SOLITARY WAVES FOR AN INTERNAL WAVE MODEL

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ABSTRACT. We show the existence of solitary wave solutions of finite energy for a model to describe the propagation of internal waves for wave speed $c$ large enough. Furthermore, some of these solutions are approximated using a Newton-type iteration combined with a collocation-spectral strategy for spatial discretization of the corresponding solitary wave equations.

1. Introduction. Travelling waves exist as a consequence of a balance between nonlinear and dispersive effects present in a system; these waves travel with a constant speed without any temporal evolution in shape or size when the frame of reference moves with the same speed of the wave. In the last decades, the study of travelling waves has grown enormously because they are present in several and varied fields of application, such as fluid mechanics, optics, acoustics, oceanography, astronomy, etc. This has attracted the interest of mathematicians and physicists due to the mathematical and numerical difficulty of analysis. Research has shown that there are travelling waves in liquid, solid, gaseous, electric current, electromagnetic fields, atmospheres of planets, crystals, plasmas, glass fibers, nervous networks. Furthermore, travelling waves can be formed at the interface between two layers of immiscible fluids of different densities, which is the phenomenon of interest in this paper. Mathematically, the travelling waves form a special class of solutions of some nonlinear, dispersive partial differential equations and to determine the existence and properties of such solutions is a fundamental problem of great interest for both pure and applied mathematicians. Some classic partial differential equations which possess these solutions are:

- Korteweg de-Vries equation (KdV) [12]: $u_t + uu_x + u_{xxx} = 0$.
- Benjamin-Bona-Mahony equation (BBM) [15]: $u_t + u_x + uu_x - u_{xxt} = 0$.
- Benney-Luke equation [4], [18]: $u_{tt} - k^2 u_{xx} + au_{xxxx} + bu_{xxtt} + cu_t u_{xx} + du_x u_{xt} = 0$.
- Benjamin-Ono equation (BO) [2]: $u_t + u_x + uu_x + H(u_{xx}) = 0$.

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Regularized Benjamin-Ono equation (rBO) [1]: \( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + uu_x - H(u_{xt}) = 0 \).

Camassa-Holm equation [5]: \( \frac{\partial u}{\partial t} + 2k u_x - u_{xxt} + 3u u_x = 2u_x u_{xx} + uu_{xxx} \).

Degasperis-Procesi equation [9]: \( \frac{\partial u}{\partial t} + cu_x + du_{xxx} - a^2 u_{xxt} - a^2 f(u_{xxx}) + 4fu_{xx} = 0 \).

Here \( Hf(x) \) denotes the Hilbert transform operator defined by

\[
H(f)(x) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{f(\tau)}{\tau - x} \, d\tau,
\]

where p.v.\( \int \) stands for the integration in the principal value sense. We refer the reader to the work by J. Duoandikoetxea [10] for more information about the Hilbert transform.

The aim of the present study is to discuss the existence of solitary wave solutions with speed \( c \) in the form \( \zeta = \xi(x - ct), u = w(x - ct) \) (where \( \xi, w \) decay to zero at infinity) of the regularized Benjamin-Ono system

\[
\begin{align*}
\zeta_t - ((1 - \alpha \zeta)u)_x &= \frac{\epsilon^2}{6} \zeta_{xxt} \\
u_t + \alpha u u_x + \left(1 - \frac{\rho_2}{\rho_1}\right) \zeta_x &= \frac{\rho_2}{\rho_1} c \mathcal{H}(u_{xt}) + \frac{\epsilon^2}{6} u_{xx},
\end{align*}
\]

recently introduced by J. C. Muñoz [16] and related to the rBO equation. Furthermore, in [16] was established existence and uniqueness of solutions in the non-periodic case and a spectral scheme was introduced to approximate solutions of the corresponding Cauchy problem.

System (1.2) describes the propagation of a weakly nonlinear internal wave propagating at the interface of two immiscible fluids with constant densities, which are contained at rest in a long channel with a horizontal rigid top and bottom, and the thickness of the lower layer is assumed to be effectively infinite (deep water limit) (see Figure 1). The constants \( \rho_1 \) and \( \rho_2 \) represent the density of the fluids and \( \rho_2/\rho_1 > 1 \) holds (for stable stratification). The constants \( \alpha \) and \( \epsilon \) are small positive real numbers (such that \( \alpha = O(\epsilon^2) \)) defined as \( \alpha = \frac{a}{h_1} \) and \( \epsilon = \frac{h_1}{L} \), that measure the intensity of nonlinear and dispersive effects, respectively. Here \( h_1 \) denotes the thickness of the upper fluid layer and the parameters \( L \) and \( a \) correspond to the characteristic wavelength and characteristic wave amplitude, respectively. The variable \( x \) represents the spatial position and \( t \) denotes the propagation time. The function \( u = u(x, t) \) is the velocity monitored at the normalized depth \( z = 1 - \sqrt{2/3} \), and \( \zeta = \zeta(x, t) \) is the wave amplitude at the point \( x \) and time \( t \), measured with respect to the rest level of the two-fluid interface.

We point out that to the best knowledge of the authors, no analytical study about existence of solitary wave solutions of system (1.2) is available in the literature. Very recently, F. Pipicano and J. C. Muñoz have only established the existence of periodic travelling wave solutions of system (1.2) with a large enough period ([17]). In the present paper, we will apply the positive operator theory introduced originally by Krasnosel’skii [13], [14] in the exploration of the existence of solitary wave solutions to system (1.2), following the ideas by T.B. Benjamin et al. [3] in the framework of solitary wave solutions of some scalar dispersive equations and H. Chen, who developed in [6] the main breakthrough in the application of this method to scalar dispersive-type equations in a periodic domain.
The second purpose of this paper is to construct approximations to solitary wave solutions of system (1.2), whose existence is guaranteed by our theoretical results. This is performed by using a numerical scheme that features a pseudospectral method with the Fourier basis to approximate the spatial structure and a Newton’s iteration for solving the system of nonlinear equations generated.

The rest of this paper is organized as follows. In Section 2, we employ the results of positive operators on cones, necessary in order to develop the existence theory of solitary wave solutions of system (1.2). In Section 3, we reformulate the problem as one of finding a fixed point of a nonlinear positive operator defined on a cone in an appropriate Fréchet space. In Section 4, we establish existence of a family of solitary wave solutions of system (1.2) parametrized by the wave speed $c$. In Section 5, we introduce the numerical solver employed to compute solitary wave solutions of system (1.2) and illustrate the theoretical results. Finally, Section 6 contains the conclusions of our work.

2. Preliminary results. In this section we include a brief review of some results from the functional analysis of positive operators whose domain constitutes a subset of a Fréchet space, following the papers by Benjamin et al. [3] and H. Chen et al. [7]. We must recall that a Frechet space $X$ is a metrizable and complete, locally-convex, linear topological space (over the real numbers). On $X$ a sequence $(p_n)_n$ of semi-norms can be defined in such a way that $p_{n+1}(x) \geq p_n(x)$ for every $x \in X$. 

