ON THE MORSE–BOTT PROPERTY OF ANALYTIC FUNCTIONS ON BANACH SPACES WITH ŁOJASIEWICZ EXPONENT ONE HALF

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Abstract. It is a consequence of the Morse–Bott Lemma (see Theorems 2.10 and 2.14) that a $C^2$ Morse–Bott function on an open neighborhood of a critical point in a Banach space obeys a Łojasiewicz gradient inequality with the optimal exponent one half. In this article we prove converses (Theorems 1.2 and Corollary 3) for analytic functions on Banach spaces: If the Łojasiewicz exponent of an analytic function is equal to one half at a critical point, then the function is Morse–Bott and thus its critical set nearby is an analytic submanifold. The main ingredients in our proofs are the Łojasiewicz gradient inequality for an analytic function on a finite-dimensional vector space \[82\] and the Morse Lemma (Theorems 4 and 5) for functions on Banach spaces with degenerate critical points that generalize previous versions in the literature, and which we also use to give streamlined proofs of the Łojasiewicz–Simon gradient inequalities for analytic functions on Banach spaces (Theorems 9 and 10).

Contents

1. Introduction 2
1.1. Morse–Bott property of analytic functions with Łojasiewicz exponent one half 5
1.2. Morse Lemma for functions on Banach spaces with degenerate critical points 9
1.3. Morse and Morse–Bott Lemmas for functions on Banach spaces 11
1.4. Łojasiewicz–Simon gradient inequalities for smooth Morse–Bott functions on Banach spaces 11
1.5. Relationship between Morse–Bott and integrability conditions 12
1.6. Łojasiewicz–Simon gradient inequalities for analytic functions on Banach spaces 13
1.7. Applications 13
1.7.1. Yamabe energy function, integrability conditions, Łojasiewicz exponents, and Morse–Bott properties 15
1.7.2. Harmonic map energy function for maps from a Riemann surface into a closed Riemannian manifold 15
1.7.3. Moduli spaces of flat connections and representation varieties 15
1.7.4. $F$-function on the space of hypersurfaces in Euclidean space 17
1.8. Structure of this article 17
1.9. Acknowledgments 17
2. Generalized Morse Lemmas for functions on Banach spaces 18

Date: This version: April 12, 2020, incorporating final galley proof corrections. Calculus of Variations and Partial Differential Equations (2020), https://doi.org/10.1007/s00526-020-01734-4.
2010 Mathematics Subject Classification. Primary 32B20, 32C05, 32C18, 32C25, 58E05; secondary 14E15, 32S45, 57R45, 58A07, 58A35.

Key words and phrases. Analytic varieties, Łojasiewicz inequalities, gradient flow, Morse theory, Morse–Bott functions, resolution of singularities, semianalytic sets and subanalytic sets, singularities.

The author was partially supported by National Science Foundation grant DMS-1510064 and the Dublin Institute for Advanced Studies.
2.1. Preliminaries on linear functional analysis

2.2. Preliminaries on nonlinear functional analysis

2.2.1. Differentiable and analytic maps on Banach spaces

2.2.2. Inverse and implicit mapping theorems for smooth and analytic maps on Banach spaces

2.2.3. Gradient maps

2.3. Morse Lemma for functions on Banach spaces with degenerate critical points

2.4. Applications to proofs of the Morse and Morse–Bott Lemmas for functions on Banach spaces

3. Łojasiewicz gradient inequality for functions on Banach spaces

3.1. Łojasiewicz gradient inequality for smooth Morse–Bott functions

3.2. Łojasiewicz gradient inequality for analytic functions

4. Analytic functions with Łojasiewicz exponent one half are Morse–Bott

Appendix A. Rate of convergence of a gradient flow for a function obeying a Łojasiewicz gradient inequality

Appendix B. Morse–Bott functions and quadratic simple normal crossing functions

Appendix C. Integrability and Morse–Bott conditions for the harmonic map energy and the area functions

C.1. Integrability and Morse–Bott conditions for the harmonic map energy function

C.2. Integrability and Morse–Bott conditions for the area function

References

1. Introduction

Let $\mathbb{K}$ be the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$, and $d \geq 1$ be an integer, and let $\mathbb{K}^d = (\mathbb{K}^d)^*$ denote the dual space. In order to better motivate our main results (Theorems 1, 2, and Corollary 3), we begin by recalling the well-known

**Theorem 1.1** (Łojasiewicz gradient inequality for an analytic function). Let $U \subset \mathbb{K}^d$ be an open neighborhood of the origin and $f : U \to \mathbb{K}$ be an analytic function. If $f(0) = 0$ and $f'(0) = 0$ then, after possibly shrinking $U$, there are constants $C \in (0, \infty)$ and $\theta \in [1/2, 1)$ such that

\[
\|f'(x)\|_{\mathbb{K}^d^*} \geq C|f(x)|^\theta, \quad \text{for all } x \in U.
\]

Łojasiewicz used the theory of semianalytic sets to prove Theorem [1, Proposition 1, p. 92 (67)] when $\mathbb{K} = \mathbb{R}$ and gave the range for $\theta$ as the interval $(0, 1)$. His article remained unpublished, but Bierstone and Milman gave a simplified and streamlined exposition of Łojasiewicz’s method in [16] for $\mathbb{K} = \mathbb{R}$ and later gave an elegant and entirely new proof in [17] of (1.1) for $\mathbb{K} = \mathbb{R}$ using resolution of singularities for analytic varieties [62] and for which they also gave a new and significantly simplified proof. In [16, 17], Bierstone and Milman state the range as for $\theta$ as the interval $(0, 1)$.

In [49], we proved Theorem [1, Theorem 1] by also appealing to resolution of singularities for analytic sets but in a different way from that of Bierstone and Milman [17]. Our approach is valid for both $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and it allowed us to give the forthcoming partial identification (1.4) of the Łojasiewicz exponent, $\theta$, and show that it is restricted to the interval $[1/2, 1)$, sharpening the

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1. The first page number refers to the version of Łojasiewicz’s original manuscript mimeographed by IHES while the page number in parentheses refers to the cited LaTeX version of his manuscript prepared by M. Coste and available on the Internet.
range \((0,1)\) provided in [16][17][82]. Resolution of singularities for analytic varieties yields the following special case of [49][Theorem 4.5] (see [49][Sections 4.3 and 4.4] for details of references to statements and proofs):

**Theorem 1.2** (Monomialization of an analytic function). Let \(U \subset \mathbb{K}^d\) be an open neighborhood of the origin and \(f : U \to \mathbb{K}\) be an analytic function. If \(f(0) = 0\) then, after possibly shrinking \(U\), there are an open neighborhood \(V \subset \mathbb{K}^d\) of the origin and an analytic map,

\[
\pi : V \ni y \mapsto x \in U,
\]

such that \(\pi(0) = 0\) and \(\pi\) restricts to an analytic diffeomorphism on the complement of the zero set, \(Z := f^{-1}(0)\),

\[
\pi : V \setminus \pi^{-1}(Z) \cong U \setminus Z,
\]

and \(\pi^* f\) is a simple normal crossing function, that is,

\[
\pi^* f(y) = y_1^{n_1} y_2^{n_2} \cdots y_d^{n_d} \quad \text{for all } y \in V,
\]

where the \(n_i\) are non-negative integers for \(i = 1, \ldots, d\).

The Łojasiewicz exponent of a monomial function can easily be computed exactly using the Generalized Young Inequality (see [49][Remark 3.1]) to give the

**Lemma 1.3** (Łojasiewicz exponent of a monomial function). (See Feehan [49][Theorem 5] or Haraux [58][Theorem 3.1].) Let \(g : \mathbb{K}^d \to \mathbb{K}\) be an analytic function given by \(g(y) = y_1^{n_1} y_2^{n_2} \cdots y_d^{n_d}\) for \(y \in \mathbb{K}^d\), where the \(n_i\) are non-negative integers for \(i = 1, \ldots, d\). If \(g(0) = 0\) and \(g'(0) = 0\), then \(g\) obeys the Łojasiewicz gradient inequality \((1.1)\) on \(U = \mathbb{K}^d\) for a constant \(C \in (0, \infty)\) and exponent

\[
\theta = 1 - \frac{1}{N} \in [1/2, 1), \quad \text{where } N := \sum_{i=1}^{d} n_i \geq 2.
\]

We apply Theorem 1.2 and Lemma 1.3 to prove Theorem 1.1 using the elementary

**Lemma 1.4** (Łojasiewicz exponents and maps). Let \(d\) and \(e\) be positive integers, \(V \subset \mathbb{K}^e\) and \(U \subset \mathbb{K}^d\) be open neighborhoods of the origins and \(\phi : V \to U\) be an open \(C^1\) map such that \(\phi(0) = 0\). If \(f : U \to \mathbb{K}\) is a \(C^1\) function such that \(\phi^* f\) obeys the Łojasiewicz gradient inequality \((1.1)\) at the origin with exponent \(\theta \geq 0\) then, after possibly shrinking \(U\), the function \(f\) obeys the Łojasiewicz gradient inequality \((1.1)\) with the same exponent \(\theta\) and a possibly smaller constant \(C \in (0, \infty)\).

Theorem 1.1 now follows as an immediate corollary of Theorem 1.2 and Lemmas 1.3 and 1.4.

The exponent \(\theta = 1/2\) is optimal in the sense that if a solution \(x(t)\), for \(t \in [0, \infty)\), to the negative gradient flow defined by \(f\),

\[
\frac{dx}{dt} = -\nabla f(x(t)), \quad x(0) = x_0 \in U,
\]

converges to a point \(x_\infty \in \text{Crit } f\) as \(t \to \infty\), then the norm of the difference, \(\|x(t) - x_\infty\|_{\mathbb{K}^d}\), converges to zero as \(t \to \infty\) like \(\exp(-ct)\) for some \(c > 0\) when \(\theta = 1/2\) but only like \(t^{-\gamma}\) for some \(\gamma > 0\) when \(\theta \in (1/2, 1);\) see Appendix A for a discussion and references.

The optimal exponent, \(\theta = 1/2\), is achieved when \(f : U \to \mathbb{K}\) is a \(C^2\) function that is Morse–Bott at the origin, that is, when the critical set \(\text{Crit } f := \{x \in U : f'(x) = 0\}\) is a connected, smooth submanifold of \(U\) (after possibly shrinking \(U\)) of dimension equal to \(\dim \text{Ker } f''(0)\). This is readily seen by applying the Morse–Bott Lemma (see Theorems 2.10 or 2.14) to produce (after
possibly shrinking $U$) a $C^2$ diffeomorphism $\Phi : V \to U$ from an open neighborhood $V \subset \mathbb{K}^d$ of the origin onto $U$ such that $\Phi(0) = 0$ and
\[
\Phi^* f(y) = \sum_{i=1}^{d-c} a_i y_i^2, \quad \text{for all } y \in V,
\]
where $c = \dim \ker f''(0)$ and $a_i \in \mathbb{K} \setminus \{0\}$ for $i = 1, \ldots, d - c$. The Łojasiewicz gradient inequality (1.1) for $f$ with exponent $\theta = 1/2$ follows immediately by direct calculation for $\Phi^* f$ and invariance of the Łojasiewicz exponent under diffeomorphisms. Proofs of the optimal Łojasiewicz gradient inequality for Morse–Bott functions on $\mathbb{R}^d$ or Banach spaces over $\mathbb{K}$ were provided by the author in [49, Theorem 3] and [48, Theorem 3] and by the author and Maridakis [50, Theorems 3 and 4] without relying on the Morse–Bott Lemma.

The main goal of this article is to explore whether the converse is true:

If $f : U \to \mathbb{K}$ is a $C^2$ function that obeys the Łojasiewicz gradient inequality (1.1) with exponent $\theta = 1/2$, then is $f$ Morse–Bott at the origin?

As we shall see in Theorems 1, 2 and Corollary 3 this converse is indeed true in great generality — for a broad class of analytic functions, $f : X \supseteq \mathcal{X} \to \mathbb{K}$, on Banach spaces $X$ over $\mathbb{K}$ and for any analytic function $f$ when $X = \mathbb{K}^d$. It is apparent from examples that $\theta \in [1/2, 1)$ provides a measure of complexity of the singularity of the critical set of $f$, at the origin. Theorems 1, 2 and Corollary 3 make this informal measure of complexity precise for analytic functions on open neighborhoods of the origin in $\mathbb{R}^d$ with arbitrary $d \geq 1$ and even Banach spaces over $\mathbb{K}$: the critical set is an analytic submanifold of the expected dimension when $\theta = 1/2$.

Intuition supporting the preceding conclusion can be obtained by examining the structure of the function $\pi^* f$ in (1.3) when $N = 2$, the lowest possible total degree of the monomial. Indeed, if $N = 2$, then (after relabeling coordinates) either $n_1 = 2$ and $n_i = 0$ for all $i \geq 2$ or $n_1 = n_2 = 1$ and $n_i = 0$ for all $i \geq 3$ and thus
\[
\pi^* f(y) = \pm y_1^2 \quad \text{or} \quad y_1 y_2, \quad \text{for all } y \in V,
\]

(1.5)

with
\[
\pi^{-1}(f^{-1}(0)) = \{y \in V : y_1 = 0\} \quad \text{or} \quad \{y \in V : y_1 = 0 \text{ or } y_2 = 0\}.
\]

In particular, the critical set of $\pi^* f$ is either the codimension-one submanifold $\{y \in V : y_1 = 0\}$ or the codimension-two submanifold $\{y \in V : y_1 = y_2 = 0\}$. These observations tell us that the condition $\theta = 1/2$ imposes strong constraints on resolution morphism $\pi$ and the structure of the analytic function $f$ itself since resolution of singularities tends to ‘increase degrees’.

Because the identification (1.3) of $\pi^* f$ as a simple normal crossing function comes from resolution of singularities for analytic sets, one might expect that methods of algebraic geometry could be used to compute $\theta$ directly in terms of $f$ and also lead to the conclusion that $f$ must be Morse–Bott, at least when $f$ is a polynomial and possibly even when $f$ is analytic. However, while the Łojasiewicz exponent has been estimated for certain classes of polynomials (see [49, Section 1] for a survey), it appears difficult to estimate the exponent in any generality, even for polynomial functions. Using the fact that $\pi^* f(y) = \pm y_1^2$ or $y_1 y_2$ when $f$ has Łojasiewicz exponent $1/2$ to directly constrain the structure of $f$ and the resolution morphism $\pi$ in the proof of resolution of singularities appears challenging, although this may provide one route to a proof of Corollary 3 using methods of algebraic geometry.

\[\text{We omit the pair of possible signs, } \pm, \text{ when } \mathbb{K} = \mathbb{C}.\]
Our approach to proving Theorems 1.2 in this article is analytic and relies on a version (see Section 1.2) of the Morse Lemma for analytic functions $f$ with degenerate critical points, together with our identification (1.3) of $\pi^*f$ when $f$ has Łojasiewicz exponent $1/2$ and $\pi$ is a resolution of singularities (1.2) for the zero set $f^{-1}(0)$.

The concept of a Morse–Bott function was introduced by Bott in [19, Definition, p. 248] and used by him in his first proof of the Bott Periodicity Theorem [20]. Morse–Bott functions were employed by Austin and Braam [11, Section 3] in their approach to developing a Morse theoretic approach to equivariant cohomology.

1.1. Morse–Bott property of analytic functions with Łojasiewicz exponent one half.

Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces over $\mathbb{K}$, and $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ denote the Banach space of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$, and Ker $A$ and Ran $A$ denote the kernel and range of $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, and $\mathcal{X}^*$ denote the continuous dual space of $\mathcal{X}$. Let $\mathcal{L}_{\text{sym}}(\mathcal{X}, \mathcal{X}^*) \subset \mathcal{L}(\mathcal{X}, \mathcal{X}^*)$ denote the closed subspace of operators $A$ that are symmetric in the sense that $\langle x, Ay \rangle_{\mathcal{X} \times \mathcal{X}^*} = \langle y, Ax \rangle_{\mathcal{X} \times \mathcal{X}^*}$ for all $x, y \in \mathcal{X}$, where $\langle \cdot, \cdot \rangle_{\mathcal{X} \times \mathcal{X}^*}$ denotes the canonical pairing, $\mathcal{X} \times \mathcal{X}^* \ni (x, \alpha) \mapsto \alpha(x) \in \mathbb{K}$. We recall the canonical identifications,

$$\mathcal{L}_{\text{sym}}(\mathcal{X}, \mathcal{X}^*) = \mathcal{L}_{\text{sym}}^2(\mathcal{X}, \mathbb{K}) = \mathcal{L}_{\text{sym}}(\mathcal{X} \otimes \mathcal{X}, \mathbb{K}),$$

where $\mathcal{L}^n(\mathcal{X}, \mathbb{K})$ (respectively, $\mathcal{L}(\mathcal{X} \otimes \mathcal{X}, \mathbb{K})$) is the Banach space of continuous $n$-linear (respectively, linear) functions, $A : \mathcal{X}^n \to \mathbb{K}$ (respectively, $A : \mathcal{X} \otimes \mathcal{X} \to \mathbb{K}$), for integers $n \geq 1$.

Let $\mathcal{X}$ be a $C^p$ Banach manifold ($p \geq 0$) modeled on a Banach space $\mathcal{X}$ (see [74, Section II.1] or [3, Definitions 3.1.1 and 3.1.3]). A subset $\mathcal{W} \subset \mathcal{X}$ is a $C^p$ Banach submanifold [74, Section II.2], [3, Definition 3.2.1] modeled on a Banach space $\mathcal{W}$ if there is a Banach space $\mathcal{N}$ such that $\mathcal{X} = \mathcal{W} \oplus \mathcal{N}$ as a direct sum of Banach spaces and, for each point $w \in \mathcal{W}$, a chart $\psi : \mathcal{X} \ni w \to \mathcal{X}$ such that $\psi(w) = 0$ and

$$\psi(\mathcal{V} \cap \mathcal{X}) = \psi(\mathcal{V}) \cap (\mathcal{W} \oplus \{0\}).$$

Observe that $T_w \mathcal{X} \cong \mathcal{X}$ and $T_w \mathcal{W} \cong \mathcal{W}$ with $T_w \mathcal{X} \equiv T_w \mathcal{W} \oplus \mathcal{N}$; in particular, $T_w \mathcal{W}$ is a closed subspace with closed complement in $T_w \mathcal{X}$.

If $\mathcal{W} \subset \mathcal{X}$ is an open subset, $f : \mathcal{W} \to \mathbb{K}$ is a $C^2$ function, and Crit $f = \{x \in \mathcal{W} : f'(x) = 0\}$ is a $C^2$, connected submanifold of $\mathcal{W}$, then the tangent space $T_x \text{Crit } f$ is contained in Ker $f''(x)$, for each $x \in \text{Crit } f$, where $f'(x) \in \mathcal{X}^*$ and $f''(x) \in \mathcal{L}_{\text{sym}}(\mathcal{X}, \mathcal{X}^*)$.

**Definition 1.5** (Morse–Bott properties). Let $\mathcal{X}$ be a Banach space over $\mathbb{K}$, and $\mathcal{U} \subset \mathcal{X}$ be an open neighborhood of the origin, and $f : \mathcal{U} \to \mathbb{K}$ be a $C^2$ function such that Crit $f$ is a $C^2$, connected submanifold.

1. If $x_0 \in \text{Crit } f$ and Ker $f''(x_0) \subset \mathcal{X}$ has a closed complement $\mathcal{X}_0$ and Ran $f''(x_0) = \mathcal{X}_0^*$, and $T_{x_0} \text{Crit } f = \text{Ker } f''(x_0)$, then $f$ is Morse–Bott at the point $x_0$.

2. If $f$ is Morse–Bott at each point $x \in \text{Crit } f$, then $f$ is Morse–Bott along Crit $f$ or a Morse–Bott function.

**Remark 1.6** (On the assumption that Ker $f''(x_0)$ has a closed complement). Because Crit $f \subset \mathcal{X}$ is a submanifold in Definition 1.5, then $T_{x_0} \text{Crit } f$ automatically has a closed complement in $\mathcal{X}$ and because $T_{x_0} \text{Crit } f = \text{Ker } f''(x_0)$ in Definition 1.5, then Ker $f''(x_0)$ automatically has a closed complement in $\mathcal{X}$. However, we include the closed complement assumption for Ker $f''(x_0)$ in our definition for the sake of emphasis.

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3I am indebted to Michael Greenblatt and András Némethi for pointing out to me that this should be a key analytical tool.

4For example, see the discussion prior to Theorem 2.10.
The Morse–Bott Lemma (see Theorem 2.10) implies that if \( f \) is Morse–Bott at a point, as in Definition 1.5 (1), then \( f \) is Morse–Bott along an open neighborhood of that point in \( \text{Crit} f \), as in Definition 1.5 (2). If \( \text{Crit} f \) in Definition 1.5 consists of isolated points and \( f''(x_0) \in \mathcal{L}(\mathcal{X}, \mathcal{X}^*) \) is invertible for each \( x_0 \in \text{Crit} f \), then \( f \) is a Morse function. The finite-dimensional analogue of Definition 1.5 (2) is well-known.

Remark 1.7 (Morse–Bott functions on Euclidean space). When \( \mathcal{X} \) is finite-dimensional, the definition of a Morse–Bott function was given by Bott [19, Definition, p. 248], [20]. See Nicolaescu [92, Definition 2.41] for a modern exposition.

When \( \mathcal{X} \) is infinite-dimensional, then one must impose hypotheses on \( f \) in addition to those of Bott in the finite-dimensional case in order to obtain a tractable version, such as Theorem 2.10 of the classical Morse–Bott Lemma (for example, Nicolaescu [92, Proposition 2.42]). Because the operator \( f''(x_0) \in \mathcal{L}(\mathcal{X}, \mathcal{X}^*) \) is symmetric, the forthcoming Lemma 2.2 (1) implies that one always has \( \text{Ran} f''(x_0) \subset \mathcal{X}_0^* \) if \( \mathcal{X}_0 \) is a closed complement of \( \text{Ker} f''(x_0) \). Thus Item (1) in Definition 1.5 imposes the non-degeneracy condition \( \text{Ran} f''(x_0) = \mathcal{X}_0^* \), like in the finite-dimensional case, given the property that \( \text{Ker} f''(x_0) \) has a closed complement.

When the operator \( f''(x_0) \in \mathcal{L}(\mathcal{X}, \mathcal{X}^*) \) is Fredholm with index zero, Lemma 2.2 (2) yields the non-degeneracy condition, \( \text{Ran} f''(x_0) = \mathcal{X}_0^* \). When the operator \( f''(x_0) \in \mathcal{L}(\mathcal{X}, \mathcal{X}^*) \) is not Fredholm but \( \mathcal{X} \) is reflexive and \( \text{Ker} f''(x_0) \) has a closed complement, Lemma 2.3 implies that the condition that \( \text{Ran} f''(x_0) = \mathcal{X}_0^* \) in Item (1) of Definition 1.5 is equivalent to the condition that \( \text{Ran} f''(x_0) \subset \mathcal{X}^* \) is a closed subspace. We now state the main results of this article.

Theorem 1 (Morse–Bott property of an analytic function with Łojasiewicz exponent one half). Let \( \mathcal{X} \) be a Banach space over \( \mathbb{K} \), and \( \mathcal{U} \subset \mathcal{X} \) be an open neighborhood of the origin, and \( f : \mathcal{U} \to \mathbb{K} \) be a non-constant analytic function such that \( f(0) = 0 \) and \( f''(0) = 0 \) and \( f''(x) \in \mathcal{L}(\mathcal{X}, \mathcal{X}^*) \) is a Fredholm operator with index zero. If there is a constant \( C \in (0, \infty) \) such that, after possibly shrinking \( \mathcal{U} \),

\[
\|f'(x)\|_{\mathcal{X}^*} \geq C|f(x)|^{1/2}, \quad \text{for all } x \in \mathcal{U},
\]

then \( f \) is a Morse–Bott function in the sense of Definition 1.5.

Hence, Theorem 1 is a converse to the simpler Theorem 6 when \( f \) is analytic. The conclusion of Theorem 1 implies that (using the version of the Morse–Bott Lemma provided by Theorem 2.10), after possibly shrinking \( \mathcal{U} \), there are an open neighborhood \( \mathcal{V} \subset \mathcal{X} \) of the origin and an analytic diffeomorphism \( \Phi : \mathcal{V} \to \mathcal{U} \) such that \( \Phi(0) = 0 \) and

\[
f(\Phi(y)) = \frac{1}{2}(y, Ay)_{\mathcal{X} \times \mathcal{X}^*}, \quad \text{for all } y \in \mathcal{V},
\]

where \( A = (f \circ \Phi)^''(0) \in \mathcal{L}_{\text{sym}}(\mathcal{X}, \mathcal{X}^*) \) and, letting \( \mathcal{H} := \text{Ker} A \subset \mathcal{X} \) denote the finite-dimensional kernel of \( A \) with closed complement \( \mathcal{X}_0 \subset \mathcal{X} \), and \( \text{Ran} A = \mathcal{X}_0^* \subset \mathcal{X}^* \) (see Lemma 2.2 (2)) denote the closed range of \( A \) with finite-dimensional complement \( \mathcal{H}^* \) in \( \mathcal{X}^* = \mathcal{X}_0^* \oplus \mathcal{H}^* \) (see Lemma 2.1), we have

\[
A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{X}_0 \oplus \mathcal{H} \to \mathcal{X}_0^* \oplus \mathcal{H}^*,
\]

where \( A_0 \in \mathcal{L}_{\text{sym}}(\mathcal{X}_0, \mathcal{X}_0^*) \) is an isomorphism of Banach spaces. Thus, \( (f \circ \Phi)'(y) = Ay \), for all \( y \in \mathcal{V} \), and \( \text{Crit} f \circ \Phi = \mathcal{V} \cap \text{Ker} A \), an analytic submanifold of \( \mathcal{V} \) of dimension equal to \( \dim \text{Ker} A \).

As explained in [50, Section 1.1], the hypotheses of Theorem 1 are restrictive since they imply that \( \mathcal{X} \) is isomorphic to its continuous dual space, \( \mathcal{X}^* \). For example, that would exclude familiar
choices such as the Banach space of $C^{2,\alpha}$ sections of a finite-rank Riemannian vector bundle over a closed finite-dimensional Riemannian manifold, as in [108, 109].

(There are examples of Banach spaces that are isomorphic to their dual spaces but are not isomorphic to Hilbert spaces. However, even the implication that $\mathcal{X}$ is a reflexive Banach space is already restrictive for some applications of infinite-dimensional Morse Theory to geometric analysis. A classical theorem of Lindenstrauss and Tzafriri [80] asserts that a real Banach space in which every closed subspace is complemented (that is, is the range of a bounded linear projection) is isomorphic to a Hilbert space.)

As in [108], one can relax the implicit restriction that $\mathcal{X} \cong \mathcal{X}^*$ by introducing an extrinsic gradient operator $\mathcal{M}(x)$ to represent the derivative $f'(x)$ for each $x \in \mathcal{U}$.

