On Hopf algebraic structures of quantum toroidal algebras

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**ABSTRACT**

We define an algebra \( \mathcal{U}_0 \) using a simplified set of generators for the quantum toroidal algebra \( \mathcal{U}_q(\mathfrak{sl}_{n+1}, \text{tor}) \) and show that there exists an epimorphism from \( \mathcal{U}_0 \) to \( \mathcal{U}_q(\mathfrak{sl}_{n+1}, \text{tor}) \). We derive a closed formula of the comultiplication on the generators of \( \mathcal{U}_0 \) that extends that of the quantum affine algebra \( \mathcal{U}_q(\hat{\mathfrak{sl}}_{n+1}) \). As a consequence, we show that \( \mathcal{U}_0 \) is a Hopf algebra for \( n = 1, 2 \) and give conjectural formulas in the general case. We further show that \( \mathcal{U}_0 \) is isomorphic to a double algebra.

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1. Introduction

The quantum toroidal algebras were first introduced via geometric realization in type A [17] and then through the McKay correspondence in the general ADE types [7] in terms of Drinfeld generators and relations. Their structures have been studied in various works [19, 35], and in particular the horizontal and vertical quantum affine subalgebras have played an important role in subsequent investigations. The quantum toroidal algebra is also closely related to the quantum Kac-Moody algebra [24], which has a Hopf algebra structure in terms of Drinfeld-Jimbo generators. It is natural to expect that the quantum toroidal algebras are also quantum groups in Drinfeld’s sense [5], and the quantum group structure should naturally extend that of quantum affine algebras.

In [8, 9, 20, 32, 34], tensor products of modules of quantum toroidal algebras are studied in terms of Drinfeld-type (infinite) comultiplication (cf [4, 23]). These works also indicate more tensor products of modules could be constructed if there exists a finite type comultiplication. However, it has been a long-standing problem to find such a (finite) Hopf algebra structure for the quantum toroidal algebra. The difficulties might be due to the fact that the quantum toroidal algebras have complicated high degree Serre relations (see [33]).

Besides the Drinfeld-Jimbo presentation, Beck [1] has given a general formula for the Hopf algebra structure of the quantum affine algebra in terms of the universal R-matrix and the braid group action. Through the Ding-Frenkel isomorphism [3] the Hopf algebra structure is also given in terms of the \( L^\pm(z) \)-generators [13]. In a different approach [28], the authors have computed the Hopf comultiplication of the two-parameter quantum affine algebras explicitly in terms of some simpler Drinfeld generators [6]. The special case of the comultiplication formula gives the same (Drinfeld-Jimbo) Hopf algebra structure for the quantum affine algebras. The goal of this paper is to generalize and study the Hopf algebra structure for the quantum toroidal algebras of type A in terms of its Drinfeld generators.

As our comultiplication stems out of the usual Drinfeld-Jimbo comultiplication over the quantum affine algebra, it is compatible with the \( q \)-characters [11]. We define an algebra \( \mathcal{U}_0 \) using a simplified set
of generators for the quantum toroidal algebra $U_q(sl_{n+1},\text{tor})$. We prove that there exists an epimorphism between $\mathcal{U}_0$ and quantum toroidal algebra $U_q(sl_{n+1},\text{tor})$. We have shown that this does give a Hopf algebra structure of $\mathcal{U}_0$ for the case of $n = 1$ and $n = 2$. We also formulate a conjectural formula of the coproduct in the general case, and it is also interesting to see if this type of comultiplication exists for other quantum toroidal algebras following [11, 12]. We remark that our conjectured formulas seem to be related with rooted trees.

Guay, Nakajima and Wendlandt [16] have recently given a Hopf algebra structure on the affine Yangian algebra $Y(\widehat{g})$ in terms of the Drinfeld generators, which generalizes the comultiplication of the current algebra $g[t]$ (see also [29]). Our result would induce a corresponding Hopf algebra structure for the double affine Yangian algebra $DY(\widehat{g})$ in view of [14, 15], which generalizes the multiplication of the loop algebra $g[ait, t^{-1}]$.

The paper is organized as follows. In Section 2, we review some basic results for the quantum affine algebra [28] to prepare for later discussions. In Section 4 we define an algebra $\mathcal{U}_0$ using a subset of Drinfeld generators for the quantum toroidal algebra $U_q(sl_{n+1},\text{tor})$ and we prove that there exists an epimorphism between $\mathcal{U}_0$ and $U_q(sl_{n+1},\text{tor})$. In Section 5, we define the comultiplication $\Delta$ on the generators of the algebra $\mathcal{U}_0$ generated by the simpler set of generators, and show that the Hopf algebra structure for $n = 1, 2$. In Section 6, we prove that $\mathcal{U}_0$ is characterized as a Drinfeld double $D(\mathcal{B}, \mathcal{B}')$ with respect to a skew-dual paring based on the comultiplication.

## 2. Quantum Affine Algebras

The quantum affine algebras have the Hopf algebra structure defined by Drinfeld and Jimbo in terms of the Chevalley generators (see also [1]). Drinfeld gave an infinite coproduct on the completion algebra (cf. [22]). Now we recall a commultiplication for the Drinfeld generators [28] coming from the two-parameter quantum affine algebras.

Let $A = (a_{ij})_{i,j \in I}$ be the generalized Cartan matrix of the (untwisted) affine Lie algebra $\hat{g}$, where $I = \{0, 1, \ldots, n\}$ and $I_0 = I \setminus \{0\}$ are the index sets of simple roots $\alpha_i$ of $\hat{g}$ and the finite dimensional simple Lie algebra $g$ respectively. The canonical bilinear form $(,)$ of $\hat{g}$ satisfies $(\alpha_i, \alpha_j) = d_i a_{ij}$ and $(\delta, \alpha_j) = (\delta, \delta) = 0$ where $\delta$ is the canonical imaginary root. So $d_i = \frac{1}{2} (\alpha_i, \alpha_i)$ and let $q_i = q^{d_i}$. Let $\mathfrak{h}$ be the Cartan subalgebra of $\hat{g}$ and $\mathfrak{h}^*$ the dual Cartan subalgebra.

**Definition 2.1.** The quantum affine algebra $U_q(\hat{g})$ is the unital associative algebra over $\mathbb{C}(q)$ generated by $e_i, f_i, K_i^{\pm 1}, (i \in I)$, $D^{\pm 1}$ and the central elements $\gamma^{\pm \frac{1}{2}}$ subject to the following relations:

$$(\chi 1) \quad K_i K_i^{-1} = K_i^{-1} K_i = 1, [K_i^{\pm 1}, K_j^{\pm 1}] = [K_i^{\pm 1}, D^{\pm 1}] = 0,$$

$$(\chi 2) \quad D e_i D^{-1} = q_i^{\delta_{0i}} e_i, \quad D f_i D^{-1} = q_i^{-\delta_{0i}} f_i,$$

$$(\chi 3) \quad K_j e_i K_j^{-1} = q_i^{a_{ij}} e_i, \quad K_j f_i K_j^{-1} = q_i^{-a_{ij}} f_i,$$

$$(\chi 4) \quad [e_i, f_j] = -\frac{\delta_{ij}}{q_i - q_i^{-1}} (K_i - K_i^{-1}).$$

$$(\chi 4) \quad (\text{ad}_i e_j)^{1-a_{ij}}(e_j) = 0, \quad (\text{ad}_i f_j)^{1-a_{ij}}(f_j) = 0, \quad i \neq j \in I,$$

where the left-adjoint $\text{ad}_i e_i$ and the right-adjoint $\text{ad}_r f_i$ are defined by

$$\text{ad}_l (a b) = \sum_{(a)} a_{(1)} b S(a_{(2)}), \quad \text{ad}_r (a b) = \sum_{(a)} S(a_{(1)}) b a_{(2)}, \quad \forall a, b \in U(\hat{g}),$$

where $\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}$ is the comultiplication of $U_q(\hat{g})$ given below.
The Hopf algebra structure of $U_q(\mathfrak{g})$ can be easily described as follows. Under the comultiplication $\Delta$, the elements $K_i, D$ are group-like elements and

$$
\Delta(e_i) = e_i \otimes 1 + K_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes K_i^{-1} + 1 \otimes f_i.
$$

The antipode $S$ is given by $S(e_i) = -K_i^{-1} e_i, S(f_i) = -f_i K_i, S(K_i) = K_i^{-1}, S(D) = D^{-1}$, and the counit $\varepsilon$ is given by $\varepsilon(e_i) = \varepsilon(f_i) = 0, \varepsilon(K_i) = \varepsilon(D) = 1$.

The $v$-bracket is defined by $[a, b]_v = ab - vba$. For $v_i \in \mathbb{C}(q, q^{-1}) \setminus \{0\}$, we define two types of multi-brackets inductively by

$$
[a_1, a_2, \ldots, a_s]_{(v_1, v_2, \ldots, v_{s-1})} = [a_1, \ldots, [a_{s-1}, a_s]_{v_1}, \ldots, v_{s-1})],
$$

$$
[a_1, a_2, \ldots, a_s]_{(v_1, v_2, \ldots, v_{s-1})} = [[a_1, a_2]_{v_1}, \ldots, a_{s-1}]_{(v_2, \ldots, v_{s-2})}.
$$

The following identities will be useful [22]:

$$
[a, [b, c]_u]_v = [[a, b]_q, c]_u q + q [ [a, c]_q, b]_u q,
$$

$$
[[a, b]_u, c]_v = [a, [b, c]_q]_u q + q [ [a, c]_q, b]_u q.
$$

The quantum affine algebra also admits Drinfeld’s new realization [5], which is the quantum analog of the current algebra realization for the affine Lie algebra.

**Definition 2.2.** [1, 6] The Drinfeld realization $U_q(\mathfrak{g})$ is an associative algebra over $\mathbb{C}(q)$ generated by $x^\pm_i(k), a_i(r), K_i^{\pm 1}, D^{\pm 1}$ ($i \in I_0, k \in \mathbb{Z}, r \in \mathbb{Z}\setminus\{0\}$) and the central elements $\gamma^\pm 1$ satisfy the following relations.

$$
D x^\pm_i(k) D^{-1} = q^k, \quad D a_i(r) D^{-1} = q^r a_i(r),
$$

$$
K_i^{\pm 1} K_i^{\mp 1} = 1, \quad K_i x^\pm_j(k) K_i^{-1} = q^{\pm a_i} x^\pm_j(k),
$$

$$
[a_i(r), K_i^{\mp 1}] = 0,
$$

$$
[a_i(r), a_j(r')] = \delta_{r+r',0} \frac{[ra_{ij}]}{r} \gamma' - \gamma^{-r},
$$

$$
[a_i(r), x^\pm_j(k)] = \pm \frac{[ra_{ij}]}{r} \gamma^{\mp r} x^\pm_j(r+k),
$$

$$
[x^\pm_i(k+1), x^\pm_j(l)]_{q^{\pm(a_i, a_j)}} = -[x^\pm_i(l+1), x^\pm_j(k)]_{q^{\pm(a_i, a_j)}},
$$

$$
[x^\pm_i(k), x^\pm_j(k')] = \sum_{0 \leq l \leq k} \delta_{ij} q^{-l} \gamma^{r-k} \phi_i(k+k') - \gamma^{r-k} \phi_i(k+k'),
$$

where $\phi_i(m), \varphi_i(-m)$ ($m \in \mathbb{Z}_{\geq 0}$) such that $\phi_i(0) = K_i$ and $\varphi_i(0) = K_i^{-1}$ are defined as below:

$$
\sum_{m=0}^{\infty} \phi_i(m) z^{-m} = K_i \exp \left( (q_i - q_i^{-1}) \sum_{\ell=1}^{\infty} a_i(r) z^{-r} \right),
$$

$$
\sum_{m=0}^{\infty} \varphi_i(-m) z^m = K_i^{-1} \exp \left( -(q_i - q_i^{-1}) \sum_{r=1}^{\infty} a_i(-r) z^r \right),
$$

$$
\text{Sym} \sum_{m_1, \ldots, m_n} (-1)^k \left[ \binom{n}{k} \right] x^\pm_i(m_1) \cdots x^\pm_i(m_k) x^\pm_j(l) x^\pm_j(m_{k+1}) \cdots x^\pm_i(m_n) = 0, \quad i \neq j,
$$

where $n = 1 - a_{ij}, [m]_i = q_i^m - q_i^{-m}, [m]_i! = [m]_i \cdots [2]_i [1]_i, \left[ \binom{m}{n} \right]_i = \frac{[m]_i!}{[n]_i! [m-n]_i!}$, and $\text{Sym}_{m_1, \ldots, m_n}$ denotes the symmetrization with respect to the indices $(m_1, \ldots, m_n)$. 
We now choose a convenient set of generators for $\mathcal{U}_q(\widehat{\mathfrak{g}})$:

$$x_1^+ (0), x_1^- (0), x_1^- (-1), x_1^+ (1), K_i^{\pm 1}, y_i^{\pm 1}, D_i^{\pm 1}, i \in I_0,$$  \hspace{1cm} (2.11)

where $y_i^{\pm 1}$ are central and the relations are: (2.4)–(2.5) for $l = 0, k = 0, \pm 1$; (2.9) for $k = k' = 0$ and $k = -k' = 1$; and

$$[x_i^+ (-\epsilon), x_i^+ (0)]_{q^2} = 0,$$

$$[x_i^- (-\epsilon), x_i^+ (0)] = 0, \text{ for } i \neq 0,$$

$$[x_i^+ (0), x_j^+ (0)] = 0, \quad [x_i^+ (-\epsilon), x_i^+ (0)] = 0, \quad \text{ for } a_{ij} = a_{1k} = 0,$$

$$\sum_{k=0}^{n-1} (-1)^k \sum_{i=0}^{n-k} (x_i^+ (-\epsilon))^{k} x_i^+ (0) (x_i^+ (0))^{n-k} = 0, \quad \text{ for } i \neq j,$$

$$\sum_{k=0}^{2} (-1)^k \left[ \binom{2}{k} \right] x_i^+ (0) x_i^+ (-\epsilon) (x_i^+ (0))^{2-k} = 0,$$

$$\sum_{k=0}^{2} (-1)^k \left[ \binom{2}{k} \right] x_i^+ (-\epsilon) (x_i^+ (0))^{2-k} = 0,$$

$$\text{Sym}_{\epsilon} \sum_{k=0}^{2} (-1)^k \left[ \binom{2}{k} \right] x_i^+ (-\epsilon) x_i^+ (0) (x_i^+ (0))^{2-k} = 0,$$

where $\text{Sym}_{m_1,m_2}$ denotes the symmetrization with respect to the indices $(m_1, m_2)$.

