On the Complexity of Recognizing Integrality and Total Dual
Integrality of the \(\{0, 1/2\}\)-Closure

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Abstract

The \(\{0, 1/2\}\)-closure of a rational polyhedron \(\{x: Ax \leq b\}\) is obtained by adding all
Gomory-Chvátal cuts that can be derived from the linear system \(Ax \leq b\) using multipliers
in \(\{0, 1/2\}\). We show that deciding whether the \(\{0, 1/2\}\)-closure coincides with the integer hull
is strongly NP-hard. A direct consequence of our proof is that, testing whether the linear
description of the \(\{0, 1/2\}\)-closure derived from \(Ax \leq b\) is totally dual integral, is strongly
NP-hard.

1 Introduction

Let \(P = \{x \in \mathbb{R}^n : Ax \leq b\}\) with \(A \in \mathbb{Z}^{m \times n}\) and \(b \in \mathbb{Z}^m\) be a rational polyhedron. The integer
hull of \(P\) is denoted by \(P_I = \text{conv}(P \cap \mathbb{Z}^n)\). Any inequality of the form \(u^T Ax \leq \lfloor u^T b \rfloor\)
where \(u \in \mathbb{R}_{\geq 0}^m\) and \(u^T A \in \mathbb{Z}^n\) is valid for \(P_I\). Inequalities of this kind are called Gomory-Chvátal
cuts [5, 16]. The intersection of all halfspaces corresponding to Gomory-Chvátal cuts yields the Gomory-Chvátal closure \(P'\) of \(P\). In fact, \([0, 1)\]-valued multipliers \(u\) suffice (see, e.g., [7]), i.e.,

\[
P' = \{x \in P : u^T Ax \leq \lfloor u^T b \rfloor, u \in [0, 1)^m, u^T A \in \mathbb{Z}^n\}.
\]

Caprara and Fischetti [4] introduced the family of Gomory-Chvátal cuts with multipliers
\(u \in \{0, 1/2\}^m\). We refer to them as \(\{0, 1/2\}\)-cuts. The \(\{0, 1/2\}\)-closure of \(P\) is defined as

\[
P_{1/2}(A, b) := \{x \in P : u^T Ax \leq \lfloor u^T b \rfloor, u \in \{0, 1/2\}^m, u^T A \in \mathbb{Z}^n\}.
\]

Note that \(P_{1/2}(A, b)\) depends on the system \(Ax \leq b\) defining the polyhedron \(P\). From the
definition, it follows that \(P_I \subseteq P' \subseteq P_{1/2}(A, b) \subseteq P\).

\(\{0, 1/2\}\)-cuts are prominent in polyhedral combinatorics; examples of classes of inequalities
that can be derived as \(\{0, 1/2\}\)-cuts include the blossom inequalities for the matching polytope [5,11] and the odd-cycle inequalities for the stable set polytope [14]. Both classes of inequalities
can be separated in polynomial time [14,20]. In general, though, separation (and, thus, optimization) over the \(\{0, 1/2\}\)-closure of polyhedra is NP-hard: Caprara and Fischetti [4] show that the following membership problem for the \(\{0, 1/2\}\)-closure is strongly coNP-complete (see also [13, Theorem 2]).

Given \(A \in \mathbb{Z}^{m \times n}, b \in \mathbb{Z}^m\) and \(\hat{x} \in \mathbb{Q}^n\) such that \(\hat{x} \in P := \{x \in \mathbb{R}^n : Ax \leq b\}\),
decide whether \(\hat{x} \in P_{1/2}(A, b)\).

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The membership problem remains strongly coNP-complete even when $Ax \leq b$ defines a polytope in the 0/1 cube, as shown by Letchford, Pokutta and Schulz [19]. It is, however, well known that testing membership in the Gomory-Chvátal closure belongs to NP $\cap$ coNP if restricted to polyhedra $P$ with $P' = P_I$ (see, e.g., [1]), which naturally includes all polyhedra $P$ whose $\{0, \frac{1}{2}\}$-closure coincides with $P_I$. For instance, the relaxation of the matching polytope given by nonnegativity and degree constraints has this property: If we add the blossom inequalities, the resulting linear system is sufficient to describe the integer hull [11], and it is even totally dual integral (TDI) [10]. This motivates the following research questions that are the subject of this paper: What is the computational complexity of recognizing rational polyhedra whose $\{0, \frac{1}{2}\}$-closure coincides with the integer hull, and of deciding whether adding all $\{0, \frac{1}{2}\}$-cuts produces a TDI system?

