Cooling of nanomechanical vibrations by Andreev injection

O.M. Bahrova,1,2 S.I. Kulinich,2 L.Y. Gorelik,3 R.I. Shekhter,4 and H.C. Park1

1Center for Theoretical Physics of Complex Systems, Institute for Basic Science, Daejeon, 34126, Republic of Korea
2B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine, 47 Nauky Ave., Kharkiv 61103, Ukraine
3Department of Physics, Chalmers University of Technology, SE-412 96 Göteborg, Sweden
4Department of Physics, University of Gothenburg, SE-412 96 Göteborg, Sweden

(Dated: February 17, 2022)

A nanoelectromechanical weak link composed of a carbon nanotube suspended between two normal electrodes in a gap between two superconducting leads is considered. The nanotube is treated as a movable single-level quantum dot in which the position-dependent superconducting order parameter is induced due to the Cooper pair tunneling. We show that electron tunneling processes significantly affect the state of the mechanical subsystem. We found that at a given direction of the applied voltage between the electrodes, the stationary state of the mechanical subsystem has a Boltzmann form with an effective temperature depended on the parameters of the device. As this takes place, the effective temperature can reach significantly small values (cooling effect). We also demonstrate that nanotube fluctuations strongly affect the dc current through the system. The latter can be used to probe the predicted effects in an experiment.

PACS numbers:

INTRODUCTION

Nanoelectromechanical (NEM) systems promise to manipulate the mechanical motion of a nano-object using electronic dynamics. There are many approaches to control nanomechanical performance providing a number of new functionalities of nano-device operations, in particular, pumping or cooling of the mechanical subsystem. One of the main approaches exploits the dc electronic flow through a nanosystem induced by either the bias voltage or temperature drop between two electronic reservoirs connected by a quantum dot (QD). In particular, pumping or cooling effect. Furthermore, if the tunneling amplitude depends on the distance between the QD and the SC leads, such exchange also provides a connection between the electronic and mechanical degrees of freedom.

In the paper, a nanoelectromechanical weak link composed of the carbon nanotube suspended above a trench in a normal metal electrode and positioned in a gap between two superconducting electrodes. The nanotube has been treated as a movable single-level quantum dot, in which the position-dependent superconducting order parameter is induced as a result of Cooper pair tunneling. It has been shown that in such a system self-sustained bending vibrations can emerge if a constant bias voltage is applied between normal and superconducting electrodes.

However, the semiclassical approach used in that paper does not allow the investigation of the operation of such a NEMS in the cooling regime. In this paper, using the reduced density matrix technique, we calculate the Wigner function characterizing a stationary state of the mechanical subsystem. We demonstrate that at certain conditions it has a Boltzmann distribution form with an effective temperature that can reach abnormally low values. The occurrence of this effect crucially depends on the direction of the bias voltage and the relative position of the quantum dot level. We also show that the nanotube fluctuations strongly affect the dc current through the system, a characteristic that can be used for direct experimental observation.
The Hamiltonian \( \hat{H}_t = H_t^n + H_t^s \), describes the normal and superconducting leads, respectively,

\[
H_t^n = \sum_{k\sigma} (\varepsilon_k - eV_b) a_{k\sigma}^\dagger a_{k\sigma},
\]

\[
H_t^s = \sum_{kj\sigma} \left( \varepsilon_k c_{kj\sigma}^\dagger c_{kj\sigma} - \Delta_s (e^{i\phi/2} c_{kj\uparrow}^\dagger c_{kj\downarrow} - e^{-i\phi/2} c_{kj\downarrow}^\dagger c_{kj\uparrow}) + \text{H.c.} \right).
\]

Here \( a_{k\sigma}^\dagger(a_{k\sigma}) \) and \( c_{kj\sigma}^\dagger(c_{kj\sigma}) \) are creation (annihilation) operators of the electron with quantum number \( k \) and spin projection \( \sigma \) in the normal and superconducting \( j = 1, 2 \) leads, respectively. \( \Delta_s, e^{i\phi/2} \) is the superconducting order parameter (in the \( j \) electrode). Note that energies \( \varepsilon_d, \varepsilon_k \) are counted from the Fermi energy of superconductors. In what follows we suppose \( \phi_1 = -\phi_2 = \phi/2 \).

