EMBEDDING MINIMAL DYNAMICAL SYSTEMS INTO HILBERT CUBES

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Abstract. We study the problem of embedding minimal dynamical systems into the shift action on the Hilbert cube \(([0,1]^N)^\mathbb{Z}\). This problem is intimately related to the theory of mean dimension, which counts the averaged number of parameters of dynamical systems. Lindenstrauss proved that minimal systems of mean dimension less than \(N/36\) can be embedded into \(([0,1]^N)^\mathbb{Z}\), and he proposed the problem of finding the optimal value of the mean dimension for the embedding. We solve this problem by proving that minimal systems of mean dimension less than \(N/2\) can be embedded into \(([0,1]^N)^\mathbb{Z}\). The value \(N/2\) is optimal. The proof uses Fourier and complex analysis.

1. Introduction

1.1. Embedding into Hilbert cubes. A tuple \((X,T)\) is called a dynamical system if \(X\) is a compact metric space and \(T\) is a homeomorphism of \(X\). Basic examples for us are the shifts on the Hilbert cubes: Let \(N\) be a natural number and consider the infinite product

\[ ([0,1]^N)^\mathbb{Z} = \cdots \times [0,1]^N \times [0,1]^N \times [0,1]^N \times \cdots. \]

We define the shift \(\sigma\) on it by

\[ \sigma((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}, \quad \text{where } x_n \in [0,1]^N. \]

\(\left( ([0,1]^N)^\mathbb{Z}, \sigma \right) \) is a dynamical system. We study the problem of embedding arbitrary dynamical systems into \(\left( ([0,1]^N)^\mathbb{Z}, \sigma \right) \). More formally, we study

**Problem 1.1** (Embedding Problem). Let \((X,T)\) be a dynamical system. Decide whether there exists a topological embedding

\[ f : X \to ([0,1]^N)^\mathbb{Z} \]

satisfying \(f \circ T = \sigma \circ f\). Such a map \(f\) is called an embedding of a dynamical system.
For example, consider an irrational rotation

\[ X = \mathbb{R}/\mathbb{Z}, \quad T(x) = x + \alpha, \quad \alpha \in \mathbb{R} \setminus \mathbb{Q}. \]

Then the map

\[ \mathbb{R}/\mathbb{Z} \to [0,1]^\mathbb{Z}, \quad x \mapsto \left( \frac{1 + \cos(2\pi(x + n\alpha))}{2} \right)_{n \in \mathbb{Z}} \]

is an embedding of the irrational rotation \((\mathbb{R}/\mathbb{Z}, T)\). This example is very simple. But in general the problem is much more involved and still not fully understood. We quickly review the history of the problem before explaining the main result.

An obvious obstruction for the embedding comes from periodic points; if \((X,T)\) has too many periodic points then it cannot be embedded into the shift on \((\mathbb{R}/\mathbb{Z})^\mathbb{Z}\). For example, the shift on \(([0,1]^2)^\mathbb{Z}\) cannot be embedded into the shift on \([0,1]^\mathbb{Z}\) because the fixed points set of the former is homeomorphic to \([0,1]^2\), which cannot be embedded into \([0,1]\). Somewhat surprisingly, Jaworski [Jaw] proved that periodic points are the only obstruction if \(X\) is finite dimensional:

**Theorem 1.2** (Jaworski, 1974). If \((X,T)\) is a finite dimensional system having no periodic points, then we can embed it into the shift on \([0,1]^\mathbb{Z}\).

The first named author [Gut12] extended this result to the case of finite dimensional systems having *reasonable* amount of periodic points. The embedding problem for finite dimensional systems is fairly well understood now. Therefore the main targets of our study are infinite dimensional systems. But completely general infinite dimensional systems are still beyond our present technologies. We have to consider some restrictions on our systems.

Probably the most fundamental dynamical systems are minimal systems. A system \((X,T)\) is said to be **minimal** if for every \(x \in X\) the orbit

\[ \ldots, T^{-3}x, T^{-2}x, T^{-1}x, x, Tx, T^2x, T^3x, \ldots \]

is dense in \(X\). Minimal systems have no periodic points unless they are finite. (Finite systems are trivial cases.) So there is no “periodic points obstruction”. Auslander [Aus, p.193] asked whether we can embed every minimal system into the shift on \([0,1]^\mathbb{Z}\). In other words, he asked whether there is another obstruction different from periodic points. This problem remained open for more than 10 years.

Lindenstrauss–Weiss [LW] solved Auslander’s problem by using the theory of mean dimension. Mean dimension is a topological invariant of dynamical systems introduced by Gromov [Gro]. It counts the *number of parameters of systems per second* like topological entropy counts the *number of bits per second for describing dynamical systems*. We review the definition in Section 2.1. The mean dimension of the shift on \(([0,1]^N)\mathbb{Z}\) is equal to \(N\). This is a rigorous statement of the intuitive idea that the system \(([0,1]^N)\mathbb{Z}\) has \(N\) parameters per second. If a system \((X,T)\) is embeddable into the shift on \(([0,1]^N)\mathbb{Z}\) then
its mean dimension (denoted by $\text{mdim}(X, T)$) is less than or equal to $N$. Lindenstrauss–Weiss [LW, Proposition 3.5] constructed a minimal system of mean dimension strictly greater than one. So this system cannot be embedded into the shift on $[0, 1]^\mathbb{Z}$ although it is minimal.

It is a big surprise that a partial converse holds ([Lin, Theorem 5.1]):

**Theorem 1.3** (Lindenstrauss, 1999). If $(X, T)$ is a minimal system with

\[ \text{mdim}(X, T) < \frac{N}{36}, \]

then we can embed it into the shift on $([0, 1]^N)^\mathbb{Z}$.

This is a wonderful theorem. But the number $N/36$ looks artificial. Quoting Lindenstrauss [Lin, p. 229] with a slight change of notations:

Another nice question that remains open is what is the largest constant $c$ such that $\text{mdim}(X, T) < cN$ implies that $(X, T)$ can be embedded in $([0, 1]^N)^\mathbb{Z}$, shift? The bound we get is that $c \geq 1/36$.

We solve this problem. The answer is $c = 1/2$. Namely

**Theorem 1.4** (Main Theorem). If $(X, T)$ is a minimal system with

\[ \text{mdim}(X, T) < \frac{N}{2}, \]

then we can embed it into the shift on $([0, 1]^N)^\mathbb{Z}$.

The value $N/2$ is optimal because Lindentsrauss and the second named author [LT, Theorem 1.3] constructed a minimal system of mean dimension $N/2$ which cannot be embedded into the shift on $([0, 1]^N)^\mathbb{Z}$. The statement of Theorem 1.4 also holds for extensions of nontrivial (i.e. infinite) minimal systems; see Corollary 3.5. Therefore the embedding problem is now well understood for nontrivial minimal systems and their extensions.

The proof of Theorem 1.4 has a fascinating feature. The nature of the statement itself is purely abstract topological dynamics. But crucial ingredients of the proof are Fourier analysis and complex function theory. Therefore the theorem exhibits a new unexpected interaction between topological dynamics and classical analysis.

1.2. **Embedding via signal processing.** Elements $x$ of $([0, 1]^N)^\mathbb{Z}$ are discrete signals valued in the $N$-dimensional cube $[0, 1]^N$:

\[ \ldots x_{-3} x_{-2} x_{-1} x_0 x_1 x_2 x_3 \ldots, \quad \text{where } x_n \in [0, 1]^N. \]

(Here “discrete” means “time-discrete”.) Informally speaking, the embedding problem asks how to encode dynamical systems into discrete signals. Our approach in Theorem
A dynamical system \( \xrightarrow{\text{encode}} \) Continuous signals \( \xrightarrow{\text{sampling}} \) Discrete signals.

First we encode a given system into \((\text{time})\)-continuous signals. Next we convert continuous signals into discrete ones by sampling. Continuous signals are more flexible than discrete ones (see Remark 3.1), and we can prove the sharp embedding result.

We prepare some definitions on signal analysis. For rapidly decreasing functions \( \varphi : \mathbb{R} \to \mathbb{C} \) we define the Fourier transforms by

\[
\mathcal{F}(\varphi)(\xi) = \int_{-\infty}^{\infty} e^{-2\pi \sqrt{-1} t \xi} \varphi(t) dt, \quad \mathcal{F}(\varphi)(t) = \int_{-\infty}^{\infty} e^{2\pi \sqrt{-1} t \xi} \varphi(\xi) d\xi.
\]

We have \( \mathcal{F}(\mathcal{F}(\varphi)) = \mathcal{F}(\varphi) = \varphi \). We extend \( \mathcal{F} \) and \( \mathcal{F} \) to tempered distributions in the standard way (Schwartz [Sch, Chapter 7]). For example, \( \mathcal{F}(1) = \delta_0 \) is the delta probability measure at the origin. Take two real numbers \( a < b \). A bounded continuous function \( \varphi : \mathbb{R} \to \mathbb{C} \) is said to be band-limited in \([a, b]\) if

\[
\text{supp} \mathcal{F}(\varphi) \subset [a, b].
\]

Here recall that \( \text{supp} \mathcal{F}(\varphi) \subset [a, b] \) means that the pairing \( \langle \mathcal{F}(\varphi), \phi \rangle \) vanishes for any rapidly decreasing function \( \phi : \mathbb{R} \to \mathbb{C} \) with \( \text{supp}(\phi) \cap [a, b] = \emptyset \). Let \( V[a, b] \) be the space of bounded continuous functions \( \varphi : \mathbb{R} \to \mathbb{C} \) band-limited in \([a, b]\). This is a Banach space with respect to the \( L^\infty \)-norm over the line \( \mathbb{R} \).

For two functions \( \varphi_1, \varphi_2 \in V[a, b] \) we define a distance between them by

\[
d(\varphi_1, \varphi_2) = \sum_{n=1}^{\infty} 2^{-n} \| \varphi_1 - \varphi_2 \|_{L^\infty([-n,n])}.
\]

We define \( B_1(V[a, b]) \) as the space \( \varphi \in V[a, b] \) satisfying \( \| \varphi \|_{L^\infty(\mathbb{R})} \leq 1 \). This is compact with respect to the distance \( d \); see Lemma 2.3 in Section 2.2. Throughout the paper, \( B_1(V[a, b]) \) is always endowed with the topology given by \( d \), which coincides with the standard topology of tempered distributions [Sch, Chapter 7, Section 4]. We define

\[
\sigma : V[a, b] \to V[a, b], \quad \sigma(\varphi)(t) = \varphi(t+1),
\]

and consider the dynamical system \((B_1(V[a, b]), \sigma)\). We call this system the shift on \( B_1(V[a, b]) \). This is related to the shifts on the Hilbert cubes by the next lemma (sampling).

**Lemma 1.5.** Let \( N \) be a natural number. Let \( c > 0 \) and consider the space \( V^\mathbb{R}[-c, c] \) of bounded continuous functions \( \varphi : \mathbb{R} \to \mathbb{R} \) satisfying \( \text{supp} \mathcal{F}(\varphi) \subset [-c, c] \). If \( c < N/2 \) then we can embed the system \((B_1(V^\mathbb{R}[-c, c]), \sigma)\) into the shift on \((([-1, 1]^N)^\mathbb{Z}) \).

**Proof.** By a sampling theorem (see Lemma 2.4 in Section 2.2), the map

\[
V^\mathbb{R}[-c, c] \to \ell^\infty \left( \frac{1}{N} \mathbb{Z} \right), \quad \varphi \mapsto \varphi|_{\frac{1}{N} \mathbb{Z}}
\]
is injective. The above statement follows from this. □

**Remark 1.6.** The mean dimensions of the shifts on $B_1(V[a,b])$ and $B_1(V^\mathbb{R}[-c,c])$ are $2(b-a)$ and $2c$ respectively. More generally, if we denote by $V(E)$ the space of bounded continuous functions in $\mathbb{R}$ band-limited in a compact subset $E \subset \mathbb{R}$ then the mean dimension of the shift on $B_1(V(E))$ is equal to $2|E|$. Here $|E|$ is the Lebesgue measure of $E$. This fact is probably helpful for clarifying the picture. But we don’t need it for the proof of Theorem 1.4. So we omit the detailed explanations in this paper.

The next result is the continuous signal version of the main theorem.

**Theorem 1.7.** If $(X,T)$ is a nontrivial minimal system with

$$\text{mdim}(X,T) < b - a,$$

then we can embed it into the shift on $B_1(V[a,b])$. Here “nontrivial” means that $X$ is an infinite set.

Theorem 1.7 is proved in Section 3.2.

**Proof of Theorem 1.4, assuming Theorem 1.7.** If $X$ is finite, then the statement is trivial. So we assume that $(X,T)$ is a nontrivial minimal system. Take $0 < a < b < N/2$ with $\text{mdim}(X,T) < b - a$. By Theorem 1.7 we can embed $(X,T)$ into the system $B_1(V[a,b])$, which becomes a subsystem of $B_1(V^\mathbb{R}[-b,b])$ by

$$B_1(V[a,b]) \to B_1(V^\mathbb{R}[-b,b]), \quad \varphi \mapsto \frac{1}{2}(\varphi + \overline{\varphi}).$$

By Lemma 1.5 we can embed $B_1(V^\mathbb{R}[-b,b])$ into the shift on $([-1,1]^N)^\mathbb{Z} \cong ([0,1]^N)^\mathbb{Z}$. □

### 1.3. Open problems

The following are the most significant questions in the direction of the paper.

- **Can one solve the embedding problem for general dynamical systems?** The case of minimal systems is fairly well understood now. But we still don’t have a clear picture for more general dynamical systems. Lindenstrauss and the second named author [LT, Conjecture 1.2] conjectured that if a dynamical system $(X,T)$ satisfies

$$\text{mdim}(X,T) < \frac{N}{2}, \quad \frac{\text{dim}\{x | T^n x = x\}}{n} < \frac{N}{2} \quad (\forall n \geq 1),$$

then we can embed it into the shift on $([0,1]^N)^\mathbb{Z}$.

- **Can one generalize the result to the case of non-commutative group actions?** Probably it is possible to generalize the result to the case of $\mathbb{Z}^k$-actions by using the techniques of [GLT] and the present paper. But the generalization to non-commutative groups seems to require substantially new ideas.
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2. Basic materials

We review mean dimension and band-limited functions in this section.

2.1. Review of mean dimension. Here we review the definition of mean dimension. For the details, see Gromov [Gro] and Lindenstrauss–Weiss [LW].

