Lagrangian Matroids:
Representations of Type $B_n$

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17 September 2002

Introduction

Coxeter matroids are combinatorial objects associated with finite Coxeter groups; they can be viewed as subsets $M$ of the factor set $W/P$ of a Coxeter group $W$ by a parabolic subgroup $P$ which satisfy a certain maximality property with respect to a family of shifted Bruhat orders on $W/P$. The classical matroids of matroid theory are exactly the Coxeter matroids for the symmetric group $\text{Sym}_n$ (which is a Coxeter group of type $A_{n-1}$) and a maximal parabolic subgroup, while the maximality property turns out to be Gale’s classical characterisation of matroids [13].

The theory of Coxeter matroids sheds new light on the classical matroid theory and brings into the consideration a wider class of combinatorial objects [7]. Of this, we can specifically mention Lagrangian matroids, which are Coxeter matroids for the hyperoctahedral group $BC_n$ and a particular maximal parabolic subgroup. Lagrangian matroids are cryptomorphically equivalent to symmetric matroids or 2-matroids of Bouchet’s papers [9] and [11]. They are also equivalent to $\Delta$-matroids [8] and to Dress and Havel’s metroids, see [12]. Because of the natural embedding of Coxeter groups $D_n < BC_n$, the even $\Delta$-matroids of Wenzel [15] are in fact Coxeter matroids for $D_n$.

The present paper belongs to a series of publications aimed at the development of the concept of orientation for Coxeter matroids which would generalise the classical oriented matroids [1]. The concept of orientation for even $\Delta$-matroids was introduced by Wenzel [14, 15] and developed by Booth [2] in a form which better fits the general theory. However, as we shall soon see, this concept does not cover all natural orientation structures on Lagrangian matroids.

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*Partially supported by The Treaty of Windsor Research Programme of the British Council in Portugal.
†Partially supported by The Treaty of Windsor Research Programme of the British Council in Portugal.
‡Partially supported by EPSRC grant GR/R53593.
When attempting to develop the theory of orientation for Coxeter matroids other than classical (ordinary) matroids, one needs to meet the fundamental requirement that the orientation axioms should reflect the geometry of the appropriate flag varieties over the field \( \mathbb{R} \) of real numbers. In the case of ordinary matroids of rank \( k \) on \( n \) elements these are the Grassmann varieties \( G_{n,k} \) of \( k \)-dimensional subspaces in \( \mathbb{R}^n \). So far the two versions of orientation of Lagrangian matroids, as developed in [3] and [2], reflected the geometry of the flag varieties of maximal isotropic subspaces in \( C_n(\mathbb{R}) = \text{Sp}_{2n}(\mathbb{R}) \) and \( D_n(\mathbb{R}) = O_{2n,n}(\mathbb{R}) \). The bilinear forms on the underlying vector space \( \mathbb{R}^{2n} \) are, correspondingly, skew-symmetric and symmetric. Not surprisingly, the corresponding theories of oriented Lagrangian matroids are very different.

However, the real Lie groups \( C_n(\mathbb{R}) = \text{Sp}_{2n}(\mathbb{R}) \) and \( B_n(\mathbb{R}) = O_{2n+1,n}(\mathbb{R}) \) have the same Weyl group \( B_n \). Lagrangian matroids represented in the flag variety of maximal isotropic subspaces in the underlying vector space of \( \mathbb{R}^{2n+1} \) have rather natural orientation properties: as we show in this paper, a \( B_n \)-represented Lagrangian matroid \( M \) can be obtained by gluing together a Lagrangian pair \( (M_1, M_2) \) of \( D_n \)-represented orthogonal Lagrangian matroids, and the corresponding orientation can be very naturally described as the orientation of the exploded sum \( M_1 \sqcup M_2 \), which turns out to be a \( D_{n+1} \)-represented Lagrangian matroid.

Hence, although \( B_n \)-represented Lagrangian matroids have properties very different from that of \( D_n \)-represented matroids, the corresponding orientation theory are essentially the same (up to some non-trivial cryptomorphism).

\( B_n \)-representations belong to a series of representations of Lagrangian matroids in groups \( O_{2n+m,n} \) of isometries of the spaces \( \mathbb{R}^{2n+m} \) endowed with non-degenerate symmetric bilinear forms which allow maximal isotropic subspaces of dimension \( n \). At this point we can only conjecture that these new representations are likely to lead to the orientation theories which can be cryptomorphically reduced to \( D_n \)-orientations.

The terminology and notation follow [7].

1 Symplectic and orthogonal matroids

Let

\[ [n] = \{1, 2, \ldots, n\} \text{ and } [n]^* = \{1^*, 2^*, \ldots, n^*\}. \]

Define the map \( * : [n] \to [n]^* \) by \( i \mapsto i^* \) and the map \( * : [n]^* \to [n] \) by \( i^* \mapsto i \). In other words, we are defining \( i^{**} = i \). Then \( * \) is an involutive permutation of the set \([n] \cup [n]^*\).

We denote \( J = [n] \cup [n]^* \). We say that a subset \( K \subset J \) is admissible if and only if \( K \cap K^* = \emptyset \). If \( B \subseteq J \), we set \( B^+ = B \cup B^* \).

