ON GLOBAL LARGE ENERGY SOLUTIONS TO THE VISCOUS SHALLOW WATER EQUATIONS

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Abstract. By exploring the smooth effect of the heat flows and the weighted-
Chemin-Lerner technique, we obtain the global solutions of large energy to the
viscous shallow water equations with initial data in the critical Besov spaces,
which improves the previous small energy type arguments [5], [13]. Moreover,
the method used here is quiet different from [5], [13].

1. Introduction and the main result. In this paper, we mainly study the global
well-posedness of the viscous shallow water equations in \(\mathbb{R}^2\):

\[
\begin{aligned}
\partial_t h + \text{div}(hu) &= 0, \\
hu_t + hu \cdot \nabla u - \nu \text{div}(2hD(u)) - \nu \nabla(h\text{div}u) + h\nabla h &= 0,
\end{aligned}
\]

\( (h, u) |_{t=0} = (h_0, u_0), \tag{1} \)

where \(h(t, x)\) is the height of fluid surface, \(u(t, x) = (u_1(t, x), u_2(t, x))\) is the
horizontal velocity vector field, \(D(u) = \frac{1}{2} (\nabla u + \nabla u^t)\) is the deformation tensor, and
\(\nu > 0\) is the viscous coefficient.

The shallow water equations are the simplest form of equation of motion that
can be used to describe the horizontal structure of the atmosphere. They describe
the evolution of an incompressible fluid in response to gravitational and rotational
accelerations. The solutions of the shallow water equations represent many types
of motion, including Rossby waves and inertia-gravity waves. One can find more
details of the derivation for the viscous shallow water equations in [2, 3]. Bui in
[12] proved the local existence and uniqueness of classical solutions to the Cauchy-
Dirichlet problem for the shallow water equations with initial data in \(C^{2+\alpha}\) by using
Lagrangian coordinates and H"older space estimates. Following the energy method
of Matsumura and Nishida [8], Kloeden in [7] and Sundbye in [11] independently
obtained the global classical solutions to the Cauchy-Dirichlet problem. Recently,
Wang and Xu in [13] proved the local solutions to system (1) for any initial data,
and global solutions for small initial data in the Sobolev space \(H^s(s > 2)\) as well
as the initial height bounded away from zero. Chen, Miao and Zhang [5] further

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extended the result of [13] with low regularity by using the weighted Besov space. More precisely, they obtained the following result:

**Theorem 1.1.** (see Chen, Miao and Zhang [5]) Let \( \bar{h}_0 \) be a positive constant and \( h_0 \geq \bar{h}_0 \). If there exists a strictly positive constant \( c \) such that

\[
\|h_0 - \bar{h}_0\|_{\dot{B}^0_{p,1}} + \|u_0\|_{\dot{B}^0_{p,1}} \leq c,
\]

then there exist a global unique solution \((h - \bar{h}_0, u)\) of (1) such that \( h \geq \frac{1}{2} \bar{h}_0 \) and

\[
\begin{align*}
\|h - \bar{h}_0\|_{C(\mathbb{R}^+; \dot{B}^1_{2,1})}, & \quad (h - \bar{h}_0)^R \in L^1(\mathbb{R}^+; \dot{B}^2_{2,1}), \\
(h - \bar{h}_0)^H & \in L^1(\mathbb{R}^+; \dot{B}^1_{2,1}), \\
u \in C(\mathbb{R}^+; \dot{B}^0_{2,1}) \cap L^1(\mathbb{R}^+; \dot{B}^2_{2,1}).
\end{align*}
\]

where \( f^L \overset{\text{def}}{=} \sum_{j < 0} \Delta_j f \) and \( f^H \overset{\text{def}}{=} \sum_{j \geq 0} \Delta_j f \).

The aim of this paper is to construct the global solutions of the shallow wave equations with initial data in \( L^p \)-type Besov spaces and large energy. Our key observation is that the energy of the solutions about the post-perturbation equations for the incompressible parts are finite although the energy of the perturbation equations are infinite. The proof relies highly on the perturbation theory and weighted-Chemin-Lerner technique. Denote \( \nu = 1 \), the projector by \( \mathcal{P} = I - Q := I - \nabla \Delta^{-1} \text{div} \) and the heat flows by \( e^{t \Delta} \). We are now in the position to state the main result of the present paper:

**Theorem 1.2.** Let \( \bar{h}_0 \) be a positive constant and \( h_0 \geq \bar{h}_0 \). For any \( 2 \leq p \leq 4 \), we assume that \( h_0 - \bar{h}_0 \in \dot{B}^2_{2,1} \cap \dot{B}^1_{2,1}(\mathbb{R}^2) \), \( \mathcal{P} u_0 \in \dot{B}^{\frac{3}{2}-1}_{p,1}(\mathbb{R}^2) \), \( \mathcal{Q} u_0 \in \dot{B}^0_{2,1}(\mathbb{R}^2) \). If there exist two positive constants \( c_0 \) and \( C_0 \) such that

\[
\|h_0 - \bar{h}_0\|_{\dot{B}^0_{p,1}} + \|\nabla e^{t \Delta} \mathcal{P} u_0\|_{L^1(\mathbb{R}^+; \dot{B}^1_{2,1})} + \|\mathcal{Q} u_0\|_{\dot{B}^0_{2,1}} \leq c_0 \exp \left( -C_0 \|\mathcal{P} u_0\|_{\dot{B}^{\frac{3}{2}-1}_{p,1}} \right),
\]

(2) then the system (1) has a unique global solution \((h - \bar{h}_0, u)\) such that

\[
\begin{align*}
(h - \bar{h}_0)^R & \in L^1(\mathbb{R}^+; \dot{B}^2_{2,1}), \\
(h - \bar{h}_0)^H & \in L^1(\mathbb{R}^+; \dot{B}^1_{2,1}), \\
u \in C(\mathbb{R}^+; \dot{B}^0_{2,1}) \cap L^1(\mathbb{R}^+; \dot{B}^2_{2,1}).
\end{align*}
\]

**Remark 1.** On one hand, for any \( 2 \leq p \leq 4 \), by Lemmas 2.3, 2.7 and 2.10, we have

\[
\begin{align*}
\int_0^t \|e^{t \Delta} \mathcal{P} u_0\|_{\dot{B}^{2}_{p,1}} \, dt & \leq C \int_0^t \left( \|e^{t \Delta} \mathcal{P} u_0\|_{\dot{B}^{\frac{3}{2}-1}_{p,1}} + \|e^{t \Delta} \mathcal{P} u_0\|_{\dot{B}^{\frac{3}{2}+1}_{p,1}} \right) \, dt \\
& \leq C \int_0^t \|e^{t \Delta} \mathcal{P} u_0\|_{\dot{B}^{\frac{3}{2}-1}_{p,1}} \, dt \\
& \leq C \|e^{t \Delta} \mathcal{P} u_0\|_{L^\infty(\dot{B}^{\frac{3}{2}}_{p,1})} \|e^{t \Delta} \mathcal{P} u_0\|_{L^1(\dot{B}^{\frac{3}{2}+1}_{p,1})} \\
& \leq C \|\mathcal{P} u_0\|_{\dot{B}^{\frac{3}{2}-1}_{p,1}}.
\end{align*}
\]

Then, if \( \|h_0 - \bar{h}_0\|_{\dot{B}^0_{2,1}} + \|\mathcal{P} u_0\|_{\dot{B}^{\frac{3}{2}-1}_{p,1}} + \|\mathcal{Q} u_0\|_{\dot{B}^0_{2,1}} \) is small enough, the condition (2) is satisfied automatically. On the other hand, due to the embedding relation
\[ \dot{B}^0_{s,1}(\mathbb{R}^2) \hookrightarrow \dot{B}^{2/3}_{p,1}(\mathbb{R}^2) \] with \( p \geq 2 \), it is easy to see that \( \| \mathcal{P}u_0 \|_{\dot{B}^0_{2,1}} \) can be arbitrarily large. Thus, our theorem extends the previous results obtained by Wang et al. in [13] and Chen et al. in [5]. Moreover, the method used here is quiet different from that of [5,13].

The remaining part of the paper is organized as follows. In Section 2 we give some preliminaries which will be used in the sequel. Sections 3 is devoted to the proof of Theorem 1.2.