Starting from the elements in (2.11), we can inductively determine other Drinfeld generators as follows. First we have

$$a_1 (\pm 1) = \pm K_1^{\pm 1} y_1^{1/2} [x_1^+ (0), x_1^- (\pm 1)].$$  \hspace{1cm} (2.19)

Then $x_1^+ (\mp 1)$ and $x_1^- (\mp 1)$ can be determined by relation (2.7), and subsequently all other $x_i^+ (\mp 1)$ and $x_i^- (\mp 1)$ are fixed. Using this idea, one can inductively prove the following result (cf. [28]).

**Proposition 2.3.** As an associative algebra, the algebra generated by the elements in (2.11) subject to the aforementioned relations is identical to $\mathcal{U}_q(\widehat{\mathfrak{g}})$.

We can describe the isomorphism between the two presentations in details.

Let $X_\theta = [e_{h-1}, e_{h-2}, \ldots, e_{i_1}, e_{i_2}, \ldots] = \alpha_{h-1} + \cdots + \alpha_{i_1} + \alpha_{i_2}, \ldots$ be the root vector associated with the maximum root $\theta = \alpha_{h-1} + \cdots + \alpha_{i_1} + \alpha_{i_2} + \alpha_{i_3}, \ldots$ of $\mathfrak{g}$. This determines a sequence:

$$i_1, i_2, \ldots, i_{h-1},$$  \hspace{1cm} (2.20)

where $i_k \in I$ and $h$ is the Coxeter number. We will call such a sequence a root chain to the maximum root. Clearly root chains of the maximum root are not unique.

With respect to such a root chain for the maximum root $\theta$, we define the following numbers: for $t = 2, \ldots, h - 1$,

$$v_{i_1} = q^{-(\alpha_{i_1} \cdot \alpha_{j_1}) + \alpha_{i_1} \cdot \alpha_{j_1} + \cdots + \alpha_{i_1} \cdot \alpha_{j_1}},$$

$$v'_{i_1} = q^{-(\alpha_{i_1} \cdot \alpha_{j_1}) + \alpha_{i_2} \cdot \alpha_{j_1} + \cdots + \alpha_{i_1} \cdot \alpha_{j_1}},$$

$$v''_{i_1} = q^{-(\alpha_{i_1} \cdot \alpha_{j_1}) + \alpha_{i_2} \cdot \alpha_{j_1} + \cdots + \alpha_{i_{h-1}} \cdot \alpha_{j_1}}.$$  

**Theorem 2.4.** [28] For each fixed root chain to the maximum root $\theta$ given in (2.20), the map $\Psi : U_q(\widehat{\mathfrak{g}}) \rightarrow U_q(\widehat{\mathfrak{g}})$ defined below is an algebra isomorphism:
where $K_\theta = K_i \cdots K_{i_{h-1}}$ and

\[ \begin{align*}
& x^-_\theta^1 = [x^-_{h-1}^0, \ldots, x^-_{i_2}^0, x^-_{i_1}^1]_{(v_{i_2}, \ldots, v_{i_{h-1}})}, \\
& x^+_\theta^1(-1) = [x^+_{i_1}^1(-1), x^+_{i_2}^0, \ldots, x^+_{h-1}^0]_{(v_{i_2}^0, \ldots, v_{i_{h-1}}^0)},
\end{align*} \]

(2.21) For example in type A, (2.21)–(2.22) read as follows:

\[ \begin{align*}
& x^-_\theta^1 = [x^-_{n-1}^0, \ldots, x^-_2^0, x^-_1^1]_{(q^{-1}, \ldots, q^{-1})}, \\
& x^+_\theta^1(-1) = [x^+_{i_1}^1(-1), x^+_{i_2}^0, \ldots, x^+_{n-1}^0]_{(q_1, q_2, \ldots, q_{n-1})}.
\end{align*} \]

The theorem was proved in [28] for the two-parameter quantum affine algebra using the new generators, which include the above as a special case. In particular, the inverse isomorphism of $\Phi$ can be given as follows.

**Theorem 2.5.** [28] Let $1 = i_1, i_2, \ldots, i_{h-1}$ be a fixed root chain to the maximum root $\theta$ given in (2.20). Then the following map $\Phi : \mathcal{U}_q(\hat{\mathfrak{g}}) \rightarrow \mathcal{U}_q(\hat{\mathfrak{g}})$ given below is an epimorphism, where for all $i \in I$

\[ \begin{align*}
\Phi(\gamma^1) &= \gamma, & \Phi(D) &= D, & \Phi(K_i) &= K_i, \\
\Phi(x^-_i(0)) &= f_i, & \Phi(x^+_i(0)) &= e_i, \\
\Phi(x^-_i(1)) &= \{e_{i_2}, e_{i_3}, \ldots, e_{i_{h-1}}, e_0 \}_1 (v_{i_2}^{-1}, \ldots, v_{i_1}^{-1}) \gamma^{-1} K_1, \\
\Phi(x^+_i(-1)) &= \gamma K_1^{-1} [f_0, f_{i_{h-1}}, f_{i_{h-2}}, \ldots, f_{i_2}]_{(v_{i_{h-1}}^{-1}, \ldots, v_{i_2}^{-1})}.
\end{align*} \]

We can now describe the Hopf structure of $\mathcal{U}_q(\hat{\mathfrak{g}})$ in terms of the simplified (Drinfeld) generators. For $i_2 \leq j_1 < \cdots < j_i \leq i_{h-1}$, define $x^+_{i,j}(1)$ and $x^-_{i,j}(-1)$ inductively by

\[ \begin{align*}
& x^-_{i,j}(1) = [x^-_{i_2}(0), \ldots, x^-_{i_1}(0), x^-_{i_1}^1]_{(v_{i_2}, \ldots, v_{i_1})}, \\
& x^+_{i,j}(-1) = [x^+_{i_1}^1(-1), x^+_{i_2}^0, \ldots, x^+_{i_1}^0]_{(v_{i_2}^{-1}, \ldots, v_{i_1}^{-1})}.
\end{align*} \]

Similarly we also define $x^+_{i,j}(0)$ and $x^-_{i,j}(0)$ for $i_2 \leq j_1 < \cdots < j_i \leq i_{h-1}$ as follows.

\[ \begin{align*}
& x^+_{i,j}(0) = [x^+_{i_2}(0), x^+_{i_2}(0), \ldots, x^+_{i_1}(0)]_{(v_{i_2}^{-1}, \ldots, v_{i_1}^{-1})}, \\
& x^-_{i,j}(0) = [x^-_{i_2}(0), x^-_{i_2}^0, \ldots, x^-_{i_1}(0)]_{(v_{i_2}^{-1}, \ldots, v_{i_1}^{-1})}.
\end{align*} \]

As before, we fix the root chain $1 = i_1, i_2, \ldots, i_{h-1}$ to the maximum root $\theta$ given in (2.20). The comultiplication $\Delta : \mathcal{U}_q(\hat{\mathfrak{g}}) \otimes \mathcal{U}_q(\hat{\mathfrak{g}})$ is explicitly given as follows: for $i \in I$

\[ \begin{align*}
\Delta(K_i) &= K_i \otimes K_i, & \Delta(\gamma^1) &= \gamma^1 \otimes \gamma^1, \\
\Delta(D^\pm 1) &= D^\pm 1 \otimes D^\pm 1, \\
\Delta(x^-_i(0)) &= x^-_i(0) \otimes 1 + K_i \otimes x^-_i(0), \\
\Delta(x^+_i(0)) &= x^+_i(0) \otimes K_i^{-1} + 1 \otimes x^+_i(0), \\
\Delta(x^+_i(1)) &= x^+_i(1) \otimes \gamma^{-1} K_1 + 1 \otimes x^+_i(1) + \sum_{i_2 \leq j_1 < \cdots < j_i \leq i_{h-1}} \xi_j x^+_{i,j}(1) \otimes x^+_i(0) \gamma^{-1} K_1, \\
\Delta(x^-_i(-1)) &= x^-_i(-1) \otimes 1 + \gamma K_1^{-1} \otimes x^-_i(-1) + \sum_{i_2 \leq j_1 < \cdots < j_i \leq i_{h-1}} \xi_j \gamma K_1 \otimes x^-_i(0) \otimes x^+_{i,j}(-1),
\end{align*} \]
where
\[ \xi_j = (v'_{j_1} - v''_{j_1}) t_{j_2}^{-1} \cdots t_{j_{i-1}}^{-1} v''_{j_{i-1}}, \]
\[ \zeta_j = (v''_{j_1} - v'_{j_1}) t_{j_2}^{-1} \cdots t_{j_{i-1}}^{-1} v'_{j_{i-1}}, \]
and \( t'_j \) is given by \( t'_j = \frac{v_k - v^{-1}_k}{q_k - q^{-1}_k} \). These formulas extend the action of the original Drinfeld-Jimbo comultiplication from the Chevelley generators to Drinfeld’s new generators, so they are different from Drinfeld’s infinite coproduct formulas (see [3, 18, 23]).

**Proposition 2.6.** The algebra \( \mathcal{U}_q(\hat{g}) \) is a Hopf algebra with the comultiplication \( \Delta \) (defined above), counit \( \varepsilon \) and antipode \( S \) given as follows: for \( i \in I \)
\[ \varepsilon(x_i^+(0)) = \varepsilon(x_i^+(1)) = 0, \]
\[ \varepsilon(K_i) = \varepsilon(D) = 1, \quad S(K_i) = K_i^{-1}, \quad S(D) = D^{-1}, \]
\[ S(x_i^+(0)) = x_i^{-1}, \quad S(x_i^-(0)) = -x_i^-(0) K_i, \]
\[ S(x_i^-(1)) = -\gamma^{-1} K_i x_i^+(1) - \sum_{i_2 \leq i < j \leq i_{h-1}} \xi_j y_j^-(0) \gamma^{-1} K_j \cdots K_i x_i^+(1), \]
\[ S(x_i^+(1)) = x_i^+(1) \gamma K_i - \sum_{i_2 \leq i < j \leq i_{h-1}} \xi_j x_i^+(1) \gamma K_j \cdots K_i^{-1} y_j^+(0), \]
where the sums run over all sequences \( j = (j_1, \ldots, j_l) \) such that \( i_2 \leq j_1 < \cdots < j_l \leq i_{h-1} \) and
\[ y_j^-(0) = a \{ x_{j_1}^-(0), x_{j_2}^-(0), \ldots, x_{j_l}^-(0) \}, \]
\[ y_j^+(0) = a^{-1} \{ x_{j_1}^+(0), x_{j_2}^+(0), \ldots, x_{j_l}^+(0) \}, \]
and for \( 1 \leq t \leq l-1 \), \( p_{j_t} = q^{(\alpha_{j_t}, \alpha_{j_{t+1}})}. \) The constant \( a \) is given by
\[ a = \prod_{t=1}^{l-1} q^{(2(\alpha_{j_t}, \alpha_{j_{t+1}}))}, \]

**Theorem 2.7.** The morphisms \( \Phi \) and \( \Psi \) are two coalgebra homomorphisms:
\[ \Delta \circ \Psi = (\Psi \otimes \Psi) \Delta, \quad \Delta \circ \Phi = (\Phi \otimes \Phi) \Delta. \]
Moreover \( \Phi \) and \( \Psi \) are mutually inverse to each other.

**Remark 2.8.** This result upgrades the algebra isomorphism into a Hopf algebra isomorphism (cf. [5]) between the Drinfeld-Jimbo definition and the Drinfeld realization. In viewing of [14, 15] it also provides a comultiplication formula for the Yangian algebra associated to any simple Lie algebra in terms of the Drinfeld-Jimbo generators. The main terms of the comultiplication formula were known for untwisted quantum affine algebras in [25, Th. 2.3] and for twisted quantum affine algebras in [26, Th. 2.2].

As our comultiplication is determined by the usual Drinfeld-Jimbo comultiplication, it is compatible with the \( q \)-characters [11, 12].

