Electric impedance tomography problem for surfaces with internal holes

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Abstract

Let \((M, g)\) be a smooth compact Riemann surface with the multicomponent boundary \(\Gamma = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_m = : \Gamma_0 \cup \bar{\Gamma}\). Let \(u = u^f\) obey \(\Delta u = 0\) in \(M\), \(u|_{\Gamma_0} = f\), \(u|_{\bar{\Gamma}} = 0\) (the grounded holes) and \(v = v^h\) obey \(\Delta v = 0\) in \(M\), \(v|_{\Gamma_0} = h\), \(\partial_{\nu} v|_{\bar{\Gamma}} = 0\) (the isolated holes). Let \(\Lambda^g_{gr}: f \rightarrow \partial_{\nu} u^f|_{\Gamma_0}\) and \(\Lambda^g_{is}: h \rightarrow \partial_{\nu} v^h|_{\Gamma_0}\) be the corresponding Dirichlet-to-Neumann map. The electric impedance tomography problem is to determine \(M\) from \(\Lambda^g_{gr}\) or \(\Lambda^g_{is}\). To solve it, an algebraic variant of the boundary control method is applied. The central role is played by the algebra \(A\) of functions holomorphic on the manifold obtained by gluing two examples of \(M\) along \(\bar{\Gamma}\). We show that \(A\) is determined by \(\Lambda^g_{gr}\) (or \(\Lambda^g_{is}\)) up to isometric isomorphism. A relevant copy \((M', g', \Gamma'_0)\) of \((M, g, \Gamma_0)\) is constructed from the Gelfand spectrum of \(A\). By construction, this copy turns out to be conformally equivalent to \((M, g, \Gamma_0)\), obeys \(\Gamma'_0 = \Gamma_0\), \(\Lambda^g_{gr}' = \Lambda^g_{gr}\), \(\Lambda^g_{is}' = \Lambda^g_{is}\) and provides a solution of the problem.

Keywords: determination of Riemann surface from its DN-map, algebraic version of boundary control method, 35R30, 46J15, 46J20, 30F15, electric impedance tomography of surfaces

(Some figures may appear in colour only in the online journal)
1. About the paper

- The subject in the scope of this paper is the electric impedance tomography (EIT) problem of surfaces with internal holes. The physical meaning of this problem is as follows. Let $M$ be a conducting shell. Up to a continuous deformation, the shell $M$ is a hemisphere with finite number of handles and holes. The shell is not accessible for the observer, but only one (exterior) component $\Gamma_0$ of its boundary $\partial M$. The remain (inaccessible) part $\tilde{\Gamma}$ of the boundary, if it exists, can be either grounded or electrically isolated. The observer can apply arbitrary potentials $f$ to the external boundary, which induce currents $\nabla u^f$ into the shell. Here $u^f$ is the potential in the interior of the shell, which obeys the Beltrami–Laplace equation. Measuring the currents $\partial_\nu u^f|_{\Gamma_0} = \Lambda f$, which flow across $\Gamma_0$, the observer gets the response operator $\Lambda$ that is the Dirichlet-to-Neumann map (DN-map) of $(M, \Gamma_0)$. The EIT consists in the reconstruction of the shape of the shell $M$ from the above measurements on $\Gamma_0$, i.e. from the (known) DN-map $\Lambda$.

- It is known that any two conformally equivalent Riemann surfaces may have the same DN-map and, hence, they are indistinguishable for the observer, which operates at the boundary [3, 12]. Therefore, the ‘reconstruction of $M’ is understood as a construction of a copy $M’$ of $M$, whose DN-map coincides with that for $M$. This is the only meaningful understanding of the EIT problem [3–5]. Some of the remarks of a ‘philosophical’ nature about such an interpretation are placed in comments at the very end of work.

- The fact that the DN map determines the Riemann surface with boundary up to conformal equivalence, is now well known [10, 12] and first established in [12]. In [3] it is obtained by a version of the boundary control method based on connections between the EIT problem and Banach algebras of holomorphic functions [3, 6, 7]. The goal of our paper is to extend this version to the surfaces with a multicomponent boundary provided that only one (‘exterior’) connected component is accessible for the measurements and observations. So, the novelty is that the technique developed in [3, 6, 7], is applied to a new important class of the EIT problems.

- Note that the determination of unknown components of boundary is a separate problem, and the literature devoted to it is hardly observable (see, e.g. [1, 2]). However, there is a specific feature of the statement that we deal with. Traditionally, the surface $M$ and its external boundary $\Gamma_0$ are assumed to be given, as well as the parameters (metric, conductivity, density, etc) of the medium, which fills $M$. Roughly speaking, we know the surface but do not know the holes into it, whereas the goal is to determine $\partial M \setminus \Gamma_0$ (the interior holes) from the measurements on $\Gamma_0$. In contrast to this setup, we do not assume the surface to be known and recover the surface $M$ together with holes $\tilde{\Gamma}$ on it.

2. Statement of problem and results

- Let $(M, g)$ be a compact oriented Riemann surface with the boundary $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_m =: \Gamma_0 \cup \tilde{\Gamma}$, each $\Gamma_j$ being diffeomorphic to a circle; $g$ the smooth metric tensor; $\Delta_g$ the Beltrami–Laplace operator.

  Consider the problem
  \[ \Delta_g u = 0 \quad \text{in } \text{int } M, \quad u = f \quad \text{on } \Gamma_0, \quad u = 0 \quad \text{on } \tilde{\Gamma} \quad (1) \]

  (int $M := M \setminus \Gamma$) and denote by $u^f$ the solution of (1) for a smooth $f$. The DN-map associated with problem (1) is given by $\Lambda_g^\nu : f \rightarrow \partial_\nu u^f|_{\Gamma_0}$, where $\nu$ is the outward normal.
The second problem under consideration is
\[
\Delta_0 \nu = 0 \quad \text{in} \ \text{int} M, \quad \nu = h \quad \text{on} \ \Gamma_0, \quad \partial_\nu \nu = 0 \quad \text{on} \ \tilde{\Gamma}; \tag{2}
\]
let \(\nu^h\) be the solution for a smooth \(h\). The DN map associated with (2) is \(\Lambda^h = h \to \partial_\nu \nu^h|_{\Gamma_0}\).

- Let any of the operators \(\Lambda = \Lambda^0_\Gamma\) or \(\Lambda = \Lambda^0_\Gamma\) be given; the EIT problem is to construct a Riemann surface \((M', g')\) such that \(\Gamma_0 \subset \partial M'\) and \(\Lambda = \Lambda'\).

  We say such a surface \((M', g')\) to be a copy of the original \((M, g)\); to provide it is the only reasonable meaning of the EIT problem [3–5]. Some of the remarks of a ‘philosophical’ nature about such an interpretation are placed in comments at the very end of the paper.

