Deep neural network approximation for high-dimensional elliptic PDEs with boundary conditions

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Abstract

In recent work it has been established that deep neural networks are capable of approximating solutions to a large class of parabolic partial differential equations without incurring the curse of dimension. However, all this work has been restricted to problems formulated on the whole Euclidean domain. On the other hand, most problems in engineering and the sciences are formulated on finite domains and subjected to boundary conditions. The present paper considers an important such model problem, namely the Poisson equation on a domain $D \subset \mathbb{R}^d$ subject to Dirichlet boundary conditions. It is shown that deep neural networks are capable of representing solutions of that problem without incurring the curse of dimension. The proofs are based on a probabilistic representation of the solution to the Poisson equation as well as a suitable sampling method.

Key words: High dimensional Approximation, Neural Network Approximation, Monte Carlo Methods

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1 Introduction

The approximation of solutions to partial differential equations (PDEs) in high dimensions by classical algorithms such as finite difference or finite element methods is burdened by the so called curse of dimension. This means that the computational cost to achieve a certain accuracy depends exponentially on the dimension of the domain with respect to the reciprocal of the accuracy as base. This is for example improved in the case of so called sparse tensor discretizations. There the logarithm of the reciprocal of the accuracy is the base, but the dependence with respect to the dimension is still exponential [31]. This curse of dimension does not appear in Monte Carlo methods, which are stochastic methods and converge in the root mean squared sense. These methods are however typically restricted to evaluating the solution of a given PDE at a single point rather than the full computational domain. The approximation of solutions to PDEs in high dimensions on the full computational domain hence remains a challenging problem.

Deep neural networks (DNNs) emerge as an approximation architecture with application in various areas of function approximation theory, which are in many cases as good as the established best in class method, cf. [4, 14, 7, 27]. They are also used in the context of uncertainty quantification to approximate mappings that result in parametrized physical systems, where each realization is computationally expensive and treated as an offline cost, cf. [21, 16, 30, 19, 9]. The weights of DNNs
are usually obtained by approximately solving an optimization problem with a given loss functional defined with computed training data, see for example [21, 22].

Recently, there has been vivid research in the approximation of solutions to PDEs in high dimensions posed on \( \mathbb{R}^d \) by DNNs, cf. [14, 17, 15, 12, 1]. In [14], the authors prove that DNNs are capable to overcome the curse of dimension in the case of certain parabolic PDEs posed on all of \( \mathbb{R}^d \). In several recent works this ability of DNNs has also been proven for certain other PDEs on all of \( \mathbb{R}^d \), also including non-linear PDEs, cf. [17, 4]. However, many applications in engineering and in the sciences require the numerical solution of PDEs with boundary conditions.

Therefore, in this work, we seek to numerically approximate solutions to elliptic PDEs in bounded domains with boundary conditions such that the curse of dimension can be overcome.

In particular we establish the first result on the approximation of solutions to PDEs with boundary conditions without curse of dimension using DNNs.

More precisely, we consider the Poisson equation

\[
-\Delta u = f \quad \text{in } D \quad \text{and } u|_{\partial D} = g.
\]

Our main result, Theorem 4.1, states that the solution \( u \) can be approximated to within accuracy \( \delta \) by a DNN of size scaling polynomially in \( d \) and \( \delta^{-1} \) whenever an analogous approximation property holds for the right hand side \( f \), the boundary condition \( g \), as well as the distance function \( x \mapsto \text{dist}(x, \partial D) \). Theorem 4.1 may thus be interpreted as a “regularity result” in the sense that the property of being representable by DNNs without the curse of dimension is conserved under the solution operator of the Poisson equation.

We explicitly establish the required DNN approximation property for \( x \mapsto \text{dist}(x, \partial D) \) for \( D \) a cube or a Euclidean ball. On the way to this result we derive a novel DNN approximation for the square root function at a spectral rate that may be of independent interest, see Lemma A.1.

Theorem 4.1 is similar in spirit to other existing works [14, 4] where Monte Carlo methods have been used to show existence of the DNN weights, cf. [13]. The approaches and techniques in the presented manuscript differ significantly for the reason that the behavior of the solution near the boundary needs to be taken into account, which complicates the analysis.

The structure of the manuscript is as follows. In Section 2 we briefly recapitulate basic facts on DNNs. In Section 3 we introduce the walk-on-the-sphere algorithm and prove basic properties. It serves as a tool in Section 4 where we show the existence of DNNs that approximate the solution to certain elliptic PDEs with boundary conditions.

2 Neural networks

Let \( \sigma : \mathbb{R} \to \mathbb{R} \) be the rectified linear unit (ReLU) activation function, which is defined by \( \sigma(x) := \max\{0, x\}, x \in \mathbb{R} \). We consider in general fully connected DNNs of depth \( L \in \mathbb{N} \). Let \( (N_i)_{i=0,\ldots,d} \) be a sequence of positive integers. Let \( A^i \in \mathbb{R}^{N_i \times N_{i-1}} \) and \( b^i \in \mathbb{R}^{N_i}, \) \( i = 1, \ldots, L \). We define the realization of the DNN \( \mathbb{R}^{N_0} \ni x \mapsto \phi^L(x) \) by

\[
\mathbb{R}^{N_0} \ni x \mapsto \phi^i(x) := A^i \sigma(\phi^{i-1}(x)) + b^i, \quad i = 2, \ldots, L, \quad \text{with } \mathbb{R}^{N_0} \ni x \mapsto \phi^1(x) := A^1 x + b^1, \quad (1)
\]

where \( \mathbb{R}^N \ni x \mapsto \sigma(x) := (\sigma(x_1), \ldots, \sigma(x_N)), N \in \mathbb{N}, \) is defined coordinatewise. The weights of the ReLU DNN \( \phi^L \) are the entries of \( (A^i, b^i)_{i=1,\ldots,L} \). The size of the ReLU DNN \( \phi^L \) is defined to be the number of non-zero weights and will be denoted by \( \text{size}(\phi^L) \). The width of the ReLU DNN \( \phi^L \) is defined by \( \text{width}(\phi^L) = \max\{N_0, \ldots, N_L\} \) and \( L \) is the depth of \( \phi^L \). Sometimes in the literature [7] it is distinguished between the architecture of the DNN and the realization, which is the function that is induced, see for example [4]. In this manuscript, we shall not make this distinction, since we are mostly interested in asymptotic upper bounds of the size of ReLU DNNs. However, we note that the realization does not uniquely determine the weights. Moreover, also the depth \( L \) in the notation of the ReLU DNN \( \phi^L \) shall not be made explicit in the following (meaning that we will drop the superscript
Let $\phi$ be the Hilbert space of real-valued square integrable functions. The Euclidean norm on functions on $D$ differs from the Lebesgue measure on $D$ by $\delta$. Furthermore, for any $\varepsilon \in (0, 1)$ define the subdomain $D_\varepsilon$ of $D$ by
\[ D_\varepsilon := \{ x \in D : \text{dist}(x, \partial D) > \varepsilon \}. \]

Throughout this manuscript, we will construct ReLU DNNs mostly by composition and addition of already existing ReLU DNNs. The asserted upper bounds in later parts of the manuscript on the size of certain DNNs then result by Lemmas 2.2 and 2.3.

### 3 Basics on the walk-on-the-sphere algorithm

In this work we consider the following elliptic PDE with Dirichlet boundary conditions,
\[ -\Delta u = f \quad \text{in} \ D \quad \text{and} \quad u|_{\partial D} = g. \]  

Here, $D \subset \mathbb{R}^d$ is a convex, bounded domain and $f$ and $g$ are continuous functions. To this end, for any $x_0 \in \mathbb{R}^d$ and $c > 0$, $B(x_0, c) := \{ x \in \mathbb{R} : |x - x_0| \leq c \}$. Let $C^k(\overline{D})$ denote the $k$-times continuously differentiable functions on the closure $\overline{D}$, $k \in \mathbb{N}$. Denote by $L^\infty(D)$ the space of essentially bounded functions on $D$ with the usual norm $v \mapsto \|v\|_{L^\infty(D)} := \text{ess sup}_{x \in D} |v(x)|$. The volume of the domain $D$ with respect to the Lebesgue measure is denoted by $|D|$ and the diameter is denoted by diam$(D)$. The Euclidean norm on $\mathbb{R}^d$ will be denoted by $x \mapsto |x|$. For any measured space $(B, \mathcal{B}, \nu)$ we denote the Hilbert space of real-valued square integrable functions by $L^2(B, \nu)$.

We shall recall some elements from the theory of Brownian motion. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space and let $W_t$, $t \geq 0$, denote a $d$-dimensional Brownian motion, which is adapted to the filtration $\mathcal{F}$, i.e., $W_t$ is $\mathcal{F}_t$-measurable. Define the stochastic process
\[ X_t = X_0 + W_t, \quad t \geq 0, \]
where $X_0$ is $\mathcal{F}_0$-measurable. Further, let us denote by $\mathbb{P}_x$ the probability measure conditioned on $X_0 = x$ for every $x \in \mathbb{R}^d$. The expectation with respect to $\mathbb{P}_x$ will be denoted by $\mathbb{E}_x(\cdot)$.

For any open, non-empty set $\tilde{D} \subset \mathbb{R}^d$, define the first exit time of the process $X_t$ starting at $x \in \tilde{D}$ from $\tilde{D}$ by
\[ \tau_{\tilde{D}} := \inf\{ t \geq 0 : X_t \notin \tilde{D} \}. \]

Furthermore, for any $\varepsilon \in (0, 1)$ define the subdomain $D_\varepsilon$ of $D$ by
\[ D_\varepsilon := \{ x \in D : \text{dist}(x, \partial D) > \varepsilon \}. \]
Lemma 3.1. For any $r > 0$ and $x \in B(0, r) \subset \mathbb{R}^d$, it holds that

\[ E_x(\tau_{B(0, r)}) = \frac{r^2 - |x|^2}{d} \]

Proof. This is explicitly [28] Proposition 3.1.8.