**Definition 1.8** (Gradient map). (See Berger [13] Section 2.5 or Huang [67] Definition 2.1.1.) Let $\mathcal{X}$ and $\tilde{\mathcal{X}}$ be Banach spaces over $\mathbb{K}$, and $\tilde{\mathcal{X}} \subset \mathcal{X}^*$ be a continuous embedding, and $\mathcal{U} \subset \mathcal{X}$ be an open subset. A continuous map $\mathcal{M} : \mathcal{U} \to \tilde{\mathcal{X}}$ is a gradient map if there is a $C^1$ potential function $f : \mathcal{U} \to \mathbb{K}$ such that

$$f'(x)v = \langle v, \mathcal{M}(x) \rangle_{\mathcal{X} \times \mathcal{X}^*}, \quad \text{for all } x \in \mathcal{U} \text{ and } v \in \mathcal{X}.$$  

(1.7)

A continuous embedding of Banach spaces $\tilde{\mathcal{X}} \subset \mathcal{X}^*$ induces a continuous embedding of Banach spaces of bounded linear operators,

$$\mathcal{L}(\mathcal{X}, \tilde{\mathcal{X}}) \subset \mathcal{L}(\mathcal{X}, \mathcal{X}^*),$$

since if $T \in \mathcal{L}(\mathcal{X}, \tilde{\mathcal{X}})$, then we obtain $T \in \mathcal{L}(\mathcal{X}, \mathcal{X}^*)$ by composing $T$ with the continuous embedding $\tilde{\mathcal{X}} \subset \mathcal{X}^*$. We can therefore define

$$\mathcal{L}_{\text{sym}}(\mathcal{X}, \tilde{\mathcal{X}}) := \mathcal{L}(\mathcal{X}, \tilde{\mathcal{X}}) \cap \mathcal{L}_{\text{sym}}(\mathcal{X}, \mathcal{X}^*).$$

Some basic properties of gradient maps are listed in Proposition 2.5, including the fact that $\mathcal{M}(x) \in \mathcal{L}_{\text{sym}}(\mathcal{X}, \tilde{\mathcal{X}})$ for all $x \in \mathcal{U}$. When $\tilde{\mathcal{X}} = \mathcal{X}^*$ in Definition 1.8, then the derivative and gradient maps coincide. If we are given a $C^1$ function $f : \mathcal{U} \to \mathbb{K}$ such that $f'(x) = \langle \cdot, \mathcal{M}(x) \rangle_{\mathcal{X} \times \mathcal{X}^*}$, for all $x \in \mathcal{U}$, for a $C^0$ map $\mathcal{M} : \mathcal{U} \to \tilde{\mathcal{X}}$, then we simply write $f'(x) = \mathcal{M}(x)$, for all $x \in \mathcal{U}$. These observations motivate the following generalization of Definition 1.5.

**Definition 1.9** (Generalized Morse–Bott properties). Let $\mathcal{X}$ and $\tilde{\mathcal{X}}$ be Banach spaces over $\mathbb{K}$, and $\tilde{\mathcal{X}} \subset \mathcal{X}^*$ be a continuous embedding, and $\mathcal{U} \subset \mathcal{X}$ be an open subset, and $f : \mathcal{U} \to \mathbb{K}$ be a $C^2$ function such that Crit $f$ is a $C^2$, connected submanifold and $f'(x) \in \tilde{\mathcal{X}}$ for all $x \in \mathcal{U}$.

1. If $x_0 \in \text{Crit } f$ and Ker $f''(x_0)$ has a closed complement and Ran $f''(x_0) = \tilde{\mathcal{X}}$ and $T_{x_0} \text{Crit } f = \text{Ker } f''(x_0)$, then $f$ is Morse–Bott at the point $x_0$;

2. If $f$ is Morse–Bott at each point $x \in \text{Crit } f$, then $f$ is Morse–Bott along Crit $f$ or a Morse–Bott function.

Because $f''(x_0) \in \mathcal{L}(\mathcal{X}, \mathcal{X}^*)$, then $\mathcal{K} := \text{Ker } f''(x_0) \subset \mathcal{X}$ is a closed subspace, the quotient $\mathcal{X}/\mathcal{K}$ is a Banach space, and the induced operator $f''(x_0) \in \mathcal{L}(\mathcal{X}/\mathcal{K}, \tilde{\mathcal{X}})$ is an isomorphism by the Open Mapping Theorem. However, in our proof of the Morse–Bott Lemma (see Theorem 2.14) for functions that are Morse–Bott at a point in the sense of Definition 1.9, we shall exploit the existence of a splitting $\mathcal{X} = \mathcal{X}_0 \oplus \mathcal{K}$, where $\mathcal{X}_0 \subset \mathcal{X}$ is a closed subspace.

Again, the Morse–Bott Lemma (see Theorem 2.14) implies that if $f$ is Morse–Bott at a point, as in Definition 1.9, then $f$ is Morse–Bott along an open neighborhood of that point in Crit $f$, as in Definition 1.9.
Theorem 2 (Generalized Morse–Bott property of an analytic function with Łojasiewicz exponent one half). Let \( X \) and \( \tilde{X} \) be Banach spaces over \( \mathbb{K} \), and \( \tilde{X} \subset X^* \) be a continuous embedding, and \( U \subset \tilde{X} \) be an open neighborhood of the origin, and \( f : U \to \mathbb{K} \) be a non-constant analytic function such that \( f(0) = 0 \) and \( f'(0) = 0 \) and \( f'(x) \in \tilde{X} \) for all \( x \in U \) and \( f''(0) \in L(\tilde{X}, \tilde{X}^*) \) is a Fredholm operator with index zero. If there is a constant \( C \in (0, \infty) \) such that, after possibly shrinking \( U \),

\[
\|f'(x)\|_{\tilde{X}} \geq C |f(x)|^{1/2}, \quad \text{for all } x \in U,
\]

then \( f \) is a Morse–Bott function in the sense of Definition 1.2.

Hence, Theorem 2 is a converse to the simpler Theorem 1 when \( f \) is analytic and, moreover, immediately yields Theorem 1 upon choosing \( \tilde{X} = X^* \).

The conclusion of Theorem 2 has an interpretation similar to that of Theorem 1. Using the more general version of the Morse–Bott Lemma provided by Theorem 2.14, after possibly shrinking \( U \), there are an open neighborhood \( V \subset X \) of the origin and an analytic diffeomorphism \( \Phi : V \to U \) such that \( \Phi(0) = 0 \) and

\[
f(\Phi(y)) = \frac{1}{2} \langle y, Ay \rangle_{X \times X^*}, \quad \text{for all } y \in V,
\]

where \( A \in L_{sym}(X, \tilde{X}) \) and, letting \( \mathcal{H} := \text{Ker} \, A \subset X \) denote the finite-dimensional kernel of \( A \) with closed complement \( \tilde{X}_0 \subset \tilde{X} \) and \( \tilde{X}_0 := \text{Ran} \, A \subset \tilde{X} \) denote the closed range of \( A \) with finite-dimensional complement \( \tilde{X} \cong \mathcal{H} = \text{Ker} \, A \), and \( \tilde{X}_0 \cong \tilde{X}_0^* \) (see Lemma 2.1), we have

\[
A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix} : \tilde{X}_0 \oplus \mathcal{H} \to \tilde{X}_0^* \oplus \tilde{X},
\]

where \( A_0 \in L(\tilde{X}_0, \tilde{X}_0^*) \) is an isomorphism of Banach spaces that is symmetric with respect to the continuous embedding \( \tilde{X}_0 \subset X_0^* \) and canonical pairing \( \tilde{X}_0 \times X_0^* \to \mathbb{K} \). Indeed, because \( \tilde{X} \subset X^* \) is a continuous embedding and the splitting \( \tilde{X} = \tilde{X}_0 \oplus \mathcal{H} \) yields a splitting \( X^* = X_0^* \oplus \mathcal{H} \) by Lemma 2.1 while \( \tilde{X} = \tilde{X}_0 \oplus \mathcal{H} \cong \tilde{X}_0 \oplus \mathcal{H} \), we obtain a continuous embedding \( \tilde{X}_0 \subset X_0^* \), as implied above. Thus, \( (f \circ \Phi)'(y) = Ay \), for all \( y \in V \), and \( \text{Crit} \, f \circ \Phi = V \cap \text{Ker} \, A \), an analytic submanifold of \( V \) of dimension equal to \( \dim \, \text{Ker} \, A \).

When we specialize Theorem 1 to \( X = \mathbb{K}^d \), we obtain the desired characterization of the optimal Łojasiewicz exponent for analytic functions on finite-dimensional vector spaces.

Corollary 3 (Morse–Bott property of an analytic function with Łojasiewicz exponent one half). Let \( d \geq 1 \) be an integer, \( U \subset \mathbb{K}^d \) be an open neighborhood of the origin, and \( f : U \to \mathbb{K} \) be a non-constant analytic function such that \( f(0) = 0 \) and \( f'(0) = 0 \). If there is a constant \( C \in (0, \infty) \) such that, after possibly shrinking \( U \), the function \( f \) obeys the Łojasiewicz gradient inequality (1.1) with exponent \( \theta = 1/2 \), then \( f \) is a Morse–Bott function.

The interpretation of Corollary 3 is simpler than that of Theorem 1. By the Morse–Bott Lemma (Theorem 2.10 with \( X = \mathbb{K}^d \) and diagonalization [66, p. 278] of the symmetric matrix \( A \) over \( \mathbb{K} \)), after possibly shrinking \( U \), there are an open neighborhood \( V \subset \mathbb{K}^d \) of the origin and an analytic diffeomorphism \( \Phi : V \to U \) with \( \Phi(0) = 0 \) and an integer \( c \) obeying \( 0 \leq c \leq d - 1 \) such that

\[
f(\Phi(y)) = \frac{1}{2} \sum_{i=1}^{d-c} a_i y_i^2, \quad \text{for all } y \in V,
\]
where \(a_i \in \mathbb{K} \setminus \{0\}\), for \(i = 1, \ldots, d - c\). Thus, \((f \circ \Phi)'(y) = (a_1y_1, \ldots, a_{d-c}y_{d-c}, 0, \ldots, 0) \in \mathbb{K}^d\) for all \(y \in V\), and \(\text{Crit } f \circ \Phi = \{y \in V : y_1 = \cdots = y_{d-c} = 0\}\), an analytic submanifold of \(V\) of dimension \(c\).

1.2. Morse Lemma for functions on Banach spaces with degenerate critical points. It is important to carefully distinguish between the Morse–Bott Lemma and the more general Morse Lemma for functions on Banach spaces with degenerate critical points (also known as the Morse Lemma with parameters or Splitting Lemma): the latter makes no assumption on whether the critical set is a submanifold or, even if it is a submanifold, whether its tangent space at each critical point is equal to the kernel of the Hessian operator at that point. We begin with the

**Theorem 4** (Morse Lemma for functions on Banach spaces with degenerate critical points). Let \(\mathcal{X}\) and \(\mathcal{Y}\) be Banach spaces over \(\mathbb{K}\), and \(\mathcal{U} \subset \mathcal{X}\) and \(\mathcal{V} \subset \mathcal{Y}\) be open neighborhoods of the origin, and

\[
 f : \mathcal{X} \times \mathcal{Y} \supset \mathcal{U} \times \mathcal{V} \ni (x, y) \mapsto f(x, y) \in \mathbb{K}
\]

be a \(C^{p+2}\) function \((p \geq 1)\) such that \(f(0, 0) = 0\) and \(D_1f(0, 0) = 0\). If \(D_1^2f(0, 0) \in \mathcal{L}(\mathcal{X}, \mathcal{X}^*)\) is invertible then, after possibly shrinking \(\mathcal{U}\) and \(\mathcal{V}\), there are an open neighborhood \(\mathcal{U}'\) of the origin and a \(C^p\) diffeomorphism

\[
 \Phi : \mathcal{U}' \times \mathcal{V} \ni (z, y) \mapsto (x, y) \in \mathcal{U} \times \mathcal{V}
\]

with \(\Phi(0, 0) = (0, 0)\) and

\[
 D\Phi(0, 0) = \begin{pmatrix}
 \text{id}_\mathcal{X} & \ast \\
 0 & \text{id}_\mathcal{Y}
\end{pmatrix} \in \mathcal{L}(\mathcal{X} \oplus \mathcal{Y})
\]

such that

\[
 f(\Phi(z, y)) = f(\Phi(0, y)) + \frac{1}{2} \langle z, Ay \rangle_{\mathcal{X} \times \mathcal{X}^*}, \quad \text{for all } (z, y) \in \mathcal{U}' \times \mathcal{V},
\]

where

\[
 A := D_1^2f(0, 0) = D_1^2(f \circ \Phi)(0, 0) \in \mathcal{L}_{\text{sym}}(\mathcal{X}, \mathcal{X}^*).
\]

If \(f\) is analytic, then \(\Phi\) is analytic.

**Remark 1.10** (Previous versions of the Morse Lemma for functions with degenerate critical points). Theorem 4 is a generalization of [55] Lemma C.6.1 from the case where \(f\) is \(C^\infty\) on \(\mathcal{H} = \mathcal{X} \times \mathcal{Y}\) and \(\mathcal{H} = \mathbb{R}^d\) with its standard inner product; and [55], Lemma 1], due to Gromoll and Meyer, from the case where \(f\) is \(C^\infty\) and \(\mathcal{H}\) is a real separable Hilbert space and the Hessian is Fredholm at the critical point; and [54], Theorem 1], due to Mawhin and Willem, from the case where \(f\) is \(C^2\) and \(\mathcal{H}\) is a real Hilbert space and the Hessian is Fredholm at the critical point. Hörmander’s proof is similar to that of Palais [94, p. 307], who allows \(\mathcal{H}\) to be a real Hilbert space but assumes that the critical point is non-degenerate. In [95], Palais uses the Moser Path Method from [88]; see [124] for another early application of [88] in the setting of Banach manifolds. Theorem 4 is also known as the Morse Lemma with parameters (see [99]) or Splitting Lemma — see Bröcker [22], Lemma 14.12 or Poston and Stewart [100] Theorems 4.5 and 6.1 — and attributed to Thom.

**Remark 1.11** (Previous versions of the Morse Lemma for functions on Banach spaces). Palais proved the Morse Lemma for smooth functions on Hilbert spaces in [94, p. 307] and later extended and simplified his proof to give the Morse Lemma for smooth functions on Banach spaces in [95, p. 968] (see also Guillemin and Sternberg [53], Chapter 1, Appendix 1) for another exposition of Palais’s proof in [95].
The version of the Morse Lemma (for functions with degenerate critical points) in Theorem 4 is not the most general possible extension of Hörmander’s [65, Lemma C.6.1] from Euclidean space to a Banach space. Rather, as in Lang’s exposition [74, Section 7.5] of the proof of Palais’s version of the Morse Lemma on Hilbert spaces [94, p. 307], our hypotheses in Theorem 4 are strong enough that replacement of Euclidean space by a Banach space over $K$ involves no new complication.

A shorter proof of Palais’s version of the Morse Lemma on Banach spaces [95, p. 968] is given by Ang and Tuan [8]; see also their article [122]. Kuo [72, Theorem, p. 364] and Tromba [121] consider $C^p$ functions on Banach spaces with isolated critical points obeying a more general notion of non-degeneracy than that of Palais [95, p. 968], avoiding the implication in [95] that the Banach space is isomorphic to its dual space via the isomorphism provided by the Hessian operator. A related extension was proposed earlier by Uhlenbeck [123], inspired by a question of Smale [113]. Antoine [9, Theorem 1] considers $C^p$ functions on Banach spaces with isolated critical points and invertible Hessian operators.

Gromoll and Meyer [55, Lemma 1] consider $C^\infty$ functions on Hilbert spaces with Fredholm Hessian operators while Mawhin and Willem [84, Theorem 1] relax the regularity requirement of Gromoll and Meyer in [55] from $C^\infty$ to $C^2$.

**Remark 1.12 (Regularity of the function $f$).** It is likely that one can adapt the arguments of Cambini [25], Kuiper [71], and Mawhin and Willem [84] to reduce the $C^{p+2}$ (with $p \geq 1$) regularity requirement on $f$ in Theorem 4 to $C^2$, as those authors allow for their versions of the Morse Lemma on Banach spaces, but the resulting proof would be lengthier and less elegant. See Remark 1.16 for further discussion.

**Remark 1.13 (Morse Lemma for functions with degenerate critical points and Lyapunov–Schmidt reduction).** Theorem 4 may be regarded as a more refined version of the technique of Lyapunov–Schmidt reduction (see Guo and Wu [57, Section 5.1], Huang [67, Proposition 2.4.1], or Nirenberg [92, Section 2.7.6]) when $\mathcal{Y}$ has finite dimension. Indeed, as we see from Lemma 3.3, Theorem 4 immediately reduces the proof of the Łojasiewicz gradient inequality for analytic functions on Banach space to the well-known Łojasiewicz gradient inequality for analytic functions on Euclidean space (see Feehan [49] for a detailed survey and references).

As noted in Section 1.1 (see the discussion prior to Theorem 2), the hypothesis in Theorem 4 that $D_1^2 f(0,0) \in L(\mathcal{X}, \mathcal{X}^*)$ is an isomorphism is strong but is relaxed in the following generalization which immediately yields Theorem 4 upon specializing to $\mathcal{X} = \mathcal{X}^*$. See Kuo [72, Theorem, p. 364], Tromba [121, 123], and Mawhin and Willem [84, Theorem 1] for related refinements, though none provide the generality of Theorem 5.

**Theorem 5 (Generalized Morse Lemma on Banach spaces with degenerate critical points).** Let $\mathcal{X}$, $\mathcal{X}$, and $\mathcal{Y}$ be Banach spaces over $K$, and $\mathcal{X} \subset \mathcal{X}^*$ be a continuous embedding, and $U \subset \mathcal{X}$ and $V \subset \mathcal{Y}$ be open neighborhoods of the origin, and

$$f : \mathcal{X} \times \mathcal{Y} \supset U \times V \ni (x,y) \mapsto f(x,y) \in K$$

be a $C^{p+2}$ function ($p \geq 1$) such that $f(0,0) = 0$ and $D_1 f(0,0) = 0$ and $D_1 f(x,y) \in \mathcal{X}$ for all $(x,y) \in U \times V$. If $D_2^2 f(0,0) \in L(\mathcal{X}, \mathcal{X})$ is invertible then, after possibly shrinking $U$ and $V$, there are an open neighborhood of the origin $U' \subset \mathcal{X}$ and a $C^p$ diffeomorphism,

$$\Phi : U' \times V \ni (z,y) \mapsto (x,y) = \Phi(z,y) \in U \times V$$
with $\Phi(0,0) = (0,0)$ and
\begin{equation}
D\Phi(0,0) = \begin{pmatrix}
\text{id}_X & \text{id}_{Y^*} \\
0 & \text{id}_Y
\end{pmatrix} \in \mathcal{L}(X \oplus Y)
\end{equation}
such that
\begin{equation}
f(\Phi(z,y)) = f(\Phi(0,y)) + \frac{1}{2}(z,Az)_{X \times X^*}, \quad \text{for all } (z,y) \in U' \times V,
\end{equation}
where
\[ A := D^2_f(0,0) = D^2_f(f \circ \Phi)(0,0) \in \mathcal{L}_{\text{sym}}(X^*, X).
\]
If $f$ is analytic, then $\Phi$ is analytic.

1.3. Morse and Morse–Bott Lemmas for functions on Banach spaces. Theorems 4 and 5 easily yield versions of the Morse Lemma and, more broadly, the Morse–Bott Lemma in varying degrees of generality, namely Theorems 2.8, 2.10, 2.14, and 2.15. We refer to Section 2.4 for their statements and short proofs.

1.4. Łojasiewicz–Simon gradient inequalities for smooth Morse–Bott functions on Banach spaces. The Morse–Bott Lemma (Theorems 2.10 and 2.14) readily leads to Łojasiewicz–Simon gradient inequalities with exponent one half for $C^{p+2}$ (with $p \geq 1$) Morse–Bott functions on Banach spaces, giving alternative proofs to those that do not rely on the Morse–Bott Lemma provided by the author in [48] Theorem 3 (when $f$ is $C^2$ and $X$ is finite-dimensional) and [48] Theorem 3 and Corollaries 4 and 5 (when $f$ is $C^2$ and $X$ is a Banach space) and by the author and Maridakis [50] Theorem 4 (when $f$ is $C^2$ and $X$ is a Banach space and $f''(0)$ is Fredholm with index zero). We begin with the following analogue of Theorem 1 and which is proved by appealing to the Morse–Bott Lemma provided by Theorem 2.10 the following Theorem 1 is similar to our [48] Corollary 5, except that here we assume that $f$ is $C^{p+2}$ for some $p \geq 1$.

**Theorem 6** (Łojasiewicz gradient inequality for $C^{p+2}$ Morse–Bott functions on Banach spaces). (Compare Feehan [48] Corollary 5.) Let $X$ be a Banach space over $\mathbb{R}$, and $U$ be an open neighborhood of the origin, and $f: U \to \mathbb{R}$ be a $C^{p+2}$ function ($p \geq 1$) such that $f(0) = 0$ and $f'(0) = 0$. If $f$ is Morse–Bott at the origin in the sense of Definition 1.3, then, after possibly shrinking $U$, there is a constant $C \in (0, \infty)$ such that
\begin{equation}
\|f'(x)\|_{X^*} \geq C|f(x)|^{1/2}, \quad \text{for all } x \in U.
\end{equation}

**Remark 1.14** (On the definition of a Morse–Bott function on a Banach space). In [48] Definition 1.5 and [50] Definition 1.10, we said that a $C^2$ function $f: U \to \mathbb{R}$ is Morse–Bott at a point $x_0 \in U$ if $\text{Crit } f$ is a $C^2$ (connected) submanifold and $T_{x_0} \text{Crit } f = \text{Ker } f''(x_0)$, but omitted the requirement that $\text{Ran } f''(x_0) = X_0^*$. In our Łojasiewicz gradient inequality [48] Corollary 5] for $C^2$ Morse–Bott functions analogous to Theorem 6 we required that $\text{Ran } f''(x_0) \subset X^*$ be a closed subspace (equivalent to $\text{Ran } f''(x_0) = X_0^*$ when $X$ is reflexive by Lemma 2.12 [2]). In the hypotheses for our [50] Theorem 4], we imposed the stronger requirement that $f''(x_0) \in \mathcal{L}(X, X^*)$ be Fredholm, so the additional condition that $\text{Ran } f''(x_0) = X_0^*$ was obeyed automatically — see Lemma 2.2 [2].

The forthcoming Theorem 7 is an analogue of Theorem 2. While Theorem 7 is similar to our [48] Corollary 4, we proved the latter result directly for functions $f$ that are only $C^2$, without

---

The property that $\text{Ker } f''(x_0) \subset X$ has a closed complement $X_0 \subset X$ is automatically obeyed for the reasons explained in Remark 1.14.\footnote{The property that $\text{Ker } f''(x_0) \subset X$ has a closed complement $X_0 \subset X$ is automatically obeyed for the reasons explained in Remark 1.14.}\footnote{The property that $\text{Ker } f''(x_0) \subset X$ has a closed complement $X_0 \subset X$ is automatically obeyed for the reasons explained in Remark 1.14.}
appealing to the Morse–Bott Lemma (provided here by Theorem 2.14); by contrast, we assume in Theorem 7 that \( f \) is \( C^{p+2} \) for some \( p \geq 1 \).

**Theorem 7** (Generalized Łojasiewicz gradient inequality for \( C^{p+2} \) Morse–Bott functions on Banach spaces). (Compare Feehan [48, Corollary 4].) Let \( \mathcal{X} \) and \( \mathcal{X} \) be Banach spaces over \( K \), and \( \mathcal{X} \subset \mathcal{X}^* \) be a continuous embedding, \( \mathcal{U} \) be an open neighborhood of the origin, and \( f : \mathcal{U} \to K \) be a \( C^{p+2} \) function (\( p \geq 1 \)) such that \( f(0) = 0 \) and \( f'(0) = 0 \) and \( f'(x) \in \mathcal{X} \) for all \( x \in \mathcal{U} \). If \( f \) is Morse–Bott at the origin in the sense of Definition 1.9 then, after possibly shrinking \( \mathcal{U} \), there is a constant \( C \in (0, \infty) \) such that
\[
\|f'(x)\|_{\mathcal{X}} \geq C|f(x)|^{1/2}, \quad \text{for all } x \in \mathcal{U}.
\]

Simon [110, Lemma 3.13.1] proved an analogue of Theorem 7 for a certain class of smooth functions on an open neighborhood of the origin in the Banach space of \( C^3 \) sections of a finite-rank, smooth Riemannian vector bundle over a closed, finite-dimensional, smooth Riemannian manifold; he describes the construction of the class of smooth functions in [110, Sections 3.11, 3.12, and 3.13]. Simon’s [110, Lemma 3.13.1] can be recovered from our results with Maridakis [50, Theorems 3 and 4], as we note in [50, Remark 1.8]. See also Haraux and Jendoubi [59] for related results.

**Remark 1.15** (On the definition of a generalized Morse–Bott function on a Banach space). In Feehan [48, Definition 1.5] and Feehan and Maridakis [50, Definition 1.10], we said that a \( C^2 \) function \( f : \mathcal{U} \to K \) is Morse–Bott at a point \( x_0 \in \mathcal{U} \) if \( \text{Crit } f \) is a \( C^2 \) (connected) submanifold and \( T_{x_0} \text{Crit } f = \text{Ker } f''(x_0) \), but omitted the requirement that \( \text{Ran } f''(x_0) \subset \mathcal{X} \). In the hypotheses for our Łojasiewicz gradient inequality [48, Corollary 4] for \( C^2 \) Morse–Bott functions analogous to Theorem 7, we required that \( \text{Ran } f''(x_0) \subset \mathcal{X} \) be a closed subspace. In the hypotheses for our Theorem 4, we imposed the stronger requirement that \( f''(x_0) \in \mathcal{L}(\mathcal{X}, \mathcal{X}) \) be Fredholm, so this additional condition was obeyed automatically — see Lemma 2.14.

Theorem 7 immediately yields Theorem 6 upon choosing \( \mathcal{X} = \mathcal{X}^* \).

**Remark 1.16** (On proofs of the Łojasiewicz–Simon gradient inequalities via Morse–Bott Lemmas). As we shall see in Section 3.11 the Łojasiewicz–Simon gradient inequalities with exponent one half for \( C^{p+2} \) Morse–Bott functions (\( p \geq 1 \)) on Banach spaces are indeed easy consequences of the Morse–Bott Lemma (Theorems 2.10 and 2.14). However, the most useful version of such a Łojasiewicz–Simon gradient inequality (namely [48, Theorem 3]), matching the generality of our forthcoming Theorem 11 does not appear to be an obvious consequence of a Morse–Bott Lemma. Second, because our primary focus in this article is on the Morse–Bott property of analytic functions with Łojasiewicz exponent one half, we have not striven to reduce the regularity requirements on \( f \) from \( C^{p+2} \) (with \( p \geq 1 \)) to \( C^2 \). Mawhin and Willem [54, Theorem 1] do establish a Morse–Bott Lemma for functions that are only \( C^2 \), but impose additional hypotheses on \( f \) that we do not require in our Theorem 2.14 namely that \( \mathcal{X} \) be a Hilbert space and (after identifying \( \mathcal{X}^* = \mathcal{X} \)) that \( f''(0) \in \mathcal{L}(\mathcal{X}) \) be Fredholm; Mawhin and Willem generalize earlier Morse–Bott Lemmas for functions that are only \( C^2 \) due to Cambini [25], Hofer [63, 64], and Kuiper [71].

### 1.5. Relationship between Morse–Bott and integrability conditions

We begin with the definition of (Jacobi vectors and integrability). Let \( \mathcal{X} \) and \( \mathcal{X} \) be Banach spaces over \( K \), and \( \mathcal{X} \subset \mathcal{X}^* \) be a continuous embedding, and \( \mathcal{U} \subset \mathcal{X} \) be an open subset, and \( f : \mathcal{U} \to K \) be a \( C^2 \) function such that \( f'(x) \in \mathcal{X} \) for all \( x \in \mathcal{U} \) and \( f'(x_0) = 0 \) for some point \( x_0 \in \mathcal{U} \). We call \( \text{Ker } f''(x_0) \subset \mathcal{X} \) the subspace of Jacobi vectors for \( f \) at the critical point \( x_0 \). We say that
v ∈ Ker $f''(x_0)$ is an integrable Jacobi vector if there exists an open neighborhood $J ⊂ \mathbb{R}$ of the origin and a $C^1$ map $u : J \rightarrow \mathcal{X}$ such that $u(0) = x_0$ and $u'(0) = v$ and $u(J) \subset \text{Crit } f$. Finally, we say that $x_0$ is an integrable critical point of $f$ if every Jacobi vector for $f$ at $x_0$ is integrable.