![Figure 1. A typical solitary wave propagating at the interface of the two-fluid system which are contained at rest in a long channel with a horizontal rigid top and bottom.](image)
and every \( n = 1, 2, 3, \ldots \) and that the formula
\[
d(x, y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \left( \frac{p_j(x - y)}{1 + p_j(x - y)} \right), \quad x, y \in X,
\]
provides a metric that generates a topology that coincides with the original topology on \( X \). In this case, we say that \( X \) is a Fréchet space with generating family of semi-norms \((p_n)_n\). Hereafter, we use the notation
\[
B_r = \{ x \in X : d(x, 0) < r \}, \quad B'_r = \{ x \in X : p_j(x) < r \}, \quad j \in \mathbb{N}.
\]
It is clear from (2.1), we have that \( X = B_1 \). In general, a set \( B \) in a topological linear space \( X \) is said to be bounded if, for any neighborhood \( U \) of 0 in \( X \), there is a \( \lambda > 0 \) such that \( \lambda B \subset U \). In the case of a Fréchet space with metric \( d \) given by (2.1), a set \( B \) in \( X \) is bounded if and only if corresponding to each positive integer \( j \) there is a \( R > 0 \) such that \( B \subset B'_R \). If \( r > 0 \), then \( B_r \) is usually not bounded (for details see [19]). A closed subset \( K \) of a Fréchet space \( X \) is a cone if, the following conditions hold true:
\[
\lambda K = \{ \lambda u : u \in K \} \subset K, \quad \text{for all } \lambda \geq 0. \tag{2.2}
\]
\[
K + K = \{ u + v : u, v \in K \} \subset K. \tag{2.3}
\]
\[
K \cap \{ -K \} = K \cap \{ -u : u \in K \} = \{ 0 \}. \tag{2.4}
\]
From (2.2) and (2.4), \( K \) must be convex. On the other hand, we also have a partial ordering on \( K \) given by
\[
x \prec y \iff y - x \in K.
\]
For any \( 0 < r < R < \infty \), let us denote
\[
K_r = K \cap B_r, \quad \partial K_r = K \cap \partial B_r, \quad \text{and} \quad K^r_r = \{ u \in K : r < d(u, 0) < R \}.
\]
An operator \( A \) defined on \( K \) is said to be positive, if \( (A(K) \subset K \). On the other hand, we say that a positive operator \( A \) on \( K \) is \( K \)-compact, if the set \( A(K_r) \) has a compact closure, for each \( r \geq 0 \). A triplet \((K, A, U)\) is said to be admissible, if
1. \( K \) is a convex subset of \( X \),
2. \( U \subset K \) is open in the relative topology on \( K \),
3. \( A \) is continuous and \( K \)-compact,
4. There are no fixed points of \( A \) on \( \partial U \), the boundary of the open set \( U \) in the relative topology on \( K \).

Form Granas’ work ([11]), there is an integer-valued function \( i(K, A, U) \) that satisfies the basic axioms of a fixed-point index. Among them, we consider the following ones:

- **(Homotopy invariant)** If \((K, A, U)\) and \((K, B, U)\) are two admissible triplets and the operator \( A \) is homotopic to the operator \( B \) on \( U \), then \( i(K, A, U) = i(K, B, U) \).
- **(The fixed point property)** If \((K, A, U)\) is admissible and \( i(K, A, U) \neq 0 \), then \( A \) has at least one fixed point in \( U \).
- **(Index of constant maps)** If \((K, A, U)\) is admissible and \( A \) is constant (i.e., there is a point \( a \in K \) such that \( Au = a \) for all \( u \in K \)), then
  \[
i(K, A, U) = \begin{cases} 1 & \text{if } a \in U, \\ 0 & \text{if } a \notin U. \end{cases}
\]
We refer the reader to [3] (see also [13], [14] and [11]) for details in the following result. It is assumed throughout that $K$ is a cone in a Frechét space with generating family of semi-norms ($p_j$) and the standard metric $d$ as in (2.1), and that $\mathcal{A} : K \to K$ is continuous and $K$-compact.

**Lemma 2.1.** Suppose that $0 < \rho < 1$ and that either
\[
Ax - x \notin K, \quad \text{for all } x \in \partial K \rho, \quad tAx \notin x \quad \text{for all } x \in \partial K \rho \quad \text{and all } t \in [0, 1].
\]
Then we have that $(K, \mathcal{A}, K \rho)$ is admissible and $i(K, \mathcal{A}, K \rho) = 1$.

**Lemma 2.2.** Suppose that $0 < \rho < 1$ and that either
\[
x - Ax \notin K, \quad \text{for all } x \in \partial K \rho,
\]
There exists $\tilde{x} \neq 0$ in $K$ such that $x - Ax \neq \lambda \tilde{x}$ for $x \in \partial K \rho$ and $\lambda \geq 0$. Then $(K, \mathcal{A}, K \rho)$ is admissible and $i(K, \mathcal{A}, K \rho) = 0$.

**Lemma 2.3.** Let $(K, \mathcal{A}, U)$ be admissible. If there exists $\tilde{x} \in K$ with $\tilde{x} \neq 0$ such that $x - Ax \neq \lambda \tilde{x}$ for $x \in \partial U$ and all $\lambda \geq 0$, then $i(K, \mathcal{A}, U) = 0$.

The following theorem is a consequence of the first two lemmas:

**Theorem 2.4.** If either (2.5) or (2.6) holds for $r$ satisfying $0 < r < 1$ and either (2.7) or (2.8) holds for $R$ satisfying $r < R < 1$. Then $\mathcal{A}$ has at least one fixed point in $K r$\textsuperscript{R}. Moreover, $i(K, \mathcal{A}, K r _\rho) = -1$.

3. **Problem setting.** Before we go further, we rescale the parameters $\alpha, \epsilon$ by considering the change of variables
\[
\zeta(x, y) = \sigma w(y, \tau), \quad u(x, y) = \sigma v(y, \tau), \quad y = ax, \quad \tau = at, \quad \epsilon a = 1, \quad \alpha \sigma = 1.
\]
For $p_0 = p_2 / \rho_1$, the system (1.2) takes the form
\[
w_t - ((1 - w) w_x)_{xx} = \frac{1}{6} w_{xxx},
\]
\[
v_t + vv_x + (1 - p_0) w_x = p_0 \mathcal{H}(v_{xx}) + \frac{1}{6} v_{xxx}.
\]
So, for the sake of simpleness, we consider system (1.2) with rescaled constants $\alpha = \epsilon = 1$:
\[
\left\{
\begin{array}{l}
\zeta_t - u_x + (\zeta u)_x = \frac{1}{6} \zeta_{xxx}, \\
u_t - (\rho_0 - 1) \zeta_x + uu_x = \rho_0 \mathcal{H}(u_{xx}) + \frac{1}{6} u_{xxx}.
\end{array}
\right.
\]
We are looking for solitary wave solutions with speed $c$ in the form
\[
(\zeta, u)(x, t) = (\xi, w)(z), \quad z = x - ct,
\]
where $\xi(z), w(z) \to 0$, as $|z| \to \infty$. We see that $(\xi, w)$ satisfies the system
\[
-c (I - \frac{1}{6} \partial_z^2) \xi = w - \xi w,
\]
\[
-c (I - \rho_0 \mathcal{H}(\partial_z) - \frac{1}{6} \partial_z^2) w = (\rho_0 - 1) \xi - \frac{w^2}{2}.
\]
In order to prove the existence of solutions of this travelling wave system, we use a continuation argument by considering for $\nu > 0$ the perturbed system
\[
\left\{
\begin{array}{l}
-c (I - \frac{1}{6} \partial_z^2) \xi = w - \xi w + \nu \xi^2, \\
-c (I - \rho_0 \mathcal{H}(\partial_z) - \frac{1}{6} \partial_z^2) w = (\rho_0 - 1) \xi - \frac{w^2}{2}.
\end{array}
\right.
\]
The first observation is that the operators
\[ A_1(c) = e \left( I - \frac{1}{6} \partial_x^2 \right), \quad A_2(c) = e \left( I - \rho_0 \mathcal{H}(\partial_x) - \frac{1}{6} \partial_x^2 \right), \]
are invertible in standard Sobolev spaces \( H^s(\mathbb{R}) \). Then we see that the solution \((\xi, w)\) for the perturbed system (3.3) takes the fixed point form
\[ \left( \begin{array}{c} \xi \\ w \end{array} \right) = \left( A_1(c)^{-1} (\xi w - \nu \xi^2) \right), \quad \left( A_2(c)^{-1} \left( \frac{1}{2} w^2 - (\rho_0 - 1) \xi \right) \right). \]
(3.4)

In other words, to show the existence of travelling waves for the perturbed system (3.3) is equivalent to establish the existence of a fixed point for the equation in the variable \( U = (\xi, w)^t \)
\[ U = A_c(U), \]
where the components of \( A_c \) are given by
\[ A_{1,c}(U) = A_1(c)^{-1} (\xi w - \nu \xi^2), \quad A_{2,c}(U) = A_2(c)^{-1} \left( \frac{1}{2} w^2 - (\rho_0 - 1) \xi \right). \]
(3.6)

