AUTOMORPHISMS AND DERIVATIONS OF AFFINE COMMUTATIVE AND PI-ALGEBRAS

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Abstract. We prove analogs of A. Selberg’s result for finitely generated subgroups of Aut(\(A\)) and of Engel’s theorem for subalgebras of Der(\(A\)) for a finitely generated associative commutative algebra \(A\) over an associative commutative ring. We prove also an analog of the theorem of W. Burnside and I. Schur about locally finiteness of torsion subgroups of Aut(\(A\)).

1. Introduction

Let \(\mathbf{A}\) be the algebra of regular (polynomial) functions on an affine algebraic variety \(V\) over an associative commutative ring \(\Phi\) with 1.

The group of \(\Phi\)-linear automorphisms Aut(\(\mathbf{A}\)) and the Lie algebra of \(\Phi\)-linear derivations Der(\(\mathbf{A}\)) are referred to as the group of polynomial automorphisms of \(V\) and the Lie algebra of vector fields on \(V\), respectively.

When the variety \(V\) is irreducible, i.e. the ring \(\mathbf{A}\) is a domain, the group Aut(\(K\)) of automorphisms of the field \(K\) of fractions of \(\mathbf{A}\) is called the group of birational automorphisms of \(V\); and the Lie algebra Der(\(K\)) of derivations of \(K\) is called the Lie algebra of rational vector fields on \(V\).

Let \(\mathbb{F}\) be the field. Then \(\mathbb{F}[x_1,\ldots,x_n]\) and \(\mathbb{F}(x_1,\ldots,x_n)\) are the polynomial algebra and the field of rational functions. The group Aut(\(\mathbb{F}(x_1,\ldots,x_n)\)) (resp. Aut(\(\mathbb{F}[x_1,\ldots,x_n]\)) and Der(\(\mathbb{F}[x_1,\ldots,x_n]\))) are called the Cremona group and the Cremona Lie algebra (resp. polynomial Cremona group and polynomial Cremona Lie algebra).

Recall that a group is called linear if it is embeddable into a group of invertible matrices over an associative commutative ring. Groups Aut(\(\mathbf{A}\)) are, generally speaking, not linear. It has been an ongoing effort of many years to understand:

*which properties of linear groups can be carried over to automorphisms groups Aut(\(\mathbf{A}\)) and to Cremona groups?*

J.-P. Serre \([37,38]\) studied finite subgroups of Cremona groups. V. L. Popov \([28]\) initiated the study of the question of whether the celebrated Jordan’s theorem...
on finite subgroups of linear groups carries over to the groups Aut(A). For some important results in this direction see [5, 8, 11, 28, 29, 31].

S. Cantat [10] proved the Tits Alternative for Cremona groups of rank 2.

In this paper, we prove analogs of A. Selberg’s result [36] (see also [2]) for finitely generated subgroups of Aut(A) and of Engel’s theorem for subalgebras of Der(A) for a finitely generated associative commutative algebra A.

We say that a group is virtually torsion free if it has a subgroup of finite index that is torsion free.

**Theorem 1.1.** Let A be a finitely generated associative commutative algebra over an associative commutative ring Φ with 1. Suppose that A does not have additive torsion. Then

(a) an arbitrary finitely generated subgroup of the group Aut(A) is virtually torsion free;

(b) if A is a finitely generated ring (i.e. Φ is the ring of integers Z), then the group Aut(A) is virtually torsion free.

**Corollary 1.2 (An analog of the theorem of W. Burnside and I. Schur; see [12, 13]).** Under the assumptions of theorem 1.1(a) every torsion subgroup of Aut(A) is locally finite.

**Corollary 1.3.** Every torsion subgroup of a polynomial Cremona group Aut(F[x₁,...,xₙ]), where F is a field of characteristic zero, has an abelian normal subgroup of finite index.

Corollary 1.3 immediately follows from corollary 1.2 and from the Jordan property of the group Aut(F[x₁,...,xₙ]); see [3, 31].

If the torsion subgroup in corollary 1.2 is torsion of bounded degree, then we don’t need any assumptions on additive torsion. Indeed, in [6], it was shown that the group Aut(A) is locally residually finite. Hence, by the positive solution of the restricted Burnside problem (see [11, 22]), the group G is locally finite.

Recall that a derivation d of an algebra A is called locally nilpotent if for an arbitrary element a ∈ A there exists an integer n(a) ≥ 1 such that dⁿ(a) = 0. For more information about locally nilpotent derivations see [12]. An algebra is called locally nilpotent if every finitely generated subalgebra is nilpotent.

Let L ⊆ Der(A) be a Lie algebra that consists of locally nilpotent derivations. The question of whether it implies that the Lie algebra L is locally nilpotent was discussed in [12, 20, 39]. In particular, A. Skutin [39] proved local nilpotency of L for a commutative domain A of finite transcendence degree and characteristic zero.

**Theorem 1.4.** Let A be a finitely generated associative commutative algebra over an associative commutative ring, and let L be a subalgebra of Der(A) that consists of locally nilpotent derivations. Then the Lie algebra L is locally nilpotent.

The assumption of finite generation of the algebra A is essential. If A is the algebra of polynomials in countably many variables over a field, then there exists a non-locally nilpotent Lie subalgebra L ⊆ Der(A) that consists of locally nilpotent derivations. The following theorem, however, imposes a finiteness condition that is weaker than finite generation.

Let A be a commutative domain. Let K be the field of fractions of A. An arbitrary derivation of the domain A extends to a derivation of the field K, Der(A) ⊆
Der(K). We have \( K \text{Der}(K) \subseteq \text{Der}(K) \), hence \( \text{Der}(K) \) can be viewed as a vector space over the field \( K \).

**Theorem 1.5.** Under the assumptions above, let \( L \subseteq \text{Der}(A) \) be a Lie ring that consists of locally nilpotent derivations. Suppose that \( \dim_K KL < \infty \). Then the Lie ring \( L \) is locally nilpotent.

A special case of this theorem was proved by A. P. Petravchuk and K. Ya. Sysak in [26].

The proof of theorem 1.5 is based on a stronger version of theorem 1.4, which is of independent interest.

Recall that a subalgebra \( B \) of an associative commutative algebra \( A \) is called an **order** in \( A \) if there exists a multiplicative semigroup \( S \subseteq B \) such that

1. every element from \( S \) is invertible in \( A \),
2. an arbitrary element \( a \in A \) can be represented as \( a = s^{-1}b \), where \( s \in S \) and \( b \in B \).

Let \( L \subseteq \text{Der}(A) \) be a subalgebra. The subset \( A_L = \{ a \in A \mid \text{for an arbitrary } d \in L \text{ there exists an integer } n(d) \geq 1 \text{ such that } d^{n(d)}(a) = 0 \} \) is a subalgebra of the algebra \( A \).

**Proposition 1.6.** Let \( A \) be a finitely generated commutative domain. Let \( L \) be a subalgebra of \( \text{Der}(A) \). If the subalgebra \( A_L \) is an order in \( A \), then the Lie algebra \( L \) is locally nilpotent.

To achieve a natural generality and to expand to noncommutative cases we extended theorems 1.1 and 1.4 to algebras with polynomial identities, i.e. PI-algebras; see [1, 7, 35].

A PI-algebra is called **representable** if it is embeddable in a matrix algebra over an associative commutative algebra. In [40], L. W. Small constructed an example of a finitely generated PI-algebra that is not representable.

**Theorem 1.7.** Let \( A \) be a finitely generated representable PI-algebra over an associative commutative ring. Suppose that \( A \) does not have additive torsion. Then

(a) an arbitrary finitely generated subgroup of the group \( \text{Aut}(A) \) is virtually torsion free;
(b) if \( A \) is a finitely generated ring, then the group \( \text{Aut}(A) \) is virtually torsion free.

**Theorem 1.8.** Let \( A \) be a finitely generated PI-algebra over an associative commutative ring. Suppose that \( A \) does not have additive torsion. Then an arbitrary torsion subgroup of \( \text{Aut}(A) \) is locally finite.

We remark that theorem 1.8 does not contain assumptions on representability.

C. Procesi [30] proved local finiteness of torsion subgroups of multiplicative groups of PI-algebras.

**Theorem 1.9.** Let \( A \) be a finitely generated PI-algebra over an associative commutative ring. Let \( L \subseteq \text{Der}(A) \) be a subalgebra that consists of locally nilpotent derivations. Then the Lie algebra \( L \) is locally nilpotent.

