CONJUGACY CLASSES OF 3-BRAID GROUP $B_3$

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Abstract. In this article we describe the summit sets in $B_3$, the smallest element in a summit set and we compute the Hilbert series corresponding to conjugacy classes. The results will be related to Birman-Menasco classification of knots with braid index three or less than three.

1. Introduction

The 3-braid group $B_3$ admits the following classical presentation given by Artin\cite{1}:

\begin{equation}
B_3 = \langle x_1, x_2 : x_2 x_1 x_2 = x_1 x_2 x_1 \rangle
\end{equation}

Elements of $B_3$ are words expressed in $x_1, x_2, x_1^{-1}$ and $x_2^{-1}$. Words expressed only in $x_1, x_2$ are called positive words and the set of all these is denoted by $\mathcal{MB}_3$. Garside\cite{8} proved that $\mathcal{MB}_3$ also admits the presentation (1) as a monoid.

\footnotesize
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The solution of conjugacy problem in $B_3$ was reduced to a problem in $MB_3$. The element $\Delta_3 = x_1x_2x_1$ in $MB_3$ is called the Garside braid. The set $MB_3^+ = MB_3 \setminus \Delta_3MB_3$ is known as the set of primes to $\Delta_3$. Elements in $B_3$ admit a unique Garside normal form $\Delta^r_3W$, where $r \in \mathbb{Z}$ and $W \in MB_3^+$. Garside proved that in a given conjugacy class, $C$ (with some exceptions, $C$ is an infinite set), the set of numbers $r$ such that $\Delta^r_3W \in C$ has a least upper bound, $\exp(C)$. The summit set of a conjugacy class $C$ is defined as

$$SS(C) = \{\Delta^r_3W \in C | r = \exp(C)\}.$$ 

and this is a finite set. The set of elements $W \in MB_3^+$ such that $\Delta^r_3W \in SS(C)$ is called the base-summit set of the class $C$. In this way, Garside’s solution of the conjugacy problem is to compute $\exp(C)$ and the base-summit set: two elements in $B_3$ are conjugate if and only if their conjugacy classes have the same exponents and the same base-summit sets. It is easy to see that all the (positive) elements in a base-summit set have the same length and for two exponents which are congruent mod 2, the corresponding base summit sets coincide or are disjoint. So we will treat separately the two types of base summit sets: $E$-summit set corresponding to even exponents and $O$-summit sets corresponding to odd exponents.

Over the years, a lot of work has been done to replace the summit set for $B_n$ by a smaller set. Elrifai and Morton introduced super summit set, a subset of summit set and still a conjugacy invariant. V. Gebhart found a smaller subset of summit set known as ultra summit set, which is also conjugacy invariant. Both super and ultra summit sets are great improvement in the sense that they reduce the
size of the set which is a conjugacy invariant. K. Murasugi [11] gave
seven different classes of $\mathcal{B}_3$ (not in Garside normal form) and showed
that an arbitrary word is conjugate to an unique element of these seven
classes. Using band presentation of $\mathcal{B}_3$, P. J. Xu [9] described explicit-
ly the normal and summit forms of words in $\mathcal{B}_3$ and found a unique
representative in the summit set of words (see §3).

In this paper we describe explicitly the words in classical generators
which are:

(i) in the Garside normal form;
(ii) summit words (elements of summit set);
(iii) super summit words (elements of super summit set);
(iv) smallest summit words (smallest elements in a given summit set).

The results of Th 1 and Th 2 seem to be well known by the experts,
but I could not find these statements in the literature.

Garside normal form of elements in $\mathcal{B}_3$ are given by the following
result:

**Theorem 1.** For $s_i > 0$, $\Delta_3^{r_1 x_1^{s_1} x_2^{s_2} \cdots x_h^{s_h}}$ is in the normal form if and
only if:

(i) $h \leq 2$, either

(ii) $h \geq 3$ and all the exponents are $\geq 2$, with the possible exceptions
of $s_1$ or $s_h$.

The next result describes the elements of $\mathcal{B}_3$ in summit sets:

**Theorem 2.** A word $\Delta_3^r W = \Delta_3^{r_1 x_1^{s_1} x_2^{s_2} \cdots x_h^{s_h}}$ in normal form is sum-
mit word if and only if:

(0) $\Delta_3^r W$ is the word: $\Delta_3^{even}$ or $\Delta_3^{odd}$ $x_i$ or $\Delta_3^{even} x_{i_1} x_{i_2}$, or
(i) \( h + 1 \equiv r \pmod{2} \), or
(ii) \( s_j \geq 2 \), for all \( 1 \leq j \leq h \).

The following theorem describes the elements of \( \mathcal{B}_3 \) in super summit sets:

**Theorem 3.** A summit word \( \Delta_3^r W = \Delta_3^r x_{i_1}^{s_1} x_{i_2}^{s_2} \cdots x_{i_h}^{s_h} \) is super summit word if and only if:

(0) \( \Delta_3^r W \) is the word: \( \Delta_3^{\text{even}} \) or \( \Delta_3^{\text{odd}} x_i \) or \( \Delta_3^{\text{even}} x_{i_1} x_{i_2}, \) or

(i) \( h + 1 \equiv r \pmod{2} \).

The smallest elements of \( \mathcal{B}_3 \) in summit sets are described by the next theorem:

**Theorem 4.** A summit word \( \Delta_3^r W = \Delta_3^r x_{i_1}^{s_1} x_{i_2}^{s_2} \cdots x_{i_h}^{s_h} \) is the smallest in its summit set if and only if:

(0) \( \Delta_3^r W \) is the word: \( \Delta_3^{\text{odd}} \) or \( \Delta_3^{\text{even}} x_1^{s_1} \) \( (s_1 \geq 0) \) either

\( \Delta_3^r W \) satisfies the following conditions:

(a) \( x_{i_1} = x_1 \) and

(b) \( h \equiv r \pmod{2} \) and

(c) \((s_1, \ldots, s_h)\) satisfies max-min condition.

