Reconstruction of density functions by $sk$-splines

A. Kushpel and J. Levesley
Department of Mathematics
University of Leicester, UK
ak412@le.ac.uk, jl1@le.ac.uk

April 18, 2014

Abstract

Reconstruction of density functions and their characteristic functions by radial basis functions with scattered data points is a popular topic in the theory of pricing of basket options.

Such functions are usually entire in $\mathbb{C}^n$ or admit an analytic extension into an appropriate tube in $\mathbb{C}^n$ and "bell-shaped" with rapidly decaying tails.

Unfortunately, the domain of such functions (which are important in practical applications) is not compact (e.g., $\mathbb{R}^n$) which creates difficulties of a fundamental nature. Frequently used approach to overcome this problem is an "appropriate" compactification (or truncation) of the domain. Then in this compact domain we can try to interpolate (or quasi-interpolate) by radial basis functions with a finite number of data points. However, solution of the respective interpolation problem is connected with inversion of "big" matrices and remains a challenging computational problem. Moreover, the accuracy of approximation can not be improved after truncation no matter how many data points we take in the truncated domain. Also, the Fourier transform of truncated characteristic function (which is a density function up to some multiplicative factor) usually is not integrable which creates additional technical difficulties. To avoid this range of problems many authors tried to construct extensions of truncated characteristic functions into a larger but still compact domain.
using algebraic polynomials and ignoring their analytic smoothness. Of course, such approach can not produce effective and saturation free algorithms.

In this article we present a different approach. The values of a given characteristic function can be computed on a rectangular mesh which allows us to solve the respective interpolation problem explicitly under very general conditions on the kernel function. Then we calculate explicitly Fourier transform of such interpolant to obtain an approximant for the density function.

1 Introduction

Consider a frictionless market with no arbitrage opportunities and a constant riskless interest rate $r > 0$. Let $S_{j,t}$, $1 \leq j \leq n$, $t \geq 0$, be $n$ asset price processes. The common spread option with maturity $T > 0$ and strike $K \geq 0$ is the contract that pays $\left(S_{1,T} - \sum_{j=2}^{n} S_{j,T} - K\right)_+$ at time $T$, where $(a)_+ := \max\{a, 0\}$. There is a wide range of such options traded across different sectors of financial markets. For instance, the crack spread and crush spread options in the commodity markets [13], [15], credit spread options in the fixed income markets, index spread options in the equity markets [2] and the spark (fuel/electricity) spread options in the energy markets [1], [14].

Assuming the existence of a risk-neutral equivalent martingale measure $\mathbb{Q}$ we get the following pricing formula for the value at time 0,

$$V = e^{-rT} \mathbb{E}^{\mathbb{Q}} [\varphi],$$

where $\varphi$ is a reward function and the expectation is taken with respect to the equivalent martingale measure. Usually, the reward function has a simple structure, hence the main problem is to approximate properly the respective density function and then to approximate $\mathbb{E}^{\mathbb{Q}} [\varphi]$.

Let $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ be two vectors in $\mathbb{R}^n$, $\langle x, y \rangle := \sum_{k=1}^{n} x_k y_k$ be the usual scalar product and $|x|^2 := \|x\|_2 := \langle x, x \rangle^{1/2}$. For an integrable on $\mathbb{R}^n$ function, i.e., $f(x) \in L_1 (\mathbb{R}^n)$ define its Fourier transform

$$\mathcal{F} f(y) = \int_{\mathbb{R}^n} \exp \left( -i \langle x, y \rangle \right) f(x) dx,$$

and its formal inverse as

$$\left( \mathcal{F}^{-1} f \right) (x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp \left( -i \langle x, y \rangle \right) f(y) dy.$$
The characteristic function of the distribution of $X_t$ of any Lévy process can be represented in the form

