LOCAL EXISTENCE, GLOBAL EXISTENCE, AND SCATTERING FOR THE NONLINEAR SCHRÖDINGER EQUATION

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Abstract. In this paper, we construct for every $\alpha > 0$ and $\lambda \in \mathbb{C}$ a space of initial values for which there exists a local solution of the nonlinear Schrödinger equation

\[
\begin{cases}
iu_t + \Delta u + \lambda |u|^\alpha u = 0 \\u(0, x) = u_0
\end{cases}
\]

on $\mathbb{R}^N$. Moreover, we construct for every $\alpha > \frac{2}{N}$ a class of (arbitrarily large) initial values for which there exists a global solution that scatters as $t \to \infty$.

1. Introduction

In this article, we study the existence of local and global solutions of the nonlinear Schrödinger equation

\[
\begin{cases}
iu_t + \Delta u + \lambda |u|^\alpha u = 0 \\u(0, x) = u_0
\end{cases}
\] (1.1)
on $\mathbb{R}^N$, where $\alpha > 0$ and $\lambda \in \mathbb{C}$, or its equivalent formulation

\[
u(t) = e^{it\Delta} u_0 + i\lambda \int_0^t e^{i(t-s)\Delta} |u(s)|^\alpha u(s) \, ds
\] (1.2)

where $(e^{it\Delta})_{t \in \mathbb{R}}$ is the Schrödinger group.

Concerning the local theory, the relevant space in which to study the Cauchy problem appears to be the Sobolev space $H^s(\mathbb{R}^N)$. Local well-posedness is well-known in $L^2$ if $\alpha < \frac{4}{N}$ (see [18]), in $H^1$ if $\alpha < \frac{4}{N-2}$ (see [8]), and in $H^2$ if $\alpha < \frac{4}{N-4}$ (see [11]). More generally, the problem is locally well-posed in $H^s$ if $0 \leq s < \frac{N}{2}$ and $\alpha < \frac{N-2s}{2}$, but under the additional condition $\alpha > [s]$ if $s > 1$ and $\alpha$ is not an even integer. (Here, $[s]$ the integer part of $s$) This condition appears because the map $u \mapsto |u|^\alpha u$ is $C^\infty$ if $\alpha$ is an even integer; it is $C^\alpha$, but not $C^{\alpha+1}$ if $\alpha$ is an odd integer; and it is $C^{[\alpha]+1}$ but not $C^{[\alpha]+2}$ if $\alpha$ is not an integer. It appears naturally. Indeed, solutions are constructed by a fixed-point argument, for which one is lead to estimate derivatives of order up to $s$ of $|u|^\alpha u$. When $s > \frac{N}{2}$ is an integer, local existence in $H^s$ is proved in [9] under the same assumption: $\alpha > [s]$ if $\alpha$ is not an even integer. This condition was improved in certain cases, see [13, 7], but not eliminated except for $s \leq 2$. For instance, it seems that no available local theory applies to the case $N = 12$ and $\alpha = 1$, and that there is no $u_0 \neq 0$ for which the existence of a local solution (in some sense) of (1.1) is known\(^1\). There is some evidence that such a regularity assumption is not purely technical, see [4].

\(^{1}\)This observation is for the case of a general $\lambda$. If $\lambda \in \mathbb{R}$ and $\lambda < 0$, then the existence of a (global) weak solution for $u_0 \in H^1(\mathbb{R}^N) \cap L^{4+2}(\mathbb{R}^N)$ follows from compactness arguments, see [14, 16].
A regularity condition also appears for the low-energy scattering problem. It is a natural conjecture that if $\alpha > \frac{4}{N}$, then small initial values (in an appropriate sense) give rise to global solutions of (1.1) that are asymptotically free, i.e. behave like a solution of the linear equation as $t \to \infty$. This property is known in dimension $N = 1, 2, 3$, see [17, 6, 10, 12]. However, in larger dimension, the available methods leave a gap. This gap is not only due to the limitations discussed above, but also concerns values of $\alpha$ close to $\frac{4}{N}$, for which local existence is not an issue. The difficulty that arises clearly appears by using the pseudo-conformal transformation through which, a global, asymptotically free solution of (1.1) corresponds (see Section 4) to a solution of the nonautonomous equation

$$\begin{cases}
iv_t + \Delta v + \lambda(1-bt) - \frac{4}{N} |v|^\alpha v = 0 \\
v(0, x) = v_0(x)
\end{cases}$$ (1.3)

which has a limit (in an appropriate space) as $t \to \frac{4}{b}$. (Here, $b > 0$ is a constant.) The problem is then to solve (1.3) on $[0, \frac{4}{b})$. Note that the assumption $\alpha > \frac{4}{N}$ implies that $(1-bt) - \frac{4}{N} |v|^\alpha$ is integrable at $\frac{4}{b}$. However, the singularity at $\frac{4}{b}$ makes it problematic to apply Strichartz’s estimates when $\alpha$ is close to $\frac{4}{N}$, see [6, 3]. One can try another approach, and use the integrability of $(1-bt) - \frac{4}{N} |v|^\alpha$ by estimating the $L^\infty$-norm of the solution, rather than applying Strichartz’s estimates. However, the only way of controlling the $L^\infty$-norm seems to be by a control of the $H^s$-norm, for $s > \frac{N}{2}$, via Sobolev’s inequality. When $N$ is large, we again face the problem of lack of regularity of the nonlinearity.

In this paper, we construct for every $\alpha > 0$ a class of initial values for which there exists a local solution of (1.1). Moreover, we construct for every $\alpha > \frac{4}{N}$ a class of initial values for which there exists a global solution of (1.1) that scatters. Before stating our results, we introduce some notation. We fix $\alpha > 0$, we consider three integers $k, m, n$ such that

$$k > \frac{N}{2}, \quad n > \max\left\{\frac{N}{2} + 1, \frac{N}{2\alpha}\right\}, \quad 2m \geq k + n + 1$$ (1.4)

and we let

$$J = 2m + 2 + k + n.$$ (1.5)

We define the space $\mathcal{X}$ by

$$\mathcal{X} = \{u \in H^J(\mathbb{R}^N); \langle x \rangle^n D^\beta u \in L^\infty(\mathbb{R}^N) \text{ for } 0 \leq |\beta| \leq 2m \}
\langle x \rangle^n D^\beta u \in L^2(\mathbb{R}^N) \text{ for } 2m + 1 \leq |\beta| \leq 2m + 2 + k,$$

and we equip $\mathcal{X}$ with the norm

$$\|u\|_{\mathcal{X}} = \sum_{j=0}^{2m} \sup_{|\beta|=j} \|\langle x \rangle^n D^\beta u\|_{L^\infty} + \sum_{\nu=0}^{k+1} \sum_{\mu=0}^{n} \sum_{|\beta|=\nu+\mu+2m+1} \|\langle x \rangle^{n-\mu} D^\beta u\|_{L^2}$$ (1.7)

where

$$\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}.$$ (1.8)

**Remark 1.1.** Here are some comments on the space $\mathcal{X}$ defined by (1.6)-(1.7).

(i) It follows from standard considerations that $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is a Banach space.

(ii) Note that $2n > N + 2$ by (1.4), so that $\|\langle x \rangle u\|_{L^2} \leq C \|\langle x \rangle^n u\|_{L^\infty}$. It easily follows that

$$\mathcal{X} \hookrightarrow H^J(\mathbb{R}^N) \cap \Sigma$$ (1.9)

where

$$\Sigma = H^J(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, |x|^2 dx).$$ (1.10)
(iii) It is immediate that \( S(\mathbb{R}^N) \subset X \). Furthermore, it is not difficult to show that \( \langle x \rangle^{-p} \in X \) if \( p \geq n \) (apply (A.2)).

Our main results are the following.

**Theorem 1.2.** Let \( \alpha > 0 \) and \( \lambda \in \mathbb{C} \). Assume (1.4)-(1.5) and let \( X \) be defined by (1.6)-(1.7). If \( u_0 \in X \) satisfies

\[
\inf_{x \in \mathbb{R}^N} \langle x \rangle^n |u_0(x)| > 0 \tag{1.11}
\]

then there exist \( T > 0 \) and a unique solution \( u \in C([-T,T],X) \) of (1.2).

