Families of $K3$ surfaces and curves of $(2,3)$-torus type

Makiko Mase

Abstract

We study families of $K3$ surfaces obtained by double covering of the projective plane branching along curves of $(2,3)$-torus type. In the first part, we study the Picard lattices of the families, and a lattice duality of them. In the second part, we describe a deformation of singularities of Gorenstein $K3$ surfaces in these families.

Running head. Families of $K3$ surfaces and curves of $(2,3)$-torus type

1 Introduction

Plane sextic curves of $(2,3)$-torus type, which is defined by a polynomial of the form $F = F_3^2 + F_3^2$ with polynomials $F_i$ of homogeneous degree $i$, that have at most simple singularities, which we simply call curves of $(2,3)$-torus type, are classified by Oka and Pho [16], and Pho [17].

A compact complex connected algebraic surface $S$ is called a Gorenstein $K3$ surface if $S$ has at most simple singularities, has trivial canonical divisor, and the irregularity of $S$ is zero. If a Gorenstein $K3$ surface is non-singular, we simply call it a $K3$ surface. Note that for any Gorenstein $K3$ surface $S$, there exists a $K3$ surface $\tilde{S}$ such that $S$ and $\tilde{S}$ are birationally equivalent.

Concerning algebraic curves on a $K3$ surface, the study of their Weierstrass semi-group and the existence of curves that admit a given semi-group are quite authentic and interesting but there are not many known results: see for instance Komeda and Watanabe [12]. On the other hand, motivated originally in theoretical physics, many concepts have been discovered related to mirror symmetry for 2-dimensional Calabi-Yau manifolds such as by Batyrev, Berglund and Hübsch, Dolgachev, Ebeling and Takahashi, and Ebeling and Ploog [2, 3, 6, 8, 7].

In this article, we are to focus on families of $K3$ surfaces that are obtained as double covering of the projective plane branching along curves of $(2,3)$-torus type, and their Picard lattices.

Consider the generic non-Galois triple covering $X$ of $\mathbb{P}^2$ branching along a curve of $(2,3)$-torus type $B$, and then take the Galois closure $\tilde{X}$ of $X$. Moreover, if $Z$ is the minimal model of the double covering $D(X/\tilde{X})$.

---

*Tokyo Metropolitan University and University of Mannheim, mtmase@arion.ocn.ne.jp

†Keywords and phrases. families of $K3$ surfaces that are double cover of the projective plane, curves of $(2,3)$-torus type, duality of Picard lattices, non-Galois triple covering of the projective plane

‡2010 MSC numbers. Primary 14J28; Secondary 14J17, 14J33, 14H30.
two cases are distinguished by an invariant $\delta_K$ surface (possibly with singularities), or a Gorenstein covering $\hat{\mathcal{C}}$ covering of $Z_P$, case, in which one has $\delta = 9$ is in fact studied in detail by Barth [1] as an analogy of Nikulin’s result [14]. It is easily seen by the classification [16, 17], that $\delta = 9$ occurs if and only if $\text{Sing} B = 9A_2$; and together with the fact that the order of the fundamental group of the singularities should be divided by 3, that $\delta = 6$ occurs when $\text{Sing} B$ is one of the followings:

$$
\begin{align*}
A_1; & 2A_4, 3A_5, A_2 + A_{14}, A_7 + A_8, \quad A_5 + E_6, \\
3E_6, & 6A_2, A_2 + A_{11}, \quad A_2 + A_8 + A_8, \quad 3A_2 + A_8, \\
2A_2 + 2A_5, & 4A_6 + A_5, \quad 2A_2 + E_6, \quad A_5 + 2E_6, \\
2A_2 + A_4 + E_6, & 2A_2 + 2E_6, \quad A_2 + A_8 + E_6, \quad 4A_2 + E_6.
\end{align*}
$$

Remark 1 The above list of $\text{Sing} B$ picked up in [17] covers all possible cases: indeed, with the aid of Proposition 1.1 [10].

The dual curve of a plane smooth cubic curve is a typical example (see [14]) of a curve of $(2,3)$-torus type $B$ with nine singularities of type $A_2$ (cusps).

Since $\delta = 9$ case is well-understood, we focus on the other case in this article.

The defining equation of the Gorenstein $K3$ surface $D(X/\mathbb{P}^2)$ is given by

$$W^2 - F(X, Y, Z) = 0,$$

where $F$ is the defining polynomial of the branch curve $B$. Being parameterised by the coefficients of the monomials in $W^2 - F(X, Y, Z)$, one can construct families of (Gorenstein) $K3$ surfaces. Such a family should be a subfamily of $K3$ surfaces parametrised by the complete anticanonical linear system of the weighted projective space $\mathbb{P}(1,1,1,3)$ with weights $(1,1,1,3)$, which is one of 95 weights corresponding to simple $K3$ singularities classified by Yonemura [18].

The aim of this article is to study these families: we first construct polytopes $\Delta_1, \Delta_2, \Delta_3$ such that the complete anticanonical linear systems of toric $K3$ surfaces associated to them parametrise families $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ of $K3$ surfaces obtained by a double covering of $\mathbb{P}^2$ branching along curves of $(2,3)$-torus type. Then, we shall give the Picard lattice of the families using toric geometry. More precisely, our main theorem is stated as follows:

**Theorem 3.1** (1) The Picard lattices of the families $\mathcal{F}_1$ and $\mathcal{F}_2$ are respectively isometric to $U \oplus (-2) \oplus (-4)$, and $U \oplus A_5$.

(2) The Picard lattice of the family $\mathcal{F}_3$ is isometric to $(-2) \oplus (2)$.

(3) The Picard lattice $\text{Pic}_{\Delta_3}$ of the family $\mathcal{F}_3$ satisfies the duality

$$U \oplus \text{Pic}_{\Delta_3} \simeq (\text{Pic}_{\Delta_3})^* \Lambda_{K3}^\vee,$$

where $\Delta_3^\vee$ is the polar dual of $\Delta_3$, $U$ is the hyperbolic lattice of rank 2, and $(L)_{K3}^\vee$ is the orthogonal complement of a primitive sublattice $L$ in the $K3$ lattice $\Lambda_{K3}$. 


We review fundamental facts of toric geometry necessary in this article in §2. The main theorem is proved in §3 after verifying invariants. In §4, we describe families that contain K3 surfaces obtained as double covering of \( \mathbb{P}^2 \) branching along curves of (2, 3)-torus type.

For a curve \( B \) of (2, 3)-torus type, when \( \text{Sing}(B) = A \) is the set of singularities of \( B \), we call \([A]\) for the curve \( B \). If \( \text{Sing}(B) \) contains more than one singularities, we denote by \( A = A' + A'' \). The singularities of type \( A_n \) is the singularity given locally by \( x^2 + y^{n+1} = 0 \) for \( n \geq 1 \), and of type \( E_6 \) is given locally by \( x^3 + y^4 = 0 \).

In an unnecessarily confusing way, we also mean by \( A_n \) the root lattice of type \( A_n \). The hyperbolic lattice of rank 2 is denoted by \( U \), and negative definite root lattice of rank 8 is denoted by \( E_8 \). The K3 lattice is defined by \( \Lambda_{K3} := U \oplus 3 \oplus E_8^2 \).

Acknowledgement. The author thanks to Professor J.Komeda who gave her an opportunity to study Weierstrass semi-groups, and hopes that the result may enrich the study of that of algebraic curves on K3 surfaces. Thanks to Professor C.Hertling for reading through the first draft.

2 Setups

A lattice is a finitely-generated \( \mathbb{Z} \)-module with a non-degenerate bilinear form. Let \( M \) be a lattice of rank 3, and \( N := \text{Hom}_\mathbb{Z}(M, \mathbb{Z}) \) be its dual. See Section 3.3 of [9] for basic facts on toric geometry.

A polytope is a convex hull of finitely-many points, say, \( v_1, \ldots, v_r \), in \( M \otimes \mathbb{R} \), and is denoted by \( \Delta = \text{Conv}\{v_1, \ldots, v_r\} \).