- To construct the copy \((M', g')\), we extend the algebraic approach which was proposed in [3] for surfaces with the single component boundary. As well as in [3], the main element of the construction is the algebra formed by the boundary values (traces) of the functions holomorphic in a manifold. However, in the present paper, the manifold is not \(M\) but its double \(\hat{M}\), which is obtained by gluing two examples of \(M\) along the ‘inner’ boundary \(\Gamma\) (see figure 1). We show that the corresponding algebra \(\hat{A}\) is determined by \(\Lambda^0_\Gamma\) or \(\Lambda^0_\Gamma\) up to an isometric isomorphism. Then we provide the relevant copy \(M'\) of \(M\) as the Gelfand spectrum \(\hat{A}\) of the algebra \(\hat{A}\) and equip \(M' = \hat{A}\) with a relevant metric (see figure 1). At last, identifying the ‘symmetrically placed’ points of \(M'\), we get a manifold \((M', g')\) obeying \(\partial M' \supset \Gamma_0\) and \(\Lambda' = \Lambda\). It is the copy \(M'\), which solves the EIT problem.

### 3. Preliminaries

- Recall that \(\Lambda^0_{\Gamma}\) and \(\Lambda^0_{\Gamma}\) are the positive selfadjoint operators in the real space \(L_2(\Gamma_0)\) defined on \(\text{Dom} \ \Lambda^0_{\Gamma} = \text{Dom} \ \Lambda^0_{\Gamma} = H^1(\Gamma_0)\). Also, \(\Lambda^0_{\Gamma}\) and \(\Lambda^0_{\Gamma}\) are pseudo-differential operators of first order. The metric on \(\Gamma_0\) can be determined from the principal symbol of \(\Lambda^0_{\Gamma}\) or \(\Lambda^0_{\Gamma}\) (see [13], pp 1105–1106). Therefore, in the subsequent we assume that length element \(ds\) on \(\Gamma_0\) is known and a continuous field of the unit tangent vectors \(\gamma\) is chosen on \(\Gamma_0\). Define

  \[
  \tilde{L}_2(\Gamma_0) := \left\{ f \in L_2(\Gamma_0) \mid \int_{\Gamma_0} f \, ds = 0 \right\}, \quad \tilde{C}^\infty(\Gamma_0) = C^\infty(\Gamma_0) \cap \tilde{L}_2(\Gamma_0).
  \]

  It is easy to see that \(\text{Ran} \ \Lambda^0_{\Gamma} = \tilde{L}_2(\Gamma_0)\) and \(\ker \ \Lambda^0_{\Gamma}\) consists of constant functions. If \(\tilde{\Gamma} \neq \emptyset\), then the relations \(\ker \ \Lambda^0_{\Gamma} = \{0\}\) and \(\text{Ran} \ \Lambda^0_{\Gamma} = \tilde{L}_2(\Gamma_0)\) hold. Note that \(\Lambda^0_{\Gamma} = \Lambda^0_{\Gamma}\) for \(\tilde{\Gamma} = \emptyset\).

- There are two orientations on \(M\) and each of them corresponds to a continuous family of rotations \(M \ni x \to \Phi(x) \in \text{End} T_x M\) with the properties

  \[
  g(\Phi(x) k, \Phi(x) l) = g(k, l), \quad g(\Phi(x) k, k) = 0, \quad k, l \in T_x M, \quad x \in M,
  \]

  which imply \(\Phi(x) = -\Phi(x)^{-1}\), \(x \in M\). Let us fix the orientation (and, hence, \(\Phi\)) by the rule \(\Phi(\gamma)\nu = \gamma\). Then such a rule determines the tangent field of unit vectors \(\gamma\) on each component \(\Gamma_{\nu, j}\), \(j = 1, \ldots, m\).

  The (real) functions \(u, u^1 \in C(M) \cap C^\infty(\text{int} M)\) are conjugate by Cauchy–Riemann if they satisfy

  \[
  \nabla u^1 = \Phi \nabla u \quad \text{in} \ \text{int} M. \tag{3}
  \]
In such a case, the function \( w = u + iu' \) is said to be holomorphic in int \( M \). Obviously, \( u \) and \( u' \) are harmonic in int \( M \) and \( u + c_1, u' + c_2 \) are conjugate for any \( c_1, c_2 \in \mathbb{R} \). The class of functions \( u \in C^\infty(M) \) which have a conjugate \( u' \in C^\infty(M) \) is infinite-dimensional.

- In what follows, \( \partial_\gamma \) denotes the tangent derivative on the boundary. The operator \( J \) in \( L_2(\Gamma_0) \) that is inverse to \( \partial_\gamma \) is the integration on \( \Gamma_0 \).

Suppose that \( \tilde{\Gamma} = \emptyset \), whereas \( u, u' \in C^\infty(M) \) are conjugate. Then (3) implies \( \partial_\gamma u = \partial_\gamma u' \). Since \( u + c_1, u' + c_2 \) are also conjugate, we can chose \( u \) and \( u' \) in such a way that their traces \( f := u|_{\Gamma_0} \) and \( p = u'|_{\Gamma_0} \) belong to \( L_2(\Gamma_0) \). Then, denoting by \( \Lambda_f \) the operator \( \Lambda_f^u = \lambda_i \), one has \( \Lambda_f f = \partial_\gamma p, \partial_\gamma f = -\lambda_i p \) and obtains

\[
[I + (\Lambda_f J^2)]\partial_\gamma f = \partial_\gamma f + \Lambda_f J \partial_\gamma f = \partial_\gamma f + \lambda_i p = 0,
\]

and hence ker\([I + (\Lambda_f J^2)]\neq \{0\}. So, \( \tilde{\Gamma} = \emptyset \) implies ker\([I + (\Lambda_f J^2)]\neq \{0\}. Moreover, dim ker\([I + (\Lambda_f J^2)] = \infty \) holds; see [3].

