Local D=4 Field Theory on $\kappa$–Deformed Minkowski Space

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We describe the local D=4 field theory on $\kappa$–deformed Minkowski space as nonlocal relativistic field theory on standard Minkowski space–time. For simplicity the case of $\kappa$-deformed scalar field $\phi$ with the interaction $\lambda \phi^4$ is considered, and the $\kappa$–deformed interaction vertex is described. It appears that fundamental mass parameter $\kappa$ plays a role of regularizing imaginary Pauli–Villars mass in $\kappa$–deformed propagator.

INTRODUCTION

The conventional field–theoretic models in elementary particle physics (e.g. standard model – Yang-Mills theory with spinorial and Higgs sector) are described by quantum field theory on standard D=4 Minkowski space. The addition of gravity effects can be realized in the following two steps:

a) We consider field–theoretic models in curved background of classical gravitational field, describing curved space–time geometry.

b) Further we quantize curved space–time geometry by considering quantized gravitational field.

At present it is not known how to handle with reasonable accuracy the quantum gravity modification of geometry, e.g. describe field theories in the background of quantized geometries. It is known moreover, that due to divergent quantum effects in Einstein gravity the notion of classical space–time can not be used at distances comparable and smaller than the Planck length $l_p$ ($l_p \approx 10^{-33}$ cm) (see e.g. [1-3]). In front of this difficulty one way of generalizing the standard framework of local relativistic fields is to replace classical space–time points by some primary extended objects (e.g. fundamental strings, p-branes etc.) providing new field theory (e.g. string field theory) which leads to finite quantum corrections. Another way, closer to the idea of quantized geometry, is to replace commuting space–time coordinates by noncommuting generators of a quantized Minkowski space. There is a hope that such a quantized space–time geometry will provide additional convergence factors or even finite quantum field theory. Indeed, as we shall show below, if we introduce mass–like deformation parameter $\kappa$, it occurs also as a regularizing imaginary Pauli–Villars large mass parameter, describing tachyonic pole in the $\kappa$–deformed propagator.

I. $\kappa$–DEFORMED $D = 4$ RELATIVISTIC SYMMETRIES

The standard space–time coordinates $x_\mu = (\vec{x}, x_0)$ can be described as translations which form the Abelian subgroup of $D = 4$ Poincaré group $P_4$ (formally one can identify the Minkowski space with the coset $P_4/O(3,1)$, where $O(3,1)$ describes the Lorentz subgroup). Similarly the properties of four relativistic momenta can be described by the four translation generators of $D = 4$ Poincaré algebra. In such a scheme one considers the relativistic Poincaré symmetries, described by the dual
pair - $D = 4$ Poincaré group and $D = 4$ Poincaré algebra - as primary geometric notions from which the properties of Minkowski space, fourmomenta as well as relativistic phase space are derived. This primary status of symmetries we shall keep also in the case when the classical relativistic framework is modified by the procedure of quantum deformations. In analogy with classical case, the quantum Minkowski space and quantum fourmomentum space can be obtained, respectively, from the translation sector of D=4 quantum–deformed Poincaré group and fourmomentum generators belonging to quantum Poincaré algebra.

In the last years there were proposed different quantum deformations of D=4 Poincaré symmetries (see e.g. [4–7]) in the form of real noncommutative and noncocommutative deformations of D=4 Poincaré algebra.

The classical Poincaré algebra is obtained in the limit $\kappa \to \infty$. The Hopf algebra (1.1a–d) has its dual form, generated by the following duality relations

\[ \langle P^\mu, \hat{x}_\rho \rangle = \delta^\mu_\rho \]  
\[ \langle M^{\mu\nu}, \Lambda_\rho \rangle = \delta^\mu_\rho \delta^\nu_\tau - \delta^\nu_\rho \delta^\mu_\tau \]  

In accordance with general scheme the dual generators ($\hat{x}_\rho, \Lambda_\rho$) describe $\kappa$–deformed Poincaré group. Using duality relations between arbitrary powers of the generators ($P^\mu, M^{\mu\nu}$) and ($x_\rho, \Lambda_\rho$) as well as the property of inner product

\[ \langle A \otimes B, \Delta(a) \rangle = \langle A \cdot B, a \rangle \]
\[ \langle \Delta(A), a \otimes b \rangle = \langle A, a \cdot b \rangle \]  

one can derive the following set of relations, defining $\kappa$–deformed Poincaré group [5,8]:

a) algebraic relations

\[ \Delta(M_i) = M_i \otimes \mathbb{I} + \mathbb{I} \otimes M_i, \]
\[ \Delta(N_i) = N_i \otimes e^{-P_0/\kappa} + \mathbb{I} \otimes N_i - \frac{1}{\kappa} \epsilon_{ijk} M_j \otimes P_k, \]
\[ \Delta(P_i) = P_i \otimes e^{-P_0/\kappa} + \mathbb{I} \otimes P_i, \]
\[ \Delta(P_0) = P_0 \otimes \mathbb{I} + \mathbb{I} \otimes P_0, \]  

\[ S(M_i) = -M_i, \]  
\[ S(N_i) = - \left( N_i + \frac{1}{\kappa} \epsilon_{ijk} M_j P_k \right) e^{P_0/\kappa}, \]
\[ S(P_i) = -P_i e^{P_0/\kappa}, \]
\[ S(P_0) = -P_0, \]  