**Theorem 1.3.** Let \( \alpha > \frac{N}{2} \) and \( \lambda \in \mathbb{C} \). Assume (1.4)-(1.5), let \( X \) be defined by (1.6)-(1.7) and \( \Sigma \) by (1.10). Suppose \( u_0 = e^{i\frac{\lambda}{N}t}v_0 \), where \( b \in \mathbb{R} \) and \( v_0 \in X \) satisfies (1.11). If \( b > 0 \) is sufficiently large, then there exists a unique, global solution \( u \in C([0,\infty),\Sigma) \cap L^\infty([0,\infty) \times \mathbb{R}^N) \) of (1.2). Moreover \( u \) scatters, i.e.,

\[
\lim_{t \to \infty} e^{-it\Delta}u(t) = u^+ \in \Sigma \text{ as } t \to \infty.
\]

In addition, \( \sup_{t \geq 0}(1+t)^{-\frac{N}{2}}||u(t)||_{L^\infty} < \infty \).

**Remark 1.4.** Here are some comments on Theorems 1.2 and 1.3.

(i) We will prove a slightly more general version of Theorem 1.2, in which the admissible initial values are not only the functions of \( X \) that satisfy (1.11), but also all the functions that are obtained by multiplying them by \( e^{i\frac{\lambda}{N}t} \), where \( b \) is any real number. This is Theorem 4.1 below, from which Theorem 1.2 follows immediately by choosing \( b = 0 \).

(ii) We note that if \( u \in C([a,b],\Sigma) \cap L^\infty((a,b) \times \mathbb{R}^N) \), then \( |u|^\alpha u \in C([a,b],\Sigma) \). Therefore, equation (1.2) makes sense in \( \Sigma \) for \( u \) as in Theorems 1.2, 1.3, and 4.1.

(iii) Note that we have the choice on the parameters \( k, m, n \) as long as they satisfy (1.4). In particular, \( n \) can be any integer satisfying the second condition in (1.4).

(iv) It follows from Remark 1.1 (iii) that Theorem 1.3 applies to the initial value

\[
u_0 = ze^{i\frac{\lambda}{N}t}((x)^{-n} + \psi) \text{ where } n > \max\left\{ \frac{N}{2} + 1, \frac{N}{2} \right\}, \psi \in S(\mathbb{R}^N) \text{ satisfies } \| (x)^n \psi \|_{L^\infty} < 1, z \in \mathbb{C} \text{ and } b > 0 \text{ is sufficiently large}. \tag{1.11}
\]

Theorem 1.2 applies if \( b = 0 \), and Theorem 4.1 applies if \( b \) is any real number.

(v) The solution constructed given by Theorem 1.3 has stronger regularity properties than stated. See Remark 4.5.

(vi) Note that there are no restrictions on the size of the initial value in Theorems 1.2 and 1.3. Besides the smoothness and decay imposed by the assumption \( u_0 \in X \) (or \( v_0 \in X \)), the only limitation is condition (1.11). Note that if \( u_0 \in X \) satisfies (1.11), then

\[
0 < \liminf_{|x| \to \infty} |x|^n |u_0(x)| \leq \limsup_{|x| \to \infty} |x|^n |u_0(x)| < \infty.
\]

(vii) The condition \( \alpha > \frac{N}{2} \) in Theorem 1.3 cannot be replaced by \( \alpha > \frac{\alpha_0}{N} \) for some \( \alpha_0 > \frac{N}{2} \). Indeed, if \( \alpha < \frac{N}{2} \) and \( \lambda \leq 0 \), then it follows from [3, Theorem 1.1] that every \( H^1 \)-solution of (1.1) blows up in finite or infinite time. Thus we see that no nontrivial solution of (1.1) can satisfy the conclusion of Theorem 1.3. Moreover, if \( \alpha < \frac{2}{N} \) and \( \lambda \geq 0 \), then all \( L^2 \) solutions are global, but they do not scatter. (See Strauss [15], Theorem 3.2 and Example 3.3, p. 68. See also [2] for the one-dimensional case.) In particular, the case \( \alpha = \frac{2}{N} \) is critical.

(viii) In the range \( \alpha_0 < \alpha < \frac{N}{2} \), where \( \alpha_0 \) is the positive zero of \( N\alpha^2 + (N - 2)\alpha = 4 \), the conclusion of Theorem 1.3 (except for the \( L^\infty \) decay estimate) follows from [6, Corollary 2.5]. Note that the assumptions in [6, Corollary 2.5]
concerning \( v_0 \) are less restrictive than in Theorem 1.3, it is only required that \( v_0 \in \Sigma \).

(ix) Theorem 1.3 does not say anything on what happens to the solution \( u \) for \( t < 0 \). In fact, one cannot in general expect that the initial values considered in Theorem 1.3 give rise to global solutions for negative times. Indeed, suppose \( \frac{4}{N} \leq \alpha < \frac{4}{N-2} \) (so that the Cauchy problem (1.2) is locally well-posed in \( \Sigma \)) and that \( \lambda \in \mathbb{R}, \lambda > 0 \). Let \( v_0 \) satisfy the assumptions of Theorem 1.3 and suppose further that \( \frac{1}{2} \| \nabla v_0 \|_{L^2}^2 + \frac{1}{4\alpha+2} \| v_0 \|_{L^{4\alpha+2}}^{4\alpha+2} < 0 \). (This can be achieved by multiplying \( v_0 \) by a sufficiently large constant.) Let \( u \) be the corresponding solution of (1.2) with \( u_0 = e^{it\Delta}v_0 \) defined on the maximal interval \((-S_{\text{max}}, T_{\text{max}})\). It follows from Theorem 1.3 (or [6, Corollary 2.5]) that if \( b > 0 \) is sufficiently large, then \( T_{\text{max}} = \infty \) and \( u \) scatters. On the other hand, it follows from [6, Remark 2.6] that for every \( b > 0 \), \( S_{\text{max}} < \infty \).

(x) We can apply Theorem 1.3 to construct solutions of (1.2) that exist for all \( t < 0 \) and scatter as \( t \to -\infty \). Indeed, it suffices to apply Theorem 1.3 to equation (1.2) with \( \lambda \) replaced by \( \overline{\lambda} \). If \( u_0 \) satisfies the assumptions of Theorem 1.3 (for \( \lambda \)) and \( u \) is the corresponding solution, then we see that \( v(t) = \overline{u}(-t) \) is a solution of (1.2) (with \( \lambda \)) for \( t < 0 \), which scatters as \( t \to -\infty \), and with initial value \( \overline{u_0} \). Of course, one cannot expect in general that \( v \) is global for positive times, since this would mean that \( u \) is global for negative times. (See (ix) above.)

Our strategy for proving Theorem 1.2 is based on the following observation: Since the possible defect of smoothness of the nonlinearity \( |u|^\alpha u \) is only at \( u = 0 \), there is no obstruction to regularity for a solution that does not vanish. This suggests to look for such solutions. This is not completely trivial, since there is no maximum principle for the Schrödinger equation, and this is why the various conditions in the definition of the space \( \mathcal{X} \) arise. Indeed, consider for instance \( v(x) = \langle x \rangle^{-n} \), where \( n > \frac{N}{2} + 1 \) so that \( \psi \in \Sigma \), and let \( v(t) = e^{it\Delta}u_0 \) be the solution of

\[
\begin{cases}
iv_t + \Delta v = 0 \\
v(0,x) = \psi(x).
\end{cases}
\]  

(1.12)

We want to estimate \( \inf_{x \in \mathbb{R}^n} \langle x \rangle^n |v(t,x)| \) and we note that

\[
v(t,x) = \psi(x) + i \int_0^t \Delta v(s,x) \, ds.
\]  

(1.13)

Therefore,

\[
\langle x \rangle^n |v(t,x)| \geq \langle x \rangle^n |\psi(x)| - \int_0^t \langle x \rangle^n |\Delta v|
\]

so that

\[
\inf_{x \in \mathbb{R}^n} \langle x \rangle^n |v(t,x)| \geq \inf_{x \in \mathbb{R}^n} \langle x \rangle^n |\psi(x)| - t \| \langle x \rangle^n \Delta v \|_{L^\infty((0,t) \times \mathbb{R}^n)}.
\]  

(1.14)