2. Preliminary. In this section, we recall some basic facts on Littlewood-Paley theory (see [1]). Throughout this paper, \( C \) stands for a generic positive constant which may vary from line to line. Let \( \mathcal{S}(\mathbb{R}^2) \) be the Schwartz class of rapidly decreasing functions. Choose two nonnegative radial functions \( \chi, \varphi \in \mathcal{S}(\mathbb{R}^2) \) supported respectively in \( B = \{ \xi \in \mathbb{R}^2; |\xi| \leq \frac{2}{3} \} \) and \( C = \{ \xi \in \mathbb{R}^2; \frac{2}{4} \leq |\xi| \leq \frac{8}{5} \} \), such that

\[ \sum_{j \in \mathbb{Z}} \varphi(2^{-j} \xi) = 1, \quad \xi \in \mathbb{R}^2 \setminus \{0\}. \]

For \( h = \mathcal{F}^{-1} \varphi \) and \( \bar{h} = \mathcal{F}^{-1} \chi \), we define the frequency localization operator as follows:

\[ \dot{\Delta}_j f = \varphi(2^{-j} D)f = 2^{2j} \int_{\mathbb{R}^2} h(2^j y)f(x - y) \, dy, \quad j \in \mathbb{Z}, \]

\[ \dot{\mathcal{S}}_j f = \chi(2^{-j} D)f = 2^{2j} \int_{\mathbb{R}^2} \bar{h}(2^j y)f(x - y) \, dy, \quad j \in \mathbb{Z}. \]

Denote by \( \mathcal{S}'_h(\mathbb{R}^2) \) the space of tempered distributions \( u \) such that

\[ \lim_{j \to -\infty} \dot{\mathcal{S}}_j u = 0 \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^2). \]

By telescoping the series, we thus have the following Littlewood-Paley decomposition

\[ u = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u, \quad u \in \mathcal{S}'_h(\mathbb{R}^2). \]

We would like to mention that the Littlewood-Paley decomposition has a nice property of quasi-orthogonality:

\[ \dot{\Delta}_j \dot{\Delta}_k u = 0 \quad \text{if} \quad |j - k| \geq 2 \quad \text{and} \quad \dot{\Delta}_j(\dot{\mathcal{S}}_{k-1} u \dot{\Delta}_k u) = 0 \quad \text{if} \quad |j - k| \geq 5. \]

Lemma 2.1. (Bernstein’s inequalities [1]) Let \( 1 \leq p \leq q \leq \infty \) and \( u \in L^p(\mathbb{R}^2) \). Then

\[ \text{supp} \hat{u} \subset \{ \xi; |\xi| \leq 2^j \} \Rightarrow \| \partial^\alpha u \|_{L^r} \leq C 2^{j|\alpha|+2j(\frac{1}{p} - \frac{1}{2})} \| u \|_{L^p}, \]

\[ \text{supp} \hat{u} \subset \{ \xi; |\xi| \approx 2^j \} \Rightarrow \| u \|_{L^r} \approx C 2^{-j|\alpha|} \sup_{|\beta| = |\alpha|} \| \partial^\beta u \|_{L^p}, \]

where the positive constant \( C \) is independent of \( f \) and \( j \).

Now we recall the definition of homogeneous Besov spaces.

Definition 2.2. Let \( s \in \mathbb{R}, 1 \leq p, r \leq \infty \), the homogenous Besov space \( \dot{B}^s_{p,r} \) is defined by

\[ \dot{B}^s_{p,r} = \{ u \in \mathcal{S}'_h(\mathbb{R}^2); \| u \|_{\dot{B}^s_{p,r}} < \infty \}, \]
where
\[ \|u\|_{\dot{B}^s_{p,r}} = \begin{cases} \left( \sum_{j \in \mathbb{Z}} 2^{jr} \|\hat{\Delta}_j u\|_{L^p}^r \right)^{\frac{1}{r}}, & r < \infty, \\ \sup_{j \in \mathbb{Z}} 2^{jr} \|\hat{\Delta}_j u\|_{L^p}, & r = \infty. \end{cases} \]

The homogenous Besov spaces obey the inclusion relations stated in the following lemma.

**Lemma 2.3.** ([1, 10]) Assume that \( s \in \mathbb{R} \) and \( p, r \in [1, \infty] \).

1. If \( 1 \leq r_1 \leq r_2 \leq \infty \), then \( \dot{B}^s_{p,r_1}(\mathbb{R}^2) \hookrightarrow \dot{B}^s_{p,r_2}(\mathbb{R}^2) \).
2. If \( 1 \leq p_1 \leq p_2 \leq \infty \), then \( \dot{B}^s_{p_1,r}(\mathbb{R}^2) \hookrightarrow \dot{B}^{s-2(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r} \mathbb{R}^2 \).
3. If \( s_1 \neq s_2 \) and \( \theta \in (0, 1) \), then \( [\dot{B}^{s_1}_{p_1,r}, \dot{B}^{s_2}_{p_2,r}]_{\theta,r} = \dot{B}^{s_1+1-(1-\theta)s_2}_{p_2,r} \).

In addition to the general time-space such as \( L^p_T(0, \mathbb{R}^2) \), we introduce a useful mixed time-space homogeneous Besov space \( \dot{L}^p_T(\dot{B}^s_{p,r}) \), which was introduced by Chemin and Lerner in [4].

**Definition 2.4.** Let \( s \in \mathbb{R} \), \( 1 \leq p, r, \rho \leq +\infty \) and \( 0 < T \leq +\infty \). The mixed time-space homogeneous Besov space \( \dot{L}^p_T(\dot{B}^s_{p,r}) \) is defined by
\[ \dot{L}^p_T(\dot{B}^s_{p,r}) = \{ u \in S'_h(\mathbb{R}^2) : \|u\|_{\dot{L}^p_T(\dot{B}^s_{p,r})} < +\infty \}, \]
where
\[ \|u\|_{\dot{L}^p_T(\dot{B}^s_{p,r})} = \left( \sum_{j \in \mathbb{Z}} 2^{jr} \left( \int_0^T \|\hat{\Delta}_j u\|_{L^p}^p \, dt \right)^{\frac{1}{p}} \right)^{\frac{1}{r}}. \]

Using the Minkowski inequality, it is easy to verify that
\[ \dot{L}^p_T(\dot{B}^s_{p,r}) \subseteq \dot{L}^p_T(\dot{B}^s_{p,r}) \text{ if } \rho < r \quad \text{and} \quad \dot{L}^p_T(\dot{B}^s_{p,r}) \subseteq \dot{L}^p_T(\dot{B}^s_{p,r}) \text{ if } \rho \geq r. \]

For the convenience of readers, we recall the following form of weighted Chemin-Lerner type norm:

**Definition 2.5.** Let \( f(t) \in L^1_{loc}(\mathbb{R}^+) \), \( f(t) \geq 0 \). We define
\[ \|u\|_{\dot{L}^q_T(\dot{B}^s_{p,r})} = \sum_{j \in \mathbb{Z}} 2^{js} \left( \int_0^T f(t) \|\hat{\Delta}_j u(t)\|_{L^q}^q \, dt \right)^{\frac{1}{q}} \]
for \( s \in \mathbb{R} \) and \( q \in [1, \infty] \).

The Bony’s decomposition is a very effective method to estimate the nonlinear terms in fluid motion equations. Here, we recall the decomposition in the homogeneous context:
\[ uv = \dot{T}_u v + \dot{T}_u + \dot{R}(u,v), \tag{3} \]
where
\[ \dot{T}_u v \overset{\text{def}}{=} \sum_{j \in \mathbb{Z}} \dot{S}_{j-1} u \dot{\Delta}_j v \quad \text{and} \quad \dot{R}(u,v) \overset{\text{def}}{=} \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u \dot{\Delta}_j v \quad \text{with} \quad \dot{\Delta}_j v \overset{\text{def}}{=} \sum_{|j-j'| \leq 1} \dot{\Delta}_{j'} v. \]

Next, we give some multilinear estimates in the Besov spaces which will be used in this paper.
Lemma 2.6. Let \( p \geq 2, s_1 \leq \frac{2}{p}, s_2 \leq \frac{2}{p} \) and \( s_1 + s_2 > 0 \). For \((u, v) \in \dot{B}_{p,1}^{s_1}(\mathbb{R}^2) \times \dot{B}_{p,1}^{s_2}(\mathbb{R}^2)\), we have

\[
\|uv\|_{\dot{B}_{2,1}^{s_1+s_2-\frac{2}{p}}} \leq C\|u\|_{\dot{B}_{p,1}^{s_1}} \|v\|_{\dot{B}_{p,1}^{s_2}}.
\]

Proof. By Bony’s decomposition, we rewrite

\[
uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u, v).
\]