Lemma 3.2. Let $D \subset \mathbb{R}^d$ be a bounded convex domain. For every $\varepsilon \in (0, 1)$ such that $D_\varepsilon \subset D$ defined in (3) is not empty and for every $x \in D_\varepsilon$,

\[ E_x(\tau_D - \tau_{D_\varepsilon}) \leq \text{diam}(D)\varepsilon. \]

Proof. By the tower property,

\[ E_x(\tau_D - \tau_{D_\varepsilon}) = E_x(E_{X_{\tau_{D_\varepsilon}}}(\tau_D - \tau_{D_\varepsilon})) = E_x(E_{X_{\tau_{D_\varepsilon}}}(\tau_D)). \]

Note that $X_t$, $t \geq 0$, under the measure $\mathbb{P}_{X_{\tau_{D_\varepsilon}}}$ has the same distribution as $X_{\tau_{D_\varepsilon}} + \tilde{W}_t$, $t \geq 0$, where $\tilde{W}_t$, $t \geq 0$, is a Brownian motion that is independent from $W_t$, $t \geq 0$. To estimate the expectation of the stopping time conditioned on $X_{\tau_{D_\varepsilon}}$, we use that there exist two parallel hyperplanes $A, B$ of dimension $d - 1$ which do not intersect with $D$. One of the hyperplane $A$ satisfies $\text{dist}(X_{\tau_{D_\varepsilon}}, A) = \varepsilon$. The other hyperplane $B$ satisfies $\text{dist}(X_{\tau_{D_\varepsilon}}, B) = \text{diam}(D)$. Specifically, there exists a unit vector $c \in \mathbb{R}^d$ and $\alpha, \beta \in \mathbb{R}$ such that $A = \{v = x + \alpha c : x \in \mathbb{R}^d, x^\top c = 0\}$ and $B = \{v = x + \beta c : x \in \mathbb{R}^d, x^\top c = 0\}$, where $\beta < \alpha$ and $\alpha - \beta = \varepsilon + \text{diam}(D)$. They exist due to the assumed convexity of the domain $D$. Also define the unbounded domain of points in between $A$ and $B$ by $D = \{x \in \mathbb{R}^d : x^\top c \in (\beta, \alpha)\}$.

It follows that $E_{X_{\tau_{D_\varepsilon}}}(\tau_D) \leq E_{X_{\tau_{D_\varepsilon}}}(\tau_D)$. Under the measure $\mathbb{P}_{X_{\tau_{D_\varepsilon}}}$, the stopping time $\tau_D$ satisfies that $\tau_D = \inf\{t \geq 0 : X_t^\top c - \alpha \notin (-\text{diam}(D), \varepsilon)\}$. Moreover since $c$ is a unit vector, under the measure $\mathbb{P}_{X_{\tau_{D_\varepsilon}}}$, the process $X_t^\top c - \alpha$ is a one-dimensional Brownian motion that starts at zero. Thus, $E_{X_{\tau_{D_\varepsilon}}}(\tau_D) = \text{diam}(D)\varepsilon$. See Figure 1 for a geometric illustration.

We recall the fact that for a one-dimensional Brownian motion starting at zero, the expected time such that it leaves the interval $[a, b]$ for $a < 0 < b$ is equal to $|ab|$, see [28] Proposition 2.2.20. Thus the claimed estimate follows.

The following result is also implied by [26] Theorems 9.13 and 9.17. We give a proof to establish some techniques to be used throughout this section.

Proposition 3.3. Suppose that $D \subset \mathbb{R}^d$ is convex and bounded. Let $f$ be Hölder continuous and let $g$ be extendable to $\overline{D}$ such that $g \in C^2(\overline{D})$. Then, for every $x \in \overline{D}$,

\[ u(x) = E_x(g(X_{\tau_{\overline{D}}})) + \frac{1}{2} E_x \left( \int_0^{\tau_{\overline{D}}} f(X_s) ds \right). \]

Proof. We will first establish a formula of the type asserted in this proposition in the interior of $D$ and then extend it to also incorporate the boundary.

The assumed convexity of the domain $D$ implies that all boundary points are regular in the sense of [11]; for details see [11] pp. 25, 27. In conjunction with [11] Theorem 4.3, it follows that the solution $u$ to (2) exists and is unique. More precisely by [11] Lemma 4.2, $u$ is twice continuously differentiable in $D$.

Let $\varepsilon > 0$ be arbitrary. Recall $D_\varepsilon := \{x \in D : \text{dist}(x, \partial D) > \varepsilon\}$. Suppose that $\varepsilon$ is sufficiently small such that $D_\varepsilon$ is not empty. By Ito’s formula (see for example [29] Theorem 17.8), for every $x \in D_\varepsilon$

\[ u(X_{\tau_{D_\varepsilon}}) - u(x) = \int_0^{\tau_{D_\varepsilon}} \nabla u(X_t) \cdot dW_t + \frac{1}{2} \int_0^{\tau_{D_\varepsilon}} \Delta u(X_t) dt. \]

The optional stopping theorem (see for example [29] Theorem A.18 and Remark A.21) implies that for an integrable martingale $M_t$, $t \geq 0$, and an integrable stopping time $\overline{t}$, it holds that $E(M_{\overline{t}}) = E(M_0)$,
Figure 1: Illustration of a realization of $X_{\tau_{D_x}}$. Here $D$ is given by an ellipse. The domain $D$ may be positioned in between two hyperplanes $A$ and $B$ such that $\text{dist}(X_{\tau_{D_x}}, \partial D) = \text{dist}(X_{\tau_{D_x}}, A) = \varepsilon$.

As a consequence, the martingale property of the stochastic integral $\int_0^t \nabla u(X_s) \cdot dW_s, t \geq 0$, see [18, Proposition 3.2.10], implies

$$ E_x \left( \int_0^{\tau_{D_x}} \nabla u(X_t) \cdot dW_t \right) = 0. \tag{4} $$

and thus

$$ u(x) = E_x(u(X_{\tau_{D_x}})) + \frac{1}{2} E_x \left( \int_0^{\tau_{D_x}} f(X_t) dt \right). $$

We seek to study the limit $\varepsilon \to 0$. The solution $u$ is Lipschitz continuous on the closure $\overline{D}$ with Lipschitz constant $L_u > 0$, which may be concluded by [5, Theorem 1.4]. There, the statement of [5, Theorem 1.4] is applied to $v = u - g$ with right hand side $f - \Delta g$. The Lipschitz continuity of $u$ yields

$$ |E_x(u(X_{\tau_D}) - u(X_{\tau_{D_x}}))| \leq L_u E_x(|X_{\tau_D} - X_{\tau_{D_x}}|). $$

Since for any two integrable stopping times $\tilde{\tau}, \tilde{\tau}$ that satisfy $\tilde{\tau} \geq \tilde{\tau}$ it holds

$$ E_x(|X_{\tilde{\tau}} - X_{\tilde{\tau}}|^2) = E(\tilde{\tau} - \tilde{\tau}), \tag{5} $$

cf. [23, Corollary 2.46 and Theorem 2.48], we obtain by Lemma [3.2]

$$ E_x(|X_{\tau_D} - X_{\tau_{D_x}}|) \leq \sqrt{E_x(|X_{\tau_D} - X_{\tau_{D_x}}|^2)} = \sqrt{E_x(\tau_D - \tau_{D_x})} \leq \sqrt{\text{diam}(D)}\varepsilon, $$

which implies that $E_x(u(X_{\tau_{D_x}})) \to E_x(g(X_{\tau_D}))$ as $\varepsilon \to 0$. Similarly, also by Lemma [3.2]

$$ \left| E_x \left( \int_{\tau_{D_x}}^{\tau_D} f(X_t) dt \right) \right| \leq \|f\|_{L^\infty(D)} E_x(\tau_D - \tau_{D_x}) \leq \|f\|_{L^\infty(D)} \text{diam}(D)\varepsilon $$

5
and thus $E_x(\int_0^{\tau_D} f(X_t)dt) \to E_x(\int_0^{\tau_D} f(X_t)dt)$ as $\varepsilon \to 0$.

We define the discrete processes $\bar{X}_k$, $k \geq 0$, and $r_k$, $k \geq 1$, which also tacitly depend on an initial starting point $x \in D$. Let $\bar{X}_0 = x$ and

$$\bar{X}_k = \bar{X}_{k-1} + Y_k r_k \quad \text{with} \quad r_k = \text{dist}(\bar{X}_{k-1}, \partial D), \quad k \geq 1. \quad (6)$$

The sequence $Y_k$, $k \geq 1$, is independent and identically distributed according to $X_{\tau_{B(0,1)}}$ with respect to $P_0$. Note that $Y_k$ is uniformly distributed on the unit sphere, $k \geq 1$, see [25, Theorem 2]. This process has been introduced in [25] and is commonly referred to as walk-on-the-sphere. The resulting random vector $\bar{X}_k$ depends on the initial point $X_0 = x$ and the random directions $Y_k$, $k' = 1, \ldots, k$. Sometimes, we shall use the notation

$$\bar{X}_k = \bar{X}_k(x, Y_1, \ldots, Y_k), \quad k \geq 0, \quad (7)$$

where this dependence is explicit.

The process $X_k$, $k \geq 0$, is related to $X_t$, $t \geq 0$ as follows. Define $\bar{r}_1 := \text{dist}(x, \partial D)$ and $\tau_1 := \tau_{B(x, \bar{r}_1)}$. For every $k \geq 2$,

$$\bar{r}_k := \text{dist}(X_{\tau_{k-1}}, \partial D) \quad \text{and} \quad \tau_k := \inf\{t \geq 0 : X_{t+\tau_{k-1}} \notin B(X_{\sum_{j=1}^{k-1} \bar{r}_j}, \bar{r}_k)\}. \quad (8)$$

Note that $r_k$ and $\bar{r}_k$ have the same distribution, $k \geq 1$. Define

$$T(k) := \sum_{j=1}^{k} \tau_j \quad \forall k \in \mathbb{N}. \quad (9)$$

As noted above, $X_{T(k)}$ is equally distributed on the boundary of the ball $B(X_{T(k-1)}, \bar{r}_k)$. Thus, by construction of the process $\bar{X}_k$, $k \geq 0$, it holds that $\bar{X}_k$ has the same distribution as $X_{T(k)}$, $k \geq 1$, see also Figure 3. Note that $\lim_{k \to \infty} X_{T(k)} = X_{\tau_D}$ $P$-a.s., cf. [25, Theorem 3.6]. The following lemma is a version of [20, Lemma 6.3] for $\alpha = 2$ (in the notation of [20]). We give a proof for the convenience of the reader.

