Pseudo null growth model and its classifications based on generalized vortex filament equation

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ABSTRACT
The pseudo null growth model in Minkowski 3-space is shaped by evolving a pseudo null curve as denoted direction, growth velocity, and stretch in this paper. Meanwhile, the idea of the generalized Hasimoto surfaces is proposed based on the vortex filament equation. The conditions and geometric structures of the generalized Hasimoto pseudo null growth surfaces are investigated accompanied by some typical examples.

1. Introduction
In 1972, Hasimoto discovered that there exists a kind of relationship between the thin vortex filament without stretching in an incompressible inviscid fluid and the nonlinear Schrödinger (NLS) equation [1]. Later, Rogers C. and Schief W. K. established that the particular vortex motions without change of form correspond to the traveling wave solutions of the NLS equation [10]. Hasimoto surface is defined as the soliton surface whose position vector \( r(s,t) \) satisfies

\[
r_s(s,t) \times r(s,t) = r(s,t),
\]

where \( s \) is the arclength measured along the vortex filament and \( t \) is the time, which is called the vortex filament equation or the smoke ring equation [1]. In the discussion process for some Hasimoto surfaces, a similar fact emerged as the following example shown.

Example 1.1. A regular surface \( r(s,t) \) formed as

\[
r(s,t) = \left( \sin t + (s - \frac{g^2}{2} - t) \cos t, \cos t - (s - \frac{g^2}{2} - t) \sin t, s + \frac{g^2}{2} \right),
\]

whose coordinate parameter curves satisfy the following relationship

\[
r_s(s,t) \times r_s(s,t) = \frac{4}{s^2 - 2s - 2t} r_s(s,t).
\]

Consider the above situation in Example 1.1, we propose the following definition naturally.

Definition 1.2. Let \( r(s,t) \) be a regular surface. If it satisfies

\[
r_s(s,t) \times r_s(s,t) = \sigma(s,t) r_s(s,t),
\]

(1.1)
then it is said to be a generalized Hasimoto surface, equality (1.1) is called generalized vortex filament equation. Where \( \sigma(s, t) \) is a nonzero smooth function, \( s \) is the arclength measured along the surface and \( t \) is the time.

**Remark 1.3.** Notice that, the generalized Hasimoto surfaces are just Hasimoto surfaces when \( \sigma(s, t) = 1 \). No matter for the Hasimoto surfaces or the generalized Hasimoto surfaces, the coordinate parameter \( s \) should be the arclength parameter and \( t \) the time parameter, then \( r(s, t) \times r_r(s, t) = \sigma(s, t)r_r(s, t) \) is not equivalent to \( r(s, t) \times r_r(s, t) = \sigma(s, t)r_r(s, t) \) when the geometric meaning is considered. In order to distinguish them, we give the following definition.

**Definition 1.4.** Let \( r(s, t) \) be a regular surface. If it satisfies

\[
r'_r(s, t) \times r(s, t) = \sigma(s, t)r_r(s, t),
\]

then it is called a 1-type generalized Hasimoto surface. If it satisfies

\[
r'_r(s, t) \times r(s, t) = \sigma(s, t)r_r(s, t),
\]

then it is called a 2-type generalized Hasimoto surface, where \( \sigma(s, t) \) is a nonzero smooth function, \( s \) is the arclength measured along the surface and \( t \) is the time.

The growth surfaces often appear in many physical, biological, and industrial processes, such as seashells, antlers, and gears. In recent years, many researchers have paid much attention on the structures of some growth surfaces. For instance, Moseley began the first attempt to describe the spiral coils of molluscan shells from the viewpoint of geometry [4]. Moulton and Goriely developed a mathematical framework to model shell growth [2, 3]. The models are designed to describe the coiled shells in mathematical terms in [8] and [9]. Those related research works not only are carried in Euclidean space but also are generalized into Minkowski space, such as Tuğ considered the surface growth with the Darboux growth velocity field on a spatial generating curve [11], Özdemir and Tuğ determined the model of the growth surface with a null generating curve [5].

In order to explore the biological structures and represent the surface growth in a local, elegant mathematical structure in Minkowski space deeply, we will build a mathematical model named as the pseudo null growth surface by evolving a pseudo null curve according to a denoted direction and growth velocity. Meanwhile, the pseudo null growth surfaces are classified into three kinds according to the growth velocity vector field lying on the normal plane, osculating plane and rectifying plane of the pseudo null curve, which is briefly called PNGS, POGS and PRGS, respectively. The three kinds of growth surfaces are classified based on the generalized vortex filament equation and they are expressed by the structure function of the pseudo null curves. Furthermore, some typical examples of generalized Hasimoto pseudo null growth surfaces are presented explicitly.

2. Preliminaries

Let \( \mathbb{E}^3 \) be a Minkowski 3-space with the standard flat metric

\[
(x, y) = -x_1y_1 + x_2y_2 + x_3y_3,
\]

where \( x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \). A vector \( \theta \in \mathbb{E}^3 \) is called spacelike, if \( (\theta, \theta) > 0 \) or \( \theta = 0 \); timelike, if \( (\theta, \theta) < 0 \); lightlike (or null), if \( (\theta, \theta) = 0(\theta \neq 0) \). A curve is called spacelike curve, timelike curve, or lightlike curve if its tangent vector is spacelike, timelike, or lightlike. A spacelike curve in \( \mathbb{E}^3 \) is said to be a pseudo null curve when its principal normal vector and binormal vector are linearly independent lightlike vectors.

**Proposition 2.1.** [6] A pseudo null curve \( a(s) \) which is parameterized by arclength in \( \mathbb{E}^3 \) can be framed by a unique Frenet frame \( \{ a = a(s), \beta = \beta(s), \gamma = \gamma(s) \} \) such that

\[
\begin{bmatrix}
a'(s) \\
\beta'(s) \\
\gamma'(s)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 \\
0 & \kappa(s) & 0 \\
1 & 0 & -\kappa(s)
\end{bmatrix}
\begin{bmatrix}
a(s) \\
\beta(s) \\
\gamma(s)
\end{bmatrix},
\]

where \( (\beta, \gamma) = (a, \gamma) = (a, a) = 1, \gamma \times a = a, a \times a = a, a \times a = a \). In sequence, \( a, \beta, \gamma \) is the tangent vector field, the principal normal vector field and the binormal vector field of \( a(s) \). The plane spanned by \( (\beta, \gamma) \), \( (a, \beta) \) and \( (a, \gamma) \) is called the normal plane, the oscillating plane and the rectifying plane of \( a(s) \), respectively. The function \( \kappa(s) \) is said to be the curvature of \( a(s) \).

**Remark 2.2.** The pseudo null curves throughout this paper are assumed to be parameterized by arclength and the pseudo null geodesic is excluded.

**Proposition 2.3.** [7] A pseudo null curve \( a(s) \) framed by \( (a, \beta, \gamma) \) in \( \mathbb{E}^3 \) can be represented by its structure function \( g = g(s) \) as follows

\[
a(s) = \frac{1}{2} \int \left( c(g - 1) + \frac{1}{c}(g + 1), 2g, c(g - 1) - \frac{1}{c}(g + 1) \right) ds.
\]

Meanwhile, its Frenet frame can be written as

\[
a = \frac{1}{2} \left( c(g - 1) + \frac{1}{c}(g + 1), 2g, c(g - 1) - \frac{1}{c}(g + 1) \right),
\]

\[
\beta = \frac{g'}{2} \left( c + \frac{1}{c}, 2, c - \frac{1}{c} \right),
\]

\[
\gamma = \frac{1}{2g} \left( c_1 + \frac{2(c^2 + 1)g - (c^2 + 1)g^2}{2c}, 2c_2 - c_3, c_3 + \frac{2(c^2 + 1)g - (c^2 + 1)g^2}{2c} \right)
\]

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and its curvature \( \kappa(s) \) is expressed as \( g''(s) = \kappa(s)g'(s) \), where \( 0 \neq c_i, (i = 1, 2, 3) \in \mathbb{R} \).

Sequently, we construct a surface model from a pseudo null curve \( a(s) \) in Minkowski 3-space.

**Definition 2.4.** Let \( a(s) \) be a pseudo null curve in \( \mathbb{E}^3_1 \), \( \rho(s, t) \) the velocity vector field on \( a(s) \) at time \( t \), \( \lambda(s) \) the stretch of \( a(s) \) along the direction of \( \rho(s, t) \). Then the evolution of \( a(s) \) deduced by \( \rho(s, t) \) is called a pseudo null growth surface, which is written as

\[
\mathbf{r}(s, t) = a(s) + \lambda(s)\rho(s, t),
\]

where \( a(s) \) is called the generating curve, \( \rho(s, t) \) and \( \lambda(s) \) is called the growth velocity vector and the scaling factor of \( \mathbf{r}(s, t) \), respectively.

