Discrete Quantum Gravity: I. Zonal spherical functions of the representations of the SO(4,R) group with respect to the SU(2) subgroup and their application to the Euclidean invariant weight for the Barrett-Crane model.

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Abstract. Starting from the defining transformations of complex matrices for the SO(4,R) group, we construct the fundamental representation and the tensor and spinor representations of the group SO(4,R). Given the commutation relations for the corresponding algebra, the unitary representations of the group in terms of the generalized Euler angles are constructed. The crucial step for the Barrett-Crane model in Quantum Gravity is the description of the amplitude for the quantum 4-simplex that is used in the state sum partition function. We obtain the zonal spherical functions for the construction of the SO(4,R) invariant weight and associate them to the triangular faces of the 4-simplices.

Keywords: SO(4,R) group, tensor representation, spin representation, quantum gravity, spin networks.

1. Discrete models in quantum gravity

The use of discrete models in Physics has become very popular, mainly for two reasons. It helps to find the solutions of some differential equations by numerical methods, which would not be possible to solve by analytic methods. Besides that, the introduction of a lattice is equivalent to the introduction of a cut-off in the momentum variable for the field in order to achieve the finite limit of the solution. In the case of relativistic field equations -like the Dirac, Klein-Gordon, and the electromagnetic interactions- we have worked out some particular cases [1].

There is another motivation for the discrete models and it is based in some philosophical presuppositions that the space-time structure is discrete. This is more attractive in the case of general relativity and quantum gravity because it makes more transparent the connection between the discrete properties of the intrinsic curvature and the background independent gravitational field.
This last approach was started rigorously by Regge in the early sixties [2]. He introduces some triangulation in a Riemannian manifold, out of which he constructs local curvature, coordinate independent, on the polyhedra. With the help of the total curvature on the vertices of the discrete manifold he constructs a finite action which, in the continuous limit, becomes the standard Hilbert-Einstein action of general relativity.

Regge himself applied his method ("Regge calculus") to quantum gravity in three dimensions [3]. In this work he assigns some representation of the $SU(2)$ group to the edges of the triangles. To be more precise, to every tetrahedron appearing in the discrete triangulation of the manifold he associates a 6j-symbol in such a way that the spin eigenvalues of the corresponding representation satisfy sum rules described by the edges and vertices of the tetrahedra. Since the value of the 6j-symbol has a continuous limit when some edges of the tetrahedra become very large, he could calculate the sum of this limit for all the 6j-symbols attached to the tetrahedra, and in this way he could compare it with the continuous Hilbert-Einstein action corresponding to an Euclidean non planar manifold.

A different approach to the discretization of space and time was taken by Penrose [4]. Given some graph representing the interaction of elementary units satisfying the rules of angular momentum without an underlying space, he constructs out of this network ("spin network") the properties of total angular momentum as a derived concept. Later this model was applied to quantum gravity in the sense of Ponzano and Regge. In general, a spin network is a triple $(\gamma, \rho, i)$ where $\gamma$ is a graph with a finite set of edges $e$, a finite set of vertices $v$, $\rho_e$ is the representation of a group $G$ attached to an edge, and $i_v$ is an intertwiner attached to each vertex. If we take the product of the amplitudes corresponding to all the edges and vertices (given in terms of the representations and intertwiners) we obtain the particular diagram of some quantum state.

Although the physical consequences of Penrose’s ideas were soon considered to be equivalent to the Ponzano-Regge approach to quantum gravity [5], the last method was taken as guiding rule in the calculation of partition functions. We can mention a few results. Turaev and Viro [6] calculated the state sum for a 3d-triangulated manifold with tetrahedra described by 6j-symbols using the $SU(2)_q$ group. This model was enlarged to 4-dimensional triangulations and was proved by Turaev, Oguri, Crane and Yetter [7] to be independent of the triangulation (the “TOCY model”).

A different approach was introduced by Boulatov [8] that led to the same partition function as the TOCY model, but with the advantage that the terms corresponding to the kinematics and the interaction
could be distinguished. For this purpose he introduced some fields defined over the elements of the groups $SO(3)$, invariant under the action of the group, and attached to the edges of the tetrahedra. The kinematical term corresponds to the self interacting field over each edge and the interaction term corresponds to the fields defined in different edges and coupled among themselves. This method (the Boulatov matrix model) was very soon enlarged to 4-dimensional triangulations by Ooguri [9]. In both models the fields over the matrix elements of the group are expanded in terms of the representations of the group and then integrated out, with the result of a partition function extended to the amplitudes over all tetrahedra, all edges and vertices of the triangulation.

A more abstract approach was taken by Barrett and Crane, generalizing Penrose’s spin networks to 4 dimensions. The novelty of this model consists in the association of representation of the $SO(4, R)$ group to the faces of the tetrahedra. We will come back to this model in section 5.

Because we are interested in the physical and mathematical properties of the Barrett-Crane model, we mention briefly some recent work about this model combined with the matrix model approach of Boulatov and Ooguri [10]. In this work the 2D quantum space-time emerges as a Feynman graph, in the manner of the 4d– matrix models. In this way a spin foam model is connected to the Feynman diagram of quantum gravity.

In these papers part I and II we try to implement the mathematical consequences of the Barrett-Crane model in both the Euclidean and the Lorentz case, We examine the group theory in relation to the triangulation of 4-dimensional manifolds in terms of 4-simplices.

In section 2 and 3 we develop the representation theory for the group $SO(4, R)$ and the algebra $so(4)$, out of which the Biedenharn-Dolginov function is constructed for the boost transformation. In section 5 we review the Barret-Crane model. We define the spherical harmonics on a coset space $SO(4, R)/SU(2)^c$, equivalent to the sphere $S^5$. The intertwiner of two spherical harmonics yields a zonal spherical function. In section 6 we introduce the triple product in $R^4$ that generalizes the vector product and can be useful for the model. In section 7 we apply our results to the evaluation and interpretation of the state sum for the spin network, which in the continuous limit tends to the Hilbert-Einstein action. Using the correspondence between bivectors and generators of $SO(4, R)$ we find a relation between the area of the triangular faces of the tetrahedra and the spin of the representation.
2. The groups $SO(4, R)$ and $SU(2) \times SU(2)$

The rotation group in 4 dimensions is the group of linear transformations that leaves the quadratic form $x_1^2 + x_2^2 + x_3^2 + x_4^2$ invariant. The well known fact that this group is locally isomorphic to $SU(2) \times SU(2)$ enables one to decompose the group action in the following way:

Take a complex matrix (not necessarily unimodular)

$$w = \begin{pmatrix} y & z \\ -\bar{z} & \bar{y} \end{pmatrix}, \quad y = x_1 + ix_2, -\bar{z} = x_3 + ix_4,$$

where $w$ satisfies $ww^+ = \det(w)$.

We define the full group action

$$(u_1, u_2) : w \rightarrow w' = (u_1)^{-1}wu_2,$$

where the inverse $(u_1)^{-1}$ is introduced in order to assure a homomorphic action. Here $(u_1, u_2) \in SU(2)^L \times SU(2)^R$ generate the left and right action, respectively,

$$u_1 = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \in SU(2)^L, \quad \alpha \bar{\alpha} + \beta \bar{\beta} = 1,$$

$$u_2 = \begin{pmatrix} \gamma & \delta \\ -\bar{\delta} & \bar{\gamma} \end{pmatrix} \in SU(2)^R, \quad \gamma \bar{\gamma} + \delta \bar{\delta} = 1.$$

The full group action satisfies:

$$w' w'^+ = \det(w') = w w^+ = \det(w),$$

or $x_1'^2 + x_2'^2 + x_3'^2 + x_4'^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$, which corresponds to the defining relation for $SO(4, R)$. More precisely, we have the relation

$$SO(4, R) = SU(2)^L \times SU(2)^R / Z_2.$$  \hspace{1cm} (4)

Here $Z_2$ is the matrix group generated by $(-1)$ times the $2 \times 2$ identity matrix $e$. Clearly for $u_1 = u_2 = -e$ the action eq.(2) keeps $w$ unchanged.