A linear ordering \( \prec \) of \( J \) is called a \( C_n \)-admissible ordering if \( i \prec j \) implies that \( j^* \prec i^* \) for all \( i, j \in J \). Equivalently, an ordering \( \prec \) on \( J \) is \( C_n \)-admissible if and only if, when the \( 2n \) elements are listed from largest to smallest, the first \( n \) elements listed form an admissible set, and the last \( n \) elements listed are
the stars of the first $n$ elements listed, but are listed in reverse order. A $D_n$-admissible ordering of $J$ is similar to a $C_n$-admissible ordering, except that the middle two elements (i.e., the $n$-th and $n + 1$-st elements in the above listing) are now incomparable.

Denote by $J_k$ the collection of all admissible $k$-subsets in $J$, for some $k \leq n$. If $\prec$ is $C_n$ or $D_n$-admissible ordering on $J$, it induces the partial ordering (which we denote by the same symbol $\prec$) on $J_k$: if $A, B \in J_k$ and

$$A = \{a_1 \prec a_2 \prec \cdots \prec a_k\} \quad \text{and} \quad B = \{b_1 \prec b_2 \prec \cdots \prec b_k\},$$

we set $A \prec B$ if

$$a_1 \prec b_1, a_2 \prec b_2, \ldots, a_k \prec b_k.$$

This partial ordering is called the Gale ordering on $J_k$ induced by $\prec$.

Now let $B \subseteq J_k$ be a collection of admissible $k$-element subsets of the set $J$. We say that $M = (\ast, B)$ is a symplectic matroid if it satisfies the following Maximality Property:

for every $C_n$-admissible order $\prec$ on $J$, the collection $B$ contains a unique maximal member, i.e. a subset $A \in B$ such that $B \prec A$ (in the Gale order induced by $\prec$), for all $B \in B$.

The collection $B$ is called the collection of bases of the symplectic matroid $M$, its elements are called bases of $M$, and the cardinality $k$ of the bases is the rank of $M$. An orthogonal matroid is defined similarly using $D_n$-admissible orderings. Ordinary matroids on $[n]$ can be defined in a similar fashion, using $A_n$-admissible orderings, which are arbitrary linear orderings on $[n]$; indeed, this is essentially the well-known greedy algorithm of matroid theory. A Lagrangian matroid (resp. Lagrangian orthogonal matroid) is a symplectic matroid (resp. orthogonal matroid) of rank $n$, the maximum possible.

One more very useful characterization of Lagrangian orthogonal matroids is the Strong Exchange Property \[5\]. A collection $B \subseteq J_n$ is the collection of bases of a Lagrangian orthogonal matroid if and only if:

For every $A, B \in B$ and $a \in A \triangle B$, there exists $b \in B \smallsetminus A$ with $b \neq a^\ast$, such that both $A \triangle \{a, b, a^\ast, b^\ast\}$ and $B \triangle \{a, b, a^\ast, b^\ast\}$ are members of $B$.

Here, $\triangle$ is the symmetric difference of sets.

2 Lagrangian pairs

Consider an admissible set of size $n - 1$. Such a set can be completed to an admissible set of size $n$ in exactly two ways, by appending either $i$ or $i^\ast$ for some $i$. The two resulting sets are called a Lagrangian pair of sets, and are characterised by the fact that their symmetric difference is exactly $\{i, i^\ast\}$.

Consider now two Lagrangian orthogonal matroids $\mathcal{M}_1, \mathcal{M}_2$ of rank $n$ and of opposite parity. We say that they form a Lagrangian pair (of Lagrangian orthogonal matroids) if they satisfy:
For every admissible ordering, the maximal bases of \( M_1 \) and \( M_2 \) under the ordering are a Lagrangian pair of sets.

We say that a pair of Lagrangian subspaces of orthogonal \( 2n \)-space form a Lagrangian pair of subspaces if their intersection is of dimension \( n - 1 \).

The following result is well-known.

**Lemma 1** A totally isotropic subspace of dimension \( n - 1 \) in orthogonal \( 2n \)-space is contained in exactly two Lagrangian subspaces (which are a Lagrangian pair).

**Theorem 2** [4, Theorem 2] A Lagrangian pair of subspaces represent a Lagrangian pair of orthogonal matroids.

For \( \sigma \in D_n \), write \( \sigma(M) \) for the maximum basis of an orthogonal matroid \( M \) under the ordering \( \prec^{\sigma} \).

**Theorem 3** [4, Theorem 3] Given a Lagrangian pair of Lagrangian matroids, \( M_1, M_2 \), set

\[
B = \{ \sigma(M_1) \cap \sigma(M_2) \mid \sigma \in D_n \}.
\]

Then \( B \) is the collection of bases of an orthogonal matroid \( M \) of rank \( n - 1 \). Furthermore, \( M_1 \) and \( M_2 \) are the unique Lagrangian orthogonal matroids obtained by completing the bases of \( M \) to \( n \)-sets of odd and even parity.

**Theorem 4** [4, Theorem 7] Let \( B_1, B_2 \) be the collections of bases of a Lagrangian pair of Lagrangian matroids. Then \( B = B_1 \cup B_2 \) is the collection of bases of a (symplectic) Lagrangian matroid.

As it is shown in [4], the converse is not true.

Let \( B_1, B_2 \) be the collections of bases of a Lagrangian pair of Lagrangian matroids of rank \( n \). We say that

\[
B = \{ B \cup (n+1) \mid B \in B_1 \} \cup \{ B \cup (n+1)^* \mid B \in B_2 \}
\]

is the exploded union of the Lagrangian pair and write \( B = B_1 \oplus B_2 \).