For a polytope \( \Delta \), a vertex is a 0-dimensional face, an edge is a 1-dimensional face, and a face is a 2-dimensional face. A polytope is called integral if every vertex is in \( M \).

Denote by \( \langle \cdot, \cdot \rangle : N \times M \to \mathbb{Z} \) a natural pairing, which is the inner product in \( \mathbb{R}^3 \). Let \( \Delta \) be a polytope in \( M \otimes \mathbb{R} \). Define the polar dual polytope of \( \Delta \) by \( \Delta^* := \{ y \in N \otimes \mathbb{R} | \langle y, x \rangle \geq -1 \text{ for all } x \in \Delta \} \).

An integral polytope \( \Delta \) is reflexive if \( \Delta \) contains the origin in the interior as its only lattice point, and the polar dual \( \Delta^* \) is also integral.

It is known [2] that a polytope \( \Delta \) is reflexive if and only if the corresponding toric variety \( \mathbb{P}_\Delta \) is Fano, in particular, the complete anticanonical linear system of \( \mathbb{P}_\Delta \) parametrises a family \( \mathcal{F}_\Delta \) of which general sections are Gorenstein K3 surfaces. Since every Gorenstein K3 surface is birationally equivalent to a unique K3 surface, we identify these two surfaces.

Suppose \( \Delta \) is a reflexive polytope, and take a general anticanonical section \( Z \) of \( \mathbb{P}_\Delta \).

It is also known [2] that there exists a MPCP desingularization, which is achieved by a simultaneous toric desingularization, and in consequence, we obtain the resulting desingularized varieties \( \mathbb{P}_\Delta \) and \( Z \) of the ambient space \( \mathbb{P}_\Delta \) and of \( Z \), respectively.

Denote by \( \mathbb{P}_\Delta \) and \( Z \) the resulting desingularized varieties. Define a restriction map \( r : H^1(\mathbb{P}_\Delta) \to H^1(Z) \) of Hodge (1,1)-parts. Note that the map \( r \) is not necessarily surjective. Define

\[
L_D(\Delta) := \text{Im}\, r \cap H^2(\mathbb{Z}, \mathbb{Z}), \quad L_0(\Delta) := (L_D(\Delta))^\perp_{H^2(\mathbb{Z}, \mathbb{Z})}.
\]
Then,
\[ L_0(\Delta) = \text{coker } r \cap H^2(\tilde{Z}, \mathbb{Z}). \]
We call the rank of \( L_0(\Delta) \) the toric correction term.

Denote by \( v_i, i = 1, \ldots, r \) the vectors starting from the origin and ending at vertices \( e_i \) of \( \Delta \). The vectors \( v_i \) being as one-simplices, one can construct a fan \( \Sigma = \Sigma(\Delta) \) associated to the polytope \( \Delta \). It is easily observed that rank \( L_0(\Delta) = 0 \) if and only if \( \Delta \) is simplicial, namely, any three of one-simplices of \( \Sigma \) generate \( M \), and it is also equivalent that the toric variety \( P_\Delta \) has at most orbifold singularities.

Define the Picard lattice of the family \( F_\Delta \) to be the Picard lattice of \( K3 \) surfaces that are minimal models of any generic sections in \( F_\Delta \), which is known to be well-defined [5]. In other words, the Picard lattice of the family is a lattice generated by the irreducible components of the restrictions to \( -K_{P_\Delta} \) of generators, which are torus-action invariant, of the Picard group of the toric 3-fold \( P_\Delta \). We denote it by \( \text{Pic}_\Delta \) and \( \rho_\Delta \) be its rank.

The toric correction term rank \( L_0(\Delta) \), \( \rho_\Delta \), and intersection numbers \( D \cdot D' \) for restricted torus-invariant divisors \( D, D' \) in \( \text{Pic}_\Delta \) can be combinatorically computed [11].

For a face \( \Gamma \) of any dimensional of \( \Delta \), denote by \( l(\Gamma) \) the number of lattice points in \( \Gamma \), and \( l^*(\Gamma) \) the number of inner lattice points in \( \Gamma \).

For an edge \( \Gamma \) of \( \Delta \), denote by \( \Gamma^* \) the dual edge of \( \Gamma \) in \( \Delta^* \). The toric correction term is computed by
\[ \text{rank } L_0(\Delta) = \sum_{\Gamma \text{ edge of } \Delta} l^*(\Gamma)l^*(\Gamma^*). \]  
(1)

The rank \( \rho_\Delta \) is computed by
\[ \rho_\Delta = \sum_{\Gamma \text{ edges of } \Delta} l(\Gamma^*) + \text{rank } L_0(\Delta) - 3. \]  
(2)

A primitive vector \( v \) that generates a ray of the fan defining the 3-fold \( P_\Delta \) determines a torus-invariant divisor \( \text{Orb}(\mathbb{R}_{\geq 0}v) \) on \( P_\Delta \). By construction, it is equivalent to take a lattice point \( v^* \) in \( \Delta^* \). The dual of \( v^* \) is a face \( F \) in \( \Delta \). It is well-known that the number of lattice points in the interior of \( F \) is equal to the genus of a smooth curve corresponding to the divisor. For \( D := \text{Orb}(\mathbb{R}_{\geq 0}v)|_{-K_{P_\Delta}} \), one has
\[ D \cdot D = D^2 = 2l^*(F) - 2. \]  
(3)

Let \( D \) and \( D' \) be the restriction to \( -K_{P_\Delta} \) of torus-invariant divisors on \( P_\Delta \) corresponding to vertices \( v \) and \( v' \), and let \( \Gamma \) be the edge of \( \Delta \) which connects the vertices \( v \) and \( v' \). The intersection number is thus obtained by
\[ D \cdot D' = l^*(\Gamma) + 1. \]  
(4)

3 Main Results

Define a lattice \( M \) by
\[ M := \{(a_0, a_1, a_2, a_3) \in \mathbb{Z}^4 \mid a_0 + a_1 + a_2 + 3a_3 \equiv 0(6)\}. \]
The lattice \( M \) is of rank 3, and one can take a basis \( \{e_1, e_2, e_3\} \) for \( M \) over \( \mathbb{Z} \), where
\[ e_1 = (1, 0, -1, 0), \quad e_2 = (0, 1, -1, 0), \quad e_3 = (0, 0, -3, 1). \]
We can associate $M$ with the set of monomials $\mathcal{M}_6$ of weighted degree 6 with weights $(1, 1, 1, 3)$ by

$$\begin{align*}
M &\rightarrow \mathcal{M}_6 \\
(a_0, a_1, a_2, a_3) &\rightarrow X^{a_0+1}Y^{a_1+1}Z^{a_2+1}W^{a_3+1},
\end{align*}$$

where the weights of $X, Y, Z, W$ are respectively $1, 1, 1, 3$, and thus there exists a correspondence between lattice points in $M$ and such monomials.

One can embed a polytope of which lattice points are labelled by monomials in $\mathcal{M}_6$ into $\mathbb{R}^3$: express elements in $M$ as a linear combination of the chosen basis $\{e_1, e_2, e_3\}$ of which the coefficients form a point in $\mathbb{R}^3$. Thus, one gets a correspondence between monomials and points in $\mathbb{R}^3$ under this choice of a basis, for which some examples are presented in Table 1.

Table 1: Correspondence of monomials and points in $M$.

| Monomial | Point |
|----------|-------|
| $W^2$ | $(-1, -1, 1)$ |
| $Y^6$ | $(-1, 5, -1)$ |
| $X^6$ | $(5, -1, -1)$ |
| $Z^6$ | $(-1, -1, -1)$ |
| $Y^2Z^3$ | $(-1, 2, -1)$ |
| $Y^2Z^4$ | $(-1, 1, -1)$ |
| $X^3Z^2$ | $(3, -1, -1)$ |
| $X^2Z^4$ | $(1, -1, -1)$ |

Proposition 3.1 The polytopes $\Delta_1, \Delta_2, \text{ and } \Delta_3$ are reflexive.