(See §2 for max-min condition). We also give in §2 an algorithm to compute this smallest element.

As an application in knot theory, in §3 we find the unique representative in the conjugacy classes according to Birman-Menasco \([4, 5]\) classification of links with braid index \( \leq 3 \) and invertibility of 3-closed braid.

Next we compute the number \( C_n \) of base-summit sets containing elements \( W \) of length \( n \); we denote the generating functions by \( H^*(t) = \)
\sum C_n t^n$, where $*$ has only two values, $E$ or $O$. In the next formulas $\mu$ is the classical Möbius function:

**Theorem 5.** (a) The Hilbert series of even base-summit sets is given by

$$H^E(t) = 1 + t + 2t^2 + \sum_{n \geq 3} \left[1 + \sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \sum_{d \mid a} \mu(d) \left(\frac{\delta(c - b)}{\delta b} - 1\right)\right] t^n$$

where $a = \gcd(n, 2k)$, $n = ac$ and $2k = ab$.

(b) The Hilbert series of odd base-summit sets is given by

$$H^O(t) = \sum_{n=0}^{5} t^n + \sum_{n \geq 6} \left[1 + \sum_{k=1}^{\left\lfloor \frac{n-2}{4} \right\rfloor} \sum_{d \mid a} \mu(d) \left(\frac{\delta(c - b)}{\delta b} - 1\right)\right] t^n$$

where $a = \gcd(n, 2k + 1)$, $n = ac$ and $2k + 1 = ab$.

**Remark 1:** Given $a$, $b$, $c$ positive integers such that $(b, c) = 1$, and $ab =$ even, we can reconstruct $n$ and $2k$.

2. Normal Form And Summit Set

The unique smallest word in the summit set of a given word $V$ in $B_3$ can be described completely. A word in normal form is called *summit word* if it belongs to its summit set of its conjugacy class and it is called *super summit word* if it belongs to its super summit set. We have the length-lexicographic order given by $x_2 > x_1$ in $\mathcal{MB}_3$. We use the notation: $\widehat{W} = x_{3-i_1} x_{3-i_2} \ldots x_{3-i_j}$ for $W = x_{i_1} x_{i_2} \ldots x_{i_j}$ used as in [2].

**Definition 1.** A summit word $\Delta^*_3W$ is called the smallest summit word if $W$ is the smallest (in length-lexicographic order) in the its base-summit set.
Lemma 1. The word $\Delta_3$ in $B_3$ has the following properties:

(i) $\Delta_3 W = \hat{W} \Delta_3$ and $W \Delta_3 = \Delta_3 \hat{W}$ for each word $W \in \mathcal{MB}_3$;

(ii) $x_1^{-1} = \Delta_3^{-1} x_1 x_2$ and $x_2^{-1} = \Delta_3^{-1} x_2 x_1$.

Definition 2. A word $V$ of $\mathcal{MB}_3$ is said to be divisible by $\Delta_3$ if and only if $V = B \Delta_3 C$ for some $B, C \in \mathcal{MB}_3$. Otherwise the word $V$ is prime to $\Delta_3$.

Remark 2: Using lemma 1 we can see that $\Delta_3 \mid V$ if and only if $V = \Delta_3 W$ and if and only if $V = U \Delta_3$ (for some $U, W \in \mathcal{MB}_3$).

Definition 3. a) A nonnegative integer $k$ is called the exponent of a word $V$ in $\mathcal{MB}_3$ if $\Delta_3^k \mid V$ but $\Delta_3^{k+1} \nmid V$.

b) An integer $k$ is called the exponent of a word $V$ in $B_3$ if $V = \Delta_3^k W$, where $W \in \mathcal{MB}_3^+$.

Proof of the Theorem. Let $h \geq 3$ and $s_j \geq 2$ for $1 < j < h$ or $h \leq 2$, then $W = x_{i_1}^{s_1} x_{i_2}^{s_2} \cdots x_{i_h}^{s_h}$ does not contain $x_1 x_2 x_1$ or $x_2 x_1 x_2$, so $W$ is unique in its diagram (see [8] for definition of diagram) and $\Delta_3^h x_{i_1}^{s_1} x_{i_2}^{s_2} \cdots x_{i_h}^{s_h}$ is the normal form. Conversely, let $\Delta_3^h x_{i_1}^{s_1} x_{i_2}^{s_2} \cdots x_{i_h}^{s_h}$ is the normal form. If $h \geq 3$ and there is a $j$ satisfying $1 < j < h$ and $s_j = 1$, then $x_{i_1}^{s_1} x_{i_2}^{s_2} \cdots x_{i_h}^{s_h}$ contains $x_1 x_2 x_1$ or $x_2 x_1 x_2$, so the word $W$ is divisible by $\Delta_3$ and we have a contradiction.

Remark 3: As a consequence of the proof the set of words in $\mathcal{MB}_3$ primes to $\Delta_3$ i.e $\mathcal{MB}_3^+$ is a subset of $k$-vector basis of the algebra $k \langle \mathcal{MB}_3 \rangle$ for a field $k$. But this is not true for braid monoid $\mathcal{MB}_n$, $n \geq 4$. (See Gröbner-Shirshov bases of $\mathcal{MB}_n$ [6] for more details).

In order to find the conjugacy class (and then the summit set) of the word $\Delta_3^h W$, Garside [8] proved that the conjugacy relation is generated by conjugations with the divisors of $\Delta_3$: $\text{Div} = \{ 1, x_1, x_2, x_1 x_2, x_2 x_1, \Delta_3 \}$. 


