BOUNDS FOR TWISTS OF GL(3) L-FUNCTIONS

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ABSTRACT. Let $\pi$ be a fixed Hecke-Maass cusp form for $\text{SL}(3, \mathbb{Z})$ and $\chi$ be a primitive Dirichlet character modulo $M$, which we assume to be a prime. Let $L(s, \pi \otimes \chi)$ be the $L$-function associated to $\pi \otimes \chi$. In this paper, introducing some variants to previous methods, we establish the bound $L(1/2 + it, \pi \otimes \chi) \ll_{\epsilon, \delta} (M(|t| + 1))^{3/4 - \delta}$ for any $\delta < 1/36$.

1. Introduction and Statement of Results

The subconvexity problem, which asks for an estimate of an automorphic $L$-function on the critical line $s = 1/2 + it$ that is better by a power saving than the bound implied by the functional equation and the Phragmen-Lindelöf principle, is a central problem in analytic number theory. Many cases have been treated in the past; see [12] for results with full generality on $\text{GL}(2)$. It has only been recently that people have started making progress on $\text{GL}(3)$ with the introduction of new techniques.

In this paper, we are interested in certain degree 3 $L$-functions. Let $\pi$ be a fixed Hecke-Maass cusp form of type $(\nu_1, \nu_2)$ for $\text{SL}(3, \mathbb{Z})$ with normalized Fourier coefficients $\lambda(m, n)$. Let $\chi$ be a primitive Dirichlet character modulo $M$. Let

$$L(s, \pi) = \sum_{n=1}^{\infty} \frac{\lambda(1, n)}{n^s} \quad \text{and} \quad L(s, \pi \otimes \chi) = \sum_{n=1}^{\infty} \frac{\lambda(1, n)\chi(n)}{n^s}$$

be the $L$-series associated with $\pi$ and $\pi \otimes \chi$; these series can be continued to entire functions of $s \in \mathbb{C}$ with functional equations. One aims to beat the convexity bound $L(1/2 + it, \pi \otimes \chi) \ll_{\epsilon} (M(|t| + 1))^{3/4 + \epsilon}$.

For the $L$-function $L(s, \pi)$, the first breakthrough was made by Li [9] who resolved the subconvexity problem in the $t$-aspect. Using a first moment method, Li showed that $L(1/2 + it, \pi) \ll (|t| + 1)^{3/4 - \delta + \epsilon}$ with $\delta = 1/16$, for the symmetric square lift $\pi$ of an $\text{SL}(2, \mathbb{Z})$ Maass cusp form. The method depends on the non-negativity of central values of certain $L$-functions, which necessitates in the self-duality assumption on the cusp form $\pi$. Li’s exponent of saving $\delta = 1/16$ was later improved to $\delta = 1/12$ by Mckee, Sun and Ye [10], and to $\delta = 1/8$ by Nunes [19]. Later, Munshi [17] generalized Li’s result [9] to arbitrary fixed cusp forms with the same exponent of saving $\delta = 1/16$, by taking an approach other than the moment method, namely, Kloosterman’s variant of the circle method, enhanced by a “conductor lowering” trick. Munshi’s approach does not have to use the assumption on the non-negativity of central values of $L$-functions, which enables him to deal with more general cusp forms.

For the case where $M$, the conductor of $\chi$, is varying, in the special case that $\pi$ is self-dual and $\chi$ is quadratic, a subconvex bound was obtained by Blomer [1]. He showed that $L(1/2, \pi \otimes \chi) \ll M^{3/4 - \delta}$, for any $\delta < 1/8$, by using the first moment method as in Li’s work. Introducing a variant of the circle method, the $\text{GL}_2$ Petersson delta method, Munshi [16, 18] established a subconvexity result $L(1/2, \pi \otimes \chi) \ll M^{3/4 - \delta}$, for any $\delta < 1/308$. Again, the approach that Munshi took does not require the non-negativity of certain $L$-functions, which removes the self-duality assumption on the forms $\pi$ and $\chi$ in Blomer’s work. Recently Holowinsky and Nelson [5] discovered a new look at Munshi’s delta method, which removes the use of Petersson trace formula in [16] altogether as well as improves the exponent of saving to any $\delta < 1/36$.

It is then natural to ask the question of establishing a subconvex bound with two simultaneously varying parameters, for example, the conductor and $t$-aspects. For the conductor aspect, we will be content by considering the special case of the twists $\pi \otimes \chi$ of a fixed cusp form $\pi$ by Dirichlet characters $\chi$ of varying
conductor $M$. Our task is then to solve the subconvexity problem for $L(1/2 + it, \pi \otimes \chi)$, simultaneously in $M$ and $t$. In the special case that $\pi$ is self-dual and $\chi$ is quadratic, a bound $L(1/2 + it, \pi \otimes \chi) < (M(|t| + 1))^{3/4 - \delta}$, for some $\delta > 0$, was obtained by Huang [6], by combining the treatment of Li and Blomer, with input from [22]. It is now desirable to ask, “Can one prove a subconvex bound for the Dirichlet twist $L$-functions $L(s, \pi \otimes \chi)$, simultaneously in the conductor and $t$-aspects, for a general SL(3, $\mathbb{Z}$) Hecke cusp form and general primitive Dirichlet characters?” Our main result answers this affirmatively.

**Theorem 1.1.** Let $\pi$ be a Hecke-Maass cusp form for SL(3, $\mathbb{Z}$) and $\chi$ be a primitive Dirichlet character modulo $M$, which we assume to be prime. Given any $\varepsilon > 0$, we have

$$L(1/2 + it, \pi \otimes \chi) < (M(|t| + 1))^{3/4 - 1/36 + \varepsilon}.$$

**Remark 1.2.** Below we will carry out the proof under the assumption $|t| > M^\varepsilon$ for any $\varepsilon > 0$. We make such assumption so as to control the error term of the stationary phase analysis in our approach. For the case $|t| < M^\varepsilon$, the bound (1) follows from the work [5], since there their bound $L(1/2 + it, \pi \otimes \chi) < t, \pi, \varepsilon$ $M^{3/4 - 1/36 + \varepsilon}$ is of polynomially dependence in $t$.

For subconvexity bounds on GL(3) in other aspects, see [2, 3, 15, 21].

Our approach is a variant of the methods introduced in the works [16] and [5]. In Section 2, we will give a brief outline of our approach for the simpler case $L(1/2 + it, \pi)$, to guide the readers through.

**Notation.** We use $e(x)$ to denote $\exp(2\pi ix)$. We denote $\varepsilon$ an arbitrary small positive constant, which might change from line to line. In this paper the notation $A \asymp B$ (sometimes even $A \approx B$) means that $B/(M|t|)^\varepsilon \leq A \leq B(M|t|)^\varepsilon$. We reserve the letters $p$ and $\ell$ to denote primes. The notations $p \sim P$ and $\ell \sim L$ denote primes in the dyadic segments $[P, 2P]$ and $[L, 2L]$ respectively.

**2. An outline of the proof**

For any $N \geq 1$, let

$$S(N) = \sum_{n \leq N} \lambda(1, n)\chi(n)n^{-it}w\left(\frac{n}{N}\right),$$

where $w$ is a smooth function with support in $[1, 2]$ satisfying $w^{(j)}(x) \ll j, 1$.

By symmetry, we assume $t > 2$ from now on. Using a standard approximate functional equation argument ([8, Theorem 5.3]) and the estimate $\sum_{n \leq X} |\lambda(1, n)| \ll X^{1+\varepsilon}$, one can derive the following.

**Lemma 2.1.** For any $\delta > 0$ and $\varepsilon > 0$, we have

$$L\left(\frac{1}{2} + it, \pi \otimes \chi\right) \ll (Mt)^\varepsilon \sup_N \frac{|S(N)|}{\sqrt{N}} + (Mt)^{3/4 - \delta/2 + \varepsilon},$$

where the supremum is taken over $N$ in the range $(Mt)^{3/2 - \delta} < N < (Mt)^{3/2 + \varepsilon}$.

From the lemma, it suffices to beat the convexity bound $S(N) \ll N^{1+\varepsilon}$, for $N$ in the range $(Mt)^{3/2 - \delta} < N < (Mt)^{3/2 + \varepsilon}$, which we henceforth assume, where $0 < \delta < 1/2$ is a constant to be optimized later.

Our approach is inspired by the work [16] and is a further variant to the recent work [5]. We now give a brief introduction to the approach in [16]. Let $p$ be a prime number, and let $k \equiv 3 \mod 4$ be a positive integer. Let $\psi$ be a character of $\mathbb{F}_p^*$ satisfying $\psi(-1) = -1 = (-1)^k$. One can consider $\psi$ as a character modulo $pM$. Let $H_k(pM, \psi)$ be an orthogonal Hecke basis of the space of cusp forms $S_k(pM, \psi)$ of level $pM$, nebentypus $\psi$ and weight $k$. For $f \in H_k(pM, \psi)$, let $\lambda_f(n)$ be its Fourier coefficients. Denote $P^* = \sum_{p|P < 2P} \sum_{\psi \mod p}(1 - \psi(-1))$. Then we have the following averaged version of the Petersson formula:

$$\delta(r, n) = \frac{1}{P^*} \sum_{p \sim P} \sum_{\psi \mod p} (1 - \psi(-1)) \sum_{f \in H_k(pM, \psi)} \omega_f^{-1}\lambda_f(r)\lambda_f(n)$$

$$- \frac{2\pi i}{P^*} \sum_{p \sim P} \sum_{\psi \mod p} (1 - \psi(-1)) \sum_{c=1}^{\infty} \frac{S_\psi(r, n; cpM)}{cpM} J_{k-1}\left(\frac{4\pi \sqrt{rn}}{cpM}\right),$$

where $c = 1, \ldots, P^*$.
where $\delta(r, n)$ denotes the Kronecker symbol, $\omega_f^{-1} = \frac{\Gamma(k-1)}{(4\pi)^{k-1} k!}$ is the spectral weight, and $S_\psi(r, n; c) = \sum_{\alpha \mod c} \psi(\alpha) e\left(\frac{\alpha r + \alpha n}{c}\right)$ is the generalized Kloosterman sum.