$$
\mathbb{E}^Q [\exp (\langle ix, X_t \rangle)] = e^{-t\psi^Q(x)}
$$

$$
= (2\pi)^n F^{-1} p^Q_t (x),
$$

where $p^Q_t (x)$ is the density function of $X_t$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}_+$ and the function $\psi^Q(x)$ is uniquely determined. This function is called the characteristic exponent. Vice versa, a Lévy process $X = \{X_t\}_{t \in \mathbb{R}_+}$ is determined uniquely by its characteristic exponent $\psi^Q(x)$. In particular, density function $p^Q_t$ can be expressed as

$$
p^Q_t (\cdot) = (2\pi)^{-n} \int_{\mathbb{R}^n} \exp \left( -i \langle \cdot, x \rangle - t\psi^Q(x) \right) dx
$$

$$
= (2\pi)^{-n} F \left( \exp \left( -t\psi^Q(x) \right) \right) (\cdot)
$$

$$
= (2\pi)^{-n} F \left( \Phi^Q (x,t) \right) (\cdot),
$$

where $\Phi^Q(x,t)$ is the characteristic function of $X = \{X_t\}_{t \in \mathbb{R}_+}$. Let $\Lambda := \{x_k\}$ be an additive group of lattice points in $\mathbb{R}^n$ and $K (\cdot)$ be a fixed kernel function (The reader should't mix the strike price $K$ with the kernel function $K (\cdot)$). Assume that the interpolant

$$
sk (\Phi^Q, x) := \sum_{\Lambda} c_k K (x - x_k)
$$

for $\Phi^Q(x,t)$ exists and unique. Then, formally, we get

$$
p^Q_t (\cdot) \approx (2\pi)^{-n} \sum_{\Lambda} c_k \left( \Phi^Q(x_k,t) \right) F \left( K (x - x_k) \right) (\cdot)
$$

$$
= (2\pi)^{-n} F \left( K \right) (\cdot) \sum_{\Lambda} c_k \left( \Phi^Q(x_k,t) \right) \exp \left( i \langle \cdot, x_k \rangle \right).
$$

In what follows we give an explicit form of $c_k \left( \Phi^Q(x_k,t) \right)$ and $F \left( K (x - x_k) \right) (\cdot)$ which will give us an approximant for the density function $p^Q_t$. Remark that in many important cases the coefficients $|c_k \left( \Phi^Q(x_k,t) \right)|$ decay exponentially fast as $|k| \to \infty$. 

3
Let us consider in more details the problem of interpolation in $\mathbb{R}^n$. Let $f$ be a continuous function on $\mathbb{R}^n$, $f \in C(\mathbb{R}^n)$ and
\[
\sum_{x_j \in \Lambda} c_j K(x_k - x_j) = f(x_k), \quad x_k, x_j \in \Lambda.
\]

Of course, it is very difficult (or in general impossible) to get an explicit solution of the interpolation problem. But, if we assume some regularity condition on the data points $\{x_k\}$ it is still possible to solve the interpolation system. We give here an explicit solution of the interpolation problem in the case of a uniform mesh on $\mathbb{R}^n$.

Let $L_p(\mathbb{R}^n)$ be the usual space of $p$-integrable functions equipped with the norm
\[
\|f\|_p = \|f\|_{L_p(\mathbb{R}^n)} := \begin{cases}
\left( \int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{1/p}, & 1 \leq p < \infty, \\
\text{ess sup}_{x \in \mathbb{R}^n} |f(x)|, & p = \infty.
\end{cases}
\]

To justify an inversion formula we will need celebrated Plancherel’s theorem.

**Theorem 1** (Plancherel) The Fourier transform is a linear continuous operator from $L_2(\mathbb{R}^n)$ onto $L_2(\mathbb{R}^n)$. The inverse Fourier transform, $F^{-1}$, can be obtained by letting
\[
(F^{-1}g)(x) = \frac{1}{(2\pi)^n} (Fg)(-x)
\]
for any $g \in L_2(\mathbb{R}^n)$.