Proof. It is clear that the origin is the only lattice point contained in the polytopes. By a direct computation, the polar dual polytopes $\Delta_1^*, \Delta_2^*, \text{ and } \Delta_3^*$ of $\Delta_1, \Delta_2, \text{ and } \Delta_3$ are as follows:

$$\begin{align*}
\Delta_1^* &:= \text{Conv } \{(0, 0, 1), (-1, -1, -3), (0, 1, 0), (1, 2, 2), (1, 0, 0)\}, \\
\Delta_2^* &:= \text{Conv } \{(0, 0, 1), (-1, -1, -3), (0, 1, 0), (3, 4, 6), (1, 0, 0)\}, \\
\Delta_3^* &:= \text{Conv } \{(0, 0, 1), (-1, -1, -3), (0, 1, 0), (1, 1, 1), (1, 0, 0)\},
\end{align*}$$

respectively. Thus $\Delta_1^*, \Delta_2^*, \Delta_3^*$ are integral as well. $\Box$

Proposition 3.2 We have $\text{rank } L_0(\Delta_1) = 1$, $\text{rank } L_0(\Delta_2) = 2$, and $\text{rank } L_0(\Delta_3) = 0$, and $\rho_{\Delta_1} = 4$, $\rho_{\Delta_2} = 7$, $\rho_{\Delta_3} = 2$, $\rho_{\Delta_1^*} = 17$, $\rho_{\Delta_2^*} = 15$, and $\rho_{\Delta_3^*} = 18$. In particular, $\rho_{\Delta_i} + \rho_{\Delta_i^*} > 20$ for $i = 1, 2$, and $\rho_{\Delta_3} + \rho_{\Delta_3^*} = 20$ hold.
Figure 1: Polytopes $\Delta_1$, $\Delta_2$, $\Delta_3$, $\Delta_1^*$, $\Delta_2^*$ and $\Delta_3^*$. 

Proof. There exists a lattice point on the edge

$$\Gamma_1 = \text{Conv}\{(-1, -1, 1), (-1, 1, -1)\}$$

of $\Delta_1$, and a lattice point on its dual edge

$$\Gamma_1^* = \text{Conv}\{(1, 0, 0), (1, 2, 2)\}$$

of $\Delta_1^*$. There is no more edge on $\Delta_1$ that contributes rank $L_0(\Delta_1)$. Thus, by the formula (1),

$$\text{rank } L_0(\Delta_1) = l'(\Gamma_1)l'(\Gamma_1^*) = 1 \cdot 1 = 1.$$ 

By the formula (2), one has $\rho_{\Delta_1} = (5+1)+1-3 = 4$, and $\rho_{\Delta_1^*} = 19+1-3 = 17$. Clearly, $\rho_{\Delta_1} + \rho_{\Delta_1^*} = 4 + 17 = 21 > 20$. 


There exists a lattice point on the edge
\[ \Gamma_2 = \text{Conv}\{(-1, -1, 1), (3, -1, -1)\} \]
of \( \Delta_2 \), and two lattice points on its dual edge
\[ \Gamma_2^* = \text{Conv}\{(3, 4, 6), (0, 1, 0)\} \]
of \( \Delta_2^* \). There is no more edge on \( \Delta_2 \) that contributes rank \( L_0(\Delta_2) \). Thus, by the formula (1),
\[ \text{rank} \ L_0(\Delta_2) = l^*(\Gamma_2) l^*(\Gamma_2^*) = 1 \cdot 2 = 2. \]

By the formula (2), one has
\[ \rho_{\Delta_2} = 5 + 3 + 2 - 3 = 7, \]
\[ \rho_{\Delta_2^*} = 21 - 3 = 18. \]
Clearly,
\[ \rho_{\Delta_2} + \rho_{\Delta_2^*} = 7 + 18 = 25 > 20. \]

There does not exist an edge on \( \Delta_3 \) that contributes rank \( L_0(\Delta_3) \). Thus, by the formula (1),
\[ \text{rank} \ L_0(\Delta_3) = 0. \]
By the formula (2), one has
\[ \rho_{\Delta_3} = 5 - 3 = 2, \]
\[ \rho_{\Delta_3^*} = 21 - 3 = 18. \]
Clearly,
\[ \rho_{\Delta_3} + \rho_{\Delta_3^*} = 2 + 18 = 20. \]
\[ \square \]

Denote by \( F_1, F_2, F_3, \) and \( F_3' \) the families of \( K3 \) surfaces parametrised by the complete anticanonical linear systems of toric Fano 3-folds \( P_{\Delta_1}, P_{\Delta_2}, P_{\Delta_3}, \) and \( P_{\Delta_3^*} \), respectively. Here, \( \Delta_3^* \) is the polar dual polytope of \( \Delta_3 \). Denote by \( \text{Pic}_\Delta \) the Picard lattice of the family of \( K3 \) surfaces that is associated to a reflexive polytope \( \Delta \).

**Remark 2** We have seen that the sum of Picard numbers \( \rho_{\Delta_3} \) and \( \rho_{\Delta_3^*} \) coincides with the rank of the unimodular lattice \( U^\oplus 2 \oplus E_8^\oplus 2 \), and that the rank of \( L_0(\Delta_3) \) is 0 means that the toric 3-fold \( P_{\Delta_3} \) is simplicial. In [13], it is concluded that if a toric Fano 3-fold \( P_\Delta \) is simplicial, then, the family \( F_\Delta \) of \( K3 \) surfaces is lattice dual in the sense that
\[ (\text{Pic}_\Delta)^\perp_{\Lambda_{K3}} \simeq U \oplus \text{Pic}_\Delta. \]
holds. Thus, in our situation here, we expect that the family \( F_3 \) constructed by the toric 3-fold \( P_\Delta \) might be lattice dual. Therefore, we have to study the dual \( \Delta_3^* \), or equivalently, the family \( F_3' \) constructed by the toric 3-fold \( P_{\Delta_3^*} \).

**Remark 3** Let \( \Delta \) be a reflexive polytope. We occasionally identify the complete anticanonical linear system of the toric Fano 3-fold \( P_\Delta \), and the family \( F_\Delta \). Indeed, a section \( f \in |\mathcal{L}_{\Delta_3}| \) determines a surface \( (f = 0) \) in \( F_\Delta \). Thus, we may also call \( (f = 0) \) a section as long as there is no confusion.

**Theorem 3.1.** (1) The Picard lattices of the families \( F_1 \) and \( F_2 \) are respectively isometric to \( U \oplus (-2) \oplus (-4) \), and \( U \oplus A_5 \).
(2) The Picard lattice of the family \( F_3 \) is isometric to \( (-2) \oplus (2) \).
(3) The Picard lattice \( \text{Pic}_{\Delta_3} \) of the family \( F_3 \) satisfies the duality
\[ U \oplus \text{Pic}_{\Delta_3}^* \simeq (\text{Pic}_{\Delta_3})^\perp_{\Lambda_{K3}}, \]
where \( \Delta_3^* \) is the polar dual of \( \Delta_3 \), \( U \) is the hyperbolic lattice of rank 2, and \((L)^\perp_{\Lambda_{K3}} \) is the orthogonal complement of a primitive sublattice \( L \) in the \( K3 \) lattice \( \Lambda_{K3} \).
Proof. (1) We label the primitive vectors that generate rays of the fan defining the 3-fold $\mathbb{P}_{\Delta_1}$, or equivalently, the lattice points in $\Delta_1^*$ as follows:

$v_1 = (0, 0, 1), \quad v_2 = (-1, -1, -3), \quad v_3 = (0, 1, 0), \quad v_4 = (1, 2, 2), \quad v_5 = (1, 0, 0), \quad v_6 = (1, 1, 1)$.

Let $D_i := \text{Orb}(\mathbb{R}_{\geq 0}v_i)|_{-K_{\mathbb{P}_{\Delta_1}}}$ for $i = 1, \ldots, 5$ be restricted torus-invariant divisors, and $D_6, D_7$ be components of the divisor $\text{Orb}(\mathbb{R}_{\geq 0}v_6)|_{-K_{\mathbb{P}_{\Delta_1}}}$.