Let $\mathcal{L}$ be the set of primes in the interval $[L, 2L]$ and let $L^* = |\mathcal{L}|$ denote the cardinality of $\mathcal{L}$. By writing his main sum of interest $\sum_{m, n = 1}^\infty \lambda(m, n)\chi(n)W\left(\frac{nm^2}{N}\right)V\left(\frac{r}{N^2}\right)$ as

$$\frac{1}{L^*} \sum_{\ell \in \mathcal{L}} \tilde{\chi}(\ell) \sum_{m, n = 1}^\infty \lambda(m, n)W\left(\frac{nm^2}{N}\right) \sum_{r = 1}^\infty \chi(r)V\left(\frac{r}{N^2}\right) \delta(r, \ell),$$

and then substituting the formula (3) with $\delta(r, \ell)$ in, Munshi expressed the sum as the summation of two terms, say $F^*$ and $O^*$. Successfully bounding $F^*$ and $O^*$ simultaneously with suitable choices of $P$ and $L$ to balance the contribution enables him to get his main result $L(1/2, \pi \otimes \chi) \ll_{\pi, \epsilon} M^{3/4-1/308+\epsilon}$.

Now we turn to our case. We give a sketch of our argument for the simpler case $L(1/2 + it, \pi)$, for which the argument will be more transparent. The general case $L(1/2 + it, \pi \otimes \chi)$ follows along the same line of proof. From Lemma 2.1, it suffices to beat the convexity bound $N$ for the smoothed sum

$$S^*(N) := \sum_{n \geq 1} \lambda(1, n)n^{-it} w\left(\frac{n}{N}\right),$$

for $t^{3/2-\delta} < N < t^{3/2+\epsilon}$. For the purpose of this sketch, we focus on the case $N \approx t^{3/2}$ and assume the Ramamuan bound $|\lambda(m, n)| \ll (mn)^{\frac{1}{2}}$.

Let $P$ and $L$ be two large parameters to be specified later. We will show that without using the Petersson formula (3) we can still write, up to some scalars,

$$S^*(N) = F + O \left(NT^{-100}\right),$$

where

$$F = \frac{1}{P^2} \sum_{p \sim P} \sum_{\ell \sim L} \sum_{r \sim t^{1/2} p/L} \lambda(1, n)e\left(\frac{np}{\ell}\right) w\left(\frac{n}{N}\right),$$

$$O = \frac{t^{1/2} \pi}{PL} \sum_{\ell \sim L} \lambda(1, n)w\left(\frac{n}{N}\right) \sum_{p \sim P} \sum_{\ell \sim L} \sum_{r \neq 0} J_\ell(r, np/\ell),$$

with

$$J_\ell(r, np/\ell) = \int_{\mathbb{R}} x^{-it} e\left(-\frac{nt}{x}\right) v(x) e\left(-\frac{rNp}{\ell}\right) dx.$$

Here $V(x)$ is a smooth compactly supported function satisfying $V^{(j)}(x) \ll 1$ for all $j \geq 0$.

Now our task is to beat the bound $N \approx t^{3/2}$ for $F$ and $O$ simultaneously. We estimate the term $O$ first. The integral $J_\ell(r, np/\ell)$ restricts the length of the $r$-sum to $0 \leq |r| \leq N^{2/3} \frac{2\ell}{P} \approx \frac{t^{1/2}L}{100}$. From the second derivative test we have $J_\ell(r, np/\ell) \ll t^{-1/2}$.

Estimating trivially using the bound $J_\ell(r, np/\ell) \ll t^{-1/2}$, we find that

$$O \ll t^{1/2} \frac{PL}{P} \frac{t^{1/2}L}{P} \frac{t^{-1/2}}{P} \ll N \frac{t^{1/2}L}{P},$$

so we need to save a little more than $t^{1/2}/L/P$ for $O$.

We apply the Cauchy–Schwarz inequality to reduce the task to saving $t^{1/2}L/P$ from

$$\left(\sum_{\ell \sim L} \sum_{r \sim t^{1/2} p/L} J_\ell(r, np/\ell)\right)^2 \sim \frac{t^{1/2}}{PL} N^{1/2} \left(\sum_{n \sim N} \sum_{p \sim P} \sum_{\ell \sim L} \sum_{r \sim R} |J_\ell(r, np/\ell)|^2\right)^{1/2},$$

or equivalently, saving $tL^2/P^2$ from the sum

$$\sum_{n \sim N} \sum_{p \sim P} \sum_{\ell \sim L} \sum_{r \sim R} |J_\ell(r, np/\ell)|^2,$$

where $1 \ll R \ll \frac{t^{1/2}L}{P}$. 

For the moment we pretend $R = t^{3/2}L$. Opening the square and applying Poisson summation to the $n$-sum, only the zero frequency contributes. For the diagonal term, we save $PLR \approx t^{1/2}L^2$, which is satisfactory as long as $t^{1/2}L^2 > tL^2/P^2$, i.e., $P \gg t^{1/4}$. For the off-diagonal, after Poisson summation we encounter the integral

$$\mathfrak{I} = \int_\mathbb{R} \mathcal{J}_i(r_1, Np_1y/\ell_1)\mathcal{J}_i(r_2, Np_2y/\ell_2)w(y)\,dy.$$  

We save a $t$ from bounding the integral, by using stationary phase and the first derivative test (which is the content of Lemma 5.1), so that the off-diagonal is satisfactory as long as $P \gg L$. Hence $O$ is fine for our purpose if $P > \max\{t^{1/4}, L\}$.

Next, we try to bound the $F$ term in (4). Estimating trivially, we have

$$F \ll \frac{1}{P^2}PLt^{1/2}/P/LN \ll Nt^{1/2},$$

so our job is to save a little more than $t^{1/2}$.

We apply Voronoi summation to the $n$-sum, to get

$$F \approx \frac{1}{P^4} \sum_{p \sim P} \sum_{\ell \sim L} \sum_{r \sim t^{1/2}P/L} \sum_{n \sim P^3} \lambda(n, 1)S(\ddot{p}, n; \ell r).$$

Using Weil’s bound for the Kloosterman sum we get $F \ll t^{5/4}P^{3/2}$, which gives us a saving of $t^{3/4}/P^{3/2}$ over the original bound $Nt^{1/2}$, and we need to save $P^{3/2}/t^{1/4}$ from the above sum. Pulling the $r$ and $n$-sums outside, and applying the Cauchy–Schwarz inequality, our job is to save $P^{3/2}/t^{1/2}$ from the sum

$$F \ll \frac{1}{P^4} \sum_{r \sim t^{1/2}P/L} \sum_{n \sim P^3} \left( \sum_{\ell \sim L} \sum_{p \sim P} \lambda(n, 1)S(\ddot{p}, n; \ell r) \right)^2.$$

Our final step is to open the square and apply Poisson summation to the $n$-sum. The diagonal is satisfactory if $PL > P^3/t^{1/2}$, that is, $L > P^2/t^{1/2}$. For the off-diagonal, the zero frequency (which vanishes unless $\ell_1 = \ell_2$) makes a contribution which is dominated by the diagonal contribution. The non-zero frequencies contribute an amount of $P^2L^4(t^{1/2}P/L)^{5/2} = P^9/2L^{3/2}t^{5/4}$, from which we earn a saving of $P^7Lt/(P^{9/2}L^{3/2}t^{5/4}) = \frac{P^{3/2}}{L^{1/4}t^{3/4}}$, which is satisfactory if $\frac{P^{3/2}}{L^{1/4}t^{3/4}} > P^3/t^{1/2}$, i.e., $LP < t^{1/2}$. Hence $F$ is fine for our purpose if $t^{1/2}/P > L > P^2/t^{1/2}$.

Now it turns out that we have a choice for the parameters $P$ and $L$ to simultaneously beat the convexity bound for $F$ and $O$, which in turn implies a subconvexity bound for $L (1/2 + it, \pi \otimes \chi)$, where $\chi$ is a primitive Dirichlet character modulo $M$.

3. Some lemmas

In this section, we collect some lemmas that we are going to use in our proof. Let $(\alpha_1, \alpha_2, \alpha_3)$ be the spectral parameters associated to the Maass form $\pi$. Let

$$G_\delta(s) := \begin{cases} 2(2\pi)^{-s}\Gamma(s)\cos(\pi s/2), & \text{if } \delta = 0, \\ 2i(2\pi)^{-s}\Gamma(s)\sin(\pi s/2), & \text{if } \delta = 1, \end{cases}$$

and let

$$G(\alpha, \delta)(s) = \prod_{j=1}^3 G_{\delta_j}(s + \alpha_j),$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, and $\delta = (\delta_1, \delta_2, \delta_3)$.

Define

$$j(\alpha, \delta)(x) = \frac{1}{2\pi i} \int_C G(\alpha, \delta)(s)x^{-s}ds, \quad x > 0,$$

where $C$ is a curved contour such that all the singularities of $G^\pm(s)$ are to the left of $C$, defined as in Def. 3.2 of [20].
Let

$$J_x(\pm x) := J_{(\alpha, \delta)}(\pm x) = \frac{1}{2} \left( j(\alpha, \delta)(x) \pm j(\alpha, \delta+e)(x) \right),$$

where $e = (1, 1, 1)$, and $\delta + e$ is taken modulo 2.

The Bessel function $J_x(\pm x)$ satisfies the following property.

