Entanglement of Two Distinguishable Atoms in a Rectangular Waveguide: Linear Approximation with Single Excitation

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We consider two two-level systems (TLSs) coupled to the vacuum of guided modes confined in a rectangular waveguide. Two TLSs are fixed at different points in the waveguide and initially share an excitation. For the energy separation of the TLSs far away from the cutoff frequencies of transverse modes, two coupled delay-differential equations are obtained for the probability amplitudes of the TLSs. The effects of the difference of TLSs' energy separations and the inter-TLS distance on the time evolution of the concurrence of the TLSs are examined.

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I. INTRODUCTION

The quantum superposition principle allows a system composed of multipartite quantum systems to have states that cannot be factorized in products of states of the individual quantum systems. This nonseparability, labeled as entanglement, is an important physical resource for applications of quantum information processing. Scalable quantum information processing in quantum computation and communication is essentially based on a quantum network [1]. A quantum network consists of quantum channels and nodes. Two-level systems (TLSs) fixed at quantum nodes are called stationary qubits which generate, store, and process quantum information. It is essentially important to generate or keep the correlation among TLSs located at different positions for protecting quantum information. The bipartite entanglement involving two TLSs is of special interest. Spatially separated TLSs talking to each other can be either mediated or destroyed via electromagnetic fields [2–5]. Complete disentanglement is achieved in finite time for two TLSs coupled individually to two vacuum cavities [6]. The entanglement exhibits revivals in time for two TLSs coupled collectively to a multimode vacuum field in free space [7].

To build large scale quantum networks, an electromagnetic field in a one-dimensional (1D) waveguide is of special interest. The electromagnetic field is confined spatially in two dimensions and propagates along the remaining one, so it consists of infinite modes for right and left-going photons of continuous varying frequencies. Spontaneously emitted waves from the TLS will interfere with the incident wave [8–14]. The coupling of the electromagnetic field to a TLS can be increased by reducing the transverse size. A waveguide with a cross section has many guided modes, e.g. transverse-magnetic (TM) modes or transverse-electric (TE) ones. However, most work only consider TLSs interacting with one guided mode of the waveguide [9–14, 16–22], which means that the transverse-size effect has been ignored. In this paper, we study the time evolution of entanglement measure for two uncoupled TLSs interacting with the electromagnetic field confined in a 1D rectangular hollow metallic waveguide. The TLSs share initially an excitation and the field is in vacuum. Such waveguide has many guided modes. There is a continuous range of frequencies and a minimum frequency (called cutoff frequency) allowed in each guided mode [15]. When the transitions of the TLSs are far away from the cutoff frequencies of guided modes, the probability amplitudes the TLSs is described by the delay differential equations by tracing out the continuum of bosonic modes in the waveguide. The spatial separation of the two TLSs introduces the position-dependent phase factor and the time delay (finite time required for light to travel from one TLS to the other) in each transverse mode. The phase factors and the time delays are different in different transverse modes. The effect of the phase factors and the time delays on the entanglement dynamics of the TLSs are studied in details by considering the TLSs interacting with single transverse mode and double transverse modes.

This paper is organized as follows. In Sec. II we introduce the model and establish the notation. In Sec. III we derive the relevant equations describing the dynamics of the system for the case of the TLSs being initially sharing an excitation and the waveguide mode in the vacuum state. In Sec. IV we analyze the behavior of the TLSs’ concurrence when the TLSs interact resonantly with the electromagnetic field of one or two guided modes. We make a conclusion in Sec. V.
II. TWO TLSS IN A RECTANGULAR WAVEGUIDE

