Cohomology of n-categories and derivations in group algebras

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1 Abstract

There is a well-known problem of describing the algebra of the derivations in group algebra. In [4], a method was proposed for describing derivations by considering the character space of a certain groupoid associated with a given group algebra. This groupoid is a groupoid action generated by the inner automorphisms of the group.

This construction allows some generalizations, which this work is devoted to.

It turns out that a given $n$--category can be associated with a set of vector spaces, the so-called $n$--characters, i.e. mappings from the set of $n$--morphisms of the $n$--category to complex numbers preserving the composition. We show that the corresponding sequence is exact, that is, in fact, we construct the cohomology of a given $n$--category. In the particular case when this category is a 2--groupoid associated with a group algebra, the complex gives a description of the derivation algebra in the group algebra.

Within the framework of this paper, we will also propose a natural modification of the construction of internal and external derivations, which significantly encapsulates the study of the space of derivations, taking into account the geometric constructions that we introduce.
1.1 General definitions

An important role for us will be played by the notion of the \(n\)-category and, in particular, the \(2\)-category (\[1\], \[3\], p. 312).

We start with the definition of the \(2\)-category.

**Definition 1.1** *2*-category \(C^2\) - is:

- A class \(\text{Obj}(C^2)\), whose elements are called objects

- A class \(\text{Hom}^1(a, b)\) for every \(a\) and \(b\), whose elements are called maps, object \(a\) is called a source, \(b\) is called a target. Every map \(\varphi\) can be written like an expression: \(\varphi : a \Rightarrow b\). \(\text{Hom}^1(C^2)\) - a class of the all 1-maps.

- A binary operation \(\circ\) is called composition of maps \(\varphi \in \text{Hom}^1(a, b)\) and \(\psi \in \text{Hom}^1(b, c)\) and written as \(\varphi \circ \psi\) or \(\varphi \psi \in \text{Hom}^1(a, c)\), governed by two axioms:
  
  1. **Associativity:** If \(\varphi : a \Rightarrow b\), \(\psi : b \Rightarrow c\) and \(\xi : c \Rightarrow d\) then 
     \[\psi \circ (\xi \circ \varphi) = (\psi \circ \xi) \circ \varphi,\]
     and
  
  2. **Identity:** For every object \(a\), there exists a map \(1_a : a \Rightarrow a\) called the identity map for \(a\), such that for every map \(\varphi : a \Rightarrow b\), we have 
     \[1_b \circ \varphi = \varphi = \varphi \circ 1_a.\]

- A class \(\text{Hom}^2(\varphi, \psi)\) for every \(\varphi, \psi \in \text{Hom}^1(a, b)\), whose elements are called 2-maps. Every 2-map \(\alpha\) can be written like an expression: \(\alpha : \varphi \Rightarrow \psi\)

\[
\begin{array}{c}
A \\
\text{\(\varphi\)}
\end{array} \quad \begin{array}{c}
\downarrow \alpha \\
\psi
\end{array} \quad \begin{array}{c}
B
\end{array}
\]

- A binary operation \(\bullet\) is called vertical composition of 2-maps \(\alpha : \varphi \Rightarrow \psi\) and \(\beta : \psi \Rightarrow \xi\) and written as \(\alpha \bullet \beta : \varphi \Rightarrow \xi\), governed by two axioms:

  1. **Associativity:** If \(\alpha : \varphi \Rightarrow \psi\), \(\beta : \psi \Rightarrow \xi\) and \(\gamma : \xi \Rightarrow \eta\) then 
     \[\alpha \bullet (\beta \bullet \gamma) = (\alpha \bullet \beta) \bullet \gamma,\]
     and

  2. **Identity:** For every 1-map \(\varphi\), there exists a 2-map \(1_\varphi : \varphi \Rightarrow \varphi\) called the horizontal identity 2-map for \(\varphi\), such that for every 2-map \(\alpha : \varphi \Rightarrow \psi\), we have 
     \[1_\psi \bullet \alpha = \alpha = \alpha \bullet 1_\varphi.\]
• A binary operation \( \circ \) is called horizontal composition of 2-maps \( \alpha : \varphi \Rightarrow \psi \) and \( \alpha' : \varphi' \Rightarrow \psi' \) and written as \( \alpha \circ \alpha' : \varphi \varphi' \Rightarrow \psi \psi' \), governed by two axioms:

1. **Associativity:** If \( \alpha : \varphi \Rightarrow \psi, \alpha' : \varphi' \Rightarrow \psi' \) and \( \alpha'' : \varphi'' \Rightarrow \psi'' \) then \( \alpha \circ (\alpha' \circ \alpha'') = (\alpha \circ \alpha') \circ \alpha'' \), and