**Lemma 3.1.** (1) Let $\rho > \max\{-\Re \alpha_1, -\Re \alpha_2, -\Re \alpha_3\}$. For $x < 1$, we have

$$x^j J_x^{(j)}(\pm x) \ll_{\alpha_1, \alpha_2, \alpha_3, \rho, j} x^{-\rho}.$$  

(2) Let $K \geq 0$ be a fixed nonnegative integer. For $x > 0$, we may write

$$J_x(\pm x^3) = \frac{e(\pm 3x)}{x} W_\pi^\pm(x) + E_x^\pm(x),$$

such that $W_\pi^\pm(x)$ and $E_x^\pm(x)$ are real analytic functions on $(0, \infty)$ satisfying

$$W_\pi^\pm(x) = \sum_{m=0}^{K-1} B^\pm_{m}(\pi) x^{-m} + O_{K, \alpha_1, \alpha_2, \alpha_3} (x^{-K}),$$

and

$$E_x^\pm(j)(x) \ll_{\alpha_1, \alpha_2, \alpha_3, j} \frac{\exp(-3\sqrt{7\pi}x)}{x},$$

for $x \gg_{\alpha_1, \alpha_2, \alpha_3} 1$, where $B^\pm_{m}(\pi)$ are constants depending on $\alpha_1$, $\alpha_2$ and $\alpha_3$.

**Proof.** See [20, Theorem 14.1]; note that our $J_x(\pm x)$ is the $J_{(\lambda, \delta)}(x^{1/3})$ in the notation of [20]. Q.E.D.

Now we recall the Voronoi formula for GL(3), in which the Bessel function $J_x(\pm x)$ appears naturally.

**Lemma 3.2** ([14]). For $(a, c) = 1$, $\bar{a}a \equiv 1 \pmod{c}$, we have

$$\sum_{n=1}^{\infty} \lambda(m, n) e\left(-\frac{na}{c}\right) w(n) = e \sum_{\pm} \sum_{m' | mc} \sum_{n=1}^{\infty} \lambda(n, m') \frac{m'^2 n}{mc^3} W^\pm \left( \frac{m'^2 n}{mc^3} \right),$$

where

$$W^\pm(x) = \int_{0}^{\infty} w(y) J_x(\mp x y) dy.$$

In particular, replacing $w(n)$ by $w\left(\frac{n}{N}\right)$ gives

$$\sum_{n=1}^{\infty} \lambda(m, n) e\left(-\frac{na}{c}\right) w\left(\frac{n}{N}\right) = e \sum_{\pm} \sum_{m' | mc} \sum_{n=1}^{\infty} \lambda(n, m') \frac{m'^2 n}{mc^3} S(\bar{a}m, \pm n; mc/m') N m'^2 n \frac{m'^2 n}{mc^3} W^\pm \left( \frac{N m'^2 n}{mc^3} \right).$$

If $w^{(j)}(y) \ll 1$, then from the oscillation of $J_x(\pm x)$ when $|x| > N^\varepsilon$, $W^\pm \left( \frac{N m'^2 n}{mc^3} \right)$ is negligibly small as long as $m'^2 n$ is such that $\frac{N m'^2 n}{mc^3} \gg N^{\varepsilon}$.

If we write

$$U^\pm(x) = x W^\pm(x),$$

then (5) becomes

$$\sum_{n=1}^{\infty} \lambda(m, n) e\left(-\frac{na}{c}\right) w(n) = e \sum_{\pm} \sum_{m' | mc} \sum_{n=1}^{\infty} \lambda(n, m') \frac{m'^2 n}{mc^3} S(\bar{a}m, \pm n; mc/m') U^\pm \left( \frac{m'^2 n}{mc^3} \right),$$

which is the usual version of Voronoi formula given in the work [14] and others.

**Remark 3.3.** Here the normalization of (5) is different from the usual version (6). With this normalization, the weight function on the right is the Hankel transform of the original Schwarz class function, matching the rank one and rank two cases. We thank Zhi Qi for making us aware of this.
Lemma 3.4 (Miller’s bound, [13]). Uniformly in $\alpha$, we have
\[
\sum_{n \leq X} \lambda(1, n)e(an) \ll_{\pi, \epsilon} X^{\frac{1}{4} + \epsilon}.
\]

Lemma 3.5 ([5, Lemma 2]). Let $s_1, s_2$ be natural numbers. Let $t_1, t_2, n$ be integers. Set
\[
C := \sum_{x \in [s_1, s_2]} S(t_1 x, 1; s_1)S(t_2 x, 1; s_2)e\left(\frac{nx}{s_1, s_2}\right).
\]
Write $s_i = w_i(s_1, s_2)$, $i = 1, 2$, and set $\Delta = w_2^2 t_1 - w_1^2 t_2$. Then
\[
|C| \ll 2^{O(\omega([s_1, s_2]))}(s_1 s_2)^{1/2}(\Delta, n, s_1, s_2)^{1/2}.
\]
where $\omega([s_1, s_2])$ denotes the number of distinct prime factors of $[s_1, s_2]$, and the implied constant in $O$-symbol is absolute.

Following [22] and [11], we say a smooth function $f(x_1, \ldots, x_n)$ on $\mathbb{R}^n$ to be inert if
\[
x_1^j \ldots x_n^j \chi^{j_1 \ldots j_n}(x_1, \ldots, x_n) \ll_{j_1, \ldots, j_n} 1.
\]

Lemma 3.6. Let $V$ be a smooth function with compact support on $\mathbb{R}_{>0}$, satisfying $V^{(j)}(x) \ll_j 1$ for all $j \geq 0$. Assume $(M, r) = 1$ and $n = N$, one has
\[
\sum_{r=1}^{N} \chi(r)r^{-it}e\left(-\frac{nr}{M}\right) V\left(\frac{r}{N}\right)
\]
\[
= \frac{N}{M^{3/2}t^{3/2}} \frac{g_{\chi}}{\sqrt{M}} \left(\frac{2\pi}{Mt}\right)^{-it} e(-t/2\pi)\chi(n)n^{-it}V_A\left(\frac{2\pi n}{N}\right) + O\left(\frac{N}{M^{3/2}t^{1+A}}\right)
\]
\[
+ \frac{1}{M} \left(\frac{N}{Mt}\right)^{1-it} \sum_{r \neq 0} S_{\chi}(r, n; M) \int_{\mathbb{R}} x^{-it}e\left(-\frac{nt}{N}\right) V(x)e\left(-\frac{rN}{M^2t}\right) dx,
\]
where $S_{\chi}(r, n; M)$ is the generalized Kloosterman sum, $V_A(x)$ is an inert function supported on $x = 1$, and $A \geq 1$ is any positive constant.

Proof. Writing
\[
e\left(-\frac{nr}{M}\right) = e\left(\frac{nr}{M}\right) e\left(-\frac{nr}{M}\right),
\]
which follows from reciprocity, and applying Poisson summation, the $r$-sum becomes
\[
\sum_{r=1}^{\infty} \chi(r)r^{-it}e\left(-\frac{nr}{M}\right) V\left(\frac{r}{N/Mt}\right)
\]
\[
= \frac{N}{M^{2t}} \sum_{r \in \mathbb{Z}} \sum_{a(M)} \chi(a) e\left(\frac{n\bar{a}}{M}\right) e\left(\frac{ar}{M}\right) \int_{\mathbb{R}} \left(\frac{N}{Mt}\right)^{-it} e\left(-\frac{nt}{N}\right) V(x)e\left(-\frac{rN}{M^2t}\right) dx.
\]
In particular, the zero frequency $r = 0$ is
\[
\frac{1}{M} \left(\frac{N}{Mt}\right)^{1-it} g_{\chi}(n) \int_{\mathbb{R}} x^{-it}e\left(-\frac{nt}{N}\right) V(x) dx.
\]
Considering the integral, by [11, Main Theorem], there is an inert function $V_A$ supported on $x_0 = 1$ such that
\[
\int_{\mathbb{R}} x^{-it}e\left(-\frac{nt}{N}\right) V(x) dx = \int_{\mathbb{R}} e\left(-\frac{t \log x}{2\pi} - \frac{nt}{N}\right) V(x) dx
\]
\[
= \frac{e(f(x_0))}{\sqrt{t}} V_A(x_0) + O_A\left(t^{-A}\right),
\]

where \( f(x) = \frac{\log x}{2\pi} - \frac{nt}{N^2} \), and \( x_0 = \frac{2\pi n}{N} \) is the unique solution for \( f'(x) = 0 \), and \( A \geq 1 \) is any large constant. Therefore,

\[
\int_{\mathbb{R}} x^{-it} e\left(-\frac{nt}{N}\right) V(x) \, dx = \left(\frac{2\pi}{N}\right)^{-it} e(-t/2\pi) \sqrt{t} n^{-it} V_A \left(\frac{2\pi n}{N}\right) + O(t^{-A}).
\]

Hence

\[
\sum_{r=1}^{\infty} \chi(r) e\left(\frac{nF}{M}\right) e\left(-\frac{nt}{M}\right) = \frac{1}{M} \left(\frac{N}{Mt}\right)^{1-it} g_X(n) \left(\frac{2\pi}{N}\right)^{-it} e(-t/2\pi) \sqrt{t} \chi(n) n^{-it} V_A \left(\frac{2\pi n}{N}\right) + O\left(\frac{N}{M^{5/2}t^{1+A}}\right)
\]

and (9) follows. Q.E.D.

