Groups in which the co-degrees of the irreducible characters are distinct

Mahdi Ebrahimi

School of Mathematics, Institute for Research in Fundamental Sciences (IPM), Tehran, Iran

ABSTRACT

Let $G$ be a finite group and let $\text{Irr}(G)$ be the set of all irreducible complex characters of $G$. For a character $\chi \in \text{Irr}(G)$, the number $\text{cod}(\chi) := |G : \ker(\chi)|/\chi(1)$ is called the co-degree of $\chi$. The set of co-degrees of all irreducible characters of $G$ is denoted by $\text{cod}(G)$. In this paper, we show that for a nontrivial finite group $G$, $|\text{Irr}(G)| = |\text{cod}(G)|$ if and only if $G$ is isomorphic to $\mathbb{Z}_2$ or $S_3$.

1. Introduction

Let $G$ be a finite group. Also let $\text{cd}(G)$ be the set of all character degrees of $G$, that is, $\text{cd}(G) = \{\chi(1) | \chi \in \text{Irr}(G)\}$, where $\text{Irr}(G)$ is the set of all complex irreducible characters of $G$. For a character $\chi$ of $G$, the co-degree of $\chi$ is defined as $\text{cod}(\chi) := |G : \ker(\chi)|/\chi(1)$ (see [12]). Clearly, if $\chi$ is irreducible, then $\text{cod}(\chi)$ is an integer divisor of $|G/\ker(\chi)|$. Set $\text{cod}(G) := \{\text{cod}(\chi) | \chi \in \text{Irr}(G)\}$. It is well known that the co-degree set $\text{cod}(G)$ may be used to provide information on the structure of the group $G$. For example, Gagola and Lewis [9] showed that $G$ is nilpotent if and only if $\chi(1)$ divides $\text{cod}(\chi)$, for every $\chi \in \text{Irr}(G)$. As another example, Isaacs in [11] showed that if $G$ is a finite group and $g \in G$, then there exists $\chi \in \text{Irr}(G)$ so that every prime divisor of $o(g)$ divides $\text{cod}(\chi)$.

Let $\text{Irr}_1(G)$ be the set of nonlinear characters in $\text{Irr}(G)$ and $\text{cd}_1(G)$ be the set of degrees of the characters in $\text{Irr}_1(G)$. If for a finite group $G$, there exists a nonnegative integer $n$ such that $|\text{cd}_1(G)| = |\text{Irr}_1(G)| = n$, then $G$ is called a $D_n$-group (see [2]). In [3], Berkovich et al. gave the classification of $D_n$-groups. In [2, 4], Berkovich and Kazarin classified all $D_1$-groups. We may replace the condition $|\text{cd}_1(G)| = |\text{Irr}_1(G)| = n$ with $|\text{cod}(G)| = |\text{Irr}(G)| = n$ and obtain a new family of finite groups. If for some nonnegative integer $n$, $|\text{cd}(G)| = |\text{Irr}(G)| = n$, then we say that $G$ is a $D'_n$-group. In this paper, we wish to classify $D'_n$-groups. Note that the symmetric group on $n$ letters is denoted by $S_n$, and a cyclic group of order $n$ by $\mathbb{Z}_n$.

Theorem A. Suppose that $G$ is a nontrivial finite group. Then $G$ is a $D'_n$-group if and only if $G$ is isomorphic to $\mathbb{Z}_2$ or $S_3$.

Theorem A will be used to obtain some information on the structure of a finite group $G$. 
Corollary B. Assume that $G$ is a non-Abelian simple group. Then for every non-Abelian subgroup $H$ of $G$, there exist two distinct characters $\chi_1, \chi_2 \in \text{Irr}(H)$ such that $\text{cod}(\chi_1) = \text{cod}(\chi_2)$, if and only if $G$ is isomorphic to either $\text{PSL}_2(3^{2m+1})$ or the simple Suzuki group $\text{Sz}(2^{2m+1})$, for some positive integer $m$.