The Hamiltonian \( H_t = H_t^n + H_t^s \) represents tunneling of electrons between the dot and the leads,

\[
H_t^n = \sum_{k\sigma} t^n_{kj}(a_{k\sigma}^\dagger d_{\sigma} + \text{H.c.}),
\]

\[
H_t^s = \sum_{kj\sigma} t^s_{kj}(x)(c_{kj\sigma}^\dagger d_{\sigma} + \text{H.c.}).
\]

Here the superconducting tunneling amplitude \( t^s_{kj}(x) = t^s_{nk}(x+\pi a/2\lambda) \) is position dependent, where \( 2\lambda \) is the characteristic tunneling length and \( a \) is the parameter of the asymmetry. For a typical CNT-based nanomechanical resonator, \( 2\lambda \sim 0.5 \text{ nm} \) [23]. In what follows we will concentrate our attention on the symmetric case, \( a = 0 \).

**DENSITY MATRIX APPROACH**

The time evolution of the system density matrix \( \hat{\rho} \) is described by the Liouville-von Neumann equation. We use the reduced density matrix approximation according to which the full density matrix of the system \( \hat{\rho} \) is factorized to the tensor product of the equilibrium density matrices of the normal and superconducting leads, and the dot density matrix as \( \hat{\rho} = \hat{\rho}_n \otimes \hat{\rho}_s \otimes \hat{\rho}_d \). Note that the reduced density operator \( \hat{\rho}_d \) acts on the Hilbert space which can be presented as the tensor product of the vibrational space of the oscillator and the electronic space of the single electron level on the QD.

In this paper we consider the stationary state of the system in the deep subgap case \( \Delta_s \gg |eV_b| \gg \Delta_d, \Gamma_n \), where \( \Delta_d = 2\pi\nu_s|\mu|^2, \Gamma_n = 2\pi\nu_s|\mu_n|^2 \) (\( \nu_s(n) \) is a density of states in the superconducting (normal) electrode). Using the standard procedure, one can trace out the leads degree of freedom and obtain the following equation for the reduced density matrix \( \hat{\rho}_d \) [24].

\[
-\dot{t} \left[ H_{\text{eff}}^d + H_v, \hat{\rho}_d \right] + \mathcal{L}_n(\hat{\rho}_d) + \mathcal{L}_\gamma(\hat{\rho}_d) = 0,
\]

where

\[
H_{\text{eff}}^d = H_d + \Delta_d(x, \phi) d_{\uparrow} d_{\downarrow} + \Delta_d^*(x, \phi) d_{\downarrow}^\dagger d_{\uparrow}^\dagger,
\]

\[
\Delta_d(x, \phi) = \Delta_d \cosh(x/\lambda + i\phi/2).
\]
In Eq. (9) $\Delta_d(x, \phi)$ is the off-diagonal order parameter induced by the superconducting proximity effect [23]. The Lindbladian term in Eq. (8), $\mathcal{L}_n\{\hat{\rho}_d\}$, is induced by the incoherent electron exchange between the normal lead and the QD. The latter in the high bias voltage regime, $|eV_b| > \varepsilon_0$, $\hbar \omega, T$, takes the form

$$\mathcal{L}_n\{\hat{\rho}\} = \Gamma_n \sum_\sigma \left\{ 2d_\sigma \hat{d}_\sigma - \{d_\sigma d_\sigma^\dagger, \hat{\rho}\} \right\}; \quad \kappa = +1;$$

$$2d_\sigma \hat{d}_\sigma^\dagger - \{d_\sigma^\dagger d_\sigma, \hat{\rho}\}, \quad \kappa = -1;$$  \hspace{1cm} (11)

where $\kappa = \text{sgn}(eV_b)$.

In Eq. (8) we phenomenologically introduce the dissipation term $\mathcal{L}_\gamma\{\hat{\rho}_d\}$ [26],

$$\mathcal{L}_\gamma\{\hat{\rho}\} = -m\omega \gamma (n_B + 1/2) [x, [x, \hat{\rho}]] - \iota (\gamma/2) [x, \{p, \hat{\rho}\}],$$

where $\gamma$ is the damping rate, $n_B$ is the Bose-Einstein distribution function,

$$n_B = \frac{1}{e^{\hbar \omega/T} - 1},$$

and $T$ is a temperature of the thermodynamic environment.

