms.

Keywords: Group of p-adic numbers, Characterization theorem, Skitovich-Darmois theorem

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1. Introduction

One of the most important characterization theorem of mathematical statistics is the Kac–Bernstein theorem. This theorem characterizes a Gaussian distribution by the independence of the sum and of the difference of two independent random variables. V.P. Skitovich and G. Darmois generalized this theorem.

Theorem A ([23, 1], see also [14, Ch. 3]). Let $\xi_j, j = 1, 2, \ldots, n, n \geq 2$, be independent random variables. Let $\alpha_j, \beta_j$ be nonzero constants. If the linear forms $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ are independent, then all random variables $\xi_j$ are Gaussian.

The classical characterization theorems of mathematical statistics were extended to different algebraic structures such as finite-dimensional and infinite-dimensional linear spaces, symmetric spaces, quantum groups (see e.g. [18], [20]–[22]). Much attention has been also devoted to the study of analogues of the Skitovich–Darmois theorem for some classes of locally compact Abelian groups. In this case coefficients of linear forms are topological automorphisms of a group. (see e.g. [3]–[6], [11, 15–19, 24] and also [7], where one can find additional references). Note that most articles were devoted to the description of groups on which the independence of two linear forms from $n$ random variables implies that all random variables are Gaussian or have distributions which can be considered as analogues of Gaussian distributions on groups (e.g. idempotent distributions). We also note that the description of distributions on groups which are characterized by the independence of linear forms depends not only on a group but also on the length of forms. For example, on the one hand, it was proved that for $n = 2$ an analogue of the Skitovich–Darmois theorem is valid for all finite groups ([6]). On the other hand, if $n > 2$ then the analogue of the Skitovich–Darmois theorem is valid only for some special finite Abelian groups ([3]). Similar results were obtained for random variables with values in discrete Abelian groups (see [11] and [5]). The case of compact Abelian groups was studied in [4] and [9].

The situation is essentially changing, if we consider $n$ linear forms of $n$ independent random variables instead of two linear forms of $n$ independent random variables. We note that one of
The first paper in which \(n\) linear forms were considered, was the paper [24], where J.H. Stapleton considered the case of compact connected Abelian groups with some restrictions on coefficients of the forms and random variables. Next it was proved in [15], that if random variables take values in a finite Abelian group then the independence of \(n\) linear forms of \(n\) independent random variables implies that random variables are idempotent. The cases of discrete and compact Abelian groups were studied in [16] and [17].

The distributions on a group of \(p\)-adic numbers \(\Omega_p\), which are characterized by the independence of two linear forms of two independent random variables were studied in [8]. Since the group \(\Omega_p\) is totally disconnected, the Gaussian distributions on it are degenerated. The role of Gaussian distributions on \(\Omega_p\) plays idempotent distributions, i.e. the set of shifts of the Haar distributions of the compact subgroups. It turned out that on this group the description of distributions which are characterized by the independence of two linear forms of two independent random variables depends on the coefficients of the linear forms. In particular, all the coefficients of linear forms for which the independence of two linear forms of two independent random variables implies that random variables are idempotent were described in [8].

In this paper we study distributions on the group of \(p\)-adic numbers \(\Omega_p\), which are characterized by the independence of three linear forms of three independent random variables. This studying is much more complicated than in [8] because the number of parameters increases. The main result of the paper is Theorem 1. In particular, we obtain in Theorem 1 the description of all the coefficients of linear forms for which the independence of three linear forms of three independent random variables implies that all random variables are idempotent.

2. Definitions and notation

We use some results of the duality theory for the locally compact Abelian groups (see [12]). For an arbitrary locally compact Abelian group \(X\) let \(Y = X^*\) be its character group, and \((x, y)\) be the value of a character \(y \in Y\) at an element \(x \in X\). If \(K\) is a closed subgroup of \(X\), we denote by \(A(Y, K) = \{y \in Y : (x, y) = 1\ \text{for all} \ x \in K\}\) its annihilator. If \(\delta : X \mapsto X\) is a continuous homomorphism, then the adjoint homomorphism \(\tilde{\delta} : Y \mapsto Y\) is defined by the formula \((x, \tilde{\delta}y) = (\delta x, y)\) for all \(x \in X\), \(y \in Y\). Denote by \(\text{Aut}(X)\) the group of topological automorphisms of the group \(X\). We note that \(\delta \in \text{Aut}(X)\) if and only if \(\tilde{\delta} \in \text{Aut}(Y)\). Denote by \(I\) the identity automorphism of a group. Denote by \(f_n\) homomorphism defined by the formula \(f_n(x) = nx\). Put \(X_{(n)} = \{x \in X : nx = 0\}\).

Let \(M^1(X)\) be the convolution semigroup of probability distributions on \(X\). For a distribution \(\mu \in M^1(X)\) denote by

\[
\hat{\mu}(y) = \int_X (x, y) d\mu(x)
\]

its characteristic function (Fourier transform), and by \(\sigma(\mu)\) the support of \(\mu\). For \(\mu \in M^1(X)\), we define the distribution \(\hat{\mu} \in M^1(X)\) by the formula \(\hat{\mu}(E) = \mu(-E)\) for any Borel set \(E \subset X\). Observe that \(\hat{\hat{\mu}}(y) = \overline{\mu(y)}\). Let \(K\) be a compact subgroup of \(X\). Denote by \(m_K\) the Haar distribution on \(K\). We note that the characteristic function of \(m_K\) is of the form

\[
\hat{m}_K(y) = \begin{cases} 
1, & y \in A(Y, K), \\
0, & y \notin A(Y, K).
\end{cases}
\]

Denote by \(I(X)\) the set of all idempotent distributions on \(X\), i.e. the set of shifts of the Haar distributions \(m_K\) of the compact subgroups \(K\) of \(X\). Let \(x \in X\). Denote by \(E_x\) the degenerated
distribution concentrated at the point \( x \). Denote by \( D(X) \) the set of all degenerated distributions on \( X \).

Let \( p \) be a prime number. We need some properties of the group of \( p \)-adic numbers \( \Omega_p \) (see [12 §10]). As a set \( \Omega_p \) coincides with the set of sequences of integers of the form \( x = (\ldots, x_{-n}, x_{-n+1}, \ldots, x_{-1}, x_0, x_1, \ldots, x_n, x_{n+1}, \ldots) \), where \( x_n \in \{0, 1, \ldots, p - 1\} \), such that \( x_n = 0 \) for \( n < n_0 \), where the number \( n_0 \) depends on \( x \). We correspond to each element \( x \in \Omega_p \) the series

\[
\sum_{k=-\infty}^{\infty} x_k p^k.
\]

Addition and multiplication of the series are defined in a natural way and they define the operations of addition and multiplication in \( \Omega_p \). With respect to these operations \( \Omega_p \) is a field. Denote by \( \Delta_p \) a subgroup of \( \Omega_p \) consisting of \( x \in \Omega_p \) such that \( x_n = 0 \) for \( n < 0 \). The subgroup \( \Delta_p \) is called the group of \( p \)-adic integers. The family of the subgroups \( \{p^k \Delta_p\}_{k=-\infty}^{\infty} \) forms an open basis at zero of the group \( \Omega_p \) and defines a topology on \( \Omega_p \). With respect to this topology the group \( \Omega_p \) is locally compact, non-compact, and totally disconnected. We note that the group \( \Omega_p \) is represented as a union

\[
\Omega_p = \bigcup_{k=-\infty}^{\infty} p^k \Delta_p.
\]

Note that any compact subgroup \( K \) of \( \Omega_p \) is of the form

\[
K = p^l \Delta_p
\]

for some \( l \). The character group \( \Omega_p^* \) of the group \( \Omega_p \) is topologically isomorphic to \( \Omega_p \). Each automorphism \( \alpha \in \text{Aut}(\Omega_p) \) is of the form \( \alpha g = x_\alpha g, g \in \Omega_p, \) where \( x_\alpha \in \Omega_p, x_\alpha \neq 0 \). For \( \alpha \in \text{Aut}(\Omega_p) \) we identify the automorphism \( \alpha \in \text{Aut}(\Omega_p) \) with the corresponding element \( x_\alpha \in \Omega_p \), i.e. when we write \( \alpha g \), we suppose that \( \alpha \in \Omega_p \). We note that \( \alpha \) is of the form \( \sum_{j=0}^{\infty} a_j \). Denote by \( \Delta_p^0 \) the subset of \( \Omega_p \) consisting of all invertible in \( \Delta_p \) elements, \( \Delta_p^0 = \{x \in \Omega_p : x_n = 0 \text{ for } n < 0 \}, x_0 \neq 0 \}. \) We note that each element \( g \in \Omega_p \) is represented in the form \( g = p^k c \), where \( k \) is an integer, and \( c \in \Delta_p^0 \). Hence, multiplication on \( c \) is a topological automorphism of the group \( \Delta_p \).

Denote by \( \mathbb{Z}(p^\infty) \) the set of rational numbers of the form \( \{k/p^n : k = 0, 1, \ldots, p^n - 1, n = 0, 1, \ldots\} \). If we define the operation in \( \mathbb{Z}(p^\infty) \) as addition modulo 1, then \( \mathbb{Z}(p^\infty) \) is transformed into an Abelian group which we consider in the discrete topology. Obviously, this group is topologically isomorphic to the multiplicative group of all \( p^n \)-th roots of unity, where \( n \) goes through the set of nonnegative integers considering in the discrete topology. For a fixed \( n \) denote by \( \mathbb{Z}(p^n) \) a subgroup of \( \mathbb{Z}(p^\infty) \) consisting of all elements of the form \( \{k/p^n : k = 0, 1, \ldots, p^n - 1\} \). Note that the group \( \mathbb{Z}(p^n) \) is topologically isomorphic to the multiplicative group of all \( p^n \)-th roots of unity considering in the discrete topology. Observe that the groups \( \mathbb{Z}(p^\infty) \) and \( \Delta_p \) are the character groups of one another.

3. Main results

Let \( X = \Omega_p \), and let \( \xi_1, \xi_2, \xi_3 \) be independent random variables with values in \( X \) and distributions \( \mu_1, \mu_2, \mu_3 \). Let \( \alpha_j, \beta_j, \gamma_j \in \text{Aut}(X) \). We consider linear forms \( L_1 = \alpha_1 \xi_1 + \alpha_2 \xi_2 + \alpha_3 \xi_3, \) \( L_2 = \beta_1 \xi_1 + \beta_2 \xi_2 + \beta_3 \xi_3 \), and \( L_3 = \gamma_1 \xi_1 + \gamma_2 \xi_2 + \gamma_3 \xi_3 \) and assume that \( L_1, L_2 \) and \( L_3 \) are independent.