**Remark 2.5.** Indeed, the growth velocity vector \( \rho(s, t) \) of \( \mathbf{r}(s, t) \) always can be decomposed by the Frenet frame \( \{a, \beta, \gamma\} \) of its pseudo null generating curve \( a(s) \). Then the surface \( \mathbf{r}(s, t) \) is rewritten as

\[
\mathbf{r}(s, t) = a(s) + \lambda(s)[\rho_1(s, t)a + \rho_2(s, t)\beta + \rho_3(s, t)\gamma],
\]

where \( \rho_1(s, t), \rho_2(s, t) \) and \( \rho_3(s, t) \) are differential functions of \( s \) and \( t \) (see Fig. 1).

In particular, the pseudo null growth surfaces can be divided into three kinds as follows.

- If \( \rho(s, t) \) lies on the normal plane of \( a(s) \), it is called a pseudo null normal growth surface (denoted as PNGS) which is expressed as
  \[
  \mathbf{r}(s, t) = a(s) + \lambda(s)[\rho_2(s, t)\beta + \rho_3(s, t)\gamma].
  \]

- If \( \rho(s, t) \) lies on the osculating plane of \( a(s) \), it is called a pseudo null osculating growth surface (denoted as POGS) which is expressed as
  \[
  \mathbf{r}(s, t) = a(s) + \lambda(s)[\rho_1(s, t)a + \rho_2(s, t)\beta].
  \]

- If \( \rho(s, t) \) lies on the rectifying plane of \( a(s) \), it is called a pseudo null rectifying growth surface (denoted as PRGS) which is expressed as
  \[
  \mathbf{r}(s, t) = a(s) + \lambda(s)[\rho_1(s, t)a + \rho_2(s, t)\beta + \rho_3(s, t)\gamma],
  \]

where \( \rho_1(s, t), \rho_2(s, t) \) and \( \rho_3(s, t) \) are differential functions of \( s \) and \( t \).

**Remark 2.6.** In this paper, we use "' \( t' \)" and "' \( \cdot' \)" to express the derivative or partial derivative respect to the parameter \( s \) and \( t \), respectively. Throughout this paper, all the geometric objects under consideration are smooth and all surfaces are connected unless otherwise stated.

### 3. The generalized Hasimoto pseudo null normal growth surfaces

Assume that \( \mathbf{r}(s, t) \) is a PNGS, then it can be represented as

\[
\mathbf{r}(s, t) = a(s) + \lambda(s)[\rho_2(s, t)\beta + \rho_3(s, t)\gamma].
\]  \( (3.1) \)

Firstly, differentiating equation \( (3.1) \) on both sides with respect to \( s \), we have

\[
r_s(s, t) = \xi_1 a + \xi_2 \beta + \xi_3 \gamma,
\]  \( (3.2) \)

where the differentiable functions \( \xi_i = \xi_i(s, t), (i = 1, 2, 3) \) are as follows

\[
\begin{align*}
\xi_1 &= 1 - \lambda \rho_3, \\
\xi_2 &= \lambda \rho_2 + \lambda \rho_3, \\
\xi_3 &= \lambda \rho_3 - \lambda \rho_2.
\end{align*}
\]  \( (3.3) \)
Furthermore, differentiating equation (3.2) on both sides with respect to \(s\), we have

\[ r_s(s,t) = \eta_1 \alpha + \eta_2 \beta + \eta_3 \gamma, \]

where the differentiable functions \(\eta_i = \eta_i(s,t), (i = 1, 2, 3)\) are as follows

\[
\begin{cases}
\eta_1 = \lambda x^t \rho_3 - 2 x' \rho_3 - 2 \lambda \rho_3', \\
\eta_2 = 1 - \lambda \rho_3 + x'' \rho_3 + 2 x' \rho_3' + \lambda \rho_3'' + 2 x' \kappa \rho_2 + 2 \lambda \kappa \rho_2' + 2 \lambda \kappa^2 \rho_2, \\
\eta_3 = x'' \rho_3 + 2 x' \rho_3' + \lambda \rho_3'' - 2 x' \kappa \rho_3 - 2 \lambda \kappa \rho_3' + 2 \lambda \kappa^2 \rho_3.
\end{cases}
\]

(3.4)

At the same time, by differentiating equation (3.1) with respect to \(t\) twice, we have

\[ r_t(s,t) = \lambda (\rho_2 \beta + \rho_3 \gamma) \]

(3.5)

and

\[ r_{tt}(s,t) = \lambda (\rho_2 \beta + \rho_3 \gamma). \]

To serve the following discussions, from Proposition 2.1, we could obtain

\[ r_s(s,t) \times r_s(s,t) = (\xi_2 \eta_3 - \xi_3 \eta_2) \alpha + (\xi_1 \eta_2 - \xi_2 \eta_1) \beta + (\xi_3 \eta_1 - \xi_1 \eta_3) \gamma \]

(3.6)

and

\[ r_t(s,t) \times r_t(s,t) = \lambda^2 (\rho_2 \beta - \rho_3 \gamma) \alpha. \]

(3.7)

3.1. The 1-type generalized Hasimoto pseudo null normal growth surfaces

Based on the definition of 1-type generalized Hasimoto surface, i.e.,

\[ r_s(s,t) \times r_s(s,t) = \sigma(s,t) r_t(s,t), \]

where \(\sigma(s,t)\) is a nonzero smooth function. From equations (3.5) and (3.6), we get

\[ (\xi_2 \eta_3 - \xi_3 \eta_2) \alpha + (\xi_1 \eta_2 - \xi_2 \eta_1) \beta + (\xi_3 \eta_1 - \xi_1 \eta_3) \gamma = \lambda \sigma (\rho_2 \beta + \rho_3 \gamma). \]

(3.8)

Obviously, the following equation system holds from equation (3.8)

\[
\begin{cases}
\xi_2 \eta_3 - \xi_3 \eta_2 = 0, \\
\xi_1 \eta_2 - \xi_2 \eta_1 = \sigma \rho_2, \\
\xi_3 \eta_1 - \xi_1 \eta_3 = \lambda \sigma \rho_3,
\end{cases}
\]

(3.9)

since \(\lambda\) and \(\sigma\) are not zero everywhere, then we can find

\[
\begin{cases}
\xi_1 \rho_2 + \xi_2 \rho_3 = 0, \\
\eta_2 \rho_3 + \eta_3 \rho_2 = 0.
\end{cases}
\]

(3.10)

Substituting equations (3.3) and (3.4) into (3.10), we obtain

\[
\begin{cases}
\rho_2 (x' \rho_2 + \lambda \rho_3' - \lambda \kappa \rho_3) + \rho_3 (x' \rho_2 + \lambda \rho_2' + \lambda \kappa \rho_3) = 0, \\
\rho_3 (x'' \rho_3 + 2 x' \rho_3' + \lambda \rho_3'' - 2 x' \kappa \rho_3 - 2 \lambda \kappa \rho_3' + 2 \lambda \kappa^2 \rho_3) + \\
\rho_3 (1 - \lambda \rho_3 + x' \rho_3 + \lambda \rho_3' + 2 x' \kappa \rho_3 + 2 \lambda \kappa \rho_3' + 2 \lambda \kappa^2 \rho_3) = 0.
\end{cases}
\]

(3.11)

Because \(r(s,t)\) is a function of \(s\) and \(t\), then \(\rho_2\) and \(\rho_3\) can not be zero simultaneously. In generality, \(\rho_2, \rho_3 \neq 0\), from equation (3.11), we have

\[
1 - \lambda \rho_3 + (\lambda \rho_2')' + k(\lambda \rho_2)'' = k[(\lambda \rho_2)' - \lambda \kappa \rho_3]' - k[(\lambda \rho_2)' - \lambda \kappa \rho_3].
\]

i.e.,

\[
1 - \lambda \rho_3 + (\lambda \rho_2')' + \lambda \kappa \rho_3' + k[(\lambda \rho_2)' - \lambda \kappa \rho_3]' = k[(\lambda \rho_2)' - \lambda \kappa \rho_3].
\]

In particular, when \(\rho_2, \rho_3 = 0\), we have the following two cases.