### 3. Quantum toroidal algebra \( \mathcal{U}_q(sl_{n+1}, \text{tor}) \)

We now generalize the Hopf algebra structure to the (one-parameter) quantum toroidal algebra. We focus on type \( A \) [17, 35], though similar results are expected for ADE types [7] and the two-parameter case [27]. However, our construction seems not work for the quantum toroidal algebra associated to \( gl(1) \) ([8, 9, 31]).
Fix the integer \( n \geq 1 \), and let \( q \) be a generic complex number. The quantum toroidal algebra \( U_q(sl_{n+1}, \text{tor}) \) is an associative algebra over \( \mathbb{C}(q) \) with generators \( x_i^{\pm}(k), a_i(r), K_i^\pm, \gamma^\pm, q^{\pm d_i}, q^{\pm d_2}, \) \((i \in I, k \in \mathbb{Z}, r \in \mathbb{Z}/\{0\})\) satisfying relations (2.6), (2.7), (2.8), (2.9), (2.10) and the following additional relations:

\[
\gamma^{\pm} \text{ are central such that, } \gamma^{\pm} \gamma^{\pm} = 1, K_i^{\pm}, a_j(r), \text{ and } q^{\pm d_i} \text{ commute among each other, (3.1)}
\]

\[
K_i x_i^{\pm}(k) K_i^{-1} = q^{\pm a_i} x_i^{\pm}(k), \quad (3.2)
\]

\[
q^{d_1} a_i(r) q^{-d_1} = q^r a_i(r), \quad q^{d_1} x_i^{\pm}(k) q^{-d_1} = q^k x_i^{\pm}(k), \quad (3.3)
\]

\[
q^{d_2} a_i(r) q^{-d_2} = a_i(r), \quad q^{d_2} x_i^{\pm}(k) q^{-d_2} = q^{\delta_0} x_i^{\pm}(k). \quad (3.4)
\]

Let \( \tau \) be the map of \( \mathbb{Z}_{n+1} \) given by \( \tau(i) = i + 1 \). Then \( \tau \) induces the diagram automorphism for the derived quantum toroidal algebra \( U_q'(sl_{n+1}, \text{tor}) \), the subalgebra without the generator \( q^{d_2} \), such that

\[
\tau(x_i^{\pm}(k)) = x_{i+1}^{\pm}(k), \quad \tau(a_i(k)) = a_{i+1}(k). \quad (3.5)
\]

Let \( U_q(n^\pm) \) be the subalgebra of \( U_q(sl_{n+1}, \text{tor}) \) generated by \( x_i^{\pm}(k) \) \((i \in I)\) respectively, and \( U_q^0(g) \) be the subalgebra of \( U_q(sl_{n+1}, \text{tor}) \) generated by \( K_i^{\pm}, \gamma^{\pm}, q^{\pm d_i}, q^{\pm d_2} \) and \( a_i(\pm r) \) \((i \in I, r \in \mathbb{N})\).

Using Lemma 3.3 of [32], it is easy to see that as a vector space \((cf. [33])\)

\[
U_q(sl_{n+1}, \text{tor}) \simeq U_q(n^-) \otimes U_q^0(g) \otimes U_q(n^+).
\]

4. The algebra \( \mathfrak{u}_0 \)

We denote by \( \langle x_i \rangle \) the free associative algebra generated by \( x_i \). We first introduce the algebra \( \mathfrak{u}_0 \) to extract a smaller set of generators from the quantum toroidal algebra.

**Definition 4.1.** The algebra \( \mathfrak{u}_0 \) is an associative algebra over \( \mathbb{C}(q) \) generated by \( x_i^{\pm}(0), x_0^{+}(-1), x_0^{-}(1), K_i^{\pm}(i \in I), q^{\pm d_i}, q^{\pm d_2}, \) and \( \gamma^{\pm} \) satisfying the following relations (4.1)–(4.11), that is

\[
\mathfrak{u}_0 := \left\{ x_i^{\pm}(0), x_0^{+}(-1), x_0^{-}(1), K_i^{\pm}, q^{\pm d_i}, q^{\pm d_2}, \gamma^{\pm} \mid i \in I \right\} / \sim.
\]

\[
\gamma^{\pm} \text{ are central such that, } \gamma^{\pm} \gamma^{\pm} = 1, K_i^{\pm}, q^{\pm d_i} \text{ and } K_i^{\pm} \text{ commute with each other, (4.1)}
\]

\[
K_i x_i^{\pm}(0) K_i^{-1} = q^{\pm a_i} x_i^{\pm}(0), \quad (4.2)
\]

\[
q^{d_1} x_i^{\pm}(0) q^{-d_1} = x_i^{\pm}(0), \quad q^{d_1} x_0^{\pm}(0) q^{-d_1} = q^{-\epsilon} x_0^{\pm}(0), \quad (4.3)
\]

\[
q^{d_2} x_i^{\pm}(0) q^{-d_2} = q^{\delta_0} x_i^{\pm}(0), \quad q^{d_2} x_0^{\pm}(0) q^{-d_2} = q x_0^{\pm}(0), \quad (4.4)
\]

\[
[ x_i^{\pm}(0), x_j^{\pm}(0) ]_{q^{-2}} = 0, \quad (4.5)
\]

\[
[ x_i^{\pm}(0), x_j^{-}(0) ] = \delta_{ij} - \frac{K_i - K_i^{-1}}{q - q^{-1}} K_i x_j^{\pm}(0) K_i^{-1} = \left[ x_0^{+}(-1), x_0^{-}(1) \right] = \frac{\gamma^{-1} K_0 - \gamma K_0^{-1}}{q - q^{-1}}, \quad (4.6)
\]

\[
[ x_i^{\pm}(0), x_k^{\pm}(0) ] = 0, \quad \text{for } i \neq k, \quad (4.7)
\]

\[
[ x_i^{\pm}(0), x_j^{\pm}(0) ] = 0, \quad [ x_i^{\pm}(0), x_k^{\pm}(0) ] = 0, \quad \text{for } a_{ij} = a_{jk} = 0, \quad (4.8)
\]

\[
x_i^{\pm}(0) x_j^{\pm}(0) x_j^{\pm}(0) - [2] x_i^{\pm}(0) x_j^{\pm}(0) x_i^{\pm}(0) + x_i^{\pm}(0) x_j^{\pm}(0) x_i^{\pm}(0) x_j^{\pm}(0) = 0, \quad a_{ij} = -1, \quad (4.9)
\]

\[
x_i^{\pm}(0) x_j^{\pm}(0) x_j^{\pm}(0) - [2] x_i^{\pm}(0) x_j^{\pm}(0) x_j^{\pm}(0) + x_i^{\pm}(0) x_j^{\pm}(0) x_j^{\pm}(0) x_i^{\pm}(0) = 0, \quad a_{ij} = -1, \quad (4.10)
\]

\[
x_i^{\pm}(0) x_j^{\pm}(0) x_j^{\pm}(0) - [2] x_i^{\pm}(0) x_j^{\pm}(0) x_j^{\pm}(0) + x_i^{\pm}(0) x_j^{\pm}(0) x_j^{\pm}(0) = 0, \quad a_{ij} = -1, \quad (4.11)
\]

where \( \epsilon = \pm \) or \( \pm 1 \), \( A = (a_{ij}) \) the Cartan matrix of \( A_n^{(1)} \).
It is clear that \( U_0 \) is finitely generated with finitely many relations, and has fewer generators and simpler relations than Drinfeld’s original form. We will show that \( U_0 \) is isomorphic to a quotient algebra of \( U_q(sl_{n+1}, \text{tor}) \). Observe that \( U_0 \) has a Chevalley anti-involution \( i : x_i^+ (k) \leftrightarrow x_i^- (-k) \), \( K_i \leftrightarrow K_i^{-1} \), \( q^{d_1} \leftrightarrow q^{-d_1} \), \( q^{d_2} \leftrightarrow q^{d_2} \), \( \gamma \leftrightarrow \gamma^{-1} \) and \( q \leftrightarrow q^{-1} \) over the complex field.

**Theorem 4.2.** There exists an epimorphism \( \pi : U_0 \rightarrow U_q(sl_{n+1}, \text{tor}) \) such that \( \pi \) is identity on the set of generators of \( U_0 \) in Definition 4.1. That is, \( U_0/\ker \pi \cong U_q(sl_{n+1}, \text{tor}) \).

To prove Theorem 4.2, we need to check \( \pi \) can be extended from the generating set to the whole algebra and preserves all the relations. To this end, we introduce the following elements:

\[
\begin{align*}
  a_0(1) &= K_0^{-1/2} \left[ x_0^+ (0), x_0^- (1) \right] \in U_0, \\
  a_0(-1) &= K_0^{-1/2} \left[ x_0^+ (-1), x_0^- (0) \right] \in U_0,
\end{align*}
\]

and use them to generate higher degree elements in a spiral way. As \( i(a_0(\pm 1)) = a_0(\mp 1) \) we only check half of the relations. The following relations involving with \( a_0(\pm 1) \) are clear.

**Proposition 4.3.** Using the above notations, the following relations are compatible with the defining relations of the quantum toroidal algebra \( U_q(sl_{n+1}, \text{tor}) \) (\( \varepsilon = \pm or \pm 1 \)):

\[
\begin{align*}
  K_0 a_0(\varepsilon) K_0^{-1} &= a_0(\varepsilon), \\
  q^{d_1} a_0(\varepsilon) q^{-d_1} &= q^{d_1} a_0(\varepsilon), \\
  q^{d_2} a_0(\varepsilon) q^{-d_2} &= a_0(\varepsilon), \\
  \left[ a_0(\varepsilon), \left. x_0^\varepsilon (-\varepsilon) \right] \right] &= \varepsilon [2] \gamma^{-\frac{\varepsilon}{2}} x_0^\varepsilon(0), \\
  \left[ a_0(-\varepsilon), \left. x_0^\varepsilon (0) \right] \right] &= \varepsilon [2] \gamma^{-\frac{\varepsilon}{2}} x_0^\varepsilon (-\varepsilon), \\
  \left[ a_0(1), a_0(-1) \right] &= [2] \frac{\gamma - \gamma^{-1}}{q - q^{-1}}.
\end{align*}
\]

Here the compatibility means that the map \( \pi : a_0(\varepsilon) \mapsto a_0(\varepsilon), \ x_0^\varepsilon (-\varepsilon) \mapsto x_0^\varepsilon (-\varepsilon) \) is a homomorphism. This meaning will be used repeatedly in the following.

**Proof.** Using the Chevalley anti-involution, we only need to show the relations for \( \varepsilon = 1 \) (viewed as + in superscript). (4.14)–(4.16) follow by direction computation.

Furthermore, for \( \varepsilon = \pm or \pm 1 \) we have

\[
x_0^\varepsilon(0) = [2]^{-1} \gamma^\frac{\varepsilon}{2} \left[ a_0(\varepsilon), x_0^\varepsilon(0) \right] \in U_0.
\]

In fact, using the above notations and (4.17), we have the following lemma.

**Lemma 4.4.** One has that for \( \varepsilon = \pm 1 \)

\[
\left[ x_0^\varepsilon (0), x_0^- (0) \right] = \gamma \left[ x_0^\varepsilon (0), x_0^- (\varepsilon) \right],
\]

which are equivalent to

\[
a_0(\pm 1) = \pm K_0^{\pm 1} \gamma^{\frac{\varepsilon}{2}} \left[ x_0^\pm (0), x_0^\pm (\pm 1) \right] = \pm K_0^{\pm 1} \gamma^{\frac{\varepsilon}{2}} \left[ x_0^\pm (\pm 1), x_0^\pm (0) \right].
\]

**Proposition 4.5.** From the above constructions, we have the following relations, which are compatible to the defining relations of the quantum toroidal algebra \( U_q(sl_{n+1}, \text{tor}) \):

\[
\begin{align*}
  K_0 x_0^\varepsilon(0) K_0^{-1} &= q^{a_0} x_0^\varepsilon(0), \\
  q^{d_1} x_0^\varepsilon(0) q^{-d_1} &= q^{d_1} x_0^\varepsilon(0), \\
  q^{d_2} x_0^\varepsilon(0) q^{-d_2} &= q x_0^\varepsilon(0), \\
  \left[ x_0^\varepsilon (1), x_0^- (0) \right]_{q^{\pm 2}} + \left[ x_0^\varepsilon (0), x_0^\varepsilon (0) \right]_{q^{\pm 2}} &= 0, \\
  \left[ a_0(-\varepsilon), x_0^\varepsilon (0) \right] &= \varepsilon [2] \gamma^{-\frac{\varepsilon}{2}} x_0^\varepsilon (0), \\
  \left[ x_0^\varepsilon (1), x_0^- (0) \right] &= \gamma K_0 - \gamma^{-1} K_0^{-1} \left[ x_0^\varepsilon (0), \frac{q - q^{-1}}{q - q^{-1}} \right].
\end{align*}
\]
Proof. By the Chevalley anti-involution, it is enough to show the relations for \( \epsilon = 1 \). To check (4.21), note that \([ x_0^+ (0), x_0^+ (-1)]_q^2 = 0\) by (4.5), then

\[
0 = \left[ a_0 (1), [ x_0^+ (0), x_0^+ (-1)]_q^2 \right]
= \left( \left[ a_0 (1), x_0^+ (0) \right]_q^2 + \left[ x_0^+ (0), [ a_0 (1), x_0^+ (-1)]_q^2 \right] \right)
= [2]y^{- \frac{1}{2}} \left( [ x_0^+ (1), x_0^+ (-1)]_q^2 + [ x_0^+ (0), x_0^+ (0)]_q^2 \right).
\]

It means that \([ x_0^+ (1), x_0^+ (-1)]_q^2 + [ x_0^+ (0), x_0^+ (0)]_q^2 = 0\), which is compatible with the defining relation (2.8). Similarly (4.22) follows from (4.13) and (4.6) via (4.19) and (4.21). (4.23) holds from direct calculation.