The Hamiltonian of the TLSSs interacting with the electromagnetic field of a rectangular waveguide consist of three parts

$$\hat{H} = \hat{H}_a + \hat{H}_f + \hat{H}_{int}. \quad (1)$$

The fist part is the free Hamiltonian of the TLSSs

$$\hat{H}_a = \frac{2}{\hbar} \sum_{l=1}^{2} \omega_l \hat{S}_l^+ \hat{S}_l^- , \quad (2)$$

where $\hat{S}_l^+ \equiv |e_l\rangle \langle g_l|$ and $\hat{S}_l^- \equiv |g_l\rangle \langle e_l|$ is the rising (lowing) atomic operator of the $l$-th TLS. $\omega_l (l = 1, 2)$ are the energy difference between the excited state $|e_l\rangle$ and the ground state $|g_l\rangle$. The rectangular hollow metallic waveguide made of perfect conductor is confined in the $x$-$y$ plane with the area $A = ab$ of its cross section, and translational invariant in the $z$ direction, as shown in Fig. 1. For the convenience of later discussion, we set $a = 2b$. The fields in the rectangular waveguide are classified as transverse magnetic (TM) or transverse electric (TE) according to whether the electric field or magnetic field transverse to the axial direction of the guide. Each guiding mode is characterized by three wave numbers $(m\pi/a, n\pi/b, k)$. Its dispersion relation is given by $\omega_{mnk} = \sqrt{\Omega_{mn}^2 + (ck)^2}$, where $\Omega_{mn} = c\sqrt{(m\pi/a)^2 + (n\pi/b)^2}$ is the cutoff frequency. We note that we only study the role of the guided modes in this paper and the evanescent modes are not considered. The free Hamiltonian of the fields reads

$$\hat{H}_f = \sum_j d\omega j k \hat{a}_j^\dagger \hat{a}_j \quad (3)$$

where $\hat{a}_j^\dagger$ ($\hat{a}_{jk}$) is the creation (annihilation) operator of the $TM_{mn}$ modes. Here, the numbers $(m, n)$ have been replaced with the sequence number $j$, i.e., $j = 1, 2, 3, \ldots$. The reason why only $TM_{mn}$ modes are considered will be given in the following. Two TLSSs, named TLS 1 and TLS 2, are separately located inside the waveguide at positions $\vec{r}_1 = (a/2, b, z_1)$ and $\vec{r}_2 = (a/2, b/2, z_2)$, the distance between the TLSSs is denoted by $d = z_2 - z_1$. We assume the dipoles of TLSSs are along the $z$ axis. In this case, only the $TM_{mn}$ guided modes with odd integer $m$ and $n$ are interacted with the TLSSs. The interaction between the TLSSs and the electromagnetic field is written as

$$\hat{H}_{int} = \frac{2}{\hbar} \sum_{l=1}^{2} \sum_j dk \frac{|g_j|}{\omega_{jk}} e^{ikzk} \hat{S}_l^+ \hat{a}_j^\dagger + h.c. \quad (4)$$

in the electric dipole and rotating wave approximations, where $g_{jk} = \Omega_{jk} \mu_l \sin \frac{m\pi}{A} \sin \frac{n\pi}{B} \epsilon_0$ and $\mu_l$ the magnitude of the dipole of the $l$-th TLS. We assume that $\mu_l$ is real. If the dipoles $\mu_l = \mu$, the parameter $g_{jk}$ is independent of the subscript $l$ and it becomes

$$g_j = \frac{\Omega_{mnk} \mu}{\sqrt{\hbar A \pi}} \sin \frac{m\pi}{2} \sin \frac{n\pi}{2} \quad \epsilon_0 \quad (5)$$

where $\epsilon_0$ is the permittivity of free space, and $j = (1, 1), (3, 1), (5, 1) \ldots \ldots$ in the ascending order.