Show that the converse is also true. Suppose that \( \tilde{\Gamma} \neq \emptyset \) and \([I + (\Lambda_f J^2)]k = 0\), where \( k \in \tilde{C}^\infty(\Gamma_0) \) and \( \Lambda_f = \Lambda_f^u \) or \( \Lambda_f^\Lambda \). Then for \( u = u^k \) and \( u' = u^{(\Lambda_f J^2)} \), one has \( \partial_\gamma u = \partial_\gamma u' \), and \( \partial_\gamma u = -\partial_\gamma u' \) on \( \Gamma_0 \). Thus, \( \nabla u^k = \Phi \nabla u \) holds on \( \Gamma_0 \). By the Poincare theorem, for any neighborhood \( U \subset M \) homeomorphic to a disk in \( \mathbb{R}^2 \) and such that \( \Gamma_0 := \partial U \cap \Gamma_0 \) is non-empty and of non-zero length, there exists a conjugate function \( u'^k \) obeying \( \nabla u'^k = \Phi \nabla u \). The last equality yields \( \partial_\gamma u' = \partial_\gamma u = \partial_\gamma u' \) and \( \partial_\gamma u = -\partial_\gamma u' \) in \( \Gamma_0 \). So, \( u' \) and \( u'' + \text{const} \) are harmonic in \( U \) and have the same Cauchy data on \( \Gamma' \). Thus, \( u' = u'' + \text{const} \) and \( \nabla u' = \Phi \nabla u \) in \( U \). Since \( U \) is arbitrary, the function \( w := u + iu' \) is holomorphic in int \( M \). If \( \Lambda_f = \Lambda_f^u \), then \( w = 0 \) on \( \Gamma \) and, due to analyticity, \( w = 0 \) on \( M \). If \( \Lambda_f = \Lambda_f^\Lambda \), then \( \partial_\gamma u' = \partial_\gamma u = 0 \) and \( \partial_\gamma u = -\partial_\gamma u' = 0 \) on \( \Gamma \). Therefore \( w = \text{const} \) on \( \Gamma \) and, by analyticity, \( w = \text{const} \) on \( M \). In both cases, \( k = u|_{\Gamma_0} = \text{const} \) and hence \( k = 0 \). So, \( \tilde{\Gamma} \neq \emptyset \) implies ker\([I + (\Lambda_f J^2)] \neq \{0\}.

So, for a given operator \( \Lambda_f = \Lambda_f^u \) or \( \Lambda_f^\Lambda \), only the following three alternative cases are realizable:

- If ker\([I + (\Lambda_f J^2)] \neq \{0\} then \( \tilde{\Gamma} = \emptyset \) and \( \Lambda_f = \Lambda_f^u = \Lambda_f^\Lambda \).
Inverse Problems

Lemma 1. For $f \in C^\infty(\Gamma_0)$ the following conditions are equivalent:

(a) $u^f \in \mathcal{H}(\mathfrak{A}_c(\mathcal{M}))$;

4. Algebras on $M$

- From now on, we suppose that $\tilde{\Gamma} \neq \emptyset$, referring the reader to [3] for the case $\tilde{\Gamma} = \emptyset$. Let

$$\mathfrak{A}_c(\mathcal{M}) := \{ w = v + iu | u, v \in C(\mathcal{M}; \mathbb{R}), \nabla u = \Phi \nabla v \text{ in } \text{int} \mathcal{M}, \text{ } u = 0 \text{ on } \tilde{\Gamma} \}$$

be the set of holomorphic continuous functions with real traces on $\tilde{\Gamma}$; we also denote $\mathfrak{A}_{c,0}(\mathcal{M}) := \mathfrak{A}_c(\mathcal{M}) \cap C^\infty(\mathcal{M}; \mathbb{R})$.

Also, note that $\mathfrak{A}_c(\mathcal{M})$ is not a complex algebra because $i \mathfrak{A}_c(\mathcal{M}) \neq \mathfrak{A}_c(\mathcal{M})$. At the same time,

$$\mathfrak{A}(\mathcal{M}) := \mathfrak{A}_c(\mathcal{M}) + i\mathfrak{A}_c(\mathcal{M}) = \{ a + ib | a, b \in \mathfrak{A}_c(\mathcal{M}) \}$$

is an algebra in $C(\mathcal{M}; \mathbb{C})$. The set $\mathfrak{A}_{c,0}(\mathcal{M})$ of smooth elements of $\mathfrak{A}(\mathcal{M})$ is a dense subalgebra of $\mathfrak{A}(\mathcal{M})$.

For any element $\zeta \in \mathfrak{A}(\mathcal{M})$, the representation $\zeta = w_1 + iw_2$ with $w_1, w_2 \in \mathfrak{A}_c(\mathcal{M})$ is unique. Indeed, if $w_1 + iw_2 = w_3 + iw_4$ with $w_3, w_4 \in \mathfrak{A}_c(\mathcal{M})$, then $\zeta = w_1 - w_3 = i(w_4 - w_2)$ is an element of $\mathfrak{A}_c(\mathcal{M}) \cap i\mathfrak{A}_c(\mathcal{M})$. Hence, $\zeta = 0$ on $\tilde{\Gamma}$ and, by analyticity, $\tilde{\zeta} = 0$ on $\mathcal{M}$. Thus, $w_1 = w_3$ and $w_2 = w_4$.

If we consider the map

$$w_1 + iw_2 \rightarrow (w_1 + iw_2)^* := w_1 - iw_2, \quad w_1, w_2 \in \mathfrak{A}_c(\mathcal{M})$$

(6)

obeys $(\zeta_1 \zeta_2)^* = \zeta_2^* \zeta_1^*$ and, hence, is an involution on the algebra $\mathfrak{A}(\mathcal{M})$.

For $\zeta \in \mathfrak{A}(\mathcal{M})$, we put

$$||\zeta|| := \max \{ ||\zeta||_{C(\mathcal{M}; \mathbb{C})}, ||\zeta^*||_{C(\mathcal{M}; \mathbb{C})} \}$$

(7)

and see that $||\zeta \eta|| = || \zeta \eta ||$ holds. Then, easily checking the property $||\zeta \eta || \leq ||\zeta|| \cdot ||\eta||$, we conclude that $\{ \mathfrak{A}(\mathcal{M}), || \cdot ||, \ast \}$ is an involutive Banach algebra.

- In accordance with the uniqueness of analytic continuation, the trace map

$$\mathfrak{A}(\mathcal{M}) \ni \zeta \xrightarrow{\text{Tr}_{\Gamma_0}} \zeta|_{\Gamma_0} \in C(\Gamma_0; \mathbb{C})$$

(8)

is an isomorphism of the algebras $\mathfrak{A}(\mathcal{M})$ and $\text{Tr}_{\Gamma_0} \mathfrak{A}(\mathcal{M})$. This isomorphism determines the involution of the trace algebra

$$\eta_1 + i\eta_2 \rightarrow (\eta_1 + i\eta_2)^* := \eta_1 - i\eta_2, \quad \eta_1, \eta_2 \in \text{Tr}_{\Gamma_0} \mathfrak{A}_c(\mathcal{M}).$$

(9)

In what follows, we prove that $\text{Tr}_{\Gamma_0}$ is an isometry between $\mathfrak{A}(\mathcal{M})$ and $\text{Tr}_{\Gamma_0} \mathfrak{A}(\mathcal{M})$, where the norm in $\text{Tr}_{\Gamma_0} \mathfrak{A}(\mathcal{M})$ is defined by (7) with $\Gamma_0$ instead of $\mathcal{M}$.

Now, we derive explicit formulas that determine $\text{Tr}_{\Gamma_0} \mathfrak{A}(\mathcal{M}) \subset C(\Gamma_0; \mathbb{C})$ in terms of the $\Lambda^{\overline{\rho}}_\mathcal{M}$ or $\Lambda^\rho_\mathcal{M}$.
inverse problems

(b)

\[ 2\Lambda^\theta_g(f \cdot J\Lambda^\theta_g f) = \partial_\gamma [(J\Lambda^\theta_g f)^2 - f^2]. \tag{10} \]

Proof.