**Theorem 5** [4, Theorem 3] Two admissible collections of \( n \)-sets \( B_1, B_2 \) are the collections of bases of a Lagrangian pair of Lagrangian matroids if and only if their exploded union is the collection of bases of a Lagrangian orthogonal matroid of rank \( n + 1 \).

### 3 Representations of type \( D_n \)

Concepts of representation of matroids have been introduced in two separate, but closely related, ways. Bouchet introduces a concept of representation by square matrices of ‘symmetric type’ (I), whereas in \( D_n \) representations are
introduced in terms of isotropic subspaces. In this paper we are concerned mainly with representations over the real numbers.

Representable symplectic matroids arise naturally from symplectic and orthogonal geometries, similarly to the way that classical matroids arise from projective geometry.

### 3.1 Symplectic and orthogonal representations

Let $V$ be a vector space with basis $E = \{e_1, \ldots, e_n, e_1^*, \ldots, e_n^*\}$.

Let $\cdot$ be a bilinear form on $V$, with the symbol $\cdot$ often suppressed as usual, with

\[
e_i e_i^* = 1 \quad \text{for all } i \in I
\]

\[
e_i e_j = 0 \quad \text{for all } i, j \in J \text{ with } i \neq j^*.
\]

**Definition 1** The pair $(V, \cdot)$ is called a *symplectic space* if $\cdot$ is antisymmetric and an *orthogonal space* if $\cdot$ is symmetric. If the vector space is of characteristic 2, it is symplectic. A subspace $U$ of $V$ is called *totally isotropic* if $\cdot$ restricted to $U$ is identically zero. A *Lagrangian subspace* is a totally isotropic subspace of maximal dimension (easily seen to be $n$).

Choose a basis $u_1, \ldots, u_k$ of a totally isotropic subspace $U$ and represent this basis in terms of $E$, so that

\[
u_i = \sum_{j=1}^{n} (a_{ij} e_j + b_{ij} e_j^*)
\]

Now we have represented $U$ as the row space of a $k \times 2n$ matrix $C = (A, B)$ with columns indexed by $J$. Let $\mathcal{B}$ be the collection of sets of column indices corresponding to non-zero $k \times k$ minors which are admissible; then

**Theorem 6** If $U$ is a totally isotropic subspace of a symplectic or orthogonal space, $\mathcal{B}$ is the collection of bases of a symplectic or orthogonal matroid, respectively. Note that the matroid is independent of the choice of basis $u_1, \ldots, u_k$ of $U$.

This is Theorem 5 in [14]; the statement for symplectic matroids only is Theorem 2 in [8].

$C$ is called a *(symplectic/orthogonal) representation* of $M = (J, *, \mathcal{B})$, and $M$ is said to be *(symplectic/orthogonally) representable*. Note that orthogonal matroids may have symplectic representations. We also note that, when considered in matrix form, the requirement that $U$ be totally isotropic is equivalent to the requirement that $AB^t$ be symmetric in the symplectic case and skew-symmetric in the orthogonal case.
Note that we can ‘embed’ a representation of a classical matroid as a representation of the canonically associated Lagrangian orthogonal matroid. We simply make the top $k$ rows of $A$ (for a matroid of rank $k$) the representation of the classical matroid, and the remaining rows of $A$ zero; and the top $k$ rows of $B$ zero, and the bottom $n-k$ rows an orthogonal complement of maximal rank of $A$. This is clearly the required representation, and is both a symplectic and an orthogonal representation simultaneously.

In the case of a general, symplectically represented, symplectic Lagrangian matroid, we assign orientations by considering essentially signs of determinants of principal minors of the above symmetric matrices. Unfortunately, in skew-symmetric matrices that produces uninteresting results, as we shall see; the correct concept is that of the Pfaffian, which we shall define in the next section.

### 3.2 Orientations

Bouchet, in [3], defines a $\Delta$-matroid as a collection $B$ of subsets of $I = [n]$, not necessarily equicardinal, satisfying the following:

**Axiom 1 (Symmetric Exchange Axiom)** For $A, B \in B$ and $i \in A \triangle B$, there exists $j \in B \triangle A$ such that $(A \triangle \{i, j\}) \in B$.

It is thus immediately apparent that a classical matroid is also a $\Delta$-matroid. Bouchet goes on to define a symmetric matroid as essentially a $\Delta$-matroid with bases extended to $n$ elements by adding to $B \in B$ all starred elements which do not appear, unstarrred, in $B$. Thus a symmetric matroid is a set $B \subseteq J_n$ satisfying:

**Axiom 2** For $A, B \in B$ and $i \in A \triangle B$, there exists $j \in B \triangle A$ such that $(A \triangle \{i, j, i^*, j^*\}) \in B$.

We shall refer to these two axioms interchangeably as ‘the symmetric exchange axiom’ depending on the structure to which we refer.

In this section we shall state Wenzel’s definition of (even) oriented $\Delta$-matroids, and extend it in the obvious way to orthogonal Lagrangian matroids. We remark parenthetically that symplectic Lagrangian matroids (and so $\Delta$-matroids, even or otherwise) may be oriented as described in [3]. We go on to discuss representations of these objects, and prove that a representable (classical) oriented matroid is representable as an oriented orthogonal matroid.