III. TIME EVOLUTION OF THE TLSS

In the case of a single excitation present in the system, the state vector of the system can be written as

$$|\psi(t)\rangle = b_1 |eg0\rangle + b_2 |ge0\rangle + \sum_j dkb_jg_j^\dagger |gg0\rangle \quad (6)$$

where $|0\rangle$ is the vacuum state of the quantum field, $b_1 (t), b_2 (t)$ is the probability amplitude for TLS $l$ being excited, $b_{jk} (t)$ the probability amplitude for the excitation in a mode $k$ of the TM$_{mn}$ guided mode. The initial state of the system is denoted by the amplitudes $b_1 (0), b_2 (0), b_{jk} (0) = 0$. The Schrödinger equation results in the following coupled equation of the amplitudes

$$b_1 = -i\omega_1 b_1 - \sum_j dkb_jg_j \frac{g_{jk}}{\omega_{jk}} e^{-ikz_1} \quad (7a)$$

$$b_2 = -i\omega_2 b_2 - \sum_j dkb_jg_{jk} \frac{g_{jk}}{\omega_{jk}} e^{-ikz_2} \quad (7b)$$

$$b_{jk} = -i\omega_{jk} b_{jk} + \frac{e^{ikz}}{\sqrt{\omega_{jk}}} (g_{jk} b_1 + g_{jk} g_{jk} e^{ikd}) \quad (7c)$$

We introduce three new variables to remove the high-frequency effect

$$b_1 (t) = B_1 (t) e^{-i\omega_A t}, \quad (8a)$$

$$b_2 (t) = B_2 (t) e^{-i\omega_A t}, \quad (8b)$$

$$b_{jk} (t) = B_{jk} (t) e^{-i\omega_{jk} t}, \quad (8c)$$

FIG. 1: (color online) Schematic illustration for an infinite waveguide of rectangular cross section $A = ab$ (a) coupling to two TLSSs (b) located at $\vec{r}_1 = (a/2, b/2, z)$ and $\vec{r}_2 = (a/2, b/2, z + d)$.
and define the mean frequency of the TLSs as well as the difference of the TLSs’ frequencies
\[ \omega_A = \frac{\omega_2 + \omega_1}{2}, \delta = \frac{\omega_1 - \omega_2}{2} \] (9)

Then, we formally integrate equation of \( B_{jk}(t) \), which is later inserted into the equations for \( B_1(t) \) and \( B_2(t) \). The probability amplitudes for one TLS being excited are determined by two coupled integro-differential equations. Assuming that the frequency \( \omega_A \) is far away from the cutoff frequencies \( \Omega_j \), we can expand \( \omega_{jk} \) around \( \omega_A \) up to the linear term
\[ \omega_{jk} = \omega_A + v_j (k - k_{j0}), \] (10)
where the wavelength of the emitted radiation \( k_{j0} = \sqrt{\omega_A^2 - \Omega_j^2}/c \) is determined by \( \omega_{jk0} = \omega_A \), and the group velocity
\[ v_j \equiv \frac{d\omega_{jk}}{dk}|_{k=k_{j0}} = \frac{c\sqrt{\omega_A^2 - \Omega_j^2}}{\omega_A} \] (11)
is different for different TM guided modes. Integrating over all wave vectors \( k \) gives rise to a linear combination of \( \delta (t - \tau_j) \) and \( \delta (t - \tau_j) \), where \( \tau_j = d/v_j \) is the time that the emitted photon travels from one TLS to the other TLS in the given transverse mode \( j \). The dynamics of two TLSs is governed by the differential equations[23–27]
\[ \begin{align*}
(\partial_t + \Gamma_1 + i\delta) B_1(t) &= -\sum_j \gamma_j e^{i\varphi_j} B_2(t_j) \Theta (t_j|2a) \\
(\partial_t + \Gamma_2 - i\delta) B_2(t) &= -\sum_j \gamma_j e^{i\varphi_j} B_1(t_j) \Theta (t_j|2b)
\end{align*} \]
where we have defined the phase \( \varphi_j = k_{j0}d \) due to the distance between the TLSs, and \( \gamma_j = g_{j1}g_{j2}/(v_j\omega_A) \) are caused by the interaction between the TLSs and the vacuum field in a given transverse mode \( j \), \( \Theta (x) \) is the Heaviside unit step function. The decay rate of the lth TLS to all TM guided modes is denoted by \( \Gamma_l = \sum_j \gamma_j \), where \( \gamma_j = g_{j1}^2g_{j2}/(v_j\omega_A) \) is the decay rate of the lth TLS to the continuum of the TM guided mode, the retard effect has been implied by the symbol \( t_j = t - \tau_j \). Eqs. [12] that the two separate TLSs are coupled after the time \( \min \tau_j \) due to the spontaneous emission from one TLS to the other by the TLSs coupled to the same modes of the vacuum field.

