Set Representation of Partial Dynamic De Morgan algebras

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Abstract—By a De Morgan algebra is meant a bounded poset equipped with an antitone involution considered as negation. Such an algebra can be considered as an algebraic axiomatization of a propositional logic satisfying the double negation law. Our aim is to introduce the so-called tense operators in every De Morgan algebra for to get an algebraic counterpart of a tense logic with negation satisfying the double negation law which need not be Boolean.

Following the standard construction of tense operators $G$ and $H$ by a frame we solve the following question: if a dynamic De Morgan algebra is given, how to find a frame such that its tense operators $G$ and $H$ can be reached by this construction.

Index Terms—De Morgan lattice, De Morgan poset, semi-tense operators, tense operators, (partial) dynamic De Morgan algebra.

INTRODUCTION

Dynamic De Morgan algebras were already investigated by the authors in [7]. The reached theory is good enough for a description of tense operators in a logic satisfying the double negation law when a frame is given as well for the task to determine a frame provided tense operators are given.

When studying partial dynamic De Morgan algebras, we are given a De Morgan poset and we solve both the questions mentioned above. For this, we have to modify our original definition by axioms which are formulated in the language of ordered sets with involution only. On the other hand, we have an advantage of using the algebraic tools introduced already in [7] which can essentially shorten our paper.

For the reader convenience we repeat that tense operators are introduced for to incorporate the time dimension in the logic under consideration. It means that our given logic is enriched by the operators $G$ and $H$, see e.g. [3] for the classical logic and [9], [10], [4] for several non-classical logics.

It is worth noticing that the operators $G$ and $H$ can be considered as certain kind of modal operators which were already studied for intuitionistic calculus by D. Wijesekera [17] and in a general setting by W.B. Ewald [12]. For the logic of quantum mechanics (see e.g. [11] for details of the so-called quantum structures), the underlying algebraic structure is e.g. an orthomodular lattice or the so-called effect algebra (see [11], [14]) and the corresponding tense logic was treated in [5], [6], [8], [15], in a bit more general setting also in [2].

The paper is organized as follows. After introducing several necessary algebraic concepts, we introduce tense operators in a De Morgan poset, i.e., in an arbitrary logic satisfying double negation law without regards what another logical connectives are considered. Moreover, in this logic neither the principle of contradiction, nor the principle of excluded middle are valid for the negation, but all De Morgan laws hold. Also we get a simple construction of tense operators which uses lattice theoretical properties of the underlying ordered set. In Section II we outline the problem of a representation of partial dynamic De Morgan algebras and we solve it for partial dynamic De Morgan algebras satisfying natural assumptions. This means that we get a procedure how to construct a corresponding frame to be in accordance with the construction from Section I. In particular, any dynamic De Morgan algebra is set representable.
I. Preliminaries and Basic Facts

We refer the reader to [1] for standard definitions and notations for lattice structures.

**Definition I.1.** A structure $A = (A; \leq', 0, 1)$ ($A = (A; \land, \lor', 0, 1)$) is called a De Morgan poset (De Morgan lattice) if $(A; \leq, 0, 1)$ is a poset ($(A; \land, \lor, 0, 1)$ is a lattice) with the top element 1 and the bottom element 0 and $'$ is a unary operation called negation with properties $a \leq b \Rightarrow b' \leq a'$ and $a = a''$.

In fact in a De Morgan poset $A$ we have $a \leq b$ iff $b' \leq a'$, because $a \leq b \Rightarrow b' \leq a' \Rightarrow a'' \Rightarrow a \leq b$.

A morphism $f: A \rightarrow B$ of bounded posets (De Morgan posets) is an order, (negation), top element and bottom element preserving map. A morphism $f: A \rightarrow B$ of bounded posets is order reflecting if $(f(a) \leq f(b)$ if and only if $a \leq b$) for all $a, b \in A$.