Definition 4. A word $V$ in normal form is called special if it is one of the form: $\Delta_3^{odd} x_i$ or $\Delta_3^{even} x_i$ or $\Delta_3^{even} x_{i_1} x_{i_2}$ or $\Delta_3^{even} x_{i_1}^s$ ($s_1 \geq 0$). Otherwise $V$ in normal form is a general word.

Definition 5. For a positive $W = x_{i_1}^{s_1} x_{i_2}^{s_2} \cdots x_{i_h}^{s_h}$ (all $s_i \geq 1$), $h$ is called the syllable length of $W$ and it is denoted by $l_s(W)$. The sum of $s_i$ for $1 \leq s_i \leq h$ is called length of $W$ and it is denoted by $|W|$

Lemma 2. (Computation)

(a) Let $\Delta_3^{2m} W = \Delta_3^{2m} x_{i_1}^{s_1} x_{i_2}^{s_2} \cdots x_{i_h}^{s_h}$ is normal form of a general word with $s_j \geq 2$ and $i_1 \neq i_h$, then, for any $A$ such that $A^\pm 1 \in \text{Div}$ the conjugate $A \Delta_3^{2m} W A^{-1}$ has exponent $< 2m$ or belongs to the set:

$$\Delta_3^{2m} \text{CSSS}^E \bigcup \Delta_3^{2m} \text{BSSS}^E,$$

where

$$\text{CSSS}^E = \{ x_{i_k}^{s_k} \cdots x_{i_h}^{s_h} x_{i_1}^{s_1} \cdots x_{i_{k-1}}^{s_{k-1}}, \tilde{x}_{i_k}^{s_k} \cdots \tilde{x}_{i_h}^{s_h} x_{i_1}^{s_1} \cdots \tilde{x}_{i_{k-1}}^{s_{k-1}} \}_{k=1,h}$$

and

$$\text{BSSS}^E = \{ x_{i_k}^{s_k-j} x_{i_{k+1}}^{s_k+1} \cdots x_{i_{k-1}}^{s_{k-1}} x_{i_k}^{j}, \tilde{x}_{i_k}^{s_k-j} \tilde{x}_{i_{k+1}}^{s_{k+1}} \cdots \tilde{x}_{i_{k-1}}^{s_{k-1}} \}_{k=1,h,j=1,s_{k-1}}$$

(b) Let $\Delta_3^{2m+1} W = \Delta_3^{2m} x_{i_1}^{s_1} x_{i_2}^{s_2} \cdots x_{i_h}^{s_h}$ is normal form of a general word with $s_j \geq 2$ and $i_1 = i_h$, then, for any $A$ such that $A^\pm 1 \in \text{Div}$ the conjugate $A \Delta_3^{2m+1} W A^{-1}$ has exponent $< 2m + 1$ or belongs to the set:

$$\Delta_3^{2m+1} \text{CSSS}^O \bigcup \Delta_3^{2m+1} \text{BSSS}^O,$$

where

$$\text{CSSS}^O = \{ x_{i_k}^{s_k} \cdots x_{i_h}^{s_h} \tilde{x}_{i_1}^{s_1} \cdots \tilde{x}_{i_{k-1}}^{s_{k-1}}, \tilde{x}_{i_k}^{s_k} \cdots \tilde{x}_{i_h}^{s_h} x_{i_1}^{s_1} \cdots \tilde{x}_{i_{k-1}}^{s_{k-1}} \}_{k=1,h}$$

and

$$\text{BSSS}^O = \{ x_{i_k}^{s_k-j} x_{i_{k+1}}^{s_k+1} \cdots x_{i_{k-1}}^{s_{k-1}} \tilde{x}_{i_k}^{j}, \tilde{x}_{i_k}^{s_k-j} \tilde{x}_{i_{k+1}}^{s_{k+1}} \cdots \tilde{x}_{i_{k-1}}^{s_{k-1}} \}_{k=1,h,j=1,s_{k-1}}$$
Proof. (a) This is just a matter of computation. One has to compute and verify $A\Delta_3^{2m} WA^{-1}$ for all $A^{\pm 1} \in \text{Div}$. For example we verify $A\Delta_3^{2m} WA^{-1}$ for $A = x_1$: There are two cases (i) $x_{ih} = x_1$; and (ii) $x_{ih} = x_2$. For (i), we have $x_1\Delta_3^{2m} W x_1^{-1} = \Delta_3^{2m} x_1 x_1^{s_1} x_2^{s_2} \cdots x_1^{s_h} x_1^{-1}$. For (ii), $x_1\Delta_3^{2m} W x_1^{-1} = \Delta_3^{2m} x_1^{s_1} x_2^{s_2} \cdots x_2^{s_h} x_1^{-1} x_2 x_1 = \Delta_3^{2m-1} x_1^{s_1} x_2^{s_2} \cdots x_1^{s_h} x_2$, the exponent $< 2m$. Similarly it is true for all other $A^{\pm 1} \in \text{Div}$.

(b) We also verify $A\Delta_3^{2m+1} WA^{-1}$ only for $A = x_1$: There are again two cases (i) $x_{ih} = x_1$; and (ii) $x_{ih} = x_2$. For (i), we have $x_1\Delta_3^{2m+1} W x_1^{-1} = \Delta_3^{2m+1} x_1 x_1^{s_1} x_2^{s_2} \cdots x_1^{s_h} x_1^{-1} = \Delta_3^{2m+1} x_2 x_1^{s_1} x_2^{s_2} \cdots x_1^{s_h} x_1^{-1}$. And (ii) give us, $x_1\Delta_3^{2m+1} W x_1^{-1} = \Delta_3^{2m+1} x_1 x_1^{s_1} x_2^{s_2} \cdots x_2^{s_h} x_1^{-1} x_2 x_1 = \Delta_3^{2m} x_1^{s_1} x_2^{s_2} \cdots x_1^{s_h} x_2 x_1$, the exponent $< 2m + 1$.