Figure 2A represents the electronic dynamics on the dot for two directions of the applied bias voltage, $\kappa = \pm 1$. Because of the considered parameter scales, not all electron processes are allowed. In the subgap regime, single-electron transitions between the dot and the superconducting leads are prohibited, and thus only an exchange of Cooper pairs occurs. Additionally, single-electron tunneling between the dot and the normal lead is enabled exclusively in one direction (from the lead to the dot, Fig. 2A) or otherwise, Fig. 2B in the high bias voltage regime.

The state of the mechanical subsystem is completely described by the reduced density matrix $\hat{\rho}_v = \text{Tr}\hat{\rho}_d$, where the tracing operation is taken over the electronic degrees of freedom on the dot. It is obvious that in the limiting case $\lambda \to \infty$ the electronic and vibronic subsystems are independent and the reduced vibronic density matrix has a form of equilibrium density matrix with the effective temperature that is determined by an environment temperature $T$. An alternative (and more visual) is the description in terms of the Wigner distribution function

$$W_v(x, p) = \frac{1}{2\pi} \int d\xi e^{-i\xi p} \left\langle x + \frac{\xi}{2} | \hat{\rho}_v | x - \frac{\xi}{2} \right\rangle$$

(we use the dimensionless variables: $x/x_0 \to x, px_0/h \to p$, where $x_0$ is the amplitude of zero-point oscillations, all energy parameters are measured in units of $\hbar \omega$, the tunneling length $\lambda$ is measured in units of $x_0, \gamma/\omega \to \gamma$).

The problem, Eqs. (9)-(12), can be solved by the perturbation expansion for the Wigner distribution function $W_v(x, p)$,

$$W_v(x, p) = W_v^{(0)}(x, p) + W_v^{(1)}(x, p) + ..., $$

using the smallness of the parameter $1/\lambda \simeq 10^{-2} - 10^{-3}$ [23] (or the parameter of electromechanical coupling, $\Delta_d/\lambda \ll 1$). We have found (see Appendix for details of calculations) that the relevant Wigner function which gives the probability distribution of the vibrational amplitudes $A = \sqrt{x^2 + p^2}$ in a stationary regime, for $A \ll \lambda$ is a solution of the stationary Fokker-Planck equation.

FIG. 2: Schematic illustration of the enabled transitions between electronic states in the quantum dot. The single-electron states change due to transitions from the empty to the single-occupied QD and then from the single-occupied to the double-occupied one (indicated by orange arrows). In the high bias voltage regime, the tunneling of electrons (a) or holes (b) with spin $\downarrow$ or $\uparrow$ is allowed only from the normal lead to the dot and forbidden in the opposite direction. Transitions between the empty and double-occupied QD are due to coupling with the superconducting leads (indicated by blue arrows).
equation,
\[ D_1 \frac{\partial}{\partial \phi} \left( A^2 W^{(0)}_\phi \right) + D_2 \frac{\partial}{\partial A} \left( A \frac{\partial W^{(0)}_\phi}{\partial A} \right) = 0. \]  \hspace{1cm} (16)

Here the drift, \( D_1 \), and diffusive, \( D_2 \), coefficients take the form
\[ D_1 = -\frac{\kappa \Delta_2^2 \Gamma_n \varepsilon_d}{\lambda^2 D_1} \sin^2(\phi/2) + \gamma, \]  \hspace{1cm} (17)
\[ D_2 = \frac{\Delta_2^2 \Gamma_n \varepsilon_d}{\lambda^2 D_1} \sin^2(\phi/2) + \gamma (n_B + 1/2), \]  \hspace{1cm} (18)

where
\[ D = \varepsilon_d^2 + \Gamma_n^2 + \Delta_2^2 \cos^2(\phi/2), \]  \hspace{1cm} (19)
\[ D_1 = (D - 1/4)^2 + \Gamma_n^2, \]  \hspace{1cm} (20)
\[ C = \frac{(D + 1/4)(D + \varepsilon_d^2 + \Gamma_n^2) - 4\Delta_2^2 \Gamma_n \cos^2(\phi/2)}{4D}. \]  \hspace{1cm} (21)

The solution of Eq. (16) at small (in comparison to \( \lambda \)) values of the amplitude has a form of the Boltzmann distribution function,
\[ W^{(0)}_\phi(x, p) = (\beta/\pi) \exp \left[ -\beta \left( x^2 + p^2 \right) \right], \]  \hspace{1cm} (22)
where the coefficient \( \beta = D_1/2D_2 \).