Hereafter, we consider the space of real valued continuous functions defined on \( \mathbb{R} \) (denoted by \( C(\mathbb{R}) \)), with the topology of uniform convergence on bounded intervals under the semi-norms
\[ p_k(w) = \max_{-k \leq x \leq k} |w(x)|, \quad k = 1, 2, ..., \]
(3.7)

In this case, the distance is given by
\[ d(f, g) = \sum_{k=1}^{\infty} \frac{1}{2^k} \left( \frac{d_k(f, g)}{d_k(f, g) + 1} \right), \quad d_k(f, g) = \sup_{x \in [-k, k]} |f(x) - g(x)|. \]
(3.8)

The open ball of radius \( r < 1 \) centered at zero and its boundary are given respectively by
\[ B_r(0) = \{ u \in C(\mathbb{R}) : d(0, u) < r \}, \quad \partial B_r(0) = \{ u \in C(\mathbb{R}) : d(0, u) = r \}. \]
(3.9)

Let \( \mathcal{K} \subset C(\mathbb{R}) \) be the cone defined as
\[ \mathcal{K} = \{ w \in C(\mathbb{R}) : w(x) = w(-x) \geq 0; \text{ w is non-increasing for } x \geq 0 \}. \]

Note the that for \( j = 1, 2, 3, ... \), we have for all \( w \in \mathcal{K} \) that
\[ p_j(w) = p_1(w), \]
and so, we have for \( 0 < r < 1 \) that
\[ d(w, 0) = \sum_{k=1}^{\infty} \frac{1}{2^k} \left( \frac{p_k(w)}{p_k(w) + 1} \right) = \frac{w(0)}{1 + w(0)} < r \iff w(0) < \frac{r}{1 - r}. \]
(3.10)

We note that the space \( Y = C(\mathbb{R}) \times C(\mathbb{R}) \) is a Fréchet space with generating family of semi-norms
\[ P_j(f, g) = \max_{x \in [-j, j]} \{|f(x)|, |g(x)|\}. \]

Now, we define the cone \( \widetilde{\mathcal{K}} \subset C(\mathbb{R}) \times C(\mathbb{R}) \) by \( \widetilde{\mathcal{K}} = -\mathcal{K} \times \mathcal{K} \). We also set
\[ B_r(0) = \left\{(f, g) \in C(\mathbb{R}) \times C(\mathbb{R}) : \widetilde{d}(f, g, 0) < r \right\}, \]
and its boundary,
\[ \partial B_r(0) = \left\{(f, g) \in C(\mathbb{R}) \times C(\mathbb{R}) : \widetilde{d}(f, g, 0) = r \right\}, \]
where \( \tilde{d} \) is defined by
\[
\tilde{d}((f,g),0) = \max\{-f(0), g(0)\} \cdot \frac{1 + \max\{-f(0), g(0)\}}{1 + r}.
\]

We observe that if \((f,g) \in \tilde{K} \cap \partial B_r(0)\), in other words, \(\tilde{d}((f,g),0) = r\), then we have that
\[
\max\{-f(0), g(0)\} = r - \frac{r}{1 + r}.
\] (3.11)

Note that in this case, we have that \(K\) is closed and is \(P_1\)-bounded, where
\[
P_1(f,g) = \max_{x \in [-1,1]} \{|f(x)|, |g(x)|\}.
\]

Hereafter, according with the notation introduced in the previous section, we set for \(r > 0\), the convex set
\[
\tilde{K}_r = \tilde{K} \cap B_r(0).
\]

4. Main results. Before we go further, we establish a general result.

Lemma 4.1. Let \(k \in C(\mathbb{R}) \cap L^1(\mathbb{R})\) be an even positive non increasing function such that \(\hat{k} \in L^1(\mathbb{R})\) is an even positive non increasing bounded function for \(w \geq 0\). Then the operator \(\mathcal{B}\) defined by
\[
\mathcal{B}(f)(x) = (k * f)(x) = \int_{\mathbb{R}} k(x-r)f(r) \, dr
\]
maps \(K\) into \(K\).

Proof. We first note that \(\mathcal{B}(f)(x)\) is bounded due to the fact that \(k \in L^1(\mathbb{R})\) and also from the Young inequality, since
\[
\max_{\mathbb{R}} |k * f| \leq f(0)||k||_{L^1(\mathbb{R})}, \quad f \in K.
\]

Now, we also have that \(\mathcal{B}(f) \geq 0\) for \(f \in K\). In fact,
\[
\mathcal{B}(f)(x) = \int_{\mathbb{R}} k(x-y)f(y) \, dy \geq 0,
\]
since \(k \geq 0\) and \(f \geq 0\). On the other hand, \(\mathcal{B}(f)\) is also an even function for \(f \in K\). In fact,
\[
\mathcal{B}(f)(-x) = \int_{\mathbb{R}} k(-x-y)f(y) \, dy
\]
\[
= \int_{\mathbb{R}} k(x+y)f(y) \, dy
\]
\[
= \int_{\mathbb{R}} k(x-z)f(z) \, dz
\]
\[
= \mathcal{B}(f)(x).
\]

We claim now that \(\mathcal{B}(f)\) for \(f \in K\) is a continuous function on \(\mathbb{R}\). In fact, first note that \(0 \leq f(y) \leq f(0)\) for any \(y \in \mathbb{R}\).
\[
|\mathcal{B}(f)(x + h) - \mathcal{B}(f)(x)| \leq \int_{\mathbb{R}} |(k(x + h - y)f(y) - (k(x - y)f(y)| \, dy
\]
\[
\leq f(0) \int_{\mathbb{R}} |k(x + h - y) - k(x - y)| \, dy
\]
\[
\leq f(0) \int_{\mathbb{R}} |k(y + h) - k(y)| \, dy. \quad (4.1)
\]
Using that $k \in L^1(\mathbb{R})$ and the dominated convergence theorem, we conclude that
\[
\lim_{h \to 0} |B(f)(x + h) - B(f)(x)| = 0,
\]
meaning that $B(f)$ is a continuous function on $\mathbb{R}$, as long as $f \in \mathcal{K}$. Finally, we need to establish $B(f)$ is a non-increasing function for $x \geq 0$, for $f \in \mathcal{K}$. So, let $f \in \mathcal{K}$ be fixed and take $x \geq 0$ and $h > 0$. Then, we have for any $r \in \mathbb{R}$ that
\[
B(f)(x) = \int_{\mathbb{R}} k(x-y)f(y) \, dy
= \int_{-\infty}^{r} k(x-y)f(y) \, dy + \int_{r}^{\infty} k(x-y)f(y) \, dy
= \int_{0}^{\infty} k(x-r-z)f(z+r) \, dz + \int_{0}^{\infty} k(x-r-z)f(z+r) \, dz,
\]
and so, we have that
\[
B(f)(x + h) = \int_{0}^{\infty} k(x+h-z-r)f(z+r) \, dz + \int_{0}^{\infty} k(x+h-z-r)f(z-r) \, dz.
\]
Using $r = -\frac{h}{2}$ in the first formula and $r = \frac{h}{2}$ in the second one, we get that
\[
B(f)(x) - B(f)(x + h)
= \int_{0}^{\infty} \left( k \left( z - x - \frac{1}{2}h \right) - k \left( x + \frac{1}{2}h + z \right) \right) \left( f \left( z - \frac{1}{2}h \right) - f \left( z + \frac{1}{2}h \right) \right) \, dz.
\]
Now, we note that
\[
\int_{z+\frac{1}{2}h}^{\infty} \left( k \left( z - x - \frac{1}{2}h \right) - k \left( x + \frac{1}{2}h + z \right) \right) \left( f \left( z - \frac{1}{2}h \right) - f \left( z + \frac{1}{2}h \right) \right) \, dz \geq 0,
\]
since $z \geq x + \frac{1}{2}h \geq \frac{1}{2}h$, and the fact that $k$ and $f$ are non increasing for $w \geq 0$. Now, for the rest of the integral, we use a similar argument, after noting that $k$ and $f$ are even functions. In fact, first note for $z \geq 0$ that
\[
f \left( \frac{1}{2}h - z \right) - f \left( z + \frac{1}{2}h \right) = f \left( z - \frac{1}{2}h \right) - f \left( z + \frac{1}{2}h \right) \geq 0,
\]
for either $z \geq \frac{1}{2}h$ or $z \leq \frac{1}{2}h$, since $f$ is an even non-increasing function for $w \geq 0$. So, from this fact, we have that
\[
\int_{0}^{x+\frac{1}{2}h} \left( k \left( x + \frac{1}{2}h - z \right) - k \left( x + \frac{1}{2}h + z \right) \right) \left( f \left( \frac{1}{2}h - z \right) - f \left( \frac{1}{2}h + z \right) \right) \, dz \geq 0,
\]
since $0 \leq z \leq x + \frac{1}{2}h$, and the fact that $k$ is an even non-increasing function for $w \geq 0$. In other words, we have shown that
\[
B(f)(x) - B(f)(x + h) \geq 0,
\]
for any $x \geq 0$ and $h > 0$, which means that $B(f)(x)$ is a non-increasing function for $x \geq 0$. \hfill \Box