Let $G$ be a finite group. A character $\chi$ of $G$ is called real valued or real, if for every $g \in G, \chi(g)$ is a real number.

Corollary C. Suppose $G$ is a finite group. Then for every nontrivial subgroup $H$ of $G$, there exist two distinct characters $\chi_1, \chi_2 \in \text{Irr}(H)$ such that $\chi_1$ and $\chi_2$ are not real valued and $\text{cod}(\chi_1) = \text{cod}(\chi_2)$, if and only if $G$ is a solvable group with odd order.

2. Preliminaries

In this section, we state some relevant results on finite groups needed to prove our main result. In this paper all groups are finite and all characters are complex. The derived subgroup of a group $G$ is denoted by $G'$, and the semi-direct product of groups $H$ and $K$ by $H \rtimes K$. Also, if $p$ is a prime number, $E(p^n)$ denotes the elementary abelian $p$-group of order $p^n$. Finally, we write $Q_8$ for the quaternion group of order 8. We begin with the classification of $D_0$-groups.

Lemma 2.1. [3]. Let $G$ be a $D_0$-group. Then one of the following cases occurs:

a. $G$ is an extra-special 2-group.

b. $G$ is a Frobenius group of order $p^n(p^n - 1)$ for some prime power $p^n$ with an abelian Frobenius kernel of order $p^n$ and a cyclic Frobenius complement.

c. $G$ is a Frobenius group of order 72 in which the Frobenius complement is isomorphic to the quaternion group of order 8.

Let $G$ be a finite group and $\chi \in \text{Irr}(G)$. The field generated by all $\chi(x), x \in G$, is denoted by $\mathbb{Q}(\chi)$. By definition, $\chi$ is called rational if $\mathbb{Q}(\chi) = \mathbb{Q}$, and a finite group $G$ is called a rational group if every irreducible character of $G$ is rational. Now we present the classification of rational Frobenius groups due to Darafsheh and Sharifi.

Lemma 2.2. [8, Theorem]. Let $G$ be a rational Frobenius group. Then $G$ is isomorphic to one of the groups $E(3^n) : \mathbb{Z}_2, E(3^{2n}) : Q_8$ or $E(5^2) : Q_8$. When $G \cong E(3^n) : \mathbb{Z}_2$, then $\mathbb{Z}_2$ acts on $E(3^n)$ by inverting every nonidentity element.

Now we present the classification of non-Abelian simple $S_3$-free groups due to Aschbacher.

Lemma 2.3. [1, Corollary 3] Let $G$ be a non-Abelian simple $S_3$-free group. Then $G$ is isomorphic to either $\text{PSL}_2(3^{2m+1})$, the projective special linear group of dimension 2 over a field with $3^{2m+1}$ elements, or $\text{Sz}(2^{2m+1})$, the simple Suzuki group over a field with $2^{2m+1}$ elements, for some $m \in \mathbb{N}$.

Let $N \neq 1$ be a normal subgroup of the finite group $G$. We say the pair $(G, N)$ is a Camina pair if it satisfies the following hypothesis:

(F2) If $x \in G - N$, $x$ is conjugate to $xy$ for all $y \in N$.

Camina pairs were first introduced by Camina [5]. For the definition of these pairs, he used in [5] a character-theoretic approach (see hypothesis (F1) in [5]), and proved that his definition is equivalent to hypothesis (F2) above.

Lemma 2.4. [6, Proposition 3.1]. Let $G$ be a finite group and let $N$ be a nontrivial normal subgroup of $G$. Then $(G, N)$ is a Camina pair if and only if for every $x \in G - N$, we have $|C_G(x)| = |C_{G/N}(xN)|$. 
Assume that $(G, N)$ is a Camina pair such that $G$ is not a Frobenius group with the Frobenius kernel $N$. The pair $(G, N)$ is called an $F2(p)$-pair if either $N$ or $G/N$ is a $p$-group, for some prime $p$. We refer to [6] and [7] for a thorough analysis of this and related topics.