### 3.3 Orientation Axioms

We shall follow Wenzel in [16] by making:

**Definition 2** A map $p : 2^I \to \mathbb{R}$ is called a twisted Pfaffian map if it satisfies the following:

1. $p$ is not identically zero.
2. For all \( A, B \subseteq I \) with \( p(A) \neq 0 \), \( p(B) \neq 0 \), we have \( \#A = \#B \mod 2 \).

3. If \( A, B \subseteq I \) and \( A \triangle B = \{i_1 < \cdots < i_l\} \) then we have

\[
\sum_{j=1}^{l} (-1)^j p(A \triangle \{i_j\}) \cdot p(B \triangle \{i_j\}) = 0.
\]

We call two twisted Pfaffian maps equivalent if they differ only by a non-zero constant scalar multiple. In fact, Wenzel makes the definition for a ‘fuzzy ring’ rather than for the real numbers, but we are interested in this paper only in representations over the real numbers. Pfaffian maps may be defined as twisted Pfaffian maps where \( p(\emptyset) = 1 \). The Pfaffian of a square matrix of odd size is defined to be 0; for a \( 2m \times 2m \) square skew-symmetric matrix, it is defined as follows:

**Definition 3** Let

\[
S'_{2m} = \left\{ \sigma \in S_{2m} \mid \sigma(2k-1) = \min_{2k-1 \leq j \leq 2m} \sigma(j) \text{ for } 1 \leq k \leq m \right\},
\]

and let \( A \) be a skew-symmetric matrix. Then the Pfaffian of \( A \) is defined by

\[
\text{Pf}((a_{ij})_{1 \leq i,j \leq 2m}) = \sum_{\sigma \in S'_{2m}} \text{sign } \sigma \prod_{k=1}^{m} a_{\sigma(2k-1) \sigma(2k)}.
\]

The Pfaffian of the empty set is 1, by definition.

It can be shown that the square of the Pfaffian of a (skew-symmetric) matrix is the determinant of that matrix.

**Theorem 7** If \( A \) is a skew-symmetric \( n \times n \) matrix, \( I_1, I_2 \subseteq I \) and \( I_1 \triangle I_2 = \{i_1, \ldots, i_l\} \) with \( i_j < i_{j+1} \) for \( 1 \leq j \leq l-1 \) then

\[
\sum_{j=1}^{l} (-1)^j p(I_1 \triangle \{i_j\})p(I_2 \triangle \{i_j\}) = 0
\]

where \( p(S) = \text{Pf}((a_{ij})_{i,j \in S}) \) for any \( S \subseteq I \).

This is Proposition 2.3 in [15].

Thus a skew-symmetric matrix with real coefficients yields a Pfaffian map, and in fact Pfaffian maps to a given ring (here, to the reals) are in \( 1-1 \) correspondence with skew-symmetric matrices over the same ring (this is Theorem 2.2 in [15]). It can be seen (see the details in [2]) that the subsets of \( I \) corresponding to non-zero values of the twisted Pfaffian map form a \( \Delta \)-matroid.

We now follow [14, Definition 2.10] in making
Definition 4 An oriented even $\Delta$-matroid is an equivalence class of maps 

$$p : 2^I \rightarrow \{+1, -1, 0\}$$ 

satisfying 

1. $p$ is not identically zero. 

2. For all $A, B \subseteq I$ with $p(A) \neq 0$, $p(B) \neq 0$, we have $\#A = \#B \mod 2$. 

3. If $A, B \subseteq I$ and $A \triangle B = \{i_1 < \cdots < i_l\}$ and for some $w \in \{+1, -1\}$ we have 

$$\kappa_j = w(-1)^j p(A \triangle \{i_j\}) \cdot p(B \triangle \{i_j\}) \geq 0$$ 

for $1 \leq j \leq l$, then $\kappa_j = 0$ for all $1 \leq j \leq l$. 

We shall often speak of a map as an oriented even $\Delta$-matroid, with the equivalence class implicitly understood. 

The bases of the oriented even $\Delta$-matroid are those subsets of $F \subseteq I$ for which $p(F) \neq 0$. We observe that every Pfaffian map yields an oriented $\Delta$-matroid by simply ignoring magnitudes. 

Lemma 8 The collection of bases of an oriented $\Delta$-matroid is a $\Delta$-matroid. 

We now make the obvious definition: Take a Lagrangian orthogonal matroid $\mathcal{B}$, with an equivalence class of signs assigned to its bases. Two sets of signs are said to be equivalent when they are either identical on all bases or opposite on all bases. We express this as an equivalence class of maps 

$$p : J_n \rightarrow \{+, -, 0\}$$ 

with 

$$\mathcal{B} = \{A \in J_n \mid p(A) \neq 0\}$$ 

and equivalence given by $p \sim -p$. Consider the corresponding even $\Delta$-matroid and equivalence class of signs $p'$ obtained by ignoring starred elements; that is, 

$p'(A) = p(B)$, where $B \in J_n$ is the unique element with $B \cap I = A$. Now we say that $p$ is an oriented orthogonal matroid exactly when $p'$ is an oriented even $\Delta$-matroid. 