Related questions for the Gomory-Chvátal closure have been studied by Cornuéjols and Li [9]. They prove that, given a rational polyhedron $P$ with $P_I = \emptyset$, deciding whether $P' = \emptyset$ is weakly NP-complete. This immediately implies weak NP-hardness of verifying $P' = P_I$. Cornuéjols, Lee and Li [8] extend these hardness results to the case when $P$ is contained in the 0/1 cube. Moreover, they show that deciding whether a constant number of Gomory-Chvátal inequalities is sufficient to obtain the integer hull is weakly NP-hard, even for polytopes in the 0/1 cube. In this paper, we establish analogous hardness results for the $\{0, \frac{1}{2}\}$-closure. Our main result is the following theorem, where $\mathbb{1}$ denotes the all-one vector.

**Theorem 1.** Given $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$ with $P := \{x \in \mathbb{R}^n : Ax \leq b\} \subseteq [0, 1]^n$, deciding whether $P_1(A, b) = P_I$ is strongly NP-hard, even when the inequalities $-x \leq 0$ and $x \leq 1$ are part of the system $Ax \leq b$.

We give a proof of this theorem in the next section. Our proof implies several further hardness results, which we explain in Section 3. In particular, deciding whether adding all $\{0, \frac{1}{2}\}$-cuts to a given linear system $Ax \leq b$ produces a TDI system, is strongly NP-hard. We also establish strong NP-hardness of the following problems: deciding whether the $\{0, \frac{1}{2}\}$-closure coincides with the Gomory-Chvátal closure; deciding whether a constant number of $\{0, \frac{1}{2}\}$-cuts suffices to obtain the integer hull. Finally, we give a hardness result for the membership problem for the $\{0, \frac{1}{2}\}$-closure, which is slightly stronger than the one of Letchford, Pokutta and Schulz [19].

2 **Proof of Theorem 1**

**Proof of Theorem 1.** We reduce from **STABLE SET**:

Let $G = (V, E)$ be a graph and $k \in \mathbb{N}, k \geq 2$. Does $G$ have a stable set of size at least $k$?

It is well known that **STABLE SET** is strongly NP-hard [18]. Note that the problem remains strongly NP-hard if restricted to graphs with minimum degree at least 2: Given an instance of **STABLE SET** specified by $G$ and $k$, we construct a new graph $G'$ by adding two dummy nodes to $G$ as well as all edges with at least one endpoint being a dummy node. Every node in $G'$ has degree at least 2, and every stable set in $G'$ of size $k \geq 2$ is a stable set in $G$ of the same size.

Consider an instance of **STABLE SET** given by $G = (V, E)$ and $k \geq 2$. By the above observation, we may assume that every node in $V$ has degree at least 2. Note that $|V| = n \geq 3$ and $|E| = m \geq 3$ in this case. Let $A := 2 \cdot \mathbb{1}^T - M^T$ where $M \in \{0, 1\}^{m \times n}$ denotes the edge-node incidence matrix of $G$ and $\mathbb{1}$ is the all-one vector of appropriate dimension. We
define a polytope $P \subseteq \mathbb{R}^m$ by the following system of inequalities:

\[ 0 \leq x \leq 1 \quad (1) \]
\[ Ax \leq 2 \cdot 1 \quad (2) \]
\[ (2k - 3)1^T x \geq 2k - 3 \quad (3) \]

**Claim 1.** $P_I = \{ x \in P : 1^T x = 1 \}$.

**Proof of Claim 1.** If we add all inequalities in (2), we obtain the valid inequality $2(n-1)1^T x \leq 2n$. Every integral point $x$ in $P$ therefore satisfies $1^T x = 1$. Since $A \in \{1, 2\}^{n \times m}$, it is easy to check that every unit vector is indeed contained in $P$. We conclude that

\[
P_I = \{ x \in [0,1]^m : 1^T x = 1 \} \supseteq \{ x \in P : 1^T x = 1 \} \supseteq P_I.
\]