In order to make a connection with $R^4$, we take only the left action $w' = u_1 w$ and express the matrix elements of $w$ as a 4-vector

$$\begin{pmatrix} y' \\ -\bar{z}' \\ z' \\ \bar{y}' \end{pmatrix} = \begin{pmatrix} \alpha & \beta & 0 & 0 \\ -\bar{\beta} & \bar{\alpha} & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} y \\ -\bar{z} \\ z \\ \bar{y} \end{pmatrix}. \hspace{1cm} (5)$$

Substituting $y = x_1 + ix_2, -\bar{z} = x_3 + ix_4$, and $\alpha = \alpha_1 + i\alpha_2, \beta = \beta_1 + i\beta_2$, we get

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} \alpha_1 & -\alpha_2 & \beta_1 & -\beta_2 \\ \alpha_2 & \alpha_1 & \beta_2 & \beta_1 \\ -\beta_1 & -\beta_2 & \alpha_1 & \alpha_2 \\ \beta_2 & -\beta_1 & -\alpha_2 & \alpha_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}. \hspace{1cm} (6)$$
Obviously, the transformation matrix is orthogonal. Similarly for the right action \( w' = w u_{2}^{+} \) we get

\[
\begin{pmatrix}
\tilde{y}' \\
\tilde{z}' \\
y' \\
z'
\end{pmatrix} = \begin{pmatrix}
\tilde{\gamma} & 0 & \tilde{\delta} & 0 \\
0 & \gamma & 0 & \delta \\
-\delta & 0 & \gamma & 0 \\
0 & -\delta & 0 & \gamma
\end{pmatrix} \begin{pmatrix}
y \\
z
\end{pmatrix},
\]

and after substituting \( \gamma = \gamma_{1} + i \gamma_{2}, \ \delta = \delta_{1} + i \delta_{2}, \) we get

\[
\begin{pmatrix}
x_{1}' \\
x_{2}' \\
x_{3}' \\
x_{4}'
\end{pmatrix} = \begin{pmatrix}
\gamma_{1} & \gamma_{2} & -\delta_{1} & \delta_{2} \\
-\gamma_{2} & \gamma_{1} & \delta_{1} & \delta_{2} \\
\delta_{1} & -\delta_{2} & \gamma_{1} & \gamma_{2} \\
-\delta_{2} & -\delta_{1} & -\gamma_{2} & \gamma_{1}
\end{pmatrix} \begin{pmatrix}
x_{1} \\
x_{2} \\
x_{3} \\
x_{4}
\end{pmatrix},
\]

where the transformation matrix is orthogonal.

If we take the full action

\[
\begin{pmatrix}
y' \\
\tilde{z}' \\
y' \\
z'
\end{pmatrix} = \begin{pmatrix}
\alpha & \beta \\
-\beta & \alpha \\
\gamma & 0 \\
0 & \delta
\end{pmatrix} \begin{pmatrix}
y \\
z \\
\tilde{z} \\
\gamma
\end{pmatrix},
\]

we get

\[
\begin{pmatrix}
y' \\
\tilde{z}' \\
y' \\
z'
\end{pmatrix} = \begin{pmatrix}
\alpha \tilde{\gamma} & \beta \bar{\gamma} & \alpha \bar{\delta} & \beta \bar{\delta} \\
-\beta \bar{\gamma} & \alpha \bar{\gamma} & -\beta \bar{\delta} & \alpha \bar{\delta} \\
-\alpha \delta & -\beta \delta & \alpha \gamma & \beta \gamma \\
\beta \delta & -\alpha \delta & -\beta \gamma & \alpha \gamma
\end{pmatrix} \begin{pmatrix}
y \\
z \\
\tilde{z} \\
\gamma
\end{pmatrix} = \begin{pmatrix}
\alpha & \beta & 0 & 0 \\
-\beta & \alpha & 0 & 0 \\
0 & 0 & \alpha & \beta \\
0 & 0 & -\beta & \alpha
\end{pmatrix} \begin{pmatrix}
\gamma & 0 & \delta & 0 \\
0 & \tilde{\gamma} & 0 & \tilde{\delta} \\
0 & -\delta & 0 & \gamma \\
0 & -\beta & 0 & \gamma
\end{pmatrix} \begin{pmatrix}
y \\
z \\
\tilde{z} \\
\gamma
\end{pmatrix},
\]

and taking \( y = x_{1} + ix_{2}, \ \tilde{z} = x_{3} + ix_{4} \) we get the general transformation matrix for the 4-dimensional vector in \( \mathbb{R}^{4} \) under the group \( \text{SO}(4, R) \) as

\[
\begin{pmatrix}
x_{1}' \\
x_{2}' \\
x_{3}' \\
x_{4}'
\end{pmatrix} = \begin{pmatrix}
\alpha_{1} & -\alpha_{2} & \beta_{1} & -\beta_{2} \\
\alpha_{2} & \alpha_{1} & \beta_{2} & \beta_{1} \\
-\beta_{1} & -\beta_{2} & \alpha_{1} & \alpha_{2} \\
\beta_{2} & -\beta_{1} & -\alpha_{2} & \alpha_{1}
\end{pmatrix} \begin{pmatrix}
\gamma_{1} & \gamma_{2} & -\delta_{1} & \delta_{2} \\
-\gamma_{2} & \gamma_{1} & \delta_{1} & \delta_{2} \\
\delta_{1} & -\delta_{2} & \gamma_{1} & \gamma_{2} \\
-\delta_{2} & -\delta_{1} & -\gamma_{2} & \gamma_{1}
\end{pmatrix} \begin{pmatrix}
x_{1} \\
x_{2} \\
x_{3} \\
x_{4}
\end{pmatrix}.
\]

Notice that the eight parameters \( \alpha_{1}, \alpha_{2}, \beta_{1}, \beta_{2}, \gamma_{1}, \gamma_{2}, \delta_{1}, \delta_{2} \) with the constraints \( \alpha_{1}^{2} + \alpha_{2}^{2} + \beta_{1}^{2} + \beta_{2}^{2} = 1, \ \gamma_{1}^{2} + \gamma_{2}^{2} + \delta_{1}^{2} + \delta_{2}^{2} = 1, \) can be considered the Cayley parameters for the \( \text{SO}(4, R) \) group [11].
3. Tensor and spinor representations of SO(4,R)

Given the fundamental 4-dimensional representation of \( SO(4,R) \) in terms of the parameters \( \alpha, \beta, \gamma, \delta \), as given in eq. 11,

\[ x'_\mu = g_{\mu\nu} x_\nu, \]

the tensor representations are defined in the usual way

\[ T_{k'_1 k'_2 \ldots k'_n} = g_{k'_1 k_1} \cdots g_{k'_n k_n} T_{k_1 k_2 \ldots k_n}, \]

\[ (k'_i, k_i = 1, 2, 3, 4). \]

For the sake of simplicity we take the second rank tensors. We can decompose them into totally symmetric and antisymmetric tensors, namely,

\[ S_{ij} \equiv x_i y_j + x_j y_i \quad \text{(totally symmetric)}, \]
\[ A_{ij} \equiv x_i y_j - x_j y_i \quad \text{(antisymmetric)}. \]

If we substract the trace from \( S_{ij} \) we get a tensor that transforms under an irreducible representation. For the antisymmetric tensor the situation is more delicate. In general we have

\[ A'_{ij} \equiv x'_i y'_j - x'_j y'_i = (g_{\ell m} g_{jm} - g_{\ell m} g_{im}) A_{\ell m}. \]