Using Lemma 2.1 and the property of quasi-orthogonality, we get that

\[
\|\dot{T}_u v\|_{\dot{B}_{2,1}^{s_1+s_2-\frac{2}{p}}}
\]

\[
= \sum_{j \in \mathbb{Z}} 2^{j(s_1+s_2-\frac{2}{p})} \| \sum_{|k-j| \leq 4} \dot{\Delta}_j (\dot{S}_{k-1} u \dot{\Delta}_k v) \|_{L^2}
\]

\[
\leq C \sum_{j \in \mathbb{Z}} 2^{j(s_1+s_2-\frac{2}{p})} \| \dot{S}_{j-1} u \|_{L^\infty} \| \dot{\Delta}_j v \|_{L^2}
\]

\[
\leq C \sum_{j \in \mathbb{Z}} 2^{j(s_1+s_2-\frac{2}{p})} \sum_{k' \leq j-2} \| \dot{\Delta}_{k'} u \|_{L^p} 2^{\frac{2}{p}j} \| \dot{\Delta}_j v \|_{L^2}
\]

\[
\leq C \sum_{j \in \mathbb{Z}} \sum_{k' \leq j-2} \left( \| \dot{\Delta}_{k'} u \|_{L^p} 2^{k's_1} \right) \left( \| \dot{\Delta}_j v \|_{L^2} 2^{j's_2} \right) 2^{(s_{j-k})(\frac{2}{p}-s_1)}.
\]

Obviously, if \( s_1 \leq \frac{2}{p} \), then by Proposition 2.3,

\[
\|\dot{T}_u v\|_{\dot{B}_{2,1}^{s_1+s_2-\frac{2}{p}}} \leq C\|u\|_{\dot{B}_{p,\infty}^{s_1}} \|v\|_{\dot{B}_{p,1}^{s_2}} \leq C\|u\|_{\dot{B}_{p,1}^{s_1}} \|v\|_{\dot{B}_{p,1}^{s_2}}.
\]

Similarly, using the fact \( s_2 \leq \frac{2}{p} \), we also obtain

\[
\|\dot{T}_u v\|_{\dot{B}_{2,1}^{s_1+s_2-\frac{2}{p}}} \leq C\|u\|_{\dot{B}_{p,1}^{s_1}} \|v\|_{\dot{B}_{p,1}^{s_2}}.
\]

Now we turn to estimate the last term. Denote \( \frac{1}{p} \equiv \frac{1}{p} + \frac{1}{q} \leq 1 \). Applying Lemma 2.1, Hölder’s inequality, Young’s inequality and the fact \( s_1 + s_2 > 0 \), one has

\[
\|\dot{R}(u, v)\|_{\dot{B}_{2,1}^{s_1+s_2-\frac{2}{p}}}
\]

\[
\leq \sum_{j \in \mathbb{Z}} 2^{j(s_1+s_2-\frac{2}{p})} \| \sum_{|k-k'| \leq 1, k, k' \geq j-3} \dot{\Delta}_j (\dot{\Delta}_k u \dot{\Delta}_{k'} v) \|_{L^2}
\]

\[
\leq C \sum_{j \in \mathbb{Z}} 2^{j(s_1+s_2-\frac{2}{p})} \sum_{|k-k'| \leq 1, k, k' \geq j-3} \| \dot{\Delta}_k u \dot{\Delta}_{k'} v \|_{L^p} 2^{\frac{2}{p}j}
\]

\[
\leq C \sum_{j \in \mathbb{Z}} \sum_{|k-k'| \leq 1, k, k' \geq j-3} \left( \| \dot{\Delta}_k u \|_{L^p} 2^{k's_1} \| \dot{\Delta}_{k'} v \|_{L^q} 2^{k's_2} \right) 2^{(s_1+s_2)(j-k)}
\]

\[
\leq C \sum_{j \in \mathbb{Z}} \sum_{|k-k'| \leq 1, k, k' \geq j-3} \left( \| \dot{\Delta}_k u \|_{L^p} 2^{k's_1} \right) \left( \| \dot{\Delta}_{k'} v \|_{L^q} 2^{k's_2} \right)
\]

\[
\leq C\|u\|_{\dot{B}_{p,1}^{s_1}} \|v\|_{\dot{B}_{p,1}^{s_2}}.
\]

This completes the proof of the lemma. \( \square \)
Lemma 2.7. For any $2 \leq p \leq 4$, $u \in \dot{B}_{p,1}^{\frac{3}{2}} \cap \dot{B}_{p,1}^{-\frac{3}{2} - 1}(\mathbb{R}^2)$, $v \in \dot{B}_{p,1}^{\frac{3}{2}} \cap \dot{B}_{p,1}^{-\frac{3}{2} - 1}(\mathbb{R}^2)$, there holds
\[
\|uv\|_{\B_{2,1}^{0}} \leq C(\|u\|_{\B_{p,1}^{\frac{3}{2}}}^2 \|v\|_{\B_{p,1}^{-\frac{3}{2} - 1}} + \|v\|_{\B_{p,1}^{\frac{3}{2}}} \|u\|_{\B_{p,1}^{-\frac{3}{2} - 1}}).
\]

Proof. We first write $uv$ as follows
\[
vw = \dot{T}_u v + \dot{T}_v u + \dot{R}(u, v).
\]
Let $\frac{1}{q} = \frac{1}{2} - \frac{1}{p}$, then, we have $2 \leq p \leq 4 \leq q$. For the first term $\dot{T}_u v$, we deduce that
\[
\|\dot{T}_u v\|_{\B_{2,1}^{0}} \leq \sum_{j \in \mathbb{Z}} \sum_{|j-k| \leq 4} \|\Delta_j (\dot{S}_{k-1} u \hat{\Delta}_k v)\|_{L^2}
\leq C \sum_{j \in \mathbb{Z}} \sum_{|j-k| \leq 4} \|\dot{S}_{k-1} u\|_{L^p} \|\hat{\Delta}_k v\|_{L^p}
\leq C \sum_{j \in \mathbb{Z}} \sum_{|j-k| \leq 4} \sum_{\ell \leq k-2} \|\Delta^\ell u\|_{L^p} \|\hat{\Delta}_k v\|_{L^p}
\leq C \sum_{j \in \mathbb{Z}} \sum_{|j-k| \leq 4} \sum_{\ell \leq k-2} 2^{\frac{2\ell}{p}} \|\Delta^\ell u\|_{L^p} \|\hat{\Delta}_k v\|_{L^p}
\leq C \|u\|_{\B_{p,1}^{\frac{2}{2}}} \sum_{j \in \mathbb{Z}} \sum_{|j-k| \leq 4} 2^{\frac{2k}{p}} \|\hat{\Delta}_k v\|_{L^p}
\leq C \|u\|_{\B_{p,1}^{\frac{2}{2}}} \|v\|_{\B_{p,1}^{\frac{2}{2}}}.
\]
Analogously, we have
\[
\|\dot{T}_v u\|_{\B_{2,1}^{0}} \leq C \|v\|_{\B_{p,1}^{\frac{2}{2}}} \|u\|_{\B_{p,1}^{\frac{2}{2}}}.
\]
For the last term $\dot{R}(u, v)$, we get
\[
\|\dot{R}(u, v)\|_{\B_{2,1}^{0}} \leq \sum_{j \in \mathbb{Z}} \sum_{k \geq j-3} \|\Delta_j (\Delta_k u \hat{\Delta}_k v)\|_{L^2}
\leq C \sum_{j \in \mathbb{Z}} \sum_{k \geq j-3} \|\Delta_j u\|_{L^p} \|\Delta_k v\|_{L^p}
\leq C \sum_{j \in \mathbb{Z}} \sum_{k \geq j-3} 2^{\frac{2j}{p} - \frac{1}{2} k} \|\Delta_j u\|_{L^p} \sum_{|\ell-k| \leq 1} \|\Delta^\ell v\|_{L^p}
\leq C \sum_{j \in \mathbb{Z}} \sum_{k \geq j-3} 2^{\frac{2j}{p} - 1 k} \|\Delta_j u\|_{L^p} \|v\|_{\B_{p,1}^{\frac{2}{2}}}
\leq C \|u\|_{\B_{p,1}^{\frac{2}{2}}} \|v\|_{\B_{p,1}^{\frac{2}{2}}}.
\]
This completes the proof of this lemma.

Lemma 2.8. (see [9, Lemma 2.4]) Denote $[\Delta_j, u \cdot \nabla] v = \Delta_j (u \cdot \nabla v) - u \cdot \nabla \Delta_j v$, there holds
\[
\sum_{j \in \mathbb{Z}} \|\Delta_j (u \cdot \nabla v)\|_{L^2} \leq C \|\nabla u\|_{L^\infty} \|v\|_{\B_{2,1}^{0}}.
\]

Lemma 2.9. (see [6, Proposition A.5]) Let $1 \leq p \leq \infty$ and
\[
- \min \left\{ \frac{2}{p}, 1 \right\} < s \leq 1 + \min \left\{ \frac{2}{p}, 1 \right\}.
\]
There exists a constant $C > 0$ depending only on $s$ such that for all $j \in \mathbb{Z}$ and $k = 1, 2$, we have
\[
\|\partial_k \Delta_j, u \cdot \nabla v\|_{L^2} \leq C_j 2^{-j(s-1)} \|\nabla u\|_{\dot{B}^s_{p,1}} \|\nabla v\|_{\dot{B}^s_{p,1}}, \tag{4}
\]
where $(c_j)_{j \in \mathbb{Z}}$ denotes a sequence such that $\|c_j\|_{L^1} \leq 1$.

