An Exploratory Method for Smooth/Transient Decomposition
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Abstract—We consider a separation problem where the observation consists of the sum of a high amplitude smooth signal and a low amplitude transient signal. We propose a method for decomposition that relies on solving instances of a ‘constrained filtering problem,’ which is posed as a convex minimization problem. We provide a fast algorithm for solving the minimization problem, and demonstrate the potential of the scheme on a vital signs monitoring experiment using radar.

Index Terms—Constrained filtering, gaussian process, signal decomposition, empirical mode decomposition, vital signs monitoring, radar.

I. INTRODUCTION

This letter develops a signal analysis tool for accurately estimating the components of a composite signal, where precise models are not available. We consider a problem where a small amplitude transient signal is mixed with a large amplitude smooth signal. Our motivating application is radar-based vital signs monitoring. In that application, radar picks up a 1D motion signal from a subject’s chest (see Fig. 1a). This signal comprises respiration and heart activity components. Respiration is slower, but significantly larger in amplitude than heart activity. The heart activity component is not easy to model because its shape depends on the antenna beam pattern, position of the radar relative to the subject, radar operating frequency (which in turn determines the amount of radiation penetrating the body, if any) [1].

A. Relevant Approaches

Arguably the simplest method for extracting heart activity information is linear-time invariant (LTI) bandpass filtering [2]. Since respiration rate (~10-20 breaths per minute) and heart rate (~40-120 beats per minute) lie in non-overlapping intervals, LTI filtering appears plausible. However, LTI filtering ignores the fact that the two components have harmonics. When we apply a bandpass filter to keep components in the range [40,120] beats per minute, we (i) allow the harmonics of respiration to be part of the heart activity estimate, (ii) lose the higher harmonics of heart activity. Due to (i), we end up with a smoother signal, and lose the impulse-train-like appearance, making the exact peak locations more ambiguous (see Fig. 6b). Due to (i), the shape of the heart activity estimate may be slightly altered, leading to additional error when locating the peaks. Therefore, we assert that LTI filtering is not ideal for this problem.

An interesting thread of relevant research is referred to as ‘morphological component analysis,’ or ‘resonance based signal processing’ - see e.g., [3], [4] for an overview for images, and [5]–[7] for applications to 1D signals. These frameworks assume that components can be parsimoniously represented in distinct bases/frames. The sparsest solution in the union of the frames leads to the sought decomposition. In our problem, however, we do not have a viable model for either component, other than the vague statement of smoothness for the high magnitude component. In our experiments, resonance based processing did not yield good results, possibly because we were not able to find significantly distinct frames that can parsimoniously represent the individual components.

Another potentially useful framework for this problem is empirical mode decomposition (EMD) [8]. EMD aims to decompose a signal into ‘intrinsic mode functions’ (IMFs) such that each IMF has a unique time-varying frequency. EMD is originally defined through an algorithm [8], but other interpretations with alternative formulations/algorithms for extracting the components have been proposed [9], [10]. EMD has the potential to alleviate the leakage of the respiration harmonics into the heart activity estimate, that is an issue for LTI filtering. However, in practice, off-the-shelf EMD does not specify which IMF belongs to which component, so is not straightforward to use. We found that the components produced by EMD may be contaminated by a mixture of respiration and heart activity (see Fig. 5). In a similar context, we also refer to [11] for a study of how EMD can be improved for the assessment of electromagnetic interference.

B. Proposed Method

We pose the problem as decomposing a signal into smooth and transient components. Let us outline our proposed approach on a toy example. We get to observe the composite signal in Fig. 2b, and would like to recover the components in Fig. 2a. As in the first step of EMD, we fit upper and lower envelopes to
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Fig. 2. A simple example to demonstrate the idea. (a) The smooth (thick line) and transient (thin line) component signals, making up the observed composite signal in (b).

Fig. 3. Given the composite observation (thick line), we fit upper and lower envelopes (thin lines), that are as smooth as possible. The smoothness of the envelopes is disrupted by the perturbations in the observation.