This representation of dimension 6 is still reducible. For simplicity take the left action of the group given in eq. 6. The linear combination of the antisymmetric tensor components are transformed among themselves in the following way:

\[
\begin{pmatrix}
A'_{12} + A'_{34} \\
A'_{21} + A'_{24} \\
A'_{23} + A'_{14}
\end{pmatrix} = \begin{pmatrix}
A_{12} + A_{34} \\
A_{31} + A_{24} \\
A_{23} + A_{14}
\end{pmatrix},
\]

\[
\begin{pmatrix}
A'_{12} - A'_{34} \\
A'_{31} - A'_{24} \\
A'_{23} - A'_{14}
\end{pmatrix} = \begin{pmatrix}
\alpha_1^2 + \alpha_2^2 - \beta_1^2 - \beta_2^2 \\
2 (\alpha_1 \beta_2 + \alpha_2 \beta_1) \\
2 (\alpha_1 \beta_1 - \alpha_2 \beta_2)
\end{pmatrix} \times \begin{pmatrix}
A_{12} - A_{34} \\
A_{31} - A_{24} \\
A_{23} - A_{14}
\end{pmatrix}.
\]

In the case of the right action given by eq. 8 the 6-dimensional representation for the antisymmetric second rank tensor decomposes into two
irreducible 3-dimensional representation of $SO(4, R)$. For this purpose one takes the linear combination of the components of the antisymmetric tensor as before:

$$
\begin{pmatrix}
A'_{23} - A'_{14} \\
A'_{31} - A'_{24} \\
A'_{12} - A'_{34}
\end{pmatrix} = \begin{pmatrix}
A_{23} - A_{14} \\
A_{31} - A_{24} \\
A_{12} - A_{34}
\end{pmatrix},
$$

(17)

$$
\begin{pmatrix}
A'_{23} + A'_{14} \\
A_{31} + A_{24} \\
A'_{12} + A'_{34}
\end{pmatrix} =
\begin{pmatrix}
\gamma_1^2 - \gamma_2^2 - \delta_1^2 + \delta_2^2 & 2(\gamma_1 \gamma_2 + \delta_1 \delta_2) & -2(\gamma_1 \delta_1 - \gamma_2 \delta_2) \\
-2(\gamma_1 \gamma_2 - \delta_1 \delta_2) & \gamma_1^2 - \gamma_2^2 - \delta_1^2 - \delta_2^2 & 2(\gamma_1 \delta_1 + \gamma_2 \delta_2) \\
2(\gamma_1 \delta_1 + \gamma_2 \delta_2) & -2(\gamma_1 \delta_2 - \gamma_2 \delta_1) & \gamma_1^2 + \gamma_2^2 - \delta_1^2 - \delta_2^2
\end{pmatrix}
\times
\begin{pmatrix}
A_{23} + A_{14} \\
A_{31} + A_{24} \\
A_{12} + A_{34}
\end{pmatrix}.
$$

(18)

Therefore the 6-dimensional representation for the antisymmetric tensor decomposes into two irreducible 3-dimensional irreducible representation of the $SO(4, R)$ group.

For the spinor representation of $SU(2)^L$ we take

$$
\begin{pmatrix}
a'_1 \\
\alpha
\end{pmatrix} = \begin{pmatrix}
\alpha & \beta \\
-\beta & \alpha
\end{pmatrix}
\begin{pmatrix}
a_1 \\
a_2
\end{pmatrix}, \quad a_1, a_2 \in \mathcal{C}
$$

(19)

Let $a^{i_1i_2...i_k}$, $(i_1, i_2, \ldots, i_k = 1, 2)$ be a set of complex numbers of dimension $2^k$ which transform under the $SU(2)^L$ group as follows:

$$
a^{i_1...i_k} = u_{i_1}^{i_1} \ldots u_{i_k}^{i_k} a^{i_1...i_k},
$$

(20)

where $u_{i_1}^{i_1}, u_{i_2}^{i_2} \ldots$ are the components of $u \in SU(2)^L$. If $a^{i_1...i_k}$ is totally symmetric in the indices $i_1 \ldots i_k$ the representation of dimension $(k + 1)$ is irreducible. In an analogous way we can define an irreducible representation of $SU(2)^R$ with respect to the totally symmetric multispinor of dimension $(\ell + 1)$.

For the general group $SO(4, R) \sim SU(2)^L \times SU(2)^R$ we can take a set of totally symmetric multispinors that transform under the $SO(4, R)$ group as

$$
a^{i_1...i_k j_1...j_\ell} = u_{i_1}^{i_1} \ldots u_{i_k}^{i_k} \bar{v}_{j_1 j_1} \ldots \bar{v}_{j_\ell j_\ell} a^{i_1...i_k j_1...j_\ell}
$$

(21)

where $u_{i_1}^{i_1} \ldots$ are the components of a general element of $SU(2)^L$ and $\bar{v}_{j_\ell j_\ell}$ are the components of a general element of $SU(2)^R$. They define
an irreducible representation of $SO(4, R)$ of dimension $(k + 1)(\ell + 1)$ and with labels (see next section)

$$
\ell_0 = \frac{k - \ell}{2}, \quad \ell_1 = \frac{k + \ell}{2} + 1.
$$

(22)

4. Representations of the algebra so(4,R)

Let $J_1, J_2, J_3$ be the generators corresponding to the rotations in the planes $(x_2, x_3), (x_3, x_1)$, and $(x_1, x_2)$ respectively, and $K_1, K_2, K_3$ the generators corresponding to the rotations (boost) in the planes $(x_1, x_4)$, $(x_2, x_4)$ and $(x_3, x_4)$ respectively. They satisfy the following commutation relations:

$$
\begin{align*}
[J_p, J_q] &= i\varepsilon_{pqr} J_r, & p, q, r &= 1, 2, 3, \\
[J_p, K_q] &= i\varepsilon_{pqr} K_r, \\
[K_p, K_q] &= i\varepsilon_{pqr} J_r.
\end{align*}
$$

(23)

If one defines $\bar{A} = \frac{1}{2} (\bar{J} + \bar{K})$, $\bar{B} = \frac{1}{2} (\bar{J} - \bar{K})$, with $J = (J_1, J_2, J_3)$, $K = (K_1, K_2, K_3)$, then

$$
\begin{align*}
[A_p, A_q] &= i\varepsilon_{pqr} A_r, & p, q, r &= 1, 2, 3, \\
[B_p, B_q] &= i\varepsilon_{pqr} B_r, \\
[A_p, B_q] &= 0,
\end{align*}
$$

(24)

that is to say, the algebra so(4) decomposes into two simple algebras su(2) + su(2)

Let $\phi_{m_1 m_2}$ be a basis where $\bar{A}^2, A_3$ and $\bar{B}^2, B_3$ are diagonal. Then a unitary irreducible representation for the sets $\{A_\pm \equiv A_1 \pm i A_2, A_3\}$ and $\{B_\pm \equiv B_1 \pm i B_2, B_3\}$ is given by

$$
\begin{align*}
A_\pm \phi_{m_1 m_2} &= \sqrt{(j_1 \pm m_1)(j_1 \pm m_1 + 1)} \phi_{m_1 \pm 1, m_2}, \\
A_3 \phi_{m_1 m_2} &= m_1 \phi_{m_1 m_2}, & -j_1 \leq m_1 \leq j_1,
\end{align*}
$$

(25)

$$
\begin{align*}
B_\pm \phi_{m_1 m_2} &= \sqrt{(j_2 \pm m_2)(j_2 \pm m_2 + 1)} \phi_{m_1 m_2 \pm 1}, \\
B_3 \phi_{m_1 m_2} &= m_2 \phi_{m_1 m_2}, & -j_2 \leq m_2 \leq j_2.
\end{align*}
$$