3.4 Oriented representations 

Theorem 9 Given an $n \times n$ square skew-symmetric real matrix $A$ and $T \subseteq I$, define $p : 2^I \rightarrow \{+1, -1, 0\}$ by setting $p(B)$ to be the sign of the Pfaffian of the principal minor indexed by $B \cap T$. Then $p$ is an oriented even $\Delta$-matroid, and the underlying $\Delta$-matroid is that represented by $A$ and $T$. 

We now move on to define a representation of an oriented orthogonal matroid.
Definition 5  Given $C$, an orthogonal representation of an orthogonal matroid $M$ over $\mathbb{R}$, we construct the oriented orthogonal matroid represented by $C$ as follows. Choose a basis $F$ of $M$, and swap columns $j$ and $j^*$ for $j \in T = F \cap I$ so that all columns of $F$ are in the right-hand $n$ places. Now perform row operations so that the right-hand $n$ columns become the identity matrix. Now the left-hand side, $A'$, is a skew-symmetric matrix. Since we have $A'$ and $T$, we have a representation of an oriented even $\Delta$-matroid. Unfortunately, this oriented even $\Delta$-matroid is dependent on the initial choice of $F$, although the underlying non-oriented $\Delta$-matroid is not, so we modify $A'$ as follows.

Set $\varepsilon_0 = 1$ and $\varepsilon_i = \begin{cases} 
\varepsilon_{i-1} & i \notin T \\
-\varepsilon_{i-1} & i \in T.
\end{cases}$ for $i > 0$. Then set $a_{ij} = \varepsilon_i \varepsilon_j a'_{ij}$. $A = (a_{ij})$ is again skew-symmetric, with rows and columns indexed by $I$, and we assign to the basis $B$ the sign of the Pfaffian of the principal minor of $A$ indexed by $(B \Delta F) \cap I$. If we consider instead that we have permuted column labels with columns, then the indices giving rise to this Pfaffian are those of the columns of $A$ labelled by elements of $B$. Note that this corresponds to the oriented even $\Delta$-matroid represented by $A, T$.

Notice that this definition may be rather simply stated as follows:

- Standard row operations are permitted.
- Swapping columns $i$ and $i^*$, and the associated column labels, is permitted after multiplying all columns $i, \ldots, n$ and $i^*, \ldots, n^*$ by $-1$.
- If the right-hand $n$ columns of the representation form an identity matrix, write $T$ for the admissible $n$-set of their column indeces. Now the left-hand $n$ columns form a skew-symmetric matrix $A$, and we assign signs as in the underlying $\Delta$-matroid represented by $A$ and $T$.

Theorem 10  The above procedure obtains an oriented orthogonal matroid, which is independent of choice of $F$.

4 Representations of type $B_n$

As usual, we write $J = [n]^+ = [n] \cup [n]^*$. We shall also use the index set $K = \{2n + 1, \ldots, 2n + m\}$.

We begin with a standard orthogonal space $V^{2n+m}$, which is a vector space $V$ over $\mathbb{K}$ with basis

$$E = \{e_1, e_2, \ldots, e_n, e_1^*, e_2^*, \ldots, e_n^*, f_{2n+1}, \ldots, f_{2n+m}\}$$

and which is endowed with a symmetric bilinear form $\langle \ , \ \rangle$ (which we shall call the scalar product) such that $\langle e_i, e_j \rangle = 0$ for all $i, j \in J, i \neq j^*$, $\langle e_i, f_k \rangle = 0$ for all $i \in J$ and $k \in K$ whereas $\langle e_i, e_i^* \rangle = 1$ for $i \in [n]$ and $\langle f_k, f_k \rangle = 1$ for $k \in K$. 
A totally isotropic subspace of $V$ is a subspace $U$ such that $\langle u, v \rangle = 0$ for all $u, v \in U$. Let $U$ be a totally isotropic subspace of $V$ of dimension $l$.

The following is one of the standard facts on symmetric bilinear forms.

**Lemma 11** Assume that $K$ is either formally real or $m = 1$. Then $l \leq n$.

Now choose a basis $\{u_1, u_2, \ldots, u_l\}$ of $U$, and expand each of these vectors in terms of the basis $E$:

$$u_i = \sum_{j=1}^{n} a_{i,j} e_j + \sum_{j=1}^{n} b_{i,j} e_j^* + \sum_{k=1}^{m} c_{i,k} f_k.$$ 

Thus we have represented the totally isotropic subspace $U$ as the row-space of a $l \times (2n + m)$ matrix $(A, B, C)$, $A = (a_{i,j})$, $B = (b_{i,j})$, $C = (c_{i,k})$ with the columns indexed by $J \cup K$, specifically, the columns of $A$ by $[n]$, those of $B$ by $[n]^*$, and those of $C$ by $K$.

A direct computation proves

**Lemma 12** A subspace $U$ of the standard orthogonal space $V$ is totally isotropic if and only if $U$ is represented by a matrix $(A, B, C)$ with

$$AB^t + BA^t + CC^t = 0. \quad (1)$$

Now, given a $k \times (2n + m)$ matrix $D = (A, B, C)$ with columns indexed by $J \cup K$, let us define a family $B \subseteq J_k$ by saying $X \in B$ if $X$ is an admissible $k$-set and the $k \times k$ minor formed by taking the columns of $D$ indexed by elements of $X$ is non-zero.

**Lemma 13** Let $D = (A, B, C)$ be a matrix defining a family $B$, and let $D'$ be a matrix which is row-equivalent to $D$. Then $D'$ defines the same family $B$. If $D$ satisfies the identity (1), then $D'$ satisfies the same identity.