Case 1. \(\rho_2 = 0, \rho_3 \neq 0\). From equation (3.11), we have

\[
\begin{cases}
(\lambda \rho_2)' = 0, \\
1 - \lambda \rho_3 + k[(\lambda \rho_2)' - \lambda \kappa \rho_3]' = 0.
\end{cases}
\]

(3.12)

\[ \lambda \rho_3 = 0, \quad k(\lambda \rho_3)' = 0. \]

(3.13)
From the above equation system, we know \( \lambda \rho_3 = 1 \), then \( \rho_3 = 0 \) is followed, it is a contradiction. Case 2. \( \rho_2 \neq 0, \rho_3 = 0 \). From equation (3.11), we have

\[
\begin{align*}
(\lambda \rho_3)' - \lambda \kappa \rho_3 &= 0, \\
[(\lambda \rho_3)' - \lambda \kappa \rho_3]' - \kappa [(\lambda \rho_3)' - \lambda \kappa \rho_3] &= 0.
\end{align*}
\]

From the above equation system, it is easy to know \( \lambda \rho_3 = e^{\int \kappa(s) ds} \) by the first equation. Furthermore, from the second equation of (3.9), we obtain

\[
\lambda \rho_2 = (1 - \lambda \rho_3)(1 - \lambda \rho_3 + (\lambda \kappa \rho_2)' + \kappa (\lambda \rho_2)' + \kappa^2 (\lambda \rho_2)' + \kappa (\lambda \rho_2)') + (\lambda \rho_3)' [(\lambda \rho_2)' + \kappa (\lambda \rho_2)].
\]

Substituting \( \lambda \rho_3 = e^{\int \kappa(s) ds} \) into the above equation, we get

\[
\lambda \rho_2 = (1 - e^{\int \kappa(s) ds})^2 + (1 - e^{\int \kappa(s) ds}) [(\lambda \rho_2)' + \lambda \kappa \rho_2]' + \kappa [(\lambda \rho_2)' + \lambda \kappa \rho_2].
\]

Summarizing the above computation process, we have

**Theorem 3.1.** Let a surface \( r(s, t) \) be 1-type generalized Hasimoto PNGS with \( \rho_2 \rho_3 \neq 0 \), then its scaling factor \( \lambda(s) \) and the components \( \rho_2(s, t), \rho_3(s, t) \) of its growth velocity vector satisfy

\[
\frac{1 - \lambda \rho_3 + ((\lambda \rho_2)' + \lambda \kappa \rho_2)'}{(\lambda \rho_2)' + \lambda \kappa \rho_2} + 2\kappa = \frac{[(\lambda \rho_3)' - \lambda \kappa \rho_3]'}{(\lambda \rho_3)' - \lambda \kappa \rho_3},
\]

where \( \kappa \) is the curvature of generating curve \( a(s) \).

**Corollary 3.2.** Let a surface \( r(s, t) \) be 1-type generalized Hasimoto PNGS with \( \rho_2 \rho_3 \neq 0 \), i.e.,

\[
r_s(s, t) \times r_t(s, t) = \sigma(s, t)r_s(s, t),
\]

then the function \( \sigma(s, t) \) is given by

\[
\sigma(s, t) = \frac{\xi_1 \eta_2 - \xi_2 \eta_1}{\lambda \rho_2} = \frac{\xi_3 \eta_1 - \xi_1 \eta_3}{\lambda \rho_3},
\]

where \( \xi_i, \eta_i (i = 1, 2, 3) \) are differentiable functions as stated in equation (3.3) and (3.4).

**Theorem 3.3.** Let a surface \( r(s, t) \) be 1-type generalized Hasimoto PNGS with \( \rho_2 \neq 0, \rho_3 = 0 \), then its scaling factor \( \lambda(s) \), the components \( \rho_2(s, t), \rho_3(s, t) \) of its growth velocity vector satisfy

\[
\begin{align*}
\lambda \rho_2 &= e^{\int \kappa(s) ds}, \\
\lambda \rho_3 &= (1 - e^{\int \kappa(s) ds})^2 + (1 - e^{\int \kappa(s) ds}) [(\lambda \rho_2)' + \lambda \kappa \rho_2]' + \kappa [(\lambda \rho_2)' + \lambda \kappa \rho_2],
\end{align*}
\]

where \( \kappa \) is the curvature of generating curve \( a(s) \), \( \sigma = \sigma(s, t) \) is a nonzero smooth function.

**Corollary 3.4.** Let a surface \( r(s, t) \) be 1-type generalized Hasimoto PNGS with \( \rho_2 \neq 0, \rho_3 = 0 \), i.e.,

\[
r_s(s, t) \times r_t(s, t) = \sigma(s, t)r_s(s, t),
\]

then the function \( \sigma(s, t) \) can be expressed as

\[
\sigma(s, t) = \frac{(1 - e^{\int \kappa(s) ds})^2 + (1 - e^{\int \kappa(s) ds}) [(\lambda \rho_2)' + \lambda \kappa \rho_2]' + \kappa [(\lambda \rho_2)' + \lambda \kappa \rho_2]}{\lambda \rho_2},
\]

where \( \kappa \) is the curvature of generating curve \( a(s) \).

### 3.2. The 2-type generalized Hasimoto pseudo null normal growth surfaces

Based on the definition of 2-type generalized Hasimoto surface, i.e.,

\[
r_t(s, t) \times r_s(s, t) = \sigma(s, t)r_t(s, t),
\]

where \( \sigma(s, t) \) is a nonzero smooth function. From equation (3.2) and (3.7), we have

\[
\lambda^2 (\rho_2 \rho_3 - \rho_1 \rho_2) \alpha = (\xi_1 \alpha + \xi_2 \beta + \xi_3 \gamma).
\]

Obviously, the following equation system holds from equation (3.12)

\[
\begin{align*}
\lambda^2 (\rho_2 \rho_3 - \rho_1 \rho_2) \alpha &= \sigma \xi_1, \\
\sigma \xi_2 &= \sigma \xi_3 = 0.
\end{align*}
\]

(3.12)
By substituting equation (3.3) into (3.13) together with $\sigma \neq 0$, we obtain
\[
\begin{align*}
\dot{\lambda}^2 (\rho_2 \rho_3 - \rho_3 \rho_2) &= \sigma (1 - \lambda \rho_3), \\
\lambda' \rho_2 + \dot{\lambda} \rho_2 + \lambda k \rho_2 &= 0, \\
\lambda' \rho_3 + \dot{\lambda} \rho_3 - \lambda k \rho_3 &= 0.
\end{align*}
\] (3.14)

From the second and the third equation of (3.14), we can get
\[
\begin{align*}
\dot{\lambda} \rho_2 &= \dot{\varphi}_1(t) e^{-\int s \, ds}, \\
\dot{\lambda} \rho_3 &= \dot{\varphi}_2(t) e^{\int s \, ds},
\end{align*}
\] (3.15)

where $\varphi_1(t)$ and $\varphi_2(t)$ are differentiable functions of $t$. Taking partial derivative on both sides of (3.15) with respect to $t$ twice, we get
\[
\begin{align*}
\lambda \ddot{\rho}_2 &= \dot{\varphi}_2(t) e^{\int s \, ds}, \\
\lambda \ddot{\rho}_3 &= \dot{\varphi}_2(t) e^{\int s \, ds}.
\end{align*}
\] (3.16)

By substituting equations (3.15) and (3.16) into the first equation of (3.14), we get
\[
\varphi_1(t) \ddot{\rho}_2(t) - \ddot{\varphi}_1(t) \rho_2(t) = \sigma (1 - \varphi_2(t) e^{\int s \, ds}).
\]

Because $r(s, t)$ is a function of $s$ and $t$, then $\dot{\rho}_2$ and $\dot{\rho}_3$ can not be zero simultaneously.

In generality, $\lambda \rho_2 \rho_3 \neq 0$. Notice that $1 - \varphi_2(t) e^{\int s \, ds} \neq 0$ according to the first equation of (3.14) and $\sigma \neq 0$, then it is easy to know
\[
\sigma(s,t) = \frac{\dot{\varphi}_1(t) \dot{\varphi}_2(t) - \ddot{\varphi}_1(t) \rho_2(t)}{1 - \varphi_2(t) e^{\int s \, ds}}.
\]

In particular, when $\rho_2 \rho_3 = 0$, we consider the following two cases.

Case 1. $\rho_2 = 0, \rho_3 \neq 0$. From the first equation of (3.14) and $\sigma \neq 0$, we have $\lambda \rho_3 = 1$ which contradicts to $\rho_3 \neq 0$.

Case 2. $\rho_2 \neq 0, \rho_3 = 0$. Similar to Case 1, we find $\lambda \rho_3 = 1$. Continuously, we have $\kappa = 0$ by the third equation of (3.14). From Remark 2.2, we know $\kappa \neq 0$, it is a contradiction.

Through the above computation process, we can get the conclusions as follows.