We change now to a new basis

$$
\psi_{JM} = \sum_{m_1 + m_2 = m} \langle j_1 m_1 j_2 m_2 | JM \rangle \phi_{m_1 m_2}
$$

(26)
that corresponds to the Gelfand-Zetlin basis for \( \mathfrak{so}(4) \),

\[
\psi_{JM} = \begin{vmatrix} j_1 + j_2 \\ J \\ j_1 - j_2 \\ M \end{vmatrix}.
\]

In this basis the representation for the generators \( \bar{J}, \bar{K} \) of \( \mathfrak{so}(4) \) are given by [12]

\[
\begin{align*}
J_\pm \psi_{JM} &= \sqrt{(J \pm M) (J \pm M + 1)} \psi_{JM \pm 1}, \\
J_3 \psi_{JM} &= M \psi_{JM}, \\
K_3 \psi_{JM} &= a_{JM} \psi_{J-1,M} + b_{JM} \psi_{JM} + a_{J+1,M} \psi_{J+1,M},
\end{align*}
\]

where

\[
a_{JM} \equiv \left( \frac{(J^2 - M^2) (J^2 - \ell_0^2) (\ell_1^2 - J^2)}{(2J-1)J^2(2J+1)} \right)^{1/2},
\]

\[
b_{JM} = \frac{M \ell_0 \ell_1}{J(J+1)},
\]

with \( \ell_0 = j_1 - j_2 \), \( \ell_1 = j_1 + j_2 + 1 \) the labels of the representations.

The representation for \( K_1, K_2 \) are obtained with the help of the commutation relations.

The Casimir operators are

\[
\begin{align*}
\left( \bar{J}^2 + \bar{K}^2 \right) \psi_{JM} &= \left( \ell_0^2 + \ell_1^2 - 1 \right) \psi_{JM}, \\
\bar{J} \cdot \bar{K} \psi_{JM} &= \ell_0 \ell_1 \psi_{JM}.
\end{align*}
\]

The representations in the bases \( \psi_{JM} \) are irreducible in the following cases

\[
\begin{align*}
\ell_0 &= j_1 - j_2 = 0, \pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}, \pm 2, \ldots, \\
\ell_1 &= j_1 + j_2 - 1 = \ell_0 + 1, \ell_0 + 2, \ldots, \\
J &= j_1 - j_2, \ldots, j_1 + j_2.
\end{align*}
\]

If we exponentiate the infinitesimal generators we obtain the finite representations of \( \text{SO}(4, R) \) given in terms of the rotation angles. An element \( U \) of \( \text{SO}(4, R) \) is given as [13]

\[
U (\varphi, \theta, \tau, \alpha, \beta, \gamma) = R_3(\varphi) R_2(\theta) S_3(\tau) R_3(\alpha) R_2(\beta) R_3(\gamma),
\]

where \( R_2 \) is the rotation matrix in the \( (x_1x_3) \) plane, \( R_3 \) the rotation matrix in the \( (x_1x_2) \) plane and \( S_3 \) the rotation ("boost") in the \( (x_3x_4) \) plane, and

\[
0 \leq \beta, \tau, \theta \leq \pi, \quad 0 \leq \alpha, \varphi, \gamma \leq 2\pi.
\]
In the basis $\psi_{jm}$ the action of $S_3$ is as follows:

$$S_3(\tau) \psi_{jm} = \sum_{j'} d_{j'JM}^{j_1j_2}(\tau) \psi_{J'M},$$  \hspace{1cm} (31)$$

where

$$d_{j'JM}^{j_1j_2}(\tau) = \sum_{m_1m_2} \langle j_1j_2m_1m_2 | JM \rangle e^{-i(m_1-m_2)\tau} \langle j_1j_2m_1m_2 | J'M \rangle,$$  \hspace{1cm} (32)$$
is the Biedenharn-Dolginov function, \cite{14} and \cite{15} IV.3.

From this function the general irreducible representation of the operator $U$ in terms of rotation angles is \cite{13}:

$$U(\varphi, \theta, \tau, \alpha, \beta, \gamma) \psi_{JM} = \sum_{J'M'} D_{j'JM'}^{j_1j_2}(\varphi, \theta, \tau, \alpha, \beta, \gamma) \psi_{J'M'},$$  \hspace{1cm} (33)$$

where

$$D_{j'JM'}^{j_1j_2}(\varphi, \theta, \tau, \alpha, \beta, \gamma) = \sum_{m''} D_{J'M''}^{j_1j_2}(\varphi, \theta, 0) d_{j'JM''}^{j_1j_2}(\tau) D_{J'M}^{J'M'}(\alpha, \beta, \gamma).$$  \hspace{1cm} (34)$$

We now give some particular values of these representations. In the case of spin $j = 1/2$ we know

$$R_3(\alpha) R_2(\beta) R_3(\gamma) = \left( \begin{array}{cc} \cos \frac{\beta}{2} e^{i\alpha + \gamma} & i \sin \frac{\beta}{2} e^{-i\left(\frac{\alpha + \gamma}{2}\right)} \\ i \sin \frac{\beta}{2} e^{i\left(\frac{\alpha + \gamma}{2}\right)} & \cos \frac{\beta}{2} e^{-i\left(\frac{\alpha + \gamma}{2}\right)} \end{array} \right).$$  \hspace{1cm} (35)$$

Introducing the variables

$$x_1 = \cos \frac{\beta}{2} \cos \frac{\alpha + \gamma}{2}, \quad x_2 = \cos \frac{\beta}{2} \sin \frac{\alpha + \gamma}{2},$$

$$x_3 = \sin \frac{\beta}{2} \sin \frac{\gamma - \alpha}{2}, \quad x_4 = \sin \frac{\beta}{2} \cos \frac{\gamma - \alpha}{2},$$

we have

$$R_3(\alpha) R_2(\beta) R_3(\gamma) = \left( \begin{array}{cc} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{array} \right).$$  \hspace{1cm} (36)$$

Similarly we have

$$R_3(\varphi) R_2(\theta) R_3(\tau) = \left( \begin{array}{cc} y_1 + iy_2 & y_3 + iy_4 \\ -y_3 + iy_4 & y_1 - iy_2 \end{array} \right),$$  \hspace{1cm} (37)$$

with

$$y_1 = \cos \frac{\theta}{2} \cos \frac{\varphi + \tau}{2}, \quad y_2 = \cos \frac{\theta}{2} \sin \frac{\varphi + \tau}{2},$$

$$y_3 = \sin \frac{\theta}{2} \sin \frac{\tau - \varphi}{2}, \quad y_4 = \sin \frac{\theta}{2} \cos \frac{\tau - \varphi}{2}.$$
For the Biedenharn-Dolginov function we have some particular values, see [15] IV.2.3,

\[ d_{j,M}^{[j+0]}(\tau) = i^{J-M}2^J \sqrt{2J + 1} \Gamma(J + 1) \times \]

\[ \times \left( \frac{\Gamma\left(M + \frac{3}{2}\right) \Gamma(j_+ - M + 1) \Gamma(j_+ - J + 1) \Gamma(J + M + 1)}{\Gamma\left(\frac{3}{2}\right) \Gamma(j_+ + M + 2) \Gamma(j_+ + J + 2) \Gamma(J - M + 1) \Gamma(M + 1)} \right)^{\frac{1}{2}} \]

\[ \times (\sin \tau)^{J-M} C_{j+1-j}^{J+1}(\cos \tau), \tag{38} \]

where \( j_+ \equiv j_1 + j_2 \), \( j_- = j_1 - j_2 = 0 \), and \( C_n^\nu(\cos \tau) \) are the Gegenbauer (ultraspherical) polynomials which are related to the Jacobi polynomials by

\[ C_n^\nu(\cos \tau) = \frac{\Gamma\left(\nu + \frac{3}{2}\right) \Gamma(2\nu + n)}{\Gamma(2\nu) \Gamma\left(\nu + n + \frac{3}{2}\right)} P_n^{(\nu-\frac{1}{2}, \nu-\frac{1}{2})}(\cos \tau), \tag{39} \]

5. Relativistic spin network in 4-dimensions

We address ourselves to the Barrett-Crane model that generalized Penrose’s spin networks from three dimensions to four dimensions [16]. They characterize the geometrical properties of 4-simplices, out of which the tessellation of the 4-dimensional manifold is made, and then attach to them the representations of \( SO(4, R) \).

