Asymptotic representations of Hamiltonian diffeomorphisms and quantization

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Abstract

We show that for a special class of geometric quantizations with “small” quantum errors, the quantum classical correspondence gives rise to an asymptotic unitary representation of the universal cover of the group of Hamiltonian diffeomorphisms. As an application, we get an obstruction to Hamiltonian actions of finitely presented groups.

1 Introduction and main results

Geometric quantization is a mathematical theory modeling the quantum classical correspondence. The latter is a fundamental physical principle stating that the quantum mechanics contains the classical mechanics in the limit when the Planck constant goes to zero. In the present paper we focus on the correspondence between Hamiltonian diffeomorphisms modeling motions of classical mechanics, and their quantum counterparts, unitary operators coming from the Schrödinger evolution. We show that for a special class of geometric quantizations with “small” quantum errors, which exist on a certain class of phase spaces (see Theorem 1.4), this correspondence gives rise to an asymptotic unitary representation of the universal cover of the group of Hamiltonian diffeomorphisms (Theorem 1.5). Interestingly enough, together with recent results from group theory [13], this yields an obstruction to Hamiltonian actions of finitely presented groups (Corollary 1.7). Let us pass to precise definitions.

Let $(M^{2n}, \omega)$ be a closed symplectic manifold. Write $\{f, g\}$ for the Poisson bracket of functions $f$ and $g$, and $\|f\| = \max|f|$ for the uniform norm of $f$. Denote by $\hat{\text{Ham}}(M, \omega)$ the universal cover of the group of Hamiltonian diffeomorphisms of $(M, \omega)$. Its elements $\hat{\phi}$ are Hamiltonian paths $\{\phi_t\}$,
$t \in [0, 1]$ with $\phi_0 = 1$, considered up to a homotopy with fixed end points. We write $\phi = \phi_1$ for the projection of $\tilde{\phi}$ to $\text{Ham}(M, \omega)$. Recall that every path $\{\phi_t\}$ is uniquely determined by a time-dependent generating Hamiltonian $f_t \in C^\infty(M)$, where the functions $f_t$ are assumed to have zero mean: $\int_M f_t \omega^n = 0$ for all $t$. We shall say that $\tilde{\phi} \in \text{Ham}(M, \omega)$ is generated by a Hamiltonian $f \in C^\infty(M \times [0, 1])$. Let us mention that the fundamental group $\pi_1(\text{Ham}(M, \omega))$ is an abelian group, and we have a central extension

$$1 \longrightarrow \pi_1(\text{Ham}(M, \omega)) \longrightarrow \tilde{\text{Ham}}(M, \omega) \longrightarrow \text{Ham}(M, \omega) \longrightarrow 1 \ .$$

In what follows we denote by $\mathcal{L}(\mathcal{H})$ the space of Hermitian operators acting on a finite-dimensional complex Hilbert space $\mathcal{H}$, and write $\mathbb{U}(\mathcal{H})$ for the unitary group of $\mathcal{H}$.

**Definition 1.1.** A fine quantization of $(M, \omega)$ consists of a sequence of positive numbers $\hbar_k$ with $\lim_{k \to \infty} k \hbar_k = 1$, a family of finite-dimensional complex Hilbert spaces $\mathcal{H}_k$ such that

$$\dim \mathcal{H}_k = \left( \frac{k}{2\pi} \right)^n \text{Vol}(M, \omega) + O(k^{n-1}) ,$$

and a family of $\mathbb{R}$-linear maps $Q_k : C^\infty(M) \to \mathcal{L}(\mathcal{H}_k)$ with $Q_k(1) = 1$, satisfying the following properties:

(P1) **(norm correspondence)** $\|Q_k(f)\|_{\text{op}} = \|f\| + O(1/k)$;

(P2) **(bracket correspondence)** $[Q_k(f), Q_k(g)] = \frac{\hbar_k}{i} Q_k(\{f, g\}) + O(1/k^3)$,

where the remainder is understood in the operator norm $\|\cdot\|_{\text{op}}$.

The wording “fine” is chosen in order to emphasize that the remainder in (P2) is $O(1/k^3)$, as opposed to $O(1/k^2)$, as it happens for a wide class of geometric quantizations. For Kähler quantizations (see Section 2 below), the order of the remainder cannot be improved to $O(1/k^3)$, see [3, p.470]. It is unknown whether the same holds true for “abstract” quantizations defined by axioms (P1) and (P2).

Recall that $(M, \omega)$ is quantizable if the cohomology class $[\omega]/(2\pi)$ is integral. The following conditions on the first Chern class $c_1(TM)$ and the cohomology class of symplectic form $[\omega]$ of a quantizable symplectic manifold are equivalent:

(C1) the line $\frac{1}{2} c_1(TM) - \mathbb{R}[\omega]$ in $H^2(M, \mathbb{R})$ intersects the lattice of integral classes $H^2(M, \mathbb{Z})/\text{torsion}$;
(C2) $c_1$ takes even values on $\text{Ker}(\omega)$, where both $c_1$ and $[\omega]$ are considered as morphisms $H_2(M, \mathbb{Z})/\text{torsion} \rightarrow \mathbb{R}$.

Indeed, (C1) yields (C2) immediately. In the opposite direction, choose a basis in $\text{Ker}(\omega)$, say $e_1, \ldots, e_{m-1}$, and extend it to a basis in $H_2(M, \mathbb{Z})/\text{torsion}$ by $e_0$. Then $\omega(e_0) = 2\pi N$, where the number $N \in \mathbb{Z}$ is defined as an integer such that $[\omega]/(2\pi N)$ is a primitive vector. To get (C1) from (C2), we choose $\lambda = (c_1(e_0) + 2p)/(2N)$, with any integer $p$.

**Definition 1.2.** We say that $(M, \omega)$ satisfies condition (C) if it satisfies one of the equivalent conditions (C1) or (C2).