Fig. 4. (a) Given the envelopes, the ‘smoothest’ signal (thick line) that lies between the envelopes (thin lines) forms our estimate of the smooth component. (b) Subtracting the smooth component estimate from the composite observation gives the estimate of the transient.

the observed signal. However, unlike in EMD, these envelopes are chosen to snugly sandwich the signal. The upper (lower) envelope is selected such that it is (i) as smooth as possible, (ii) nowhere less (greater) than the observed signal, and (iii) as close as possible to the original signal (see Fig. 3). Note that the gap between the envelopes is reduced if the transient signal magnitude is small. Given the envelopes, the smooth component is estimated as the ‘smoothest signal’ that lies between the envelopes (see Fig. 4). In seeking the smoothest signal, we no longer require the estimate to be close to the original composite signal, as the envelopes contain information about the composite signal.

The proposed method may be interpreted as LTI filtering under a nonlinear constraint (the output is constrained to lie between the envelopes). Since the envelopes are snug, their shape resembles that of the original smooth signal, except for disturbances caused by the transient signal. This in turn leads to a smooth signal estimate that preserves the harmonics (with the correct phase) of the underlying smooth signal.

One feature we want to emphasize is all steps of the method are realized by solving an instance of a ‘constrained filtering problem’. The constrained filtering problem to be introduced is a quadratic program, but amounts to applying a nonlinear operation on the input. By considering a specific dual of the problem, we make use of the problem’s structure to devise a fast algorithm that is computationally favorable to direct off-the-shelf quadratic program solvers.

C. Outline

We introduce the constrained filtering problem in Section II. In Section III, we consider the dual of this problem and derive an efficient algorithm for solving it. An experiment demonstrating the utility of the proposed algorithm on a real signal is described in Section IV. Section V contains remarks about noise.

II. FORMULATION

In this section, we introduce the ‘constrained filtering problem’, and discuss how to use it to achieve a decomposition.

A. Constrained Filtering

Part of our problem requires us to estimate a smooth signal. Gaussian processes are usually used as a prior distribution over smooth random signals, and are useful for deriving principled formulations [12], [13]. Specifically, suppose a signal of interest $x$ is modelled as a stationary zero-mean Gaussian process with covariance

$$C_\sigma(x_n, x_m) = \epsilon \delta(n - m)$$

$$+ \begin{cases} 0, & \text{if } \exp\left(-\frac{(n - m)^2}{\sigma^2}\right) < \tau. \\ \exp\left(-\frac{(n - m)^2}{\sigma^2}\right), & \text{otherwise.} \end{cases}$$

(1)

where $\tau > 0$ is a threshold, $\epsilon$ is a small constant.\(^1\) Note that, if $|n - m|$ is large, then $C_\sigma(x_n, x_m) = 0$. For $x \in \mathbb{R}^N$, $C_\sigma$ is an $N \times N$ Toeplitz matrix. Using $C_\sigma$, let us define $S_\sigma(x) = \frac{1}{2}x^TC_\sigma^{-1}x$.

If $y$ denotes noisy observations of a smooth signal, a denoising formulation based on Gaussian processes could be,

$$\arg\min_x \frac{1}{2}\|y - x\|^2_2 + S_\sigma(x).$$

(2)

\(^1\)The addition of the term $\epsilon \delta(n - m)$ ensures that the covariance matrix is positive semi definite if $\epsilon$ is sufficiently large – see [14] for a further discussion, and alternatives to ensure positive definiteness.
**A. A Dual Problem**

Let us denote the constraint set as \( B = \{ x : a_n \leq x_n \leq b_n \} \). We can write (3) as

\[
\min_x \frac{\lambda}{2} \| y - x \|_2^2 + \sigma \| x \|_1 \quad \text{subject to} \quad a_i \leq x_i \leq b_i,
\]

where \( y \in \mathbb{R}^N \) is an observation, \( \lambda \in \mathbb{R}_+ \) is a weight parameter, \( a_i < b_i \)'s are given constants. We denote the minimizer of (3) as \( \hat{x}_{\lambda,\sigma,a,b} \). This problem seeks the ‘smoothest’ signal in a given interval, that is close to \( y \). Relying on our previous interpretation of (2), we regard the mapping \( y \to \hat{x}_{\lambda,\sigma,a,b} \) as a constrained filtering operation.