A geometric 4-simplex \( S^4 \) in Euclidean space is given by the embedding of an ordered set of 5 points \((0, x, y, z, t)\) in \( \mathbb{R}^4 \) which is required to be non-degenerate (the points should not lie in any hyperplane). Each triangle in it determines a bivector constructed out of the vectors for the edges. Barrett and Crane proved that classically, a geometric 4-simplex in Euclidean space is completely characterized (up to parallel translation and inversion through the origin) by a set of 10 bivectors \( b_i \), each corresponding to a triangle in the 4-simplex and satisfying the following properties:

i) the bivector changes sign if the orientation of the triangle is changed;

ii) each bivector is simple, i.e. is given by the wedge product of two vectors for the edges;

iii) if two triangles share a common edge, the sum of the two bivector is simple;
iv) the sum (considering orientation) of the 4 bivectors corresponding to the faces of a tetrahedron is zero;

v) for six triangles sharing the same vertex, the six corresponding bivectors are linearly independent;

vi) the bivectors (thought of as operators) corresponding to triangles meeting at a vertex of a tetrahedron satisfy $|tr b_1 [b_2, b_3]| > 0$, i.e. the tetrahedron has non-zero volume.

Then Barrett and Crane define the quantum 4-simplex with the help of bivectors (thought as elements of the Lie algebra $SO(4, R)$). They associate a representation to each triangle and a tensor to each tetrahedron. The representations chosen should satisfy the following conditions, corresponding to the geometrical ones:

i) different orientations of a triangle correspond to dual representations;

ii) the representations of the triangle are “simple” representations of $SO(4, R)$, i.e. $j_1 = j_2$;

iii) given two triangles, if we decompose the pair of representations of the tetrahedra bounded by it into its Clebsch-Gordan series, the tensor for the tetrahedron is decomposed into summands which are non-zero only for simple representations;

iv) the tensor for the tetrahedron is invariant under $SO(4, R)$.

5.1. SPIN FOAM MODELS AND THE BARRETT-CRANE MODEL.

We revise the geometrical analysis of Barrett and Crane and follow Reisenberger and Rovelli [21] p. 2. Consider a simplicial complex in $R^4$ and fix in it a 4-simplex $S^4$. This 4-simplex is bounded by five 3-simplices or tetrahedra, by ten 2-simplices or triangles, by ten 1-simplices or edges, and has five vertices. Any triangle belonging to $S^4$ bounds and determines exactly two tetrahedra of $S^4$, as can be seen by inspection of Fig. 1.

For the dualization of the spin network we follow Reisenberger and Rovelli [21] pp. 2-4 which is in line with the standard dualization of cell complexes [20] pp. 377-382. In the language of [21] the simplicial complex is denoted as $\Delta$ and its dual 2-skeleton as $J(\Delta)$. We denote dual objects by *. The dual to the 4-simplex is a vertex $v^*$, the dual to the five tetrahedra of $S^4$ are five edges $e^*$, and the duals to the ten triangles are ten 2-faces $f^*$. The dual boundaries corresponding to
a fixed 4-simplex $S^4$ all share a single dual vertex $v^*$. A dual vertex bounds five dual edges and ten dual faces. A single dual edge $e^*$ at a vertex $v^*$ bounds four faces $f^*$. A single dual face $f^* = f_{kl}^*$ at a dual vertex $v^*$ has exactly two bounding dual edges $(e_k^*, e_l^*), k < l, k = 1, 2, \ldots, 4$ and therefore can be labelled by the pair $(k, l), k < l$.

Following [21], the coloring of a spin network $\Delta$ is the assignment of pairs $c = \{\rho(g), b\}$ to geometric boundaries of $\Delta$, with $\rho(g)$ an irrep of the chosen group $G$ for an element $g \in G$, and $b$ intertwiners. Reisenberger and Rovelli [21] assign the irreps $\rho(g)$ to the ten dual faces $f_{ij}^*$, and the intertwiners to the edges $e_k^*, l = 1, \ldots, 5$ of each fixed vertex $v^*(J(\Delta))$. The geometric property that a dual edge at a dual vertex bounds four faces is converted by the coloring into the requirement that the intertwiner for this edge couples the four irreps associated to the four faces to an invariant under right action. Reisenberger and Rovelli [21] p.3 claim that in the TOCY (Turaev-Ooguri-Crane-Yetter) models this intertwiner is reduced to the intertwining of pairs. In their explanation of this pairing on [21] p. 4 they use twenty instead of ten representations and group elements, labelled in pairs as $(g_{ij}^*, g_{ji}^*), i < j$.

Their pairwise intertwiner for a fixed face takes the form, [21] eq. (19),

$$V(g_{ij}^*) = W(g_{ij}^* (g_{ji}^*)^{-1}).$$  \hfill (40)

We shall show in part II section 4 for general groups that a group representation depending on $g_1(g_2)^{-1}$ like in eq. 40 arises from the intertwining of two irreps to an invariant under the right action $(g_1, g_2) \rightarrow (g_1 q, g_2 q), q \in G$. For the group $SO(4, R)$ we get this function in terms of the Gelfand-Zetlin representation eq. 43 in the bracket notation,

$$((j_1, j_2), J'M' | T_{g_1} T_{(g_2)^{-1}} | (j_1, j_2), J'M')$$  \hfill (41)

Unfortunately the doubling of the number of irreps and group elements proposed in [21] and their pairing has no natural counterpart in the geometry of the spin network. If we modify the coloring of $J(\Delta)$ such that irreps are attached to dual edges and intertwiners to dual faces at a dual vertex, the representations and group elements would pair naturally, and the intertwiners would couple the five irreps, functions of five group elements, in ten pairs in a form as in eq. 40. This modified coloring would naturally represent the geometric property that any pair of dual edges bounds exactly one dual face.

A second observation arises from the use of the trace of representations in [21]. If from eq. 41 we take the trace of the representation, we obtain

$$\text{Trace}(D^{j_1 j_2}(g_1(g_2)^{-1})) = \chi^{j_1 j_2}(g_1(g_2)^{-1}),$$  \hfill (42)

that is, the character $\chi^{j_1 j_2}$ of the irrep. It is easy to see that this expression now is invariant not only under right action but also under
the left action \((g_1, g_2) \rightarrow (q g_1, q g_2), q \in G\). A weaker alternative to this trace formation leads to zonal spherical functions, as we explain in the next subsection.

5.2. **Spherical harmonics, simple representations and spherical functions.**

The spherical harmonics are functions on a coset or quotient space \(SO(4, R)/SU(2)^c \sim S^3\). We shall derive the spherical harmonics from particular representations on a coset space by the condition that they be left-invariant under \(SU(2)^c\). To determine the stability group consider in eq. 1 the point \(P_0: (x_1, x_2, x_3, x_4) = (1, 0, 0, 0)\) of the sphere \(S^3 \subseteq R^4\). In the matrix notation eq. 1, the point \(P_0\) corresponds to the unit matrix \(w_0 = e\). With respect to the actions eq. 2, this point is stable under any action \(w_0 \rightarrow u^{-1} w_0 u\). These elements form a subgroup \(SU(2)^c \subset SO(4, R)\) equivalent to \(SU(2)\) with elements \((v_1, v_2)\). The corresponding coset space \(SO(4, R)/SU(2)^c\) can be parametrized by choosing in eq. 1 \(w = u' = SU(2)^R\), see eq. 45 below.