1. \( \Rightarrow \) 2. Let \( w = u^I + iv \) belong to \( i\mathcal{A}^\infty(M) \); then \( u^I = 0 \) on \( \overline{\Gamma} \). Denote \( h = v|_{\Gamma_0} \). The function \( w^2 \) is holomorphic in \( M \) and, hence, \( \Im w^2 = 2u^I v \) is harmonic in \( M \). Moreover, \( 2u^I v = 0 \) on \( \overline{\Gamma} \). Thus, \( 2u^I v = u^2 h \) and

\[ 2\Lambda^\theta_g(f h) = 2\partial_\gamma(u^I v) = 2(f \partial_\gamma v + h\Lambda^\theta_g f). \tag{11} \]

In addition, the Cauchy-Riemann conditions \( \Phi \nabla u^I = \nabla v \) in \( M \) imply

\[ \Lambda^\theta_g f = \partial_\gamma u^I = \partial_\gamma h, \quad \partial_\gamma v = -\partial_\gamma f \quad \text{on } \Gamma_0. \tag{12} \]

Integrating the first equality of (12), one gets

\[ \int_{\Gamma_0} \partial_\gamma v \, ds = -\int_{\Gamma_0} \partial_\gamma f \, ds \quad \text{on } \Gamma_0. \]

Comparing (15) with condition 2., we obtain

\[ \partial_\gamma v = \frac{1}{2} \partial_\gamma [(J\Lambda^\theta_g f)^2 - f^2]. \tag{15} \]

Comparing (15) with condition 2., we obtain

\[ \partial_\gamma (u^I v) = \Lambda^\theta_g (f \cdot J\Lambda^\theta_g f) = \partial_\gamma u^I J\Lambda^\theta_g f. \]

Since \( w_U := u^I + iv_U \) is holomorphic in \( U \), the function \( w_U^2 \) is also holomorphic in \( U \) and, hence, \( \frac{1}{2} \Im w_U^2 = u^I v_U \) is harmonic in \( U \). So, \( u^I v_U \) and \( u^I J\Lambda^\theta_g f \) are both harmonic in \( U \) and have the same Cauchy data on \( \Gamma' \). Thus, \( u^I v_U = u^I J\Lambda^\theta_g f \) in \( U \). Therefore, outside the (possible) zeros of \( u^I \) in \( U \), the function

\[ w := u^I + i \frac{u^I J\Lambda^\theta_g f}{u^I} \]

coincides with the function \( w_U \) holomorphic in \( U \). Since \( U \) is arbitrary, \( w \) is holomorphic in \( M \). Moreover, \( \Re w = u^I = 0 \) on \( \overline{\Gamma} \) and \( w \in i\mathcal{A}^\infty(M) \). The latter means that \( u^I \in \Re(i\mathcal{A}^\infty(M)) \) holds.

\( \square \)
As a consequence of lemma 1, we have the representation

\[ \text{Tr}_{\Gamma_0} \mathfrak{A}_c(M) = \{ \eta = j \Lambda_g^b f - if + c \mid f \in C^\infty(\Gamma_0) \text{ obeys (10)}, \ c \in \mathbb{R} \} \quad (16) \]

and conclude that \( \Lambda_g^b \) determines \( \text{Tr}_{\Gamma_0} \mathfrak{A}(M) \).

**Lemma 2.** For \( h \in C^\infty(\Gamma_0) \) the following conditions are equivalent:

(a) \( \psi^h \in \mathfrak{R}(\mathfrak{A}_c(M)) \); 

(b) There is a (real) number \( c_h \) such that

\[
\frac{1}{2} \Lambda_g^b [h^2 - (J \Lambda_g^b h)^2] - h \Lambda_g^b h - J \Lambda_g^b h \cdot \partial_g h = c_h (\partial_g - \Lambda_g^b J \Lambda_g^b) h \quad \text{on } \Gamma_0. \quad (17)
\]

**Proof.** Obviously, the statement is valid for \( h = \text{const} \). In what follows we deal with \( h \neq \text{const} \).

1. \( \Rightarrow \) 2. Let the function \( w = \psi^h + i u \) belong to \( \mathfrak{A}_c(M) \); then \( u = 0 \) on \( \tilde{\Gamma} \). Denote \( f = u|_{\Gamma_0} \).

The function \( w^2 \) is holomorphic in \( M \) and, hence, \( \mathfrak{H} w^2 = (\psi^h)^2 - u^2 \) is harmonic in \( M \). Moreover, \( \partial_g \psi^h = \partial_g u = 0 \) on \( \tilde{\Gamma} \), whence we have

\[
\partial_g [(\psi^h)^2 - u^2] = 2(\psi^h \partial_g \psi^h - u \partial_g u) = 0 \quad \text{on } \tilde{\Gamma}.
\]

Thus, \( (\psi^h)^2 - u^2 = \psi^h - f^2 \) holds and we have

\[
\Lambda_g^b (h^2 - f^2) = \partial_g [(\psi^h)^2 - u^2] = 2(h \Lambda_g^b h - f \partial_g u) \quad \text{on } \Gamma_0. \quad (18)
\]

In addition, the Cauchy–Riemann conditions \( \Phi \nabla \psi^h = \nabla u \) in \( M \) imply

\[
\partial_g f = \Lambda_g^b h, \quad \partial_g u = -\partial_g h \quad \text{on } \Gamma_0. \quad (19)
\]

The first equality of (19) means that \( f = J \Lambda_g^b h + c_h \) with some \( c_h \in \mathbb{R} \). Taking into account this and the second equality of (19), we rewrite (18) as

\[
\Lambda_g^b (h^2 - f^2) = 2(h \Lambda_g^b h + J \Lambda_g^b h + c_h) \partial_g h].
\]

Since \( \Lambda_g^b (c_h^2) = 0 \), the latter implies (17).

2. \( \Rightarrow \) 1. Since \( \Lambda_g^b (c_h^2) = 0 \), (17) can be rewritten as

\[
\Lambda_g^b (h^2 - f^2) = 2(h \Lambda_g^b h + f \partial_g h), \quad (20)
\]

where \( f := J \Lambda_g^b h + c_h \). Choose a segment \( \Gamma' \subset \Gamma_0 \) and an arbitrary simple connected neighbourhood \( U \subset M \) whose boundary contains \( \Gamma' \). Then there is a harmonic in \( U \) function \( u_U \) provided \( \Phi \nabla \psi^h = \nabla u_U \) in \( U \); such a function is defined up to a constant. In particular,

\[
\partial_g u_U = \partial_g \psi^h = \Lambda_g^b h = \partial_g f, \quad -\partial_g h = \partial_g u_U \quad \text{on } \Gamma'. \quad (21)
\]

In view of the first equality of (21), \( u_U \) can be chosen is such a way that \( u_U = f \) on \( \Gamma' \).