Let \((X, d)\) be a compact metric space with a continuous function \(\rho : X \times X \to \mathbb{R}\). Let \(Y\) be a topological space. For \(\varepsilon > 0\), a continuous map \(f : X \to Y\) is called an \(\varepsilon\)-embedding with respect to \(\rho\) if it satisfies

\[ f(x) = f(y) \implies \rho(x, y) < \varepsilon. \]

Note that this is an open condition for \(f\) in the compact-open topology. We usually consider the case of \(\rho = d\), but sometimes \(\rho\) is a semi-distance different from \(d\).

We define \(\text{Widim}_\varepsilon(X, d)\) as the minimum integer \(n\) such that there exist an \(n\)-dimensional finite simplicial complex \(P\) and an \(\varepsilon\)-embedding \(f : X \to P\) with respect to the distance \(d\). It is classically known that the topological dimension \(\dim X\) is recovered by

\[ \dim X = \lim_{\varepsilon \to 0} \text{Widim}_\varepsilon(X, d). \]

Let \(T : X \to X\) be a homeomorphism. For a natural number \(N\) we define a distance \(d_N\) on \(X\) by

\[ d_N(x, y) = \max_{0 \leq n < N} d(T^n x, T^n y). \]

We define the mean dimension of the dynamical system \((X, T)\) by

\[ \text{mdim}(X, T) = \lim_{\varepsilon \to 0} \left( \lim_{N \to \infty} \frac{\text{Widim}_\varepsilon(X, d_N)}{N} \right). \]

This limit exists because the function \(N \mapsto \text{Widim}_\varepsilon(X, d_N)\) is subadditive. The mean dimension is a topological invariant of the dynamical system \((X, T)\), namely, it is independent of the choice of the distance \(d\).

2.2. Review of band-limited functions. Here we review basic properties of band-limited functions. All the results in this subsection are classically known. But some of them are not very popular in general mathematical community. So we provide self-contained proofs, assuming (hopefully) only well-known results. For systematic treatments, see Beurling [Beu, pp. 341-365] and Nikol’skii [Nik, Chapter 3].
Lemma 2.1. Let $a > 0$, and let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function satisfying
\[(2.1) \quad \exists C > 0 \forall x, y \in \mathbb{R} : |f(x + y\sqrt{-1})| \leq Ce^{2\pi |y|}.
\]
Then it satisfies
\[|f(x + y\sqrt{-1})| \leq e^{2\pi |y|} \|f\|_{L^\infty(\mathbb{R})}.
\]
Here $\|f\|_{L^\infty(\mathbb{R})}$ is the supremum of $|f|$ over the real line.

Proof. Let $\varepsilon > 0$ and set $g(z) = e^{2\pi(a+\varepsilon)z\sqrt{-1}}f(z)$. For $y \geq 0$
\[|g(x + y\sqrt{-1})| = e^{-2\pi(a+\varepsilon)y}|f(x + y\sqrt{-1})| \leq Ce^{-2\pi\varepsilon y} \to 0 \quad (y \to \infty).
\]
Set $L = \{y\sqrt{-1} | y > 0\} \subset \mathbb{C}$. By the Phragmén–Lindelöf principle ([DM, Section 3.1.7])
\[
\sup_{y \geq 0} |g(x + y\sqrt{-1})| \leq \max \left(\|g\|_{L^\infty(\mathbb{R})}, \|g\|_{L^\infty(L)}\right).
\]
If $\|g\|_{L^\infty(\mathbb{R})} < \|g\|_{L^\infty(L)}$ then $g$ attains $\sup_{y \geq 0} |g(x + y\sqrt{-1})|$ over $L$. But this contradicts the maximum principle. Therefore
\[\sup_{y \geq 0} |g(x + y\sqrt{-1})| \leq \|g\|_{L^\infty(\mathbb{R})} = \|f\|_{L^\infty(\mathbb{R})}.
\]
Thus we get $|f(x + y\sqrt{-1})| \leq e^{2\pi(a+\varepsilon)y} \|f\|_{L^\infty(\mathbb{R})}$. Letting $\varepsilon \to 0$, we get the desired result for $y \geq 0$. The case $y < 0$ is similar. \hfill $\Box$

Lemma 2.2. Let $a > 0$, and let $f : \mathbb{R} \to \mathbb{C}$ be a bounded continuous function. The following two conditions are equivalent.
(1) The Fourier transform $\mathcal{F}(f)$ is supported in $[-a, a]$.
(2) We can extend $f$ to a holomorphic function in $\mathbb{C}$ satisfying (2.1).

Proof. If we additionally assume $f \in L^2(\mathbb{R})$, then the above equivalence is a standard theorem of Paley–Wiener [DM, Section 3.3]. So the problem is how to extend the Paley–Wiener theorem to the case of bounded continuous functions. For a more general result, see Schwartz [Sch] Chapter 7, Section 8.

Let $\psi(\xi)$ be a nonnegative smooth function in $\mathbb{R}$ satisfying
\[
\text{supp}(\psi) \subset [-1, 1], \quad \int_{-1}^{1} \psi(\xi) d\xi = 1.
\]
Set $\varphi = \mathcal{F}(\psi)$. This is a rapidly decreasing function with $\varphi(0) = 1$ and $|\varphi(x)| \leq 1$. For $\varepsilon > 0$ we set $\varphi_\varepsilon(x) = \varphi(\varepsilon x)$. This satisfies $\mathcal{F}(\varphi_\varepsilon)(\xi) = \psi(\xi/\varepsilon)/\varepsilon$. The function $\varphi_\varepsilon$ can be extended to a holomorphic function in $\mathbb{C}$ satisfying $|\varphi_\varepsilon(x + y\sqrt{-1})| \leq e^{2\pi\varepsilon |y|}$. We have $\varphi_\varepsilon \to 1$ ($\varepsilon \to 0$) uniformly over every compact subset of $\mathbb{C}$. Set $f_\varepsilon(x) = \varphi_\varepsilon(x)f(x)$. Note that $f_\varepsilon$ is a $L^2$ function.
Suppose $f$ satisfies the condition (1). The Fourier transform $\mathcal{F}(f_\varepsilon) = \mathcal{F}(\varphi_\varepsilon) * \mathcal{F}(f)$ is supported in $[-a - \varepsilon, a + \varepsilon]$. Since $f_\varepsilon \in L^2(\mathbb{R})$, we can apply to it the standard Paley–Wiener theorem (indeed this is a trivial part of their theorem) and conclude that $f_\varepsilon$ can be extended to a holomorphic function in $C$ satisfying

$$|f_\varepsilon(x + y\sqrt{-1})| \leq e^{2\pi(a+\varepsilon)|y|} \|f_\varepsilon\|_{L^\infty(\mathbb{R})} \quad \text{(by Lemma 2.1 and } |\varphi_\varepsilon| \leq 1).$$

Then we extend $f$ to a meromorphic function in $C$ by

$$f(z) = \varphi_\varepsilon(z)^{-1}f_\varepsilon(z) \quad \text{(this is independent of } \varepsilon \text{ because of the unique continuation).}$$

Since $\varphi_\varepsilon \to 1$ uniformly over every compact subset of $C$, we get

$$|f(x + y\sqrt{-1})| \leq e^{2\pi|y|} \|f\|_{L^\infty(\mathbb{R})}.$$

Thus $f$ becomes a holomorphic function satisfying (2.1).

Next suppose $f$ satisfies (2). Then the function $f_\varepsilon$ becomes a holomorphic function in $C$ satisfying $|f_\varepsilon(x + y\sqrt{-1})| \leq \text{const} \cdot e^{2\pi(a+\varepsilon)|y|}$. By the (difficult part of) Paley–Wiener theorem we get $\text{supp}(\mathcal{F}(f_\varepsilon)) \subset [-a - \varepsilon, a + \varepsilon]$. The functions $f_\varepsilon$ converge to $f$ in the sense of tempered distributions as $\varepsilon \to 0$. Thus $f$ satisfies $\text{supp}(\mathcal{F}(f)) \subset [-a, a]$. □

**Lemma 2.3.** Let $a < b$. The space $B_1(V[a, b])$ introduced in Section 1.2 is compact with respect to the distance $d$ in (1.1).

**Proof.** First note that a sequence $\{f_n\} \subset B_1(V[a, b])$ converges to $f$ in the distance $d$ if and only if for any compact subset $K \subset \mathbb{R}

$$\lim_{n \to \infty} \|f_n - f\|_{L^\infty(K)} = 0.$$

Set $c = 2\pi \max(|a|, |b|)$. By Lemmas 2.1 and 2.2 every $f \in B_1(V[a, b])$ can be extended to a holomorphic function in $C$ satisfying

$$\forall x, y \in \mathbb{R}: |f(x + y\sqrt{-1})| \leq e^{c|y|}.$$ 

Then the compactness of $B_1(V[a, b])$ follows from the standard normal family argument (Ahlfors [Ahl, Chapter 5, Section 5.4]): If $\{f_n\}$ is a sequence of holomorphic functions in $C$ uniformly bounded over every compact subset of $C$, then a suitable subsequence converges to a holomorphic function uniformly over every compact subset. □

**Lemma 2.4** (Sampling theorem). Let $a$ and $d$ be positive numbers with $2ad < 1$. Set $\Lambda = d\mathbb{Z} \subset \mathbb{R}$. Then the following map is injective:

$$V[-a, a] \to \ell^\infty(\Lambda), \quad f \mapsto f|_\Lambda = (f(\lambda))_{\lambda \in \Lambda}.$$ 

Note that this statement is optimal because the function $\sin(2\pi x)$ belongs to $V[-1, 1]$ and vanishes over $(1/2)\mathbb{Z}$. 

Proof. Suppose there exists a nonzero $f \in V[-a,a]$ satisfying $f|_{\Lambda} = 0$. By the first main theorem of Nevanlinna (Hayman [Hay, Section 1.3] and Noguchi–Winkelmann [NW, Section 1.1]), for $r > 1$

\begin{equation}
\int_1^r \frac{dt}{t} \leq \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{\theta e^{-T}})| d\theta + \text{const}.
\end{equation}

Here $\log^+ x = \max(0, \log x)$. From $f|_{\Lambda} = 0$, the left-hand side is bounded from below by

\begin{equation}
\int_1^r \frac{2t dt}{t} + O(\log r) = \frac{2r}{d} + O(\log r).
\end{equation}

From Lemma 2.2

\begin{equation}
\log^+ |f(re^{\sqrt{-1}\theta})| \leq 2\pi ar |\sin \theta| + \text{const}.
\end{equation}

Hence the right-hand side of (2.2) is bounded by

\begin{equation}
ar \int_0^{2\pi} |\sin \theta| d\theta + \text{const} = 4ar + \text{const}.
\end{equation}

Thus

\begin{equation}
\frac{2r}{d} \leq 4ar + O(\log r).
\end{equation}

Letting $r \to \infty$, we get $2ad \geq 1$, which contradicts the assumption. \hfill \Box

3. Technical main theorem and the proof of Theorem 1.7

Here we formulate Theorem 3.1 which is technically the most important result of the paper. Theorem 1.7 in Section 1.2 follows from this. The proof of Theorem 3.1 occupies all the rest of the paper. In Section 3.3 we explains the ideas of the proof.

3.1. Technical main theorem. Let $(Y, S)$ be a dynamical system. $(Y, S)$ is said to have the marker property if for any natural number $N$ there exists an open set $U \subset Y$ satisfying

\begin{equation}
U \cap S^n U = \emptyset \quad (\forall 1 \leq n \leq N), \quad Y = \bigcup_{n \in \mathbb{Z}} S^n U.
\end{equation}

It is known that large classes of dynamical systems satisfy this condition. Nontrivial minimal systems obviously have the marker property. If $(Y, S)$ is a finite dimensional system having no periodic points, then it satisfies the marker property; see [Gut12, Theorem 6.1]. The authors don’t know an example of dynamical systems which have no periodic points but don’t have the marker property.

Let $(X, T)$ and $(Y, S)$ be dynamical systems with a continuous surjection $\Phi : X \to Y$ satisfying $\Phi \circ T = S \circ \Phi$. We call $(X, T)$ an extension of $(Y, S)$, and $(Y, S)$ a factor of $(X, T)$. Take two real numbers $a < b$. We denote by $C_T(X, B_1(V[a, b]))$ the space of continuous maps $f : X \to B_1(V[a, b])$ satisfying $f \circ T = \sigma \circ f$. Here the topology of $B_1(V[a, b])$ is given by the distance $d$ in (1.1). Note that $C_T(X, B_1(V[a, b]))$ is always
nonempty because it contains the trivial map \( f(x) = 0 \). The space \( C_T(X, B_1(V[a, b])) \) becomes a complete metric space with respect to the uniform distance

\[
\sup_{x \in X} d(f(x), g(x)), \quad (f, g \in C_T(X, B_1(V[a, b]))).
\]

**Theorem 3.1.** Under the above settings, suppose that \( \text{mdim}(X, T) < b - a \) and \((Y, S)\) has the marker property. Then for a dense \( G_\delta \) subset of \( f \in C_T(X, B_1(V[a, b])) \) the map

\[
(f, \Phi) : X \to B_1(V[a, b]) \times Y, \quad x \mapsto (f(x), \Phi(x))
\]

is an embedding.

Every dense \( G_\delta \) subset of \( C_T(X, B_1(V[a, b])) \) is non-empty by the Baire category theorem. So the theorem implies that \((X, T)\) can be embedded into the product system \( B_1(V[a, b]) \times Y \).

### 3.2. Proof of Theorem 1.7.

Let \( a < b \) be real numbers throughout this subsection.

**Lemma 3.2.** Let \((X, T)\) be a dynamical system with a non-periodic point \( p \in X \). Then for a dense \( G_\delta \) subset of \( f \in C_T(X, B_1(V[a, b])) \) the function \( f(p) \) is not shift-periodic, i.e. for any nonzero integer \( n \) there exists a real number \( t \) satisfying \( f(p)(t + n) \neq f(p)(t) \).

**Proof.** We show that the following set is open and dense for any natural number \( n \):

\[
\{ f \in C_T(X, B_1(V[a, b])) | f(p) \neq f(T^n p) \}.
\]

This is obviously open. So it is enough to show its density. Take any \( f \in C_T(X, B_1(V[a, b])) \) and suppose \( f(p) = f(T^n p) \).