2. **Identity:** For every object \( a \), there exists a 2-map \( 1_1 : 1_a \Rightarrow 1_a \) called the vertical identity 2-map for \( a \), such that for every 2-map \( \alpha : \varphi \Rightarrow \psi \), we have \( 1_a \circ \alpha = \alpha = \alpha \circ 1_a \).

\[
\begin{array}{ccc}
A & \xrightarrow{\varphi} & B \\
\downarrow^{\alpha} & & \downarrow^{\alpha'} \\
A & \xrightarrow{\psi} & B & \xrightarrow{\psi'} & C \\
& & & & \\
A & \xrightarrow{\varphi \varphi'} & C \\
& & \downarrow^{\alpha \circ \alpha'} \\
A & \xrightarrow{\psi \psi'} & C
\end{array}
\]

• A middle four exchange equations for horizontal and vertical compositions

\[
(\alpha \circ \alpha') \bullet (\beta \circ \beta') = (\alpha \bullet \beta) \circ (\alpha' \bullet \beta').
\]

In the other words the represented diagram is commutative

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & B \\
\downarrow^{\alpha'} & & \downarrow^{\alpha'} \\
A & \xrightarrow{\beta} & B & \xrightarrow{\beta'} & C \\
& & & & \\
A & \xrightarrow{\beta} & B & \xrightarrow{\beta'} & C
\end{array}
\]

This definition can be expand to higher dimensions. Here we introduce n-category definition. ([1], [3] p. 17). All the categories considered below are small.

**Definition 1.2** n-category \( C^n \) - is a set of the sets \( \text{Hom}^0(C^n), \text{Hom}^1(C^n), \text{Hom}^2(C^n), \ldots, \text{Hom}^n(C^n) \), where:

• Elements of the set \( \text{Hom}^m(C^n) \) are called m-maps
• For each $0 \leq k < m \leq n$ sets $\text{Hom}^k(C^n)$ and $\text{Hom}^m(C^n)$ creates a category, where $\text{Hom}^k(C^n)$ is an object class, $\text{Hom}^m(C^n)$ is a set of maps. Composition of the maps $\varphi$ and $\psi \in \text{Hom}^m(C^n)$ writes as $\varphi \circ_k \psi$.

• For each $0 \leq h < k < m \leq n$ sets $\text{Hom}^h(C^n)$, $\text{Hom}^k(C^n)$ and $\text{Hom}^m(C^n)$ creates a 2-category.

0–maps are usually called objects, and 1–maps usually called maps.

**Definition 1.3** An $n$-groupoid is an $n$-category whose all maps have inverse.

**Definition 1.4** A totally-connected $n$-groupoid is an $n$-groupoid whose all possible pairs $\text{Hom}^k(C^n)$ and $\text{Hom}^m(C^n)$ form a totally-connected 1-groupoid, which is a groupoid that has at least one map between objects where a map can exist.

We consider an arbitrary m-morphism $\alpha : \varphi \Rightarrow \psi$. Define the maps $s^m, t^m : \text{Hom}^m(C^n) \to \text{Hom}^{m-1}(C^n)$ such, that

\[
\begin{align*}
s^m(\alpha) &= \varphi \\
t^m(\alpha) &= \psi
\end{align*}
\]

1.2 Characters

Now we define the concept of $k$—character on the $n$—category, counting $k \leq n$. As in the case of the definition of $n$–categories, we will give the definition sequentially, beginning with small $n$.

**Definition 1.5** A map $\chi_0 : G \to \mathbb{C}$ is called 0-character on the object set $G$.

The concept of 1–character has already been applied to the description of the algebra of derivations ([1] p. 17)

**Definition 1.6** A map $\chi_1 : \text{Hom}^1(C) \to \mathbb{C}$, such that for every 1-maps $\varphi$ and $\psi$
\[ \chi(\varphi \circ \psi) = \chi(\varphi) + \chi(\psi) \]

is called 1-character on the 1-category

As in the case of maps, 1-characters are usually called simply characters.