For the present purpose we use the Gelfand-Zetlin irrep of \(SO(4, R)\) as constructed in section 4. We write these irreps for \((u_1, u_2) \in SO(4, R)\) in a bracket notation

\[
\langle (j_1 j_2), J'M' | T_{(u_1, u_2)} | (j_1 j_2), JM \rangle \equiv \sum_{m_1' m_2' m_1 m_2} \langle j_1 m_1' j_2 m_2' | J'M' \rangle D^{j_1}_{m_1'} (u_1) D^{j_2}_{m_2'} (u_2) \langle j_1 m_1 j_2 m_2 | JM \rangle. \tag{43}
\]

Consider now the restriction of the irrep eq. 43 to the action of the subgroup \(SU(2)^c\) with elements \((u_1, u_2) \rightarrow (v_1, v_1)\). We obtain

\[
\langle (j_1 j_2), J'M' | T_{(v_1, v_1)} | (j_1 j_2), JM \rangle = \delta_{J'J} \delta_{M'M} D^{j_1}_{m_1} (v_1). \tag{44}
\]

In other words, the Gelfand-Zetlin basis is explicitly reduced with respect to the stability subgroup \(SU(2)^c\). Next we rewrite a general element of \(SO(4, R)\) in the form

\[
(u_1, u_2) = (v_1, v_1) (e, v_2) = (v_1, v_1 v_2), \quad v_2 \in SU(2)^R. \tag{45}
\]

These equations show that the cosets of the stability group \(SU(2)^c < SO(4, R)\) are in one-to-one correspondence to the elements \((e, v_2)\) of the subgroup \(SU(2)^R < SO(4, R)\) of eqs. 2, 3.

Evaluation in the new basis yields in particular

\[
\langle (j_1 j_2), J'M' | T_{(e, u_2)} | (j_1 j_2), JM \rangle \tag{46}
\]
\[= \sum_{m'_1 m_1 m'_2 m_2} \delta_{m'_1 m_1} D_{m'_2 m_2}^{j_2} (v_2) \langle j_1 m'_1 j_2 m'_2 | J' M' \rangle \langle j_1 m_1 j_2 m_2 | J M \rangle.\]

It follows that the full representation under restriction to \(SU(2)^R\) is given in terms of the irrep \(D^{j_2}(v_2)\) of \(SU(2)^R\). If we choose in eq. 46 \((j' m') = (00)\), we assure from eq. 44 that all the matrix elements

\[\langle (j_1 j_2) 00 | T(e, v_2) | (j_1 j_2) J M \rangle = \delta_{j_1 j_2} \sum_{m'_1 m_1 m'_2 m_2} \delta_{m'_1 m_1} D_{m'_2 m_2}^{j_2} (v_2) \langle j_2 m'_1 j_2 m'_2 | 00 \rangle \langle j_2 m_1 j_2 m_2 | J M \rangle = \delta_{j_1 j_2} \sum_{m'_1 m_1 m'_2 m_2} (-1)^{j_2 - m'_1} \delta_{m'_1 m_1} D_{m'_2 m_2}^{j_2} (v_2) \delta_{m'_1, -m'_2} \frac{1}{\sqrt{2j_2 + 1}} \langle j_2 m_1 j_2 m_2 | J M \rangle = \delta_{j_1 j_2} \frac{1}{\sqrt{2j_2 + 1}} \sum_{m_1 m_2} (-1)^{j_2 - m_1} D_{-m_1 m_2}^{j_2} (v_2) \langle j_2 m_1 j_2 m_2 | J M \rangle\]

are invariant under left action with elements \((v_2, v_2) \in SU(2)^c\). By definition these are the spherical harmonics on \(SO(4, R)/SU(2)^c\).

We summarize these results for spherical harmonics on \(SO(4, R)/SU(2)^c\) in

1 Theorem: Spherical harmonics of \(SO(4, R)\):

(a) Domain: The spherical harmonics are defined on the coset space for the stability group \(SU(2)^c\) of the sphere \(S^3\). This coset space from eq. 45 can be taken in the form \(SU(2)^R\).

(b) Characterization: The spherical harmonics on this coset space are given by the matrix elements eq. 47 of simple irreps.

(c) Transformation properties: Under right action of \(SO(4, R)\), the spherical harmonics eq. 47 transform according to simple irreps \(D^{j_2 j_2}\), which in the Gelfand Zetlin basis are given by eq. 47 with \(j_1 = j_2\). Any left action by \((v_1, v_1) \in SU(2)^c\) leaves the expressions eq. 47, taken as matrix elements of the full irrep, invariant.

(d) Measure: The spherical harmonics form a complete orthonormal set on the coset space \(SO(4, R)/SU(2)^c\). The measure on \(SO(4, R)\) from eq. 3 is the product of two measures for groups \(SU(2)\). It follows that the measure on the coset space \(SO(4, R)/SU^c(2)\) has the form of a measure \(d\mu(u)\) on \(SU(2)^R\).

The coloring of the spin network in [21] attaches irreps \(\rho(g)\) of the group \(G\) and intertwiners to geometric boundaries. For given group element \(g \in G\), the full representation is fixed by an irrep label \(\lambda\) and sets
of row and column labels. This coloring scheme can easily be modified by attaching only subsets of matrix elements to a geometric boundary. The particular choice of matrix elements eq. 47 implies that spherical harmonics are attached. The use of a coset space $SO(4, R)/SU(2)^c$ and of functions on these for spin networks is advocated by Freidel et al. [22] pp. 14-16. We agree with these authors but strictly distinguish between spherical harmonics and simple representations which determine their transformation properties. Spherical harmonics by eq. 47 are particular matrix elements of simple irreps and live on the coset space $SO(4, R)/SU(2)$, not on the full group space of $SO(4, R)$.

The results of Theorem 1 allow us to comment on the Kronecker product of simple irreps of $SO(4, R)$. Reisenberger and Rovelli [21] p. 3 noted correctly that the Kronecker product of two simple irreps of $SO(4, R)$ contain both simple and non-simple irreps. To avoid the non-simple ones they introduce projectors. If we replace simple irreps by the spherical harmonics of eq. 47, the situation changes. A product of two spherical harmonics is still a function on the same coset space $SO(4, R)/SU(2)^c$. Since the spherical harmonics form a complete set on this coset space, such a product can be expanded again exclusively in spherical harmonics. The expansion coefficients are those particular coupling coefficients which relate products of simple irreps to simple irreps. The non-simple irreps automatically drop out of these expansions. The measure on the coset space from Theorem 1(d) is only a factor of the measure on the full group space and equivalent to the measure on the single group $SU(2)^R$.

5.3. **Spherical harmonics and zonal spherical functions.**

The intertwiners appearing in the spin networks correspond to right-hand coupling of pairs of irreps to a function invariant under right action, see part II section 4. This right-hand coupling applies as well to the coupling of pairs of spherical harmonics. Taking these in the form of eq. 47 as functions of group elements $(g_1, g_2)$ yields the expression

$$f^{(j_2 j_2)}(g_1(g_2)^{-1}) := \langle j_2 j_2 00|g_1(g_2)^{-1}|j_2 j_2 00\rangle. \quad (48)$$

The function of $g_1(g_2)^{-1}$ on the righthand side of eq. 48 is a zonal spherical function on $SO(4, R)$ with respect to the subgroup $SU(2)^c$. For a general group $G$ with subgroup $SU(2)$ we refer to part II section 8. A zonal spherical is a matrix elements of an irrep $D^\lambda(g)$ characterized by the invariance both under left- and right-action with $h \in H$. The Gelfand-Zetlin irrep eq. 43 of $SO(4, R)$ is adapted to the subgroup $SU(2)^c < SO(4, R)$ with subgroup representation labels $(J'M'), (JM)$. 