**Lemma 3.4.** Let the assumptions of Proposition 3.3 be satisfied. There holds that

$$u(x) = E_x(g(X_{\tau_D})) + E_x\left(\sum_{k \geq 1} \tau_k^2 K_1(f(\bar{X}_{k-1} + r_k'))\right),$$

where for any continuous function $v : B(0,1) \to \mathbb{R}$

$$K_1(v) := \frac{1}{2} E_0\left(\int_0^{\tau_{B(0,1)}} v(X_t)dt\right). \quad (10)$$

**Proof.** This proof builds on the representation of $u$ from Proposition 3.3. In particular, the term $E_x(\int_0^{\tau_D} f(X_t)dt)$ shall be represented by a sum of consecutive solutions of the Poisson equation on a ball.

The strong Markov property of the Brownian motion yields

$$E_x\left(\int_0^{\tau_D} f(X_t)dt\right) = E_x\left(\sum_{k \geq 1} \int_{\tau_k}^{\tau_{k-1}} f(X_t)dt\right) = E_x\left(\sum_{k \geq 1} E_{\bar{X}_{k-1}}\left(\int_0^{\tau_k} f(X_t)dt\right)\right), \quad (11)$$

where we recall the definition of the stopping times $\tau_k$, $k \geq 1$, from (3) and set $\tau_0 = 0$. Note that the scaling property of Brownian motion states that $cX_{c^{-2}t}$ has the same distribution as $X_t$ for every
Figure 2: Illustration of the process $X_k$, $k \geq 0$, in the case of a rectangular domain. One realization of $X_t$, $t \in [0, T(5)]$, and $X_k$, $k = 0, \ldots, 5$, is plotted as numbered annotations.

c > 0 and every $t \geq 0$, conditioned on $X_0 = x$, $x \in \mathbb{R}^d$. In conjunction with the strong Markov property for every $x' \in D$ and every $r > 0$ such that $B(x', r) \subset D$,

$$
E_{x'}\left( \int_0^{T_{B(x', r)}} f(X_t) dt \right) = E_0 \left( \int_0^{T_{B(0, r)}} f(x' + X_t) dt \right) = r^2 E_0 \left( \int_0^{T_{B(0, 1)}} f(x' + rX_t) dt \right).
$$

The assertion follows by inserting the previous equality into (11) with $x' = X_k$ and $r = r_k$ and Proposition 3.3.

Note that the functional $v \mapsto K_1(v)$ denotes the solution to the Poisson equation with homogeneous Dirichlet boundary conditions on the unit ball evaluated at the origin. An explicit formula for $K_1$ is a classical result by Boggio [3], see also [8]. Specifically by [8, Lemma 2.27] for $d \geq 3$,

$$
K_1(v) = \frac{\Gamma(1 + d/2)}{d\pi^{d/2}} \cdot \frac{1}{d-2} \int_{|y| \leq 1} v(y)(|y|^{2-d} - 1)dy.
$$

For every $\varepsilon > 0$, let us define the random index $N(\varepsilon)$ by

$$
N(\varepsilon) := \inf\{k \geq 1 : \text{dist}(X_k, \partial D) \leq \varepsilon\}.
$$

Proposition 3.5. Let the assumptions of Proposition 3.3 be satisfied. Let $\bar{N}$ be an integer valued random variable that satisfies $\bar{N} \geq N(\varepsilon)$ $\mathbb{P}$-a.s. It holds that

$$
\left| E_x \left( \int_0^{T_D} f(X_s) ds \right) - E_x \left( \sum_{k=1}^{\bar{N}} r_k^2 K_1(f(X_{k-1} + r_k \cdot)) \right) \right| \leq \text{diam}(D) \|f\|_{L^\infty(D)} \varepsilon
$$

(13)
and

$$|E_x(g(X_{T_D})) - E_x(g(X_N))| \leq \frac{1}{2} \|\Delta g\|_{L^\infty(D)} \text{diam}(D) \varepsilon.$$  \hfill (14)

**Proof.** As a consequence of Lemma 3.4 and its proof, see [11],

$$E_x \left( \int_0^{T_D} f(X_s) ds - \sum_{k=1}^N r_k^2 K_1(f(\bar{X}_{k-1} + r_k')) \right) = E_x \left( \int_{\mathcal{I}(N)} f(X_s) ds \right)$$

The assumption \( \bar{N} \geq N(\varepsilon) \) implies that \( \mathcal{I}(\bar{N}) \geq \tau_{D_1} \), where we recall the definition of \( \mathcal{I} \) in [9]. Thus,

$$\left| E_x \left( \int_0^{T_D} f(X_s) ds - \sum_{k=1}^{\bar{N}} r_k^2 K_1(f(\bar{X}_{k-1} + r_k')) \right) \right| \leq \|f\|_{L^\infty(D)} E_x(\tau_D - \tau_{D_1})$$

The first assertion (13) follows by Lemma 3.2. To show the second assertion (14), we may apply Ito’s lemma (see for example [29, Theorem 17.8]), which implies

$$g(X_{T_D}) - g(X_{\mathcal{I}(N(\varepsilon))}) = \int_{\mathcal{I}(N(\varepsilon))} \nabla g(X_s) \cdot dW_s + \frac{1}{2} \int_{\mathcal{I}(N(\varepsilon))} \Delta g(X_s) ds.$$  

By (4),

$$E_x \left( \int_{\mathcal{I}(N(\varepsilon))} \nabla g(X_s) \cdot dW_s \right) = 0.$$

The assertion follows by

$$\left| E_x \left( \int_{\mathcal{I}(N(\varepsilon))} \Delta g(X_s) ds \right) \right| \leq \|\Delta g\|_{L^\infty(D)} E_x(\tau_D - \tau_{D_1}) \leq \|\Delta g\|_{L^\infty(D)} \text{diam}(D) \varepsilon,$$

which is consequence of Lemma 3.2.

The statement of the following lemma is in principle known. We provide a proof for the convenience of the reader.

**Lemma 3.6.** Let \( D \subset \mathbb{R}^d \) be a bounded, convex domain. For any \( \varepsilon > 0 \) such that \( D_\varepsilon \) is non-empty, it holds that

$$E \left( \sup_{x \in D} N(\varepsilon) \right) \leq (\text{diam}(D))^2 \varepsilon^{-2}.$$  

**Proof.** We recall that for \( x \in D \) such that \( \text{dist}(x, \partial D) < \varepsilon \), \( N(\varepsilon) = 1 \). Thus, by the Markov property and Lemma 3.1 for every \( k \geq 1 \),

$$E_{X_{\mathcal{I}(N-1)}}(\tau_k) = \frac{r_k^2}{d} \ \mathbb{P}-\text{a.s.},$$

where \( \tau_k \) is defined in [3]. Note that since \( D_\varepsilon \) is not empty, it follows that \( \varepsilon \leq \text{diam}(D)/2 \). Since \( r_k \geq \varepsilon \) for every \( k \leq N(\varepsilon) \) and \( x \in D_\varepsilon \) \( \mathbb{P} \)-a.s.,

$$E \left( \sup_{x \in D} N(\varepsilon) \right) \leq E \left( \sup_{x \in D} \sum_{k \geq 1} \mathbb{1}_{\{r_k \geq \varepsilon\}} \right) \leq d \varepsilon^{-2} E \left( \sup_{x \in D} \sum_{k \geq 1} E_{X_{\mathcal{I}(N-1)}}(\tau_k) \mathbb{1}_{\{r_k \geq \varepsilon\}} \right) \leq d \varepsilon^{-2} E \left( \sup_{x \in D} \tau_D \right).$$

Since \( \{t > 0 : W_t \notin B(0, \text{diam}(D))\} \subset \{t > 0 : x + W_t \notin D\} \) for every \( x \in D \) (using convexity of \( D \)), it holds that \( \sup_{x \in D} \tau_D \leq \inf \{t > 0 : W_t \notin B(0, \text{diam}(D))\} \). By Lemma 3.1

$$E \left( \sup_{x \in D} \tau_D \right) \leq \frac{\text{diam}(D)^2}{d},$$

which concludes the proof of this lemma. \( \square \)
4 Approximation by deep neural networks without curse of dimension

Deep neural networks allow to accommodate composition of mappings in their structure or architecture. The repeated occurrence of linear maps (here expectation $E_x(\cdot)$) and compositions of maps in the Feynman–Kac representation of a solution to an elliptic or also to a parabolic PDE was found to suit the architecture of deep neural networks in [14 Proposition 3.4]. In this section the applicability of DNNs shall be extended to PDEs with boundary conditions. The main obstruction in the analysis is the stopping time $\tau_D$, which also depends on $x \in D$; the point where the process $X_i$ is started.

Recall that we aim to approximate solutions to the prototypical elliptic PDE
\[-\Delta u = f \quad \text{in } D \quad \text{and} \quad u|_{\partial D} = g\]  
by DNNs with ReLU activation function. The basics on stochastic sampling methods introduced in Section 3 shall serve as tools in the proofs of this section. The following theorem constitutes our main result.