**Lemma 4.2.** For $\rho_0 > 1$, the operator $\mathcal{A}_c$ defined by (3.5) maps continuously $\bar{\mathcal{K}}$ into $\bar{\mathcal{K}}$. For each $0 < r < 1$, the set $\mathcal{A}_c(\bar{\mathcal{K}}_r)$ is a relative compact subset of $\bar{\mathcal{K}}$.
Proof. To prove the first statement is equivalent to establish that the operators $A_1(c)$ and $A_1(c)$ map continuously $\mathcal{K}$ into $-\mathcal{K}$ and $\mathcal{K}$, respectively. First we define the functions $k_{i,c} = \frac{1}{c} k_i$ in terms of its Fourier transform $\hat{k}_{i,c}$ by

$$\hat{k}_{1,c}(y) = \frac{1}{c (1 + \frac{1}{y^2})} \geq 0, \quad \hat{k}_{2,c}(y) = \frac{1}{c (1 + \rho_0 |y| + \frac{1}{y^2})} \geq 0,$$

meaning that $\hat{A}_1(c) = \hat{k}_{i,c}$, $i = 1, 2$. We also note that $\hat{k}_{i,c}$ are positive, even, monotone decreasing on $(0, \infty)$, and $\hat{k}_{i,c}(y) = O(|y|^{-2})$ as $|y| \to \infty$. Then we conclude that $\hat{k}_{i,c} \in L^1(\mathbb{R})$. Moreover, $k_{i,c} = \mathcal{F}^{-1} \hat{k}_{i,c}$ is a real valued function, even, bounded, continuous and vanishing as $|x| \to \infty$.

On the other hand, from the Bateman Manuscript Project [20] we have that

$$k_{1,c}(x) = \frac{2}{\pi} \int_0^\infty \frac{\cos(xy)}{c (1 + \frac{1}{y^2})} \, dy = \frac{6}{c\pi} \int_0^\infty \frac{\cos(xy)}{(y^2 + 6)} \, dy = \frac{3}{\sqrt{6} c e^{-\sqrt{6}|x|}},$$

and also that

$$\int_0^\infty \frac{\cos(xy)}{(y + \rho)} \, dy = -\sin(\rho|x|) \sin(\rho|x|) - \text{Ci}(\rho|x|) \cos(\rho x), \quad \rho > 0.$$

We note that $\text{Si}(x) = \text{Si}(x) - \frac{x}{2}$. Using this, we conclude for

$$r_1 = 3\rho_0 - \sqrt{9\rho_0^2 - 6}, \quad r_2 = 3\rho_0 + \sqrt{9\rho_0^2 - 6} \quad (\rho_0 > 1)$$

that

$$k_{2,c}(x) = \frac{6}{c\pi} \int_0^\infty \frac{\cos(xy)}{(y^2 + 6\rho_0 y + 6)} \, dy = C(\rho_0) \int_0^\infty \cos(xy) \left( \frac{1}{y + r_1} - \frac{1}{y + r_2} \right) \, dy = C(\rho_0) \left( \text{Si}(r_2|x|) \sin(r_2|x|) - \text{Si}(r_1|x|) \sin(r_1|x|) + \text{Ci}(r_2|x|) \cos(r_2 x) - \text{Ci}(r_1|x|) \cos(r_1 x) \right),$$

where $C(\rho_0) = \frac{3}{c\pi \sqrt{9\rho_0^2 - 6}}$. We observe that $k_{1,c}, k_{2,c} \in L^1(\mathbb{R})$ are positive functions and also from the Young inequality that

$$\max_{\mathbb{R}} |k_{i,c} \ast f| \leq f(0)|k_{i,c}|_{L^1(\mathbb{R})}, \quad f \in \mathcal{K}.$$

We observe that $k_{1,c}, k_{2,c} \in L^1(\mathbb{R})$ are positive functions and also from the Young inequality that

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If we set $B_{i,c}(f) = k_{i,c} \ast f$, then from Lemma (4.1), we have that $B_{i,c}(f) \in \mathcal{K}$ for any $f \in \mathcal{K}$. In other words, $B_{i,c}(f)$ is an even, continuous, non-increasing, positive, bounded function. Now, let $(\xi, w) \in \mathcal{K}$, then we have that $w - \xi w, \frac{1}{2} w^2 - \xi \in \mathcal{K}$, and also that

$$A_{1,c}(\xi, w) = -B_{1,c}(w - \xi w + \nu \xi^2), \quad A_{2,c}(\xi, w) = B_{2,c} \left( \frac{1}{2} w^2 - (\rho_0 - 1) \xi \right),$$

which means from Lemma (4.1) that

$$A_{1,c}(\xi, w) \leq 0, \quad A_{2,c}(\xi, w) \geq 0,$$
and also that $\mathcal{A}_c(\xi, w)$ for $(\xi, w) \in \widetilde{K}$ is continuous on $\mathbb{R}$. Moreover, we also have that $\mathcal{A}_1(c)(\xi, w)$ and $\mathcal{A}_2(c)(\xi, w)$ are non-decreasing and non-increasing for $x \geq 0$, respectively, for $(\xi, w) \in \bar{K}$.

Finally, we want to prove that $\mathcal{A}_c = (\mathcal{A}_{1,c}, \mathcal{A}_{2,c})$ maps $\bar{K}$ continuously to $\tilde{K}$, where

$$\mathcal{A}_{1,c}(\xi, w) = -\mathcal{B}_{1,c}(w - \xi w + \nu \xi^2), \quad \mathcal{A}_{2,c}(\xi, w) = \mathcal{B}_{2,c}\left(\frac{1}{2}w^2 - (\rho_0 - 1)\xi\right).$$

To see this, we must recall that convergence in $(C(\mathbb{R}) \times C(\mathbb{R}), \bar{d})$ is equivalent to uniform convergence on closed bounded intervals $I \subset \mathbb{R}$. Assume that $(\xi_n, w_n) \to (\xi, w)$ in $(C(\mathbb{R}) \times C(\mathbb{R}), \bar{d})$, as $n \to \infty$. Let $I = [a, b] \subset \mathbb{R}$ be a fixed closed bounded interval. For $\epsilon > 0$ given, we choose $\alpha > 0$ sufficiently large such that

$$\int_{-\infty}^{b-\alpha} k_{1,c}(r) \, dr + \int_{a+\alpha}^{\infty} k_{1,c}(r) \, dr < \epsilon.$$

Using this fact, we see easily that

$$\max_{x \in I} \int_{|r| \geq \alpha} k_{1,c}(x-r) \, dr \leq \int_{-\infty}^{b-\alpha} k_{1,c}(r) \, dr + \int_{a+\alpha}^{\infty} k_{1,c}(r) \, dr < \epsilon. \quad (4.3)$$