Let us describe first the one-dimensional situation on $\mathbb{T}^1$. For a given $m \in \mathbb{N}$ let $\Lambda_m = \{0 = x_0 < \cdots < x_{n-1} < x_m = 2\pi\}$ be an arbitrary partition of $[0,2\pi)$ and $K$ be a continuous function. Then
\[
s_k(x) = \sum_{x_k \in \Lambda_m} c_k K(x - x_k), \quad c_k \in \mathbb{R}.
\]

Denote by $SK(\Lambda_m)$ the space of $sk$-splines, i.e.
\[
SK(\Lambda_m) = \text{span}\{K(x - x_m), x_k \in \Lambda_m\}.
\]
See [7] for more information. Let $y \in \mathbb{R}$ be a fixed parameter and $y_k = y + x_k$, $1 \leq k \leq m$ be the points of interpolation. If the interpolation problem has a unique solution then the spline interpolant $sk(x, y, \Lambda_m, f) = sk(x, f)$ with
knots $x_k$, $1 \leq k \leq m$ and points of interpolation $y_k$ can be written in the form

$$sk(x) = \sum_{k \in \Lambda_m} f(y_k) \tilde{sk}(x),$$

where

$$\tilde{sk}(y_s) = \begin{cases} 1, & k = s, \\ 0, & k \neq s. \end{cases}$$

are the fundamental $sk$-splines. It is important in various applications to have an explicit form of the Fourier series expansions for the fundamental $sk$-splines. As a motivating example consider $sk$-splines on the uniform greed $\Lambda_m = \{x_k = 2\pi/m, 1 \leq k \leq m\}$, with the points of interpolation $y_1, \cdots, y_m$ where $y \in \mathbb{R}$ is a fixed parameter,

$$sk(x) = c_0 + \sum_{k=1}^{m} c_k K(x - x_k), \quad \sum_{k=1}^{m} c_k = 0, \quad c_k \in \mathbb{R}, \quad 1 \leq k \leq m,$$

In this case fundamental splines are just the shifts of $\tilde{sk}(\cdot)$, where

$$\tilde{sk}(y_k) = \begin{cases} 1, & k \equiv 0(\text{mod} n), \\ 0, & \text{otherwise}. \end{cases}$$

Let, in particular,

$$K(x) = D_r(x) = \sum_{k=1}^{\infty} \frac{1}{k^r} \cos \left(kx + \frac{r\pi}{2} \right), \quad r \in \mathbb{N}$$

be the Bernoulli monospline. Then the space $SK(\Lambda_m)$ is the space of polynomial splines of order $r - 1$, defect 1 with knots $x_k$, $1 \leq k \leq m$ and points of interpolation $y_k$, $1 \leq k \leq m$.

First Fourier series expansions of fundamental splines have been obtained by Golomb [3] in the case $r = 4$, and $y = 0$, i.e. in the case of cubic splines. It was shown that

$$\tilde{sk}(x, 0) = \tilde{sk}(x) = \frac{1}{m} + \frac{1}{m} \sum_{j=1}^{m-1} \frac{\rho_j(x)}{\rho_j(0)},$$

where

$$\rho_j(x) = \sum_{\nu=1}^{m} \cos \left(\frac{2\pi \nu j}{m} \right) D_4 \left(x - \frac{2\pi \nu}{m} \right).$$
In the case of a general kernel function \( K \in C(T^1) \) and an arbitrary \( y \in \mathbb{R} \) the respective results were established in [7]. Namely, it was shown that

\[
\tilde{s}_k(x, y) = \tilde{s}_k(x) = \frac{1}{m} + \frac{1}{m} \sum_{j=1}^{m-1} \frac{\rho_j(x)\rho_j(y) + \sigma_j(x)\sigma_j(y)}{\rho_j^2(y) + \sigma_j^2(y)},
\]

where

\[
\rho_j(x) = \sum_{\nu=1}^{m} \cos \left( \frac{2\pi \nu j}{m} \right) K \left( x - \frac{2\pi \nu}{m} \right), \quad \sigma_j(x) = \sum_{\nu=1}^{m} \sin \left( \frac{2\pi \nu j}{m} \right) K \left( x - \frac{2\pi \nu}{m} \right).
\]

Of course, to guarantee existence of fundamental splines for a given \( y \) we need to assume that \( \max\{\rho_j^2(y), \sigma_j^2(y), 1 \leq j \leq m\} > 0 \). A detailed study of such kind of conditions in terms of Fourier coefficients of the kernel function \( K \) can be found in [7], [5]. Different analogs of these results in multidimensional settings, on \( T^d \) can be found in [4], [8], [12]. Remark that the problem of convergence of \( sk \)-spline interpolants and quasi-interpolants was considered in [6], [9], [12], [10], [11] where it was shown that the rate of convergence of \( sk \)-splines has the same order as the respective \( n \)-widths.