**Theorem 4.1.** Let the assumptions of Proposition 3.3 be satisfied. Suppose that for every $\delta, \delta_f, \delta_g \in (0, 1)$, there exist ReLU DNNs $\phi_{\text{dist}, \delta}$, $\phi_{f, \delta_f}$, and $\phi_{g, \delta_g}$ such that
\[
\sup_{x \in D} |\text{dist}(x, \partial D) - \phi_{\text{dist}, \delta}(x)| \leq \delta
\]  
and
\[
\|f - \phi_{f, \delta_f}\|_{L^\infty(D)} \leq \delta_f \tag{16}
\]
\[
\|g - \phi_{g, \delta_g}\|_{L^\infty(D)} \leq \delta_g \tag{17}
\]
and $\text{size}(\phi_{\text{dist}, \delta}) = O(d^a|\log(\delta^{-1})|^b)$, $\text{size}(\phi_{f, \delta_f}) = O(d^a\delta_f^{-b})$, and $\text{size}(\phi_{g, \delta_g}) = O(d^a\delta_g^{-b})$ for some $a, b \in (1, \infty)$ which do not depend on $d$. Let additionally $f$ and $g$ be Lipschitz continuous on $\overline{D}$. For every $\tilde{\delta} \in (0, 1)$, there exists a ReLU DNN $\phi_{u, \tilde{\delta}}$ such that
\[
\|u - \phi_{u, \tilde{\delta}}\|_{L^2(D)} \leq \tilde{\delta}
\]
with $\text{size}(\phi_{u, \tilde{\delta}}) = O(d^a\tilde{\delta}^{-12-8b}|D|^{12+8b})$. The tacit constants in the Landau symbols depend on $\|f\|_{L^\infty(D)}$, $\|\Delta g\|_{L^\infty(D)}$, the Lipschitz constants of $f$ and $g$, and on $\text{diam}(D)$.

The proof of Theorem 4.1 will be postponed to end of this section after two intermediate propositions have been proven.

**Proposition 4.2.** Suppose that $g \in C^2(\overline{D})$, $x \mapsto \text{dist}(x, \partial D)$ can be realized by a ReLU DNN $\phi_{\text{dist}}$, and for any $\delta_g \in (0, 1)$ there exists a ReLU DNN $\phi_{g, \delta_g}$ such that
\[
\|g - \phi_{g, \delta_g}\|_{L^\infty(D)} \leq \delta_g.
\]
For every $\tilde{\delta} \in (0, 1)$, there exists a ReLU DNN $\phi_{1, \tilde{\delta}}$ such that
\[
\sqrt{\int_D \|E_x(g(X_{\tau_D})) - \phi_{1, \tilde{\delta}}(x)\|^2 dx} \leq \tilde{\delta}.
\]
Furthermore, there exist $M = \lfloor c\tilde{\delta}^{-2}(1 + |D|)\rfloor$, $\bar{N}_i$, $i = 1, \ldots, M$, and unit vectors $Y_{i,k}$, $k = 1, \ldots, \bar{N}_i$, $i = 1, \ldots, M$, such that for every $x \in D$,
\[
\phi_{1, \tilde{\delta}}(x) = \frac{1}{M} \sum_{i=1}^M (\phi_{g, \delta_g}(\bar{X}_{\bar{N}_i}(x, Y_{i,1}, \ldots, Y_{i,\bar{N}_i}))).
\]


The accuracy $\delta_g$ of the ReLU DNN $\phi_{g,\delta}$ satisfies $\delta_g = c' \delta/(1 + \sqrt{|D|})$. The numbers $N_i$, $i = 1, \ldots, M$, satisfy that

$$\sum_{i=1}^{M} N_i \leq CM^2 (M^2 + |D|).$$

(18)

The constants $c, c', C > 0$ only depend on $\|\Delta g\|_{L^\infty(D)}$ and on $\text{diam}(D)$.

Proof. Let $\varepsilon \in (0, 1)$ and $\delta_g \in (0, 1)$ be arbitrary, which will be determined in the following. Define the random variable $\bar{N}(\varepsilon) := \sup_{x \in D} N(\varepsilon)$. By Proposition 3.5

$$\sup_{x \in D} |E_x(g(X_{\tau_D})) - E_x(g(X_{\bar{N}(\varepsilon)}))| \leq \frac{1}{2} \|\Delta g\|_{L^\infty(D)} \text{diam}(D) \varepsilon.$$

(19)

The assumed approximability of $g$ by the ReLU DNN $\phi_{g,\delta}$ results in

$$\sup_{x \in D} |E_x(g(X_{\bar{N}(\varepsilon)})) - E_x(\phi_{g,\delta}(X_{\bar{N}(\varepsilon)}))| \leq \delta_g.$$  

(20)

Denote by $E_M(\cdot)$, $M \in \mathbb{N}$, a Monte Carlo estimator with respect to the random variables $\bar{N}(\varepsilon)$ and the sequence $\bar{X}_k$, $k \geq 0$, i.e., for every square integrable function $\varphi : D^{N_0} \times \mathbb{N} \to \mathbb{R}$

$$E_M(\varphi) := \frac{1}{M} \sum_{i=1}^{M} \varphi(\bar{X}^{(i)}_{k \geq 0}, \bar{N}^{(i)}(\varepsilon)).$$

(21)

where $((\bar{X}^{(i)}_{k \geq 0}, \bar{N}^{(i)}(\varepsilon))$, $i = 1, \ldots, M$, are mutually independent and have the same distribution as $((\bar{X}_k)_{k \geq 0}, \bar{N}(\varepsilon))$. It is well-known that that for any square integrable $\varphi$

$$\|E_x(\varphi((\bar{X}_k)_{k \geq 0}, \bar{N}(\varepsilon))) - E_M(\varphi(\bar{X}_k, \bar{N}(\varepsilon))))\|_{L^2(D)} \leq \sqrt{\frac{E_x((\varphi((\bar{X}_k)_{k \geq 0}, \bar{N}(\varepsilon))) - E_x(\varphi((\bar{X}_k)_{k \geq 0}, \bar{N}(\varepsilon))))^2}{M}}.$$

(22)

Thus, by (19) and (20) and by Lemma 3.6

$$E \left( \int_D |E_x(g(X_{\tau_D})) - E_M(\phi_{g,\delta}(X_{\bar{N}(\varepsilon)}))|^2 \, dx + \varepsilon^2 \right) \left[ E \left( \sqrt{\bar{N}(\varepsilon)} \right) - E_M \left( \sqrt{\bar{N}(\varepsilon)} \right) \right]^2 \leq \frac{1}{2} \|\Delta g\|_{L^\infty(D)}^2 \text{diam}(D)^2 |D| \varepsilon^2 + 2 |D| \delta_g^2 + \frac{\text{diam}(D)^2}{M} := \text{error}^2_{M,\varepsilon,\delta_g}.$$  

(23)

The fact that for a positive random variable $Z$ such that $E(Z) \leq c$, there exists a set of positive probability $A \subset \Omega$ such that $Z(\omega) \leq c$ for every $\omega \in A$ implies there exist $N_i$ and direction vectors $Y_{i,k}, i = 1, \ldots, M$, $k = 1, \ldots, N_i$, such that

$$\int_D \left| u - \frac{1}{M} \sum_{i=1}^{M} (\phi_{g,\delta}(\bar{X}_{N_i}(\varepsilon))) \right|^2 \, dx + \varepsilon^2 \left[ E \left( \sqrt{\bar{N}(\varepsilon)} \right) - \frac{1}{M} \sum_{i=1}^{M} \left( \sqrt{N_i} \right) \right]^2 \leq \text{error}^2_{M,\varepsilon,\delta_g}.$$  

(24)

In conjunction with the previous estimate, the Jensen inequality, Lemma 3.6, and the elementary estimate that $\sqrt{\sum_{i=1}^{M} c_i} \leq \sum_{i=1}^{M} \sqrt{c_i}$ for any positive numbers $c_i, i = 1, \ldots, M$, imply

$$\sum_{i=1}^{M} N_i \leq 2M^2 \varepsilon^{-2} \mathbb{E}(\bar{N}(\varepsilon)) + 2M^2 \varepsilon^{-2} \text{error}^2_{M,\varepsilon,\delta_g} \leq 2M^2 \varepsilon^{-4} \text{diam}(D)^2 + 2M^2 \varepsilon^{-2} \text{error}^2_{M,\varepsilon,\delta_g}.$$  

We choose the parameters $\delta_g = \varepsilon$ and $M = \lceil \varepsilon^{-2} \rceil$. The assertion follows by inserting the expression for $\text{error}^2_{M,\varepsilon,\delta_g}$ from (23) into the previous estimate. Define the ReLU DNN $\phi_{u,\delta}$ by its realization

$$\phi_{u,\delta} := \frac{1}{M} \sum_{i=1}^{M} (\phi_{g,\delta}(\bar{X}_{N_i})).$$
Then, as a consequence of \[24\] there exists a constant \(C > 0\), which only depends on \(\|\Delta g\|_{L^\infty(D)}\) and on \(\text{diam}(D)\) such that
\[
\sqrt{\int_D \left| \mathbb{E}_x (g(X_{\tau_D})) - \phi_{u,\tilde{\epsilon}}(x) \right|^2 dx} \leq C \varepsilon (1 + \sqrt{|D|}).
\]
We choose \(\varepsilon = \tilde{\delta}/(C(1 + \sqrt{|D|}))\), which proves the assertion of this proposition.

