Theory of valuations on manifolds, II.

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Abstract

This article is the second part in the series of articles where we are developing
theory of valuations on manifolds. Roughly speaking valuations could be thought as
finitely additive measures on a class of nice subsets of a manifold which satisfy some
additional assumptions.

The goal of this article is to introduce a notion of a smooth valuation on an arbitrary
smooth manifold and establish some of the basic properties of it.

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0 Introduction.

This article is the second part in the series of articles where we are developing theory of valuations on manifolds. Roughly speaking valuations could be thought as finitely additive measures on a class of nice subsets of a manifold which satisfy some additional assumptions.

The goal of this article is to introduce the notion of a smooth valuation on an arbitrary smooth manifold and establish some of the basic properties of it. Let us describe this notion with several oversimplifications referring for the details to the main text.

Let \( X \) be a smooth manifold of dimension \( n \). Let us denote by \( \mathcal{P}(X) \) the family of simple differentiable subpolyhedra of \( X \) (see Subsection 2.1). \( \mathcal{P}(X) \) serves as a natural class of "nice" sets. For any set \( P \in \mathcal{P}(X) \) one defines a characteristic cycle \( \text{CC}(P) \) which is a closed cycle of dimension \( n \) in the cotangent bundle \( T^*X \) (Definition 2.4.1). (Note that if \( P \) is a smooth submanifold of \( X \) then \( \text{CC}(P) \) coincides with the conormal bundle of \( P \).)

A smooth valuation \( \phi \) is a complex valued finitely additive functional (measure) on \( \mathcal{P}(X) \) which satisfies some additional properties. One of the main such properties is continuity of \( \phi \) with respect to convergence in the sense of currents of the characteristic cycles of subsets from \( \mathcal{P}(X) \). Most of the other properties were introduced essentially for technical reasons and their necessity is not very clear for the moment.

0.1.1 Remark. The class \( \mathcal{P}(X) \) is not closed neither under finite unions nor under finite intersections. Thus the notion of a finitely additive functional on \( \mathcal{P}(X) \) should be defined more formally. This is done in Subsections 2.2 and 2.3 using the notion of a subdivision of a differentiable polyhedron.

Thus we get the space \( V^{\infty}(X) \) of smooth valuations on \( X \). It is a Fréchet space. The group of diffeomorphisms acts continuously on \( V^{\infty}(X) \). It is important to notice that if \( X \) is an affine space then the subspace of translation invariant elements from \( V^{\infty}(X) \) coincides with the space \( \text{Val}^{\text{sm}}(X) \) introduced and studied by the author in [2]; the last space is a dense subspace of the space \( \text{Val}(X) \) of continuous translation invariant valuations on convex subsets of \( X \) which is the classical object. For the classical theory of valuations we refer to the surveys McMullen-Schneider [14] and McMullen [13].

Next, the notion of smooth valuation is a local notion. More precisely for any open subset \( U \subset X \) the correspondence \( U \mapsto V^{\infty}(U) \) is a sheaf on \( X \) (when the restriction maps are obvious). This sheaf is denoted by \( \mathcal{V}_X^{\infty} \). Thus \( V^{\infty}(X) \) is equal to the space of global sections \( \Gamma(X, \mathcal{V}_X^{\infty}) = \mathcal{V}_X^{\infty}(X) \).

The sheaf \( \mathcal{V}_X^{\infty} \) has a canonical filtration by subsheaves of vector spaces

\[
\mathcal{V}_X^{\infty} = \mathcal{W}_0 \supset \mathcal{W}_1 \supset \cdots \supset \mathcal{W}_n
\]

where \( n = \dim X \). \( \mathcal{W}_n \) coincides with the sheaf of smooth densities (measures) on \( X \). For any open subset \( U \subset X \) and any \( i = 0, 1, \ldots, n \), \( \mathcal{W}_i(U) \) is a closed subspace of \( \mathcal{V}_X^{\infty}(U) \).

It turns out that the associated graded sheaf \( \text{gr}_W \mathcal{V}_X^{\infty} := \bigoplus_{i=0}^n \mathcal{W}_i/\mathcal{W}_{i+1} \) admits a simple description in terms of translation invariant valuations. To state it let us denote by \( \text{Val}(TX) \) the (infinite dimensional) vector bundle over \( X \) such that its fiber over a point \( x \in X \) is equal to the space \( \text{Val}^{\text{sm}}(T_x X) \) of smooth translation invariant valuations of the tangent space \( T_x X \). By McMullen’s Theorem [1.3.3] the space \( \text{Val}^{\text{sm}}(T_x X) \) has natural grading by
the degree of homogeneity which must be an integer between 0 and $n$. Thus $Val(TX)$ is a graded vector bundle. Let us denote by $Val(TX)$ the sheaf $\mathcal{U} \mapsto C^\infty(\mathcal{U}, Val(TX))$ where the last space denotes the space of infinitely smooth sections of $Val(TX)$ over $\mathcal{U}$. The next result is Corollary 3.1.7.