Let \( \psi(\xi) \) be a nonnegative smooth function in \( \mathbb{R} \) satisfying

\[
\text{supp}(\psi) \subset \left[ -\frac{b - a}{2}, \frac{b - a}{2} \right], \quad \int_{-\infty}^{\infty} \psi(\xi) d\xi = 1.
\]

Define \( \varphi \in V[a, b] \) by

\[
\varphi(t) = \exp \left(2\pi \sqrt{-1} \left(\frac{a + b}{2}\right) t\right) \mathcal{F}(\psi)(t).
\]

This satisfies \( |\varphi(t)| \leq 1 \) and \( \varphi(0) = 1 \). The function \( \varphi \) is rapidly decreasing. In particular we can find \( K > 0 \) such that \( \varphi(t) \leq K/(1 + t^2) \). Take a sufficiently large natural number \( N = N(K) \). Since \( p \in X \) is not periodic, there exists an open neighborhood \( U \subset X \) of \( p \) satisfying \( U \cap T^k U = \emptyset \) for \( 1 \leq k \leq 2nN \). Let \( \alpha : X \to [0, 1] \) be a continuous function satisfying \( \text{supp}(\alpha) \subset U \) and \( \alpha(p) = 1 \). For \( x \in X \) we define \( g(x) \in V[a, b] \) by

\[
g(x)(t) = \sum_{k \in \mathbb{Z}} \alpha(T^k x) \varphi(t - k).
\]

This is equivariant: \( g(Tx)(t) = g(x)(t + 1) \). Since \( N \gg 1 \) we can assume \( |g(x)(t)| \leq 2 \), \( g(p)(0) > 1/2 \) and \( g(p)(nN) < 1/2 \).
Let \( \varepsilon \) be a small positive number. We define \( h \in C_T(X, B_1(V[a,b])) \) by
\[
 h(x) = \left(1 - \frac{\varepsilon}{2}\right)f(x) + \frac{\varepsilon}{4}g(x).
\]
This satisfies \(|h(x) - f(x)| \leq \varepsilon\).

\[
 h(p)(nN) = \left(1 - \frac{\varepsilon}{2}\right)f(p)(0) + \frac{\varepsilon}{4}g(p)(0) = \left(1 - \frac{\varepsilon}{2}\right)f(p)(0) + \frac{\varepsilon}{4}g(p)(nN).
\]
Since \( g(p)(0) > 1/2 \) and \( g(p)(nN) < 1/2 \), this is smaller than
\[
 h(p)(0) = \left(1 - \frac{\varepsilon}{2}\right)f(p)(0) + \frac{\varepsilon}{4}g(p)(0).
\]
Thus \( h(T^n p) \neq h(p) \). In particular \( h(T^n p) \neq h(p) \).

□

Theorem 1.7 is a special case of the next corollary.

**Corollary 3.3.** If \((X,T)\) is an extension of a nontrivial minimal system with
\[
 \text{mdim}(X,T) < b - a,
\]
then we can embed it into the shift on \( B_1(V[a,b]) \).

**Proof.** \((X,T)\) has a nontrivial minimal factor \((Y,S)\). Take \( a < c_1 < c_2 < b \) with \text{mdim}(X,T) < \( c_1 - a \). From Lemma 3.2 there exists an equivariant continuous map \( g : Y \rightarrow B_1(V[c_2,b]) \) such that \( Z = g(Y) \) is a nontrivial minimal system with respect to the shift. Note that \( Z \) also becomes a factor of \( X \) and that it has the marker property. Applying Theorem 3.1 to the factor map \( X \rightarrow Z \), we can find an embedding of \((X,T)\) into the system \( B_1(V[a,c_1]) \times Z \), which becomes a subsystem of \( B_1(V[a,b]) \) by the embedding
\[
 B_1(V[a,c_1]) \times B_1(V[c_2,b]) 
 \rightarrow B_1(V[a,b]),\quad (\varphi_1, \varphi_2) \mapsto \frac{1}{2}(\varphi_1 + \varphi_2).
\]
□

**Remark 3.4.** Although the above proof of Corollary 3.3 is very simple, it exhibits the reason why continuous signals are more flexible than discrete ones. The main trick above is to *take two disjoint bands* \([a,c_1]\) and \([c_2,b]\). This is possible because \( a \) and \( b \) are continuous parameters. But the Hilbert cubes \((0,1)^N\) have only the discrete parameter \( N \). So we cannot apply the same trick.

By the same argument as in the proof of (Theorem 1.7 \( \Rightarrow \) Theorem 1.4), we can deduce the next corollary from Corollary 3.3.

**Corollary 3.5.** Let \( N \) be a natural number, and \((X,T)\) an extension of a nontrivial minimal system with \text{mdim}(X,T) < N/2. Then we can embed \((X,T)\) into the shift on \((0,1)^N\).
3.3. Ideas of the proof. Here we explain our approach to Theorem 3.1. The proof is technically involved. So we explain the proof of the toy-model and discuss what are the main difficulties in the case of Theorem 3.1.

Suppose \((Y, S)\) is a zero dimensional dynamical system with the marker property. Here the zero dimensionality means that clopen sets (closed and open sets) form an open basis of the topology. Let \(\Phi : (X, T) \to (Y, S)\) be an extension. We denote by \(C_T (X, [0, 1]^Z)\) the space of equivariant continuous maps \(f : X \to [0, 1]^Z\). The next theorem is proved in [GT, Theorem 1.5].

**Theorem 3.6.** If \(\text{mdim}(X, T) < 1/2\) then for a dense \(G_\delta\) subset of \(f \in C_T (X, [0, 1]^Z)\) the map

\[ (f, \Phi) : X \to [0, 1]^Z \times Y, \quad x \mapsto (f(x), \Phi(x)) \]

is an embedding.

We briefly explain the proof of this theorem, which is a prototype of the proof of Theorem 3.1. Let \(d\) be a distance on \(X\). Note the obvious equivalence:

embedding \(\iff\) \(\varepsilon\)-embedding for all \(\varepsilon > 0\).

Therefore it is enough to prove that the following set is dense and \(G_\delta\):

\[
\bigcap_{n=1}^{\infty} \left\{ f \in C_T (X, [0, 1]^Z) \mid (f, \Phi) \text{ is a } (1/n)\text{-embedding w.r.t. } d \right\}.
\]

This is obviously \(G_\delta\). So our main task is to prove the next proposition.

**Proposition 3.7.** For any \(\delta > 0\) and \(f \in C (X, [0, 1]^Z)\) there exists \(g \in C (X, [0, 1]^Z)\) satisfying the following.

1. For all \(x \in X\) and \(t \in \mathbb{Z}\), \(|f(x)(t) - g(x)(t)| < \delta\).
2. \((g, \Phi) : X \to [0, 1]^Z \times Y\) is a \(\delta\)-embedding with respect to \(d\).

**Proof.** Take \(0 < \varepsilon < \delta\) such that

\[d(x, y) < \varepsilon \implies |f(x)(0) - f(y)(0)| < \delta.\]

From \(\text{mdim}(X, T) < 1/2\), we can find \(N > 0\) such that

\[\operatorname{Widim}_\varepsilon (X, d_n) < \frac{n}{2} \quad (\forall n \geq N).\]

Then for every \(n \geq N\) we can construct an \(\varepsilon\)-embedding with respect to \(d_n\)

\[G_n : X \to [0, 1]^n = [0, 1]^{0,1,2,...,n-1}\]

satisfying \(|G_n(x)(t) - f(x)(t)| < \delta\) for all \(x \in X\) and \(t = 0, 1, 2, \ldots, n - 1\). For the reason why such \(G_n\) exists, see Lemma 4.1 and Corollary 4.4.

From the assumption on \(Y\), there exists a clopen set \(U \subset Y\) satisfying

\[U \cap S^n U = \emptyset \quad (\forall 1 \leq n \leq N), \quad Y = \bigcup_{n \in \mathbb{Z}} S^n U.\]
Take \( x \in X \). Let \( E(x) \) be the set of integers \( n \) satisfying \( \Phi(T^n x) = S^n \Phi(x) \in U \). For each \( n \in E(x) \) we define the interval \( I(x, n) \subset \mathbb{Z} \) by

\[
I(x, n) = \{ k \in \mathbb{Z} | \forall m \in E(x) : |k - n| \leq |k - m| \}.
\]

This is a kind of “Voronoi diagram construction”. (Voronoi diagram was first used by [Gut11] in the context of mean dimension.) From the assumption on \( U \), the interval \( I(x, n) \) is always finite and \#\( I(x, n) \) > \( N \). We denote by \( \alpha_{x,n} \) and \( \beta_{x,n} \) the left and right end-points of \( I(x, n) \) respectively.

We define \( g(x) \in [0, 1]^\mathbb{Z} \) by

\[
g(x)(t) = G_{\#I(x,n)-1}(T^{\alpha_{x,n}}x)(t - \alpha_{x,n}),
\]

where \( n \) is the integer in \( E(x) \) satisfying \( \alpha_{x,n} \leq t < \beta_{x,n} \). Roughly speaking, we attached the “perturbation map” \( G_{\#I(x,n)-1} \) to each interval \( I(x, n) \). It is direct to check that \( g : X \to [0, 1]^\mathbb{Z} \) is continuous and equivariant. We have

\[
|g(x)(t) - f(x)(t)| = |G_{\#I(x,n)-1}(T^{\alpha_{x,n}}x)(t - \alpha_{x,n}) - f(T^{\alpha_{x,n}}x)(t - \alpha_{x,n})| < \delta.
\]

We prove that \((g, \Phi) : X \to [0, 1]^\mathbb{Z} \times Y \) is a \( \delta \)-embedding with respect to \( d \). Suppose \((g(x), \Phi(x)) = (g(x'), \Phi(x'))\) for some \( x, x' \in X \). The equation \( \Phi(x) = \Phi(x') \) means that \( E(x) = E(x') \) and \( I(x, n) = I(x', n) \) for all \( n \in E(x) \). Take \( n \in E(x) \) with \( \alpha_{x,n} \leq 0 < \beta_{x,n} \).

From \( g(x) = g(x') \)

\[
G_{\#I(x,n)-1}(T^{\alpha_{x,n}}x) = G_{\#I(x,n)-1}(T^{\alpha_{x,n}}x').
\]

Since the map \( G_{\#I(x,n)-1} \) is an \( \varepsilon \)-embedding with respect to \( d_{\#I(x,n)-1} \), we get

\[
d(x, x') \leq d_{\#I(x,n)-1}(T^{\alpha_{x,n}}x, T^{\alpha_{x,n}}x') < \varepsilon < \delta.
\]

Here we have used \( \alpha_{x,n} \leq 0 < \beta_{x,n} \) in the first inequality.

The above proof is simple. But if we try a similar approach to Theorem 3.1, then we encounter the following four difficulties.

- **Difficulty 1.** Theorem 3.6 deals with discrete signals. But Theorem 3.1 deals with continuous ones. So we need to convert the above procedure to the continuous setting. This is a rather straightforward issue. The main ingredient is a certain interpolation function prepared in Section 5.
- **Difficulty 2.** A crucial fact in the above proof of Theorem 3.6 is that the set \( U \) is clopen, which implies that the map \( g \) continuously depends on \( x \in X \). We cannot hope this in Theorem 3.1. We overcome this difficulty by *going one dimension higher*. We consider a certain Voronoi diagram in the plane \( \mathbb{R}^2 \) and construct a tiling of the line

\[
\mathbb{R} = \bigcup_{n \in \mathbb{Z}} I(x, n)
\]
from the Voronoi diagram. This is a tricky idea first introduced by Lindenstrauss and the authors [GLT]. We explain it in Section 6.

- **Difficulty 3.** All the intervals \( I(x, n) \) are sufficiently long (\( \#I(x, n) > N \)) in the proof of Theorem 3.6. But some intervals \( I(x, n) \) in (3.1) may be short. We cannot construct a good perturbation over such short intervals. This is the most crucial difficulty. The key observation is that most part of the line is cover by sufficiently long intervals in (3.1). Then we take tax from these long intervals and use it for helping short intervals. We explain this heuristic idea more precisely in Section 6.

- **Difficulty 4.** In the proof of Theorem 3.6, \( I(x, n) \) are subsets of \( \mathbb{Z} \). In particular they are locally constant with respect to \( x \in X \). So it was easy to attach the perturbation maps \( G_{\#I(x,n)}^{-1} \) to \( I(x, n) \). But the intervals \( I(x, n) \) in (3.1) may continuously vary. Then it is more difficult to attach appropriate perturbation maps to \( I(x, n) \) even if they are sufficiently long. We have to construct _adjustable perturbation maps which can fit intervals of various length_. This is a quite technical issue. The construction is given in Section 7.2, which is based on Section 4.

After resolving all these difficulties, we finish the proof of Theorem 3.1 in Section 7.3.

4. Embeddings of simplicial complexes

Here we prepare the method of constructing good maps from simplicial complex. Every simplicial complex is assumed to be finite, i.e., it has only finitely many simplices. The main result of this section is Lemma 4.5.

In this section we use some standard ideas of real algebraic geometry. A reference is Bochnak–Coste–Roy [BCR]. We will repeatedly use the Tarski–Seidenberg principle [BCR, Proposition 2.2.7]: _The image of a semi-algebraic set under a semi-algebraic map is also semi-algebraic._ The dimension of semi-algebraic sets is the algebraic dimension [BCR, Definition 2.8.1], which does not behave pathologically. (Indeed algebraic dimension coincides with topological dimension [BCR, Theorem 2.3.6, Corollary 2.8.9]. But we don’t need this fact.)

Let \( P \) be a simplicial complex, and \( V \) a real vector space. A map \( f : P \to V \) is said to be _simplicial_ if for every simplex \( \Delta \subset P \) it has the form

\[
f \left( \sum_{k=0}^{n} \lambda_k v_k \right) = \sum_{k=0}^{n} \lambda_k f(v_k), \quad \left( \lambda_k \geq 0, \sum_{k=0}^{n} \lambda_k = 1 \right),
\]

on \( \Delta \), where \( v_0, \ldots, v_n \) are the vertices of \( \Delta \). The next lemma establishes the technique to approximate arbitrary continuous maps by simplicial ones.

**Lemma 4.1.** Suppose \( V \) is endowed with a norm \( \| \cdot \| \). Let \( (X, d) \) be a compact metric space with a continuous map \( f : X \to V \). Let \( \varepsilon \) and \( \delta \) be positive numbers satisfying

\[
d(x, y) < \varepsilon \implies \| f(x) - f(y) \| < \delta.
\]
Let $P$ be a simplicial complex, and $\pi : X \to P$ an $\varepsilon$-embedding with respect to $d$. Then, after replacing $P$ by a sufficiently finer subdivision, there exists a simplicial map $g : P \to V$ satisfying
\[ \|f(x) - g(\pi(x))\| < \delta, \quad \forall x \in X. \]

**Proof.** This is proved in [GLT, Lemma 2.1]. But we reproduce it here for the completeness.

For a vertex $v$ of $P$ we denote by $O(v)$ the open star around it (the union of the relative interiors of simplices containing $v$). We can subdivide $P$ sufficiently finer so that $\text{diam} \pi^{-1}(O(v)) < \varepsilon$ for all vertices $v$ of $P$. Take any vertex $v \in P$. If $\pi^{-1}(O(v)) \neq \emptyset$ then we choose a point $x_v \in \pi^{-1}(O(v))$ and set $g(v) = f(x_v)$. If $\pi^{-1}(O(v)) = \emptyset$, then we choose $g(v) \in V$ arbitrarily. We define $g : P \to V$ by extending it linearly on every simplex.

