Analytic Bethe ansatz and functional equations for
Lie superalgebra $sl(r + 1|s + 1)$

Zengo Tsuboi
Institute of Physics, Graduate School of Arts and Sciences
University of Tokyo, Komaba
3-8-1 Komaba, Meguro-ku, Tokyo 153, JAPAN

Abstract. From the point of view of the Young superdiagram method, an analytic
Bethe ansatz is carried out for Lie superalgebra $sl(r + 1|s + 1)$. For the transfer matrix
eigenvalue formulae in dressed vacuum form, we present some expressions, which
are quantum analogue of Jacobi-Trudi and Giambelli formulae for Lie superalgebra
$sl(r + 1|s + 1)$. We also propose transfer matrix functional relations, which are Hirota
bilinear difference equation with some constraints.

Report number: UT-Komaba/97-8
Journal-ref: J. Phys. A: Math. Gen. 30 (1997) 7975-7991
DOI: 10.1088/0305-4470/30/22/031
1. Introduction

In [KNS1], a class of functional relations, the T-system, was proposed. It is a family of functional relations for a set of commuting transfer matrices of solvable lattice models associated with any quantum affine algebras \( U_q(G_r^{(1)}) \). Using T-system, we can calculate various physical quantities [KNS2] such as the correlation lengths of the vertex models and central charges of RSOS models. The T-system is not only a family of transfer matrix functional relations but also two-dimensional Toda field equation on discrete space time. And it has beautiful pfaffian and determinant solutions [KOS,KNH,TK] (see also, [T]).

In [KS1], analytic Bethe ansatz [R1] was carried out for fundamental representations of the Yangians \( Y(G)[D] \), where \( G = B_r, C_r \) and \( D_r \). That is, eigenvalue formulas in dressed vacuum form were proposed for the transfer matrices of solvable vertex models. These formulae are Yangian analogues of the Young tableaux for \( G \) and satisfy certain semi-standard like conditions. It had been proven that they are free of poles under the Bethe ansatz equation. Furthermore, for \( G = B_r \) case, these formulae were extended to the case of finite dimensional modules labeled by skew-Young diagrams \( \lambda \subset \mu \) [KOS]. In analytic Bethe ansatz context, above-mentioned solutions of the T-system correspond to the eigenvalue formulae of the transfer matrices in dressed vacuum form labeled by rectangular-Young diagrams \( \lambda = \phi, \mu = (m^a) \) (see also, [BR,KLWZ,K,KS2,S2]).

The purpose of this paper is to extend similar analyses to Lie superalgebra \( G = sl(r + 1|s + 1) \) [Ka] case (see also [C] for comprehensible account on Lie superalgebras). Throughout this paper, we frequently use similar notation presented in [KS1,KOS,TK]. Studying supersymmetric integrable models is important not only in mathematical physics but also in condensed matter physics (cf.[EK,FK,KE,S1,ZB]). For example, the supersymmetric \( t-J \) model received much attention in connection with high \( T_c \) superconductivity. In the supersymmetric models, the R-matrix satisfies the graded Yang-Baxter equation [KulSk]. The transfer matrix is defined as a super trace of monodromy matrix. As a result, extra signs appear in the Bethe ansatz equation and eigenvalue formula of the transfer matrix.

There are several inequivalent choices of simple root system for Lie superalgebra. We treat so-called distinguished simple root system [Ka] in the main text. We introduce the Young superdiagram [BB1], which is associated with a covariant tensor representation. To be exact, this Young superdiagram is different from the classical one in that it carries spectral parameter \( u \). In contrast to ordinary Young diagram, there is no restriction on the number of rows. We define semi-standard like tableau on it. Using this tableau, we introduce the function \( T_{\lambda \subset \mu}^{(a)}(u) \) (2.15). This should be fusion transfer matrix of dressed vacuum form in the analytic Bethe ansatz. We prove pole-freeness of \( T^{(a)}(u) = T_{(1^a)}(u) \), crucial property of analytic Bethe ansatz. Due to the same mechanism presented in [KOS], the function \( T_{\lambda \subset \mu}(u) \) has determinant expression whose matrix elements are only the functions associated with Young superdiagrams with shape \( \lambda = \phi; \mu = (m) \) or \( (1^a) \). It can be viewed as quantum analogue of Jacobi-Trudi and
Giambelli formulae for Lie superalgebra \( sl(r + 1|s + 1) \). Then one can easily show that the function \( T_{\lambda \subseteq \mu}(u) \) is free of poles under the Bethe ansatz equation (2.6a). Among the above-mentioned eigenvalue formulae of transfer matrix in dressed vacuum form associated with rectangular Young superdiagrams, we present a class of transfer matrix functional relations. It is a special case of Hirota bilinear difference equation [H].

Deguchi and Martin [DM] discussed the spectrum of fusion model from the point of view of representation theory (see also, [MR]). The present paper will partially give us elemental account on their result from the point of view of the analytic Bethe ansatz.

The outline of this paper is given as follows. In section 2, we execute analytic Bethe ansatz based upon the Bethe ansatz equation (2.6a) associated with the distinguished simple roots. The observation that the Bethe ansatz equation can be expressed by root system of Lie algebra is traced back to [RW] (see also, [Kul] for \( sl(r + 1|s + 1) \) case). Moreover, Kuniba et.al.[KOS] conjectured that left hand side of the Bethe ansatz equation (2.6a) can be written as a ratio of certain ‘Drinfeld polynomials’ [D]. We introduce the function \( T_{\lambda \subseteq \mu}(u) \), which should be the transfer matrix whose auxiliary space is finite dimensional module of super Yangian \( Y(sl(r + 1|s + 1)) \) [N] or quantum affine superalgebra \( U_q(sl(r + 1|s + 1)) \) [Y], labeled by skew-Young superdiagram \( \lambda \subseteq \mu \). The origin of the function \( T^1(u) \) goes back to the eigenvalue formula of transfer matrix of Perk-Schultz model [PS1,PS2,Sc], which is a multi-component generalization of the six-vertex model (see also [Kul]). In addition, the function \( T^1(u) \) reduces to the eigenvalue formula of transfer matrix derived by algebraic Bethe ansatz (For example, [FK]: \( r = 1, s = 0 \) case; [EK]: \( r = 0, s = 1 \) case; [EKS1,EKS2]: \( r = s = 1 \) case). In section 3, we propose functional relations, the T-system, associated with the transfer matrices in dressed vacuum form defined in the previous section. Section 4 is devoted to summary and discussion. In appendix A, we briefly mentioned relation between fundamental \( L \) operator and the transfer matrix. In this paper, we treat mainly the expressions related to covariant representations. For contravariant ones, we present several expressions in Appendix B. Appendix C and D provide some expressions related to non-distinguished simple roots of \( sl(1|2) \). Appendix E explains how to represent the eigenvalue formulae of transfer matrices in dressed vacuum form \( T_m(u) \) and \( T^a(u) \) in terms of the functions \( A_m(u), A^a(u), B_m(u) \) and \( B^a(u) \), which are analogous to the fusion transfer matrices of \( U_q(G^{(1)}) \) vertex models \( (G = sl_{r+1}, sl_{s+1}) \).