The main aim of this article is to establish representations of cardinal \( sk \)-splines on a uniform mesh in \( \mathbb{R}^n \) and to apply these results to the problem of recovery of density functions which are important in the theory of pricing.

## 2 Interpolation by \( sk \)-splines on \( \mathbb{R}^n \)

Let \( a = (a_1, \cdots, a_n), a_k > 0, 1 \leq k \leq n \) be a fixed mesh parameter \( m = (m_1, \cdots, m_n) \in \mathbb{Z}^n \) and

\[
\Omega_a := \{(a_1m_1, \cdots, a_nm_n) | m \in \mathbb{Z}^n\} \subset \mathbb{R}^n
\]

be a mesh in \( \mathbb{R}^n \). Let \( A := \text{diag}(a_1, \cdots, a_n) \), then the mesh points are \( x_m := Am^T \). For a fixed continuous kernel function \( K \), the space \( SK(\Omega_a) \) of \( sk \)-splines on \( \Omega_a \) is the space of functions representable in the form

\[
sk(x) = \sum_{m \in \mathbb{Z}^n} c_m K(x - x_m),
\]

where \( c_m \in \mathbb{R} \). Let \( f(x) \) be a continuous function, \( f : \mathbb{R}^n \to \mathbb{R} \). Consider the problem of interpolation

\[
sk(x_m) = f(x_m),
\]
where
\[ s_k(x_m) = \sum_{m \in \mathbb{Z}^n} c_m^* K(x - x_m), c_m^* \in \mathbb{R}. \]

Even in the one-dimensional case the problem of interpolation not always has a solution. If the solution exists then \( s_k \)-spline interpolant can be written in the form
\[ s_k(x) = \sum_{m \in \mathbb{Z}^n} f(x_m) \tilde{s}_k(x - x_m), \]
where \( \tilde{s}_k(x - x_m) \) are fundamental \( s_k \)-splines, i.e.
\[ \tilde{s}_k(x_m) = \begin{cases} 1, & m = 0, \\ 0, & m \neq 0. \end{cases} \]

**Theorem 2** Let \( K : \mathbb{R}^n \to \mathbb{R} \) be such that \( K \in L_2(\mathbb{R}^n) \cap C(\mathbb{R}^n), \)
\[ \sum_{m \in \mathbb{Z}^n} F(K)(z + 2\pi A^{-1}m^T) \neq 0, \forall z \in 2\pi Q_a, \]
where \( Q_a := \{ x | x = (x_1, \cdots, x_n) \in \mathbb{R}^n, 0 \leq x_k \leq 1/a_k, 1 \leq k \leq n \}, \) and the function
\[ \frac{1}{\sum_{m \in \mathbb{Z}^n} F(K)(z + 2\pi A^{-1}m^T)} \]
can be represented by its Fourier series, i.e. for any \( z \in \mathbb{R}^n, \)
\[ \frac{1}{\sum_{m \in \mathbb{Z}^n} F(K)(z + 2\pi A^{-1}m^T)} = \sum_{s \in \mathbb{Z}^n} \alpha_s \exp \left( -i \langle As^T, z \rangle \right) \]
then
\[ \tilde{s}_k(x) = \frac{\det(A)}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{F(K)(z)}{\sum_{m \in \mathbb{Z}^n} F(K)(z + 2\pi A^{-1}m^T)} \exp(i \langle z, x \rangle) \, dz \]
and this representation is unique.