Let $h: A \rightarrow B$ be a partial mapping of De Morgan posets. We say that the partial mapping $h^\partial: A \rightarrow B$ is the dual of $h$ if $h^\partial(a)$ is defined for all $a \in A$ such that $a' \in \text{dom } h$ in which case

$$h^\partial(a) = h(a').$$

![Fig. 1. The poset of Example I.2](image)

**Example I.2.** The De Morgan poset $M = (M; \leq, \lor', 0, 1), M = \{0, a, b, c, d, 1\}$ displayed by the Hasse diagram in Figure 1 is the smallest non-lattice De Morgan poset.

**Observation I.3** ([6]). Let $A, B$ be bounded posets (De Morgan posets), $T$ a set of morphisms from $A$ to $B$ of bounded posets (De Morgan posets). The following conditions are equivalent:

(i) $((\forall t \in T) t(a) \leq t(b)) \Rightarrow a \leq b$ for any elements $a, b \in A$;

(ii) The morphism $i^T_A: A \rightarrow B^T$ defined by $i^T_A(a) = (t(a))_{t \in T}$ for all $a \in A$ is order reflecting.

We then say that $T$ is a full set of order preserving maps with respect to $B$. We may in this case identify $A$ with a subposet (sub-De Morgan poset) of $B^T$ since $i^T_A$ is an order reflecting morphism of bounded posets (De Morgan posets).

A pair $(f, g)$ of order-preserving mappings $f: A \rightarrow B$ and $g: B \rightarrow A$ between posets $A$ and $B$ is a Galois connection or an adjunction between $A$ and $B$ provided that $f(a) \leq b$ if and only if $a \leq g(b)$ for all $a, b \in A, B$. In an adjunction $(f, g)$ the mapping $f$ is called the left adjoint and the mapping $g$ is called the right adjoint. The pair $(f, g)$ of order-preserving mappings $f: A \rightarrow B$ and $g: B \rightarrow A$ is an adjunction if and only if $a \leq g(f(a))$ and $f(g(b)) \leq b$ for all $a, b \in A, B$.

The second concept which will be used are so-called tense operators. They are in certain sense quantifiers which quantify over the time dimension of the logic under consideration. These tense operators were firstly introduced as operators on Boolean algebras (see [3] for an overview). Chajda and Paseka introduced in [7] the notion of a dynamic De Morgan algebra.

The following notion of a partial dynamic De Morgan algebra is stronger than the notion introduced in [7] but for dynamic De Morgan algebras both notions coincide. Note only that our condition (P1) combined with the condition (T4) for tense De Morgan algebras in the sense of [13] yields our condition (P4).

**Definition I.4.** By a partial dynamic De Morgan algebra is meant a triple $D = (A; G, H)$ such that $A = (A; \leq', 0, 1)$ is a De Morgan poset with negation $'$ and $G, H$ are partial mappings of $A$ into itself satisfying

(P1) $G(0) = 0, G(1) = 1, H(0) = 0$ and $H(1) = 1$.

(P2) $x \leq y$ implies $G(x) \leq G(y)$ whenever $G(x), G(y)$ exist, and $H(x) \leq H(y)$ whenever $H(x), H(y)$ exist.

(P3) $x \leq GP(x)$ whenever $H(x')$ exists, $P(x) = H^\partial(x)$ and $GP(x)$ exists, and $x \leq HF(x)$ whenever $G(x')$ exists, $F(x) = G^\partial(a)$ and $HF(x)$ exists.
logical calculus. We say that \( G \) and \( H \) exist, and \( H(x) \leq P(y) \) whenever \( H(y') \) and \( H(x) \) exist.

Just defined \( G \) and \( H \) will be called **tense operators** of a partial dynamic De Morgan algebra \( D \). If both \( G \) and \( H \) are total we will speak about a **dynamic De Morgan algebra**.

If we omit the condition (P3), i.e., only the conditions (P1), (P2) and (P4) are satisfied we say that \( G \) and \( H \) are **semi-tense operators** on \( A \).