We can consider new independent random variables \( \xi'_1 = \alpha'_1 \xi_1, \xi'_2 = \alpha'_2 \xi_2, \xi'_3 = \alpha'_3 \xi_3 \), and reduce the problem of description of possible distributions \( \mu_1, \mu_2, \mu_3 \) to the case when

\[
L_1 = \xi_1 + \xi_2 + \xi_3,
\]

\[
L_2 = \alpha'_1 \xi_1 + \alpha'_2 \xi_2 + \alpha'_3 \xi_3,
\]

\[
L_3 = \beta'_1 \xi_1 + \beta'_2 \xi_2 + \beta'_3 \xi_3.
\]

Since \( \alpha'_j, \beta'_j \in \text{Aut}(X) \), we have \( \alpha'_j = p^j c_j, \beta'_j = p^j d_j \), where all \( c_j, d_j \in \Delta_p^0, k_j, l_j \in \mathbb{Z} \).

By renumbering random variables, we can assume that there are two possible cases:

1. \( k_1 = \min\{k_1, k_2, k_3\}, l_1 = \min\{l_1, l_2, l_3\} \).
2. \( k_1 = \min\{k_1, k_2, k_3\}, \quad l_2 = \min\{l_1, l_2, l_3\} \).

Since \( L_1, L_2 \) and \( L_3 \) are independent if and only if either \( L_1, L_2' = \alpha_1^{-1} L_2 \) and \( L_3' = \beta_1^{-1} L_3 \), or \( L_1, L_2' = \alpha_1^{-1} L_2 \) and \( L_3' = \beta_2^{-1} L_3 \) are independent, the problem of description of possible distributions \( \mu_1, \mu_2, \mu_3 \) is reduced to the case when

\[
\begin{align*}
L_1 &= \xi_1 + \xi_2 + \xi_3, \\
L_2 &= \xi_1 + \delta_1 \xi_2 + \delta_2 \xi_3, \\
L_3 &= \xi_1 + \varepsilon_1 \xi_2 + \varepsilon_2 \xi_3,
\end{align*}
\]

or to the case when

\[
\begin{align*}
L_1 &= \xi_1 + \xi_2 + \xi_3, \\
L_2 &= \delta_1 \xi_1 + \xi_2 + \delta_2 \xi_3, \\
L_3 &= \xi_1 + \varepsilon_1 \xi_2 + \varepsilon_2 \xi_3.
\end{align*}
\]

Note that it follows from the choice of automorphisms that

\[
\delta_j = p^k c_j, \quad \varepsilon_j = p^l d_j, \quad c_j, d_j \in \Delta_p^0, k_j \geq 0, l_j \geq 0, j = 1, 2, \tag{2}
\]

in both cases. Let \( k = \min\{k_1, k_2, l_1, l_2\} \).

Denote

\[
\Lambda_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \delta_1 & \delta_2 \\ 1 & \varepsilon_1 & \varepsilon_2 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 1 & 1 & 1 \\ \delta_1 & 1 & \delta_2 \\ \varepsilon_1 & \varepsilon_2 \end{pmatrix}
\]

Let \( \Lambda = \Lambda_1 \) or \( \Lambda = \Lambda_2 \). We will keep the designation \( \Lambda_1 \) and \( \Lambda_2 \) throughout the article. Then \( \det \Lambda = p^q \lambda \), where \( q \geq 0, \lambda \in \Delta_p^0 \).

The main result of the article is to get necessary and sufficient conditions on coefficients of the linear forms \( L_j \) when the independence of \( L_j \) implies that all \( \mu_j \) are independent.

**Theorem 1.** Let \( X = \Omega_p \). Let \( \xi_1, \xi_2, \xi_3 \) be independent random variables with values in \( X \) and distributions \( \mu_1, \mu_2, \mu_3 \). Let \( \delta_j, \varepsilon_j \in \text{Aut}(X) \), where \( \delta_j = p^k c_j, \varepsilon_j = p^l d_j \), where \( c_j, d_j \in \Delta_p^0, k_j \geq 0, l_j \geq 0, j = 1, 2 \). Put \( k = \min\{k_1, k_2, l_1, l_2\} \).

Consider a random vector \((L_1, L_2, L_3) = \Lambda(\xi_1, \xi_2, \xi_3)\), where either \( \Lambda = \Lambda_1 \) or \( \Lambda = \Lambda_2 \). Let \( \det \Lambda = p^q \lambda \), where \( q \geq 0, \lambda \in \Delta_p^0 \). Suppose that the random vector \((L_1, L_2, L_3)\) has independent components. Then the following statements hold.

**A.** Assume that \( q = 0 \).

All \( \mu_j \in I(X) \) iff each row and each column of the matrix \( \Lambda \) contains at least two elements from \( \Delta_p^0 \).

**B.** Assume that \( q > 0 \).

**B.1.** Assume that at least one of the elements \( \delta_j, \varepsilon_j \in \Delta_p^0 \). Then all \( \mu_j \in D(X) \).

**B.2.** Assume that all \( \delta_j, \varepsilon_j \notin \Delta_p^0 \). All \( \mu_j \in I(X) \) iff either \( q > k \) (in this case all \( \mu_j \in D(X) \)) or \( q = k \) and either \( k_1 = l_2 = k \) or \( k_2 = l_1 = k \).

**Remark 1.** Case B.2 is possible only when \( \Lambda = \Lambda_1 \).

To prove Theorem 1 we need some lemmas.

**Lemma 1 ([15]).** Let \( X \) be a second countable locally compact Abelian group, \( Y \) be its character group. Let \( \alpha_j, \beta_j, \gamma_j \) be continuous endomorphisms of \( X \), \( \xi_1, \xi_2, \xi_3 \) be independent
random variables with values in $X$ and distributions $\mu_1, \mu_2, \mu_3$. Let $L_1 = \alpha_1 \xi_1 + \alpha_2 \xi_2 + \alpha_3 \xi_3$, $L_2 = \beta_1 \xi_1 + \beta_2 \xi_2 + \beta_3 \xi_3$ and $L_3 = \gamma_1 \xi_1 + \gamma_2 \xi_2 + \gamma_3 \xi_3$. The linear forms $L_1$, $L_2$ and $L_3$ are independent iff the characteristic functions $\hat{\mu}_j(y)$ satisfy the equation

$$\hat{\mu}_1(\alpha_1 u + \beta_1 v + \gamma_1 w)\hat{\mu}_2(\alpha_2 u + \beta_2 v + \gamma_2 w)\hat{\mu}_3(\alpha_3 u + \beta_3 v + \gamma_3 w) =$$

$$= \hat{\mu}_1(\alpha_1 u)\hat{\mu}_2(\alpha_2 u)\hat{\mu}_3(\alpha_3 u)\hat{\mu}_1(\beta_1 v)\hat{\mu}_2(\beta_2 v)\hat{\mu}_3(\beta_3 v)\hat{\mu}_1(\gamma_1 w)\hat{\mu}_2(\gamma_2 w)\hat{\mu}_3(\gamma_3 w), \ u,v,w \in Y. \quad (3)$$

The proof of the next lemma is analogous to the proof of Lemma 2 of the paper [8].

**Lemma 2.** Let $X = \Omega_p$, $Y$ be its character group. Let $\xi_1, \xi_2, \xi_3$ be independent random variables with values in $X$ and distributions $\mu_1, \mu_2, \mu_3$ such that $\hat{\mu}_j(y) \geq 0$, $j = 1, 2, 3$. Let $\alpha_j, \beta_j \in \text{Aut}(X)$ and $\alpha_j = p_k c_j, \beta_j = p_l d_j \in \text{Aut}(X)$, where all $k_j \geq 0$, $l_j \geq 0$ and $c_j, d_j \in \Delta_0$. Assume that the linear forms $L_1 = \xi_1 + \xi_2 + \xi_3$, $L_2 = \alpha_1 \xi_1 + \alpha_2 \xi_2 + \alpha_3 \xi_3$ and $L_3 = \beta_1 \xi_1 + \beta_2 \xi_2 + \beta_3 \xi_3$ are independent. Then there exists a subgroup $B = p^n \Delta_p$ of the group $Y$ such that all $\sigma(\mu_j) \subset B$.

**Proof.** We use the fact that the family of the subgroups $\{p^i \Delta_p\}_{i=-\infty}^{\infty}$ forms an open basis at zero of the group $Y$. Since $\hat{\mu}_1(0) = \hat{\mu}_2(0) = \hat{\mu}_3(0) = 1$, we can choose $m \geq 0$ in such a way that $\hat{\mu}_j(y) > 0$ for $y \in L = p^m \Delta_p$, $j = 1, 2, 3$. Note that $\alpha_j(L) \subset L$, $j = 1, 2, 3$. Put $\psi_j(y) = -\log \hat{\mu}_j(y)$, $y \in L$, $j = 1, 2, 3$.

By Lemma 1 the characteristic functions $\hat{\mu}_j(y)$ satisfy equation (3). Taking into account that $\tilde{\alpha}_j = \alpha_j, \tilde{\beta}_j = \beta_j$, we get from (3) the following equation

$$\psi_1(u + v + w) + \psi_2(\alpha_1 u + \alpha_2 v + \alpha_3 w) + \psi_3(\beta_1 u + \beta_2 v + \beta_3 w) =$$

$$= \psi_1(u) + \psi_2(\alpha_1 u) + \psi_3(\beta_1 u) + \psi_1(v) + \psi_2(\alpha_2 v) + \psi_3(\beta_2 v) + \psi_1(w) + \psi_2(\alpha_3 w) + \psi_3(\beta_3 w), \ u,v,w \in L. \quad (4)$$

Put $u = 0$ in (4) and integrate the obtained equation over the group $L$ with respect to the Haar distribution $dm_L(u)$. Using the fact that the Haar distribution $m_L$ is $L$-invariant, we obtain

$$\psi_1(v) + \psi_2(\alpha_2 v) + \psi_3(\beta_2 v) = 0, \ v \in L. \quad (5)$$

Since $\psi_j(y) \geq 0$ on $Y$, it follows from (5) that $\psi_1(v) = \psi_2(\alpha_2 v) = \psi_3(\beta_2 v) = 0$ for $v \in L$. Thus $\hat{\mu}_1(y) = \hat{\mu}_2(\alpha_2 y) = \hat{\mu}_3(\beta_2 y) = 1$, $y \in L$. Put $B = L \cap \alpha_2(L) \cap \beta_2(L)$, i.e. $B = p^n \Delta_p$, where $n = \max\{k_2, l_2\}$. Then $B$ is a required subgroup. Lemma 2 is proved.