The problem (3) is simple but flexible enough to realize all steps of the proposed method, outlined in the Introduction. Specifically, if we set \( a = y, b = \infty \), we obtain an upper envelope, \( u \). For \( a = -\infty, b = y \), we obtain a lower envelope, \( \ell \). Finally, setting \( a = \ell, b = u \), we obtain the estimate of the smooth component. Subtracting the smooth component from the composite observation, we obtain the transient component.

We next discuss briefly how to set the parameters in (3).

**B. Parameters of the Formulation**

We expect different behavior from the envelopes and the smooth signal estimate. In order to make the envelopes fit tightly, we can increase the value of \( \lambda \), or penalize deviation from smoothness less by reducing \( \sigma \). Once we have the envelopes, to reduce the influence of the underlying observation \( y \), we reduce \( \lambda \), and possibly increase \( \sigma \), to obtain a smoother signal. These considerations lead to Algorithm 1.

We next discuss how to efficiently solve (3).

### III. Reformulating the Problem

The problem (3) is a quadratic program [16]. Direct approaches to this problem require multiplications with \( C_{\sigma}^{-1} \) (see e.g., Chp. 16 of [17]). Unfortunately, the Toeplitz structure of \( C_{\sigma} \) is lost during inversion, and \( C_{\sigma}^{-1} \) is not available in closed form. Further, even if we had \( C_{\sigma}^{-1} \), lack of structure prevents us to realize multiplication with \( C_{\sigma}^{-1} \) efficiently. To exploit the Toeplitz structure of \( C_{\sigma} \), we consider a dual of (3), and derive a splitting algorithm that uses FFTs.

**Algorithm 1: Smooth Component Estimation.**

**Require:** Input signal \( y \)

1: Set \( 0 < \lambda_1 \ll \lambda_0, 0 < \sigma_0 \leq \sigma_1 \)
2: \( \ell \leftarrow \hat{x}_p \) for \( p = \{ y, \lambda_0, \sigma_0, \min(y), y \} \) \% lower envelop.
3: \( u \leftarrow \hat{x}_p \) for \( p = \{ y, \lambda_0, \sigma_0, y, \max(y) \} \) \% upper envelop.
4: \( x^* \leftarrow \hat{x}_p \) for \( p = \{ y, \lambda_1, \sigma_1, \ell, u \} \) \% smooth component
5: \( t^* \leftarrow y - x^* \) \% transient component

This formulation coincides with that of maximum a posteriori (MAP) estimation [15], where the first term is the likelihood, provided the noise is standard Gaussian. The solution of (2) is \( (I + C_{\sigma}^{-1})^{-1} y \). If \( y \) were an infinite length discrete-time signal, this operation would be equivalent to LTI lowpass filtering with a kernel determined by \( C_{\sigma} \). For finite-length \( y \), this is no longer LTI filtering exactly, but only approximately.

Consider now the following variation on (2):

\[
\min_x \frac{\lambda}{2} \| y - x \|_2^2 + S_\sigma(x) \text{ subject to } a_i \leq x_i \leq b_i,
\]

where \( y \in \mathbb{R}^N \) is an observation, \( \lambda \in \mathbb{R}_+ \) is a weight parameter, \( a_i < b_i \)'s are given constants. We denote the minimizer of (3) as \( \hat{x}_{\lambda,\sigma,a,b} \). This problem seeks the ‘smoothest’ signal in a given interval, that is close to \( y \). Relying on our previous interpretation of (2), we regard the mapping \( y \to \hat{x}_{\lambda,\sigma,a,b} \) as a constrained filtering operation.

The problem (3) is simple but flexible enough to realize all steps of the proposed method, outlined in the Introduction. Specifically, if we set \( a = y, b = \infty \), we obtain an upper envelope, \( u \). For \( a = -\infty, b = y \), we obtain a lower envelope, \( \ell \). Finally, setting \( a = \ell, b = u \), we obtain the estimate of the smooth component. Subtracting the smooth component from the composite observation, we obtain the transient component.