For example the sets $CSSS^E$ and $BSSS^E$ for $V = \Delta_3^4 x_2^2 x_1^2$ are given as below:

$CSSS^E = \{x_2^2 x_1, x_1^2 x_2, x_1 x_2^2, x_2^2 x_1\}$;

$BSSS^E = \{x_2^3 x_1^2 x_2, x_2^2 x_1^2 x_2, x_2 x_1^2 x_2^2, x_1 x_2 x_1^2, x_1^3 x_2^2 x_1, x_1 x_2^2 x_1^2, x_1 x_2 x_1^3, x_2 x_1^2 x_2\}$.

The following Corollary is a consequence of the Lemma.

Corollary 1. Let $\Delta_3^r W = \Delta_3^r x_1^{s_1} x_2^{s_2} \cdots x_1^{s_h}$ be normal form of a general word; then $W$ belongs to a base-summit set if one of the following conditions is satisfied:

(i) $h + 1 \equiv r \pmod{2}$ or

(ii) $s_j \geq 2$, for all $1 \leq j \leq h$.

Proof of the Theorem. Let $\Delta_3^r W = \Delta_3^r x_1^{s_1} x_2^{s_2} \cdots x_1^{s_h}$ in $B_3$ be a summit word. Suppose that $h \equiv r \pmod{2}$ and $s_1 = 1$ then $x_{ih} \Delta_3^r W x_{ih}^{-1} = \Delta_3^{r+1} W_1$, but $W$ belongs to a base-summit set so we obtain contradiction. The converse is true by Corollary. It is also easy to check that $\Delta_3^{even} x_1$, $\Delta_3^{odd} x_1$ and $\Delta_3^{even} x_1 x_2$ are summit words.
Definition 6. The word $V = \Delta^r A_1 \cdots A_q$ of $B_3$ in normal form with $A_i \in \text{Div}$ is called left-canonical form of $V$ if $A_{j+1} = x_k B$ implies that $A_j = Cx_k$ for some $B, C \in \mathcal{MB}_3^+$. 

Definition 7. The number of $A_i$s in left-canonical form of $V = \Delta^r A_1 \cdots A_q$ is called canonical-length of $V$ denoted by $l_c(V)$.

The subset of a summit set of a word known as super summit set is defined as:

$$SSS(C) = \{ \Delta^r W \in SS \mid W \text{ has minimal canonical-length} \}.$$  

(See [7] for more detail about left-canonical form and super summit set)

Lemma 3. Let $W = x_{i_1}^{s_1} x_{i_2}^{s_2} \cdots x_{i_h}^{s_h}$ belongs to $CSSS^*$ then the syllable length and canonical length are given by the table:

|       | $CSSS^*$ | $BSSS^*$ |
|-------|----------|----------|
| $l_s(W)$ | $h$      | $h + 1$  |
| $l_c(W)$ | $L$      | $L - 1$  |

where $L = \sum s_h - (h - 1)$ and $* \in \{E, O\}$.

Proof. The description of $BSSS^*$ in Lemma 2 implies $l_s(W) = h + 1$. $W$ is written as a product of divisors of $\Delta_3$ as below:

$$\underbrace{(x_{i_1})(x_{i_1}) \cdots (x_{i_1})}_{s_1-1 \text{ times}} \underbrace{(x_{i_1} x_{i_2})}_{s_2-2 \text{ times}} \underbrace{(x_{i_2})}_{s_3-1 \text{ times}} \cdots \underbrace{(x_{i_2} x_{i_3})}_{s_4-1 \text{ times}} \cdots \underbrace{(x_{i_h})}_{s_{h-1} \text{ times}} \cdots \underbrace{(x_{i_h})}_{s_{h-1} \text{ times}}.$$  

Therefore $l_c(W)$ of $CSSS^* = \sum s_h - (h - 1)$ and obviously $l_c(W)$ of $BSSS^* = \sum s_h - (h - 2)$ ($BSSS^*$ has one more divisors of length 2 than $CSSS^*$). \qed
Proof of the Theorem 3: By Lemma 3 the super summit words of a general word are precisely the words of $BSSS^*$ and hence they must satisfy (i) of Th 2. It is also easy that $\Delta_3^{even}, \Delta_3^{odd}x_i$ and $\Delta_3^{even}x_1x_2$ are super summit words. □

Super summit set and ultra summit set are the same in $B_3[13]$. Therefore the Theorem 3 is also valid for ultra summit words.

The following Corollaries are consequences of the above Lemma 3.

Corollary 2. The following are true for a general summit word $\Delta_3^rW$:
(a) If $\Delta_3^rW_1$ is in summit set of $\Delta_3^rW$ then $|l_s(W_1) - l_s(W)| \leq 1$;
(b) The summit set of $\Delta_3^rW$ has at least one element $\Delta_3^rW_2$ such that $l_s(W_2) \equiv r \pmod{2}$;
(c) The summit set of $\Delta_3^rW$ has at least one element $\Delta_3^rW_3$ such that $l_s(W_3) \equiv r + 1 \pmod{2}$.

Corollary 3. If $\Delta_3^rW_1$ and $\Delta_3^rW_2$ are general words from the same summit set then $l_s(W_1) > l_s(W_2)$ if and only if $l_c(W_1) < l_c(W_2)$.

Remark 4: If $SSS(C)$ of a word in $B_3$ is a proper subset of its $SS(C)$ then the smallest summit word lies in the complement of $SSS(C)$.