**Lemma 3 ([16]).** Let $X = \Delta_p$, let $\xi_i, i = 1, ..., n$, be independent random variables with values in $X$ and with distributions $\mu_i$. Let $\alpha_{ij} \in \text{Aut}(X)$. If the linear form $L_j = \sum^n_{i=1} \alpha_{ij} \xi_i$, $j = 1, ..., n$, are independent then all $\mu_i \in I(X)$.

The following lemma is a partial case of the general theorem of the paper [17].

**Lemma 4 ([17]).** Let $X$ be a finite Abelian group. Let $\alpha_j, \beta_j, \gamma_j$ be endomorphisms of the group $X$. Put

$$\Lambda = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}. \quad (6)$$


Assume that each column of the matrix \( \Lambda \) contain at least two automorphisms of the group \( X \), and the matrix \( \Lambda \in \text{Aut}(X^3) \).

Let \( \xi_1, \xi_2, \xi_3 \) be independent random variables with values in \( X \) and distributions \( \mu_1, \mu_2, \mu_3 \) such that each support \( \sigma(\mu_j) \) is not contained in a coset of a proper subgroup of the group \( X \). If the linear forms \( L_1 = \alpha_1 \xi_1 + \alpha_2 \xi_2 + \alpha_3 \xi_3 \), \( L_2 = \beta_1 \xi_1 + \beta_2 \xi_2 + \beta_3 \xi_3 \) and \( L_3 = \gamma_1 \xi_1 + \gamma_2 \xi_2 + \gamma_3 \xi_3 \) are independent, then all distributions \( \mu_1 = \mu_2 = \mu_3 = m_X \).

It is convenient for us to formulate as a lemma the following well-known statement (see e.g. \cite[Proposition 2.13]{7}).

**Lemma 5.** Let \( X \) be a locally compact Abelian group, \( Y \) be its character group. Let \( \mu \in M^1(X) \). Then the set \( E = \{ y \in Y : \hat{\mu}(y) = 1 \} \) is a closed subgroup of \( Y \), the characteristic function \( \hat{\mu}(y) \) is \( E \)-invariant, i.e., \( \hat{\mu}(y) \) takes a constant value on each coset of the group \( Y \) with respect to \( E \), and \( \sigma(\mu) \subset A(X, E) \).

**Lemma 6.** Let \( X = \mathbb{Z}(p^n) \), \( Y \) be its character group. Let \( \alpha_j, \beta_j, \gamma_j \) be endomorphisms of the group \( X \). Put

\[
\Lambda = \begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 \\
\beta_1 & \beta_2 & \beta_3 \\
\gamma_1 & \gamma_2 & \gamma_3
\end{pmatrix}
\]  

(7)

Assume that each row and column of the matrix \( \Lambda \) contains at least two automorphisms of the group \( X \), and the matrix \( \Lambda \in \text{Aut}(X^3) \).

Let \( \xi_1, \xi_2, \xi_3 \) be independent random variables with values in \( X \) and distributions \( \mu_1, \mu_2, \mu_3 \) such that \( \hat{\mu}_j(y) \geq 0 \), \( j = 1, 2, 3 \), and at least one support \( \sigma(\mu_j) \) is not contained in a coset of a proper subgroup of \( X \). If the linear forms \( L_1 = \alpha_1 \xi_1 + \alpha_2 \xi_2 + \alpha_3 \xi_3 \), \( L_2 = \beta_1 \xi_1 + \beta_2 \xi_2 + \beta_3 \xi_3 \) and \( L_3 = \gamma_1 \xi_1 + \gamma_2 \xi_2 + \gamma_3 \xi_3 \) are independent, then \( \mu_1 = \mu_2 = \mu_3 = m_X \).

**Proof.** Since \( X = \mathbb{Z}(p^n) \), we have \( Y \cong \mathbb{Z}(p^n) \). To avoid introducing new notation we will suppose that \( Y = \mathbb{Z}(p^n) \). Lemma 1 implies that the characteristic functions \( \hat{\mu}_j(y) \) satisfy equation \( (3) \).

Suppose that there exists such element \( y_0 \in Y \) that one of the functions \( \hat{\mu}_j(y) \) is equal to \( 1 \) at this element. We may assume without loss of generality that \( \hat{\mu}_1(y_0) = 1 \). Since the set of elements where a characteristic function is equal to one is a subgroup, and any nonzero subgroup of \( Y \) contains the subgroup \( \mathbb{Z}(p) \), we obtain that \( \hat{\mu}_1(y) = 1 \) for \( y \in \mathbb{Z}(p) \). Since \( \mathbb{Z}(p) \) is a characteristic subgroup, we can consider the restriction of equation \( (3) \) to \( \mathbb{Z}(p) \). Thus, we have

\[
\hat{\mu}_2(\alpha_2 u + \beta_2 v + \gamma_2 w)\hat{\mu}_3(\alpha_3 u + \beta_3 v + \gamma_3 w) = \\
= \hat{\mu}_2(\alpha_2 u)\hat{\mu}_3(\alpha_3 u)\hat{\mu}_2(\beta_2 v)\hat{\mu}_3(\beta_3 v)\hat{\mu}_2(\gamma_2 w)\hat{\mu}_3(\gamma_3 w), \quad u, v, w \in \mathbb{Z}(p).
\]  

(8)

Suppose that all coefficients \( \alpha_j, \beta_j, \gamma_j \) \( (j = 2, 3) \) are automorphisms of the group \( Y \). It is obvious that there exists a non-zero element \( (u_0, v_0, w_0) \) such that

\[
\begin{aligned}
\alpha_3 u_0 + \beta_2 v_0 + \gamma_2 w_0 &= 0 \\
\alpha_3 u_0 + \beta_3 v_0 + \gamma_3 w_0 &= 0
\end{aligned}
\]  

(9)

Put \( u = u_0 \), \( v = v_0 \), \( w = w_0 \) in \( (8) \). We obtain

\[
1 = \hat{\mu}_2(\alpha_2 u_0)\hat{\mu}_3(\alpha_3 u_0)\hat{\mu}_2(\beta_2 v_0)\hat{\mu}_3(\beta_3 v_0)\hat{\mu}_2(\gamma_2 w_0)\hat{\mu}_3(\gamma_3 w_0).
\]  

(10)
Since the element \((u_0, v_0, w_0)\) is non-zero, and the set of elements where a characteristic function is equal to 1 is a subgroup, it follows from (10) that \(\hat{\mu}_2(y) = \hat{\mu}_3(y) = 1\) for \(y \in \mathbb{Z}(p)\). Thus Lemma 5 implies that all supports \(\sigma(\mu_j) \subset A(X, \mathbb{Z}(p))\). We obtain the contradiction to the conditions of the lemma.

Suppose that the only one of the coefficients \(\alpha_j, \beta_j, \gamma_j\) \((j = 2, 3)\) is not an automorphism, and all the rest are automorphisms of the group \(Y\). Without loss of generality we can assume that \(\gamma_3 \notin \text{Aut}(X)\). Then equation (8) takes the form

\[
\hat{\mu}_2(\alpha_2 u + \beta_2 v + \gamma_2 w) \hat{\mu}_3(\alpha_3 u + \beta_3 v) = \hat{\mu}_2(\alpha_2 u) \hat{\mu}_3(\alpha_3 u) \hat{\mu}_2(\beta_2 v) \hat{\mu}_3(\beta_3 v) \hat{\mu}_2(\gamma_2 w), \quad u, v, w \in \mathbb{Z}(p).
\]

Put \(u = -\alpha_3^{-1} \beta_3 y, v = y, w = \gamma_2^{-1}(\alpha_2 \alpha_3^{-1} \beta_3 - \beta_2)y\) in (11). We obtain

\[
1 = \hat{\mu}_2(-\alpha_3^{-1} \beta_3 \alpha_2 y) \hat{\mu}_3(-\beta_3 y) \hat{\mu}_2(\beta_2 y) \hat{\mu}_3(\beta_3 y) \hat{\mu}_2((\alpha_2 \alpha_3^{-1} \beta_3 - \beta_2)y), \quad y \in \mathbb{Z}(p).
\]  

It follows from this that \(\hat{\mu}_2(y) = \hat{\mu}_3(y) = 1\) for \(y \in \mathbb{Z}(p)\). Thus Lemma 5 implies that all supports \(\sigma(\mu_j) \subset A(X, \mathbb{Z}(p))\). We obtain the contradiction to the conditions of the lemma.

Suppose that two of the coefficients \(\alpha_j, \beta_j, \gamma_j\) \((j = 2, 3)\) are not automorphisms, and all the rest coefficients are automorphisms of the group \(Y\). By the conditions on \(\Lambda\) it is possible when automorphisms are coefficients for different variables. Without loss of generality we can assume that \(\beta_2, \gamma_3 \notin \text{Aut}(X)\). Then equation (8) takes the form

\[
\hat{\mu}_2(\alpha_2 u + \gamma_2 w) \hat{\mu}_3(\alpha_3 u + \beta_3 v) = \hat{\mu}_2(\alpha_2 u) \hat{\mu}_3(\alpha_3 u) \hat{\mu}_2(\beta_2 y) \hat{\mu}_3(\beta_3 y) \hat{\mu}_2(\gamma_2 w), \quad u, v, w \in \mathbb{Z}(p).
\]

Put \(u = y, v = -\beta_3^{-1} \alpha_3 y, w = -\gamma_2^{-1} \alpha_2 y\) in (13). We obtain

\[
1 = \hat{\mu}_2(\alpha_2 y) \hat{\mu}_3(\alpha_3 y) \hat{\mu}_3(-\alpha_3 y) \hat{\mu}_2(-\alpha_2 y), \quad y \in \mathbb{Z}(p).
\]

It follows from this that \(\hat{\mu}_2(y) = \hat{\mu}_3(y) = 1\) for \(y \in \mathbb{Z}(p)\). Thus, Lemma 5 implies that all supports \(\sigma(\mu_j) \subset A(X, \mathbb{Z}(p))\). We obtain the contradiction to the conditions of the lemma.