If \( (A_1; G_1, H_1) \) and \( (A_2; G_2, H_2) \) are partial dynamic algebras, then a **morphism of partial dynamic algebras** \( f: (A_1; G_1, H_1) \rightarrow (A_2; G_2, H_2) \) is a morphism of De Morgan posets such that \( f(G_1(a)) = G_2(f(a)) \), for any \( a \in A_1 \) such \( G_1(a) \) is defined and \( f(H_1(b)) = H_2(f(b)) \), for any \( b \in A_1 \) such \( H_1(b) \) is defined.

Partial dynamic De Morgan algebra \( D = (A; G, H) \) is called **complete** if its reduct \((A; \leq, 0, 1)\) is a complete lattice.

The semantical interpretation of these tense operators \( G \) and \( H \) is as follows. Consider a pair \((T, \leq)\) where \( T \) is a non-void set and \( \leq \) is a partial order on \( T \). Let \( s \in T \) and \( f(s) \) be a formula of a given logical calculus. We say that \( G(f(t)) \) is **valid** if for any \( s \geq t \) the formula \( f(s) \) is valid. Analogously, \( H(f(t)) \) is valid if \( f(s) \) is valid for each \( s \leq t \). Thus the unary operators \( G \) and \( H \) constitute an algebraic counterpart of the tense operations “it is always going to be the case that” and “it has always been the case that”, respectively. Similarly, the operators \( F \) and \( P \) can be considered in certain sense as existential quantifiers “it will at some time be the case that” and “it has at some time been the case that”.

In what follows we want to provide a meaningful procedure giving tense operators on every De Morgan poset which will be in accordance with an intuitive idea of time dependency.

By a **frame** (see e.g. [10]) is meant a couple \( (T, R) \) where \( T \) is a non-void set and \( R \) is a binary relation on \( T \). Furthermore, we say that \( R \) is **serial** for all \( x \in T \) there is \( y \in T \) such that \( x \mathrel{R} y \). In particular, every reflexive relation is serial. The set \( T \) is considered to be a **time scale**, the relation \( R \) expresses a relationship “to be before” and “to be after”. Having a De Morgan poset \( A = (A; \leq, 0, 1) \) and a non-void set \( T \), we can produce the direct power \( A^T = (A^T; \leq', o, j) \) where the relation \( \leq \) and the operation \( ' \) are defined and evaluated on \( p, q \in A^T \) componentwise, i.e. \( p \leq q \) if \( p(t) \leq q(t) \) for each \( t \in T \) and \( p'(t) = p(t)' \) for each \( t \in T \). Moreover, \( o \) and \( j \) are such elements of \( A^T \) that \( o(t) = 0 \) and \( j(t) = 1 \) for all \( t \in T \).

**Theorem I.5.** [7] Theorem II.7,Corollary II.9] Let \( M = (M; \leq, 0, 1) \) be a complete De Morgan algebra and let \( (T, R) \) be a frame. Define mappings \( \hat{G}, \hat{H} \) of \( M^T \) into itself as follows: For all \( p \in M^T \) and all \( x \in T \),

\[
\hat{G}(p)(x) = \bigwedge_M \{ p(y) \mid x \mathrel{R} y \} \quad \text{and} \quad \hat{H}(p)(x) = \bigwedge_M \{ p(y) \mid y \mathrel{R} x \}.
\]

Then

(a) \( \hat{G}, \hat{H} \) are total operators on \( M^T \).
(b) If \( R \) is serial then \( \hat{G} \) is a semi-tense operator.
(c) If \( R^{-1} \) is serial then \( \hat{H} \) is a semi-tense operator.
(d) If \( R \) and \( R^{-1} \) are serial then \( D = (M^T; \hat{G}, \hat{H}) \) is a dynamic De Morgan algebra.

We say that the operators \( \hat{G} \) and \( \hat{H} \) on \( M^T \) are **constructed by means of** \((T, R)\).