Our Definition 1.17 is partly based on the integrability condition (⋆) described by Adams and Simon in [3] pp. 229–230 and inspired by an earlier definition due to Allard and Almgren [6]: According to [4], a critical point $x_0$ is integrable if

(⋆) For all $v ∈ \text{Ker } f''(x_0)$, there exists $u ∈ C^0((0,1); \mathcal{X})$ such that $O(u) ⊂ \text{Crit } f$

and $\lim_{t \downarrow 0} u(t) = 0$ (in $\mathcal{X}$) and $\lim_{t \downarrow 0} u(t)/t = v$ (in $\mathcal{Y}$),

where $O(u) := \{u(t) : t ∈ (0,1)\}$ and $\mathcal{Y}$ is a Banach space with continuous embeddings $\mathcal{X} ⊂ \mathcal{Y} ⊂ \mathcal{X}^*$ as in the hypotheses of Theorem 1.11 (Adams and Simon choose $\mathcal{Y}$ to be a certain Hilbert space but do not otherwise precisely specify the regularity properties of the path $u$ in their definition.)

Clearly, the property that a $C^2$ function be Morse–Bott at a critical point is closely related to integrability of that critical point. Indeed, by comparing Definitions 1.9 and 1.17 we obtain the

Lemma 1.18 (Morse–Bott property implies integrability for critical points of $C^2$ functions on Banach spaces). Let $\mathcal{X}$ and $\mathcal{X}$ be Banach spaces over $\mathbb{K}$, and $\mathcal{X} ⊂ \mathcal{X}^*$ be a continuous embedding, and $\mathcal{U} ⊂ \mathcal{X}$ be an open subset, and $f : \mathcal{U} → \mathbb{K}$ be a $C^2$ function such that $f'(x) ∈ \mathcal{X}$ for all $x ∈ \mathcal{U}$. If $f$ is Morse–Bott at a point $x_0 ∈ \text{Crit } f$ in the sense of Definition 1.17, then $x_0$ is an integrable critical point in the sense of Definition 1.17.

The converse to Lemma 1.18 is far more subtle since the integrability condition for a critical point is weaker than the Morse–Bott condition. The following result is proved by Simon for a specific class of analytic functions [6, 108, 109] on certain Banach spaces (given by $C^{2,α}$ sections of a Riemannian vector bundle over a closed Riemannian manifold), but his method of proof extends with little change to give the slightly more general

Theorem 8 (Integrability implies Morse–Bott property for critical points of analytic functions on Banach spaces). Let $\mathcal{X}$ and $\mathcal{X}$ be Banach spaces over $\mathbb{K}$, and $\mathcal{X} ⊂ \mathcal{X}^*$ be a continuous embedding, $\mathcal{U}$ be an open neighborhood of a point $x_0$, and $f : \mathcal{U} → \mathbb{K}$ be an analytic function such that $f'(x) ∈ \mathcal{X}$ for all $x ∈ \mathcal{U}$. If $f'(x_0) = 0$ and $f''(x_0) ∈ \mathcal{L}(\mathcal{X}, \mathcal{X})$ is a Fredholm operator with index zero and the critical point $x_0$ is integrable in the sense of Definition 1.17, then $f$ is Morse–Bott at $x_0$ in the sense of Definition 1.17.

We refer to Appendix C for references to the literature where versions of Theorem 8 are stated, an outline of the proof based on those references, and a discussion of integrability and Morse–Bott conditions for the harmonic map energy and area functions, together with examples. We shall give a detailed proof of a more general version of Theorem 8 elsewhere [10].

1.6. \textbf{Lojasiewicz–Simon gradient inequalities for analytic functions on Banach spaces}. Theorems 3 and 8 can be used to give new proofs of the Lojasiewicz–Simon gradient inequalities for analytic functions on Banach spaces proved earlier by the author and Maridakis in [50]; moreover, our new proofs allow us to slightly weaken the hypotheses that we assumed in [50].

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7I am grateful to Otis Chodosh for reminding me of this condition.
8I am grateful to Leon Simon for explaining the relevant key ideas from [4, 108, 109].
9By relaxing the hypotheses that the continuous embedding $\mathcal{X} ⊂ \mathcal{X}^*$ be definite, that one has a continuous embedding $\mathcal{X} ⊂ \mathcal{X}$, and that $\mathbb{K} = \mathbb{R}$.
Theorem 9 (Łojasiewicz–Simon gradient inequality for analytic functions on Banach spaces). (Compare Feehan and Maridakis [50] Theorem 1.) Let $\mathcal{X}$ be a Banach space over $\mathbb{K}$, and $\mathcal{U} \subset \mathcal{X}$ be an open neighborhood of the origin, and $f : \mathcal{U} \to \mathbb{K}$ be an analytic function such that $f(0) = 0$ and $f'(0) = 0$. If $f''(0) \in \mathcal{L}(\mathcal{X}, \mathcal{X}^*)$ is a Fredholm operator with index zero then, after possibly shrinking $\mathcal{U}$, there are constants $C \in (0, \infty)$ and $\theta \in [1/2, 1)$ such that
\begin{equation}
\|f'(x)\|_{\mathcal{X}} \geq C |f(x)|^\theta, \quad \text{for all } x \in \mathcal{U}.
\end{equation}

The following generalization of Theorem 9 relaxes the strong hypothesis that $f''(0) \in \mathcal{L}(\mathcal{X}, \mathcal{X}^*)$ be Fredholm and immediately yields Theorem 10 upon specializing to $\mathcal{X} = \mathcal{X}^*$.

Theorem 10 (Generalized Łojasiewicz–Simon gradient inequality for analytic functions on Banach spaces). (Compare Feehan and Maridakis [50] Theorem 2.) Let $\mathcal{X}$ and $\mathcal{X}'$ be Banach spaces over $\mathbb{K}$ with a continuous embedding $\mathcal{X}' \subset \mathcal{X}^*$, and $\mathcal{U} \subset \mathcal{X}'$ be an open neighborhood of the origin, $f : \mathcal{U} \to \mathbb{K}$ be an analytic function with $f(0) = 0$ and $f'(0) = 0$ and $f''(x) \in \mathcal{X}'$ for all $x \in \mathcal{U}$. If $f''(0) \in \mathcal{L}(\mathcal{X}, \mathcal{X}')$ is Fredholm with index zero then, after possibly shrinking $\mathcal{U}$, there are constants $C \in (0, \infty)$ and $\theta \in [1/2, 1)$ such that
\begin{equation}
\|f'(x)\|_{\mathcal{X}} \geq C |f(x)|^\theta, \quad \text{for all } x \in \mathcal{U}.
\end{equation}

Theorem 11 is deduced from Theorem 9 in Section 8. While Theorem 10 is sufficient for many applications in geometric analysis, it also excludes some important examples (see [50] Section 1.2 for a discussion of such examples), including Simon’s [108, Theorem 3], so we shall recall a useful generalization of Theorem 10. We say that a bilinear form $b : \mathcal{X} \times \mathcal{X} \to \mathbb{K}$ is definite if $b(x, x) \neq 0$ for all $x \in \mathcal{X} \setminus \{0\}$. We say that a continuous embedding $\mathcal{J} : \mathcal{X} \to \mathcal{X}'$ of a Banach space into its continuous dual space is definite if the pullback of the canonical pairing, $\mathcal{X} \times \mathcal{X}' \ni (x, y) \mapsto \langle x, f(y) \rangle_{\mathcal{X} \times \mathcal{X}'} \in \mathbb{K}$, is a definite bilinear form. The following generalization of Theorem 10 does not appear to be a simple consequence of a Morse Lemma for degenerate critical points like Theorem 9.

Theorem 11 (Generalized Łojasiewicz–Simon gradient inequality for analytic functions on Banach spaces). (See Feehan and Maridakis [50] Theorem 3 for the case $\mathbb{K} = \mathbb{R}$.) Let $\mathcal{X}$ and $\mathcal{X}'$ be Banach spaces over $\mathbb{K}$ with continuous embeddings $\mathcal{X} \subset \mathcal{X}' \subset \mathcal{X}^*$ and such that the embedding $\mathcal{X} \subset \mathcal{X}^*$ is definite. Let $\mathcal{U} \subset \mathcal{X}'$ be an open subset and $f : \mathcal{U} \to \mathbb{K}$ be an analytic function such that $f(0) = 0$ and $f'(0) = 0$. Let
\begin{align*}
\mathcal{X} &\subset \mathcal{G} \subset \mathcal{X}, \\
\mathcal{X}' &\subset \mathcal{G} \subset \mathcal{X}',
\end{align*}
be continuous embeddings of Banach spaces such that the compositions
\begin{align*}
\mathcal{X} &\subset \mathcal{G} \subset \mathcal{X}, \\
\mathcal{X}' &\subset \mathcal{G} \subset \mathcal{X}',
\end{align*}
induce the same embedding $\mathcal{X} \subset \mathcal{G}$. Let $\mathcal{M} : \mathcal{U} \to \mathcal{X}'$ be a gradient map for $f$ in the sense of Definition 11. Suppose that for each $x \in \mathcal{U}$, the bounded, linear operator
\begin{equation}
\mathcal{M}'(x) : \mathcal{X} \to \mathcal{X}'
\end{equation}
has an extension
\begin{equation}
\mathcal{M}_1(x) : \mathcal{G} \to \mathcal{G}
\end{equation}
such that the map
\begin{equation}
\mathcal{U} \ni x \mapsto \mathcal{M}_1(x) \in \mathcal{L}(\mathcal{G}, \mathcal{G})
\end{equation}
is continuous.

If $\mathcal{M}'(0) : \mathcal{X} \to \mathcal{X}'$ and $\mathcal{M}_1(0) : \mathcal{G} \to \mathcal{G}$ are Fredholm operators with index zero then, after possibly shrinking $\mathcal{U}$, there are constants $C \in (0, \infty)$ and $\theta \in [1/2, 1)$ such that
\begin{equation}
\|\mathcal{M}(x)\|_{\mathcal{G}} \geq C |f(x)|^\theta, \quad \text{for all } x \in \mathcal{U}.
\end{equation}
Suppose now that \( \tilde{G} = H \), a Hilbert space, so that the embedding \( \tilde{G} \subset H \) in Theorem 11 factors through \( \tilde{G} \subset H \cong H^* \), and therefore
\[
f'(x)v = \langle v, \mathcal{M}(x) \rangle_{X \times X^*} = \langle v, \mathcal{M}(x) \rangle_H, \quad \text{for all } x \in U \text{ and } v \in X,
\]
using the continuous embeddings \( \tilde{X} \subset H \subset X^* \). As we note in Remark 1.19, the hypothesis in Theorem 11 that the embedding \( X \subset X^* \) is definite is implied by the assumption that \( X \subset H \) is a continuous embedding into a Hilbert space. Theorem 11 then yields
\[
\|\mathcal{M}(x)\|_H \geq C|f(x)|^\theta, \quad \text{for all } x \in U,
\]
as desired.

**Remark 1.19 (Comments on the embedding hypothesis in Theorem 11).** (See Feehan and Mariadakis [50, Remark 1.1].) The hypothesis in Theorem 11 on the continuous embedding \( X \subset X^* \) is easily achieved given a continuous embedding \( \varepsilon \) of \( X \) into a Hilbert space \( H \). Indeed, because \( \langle y, \varepsilon(x) \rangle_{X \times X^*} = (\varepsilon(y), \varepsilon(x))_H \) for all \( x, y \in X \), then \( \langle x, \varepsilon(x) \rangle_{X \times X^*} = 0 \) implies \( x = 0 \); see [21, Remark 3, page 136] or [51, Lemma D.1] for details.

### 1.7. Applications

Due to the difficulty in computing the Łojasiewicz exponent, one should not in general expect Theorems 1 or 2 to provide a useful way to prove the Morse–Bott property of an analytic function with Hessian operator that is Fredholm of index zero. Nonetheless, they provide insight to applications in geometric analysis and we survey a few such applications here.

#### 1.7.1. Yamabe energy function, integrability conditions, Łojasiewicz exponents, and Morse–Bott properties.

Carlotto, Chodosh, and Rubinstein [27] study the existence of ‘slowly-converging’ (volume-normalized) gradient flows for the Yamabe energy function on Riemannian metrics over a closed manifold of dimension three or more with the aid of results due to a) Adams and Simon [4] on the relationship between integrability and certain types of non-integrability and rates of convergence of geometric flows, and b) Chill [29] on the Łojasiewicz–Simon gradient inequality for functions on Banach spaces. In particular, for a certain class of geometric flows, Adams and Simon show that the integrability condition (\( \star \)) implies an exponential rate of convergence [4, Theorem 1 (i)] and in a certain subcase where integrability fails [4, Theorem 1 (ii)], the flow converges according to a negative power law and thus is slowly converging in the terminology of [27]. We refer the reader to Appendix A for an exposition of our general result [45, Theorem 3] on the relationship between the rate of convergence for the gradient flow of a function obeying a Łojasiewicz–Simon gradient inequality near a critical point and the value of the Łojasiewicz exponent.

When \( \mathcal{E} \) is the Yamabe (or Einstein–Hilbert) energy function, Carlotto, Chodosh, and Rubinstein show that the Adams–Simon integrability condition (\( \star \)) implies that the Łojasiewicz exponent for \( \mathcal{E} \) at a critical point is equal to one half [27, Proposition 13], for a suitable choice of Banach spaces, and that in turn indicates (by the main results of this article) that \( \mathcal{E} \) should be Morse–Bott at the critical point. More generally, when \( \mathcal{E} \) is an analytic function on a Banach space obeying hypotheses similar to those of Theorems 1, 2, or perhaps even Theorem 11, we would expect the Adams–Simon integrability condition (\( \star \)) for a critical point to imply that \( \mathcal{E} \) is Morse–Bott at that point by generalizing the proof due to Kwon [73] of her Theorem C.2.

#### 1.7.2. Harmonic map energy function for maps from a Riemann surface into a closed Riemannian manifold.

For background on harmonic maps, we refer to Hélein [60], Jost [69], Simon [110], and Struwe [117]. Let \( (M, g) \) and \( (N, h) \) be a pair of closed, smooth Riemannian manifolds. One
defines the harmonic map energy function by

\[
E_{g,h}(f) := \frac{1}{2} \int_M |df|_{g,h}^2 \, d\text{vol}_g
\]

for smooth maps \( f : M \to N \), where \( df : TM \to TN \) is the differential map.

For the harmonic map energy function, a Łojasiewicz gradient inequality with exponent one half,

\[
\|E'(f)\|_{L^p(S^2)} \geq C\|E(f) - E(f_\infty)\|^{1/2},
\]

has been obtained by Kwon [73, Theorem 4.2] for maps \( f : S^2 \to N \), where \( N \) is a closed Riemannian manifold and \( f \) is close to a harmonic map \( f_\infty \) in the sense that

\[
\|f - f_\infty\|_{W^{2,p}(S^2)} < \sigma,
\]

where \( p \) is restricted to the range \( 1 < p \leq 2 \), and \( f_\infty \) is assumed to be integrable in the sense of [73, Definitions 4.3 or 4.4 and Proposition 4.1]. Her proof of [73, Proposition 4.1] quotes results of Simon [109, pp. 270–272] and Adams and Simon [4, Lemma 1]. The result [81, Lemma 3.3] due to Liu and Yang is another example of a Łojasiewicz gradient inequality with exponent one half for the harmonic map energy function, but restricted to the setting of maps \( f : S^2 \to N \), where \( N \) is a Kähler manifold of complex dimension \( n \geq 1 \) and nonnegative bisectional curvature, and the energy \( \mathcal{E}(f) \) is sufficiently small. The result of Liu and Yang generalizes that of Topping [120, Lemma 1], who assumes that \( N = S^2 \).

Milnor observes [86, Footnote to Problem 3-c] that the space of holomorphic maps of degree \( d \) from \( \mathbb{CP}^1 \) to \( \mathbb{CP}^1 \) is a non-compact complex manifold of dimension \( 2d + 1 \). However, he notes [86, Footnote to Problem 3-c] that there is an example (due to J. Harris) of a Riemann surface \( \Sigma \) of genus 5 such that the space of holomorphic maps from \( \Sigma \) into \( \mathbb{CP}^1 \) has singularities. In general, the space of harmonic maps of degree \( d \) from \( S^2 \) into \( S^{2n} \) (with \( n \geq 1 \)) will not be a smooth manifold [53]. We survey some positive results for spaces of harmonic maps in Appendix C.1.

The version of the ‘Bumpy Metric Theorem’ proved by Moore as [87, Theorem 5.1.1] states that if \( M \) is a compact manifold of dimension at least three and the Riemannian metric is generic, then all minimal two-spheres in \( M \) are as nondegenerate as allowed by the group of conformal automorphisms of \( S^2 \), that is, they lie on nondegenerate critical submanifolds of \( \text{Map}(S^2, M) \), each such submanifold being an orbit for the symmetry group \( \text{PSL}(2, \mathbb{C}) \).

1.7.3. Moduli spaces of flat connections and representation varieties. When a base manifold \( X \) is compact and Kähler\(^{10}\), and \( G \) is a complex reductive Lie group, Simpson proved that the singularities in the moduli space of flat connections are at worst quadratic at any reductive representation of the fundamental group [112, Corollary 2.4]; when \( G \) is a compact Lie group, this result is due to Goldman and Millson [54, Theorem 1]. When the base manifold \( X \) is not compact or Kähler then the singularities in the moduli space of flat connections may be worse. Indeed, this can occur for representation varieties for fundamental groups of certain closed, smooth three-dimensional manifolds. Goldman and Millson [54, Section 9.1] choose \( X = H/\Gamma \), where \( H \) is the three-dimensional real Heisenberg group and \( \Gamma \subset H \) is a lattice, so that \( X \) is the total space of an oriented circle bundle over a two-torus with non-zero Euler class. If \( G \) is an algebraic Lie group that is not two-step nilpotent and \( \rho : \Gamma \to G \) is the trivial representation, then the representation variety \( \mathcal{R}(\Gamma, G) \) is not quadratic at \( \rho \); the analytic germ of \( \mathcal{R}(\Gamma, G) \) is isomorphic to a cubic cone. A compact Lie group with a simple Lie algebra, such as \( \text{SU}(n) \) for \( n \geq 2 \), is not nilpotent and so we may choose \( G = \text{SU}(n) \) with \( n \geq 2 \) in the Goldman–Millson counterexample.

\(^{10}\)I am grateful to Graeme Wilkin for drawing my attention to the results of Simpson and Goldman–Millson described here.
Recall [59, Proposition 2.2.3] that the gauge-equivalence classes of flat connections on a principal \( G \)-bundle over a connected manifold \( X \) are in one-to-one correspondence with the conjugacy classes of representations \( \pi_1(X) \rightarrow G \).

In our articles [48, 47], we explore the Morse–Bott properties of the Yang–Mills energy function,

\[
\mathcal{E}(A) := \frac{1}{2} \int_X |F_A|^2 \, d\text{vol}_g,
\]

on the affine space of \( W^{1,q} \) connections \( A \) on a principal \( G \)-bundle (for a compact Lie group \( G \)) over a closed Riemannian manifold \((X, g)\) of dimension \( d \geq 2 \) (and \( q \in [2, \infty) \) obeying \( q > d/2 \)). In particular, we explore the Morse–Bott properties of \( \mathcal{E} \) in (1.20) near the subspace of flat connections.

### 1.7.4. \( F \)-function on the space of hypersurfaces in Euclidean space.

Colding and Minicozzi [31, 32] have given proofs of Łojasiewicz–Simon gradient and distance inequalities [33, Equations (5.9) and (5.10)] that do not involve Lyapunov–Schmidt reduction to a finite-dimensional gradient inequality, as in the original paradigm due to Simon [108]. Their gradient inequality applies to the \( F \) function [33, Section 2.4] on the space of hypersurfaces \( \Sigma \subset \mathbb{R}^{d+1} \) and is analogous to (1.1) with \( \theta = 2/3 \). Their cited articles contain detailed technical statements of their inequalities while their article with Pedersen [33] contains a less technical summary of some of their main results.

### 1.8. Structure of this article.

In Section 2, we prove the Morse Lemma for functions on Banach spaces with degenerate critical points (Theorem 5) and then deduce some corollaries, including the Morse Lemma for functions on Banach spaces with non-degenerate critical points (Theorem 2.8), and the Morse–Bott Lemma for functions on Banach spaces (Theorems 2.10 and 2.14). In Section 3, we apply Theorem 2.14 to prove the Łojasiewicz gradient inequality for \( C^{p+2} \) Morse–Bott functions on Banach spaces (Theorem 7) and apply Theorem 5 to prove the Łojasiewicz gradient inequality for analytic functions on Banach spaces (Theorem 10). Finally, in Section 4 we complete the proof of the Morse–Bott property of an analytic function with Łojasiewicz exponent one half (Theorem 11). In Appendix A we discuss our general result [45, Theorem 3] on the rate of convergence of a gradient flow for a function obeying a Łojasiewicz gradient inequality. In Appendix B we describe the relationship between Morse–Bott functions and quadratic simple normal crossing functions. In Appendix C we outline the proof of Theorem 8 and survey results on integrability conditions and the Morse–Bott property for critical points of the harmonic map energy and area functions.

### 1.9. Acknowledgments.

I am indebted to Michael Greenblatt and András Némethi for independently pointing out to me that, for functions on Euclidean space, the Morse Lemma for functions with degenerate critical points (also known as the Morse Lemma with parameters or Splitting Lemma) should be the key ingredient needed to prove the main result of this article in the finite-dimensional case (Corollary 3). I am extremely grateful to Brian White for explaining results of his [125, 126, 127] and others on minimal surfaces and integrability of Jacobi fields and to Leon Simon for explaining his results and results with Adams on integrability of Jacobi fields in [41, 108, 109]. I also thank Carles Bivià-Ausina, Otis Chodosh, Tristan Collins, Santiago Encinas, Luis Fernandez, Antonella Grassi, David Hurtubise, Johan de Jong, Daniel Ketover, Qingyue Liu, Doug Moore, Yanir Rubinstein, Siddhartha Sahi, Ovidiu Savin, Peter Topping, Graeme Wilkin, and Jarek Włodarczyk for helpful communications, discussions, or questions during the preparation of this article. I am grateful to the National Science Foundation for their support and the Dublin Institute for Advanced Studies and Yi-Jen Lee and the Institute of Mathematical Sciences at the Chinese University of Hong Kong for their hospitality and support. Lastly, I
am most grateful to the anonymous referee for numerous comments and suggestions that helped improve this article.

2. Generalized Morse Lemmas for functions on Banach spaces

In Sections 2.1 and 2.2 respectively, we collect some basic observations from linear and nonlinear functional analysis that we require in this article. In Section 2.3 we prove the Morse Lemma for functions on Banach spaces with degenerate critical points (Theorem 3) and in Section 2.4 we deduce some corollaries, including the Morse Lemma for functions on Banach spaces with non-degenerate critical points (Theorem 2.8), and the Morse–Bott Lemma for functions on Banach spaces (Theorems 2.10 and 2.13).

2.1. Preliminaries on linear functional analysis. In this subsection, we gather a few elementary observations from linear functional analysis. We begin with the following useful

Lemma 2.1 (Dual space of a direct sum of Banach spaces). (See [4].) If \( X, Y \) are Banach spaces over \( \mathbb{K} \), then \( (X \oplus Y)^* = X^* \oplus Y^* \).

Proof. Let \( Z := X \oplus Y \), with product norm \( \| (x, y) \|_{X \oplus Y} := \| x \|_X + \| y \|_Y \), continuous projection operators \( \pi_X : Z \to X \) and \( \pi_Y : Z \to Y \), continuous injection operators \( \iota_X : X \to Z \) and \( \iota_Y : Y \to Z \), and define \( T : Z^* \to X^* \oplus Y^* \) by \( Tz^* := (z^* \iota_X, z^* \iota_Y) \). We observe that \( T \) is bounded because

\[
\| Tz^* \|_{X^* \oplus Y^*} = \| z^* \iota_X \|_{X^*} + \| z^* \iota_Y \|_{Y^*} \leq 2\| z^* \|_{Z^*},
\]

noting that

\[
\| z^* \iota_X \|_{X^*} = \sup_{x \in X \setminus \{0\}} \frac{|z^*(\iota_X(x))|}{\| x \|_X} \leq \sup_{z \in Z \setminus \{0\}} \frac{|z^*(z)|}{\| z \|_Z} = \| z^* \|_{Z^*},
\]

and similarly \( \| z^* \pi_Y \|_{Y^*} \leq \| z^* \|_{Z^*} \). The operator \( T \) is injective since if \( Tz^* = 0 \), then \( (z^*(\iota_X(x)), z^*(\iota_Y(y))) = 0 \) for all \( (x, y) \in X \oplus Y \) and so \( z^*(z) = 0 \) for all \( z = x + y \in Z \) and hence \( z = 0 \). The operator \( T \) is surjective since if \( x^* \in X^* \) and \( y^* \in Y^* \) and we define \( z^*(z) := x^*(x) + y^*(y) \) for \( z = x + y \in Z \), then \( z^* \in Z^* \). The operator \( T^{-1} \) is bounded by the Open Mapping Theorem or by observing that \( T^{-1}(x^*, y^*) = x^* \pi_X + y^* \pi_Y \in Z^* \). Therefore, \( T \) is an isomorphism of Banach spaces.

In the proof of Lemma 2.1 we note that the adjoint map \( \iota_X^* : Z^* \to X^* \) is continuous and \( (\iota_X^* z^*)(x) = z^*(\iota_X(x)) \) for all \( x \in X \), so if \( x^* \in X^* \), then \( (\iota_X^* x^*)(x) = x^*(\iota_X(x)) = x^*(x) \) for all \( x \in X \). Thus, \( \pi_X^* = \iota_X^* : Z^* \to X^* \) and \( \pi_Y^* = \iota_Y^* : Z^* \to Y^* \) are the induced projection operators. The following lemma helps motivate Definition 1.5 (1) but is not used elsewhere in this article.

Lemma 2.2 (Range of a symmetric operator whose kernel has closed complement). Let \( \mathcal{X} \) be a Banach space over \( \mathbb{K} \). If \( A \in \mathcal{L}(\mathcal{X}, \mathcal{X}^*) \) is symmetric and \( \text{ker} A \) has a closed complement in \( \mathcal{X} \), then the following hold:

1. \( \text{ran} A \subset \mathcal{X}^*_0 \);
2. If \( A \) is Fredholm with index zero, then \( \text{ran} A = \mathcal{X}^*_0 \).

Proof. By hypothesis on \( \mathcal{H} := \text{ker} A \), we have \( \mathcal{X} = \mathcal{X}_0 \oplus \mathcal{H} \) and thus \( \mathcal{X}^* = \mathcal{X}_0^* \oplus \mathcal{H}^* \) by Lemma 2.1. Suppose \( \alpha \in \mathcal{X}^* \) belongs to \( \mathcal{X}_0^* \cap \text{ran} A \), so \( \alpha = Ax \) for some \( x \in \mathcal{X} \). If \( \xi \in \mathcal{H} \), then \( (\xi, \alpha)_{\mathcal{X} \times \mathcal{X}^*} = \langle \xi, Ax \rangle_{\mathcal{X} \times \mathcal{Y}^*} = \langle x, A\xi \rangle_{\mathcal{X} \times \mathcal{Y}^*} = 0 \) as \( \xi \in \text{ker} A \). Because \( \xi \in \mathcal{H} \) was arbitrary, we see that \( \alpha = 0 \) on \( \mathcal{H} \) and thus \( \alpha = 0 \in \mathcal{X}^* \).

Hence, \( \text{ran} A \subset \mathcal{X}_0^* \), as claimed in Item (1).
If $A$ is Fredholm, then $\mathcal{H}$ is finite-dimensional and thus has a closed complement $\mathcal{H}_0$ by [103, Lemma 4.21 (a)]. Item (1) implies that $\text{Ran} A \subset \mathcal{H}_0^*$. Because $A$ has index zero, then $\dim \text{Ker} A = \dim (\mathcal{H}^* / \text{Ran} A)$ and because $\mathcal{H}^* / \text{Ran} A$ is finite-dimensional, $\text{Ran} A$ has a closed complement, say $\mathcal{M}$, with $\mathcal{H}^* = \text{Ran} A \oplus \mathcal{M}$ by [103, Lemma 4.21 (b)]. But $\dim \mathcal{H}^* = \dim \mathcal{H} = \dim \mathcal{M}$ and $\mathcal{H}^* = (\mathcal{H}_0 \oplus \mathcal{H})^* = \mathcal{H}_0^* \oplus \mathcal{H}^*$ by Lemma 2.1, so $\text{Ran} A = \mathcal{H}_0^*$, as claimed in Item (2).