**Theorem 3.5.** Let a surface $r(s, t)$ be 2-type generalized Hasimoto PNSG with $\rho_2 \rho_3 \neq 0$, then its scaling factor $\sigma(s, t)$ and the components $\rho_2(s, t), \rho_3(s, t)$ of its growth velocity vector satisfy
\[
\begin{align*}
\dot{\lambda} \rho_2 &= \dot{\varphi}_1(t) e^{-\int s \, ds}, \\
\dot{\lambda} \rho_3 &= \dot{\varphi}_2(t) e^{\int s \, ds},
\end{align*}
\]
where $\kappa$ is the curvature of generating curve $a(s)$, $\varphi_1(t), \varphi_2(t)$ are differentiable functions.

**Corollary 3.6.** Let a surface $r(s, t)$ be 2-type generalized Hasimoto PNSG with $\rho_2 \rho_3 \neq 0$, i.e.,
\[
r_1(s, t) \times r_2(s, t) = \sigma(s, t)s(s, t),
\]
then the function $\sigma(s, t)$ can be represented as
\[
\sigma(s,t) = \frac{\dot{\varphi}_1(t) \dot{\varphi}_2(t) - \ddot{\varphi}_1(t) \rho_2(t)}{1 - \varphi_2(t) e^{\int s \, ds}}.
\]
where $\varphi_1(t), \varphi_2(t)$ are differentiable functions, $\kappa$ is the curvature of generating curve $a(s)$.

In what follows, we characterize the PNSG by the aid of the structure function $g = g(s)$ of its generating curve $a(s)$. From Proposition 2.3 and equation (3.1), we have

**Theorem 3.7.** Let $r(s, t) = \{r_1(s, t), r_2(s, t), r_3(s, t)\}$ be a PNSG in $\mathbb{E}^3$. Then it is expressed as
\[
\begin{align*}
r_1(s, t) &= \frac{1}{2g} \left[ e^{c_1^1} + 1 \right] \left[ g ds + (1 - c^2) s + (c^2 + 1) g' \lambda \rho_2 + \frac{2 cc_1 + 2c^2 - 1}{g' \lambda \rho_3} \right], \\
r_2(s, t) &= \frac{1}{g} \left[ g ds + g' \lambda \rho_2 + \frac{2 c^2 - 1}{g' \lambda \rho_3} \right], \\
r_3(s, t) &= \frac{1}{2g} \left[ e^{c_1^1} + 1 \right] \left[ g ds + (1 + c^2) s + (c^2 - 1) g' \lambda \rho_2 + \frac{2 cc_1 + 2c^2 + 1}{g' \lambda \rho_3} \right],
\end{align*}
\]
where $g$ is the structure function of generating curve $a(s)$, $\lambda, \rho_2, \rho_3$ are the scaling factor and the components of its growth velocity vector, $0 \neq c, c_i, i = 1, 2, 3 \in \mathbb{R}$.

**Corollary 3.8.** Let a surface $r(s, t)$ be 1-type generalized Hasimoto PNSG with $\rho_2 \rho_3 \not\equiv 0$, then $\lambda \rho_2$ and $\lambda \rho_3$ in Theorem 3.7 are determined by Theorem 3.1.

From Proposition 2.3, we know $e^{\int s \, ds} = g'$, by Theorem 3.3 and Theorem 3.5, we obtain
Corollary 3.9. Let a surface \( r(s,t) \) be 1-type generalized Hasimoto PNGS with \( \rho_2 \neq 0, \rho_3 = 0 \), then \( \lambda \rho_2 = g' \) and \( \lambda \rho_2 \) in Theorem 3.7 satisfies
\[
\lambda \rho_2 = (1 - g')^2 + (1 - g')(\lambda \rho_2')' + \frac{g''}{g'} \lambda \rho_2' + \frac{g'''}{g''} (\lambda \rho_2') + \frac{g'''}{g''} \lambda \rho_2.
\]
where \( \sigma = \sigma(s,t) \) is a nonzero smooth function.

Corollary 3.10. Let a surface \( r(s,t) \) be 2-type generalized Hasimoto PNGS with \( \rho_2 \rho_3 \neq 0 \), then it is expressed as
\[
\begin{align*}
\rho_2(s,t) &= \frac{1}{2c} \left[ (c^2 + 1) \int gd s + (1 - c^2)s + (c^2 + 1) \varphi_2(t) + \frac{2cc_1 + 2(c^2 - 1)g - (c^2 + 1)g^2}{2} \varphi_2(t) \right], \\
\rho_3(s,t) &= \frac{1}{2c} \left[ (c^2 - 1) \int gd s + (1 + c^2)s + (1 - c^2) \varphi_2(t) + \frac{2cc_1 - 2(c^2 + 1)g - (c^2 - 1)g^2}{2} \varphi_2(t) \right],
\end{align*}
\]
where \( \varphi_1(t), \varphi_2(t) \) are differentiable functions and \( 0 \neq c, c_i, i = 1, 2, 3 \in \mathbb{R} \).

Example 3.11. Taking a pseudo null curve \( a(s) = - (2 \cos \frac{4t}{\sqrt{2}}, 2 \cos \frac{4t}{\sqrt{2}}, s) \) whose curvature
\[
\kappa(s) = - \frac{1}{\sqrt{2}} \tan \frac{s}{\sqrt{2}}.
\]
By Proposition 2.3, its structure function \( g(s) \) satisfies \( \frac{g''(s)}{g'(s)} = - \frac{1}{\sqrt{2}} \tan \frac{s}{\sqrt{2}} \), solving this differential equation, we obtain \( g(s) = \sqrt{2} \sin \frac{s}{\sqrt{2}} \). At the same time, we let \( c = 1, c_1 = -1, c_2 = c_3 = \frac{1}{2} \).

• If \( r(s,t) \) is a 1-type generalized Hasimoto PNGS generated by \( a(s) \) with \( \rho_2 \rho_3 \neq 0 \), by Theorem 3.1, we let
\[
\begin{align*}
\lambda \rho_2 &= \cos t \sec \frac{s}{\sqrt{2}}, \\
\lambda \rho_3 &= - t \cos \frac{s}{\sqrt{2}}.
\end{align*}
\]
From Corollary 3.8, the surface \( r(s,t) \) is written as (see Fig. 2)
\[
\begin{align*}
\rho_1(s,t) &= \cos t - 2 \cos \frac{s}{\sqrt{2}} - t \left( \cos^2 \frac{s}{\sqrt{2}} - \frac{3}{2} \right), \\
\rho_2(s,t) &= \cos t - 2 \cos \frac{s}{\sqrt{2}} - t \left( \cos^2 \frac{s}{\sqrt{2}} - \frac{1}{2} \right), \\
\rho_3(s,t) &= \left( s + \sqrt{2} \sin \frac{s}{\sqrt{2}} \right),
\end{align*}
\]
which satisfies
\[
r_{1}(s,t) \times r_{2}(s,t) = \frac{t \cos^2 \frac{s}{\sqrt{2}} + \cos \frac{s}{\sqrt{2}}}{\sin t} r_{3}(s,t).
\]
• If \( r(s,t) \) is a 1-type generalized Hasimoto PNGS generated by \( a(s) \) with \( \rho_2 \neq 0, \rho_3 = 0 \), by Corollary 3.9, we get \( \lambda \rho_3 = \cos \frac{t}{\sqrt{2}} \). Taking \( \lambda \rho_2 = s \cos t \), then \( r(s,t) \) is written as (see Fig. 3)
\[
\begin{align*}
\rho_1(s,t) &= (s \cos t - 2) \cos \frac{s}{\sqrt{2}} + \cos^2 \frac{s}{\sqrt{2}} - \frac{3}{2}, \\
\rho_2(s,t) &= (s \cos t - 2) \cos \frac{s}{\sqrt{2}} + \cos^2 \frac{s}{\sqrt{2}} - \frac{1}{2}, \\
\rho_3(s,t) &= \sqrt{2} \sin \frac{s}{\sqrt{2}} - 2s,
\end{align*}
\]
which satisfies
\[
r_{1}(s,t) \times r_{2}(s,t) = \left( \frac{2 \sqrt{2} \tan \frac{s}{\sqrt{2}} - \sqrt{2} \sin \frac{s}{\sqrt{2}} - 2 \sec \frac{s}{\sqrt{2}} + s}{2s} \right) \cot t - 8 \sin^3 \frac{s}{\sqrt{2}} \csc t \right) r_{3}(s,t).
\]
• If \( r(s,t) \) is a 2-type generalized Hasimoto PNGS generated by \( a(s) \) with \( \rho_2 \rho_3 \neq 0 \), we can let
\[
\varphi_1(t) = \frac{s}{t}, \quad \varphi_2(t) = st,
\]
then by Corollary 3.10, the surface \( r(s,t) \) can be written as (see Fig. 4)
\[
\begin{align*}
    r_1(s,t) &= \frac{s}{t} - 2\cos\frac{s}{\sqrt{2}} + \left(\cos^2\frac{s}{\sqrt{2}} - \frac{3}{8}\right)st, \\
    r_2(s,t) &= \frac{s}{t} - 2\cos\frac{s}{\sqrt{2}} + \left(\cos^2\frac{s}{\sqrt{2}} - \frac{1}{8}\right)st, \\
    r_3(s,t) &= \sqrt{2st}\sin\frac{s}{\sqrt{2}} - s,
\end{align*}
\]
which satisfies
\[
    r_1(s,t) \times r_2(s,t) = \left(\frac{2s^2}{t^3 \left(st \cos\frac{s}{\sqrt{2}} - 1\right)}\right) r_3(s,t).
\]

Fig. 2. The 1-type generalized Hasimoto PNG S with $\dot{\rho}_2 \neq 0$.