Condition (C) may be viewed as generalisation of the existence of metaplectic structure. It is more general: all complex projective spaces satisfy condition (C) because their second cohomology groups are one-dimensional. However, only the projective spaces with an odd complex dimension have a metaplectic structure.

**Example 1.3.** Take $M$ to be $\mathbb{C}P^2$ blown up at one point. Let $L, E$ be the basis in $H_2(M, \mathbb{Z})$ with $L$ being the class of a general line and $E$ of the exceptional divisor, respectively. There exist sympelctic forms on $M$ with $\omega(L) = 2\pi m$, $\omega(E) = 2\pi n$, for any integral $m > n > 0$. We have $c_1(nL - mE) = 3n - m$, and hence (C2) is satisfied iff $m = n \mod 2$.

**Theorem 1.4.** Every quantizable closed symplectic manifold $M$ satisfying condition (C)) admits a fine quantization.

The proof is given in Section 2.

Let $Q_k$ be a fine quantization. For a Hamiltonian $f_t$ as above consider the unitary quantum equation $U_k(t): \mathcal{H}_k \rightarrow \mathcal{H}_k$ described by the Schroedinger equation

$$\dot{U}_k(t) = -i \hbar Q_k(f_t)U_k(t), \quad U_k(0) = 1.$$  \hspace{1cm} (3)

One can view the time-one map $U_k = U_k(1)$ as a quantization of the element $\tilde{\phi}$ represented by $f_t$ \cite{12}.

Consider family of maps $\mu := \{\mu_k\}$,

$$\mu_k : \tilde{\text{Ham}}(M, \omega) \rightarrow \mathbb{U}({\mathcal{H}_k}), \quad \tilde{\phi} \mapsto U_k.$$  

Let us emphasize that $\mu_k(\tilde{\phi})$ depends on the specific choice of a Hamiltonian path joining the identity with $\tilde{\phi}$, in the class of paths homotopic with fixed endpoints.
Theorem 1.5.

(i) The unitaries $\mu_k(\tilde{\phi})$ and $\mu'_k(\tilde{\phi})$ defined via two different choices of paths homotopic with fixed endpoints representing $\phi \in \widetilde{\text{Ham}}(M,\omega)$, satisfy

$$\|\mu_k(\tilde{\phi}) - \mu'_k(\tilde{\phi})\|_{op} = O(1/k).$$

(ii) For every $\tilde{\phi}, \tilde{\psi} \in \widetilde{\text{Ham}}(M,\omega)$

$$\|\mu_k(\tilde{\phi})\mu_k(\tilde{\psi}) - \mu_k(\tilde{\phi}\tilde{\psi})\|_{op} = O(1/k).$$

(iii) If $\phi \neq 1$,

$$\|\mu_k(\tilde{\phi}) - 1\|_{op} \geq 1/2 + O(1/k).$$

The proof is given in Section 3.2.

The collection of maps $\mu_k$ gives rise to an interesting algebraic object. In order to describe it, we need some preliminaries from [13]. For $p \geq 1$ and an operator $A : \mathcal{H} \to \mathcal{H}$ acting on a $d$-dimensional Hilbert space $\mathcal{H}$ denote by $\|A\|_p$ its $p$-th Schatten norm given by

$$\|A\|_p = \left(\text{tr}\left((\sqrt{A^*A})^p\right)\right)^{1/p}.$$ 

Recall that

$$\|A\|_{op} \leq \|A\|_p \leq d^{1/p}\|A\|_{op}. \tag{7}$$

Definition 1.6 ([13]). A group $\Gamma$ is called $p$-norm approximated if there exists a family of maps $\rho_k : \Gamma \to \mathbb{U}(\mathcal{H}_k)$, where $\mathcal{H}_k$ is a sequence of Hilbert spaces of growing dimension, such that

$$\lim \|\rho_k(x)\rho_k(y) - \rho_k(xy)\|_p = 0, \quad \forall x, y \in \Gamma,$$  

and

$$\limsup \|\rho_k(x) - 1\|_p > 0, \quad \forall x \in \Gamma, x \neq 1.$$  

We call any sequence of maps $\rho_k$ satisfying (8) an asymptotic representation of $\Gamma$ in the sequence of unitary groups equipped with the $p$-norms.

Theorem 1.5 combined with estimate (7) and formula (2) immediately yields the following result.
Corollary 1.7. Assume that a $2n$-dimensional closed symplectic manifold $M$ admits a fine quantization. Let $\Gamma \subset \widetilde{\text{Ham}}(M,\omega)$ be a finitely presented subgroup with

$$\Gamma \cap \pi_1(\text{Ham}(M,\omega)) = \{1\}. \quad (10)$$

Then $\Gamma$ is $p$-norm approximated for every $p > n$.

Denote by $\mathcal{LO}_p$ the class of finitely presented groups with are not $p$-norm approximated. Existence of such groups for $p > 1$ was established by Lubotzky and Oppenheim in [13]. For instance, certain finite central extensions of lattices in simple $\ell$-adic Lie groups belong to this class.

Corollary 1.7 yields obstructions to actions of groups from $\mathcal{LO}_p$ on certain symplectic manifolds.

Example 1.8. Let $M$ be a closed oriented surface of genus $\geq 2$ equipped with an area form $\omega$. Then $\pi_1(\text{Ham}(M,\omega)) = 1$ (e.g. see [15]). Furthermore, $H^2(M,\mathbb{Z}) = \mathbb{Z}$, and hence $M$ satisfies condition (C) of Theorem 1.4. Thus no group of class $\mathcal{LO}_p$ admits a Hamiltonian action on $(M,\omega)$.

Question 1.9. Can groups from the class $\mathcal{LO}_p$ act by volume-preserving diffeomorphisms on closed manifolds?

Denote by $K_p \subset \pi_1(\text{Ham}(M,\omega))$ the subgroup formed by elements $\tilde{\phi} \in \widetilde{\text{Ham}}(M,\omega)$ with $\lim_{k \to \infty} \| \mu_k(\tilde{\phi}) - 1 \|_p = 0$. Assumption (10) in Corollary 1.7 can be replaced to

$$\Gamma \cap K_p = \{1\}. \quad (11)$$

It would be interesting to explore the subgroup $K_p$.