The expressions, Eqs. (17), (18), define the framework of validity of our consideration. It follows from Eqs. (17)-(21) that in the region which is related to the maximal cooling effect (the range of the values of parameters \( \phi, \varepsilon_d \)) near the point \( \varepsilon_d = 1/2, \phi = \pi \) the value of the level width is restricted from below, \( \Gamma_n \geq \Gamma^{(0)}_n = \Delta_d^2/\lambda^2 \).

**GROUND-STATE COOLING**

Nowadays, nanomechanical resonators with a significant value of the quality factor are achieved in experiments \[27,28\]. For such a case, the electromechanical coupling dominates the coupling with the thermodynamic environment, \( 1/\lambda \gg \gamma \). Thus, let us consider the case \( \gamma \to 0 \). From Eqs. (17)-(21) it follows that the sign of the coefficient \( \beta \) is determined by the sign of \( \kappa \varepsilon_d \). If \( \kappa \varepsilon_d \) is positive, \( \beta \) becomes negative. This situation corresponds to mechanical instability of the system and it was discussed in Ref. [21]. In what follows we consider the vibronic (stable) regime, when \( \kappa = -1, \varepsilon_d > 0 \) (the same for \( \kappa = +1, \varepsilon_d < 0 \)).

The coefficient \( \beta \) determines the probability \( P_0 \) that the system is in its ground state. In terms of Wigner distribution functions this probability takes a form
\[ P_0 = 2\pi \int dx dp W^{(0)}_\phi(x, p)W_0(x, p) = \frac{2\beta}{\beta + 1}, \]  \hspace{1cm} (23)
where \( W_0(x, p) = (1/\pi) \exp[-(x^2 + p^2)] \) is the Wigner function of the harmonic oscillator ground state. Note that according to Heisenberg’s uncertainty principle the maximal value of parameter \( \beta \) is equal to unity, \( \beta < \beta_{\text{max}} = 1 \).

Dependences of the probability \( P_0 \) as a function of the superconducting phase difference \( \phi \) for different values of the quantum dot energy level: \( \varepsilon_d = 0.1 \) (blue dashed curve), 0.56 (black thick), 1.5 (red dotted), 5 (orange dot-dashed). The black dashed line indicates the maximal value of the occupation probability. Inset: zoomed central region where the cooling reaches its maximum at \( \phi = \pi \). Other parameters:
\( \Gamma_n = 0.2; \Delta_d = 25; \lambda = 100; \gamma = 10^{-5}, T = 15 \).

**DIRECT ELECTRIC CURRENT**

The effects of cooling or heating of the mechanical vibrations can be explored by dc current measurements. The Wigner distribution function gives the possibility to calculate various physical quantities. The supercurrent in the \( j \) superconducting lead is determined by the change of the number of Cooper pairs and can be presented as
\[ I^{(s)}_j = \frac{2e}{h} \text{Tr} \left( \frac{\partial H^{\text{eff}}_d}{\partial \phi_j} \rho_d \right). \]  \hspace{1cm} (24)
Due to the geometry of our system, the normal current is equal to the sum of the partial currents corresponding to the superconducting electrodes, \( I_n = I_n^{(s)} + I_n^{(c)} \). In terms of Wigner functions of the operators (see Appendix for details)

\[
\hat{p}_1 = \text{Tr} \left[ \left( \hat{d} \hat{d}^\dagger + \hat{d}^\dagger \hat{d} \right) \hat{ρ}_d \right], \\
\hat{p}_2 = \text{i} \text{Tr} \left[ \left( \hat{d}^\dagger \hat{d}^\dagger - \hat{d} \hat{d} \right) \hat{ρ}_d \right], 
\]

the expression for the current, Eq. \( \text{(24)} \), takes a form

\[
I_n = e \omega \int dx dp \left[ \Delta_d \sin(\phi/2) \sinh(x/\lambda) W_1 + \Delta_d \cos(\phi/2) \cosh(x/\lambda) W_2 \right]. \tag{27}
\]