2. Analytic Bethe ansatz

Lie superalgebra [Ka] is a \( \mathbb{Z}_2 \) graded algebra \( G = G_0 \oplus G_1 \) with a product \([ , , ]\), whose homogeneous elements \( a \in G_\alpha, b \in G_\beta \ (\alpha, \beta \in \mathbb{Z}_2 = \{0, 1\}) \) and \( c \in G \) satisfy the following relations.

\[
[a, b] \in G_{\alpha + \beta}, \\
[a, b] = -(-1)^{\alpha \beta} [b, a], \\
[a, [b, c]] = [[a, b], c] + (-1)^{\alpha \beta} [b, [a, c]].
\] (2.1)
Analytic Bethe ansatz

The set of non-zero roots can be divided into the set of non-zero even roots (bosonic roots) \( \Delta'_0 \) and the set of odd roots (fermionic roots) \( \Delta_1 \). For \( \mathfrak{sl}(r+1|s+1) \) case, they read

\[
\Delta'_0 = \{ \epsilon_i - \epsilon_j \} \cup \{ \delta_i - \delta_j \}, \quad \Delta_1 = \{ \pm (\epsilon_i - \delta_j) \}
\]

where \( \epsilon_1, \ldots, \epsilon_{r+1}; \delta_1, \ldots, \delta_{s+1} \) are basis of dual space of the Cartan subalgebra with the bilinear form (\( \cdot \)) such that

\[
(\epsilon_i | \epsilon_j) = \delta_{ij}, \quad (\epsilon_i | \delta_j) = (\delta_i | \epsilon_j) = 0, \quad (\delta_i | \delta_j) = -\delta_{ij}.
\]

There are several choices of simple root system reflecting choices of Borel subalgebra. The simplest system of simple roots is so called distinguished one [Ka] (see, figure 1). Let \( \{ \alpha_1, \ldots, \alpha_{r+s+1} \} \) be the distinguished simple roots of Lie superalgebra \( \mathfrak{sl}(r+1|s+1) \)

\[
\begin{align*}
\alpha_i &= \epsilon_i - \epsilon_{i+1} \quad i = 1, 2, \ldots, r, \\
\alpha_{r+1} &= \epsilon_{r+1} - \delta_1 \\
\alpha_{j+r+1} &= \delta_j - \delta_{j+1}, \quad j = 1, 2, \ldots, s
\end{align*}
\]

and with the grading

\[
\deg(\alpha_a) = \begin{cases} 
0 & \text{for even root} \\
1 & \text{for odd root}
\end{cases}
\]

Especially for distinguished simple root, we have \( \deg(\alpha_a) = \delta_{a,r+1} \).

We consider the following type of the Bethe ansatz equation (cf. [Kul,RW,KOS]).

\[
-\frac{P_a(u_k^{(a)} + \frac{1}{t_a})}{P_a(u_k^{(a)} - \frac{1}{t_a})} = (-1)^{\deg(\alpha_a)} \prod_{b=1}^{r+s+1} \frac{Q_b(u_k^{(a)} + (\alpha_a | \alpha_b))}{Q_b(u_k^{(a)} - (\alpha_a | \alpha_b))},
\]

\[
Q_a(u) = \prod_{j=1}^{N_a} [u - u_j^{(a)}],
\]

\[
P_a(u) = \prod_{j=1}^{N} P_a^{(j)}(u),
\]

\[
P_a^{(j)}(u) = [u - w_j]^{\delta_{a,1}},
\]

where \( [u] = (q^u - q^{-u})/(q - q^{-1}) \); \( N_a \in \mathbb{Z}_{\geq 0} \); \( u, w_j \in \mathbb{C} \); \( a, k \in \mathbb{Z} \) (\( 1 \leq a \leq r + s + 1, 1 \leq k \leq N_a \)); \( t_a = 1 \) for \( 1 \leq a \leq r + 1 \); \( t_a = -1 \) for \( r + 2 \leq a \leq r + s + 1 \). In this paper, we suppose that \( q \) is generic. The left hand side of the Bethe ansatz equation (2.6a) is related to the quantum space. We suppose that it is given by the ratio of some ‘Drinfeld polynomials’ labeled by skew-Young diagrams \( \mu \subset \mu \) (cf.[KOS]). For

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Dynkin_diagram.png}
\caption{Dynkin diagram for the Lie superalgebra \( \mathfrak{sl}(r+1|s+1) \) corresponding to the distinguished simple roots: white circle denotes even root \( \alpha_i \); grey (a cross) circle denotes odd root \( \alpha_j \) with \( \langle \alpha_j | \alpha_j \rangle = 0 \).}
\end{figure}
simplicity, we consider only the case $\tilde{\lambda} = \phi, \tilde{\mu} = (1)$. The generalization to the case for any skew-Young diagram will be achieved by the empirical procedures mentioned in [KOS]. The factor $(-1)^{\deg(a_{a})}$ of the Bethe ansatz equation \(2.6\) appears so as to make the transfer matrix to be a super trace of monodromy matrix. We define the sets

$$J = \{1, 2, \ldots, r + s + 2\}, \quad J_+ = \{1, 2, \ldots, r + 1\},$$
$$J_- = \{r + 2, r + 3, \ldots, r + s + 2\},$$

with the total order

$$1 \prec 2 \prec \cdots \prec r + s + 2$$

and with the grading

$$p(a) = \begin{cases} 0 & \text{for } a \in J_+ \\ 1 & \text{for } a \in J_- \end{cases}$$

For $a \in J$, set

$$z(a; u) = \psi_a(u) \frac{Q_a-1(u + a + 1)Q_a(u + a - 2)}{Q_a-1(u + a - 1)Q_a(u + a)} \quad \text{for } a \in J_+,$$
$$z(a; u) = \psi_a(u) \frac{Q_a-1(u + 2r - a + 1)Q_a(u + 2r - a + 4)}{Q_a-1(u + 2r - a + 3)Q_a(u + 2r - a + 2)} \quad \text{for } a \in J_-,$$

where $Q_0(u) = 1, Q_{r+s+2}(u) = 1$ and

$$\psi_a(u) = \begin{cases} P_1(u + 2) & \text{for } a = 1 \\ P_1(u) & \text{for } a \in J - \{1\} . \end{cases}$$

In this paper, we often express the function $z(a; u)$ as the box $[a]_u$, whose spectral parameter $u$ will often be abbreviated. Under the Bethe ansatz equation, we have

$$\text{Res}_{u=-b+u_k^b} (z(b; u) + z(b + 1; u)) = 0 \quad 1 \leq b \leq r \quad (2.12a)$$
$$\text{Res}_{u=-r-1+u_k^{r+1}} (z(r + 1; u) - z(r + 2; u)) = 0 \quad (2.12b)$$
$$\text{Res}_{u=-2r-2+b+u_k^b} (z(b; u) + z(b + 1; u)) = 0 \quad r + 2 \leq b \leq r + s + 1 \quad (2.12c)$$

We will use the functions $T^a(u)$ and $T_m(u)$ ($a \in \mathbb{Z} ; m \in \mathbb{Z} ; u \in \mathbb{C}$) determined by the following generating series

$$(1 + z(r + s + 2; u)X)^{-1} \cdots (1 + z(r + 2; u)X)^{-1} (1 + z(r + 1; u)X) \cdots (1 + z(1; u)X)$$

$$= \sum_{a=-\infty}^{\infty} \mathcal{F}^a(u + a - 1)T^a(u + a - 1)X^a,$$

$$\mathcal{F}^a(u) = \begin{cases} 1 & \text{for } a \geq 2 \\ \frac{1}{P_1(u-1)^{a-1}} & \text{for } a = 1 \\ \frac{1}{P_1(u-1)} & \text{for } a = 0 \\ 0 & \text{for } a \leq -1 \end{cases} \quad (2.13b)$$

$$(1 - z(1; u)X)^{-1} \cdots (1 - z(r + 1; u)X)^{-1} (1 - z(r + 2; u)X) \cdots (1 - z(r + s + 2; u)X)$$

$$= \sum_{m=-\infty}^{\infty} T_m(u + m - 1)X^m,$$
where $X$ is a shift operator $X = e^{2\hbar u}$. In particular, we have $T_0(u) = P_1(u - 1)$; $T_0(u) = 1$; $T_a(u) = 0$ for $a < 0$; $T_m(u) = 0$ for $m < 0$. We remark that the origin of the function $T^1(u)$ and the Bethe ansatz equation (2.6a) traces back to the eigenvalue formula of transfer matrix and the Bethe ansatz equation of Perk-Schultz model [Sc] except the vacuum part, some gauge factors and extra signs after some redefinition. (See also, [Kul]).