Now we construct all degree-\( k \) elements \( x_0^\pm (k), x_0^\pm (-k), a_0 (\pm k) \) involving with index \( i = 0 \) by induction on the degree as follows. For \( \epsilon = \pm \) or \( \pm 1 \), we denote that,

\[
x_0^\pm (\epsilon k) = \pm [2]^{-1} y^{\pm \frac{1}{2}} \left[ a_0 (\epsilon), x_0^\pm (\epsilon (k - 1)) \right] \in \mathcal{U}_0,
\]

(4.24)

\[
\phi_i (k) = (q - q^{-1}) y^{\frac{\pm 1}{2}} \left[ x_0^\pm (k - 1), x_0^\pm (1) \right] \in \mathcal{U}_0,
\]

(4.25)

\[
\varphi_i (-k) = -(q - q^{-1}) y^{\frac{\pm 1}{2}} \left[ x_0^\pm (-1), x_0^\pm (-k + 1) \right] \in \mathcal{U}_0,
\]

(4.26)

where \( a_0 (\pm k) \) are defined by \( \phi_0 (k) \) and \( \varphi_0 (-k) \) \( (k \geq 0) \) as follows:

\[
\sum_{m=0}^{\infty} \phi_0 (m) z^{\pm r} = K_0^{\pm 1} \exp \left( \pm (q - q^{-1}) \sum_{r=1}^{\infty} a_0 (0) z^{\pm r} \right).
\]

A partition \( \lambda \) of \( k \), denoted \( \lambda \vdash k \), is a decreasing sequence of positive integers \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0 \) such that \( \lambda_1 + \lambda_2 + \cdots + \lambda_l = k \), where \( l (\lambda) = l \) is called the number of parts. \( \lambda \) can also be written as \( (1^{m_1} 2^{m_2} \cdots) \) with multiplicity of \( i \) being \( m_i \). Then we obtain the following formulas between \( a_0 (\pm k) \) and \( \phi_0 (k) \) or \( \varphi_0 (-k) \):

\[
\phi_0 (\pm k) = K_0^{\pm 1} \sum_{\lambda \vdash k} \frac{(q^{\pm 1} - q^{-1})^l (\lambda)}{m_\lambda !} a_0 (\pm \lambda),
\]

(4.27)

where \( m_\lambda = \prod_{i \geq 1} m_i ! \) and \( a_0 (\pm \lambda) = a_0 (\pm \lambda_1) a_0 (\pm \lambda_2) \cdots \) \( \epsilon (\phi_0 (k)) = \varphi_0 (-k) \)

The following result can be directly checked, but we will give a general proof in Proposition 4.12.

Lemma 4.6. For \( \epsilon = \pm \) or \( \pm 1 \), one has that

\[
[a_0 (2\epsilon), x_0^\pm (-\epsilon)] = \frac{[4]}{2} y^{\pm 1} x_0^\pm (\epsilon),
\]

(4.28)

\[
[a_0 (2\epsilon), x_0^\pm (0)] = \frac{[4]}{2} y^{\pm 1} x_0^\pm (2\epsilon),
\]

(4.29)

\[
[a_0 (2), a_0 (-1)] = 0,
\]

(4.30)

\[
[a_0 (2), a_0 (-2)] = \frac{[4]}{2} \frac{y^2 - y^{-2}}{q - q^{-1}}.
\]

(4.31)

Using these relations, we have the following result.

Proposition 4.7. For \( \epsilon = \pm \) or \( \pm 1 \), it holds that,

\[
[x_0^\pm (2), x_0^\pm (-1)]_q^{\pm 2} + [ x_0^\pm (0), x_0^\pm (1) ]_q^{\pm 2} = 0,
\]

(4.32)

\[
[x_0^\pm (\epsilon), x_0^\pm (0)]_q^2 = 0,
\]

(4.33)

which are compatible with the defining relation (2.8).
\textbf{Proof.} Using the Chevalley anti-involution \(\iota\), it is enough to show the case of \(\epsilon = +\). To check (4.32), note that \([x_0^+(0), x_0^-(1)]_{q^2} = 0\). Thus it follows from (4.28) and (4.29) that

\[0 = [a_0, [x_0^+(0), x_0^-(1)]]_{q^2} = -[1] \gamma^{-1} ([x_0^+(2), x_0^-(1)]_{q^2} + [x_0^+(0), x_0^+(1)]_{q^2}),\]

which implies that \([x_0^+(2), x_0^-(1)]_{q^2} + [x_0^+(0), x_0^+(1)]_{q^2} = 0\).

Similarly by (4.21),

\[0 = [a_0, [x_0^+(1), x_0^-(1)]]_{q^2} + [x_0^+(0), x_0^+(0)]_{q^2}\]

\[= 2[1] \gamma^{-\frac{1}{2}} ([x_0^+(2), x_0^-(1)]_{q^2} + [x_0^+(1), x_0^+(0)]_{q^2} + [x_0^+(0), x_0^+(0)]_{q^2}]\]

\[= 2[1] \gamma^{-\frac{1}{2}} [x_0^+(1), x_0^+(0)]_{q^2},\]

where we have used (4.32). It yields that \([x_0^+(1), x_0^+(0)]_{q^2} = 0\).

So far we have shown that \(x_i^\pm(0), x_i^\pm(1), x_i^\pm(-1), a_0(\pm 1)\) \((i \in I)\) in the algebra \(\mathcal{U}_0\) satisfy relations (3.1) to (3.4) and (2.6) to (2.10). Then we can use induction on the degree \(k\) to show that \(x_i^\pm(k), a_0(\pm k)\) are all in the algebra and satisfy the Drinfeld relations. To this end, suppose all degree-\(k\) \((k \leq n)\) elements \(x_i^\pm(\epsilon k), a_0(\epsilon k), x_i^\pm(0)\) \((i \in I)\) are in the algebra \(\mathcal{U}_0\), and satisfy relations (3.1) to (3.4) and (2.6) to (2.10). Next we need to verify that all the elements \(x_i^\pm(\epsilon(n + 1)), a_0(\epsilon(n + 1))\) \((i \in I)\) satisfy the relations (3.1) to (3.4) and (2.6) to (2.10). We need a couple of lemmas for this.

\textbf{Lemma 4.8.} One has that

\[[a_0(1), \varphi_0(n)] = 0, \quad [a_0(-1), \varphi_0(-n)] = 0.\]

\textbf{Lemma 4.9.} It is clear that,

\[
\frac{1}{(q - q^{-1})} \varphi_0(n + 1) = \gamma^{-\frac{n+1}{2}} [x_0^+(n + 1), x_0^-(0)] = \gamma^{-\frac{n}{2}} [x_0^+(n), x_0^-(1)]
\]

\[= \cdots = \gamma^{-\frac{n+1}{2}} [x_0^+(1), x_0^-(n)] = \gamma^{-\frac{n+1}{2}} [x_0^+(0), x_0^-(n + 1)],\]

\[
\frac{-1}{(q - q^{-1})} \varphi_0(-n - 1) = \gamma^{-\frac{-n-1}{2}} [x_0^+(n - 1), x_0^-(0)] = \gamma^{\frac{-n}{2}} [x_0^+(n), x_0^-(1)],
\]

\[= \cdots = \gamma^{-\frac{n+1}{2}} [x_0^+(-1), x_0^-(n)] = \gamma^{\frac{n+1}{2}} [x_0^+(0), x_0^-(n - 1)].\]

\textbf{Proof.} In fact, (4.36) can be obtained from (4.35) by the Chevally anti-involution \(\iota\). (4.35) follows from inductive hypothesis.

We turn to the following proposition.

\textbf{Proposition 4.10.} From the above constructions, we have the following relations for \(\epsilon = \pm\) or \(\pm 1\), \(-n + 1 \leq l_1 \leq n - 1\) and \(-n \leq l_2 \leq n,

\[
K_1[x_0^+(\epsilon(n+1)), K_1^{-1} = q^{\epsilon a_0} x_0^+(\epsilon(n + 1)),
\]

\[
q^{\epsilon 1} x_0^+(\epsilon(n+1)) q^{-\epsilon 1} = q^{\epsilon(n+1)} x_0^+(\epsilon(n + 1)),
\]

\[
q^{\epsilon 2} x_0^+(\epsilon(n+1)) q^{-\epsilon 2} = q x_0^+(\epsilon(n + 1)),
\]

\[
[x_0^+(n+1), x_0^+(n-1)]_{q^{\epsilon 2}} + [x_0^+(n), x_0^+(n)]_{q^{\epsilon 2}} = 0,
\]

\[
[x_0^+(n+2), x_0^+(n-1)]_{q^{\epsilon 2}} + [x_0^+(n), x_0^+(n+1)]_{q^{\epsilon 2}} = 0,
\]

\[
[x_0^+(n+1), x_0^+(n)]_{q^{\epsilon 2}} = 0,
\]

\[
[x_0^+(n+1), x_0^+(n)]_{q^{\epsilon 2}} = 0.
\]
\[ [x_0^+ (n + 1), x_0^- (l_2)] = [2] \gamma^{-\frac{1}{2}} \left( [[a_0(1), x_0^+(l_2)], x_0^+(n)] + [a_0(1), [x_0^+(n), x_0^-(l_2)]] \right) = -\gamma [x_0^-(l_2 + 1), x_0^+(n)] = \frac{\gamma^{n+1} K_0}{q - q^{-1}}. \]

For (4.45), one has that,
\[ [x_0^+(n + 1), x_0^- (-n - 1)] = \gamma^{n+1} K_0 - \gamma^{-n-1} K_0^{-1} = \frac{\gamma^{n+1} K_0}{q - q^{-1}}. \]

Denote that \( \bar{\phi}_0(k) = \frac{K_0^{-1}}{q - q^{-1}} \phi_0(k) \) and \( \bar{\phi}_0(-k) = \frac{K_0}{q - q^{-1}} \bar{\phi}_0(-k) \), we have the following relations.

**Proposition 4.11.** The following relations hold for \( d = \gamma^{-\frac{1}{2}} q^2 \),
\[ [\bar{\phi}_0(r), x_0^+(m)] = [2] \gamma^{-\frac{1}{2}} \left( \sum_{t=1}^{r-1} (q - q^{-1}) d^t r \bar{\phi}_0(r-t) + d^t x_0^+(r + m) \right), \]
\[ [\bar{\phi}_0(-r), x_0^- (-m)] = -[2] \gamma^{-\frac{1}{2}} \left( \sum_{t=1}^{r-1} (q - q^{-1}) d^t r \bar{\phi}_0(-r-t) x_0^- (-m - t) + d^t x_0^- (-r - m) \right). \]
Proof. By induction, it is enough to check the case of \( m = -1 \), other cases are similar. \[
\phi_0(n+1), x_0^+(1) = \gamma^{\frac{n+1}{2}} K_0^{-1} \left( [x_0^+(n), x_0^-(1), x_0^+(1)] \right)_{q^2} + [x_0^+(n), x_0^+(1)]_{q^2}, x_0^-(1) \right) \]
\[
d \left( [\phi_0(n), x_0^+(0)] + (q-q^{-1}) q^{-2}[2] x_0^+(0) \phi_0(n) \right) = \cdots \]
\[
d^n \left( [\phi_0(1), x_0^+(n-1)] + (q-q^{-1}) q^{-2}[2] x_0^+(n-1) \phi_0(1) \right) \]
\[
+ \sum_{t=1}^{n-1} d^n(q-q^{-1}) q^{-2}[2] x_0^+(t-1) \phi_0(n+1-t) \]
\[
= [2] \gamma^{-\frac{n}{2}} \sum_{t=1}^{n} d^n(q-q^{-1}) x_0^+(t-1) \phi_0(n+1-t) + d^n x_0^+(n). \]

Now we convert the relations to those with \( a_0(n+1) \). First it is easily seen that,
\[
ka_0(k) = k\bar{\phi}_0(k) - (q-q^{-1}) \sum_{t=1}^{k-1} t\bar{\phi}_0(k-t) a_0(t), \quad (4.48) \\
ka_0(-k) = k\bar{\phi}_0(-k) - (q-q^{-1}) \sum_{t=1}^{k-1} t a_0(-t) \bar{\phi}_0(-k+t). \quad (4.49) 
\]

Proposition 4.12. The following relations hold for \( \epsilon = \pm 1 \) or \( \pm 1 \),
\[
[a_0(\epsilon(n+1)), x_0^+(-\epsilon)] = \frac{[2(n+1)]}{n+1} \gamma^{-\frac{n+1}{2}} x_0^\epsilon(\epsilon n), \quad (4.50) \\
[a_0(\epsilon(n+1)), x_0^-(\epsilon n)] = \frac{[2(n+1)]}{n+1} \gamma^{-\frac{n+1}{2}} x_0^\epsilon(-\epsilon), \quad (4.51) \\
[a_0(n+1), a_0(-n-1)] = \frac{[2(n+1)]}{n+1} \gamma^{n+1} - \gamma^{-(n+1)} - \frac{q-q^{-1}}{q-q^{-1}}. \quad (4.52) 
\]

Proof. Note that (4.51) is similar to (4.50), so we only worry about (4.50). It follows from Proposition 4.11 that \[
[a_0(n+1), x_0^+(1)] \]
\[
= [\phi_0(n+1), x_0^+(1)] - (q-q^{-1}) \sum_{t=1}^{n} t[\bar{\phi}_0(n+1-t) a_0(t), x_0^+(1)] \]
\[
= \frac{[2(n+1)]}{(n+1)!} \gamma^{-\frac{n+1}{2}} x_0^+(n). \]

For (4.52), it follows from (4.50)–(4.51), and (4.27) by inductive hypothesis that \[
[a_0(n+1), a_0(-n-1)] \]
\[
= \gamma^{-\frac{n+1}{2}} K_0[a_0(n+1), [x_0^+(1), x_0^-(1)]] - \sum_{\lambda \neq n+1} \frac{(q-q^{-1})^{(\lambda_n-1)}}{m_\lambda!} [a_0(n+1), a_0(\lambda)] \]
\[
= \gamma^{-\frac{n+1}{2}} K_0([a_0(n+1), x_0^+(1)], x_0^-(1) \ ] + [x_0^+(1), [a_0(n+1), x_0^-(1)]] \]
\[
= \frac{[2(n+1)]}{n+1} \gamma^{n+1} - \gamma^{-(n+1)} - \frac{q-q^{-1}}{q-q^{-1}}. \]

\[\]
From Propositions 4.10 and 4.12, we have checked that all elements $x^\pm_0(\epsilon(n+1))$, $a_0(\epsilon(n+1))$ satisfy the defining relations (3.1)–(3.4) and (2.6)–(2.10). By induction it follows that all elements $x^\pm_k(\epsilon k)$, $a_k(\epsilon k)$ for $k \in \mathbb{Z}/[0]$ satisfy the defining relations (3.1)–(3.4) and (2.6)–(2.10). In other words, the algebra $\mathcal{U}_0$ contains a subalgebra $U_q(\hat{sl}_2)_{0}$ isomorphic to $U_q(\hat{sl}_2)$.