**Proof** Since
\[ \frac{1}{\sum_{m \in \mathbb{Z}^n} F(K)(z + 2\pi A^{-1}m^T)} = \sum_{s \in \mathbb{Z}^n} \alpha_s \exp \left( -i \langle As^T, z \rangle \right) \]
then
\[ \tilde{s}_k(x) = \frac{\det(A)}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{F(K)(z)}{\sum_{m \in \mathbb{Z}^n} F(K)(z + 2\pi A^{-1}m^T)} \exp(i \langle x, z \rangle) \, dz \]
\[ \det(A) \int_{\mathbb{R}^n} F(K)(z) \left( \sum_{s \in \mathbb{Z}^n} \alpha_s \exp(-i \langle As^T, z \rangle) \right) \exp(i \langle x, z \rangle) \, dz \]
\[ = \det(A) \sum_{s \in \mathbb{Z}^n} \alpha_s \int_{\mathbb{R}^n} F(K)(z) \exp(i \langle x - As^T, z \rangle) \, dz. \]

Since \( K \in L_2(\mathbb{R}^n) \) then by Plancherel's theorem
\[ \widetilde{s}(x) = \det(A) \sum_{s \in \mathbb{Z}^n} \alpha_s K(x - As^T), \]
so that \( \widetilde{s}(x) \in SK(\Omega_a) \). Hence
\[ \widetilde{s}(Am^T) = \det(A) \int_{\mathbb{R}^n} \frac{F(K)(z)}{\sum_{m \in \mathbb{Z}^n} F(K)(z + 2\pi A^{-1}m^T)} \exp(i \langle z, Am^T \rangle) \, dz \]
\[ = \det(A) \sum_{l \in \mathbb{Z}^n} \int_{2\pi A^{-1}l + 2\pi Q_a} \frac{F(K)(z)}{\sum_{m \in \mathbb{Z}^n} F(K)(z + 2\pi A^{-1}m^T)} \exp(i \langle z + 2\pi A^{-1}l, Am^T \rangle) \, dz \]
\[ = \det(A) \sum_{l \in \mathbb{Z}^n} \int_{2\pi Q_a} \frac{F(K)(z + 2\pi A^{-1}l^T)}{\sum_{m \in \mathbb{Z}^n} F(K)(z + 2\pi A^{-1}m^T)} \exp(i \langle z + 2\pi A^{-1}l, Am^T \rangle) \, dz \]
\[ = \det(A) \int_{2\pi Q_a} \exp(i \langle z, Am^T \rangle + 2\pi i \langle l, m \rangle) \, dz \]
\[ = \det(A) \int_{2\pi Q_a} \exp(i \sum_{k=1}^n a_k m_k z_k) \, dz \]

8
\[ \det(A) \prod_{k=1}^{n} \int_{0}^{2\pi/a_k} \exp(ia_k m_k z_k) dz_k \]

\[ = \begin{cases} 1, & m_k = 0, 1 \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases} \]

Finally, we need to show that the representation of fundamental \( s_k \)-spline is unique. It is sufficient to show that the functions \( \tilde{s}_k(x - x_m), x_m \in \Omega_a \) are linearly independent. Let \( a_m \in \mathbb{R}, m \in \mathbb{Z}^n \) be such that not all \( a_m \) are zero. Let, in particular, \( a_s \neq 0 \) for some \( s \in \mathbb{Z}^n \). Consider a linear operator,

\[ A : C(\mathbb{R}^n) \rightarrow \mathbb{R} \\
\quad f(\cdot) \mapsto f(x_s). \]

Assume that \( \sum_{m \in \mathbb{Z}^n} a_m \tilde{s}_k(x - x_m) \equiv 0 \) then

\[ 0 = A \left( \sum_{m \in \mathbb{Z}^n} a_m \tilde{s}_k(x - x_m) \right) = a_s A \left( \tilde{s}_k(x - x_s) \right) = a_s, \]

which is a contradiction. \( \square \)

**Theorem 3** In the assumptions of Theorem 2

\[ p_t^Q(\cdot) \approx \left( \frac{\det(A) F(K)(\cdot)}{(2\pi)^n \sum_{m \in \mathbb{Z}^n} F(K)(\cdot + 2\pi A^{-1} m^T)} \right) \left( \sum_{m \in \mathbb{Z}^n} \Phi^Q(x_m, t) \exp(i \langle \cdot, x_m \rangle)(\cdot) \right), \]

as \( \max\{a_k, 1 \leq k \leq n\} \rightarrow 0 \).