We next discuss briefly how to set the parameters in (3).

**B. A Modified Problem**

We will obtain a fast algorithm for (7) by adapting the Douglas-Rachford (DR) algorithm [19]–[21]. In principle, any convex splitting algorithm can be used for tackling (4) or (7) [19], [20], [22]. We opt for the DR algorithm because its steps involve solutions of non-trivial problems, but can be efficiently realized in our case, and it does not require variable splitting.

A crucial ingredient for DR is the proximity operator [20].

**Definition 1:** For a convex \( g : \mathbb{R}^n \to \mathbb{R} \), and \( \alpha > 0 \), the proximity operator for \( g \), namely \( J_{\alpha g} : \mathbb{R}^n \to \mathbb{R}^n \), is defined as

\[
J_{\alpha g}(z) = \arg \min_x \frac{1}{2} \| x - z \|_2^2 + \alpha g(x).
\]

For a convex problem involving the sum of two functions,

\[
\min_x f(x) + g(x),
\]

the DR iterations are of the form [20]

\[
u^n = \gamma u^{n-1} + (1 - \gamma) \left( N_{\alpha g} \left( N_{\alpha f}(u^{n-1}) \right) \right),
\]

for \( 0 < \gamma < 1 \), where \( N_{\alpha f}(\cdot) := (2J_{\alpha f} - I)(\cdot), N_{\alpha g}(\cdot) := (2J_{\alpha g} - I)(\cdot) \). This sequence converges to a point \( u \) such that \( x = J_{\alpha f}(u) \) solves (10).

To employ DR, we need to split the function in (7). However, straightforward splitting requires applying \( (I + \alpha C_{\sigma})^{-1} \) at each iteration, and we face the same issue of inverting \( C_{\sigma} \). To avoid this, we consider a modified problem, following [23].

Specifically, we note that \( C_{\sigma} \) can be embedded in a larger circulant matrix \( \tilde{C}_{\sigma} \) (see e.g. Sec. V in [23]) as

\[
\tilde{C}_{\sigma} = \begin{bmatrix}
C_{\sigma} & D_0 \\
D_1 & D_2
\end{bmatrix},
\]
Algorithm 2: Computation of $\hat{x}_{y,\lambda,\sigma,a,b}$ That Solves (3).

1: Set $0 < \gamma < 1, 0 < \alpha$.
2: Given $N \times N$ Toeplitz $C_{\sigma}$, find the smallest $(N + K) \times (N + K)$ circulant $\overline{C}_{\sigma}$ s.t. (12) holds.
3: Initialize $u \in \mathbb{R}^N$, $\tilde{u} \in \mathbb{R}^N$.
4: $c \leftarrow \lambda(y - (1 + \alpha/\lambda)a)$
5: $d \leftarrow \lambda(y - (1 + \alpha/\lambda)b)$
6: repeat
7: $\left[ \begin{array}{c} \tilde{u} \\ \tilde{z} \end{array} \right] \leftarrow \left(2(I + \alpha\overline{C}_{\sigma})^{-1} - I \right) \left[ \begin{array}{c} \tilde{u} \\ \tilde{z} \end{array} \right]$ %using FFT
8: $t_n \leftarrow \begin{cases} \frac{2\alpha y_n + (1 - \alpha/\lambda)t_n}{1 + \alpha/\lambda}, & \text{if } d_n \leq t_n \leq c_n, \\
\frac{t_n + 2\alpha a_n}{1 + \alpha/\lambda}, & \text{if } c_n < t_n,
\end{cases}$ for $n = 1, \ldots, N$
9: $u \leftarrow \gamma u + (1 - \gamma)t$
10: $\tilde{u} \leftarrow \tilde{u} - (1 - \gamma)\tilde{t}$
11: until some convergence criterion is met
12: $\left[ \begin{array}{c} \tilde{z} \\ \tilde{z} \end{array} \right] \leftarrow \left(2(I + \alpha\overline{C}_{\sigma})^{-1} - I \right) \left[ \begin{array}{c} \tilde{u} \\ \tilde{z} \end{array} \right]$ %using FFT
13: $\hat{x}_{y,\lambda,\sigma,a,b} \leftarrow P_B(y - z/\lambda)$