**II. SET REPRESENTATION OF PARTIAL DYNAMIC DE MORGAN ALGEBRAS**

In Theorem [15] we presented a construction of natural tense operators when a De Morgan poset and a frame are given. However, we can ask, for a given partial dynamic De Morgan algebra \( (A; G, H) \), whether there exist a frame \((T, R)\) and a complete De Morgan lattice \( M = (M; \leq, 0, 1) \) such that the tense operators \( G, H \) can be derived by this construction where \( (A; G, H) \) is embedded into the power algebra \( (M^T; \hat{G}, \hat{H}) \). Hence, we ask, that there exists a suitable set \( T \) and a binary relation \( R \) on \( T \) such that if every element \( p \) of \( A \) is in the form \((p(t))_{t \in T} \) in \( M^T \) then \( G(p)(s) = \bigwedge_M \{ p(t) \mid s \mathrel{R} t \} \) for all \( p \in \text{dom} \, G \) and \( s \in T \), and \( H(p)(s) = \bigwedge_M \{ p(t) \mid t \mathrel{R} s \} \) for all \( p \in \text{dom} \, H \) and \( s \in T \). If such a representation exists then one can recognize the time variability of elements of \( A \) expressed as time dependent functions \( p: T \rightarrow M \) and \( (A; G, H) \) is said to be **representable in** \( M \) with respect to \( T \).
In what follows, we will show that there is a set representation theorem for (partial) dynamic De Morgan algebras.

Let us start with the following example.

**Example II.1.** Let $2 = \{\{0, 1\}; \lor, \land, 0, 1\}$ be a two-element Boolean algebra. We will denote by $M_2 = (M_2; \leq, 0, 1)$ a complete De Morgan lattice such that $M_2 = \{0, 1\} \times \{0, 1\}$, $(M_2; \lor, \land, 0, 1)$ is a lattice reduct of the Boolean algebra $2 \times 2$ with the induced order $\leq$ and the negation on $M_2$ is defined by $(a, b) = (b', a')$. Let $(T, R)$ be a frame. Let the operators $\widehat{G}$ and $H$ on $M_T^D$ be constructed by means of $(T, R)$. Then by Theorem I.5 we have that $(M_T^D; \widehat{G}, H)$ is a complete dynamic De Morgan algebra. Moreover, $M_T^D$ is isomorphic to a lattice to a De Morgan algebra of sets.

![Fig. 2. Figure of the underlying poset of the complete De Morgan lattice $M_2$ from Example II.1](image)

Recall that the four-element De Morgan poset $M_2$, considered as a distributive De Morgan lattice, generates the variety of all distributive De Morgan lattices (see e.g. [1, 16]). This result was a motivation for our study of the representation theorem of De Morgan posets.

For any De Morgan poset $A = (A; \leq, 0, 1)$, we will denote by $T_A^D$ a set of morphisms of De Morgan poset into the four-element De Morgan poset $M_2$. The elements $\kappa_D: A \to M_2$ of $T_A^D$ (indexed by proper down-sets $D$) correspond to morphisms of bounded posets $h_D: A \to \{0, 1\}$ such that $h_D(a) = 0 \iff a \in D$ are morphisms of De Morgan posets defined by the prescription $\kappa_D(a) = (h_D(a), h_D(a)^\partial)$ for all $a \in A$.

As we will see later, this set $T_A^D$ will serve as our time scale. Hence, our next task is to determine a binary relation $R$ on $T_A^D$ such that the couple $(T_A^D, R)$ will be our appropriate frame.

Let us denote, for any proper down-set $D$ of $A$, by $\partial(D)$ the set $A \setminus \{d' \mid d \in D\}$. Then $\partial(D)$ is again a proper down-set of $A$ such that $h_D^\partial = h_{\partial(D)}$ and we have $h_D = h_{\partial(D)}^\partial$.

To simplify the notation we will use for elements of $T_A^D$ letters $s$ and $t$ whenever we will need not their concrete representation via down-sets.

**Proposition II.2.** Let $A = (A; \leq, 0, 1)$ be a De Morgan poset. Then the map $i_A: A \to M_2^A$ given by $i_A(a)(s) = s(a)$ for all $a \in A$ and all $s \in T_A^D$ is an order-reflecting morphism of De Morgan posets such that $i_A(A)$ is a De Morgan subposet of $M_2^A$.