b) coproducts

\[ \Delta(\hat{x}^\mu) = \Lambda_\rho^\mu \otimes \hat{x}^\rho + \hat{x}^\mu \otimes \mathbb{I}, \]
\[ \Delta(\Lambda_\rho^\mu) = \Lambda_\rho^\mu \otimes \Lambda_\rho^\mu, \]  

c) antipodes and counits

1The $\kappa$–deformed algebra can be written with the use of different basic generators. The one presented below uses so-called bicrossproduct basis [8] with the quantum deformation in the algebraic sector occurring entirely in the “cross” commutation relations of the Abelian fourmomentum generators with the classical Lorentz generators.
\[ S(\Lambda_{\mu}^{a}) = -\Lambda_{\mu}^{a}, \quad S(x^{\mu}) = -x^{\mu}, \]
\[ \epsilon(\Lambda_{\mu}^{a}) = \delta_{\mu}^{a}, \quad \epsilon(x^{\mu}) = 0. \quad (1.4c) \]
It should be observed that the relations \((1.2b)\) and \((1.2c)\) remains undeformed, as for classical Poincaré group.

Our aim in this paper is to consider the field theory on \(\kappa\)-deformed Minkowski space, described by the translation generators \(\hat{x}_{\mu}\) of the \(\kappa\)-deformed Poincaré group (see \((1.4a\text{--}c)\)) after the contraction mapping \(\Lambda_{\mu}^{a} \rightarrow \delta_{\mu}^{a}\).

The \(\kappa\)-deformed Hopf algebra \(H_{x}\) describing the quantum Minkowski space—time is generated by the coordinates \(\hat{x}_{\mu}\) and determined by the following basic relations:
\[ [\hat{x}_{0}, \hat{x}_{i}] = \frac{i}{\kappa} \hat{x}_{i}, \quad [\hat{x}_{i}, \hat{x}_{j}] = 0, \quad (1.5a) \]
\[ \Delta(\hat{x}_{\mu}) = \hat{x}_{\mu} \otimes \mathbb{1} + \mathbb{1} \otimes \hat{x}_{\mu}, \quad (1.5b) \]
The dual Hopf algebra \(H_{p}\) of functions on \(\kappa\)-deformed fourmomenta is described by the Hopf subalgebra of the \(\kappa\)-deformed Poincaré algebra \((1.1a\text{--}c)\) as follows
\[ [p_{\mu}, p_{\nu}] = 0, \quad (1.6a) \]
\[ \Delta(p_{i}) = p_{i} \otimes e^{-p_{0}/\kappa} + \mathbb{1} \otimes p_{i}, \]
\[ \Delta(p_{0}) = p_{0} \otimes \mathbb{1} + \mathbb{1} \otimes p_{0}. \quad (1.6b) \]
The antipodes and counits in \(H_{x}\) and \(H_{p}\) remain classical.

An important tool will be the use of \(\kappa\)-deformed Fourier transform, describing the fields on noncommutative Minkowski space with generators \(\hat{x} = (\hat{x}_{i}, \hat{x}_{0})\) in the following way \((p\hat{x} \equiv p_{i}\hat{x}_{i} - p_{0}\hat{x}_{0})^{2}\)
\[ \Phi(\hat{x}) = \frac{1}{(2\pi)^{4}} \int d^{4}p \tilde{\Phi}(p) : e^{ip\hat{x}} : \quad (1.7) \]
\(\tilde{\Phi}(p)\) is a classical function on commuting fourmomentum space \(p = (p_{i}, p_{0})\) and \((p\tilde{x} \equiv p_{i}\hat{x}_{i})\)
\[ : e^{ip\hat{x}} : = e^{i\hat{p}_{0}\hat{x}_{0}} e^{ip\hat{x}}. \quad (1.8) \]

The \(\kappa\)-deformed exponential \((1.4b)\) describes the canonical element \([9,10]\) or \(T\)-matrix \([11,12]\) for the pair of dual Hopf algebras \(H_{x}\) and \(H_{p}\).

From \((1.6b)\) and \((1.8)\) follows that
\[ : e^{i\hat{p}\hat{x}} : = : e^{iq\hat{x}} : = : e^{i\Delta^{(2)}(p,q)\hat{x}} : \quad (1.9) \]
where \(\Delta^{(2)}(p,q) = (p_{i}e^{-q_{0}/\kappa} + q_{i}p_{0} + q_{0})\) is the fourmomentum addition rule described by the coproduct \((1.6b)\).

From the Fourier transform \(\tilde{\Phi}(p)\) one can obtain also a standard relativistic field \(\phi(x)\) on classical Minkowski space \(x \equiv (x_{i}, x_{0})\) by performing the classical Fourier transform
\[ \phi(x) = \frac{1}{(2\pi)^{4}} \int d^{4}p \tilde{\Phi}(p) e^{ipx}. \quad (1.10) \]

It is easy to see that in the limit \(\kappa \rightarrow \infty\) we get \(\hat{x}_{\mu} \rightarrow x_{\mu}\) and the formula \((1.7)\) describing “quantum” Fourier transform can be identified with the classical one, given by \((1.10)\).

We shall describe the \(\kappa\)-deformation of relativistic local field theory in the following three steps:

1) We replace in conventional local relativistic field theory the classical Minkowski coordinates \(x_{\mu}\) by quantum Minkowski coordinates \(\hat{x}_{\mu}\), and relativistic-covariant differential operators defining free fields by corresponding \(\kappa\)-covariant differential operators on \(\kappa\)-deformed Minkowski space. The \(\kappa\)-deformed Lagrangean is the product of fields \((1.7)\) and its derivatives and becomes an element of noncommutative Hopf algebra defined by \((1.5)\).