We have the following generalization of [51, Lemma D.3]; note that Lemma 2.3 (2) does not directly generalize Lemma 2.2 (2), since $\mathcal{H}$ is assumed to be reflexive in Lemma 2.3 and while it also helps motivate Definition 1.5 (1), it is not used elsewhere in this article.

**Lemma 2.3** (Isomorphism properties of a symmetric operator). Let $\mathcal{H}$ be a reflexive Banach space over $\mathbb{K}$. If $A \in \mathcal{L}(\mathcal{H}, \mathcal{H}^*)$ is symmetric with closed range, then the following hold:

1. If $\text{Ker} A = \{0\}$, then $\text{Ran} A = \mathcal{H}^*$;
2. If $\text{Ker} A$ has a closed complement $\mathcal{H}_0 \subset \mathcal{H}$, then $\text{Ran} A \cong \mathcal{H}_0^*$.

**Proof.** If $M \subset \mathcal{H}^*$ is a subspace, we recall from [103, Section 4.6] that its annihilator is

$$M^\perp := \{ \phi \in \mathcal{H}^{**} : \langle \alpha, \phi \rangle = 0, \text{ for all } \alpha \in M \},$$

where $\langle \cdot, \cdot \rangle : \mathcal{H}^* \times \mathcal{H}^{**} \to \mathbb{K}$ denotes the canonical pairing, and that by [103, Theorem 4.12]

$$\text{(Ran} A)^\perp = \text{Ker} A^*, \quad \text{(2.1)}$$

where $A^* : \mathcal{H}^{**} \to \mathcal{H}^*$ is the adjoint operator defined by

$$\langle x, A^* \phi \rangle := \langle Ax, \phi \rangle, \quad \text{for all } x \in \mathcal{H}, \phi \in \mathcal{H}^{**}. \quad \text{(2.1)}$$

If $J : \mathcal{H} \to \mathcal{H}^{**}$ is the canonical map defined by $J(x)\alpha = \alpha(x)$ for all $x \in \mathcal{H}$ and $\alpha \in \mathcal{H}^*$, then $J$ is an isomorphism by hypothesis that $\mathcal{H}$ is reflexive and thus

$$\langle y, A^* J(x) \rangle = \langle Ay, J(x) \rangle = \langle x, Ay \rangle, \quad \text{for all } x, y \in \mathcal{H},$$

that is,

$$\langle y, A^* J(x) \rangle = \langle x, Ay \rangle, \quad \text{for all } x, y \in \mathcal{H}, \quad \text{(2.2)}$$

where $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H}^* \to \mathbb{K}$ also denotes the canonical pairing. Hence,

$$\text{Ker} A^* = \{ \phi \in \mathcal{H}^{**} : \langle y, A^* \phi \rangle = 0, \text{ for all } y \in \mathcal{H} \} = J (\{ x \in \mathcal{H} : \langle y, A^* J(x) \rangle = 0, \text{ for all } y \in \mathcal{H} \}) \quad \text{(by reflexivity of } \mathcal{H})$$

$$= J (\{ x \in \mathcal{H} : \langle y, A^* J(x) \rangle = 0, \text{ for all } y \in \mathcal{H} \}) \quad \text{(by } \text{(2.2)})$$

$$= J (\{ x \in \mathcal{H} : \langle y, Ax \rangle = 0, \text{ for all } y \in \mathcal{H} \}) \quad \text{(by symmetry of } A),$$

that is,

$$\text{Ker} A^* = J (\text{Ker} A) \cong \text{Ker} A. \quad \text{(2.3)}$$

Consider Item (1). Because $\text{Ker} A = \{0\}$ by assumption, then (2.1) and (2.3) imply that

$$\text{(Ran} A)^\perp = \{0\}. \quad \text{(2.4)}$$
If $\overline{\text{Ran } A}$ denotes the (norm) closure of $\text{Ran } A$ in $\mathcal{X}^*$, then

$$\text{Ran } A = \overline{\text{Ran } A} \quad \text{(as $\text{Ran } A$ closed by hypothesis)}$$

$$= \perp \left( (\text{Ran } A)^\perp \right) \quad \text{(by [103, Theorem 4.7(a)])}$$

$$= \perp \{0\} \quad \text{(by (2.4))}$$

$$= \mathcal{X}^*,$$

where if $N \subset X^{**}$ is a subspace, we recall from [103, Section 4.6] that its annihilator is

$$\perp N := \{ \alpha \in \mathcal{X}^* : \langle \alpha, \phi \rangle = 0, \text{ for all } \phi \in N \}.$$

This establishes Item (1).

Consider Item (2). By modifying the argument yielding Item (1), we now obtain

$$\text{Ran } A = \perp \left( (\text{Ran } A)^\perp \right) \quad \text{(by [103, Theorem 4.7(a)] and closedness of $\text{Ran } A$)}$$

$$= \perp (\ker A^*) \quad \text{(by (2.1))}$$

$$= \perp (J(\ker A)) \quad \text{(by (2.3))}.$$

But

$$\perp (J(\ker A)) = \{ \alpha \in \mathcal{X}^* : \langle \alpha, Jx \rangle = 0, \text{ for all } x \in \ker A \}$$

$$= \{ \alpha \in \mathcal{X}^* : \langle x, \alpha \rangle = 0, \text{ for all } x \in \ker A \}$$

$$= (\ker A)^\perp,$$

where if $L \subset \mathcal{X}$ is a subspace, we recall from [103, Section 4.6] that its annihilator is

$$L^\perp := \{ \alpha \in \mathcal{X}^* : \langle x, \alpha \rangle = 0, \text{ for all } x \in L \},$$

where $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X}^* \to \mathbb{K}$ denotes the canonical pairing. Therefore, by combining the preceding identities, we obtain

$$\text{Ran } A = (\ker A)^\perp.$$

Since $\ker A \subset \mathcal{X}$ is a closed subspace, the quotient space $\mathcal{X} / \ker A$ is a Banach space (by [21, Proposition 11.8]). By hypothesis, $\ker A$ has a closed complement $\mathcal{X}_0 \subset \mathcal{X}$, so $\mathcal{X} = \mathcal{X}_0 \oplus \ker A$ and $\mathcal{X} / \ker A \cong \mathcal{X}_0$. Hence, by [21, Proposition 11.9] there is an isomorphism of Banach spaces,

$$\mathcal{X}_0^* \cong (\ker A)^\perp.$$

Consequently, we find that

$$\text{Ran } A \cong \mathcal{X}_0^*,$$

as claimed. This establishes Item (2) and completes the proof of Lemma 2.3. □

We have the following generalization of Lemma 2.2 which helps motivate Definition [1.9 1] but is not used elsewhere in this article.

**Lemma 2.4** (Isomorphism properties of a Fredholm operator). Let $\mathcal{X}$ and $\tilde{\mathcal{X}}$ be Banach spaces over $\mathbb{K}$. If $T \in L(\mathcal{X}, \tilde{\mathcal{X}})$ is Fredholm, then $\ker T$ has a closed complement $\mathcal{X}_0$ in $\mathcal{X}$ such that the following hold:

1. $\text{Ran } T \cong \mathcal{X}_0$ and $\tilde{\mathcal{X}} \cong \mathcal{X}_0 \oplus \ker T^*$;
2. If $\text{Index } T = 0$, then $\tilde{\mathcal{X}} \cong \mathcal{X}_0 \oplus \ker T$. 

---
Proof. Consider Item (1). Since $T$ is Fredholm, then $\text{Ker} \, T$ is finite-dimensional and thus has a closed complement $\mathcal{X}_0 \subset \mathcal{X}$ such that $\mathcal{X} = \mathcal{X}_0 \oplus \text{Ker} \, T$ by \cite[Lemma 4.21 (a)]{103}. Similarly, because $T$ is Fredholm, $\text{Ran} \, T$ is a closed subspace of $\mathcal{X}$ and $\text{Coker} \, T = \mathcal{X} / \text{Ran} \, T$ is finite-dimensional, so $\text{Ran} \, T$ has a closed complement $\mathcal{K} \subset \mathcal{X}$ such that $\mathcal{X} = \text{Ran} \, T \oplus \mathcal{K}$ by \cite[Lemma 4.21 (b)]{103}, and $\mathcal{X} / \text{Ran} \, T = \mathcal{K}$. Since $\text{Ran} \, T$ is a Banach space and $T : \mathcal{X}_0 \to \text{Ran} \, T$ bijective and bounded, then $T$ is an isomorphism from $\mathcal{X}_0$ onto $\text{Ran} \, T$ by the Open Mapping Theorem. By \cite[Proposition 11.9]{21}, we have $\mathcal{K}^* = (\text{Ran} \, T)^\perp$, and $(\text{Ran} \, T)^\perp = \text{Ker} \, T^*$ by \cite[Theorem 4.12]{103}, and $\mathcal{X} \cong \mathcal{X}_0 \oplus \text{Ker} \, T^*$, as claimed.

Consider Item (2). If $\text{Index} \, T = 0$, then $\dim \text{Ker} \, T^* = \dim \text{Ker} \, T$ and $\text{Ker} \, T \cong \text{Ker} \, T^*$ by finite-dimensionality and $\mathcal{K} \cong \mathcal{X}_0 \oplus \text{Ker} \, T$.

2.2. Preliminaries on nonlinear functional analysis. In this subsection, we gather a few observations from nonlinear functional analysis.

2.2.1. Differentiable and analytic maps on Banach spaces. We refer to Huang \cite[Section 2.1A]{67}; see also Berger \cite[Section 2.3]{15}. Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces over $\mathbb{K}$, let $\mathcal{U} \subset \mathcal{X}$ be an open subset, and $\mathcal{F} : \mathcal{U} \to \mathcal{Y}$ be a map. Recall that $\mathcal{F}$ is Fréchet differentiable at a point $x \in \mathcal{U}$ with a derivative, $\mathcal{F}'(x) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, if
\[
\lim_{y \to 0} \frac{1}{\|y\|_\mathcal{X}} \|\mathcal{F}(x + y) - \mathcal{F}(x) - \mathcal{F}'(x)y\|_\mathcal{Y} = 0.
\]

Recall from Berger \cite[Definition 2.3.1]{15}, Deimling \cite[Definition 15.1]{36}, or Zeidler \cite[Definition 8.8]{133} that $\mathcal{F}$ is analytic at $x \in \mathcal{U}$ if there exists a constant $r > 0$ and a sequence of continuous symmetric linear maps $L_n : \otimes^n \mathcal{X} \to \mathcal{Y}$ such that $\sum_{n \geq 1} \|L_n\| r^n < \infty$ and there is a positive constant $\delta = \delta(x)$ such that
\[
(2.5) \quad \mathcal{F}(x + y) = \mathcal{F}(x) + \sum_{n \geq 1} L_n(y^n), \quad \text{for all } y \in \mathcal{X} \text{ with } \|y\|_\mathcal{X} < \delta,
\]

where $y^n = (y, \ldots, y) \in \mathcal{X} \times \cdots \times \mathcal{X}$ ($n$-fold product). If $\mathcal{F}$ is differentiable (respectively, analytic) at every point $x \in \mathcal{U}$, then $\mathcal{F}$ is differentiable (respectively, analytic) on $\mathcal{U}$. It is a useful observation that if $\mathcal{F}$ is analytic at $x \in \mathcal{X}$, then it is analytic on a ball $B_x(\varepsilon)$ (see Whittlesey \cite[p. 1078]{128}).

2.2.2. Inverse and implicit mapping theorems for smooth and analytic maps on Banach spaces. Statements and proofs of the Inverse Mapping Theorem for $C^k$ maps of Banach spaces are provided by Abraham, Marsden, and Ratiu \cite[Theorem 2.5.2]{3}, Deimling \cite[Theorem 4.15.2]{36}, and Zeidler \cite[Theorem 4.4]{133}; statements and proofs of the Inverse Mapping Theorem for analytic maps of Banach spaces are provided by Berger \cite[Corollary 3.3.2]{15} (complex), Deimling \cite[Theorem 4.15.3]{36} (real or complex), and Zeidler \cite[Corollary 4.37]{133} (real or complex). The corresponding Implicit Mapping Theorems for $C^k$ or analytic maps are proved in the standard way as corollaries, for example \cite[Theorem 2.5.7]{3} and \cite[Theorem 4.4]{133}.

2.2.3. Gradient maps. We recall the following basic facts concerning gradient maps.

**Proposition 2.5** (Properties of gradient maps). (See Huang \cite[Proposition 2.1.2]{67}.) Let $\mathcal{U}$ be an open subset of a Banach space, $\mathcal{X}$, let $\mathcal{Y}$ be continuously embedded in $\mathcal{X}^*$, and let $\mathcal{M} : \mathcal{U} \to \mathcal{Y} \subset \mathcal{X}^*$ be a continuous map. Then the following hold.

(1) If $\mathcal{M}$ is a gradient map for $\mathcal{E}$, then
\[
\mathcal{E}(x_1) - \mathcal{E}(x_0) = \int_0^1 \langle x_1 - x_0, \mathcal{M}(tx_1 + (1-t)x_0) \rangle_{\mathcal{X} \times \mathcal{X}^*} \, dt, \quad \text{for all } x_0, x_1 \in \mathcal{U}.
\]
(2) If \( M \) is of class \( C^1 \), then \( M \) is a gradient map if and only if all of its Fréchet derivatives, \( M'(x) \) for \( x \in U \), are symmetric in the sense that
\[
\langle w, M'(x)v \rangle_{\mathcal{X} \times \mathcal{X}^*} = \langle v, M'(x)w \rangle_{\mathcal{X} \times \mathcal{X}^*}, \quad \text{for all } x \in U \text{ and } v, w \in \mathcal{X}.
\]
(3) If \( M \) is an analytic gradient map, then any potential \( \mathcal{E} : U \to \mathbb{R} \) for \( M \) is analytic.

2.3. Morse Lemma for functions on Banach spaces with degenerate critical points.

In this subsection, we prove Theorem 5 and hence Theorem 4 by taking \( \mathcal{X} = \mathcal{X}^* \).

Given Banach spaces \( \mathcal{X} \) and \( \mathcal{Z} \) over \( \mathbb{K} \), an open subset \( U \subset \mathcal{X} \), and a smooth map, \( f : U \to \mathcal{Y}, \) and an integer \( n \geq 0 \), we partly follow Zeidler [555, Sections 4.3–4.5] and let \( D^nf(x) = f^{(n)}(x) \in \mathcal{L}^n(\mathcal{X}, \mathcal{Z}) = \mathcal{L}(\otimes^n \mathcal{X}, \mathcal{Z}) \) denote the derivatives of order \( n \) at a point \( x \in U \).

If \( \mathcal{X} = \mathcal{X}_i \times \mathcal{X}_2 \), a product of Banach spaces \( \mathcal{X}_i \) over \( \mathbb{K} \) for \( i = 1, 2 \), we let
\[
D_i f(x_1, x_2) = f_{x_i}(x_1, x_2) \in \mathcal{L}(\mathcal{X}_i, \mathcal{Z})
\]
and
\[
D_{ij} f(x_1, x_2) = f_{x_ix_j}(x_1, x_2) \in \mathcal{L}(\mathcal{X}_i \otimes \mathcal{X}_j, \mathcal{Z})
\]
denote the first and second-order partial derivatives at a point \( (x_1, x_2) \in U \); we may also write \( D_i^2 f(x_1, x_2) \) in place of \( D_i f(x_1, x_2) \) for \( i = 1, 2 \). We let \( \text{GL}(\mathcal{X}) \subset \mathcal{L}(\mathcal{X}) \) denote the group of invertible operators on \( \mathcal{X} \).

Proof of Theorem 5 We generalize Ang and Tuan’s proof of the [8] Morse–Palais Lemma, where \( f \) is \( C^{p+2} \) with \( p \geq 1 \) and \( \mathcal{Y} = \{0\} \) and \( 0 \in \mathcal{X} \) is a non-degenerate critical point, Hörmander’s proof of [555, Lemma C.6.1], where \( f \) is \( C^\infty \) and \( (0, 0) \in \mathcal{X} \times \mathcal{Y} \) is a degenerate critical point and \( \mathcal{X} = \mathbb{R}^n \) and \( \mathcal{Y} = \mathbb{R}^m \), and Lang’s proof of [554, Theorem 7.5.1], where \( f \) is \( C^{p+2} \) with \( p \geq 1 \) and \( \mathcal{Y} = \{0\} \) and \( 0 \in \mathcal{X} \) is a non-degenerate critical point and \( \mathcal{X} \) is a real Hilbert space.

Consider the \( C^{p+1} \) map,
\[
M : \mathcal{X} \times \mathcal{Y} \ni (x, y) \mapsto M(x, y) := D_1 f(x, y) \in \mathcal{X},
\]
and observe that its partial derivative with respect to \( x \), that is, the \( C^p \) map,
\[
D_1 M : \mathcal{U} \times \mathcal{Y} \ni (x, y) \mapsto D_1 M(x, y) : D_1 f(x, y) \eta = D_1^2 f(x, y) \eta \in \mathcal{X},
\]
gives an isomorphism,
\[
\mathcal{X} \ni \eta \mapsto D_1 M(0, 0) \eta \in \mathcal{X},
\]
by our hypothesis on \( D_2^2 f(0, 0) = D_1 M(0, 0) \). By the Implicit Mapping Theorem, after possibly shrinking \( \mathcal{U} \) and \( \mathcal{V} \), there exists a \( C^{p+1} \) map,
\[
\psi : \mathcal{Y} \ni y \mapsto w = \psi(y) \in \mathcal{U} \subset \mathcal{X},
\]
with \( \psi(0) = 0 \), such that \( M(x, y) = 0 \) if and only if \( x = \psi(y) \), for each \( y \in \mathcal{V} \); moreover,
\[
D_y M(\psi(y), y) = 0 = D_1 M(0) \psi(y) + D_2 M(\psi(y), y),
\]
where \( D_y M(\psi(y), y) \) denotes the derivative of the one-variable map \( M(\psi(y), y) \) with respect to \( y \), and so
\[
D \psi(y) = - (D_1 M(\psi(y), y))^{-1} D_y M(\psi(y), y) \in \mathcal{L}(\mathcal{Y}, \mathcal{X}), \quad \text{for all } y \in \mathcal{V}.
\]
Define a \( C^{p+1} \) map,
\[
\Psi : \mathcal{U} \times \mathcal{V} \ni (w, y) \mapsto (x, y) = \Psi(w, y) := (w + \psi(y), y) \in \mathcal{U} \times \mathcal{V},
\]
a \( C^{p+1} \) function \( \tilde{f} \), and a \( C^{p+1} \) map \( \tilde{\mathcal{M}} \) by
\[
\tilde{f}(w, y) := f \circ \Psi(w, y) = f(w + \psi(y), y),
\]
\[
\tilde{\mathcal{M}}(w, y) := D_1 \tilde{f}(w, y) = D_1 f(w + \psi(y), y)
= \mathcal{M}(w + \psi(y), y), \text{ for all } (w, y) \in \mathcal{U} \times \mathcal{V},
\]
noting that \( D_1 \Psi(w, y) = \id_{\mathcal{X}} \), since \( \Psi(w, y) = (w + \psi(y), y) \), and
\[
D_1 \tilde{f}(w, y) = D_1 (f \circ \Psi)(w, y) = D_1 f(\Psi(w, y)) \circ D_1 \Psi(w, y)
= D_1 f(w + \psi(y), y) \circ \id_{\mathcal{X}} = D_1 f(w + \psi(y), y) = D_1 f(x, y),
\]
from the Chain Rule. The map \( \Psi \) has derivative
\[
D\Psi(w, y) = \begin{pmatrix}
\id_{\mathcal{X}} & D\psi(y) \\
0 & \id_{\mathcal{V}}
\end{pmatrix} \in \mathcal{L}(\mathcal{X} \oplus \mathcal{V}), \text{ for all } (w, y) \in \mathcal{U} \times \mathcal{V},
\]
an invertible operator at each \((w, y) \in \mathcal{U} \times \mathcal{V}\). In particular, after possibly shrinking \( \mathcal{U} \) and \( \mathcal{V} \), the map \( \Psi \) is a \( C^{p+1} \) diffeomorphism of an open neighborhood of the origin in \( \mathcal{X} \times \mathcal{V} \) by the Inverse Mapping Theorem. Therefore, the identity
\[
\mathcal{M}(\psi(y), y) = 0, \text{ for all } y \in \mathcal{V},
\]
is equivalent to
\[
\tilde{\mathcal{M}}(0, y) = 0, \text{ for all } y \in \mathcal{V},
\]
since \( \tilde{\mathcal{M}}(0, y) = \mathcal{M}(\psi(y), y) \circ D\Psi(0, y) \) and \( D\Psi(0, y) \) is invertible, for all \( y \in \mathcal{V} \). The Chain Rule and calculations similar to previous ones giving \( D_1 \tilde{f}(w, y) = D_1 f(x, y) \) also yield
\[
D_1^2 \tilde{f}(w, y) = D_1^2 f(x, y) \text{ and thus } D_1^2 \tilde{f}(0, 0) = D_1^2 f(0, 0) = A \in \mathcal{L}_{\text{sym}}(\mathcal{X}, \mathcal{X}^*),
\]
recalling the definition of \( A \) in the statement of Theorem 5. By shrinking \( \mathcal{U} \) if necessary, we may assume that \( \mathcal{U} \) is convex and so by the second-order Taylor Formula [25, Section 1.4] we have
\[
\tilde{f}(w, y) = \tilde{f}(0, y) + D_1 \tilde{f}(0, y)w + \int_0^1 (1 - t)D_1^2 \tilde{f}(tw, y)w^2 dt, \text{ for all } (w, y) \in \mathcal{U} \times \mathcal{V},
\]
that is,
\[
\tilde{f}(w, y) = \tilde{f}(0, y) + \langle w, \tilde{\mathcal{M}}(0, y) \rangle + \int_0^1 (1 - t) \left\langle w, D_1 \tilde{\mathcal{M}}(tw, y)w \right\rangle_{\mathcal{X} \times \mathcal{X}^*} dt,
\]
for all \((w, y) \in \mathcal{U} \times \mathcal{V}\).

Therefore, using \( \tilde{\mathcal{M}}(0, y) = 0 \) for all \( y \in \mathcal{V} \),
\[
\tilde{f}(w, y) = \tilde{f}(0, y) + \frac{1}{2} \langle w, B(w, y)w \rangle_{\mathcal{X} \times \mathcal{X}^*}, \text{ for all } (w, y) \in \mathcal{U} \times \mathcal{V},
\]
where
\[
B(w, y) := 2 \int_0^1 (1 - t)D_1 \tilde{\mathcal{M}}(tw, y) dt, \text{ for all } (w, y) \in \mathcal{U} \times \mathcal{V}.
\]
The expression (2.8) for \( B \) defines a \( C^p \) map,
\[
\mathcal{X} \times \mathcal{V} \supset \mathcal{U} \times \mathcal{V} \ni (w, y) \mapsto B(w, y) \in \mathcal{L}_{\text{sym}}(\mathcal{X}, \mathcal{X}^*),
\]
such that \( B(0, 0) = D_1 \tilde{\mathcal{M}}(0, 0) = A \). We now generalize an argument due to Ang and Tuan (see [8, Lemma 1]) from the case \( \mathcal{X} = \mathcal{X}^* \) to the case \( \mathcal{X} \subset \mathcal{X}^* \) and make the
\textbf{Definition 2.6} (A closed subspace of the space of continuous, linear operators). Let \( \mathcal{L}_A(\mathcal{X}) \subset \mathcal{L}(\mathcal{X}) \) denote the closed subspace of operators \( R \in \mathcal{L}(\mathcal{X}) \) whose adjoints \( R^* \in \mathcal{L}(\mathcal{X}^*) \) restrict to operators \( R^* \upharpoonright \mathcal{X} \in \mathcal{L}(\mathcal{X}) \) after composition with the embedding \( \mathcal{X} \subset \mathcal{X}^* \) and obey
\begin{equation}
R^* A = AR \in \mathcal{L}(\mathcal{X}, \tilde{\mathcal{X}}).
\end{equation}

We would like to write \( z = R(w, y)w \in \mathcal{X} \) where, after possibly further shrinking \( \mathcal{U} \) and \( \mathcal{V} \),
\begin{equation}
\mathcal{X} \times \mathcal{U} \supset \mathcal{U} \times \mathcal{V} \ni (w, y) \mapsto R(w, y) \in \text{GL}(\mathcal{X}) \cap \mathcal{L}_A(\mathcal{X})
\end{equation}
is a \( C^p \)-map such that \( R(0, 0) = \text{id}_\mathcal{X} \) and
\begin{equation}
\langle w, B(w, y)w \rangle_{\mathcal{X} \times \mathcal{X}^*} = \langle R(w, y)w, AR(w, y)w \rangle_{\mathcal{X} \times \mathcal{X}^*}, \quad \text{for all} \ (w, y) \in \mathcal{U} \times \mathcal{V},
\end{equation}
where \( B(w, y) \) is as in (2.8) and we recall that there is a continuous embedding \( \mathcal{X} \subset \mathcal{X}^* \) by hypothesis. (After preparing the required foundations, the map \( R \) will be contructed in the forthcoming Equations (2.15).) The identity (2.10) follows if we can write
\begin{equation}
B(w, y) = R(w, y)^* AR(w, y), \quad \text{for all} \ (w, y) \in \mathcal{U} \times \mathcal{V}.
\end{equation}
Equation (2.11) is valid at \((w, y) = (0, 0)\) with \( R(0, 0) = \text{id}_\mathcal{X} \) and \( B(0, 0) = A \). We have the following generalization of Ang and Tuan [8, Lemma 1].

\textbf{Claim 2.7} (Isomorphism onto a space of continuous, linear symmetric operators). \textit{The following linear map is an isomorphism of Banach spaces},
\begin{equation}
\mathcal{L}_A(\mathcal{X}) \ni Q \mapsto Q^* A + AQ \in \mathcal{L}_{\text{sym}}(\mathcal{X}, \tilde{\mathcal{X}}).
\end{equation}

\textit{Proof.} We first observe that the map (2.12) is well-defined by virtue of the Definition 2.6 of the subspace \( \mathcal{L}_A(\mathcal{X}) \). Second, we show that the map (2.12) is surjective. If \( C \in \mathcal{L}_{\text{sym}}(\mathcal{X}, \tilde{\mathcal{X}}) \), set
\begin{equation}
Q := \frac{1}{2} A^{-1} C \in \mathcal{L}(\mathcal{X}).
\end{equation}
The adjoint of \( Q \) is \( Q^* = \frac{1}{2} C^* (A^{-1})^* \in \mathcal{L}(\mathcal{X}^*) \). Now, \( A^* = A \) and \( C^* = C \in \mathcal{L}(\mathcal{X}, \tilde{\mathcal{X}}) \) by our earlier discussion of properties of operators in \( \mathcal{L}_{\text{sym}}(\mathcal{X}, \tilde{\mathcal{X}}) \) and thus also \((A^*)^{-1} = A^{-1} \in \mathcal{L}(\mathcal{X}, \tilde{\mathcal{X}}) \). But\(^{11} \) \((A^{-1})^* = (A^{-1}) \in \mathcal{L}(\mathcal{X}^*, \tilde{\mathcal{X}}^*) \) and thus \((A^{-1})^* = A^{-1} \in \mathcal{L}(\mathcal{X}, \tilde{\mathcal{X}}) \). By combining these observations, we see that \( Q^* = \frac{1}{2} C^* (A^{-1})^* = \frac{1}{2} CA^{-1} \in \mathcal{L}(\mathcal{X}) \), so \( Q \in \mathcal{L}_A(\mathcal{X}) \), as required, and
\begin{equation}
Q^* A + AQ = \frac{1}{2} \left( C^* (A^{-1})^* A + AA^{-1} C \right) = \frac{1}{2} \left( CA^{-1} A + AA^{-1} C \right) = C \in \mathcal{L}_{\text{sym}}(\mathcal{X}, \tilde{\mathcal{X}}),
\end{equation}
completing the proof of surjectivity. Third, we show that the map (2.12) is injective. If \( AQ + Q^* A = 0 \), then \( AQ = -Q^* A \in \mathcal{L}(\mathcal{X}, \tilde{\mathcal{X}}) \) while \( AQ = Q^* A \) by (2.9) and thus \( AQ = 0 \) and so \( Q = 0 \in \mathcal{L}_A(\mathcal{X}) \) since \( A \) is invertible. Clearly, the map (2.12) is continuous and its inverse is also continuous by the Open Mapping Theorem. This completes the proof of Claim 2.7. \( \square \)

The derivative of the quadratic map,
\begin{equation}
\mathcal{D} : \mathcal{L}_A(\mathcal{X}) \ni P \mapsto P^* AP \in \mathcal{L}_{\text{sym}}(\mathcal{X}, \tilde{\mathcal{X}}),
\end{equation}
at \( P \) in the direction \( Q \) is given by
\begin{equation}
\mathcal{D} \mathcal{D}(P) : \mathcal{L}_A(\mathcal{X}) \ni Q \mapsto Q^* AP + P^* AQ \in \mathcal{L}_{\text{sym}}(\mathcal{X}, \tilde{\mathcal{X}}).
\end{equation}

\( ^{11} \)To avoid notational clutter, we omit explicit notation, such as \( \iota : \mathcal{X} \subset \mathcal{X}^* \), for the continuous embedding.