**Proof.** First, let us show that, for any proper down-sets $D$ of $A$, the mapping $\kappa_D: A \to M_2$ is a morphism of De Morgan posets. Since both $h_D$ and $h_D^\partial$ are order-preserving we have that $\kappa_D$ is order-preserving. Now, let us compute

$$h_D^\partial(0) = (h_D(0'))' = (h_D(1))' = 1' = 0,$$

$$h_D^\partial(1) = (h_D(1'))' = (h_D(0'))' = 0' = 1.$$

It follows that both $h_D$ and $h_D^\partial$ preserve 0 and 1, i.e., they are morphisms of bounded posets. Hence $\kappa_D$ is a morphism of bounded posets.

Let $a \in A$. Let us check that $\kappa_D(a') = \kappa_D(a)$.

We have

$$\kappa_D(a') = (h_D(a), h_D^\partial(a))' = (h_D(a'), h_D(a')') = (h_D(a'), h_D^\partial(a')) = \kappa_D(a').$$

This yields that $\kappa_D$ is a morphism of De Morgan posets.

It remains to check that $T_A^D$ a full set of morphisms. Let $a, b \in A$ such that $s(a) \leq s(b)$ for all $s \in T_A^D$. In particular, $h_D(a) \leq h_D(b)$ for all proper down-sets $D$ of $A$. Put $D = \{x \in A \mid x \leq b\}$. Since $h_D(b) = 0$ we have $h_D(a) = 0$, i.e., $a \in D$ and hence $a \leq b$.

The next theorem solves our problem of finding the binary relation $R$ on $T_A^D$. In fact, we are restricted here on a semi-tense operator $G$ only.

**Theorem II.3.** Let $A = (A; \leq, 0, 1)$ be a De Morgan poset, $G: A \to A$ a semi-tense operator on $A$. Let us put

$$R_G = \{(s, t) \in T_A^D \times T_A^D \mid (\forall x \in \text{dom}(G))(s(G(x)) \leq t(x))\}.$$
Then \((T_A^{DMP}, R_G)\) is a frame with \(R\) being serial. Let \(\hat{G}\) be the operator constructed by means of the frame \((T_A^{DMP}, R_G)\) and let us put \(\hat{F} = \hat{G}^\circ\). Then the mapping \(i_A\) is an order-reflecting morphism of De Morgan posets into the complete De Morgan lattice \(M_2^{TA}\) such that the following diagram commutes:

\[
\begin{array}{ccc}
A & \xrightarrow{F} & A \\
\downarrow i_A & & \downarrow i_A \\
M_2^{T_A^{DMP}} & \xrightarrow{\hat{F}} & M_2^{T_A^{DMP}} \\
\end{array}
\]

\[
\begin{array}{ccc}
A & \xrightarrow{G} & A \\
\downarrow i_A & & \downarrow i_A \\
M_2^{T_A^{DMP}} & \xrightarrow{\hat{G}} & M_2^{T_A^{DMP}} \\
\end{array}
\]

**Proof.** First, let us verify that the following holds:

1) for all \(b \in \text{dom}(G)\) and for all \(s \in T_A^{DMP}\),
\[s(G(b)) = \bigwedge_{M_2} \{t(b) \mid s \subseteq R_G t\},\]
2) for all \(b \in \text{dom}(F)\) and for all \(s \in T_A^{DMP}\),
\[s(F(b)) = \bigvee_{M_2} \{s(b) \mid s \subseteq R_G t\}.

Let us first check statement 1.

Assume that \(b \in \text{dom}(G)\) and \(s \in T_A^{DMP}\), \(s = \kappa_D\) where \(D\) is a proper down-set of \(A\). Then, for all \(t \in T_B^{DMP}\) such that \(s \subseteq R_G t\), \(s(G(b)) \leq t(b)\). Hence

\[(h_D(G(b)), h_D^\circ(G(b))) \subseteq \bigwedge_{M_2} \{t(b) \mid s \subseteq R_G t\}.