One computes the self-intersection numbers by the formula (3),

- $D_1^2 = 2 \cdot 8 - 2 = 14,$
- $D_2^2 = 2 \cdot 2 - 2 = 2,$
- $D_3^2 = 2 \cdot 0 - 2 = -2,$
- $D_4^2 = 2 \cdot 1 - 2 = 0,$
- $D_5 = D_7 = -2.$

One also has the graph of intersections among these divisors by the formula (4) as in Figure 2.

![Figure 2: Nodes are divisors on a general section in $\mathcal{F}_1$. There are $n$ lines connecting $D_i$ and $D_j$ if $D_i \cdot D_j = n$.](image)

By solving the linear system

$$\sum_{i=1}^{7} (v_i, e_j)D_i = 0,$$

for $j = 1, 2, 3$, where $e_j$ is the $j$-th column of the identity matrix of size 3, and $(x, y)$ is the inner product on $\mathbb{R}^3$, one obtains a set of linearly-independent divisors $B = \{D_4, D_5, D_6, D_7\}$, the intersection matrix with respect to which is

$$A_B = \begin{pmatrix}
-2 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 \\
1 & 1 & -2 & 0 \\
1 & 1 & 0 & -2 \\
\end{pmatrix}.$$ 

By a translation $PA_B^tP$ with

$$P = \begin{pmatrix}
0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 \\
0 & -2 & -1 & 1 \\
\end{pmatrix},$$
one gets a new basis
\[ B' = \{ D_5 + D_6, D_5, D_4 - D_5, -2D_5 - D_6 + D_7 \} \]
with respect to which the intersection matrix is
\[
A_{B'} = \begin{pmatrix}
0 & 1 & O \\
1 & 0 & -2 & 0 \\
O & 0 & -4 \\
\end{pmatrix}.
\]

Thus, \( \text{Pic}_\Delta = U \oplus \langle -2 \rangle \oplus \langle -4 \rangle \), which is clearly primitively embedded into the \( K3 \) lattice.

We label the primitive vectors that generate rays of the fan defining the 3-fold \( \mathbb{P}_\Delta \), or equivalently, the lattice points in \( \Delta_2^* \) as follows:

\[
\begin{align*}
v_1 &= (0, 0, 1), & v_2 &= (-1, -1, -3), & v_3 &= (0, 1, 0), \\
v_4 &= (3, 4, 6), & v_5 &= (1, 0, 0), & v_6 &= (2, 2, 3), \\
v_7 &= (1, 2, 2), & v_8 &= (2, 3, 4).
\end{align*}
\]

Let \( D_i := \text{Orb}(\mathbb{R} v_i) |_{-K_\Delta} \) for \( i = 1, \ldots, 6 \) be restricted torus-invariant divisors, and \( D_7, D'_7 \) and \( D_8, D'_8 \) be components of the divisor \( \text{Orb}(\mathbb{R} v_7) |_{-K_\Delta} \) and \( \text{Orb}(\mathbb{R} v_8) |_{-K_\Delta} \), respectively. One computes the self-intersection numbers by the formula (3),

\[
\begin{align*}
D_1^2 &= 2 \cdot 8 - 2 = 14, & D_2^2 &= 2 \cdot 2 - 2 = 2, & D_3^2 &= 2 \cdot 0 - 2 = -2, \\
D_4^2 &= 2 \cdot 0 - 2 = -2, & D_5^2 &= 2 \cdot 1 - 2 = 0, & D_6^2 &= D_7^2 = D_8^2 = D'_7^2 = D'_8^2 = -2.
\end{align*}
\]

One also has the graph of intersections among these divisors by the formula (4) as in Figure 3.

---

Figure 3: Nodes are divisors on a general section in \( \mathcal{F}_2 \). There are \( n \) lines connecting \( D_i \) and \( D_j \) if \( D_i D_j = n \).
By solving the linear system
\[ \sum_{i=1}^{8} (v_i, e_j) D_i = 0, \]
for \( j = 1, 2, 3 \), where \( e_j \) is the \( j \)-th column of the identity matrix of size 3, and \( (x, y) \) is the inner product on \( \mathbb{R}^3 \), one obtains a set of linearly-independent divisors \( \mathcal{B} = \{ D_4, D_5, D_6, D_7, D_7', D_8, D_8' \} \), the intersection matrix with respect to which is
\[
A_B = \begin{pmatrix}
-2 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -2 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & -2 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & -2
\end{pmatrix}.
\]
By a translation \( P A_B P^t \) with
\[
P = \begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix},
\]
one gets a new basis
\[ \mathcal{B}' = \{ D_5 + D_6, D_5, D_7', D_8, D_4 - D_5, D_8, D_7 \} \]
with respect to which the intersection matrix is
\[
A_{B'} = \begin{pmatrix}
0 & 1 & O \\
1 & 0 & -2 & 1 & 0 & 0 & 0 \\
O & 0 & 1 & -2 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & -2
\end{pmatrix}.
\]
Thus, \( \text{Pic}_{\Delta_2} = U \oplus A_5 \), which is clearly primitively embedded into the \( K3 \) lattice.

(2) We label the primitive vectors that generate rays of the fan defining the 3-fold \( \mathcal{F}_{\Delta_3} \), or equivalently, the lattice points in \( \Delta_3^* \) as follows:
\[
v_1 = (0, 0, 1), \quad v_2 = (-1, -1, -3), \quad v_3 = (0, 1, 0), \\
v_4 = (1, 1, 1), \quad v_5 = (1, 0, 0).
\]
Let \( D_i := \text{Orb}(\mathbb{R}_{\geq 0} v_i) \) for \( i = 1, \ldots, 5 \) be restricted torus-invariant divisors. One computes the self-intersection numbers by the formula (3),
\[
D_1^2 = 2 \cdot 9 - 2 = 16, \quad D_2^2 = 2 \cdot 3 - 2 = 4, \quad D_3^2 = 2 \cdot 1 - 2 = 0, \\
D_4^2 = 2 \cdot 0 - 2 = -2, \quad D_5^2 = 2 \cdot 1 - 2 = 0.
\]
Figure 4: Nodes are divisors on a general section in $\mathcal{F}_3$. There are $n$ lines connecting $D_i$ and $D_j$ if $D_i \cdot D_j = n$.

One also has the graph of intersections among these divisors by the formula (4) as in Figure 4.

By solving the linear system

$$\sum_{i=1}^{5} t^i v_i, e_j D_i = 0,$$

for $j = 1, 2, 3$, where $e_j$ is the $j$-th column of the identity matrix of size $3$, and $(x, y)$ is the inner product on $\mathbb{R}^3$, one obtains a set of linearly-independent divisors $B = \{D_4, D_5\}$, the intersection matrix with respect to which is

$$A_B = \begin{pmatrix} -2 & 2 \\ 2 & 0 \end{pmatrix}.$$ 

By a translation $P A_B^t P$ with

$$P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

one gets a new basis

$$B' = \{D_4, D_4 + D_5\}$$

with respect to which the intersection matrix is

$$A_{B'} = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}.$$ 

Thus, $\text{Pic}_{\Delta_3} = \langle -2 \rangle \oplus \langle 2 \rangle$, which is clearly primitively embedded into the $K3$ lattice.