**Proof** Observe that

\[ \tilde{s}_k(x) = \frac{\det(A)}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{F(K)(z)}{\sum_{m \in \mathbb{Z}^n} F(K)(z + 2\pi A^{-1} m^T)} \exp(i \langle z, x \rangle) dz \]

\[ = \det(A) F^{-1} \left( \frac{F(K)(z)}{\sum_{m \in \mathbb{Z}^n} F(K)(z + 2\pi A^{-1} m^T)} \right)(x), \forall x_s \in \Omega_a. \]

Hence

\[ p_t^Q(\cdot) = (2\pi)^{-n} F \left( \Phi^Q(x,t) \right)(\cdot) \]

\[ \approx (2\pi)^{-n} F \left( \sum_{m \in \mathbb{Z}^n} \Phi^Q(x_m, t) \tilde{s}_k(x - x_m) \right)(\cdot) \]

9
\[
\begin{align*}
&= \frac{\det (A)}{(2\pi)^n} \mathbf{F} \left( \sum_{m \in \mathbb{Z}^n} \Phi^Q (x_m, t) \mathbf{F}^{-1} \left( \frac{\mathbf{F} (K) (z)}{\sum_{m \in \mathbb{Z}^n} \mathbf{F} (K) (z + 2\pi A^{-1} m^T)} \right) (x - x_m) \right) \\
&= \frac{\det (A)}{(2\pi)^n} \left( \sum_{m \in \mathbb{Z}^n} \mathbf{F} (K) (z) \right) \left( \sum_{m \in \mathbb{Z}^n} \Phi^Q (x_m, t) \exp (i \langle \cdot, x_m \rangle) \right),
\end{align*}
\]

since
\[
\mathbf{F} (K) (z) \sum_{m \in \mathbb{Z}^n} \mathbf{F} (K) (z + 2\pi A^{-1} m^T) \in L_2 (\mathbb{R}^n).
\]

\[\square\]

**Example 1** Let \( K (x) \) be a Gaussian of the form

\[ K (x) = K (x_1, \ldots, x_n) = \exp \left( -\sum_{k=1}^{n} b_k x_k^2 \right) \]

and \( B = \text{diag} (b_1, \ldots, b_n) \), then

\[ \mathbf{F} (K) (z) = \pi^{n/2} \left( \det (B) \right)^{-1/2} \exp \left( -\sum_{k=1}^{n} \frac{z_k^2}{4b_k} \right). \]

Applying Poisson summation formula and Plancherel’s theorem we get

\[
\begin{align*}
&\sum_{m \in \mathbb{Z}^n} \mathbf{F} (K) (z + 2\pi A^{-1} m^T) \\
&= \sum_{m \in \mathbb{Z}^n} \prod_{k=1}^{n} \left( \mathbf{F} \exp \left( -b_k y_k^2 \right) \right) \left( z_k + \frac{2\pi m_k}{a_k} \right) \\
&= \prod_{k=1}^{n} \sum_{m_k \in \mathbb{Z}} \left( \mathbf{F} \exp \left( -b_k y_k^2 \right) \right) \left( z_k + \frac{2\pi m_k}{a_k} \right) \\
&= \prod_{k=1}^{n} \left( \frac{a_k}{2\pi} \right) \sum_{m_k \in \mathbb{Z}} \exp \left( i a_k m_k z_k \right) \mathbf{F} \circ \mathbf{F} \left( \exp \left( -b_k \left( \frac{a_k m_k}{2\pi} \right)^2 \right) \right) \\
&= \prod_{k=1}^{n} \left( \frac{a_k}{2\pi} \right) (2\pi) \sum_{m_k \in \mathbb{Z}} \exp \left( i a_k m_k z_k \right) \exp \left( -b_k \left( \frac{a_k m_k}{2\pi} \right)^2 \right) \\
&= \det (A) \prod_{k=1}^{n} \sum_{m_k \in \mathbb{Z}} \exp \left( i a_k m_k z_k \right) \exp \left( -b_k \left( \frac{a_k m_k}{2\pi} \right)^2 \right).
\end{align*}
\]
Hence, in this case