Fig. 3. The 1-type generalized Hasimoto PNG S with $\dot{\rho}_2 \neq 0$ and $\dot{\rho}_3 = 0$.

Fig. 4. The 2-type generalized Hasimoto PNG S with $\dot{\rho}_2 \neq 0$. 
4. The generalized Hasimoto pseudo null osculating growth surfaces

Assume that \( r(s, t) \) is a POGS, then it can be represented as

\[
 r(s, t) = a(s) + \lambda(s)[\rho_1(s, t)a + \rho_2(s, t)\beta].
\] (4.1)

First of all, by differentiating equation (4.1) on both sides with respect to \( s \), we have

\[
 r_s(s, t) = \xi_1a + \xi_2\beta,
\] (4.2)

where \( \xi_i = \xi_i(s, t), (i = 1, 2) \) are differentiable functions of \( s \) and \( t \) as follows

\[
 \begin{aligned}
 \xi_1 &= 1 + \lambda' \rho_1 + \lambda\rho_1', \\
 \xi_2 &= \lambda \rho_1 + \lambda' \rho_2 + \lambda\rho_2'.
\end{aligned}
\] (4.3)

Furthermore, by differentiating equation (4.2) on both sides with respect to \( s \), we have

\[
 r_{ss}(s, t) = \eta_1a + \eta_2\beta,
\]

where \( \eta_i = \eta_i(s, t), (i = 1, 2) \) are differentiable functions of \( s \) and \( t \) as follows

\[
 \begin{aligned}
 \eta_1 &= \lambda'' \rho_1 + 2\lambda' \rho_1' + \lambda\rho_1'', \\
 \eta_2 &= 1 + 2(\lambda\rho_1)' + (\lambda\rho_2)' + 2\lambda'\rho_2 + \lambda\rho_2'' + 2\lambda \rho_2' + \lambda\rho_1 + \lambda\rho_1'.
\end{aligned}
\] (4.4)

At the same time, by differentiating equation (4.1) with respect to \( t \) twice, we have

\[
 r_t(s, t) = \lambda(\dot{\rho}_1a + \dot{\rho}_2\beta)
\] (4.5)

and

\[
 r_{tt}(s, t) = \lambda(\ddot{\rho}_1a + \ddot{\rho}_2\beta).
\]

To serve the following discussions, from Proposition 2.1, we could obtain

\[
 r_s(s, t) \times r_{ss}(s, t) = (\xi_1\eta_2 - \xi_2\eta_1)\beta
\] (4.6)

and

\[
 r_t(s, t) \times r_{tt}(s, t) = \lambda^2(\dot{\rho}_1\rho_2 - \dot{\rho}_2\rho_1)\beta.
\] (4.7)

4.1. The 1-type generalized Hasimoto pseudo null osculating growth surfaces

Based on the definition of 1-type generalized Hasimoto surface, i.e.,

\[
 r_s(s, t) \times r_{ss}(s, t) = \sigma(s, t)r_t(s, t),
\]

where \( \sigma(s, t) \) is a nonzero smooth function. From equations (4.5) and (4.6), we get

\[
 (\xi_1\eta_2 - \xi_2\eta_1)\beta = \lambda\sigma(\dot{\rho}_1a + \ddot{\rho}_2\beta).
\] (4.8)

Obviously, the following equation system holds from equation (4.8)

\[
 \begin{aligned}
 \lambda\sigma\rho_1 &= 0, \\
 \lambda\sigma \rho_2 &= \xi_1\eta_2 - \xi_2\eta_1.
\end{aligned}
\] (4.9)

From the above equation system together with \( \lambda \) and \( \sigma \) are not zero everywhere, we obtain \( \rho_1 = 0 \), i.e., \( \rho_1 = \rho_1(s) \). Substituting equations (4.3) and (4.4) into (4.9), we have after arrangement

\[
 \lambda\sigma \rho_2 = [1 + (\lambda\rho_1)']^2 - (\lambda\rho_1)(\lambda\rho_1)' + [1 + (\lambda\rho_1)'][(\lambda\rho_1)' + \lambda\rho_2] +
\]

\[
 [1 + (\lambda\rho_1)'][(\lambda\rho_2)' + \lambda\rho_1] + [\lambda \rho_1' - (\lambda\rho_1)'][(\lambda\rho_2)' + \lambda\rho_2].
\]

Summarizing the above computation process, we have

**Theorem 4.1.** Let a surface \( r(s, t) \) be 1-type generalized Hasimoto POGS, then its scaling factor \( \lambda(s) \) and the components \( \rho_1(s, t), \rho_2(s, t) \) of its growth velocity vector satisfy

\[
 \begin{aligned}
 \rho_1 &= \rho_1(s), \\
 \lambda\sigma \rho_2 &= [1 + (\lambda\rho_1)]^2 - (\lambda\rho_1)(\lambda\rho_1)' + [1 + (\lambda\rho_1)'][(\lambda\rho_1)' + \lambda\rho_2] +
\]

\[
 [1 + (\lambda\rho_1)'][(\lambda\rho_2)' + \lambda\rho_1] + [\lambda \rho_1' - (\lambda\rho_1)'][(\lambda\rho_2)' + \lambda\rho_2],
\]

where \( \kappa \) is the curvature of generating curve \( a(s) \), \( \sigma = \sigma(s, t) \) is a nonzero smooth function.
Corollary 4.2. Let a surface \( r(s,t) \) be 1-type generalized Hasimoto POGS, i.e.,
\[
r_s(s,t) \times r_s(s,t) = \sigma(s,t)p(s,t),
\]
then the function \( \sigma(s,t) \) can be expressed as
\[
\sigma(s,t) = \frac{\xi \eta_2 - \xi_2 \eta}{\lambda \rho_2},
\]
where \( \xi, \eta, (i = 1, 2) \) are differentiable functions as stated in equation (4.3) and (4.4).

4.2. The 2-type generalized Hasimoto pseudo null osculating growth surfaces

Based on the definition of 2-type generalized Hasimoto surface, i.e.,
\[
r_s(s,t) \times r_s(s,t) = \sigma(s,t)p(s,t),
\]
where \( \sigma(s,t) \) is a nonzero smooth function. From equations (4.2) and (4.7), we have
\[
\lambda^2(\rho_1 \rho_2 - \rho_2 \rho_1) \beta = \sigma(\xi \eta + \xi_2 \beta).
\]
(4.10)

Obviously, the following equation system holds from equation (4.10)
\[
\begin{align*}
\lambda^2(\rho_1 \rho_2 - \rho_2 \rho_1) &= \sigma \xi_2, \\
\sigma \xi_1 &= 0.
\end{align*}
\]
(4.11)

Substituting equation (4.3) into (4.11) together with \( \sigma \neq 0 \), we have
\[
\begin{align*}
\lambda^2(\rho_1 \rho_2 - \rho_2 \rho_1) &= \sigma(\lambda \rho_1 + \lambda' \rho_2 + \lambda \rho_2'), \\
1 + \lambda' \rho_1 + \lambda \rho_2 &= 0.
\end{align*}
\]
(4.12)

We have \( \lambda \rho_1 = \psi(t) + s \) by the second equation of (4.12), where \( \psi = \psi(t) \) is a differentiable function of \( t \). Continuously, substituting \( \lambda \rho_1 \) into the first equation of (4.12), we obtain
\[
\lambda(\psi \rho_2 - \rho_2 \psi) = \sigma((\lambda \rho_2)' + \lambda \rho_2 + \psi - s).
\]
(4.13)

Because \( r(s,t) \) is a function of \( s \) and \( t \), then \( \rho_1 \) and \( \rho_2 \) cannot be zero simultaneously. In generality, \( \rho_1, \rho_2 \neq 0 \). From equation (4.13), we have the following two cases.