**0.1.2 Theorem.** There exists a canonical isomorphism of graded sheaves

$$gr_W V^\infty_X \simeq \overline{Val}(TX).$$

This theorem provides a description of smooth valuations since translation invariant valuations are studied much better. Proposition 3.1.9 gives yet another description of smooth valuations in terms of integration with respect to the characteristic cycle. Combined with Lemma 2.4.8 it says the following.

**0.1.3 Theorem.** Let $\phi$ be a smooth valuation on $X$. Then there exists a section $\eta \in C^\infty(T^*X, \Omega^n \otimes p^*o)$ such that for any $P \in \mathcal{P}(X)$ one has

$$\phi(P) = \int_{CC(P)} \eta$$

where $p : T^*X \to X$ is the canonical projection, $\Omega^n$ denotes the vector bundle of $n$-forms on $T^*X$, $o$ denotes the orientation bundle on $X$, and $C^\infty(T^*X, \Omega^n \otimes p^*o)$ denotes the space of infinitely smooth sections of the bundle $\Omega^n \otimes p^*o$ such that the restriction of the projection $p$ to the support of this section is proper.

Conversely any expression of the above form is a smooth valuation.

The sheaf $V^\infty_X$ has yet another interesting structure which we call the Euler-Verdier involution and denote by $\sigma$. This is a non-trivial automorphism of sheaf $\sigma : V^\infty_X \to V^\infty_X$. The next result is Theorem 3.3.2.

**0.1.4 Theorem.** (i) The Euler-Verdier involution $\sigma$ preserves the filtration $W_\bullet$.

(ii) The induced involution on $gr_W V^\infty_X \simeq \overline{Val}(TX)$ comes from the involution on the bundle $Val(TX)$ defined as $\phi \mapsto [K \mapsto (-1)^{deg\phi}(-K)]$ for any $\phi \in Val(T_xX)$ for any $x \in X$, and where $deg\phi$ is the degree of homogeneity of $\phi$.

Thus the sheaf $V^\infty_X$ of smooth valuations decomposes under the action of the Euler-Verdier involution into two subsheaves $V^{\infty,+}_X$ and $V^{\infty,-}_X$ corresponding to eigenvalues 1 and -1 of $\sigma$ respectively. Thus

$$V^\infty_X = V^{\infty,+}_X \oplus V^{\infty,-}_X.$$
obtain a description of smooth valuations in terms of the integration with respect to the characteristic cycle. In Subsection 3.2 we define the natural structure of Fréchet space on the space of smooth valuations. In Subsection 3.3 we introduce the Euler-Verdier involution on smooth valuations.

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1 Background

In Subsection 1.1 we remind some very basic definitions and facts from representation theory. In Subsection 1.2 we remind basic facts from the sheaf theory. In Subsection 1.3 we remind some facts from the valuation theory. This section does not contain new results.

1.1 Some representation theory.

1.1.1 Definition. Let $\rho$ be a continuous representation of a Lie group $G$ in a Fréchet space $F$. A vector $\xi \in F$ is called $G$-smooth if the map $g \mapsto \rho(g)\xi$ is an infinitely differentiable map from $G$ to $F$.

It is well known (see e.g. [17], Section 1.6) that the subset $F^{sm}$ of smooth vectors is a $G$-invariant linear subspace dense in $F$. Moreover it has a natural topology of a Fréchet space (which is stronger than that induced from $F$), and the representation of $G$ in $F^{sm}$ is continuous. Moreover all vectors in $F^{sm}$ are $G$-smooth.

1.2 Sheaf theory.

The definitions of this subsection are taken from Godement’s book [9].

Let $X$ be a topological space. Let $\Phi$ be a family of closed subsets of $X$.

1.2.1 Definition ([9], Section 3.2). The family $\Phi$ is called paracompactifiable if

1. any $S \in \Phi$ is closed and paracompact;
2. $\Phi$ is closed under finite unions;
3. any closed subset of any $S \in \Phi$ also belongs to $\Phi$;
4. any $S \in \Phi$ has a neighborhood belonging to $\Phi$.