DisQ_I085.tex; 28/04/2008; 22:05; p.16
2 Def: A zonal spherical function of \( g = (u_1, u_2) \) for the subgroup \( SU(2)^c < SO(4, R) \) is given in the Gelfand-Zetlin basis eq. 43 by
\[
f^{(j_2j_2)}(u_1, u_2) := \langle j_2j_2|T_{(u_1, u_2)}|j_2j_200\rangle.
\]

The expression eq. 49, in contrast to the trace eq. 42, is not invariant under general left actions. It has the weaker invariance
\[
f^{(j_2j_2)}((h_1)^{-1}gh_2) = f^{(j_2j_2)}(g), \ (h_1, h_2) \in SU(2)^c.
\]
and so it lives on the double cosets of \( SO(4, R) \) with respect to \( SU(2)^c \).

The zonal spherical functions eq. 49 must be distinguished from the spherical functions discussed by Godement in [18].

By use of the angular parameters introduced in eq. 30, we obtain the zonal spherical function eq. 49 in terms of the single parameter \( \tau \):
\[
f^{(j_2j_2)}(\tau) = \sum_{m_1 + m_2 = 0} \langle j_2m_1j_2m_2|00\rangle \exp(-i(m_1 - m_2)\tau)\langle j_2m_1j_2m_2|00\rangle
\]
\[
= \frac{1}{2j_2 + 1} \sum_{m_1 = -j_1}^{j_1} \exp(-i2m_1\tau) = \frac{1}{2j_2 + 1} \sin((2j_1 + 1)\tau) \sin \tau
\]

3 Theorem: The zonal spherical functions for simple irreps of \( SO(4, R) \) with subgroup \( SU(2)^c \) given by eq. 49 become the functions eq. 51 of the parameter \( \tau \).

Pairs of spherical harmonics can still be intertwined to invariants under the right action of \( SO(4, R) \). The result of this intertwining is a zonal spherical function of the type eq. 49 of the product \((g_1(g_2))^{-1}\) of two group elements and by eq. 51 can be given as a function of the angular parameter \( \tau \) for the group element \( g_1(g_2))^{-1}\).

4 Theorem: If, in agreement with [22], not full simple irreps but spherical harmonics are attached to boundaries of the spin network, any pairwise intertwiner becomes a zonal spherical function \( f^{(j_2j_2)}(g_1(g_2))^{-1}\) eq. 49.

6. The triple product in \( R^4 \)

Before we apply the representation theory developed in previous sections to the Barrett-Crane model we introduce some geometrical properties based in the triple product that generalizes the vector (cross) product in \( R^3 \). Given three vectors in \( R^4 \), we define the triple product:
\[
u \wedge v \wedge w = -v \wedge u \wedge w = -u \wedge w \wedge v = -w \wedge v \wedge u = v \wedge w \wedge u = w \wedge u \wedge v,
\]
\[
u \wedge u \wedge v = u \wedge v \wedge u = v \wedge u \wedge u = 0. \tag{52}
\]
If the vectors in $\mathbb{R}^4$ have cartesian coordinates
\[ u = (u_1, u_2, u_3, u_4), \quad v = (v_1, v_2, v_3, v_4), \quad w = (w_1, w_2, w_3, w_4), \]
we define an orthonormal basis in $\mathbb{R}^4$
\[ \hat{i} = (1, 0, 0, 0), \quad \hat{j} = (0, 1, 0, 0), \quad \hat{k} = (0, 0, 1, 0), \quad \hat{\ell} = (0, 0, 0, 1). \]
The triple product of these vectors satisfies
\[ \hat{i} \wedge \hat{j} \wedge \hat{k} = -\hat{\ell}, \quad \hat{j} \wedge \hat{k} \wedge \hat{\ell} = \hat{i}, \quad \hat{k} \wedge \hat{\ell} \wedge \hat{i} = -\hat{j}, \quad \hat{i} \wedge \hat{j} \wedge \hat{\ell} = \hat{k}. \]
In coordinates the triple product is given by the determinant
\[ u \wedge v \wedge w = \begin{vmatrix} i & j & k & \ell \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{vmatrix}. \tag{53} \]
The scalar quadruple product is defined by
\[ a \cdot (b \wedge c \wedge d) = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = [abcd] = -[abdc] = [acbd] = -[acdb] \text{ and so on.} \tag{54} \]
It follows: $a \cdot a \wedge b \wedge c = b \wedge a \wedge b \wedge c = c \wedge a \wedge b \wedge c = 0$.

We can use the properties of the three vector for the description of the 4-simplex. Let \( \{0, x, y, z, t\} \) be the 4-simplex in $\mathbb{R}^4$. Two tetrahedra have a common face
\[ \{0, x, y, z\} \cap \{0, x, y, t\} = \{0, x, y\}. \]
Each tetrahedron is embedded in an hyperplane characterized by a vector perpendicular to all the vectors forming the tetrahedron. For instance,
\[ \{0, x, y, z\} \text{ is characterized by } a = x \wedge y \wedge z, \]
\[ \{0, x, y, t\} \text{ is characterized by } b = x \wedge y \wedge t. \]
Fig. 1. A simplex $S^4$ in $R^4$ seen in a projection to a two dimensional plane.

The vector $a$ satisfies $a \cdot x = a \cdot y = a \cdot z = 0$,
the vector $b$ satisfies $b \cdot x = b \cdot y = b \cdot t = 0$.

The triangle $\{0, x, y\}$ shared by the two tetrahedra is characterized by the bivector $x \wedge y$. The plane where the triangle is embedded is defined by the two vectors $a, b$, forming the angle $\phi$, given by

$$\cos \phi = a \cdot b.$$ 

The bivector $a \wedge b$ can be calculated with the help of bivectors $x \wedge y$, namely,

$$a \wedge b = [x y z t] \ast (x \wedge y).$$ 

Obviously $a \wedge b$ is perpendicular to $x \wedge y$

$$\langle a \wedge b, x \wedge y \rangle = (a \cdot x) (b \cdot y) - (a \cdot y) (b \cdot x) = 0. \quad (55)$$

For completeness we add some useful properties of bivectors in $R^4$. The six components of a bivector can be written as

$$B_{\mu\nu} = x_\mu y_\nu - x_\nu y_\mu, \quad \mu, \nu = 1, 2, 3, 4, \quad B = (J, K),$$

$$J_1 = (x_2y_3 - x_3y_2), \quad J_2 = (x_3y_1 - x_1y_3), \quad J_3 = (x_1y_2 - x_2y_1),$$

$$K_1 = (x_1y_4 - x_4y_1), \quad K_2 = (x_2y_4 - x_4y_2), \quad K_3 = (x_3y_4 - x_4y_3).$$

The six components of the dual of a bivector are

$$B_{\alpha \beta} = \frac{1}{2} B_{\mu \nu} \epsilon_{\mu \nu \alpha \beta}, \quad \ast B = (K, J).$$

We take the linear combinations of $J, K$

$$M = \frac{1}{2} (J + K), \quad N = \frac{1}{2} (J - K). \quad (56)$$

They form the bivector $(M, N)$, whose dual is:

$$\ast (M, N) = (M, -N), \quad (57)$$

therefore $M$ can be considered the self-dual part, $N$ the antiselfdual part of the bivector $(M, N)$. $M$ and $N$ coincides with the basis for the irreducible tensor representations of section 3. The norm of the bivectors can be explicitly calculated.

$$\|B\|^2 = \langle B, B \rangle = J^2 + K^2 = \|x\|^2 \|y\|^2 - |x, y|^2 =$$

$$= \|x\|^2 \|y\|^2 \sin^2 \phi(x, y) = 4(area)^2 \{0, x, y\}, \quad (58)$$

$$\|\ast B\|^2 = \langle \ast B, \ast B \rangle = J^2 + K^2 = \|B\|^2. \quad (59)$$

Finally, the scalar product of two vectors in $R^4$ can be expressed in terms of the corresponding $SU(2)$ matrices
Let $X \Leftrightarrow \left( \begin{array}{cc} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{array} \right)$, $Y \Leftrightarrow \left( \begin{array}{cc} y_1 + iy_2 & y_3 + iy_4 \\ -y_3 + iy_4 & y_1 - iy_2 \end{array} \right)$.