2) The new action is obtained by integration of local products of \(\kappa\)-deformed fields and their derivatives over \(\kappa\)-deformed Minkowski space.

3) In order to perform the integration we substitute in the \(\kappa\)-deformed Lagrangean the Fourier transforms \((1.7)\), apply the formula \((1.9)\), and replace the integration over \(\kappa\)-deformed Minkowski space by the \(\kappa\)-deformed convolution integrals in fourmomentum space. We can calculate all occurring \(\kappa\)-deformed integrals \(\iint d^{4}\hat{x}\) by using the formula
\[ \frac{1}{(2\pi)^{4}} \iint d^{4}\hat{x} : e^{ip\hat{x}} : = \delta^{4}(p). \quad (1.11) \]
which implies for example that
\[ \iint \phi(\hat{x}) d^{4}x = \int d^{4}p \delta^{4}(p) \tilde{\Phi}(p) = \tilde{\Phi}(0) \quad (1.12a) \]
\[ \iint \phi^{2}(\hat{x}) d^{4}x = \int d^{4}p_{1} \int d^{4}p_{2} \tilde{\Phi}(p_{1}) \tilde{\Phi}(p_{2}) \delta(\Delta^{(2)}(p_{1}, p_{2})) \]
\[ = \int d^{4}p \tilde{\Phi}(\bar{p}, p_{0}) \left(-\bar{p}e^{-p_{0}/\kappa}, -p_{0}\right) \quad (1.12b) \]
and in general case
\[ \iint d^{4}\hat{x} \Phi^{n}(\hat{x}) = \int d^{4}p^{(1)} \int d^{4}p^{(n)} \Phi(p^{(1)}) \]
\[ \ldots \Phi(p^{(n)}) \delta^{(4)}(\Delta_{\mu}(p^{(1)}, \ldots, p^{(n)})) \quad (1.12c) \]

\(^{2}\) One can also use in \((1.7)\) the measure \(\Omega^{4} = d^{4}p e^{-3\Re p_{0}/\kappa}\) which is invariant under the shift in fourmomentum space described by the coproduct \((1.6b)\).
where $\Delta^{(n)}$ is the iterated coproduct (1.4b)

\[
\Delta^{(n)}(p^{(1)}, \ldots, p^{(n)}) = p^{(1)}_0 + \ldots + p^{(n)}_0
\]

\[
\tilde{\Delta}^{(n)}(\tilde{p}^{(1)}, \ldots, \tilde{p}^{(n)}) = \sum_{k=1}^{n} \tilde{p}^{(k)} e^{-\frac{\kappa}{\hbar} \sum_{i=k+1}^{n} p^{(i)}}
\]

The integral (1.12a) in the two-dimensional case in different contexts has been used by Majid [13,14].

3) In order to interpret the $\kappa$-deformation in standard Minkowski space we use the formula (1.10). The $\kappa$-deformed convolution integrals of Fourier transforms $\hat{\phi}(p)$ become the $\kappa$-dependent nonlocal vertices of fields $\phi(x)$ in standard Minkowski space.

**II. $\kappa$-DEFORMED MINKOWSKI SPACE: DIFFERENTIAL BICOVARIANT CALCULUS, VECTOR FIELDS AND INTEGRATION**

In order to describe the field equations on $\kappa$-deformed Minkowski space one should consider corresponding differential calculus and its covariance properties under the action of $\kappa$-deformed Poincaré group.

a) Differential bicovariant calculus.

On $\kappa$-deformed $H_\kappa$ with four selfadjoint generators $\hat{x}_\mu$, one can construct a five-dimensional bicovariant differential calculus with the basis [15,16,17]

\[
\tau^\mu = dx^\mu, \quad \tau^5 = [dx^\mu, \hat{x}_\mu] + \frac{3i}{\kappa} dx^0,
\]

satisfying the relations

\[
[\tau^\mu, \hat{x}^\nu] = \frac{i}{\kappa} \eta^0\tau^\nu - \frac{i}{\kappa} \eta^\nu\tau^0 + \frac{1}{4} \eta^\mu \tau^0,
\]

\[
[\tau^5, \hat{x}^\nu] = -\frac{4}{\kappa^2} \tau^\nu, \quad (A = 0, 1, 2, 3, 4, 5)
\]

and $(A = 0, 1, 2, 3, 4, 5)$

\[
\tau^A \wedge \tau^B = -\tau^A \wedge \tau^B,
\]

\[
d\tau^\mu = 0 \quad d\tau^5 = -2\tau^\mu \wedge \tau^\mu. \quad (2.3)
\]

The $\kappa$-Minkowski space (1.5a–b) carries the left covariant action of $\kappa$-Poincaré group (1.4a–c)

\[
\rho_L(\hat{x}^\mu) = \Lambda^\mu_\nu \otimes \hat{x}^\nu + a^\mu \otimes 1
\]

and the relations (2.2–3) are covariant under the following transformations of differentials

\[
\tilde{\rho}_L(\tau^\mu) = \Lambda^\mu_\nu \otimes \tau^\nu, \quad \tilde{\rho}_L(\tau^5) = 1 \otimes \tau^5. \quad (2.5)
\]

One can also define the covariant right action of another Poincaré group

\[
\rho_R(\hat{x}^\mu) = \hat{x}^\nu \otimes \Lambda^\mu_\nu - 1 \otimes a^\nu \Lambda^\mu_\nu, \quad (2.6)
\]

obtained by the change $\kappa \to -\kappa$ in the formulae (1.2). Such right covariance quantum group provides the following covariance of the relations (2.2–3)

\[
\tilde{\rho}_L(\tau^\mu) = \tau^\nu \otimes \Lambda^\mu_\nu, \quad \tilde{\rho}_L(\tau^5) = \tau^5 \otimes 1, \quad (2.7)
\]

b) $\kappa$-deformed vector fields.