**Definition 1.7** 1-character \( \chi_1 \), such that \( \forall a, \forall \varphi \in \text{Hom}_1(a, a) \Rightarrow \chi_1(\varphi) = 0 \) is called trivial on loops

**Definition 1.8** A map \( \chi_2 : \text{Hom}_2(C^2) \to \mathbb{C} \) such that for every \( \alpha, \alpha', \beta \)

\[
\begin{align*}
\chi_2(\alpha \circ \alpha') &= \chi_2(\alpha) + \chi_2(\alpha') \\
\chi_2(\alpha \bullet \beta) &= \chi_2(\alpha) + \chi_2(\beta)
\end{align*}
\]

\[
\begin{array}{c c c}
A & \longrightarrow & B & \longrightarrow & C \\
\alpha & \quad & \alpha' & & \\
\gamma & \quad & \gamma & & \\
B & \longrightarrow & C & & \\
\beta & \quad & \beta' & & \\
A & \longrightarrow & B & \longrightarrow & C \\
\gamma & \quad & \gamma & & \\
\end{array}
\]

is called a 2-character on the 2-category

**Definition 1.9** A 2-groupoid \( \Gamma^2 \), such that

1. \( \Gamma^2 \) - totally-connected

2. A set \( \text{Hom}_2(\varphi, \psi) \) contains exactly one 2-map if there can exist a 2-map between \( \varphi \) and \( \psi \).

is called primitive 2-groupoid

Let \( X_k \) be a set of all k-characters on the \( \Gamma^2 \). Let \( \varphi_0 : \mathbb{C} \to X_0 \) be a map such that

\[ [\varphi_0(z)](g) = z \]  \hspace{1cm} (4)

where \( z \in \mathbb{C}, g \in G \)

Let \( \varphi_n : X_{n-1} \to X_n \) for \( n = 1, 2 \) be a map such that

\[ [\varphi_n(\chi)](\varphi) = \chi(t^n(\varphi)) - \chi(s^n(\varphi)) \] \hspace{1cm} (5)

In the following statements we show the correctness of this definition.
Proposal 1.1 A map $\varphi_1 : X_0 \to X_1$ is a homomorphism of the linear spaces and $\varphi_1(\chi)$ is trivial on loops.

Proof.

1. $\varphi_1(\chi_0)$ is a 1-character, since for 1-maps $\varphi \in \text{Hom}^1(a, b)$ and $\varphi' \in \text{Hom}^1(b, c)$

   $\varphi_1(\chi_0)(\varphi \circ \varphi') = \chi_0(\varphi(\varphi')) = \chi_0(c) + \chi_0(b) - \chi_0(b) - \chi_0(a) = \varphi_1(\chi_0(\varphi)) + \varphi_1(\chi_0)(\varphi')$

2. $\varphi_1(\cdot)$ is a homomorphism of the linear spaces, since

   $\varphi_1(\chi_n(1) + \chi_n(2)) = \varphi_1(\chi_n(1)) + \varphi_1(\chi_n(2))$

3. $\varphi_1(\chi_0)$ is trivial on loops, since for every $\varphi \in \text{Hom}^1(a, a)$

   $\varphi_1(\chi_0)(\varphi) = \chi_0(a) - \chi_0(a) = 0$

Proof 1.2 A map $\varphi_2 : X_1 \to X_2$ is a homomorphism of the linear spaces and $\varphi_2$ is an epimorphism.

Proof.

1. $\varphi_2(\chi_1)$ is a 2-character, since for $\varphi, \psi, \eta \in \text{Hom}^1(a, b), \varphi', \psi' \in \text{Hom}^1(b, c)$ and $\alpha : \varphi \Rightarrow \psi, \alpha' : \varphi' \Rightarrow \psi', \beta : \psi \Rightarrow \eta$

   $\varphi_2(\chi_1)(\alpha \circ \alpha') = \chi_1(\psi') + \chi_1(\psi) - \chi_1(\varphi) - \chi_1(\varphi') = \varphi_2(\chi_1)(\alpha) + \varphi_2(\chi_1)(\alpha')$

   $\varphi_2(\chi_1)(\alpha \cdot \beta) = \chi_1(\eta) - \chi_1(\varphi) = \varphi_2(\chi_1)(\alpha) + \varphi_2(\chi_1)(\beta)$

2. $\varphi_2(\cdot)$ is a homomorphism of the linear spaces, since

   $\varphi_2(\lambda \chi_1(1) + \mu \chi_1(2)) = \lambda \varphi_2(\chi_1(1)) + \mu \varphi_2(\chi_1(2))$

3. $\varphi_2(\cdot)$ is an epimorphism.