2. Preliminaries

In this section, we review some facts that will be used in proofs.
2.1. Theorems 1.1, 1.4, 1.8 and 1.9 were formulated for finitely generated associative commutative algebras over an associative commutative ring Φ. We will show that it is sufficient to assume Φ = \( \mathbb{Z} \), that is to prove the theorems for finitely generated rings. In particular, theorems 1.1(b) and 1.7(b) imply theorems 1.1(a) and 1.7(a), respectively. We will do it for theorem 1.9. The arguments for theorems 1.1, 1.4 and 1.8 are absolutely similar.

Let Φ be an associative commutative ring and let \( A \) be an associative PI-algebra over Φ (see 2.2) generated by elements \( a_1, \ldots, a_m \); and \( A \ni 1 \). Let \( L \subseteq \operatorname{Der}_\Phi(A) \) be a Lie subalgebra generated by derivations \( d_1, \ldots, d_n \). Suppose that every derivation of the \( \Phi \)-algebra \( L \) is locally nilpotent. Let \( \Phi(x_1, \ldots, x_m) \) be the free associative \( \Phi \)-algebra in free generators \( x_1, \ldots, x_m \). Then there exist elements \( f_{ij}(x_1, \ldots, x_m), 1 \leq i \leq n, 1 \leq j \leq m \), such that \( d_i(a_j) = f_{ij}(a_1, \ldots, a_m) \).

Let \( A_1 \) be the subring of \( A \) generated by elements \( 1, a_1, \ldots, a_m \) and by all coefficients of the elements \( f_{ij}(x_1, \ldots, x_m) \). It is straightforward that the subring \( A_1 \) is invariant under \( d_1, \ldots, d_n \). Assuming that theorem 1.9 is true for \( \Phi = \mathbb{Z} \), there exists an integer \( r \geq 1 \) such that \( L'(A_1) = (0) \). In particular, \( L'(a_i) = (0), 1 \leq i \leq m \). Since the elements \( a_1, \ldots, a_m \) generate the \( \Phi \)-algebra \( A \) we conclude that \( L' = (0) \).

Let us review some basic definitions and facts about PI-algebras that can be found in the books [1, 7, 35].

2.2. An associative algebra over an associative commutative ring \( \Phi \ni 1 \) is said to be PI if there exists an element

\[
f(x_1, \ldots, x_n) = x_1 \cdots x_n + \sum_{1 \neq \sigma \in S_n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}
\]

of the free associative algebra \( \Phi(x_1, \ldots, x_n) \) such that \( f(a_1, \ldots, a_n) = 0 \) for arbitrary elements \( a_1, \ldots, a_n \in A \); hereafter \( S_n \) is the group of permutations of the set \( \{1, \ldots, n\} \). In this case we say that the algebra \( A \) satisfies the identity \( f(x_1, \ldots, x_n) = 0 \).

If \( A \) is a PI-algebra, then it satisfies an identity with all the coefficients \( \alpha_\sigma, 1 \neq \sigma \in S_n \), lying in \( \mathbb{Z} \). In other words, every PI-algebra is PI over \( \mathbb{Z} \), i.e. PI as a ring.

2.3. A ring \( A \) is called prime if the product of any two nonzero ideals is different from zero. If \( A \) is a prime PI-ring, then the center

\[
Z = \{ a \in A \mid ab = ba \text{ for an arbitrary element } b \in A \} \neq (0)
\]

and the ring of fractions \( (Z \setminus \{0\})^{-1}A \) is a finite-dimensional central simple algebra over the field of fractions of the domain \( Z \); see [22, 34].

2.4. A ring \( A \) is called semiprime if it does not contain nonzero nilpotent ideals. Let \( A \) be a finitely generated semiprime PI-ring. Let \( Z \) be the center of \( A \) and let \( Z^* \) denote the set of elements from \( Z \) that are not zero divisors. Then the ring of fractions \( (Z^*)^{-1}A \) is a finite direct sum of simple finite-dimensional (over their centers) algebras.

2.5. An element \( a \in L \) of a Lie algebra \( L \) is called ad-nilpotent if the operator

\[
ad(a) : L \to L, \quad ad(a) : x \mapsto [a, x],
\]

is nilpotent.
Suppose that a Lie algebra $L$ is generated by elements $a_1, \ldots, a_m$. Commutators in $a_1, \ldots, a_m$ are defined via the following rules:

(i) an arbitrary generator $a_i$, $1 \leq i \leq m$, is a commutator in $a_1, \ldots, a_m$;
(ii) if $\rho'$ and $\rho''$ are commutators in $a_1, \ldots, a_m$, then $\rho = [\rho', \rho'']$ is a commutator in $a_1, \ldots, a_m$.

An element $a \in L$ is called a commutator in $a_1, \ldots, a_m$ if it is a commutator because of (i) and (ii).

A Lie algebra $L$ over an associative commutative ring $\Phi \ni 1$ is called PI (satisfies a polynomial identity) if there exists a multilinear element of the free Lie algebra $f(x_0, x_1, \ldots, x_n) = (\text{ad}(x_1) \cdots \text{ad}(x_n) + \sum_{1 \neq \sigma \in S_n} \alpha_{\sigma} \text{ad}(x_{\sigma(1)}) \cdots \text{ad}(x_{\sigma(n)})) x_0, \alpha_{\sigma} \in \Phi,$

such that $f(a_0, a_1, \ldots, a_n) = 0$ for arbitrary elements $a_0, a_1, \ldots, a_n \in L$.

The following theorem was proved in [42].

**Theorem ([42])**. Let $L$ be a Lie PI-algebra over an associative commutative ring generated by elements $a_1, \ldots, a_m$. Suppose that every commutator in $a_1, \ldots, a_m$ is ad-nilpotent. Then the Lie algebra $L$ is nilpotent.

### 3. Groups of Automorphisms

**Lemma 3.1.** Let $A$ be a finitely generated commutative domain without additive torsion. Then the group $\text{Aut}(A)$ is virtually torsion free.

**Proof.** Let $I$ be a maximal ideal of the ring $A$. The field $A/I$ is finitely generated, hence $A/I$ is a finite field, $A/I \cong GF(p')$. Let $P$ be the set of all ideals $P \triangleleft A$ such that $A/P \cong GF(p')$. Let $P_0$ be the ideal of the ring $A$ generated by all elements $a^{p'} - a, a \in A$, and by the prime number $p$. It is easy to see that the ring $A/P_0$ is finite, $P_0 \subseteq \cap_{P \in P} P$. This implies that the set $P$ is finite.

Automorphisms of the ring $A$ permute ideals from $P$. The ideal $I$ belongs to $P$. Hence, there exists a subgroup $H_1 \leq \text{Aut}(A), |\text{Aut}(A) : H_1| < \infty$, that leaves the ideal $I$ invariant. We have $|A : I^2| < \infty$. Therefore, there exists a subgroup $H_2 \leq H_1, |\text{Aut}(A) : H_2| < \infty$, such that

$$(1 - h)(A) \subseteq I^2$$

for an arbitrary element $h \in H_2$. Furthermore, if $a_1, \ldots, a_k \in I$, then

$$(h - 1)(a_1 \cdots a_k) = (h(a_1) - a_1 + a_1) \cdots (h(a_k) - a_k + a_k) - a_1 \cdots a_k = \sum b_1 \cdots b_k,$$

where each $b_i = (h - 1)(a_i)$ or $a_i$ and in each summand at least one element $b_i$ is equal to $(h - 1)(a_i)$. This implies that

$$(1 - h)(I^k) \subseteq I^{k+1}.$$ 

By the Krull intersection theorem (see [4]), we have

$$\bigcap_{k \geq 1} I^k = (0).$$

If an element from $H_2$ has finite order, then this order must be a power of the prime number $p$.

Consider the ring $\tilde{A} = \langle 1/p, A \rangle \subseteq A \otimes \mathbb{Q}$,
from a domain without additive torsion. By lemma 3.1, there exists a subgroup $H_3 \trianglelefteq \text{Aut}(A)$ of a finite index such that $(1 - h)(J^k) \subseteq J^{k+1}$, $k \geq 0$, for an arbitrary element $h \in H_3$. Hence, if an element from $H_3$ has finite order, then this order must be a power of the prime number $q$.