In order to define the vector fields describing left or right partial derivatives $\partial_A$ acting on functions on $\kappa$-deformed Minkowski space we write:

\[
df = \partial_A f \cdot \tau^A = \tau^A \partial_A f. \quad (2.8)
\]

In particular, using Leibnitz rule $d(fg) = dfg + fdg$ we obtain that

\[
d = e^{iap} := \chi_A(p_\mu)e^{iap} : \tau^A, \quad (2.9)
\]

where

\[
\chi_i = e^{ip_i/\kappa} p_i, \\
\chi_0 = \frac{1}{\kappa}(e^{p_0/\kappa} - 1) + \frac{1}{2\kappa} M^2_\kappa(p), \\
\chi_5 = -\frac{1}{8} M^2_\kappa(p)
\]

and

\[
M^2_\kappa(p) = e^{p_0/\kappa} \bar{p}^2 - \left(2\kappa \sinh\frac{p_0}{2\kappa}\right)^2
\]

\[
= e^{p_0/\kappa} \bar{p}^2 + 2\kappa^2 \left(1 - \cosh\frac{p_0}{\kappa}\right), \quad (2.11)
\]

where in the limit $\kappa \to \infty$ we get $M^2_\kappa(p) \to \bar{p}^2 - p_0^2 = -m^2$. Using Fourier transform (1.3) and the relation $p_\mu : e^{iap} := \frac{1}{i} \partial_\mu e^{iap}$ one obtains

\[
\partial_A \Phi(\hat{x}) = : \chi_A \left(\frac{1}{i} \frac{\partial}{\partial x^\mu}\right) \Phi(\hat{x}). \quad (2.12)
\]

Using duality relations one can relate the derivatives on $\kappa$-deformed Minkowski space with the fourmomentum generators (1.4a–b). In bicrosproduct basis one obtains

\[
\langle p_\mu f(p), : \phi(\hat{x}) :: \rangle = (f(p), : i \frac{\partial}{\partial x^\mu} \Phi(\hat{x})), \quad (2.13)
\]
where $P_\mu \in H_p$, or equivalently

$$P_\mu : \phi(\hat{x}) := \frac{1}{i} \frac{\partial}{\partial x^\mu} \phi(\hat{x}) :$$  \hfill (2.14)

We see that one can express the vector fields in terms of the fourmomentum generators $P_\mu$ of $\kappa$–deformed Poincaré algebra (1.1):

$$\partial_A \Phi(\hat{x}) = \chi_A(P_\mu) : \phi(\hat{x}) :$$  \hfill (2.15)

where $\chi_A$ are given by relations (2.10).

c) $\kappa$–invariant integration over $\kappa$–deformed Minkowski space.

The relation (1.12a) and the definition (1.8) lead to the equality

$$\frac{1}{(2\pi)^4} \int d^4 \hat{x} \Phi(\hat{x}) = \tilde{\Phi}(0) = \frac{1}{(2\pi)^4} \int d^4 x \phi(x).$$  \hfill (2.16)

In order to show the $\kappa$–Poincaré invariance of (2.16) we should prove that

$$\int d^4 \hat{x} \Phi(\hat{x}) (\hat{x}^\mu \otimes A_\nu - I \otimes a^\mu A_\nu) = \tilde{\Phi}(0) \cdot 1.$$  \hfill (2.17)

The relation (2.17) is shown in the Appendix.

It is easy to see from (1.12b–d) that the relation (2.16) can not be generalized to the powers of functions; in particular

$$\frac{1}{(2\pi)^4} \int d^4 \hat{x} \Phi^n(\hat{x}) = \frac{1}{(2\pi)^4} \int d^4 x \phi^n(x) \quad n \geq 2.$$  \hfill (2.18)

d) Hermitean conjugation, adjoint derivatives

$$\hat{\partial}_A^\mu.$$  \hfill (2.20)

If we denote

$$\Phi^\dagger(\hat{x}) = \frac{1}{(2\pi)^4} \int d^4 p \tilde{\Phi}^\dagger(p) : e^{ip\hat{x}} :.$$  \hfill (2.19)

one easily calculates that

$$\tilde{\Phi}^\dagger(\hat{p}, \hat{p}_0) = e^{-3p_0/\kappa} \tilde{\Phi}(-e^{p_0/\kappa} \hat{p}, -\hat{p}_0).$$  \hfill (2.20)

The Fourier transform (2.19) substituted in (1.10) provide the following notion of $\kappa$–deformed adjoint operation in standard Minkowski space:

$$\phi^\dagger(x) = e^{-i \hat{\partial}_\mu / \kappa (3 + \hat{x} \nabla) \phi^\dagger(x) = (\hat{x}, x_0 - \frac{3i}{2\kappa})},$$  \hfill (2.21)

where $R_\kappa = \exp \frac{1}{\kappa} \partial_0 D$ and $D = -i (\vec{x} \nabla + \frac{3i}{2})$. The formula (2.19) also defines the notion of adjoint derivatives, satisfying the relation corresponding to the integration by parts:

$$\int d^4 \hat{x} \Phi^\dagger(\hat{x}) (\hat{\partial}_A \Phi(\hat{x})) = \int d^4 \hat{x} (\hat{\partial}_A^\dagger \Phi^\dagger(\hat{x})) \Phi(\hat{x}).$$  \hfill (2.22)