To get the other inequality assume that \(h_D(G(b)) = 0\) or \(h_D^\circ(G(b)) = 0\).

Assume that \(h_D(G(b)) = 0\). Put \(V = \{z \in A \mid (\exists x \in \text{dom}(G))(z \geq x \text{ and } h_D(G(x)) = 1\}\) and \(X = \{z \in A \mid (\exists y \in \text{dom}(F))(z \leq y \text{ and } h_D(F(y)) = 0\}\). Then \(V\) is a proper upper subset of \(A\), \(1 \in V\), and \(X\) is a proper down-set of \(A\). \(0 \in X\) such that \(X \cap V = \emptyset\) and \(b \in X\). To verify this, assume that there is an element \(z \in X \cap V\). Then there is \(x \in \text{dom}(G)\), \(x \leq z\) such that \(h_D(G(x)) = 1\). Also, we have that there is \(y \in \text{dom}(F)\), \(z \leq y\) such that \(h_D(F(y)) = 0\) or \(x \leq z \leq b\). It follows that \(x \leq y\) and \(1 = h_D(G(x)) \leq h_D(F(y)) = 0\) or \(h_D(G(b)) = 1\), a contradiction.

Let \(U\) be a maximal down-set of \(A\) including \(X\) such that \(V \cap U = \emptyset\). Hence \(U\) determines a morphism \(h_U: A \rightarrow \{0, 1\}\) of bounded posets such that \(h_U(z) = 0\) for all \(z \in X\) and \(h_U(z) = 1\) for all \(z \in V\), i.e., \(h_D(G(x)) \leq h_U(x)\) for all \(x \in \text{dom}(G)\). Let us check that \(h_D^\circ(G(x)) \leq h_U^\circ(x)\) for all \(x \in \text{dom}(G)\). Assume that \(1 = h_D^\circ(G(x)) = h_D(F(x'))'\). Then \(h_D(F(x')) = 0\), i.e., \(x' \in X\). It follows that \(h_U(x') = 0\), i.e., \(h_U^\circ(x) = 1\). But this yields that \((\forall x \in \text{dom}(G))(s(G(x)) \leq \kappa_U(x))\), i.e., \(s \subseteq R_G \kappa_U = (h_U^\circ, h_U^\circ\circ)\) and \(h_U(b) = 0\).

Assume now that \(h_D^\circ(G(b)) = h_{\partial(D)}(G(b)) = 0\). As above, there is a maximal downset \(W\) of \(A\) such that \(h_{\partial(W)}^\circ(b) = h_{\partial(W)}(b) = 0\),

\[h_{\partial(D)}(G(x)) \leq h_W(x) \text{ and } h_{\partial(D)}^\circ(G(x)) \leq h_W^\circ(x)\]

for all \(x \in \text{dom}(G)\). It follows that \(h_D(G(x)) \leq h_{\partial(W)}(x)\) and \(h_D^\circ(G(x)) \leq h_{\partial(W)}^\circ(x)\) for all \(x \in B\), i.e., \(s \subseteq R_G \kappa_{\partial(W)} = (h_{\partial(W)}^\circ, h_{\partial(W)}^\circ\circ)\) and \(h_{\partial(W)}(b) = 0\).

Consequently, \(s(G(b)) \subseteq \bigwedge_{M_2} \{t(b) \mid s \subseteq R_G t\}\).

Let us check Statement 2. We have

\[s(F(b)) = s(G(b'))' = s(G(b'))' = \bigwedge_{M_2} \{t(b') \mid s \subseteq R_G t\} = \bigvee_{M_2} \{t(b) \mid s \subseteq R_G t\} = \bigvee_{M_2} \{t(b) \mid s \subseteq R_G t\}.

It remains to verify that \(R_G\) is serial. Let \(s \in T_A^{DMP}\). We know from (P1) that \(0 = s(G(0)) = \bigwedge_{M_2} \{t(0) \mid s \subseteq R_G t\}\). The set \(\{t \in T_A^{DMP} \mid s \subseteq R_G t\}\) is non-empty (otherwise one has \(0 = s(G(0)) = 1\), a contradiction).