Then
\[
\frac{1}{2} \text{Tr} (XY^*) = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4. \tag{60}
\]

7. Evaluation of the state sum for the 4-dimensional spin network

We adopt the geometry of the spin network as explained at the end of section 5.1. In order to evaluate the state sum for a particular triangulation of the total $R^4$ space by 4-simplices, we assign an element $g_k \in SO(4,R)$ and representation $\rho_k(g_k)$ to each tetrahedron ($k = 1, 2, 3, 4, 5$) of $S^4$ and an intertwiner of $SO(4,R)$ to each triangle of $S^4$ shared by two tetrahedra. From this triangulation we obtain a dual 2-complex where two dual edges correspond to the two tetrahedra and a dual face to the triangle, with the two edges bounding the dual face. Dually we attach the representations $\rho_k(g_k)$ and $\rho_l(g_l)$ of $SO(4,R)$ to the edges $e_k^*$ and $e_l^*$ and contract both representations at the dual face $f^{*}_{kl}$, giving
\[
f^{(j_2j_2)}(g_kg_l^{-1}). \tag{61}\]

Here $f^{(j_2j_2)}$ is the contraction of the two simple representations of $SO(4,R)$ to an invariant under right action, compare section 5.3 and part II section 4. This contraction is shown to require that the two representations of $SO(4,R)$ be equivalent, $\rho_k \sim \rho_l \sim \rho_{kl}$. Since each element $g \in SO(4,R)$ is a pair $(u_1, u_2)$ of elements of $SU(2)$ and the representations are simple, the expression eq. 49 reduces to a product of two expressions in terms of $SU(2)$ with the same representation $j_2$. The expression eq. 49 has one more implication which we pointed out in section 5.3: It is valid only if the irreps $\rho(g)$ attached to the tetrahedra are replaced by the spherical harmonics eq. 47. Then the intertwiner of a pair of spherical harmonics becomes a zonal spherical function eq. 49.

The state sum for the 2-dimensional complex (the Feynman graph of the model) is obtained by taking the product expression eq. 61 for all the edges of the graph and integrating over all the copies of elements of $SO(4,R)$. Barrett and Crane construct a state sum $Z_{BC}$ for the quantum 4-simplex in terms of amplitudes $A$, functions of the colorings and intertwiners attached to simplices of the spin network:
\[
Z_{BC} = \sum_{J} \prod_{\text{trig.}} A_{\text{tr}} \prod_{\text{tetrahedra}} A_{\text{tetr}} \prod_{\text{4-simplices}} A_{\text{simp}}. \tag{62}\]
where the sum extends to all possible values of the representations \( J \).
All the amplitudes \( A \) can be expressed by intertwiners of pairs of irreps and group elements, and by corresponding zonal spherical functions. Due to the properties of zonal spherical functions, the expression eq. 62 is in addition invariant under left and right multiplication with arbitrary elements of \( SU(2)^c < SO(4,R) \). We can obtain a particular value of eq. 51 for \( j_2 = 1/2 \) if we take the elements \( g_k \) and \( g_l \) as pairs of unit vectors in \( R_4 \), say, \( x \) and \( y \), and use eqs. 36, 37 to obtain

\[
f^{(12)}(xy^+) = x \cdot y = \cos(\varphi)
\]

where \( \varphi \) is the angle between the vectors \( x \) and \( y \).

The two vectors \((x,y)\) are perpendicular to the hyperplanes where the tetrahedra \( k \) and \( l \) are embedded, and correspond to the vectors perpendicular to the face shared by the two tetrahedra, as explained in [17].

With eq. 49 it is still possible to give a geometrical interpretation of the probability amplitude encompassed in the zonal spherical function. In fact, the spin dependent factor appearing in the exponential of eq. 51,

\[
e^{i(2j_k+1)\tau_{kl}},
\]

corresponding to two tetrahedra \( k,\ell \) intersecting in the triangle \( k\ell \), can be interpreted as the product of the angle between the two vectors \( g_k, g_\ell \) perpendicular to the triangle and the area \( A_{k\ell} \) of the intersecting triangle.

For the proof we identify the component of the antisymmetric tensor \((J,K)\) with the components of the infinitesimal generators of the \( SO(4,R) \) group

\[
J_{\mu\nu} \equiv i \left( x_\mu \frac{\partial}{\partial x_\nu} - x_\nu \frac{\partial}{\partial x_\mu} \right).
\]

From eq. 58 and eq. 59 we have \( \|B\|^2 = 4 (A_{k\ell})^2 = 2 (\bar{M}^2 + \bar{N}^2) \)

But \( \bar{M}^2 \) and \( \bar{N}^2 \) are the Casimir operators of the \( SU(2) \times SU(2) \) group with eigenvalues \( j_1 (j_1 + 1) \) and \( j_2 (j_2 + 1) \).

For large values of \( j_1 = j_2 = j_{k\ell} \) we have

\[
2 \left( \bar{M}^2 + \bar{N}^2 \right) \cong 4 j_{k\ell}^2 + 4 j_{k\ell} + 1 = (2j_{k\ell} + 1)^2,
\]

therefore \( \frac{1}{2} (2j_{k\ell} + 1) = A_{k\ell} \) where \( A_{k\ell} \) is the area of the triangle characterized by the two vectors \( g_k \) and \( g_\ell \) and \( j_{k\ell} \) is the spin corresponding to the representation \( \rho_{k\ell} \) associated to the triangle \( k\ell \). Substituting this result in eq. 51 we obtain the asymptotic value of the amplitude given by Barrett and Williams [19].
8. Conclusion.

Starting from the Barrett-Crane model, we examine the geometry and quantization of spin networks in Euclidean space $\mathbb{R}^4$. We find that alternative choices are possible. We follow in part [21] and quantize a simplicial spin network by attaching to its boundaries the irreps and intertwiners of the group $SO(4, \mathbb{R})$. The intertwiners usually are required to be invariant under right action. We point out the equal importance of left action. A large class of models as [17-23] employs right action invariant intertwiners only between pairs of irreps. Invariance in addition under left action can be achieved from full irreps by the formation of traces. As an alternative quantization, we follow [22] and examine spherical harmonics and their right action invariant intertwiners attached to boundaries of the spin network. The Gelfand-Zetlin basis of the irreps of $SO(4, \mathbb{R})$ is the appropriate tool for the analysis. Spherical harmonics by their transformation properties select only simple representations. Since spherical harmonics live on the coset space $SO(4, \mathbb{R})/SU(2)^c$, not on the full group space, their intertwiners relate simple representations exclusively to simple representations. The pairwise right-invariant intertwiners of spherical harmonics in the Gelfand-Zetlin basis become zonal spherical functions. We construct these explicitly and write them in terms of a single group parameter. Moreover the zonal spherical functions admit a corresponding geometrical interpretation in terms of the area of triangles. In part II we shall develop a similar analysis for relativistic spin networks in Minkowski space.

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