Let us recall the parabolic regularity estimate for the heat equation to end this section.

**Lemma 2.10.** (see [1]) Let $s \in \mathbb{R}$, $T > 0$, $1 \leq p \leq \infty$ and $1 \leq q_2 \leq q_1 \leq \infty$. Let $u$ satisfy the heat equation
\[
\partial_t u - \Delta u = f, \quad u|_{t=0} = u_0.
\]
Then there holds
\[
\|u\|_{L^p_t (L^{q_1}_{x})} \leq C(\|u_0\|_{\dot{B}^s_{p,1}} + \|f\|_{L^p_t (\dot{B}^{s-2+}_{p,1})}).
\]

In order to deal with composition functions in the Besov spaces, we also need the following proposition:

**Lemma 2.11.** (see [1]) For $\alpha < 0 < \beta$, let $G$ be a smooth function defined on the open interval $(\alpha, \beta)$ so that $G(0) = 0$. Assume that $f : \mathbb{R} \to \mathcal{I} \subset (\alpha, \beta)$ for an interval $\mathcal{I}$. Then we have the estimate
\[
\|G(f)\|_{\dot{B}^s_{p,1}} \leq C\|f\|_{\dot{B}^s_{p,1}} \quad \text{for} \quad 1 \leq p \leq \infty, \quad s > 0.
\]

3. **Proof of Theorem 1.2.** Given $h_0 - \tilde{h}_0 \in \dot{B}^0_{2,1} \cap \dot{B}^1_{2,1}(\mathbb{R}^2)$, $\mathcal{P}u_0 \in \dot{B}^{-1}_{p,1}(\mathbb{R}^2)$, $\mathcal{Q}u_0 \in \dot{B}^0_{2,1}(\mathbb{R}^2)$, it follows from a similar argument as in [5] that there exists a positive time $T$ so that (1) has a unique solutions $(h - \tilde{h}_0, u)$ with
\[
h - \tilde{h}_0 \in C((0,T]; \dot{B}^0_{2,1} \cap \dot{B}^1_{2,1}),
\]
\[
\mathcal{P}u \in C((0,T]; \dot{B}^{-1}_{p,1}) \cap L^1((0,T]; \dot{B}^{1+}_{p,1}), \quad \mathcal{Q}u \in C((0,T]; \dot{B}^0_{2,1}) \cap L^1((0,T]; \dot{B}^2_{2,1}). \tag{5}
\]

Let $T^*$ be the largest time so that there holds (5). Hence to prove Theorem 1.2, we only need to prove that $T^* = \infty$.

Without loss of generality, we assume that $\tilde{h}_0 = 1$. Replacing $h$ by $h + 1$ in (1), we reformulate the system (1) as
\[
\begin{cases}
\partial_t h + \text{div}\ u + \text{div} \ (hu) = 0, \\
\partial_t u + u \cdot \nabla u - \Delta u - 2\nabla \text{div} u + \nabla h = (2D(u) + \text{div} u) \nabla \ln(h + 1), \\
(h,u)|_{t=0} = (h_0,u_0).
\end{cases} \tag{6}
\]

To prove $T^* = \infty$, we need to produce some *a priori* estimates for the incompressible part and the compressible part of the solutions.

Throughout we make the assumption that
\[
\sup_{t \in \mathbb{R}_+, x \in \mathbb{R}^2} |h(t,x)| \leq \frac{1}{2} \tag{7}
\]
which will enable us to use freely the composition estimate stated in Proposition 2.11. Note that as $\dot{B}^0_{p,1} \hookrightarrow \mathcal{L}^\infty$, condition (7) will be ensured by the fact that the constructed solution has small norm.
3.1. Estimates for incompressible part of the solutions. Applying project operator \( P \) to the second equation in (6), we can get the incompressible part of the shallow wave equations as follows:

\[
\begin{align*}
\begin{cases}
\partial_t Pu + P(u \cdot \nabla u) - \Delta P u &= P\left((2D(u) + \text{div } u) \nabla \ln(h + 1)\right), \\
\text{div } P u &= 0, \\
P u|_{t=0} &= Pu_0.
\end{cases}
\end{align*}
\]

Assuming \( Pu_F \) is the solution of the following Cauchy problem of heat equation:

\[
\begin{align*}
\begin{cases}
\partial_t Pu_F - \DeltaPu_F &= 0, \\
Pu_F|_{t=0} &= Pu_0.
\end{cases}
\end{align*}
\]

Denote \( P\bar{u} = Pu - Pu_F \). Subtracting (9) from (8) gives

\[
\begin{align*}
\begin{cases}
\partial_t P\bar{u} - \Delta P\bar{u} &= PM, \\
\text{div } P\bar{u} &= 0, \\
P\bar{u}|_{t=0} &= 0,
\end{cases}
\end{align*}
\]

where

\[
PM = -P(Pu_F \cdot \nabla Pu) - P(P\bar{u} \cdot \nabla Pu_F) - P(Pu_F \cdot \nabla Qu) - P(Q\bar{u} \cdot \nabla Pu) \\
- P(P\bar{u} \cdot \nabla Qu) - P(Q\bar{u} \cdot \nabla Pu) - P(Qu \cdot \nabla Pu) \\
+ P\left((2D(u) + \text{div } u) \nabla \ln(h + 1)\right) - P(Pu_F \cdot \nabla Pu_F).
\]

Throughout this paper, we denote

\[
f(t) \overset{\text{def}}{=} \|Pu_F\|^2_{B^{2,1}_{p,1}} + \|Pu_F\|_{B^{2,1}_{p,1}}, \quad h_\lambda \overset{\text{def}}{=} h \exp\left\{ -\lambda \int_0^t f(\tau) \, d\tau \right\},
\]

with \( \lambda > 0 \) and similar notations for \( P\bar{u}_\lambda, Qu_\lambda \).

Applying Lemma 2.10 to the Cauchy problem (9) gives

\[
\int_0^t f(\tau) \, d\tau = \int_0^t \left( \|Pu_F\|^2_{B^{2,1}_{p,1}} + \|Pu_F\|_{B^{2,1}_{p,1}} \right) \, d\tau \\
\leq \|Pu_0\|_{B^{2,1}_{p,1}}^2 \left( 1 + \|Pu_0\|^2_{B^{2,1}_{p,1}} \right).
\]

So \( h_\lambda, P\bar{u}_\lambda \) and \( Qu_\lambda \) are well defined.

Now, we begin to present the estimates for the incompressible part of the solutions.

Multiplying by \( \exp\left\{ -\lambda \int_0^t f(\tau) \, d\tau \right\} \) on both hand side of the first equation in (10) and applying \( \Delta_j \), we obtain