We now must estimate the last term on the right-hand side of (1.14). Note that we cannot simply use Sobolev’s embedding \( H^s \hookrightarrow L^\infty \) for \( s > \frac{N}{2} \). Indeed, this would require \( \langle x \rangle^n \Delta \psi \in L^2(\mathbb{R}^n) \), i.e. \( \langle x \rangle^{-2} \in L^2(\mathbb{R}^n) \), which fails if \( N \geq 4 \). On the other hand, we note that \( |\langle x \rangle^n \Delta^{k+1} \psi| \leq C \langle x \rangle^{-2k-2} \), which belongs to \( L^2(\mathbb{R}^N) \) if \( k \) is sufficiently large. Therefore, instead of applying (1.13), we apply Taylor’s formula with integral remainder involving derivatives of \( v \) of sufficiently large order, and this leads to estimating \( \langle x \rangle^n \Delta^{k+1} v(t) \) in the Sobolev space \( H^s \) where \( s > \frac{N}{2} \) and \( k \) is sufficiently large. This first step is achieved in Lemma 2.2 below. In order to estimate \( \| \langle x \rangle^n \Delta^{k+1} v(t) \|_{H^s} \), we use energy estimates. Every integration by parts will decrease by 1 the power of \( \langle x \rangle \) which is involved in the estimate, but
will at the same time increase by 1 the number of derivatives. This second step is achieved in Lemma 2.3 below, and this explains why the definition of the space $X$ involves weighted $L^\infty$-norms of the derivatives of the function up to a certain order, then weighted $L^2$-norms of the derivatives of higher order. The combination of Lemmas 2.2 and 2.3 yields Proposition 2.1 below, which is the main linear estimate we use in this paper. It shows in particular that for $\psi$ as above, $\inf_{x \in \mathbb{R}^N} (x)^{\beta} |\psi(t,x)|$ remains positive for all sufficiently small $t$. The proof of Theorem 4.1 is then a simple contraction mapping argument applied to equation (1.1). This argument requires, as usual, a linear estimate (Proposition 2.1 is our case) and a nonlinear estimate. The nonlinear estimate is provided by Proposition 3.1 below, which yields an estimate of $|u|^\alpha u$ in the space $X$, assuming $u \in X$ satisfies $|u(x)| \geq c(x)^{-\alpha}$ for some $c > 0$. This justifies the introduction of the space $X$, which is well-suited for both the Schrödinger group and the nonlinearity.

The proof of Theorem 1.3 requires one more argument, which is inspired from [6]. It consists in applying the pseudo-conformal transformation to equation (1.1). A global solution of (1.1) which scatters, corresponds to a solution of the non-autonomous equation (1.3) which is defined on $[0, \frac{1}{b}]$. If $\alpha > \frac{2}{b}$, then the time-dependent factor in (1.3) is integrable at $\frac{1}{b}$, so that can apply a standard contraction argument (based on Propositions 2.1 and 3.1) to construct a solution on $[0, \frac{1}{b})$, provided $b$ is sufficiently large.

The rest of this paper is organized as follows. In Section 2 we establish Proposition 2.1, which measures the action of the Schrödinger group $(e^{it\Delta})_{t \in \mathbb{R}}$ on the space $X$. Section 3 is devoted to the nonlinear estimate, i.e. the estimate of $|u|^\alpha u$ in $X$. The proofs of Theorems 4.1 and 1.3 are completed in Section 4. Finally, we recall in Appendix A some elementary estimates which we use in the paper.

**Notation.** We denote by $L^p(U)$, for $1 \leq p \leq \infty$ and $U = \mathbb{R}^N$ or $U = (0,T) \times \mathbb{R}^N$, $0 < t \leq \infty$, the usual (complex valued) Lebesgue spaces. We use the standard notation that $\|u\|_{L^p} = \infty$ if $u \in L^p_{loc}(U)$ and $u \notin L^p(U)$. $H^s(\mathbb{R}^N)$, $s \in \mathbb{R}$, is the usual (complex valued) Sobolev space. (See e.g. [1] for the definitions and properties of these spaces.) We denote by $(e^{it\Delta})_{t \in \mathbb{R}}$ the Schrödinger group on $\mathbb{R}^N$. As is well known, $(e^{it\Delta})_{t \in \mathbb{R}}$ is a group of isometries on $L^2(\mathbb{R}^N)$, and on $H^s(\mathbb{R}^N)$ for all $s \in \mathbb{R}$.

## 2. Weighted estimates for the linear Schrödinger equation

Our main result of this section is the following estimate of the action of the Schrödinger group $(e^{it\Delta})_{t \in \mathbb{R}}$ on the space $X$.

**Proposition 2.1.** Assume (1.4)-(1.5) with $\alpha = 1$, and let the space $X$ be defined by (1.6)-(1.7). Given $\psi \in X$, it follows that $e^{it\Delta}\psi \in C(\mathbb{R},X)$. Moreover, there exists a constant $C$ such that

\[
\|e^{it\Delta}\psi\|_X \leq C(1 + |t|)^{m+n+1}\|\psi\|_X
\]  

(2.1)

for all $t \in \mathbb{R}$ and all $\psi \in X$. In addition,

\[
\sup_{|\beta| \leq 2m} \|\langle x \rangle^\beta D^\beta (e^{it\Delta}\psi - \psi)\|_{L^\infty} \leq C|t|(1 + |t|)^{m+n+1}\|\psi\|_X
\]

(2.2)

for all $t \in \mathbb{R}$ and all $\psi \in X$.

Before proving Proposition 2.1, we first establish the following weighted $L^\infty$ estimate.
Lemma 2.2. Assume (1.4)-(1.5) with $\alpha = 1$. There exists a constant $C$ such that
\[
\sum_{j=0}^{2m} \sup_{|\beta|=j} \| x^n D^\beta e^{ix\Delta} \psi \|_{L^\infty((0,t)\times \mathbb{R}^N)} \leq C(1 + t)^m \sum_{j=0}^{2m} \sup_{|\beta|=j} \| x^n D^\beta \psi \|_{L^\infty}
+ C t(1 + t)^m \sup_{|\beta|=2m+2} \| x^n D^\beta e^{ix\Delta} \psi \|_{L^\infty((0,t)\times \mathbb{R}^N)}.
\] (2.3)
for all $t \geq 0$ and all $\psi \in H^{J}(\mathbb{R}^N)$.

Proof. Set $v(t) = e^{ix\Delta} \psi$. Since $\Delta^j \psi \in H^{J-2j}(\mathbb{R}^N)$ for $0 \leq j \leq m + 1$, we have $v \in C^J([0,\infty), H^{J-2j}(\mathbb{R}^N))$ and $\frac{d^j}{dt^j} = i^j \Delta^j v(t)$ for all $0 \leq j \leq m + 1$. Given $0 \leq \ell \leq m$, we apply Taylor’s formula with integral remainder involving the derivative of order $m - \ell + 1$ to the function $v$, and we obtain
\[
v(t) = \sum_{j=0}^{m-\ell} \frac{(it)^j}{j!} \Delta^j \psi + \frac{m-\ell+1}{(m-\ell)!} \int_0^t (t-s)^{m-\ell} \Delta^{m-\ell+1} v(s) \, ds
\] (2.4)
for all $t \geq 0$. Applying now $D^\beta$ with $|\beta| = 2\ell$, we deduce that
\[
D^\beta v(t) = \sum_{j=0}^{m-\ell} \frac{(it)^j}{j!} D^\beta \Delta^j \psi + \frac{m-\ell+1}{(m-\ell)!} \int_0^t (t-s)^{m-\ell} D^\beta \Delta^{m-\ell+1} v(s) \, ds.
\] (2.5)
Identity (2.5) holds in $C([0,\infty), H^{J}(\mathbb{R}^N))$, hence in $C([0,\infty) \times \mathbb{R}^N)$ by Sobolev’s embedding $H^k \subset C(\mathbb{R}^N)$. Multiplying by $|x|^n$ and taking the supremum in $x$, then in $t$, we obtain
\[
\sup_{|\beta|=2\ell} \| |x|^n D^\beta e^{ix\Delta} \psi \|_{L^\infty((0,t)\times \mathbb{R}^N)} \leq C(1 + t)^m \sum_{j=0}^{2m} \sup_{|\beta|=j} \| |x|^n D^\beta \psi \|_{L^\infty}
+ C t(1 + t)^m \sup_{|\beta|=2m+2} \| |x|^n D^\beta e^{ix\Delta} \psi \|_{L^\infty((0,t)\times \mathbb{R}^N)}.
\] (2.6)
Since the right-hand side of the above inequality is independent of $0 \leq \ell \leq m$, it follows by summing up in $\ell$ that
\[
\sum_{j=0}^m \sup_{|\beta|=2j} \| |x|^n D^\beta e^{ix\Delta} \psi \|_{L^\infty((0,t)\times \mathbb{R}^N)} \leq C(1 + t)^m \sum_{j=0}^{2m} \sup_{|\beta|=j} \| |x|^n D^\beta \psi \|_{L^\infty}
+ C t(1 + t)^m \sup_{|\beta|=2m+2} \| |x|^n D^\beta e^{ix\Delta} \psi \|_{L^\infty((0,t)\times \mathbb{R}^N)}.
\] (2.7)
By the interpolation estimate (A.4), derivatives of odd order in the left-hand side of (2.3) are estimated by the left-hand side of (2.7), and we conclude that (2.3) holds.