(3) We label the primitive vectors that generate rays of the fan defining
the 3-fold \( \mathbb{P}_{\Delta_3} \), or equivalently, the lattice points in \( \Delta_3 \) as follows:

\[
\begin{align*}
    m_1 &= (-1, -1, 1), & m_2 &= (0, -1, 0), & m_3 &= (2, -1, 0), \\
    m_4 &= (-1, 2, 0), & m_5 &= (-1, 0, 0), & m_6 &= (1, -1, -1), \\
    m_7 &= (2, -1, -1), & m_8 &= (3, -1, -1), & m_9 &= (4, -1, -1), \\
    m_{10} &= (5, -1, -1), & m_{11} &= (4, 0, -1), & m_{12} &= (3, 1, -1), \\
    m_{13} &= (2, 2, -1), & m_{14} &= (1, 3, -1), & m_{15} &= (0, 4, -1), \\
    m_{16} &= (-1, 5, -1), & m_{17} &= (-1, 4, -1), & m_{18} &= (-1, 3, -1), \\
    m_{19} &= (-1, 2, -1), & m_{20} &= (-1, 1, -1), & m_{21} &= (0, 0, -1).
\end{align*}
\]

Let \( M_i := \text{Orb}(\mathbb{R}_{\geq 0} m_i)|_{-K_{\Delta_3}} \) for \( i = 1, \ldots, 21 \) be restricted torus-invariant divisors. One computes the self-intersection numbers by the formula (3),

\[
M^2_{21} = 2 \cdot 1 - 2 = 0, \quad M^2_j = 2 \cdot 0 - 2 = -2, \quad \forall \ i = 2, \ldots, 21.
\]

One also has the graph of intersection numbers among these divisors by the formula (4) as in Figure 5.

![Figure 5: Nodes are divisors on a general section in \( \mathcal{F}_{\Delta_3} \). There are \( n \) lines connecting \( D_i \) and \( D_j \) if \( D_i \cdot D_j = n \).](image)

By solving the linear system

\[
\sum_{i=1}^{21} (t m_i, e_j) D_i = 0,
\]

for \( j = 1, 2, 3 \), where \( e_j \) is the \( j \)-th column of the identity matrix of size 3, and \( (x, y) \) is the inner product on \( \mathbb{R}^3 \), one obtains a set of linearly-independent divisors \( \mathcal{B} = \{ M_1, \ldots, M_{18} \} \), the intersection matrix with
isometric to the discriminant group of Pic $\text{F}_2$ that generate the group since all the coefficients of that there exist two distinct elements in the discriminant group of order the Lemma below,

Thus, the lattice Pic $\text{K}$ discriminant form of $\text{S}$ embedded into the $\text{K}$ lattice if and only if $\text{D}$ are not lattice dual

Remark 4 Since $\mu_1$ and $\mu_2$ are not lattice dual in the sense of Dolgachev \cite{Dolgachev1997}, the families $\mathcal{F}_1 = \mathcal{F}_{\Delta_1}$ and $\mathcal{F}_{\Delta_1^*}$, and $\mathcal{F}_2 = \mathcal{F}_{\Delta_2}$ and $\mathcal{F}_{\Delta_2^*}$ are not lattice dual. Indeed, there exist smooth curves $s$ represented by the divisor $\text{D}_s$, and $f$ by $\text{D}_f$ such that $s$ is genus 1, $f$ is genus 0, and $s.f = 2$, thus, this is a 2-section.

Remark 5 Since the Picard lattices Pic$\Delta_1$ and Pic$\Delta_2$ both contain the hyperbolic lattice $\text{U}$ of rank 2 as a sublattice, general sections of families $\mathcal{F}_1$ and $\mathcal{F}_2$ have a structure of Jacobian elliptic surface.

Since the Picard lattice Pic$\Delta_1$ contains divisors $\text{D}_s$ and $\text{D}_f$ satisfying $\text{D}_s^2 = -2$, $\text{D}_f^2 = 0$, and $\text{D}_s \cdot \text{D}_f = 2$, general sections of the family has a structure of elliptic surface, but, not Jacobian. Indeed, there exist smooth curves $s$ represented by the divisor $\text{D}_s$, and $f$ by $\text{D}_f$ such that $s$ is genus 1, $f$ is genus 0, and $s.f = 2$, thus, this is a 2-section.
4 Families of double sextic $K3$ surfaces branching along curves of $(2,3)$-torus type

In this section, we introduce and study the families $F_1$, $F_2$, and $F_3$ that contain $K3$ surfaces that are double covering of $\mathbb{P}^2$ branching along curves of $(2,3)$-torus type. Since every Gorenstein $K3$ surface is birationally equivalent to a unique $K3$ surface, we identify these two surfaces.

**Proposition 4.1.** The family $F_1$ contains $K3$ surfaces that are double covering of $\mathbb{P}^2$ branching along curves of $(2,3)$-torus type with singularities

\[ A_{17}, \ A_2 + A_{14}, \ A_5 + A_{11}, \ E_6 + A_{11}, \ 2A_8, \ 2A_2 + A_{11}, \ A_2 + A_5 + A_8, \ 3A_5, \ 3A_2 + A_8, \ 2A_2 + 2A_5, \ \text{and} \ 4A_2 + A_5. \]

**Proof.**

\[ [A_{17}] \] The curve is defined by $F(X,Y,Z) := F_3^2 + F_7^2 = 0$ [17] with

\[ F_3(X,Y,Z) = YZ - X^2, \quad F_7(X,Y,Z) = -X^2Z + YZ^2 + Y^3. \]

The polynomial $F$ contains monomials $X^6$, $Y^6$, $X^4Z^2$, $Y^4Z^2$, $Y^3Z^3$, and $Y^2Z^4$. Thus the Gorenstein $K3$ surface $D(X/\mathbb{P}^2)$ is a section in $F_1$. Moreover, by a direct computation (e.g. with Singular), it is verified that the curve has one singularity of type $A_{17}$ at $(0 : 0 : 1)$.

\[ [A_2 + A_{14}] \] The curve is defined by $F(X,Y,Z,t_3) := F_3^2 + F_3^2 = 0$ [17] with

\[ F_3(X,Y,Z) = YZ - X^2, \quad F_3(X,Y,Z,t_3) = Y^3 + t_3XY^2 - X^3Z + YZ^2. \]

The polynomial $F$ contains monomials $X^6$, $Y^6$, $X^2Y^4$, $X^4Z^2$, $XY^5$, $Y^4Z^2$, $Y^3Z^3$, and $Y^2Z^4$. Thus the Gorenstein $K3$ surface $D(X/\mathbb{P}^2)$ is a section in $F_1$. Moreover, by a direct computation (e.g. with Singular), it is verified that there exists a curve with singularities of type $A_{14}$ at $(0 : 0 : 1)$, and of type $A_2$ at $(-1 : 1 : 1)$. For instance, setting $t_3 = 1$ would do.

\[ [A_5 + A_{11}] \] The curve is defined by $F(X,Y,Z,t_1,t_2,s) := F_3^2 + F_3^2 = 0$ [17] with

\[ F_2(X,Y,Z) = YZ - X^2, \quad F_3(X,Y,Z,t_1,t_2,s) = -t_2X^3 + (t_1 + t_2 - s - 1)Y^3 + X^2Y - t_1X^2Z + 2(t_1 + t_2 - s - 1)XY^2 + t_1YZ^2 - (t_1 + t_2 - s)Y^2Z. \]

The polynomial $F$ contains monomials $X^6$, $Y^6$, $X^3Y^3$, $X^3Y^3$, $X^2Y^4$, $X^3Y^3$, $X^2Y^4$, $XY^5$, $X^4Z^2$, $X^5Z$, $Y^4Z^2$, $Y^3Z^3$, and $Y^2Z^4$. Thus the Gorenstein $K3$ surface $D(X/\mathbb{P}^2)$ is a section in $F_1$. Moreover, by a direct computation (e.g. with Singular), it is verified that there exists a curve with singularities of type $A_{11}$ at $(0 : 0 : 1)$, and of type $A_5$ at $(1 : 1 : 1)$. For instance, setting $t_1 = t_2 = 1$ and $s = 0$ would do.

\[ [E_6 + A_{11}] \] The curve is defined by $F(X,Y,Z,t_1,t_2) := F_3^2 + F_3^2 = 0$ [17] with

\[ F_2(X,Y,Z) = YZ - X^2, \quad F_3(X,Y,Z,t_1,t_2) = -t_2X^3 + (-t_1 - t_2 + 1)Y^3 + X^2Y + 2(t_1 + t_2 - 1)XY^2 + t_1YZ^2 + (t_1 + t_2)Y^2Z - t_1X^2Z. \]

The polynomial $F$ contains monomials $X^6$, $Y^6$, $X^5Y$, $X^5Z$, $X^4Y^2$, $X^4Z^2$, $X^5Y^3$, $X^2Y^4$, $XY^5$, $X^5Z$, $Y^4Z^2$, $Y^3Z^3$, and $Y^2Z^4$. Thus the Gorenstein
K3 surface \(D(X/P^2)\) is a section in \(F_1\). Moreover, by a direct computation (e.g. with Singular), it is verified that there exists a curve with singularities of type \(A_{11}\) at \((0 : 0 : 1)\), and of type \(E_8\) at \((1 : 1 : 1)\). For instance, setting \(t_1 = -1\) and \(t_2 = 1\) would do.