We have considered all possibilities.

Thus we obtain that all \(\hat{\mu}_j(y) < 1\) for \(y \in Y \setminus \{0\}\). Now we are in the conditions of Lemma 4. The statement of the lemma follows from Lemma 4.

\[\square\]

**Lemma 7.** Let \(X\) be a locally compact Abelian group, \(Y\) be its character group. Let \(H\) be an open subgroup of \(Y\). Let \(f_0(y)\) be a continuous positive definite function on \(H\), and \(f(y)\) be the function on \(Y\) such that

\[
f(y) = \begin{cases} f_0(y), & y \in H; \\ 0, & y \notin H. \end{cases}
\]

Then \(f(y)\) is a continuous positive definite function, and there exists a distribution \(\mu \in M^1(X)\) such that \(\hat{\mu}(y) = f(y)\).  

**Proof.** The function \(f(y)\) is a positive definite function on \(Y\) ([13, §32]). Since \(H\) is open, the function \(f(y)\) is continuous. By the Bochner theorem there exists a distribution \(\mu\) such that \(\hat{\mu}(y) = f(y)\).

\[\square\]

The following lemma is obvious and follows immediately from the form of endomorphisms of \(\Delta_p\). We will constantly use this lemma, but not refer to it.
Lemma 8. Let \( X = \Delta_p \). Let \( \alpha \in Aut(X) \), let \( \beta \) be an endomorphism of \( X \) such that \( \beta \not\in Aut(X) \). Then \( \alpha - \beta \in Aut(X) \).

Lemma 9. Let \( X = \Delta_p \). Let \( \xi_1, \xi_2, \xi_3 \) be independent random variables with values in \( X \) and with distributions \( \mu_1, \mu_2, \mu_3 \) such that \( \hat{\mu}_j(y) \geq 0 \), \( j = 1, 2, 3 \). Suppose that at least one of the supports \( \sigma(\mu_j) \) is not contained in a proper subgroup of the group \( X \). Let \( c_j, d_j \in Aut(X) \). Consider linear forms \( L_1 = \xi_1 + \xi_2 + \xi_3, L_2 = \xi_1 + p^{k_1}c_1\xi_2 + p^{k_2}c_2\xi_3, \) and \( L_3 = \xi_1 + p^{k_1}d_1\xi_2 + p^{k_2}d_2\xi_3, \) where all \( k_j > 0, l_j > 0 \). Put \( k = \min\{k_1, k_2, l_1, l_2\} \). Suppose that \( L_1, L_2 \) and \( L_3 \) are independent. All \( \mu_j \in I(X) \) iff either \( k_1 = l_2 = k \) or \( k_2 = l_1 = k \).

Proof.

Let \( l, m, n \in \mathbb{N} \) such that \( 0 < k < l < m < n \). Up to notation of the random variables \( \xi_2 \) and \( \xi_3 \) and up to permutation of the linear forms \( L_2 \) and \( L_3 \) there are possible the following cases for \( k_1, l_2 \).

Since \( X = \Delta_p \), we have \( Y \cong \mathbb{Z}(p^\infty) \). To avoid introducing new notation we will suppose that \( Y = \mathbb{Z}(p^\infty) \).

Put \( f(y) = \hat{\mu}_1(y), g(y) = \hat{\mu}_2(y), h(y) = \hat{\mu}_3(y) \). By Lemma 1 the functions \( f(y), g(y), h(y) \) satisfy (3) which takes the form

\[
\begin{align*}
    f(u + v + w)g(u + p^{k_1}c_1v + p^{k_2}c_2v + p^{l_1}d_1w)h(u + p^{k_1}c_1v + p^{l_2}d_2w) &= \\
    = f(u)f(v)f(w)g(u)g(p^{k_1}c_1v)g(p^{l_1}d_1w)h(u)h(p^{k_1}c_1v)h(p^{l_2}d_2w), u, v, w \in Y, \\
    \text{(16)}
\end{align*}
\]

where \( f(y) = \hat{\mu}_1(y), g(y) = \hat{\mu}_2(y), h(y) = \hat{\mu}_3(y) \), holds.

We note that if at least one of the supports \( \sigma(\mu_j) \) is not contained in a proper subgroup of \( X \), then by Lemma 5 \( \{ y \in Y : f(y) = g(y) = h(y) = 1 \} = \{ 0 \} \).

Put

\[
\Lambda = \begin{pmatrix}
    1 & 1 & 1 \\
    1 & p^{k_1}c_1 & p^{k_2}c_2 \\
    1 & p^{l_1}d_1 & p^{l_2}d_2
\end{pmatrix}.
\]

We have

\[
\det \Lambda = p^{k_1 + l_2}c_1d_2 - p^{k_2 + l_1}c_2d_1 + p^{k_2}c_2 - p^{k_1}c_1 + p^{l_2}d_1 - p^{l_1}d_2.
\]

Note that \( \det \Lambda \) can be represented in the form

\[
\det \Lambda = p^{q} \lambda,
\]

where \( \lambda \in Aut(Y), q > 0 \). In this case \( q \geq k \).

Putting \( u = (p^{k_1 + l_2}c_1d_2 - p^{k_2 + l_1}c_2d_1)y, v = (p^{l_2}d_2 - p^{l_1}d_1)y, w = (p^{k_2}c_2 - p^{k_1}c_1)y, \) \( y \in Y \), in (16), we obtain

\[
\begin{align*}
    f(p^{q} \lambda y) &= f((p^{k_1 + l_2}c_1d_2 - p^{k_2 + l_1}c_2d_1)y)f((p^{l_2}d_2 - p^{l_1}d_1)y)f((p^{k_2}c_2 - p^{k_1}c_1)y), \\
    g((p^{k_1 + l_2}c_1d_2 - p^{k_2 + l_1}c_2d_1)y)g(p^{k_1}c_1(p^{l_2}d_2 - p^{l_1}d_1)y)g(p^{l_1}d_1(p^{k_2}c_2 - p^{k_1}c_1)y)
\end{align*}
\]
\[ h((p^{k_1+l_2}c_1d_2 - p^{k_2+l_1}c_2d_1)y)h(p^{k_2}c_2(p^{l_2}d_2 - p^{l_1}d_1)y)h(p^{l_2}d_2(p^{k_2}c_2 - p^{k_1}c_1)y) \] (19)

Putting \( u = (p^{k_2}c_2 - p^{l_2}d_2)y, \ v = (p^{l_2}d_2 - I)y, \ w = (I - p^{k_2}c_2)y, \ y \in Y, \) in (16), we obtain

\[ g(p^l \lambda y) = f((p^{l_2}d_2 - p^{k_2}c_2)y)f((p^{k_2}c_2 - I)y) \] (20)

We check that in fact \( q = k, \) i.e. the case of \( q > k \) is impossible.

Actually, suppose that \( q > k. \)

Put \( y \in Y(p^k) \) in (20). Then \( p^l \lambda y = 0, \) and the left side of (20) is equal to 1. Hence \( g(p^k c_1 (p^{l_2}d_2 - I)y) = 1. \) Since \( p^{l_2}d_2 - I \in Aut(Y) \), it follows from this that \( g(y) = 1 \) for \( y \in Y(p^{k_2}) \). Let \( y \in Y(p^{k_2}d_2 - k) \) in (20). Reasoning similarly we obtain that \( g(y) = 1 \) for \( y \in Y(p^{k_2}d_2 - k) \). Analogously we obtain that \( g(y) = 1 \) for \( y \in Y(p^{k_2}d_2 - k) \). Since \( Y = \bigcup_{N} Y(N) \) and \( Y(N) \subset Y(N+1), \) we obtain that \( g(y) = 1 \) for \( y \in Y. \) Then (21) implies that \( f(y) = h(y) = 1 \) for \( y \in Y. \) Thus, \( \mu_1 = \mu_2 = \mu_3 = E_0. \) We obtain the contradiction to the conditions of the lemma.

Thus, \( q = k. \)

Note that it follows from equation (20) that

\[ f(y) = 1, \ y \in Y(p^k) . \] (22)

Really, if \( y \in Y(p^k) \), then \( g(p^k \lambda y) = 1. \) Since \( p^{l_2}d_2 - I \in Aut(Y) \) and a subgroup \( Y(p^k) \) is characteristic, it follows from (20) that equality (22) holds true.

It follows from Lemma 1 that Lemma 7 in cases 1,2,5,8 will be proved if we show that all solutions of equation (16) have form (11) (the functions are not necessarily equal). In cases 3-4, 6-7, 9-21 of Lemma 7 we construct positive definite solutions \( f(y), g(y), h(y) \) of equation (16) such that at least one of the functions \( f(y), g(y), h(y) \) cannot be represented in the form (11). By the Bochner theorem there exist distributions \( \mu_i \) such that \( f(y) = \mu_1(y), \ g(y) = \mu_2(y), \ h(y) = \mu_3(y). \) Then Lemma 1 implies Lemma 7 in cases 3-4, 6-7, 9-21.

We consider all possible cases.