   $\forall \chi_2 \in X_2 \exists \chi_1 \in X_1 : \varphi_2(\chi_1) = \chi_2$. Let create $\chi_1$ on given $\chi_2$. Let $\chi_1(1_a) = 0 \forall a \in \text{Obj}(\Gamma^2)$. Fix the object $a$. Let $F$ - be a set, where only one representative from each $\text{Hom}^1(a, b)$ (The choice of the set $F$ is not unique). Let add in $F$ an inverse to every 1-map. Then add all of its compositions. As a result, $\forall a, b \in \text{Obj}(\Gamma^2) \exists \varphi \in \text{Hom}^1(a, b)$ such that $\varphi \in F$, also $\varphi, \psi \in F \to \varphi \circ \psi \in F$, if composition exists. Let $\chi_1(\varphi) = 0 \forall \varphi \in F$. Then if $\psi \notin F$, to $\exists \varphi \in F, \alpha \in \text{Hom}^2(\varphi, \psi)$ such that $\alpha : \varphi \Rightarrow \psi$. Let $\chi_1(\psi) = \chi_2(\alpha)$. Adjusted $\chi_1$ is a 1-character on $\Gamma^2$, such $\chi_1(\varphi \circ \psi) = \chi_1(\varphi) + \chi_1(\psi)$, since...
For this sequence to be exact, the following relations must be satisfied

\[ \text{Im } \varphi_i = \ker \varphi_{i+1}, \text{ } i \in \{s, 0, 1, 2, t\}. \]

Let’s check it out.

1. \( \text{Im } \varphi_t = \{0\} \) as well, since \( \varphi_0(z) = 0 \leftrightarrow z = 0 \) by design. A zero character is a character everywhere equal to zero.

2. \( \text{Im } \varphi_0 \) is a set of the 0-characters such that \( \chi_0 \in \text{Im } \varphi_0 \leftrightarrow \chi_0(g) = \chi_0(h) \) for every \( g, h \in G \). Since \( [\varphi_1(\chi_0)](\varphi) = \chi_0(h) - \chi_0(g) \) for \( \varphi \in \text{Hom}(g, h) \), then \( [\varphi_1(\chi_0)](\varphi) = 0 \leftrightarrow \chi_0 \in \text{Im } \varphi_0 \).

Let prove the following important proposal, which will be proved later for an arbitrary totally-connected n-groupoid \( \Gamma^n \).

**Lemma 1.1** A short sequence of algebraic objects and homomorphisms

\[ 0 \overset{\varphi_s}{\rightarrow} \mathbb{C} \overset{\varphi_0}{\rightarrow} X_0 \overset{\varphi_1}{\rightarrow} X_1 \overset{\varphi_2}{\rightarrow} X_2 \overset{\varphi_t}{\rightarrow} 0 \]

is an exact short sequence.

**Proof.** For this sequence to be exact, the following relations must be satisfied

\[ \chi_2 \text{ is a 2-character, then} \]

\[ \chi_2(\alpha \circ \beta) = \chi_2(\alpha) + \chi_2(\beta) \]

Then by construction of \( \chi_1 \):

\[ \chi_2(\alpha \circ \beta) = \chi_1(\varphi \circ \psi) \]
\[ \chi_2(\alpha) = \chi_1(\varphi), \chi_2(\beta) = \chi_1(\psi) \]
\[ \chi_1(\varphi \circ \psi) = \chi_1(\varphi) + \chi_1(\psi) \]

It follows that \( \varphi_2 \) is an epimorphism.
3. Im $\varphi_1$ - set of the trivial on loops 1-characters. Let’s show that $\varphi_2(\chi_1) = 0 \iff \chi_1$ is trivial on loops.

- Let’s first verify that if $\varphi_1(\chi_1) = 0$, leads the fact that $\chi$ is trivial on loops. If $\varphi_1(\chi_1) = 0$, then $\forall \alpha : \varphi \Rightarrow \psi$ satisfied $\chi_1(\varphi) = \chi_1(\psi)$. Let’s fix the objects $a, b$. Let’s consider $\xi \in \text{Hom}_1(a,a)$, such that $\xi = (\varphi \circ \psi)^{-1}$, $\psi^{-1}$ exists, since $\Gamma^2$ is a groupoid. Then $\chi_1(\xi) = \chi_1(\varphi) + \chi_1(\psi^{-1})$. Let’s fix the objects $a \neq b$. Let’s consider $\xi \in \text{Hom}_1(a,a)$, such that $\xi = (\varphi \circ \psi)^{-1} \psi^{-1}$ exists, since $\Gamma^2$ is a groupoid. Then $\chi_1(\xi) = \chi_1(\varphi) + \chi_1(\psi^{-1})$. Let’s consider $\xi \in \text{Hom}_1(a,a)$, such that $\xi = (\varphi \circ \psi)^{-1} \psi^{-1}$ exists, since $\Gamma^2$ is a groupoid. Then $\chi_1(\xi) = \chi_1(\varphi) + \chi_1(\psi^{-1})$.