We now want to estimate the last term in the right-hand side of (2.3) by Sobolev’s embbebing. In order to do this, we establish a weighted $L^2$ estimate (Lemma 2.3 below), for which we introduce the following notation. Assuming (1.4)-(1.5) with $\alpha = 1$, we define the space
\[
\mathcal{Y} = \{ u \in H^{J}(\mathbb{R}^N); |x|^n D^\beta u \in L^2(\mathbb{R}^N) \text{ for } 2m + 1 \leq |\beta| \leq 2m + 2 + k, \langle x \rangle^{J-|\beta|} D^\beta u \in L^2(\mathbb{R}^N) \text{ for } 2m + 2 + k < |\beta| \leq J \}
\] (2.8)
and we equip $\mathcal{Y}$ with the norm
\[
\|u\|_{\mathcal{Y}} = \sum_{j=0}^{2m} \sup_{|\beta|=j} \| D^\beta u \|_{L^2} + \sum_{\nu=0}^{k+1} \sum_{\mu=0}^{n} \sum_{|\beta|=\nu+\mu+2m+1} \| |x|^n D^\beta u \|_{L^2}.
\] (2.9)
Note that by (1.9), $\mathcal{X} \hookrightarrow \mathcal{Y}$. Standard considerations show that $(\mathcal{Y}, \| \cdot \|_{\mathcal{Y}})$ is a Banach space and that $S(\mathbb{R}^N)$ is dense in $\mathcal{Y}$.
Lemma 2.3. Assume (1.4)-(1.5) with \( \alpha = 1 \), and let the space \( \mathcal{Y} \) be defined by (2.8)-(2.9). Given \( \psi \in \mathcal{Y} \), it follows that \( e^{it\Delta} \psi \in C([0, \infty), \mathcal{Y}) \). Moreover, there exists a constant \( C \) such that

\[
\|e^{it\Delta} \psi\|_{\mathcal{Y}} \leq C(1 + t)^n \|\psi\|_{\mathcal{Y}} \tag{2.10}
\]

for all \( t \geq 0 \) and all \( \psi \in \mathcal{Y} \).

Proof. Given \( \psi \in S(\mathbb{R}^n) \), we have \( e^{it\Delta} \psi \in C([0, \infty), S(\mathbb{R}^n)) \subset C([0, \infty), \mathcal{Y}) \). Therefore, by density of \( S(\mathbb{R}^n) \) in \( \mathcal{Y} \), the result follows if we prove estimate (2.10) for \( \psi \in S(\mathbb{R}^n) \). So we consider \( \psi \in S(\mathbb{R}^n) \) and we set \( v(t) = e^{it\Delta} \psi \). Since the Schrödinger group is isometric on \( H^{2m}(\mathbb{R}^n) \), we need only estimate the weighted terms in \( \|e^{it\Delta} \psi\|_{\mathcal{Y}} \). We claim that, given any \( \ell \in \mathbb{N} \),

\[
\|\langle x \rangle^\ell v(t)\|_{L^2} \leq (1 + t)^\ell \sum_{\mu = 0} \sum_{|\beta| = \mu} \|\langle x \rangle^{\ell - \mu} \tilde{D}^\beta \psi\|_{L^2} \tag{2.11}
\]

where \( C \) is independent of \( \psi \). We note that (2.11) is immediate for \( \ell = 0 \) (since \( (e^{it\Delta})_{t \in \mathbb{R}} \) is isometric on \( L^2(\mathbb{R}^n) \)). We now proceed by induction on \( \ell \), so we suppose (2.11) holds up to some \( \ell \geq 1 \). We prove that

\[
\|\langle x \rangle^{\ell + 1} v(t)\|_{L^2} \leq \|\langle x \rangle^{\ell + 1} \psi\|_{L^2} + 2(\ell + 1)\|\langle x \rangle^\ell \nabla v\|_{L^2(0,1), L^2}. \tag{2.12}
\]

This follows from an elementary integration by parts. Indeed, multiplying (1.12) by \( \langle x \rangle^{2\ell + 2} \bar{\gamma} \) and taking the imaginary part, we obtain after integration by parts over \( \mathbb{R}^n \)

\[
\frac{1}{2} \frac{d}{dt} \|\langle x \rangle^{\ell + 1} v(t)\|_{L^2}^2 = \Im \int_{\mathbb{R}^n} \nabla v \cdot \nabla (\langle x \rangle^{2\ell + 2} \bar{\gamma}) = \Im \int_{\mathbb{R}^n} \nabla v \cdot \nabla (\langle x \rangle^{2\ell + 2})
\]

\[
\leq 2(\ell + 2) \int_{\mathbb{R}^n} \langle x \rangle^{2\ell + 2} |v| |\nabla v|
\]

\[
\leq 2(\ell + 2)\|\langle x \rangle^{\ell + 1} v(t)\|_{L^2} \|\langle x \rangle^\ell \nabla v(t)\|_{L^2}.
\]

where we used (A.3) in the next-to-last inequality, and (2.12) follows. We now estimate the last term on the right-hand side of (2.12) by applying (2.11) at the level \( \ell \), but with \( \psi \) replaced by \( \nabla \psi \), and we obtain

\[
\|\langle x \rangle^{\ell + 1} v(t)\|_{L^2} \leq \|\langle x \rangle^{\ell + 1} \psi\|_{L^2} + C(1 + t)^\ell \sum_{\mu = 0} \sum_{|\beta| = \mu + 1} \|\langle x \rangle^{\ell - \mu} \tilde{D}^\beta \psi\|_{L^2}.
\]

Thus (2.11) holds at the level \( \ell + 1 \), which proves (2.11) for all \( \ell \geq 0 \).

We now fix \( 0 \leq \nu \leq k + 1, 0 \leq \mu \leq n \), we consider a multi-index \( \beta \) such that \( |\beta| = \nu + \mu + 2m + 1 \), and we apply (2.11) with \( \psi \) replaced by \( \tilde{D}^\beta \psi \), and \( \ell = n - \mu \). It follows that

\[
\|\langle x \rangle^{n - \mu} \tilde{D}^\beta v(t)\|_{L^2} \leq C(1 + t)^n \sum_{\nu = 0}^{n - \mu} \sum_{|\gamma| = \nu + \mu + 2m + 1} \|\langle x \rangle^{n - \beta} \tilde{D}^\gamma \psi\|_{L^2}
\]

\[
= C(1 + t)^n \sum_{\nu = \mu}^{n} \sum_{|\gamma| = \nu + \mu + 2m + 1} \|\langle x \rangle^{n - \gamma} \tilde{D}^\nu \psi\|_{L^2} \tag{2.13}
\]

\[
\leq C(1 + t)^n \sum_{\nu = \mu}^{n} \sum_{|\gamma| = \nu + \mu + 2m + 1} \|\langle x \rangle^{n - \gamma} \tilde{D}^\nu \psi\|_{L^2}
\]

\[
\leq C(1 + t)^n \|\psi\|_{\mathcal{Y}}.
\]

Thus we see that every weighted term in \( \|v(t)\|_{\mathcal{Y}} \) is estimated by \( C(1 + t)^n \|\psi\|_{\mathcal{Y}} \). This shows (2.10) and completes the proof. \( \square \)
Note that only estimate the terms involving \( L \) and \((\mu,T)\). We need only establish the various properties for \( X \to X \) where we used \((\inf)\). It follows in particular from Proposition 2.1 that \( \psi \) satisfies \((2.10)\) in the last inequality. Estimate (2.1) (along with the property \( v(t) \in X \) for \( t \geq 0 \)) follows from (2.10), (2.14), (2.15) and (2.16) (applied with \( \mu = 2m+2 \)).