Another application of Theorem 1.5 deals with the following stability question: given a subgroup $\Gamma \subset \widetilde{\text{Ham}}(M,\omega)$, is its quantization $\mu_k|_{\Gamma} : \Gamma \to \mathbb{U}(\mathcal{H}_k)$ close to a genuine representation? It follows that the answer is affirmative for the class of $p$-norm stable groups defined as follows [13, 8]. Here we include the case $p = \infty$, i.e. of the operator norm. Let $\Gamma$ be a finitely presented group defined by finite collections of generators $S$ and relations $R$, considered as subsets of the free group $\mathbb{F}_S$ generated by $S$. The $p$-norm stability means that for every $\epsilon > 0$ there exists $\delta > 0$ such that for every finite-dimensional Hilbert space $\mathcal{H}$ and every homomorphism $t : \mathbb{F}_S \to \mathbb{U}(\mathcal{H})$ with

$$\max_{r \in R} \| t(r) - 1 \|_p \leq \delta,$$

there exists a homomorphism $\rho : \Gamma \to \mathbb{U}(\mathcal{H})$ whose lift $\tilde{\rho} : \mathbb{F}_S \to \mathbb{U}(\mathcal{H})$ satisfies

$$\max_{s \in S} \| t(s) - \tilde{\rho}(s) \|_p < \epsilon.$$
Let us mention that all finite groups are operator norm stable by [9, 11].

**Corollary 1.10.** Assume that a $2n$-dimensional closed symplectic manifold $M$ admits a fine quantization. Let $\Gamma = \langle S|R \rangle \subset \widehat{\operatorname{Ham}}(M,\omega)$ be a finitely presented $p$-norm stable subgroup, where $p > n$. There exists a family of homomorphisms $\rho_k : \Gamma \to \mathbb{U}(H_k)$ such that

$$\max_{s \in S} \|\mu_k(s) - \rho_k(s)\|_p \to 0, \quad k \to \infty.$$ 

**Remark 1.11.** Some examples of finite subgroups of $\widehat{\operatorname{Ham}}(M,\omega)$ come from the following construction. Let $F \subset \operatorname{Ham}(M,\omega)$ be a finite group acting in a Hamiltonian way on a closed quantizable symplectic manifold $(M,\omega)$. For instance, any unitary representation of $F$ on a finite-dimensional complex Hilbert space $V$ yields an action of $F$ on the projectivization $\mathbb{P}(V)$. Denote by $\widetilde{F} \subset \operatorname{Ham}(M,\omega)$ as the full lift of $F$. If $F$ is perfect, there exists a finite abelian extension $G$ of $F$, called the universal extension [17] such that the following diagram commutes:

\[
\begin{array}{ccc}
G & \longrightarrow & F \\
\downarrow & & \downarrow \tau \\
\widetilde{F} & \longrightarrow & F
\end{array}
\]

This provides a monomorphism of $G$ into $\operatorname{Ham}(M,\omega)$.

What can we say about the restriction of the approximate representation $\mu_k$ to the fundamental group $\pi_1(\operatorname{Ham}(M,\omega)) \subset \operatorname{Ham}(M,\omega)$? The following enhancement of Theorem 1.4 sheds light on this question.

**Theorem 1.12.** Every Kähler closed symplectic manifold $M$ satisfying condition (C) admits a fine quantization which satisfies

$$\mu_k(\gamma) = e^{i r_k(\gamma)} 1 + \mathcal{O}(1/k),$$

where $r_k : \pi_1(\operatorname{Ham}(M,\omega)) \to \mathbb{R}/(2\pi\mathbb{Z})$ is a sequence of homomorphisms.

The proof is given in Section 4. The homomorphisms $r_k$ will be explicitly described in terms of action and Maslov invariants. The result follows from [7], which is developed in the Kähler setting. But there is no serious reason to think that the Kähler assumption is essential here.
Denote by \( \mathbb{P}U(\mathcal{H}_k) = \mathbb{U}(\mathcal{H}_k)/S^1 \) the projectivization of the unitary group of the Hilbert space \( \mathcal{H}_k \). We equip this group with the quotient metric 
\[
\delta_p([A],[B]) = \inf_\theta \|A - e^{i\theta}B\|_p.
\]
Extending in a straightforward way Definition 1.6 to projective representations, we get the notion of an \textit{asymptotic projective representation} of a finitely presented group in the sequence of projective unitary groups equipped with the metric \( \delta_p \).

With this language, the asymptotic unitary representation \( \mu_k \) from Theorem 1.12 descends to an asymptotic projective representation \( \nu_k : \text{Ham}(M,\omega) \to \mathbb{P}U(\mathcal{H}_k), \; \phi \mapsto [\mu_k(\tilde{\phi})] \), where \( \tilde{\phi} \) is any lift of \( \phi \). The proof is analogous to the one of Theorem 1.5, see also Remark 3.2 below. In particular, every finitely presented subgroup of \( \text{Ham}(M,\omega) \) admits a non-trivial asymptotic projective representation.

\textbf{Question 1.13.} Can one extend Lubotzky-Oppenheim theory to projective representations and to get examples of groups for which this constraint is not void, i.e., which do not admit non-trivial asymptotic projective representations?

Let us note also that for any finite subgroup \( F \subset \text{Ham}(M,\omega) \), the restriction \( \nu_k|_G \) is close to a genuine projective representation, see [9].

A few bibliographical remarks are in order. For Kähler quantization with metaplectic correction an asymptotic representation of the quantomorphisms group of a prequantum circle bundle over a closed symplectic manifold is constructed by Charles in [3]. In the present paper we generalize this result in two directions: first, we prove it for arbitrary fine quantizations, and second, for Kähler quantization, we impose Condition (C) instead of the assumption that the canonical bundle admits a square root.