The $\{0, \frac{1}{2}\}$-cuts that can be derived from (1)–(3) are all the inequalities of the following two types with $u \in \{0, \frac{1}{2}\}^n$ and $v \in \{0, \frac{1}{2}\}^m$:

\[
\sum_{i=1}^m (2u^T 1 + [v_i - (Mu)_i]) x_i \leq 2u^T 1 + [v^T 1] \quad (4)
\]
\[
\sum_{i=1}^m (2u^T 1 - (k - 1) + [\frac{1}{2} + v_i - (Mu)_i]) x_i \leq 2u^T 1 - (k - 1) + [\frac{1}{2} + v^T 1] \quad (5)
\]

The first type (4) defines all cuts that are derived only from (1) and (2), whereas the second type (5) also uses inequality (3). The vector $u$ is the vector of multipliers for inequalities (2) while $v$ collects the multipliers for the upper bounds in (1).

In what follows, $P_{\frac{1}{2}}$ denotes the $(0, \frac{1}{2})$-closure of $P$ defined by (1)–(3) together with (4) and (5) for all $u \in \{0, \frac{1}{2}\}^n$ and $v \in \{0, \frac{1}{2}\}^m$.

**Claim 2.** $P_{\frac{1}{2}} = P_I$ if and only if there is a $\{0, \frac{1}{2}\}$-cut equivalent to $1^T x \leq 1$.

**Proof of Claim 2.** If there is such a cut, then $P_{\frac{1}{2}} \subseteq \{ x \in P : 1^T x \leq 1 \} = P_I$ by Claim 1. To see the “only if” part, consider the vector $y = (\frac{n}{m} + \varepsilon)1$ for some small $\varepsilon > 0$. Clearly, $y \notin P_I$ since $1^T y > 1$. We claim that there is a choice for $\varepsilon$ such that $y \in P$ and $y$ satisfies all $\{0, \frac{1}{2}\}$-cuts except those that are equivalent to $1^T x \leq 1$. First observe that every cut (of either type (4) or (5)) as well as every inequality in (2) and (3) may be written as $a^T x \leq \alpha$ for some $a \in \mathbb{Z}^m, \alpha \in \mathbb{Z}$ where $a_i \leq \alpha$ for all $i \in [m]$ and $\alpha \leq m + n$. If $\alpha \leq 0$, we clearly have $a^T y \leq \alpha$ since $y \geq \frac{1}{m}$. If $\alpha > 0$ and $a^T x \leq \alpha$ is not equivalent to $1^T x \leq 1$, then $a_i < \alpha$ for at least one $i \in [m]$. It follows that $a^T y \leq \alpha - \frac{1}{m} + \varepsilon (m \alpha - \alpha - 1)$. For instance, taking $\varepsilon := \frac{2}{m^2 (m + n)}$ yields $a^T y \leq \alpha$ as desired.

In particular, the proof of Claim 2 shows that the inequality $1^T x \leq 1$ is not valid for $P$.

**Claim 3.** No cut of type (4) is equivalent to $1^T x \leq 1$.

**Proof of Claim 3.** Let $u \in \{0, \frac{1}{2}\}^n$ and $v \in \{0, \frac{1}{2}\}^m$. If $u = 0$, (4) is dominated by the sum of the inequalities $[v_i - (Mu)_i] x_i \leq 0$ for all $i \in [m]$. Note that these are valid for $P$ since $[v_i - (Mu)_i] \leq 0$ for all $i \in [m]$. If $v = 0$, the cut (4) is a trivial cut which is only derived from inequalities in the description of $P$ with even right-hand sides. Hence, we may assume that both $u \neq 0$ and $v \neq 0$. It suffices to show that $[v_i - (Mu)_i] < [v^T 1]$ for at least one $i \in [m]$. If $v^T 1 \geq 1$, there is nothing to show. Now let $v^T 1 = \frac{1}{2}$ and suppose for the sake of contradiction that $[v_i - (Mu)_i] \geq 0$ for all $i \in [m]$. It follows that $Mu \leq v$. Since every column of $M$ has at least two nonzero entries by assumption, we obtain $u = 0$, a contradiction.

\[
\diamond
\]
Claim 4. A cut of type \([5]\) induced by \(u \in \{0, \frac{1}{2}\}^n\) and \(v \in \{0, \frac{1}{2}\}^m\) is equivalent to \(1^T x \leq 1\) if and only if \(v = 0, 2Mu \leq 1,\) and \(2u^T 1 \geq k\).