On the other hand, due to the convergence $\xi_n \to \xi_0$ and $w_n \to w_0$ in $(C(\mathbb{R}), \bar{d})$, we have that $\xi_n \to \xi_0$ and $w_n \to w_0$ converge uniformly on $I_\alpha = [-\alpha, \alpha]$, meaning that there is $n_0 \in \mathbb{N}$ such that

$$|\xi_n(y) - \xi_0(y)| + |w_n(y) - w_0(y)| < \epsilon, \quad \text{for all } y \in I_\alpha. \quad (4.4)$$

Due to the fact that $\xi_n \to \xi_0$ and $w_n \to w_0$ in $(C(\mathbb{R}), \bar{d})$, given $0 < r < 1$ (fixed), there is $n_1 \in \mathbb{N}$ such that for $n \geq n_1$, we have that

$$||\xi_n - \xi_0||_{C(I_\alpha)} + ||w_n - w_0||_{C(I_\alpha)} < r < 1,$$

and so, since $\xi_n, w_n, \xi_0, w_0 \in \bar{K}$, we have from (3.11) that for $y \in \mathbb{R}$ and $n \geq n_1$,

$$|\xi_n(y) - \xi_0(y)| + |w_n(y) - w_0(y)| < \epsilon, \quad \text{for all } y \in I_\alpha, \quad (4.5)$$

Now, note that

$$|A_{2,c}(\xi_n, w_n)(x) - A_{2,c}(\xi_0, w_0)(x)|$$

$$\leq \left( \int_{|y| \leq \alpha} + \int_{|y| \geq \alpha} \right) k_{2,c}(x-y) \left( \frac{1}{2}w_n^2(y) - w_0^2(y) \right) \, dy.$$

Now, using previous estimates, we have for $n \geq \max\{n_0, n_1\}$ that

$$\int_{|y| \leq \alpha} k_{2,c}(x-y) \left( \frac{1}{2}w_n^2(y) - w_0^2(y) \right) \, dy$$

$$\leq L(||\xi_n - \xi_0||_{C(I_\alpha)} + ||w_n - w_0||_{C(I_\alpha)}) \int_{\mathbb{R}} k_{2,c}(r) \, dr$$

$$\leq \epsilon C_1 \int_{\mathbb{R}} k_{2,c}(r) \, dr,$$

where $H_0$ is defined as

$$H_0 = ||\xi_0||_{C(I_\alpha)} + ||w_0||_{C(I_\alpha)}.$$
On the other hand, we know that there exists $C_2 > 0$ such that
\[
|w_n(y) - w_0(y)| + |w_n(y) + w_0(y)| + |\xi_n(y) - \xi_0(y)|
\leq 2|w_n(y)| + 2|w_0(y)| + |\xi_n(y)| + |\xi_0(y)| < C_2.
\]
So, using this fact and estimate (4.3), we conclude for $n \geq \max\{n_0, n_1\}$ that
\[
\int_{|y| \geq \alpha} k_{2,c}(x - y) \left( \frac{1}{2} |w_n(y) - w_0(y)| + (\rho_0 - 1)|\xi_n(y) - \xi_0(y)| \right) dy
\leq C_1 \int_{|y| \geq \alpha} k_{2,c}(y) dy < C_1 \epsilon.
\]
In other words, we have shown $A_{2,c}(\xi_n, w_n) \rightarrow A_{2,c}(\xi_0, w_0)$ in $(C(\mathbb{R}) \times C(\mathbb{R}), \tilde{d})$, since we have that $|A_{2,c}(\xi_n, w_n)(x) - A_{2,c}(\xi_0, w_0)(x)| < \epsilon C(L)$, for all $x \in I_\alpha$ and $n \geq \max\{n_0, n_1\}$. The same argument applies to the operator $A_{1,c}$, since
\[
|A_{1,c}(\xi_n, w_n)(x) - A_{1,c}(\xi_0, w_0)(x)|
\leq \left( \int_{|y| \leq \alpha} + \int_{|y| \geq \alpha} \right) k_{1,c}(x - y) \left( |w_n(y) - w_0(y)| (1 + |\xi_n(y)|) + |\xi_n(y) - \xi_0(y)| |w_0(y)| + \nu|\xi_n(y) + \xi_0(y)||\xi_n(y) - \xi_0(y)| \right) dy.
\]
In other words, $A_c$ maps $\overline{\mathcal{K}}$ continuously to $\overline{\mathcal{K}}$.

It remains proving that $A_c(\overline{\mathcal{K}_r})$ is relative compact subset of $\overline{\mathcal{K}}$, which is equivalent that $A_i(c)(\overline{\mathcal{K}_r})$ is relative compact subset of $\overline{\mathcal{K}}$. To see this, we use the Arzela-Ascoli Theorem to establish the compactness of the families $\mathcal{M}_i$ in $C(\mathbb{R})$,
\[
\mathcal{M}_1 = \left\{ v \in C(\mathbb{R}) : v = -B_{1,c}(w - \xi w + \nu \xi^2), \ (\xi, w) \in \overline{\mathcal{K}_r} \right\}
\]
\[
\mathcal{M}_2 = \left\{ v \in C(\mathbb{R}) : v = B_{2,c} \left( \frac{1}{2} w^2 - (\rho_0 - 1)\xi \right), \ (\xi, w) \in \overline{\mathcal{K}_r} \right\}.
\]
Let $(\xi, w) \in \overline{\mathcal{K}_r}$ be such that $v = B_{i,c}(f_i)$ with $f_1 = -(w - \xi w + \nu \xi^2)$ and $f_2 = \frac{1}{2} w^2 - (\rho_0 - 1)\xi$. We first note that for $f_i \in C(\mathbb{R})$, we have that $B_{i,c}(f_i) : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function such that (see Lemma (4.1))
\[
|B_{i,c}(f)(x + h) - B_{i,c}(f)(x)| \leq \int_{\mathbb{R}} |k_{i,c}(x + h - y) - k_{i,c}(x - y)||f(y)||dy
\leq f(0) \int_{\mathbb{R}} |k_{i,c}(z + h) - k_{i,c}(z)| dz \rightarrow 0,
\]
as $h \rightarrow 0$ (uniformly in $h$), meaning that $B_{i,c}(f)$ is equicontinuous in $\mathbb{R}$ due to uniformity of the last estimates in $h$. Moreover, the families $\mathcal{M}_i$ are equicontinuous in $\mathbb{R}$, since $B_{i,c}(f_i)$ are equicontinuous in $\mathbb{R}$, and the uniform estimate for $f(0)$ in (3.10). On the other hand, for each $x \in \mathbb{R}$, the set
\[
\mathcal{M}_i(x) = \{ v(x) : v \in \mathcal{M}_i \}
\]
has a compact closure in $\mathbb{R}$ since $\mathcal{M}_i(x) \subset [0, C_i(r)]$ for any $x \in \mathbb{R}$. Here $C_i(r)$, $i = 1, 2$ are constants which depend only on $r$. From the Arzela-Ascoli Theorem, the families $\mathcal{M}_i$ are normal (see Theorem VII. 1.23) in [8]. In other words, we have shown that the set $A_c(\overline{\mathcal{K}_r})$ is a relative compact subset of $\overline{\mathcal{K}}$. \qed
Lemma 4.3. Let $0 < \nu < 1$, $0 < r < \frac{c-r_0}{2+c-r_0} < R < 1$, where $r_0 = \max\{1, \rho_0 - 1\}$. If $c > r_0$ is large enough and $\rho_0 > \frac{3}{2}$, then

a) $U \neq tA_c(U)$ for each $U \in K \cap \partial B_r(0)$, and $t \in [0, 1]$.

b) $U - A_c(U) \neq aV$ for each $U \in K \cap \partial B_R(0)$, and $a \geq 0$, where $V$ is the constant function on $\mathbb{R}$ given by $V = (-1, 1)^j$.