Take $x \in X$. Let $v_0, \ldots, v_n$ be the vertices of $P$ satisfying $\pi(x) \in O(v_k)$. We have $d(x, x_{v_k}) < \varepsilon$ and hence $\|f(x) - f(x_{v_k})\| < \delta$. The point $g(\pi(x))$ is a convex combination of $f(x_{v_0}), \ldots, f(x_{v_n})$. Thus $\|f(x) - g(\pi(x))\| < \delta$. \hfill \Box

Let $D$ be a natural number, and $P$ a simplicial complex of dimension $n$. We denote by $V(P)$ the set of vertices of $P$. We naturally consider $P \subset \mathbb{R}^{V(P)}$. The space of simplicial maps from $P$ to $\mathbb{R}^D$ is identified with the space $\text{Hom}(\mathbb{R}^{V(P)}, \mathbb{R}^D)$ of linear maps from the vector space $\mathbb{R}^{V(P)}$ to $\mathbb{R}^D$. This is endowed with the structure of a real algebraic manifold. Its topology is the standard Euclidean topology (not the Zariski topology).

**Lemma 4.2.** Let $X$ be the space of simplicial maps $f : P \to \mathbb{R}^D$ which are not embeddings. This is a semi-algebraic set in $\text{Hom}(\mathbb{R}^{V(P)}, \mathbb{R}^D)$, and its codimension is greater than or equal to $D - 2n$.

**Proof.** Consider
\[ \{(x, y, f) \in P \times P \times \text{Hom}(\mathbb{R}^{V(P)}, \mathbb{R}^D) | x \neq y, f(x) = f(y)\}. \]

This is a semi-algebraic set in $\mathbb{R}^{V(P)} \times \mathbb{R}^{V(P)} \times \text{Hom}(\mathbb{R}^{V(P)}, \mathbb{R}^D)$. Its projection to the factor $\text{Hom}(\mathbb{R}^{V(P)}, \mathbb{R}^D)$ is equal to $X$. Thus $X$ is semi-algebraic by the Tarski–Seidenberg principle. Let $A \subset V(P)$. We define $X_A$ as the space of simplicial maps $f : P \to \mathbb{R}^D$ such that $f(v)$ $(v \in A)$ are affinely dependent. Its codimension is greater than or equal to $D - \#A + 2$ by Sublemma 4.3 below. The space $X$ is contained in
\[ \bigcup_{A \subset V(P), \#A \leq 2n+2} X_A. \]

Hence its codimension is greater than or equal to $D - 2n$.

**Sublemma 4.3.** Let $N$ be a natural number. We consider the space $Y$ of non-injective linear maps $F : \mathbb{R}^N \to \mathbb{R}^D$. Then its codimension in $\text{Hom}(\mathbb{R}^N, \mathbb{R}^D)$ is greater than or equal to $D - N + 1$. 

Proof. Set
\[ Z = \{(F, x) \in \text{Hom}(\mathbb{R}^N, \mathbb{R}^D) \times \mathbb{P}^{N-1}(\mathbb{R}) \mid Fx = 0\}. \]
Here \( \mathbb{P}^{N-1}(\mathbb{R}) \) is the \( N-1 \) dimensional projective space. Let \( \pi_1 \) and \( \pi_2 \) be the projections from \( Z \) to \( \text{Hom}(\mathbb{R}^N, \mathbb{R}^D) \) and \( \mathbb{P}^{N-1}(\mathbb{R}) \) respectively. For each \( x \in \mathbb{P}^{N-1}(\mathbb{R}) \) the space \( \pi_2^{-1}(x) \) has dimension \( D(N-1) \). Thus \( \dim Z \leq N - 1 + D(N - 1) = (D + 1)(N - 1) \).

Then the codimension of \( Y = \pi_1(Z) \) is greater than or equal to \( ND - (D + 1)(N - 1) = D - N + 1 \).

□

Corollary 4.4. If \( D \geq 2n+1 \) then embeddings \( f : P \to \mathbb{R}^D \) are dense in \( \text{Hom}(\mathbb{R}^V(P), \mathbb{R}^D) \).

Lemma 4.5. Suppose \( D \geq 2n + 2 \), and let \( g : P \to \mathbb{R}^D \) be a simplicial map. Then for an open dense subset of simplicial maps \( f \in \text{Hom}(\mathbb{R}^V(P), \mathbb{R}^D) \), the maps
\[ (1 - t)f + tg : P \to \mathbb{R}^D, \quad x \mapsto (1 - t)f(x) + tg(x) \]
become embeddings for all \( 0 \leq t < 1 \).

Proof. Let \( Z \) be the space of simplicial maps \( f : P \to \mathbb{R}^D \) such that \( f + g : P \to \mathbb{R}^D \) is not an embedding. By Lemma 4.2, this is semi-algebraic and its codimension in \( \text{Hom}(\mathbb{R}^V(P), \mathbb{R}^D) \) is greater than or equal to \( D - 2n \geq 2 \). Consider
\[ \bigcup_{a \geq 0} a Z = \bigcup_{a \geq 0} \{af \mid f \in Z\} \subset \text{Hom}(\mathbb{R}^V(P), \mathbb{R}^D). \]
This is the image of the semi-algebraic map
\[ [0, \infty) \times Z \to \text{Hom}(\mathbb{R}^V(P), \mathbb{R}^D), \quad (a, f) \mapsto af. \]
So it is semi-algebraic by the Tarski–Seidenberg principle. Its codimension is greater than or equal to \( D - 2n - 1 \geq 1 \); see [BCR, Theorem 2.8.8]. Here we used real algebraic geometry essentially. We cannot hope a reasonable behavior of the topological dimension of \( \bigcup_{a \geq 0} a Z \) if \( Z \) is a fractal.

Then the codimension of the union
\[ (4.1) \{ f : P \to \mathbb{R}^D \mid \text{simplicial non-embedding} \} \cup \bigcup_{a \geq 0} a Z. \]
is also greater than or equal to 1. In particular this is nowhere dense because the dimension of semi-algebraic sets does not increase under the operation of closure [BCR, Proposition 2.8.2]. Any simplicial map \( f : P \to \mathbb{R}^D \) in the complement of (4.1) satisfies the required property. □
5. Interpolation

In this section we prepare the technique of interpolations. This is used for converting
discrete signals into continuous ones. Every idea here is due to Beurling [Beu, pp. 351-
365]. We follow his argument. The construction in this section is somewhat ad hoc, and
a more sophisticated approach is possible. But we prefer the ad hoc approach because it
is more elementary.

Let \( l \) and \( \rho \) be positive numbers with \( l\rho \in \mathbb{Z} \).

**Notation 5.1.** For two quantities \( x \) and \( y \) we write
\[
x \lesssim y
\]
if there exists a positive constant \( C \) depending only on \( l \) and \( \rho \) satisfying
\[
x \leq Cy.
\]

Let \( \Lambda \subset \mathbb{R} \) be a multiset. "Multi" means that some points may have multiplicity. For
integers \( n \) we set \( \Lambda_n = \Lambda \cap [nl, (n+1)l) \). The notation \( \Lambda_n \) is used only in this section.

**Condition 5.2.**
1. \( \inf_{\lambda \neq 0} |\lambda| \geq 1/\rho. \)
2. For all integers \( n \), we have \( \#\Lambda_n \leq l\rho \). Here \( \#(\cdot) \) is the counting with multiplicity. For
   example, \( \#\{1, 1, 1, 2, 3\} = 5. \)
3. For all nonzero integers \( n \), we have \( \#\Lambda_n = l\rho. \)

**Lemma 5.3.** Suppose \( \Lambda \) satisfies Condition 5.2 (1), (2), (3). Then
\[
f(z) = \lim_{A \to \infty} \prod_{\lambda \in \Lambda, 0 < |\lambda| < A} \left( 1 - \frac{z}{\lambda} \right)
\]
defines a holomorphic function in \( \mathbb{C} \) satisfying \( f(0) = 1 \) and \( f(\lambda) = 0 \) for all nonzero
\( \lambda \in \Lambda. \) Moreover for all \( z \in \mathbb{C} \)
\[
|f(z)| \lesssim (1 + |z|)^{5l\rho e^{|y|}}, \quad (y \text{ is the imaginary part of } z).
\]
The above product takes the multiplicity into account. For example if a nonzero \( \lambda \) appears
twice in \( \Lambda \) then the factor \( (1 - z/\lambda) \) appears twice in the product.

**Proof.** We should keep in mind the following simple fact, which is a toy-model of the
statement. The function
\[
\frac{\sin z}{z} = \lim_{A \to \infty} \prod_{0 < |n| < A} \left( 1 - \frac{z}{n\pi} \right)
\]
is holomorphic, and its growth is \( O(e^{|y|}) \).

We use the notation \( \sum^* \) and \( \prod^* \) for indicating sum and product over \( \lambda \neq 0. \) For
example we write
\[
f(z) = \lim_{A \to \infty} \prod_{\lambda \in \Lambda, |\lambda| < A}^* \left( 1 - \frac{z}{\lambda} \right).
\]
First we need to show the convergence of $f(z)$. A slightly delicate point is that this is a conditional convergence, i.e. the sum $\sum_{\lambda \in \Lambda} 1/|\lambda|$ diverges.

For $|z/\lambda| < 1$

$$\log \left(1 - \frac{z}{\lambda}\right) = - \frac{z}{\lambda} + \frac{z^2}{2\lambda^2} - \frac{z^3}{3\lambda^3} - \ldots,$$

From Condition 5.2 (2), we can easily prove that for any $B > 0$

$$\sum_{\lambda \in \Lambda, |\lambda| > B} \frac{1}{|\lambda|^k} \sim \frac{1}{B^{k-1}} \quad (k \geq 2).$$

From Condition 5.2 (3), for any $n \geq 2$

$$\left| \sum_{\lambda \in \Lambda_n} \frac{1}{\lambda} + \sum_{\lambda \in \Lambda_{-n}} \frac{1}{\lambda} \right| \lesssim \frac{1}{n^2}. \quad (5.3)$$

This implies the convergence of

$$\lim_{A \to \infty} \sum_{\lambda \in \Lambda, |\lambda| < A} \frac{1}{\lambda}.$$ 

Therefore $f(z)$ becomes a holomorphic function satisfying $f(0) = 1$ and $f(\lambda) = 0$ for all nonzero $\lambda \in \Lambda$.

Next we estimate the growth of $f$ on the real line. Suppose $x > 0$ and let $k$ be the integer with $kl \leq x < (k+1)l$. We assume $k > 0$. The case $k = 0$ is easier and can be discussed in a similar way.

- For $\lambda \in \Lambda_n$ with $n \leq -2$ or $n \geq k + 1$, we have $|1 - x/\lambda| \leq 1 - x/(n+1)l$. So

$$\prod_{\lambda \in \Lambda_n} |1 - \frac{x}{\lambda}| \leq \left| 1 - \frac{x}{(n+1)l} \right|^{l \rho}.$$ 

- For $\lambda \in \Lambda_n$ with $1 \leq n < k$, we have $|1 - x/\lambda| \leq x/(nl) - 1$. So

$$\prod_{\lambda \in \Lambda_n} |1 - \frac{x}{\lambda}| \leq \left| 1 - \frac{x}{nl} \right|^{l \rho}.$$ 

The factors for $n = -1, 0, k$ should be treated exceptionally. The modulus $|f(x)|$ is bounded by

$$\prod_{\lambda \in \Lambda_{-1} \cup \Lambda_0 \cup \Lambda_k} |1 - \frac{x}{\lambda}| \cdot \lim_{A \to \infty} \prod_{|\lambda| < A, n \neq 0, k_1 (k+1)} |1 - \frac{x}{nl}|^{l \rho}$$

$$= \prod_{\lambda \in \Lambda_{-1} \cup \Lambda_0 \cup \Lambda_k} |1 - \frac{x}{\lambda}| \cdot \frac{\sin \frac{\pi x}{\lambda}}{\pi x} \left(1 - \frac{x}{kl}\right) \left(1 - \frac{x}{(k+1)l}\right)$$

The first factor is easy to estimate:

$$\prod_{\lambda \in \Lambda_{-1} \cup \Lambda_0 \cup \Lambda_k} |1 - \frac{x}{\lambda}| \lesssim (1 + x)^{3l \rho}.$$
Set \( t = x/l \).

\[
\frac{\sin \frac{\pi x}{t}}{rac{\pi x}{t} \left(1 - \frac{x}{kl}ight) \left(1 - \frac{x}{(k+1)l}\right)} = \frac{k(k+1) \sin \pi t}{\pi t(k-t)(k+1-t)}.
\]

From the mean value theorem,

\[
\left| \frac{\sin \pi t}{t} \right| \leq \pi, \quad \left| \frac{\sin \pi t}{k-t} \right| \leq \pi, \quad \left| \frac{\sin \pi t}{k+1-t} \right| \leq \pi.
\]

Hence

\[
\frac{|k(k+1) \sin \pi t|}{\pi t(k-t)(k+1-t)} \lesssim k(k+1) \lesssim (1+x)^2.
\]

Therefore

\[
|f(x)| \lesssim (1+x)^{5\rho}.
\]

The case \( x < 0 \) is the same and we get

\[
|f(x)| \lesssim (1+|x|)^{5\rho}.
\]

Next we estimate \(|f(y\sqrt{-1})|\). Suppose \( y > 0 \). For \( r > 0 \) we set \( n(r) = \#(\Lambda \cap (-r,r)) \).