- Inversely, $\varphi_1(\chi) = 0 \iff \chi$ is trivial on loops. Let’s fix the objects $a \neq b$. Let’s consider $\xi \in \text{Hom}_1(a,a)$, such that $\xi = (\varphi \circ \psi)^{-1} \psi^{-1}$ exists, since $\Gamma^2$ is a groupoid. Then $\chi_1(\xi) = \chi_1(\varphi) + \chi_1(\psi^{-1})$. Let’s fix the objects $a \neq b$. Let’s consider $\xi \in \text{Hom}_1(a,a)$, such that $\xi = (\varphi \circ \psi)^{-1} \psi^{-1}$ exists, since $\Gamma^2$ is a groupoid. Then $\chi_1(\xi) = \chi_1(\varphi) + \chi_1(\psi^{-1})$. Let’s fix the objects $a \neq b$. Let’s consider $\xi \in \text{Hom}_1(a,a)$, such that $\xi = (\varphi \circ \psi)^{-1} \psi^{-1}$ exists, since $\Gamma^2$ is a groupoid. Then $\chi_1(\xi) = \chi_1(\varphi) + \chi_1(\psi^{-1})$.

4. ker $\varphi_t = X_2$ by design, Im $\varphi_2 = X_2$, since $\varphi_2$ - epimorphism.

Our next goal is to generalize this lemma to the general case of $n$–categories.

**Definition 1.10** A map $\chi_m : \text{Hom}^m(C^n) \to \mathbb{C}, m \geq 1$, such that for every $\varphi, \psi \in \text{Hom}^m(C^n)$ and for every $0 \leq k < m \leq n$, where composition $\varphi \circ_k \psi$ is possible,

$$\chi_m(\varphi \circ_k \psi) = \chi_m(\varphi) + \chi_m(\psi)$$

is called $m$-character on the $n$-category $C^n$.

Let $X_k$ be a set of all $k$-characters on the groupoid $\Gamma^\infty$.

Let $\varphi_m : X_{m-1} \to X_m$ for $m \geq 1$ be a map such that

$$\varphi_m(\chi)(\varphi) = \chi(t^n(\varphi)) - \chi(s^n(\varphi)) \quad (9)$$

In the following statements we show the correctness of this definition.

**Proposal 1.3** A map $\varphi_m$ is a homomorphism on the linear spaces.

**Proof.**

1. Let fix $\text{Hom}^k(\Gamma^n), 0 \leq k < m \leq n$. Let $\varphi, \psi \in \text{Hom}^m(\Gamma^n)$ such that $\varphi : a \to b, \psi : b \to c, \varphi \circ_k \psi : a \to c$ for $a, b, c \in \text{Hom}^k(\Gamma^n)$. Let’s proof that $\varphi_m(\cdot)$ is a $m$-character.
\[
\chi_m(\varphi \circ_k \psi) = \chi_{m-1}(c) - \chi_{m-1}(a) \\
\chi_{m-1}(c) - \chi_{m-1}(a) = \left[\chi_{m-1}(c) - \chi_{m-1}(b)\right] + \left[\chi_{m-1}(b) - \chi_{m-1}(a)\right] \\
\chi_m(\varphi \circ_k \psi) = \chi_m(\varphi) + \chi_m(\psi)
\]

2. \(\varphi_m(\cdot)\) is a homomorphism, since for every \(\chi^{(1)}_{m-1}, \chi^{(2)}_{m-1} \in X_{m-1}\) the following equation is performed

\[
\varphi_m(\chi^{(1)}_{m-1} + \chi^{(2)}_{m-1}) = \varphi_m(\chi^{(1)}_{m-1}) + \varphi_m(\chi^{(2)}_{m-1})
\]

Let \(\varphi_1 : C \rightarrow X_0\) be a map, such that

\[
[\varphi_1(z)](g) = z
\]

where \(z \in C, g \in G\)

**Theorem 1.1** A sequence of algebraic objects and homomorphisms

\[
0 \rightarrow C \xrightarrow{\varphi_0} X_0 \xrightarrow{\varphi_1} X_1 \rightarrow \cdots \rightarrow X_{m-1} \xrightarrow{\varphi_m} X_m \rightarrow \cdots
\]

is an exact sequence.