Proof. a) We argue by contradiction. Assume that there exist $(\xi, w) = U \in K \cap \partial B_r(0)$ and $t \in [0, 1]$ such that $\bar{d}(U, 0) = r$ and

$$U = tA_c(U).$$

Then we have that $w - w\xi + \nu \xi^2 \in K$ and $\frac{1}{2} w^2 - \xi(\rho_0 - 1) \in K$, and so

$$0 \leq w(y) - w(y)\xi(y) + \nu \xi^2(y) \leq w(0) - w(0)\xi(0) + \nu \xi^2(0),$$

$$0 \leq \frac{1}{2} w^2(y) - (\rho_0 - 1)\xi(y) \leq \frac{1}{2} w^2(0) - (\rho_0 - 1)\xi(0).$$

From these facts, we have that

$$-\xi(0) = t \int_{\mathbb{R}} k_{1,c}(-y)(w(y) - w(y)\xi(y) + \nu \xi^2(y)) dy$$

$$\leq (w(0) - w(0)\xi(0) + \nu \xi^2(0)) \int_{\mathbb{R}} k_{1,c}(y) dy$$

$$\leq \frac{1}{c} (w(0) - w(0)\xi(0) + \nu \xi^2(0)),$$

$$w(0) = t \int_{\mathbb{R}} k_{2,c}(-y) \left( \frac{1}{2} w^2(y) - (\rho_0 - 1)\xi(y) \right) dy$$

$$\leq \left( \frac{1}{2} w^2(0) - (\rho_0 - 1)\xi(0) \right) \int_{\mathbb{R}} k_{2,c}(y) dy$$

$$\leq \frac{1}{c} \left( \frac{1}{2} w^2(0) - (\rho_0 - 1)\xi(0) \right),$$

where we are using (see the Fourier transform) that

$$\int_{\mathbb{R}} k_{1,c}(y) dy = \int_{\mathbb{R}} k_{2,c}(y) dy = \frac{1}{c}.$$

So, using the estimate (3.10), we conclude that

$$-\xi(0) \leq \frac{1}{c} \left( \frac{r}{1-r} + (1 + \nu) \left( \frac{r}{1-r} \right)^2 \right),$$

$$w(0) \leq \frac{1}{c} \left( (\rho_0 - 1) \frac{r}{1-r} + \frac{1}{2} \left( \frac{r}{1-r} \right)^2 \right).$$

Moreover, in virtue of

$$\max\{-\xi(0), w(0)\} = \frac{r}{1-r},$$

we have for $0 < \nu < 1$ and $r_0 = \max\{1, \rho_0 - 1\}$ that

$$c \leq \left( r_0 + \frac{2r}{1-r} \right) \iff r > \frac{c-r_0}{2+c-r_0},$$

which is a contradiction since we are assuming that $r < \frac{c-r_0}{2+c-r_0}$. 
b) Assume that there are \((\xi, w) = U \in \tilde{K}\) and \(a \geq 0\) such that \(\tilde{d}(U, 0) = R\) and

\[
U = a \begin{pmatrix} -1 \\ 1 \end{pmatrix} + A_c(U). \tag{4.6}
\]

For \(i = 1, 2\), we define \(\alpha_i\) as

\[
\alpha_i = \int_{0}^{2} k_i(y) dy.
\]

Let \(f_1 = w - \xi w + \nu \xi^2\) and \(f_2 = \frac{w^2}{2} - (\rho_0 - 1)\xi\), then from the conditions on \(U\) and \(V\), we have that

\[
a - \int_{0}^{1} A_{1,c}(\xi, w)(x) dx = - \int_{0}^{1} \xi(x) dx \leq \left( \int_{0}^{1} \xi^2(x) dx \right)^{1/2}, \tag{4.7}
\]

\[
a + \int_{0}^{1} A_{2,c}(\xi, w)(x) dx = \int_{0}^{1} w(x) dx \leq \left( \int_{0}^{1} w^2(x) dx \right)^{1/2}. \tag{4.8}
\]

On the other hand, we note for \(i = 1, 2\) that

\[
(-1)^i \int_{0}^{1} A_{i,c}(\xi, w)(x) dx = \int_{0}^{1} \int_{R} k_{i,c}(x - y) f_i(y) dy dx
\]

\[
\geq \int_{0}^{1} \left( \int_{0}^{1} (k_{i,c}(x - y) + k_{i,c}(x + y)) dx \right) f_i(y) dy
\]

\[
\geq \left( \int_{0}^{2} k_{i,c}(y) dy \right) \int_{0}^{1} f_i(y) dy
\]

\[
\geq \frac{\alpha_i}{c} \int_{0}^{1} f_i(y) dy,
\]

where we are using that \(k_{i,c}\) is a positive even function. So, replacing these inequalities in previous equations, we conclude that

\[
a + \frac{\alpha_1}{c} \int_{0}^{1} (w(x) - \xi(x) w(x) + \nu \xi^2(x)) dx \leq \left( \int_{0}^{1} \xi^2(x) dx \right)^{1/2} \tag{4.9}
\]

\[
a + \frac{\alpha_2}{c} \int_{0}^{1} \left( \frac{1}{2} w^2(x) - (\rho_0 - 1)\xi(x) \right) dx = \int_{0}^{1} w(x) dx \leq \left( \int_{0}^{1} w^2(x) dx \right)^{1/2}. \tag{4.10}
\]

As a consequence of \(\xi(x) \leq 0\), we conclude from inequalities (4.9), (4.10), we have that

\[
\nu \alpha_1 S_1^2 - c S_1 + ac \leq 0, \quad \alpha_2 S_2^2 - 2c S_2 + 2ac \leq 0,
\]

where \(S_1\) and \(S_2\) are defined as

\[
S_2^2 = \int_{0}^{1} w^2(y) dy, \quad S_1^2 = \int_{0}^{1} \xi^2(y) dy.
\]

From previous inequalities, we conclude that

\[
S_1 \leq \frac{c}{\nu \alpha_1}, \quad S_2 \leq \frac{2c}{\alpha_2} \quad \text{and} \quad a \leq \frac{c}{2\alpha_2}. \tag{4.11}
\]
On the other hand, since $k_{i,c} = \frac{1}{c} k_i$ (see equations (4.2)), $i = 1, 2$ are non-increasing functions for $w \geq 0$, there is a periodic function $\tilde{k}_{i,c}$ of period 2 such that

$$\tilde{k}_{i,c}(x) = \sum_{m=-\infty}^{\infty} k_{i,c}(x - 2m) = \frac{1}{c} \sum_{m=-\infty}^{\infty} k_i(x - 2m), \quad (4.12)$$

satisfying that

$$\tilde{k}_{0,i,c} = \max_{-1 \leq x \leq 1} \tilde{k}_{i,c}(x) = \frac{1}{c} \sum_{m=-\infty}^{\infty} k_i(2m) = \tilde{k}_{0,i,c}.$$ 

Therefore it follows that

$$(-1)^i A_{i,c}(\xi, w) = \int_\mathbb{R} k_{i,c}(x - y) f_i(y) \, dy = \sum_{m=-\infty}^{\infty} \int_{1+2m}^{1+2m} k_{i,c}(x - y) f_i(y) \, dy \leq \frac{2 \tilde{k}_{0,i,c}}{c} \int_0^1 f_i(y) \, dy.$$ 

Using previous estimates, we conclude that

$$-\xi(x) \leq a + \frac{2 \tilde{k}_{0,1}}{c} \int_0^1 (w(y) - w(y)\xi(y) + \nu \xi^2(y)) \, dy \leq \frac{c}{2\alpha_2} + \frac{2 \tilde{k}_{0,1}}{c} \left( S_2 + \frac{1}{2} S_2^2 + \left( \frac{1}{2} + \nu \right) S_1^2 \right) \leq \frac{c}{2\alpha_2} + \tilde{k}_{0,1} \left( \frac{4}{\alpha_2} + \frac{4c}{\alpha_2} + \frac{3c}{2\nu^2 \alpha_1^2} \right) \leq \gamma_1 c,$$

for some $\gamma_1 > 1$. In a similar fashion, we have for some $\gamma_2 > 1$ that

$$w(x) \leq \gamma_2 c.$$

As a consequence of these facts, we have for $\gamma = \max\{\gamma_1, \gamma_2\} > 1$ that

$$\max\{-\xi(0), w(0)\} = \frac{R}{1 - R} \leq \gamma c.$$

Now, if $R > \frac{\gamma c}{1 + \gamma c}$, we conclude that $\tilde{d}(\xi(w), 0) > R$, meaning that $(\xi, w) \notin \tilde{K} \cap \partial B_R(0)$. Note that $R > \frac{\gamma c}{1 + \gamma c} > \frac{c - r_0}{2 + c - r_0}$.  