**Proof** Elementary row operations do not change the dependencies among columns of $D$, hence they do not change which $k \times k$ minors are non-zero. Furthermore, they do not change the row-space of $D$, hence the total isotropy of the corresponding subspace $U$ of $V$, and therefore the identity (1).

**Theorem 14** Assume that either $K$ is formally real, or $m = 1$. If $U$ is totally isotropic, then $B$ is the collection of bases of a symplectic matroid.

**Proof** Let $D$ be the matrix corresponding to $U$, and let $\prec$ be an admissible order on $J$. We must show that $B$ has a unique maximal member. Let $A$ be the collection of all $k$-elements subsets of $J$ (admissible or not) such that the corresponding $k \times k$ minor of $D$ is non-zero. In fact we will show that $A$ has a unique maximal member, and that this member is also in $B$, and is therefore clearly the unique maximal member of $B$, since $B$ is a subcollection of $A$. 

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Let us reorder the columns of $D$ according to the order $\prec$ on their indices, starting with the largest index. Let $E$ be the row-echelon form of that matrix. Let $A$ be the set of indices of the $k$ pivot columns of $E$. Clearly $A$ is the unique maximal member of $A$. Suppose that $A$ is not admissible. Thus we may assume that $j, j^* \in A$, and consider the two rows $r_t$ and $r_q$ of $E$ in which the non-zero entries of the columns indexed by $j$ and $j^*$ occur:

$$
\begin{pmatrix}
    a & b & \ldots & j & \ldots & t & t^* & \ldots & j^* & \ldots & b^* & 2n + 1 & \ldots & 2n + m \\
    0 & 0 & \ldots & 1 & \ldots & * & * & \ldots & 0 & \ldots & * & * & \ldots & * \\
    0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 1 & \ldots & * & * & \ldots & * \\
\end{pmatrix}.
$$

Since $D$ is isotropic, the scalar product $\langle r_q, r_q \rangle$ of the $q$th row with itself is zero. On the other hand, tracing the way in which the scalar product is calculated, we immediately see that it equals

$$
\langle r_q, r_q \rangle = c^2_{q,2n+1} + \cdots + c^2_{q,2n+m}.
$$

Since either $K$ is formally real or $m = 1$, we conclude that all $c_{q,k} = 0$, and our two rows look like this:

$$
\begin{pmatrix}
    a & b & \ldots & j & \ldots & t & t^* & \ldots & j^* & \ldots & b^* & 2n + 1 & \ldots & 2n + m \\
    0 & 0 & \ldots & 1 & \ldots & * & * & \ldots & 0 & \ldots & * & * & \ldots & * \\
    0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 1 & \ldots & * & * & \ldots & 0 \\
\end{pmatrix}.
$$

But now it is easy to see that $\langle r_q, r_q \rangle = 1$, which contradicts our assumption that $U$ is isotropic.

A symplectic matroid $B$ which arises from a matrix $(A, B, C)$, with $AB^t + BA^t + CC^t = 0$, is called a $O(2n+m,n)$-representable symplectic matroid, and $(A, B, C)$ (with its columns indexed by $J \cup K$) is a representation or coordinatisation of it (over the field $K$). If $m = 1$, we shall call $B$ a $B_n$-representable symplectic matroid.

**Theorem 15** The union of the Lagrangian pair of Lagrangian matroids represented by a Lagrangian pair of subspaces is a $B_n$-represented Lagrangian matroid. Furthermore, every $B_n$-represented Lagrangian matroid either arises in this way, or is itself a represented Lagrangian orthogonal matroid.

**Proof** Consider a pair of Lagrangian subspaces. We may represent them as

$$
\begin{pmatrix}
    A \\
    x
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
    A \\
    y
\end{pmatrix}
$$

where $A$ is an $(n - 1) \times 2n$ matrix and $x$ and $y$ are $1 \times 2n$ row vectors. Now, every row of $A$ is orthogonal to itself, every other row of $A$ and each of $x$ and $y$. Choose some $c, \alpha, \beta \in K$ such that $\langle \alpha x, \beta y \rangle = -\frac{1}{2}c^2$. Now the matrix

$$
\begin{pmatrix}
    A & 0 \\
    \alpha x + \beta y & c
\end{pmatrix}
$$

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is a totally isotropic subspace of $V$. Consider an $n$-subset of the column indices $1, \ldots, 2n$, and the $n \times n$ minor corresponding to these columns. Its determinant is the sum of the determinants of the corresponding minors in the Lagrangian pair, not both of which are non-zero (since they represent matroids of opposite parity). Thus, the collection of bases produced is exactly the union of the collections produced from the Lagrangian pair.

For the converse, suppose we are given a $B_n$-representation. If the $(2n + 1)$th column is empty, then removing it we have an orthogonal representation of a Lagrangian orthogonal matroid, and are done. Otherwise, we perform elementary row operations to obtain the form

$$\begin{pmatrix} A & 0 \\ z & 1 \end{pmatrix}.$$ 

Since $A$ spans a totally isotropic subspace $U$ of dimension $n - 1$, it is contained in a unique pair of Lagrangian subspaces, which form a Lagrangian pair, by Lemma 2 again, we shall write $x, y$ for row vectors completing $A$ to matrices spanning each of these two spaces. Since $U$ is $n - 1$-dimensional, its annihilator is of dimension $n + 1$, and so is generated by the rows of $A$ and the vectors $x$ and $y$. Since $z$ is orthogonal to every row of $A$ and not contained in $U$, it can be expressed in the form $\alpha x + \beta y$ (after some row operation), and so the Lagrangian pair of subspaces represent a Lagrangian pair of matroids whose union is our $B_n$-represented Lagrangian matroid by the same argument as above.