From the lemma, assuming \((M, \ell r) = 1\) and \(n \equiv N\), one has

\[
\sum_{r=1}^{\infty} \chi(r) e\left(-\frac{npM}{\ell r}\right) V\left(\frac{r}{N^2}\right) = \frac{Np}{M^{3/2}t^{1+\varepsilon}} g_X \chi(n) \left(\frac{2\pi p}{M^2t}\right)^{-it} e(-t/2\pi) \chi(p\ell) \chi(n) n^{-it} V_A \left(\frac{2\pi n}{N}\right)
\]

\[
+ \frac{1}{M} \left(\frac{Np}{M^2t}\right)^{1-it} \sum_{r \neq 0} S_X(r, np\ell; M) J_H(r, np/\ell; M) + O\left(\frac{Np}{M^{3/2}t^{1+A}}\right),
\]

where

\[
J_H(r, np/\ell; M) := \int_{\mathbb{R}} x^{-it} e\left(-\frac{nt}{N}\right) V(x) e\left(-\frac{rNp}{M^2t}\right) dx.
\]

Remark 3.7. The identity (9) is a further variant to the following key identity in [5, (3.6)].

\[
\chi(n) = \frac{M}{R} g_X \sum_{r \in \mathbb{Z}} \chi(r) e\left(\frac{nF}{M}\right) V\left(\frac{r}{R}\right) - \frac{1}{g_X} \sum_{r \neq 0} S_X(r, n; M) \hat{V}\left(\frac{R}{M}\right),
\]

where \(\hat{V}\) denotes the Fourier transform of the Schwartz function \(V\) which is normalized such that \(\hat{V}(0) = 1\), and \(R > 0\) is a parameter. With suitable amplification, one can express the smoothed sum \(\sum_{n \geq 1} \chi(1, n) \chi(n) w\left(\frac{r}{N}\right)\) as \(F + O\).Balancing the contribution of \(F\) and \(O\) properly, the authors of [5] obtained \(L(1/2, \pi \otimes \chi) \ll M^{3/4 - 1/36 + \varepsilon}\).

Lemma 3.8. For any \(\varepsilon > 0\), one has

\[
\sum_{P_1 \sim P_2 \sim P_{\ell_1 \sim L \ell_2 \sim L_{R \sim R_{1\sim R}}} \sum_{\ell_{1\sim R \neq \ell_{2\sim R}}} \frac{1}{|\ell_{1\sim R \neq \ell_{2\sim R}| \mathbb{R}^+} \ll (LPR)^{1+\varepsilon} + \min\{L^{2+\varepsilon}, P^{2+\varepsilon}, R^{2+\varepsilon}\}.
\]

Proof. The sum is bounded by

\[
\sum_{\ell_1 \sim L \ell_2 \sim L_{m_1 \sim P_R m_2 \sim P_R}} \sum_{\ell_{1\sim R \neq \ell_{2\sim R}}} \frac{\tau(m_1) \tau(m_2)}{|\ell_{1\sim R \neq \ell_{2\sim R}| \mathbb{R}^+} \ll \sum_{1 \leq d \leq 2L} \frac{1}{d} \sum_{\ell_1 \sim L/d \ell_2 \sim L/d m_1 \sim P_R m_2 \sim P_R} \frac{\tau(m_1) \tau(m_2)}{|\ell_{1\sim R \neq \ell_{2\sim R}| \mathbb{R}^+} \ll \sum_{1 \leq d \leq 2L} \frac{1}{d} \sum_{\ell_1 \sim L/d \ell_2 \sim L/d m_1 \sim P_R m_2 \sim P_R} \frac{\tau(m_1) \tau(m_2)}{|\ell_{1\sim R \neq \ell_{2\sim R}| \mathbb{R}^+}},
\]

where \(\tau\) denotes the divisor function.

Given \(d\), for a fixed pair \((\ell_1, \ell_2)\), and a fixed \(i\) with \(1 \leq |i| \ll LPR/d\), suppose \((m'_2, m'_1)\) is a solution of the equation \(\ell_1 m_2 - \ell_2 m_1 = i\). Since \((\ell_1, \ell_2) = 1\), all the other solutions must be of the form \((m'_{2, m'_{1}} + \ldots) +\)
\[ \sum_{\ell_1 \sim L} \sum_{\ell_2 \sim L} \sum_{m_1 \sim PR} \sum_{m_2 \sim PR} \sum_{\ell_1 m_2 \neq \ell_2 m_1} \frac{1}{|\ell_1 m_2 - \ell_2 m_1|} \leq \sum_{1 \leq \ell_1 / d \leq L} \sum_{1 \leq \ell_2 / d \leq L} \sum_{1 \leq \ell_2 / d \leq L} \sum_{1 \leq \ell_2 / d \leq L} \sum_{d \leq LPR/d} \frac{1}{d} \left( 1 + \frac{PRd}{L} \right) \leq \frac{LPR + L^2}{\epsilon}. \]

Clearly, the roles of \( \ell_1, p_1 \) and \( r_1 \) are symmetric in the above argument. One can replace the bound \( O(L^{2+\epsilon}) \) by \( O(P^{2+\epsilon}) \) or \( O(R^{2+\epsilon}) \). The lemma follows.

Using the same argument, we also have

\[ \sum_{p_1 \sim P} \sum_{p_2 \sim P} \sum_{\ell_1 \sim L} \sum_{\ell_2 \sim L} \sum_{r_1 \sim R} \sum_{r_2 \sim R} \frac{1}{|r_1 p_1 \ell_2 - r_2 p_2 \ell_1|} \leq (LPR)^{1+\epsilon}/M + \min\{L^{2+\epsilon}, P^{2+\epsilon}, R^{2+\epsilon}\}/M. \]

Q.E.D.

Other relevant lemmas will be stated during the course of the proof.

4. Reducing \( S(N) \) to \( F_1 \) and \( O \)

Our basic strategy is to introduce more ‘points’ of summation to mimic the smoothed sum \( S(N) \) (2), which is our main object of study. Throughout the paper we assume that \(|t| > M^\varepsilon\) for any \( \varepsilon > 0 \).

Let \( P \) and \( L \) be two large parameters. We begin by introducing the following sum

\[ F_1 = \frac{M^{3/2}P^{3/2}}{NP^2} \sum_{p \sim P} \chi(p)p^{\ell} \sum_{\ell \sim L} \sum_{n=1}^{\infty} \lambda(1,n)\chi(n)n^{-it}w \left( \frac{n}{N} \right) V \left( \frac{r}{Nt/M} \right) \sum_{n=1}^{\infty} \lambda(1,n)\chi(n)w \left( \frac{n}{N} \right) V \left( \frac{r}{Nt/M} \right) w \left( \frac{n}{N} \right), \]

where \( p \sim P \) and \( \ell \sim L \) denote primes in the dyadic segments \([P, 2P]\) and \([L, 2L]\), respectively; \( w \) and \( V \) are smooth functions with compact supports on \( \mathbb{R}^+ \) satisfying \( w^{(j)}(x), V^{(j)}(x) \leq 1 \) for all \( j \geq 0 \).

We shall see that if one applies Poisson summation to the \( r \)-sum (which is the content of Lemma 3.6), then contribution of the zero frequency \( r = 0 \) will give rise to the sum \( S(N) \) that we are initially interested in. In order to bound \( S(N) \), it suffices to bound \( F_1 \) and the sum arising from the non-zero frequencies \( r \neq 0 \) (if we apply Poisson summation to the \( r \)-sum), which we denote by \( O \). This observation is initially due to Holowinsky and Nelson [5, B.4], in their work in the Dirichlet character twist case.

Plugging the identity (10) in, we get

\[ F_1 = (2\pi/Mt)^{-it} e \left( -t/2\pi \right) \frac{dy}{M^1/2} \sum_{p \sim P} \frac{p}{P^2} \sum_{\ell \sim L} \sum_{n=1}^{\infty} \lambda(1,n)\chi(n)n^{-it}w \left( \frac{n}{N} \right) V \left( \frac{r}{Nt/M} \right) \sum_{n=1}^{\infty} \lambda(1,n)\chi(n)w \left( \frac{n}{N} \right) V \left( \frac{r}{Nt/M} \right) w \left( \frac{n}{N} \right), \]

\[ + (N/Mt)^{-it} \frac{t^{1/2}}{P^2M^{1/2}} \sum_{n=1}^{\infty} \lambda(1,n)w \left( \frac{n}{N} \right) \sum_{p \sim P} \chi(p) \sum_{\ell \sim L} \lambda(\ell)/\ell \]

\[ \sum_{r \neq 0} S_\chi(r, np\ell; M) J_\chi(r, np\ell; M) + O(NT^{1/2-A}) \]

\[ = \frac{1}{\log P \log L} \sum_{n=1}^{\infty} \lambda(1,n)\chi(n)n^{-it}w \left( \frac{n}{N} \right) V \left( \frac{2\pi n}{N} \right) + O(NT^{1/2-A}) \]

\[ + \frac{t^{1/2}}{M^{1/2}PL} \sum_{n=1}^{\infty} \lambda(1,n)w \left( \frac{n}{N} \right) \sum_{p \sim P} \chi(p) \sum_{\ell \sim L} \lambda(\ell) \sum_{r \neq 0} S_\chi(r, np\ell; M) J_\chi(r, np\ell; M). \]

We have shown the following.

**Lemma 4.1.** Asymptotically, one has

\[ (13) \quad \frac{1}{\log P \log L} \sum_{n=1}^{\infty} \lambda(1,n)\chi(n)n^{-it}w \left( \frac{n}{N} \right) V \left( \frac{2\pi n}{N} \right) = F_1 + O + O \left( NT^{1/2-A} \right), \]
with
\[ F_1 = \frac{M^{3/2} \overline{J}}{NP^2} \sum_{p \sim P} \chi(p) \sum_{\ell \sim L} \chi(\ell) e^{-it} \sum_{r=1}^{\infty} \frac{\chi(r) r^{-it} V \left( \frac{r}{NP/\ell t} \right)}{r} \sum_{n=1}^{\infty} \lambda(1, n) e \left( -\frac{npM}{t} \right) w \left( \frac{n}{N} \right), \]

and
\[ \mathcal{O} = \frac{t^{1/2}}{M^{1/2} PL} \sum_{n=1}^{\infty} \lambda(1, n) w \left( \frac{n}{N} \right) \sum_{p \sim P} \chi(p) \sum_{\ell \sim L} \chi(\ell) \sum_{r \neq 0} S(\chi, r, np\ell; M) J_{\mathfrak{A}}(r, np/\ell; M), \]

where \( A \geq 1 \) is any constant, \( V_A(x) \) is an inert function (see (8)) depending on \( A \), supported on \( x = 1 \) and \( J_{\mathfrak{A}}(r, np/\ell; M) \) is defined in (11).