Let $\lambda \subset \mu$ be a skew-Young superdiagram labeled by the sequences of non-negative integers $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ such that $\mu_i \geq \lambda_i : i = 1, 2, \ldots$; $\lambda_1 \geq \lambda_2 \geq \ldots \geq 0$; $\mu_1 \geq \mu_2 \geq \ldots \geq 0$ and $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ be the conjugate of $\lambda$ (see, figure 2 and 3). On this skew-Young superdiagram $\lambda \subset \mu$, we assign a coordinates $(i, j) \in \mathbb{Z}^2$ such that the row index $i$ increases as we go downwards and the column index $j$ increases as we go from left to right and that $(1, 1)$ is on the top left corner of $\mu$. Define an admissible tableau $b$ on the skew-Young superdiagram $\lambda \subset \mu$ as a set of element $b(i, j) \in J$ labeled by the coordinates $(i, j)$ mentioned above, obeying the following rule (admissibility conditions).

(i) For any elements of $J_+$, $b(i, j) \prec b(i + 1, j)$ (2.14a)

(ii) for any elements of $J_-$, $b(i, j) \prec b(i, j + 1)$, (2.14b)
(iii) and for any elements of $J$,

$$b(i, j) \preceq b(i, j + 1), \quad b(i, j) \preceq b(i + 1, j).$$

Let $B(\lambda \subset \mu)$ be the set of admissible tableaux on $\lambda \subset \mu$. For any skew-Young superdiagram $\lambda \subset \mu$, define the function $T_{\lambda \subset \mu}(u)$ as follows

$$T_{\lambda \subset \mu}(u) = \frac{1}{\mathcal{F}_{\lambda \subset \mu}(u)} \sum_{b \in B(\lambda \subset \mu)} \prod_{(i, j) \in (\lambda \subset \mu)} (-1)^{\nu(b(i, j))}z(b(i, j); u - \mu_1 + \mu'_1 - 2i + 2j)$$

where the product is taken over the coordinates $(i, j)$ on $\lambda \subset \mu$ and

$$\mathcal{F}_{\lambda \subset \mu}(u) = \prod_{j=1}^{\mu_1} \mathcal{F}_j^{-\lambda_j'}(u + \mu'_1 - \mu_1 - \mu'_j - \lambda'_j + 2j + 1).$$

In particular, for an empty diagram $\phi$, set $T_\phi(u) = \mathcal{F}_\phi(u) = 1$. The following relations should be valid by the same reason mentioned in [KOS], that is, they will be verified by induction on $\mu_1$ or $\mu'_1$.

$$T_{\lambda \subset \mu}(u) = \text{det}_{1 \leq i, j \leq \mu_1}(T_\phi^{-\lambda_j-i+j}(u - \mu_1 + \mu'_1 - \mu'_i - \lambda'_j + i + j - 1))$$

$$= \text{det}_{1 \leq i, j \leq \mu_1}(T_{\phi^{-\lambda_i} - \lambda_i - j}(u - \mu_1 + \mu'_1 + \mu_j + \lambda_i - i + j - 1))$$

For example, $\lambda = \phi, \mu = (2^2), r = 1, s = 0$ case, we have

$$T_{(2^2)}(u) = \frac{1}{\mathcal{F}_{(2^2)}(u)} \left[ \begin{array}{c|c} 1 & 1 \\ \hline 2 & 1 \\ 2 & 3 \\ \hline 1 & 2 \\ 3 & 3 \end{array} \right]$$

$$= P_1(u + 2)P_1(u + 4) \frac{Q_2(u - 2)Q_1(u + 1)Q_2(u + 2)}{Q_2(u + 2)Q_1(u + 3)Q_2(u + 2)} - P_1(u + 2)P_1(u + 4) \frac{Q_1(u + 1)Q_2(u - 2)}{Q_1(u + 3)Q_2(u + 2)}$$

$$- P_1(u + 2)^2\frac{Q_1(u + 5)Q_2(u - 2)}{Q_1(u + 3)Q_2(u + 4)} + P_1(u + 2)^2 \frac{Q_2(u - 2)}{Q_2(u + 4)}$$

$$= \begin{vmatrix} T^2(u - 1) & T^3(u) \\ T^1(u) & T^2(u + 1) \end{vmatrix}$$

where

$$T^1(u) = \left[ \begin{array}{c} 1 \\ 2 \\ -3 \end{array} \right]$$

$$= P_1(u + 2) \frac{Q_1(u - 1)}{Q_1(u + 1)} + P_1(u) \frac{Q_1(u + 3)Q_2(u)}{Q_1(u + 1)Q_2(u + 2)} - P_1(u) \frac{Q_2(u)}{Q_2(u + 2)}.$$  

$$T^2(u) = \frac{1}{\mathcal{F}^2(u)} \left[ \begin{array}{c|c} 1 & 1 \\ \hline 2 & 3 \\ 3 & 3 \end{array} \right]$$

$$= P_1(u + 3) \frac{Q_2(u - 1)}{Q_2(u + 1)} - P_1(u + 3) \frac{Q_1(u)Q_2(u - 1)}{Q_1(u + 2)Q_2(u + 1)}$$

$$- P_1(u + 1) \frac{Q_1(u + 4)Q_2(u - 1)}{Q_1(u + 2)Q_2(u + 3)} + P_1(u + 1) \frac{Q_2(u - 1)}{Q_2(u + 3)}.$$  

$$T^3(u) = \frac{1}{\mathcal{F}^3(u)} \left[ \begin{array}{c|c} 1 & 1 \\ \hline 2 & 3 \\ 3 & 3 \end{array} \right]$$

$$= -P_1(u + 4) \frac{Q_2(u - 2)}{Q_2(u + 2)} + P_1(u + 4) \frac{Q_1(u + 1)Q_2(u - 2)}{Q_1(u + 3)Q_2(u + 2)}.$$
Analytic Bethe ansatz

\[ + P_1(u + 2) \frac{Q_1(u + 5) Q_2(u - 2)}{Q_1(u + 3) Q_2(u + 4)} - P_1(u + 2) \frac{Q_2(u - 2)}{Q_2(u + 4)}. \]

Remark 1: If we drop the \( u \) dependence of (2.17a) and (2.17b), they reduce to classical Jacobi-Trudi and Giambelli formulae for \( sl(r+1|s+1) \) [BB1,PT], which bring us classical (super) characters.

Remark 2: In the case \( \lambda = \phi \) and \( s = -1 \), (2.17a) and (2.17b) correspond to quantum analogue of Jacobi-Trudi and Giambelli formulae for \( sl_r \) [BR].

Remark 3: (2.17a) and (2.17b) have the same form as the quantum Jacobi-Trudi and Giambelli formulae for \( U_q(B_n^{(1)}) \) in [KOS], but the function \( T^a(u) \) is quite different.

Theorem 2.1 For any integer \( a \), the function \( T^a(u) \) is free of poles under the condition that the Bethe ansatz equation (2.6a) is valid.

At first, we present a lemma which is necessary for the proof of Theorem 2.1. Lemma 2.2 is \( sl(r+1|s+1) \) version of Lemma 3.3.2. in [KS1] and follows straightforwardly from the definitions of \( z(a; u) \) (2.10).