5 Orientations of type $B_n$

Consider a $B_n$-represented matroid $\mathcal{M}$ with basis collection $\mathcal{B}$ which has the basis $[n]^\ast$. (If not, we can simply swap columns $i$ and $i^\ast$ for appropriate choices of $i$ to obtain such a basis.) After row operations, its representation has the form

$$\begin{pmatrix} A & I_n & c \end{pmatrix},$$

where $A$ is an $n \times n$ matrix and $c$ a $n \times 1$ column vector. Since the row space of the matrix is a totally isotropic subspace, we obtain

$$A + A^T + cc^T = 0, \quad \text{yielding} \quad S + S^T = 0 \text{ where } S = A - \frac{1}{2}cc^T.$$ 

Thus bases of $\mathcal{M}$ correspond to non-zero determinants of diagonal minors of the matrix $A = S + \frac{1}{2}cc^T$.

Theorem 16 The determinant of the minor of $A = S + \frac{1}{2}cc^T$ indexed by $I$ is exactly the determinant of the minor of the skew-symmetric matrix

$$\begin{pmatrix} S & c \\ -c^T & 0 \end{pmatrix}$$

indexed by $I$ (if it is of even cardinality) or $I \cup (n+1)$ (if $I$ is of odd cardinality).
**Proof** For notational convenience, we shall notate the appropriate minors of \(A\), \(S\) and \(c\) as though they were the full matrices.

Clearly,

\[
\det A = \det \begin{pmatrix} S + cc^T & 0 \\ 0 & 1 \end{pmatrix} = \det \begin{pmatrix} S + cc^T & c \\ 0 & 1 \end{pmatrix} \quad \text{(by row operations)}
\]

\[
= \det \begin{pmatrix} S & c \\ -c^T & 1 \end{pmatrix} \quad \text{(by column operations)}
\]

\[
= \det \begin{pmatrix} S & c \\ -c^T & 0 \end{pmatrix} + \det \begin{pmatrix} S & c \\ 0 & 1 \end{pmatrix} \quad \text{(by standard identities)}
\]

\[
= \det \begin{pmatrix} S & c \\ -c^T & 0 \end{pmatrix} + \det (S) \quad \text{(by expansion)}.
\]

Now, both these determinants are skew-symmetric, and so are non-zero only if of even cardinality. This completes the proof. \(\Box\)

**Corollary 17** Append \(n+1\) and \((n+1)^*\) to the bases of \(\mathcal{M}\) so that all resulting sets have an even number of unstarred elements. Since this is one of the two possibilities for the exploded sum of the Lagrangian pair of matroids corresponding to \(\mathcal{M}\), this produces the collection of bases of an orthogonal Lagrangian matroid which contains the basis \([n+1]^*\). Then this orthogonal Lagrangian matroid is represented by

\[
\left( \begin{array}{cc} S & c \\ -c^T & I_n \\ 0 & 0 & 1 \end{array} \right).
\]

Since, as with orthogonal matroids, our determinants arise from skew-symmetric matrices, the signs of these matrices cannot possibly be interesting; they are determined only by rank. We again turn to the Pfaffian.

**Definition 6** Given a \(B_n\)-representation of a Lagrangian matroid \(\mathcal{M}\), we define the signs of the bases of \(\mathcal{M}\) according to the following procedure:

- If \(\mathcal{M}\) is represented by a matrix of the form \((A, I_n, c)\), with the columns of \(A\) indexed by \(D^*\) and those of \(I\) by \(D\), for some \(D \in J_n\), then

\[
\left( \begin{array}{ccc} S & c & I_n \\ -c^T & 0 & 0 & 1 \end{array} \right),
\]

represents an orthogonal Lagrangian matroid which is an explosion of \(\mathcal{M}\), with columns labelled \(D^*, w^*, D, w\), where \(w \in \{n + 1, (n + 1)^*\}\) is chosen so that \((D \cup w) \cap [n + 1]\) has even cardinality. Now the signs of the bases of \(\mathcal{M}\) are the signs of the corresponding bases of this new matroid.
• Given a representation of \( \mathcal{M} \), we may swap columns and column labels \( i \) and \( i^* \), provided we multiply columns with labels 
\[
i, i^*, i + 1, (i + 1)^*, \ldots, n, n^*
\]
by \(-1\).

• We can perform any standard row operations.

**Theorem 18** The signs given by the procedure above are independent of the choice of \( D \), up to global sign change.