For any given \( \varepsilon > 0 \), we can make the error term \( O \left( N^{-1/2-A} \right) \) to be negligibly small by assuming \( t > M^\varepsilon \) and taking \( A \) to be sufficiently large. From the lemma, to bound
\[ \sum_{n=1}^{\infty} \lambda(1, n) \chi(n) n^{-it} w \left( \frac{n}{N} \right) V_A \left( \frac{2\pi n}{N} \right), \]

which is essentially our original object of study \( S(N) \), it suffices to bound the terms \( F_1 \) and \( \mathcal{O} \). We shall do this in the next two sections separately.

5. Treatment of \( \mathcal{O} \)

This section is devoted to giving a nontrivial bound for the sum
\[ \mathcal{O} = \frac{t^{1/2}}{M^{1/2} PL} \sum_{n=1}^{\infty} \lambda(1, n) w \left( \frac{n}{N} \right) \sum_{p \sim P} \chi(p) \sum_{\ell \sim L} \chi(\ell) \sum_{r \sim R} S(\chi, r, np\ell; M) J_{\mathfrak{A}}(r, np/\ell; M), \]

introduced in (14), where \( J_{\mathfrak{A}}(r, np/\ell; M) \) is defined in (11).

For \( r \neq 0 \), integrating by parts implies that the integral \( J_{\mathfrak{A}}(r, np/\ell; M) \) is negligibly small, unless \( 0 \neq |r| < NA^{-1}M^{2} \frac{L}{N} \) (by [4, Lemma 8.1]). Moreover, using the second derivative test ([7, Lemma 5.1.3]) we find that \( J_{\mathfrak{A}}(r, np/\ell; M) \ll t^{-1/2} \).

To estimate \( \mathcal{O} \), it suffices to bound the sum
\[ \mathcal{O}(R) := \frac{t^{1/2}}{M^{1/2} PL} \sum_{n=1}^{\infty} \lambda(1, n) w \left( \frac{n}{N} \right) \sum_{p \sim P} \chi(p) \sum_{\ell \sim L} \chi(\ell) \sum_{r \sim R} S(\chi, r, np\ell; M) J_{\mathfrak{A}}(r, np/\ell; M), \]

where \( R \) satisfies
\[ 1 < R < NA^{-1}M^{2} \frac{L}{NP}. \]

By the Cauchy–Schwarz inequality and the Rankin–Selberg estimate \( \sum_{n=1}^{D} \lambda(1, n)^2 w \left( \frac{n}{N} \right) \ll N^{1+\varepsilon} \),

\[ \mathcal{O}(R) \ll \frac{N^{1/2+\varepsilon} t^{1/2}}{M^{1/2} PL} \left( \sum_{n=1}^{D} \left| \sum_{p \sim P} \chi(p) \sum_{\ell \sim L} \chi(\ell) \sum_{r \sim R} S(\chi, r, np\ell; M) J_{\mathfrak{A}}(r, np/\ell; M) \right|^2 w \left( \frac{n}{N} \right) \right)^{1/2} \]

\[ = \frac{N^{1/2+\varepsilon} t^{1/2}}{M^{1/2} PL} \left( \sum_{p_{1} \sim P, p_{2} \sim P} \sum_{\ell_{1} \sim L, \ell_{2} \sim L} \sum_{r_{1} \sim R, r_{2} \sim R} S(\chi, r_{1}, np_{1}\ell_{1}; M) S(\chi, r_{2}, np_{2}\ell_{2}; M) J_{\mathfrak{A}}(r_{1}, np_{1}/\ell_{1}; M) J_{\mathfrak{A}}(r_{2}, np_{2}/\ell_{2}; M) w \left( \frac{n}{N} \right) \right)^{1/2}. \]
Next, we apply Poisson summation to the $n$-sum, yielding
\begin{align*}
&\sum_{n=1}^{\infty} S_\chi(r_1, np_1 \ell_1; M) S_\chi(r_2, np_2 \ell_2; M) \overline{\mathcal{J}_it(r_1, np_1 \ell_1; M) \mathcal{J}_it(r_2, np_2 \ell_2; M) w \left( \frac{n}{N} \right)} \\
&= \frac{N}{M} \sum_{n \in \mathbb{Z} a(M)} S_\chi(r_1, ap_1 \ell_1; M) S_\chi(r_2, ap_2 \ell_2; M) e \left( \frac{an}{M} \right) \\
&\quad \int_\mathbb{R} \mathcal{J}_it(r_1, Np_1 y/\ell_1; M) \overline{\mathcal{J}_it(r_2, Np_2 y/\ell_2; M) w(y) e \left( -\frac{nN}{M} y \right) dy}.
\end{align*}

Taking into account the oscillations of $\mathcal{J}_it(r_1, Np_1 y/\ell_1; M)$ and $\mathcal{J}_it(r_2, Np_2 y/\ell_2; M)$, the integral is arbitrarily small for $n \neq 0$ (since $N \gg (Mt)^{1+\varepsilon}$). Hence there is only zero frequency after Poisson in $n$:
\begin{align}
&\sum_{n=1}^{\infty} S_\chi(r_1, np_1 \ell_1; M) S_\chi(r_2, np_2 \ell_2; M) \mathcal{J}_it(r_1, np_1 / \ell_1; M) \mathcal{J}_it(r_2, np_2 / \ell_2; M) w \left( \frac{n}{N} \right) \\
&= \frac{N}{M} \mathcal{C} \mathcal{J} + O(N^{-2018}),
\end{align}

where
\begin{align*}
\mathcal{C} &= \sum_{a(M)} S_\chi(r_1, ap_1 \ell_1; M) \overline{S_\chi(r_2, ap_2 \ell_2; M)} = M \chi \left( p_1 \overline{p_2} \ell_1 \ell_2 \right) \sum_{\beta(M)} e \left( \frac{(r_1 - r_2 \overline{p_1} p_2 \ell_1 \ell_2) \beta}{M} \right) \\
&= M \chi \left( p_1 \overline{p_2} \ell_1 \ell_2 \right) \left[ M \delta_{\ell_2 r_1 p_1 = \ell_1 r_2 p_2} (M) - 1 \right],
\end{align*}

and
\begin{equation}
\mathcal{J} = \int_\mathbb{R} \mathcal{J}_it(r_1, Np_1 y/\ell_1; M) \overline{\mathcal{J}_it(r_2, Np_2 y/\ell_2; M) w(y) dy}.
\end{equation}

One readily sees that
\begin{equation}
\mathcal{C} = \begin{cases} O(M^2), & \ell_1 r_2 p_2 = \ell_2 r_1 p_1 (M) \\
O(M), & \text{otherwise.}
\end{cases}
\end{equation}

For the integral $\mathcal{J}$, if we use the previously mentioned second derivative bound $\mathcal{J}_it(r, np / \ell; M) \ll t^{-1/2}$, we get $\mathcal{J} \ll t^{-1}$. However, there are more cancellations beyond $O(t^{-1})$, as long as the parameters $(r_i, p_i, \ell_i)$ satisfy $r_1 p_1 \ell_2 \neq r_2 p_2 \ell_1$. Indeed, we have the following bound.

**Lemma 5.1.** For $\mathcal{J}$ defined as in (17), we have
\begin{equation}
\mathcal{J} \ll \min \left\{ t^{-1}, \frac{M^2 \ell_1 \ell_2}{N |\ell_1 r_2 p_2 - \ell_2 r_1 p_1|} \right\}.
\end{equation}

**Proof.** From (11), we write
\begin{equation}
\mathcal{J}_it(r, Npy / \ell; M) = \int_\mathbb{R} e \left( f(x) \right) V(x) dx,
\end{equation}
where $f(x) = - \frac{t \log x}{2\pi} - \frac{yt}{x} - \frac{r_N p}{M^2 \ell} x$. Set $f'(x_0) = 0$ and solve for $x_0$ to find the stationary point. There are several cases, but for our demonstration we concentrate on the following case:
\begin{equation}
x_0 = -1 + \frac{1}{2\pi} \sqrt{1 + \frac{16\pi^2 r Npy}{M^2 \ell^2}}.
\end{equation}

Expanding the integral $\mathcal{J}_it(r, Npy / \ell; M)$ at the stationary point $x_0$ (by [11, Main Theorem]), we get
\begin{equation}
\mathcal{J}_it(r, Npy / \ell; M) = \left( -1 + \sqrt{1 + \frac{16\pi^2 r Npy}{M^2 \ell^2}} \right)^{-it} e \left( -\frac{t}{2\pi} \sqrt{1 + \frac{16\pi^2 r Npy}{M^2 \ell^2}} \right) \left( \frac{4\pi^2 Npy}{M^{2+it}} \right)^{it} V(x_0) + O_B (t^{-B}).
\end{equation}
where $\tilde{V}(x) = \tilde{V}_B(x)$ is an inert function (see (8)) supported on $x = 1$, and $B \geq 1$ is any constant. Therefore
\begin{equation}
\begin{split}
\hat{J} = & \frac{1}{t} \left( \frac{r_1 N p_1}{M^2 t_1} \right)^{\mu t} r_2 N p_2 \frac{1}{M^2 t_2} \int_\mathbb{R} w(y) \tilde{V}(x_{0,1}) \tilde{V}(x_{0,2}) \left( \frac{-1 + \sqrt{1 + \frac{16\pi^2 r_1 N p_1}{M^2 t_1}} y}{-1 + \sqrt{1 + \frac{16\pi^2 r_2 N p_2}{M^2 t_2}} y} \right)^{-\mu t} \\
& \cdot e \left( -\frac{t}{2\pi} \frac{1}{2\pi} \sqrt{1 + \frac{16\pi^2 r_1 N p_1}{M^2 t_1}} y + \frac{t}{2\pi} \sqrt{1 + \frac{16\pi^2 r_2 N p_2}{M^2 t_2}} y \right) dy + O \left( t^{-B} \right),
\end{split}
\end{equation}
where $x_{0,i} = (-1 + \frac{1}{2\pi} \sqrt{1 + \frac{16\pi^2 r_i N p_i}{M^2 t_i}} y) / \frac{4\pi r_i N p_i}{M^2 t_i}$, $i = 1, 2$. Denote $z_1 = \frac{16\pi^2 r_1 N p_1}{M^2 t_1}$ and $z_2 = \frac{16\pi^2 r_2 N p_2}{M^2 t_2}$. Considering the compactness of the support of the weight function $\tilde{V}$, we necessarily have $z_1 = 1, z_2 = 1$. If $z_1 = z_2$, then we have $\hat{J} \ll t^{-1}$ by estimating trivially. From now on we assume that $z_1 \neq z_2$.

