Cosmological Effective Hamiltonian from full Loop Quantum Gravity Dynamics

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The concept of effective dynamics has proven successful in LQC, a loop-inspired quantization of cosmological spacetimes. We apply the same idea in the full theory, by computing the expectation value of the scalar constraint with respect to some coherent states peaked on the phase-space variables of flat Robertson-Walker spacetime. We comment on the relation with effective LQC and find a deviation stemming from the Lorentzian part of the Hamiltonian.

I. INTRODUCTION

Loop Quantum Gravity (LQG) has over the last decades evolved into a rich and mathematically well-defined theory of quantum gravity. While its backbone has been set up through a lot of work in the framework of the full theory \cite{[1–3]}, simultaneously its symmetry reduced sector has pushed the field into the direction of observable predictions and managed to resolve some of the open questions from the classical theory. For example, in Loop Quantum Cosmology (LQC) the Big Bang singularity is resolved by the “Big Bounce scenario” \cite{[4–6]}.

In this approach one deals with states representing 3-dimensional geometries: the data on a spatial Cauchy slice of the spacetime. While different choices of such slices do not affect physical quantities per se, depending on the system there exist convenient choices. This amounts to performing a gauge-fixing of the constraints via some degrees of freedom (usually provided by matter), which therefore play the role of physical clocks and rods (however depending on the matter choice the observables may differ). If these degrees of freedom are sufficiently well behaved, then the resulting physical Hamiltonian is independent of them \cite{[7, 8]}.

At this point, the theory is reduced to a \((\text{infinite-dimensional})\) version of quantum mechanics: a quantum \(3\)-geometry is a physical state, \(\ket{\psi}\), and its evolution (wrt the physical time \(\phi\)) is given by a physical Hamiltonian \(H\) (derived from the scalar constraint) by

\[
|\psi(\phi)\rangle = e^{-i\phi H} |\psi(0)\rangle
\]

One can then ask how quantities of interest (e.g., geometrical operators such as volume \(V\) of the universe) change in time:

\[
V_\phi(\phi) := \langle \psi(\phi) | \hat{V} | \psi(\phi) \rangle
\]

Computations such as (2) are numerically possible in the context of LQC, and in the seminal paper \cite{[9]} it was shown that the quantum evolution of the expectation value of observables (in particular, the volume of the universe) on certain coherent states labelled by loop variables \((p, c)\) follows closely some “quantum-corrected” trajectories in phase space.\(^1\) These trajectories correspond to the integral curves of an effective Hamiltonian. Such Hamiltonian is not the classical cosmological one – which in terms of \((p, c)\) reads

\[
H_{cl}(p, c) = -\frac{6}{\kappa \beta^2} \sqrt{p c^2}
\]

(with \(\kappa = 16 \pi G / c^3\) the gravitational coupling constant and \(\beta\) the Immirzi parameter) – but rather the phase-space function obtained by taking the expectation value of the quantum Hamiltonian \(\hat{H}_{LQC}\) on a semiclassical coherent state, \(\psi(p, c)\):

\[
H_{\text{eff}}(p, c) := \langle \psi(p, c) | \hat{H}_{LQC} | \psi(p, c) \rangle = -\frac{6}{\kappa \beta^2} \sqrt{p} \frac{\sin^2(\mu)}{\mu^2}
\]

where \(\mu\) is a parameter of the quantum theory. The authors of \cite{[9]} refer to this as the “effective Hamiltonian”, and to the evolution it produces (which is equivalent to a corrected version of Friedmann equations) as “effective dynamics”. In particular, it is possible to show that \(H_{\text{eff}}\) bridges a contracting universe in the far past (which would evolve towards a Big Crunch according to \(H_{cl}\)) with an expanding branch in the far future (which would come from a Big Bang according to \(H_{cl}\)) via an intermediate “bouncing” region during which the energy density reaches the Planckian regime \cite{[9]}. The cosmological singularity is then averted by this so-called “Big Bounce”.