**Lemma 2.5.** Suppose $(G, N)$ is an $F2(p)$-pair, for some prime $p$.

a. [6, Corollary 3.5]. If $P$ is a Sylow $p$-subgroup of $G$, then $P$ is non-Abelian.

b. [7, Lemma 2.1]. If $G/N$ is a $p$-group, then $G$ has a normal $p$-complement.

### 3. Proof of main results

In this section, we wish to prove our main results.

**Lemma 3.1.** Let $G$ be a non-trivial $D_{0}$-group. Then

a. For every normal subgroup $N$ of $G$, $G/N$ is a $D_{0}$-group.

b. If $\chi \in \text{Irr}(G)$, then $\chi$ is rational valued.

c. $|G : G'| = 2$.

d. If $\chi \in \text{Irr}(G) - \{1_{G}\}$, then $\text{ker}(\chi) \subseteq G'$.

e. $(G, G')$ is a Camina pair.

**Proof.**

(a) Since $\text{ cod}(G/N) \subseteq \text{ cod}(G)$ and $\text{Irr}(G/N) \subseteq \text{Irr}(G)$, we have nothing to prove.

(b) Let $\epsilon$ be a $|G|$ th root of unity and $\sigma \in \text{Gal}(\mathbb{Q}(\epsilon)/\mathbb{Q})$. Then $\chi^\sigma(1) = \chi(1)$. It is easy to see that $\ker(\chi^\sigma) = \ker(\chi)$. Thus $\text{ cod}(\chi^\sigma) = \text{ cod}(\chi)$. As $G$ is a $D_{0}$-group, $\chi^\sigma = \chi$ and so $\chi(x) = (\chi(x))^\theta = \chi(x)$ for all $x \in G$. Hence $\chi(x)$ is rational for all $x \in G$.

(c) Assume that $G = G'$. Let $N$ be a maximal normal subgroup of $G$. By (a), $G/N$ is a non-Abelian simple $D_{0}$-group. Hence $G/N$ is a non-Abelian simple $D_{0}$-group and so Lemma 2.1 leads us to a contradiction. Thus, $G' < G$. By part (b), $G/G'$ is an abelian rational group. Hence it is an elementary abelian 2-group. Thus every non-trivial $\lambda \in \text{Irr}(G/G')$ has co-degree 2 forcing $|G : G'| = 2$.

(d) If $\chi$ is linear, then as $|G : G'| = 2$, we have $\ker(\chi) = G'$. Thus we may assume that $\chi$ is nonlinear. We claim that $\ker(\chi)G'$ is a proper subgroup of $G$. On the contrary, suppose $\ker(\chi)G' = G$. Let $N := \ker(\chi) \cap G'$. Then $G/N \cong \ker(\chi)/N \times G'/N$. It is a contradiction as $G/N$ is a $D_{0}$-group. Thus $\ker(\chi)G'$ is a proper subgroup of $G$. By (a), $G/\ker(\chi)$ is a $D_{0}$-group. Hence using (c), $|G : \ker(\chi)G'| = 2 = |G : G'|$ and so $\ker(\chi) \subseteq G'$.

(e) Since $|G : G'| = 2$, $G$ has a unique non-principal linear character $\lambda$. Let $\chi$ be a nonlinear irreducible character of $G$. Then $\lambda \chi \in \text{Irr}(G)$ and $\langle \lambda \chi \rangle(1) = \chi(1)$. By (d), $\ker(\lambda)$ and $\ker(\lambda \chi)$ are subgroups of $G$. Thus we deduce that $\ker(\lambda \chi) = \ker(\chi)$ and so $\text{ cod}(\lambda \chi) = \text{ cod}(\chi)$. Thus as $G$ is $D_{0}$-group, $\lambda \chi = \chi$. Therefore, $\chi$ vanishes on $G - \ker(\lambda) = G - G'$. Now let $g \in G - G'$. As $G/G'$ is abelian, we have $\text{C}_{G/G'}(gG') = G/G'$. On the other hand $\chi(g) = 0$ for all nonlinear $\chi \in \text{Irr}(G)$, and consequently $|\text{C}_{G}(g)| = 1 + |\lambda(g)|^2 = |G : G'|$. Thus, using Lemma 2.4, $(G, G')$ is a Camina pair.