\([2A_8]\) The curve is defined by \(F(X,Y,Z,t_1,t_2) := F_2^3 + F_3^2 = 0\) \(\text{(17)}\) with

\[
F_2(X,Y,Z) = YZ - X^2,
F_3(X,Y,Z,t_1,t_2) = -(t_2 + 1)X^3 + t_2 Y^3 + 3t_2X^2Y
- 3t_2 XY^2 - t_1X^2Z + t_1 YZ^2.
\]

The polynomial \(F\) contains monomials \(X^6, Y^6, X^5Y, X^4Y^2, X^4Z^2, X^3Y^3, X^2Y^4, XY^5, X^2Z^3, Y^4Z^3,\) and \(Y^3Z^4\). Thus the Gorenstein K3 surface \(D(X/P^2)\) is a section in \(F_1\). Moreover, by a direct computation (e.g. with Singular), it is verified that there exists a curve with singularities of type \(A_8\) at \((0 : 0 : 1)\) and \((1 : 1 : 1)\). For instance, setting \(t_1 = t_2 = 1\) would do.

\([2A_2 + A_{11}]\) The curve is defined by \(F(X,Y,Z,t_1,t_2,t_3) := F_2^3 + F_3^2 = 0\) \(\text{(17)}\) with

\[
F_2(X,Y,Z) = YZ - X^2,
F_3(X,Y,Z,t_1,t_2,t_3) = Y^3 + t_4 X^2 Y - t_5 XY^2 + t_6 YZ^2.
\]

The polynomial \(F\) contains monomials \(X^6, Y^6, X^5Y, X^4Y^2, X^3Y^3, X^2Y^4, XY^5, X^2Z^3, Y^4Z^3,\) and \(Y^2Z^4\). Thus the Gorenstein K3 surface \(D(X/P^2)\) is a section in \(F_1\). Moreover, by a direct computation (e.g. with Singular), it is verified that there exists a curve with singularities type \(A_{12}\) at \((0 : 0 : 1)\), and \(A_3\) at \((-1 : 1 : 1)\) and \((-2 : 4 : 1)\). For instance, setting \(t_1 = t_2 = 1\) and \(t_5 = 1\) would do.

\([A_2 + A_5 + A_8]\) The curve is defined by \(F(X,Y,Z,t_1,t_2,s) := F_2^3 + F_3^2 = 0\) \(\text{(17)}\) with

\[
F_2(X,Y,Z) = YZ - X^2,
F_3(X,Y,Z,t_1,t_2,s) = \left(-t_2 - \frac{23}{27}\right) X^3 - \left(t_2 - \frac{4}{27}\right) Y^3
+ \left(t_1 + t_2 + \frac{23}{27} - s\right) X^2 Y + \left(t_2 - \frac{4}{27}\right) XY^2
- t_1 X^2 Z + XY^2 + t_1 YZ^2 + (-t_1 - 1 + s) Y^2 Z.
\]

The polynomial \(F\) contains monomials \(X^6, Y^6, X^5Y, X^4Y^2, X^3Y^3, X^2Y^4, XY^5, X^2Z^3, Y^4Z^3, Y^2Z^4,\) and \(Y^4Z^3\). Thus the Gorenstein K3 surface \(D(X/P^2)\) is a section in \(F_1\). Moreover, by a direct computation (e.g. with Singular), it is verified that there exists a curve with singularities of type \(A_8\) at \((0 : 0 : 1)\), of type \(A_2\) at \((-1 : 1 : 1)\), and of type \(A_3\) at \((1 : 1 : 1)\). For instance, setting \(t_1 = t_2 = s = 1\) would do.

\([3A_8]\) The curve is defined by \(F(X,Y,Z,t_1,t_2,t_3) := F_2^3 + F_3^2 = 0\) \(\text{(17)}\) with

\[
F_2(X,Y,Z) = YZ - X^2,
F_3(X,Y,Z,t_1,t_2,t_3) = -t_3 X^3 + X^2 Y + \left(-t_1 - \frac{t_2}{2} + t_3 - 1\right) X^2 Z + t_3 XYZ
+ \left(\frac{t_2}{2} - 1 + t_3\right) Y^3 + (t_2 + 1 - 2t_3) Y^2 Z + t_1 YZ^2.
\]

The polynomial \(F\) contains monomials \(X^6, Y^6, X^5Y, X^4Z, X^4Y^2, X^4Z^2, X^3Y^3, X^2Y^4, Y^5Z, Y^4Z^2, Y^3Z^3,\) and \(Y^2Z^4\). Thus the Gorenstein K3
surface $D(X/P^3)$ is a section in $F_1$. Moreover, by a direct computation (e.g. with Singular), it is verified that there exists a curve with singularities type $A_8$ at $(0 : 0 : 1)$, of type $A_5$ at $(0 : 0 : 1), (1 : 1 : 1)$, and $(-1 : 1 : 1)$. For instance, setting $t_1 = t_2 = t_3 = 1$ would do.

$[3A_2 + A_8]$ The curve is defined by $F(X, Y, Z, t_3, t_4, t_5) := F_2^3 + F_3^3 = 0$ with

$$F_2(X, Y, Z) = YZ - X^2,$$

$$F_3(X, Y, Z, t_3, t_4, t_5) = t_3X^3 + Y^3 + t_4X^2Y - X^2Z + Y^2Z + t_5XY^2.$$

The polynomial $F$ contains monomials $X^6, Y^6, X^5Y, X^5Z, X^4Y^2, X^4Z^2,$ $X^3Y^3, X^3Y^2, XY^3, Y^3Z^2, Y^3Z^3,$ and $Y^2Z^4$. Thus the Gorenstein K3 surface $D(X/P^3)$ is a section in $F_1$. Moreover, by a direct computation (e.g. with Singular), it is verified that there exists a curve with singularities of type $A_8$ at $(0 : 0 : 1)$, and of type $A_2$ at $(-1 : 1 : 1), (-2 : 4 : 1)$, and $(2 : 4 : 1)$. For instance, setting $t_3 = t_4 = -4$ and $t_5 = 1$ would do.

$[2A_2 + 2A_6]$ The curve is defined by $F(X, Y, Z, t_1, t_2) := F_2^3 + F_3^3 = 0$ with

$$F_2(X, Y, Z) = YZ - X^2,$$

$$F_3(X, Y, Z, t_1, t_2) = 3X^3 + Y^3 - (t_1 + 2)X^2Z + (t_1 - t_2 - 1)X^2Y - 3XY^2 + t_1YZ^2 + (-t_1 + 2 + t_2)Y^2Z.$$

The polynomial $F$ contains monomials $X^6, Y^6, X^5Y, X^5Z, X^4Y^2, X^4Z^2,$ $X^3Y^3, X^3Y^2, XY^3, Y^3Z^2, Y^3Z^3,$ and $Y^2Z^4$. Thus the Gorenstein K3 surface $D(X/P^3)$ is a section in $F_1$. Moreover, by a direct computation (e.g. with Singular), it is verified that there exists a curve with singularities type $A_5$ at $(0 : 0 : 1)$ and $(1 : 1 : 1)$, and of type $A_2$ at $(-1 : 1 : 1)$ and $(2 : 4 : 1)$. For instance, setting $t_1 = t_2 = 1$ would do.