**Case 1.** \( k_1 = k_2 = l_1 = l_2 = k. \) Since \( q = k, \) equation (20) takes the form

\[ g(p^k \lambda y) = f((p^k d_2 - c_2)y)f((p^k c_2 - I)y) \]
\[ g(p^k (d_2 - c_2)y)g(p^k c_1 (p^k d_2 - I)y)g(p^k d_1 (p^k c_2 - I)y) \]
\[ h((p^k (d_2 - c_2)y)h((p^k c_2 (p^k d_2 - I)y)h(p^k d_2 (p^k c_2 - I)y)). \] (23)

We get from (23) that

\[ g(p^k \lambda y) \leq g(p^k c_1 (p^k d_2 - I)y), \ y \in Y. \] (24)

It follows from this that

\[ g(y) \leq g(\lambda^{-1} c_1 (p^k d_2 - I)y), \ y \in Y. \] (25)
Since each $y \in Y_{(p^n)}$ for some $n$, $\lambda^{-1}c_1(p^kd_2 - I) \in Aut(Y)$, and $Y_{(p^n)}$ is a characteristic subgroup, there exist such number $n'$ for every $y$ that $(\lambda^{-1}c_1(p^kd_2 - I))^{n'}y = y$. Then it follows from (24) that
\[
g(y) \leq g(\lambda^{-1}c_1(p^kd_2 - I)y) \leq \ldots \leq g((\lambda^{-1}c_1(p^kd_2 - I))^{n'}y) = g(y), \ y \in Y.
\]
So,
\[
g(y) = g(\lambda^{-1}c_1(p^kd_2 - I)y) = \ldots = g((\lambda^{-1}c_1(p^kd_2 - I))^{n'}y) = g(y), \ y \in Y.
\]
Thus, inequality (24) becomes an equality
\[
g(p^k\lambda y) = g(p^k c_1 (p^kd_2 - I)y), \ y \in Y.
\]
Suppose that $g(p^k\lambda y_0) \neq 0$ for some $y_0 \in Y$ such that $p^k\lambda y_0 \neq 0$. Then it follows from (23) that
\[
1 = f(p^k(d_2 - c_2)y_0)f((p^kd_2 - I)y_0)f((p^kc_2 - I)y_0)
g(p^k(d_2 - c_2)y_0)g(p^kd_1(p^kc_2 - I)y_0)
h(p^k(d_2 - c_2)y_0)h(p^kd_2(p^kc_2 - I)y_0)h(p^kd_2(p^kc_2 - I)y_0).
\]
Since $p^k\lambda y_0 \neq 0$, we have $p^kd_1(p^kc_2 - I)y_0 \neq 0$ and $p^kd_2(p^kc_2 - I)y_0 \neq 0$. So it follows from (24) that $g(p^kd_1(p^kc_2 - I)y_0) = 1$ and $h(p^kd_2(p^kc_2 - I)y_0) = 1$. By Lemma 5 the set of elements where a characteristic function is equal to 1 is a subgroup, and each subgroup of $Y$ contains $Y_{(p)}$, we get that $g(y) = h(y) = 1$ for $y \in Y_{(p)}$. Taking into account (22), we get contradiction to the conditions of the lemma. So, $g(y) = 0$ for $y \in Y \setminus \{0\}$. Then it follows from (21) that $h(y) = 0$ for $y \in Y \setminus \{0\}$. We get from (19) that $f(y) = 0$ for $y \in Y \setminus Y_{(p^k)}$.

Thus, we obtain that
\[
f(y) = \begin{cases} 1, & y \in Y_{(p^k)}; \\ 0, & y \notin Y_{(p^k)}, \end{cases} \quad g(y) = h(y) = \begin{cases} 1, & y = 0; \\ 0, & y \neq 0. \end{cases}
\]

It follows from (11) and (28) that $\mu_1 = m_{p^kc_2}$, $\mu_2 = \mu_3 = m_{c_2}$.

Cases 2, 5, 8. Reasoning as in case 1 we study the cases 2, 5, 8.

Case 3. $k_1 = l_1 = k$, $k_2 = l_2 = l$. Put
\[
f(y) = \begin{cases} 1, & y \in Y_{(p^{k+1})}; \\ 0, & y \notin Y_{(p^{k+1})}, \end{cases}
\]
\[
g(y) = \begin{cases} 1, & y \in Y_{(p)}; \\ 0, & y \notin Y_{(p)}, \end{cases}
\]
\[
h(y) = \begin{cases} h_0(y), & y \in Y_{(p)}; \\ 0, & y \notin Y_{(p)}, \end{cases}
\]

where $h_0(y)$ is an arbitrary characteristic function on $Y_{(p)}$. By Lemma 7 the function $h(y)$ is a positive definite function on $Y$.

We check that the functions (29), (30) and (31) satisfy equation (16).

Let $u \in Y_{(p)}$ in (16). Taking into account Lemma 5, (29) and (30), we get
\[
f(v + w)g(p^kc_1v + p^kd_1w)h(u + p^l c_2v + p^l d_2w) = f(v)f(w)g(p^kc_1v)g(p^kd_1w)h(u)h(p^l c_2v)h(p^l d_2w), \quad v, w \in Y.
\]
Let $v, w \in Y(\nu^{k+1})$. It follows from Lemma 5 and (29) that $f(v + w) = f(v) = f(w) = 1$. Since $p^k c_1 v, p^k d_1 w \in Y(\nu)$, it follows from Lemma 5 and (30) that $g(p^k c_1 v + p^k d_1 w) = g(p^k c_1 v) = g(p^k d_1 w) = 1$. Since $p' c_2 v = p' d_2 w = 0$, we get $h(u + p' c_2 v + p' d_2 w) = h(u), h(p' c_2 v) = h(p' d_2 w) = 1$. Thus, equation (32) holds true.

Let $v \in Y(\nu^{k+1}), w \notin Y(\nu^{k+1})$. Then $p^k c_1 v \in Y(\nu)$ and $p^k d_1 w \notin Y(\nu)$. It follows from Lemma 5 and (30) that $g(p^k c_1 v + p^k d_1 w) = g(p^k d_1 w) = 0$, and equation (32) holds true.

Let $v \notin Y(\nu^{k+1})$. Then $p^k c_1 v \notin Y(\nu)$ and the right side of (32) is equal to 0.

If $w \in Y(\nu^{k+1})$ then $g(p^k c_1 v + p^k d_1 w) = g(p^k c_1 v) = 0$. So, the left side of (32) is also equal to 0. Equation (32) holds true.

Let $w \notin Y(\nu^{k+1})$. Suppose that the left side of (32) does not equal to 0. It follows from (29) and (30) that

$$v + w \in Y(\nu^{k+1}),$$

$$p^k c_1 v + p^k d_1 w \in Y(\nu).$$

(33) It follows from (33) that $p^k(v + w) \in Y(\nu)$. Taking into account (34) this implies that $p^k(d_1 - c_1)w \in Y(\nu)$. Since $q = k$, it is easy to see from (17) and (18) that $d_1 - c_1 \in Aut(Y)$. Therefore we obtain that $w \in Y(\nu^{k+1})$. So we obtain the contradiction, and the left side of equation (32) is also equal to 0. Thus, we proved that if $u \in Y(\nu)$ then the functions (29), (30) and (31) satisfy equation (16).

Let $u \notin Y(\nu)$ in (16). It follows from (29) that $g(u) = 0$. The right side of (16) is equal to zero. Suppose that the left side of (16) does not equal to zero. It follows from (29), (31) and (31) that

$$u + v + w \in Y(\nu^{k+1}),$$

$$u + p^k c_1 v + p^k d_1 w \in Y(\nu),$$

$$u + p' c_2 v + p' d_2 w \in Y(\nu).$$

(36) It follows from (35) that $p^k u + p^k v + p^k w \in Y(\nu)$. Taking into account (36) this implies that

$$(p^k c_1 - I)u + p^k(c_1 - d_1)w \in Y(\nu).$$

(38) It follows from (35) and the condition $l > k$ that $p'u + p'v + p'w = 0$. Taking into account (37) this implies that

$$(p' c_2 - I)u + p'(c_2 - d_2)w \in Y(\nu).$$

(39) Note that $p^k c_1 - I, p' c_2 - I \in Aut(Y)$. It follows from (17), (18), (38) and (39) that $p^k \lambda w \in Y(\nu)$. Since $\lambda \in Aut(Y)$, we get that $p^k w \in Y(\nu)$. Since $c_1 - d_1 \in Aut(Y)$, we get that $p^k(c_1 - d_1)w \in Y(\nu)$, and (38) implies that $(p^k c_1 - I)u \in Y(\nu)$. Since $p^k c_1 - I \in Aut(Y)$, we get that $u \in Y(\nu)$. So, the obtained contradiction shows that the left side of equation (16) is also equal to 0.

Case 4. $k_1 = k_2 = k, l_1 = l_2 = l$. Put

$$f(y) = \begin{cases} f_0(y), & y \in Y(\nu^k); \\ 0, & y \notin Y(\nu^k), \end{cases}$$

(40) where $f_0(y)$ is an arbitrary characteristic function on $Y(\nu^k)$ such that $f_0(y) = 1$ for $y \in Y(\nu^k)$. By Lemma 7 the function $f(y)$ is a positive definite function on $Y$.

Also put

$$g(y) = h(y) = \begin{cases} 1, & y = 0; \\ 0, & y \neq 0. \end{cases}$$

(41)
We check that the functions (40) and (41) satisfy equation (16).
Let \( u = 0 \) in (16). We get
\[
 f(v + w)g(p^k c_1 v + p^l d_1 w)h(p^k c_2 v + p^l d_2 w) = \\
 = f(v)f(w)g(p^k c_1 v)g(p^l d_1 w)h(p^k c_2 v)h(p^l d_2 w), \quad v, w \in Y. 
\]
Equation (42) holds true.

If \( v \in Y(p^k) \) then \( p^k c_1 v = 0 \) and by Lemma 5 \( f(v) = 1 \), \( f(v + w) = f(w) \). Equation (42) holds true.

If \( v \notin Y(p^k) \) then \( p^k c_1 v \neq 0 \) and \( g(p^k c_1 v) = 0 \). Thus, the right side of (42) is equal to 0.

If \( w \in Y(p^l) \) then \( p^l d_1 w = 0 \) and \( g(p^k c_1 v + p^l d_1 w) = g(p^k c_1 v) = 0 \). So, the left side is also equal to 0, and equation (42) holds true.

Let \( v \notin Y(p^l) \). Suppose that the left side of (42) does not equal to 0. It follows from (40) and (41) that
\[
 v + w \in Y(p^l), \\
p^k c_1 v + p^l d_1 w = 0.
\]
It follows from (43) that \( p^l(v + w) = 0 \). Taking into account (44) this implies that \( p^k(p^l^{-k}d_1 - c_1)v = 0 \). Since \( k < l \), \( p^l^{-k}d_1 - c_1 \in Aut(Y) \). Thus, we get that \( v \in Y(p^k) \). So, the obtained contradiction shows that the left side of equation (42) is also equal to 0. Thus, we proved that if \( u = 0 \) then the functions (40) and (41) satisfy equation (16).