It is easy to check that

$$\hat{\partial}_A^\dagger \Phi(\hat{x}) = \frac{1}{(2\pi)^4} \int d^4 p e^{-3p_0/\kappa} \chi_A \times (e^{p_0/\kappa} \hat{p}, -\hat{p}_0) \tilde{\Phi}(p) : e^{ip\hat{x}} :.$$  \hfill (2.23)

where the functions $\chi_A(p)$ are given by (2.10).

III. $\kappa$–DEFORMED $\lambda \phi^4$ THEORY

In order to write $\kappa$–deformed KG free equation we observe that (see (2.10), $\mu = 0, 1, 2, 3$)

$$\chi_\mu \chi^\mu = M^2_\kappa(p) \left(1 - \frac{M^2_\kappa(p)}{4\kappa^2}\right).$$  \hfill (3.1)

We write the free $\kappa$–deformed KG action as follows:

$$S_0 = \int d^4 \hat{x} \left[ (\hat{\partial}_\mu \Phi^\dagger(\hat{x})) \partial^\mu \Phi(\hat{x}) \right.$$

$$\left. - m^2 \Phi^\dagger(\hat{x}) \Phi(\hat{x}) \right] = \int d^4 \hat{x} \Phi^\dagger(\hat{x}) (\Box - m^2) \Phi(\hat{x}),$$  \hfill (3.2)

where $\Box \equiv \hat{\partial}_\mu \hat{\partial}^\mu$, and we assume the following interaction

$$S_{int} = \frac{\lambda}{4} \int d^4 \hat{x} (\Phi^\dagger(\hat{x}) \Phi(\hat{x}))^2.$$  \hfill (3.3)

By adding to the fields $\Phi$, $\Phi^\dagger$ classical variations we obtain the following field equation:

$$\left(\Box - m^2\right) \Phi = \frac{\lambda}{4} \left[ \Phi (\Phi \Phi^\dagger) + (\Phi^\dagger \Phi) \Phi^\dagger \right].$$  \hfill (3.4)

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The idea that integration over noncommutative space of suitably ordered function is equal to the classical integral first appeared in [13,17].

We shall use the right action of the $\kappa$–deformed Poincaré symmetry.

We follow the technique applied e.g. to quantized fields in [18], where the quantum fields have c-number variations.
Further, we shall assume for simplicity that $\Phi^+(\hat{x}) = \Phi^*(\hat{x})$, i.e. we assume that $\Phi^+ (\vec{p}, p_0) = \Phi (\vec{p}, p_0)$ (see (2.19)). We obtain for real $\kappa$-deformed KG fields

$$S_0 = \frac{1}{2} \int d^4x \hat{\Phi}(\hat{x}) \left( \Box - m^2 \right) \hat{\Phi}(\hat{x})$$

(3.5)

$$= \frac{1}{2} \int d^4p \hat{\Phi}(-p) \left( M^2(p) \left( 1 - \frac{M^2(p)}{4\kappa^2} \right) - m^2 \right) \hat{\Phi}(p)$$

The classical $\kappa$-deformed free KG field (see (1.10) and (3.2)) is described by the Lagrangean ($\tilde{M}_2 = -M^2 \left( \frac{1}{\kappa} \nabla, \frac{1}{\kappa} \partial_0 \right)$)

$$S_0 = \int d^4x \phi(x) \left[ \tilde{M}_2^2 \left( 1 + \frac{\tilde{M}_2^2}{4\kappa^2} \right) - m^2 \right] \phi(x)$$

(3.6)

where

$$\tilde{M}_2^2 = M^2 e^{-\frac{i\kappa}{\hbar}} - \left( 2\kappa \sin \frac{\partial_0}{2\kappa} \right)^2$$

$$= M^2 e^{-\frac{i\kappa}{\hbar}} - 2\kappa^2 \left( 1 - \cos \frac{\partial_0}{\kappa} \right)$$

(3.7)

and the free field equation takes the form

$$\left( \tilde{M}_2^2 - m^2_{\kappa,+} \right) \left( \tilde{M}_2^2 - m^2_{\kappa,-} \right) \phi = 0$$

where $m^2_{\kappa,\pm} = -2\kappa^2 \left( 1 \mp \sqrt{1 + \frac{m^2_{\kappa}}{\kappa^2}} \right)$, i.e.

$$m^2_{\kappa,+} = m^2 - \frac{m^4}{4\kappa^2} + O \left( \frac{1}{\kappa^4} \right)$$

(3.9a)

$$m^2_{\kappa,-} = -4\kappa^2 - m^2 + \frac{m^4}{4\kappa^2} + O \left( \frac{1}{\kappa^4} \right)$$

(3.9b)