IV. ENTANGLEMENT DYNAMICS OF THE TLS

To measure the amount of the entanglement, we use concurrence as the quantifier[28]. By taking a partial trace over the degrees of freedom of the waveguide, the density matrix of the two TLSs is of an X-form in the two-qubit standard basis \{11, 00, 10, 01\}. The concurrence for this type of state can be calculated easily as
\[ C(t) = \max(0, 2|B_1(t)B_2^*(t)|) \] (13)
for TLSs initially sharing single excitation.

A. single transverse mode

A TLS in its excited state radiates waves into the continua of the modes which are resonant with the TLS. If all TLSs’ energy separations lie within the frequency band between \( \Omega_{11} \) and \( \Omega_{31} \) and are far way from the cutoff frequencies \( \Omega_{11} \) and \( \Omega_{31} \), they only emit photons into the TM guided mode. The equations for the amplitudes of the TLSs read
\[ \begin{align*}
(\partial_t - i\xi_1) B_1(t) &= -\alpha_1 B_2(t) \Theta (t_1|1a) \\
(\partial_t - i\xi_2) B_2(t) &= -\alpha_1 B_1(t) \Theta (t_1|1b)
\end{align*} \]
where \( \xi_1 = i\gamma_{11} - \delta, \xi_2 = i\gamma_{21} + \delta \) and \( \alpha_1 = \gamma_1 e^{i\varphi_1} \). By assuming that the TLSs are excited initially and there is no photons in the field, the Laplace transform of Eqs. [14] leads to
\[ \begin{align*}
B_1(s) &= \frac{(s - i\xi_2) B_1(0) - \alpha_1 e^{-s\tau_1} B_2(0)}{(s - i\xi_2) (s - i\xi_1) - (\alpha_1 e^{-s\tau_1})^2} \Theta (t_1|2a) \\
B_2(s) &= \frac{(s - i\xi_1) B_2(0) - \alpha_1 e^{-s\tau_1} B_1(0)}{(s - i\xi_1) (s - i\xi_2) - (\alpha_1 e^{-s\tau_1})^2} \Theta (t_1|2b)
\end{align*} \]
The integrand in the inverse Laplace transform yields the time-dependent amplitudes of the TLSs. Defining \( t^{(n)} = t - n\tau_1 \), the integrand can be expanded into a power series
\[ \begin{align*}
B_1(t) &= \sum_{n=0}^{\infty} B_1(0) \Theta (t^{(2n)}) [A_n (\xi_1) + B_n (\xi_1)] \Theta (t^{(2n+1)}) [C_n (\xi_1) + C_n (\xi_1)] \\
&\quad - \sum_{n=0}^{\infty} B_2(0) \Theta (t^{(2n+1)}) [C_n (\xi_2) + C_n (\xi_1)] \\
B_2(t) &= \sum_{n=0}^{\infty} B_2(0) \Theta (t^{(2n)}) [A_n (\xi_1) + B_n (\xi_2)] \Theta (t^{(2n+1)}) [C_n (\xi_1) + C_n (\xi_2)] \\
&\quad - \sum_{n=0}^{\infty} B_1(0) \Theta (t^{(2n+1)}) [C_n (\xi_1) + C_n (\xi_2)],
\end{align*} \]
where the functions are defined as
\[ \begin{align*}
A_n (\xi_k) &= \lim_{z \to \xi_k} \frac{d^{n-1}}{dz^{n-1}} \frac{(-i\alpha_1)^{2n} e^{-i\xi_k^{(2n)}}}{(n-1)! (z - \xi_k)^{n+1}} \Theta (t^{(2n)}) \\
B_n (\xi_k) &= \lim_{z \to \xi_k} \frac{d^n}{dz^{n}} \frac{(-i\alpha_1)^{2n+1} e^{-i\xi_k^{(2n+1)}}}{n! (z - \xi_k)^{n+1}} \\
C_n (\xi_k) &= \lim_{z \to \xi_k} \frac{d^n}{dz^{n}} \frac{(-i\gamma_1 e^{i\varphi_1})^{2n+1} e^{-i\xi_k^{(2n+1)}}}{n! (z - \xi_k)^{n+1}},
\end{align*} \]
with subscripts \( k,l \in \{1,2\} \) and \( k \neq l \). The TLSs show different behavior depending on the retardation times \( n\tau_1 \) required for light to travel between two TLSs located at a finite distance, the phase \( \varphi_1 \), the difference \( \delta \) of the TLSs’
frequencies and the difference $\gamma_1 - \gamma_2$ of the decaying factors. For $t \in [0, \tau_1]$, only one term appears