The second equality of (21) implies

\[
\partial_g [(\psi^h)^2 - u_U^2] = 2(h \Lambda_g^b h - f \partial_g u_U) = 2(h \Lambda_g^b h + f \partial_g h) \quad \text{on } \Gamma'. \quad (22)
\]

Comparing (22) with (20), we obtain

\[
\partial_g [(\psi^h)^2 - u_U^2] = \Lambda_g^b (h^2 - f^2) = \partial_g \psi^h - f^2 \quad \text{on } \Gamma'.
\]
Since \( \psi^h + iu^h := w_U \) is holomorphic in \( U \), the function \( u^2_U \) is also holomorphic in \( U \) and, hence, \( \Re w_U = (\psi^h)^2 - u^2_U \) is harmonic in \( U \). So, \( (\psi^h)^2 - u^2_U \) and \( \psi^h - j^2 \) satisfy the Laplace–Beltrami equation in \( U \) and have the same Cauchy data on \( \Gamma' \). Thus, \( (\psi^h)^2 - u^2_U = \psi^h - j^2 \) in \( U \).

Introduce the function \( u \) on \( M \) by the rule \( u(x) := u_{\tilde{U}}(x) \), where \( U \) be a neighbourhood diffeomorphic to a disc in \( \mathbb{R}^2 \) and such that \( x \in U \) and \( \partial U \supset \Gamma' \). Let us check that such a definition does not depend on the choice of \( U \). Suppose that \( U_1, U_2 \) are two neighbourhoods defined as above. Then \( u^2_{U_1} = (\psi^h)^2 - u^2_{U_1} = u^2_{U_2} \) and, hence, \( u_{U_1} = su_{U_2} \) on \( U_1 \cap U_2 \), where \( s = 1 \) or \( s = -1 \). Since, the Cauchy–Riemann conditions

\[
\nabla u_{U_1} = \Phi \nabla u_{U_2} = \nabla u_{U_1} = s \nabla u_{U_1}
\]

hold on \( U_1 \cap U_2 \) and \( h, \psi^h \) are non-constant, the last formula yields \( s = 1 \). Thus, \( u_{U_1} = u_{U_2} \) on \( U_1 \cap U_2 \). So, \( u \) is well defined and the function \( w := \psi^h + iu \) is holomorphic in \( M \). From the Cauchy–Riemann conditions \( \Phi \nabla \psi^h = \nabla u \) it follows that \( \partial_s u = 0 \) on \( \Gamma \). The latter implies that \( \partial_s u \) is not identical zero on \( \Gamma \) for any \( j = 1, \ldots, m \). Indeed, the opposite means that \( \nabla u = 0 \) on \( \Gamma_j \) and then \( u = \text{const} \) on \( M \). Then \( \nabla \psi^h = -\Phi \nabla u = 0 \) in \( M \), which contradicts the assumption \( h \neq \text{const} \) on \( \Gamma_0 \). As a result, the equality

\[
0 = \partial_s (\psi^h - j^2) = \partial_s ((\psi^h)^2 - u^2) = 2(\psi^h \partial_s \psi^h - u \partial_s u) = 2u \partial_s u \quad \text{on } \Gamma
\]

yields \( u|_{\Gamma} = 0 \). \( \square \)

Lemma 2 leads to the representation

\[
\text{Tr}_{\Gamma_0} \Lambda_{\tilde{y}}^\infty(M) = \left\{ \eta = h + i(JA^y h + c_h)h | h \in C^\infty(\Gamma_0) \text{ and } c_h \in \mathbb{R} \text{ obey (17)} \right\},
\]

(23)

where \( H_{\text{const}} = 0 \) and \( H_f = 1 \) for any nonconstant \( f \). From (16) and (23) it follows that

\[
\text{Tr}_{\Gamma_0} \Lambda_y^\infty(M) = \left\{ \eta = f_1 + JA^y f_2 + i(JA^y f_1 - f_2) + c \mid \right.
\]

\[
\times f_1, f_2 \in C^\infty(\Gamma_0) \text{ obey (10), } \quad c \in \mathbb{C} \}
\]

(24)

Since \( \text{Tr}_{\Gamma_0} \Lambda_y^\infty(M) \) is dense in \( \text{Tr}_{\Gamma_0} \Lambda(M) \), we arrive at the following important fact.

**Proposition 1.** Each of the operators \( \Lambda_y^\infty \) and \( \Lambda_u^\infty \) determines the algebra \( \text{Tr}_{\Gamma_0} \Lambda(M) \) via (24).

5. Manifold \( \mathcal{M} \)

- Take two examples \( M_+ := M \times \{+1\} \) and \( M_- := M \times \{-1\} \) of \( M \), and factorize \( M_+ \cup M_- \) by the equivalence

\[
\begin{align*}
  x \times \{+1\} &\sim x \times \{+1\} \quad \text{if } x \in M \setminus \Gamma, \\
  x \times \{-1\} &\sim x \times \{-1\} \quad \text{if } x \in M \setminus \Gamma, \\
  x \times \{+1\} &\sim x \times \{-1\} \quad \text{if } x \in \Gamma
\end{align*}
\]
Choose a smooth atlas on $M$ consisting of the charts of the following two types. The charts $(U_j, \phi_j)$ of the first type obey $\bar{U}_j \cap \bar{\Gamma} = \emptyset$, whereas $\phi_j(U_j)$ belongs to the half-plane $\Pi_+ = \{(x_1, x_2) | x_2 > 0\}$. The charts $(U_j, \phi_j)$ of the second type are chosen so that $\phi_j(U_j) \subset \Pi_+$ holds and $\bar{U}_j \cap \bar{\Gamma}$ is a segment $\sigma_j$ of $\bar{\Gamma}$ such that $\phi_j(\sigma_j)$ is a segment $[a_j, b_j]$ of the axis $x_2 = 0$.

Now, let us construct the smooth atlas on $M$. For the chart $(U_j, \phi_j)$ of the first type on $M$, one constructs two charts $(\bar{U}_j^+, \phi_j^+)$ on $\bar{M}$, where $\bar{U}_j^+ = \pi^{-1}(U_j) \cap \bar{M}_\pm$, $\phi_j^+ = \phi_j \circ \pi$, and $\phi_j^+ = \kappa \circ \phi_j \circ \pi$; here $\kappa(x_1, x_2) := (x_1, -x_2)$. Each chart $(U_j, \phi_j)$ of the second type on $M$ determines the chart $(\bar{U}_j^0, \phi_j^0)$ on $\bar{M}$, where $\bar{U}_j^0 = \pi^{-1}(U_j \cup \sigma_j)$ and

$$\phi_j^0(x) := \begin{cases} \phi_j \circ \pi, & x \in \bar{U}_j^0 \cap \bar{M}_\pm, \\ \kappa \circ \phi_j \circ \pi(x), & x \in \bar{U}_j^0 \cap \bar{M}_{\mp}. \end{cases}$$

As is easy to verify, all together the charts $(\bar{U}_j^+, \phi_j^+)$ and $(\bar{U}_j^0, \phi_j^0)$ make up a smooth atlas on $\bar{M}$. So, we have the smooth manifold $M$ with the boundary

$$\partial M = \Gamma_0^+ \cup \Gamma_0^-,$$

where $\Gamma_0^\pm := \pi^{-1}(\Gamma_0 \times \{\pm 1\})$.