Proof of Claim 4. Suppose first that \(v \neq 0\). Then, for every \(i \in [m]\), we have \(\left[\frac{1}{2} + v_i - (Mu)_i\right] \leq 1 \leq \left[\frac{1}{2} + v^T 1\right].\) This holds with equality for all \(i \in [m]\) simultaneously only if \(v_i = \frac{1}{2}\) and \(v^T 1 \leq 1\), contradicting \(m \geq 3\). Thus, no inequality of the form \([5]\) with \(v \neq 0\) has identical coefficients that coincide with the right-hand side. We may therefore assume that \(v = 0\).

If \(2u^T 1 \leq k - 1\), inequality \([5]\) is redundant: It is the sum of the inequalities \((2u^T 1 - (k - 1))1^T x \leq 2u^T 1 - (k - 1)\) and \(\left[\frac{1}{2} - (Mu)_i\right] x_i \leq 0\) for all \(i \in [m]\), all of which are valid for \(P\). Assuming that \(2u^T 1 \geq k\), inequality \([5]\) is equivalent to \(1^T x \leq 1\) if and only if \((Mu)_i \leq \frac{1}{2}\) for all \(i \in [m]\).

Putting together Claims 2 to 4, we conclude that \(P_{1/2} = P_1\) if and only if there exists some \(u \in \{0, \frac{1}{2}\}^n\) such that \(2u\) is the incidence vector of a stable set in \(G\) of size at least \(k\). □

3 Further hardness results

A careful analysis of the proof of Theorem 1 shows that, if the polytopes \(P\) constructed in the reduction satisfy \(P_{1/2} = P_1\), there is a single \(\{0, \frac{1}{2}\}\)-cut that certifies this (see Claim 2). This observation immediately implies the following corollary.

Corollary 1. Let \(k \in \mathbb{N}\) be a fixed constant. Given \(A \in \mathbb{Z}^{m \times n}\) and \(b \in \mathbb{Z}^m\) with \(P := \{x \in \mathbb{R}^n : Ax \leq b\} \subseteq [0, 1]^n\), deciding whether one can obtain \(P_1\) by adding at most \(k\) \(\{0, \frac{1}{2}\}\)-cuts is strongly NP-hard, even when \(k = 1\), and \(-x \leq 0\) and \(x \leq 1\) are part of the system \(Ax \leq b\).

Moreover, let us remark that \(P' = P_1\) for the polytopes \(P\) arising from the reduction. This follows from the fact that for \(n \geq 3\), the inequality \(1^T x \leq \lfloor 2n/2(n-1) \rfloor = 1\) is a Gomory-Chvátal cut for \(P\), see the proof of Claim 4.

Corollary 2. Given \(A \in \mathbb{Z}^{m \times n}\) and \(b \in \mathbb{Z}^m\) with \(P := \{x \in \mathbb{R}^n : Ax \leq b\} \subseteq [0, 1]^n\), deciding whether \(P_{1/2}(A, b) = P'\) is strongly NP-hard, even when \(-x \leq 0\) and \(x \leq 1\) are part of the system \(Ax \leq b\).

The linear systems arising from our reduction have another interesting property. The inequality description \([1]–[5]\) of \(P_{1/2}\) in the proof of Theorem 1 is a TDI system if and only if \(P_{1/2} = P_1\). This can be seen as follows. Since any polyhedron defined by a TDI system with integer right-hand sides is integral \([12]\), it suffices to show the “if” part. Suppose that \(P_{1/2} = P_1\).

By the proof of Theorem 1, there exist vectors \(u', u'' \in \{0, \frac{1}{2}\}^n\) such that \(2Mu' \leq 1, 2Mu'' \leq 1, \) \(2(u')^T 1 = k\), and \(2(u'')^T 1 = k - 2 \geq 0\) (see Claim 4). The cuts of type \([3]\) derived with \(u'\) and \(u''\) (where we take \(v = 0\)) are the inequalities \(1^T x \leq 1\) and \(-1^T x \leq -1\), respectively. The system defined by these two inequalities and \(x \geq 0\) is a subsystem of \([1]–[5]\) that is sufficient to describe \(P_{1/2}\) (see Claim 1) and that is readily seen to be TDI: Let \(c \in \mathbb{Z}^m\). We can assume w.l.o.g. that \(c_1\) is the largest coefficient of \(c\). It follows that \(\max\{c^T x : x \in P_{1/2}\} = c_1\). It suffices to show that the inequality \(c^T x \leq c_1\) is a nonnegative integer linear combination of the selected subsystem. Indeed, it is the sum of \(c_1 1^T x \leq c_1\) (which is a nonnegative integer multiple of \(1^T x \leq 1\) or \(-1^T x \leq -1\)) and \(-(c_1 - c_i)x_i \leq 0\) for all \(i \in [m]\). The above argument shows the following result.