Next, consider a multi-index \( \beta \) with \( |\beta| \leq 2m \). If \( |\beta| + 2 \geq 2m+1 \), then it follows from (1.13), (2.15) and (2.16) (applied with \( \mu = |\beta| + 2 \)) that

\[
\|\langle x \rangle^{\gamma} D^{\beta}(v(t) - \psi)\|_{L^{\infty}} \leq \int_{0}^{t} \|\langle x \rangle^{\gamma} D^{\beta} \Delta v(s)\|_{L^{\infty}} ds \leq C(1 + t)^{n}\|\psi\|_{Y}.
\]

If \( |\beta| + 2 \leq 2m \), then \( \|\langle x \rangle^{\gamma} D^{\beta} \Delta v(s)\|_{L^{\infty}} \leq \|v(s)\|_{X} \). Therefore, it follows from (1.13) and (2.1) that \( \|\langle x \rangle^{\gamma} D^{\beta}(v(t) - \psi)\|_{L^{\infty}} \leq C(1 + t)^{n+\gamma+1} \), which completes the proof of (2.2).

We finally show that \( v \in C([0,\infty), X) \). By the semigroup property, we need only show continuity at \( t = 0 \). Moreover, \( v \in C([0,\infty), Y) \) by Lemma 2.3, so we need only estimate the terms involving \( L^{\infty} \) norms. This follows from (2.2).

\[\square\]

**Remark 2.4.** It follows in particular from Proposition 2.1 that if \( \psi \in X \) satisfies \( \inf_{x \in \mathbb{R}^{N}} \langle x \rangle^{n}\|\psi(x)\| > 0 \), then \( \inf_{x \in \mathbb{R}^{N}} \langle x \rangle^{n}|e^{it\Delta}\psi(x)| > 0 \) provided \( |t| \) is sufficiently small. We do not know if this small time requirement is necessary.
3. A nonlinear estimate

We establish an estimate of \(|u|^\alpha u \) in the space \( \mathcal{X} \).

**Proposition 3.1.** Let \( \alpha > 0 \). Assume (1.4)-(1.5), and let the space \( \mathcal{X} \) be defined by (1.6)-(1.7). For every \( \eta > 0 \) and \( u \in \mathcal{X} \) such that

\[
\eta \inf_{x \in \mathbb{R}^N} \langle (x)\alpha |u(x)| \rangle \geq 1
\]

it follows that \(|u|^\alpha u \in \mathcal{X} \). Moreover, there exists a constant \( C \) such that

\[
\| |u|^\alpha u \|_{\mathcal{X}} \leq C(1 + \eta \|u\|_{\mathcal{X}})^{2J} \|u\|_{\mathcal{X}}^{\alpha + 1}
\]

for all \( \eta > 0 \) and \( u \in \mathcal{X} \) satisfying (3.1). In addition,

\[
\| |u_1|^\alpha u_1 - |u_2|^\alpha u_2 \|_{\mathcal{X}}^2 \leq C((1 + \eta(\|u_1\|_{\mathcal{X}} + \|u_2\|_{\mathcal{X}}))(\|u_1\|_{\mathcal{X}} + \|u_2\|_{\mathcal{X}})\|u_1 - u_2\|_{\mathcal{X}}^{2J + 1})
\]

for all \( \eta > 0 \) and \( u_1, u_2 \in \mathcal{X} \) satisfying (3.1).

**Proof.** We first calculate \( D^\beta(|u|^\alpha u) \) with \( 1 \leq |\beta| \leq J \). We observe that

\[
D^\beta(|u|^\alpha u) = \sum_{\gamma + \rho = \beta} c_{\gamma, \rho} D^\gamma(|u|^\alpha) D^\rho u,
\]

with the coefficients \( c_{\gamma, \rho} \) given by Leibniz’s rule. Since \(|u|^\alpha = (u|u|)\alpha \), we see that the development of \( D^\beta(|u|^\alpha u) \) contains on the one hand the term

\[
A = |u|^\alpha D^\beta u
\]

and on the other hand, terms of the form

\[
B = |u|^{\alpha - 2p} D^\rho u \prod_{j=1}^p D^{\gamma_j - j} u D^{\gamma_j - j} p
\]

where

\[
\gamma + \rho = \beta, \quad 1 \leq p \leq |\gamma|, \quad |\gamma_j + \gamma_{2j}| \geq 1, \quad \sum_{j=0}^p (\gamma_{1j} + \gamma_{2j}) = \gamma.
\]

We now proceed in two steps.

**Step 1.** Proof of (3.2). If \( |\beta| \leq 2m \), we need to estimate the terms \( \langle x \rangle^m A \) and \( \langle x \rangle^m B \) in \( L^\infty \). If \( 2m + 1 \leq |\beta| \leq 2m + 2 + k \), we need to estimate the terms \( \langle x \rangle^m A \) and \( \langle x \rangle^m B \) in \( L^2 \). If \( 2m + 3 + k \leq |\beta| \leq J \), we need to estimate the terms \( \langle x \rangle^{J - |\beta|} A \) and \( \langle x \rangle^{J - |\beta|} B \) in \( L^2 \).

We note that the term (3.5) is very easy to handle, and gives contributions estimated by \( \|u\|_{\mathcal{X}}^2 \|u\|_{\mathcal{X}} \), hence by the right-hand side of (3.2).

We now concentrate on the terms (3.6) and we observe that, due to the lower bound (3.1)

\[
|u|^{\alpha - 2p} \leq \eta^{2p} |x|^{2p m} |u|^\alpha \leq \eta^{2p} \langle x \rangle^{(2p - \alpha)n} \|u\|_{\mathcal{X}}^\alpha.
\]

so that

\[
|B| \leq \eta^{2p} \langle x \rangle^{2p m} |u|^\alpha \leq \eta^{2p} \langle x \rangle^{(2p - \alpha)n} \|u\|_{\mathcal{X}}^{\alpha} \prod_{j=1}^p |D^{\gamma_j - j} u| |D^{\gamma_j - j} u|.
\]

We now consider three different cases.

**Case 1.** Suppose \( |\beta| \leq 2m \), so that we need to estimate \( \|\langle x \rangle^m B\|_{L^\infty} \). It follows that all the derivatives in the right-hand side of (3.9) are also of order \( \leq 2m \), hence estimated by \( \langle x \rangle^{-n} \|u\|_{\mathcal{X}} \); and so (3.9) yields

\[
|B| \leq (\eta \|u\|_{\mathcal{X}})^{2p} \langle x \rangle^{-(\alpha + 1)n} \|u\|_{\mathcal{X}}^{\alpha + 1}.
\]
Therefore,
\[ \| (x)^n B \|_{L^\infty} \leq (\eta \| u \|_X)^{2p} \| u \|_X^{n+1} \] (3.11)
which is controlled by the right-hand side of (3.2).

**Case 2.** Suppose \( 2m + 1 \leq |\beta| \leq 2m + 2 + k \), so that we need to estimate \( \| (x)^n B \|_{L^2} \). Assume one of the derivatives in the right-hand side of (3.9) is of order \( \geq 2m + 1 \), for instance \( |\gamma_1| \geq 2m + 1 \). Since the sum of all derivatives has order \( |\beta| \), and \( 4m + 2 > 2m + 2 + k \geq |\beta| \) (by the third inequality in (1.4)), it follows that all other derivatives have order \( \leq 2m \), hence are estimated by \( (x)^{-n} \| u \|_X \). Therefore, (3.9) yields
\[ |B| \leq (\eta \| u \|_X)^{2p} (x)^{-\alpha n} \| u \|_X^\alpha |D^{\gamma_1} u|. \] (3.12)

Since \( 2m + 1 \leq |\gamma_1| \leq 2m + 2 + k \), we have \( \| (x)^n D^{\gamma_1} u \|_{L^2} \leq \| u \|_X \), so we see that \( \| (x)^n B \|_{L^2} \) is estimated by the right-hand side of (3.2). If all the derivatives in the right-hand side of (3.9) are of order \( \leq 2m \), then they are estimated by \( (x)^{-n} \| u \|_X \), and we obtain again (3.10), which yields
\[ (x)^{n+1} |B| \leq (\eta \| u \|_X)^{2p} (x)^{-\alpha n} \| u \|_X^{\alpha+1}. \] (3.13)

Since \( \alpha n > \frac{N}{2} \) by the second inequality in (1.4), the right-hand side of the above inequality belongs to \( L^2(\mathbb{R}^N) \), and we obtain again an estimate by the right-hand side of (3.2).