Now, $H_2 \cap H_3$ is a torsion free subgroup of $\text{Aut}(A)$. This completes the proof of the lemma.

**Lemma 3.2.** Let $A$ be a semiprime finitely generated associative commutative ring without additive torsion. Then the group $\text{Aut}(A)$ is virtually torsion free.

**Proof.** Let $S \subseteq A$ be the set of all nonzero elements that are not zero divisors. Then the ring of fractions $S^{-1}A$ is a direct sum of fields, $S^{-1}A = \mathbb{F}_1 \oplus \cdots \oplus \mathbb{F}_k$. An arbitrary automorphism of the ring $A$ extends to an automorphism of $S^{-1}A$. Hence, there exists a subgroup $H \trianglelefteq \text{Aut}(A)$ of finite index such that every automorphism from $H$ leaves the summands $\mathbb{F}_1, \ldots, \mathbb{F}_k$ invariant. For each $i, 1 \leq i \leq k$, the factor-ring

$$K = A/A \cap (\mathbb{F}_1 \oplus \cdots \oplus \mathbb{F}_{i-1} \oplus \mathbb{F}_{i+1} \oplus \cdots \oplus \mathbb{F}_k)$$

is a domain without additive torsion. By lemma 3.1 there exists a subgroup $H_i \trianglelefteq H$ of finite index such that the image of $H_i$ in $\text{Aut}(K)$ is torsion free. This implies that the group $\cap_{i=1}^k H_i$ is torsion free. Indeed, if an element $h \in \cap_{i=1}^k H_i$ has finite order, then $h$ acts identically modulo $K$, and we get

$$(1 - h)(A) \subseteq \bigcap_{i=1}^k (\mathbb{F}_1 \oplus \cdots \oplus \mathbb{F}_{i-1} \oplus \mathbb{F}_{i+1} \oplus \cdots \oplus \mathbb{F}_k) = (0).$$

This completes the proof of the lemma.

**Proof of theorem 1.7(b).** Let $A$ be a finitely generated representable PI-ring that does not have additive torsion. A. I. Malcev [21] showed that the ring $A$ is embeddable in a matrix algebra over a field of characteristic zero, $A \hookrightarrow M_n(\mathbb{F})$, char $\mathbb{F} = 0$. Let $a_1, \ldots, a_m$ be generators of the ring $A$, and let $\mathbb{Z}[X]$ be the free associative ring on free generators $x_1, \ldots, x_m$. If $R \subseteq \mathbb{Z}(x_1, \ldots, x_m)$ is a set of defining relations of the ring $A$ in the generators $a_1, \ldots, a_m$, then $A \simeq \langle x_1, \ldots, x_m \mid R = (0) \rangle$.

Let $n, m \geq 2$. Consider $m$ generic $n \times n$ matrices

$$X_k = (x_{ij}^{(k)})_{1 \leq i, j \leq n}, 1 \leq k \leq m.$$ 

These are $n \times n$ matrices over the polynomial ring $\mathbb{Z}[X]$, where

$$X = \{x_{ij}^{(k)} \mid 1 \leq i, j \leq n, 1 \leq k \leq m\}$$

is the set of variables. The ring $G(m, n)$ generated by generic matrices $X_1, \ldots, X_m$ is a domain and it is PI; see [3].

For a relation $r \in R$ let

$$r(X_1, \ldots, X_m) = (r_{ij}(X))_{1 \leq i, j \leq n}, \quad r_{ij}(X) \in \mathbb{Z}[X].$$

Consider the associative commutative ring $U$ presented by generators $X$ and relations $r_{ij}(X) = 0$, $r \in R$, $1 \leq i, j \leq n$, i.e.

$$U = \mathbb{Z}[X]/I, \quad I = \text{id}_{\mathbb{Z}[X]}(r_{ij}(X), \ r \in R, \ 1 \leq i, j \leq n).$$
Since the ring $A$ is embeddable in $M_n(F)$ it follows that the homomorphism
\[ u : A \to M_n(U), \quad u(a_k) = X_k + I \in M_n(U), \quad 1 \leq k \leq m, \]
is an embedding. Moreover, the ring $U$ has the following universal property:
if $C$ is an associative commutative ring and $\varphi : A \to M_n(C)$ is an embedding, then
there exists a unique homomorphism $U \to C$ that makes the diagram
\[
\begin{array}{ccc}
A & \xrightarrow{u} & M_n(U) \\
\varphi \downarrow & & \downarrow \\
M_n(C) & & M_n(C)
\end{array}
\]
commutative.

This implies that every automorphism of the ring $A$ gives rise to an automorphism of the ring $U$. Let
\[ T(U) = \{ x \in U \mid \text{there exists an integer } k \geq 1 \text{ such that } kx = 0 \} \]
be the torsion part of the ring $U$. Let $J(U/T(U))$ be the radical of the ring $U/T(U)$,
\[ J(U/T(U)) = J/T(U), \]
where
\[ (0) \subseteq T(U) \subseteq J \triangleleft U, \quad \overline{U} = U/J. \]
The factor-ring $\overline{U}$ is semiprime and does not have additive torsion. An arbitrary
automorphism of the ring $A$ gives rise to an automorphism of $\overline{U}$.

Since the ring $A$ is embeddable in $M_n(\mathbb{F})$, char $\mathbb{F} = 0$, it follows that $A$ is em-
beddable in $M_n(\overline{U})$ and the group Aut($A$) is embeddable in Aut($\overline{U}$). By lemma
\[ 3.2 \] the group Aut($\overline{U}$) is virtually torsion free and so is Aut($A$). This completes the
proof of theorem 1.7(b).

Recall that theorem 1.7(b) implies theorems 1.1 and 1.7(a).

We will discuss the annoying representability assumption in theorem 1.7. Let $A$
be a finitely generated PI-algebra over the field of rational numbers $\mathbb{Q}$, and let $J$
be the Jacobson radical of the algebra $A$. By [9], the Jacobson radical of a finitely
generated PI-ring is nilpotent. So, the radical $J$ is nilpotent. The stabilizer of
the descending chain $A \supset J \supset J^2 \supset \cdots$ in Aut($A$) is torsion free. Indeed, let $\varphi \in$ Aut($A$) and $(1 - \varphi)J^i \subseteq J^{i+1}, i \geq 0$. We assume that $\varphi^n = 1$. Then we have
\[ \varphi^n = (\varphi - 1 + 1)^n = \sum_{i=2}^{n} \binom{n}{i} (\varphi - 1)^i + n(\varphi - 1) + 1. \]
Hence,
\[ n(1 - \varphi) = \sum_{i=2}^{n} \binom{n}{i} (\varphi - 1)^i. \]
Suppose that $a \in A$ and $(1 - \varphi)a \neq 0$. Let $(1 - \varphi)a \in J^k \setminus J^{k+1}$. By the above,
$n(1 - \varphi)a \in (\varphi - 1)J^k \subseteq J^{k+1}$, a contradiction.

If the group Aut($A/J^2$) is virtually torsion free, then so is the group Aut($A$).
Indeed, let $H$ be a torsion free subgroup of finite index in Aut($A/J^2$) and let $\overline{H}$ be
the preimage of $H$ under the homomorphism Aut($A$) $\to$ Aut($A/J^2$). If $h \in \overline{H}$ is
a torsion element, then $h$ acts identically modulo $J^2$, hence $h$ stabilizes the chain
$A \supset J \supset J^2 \supset \cdots$ and $h = 1$. We proved that the subgroup $\overline{H}$ of Aut($A$) is torsion free.
In all known examples of nonrepresentable finitely generated PI-algebras the Jacobson radical is nilpotent of degree $\geq 3$.

**Conjecture.** A finitely generated PI-algebra with $J^2 = 0$ is representable.

If this conjecture is true, then the representability assumption in theorem 1.7 can be dropped.

The analog of Selberg’s theorem holds for automorphism groups of some algebras that are far from being PI.

**Proposition 3.3.** Let $A = \mathbb{Z}(x_1, \ldots, x_m), m \geq 2$, be the free associative ring on free generators $x_1, \ldots, x_m$. The group of automorphisms $\text{Aut}(A)$ is virtually torsion free.