\( ^{12} \)Because \( AA^{-1} = \text{id}_\mathcal{X} = A^{-1} A \) and by [103] Exercise 4.8, one has \((A^{-1})^* A^* = \text{id}_\mathcal{X} = A^*(A^{-1})^* \), so \((A^*)^{-1} = (A^{-1})^* \).
Note that the map (2.13) is well-defined. Indeed, \((P^*A)^* = P^*A^*P^{**} \in \mathcal{L}(\mathcal{X}^{**})\), where \(P \in \mathcal{L}(\mathcal{X})\) has adjoint operator \(P^* \in \mathcal{L}(\mathcal{X}^{**})\) and bidual operator \(P^{**} \in \mathcal{L}(\mathcal{X}^{**})\). But \(P^{**} = P\) (for example, see Brezis [21, Theorem 3.24] or Pietsch [85, Chapter 0, Section A.3.6]) and thus \((P^*A)^* = P^*A = P^*AP = P^* \in \mathcal{L}(\mathcal{X}, \mathcal{X}')\) and \(P^*A\) is symmetric. When \(P\) is the identity operator, we have \(D2(id_{\mathcal{X}}) = Q^*A + AQ\) and this operator is an isomorphism by Claim 2.7.

We now adapt the proof of Ang and Tuan [8, Lemma 2] and the remainder of the proof of Hörmander [65, Lemma C.6.1]. The Implicit Mapping Theorem for analytic maps [12] provides open neighborhoods, \(\mathcal{O}_{id} \subset \mathcal{L}_A(\mathcal{X})\) of the identity operator \(id_{\mathcal{X}}\) and \(\mathcal{O}_A \subset \mathcal{L}_{sym}(\mathcal{X}, \mathcal{X}')\) of the operator \(A\), such that the restriction of the analytic map (2.13),

\[
\mathcal{L}_A(\mathcal{X}) \ni \mathcal{O}_{id} \ni Q \mapsto Q^*AQ \in \mathcal{O}_A \subset \mathcal{L}_{sym}(\mathcal{X}, \mathcal{X}'),
\]

is an analytic diffeomorphism onto its image, with analytic inverse,

\[
F : \mathcal{L}_{sym}(\mathcal{X}, \mathcal{X}') \ni \mathcal{O}_A \ni S \mapsto F(S) \in \mathcal{O}_{id} \subset \mathcal{L}_A(\mathcal{X}),
\]

such that \(F(A) = id_{\mathcal{X}}\). Therefore, Equation (2.10) is fulfilled when we choose

\[
R = F(B),
\]

where \(B\) is as in (2.13). Substituting \(z = R(w, y)w\) in Equation (2.10) and combining this identity with our previous expression (2.7) for \(\tilde{f}(w, y)\) yields

\[
\tilde{f}(w, y) = \tilde{f}(0, y) + \frac{1}{2}(w, B(w, y)w)_{\mathcal{X} \times \mathcal{X}},
\]

\[
= \tilde{f}(0, y) + \frac{1}{2}(R(w, y)w, AR(w, y)w)_{\mathcal{X} \times \mathcal{X}},
\]

\[
= \tilde{f}(0, y) + \frac{1}{2}(z, Az)_{\mathcal{X} \times \mathcal{X}}, \quad \text{for all } (w, y) \in \mathcal{U} \times \mathcal{V}.
\]

Observe that the \(C^0\) map,

\[
\mathcal{U} \times \mathcal{V} \ni (w, y) \mapsto (R(w, y)w, y) \in \mathcal{X} \times \mathcal{Y},
\]

has derivative at the origin,

\[
(\text{id}_{\mathcal{X}} & 0 \\
0 & \text{id}_{\mathcal{Y}}) \in \mathcal{L}(\mathcal{X} \oplus \mathcal{Y}),
\]

since \(R(0, 0) = id_{\mathcal{X}}\), and thus is locally invertible. Hence, after possibly further shrinking \(\mathcal{U}\) and applying the Inverse Mapping Theorem, the map

\[
\mathcal{U} \times \mathcal{V} \ni (w, y) \mapsto (z, y) = (R(w, y)w, y) \in \mathcal{U} \times \mathcal{V}
\]

is a \(C^0\) diffeomorphism onto \(\mathcal{U}' \times \mathcal{V}\), where \(\mathcal{U}'\) is an open neighborhood of the origin in \(\mathcal{X}\). We denote its \(C^0\) inverse map by

\[
\Xi : \mathcal{U}' \times \mathcal{V} \ni (z, y) \mapsto (w, y) \in \mathcal{U} \times \mathcal{V},
\]

and note that \(\Xi(0, 0) = (0, 0)\) with derivative at the origin,

\[
D\Xi(0, 0) = (\text{id}_{\mathcal{X}} & 0 \\
0 & \text{id}_{\mathcal{Y}}) \in \mathcal{L}(\mathcal{X} \oplus \mathcal{Y}),
\]

Lang [73, Theorem 5.2] and Palais [95, p. 969] use a power series argument to define \(F\) rather than apply the Implicit Mapping Theorem for analytic maps.
Thus at $t_\gamma$ in Crit $f$, we have
$$f_t(z) = 0, \quad \text{for all } (z, y) \in U' \times V.$$
But $f_t(w, y) = f_t(P(w, y))$ and setting $(x, y) = P(w, y) = P_t(z, y) = t\Phi(z, y)$, we obtain
$$f_t(z) = f_t(z) + \frac{1}{2} \langle z, Az \rangle, \quad \text{for all } (z, y) \in U' \times V,$$
which is the desired relation (1.12). Equations (2.6) and (2.17) and the Chain Rule give
$$D_\Phi(0) = \left( \begin{array}{cc} \text{id}_{\mathcal{X}} & * \\ 0 & \text{id}_{\mathcal{Y}} \end{array} \right) \in \mathcal{L}(\mathcal{X} + \mathcal{Y}),$$
which is (1.11). The conclusion on analyticity of $\Phi$ follows by replacing the role of the Inverse Mapping Theorem for $C^p$ maps in the preceding arguments by its counterpart for analytic maps when $f$ is analytic (see Section 2.2.2). The proof of Theorem 2.8 is complete. 

2.4. Applications to proofs of the Morse and Morse–Bott Lemmas for functions on Banach spaces. We begin by recalling the

**Theorem 2.8** (Morse Lemma for functions on Banach spaces with non-degenerate critical points). (See Palais [91, p. 307], [95, p. 968].) Let $\mathcal{X}$ be a Banach space over $\mathbb{K}$, and $U \subset \mathcal{X}$ be an open neighborhood of the origin, and $f : \mathcal{X} \to \mathbb{K}$ be a $C^{p+2}$ function ($p \geq 1$) such that $f(0) = 0$ and $f'(0) = 0$. If $f''(0) \in \mathcal{L}(\mathcal{X}, \mathcal{X}^*)$ is invertible, then there are an open neighborhood of the origin $V \subset \mathcal{X}$ and a $C^p$ diffeomorphism, $\mathcal{X} \ni y \mapsto x = \Phi(y) \in \mathcal{X}$ with $\Phi(0) = 0$ and $D\Phi(0) = \text{id}_{\mathcal{X}}$, such that

$$f(\Phi(z)) = f'(\Phi(z)) + \left( \begin{array}{c} \text{id} \\ \text{id} \end{array} \right) \langle z, Az \rangle + \left( \begin{array}{c} \text{id} \\ \text{id} \end{array} \right) \langle z, Bz \rangle,$$

where $A := f''(0) = (f \circ \Phi)'(0) \in \mathcal{L}_{\text{sym}}(\mathcal{X}, \mathcal{X}^*)$.

If $f$ is analytic, then $\Phi$ is analytic. If $\mathcal{X}$ is a Hilbert space with norm $\| \cdot \|$, then one may further choose $\Phi$ and an orthogonal decomposition $\mathcal{X} = \mathcal{X}_+ \oplus \mathcal{X}_-$ such that

$$f(\Phi(z)) = \frac{1}{2} \left( \| z_+ \|^2 - \| z_- \|^2 \right), \quad \text{for all } z = z_+ + z_- \in \mathcal{X}.$$

Theorem 2.8 is an immediate consequence of the more general Theorem 2.10 and which is proved below (see also Lang [73, Corollary 5.3]).

**Remark 2.9** (Tangent space to the critical set as a subspace of the kernel of the Hessian operator). If the critical set Crit $f$ of a smooth function $f : \mathcal{X} \to \mathbb{R}$ is a smooth submanifold of $\mathcal{X}$ and $x_0 \in \text{Crit } f$, then $T_{x_0} \text{Crit } f \subset \text{Ker } f''(x_0)$. Indeed, if $v \in T_{x_0} \text{Crit } f$ and $\gamma(t)$ is a smooth curve in Crit $f$ with $\gamma(0) = x_0$ and $\gamma'(0) = v$, where $t \in (\varepsilon, \varepsilon)$, then $f'(\gamma(t)) = 0 \in \mathcal{X}^*$, since $\gamma(t) \in \text{Crit } f$, and so the Chain Rule gives

$$(f \circ \gamma)'(t) = f''(\gamma(t))\gamma'(t) = 0 \in \mathcal{X}^*.$$ 

Thus at $t = 0$, we have $f(\gamma(0))''\gamma'(0) = f''(x_0)v = 0 \in \mathcal{X}^*$ and hence $v \in \text{Ker } f''(x_0)$.

We have the following generalization of Theorem 2.8

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14In other words, $f$ is Morse at the point $0 \in \mathcal{X}$. 

---
Theorem 2.10 (Morse–Bott Lemma for functions on Banach spaces). Let $\mathcal{X}$ be a Banach space over $K$, and $\mathcal{U} \subset \mathcal{X}$ be an open neighborhood of the origin, and $f : \mathcal{U} \to K$ be a $C^{p+2}$ function ($p \geq 1$) such that $f(0) = 0$. If $f$ is Morse–Bott at the origin in the sense of Definition 1.5 (7), then, after possibly shrinking $\mathcal{U}$, there are an open neighborhood of the origin $\mathcal{V} \subset \mathcal{X}$ and a $C^p$ diffeomorphism, $\mathcal{V} \ni y \mapsto x = \Phi(y) \in \mathcal{U}$ with $\Phi(0) = 0$ and $D\Phi(0) = \text{id}_\mathcal{X}$, such that

$$f(\Phi(y)) = \frac{1}{2} \langle y, Ay \rangle_{\mathcal{X} \times \mathcal{X}^*}, \quad \text{for all } y \in \mathcal{V},$$

where

$$A := f''(0) = (f \circ \Phi)''(0) \in \mathcal{L}_{\text{sym}}(\mathcal{X}, \mathcal{X}^*).$$

If $f$ is analytic, then $\Phi$ is analytic.

Remark 2.11 (Morse–Bott Lemma for functions on Banach spaces and local coordinates). By Definition 1.5 (1), the closed subspace $\mathcal{K} = \text{Ker} f''(0) \subset \mathcal{X}$ has a closed complement $\mathcal{X}_0 \subset \mathcal{X}$ such that $\mathcal{X} = \mathcal{X}_0 \oplus \mathcal{K}$ (and so $\mathcal{X}^* = \mathcal{X}_0^* \oplus \mathcal{K}^*$) by Lemma 2.1. If $\pi \in \mathcal{L}(\mathcal{X}, \mathcal{X}_0)$ and $\iota^* \in \mathcal{L}(\mathcal{X}_0^*, \mathcal{X}_0^*)$ are the continuous projections (where $\iota : \mathcal{X}_0 \to \mathcal{X}$ is the continuous injection), then (2.20) becomes

$$f(\Phi(y)) = \frac{1}{2} \langle \pi y, A_0 \pi y \rangle_{\mathcal{X} \times \mathcal{X}^*}, \quad \text{for all } y \in \mathcal{V},$$

where $A_0 := \iota^* \pi \in \mathcal{L}_{\text{sym}}(\mathcal{X}_0, \mathcal{X}_0^*)$ is an isomorphism. Indeed, if we write $x = (w, \xi) \in \mathcal{X}_0 \oplus \mathcal{K}$, then $y = \Phi(x) = (z, \xi) \in \mathcal{X}_0 \oplus \mathcal{K}$ for all $x \in \mathcal{U}$ and $A_0 = Df(0, 0) = D^2(f \circ \Phi)(0, 0)$ and (2.20) becomes

$$f(\Phi(z, \xi)) = \frac{1}{2} \langle z, A_0 z \rangle_{\mathcal{X} \times \mathcal{X}^*}, \quad \text{for all } (z, \xi) \in \mathcal{V} \cap (\mathcal{X}_0 \oplus \mathcal{K}),$$

for coordinates adapted to the direct sum decomposition.

Remark 2.12 (Morse–Bott Lemma for functions on Hilbert spaces). Suppose now that $\mathcal{X}$ is a Hilbert space over $K$ and identify $\mathcal{X}^* \cong \mathcal{X}$, so $\mathcal{A} \in \mathcal{L}(\mathcal{X})$ is self-adjoint (since $\mathcal{A} \in \mathcal{L}(\mathcal{X}, \mathcal{X}^*)$ is symmetric) and thus has spectrum $\sigma(\mathcal{A}) \subset \mathbb{R}$ by [103] Theorem 12.15 (b)]. By the Spectral Theorem for bounded normal operators on a Hilbert space [103] pp. 321–327], there are an orthogonal decomposition into closed invariant subspaces, $\mathcal{X} = \mathcal{X}_0^+ \oplus \mathcal{X}_0^- \oplus \mathcal{K}$ corresponding to the Borel subsets $(0, \infty)$, $(-\infty, 0)$, and $\{0\}$ of $\sigma(\mathcal{A})$, continuous projections $\pi_\pm \in \mathcal{L}(\mathcal{X}, \mathcal{X}_0^\pm)$, and injections $\iota_\pm \in \mathcal{L}(\mathcal{X}_0^\pm, \mathcal{X})$, and invertible positive operators $A^\pm := \pi_\pm \iota_\pm \in \mathcal{L}(\mathcal{X}_0^\pm)$ and $A^- := -\pi_- \iota_- \in \mathcal{L}(\mathcal{X}_0^-, \mathcal{X})$, such that

$$f(\Phi(z, \xi)) = \frac{1}{2} \langle z^+, A^+ z^+ \rangle_{\mathcal{X}} - \frac{1}{2} \langle z^-, A^- z^- \rangle_{\mathcal{X}} \quad \text{for all } (z, \xi) \in \mathcal{V} \cap (\mathcal{X}_0 \oplus \mathcal{K}),$$

where $z^\pm = \pi^\pm z$ and $\langle \cdot, \cdot \rangle_{\mathcal{X}}$ denotes the inner product on $\mathcal{X}$. The operators $A^\pm$ have (unique) invertible positive square roots $S^\pm$ [103] Theorem 12.33] and so we may define a norm on $\mathcal{X}_0$ that is equivalent to $\| \cdot \|_{\mathcal{X}}$ by setting $\|z\|_{S} = \|S^\pm z\|_{\mathcal{X}}$ for all $z \in \mathcal{X}_0$, so that (2.21) becomes

$$f(\Phi(z, \xi)) = \frac{1}{2} \left( \|z^+\|_S^2 - \|z^-\|_S^2 \right), \quad \text{for all } (z, \xi) \in \mathcal{V} \cap (\mathcal{X}_0 \oplus \mathcal{K}),$$

as asserted in the special case ($\mathcal{K} = 0$) provided by Theorem 2.8.

Remark 2.13 (Expositions of the proofs of the Morse and Morse–Bott Lemmas for functions on Euclidean space). Nicolaescu provides a proof [92] Theorem 1.12 of the Morse Lemma for $C^\infty$ functions on Euclidean space (Theorem 2.8 with $\mathcal{X} = \mathbb{R}^d$) based on that of Arnold, Gusein-Zade, and Varchenko [10] Section 6.4] and remarks that his proof extends to yield the Morse–Bott Lemma for $C^\infty$ functions on Euclidean space (Theorem 2.10 with $\mathcal{X} = \mathbb{R}^d$) in [92] Proposition
with Morse–Bott at the origin in the sense of Definition 1.9 (1), then, after possibly shrinking neighborhood of the origin, and $f$ closed complement and $\mathcal{X}$ where Theorem 2.14 provides a $C^p$ diffeomorphism, $\mathcal{V} \ni y \mapsto x = \Phi(y) \in \mathcal{Y}$ with $\Phi(0) = 0$, such that

\begin{equation}
\tag{2.23}
f(\Phi(y)) = \frac{1}{2}(y, Ay)_{\mathcal{X} \times \mathcal{X}^*}, \quad \text{for all } y \in \mathcal{V},
\end{equation}

where

$A := f''(0) = (f \circ \Phi)''(0) \in \mathcal{L}_{\text{sym}}(\mathcal{X}, \mathcal{X})$

and, for $\mathcal{X} = \mathcal{X}_0 \oplus \mathcal{X}$ with $\mathcal{X} := \text{Ker } f''(0)$ and a closed complement $\mathcal{X}_0$,

\begin{equation}
\tag{2.24}
D\Phi(0, 0) = \begin{pmatrix}
id_{\mathcal{X}_0} & * \\
0 & \text{id}_{\mathcal{X}}
\end{pmatrix} \in \mathcal{L}(\mathcal{X}_0 \oplus \mathcal{X}).
\end{equation}

If $f$ is analytic, then $\Phi$ is analytic.

**Proofs of Theorem 2.10 and 2.14.** Observe that Theorem 2.10 follows immediately from Theorem 2.14 by restricting to the case $\mathcal{X} = \mathcal{X}^*$, so we focus on the more general case.

Because $f$ is $C^{p+2}$ and Morse–Bott at the origin, $\text{Crit } f \subset \mathcal{Y}$ is a $C^2$ submanifold by Definition 1.9 (1), and thus a $C^{p+2}$ submanifold by the Implicit Mapping Theorem. Moreover, by Definition 1.9 (1), there is a direct sum decomposition $\mathcal{X} = \mathcal{X}_0 \oplus \mathcal{X}$, where $\mathcal{X} = \text{Ker } f''(0)$ and $\mathcal{X}_0$ is a closed complement and $\text{Crit } f = \mathcal{X}$. Hence, after possibly shrinking $\mathcal{Y}$, the Implicit Mapping Theorem provides a $C^{p+2}$ diffeomorphism $\Xi$ from an open neighborhood $\mathcal{V}$ of the origin in $\mathcal{X}$ onto $\mathcal{Y}$ such that $\Xi(0) = 0$ and $D\Xi(0) = \text{id}_{\mathcal{X}}$ with

$$\text{Crit } f \circ \Xi = \mathcal{V} \cap (\{0\} \oplus \mathcal{X}).$$

Therefore, we may assume without loss of generality that

$$\text{Crit } f = \mathcal{Y} \cap (\{0\} \oplus \mathcal{X}).$$

Furthermore, Definition 1.9 (1) provides that $\text{Ran } f''(0) = \mathcal{X}^*$. Hence, Theorem 5 implies that, after possibly shrinking $\mathcal{Y}$, there exists a $C^p$ diffeomorphism,

$$\Phi : \mathcal{Y} \cap (\mathcal{X}_0 \oplus \mathcal{X}) \ni (z, \xi) \mapsto x = \Phi(z, \xi) \in \mathcal{X} = \mathcal{X}_0 \oplus \mathcal{X},$$

such that $\Phi(0, 0) = 0$ and $D\Phi(0, 0)$ is as in (2.24) with

$$f(\Phi(z, \xi)) = \frac{1}{2}(z, A_0 z)_{\mathcal{X} \times \mathcal{X}^*} + g(\xi), \quad \text{for all } (z, \xi) \in \mathcal{Y} \cap (\mathcal{X}_0 \oplus \mathcal{X}),$$

where $g(\xi) := f(\Phi(0, \xi))$, and

$$A_0 := D^2 f(0) \upharpoonright \mathcal{X}_0 = D^2(f \circ \Phi)(0, 0) \in \mathcal{L}_{\text{sym}}(\mathcal{X}_0, \mathcal{X})$$

is an isomorphism by the Open Mapping Theorem. We observe that

$$D(f \circ \Phi)(z, \xi) = A_0 z + Dg(\xi) \in \mathcal{X}^*.$$

Hence, $(z, \xi) \in \text{Crit } f \circ \Phi \iff z = 0$ and $Dg(\xi) = 0$, that is $\xi \in \text{Crit } g$, where $g : \mathcal{Y} \cap \mathcal{X} \to \mathbb{K}$ is a $C^p$ function with $g(0) = 0$. Therefore, $\text{Crit } f \circ \Phi = \text{Crit } g$. In particular, $\text{Crit } g$ is a $C^p$ submanifold
of $\mathcal{V} \cap \mathcal{X}$, since $\text{Crit } f \circ \Phi$ is a $C^{p+2}$ submanifold of $\mathcal{V}$, and $\dim \text{Crit } f \circ \Phi = \dim \text{Crit } g$ with $T_0 \text{Crit } g = T_0 \text{Crit } f \circ \Phi = \mathcal{X}$. Since $0 \in \text{Crit } g$ and $g(0) = 0$, there is a connected open neighborhood of the origin in $\mathcal{X}$ such that $g \equiv 0$ and by shrinking $\mathcal{V}$ if necessary, we may assume that $g \equiv 0$ on $\mathcal{V} \cap \mathcal{X}$. Hence,

$$D(f \circ \Phi)(z, \xi) = A_0 z = A(z, \xi)$$

by writing $A \in \mathcal{L}(\mathcal{X}_0 \oplus \mathcal{X}, \mathcal{X})$ as

$$A(z, \xi) = A_0 z, \quad \text{for all } (z, \xi) \in \mathcal{X}_0 \oplus \mathcal{X}.$$  

If $f$ is analytic, then $\Phi$ is analytic by the Implicit Mapping Theorem for analytic functions. The proofs of Theorem 2.10 and 2.14 are complete. \hfill \Box

When $\mathcal{X} = \mathbb{C}^d$, then Theorem 2.10 yields the

**Corollary 2.15** (Holomorphic Morse–Bott Lemma for functions on $\mathbb{C}^d$). (See [99] for a statement in the case $c = 0$ and Petro [97] Lemma 3.8] for a statement in the case $c \geq 0$; compare Seidel [105] Lemma 1.6.) Let $d \geq 2$ be an integer, $U \subset \mathbb{C}^d$ be an open neighborhood of the origin, and $f : U \ni x \mapsto f(x) \in \mathbb{C}$ be a holomorphic function such that $f(0) = 0$ and $f'(0) = 0$. Assume that $\text{Crit } f$ is a complex submanifold of $U$ with complex tangent space $T_0 \text{Crit } f = \text{Ker } f''(0)$ of dimension $c \geq 0$ at the origin. Then, after possibly shrinking $U$, there are an open neighborhood $V \subset \mathbb{C}^d$ of the origin and a complex analytic diffeomorphism,

$$V \ni (w_1, \ldots, w_d) \mapsto (x_1, \ldots, x_d) = \Phi(w_1, \ldots, w_d) \in \mathbb{C}^d,$$

onto an open neighborhood of the origin in $\mathbb{C}^d$ such that

$$\Phi^{-1}(U \cap \text{Crit } \mathcal{X}) = V \cap (\mathbb{C}^c \cap 0) \subset \mathbb{C}^c \times \mathbb{C}^{d-c}$$

with $\Phi(0) = 0$ and

$$D\Phi(0) = \begin{pmatrix} \text{id}_{d-c} & \star \\ 0 & \text{id}_c \end{pmatrix} \in \text{GL}(d, \mathbb{C}),$$

where $\text{id}_{d-c} \in \text{GL}(d-c, \mathbb{C})$ and $\text{id}_c \in \text{GL}(c, \mathbb{C})$ and

$$f(\Phi(w_1, \ldots, w_d)) = w_1^2 + \cdots + w_{d-c}^2, \quad \text{for all } w = (w_1, \ldots, w_d) \in U.$$  

3. Łojasiewicz gradient inequality for functions on Banach spaces

In Section 3.1, we use the Morse–Bott Lemma for $C^{p+2}$ functions ($p \geq 1$) (see Theorems 2.10 and 2.14) to give a concise proof of the Łojasiewicz gradient inequality for $C^{p+2}$ Morse–Bott functions on Banach spaces (see Theorems 6 and 7; in Section 3.2 we apply the Morse Lemma for analytic functions with degenerate critical points (see Theorems 4 and 5) to give an elegant proof of the Łojasiewicz gradient inequality for analytic functions on Banach spaces (see Theorems 9 and 10).

3.1. Łojasiewicz gradient inequality for smooth Morse–Bott functions. In this subsection, we prove Theorem 7 and hence Theorem 6 upon choosing $\mathcal{X} = \mathcal{X}^*$. We begin with the

**Lemma 3.1** (Łojasiewicz gradient inequality for quadratic forms). Let $\mathcal{X}$ and $\mathcal{X}^*$ be Banach spaces over $\mathbb{K}$ with continuous embedding $\mathcal{X} \subset \mathcal{X}^*$. If

$$Q : \mathcal{X} \ni x \mapsto Q(x) = \frac{1}{2} (x, Ax)_{\mathcal{X} \times \mathcal{X}^*} \in \mathbb{K}$$
is defined by a symmetric operator $A \in \mathcal{L}_{\text{sym}}(\mathcal{X}, \mathcal{X}'\mathcal{X})$, whose kernel is complemented in $\mathcal{X}$ and whose range is $\mathcal{X}'\mathcal{X}$, then $Q$ has Łojasiewicz exponent $1/2$, that is, there is a constant $C \in (0, \infty)$ such that

$$\|Q'(x)\|_{\mathcal{X}'\mathcal{X}} \geq C Q(x)^{1/2}, \quad \text{for all } x \in \mathcal{X}. \quad (3.1)$$

Proof. The derivative of $Q : \mathcal{X} \to \mathbb{R}$ is given by

$$Q'(x)v = \frac{1}{2} \langle v, Av \rangle_{\mathcal{X}'\mathcal{X}} + \frac{1}{2} \langle x, Av \rangle_{\mathcal{X}'\mathcal{X}} = \langle v, Ax \rangle_{\mathcal{X}'\mathcal{X}} = Ax(v), \quad \text{for all } x, v \in \mathcal{X},$$

so $Q'(x) = Ax \in \mathcal{X}'\mathcal{X}$ for all $x \in \mathcal{X}$. By hypothesis, $\mathcal{X} = \mathcal{X}_0 \oplus \mathcal{K}$ as a direct sum of Banach spaces, where $\mathcal{K} := \text{Ker } A$ and $\mathcal{K}_0 \subset \mathcal{X}$ is a closed subspace, and $\text{Ran } A = \mathcal{X}'\mathcal{X}$, so that $A \in \mathcal{L}(\mathcal{X}_0, \mathcal{X}')$ is an isomorphism of Banach spaces by the Open Mapping Theorem. Note that for $x = z + \xi \in \mathcal{X}_0 \oplus \mathcal{K}$, we have

$$Q(z + \xi) = \frac{1}{2} \langle z + \xi, A(z + \xi) \rangle_{\mathcal{X}'\mathcal{X}} \leq \frac{1}{2} \langle z + \xi, Az \rangle_{\mathcal{X}'\mathcal{X}} = \frac{1}{2} \langle z, Az \rangle_{\mathcal{X}'\mathcal{X}} = Q(z),$$

while

$$Q'(z + \xi) = A(z + \xi) = Az = Q'(z).$$

Hence, it suffices to prove that the Łojasiewicz gradient inequality (3.1) holds for all $x \in \mathcal{X}_0$. For such $x \in \mathcal{X}_0$, we have

$$\|Q'(x)\|_{\mathcal{X}'\mathcal{X}} = \|Ax\|_{\mathcal{X}'\mathcal{X}} \geq \lambda \|x\|_{\mathcal{X}},$$

by writing

$$\|x\|_{\mathcal{X}} = \|A^{-1}Ax\|_{\mathcal{X}} \leq \|A^{-1}\|_{\mathcal{L}(\mathcal{X}_0, \mathcal{X})} \|Ax\|_{\mathcal{X}'\mathcal{X}}$$

and denoting $\lambda := \|A^{-1}\|_{\mathcal{L}(\mathcal{X}_0, \mathcal{X})} \in (0, \infty)$. On the other hand, for any $x \in \mathcal{X}'\mathcal{X}$,

$$|Q(x)| \leq \frac{1}{2} \|x, Ax\|_{\mathcal{X}'\mathcal{X}} \leq \frac{1}{2} \|x\|_{\mathcal{X}} \|Ax\|_{\mathcal{X}'\mathcal{X}} \leq \frac{\kappa}{2} \|x\|_{\mathcal{X}} \|Ax\|_{\mathcal{X}'\mathcal{X}} \leq \frac{\kappa \lambda}{2} \|x\|_{\mathcal{X}'\mathcal{X}},$$

where we denote $\kappa := \|A\|_{\mathcal{L}(\mathcal{X}, \mathcal{X}')}$ with the same exponent $\kappa \lambda \in (0, \infty)$ and where $\kappa$ is the norm of the continuous embedding $\mathcal{X}' \subset \mathcal{X}'\mathcal{X}$. Therefore,

$$\|Q'(x)\|_{\mathcal{X}'\mathcal{X}} \geq \lambda \|x\|_{\mathcal{X}} \geq \lambda \left(2|Q(x)|/\kappa \lambda \right)^{1/2} = \lambda \sqrt{2/\kappa \lambda} |Q(x)|^{1/2},$$

for all $x \in \mathcal{X}_0$ and this yields the Łojasiewicz gradient inequality (3.1) for all $x \in \mathcal{X}$. \hfill \Box

We have the following generalization of Lemma 1.4.