Case 1: \( (\lambda \rho_2)' + \lambda \rho_2 + \psi - s \neq 0 \), then the function \( \sigma \) is obtained as by equation (4.13)
\[
\sigma = \frac{\lambda(\psi \rho_2 - \rho_2 \psi)}{(\lambda \rho_2)' + \lambda \rho_2 + \psi - s}.
\]

Case 2: \( (\lambda \rho_2)' + \lambda \rho_2 + \psi - s = 0 \), then \( \rho_2 = f(s) \rho_1 \) by the first equation of (4.12), therefore, \( \rho_2 = f(s) \rho_1 + h(s) \), where \( f(s) \neq 0, h(s) \) are differentiable functions. Taking partial derivative on both sides of \( (\lambda \rho_2)' + \lambda \rho_2 + \psi - s = 0 \) with respect to \( t \), we have
\[
\lambda \rho_1 (1 + \kappa f + f') = 0,
\]
therefore \( 1 + \kappa f + f' = 0 \) since \( \rho_1 \rho_2 \neq 0 \). Continuously, we have
\[
(\lambda \rho_2)' + \kappa(\lambda \rho_2) - f = 0.
\]
Solving this differential equation, we get
\[
f(s) = \frac{\int e^f \chi ds}{e^f \chi ds} \text{ and } \lambda h(s) = - \frac{\int \left( \int e^f \chi ds \right) ds}{e^f \chi ds},
\]
thus, we obtain
\[
\lambda \rho_2 = \frac{(\psi - s) \int e^f \chi ds + \int \left( \int e^f \chi ds \right) ds}{e^f \chi ds}.
\]
Moreover, from Proposition 2.3, we have \( e^f \chi ds = g' \), then \( \lambda \rho_2 \) can be expressed as
\[
\lambda \rho_2 = \frac{(\psi - s) g + \int g ds}{g'}.
\]
Substituting \( \lambda \rho_1 = \psi(t) - s \) and \( \lambda \rho_2 \) into the equation (4.1), from Proposition 2.3, we find
\[
r(s,t) = \left( \frac{1 - c^2}{2c}, 0, \frac{(1 + c^2) \psi}{2c} \right),
\]
where \( \psi = \psi(t) \) is a differentiable function. Obviously, the case is impossible as described above.

In particular, \( \rho_1, \rho_2 = 0 \), we have the following two cases.

Case 1. \( \rho_1 \neq 0, \rho_2 = 0 \). From the first equation of (4.12), we have \( (\lambda \rho_2)' + \lambda \rho_2 + \lambda \rho_1 = 0 \). Taking the partial derivative with respect to \( t \), \( \rho_1 = 0 \) is obtained. It is a contradiction.
Case 2. \( \rho_1 = 0, \rho_2 \neq 0 \). We know \( \lambda \rho_1 = -s \) by the second equation of \((4.12)\). Continuously, we have \((\lambda \rho_2)' + \lambda \kappa \rho_2 - s = 0 \) from \( \sigma \neq 0 \), then it is easy to get
\[
\lambda \rho_2 = \frac{\int s e^t \, ds + \phi(t)}{e^t \, ds},
\]
where \( \phi(t) \) is a differentiable function of \( t \). According to \( e^t \, ds = g' \), \( \lambda \rho_2 \) can be expressed as
\[
\lambda \rho_2 = \frac{sg - \int g \, ds + \phi(t)}{g'}.
\]
Substituting \( \lambda \rho_1 = -s \) and \( \lambda \rho_2 \) into the equation \((4.1)\), from Proposition 2.3, we find
\[
r(s, t) = \left(\frac{\phi(t)}{2c}, \frac{(c^2 - 1)\phi(t)}{2c}, \frac{(c^2 + 1)\phi(t)}{2c}\right),
\]
where \( \phi(t) \) is a nonzero differentiable function of \( t \). Similarly, the case is also impossible.

**Theorem 4.3.** Let a surface \( r(s, t) \) be 2-type generalized Hasimoto POGS with \( \rho_1 \rho_2 \neq 0 \) and \( \rho_2 \neq f(s)\rho_1 \), then its scaling factor \( \lambda(s) \) and the components \( \rho_1(s, t), \rho_2(s, t) \) of its growth velocity vector satisfy
\[
\begin{align*}
\dot{\lambda} \rho_1 &= \psi(t) - s, \\
\dot{\lambda} \rho_2 &= \sigma[(\lambda \rho_2)' + \lambda \kappa \rho_2 + \psi - s].
\end{align*}
\]
where \( \sigma \) is a nonzero smooth function of \( s \) and \( t \), \( \psi(t) \) is a nonzero differentiable function and \( \kappa \) is the curvature of generating curve \( a(s) \).

**Corollary 4.4.** Let a surface \( r(s, t) \) be 2-type generalized Hasimoto POGS with \( \rho_1 \rho_2 \neq 0 \) and \( \rho_2 \neq f(s)\rho_1 \) which satisfies
\[
r_1(s, t) \times r_2(s, t) = \sigma(s, t)r_3(s, t),
\]
then the function \( \sigma(s, t) \) can be expressed as
\[
\sigma(s, t) = \frac{\lambda(\rho_2' - \rho_2\psi)}{(\lambda \rho_2)' + \lambda \kappa \rho_2 + \psi - s},
\]
where \( \psi = \psi(t), f(s) \) are nonzero differentiable functions, \( \kappa \) is the curvature of generating curve.

Based on Proposition 2.3 and equation \((4.1)\), we have

**Theorem 4.5.** Let \( r(s, t) = (r_1(s, t), r_2(s, t), r_3(s, t)) \) be a POGS in \( E^3 \). Then it is expressed as
\[
\begin{align*}
r_1(s, t) &= \frac{1}{2c} \left[ (c^2 + 1) \int g \, ds + (1 - c^2)s + (c^2 g + g - c^2 + 1)\lambda \rho_1 + (c^2 - 1)g' \lambda \rho_2 \right], \\
r_2(s, t) &= \int g \, ds + \lambda \rho_1 + g' \lambda \rho_2, \\
r_3(s, t) &= \frac{1}{2c} \left[ (c^2 - 1) \int g \, ds - (1 + c^2)s + (c^2 g - g - c^2 - 1)\lambda \rho_1 + (c^2 - 1)g' \lambda \rho_2 \right],
\end{align*}
\]
where \( g \) is the structure function of generating curve \( a(s) \), \( \lambda, \rho_1, \rho_2 \) are the scaling factor and the components of its growth velocity vector, \( 0 \neq c \in \mathbb{R} \).

**Corollary 4.6.** Let a surface \( r(s, t) \) be 1-type generalized Hasimoto POGS, then \( \rho_1 = \rho_1(s) \) and \( \lambda \rho_2 \) in Theorem 4.5 are determined by Theorem 4.1.

**Corollary 4.7.** Let a surface \( r(s, t) \) be 2-type generalized Hasimoto POGS with \( \rho_1 \rho_2 \neq 0 \) and \( \rho_2 \neq f(s)\rho_1 \), then \( \lambda \rho_1 = \psi(t) - s \) and \( \lambda \rho_2 \) in Theorem 4.5 are determined by Theorem 4.3.

**Example 4.8.** Consider a POGS \( r(s, t) \) with the generating pseudo null curve as stated in Example 3.11.

- If \( r(s, t) \) is a 1-type generalized Hasimoto POGS, by Theorem 4.1, we can let \( \lambda \rho_1 = s, \lambda \rho_2 = st \). From Corollary 4.6, the surface can be written as (see Fig. 5)
\[
r(s, t) = \left( (st - 2) \cos \frac{s}{\sqrt{2}} + \sqrt{2} s \sin \frac{s}{\sqrt{2}}, (st - 2) \cos \frac{s}{\sqrt{2}} + \sqrt{2} s \sin \frac{s}{\sqrt{2}}, -2s \right),
\]
which satisfies
\[
r_1(s, t) \times r_2(s, t) = \frac{6 - st - \sqrt{2}(s + 2t)\tan \frac{\sqrt{2}}{2}}{s} r_3(s, t).
\]
- If \( r(s, t) \) is a 2-type generalized Hasimoto POGS with \( \rho_1 \rho_2 \neq 0 \) and \( \rho_2 \neq f(s)\rho_1 \), where \( f(s) \) is a nonzero smooth function, then by Theorem 4.3, we can let \( \psi(t) = \sin t \) and \( \lambda \rho_2 = \cos t \sec \frac{\sqrt{2}}{2} \). From Corollary 4.7, the surface \( r(s, t) \) can be written as (see Fig. 6)
which satisfies
\[ r(s, t) \times r_t(s, t) = \sec \frac{\sqrt{2}}{s - \sin t} r(s, t). \]