Let \( u \neq 0 \) in (16). Then \( g(u) = 0 \) and the right side of (16) is equal to 0. Suppose that the left side of (16) does not equal to 0. Then
\[
 u + v + w \in \mathbb{Z}(p^l), \\
u + p^k c_1 v + p^l d_1 w = 0, \\
u + p^k c_2 v + p^l d_2 w = 0.
\]
It follows from (45) that \( p^l(u + v + w) = 0 \). Taking into account (46) and (47) this implies that
\[
 (p^l d_1 - I)u + (p^l d_1 - p^k c_1)v = 0, \\
 (p^l d_2 - I)u + (p^l d_2 - p^k c_1)v = 0.
\]
It follows from (48) that \( p^k \lambda v = 0 \). Hence \( v \in Y(p^k) \). Then we obtain from (45) and (46) that
\[
 p^l(u + w) = 0, \\
u + p^l d_1 w = 0.
\]
It follows from (49) that \( (p^l d_1 - I)u = 0 \). Since \( p^l d_1 - I \in Aut(Y) \), we get that \( u = 0 \). So, the obtained contradiction shows that the left side of equation (16) is also equal to 0. So, the obtained contradiction shows that the left side of equation (16) is also equal to 0.

**Case 6.** The study of this case is the same as the study of case 3.

**Case 7.** The study of this case is the same as the study of case 4.

**Case 9.** \( k_1 = k \), \( k_2 = l_1 = l_2 = l \). We check that the functions (40) and (41) satisfy equation (16).

Let \( u = 0 \) in (16). Then equation (16) takes the form
\[
 f(v + w)g(p^k c_1 v + p^l d_1 w)h(p^k c_2 v + p^l d_2 w) = \\

\]
\[ f(v) = f(w)g(p^kc_1v)g(p'd_1w)h(p'c_2v)h(p'd_2w), \quad v, w \in Y. \] \hfill (50)

Suppose that either \( v \not\in Y(p^k) \), or \( w \not\in Y(p') \). Then \( f(w)g(p^kc_1v) = 0 \), and the right side of \( (50) \) is equal to 0. Suppose that the left side of \( (50) \) does not equal to 0. Then

\[ v + w \in Y(p'), \hfill (51) \]
\[ p^k c_1v + p'd_1w = 0. \hfill (52) \]

It follows from \( (51) \) that \( p'(v + w) = 0 \). Taking into account \( (52) \) this implies that

\[ p^k(p^{l-k}d_1 - c_1)v = 0, \]
\[ p'(c_1 - p^{l-k}d_1)w = 0. \hfill (53) \]

Since \( p^{l-k}d_1 - c_1, c_1 - p^{l-k}d_1 \in Aut(Y) \), we have \( v \in Y(p^k), w \in Y(p') \). So, the obtained contradiction implies that the left side of equation \( (50) \) is also equal to 0.

Let \( u \neq 0 \) in \( (16) \). Then \( h(u) = 0 \), and the right side of \( (16) \) is equal to 0.

Suppose that the left side of \( (16) \) does not equal to zero. Then

\[ u + v + w \in Y(p'), \hfill (54) \]
\[ u + p^k c_1v + p'd_1w = 0, \hfill (55) \]
\[ u + p'l c_2v + p'd_2w = 0. \hfill (56) \]

It follows from \( (54) \) that \( u + v + w \in Y(p') \). Taking \( (55) \) and \( (56) \) into account this implies that

\[ (p'd_1 - I)u + (p'd_1 - p^kc_1)v = 0, \]
\[ (p'd_2 - I)u + (p'd_2 - p'l c_2)v = 0. \hfill (57) \]

It follows from this that \( p^k \lambda v = 0 \). Hence \( v \in Y(p^k) \). Since \( k < l \) and \( p'd_1 - I \in Aut(Y) \), it follows from \( (57) \) that \( u = 0 \). So, the obtained contradiction implies that the left side of equation \( (16) \) is also equal to 0.

**Case 10.** \( k_1 = k, k_2 = l_1 = l, l_2 = m \). We check that the functions \( (40) \) and \( (41) \) satisfy equation \( (16) \).

Let \( u = 0 \) in \( (16) \). Then equation \( (16) \) takes the form

\[ f(v + w)g(p^kc_1v + p'd_1w)h(p'c_2v + p'^md_2w) = \]
\[ = f(v)f(w)g(p^kc_1v)g(p'd_1w)h(p'c_2v)h(p'^md_2w), \quad v, w \in Y. \] \hfill (58)

If \( v \in Y(p^k) \) then \( p^kc_1v = p'c_2v = 0 \) and by Lemma 5 \( f(v) = 1, f(v + w) = f(w) \). Equation \( (42) \) holds true.

Let \( v \not\in Y(p^k) \). Then \( g(p^kc_1v) = 0 \), and a right side of equation \( (58) \) is equal to 0. Suppose that the left side of \( (58) \) does not equal to 0. Then

\[ v + w \in Y(p'), \hfill (59) \]
\[ p^k c_1 v + p^l d_1 w = 0. \]  
\( (60) \)

It follows from (59) that \( p^l (v+w) = 0 \). Taking into account (60) this implies that \( p^k (p^{l-k} d_1 - c_1) v = 0 \). Since \( p^{l-k} d_1 - c_1 \in Aut(Y) \), we get \( v \in Y_{(p^k)} \). We obtain the contradiction.

Let \( u \neq 0 \) in (16). Then \( g(u) = 0 \), and the right side of equation (16) is equal to 0. Suppose that the left side of (16) does not equal to zero. Then

\[ u + v + w \in Y_{(p^l)}, \]  
\( (61) \)

\[ u + p^k c_1 v + p^l d_1 w = 0, \]  
\( (62) \)

\[ u + p^l c_2 v + p^m d_2 w = 0. \]  
\( (63) \)

It follows from (61) that \( p^l (u + v + w) = 0 \). Taking into account (62) and (63) this implies that

\[ (p^l d_1 - I)u + p^k (p^{l-k} d_1 - c_1) v = 0, \]

\[ (64) \]

\[ (p^m d_2 - I)u + p^l (p^{m-l} d_2 - c_2) v = 0. \]

It follows from (64) that \( p^k \lambda v = 0 \). Hence \( v \in Y_{(p^k)} \). Then it follows from (62) and (63) that

\[ u + p^l d_1 w = 0, \]

\[ u + p^m d_2 w = 0. \]  
\( (65) \)

It follows from this that \( p^l (d_1 - p^{m-l} d_2) w = 0 \). Since \( d_1 - p^{m-l} d_2 \in Aut(Y) \), we get \( w \in Y_{(p^l)} \). Then it follows from the first equality of (65) that \( u = 0 \). We obtain the contradiction. So, the left side of (16) is equal to zero too.

**Case 11.** The study of this case is the same as the study of case 9.

**Cases 12-21.** The study of these cases is the same as the study of case 10.

**Proof of Theorem 1.** Let \( Y \) be a character group of \( X \). By Lemma 1 the characteristic functions \( \hat{\mu}_j(y) \) satisfy equation (3). Put \( \hat{\nu}_j = \mu_1 \ast \hat{\mu}_j \). We have \( \hat{\nu}_j(y) = |\hat{\mu}_j(y)|^2 \geq 0, j = 1, 2, 3 \). It is obvious that the characteristic functions \( \hat{\nu}_j(y) \) satisfy equation (3) as well. Hence, when we prove Theorem 1, we may assume without loss of generality that \( \hat{\mu}_j(y) \geq 0, j = 1, 2, 3 \), because \( \mu_j \) and \( \nu_j \) are either degenerated distributions or idempotent distributions simultaneously.

Taking into account the form of automorphisms (2), Lemma 2 implies that all \( \sigma(\mu_j) \subset B \), where \( B = p^n \Delta_p, n \geq 0 \), in \( X \). Suppose that \( \mu_j \) are not degenerate distributions. We choose the maximum number \( N \) such that \( \sigma(\mu_j) \subset B \), where \( B = p^N \Delta_p \) in \( X \).

Thus, the problem of description of distributions \( \mu_1, \mu_2, \mu_3 \) on the group \( X = \Omega_p \) is reduces to the problem of description of distributions \( \nu_1, \nu_2, \nu_3 \) on the group \( X = \Delta_p \), where the coefficients of the linear forms are homomorphisms of the group \( X = \Delta_p \). Also at least one support \( \sigma(\mu_j) \) is not contained in a proper subgroup of \( X \).

Let \( X = \Delta_p \). We have \( Y \approx \mathbb{Z}(p^\infty) \).

It follows from Lemma 3 that if all coefficients of the linear forms are automorphisms of \( X = \Delta_p \), then all distributions are idempotent. It is easy to verify that if \( \Lambda \) is not an automorphism of the group \( Y^3 \), i.e. \( q > 0 \), then all distributions \( \mu_j \) are degenerated.

So, we will consider the case when at least one of the automorphisms is not an automorphism of the group \( X = \Delta_p \).
Denote \( f(y) = \hat{\nu}_1(y), \ g(y) = \hat{\nu}_2(y), \ h(y) = \hat{\nu}_3(y). \) Note that \( f(y) \geq 0, \ g(y) \geq 0, \ h(y) \geq 0. \) Note that at least one support \( \sigma(\mu_j) \) is not contained in a proper subgroup of \( X \) if and only if

\[
\{ y \in Y : f(y) = g(y) = h(y) = 1 \} = \{ 0 \}. \tag{66}
\]

Lemma 1 implies that the functions \( f(y), g(y), h(y) \) satisfy equation (3).

I. Consider the case when \( \Lambda = \Lambda_1. \) Then equation (3) takes the form

\[
f(u + v + w)g(u + \delta_1 v + \varepsilon_1 w)h(u + \delta_2 v + \varepsilon_2 w) = f(u)g(u)f(v)g(\delta_1 v)h(\delta_2 v)f(w)g(\varepsilon_1 w)h(\varepsilon_2 w), \quad u, v, w \in Y. \tag{67}
\]

Note that

\[
det \Lambda = \delta_1 \varepsilon_2 - \delta_2 \varepsilon_1 - \delta_1 + \delta_2 + \varepsilon_1 - \varepsilon_2. \tag{68}
\]

Without restricting the generality, we can assume that the homomorphism \( \varepsilon_2 \) is not an automorphism of the group \( Y. \)

I.A. Suppose that \( q = 0, \) i.e. \( \Lambda \in Aut(Y^3). \) In this case at least one of the homomorphisms \( \delta_1, \delta_2, \varepsilon_1 \) is an automorphism.