**Lemma 3.2** (Łojasiewicz exponents and maps). Let $\mathcal{X}, \mathcal{X}'\mathcal{X}$, and $\mathcal{Y}$ be Banach spaces over $\mathbb{R}$ with continuous embedding $\mathcal{X}' \subset \mathcal{X}'\mathcal{X}$, and $\mathcal{V} \subset \mathcal{Y}$ and $\mathcal{W} \subset \mathcal{X}$ be open neighborhoods of the origins, and $\Phi : \mathcal{Y} \to \mathcal{W}$ be an open $C^1$ map such that $\Phi(0) = 0$. Let $f : \mathcal{W} \to \mathbb{R}$ be a $C^1$ function such that $f(0) = 0$ and $f'(x) \in \mathcal{X}'\mathcal{X}$ for all $x \in \mathcal{W}$. If $f \circ \Phi$ obeys the Łojasiewicz gradient inequality

$$\|(f \circ \Phi)'(y)\|_{\mathcal{Y}} \geq C_\theta |(f \circ \Phi)(y)|^\theta, \quad \text{for all } y \in \mathcal{V},$$

with exponent $\theta \geq 0$ then, after possibly shrinking $\mathcal{W}$, the function $f(x)$ obeys the Łojasiewicz gradient inequality (1.16) with the same exponent $\theta$ and a possibly smaller constant $C \in (0, \infty)$, for all $x \in \mathcal{W}$.
By combining the inequalities (3.3) and (3.4), we obtain

\[ \| (f \circ \Phi)'(y) \|_{\mathcal{Y}^*} \geq C |f(x)|^\theta, \]  

for all \( x \in \mathcal{W} \) and \( y \in \Phi^{-1}(x) \).

The Chain Rule provides

\[ (f \circ \Phi)'(y) = f'(\Phi(y)) \circ \Phi'(y) \in \mathcal{Y}^* \]

with \( \Phi'(y) \in \mathcal{L}(\mathcal{W}, \mathcal{X}^*) \) and \( f'(\Phi(y)) \in \mathcal{X}^* \subset \mathcal{X}^* \), so

\[ \| (f \circ \Phi)'(y) \|_{\mathcal{Y}^*} = \| f'(\Phi(y)) \circ \Phi'(y) \|_{\mathcal{Y}^*} \leq \| f'(\Phi(y)) \|_{\mathcal{X}^*} \| \Phi'(y) \|_{\mathcal{L}(\mathcal{W}, \mathcal{X}^*)} \leq \kappa M \| f'(\Phi(y)) \|_{\mathcal{X}} \]

for all \( y \in \mathcal{W} \),

where \( M := \sup_{y \in \mathcal{X}} \| \Phi'(y) \|_{\mathcal{L}(\mathcal{W}, \mathcal{X}^*)} \) and \( M < \infty \) (possibly after shrinking \( \mathcal{V} \)) and \( \kappa \) is the norm of the continuous embedding \( \mathcal{X}^* \subset \mathcal{X}^* \). Because \( \Phi(y) = x \in \mathcal{W} \), the preceding inequality simplifies to give

\[ \| (f \circ \Phi)'(y) \|_{\mathcal{Y}^*} \leq \kappa M \| f'(x) \|_{\mathcal{X}^*}, \]

for all \( y \in \mathcal{V} \).

By combining the inequalities (3.3) and (3.4), we obtain

\[ \| f'(x) \|_{\mathcal{X}^*} \geq (C/(\kappa M)) |f(x)|^\theta, \]

for all \( x \in \mathcal{W} \),

which is (1.16) with constant \( C/(\kappa M) \), as desired. \( \square \)

**Proof of Theorem 7.** By hypothesis, \( f \) is a \( C^{p+1} \) Morse–Bott function at the origin and so, possibly after shrinking \( \mathcal{V} \), Theorem 2.14 provides an open neighborhood \( \mathcal{V} \) of the origin in \( \mathcal{X} \) and a \( C^p \) diffeomorphism \( \Phi : \mathcal{V} \to \mathcal{W} \) such that \( \Phi(0) = 0 \) and

\[ (f \circ \Phi)(y) = \langle y, Ay \rangle_{\mathcal{X} \times \mathcal{X}^*}, \]

for all \( y \in \mathcal{V} \),

where \( A = f''(0) = (f \circ \Phi)''(0) \in \mathcal{L}(\mathcal{X} \times \mathcal{X}^*, \mathcal{X} \times \mathcal{X}^*) \). By Definition 1.9 (1), the kernel of \( A \) has a closed complement in \( \mathcal{X} \) and the range of \( A \) is \( \mathcal{X}^* \). Lemma 3.1 then asserts that the quadratic function, \( Q(y) = \langle y, Ay \rangle_{\mathcal{X} \times \mathcal{X}^*} \) for all \( y \in \mathcal{X} \), has Łojasiewicz exponent 1/2, while Lemma 3.2 implies that the functions \( f \) and \( f \circ \Phi \) have the same Łojasiewicz exponent, namely 1/2. This completes the proof of Theorem 7. \( \square \)

### 3.2. Łojasiewicz gradient inequality for analytic functions.

In this subsection, we apply Theorem 5 to prove Theorem 10 and hence Theorem 9 upon choosing \( \mathcal{X} = \mathcal{X}^* \). We first have the elementary

**Lemma 3.3** (Invariance of Łojasiewicz exponent under direct sum addition or subtraction of a quadratic form). Let \( \mathcal{X}, \mathcal{X}', \mathcal{Y}, \) and \( \mathcal{W} \) be Banach spaces over \( \mathbb{K} \) with continuous embeddings \( \mathcal{X} \subset \mathcal{X}' \) and \( \mathcal{Y} \subset \mathcal{Y}' \), and \( \theta \in [1/2, 1) \) be a constant, \( \mathcal{W} \subset \mathcal{X} \) be an open neighborhood of the origin, \( f : \mathcal{X} \supset \mathcal{W} \to \mathbb{K} \) be a \( C^2 \) function with \( f(0) = 0 \) and \( f'(0) = 0 \) and \( f'(x) \in \mathcal{X} \) for all \( x \in \mathcal{W} \), and

\[
Q : \mathcal{Y} \ni y \mapsto Q(y) = \frac{1}{2} \langle y, Ay \rangle_{\mathcal{Y} \times \mathcal{Y}^*} \in \mathbb{K}
\]

be defined by an operator \( A \in \mathcal{L}(\mathcal{Y}, \mathcal{Y}') \) whose kernel is complemented in \( \mathcal{Y} \) and whose range is \( \mathcal{Y}' \). If \( f_Q : \mathcal{W} \times \mathcal{Y} \to \mathbb{K} \) is a \( C^2 \) function defined by \( f_Q(x, y) := f(x) + Q(y) \) for \( (x, y) \in \mathcal{W} \times \mathcal{Y} \),
then there are constants $C, C_0 \in (0, \infty)$ and an open neighborhood $\mathcal{V} \subset \mathcal{W}$ of the origin such that, after possibly shrinking $\mathcal{W}$, the following holds: $f$ has Lojasiewicz exponent $\theta$ on $\mathcal{W}$, that is,

$$\|f'(x)\|_{\mathcal{W}} \geq C_0 |f(x)|^\theta,$$

for all $x \in \mathcal{W}$, if and only if $f_Q$ has Lojasiewicz exponent $\theta$ on $\mathcal{W} \times \mathcal{V}$, that is,

$$\|f_Q'(x, y)\|_{\mathcal{W} \oplus \mathcal{V}} \geq C |f_Q(x, y)|^\theta,$$

for all $(x, y) \in \mathcal{W} \times \mathcal{V}$.

Proof. Let $\alpha := 1/\theta \in (1, 2]$ and suppose that Inequality (3.5) holds. Since $f'(0) = 0$ and $Q'(0) = 0$, we may assume that $\|f'(x) \oplus Q'(y)\|_{\mathcal{W} \oplus \mathcal{V}} \leq 1$ for all $(x, y) \in \mathcal{W} \times \mathcal{V}$, for small enough $\mathcal{V}$ and after possibly shrinking $\mathcal{W}$. Observe that for all $(x, y) \in \mathcal{W} \times \mathcal{V}$,

$$|f_Q(x, y)| \leq |f(x)| + |Q(y)|$$

$$\leq C_0 \|f'(x)\|_{\mathcal{W}}^\alpha + C_1 |Q'(y)|^2$$

(by Lemma 3.1 and Inequality (3.5))

$$\leq C \left(\|f'(x)\|_{\mathcal{W}}^\alpha + |Q'(y)|^2\right)$$

$$\leq C \left(\|f'(x) \oplus Q'(y)\|_{\mathcal{W} \oplus \mathcal{V}}^\alpha + \|f'(x) \oplus Q'(y)\|_{\mathcal{W} \oplus \mathcal{V}}^2\right)$$

$$\leq C \|f'(x) \oplus Q'(y)\|_{\mathcal{W} \oplus \mathcal{V}}^\alpha$$

(as $\|f'(x) \oplus Q'(y)\|_{\mathcal{W} \oplus \mathcal{V}} \leq 1$ and $\alpha \in (1, 2]$)

$$= C \|f'_Q(x, y)\|_{\mathcal{W} \oplus \mathcal{V}}^\alpha,$$

where $C = \max\{C_0, C_1\}$ and $f'_Q(x, y) = f'(x) \oplus Q'(y)$. Taking the $1/\alpha$ root of the preceding inequality yields Inequality (3.6).

Conversely, suppose that Inequality (3.6) holds. For all $x \in \mathcal{W}$,

$$|f(x)| = |f_Q(x, 0)|$$

$$\leq C \|f'_Q(x, 0)\|_{\mathcal{W} \oplus \mathcal{V}}^\alpha$$

(by Inequality (3.6))

$$= C \|f'(x) \oplus Q'(0)\|_{\mathcal{W} \oplus \mathcal{V}}^\alpha$$

$$= C \left(\|f'(x)\|_{\mathcal{W}} + |Q'(0)|_{\mathcal{V}}\right)^\alpha$$

$$= C \|f'(x)\|_{\mathcal{W}}^\alpha$$

(since $Q'(0) = 0$),

which gives Inequality (3.5) after taking the $1/\alpha$ root. This completes the proof of Lemma 3.3. \hfill \Box

We can now give the

**Proof of Theorem 11.** The operator $f''(0) \in \mathcal{L}_{\text{sym}}(\mathcal{X}, \mathcal{X})$ is Fredholm by hypothesis, with finite-dimensional kernel $\mathcal{K} := \text{Ker} f''(0)$ and closed complement $\mathcal{K}_0 \subset \mathcal{X}$ such that $\mathcal{X} = \mathcal{K}_0 \oplus \mathcal{K}$. Similarly, let $\tilde{\mathcal{K}}_0 := \text{Ran} f''(0) \subset \tilde{\mathcal{X}}$ denote the closed range of $f''(0)$ with finite-dimensional complement $\tilde{\mathcal{K}}_0 \cong \mathcal{K} = \text{Ker} f''(0)$ and isomorphism $\tilde{\mathcal{K}}_0 \cong \mathcal{K}_0$ (see Lemma 2.1). Therefore, writing $x = (w, \xi) \in \mathcal{X} = \mathcal{K}_0 \oplus \mathcal{K}$,

$$f''(0, 0) = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{K}_0 \oplus \mathcal{K} \to \tilde{\mathcal{K}}_0 \oplus \tilde{\mathcal{K}},$$

where $A_0 = D_0^2 f(0, 0) \in \mathcal{L}(\mathcal{X}_0, \tilde{\mathcal{X}}_0)$ is symmetric with respect to the continuous embedding $\mathcal{K}_0 \subset \mathcal{X}_0^*$ and canonical pairing $\mathcal{X}_0 \times \mathcal{X}_0^* \to \mathbb{K}$. Moreover, $A_0$ is bijective and continuous by construction, so it is invertible by the Open Mapping Theorem.

By hypothesis, $f$ is analytic and so, possibly after shrinking $\mathcal{W}$, Theorem 5 provides an open neighborhood $\mathcal{V}$ of the origin in $\mathcal{X}$ and an analytic diffeomorphism $\Phi : \mathcal{V} \to \mathcal{W}$ such that
$\Phi(0, 0) = (0, 0)$ and
\[ f \circ \Phi(z, \xi) = g(\xi) + \langle z, A_0 z \rangle_{x_0^*}, \]
for all $y = (z, \xi) \in \mathcal{V}$,
where $A_0 = D^2_1(f \circ \Phi)(0, 0) \in \mathcal{L}_{sym}(\mathcal{X}_0, \mathcal{X}_0')$ and $g(\xi) := f(\Phi(0, \xi))$ for all $\xi$ in $V$, an open neighborhood of the origin in $\mathcal{K}$ defined as the image of the projection of $\mathcal{V} \subset \mathcal{X} = \mathcal{X}_0 \oplus \mathcal{K}$ onto the factor $\mathcal{K}$. Lemma 3.1 then asserts that the quadratic function, $Q(z) := \langle z, A_0 z \rangle_{x_0^*}$ for $z \in \mathcal{X}_0$, has Łojasiewicz exponent $1/2$ on an open neighborhood of the origin, while Lemmas 3.2 and 3.3 imply that the functions $f : \mathcal{V} \to \mathbb{K}$ and $f \circ \Phi : \mathcal{V} \to \mathbb{K}$ and $g : \mathcal{V} \to \mathbb{K}$ have the same Łojasiewicz exponent. But $\mathcal{K}$ is a finite-dimensional vector space over $\mathbb{K}$ and $g$ is analytic and thus obeys the classical Łojasiewicz gradient inequality for some exponent $\theta \in [1/2, 1)$ by Theorem 1.1. This completes the proof of Theorem 10. 

\[ \square \]

4. Analytic functions with Łojasiewicz exponent one half are Morse–Bott

Our goal in this section is to complete the proof of Theorem 2 and hence Theorem 1 upon choosing $\mathcal{X} = \mathcal{X}^*$. 

Proof of Theorem 2 Recall that $f''(0) \in \mathcal{L}_{sym}(\mathcal{X}, \mathcal{X}^*)$ is a Fredholm operator by hypothesis. Let $\mathcal{K} := \text{Ker} f''(0) \subset \mathcal{X}$ denote the finite-dimensional kernel with closed complement $\mathcal{X}_0$, so $\mathcal{X} = \mathcal{X}_0 \oplus \mathcal{K}$, and let $\mathcal{X}_0' := \text{Ran} f''(0) \subset \mathcal{X}^*$ denote the closed range, with finite-dimensional complement $\mathcal{K}^*$, so $\mathcal{X}^* = \mathcal{X}_0' \oplus \mathcal{K}^*$.

We apply the Morse Lemma for functions on Banach spaces with degenerate critical points (Theorem 5) to $f$ to produce — after possibly shrinking the open neighborhood $\mathcal{V}$ of the origin in $\mathcal{X}^*$ — an analytic diffeomorphism $\Phi$ from an open neighborhood $\mathcal{V}$ of the origin in $\mathcal{X} = \mathcal{X}_0 \oplus \mathcal{K}$ onto $\mathcal{X}$ such that
\[ \Phi^* f(z, \xi) = \frac{1}{2} \langle z, A_0 z \rangle_{x_0^*} + \Phi^* f(0, \xi), \]
for all $y = (z, \xi) \in \mathcal{V} \subset \mathcal{X}_0 \oplus \mathcal{K}$,
where $A_0 := D^2_1(\Phi^* f)(0, 0) \in \mathcal{L}_{sym}(\mathcal{X}_0, \mathcal{X}_0')$ and we recall that $\mathcal{X}_0 \subset \mathcal{X}_0^*$ is a continuous embedding, as proved in the paragraphs immediately following the statement of Theorem 2. Again, the operator $A_0$ is bijective and continuous by construction, so it is invertible by the Open Mapping Theorem. The function $g := f(\Phi(0, \cdot))$ is analytic on an open neighborhood $V$ of the origin in $\mathcal{K}$ defined as the image of the projection of $\mathcal{V} \subset \mathcal{X} = \mathcal{X}_0 \oplus \mathcal{K}$ onto the factor $\mathcal{K}$. By construction, $g(0) = 0$ and $g'(0) = 0$.

By shrinking $V$ if necessary, we may assume without loss of generality that $V$ is connected. If $g$ is identically zero on $V$, then we are done. Otherwise, if $g$ is not identically zero on $V$, our hypothesis that $f$ has Łojasiewicz exponent $1/2$ and Lemmas 3.2 and 3.3 imply that $g$ has Łojasiewicz exponent $1/2$ as well.

If $\text{Ker} g''(0) = \{0\}$, then $C\ker g''(0) = \{0\}$, since $g''(0) \in \mathcal{L}(\mathcal{K}, \mathcal{K}^*)$ is symmetric and thus $g''(0) \in \mathcal{L}(\mathcal{K}, \mathcal{K}^*)$ is invertible. Hence, $g$ is a Morse–Bott function — in fact a Morse function with $\text{Crit} g = \{0\}$ — and thus $\Phi^* f$ is a Morse–Bott function.

Proof of Theorem 10. Recall that $\Phi(0, 0) = (0, 0)$ and $f \circ \Phi(z, \xi) = g(\xi) + \langle z, A_0 z \rangle_{x_0^*}$, for all $y = (z, \xi) \in \mathcal{V}$,
where $A_0 = D^2_1(f \circ \Phi)(0, 0) \in \mathcal{L}_{sym}(\mathcal{X}_0, \mathcal{X}_0')$ and $g(\xi) := f(\Phi(0, \xi))$ for all $\xi$ in $V$, an open neighborhood of the origin in $\mathcal{K}$ defined as the image of the projection of $\mathcal{V} \subset \mathcal{X} = \mathcal{X}_0 \oplus \mathcal{K}$ onto the factor $\mathcal{K}$. Lemma 3.1 then asserts that the quadratic function, $Q(z) := \langle z, A_0 z \rangle_{x_0^*}$ for $z \in \mathcal{X}_0$, has Łojasiewicz exponent $1/2$ on an open neighborhood of the origin, while Lemmas 3.2 and 3.3 imply that the functions $f : \mathcal{V} \to \mathbb{K}$ and $f \circ \Phi : \mathcal{V} \to \mathbb{K}$ and $g : \mathcal{V} \to \mathbb{K}$ have the same Łojasiewicz exponent. But $\mathcal{K}$ is a finite-dimensional vector space over $\mathbb{K}$ and $g$ is analytic and thus obeys the classical Łojasiewicz gradient inequality for some exponent $\theta \in [1/2, 1)$ by Theorem 1.1. This completes the proof of Theorem 10. 

\[ \square \]

15Note that $\mathcal{K}$ is finite-dimensional.
If Ker $g''(0) \neq \{0\}$, then there exists $v \in \mathcal{K}$ such that $\|v\|_{\mathcal{K}} = 1$ and, since $g$ is analytic, an integer $m \geq 3$ such that $g^{(m)}(0)v^m \neq 0$. The Taylor Formula then yields

$$g(tv) = \frac{1}{m!}g^{(m)}(0)v^mt^m + \frac{1}{(m+1)!} \int_0^t g^{(m+1)}(sv)v^{m+1}s^m\,ds,$$

$$g'(tv) = \frac{1}{(m)!}g^{(m)}(0)v^{m-1}t^{m-1} + \frac{1}{m!} \int_0^t g^{(m+1)}(sv)v^{m}s^{m-1}\,ds,$$

for all $t \in \mathbb{K}$ such that $tv \in V$. Therefore, after possibly further shrinking $\mathcal{K}$ and hence $V$, and noting that $\|\xi\|_{\mathcal{K}} = \|tv\|_{\mathcal{K}} = |t|$ for all $\xi = tv \in V \cap \mathbb{K}v$, we have

$$\frac{1}{C_0} ||\xi||^m_{\mathcal{K}} \leq |g(\xi)| \leq C_0 ||\xi||^m_{\mathcal{K}},$$

$$\frac{1}{C_1} ||\xi||^{m-1} \leq ||g'(\xi)||_{\mathcal{K}} \leq C_1 ||\xi||^{m-1},$$

for positive constants $C_0, C_1$ depending at most on $m$ and $|g^{(m)}(0)v^m|$ and $\sup_{\xi \in V} ||g^{(m+1)}(\xi)||$, where $g^{(m+1)}(\xi) \in \otimes^{m+1} \mathcal{K}^\ast$. Because $g$ has Łojasiewicz exponent 1\(/\2\), then the Łojasiewicz gradient inequality (A.1) yields, after possibly further shrinking $V$,

$$||g'(\xi)||_{\mathcal{K}} \geq C_2|g(\xi)|^{1/2}, \quad \text{for all } \xi \in V,$$

for some positive constant $C_2$. By combining the preceding inequality with the inequalities (A.1) and (A.2), we obtain

$$C_1 ||\xi||_\mathcal{K}^{m-1} \geq C_2 \left( C_0^{-1} ||\xi||_\mathcal{K}^m \right)^{1/2}, \quad \text{for all } \xi \in \mathcal{K},$$

that is,

$$||\xi||_\mathcal{K}^{m-1} \geq C_0^{-1/2} C_1^{-1} C_2 ||\xi||_\mathcal{K}^{m/2}, \quad \text{for all } \xi \in \mathcal{K}.$$

Consequently, since $V$ is an open neighborhood of the origin in $\mathcal{K}$, we must have $m - 1 \leq m/2$ or equivalently, $2m - 2 \leq m$, that is, $m \leq 2$ and contradicting our assumption that $m \geq 3$ and $\mathcal{K} \neq \{0\}$. Hence, we must have Ker $g''(0) = \{0\}$ and this completes the proof of Theorem 2. □

## Appendix A. Rate of Convergence of a Gradient Flow for a Function obeying a Łojasiewicz Gradient Inequality

We recall the following enhancement of Huang [67] Theorem 3.4.8.1.

**Theorem A.1** (Convergence rate under the validity of a Łojasiewicz gradient inequality). (See Feehan [15] Theorem 3.) Let $\mathcal{K}$ be an open subset of a real Banach space $\mathcal{K}$ that is continuously embedded and dense in a Hilbert space $\mathcal{H}$. Let $\mathcal{E} : \mathcal{U} \to \mathbb{R}$ be an analytic function with gradient map $\mathcal{E}' : \mathcal{U} \to \mathcal{H}$ and $x_\infty \in \mathcal{U}$ be a critical point, that is, $\mathcal{E}'(x_\infty) = 0$. Assume that there are constants $c \in (0, \infty)$, and $\sigma \in (0, 1]$, and $\theta \in [1/2, 1)$ such that

$$||\mathcal{E}'(x)||_{\mathcal{H}} \geq c||\mathcal{E}(x) - \mathcal{E}(x_\infty)||^\theta, \quad \text{for all } x \in \mathcal{U}_\sigma,$$

where $\mathcal{U}_\sigma := \{x \in \mathcal{K} : ||x - x_\infty||_{\mathcal{H}} < \sigma\}$. Let $u \in C^\infty([0, \infty); \mathcal{K})$ be a solution to the gradient system

$$\dot{u}(t) = -\mathcal{E}'(u(t)), \quad t \in (0, \infty),$$

and assume that the orbit $O(u) := \{u(t) : t \geq 0\} \subset \mathcal{K}$ obeys $O(u) \subset \mathcal{U}_\sigma$. Then there exists $u_\infty \in \mathcal{K}$ such that

$$||u(t) - u_\infty||_{\mathcal{K}} \leq \Psi(t), \quad t \geq 0,$$
where

\begin{equation}
\Psi(t) := \begin{cases} 
\frac{1}{c(1-\theta)} \left( c^2 (2\theta - 1)t + (\gamma - a)^{1-2\theta} \right)^{-(1-\theta)/(2\theta-1)}, & 1/2 < \theta < 1, \\
\frac{2}{c} \sqrt{\gamma - a} \exp(-c^2 t/2), & \theta = 1/2, 
\end{cases}
\end{equation}

and \(a, \gamma\) are constants such that \(\gamma > a\) and

\[a \leq \mathcal{E}(v) \leq \gamma, \quad \text{for all } v \in \mathcal{V}.
\]

If in addition \(u\) obeys Hypothesis A.2, then \(u_\infty \in \mathcal{X}\) and

\begin{equation}
\|u(t + 1) - u_\infty\|_\mathcal{X} \leq 2C_1 \Psi(t), \quad t \geq 0,
\end{equation}

where \(C_1 \in [1, \infty)\) is the constant in Hypothesis A.2 for \(\delta = 1\).

We recall the

**Hypothesis A.2 (A priori interior estimate for a trajectory).** (See Feehan [45, Hypothesis 2.1].)

Let \(\mathcal{X}\) be a Banach space that is continuously embedded in a Hilbert space \(\mathcal{V}\). If \(\delta \in (0, \infty)\) is a constant, then there is a constant \(C_1 = C_1(\delta) \in [1, \infty)\) with the following significance. If \(S, T \in \mathbb{R}\) are constants obeying \(S + \delta \leq T\) and \(u \in C^\infty([S, T); \mathcal{X})\), we say that \(\dot{u} \in C^\infty([S, T); \mathcal{X})\) obeys an a priori interior estimate on \((0, T]\) if

\begin{equation}
\int_{S+\delta}^T \|\dot{u}(t)\|_\mathcal{X} \, dt \leq C_1 \int_S^T \|\dot{u}(t)\|_\mathcal{V} \, dt.
\end{equation}

In applications, \(u \in C^\infty([S, T); \mathcal{X})\) in Hypothesis A.2 will often be a solution to a quasi-linear parabolic partial differential system, from which an a priori estimate (A.6) may be deduced. For example, Hypothesis A.2 is verified by Feehan [45, Lemma 17.12] for a nonlinear evolution equation on a Banach space \(\mathcal{V}\) of the form (see Caps [26], Henry [61], Pazy [96], Sell and You [106], Tanabe [118, 119] or Yagi [131])

\begin{equation}
\frac{du}{dt} + \mathcal{A}u = \mathcal{F}(t, u(t)), \quad t \geq 0, \quad u(0) = u_0,
\end{equation}

where \(\mathcal{A}\) is a positive, sectorial, unbounded operator on a Banach space \(\mathcal{W}\) with domain \(\mathcal{V}^2 \subset \mathcal{W}\) and the nonlinearity \(\mathcal{F}\) has suitable properties.