Lemma 2.2 For any \( b \in J_+ - \{r + 1\} \), the function

\[
\begin{array}{c}
| b \\
| b + 1 \\
| u \\
| u - 2
\end{array}
\]

(2.22)

does not contain the function \( Q_b \) (2.6b).

Proof of Theorem 2.1 For simplicity, we assume that the vacuum parts are formally trivial, that is, the left hand side of the Bethe ansatz equation (2.6a) is constantly \(-1\). We prove that \( T^a(u) \) is free of color \( b \) pole, namely, \( \text{Res}_{u=u_k} \cdots T^a(u) = 0 \) for any \( b \in J - \{r + s + 2\} \) under the condition that the Bethe ansatz equation (2.6a) is valid.

The function \( z(c; u) = \begin{array}{c} c \end{array} \) with \( c \in J \) has the color \( b \) pole only for \( c = b \) or \( b + 1 \), so we shall trace only \( \begin{array}{c} b \\
| b + 1 \end{array} \) or \( \begin{array}{c} b \\
| b + 1 \end{array} \). Denote \( S_k \) the partial sum of \( T^a(u) \), which contains \( k \) boxes among \( \begin{array}{c} b \\
| b + 1 \end{array} \). Apparently, \( S_0 \) does not have color \( b \) pole. This is also the case with \( S_2 \) for \( b \in J_+ - \{r + 1\} \) since the admissible tableaux have the same subdiagrams as in (2.22) and thus do not involve \( Q_b \) by lemma 2.2. Now we examine \( S_1 \) which is the summation of the tableaux of the form

\[
\begin{array}{c}
| \xi \\
| b \\
| \zeta \\
| b + 1 \\
| \zeta
\end{array}
\]

where \( \begin{array}{c} \xi \\
| \zeta \end{array} \) and \( \begin{array}{c} \xi \\
| \zeta \end{array} \) are columns with total length \( a - 1 \) and they do not involve \( b \) and \( b + 1 \). Thanks to the relations (2.12a, 2.12a), color \( b \) residues in these tableaux (2.23) cancel each other under the Bethe ansatz equation (2.6a). Then we deal with \( S_k \) only for \( 3 \leq k \leq a \) and \( k = 2 \) with \( b \in J_+ - \{r + 1\} - \{r + s + 2\} \) from now on. In this case, only the case for \( b \in \{r + 1\} \cup J_- - \{r + s + 2\} \) should be considered because, in the
Analytic Bethe ansatz

case for \( b \in J_+ - \{ r + 1 \} \), \( b \) or \( b + 1 \) appear at most twice in one column. The case \( b = r + 1 \): \( S_k(k \geq 2) \) is the summation of the tableaux of the form

\[
\begin{array}{c|c}
& v \\
\xi & r + 1 \\
r + 2 & v - 2 \\
\vdots & \\
r + 2 & v - 2k + 2 \\
\zeta & \\
\end{array} = \frac{Q_{r+1}(v + r - 2k + 1)Q_{r+2}(v + r)}{Q_{r+1}(v + r + 1)Q_{r+2}(v + r + 2)}X_3
\tag{2.24}
\]

and

\[
\begin{array}{c|c}
& v \\
\xi & r + 2 \\
r + 2 & v - 2 \\
\vdots & \\
r + 2 & v - 2k + 2 \\
\zeta & \\
\end{array} = \frac{Q_{r+1}(v + r - 2k + 1)Q_r(v + r)}{Q_{r+1}(v + r + 1)Q_r(v + r + 2)}X_3
\tag{2.25}
\]

where \( \xi \) and \( \zeta \) are columns with total length \( a - k \), which do not contain \( r + 1 \) and \( r + 2 \); \( v = u + h_1 \): \( h_1 \) is some shift parameter; the function \( X_3 \) does not contain the function \( Q_{r+1} \). Obviously, color \( b = r + 1 \) residues in the (2.24) and (2.25) cancel each other under the Bethe ansatz equation (2.6a).

The case \( b \in J_+ - \{ r + s + 2 \} \): \( S_k(k \geq 2) \) is the summation of the tableaux of the form

\[
f(k, n, \xi, \zeta, u) := \begin{array}{c|c}
& v \\
\xi & b \\
\vdots & \\
\zeta & b + 1 \\
\end{array} = \frac{Q_{b-1}(v + 2r + 3 - 2n - b)Q_b(v + 2r + 4 - b)}{Q_{b-1}(v + 2r + 3 - b)Q_b(v + 2r + 4 - 2n - b)}
\times \frac{Q_b(v + 2r + 2 - 2k - b)Q_{b+1}(v + 2r + 3 - 2n - b)}{Q_b(v + 2r + 2 - 2n - b)Q_{b+1}(v + 2r + 3 - 2k - b)}X_4, \quad 0 \leq n \leq k
\tag{2.26}
\]

where \( \xi \) and \( \zeta \) are columns with total length \( a - k \), which do not contain \( b \) and \( b + 1 \); \( v = u + h_2 \): \( h_2 \) is some shift parameter and is independent of \( n \); the function \( X_4 \) does not have color \( b \) pole and is independent of \( n \). \( f(k, n, \xi, \zeta, u) \) has color \( b \) poles at \( u = -h_2 - 2r - 2 + b + 2n + u_p^{(b)} \) and \( u = -h_2 - 2r - 4 + b + 2n + u_p^{(b)} \) for \( 1 \leq n \leq k - 1 \); at \( u = -h_2 - 2r - 2 + b + u_p^{(b)} \) for \( n = 0 \); at \( u = -h_2 - 2r - 4 + b + 2k + u_p^{(b)} \) for \( n = k \). Evidently, color \( b \) residue at \( u = -h_2 - 2r - 2 + b + 2n + u_p^{(b)} \) in \( f(k, n, \xi, \zeta, u) \) and \( f(k, n + 1, \xi, \zeta, u) \) cancel each other under the Bethe ansatz equation (2.6a). Thus,
under the Bethe ansatz equation \( (2.6a) \), \( \sum_{n=0}^{k} f(k, n, \xi, \zeta, u) \) is free of color \( b \) poles, so is \( S_k \).

Applying Theorem 2.1 to \( (2.17a) \), one can show that \( T_{\lambda\subset\mu}(u) \) is free of poles under the Bethe ansatz equation \( (2.6a) \). The function \( T_{\lambda\subset\mu}(u) \) should express the eigenvalue of the transfer matrix whose auxiliary space \( W_{\lambda\subset\mu}(u) \) is labeled by the skew-Young superdiagram with shape \( \lambda \subset \mu \). We assume that \( W_{\lambda\subset\mu}(u) \) is a finite dimensional module of the super Yangian \( \mathcal{Y}(sl(r+1|s+1)) \) \([N]\) (or quantum super affine algebra \( U_q(sl(r+1|s+1)^{(1)}) \) \([Y]\) in the trigonometric case). On the other hand, for \( \lambda = \phi \) case, highest weight representation of Lie superalgebra \( sl(r+1|s+1) \), which is a classical counterpart of \( W_{\mu}(u) \), is characterized by the highest weight whose Kac-Dynkin labels \( a_1, a_2, \ldots, a_{r+s+1} \) \([BMR]\) are given as follows:

\[
\begin{align*}
  a_j &= \mu_j - \mu_{j+1} & 1 \leq j \leq r \\
  a_{r+1} &= \mu_{r+1} + \eta_1 \\
  a_{j+r+1} &= \eta_j - \eta_{j+1} & 1 \leq j \leq s
\end{align*}
\]

where \( \eta_j = \max\{\mu'_j - r - 1, 0\} \); \( \mu_{r+2} \leq s + 1 \) for covariant case. One can read the relations \( (2.27) \) from the ‘top term’ \([KS1,KOS]\) in \( (2.15) \) for large \( q^u \) (see, figure 4). The ‘top term’ in \( (2.15) \) is the term labeled by the tableau \( b \) such that

\[
b(i, j) = \begin{cases} 
  i & \text{for } 1 \leq j \leq \mu_i \text{ and } 1 \leq i \leq r + 1 \\
  r + j + 1 & \text{for } 1 \leq j \leq \mu_i \text{ and } r + 2 \leq i \leq \mu'_1.
\end{cases}
\]