**Proof.** Let $p$ be a prime number. Let $I_p$ be the ideal of the algebra $A$ generated by $p$ and by all elements $a^p - a, a \in A$. The ideal $I_p$ is invariant under all automorphisms, the factor-ring $A/I_p$ is finite and constant terms of all elements in $I_p$ are divisible by $p$. Hence,

$$\bigcap_{i \geq 1} I_p^i = (0).$$

The subgroup

$$H_1 = \ker(\text{Aut}(A) \to \text{Aut}(A/I_p^2))$$

has finite index in $\text{Aut}(A)$ and every element of finite order in $H_1$ has an order, which is a power of $p$. Now, choose a prime number $q, p \neq q$. The subgroup

$$H_2 = \ker(\text{Aut}(A) \to \text{Aut}(A/I_q^2))$$

also has finite index in $\text{Aut}(A)$ and every element of finite order in $H_2$ has an order which is a power of $q$. The subgroup $H_1 \cap H_2$ is torsion free and has finite index in $\text{Aut}(A)$. This completes the proof of the proposition.

**Lemma 3.4.** Let $A$ be a PI-algebra. Let $_AM$ be a finitely generated left $A$-module. Then the algebra of $A$-module endomorphisms of the module $_AM$ is PI.

**Proof.** Let $M = \sum_{i=1}^n Am_i$. Consider the free $A$-module $V$ on free generators $x_1, \ldots, x_n$:

$$V = \sum_{i=1}^n Ax_i,$$

and the homomorphism

$$f : V \to M, \quad x_i \mapsto m_i, \quad 1 \leq i \leq n.$$ 

Denote its kernel as $V_0$. Let

$$E_1 = \{ \varphi \in \text{End}_A(V) \mid \varphi(V_0) \subseteq V_0 \}, \quad E_2 = \{ \varphi \in \text{End}_A(V) \mid \varphi(V) \subseteq V_0 \}.$$ 

Then

$$\text{End}_A(M) \simeq E_1/E_2.$$ 

The algebra $\text{End}_A(V)$ is isomorphic to the algebra of $n \times n$ matrices over $A$. Hence, $\text{End}_A(V)$ is a PI-algebra. This implies that $E_1$ and $E_1/E_2$ are PI-algebras. \qed
Proof of theorem 1.8. Let $A$ be a finitely generated PI-algebra over $\mathbb{Q}$, and let $G$ be a finitely generated torsion subgroup of $\text{Aut}(A)$. Consider the Jacobson radical $J$ of the algebra $A$. The semisimple algebra $\overline{A} = A/J$ is representable; see [16]. Hence, by theorem 1.7(a), the group $\text{Aut}(\overline{A})$ has Selberg’s property, and the image of the group $G$ in $\text{Aut}(\overline{A})$ is finite. In other words, the subgroup $H = \{ \varphi \in G \mid (1 - \varphi)(A) \subseteq J \}$ has finite index in $G$.

Consider the subgroup
$$K = \{ \varphi \in \text{Aut}(A) \mid (1 - \varphi)(A) \subseteq J^2 \}.$$We showed that this subgroup centralizes the descending chain $A \supset J \supset J^2 \ldots$, hence $K$ is a torsion free group. Therefore, $G \cap K = (1)$, and the homomorphism $G \to \text{Aut}(A/J^2)$ is an embedding. Without loss of generality, we will assume that $J^2 = (0)$. The radical $J$ can be viewed as an $\overline{A}$-bimodule.

Let $a_1, \ldots, a_m$ be generators of the algebra $A$, and let $h_1, \ldots, h_r$ be generators of the subgroup $H$. We have $(1 - h_i)(A) \subseteq J, J^2 = 0$, hence $1 - h_i$ is a derivation of the algebra $A$. This implies that $(1 - h_i)(A)$ lies in the $\overline{A}$-subbimodule of $J$ generated by elements $(1 - h_i)(a_1), \ldots, (1 - h_i)(a_m)$. Let $J'$ be the $\overline{A}$-subbimodule of $J$ generated by elements $(1 - h_i)(a_j), 1 \leq i \leq r, 1 \leq j \leq m$. The finitely generated subbimodule $J'$ is invariant with respect to the action of $H$. For an automorphism $h \in H$, consider the restriction $\text{Res}(h)$ of $h$ to $J'$. This restriction is a bimodule automorphism of the $\overline{A}$-bimodule $J'$. The mapping
$$\varphi : H \to GL(\overline{A}/J'), \ h \mapsto \text{Res}(h),$$
is a homomorphism to the group of bimodule automorphisms $GL(\overline{A}/J')$. The $\overline{A}$-bimodule $J'$ is a left module over the algebra $\overline{A} \otimes_\mathbb{Q} \overline{A}^{op}$ and
$$GL(\overline{A}/J') = GL(\overline{A} \otimes_\mathbb{Q} \overline{A}^{op}(J')).$$The algebra $\overline{A} \otimes_\mathbb{Q} \overline{A}^{op}$ is PI; see [33]. By lemma 3.4, the algebra
$$\text{End}(\overline{A} \otimes_\mathbb{Q} \overline{A}^{op}(J'))$$
is PI as well. Thus, $\varphi(H)$ is a finitely generated torsion subgroup of the multiplicative group of a PI-algebra. By the result of C. Procesi [30], the group $\varphi(H)$ is finite. The kernel $H' = \ker \varphi$ is a subgroup of finite index in $G$ and for an arbitrary element $h \in H'$ we have $(1 - h)(A) \subseteq J', (1 - h)(J') = (0)$. Let $h^k = 1, k \geq 1$. We have
$$1 - h^k = k(1 - h) \bmod (1 - h)^2.$$This implies $k(1 - h)(A) = 0$ and, therefore, $h = 1, H' = (1)$. Hence, $|G| < \infty$. This completes the proof of the theorem. □

4. Lie rings of locally nilpotent derivations

**Proposition 4.1.** Let $A$ be a finitely generated PI-ring. Then the Lie ring $\text{Der}(A)$ is PI.

**Proof.** For an integer $n \geq 2$ consider the following elements of the free Lie ring
$$P_n(x_0, x_1, \ldots, x_n) = \sum_{\sigma \in S_n} (-1)^{|\sigma|} \text{ad}(x_{\sigma(1)}) \cdots \text{ad}(x_{\sigma(n)})x_0.$$For an associative commutative ring $\Phi$ let $W_\Phi(n)$ denote the Lie $\Phi$-algebra of $\Phi$-linear derivations of the polynomial algebra $\Phi[x_1, \ldots, x_n]$. In [32], Yu.P.Razmyslov
proved that for a field $F$ of characteristic zero the Lie algebra $W_p(n)$ satisfies the identity $P_N = 0$, where $N = (n + 1)^2$. The Lie ring $W_Z(n)$ is a subring of the $\mathbb{Q}$-algebra $W_0(n)$. Hence, $W_Z(n)$ satisfies the identity $P_N = 0$. Let $A$ be a PI-ring generated by elements $a_1, \ldots, a_m$. Since $A$ is a finitely generated PI-ring, it follows that $A$ is an epimorphic image of the ring of generic matrices $G(m, n)$ for some integers $m, n \geq 2$; see [7, 19]. Let

$$G(m, n) \rightarrow A, \quad X_k = (x_{ij}^{(k)})_{1 \leq i, j \leq n} \mapsto a_k, \quad 1 \leq k \leq m,$$

be an epimorphism. Let $N = (n^2m + 1)^2$. We will show that the Lie ring $\text{Der}(A)$ satisfies the identity $P_N = 0$. Denote

$$X = \{ x_{ij}^{(k)} \mid 1 \leq i, j \leq n, \quad 1 \leq k \leq m \}.$$

Choose derivations $d_0, d_1, \ldots, d_N \in \text{Der}(A)$. There exist elements $f_{st}(x_1, \ldots, x_m)$ of the free associative ring $\mathbb{Z}[x_1, \ldots, x_m]$, $0 \leq s \leq N$, $1 \leq t \leq m$, such that

$$d_s(a_i) = f_{st}(a_1, \ldots, a_m).$$

Let

$$f_{st}(X_1, \ldots, X_m) = (g_{ij}^{st}(X))_{1 \leq i, j \leq n};$$

where $g_{ij}^{st}(X) \in \mathbb{Z}[X]$ are entries of the matrix $f_{st}(X_1, \ldots, X_m)$. Consider derivations $\tilde{d}_s$ of the ring $\mathbb{Z}[X]$,

$$\tilde{d}_s(x_{ij}^{(t)}) = g_{ij}^{st}(X), \quad 1 \leq i, j \leq n, \quad 0 \leq s \leq N, \quad 1 \leq t \leq m.$$

Let $L$ be the Lie subring generated by the derivations $\tilde{d}_s, 0 \leq s \leq N$ in $\text{Der}(\mathbb{Z}[X])$. The mapping $\tilde{d}_s \rightarrow d_s, 0 \leq s \leq N$, extends to a homomorphism $L \rightarrow \text{Der}(A)$. This implies $P_N(d_0, d_1, \ldots, d_N) = 0$ and completes the proof of the proposition. \hfill \Box

Now, our aim is to prove theorem 1.9. In view of 2.1 we will assume that the finitely generated PI-algebra $A$ of theorem 1.9 is a finitely generated ring.