\(^1\) The relation between \((p, c)\) and the Ashtekar-Barbero variables is obtained by fixing a fiducial Minkowski metric. One then finds that the connection and the densitized triad are given by

\[
\Lambda^i_a = \alpha \delta^i_a, \quad E^a_i = p \delta^a_i
\]

where we set the coordinate volume of the universe to 1 for simplicity. The variables \(c\) and \(p\), which will depend on the (cosmological) time, thus encode all the dynamics of the model.
II. HAMILTONIAN OPERATOR IN LQG

Inspired by this success, one may ask whether the same qualitative behavior is also found in the full version of the theory. In order to make this comparison with LQC, we choose an embedded cubic graph $\gamma$ and use its 3 directions to define the 3 axes of coordinates of a compact manifold $\sigma$ with periodic boundary conditions, i.e., a 3-Torus. Hence $\gamma$ has a finite number of vertices, $N^3$, and setting its coordinate volume to 1 we find that the coordinate distance between two nearby vertices is $\mu := 1/N^3$. The kinematical Hilbert space is $\mathcal{H} := \otimes H_e$ where on every edge $e$ we have $H_e = L_2(SU(2), d\mu_H)$ with $d\mu_H$ being the Haar-measure. On each $H_e$ we have the action of the holonomy operator (associated with connection $A^I_a = \Gamma^I_a + \beta K^I_a e_I^a$) and the flux operator (associated with densitized triad $E_a^i$): for $f_e \in H_e$,

$$\hat{h}_{mn}(e) f_e(g) = D_{mn}^{(1)}(g) f_e(g)$$

(6)

and

$$-\frac{4}{i\hbar \beta} \hat{E}^k(e) f_e(g) = R^k f_e(g) := \frac{d}{ds} |_{s=0} f_e(e^{-i\sigma_k g})$$

(7)

where $D_{mn}^{(1)}$ is the Wigner matrix of group element $g$ in spin-$1/2$ $SU(2)$-irrep, $R^k$ is the right-invariant vector field, and $\sigma_k$ are the Pauli matrices.

In this framework of the Ashtekar-Barbero variables [10–14], the classical scalar constraint $H$ can be written as

$$H = H_E - (\beta^2 + 1) H_L$$

(8)

where

$$H_E = F^{I} a_{\nu IJK} \frac{E^a_a E^b_b}{\sqrt{\det(E)}}$$

(9)

$$H_L = \epsilon_{IMN} K^M_a K^N_b \frac{E^a_a E^b_b}{\sqrt{\det(E)}}$$

(10)

are called the Euclidean and Lorentzian part respectively. Here, $F$ denotes the Lie algebra valued curvature of connection $A$. Now, Dirac quantization scheme is employed to promote the function $H$ to an operator on $\mathcal{H}$, whose details depend on the choice of regularization. In this paper we focus on a non-graph-changing version of [15, 16], which was first proposed in [17]. Thus, using the Thiemann identities, we obtain the following operator:

$$\hat{H} := \frac{1}{2} \left( \hat{H}_E + \hat{H}_L \right) - \frac{\beta^2 + 1}{2} \left( \hat{H}_L + \hat{H}_E \right)$$

(11)

where

$$\hat{H}_E(N_v) = \frac{32}{3i\hbar \beta} \sum_{e \in V(\gamma)} N_v^{20} \sum_{e', \mu' \in \gamma} \epsilon(e, e', \mu') \frac{1}{2} \times$$

$$\times Tr \left( \left( \hat{h}(\square_{ee'}) - \hat{h}(\square_{ee'})^{-1} \right) \hat{h}(e') \hat{h}(e''')^{-1} \hat{V}_e \right)$$

(12)

and

$$\hat{H}_L(N_v) = \frac{128}{3i\hbar \beta^5} \sum_{e \in V(\gamma)} N_v^{20} \sum_{e', \mu' \in \gamma} \epsilon(e, e', \mu') \times$$

$$\times Tr \left( \left( \left( \hat{h}(\square_{ee'}) - \hat{h}(\square_{ee'})^{-1} \right) \hat{h}(e') \hat{h}(e''')^{-1} \hat{V}_e \right) \right)$$

(13)

Here, $\epsilon_e$ is the lapse function, $V(\gamma)$ the set of all vertices of the graph $\gamma$, $\hat{h}(\square_{ee'})$ the holonomy operator along the plaquette starting at $e$ along $e'$ and returning along $e''$, and $V_e$ the Ashtekar-Lewandowski volume operator [18, 19].