Suppose that only one of the homomorphism from \( \delta_1, \delta_2, \varepsilon_1 \) is an automorphism. Without restricting the generality, we can assume that \( \delta_1 \in Aut(Y), \delta_2, \varepsilon_1 \not\in Aut(Y). \)

Put

\[
f(y) = \begin{cases} f_0(y), & y \in Y_{(p)}; \\ 0, & y \not\in Y_{(p)}. \end{cases} \quad g(y) = h(y) = \begin{cases} 1, & y = 0; \\ 0, & y \neq 0, \end{cases} \tag{69}
\]

where \( f_0(y) \) is an arbitrary characteristic function on \( Y_{(p)}. \) By Lemma 7 the function \( f(y) \) is a positive definite function on \( Y \) and there exists a distribution \( \mu_1 \) such that \( \hat{\mu}_1(y) = f(y). \) It is clear that \( f_0(y) \) can be chosen in such a way that \( \mu_1 \not\in I(X). \) Note that \( \mu_2 = \mu_3 = m_X. \)

We check that the functions (69) satisfy equation (67). If \( u = v = 0, \) then the equation holds true. Let either \( u \neq 0, \) or \( v \neq 0. \) Then \( g(u)g(\delta_1 v) = 0 \) and the right side of equation (67) is equal to 0.

Suppose that the left side of (67) does not equal to 0. It follows from (69) that

\[
u + v + w \in Y_{(p)}, \tag{70}
\]

\[
u + \delta_1 v + \varepsilon_1 w = 0, \tag{71}
\]

\[
u + \delta_2 v + \varepsilon_2 w = 0. \tag{72}
\]

Since \( \Lambda \in Aut(Y^3), \) it follows from (70), (71) and (72) that \( (u, v, w) \in Y^3_{(p)}, \) i.e. \( u, v, w \in Y_{(p)}. \)

Since \( \delta_2, \varepsilon_2 \not\in Aut(Y), \) it follows from (72) that \( u = 0. \) Since \( \varepsilon_1 \not\in Aut(Y), \) we get \( \varepsilon_1 w = 0. \) Taking it into account it follows from (71) that \( \delta_1 v = 0. \) Since \( \delta_1 \in Aut(Y), \) we get \( v = 0. \) We obtain that \( u = v = 0 \) that contradict to the assumption. So, the left side of (67) is also equal to 0.

Suppose now that at least two of the homomorphisms \( \delta_1, \delta_2, \varepsilon_1 \) are automorphisms.

Suppose that \( \delta_1, \delta_2 \in Aut(Y), \varepsilon_1 \not\in Aut(Y). \) Note that in this case \( \Lambda \) is an automorphism of the group \( Y^3 \) if \( \delta_1 - \delta_2 \in Aut(Y). \) Then the functions (69) satisfy equation (67). The verification is the same as in previous case.

Suppose that \( \delta_1, \varepsilon_1 \in Aut(Y), \delta_2 \not\in Aut(Y). \) Put

\[
f(y) = g(y) = \begin{cases} 1, & y \in Y_{(p)}; \\ 0, & y \not\in Y_{(p)}; \end{cases} \quad h(y) = \begin{cases} h_0(y), & y \in Y_{(p)}; \\ 0, & y \not\in Y_{(p)}; \end{cases} \tag{73}
\]
where \( h_0(y) \) is an arbitrary characteristic function on \( Y_{(p)} \). Lemma 7 implies that the function \( h(y) \) is a positive definite function on \( Y \) and there exists a distribution \( \mu_3 \) such that \( \hat{\mu}_3(y) = h(y) \). It is clear that \( h_0(y) \) can be chosen in such a way that \( \mu_3 \notin \mathcal{I}(X) \). Note that \( \mu_1 = \mu_2 = m_{X_{(p)}} \).

We check that functions (73) satisfy equation (67). If \( u, v, w \in Y_{(p)} \), then \( \delta_2v = \varepsilon_2v = 0 \) and equation (67) holds true. Let either \( u \notin Y_{(p)} \), or \( v \notin Y_{(p)} \), or \( w \notin Y_{(p)} \). Then \( f(u)f(v)f(w) = 0 \) and the right side of equation (67) is equal to 0. Suppose that the left side of (67) does not equal to 0. Then

\[
\Lambda \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in (Y^3)_{(p)}. \tag{74}
\]

Since \( \Lambda \in \text{Aut}(Y^3) \), it follows from (74) that \( (u, v, w) \in Y^3_{(p)} \). Hence \( u, v, w \in Y_{(p)} \). We obtain the contradiction. So, the left side of (67) is also equal to 0.

Suppose that \( \delta_2, \varepsilon_1 \in \text{Aut}(Y), \delta_1 \notin \text{Aut}(Y) \) or \( \delta_1, \delta_2, \varepsilon_1 \in \text{Aut}(Y) \). Note that each subgroup of \( Y \) is finite and characteristic. If we consider the restriction of equation (67) on any subgroup of \( Y \), then we are in the conditions of Lemma 6. Lemma 6 implies the statement of Theorem 1 in this case.

1.B. Suppose that \( q > 0 \), i.e. \( \Lambda \notin \text{Aut}(Y^3) \).

Suppose that \( \delta_1, \delta_2, \varepsilon_1, \varepsilon_2 \notin \text{Aut}(Y) \). Lemma 9 implies the statement of Theorem 1 in this case.

Suppose now that at least one of the homomorphisms \( \delta_1, \delta_2, \varepsilon_1, \varepsilon_2 \) is an automorphism of the group \( Y \). Since \( \Lambda \notin \text{Aut}(Y^3) \), it is easy to see that in this case at least two homomorphisms from \( \delta_1, \delta_2, \varepsilon_1, \varepsilon_2 \) are automorphisms.

It is clear that \( \text{Ker}\Lambda \not\subset Y \times \{0\} \times \{0\}, \text{Ker}\Lambda \not\subset \{0\} \times Y \times \{0\}, \text{Ker}\Lambda \not\subset \{0\} \times \{0\} \times Y \). At first we suppose that \( \text{Ker}\Lambda \not\subset \{0\} \times Y \times Y \). Then there exists an element \((u_0, v_0, w_0) \in \text{Ker}\Lambda \) where \( u_0 \neq 0 \). Put \( u = u_0, v = v_0, w = w_0 \) in (67). We obtain

\[
1 = f(u_0)g(u_0)h(u_0)f(v_0)g(\delta_1v_0)h(\delta_2v_0)f(w_0)g(\varepsilon_1w_0)h(\varepsilon_2w_0). \tag{75}
\]

It follows from (75) that \( f(u_0) = g(u_0) = h(u_0) = 1 \). Hence \( f(y) = g(y) = h(y) = 1 \) when \( y \) belongs to the subgroup generated by \( u_0 \). So we get the contradiction to the condition (66). Thus, in this case all distributions are degenerated.

We suppose now that \( \text{Ker}\Lambda \subset \{0\} \times Y \times Y \).

Then there exists an element \((0, v_0, -v_0) \in \text{Ker}\Lambda \), where \( v_0 \neq 0 \). Put \( u = 0, v = v_0, w = -v_0 \) in (67). We obtain

\[
1 = f^2(v_0)g(\delta_1v_0)h(\delta_2v_0)g(\varepsilon_1v_0)h(\varepsilon_2v_0). \tag{76}
\]

Suppose that \( \delta_1, \delta_2, \in \text{Aut}(Y) \). It follows from (76) that \( f(v_0) = g(\delta_1v_0) = h(\delta_2v_0) = 1 \). Note that any subgroup of \( Y \) is characteristic and contains \( Y_{(p)} \). Taking into account Lemma 5 we get that \( f(y) = g(y) = h(y) = 1 \) on \( Y_{(p)} \). So we get the contradiction to the condition (66). Thus, in this case all distributions are degenerated.

Suppose that \( \delta_2, \varepsilon_1 \in \text{Aut}(Y), \delta_2 \notin \text{Aut}(Y) \). It follows from (76) that \( f(v_0) = g(\varepsilon_1v_0) = h(\delta_2v_0) = 1 \). As in previous case we get that all distributions are degenerated.

Suppose that \( \delta_1, \varepsilon_1 \in \text{Aut}(Y), \delta_2 \notin \text{Aut}(Y) \). Put \( u = (\delta_1\varepsilon_2 - \delta_2\varepsilon_1)y, v = (\varepsilon_2 - \varepsilon_1)y, w = (\delta_2 - \delta_1)y \), \( y \in Y \) in (67). We obtain that

\[
\begin{align*}
 f(p^d\lambda y) &= f((\delta_1\varepsilon_2 - \delta_2\varepsilon_1)y)f((\varepsilon_2 - \varepsilon_1)y)f((\delta_2 - \delta_1)y) \\
 g((\delta_1\varepsilon_2 - \delta_2\varepsilon_1)y)g(\delta_1(\varepsilon_2 - \varepsilon_1)y)g(\varepsilon_1(\delta_2 - \delta_1)y)
\end{align*}
\]
\[ h((\delta_1 \varepsilon_2 - \delta_2 \varepsilon_1)y)h(\delta_2(\varepsilon_2 - \varepsilon_1)y)h(\varepsilon_2(\delta_2 - \delta_1)y). \] (77)

Put \( y \in Y(p^\nu) \) in (77). Then \( p^\nu \lambda y = 0 \), and the left side of (77) is equal to 1. Hence \( f((\delta_2 - \delta_1)y) = 1 \). Since \( \delta_2 - \delta_1 \in \text{Aut}(Y) \), it follows from this that \( f(y) = 1 \) for \( y \in Y(p^\nu) \). Let \( y \in Y(p^{2\nu}) \) in (77). Reasoning similarly we obtain that \( f(y) = 1 \) for \( y \in Y(p^{2\nu}) \). Analogously we obtain that \( f(y) = 1 \) for \( y \in Y(p^{N\nu}) \) for any natural \( N \). Since \( Y = \bigcup_N Y(N) \) and \( Y(N) \subset Y(N+1) \), we obtain that \( f(y) = 1 \) for \( y \in Y \). Then (77) implies that \( g(y) = h(y) = 1 \) for \( y \in Y \). So we get the contradiction to the condition (76). Thus, in this case all distributions are degenerated. We obtain the contradiction to the conditions of the theorem.

Thus, Theorem 1 is proved for \( \Lambda = \Lambda_1 \).