$$B_i(t) = B_i(0)e^{i\xi_it}.$$  \hspace{1cm} (18)

Two TLSs decays as if they are isolated in the waveguide, so the concurrence decay from $|B_1(0)B_2(0)|$ with a rate $\gamma_1 + \gamma_2$. As long as $(\gamma_1 + \gamma_2)\tau_1 > 1$, two TLSs emit photons independently, the photon travels along the waveguide for time $\tau_1$, then the part toward the other TLS will be absorbed, and the TLSs is partially excited, later TLSs reemit the photon again, the whole process of emission and absorption was repeated as time goes on, however no interference occurs, so the phase $\varphi_1$ has no effect on the entanglement. Eq. (18) is also the solution of Eq. (14) with $\tau_1 \to \infty$. If the TLSs decay slowly so that $(\gamma_1 + \gamma_2)\tau_1$ is smaller than or equal to 1, it is possible for a TLS to be aware of the other very soon, interference can be produced by multiple reemissions and reabsorptions of light, which leads to an oscillatory energy exchange between the TLSs. It can be easily seen that the atomic upper state population contains two terms for $t \in [\tau_1, 2\tau_1]$

$$B_1(t) = B_1(0)e^{i\xi_1t} - B_2(0)\alpha_1\frac{e^{i\xi_2(t-\tau_1)} - e^{i\xi_1(t-\tau_1)}}{i(\xi_2 - \xi_1)}$$

$$B_2(t) = B_2(0)e^{i\xi_2t} - B_1(0)\alpha_1\frac{e^{i\xi_2(t-\tau_1)} - e^{i\xi_1(t-\tau_1)}}{i(\xi_2 - \xi_1)}$$

The second term in the last equation shows that the TLS is aware of the other, and presents the absorption and reemission of the other TLS, so interference is possible. As time goes on, multiple reemissions and reabsorptions of photons appear which implied in the summation in Eq. (16), then the phase $\varphi_1$ has the influence on the energy exchange between TLSs.