Charles showed that quantization enables one to reconstruct Shelukhin’s quasi-morphism on \( \overline{\text{Ham}}(M,\omega) \) [6]. Ioos, Kazhdan and Polterovich explored a link between quantization and almost representations of Lie algebras [10].

\section{Constructing fine quantizations}

In this section we prove Theorem 1.4 by constructing a fine quantization, which will be denoted by \( \text{Op}_k \).

In the usual Toeplitz-Kähler quantization, we consider a compact Kähler manifold \( (M,\omega) \) equipped with a holomorphic Hermitian line bundle \( L \) whose Chern connection has curvature \( \frac{i}{2}\omega \). The quantum space is defined

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as the space $\mathcal{H}_k$ of holomorphic sections of $L^k \otimes L'$, where $L'$ is an auxilliary Hermitian holomorphic line bundle. Here, the parameter $k$ is a positive integer. The large $k$ limit is the semiclassical limit where in first approximation the quantum mechanics reduces to the classical mechanics of $M$ considered as the classical phase space. In this context, a standard way to define a quantum observable from a classical one is the Berezin-Toeplitz quantization: for any $f \in C^\infty(M, \mathbb{R})$, we let $T_k(f)$ be the endomorphism of $\mathcal{H}_k$ such that

$$\langle T_k(f)\psi, \psi' \rangle = \langle f\psi, \psi' \rangle$$

for any $\psi, \psi' \in \mathcal{H}_k$. Here the scalar product of $C^\infty(M, L^k \otimes L')$ is given by integrating the pointwise scalar product against the Liouville volume form.

The basic properties of these operators are the following equalities which holds for any $f, g \in C^\infty(M)$

$$T_k(fg) = T_k(f)T_k(g) + O(k^{-1})$$

$$[T_k(f), T_k(g)] = (ik)^{-1}T_k(\{f, g\}) + O(k^{-2})$$

$$\text{tr}(T_k(f)) = \left(\frac{k}{2\pi}\right)^n \int_M f \mu + O(k^{n-1})$$

see [2], [1]. Furthermore $\|T_k(f)\|_\text{op} = \|f\| + O(k^{-1})$. Observe that in the bracket correspondence (second line of (14)), the remainder is a $O(k^{-2})$, so we miss the fine quantization condition given in Definition 1.1.

The first order correction to (14) have been computed in [3]. Introduce for any $f \in C^\infty(M)$, the operator

$$\text{Op}_k(f) \coloneqq T_k(f - (2k)^{-1}\Delta f)$$

where $\Delta$ is the holomorphic Laplacian of $M$ (in complex coordinates $\Delta f = \sum G^{ij} \partial_z^i \partial_{\bar{z}^j}$ with $(G^{ij})$ the inverse of $(G_{ij})$ given by $\omega = i \sum G_{ij} dz_i \wedge d\bar{z}_j$). Since $\text{Op}_k(f) = T_k(f) + O(k^{-1})$, the operators $\text{Op}_k(f)$ satisfy (14) as well. The novelty is that we have now some explicit formulas for the first corrections

$$\text{Op}_k(f) \text{Op}_k(g) = \text{Op}_k(fg) + \frac{i}{2k} \text{Op}_k(\{f, g\}) + O(k^{-2})$$

$$[\text{Op}_k(f), \text{Op}_k(g)] = (ik)^{-1} \text{Op}_k(\{f, g\}) - k^{-1}\omega_1(X_f, X_g) + O(k^{-3})$$

$$\text{tr}(\text{Op}_k(f)) = \left(\frac{k}{2\pi}\right)^n \int_M f(\omega + k^{-1}\omega_1)^n/n! + O(k^{-2})$$
see [3]. Here \( \omega_1 = i(\Theta' - \frac{1}{2}\Theta_K) \) where \( \Theta' \) and \( \Theta_K \) are the Chern curvature of \( L' \) and the canonical bundle \( K \) respectively. In complex coordinates as above, \( \Theta_K = \partial\bar{\partial}\ln \det(G_{ij}) \)

In the case where \( M \) has a metaplectic structure, one can choose for \( L' \) a square root of the canonical bundle, so that \( \omega_1 = 0 \) and we get our fine quantization. More generally, to prove the existence of fine quantizations under assumption (C), we construct a convenient auxiliary bundle \( L' \).

**Lemma 2.1.** Assume condition (C). Then there exists a holomorphic Hermitian line bundle \( L' \) such that \( \omega_1 = \lambda \omega \) with \( \lambda \in \mathbb{Q} \).

**Proof.** The basic observation we need is that for any line bundle \( D \) and integer \( m \) such that \( D^m \) is equipped with a Hermitian and holomorphic structures, \( D \) has natural holomorphic and Hermitian structures inducing the ones of \( D^m \). Furthermore the Chern curvature of \( D \) is \( \frac{1}{m} \) times the Chern curvature of \( D^m \).

Now, the assumption that \( \frac{1}{2}\pi \mathbb{R}^2(\omega) + \mathbb{R}[\omega] \) intersects the lattice of integral classes means that there exists a line bundle \( L' \) such that \( c_1^\mathbb{R}(L') = \frac{1}{2}\pi c_1^\mathbb{R}(K) + \lambda c_1^\mathbb{R}(L) \). Since \( c_1^\mathbb{R}(L) \neq 0 \), \( \lambda = p/q \) is rational. So \( (L')^{2q} = K^q \otimes L^{2p} \otimes T \) where \( T \) is a torsion line bundle, i.e. \( T^m = 1 \) for some \( m \in \mathbb{N} \). We endow \( T \) with the holomorphic and Hermitian structures such that \( T^m \) becomes the trivial Hermitian and holomorphic line bundle, so that the Chern curvature of \( T \) is zero. Then we endow \( L' \) with the Hermitian and holomorphic structure compatible with the isomorphism \( (L')^{2q} = K^q \otimes L^{2p} \otimes T \). So the Chern curvature \( \Theta' \), \( \Theta \) and \( \Theta_K \) of \( L' \), \( L \) and \( K \) satisfy \( \Theta' = \frac{1}{2}\Theta_K + \lambda \Theta \). So \( \omega_1 = i\lambda \Theta = \lambda \omega \).