II. Consider the case when \( \Lambda = \Lambda_2 \). Then equation (3) takes the form

\[
f(u + \delta_1 v + w)g(u + v + \varepsilon_1 w)h(u + \delta_2 v + \varepsilon_2 w) = \]
\[= f(u)g(u)h(u)f(\delta_1 v)g(v)h(\delta_2 v)f(w)g(\varepsilon_1 w)h(\varepsilon_2 w), \quad u, v, w \in Y. \] (78)

Note that
\[
\det \Lambda = \delta_1 \varepsilon_1 - \delta_2 \varepsilon_1 - \delta_1 \varepsilon_2 + \delta_2 + \varepsilon_2 - 1. \] (79)

II.A. Suppose that \( q = 0 \), i.e. \( \Lambda \in \text{Aut}(Y^3) \).

Let each row and each column of the matrix \( \Lambda \) contain at least two automorphisms of the group \( Y \), i.e. there is at least one automorphism in each couple \((\delta_1, \delta_2), (\varepsilon_1, \varepsilon_2), (\delta_2, \varepsilon_2)\). Note that each subgroup of \( Y \) is finite and characteristic. If we consider the restriction of equation (78) to any subgroup of \( Y \), then we are in the conditions of Lemma 6. Lemma 6 implies the statement of Theorem 1 in this case.

Let there exist a row or a column of the matrix \( \Lambda \) which does not contain two automorphisms of the group \( Y \).

Let \( \delta_1, \delta_2 \notin \text{Aut}(Y) \). Note that it follows from (79) that in this case \( I - \varepsilon_2 \notin \text{Aut}(Y) \). Put
\[
f(y) = h(y) = \left\{\begin{array}{ll}
1, & y = 0; \\
0, & y \neq 0.
\end{array}\right. \]
where \( g_0(y) \) is an arbitrary characteristic function on \( Y(p) \). By Lemma 7 the function \( g(y) \) is a positive definite function on \( Y \) and there exists a distribution \( \mu_2 \) such that \( \mu_2(y) = g(y) \). It is clear that \( g_0(y) \) can be chosen in such a way that \( \mu_2 \notin I(X) \). Note that \( \mu_1 = \mu_3 = m_X \).

We check that the functions (80) satisfy equation (78). If \( u = w = 0 \), then the equation (78) becomes an equality. Let either \( u \neq 0 \) or \( w \neq 0 \). Then \( f(u)f(w) = 0 \) and the right side of (78) is equal to 0. Suppose that the left side of (78) does not equal to 0. Then
\[
u + \delta_1 v + w = 0, \] (81)
\[
u + v + \varepsilon_1 w \in Y(p), \] (82)
\[
u + \delta_2 v + \varepsilon_2 w = 0. \] (83)

Since \( \Lambda \in \text{Aut}(Y^3) \), it follows from (81), (82) and (83) that \((u, v, w) \in Y^3(p) \). Hence, \( u, v, w \in Y(p) \). Since \( \delta_1, \delta_2 \notin \text{Aut}(Y) \), we get from (81) and (83) that
\[
u + w = 0, \] (84)
\[
u + \varepsilon_2 w = 0. \] (85)
It follows from (84) and (85) that \((1 - \varepsilon_2)w = 0\). Since \(I - \varepsilon_2 \in Aut(Y)\), we get that \(w = 0\) and hence from (84) that \(u = 0\). We obtain the contradiction to the assumption. So, the left side of (78) is also equal to 0.

Let \(\varepsilon_1, \varepsilon_2 \notin Aut(Y)\). Note that it follows from (79) that in this case \(1 - \delta_2 \in Aut(Y)\). Put

\[
f(y) = \begin{cases} f_0(y) , & y \in Y_{(p)}; \\ 0 , & y \notin Y_{(p)}. \end{cases} \quad \quad g(y) = h(y) = \begin{cases} 1 , & y = 0; \\ 0 , & y \neq 0, \end{cases}
\]

(86)

where \(f_0(y)\) is an arbitrary characteristic function on \(Y_{(p)}\). By Lemma 7 the function \(f(y)\) is a positive definite function on \(Y\) and there exists a distribution \(\mu_1\) such that \(\tilde{\mu}_1(y) = f(y)\). It is clear that \(f_0(y)\) can be chosen in such a way that \(\mu_1 \notin I(X)\). Note that \(\mu_2 = \mu_3 = m_X\).

We check that the functions (86) satisfy equation (78). If \(u = v = 0\), then the equation (78) holds true. Let either \(u \neq 0\), or \(v \neq 0\). Then the right side of (78) is equal to 0. Suppose that the left side of (78) does not equal to 0. Then

\[
u + \delta_1 v + w \in Y_{(p)},
\]

(87)

\[
u + v + \varepsilon_1 w = 0,
\]

(88)

\[
u + \delta_2 v + \varepsilon_2 w = 0.
\]

(89)

Since \(\Lambda \in Aut(Y^3)\), it follows from (87), (88) and (89) that \((u, v, w) \in Y^3_{(p)}\). Hence, \(u, v, w \in Y_{(p)}\).

Since \(\varepsilon_1, \varepsilon_2 \notin Aut(Y)\), we get from (88) and (89) that

\[
u + v = 0,
\]

(90)

\[
u + \delta_2 v = 0.
\]

(91)

It follows from (90) and (91) that \((1 - \delta_2)v = 0\). Since \(I - \delta_2 \in Aut(Y)\), we get that \(v = 0\) and hence from (90) that \(u = 0\). We obtain the contradiction to the assumption. So, the left side of (78) is also equal to 0.

Let \(\varepsilon_2 \notin Aut(Y)\). Note that it follows from (79) that in this case \(1 - \delta_1 \varepsilon_1 \in Aut(Y)\). The verification that the functions (73) satisfy equation (78) is the same as in case I.A \((\delta_1, \varepsilon_1 \in Aut(Y), \delta_2 \notin Aut(Y))\).

II.B. Suppose that \(q > 0\), i.e. \(\Lambda \notin Aut(Y^3)\). Note that it follows from (79) that in this case either \(\delta_1 \varepsilon_1 \in Aut(Y)\), or \(\delta_2 \in Aut(Y)\), or \(\varepsilon_2 \in Aut(Y)\).

Suppose that either \(\delta_2 \in Aut(Y)\), or \(\varepsilon_2 \in Aut(Y)\). It is easy to see that \(Ker\Lambda \not\subset Y \times \{0\} \times \{0\}, Ker\Lambda \not\subset \{0\} \times Y \times \{0\}, Ker\Lambda \not\subset \{0\} \times \{0\} \times Y\). Hence, there exists an element \((u_0, v_0, w_0) \in Ker\Lambda\) such that at least two coordinates do not equal to 0. Put \(u = u_0, v = v_0, w = w_0\) in (78). We obtain

\[
1 = f(u_0)g(u_0)h(u_0)f(\delta_1 v_0)g(v_0)h(\delta_2 v_0)f(w_0)g(\varepsilon_1 w_0)h(\varepsilon_2 w_0).
\]

(92)

If \(u_0 \neq 0\), we get that \(f(y) = g(y) = h(y) = 1\) when \(y\) belongs a subgroup generated by \(u_0\). So we get the contradiction to the condition (60).

Let \(u_0 = 0\). Then it follows from (92) that \(1 = f(\delta_1 v_0)g(v_0)h(\delta_2 v_0)f(w_0)g(\varepsilon_1 w_0)h(\varepsilon_2 w_0)\). Since each subgroup of \(Y\) contains \(Y_{(p)}\) and either \(\delta_2 \in Aut(Y)\), or \(\varepsilon_2 \in Aut(Y)\), Lemma 5 implies that \(f(y) = g(y) = h(y) = 1\) on \(Y_{(p)}\). So we get the contradiction to the condition (60). Thus in these cases all distributions are degenerated.

Suppose that \(\delta_1 \varepsilon_1 \in Aut(Y)\), \(\delta_2 \notin Aut(Y)\), \(\varepsilon_2 \notin Aut(Y)\).
Put $u = (\varepsilon_2 - \delta_2 \varepsilon_1)y$, $v = (\varepsilon_1 - \varepsilon_2)y$, $w = (\delta_2 - I)y$, $y \in Y$ in (78). We obtain that

$$
\begin{align*}
f(p^\delta \lambda y) &= f((\varepsilon_2 - \delta_2 \varepsilon_1)y)f(\delta_1(\varepsilon_1 - \varepsilon_2)y)f((\delta_2 - I)y) \\
g((\varepsilon_2 - \delta_2 \varepsilon_1)y)g((\varepsilon_1 - \varepsilon_2)y)g(\varepsilon_1(\delta_2 - I)y) \\
h((\varepsilon_2 - \delta_2 \varepsilon_1)y)h(\delta_2(\varepsilon_1 - \varepsilon_2)y)h(\varepsilon_2(\delta_2 - I)y).
\end{align*}
$$

(93)

Put $y \in Y_{(p\delta)}$ in (93). Then $p^\delta \lambda y = 0$, and the left side of (93) is equal to 1. Hence $f((\delta_2 - I)y) = 1$. Since $\delta_2 - I \in \text{Aut}(Y)$, it follows from this that $f(y) = 1$ for $y \in Y_{(p\delta)}$. Let $y \in Y_{(p^2\delta)}$ in (93). Reasoning similarly we obtain that $f(y) = 1$ for $y \in Y_{(p^{2\delta})}$. Analogously we obtain that $f(y) = 1$ for $y \in Y_{(p^\lambda \delta)}$ for any natural $N$. Since $Y = \bigcup_N Y(N)$, we obtain that $f(y) = 1$ for $y \in Y$. Then (93) implies that $g(y) = h(y) = 1$ for $y \in Y$. So we get the contradiction to the condition on supports of $\mu_j$. Thus, in this case all distributions are degenerated. We obtain the contradiction to the conditions of the theorem.

\textbf{Remark 2.} Statement A of Theorem 1 can not be strengthened to the statement that all distributions $\mu_1, \mu_2, \mu_3$ are degenerated. Namely, it is easy to verify that if $\mu_1 = \mu_2 = \mu_3 = m_{\Delta_p}$ then components of the random vector $(L_1, L_2, L_3)$ are independent.

Statement B.2 of Theorem 1 in case $q = k$ and or $k_1 = k_2 = k$, or $k_2 = l_1 = k$ also can not be strengthened to the statement that all distributions $\mu_1, \mu_2, \mu_3$ are degenerate.

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