This is bounded by

\[
n(r) \leq C + 2\rho r
\]

where \( C \) is a positive constant depending only on \( l \) and \( \rho \). We have \(|f(y\sqrt{-1})|^2 = \prod_{\lambda \in \Lambda} (1 + y^2/\lambda^2)\). Hence

\[
2 \log |f(y\sqrt{-1})| = \sum_{\lambda \in \Lambda} \log \left(1 + \frac{y^2}{\lambda^2}\right) = \int_0^\infty \log \left(1 + \frac{y^2}{r^2}\right) dn(r).
\]

Using the integration by parts, this is equal to

\[
2y^2 \int_0^\infty \frac{n(r)}{r(r^2 + y^2)} dr.
\]

From \( n(r) \leq C + 2\rho r \),

\[
\log |f(y\sqrt{-1})| \leq Cy^2 \int_0^\infty \frac{dr}{r(r^2 + y^2)} + 2\rho y^2 \int_0^\infty \frac{dr}{r^2 + y^2} = C \int_0^\infty \frac{dr}{r(r^2 + 1)} + \pi \rho y.
\]

Thus

\[
|f(y\sqrt{-1})| \lesssim e^{\pi \rho y}.
\]

The case \( y < 0 \) is the same. So we get

\[
|f(y\sqrt{-1})| \lesssim e^{\pi \rho |y|}.
\]

Finally we show that \(|f(z)|\) grows at most exponentially. Let \( z = x + y\sqrt{-1} \). We consider the case \( x, y > 0 \). Other cases are the same. Let \( k \) be the integer with \( kl \leq x < (k+1)l \). Set

\[
\Lambda' = \Lambda \setminus (\Lambda_{k-1} \cup \Lambda_k \cup \Lambda_{k+1}).
\]
We estimate
\[ \prod_{\lambda \in \Lambda_k} \left| 1 - \frac{z}{\lambda} \right| \lesssim (1 + |z|)^{3l_\rho}. \]

\[
\lim_{A \to \infty} \prod_{\lambda \in \Lambda \setminus \Lambda_k} \left| 1 - \frac{z}{\lambda} \right|^2 = \lim_{A \to \infty} \prod_{\lambda \in \Lambda \setminus \Lambda_k} \left\{ \left( 1 - \frac{x}{\lambda} \right)^2 + \frac{y^2}{\lambda^2} \right\}
\]
\[
= \left\{ \lim_{A \to \infty} \prod_{\lambda \in \Lambda \setminus \Lambda_k} \left( 1 - \frac{x}{\lambda} \right)^2 \right\} \cdot \prod_{\lambda \in \Lambda} \left\{ 1 + \frac{y^2}{(\lambda - x)^2} \right\}.
\]

As in the proof of \(|f(x)| \lesssim (1 + |x|)^{5l_\rho}\) we estimate
\[
\lim_{A \to \infty} \prod_{\lambda \in \Lambda \setminus \Lambda_k} \left( 1 - \frac{x}{\lambda} \right)^2 \lesssim (1 + x)^{12l_\rho}.
\]

As in \(|f(y\sqrt{-1})| \lesssim e^{\pi \rho |y|}\)
\[
\prod_{\lambda \in \Lambda \setminus \Lambda_k} \left\{ 1 + \frac{y^2}{(\lambda - x)^2} \right\} \lesssim e^{2\pi \rho |y|}.
\]
Therefore we conclude that \(|f(z)|\) grows at most exponentially.

We have proved that \(f(z)\) is of exponential type with \(|f(x)| \lesssim (1 + |x|)^{5l_\rho}\) and \(|f(y\sqrt{-1})| \lesssim e^{\pi \rho |y|}\). Then we can prove \((5.1)\) by the Phragmén–Lindelöf principle. For example, in the first quadrant \((x, y \geq 0)\), we apply the Phragmén–Lindelöf principle to the function
\[
(1 + z)^{-5l_\rho e^{\pi \rho \sqrt{-1} z}} f(z)
\]
and conclude \((5.1)\).

\[ \square \]

**Lemma 5.4.** For any positive numbers \(r, \varepsilon\) there exists \(B_1 = B_1(r, \varepsilon, l, \rho) > 0\) satisfying the following. Suppose \(\Lambda \subset \mathbb{R}\) satisfies Conditions \(5.2\) \((1), (2), (3)\). Then
\[
\left| 1 - \lim_{A \to \infty} \prod_{\lambda \in \Lambda, \lambda \in \Lambda \setminus \Lambda_k} \left( 1 - \frac{z}{\lambda} \right) \right| < \varepsilon \quad (|z| \leq r).
\]

**Proof.** For \(|z/\lambda| < 1\)
\[
\log \left( 1 - \frac{z}{\lambda} \right) = -\frac{z}{\lambda} - \frac{z^2}{2\lambda^2} - \frac{z^3}{3\lambda^3} - \ldots.
\]
From \((5.2)\) and \((5.3)\), for \(B_1 > 0\)
\[
\lim_{A \to \infty} \sum_{\lambda \in \Lambda, \lambda \in \Lambda \setminus \Lambda_k} \frac{1}{\lambda} \lesssim \frac{1}{B_1}, \quad \sum_{\lambda \in \Lambda, |\lambda| > B_1} \frac{1}{|\lambda|^k} \lesssim \frac{1}{B_1^{k-1}} \quad (k \geq 2).
\]
Thus for sufficiently large \(B_1\)
\[
\left| \lim_{A \to \infty} \sum_{\lambda \in \Lambda, \lambda \in \Lambda \setminus \Lambda_k} \log \left( 1 - \frac{z}{\lambda} \right) \right| \lesssim \frac{|z| + |z|^2}{B_1}.
\]

\[ \square \]
We need to relax the conditions on $\Lambda$. Let $\Lambda \subset \mathbb{R}$ be a multiset satisfying Conditions 5.2 (1) and (2) but not necessarily (3). For each nonzero integer $n$ we add $nl$ to $\Lambda$ with multiplicity $(l \rho - \# \Lambda_n)$. We denote by $\Lambda^+$ the resulting multiset and call it the saturation of $\Lambda$. This satisfies $\Lambda \subset \Lambda^+$ and all the three conditions of Condition 5.2. (The construction of $\Lambda^+$ is the most ad hoc part of the argument.)

Let $\tau$ be a positive number. Let $\psi(\xi)$ be a nonnegative smooth function in $\mathbb{R}$ satisfying $\mathop{\text{supp}}(\psi) \subset \left[ -\frac{\tau}{2}, \frac{\tau}{2} \right]$, $\int_{-\infty}^{\infty} \psi(\xi) d\xi = 1$.

Then the inverse Fourier transform $\mathcal{F}(\psi)$ is a rapidly decreasing function satisfying $\mathcal{F}(\psi)(0) = 1$, $\left| \mathcal{F}(\psi)(x + y \sqrt{-1}) \right| \leq e^{\pi \tau |y|}$.

We define a function $\varphi_{\Lambda}$ by

$$
\varphi_{\Lambda}(x) = \mathcal{F}(\psi)(x) \left\{ \lim_{A \to \infty} \prod_{\lambda \in \Lambda^+, 0 < |\lambda| < A} \left( 1 - \frac{x}{\lambda} \right) \right\}.
$$

From Lemmas 2.2 and 5.3

- $\varphi_{\Lambda}$ belongs to the Banach space $V[-(\rho + \tau)/2, (\rho + \tau)/2]$.
- $\varphi_{\Lambda}(0) = 1$ and $\varphi_{\Lambda}(\lambda) = 0$ for all nonzero $\lambda \in \Lambda$.
- $\varphi_{\Lambda}$ is a rapidly decreasing. In particular there exists $K > 0$ depending only on $l, \rho, \tau$ such that

$$
|\varphi_{\Lambda}(x)| \leq \frac{K}{1 + |x|^2}.
$$

Note that $\varphi_{\Lambda}$ depends on $l$, $\rho$ and $\tau$ although they are not explicitly written in the notation. In the proof of Theorem 3.1 the numbers $l$, $\rho$ and $\tau$ are fixed in the beginning of the argument. So this does not cause a confusion.

**Lemma 5.5.** For any positive numbers $r$ and $\varepsilon$ there exists $B_2 = B_2(r, \varepsilon, l, \rho, \tau) > 0$ satisfying the following. Suppose $\Lambda, \Lambda' \subset \mathbb{R}$ satisfy Conditions 5.2 (1) and (2). If $\Lambda \cap [-B_2, B_2] = \Lambda' \cap [-B_2, B_2]$ then

$$
|\varphi_{\Lambda}(x) - \varphi_{\Lambda'}(x)| < \varepsilon \quad (|x| \leq r).
$$

**Proof.** If $\Lambda \cap [-B_2, B_2] = \Lambda' \cap [-B_2, B_2]$ then the saturations satisfy $\Lambda^+ \cap [-B_2 + l, B_2 - l] = (\Lambda')^+ \cap [-B_2 + l, B_2 - l]$. Thus for $B_2 > l$

$$
\frac{\varphi_{\Lambda'}(x)}{\varphi_{\Lambda}(x)} = \lim_{A \to \infty} \prod_{\lambda \in (\Lambda')^+, l < |\lambda| < A} \left( 1 - \frac{x}{\lambda} \right) \prod_{\lambda \in \Lambda^+, l < |\lambda| < A} \left( 1 - \frac{x}{\lambda} \right).
$$

From Lemma 5.4 for sufficiently large $B_2$

$$
\left| 1 - \frac{\varphi_{\Lambda'}(x)}{\varphi_{\Lambda}(x)} \right| < \frac{\varepsilon}{K} \quad (|x| \leq r).
$$
Then by (5.5)
\[ |\varphi_\Lambda(x) - \varphi_\Lambda'(x)| = \left| \frac{\varphi_\Lambda(x)}{\varphi_\Lambda(x)} - \frac{\varphi_\Lambda'(x)}{\varphi_\Lambda(x)} \right| < \varepsilon \quad (|x| \leq r). \]

□

6. Voronoi diagram and weight functions

Here we introduce a tiling of \( \mathbb{R} \). This will be the basis of our perturbation procedure. The key ingredient of the construction is dynamical Voronoi diagram. This is first introduced by Lindenstrauss and the authors [GLT]. We will consider a Voronoi diagram in the plane and cut it by the real line. This gives a tiling of \( \mathbb{R} \), which has several nice properties revealed in this section. We would like to remark that our use of Voronoi diagram is conceptually influenced by the works of Lightwood [Lig03, Lig04], which study the \( \mathbb{Z}^2 \)-version of the Krieger embedding theorem in symbolic dynamics.

Throughout this section, \((Y,S)\) is a dynamical system with the marker property. Let \( C, L_0, L_1 \) be positive numbers. We fix a natural number \( L \) satisfying
\[(6.1) \quad L > 4L_1 + 1 + 4CL_0(4L_0 + 3).\]
The marker property condition is used in the next lemma.

**Lemma 6.1.** There exist an integer \( M > L \) and a continuous function \( h : Y \to [0,1] \) satisfying the following two conditions.
1. \( \text{supp}(h) \cap S^n(\text{supp}(h)) = \emptyset \) for all \( 1 \leq n \leq L \).
2. \( Y = \bigcup_{n=0}^{M-1} S^n(h^{-1}(1)) \).

**Proof.** By the definition of the marker property, there exists an open set \( U \subset Y \) satisfying
\[ U \cap S^nU = \emptyset \quad (1 \leq n \leq L), \quad Y = \bigcup_{n \in \mathbb{Z}} S^nU. \]

We can find \( M > L \) and a compact set \( K \subset U \) satisfying
\[ Y = \bigcup_{n=0}^{M-1} S^nK. \]
Take a continuous function \( h : Y \to [0,1] \) satisfying \( \text{supp}(h) \subset U \) and \( h = 1 \) on \( K \). Then this satisfies the required properties. □

Take \( x \in Y \). We consider the Voronoi diagram with respect to the set
\[ \left\{ \left( n, \frac{1}{h(S^nx)} \right) \mid n \in \mathbb{Z}, h(S^n x) \neq 0 \right\} \subset \mathbb{R}^2. \]

For \( n \in \mathbb{Z} \) with \( h(S^n x) \neq 0 \) we define
\[ V(x,n) = \{ u \in \mathbb{R}^2 \mid \forall m \in \mathbb{Z} : |u - (n, 1/h(S^n x))| \leq |u - (m, 1/h(S^m x))| \}. \]
If \( h(S^nx) = 0 \) then we set \( V(x,n) = \emptyset \). These form a Voronoi partitioning of \( \mathbb{R}^2 \):

\[
\mathbb{R}^2 = \bigcup_{n \in \mathbb{Z}} V(x,n).
\]

We consider \( \mathbb{R} = \mathbb{R} \times \{0\} \) as a subset of \( \mathbb{R}^2 \) and set \( I(x,n) = \mathbb{R} \cap V(x,n) \). These intervals form a tiling of \( \mathbb{R} \):

\[
\mathbb{R} = \bigcup_{n \in \mathbb{Z}} I(x,n).
\]

See figure 6.1. This construction is naturally *dynamical*. Namely we have

\[
I(Sx,n) = -1 + I(x,n+1).
\]

The key point of the construction is that the interval \( I(x,n) \) *depends continuously on* \( x \in Y \): Suppose \( I(x,n) \) is not a point. Then for any \( \varepsilon > 0 \) the Hausdorff distance between \( I(x,n) \) and \( I(y,n) \) is less than \( \varepsilon \) if \( y \in Y \) is sufficiently close to \( x \).

We define

\[
\partial(x) = \bigcup_{n \in \mathbb{Z}} \partial I(x,n).
\]

(Of course, \( \partial[a,b] = \{a,b\} \).) For \( r > 0 \) we set \( \partial_r[a,b] = [a-r,a+r] \cup [b-r,b+r] \). (\( \partial_r \emptyset = \emptyset \).) Set

\[
\partial(x,r) = \bigcup_{n \in \mathbb{Z}} \partial_r I(x,n).
\]
Lemma 6.2. (1) $I(x, n) \subset (n - M/2, n + M/2)$.

(2) For any $r > 0$

$$\limsup_{R \to \infty} \frac{1}{R} \sup_{a \in \mathbb{R}, x \in Y} #([a, a + R] \cap \partial(x, r) \cap \mathbb{Z}) \leq \frac{4r + 2}{L}.$$  

Moreover (we denote by $| \cdot |$ the Lebesgue measure of $\mathbb{R}$)

$$\limsup_{R \to \infty} \frac{1}{R} \sup_{a \in \mathbb{R}, x \in Y} |[a, a + R] \cap \partial(x, r)| \leq \frac{4r}{L}.$$  

Proof. (1) By Lemma 6.1 (2), there exist integers $l \leq n \leq m$ with $h(S^l x) = h(S^m x) = 1$ and $n - l, m - n < M$. Let $t \in I(x, n)$. If $t \geq n$ then

$$|(t, 0) - (n, 1/h(S^n x))| \leq |(t, 0) - (m, 1)|$$

implies $t - n < M/2$. In the same way, if $t < n$ then we get $n - t < M/2$.