\[
\partial_t \Delta_j P\bar{u}_\lambda + \lambda f(t) \Delta_j P\bar{u}_\lambda - \Delta \Delta_j P\bar{u}_\lambda = \Delta_j (PM)_\lambda.
\]
Taking the $L^2$ inner product of the resultant equation with $\Delta_j P\bar{u}_\lambda$ gives
\[
\frac{d}{dt} \| \Delta_j P\bar{u}_\lambda \|_{L^2}^2 + \lambda f(t) \| \Delta_j P\bar{u}_\lambda \|_{L^2}^2 + c \| P\bar{u}_\lambda \|_{L^2}^2 \leq \| \Delta_j (PM) \|_{L^2} \| \Delta_j P\bar{u}_\lambda \|_{L^2}. \]
Then we have
\[
\frac{d}{dt} \| \Delta_j P\bar{u}_\lambda \|_{L^2}^2 + \lambda f(t) \| \Delta_j P\bar{u}_\lambda \|_{L^2}^2 + c \| P\bar{u}_\lambda \|_{L^2}^2 \leq \| \Delta_j (PM) \|_{L^2}. \]
Integrating the above inequality over $[0,t]$ and summing up the resultant inequality with respect to $j$, we arrive at
\[
\| P\bar{u}_\lambda \|_{L^\infty_t(B^2_{2,1})} + \lambda \| P\bar{u}_\lambda \|_{L^1_t(B^0_{2,1})} + c \| P\bar{u}_\lambda \|_{L^1_t(B^2_{2,1})} \leq C \int_0^t \| (PM) \|_{B^0_{2,1}} \, dt, \tag{13}
\]
where we use the Minkowski inequality and the fact $P\bar{u}|_{t=0} = 0$. To handle the first two terms in $(PM)_\lambda$, we start with the Lemma 2.6 and interpolation inequality that
\[
\int_0^t \left( \| P(u_F \cdot \nabla P\bar{u}) \|_{B^0_{2,1}} + \| P(Q \bar{u} \cdot \nabla P\bar{u}) \|_{B^0_{2,1}} \right) \, dt \leq C \int_0^t \left( \| P(u_F) \|_{B^0_{p,1}} \| \nabla P\bar{u} \|_{B^0_{2,1}} + \| P\bar{u}_\lambda \|_{B^0_{2,1}} \| \nabla P\bar{u} \|_{B^0_{2,1}} \right) \, dt \leq C \int_0^t \| P(u_F) \|_{B^0_{p,1}} \| \nabla P\bar{u} \|_{B^0_{2,1}} \, dt + C \| P\bar{u}_\lambda \|_{L^0_t(B^0_{2,1})} \leq C \| P\bar{u}_\lambda \|_{L^1_t(B^0_{2,1})} + C \| P\bar{u}_\lambda \|_{L^0_t(B^0_{2,1})}. \tag{14}
\]
Similarly, we get
\[
\int_0^t \left( \| P(u_F \cdot \nabla Q u) \|_{B^0_{2,1}} + \| P(Q \bar{u} \cdot \nabla P\bar{u}) \|_{B^0_{2,1}} \right) \, dt \leq C \int_0^t \left( \| P(u_F) \|_{B^0_{p,1}} \| \nabla Q u \|_{B^0_{2,1}} + \| Q u_\lambda \|_{B^0_{2,1}} \| \nabla P\bar{u} \|_{B^0_{2,1}} \right) \, dt \leq C \int_0^t \| P(u_F) \|_{B^0_{p,1}} \| \nabla Q u \|_{B^0_{2,1}} \, dt + C \| Q u_\lambda \|_{L^0_t(B^0_{2,1})} \leq C \| Q u_\lambda \|_{L^1_t(B^0_{2,1})} + \frac{1}{2} \| Q u_\lambda \|_{L^1_t(B^0_{2,1})}, \tag{15}
\]
and
\[
\int_0^t \left( \| P(Q \bar{u} \cdot \nabla Q u) \|_{B^0_{2,1}} + \| P(u \cdot \nabla Q u) \|_{B^0_{2,1}} \right) \, dt \leq C \int_0^t \| (P\bar{u}, Q u) \|_{B^0_{2,1}} \| (\nabla P\bar{u}_\lambda, \nabla Q u_\lambda) \|_{B^0_{2,1}} \, dt \leq C \| (P\bar{u}, Q u) \|_{L^\infty_t(B^0_{2,1})} \| (P\bar{u}_\lambda, Q u_\lambda) \|_{L^1_t(B^0_{2,1})}. \tag{16}
\]
Applying Lemma 2.6, Lemma 2.11 and the embedding relation \( \dot{B}^{1}_{2,1}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2) \) yields

\[
\int_0^t \left\| \mathcal{P}((2D(u) + \text{div } u) \nabla \ln(h + 1)) \right\|_{\dot{B}^{2}_{p,1}} \, d\tau \leq C \int_0^t \left( \left\| \nabla \mathcal{P} u_F \right\|_{\dot{B}^{2}_{p,1}} + \left\| \nabla \mathcal{P} \bar{u} \right\|_{\dot{B}^{2}_{p,1}} + \left\| \nabla \mathcal{P} \bar{u} \right\|_{\dot{B}^{2}_{p,1}} \right) \left\| \nabla \ln(h + 1) \right\|_{\dot{B}^{2}_{p,1}} \, d\tau.
\]

Choosing \( \lambda \geq 4(C + 1) \), we can obtain the estimate of the incompressible part of the solutions

\[
\left\| \mathcal{P} \bar{u}_\lambda \right\|_{L^\infty_t(\dot{B}^{2}_{p,1})} + \frac{3\lambda}{4} \left\| \mathcal{P} \bar{u}_\lambda \right\|_{L^1_t(\dot{B}^{0}_{p,1})} + \frac{C}{2} \left\| \mathcal{P} \bar{u}_\lambda \right\|_{L^1_t(\dot{B}^{2}_{p,1})} 
\leq C \left( \left\| \mathcal{P} u_\lambda \right\|_{L^1_t(\dot{B}^{2}_{p,1})} + \frac{1}{2} \left\| \mathcal{P} u_\lambda \right\|_{L^1_t(\dot{B}^{2}_{p,1})} + C \left\| \mathcal{P} u_F \cdot \nabla \mathcal{P} u_F \right\|_{L^1_t(\dot{B}^{2}_{p,1})} 
+ C \left( \left\| \mathcal{P} \bar{u}_\lambda \right\|_{L^1_t(\dot{B}^{0}_{p,1})} \right) \left( \left\| \mathcal{P} \bar{u}_\lambda \right\|_{L^1_t(\dot{B}^{2}_{p,1})} \right) \left( \left\| \mathcal{P} \bar{u}_\lambda \right\|_{L^1_t(\dot{B}^{2}_{p,1})} \right) \right).
\]

3.2. Estimates for compressible part of the solutions. In this section, we mainly study the estimates for compressible part of the solutions. We first get by applying operator \( \mathcal{Q} \) to the second equation of (6) that

\[
\begin{align*}
\partial_t h + u \cdot \nabla h + \text{div } \mathcal{Q} u &= -h \text{div } u, \\
\partial_t \mathcal{Q} u + u \cdot \nabla \mathcal{Q} u - \Delta \mathcal{Q} u - 2\nabla \text{div } \mathcal{Q} u + \nabla h &= \mathcal{Q} G,
\end{align*}
\]

with

\[ G = -[\mathcal{Q}, u \cdot \nabla] u - (2D(u) + \text{div } u) \nabla \ln(h + 1). \]

Applying \( \hat{\Delta}_j \) to the first equation in (20) gives

\[
\partial_t \hat{\Delta}_j h + u \cdot \nabla \hat{\Delta}_j h + \text{div } \mathcal{Q} \hat{\Delta}_j u + [\hat{\Delta}_j, u \cdot \nabla] h = -\hat{\Delta}_j (h \text{div } u).
\]

Taking \( L^2 \) inner product of \( \hat{\Delta}_j h \) with (21) and using integrating by parts, we have