**Proposition 4.3.** Let \(d \geq 3\). Suppose that \(f\) is Lipschitz continuous on \(D\), \(x \mapsto \text{dist}(x, \partial D)\) can be realized by a ReLU DNN \(\phi_{\text{dist}}\), and for any \(\delta_f > 0\) there exists a ReLU DNN \(\phi_{f,\delta_f}\) such that
\[
\|f - \phi_{f,\delta_f}\|_{L^\infty(D)} \leq \delta_f.
\]
For every \(\tilde{\delta} \in (0,1)\), there exists a ReLU DNN \(\phi_{2,\tilde{\delta}}\) such that
\[
\sqrt{\int_D \left| \mathbb{E}_x \left( \int_0^{\tau_D} f(X_t) dt \right) - \phi_{2,\tilde{\delta}}(x) \right|^2 dx} \leq \tilde{\delta}.
\]
Furthermore, there exist \(M_1 = M_2 = [c\delta^{-2}(1 + |D|^2)]\), \(N_i\), unit vectors \(Y_{i,k}\), and elements of the unit ball \(y_{i,j,k}\), \(i = 1, \ldots, M_1\), \(k = 1, \ldots, N_i\), \(i = 1, \ldots, M\) such that for every \(x \in D\),
\[
\phi_{2,\tilde{\delta}}(x) = \frac{1}{M_1} \sum_{i=1}^{M_1} \sum_{k=1}^{N_i} \tilde{x}_\delta \left( \tilde{x}_\delta(\phi_{\text{dist}}(X_{k-1}), \phi_{\text{dist}}(X_{k-1})), \frac{1}{M_2} \sum_{j=1}^{M_2} \phi_{f,\delta_f}(X_{k-1} + \phi_{\text{dist}}(X_{k-1})y_{i,j,k}) \right),
\]
where \(X_k = X_k(x, Y_{1,1}, \ldots, Y_{i,k})\). The accuracy \(\delta_f\) of the ReLU DNN \(\phi_{f,\delta_f}\) satisfies \(\delta_f = c'\delta/(1 + |D|)\) and the accuracy \(\tilde{\delta} > 0\) of the ReLU DNN \(\tilde{x}_\delta\) satisfies \(\tilde{\delta} = c''\delta/(1 + |D|)^{\gamma}\). The numbers \(N_i\) satisfy that
\[
\sum_{i=1}^{M_1} N_i \leq CM_1^2 (M_1^2 + |D|).
\]
The constants \(c, c', c'', C > 0\) depend only on \(\|f\|_{L^\infty(D)}\) and on \(\text{diam}(D)\).

**Proof.** Let \(\varepsilon \in (0,1)\) and \(\delta_f \in (0,1)\) be arbitrary and sufficiently small. The value of these two numbers will be chosen at a later stage in the following proof. The effect of the approximation of the right hand side \(f\) by \(\phi_{f,\delta_f}\) is estimated using Proposition \[3.3\] and Lemma \[3.1\] i.e.,
\[
\sup_{x \in D} \left| \mathbb{E}_x \left( \int_0^{\tau_D} f(X_t) dt \right) - \mathbb{E}_x \left( \int_0^{\tau_D} \phi_{f,\delta_f}(X_t) dt \right) \right| = \mathbb{E}_x (\tau_D) \|f - \phi_{f,\delta_f}\|_{L^\infty(D)} \leq \frac{\text{diam}(D)^2/4}{d} \delta_f,
\]
where the domain \(D\) may be embedded into a ball with radius \(\text{diam}(D)/2\) in order to apply Lemma \[3.1\].
Define the random number \(N(\varepsilon) = \sup_{x \in D} N(\varepsilon)\). By Proposition \[3.3\] for every \(x \in D\)
\[
\sup_{x \in D} \left| \mathbb{E}_x \left( \int_0^{\tau_D} \phi_{f,\delta_f}(X_t) dt \right) - \mathbb{E}_x \left( \sum_{k=1}^{N(\varepsilon)} r_k^{2} K_1(\phi_{f,\delta_f}(X_{k-1} + r_k \cdot)) \right) \right| \leq \|\phi_{f,\delta_f}\|_{L^\infty(D)} \text{diam}(D) \varepsilon.
\]
Let us introduce two Monte Carlo estimators \(E_{M_1}(\cdot)\), \(M_1 \in \mathbb{N}\), and \(E_{M_2}(\cdot)\), \(M_2 \in \mathbb{N}\), on the probability space \((\Omega, \mathcal{A}, \mathbb{P})\). The estimators \(E_{M_1}(\cdot)\), \(M_1 \in \mathbb{N}\), are with respect to random variables \(N(\varepsilon)\) and the sequence \(X_k, k \geq 0\), see \[21\], and satisfy \[22\]. The estimators \(E_{M_2}(\cdot)\), \(M_2 \in \mathbb{N}\), shall approximate the functional \(v \mapsto K_1(v)\). As a preparation, for \(d \geq 3\), by \[12\]
\[
K_1(v) = \frac{\Gamma(1 + d/2)}{d \pi^{d/2}} \frac{1}{d - 2} \int_{B(0,1)} v(y)(|y|^{2-d} - 1) dy
\]
It holds that
\[
K_1(1) = \frac{\Gamma(1 + d/2)}{d\pi^{d/2}(d - 2)} \int_{B(0,1)} |y|^{2-d} - 1) \, dy = \frac{\Gamma(1 + d/2)}{d\pi^{d/2}(d - 2) \left( \frac{1}{2} |\partial B(0,1)| - |B(0,1)| \right)}
\]
\[
= \frac{\Gamma(1 + d/2)}{d\pi^{d/2}(d - 2)} |B(0,1)| \left( \frac{d}{2} - 1 \right)
\]
\[
= \frac{d/2 - 1}{d(d-2)} = \frac{1}{2d} =: \kappa_d
\]
where we inserted the relation $|\partial B(0,1)| = d|B(0,1)|$ and the value for the volume of the unit $d$-sphere, i.e., $|B(0,1)| = \pi^{d/2}/\Gamma(1 + d/2)$. Thus,

\[
\mu(dy) := \kappa_d^{-1} \frac{\Gamma(1 + d/2)}{d\pi^{d/2}(d - 2)} (|y|^{2-d} - 1) \, dy
\]
is a probability measure on $B(0,1)$. The Monte Carlo estimators $E_{M_2}(\cdot)$, $M_2 \in \mathbb{N}$, are with respect to the probability measure $\mu$, i.e., for every $M_2 \in \mathbb{N}$, let $y_i$, $i = 1, \ldots, M_2$ be independent random variables distributed according to $\mu$ such that for every $v \in L^2(B(0,1), \mu)$,

\[
\sqrt{\mathbb{E}(|K_1(v) - \kappa_d E_{M_2}(v)|^2)} = \frac{\|v - \int_{B(0,1)} v \mu(dy)\|_{L^2(B(0,1), \mu)}}{\sqrt{M_2}}
\]
(27)

where

\[
E_{M_2}(v) = \frac{1}{M_2} \sum_{j=1}^{M_2} v(y_j).
\]
(28)

We split the error into the contributions from the approximation of the expectation $\mathbb{E}_x(\cdot)$ and from the approximation of the integral in the functional $v \mapsto K_1(v)$, i.e., by the triangle inequality

\[
\left\| \mathbb{E}_x \left( \sum_{k=1}^{\bar{N}(\varepsilon)} r_k^2 K_1(\phi_{f, \delta_j}(\bar{X}_{k-1} + r_k \cdot)) \right) - E_{M_1} \left( \sum_{k=1}^{\bar{N}(\varepsilon)} r_k^2 E_{M_2}(\phi_{f, \delta_j}(\bar{X}_{k-1} + r_k \cdot)) \right) \right\|_{L^2(\Omega \times D, P \otimes dx)}
\]
\[
\leq \left\| \mathbb{E}_x \left( \sum_{k=1}^{\bar{N}(\varepsilon)} r_k^2 K_1(\phi_{f, \delta_j}(\bar{X}_{k-1} + r_k \cdot)) \right) - \mathbb{E}_x \left( \sum_{k=1}^{\bar{N}(\varepsilon)} r_k^2 E_{M_2}(\phi_{f, \delta_j}(\bar{X}_{k-1} + r_k \cdot)) \right) \right\|_{L^2(\Omega \times D, P \otimes dx)}
\]
\[
+ \left\| \mathbb{E}_x \left( \sum_{k=1}^{\bar{N}(\varepsilon)} r_k^2 E_{M_2}(\phi_{f, \delta_j}(\bar{X}_{k-1} + r_k \cdot)) \right) - E_{M_1} \left( \sum_{k=1}^{\bar{N}(\varepsilon)} r_k^2 E_{M_2}(\phi_{f, \delta_j}(\bar{X}_{k-1} + r_k \cdot)) \right) \right\|_{L^2(\Omega \times D, P \otimes dx)}
\]

(28)

The Monte Carlo estimators $E_{M_1}(\cdot)$ and $E_{M_2}(\cdot)$ are independent and also independent from the sequence of random directions $Y_k$, $k \geq 1$, introduced in $\square$.

We estimate by (27) that for every $x \in D$,

\[
\left\| \sum_{k=1}^{\bar{N}(\varepsilon)} r_k^2 \left( K_1(\phi_{f, \delta_j}(\bar{X}_{k-1} + r_k \cdot)) - E_{M_2}(\phi_{f, \delta_j}(\bar{X}_{k-1} + r_k \cdot)) \right) \right\|_{L^2(\Omega, \mathcal{F})}
\]
\[
\leq \sum_{k=1}^{\bar{N}(\varepsilon)} r_k^2 \kappa_d \|\phi_{f, \delta_j}(\bar{X}_{k-1} + r_k \cdot)\|_{L^2(B(0,1), \mu)}
\]

\[12\]
Thus, every $M$.

Recall the elementary observation that for a positive random variable $X$, $E(X^2) \leq \text{Var}(X) + (E(X))^2$.

Moreover, by Lemma 3.6 it holds that $E(X) \leq \text{Var}(X)$.