The direct calculations of Eq. \( \text{(27)} \) results in

\[
I_n = I_0 \frac{\Delta_d^2 \cos^2(\phi/2)}{\Gamma_n^2 + \varepsilon_d^2 + \Delta_d^2 \cos^2(\phi/2)} + \mathcal{O}(1/\lambda^2), \tag{28}
\]

where \( I_0 = e \Gamma_n / \hbar \) (in dimension units). The leading term in the expression, Eq. \( \text{(28)} \), tends to zero in the limit \( \phi \to \pi \). Thus, at \( \phi = \pi \) the current is determined by the mechanical fluctuations and in the leading order of the electromechanical coupling parameter it reads as

\[
I_n = I_0 \left( \frac{\Delta_d}{\lambda} \right)^2 \frac{(\Gamma_n^2 + \varepsilon_d^2 + 1/4) \langle x^2 \rangle + \varepsilon_d/2}{(\Gamma_n^2 + \varepsilon_d^2 - 1/4)^2 + \Gamma_n^2}, \tag{29}
\]

where the \( \langle \ldots \rangle \) denote the average value in the phase space with \( W_0^{(0)}(x,p) \) and \( \langle x^2 \rangle = (2\beta)^{-1} \).

Figure 4 shows the dependence of the electric current on the quantum dot level energy \( \varepsilon_d \) for different values of \( \Gamma_n \) at \( \phi = \pi \). We see that in the cooling regime the dependence of the electric current has a pronounced minimum-maximum structure, that disappears in the "heating" regime \( (P_0 \lesssim 0.5) \). This fact can serve as a criterion that the system is in the cooling regime.

## CONCLUSIONS

We have considered the nanomechanical weak link that involves a carbon nanotube suspended between two normal leads and biased by a constant voltage. The nanotube, which is treated as a single-level quantum dot, performs bending vibrations in a gap between two superconducting electrodes. The coupling between the electronic and mechanical degrees of freedom is induced due to the superconducting proximity effect which exhibits in the appearance of the position-dependent dot order parameter. Using the density matrix approximation, we have found that at certain direction of the applied bias voltage, the stationary state of the mechanical subsystem has a Boltzmann form. Moreover, the probability to find the system in the ground state has been demonstrated to be \( P_0 \lesssim 1 \). The latter is related to the cooling regime of the considered system. Additionally, the probability depends on the superconducting phase difference and the relative position of the QD energy level in a key manner. Also, we have discussed that the direct electric current behaviour mirrors the stationary state of the system. It can be served for an experimental detection of the predicted effects.

**Acknowledgements**

O.M.B. thanks A.V. Parafilo for helpful discussions. Authors acknowledge the financial support from the IBS in Republic of Korea (IBS-R024-D1) and the NAS of Ukraine (grant F 26-4) (S.I.K.).

## APPENDIX: EQUATIONS FOR THE WIGNER DISTRIBUTION FUNCTION

The QD density matrix \( \hat{ρ}_d \) acts in the Hilbert space that can be presented as a tensor product of the vibrational space of the harmonic oscillator and the Fock space of the single-level QD which is spanned on the state vectors \( |0\rangle, \hat{d}^\dagger |d^\dagger \rangle |0\rangle = |\uparrow \downarrow \rangle, \hat{d}^\dagger |\downarrow \rangle |0\rangle = |\uparrow \rangle \equiv |2\rangle \). We have got the following system of equations of motion for electronic components of the density matrix, \( \hat{ρ}_d \) (\( \kappa = +1 \)),