Results on the rate of convergence of a gradient flow defined by a function obeying a Łojasiewicz gradient inequality in specific examples have been proved earlier — see Simon [108] and Adams and Simon [4] for a restricted class of analytic energy functions arising in geometric analysis and Råde [102, Proposition 7.4] for the Yang-Mills energy function on connections on principal bundles over a closed smooth manifold of dimension two or three. For a recent example, see Carlotto, Chodosh, and Rubinstein [27, Theorem 1] for the Yamabe function on Riemannian metrics over closed smooth manifolds of dimension greater than or equal to three.

**Appendix B. Morse–Bott functions and quadratic simple normal crossing functions**

We are often asked about the relationship between Morse–Bott functions and quadratic simple normal crossing functions as in [15], so we explain the relationship in this section for \(\mathbb{K} = \mathbb{R}\); the analogous discussion applies for \(\mathbb{K} = \mathbb{C}\).

For an integer \(p \geq 1\) and writing \(\mathbb{R}^* = \mathbb{R} \setminus \{0\}\), we let \(\mathbb{R}^{p-1} = (\mathbb{R}^p \setminus \{0\})/\mathbb{R}^* = S^{p-1}/\{\pm 1\}\) denote real projective space, so \(\mathbb{R}^0 \cong \{1\}\) and \(\mathbb{R}^1 \cong S^1\) while \(\mathbb{R}^{p-1}\) with \(p \geq 3\) is obtained by identifying antipodal points of the sphere \(S^{p-1}\).
Definition B.1 (Blowup at a point and exceptional divisor). (See Krantz and Parks [70, Definition 6.2.2].) Let \( p \geq 1 \) be an integer and \( W \) be an open neighborhood of the origin in \( \mathbb{R}^p \). The blowup of \( W \) at the origin is the set
\[
\tilde{W} := \{(x, \ell) \in W \times \mathbb{R}^{p-1} : x \in \ell\},
\]
where \( \pi : \tilde{W} \ni (x, \ell) \mapsto x \in W \) is the blowup map and \( E := \pi^{-1}(0) \subset \tilde{W} \) is the exceptional divisor.

The set \( \tilde{W} \) is a real analytic manifold and the quotient map \( \pi : \tilde{W} \to W \) is real analytic and restricts to a real analytic diffeomorphism \( \pi : \tilde{W} \setminus E \cong W \setminus \{0\} \). By viewing \( \mathbb{R}^{p-1} = \mathbb{S}^{p-1}/\{\pm 1\} \) and \( \mathbb{R}^p \setminus \{0\} = \mathbb{R}_+ \times \mathbb{S}^{p-1} \), we may also write
\[
\tilde{W} = \{(x, [u]) \in W \times \mathbb{S}^{p-1}/\{\pm 1\} : x \in \mathbb{R}u\}
\]
where \([u] = \{\pm u\} \) and, in the last line, the blowup map is \( \pi : \tilde{W} \ni [s, u] \mapsto su \in \mathbb{W} \) and \( \pi^{-1}(0) = \{0\} \times \mathbb{S}^{p-1}/\{\pm 1\} \subset \mathbb{R}^{p-1} \) is the exceptional divisor.

If \( W \) is an open neighborhood of the origin in \( \mathbb{R}^p \) or the half-space \( \mathbb{H}^p = \{x \in \mathbb{R}^p : x_p \geq 0\} \), then we could alternatively define the blowup of \( W \) at the origin to be the real analytic manifold with boundary,
\[
\tilde{W} = \{(r, u) \in [0, \infty) \times \mathbb{S}^{p-1} : ru \in \mathbb{W}\},
\]
following the usual definition of polar coordinates on \( \mathbb{R}^p \setminus \{0\} \). The map \( \pi : \tilde{W} \ni (r, u) \mapsto ru \in W \) is the blowup map and \( \pi^{-1}(0) = \{0\} \times \mathbb{S}^{p-1} \subset \mathbb{S}^{p-1} \) is now the exceptional divisor.

Suppose now that \( U \subset \mathbb{R}^d \) is an open neighborhood of the origin and \( f : U \to \mathbb{R} \) is a \( C^2 \) function with \( f(0) = 0 \) and \( f'(0) = 0 \) and that is Morse–Bott at the origin in the sense of Definition 1.5 [1]. Thus, after possibly shrinking \( U \), we have that \( \text{Crit} f \) is a \( C^2 \) submanifold of \( U \) of dimension \( c = \dim \text{Ker} f''(0) \). Moreover, we may further assume that \( U \) is connected and so \( \text{Crit} f \subset f^{-1}(0) \).

Theorem 2.10 and Remark 2.12 (the Morse–Bott Lemma) imply, after possibly shrinking \( U \), that one can find an neighborhood \( V \) of the origin in \( \mathbb{R}^d \) and a \( C^2 \) diffeomorphism \( \Phi : \mathbb{R}^d \ni V \ni y \mapsto x \in U \subset \mathbb{R}^d \) such that \( \Phi(0) = 0 \) and
\[
f \circ \Phi(y) = \sum_{i=1}^{p} y_i^2 - \sum_{i=p+1}^{p+n} y_i^2, \quad \text{for all } y \in V.
\]

Note that \( p + n = d - c \) and
\[
\text{Crit} f \circ \Phi = V \cap \bigcap_{i=1}^{d-c} \{y_i = 0\}.
\]
If \( n = 0 \) and thus \( 1 \leq p = d - c \), we may write \((y_1, \ldots, y_p) = su \), for \( s \in [0, \infty) \) and \( u \in \mathbb{S}^{p-1} \subset \mathbb{R}^p \), so that
\[
f \circ \varpi(s, u, y_{p+1}, \cdots, y_d) = s^2, \quad \text{for all } (su, y_{p+1}, \cdots, y_d) \in U,
\]
\[\text{[16]While for clarity we have restricted our attention in this article to functions } f \text{ which are } C^{p+2} \text{ with } p \geq 1, \text{ the Morse–Bott Lemma holds for } C^2 \text{ functions on Euclidean space: see Banyaga and Hurtubise [13, Theorem 2].}\]
where we define
\[ \varpi(s, u, y_{p+1}, \ldots, y_d) := \Phi(su, y_{p+1}, \ldots, y_d). \]

We see that \( \varpi \) gives a \( C^2 \) map from an open neighborhood \( V \) of the origin in \([0, \infty) \times S^{p-1} \times \mathbb{R}^{d-p} \) onto \( U \subset \mathbb{R}^{d} \) such that \( \varpi(0) = 0 \) and
\[ \varpi(\{s = 0\} \cap V) = U \cap \bigcap_{i=1}^{n} \{y_i = 0\} = U \cap \bigcap_{i=1}^{d-c} \{y_i = 0\}, \]
and \( \varpi \) is a diffeomorphism from \( V \setminus \{s = 0\} \) onto its image.

Similarly, if \( p = 0 \) and thus \( 1 \leq n = d - c \), we may write \((y_1, \ldots, y_n) = tv, \) for \( t \in [0, \infty) \) and \( v \in S^{n-1} \subset \mathbb{R}^{n} \), so that
\[ f \circ \varpi(t, v, y_{n+1}, \ldots, y_d) = -t^2, \quad \text{for all } (tv, y_{n+1}, \ldots, y_d) \in U, \]
where we define \( \varpi(t, v, y_{n+1}, \ldots, y_d) = \Phi(tv, y_{n+1}, \ldots, y_d). \) We see that \( \varpi \) gives a \( C^2 \) map from an open neighborhood of the origin in \([0, \infty) \times S^{p-1} \times \mathbb{R}^{d-p} \) into \( \mathbb{R}^{d} \) such that \( \varpi(0) = 0 \) and
\[ \varpi(\{t = 0\} \cap V) = U \cap \bigcap_{i=1}^{n} \{y_i = 0\} = U \cap \bigcap_{i=1}^{d-c} \{y_i = 0\}, \]
and \( \varpi \) is a diffeomorphism from \( V \setminus \{t = 0\} \) onto its image.

Finally, if \( n \geq 1 \) and \( p \geq 1 \), we may write \((y_1, \ldots, y_p) = su \) and \((y_{p+1}, \ldots, y_{p+n}) = tv, \) for \( s, t \in [0, \infty) \) and \( u \in S^{p-1} \) and \( v \in S^{n-1} \), so that
\[ f \circ \varpi(s, t, u, v, y_{p+n+1}, \ldots, y_d) = s^2 - t^2, \quad \text{for all } (su, tv, y_{p+n+1}, \ldots, y_d) \in U, \]
where we define \( \varpi(s, t, u, v, y_{p+n+1}, \ldots, y_d) := \Phi(su, tv, y_{p+n+1}, \ldots, y_d). \)

We see that \( \varpi \) gives a \( C^2 \) map from an open neighborhood of the origin in
\([0, \infty) \times [0, \infty) \times S^{p-1} \times S^{n-1} \times \mathbb{R}^{d-n-p} \)
into \( \mathbb{R}^{d} \) such that \( \varpi(0) = 0 \) and
\[ \varpi(\{s = 0\} \cap \{t = 0\} \cap W) = V \cap \bigcap_{i=1}^{n+p} \{y_i = 0\} = V \cap \bigcap_{i=1}^{d-c} \{y_i = 0\}, \]
after possibly shrinking \( V \) and \( \varpi \) is a diffeomorphism from \( W \setminus (\{s = 0\} \cup \{t = 0\}) \) onto its image.

Define a diffeomorphism of \( \mathbb{R}^2 \) by \((t_1, t_2) \mapsto (s, t) = \varphi(t_1, t_2) \) where \( t_1 = s + t \) and \( t_2 = s - t \), so that \( s = \frac{1}{2}(t_1 + t_2) \) and \( t = \frac{1}{2}(t_1 - t_2) \). Hence, we obtain
\[ f \circ \Pi(t_1, t_2, u, v, y_{p+n+1}, \ldots, y_d) = t_1t_2, \quad \text{for all } (\varphi(t_1, t_2)u, \varphi(t_1, t_2)v, y_{p+n+1}, \ldots, y_d) \in U, \]
where we define
\[ \Pi(t_1, t_2, u, v, y_{p+n+1}, \ldots, y_d) := \Phi(\varphi(t_1, t_2)u, \varphi(t_1, t_2)v, y_{p+n+1}, \ldots, y_d). \]

We see that \( \Pi \) gives a \( C^2 \) map from an open neighborhood of the origin in
\( \{t_1, t_2 \in [0, \infty) \times \mathbb{R} : |t_2| \leq t_1\} \times S^{p-1} \times S^{n-1} \times \mathbb{R}^{d-n-p} \)
into \( \mathbb{R}^{d} \) such that \( \Pi(0) = 0 \) and
\[ \Pi(\{t_1 = 0\} \cap \{t_2 = 0\} \cap V) = U \cap \bigcap_{i=1}^{d-c} \{y_i = 0\}, \]
and \( \Pi \) is a diffeomorphism from \( V \setminus (\{t_1 = 0\} \cup \{t_2 = 0\}) \) onto its image.
In the preceding discussion we could have replaced the roles of the blowups \([0, \infty) \times S^{p-1}\) or \([0, \infty) \times S^{n-1}\) by \((\mathbb{R} \times S^{p-1})/\{\pm 1\}\) or \((\mathbb{R} \times S^{n-1})/\{\pm 1\}\) and the roles of the exceptional divisors, \(S^{p-1}\) or \(S^{n-1}\) by \(\mathbb{RP}^{p-1}\) or \(\mathbb{RP}^{n-1}\), the only difference being an increase in notational complexity. In summary, we have proved the

**Proposition B.2** (Pull-back of a Morse–Bott function to a quadratic simple normal crossing function). Let \(d \geq 2\) be an integer, \(U \subset \mathbb{R}^d\) be an open neighborhood of the origin, and \(f : U \to \mathbb{R}\) be a \(C^2\) function that is Morse–Bott at the origin and obeys \(f(0) = 0\). Then, after possibly shrinking \(U\), there are an open neighborhood \(V\) of the origin in \(\mathbb{R}^d\) and a \(C^2\) map \(\pi : V \to U\) such that \(\pi\) restricts to a diffeomorphism from \(V \setminus \{y_1 = 0\}\) or \(V \setminus (\{y_1 = 0\} \cup \{y_2 = 0\})\) onto its image and

\[
\pi^* f(y) = \pm y_1^2 \quad \text{or} \quad y_1 y_2, \quad \text{for all } y = (y_1, \ldots, y_d) \in V,
\]

and \(\pi(\text{Crit } f \circ \pi) = \text{Crit } f\), where \(\text{Crit } f \circ \pi = \{y_1 = 0\} \cap V\) or \((\{y_1 = 0\} \cup \{y_2 = 0\}) \cap V\).

**Appendix C. Integrability and Morse–Bott conditions for the harmonic map energy and the area functions**

In Section 1.3 we defined the concepts of Jacobi vector, integrable Jacobi vector, and integrable critical point (see Definition 1.17). We noted (see Lemma 1.18) that if a function is Morse–Bott at a critical point, then that critical point is integrable. Theorem 8 has been proved by Simon for a specific class of analytic functions on certain Banach spaces (given by \(C^{2, \alpha}\) sections of a Riemannian vector bundle over a closed Riemannian manifold) that includes the harmonic map energy and the area functions. We shall give a proof of a more general version of Theorem 8 elsewhere [46], but we outline here how Theorem 8 may be proved; in addition to the references cited below, we also refer the reader to Simon [110, Sections 3.11–3.14 and 3.13.16] for further expository details.

**Outline of proof of Theorem 8** In order to avoid notational conflict with the remainder of this section, we let \(E\) denote the analytic function considered in Theorem 8. As in Simon’s proof of his infinite-dimensional version [108, Theorem 3] of the Łojasiewicz gradient inequality, one first applies Lyapunov–Schmidt reduction as in [108, Section 2] to the function \(E : \mathcal{U} \to \mathbb{R}\). This step yields an analytic embedding \(\Psi : \mathcal{V} \cap \mathcal{K} \to \mathcal{X}\) of the intersection with the kernel \(\mathcal{K} := \text{Ker } E''(x_0)\) and an open neighborhood \(\mathcal{V}\) of the origin in \(\mathcal{X}\), together with an analytic function \(\Gamma = E \circ \Psi : \mathcal{V} \cap \mathcal{K} \to \mathbb{R}\) (see Adams and Simon [4, p. 230], Feehan and Maridakis [50, Lemmas 2.3 and 2.5], Simon [108, pp. 538–539], or Simon [109, Part II, Section 6]). By hypothesis, \(x_0\) is an integrable critical point in the sense of Definition 1.17 and so, after possibly shrinking \(\mathcal{V}\), the function \(\Gamma\) is constant on \(\mathcal{V} \cap \mathcal{K}\) by Adams and Simon [4, Lemma 1, p. 231].

One can now show that \(E'(x) = 0\) for all \(x \in \Psi(\mathcal{V} \cap \mathcal{K})\), essentially by reversing our proof of [50, Lemma 2.5] or arguing as in Simon [108, p. 539], and thus

\[
\Psi(\mathcal{V} \cap \mathcal{K}) \subseteq \text{Crit } E.
\]

By hypothesis, \(E''(x_0) \in \mathcal{L}(\mathcal{X}, \mathcal{F})\) is Fredholm with index zero and thus we have \(\mathcal{X} = \mathcal{X}_0 \oplus \mathcal{K}\) and \(\mathcal{F} \cong \mathcal{X}_0 \oplus \mathcal{K}\) (see Lemma 2.4 (2)). In particular,

\[
\text{Ran } E''(x_0) + \mathcal{K} = \mathcal{F}
\]

and so, after possibly shrinking \(\mathcal{U}\), the analytic gradient map \(E' : \mathcal{U} \to \mathcal{F}\) is transverse to the (linear) submanifold \(\mathcal{K} \subset \mathcal{F}\) and hence the preimage \((E')^{-1}(\mathcal{K})\) is an open analytic submanifold
of $\mathcal{K}$ by the Preimage Theorem from differential topology \cite{74} Proposition II.2.4. We thus have inclusions
\[
\Psi(\mathcal{Y} \cap \mathcal{K}) \subseteq \text{Crit } \mathcal{E} \subseteq (\mathcal{E}')^{-1}(\mathcal{K}),
\]
noting that Crit $\mathcal{E} \equiv (\mathcal{E}')^{-1}(0)$. Furthermore, we have
\[
T_{x_0}\Psi(\mathcal{Y} \cap \mathcal{K}) = \mathcal{K} = T_{x_0}(\mathcal{E}')^{-1}(\mathcal{K}),
\]
where the first equality follows from the construction of $\Psi$ (see \cite{50} Lemma 2.3) and the second from the observations below:
\[
T_{x_0}(\mathcal{E}')^{-1}(\mathcal{K}) = (\mathcal{E}''(x_0))^{-1}(T_{\mathcal{E}'(x_0)}\mathcal{K}) = (\mathcal{E}''(x_0))^{-1}(\mathcal{K})
\]
\[
= (\mathcal{E}''(x_0))^{-1}(0) \quad \text{(since $\mathcal{K} = \text{Ran } \mathcal{E}''(x_0) \oplus \mathcal{K}$)}
\]
\[
= \mathcal{K} \quad \text{(by definition)}.
\]
Hence, after possibly shrinking $\mathcal{Y}$ or $\mathcal{Y}'$, we have $\Psi(\mathcal{Y} \cap \mathcal{K}) = (\mathcal{E}')^{-1}(\mathcal{K})$ and consequently
\[
\Psi(\mathcal{Y} \cap \mathcal{K}) = \text{Crit } \mathcal{E} = (\mathcal{E}')^{-1}(\mathcal{K}).
\]
In particular, Crit $\mathcal{E}$ is an open analytic submanifold of $\mathcal{K}$ with tangent space $T_{x_0}\text{Crit } \mathcal{E} = \text{Ker } \mathcal{E}''(x_0)$ and so $\mathcal{E}$ is Morse–Bott at $x_0$ in the sense of Definition 1.9. \hfill $\square$

C.1. Integrability and Morse–Bott conditions for the harmonic map energy function.
Following Lemaire and Wood \cite{77} Section 1, we review the concept of integrability of a Jacobi field along a harmonic map and describe the relation between integrability and the Morse–Bott condition for the harmonic map energy function at a harmonic map. We then list a few examples where integrability is known for harmonic maps.\footnote{This appendix is a revised version of Feehan and Maridakis \cite{51} Appendix A.}

We begin by recalling the second variation of the energy for the harmonic energy function $\mathcal{E}$ discussed in Section 1.7.2. For a smooth two-parameter variation $f_{t,s} : M \to N$ of a map $f : M \to N$ with $f_{0,0} = f$ and $\partial f_{t,s}/\partial t|_{(0,0)} = v$ and $\partial^2 f_{t,s}/\partial s|_{(0,0)} = w$, the Hessian of $\mathcal{E}$ at $f$ is defined by
\[
\mathcal{E}''(f)(v,w) := \left. \frac{\partial^2 \mathcal{E}(f_{t,s})}{\partial t \partial s} \right|_{(0,0)}
\]
where $\mathcal{E}$ is as in (1.19). One has
\[
\mathcal{E}''(f)(v,w) = \langle J_f(v), w \rangle_{L^2(M,g)},
\]
where
\[
J_f(v) := \Delta v - \text{tr } R^N(df,v)df
\]
is called the Jacobi operator, a self-adjoint linear elliptic differential operator. Here, $\Delta$ denotes the Laplacian induced on $f^{-1}TN$ and the sign conventions for $\Delta$ and the curvature $R^N$ are those of Eells and Lemaire \cite{11}.

Let $v$ be a vector field along $f$, that is, a smooth section of $f^{-1}TN$, where $f : M \to N$ is a smooth map. Then $v$ is called a Jacobi field (for the energy) if $J_f(v) = 0$. The space of Jacobi fields $\text{Ker } J_f$ is finite-dimensional and its dimension is called the $(\mathcal{E}')$-nullity of $f$.

**Definition C.1** (Integrability of a Jacobi field along a harmonic map). (See Lemaire and Wood \cite{77} Definition 1.2.) A Jacobi field $v$ along a harmonic map $f_0 : M \to N$ is said to be integrable if there is a smooth family of harmonic maps, $f_t : M \to N$ for $t \in (-\varepsilon, \varepsilon)$, such that $f_t|_{t=0} = f_0$ and $v = \partial f_t/\partial t|_{t=0}$.
The following result is stated by Kwon in her Ph.D. thesis \cite{KwonThesis} (directed by Simon); it can be deduced from Theorem \ref{Thm:C.2} by applying, for example, methods of Feehan and Maridakis \cite{FeehanMaridakis}.  

**Theorem C.2 (Integrability of Jacobi fields and manifolds of harmonic maps).** \cite{KwonThesis} Let $d \geq 2$ be an integer and $\alpha \in (0,1)$ be a constant.  Let $(M,g)$ and $(N,h)$ be closed, smooth Riemannian manifolds, with smooth isometric embedding $N \subset \mathbb{R}^n$ for some integer $n$.  If $f_0 \in C^\infty(M;N)$ is a harmonic map, so $\mathcal{E}'(f_0) = 0$, then the following hold:

1. If there is a constant $\delta = \delta(f_0,g,h,n,\alpha) \in (0,1]$ such that

$$U_{f_0,\delta} := \left\{ f \in C^{2,\alpha}(M;N) : \|f - f_0\|_{C^{2,\alpha}(M;\mathbb{R}^n)} < \delta \text{ and } \mathcal{E}'(f) = 0 \right\}$$

is an open smooth manifold with tangent space $T_{f_0}U_{f_0,\delta} = \text{Ker} \mathcal{E}''(f_0)$ at $f_0$, then every Jacobi vector field in $\text{Ker} \mathcal{E}''(f_0)$ is integrable.

2. If $(N,h)$ is real analytic, the isometric embedding $N \subset \mathbb{R}^n$ is real analytic, and every Jacobi vector field in $\text{Ker} \mathcal{E}''(f_0)$ is integrable, then there is a constant $\delta = \delta(f_0,g,h,n,\alpha) \in (0,1]$ such that the set $U_{f_0,\delta}$ in (C.1) is an open smooth manifold with tangent space $T_{f_0}U_{f_0,\delta} = \text{Ker} \mathcal{E}''(f_0)$ at $f_0$.

It follows that for real-analytic target manifolds, all Jacobi fields along all harmonic maps are integrable if and only if the space of harmonic maps is a manifold whose tangent bundle is given by the Jacobi fields \cite[p. 470]{AdamsSimon}.  By Definition \ref{Def:1.5}, the conclusion of Theorem \ref{Thm:C.2} \ref{Thm:C.2} is equivalent to the assertion that all Jacobi fields along $f_0$ are integrable if and only if the harmonic map energy function $\mathcal{E}$ is Morse–Bott at $f_0$.  

For a further discussion of integrability and additional references, see Adams and Simon \cite[Section 1]{AdamsSimon}, Kwon \cite[Section 4.1]{KwonThesis}, and Simon \cite[pp. 270–272]{Simon}.

According to \cite[Theorem 1.3]{KwonThesis}, any Jacobi field along a harmonic map from $S^2$ to $\mathbb{C}P^2$ is integrable, where the two-sphere $S^2$ has its unique conformal structure and the complex projective space $\mathbb{C}P^2$ has its standard Fubini-Study metric of holomorphic sectional curvature 1; see Crawford \cite{Crawford2} for additional results.

From the list of examples provided by Lemaire and Wood \cite[p. 471]{LemaireWood}, there are few other examples of families of harmonic maps that are guaranteed to be integrable, with the list including harmonic maps from $S^2$ to $S^2$ but excluding harmonic maps from $S^2$ to $S^3$ or $S^4$.  

Fernández \cite{Fernandez} has proved that the space $\text{Harm}_d(S^2,S^{2n})$ of degree-$d$ harmonic maps from $S^2$ into $S^{2n}$ has dimension $2d + n^2$.  However, thus far, integrability for such maps is known only when $n = 1$.  Bolton and Fernandez \cite{BoltonFernandez} provide a nice survey of what is known regarding regularity of $\text{Harm}_d(S^2,S^{2n})$: they recall that $\text{Harm}_d(S^2,S^2)$ is known to be a smooth manifold, outline a proof that $\text{Harm}_d(S^2,S^6)$ is also a smooth manifold, and survey results on the structure of $\text{Harm}_d(S^2,S^4)$ and why that space is not a smooth manifold.

**C.2. Integrability and Morse–Bott conditions for the area function.** Suppose that $m, n \geq 1$ and $r \geq 2$ are integers and $M$ is a closed, connected, oriented, smooth manifold of dimension $m$.  We let $C^{r,\alpha}(M;\mathbb{R}^n)$ denote the Banach space of $C^{r,\alpha}$ maps from $M$ into $\mathbb{R}^n$, where $\alpha \in [0,1]$, and let $\text{Imm}^{r,\alpha}(M;\mathbb{R}^n) \subset C^{r,\alpha}(M;\mathbb{R}^n)$ denote the open subset of $C^{r,\alpha}$ immersions, and let $\text{Emb}^{r,\alpha}(M;\mathbb{R}^n) \subset C^{r,\alpha}(M;\mathbb{R}^n)$ denote the open subset of $C^{r,\alpha}$ embeddings.  If $\Phi \in C^{r,\alpha}(M;\mathbb{R}^d)$ is an embedding, then $g_{\Phi} := \Phi^\ast g$ is a Riemannian metric on $M$ while if $\Phi$ is an immersion, then $g_{\Phi}$ may be singular.  We now consider the area or volume function,

$$\text{Imm}^{r,\alpha}(M;\mathbb{R}^n) \ni \Phi \mapsto \mathcal{E}(\Phi) := \text{Vol}(M,g_{\Phi}) \in [0,\infty).$$
MORSE–BOTT PROPERTY FOR ANALYTIC FUNCTIONS WITH ŁOJASIEWICZ EXPONENT ONE HALF

Then $\Phi(M)$ is called a critical immersed submanifold or (as customary) a minimal immersed submanifold if $\mathcal{E}'(\Phi) = 0$, where

$$\mathcal{E}'(\Phi)\eta = \left. \frac{d}{dt} \text{Vol}(M, g_{\Phi + t\Phi_0}) \right|_{t=0}$$

for all vector fields $\eta \in C^{r,\alpha}(TM)$. One can show that

$$\mathcal{E}'(\Phi)\eta = (\eta, \mathcal{E}'(\Phi))_{L^2(M)},$$

with an explicit expression for the gradient $\mathcal{E}'(\Phi)$ provided by the first variation formula — see Calegari [24, Proposition 2.1], Colding and Minicozzi [30, pp. 154–155], Dajczer and Tojeiro [35, Proposition 3.1], Lawson [76], Schoen [104, Section 2.1], or Xin [130, Theorem 1.2.2 and Remark 1.2.5].

An explicit expression for the Hessian $\mathcal{E}''(\Phi)$ at a critical point $\Phi$ is provided by the second variation formula — see Calegari [24, Proposition 3.1], Colding and Minicozzi [30, pp. 154–155], Lawson [76], Schoen [104, Section 2.1], and Xin [130, Theorem 6.1.1].

More generally, if we replace $\mathbb{R}^n$ in the preceding discussion by a connected, smooth manifold $N$ without boundary, then it is known that $C^r(M; N)$ is a smooth Banach manifold — see Abraham [1], Bruveris [23], Eichhorn [42], Eliasson [43], or Wittmann [129]. (It is highly likely that published proofs of this result extend to show that $C^{r,\alpha}(M; N)$ is a Banach manifold when $\alpha \in [0, 1]$ and, furthermore, that $W^{k,p}(M; N)$ is Banach manifold for $k \in \mathbb{N}$ and $p \in [1, \infty)$, at least for $k \geq 2$ and $kp > m$, taking note of the Sobolev Embedding Theorem [5, Theorem 4.12].)