III. COHERENT STATES AND VOLUME

To follow the program of effective dynamics, we now must choose a set of coherent states $\Psi(A,E) \in \mathcal{H}$ and evaluate the expectation value of $\hat{H}$ on them. A choice which is peaked on both holonomies and fluxes is the complexiﬁer coherent states, developed by Thiemann and Winkler [20–23] on the basis of Hall’s work [24, 25]. These are labelled on every edge by $h_e \in SL(2, \mathbb{C})$, which can be written in the holomorphic decomposition as:

$$h_e = n_e e^{-i\sigma_3/2} n'_e$$

(14)

with $n_e, n'_e \in SU(2)$ and $z_e \in \mathbb{C}$. Then, we can write explicitly for the coherent state

$$\Psi(A,E) := \bigotimes_{e \in \gamma} \psi_{e, h_e}$$

(15)

where

$$\psi_{e, h_e}(g) = \frac{1}{\sqrt{N}} \sum_{j=0}^{\infty} d_j e^{-j(j+1)t} \sum_{m=-j}^j e^{izm} D^{(j)}_{mm} (n_e g n'_e)$$

(16)

with $N^2 = |\psi_{e, h_e}|^2$ the normalization of the state and $d_j = 2j + 1$ the dimension of spin-$j$ $SU(2)$-irrep. The dimensionless quantity $t \in \mathbb{R}^+$ is the semiclassicality parameter.$^2$

The choice of these coherent states is not only justified by the fact that they are peaked in the elementary operators, but also that it simplifies the analysis of the

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$^2$ For reasons to be discussed in an extended companion paper, $t$ should be identified with a ratio between areas:

$$t = \frac{t_p^2}{a^2}$$

(17)

where $t_p^2 = \hbar c$ is Planck length and $a$ is another scale of units of length that the theory should provide. For example, if $\Lambda$ is the cosmological constant, we can set $a = \Lambda^{-\frac{1}{2}}$, so we find $t = t_p^2 \Lambda \sim 10^{-129}$. 
quantum Hamiltonian: the AL-volume operator appearing in it is at every vertex \( v \) explicitly given by

\[
\hat{V}_v = \sqrt{|\hat{Q}_v|}
\]

where

\[
\hat{Q}_v = i\frac{\hbar c^3}{240^3} \sum_{e \in \mathcal{E}(\hat{g}) \cap \mathcal{E}(\hat{g}')} \epsilon(e, e', e'') \epsilon_{ijk} \hat{R}^i(e) \hat{R}^j(e') \hat{R}^k(e'')
\]

with \( \epsilon(e, e', e'') := \text{sgn}(\det(\dot{\hat{g}}, e', e'')) \). Since the square root is understood in the sense of the spectral theorem, knowledge of the full spectrum of \( \hat{Q}_v \) is required before we can say how \( \hat{V}_v \) acts in general. Unfortunately, a lot of analysis on the spectrum of (19) has been done [26, 27], a general formula for its eigenstates is still unknown. However, since we are interested in the expectation value on complexifier coherent states, we can make use of the result presented in [28]: there it was shown that for every polynomial \( P \) one has

\[
\langle \Psi_{(A,E)} | P(\hat{V}_v, \hat{h}) | \Psi_{(A,E)} \rangle \approx \langle \Psi_{(A,E)} | P(\hat{V}_{k,v}^{GT}, \hat{h}) | \Psi_{(A,E)} \rangle
\]

(20)

where the error is of order \( O(t^{k+1}) \) and the \( k \)-th Giesel-Thiernann volume operator, \( \hat{V}_{k,v}^{GT} \), is explicitly given by

\[
\hat{V}_{k,v}^{GT} := \langle \hat{Q}_v \rangle^{1/2} \times \left[ 1 + \sum_{n=1}^{2k+1} (-1)^{n+1} \frac{n+1}{4^4} \frac{3}{4} \left( n - 1 - \frac{1}{4} \right) \left( \frac{\hat{Q}_v^2}{\langle \hat{Q}_v \rangle^2} - 1 \right)^n \right]
\]

(21)

with \( \langle \hat{Q}_v \rangle \) a shorthand for the expectation value of \( \hat{Q}_v \) wrt to the complexifier coherent state \( \Psi_{(A,E)} \). Applying this theorem to \( \hat{H} \), we obtain a computable operator and retain control on the error we make in terms of powers of the semiclassicality parameter \( t \).