Next we will show that the quantum toroidal algebra $\mathcal{U}_0$ also contains another subalgebra $U_q(\hat{sl}_2)_1$ isomorphic to $U_q(\hat{sl}_2)$, i.e., the subalgebra generated by $x^+_1(-1)$, $x^-_1(1)$ etc satisfy exactly the same relations as those of $x^+_0(-1)$, $x^-_0(1)$ (cf. Proposition 4.14). Moreover, we will also derive the relations between $U_q(\hat{sl}_2)_0$ and $U_q(\hat{sl}_2)_1$.

For $\epsilon = \pm 1$ or $\omega$, we define that

$$x^\pm_1(\epsilon) = \pm q^{1/2}\left[ a_0(\epsilon), x^\pm_1(0) \right] \in \mathcal{U}_0, \quad (4.53)$$

$$a_1(1) = q^{1/2}K^{-1}_1\left[ x^+_1(0), x^-_1(1) \right] \in \mathcal{U}_0, \quad (4.54)$$

$$a_1(-1) = q^{-1/2}K_1\left[ x^+_1(-1), x^-_1(0) \right] \in \mathcal{U}_0. \quad (4.55)$$

Similar to Lemma 4.4, we have the following relations.

**Lemma 4.13.** It is easy to see that for $\epsilon = \pm 1$,

$$\left[ x^\pm_1(\epsilon), x^-_1(0) \right] = q\left[ x^+_1(0), x^-_1(\epsilon) \right], \quad (4.56)$$

$$\left[ a_1(1), a_1(-1) \right] = [2]q^{-1} = \frac{q^2 - q^{-1}}{q - q^{-1}}, \quad (4.57)$$

$$a_1(1) = K^{-1}_1q^{1/2}\left[ x^+_1(0), x^-_1(1) \right] = K^{-1}_1q^{-1/2}\left[ x^+_1(1), x^-_1(0) \right], \quad (4.58)$$

$$a_1(-1) = K_1q^{-1/2}\left[ x^+_1(-1), x^-_1(0) \right] = K_1q^{1/2}\left[ x^+_1(0), x^-_1(-1) \right]. \quad (4.59)$$

**Proposition 4.14.** From the above constructions, we have the following relations, which are compatible to the defining relations of the quantum toroidal algebra $U_q(sl_{n+1,\text{tor}})$ for $\epsilon = \pm$ or $\pm 1$.

$$K_1x^\pm_1(\epsilon)K^{-1}_1 = q^{\pm a_1}x^\pm_1(1), \quad q^{d_1}x^\pm_1(\epsilon)q^{-d_1} = q^\epsilon x^\pm_1(1), \quad q^{d_2}x^\pm_1(\epsilon)q^{-d_2} = q^\epsilon x^\pm_1(1), \quad (4.60)$$

$$\left[ a_0(1), a_1(-1) \right] = -\frac{q - q^{-1}}{q - q^{-1}}, \quad (4.61)$$

$$\left[ x^+_1(1), x^-_0(0) \right]_{q^\pm 1} + \left[ x^+_0(1), x^+_0(0) \right]_{q^\pm 1} = 0, \quad (4.62)$$

$$\left[ x^-_1(\epsilon), x^+_1(0) \right]_{q^{-2}} = 0, \quad (4.63)$$

$$\left[ a_0(\epsilon), x^+_1(\epsilon) \right] = -q^{+\epsilon}x^+_1(0), \quad (4.64)$$

$$\left[ x^-_1(\epsilon), x^+_1(\epsilon) \right] = 0, \quad \text{for} \ i \neq 1, \quad (4.65)$$

$$\left[ x^+_1(-1), x^-_1(1) \right] = \frac{\gamma^{-1}K_1 - \gamma K^{-1}_1}{q - q^{-1}}, \quad (4.66)$$

$$\left[ x^+_0(0), x^-_0(0), x^+_1(1) \right]_{q^{-1}} = 0. \quad (4.67)$$

**Proof.** Using the Chevalley anti-involution $\iota$, we only check for the case of $\epsilon = \pm$. (4.60) holds directly by definition. (4.61) follows from (4.55). (4.62) can be easily checked as follows.

$$\left[ x^-_1(1), x^-_0(0) \right]_{q} = q^{-1/2}\left[ \left[ a_0(1), x^-_1(0) \right], x^-_0(0) \right]_{q}$$

$$= K^{-1}_0\left[ \left[ x^-_0(0), x^-_0(1) \right], x^-_1(0) \right]_{q} x^-_0(0) \right]_{q^{-1}}.$$
\[
= K_0^{-1}\left(\left[[x_0^+(0), x_0^-(0)]_1, x_0^-(0)\right]_q^q + q\left([[x_0^+(0), x_1^-(0)], x_0^-(0)\right]_q^q)_{q^{-1}}\right)
\]
\[
= K_0^{-1}\left(\left[[x_0^+(0), [x_0^-(0), x_1^-(0)]_q, x_0^-(0)\right]_q^q + q^{-1}\left([[x_0^+(0), x_0^-(0)], [x_0^-(0), x_1^-(0)]_q^q)_{q^{-1}}\right)\right)
\]
\[
= -[x_0^-(0), x_1^-(0)]_q,
\]

which is compatible with the defining relation (2.6), (4.63) follows from (4.53) and (4.13) by using (4.6) and (4.10). (4.64)–(4.66) are easy, which can be verified similarly. Relation (4.67) is compatible with the defining relation (2.10) for \(a_i = -1\). It follows from (4.53) that

\[
\gamma^{-\frac{1}{2}}\left([x_0^-(0), [x_0^-(0), a_0(1)], x_1^-(0)]_q^{-1}\right) + \gamma^{-\frac{1}{2}}\left([x_0^-(0), a_0(1), [x_0^-(0), x_1^-(0)]_q^{-1}\right) = 0,
\]

where we have used the Serre relation (4.10).

Similarly we can construct the subalgebras \(U_q(\hat{\mathfrak{sl}}_2)\), \(i = 2, \ldots, n\), that is, we can obtain all other Drinfeld generators \(x_i^\pm(k), a_i(r)\) and verify the interrelations among them.

On the other hand, as the Dynkin diagram of affine Lie algebra \(\mathfrak{sl}_{n+1}\) is a cycle, for \(\epsilon = \pm 1\) or \(\pm\), we define that

\[
j_n^\pm(\epsilon) = \pm y^{\pm\frac{1}{2}}, \quad \hat{b}_n(1) = y^\frac{1}{2}K_1^{-1}\left[x_n^+(0), j_n^-(1)\right],
\]

\[
\hat{b}_n(-1) = y^{-\frac{1}{2}}K_1\left[j_n^+(1), x_n^-(0)\right].
\]

It is easy to see that \(x_n^\pm(\epsilon) - j_n^\pm(\epsilon) \in \text{Ker} \pi\) for \(\epsilon = \pm 1\) or \(\pm\). Therefore by induction on degree we have proved \textbf{Theorem 4.2}.

\textbf{Remark 4.15.} The quantum toroidal algebra contains two subalgebras \(A_1\) and \(A_2\) isomorphic to the quantum affine algebra \(U_q(\hat{\mathfrak{sl}}_n)\). These two subalgebras are generated by Serre generators and partial Drinfeld generators respectively.

\[
A_1 := \left\{x_i^\pm(0), K_i^\pm, q^{d_j}, y^{\pm\frac{1}{2}} \mid i \in I, j \in I\right\},
\]

\[
A_2 := \left\{x_i^\pm(0), x_0^-(0) \right\} \left\{x_i^\pm(1), x_0^-(1), K_i^\pm, q^{d_j}, y^{\pm\frac{1}{2}} \mid i \in I\right\}.
\]

It is easy to see that the subalgebra \(A_1\) is isomorphic to the Drinfeld-Jimbo realization of the quantum affine algebra \(U_q(\hat{\mathfrak{sl}}_{n+1})\), and \(A_2\) is isomorphic to its Drinfeld realization.

Denote by \(\mathfrak{u}_1\) the subalgebra of \(U_q(\mathfrak{sl}_{n+1}, \text{tor})\) generated by \(x_j^\pm(0), x_j^\pm(1), x_0^-(0) \right\} \left\{x_j^\pm(1), x_0^-(1), K_j^\pm, q^{d_j}, q^{d_j}, y^{\pm\frac{1}{2}} \mid i \in I\right\}.

It then can be shown that \(\mathfrak{u}_1\) and \(\mathfrak{u}_0\) are isomorphic.
5. Hopf algebra structure of $\mathcal{U}_0$

In this section we show that there is a similar Hopf algebra structure on $\mathcal{U}_0$ for $n = 1, 2$ extending from its vertical and horizontal quantum affine subalgebras, and that we conjecture that $\mathcal{U}_0$ for general $n$ is also a Hopf algebra under our formulas.

First we define the actions of $\Delta$ on $x_0^\pm(1)$ and $x_0^-(-1)$ for the case of $n = 1$ as follows:

$$
\Delta(x_0^+(1)) = x_0^+(1) \otimes 1 + \gamma K_0^{-1} \otimes x_0^+(1) - (q - q^{-1}) \gamma K_0^{-1} x_1(0) \otimes [x_0^+(1), x_0^+(0)]_q,
$$

$$
\Delta(x_0^-(1)) = x_0^-(1) \otimes \gamma^{-1} K_0 + 1 \otimes x_0^-(1) - (q - q^{-1}) [x_1(0), x_0^-(1)]_{q^{-1}} \otimes x_1^+(0) \gamma^{-1} K_0.
$$

The actions of $\Delta$ on $x_0^-(-1)$ and $x_0^+(1)$ for the case of $n = 2$ is given as follows:

$$
\Delta(x_0^+(1)) = x_0^+(1) \otimes 1 + \gamma K_0^{-1} \otimes x_0^+(1) - (q - q^{-1}) \gamma K_0^{-1} x_1(0) \otimes [x_0^+(1), x_0^+(0)]_q + (q - q^{-1}) \gamma K_0^{-1} [x_2(0), x_0^-(0)]_{q^{-1}} \otimes [x_0^+(1), x_0^+(0)]_q x_2^+(0)_q, 
$$

$$
\Delta(x_0^-(1)) = x_0^-(1) \otimes \gamma^{-1} K_0 + 1 \otimes x_0^-(1) - (q - q^{-1}) [x_1(0), x_0^-(1)]_{q^{-1}} \otimes x_1^+(0) \gamma^{-1} K_0
$$

$$
- (q - q^{-1}) [x_2(0), [x_1(0), x_0^-(1)]_{q^{-1}}] \otimes [x_1^+(0), x_2^+(0)]_q \gamma^{-1} K_0.
$$

Definition 5.1. Under the above action of $\Delta$, we now define a comultiplication map $\Delta : \mathcal{U}_0 \rightarrow \mathcal{U}_0 \otimes \mathcal{U}_0$ for $n = 1$ and $n = 2$ on the generators as follows.

$$
\Delta(K_i) = K_i \otimes K_i, \quad \Delta(\gamma^{\pm \frac{1}{2}}) = \gamma^{\pm \frac{1}{2}} \otimes \gamma^{\pm \frac{1}{2}}, \quad \Delta(q^{d_1}) = q^{d_1} \otimes q^{d_1},
$$

$$
\Delta(x_i^+(0)) = x_i^+(0) \otimes 1 + K_i \otimes x_i^+(0), \quad \Delta(x_i^-(0)) = x_i^-(0) \otimes K_i^{-1} + 1 \otimes x_i^-(0).
$$

Proposition 5.2. Under the above map $\Delta$, the algebra $\mathcal{U}_0$ for $n = 1$ or $n = 2$ becomes a bialgebra.

Proof. The proof will be divided into three cases $n = 1$ and $n = 2$ of $\mathcal{U}_0$.

(1) For the case of $U_q(sl_2, tor)$, we focus on checking that $[\Delta(x_0^-(1)), \Delta(x_0^+(0))]_{q^{-2}} = 0$.

$$
[\Delta(x_0^+(1)), \Delta(x_0^-(0))]_{q^{-2}} = [x_0^-(1), x_0^+(0)]_{q^{-2}} \otimes \gamma^{-1} - (q - q^{-1}) [x_1^-(0), x_0^-(1)]_{q^{-1}} \otimes x_0^+(0) \gamma^{-1}
$$

$$
+ 1 \otimes [x_0^-(1), x_0^+(0)]_{q^{-2}} - (q - q^{-1}) q^{-2} [x_0^-(0), x_0^-(1)]_{q^{-1}} \otimes [x_0^+(0), x_0^-(0)] \gamma^{-1} K_0 = 0,
$$

where the first and third term are killed by (4.5), and the second one is zero by (4.9), the last item vanishes by (4.6).