In the case where \( \omega_1 = \lambda \omega \), the second and third equations of (16) reads

\[
[\text{Op}_k(f), \text{Op}_k(g)] = (i(k + \lambda))^{-1} \text{Op}_k(\{f, g\}) + O(k^{-3})
\]

\[
\text{tr}(\text{Op}_k(f)) = \left( \frac{k + \lambda}{2\pi} \right)^n \int_M f \mu + O(k^{n-2})
\]

which proves Theorem 1.4 for a \( \text{Kähler} \) manifold with \( h_k = (k + \lambda)^{-1} \).

Let us generalize this to symplectic manifolds. So we start with a symplectic compact manifold \( (M, \omega) \) such that \( \frac{1}{2\pi}[\omega] \) is integral. We introduce a Hermitian line bundle \( L \) with Chern class \( \frac{1}{2\pi}[\omega] \) and a second Hermitian line bundle \( L' \). We denote by \( \Omega_1 \in H^2(M, \mathbb{R}) \) the cohomology class

\[
\Omega_1 = \frac{1}{2\pi}(c_1^\mathbb{R}(L') - \frac{1}{2}c_1^\mathbb{R}(K)).
\]

Here, the canonical bundle \( K \) is defined through any almost complex structure compatible with \( \omega \). It is well known that the Chern class of \( K \) only
depends on \( \omega \). If \( \mathcal{H}_k \) is a finite dimensional subspace of \( \mathcal{C}^\infty(M, L^k \otimes L') \), we can define as before the Toeplitz operators \( T_k(f) \) by (13). Then we have the following results:

1. by [4], cf. also [2], [14], one can choose the family \( (\mathcal{H}_k) \) so that the operators \( T_k(f) \) satisfy (14).

2. by [5], there exists a real differential operator \( P : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M) \) such that \( \text{Op}_k(f) = T_k(f) + k^{-1}T_k(Pf) \) satisfies (16) with \( \omega_1 \) a representative of \( \Omega_1 \). Furthermore, by adding to \( P \) a vector field, one modifies \( \omega_1 \) by an exact form. Choosing conveniently this vector field, we can obtain any representative of \( \Omega_1 \).

If condition (C) holds, we can choose \( L' \) so that \( \Omega_1 = \lambda[\omega] \) for some \( \lambda \in \mathbb{Q} \). Choosing \( P \) so that \( \omega_1 = \lambda \omega \), we obtain equations (17).

3 Quantum dynamics

First of all let us fix the sign conventions: The Hamiltonian vector field of a function \( f \in \mathcal{C}^\infty(M) \) is defined by \( i_{\text{grad} f} \omega = -df \), and the Poisson bracket is given by \( \{f, g\} = \text{L}_{\text{grad} f}g \), where \( \text{L} \) stands for the Lie derivative.

3.1 The Egorov theorem for fine quantizations

We start with the Egorov theorem for fine quantizations. Let \( f_t \) be a classical Hamiltonian generating the Hamiltonian flow \( \phi_t \), and let \( U_k(t) \) be the corresponding quantum evolution.

**Theorem 3.1.** For every function \( g \in \mathcal{C}^\infty(M) \)

\[
\|Q_k(g \circ \phi^{-1}) - U_k Q_k(g) U_k^{-1}\|_{\text{op}} = O\left(\frac{1}{k^2}\right),
\]

(18)

where the remainder depends on \( f \) and \( g \).

This formula readily follows from [12, Proposition 2.7.1]. Let us emphasize that the quantum map \( U_k \) depends on the Hamiltonian \( f \) generating the diffeomorphism \( \phi \). This dependence will be analyzed later.

**Proof of the Egorov theorem [18]:**

Recall that if \( \phi_t \) is the Hamiltonian flow generated by a time-dependent Hamiltonian \( f_t(x) \), the flow \( \phi_t^{-1} \) is generated by \( \tilde{f}_t := -f_t \circ \phi_t \). It follows that for any function \( g \in \mathcal{C}^\infty(M) \)

\[
\frac{d}{dt}g \circ \phi^{-t} = (\phi^{-t})^* (L_{\text{grad} f_t} g) = (\phi^{-t})^* \{\tilde{f}_t, g\} = -\{f_t, g \circ \phi^{-t}\}.
\]

(19)
Next, turn to the analysis of the Schrödinger equation \( \dot{\xi} = -\frac{i}{\hbar}Q_k(f_t)\xi \).

Introduce the family of unitary operators
\[
U(s, t) : \mathcal{H}_k \to \mathcal{H}_k, \xi(s) \mapsto \xi(t)
\]
which sends the solution at time \( s \) to the solution at time \( t \). Observe that \( U(0, t) = U_k(t) \) is the Schrödinger evolution, \( U(t, t)^{-1} = U(t, s)^* \). The Schrödinger equation yields
\[
\frac{\partial}{\partial s} U(t, s) = \frac{i}{\hbar} Q_k(f_s) U(t, s), \quad \frac{\partial}{\partial s} U(s, t) = \frac{i}{\hbar} U(s, t) Q_k(f_s). \quad (20)
\]

Put now \( B(s) := U(s, 1)Q_k(g \circ \phi^{-1}_s)U(1, s) \), so that \( B(0) = U_kQ_k(g)U_k = -1 \) and \( B(1) = Q_k(g \circ \phi^{-1}_1) \). From (19) and (20) we get that
\[
\frac{dB}{ds} = \frac{i}{\hbar} \left[ Q_k(f_s), Q_k(g \circ \phi^{-1}_s) \right] - Q_k(\{f_s, g \circ \phi^{-1}_s\}) U(1, s). \quad (21)
\]

Observe that the functions \( f_s \) and \( g \circ \phi^{-1}_s \), \( s \in [0; 1] \) form a compact family with respect to \( C^\infty \)-topology, and hence by bracket correspondence (P2)
\[
\max_s \| dB/ds \|_{op} = O(1/k^2). \]

Thus
\[
\| Q_k(g \circ \phi^{-1}) - U_kQ_k(g)U_k^{-1} \|_{op} = \| \int_0^1 dB/ds(s) ds \|_{op} = O(1/k^2),
\]
as required.