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \left\| \hat{\Delta}_j h \right\|_{L^2}^2 + \int_{\mathbb{R}^2} \hat{\Delta}_j h \cdot \text{div } \hat{\Delta}_j \mathcal{Q} u \, dx \\
= \frac{1}{2} \int_{\mathbb{R}^2} \text{div } u [\hat{\Delta}_j h]^2 \, dx - \int_{\mathbb{R}^2} [\hat{\Delta}_j, u \cdot \nabla] h \cdot \hat{\Delta}_j h \, dx \\
- \int_{\mathbb{R}^2} \hat{\Delta}_j (h \text{div } u) \cdot \hat{\Delta}_j h \, dx.
\end{align*}
\]
Applying \( \dot{\Delta}_j \) to the second equation in (20) gives
\[
\partial_t \Delta_j Q u + u \cdot \nabla \Delta_j Q u - 3 \Delta \Delta_j Q u + \nabla \Delta_j h = \Delta_j Q G - [\Delta_j, u \cdot \nabla] Q u. \tag{23}
\]
We can get by using a similar derivation of (22) that
\[
\frac{1}{2} \frac{d}{dt} \| \Delta_j Q u \|_{L^2}^2 + 3 \| \nabla \Delta_j Q u \|_{L^2}^2 - \int_{\mathbb{R}^2} \Delta_j h \cdot \text{div} \Delta_j Q u \, dx
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^2} \text{div} u |\Delta_j Q u|^2 \, dx - \int_{\mathbb{R}^2} [\Delta_j, u \cdot \nabla] Q u \cdot \Delta_j Q u \, dx
\]
\[
+ \int_{\mathbb{R}^2} \Delta_j Q G \cdot \Delta_j Q u \, dx. \tag{24}
\]
Applying the gradient \( \nabla \) on (21), we have
\[
\partial_t \nabla \Delta_j h + u \cdot \nabla \nabla \Delta_j h + \nabla \text{div} \Delta_j Q u = -[\Delta_j \nabla, u \cdot \nabla] h - \nabla \dot{\Delta}_j (\text{div} u). \tag{25}
\]
Taking \( L^2 \) inner product of \( \nabla \Delta_j h \) with the previous equation implies
\[
\frac{3}{2} \frac{d}{dt} \| \nabla \Delta_j h \|_{L^2}^2 + 3 \int_{\mathbb{R}^2} \nabla \Delta_j h \cdot \nabla \text{div} \Delta_j Q u \, dx
\]
\[
= \frac{3}{2} \int_{\mathbb{R}^2} \text{div} u |\nabla \Delta_j h|^2 \, dx - 3 \int_{\mathbb{R}^2} [\Delta_j, u \cdot \nabla] h \cdot \nabla \Delta_j h \, dx
\]
\[
- 3 \int_{\mathbb{R}^2} \nabla \Delta_j (\text{div} u) \cdot \nabla \Delta_j h \, dx. \tag{26}
\]
Testing (23) by \( \nabla \Delta_j h \) and (25) by \( \Delta_j Q u \), we arrive at
\[
\frac{d}{dt} \int_{\mathbb{R}^2} \Delta_j Q u \cdot \nabla \Delta_j h \, dx + \int_{\mathbb{R}^2} |\nabla \Delta_j h|^2 \, dx
\]
\[
+ \int_{\mathbb{R}^2} \nabla \text{div} \Delta_j Q u \cdot \Delta_j Q u \, dx - 3 \int_{\mathbb{R}^2} \Delta \Delta_j Q u \cdot \nabla \Delta_j h \, dx
\]
\[
= - \int_{\mathbb{R}^2} u \cdot \nabla (\Delta_j Q u \nabla \Delta_j h) \, dx + \int_{\mathbb{R}^2} \left( \Delta_j Q G - [\Delta_j, u \cdot \nabla] Q u \right) \cdot \nabla \Delta_j h \, dx
\]
\[
+ \int_{\mathbb{R}^2} \left( -[\Delta_j \nabla, u \cdot \nabla] h - \nabla \dot{\Delta}_j (\text{div} u) \right) \cdot \Delta_j Q u \, dx. \tag{27}
\]
By summing up (22), (24), (26) and (27), we get
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} \left( |\Delta_j h|^2 + |\Delta_j Q u|^2 + 2 \Delta_j Q u \cdot \nabla \Delta_j h + 3 |\nabla \Delta_j h|^2 \right) \, dx
\]
\[
+ \int_{\mathbb{R}^2} \left( 2|\Delta_j \nabla Q u|^2 + |\nabla \Delta_j h|^2 \right) \, dx
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^2} \text{div} u \left( |\Delta_j h|^2 + |\Delta_j Q u|^2 + 3 |\nabla \Delta_j h|^2 + 2 \Delta_j Q u \nabla \Delta_j h \right) \, dx
\]
\[
- \int_{\mathbb{R}^2} \left( [\Delta_j, u \cdot \nabla] h + \Delta_j (\text{div} u) \right) \cdot \Delta_j h \, dx
\]
\[
- 3 \int_{\mathbb{R}^2} \left( [\Delta_j \nabla, u \cdot \nabla] h + \nabla \Delta_j (\text{div} u) \right) \cdot \nabla \Delta_j h \, dx
\]
\[
+ \int_{\mathbb{R}^2} \left( \Delta_j Q G - [\Delta_j, u \cdot \nabla] Q u \right) \cdot \left( \nabla \Delta_j h + \Delta_j Q u \right) \, dx
\]
\[
- \int_{\mathbb{R}^2} \left( [\Delta_j \nabla, u \cdot \nabla] h + \nabla \Delta_j (\text{div} u) \right) \cdot \Delta_j Q u \, dx. \tag{28}
\]
It is readily seen that for all \( j \in \mathbb{Z} \)
\[
\int_{\mathbb{R}^2} \left( |\hat{\Delta}_j h|^2 + |\hat{\Delta}_j Q u|^2 + 2 \hat{\Delta}_j Q u \cdot \nabla \hat{\Delta}_j h + 3 |\nabla \hat{\Delta}_j h|^2 \right) \, dx
\approx \| (\hat{\Delta}_j Q u, \hat{\Delta}_j h, \nabla \hat{\Delta}_j h) \|^2_{L^2}
\]
and
\[
\int_{\mathbb{R}^2} (2|\nabla \hat{\Delta}_j Q u|^2 + |\nabla \hat{\Delta}_j h|^2) \, dx \geq c \min(2^{2j}, 2^{-2}) \| (\hat{\Delta}_j Q u, \hat{\Delta}_j h, \nabla \hat{\Delta}_j h) \|^2_{L^2}
\]
for a constant \( c \).
Therefore, we deduce from (28) that
\[
\frac{d}{dt} \| (\hat{\Delta}_j Q u, \hat{\Delta}_j h, \nabla \hat{\Delta}_j h) \|_{L^2} + c \min(2^{2j}, 2^{-2}) \| (\hat{\Delta}_j Q u, \hat{\Delta}_j h, \nabla \hat{\Delta}_j h) \|_{L^2}
\leq C \| \nabla u \|_{L^\infty} \| (\hat{\Delta}_j Q u, \hat{\Delta}_j h, \nabla \hat{\Delta}_j h) \|_{L^2}
\]
\[
+ C \left( \| (\hat{\Delta}_j (h \text{div } u), \nabla \hat{\Delta}_j (h \text{div } u)) \|_{L^2}
\right.
\]
\[
+ \| \hat{\Delta}_j Q G \|_{L^2} + \| [\hat{\Delta}_j, u \cdot \nabla] h \|_{L^2} + \| (\hat{\Delta}_j, u \cdot \nabla) h \|_{L^2} + \| (\hat{\Delta}_j, u \cdot \nabla) Q u \|_{L^2} \right).
\]
Multiplying by \( \exp \left( -\lambda \int_0^t f(\tau) \, d\tau \right) \) on both hand side of the above equation yields
\[
\frac{d}{dt} \| (\hat{\Delta}_j Q u, \hat{\Delta}_j h, \nabla \hat{\Delta}_j h) \|_{L^2} + \lambda f(t) \| (\hat{\Delta}_j Q u, \hat{\Delta}_j h, \nabla \hat{\Delta}_j h) \|_{L^2}
\]
\[
+ c \min(2^{2j}, 2^{-2}) \| (\hat{\Delta}_j Q u, \hat{\Delta}_j h, \nabla \hat{\Delta}_j h) \|_{L^2}
\leq C \| \nabla u \|_{L^\infty} \| (\hat{\Delta}_j Q u, \hat{\Delta}_j h, \nabla \hat{\Delta}_j h) \|_{L^2}
\]
\[
+ C \| (\hat{\Delta}_j (h \text{div } u), \nabla \hat{\Delta}_j (h \text{div } u)) \|_{L^2}
\]
\[
+ C \left( \| \hat{\Delta}_j Q G \|_{L^2} + \| [\hat{\Delta}_j, u \cdot \nabla] h \|_{L^2} + \| (\hat{\Delta}_j, u \cdot \nabla) h \|_{L^2}
\right.
\]
\[
+ \| (\hat{\Delta}_j, u \cdot \nabla) Q u \|_{L^2} \right).
\]
From (24), we get
\[
\frac{d}{dt} \| \hat{\Delta}_j Q u \|_{L^2} + \lambda f(t) \| \hat{\Delta}_j Q u \|_{L^2} + 2^{2j+1} \| \hat{\Delta}_j Q u \|_{L^2}
\leq C \| \nabla u \|_{L^\infty} \| \hat{\Delta}_j Q u \|_{L^2} + C \| \nabla \hat{\Delta}_j h \|_{L^2} + C \| [\hat{\Delta}_j, u \cdot \nabla] Q u \|_{L^2} + C \| \hat{\Delta}_j Q G \|_{L^2}.
\]
This together with (31) yields
\[
\| (h \lambda, Q u \lambda) \|_{L^\infty(B^2_{2,1})} + \| h \lambda \|_{L^\infty(B^2_{2,1})} + \lambda \| (h \lambda, Q u \lambda) \|_{L^2(B^2_{2,1})}
\]
\[
+ \lambda \| h \lambda \|_{L^2(B^2_{2,1})} + \int_0^t \left( \| (h \lambda, Q u \lambda) \|_{B^2_{2,1}} + \| h \lambda \|_{B^2_{2,1}} \right) \, d\tau
\leq \| (h_0, Q u_0) \|_{B^2_{2,1}} + \| h_0 \|_{B^2_{2,1}} + C \int_0^t \| \nabla u \|_{L^\infty} \| (Q u \lambda, h \lambda, \nabla h \lambda) \|_{B^2_{2,1}} \, d\tau
\]
\[
+ C \int_0^t \sum_{j \in \mathbb{Z}} \| [\hat{\Delta}_j, u \cdot \nabla] h \lambda \|_{L^2} + \| [\hat{\Delta}_j, u \cdot \nabla] Q u \lambda \|_{L^2} + \| [\hat{\Delta}_j \nabla, u \cdot \nabla] h \lambda \|_{L^2}
\]
\[
+ \| (\hat{\Delta}_j (h \text{div } u \lambda), \nabla \hat{\Delta}_j (h \text{div } u \lambda), \hat{\Delta}_j Q G \lambda) \|_{L^2} \, d\tau.
\]
Note that the embedding relation $\dot{B}^{\frac{2}{p}}_{p,1}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$ with $p \geq 2$, Lemmas 2.3, 2.6 and the definition of the weighted-Chemin-Lerner-norm, we get
\[
\int_0^t \| \nabla u \|_{L^\infty} \| (Q u_\lambda, h_\lambda, \nabla h_\lambda) \|_{\dot{B}^0_{2,1}} \, d\tau \\
\leq C \int_0^t \| (Q u_\lambda, h_\lambda, \nabla h_\lambda) \|_{\dot{B}^0_{2,1}} \, d\tau \\
\leq C \int_0^t \| (Q u_\lambda, h_\lambda, \nabla h_\lambda) \|_{\dot{B}^0_{2,1}} \, d\tau \\
\leq C \int_0^t \| (Q u_\lambda, h_\lambda, \nabla h_\lambda) \|_{\dot{B}^0_{2,1}} \, d\tau \\
\leq C \int_0^t \| (Q u_\lambda, h_\lambda, \nabla h_\lambda) \|_{L^1_t(B^{0}_{2,1})} + C \| (Q u, h, \nabla h) \|_{L^\infty_t(B^{0}_{2,1})} \| (Q u_\lambda, h_\lambda, \nabla h_\lambda) \|_{L^1_t(B^{0}_{2,1})}.
\]