**Theorem 4.4.** Let $0 < \nu < 1/c$, $\rho_0$, $r_0$, $r$ and $R$ as in Lemma (4.3) and let $c > \max\{1, \sqrt{2}(\rho_0 - 1)\}$ large enough. Then the operator $A_c$ has a non-trivial fixed point in the cone $\tilde{K}$. Equivalently, there exists a non-trivial solitary wave solution of system (3.3). Moreover, the fixed point index of the operator $A_c$ on $\tilde{K}_R$ is $i(\tilde{K}, A_c, \tilde{K}_R) = -1$.

**Proof.** First we remark that for $c > 1$, constant solutions $(\xi, w) \in \tilde{K}$ of system (3.3) must satisfy the system

$$\nu \xi^2 + (c - w)\xi + w = 0,$$

$$w^2 - 2cw - 2(\rho_0 - 1) \xi = 0.$$
Using that \( w > 0 \) and \( \xi < 0 \), we conclude from the first equation that \( w < c \). So, solving \( \xi \) in the second equation, we replace \( \xi^2 \) from the first one to see that \( w \) must satisfies the cubic equation \( p(w) = 0 \), where
\[
p(w) = \nu w^3 - (4\nu + 2(\rho_0 - 1))w^2 + (4c^2\nu + 6(\rho_0 - 1)c)w + 4(\rho_0 - 1)^2 - 4(\rho_0 - 1)c^2.
\]
Now a direct computation shows that
\[
p(0) = 4(\rho_0 - 1)^2 - 4(\rho_0 - 1)c^2 < 0, \quad p(c) = \nu c^3 + 4(\rho_0 - 1)^2 > 0,
\]
and that for any \( \alpha \),
\[
p(\alpha c) = c^2 \left( \alpha(\alpha - 2)^2\nu c - 2((\rho_0 - 1)(\alpha - 2)(\alpha - 1) - 1) \right) + 2(2(\rho_0 - 1)^2 - c^2).
\]
So, if we choose \( \nu < \frac{1}{c^3}, \rho_0 > \frac{3}{2}, c > \sqrt{2(\rho_0 - 1)} \) and \( \alpha > 0 \) small enough such that
\[
\alpha(\alpha - 2)^2 - 2((\rho_0 - 1)(\alpha - 2)(\alpha - 1) - 1) < 0,
\]
then, we conclude that \( p(\alpha c) < 0 \). In other words, we have that \( w \in (\alpha c, c) \). In particular, if we choose \( c \) large enough, we have that the constant fixed points \((0,0)\) and \((\xi, w)\) do not belong to \( \tilde{K}^R \). So, we exclude constant solutions in the cone. Therefore the existence of a non-trivial fixed point in this set of the operator \( A_c \) is a consequence of Lemma (4.3) and Theorem (2.4).

Finally, we are able to establish the existence of nontrivial travelling wave solutions for the system (3.2). We set \( \mathcal{A}_{c,0} \) and \( \mathcal{A}_i(c,0) \) as the operator \( \mathcal{A}_c \) and \( \mathcal{A}_{i,c} \), when \( \nu = 0 \) respectively. Note that using similar arguments as in Lemma (4.2) with \( \nu = 0 \), we have for \( \rho_0 > 1 \) that the operator \( \mathcal{A}_{c,0} \) maps continuously \( \tilde{K} \) into \( \tilde{K} \) and also that for each \( 0 < r < 1 \), the set \( \mathcal{A}_{c,0}(\tilde{K}_r) \) is a relative compact subset of \( \tilde{K} \).

**Theorem 4.5.** Let \( \rho_0, c, r_0, r \) and \( R \) as in Theorem (4.4). There exists a non-trivial solitary wave solution of system (3.2). Moreover, the fixed point index of the operator \( \mathcal{A}_{c,0} \) on \( \tilde{K}^R \) is \( i(\tilde{K}, \mathcal{A}_{c,0}, \tilde{K}^R) = -1 \).

**Proof.** Let \( \nu_n = 1/n < 1/c \) and let \( (\xi_n, w_n) \in \tilde{K}^R \) be a fixed point for \( \mathcal{A}_c \) guaranteed by Theorem 4.4. Using that \( \mathcal{A}_c \tilde{K}^R \) is relative compact in \( \tilde{K} \), there is a subsequence (denoted the same) \( (\xi_n, w_n) \) and \( (\xi_0, w_0) \in \tilde{K}^R \) such that \( (\xi_n, w_n) \to (\xi_0, w_0) \) with respect to the metric \( \tilde{d} \). Moreover, \( (\xi_0, w_0) \) is a fixed point for the operator \( \mathcal{A}_{c,0} \) on \( \tilde{K}^R \). In fact, for any \( (\xi, w) \in \tilde{K} \), we have that
\[
|\mathcal{A}_2(c,0)(\xi, w) - \mathcal{A}_{2,c}(\xi, w)| = 0
\]
\[
|\mathcal{A}_1(c,0)(\xi, w) - \mathcal{A}_{1,c}(\xi, w)| = \nu \int_{\mathbb{R}} k_{1,c}(x - y)\xi^2(y) \, dy \\
\leq \nu M\xi^2(0),
\]
where \( M = \int_{\mathbb{R}} k_{1,c}(x - y) \, dy \). So, we also have that
\[
|\mathcal{A}_{1,c}(\xi_n, w_n) - \mathcal{A}_{1,c}(0)(\xi_0, w_0)| \\
\leq |\mathcal{A}_{1,c}(\xi_n, w_n) - \mathcal{A}_{1,c}(0)(\xi_n, w_n)| + |\mathcal{A}_{1,c}(\xi_n, w_n) - \mathcal{A}_{1,c}(0)(\xi_0, w_0)|.
\]
Then using that \( (\xi_n, w_n) \to (\xi_0, w_0) \) with respect to the metric \( \tilde{d} \), that \( \mathcal{A}_i(c,0) \) is continuous on \( \tilde{K} \) and from previous estimates, we conclude, after taking limit as \( n \to \infty \) and for \( i = 1, 2 \),
\[
\lim_{n \to \infty} |\mathcal{A}_i(c,0)(\xi_0, w_0) - \mathcal{A}_{i,c}(\xi_n, w_n)| = 0,
\]
and so,
\[ A_{c,0}(\xi_0, w_0) = (\xi_0, w_0). \]
In other words, \((\xi_0, w_0)\) is a solitary wave solution of system (3.2) (or equivalently of system (3.3) with \(\nu = 0\)). As done for \(\nu \neq 0\), we note that there are two trivial constant solutions \(\xi = w = 0\) and \(\xi = p_{0,c}, u = q_{0,c}\) of the unperturbed system (3.2) given by
\[ q_{0,c} = \frac{3c - \sqrt{c^2 + 8(\rho_0 - 1)}}{2}, \quad p_{0,c} = \frac{q_{0,c}}{q_{0,c} - c}. \]
We note that \(u = q_{0,c} > 0\) and \(\xi = p_{0,c} < 0\) for \(c > 1\), since
\[ q_{0,c} - c = c - \sqrt{c^2 + 8(\rho_0 - 1)} < 0. \]
Equivalently, the constant functions \((0, 0)\) and \((p_{0,c}, q_{0,c})\) are fixed points of the operator \(A_{c,0}\) in the cone \(\tilde{K}\). However observe that \(q_{0,c} \to \infty\), \(c \to \infty\), and thus
\[ \tilde{d}((p_{0,c}, q_{0,c}), 0) = \max\{-p_{0,c}, q_{0,c}\} \to 1, \]
as \(c \to \infty\). Therefore, if \(c\) is large enough and \(r, R\) are selected as in Lemma (4.3), these constant solutions do not belong to the annulus \(\tilde{K}_R^r = \{ U \in \tilde{K} : r < \tilde{d}(U, 0) < R\}\).