Remark 5: The cardinality of $SSS(C)$ $\geq$ the cardinality of its Complement.

For example if $\Delta_3^{2m}W = \Delta_3^{2m}x_1^{s_1}x_2^{s_2} \cdots x_h^{s_h}$ for $h = even$ is a general summit word such that the set of $s_i$ is nonperiodical then the cardinality of $SSS(C)$ is precisely $2 \sum_{i=1}^h (s_i - 1)$ which is much smaller than the cardinality $2 \sum_{i=1}^h 1 = 2h$ of its complement.

The smallest summit word in the summit set of a general summit word $\Delta_3^rW = \Delta_3^r x_1^{t_1}x_2^{t_2} \cdots x_h^{t_h}$ is found in the following way:
Algorithm 1. By Corollary 2 $\Delta_3^r W$ is conjugate to $\Delta_3^r x_{s_1}^{s_1} x_{s_2}^{s_2} \cdots x_{s_h}^{s_h}$ such that $h \equiv r \pmod{2}$.

**step(1)** Find $s_k = \max\{s_j\}_{j=1,h}$. If $s_k$ is unique, then GO TO step 2. Otherwise find $s_k$ such that $s_{k+1} < s_{l+1}$ for all $s_l = \max\{s_j\}$ and if $s_{k+1}$ is unique, GO TO step 2. If this is not the case, then find $s_k$ minimal with $s_k$ maximal such that $s_{k+2} > s_{l+2}$ for all $s_l = \max\{s_j\}$ and if $s_{k+2}$ is unique, GO TO step 2. Otherwise repeat the process; if the sequence of exponents is periodical, apply the above algorithm for a single period. Since $\{s_j\}$ is finite the process is also finite.

**step(2)** If $r$ is even then $\Delta_3^r x_{s_1}^{s_1} x_{s_2}^{s_2} \cdots x_{s_h}^{s_h} x_{s_1}^{s_1} x_{s_2}^{s_1} \cdots x_{s_h}^{s_1}$ is the smallest, if $r$ is odd $\Delta_3^r x_{s_1}^{s_1} x_{s_2}^{s_2} \cdots x_{s_h}^{s_h} x_{s_1}^{s_1} x_{s_2}^{s_1} \cdots x_{s_h}^{s_1}$ is the smallest.

**Definition 8.** The words obtained as outputs of the previous algorithm are said to satisfy max-min condition.

**Proof of the Theorem 4:** The conditions (a), (b) and (c) of the theorem are obvious consequences of the description of $CSSS^*$ and $BSSS^*$ for $* \in \{E, O\}$. It is also easy to see that the smallest summit word in the summit set of $\Delta_3^{odd}$ and $\Delta_3^{even} x_{s_1}^{s_1}$ ($s_1 \geq 1$) is $\Delta_3^{odd}$ and $\Delta_3^{even} x_{s_1}^{s_1}$ respectively.

The word obtained by algorithm 1 satisfied conditions of Theorem 4 and compute the smallest summit word of a given summit word. For example the summit words $V = \Delta_3^{2m} x_1^2 x_2^4 x_3^5 x_4^2 x_5^2 x_6^5 x_7^3 x_1$ and $W = \Delta_3^{2m} x_2^3 x_3^5 x_4^2 x_5^3 x_6^2 x_7^4 x_1$. By Corrolary 2 $V \sim \Delta_3^{2m} x_1^3 x_2^4 x_3^5 x_4^2 x_5^2 x_6^5 x_7^3 x_1$ and the sequence of exponents is nonperiodical so the smallest word in its summit set is $\Delta_3^{2m} x_1^5 x_2^4 x_3^5 x_4^3 x_5^3 x_6^2$. On the other hand the sequence of exponents in $W$ is periodical, so the smallest word in its summit set
is $\Delta_3^{2m} x_1^5 x_2^3 x_3^2 x_1^5 x_3^3 x_2^2$.

**Definition 9.** The unique representative $(r; (s_1, s_2, \ldots, s_h))$ of a summit set of a word $V$ corresponding to the smallest summit word $\Delta_3^r W = \Delta_3^r x_{i_1}^s x_{i_2}^s \cdots x_{i_h}^s$. This unique representative (conjugacy invariant) of summit set is called Artin-invariant and it is denoted by $A^*(V)$.

**Testing Conjugacy of Elements:** Conjugacy of elements $\alpha$ and $\beta$ of $B_3$ can be tested as follow:

**Step 1.** Write $\alpha$ and $\beta$ in normal form: $\alpha = \Delta_3^{r_1} W_1$, $\beta = \Delta_3^{r_2} W_2$.

**Step 2.** Check whether $\alpha$ and $\beta$ are summit words or not according to Theorem 2. If they are not summit words then conjugate with $A^{\pm 1} \in \text{Div}$ and increase the exponents, like Garside done in [8].

**Step 3.** If the summit words of $\alpha$ and $\beta$ have different exponents, then they are not conjugate. Otherwise find the Artin-invariants of both of them. If they have the same Artin-invariants then $\alpha$ and $\beta$ are conjugates, otherwise not.

### 3. Knots of Braid Index $\leq 3$

Birman and Menasco [4, 5] gave the following classification theorem and invertibility theorem about 3-closed braids:

**Theorem 6.** (*Birman and Menasco [4, 5]*) Let $\mathcal{L}$ be a link type which is represented by the closure of a 3-braid $L$. Then the following holds:

(a) $\mathcal{L}$ has braid index 3 and every 3-braid which represents $\mathcal{L}$ is conjugate to $L$.