\[
\]
\[ \begin{align*}
\partial_t \rho_0 &= -i [H_v, \rho_0] - 4 \Gamma_n \rho_0 - i \Delta_d(x, \phi) \rho_{20} + i \rho_{02} \Delta_d^*(x, \phi) + \mathcal{L}_\gamma \{ \rho_0 \}, \\
\partial_t \rho_1 &= -i [H_v, \rho_1] + 2 \Gamma_n (\rho_0 - \rho_1) + \mathcal{L}_\gamma \{ \rho_1 \}, \\
\partial_t \rho_2 &= -i [H_v, \rho_2] + 2 \Gamma_n (\rho_0 - \rho_2) + \mathcal{L}_\gamma \{ \rho_2 \}, \\
\partial_t \rho_{02} &= -i [H_v, \rho_{02}] + 2 \varepsilon_d \rho_{02} - 2 \Gamma_n \rho_{02} - i \Delta_d(x, \phi) \rho_2 + i \rho_0 \Delta_d^*(x, \phi) + \mathcal{L}_\gamma \{ \rho_{02} \}, \\
\partial_t \rho_{12} &= -i [H_v, \rho_{12}] - 2 \varepsilon_d \rho_{12} - 2 \Gamma_n \rho_{12} - i \Delta_d^*(x, \phi) \rho_2 + i \rho_{12} \Delta_d(x, \phi) + \mathcal{L}_\gamma \{ \rho_{12} \}, \\
\partial_t \rho_{20} &= -i [H_v, \rho_{20}] - 2 \varepsilon_d \rho_{20} - 2 \Gamma_n \rho_{20} - i \Delta_d^*(x, \phi) \rho_2 + i \rho_{20} \Delta_d(x, \phi) + \mathcal{L}_\gamma \{ \rho_{20} \}.
\end{align*} \]  

(A.1)  

(A.2)  

(A.3)  

(A.4)  

(A.5)  

(A.6)

To find the equations in case of the opposite direction of the bias voltage, \( \kappa = -1 \), one needs to switch \( 0 \leftrightarrow 2 \). The consequent analysis of system, Eqs. (A.1)–(A.6), is that to use the Wigner representation in the oscillator space, Eq. (14). We are interested in a steady state regime of the mechanical subsystem in the limit when the parameter \( 1/\lambda \) is small. To find the solution of Eqs. (A.1)–(A.6) to leading order in this parameter, it is convenient to introduce the linear combinations of the Wigner distribution functions as follows,

\[ \begin{align*}
W_v &= W_0 + W_1 + W_2, \\
W_0 &= W_0 + W_2, W_1 = W_{20} + W_{02}, \\
W_2 &= i(W_{02} - W_{20}), W_3 = W_0 - W_2. \quad (A.7)
\end{align*} \]

In addition, it is convenient to change from \((x, p)\) to polar coordinates \((A, \varphi)\) so that \( x - \bar{x} = A \sin \varphi \) and \( p = A \cos \varphi \), where \( \bar{x} \sim (1/\lambda) \) is an equilibrium position of the dot. In the polar coordinates, the steady state equation for the Wigner distribution function that describes the mechanical degree of freedom, \( W_v(A, \varphi) \), is given by the equation (up to terms of the second order in the parameter \( 1/\lambda \)),

\[ \begin{align*}
- \frac{\partial W_v}{\partial \varphi} + \bar{x} \bar{T} W_v + \gamma (n_B + 1/2) \bar{T}^2 W_v &= \gamma \left( W_v + A \cos \varphi \bar{T} W_v \right) = 0.
\end{align*} \]  

(A.8)

In Eq. (A.8) the differential operator \( \bar{T} \) is defined according to the expression,

\[ \bar{T} = \cos \varphi \frac{\partial}{\partial A} - \sin \varphi \frac{\partial}{\partial \varphi}. \]  

(A.9)

Eq. (A.8) for the Wigner function \( W_v(A, \varphi) \) is coupled to the steady state equation for the vector-function \( \vec{W} = (W_0, W_1, W_2, W_3)^T \) that takes the following form (up to terms of the first order in the parameter \( 1/\lambda \)),

\[ \begin{align*}
- \frac{\partial \vec{W}}{\partial \varphi} + 2 \mathcal{M} \vec{W} &= \vec{F}, \quad (A.10)
\end{align*} \]

\[ \mathcal{M} = \begin{pmatrix}
-2 \Gamma_n & 0 & 0 & -\kappa \Gamma_n \\
0 & -\Gamma_n & 0 & 0 \\
-\varepsilon_d & -\Gamma_n & -\Delta_d \cos(\phi/2) & 0 \\
0 & \Delta_d \cos(\phi/2) & -\Gamma_n & \frac{\bar{T} W_2}{2 A \sin(\varphi/2)} \\
\end{pmatrix}, \]