From now on we will always assume that $\Phi$ is a paracompactifiable family of subsets of $X$.

1.2.2 Example. (1) If $X$ is a locally compact paracompact topological space then the family of all closed subsets is paracompactifiable.

(2) If $X$ is a locally compact paracompact topological space then the family of all compact subsets is paracompactifiable.
From now on we will always assume that $X$ is locally compact and paracompact.

1.2.3 Definition ([9], Section 3.5). (1) A sheaf $F$ on $X$ is called $\Phi$-soft if for any $S', S'' \in \Phi$ with $S' \supset S''$ the restriction map $F(S') \to F(S'')$ is surjective.

(2) A sheaf $F$ on $X$ is called soft if it is $\Phi$-soft where $\Phi$ is the family of all closed subsets of $X$.

For a sheaf $F$ on $X$ let us denote by $\Gamma_{\Phi}(F)$ the set of global sections of $F$ with the support in $\Phi$. The functor $F \mapsto \Gamma_{\Phi}(F)$ is left exact on the category of sheaves of abelian groups. Denote as usual by $H^i_{\Phi}(X,F)$ its right derived functor.

1.2.4 Theorem ([9], Theorem 3.5.4). Let $0 \to L^0 \to L^1 \to \ldots$ be an exact sequence of $\Phi$-soft sheaves of abelian groups. Then the following sequence is exact:

$$0 \to \Gamma_{\Phi}(L^0) \to \Gamma_{\Phi}(L^1) \to \ldots$$

1.2.5 Theorem ([9], Theorem 4.4.3). If $F$ is a $\Phi$-soft sheaf then $H^i_{\Phi}(X,F) = 0$ for all $i > 0$.

1.2.6 Theorem ([9], Theorem 3.7.1). Let $A$ be a sheaf of unital rings on $X$. If $A$ is $\Phi$-soft then any $A$-module is $\Phi$-soft.

1.2.7 Theorem ([9], Theorem 3.7.2). Let $A$ be a sheaf of unital rings over a paracompact space $X$. Then $A$ is soft if and only if any point of $X$ has a neighborhood $U$ such that for any disjoint closed subsets $S, T \subset U$ there exists a section of $A$ over $U$ which is equal to 1 on $S$ and to 0 on $T$.

1.2.8 Definition ([9], Section 3.7). Let $L$ be a sheaf of abelian groups on $X$. $L$ is called fine (resp. $\Phi$-fine) if the sheaf of rings $\text{Hom}_\mathbb{Z}(L, L)$ is soft (resp. $L|_S$ is fine for all $S \in \Phi$). Any $\Phi$-fine sheaf is $\Phi$-soft (by Theorem 1.2.6).

1.2.9 Lemma. Let $A$ be a sheaf of unital rings on a locally compact paracompact space $X$. Then $A$ is $\Phi$-fine if and only if it is $\Phi$-soft.

Proof. By the remark before this lemma it remains to prove that if $A$ is $\Phi$-soft then it is $\Phi$-fine. Restricting to $S \in \Phi$, it suffices to prove that if $A$ is soft then it is fine. Note that a sheaf $L$ of abelian groups is fine if and only if for any disjoint closed subsets $A$ and $B$ of $X$ there exists a morphism $L \to L$ inducing the identity map over a neighborhood of $A$ and the zero map over a neighborhood of $B$. Thus in the case of a sheaf of rings $A$, the last condition is satisfied provided $A$ has a section over $X$ which is equal to one in a neighborhood of $A$ and is equal to zero in a neighborhood of $B$. The last condition is equivalent to the fact that $A$ is soft. Q.E.D.

1.2.10 Theorem ([9], Theorem 3.7.3). If $L$ is a $\Phi$-fine sheaf of abelian groups then for any sheaf $M$ of abelian groups $L \otimes_\mathbb{Z} M$ is $\Phi$-fine (and hence $\Phi$-soft).

1.2.11 Example ([9], Section 3.7). Let $X$ be a smooth paracompact manifold. Let $\mathcal{O}_X$ denote the sheaf of $C^\infty$-functions on $X$. Then $\mathcal{O}_X$ is fine and hence soft. Hence any $\mathcal{O}_X$-module $M$ is fine and soft. It follows that $H^i(X, M) = H^i_c(X, M) = 0$ for all $i > 0$. 

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For the definition of operations $f_*, f^*, f_!$ on sheaves see e.g. the book [11] (in their notation $f^*$ is denoted by $f^{-1}$).