(b) $\mathcal{L}$ has braid index 3, and is represented by exactly two distinct
conjugacy classes of closed 3-braids. This happens if and only if the
conjugacy class of \( L \) contains a braid whose associated (open) braid is
conjugate to \( x_1^u x_2^v x_1^w x_2^\varepsilon \) for some \( u, v, w \in \mathbb{Z} \), \( \varepsilon = \pm 1 \).

(c) The braid \( L \) is conjugate to \( x_1^k x_2^{\pm 1} \) for some \( k \in \mathbb{Z} \). These are
precisely the links which are defined by closed 3-braids, but have index
less than 3.

**Theorem 7. (Birman and Menasco [4, 5])** Let \( K \) be a link of
braid index 3 with oriented 3-braid representative \( \overrightarrow{K} \). Then \( K \) is non-
invertible if and only if \( \overrightarrow{K} \) and \( \overleftarrow{K} \) are in distinct conjugacy classes,
and the class of \( \overrightarrow{K} \) does not contain a representative Whose associated
(open) braid is conjugate to \( x_1^u x_2^v x_1^w x_2^\varepsilon \) for some \( u, v, w \in \mathbb{Z} \), \( \varepsilon = \pm 1 \).

In [9] P.J Xu defined a unique representative (Xu-invariant) in the
summit set of a word following the band presentation given as:

\[
B_3 = \left\langle a_1, a_2, a_3 : a_2 a_1 = a_3 a_2 = a_1 a_3 \right\rangle,
\]

where \( a_1 = x_1 \), \( a_2 = x_2 \) and \( a_3 = x_1^{-1} x_2 x_1 \). Like Xu-invariant, we
defined Artin-invariant in the classical generators in the previous sec-
tion. The advantage of working with Artin-invariant \( A^*(V) \) instead
of Xu-invariant \( X^*(V) \) or vise versa is not completely understood. A
computer program for finding the Artin-invariant of an arbitrary word
in \( B_3 \) is already available at [14]. These invariants are important for the
implementation of classification and invertibility theorems mentioned
above. The following Tables I and II given by Birman and Menasco [?] in
Xu-invariants are translated in terms of the Artin-invariants.
Notation:  $2^k$ stands for $k$-tuples of the form $(2, 2, \cdots, 2)$ in the following tables.

| $V$                | smallest summit word | $A^*(V)$                  |
|--------------------|----------------------|---------------------------|
| $x_2$              | $x_1$                | $(0; (1))$                |
| $x_1x_2$           | $x_1x_2$             | $(0; (1, 1))$             |
| $x_1^kx_2$, $k \geq 2$ | $\Delta_3 x_1^{k-2}$ | $(1; (k - 2))$           |
| $x_1^kx_2^{-1}$, $k \geq 0$ | $\Delta_3^{-1} x_1^{k+2}$ | $(-1; (k + 2))$       |
| $x_1^kx_2$, $k < 0$ | $\Delta_3^{-k} x_1 x_2 x_1^{2} x_2^{2} \cdots |_{|k| - 1 \text{ times}}$ | $(k; (3, 2^{|k| - 1}))$ |
| $x_1^{-1}x_2^{-1}$ | $\Delta_3^{-1} x_1$  | $(-1; (1))$               |
| $x_1^{-2}x_2^{-1}$ | $\Delta^{-1}$       | $(-1; (1))$               |
| $x_1^{-3}x_2^{-1}$ | $\Delta_3^{-2} x_1 x_2$ | $(-2; (1, 1))$           |
| $x_1^{-4}x_2^{-1}$ | $\Delta_3^{-2} x_1$  | $(-2; (1))$               |
| $x_1^kx_2^{-1}$, $k \leq -5$ | $\Delta_3^{k+2} x_1^{3} x_2^{2} x_1^{2} x_2^{2} \cdots |_{|k| - 5 \text{ times}}$ | $(k + 2; (3, 2^{|k| - 5}))$ |

Table I: Artin-invariants of links of braid index $\leq 3$.  

| $\varepsilon$ | $V$                | $A^*(V)$                  |
|----------------|--------------------|---------------------------|
| $+$            | $x_2^p x_1^q x_2^{-1}$ | $(1; (p+1, q, r-1))$ or $(1; (q, r-1, p-1))$ or $(1; (r-1, p-1, q))$ |
| $+$            | $x_1^{-p} x_2^q x_2^{-1}$ | $(-p; (r, 2^p, q+1))$ for $r \geq q+1$ and $(-p; (q+1, r, 2^p))$ for $r < q+1$ |
| $+$            | $x_1^p x_2^{-q} x_1 x_2^{-1}$ | $(-q+1; (p, 2^q-1, r))$ for $p \geq r$ and $(-q+1; (r, p, 2^{q-1}))$ for $p < r$ |
| $+$            | $x_1^p x_2^q x_1^{-r} x_2$ | $(-r; (q+1, 2^p, p))$ for $q+1 \geq p$ and $(-r; (p, q+1, 2^p))$ for $q+1 < p$ |
| $-$            | $x_1^{-p} x_2^q x_1 x_2^{-1}$ | $(-1; (p+1, q, r+1))$ or $(-1; (q, r+1, p+1))$ or $(-1; (r+1, p+1, q))$ |
| $-$            | $x_1^{-p} x_2^q x_1^{-r} x_2$ | $(-p; (r+2, 2^p-2, q+1))$ for $r+1 \geq q$ and $(-p; (q+1, r+2, 2^p-2))$ for $r+1 < q$ |
| $-$            | $x_1^p x_2^{-q} x_1^{-r} x_2^{-1}$ | $(-q-1; (p+2, 2^{q-1}, r+2))$ for $p \geq r$ and $((-q-1; (r+2, p+2, 2^{q-1}))$ for $p < r$ |
| $-$            | $x_1^p x_2^q x_1^{-r} x_2^{-1}$ | $(-r; (q+1, 2^{q-2}, p, 2^p))$ for $q \geq p+1$ and $(-r; (p+2, q+1, 2^{q-2}))$ for $q < p+1$ |

Table II: Artin-invariants of links in Th 6b and Th 7.
4. Hilbert series

The number of both smallest E-summit words and smallest O-summit words in $B_3$ for a given length $n$ are computed in this section. Hilbert series corresponding to conjugacy classes are then obtained as Theorem 5. The growing functions of these series is discussed at the end of the section. In [9] Xu gave Hilbert series for the conjugacy classes of minimal word length which are different of the series given in this section.