Next, we further check that the action of $\Delta$ keeps the relation (4.6).

$$
[\Delta(x_1^+(0)), \Delta(x_0^-(1))]
$$

$$
= [x_1^+(0), x_0^+(0)] \otimes \gamma^{-1} K_0 + K_1 \otimes [x_1^+(0), x_0^-(1)] + (q - q^{-1}) x_0^-(1) K_1 \otimes x_1^+(0) \gamma^{-1} K_0
$$

$$
- (q - q^{-1}) x_0^-(1) K_1 \otimes x_1^+(0) \gamma^{-1} K_0 = 0.
$$

On the other hand, it is easy to see that

$$
[\Delta(x_0^+(1)), \Delta(x_0^-(1))]
$$

$$
= [x_0^+(1), x_0^-(1)] \otimes \gamma^{-1} K_0 + \gamma K_0^{-1} \otimes [x_0^+(1), x_0^-(1)]
$$

$$
- (q - q^{-1}) [x_1^+(0), [x_0^+(1), x_0^-(1)]]_{q^{-1}} \otimes x_0^+(1) \gamma^{-1} K_0
$$

$$
+ (q - q^{-1}) \gamma K_0^{-1} x_1^+(0) \otimes [x_0^+(1), x_0^-(1)], x_1^+(0)]_q
$$

$$
- (q - q^{-1}) [\gamma K_0^{-1} \otimes x_0^+(1), [x_0^-(1), x_0^-(1)]_{q^{-1}} \otimes x_1^+(0) \gamma^{-1} K_0]
$$

$$
+ (q - q^{-1}) [\gamma K_0^{-1} x_1^+(0) \otimes [x_0^+(1), x_0^-(1)], x_1^+(0)]_q, x_1^+(0) \otimes \gamma^{-1} K_0
$$

$$
- (q - q^{-1}) q^{-2} [\gamma K_0^{-1} x_1^+(0) \otimes [x_0^+(1), x_1^+(0)]_q, [x_0^-(1), x_0^-(1)]_{q^{-1}} \otimes x_1^+(0) \gamma^{-1} K_0].
$$
By direct computation using (2.9) and (2.6), the third and fourth terms cancel each other, and the fifth term vanishes. As for the last term, we calculate by (2.6) and the Serre relations as follows.

$$\begin{align*}
&= -(q - q^{-1}) \gamma K_0 \otimes [x_0^+(0), x_0^-(0)]_{q^{-1}} \otimes x_1^+(0) \gamma^{-1} K_0 \\
&= -q (q - q^{-1}) \gamma [x_0^+(0), x_0^-(0)]_{q^{-1}} \otimes \gamma^{-1} K_0 \\
&= -q (q - q^{-1})^2 \gamma [x_0^+(0), x_0^-(0)]_{q^{-1}} \otimes x_1^+(0) \gamma^{-1} K_0
\end{align*}$$

Therefore, one has that $[\Delta(x_0^+(-1)), \Delta(x_0^-(1))] = \frac{\gamma K_0 - \Delta(\gamma K_0^{-1})}{q - q^{-1}}$.

Finally, we would like to check Serre relation $[\Delta(x_1^-(0)), [\Delta(x_1^-(0)), \Delta(x_0^-(0))]]_{q^{-1}} = 0$. In fact, it follows from the construction of $\Delta(x_1^-(0))$ and $\Delta(x_0^-(0))$ that

$$[\Delta(x_0^-(1)), [\Delta(x_0^-(0)), \Delta(x_0^-(0))]]_{q^{-1}} = 0.$$  

(II) For the case of $n = 2$, similarly it suffices to check the relations involving $x_0^+(-1)$ and $x_0^+(1)$. More precisely, we first verify that $[\Delta(x_0^-(1)), [\Delta(x_0^-(0))]]_{q^{-2}} = 0$. Note that $\Delta(x_0^-(0)) = x_0^-(0) \otimes K_0^{-1} + 1 \otimes x_0^-(0)$ and some items are killed by (4.6) directly:

$$\begin{align*}
&= -q (q - q^{-1})^2 \gamma [x_0^+(0), x_0^-(0)]_{q^{-1}} \otimes x_1^+(0) \gamma^{-1} \\
&= -q (q - q^{-1})^2 \gamma [x_0^+(0), x_0^-(0)]_{q^{-1}} \otimes x_1^+(0) \gamma^{-1} \\
&= -q (q - q^{-1})^2 \gamma [x_0^+(0), x_0^-(0)]_{q^{-1}} \otimes x_1^+(0) \gamma^{-1}
\end{align*}$$

where the terms cancel each other by the Serre relations.

Now we turn to $[\Delta(x_1^+(0)), \Delta(x_0^-(1))] = 0$. Note that $\Delta(x_1^+(0)) = x_1^+(0) \otimes 1 + K_1 \otimes x_1^+(0)$, the left-hand side can be computed by the Serre relations and (5.11)–(5.12).

$$\begin{align*}
&= -(q - q^{-1}) [x_0^-(0), x_0^+(0)]_{q^{-1}} K_1 \otimes x_1^+(0) \gamma^{-1} K_0 \\
&= -(q - q^{-1})^2 \frac{q}{2} [x_0^-(0), x_0^+(0)]_{q^{-1}} K_1 \otimes x_1^+(0) \gamma^{-1} K_0 \\
&= -(q - q^{-1})^2 \frac{q}{2} [x_0^-(0), x_0^+(0)]_{q^{-1}} K_1 \otimes x_1^+(0) \gamma^{-1} K_0
\end{align*}$$
Next we need to verify that

\[
\left( q - q^{-1} \right) q^2 [x_0^- (0), x_0^+ (1)]_{q^{-1}} K_1 \otimes [x_1^+ (0), x_2^+ (0)]_{q^{-2}} = 0,
\]

where we have used the following two relations (which are based on the Serre relations):

\[
[x_1^+ (0), x_1^+ (0), x_2^+ (0), x_2^+ (0)]_{(q, q^{-1}, q^{-2}, 1)} = 0,
\]

\[
[x_1^+ (0), x_2^+ (0), x_1^+ (0), x_2^+ (0)]_{(q, q^{-1}, q^{-2}, 1)} = 0.
\]

Similarly it can be checked that \[ \Delta(x_2^+ (0)), \Delta(x_0^- (1)) \] = 0.

Next we need to verify that

\[
\left[ \Delta(x_0^- (-1)), \Delta(x_0^- (1)) \right] = \frac{\Delta(y K_0^{-1}) - \Delta(y K_0^{-1})}{q - q^{-1}}.
\]

It goes as follows by the actions of \( \delta \) on the generators \( x_0^+ (-1) \) and \( x_0^- (1) \).

\[
\left[ \Delta(x_0^- (-1)), \Delta(x_0^- (1)) \right] = \\
\left[ \Delta(x_0^- (-1)), x_0^- (1) \otimes y K_0 + 1 \otimes x_0^+ (1) - (q - q^{-1}) [x_1^- (0), x_0^- (1)]_{q^{-1}} \otimes x_1^+ (0) y K_0 \right]
\]

\[
= \left[ x_0^+ (-1), x_0^- (1) \right] \otimes y K_0 + y K_0^{-1} \otimes \left[ x_0^+ (-1), x_0^- (1) \right]
\]

\[
- (q - q^{-1}) \left[ [x_0^+ (-1), [x_1^- (0), x_0^- (1)]_{q^{-1}}] \otimes x_1^+ (0) y K_0 \right]
\]

\[
+ (q - q^{-1}) y K_0^{-1} x_1^- (0) \otimes \left[ x_0^- (1), x_1^+ (0) \right]_{q^{-1}}
\]

\[
+ (q - q^{-1}) y K_0^{-1} [x_2^- (0), x_1^- (0)]_{q^{-1}} \otimes \left[ [x_1^- (0), x_1^+ (0)]_{q^{-1}}, x_2^- (0) \right]_{q^{-1}}, x_0^- (1) \right].
\]

In fact, by direct computation using (2.9) and (2.6) similar to Case (I), the last terms are killed due to the Serre relations. We take one term as an example,

\[
y K_0^{-1} \left( x_1^- (0) \cdot [x_1^- (0), x_0^- (1)]_{q^{-1}} \otimes [x_0^- (-1), x_1^+ (0)]_{q} \cdot x_1^+ (0) \right)
\]

\[
- [x_1^- (0), x_0^- (1)]_{q^{-1}} \cdot x_1^- (0) \otimes x_1^+ (0) \cdot [x_0^+ (-1), x_1^+ (0)]_{q} \gamma K_0
\]

\[
= y K_0^{-1} \left( \left[ x_1^- (0), x_1^- (0), x_0^- (1) \right]_{(q^{-1}, q)} \otimes [x_0^- (-1), x_1^+ (0)]_{q^{-1}} \cdot x_1^+ (0) \right)
\]

\[
- [x_1^- (0), x_0^- (1)]_{q^{-1}} \cdot x_1^- (0) \otimes [x_1^+ (0), x_0^- (-1), x_1^+ (0)]_{(q, q^{-1})} \gamma K_0 = 0.
\]

Hence we have proved that \[ \Delta(x_0^- (-1)), \Delta(x_0^- (1)) \] = \[ \frac{\Delta(y K_0^{-1}) - \Delta(y K_0^{-1})}{q - q^{-1}} \].

For the quantum toroidal algebra \( U_q(sl_2, tor) \), we define that

\[
S(x^-_0 (-1)) = -y^{-1} K_0 x^-_0 (-1) + (q - q^{-1}) q^2 y^{-1} K_0 K_1 x^-_0 (0) [x_1^+ (0), x_1^+ (0)]_{q^{-1}}, x_0^- (0) q,
\]

\[
S(x^-_0 (1)) = -x^-_0 (1) y K_0^{-1} - (q - q^{-1}) q^{-2} [x_1^- (0), x_0^- (1)]_{q^{-1}} x_1^+ (0) y K_0^{-1} K_1^{-1}.
\]

For the quantum toroidal algebra \( U_q(sl_3, tor) \), we get that

\[
S(x^-_0 (-1)) = -y^{-1} K_0 x^-_0 (-1) + (q - q^{-1}) q^2 y^{-1} K_0 K_1 x^-_0 (0) [x_1^+ (0), x_1^+ (0)]_{q^{-1}}, x_0^- (0) q,
\]

\[
- (q - q^{-1}) q^3 y^{-1} K_0 K_1 K_2 [x_1^- (0), x_2^- (0)]_{q^{-1}} [x_0^- (0), x_1^- (0)]_{q^{-1}}, x_1^+ (0) q, x_2^+ (0) q.
\]
\[ S(x_0^-(1)) = -x_0^-(1)\gamma K_0^{-1} - (q - q^{-1})q^{-2}[x_1^-(0), x_0^-(1)]q^{-1}x_1^+(0)\gamma K_0^{-1} K_1^{-1} \]
\[-(q - q^{-1})q^{-3}[x_2^-(0), [x_1^-(0), x_0^-(1)]q^{-1}]q^{-1}[x_2^+(0), x_1^+(0)]q\gamma K_0^{-1} K_1^{-1} K_2^{-1}. \]

The following result is directly computed by the comultiplication and definition.

**Proposition 5.3.** Under the action of the antipode \(s\) on \(x_0^+(1)\) and \(x_0^-(1)\), the algebra \(\mathcal{U}_0\) for \(n = 1\) or \(n = 2\) is a Hopf algebra with the comultiplication \(\Delta\) given above, the counit \(\varepsilon\) and the antipode \(S\) defined below. For \(i \in I\),

\[ \varepsilon(x_i^+(0)) = \varepsilon(x_i^-(0)) = \varepsilon(x_0^+(1)) = \varepsilon(x_0^-(1)) = 0, \]
\[ \varepsilon(y^{\pm 1}) = \varepsilon(K_i) = \varepsilon(K_i^{-1}) = \varepsilon(q^d) = \varepsilon(q^{-d}) = 1, \]
\[ S(y^\pm 1) = y^\mp 1, \quad S(K_i^\pm 1) = K_i^\mp 1, \quad S(q^d) = q^{-d}, \quad S(q^{-d}) = q^{d}, \]
\[ S(x_i^+(0)) = -K_i^{-1} x_i^+(0), \quad S(x_i^-(0)) = -x_i^-(0) K_i. \]

For general case, we begin with the notations. Denote \(\Pi_0\) by the set of all real root vectors of toroidal Lie algebra \((sl_{n+1}, tor)\) generated by simple roots beginning with \(\alpha_0\), or all real roots supported at \(\alpha_0\) with multiplicity one. Note that \(\Pi_0\) is a finite set, in fact, if one bounds the coefficients of any fixed simple root, then there are only finitely many such affine roots. For any real root \(\alpha_{\beta_0} \in \Pi_0\) suppose \(\alpha_{\beta_0}\) obtained from the simple roots \(\alpha_0, \alpha_i, \ldots, \alpha_k\), that is, \(\alpha_{\beta_0} = \alpha_0 + \alpha_i + \cdots + \alpha_k\), where \(i, \ldots, k \in \{1, \ldots, n\}\).

We remark that the roots in \(\Pi_0\) are in fact real roots of some affine Lie algebra in type A. In fact, the real roots \(\alpha_{\beta_0} \in \Pi_0\) can always be written in three possible ways:

\[ \alpha_{\beta_0} = \alpha_0 + \alpha_1 + \alpha_2 \cdots + \alpha_k_1 \in \Pi_1, \]
\[ \alpha_{\beta_0} = \alpha_0 + \alpha_n + \alpha_{n-1} \cdots + \alpha_k_2 \in \Pi_2, \]
\[ \alpha_{\beta_0} = \alpha_0 + \alpha_1 + \alpha_n + \cdots + \alpha_k_3 \in \Pi_3, \]

where \(k_1 = 2, \ldots, n - 1\) for \(i = 1, 2, 3\). They respectively correspond to three cases of positive roots passing through a fixed simple root. For example, in type \(A_5\) any root passing through \(\alpha_2\) are: \(\alpha_2 + \alpha_3 + \cdots, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3 + \cdots\).

Therefore, \(\Pi_0 = \Pi_1 \cup \Pi_2 \cup \Pi_3\). Consider the real root system \(\Pi_0\) by induction, except for the special cases of \(n = 1\) and \(n = 2\), the real root system \(\Pi_0\) of the cases of \(n > 2\) would degenerate to the case for \(n = 3\), because \(\Pi_1\) can be obtained by the same way for \(i = 1, 2, 3\). Therefore, the proof of Proposition 5.2 will be divided into three cases \(n = 1, n = 2, \) and \(n = 3\).