(b) Next, $\mathbb{M}$ is endowed with the metric $g := \pi^* g$. By its definition, such a metric is invariant with respect to the involution $\tau$. Also, $g$ is smooth outside $\mathbb{M}_0$. However, on the whole $\mathbb{M}$ it is only Lipschitz continuous in view of possible break of smoothness on $\mathbb{M}_0$.

Given metric $g$, one defines the continuous field of rotations $\Phi$ (which is a tensor field on $\mathbb{M}$) such that $\Phi|_{\mathbb{M}_\pm} = \pi^* \Phi$ and $\tau^* \Phi = -\Phi$ hold.

(c) One can endow the manifold $M$ with a biholomorphic atlas as follows. Let $x$ be an arbitrary point of $\mathbb{M}$, and $(U, \psi)$ is a chart obeying $x \in U$. By the Vecua theorem (see chapter 2 [16]), there exists a chart $(U_x, \psi_x)$, $x \in U_x \subset U$ with isothermal coordinates $\psi_x$ corresponding to the metric $g$ and such that $\psi_x \circ \psi_x^{-1}$ and $\psi \circ \psi_x^{-1}$ are continuously differentiable. In these coordinates, the tensor $g$ is of the form $g^{ij} = \rho(\cdot) \delta^{ij}$ with a Lipschitz $\rho > 0$. Respectively, assuming $(U, \psi)$ to be properly oriented by $\Phi$, the matrix $\Phi^j_i$ in the isothermal coordinates takes the form $\Phi^j_1 = \Phi^j_3 = 0$, $\Phi^j_2 = -\Phi^j_1 = 1$.

As a consequence, if $w = v + i u$ satisfies $\Delta_x v = \Delta_x u = 0$ and $\nabla_x u = \Phi \nabla_x v$ in $U_x$, then the functions $v \circ \psi_x^{-1}$ and $u \circ \psi_x^{-1}$ turn out to be harmonic and connected via the Cauchy–Riemann
conditions in \( \psi_\lambda(U_x) \subset \mathbb{R}^2 \). Thus, \( w \circ \psi_\lambda^{-1} \) is holomorphic in \( \psi_\lambda(U_x) \subset \mathbb{C} \) in the classical sense. Respectively, \( w \) is said to be holomorphic in \( U_x \).

At last, covering \( \mathbb{M} \) by the properly oriented isothermal charts \( (U_x, \psi_x) \) and easily checking their compatibility, one obtains the biholomorphic atlas consistent with the metric \( g \). Now we get the opportunity to talk about holomorphic functions on \( \mathbb{M} \).

6. Algebra on \( \mathbb{M} \)

- The basic element of our approach is the algebra of holomorphic functions

\[
\mathfrak{A}(\mathbb{M}) := \left\{ w = v + iu \mid u, v \in C(\mathbb{M}; \mathbb{R}); \nabla u = \dot{\Phi} \nabla v \ \text{in int} \mathbb{M} \right\}.
\]

It is a closed subalgebra of \( C(\mathbb{M}; \mathbb{C}) \). The set \( \mathfrak{A}^\infty(\mathbb{M}) = \mathfrak{A}(\mathbb{M}) \cap C^\infty(\mathbb{M}) \) is a dense subalgebra of \( \mathfrak{A}(\mathbb{M}) \).

The map

\[
w \rightarrow w^* := w \circ \tau
\]

is an involution on \( \mathfrak{A}(\mathbb{M}) \), and \( \mathfrak{A}(\mathbb{M}) \) is an involutive Banach algebra. Defining

\[
\mathfrak{A}_*(\mathbb{M}) := \{ w \in \mathfrak{A}(\mathbb{M}) \mid w^* = w \}
\]

the set of Hermitian elements of \( \mathfrak{A}(\mathbb{M}) \), we have \( \mathfrak{A}(\mathbb{M}) = \mathfrak{A}_*(\mathbb{M}) + i\mathfrak{A}_*(\mathbb{M}) \). Each \( w \in \mathfrak{A}(\mathbb{M}) \) is represented in the form \( w = w_1 + iw_2 \), where \( w_1 = (w + w^*)/2 \) and \( w_2 = (w - w^*)/2i \) are Hermitian.

- By (25), one has \( w^* = \overline{w} \) on \( \mathbb{M}_0 \), and any function \( w \in \mathfrak{A}_*(\mathbb{M}) \) is real on \( \mathbb{M}_0 \). Therefore, restricting \( w \in \mathfrak{A}_*(\mathbb{M}) \) on \( \mathbb{M}_+ \), one gets \( w|_{|\mathbb{M}_+} = w \circ \pi \) with \( w \in \mathfrak{A}_*(\mathbb{M}) \) (see (4) for definition). The converse is also true: in view of \( \tau^* g = g \), any function of the form \( w \circ \pi \) with \( w \in \mathfrak{A}_*(\mathbb{M}) \), which is given on \( \mathbb{M}_+ \), admits a (unique) holomorphic continuation \( w \) to \( \mathbb{M} \) that obeys \( w \circ \tau = \overline{w} \). Thus, the algebra \( \mathfrak{A}(\mathbb{M}) \) defined in (5) and the algebra \( \mathfrak{A}(\mathbb{M}) \) are related via the restriction:

\[
\mathfrak{A}(\mathbb{M})|_{|\mathbb{M}_+} := \{ w|_{|\mathbb{M}_+} \mid w \in \mathfrak{A}(\mathbb{M}) \} = \{ w \circ \pi \mid w \in \mathfrak{A}(\mathbb{M}) \} =: \mathfrak{A}(\mathbb{M}) \circ \pi.
\]

In the meantime, by definition (7), one has

\[
\|w_1 + iw_2\|_{C(\mathbb{M}; \mathbb{C})} = \max\{\|w_1 + iw_2\|_{C(\mathbb{M}; \mathbb{C})}, \|w_1 \circ \tau + iw_2 \circ \tau\|_{C(\mathbb{M}; \mathbb{C})}\}
\]

\[
= \max\{\|w_1 + iw_2\|_{C(\mathbb{M}; \mathbb{C})}, \|w_1 - iw_2\|_{C(\mathbb{M}; \mathbb{C})}\}
\]

\[
= \|w_1 + iw_2\|.
\]

where \( w_1, w_2 \) are elements of \( \mathfrak{A}_*(\mathbb{M}) \), whereas \( w_1, w_2 \) are their ‘restrictions’: \( w_1, w_2 \mid_{|\mathbb{M}_+} = w_1, w_2 \circ \pi \). Similarly, (6) implies \( (w_1 + iw_2)^*|_{|\mathbb{M}} = [w_1 - iw_2] \circ \pi \). Thus, the restriction provides an isometric isomorphism of the involutive Banach algebra \( \mathfrak{A}(\mathbb{M}) \) onto the involutive Banach algebra \( \mathfrak{A}(\mathbb{M}) \).