**Proposition 1.** The number of smallest E-summit words $W$ of length $|W| = n \geq 3$ is given by

$$1 + \sum_{k=1}^{\left\lceil \frac{n}{2k} \right\rceil} \frac{1}{2k} \sum_{d|a} d \sum_{d|d} \mu\left(\frac{d}{\delta}\right) \left(\frac{\delta(c - b) - 1}{\delta b - 1}\right),$$

where $a = \gcd(n, 2k)$, $n = ac$ and $2k = ab$.

**Proof.** Let $W = x_{i_1}^{s_1} x_{i_2}^{s_2} \cdots x_{i_h}^{s_h}$, $|W| = n \geq 3$, be a smallest E-summit word; Theorem 4 the following are true:

(i) either $h = 1$ or $h = 2k$, $k \in \mathbb{Z}^+$;

(ii) for $h = 1$, there is only one smallest E-summit word: $\{x_1^n\}$;

(iii) $s_i \geq 2 \quad \forall \ 1 \leq j \leq 2k$;

(iv) $\sum_{j=1}^{2k} s_j = n$.

Now we calculate the number of smallest E-summit words for a fixed $k$. First the cardinality of the following set of exponents

$$E(n, k) = \left\{(s_1, s_2, \ldots, s_{2k}, s_{2k}) \in \mathbb{N} \mid s_i \geq 2 \text{ and } \sum_{j=1}^{2k} s_j = n\right\}$$

is given by

$$e(n, k) = \binom{n - 2k - 1}{2k - 1} = \binom{a(c - b) - 1}{ab - 1}$$
(look at the equation $\sum_{i=1}^{2k} (s_i - 1) = n - 2k$, where $(s_i - 1)$ are positive integers). Now we replace the pair $(n, k)$ by the triple $(a, b, c)$, where $(b, c) = 1$, and, accordingly, $E(n, k)$ by $E(a, b, c)$, and $e(n, k)$ by $e(a, b, c)$. Letting $2k = qp$ we introduce two new sets:

$E(a, b, c, q) = \{(s_1, s_2, \ldots, s_{2k}) \in E(a, b, c) \text{ with minimal period of length } p\}$,

(if $q \nmid ab$ or $q \nmid ac$, this set is empty), and also:

$M(b, c, q) = \{(s_1, s_2, \ldots, s_p) \text{ is not periodic and } \sum_{i=1}^{p} s_i = c \frac{p}{b}\}$.

Now it is clear that $E(a, b, c) = \coprod_{d \mid a} E(a, b, c, d)$ and $E(a, b, c, q) \approx M(b, c, q)$ so $E(a, b, c) \approx \coprod_{d \mid a} M(b, c, d)$. Therefore $e(a, b, c) = \sum_{d \mid a} m(b, c, d)$, where $m(b, c, d)$ is the cardinality of $M(b, c, d)$. By applying Möbius inversion formula, we have

$$m(b, c, d) = \sum_{\delta \mid d} \mu\left(\frac{d}{\delta}\right) e(\delta, b, c) = \sum_{\delta \mid d} \mu\left(\frac{d}{\delta}\right) \left(\delta(c - b) - 1\right) \left(\delta b - 1\right).$$

$C_{n,k}$, the set of non-periodic sequences of length $b(\frac{n}{d})$, up to cyclic permutations, has the cardinality given by

$$c_{n,k} = \frac{1}{b a} \sum_{\delta \mid d} \mu\left(\frac{d}{\delta}\right) \left(\delta(c - b) - 1\right).$$

Hence the number of smallest E-summit words of length $\geq 3$

$$1 + \sum_{k=1}^{\left\lfloor \frac{n}{4} \right\rfloor} \frac{1}{2k} \sum_{d \mid a} d \sum_{\delta \mid d} \mu\left(\frac{d}{\delta}\right) \left(\delta(c - b) - 1\right) \left(\delta b - 1\right).$$

□

**Proposition 2.** The number of smallest O-summit words of length $W$ of length $|W| = n \geq 6$ is given by

$$1 + \sum_{k=1}^{\left\lfloor \frac{n-2}{3} \right\rfloor} \frac{1}{2k + 1} \sum_{d \mid a} d \sum_{\delta \mid d} \mu\left(\frac{d}{\delta}\right) \left(\delta(c - b) - 1\right) \left(\delta b - 1\right).$$
where \( a = \gcd(n, 2k + 1), \ n = ac \) and \( 2k + 1 = ab \).

**Proof.** Let \( W = x_{i_1}^{s_1} x_{i_2}^{s_2} \cdots x_{i_h}^{s_h} \) and \( |W| = n \geq 6 \) be a smallest O-summit word, then the following are true by Theorem 4:

(i) either \( h = 1 \) or \( h = 2k + 1, \ k \in \mathbb{Z}_+ \);

(ii) for \( h = 1 \), there is only one smallest O-summit word: \( \{x_1^n\} \);

(iii) \( s_i \geq 2, \ \forall \ 1 \leq j \leq 2k + 1 \);

(iv) \( \sum_{j=1}^{2k+1} s_j = n \).