$[4A_2 + A_6]$ The curve is defined by $F(X, Y, Z, t_3, t_4, t_5) := F_2^3 + F_3^3 = 0$ with

$$F_2(X, Y, Z) = YZ - X^2,$$

$$F_3(X, Y, Z, t_3, t_4, t_5) = t_3X^3 + Y^3 + t_4X^2Y + (t_2 - 1)X^2Z + t_5XY^2 + Y^2Z.$$

The polynomial $F$ contains monomials $X^6, Y^6, X^5Y, X^5Z, X^4Y^2, X^4Z^2,$ $X^3Y^3, X^3Y^2, XY^3, Y^3Z^2, Y^3Z^3,$ and $Y^2Z^4$. Thus the Gorenstein K3 surface $D(X/P^3)$ is a section in $F_1$. Moreover, by a direct computation (e.g. with Singular), it is verified that there exists a curve with singularities of type $A_2$ at $(0 : 0 : 1)$, and of type $A_2$ at $(-1 : 1 : 1), (1 : 1 : 1), (-2 : 4 : 1)$, and $(2 : 4 : 1)$. For instance, setting $(t_2, t_3, t_4, t_5) = (4, 0, -5, 0)$ would do.

Thus the assertion is verified. □

**Proposition 4.2** The family $F_2$ contains K3 surfaces that are double covering of $P^3$ branching along curves of $(2,3)$-torus type with singularities $2A_2 + E_6$, $A_5 + 2E_6$, $3E_6$, $2A_2 + A_5 + E_6$, $2A_2 + 2E_6$, and $A_2 + E_6 + A_8$.

**Proof.**
The polynomial $F_0$ with $t$ singular, it is verified that there exists a curve with singularities of type $E_6$ at $(0 : 0 : 1)$, and of type $A_5$ at $(1 : 1 : 1)$ and $(-1 : 1 : 1)$. For instance, setting $t_1 = t_2 = 1$ would do.

$[2A_5 + E_6]$ The curve is defined by $F(X, Y, Z, t_1, t_2) := F_2^3 + F_3^3 = 0$ with

$$F_2(X, Y, Z) = YZ - X^2,$$
$$F_3(X, Y, Z, t_1, t_2) = -t_3X^3 + X^2Y - \left(1 + \frac{t_2}{2} - t_3\right)X^2Z + t_3XYZ - \left(1 + t_2 - 2t_3\right)Y^2Z.$$

The polynomial $F$ contains monomials $X^6, Y^6, X^3Y, X^5Z, X^4Y^2, X^4Z^2, X^3Y^3, X^2Y^4, Y^4Z, Y^4Z^2$, and $Y^3Z^3$. Thus the Gorenstein $K3$ surface $D(X/Y^2)$ is a section in $F_2$. Moreover, by a direct computation (e.g. with Singular), it is verified that there exists a curve with singularities of type $E_6$ at $(0 : 0 : 1)$, and of type $A_5$ at $(1 : 1 : 1)$ and $(-1 : 1 : 1)$. For instance, setting $t_3 = -1$ would do.

$[A_5 + 2E_6]$ The curve is defined by $F(X, Y, Z, t_3) := F_2^3 + F_3^3 = 0$ with

$$F_2(X, Y, Z) = YZ - X^2,$$
$$F_3(X, Y, Z, t_3) = -t_3X^3 - (1 - t_3)Y^3 + X^2Y - (1 - t_3)X^2Z + (1 - 2t_3)Y^2Z + t_3XYZ.$$

The polynomial $F$ contains monomials $X^6, Y^6, X^5Y, X^5Z, X^4Y^2, X^3Y^3, X^2Y^4, Y^4Z, Y^4Z^2$, and $Y^3Z$. Thus the Gorenstein $K3$ surface $D(X/Y^2)$ is a section in $F_2$. Moreover, by a direct computation (e.g. with Singular), it is verified that there exists a curve with singularities of type $A_5$ at $(0 : 0 : 1), (1 : 1 : 1)$ and $(-1 : 1 : 1)$. For instance, setting $t_3 = -1$ would do.

$[3E_6]$ The curve is defined by $F(X, Y, Z) := F_2^3 + F_3^3 = 0$ with

$$F_2(X, Y, Z) = YZ - X^2,$$
$$F_3(X, Y, Z) = X^2Y - X^2Z - Y^3 + Y^2Z.$$

The polynomial $F$ contains monomials $X^6, Y^6, X^4Y^2, X^4Z^2, X^3Y^3, Y^4Z, Y^4Z^2$, and $Y^3Z^3$. Thus the Gorenstein $K3$ surface $D(X/Y^2)$ is a section in $F_2$. Moreover, by a direct computation (e.g. with Singular), it is verified that the curve has singularities of type $E_6$ at $(0 : 0 : 1), (1 : 1 : 1)$ and $(-1 : 1 : 1)$. For instance, setting $t_3 = -1$ would do.

$[2A_2 + A_5 + E_6]$ The curve is defined by $F(X, Y, Z, t_2) := F_2^3 + F_3^3 = 0$ with

$$F_2(X, Y, Z) = YZ - X^2,$$
$$F_3(X, Y, Z, t_2) = 3X^3 + Y^3 - (1 + t_3)X^2Y - 3XY^2 - 2X^2Z + (2 + t_2)Y^2Z.$$

The polynomial $F$ contains monomials $X^6, Y^6, X^5Y, X^5Z, X^4Y^2, X^4Z^2, X^3Y^3, X^2Y^4, XY^5, Y^4Z, Y^4Z^2$, and $Y^3Z^3$. Thus the Gorenstein $K3$ surface $D(X/Y^2)$ is a section in $F_2$. Moreover, by a direct computation (e.g. with Singular), it is verified that there exists a curve with singularities of type $E_6$ at $(0 : 0 : 1)$, of type $A_2$ at $(-1 : 1 : 1)$ and $(2 : 4 : 1)$, and of type $A_5$ at $(1 : 1 : 1)$. For instance, setting $t_2 = 1$ would do.

$[2A_2 + 2E_6]$ The curve is defined by $F(X, Y, Z) := F_2^3 + F_3^3 = 0$ with

$$F_2(X, Y, Z) = YZ - X^2,$$
$$F_3(X, Y, Z) = 3X^3 + Y^3 - X^2Y - 3XY^2 - 2X^2Z + 2Y^2Z.$$

The polynomial $F$ contains monomials $X^6, Y^6, X^5Y, X^5Z, X^4Y^2, X^4Z^2, X^3Y^3, X^2Y^4, XY^5, Y^4Z, Y^4Z^2$, and $Y^3Z^3$. Thus the Gorenstein $K3$ surface $D(X/Y^2)$ is a section in $F_2$. Moreover, by a direct computation (e.g. with Singular), it is verified that there exists a curve with singularities of type $E_6$ at $(0 : 0 : 1)$, of type $A_2$ at $(-1 : 1 : 1)$ and $(2 : 4 : 1)$, and of type $A_5$ at $(1 : 1 : 1)$. For instance, setting $t_3 = -1$ would do.

$[2A_5 + E_6]$ The curve is defined by $F(X, Y, Z, t_3) := F_2^3 + F_3^3 = 0$ with

$$F_2(X, Y, Z) = YZ - X^2,$$
The polynomial $F(X)$ is a section in $\mathcal{F}_2$. Moreover, by a direct computation (e.g. with Singular), it is verified that the curve has singularities of type $E_6$ at $(0 : 0 : 1)$ and $(1 : 1 : 1)$, and of type $A_2$ at $(-1 : 1 : 1)$ and $(2 : 4 : 1)$.