- Obviously, the map

\[
w \rightarrow \eta := w|_{|\partial \mathbb{M}}
\]

is a homomorphism of algebras \( \mathfrak{A}(\mathbb{M}) \) and \( \text{Tr} \mathfrak{A}(\mathbb{M}) \). The involution on \( \mathfrak{A}(\mathbb{M}) \) induces the corresponding involution \( \eta \rightarrow \eta^* := \overline{\eta} \circ \tau \) on \( \text{Tr} \mathfrak{A}(\mathbb{M}) \). Due to the maximal principle
Inverse Problems 37 (2021) 105013
A V Badanin et al

\[ \|w\|_{C(M,\mathbb{C})} = \|\text{Tr } w\|_{C(M,\mathbb{C})} \quad (w \in \mathfrak{A}(M)) \] \text{Tr } \mathfrak{A}(M) \text{ is a closed subalgebra of } C(\partial M; \mathbb{C}) \text{ and} \text{Tr } : \mathfrak{A}(M) \mapsto \text{Tr } \mathfrak{A}(M) \text{ is an isomorphism of involutive Banach algebras.}

Next, let \( w = w_1 + iw_2 \), where \( w_1, w_2 \in \mathfrak{A}_c(M) \) obey \( w_j|_{M^+} = w_j \circ \pi \) with \( w_j \in \mathfrak{A}(M) \). Then one has
\[ w|_{\Gamma_0^+} = [(w_1 + iw_2)|_{\Gamma_0^+}] \circ \pi \quad \text{and} \quad w^*|_{\Gamma_0^+} = [(w_1 - iw_2)|_{\Gamma_0^+}] \circ \pi \circ \tau. \]

The Gelfand transform \( \Lambda_\beta \) defines the metrics \( g \) is a Riemann surface that is biholomorphically equivalent to \( \hat{\mathfrak{A}} \). Since the biholomorphic atlas on \( \hat{\mathfrak{A}} \) and \( \mathfrak{A}(M) \) is the function defined on the spectrum by the rule \( \hat{b}(\chi) = \chi(b) \) (\( \chi \in \hat{\mathfrak{A}} \)). The spectrum \( \hat{\mathfrak{A}} \) is endowed with the canonical Gelfand topology in which all Gelfand transforms are continuous functions on \( \hat{\mathfrak{A}} \). With such a topology, \( \hat{\mathfrak{A}} \) is a Hausdorff compact space. If \( \beta : \mathfrak{A} \mapsto \mathfrak{B} \) is isometrical isomorphism of algebras \( \mathfrak{A} \) and \( \mathfrak{B} \), then the map \( \chi \mapsto \chi \circ \beta^{-1} \) is homeomorphism of \( \hat{\mathfrak{A}} \) and \( \mathfrak{B} \).

7. Recovering \((M, g)\)

- First, we briefly recall a basic information on commutative Banach algebras (see, e.g. [9, 14]) which will be used in the reconstruction procedure.

Let \( \mathfrak{A} \) be a (complex) commutative Banach algebra. A homomorphism \( \chi : \mathfrak{A} \rightarrow \mathbb{C} \) is called a character on \( \mathfrak{A} \). The set of all nonzero characters on \( \mathfrak{A} \) is called the spectrum of \( \mathfrak{A} \) and denoted by \( \hat{\mathfrak{A}} \). For any element \( b \) of \( \mathfrak{A} \), one associates its Gelfand transform \( \hat{b} \) that is the function defined on the spectrum by the rule \( \hat{b}(\chi) = \chi(b) \) (\( \chi \in \hat{\mathfrak{A}} \)). The spectrum \( \hat{\mathfrak{A}} \) is endowed with the canonical Gelfand topology in which all Gelfand transforms are continuous functions on \( \hat{\mathfrak{A}} \). With such a topology, \( \hat{\mathfrak{A}} \) is a Hausdorff compact space. If \( \beta : \mathfrak{A} \mapsto \mathfrak{B} \) is isometrical isomorphism of algebras \( \mathfrak{A} \) and \( \mathfrak{B} \), then the map \( \chi \mapsto \chi \circ \beta^{-1} \) is homeomorphism of \( \hat{\mathfrak{A}} \) and \( \mathfrak{B} \).

Obviously, the Dirac measures \( \delta_x : w \mapsto w(x) \) (\( x \in \mathbb{M} \)) are characters on the algebra \( \mathfrak{A}(\mathbb{M}) \). The important fact is that the converse is also true: each character on \( \mathfrak{A}(\mathbb{M}) \) is the Dirac measure \( \delta_x \) of some point \( x \in \mathbb{M} \) and the map \( x \mapsto \delta_x \) is a homeomorphism of \( \mathbb{M} \) and \( \mathfrak{A}(\mathbb{M}) \) [14, 15]. Since the algebras \( \mathfrak{A}(\mathbb{M}) \) and \( \text{Tr}_{\Gamma_0} \mathfrak{A}(M) \) are isometrically isomorphic, one has
\[ \mathbb{M} \cong \text{Tr}_{\Gamma_0} \mathfrak{A}(M) =: \mathbb{M}', \]
where the notation \( S \cong T \) means the homeomorphism of the topological spaces \( S \) and \( T \).

With homeomorphism (27), a pull-back of the holomorphic function \( w \in \mathfrak{A}(\mathbb{M}) \) is the Gelfand transform \( \tilde{\eta} \) of its trace \( \eta := w|_{\Gamma_0^+} \). Note that the biholomorphic atlas on \( \mathbb{M} \) can be composed from the reserve of local coordinates \( x \mapsto w(x) \) with \( w \in \mathfrak{A}(\mathbb{M}) \). Thus, the Gelfand transforms \( \tilde{\eta} \in \mathfrak{A}(\mathbb{M}) \) provide the biholomorphic atlas on \( \mathbb{M}' \). With this atlas, \( \mathbb{M}' \) is a Riemann surface that is biholomorphically equivalent to \( \mathbb{M} \). Since the biholomorphic structure defines the metrics \( g' \) on \( \mathbb{M}' \) up to conformal equivalence, the surfaces \( (\mathbb{M}', g') \)
and \((M, g)\) are conformally equivalent. For this reason, we take the spectrum \(\tilde{M}'\) as a copy of the (unknown) surface \(M\).