In what follows, we show that if \(G\) and \(H\) are semi-tense operators such that the induced relations \(R_G\) and \(R_H\) satisfy a natural condition \(R_G = (R_H)^{-1}\) then the obtained frame is just the one we asked for.

**Theorem II.4.** Let \(A = (A; \leq, ', 0, 1)\) be a De Morgan poset, \(G, H: A \rightarrow A\) be semi-tense operators on \(A\) such that \(R_G = (R_H)^{-1}\). \(H\) is a frame with \(R_G\) and \(R_H\) serial. Let \((M_2^{TA}, \hat{\partial}, \hat{H})\) be the dynamic De Morgan algebra constructed by means of the frame \((T_A^{DMP}, R_G)\). Then the mapping \(i_A\) is an order-reflecting morphism of De Morgan posets into the complete De Morgan lattice \(M_2^{TA}\) such that the following diagram commutes:

\[
\begin{array}{ccc}
A & \xrightarrow{H} & A \\
\downarrow i_A & & \downarrow i_A \\
M_2^{T_A^{DMP}} & \xrightarrow{\hat{H}} & M_2^{T_A^{DMP}} \\
\end{array}
\]
Proof. It immediately follows from Theorem II.5 and Theorem II.3.

The following theorem gives us a complete solution of our problem established in the beginning of this section. This is a new result showing that the partial dynamic De Morgan algebra can be equipped with the corresponding frame similarly as it is known for Boolean algebras in [3] at least in a case when the tense operators $G$ and $H$ are interrelated. In particular, any dynamic De Morgan algebra has such a frame.

**Theorem II.5.** Let $(A; G, H)$ be a partial dynamic De Morgan algebra such that

(a) $x \in \text{dom}(G)$ implies $G(x) \in \text{dom}(H)$,
(b) $x \in \text{dom}(H)$ implies $H(x) \in \text{dom}(G)$.

Then $(T_A^{\text{DMP}}, R_G)$ is a frame with $R_G$ and $(R_G)^{-1}$ serial such that $(A; G, H)$ can be embedded into $(M_2^{\text{DMP}}, \hat{G}, \hat{H})$.

**Proof.** It is enough to check that $R_G = (R_H)^{-1}$.

Let $(s, t) \in R_G$, i.e., $(\forall x \in \text{dom}(G))(s(G(x)) \leq t(x))$. We have to check that $(t, s) \in R_H$. Let $y \in \text{dom}(H)$. Then by assumption (b) we obtain that $H(y) \in \text{dom}(G)$. It follows that $G(H(y)) = F(H(y))$ is defined and by axiom (P3) we get that $G(H(y)) \leq y$, i.e., $y' \leq G(H(y))$. Since $(s, t) \in R_G$ we obtain that $s(y') \leq s(G(H(y))) \leq t(H(y))$.

But $s$ and $t$ are morphisms of bounded De Morgan posets which yields that $t(H(y)) \leq s(y)$. Hence $R_G \subseteq (R_H)^{-1}$. A symmetry argument gives us that $R_H \subseteq (R_G)^{-1}$, i.e., $R_G = (R_H)^{-1}$.

From Theorem II.5 we obtain the following.

**Corollary II.6.** Let $(A; G, H)$ be a dynamic De Morgan algebra. Then $(T_A^{\text{DMP}}, R_G)$ is a frame with $R_G$ and $(R_G)^{-1}$ serial such that $(A; G, H)$ can be embedded into $(M_2^{\text{DMP}}, \hat{G}, \hat{H})$.

**Remark II.7.** Usually, one uses complex algebras associated with the given model to establish a discrete duality between algebraic and relational models. Figallo and Pelaitay in [13] established a discrete duality between tense distributive De Morgan algebras and so-called tense De Morgan spaces. Since we are only interested in the representation of tense operators we use relational models without any additional structure.

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