\section*{3.2 Proof of Theorem 1.5}

Suppose that we have two Hamiltonian paths \( \gamma_0 = \phi_{0,t} \) and \( \gamma_1 = \phi_{1,t} \), \( t \in [0; 1] \) with \( \phi_{0,1} = \phi_{1,1} = \phi \), which are homotopic with fixed end points through a family \( \phi_{t,s} \), \( s \in [0, 1] \). Denote by \( U_k(\phi_{1,j}) \) the time one map of the Schrödinger evolution obtained by the quantization of \( \gamma_j \). We claim that
\[
\| U_k(\phi_{1,1}) - U_k(\phi_{1,0}) \|_{op} = O(1/k). \quad (21)
\]

To see this, look at the family \( \phi_{t,s} \) and denote by \( p_{t,s} \) the generating Hamiltonian when \( s \) is fixed, \( t \) varies, and by \( q_{t,s} \) the Hamiltonian when \( t \) is fixed, \( s \) varies. All the Hamiltonians are assumed to have zero mean. Then
\[
\partial_s(p) = \partial_t q + \{ p, q \}. \quad (22)
\]

Put \( A = h_k^{-1}Q_k(p), C = h_k^{-1}Q_k(q) \). Let \( U(t, s) \) be the unitary evolution of
\[
\partial_t U = -iAU
\]
with \( U(0, s) = 1 \). Note that

\[
U_k(\phi_{1,1}) = U(1, 1), \quad U_k(\phi_{1,0}) = U(1, 0).
\]

Define \( B \) by

\[
\partial_s U = -iBU. \tag{23}
\]

Then

\[
\partial_s \partial_t U = -iA\partial_s U - i\partial_s AU = -iABU - i\partial_s AU,
\]

\[
\partial_t \partial_s U = -iB\partial_t U - i\partial_t BU = -iBAU - i\partial_t BU.
\]

Subtracting and rearranging we get

\[
\partial_t B = \partial_s A - i[A, B].
\]

Further, by (22)

\[
\partial_t C = h_k^{-1}Q_k(\partial_t q) = h_k^{-1}Q_k(\partial_s p) + h_k^{-1}Q_k(\{p, q\}) = \partial_s A + h_k^{-1}Q_k(\{p, q\}).
\]

Thus

\[
\partial_t (B - C) = h_k^{-2}(-i[Q_k(p)Q_k(q)] - h_kQ_k(\{p, q\})) = O(1/k),
\]

by bracket correspondence (P2). Observe that \( \partial_s U(0, s) = 0, \) so \( B(0, s) = 0. \)

Further, \( q(0, s) = 0, \) so \( C(0, s) = 0. \) Thus

\[
\|B(1, s) - C(1, s)\|_{op} = O(1/k).
\]

But \( C(1, s) = 0 \) since \( q(1, s) = 0. \) Thus \( \|B(1, s)\|_{op} = O(1/k) \) and hence by (23)

\[
\|U(1, 1) - U(1, 0)\|_{op} = O(1/k),
\]

and (21) follows. This proves item (i) of the theorem.

Let’s analyze the quantization of the product of two Hamiltonian paths.

Let \( \phi_t \) and \( \psi_t \) be two paths generated by normalized Hamiltonians \( f_t \) and \( g_t \) respectively, and denote \( \theta_t = \phi_t\psi_t. \) Consider the corresponding Schroedinger evolutions

\[
\dot{U}_k = -i\hbar_k^{-1}Q_k(f_t)U_k, \quad U_k(0) = 1,
\]

\[
\dot{V}_k = -i\hbar_k^{-1}Q_k(g_t)V_k, \quad V_k(0) = 1.
\]

Put

\[
S(t) = Q_k(f_t) + U_k(t)Q_k(g_t)U_k(t)^{-1}, \quad W_k(t) = U_k(t)V_k(t).
\]
Observe that
\[
\dot{W}_k = -i\hbar^{-1} S(t) W.
\] (24)

Since \(\theta_t\) is generated by \(h_t := f_t + g_t \circ \phi_t^{-1}\), the Egorov theorem (Theorem 3.1) yields
\[
Q_k(h_t) = S(t) + \mathcal{O}(1/k^2).
\]

Denote by \(Z_k(t)\) the Schrödinger evolution of \(\theta_t\), that is
\[
\dot{Z}_k = -i\hbar^{-1} Q_k(h_t) Z_k = (-i\hbar^{-1} S(t) + \mathcal{O}(1/k)) Z_k, Z_k(0) = 1.
\]

Comparing this equation with (24) we conclude that
\[
\|U_k(1)V_k(1) - Z_k(1)\|_{op} = \mathcal{O}(1/k).
\]

Thus \(\mu_k\) is an almost-representation, which proves item (ii) of the theorem.