\[\text{IV. EFFECTIVE HAMILTONIAN}\]

In principle, we have all we need: we can compute the effective Hamiltonian from the full theory (on a cubic graph), \( H_{\text{eff}}(A, E) \). Given the generic form of the complexifier coherent states, however, this is a very hard task. For this reason (and to allow comparison with LQC effective dynamics), we limit ourselves to the homogeneous isotropic case. This is achieved by choosing very specific labels \( h_{\eta} \) for our coherent states (16): using the notation in (14), we choose \( z_\eta = z = \xi + i\eta \) a complex number for all \( e \) and \( n_\eta = n_{(\hat{g})} \) with \( n_{(\hat{g})} \) being the \( SU(2) \) element that rotates the \( z \)-axis into the tangent \( \dot{\hat{g}} \) to edge \( e \). Then, it is a matter of computation (which will be presented in an extended companion paper to appear soon) to check that these states are indeed peaked on the elementary operators:

\[
\langle \hat{h}_{mn}(e) \rangle = D^{(2)}_{mn}(n_e e^{-i\xi \sigma_3 / 2} n'_e) [1 + \mathcal{O}(t)]
\]

(22)

\[
\langle \hat{E}_k(e) \rangle = a^2 \beta \eta D^{(1)}_{k0}(n_e) [1 + \mathcal{O}(t)]
\]

(23)

and their spread goes like \( t \) (no summation over repeated indices)

\[
\frac{\langle \hat{h}_{mn} \hat{h}_{mn} \rangle}{\langle \hat{h}_{mn} \rangle^2} - 1 = tf(\eta, \xi, n_e)
\]

(24)

\[
\frac{\langle \hat{E}_k \hat{E}_k \rangle}{\langle \hat{E}_k \rangle^2} - 1 = t\tilde{g}(\eta, n_e)
\]

(25)

with analytic functions \( f \) and \( \tilde{g} \). This sharp peakedness also allows us to deal with the Gauss-constraint of general relativity: since we are interested in observables which are themselves gauge-invariant (i.e., \( U(g) \hat{H} U(g) = \hat{H} \) for any gauge transformation \( U(g) \)), and since the coherent states are gauge-covariant (i.e., \( U(g) \psi_{(\hat{g})} = \psi_{(\hat{g})} \)), then one can see that group averaging produces

\[
\int dg \int dg' \langle \psi_{(\hat{h})} | U(g) \hat{H} U(g') | \psi_{(\hat{h})} \rangle \approx \langle \psi_{(\hat{h})} | \hat{H} | \psi_{(\hat{h})} \rangle
\]

(26)

with an error of order \( O(t) \). (Further details can be found in [29, 30]). This guarantees that our results have physical significance without having to solve the Gauss-constraint.

Looking at (22) and (23), we see that they coincide with the holonomy and flux computed in a classical flat Robertson-Walker spacetime described by \( (p, c) \) if we perform the identification

\[
\xi = \mu c, \quad \eta = \frac{\mu^2 p}{a^2 \beta}
\]

(27)

Under this identification, one also finds the following result for the expectation value of volume:

\[
\langle \hat{V} \rangle = N^3 p^3 p^{3/2} [1 + \mathcal{O}(t)]
\]

(28)

which confirms that we should set \( N\mu = 1 \) (so that the physical volume of the universe is given by \( p^{3/2} \), the classical result). Under this identification, the expectation value of Hamiltonian is found to be (details of this computation will be in the companion paper to appear soon)

\[
H_{\text{eff}}(p, c) := \langle \hat{H} \rangle = \text{Re} \left( \langle \hat{H}_E \rangle - (\beta^2 + 1) \langle \hat{H}_L \rangle \right)
\]

(29)

with

\[
\langle \hat{H}_E \rangle \propto \sqrt{\frac{p}{\kappa}} \frac{\sin(c \mu)^2}{\mu^2} + \mathcal{O}(t)
\]

\[
\langle \hat{H}_L \rangle \propto \sqrt{\frac{p}{\kappa}} \frac{\sin(2c \mu)^2}{4\mu^2} + \mathcal{O}(t)
\]

(30)