Similarly, by Lemma 2.8, we have
\[
\int_0^t \sum_{j \in \mathbb{Z}} \| \dot{\Delta}_j, u \cdot \nabla \|_{L^2} + \| \dot{\Delta}_j, u \cdot \nabla \|_{L^2} \, d\tau \\
\leq C \int_0^t \| \nabla u \|_{L^\infty} \| (Q u_\lambda, h_\lambda) \|_{\dot{B}^0_{2,1}} \, d\tau \\
\leq C \| (Q u_\lambda, h_\lambda) \|_{L^1_t(B^{0}_{2,1})} + C \| (Q u, h) \|_{L^\infty_t(B^{0}_{2,1})} \| (Q u_\lambda, h_\lambda) \|_{L^1_t(B^{0}_{2,1})}.
\]

Applying Lemma 2.9 and the embedding relation $\dot{B}^{\frac{2}{p}}_{2,1}(\mathbb{R}^2) \hookrightarrow \dot{B}^{\frac{2}{p}+1}_{p,1}(\mathbb{R}^2)$ with $p \geq 2$, we also have
\[
\int_0^t \sum_{j \in \mathbb{Z}} \| \dot{\Delta}_j, u \cdot \nabla \|_{L^2} \, d\tau \\
\leq C \int_0^t \| \nabla u \|_{\dot{B}^{\frac{2}{p}+1}_{p,1}} \| \nabla h_\lambda \|_{\dot{B}^0_{2,1}} \, d\tau \\
\leq C \int_0^t \| (Q u_\lambda) \|_{L^1_t(B^{0}_{2,1})} + C \| (Q u, h) \|_{L^\infty_t(B^{0}_{2,1})} \| (Q u_\lambda) \|_{L^1_t(B^{0}_{2,1})}.
\]

Using Lemma 2.6, we easily obtain
\[
\int_0^t \| (\text{div} Q u_\lambda, \nabla \text{div} Q u_\lambda) \|_{\dot{B}^0_{2,1}} \, d\tau \\
\leq C \int_0^t \| \text{div} Q u_\lambda \|_{\dot{B}^1_{2,1}} \, d\tau + C \int_0^t \| \text{div} Q u_\lambda \|_{\dot{B}^1_{2,1}} \, d\tau \\
\leq C \int_0^t \| \text{div} Q u_\lambda \|_{\dot{B}^1_{2,1}} \, d\tau + C \int_0^t \| \text{div} Q u_\lambda \|_{\dot{B}^1_{2,1}} \, d\tau \\
\leq C \| (Q u_\lambda) \|_{L^1_t(B^{0}_{2,1})} + C \| (Q u_\lambda) \|_{L^1_t(B^{0}_{2,1})}.
\]
To get the estimates for the compressible part of the solutions, we have to deal with the last term about $QG$ in (32). In fact, from $u = Pu + Qu = (Pu_F + Pu) + Qu$, one can deduce that

\[ [Q, u \cdot \nabla]u = [Q, (Qu + Pu) \cdot \nabla](Qu + Pu) \]
\[ = [Q, (Qu) \cdot \nabla]Pu + [Q, (Qu) \cdot \nabla]Qu \]
\[ + [Q, (Pu) \cdot \nabla]Qu + [Q, (Pu) \cdot \nabla]Pu \]
\[ = [Q, (Qu) \cdot \nabla]Pu + [Q, (Qu) \cdot \nabla]Pu_F + [Q, (Qu) \cdot \nabla]Qu \]
\[ + [Q, (Pu) \cdot \nabla]Qu + [Q, (Pu_F) \cdot \nabla]Qu + [Q, (Pu) \cdot \nabla]Pu \]
\[ + [Q, (Pu) \cdot \nabla]Pu_F + [Q, (Pu_F) \cdot \nabla]Pu + Q(Qu_F \cdot \nabla)Pu_F. \quad (37) \]

With the aid of Lemma 2.6, we have

\[ \int_0^t \| [Q, (Qu) \cdot \nabla]Pu \|_{L^1_{t,1}} \, dt \leq C \int_0^t \| (Qu) \cdot \nabla Pu \|_{L^1_{t,1}} \, dt \]
\[ \leq C \int_0^t \| Qu \|_{L^\infty_t(B_{p,1}^0)} \| \nabla Pu \|_{L^1_t(B_{p,1}^0)} \, dt \]
\[ \leq C \| Qu \|_{L^\infty_t(B_{p,1}^0)} \| Pu \|_{L^1_t(B_{p,1}^0)}. \quad (38) \]

Following the same line, one has

\[ \int_0^t \left( \| [Q, (Qu) \cdot \nabla]Qu \|_{B_{p,1}^0} + \| [Q, (Pu) \cdot \nabla]Qu \|_{B_{p,1}^0} \right) \, dt \]
\[ \leq C \| (Pu, Qu) \|_{L^\infty_t(B_{p,1}^0)} \| Qu \|_{L^1_t(B_{p,1}^0)} + \| Pu \|_{L^\infty_t(B_{p,1}^0)} \| Pu \|_{L^1_t(B_{p,1}^0)} \]
\[ \leq C \| (Pu, Qu) \|_{L^\infty_t(B_{p,1}^0)} \| (Pu, Qu) \|_{L^1_t(B_{p,1}^0)}. \quad (39) \]

By virtue of Lemma 2.6 and the interpolation inequality, we arrive at

\[ \int_0^t \| [Q, (Pu_F) \cdot \nabla]Qu \|_{B_{p,1}^0} + \| [Q, (Pu_F) \cdot \nabla]Pu \|_{B_{p,1}^0} \, dt \]
\[ \leq C \int_0^t \| Pu_F \|_{B_{p,1}^0} \left( \| Qu \|_{B_{p,1}^0} + \| Pu \|_{B_{p,1}^0} \right) \, dt \]
\[ \leq C \int_0^t \| Pu_F \|_{B_{p,1}^0} \left( \| Qu \|_{B_{p,1}^0} + \| Pu \|_{B_{p,1}^0} \right) \, dt \]
\[ \leq \frac{C}{4} \| (Pu, Qu) \|_{L^1_t(B_{p,1}^0)} + C \int_0^t \| Pu_F \|_{B_{p,1}^0} \left( \| Qu \|_{B_{p,1}^0} + \| Pu \|_{B_{p,1}^0} \right) \, dt \]
\[ \leq \frac{C}{4} \| (Pu, Qu) \|_{L^1_t(B_{p,1}^0)} + C \| Qu \|_{L^1_t(B_{p,1}^0)} + C \| Pu \|_{L^1_t(B_{p,1}^0)}. \quad (40) \]
3.3. By summing up (37)–(41), we obtain
\[
\int_0^t \left( ||Q, (Q u_\lambda) \cdot \nabla P u_F||_{\dot{B}^{2,1}_t} + ||Q, (P \tilde{u}_\lambda) \cdot \nabla P u_F||_{\dot{B}^{2,1}_t} \right) d\tau \\
\leq C \int_0^t \left( ||Q u_\lambda||_{\dot{B}^{2,1}_t} + ||P \tilde{u}_\lambda||_{\dot{B}^{2,1}_t} \right) ||\nabla P u_F||_{\dot{B}^{2,1}_t} d\tau \\
\leq C ||Q u_\lambda||_{L^1_t(\dot{B}^{2,1}_2)} + C ||P \tilde{u}_\lambda||_{L^1_t(\dot{B}^{2,1}_2)}. \quad (41)
\]