**Proof.** For this sequence to be exact, the following relations must be satisfied:

\[
\text{Im } \varphi_i = \ker \varphi_{i+1}, \ i \in \mathbb{N}_0
\]

Let’s check it out:

1. \(\text{Im } \varphi = \{0\}\). Also \(\ker \varphi_0 = \{0\}\), since \(\varphi_0(z) = 0 \iff z = 0\) by design.

2. A category, where a set \(\text{Hom}^{m-2}(\Gamma^n)\) is an object set, \(\text{Hom}^{m-1}(\Gamma^n)\) – is a set of 1-morphisms and \(\text{Hom}^m(\Gamma^n)\) – is a set of 2-morphisms, is a 2-category by definition. And it is a 2-groupoid in our case.

\[
X_{m-2} \xrightarrow{\varphi^{m-1}_{m-2}} X_{m-1} \xrightarrow{\varphi_m} X_m
\]

We apply Lemma 1.1 to this section. This groupoid is not primitive, but in this case the epimorphicity of the morphism \(\varphi_m\) is not important, so the lemma is applied correctly.

In terms of the constructed sequence, it is possible to describe a primitive 2-groupoid:

**Proposal 1.4** If for some \(n\)-groupoid, the 2-groupoid formed by the sets \(\text{Hom}^0(\Gamma^n), \text{Hom}^1(\Gamma^n)\) and \(\text{Hom}^2(\Gamma^n)\) is primitive, then the cohomology \((\ker \varphi_m \setminus \text{Im } \varphi_{m-1})\) starting with \(m = 3\) are trivial.
2 The 2-category example

Let’s create a 2-groupoid $\Gamma^2$ based on infinite noncommutative group $G$ in the following way ([4] p. 18):

- $\text{Obj}(\Gamma^2) = G$
- $\text{Hom}(a,b) = \{(u,v) \in G \times G | v^{-1}u = a, uv^{-1} = b\}$ for every $a, b \in \text{Obj}(\Gamma^2)$
- Composition of maps $\varphi = (u_1, v_1) \in \text{Hom}(a,b), \psi = (u_2, v_2) \in \text{Hom}(b,c)$ is a map $\varphi \circ \psi \in \text{Hom}(a,c)$ such that:
  $$\varphi \circ \psi = (u_2v_1, v_2v_1)$$
- A single 2-map $\alpha : \varphi \Rightarrow \psi$ is defined for every $\varphi, \psi \in \text{Hom}(a,b)$

$\Gamma^2$ can be represented as a disjoint union ([4] p. 19):

$$\Gamma^2 = \bigsqcup_{[u] \in [G]} \Gamma^2_{[u]}$$

3 The connection between the $m$-characters and derivations of the group algebra

3.1 General definitions

Let $d$ be a differentiation operator. Consider element $u$ of the group algebra $\mathbb{C}[G]$, which can be represented as $u = \sum_{g \in G} \lambda^g g$, where sum is finite. Then element $d(u)$ can be represented as ([4] p. 17)

$$d(u) = \sum_{g \in G} \left( \sum_{h \in G} d^h g^h \right) g,$$

where $d^h \in \mathbb{C}$ – coefficients that depend only on the derivation $d$.

Let $\chi_d : \text{Hom}(\Gamma) \rightarrow \mathbb{C}$ be a map, based on the derivation $d$, defined as

$$\chi_d((h, g)) = d^h_g.$$  (12)

Proposal 3.1 A map $\chi_d$ is a $1$-character.
Definition 3.1 A 1-character $\chi$ – such that for every group element $v \in G$

$$\chi(x, v) = 0,$$

almost for every $x \in G$ is called locally finite 1-character.

1-character $\chi$ sets the derivation if and only if it is locally finite ([1]).

Let $X_{fin}^1$ be a set of the all locally finite 1-characters on the $\Gamma^2$.

Let $X_{0}^{fin}$ be a set such that

$$X_{0}^{fin} = \{\chi_0 \in X_0 | \varphi_2(\chi_0) \in X_1^{fin}\}$$

Let $X_{2}^{fin}$ be a set such that

$$X_{2}^{fin} = \varphi_3(X_1^{fin})$$

Considered 2-groupoid $\Gamma^2$ is primitive, and the following short sequence is exact.

$$0 \xrightarrow{\phi} \mathbb{C} \xrightarrow{\varphi_0} X_0^{fin} \xrightarrow{\varphi_1} X_1^{fin} \xrightarrow{\varphi_2} X_2^{fin} \xrightarrow{\varphi_3} 0$$  \hspace{1cm} (13)

3.2 Examples of the derivations

A derivation is called internal if it is given by a formula ([2], p. 21)

$$d_a : x \rightarrow [a, x], a \in \mathbb{C}[G].$$

The Lie subalgebra of the inner derivations $\text{Der}_{inn} \subset \text{Der}$ is an ideal.