Then, for large \( q^u \), we have

\[
\prod_{(i,j)\in\mu} (-1)^{p(b(i,j))} z(b(i, j); u + \mu'_1 - \mu_1 - 2i + 2j)
= (-1)^{\sum_{i=r+2}^{\mu'_1} \mu_i} \left\{ \prod_{i=1}^{r+1} \prod_{j=1}^{\mu_i} z(i; u + \mu'_1 - \mu_1 - 2i + 2j) \right\}
\times \left\{ \prod_{j=1}^{\mu_{r+2}} \prod_{i=r+2}^{\mu'_j} z(r + j + 1; u + \mu'_1 - \mu_1 - 2i + 2j) \right\}
\approx (-1)^{\sum_{i=r+2}^{\mu'_1} \mu_i} q^{-2} \sum N_{b_{ab}t_b}. \tag{2.29}
\]
Here we omit the vacuum part $\psi_a$. The ‘top term’ is considered to be related with the ‘highest weight vector’. See [KS1,KOS], for more details.

3. Functional equations

Consider the following Jacobi identity:

$$D \begin{bmatrix} b & c \\ b & c \end{bmatrix} - D \begin{bmatrix} b & c \\ b & c \end{bmatrix} = D \begin{bmatrix} b & c \\ b & c \end{bmatrix} D, \quad b \neq c$$

(3.1)

where $D$ is the determinant of a matrix and $D \begin{bmatrix} a_1 & a_2 & \ldots \\ b_1 & b_2 & \ldots \end{bmatrix}$ is its minor removing $a_\alpha$’s rows and $b_\beta$’s columns. Set $\lambda = \phi, \mu = (m^a)$ in (2.17a). From the relation (3.1), we have

$$T_m^a(u - 1)T_m^a(u + 1) = T_{m+1}^a(u)T_{m-1}^a(u) + g_m^a(u)T_{m}^{a-1}(u)T_{m}^{a+1}(u)$$

(3.2)

where $a, m \geq 1$; $T_m^a(u) = T_{(m^a)}(u)$: $a, m \geq 1$; $T_m^0(u) = 1$: $m \geq 0$; $T_0^a(u) = 1$: $a \geq 0$; \(g_m^1(u) = \prod_{j=1}^{m} P_1(u - m + 2j - 2); m \geq 1\); $g_m^a(u) = 1$: $a \geq 2$ and $m \geq 0$, or $a = 1$ and $m = 0$. Note that the following relation holds:

$$g_m^a(u + 1)g_m^a(u - 1) = g_{m+1}^a(u)g_{m-1}^a(u) \quad \text{for} \quad a, m \geq 1.$$  

(3.3)

The functional equation (3.2) is a special case of Hirota bilinear difference equation [H]. In addition, there are some restrictions on it, which we consider below.

Theorem 3.1 $T_{\lambda \subset \mu}^a(u) = 0$ if $\lambda \subset \mu$ contains a rectangular subdiagram with $r + 2$ rows and $s + 2$ columns. (see, [DM,MR])

Proof. We assume the coordinate of the top left corner of this subdiagram is $(i_1, j_1)$. Consider the tableau $b$ on this Young superdiagram $\lambda \subset \mu$. Fill the first column of this subdiagram from the top to the bottom by the elements of $b(i, j_1) \in J$: $i_1 \leq i \leq i_1 + r + 1$, so as to meet the admissibility conditions (i), (ii) and (iii). We find $b(i_1 + r + 1, j_1) \in J$. Then we have $r + 2 \leq b(i_1 + r + 1, j_1) \prec b(i_1 + r + 1, j_1 + 1) \prec \ldots \prec b(i_1 + r + 1, j_1 + s + 1)$. This contradicts the condition $b(i_1 + r + 1, j_1 + s + 1) \leq r + s + 2$. \qed

As a corollary, we have

$$T_m^a(u) = 0 \quad \text{for} \quad a \geq r + 2 \quad \text{and} \quad m \geq s + 2.$$  

(3.4)

Consider the admissible tableaux on the Young superdiagram with shape $(m^{r+1})$. From the admissibility conditions (i), (ii) and (iii), only such tableaux as $b(i, j) = i$ for $1 \leq i \leq r + 1$ and $1 \leq j \leq m - s - 1$ are admissible. Then we have,

$$T_{m^{r+1}}^1(u) = T_{(m^{r+1})}(u)$$

$$= \frac{1}{F_{(m^{r+1})}(u)} \sum_{b \in B((m^{r+1}) (i, j) \in (m^{r+1}))} (-1)^{p(b(i, j))} z(b(i, j); u + r + 1 - m - 2i + 2j)$$

$$= \frac{1}{F_{(m^{r+1})}(u)} \prod_{i=1}^{r+1} \prod_{j=1}^{m-s-1} (-1)^{p(i)} z(i; u + r + 1 - m - 2i + 2j)$$
\[ \times \sum_{b \in B((s+1)^{r+1})} \prod_{i=1}^{r+1} \prod_{j=m-s}^{m} (-1)^{\rho(b(i,j))} z(b(i,j); u + r + 1 - m - 2i + 2j) \]
\[ = \mathcal{F}^{m-s}(u + r - s + 2) \frac{Q_{r+1}(u - m)}{Q_{r+1}(u + m - 2s - 2)} \times T_{s+1}^{r+1}(u + m - s - 1), \quad m \geq s + 1. \quad (3.5a) \]

Similarly, we have
\[ T_{s+1}^{a}(u) = (-1)^{(s+1)(a-r-1)} \frac{Q_{r+1}(u - a - s + r)}{Q_{r+1}(u + a - s - r - 2)} \times T_{s+1}^{r+1}(u + a - r - 1), \quad a \geq r + 1. \quad (3.5b) \]

From the relations (3.5a) and (3.5b), we obtain

**Theorem 3.2** For \( a \geq 1 \) and \( r \geq 0 \), the following relation is valid.
\[ T_{a+s}^{r+1}(u) = (-1)^{(a+1)(a-1)} \mathcal{F}^{a}(u + r - s + 2) T_{s+1}^{r+1}(u). \quad (3.6) \]

Applying the relation (3.4) to (3.2), we obtain
\[ T_{m+1}^{r+1}(u - 1) T_{m}^{r+1}(u + 1) = T_{m+1}^{r+1}(u) T_{m-1}^{r+1}(u) \quad m \geq s + 2, \quad (3.7a) \]
\[ T_{s+1}^{a}(u - 1) T_{s+1}^{a}(u + 1) = g_{s+1}^{a}(u) T_{s+1}^{a-1}(u) T_{s+1}^{a+1}(u) \quad a \geq r + 2. \quad (3.7b) \]

Thanks to Theorem 3.2 (3.7a) is equivalent to (3.7b). From Theorem 3.2, we also have
\[ T_{s+1}^{r+1}(u - 1) T_{s+1}^{r+1}(u + 1) = T_{s+2}^{r+1}(u) (T_{s+1}^{r+1}(u) + (-1)^{s+1} \frac{T_{s+1}^{r+1}(u)}{\mathcal{F}^{2}(u + r - s + 2)}). \quad (3.8) \]

Remark: In the relation (3.5a), we assume that the parameter \( m \) takes only integer value. However, there is a possibility of \( m \) taking non-integer values, except some ‘singular point’, for example, on which right hand side of (3.5a) contains constant terms, by ‘analytic continuation’. We can easily observe this fact from the right hand side of (3.5a) as long as normalization factor \( \mathcal{F}^{m-s}(u) \) is disregarded. This seems to correspond to the fact that \( r+1 \) th Kac-Dynkin label \((2.27)\) \( \alpha_{r+1} \) can take non-integer value [Ka]. Furthermore, these circumstances seem to be connected with the lattice models based upon the solution of the graded Young-Baxter equation, which depends on non-additive continuous parameter (see for example, [M,PF]).