**Proof** Since the rules for swapping columns are the same as those in the resulting \( D_n \)-orientation from definition 5, it is enough to prove that swapping only columns \( n \) and \( n^* \) gives the correct relative signs. So, we assume that we have a \( B_n \)-representation in canonical form:
\[
\begin{pmatrix}
S - \frac{1}{2}x^2bb^T & a - \frac{1}{2}x^2b & I_{n-1} & 0 & xb \\
-a^T - \frac{1}{2}x^2b^T & -\frac{1}{2}x^2 & 0 & 1 & x \\
-\frac{1}{2}x^2 & 0 & 1 & x & 0
\end{pmatrix}.
\]
Here \( S \) is an \((n-1)\)-square skew symmetric matrix, \( a \) and \( b \) are column vectors of dimension \( n - 1 \), and \( x \) is a constant; any \( B_n \)-representation in which both \([n]^*\) and \([n-1]^* \cup n \) are bases can be written in this way by performing row operations to obtain an identity matrix in the (necessarily independent) set of columns \([n]^*\). The rest of the structure shown follows from considering the row-orthogonality of the matrix.

From the definition, our signs come from the \((n+1)\)-square skew symmetric matrix
\[
\begin{pmatrix}
S & a & xb \\
-a^T & 0 & x \\
-xb^T & -x & 0
\end{pmatrix}.
\]

Now we consider swapping columns \( n \) and \( n^* \) before expanding the matrix. Our representation becomes
\[
\begin{pmatrix}
S - \frac{1}{2}x^2bb^T & a - \frac{1}{2}x^2b & I_{n-1} & 0 & xb \\
-a^T - \frac{1}{2}x^2b^T & -\frac{1}{2}x^2 & 0 & 1 & x \\
-\frac{1}{2}x^2 & 0 & 1 & x & 0
\end{pmatrix}.
\]
Inverting this right-hand-side, we obtain
\[
\begin{pmatrix}
T - \frac{1}{2}y^2bb^T & b - \frac{1}{2}y^2a & I_{n-1} & 0 & ya \\
-b^T - \frac{1}{2}y^2a^T & -\frac{1}{2}y^2 & 0 & 1 & y
\end{pmatrix},
\]
where \( y = 2/x \) and \( T = S + ba^T - ab^T \), another skew-symmetric matrix. Thus the signs now come from
\[
\begin{pmatrix}
T & b & ya \\
-b^T & 0 & y \\
-ya^T & -y & 0
\end{pmatrix}.
\]
Here the columns are indexed by \( 1, \ldots, n-1, n^*, (n+1)^* \). Consider the sign of a basis, in each of these two \( n+1 \)-square matrices. We take four cases:
1. The basis contains neither \( n \) nor \( n + 1 \). Thus the signs are obtained from Pfaffians of matrices of the form \( S \) and
\[
\begin{pmatrix}
T & b & ya \\
-b^T & 0 & y \\
-ya^T & -y & 0
\end{pmatrix}.
\]
respectively. Now, since Pfaffians are unchanged by adding a multiple of row \( i \) to row \( j \) and the same multiple of column \( i \) to column \( j \), we obtain
\[
Pf \begin{pmatrix}
T & b & ya \\
-b^T & 0 & y \\
-ya^T & -y & 0
\end{pmatrix} = Pf \begin{pmatrix}
S & b & 0 \\
-b^T & 0 & y \\
0 & -y & 0
\end{pmatrix}
\]
which, upon expansion, is \( y \text{Pf}(S) \). So the sign is multiplied by the sign of \( x \) (which is also the sign of \( y \)).

2. The basis contains \( n \) but not \( n + 1 \). Thus the signs are given by Pfaffians of matrices of the forms
\[
\begin{pmatrix}
S & a \\
-a^T & 0
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
T & ya \\
-ya^T & 0
\end{pmatrix}.
\]
Again, using row operations, we obtain
\[
Pf \begin{pmatrix}
T & ya \\
-ya^T & 0
\end{pmatrix} = Pf \begin{pmatrix}
S & ya \\
-ya^T & 0
\end{pmatrix},
\]
and so again the sign is multiplied by the sign of \( y \) (which is also the sign of \( x \)).

3. The basis contains \( n + 1 \) but not \( n \); this is similar to case 2.

4. The basis contains both \( n \) and \( n + 1 \); this is similar to case 1.

This completes the proof. \( \diamond \)

The above results mean that, since \( B_n \)-represented matroids correspond to \( D_n \)-represented matroids of one dimension larger with essentially the same bases, there is nothing new to be gained by studying their orientations.

It is natural to wonder, given that our \( B_n \)-represented matroid is built from a Lagrangian pair of subspaces, to what extent the signs of the oriented orthogonal Lagrangian matroids thus represented are preserved.

**Theorem 19** The signs of the Lagrangian pair of (represented, and so oriented) matroids constituting a \( B_n \)-represented matroid are the same as the signs of the corresponding bases in the \( B_n \)-representation, up to changing sign throughout either constituent.
Proof  Again, we assume that \([n]^*\) and \([n-1]^* \cup n\) are bases. Thus the orthogonal pair are represented by subspaces

\[
\begin{pmatrix}
S + ba^T & a & I_{n-1} & -b \\
-a^T & 0 & 0 & 1
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
S + ba^T & a & I_{n-1} & -b \\
b^T & 1 & 0 & 0
\end{pmatrix},
\]

using the same notation as in the previous proof. The signs of these matroids are Pfaffian minors of the matrices

\[
\begin{pmatrix}
S & a \\
-a^T & 0
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
T & b \\
-b^T & 0
\end{pmatrix}
\]

indexed by \([n]\) and \([n-1] \cup n^*\) respectively. Attaching the two together to form a \(B_n\)-representation, we get exactly the representation in canonical form from the previous proof, and the proof that the signs match up is similar to the calculations there also.

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