Accordingly we define the quantum root vectors \(x_{\alpha_{\beta_0}}^{-}(1), x_{\alpha_{\beta_0}}^{+}(-1), y_{\alpha_{\beta_0}}^{-}(1)\) and \(y_{\alpha_{\beta_0}}^{+}(-1)\) associated to the root \(\alpha_{\beta_0}\) as follows:

\[ x_{\alpha_{\beta_0}}^{-}(1) = [x_{i_k_1}^{-}(0), x_{i_k_2}^{-}(0), \ldots, x_{i_1}^{-}(0), x_{0}^{-}(1)]_{(q^{m_1}, \ldots, q^{m_k})}, \]
\[ x_{\alpha_{\beta_0}}^{+}(-1) = [x_{0}^{-}(1), x_{i_1}^{-}(0), x_{i_2}^{-}(0), \ldots, x_{i_k_1}^{-}(0), x_{i_k_2}^{-}(0)]_{(q^{m_1}, \ldots, q^{m_k})}, \]
\[ y_{\alpha_{\beta_0}}^{-}(1) = [x_{i_k_1}^{-}(0), x_{i_k_2}^{-}(0), \ldots, x_{i_1}^{-}(0), x_{0}^{-}(1)]_{(q^{m_k}, \ldots, q^{m_1})}, \]
\[ y_{\alpha_{\beta_0}}^{+}(-1) = [x_{0}^{-}(1), x_{i_1}^{-}(0), x_{i_2}^{-}(0), \ldots, x_{i_k_1}^{-}(0), x_{i_k_2}^{-}(0)]_{(q^{m_k}, \ldots, q^{m_1})}, \]
where for $j = 1, \ldots, k$

$$u_{ij} = ((\alpha_0, \alpha_i) + (\alpha_0, \alpha_{i_2}) \ldots + (\alpha_0, \alpha_{i_j})), \quad v_{ij} = -((\alpha_i, \alpha_{i-1}) + (\alpha_i, \alpha_{i-2}) \ldots + (\alpha_i, \alpha_j)).$$

Define $\xi_{\alpha,0} = \frac{1}{\langle -u_{ij} \rangle}$.

We define the diagram automorphism $\sigma$ of the algebra $\mathcal{U}_0$ as follows: $\sigma(i) = n + 1 - i$ for $1 \leq i \leq n$. Note that for any $\alpha_{\beta,0} \in \Pi_0$, and $\sigma(\alpha_{\beta,0}) \in \Pi_0$.

The following are easy consequences of (2.9) and (3.1).

**Lemma 5.4.** With the above notations, one gets that

$$[x^-_0(1), x^+_\alpha(1)] = \langle -u_{ik} \rangle [x^+_i(0), \ldots, x^+_i(0), x^{-}_i(0)]_{(q^{-1}_{i_{k-1}}, \ldots, q^{-1}_{i_{j}})} y^{-1} K_0, (5.5)$$

$$[y^-_w(1), x^+_0(-1)] = [u_i]_y^{-1} \gamma K_0^{-1} [x^{-}_i(0), \ldots, x^{-}_i(0), x^+_i(0)]_{(q^{-1}_{i_{k-1}}, \ldots, q^{-1}_{i_{j}})}, (5.6)$$

$$[y^-_w(1), x^+_0(1)] = q^{u_k - u_i} [u_i]_y^{-1} K_0 x^+_i(0), \ldots, x^+_i(0)]_{(q^{-1}_{i_{k-1}}, q^{-1}_{i_{j}})}, (5.7)$$

$$[y^-_w(1), x^-_0(-1)] = q^{-u_k - u_i} [u_i]_y^{-1} K_0 [x^-_i(0), \ldots, x^-_i(0)]_{(q^{-1}_{i_{k-1}}, q^{-1}_{i_{j}})} y^{-1} K_0. (5.8)$$

We now define a comultiplication map $\mathcal{U}_0 \longrightarrow \mathcal{U}_0 \otimes \mathcal{U}_0$ on the generators as follows.

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(\gamma^\pm \frac{1}{2}) = \gamma^\pm \frac{1}{2} \otimes \gamma^\pm \frac{1}{2}, \quad \Delta(q^{d_1}) = q^{d_1} \otimes q^{d_1}, \quad (5.9)$$

$$\Delta(x^+_i(0)) = x^+_i(0) \otimes 1 + K_i \otimes x^+_i(0), \quad \Delta(x^-_i(0)) = x^-_i(0) \otimes K_i^{-1} + 1 \otimes x^-_i(0), \quad (5.10)$$

$$\Delta(x^-_0(1)) = x^-_0(1) \otimes y^{-1} K_0^{-1} + 1 \otimes x^-_0(1)$$

$$- (q - q^{-1}) \sum_{\alpha_{\beta,0} \in \Pi_0} \xi_{\alpha,0} x^-_\alpha^- (1) \otimes [x^-_0(1), x^+_\alpha(1)], \quad (5.11)$$

$$\Delta(x^+_0(-1)) = x^+_0(-1) \otimes 1 + \gamma K_0^{-1} \otimes x^+_0(-1)$$

$$- (q - q^{-1}) \sum_{\alpha_{\beta,0} \in \Pi_0} \xi_{\alpha,0} [y^-_\alpha(1), x^+_0(-1)] \otimes y^-_\alpha(1). \quad (5.12)$$

**Remark 5.5.** Note that $\Pi_0$ is finite. The coproduct formulas can also be formulated as a sum over different paths along the Dynkin diagram of type $A_n^{(1)}$.

**Conjecture:** The algebra $\mathcal{U}_0$ for $n \geq 1$ is a Hopf algebra under the comultiplication formulas.

The conjecture holds when $n = 1, 2$ due to Proposition 5.2.

### 6. Quantum double algebra structure of $\mathcal{U}_0$

We give the Drinfeld double structure for the algebra $\mathcal{U}_0$ in this section.

**Definition 6.1.** A bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{B} \times \mathfrak{B} \longrightarrow \mathbb{K}$ is called a skew-dual pairing of two Hopf algebras $\mathfrak{A}$ and $\mathfrak{B}$, if it satisfies

$$\langle b, 1_{\mathcal{A}} \rangle = \varepsilon_{\mathfrak{B}}(b), \quad \langle 1_{\mathfrak{B}}, a \rangle = \varepsilon_{\mathfrak{A}}(a),$$

$$\langle b, a_1 a_2 \rangle = \langle \Delta_{\mathfrak{B}}^{op}(b), a_1 \otimes a_2 \rangle, \quad \langle b_1 b_2, a \rangle = \langle b_1 \otimes b_2, \Delta_{\mathfrak{A}}(a) \rangle,$$

for all $a, a_1, a_2 \in \mathfrak{A}$ and $b, b_1, b_2 \in \mathfrak{B}$, where $\varepsilon_{\mathfrak{A}}, \varepsilon_{\mathfrak{B}}$ denote the counites of $\mathfrak{A}, \mathfrak{B}$, respectively, and $\Delta_{\mathfrak{A}}, \Delta_{\mathfrak{B}}$ are the respective comultiplications.

In [18], Grossé defined a weak Hopf pairing for the quantum affine algebras within a much larger topological Hopf algebra. It was not clear if the formulas there are closed on the quantum affine algebras.
Our Hopf pairing is for the quantum toroidal algebra and closed on the subalgebra of the quantum affine algebra.

A direct consequence of the definition is that
\[ \langle S_B(f), a \rangle = \langle f, S_A(a) \rangle, \quad f \in B, \quad a \in A, \]
where $S_B, S_A$ denote the antipodes of $B$ and $A$, respectively.

**Definition 6.2.** For any two skew-paired Hopf algebras $A$ and $B$ (see [30]) by $\langle , \rangle$, there exists a Drinfeld quantum double $D(A, B)$ which is a Hopf algebra whose underlying coalgebra is $A \otimes B$ with the tensor product coalgebra structure, and its algebra structure is defined by
\[ (a \otimes b)(a' \otimes b') = \sum (S_B(b_{(1)}), a'_{(1)})\langle b_{(2)}, a'_{(3)}\rangle a_{(2)}a'_{(2)} \otimes b_{(2)}b', \]
for $a, a' \in A$ and $b, b' \in B$, and whose antipode $S$ is given by
\[ S(a \otimes b) = (1 \otimes S_B(b))(S_A(a) \otimes 1). \]

Let $B$ (resp. $B'$) denote the Hopf (Borel-type) algebra of $U_q(sl_n)$ for $n = 1$ or $n = 2$ generated by $x^+_j(0)$, $x^-_j(1)$, $K_j^{\pm 1}$, $q^{d_1}$, $q^{d_2}$, $\gamma^{\pm 1}$ (resp. $x^-_j(0)$, $x^+_j(-1)$, $K_j^{\pm 1}$, $q^{d_1}$, $q^{d_2}$, $\gamma^{\pm 1}$) with $j \in I$.

**Proposition 6.3.** There exists a unique skew-dual pairing $\langle , \rangle : B' \times B \rightarrow \mathbb{C}(q^{\pm 1})$ of the Hopf subalgebras $B$ and $B'$ such that $(i \in I)$:
\[
\langle x^-_j(0), x^+_j(0) \rangle = \left\{ \begin{array}{ll}
1 & \text{if } j = 0 \\
\frac{1}{q^{-1} - q} & \text{if } j \neq 0
\end{array} \right. \tag{6.1}
\]
\[
\langle x^-_0(1), x^+_0(-1) \rangle = \frac{1}{q^{-1} - q}, \tag{6.2}
\]
\[
\langle K_i^{-1}, K_j^\pm \rangle = \langle K_i, K_j^{-1} \rangle = \langle K_i, K_j \rangle^\pm = q^{\pm 2 \delta_{ij}}, \tag{6.3}
\]
\[
\langle \gamma^{\pm 1}, \gamma^{\pm 1} \rangle = 1 = \{ \gamma^{\pm 2}, \gamma^{\mp 1} \}, \tag{6.4}
\]
\[
\langle q^{d_1}, q^{d_1} \rangle = \langle q^{d_1}, q^{d_1} \rangle = 1 = \langle q^{d_2}, q^{d_1} \rangle = \langle q^{d_2}, q^{d_2} \rangle, \tag{6.5}
\]
\[
\langle \gamma^{\pm 2}, K_i^\pm \rangle = \langle K_i^{-1}, \gamma^{\pm 2} \rangle = 1 = \langle K_i^{-1}, \gamma^{\pm 2} \rangle = \langle \gamma^{\pm 2}, K_i \rangle, \tag{6.6}
\]
\[
\langle q^{d_1}, K_i^{-1} \rangle = \langle q^{d_1}, K_i^{-1} \rangle = q^{\delta_{ij}}, \tag{6.7}
\]
\[
\langle K_i^{-1}, q^{d_1} \rangle = \langle K_i^{-1}, q^{d_1} \rangle = q^{-\delta_{ij}}, \tag{6.8}
\]
\[
\langle q^{d_1}, \gamma^{\pm 1} \rangle = \langle q^{d_2}, \gamma^{\pm 1} \rangle = q^{\pm \delta_{ij}}, \tag{6.9}
\]
and all other pairs of generators are 0. Moreover, $\langle S(b'), S(b) \rangle = \langle b', b \rangle$ for $b' \in B'$, $b \in B$.

**Proof.** The uniqueness is clear, as any skew-dual pairing of a bialgebra is determined by the value on the generators. We proceed to prove the existence of the Hopf pairing.

The pairing defined on the generators as given in (6.1)–(6.9) can be extended to a bilinear form on $B' \times B$ such that the defining properties in Definition 6.2 hold. We will verify that the relations in $B$ and $B'$ are preserved, ensuring that the form is well-defined and is a skew-dual pairing of $B$ and $B'$.

First of all, it is straightforward to check that the bilinear form preserves the relation (3.1) for $K_i^{\pm 1}$, $\gamma^{\pm 1} \in B$ and $K_i^{\pm 1}, \gamma^{\pm 1} \in B'$. Similar it also preserves (3.2) in $B$ or $B'$ respectively. So we are left to verify that the bilinear form preserves the quantum Serre relations in $B$ and $B'$.