4. Summary and discussion

In this paper, we have executed analytic Bethe ansatz for Lie superalgebra \( sl(r+1|s+1) \). Pole-freeness of eigenvalue formula of transfer matrix in dressed vacuum form was shown for a wide class of finite dimensional representations labeled by skew-Young superdiagrams. Functional relation has been given especially for the eigenvalue formulae of transfer matrices in dressed vacuum form labeled by rectangular Young superdiagrams, which is a special case of Hirota bilinear difference equation with some restrictive relations.

It should be emphasized that our method presented in this paper is also applicable even if such factors like extra sign (different from that of (2.6a)), gauge factor, etc.
The function $T$ matrix where we assume $T$ matrix

In this section, we define the transfer matrix along the same line presented in \[EK\]. Let $L(u)_{ab}^{a'}$ be the $L$ operator [KulSk,PS1,PS2,Sc,BS] such that

$$L(u)_{aa}^{ab} = [u + 2(-1)^{p(a)}], L(u)_{bb}^{ba} = [u], L(u)_{ab}^{ba} = [2(-1)^{p(a)p(b)}]q^{sign(a-b)u}$$

where we assume $a \neq b; a, b \in J$. The monodromy matrix $\mathcal{J}(u)$ is defined as

$$\mathcal{J}(u)_{a_1...a_N}^{a_1...a_N, \gamma_N} = \sum_{a_1,...,a_N} L(u)_{aa_N}^{a_N} L(u)_{a_N a_N-1}^{a_N a_N-1} \cdots L(u)_{a_2a_1}^{a_2a_1} L(u)_{a_1b}^{a_1b}$$

$$\times (-1)^{\sum_{i=2}^{N} p(\gamma_i)+p(\beta_i)} \sum_{j=1}^{N} p(\gamma_j)$$

The transfer matrix is defined as supertrace of the monodromy matrix

$$t(u)_{\beta_1...\beta_N}^{\gamma_1...\gamma_N} = \sum_{a_1=1}^{r+s+2} (-1)^{p(a)} \mathcal{J}(u)_{a_1...a_N}^{a_1...a_N, \gamma_N}$$

Thanks to the intertwining relation, the commutativity relation $[t(u), t(v)] = 0$ follows. The function $T^1(u)$ defined in \[2.13\] will coincide with the the spectrum of the transfer matrix $t(u)$ under the Bethe ansatz equation \[2.6\] for relevant $N_j$. For example, for

**Acknowledgments**

The author would like to thank Professor A Kuniba for continual encouragement, useful advice and comments on the manuscript. He also thanks Dr J Suzuki for helpful discussions and pointing out some mistake in the earlier version of the manuscript; Professor T Deguchi for useful comments.

**Appendix A. Example of the $L$ operator and transfer matrix**

In this section, we define the transfer matrix along the same line presented in [EK]. Let $L(u)_{a_1a_2}^{a_1a_2}$ be the $L$ operator [KulSk,PS1,PS2,Sc,BS] such that

$$L(u)_{aa}^{ab} = [u + 2(-1)^{p(a)}], L(u)_{bb}^{ba} = [u], L(u)_{ab}^{ba} = [2(-1)^{p(a)p(b)}]q^{sign(a-b)u}$$

where we assume $a \neq b; a, b \in J$. The monodromy matrix $\mathcal{J}(u)$ is defined as

$$\mathcal{J}(u)_{a_1...a_N}^{a_1...a_N, \gamma_N} = \sum_{a_1,...,a_N} L(u)_{aa_N}^{a_N} L(u)_{a_N a_N-1}^{a_N a_N-1} \cdots L(u)_{a_2a_1}^{a_2a_1} L(u)_{a_1b}^{a_1b}$$

$$\times (-1)^{\sum_{i=2}^{N} p(\gamma_i)+p(\beta_i)} \sum_{j=1}^{N} p(\gamma_j)$$

The transfer matrix is defined as supertrace of the monodromy matrix

$$t(u)_{\beta_1...\beta_N}^{\gamma_1...\gamma_N} = \sum_{a_1=1}^{r+s+2} (-1)^{p(a)} \mathcal{J}(u)_{a_1...a_N}^{a_1...a_N, \gamma_N}$$

Thanks to the intertwining relation, the commutativity relation $[t(u), t(v)] = 0$ follows. The function $T^1(u)$ defined in \[2.13\] will coincide with the the spectrum of the transfer matrix $t(u)$ under the Bethe ansatz equation \[2.6\] for relevant $N_j$. For example, for
In the main text, we have treated mainly the expressions related to covariant representations. We often mark the expression related to contravariant representation with a dot. In contravariant case, the relations (2.7), (2.8), (2.9), (2.10) and (2.27) become respectively as follows:

\[
\begin{align*}
\hat{J} &= \{-1, -2, \ldots, -r - s - 2\}, \\
\hat{J}^- &= \{-r - 2, -r - 3, \ldots, -r - s - 2\}, \\
\hat{J}^+ &= \{-1, -2, \ldots, -r - 1\}, \\
- r - s - 2 < -r - s - 1 < \cdots < -1,
\end{align*}
\]

\[
\begin{cases}
0 \text{ for } a \in \hat{J}^+, \\
1 \text{ for } a \in \hat{J}^-.
\end{cases}
\]

\[
\dot{z}(a; u) = \psi_a(u) \frac{Q_{-a-1}(u + r - s + a - 1)Q_{-a}(u + r - s + a + 2)}{Q_{-a-1}(u + r - s + a + 1)Q_{-a}(u + r - s + a)}
\]

\[
\dot{z}(a; u) = \psi_a(u) \frac{Q_{-a-1}(u - r - s - a - 1)Q_{-a}(u - r - s - a - 4)}{Q_{-a-1}(u - r - s - a - 3)Q_{-a}(u - r - s - a - 2)}
\]

\[
a_{r+1-j} = \xi_j - \xi_{j+1} \quad \text{for} \quad 1 \leq j \leq r, \\
a_{r+1} = -\xi_1 - \mu'_{s+1}, \\
a_{r+s+2-j} = \mu'_j - \mu'_{j+1} \quad \text{for} \quad 1 \leq j \leq s,
\]

where \(\xi_j = \max\{\mu_j - s - 1, 0\}; \mu'_{s+2} \leq r + 1\). The function (2.6d) and (2.11) take the form

\[
P_a^{(j)}(u) = [u - w_j]^{\delta_a,r+1}, \quad \psi_a(u) = \begin{cases}
P_{r+s+1}(u - 2) & \text{for } a = -r - s - 2, \\
P_{r+s+1}(u) & \text{for } a \in \hat{J} - \{-r - s - 2\}
\end{cases}
\]

if the quantum space is labeled by the contravariant Young superdiagram with shape \(\hat{\mu} = (1^1)\);

\[
P_a^{(j)}(u) = [u - w_j]^{\delta_a,1}, \quad \psi_a(u) = \begin{cases}
P_1(u + r - s - 2) & \text{for } a = -1, \\
P_1(u + r - s) & \text{for } a \in \hat{J} - \{-1\}
\end{cases}
\]

if the quantum space is labeled by the covariant Young superdiagram with shape \(\hat{\mu} = (1^1)\).