Corollary 3. Let \(A \in \mathbb{Z}^{m \times n}\) and \(b \in \mathbb{Z}^m\). Deciding whether the system given by \(Ax \leq b\) and all \(\{0, \frac{1}{2}\}\)-cuts derived from it is TDI, is strongly NP-hard, even when \(-x \leq 0\) and \(x \leq 1\) are part of the system \(Ax \leq b\).
Further note that the presence of the constraints \( x \leq 1 \) in (1) is not essential for our reduction in the proof of Theorem 1. In fact, the upper bounds are redundant: For every \( i \in [m] \), consider a row of \( A \) such that the entry in column \( i \) is equal to 2. Such a row exists because \( n \geq 3 \). The corresponding inequality in (2) together with the nonnegativity constraints \( -x_j \leq 0 \) (possibly twice) for all \( j \neq i \) yields \( 2x_i \leq 2 \) for all \( x \in P \). As the only relevant cuts among (4) and (5) are those with \( v = 0 \), we conclude that all of the above results still hold true when the upper bounds \( x \leq 1 \) are not part of the input.

Another byproduct of our proof of Theorem 1 is that the membership problem for the \{0, \frac{1}{2}\}-closure of polytopes in the 0/1 cube is strongly coNP-complete. This has already been shown by Letchford, Pokutta and Schulz [19]. However, neither of the two different reductions given in [19] constructs linear systems that include both nonnegativity constraints and upper bounds on every variable. When these constraints are required to be part of the input, membership testing remains strongly coNP-complete, as the following result shows.

**Corollary 4.** The membership problem for the \{0, \frac{1}{2}\}-closure of polytopes contained in the 0/1 cube is strongly coNP-complete, even when the inequalities \(-x \leq 0 \) and \( x \leq 1 \) are part of the input.

**Proof.** The problem clearly belongs to coNP. To show hardness, we use the same reduction from Stable Set as in the proof of Theorem 1. The vector \( y \) defined in the proof of Claim 2 satisfies \( y \notin P_{\frac{1}{2}} \) if and only if the instance of Stable Set is a “yes” instance. The encoding length of \( y \) is polynomial in \( m \) and \( n \) if we choose \( \varepsilon \) as in Claim 2.

4 Concluding remarks

It is worth pointing out that the problem of recognizing integrality of the \{0, \frac{1}{2}\}-closure is in coNP when the membership problem for the \{0, \frac{1}{2}\}-closure can be solved in polynomial time: If \( P = \{ x : Ax \leq b \} \) is a rational polyhedron with \( P_{\frac{1}{2}}(A,b) \neq P_I \), it suffices to exhibit a fractional vertex \( \hat{x} \) of \( P_{\frac{1}{2}}(A,b) \) along with a corresponding basis. Then one can verify in polynomial time that \( \hat{x} \in P_{\frac{1}{2}}(A,b) \) and that \( \hat{x} \) is indeed a vertex. This observation can be found in [17, Chapter 9] where it is stated in the context of recognizing t-perfect graphs. These are the graphs whose stable set polytope is determined by nonnegativity and edge constraints together with the odd-cycle inequalities [6]. In fact, the odd-cycle inequalities can be derived as \{0, \frac{1}{2}\}-cuts from the other two classes of inequalities [14]. This means that a graph is t-perfect if and only if the \{0, \frac{1}{2}\}-closure of the relaxation of its stable set polytope given by nonnegativity and edge constraints is integral. Since a separating odd-cycle inequality can be found in polynomial time [14], recognizing t-perfection is in coNP. Whether this problem is in NP or in P is not known (see [17, Chapter 9]). However, some classes of t-perfect graphs are known to be polynomial-time recognizable, including claw-free t-perfect graphs [2] and bad-K\( _4 \)-free graphs [15]. Interestingly, for these two classes of graphs, the linear system in [6] that determines the stable set polytope is TDI [3, 21]. It is not known whether this holds true for t-perfect graphs in general (see [22]).

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