**Case 3.** Suppose \( 2m + 3 + k \leq |\beta| \leq J \), so that we need to estimate \( (x)^{J-|\beta|} B \) in \( L^2 \). This is very similar to Case 2. Assume one of the derivatives in the right-hand side of (3.9) is of order \( \geq 2m + 1 \), for instance \( |\gamma_1| \geq 2m + 1 \). Since the sum of all derivatives has order \( |\beta| \), and \( 4m + 2 > J \geq |\beta| \) (by the third inequality in (1.4)), it follows that all other derivatives have order \( \leq 2m \), hence are estimated by \( (x)^{-n} \| u \|_X \). Therefore, (3.9) yields estimate (3.12). If \( 2m + 1 \leq |\gamma_1| \leq 2m + 2 + k \), we have \( \| (x)^{J-|\beta|} D^{\gamma_1} u \|_{L^2} \leq \| (x)^n D^{\gamma_1} u \|_{L^2} \leq \| u \|_X \), so we deduce from (3.12) that \( \| (x)^{J-|\beta|} B \|_{L^2} \) is estimated by the right-hand side of (3.2). If \( 2m + 3 + k \leq |\gamma_1| \leq |\beta| \), then \( \| (x)^{J-|\beta|} D^{\gamma_1} u \|_{L^2} \leq \| (x)^{J-|\beta|} D^{\gamma_1} u \|_{L^2} \leq \| u \|_X \), and we conclude as just above. Finally, if all the derivatives in the right-hand side of (3.9) are of order \( \leq 2m \), then they are estimated by \( (x)^{-n} \| u \|_X \), and we obtain again (3.10), which yields estimate (3.13). Since \( \alpha n > \frac{N}{2} \) by the second inequality in (1.4), the right-hand side of (3.13) belongs to \( L^2(\mathbb{R}^N) \), and we obtain again an estimate by the right-hand side of (3.2). This completes the proof of (3.2).

**Step 2.** Proof of (3.3). We use the expressions (3.5) and (3.6) for both \( u_1 \) and \( u_2 \) and we form the difference. Suppose for instance that
\[ \| u_2 \|_X \leq \| u_1 \|_X. \] (3.14)
Concerning (3.5), this yields \( |u_1|^\alpha D^\beta u_1 - |u_2|^\alpha D^\beta u_2 \), which we write \( |u_1|^\alpha (D^\beta u_1 - D^\beta u_2) + (|u_1|^\alpha - |u_2|^\alpha) D^\beta u_2 \). Arguing as in Step 1, we see that the first term is estimated in the appropriate weighted spaces by \( \| u_1 \|_X^\alpha \| u_1 - u_2 \|_X \), hence by the right-hand side of (3.3). The second term is estimated by \( \| |u_1|^\alpha - |u_2|^\alpha \|_{L^\infty} \| u_2 \|_X \). We note that by (3.1) and (3.14)
\[ |u_1|^\alpha - |u_2|^\alpha \leq C \| (u_1)^{\frac{1}{2}} + |u_2|^{-\frac{1}{2}} \| (|u_1| + |u_2|)^\alpha \| u_1 - u_2 \| \leq C \eta \| u_1 \|_X \| u_1 + u_2 \| \| u_1 - u_2 \|_X \]
which yields again a control by the right-hand side of (3.3). We now examine the terms coming from the expression (3.6). We note that (3.6) is equal to \( |u|^{\alpha-2p} \)
multiplied by a multilinear expression of $u$. Therefore, the difference between the expressions for $u_1$ and $u_2$ can then be written as

$$
|u_1|^{\alpha-2p} - |u_2|^{\alpha-2p})D^\alpha u_2 \prod_{j=1}^p D^{\gamma_{1,j}} u_1 D^{\gamma_{2,j}} w_{2,j} \tag{3.15}
$$

plus a sum of terms of the form

$$
|u_1|^{\alpha-2p} D^\alpha w \prod_{j=1}^p D^{\gamma_{1,j}} w_{1,j} D^{\gamma_{2,j}} w_{2,j} \tag{3.16}
$$

where $w, w_{1,j}, w_{2,j}$ are all equal to either $u_1$ or $u_2$, except one of them which is equal to $u_1 - u_2$. The terms (3.16) can easily be estimated as in Step 1 (Cases 2 and 3), and are controlled by the right-hand side of (3.3). Finally, it remains to estimate the term (3.15). We have (using again (3.14))

$$
| |u_1|^{\alpha-2p} - |u_2|^{\alpha-2p}| \leq C(|u_1|^{1-2p-1} + |u_2|^{1-2p-1})(|u_1| + |u_2|)^\alpha |u_1 - u_2|
$$

$$
\leq C\eta \gamma^{2p+1}(x)^{(2p+1)n}(|u_1| + |u_2|)^n |u_1 - u_2|_\mu
$$

$$
= C(\eta) |u_1||\mu|^2 \gamma^{2p+1}(x)^{(2p-\alpha)n} |u_1|^{\alpha-2p-1} |u_1 - u_2|_\mu.
$$

We can use this estimate (along with (3.14)) in exactly the same way as we used estimate (3.8), and we can conclude as in Step 1. This completes the proof. \hfill \square

4. Proofs of Theorems 1.2 and 1.3

We will prove the following result, slightly more general than Theorem 1.2.

**Theorem 4.1.** Let $\alpha > 0$ and $\lambda \in \mathbb{C}$. Assume (1.4)-(1.5), let $\mathcal{X}$ be defined by (1.6)-(1.7) and $\Sigma$ by (1.10). Suppose $u_0 = e^{\frac{bi\cdot x}{2}} v_0$ where $b \in \mathbb{R}$, and $v_0 \in \mathcal{X}$ satisfies (1.11). It follows that there exist $T > 0$ and a unique solution $u \in C([-T, T], \Sigma) \cap L^\infty((-T, T) \times \mathbb{R}^N)$ of (1.2). Moreover, the map $t \mapsto e^{-\frac{bi\cdot x}{2}} u(t, x)$ is continuous $[-T, T] \rightarrow \mathcal{X}$.

We let $b \in \mathbb{R}$ and we consider equation (1.3), or its equivalent integral form

$$
v(t) = e^{it\Delta} v_0 + i\lambda \int_0^t (1 - bs)^{-\frac{\Delta}{2\Delta}} e^{i(t-s)\Delta} |v|^\alpha v ds. \tag{4.1}
$$

We prove existence results for (4.1), of which Theorems 4.1 and 1.3 are immediate consequences, by using the pseudo-conformal transformation.

**Proposition 4.2.** Let $\alpha > 0$ and $\lambda \in \mathbb{C}$. Assume (1.4)-(1.5), and let the space $\mathcal{X}$ be defined by (1.6)-(1.7). Given any $b \in \mathbb{R}$ and $v_0 \in \mathcal{X}$ satisfying (1.11), there exist $0 < T < \frac{1}{|\lambda|}$ and a solution $v \in C([-T, T], \mathcal{X})$ of (4.1).