The quotient set $\text{Der}_{out} := \text{Der} \setminus \text{Der}_{inn}$ is called a set of the outer derivations.

Consider an element $a \in G$. Let $\chi^a : \text{Hom}(\Gamma) \rightarrow \mathbb{C}$ be a map defined as follows. If $b \neq a$ – is an element of the group $G$, then for every map $\phi \in \text{Hom}(a, b)$ with the source $a$ and the target $b$ let $\chi^a(\phi) = 1$, for every map $\psi \in \text{Hom}(b, a)$ let $\chi^a(\psi) = -1$. For the rest maps let $\chi^a$ equals to zero.

Proposal 3.2 The character $\chi^a$ is the 1-character. The 1-character $\chi^a$ is the 1-character defined by an inner differentiation:

$$d_a : x \rightarrow [x, a]$$  \hspace{1cm} (14)
3.3 Algebra of the 2-characters

The linear space $\text{Der}$ is a Lie algebra with the commutator ([6], p. 206):

$$[d_1, d_2] = d_1 d_2 - d_2 d_1$$

**Proposal 3.3** The values of the 1-character $\chi_{[d_1, d_2]} = \{\chi_{d_1}, \chi_{d_2}\}$ are defined by $\chi_{d_1}$ and $\chi_{d_2}$ as follows

$$\{\chi_{d_1}, \chi_{d_2}\}(a, g) = \sum_{h \in G} \chi_{d_1}(a, h) \chi_{d_2}(h, g) - \chi_{d_2}(a, h) \chi_{d_1}(h, g) \quad (15)$$

**Proof.** Let $g \in G$, then:

$$d_1(g) = \sum_{h \in G} \chi_{d_1}(h, g) h$$
$$d_2(g) = \sum_{h \in G} \chi_{d_2}(h, g) h$$

$$[d_1, d_2](g) = \sum_{h \in G} \{\chi_{d_1}, \chi_{d_2}\}(a, g) a$$

Represent the expression for the commutator by definition:

$$[d_1, d_2](g) = d_1 d_2(g) - d_2 d_1$$
$$d_1 d_2(g) = \sum_{h \in G} \chi_{d_2}(h, g) (\sum_{a \in G} \chi_{d_1}(a, h) a)$$
$$d_2 d_1(g) = \sum_{h \in G} \chi_{d_1}(h, g) (\sum_{a \in G} \chi_{d_2}(a, h) a)$$

Change the sum order in the last expressions:

$$[d_1, d_2](h) = \sum_{a \in G} \left( \sum_{h \in G} \chi_{d_2}(h, g) \chi_{d_1}(a, h) - \chi_{d_2}(a, h) \chi_{d_1}(h, g) \right) a$$

The formula for $\{\chi_{d_1}, \chi_{d_2}\}(a, h)$ is a coefficient of $a$, i.e.

$$\{\chi_{d_1}, \chi_{d_2}\}(a, g) = \sum_{h \in G} \chi_{d_1}(a, h) \chi_{d_2}(h, g) - \chi_{d_2}(a, h) \chi_{d_1}(h, g)$$

Define a weak inner derivation space $D^*_{\text{Inn}}$ as follows:

$$D^*_{\text{Inn}} = \{d \in \text{Der} \mid \chi_d \text{ - trivial on loops}\}$$

**Theorem 3.1** Linear space $\text{Der}^*_{\text{Inn}} \subset \text{Der}$ is an ideal:

$$d_0 \in \text{Der}^*_{\text{Inn}}, d \in \text{Der} \Rightarrow [d_0, d], [d, d_0] \in \text{Der}^*_{\text{Inn}}$$

**Proof.** Let's proof that $\text{Der}^*_{\text{Inn}} \subset \text{Der}$ is a subalgebra, i.e.
\[ d_1, d_2 \in D_{inn}^* \Rightarrow [d_1, d_2] \in D_{inn}^* \]

1-character \( \chi_{d_1} \) can be represented as:
\[
\chi_{d_1} = \sum_{a \in G} \lambda^a \chi^a, \lambda^a \in \mathbb{C}
\]

since \( \chi_{d_1} \) is trivial on loops, where \( \chi^a \) is defined by formula (14)

Similarly with a 1-character \( \chi_{d_2} \):
\[
\chi_{d_2} = \sum_{b \in G} \mu^b \chi^b, \mu^b \in \mathbb{C}
\]