\[ \vec{F} = -\bar{x} \bar{T} \vec{W} - 2 \Gamma_n W_v \begin{pmatrix}
1 \\
0 \\
0 \\
-\kappa
\end{pmatrix} + \Delta_d \sin(\phi/2) \begin{pmatrix}
\bar{T} W_2 \\
\frac{\bar{T} W_0}{2 A \sin(\varphi/2)} \\
-2 A \sin(\varphi/2) \\
\end{pmatrix}. \]

Eqs. (A.8)–(A.11) have to be solved subject to the periodic boundary conditions, \( W_i(A, \varphi + 2\pi) = W_i(A, \varphi) \), \( \vec{W}(A, \varphi + 2\pi) = \vec{W}(A, \varphi) \).

We solve these equations by perturbation expansions,

\[ \begin{align*}
W_i(A, \varphi) &= W_i^{(0)}(A, \varphi) + W_i^{(1)}(A, \varphi) + ..., \quad (A.11)
\end{align*} \]

\((i = v, 0, 1, 2, 3)\), where \( W_i^{(n)} \) is of \( n \)-th order in \( 1/\lambda \).

It is evident from Eqs. (A.8)–(A.11) that the functions \( W_v^{(n)}(A, \varphi), \vec{W}^{(n)}(A, \varphi) \) do not depend on \( \varphi \). Hence, \( W_v^{(0)}(A, \varphi) = W_v^{(0)}(A) \) and

\[ \begin{align*}
W_0^{(0)} &= \frac{\varepsilon_d^2 + \Gamma_n^2 + (\Delta_d^2/2) \cos(\phi/2)}{D} W_v^{(0)}, \\
W_1^{(0)} &= \frac{\Delta_d \varepsilon_d \cos(\phi/2)}{D} W_v^{(0)}, \\
W_2^{(0)} &= \frac{\Delta_d \Gamma_n \cos(\phi/2)}{D} W_v^{(0)}, \\
W_3^{(0)} &= -\kappa \frac{\varepsilon_d^2 + \Gamma_n^2}{D} W_v^{(0)}, \]

(A.12)  

(A.13)  

(A.14)  

(A.15)

where the coefficient \( D \) is defined by Eq. (13).

From the requirement, \( W_v^{(1)}(A, \varphi) = W_v^{(1)}(A) \), to first order in the perturbation theory, Eq. (A.8) determines the equilibrium position of the dot,

\[ \bar{x} = \kappa \frac{\Delta_d^2}{\lambda D} \sin(\phi/2) \cos(\phi/2). \]  

(A.16)
To second order in perturbation theory, Eq. (A.8) after averaging over $\varphi$ variable takes the form,

$$ -\Delta_0 \sin(\phi/2) \frac{\partial}{\partial A} \left( A \left( \cos \frac{\varphi}{2} W_2^{(1)} \right) \right) + \frac{\gamma}{2A} \frac{\partial}{\partial A} \left( A^2 W_v^{(0)} \right) + \frac{\gamma (n_B + 1/2)}{2A} \frac{\partial}{\partial A} \left( A^2 \frac{\partial W_v^{(0)}}{\partial A} \right) = 0. \quad (A.17) $$

Here the brackets, $\langle f(A, \varphi) \rangle$, in Eq. (A.17) denote the zeroth Fourier component of the $2\pi$-periodic function $f(A, \varphi)$; when deriving Eq. (A.17) we used the property,

$$ \langle \hat{T} f(A, \varphi) \rangle = \frac{1}{A} \frac{\partial}{\partial A} \left( A \langle \cos \varphi f(A, \varphi) \rangle \right). \quad (A.18) $$

Therefore, to get a closed equation for $W_v^{(0)}(A)$, one needs to know the function $W_2^{(1)}(A, \varphi)$. To first order in perturbation theory, this function can be determined from Eqs. (A.10)-(A.11). As a result, one gets the stationary Fokker-Planck equation for the oscillator Wigner distribution function $W_v^{(0)}(A)$, Eq. (16).