(2) By the above (1)

$$[a, a + R] \subset \bigcup_{-M/2 < n - a < R + M/2} I(x, n).$$

Therefore $[a, a + R] \cap \partial(x, r)$ is contained in

$$[a, a + r] \cup [a + R - r, a + R] \cup \bigcup_{-M/2 < n - a < R + M/2} \partial_r I(x, n).$$

The number of integers $n \in (a - M/2, a + R + M/2)$ satisfying $h(S^n x) \neq 0$ is bounded by

$$1 + \frac{R + M}{L}$$

because $\text{supp}(h) \cap S^m(\text{supp}(h)) = \emptyset$ for $1 \leq m \leq L$. Thus $\#([a, a + R] \cap \partial(x, r) \cap \mathbb{Z})$ is bounded by

$$2(r + 1) + 2(2r + 1) \left(1 + \frac{R + M}{L}\right).$$

Dividing this by $R$ and letting $R \to \infty$, we get the result. Another statement can be proved in the same way. \(\square\)

Now we have come to the core of the argument. In the proof of Theorem 3.1 we will have the following dichotomy: Take $m \in \mathbb{Z}$. The point $m$ is said to be wild if $m \in \partial(x, L_0 - 4)$. Otherwise it is tame. Here $L_0 - 4$ is just a technical number. Readers may think that a point is wild if it is close to $\partial(x)$. Tame points can be handled easily. A main problem is how to deal with wild points.

The following is the idea behind the above dichotomy. It is not difficult to control band-limited functions over sufficiently long intervals. But it is impossible to control them over short intervals because of their band-limited nature. (Intuitively speaking, band-limited functions cannot have very small fluctuation.) As a consequence, if the length of $I(x, n)$ is sufficiently larger than $L_0$ (which will be chosen appropriately later), then we can construct a good perturbation of band-limited functions over it. But if it is less than $L_0$, ...
then we cannot construct a perturbation there. So the problem is how to deal with short intervals.

We overcome the difficulty by introducing tax system. Long intervals are good for our perturbation procedure. But some of them are unnecessarily long. We consider \( I(x, n) \) too long if \( |I(x, n)| > L_1 \). Then we take \( \text{tax} = |I(x, n)| - L_1 \) from too long intervals, and use it for the care of wild points. If every lattice point becomes happy, then the proof is done.

The next lemma is the basis of our tax system. Intuitively, it means that the sum of tax is larger than the cost of social security. For \( x \in \mathbb{R} \) we set \( x^+ = \max(x, 0) \).

**Lemma 6.3.** There exists an integer \( R > M \) such that for all \( x \in Y \) and \( a \in \mathbb{R} \)
\[
(6.2) \quad \sum_{a \leq n \leq a+R} (|I(x, n)| - L_1)^+ \geq C \sum_{a \leq n \leq a+R} (L_0 - \text{dist}(n, \partial(x)))^+,
\]
where \( \text{dist}(n, \partial(x)) = \min_{t \in \partial(x)} |n - t| \). In the above two sums, \( n \) runs over \( \mathbb{Z} \cap [a, a + R] \).

Intuitively the left-hand side is the sum of tax in the region \( a \leq n \leq a + R \) and the right-hand side is the cost of social security there.

**Proof.** By Lemma 6.2 (2), the right-hand side of (6.2) is bounded by
\[
CL_0 \#(\mathbb{Z} \cap [a, a + R] \cap \partial(x, L_0)) \leq C L_0 R \frac{4L_0 + 3}{L}
\]
for \( R \gg 1 \). Let \( A(x) \) be the set of integers \( n \) with \( |I(x, n)| \geq 2L_1 \). For \( n \in A(x) \) we have \( (|I(x, n)| - L_1)^+ \geq |I(x, n)|/2 \). So the left-hand side of (6.2) is bounded from below by
\[
\frac{1}{2} \sum_{n \in A(x) \cap [a, a + R]} |I(x, n)|.
\]

By Lemma 6.2 (1) we have
\[
[a + M/2, a + R - M/2] \subset \bigcup_{a \leq n \leq a+R} I(x, n).
\]

Hence
\[
[a + M/2, a + R - M/2] \setminus \partial(x, L_1) \subset \bigcup_{n \in A(x) \cap [a, a + R]} I(x, n),
\]

\[
\sum_{n \in A(x) \cap [a, a + R]} |I(x, n)| \geq R - M - |[a + M/2, a + R - M/2] \cap \partial(x, L_1)|
\]

\[
\geq R - M - (R - M) \frac{4L_1 + 1}{L} \quad \text{for } R \gg 1 \text{ by Lemma 6.2 (2)}
\]

\[
\geq \frac{R}{2} \left( 1 - \frac{4L_1 + 1}{L} \right) \quad (R \gg 1)
\]

\[
\geq 2C L_0 R \frac{4L_0 + 3}{L} \quad \text{by (6.1)}.
\]

Combing the above estimates, we get (6.2). \( \square \)

The next lemma is our tax system.
Lemma 6.4. There exists a continuous map (called weight)
\[ Y \to ([0,1]^{R+1})^\mathbb{Z}, \quad x \mapsto w(x) = (w_n)_{n \in \mathbb{Z}}, \quad w_n = (w_{n0}, w_{n1}, \ldots, w_{nR}) \in [0,1]^{R+1}, \]
satisfying the following.

1. The map is equivariant: \( w_n(Sx) = w_{n+1}(x) \).
2. If \(|I(x,n)| \leq L_1\) then \( w_n = (0, \ldots, 0) \).
3. For all \( n \in \mathbb{Z} \)
\[ \# \{ m \mid w_{nm} > 0 \} \leq 1 + \frac{1}{C}(|I(x,n)| - L_1)^+. \]
4. For any \( m \in \mathbb{Z} \cap \partial(x, L_0 - 4) \) there exists an integer \( n \) with \( m - R \leq n \leq m \) satisfying
\[ w_{n,m-n} = 1. \]

Before proving the lemma, we explain its intuitive meaning. The weight \( w_n \) is the tax paid by the interval \( I(x,n) \). The entry \( w_{nm} \) is the (rescaled) money taken from \( I(x,n) \) which is used for the care of the point \( n + m \in \mathbb{Z} \). A point \( l \in \mathbb{Z} \) becomes happy if there exist \( n \) and \( m \in [0,R] \) satisfying \( n + m = l \) and \( w_{nm} = 1 \). The condition (2) means that intervals of length \( \leq L_1 \) do not pay tax. The condition (3) (roughly) means that the tax taken from \( I(x,n) \) does not exceed \(|I(x,n)| - L_1 \). The condition (4) means that we achieve the perfect social welfare, namely every wild point becomes happy. (\( w_{n,m-n} = 1 \) implies that the point \( m \) becomes happy.)

Proof. Take \( x \in Y \). Set \( a_n^{(0)} = (|I(x,n)| - L_1)^+/C \) and \( b_n^{(0)} = (L_0 - \text{dist}(n, \partial(x)))^+ \). We define \( v_{nm} \geq 0 \) for \( n \in \mathbb{Z} \) and \( m \geq 0 \) inductively (with respect to \( m \)) by
\[ v_{nm} = \min(a_n^{(m)}, b_n^{(m)}), \quad a_n^{(m+1)} = a_n^{(m)} - v_{nm}, \quad b_n^{(m+1)} = b_n^{(m)} - v_{n-m,m}. \]

The heuristic idea behind this process is as follows: The intervals \( I(x,n) \) are donors of tax, and the integral points on the line are receivers. \( a_n^{(0)} \) is the money that the interval \( I(x,n) \) can pay as the tax, and \( b_n^{(0)} \) is the money we need for the care of the point \( n \). At the \( m \)-th step of the process, the interval \( I(x,n) \) pays \( v_{nm} \) and we use it for the point \( n + m \). After the \( m \)-th step, \( I(x,n) \) still have the extra money \( a_n^{(m+1)} \) and points \( n \) still need \( b_n^{(m+1)} \). At each step, \( I(x,n) \) pays as much as possible. Namely, if \( a_n^{(m)} \geq b_n^{(m)} \) then \( I(x,n) \) pays \( b_n^{(m+1)} \) and the point \( n + m \) becomes satisfied. If \( a_n^{(m)} < b_n^{(m)} \) then \( I(x,n) \) pays \( a_n^{(m)} \) and loses all its ability to help integral points.

Every point is satisfied after the \( R \)-th step:
\[ b_n^{(m)} = v_{nm} = 0 \quad (m > R). \]

This is because the condition \( b_n^{(m)} > 0 \) implies
\[ \sum_{n-m+1 \leq k < n} a_k^{(0)} < \sum_{n-m+1 \leq k < n} b_k^{(0)}, \]
which is impossible for \( m = R + 1 \) by Lemma 6.3.
We set $v_n = (v_{n0}, \ldots, v_{nR})$. This construction is equivariant: $v_n(Sx) = v_{n+1}(x)$. Moreover for all integers $n$

\begin{equation}
C \sum_{m=0}^{R} v_{nm} \leq (|I(x,n)| - L_1)^+, \quad \sum_{m=0}^{R} v_{n-m,m} = (L_0 - \text{dist}(n, \partial(x)))^+.
\end{equation}

The former inequality holds because $\sum_{m=0}^{R} v_{nm}$ is the tax paid by $I(x,n)$ and does not exceed $a_n(0)$. The latter equality holds because every point is satisfied after the $R$-th step.

We choose continuous functions $\alpha : \mathbb{R} \to [1, R]$ and $\beta : \mathbb{R} \to [0, 1]$ satisfying

$$\alpha(0) = R, \quad \alpha(t) = 1 \quad (t \geq 1), \quad \beta(t) = 0 \quad (t \leq 1), \quad \beta(t) = 1 \quad (t \geq 2).$$

For $(x_0, x_1, \ldots, x_R) \in \mathbb{R}^{R+1}$ we define $A(x_0, x_1, \ldots, x_R) = (y_0, y_1, \ldots, y_R) \in \mathbb{R}^{R+1}$ by

$$y_R = Rx_R, \quad y_{R-1} = \alpha(y_R)x_{R-1}, \quad y_{R-2} = \alpha(\max(y_R, y_{R-1}))(x_{R-2}), \ldots, \quad y_0 = \alpha(\max(y_R, \ldots, y_1))x_0.$$ 

Note that this definition implies

\begin{equation}
\#\{m | y_m > 1\} \leq 1 + \#\{m | x_m > 1\}
\end{equation}

Define $B : \mathbb{R}^{R+1} \to [0, 1]^{R+1}$ by $B(y_0, \ldots, y_R) = (\beta(y_0), \ldots, \beta(y_R))$. We define $w(x) = (w_n)_{n \in \mathbb{Z}}$ by $w_n = B(A(v_n))$. We check the required conditions. The continuity and equivariance are obvious. The condition (2) follows from the former inequality of (6.3).

This inequality with the help of (6.4) also implies the condition (3):

$$\#\{m | w_{nm} > 0\} \leq 1 + \#\{m | v_{nm} > 1\} \leq 1 + \frac{1}{C}(|I(x,n)| - L_1)^+.$$ 

For the condition (4), take $m \in \mathbb{Z} \cap \partial(x, L_0 - 4)$. Set

$$n_0 = \min\{n | v_{n,m-n} > 0\}.$$ 

We have $m - R \leq n_0 \leq m$. If $v_{n_0,m-n_0} \geq 2$ then $w_{n_0,m-n_0} = 1$. So we assume $v_{n_0,m-n_0} < 2$. From the latter equality of (6.3) and $\text{dist}(m, \partial(x)) \leq L_0 - 4$,

$$\sum_{n=n_0+1}^{m} v_{n,m-n} = (L_0 - \text{dist}(m, \partial(x)))^+ - v_{n_0,m-n_0} > 2.$$ 

Since $m - n_0 \leq R$, there exists $n_0 < n_1 \leq m$ satisfying $v_{n_1,m-n_1} > 2/R$. The condition $v_{n_0,m-n_0} > 0$ with $n_0 < n_1$ implies that the point $m$ is not satisfied after the $(m - n_1)$-th step and that the interval $I(x, n_1)$ finishes to pay all its tax at the $(m - n_1)$-th step. Thus we have $v_{n_1,k} = 0$ for $k > m - n_1$. Then the definitions of $A$ and $B$ imply $w_{n_1,m-n_1} = 1$. This shows the condition (4). \qed
7. Proof of Theorem 3.1

In this section we combine all the preparations and prove Theorem 3.1. Throughout this section we assume the following.

- \( a < b \) are two real numbers.
- \((Y, S)\) is a dynamical system having the marker property.
- \( \Phi : (X, T) \to (Y, S) \) is an extension with \( \text{mdim}(X, T) < b - a \).

For the convenience of readers, we restate Theorem 3.1:

**Theorem 7.1** (=Theorem 3.1). For a dense \( G_\delta \) subset of \( f \in C_T(X, B_1(V[a,b])) \) the map

\[
(f, \Phi) : X \to B_1(V[a,b]) \times Y, \quad x \mapsto (f(x), \Phi(x))
\]

is an embedding.

7.1. Setting of the proof. We fix positive numbers \( l, \rho, \tau \) satisfying the following.

- \( \rho \in \mathbb{Q} \) and \( \text{mdim}(X, T) < \rho < b - a \).
- \( l \in \mathbb{N} \) and \( l\rho \in \mathbb{N} \).
- \( \rho + \tau < b - a \).

We use these \( l, \rho, \tau \) for the construction of the interpolation function \( \varphi_\Lambda \) in (5.4). Let \( K = K(l, \rho, \tau) \) be the positive number introduced in (5.5). We denote the distance on \( X \) by \( d \). Recall that for a natural number \( N \) we defined

\[
d_N(x, y) = \max_{0 \leq n < N} d(T^n x, T^n y).
\]

For the proof of Theorem 3.1 it is enough to prove that the set

\[
\bigcap_{n=1}^{\infty} \{ f \in C_T(X, B_1(V[a,b])) | (f, \Phi) \text{ is a } (1/n)\text{-embedding w.r.t. } d \}
\]

is dense and \( G_\delta \) in \( C_T(X, B_1(V[a,b])) \). This is obviously \( G_\delta \) because "\((1/n)\)-embedding" is an open condition. So the task is to prove the next proposition. Its proof occupies all the rest of the paper.

**Proposition 7.2.** For any positive number \( \delta \) and \( f \in C_T(X, B_1(V[a,b])) \), there exists \( g \in C_T(X, B_1(V[a,b])) \) satisfying the following two conditions.

1. For all \( x \in X \) and \( t \in \mathbb{R} \), \( |f(x)(t) - g(x)(t)| < \delta \).
2. \( (g, \Phi) : X \to B_1(V[a,b]) \times Y \) is a \( \delta \)-embedding with respect to \( d \).