5. The generalized Hasimoto pseudo null rectifying growth surfaces

Assume that \( r(s, t) \) is a PRGS, then it can be represented as
\[
    r(s, t) = a(s) + \lambda(s) [p_1(s, t) \alpha + p_3(s, t) \gamma].
\]  
(5.1)

First of all, by differentiating equation (5.1) on both sides with respect to \( s \), we have
\[
    r_s(s, t) = \xi_1 \alpha + \xi_2 \beta + \xi_3 \gamma,
\]  
(5.2)

where the differentiable functions \( \xi_i = \xi_i(s, t), (i = 1, 2, 3) \) are as follows
\[
    \begin{cases}
    \xi_1 = 1 + \lambda' p_1 + \lambda \rho_3' - \lambda \rho_3, \\
    \xi_2 = \lambda \rho_1, \\
    \xi_3 = \lambda' p_3 + \lambda \rho_3' - \lambda \rho_3.
    \end{cases}
\]  
(5.3)

Furthermore, differentiating equation (5.2) with respect to \( s \), we have
\[
    r_{ss}(s, t) = \eta_1 \alpha + \eta_2 \beta + \eta_3 \gamma,
\]
where the differentiable functions \( \eta_i = \eta_i(s, t), (i = 1, 2, 3) \) are as follows

\[
\begin{align*}
\eta_1 &= \lambda'' \rho_1 + 2 \lambda' \rho'_1 + \lambda \rho''_1 - 2 \lambda \rho_2 - 2 \lambda \rho_3 + \lambda \kappa \rho_3, \\
\eta_2 &= 1 + 2 \lambda' \rho_1 + 2 \lambda \rho'_1 + \lambda \kappa \rho_1 - \lambda \rho_1, \\
\eta_3 &= \lambda'' \rho_3 + 2 \lambda' \rho'_3 + \lambda \rho''_3 - 2 \lambda \kappa \rho_3 - \lambda \kappa^2 \rho_3.
\end{align*}
\]

(5.4)

At the same time, by differentiating equation (5.1) with respect to \( t \) twice, we have

\[
r_t(s, t) = \lambda (\rho_1 a + \rho_1 \gamma)
\]

(5.5)

and

\[
r_a(s, t) = \lambda (\rho_1 a + \rho_1 \gamma).
\]

To serve the following discussions, from Proposition 2.1, we could obtain

\[
r(s, t) \times r_a(s, t) = (\xi_1 \eta_3 - \xi_2 \eta_2)a + (\xi_1 \eta_2 - \xi_2 \eta_1)\beta + (\xi_3 \eta_3 - \xi_1 \eta_3)\gamma
\]

(5.6)

and

\[
r_t(s, t) \times r_a(s, t) = \lambda^2 (\rho_1 \rho_1 - \rho_3 \rho_3)\gamma.
\]

(5.7)

### 5.1. The 1-type generalized Hasimoto pseudo null rectifying growth surfaces

Based on the definition of 1-type generalized Hasimoto surface, i.e.,

\[
r(s, t) \times r_a(s, t) = \sigma(s, t)r(s, t),
\]

where \( \sigma(s, t) \) is a nonzero smooth function. From equations (5.5) and (5.6), we get

\[
(\xi_1 \eta_3 - \xi_2 \eta_2)a + (\xi_1 \eta_2 - \xi_2 \eta_1)\beta + (\xi_3 \eta_3 - \xi_1 \eta_3)\gamma = \lambda \sigma (\rho_1 a + \rho_1 \gamma).
\]

(5.8)

Obviously, the following equation system holds from equation (5.8)

\[
\begin{align*}
\xi_1 \rho_1 + \xi_2 \rho_3 &= 0, \\
\xi_1 \eta_1 + \xi_2 \eta_3 &= 0, \\
\xi_3 \eta_1 + \xi_1 \eta_3 &= \lambda \kappa \rho_1.
\end{align*}
\]

(5.9)

Substituting equations (5.3) and (5.4) into (5.9), we obtain

\[
\begin{align*}
\left\{ \begin{array}{l}
(1 + \lambda' \rho_1 + \lambda \rho'_1 - \lambda \rho_3)\rho_1 + \lambda \rho_1 \rho_3 = 0, \\
(\lambda'' \rho_1 + 2 \lambda' \rho'_1 + \lambda \rho''_1 - 2 \lambda \rho_3 - 2 \lambda \rho_3 + \lambda \kappa \rho_3)\rho_1 + [1 + 2(\lambda \rho_1)']\gamma + \lambda \kappa \rho_1 - \lambda \rho_3 = 0.
\end{array} \right.
\end{align*}
\]

(5.10)

Because \( r(s, t) \) is a function of \( s \) and \( t \), then \( \rho_1 \) and \( \rho_3 \) can not be zero simultaneously. In generality, \( \rho_1 \rho_3 \neq 0 \). From equation system (5.10), we have

\[
1 + 2(\lambda \rho_1)\gamma - \lambda \rho_3 = \frac{\lambda'' \rho_1 + 2 \lambda' \rho'_1 + \lambda \rho''_1 - 2(\lambda' \rho_3 + \lambda \rho'_3 + \lambda \kappa \rho_3)}{1 + \lambda' \rho_1 + \lambda \rho'_1 - \lambda \rho_3}.
\]

i.e.,

\[
1 + 2(\lambda \rho_1)\gamma - \lambda \rho_3 = \frac{\lambda(\rho_1)'' - 2(\lambda \rho_1)' + \lambda \kappa \rho_3}{1 + (\lambda \rho_1)' + \lambda \rho'_3 - \lambda \rho_3}.
\]

In particular, when \( \lambda \rho_1 \rho_3 = 0 \), we have the following two cases.

Case 1. \( \rho_1 = 0 \), \( \rho_3 \neq 0 \). From equation system (5.10), we get

\[
\begin{align*}
\lambda \rho_1 &= 0, \\
1 + 2(\lambda \rho_1)' + \lambda \kappa \rho_1 - \lambda \rho_3 &= 0.
\end{align*}
\]

From the above equation system, we know \( \rho_1 = 0 \) and \( \lambda \rho_3 = 1 \), which contradicts to \( \rho_3 \neq 0 \).

Case 2. \( \rho_1 \neq 0 \), \( \rho_3 = 0 \). From equation system (5.10), we have

\[
\begin{align*}
1 + \lambda' \rho_1 + \lambda \rho'_1 - \lambda \rho_3 &= 0, \\
\lambda'' \rho_1 + 2 \lambda' \rho'_1 + \lambda \rho''_1 - 2(\lambda \rho_3 + \lambda \rho'_3 + \lambda \kappa \rho_3) &= 0.
\end{align*}
\]

(5.11)
Taking the partial derivative of the first equation of (5.11) with respect to $s$, it is easy to know $(\dot{\lambda}\rho_1)' = (\dot{\lambda}\rho_3)'$. Substituting it into the second equation of (5.11), we have $(\dot{\lambda}\rho_2)' = \lambda\rho_1$, then $\lambda\rho_2 = e^{\int \lambda ds}$. Continuously, $\lambda\rho_1 = \int e^{\int \lambda ds} ds = s + o(t)$ by the first equation of (5.11), where $o(t)$ is a differentiable function of $t$.

Summarizing the above deduction, we have the following conclusions.

**Theorem 5.1.** Let a surface $r(s,t)$ be 1-type generalized Hasimoto PRGS with $\rho_1, \rho_3 \neq 0$, then its scaling factor $\lambda(s)$ and the components $\rho_1(s,t), \rho_3(s,t)$ of its growth velocity vector satisfy

\[
\frac{1 + 2(\dot{\lambda}\rho_2)' - \dot{\lambda}\rho_2}{\dot{\lambda}\rho_1} + \kappa = \frac{(\dot{\lambda}\rho_1)' - 2(\dot{\lambda}\rho_2)' + \lambda\rho_2}{1 + (\dot{\lambda}\rho_1)' - \dot{\lambda}\rho_3},
\]

where $\kappa$ is the curvature of generating curve $a(s)$.

**Corollary 5.2.** Let a surface $r(s,t)$ be 1-type generalized Hasimoto PRGS with $\rho_1, \rho_3 \neq 0$, i.e.,

\[
\dot{r}(s,t) \times r_{ss}(s,t) = \sigma(s,t) r_a(s,t),
\]

then the function $\sigma(s,t)$ can be expressed as

\[
\sigma(s,t) = \frac{\xi_3 n_3 - \xi_1 n_1}{\dot{\lambda}\rho_1} = \frac{\xi_2 n_2 - \xi_1 n_1}{\dot{\lambda}\rho_2},
\]

where $\xi_i, n_i, (i = 1, 2, 3)$ are differentiable functions as stated in (5.3) and (5.4).