Finally, assume that a Hamiltonian \(f_t\) generates a Hamiltonian path \(\phi_t\) with \(\phi_1 \neq 1\). Thus \(\phi_1\) displaces an open set \(Y \subset M\): \(\phi_1(Y) \cap Y = \emptyset\). Take a non-vanishing function \(g\) supported in \(\phi_1(Y)\). Observe that
\[
\|g \circ \phi^{-1} - g\| = \|g\|. \quad (25)
\]

Put \(A_k := Q_k(g)\). Let \(U_k\) be the unitary operator quantizing \(\phi_1\). By the Egorov theorem, \(Q_k(g \circ \phi^{-1}) = U_k A_k U_k^{-1} + \mathcal{O}(1/k^2)\). It follows from (25) and (P1) that \(\|U_k A_k U_k^{-1} - A\|_{op} = \|A\|_{op} + \mathcal{O}(1/k)\). Estimating
\[
\|A\|_{op} + \mathcal{O}(1/k) = \|U_k A_k U_k^{-1} - A\|_{op} =
\]
\[
\|U_k A_k U_k^{-1} - U_k A + U_k A - A\|_{op} \leq 2\|A\|_{op} \cdot \|1 - U_k\|_{op},
\]
we get that \(\|1 - U_k\|_{op} \geq 1/2 + \mathcal{O}(1/k)\), which proves item (iii) of the theorem.

Remark 3.2. Replacing \(U_k\) by \(e^{i\theta} U_k\) in the proof of (iii), we get that
\[
\|U_k - e^{i\theta} 1\|_{op} \geq 1/2 + \mathcal{O}(1/k)
\]
for every phase \(\theta\). This implies that the approximate projective representation \(\nu_k\) appearing right after Theorem 1.12 satisfies, for every \(\phi \in \text{Ham}(M, \omega)\),
\[
\delta_p(\nu_k(\phi), 1) \geq \text{const} > 0, \quad \forall k \in \mathbb{N},
\]
provided \(\phi \neq 1\).
4 Loop quantization

In this section we prove Theorem 1.12 from the introduction. A more detailed formulation of this result appears in Theorem 4.1 below.

4.1 Action and Maslov index

Let \((M, \omega)\) be a compact symplectic manifold equipped with a prequantum line bundle \(L\) and an auxiliary line bundle \(L'\) such that
\[
c_1^R(L') = \lambda c_1^R(L) + \frac{1}{2} c_1^R(K) \tag{26}
\]
where \(K\) is the canonical line bundle.

Since \(\frac{1}{i} \omega\) is the curvature of \(L\), the periods of \(\omega\) are multiple of \(2\pi\), so the action of any contractible periodic trajectory \(\gamma(t), t \in [0, T]\) of a Hamiltonian \((H_t)\) is well-defined modulo \(2\pi \mathbb{Z}\) and given by the usual formula
\[
A(\gamma) = \int_D \omega - \int_0^T H_t(\gamma(t)) dt \tag{27}
\]
where \(D\) is a disc with boundary \(\gamma\). We can even define the action modulo \(2\pi\) of any periodic trajectory, by using parallel transport in \(L\) instead of the integral of \(\omega\).

If \((H_t)\) generates a loop \(\mathcal{L} = (\phi_t, t \in [0, 1])\) of Hamiltonian diffeomorphisms, then our assumption on \(L'\) allows to define a mixed action-Maslov invariant as follows \([16]\). By Floer theory, any trajectory \(\phi_t(x), t \in [0, 1]\) is the boundary of a disc \(D\). We set
\[
I(\mathcal{L}) = \lambda (\int_D \omega - \int_0^1 H_t(\phi_t(x)) dt) + \pi m(\psi) \tag{28}
\]
where \(\psi\) is the loop of \(\text{Sp}(2n)\) obtained by trivialising the symplectic bundle \(TM\) over \(D\) and defining \(\psi(t) := T_x \phi_t, m(\psi) = 0\) or \(1\) according to the class of \(\psi\) in \(\pi_1(\text{Sp}(2n)) = \mathbb{Z}\) is even or odd. One readily checks that \(I(\mathcal{L})\) is well defined modulo \(2\pi \mathbb{Z}\).

4.2 Quantization of a Hamiltonian loop

Assume now that \((M, \omega)\) is Kähler, that \(L\) and \(L'\) are holomorphic hermitian line bundles with Chern curvatures \(\Theta\) and \(\Theta'\) satisfying \(\Theta = \frac{1}{i} \omega, \Theta' = \lambda \Theta + \frac{1}{2} \Theta_K\). Consider the space \(\mathcal{H}_k\) of holomorphic sections of \(L^k \otimes L'\). For any \(f \in C^\infty(M, \mathbb{R})\), we define the operator \(\text{Op}_k(f)\) as in \([15]\).
Let \((H_t)\) be a Hamiltonian of \(M\) generating a loop \(L = (\phi_t, t \in [0, 1])\). Introduce the quantum propagator \(U_{t,k}\),

\[
\frac{1}{i(k + \lambda)} \partial_t U_{k,t} + \text{Op}_k(H_t)U_{k,t} = 0, \quad U_{k,0} = 1
\]

We assume from now on that \(M\) is connected, so the periodic trajectories \((\phi_t(x), t \in [0, 1])\) have all the same action, denoted by \(A(L)\).

**Theorem 4.1.** We have \(U_{k,1} = e^{ikA(L) + iI(L)} + \mathcal{O}(k^{-1})\).

**Proof.** We can rewrite the Schrödinger equation as

\[
\frac{1}{ik} \partial_t U_{k,t} + (1 + \lambda k) \text{Op}_k(H_t)U_{k,t} = 0
\]

Then, by \([7, Theorem 4.2]\) the Schwartz kernel of \(U_{k,t}\) is a Lagrangian state associated to the graph of \(\phi_t\). We refer to \([7]\) for the precise definitions.

What is important to us here is that since \(\phi_1\) is the identity, \(U_{k,1} = e^{ik\theta}T_k(\sigma) + \mathcal{O}(k^{-1})\) (29)

where \(\theta\) is a real number, \(\sigma \in \mathcal{C}^\infty(M)\) and \(T_k(\sigma)\) is the Berezin-Toeplitz operator with multiplicator \(\sigma\) defined as in section 2.

Furthermore, we can compute \(\theta\) and \(\sigma\) by introducing a half-form bundle (i.e., the square root of the canonical bundle) denoted by \(\delta\). It is possible that such a bundle does not exist on \(M\) but we only need it on the trajectory \(\gamma\) of a given point \(x\). In this case we take a disk \(D\) with boundary \(\gamma\) and choose the square root \(\delta\) which extends to \(D\).