We notice several things: (1) at leading order, \( H_E \) is consistent with the result obtained in LQC, equation (5), though there are corrections of order \( O(t) \); (2) in the classical and continuum limit \( (t, \mu \to 0) \), \( H_L \) and \( H_E \) coincide (up to a numerical factor), which is consistent with classical cosmology; (3) if \( \mu \neq 0 \), however, the functional form of \( H_L \) is different from that of \( H_E \) in regard of the \( c \)-dependence, due to the different factor in the argument of sine. This is a modification to the established
effective LQC Hamiltonian. A corresponding modification to the quantum LQC Hamiltonian $H_{LQC}$ leads to a difference equation of higher order, whose quantum dynamics is currently being numerically investigated. However, we should note that the Euclidean Hamiltonian in (29) reproduces LQC Hamiltonian in the so-called $\mu$-scheme, which is replaced in modern cosmology by the $\bar{\mu}$-scheme. In this regard, the authors of [31] have found an interesting way to extend the setting to the $\bar{\mu}$-scheme by considering a superposition of different graphs (which is ultimately believed to be the most correct type of semiclassical state). Even though this was developed in the framework of Quantum Reduced Loop Gravity – a gauge fixed version of LQG – their results might be transferable to our setting.

V. A TOY MODEL

What we derived in (29) and (30) is the expectation value of the gravitational part of the LQG scalar constraint on a family of complexifier coherent states adapted to homogeneous isotropic cosmology. We can now use it as an effective Hamiltonian on the phase space parametrized by $(p, c)$ and study the dynamics it produces. To facilitate the comparison with LQC effective dynamics, we choose as matter content a massless scalar field $\phi$, so that the effective scalar constraint reads

$$H = H_{\text{eff}} + H_{\phi}, \quad H_{\phi} = \frac{\pi_{\phi}^2}{2p^{3/2}}$$

with $\pi_{\phi}$ the momentum of $\phi$. The evolution of any phase space function $f$ wrt cosmological time $t$ is now obtained by Hamilton’s equation $\dot{f} = \{f, H\}$, and in particular we see that $\pi_{\phi}$ is a constant of motion. We can now compute the system of $\dot{p}$ and $\dot{c}$, which can be numerically solved, leading to the phase space trajectories labelled by the constant of motion $\pi_{\phi}$. The evolution of volume $v = p^{3/2}$ wrt cosmological time $t$ and physical time $\phi$ is plotted in figure 1 for a simple example.

As the plots show, the behavior of the universe at late time is the same as in LQC (and classical cosmology), but is quite different in the early universe: while a bounce still occurs, in our model it is highly non-symmetric and in the far past the universe does not obey classical Friedmann equations for a contracting spacetime. This is a major departure from the standard big bounce picture of LQC, and it is entirely due to the introduction of the Lorentzian term coming from the full theory.\(^3\)

VI. CONCLUSIONS

The idea to obtain an effective scalar constraint (and hence effective dynamics) from the quantum theory has a long road of success. In the seminal paper [9], the effective Hamiltonian was shown to reproduce the quantum evolution of coherent states, in particular removing the cosmologic singularity. First steps of repeating the effective dynamics program in a more complicated setting were taken in [33–35], in the context of a gauge-fixed version of the full theory. These results confirmed the effective LQC Hamiltonian (5).

In this paper, we have pushed the program further,
by calculating the effective scalar constraint in full LQG without assuming any simplifications, and considering both Euclidean and Lorentzian contributions. We found that the Euclidean part is in agreement with LQG at leading order (the corrections are proportional to the semiclassicality parameter $t$ of the complexifier coherent states we used). The Lorentzian part, on the other hand, introduces a different dependence on the variable $c$. While this dependence was already discussed within LQC [32], it was not emphasized that it leads to a departure from the standard bounce picture. We have shown that this is the case in a simple toy model.

The main message we want to communicate is twofold. On one hand, we have seen how from the full theory one can derive the effective dynamics of a reduced symmetry model (encoded in the special choice of coherent state labels), and that this is in agreement with LQC (and classical cosmology) at late times, whereas important modifications appear in the early universe. On the other hand, we have shown that the choice of coherent states and the regularization procedure by which one obtains the Hamiltonian operator have crucial impact even on the semiclassical dynamics. Unless these details are fixed, no reliable quantitative and finer qualitative predictions can be made. It is thus inescapable to integrate methods (such as the renormalization group or consistency with Dirac algebra) into the quantization process in order to fix any discretization errors.

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