[1] K.L. Ekinci, and M.L. Roukes, Rev. Sci. Instrum. 76, 061101 (2005).
[2] A.N. Cleland, Foundations of Nanomechanics (Springer, New York, 2002).
[3] L.Y. Gorelik, A. Isacsson, M.V. Voinova, B. Kasemo, R.I. Shekhter, and M. Jonson, Phys. Rev. Lett. 80, 4526 (1998).
[4] G. Rastelli, W. Belzig Eur. Phys. J. Special Topics 227, 1885 (2019).
[5] C. Urgell, W. Yang, S.L. De Bonis, C. Samanta, M.J. Esplandiu, Q. Dong, Y. Jin, and A. Batchtold, Nature Physics 16, 32 (2020).
[6] K. Willick, and J. Baugh, Phys. Rev. Research 2, 033040 (2020).
[7] G.A. Steele, A. Huttel, B. Witkamp, M. Poot, H.B. Meerveld, L.P. Kouwenhoven, and H.S.J. van der Zant, Science 325, 1103 (2009).
[8] O.A. Ilinskaya, S.I. Kulinich, I.V. Krive, R.I. Shekhter, H.C. Park, M. Jonson New J. Phys. 20, 063036 (2018).
[9] T.T. Heikkilä and Y.V. Nazarov Phys. Rev. Lett. 102, 130605 (2009).

[10] O.A. Ilinskaya, D. Radic, H.C. Park, I.V. Krive, R.I. Shekhter, M. Jonson Physica E 122, 114151 (2020).
[11] A. Vikström, A.M. Eriksson, S.I. Kulinich, and L.Y. Gorelik, Phys.Rev.Lett. 117, 247701 (2016).
[12] S.I. Kulinich, and L.Y. Gorelik, Low Temp. Phys./Fiz. Nizk. Temp., 44, 1582 (2018).
[13] A.V. Paraﬁlo, L.Y. Gorelik, M.V. Fistul, H.C. Park, and R.I. Shekhter Phys. Rev. B 102, 235402 (2020).
[14] A.V. Paraﬁlo, S.I. Kulinich, L.Y. Gorelik, M.N. Kiselev, R.I. Shekhter, and M. Jonson Phys. Rev. Lett. 117, 057202 (2016).
[15] P. Stadler, W. Belzig, G. Rastelli Phys. Rev. Lett. 117, 197202 (2016).
[16] D. Fedorets, L.Y. Gorelik, R.I. Shekhter and M. Jonson Europhys. Lett., 58, 1, 99 (2002).
[17] S.I. Kulinich, L.Y. Gorelik, A.N. Kalinenko, I.V. Krive, R.I. Shekhter, Y.W. Park and M. Jonson Phys Rev. Lett. 112, 17206 (2014).
[18] J. Atalaya, L.Y. Gorelik Phys.Rev.B 85, 245309 (2012).
[19] A.F. Andreev, Sov. Phys. JETP. 19, 1228 (1964).
[20] I.O. Kulik, Zh. Eksp. Teor. Fiz. 57, 1745 (1969).
[21] O.M. Bahrova, S.I. Kulinich, L.Y. Gorelik, R.I. Shekhter, H.C. Park, arXiv: 2112.00210 [cond-mat.mes-hall] (2021).
[22] J. Gramich, A. Baumgarther, C. Schonenberger Phys. Rev. Lett. 115, 216801 (2015).
[23] A.F. Morpurgo, J. Kong, C.M. Marcus, H. Dai Science Vol. 286, Issue 5438, 263 (1999).
[24] T. Novotný, A. Donarini, A.-P. Jauho Phys. Rev. Lett. 90, 256801 (2003).
[25] A.V. Rozhkov, D.P. Arovas Phys. Rev. B 62, 6687 (2000).
[26] H.-P. Breuer, F. Petruccione The theory of open quantum systems (Oxford, New York, 2002).
[27] J. Moser, A. Eicher, J. Guttinger, M.I. Dykman, A. Batchtold Nature Nanotech. 9, 1007 (2014).
[28] E.A. Laird, F. Pei, W. Tang, G.A. Steele, and L.P. Kouwenhoven Nano Lett. 12, 193 (2012).