If the quantum space is labeled by the contravariant Young superdiagram, in contrast to covariant case, the parameter \(t_{r+1}\) in left hand side of the Bethe ansatz

\[
T^1(u) = \begin{bmatrix} 1 & -2 & -3 \end{bmatrix}
\]
equation (2.6a) will be \(-1\), since \(r + 1\) th Kac-Dynkin label takes negative value for contravariant Young superdiagram \([BMR]\). For \(-a \in J\) and (B.4) with (B.7), the following relation holds

\[
z(a; u) = (-1)^N \dot{z}(-a; s - r - u)|_{u_k^(-), \ldots, u_k^(-), \ldots, w_i \rightarrow -w_i}.
\]

Note that this relation reduces to the crossing symmetry \([R2]\) for \(sl_{r+1}\), if we set \(s = -1\) (see, also \([KS1]\)). Pole freeness of the function \(\dot{T}_{\lambda\mu}^\theta(u)\) under the Bethe ansatz equation (2.6a) can be proved in the same way as Theorem 2.1.

Appendix C. Example of non-distinguished simple roots case:

Let \(\alpha_1\) and \(\alpha_2\) be the simple roots of \(sl(1|2)\) normalized so that \((\alpha_1|\alpha_1) = (\alpha_2|\alpha_2) = 0\) and \((\alpha_1|\alpha_2) = (\alpha_2|\alpha_1) = -1\) (see figure C1). In this case, the sets \((2.7)\) and \((B.1)\) become \(J_+ = \{2\}\), \(J_- = \{1, 3\}\), \(\dot{J}_+ = \{-2\}\), \(\dot{J}_- = \{-1, -3\}\). The function \(z(a; u) = [a]_u\), \((a \in J)\) has the form

\[
\begin{align*}
1 &= [u - 2]^N \frac{Q_1(u + 1)}{Q_1(u - 1)}, \\
2 &= [u]^N \frac{Q_1(u + 1)Q_2(u - 2)}{Q_1(u - 1)Q_2(u)}, \\
3 &= [u]^N \frac{Q_2(u - 2)}{Q_2(u)}
\end{align*}
\]

and the function \(\dot{z}(a; u) = [a]_u\), \((a \in \dot{J})\) has the form

\[
\begin{align*}
-3 &= [u - 2]^N \frac{Q_2(u + 1)}{Q_2(u - 1)}, \\
-2 &= [u]^N \frac{Q_1(u - 2)Q_2(u + 1)}{Q_1(u)Q_2(u - 1)}, \\
-1 &= [u]^N \frac{Q_1(u - 2)}{Q_1(u)}
\end{align*}
\]

Here we assume the quantum spaces are labeled by Young superdiagrams with shapes \(\tilde{\mu} = (1^3)\) and \(\tilde{\nu} = (2^1)\) respectively; for simplicity, inhomogeneity parameters \(w_i\) are set to 0. For example, for \(\lambda = \phi; \mu = (2^1)\), (2.15) has the form

\[
\begin{align*}
T^1_2(u) &= -\frac{1}{[u - 1]^N} \left[1 \ 2 \ + \ 1 \ 3 \ + \ 2 \ 2 \ - \ 2 \ 3\right] \\
&= -[u - 3]^N[u + 1]^N \frac{Q_1(u + 2)Q_2(u - 1)}{Q_1(u - 2)Q_2(u + 1)} + [u - 3]^N[u + 1]^N \frac{Q_1(u)Q_2(u - 1)}{Q_1(u - 2)Q_2(u + 1)} \\
&\quad + [u - 1]^N[u + 1]^N \frac{Q_1(u + 2)Q_2(u - 3)}{Q_1(u - 2)Q_2(u + 1)} \\
&\quad - [u - 1]^N[u + 1]^N \frac{Q_1(u)Q_2(u - 3)}{Q_1(u - 2)Q_2(u + 1)}
\end{align*}
\]

and for \(\lambda = \phi; \mu = (1^2)\), (2.15) has the form

\[
\begin{align*}
T^2_1(u) &= \frac{1}{[u - 1]^N} \left[1 \ 1 \ - \ 2 \ 2 \ + \ 2 \ 3 \ + \ 3 \ 3\right]
\end{align*}
\]
Appendix D. Example of non-distinguished simple roots case: $p(1)=p(2)=1, p(3)=0$ grading

Let $\alpha_1$ and $\alpha_2$ be the simple roots of $sl(1|2)$ normalized so that $(\alpha_1|\alpha_1) = -2, (\alpha_2|\alpha_2) = 0$ and $(\alpha_1|\alpha_2) = (\alpha_2|\alpha_1) = 1$ (see figure D1). In this case, the sets $\{2,7\}$ and $\{2,1\}$ become $J_+ = \{3\}, J_- = \{1, 2\}, \hat{J}_+ = \{-3\}, \hat{J}_- = \{-1, -2\}$. The function $z(a; u) = \begin{array}{c}\square\end{array}_a$ $(a \in \hat{J})$ has the form

\[
\begin{array}{cccc}
1 = [u - 2]^N \frac{Q_1(u + 1)}{Q_1(u - 1)}, & 2 = [u]^N \frac{Q_1(u - 3)Q_2(u)}{Q_1(u - 1)Q_2(u - 2)}, & 3 = [u]^N \frac{Q_2(u)}{Q_2(u - 2)}
\end{array}
\] (D.1)

and the function $\hat{z}(a; u) = \begin{array}{c}\square\end{array}_a$ $(a \in \hat{J})$ has the form

\[
\begin{array}{cccc}
\hat{3} = [u + 2]^N \frac{Q_2(u - 1)}{Q_2(u + 1)}, & \hat{2} = [u]^N \frac{Q_1(u + 2)Q_2(u - 1)}{Q_1(u)Q_2(u + 1)}, & \hat{1} = [u]^N \frac{Q_1(u - 2)}{Q_1(u)}
\end{array}
\] (D.2)

Here we assume the quantum spaces are labeled by Young superdiagrams with shapes $\hat{\mu} = (1^1)$ and $\hat{\lambda} = (1^1)$ respectively; for simplicity, inhomogeneity parameters $w_i$ are set to 0. For example, for $\lambda = \phi; \mu = (2^1)$, (2.15) has the form

\[
\begin{array}{cccc}
\mathcal{T}_2^1(u) = \begin{array}{c}1 \\ 2 \\ 3 \\ 3 \\ 3 \end{array}, & \mathcal{\hat{T}}_2^1(u) = \begin{array}{c}1 \\ 2 \\ 3 \\ 3 \\ 3 \end{array}
\end{array}
\]

\[
\begin{array}{cccc}
= [u - 3]^N[u + 1]^N \frac{Q_2(u + 1)}{Q_2(u - 1)} - [u - 3]^N[u + 1]^N \frac{Q_1(u)Q_2(u + 1)}{Q_1(u - 2)Q_2(u - 1)} - [u - 1]^N[u + 1]^N \frac{Q_1(u - 4)Q_2(u + 1)}{Q_1(u - 2)Q_2(u - 3)} + [u - 1]^N[u + 1]^N \frac{Q_2(u + 1)}{Q_2(u - 3)}
\end{array}
\] (D.3)
and for \( \lambda = \phi; \mu = (1^2) \), (2.15) has the form

\[
T^2(u) = \frac{1}{|u - 1|^N} \left( \begin{array}{cccc}
1 & 1 & 1 & 2 \\
1 & 2 & 3 & -2 \\
\end{array} \right)
\]

\[
= [u - 3]^N \frac{Q_1(u + 2)}{Q_1(u - 2)} + [u - 1]^N \frac{Q_1(u - 4)Q_1(u + 2)Q_2(u - 1)}{Q_1(u - 2)Q_1(u)Q_2(u - 3)}
\]

\[
- [u - 1]^N \frac{Q_1(u + 2)Q_2(u - 1)}{Q_1(u)Q_2(u - 3)} + [u + 1]^N \frac{Q_1(u - 4)Q_2(u + 1)}{Q_1(u)Q_2(u - 3)}
\]