**Lemma 3.2.** Let $G$ be a Frobenius $D_{0}'$-group with the Frobenius kernel $G'$. Then $G \cong S_{3}$.

**Proof.** Using (c) of Lemma 3.1, $|G : G'| = 2$. Let $x \in G$ be an element of order 2. By Lemma 3.1 (b), $G$ is rational. Hence by lemma 2.2, $G \cong G' : \langle x \rangle$, where for some integer $n \geq 1, G' \cong E(3^{n})$ and $\langle x \rangle$ is the group generated by $x$. Now let $\lambda_{1}, \lambda_{2} \in \text{Irr}(G') - \{1_{G'}\}$. Then for every, $i \in \{1, 2\}, \chi_{i} := \lambda_{i}^{G} \in \text{Irr}(G)$ and using lemma 2.2, it is easy to see that $\text{ cod}(\chi_{1}) = \text{ cod}(\chi_{2})$. Hence as $G$ is a $D_{0}'$-group, $n = 1$ and $G \cong S_{3}$.
Proof of Theorem A. Suppose that $G$ is a nontrivial $D_0'$-group. By Lemma 3.1 (c) and (e), $|G : G'| = 2$ and $(G, G')$ is a Camina pair. If for some prime $p$, $G$ is a $p$-group, then as $|G : G'| = 2$, we deduce that $G \cong \mathbb{Z}_2$. Also if $G$ is a Frobenius group with the Frobenius kernel $G'$, then Lemma 3.2 implies that $G \cong S_3$. Thus we may assume that $G$ is neither a $p$-group nor a Frobenius group with the Frobenius kernel $G'$. Hence $(G, G')$ is an $F(2)$-pair. Let $P$ be a Sylow 2-subgroup of $G$. By Lemma 2.5 (a), $P$ is non-Abelian. Also using Lemma 2.5 (b), $G$ has a normal 2-complement, say $N$. Thus, by Lemma 3.1 (a), $G/N \cong P$ is a $D_0'$-group. Hence $P \cong \mathbb{Z}_2$ which is impossible. Conversely, assume that $G$ is isomorphic to $\mathbb{Z}_2$ or $S_3$. Then it is easy to see that $G$ is a $D_0'$-group and it completes the proof. □

Proof of Corollary B Using Theorem A and Lemma 2.3, we are done. □

Proof of Corollary C If for every nontrivial subgroup $H$ of $G$, there exist two distinct characters $\chi_1, \chi_2 \in \text{Irr}(H)$ such that $\chi_1$ and $\chi_2$ are not real valued and $\text{cod}(\chi_1) = \text{cod}(\chi_2)$, then using Theorem A, $|G|$ is odd and so by Fett-Thompson’s Theorem, $G$ is a solvable group with odd order as desired. Conversely, suppose that $G$ is a solvable group with odd order and $H$ is a nontrivial subgroup of $G$. Since $|H|$ is odd, Theorem A implies that there exist two distinct characters $\chi_1, \chi_2 \in \text{Irr}(H)$ such that $\text{cod}(\chi_1) = \text{cod}(\chi_2)$. If $\text{cod}(\chi_1) = \text{cod}(\chi_2) = 1$, then it is easy to see that $\chi_1 = \chi_2 = 1_H$ which is impossible. Thus $\text{cod}(\chi_1) = \text{cod}(\chi_2) \neq 1$ and both $\chi_1$ and $\chi_2$ are non-trivial. Hence using Burnside’s Theorem (for example, see [10, Problem 3.16]), $\chi_1$ and $\chi_2$ are not real valued. This completes the proof. □

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