Let's prove theorem 1.9 and proposition 1.6 for the case of prime characteristics.

Let $A$ be a finitely generated PI-ring and let $L \subseteq \text{Der}(A)$ be a Lie ring that consists of locally nilpotent derivations. Suppose further that there exists a prime number $p \geq 2$ such that $pA = (0)$.

Let $a_1, \ldots, a_m$ be generators of the ring $A$. Let $d \in L$. There exists a power $p^k$ of the prime number $p$ such that

$$d^{p^k}(a_i) = 0, \quad 1 \leq i \leq m.$$

The power $d^{p^k}$ is again a derivation of the ring $A$. Hence $d^{p^k} = 0$. This implies that $\text{ad}(d)^{p^k} = 0$ in the Lie ring $L$. By proposition 2.1, the Lie ring $L$ is PI, and by results of [33] (see 2.4), the Lie ring $L$ is locally nilpotent. Moreover, every finitely generated subalgebra $L_1$ of $L$ acts on $A$ nilpotently, i.e. there exists an integer $s \geq 1$ such that

$$L_1 \cdots L_1 A = (0).$$

This proves theorem 1.9 in the case of a prime characteristic.

Now, let $A$ be an associative commutative ring generated by elements $a_1, \ldots, a_m$, let $p$ be a prime number such that $pA = (0)$, and let $L \subseteq \text{Der}(A)$ be a Lie subring of $\text{Der}(A)$. Suppose that the subring $A_L$ is an order in $A$. Then $a_i = b^{-1}_i c_i, 1 \leq i \leq m,$
where \( b_i, c_i \in A_L \). For an arbitrary derivation \( d \in L \) there exists a power \( p^k \) such that \( dp^k(b_i) = dp^k(c_i) = 0, 1 \leq i \leq m \). Then \( dp^k(a_i) = 0, 1 \leq i \leq m \), and, therefore, \( dp^k = 0 \). Again, by \([43]\), the ring \( L \) is locally nilpotent. This proves proposition \([1.6]\) in the case of prime characteristic.

A Lie ring \( L \) is called weakly Engel if for arbitrary elements \( a, b \in L \) there exists an integer \( n(a, b) \geq 1 \) such that

\[
\text{ad}(a)_{n(a, b)}b = 0.
\]

B. I. Plotkin \([27]\) proved that a weakly Engel Lie ring has a locally nilpotent radical. In other words, if \( L \) is a weakly Engel Lie ring, then \( L \) contains the largest locally nilpotent ideal \( I \) such that the factor-ring \( L/I \) does not contain nonzero locally nilpotent ideals. We denote \( I = \text{Loc}(L) \).

**Lemma 4.2.** Let \( A \) be a finitely generated ring and let a Lie ring \( L \subseteq \text{Der}(A) \) consist of locally nilpotent derivations. Then the Lie ring \( L \) is weakly Engel.

**Proof.** Let the ring \( A \) be generated by elements \( a_1, \ldots, a_m \). Let \( d_1, d_2 \in L \). There exists an integer \( n \geq 1 \) such that \( d_1^n(a_i) = 0, 1 \leq i \leq m \). Since the set

\[
\{ d_2d_1^i(a_j), \quad 0 \leq i \leq n - 1, \quad 1 \leq j \leq m \}
\]

is finite there exists an integer \( k \geq 1 \) such that

\[
d_1kd_2d_1^i(a_j) = 0, \quad 0 \leq i \leq n - 1, \quad 1 \leq j \leq m.
\]

We have

\[
\text{ad}(d_1)^sd_2 = \sum_{i+j=s} (-1)^j \binom{s}{i}^i d_1^s d_2^i.
\]

Hence

\[
\text{ad}(d_1)^{n+k-1}d_2(a_j) = 0, 1 \leq j \leq m.
\]

This implies \( \text{ad}(d_1)^{n+k-1}d_2 = 0 \) and completes the proof of the lemma. \( \square \)

**Lemma 4.3.** Let \( A \) be a finitely generated associative commutative ring. Let \( L \subseteq \text{Der}(A) \) be a Lie ring of derivations such that the subring \( A_L \) is an order in \( A \). Then the Lie ring \( L \) is weakly Engel.

**Proof.** Let \( a_1, \ldots, a_m \) be generators of the ring \( A \), let \( a_i = b_i^{-1}c_i, 1 \leq i \leq m \), where \( b_i, c_i \in A_L \). Choose derivations \( d_1, d_2 \in L \). In the proof of lemma \([4.2] \) we showed that there exists an integer \( s \geq 1 \) such that

\[
\text{ad}(d_1)^sd_2(b_i) = \text{ad}(d_1)^sd_2(c_i) = 0, 1 \leq i \leq m.
\]

Since \( d' = \text{ad}(d_1)^sd_2 \) is a derivation of the algebra \( A \) it follows that \( d'(a_i) = 0, 1 \leq i \leq m \), and therefore \( d' = 0 \). This completes the proof of the lemma. \( \square \)

**Lemma 4.4.** Let \( A \) be a finitely generated semiprime PI-ring. Then there exists a family of homomorphisms \( A \to M_n(\mathbb{Z}/p\mathbb{Z}) \) into matrix rings over prime fields that approximates \( A \).

**Proof.** The ring \( A \) is representable \([16]\), i.e. it is embeddable into a ring of matrices over a finitely generated associative commutative semiprime ring \( C \), \( A \to M_n(C) \). Hilbert’s Nullstellensatz \([4]\) implies that \( C \) is a subdirect product of finite fields. Hence, there exists a family of homomorphisms \( \varphi_i : A \to M_n(\mathbb{F}_i) \), where \( \mathbb{F}_i \) are finite fields such that \( \cap_i \ker \varphi_i = (0) \). If \( \text{char } \mathbb{F}_i = p \), then the field \( \mathbb{F}_i \) is embeddable into a ring of matrices over \( \mathbb{Z}/p\mathbb{Z} \). This completes the proof of the lemma. \( \square \)
Lemma 4.5. Let $A$ be a finitely generated prime PI-ring. Let $Z$ be the center of $A$ and let $K$ be the field of fractions of the commutative domain $Z$. Then $\dim_K K\text{Der}(A) < \infty$.

Proof. Let $a_1, \ldots, a_m$ be generators of the ring $A$. As we have remarked in 2.3 the ring of fractions $\bar{A} = (Z \setminus \{0\})^{-1}A$ is a finite-dimensional central simple algebra over the field $K$. Let $\dim_K \bar{A} = s$. We will show that $\dim_K K\text{Der}(A) \leq ms$. Choose $ms + 1$ derivations $d_1, \ldots, d_{ms+1}$ of the ring $A$. Consider the vector space

$$V = \bar{A} \oplus \cdots \oplus \bar{A}$$

over the field $K$, $\dim_K V = ms$, and vectors $v_i = (d_i(a_1), \ldots, d_i(a_m)) \in V, 1 \leq i \leq ms + 1$. There exist coefficients $k_1, \ldots, k_{ms+1} \in K$, not all equal to 0, such that

$$\sum_{i=1}^{ms+1} k_i v_i = 0.$$ 

This implies $d(a_i) = 0, 1 \leq i \leq m$, where $d = \sum_{i=1}^{ms+1} k_i d_i$. Since $d$ is a derivation of the ring $\bar{A}$ and elements $a_1, \ldots, a_m$ generate $A$ as a ring it follows that $d(A) = \{0\}$. This implies that $d(K) = 0$ and completes the proof of the lemma.

Now, we will prove theorem 1.9 and proposition 1.6 in the case when the algebra $A$ is prime.