We see that the first mass $m^2_{\kappa,+}$ describes the physical spectrum, while the second one $m^2_{\kappa,-}$ represents the tachyonic regularization mass, in the spirit of Pauli--Villars regularization. Indeed, writing the causal propagator, corresponding to (3.8) we obtain

$$\Delta^F (\vec{p}, p_0) =$$

(3.10)

$$\left[ M^2(p) \left( \frac{1}{4\kappa^2} \right) - m^2 + i\epsilon \right]^{-1}$$

$$= \left( M^2(p) - m^2_{\kappa,+} + i\epsilon \right) \left( M^2(p) - m^2_{\kappa,-} + i\epsilon \right)$$

$$= \frac{1}{\sqrt{1 + \frac{m^2_{\kappa}}{\kappa^2}}} \left( \frac{1}{M^2(p) - m^2_{\kappa,+} + i\epsilon} - \frac{1}{M^2(p) - m^2_{\kappa,-} + i\epsilon} \right)$$

Defining

$$\Delta^F_{\kappa}(\vec{x}, p_0) =$$

(3.11)

$$= \frac{1}{(2\pi)^4} \int d^3\vec{p} e^{i\vec{p}\cdot\vec{x}} \int_0^{+\infty} dp_0 \frac{e^{-ip_0 x_0}}{M^2(p) - m^2_{\kappa} + i\epsilon}$$

we get

$$\Delta^F_{\kappa}(\vec{x}, x_0) =$$

(3.12)

$$= \frac{1}{(2\pi)^4} \int d^3\vec{p} e^{i\vec{p}\cdot\vec{x}} \int_0^{+\infty} dp_0 e^{-ip_0 x_0} \Delta^F (\vec{p}, p_0)$$

$$= \frac{1}{\sqrt{1 + \frac{m^2_{\kappa}}{\kappa^2}}} \left( \Delta^F_{\kappa,+}(\vec{x}, x_0) - \Delta^F_{\kappa,-}(\vec{x}, x_0) \right)$$

In order to calculate short distance behaviour of (3.11–12) let us observe that ($M^2(p) - m^2_{\kappa} - 1$) in (3.11) contains the following double infinite sequence of poles:

$$\Delta^F_{\kappa}(\vec{x}, x_0) =$$

(3.13)

$$\sum_{n=0}^{\infty} e^{-i\kappa x_0 (p_{0,n}^{(\pm)} + 2i\pi n \kappa)} = \frac{e^{-i\kappa x_0 p_{0}^{(\pm)}}}{(1 - e^{2i\pi \kappa x_0})}$$

(3.14)

and using the relation

$$\int_0^{\infty} dp e^{i\vec{p}\cdot\vec{x}} \left( \frac{p}{\kappa} \right)^{ix_0} = \kappa (-i\kappa |\vec{x}|)^{1 - i\kappa x_0} \Gamma (1 + i\kappa x_0)$$

(3.15)

one gets, after some tedious calculations, for small $|\vec{x}|$, $x_0$ the formula

$$\Delta^F_{\kappa}(\vec{x}, x_0) \sim \frac{-i\kappa \cosh \left( \frac{\pi x_0}{\kappa} \right) \Gamma (1 + i\kappa x_0)}{\left( 2\pi \right)^2 \sqrt{1 + \frac{m^2_{\kappa}}{\kappa^2} |\vec{x}| \left( \kappa |\vec{x}| \right)^{1 + i\kappa x_0}}$$

(3.16)

In order to calculate the corrections due to interactions we should rewrite the interaction term (3.3) in momentum space. For real $\kappa$–deformed scalar field one gets

$$S_{\text{int}} = \frac{\lambda}{4} \int d^4x \hat{\Phi}^4 (\hat{x})$$

$$= \frac{\lambda}{4} \int d^4p^{(1)} \ldots d^4p^{(4)}$$

6
where \( i.e., \Phi(\hat{x}, \hat{p}) \).

One gets further

\[
S_{\text{int}} = \frac{1}{4} \int d^4p^{(1)} d^4p^{(2)} d^4p^{(3)} \Phi(p^{(1)}, p^{(2)}, p^{(3)}, p^{(4)})
\]

where \( \Delta^{(4)} = p^{(1)} + p^{(2)} + p^{(3)} + p^{(4)} \),

\[
\Delta^{(4)} = \sum_{k=1}^{4} p^{(k)} - \frac{1}{\kappa} \sum_{i=k+1}^{4} p^{(i)}.
\]  

One obtains

\[
S_{\text{int}} = \frac{\lambda}{4} \int d^4q^{(1)} d^4q^{(2)} d^4q^{(3)} e^{3\kappa(\hat{q}^{(1)} + \hat{q}^{(2)} + \hat{q}^{(3)})} \Phi(q^{(1)}, q^{(2)}, q^{(3)}) \times \Phi(-q^{(1)} + q^{(2)} + q^{(3)}) e^{(q^{(1)} + q^{(2)} + q^{(3)})} / \kappa, -(q^{(1)} + q^{(2)} + q^{(3)})
\]

where

\[
\phi^4_{\kappa}(x) = e^{1/\kappa} \left[ (\hat{x}^{(1)} + \hat{x}^{(2)} + \hat{x}^{(3)}) - \frac{6i}{\kappa} \right] \Phi(\hat{x}^{(1)}, \hat{x}^{(2)}, \hat{x}^{(3)} - \frac{3i}{\kappa}) \Phi(\hat{x}^{(4)}, 0) \Phi(\hat{x}^{(1)}, 0) \Phi(\hat{x}^{(2)}, 0) \Phi(\hat{x}^{(3)}, 0) \Phi(\hat{x}^{(4)}, 0) \bigg|_{x^{(i)} = x}
\]