We can rewrite the integral in (19) as
\begin{equation}
\int_\mathbb{R} w_1(y) e \left( -\frac{t}{2\pi} \phi(y) \right) dy,
\end{equation}
where $w_1(y) = w(y) \tilde{V}(x_{0,1}) \tilde{V}(x_{0,2})$, and
\begin{equation}
\phi(y) = \log \left( \frac{-1 + \sqrt{1 + z_1 y}}{-1 + \sqrt{1 + z_2 y}} \right) + \sqrt{1 + z_1 y} - \sqrt{1 + z_2 y}.
\end{equation}

It turns out that
\begin{equation}
\frac{\partial}{\partial y} \phi(y) = \frac{1}{2y} \sqrt{1 + z_1 y} - \frac{1}{2y} \sqrt{1 + z_2 y} = \frac{z_1 - z_2}{2(\sqrt{1 + z_1 y} + \sqrt{1 + z_2 y})}.
\end{equation}

Since $z_1 y \approx 1$ and $z_2 y \approx 1$, we have $\frac{\partial}{\partial y} \phi(y) \gg |z_1 - z_2|$. Now by the first derivative test ([7, Lemma 5.1.2]),
\begin{equation}
\int_\mathbb{R} w_1(y) e \left( -\frac{t}{2\pi} \phi(y) \right) dy \ll \frac{1}{\ell \sqrt{|z_1 - z_2|}} \ll \frac{M^2 t \ell \ell_1 \ell_2}{N |\ell_1 r_2 p_2 - \ell_2 r_1 p_1|},
\end{equation}
upon plugging in $z_1 = \frac{16\pi^2 r_1 N p_1}{M^2 t_1}$ and $z_2 = \frac{16\pi^2 r_2 N p_2}{M^2 t_2}$. The lemma readily follows. Q.E.D.

Remark 5.2. For $\ell_1 r_2 p_2 \neq \ell_2 r_1 p_1$, typically
\begin{equation}
\frac{M^2 \ell_1 \ell_2}{N |\ell_1 r_2 p_2 - \ell_2 r_1 p_1|} \approx t^{-2},
\end{equation}
so that the second bound of the lemma shows that we save an extra $t$ over the ‘trivial bound’ $t^{-1}$. The estimation of this lemma is an analytic analogue of the bound (18).

Now we return to the estimate of $O(R)$, in (15). Plugging the $n$-sum (16) into $O(R)$, up to a negligible error, we have
\begin{equation}
O(R) \ll \frac{N^{1/2 + \epsilon} t^{1/2}}{M^{1/2} P L} \left( \sum_{p_1 \sim P} \sum_{p_2 \sim P} \sum_{\ell_1 \sim L} \sum_{\ell_2 \sim L} \sum_{r_1 \sim R} \sum_{r_2 \sim R} N |\epsilon| |3| \right)^{1/2} \ll \frac{N^{1+\epsilon} t^{1/2}}{M P L} \left( \sum_{p_1 \sim P} \sum_{p_2 \sim P} \sum_{\ell_1 \sim L} \sum_{\ell_2 \sim L} \sum_{r_1 \sim R} \sum_{r_2 \sim R} M^2 |3| \right)^{1/2} \\
+ \sum_{p_1 \sim P} \sum_{p_2 \sim P} \sum_{\ell_1 \sim L} \sum_{\ell_2 \sim L} \sum_{r_1 \sim R} \sum_{r_2 \sim R} M \frac{M^2 \ell_1 \ell_2}{N |\ell_1 r_2 p_2 - \ell_2 r_1 p_1|} \left( \sum_{\ell_1 r_2 p_2 = \ell_2 r_1 p_1(M)} \right)^{1/2},
\end{equation}
by using (18) and Lemma 5.1. We remind the reader that $R$ satisfies $1 \leq R \ll N^\epsilon M^2 t^2 / NP$. 
Using Lemma 5.1 again, the first term inside the parentheses is bounded by
\[
M^2 \sum_{p_1 \sim P} \sum_{p_2 \sim P} \sum_{\ell_1 \sim L} \sum_{\ell_2 \sim L} \sum_{r_1 \sim R} \sum_{r_2 \sim R} t^{-1} + \frac{M^4 L^2}{N} \sum_{p_1 \sim P} \sum_{p_2 \sim P} \sum_{\ell_1 \sim L} \sum_{\ell_2 \sim L} \sum_{r_1 \sim R} \sum_{r_2 \sim R} \sum_{\ell_1 \sim \ell_2 \neq \ell_2} \frac{1}{|\ell_1 r_2 p_2 - \ell_2 r_1 p_1|},
\]
which is further dominated by
\[
\ll N^\varepsilon M^2 t^{-1} P L R + N^\varepsilon \frac{M^4 L^2}{N} \left( \frac{L P R}{M} + \frac{L^2}{M} \right)
\ll N^\varepsilon \frac{M^4 t^2 L^4}{N} + N^\varepsilon \frac{M^5 t^2 L^4}{N^2},
\]
by using Lemma 3.8 and by noting that \( R \ll N^\varepsilon \frac{M^2 t^2 L}{NP} \).

Similarly, the second term inside the parentheses of (20) is bounded by
\[
\frac{M^3 L^2}{N} \sum_{p_1 \sim P} \sum_{p_2 \sim P} \sum_{\ell_1 \sim L} \sum_{\ell_2 \sim L} \sum_{r_1 \sim R} \sum_{r_2 \sim R} \sum_{\ell_1 \sim \ell_2 \neq \ell_2} \frac{1}{|\ell_1 r_2 p_2 - \ell_2 r_1 p_1|} \ll N^\varepsilon \frac{M^3 L^2}{N} (PLR + L^2) \ll N^\varepsilon \frac{M^5 t^2 L^4}{N^2},
\]
upon using Lemma 3.8.

Returning to the estimate of \( \mathcal{O}(R) \), we have shown for any \( 1 \leq R \ll N^\varepsilon \frac{M^2 t^2 L}{NP} \), that
\[
\mathcal{O}(R) \ll \frac{N^{1 + \varepsilon} t^{1/2}}{MPL} \left( \frac{M^4 L^2}{N} + \frac{M^5 t^2 L^4}{N^2} \right)^{1/2} \ll \frac{N^{1/2 + \varepsilon} t M}{P} + N^\varepsilon \frac{M^3 t^3 L}{P}.
\]

We summarize the main result of this section.

**Proposition 5.3.** For any \( \varepsilon > 0 \), we have the bound
\[
\mathcal{O} \ll \frac{N^{1/2 + \varepsilon} t M}{P} + N^\varepsilon \frac{M^3 t^3 L}{P},
\]
for \( \mathcal{O} \) defined as in (14).

**Remark 5.4.** If we only use the ‘trivial’ bound \( \mathcal{O} \ll t^{-1} \) for the estimate of the integral \( \mathcal{O} \), then one will see that for the second term we get \( \mathcal{O} \ll N^\varepsilon \frac{M^3 t^3 L}{NP} \) instead. It is thus crucial to use Lemma 5.1 to get an extra \( t^{1/2} \) saving in order to beat the convexity bound in the \( t \)-aspect.

6. **TREATMENT OF \( \mathcal{F}_1 \)**

The purpose of this section is to give a nontrivial bound for
\[
\mathcal{F}_1 = \frac{M^{3/2} t^{3/2}}{NP^2} \sum_{p \sim P} \sum_{\ell \sim L} \chi(p) \chi(\ell) L^{-it} \sum_{n=1}^\infty \lambda(1, n) e \left( \frac{n \ell M}{M t} \right) \frac{N}{M t} \psi \left( \frac{n}{N} \right),
\]
defined in (12), where \( \psi \) and \( V \) are smooth compactly supported functions with bounded derivatives.

Bounding the sum directly with Miller’s bound (7), we have \( \mathcal{F}_1 \ll N^{3/4 + \varepsilon} (Mt)^{3/2} \), which is not satisfactory yet for our purpose.