Now the number of smallest O-summit words for a fixed \( k \) can be calculated as in proposition 1. \( \square \)

**Proof of the Theorem 5** The only E-summit set for \( |W| = 0 \) is \( \{e\} \), the E-summit set for \( |W| = 1 \) is \( \{x_1, x_2\} \) and for \( |W| = 2 \), the two E-summit sets are \( \{x_2^1, x_2^2\} \) and \( \{x_1x_2, x_2x_1\} \). This list of E-summit sets and proposition 1 implies Theorem 5a.

Similarly, the O-summit sets for \( |W| = 0, 1, 2, 3, 4 \) and 5 are \( \{e\}, \{x_1, x_2\}, \{x_1^2, x_2x_1^2, x_2^2\}, \{x_1^3, x_1^2x_2, x_1x_2^2, x_2x_1^2, x_2x_2^2, x_2^3\}, \{x_1^4, x_1^3x_2, x_1^2x_2^2, x_2x_1^3, x_2^3, x_2^2x_1, x_2^2, x_2x_1^2, x_2x_2^2, x_2^3x_1, x_2^3, x_2^2x_1, x_2^5\} \) respectively. This list of O-summit sets and proposition 2 implies Theorem 5b. \( \square \)

The first few terms of the two series are:

\[
H^E(t) = 1 + t + 2t^2 + t^3 + 2t^4 + 2t^5 + 3t^6 + 3t^7 + 5t^8 + 5t^9 + 8t^{10} + 10t^{11} + 17t^{12} + \cdots
\]

and

\[
H^O(t) = 1 + t + t^2 + t^3 + t^4 + t^5 + 2t^6 + 2t^7 + 3t^8 + 5t^9 + 7t^{10} + 9t^{11} + 14t^{12} + \cdots
\]

Next the nature of growth of these Hilbert series is discussed. All the symbols and notations used onward are those of Theorem 5a and the proof of Proposition 1.
**Lemma 4.** The following holds for $n \geq 8$:

(i) $\frac{2^{\lceil \frac{n}{8} \rceil}}{2^{\lceil \frac{n}{8} \rceil}(2^{\lceil \frac{n}{8} \rceil} + 1)} \leq \sum_{k=1}^{\lceil \frac{n}{4} \rceil} \frac{1}{2^k} e(n, k)$;

(ii) $\sum_{k=1}^{\lceil \frac{n}{4} \rceil} e(n, k) \leq 2^{n-3}$.

**Proof.** Since $e(n, \lfloor \frac{n}{8} \rfloor)$ is a part of $\sum_{k=1}^{\lfloor \frac{n}{4} \rfloor} e(n, k)$ so

$$e(n, \lfloor \frac{n}{8} \rfloor) \leq \sum_{k=1}^{\lfloor \frac{n}{4} \rfloor} e(n, k)$$

Now we prove that

(3) $\binom{2^{\lceil \frac{n}{8} \rceil}}{\lfloor \frac{n}{8} \rfloor} \leq e(n, \lfloor \frac{n}{8} \rfloor)$.

Let $\lfloor \frac{n}{8} \rfloor = m$ then $n \in \{8m, 8m + 1, \cdots, 8m + 7\}$. First (2) is true for $n = 8m$ i.e.

$$\binom{n - 2\lfloor \frac{n}{8} \rfloor - 1}{2\lfloor \frac{n}{8} \rfloor - 1} = \binom{6m - 1}{2m - 1} \geq \binom{6m - 1}{m} \geq \binom{2m}{m}.$$  

The inequality (2) holds for all $n \in \{8m, 8m + 1, \cdots, 8m + 7\}$ because $\lfloor \frac{n}{8} \rfloor$ is constant and for $n < n'$ we have $e(n, \lfloor \frac{n}{8} \rfloor) < e(n', \lfloor \frac{n'}{8} \rfloor)$. The inequality $\frac{2^{2m}}{2m+1} \leq \binom{2m}{m}$ completes the proof (i) of the Lemma.

By the inequality

$$\binom{n}{k} \leq 2^{n-1}$$

We have

$$\sum_{k=1}^{\lfloor \frac{n}{4} \rfloor} e(n, k) \leq \sum_{k=1}^{\lceil \frac{n}{4} \rceil} 2^{n-2k-2} \leq 2^{n-3}.$$ 

□

**Corollary 4.** The Hilbert series $H^E(t)$ and $H^O(t)$ grow exponentially.
Proof. Consider the finite set \( E(n, k, q) \) of sequences of exponents having minimal period \( p \leq 2k \) given by

\[
E(n, k, q) = \{(s_1, s_2, \ldots, s_{2k-1}, s_{2k}) \in \mathbb{N} \mid s_i \geq 2 \text{ and } \sum_{j=1}^{2k} s_j = n\}
\]

where \( |E(n, k, q)| = e(n, k, q) \). By definition \( c_{n,k,q} \) is given by \( \frac{e(n,k,q)}{p} \).

We also know that \( e(n, k) = \sum_{q \mid a} e(n, k, q) \), where \( a = gcd(n, 2k) \) and \( 2k = qp \). Therefore we have:

\[
\frac{1}{2k} e(n, k) \leq c_{n,k} \leq e(n, k).
\]

The above inequality clearly shows that the coefficient \( C_n \) of \( H^E(t) \) satisfy

\[
\sum_{k=1}^{[\frac{n}{2}] \leq C_n \leq \sum_{k=1}^{[\frac{n}{2}] e(n, k)}.
\]

The result follows by Lemma 4. Similarly, it can be proved that \( H^O(t) \) also grows exponentially.

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The computer program is available at [14]. This program is able to compute the following for arbitrary words in \( B_3 \):

- Artin-invariant (or the smallest summit word)
- Garside Normal Form
- Summit word
- Test equality of two words.
- Test conjugacy of two words.
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