To estimate $I$, note that $|E_M(\phi_f(x))| \leq \|\phi_f\|_{L^\infty(D)}$. Since $I^2 = E_{X_i}(\tau_k)$ by Lemma 3.1, Jensen’s inequality implies

$$E_x\left(\sum_{k \geq 1} r_k^2\right) = E_x\left(\sum_{k \geq 1} |X_i(k) - X_i(k-1)|^2\right) = E_x(\tau^2) \leq \frac{\text{diam}(D)^2/4}{d}.$$

Thus,

$$I \leq \sqrt{|D|}E_x(\tau^2) \leq \frac{\text{diam}(D)^2/2}{d} \|\phi_f\|_{L^\infty(D)} \sqrt{M_2}.$$

To estimate $II$, note that $|E_M(\phi_f(x))| \leq \|\phi_f\|_{L^\infty(D)}$. Since $I^2 = E_{X_i}(\tau_k)$ by Lemma 3.1, Jensen’s inequality implies

$$\mathbb{E}_x\left(\sum_{k \geq 1} r_k^2\right) = \mathbb{E}_x\left(\sum_{k \geq 1} \mathbb{E}_{X_i(k)}(\tau_k)^2\right) \leq d \mathbb{E}_x(\tau^2) \leq 2\text{diam}(D)^4 \left(2 - \frac{d}{d+2}\right),$$

where we used that $\mathbb{E}_0(\tau_B^2) = 2r^2(2/d - 1/(d + 2))$ for any $r > 0$, which follows for example from [10, Equation (B)]. We conclude that

$$II \leq \|\phi_f\|_{L^\infty(D)}\text{diam}(D)^2 \sqrt{\frac{4 - 2d/(d + 2)}{M_1}}.$$
Thus, for every $j = 1, \ldots, M_2$ such that
\[
\int_D \left| \mathbb{E}_x \left( \int_0^{r_D} f(X_t) dx \right) - \frac{1}{M_1} \sum_{i=1}^{M_1} \sum_{k=1}^{N_i} r_k \sum_{j=1}^{M_2} \phi_{f,\delta_j}(\bar{X}_{k-1} + r_k y_{i,j,k}) \right|^2 \, dx \leq \text{error}^2_{M_1,M_2,\varepsilon,\delta_f} \tag{33}
\]
and
\[
\sum_{i=1}^{M_1} \bar{N}_i \leq 2M_1^2 \varepsilon^{-2} \mathbb{E}(\bar{N}(\varepsilon)) + 2M_1^2 \varepsilon^{-2} \text{error}^2_{M_1,M_2,\varepsilon,\delta_f}, \tag{34}
\]
where the latter estimate follows with the Jensen inequality and the elementary estimate that $\sqrt{\sum_{i=1}^{M_1} c_i} \leq \sum_{i=1}^{M_1} \sqrt{c_i}$ for any positive numbers $c_i$, $i = 1, \ldots, M_1$. The assertion (25) follows by inserting the expression for error $\text{error}^2_{M_1,M_2,\varepsilon,\delta_f}$ from (32) into the previous estimate.

Let us define the DNN $\phi_{2,\delta}$ by its realization, i.e., for every $x \in D$
\[
\phi_{2,\delta}(x) = \frac{1}{M_1} \sum_{i=1}^{M_1} \sum_{k=1}^{N_i} \bar{x}_i \left( \tilde{x}_i(\text{dist}(\bar{X}_{k-1}), \phi_{\text{dist}}(\bar{X}_{k-1})), \frac{1}{M_2} \sum_{j=1}^{M_2} \phi_{f,\delta_j}(\bar{X}_{k-1} + \phi_{\text{dist}}(\bar{X}_{k-1}) y_{i,j,k}) \right), \tag{35}
\]
where $\tilde{x}_i$ is the ReLU DNN from Lemma 2.1 that approximates the product of two scalars with accuracy $\delta_i$ and size($\tilde{x}_i$) = $O(\log(\delta_i^{-1}))$. The parameters $\varepsilon$, $\delta_f$, $M_1$, and $M_2$ are chosen to equilibrate error contributions in (33). Specifically, we choose $\delta_f = \varepsilon$ and $M_1 = M_2 = [\varepsilon^{-2}]$. Thus, by (33) and (25) we conclude that
\[
\sqrt{\int_D \left| \mathbb{E}_x \left( \int_0^{r_D} f(X_t) dx \right) - \phi_{2,\delta}(x) \right|^2 \, dx} \leq \text{error}_{M_1,M_2,\varepsilon,\delta_f} + \frac{1}{M_1} \sum_{i=1}^{M_1} \bar{N}_i \delta_i \tag{36}
\]

where $C' > 0$ is a generic constant that only depends on $\text{diam}(D)$ and $\|f\|_{L^\infty(D)}$. Consequently, we choose $\delta = \delta_1^7$ and finally $\delta_f = \delta/(2C'(1 + |D|))$, which then also yields $\delta = [\delta/(2C'(1 + |D|))]^7$. \qed

**Proof of Theorem 4.1.** In this proof, we also include the approximation of the distance function to the boundary by a ReLU DNN $\phi_{\text{dist},\delta}$, where $\delta > 0$ is still to be chosen. We may restrict ourselves to the case $d \geq 3$. For $d = 1, 2$, the statement follows by [33, Theorem 1], since the solution $u$ is Lipschitz continuous on $\bar{D}$ as observed in the proof of Proposition 3.3 and may be extended Lipschitz-continuously to a suitable box that is a superset of $D$. For every $x \in D$, we define the process $\bar{X}_k$, $k \geq 0$, by
\[
\bar{X}_k = \bar{X}_{k-1} + Y_k \phi_{\text{dist},\delta}(\bar{X}_{k-1}) \quad \text{and} \quad \bar{X}_0 = x.
\]
The assumed accuracy of the ReLU DNN $\phi_{\text{diam},\delta}$ implies
\[
|\bar{X}_k - \bar{X}_k| \leq |\bar{X}_{k-1} - \bar{X}_{k-1}| + |\text{dist}(\bar{X}_{k-1}, \partial D) - \phi_{\text{dist},\delta}(\bar{X}_{k-1})|
\]
\[
\leq |\bar{X}_{k-1} - \bar{X}_{k-1}|
\]
\[
\quad + |\text{dist}(\bar{X}_{k-1}, \partial D) - \text{dist}(\bar{X}_{k-1}, \partial D)| + |\text{dist}(\bar{X}_{k-1}, \partial D) - \phi_{\text{dist},\delta}(\bar{X}_{k-1})|
\]
\[
\leq 2|\bar{X}_{k-1} - \bar{X}_{k-1}| + \delta,
\]
where we used that the distance function is Lipschitz continuous with Lipschitz constant equal to one. Thus, for every $x \in D$,
\[
|\bar{X}_k - \bar{X}_k| \leq 2^k \delta \quad \text{and} \quad |r_k - \phi_{\text{dist},\delta}(\bar{X}_{k-1})| \leq \delta(1 + 2^{k-1}).
\]
By Proposition 4.2 for every $\delta_1 > 0$ the function $\phi_{1,\delta_1}$ satisfies that
\[
\sqrt{\int_D |E_x (g(\tau_D)) - \phi_{1,\delta_1}(x)|^2 \, dx} \leq \delta_1. \tag{37}
\]
Also according to Proposition 4.2 $\phi_{1,\delta_1}$ depends on weight parameters $M = [c\delta_1^{-2}(1 + \sqrt{|D|})]$, $\bar{N}_i$, $i = 1, \ldots, M$, and unit vectors $Y_{i,k}$, $k = 1, \ldots, \bar{N}_i$; $i = 1, \ldots, M$. However, $\phi_{1,\delta}$ does not constitute a ReLU DNN here, since the assumption on the distance function in Proposition 4.2 is weakened in the theorem to be proved here. Define the ReLU DNN $\phi_{1,\delta}$ by its realization
\[
\phi_{1,\delta}(x) = \frac{1}{M} \sum_{i=1}^{M} (\phi_{g,\delta}(\bar{X}_{N_i}(x,Y_{i,1},\ldots,Y_{i,N_i}))).
\]
For any $x, y \in \mathcal{D}$
\[
|\phi_{g,\delta}(x) - \phi_{g,\delta}(y)| \leq 2\delta + L_g|x - y|, \tag{38}
\]
where $L_g$ denotes the Lipschitz constant of $g$. By the triangle inequality and the estimates (37) and (38)
\[
\sqrt{\int_D |E_x (g(\tau_D)) - \phi_{1,\delta}(x)|^2 \, dx} \leq \delta_1 + \sqrt{|D|} \left(2\delta + \frac{1}{M} \sum_{i=1}^{M} 2\bar{N}_i\delta \right).
\]
We equilibrate the error contributions by the choices $\delta_g = \delta_1$ and $\delta = 2^{-C\text{diam}(D)^2}M\delta_1$, where $C > 0$ is the constant from (18). Thus,
\[
\sqrt{\int_D |E_x (g(\tau_D)) - \phi_{1,\delta}(x)|^2 \, dx} \leq \delta_1(1 + 3\sqrt{|D|}). \tag{39}
\]
Recall the ReLU DNN $\tilde{x}\delta$ from Lemma 2.1 which approximates the product of two scalars on $[-m, m]^2$ to accuracy $\tilde{\delta}$; $m$ may be chosen appropriately such that it upper bounds $r_k$, $r_k^2 + \tilde{\delta}$, and $\|\phi_{f,\delta}\|_{L\infty(D)}$. It satisfies for any $a, c \in [-\sqrt{m + \delta}, \sqrt{m - \delta}]$ and $b, d \in [-m, m]$.
\[
|\tilde{x}\delta(\tilde{x}\delta(a,a),b) - \tilde{x}\delta(\tilde{x}\delta(c,c),d)| \leq 2\tilde{\delta} + m(|\tilde{x}\delta(a,a) - \tilde{x}\delta(c,c)| + |b - d|) \leq 2(1 + m)m + 2m^2|a - c| + m|b - d|. \tag{40}
\]
Moreover, for any $x, y \in \mathcal{D}$
\[
|\phi_{f,\delta}(x) - \phi_{f,\delta}(y)| \leq 2\delta + L_f|x - y|, \tag{41}
\]
where $L_f$ denotes the Lipschitz constant of $f$. By Proposition 4.3 for every $\delta_2$ the function $\phi_{2,\delta_2}$ satisfies that
\[
\sqrt{\int_D E_x \left(\int_0^{\tau_D} f(X_t)dt\right) - \phi_{2,\delta_2}(x)^2 \, dx} \leq \delta_2. \tag{42}
\]
Also according to Proposition 4.3 $\phi_{2,\delta_2}$ depends on weight parameters $M_1 = M_2 = [c\delta_2^{-2}(1 + |D|^2)]$, $\bar{N}_i$, unit vectors $Y_{i,k}$, and elements of the unit ball $y_{i,j,k}$, $i = 1, \ldots, M_1$, $k = 1, \ldots, \bar{N}_i$, $j = 1, \ldots, M_2$. However, $\phi_{2,\delta}$ does not constitute a ReLU DNN here, since the assumption on the distance function in Proposition 4.3 is weakened in the theorem to be proved here. Recall $r_k = \text{dist}(\bar{X}_{k-1}, \partial D)$ and $\tilde{X}_{k-1} = \bar{X}_{k-1}(x_{i,1},\ldots,x_{i,k-1})$. The estimates (40) and (41) imply for $k = 1, \ldots, \bar{N}_i$, $i = 1, \ldots, M_1$
\[
\left|\tilde{x}\delta \left(\tilde{x}\delta(r_k,r_k), \frac{1}{M} \sum_{j=1}^{M_2} \phi_{f,\delta}(\bar{X}_{k-1} + r_ky_{i,j,k})\right) - \tilde{x}\delta \left(\tilde{x}\delta(\bar{r}_k, \bar{r}_k), \frac{1}{M} \sum_{j=1}^{M_2} \phi_{f,\delta}(\bar{X}_{k-1} + \bar{r}_ky_{i,j,k})\right)\right| \leq 2(1 + m)m + 2m^2|\bar{r}_k - \bar{r}_k| + m(2\delta_f + L_f[|\bar{X}_{k-1} - \bar{X}_{k-1}| + |r_k - \bar{r}_k|]) \leq 2(1 + m)m + 2m^2|\bar{r}_k - \bar{r}_k| + m(2\delta_f + L_f(1 + 2k\delta)) \tag{43}
\]
Define the ReLU DNN $\phi_{u,\delta}^2$ by its realization