We shall apply a Voronoi summation to the \( n \)-sum. To this end, one may assume \( (p, r) = 1 \) in \( \mathcal{F}_1 \), as the contribution from the terms \( (p, r) > 1 \) is negligible, compared to the generic terms \( (p, r) = 1 \). We briefly justify this. Denote the terms with \( p | r \) in \( \mathcal{F}_1 \) by \( \mathcal{F}_1^e \). Then,
\[
\mathcal{F}_1^e = \frac{M^{3/2} t^{3/2}}{NP^2} \sum_{p \sim P} \sum_{\ell \sim L} \chi(p) \chi(\ell) L^{-it} \sum_{n=1}^\infty \lambda(1, n) e \left( \frac{n \ell M}{M t} \right) \frac{N}{M t} \psi \left( \frac{n}{N} \right).
\]

An application of Voronoi summation (6) takes the \( n \)-sum to the following dual sum
\[
\ell \sum_{\pm} \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{\lambda(n, m)}{m n} S(M, \pm n; \ell r/m) U_{\pm} \left( \frac{m^2 n}{(\ell r)^3 / N} \right),
\]
where the new length can be truncated at \( m^2 n < N^\varepsilon \ell r)^3 / N \), at the cost of a negligible error.
Hence we can estimate $F_1^2$ as follows.

$$F_1^2 = \frac{M^{3/2} \ell^{3/2}}{NP \log P} \sum_{\ell \sim L} \chi(\ell) \epsilon^{-it} \sum_{r=1}^\infty \chi(r) r^{-it} V \left( \frac{r}{N/M \ell t} \right)$$

$$\leq \ell \epsilon \sum_{m | \ell r} \sum_{n=1}^\infty \frac{\lambda(m,n)}{mn} S(M, n; \ell r/m) U^+ \left( \frac{m^2 n}{(\ell r)^3/N} \right)$$

$$\leq \frac{N^2 \ell}{NP} \sum_{\ell \sim L / \sim N / M \ell t} \sum_{m^2 n < (\ell r)^3/N} \frac{|\lambda(m,n)|}{mn} \left( \frac{\ell r}{m} \right)^{1/2}$$

upon using Weil’s bound, which is satisfactory for our purpose.

From now on we assume that $(p, \ell r) = 1$. Application of Voronoi summation (6) to the $n$-sum yields

$$\sum_{n=1}^\infty \lambda(1,n) e \left( \frac{npM}{\ell r} \right) w \left( \frac{n}{N} \right) = \ell \epsilon \sum_{m | \ell r} \sum_{n=1}^\infty \frac{\lambda(n,m)}{mn} S(pM, n; \ell r/m) U^\pm \left( \frac{m^2 n N}{(\ell r)^3} \right).$$

Here contribution from the terms with $m^2 n \gg N^2 (\ell r)^3 / N$ is negligibly small. Thus, we can truncate the $(m,n)$-sum at $m^2 n \ll N^{2+\varepsilon} P^3 / M^3 \ell^3$, at the cost of a negligible error. For those $m^2 n \ll N^{2+\varepsilon} P^3 / M^3 \ell^3$, the result of Jacquet and Shalika gives us the bound

$$U^\pm \left( \frac{m^2 n N}{\ell^3 \ell^3} \right) \ll \sqrt{\frac{m^2 n N}{\ell^3 \ell^3}},$$

while in general we have $y^1 U^{\pm,(1)} (y) \ll \sqrt{y}$.

Considering for example, the plus case, we have

$$F_1 = \frac{M^{3/2} \ell^{3/2}}{NP^2} \sum_{p \sim P} \hat{\chi}(p) p^{-it} \sum_{\ell \sim L} \chi(\ell) \epsilon^{-it} \sum_{r=1}^\infty \chi(r) r^{-it} V \left( \frac{r}{NP / M \ell \ell t} \right)$$

$$\leq \frac{(M \ell)^{1/2}}{P} \sum_{r=1}^\infty \chi(r) r^{-it} V \left( \frac{r}{NP / M \ell \ell t} \right) \sum_{\ell \sim L} \chi(\ell) \epsilon^{-it}$$

$$\sum_{m | \ell r} \sum_{n=1}^\infty \frac{\lambda(n,m)}{mn} \sum_{p \sim P} \hat{\chi}(p) p^{-it} S(pM, n; \ell r/m) U^+ \left( \frac{m^2 n N}{\ell^3 \ell^3} \right) + O \left( \frac{N^{3/2+\varepsilon}}{P \ell M t} \right).$$

Pulling the $\ell$-sum inside the $(m,n)$-sum and applying the Cauchy–Schwarz inequality to $F_1$, we obtain

$$F_1 \leq \frac{(M \ell)^{1/2}}{P} \sum_{r=1}^\infty \chi(r) r^{-it} V \left( \frac{r}{NP / M \ell \ell t} \right) \sum_{m,n \geq 1} \frac{\lambda(n,m)}{mn}$$

$$\sum_{m | \ell r} \sum_{n=1}^\infty \frac{\lambda(n,m)}{mn} \sum_{p \sim P} \hat{\chi}(p) p^{-it} S(pM, n; \ell r/m) U^+ \left( \frac{m^2 n N}{\ell^3 \ell^3} \right) + O \left( \frac{N^{3/2+\varepsilon}}{P \ell M t} \right)$$

$$\ll \frac{N^{1/2+\varepsilon}}{P^{3/2} L^{1/2}} \sum_{}(1/2) + \frac{N^{3/2+\varepsilon}}{P \ell M t}.$$
where

\[ \Sigma := \sum_{r \sim NP/MLt, (r,M) = 1} \sum_{m,n} \frac{1}{mn} \sum_{\ell_1 \sim L, \ell_2 \sim L} \sum_{p \sim p} \sum_{\ell r, m|\ell r} \sum_{\ell p} \sum_{\ell q} \sum_{\ell s} |\chi(\ell_1)\ell^{-it} + \chi(\ell_2)\ell^{-it} + \chi(p)\ell^{-it} + \chi(q)\ell^{-it} + \chi(s)\ell^{-it}|^2 S(pM, n; \ell r/m)U^+ \left( \frac{m^2nN}{\ell_1 r^3} \right)^2, \]

by noting that

\[ \sum_{m,n} \frac{|\lambda(n,m)|^2}{mn} \ll N^\varepsilon. \]

Now it remains to estimate \( \Sigma \). Opening the square and interchanging the order of summations, we find

\[ \Sigma \ll \sum_{r \sim NP/MLt} \sum_{m \sim N^{3/2}M^{3/2}L^3} \frac{1}{r} \sum_{\ell_1 \sim L, \ell_2 \sim L} \sum_{p_1 \sim P} \sum_{p_2 \sim P} \sum_{p_3 \sim P} |S(p_1 M, n; \ell_1 r/m)S(p_2 M, n; \ell_2 r/m)U^+ \left( \frac{m^2nN}{\ell_1 r^3} \right) \left( \frac{m^2nN}{\ell_2 r^3} \right) F \left( \frac{n}{N_m} \right)|. \]

Our next step is to apply Poisson summation to the \( n \)-sum. To this end, one can insert an nonnegative smooth function \( F \) which is supported on, say \([1/2, 3] \), and constantly 1 on \([1, 2] \), into the \( n \)-sum.

We apply Poisson summation with modulus \([\ell_1 r/m, \ell_2 r/m] \), to get

\[ \sum_{n \sim N^{3/2}M^{3/2}L^3} \frac{1}{r} \sum_{n \in \mathbb{Z}} C_{\ell_1, \ell_2}(n) T(n, \ell_1, \ell_2), \]

where

\[ N_m \ll N^2 P^3/m^2 M^3 \ell^3, \]

and

\[ C_{\ell_1, \ell_2}(n) = \sum_{a([\ell_1, \ell_2] \cap m)} S(p_1 M, a; \ell_1 r/m)S(p_2 M, a; \ell_2 r/m) e \left( \frac{an}{[\ell_1, \ell_2] r/m} \right), \]

and

\[ T(n, \ell_1, \ell_2) = \int_{\mathbb{R}} F(x) U^+ \left( \frac{m^2N_m N x}{\ell_1 r^3} \right) U^+ \left( \frac{m^2N_m N x}{\ell_2 r^3} \right) e \left( \frac{nx}{[\ell_1, \ell_2] r/m} \right) dx. \]

By integrating by parts repeatedly, the integral \( T(n, \ell_1, \ell_2) \) is negligibly small, unless \(|n| \ll N^\varepsilon \frac{[\ell_1, \ell_2] r/m}{N_m} \ll N^{NPL/Mt} N_m \). Therefore, we can truncate the dual \( n \)-sum at \( N^{NPL/Mt} N_m \) (with the convention that if this is \( < 1 \), then only the zero frequency \( n = 0 \) survives), at the cost of a negligible error. While in the range \(|n| \ll N^\varepsilon \frac{[\ell_1, \ell_2] r/m}{N_m} \ll N^{NPL/Mt} N_m \), we use the bound \( y^2 U^{+,\ell}(y) \ll \sqrt{y} \) to obtain

\[ \Gamma(n, \ell_1, \ell_2) \ll \frac{m^2N_m N}{(NP/Mt)^3}. \]

In particular, \( \Gamma(n, \ell_1, \ell_2) \ll 1 \), by consideration of the bound (22).

Let us also observe in particular that

\[ N_1 \ll N^2 P^3/M^4 \ell^3, \]

for later convenience.

We arrive at

\[ F_1 \ll \frac{N^{1/2+\varepsilon}}{P^{1/2}L^{1/2}} \Omega^{1/2} + \frac{N^{3/2+\varepsilon}}{PMt}. \]
where
\[
\Omega = \sum_{r \sim NP/MLt} \sum_{m < NP^{3/2}/M^{3/2}r^2} \sum_{\ell_1 \sim L_{(r/m)}} \sum_{\ell_2 \sim L_{(2r/m)}} \sum_{p_1 \sim P} \sum_{p_2 \sim P} \sum_{\nu | n < NP/L/N_mM_t} \frac{1}{(\ell_1, \ell_2)_{r^2}} |C_{\ell_1, \ell_2}(n)\tau(n, \ell_1, \ell_2)|.
\]

We have essentially square-root cancellation for the character sum \(C_{\ell_1, \ell_2}(n)\), defined in (23). The details of this calculation were carried out in [5]. We have collected their results relevant to our present setting in Lemma 3.5.