**Theorem 5.3.** Let a surface $r(s,t)$ be 1-type generalized Hasimoto PRGS with $\rho_1 \neq 0, \rho_3 = 0$, then its scaling factor $\lambda(s)$, the components $\rho_1(s,t), \rho_3(s,t)$ of its growth velocity vector satisfy

\[
\begin{align*}
\dot{\lambda}\rho_1 &= \int e^{\int \lambda ds} ds - s + o(t), \\
\dot{\lambda}\rho_3 &= e^{\int \lambda ds},
\end{align*}
\]

where $o(t)$ is a nonzero smooth function of $t$, $\kappa$ is the curvature of generating curve $a(s)$.

**5.2. The 2-type generalized Hasimoto pseudo null rectifying growth surfaces**

Based on the definition of 2-type generalized Hasimoto surface, i.e.,

\[
r_a(s,t) \times r_a(s,t) = \sigma(s,t) r_a(s,t),
\]

where $\sigma(s,t)$ is a nonzero smooth function. From equation (5.2) and (5.7), we have

\[
\lambda^2 (\rho_3 \rho_1 - \rho_1 \rho_3)' = \sigma(\xi, \beta + \xi_2\gamma).
\]  

(5.12)

Obviously, the following equation system holds from equation (5.12)

\[
\begin{align*}
\lambda^2 (\rho_3 \rho_1 - \rho_1 \rho_3) &= \sigma(\xi, \beta + \xi_2\gamma), \\
\sigma(\xi) &= \sigma(\xi_2) = 0.
\end{align*}
\]

(5.13)

By substituting equation (5.3) into (5.13) together with $\sigma \neq 0$ everywhere, we have

\[
\begin{align*}
\lambda^2 (\rho_3 \rho_1 - \rho_1 \rho_3) &= \sigma(\xi, \beta + \xi_2\gamma), \\
1 + \lambda' \rho_1 + \lambda \rho_1' - \lambda \rho_3 &= 0, \\
\dot{\lambda}\rho_1 &= 0.
\end{align*}
\]  

(5.14)

From the second and the third equation of (5.14), we get $\rho_1 = 0$ and $\lambda\rho_3 = 1$, it is impossible.

**Theorem 5.4.** There does not exist 2-type generalized Hasimoto PRGS in $\mathbb{E}^3_1$.

Based on Proposition 2.3 and equation (5.1), we obtain

**Theorem 5.5.** Let $r(s,t) = (r_1(s,t), r_2(s,t), r_3(s,t))$ be a PRGS in $\mathbb{E}^3_1$. Then it is expressed as

\[
\begin{align*}
r_1(s,t) &= \frac{1}{2c} \left[ c^2 + 1 \right] g ds + \left( -c^2 \right) s + \left( -c^2 g + g^2 - c^2 + 1 \right) \lambda \rho_1 + \frac{2c^2 + 2(c^2 - 1)g - (c^2 - 1)g^2}{2g'} \dot{\lambda}\rho_1, \\
r_2(s,t) &= \int g ds + 4 \dot{\lambda}\rho_1 + \frac{2c^2 - g^2}{2g'} \lambda\rho_2, \\
r_3(s,t) &= \frac{1}{2c} \left[ c^2 - 1 \right] g ds - \left( -c^2 \right) s + \left( -c^2 g - g^2 + c^2 - 1 \right) \lambda\rho_1 + \frac{2c^2 + 2(c^2 - 1)g - (c^2 - 1)g^2}{2g'} \dot{\lambda}\rho_3.
\end{align*}
\]
where $g$ is the structure function of generating curve $a(s)$, $\lambda$, $\rho_1$, $\rho_3$ are the scaling factor and the components of its growth velocity vector, $0 \neq c, c_i, (i=1,2,3) \in \mathbb{R}$.

Corollary 5.6. Let a surface $r(s,t)$ be 1-type generalized Hasimoto PRGS with $\rho_1, \rho_3 \neq 0$, then $\lambda \rho_1$ and $\lambda \rho_3$ in Theorem 5.5 are determined by Theorem 5.1.

Corollary 5.7. Let a surface $r(s,t)$ be 1-type generalized Hasimoto PRGS with $\rho_1 \neq 0$, $\rho_3 = 0$, then $\lambda \rho_1 = g - s + \omega(t)$ and $\lambda \rho_3 = g'$ in Theorem 5.5. Where $g$ is the structure function of the generating curve $a(s)$, $\omega(t)$ is a nonzero smooth function of $t$.

Example 5.8. Taking a pseudo null curve $a(s) = (e^s, e^t, -s)$ whose curvature $\kappa(s) = 1$. By Proposition 2.3, the structure function $g(s) = e^t$. If $r(s,t)$ is a 1-type generalized Hasimoto PRGS with $\rho_1, \rho_3 \neq 0$, by Theorem 5.1, we can let $\lambda \rho_1 = te^s$. Thus

$$
\lambda \rho_1 = \frac{\sqrt{e^2 \sqrt{2}}}{\sqrt{e^2 \sqrt{2} + \sqrt{e^2 + \sqrt{2}}}} \cdot ( \sqrt{e^2 - 2 \sqrt{2}} + \sin t \sqrt{e^2 + \sqrt{2}} + \sqrt{2} )
$$

From Corollary 5.6, the surface $r(s,t)$ can be written as (see Fig. 7)

$$
\begin{align*}
\rho_1(s,t) &= e^t + \frac{e^t (1 + e^{2t})}{2} \left[ \sqrt{e^2 \sqrt{2}} \left( \sqrt{e^2 - 2 \sqrt{2}} + \sin t \left( \sqrt{e^2 + \sqrt{2}} \right) \right) \right] \\
\rho_2(s,t) &= e^t (1 + e^{2t}) \left[ \sqrt{e^2 \sqrt{2}} \left( \sqrt{e^2 - 2 \sqrt{2}} + \sin t \left( \sqrt{e^2 + \sqrt{2}} \right) \right) \right] \\
\rho_3(s,t) &= e^t (1 + 4e^t) \left[ \sqrt{e^2 \sqrt{2}} \left( \sqrt{e^2 - 2 \sqrt{2}} + \sin t \left( \sqrt{e^2 + \sqrt{2}} \right) \right) \right] - s - e^t.
\end{align*}
$$

Fig. 7. The 1-type generalized Hasimoto PRGS with $\rho_1, \rho_3 \neq 0$.

Example 5.9. Consider a PRGS $r(s,t)$ with the generating pseudo null curve as stated in Example 3.11. If $r(s,t)$ is a 1-type generalized Hasimoto PRGS with $\rho_1 \neq 0$, $\rho_3 = 0$, from Corollary 5.7, we have $\lambda \rho_3 = \cos \frac{s}{\sqrt{2}}$. Assume that $\omega(t) = t$, then the surface $r(s,t)$ can be written as (see Fig. 8)

$$
\begin{align*}
\rho_1(s,t) &= \cos^2 \frac{s}{\sqrt{2}} - 2 \cos \frac{s}{\sqrt{2}} + 2 \sin^2 \frac{s}{\sqrt{2}} + \sqrt{2}(t - s) \sin \frac{s}{\sqrt{2}} = \frac{3}{2} \\
\rho_2(s,t) &= \cos^2 \frac{s}{\sqrt{2}} - 2 \cos \frac{s}{\sqrt{2}} + 2 \sin^2 \frac{s}{\sqrt{2}} + \sqrt{2}(t - s) \sin \frac{s}{\sqrt{2}} = \frac{1}{2} \\
\rho_3(s,t) &= -t
\end{align*}
$$

which satisfies $\rho_3(s,t) \times \rho_3(s,t) = \sigma(s,t) \rho_3(s,t)$, where

$$
\sigma(s,t) = \frac{1}{2} \left[ \sec^2 \frac{s}{\sqrt{2}} \left( t - s + \sqrt{2} \sin \frac{s}{\sqrt{2}} + \sqrt{2}(t - s) \sin \frac{s}{\sqrt{2}} \right) \right] - \cos \frac{1}{\sqrt{2}} (t - s + \sqrt{2} \sin \frac{s}{\sqrt{2}}).
$$
Fig. 8. The 1-type generalized Hasimoto PRGS with $\dot{\rho}_1 \neq 0$, $\dot{\rho}_3 = 0$.

Declarations

Author contribution statement

Jinhua Qian: Conceived and designed the experiments; performed the experiments; analyzed and interpreted the data; wrote the paper. Yao Guo: analyzed and interpreted the data; contributed reagents, materials, analysis tools or data; wrote the paper. Young Ho Kim: Contributed reagents, materials, analysis tools or data; wrote the paper.

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Additional information

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