**Proof.** We use a standard contraction mapping argument, based on the linear estimates of Proposition 2.1 and the nonlinear estimates of Proposition 3.1. We let

$$
\eta > 0, \quad K > 0, \quad 0 < T < \max\{\frac{1}{|\lambda|}, 1\}
$$

and we define the set $\mathcal{E}$ by

$$
\mathcal{E} = \{v \in C([-T, T], \mathcal{X}); \|v\|_{L^\infty([-T, T], \mathcal{X})} \leq K \text{ and } \eta(x)^\alpha |v(t, x)| \geq 1 \text{ for } -T < t < T, x \in \mathbb{R}^N\}. \tag{4.2}
$$
It follows that $\mathcal{E}$ equipped with the distance $d(u,v) = ||u - v||_{L^\infty([−T,T],\mathcal{X})}$ is a complete metric space. Given $v \in \mathcal{E}$ and $v_0 \in \mathcal{X}$, we set

$$\Phi_v(t) = i\lambda \int_0^t (1 - bs) - \frac{\bar{\nu}}{2} e^{i(t-s)\Delta} |v|^\alpha v \, ds$$

(4.3)

$$\Psi_{v_0,v}(t) = e^{it\Delta} v_0 + \Phi_v(t)$$

(4.4)

for $−T < t < T$. We observe that the definition of $\mathcal{E}$ together with Proposition 3.1 imply that if $u \in \mathcal{E}$, then $|u|^\alpha u \in C([-T,T],\mathcal{E})$ and

$$||u|^\alpha u||_{L^\infty([-T,T],\mathcal{X})} \leq C(1 + \eta K)^{2J} K^{\alpha+1}.$$  (4.5)

Using Proposition 2.1, we deduce that the map $s \mapsto e^{i\lambda \Delta} |u(s)|^\alpha u(s)$ belongs to $C([-T,T],\mathcal{X})$, so that (still using Proposition 2.1) $\Phi_v \in C([-T,T],\mathcal{X})$. In addition, we deduce from (4.5) and (2.1) that

$$||\Phi_v||_{L^\infty([-T,T],\mathcal{X})} \leq C|\lambda| f(T) 2^J (1 + \eta K)^{2J} K^{\alpha+1}$$

(4.6)

and

$$||\Psi_{v_0,v}||_{L^\infty([-T,T],\mathcal{X})} \leq C 2^J (||v_0||_\mathcal{X} + |\lambda| f(T) (1 + \eta K)^{2J} K^{\alpha+1})$$

(4.7)

where

$$f(T) = \int_0^T \max\{(1 - bs) - \frac{\bar{\nu}}{2}, (1 + bs) - \frac{\bar{\nu}}{2}\} \, ds \rightarrow 0.$$  

(4.8)

By a similar argument, it follows from (3.3) that if $v, w \in \mathcal{E}$, then

$$||\Phi_v - \Phi_w||_{L^\infty([-T,T],\mathcal{X})} \leq C|\lambda| f(T) 2^J (1 + \eta K)^{2J+1} K^{\alpha} d(v,w).$$  

(4.9)

Next, we deduce from (2.2) and (4.6) that

$$\langle x \rangle^n |\Psi_{v_0,v}(t,x)| \geq \inf_{x \in \mathbb{R}^N} \langle x \rangle^n |v_0(x)| - CT 2^J ||v_0||_\mathcal{X} - ||\Phi_v(t)||_\mathcal{X}$$

$$\geq \inf_{x \in \mathbb{R}^N} \langle x \rangle^n |v_0(x)| - CT 2^J (||v_0||_\mathcal{X} + |\lambda| f(T) (1 + \eta K)^{2J} K^{\alpha+1})$$

(4.10)

for all $−T \leq t \leq T$ and $x \in \mathbb{R}^N$. We now argue as follows. We denote by $\widetilde{C}$ the supremum of the constants $C$ in (4.6)–(4.10) and we consider $v_0 \in \mathcal{X}$ such that

$$\inf_{x \in \mathbb{R}^N} \langle x \rangle^n |v_0(x)| > 0.$$  

(4.11)

We set

$$\eta = 2\left(\inf_{x \in \mathbb{R}^N} \langle x \rangle^n |v_0(x)|\right)^{-1}$$

$$K = 2\widetilde{C} 2^J ||v_0||_\mathcal{X}.$$  

(4.12)

It follows in particular that $v(t) \equiv v_0$ belongs to $\mathcal{E}$, so that $\mathcal{E} \neq \emptyset$. We fix $T$ sufficiently small so that

$$\widetilde{C} 2^J |\lambda| f(T) (1 + \eta K)^{2J+1} K^{\alpha} \leq \frac{1}{2}$$

(4.13)

$$\widetilde{C} T 2^J (||v_0||_\mathcal{X} + |\lambda| f(T) (1 + \eta K)^{2J} K^{\alpha+1}) \leq \frac{1}{\eta}.$$  

(4.14)

Inequalities (4.7), (4.11) and (4.13) imply that $||\Psi_{v_0,v}||_{L^\infty([-T,T],\mathcal{X})} \leq K$. Moreover, (4.10), (4.11) and (4.14) imply that

$$\eta (\langle x \rangle^n |\Psi_{v_0,v}(t,x)|) \geq 1$$

for $−T \leq t \leq T$ and $x \in \mathbb{R}^N$. Thus we see that $\Psi_{v_0,v} \in \mathcal{E}$ for all $v \in \mathcal{E}$. In addition, it follows from (4.9) and (4.13) that the map $v \mapsto \Psi_{v_0,v}$ is a strict contraction $\mathcal{E} \rightarrow \mathcal{E}$. Therefore, it has a fixed point, which is a solution of (4.1) on $[-T,T]$. 

**Proof of Theorem 4.1.** Given $v_0 \in \mathcal{X}$ satisfying (1.11) and $b \in \mathbb{R}$, let $0 < T < \frac{1}{b}$ and $v \in C([-T,T],\mathcal{X})$ be the solution of (4.1) given by Proposition 4.2. Let $u$ be defined by

$$u(t,x) = (1 + bt)^{-\frac{\nu}{2}} e^{\frac{\nu}{4} |x|^2} v\left(\frac{t}{1 + bt}, \frac{x}{1 + bt}\right)$$

(4.15)
The proof is similar to the proof of Proposition 4.2. Moreover, (4.19) implies that $\eta(x)^n|v(t,x)| \geq 0$ for $0 \leq t \leq \frac{1}{\eta}$ and $x \in \mathbb{R}^N$. Arguing as in the proof of Proposition 4.2, we see that

$$\|v^n\|_{L^\infty(0,\frac{1}{\eta},x)} \leq C(1 + \eta K)^{2J} K^{\alpha+1}$$

that $\Phi_v \in C([0,\frac{1}{\eta}], \mathcal{X})$ and that

$$\|\Phi_v\|_{L^\infty(0,\frac{1}{\eta},x)} \leq C|\lambda|^\frac{1}{\theta}(1 + \frac{1}{\theta})^J(1 + \eta K)^{2J} K^{\alpha+1}$$

$$\|\Psi_{v_0}v\|_{L^\infty(0,\frac{1}{\eta},x)} \leq C(1 + \frac{1}{\eta})^J(\|v_0\|_x + |\lambda|^\frac{1}{\theta}(1 + \eta K)^{2J} K^{\alpha+1})$$

$$\|\Phi_v - \Phi_{v_0}\|_{L^\infty(0,\frac{1}{\eta},x)} \leq C|\lambda|^\frac{1}{\theta}(1 + \frac{1}{\theta})^J(1 + \eta K)^{2J+1} K^{\alpha}d(v, v_0).$$

(In the last three inequalities, we use the identity $\int_0^\frac{1}{\eta} (1 - bs)^{-\frac{4N\alpha}{(N\alpha - 2)\theta}} ds = \frac{2}{(N\alpha - 2)\theta}$.)

Next, it follows from (2.2) and (4.18) that

$$\langle x \rangle^n|v_0(x)| \geq \inf_{x \in \mathbb{R}^N} \inf_{t \in \mathbb{R}} \langle x \rangle^n|v_0(x)| - C(1 + \eta)\|v_0\|_x - \|\Phi_v(t)|_x$$

$$\geq \inf_{x \in \mathbb{R}^N} \langle x \rangle^n|v_0(x)| - C\bigg(\|v_0\|_x + |\lambda|(1 + \eta K)^{2J} K^{\alpha+1}\bigg).$$

We now argue as follows. We denote by $\tilde{C}$ the supremum of the constants $C$ in (4.18)-(4.21). We consider $v_0 \in \mathcal{X}$ such that $\inf_{x \in \mathbb{R}^N} \langle x \rangle^n|v_0(x)| > 0$. (Such $v_0$ exist, , see Remark 1.1 (iii)), we let $\eta$ be defined by (4.11) and we set

$$K = 4\tilde{C}\|v_0\|_x.$$  

It follows in particular that $v(t) \equiv v_0$ belongs to $\mathcal{E}$, so that $\mathcal{E} \neq \emptyset$. We consider $b > 0$ sufficiently large so that

$$|\lambda|^\frac{1}{\theta}(1 + \eta K)^{2J} K^{\alpha+1} \leq \|v_0\|_x$$

$$2\eta\tilde{C}^\frac{1}{\theta}(\|v_0\|_x + |\lambda|(1 + \eta K)^{2J} K^{\alpha+1}) \leq 1$$

$$4\tilde{C}\bigg|\lambda|^\frac{1}{\theta}(1 + \eta K)^{2J+1} K^{\alpha} \leq 1.$$  