By summing up (37)–(41), we obtain
\[
\int_0^t ||(Q, u) \cdot \nabla u_\lambda||_{\dot{B}^{2,1}_2} d\tau \\
\leq \frac{c}{4} ||(P \tilde{u}_\lambda, Q u_\lambda)||_{L^1_t(\dot{B}^{2,1}_2)} + C ||(P u_F, \nabla P u_F)\lambda||_{L^1_t(\dot{B}^{2,1}_2)} \\
+ C ||(P \tilde{u}_\lambda, Q u_\lambda)||_{L^1_t(\dot{B}^{2,1}_2)} + C ||(P \tilde{u}, Q u)||_{L^\infty_t(\dot{B}^{2,1}_2)} ||(P \tilde{u}_\lambda, Q u_\lambda)||_{L^1_t(\dot{B}^{2,1}_2)}. \quad (42)
\]
Combining with (17) and (42), one can get
\[
\int_0^t ||Q G_\lambda||_{\dot{B}^{2,1}_2} d\tau \\
\leq \frac{c}{4} ||(P \tilde{u}_\lambda, Q u_\lambda)||_{L^1_t(\dot{B}^{2,1}_2)} + C ||P u_F, \nabla P u_F||_{L^1_t(\dot{B}^{2,1}_2)} \\
+ C ||(P \tilde{u}_\lambda, Q u_\lambda)||_{L^1_t(\dot{B}^{2,1}_2)} + C ||h||_{L^\infty_t(\dot{B}^{2,1}_2)} + C ||(P \tilde{u}, Q u)||_{L^\infty_t(\dot{B}^{2,1}_2)} ||(P \tilde{u}_\lambda, Q u_\lambda)||_{L^1_t(\dot{B}^{2,1}_2)}. \quad (43)
\]

Inserting (33)–(36) and (43) into (32) and choosing \( \lambda \geq 4(C + 1) \), we have
\[
||h_\lambda, Q u_\lambda||_{L^\infty_t(\dot{B}^{2,1}_2)} + ||h_\lambda||_{L^\infty_t(\dot{B}^{2,1}_2)} + \frac{3\lambda}{4} \big( ||h_\lambda, Q u_\lambda||_{L^1_t(\dot{B}^{2,1}_2)} \big) \\
+ \frac{3\lambda}{4} \big( ||h_\lambda||_{L^1_t(\dot{B}^{2,1}_2)} \big) + \int_0^t \left( \big( ||h_\lambda^L, Q u_\lambda||_{\dot{B}^{2,1}_2} + ||h_\lambda^H||_{\dot{B}^{2,1}_2} \big) \right) d\tau \\
\leq ||(h_0, Q u_0)||_{\dot{B}^{2,1}_2} + ||h_0||_{\dot{B}^{2,1}_2} + C ||P u_F, \nabla P u_F||_{L^1_t(\dot{B}^{2,1}_2)} \\
+ \frac{c}{4} ||P \tilde{u}_\lambda||_{L^1_t(\dot{B}^{2,1}_2)} + C ||P \tilde{u}_\lambda||_{L^1_t(\dot{B}^{2,1}_2)} + C ||h_\lambda||_{L^1_t(\dot{B}^{2,1}_2)} \\
+ C \left( ||h||_{L^\infty_t(\dot{B}^{2,1}_2)} + ||h, P \tilde{u}, Q u||_{L^\infty_t(\dot{B}^{2,1}_2)} \right) ||(P \tilde{u}_\lambda, Q u_\lambda)||_{L^1_t(\dot{B}^{2,1}_2)}. \quad (44)
\]
Combining with (19) and (44), we finally conclude that
\[
||h_\lambda, P \tilde{u}_\lambda, Q u_\lambda||_{L^\infty_t(\dot{B}^{2,1}_2)} + ||h_\lambda||_{L^\infty_t(\dot{B}^{2,1}_2)} \\
+ ||(h_\lambda^L, P \tilde{u}_\lambda, Q u_\lambda)||_{L^1_t(\dot{B}^{2,1}_2)} + ||h_\lambda^H||_{L^1_t(\dot{B}^{2,1}_2)} \\
\leq ||(h_0, Q u_0)||_{\dot{B}^{2,1}_2} + ||h_0||_{\dot{B}^{2,1}_2} + C ||P u_F, \nabla P u_F||_{L^1_t(\dot{B}^{2,1}_2)} \\
+ C \left( ||(h, P \tilde{u}, Q u)||_{L^\infty_t(\dot{B}^{2,1}_2)} + ||h||_{L^\infty_t(\dot{B}^{2,1}_2)} \right) ||(P \tilde{u}_\lambda, Q u_\lambda)||_{L^1_t(\dot{B}^{2,1}_2)}. \quad (45)
\]

3.3. Continuity argument. To complete the proof, we shall use the method of continuity. Choosing a small positive constant \( c_0 \in (0, 1) \), we define
\[
T^{**} = \sup \left\{ t \in [0, T^*] : ||(h, P \tilde{u}, Q u)||_{L^\infty_t(\dot{B}^{2,1}_2)} + ||h||_{L^\infty_t(\dot{B}^{2,1}_2)} + ||(h^L, P \tilde{u}, Q u)||_{L^1_t(\dot{B}^{2,1}_2)} + ||h^H||_{L^1_t(\dot{B}^{2,1}_2)} \leq c_0 \right\}. \quad (46)
\]
In what follows, we will prove that $T^{**} = T^*$ under the assumptions of (2).

If not, assuming that $T^{**} < T^*$, for any $t \leq T^{**}$, we get from (45) that

$$
\|(h, P\bar{u}, Qu)\|_{L^\infty_t(\dot{B}^0_{2,1})} + \|h\|_{L^\infty_t(\dot{B}^0_{2,1})} \\
+ \|(h^L, P\bar{u}, Qu)\|_{L^1_t(\dot{B}^2_{2,1})} + \|h^L\|_{L^1_t(\dot{B}^2_{2,1})} \\
\leq \|(h_0, Qu_0)\|_{\dot{B}^0_{2,1}} + \|h_0\|_{\dot{B}^2_{2,1}} + C\|Pu_F \cdot \nabla Pu_F\|_{L^1_t(\dot{B}^2_{2,1})}. 
$$

(47)

As a consequence,

$$
\|(h, P\bar{u}, Qu)\|_{L^\infty_t(\dot{B}^0_{2,1})} + \|h\|_{L^\infty_t(\dot{B}^0_{2,1})} + \|(h^L, P\bar{u}, Qu)\|_{L^1_t(\dot{B}^2_{2,1})} + \|h^H\|_{L^1_t(\dot{B}^2_{2,1})} \\
\leq \left(\|(h_0, Qu_0)\|_{\dot{B}^0_{2,1}} + \|h_0\|_{\dot{B}^2_{2,1}} + C\|Pu_F \cdot \nabla Pu_F\|_{L^1_t(\dot{B}^2_{2,1})}\right) \exp(\lambda \int_0^T f(\tau) d\tau) \\
\leq \left(\|(h_0, Qu_0)\|_{\dot{B}^0_{2,1}} + \|h_0\|_{\dot{B}^2_{2,1}} + C\|Pu_F \cdot \nabla Pu_F\|_{L^1_t(\dot{B}^2_{2,1})}\right) \\
\times \exp \left(C\|Pu_0\|_{\dot{B}^{\frac{2}{3}-1}_{3,1}} (1 + \|Pu_0\|_{\dot{B}^{\frac{2}{3}-1}_{3,1}})\right) 
$$

(48)

for $t \leq T^{**}$, where we used the estimate (11). If the smallness condition (2) is satisfied, then we deduce from (48) that

$$
\|(h, P\bar{u}, Qu)\|_{L^\infty_t(\dot{B}^0_{2,1})} + \|h\|_{L^\infty_t(\dot{B}^0_{2,1})} + \|(h^L, P\bar{u}, Qu)\|_{L^1_t(\dot{B}^2_{2,1})} + \|h^H\|_{L^1_t(\dot{B}^2_{2,1})} \leq \frac{\epsilon_0}{2} 
$$

for $t \leq T^{**}$, which contradicts with (46). Whence we conclude that $T^{**} = \infty$ and the conclusion of Theorem 1.2 follows.

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