5. **Numerical results.** Explicit solitary wave solutions of system (1.2) are not known. We recall that a solitary wave solution \((\zeta, u)\) of system (1.2) must satisfy the following equations:
\[
\frac{c^2}{6} \zeta'' + cu + \alpha u \zeta = 0, \\
\frac{c^2}{6} u'' + \frac{\rho_2}{\rho_1} c \mathcal{H}(u') - cu + \alpha u^2 + \left(1 - \frac{\rho_2}{\rho_1}\right) \zeta = 0. \tag{5.1}
\]
To approximate these solutions of system (1.2), let us introduce truncated cosine expansions for \(\zeta\) and \(u\):
\[
\zeta(x) \approx \zeta_0 + \sum_{n=1}^{N/2} \zeta_n \cos \left(\frac{n\pi}{l} x\right), \\
u(x) \approx u_0 + \sum_{n=1}^{N/2} u_n \cos \left(\frac{n\pi}{l} x\right), \tag{5.2}
\]
where
\[
\zeta_0 = \frac{1}{l} \int_0^l \zeta(x) dx = \frac{1}{2l} \int_0^{2l} \zeta(x) dx, \\
\zeta_n = \frac{2}{l} \int_0^l \zeta(x) \cos \left(\frac{n\pi x}{l}\right) dx = \frac{2}{2l} \int_0^{2l} \zeta(x) \cos \left(\frac{n\pi x}{l}\right) dx, \tag{5.3}
\]
and analogous expressions for \(u\). This strategy can be used for approximating solutions decaying to zero at infinity, provided that the period \(2l\) is taken large enough.

By substituting expressions (5.2) into equations (5.1), evaluating them at the \(N/2 + 1\) collocation points
\[ x_j = \frac{2l(j - 1)}{N}, \quad j = 1, ..., N/2 + 1, \]
and using the property of the Hilbert transform

\[ \mathcal{H}(e^{ikx}) = i \text{sign}(k)e^{ikx}, \quad k \in \mathbb{Z}, \]

we obtain a system of \( N + 2 \) nonlinear equations in the form

\[ F(\zeta_0, \zeta_1, \ldots, \zeta_{N/2}, u_0, u_1, \ldots, u_{N/2}) = 0, \quad (5.4) \]

where the \( N + 2 \) coefficients \( \zeta_n, u_n \) are the unknowns. Nonlinear system (5.4) can be solved by Newton’s iteration. Computation of the cosine series in (5.2) and the integrals in (5.3) is performed using the FFT (Fast Fourier Transform) algorithm.

The Jacobian of the vector field \( F : \mathbb{R}^{N+2} \rightarrow \mathbb{R}^{N+2} \) is approximated by the second-order accurate formula

\[ J_{i,j}F(x) \approx \frac{F_i(x + he_j) - F_i(x - he_j)}{2h}, \quad j = 1, \ldots, N + 2. \]

where \( e_j = (0, \ldots, 1, \ldots, 0) \) and \( h = 0.01 \). We stop the iteration procedure when the relative error between two successive approximations and the value of the vector field \( F \) are smaller than \( 10^{-12} \).

5.1. Description of the numerical experiments. In this section the numerical scheme explained above is employed to compute approximations to some solitary wave solutions of system (1.2). In all experiments we take \( \alpha = \epsilon = 1, \rho_1 = 1, \rho_2 = 2 \) and thus \( \rho_0 = \rho_2/\rho_1 = 2 \). Moreover we set the number of FFT points to \( N = 2^{10} \).

Newton’s iteration is then performed using as initial data the following profiles:

\[ u_0(x) = \frac{a}{1 + \left( \frac{x-a_0}{\lambda} \right)^2}, \quad \zeta_0(x) = -\frac{1}{\sqrt{\rho_0-1}}u_0(x), \quad (5.5) \]

where

\[ \lambda = \frac{4c_2}{ac_1}, \quad c = c_0 + \frac{c_1}{4}a, \quad c_0 = \sqrt{\rho_0-1}, \quad c_1 = \frac{3\alpha}{2}, \quad c_2 = \frac{\rho_0}{2}c\sqrt{\rho_0-1}. \]

The expression given in (5.5) corresponds to an approximate solitary wave solution (for \( \alpha, \epsilon \) small) of system (1.2) derived by Muñoz in [16].

In Figure 2 is displayed a solitary wave computed using the numerical scheme explained above using 6 Newton’s iterations. The spatial computational domain is the interval \([0, 2\ell] = [0, 600]\) and \( a = 2 \) in the initial step (5.5).

To verify that we have computed really a solitary wave of system (1.2), we run the numerical solver introduced by Muñoz in [16] with time step \( \Delta t = 0.01 \) and \( 2^{10} \) FFT points for spatial discretization, to compute the time evolution of the full system (1.2) using the profiles \((\zeta, u)\) (shown in Figure 2) as initial data. In Figure 3 we show the numerical solution \((\zeta, u)\) given by the solver in [16] at \( t = 50 \) superimposed with the profiles in Figure 2 translated a distance of \( 50c = 87.5 \) units to the right. We observe that the profiles propagate approximately without change of shape with the expected speed velocity.

In Figures 4, 5, 6 we plot the profiles \((\zeta, u)\) obtained after 6 Newton’s iterations for different values of the amplitude \( a \) and wave speed \( c \). The numerical parameters are the same as in the previous computer simulations. Again we checked the results obtained using the numerical solver in [16]. The errors obtained were of about \( 1e-4 \) in all numerical experiments. Finally in Figure 7 we repeat the experiments but instead using the interval \([0, 2\ell] = [0, 4800]\) as the spatial computational domain.

We point out that the simulations presented show numerical evidence of the existence of solitary wave solutions of system (3.1) even when the wave speed \( c \) is
near 1, although our theoretical results guarantee the existence of such solutions only for a large enough wave speed $c$.

**Figure 2.** Solitary wave solution of system (1.2) for $\alpha = \beta = 1$, $\rho_0 = 2$, $a = 2$, wave speed $c = 1.75$, obtained after 6 Newton’s iterations.

**Figure 3.** Evolution of the solitary wave solution in Figure 2 of system (1.2) for $\alpha = \beta = 1$, $\rho_0 = 2$, wave speed $c = 1.75$. In points: Solitary wave computed in Figure 2 translated a distance of $50c = 87.5$ units to the right. Solid line: Numerical solution $(\zeta, u)$ computed with the scheme in [16] at time $t = 50$. The difference between the profiles is about $1e-3$.

6. **Conclusions.** In this paper using the theory of positive operators in a cone in a Fréchet space [13], [14], we established the existence of solitary waves of the regularized Benjamin-Ono system (3.1) provided that the wave speed $c$ is large enough. To achieve this, we extended the techniques in [3] for the case of a family of scalar dispersive equations. We further illustrated the geometry of these solutions by approximating them through a numerical solver, which involves a Newton-type iteration together with a spectral discretization for the spatial variable. Our numerical simulations give evidence of the existence of solitary waves even when the wave speed $c$ is near 1. Further research is needed to extend our results to this speed regime and study other issues of great interest, such as orbital stability under small initial disturbances of travelling wave solutions and relationship between periodic and non-periodic travelling wave solutions of system (3.1).
Figure 4. Solitary wave solution of system (1.2) for $\alpha = \beta = 1, \rho_0 = 2, a = 1.5$, wave speed $c = 1.5625$, obtained after 6 Newton’s iterations.

Figure 5. Solitary wave solution of system (1.2) for $\alpha = \beta = 1, \rho_0 = 2, a = 1$, wave speed $c = 1.375$, obtained after 6 Newton’s iterations.

Figure 6. Solitary wave solution of system (1.2) for $\alpha = \beta = 1, \rho_0 = 2, a = 1$, wave speed $c = 1.5$, obtained after 6 Newton’s iterations.

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Figure 7. Solitary wave solution of system (1.2) for $\alpha = \beta = 1$, $\rho_0 = 2$, $a = 0.1$, wave speed $c = 1.0375$, obtained after 4 Newton’s iterations.

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