Because of the bilinearity of the commutator,
\[
\{ \chi_{d_1}, \chi_{d_2} \} = \sum_{a \in G} \sum_{b \in G} \lambda^a \mu^b \{ \chi^a, \chi^b \}
\]

The commutator of the \( d^a \) and \( d^b \) can be represented as follows:
\[
[d^a, d^b] = d^{ab} - d^{ba}
\]

Then define the 1-character \( \{ \chi^a, \chi^b \} \) by the following formula
\[
\{ \chi^a, \chi^b \} = \chi^{ab} - \chi^{ba}
\]

Obtain the final expression for \( \{ \chi_{d_1}, \chi_{d_2} \} \):
\[
\{ \chi_{d_1}, \chi_{d_2} \} = \sum_{a \in G} \sum_{b \in G} \lambda^a \mu^b \chi^{ab} - \sum_{a \in G} \sum_{b \in G} \lambda^a \mu^b \chi^{ba}
\]

\( \{ \chi_{d_1}, \chi_{d_2} \} \in \text{Der}_{inn}^* \), to proof that, consider the value of the character on the loop \((uz, z), z \in Z_G(u)\)
\[
\{ \chi_{d_1}, \chi_{d_2} \}(uz, z) = \sum_{ab=uz^{-1}} \lambda^a \mu^b - \sum_{ab=u} \lambda^a \mu^b - \sum_{ba=u} \lambda^a \mu^b + \sum_{ba=uz^{-1}} \lambda^a \mu^b = 0
\]

Now pass directly to the proof of the theorem. Represent \( \chi_{d_0} \) as
\[
\chi_{d_0} = \sum_{a \in G} \lambda^a \chi^a
\]

Proof theorem for \( \chi^a \) and extend the result to the \( \chi_{d_0} \), using the bilinearity of the commutator. Consider 1-character \( \{ \chi_d, \chi^a \} \). Proof that it is trivial on loops, i.e. \( \forall b \in G \) and for \( \forall z \in Z_G(b) \) is performed \( \{ \chi_d, \chi^a \}(bz, z) = 0 \). By the formula (15):
\[
\{ \chi_d, \chi^a \}(bz, z) = \sum_{h \in G} \chi_d(bz, h) \chi^a(h, z) - \chi^a(bz, h) \chi_d(h, z)
\]
χ^a(h, z) ≠ 0 only in 2 cases: when h = za and h = az. χ^a(bz, h) ≠ 0 only in 2 cases: when h = bza^{-1} and h = a^{-1}bz. It means that

\{χ_d, χ^a\}(bz, z) = χ_d(bz, za) − χ_d(bz, az) + χ_d(a^{-1}bz, z) − χ_d(bza^{-1}, z)

But also

(a^{-1}bz, z) ◦ (bz, za) = (bz, az) ◦ (bza^{-1}, z)

That means

χ_d(bz, za) + χ_d(a^{-1}bz, z) = χ_d(bz, az) + χ_d(bza^{-1}, z)

\{χ_d, χ^a\}(bz, z) = 0

Q.E.D.

In accordance with the last statement, the space of weak external derivations becomes meaningful.

\text{Der}^*_\text{Out} := \text{Der} \setminus \text{Der}^*_\text{Inn}

Similarly, the Lie subalgebra \(X^f_{\text{fin}}\subset X^f_{\text{fin}}\) is an ideal. Thus redefine space of the 2-characters as follows:

\(X^f_{\text{fin}} = X^f_{\text{fin}} \setminus X^f_{\text{fin}}\)

**Theorem 3.2** Lie algebra \((X^f_{\text{fin}}, \{\cdot, \cdot\})\) is isomorphic to the Lie algebra \((\text{Der}^*_\text{Out}, [\cdot, \cdot])\).

**Proof.** Let \(F\) be an isomorphism between \(\text{Der}\) and \(X^f_{\text{fin}}\), which is defined if paragraph 3.1. Thus define an isomorphism \(F^*\) as follows

\[F^* : \text{Der}^*_\text{Out} \rightarrow X^f_{\text{fin}}\]

\[F^*(d + \text{Der}^*_\text{Inn}) = F(d) + X^f_{\text{fin}}\]

Show that it preserves the commutation operation. By the formula (15):

\[\{F^*(d_1 + \text{Der}^*_\text{Inn}), F^*(d_2 + \text{Der}^*_\text{Inn})\} = \{F(d_1), F(d_2)\} + X^f_{\text{fin}} = F^*([d_1, d_2] + \text{Der}^*_\text{Inn})\]
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