for some $D_1$’s, where the sizes of smallest $D_1$’s are determined by the band-size of the Toeplitz $C_{\sigma}$. Notice now that

$$z^T C_{\sigma} z = \left[ \begin{array}{c} z \\ \tilde{z} \end{array} \right]^T \overline{C}_{\sigma} \left[ \begin{array}{c} z \\ \tilde{z} \end{array} \right], \quad \text{for } \tilde{z} = 0. \quad (13)$$

This motivates the following modification of (7):

$$\min_{z,\tilde{z}} \left\{ h(z, \tilde{z}) := \frac{1}{2} \left[ \begin{array}{c} z \\ \tilde{z} \end{array} \right]^T \overline{C}_{\sigma} \left[ \begin{array}{c} z \\ \tilde{z} \end{array} \right] + i_0(\tilde{z}) + \frac{\lambda}{2} \left(2\left(y - \frac{z}{\lambda}\right) - P_B\left(y - \frac{z}{\lambda}\right), P_B\left(y - \frac{z}{\lambda}\right)\right) \right\}, \quad (14)$$

where $i_0(\tilde{z})$ is the indicator function for the set $\{0\}$.

We summarize the discussion in a proposition.

**Proposition 1:** If $(z^*, \tilde{z}^*)$ is a solution to (14), then $x^* = P_B\left(y - \frac{z^*}{\lambda}\right) = C_{\sigma} z^*$ is a solution to (3).

The DR algorithm on a specific splitting of this cost function leads to Algorithm 2 (see Appendix B for the details). Although this algorithm addresses a problem with more variables than (3), the ability to use the Toeplitz structure makes up for the increase in dimension – see [23] for a discussion.

**IV. Demonstration of the Method on Real Data**

We consider a vital signs monitoring experiment with real data, obtained from radar.\(^2\) The subject was stationary during signal acquisition, and kept a steady breathing pattern for about five minutes. An excerpt from the phase signal from radar is shown in Fig. 1. We compare the results of the proposed algorithm, bandpass filtering, as well as EMD with this signal. Some details of the experiment can be found in Appendix C.

The first six IMFs from EMD are shown in Fig. 5.\(^3\) The remaining IMFs all have higher amplitudes, and slow variation, and therefore are not correlated with heart activity. Note that, no single IMF really captures a plausible heart activity. To estimate a heart activity signal, we consider the sum from second up to fifth IMF, because the first IMF is noise like, and the sixth IMF contains a high amplitude segment which cannot be coming from heart activity. The resulting heart activity estimate is shown in Fig. 6a.

Fig 6b shows the output of a linear phase bandpass filter applied to the input. The heart activity signal estimated with the proposed method is shown in Fig 6c. In Fig. 6a–c, vertical bars indicate the peaks of the heart activity signal obtained with the proposed method. We see that, occasionally, the EMD estimate is in sync with the proposed method, but in general, the behavior of the EMD estimate is not consistent. Bandpass filtering produces a fairly reasonable estimate, but it occasionally misses beats, and its peaks deviate around those of the proposed method. Overall, we found that the standard deviation of beat to beat intervals obtained from the bandpass filtered estimate is higher than usual. We therefore believe that the deviation of the peaks is due to the additional bias introduced by the harmonics of respiration.

**V. Discussion**

One issue we have not addressed is observation noise. Lacking a further model on noise, the proposed method is likely to produce noisy estimates of the transient component. Therefore, we recommend denoising as a preprocessing stage. As long as the transient component is not buried in noise, we expect the method to provide useful estimates, as demonstrated in the experiment with a real signal.

**Acknowledgement**

The author thanks Sundar Palani, Analog Devices Inc., for his help in obtaining the radar signal, and the anonymous reviewers for their suggestions.

\(^2\) An additional experiment with synthetic data is available in [24].

\(^3\) We used the Matlab EMD code by G. Rilling, associated with [25].
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