describes the nonlocal \( \kappa \)-deformed \( \lambda \phi^4 \) vertex. If we restrict ourselves to the leading term linear in \( \frac{1}{\kappa} \) we get

\[
S_{\text{int}} = S^{(0)}_{\text{int}} + \frac{1}{\kappa} S^{(1)}_{\text{int}} + \ldots
\]

where

\[
S^{(1)}_{\text{int}} = \frac{\lambda}{4} \int d^4p^{(1)} d^4p^{(2)} d^4p^{(3)} \left( p^{(1)} \bar{p}^{(1)} + \left( p^{(1)} + p^{(2)} \right) \bar{p}^{(2)} + \left( p^{(1)} + p^{(2)} + p^{(3)} \right) \bar{p}^{(3)} \right)
\]

\[
\times \Phi(\bar{p}^{(1)}, p^{(1)}) \Phi(\bar{p}^{(2)}, p^{(2)}) \Phi(\bar{p}^{(3)}, p^{(3)}) \frac{\partial}{\partial \bar{p}} \phi(\bar{p}, -p^{(1)} - p^{(2)} - p^{(3)}) \bigg|_{\bar{p} = -\bar{p}_1 - \bar{p}_2 - \bar{p}_3}
\]

i.e., one can write

\[
\phi^4_{\kappa}(x) = \phi^4(x) + \frac{x^{(1)}}{\kappa} \left( \hat{x}^{(1)} + \hat{x}^{(2)} + \hat{x}^{(3)} \right) \Phi(\hat{x}^{(1)}, x^{(2)}, 0) \Phi(\hat{x}^{(3)}, x^{(4)}) \Phi(\hat{x}^{(1)}, x^{(3)}, 0) \Phi(\hat{x}^{(4)}, x^{(4)}) \bigg|_{x^{(i)} = x}
\]

\[
+ O\left( \frac{1}{\kappa^2} \right).
\]

\footnote{It should be pointed out that the coassociativity of the co-product \( \Delta^{(4)} \) provides the associativity of the product \( \Phi^4(\hat{x}) \), i.e., \( \Phi(\hat{x})\Phi^2(\hat{x}) = \Phi^2(\hat{x})\Phi(\hat{x}) \) etc.}
It should be mentioned, however, that the expansion in \( \frac{1}{\kappa} \) provides the terms proportional to the powers of \( \kappa \) and the nature of the nonlocality described by the exponential of differential operator is not well described by such a power expansion.\(^8\)

The formulae (3.21) and (3.23) describing nonlocal interaction vertex can be used in the case when the field \( \Phi(x) \) is classical (the Fourier components \( \Phi(x) \) commuting) as well as in the quantized case. Using perturbative expansion in coupling constant \( \lambda \) one can derive from (3.12) and (3.21) the modified rules for \( \kappa \)-deformed Feynman diagrams, with

i) the internal lines described by the propagator (3.12)

ii) the vertices introducing \( \kappa \)-deformed four-momentum conservation law (see (3.13))

\[
\delta^4(p_1 + \ldots + p_4) \to \Delta^{(4)}(p_1, p_2, p_3, p_4) .
\]

Unfortunately due to (3.2a) the \( \kappa \)-deformed Feynman diagrams will have an unusual property: the fourmomentum through virtual lines is not conserved. For example if we consider the self–energy diagram in \( \kappa \)-deformed \( \lambda \phi^4 \) theory the standard undeformed formula

\[
\delta^4(p_q, q, q, q) \to \delta^4(q, q, q, p)
\]

is very inaccuracy.

We would like to point out that in this paper we present a new scheme providing the rules how to calculate the corrections to the fourdimensional local relativistic interacting field theory in the presence of quantum deformations\(^9\). Our scheme can be described by the following diagram:

At present we consider this nonconservation of the fourmomenta at \( \kappa \)-deformed vertices as a serious difficulty.

IV. CONCLUSIONS

We would like to point out that in this paper we present a new scheme providing the rules how to calculate the corrections to the fourdimensional local relativistic interacting field theory in the presence of quantum deformations\(^9\). Our scheme can be described by the following diagram:

The scheme is valid for any deformation of D=4 Poincaré symmetries with commutative fourmomenta. Such deformations can be described by the Poincaré quantum group with the following bicrossproduct structure:

\[
\mathcal{P}^{(q)} = \text{O}(3,1) \ltimes T_4^{(q)} ,
\]

where

- \( \text{O}(3,1) \) is the classical Lorentz algebra, with primitive coproducts (classical Hopf-Lie algebra).

- \( T_4^{(q)} \) describes the Hopf algebra of translations (see (1.5a-b) in our case) deformed in a way preserving the primitive coproduct (1.5b).

It appears that the deformed space–time translations \( T_4^{(q)} \) should be described by a set of relations

\[
[x_\mu, x_\nu] = \frac{1}{\kappa} C^\rho_{\mu \nu} x_\rho + \frac{1}{\kappa^2} T_{\mu \nu} ,
\]

\(8\)In similar way the finite shift operator \( e^{a \partial x} \) is very inaccurately approximated by the powers \((a \partial x)^n\).

\(9\)It should be mentioned that the deformation of local interaction vertices in the presence of space–time coordinates commuting to a nonvanishing c-number (see [1]) were considered by Filk [22]. Recently also the deformations of two–dimensional theories were considered in [23].
with suitable conditions for the dimensionless coefficients $C^\rho_{\mu\nu}$, $T_{\mu\nu}$, which can be obtained from the results presented in [7]. The $\kappa$–deformation described in this paper is distinguished by the property that it preserves the classical nonrelativistic $O(3)$ symmetries.