Inequalities (4.19), (4.23), (4.24) and (4.22) imply that $\|\Psi_{v_0}v\|_{L^\infty(0,\frac{1}{\eta},x)} \leq K$. Moreover, (4.21), (4.11), (4.23) and (4.25) imply that $\eta(x)^n|\Psi_{v_0}(t,x)| \geq 1$ for $0 \leq t \leq \frac{1}{\eta}$ and $x \in \mathbb{R}^N$. Thus we see that $\Psi_{v_0}v \in \mathcal{E}$ for all $v \in \mathcal{E}$. In addition, it
follows from (4.20), (4.23) and (4.26) that the map \( v \mapsto \Psi_{v_0, v} \) is a strict contraction \( \mathcal{E} \to \mathcal{E} \). Therefore, it has a fixed point, which is a solution of (4.1) on \([0, \frac{1}{T}]\). □

**Remark 4.4.** One can solve under the same conditions the problem with final value \( \psi \) at time \( \frac{1}{T} \), i.e.

\[
v(t) = e^{i(t-\frac{1}{T})\Delta} \psi + i\lambda \int_t^{\frac{1}{T}} (1 - bs) \frac{(-1)^s}{s!} e^{i(t-s)\Delta} |v|^\alpha v \, ds
\]

(4.27)

**Proof of Theorem 1.3.** Given \( v_0 \in \mathcal{X} \) satisfying (1.11), let \( b > 0 \) be sufficiently large so that there exists a solution \( v \in C([0, \frac{1}{T}], \mathcal{X}) \) of (4.1), by Proposition 4.3. Let \( u \) be defined by (4.15) for \( 0 \leq t < \infty \) and \( x \in \mathbb{R}^N \). It follows from elementary calculations that

\[
u(0) = \frac{e^{ibx^2} - e^{-ibx^2}}{2} v_0, \quad \text{and} \quad e^{-it\Delta} u(t) \to u^+
\]

in \( \mathcal{X} \) as \( t \to \infty \), where \( u^+ = \frac{e^{ibx^2} - e^{-ibx^2}}{2} v \). See e.g. [6, Proposition 3.14]. In addition, since \( \sup_{0 \leq t \leq 1} \| v(t) \|_{L^\infty} < \infty \), it follows from (4.15) that \( \sup_{t \geq 0} (1 + t)^\frac{\alpha}{2} \| u(t) \|_{L^\infty} < \infty \). Finally, uniqueness in \( C([0, \frac{1}{T}], \mathcal{X}) \cap L^\infty((0, \frac{1}{T}) \times \mathbb{R}^N) \) is immediate (because of the \( L^\infty \) bound), and the proof is complete. □

**Remark 4.5.** Note that the solution \( u \) of (1.2) in Theorem 1.3 is obtained by applying the pseudo-conformal transformation to the solution \( v \) of (4.1) constructed in Proposition 4.3. It follows that it has stronger regularity properties than stated in Theorem 1.3. Indeed, it follows easily from formula (4.15) that the map \( t \mapsto e^{-i(x^2 + |x|^\alpha) t} u(t, x) \) is continuous \( [0, \infty) \to \mathcal{X} \). In addition, the scattering state \( u^+ \) satisfies \( e^{-i(x^2 + |x|^\alpha) t} u^+ \in \mathcal{X} \).

**APPENDIX A. SOME ELEMENTARY ESTIMATES**

In this section, we collect a point-wise estimate (Lemma A.2) and an interpolation estimate (Lemma A.3) which we use in this paper. For the proof of Lemma A.2, we will use the following observation.

**Remark A.1.** Let \( f : \mathbb{R} \to \mathbb{R} \) and \( g : \mathbb{R}^N \to \mathbb{R} \). Given a multi-index \( \alpha \) with \( |\alpha| \geq 1 \), \( D^\alpha f(g(x)) \) is a sum of terms of the form \( f^{(m)}(g(x)) \prod_{k=1}^m \beta_k g(x) \) where \( 1 \leq m \leq |\alpha| \), and the \( \beta_k \) are multi-indices such that \( |\beta_k| \geq 1 \) and \( |\beta_1| + \cdots + |\beta_m| = |\alpha| \).

**Lemma A.2.** Given \( \eta \in \mathbb{R} \) and a multi-index \( \alpha \), there exists a constant \( C \) such that

\[
|D^\alpha((x)\eta^u)| \leq C \sum_{n=0}^{|\alpha|} (\eta)^{-|\alpha|+n} \sum_{|\beta|=n} |D^\beta u|
\]

(A.1)

for all \( u \in C^{[\alpha]}(\mathbb{R}^N, \mathbb{C}) \).

**Proof.** We first claim that

\[
|D^\alpha(x)| \leq C\eta^{-|\alpha|}.
\]

(A.2)

Indeed, we apply Remark A.1 with \( f(t) = (1 + t)^{\frac{\alpha}{2}} \) and \( g(x) = |x|^2 \), so that \( \langle x \rangle = f(g(x)) \). We note that

\[
f^{(\ell)}(t) = c_\ell (1 + t)^{\frac{\alpha}{2} - \ell}
\]

and that \( \partial_j g = 2x_j, \partial_{ij} g = 2\delta_{ij}, D^\beta g = 0 \) if \( |\beta| \geq 3 \). In particular,

\[
|D^\beta g(x)| \leq c_\alpha |x|^{2-|\beta|}
\]
so that the generic term in the development of $D^\alpha(x)$ given by Remark A.1 can be estimated by $(x)^{1-2\delta}\prod_{k=1}^\ell |x|^{2-|\beta_k|}$. Since $|\beta_1| + \cdots + |\beta_\ell| = |\alpha|$, we obtain (A.2). We now consider a real number $\eta$ and we apply Remark A.1 with $f(t) = t^\eta$ and $g(x) = (x)$. Using (A.2), we deduce easily that

$$|D^\alpha(x)^\eta| \leq C_{\eta,\alpha} (x)^{\eta - |\alpha|}$$

and (A.1) follows by applying Leibniz’s rule. □

**Lemma A.3.** Given $j \in \mathbb{N}$ and $\nu \in \mathbb{R}$, there exists a constant $C$ such that

$$\sup_{|\beta|=j+1} \|\langle x \rangle^\nu D^\beta u\|_{L^\infty} \leq C (\sup_{|\beta|\leq j} \|\langle x \rangle^\nu D^\beta u\|_{L^\infty} + \sup_{|\alpha|=2} \|\langle x \rangle^\nu D^\alpha u\|_{L^\infty})$$

for all $u \in C^{j+2}(\mathbb{R}^N)$.

**Proof.** It suffices to show that

$$\|\langle x \rangle^\nu \nabla u\|_{L^\infty} \leq C (\|\langle x \rangle^\nu u\|_{L^\infty} + \sup_{|\alpha|=2} \|\langle x \rangle^\nu D^\alpha u\|_{L^\infty}).$$

Given $x, y \in \mathbb{R}^N$, we have

$$u(y) - u(x) = \int_0^1 \frac{d}{ds} u(x + s(y-x)) \, ds = \int_0^1 (y-x) \cdot \nabla u(x + s(y-x)) \, ds$$

and

$$\nabla u(x + s(y-x)) - \nabla u(x) = \int_0^1 s(y-x) \cdot \nabla^2 u(x + s(y-x)) \, ds$$

which imply

$$u(y) - u(x) = \int_0^1 (y-x) \cdot \nabla u(x) \, ds$$

$$+ \int_0^1 \int_0^1 (y-x) \cdot [s(y-x) \cdot \nabla^2 u(x + s(y-x))] \, d\sigma \, ds.$$

If $\nabla u(x) \neq 0$, we let $y = x + \frac{\nabla u(x)}{|\nabla u(x)|}$, and we obtain

$$u(y) - u(x) = |\nabla u(x)|$$

$$+ \int_0^1 \int_0^1 (y-x) \cdot [\nabla^2 u(x + s(y-x))] \, d\sigma \, ds$$

so that

$$|\nabla u(x)| \leq 2 \sup_{|y-x|\leq 1} |u(y)| + \sup_{|y-x|\leq 1} \sup_{|\beta|=2} |D^\beta u(y)|.$$  

Estimate (A.5) easily follows. □

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