To show the effect of the energy difference $\delta$ of the uncoupled state $|eg\rangle$ and $|ge\rangle$ on the entanglement dynamics, we consider the dipoles $\mu_1 = \mu$, so the damping rates $\gamma_1 = \gamma_2 = \gamma_1$. It is known in Ref. [29] that there is a dark state for two identical TLSs which is completely isolated and evolves independently, it can preserve the concurrence, the dark state could be the symmetry state $|s\rangle = (|eg\rangle + |ge\rangle)/\sqrt{2}$ and the antisymmetry state $|a\rangle = (|eg\rangle - |ge\rangle)/\sqrt{2}$ could have a maximum damping rate $2\gamma$ when the phase $\varphi_1 = (2n + 1)\pi$; the antisymmetry state $|a\rangle$ is the dark state and the symmetry state $|s\rangle$ could have a maximum damping rate $2\gamma$ when the phase $\varphi_1 = 2n\pi$. In terms of the amplitudes $C_a(t)$ and $C_s(t)$ of the antisymmetry and symmetry state, Eq. (14) can be written as

$$(\partial_t + \gamma_1)C_s(t) = -i\delta C_a(t) - \alpha_1 C_s(t_1)\Theta(t_1)\chi,$$  \hspace{1cm} (20a)

$$(\partial_t + \gamma_1)C_a(t) = -i\delta C_s(t) + \alpha_1 C_a(t_1)\Theta(t_1)\chi.$$  \hspace{1cm} (20b)

In Fig. 2 we plot the concurrence as a function of the time in unit of $\tau_1$ with the TLSs initially in the antisymmetry state $|a\rangle$ for $\gamma_1\lambda_1/v_1 = 1.5 \times 10^{-16}$ where the wavelength $\lambda_1k_{10} = 2\pi$. The evolution of $C(t)$ is profoundly affected by the difference $\delta$, phase $\varphi_1$ and delay time $\tau_1$, where $\varphi_1$ and $\tau_1$ are introduced by the inter-TLS distance. It can be observed from Fig. 2(a) that the evolution of $C(t)$ is independent of the finite propagating time of the light for the delay time $\tau_1 \ll \gamma_1^{-1}$, the two TLSs act collectively. When $\delta = 0$ the antisymmetry state $|a\rangle$ is a dark state which preserve the entanglement among the TLSs; as $\delta$ increases but still smaller than $2\gamma$, the concurrence decreases monotonously as time increases, as $\delta$ is larger than $2\gamma$, the concurrence decreases non-monotonically. The dependence of the entanglement on $\delta$ in Fig. 2(a) can be understood by letting $\tau_1 \to 0$. In
this case, Eq. (20) becomes
\[
\begin{align*}
\partial_t C_s(t) &= -i\delta C_s(t) - 2\gamma_1 C_s(t) \\
\partial_t C_a(t) &= -i\delta C_a(t)
\end{align*}
\]

As long as \( \delta \neq 0 \), the energy difference \( \delta \) of two TLSs introduces the coupling between state \( |s\rangle \) and \( |a\rangle \). Symmetry state \( |s\rangle \) is not only coupled to antisymmetry state \( |a\rangle \) but also coupled to the broad continua of the field, the coupling of the state \( |s\rangle \) to the field introduces the dissipation, which characterized by the damping rate \( 2\gamma \). Energy loss occurs when \( |s\rangle \) is populated. When \( 2\gamma > \delta \), the loss out of the two TLSs is the dominant coupling, the initially unoccupied state \( |a\rangle \) exchanges energy with state \( |s\rangle \), but the energy in state \( |s\rangle \) loses to the field quickly, so it can not be back to state \( |a\rangle \), this is why the concurrence is a monotonically decreasing function of time. When \( 2\gamma < \delta \), the population in state \( |s\rangle \) can transferred back to state \( |a\rangle \), so the concurrence undergoes oscillations before decaying to zero. As the inter-TLS separation increases a little bit to meet \( \tau_1 \sim \gamma_1^{-1} \) in Fig. 2(b), the interference produced by multiple remissions and reabsorptions of photon results in an oscillatory entanglement even when \( \delta = 0 \). However the exchange of population reduces the magnitude of the concurrence. Panel (c) of Fig. 2 illustrates the dynamics of entanglement for a larger inter-TLS distance with \( \tau_1 \gg \gamma_1^{-1} \). It can be observed that at time interval \([0, \tau_1]\), each initially excited TLS emits light to the waveguide, and the entanglement decays exponentially from unity to zero. The radiation field emitted into the waveguide returns to the TLSs after \( \tau_1 \), then the entanglement is created. But, the periodic maxima of the concurrence are in magnitude as time increases due to the energy loss carried away by the forward-going and the backward-going waves. Population exchange introduced by the energy difference \( \delta \) further lower the periodic maxima of the concurrence, however, oscillations can be observed when \( \delta > 2\gamma \) after time \( \tau_1 \). We would like to note that non-vanishing \( \delta \) can also raise the transient behaviors of the concurrence if the TLSs are initially in the symmetry state \( |s\rangle \) with \( \varphi_1 = 2n\pi \), as shown in Fig. 3.