As above, let $A$ be a finitely generated prime PI-ring, let $Z = Z(A)$ be the center of the ring $A$, and let $K = (Z \setminus \{0\})^{-1}Z$ be the field of fractions of the domain $Z$. Suppose that a Lie ring $L \subseteq \text{Der}(A)$ consists of locally nilpotent derivations. For a derivation $d \in L$ let $\text{id}_L(d)$ denote the ideal of the Lie ring $L$ generated by the element $d$. Consider the descending chain of ideals

$$I_1 = L, \quad I_{i+1} = \sum_{d \in I_i} [\text{id}_L(d), \text{id}_L(d)].$$

Since $\dim_K KL < \infty$, by lemma 4.5 it follows that the descending chain

$$KI_1 \supseteq KI_2 \supseteq \cdots$$

stabilizes. Let $KI_I = KI_{I+1} = \cdots$. We will show that $I_I = \{0\}$. Indeed, there exists a finite collection of derivations $d_1, \ldots, d_r \in I_I$ such that

$$KI_{I+1} = \sum_{i=1}^r K[\text{id}_L(d_i), \text{id}_L(d_i)].$$

Recall that

$$(4.1) \quad \text{id}_L(d_i) = Zd_i + \sum_{t \geq 1} [\ldots [d_i, L], L], \ldots, L].$$

Let

$$(4.2) \quad d \in [\text{id}_L(d_i), \text{id}_L(d_i)].$$

Expanding the commutators on the right-hand sides of (4.1) and (4.2) we get

$$d = \sum_{(\ast_1)} d_1 \ldots d_i \ldots, \ldots (\ast_3),$$

where $(\ast_1)$ is a product of derivations from $L$ and, possibly, a multiplication by an element from $K$, $(\ast_2)$ and $(\ast_3)$ are products, may be empty, of derivations.
from $L$. Hence, $d = \sum d_i$, where each summand has a nonempty product of derivations from $L$ to the right of $d_i$.

Since $d_1, \ldots, d_r \in \sum K[\text{id}_L(d_j), \text{id}_L(d_j)]$, we have

$$(4.3) \quad d_i = \sum k_{ijt} u_{ijt} d_j v_{ijt}, \quad 1 \leq i \leq r,$$

where $k_{ijt} \in K; u_{ijt}, v_{ijt}$ are products of derivations from $L$; $v_{ijt}$ are nonempty products of derivations from $L$.

Let $b$ be a common denominator of all elements $k_{ijt}$, that is $k_{ijt} \in b^{-1}Z$. Consider the finitely generated prime PI-ring $A_1 = \langle b^{-1}, A \rangle$. The ring $A_1$ is invariant under $\text{Der}(A)$. Suppose that there exists an element $a \in A$ such that $d_i(a) \neq 0$. By lemma \ref{lem:4.4}, there exists a family of homomorphisms $\varphi : A_1 \to M_n(Z/pZ)$ that approximates the ring $A_1$. Hence, there exists a prime number $p$ such that $d_i(a) \notin pA_1$.

Consider the subring $L'$ of the Lie ring $L$ generated by all derivations that are involved in the products $v_{ijt}$. Clearly, $L'$ is a finitely generated Lie ring.

We have shown above that theorem \ref{thm:1.9} is true for rings of prime characteristics. Applying this result to the ring $A/pA$, we conclude that the ring $L'$ acts nilpotently on $A/pA$. In other words, there exists an integer $s \geq 1$ such that

$$(4.4) \quad \frac{L'}{pA} \subseteq \frac{L'}{s}(A) \subseteq pA.$$

Iterating (4.4) $s$ times, we get

$$d_i = \sum u_{i} d_j v_{i1j1t1} \cdots v_{isjt},$$

where $u_i \in A_1$. By (4.3), we get

$$v_{i1j1t1} \cdots v_{isjt} A \subseteq pA \subseteq pA_1$$

and, therefore, $d_i(a) \in pA_1, 1 \leq i \leq r$, a contradiction. We showed that $I_i = (0)$. Recall that, by B. I. Plotkin’s theorem \cite{27}, the ring $L$ has a locally nilpotent radical $\text{Loc}(L)$. Let $i \geq 1$ be a minimal positive integer such that $I_i \subseteq \text{Loc}(L), i \leq l$. Suppose that $i \geq 2$. For an arbitrary element $a \in I_{i-1}$ the ideal $\text{id}_L(a)$ is abelian modulo $I_i$. Since the factor-ring $L/\text{Loc}(L)$ does not contain nonzero abelian ideals it follows that $a \in \text{Loc}(L), I_{i-1} \subseteq \text{Loc}(L)$, a contradiction.

We showed that $L = I_1 \subseteq \text{Loc}(L)$, in other words, the ring $L$ is locally nilpotent. This completes the proof of theorem \ref{thm:1.9} in the case when the ring $A$ is prime.

To finish the proof of proposition \ref{prop:1.6} we need just to repeat the arguments above. Let $A$ be a commutative domain, $L \subseteq \text{Der}(A)$ and $A_L$ is an order in $A$. We see that the subring $(A_1)_L$ is an order in the ring $A_1$ and, therefore, for any prime number $p$ the subring $(A_1/pA_1)_L$ is an order in $A_1/pA_1$. In the case of a prime characteristic proposition \ref{prop:1.6} was proved for an arbitrary finitely generated associative commutative ring, not necessarily a domain. Hence, we can apply it to $A_1/pA_1$, and finish the proof of proposition \ref{prop:1.6} following the proof of theorem \ref{thm:1.9} verbatim.

To tackle the semiprime case we will need the following lemma.

**Lemma 4.6.** Let $A$ be a finitely generated semiprime ring. Then there exists a finite family of ideals $I_1, \ldots, I_n < A$ such that each ideal $I_i, 1 \leq i \leq n$, is invariant
under $\text{Der}(A)$; each factor-ring $A/I_i$ is prime, and

$$\bigcap_{j=1}^n I_j = (0).$$

**Proof.** As we have mentioned in [2.4] the ring of fractions $\tilde{A} = (Z^*)^{-1}A$, where $Z^*$ is the set of all nonzero central elements of $A$ that are not zero divisors, is a direct sum $\tilde{A} = \tilde{A}_1 \oplus \cdots \oplus \tilde{A}_n$ of simple finite-dimensional over their centers algebras. Let $I_i = A \cap (\tilde{A}_1 + \cdots + \tilde{A}_{i-1} + \tilde{A}_{i+1} + \cdots + \tilde{A}_n)$, $1 \leq i \leq n$.

All direct summands $\tilde{A}_i$ are invariant under $\text{Der}(\tilde{A})$. An arbitrary derivation of the ring $A$ extends to a derivation of $\tilde{A}$. This implies that each ideal $I_i$ is invariant under $\text{Der}(A)$.

Let us prove that each factor-ring $A/I_i$ is prime. Suppose that $a, b \in A$ and $aAb \subseteq I_i$. We need to show that $a \in I_i$ or $b \in I_i$. The inclusion above implies that

$$aAb \subseteq \tilde{A}_1 + \cdots + \tilde{A}_{i-1} + \tilde{A}_{i+1} + \cdots + \tilde{A}_n.$$  

The factor-ring

$$\tilde{A}/(\tilde{A}_1 + \cdots + \tilde{A}_{i-1} + \tilde{A}_{i+1} + \cdots + \tilde{A}_n) \simeq \tilde{A}_i$$

is simple. Hence, at least one of the elements $a, b$ lies in $I_i$. It is straightforward that $I_1 \cap \cdots \cap I_n = (0)$. This completes the proof of the lemma. □

Now, we are ready to prove theorem 1.9 in the case when the ring $A$ is semiprime. Let $A$ be a finitely generated semiprime PI-ring. Let $L$ be a finitely generated Lie subring $L \subseteq \text{Der}(A)$ that consists of locally nilpotent derivations. Let $I_1, \ldots, I_n$ be the ideals of lemma 4.6. We showed above that there exists $r \geq 1$ such that

$$L^r(A/I_i) = (0), 1 \leq i \leq n.$$  

Hence,

$$L^r(A) \subseteq \bigcap_{i=1}^n I_i = (0) \quad \text{and} \quad L^r = (0).$$

This completes the proof of theorem 1.9 for semisimple rings.