(D.4)

We note that the function \( \dot{T}^1(u) \) associated with the contravariant Young superdiagram with shape \( \dot{\lambda} = \phi; \dot{\mu} = (1^1) \)

\[
\dot{T}^1(u) = \begin{array}{ccc}
-3 & -2 & -1
\end{array}
\]

(D.5)

coincides with Lai’s solution [L] on supersymmetric \( t - J \) model presented in [EK] in the limit \( q \to 1 \) except overall scalar factor after some redefinition. Pole freeness of the functions \( T^a(u) \) and \( \dot{T}^a(u) \) under the Bethe ansatz equation (2.6) can be proved in the same way as Theorem 2.1.

Appendix E. Other representation of \( T^a \) and \( T_m \)

For simplicity, we assume the vacuum part is formally trivial. Define the functions \( A^a, B^a, A_m \) and \( B_m \) by the generating series such that

\[
\sum_{k=-\infty}^{\infty} A_k(u + k - 1)X^k = (1 - z(1; u)X)^{-1} \cdots (1 - z(r + 1; u)X)^{-1}
\]

(E.1)

\[
\sum_{l=-\infty}^{\infty} B^l(u + l - 1)X^l = (1 - z(r + 2; u)X) \cdots (1 - z(r + s + 2; u)X)
\]

(E.2)

\[
\sum_{k=-\infty}^{\infty} B_k(u + k - 1)X^k = (1 + z(r + s + 2; u)X)^{-1} \cdots (1 + z(r + 2; u)X)^{-1}
\]

(E.3)

\[
\sum_{l=-\infty}^{\infty} A^l(u + l - 1)X^l = (1 + z(r + 1; u)X) \cdots (1 + z(1; u)X)
\]

(E.4)

Combining these relations, we obtain

\[
T^a(u) = \sum_{l=0}^{\min(r+1, a)} B_{a-l}(u - l)A^l(u + a - l)
\]

(E.5)

\[
T_m(u) = \sum_{l=0}^{\min(s+1, m)} A_{m-l}(u - l)B^l(u + m - l).
\]

(E.6)

Note that these functions \( A_m(u) \) and \( A^a(u) \) are analogous to eigenvalue formulae of transfer matrices in dressed vacuum form of fusion \( U_q(sl_{r+1}^{(1)}) \) vertex model labeled by Young diagrams with shapes \( (m^1) \) and \( (1^a) \) respectively. We also note that the functions
$B^a(u)$ and $B_m(u)$ are analogous to eigenvalue formulae of transfer matrices in dressed vacuum form of fusion $U_q(s^{(1)}_{\pm 1})$ vertex model labeled by Young diagrams with shapes $(1^a)$ and $(m^1)$ respectively.

References

[BB1] Balantekin A B and Bars I 1981 J. Math. Phys. 22 1149
[BB2] Balantekin A B and Bars I 1981 J. Math. Phys. 22 1810
[BMR] Bars I, Morel B and Ruegg H 1983 J. Math. Phys. 24 2253
[BR] Bazhanov V V and Reshetikhin N 1990 J. Phys. A: Math. Gen. 23 1477
[BS] Bazhanov V V and Shadrikov A G 1987 Theor. Math. Phys. 73 1302
[C] Cornwell J F 1989 GROUP THEORY IN PHYSICS Vol 3 Supersymmetries and Infinite-Dimensional Algebras (Academic press, New York)
[D] Drinfel’d V G 1988 Sov.Math.Dokl 36 212
[DM] Deguchi T and Martin P P 1992 Int. J. Mod. Phys. A 7 Suppl. 1A 165
[EK] Essler F H L and Korepin V E 1992 Phys. Rev. B 46 9147
[EKS1] Essler F H L, Korepin V E and Schoutens K 1992 Phys. Rev. Lett. 68 2960
[EKS2] Essler F H L, Korepin V E and Schoutens K cond-mat/9211001 1994 Int. J. Mod. Phys. B 8 3205
[FK] Föerster A and Karowski M 1993 Nucl. Phys. B 396 611
[H] Hirota R 1981 J. Phys. Soc. Japan 50 3787
[Ka] Kac V 1977 Adv. Math. 26 8
[KE] Korepin V E and Essler F H L eds. 1994 Exactly solvable models of strongly correlated electrons (World Scientific, Singapore)
[KLWZ] Krichever I, Lipan O, Wiegmann P and Zabrodin A 1997 Commun. Math. Phys. 188 267
[K] Kuniba A 1994 J. Phys. A: Math. Gen. 27 L113
[KNH] Kuniba A, Nakamura S and Hirota R 1996 J. Phys. A: Math. Gen. 29 1759
[KNS1] Kuniba A, Nakanishi T and Suzuki J 1994 Int. J. Mod. Phys. A 9 5215
[KNS2] Kuniba A, Nakanishi T and Suzuki J 1994 Int. J. Mod. Phys. A 9 5267
[KOS] Kuniba A, Ohta Y and Suzuki J 1995 J. Phys. A: Math. Gen. 28 6211
[KS1] Kuniba A and Suzuki J 1995 Commun. Math. Phys. 173 225
[KS2] Kuniba A and Suzuki J 1995 J. Phys. A: Math. Gen. 28 711
[Kul] Kulish P P 1986 J. Sov. Math 35 2648
[KulSk] Kulish P P and Sklyanin E K 1982 J. Sov. Math 19 1596
[L] Lai C K 1974 J. Math. Phys. 15 1675
[M] Maassarani Z 1995 J. Phys. A: Math. Gen. 28 1305
[MR] Martin P and Rittenberg V 1992 Int. J. Mod. Phys. A 7 Suppl. 1B 707
[N] Nazarov M L 1991 Lett. Math. Phys 21 123
[P] Pfannmüller M P and Frahm H 1996 Nucl. Phys. B479 575
[PS1] Perk J H H and Schultz CL in 1983 Nonlinear Integrable Systems -Classical Theory and Quantum Theory ed Jimbo M and Miwa T (World Scientific, Singapore)
[PS2] Perk J H H and Schultz CL 1981 Phys. Lett. 84A 407
[PT] Pragacz P and Thorup A 1992 Adv. Math. 95 8
[R1] Reshetikhin N Yu 1983 Sov. Phys. JETP 57 691
[R2] Reshetikhin N Yu 1987 Lett. Math. Phys. 14 235
[RW] Reshetikhin N Yu and Wiegmann P B 1987 Phys. Lett. B189 125
[Sc] Schultz C L 1983 Physica A122 71
[S1] Suzuki J 1992 J. Phys. A: Math. Gen. 25 1769
[S2] Suzuki J 1994 Phys. Lett. A195 190
[Su] Sutherland B 1975 Phys. Rev. 12B 3795
[T] Tsuboi Z  solv-int/9610011; 1997 J. Phys. Soc. Japan 66 3391
[TK] Tsuboi Z and Kuniba A 1996 J. Phys. A: Math. Gen. 29 7785
[Y] Yamane H prepent q-alg/9603015
[ZB] Zhou Y K and Batchelor M T 1997 Nucl. Phys. 490B 576