**B. two transverse modes**

As the energy splitting of both TLS increases so that they are much larger than the cutoff frequency \( \Omega_{31} \) and much smaller than \( \Omega_{51} \), the TLSs interact with the field of both TM13 and TM31 guided modes. For dipoles \( \mu = \mu \), the equations for the amplitudes of the symmetry and antisymmetry states reads
\[
\begin{align*}
\partial_t C_s(t) + \Gamma C_s(t) + i\delta C_a(t) &= -\alpha_1 C_s(t_1) \Theta (t_1) - \alpha_2 C_s(t_2) \Theta (t_2) \\
\partial_t C_a(t) + \Gamma C_a(t) + i\delta C_s(t) &= \alpha_1 C_a(t_1) \Theta (t_1) + \alpha_2 C_a(t_2) \Theta (t_2)
\end{align*}
\]

where \( \Gamma = \gamma_1 + \gamma_2 \) and \( \alpha_j = \gamma_j e^{i\varphi_j} \) \( (j = 1, 2) \). The definitions of delay time \( \tau_j \) and phase \( \varphi_j \) indicate that \( \tau_j < \tau_{j+1} \) and \( \varphi_j < \varphi_{j+1} \) for a given TLS’s separation. Through an inspection of Eq. (21) for \( d = 0 \), it can be found that the antisymmetry state \( |a\rangle \) is still a dark state and the symmetry state \( |s\rangle \) has a maximum damping rate \( 2\Gamma \). As \( d \) increases a little bit but still satisfying \( \varphi_1 = 2n\pi \) and the energy difference \( \delta = 0 \), state \( |a\rangle \) is no longer a dark state, it damps with a damping rate \( \gamma_2 - \text{Re} \omega_2 \) in which is smaller than \( 2\Gamma \), but there is an energy splitting \( 2\text{Im} \omega_2 \) between the two states. When \( \delta \neq 0 \), the two states are coupled to each other, so the oscillation strength is changed from \( \Omega = \delta \) in one guiding mode to \( \Omega = \sqrt{\delta^2 + (\text{Im} \omega_2)^2} \) in two guiding modes, the damping rates of the two states are also changed. As long as the oscillation strength \( \Omega \) is larger than the sum \( 2\Gamma \), the population will oscillate obviously when the effect of the phase on the dynamics is more important than the delay time.

In Fig. 4, we have plotted the concurrence between the TLSs as a function of the dimensionless time \( t/\tau_1 \) with the TLSs initially in the antisymmetry state \( |a\rangle \). Panel (a) shows the entanglement dynamics when the inter-TLS distance \( d = 0 \) with different \( \delta \), it can be found that the concurrence remain its initial value when \( \delta = 0 \), however, as \( \delta \) increases until \( 2\Gamma \), the faster the population changes between the two states, the faster the concurrence decays. As \( \delta \) increases further, i.e. more than \( 2\Gamma \), there is an oscillation. In panel (b), the delay time \( \tau_2 \ll \gamma_2^{-1} \), which means there is no delay in the absorption of the energy by another TLS in both TM11 and TM31 modes. The antisymmetry state interacts with the field in TM31 guiding mode due to the phase \( \varphi_2 \neq 2n\pi \), the concurrence undergoes an exponential decay when \( \delta = 0 \). Although the concurrence is further decreased by the increasing of \( \delta \), its evolution deviates from the exponential decay. The dotted blue line in panel (c) exhibits a behavior different from that in panel (b), which indicates that phases and delay times play an equal role. In the interval \([0, \tau_2]\), the concurrence exponentially decay with a rate \( \Gamma \) up to time \( \tau_1 \). After this, it still decreases but de-
The antisymmetry state