$$x \mapsto \phi_{u,\delta}^2(x) = \frac{1}{M_1} \sum_{i=1}^{M_1} \sum_{k=1}^{\bar{N}_i} \bar{x}_{\delta} \left( \bar{r}_k \bar{r}_k, \frac{1}{M_2} \sum_{j=1}^{M_2} \phi_{f,\delta_j} \left( \bar{X}_{k-1} + \bar{r}_k y_{i,j,k} \right) \right).$$

By the triangle inequality, (42), and (43)

$$\sqrt{\int_D \left| \mathbb{E}_x \left( \int_0^{\tau_D} f(X_s)ds \right) - \phi_{u,\delta}^2(x) \right|^2} \, dx \leq \delta_2 + C \sqrt{|D|} \frac{1}{M_1} \sum_{i=1}^{M_1} \bar{N}_i \left( \delta + \delta_f + 2\bar{N}_i \delta \right),$$

where the constant $C' > 0$ only depends on $m$ and $L_f$. We equilibrate the error contributions by the adjustments $\delta_f = \tilde{\delta} = \delta_2 M_1^{-3} C^{-1} \text{diam}(D)^{-2}$ and $\delta = M_1^{-3} C^{-1} \text{diam}(D)^{-2} - M_1^4 \text{diam}(D)^2$, where $C > 0$ is the generic constant from (25). These adjustments have potentially decreased the already chosen values of $\delta_f$ and $\delta$. Thus,

$$\sqrt{\int_D \left| \mathbb{E}_x \left( \int_0^{\tau_D} f(X_s)ds \right) - \phi_{u,\delta}^2(x) \right|^2} \, dx \leq \delta_2 \left( 1 + C' \sqrt{|D|} \right). \quad (44)$$

We define the ReLU DNN $\phi_{u,\delta}^1$ by

$$\phi_{u,\delta} = \phi_{u,\delta}^1 + \phi_{u,\delta}^2$$

and chose the remaining two parameter $\delta_1$ and $\delta_2$ such that $\delta_1 = \tilde{\delta}/(2 + 6 \sqrt{|D|})$ and $\delta_2 = \tilde{\delta}/(2 + 2C' \sqrt{|D|})$. Thus, by the estimates (39) and (44)

$$\| u - \phi_{u,\delta}^1 \|_{L^2(D)} \leq \tilde{\delta}.$$

It remains to estimate the size of the DNN $\phi_{u,\delta}^1$ and $\phi_{u,\delta}^2$. We will apply Lemmas 2.2 and 2.3 in order to estimate the size of the ReLU DNN $\phi_{u,\delta}^2$, which is defined by addition and composition of ReLU DNNs. Note that for the chosen parameters, it holds that $\text{size}(\phi_{f,\delta_j}) = \mathcal{O}(d^6 \tilde{\delta}^{-7b}(1 + |D|^{7b}))$ and $\text{size}(\phi_{\text{dist},\delta}) = \mathcal{O}(d^6 \tilde{\delta}^{-8b}(1 + |D|^{8b}))$. In conjunction with (25), the size of the ReLU DNN $\phi_{u,\delta}^2$ is bounded by

$$\text{size}(\phi_{u,\delta}) \leq C_1 \sum_{i=1}^{M_1} \sum_{k=1}^{\bar{N}_i} \text{size} (\bar{x}_{\delta}) + M_2 [k \text{ size} (\phi_{\text{dist},\delta}) + \text{size}(\phi_{f,\delta_j})]$$

$$\leq C_2 \sum_{i=1}^{M_1} \bar{N}_i \text{size} (\bar{x}_{\delta}) + M_2 \text{size}(\phi_{\text{dist},\delta})$$

$$\leq C_3 [M_1^2 (M_1^3 + |D|) \text{size}(\phi_{f,\delta_j}) + M_2^2 (M_2^2 + |D|) \text{size}(\phi_{\text{dist},\delta})]$$

$$\leq C_4 d^6 \tilde{\delta}^{-12-8b}(1 + |D|^{12+8b}),$$

where $C_1, C_2, C_3, C_4$ are generic constants. The size of the ReLU DNN $\phi_{u,\delta}^1$ will be asymptotically dominated by the size of the ReLU DNN $\phi_{u,\delta}^2$.

**Remark 4.4.** The assumption in the previous theorem on the availability of a DNN that approximates the distance function to the boundary may be verified for example in the case that $D = B(0,1) \subset \mathbb{R}^d$. The distance function to the boundary of $D$ is given by $D \ni x \mapsto \text{dist}(x, \partial D) = 1 - |x|$, where we recall that $| \cdot |$ denotes the Euclidean norm. Then, $\phi_{\text{dist},\delta}(x) := 1 - \phi_{\sqrt{\delta_1}} (\sum_{i=1}^{M_2} \phi_{\text{eq},\delta_2}(x))$, where $\phi_{\sqrt{\delta_1}}$ is the DNN that is defined in Lemma A.1 and $\phi_{\text{eq},\delta_2}$ is the DNN in [33] that approximates the square of
Thus, we choose $\delta$. The size of the ReLU DNN in Theorem 4.1. The exponent $12 + 8b$ may be reduced when a tighter bound on $E(\sup_{x \in D} N(\varepsilon))$ would be available, see Lemma 3.6. In the literature, the bound $E_x(N(\varepsilon)) = O(\lceil d \log(\varepsilon^{-1}) \rceil)$ was indicated, cf. [24]. However, it did not seem to be obvious to apply the proposed techniques to also interchange supremum over $x \in D$ and expectation, which is essential in our approach.

5 Conclusions

We have established the existence of numerical approximations of solutions to elliptic PDEs with boundary conditions by DNNs. It is common to obtain the weights of the DNN by an optimization procedures on sampled training data. The generalization error that the DNN has on different data points in the domain may also be controlled is ideally also free from the curse of dimension in terms of the size of the sampled training data. This has been analyzed for certain parabolic PDEs on $\mathbb{R}^d$ in [2]. The extension to PDEs with boundary conditions is subject of future work. Moreover, our results apply to the Poisson equation with non-homogeneous Dirichlet boundary conditions but more general elliptic PDEs could be treated by similar methods.

A Neural network approximation of the square root

In this appendix we provide a constructive DNN approximation to the square root function that converges at a spectral rate. We use this result to establish spectral DNN approximability of the distance function of Euclidean balls but the result may be of independent interest.

Lemma A.1. For every $\delta \in (0, 1)$, there exists a ReLU DNN $\phi_{\sqrt{\cdot}, \delta}$ such that

$$\sup_{x \in [0, 2]} |\sqrt{x} - \phi_{\sqrt{\cdot}, \delta}(x)| \leq \delta$$

with $\text{size}(\phi_{\sqrt{\cdot}, \delta}) = O(\lceil \log(\delta^{-1}) \rceil^2)$.