Fix \( \delta > 0 \) and \( f \in C_T(X, B_1(V[a,b])) \). We can assume \( |f(x)(t)| \leq 1 - \delta \) for all \( x \in X \) and \( t \in \mathbb{R} \) by replacing \( f \) with \((1 - \delta)f\) if necessary. We choose \( \delta' > 0 \) so that if a subset \( \Lambda \subset \mathbb{R} \) satisfies

\[
|\lambda - \lambda'| \geq \frac{1}{\rho}, \quad (\forall \lambda, \lambda' \in \Lambda \text{ with } \lambda \neq \lambda'),
\]

then...
then
\begin{equation}
\delta' \cdot \sum_{\lambda \in \Lambda} \frac{K}{1 + |t - \lambda|^2} < \delta \quad (\forall t \in \mathbb{R}).
\end{equation}

We choose $0 < \varepsilon < \delta$ so that
\begin{equation}
d(x, y) < \varepsilon \iff \forall t \in [0, 1] : |f(x)(t) - f(y)(t)| < \delta'.
\end{equation}

We can find a simplicial complex $Q$ with an $\varepsilon$-embedding $\pi : X \to Q$ with respect to $d$. Let $CQ = [0, 1] \times Q/\{0\} \times Q$ be the cone over $Q$. For $(t, x) \in [0, 1] \times Q$ we denote its equivalence class by $tx \in CQ$. We set $* = 0x$ (the vertex of the cone). The cone $CQ$ will be used for the care of wild points.

From $\text{mdim}(X, T) < \rho$ there are an integer $N > 1$ with $\rho N \in \mathbb{N}$, a simplicial complex $P$ of dimension less than $\rho N$ and an $\varepsilon$-embedding $\Pi : X \to P$ with respect to $d_N$. For a natural number $n$ we set
\[ \Pi_n : X \to P^n, \quad x \mapsto (\Pi(x), \Pi(T^N x), \ldots, \Pi(T^{(n-1)N} x)), \]
which is an $\varepsilon$-embedding with respect to $d_{nN}$. The space $P^n$ will be used for constructing perturbations over long intervals. The number $n$ will be chosen so large that the perturbations can fit intervals of various length.

We choose natural numbers $C_1, C_2$ and a sequence of integers $2 < n_0 < n_1 < n_2 < \ldots$ satisfying $n_k < C_1 k + C_2$ and
\begin{equation}
\forall n \geq n_k : n \text{dim } P + k \text{dim } CQ + 1 \leq (n - 1) \rho N.
\end{equation}

Here we have used $\text{dim } P < \rho N$.

We set
\[ C = C_1 N, \quad L_0 = n_0 N + 4, \quad L_1 = C_1 N + C_2 N + 2N. \]
We apply to $(Y, S)$ the construction of Section 6 with respect to these $C, L_0, L_1$. Then we get natural numbers
\[ R > M > L > 4L_1 + 1 + 4CL_0(4L_0 + 3), \]
the tiling $\mathbb{R} = \bigcup_{n \in \mathbb{Z}} I(x, n)$ and the weight $w(x) = (w_n)_{n \in \mathbb{Z}} \in ([0, 1]^{R+1})^\mathbb{Z}$ for each $x \in Y$.

**Lemma 7.3.** Let $x \in Y$ and $n \in \mathbb{Z}$ with $I(x, n) \neq \emptyset$. Set
\[ I(x, n) = [\alpha, \beta], \quad r = \left\lceil \frac{\alpha - n}{N} \right\rceil, \quad s = \left\lfloor \frac{\beta - n}{N} \right\rfloor. \]

See Figure 7.1. If $s - r > n_0$ then
\[ \# \{m | w_{nm}(x) > 0\} \leq \max \{k | n_k < s - r\}. \]
Figure 7.1. $n + rN$ and $n + sN$ on $I(x, n)$.

Proof. If $|I(x, n)| \leq L_1$ then $w_n = (0, \ldots, 0)$ by Lemma 6.4 (2). So we assume $|I(x, n)| > L_1$. We have $|I(x, n)| < (s - r)N + 2N$. By Lemma 6.4 (3), the number of $m$ with $w_{nm}(x) > 0$ is bounded by

$$\left\lfloor 1 + \frac{1}{C_1}(|I(x, n)| - L_1) \right\rfloor \leq \left\lfloor 1 + \frac{(s - r)N + 2N - L_1}{C_1N} \right\rfloor = \left\lfloor \frac{s - r - C_2}{C_1} \right\rfloor.$$ 

Here we have used $C = C_1N$ and $L_1 = C_1N + C_2N + 2N$. From $n_k < C_1k + C_2$

$$\left\lfloor \frac{s - r - C_2}{C_1} \right\rfloor = \max\{k | C_1k + C_2 \leq s - r\} \leq \max\{k | n_k < s - r\} \leq \max\{k | w_{nm}(x) > 0\}.$$

\[\Box\]

We set $W = (CQ)^{R+1} = (CQ)_{0,1,2,\ldots,R}$. For $0 \leq k \leq R$ we define $W_k \subset W$ as the set of $(x_n)_{n=0}^R$ satisfying $x_n = *$ except for at most $k$ entries. Hence $\{(*,\ldots,*)\} = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_{R+1} = W$. We have $\dim W_k = k \dim CQ$.

Consider the disjoint union $P \sqcup CQ$ and take a distance $D$ on it. We consider $Q = \{1\} \times Q$ as a subspace of $CQ$. So $D$ also gives a distance on $Q$. There exists $\varepsilon' > 0$ such that for $x, y \in X$

$$(7.4) \quad D(\pi(x), \pi(y)) < \varepsilon' \implies d(x, y) < \varepsilon, \quad D(\Pi(x), \Pi(y)) < \varepsilon' \implies d_N(x, y) < \varepsilon.$$

Let $0 \leq m_1 \leq m_2 < n$. We define a semi-distance $D_{m_2}^{m_1}$ on $P^n \times W$ by

$$D_{m_1}^{m_2}((x, y), (x', y')) = \max \left(D(x_{m_1}, x'_{m_1}), \ldots, D(x_{m_2}, x'_{m_2}), D(y_0, y'_0), \ldots, D(y_R, y'_R)\right),$$

where

$x = (x_0, \ldots, x_{n-1}), x' = (x'_0, \ldots, x'_{n-1}) \in P^n, \quad y = (y_0, \ldots, y_R), y' = (y'_0, \ldots, y'_R) \in W.$

The dependence of $D_{m_2}^{m_1}$ on $n$ is not explicitly written in this notation. But we believe that it does not cause a confusion.
7.2. **Successive perturbations.** For a finite set $A$ we define $\mathbb{C}[A]$ as the vector space of all maps from $A$ to $\mathbb{C}$. This is isomorphic to $\mathbb{C}^{|A|}$. The following lemma is based on Lemma 4.5.

**Lemma 7.4.** For all integers $n \geq 1$ and $-M \leq r \leq M$ we can construct simplicial maps

$$F_{n,r} : P^n \times W \to \mathbb{C} \left[ \left( \frac{1}{\rho} \right) \cap [0,nN] \right]$$

satisfying the following.

(1) For all $x \in X$, $y \in W$ and $t \in (1/\rho)Z \cap [0,nN)$

$$|F_{n,r}(\Pi_n(x), y)(t) - f(x)(t)| < \delta'.$$

(2) Let $0 \leq k \leq R$ and $n'$ be integers with $n_k \leq n' \leq n$. For any $-M \leq r < M$ and $0 \leq c \leq 1$ the following map is an $\varepsilon'$-embedding with respect to $D_1^n$:

$$P^{n+1} \times W_k \to \mathbb{C} \left[ \left( \frac{1}{\rho} \right) \cap [N,n'N] \right]$$

$$(x_0, \ldots, x_n, y) \mapsto (1 - c)F_{n,r}(x_0, \ldots, x_{n-1}, y)|_{(1/\rho)Z \cap [N,n'N]} + cF_{n,r+1}(x_1, \ldots, x_n, y)|_{(1/\rho)Z \cap [0,(n'-1)N]}.$$

The right-hand side is the function whose value of $t \in (1/\rho)Z \cap [N,n'N]$ is

$$(1 - c)F_{n,r}(x_0, \ldots, x_{n-1}, y)(t) + cF_{n,r+1}(x_1, \ldots, x_n, y)(t - N).$$

Note that the variables of $F_{n,r+1}$ are $x_1, \ldots, x_n, y$ (not $x_0, \ldots, x_{n-1}, y$).

**Proof.** First note that the above two conditions (1) and (2) are stable under sufficiently small perturbations of $F_{n,r}$. The maps $F_{n,r}$ will be constructed by successive perturbations. Once the maps satisfy the conditions, their small perturbations also satisfy them.

By Lemma 4.1 and the choice of $\varepsilon$ in (7.2), there exists a simplicial map

$$F : P \to \mathbb{C} \left[ \left( \frac{1}{\rho} \right) \cap [0,N] \right]$$

satisfying $|F(\Pi(x))(t) - f(x)(t)| < \delta'$ for all $x \in X$ and $t \in (1/\rho)Z \cap [0,N)$. For $n < n_0$ we set $F_{n,r}(x, y) = (F(x_0), \ldots, F(x_{n-1}))$ for $x = (x_0, \ldots, x_{n-1}) \in P^n$ and $y \in W$. This notation means that

$$F_{n,r}(x, y)(i) = F(x_i)(i - N), \quad (0 \leq i < n, t \in (1/\rho)Z \cap [iN,(i + 1)N]).$$

We will use similar notations below. These $F_{n,r}$ satisfy the required conditions since the condition (2) is empty for $n < n_0$. So we assume $n \geq n_0$ and that we have constructed $F_{n-1,r}$ for all $-M \leq r \leq M$. We try to construct $F_{n,r}$.

Consider

$$(F_{n-1,r}, F) : P^n \times W \to \mathbb{C} \left[ \left( \frac{1}{\rho} \right) \cap [0,nN] \right],$$

$$(x_0, \ldots, x_{n-1}, y) \mapsto (F_{n-1,r}(x_0, \ldots, x_{n-2}, y), F(x_{n-1})).$$
These satisfy the condition (1) and also the condition (2) for \( n_k \leq n' \leq n - 1 \). So we will construct \( F_{n,r} \) by slightly perturbing \((F_{n-1,r}, F)\). Consider the following condition:

(3) Take integers \(-M \leq r < M\), \(0 \leq k \leq R\) with \( n_k \leq n \) and a real number \(0 \leq c \leq 1\). The following map is an \( \varepsilon' \)-embedding with respect to \( D_{11}^{n-1} \).

\[
P^n \times W_k \to \mathbb{C} \left[ \left( \frac{1}{\rho} \mathbb{Z} \right) \cap [N, nN] \right]
\]

\[
((x_0, \ldots, x_{n-1}), y) \mapsto (1-c)F_{n,r}(x_0, \ldots, x_{n-1}, y)\big|_{\left( \frac{1}{\rho} \mathbb{Z} \cap [N, nN] \right)} + cF_{n-1,r+1}(x_1, \ldots, x_{n-1}, y)\big|_{\left( \frac{1}{\rho} \mathbb{Z} \cap [0,(n-1)N] \right)}.
\]

The main difference between the conditions (2) and (3) is that \( F_{n,r+1}(x_1, \ldots, x_n, y) \) in (2) is replaced with \( F_{n-1,r+1}(x_1, \ldots, x_{n-1}, y) \) in (3).

Note that the real dimension of \( \mathbb{C}[\left( \frac{1}{\rho} \mathbb{Z} \cap [N, nN] \right)\] is \(2(n-1)\rho N\). By Corollary 4.4 and the choice of \( n_k \) in (7.3), we can assume that the condition (3) is satisfied for \( c = 1 \) after replacing the maps \( F_{n-1,r+1} \) by small perturbations (if necessary).

By using Lemma 4.3 and (7.3), we can construct \( F_{n-M} \) as a small perturbation of \((F_{n-1,-M}, F)\) so that it satisfies the condition (3) for \( r = -M \). Then, if \( F_{n-M+1} \) is a sufficiently small perturbation of \((F_{n-1,-M+1}, F)\), the condition (2) is satisfied for \( r = -M \).

Moreover we can assume that it satisfies the condition (3) for \( r = -M + 1 \) by the same reason. By continuing this process, we can construct \( F_{n,r} \) inductively (with respect to \( r \)) so that they satisfy the required properties.

For \(-M \leq r \leq M\) we set

\[
G_r = F_{M,r} : P^M \times W \to \mathbb{C} \left[ \left( \frac{1}{\rho} \mathbb{Z} \right) \cap [0, MN] \right].
\]

Indeed any \( F_{n,r} \) will do the same work if \( n \) is sufficiently large. We use the choice \( F_{M,r} \) because \( |I(x,n)| < M \) by Lemma 6.2 (1).

7.3. Construction of the map \( g \). Take \( x \in X \). We will define \( g(x) \in B_1(V[a,b]) \). We define \( E(x) \) as the set of integers \( n \) with \( I(\Phi(x), n) \neq \emptyset \). Take \( n \in E(x) \). We set

\[
I(\Phi(x), n) = [\alpha_{x,n}, \beta_{x,n}], \quad r_{x,n} = \left\lfloor \frac{\alpha_{x,n} - n}{N} \right\rfloor, \quad s_{x,n} = \left\lfloor \frac{\beta_{x,n} - n}{N} \right\rfloor.
\]

We define \(0 \leq c_{x,n}, c'_{x,n} < 1\) by

\[
c_{x,n} = \frac{n + r_{x,n}N - \alpha_{x,n}}{N}, \quad c'_{x,n} = \frac{\beta_{x,n} - n - s_{x,n}N}{N}.
\]

See Figure 7.2.