We refer to Michor and Mumford [85, Section 2.1] for their analysis of these spaces in the $C^\infty$ category.

Recall from Dajczer and Tojeiro [35, Corollary 3.7] or Xin [130, Corollary 1.3.4] that there exists no minimal isometric immersion $\Phi : M^m \to \mathbb{R}^n$ of a compact Riemannian manifold without boundary. Hence, we restrict our attention to cases where $M$ and $N$ are closed or $M$ and $N$ are complete or $M$ is compact with boundary and $N$ is complete.

One could again derive an analogue of Theorem C.2 (giving the relationship between integrability of Jacobi vector fields and the Morse–Bott property of an immersed minimal submanifold) from Theorem 8 or derive an analogue of Theorem C.2 for immersed minimal submanifolds from prior, more general results of Simon [108, 109] and Adams and Simon [4].

Adams and Simon list examples of minimal submanifolds whose Jacobi vector fields are all integrable as well as examples that have nontrivial Jacobi vector fields that are not integrable [4, pp. 249–252]. See also Allard and Almgren [5, Section 6], Nagura [89, 90, 91], Simons [111], and Smith [115, 114] (via Remark C.3) for related examples.

White [126, 127] has shown that for generic $C^r$ Riemannian metrics on a manifold $N$, there are no closed, immersed, minimal submanifolds $M \subset N$ with nontrivial Jacobi fields; the case of geodesics, including immersed geodesics, was proved earlier by Abraham [2].

Remark C.3 (On the relationship between harmonic maps and minimal surfaces). It is useful to recall the relationship between harmonic maps $f$ from a closed, smooth Riemann surface $(\Sigma, g)$ into a closed Riemannian manifold $(N, h)$, and immersed minimal surfaces in $(N, h)$, since that relationship enriches our supply of examples. Chern and Goldberg [28, Proposition 5.1] show that if $\Sigma = S^2$ and $f$ is a harmonic immersion, then $f$ is a minimal immersion. More generally, though they assume $(N, h) = \mathbb{R}^3$ with its standard metric and allow $(\Sigma, g)$ to be a Riemann surface with boundary, Dierkes, Hildebrandt, and Sauvigny prove [38, Theorem 2.6.1] that a conformal map $f$ is minimal if and only if it is harmonic. According to their [38, Definition 2.6.1], they may replace $\mathbb{R}^3$ by $\mathbb{R}^n$ for any $n \geq 2$ and more generally, by any Riemannian manifold $(N, h)$ of dimension...
Moore [87, Theorem 4.2.2] proves a similar result, namely that (the image of) a (weakly) conformal harmonic map \( f : (\Sigma, g) \rightarrow (N, h) \) is a minimal surface. By restricting to \( \Sigma = S^2 \), Moore [87, Proposition 4.2.3] recovers the result of Chern and Goldberg: a harmonic two-sphere, \( f : (S^2, g_{\text{round}}) \rightarrow (N, h) \), is automatically weakly conformal and hence a parametrized minimal surface. See also [38, pp. 36, 77, 249–250, and 309–311] and their discussion of Lichtenstein's Theorem on reparameterizing maps of the disk and [38, pp. 249–250] for the relationship between area and energy integrals and the minimization problem.

**Remark C.4 (On the interpretation of mean curvature flow as a gradient flow).** While there is a wealth of references on mean curvature flow, relatively few treat it as gradient flow for the area (volume) function, thus making it less accessible to gradient flow methods pioneered by Simon [108] [109]. For interpretations of mean curvature flow as a gradient system, we refer the reader to Bellettini [14, Remark 2.8 and Section 2.3], Colding, Minicozzi, and Pedersen [33, Section 1], Ilmanen [68], Mantegazza [83, p. 7, second paragraph], Ritoré and Sinestrari [101, Equation (4.3)], Smoczyk [116], and Zaal [132]. Shi and Vorotnikov [107] provide a useful recent reference, with a view to applications. For introductions to mean curvature flow, we refer to Ecker [40], Mantegazza [83], Ritoré and Sinestrari [101].

For applications of the DeTurck trick [37] to convert mean curvature flow to a nonlinear parabolic partial differential equation and establish short-time existence, we refer to Andrews and Baker [7], Baker [12], and Leng, Zhao, and Zhao [79].

As in the case of Ricci flow, the interpretation of mean curvature flow as a gradient system can lead to the introduction of a time-varying family of Hilbert spaces — a family of \( L^2 \) spaces defined by a measure that depends on the time-varying family of immersions [83, Section 1.2, page 7].

**References**

[1] Ralph H. Abraham, *Lectures of Smale on differential topology*, unpublished manuscript, 1963.

[2] Ralph H. Abraham, *Bumpy metrics*, Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968), Amer. Math. Soc., Providence, R.I., 1970, pp. 1–3. MR 0271994 (42 #6875)

[3] Ralph H. Abraham, Jerrold E. Marsden, and Tudor S. Ratiu, *Manifolds, tensor analysis, and applications*, second ed., Springer, New York, 1988. MR 960687 (89f:58001)

[4] David Adams and Leon Simon, *Rates of asymptotic convergence near isolated singularities of geometric extrema*, Indiana Univ. Math. J. 37 (1988), 225–254. MR 963501 (90b:58046)

[5] Robert A. Adams and John J. F. Fournier, *Sobolev spaces*, second ed., Elsevier/Academic Press, Amsterdam, 2003. MR 2424078 (2009e:46025)

[6] William K. Allard and Frederick J. Almgren, Jr., *On the radial behavior of minimal surfaces and the uniqueness of their tangent cones*, Ann. of Math. (2) 113 (1981), no. 2, 215–265. MR 607893

[7] Ben Andrews and Charles Baker, *Mean curvature flow of pinched submanifolds to spheres*, J. Differential Geom. 85 (2010), no. 3, 357–395. MR 2739807 (2012a:53122)

[8] Dang D. Ang and Vu T. Tuan, *An elementary proof of the Morse-Palais lemma for Banach spaces*, Proc. Amer. Math. Soc. 39 (1973), 642–644. MR 0319225

[9] Philippe Antoine, *Lemme de Morse et calcul des variations*, Bull. Soc. Math. France Mém. (1979), no. 60, 25–29. Analyse non convexe (Proc. Colloq., Pau, 1977). MR 562253

[10] Vladimir I. Arnol’d, Sabir M. Gusein-Zade, and Alexander N. Varchenko, *Singularities of differentiable maps. Volume 1*, Modern Birkhäuser Classics, Birkhäuser/Springer, New York, 2012, Classification of critical points, caustics and wave fronts, Translated from the Russian by Ian Porteous based on a previous translation by Mark Reynolds, Reprint of the 1985 edition. MR 2896292

[11] David M. Austin and Peter J. Braam, *Morse–Bott theory and equivariant cohomology*, The Floer memorial volume, Progr. Math., vol. 133, Birkhäuser, Basel, 1995, pp. 123–183. MR 1362827 (96i:57037)

[12] Charles Baker, *The mean curvature flow of submanifolds of high codimension*, Ph.D. thesis, Australian National University, Canberra, November 2010, arXiv:1104.4409.
Augustin Banyaga and David E. Hurtubise, A proof of the Morse–Bott lemma, Expo. Math. 22 (2004), no. 4, 365–373. MR 2075744

Giovanni Bellettini, Lecture notes on mean curvature flow, barriers and singular perturbations, Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)], vol. 12, Edizioni della Normale, Pisa, 2013. MR 3155251

Marcel Berger, Nonlinearity and functional analysis, Academic Press, New York, 1977. MR 0488101 (58 #7671)

Edward Bierstone and Pierre D. Milman, Semianalytic and subanalytic sets, Inst. Hautes Études Sci. Publ. Math. (1988), no. 67, 5–42. MR 972342 (89k:32011)

Edward Bierstone and Pierre D. Milman, Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant, Invent. Math. 128 (1997), no. 2, 207–302. MR 1440306

John Bolton and Luis Fernández, On the regularity of the space of harmonic 2-spheres in the 4-sphere, Harmonic maps and differential geometry, Contemp. Math., vol. 542, Amer. Math. Soc., Providence, RI, 2011, pp. 187–194. MR 2796649

Raoul H. Bott, Nondegenerate critical manifolds, Ann. of Math. (2) 60 (1954), 248–261. MR 0064399 (16,276f)

Raoul H. Bott, The stable homotopy of the classical groups, Ann. of Math. (2) 70 (1959), 313–337. MR 0110104

H. Brézis, Functional analysis, Sobolev spaces and partial differential equations, Universitext, Springer, New York, 2011. MR 2759829 (2012a:35002)

Theodor Bröcker, Differentiable germs and catastrophes, Cambridge University Press, Cambridge-New York-Melbourne, 1975, Translated from the German, last chapter and bibliography by L. Lander, London Mathematical Society Lecture Note Series, No. 17. MR 0494220

Martins Bruveris, Notes on Riemannian geometry of manifolds of maps, unpublished manuscript, 2016.

Danny Calegari, Topics in differential geometry minimal submanifolds, unpublished lecture notes.

Alberto Cambini, Sul lemma di M. Morse, Boll. Un. Mat. Ital. (4) 7 (1973), 87–93. MR 0315738

Oliver Caps, Evolution equations in scales of Banach spaces, Teubner-Texte zur Mathematik [Teubner Texts in Mathematics], vol. 140, B. G. Teubner, Stuttgart, 2002. MR 2352809 (2008m:34129)

Alessandro Carlotto, Otis Chodosh, and Yannir A. Rubinstein, Slowly converging Yamabe flows, Geom. Topol. 19 (2015), no. 3, 1523–1568, arXiv:1401.3738. MR 3352243

Shiing Shen Chern and Samuel I. Goldberg, On the volume decreasing property of a class of real harmonic mappings, Amer. J. Math. 97 (1975), 133–147. MR 0367860

Ralph Chill, On the Łojasiewicz–Simon gradient inequality, J. Funct. Anal. 201 (2003), 572–601. MR 1986700 (2005c:26019)

Tobias H. Colding and William P. Minicozzi, II, Minimal submanifolds, Bull. London Math. Soc. 38 (2006), no. 3, 353–395. MR 2230932

Tobias H. Colding and William P. Minicozzi, II, Łojasiewicz inequalities and applications, Surveys in Differential Geometry XIX (2014), 63–82, arXiv:1402.5087.

Tobias H. Colding and William P. Minicozzi, II, Uniqueness of blowups and Łojasiewicz inequalities, Ann. of Math. (2) 182 (2015), no. 1, 221–285. MR 3374960

Tobias H. Colding, William P. Minicozzi, II, and E. K. Pedersen, Mean curvature flow, Bull. Amer. Math. Soc. (N.S.) 52 (2015), no. 2, 297–333. MR 3312634

Thomas A. Crawford, The space of harmonic maps from the 2-sphere to the complex projective plane, Canad. Math. Bull. 40 (1997), no. 3, 285–295. MR 1464837

Marcos Dajczer and Rudy Tojeiro, Submanifold theory, Universitext, Springer, New York, 2019. MR 3969932

Klaus Deimling, Nonlinear functional analysis, Springer–Verlag, Berlin, 1985. MR 787404 (86j:47001)

Dennis M. DeTurck, Deforming metrics in the direction of their Ricci tensors, J. Differential Geom. 18 (1983), 157–162. MR 067987 (85j:53050)

Ulrich Dierkes, Stefan Hildebrandt, and Friedrich Sauvigny, Minimal surfaces, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 339, Springer, Heidelberg, 2010, With assistance and contributions by A. Küster and R. Jakob. MR 2566897

Simon K. Donaldson and Peter B. Kronheimer, The geometry of four-manifolds, Oxford University Press, New York, 1990.

Klaus Ecker, Regularity theory for mean curvature flow, Progress in Nonlinear Differential Equations and their Applications, 57, Birkhäuser Boston, Inc., Boston, MA, 2004. MR 2024995 (2005b:53108)
[41] James Eells and Luc Lemaire, *Selected topics in harmonic maps*, CBMS Regional Conference Series in Mathematics, vol. 50, American Mathematical Society, Providence, RI, 1983. MR 703510 (85g:58030)

[42] Jürgen Eichhorn, *The manifold structure of maps between open manifolds*, Ann. Global Anal. Geom. 11 (1993), 253–300. MR 1237457 (95b:58024)

[43] Halldór I. Eliasson, *Geometry of manifolds of maps*, J. Differential Geometry 1 (1967), 169–194. MR 0226681

[44] Math Stack Exchange, *The dual of the direct sum*, Internet, October 20, 2012, https://math.stackexchange.com/questions/217596/the-dual-of-the-direct-sum

[45] Paul M. N. Feehan, *Global existence and convergence of solutions to gradient systems and applications to Yang–Mills gradient flow*, arXiv:1409.1525v4, xx+475 pages.

[46] Paul M. N. Feehan, *Integrability of Jacobi vectors for analytic functions on Banach manifolds*, preprint.

[47] Paul M. N. Feehan, *Morse theory for the Yang–Mills energy function near flat connections*, arXiv:1906.03954.

[48] Paul M. N. Feehan, *Optimal Łojasiewicz–Simon inequalities and Morse–Bott Yang–Mills energy functions*, arXiv:1706.09349.

[49] Paul M. N. Feehan, *Resolution of singularities and geometric proofs of the Łojasiewicz inequalities*, Geom. Topol. 23 (2019), no. 7, 3273–3313, arXiv:1708.09775. MR 4046966

[50] Paul M. N. Feehan and Manousos Maridakis, *Łojasiewicz–Simon gradient inequalities for analytic and Morse–Bott functions on Banach spaces*, J. Reine Angew. Math., in press, arXiv:1510.03817v8.

[51] Paul M. N. Feehan and Manousos Maridakis, *Łojasiewicz–Simon gradient inequalities for analytic and Morse–Bott functions on Banach spaces and applications to harmonic maps*, arXiv:1510.03817v5.

[52] Paul M. N. Feehan and Manousos Maridakis, *Łojasiewicz–Simon gradient inequalities for harmonic maps*, arXiv:1903.01953.

[53] Luis Fernández, *The dimension and structure of the space of harmonic 2-spheres in the m-sphere*, Ann. of Math. (2) 175 (2012), no. 3, 1093–1125. MR 2912703

[54] William M. Goldman and John J. Millson, *The deformation theory of representations of fundamental groups of compact Kähler manifolds*, Inst. Hautes Études Sci. Publ. Math. (1988), no. 67, 43–96. MR 972343

[55] Detlef Gromoll and Wolfgang T. Meyer, *On differentiable functions with isolated critical points*, Ann. of Math. (2) 79 (1964), 361–369. MR 0246329

[56] Victor Guillemin and Shlomo Sternberg, *Geometric asymptotics*, American Mathematical Society, Providence, R.I., 1977. Mathematical Surveys, No. 14. MR 0516965

[57] Shangjiang Guo and Jianhong Wu, *Bifurcation theory of functional differential equations*, Applied Mathematical Sciences, vol. 126, American Mathematical Society, Providence, RI, 2006. MR 2226672 (2007b:35035)

[58] Alain Haraux, *Positively homogeneous functions and the Łojasiewicz gradient inequality*, Ann. Polon. Math. 87 (2005), 165–174. MR 2208543

[59] Alain Haraux and M. A. Jendoubi, *The Łojasiewicz gradient inequality in the infinite-dimensional Hilbert space framework*, J. Funct. Anal. 260 (2011), 2826–2842. MR 2772353 (2012c:47168)

[60] Frédéric Hélein, *Harmonic maps, conservation laws and moving frames*, second ed., Cambridge Tracts in Mathematics, vol. 150, Cambridge University Press, 2002. MR 1913803 (2003g:58024)

[61] Daniel Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics, vol. 840, Springer–Verlag, Berlin-New York, 1981. MR 610244 (83j:35084)

[62] Heisuke Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II*, Ann. of Math. (2) 79 (1964), 109–203; ibid. (2) 79 (1964), 205–326. MR 0199184

[63] Helmut H. Hofer, *A note on the topological degree at a critical point of mountain-pass-type*, Proc. Amer. Math. Soc. 90 (1984), no. 2, 309–315. MR 727256

[64] Helmut H. Hofer, *The topological degree at a critical point of mountain-pass type*, Nonlinear functional analysis and its applications, Part 1 (Berkeley, Calif., 1983), Proc. Sympos. Pure Math., vol. 45, Amer. Math. Soc., Providence, RI, 1986, pp. 501–509. MR 843584

[65] Lars Hörmander, *The analysis of linear partial differential operators, III. Pseudo-differential operators*, Springer, Berlin, 2007. MR 2304165 (2007k:35009)

[66] Roger A. Horn and Charles R. Johnson, *Topics in matrix analysis*, Cambridge University Press, Cambridge, 1994, Corrected reprint of the 1991 original. MR 1288752 (95c:15001)

[67] Sen-Zhong Huang, *Gradient inequalities*, Mathematical Surveys and Monographs, vol. 126, American Mathematical Society, Providence, RI, 2006. MR 2226672 (2007b:58035)

[68] Tom Ilmanen, *Elliptic regularization and partial regularity for motion by mean curvature*, Mem. Amer. Math. Soc. 108 (1994), no. 520. MR 1196160 (95d:49060)
MORSE–BOTT PROPERTY FOR ANALYTIC FUNCTIONS WITH ŁOJASIEWICZ EXPONENT ONE HALF

[96] Jürgen Jost, Riemannian geometry and geometric analysis, sixth ed., Universitext, Springer, Heidelberg, 2011. MR 2829653

[97] Steven G. Krantz and Harold R. Parks, A primer of real analytic functions, second ed., Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Boston, Inc., Boston, MA, 2002. MR 1916029

[98] Nicolaas H. Kuiper, C¹-equivalence of functions near isolated critical points, (1972), 199–218. Ann. of Math. Studies, No. 69. MR 0413161

[99] Hui Hsiung Kuo, The Morse-Palais lemma on Banach spaces, Bull. Amer. Math. Soc. 80 (1974), 363–365. MR 0334274

[100] Heaseung Kwon, Asymptotic convergence of harmonic map heat flow, Ph.D. thesis, Stanford University, Palo Alto, CA, 2002. MR 2703296

[101] Serge Lang, Fundamentals of differential geometry, Graduate Texts in Mathematics, vol. 191, Springer–Verlag, New York, 1999. MR 1666820

[102] Serge Lang, Introduction to differentiable manifolds, second ed., Universitext, Springer-Verlag, New York, 2002. MR 1931083

[103] H. Blaine Lawson, Jr., Lectures on minimal submanifolds. Vol. I, second ed., Mathematics Lecture Series, vol. 9, Publish or Perish, Inc., Wilmington, Del., 1980. MR 576752

[104] Luc Lemaire and John C. Wood, Jacobi fields along harmonic 2-spheres in CP² are integrable, J. London Math. Soc. (2) 66 (2002), no. 2, 468–486. MR 1920415 (2003k:58022)

[105] Luc Lemaire and John C. Wood, Jacobi fields along harmonic 2-spheres in 3- and 4-spheres are not all integrable, Tohoku Math. J. (2) 61 (2009), no. 2, 165–204. MR 2541404 (2010g:53117)

[106] Yan Leng, Entao Zhao, and Haoran Zhao, Notes on the extension of the mean curvature flow, Pacific J. Math. 269 (2014), no. 2, 385–392. MR 3238481

[107] Joram Lindenstrauss and Lior Tzafriri, On the complemented subspaces problem, Israel J. Math. 9 (1971), 263–269. MR 0276734

[108] Qingyue Liu and Yunyan Yang, Rigidity of the harmonic map heat flow from the sphere to compact Kähler manifolds, Ark. Mat. 48 (2010), 121–130. MR 2594589 (2011a:53066)

[109] Stanisław Łojasiewicz, Ensembles semi-analytiques, (1965), Publ. Inst. Hautes Etudes Sci., Bures-sur-Yvette. LaTeX version by M. Coste, August 29, 2006 based on mimeographed course notes by S. Łojasiewicz, available at perso.univ-rennes1.fr/michel.coste/Lojasiewicz.pdf

[110] Carlo Mantegazza, Lecture notes on mean curvature flow, Progress in Mathematics, vol. 290, Birkhäuser/Springer Basel AG, Basel, 2011. MR 2815949

[111] Jean Mawhin and Michel Willem, On the generalized Morse lemma, Bull. Soc. Math. Belg. Sér. B 37 (1985), no. 2, 23–29. MR 845402

[112] Peter W. Michor and David Mumford, Vanishing geodesic distance on spaces of submanifolds and diffeomorphisms, Doc. Math. 10 (2005), 217–245. MR 2148075

[113] John W. Milnor, Dynamics in one complex variable, third ed., Annals of Mathematics Studies, vol. 160, Princeton University Press, Princeton, NJ, 2006. MR 2193309

[114] John D. Moore, Introduction to global analysis, Graduate Studies in Mathematics, vol. 187, American Mathematical Society, Providence, RI, 2017, Minimal surfaces in Riemannian manifolds. MR 3729450

[115] Jürgen K. Moser, On the volume elements on a manifold, Trans. Amer. Math. Soc. 120 (1965), 286–294. MR 0182927

[116] Toshinobu Nagura, On the Jacobi differential operators associated to minimal isometric immersions of symmetric spaces into spheres. I, Osaka Math. J. 18 (1981), no. 1, 115–145. MR 069982

[117] Toshinobu Nagura, On the Jacobi differential operators associated to minimal isometric immersions of symmetric spaces into spheres. II, Osaka Math. J. 19 (1982), no. 1, 79–124. MR 656235

[118] Toshinobu Nagura, On the Jacobi differential operators associated to minimal isometric immersions of symmetric spaces into spheres. III, Osaka Math. J. 19 (1982), no. 2, 241–281. MR 667489

[119] Liviu I. Nicolaescu, An invitation to Morse theory, second ed., Universitext, Springer, New York, 2011. MR 2883440 (2012:58007)

[120] Louis Nirenberg, Topics in nonlinear functional analysis, Courant Lecture Notes in Mathematics, vol. 6, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2001, Chapter 6 by E. Zehnder, Notes by R. A. Artino, Revised reprint of the 1974 original. MR 1850453

[121] Richard S. Palais, Morse theory on Hilbert manifolds, Topology 2 (1963), 299–340. MR 0158410 (28 #1633)
Richard S. Palais, *The Morse lemma for Banach spaces*, Bull. Amer. Math. Soc. **75** (1969), 968–971. MR 0253378

Amnon Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, vol. 44, Springer–Verlag, New York, 1983. MR 701486 (85g:47061)

Matthew Petro, *Moduli spaces of Riemann surfaces*, Ph.D. thesis, The University of Wisconsin–Madison, Madison, WI, 2008, p. 119. MR 2712204

Albrecht Pietsch, *Operator ideals*, North-Holland Mathematical Library, vol. 20, North-Holland Publishing Co., Amsterdam-New York, 1980, Translated from German by the author. MR 582655

Mikhail M. Postnikov and Yuli B. Rudyak, *Morse lemma*, Encyclopedia of Mathematics, Springer Verlag and the European Mathematical Society, [http://www.encyclopediaofmath.org/index.php?title=Morse_lemma&oldid=32324](http://www.encyclopediaofmath.org/index.php?title=Morse_lemma&oldid=32324).

Tim Poston and Ian Stewart, *Catastrophe theory and its applications*, Dover Publications, Inc., Mineola, NY, 1996, With an appendix by D. R. Olsen, S. R. Carter and A. Rockwood, Reprint of the 1978 original. MR 1426131

Manuel Ritoré and Carlo Sinestrari, *Mean curvature flow and isoperimetric inequalities*, Advanced Courses in Mathematics. CRM Barcelona, Birkhäuser Verlag, Basel, 2010, Edited by Vicente Miquel and Joan Porti. MR 2590630

Johan Råde, *On the Yang–Mills heat equation in two and three dimensions*, J. Reine Angew. Math. **431** (1992), 123–163. MR 1179335 (94a:58041)

Walter Rudin, *Functional analysis*, second ed., International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., New York, 1991. MR 1157815

Richard M. Schoen, *Theorems on regularity and singularity of energy minimizing maps*, Lectures in Mathematics ETH Zürich, Birkhäuser, Basel, 1996. MR 1399562 (98c:58042)

James Simons, *Minimal varieties in riemannian manifolds*, Ann. of Math. (2) **88** (1968), 62–105. MR 233295

Carlos T. Simpson, *Higgs bundles and local systems*, Inst. Hautes Études Sci. Publ. Math. (1992), no. 75, 5–95. MR 1179076

Stephen J. Smale, *Morse theory and a non-linear generalization of the Dirichlet problem*, Ann. of Math. (2) **80** (1964), 382–396. MR 0165539 (29 #2820)

R. T. Smith, *Harmonic mappings of spheres*, Amer. J. Math. **97** (1975), 364–385. MR 39127

R. T. Smith, *The second variation formula for harmonic mappings*, Proc. Amer. Math. Soc. **47** (1975), 229–236. MR 375386

Knut Smoczyk, *Mean curvature flow in higher codimension: introduction and survey*, Global differential geometry, Springer Proc. Math., vol. 17, Springer, Heidelberg, 2012, pp. 231–274. MR 3289845

Michael Struwe, *Variational methods*, fourth ed., Springer, Berlin, 2008. MR 2431434 (2009g:49002)

Hiroki Tanabe, *Equations of evolution*, Monographs and Studies in Mathematics, vol. 6, Pitman (Advanced Publishing Program), Boston, MA, 1979, Translated from the Japanese by N. Mugibayashi and H. Haneda. MR 533824 (82g:47032)

Hiroki Tanabe, *Functional analytic methods for partial differential equations*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 204, Marcel Dekker Inc., New York, 1997. MR 1413304 (97i:35002)

Peter M. Topping, *Rigidity in the harmonic map heat flow*, J. Differential Geom. **45** (1997), 593–610. MR 1472890 (99d:58050)

Anthony J. Tromba, *The Morse lemma on Banach spaces*, Proc. Amer. Math. Soc. **34** (1972), 396–402. MR 0295395
[122] Vu T. Tuan and Dang D. Ang, *A representation theorem for differentiable functions*, Proc. Amer. Math. Soc. **75** (1979), no. 2, 343–350. MR 532164
[123] Karen K. Uhlenbeck, *Morse theory on Banach manifolds*, Bull. Amer. Math. Soc. **76** (1970), 105–106. MR 0253381
[124] Alan D. Weinstein, *Symplectic structures on Banach manifolds*, Bull. Amer. Math. Soc. **75** (1969), 1040–1041. MR 0245052
[125] Brian White, *The space of $m$-dimensional surfaces that are stationary for a parametric elliptic functional*, Indiana Univ. Math. J. **36** (1987), no. 3, 567–602. MR 905611
[126] Brian White, *The space of minimal submanifolds for varying Riemannian metrics*, Indiana Univ. Math. J. **40** (1991), no. 1, 161–200. MR 1101226
[127] Brian White, *On the bumpy metrics theorem for minimal submanifolds*, Amer. J. Math. **139** (2017), no. 4, 1149–1155. MR 3689325
[128] Emmet F. Whittlesey, *Analytic functions in Banach spaces*, Proc. Amer. Math. Soc. **16** (1965), 1077–1083. MR 0184092 (32 #1566)
[129] Johannes Wittmann, *The Banach manifold $C^k(M,N)$*, Differential Geom. Appl. **63** (2019), 166–185. MR 3903188
[130] Yuanlong Xin, *Minimal submanifolds and related topics*, Nankai Tracts in Mathematics, vol. 8, World Scientific Publishing Co., Inc., River Edge, NJ, 2003. MR 2035469
[131] Atsushi Yagi, *Abstract parabolic evolution equations and their applications*, Springer Monographs in Mathematics, Springer–Verlag, Berlin, 2010. MR 2573296 (2011c:35008)
[132] Martijn M. Zaal, *The gradient structure of the mean curvature flow*, Adv. Calc. Var. **8** (2015), no. 3, 183–202. MR 3365740
[133] Eberhard Zeidler, *Nonlinear functional analysis and its applications, I. Fixed-point theorems*, Springer, New York, 1986. MR 816732 (87f:47083)

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