$[A_2 + E_6 + A_8]$ The curve is defined by $F(X,Y,Z,t_1,t_2) := F_2 + F_3 = 0 \ [17]$ with

\[
F_2(X,Y,Z) = YZ - X^2, \quad F_3(X,Y,Z,t_1,t_2) = -\left(t_2 + \frac{23}{27}\right)X^3 + \left(\frac{4}{27} - t_2\right)Y^3 + \left(t_1 + t_2 + \frac{23}{27}\right)X^2Y + \left(t_2 - \frac{4}{27}\right)XY^2 - t_1X^2Z - (t_1 + 1)Y^2Z + t_1YZ + XYZ.
\]

The polynomial $F$ contains monomials $X^6$, $Y^6$, $X^5Y$, $X^4Y^2$, $X^3Y^3$, $X^2Y^4$, $X^2Z$, $XY^5$, $X^3Z$, $Y^5Z$, $Y^4Z^2$, and $Y^3Z^3$. Thus the Gorenstein $K_3$ surface $D(X/E^2)$ is a section in $\mathcal{F}_2$. Moreover, by a direct computation (e.g. with Singular), it is verified that there exists a curve with singularities of type $A_2$ at $(0 : 0 : 1)$, of type $A_2$ at $(-1 : 1 : 1)$, and of type $E_6$ at $(1 : 1 : 1)$. For instance, setting $t_1 = t_2 = 1$ would do.

Thus the assertion is verified. □

**Proposition 4.3** The family $\mathcal{F}_3$ contains $K_3$ surfaces that are double covering of $\mathbb{P}^2$ branching along the curve of $(2,3)$-torus type with singularities $6A_2 + E_6$.

The general $K_3$ surfaces that are double covering of $\mathbb{P}^2$ branching along the curve of $(2,3)$-torus type with singularities $6A_2$ cannot be contained in any of the three families, but in the full family of the weighted projective space with weight $(1,1,1,3)$.

**Proof.**

$[4A_2 + E_6]$ The curve is defined by $F(X,Y,Z) := F_2^3 + F_3^3 = 0 \ [17]$ with

\[
F_2(X,Y,Z) = YZ - X^2, \quad F_3(X,Y,Z) = Y^3 - XY^2 - 4XZ^2 - 5YZ^2 + 5XYZ.
\]

The polynomial $F$ contains monomials $X^6$, $Y^6$, $X^2Y^4$, $X^2Z$, $XY^5$, $X^3Z$, $Y^4Z^2$, $Y^3Z^3$, and $Y^2Z^4$. Thus the Gorenstein $K_3$ surface $D(X/E)$ is a section in $\mathcal{F}_3$. Moreover, by a direct computation (e.g. with Singular), it is verified that the curve has singularities of type $E_6$ at $(1 : 1 : 1)$, and $A_2$ at $(0 : 0 : 1)$, $(-1 : 1 : 1)$, $(-2 : 4 : 1)$ and $(2 : 4 : 1)$.

$[6A_2]$ The curve is defined by $F(X,Y,Z) := F_2^3 + F_3^3 = 0 \ [17]$ with

\[
F_2(X,Y,Z) = YZ - X^2, \quad F_3(X,Y,Z) = X^3 + Y^3 + Z^3.
\]

The polynomial $F$ contains monomials $X^6$, $Y^6$, $Z^6$, $X^3Y^3$, $X^3Z^3$, and $Y^3Z^3$. Thus the Gorenstein $K_3$ surface $D(X/E^2)$ is a section neither in $\mathcal{F}_1$ nor $\mathcal{F}_2$ nor $\mathcal{F}_3$, but in the full family of the weighted projective space with weight $(1,1,1,3)$. Moreover, by a direct computation (e.g. with Mathematica), it is verified that by Bézout’s theorem, the conic $C_2 := (F_2 = 0)$ and the cubic $C_3 := (F_3 = 0)$ transversely intersect at the six points

\[
(e^{2/3\pi} : e^{4/3\pi} : 1), (e^{2/3\pi} : e^{4/3\pi} : 1), (e^{2/3\pi} : e^{4/3\pi} : 1), (e^{2/3\pi} : e^{4/3\pi} : 1), (e^{2/3\pi} : e^{4/3\pi} : 1), (e^{2/3\pi} : e^{4/3\pi} : 1),
\]

\[
(e^{4/3\pi} : e^{4/3\pi} : 1), (e^{4/3\pi} : e^{4/3\pi} : 1), (e^{4/3\pi} : e^{4/3\pi} : 1), (e^{4/3\pi} : e^{4/3\pi} : 1).
\]
at which points the curve \((F = 0)\) has six singularities of type \(A_2\). Note that the curve is the plane sextic that are studied by Zariski [19].

Let \((F' = 0)\) be any deformation that preserves the type of singularity of the curve \((F = 0)\). In this case, the polynomial \(F'\) is a projective transformation of \(F\). Thus, \(F'\) also contains monomials \(X^6, Y^6,\) and \(Z^6\). Thus, the \(K3\) surface obtained by the double covering of \(\mathbb{P}^2\) branching along the curve \((F' = 0)\) cannot be contained in any families \(F_i, i = 1, 2, 3\).

The assertion is verified. □

References

[1] W. Barth, K3 surfaces with nine cusps, Geometriae Dedicata 72 (1998), 171–178.
[2] V.V. Batyrev, Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties, J. Alg. Geom. 3 (1994), 493–535.
[3] P. Berglund and T. Hübsch, A generalized construction of mirror manifolds, Nuclear Physics B 393 (1993), 377–391.
[4] E. Brieskorn and H. Knörrer, Plane Algebraic Curves, Birkhäuser, 1986.
[5] U. Bruzzo and A. Grassi, Picard group of hypersurfaces in toric 3-folds, International Journal of Mathematics 23 (2012), 1–14.
[6] I. Dolgachev, Mirror symmetry for lattice polarized \(K3\) surfaces, J. Math. Sci. 81 (1996), 2599–2630.
[7] W. Ebeling and D. Ploog, A geometric construction of Coxeter-Dynkin diagrams of bimodal singularities, Manuscripta Math. 140 (2013), 195–212.
[8] W. Ebeling and A. Takahashi, Strange duality of weighted homogeneous polynomials, Compos. Math. 147 (2011), 1413–1433.
[9] W. Fulton, Introduction to toric varieties, second edition, Ann. Math. Stud. 131, Princeton University Press, Princeton, 1997.
[10] H. Ishida and H. Tokunaga, Triple covers of algebraic surfaces and a generalization of Zariski’s example, Advanced Studies in Pure Mathematics (Singularities –Niigata-Toyama 2007) 56 (2009), 169–185.
[11] M. Kobayashi, Duality of weights, mirror symmetry and Arnold’s strange duality, Tokyo J. Math. 31 (2008), 225–251.
[12] J. Komeda and K. Watanabe, On extensions of a double covering of plane curves and Weierstrass semigroups of the double covering type, Semigroup Forum 91 (2015), 517–523.
[13] M. Mase, Lattice duality for families of \(K3\) surfaces associated to transpose duality, Manuscripta Math. 155 (2018), 61–76.
[14] V.V. Nikulin, On Kummer surfaces, Math. USSR-Izv. 9 (1975), 261–275.
[15] V.V. Nikulin, Integral symmetric bilinear forms and some of their applications, Math. USSR-Izv. 14 (1980), 103–167.
[16] M. Oka and D. T. Pho, Classification of Sextics of Torus Type, Tokyo J. Math. 25 (2002), 399–433.
[17] D. T. Pho, Classification of singularities on torus curves of type \((2, 3)\), Kodai Math. J. 24 (2000), 259–284.
[18] T. YONEMURA, Hypersurface simple $K3$ singularities, Tôhoku Math. J. 42 (1990), 351–380.

[19] O. ZARISKI, On the problem of existence of algebraic functions of two variables possessing a given branch curve, American J. Math. 51 (1929), 305–328.

Makiko Mase

Postal address 1:
Department of Mathematics and Information Sciences
Tokyo Metropolitan University,
1-1 Hachioji-shi, Minami-osawa, Tokyo, Japan, 192-0397.

Postal address 2:
Lehrstuhl für Mathematik
Universität Mannheim,
B6, 26 68131 Mannheim, Germany.

email address: mtmase@arion.ocn.ne.jp