Then by \([7, Theorem 1.1]\]

\[
U_{k,t}(\phi_t(x),x) \sim \left(\frac{k}{2\pi}\right)^n e^{\frac{i}{2} \int_0^t H_{\phi_t}^{\text{sub}}(\phi_t(x))} dt [\phi_t^L(x)]^\otimes k \otimes T_t^{L1}(x) \otimes [D_t(x)]^{1/2}.
\]

Here \(\phi_t^L\) is the prequantum lift of \(\phi_t\) to \(L\), and \(H_{\phi_t}^{\text{sub}} = \lambda H_t\) is the subprincipal symbol of \((1 + \frac{1}{2}) \text{Op}_k(H_t)\). The second term \(T_t^{L1}(x) : L_1|_x \to L_1|_{\phi_t(x)}\) is the parallel transport in the line bundle \(L_1 = L' \otimes \delta^{-1}\). It is the multiplication by \(\exp(i\lambda \int_\omega)\) because the curvature of \(L_1\) is \(\Theta' - \frac{1}{2} \Theta_K = \lambda \Theta = \frac{\lambda}{i} \omega\). The last term is the square root of an isomorphism \(D_t(x) : K_x \to K_{\phi_t(x)}\) defined by

\[
D_t(x)(\alpha)((T_x \phi_t)^{1,0}u) = \alpha(u), \quad \forall \alpha \in K_x, u \in \det T_x^{1,0}M.
\]

Here the square root is chosen so as to be continuous and equal to 1 at \(t = 0\).
On the other hand, by (29), \( U_{k,1}(x, x) = (k/2\pi)^n e^{ik\theta}(\sigma(x) + O(k^{-1})) \).

Now \( \phi^L_1(x) = e^{iA(L)} \) implies that \( \theta = A(L) \) and it remains to prove that
\[
e^{\frac{1}{2} \int_0^1 H^\nu_{ab}(\phi_t(x)) \, dt \, \mathcal{T}_1^L(x) \otimes [\mathcal{D}_1(x)]^{1/2} = e^{iI(L)} \]  

(30)

Since \( T_x\phi_1 \) is the identity of \( T_xM \), \( \mathcal{D}_1(x) \) is the identity of \( K_x \) so
\[
(\mathcal{D}_1(x))^{1/2} = \pm 1_{\delta_x}.
\]

To determine the sign, we trivialize \( TM \) along \( \gamma \) with a symplectic frame, so that \( (T_x\phi_t) \) becomes a loop \( \alpha \) of symplectic matrices based at the identity and in the corresponding trivialisation of \( K \), \( \mathcal{D}_t(x) \) is the multiplication by a complex number. The sign we search depends only on the homotopy class of \( \alpha \). Since \( \text{Sp}(2n) \) deformation retracts to its subgroup \( \text{U}(n) \), we can assume that \( \alpha \) is a loop of \( \text{U}(n) \), in which case \( \mathcal{D}_t(x) \) is the complex determinant of \( \alpha(t) \). Thus, our sign is positive or negative according to the class of \( \alpha \) in \( \pi_1(\text{Sp}(2n)) = \mathbb{Z} \) is even or odd. We conclude that each factor in (30) corresponds to a summand in (28), which completes the proof.

\[\Box\]

References

[1] Bordemann, M., Meinrenken, E., and Schlichenmaier, M., Toeplitz quantization of Kähler manifolds and \( \text{gl}(N) \), \( N \to \infty \) limits, Comm. Math. Phys. 165 (1994), 281–296.

[2] Boutet de Monvel, L., and Guillemin, V., The spectral theory of Toeplitz operators, vol. 99, Annals of Mathematics Studies, Princeton University Press, Princeton, NJ, 1981.

[3] Charles, L., Semi-classical properties of geometric quantization with metaplectic correction, Comm. Math. Phys. 270 (2007),445–480.

[4] Charles, L., On the quantization of compact symplectic manifold, J Geom. Anal. 26 (2016), 2664–2710.

[5] Charles, L., Subprincipal symbol for Toeplitz operators. Lett. Math. Phys. 106 (2016), no. 12, 1673-1694.

[6] Charles, L., On a quasimorphism of Hamiltonian diffeomorphisms and quantization, preprint arXiv:1910.05073, 2019.

[7] Charles, L., Le Floch, Y. Quantum propagation for Berezin-Toeplitz operators, preprint hal-02935681, 2020.
[8] De Chiffre, M., Glebsky, L., Lubotzky, A., Thom, A., Stability, cohomology vanishing, and nonapproximable groups, Forum of Mathematics, Sigma 8 (2020), e18.

[9] Grove, K., Karcher, H. and Ruh, E.A., Jacobi fields and Finsler metrics on compact Lie groups with an application to differentiable pinching problems, Math. Ann., 211 (1974), 7–21.

[10] Ioos, L., Kazhdan, D., Polterovich, L., Almost representations of Lie algebras and quantization, preprint arXiv:2005.11693, 2020

[11] Kazhdan, D., On $\epsilon$-representations, Israel J. Math. 43 (1982), 315–323.

[12] Landsman, N.P., Mathematical Topics between Classical and Quantum Mechanics, Springer, 2012.

[13] Lubotzky, A., Oppenheim, I., Non $p$-norm approximated groups, preprint arXiv:1807.06790, 2018.

[14] Ma, X., and Marinescu, G., Holomorphic Morse inequalities and Bergman kernels, volume 254, Progress in Mathematics, Birkhäuser Verlag, Basel, 2007.

[15] Polterovich, L., The geometry of the Group of Symplectic Diffeomorphisms, Birkhäuser; 2012

[16] Polterovich, L., Hamiltonian loops and Arnold’s principle, Translations of the American Mathematical Society-Series 2, 180 (1997), 181–188.

[17] Rosenberg, J., Algebraic K-theory and its Applications, Springer, 1995.

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