In conclusion we would like to point out the following two problems which we found while considering our corrections:

i) The modification of classical fourmomentum conservation law. The corrections to the local vertices are nonlocal in time and from the point of view of classical relativistic invariance are not translation–invariant (see e.g. (3.21) and (3.24)). This property occurs always when in (4.2) $C^\rho_{\mu\nu} \neq 0^{10}$

ii) The nonsymmetric coproduct of three–momenta requires some novel approach to the problem of statistics and the notion of bosons and fermions in the presence of $\kappa$–deformation.

The first difficulty means that the $\kappa$–deformed field theory looks quite different from the undeformed one, and it is not yet clear how such modified field theory and modified energy–momentum conservation laws could be useful in fundamental interactions theory. This paper should be treated rather as an indication of new features and problems which are met when we modify local field theory in a way covariant under quantum–deformed Poincaré symmetries. In particular because the Abelian addition law for the fourmomenta described by their coproducts are modified (see (1.6b) and (1.12c) the nonlocalities are necessarily required by the $\kappa$–deformed form of translational invariance.

The second difficulty we consider to be rather of technical nature, which should be solved by the introduction of some highly nontrivial unitary operator, representing the exchange of space–time position of two $\kappa$–deformed bosonic or fermionic particles (for the solution of an analogous but much simpler problem see [24]).

It should be mentioned that preliminary results of this paper have been presented at XXII International Colloquium on Group–Theoretic Methods in July 1998 (Hobart, Tasmania).

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APPENDIX

After tedious calculations one can show that

\[ e^{-ip_\mu(\hat{x}^\nu \otimes \Lambda_\nu^\rho - I \otimes a^\rho \Lambda_\mu^\rho)} = e^{i\rho \otimes a^\rho \Lambda_\mu^\rho} e^{i\rho \otimes a^\rho \Lambda_\kappa^\rho} : e^{-i(\hat{x}^\nu \otimes I)\phi_\rho(p, \Lambda)}, \quad (A.1) \]

where

\[ \phi_\rho(p, \Lambda) = \kappa \ln \left\{ \cosh \left( \frac{p_0}{\kappa} \otimes I \right) + \left( \sinh \left( \frac{p_0}{\kappa} \otimes I \right) \otimes H_k^0 \right) \right\} \]

\[ \times \left( \frac{p^2}{2\kappa^2} \otimes (\Lambda_0^0 - 1) + \frac{p_0}{\kappa} \otimes \Lambda_0^0 + I \otimes I \right) \], \quad (A.2a) \]

\[ \phi_k(p, \Lambda) = \kappa \left( \sinh \left( \frac{p_0}{\kappa} \otimes I \right) \otimes H_k^0 \right) \left\{ \cosh \left( \frac{p_0}{\kappa} \otimes I \right) + \left( \sinh \left( \frac{p_0}{\kappa} \otimes I \right) \otimes H_k^0 \right) \right\} \]

\[ \left( \frac{p^2}{2\kappa^2} \otimes (\Lambda_0^0 - 1) + \frac{p_0}{\kappa} \otimes \Lambda_0^0 + 2 \otimes I \right) \], \quad (A.2b) \]

\[ H_k^0 = \frac{-2p_0}{\kappa} \otimes \Lambda_0^0 + \frac{p^2}{2\kappa^2} \otimes (\Lambda_0^0 - 1) + \frac{p_0}{\kappa} \otimes \Lambda_0^0 + 2 \otimes \Lambda_0^0 \]

\[ + 2 \otimes \Lambda_0^0 \]

We obtain

\[ \Phi (\hat{x}^\nu \otimes \Lambda_\nu^\rho - I \otimes a^\rho \Lambda_\mu^\rho) \]

\[ 10^\text{The case of (4.2) with } C^\rho_{\mu\nu} = 0 \text{ was considered in [1]. In such a case (see [22]) the classical fourmomentum conservation at the vertices of deformed Feynman diagrams is valid.} \]
\begin{equation}
\int d^4 p \, \Phi(p) e^{i p_\mu \otimes a^\mu \Lambda_\nu^0 \otimes e^{i p_\nu \otimes a^\nu \Lambda_k^k}} e^{-i(\hat{x}' \otimes \mathbb{1}) \phi_\mu(p,\Lambda)} = (2\pi)^4 \delta^4(p) \otimes \mathbb{1} \tag{A.4}
\end{equation}

Using the explicit formulae (A.2–3) and commutativity of \( x^\mu \) with \( \phi_\mu \) one gets
\begin{equation}
\int d^4 \hat{x} : e^{-i \hat{x} \phi(p,\Lambda) \otimes \mathbb{1}} := (2\pi)^4 \delta^4(\phi(p,\Lambda)) \otimes \mathbb{1} = (2\pi)^4 J \left( \frac{\partial \phi_\mu}{\partial p_\nu} \right) \delta^4(p) \otimes \mathbb{1} \tag{A.5}
\end{equation}

Because \( \frac{\partial \phi_\mu}{\partial p_\nu} \big|_{p_\mu=0} = 1 \otimes \Lambda_\nu^\mu \), we get \( J \big|_{p_\mu=0} = 1 \). Inserting (A.1) and (A.4) in (2.17) one shows that the integral (2.16) has the same value for all frames described by \( \kappa \)-deformed Poincaré group.

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