The fact of the crucial value for the subsequent is that each of \(\text{DN}-\text{maps} \Lambda_g^\text{gr} \) or \(\Lambda_g^\text{const} \) does determine the spectrum \(\tilde{M}'\) by the scheme (26).

- Recall the statement of the EIT problem. Assume that the curve \(\Gamma_0\) and the operator \(\Lambda : C^\infty(\Gamma_0, \mathbb{R}) \rightarrow C^\infty(\Gamma_0, \mathbb{R})\) are given. Moreover, it is a priori known that \(\Gamma_0\) is a component of the boundary \(\partial M\) of some unknown manifold \((M, g)\) and \(\Lambda\) is one of its \(\text{DN}-\text{maps} \Lambda_g^\text{gr} \) or \(\Lambda_g^\text{const} \). One needs to construct a surface \((M', g')\) such that \(\partial M' \supset \Gamma_0\) and \(\Lambda_g^\text{gr} = \Lambda\) (or \(\Lambda_g^\text{const} = \Lambda\)) holds. Such a copy of \((M, g)\) is regarded as a solution of the EIT problem. We claim and repeat that this is the only relevant understanding of to recover the unknown surface \([3−5]\).

Now, we describe a procedure that constructs the required copy \((M', g')\). We omit some details which can be found in papers \([3, 4]\).

**Step 1.** Given \(\Lambda\), one recovers the metric \(ds\) and the integration \(J\) on \(\Gamma_0\). Checking the relation \(\ker(I + (\Lambda J)^2) = \{0\}\), one detects the presence (or absence) of another connected components of \(\partial M\). We also establish that \(\Lambda = \Lambda_g^\text{gr}\) if \(\ker \Lambda = \{0\}\) or \(\Lambda = \Lambda_g^\text{const}\) if \(\ker \Lambda = \{\text{const}\}\).

**Step 2.** The algebra \(\{\text{Tr}_{\Gamma_0} \hat{\Lambda}(M), \ast\}\) is determined by (24), the involution being determined by (9). Then one finds its spectrum \(\text{Tr}_{\Gamma_0} \hat{\Lambda}(M) = M'\) and the Gelfand transforms \(\tilde{\eta}\) of the elements \(\eta\) of \(\text{Tr}_{\Gamma_0} \hat{\Lambda}(M)\). Taking \(\tilde{\eta}\) as local coordinates on \(M'\), one endows \(M'\) with the biholomorphic atlas. As explained above, the manifold \(M'\) with such atlas is biholomorphically equivalent to the covering \(\tilde{M}\) of \(M\).

**Step 3.** Since \(M'\) is homeomorphic to \(M\), its boundary \(\partial M'\) consists of two connected components, each of which is homeomorphic to \(\Gamma_0\). Each point \(x \in \Gamma_0\) defines the character \(\kappa_x(\eta) := \eta(x) (\eta \in \text{Tr}_{\Gamma_0} \hat{\Lambda}(M))\). So, one identifies \(\Gamma_0\) with one connected component of \(\partial M'\) by \(x \equiv \kappa_x\). Also, note that the whole \(\partial M'\) can be identified as the Shilov boundary of the algebra \(\text{Tr}_{\Gamma_0} \hat{\Lambda}(M)\) \([9]\).

**Step 4.** The ‘space’ involution \(\tau\) on \(M\) induces the involution \(\tau'\) on the spectrum \(\hat{\mathfrak{M}}(\tilde{M}) \cong M'\) by \(\tau' : \delta_x \mapsto \delta_{\tau(x)}\), so that one has

\[
[\tau'(\delta_x)](w) = \delta_{\tau(x)}(w) = w(\tau(x)) = \overline{w(x)} = \delta_x(w^*) \quad x \in \tilde{M}, \quad w \in \hat{\mathfrak{M}}.
\]

This motivates the following definition of the involution \(\tau'\) on \(M'\):

\[
[\tau'(\chi)](\eta) := \chi(\overline{\eta}), \quad \eta \in \text{Tr}_{\Gamma_0} \hat{\Lambda}(M), \quad \chi \in M'.
\]

It is a copy of the involution \(\tau\) in \(M\) in view of (28). By the same (28), the copy of \(\tilde{\Gamma}\) is the subset \(\tilde{\Gamma}'\) of Hermitian characters in \(M'\).

Denote by \(M'\) the connected component of \(\tilde{M}' \setminus \tilde{\Gamma}'\) that contains \(\Gamma_0\). By definition of \(\tilde{M}\), the surface \(\tilde{M} := M' \cup \tilde{\Gamma}'\) is a biholomorphically equivalent to \(M\).

**Step 5.** The biholomorphic atlas on \(M'\) determines the metrics \(g'\) up to the conformal factor. We refer the reader to [3, 8, 12], page 16, where the tricks for finding of \(g'\) are described explicitly. The metrics \(g'\) can be chosen in such a way that the length element \(ds'\) on \(\Gamma_0\) induced by \(g'\) coincides with the (known) length element \(ds\). Indeed, if the condition \(ds' = ds\) does not hold, it can be satisfied for by choosing of the new metrics \(\rho g'\), where \(\rho\) is a smooth positive function on \(M'\) such that \(\rho ds' = ds\).

By construction, the manifold \((M', g')\) that is conformally equivalent to the original \((M, g)\) and the boundaries \(\partial M\) and \(\partial M'\) have a joint connected component \(\Gamma_0, ds\). This yields \(\Lambda_g^\text{gr} = \Lambda_{g'}^\text{gr}, \Lambda_g^\text{const} = \Lambda_{g'}^\text{const}\). So, \((M', g')\) is a solution of the EIT problem.
• In conclusion, we give a look on the EIT problem in the light of very general principles of the systems theory (see [11], chapter 10.6, abstract realization theory). Such a look explains and clarifies the statement of the problem that we deal with. For the external observer, which solves the problem, the surface $M$ (except $\Gamma_0$) is an unreachable ‘black box’. So, the observer has to operate not with the intrinsic attributes of $M$ but with the reaction of $M$ on the input signals (potentials applied to $\Gamma_0$). In such a situation, by the procedure above, the observer uses the input/output map $\Lambda$ to determine not the original $M$ but its model, i.e. a surface $M'$ that responds on the external signals in the same way as $M$. This is the most that the observer can hope for.

The only chance to recover the original itself is to use some additional (to $\Lambda$) \textit{a priori} information about it. For instance, assume that the shell $M$ is placed in $\mathbb{R}^3$ and is rigid, i.e. its shape determines $M$ uniquely, up to trivial isometries (shift, rotation, mirror reflection). Then, embedding the copy $M'$ into $\mathbb{R}^3$, the observer can visualize $M$.

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Data availability statement

No new data were created or analysed in this study.

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