**Lemma 4.7.** Let $A$ be a finitely generated PI-ring and let $L \subseteq \text{Der}(A)$ be a Lie ring that consists of locally nilpotent derivations. Let $I \preceq A$ be a differentially invariant ideal such that $I^2 = (0)$ and the image of the Lie ring $L$ in $\text{Der}(A/I)$ is locally nilpotent. Then the Lie ring $L$ is locally nilpotent.

**Proof.** Choose derivations $d_1, \ldots, d_n \in L$. We need to show that the Lie ring $L'$ generated by $d_1, \ldots, d_n$ is nilpotent. By the assumption of lemma 4.6, there exists $r \geq 1$ such that $L^r(A) \subseteq I$. Let $d \in L^r$ and let $a_1, \ldots, a_m$ be generators of the ring $A$. There exists an integer $l \geq 1$ such that $d^l(a_j) = 0$, $1 \leq j \leq m$. Let $v = a_{i_1} \cdots a_{i_s}$ be a product of generators in the ring $A$. Since $d(a_i)Ad(a_j) \subseteq I^2 = (0)$ it follows that

$$d^l(a_{i_1} \cdots a_{i_s}) = d^l(a_{i_1})a_{i_2} \cdots a_{i_s} + a_{i_1}d^l(a_{i_2}) \cdots a_{i_s} + \cdots + a_{i_1} \cdots a_{i_{s-1}}d^l(a_{i_s}) = 0.$$  

Hence $d^l = 0$.

Since the ring $L$ is weakly Engel by lemma 4.2, B. I. Plotkin’s theorem [27] implies that the Lie ring $L''$ is finitely generated. Hence, by [33] (see also [2.4]), the
Lie ring $L'$ is nilpotent and the Lie ring $L$ is solvable. Again by B. I. Plotkin’s theorem, the Lie ring $L'$ is nilpotent. This completes the proof of the lemma. □

Let us prove theorem 1.9 in the case when the ring $A$ does not have additive torsion.

Let $J$ be the Jacobson radical of the ring $A$. By 9, the radical $J$ is nilpotent. Let $J^n = (0), J^{n-1} \neq (0), n \geq 2$. It is well known that if the ring $A$ does not have additive torsion, then the radical $J$ is differentially invariant.

Let $I = \{a \in A \mid \text{there exists an integer } s \geq 1 \text{ such that } sa \in J^{n-1}\}.$

The ideal $I$ is differentially invariant. We claim that $I^2 = (0)$. Indeed, let $a, b \in I$. There exist integers $s_1, s_2 \geq 1$ such that $s_1a \in J^{n-1}, s_2b \in J^{n-1}$. Hence $s_1s_2ab \in (J^{n-1})^2 = (0)$. Since the ring $A$ does not have additive torsion it follows that $ab = 0$.

The Jacobson radical of the ring $A/I$ is $J/I, (J/I)^{n-1} = (0)$. The ring $A/I$ obviously does not have additive torsion. Hence, by inductive assumption on $n$, the image of $L$ in $\text{Der}(A/I)$ is locally nilpotent; and by lemma 4.7 the ring $L$ is locally nilpotent.

Now, we are ready to finish the proof of theorem 1.9.

Let $a_1, \ldots, a_m$ be generators of a PI-ring $A$. Let $L \subseteq \text{Der}(A)$ be a finitely generated Lie subring such that every derivation from $L$ is locally nilpotent. Let $T(A)$ be the ideal of $A$ that consists of elements of a finite additive order. Clearly, $T(A)$ is differentially invariant. The factor-ring $A/T(A)$ does not have an additive torsion. Hence, by the proof of theorem 1.9 in the case when the ring $A$ does not have additive torsion, the image of the ring $L$ in $\text{Der}(A/T(A))$ is nilpotent. Therefore, there exists $r \geq 1$ such that for any derivation $d \in L'$ we have $d(A) \subseteq T(A)$. Since the ring $L$ is finitely generated and weakly Engel by lemma 1.2 it follows from B. I. Plotkin’s theorem 27 that the Lie ring $L'$ is finitely generated.

We aim to show that the Lie ring $L'$ is nilpotent. Let $d'_1, \ldots, d'_j$ be generators of $L'$. There exists an integer $n \geq 1$ such that

$$nd'_i(a_j) = 0, \quad 1 \leq i \leq l, \quad 1 \leq j \leq m.$$ 

Hence, $nL'(A) = (0)$. For a prime number $p$, consider the ideal

$$I_p = \{a \in A \mid \text{there exists an integer } t \geq 1 \text{ such that } p^t a = 0\}.$$

Let $a \in I_p, d \in L'$. Then $nd(a) = 0$ and $p^t d(a) = 0$ for some $t \geq 1$. Hence, for a prime number $p$ not dividing $n$, we have $L'I_p = (0)$. This allows us to consider the factor-ring $A/\sum_{p|n} I_p$ instead of $A$. In other words, we will assume that for a prime number $p$ not dividing $n$ the ring $A$ does not have a $p$-torsion.

Let $p_1, \ldots, p_s$ be all distinct prime divisors of $n$. Then

$$T(A) = I_{p_1} \oplus \cdots \oplus I_{p_s}.$$ 

Let $s \geq 2$. Inducting on the integer $n$ we can assume that the image of the Lie ring $L$ in each $\text{Der}(A/I_{p_i})$ is nilpotent. In other words, there exists a number $r_i \geq 1$ such that $L^{r_i}(A) \subseteq I_{p_i}$. This implies

$$L^{\text{max}(r_1, r_2)}(A) \subseteq I_{p_1} \cap I_{p_2} = (0).$$ 

Therefore, we assume that $T(A) = I_p$ for some prime number $p$. The ideal $pI_p$ lies in the Jacobson radical of $A$ and $pI_p$ is differentially invariant. Let $(pI_p)^q = (0), q \geq 1$. If $q \geq 2$, then inducting on $q$ we can assume that the image of the Lie ring $L$ in
Der\((A/(pI_p)^{q-1})\) is nilpotent. Hence, the ideal \((pI_p)^{q-1}\) satisfies the assumptions of lemma 4.7 Suppose, therefore, that \(q = 1\), \(pI_p = (0)\), \(n = p\). Now, we have \(pL^r(A) = (0)\). This implies that for an arbitrary derivation \(d \in L^r\) every \(p\)-power \(dp^k\) is again a derivation. Indeed,

\[
dp^k(ab) = \sum_{i=0}^{p^k} \binom{p^k}{i} dp^k(a)d^{p^k-1}(b)
\]

for arbitrary elements \(a, b \in A\). If \(0 < i < p^k\), then the binomial coefficient \(\binom{p^k}{i}\) is divisible by \(p\), hence

\[
\binom{p^k}{i} dp^k(a) = 0,
\]

which implies \(dp^k(ab) = dp^k(a)b + adp^k(b)\).

Choosing \(d \in L^r\) and arguing as above, we find \(p^k\) such that \(dp^k(a_j) = 0, 1 \leq j \leq m\), therefore, \(dp^k = 0\). The Lie ring \(L^r\) is finitely generated, PI, and an arbitrary derivation from \(L^r\) is nilpotent. By [43], the Lie ring \(L^r\) is nilpotent. The ring \(L\) is solvable, hence, by the result of B. I. Plotkin [27], it is nilpotent. This completes the proof of theorem 4.9.

Now, our aim is to prove theorem 1.5. In the rest of this section, we assume that \(A\) is a commutative domain; \(L \subseteq \text{Der}(A)\) is a Lie ring that consists of locally nilpotent derivations; \(K\) is the field of fractions of the domain \(A\), and \(\dim_K KL < \infty\). Our aim is to prove that the Lie ring \(L\) is locally nilpotent. Let

\[
KL = \sum_{i=1}^{n} Kd_i, \quad d_i \in L, \quad \text{and} \quad [d_i, d_j] = \sum_{t=1}^{n} c_{ijt}d_t, \quad c_{ijt} = \frac{a_{ijt}}{b_{ijt}},
\]

where \(a_{ijt}, b_{ijt} \in A\). Enlarging the set \(\{d_1, \ldots, d_n\}\) if necessary we will assume that the derivations \(d_1, \ldots, d_n\) generate \(L\), that is, \(L = \text{Lie}_K(d_1, \ldots, d_n)\). Let \(d_1 \cdots d_m\) be a product in the associative ring of additive endomorphisms of the field \(K\). We call this product ordered if \(i_1 \leq i_2 \leq \cdots \leq i_m\). Let \(\mathcal{P}\) denote the set of all ordered products of derivations \(d_1, \ldots, d_n\) including the empty product, i.e. the identity operator.