Proof. The idea of the proof is that ReLU DNNs are able to approximate the product of two scalars well, see [33]. For every $x \in [0, 2]$ and every $n \in \mathbb{N}$ define the sequences

$$s_{n+1} = s_n - \frac{s_n c_n}{2} \quad \text{and} \quad c_{n+1} = c_n^2 c_n - \frac{3}{4}$$

(46)
with \( s_0 = x \) and \( c_0 = x - 1 \). This scheme seems to be introduced in [13]. Following [13], it holds that for every \( n \in \mathbb{N} \), \( 1 + c_{n+1} = (1 + c_n)(1 - \frac{c_n}{2})^2 \), which implies by induction that for every \( n \in \mathbb{N} \)
\[
x(1 + c_n) = s_n^2. \tag{47}
\]
It is easy to see that \(|(c_n - 3)/4| \leq 1 \), and thus (by induction) for every \( n \in \mathbb{N} \)
\[
|c_n| \leq |c_{n-1}|^2 \leq |c_0|^{2^n-1} |c_0|^{2^n-1} = |c_0|^{2^n},
\]
which implies with (47) that for every \( n \in \mathbb{N} \)
\[
|x - s_n^2| \leq |c_0|^{2^n}. \tag{49}
\]
Thus, for every \( x \in [0,2] \), \( s_n \to \sqrt{x} \) as \( n \to \infty \). However, this convergence is not uniform with respect to \( x \in [0,2] \). For that reason, we introduce a shift by \( \delta^2 \) for some \( \delta \in (0,1) \). Specifically, we set for every \( x \in [0,2] \),
\[
s_0 = x + \delta^2 \quad \text{and} \quad c_0 = s_0 - 1.
\]
Suppose that \( x \in [0,2] \). By (49), for every \( n \in \mathbb{N} \)
\[
|\sqrt{x + \delta^2} - s_n| \leq \frac{|x + \delta^2 - s_n^2|}{x + \delta^2 + s_n} \leq \frac{|c_0|^{2^n}}{2(x + \delta^2)} \leq \frac{(1 - \delta^2)^{2^n}}{2\delta^2}.
\]
The condition \((1 - \delta^2)^{2^n}/2\delta^2 \leq \delta \) is satisfied if \( 2^n \geq \log(1/2) + 3\log(\delta^{-1})\delta^{-2} \), where we used the fact that \( \log(1/(1 - \delta^2)) \geq \delta^2/(1 - \delta^2) \geq \delta^2 \). Since \( |\sqrt{x - \sqrt{x + \delta^2}}| \leq \delta \) for any \( x \in [0,2] \),
\[
\sup_{x \in [0,2]} |\sqrt{x} - s_n| \leq 2\delta \quad \text{for} \quad n \geq \frac{\log(1/2) + 3\log(\delta^{-1})/2\log(\delta^{-1})}{\log(2)}.
\tag{50}
\]
The second step of the proof is to account for errors that occur in multiplications in the scheme (46), which are approximated by ReLU DNNs. Let \( \tilde{c}_n \) and \( \tilde{s}_n \), \( n \in \mathbb{N} \), denote realizations of DNNs that are defined by
\[
\tilde{c}_n = \tilde{x}_{\varepsilon/2} \left( \tilde{x}_{\varepsilon/2}(\tilde{c}_{n-1}, \tilde{c}_{n-1}), \frac{\tilde{c}_{n-1} - 3}{4} \right) \quad \text{and} \quad \tilde{s}_n = \tilde{x}_{\varepsilon} \left( \tilde{s}_{n-1}, 1 - \frac{\tilde{c}_{n-1}}{2} \right)
\]
with \( \tilde{c}_0 = c_0 \) and \( \tilde{s}_0 = s_0 \). The DNN \( \tilde{x}_{\varepsilon/2} \) denotes the ReLU DNN from Lemma 2.1 that approximate the product of two scalars on \([-1,1]^2\) with accuracy \( \varepsilon/2 \). Thus, it holds that
\[
|\tilde{c}_n| \leq |c_n|^{2^n-2^n-1} + n\varepsilon \quad \forall n \in \mathbb{N}
\]
We seek an upper bound of \(|\tilde{c}_n|\) that corresponds to (48). Let us assume that \( \sqrt{\varepsilon} \leq 1/(N-1) \) for some \( N \in \mathbb{N} \) and let \( \eta \in (0,1) \) satisfy \((1 + \eta)(1 - \delta^2) \leq 1 \) and additionally let \( \eta \) and \( \varepsilon \) satisfy \((1 + \eta^{-1})\sqrt{\varepsilon} \leq 1 \). We seek to show that
\[
|\tilde{c}_n| \leq |c_0|^{2^n-2^n-1} + n\varepsilon \quad n = 1, \ldots, N.
\]
Let \( p_n = 2^n - 2^n-1 + 1, n \in \mathbb{N} \). It holds that \( p_n = 2p_{n-1} - 1, n \geq 2 \). Indeed by induction with respect to \( n = 2, \ldots, N \), under these conditions, by Young’s inequality,
\[
|\tilde{c}_n| \leq (|c_0|^{p_n-1} + (n-1)\varepsilon)^2 + \varepsilon
\leq (1 + \eta)|c_0|^{2p_{n-1}-1} + (1 + \eta^{-1})\sqrt{\varepsilon}(n-1)^2 + \varepsilon
\leq |c_0|^{2p_{n-1}-1} + \sqrt{\varepsilon}(n-1)^2 + \varepsilon
\leq |c_0|^{p_n} + n\varepsilon.
\]
Since \( p_n \geq 2^n-1, n \in \mathbb{N}, \)
\[
|\tilde{c}_n| \leq |c_0|^{2^n-1} + n\varepsilon \quad n = 1, \ldots, N.
\]

In conclusion, combining with (50) we have estimated that

\[
|y^3 y - \frac{3}{4} - z^2 \frac{z - 3}{4}| \leq \frac{3}{4} (2 - \bar{c}) |y - z| \quad \forall y, z \in [-\bar{c}, 1]
\]

for any \( \bar{c} \in (0, 1) \), implies that

\[
|c_n - \bar{c}_n| \leq \frac{9}{4} |c_0|^{2n-2} |c_{n-1} - \bar{c}_{n-1}| + \frac{9}{2} n \varepsilon \quad n = 1, \ldots, N.
\]  

(51)

Denote \( b_n = |c_n - \bar{c}_n|, n = 1, \ldots, N \). We seek to prove by induction that

\[
b_n \leq \bar{\varepsilon} \left( 1 + \sum_{i=1}^{n-1} \prod_{j=i}^{n-1} \frac{9}{4} |c_0|^{2j-1} \right) \quad n = 1, \ldots, N,
\]

where \( \bar{\varepsilon} = 9 \varepsilon / 2 \). Indeed, by (51) for \( n = 2, \ldots, N \),

\[
b_n \leq \frac{9}{4} |c_0|^{2n-2} b_{n-1} + \bar{\varepsilon} \leq \frac{9}{4} |c_0|^{2n-2} \bar{\varepsilon} \left( 1 + \sum_{i=1}^{n-2} \prod_{j=i}^{n-2} \frac{9}{4} |c_0|^{2j-1} \right) + \bar{\varepsilon}
\]

\[
= \bar{\varepsilon} \left( 1 + \sum_{i=1}^{n-1} \prod_{j=i}^{n-1} \frac{9}{4} |c_0|^{2j-1} \right).
\]

Another tool is the following estimate for any \( a, b, c > 1 \),

\[
\int_0^\infty a^y e^{-by} dy = \int_1^\infty z^{\log(a)/\log(b)} e^{-\log(c)z} \frac{dz}{\log(b)} \leq \frac{k!}{\log(c)^{k+1} \log(b)},
\]

where \( k = \lceil \log(a) / \log(b) \rceil \) and we used the transformation \( z = b^y \). The estimate of this integral implies

\[
\sum_{i=1}^{n-1} \prod_{j=i}^{n-1} \frac{9}{4} |c_0|^{2j-1} \leq \sum_{i=0}^{n-2} \left( \frac{9}{4} \right)^j |c_0|^{2j} \leq 1 + \int_0^\infty \left( \frac{9}{4} \right)^t |c_0|^t dt \leq 1 + \frac{2}{\log(|c_0|^{-1})^3 \log(2)} \leq 1 + \frac{4}{\log(2)} \delta^{-6},
\]

where we used that \( \log(1 + x) > x/2 \) for every \( x \in [0, 1/2] \) (assuming \( \delta \in (0, \sqrt{1/3}) \)). Thus,

\[
|c_n - \bar{c}_n| \leq \bar{\varepsilon} (2 + 4 \delta^{-6} / \log(2)) \quad n = 1, \ldots, N.
\]  

(52)

We can now estimate the total error

\[
|s_n - \tilde{s}_n| \leq |s_{n-1} (1 - c_{n-1}/2) - \bar{s}_{n-1} (1 - \bar{c}_{n-1}/2)| + \varepsilon / 2 \leq |c_{n-1} - \bar{c}_{n-1}| + |s_{n-1} - \bar{s}_{n-1}| + \varepsilon / 2,
\]

where we used that \( s_{n-1}/2 \leq 1 \) and \( (1 - \bar{c}_{n-1}/2) \leq 1 \). The previous estimate (52) implies that

\[
|s_n - \tilde{s}_n| \leq n |\bar{\varepsilon} (2 + 4 \delta^{-6} / \log(2)) + \varepsilon / 2| \quad n = 1, \ldots, N.
\]

In conclusion, combining with (50) we have estimated that

\[
\sup_{x \in [0, 2]} |\sqrt{x} - \tilde{s}_N| \leq 2 \delta + \varepsilon N [(9/2) N (2 + 4 \delta^{-6} / \log(2)) + 1 / 2]
\]

for \( \geq (\log(\log(1/2) + 3 \log(\delta^{-1})) + 2 \log(\delta^{-1})) / 2 \).

It is left now to choose the parameters \( \delta, \varepsilon, N \), and \( \eta \) in a suitable way to estimate the total size of the DNN \( \tilde{s}_N \). For the given target accuracy \( \delta \), we choose \( \delta = \delta/4 \) and \( N = \lceil (\log(\log(1/2) + 19

(50)
\[3 \log(\delta^{-1}) + 2 \log(\delta^{-1})/2\]. Thus, there exists a generic constant \(C > 0\) that neither depends on \(\delta\) nor on \(\varepsilon\) such that
\[
\sup_{x \in [0,2]} |\sqrt{x} - \tilde{s}_N| \leq \frac{\delta}{2} + C\varepsilon\delta^{-7}.
\]
The choice \(\varepsilon \leq \bar{\delta}/2(\delta/4)^7/C\) implies that
\[
\sup_{x \in [0,2]} |\sqrt{x} - \tilde{s}_N| \leq \bar{\delta}.
\]

Let \(\phi_{\sqrt{\cdot},\bar{\delta}}\) be the DNN that corresponds to \(\tilde{s}_N\), i.e., \(\phi_{\sqrt{\cdot},\bar{\delta}}(x) = \tilde{s}_N\) for every \(x \in [0, 2]\). It readily follows (see also Lemmas 2.2 and 2.3) that size\((\phi_{\sqrt{\cdot},\bar{\delta}}) = \mathcal{O}(N[\log(\varepsilon^{-1})]) = \mathcal{O}(\log(\delta^{-1})|2\), which completes the proof of the lemma.

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