Bounding our sum (23) using Lemma 3.5, we get
\[
|C_{\ell_1, \ell_2}(n)| \leq 2^{O(\omega(r))} \left( \frac{r}{m} \right)^{3/2} \frac{\ell_1 \ell_2}{(\ell_1, \ell_2)^{1/2}} (\Delta, n, \ell_1 r/m, \ell_2 r/m) (n, \ell_1 r/m, \ell_2 r/m)^{1/2},
\]
where
\[
\Delta := \bar{p}_1 \ell_2^2 - \bar{p}_2 \ell_1^2 \quad (\ell_1, \ell_2)^2 M_t
\]
and \(\bar{p}_1\) and \(\bar{p}_2\) denote the multiplicative inverses of \(p_1\) and \(p_2\) modulo \(\ell_1 r/m\) and \(\ell_2 r/m\), respectively.

We write
\[
\Omega = \Omega_0 + \Omega_1,
\]
where \(\Omega_0\) denotes contribution from the terms \(n = \Delta = 0\), and \(\Omega_1\) denotes the complement.

**Remark 6.1.** In fact, \(\Omega_0\) is the diagonal contribution \((\ell_1, p_1) = (\ell_2, p_2)\) to the sum (21), and \(\Omega_1\) is the off-diagonal contribution.

If \(\Delta = 0\), then \(\bar{p}_1 \ell_2^2 - \bar{p}_2 \ell_1^2 = 0\). Necessarily, \(\ell_1 = \ell_2 := \ell\) and \(p_1 = p_2 := p\). Under this condition,
\[
|C_{\ell, \ell}(n)| \leq 2^{O(\omega(r))} \left( \frac{r}{m} \right)^{3/2} \left( \frac{\ell r}{m} \right)^{1/2}.
\]
In particular, \(|C_{\ell, \ell}(0)| \leq 2^{O(\omega(r))} \left( \frac{r}{m} \right)^2\). Therefore,
\[
\Omega_0 \ll \sum_{r \sim NP/MLt} \sum_{\ell_1 \sim L_{(r/m)}} \sum_{\ell_2 \sim L_{(2r/m)}} \sum_{p_1 \sim P} \sum_{p_2 \sim P} 2^{O(\omega(r))} \left( \frac{r}{m} \right)^{3/2} \left( \frac{\ell r}{m} \right)^{1/2} |\tau(0, \ell, \ell)| \ll \frac{N^{2+\varepsilon} P^3}{M^{3/2} \ell^2}.
\]

Meanwhile for \(\Omega_1\), we further write
\[
\Omega_1 = \Omega_{1a} + \Omega_{1b},
\]
where \(\Omega_{1a}\) denotes the contribution coming from the \(n \neq 0\) terms, and \(\Omega_{1b}\) denotes the contribution of the zero frequency: \(n = 0, \Delta \neq 0\). Plugging the bounds (24) and (26) in, we first see that
\[
\Omega_{1a} \ll \sum_{r \sim NP/MLt} \sum_{\ell_1 \sim L_{(r/m)}} \sum_{\ell_2 \sim L_{(2r/m)}} \sum_{p_1 \sim P} \sum_{p_2 \sim P} \sum_{\nu | n < NP/L/N_mM_t} \frac{1}{(\ell_1, \ell_2)_{r^2}} |C_{\ell_1, \ell_2}(n)\tau(n, \ell_1, \ell_2)|
\ll N^\varepsilon \sum_{r \sim NP/MLt} \sum_{\ell_1 \sim L_{(r/m)}} \sum_{\ell_2 \sim L_{(2r/m)}} \sum_{p_1 \sim P} \sum_{p_2 \sim P} \sum_{\nu | n < NP/L/N_mM_t} r^{1/2} (\Delta, n, r/m) m^2 N_m^2 \frac{N_1 N (NP/Mt)^3}{m^{3/2} (n, r/m)^{1/2}}
\ll N^\varepsilon \frac{NP}{MLt} \left( \frac{NP}{MLt} \right)^{1/2} \frac{N_1 N (NP/Mt)^3}{(NP/Mt)^{3/2}}.
\]

Now we treat the case of \(\Omega_{1b}\), which by our definition is
\[
\Omega_{1b} = \sum_{r \sim NP/MLt} \sum_{m < NP^{3/2}/M^{3/2}r^2} \sum_{\ell_1 \sim L_{(r/m)}} \sum_{\ell_2 \sim L_{(2r/m)}} \sum_{p_1 \sim P} \sum_{p_2 \sim P} \frac{1}{(\ell_1, \ell_2)_{r^2}} |C_{\ell_1, \ell_2}(0)\tau(0, \ell_1, \ell_2)|.
\]
A direct evaluation of \(C_{\ell_1, \ell_2}(0)\) from the definition (23) shows that it vanishes unless \(\ell_1 = \ell_2 := \ell\). In the later case we have
\[
C_{\ell, \ell}(0) = \frac{\ell r}{m} \sum_{\beta (\ell r/m)} \left( \frac{\bar{p}_1 - \bar{p}_2}{\ell r/m} \right). \]
Recall for \( \ell_1 = \ell_2 = \ell \), \( \Delta = (\tilde{p}_1 - \tilde{p}_2)M \), where \( \tilde{p}_1 \) and \( \tilde{p}_2 \) are the multiplicative inverses of \( p_1 \) and \( p_2 \) modulo \( \ell r/m \), respectively. As \( \Delta \neq 0 \), we have \( \tilde{p}_1 \neq \tilde{p}_2 \), and hence in particular, \( \tilde{p}_1 \neq \tilde{p}_2 \mod \ell r/m \). Therefore

\[
|C_{\ell,\ell}(0)| \leq \frac{\ell r}{m} \tau(\ell r/m),
\]
where \( \tau \) denotes the divisor function.

We thus have

\[
\Omega_{1b} \ll N^\varepsilon \sum_{r \sim N/P} \sum_{\ell \sim L} \sum_{m | \ell r/m} \sum_{p_1 \sim P} \sum_{p_2 \sim P} \frac{1}{\ell r} \frac{\ell r}{m} |T(0, \ell, \ell)| \ll \frac{N^{1+\varepsilon} P^3}{Mt}.
\]

This is dominated by the diagonal contribution \( \Omega_0 \) \((27)\), since \( Mt < N \).

Hence we obtain the bound

\[
\Omega = \Omega_0 + \Omega_{1a} + \Omega_{1b}
\]
\((28)\)

\[
\ll \frac{N^{2+\varepsilon} P^3}{M^2 t^2} + N^\varepsilon (NMt)^{1/2}(PL)^{3/2}.
\]

Combining \((25)\) and \((28)\), we retrieve the bound on \( F_1 \) in the following.

**Proposition 6.2.** For any given \( \varepsilon > 0 \),

\[
F_1 \ll \frac{N^{3/2+\varepsilon} P}{Mt L^{1/2}} + N^{3/4+\varepsilon} (MtPL)^{1/4}.
\]

**Remark 6.3.** We will assume \( L < P \), so that the term \( O \left( \frac{N^{3/2+\varepsilon}}{P Mt} \right) \) in \((25)\) is negligible.

### 7. The choices of the parameters \( P \) and \( L \)

Recall from Proposition 6.2, one has

\[
F_1 \ll \frac{N^{3/2+\varepsilon} P}{Mt L^{1/2}} + N^{3/4+\varepsilon} (MtPL)^{1/4},
\]

while Proposition 5.3 gives

\[
O \ll \frac{N^{1/2+\varepsilon} Mt}{P} + N^\varepsilon \frac{M^{3/2} t\sqrt{L}}{P}.
\]

Plugging these bounds into \((13)\),

\[
S(N) \ll \frac{N^{3/2+\varepsilon} P}{Mt L^{1/2}} + \frac{N^{3/4+\varepsilon} (MtPL)^{1/4}}{P} + \frac{N^{1/2+\varepsilon} Mt}{P} + N^\varepsilon \frac{M^{3/2} t\sqrt{L}}{P}.
\]

Substituting this into Lemma 2.1 and noting that \((Mt)^{3/2-\delta} < N < (Mt)^{3/2+\varepsilon} \), one gets

\[
\begin{aligned}
L \left( \frac{1}{2} + it, \pi \otimes \chi \right) &\ll \frac{(Mt)^{1/2+\varepsilon} P L^{1/2}}{L^{1/2}} + \frac{(Mt)^{3/4+\varepsilon} (PL)^{1/4}}{P} + \frac{(Mt)^{1+\varepsilon} P}{P} + \frac{(Mt)^{3/4+\delta/2+\varepsilon} L}{P} + (Mt)^{3/4-\delta/2+\varepsilon} \\
&= \frac{(Mt)^{5/8+\varepsilon} P^{1/4}}{L^{1/2}} \left( \frac{P^{3/4}}{(Mt)^{1/8}} + L^{3/4} \right) + (Mt)^{\varepsilon} \left( \frac{Mt}{P} + (Mt)^{3/4-\delta/2} \right),
\end{aligned}
\]

upon assuming \( L < (Mt)^{1/4-\delta/2} \).

Equate the first two terms by letting \( L = P(Mt)^{-1/6} \) to get

\[
L \left( \frac{1}{2} + it, \pi \otimes \chi \right) \ll (Mt)^{7/12+\varepsilon} P^{1/2} + (Mt)^{1+\varepsilon}/P + (Mt)^{3/4-\delta/2+\varepsilon}.
\]

Letting \( P = (Mt)^{5/18} \),

\[
L \left( \frac{1}{2} + it, \pi \otimes \chi \right) \ll (Mt)^{13/18+\varepsilon} + (Mt)^{3/4-\delta/2+\varepsilon}.
\]

\((29)\)

Finally, by choosing \( \delta = 1/18 \), \((29)\) implies that

\[
L \left( \frac{1}{2} + it, \pi \otimes \chi \right) \ll (Mt)^{3/4-1/36+\varepsilon}.
\]
Note that with such choices, \( L = (M\ell)^{1/9} \) satisfies the assumption \( L < (M\ell)^{1/4-5/2} = (M\ell)^{2/9} \). Theorem 1.1 follows.

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