**Lemma 4.8.** For an arbitrary element \(a \in A\) the set of ordered products \(v = d_{i_1} \cdots d_{i_m} \in \mathcal{P}\) such that \(v(a) \neq 0\), is finite.

**Proof.** Let

\[
v = d_1^{k_1}d_2^{k_2} \cdots d_n^{k_n}, \quad \text{where} \quad k_i \text{ are nonnegative integers}.
\]

There exists an integer \(q_n \geq 1\) such that \(d_{i_n}^{q_n}(a) = 0\). Hence, if \(v(a) \neq 0\), then \(k_n < q_n\). Similarly, there exists \(q_{n-1} \geq 1\) such that

\[
d_{i_{n-1}}^{q_{n-1}}d_{i_n}^{q_n}(a) = 0 \quad \text{for all} \quad 0 \leq i \leq q_{i-1}.
\]

Hence, \(v(a) \neq 0\) implies \(k_n < q_n, k_{n-1} < q_{n-1}\) and so on. This completes the proof of the lemma.

Consider the set \(C = \{c_{ijt}\}_{i,j,t} \subset K\); see [155].

**Lemma 4.9.** An arbitrary product \(d_{i_1} \cdots d_{i_r}\) can be represented as

\[
d_{i_1} \cdots d_{i_r} = \sum \pm (v_1(c_1)) \cdots (v_s(c_s)) v_0,
\]
where in each summand the operators \( v_0, v_1, \ldots, v_s \) lie in \( \mathcal{P} \) and elements \( c_1, \ldots, c_s \) lie in \( C \).

**Proof.** For a product \( v = d_i \cdots d_k \) let \( l \) be the number of \( 1 \leq k \leq r - 1 \), such that \( i_k > i_{k+1} \). Let \( \nu(v) = (r, l) \). We will compare pairs \( (r, l) \) lexicographically and use induction on \( \nu(v) \). Let \( i = i_k > i_{k+1} = j \). Then

\[
d_i d_j = d_j d_i + \sum_{t} c_{ijt} d_t.
\]

Clearly,

\[
\nu(d_i \cdots d_{i-1} d_j d_i d_{i+1} \cdots d_r) < \nu(v).
\]

Consider the product

\[
d_i \cdots d_{i-1} c_{ijt} d_i d_{i+1} \cdots d_r.
\]

Commuting the element \( c_{ijt} \) with derivations \( d_i, \ldots, d_{i-1} \) we get

\[
d_i \cdots d_{i-1} c_{ijt} = \sum (v'(c_{ijt})) v'\nu,
\]

where \( v', v'' \) are products of derivations \( d_i, \ldots, d_{i-1} \) of total length \( k - 1 \). Hence,

\[
d_i \cdots d_{i-1} c_{ijt} d_i d_{i+1} \cdots d_r = \sum \pm(v'(c_{ijt})) v'' d_i d_{i+1} \cdots d_r.
\]

In each summand the lengths of products \( v' \) and \( v'' d_i d_{i+1} \cdots d_r \) are less than \( r \). Applying the induction assumption to these products, we get the assertion of the lemma. \( \square \)

Consider the subring \( \tilde{A} \) of the field \( K \) generated by the elements

\[
v(a_{ijt}), \quad v(b_{ijt}), \quad v(b_{ijt})^{-1}; \quad v \in \mathcal{P}; \quad i, j, t \geq 1.
\]

By lemma 4.8 the ring \( \tilde{A} \) is finitely generated.

**Lemma 4.10.** The subring \( \tilde{A} \) is invariant under the action of \( L \).

**Proof.** For an arbitrarily ordered product of derivations \( v \in \mathcal{P} \) we have

\[
v(b_{ijt}^{-1}) = \sum b_{ijt}^{-1} v(b_{ijt}^{-1}) = \sum_1 \frac{1}{b_{ijt}^{-1}} v_1 v_2 \cdots v_s v_{ijt},
\]

where \( m \geq 1; v_1, \ldots, v_s \in \mathcal{P} \), and

\[
v(c_{ijt}) = v(a_{ijt} \cdot b_{ijt}^{-1}) = \sum v'(a_{ijt}) v''(b_{ijt}^{-1}).
\]

These equalities imply \( v(c_{ijt}) \in \tilde{A} \). Now, by lemma 4.9 the ring \( \tilde{A} \) is invariant under the action of \( L \). \( \square \)

The ring \( \tilde{A} \) is generated by elements \( v(a_{ijt}), v(b_{ijt}) \in A \cap \tilde{A} \) and elements \( v(b_{ijt})^{-1} \). Hence, an arbitrary element of the ring \( \tilde{A} \) can be represented as a ratio \( a/b \) where \( a, b \in A \cap \tilde{A} \). Hence, \( A \cap \tilde{A} \) is an order in the ring \( \tilde{A} \), and the multiplicative semigroup \( S \) being generated by elements \( v(b_{ijt}) \neq 0 \).

By proposition 4.6 the image of the ring \( L \) in \( \text{End}_k(\tilde{A}) \) is a nilpotent Lie ring. Hence, there exists an integer \( r \geq 1 \) such that \( L^r(\tilde{A}) = (0) \). By lemma 4.3 and Plotkin’s theorem [27], the Lie ring \( L^r \) is finitely generated. Consider the subfield

\[
K_0 = \{ \alpha \in K \mid L^r(\alpha) = (0) \}.
\]
The $K_0$-algebra $A' = K_0 A \subseteq K$ is a domain. The field $K_0$ is invariant under the action of $L$, hence the $K_0$-algebra $A'$ is invariant as well.

Let $L'$ be the image of the Lie ring $L'$ in $\text{End}_Z(A')$. Since all the coefficients $c_{ijt}$ lie in $K_0$ the product $K_0 L$ is a Lie ring and a finite-dimensional vector space over $K_0$. This implies that $K_0 L'$ is a finite-dimensional $K_0$-algebra. Now, Petravchuk-Sysak theorem (see [20]) implies that $L'$ is a nilpotent Lie ring. Again, by lemma 4.3 and B. I. Plotkin’s theorem, the Lie ring $L$ is nilpotent. This completes the proof of theorem 1.5.

We will finish with examples showing that corollary 1.2 of theorem 1.1 and theorem 1.4 are wrong for countably generated algebras. Let $F$ be an arbitrary field and let $A = F[x_1, x_2, \ldots]$ be the polynomial algebra on countable many generators. We will construct

(i) a Lie algebra $L \subseteq \text{Der}(A)$ that consists of locally nilpotent derivations and is not locally nilpotent,

(ii) a torsion group $G < \text{Aut}(A)$ that is not locally finite.

Consider the countable-dimensional vector space $V = \sum_{i \geq 1} F x_i$. There exists a countable finitely generated Lie algebra $L$ such that every operator $\text{ad}(a), a \in L,$ is nilpotent, and the algebra $L$ has zero center (see [14, 20]). The mapping $L \to \mathfrak{gl}(L)$, $a \mapsto \text{ad}(a), a \in L,$ is an embedding of the Lie algebra $L$ in $\mathfrak{gl}(L)$ and every linear transformation $\text{ad}(a)$ from the image of $L$ is nilpotent. Therefore, we can suppose that $L \subseteq \mathfrak{gl}(V)$ and every linear transformation from $L$ is nilpotent. An arbitrary linear transformation on $V$ is a restriction of a derivation from

$$\sum_{i \geq 1} V \frac{\partial}{\partial x_i}.$$ 

Hence, we assume that

$$L \subseteq \sum_{i \geq 1} V \frac{\partial}{\partial x_i} \subseteq \text{Der}(A).$$

Since every derivation from $L$ acts nilpotently on $V$ it follows that it acts locally nilpotently on $A$. Similarly, there exists a finitely generated torsion group $G < \text{Aut}(V)$ that is not locally finite (see [14, 20, 24, 25]). Every linear transformation $\varphi \in GL(V)$ uniquely extends to an automorphism $\tilde{\varphi} \in \text{Aut}(A)$ Thus the mapping $GL(V) \to \text{Aut}(A), \varphi \mapsto \tilde{\varphi},$ is an embedding of groups. Hence, $G$ is a torsion not locally finite subgroup of $\text{Aut}(A)$.

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