Let \( \delta_0 \) and \( \delta_1 \) be the delta measures on the two-points space \( \{0, 1\} \) concentrated at 0 and 1 respectively. We define a probability measure on \( \{0, 1\} \times \{0, 1\} \) by

\[
\mu_{x,n} = (c_{x,n}\delta_0 + (1-c_{x,n})\delta_1) \times (c'_{x,n}\delta_0 + (1-c'_{x,n})\delta_1).
\]
We define a probability measure on $\prod_{n \in E(x)} \{0,1\}^2$ by
\[
\mu_x = \prod_{n \in E(x)} \mu_{x,n}.
\]

Take
\[
\theta \in \prod_{n \in E(x)} \{0,1\}^2, \quad \theta = ((\theta_n, \theta'_n))_{n \in E(x)}, \quad \theta_n \in \{0,1\}, \theta'_n \in \{0,1\}.
\]

For $n \in E(x)$ we define
\[
\Lambda(x, \theta, n) = n + \left( \left( \frac{1}{\rho} \mathbb{Z} \right) \cap [(r_{x,n} + \theta_n)N, (s_{x,n} - \theta'_n)N] \right) \subset I(\Phi(x), n).
\]

When $r_{x,n} + \theta_n \geq s_{x,n} - \theta'_n$, this is empty. See Figure 7.3. We define a subset of $\mathbb{R}$ by
\[
\Lambda(x, \theta) = \bigcup_{n \in E(x)} \Lambda(x, \theta, n).
\]

The distance between any two distinct points of $\Lambda(x, \theta)$ is $\geq 1/\rho$. So for any $\lambda \in \Lambda(x, \theta)$ the set $-\lambda + \Lambda(x, \theta)$ satisfies Conditions 5.2 (1) and (2). Let $\varphi_{-\lambda + \Lambda(x, \theta)} \in V[-(\rho +
Let \( \tau/2, (\rho + \tau)/2 \) be the interpolation function introduced in (5.4). We define \( \varphi_{x,\theta,\lambda} \) by
\[
\varphi_{x,\theta,\lambda}(t) = \exp \left( 2\pi \sqrt{-1} \frac{a + b}{2} (t - \lambda) \right) \varphi_{-\lambda + \Lambda(x,\theta)}(t - \lambda).
\]
This satisfies
- \( \varphi_{x,\theta,\lambda} \in V[a, b] \) because \( \rho + \tau < b - a \).
- \( \varphi_{x,\theta,\lambda}(\lambda) = 1 \) and \( \varphi_{x,\theta,\lambda}(\lambda') = 0 \) for all \( \lambda' \in \Lambda(x,\theta) \setminus \{\lambda\} \).
- \( \varphi_{x,\theta,\lambda} \) is rapidly decreasing and
\[
|\varphi_{x,\theta,\lambda}(t)| \leq \frac{K}{1 + |t - \lambda|^2}.
\]
Let \( w(\Phi(x)) = (w_n)_{n \in \mathbb{Z}} \), \( w = (w_{n0}, \ldots, w_{nR}) \in [0, 1]^R \), be the weight introduced in Lemma 6.4. Let \( n \in E(x) \). We set
\[
y_{x,n} = (w_{n0}\pi(T^n x), w_{n1}\pi(T^{n+1} x), \ldots, w_{nR}\pi(T^{n+R} x)) \in W = (CQ)^R.
\]
For \( \lambda \in \Lambda(x,\theta, n) \) we set
\[
u(x, \theta, n, \lambda) = G_{r_{x,n} + \theta_n} \left( \Pi_M(T^{n+(r_{x,n}+\theta_n)N} x), y_{x,n} \right) (\lambda - n - (r_{x,n} + \theta_n)N) - f(x)(\lambda).
\]
Note
\[
f(x)(\lambda) = f(T^{n+(r_{x,n}+\theta_n)N} x) (\lambda - n - (r_{x,n} + \theta_n)N).
\]
Hence by Lemma 7.4 (1)
\[
|\nu(x, \theta, n, \lambda)| < \delta'.
\]
We define a function \( g(x, \theta) \) in \( V[a, b] \) by
\[
g(x, \theta)(t) = f(x)(t) + \sum_{n \in E(x)} \sum_{\lambda \in \Lambda(x,\theta, n)} \nu(x, \theta, n, \lambda) \varphi_{x,\theta,\lambda}(t).
\]
From (7.1), (7.5) and (7.6)
\[
|g(x, \theta)(t) - f(x)(t)| < \delta.
\]
Finally we define \( g(x) \in V[a, b] \) by
\[
g(x) = \int_{\Pi_{n \in E(x)} \{0,1\}^2} g(x, \theta) \, d\mu_x(\theta).
\]
This satisfies \( |g(x)(t) - f(x)(t)| < \delta \). Since \( |f(x)(t)| \leq 1 - \delta \), we have \( g(x) \in B_1(V[a, b]) \).

For every \( n \in E(x) \) with \( r_{x,n} + 1 < s_{x,n} - 1 \)
\[
g(x) \mid_{n+((1/\rho)\mathbb{Z}) \cap (s_{x,n}-1)N} = c_{x,n} G_{r_{x,n}} \left( \Pi_M(T^{n+(r_{x,n}+\theta_n)N} x), y_{x,n} \right) \mid_{(1/\rho)\mathbb{Z} \cap N(s_{x,n}-r_{x,n}-2)N}
+ (1 - c_{x,n}) G_{r_{x,n}+1} \left( \Pi_M(T^{n+(r_{x,n}+1)N} x), y_{x,n} \right) \mid_{(1/\rho)\mathbb{Z} \cap (0, s_{x,n}-r_{x,n}-2)N}.
\]

Lemma 7.5. The map
\[
X \ni x \mapsto g(x) \in B_1(V[a, b])
\]
is equivariant and continuous.
Proof. The check of the equivariance is direct. We have $I(\Phi(Tx), n) = -1 + I(\Phi(x), n+1)$. Hence $E(Tx) = -1 + E(x)$ and for $n \in E(Tx)$
\[
    r_{Tx,n} = r_{x,n+1}, \quad s_{Tx,n} = s_{x,n+1}, \quad c_{Tx,n} = c_{x,n+1}, \quad c'_{Tx,n} = c'_{x,n+1}.
\]
We have a one to one correspondence between $\prod_{n \in E(x)} \{0, 1\}^2$ and $\prod_{n \in E(Tx)} \{0, 1\}^2$ by
\[
    \theta \longleftrightarrow \tilde{\theta}, \quad (\tilde{\theta}_n, \tilde{\theta}'_n) = (\theta_{n+1}, \theta'_{n+1}).
\]
Under this identification, we have $\mu_{Tx} = \mu_x$. We can check the following.
\[
    \Lambda(Tx, \tilde{\theta}, n) = -1 + \Lambda(x, \theta, n+1), \quad \Lambda(Tx, \tilde{\theta}) = -1 + \Lambda(x, \theta), \quad \varphi_{Tx, \tilde{\theta}, \lambda}(t) = \varphi_{x, \theta, \lambda+1}(t+1),
\]
\[
    y_{Tx,n} = y_{x,n+1} \quad \text{by Lemma \ref{lem:transient} (1),} \quad u(Tx, \tilde{\theta}, n, \lambda) = u(x, \theta, n+1, \lambda+1).
\]
Then
\[
    g(Tx, \tilde{\theta})(t) = g(x, \theta)(t + 1), \quad g(Tx)(t) = g(x)(t + 1).
\]

The proof of the continuity is slightly involved. Let $x \in X$. Discontinuity appears in the two places of the above construction.

- If $I(\Phi(x), n)$ is one point, then it may become empty after $x$ moves slightly.
- The integers $r_{x,n}$ and $s_{x,n}$ may jump when $c_{x,n} = 0$ or $c'_{x,n} = 0$.

The first issue causes no problem because $\Lambda(x, \theta, n)$ is empty and does not contribute to the value of $g(x)$ if $|I(\Phi(x), n)| = 0$. The second issue is more serious and causes a problem that $g(x, \theta)$ does not depend continuously on $x$. We introduced the probability measure $\mu_{x,n}$ for dealing with this problem. Let $C$ and $C'$ be the sets of integers $n \in E(x)$ satisfying $c_{x,n} = 0$ and $c'_{x,n} = 0$ respectively. These are the positions where the difficulty occurs.

Let $A$ and $\eta$ be positive numbers. Suppose $x' \in X$ is sufficiently close to $x$. We want to show $|g(x')(t) - g(x)(t)| < \eta$ for $|t| \leq A$. Let $B > 0$ be a sufficiently large number. We can assume $E(x') \cap [-A - B, A + B] \subseteq E(x) \cap [-A - B, A + B]$ and that every integer $n$ in the difference of these two sets satisfies $|I(\Phi(x), n)| = 0$. This means that these two sets are essentially equal.

Take $\theta \in \prod_{n \in E(x') \cap \{0, 1\}} \{0, 1\}^2$. We define $\Theta(x', \theta) \in \prod_{n \in E(x')} \{0, 1\}^2$ as follows.

- For $|n| > A + B$ we set $(\Theta(x', \theta)_n, \Theta(x', \theta)'_n) = (0, 0)$.
- Let $|n| \leq A + B$. If $n \not\in C$ then $\Theta(x', \theta)_n = \theta_n$. If $n \not\in C'$ then $\Theta(x', \theta)'_n = \theta'_n$.
- For $n \in C \cap [-A - B, A + B]$, we define $\Theta(x', \theta)_n \in \{0, 1\}$ by
  \[
  r_{x',n} + \Theta(x', \theta)_n = r_{x,n} + 1.
  \]
- For $n \in C' \cap [-A - B, A + B]$ we define $\Theta(x', \theta)'_n \in \{0, 1\}$ by
  \[
  s_{x',n} - \Theta(x', \theta)'_n = s_{x,n} - 1.
  \]
Note that for \( n \in \mathcal{C} \cap [-A-B, A+B] \) the number \( c_{x',n} \) is very close to 0 or 1, and that the measure \( c_{x',n} \delta_0 + (1 - c_{x',n}) \delta_1 \) is almost equal to the delta measure at \( \Theta(x', \theta) \). Similarly for \( n \in \mathcal{C}' \cap [-A-B, A+B] \). Then we can assume that for \( |t| \leq A \)

$$
\left| \int_{\Pi_E(\mathcal{C}) \times \{0,1\}^2} g(x', \theta)(t)d\mu_{x'}(\theta) - \int_{\Pi_E(\mathcal{C}) \times \{0,1\}^2} g(x', \Theta(x', \theta))(t)d\mu_{x'}(\theta) \right| < \eta/3.
$$

Here we have also used Lemma 5.5 (with the assumption \( B \gg 1 \)) and (7.5). Since the two sets \( E(x) \cap [-A-B, A+B] \) and \( E(x') \cap [-A-B, A+B] \) are essentially equal, applying Lemma 5.5 and (7.5) again, we get

$$
\left| \int_{\Pi_E(\mathcal{C}) \times \{0,1\}^2} g(x', \Theta(x', \theta))(t)d\mu_{x'}(\theta) - \int_{\Pi_E(\mathcal{C}) \times \{0,1\}^2} g(x, \Theta(x, \theta))(t)d\mu_x(\theta) \right| < \eta/3
$$

for \( |t| \leq A \). Thus \( |g(x')(t) - g(x)(t)| < \eta \) for \( |t| \leq A \). \(\square\)

The rest of the task is to show that the map

\( (g, \Phi) : X \to B_1(V[a, b]) \times Y, \quad x \mapsto (g(x), \Phi(x)) \)

is a \( \delta \)-embedding with respect to \( \delta \). Suppose \( x, x' \in X \) satisfy \( g(x), \Phi(x) = (g(x'), \Phi(x')) \). We want to show \( d(x, x') < \delta \). Let \( w(\Phi(x)) = w(\Phi(x')) = (w_n)_{n \in \mathbb{Z}}. \) We divide the argument into two cases, according to whether the origin is tame or wild.

**Case 1.** Suppose \( \text{dist}(0, \partial (\Phi(x))) > L_0 - 4 = n_0 N > 2N. \) Take an integer \( n \) with \( 0 \in \Pi_{\Phi(x), n} \). Then \( |I(\Phi(x), n)| > 2n_0 N \) and hence \( s_{x, n} - r_{x, n} > n_0 \). Let \( k \) be the maximum integer satisfying \( n_k < s_{x, n} - r_{x, n} \). By Lemma 7.3 the points \( y_{x, n} \) and \( y_{x', n} \) belong to \( W_k \). Then by (7.4) and Lemma 7.4 (2) (with \( n' = s_{x, n} - r_{x, n} - 1 \geq n_k \) ), we get

\[
D_{1}^{s_{x, n} - r_{x, n} - 2} \left( \left( \Pi_M(T^{n + r_{x, n}}, x), y_{x, n} \right), \left( \Pi_M(T^{n + r_{x, n} N}, x'), y_{x', n} \right) \right) < \varepsilon'.
\]

From the second condition on \( \varepsilon' \) in (7.4), this implies that for all integers \( i \) with \( n + (r_{x, n} + 1) N \leq i < n + (s_{x, n} - 1) N \)

\[
d(T^i x, T^i x') < \varepsilon.
\]

Since \( \text{dist}(0, \partial I(\Phi(x), n)) > n_0 N > 2N \), the origin is contained in \( [n + (r_{x, n} + 1) N, n + (s_{x, n} - 1) N] \). Thus we get \( d(x, x') < \varepsilon < \delta \). Note that the points \( y_{x, n} \) and \( y_{x', n} \) do not play any role in this argument. They will become crucial in Case 2.

**Case 2.** Suppose \( \text{dist}(0, \partial (\Phi(x))) \leq L_0 - 4. \) By Lemma 6.3 (4) there exists an integer \( n \in [-R, 0] \) with \( w_{n, -n} = 1 \). By Lemma 6.3 (2) this implies \( |I(\Phi(x), n)| > L_1 > C_2 N + 2N. \) Then \( s_{x, n} - r_{x, n} > C_2 > n_0. \) Let \( k \) be the maximum integer satisfying \( n_k < s_{x, n} - r_{x, n} \). By Lemma 7.3 \( y_{x, n}, y_{x', n} \in W_k \). By (7.4) and Lemma 7.4 (2)

\[
D_{1}^{s_{x, n} - r_{x, n} - 2} \left( \left( \Pi_M(T^{n + r_{x, n}}, x), y_{x, n} \right), \left( \Pi_M(T^{n + r_{x, n} N}, x'), y_{x', n} \right) \right) < \varepsilon'.
\]

This is the same as in Case 1. But the next step is different. Since \( w_{n, -n} = 1 \), the points \( \pi(x) = w_{n, -n} \pi(x) \) and \( \pi(x') = w_{n, -n} \pi(x') \) appear as the \( n \)-th entries of \( y_{x, n} \) and \( y_{x', n} \).
respectively. Therefore
\[ D(\pi(x), \pi(x')) \leq D^{sx,n-rx,n-2} \left( \left( \prod_M (T^{n+rx,n}x), y_{x,n} \right), \left( \prod_M (T^{n+rx,n}x'), y_{x',n} \right) \right) < \varepsilon'. \]

By the first condition on \( \varepsilon' \) in (7.4), we finally get \( d(x, x') < \varepsilon < \delta \).

We have completed the proof of Proposition 7.2. Thus Theorem 3.1 has been proved.

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