\[
\frac{\det(A) \mathbf{F}(K)(z)}{(2\pi)^n \sum_{m \in \mathbb{Z}^n} \mathbf{F}(K)(z + 2\pi A^{-1}m^T)}
\]

\[
= \frac{\pi^{n/2} (\det(B))^{-1/2} \exp \left(-\sum_{k=1}^{n} \frac{z_k^2}{b_k} \right)}{(2\pi)^n \prod_{k=1}^{n} \sum_{m_k \in \mathbb{Z}} \exp(i a_k m_k z_k) \exp \left(-b_k \left(\frac{a_k m_k}{2\pi} \right)^2 \right)}.
\]

References

[1] Deng, S., Stochastic models of energy commodity prices and their applications: mean reversion with jumps and spikes. Working paper, Georgia Institute of Technology, October 1999.

[2] Duan, J. C., Pliska, S. R., Option valuation with co-integrated asset prices. Working paper, Department of Finance, Hong Kong University of Science and Technology, January 1999.

[3] Golomb, M., Approximation by periodic spline interpolants on uniform meshes, *J. Approx Theory* 1 (1968), 26–65.

[4] Gomes, S. M., Kushpel, A. K., Levesley, J., Ragozin, D. L. Interpolation on the Torus using sk-Splines with Number Theoretic Knots, *J. of Approx. Theory*, 98, 1999, 56–71.

[5] Kushpel, A. K., Extremal properties of splines and diameters of classes of periodic functions in the space \(C_{2\pi}\), PREPRINT, 84.15, Kiev, Inst. Math. Acad. Nauk Ukrain. SSR, 1984, 1-44.

[6] Kushpel, A. K., Rate of convergence of the interpolation sk-splines on classes of convolutions, In: *Investigations in Approximation Theory*, Inst. Math. Acad. Nauk. Ukrain. SSR, Kiev, (1987), 50–58.

[7] Kushpel, A. K., Sharp Estimates of the Widths of Convolution Classes, *Math. USSR Izvestiya, American Mathematical Society*, 33, 3, (1989), 631–649.
[8] Kushpel, A. K., Levesley, J., Light, W. Approximation of smooth functions by $sk$-splines, In: *Advanced Topics in Multivariate Approximation*, F. Fontanella, K. Jetter and P.-J. Laurant (eds), World Scientific Publishing, (1996), 155–180.

[9] Kushpel, A. K., Grandison, C. J., Ha, D. M., Optimal $sk$-Spline Approximation and Reconstruction on the Torus and Sphere, *International Journal of Pure and Applied Mathematics*, 29, 2, (2006), 469–490.

[10] Kushpel, A. K., Convergence of $sk$-Splines in $L_q – I$, *International Journal of Pure and Applied Mathematics*, 45, 1, (2008), 87–101.

[11] Kushpel, A. K., Convergence of $sk$-Splines in $L_q – II$, *International Journal of Pure and Applied Mathematics*, 45, 1, (2008), 103–119.

[12] Levesley, J., Kushpel, A. K., Generalised $sk$-Spline Interpolation on Compact Abelian Groups, *J. of Approx. Theory*, 97, (1999), 311-333.

[13] Mbanefo, A. Co-movement term structure and the valuation of crack energy spread options. In *Mathematics of Derivatives Securities*. M. A. H. Dempster and S. R. Pliska, eds. Cambridge University Press, 88-102, 1997.

[14] Pilipovic, D., Wengler, J., Basis for Boptions. Energy and Power risk Management, December 1998, 28-29.

[15] Shimko, D. C., Options on futures spreads: hedging, speculation, and valuation. *The Journal of Futures Markets*, 14, 2, 183-213. 1709–1718.