Using [21], we only need to verify the relations involving generators $x_i^{\pm 1}(\mp 1)$ in $B$ and $B'$. Namely, we need to check that
\[
\langle X, (x_i^+(0))^2 x_i^+(0) - 2[x_i^+(0)x_i^+(0)] - x_i^+(0) + x_i^+(0)(x_i^+(0))^2 \rangle = 0,
\]
for any word \( X \) in the generators of \( \mathcal{B}' \). By definition, this equals to

\[
\langle \Delta^{(2)}(X), x_i^+(0) \otimes x_i^+(0) \otimes x_0^-(1) \rangle - [2|x_i^+(0) \otimes x_0^+(1) \otimes x_i^+(0)| + x_0^+(1) \otimes x_i^+(0) \otimes x_i^+(0)),
\]

where the \( \Delta \) corresponds to \( \Delta^{op}_{\mathcal{B}'} \). If one of these terms were nonzero, \( X \) would have involved exactly two \( x_i^-(0) \) factors, one \( x_i^+(1) \) factor, and arbitrarily many \( K_j^{\pm 1} (j \in I) \) or \( \gamma^{\pm \frac{1}{2}} \) factors. For simplicity, we first consider three cases:

(i) If \( X = x_i^-(0)^2 x_i^-(0) \), then \( \Delta^{(2)}(X) \) is equal to

\[
(K_0^{-1} \otimes K_0^{-1} \otimes x_i^-(0) + K_0^{-1} \otimes x_i^-(0) \otimes 1 + x_i^-(0) \otimes 1 \otimes 1)^2 (K_0 \otimes K_0 \otimes x_0^-(1) + K_0 \otimes x_0^-(1) \otimes 1 + x_0^-(1) \otimes 1 \otimes 1 - \sum_{\ell=1}^{n-1} (q - q^{-1}) x_{\ell,-1}^+(0) K_0 \otimes x_{\ell,0}^-(1) \otimes 1 - \sum_{\ell=1}^{n-1} (q - q^{-1}) \Delta (x_{\ell,-1}^+(0) K_0) \otimes x_{\ell,0}^-(1)).
\]

The relevant terms of \( \Delta^{(2)}(X) \) are

\[
x_1^-(0) K_1^{-1} K_0 \otimes x_1^-(0) K_0 \otimes x_0^-(1) + K_1^{-1} x_1^-(0) K_0 \otimes x_1^-(0) K_0 \otimes x_0^-(1) + x_1^-(0) K_1^{-1} K_0 \otimes K_1^{-1} x_0^-(1) \otimes x_1^-(0) + K_1^{-1} x_1^-(0) K_0 \otimes K_1^{-1} x_0^-(1) \otimes x_1^-(0) + K_1^{-1} K_1^{-1} x_0^-(1) \otimes x_1^-(0) \otimes x_1^-(0) + K_1^{-1} K_1^{-1} x_0^-(1) \otimes K_1^{-1} x_1^-(0) \otimes x_1^-(0).
\]

Therefore (6.10) becomes

\[
\langle x_1^-(0) K_1^{-1} K_0, x_1^+(0) \rangle \langle x_1^-(0) K_0, x_1^+(0) \rangle \langle x_0^-(1), x_0^+(1) \rangle - \langle x_1^-(0) K_1^{-1} K_0, x_1^+(0) \rangle \langle x_1^-(0) K_0, x_1^+(0) \rangle \langle x_0^-(1), x_0^+(1) \rangle - \langle x_1^-(0) K_1^{-1} K_0, x_1^+(0) \rangle \langle x_1^-(0) K_0, x_1^+(0) \rangle \langle x_0^-(1), x_0^+(1) \rangle
\]

\[
+ \langle x_1^-(0) K_0, x_1^+(0) \rangle \langle x_1^-(0) K_0, x_1^+(0) \rangle \langle x_0^-(1), x_0^+(1) \rangle - \langle x_1^-(0) K_0, x_1^+(0) \rangle \langle x_1^-(0) K_0, x_1^+(0) \rangle \langle x_0^-(1), x_0^+(1) \rangle
\]

\[
+ \langle x_1^-(0) K_0, x_1^+(0) \rangle \langle K_1^{-1} K_0, x_1^-(0) \rangle \langle x_1^-(0), x_1^+(0) \rangle - \langle x_1^-(0) K_0, x_1^+(0) \rangle \langle K_1^{-1} K_0, x_1^-(0) \rangle \langle x_1^-(0), x_1^+(0) \rangle
\]

\[
+ \langle (K_1^{-1} K_1^{-1} x_0^-(1), x_1^-(0) \rangle \langle K_1^{-1} x_1^-(0), x_0^+(1) \rangle \rangle + \langle (K_1^{-1} K_1^{-1} x_0^-(1), x_1^-(0) \rangle \langle K_1^{-1} x_1^-(0), x_0^+(1) \rangle \rangle
\]

\[
+ \langle (K_1^{-1} x_1^-(0), x_1^-(0) \rangle \langle x_1^-(0) x_0^-(1), x_1^+(0) \rangle \rangle + \langle (K_1^{-1} x_1^-(0), x_1^-(0) \rangle \langle x_1^-(0) x_0^-(1), x_1^+(0) \rangle \rangle
\]

\[
= \frac{1}{(q^{-1}-q)^3} \left[ 1 + q^2 - (q + q^{-1})(q + q^{-1}) + (q^2 + q^{-2}) \right] = 0.
\]

(ii) When \( X = x_i^-(0) x_0^-(1) x_i^-(0) \), it is easy to get the relevant terms of \( \Delta^{(2)}(X) \):

\[
K_1^{-1} K_0 x_1^-(0) \otimes x_1^-(0) K_0 \otimes x_0^-(1) + x_1^-(0) K_0 K_1^{-1} \otimes K_0 x_1^-(0) \otimes x_0^-(1) + K_1^{-1} K_0 x_1^-(0) \otimes K_1^{-1} x_0^-(1) \otimes x_1^-(0) + x_1^-(0) K_0 K_1^{-1} \otimes x_0^-(1) K_1^{-1} \otimes x_1^-(0) + K_1^{-1} x_0^-(1) K_1^{-1} \otimes x_1^-(0) \otimes x_1^-(0) + K_1^{-1} x_0^-(1) \otimes K_1^{-1} x_1^-(0) \otimes x_1^-(0) + K_1^{-1} x_0^-(1) K_1^{-1} \otimes x_1^-(0) \otimes x_1^-(0).
\]

Thus, (6.10) becomes

\[
\frac{1}{(q^{-1}-q)^3} \left[ \langle K_1^{-1} K_1^{-1} x_1^-(0), x_1^-(0) \rangle \langle K_0, K_1 \rangle + \langle K_0, K_1 \rangle - (q + q^{-1}) \langle K_1^{-1} K_1^{-1} x_1^-(0), x_1^-(0) \rangle + \langle K_1^{-1} K_1^{-1} x_1^-(0), x_1^-(0) \rangle \langle K_0, K_1 \rangle + \langle K_1^{-1} x_1^-(0), x_1^-(0) \rangle \langle K_0, K_1 \rangle + \langle K_1^{-1} x_1^-(0), x_1^-(0) \rangle \langle K_1^{-1}, K_0 \rangle + \langle K_1^{-1}, K_0 \rangle \langle K_1^{-1}, K_1 \rangle + \langle K_1^{-1}, K_0 \rangle \rangle
\]

\[
= \frac{1}{(q^{-1}-q)^3} \left[ (q^2 + q) - (q + q^{-1})(q^2 + 1) + (q + q^{-1}) \right] = 0.
\]

(iii) If \( X = x_0^-(1) x_i^-(0) x_i^-(0) \), one can similarly get that (6.10) vanishes.

Finally, if \( X \) is any word involving exactly two \( x_i^-(0) \) factors, one \( x_0^-(1) \) factor, and arbitrarily many factors \( K_j^{\pm 1} (j \in I) \) and \( \gamma^{\pm \frac{1}{2}} \), then (2.2) will just be a scalar multiple of one of the quantities we have already calculated, and then it will be zero.

Analogous calculations show that the relations in \( \mathcal{B}' \) are preserved.
The following can be proved similarly as [2].

**Theorem 6.4.** $\mathcal{D}(B, B')$ is isomorphic to $\mathcal{U}_0$ for $n = 1$ or $n = 2$ as a Hopf algebra.

**Remark 6.5.** Using the Hopf double structure, one can show the existence of a universal R-matrix for the Hopf algebra $\mathcal{U}_0$. The universal R-matrix [5] for the Hopf algebra $U$ is an invertible element $R$ of $U \hat{\otimes} U$ satisfying the conditions

\[
\Delta(x) = R \Delta(x) R^{-1}, \quad \forall x \in U,
\]

\[
(\Delta \otimes \text{id}) R = R_{13} R_{23}, \quad (\text{id} \otimes \Delta) R = R_{13} R_{12},
\]

where $\Delta = \Delta^{op}$ is the opposite comultiplication in $U$. In this case $U$ is a quantum double of a Hopf algebra $U^+, U \simeq U^+ \otimes U^-, U^-$ being dual to $U^+$ with an opposite comultiplication, then $U$ admits a canonical presentation of the universal R-matrix $R = \sum e_i \otimes f_i$, where $e_i$ and $f_i$ are dual bases of $U^+$ and $U^-$. 

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**References**

1. Beck, J. (1994). Braid group action and quantum affine algebras. Commun. Math. Phys. 165:555–568.
2. Bergeron, N., Gao, Y., Hu, N. (2006). Drinfeld doubles and Lusztig's symmetries of two-parameter quantum groups. J. Algebra 301:378–405.
3. Ding, J., Frenkel, I. B. (1993). Isomorphism of two realizations of quantum affine algebra $U_q(gl(n))$. Commun. Math. Phys. 156:277–300.
4. Ding, J., Iohara, K. (1997). Generalization of Drinfeld quantum affine algebras. Lett. Math. Phys. 41:181–193.
5. Drinfeld, V. G. (1987). Quantum groups. Proceedings of the ICM, Berkeley, 1986, pp. 798–820. Providence: American Mathematical Society.
6. Drinfeld, V. G. (1988). A new realization of Yangians and quantized affine algebras. Soviet Math. Dokl. 36:212–216.
7. Frenkel, I. B., Jing, N., Wang, W. (2000). Quantum vertex representations via finite groups and the McKay correspondence. Commun. Math. Phys. 211:365–393.
8. Feigin, B., Jimbo, M., Miwa, T., Mukhin, E. (2016). Branching rules for quantum toroidal $gl(n)$. Adv. Math. 300:229–274.
9. Feigin, B., Jimbo, M., Miwa, T., Mukhin, E. (2013). Representations of quantum toroidal $gl_n$. J. Algebra 380:78–108.
10. Feigin, B., Jimbo, M., Miwa, T., Mukhin, E. (2015). Quantum toroidal $gl_1$ and Bethe ansatz. J. Phys. A: Math. Theor. 48:244001.
11. Frenkel, E., Yu. Reshetikhin, N. (1999). The q-characters of Representations of Quantum Affine Algebras and Deformations of W-algebras, Recent Developments in Quantum Affine Algebras and Related Topics (Raleigh, NC, 1998), Contemporary Mathematics, Vol. 248. American Mathematical Society: Providence, RI, pp. 163–205.
12. Frenkel, E., Mukhin, E. (2001). Combinatorics of q-characters of finite-dimensional representations of quantum affine algebras. Commun. Math. Phys. 216:23–57.
13. Frenkel, I. B., Yu. Reshetikhin, N. (1992). Quantum affine algebras and holonomic difference equations. Commun. Math. Phys. 146:1–60.
14. Gautam, S., Toledano-Laredo, V. (2013). Yangians and quantum loop algebras. Selecta Math., (N.S.) 19:271–336.
15. Guay, N., Ma, X. (2012). From quantum loop algebras to Yangians. J. Lond. Math. Soc. 86:683–700.
16. Guay, N., Nakajima, H., Wendlandt, C. (2018). Coproduct for the Yangian of an affine Kac-Moody algebra. Adv. Math. 338:865–911.
17. Ginzburg, V., Kapranov, M., Vasserot, E. (1995). Langlands reciprocity for algebraic surfaces. Math. Res. Lett. 2:147–160.
18. Grosse, P. (2007). On quantum shuffle and quantum affine algebras. J. Algebra 318:495–519.
[19] Hernandez, D. (2007). Drinfeld coproduct, quantum fusion tensor category and applications. *Proceedings of the London Mathematical Society* 95:567–608.

[20] Hernandez, D. (2009). Quantum toroidal algebras and their representations. *Selecta Math. (N.S.)*, 14:701–725.

[21] Hu, N., Rosso, M., Zhang, H. (2008). Two-parameter quantum affine algebra $U_{r,s}(\hat{sl}_n)$, Drinfeld realization and quantum affine Lyndon basis. *Commun. Math. Phys.* 278:453–486.

[22] Jing, N. (1998). On Drinfeld realization of quantum affine algebras. *Proc. Monster and Lie Algebras, Ohio Publ., Gruyter Verlag*. 7:195–206.

[23] Jing, N. (1996). Higher level representations of the quantum affine algebra $U_q(\hat{sl}(2))$. *J. Algebra* 182:448–468.

[24] Jing, N. (1998). Quantum Kac-Moody algebras and vertex representations. *Lett. Math. Phys.* 4:261–271.

[25] Jing, N., Kang, S.-J., Koyama, Y. (1995). Vertex operators between level one irreducible representations of the quantum affine algebra $U_q(D^{(1)})$. *Commun. Math. Phys.* 174:367–392.

[26] Jing, N., Misra, K. C. (1999). Vertex operators for twisted quantum affine algebras. *Trans. Amer. Math. Soc.* 351:1663–1690.

[27] Jing, N., Zhang, H. (2011). Two-parameter quantum vertex representations via finite groups and the McKay correspondence. *Trans. Amer. Math. Soc.* 363:3769–3797.

[28] Jing, N., Zhang, H. (2016). Two-parameter twisted quantum affine algebras. *J. Math. Phys.* 57:091702.

[29] Khoroshkin, S. M., Tolstoy, V. N. (1996). Yangian double. *Lett. Math. Phys.* 36:373–402.

[30] Klimyk, A., Schmüdgen, K. (1997). Quantum groups and their representations. Berlin-Heidelberg-New York: Springer-Verlag.

[31] Maulik, D., Okounkov, A. (2019). Quantum groups and quantum cohomology. *Astérisque* 408:209.

[32] Miki, K. (2000). Representations of quantum toroidal algebra $U_q(sl_{n+1}, tor)(n > 2)$. *J. Math. Phys.* 41:7079–7098.

[33] Miki, K. (2016). Actions of the quantum toroidal algebra of type $sl_2$ on the space of vertex operators for $U_q(\hat{gl}_2)$ modules. *J. Math. Phys.* 57:071701.

[34] Miki, K. (2005). Some quotient algebras arising from the quantum toroidal algebra $U_q(sl_2(C_\gamma))$. *Osaka J. Math.* 42:885–929.

[35] Varagnolo, M., Vasserot, E. (1996). Schur duality in the toroidal setting. *Commun. Math. Phys.* 182:469–484.