This equation can also explain the exponential decay of phase

viates from the exponential decay, which means that the phase \( \varphi_1 \) begins to have an effect until time \( t = \tau_2 \). After time \( \tau_2 \), the dynamics can be dramatically affected by the phases \( \varphi_j \), delay times \( \tau_j \) when \( \delta = 0 \). However, the energy difference \( \delta \) can increase the entanglement. We note that in the interval \([0, \tau_1]\), the exponential decay of the concurrence is independent of \( \delta \), but it is possible for the population of state \( |a\rangle \) to present an oscillating behavior as shown in Fig. 4 since the amplitudes obey the following equation

\[
\begin{align*}
\partial_t C_a(t) + \Gamma C_a(t) + i\delta C_a(t) &= 0 & (22a) \\
\partial_t C_b(t) + \Gamma C_b(t) + i\delta C_b(t) &= 0 & (22b)
\end{align*}
\]

This equation can also explain the exponential decay of panel (d) of Fig. 4 in the time interval \([0, \tau_1]\). Actually, Eq. (22) also describes the dynamics of TLSs when the delay time \( \tau_1 \to \infty \), it indicates that two TLSs emit photons independently, and the emitted photon travels along the waveguide and is never absorbed by the TLSs. For two TLSs far apart as shown in Fig. 4(d), the phase \( \varphi_2 \) do not make an sense, and the increase of \( \delta \) lowers the magnitude of the concurrence. Although the emitted photon could be absorbed by the TLS resulting the birth of entanglement, but the revival time becomes \( p\tau_1 + q\tau_2 \) with \( p, q \) are integer, which can be obtained by performing the Laplace transformation on Eq. (12) with \( j = 1, 2 \).

V. CONCLUSION

We have studied the entanglement dynamics of two distinguishable TLSs characterized by energy difference \( \delta \) located inside a rectangular hollow metallic waveguide of transverse dimensions \( a \) and \( b \). The effects of energy difference \( \delta \) and the inter-TLS distance on the time evolution of the concurrence of the TLSs are examined in the single excitation subspace when the energy separation of the TLS is far away from the cutoff frequencies of the transverse mode. The inter-TLS distance induces phase factors and delay times in the delay differential equations. The energy difference introduces the coupling between the symmetry and antisymmetry state. For the inter-TLS distance \( d = 0 \), the entanglement can be trapped in the antisymmetry state when \( \delta = 0 \) since the antisymmetry state is decoupled with the guiding mode, however, the population exchange induced by non-vanishing \( \delta \) decreases the entanglement from one to zero. As \( d \) increases to satisfy \( \max\{\tau_j\} \ll \gamma_j^{-1} \), two TLSs behave collectively. It is well known that a change of phase leads to an enhanced or inhibited exponential decay of the concurrence, however, \( \delta \) makes the dynamics of the concurrence deviating from the exponential decay. As \( d \) increases further so that \( \max\{\tau_j\} \approx \gamma_j^{-1} \), although the interference produced by multiple reemissions and reabsorptions of photon results in the dynamic behavior of the entanglement deviating from the exponential decay, non-zero \( \delta \) can raise the entanglement in transient as time increases. When \( \tau_j \gg \gamma_j^{-1} \), an increasing of \( \delta \) only lower the entanglement.
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