1.2.12 Lemma. Let $X$ be a locally compact paracompact topological space. Let $Z \subset X$ be a closed subset of $X$. Consider the imbeddings

$$Z \overset{i}{\hookrightarrow} X \overset{j}{\hookrightarrow} U := X \setminus Z.$$  

Let $\mathcal{F}$ be a sheaf on $X$.

1) If $H^1(U, j_! j^* \mathcal{F}) = 0$ then any section of $\mathcal{F}$ over $Z$ extends to a section over $X$.

2) If any section of $\mathcal{F}$ over $Z$ extends to a section over $X$ and $H^1(X, \mathcal{F}) = 0$ then $H^1(U, j_! j^* \mathcal{F}) = 0$.

3) Let $\mathcal{A}$ be a soft sheaf of unital rings on $X$. Then for any $\mathcal{A}|_U$-module $\mathcal{M}$ one has $H^i(X, j_! \mathcal{M}) = 0$ for all $i > 0$.

Proof. 1) We have an exact sequence of sheaves

$$0 \rightarrow j_! j^* \mathcal{F} \rightarrow \mathcal{F} \rightarrow i_* i^* \mathcal{F} \rightarrow 0$$  

(see e.g. the exact sequence in Proposition 2.3.6(v) of [11] combined with Propositions 2.3.6(iv) and 2.5.4(ii) of [11]). Hence the following sequence is exact

$$\Gamma(X, \mathcal{F}) \rightarrow \Gamma(X, i_* i^* \mathcal{F}) \rightarrow H^1(X, j_! j^* \mathcal{F}) = 0.$$  

Since $\Gamma(X, i_* i^* \mathcal{F}) = \Gamma(Z, i^* \mathcal{F})$ the result follows.

2) From the same exact sequence and our assumptions we obtain an exact sequence

$$\Gamma(X, \mathcal{F}) \onto \Gamma(Z, i^* \mathcal{F}) \rightarrow H^1(X, j_! j^* \mathcal{F}) \rightarrow H^1(X, \mathcal{F}) = 0.$$  

This implies the statement.

3) Indeed $j_! \mathcal{M}$ is an $\mathcal{A}$-module. Hence it is acyclic by Theorems 1.2.6 and 1.2.5. Q.E.D.

1.2.13 Proposition. Let $X$ be a smooth manifold. Let $\mathcal{O}_X$ denote the sheaf of $C^\infty$-functions on $X$. Let $\mathcal{V}$ be a sheaf on $X$ which admits a finite filtration by subsheaves

$$\mathcal{V} = \mathcal{W}_0 \supset \mathcal{W}_1 \supset \cdots \supset \mathcal{W}_N \supset \mathcal{W}_{N+1} = 0$$

such that the quotients $\mathcal{W}_k/\mathcal{W}_{k+1}$ have a structure of $\mathcal{O}_X$-modules.

Then $\mathcal{V}$ is soft.

Proof. Let $Z$ be any closed subset of $X$. We have to show that any section of $\mathcal{V}$ over $Z$ extends to a section over $X$. By Lemma 1.2.12(1) it is enough to check that for any open imbedding $j: U \hookrightarrow X$ one has $H^i(X, j_* j^* \mathcal{V}) = 0$ for $i > 0$. Observe that $j_!$ and $j^*$ are exact functors (see e.g. 2.5.4 and 2.3.2 of [11] respectively). So we have a filtration

$$j_! j^* \mathcal{V} = j_! j^* \mathcal{W}_0 \supset j_! j^* \mathcal{W}_1 \supset \cdots \supset j_! j^* \mathcal{W}_N \supset j_! j^* \mathcal{W}_{N+1} = 0.$$  

By induction and the long exact sequence, it is enough to check that $H^i(X, j_! j^* \mathcal{W}_k/j_! j^* \mathcal{W}_{k+1}) = 0$ for $i > 0$. But since the functor $j_! j^*$ is exact we have $j_! j^* \mathcal{W}_k/j_! j^* \mathcal{W}_{k+1} = j_! j^* (\mathcal{W}_k/\mathcal{W}_{k+1})$. Now the result follows from Lemma 1.2.12(3). Q.E.D.
1.3 Valuation theory.

In this subsection we remind some facts from the theory of valuations. Let $V$ be a finite dimensional real vector space, $n = \dim V$. Let $\mathcal{K}(V)$ denote the class of all convex compact subsets of $V$. Equipped with the Hausdorff metric, the space $\mathcal{K}(V)$ is a locally compact space.

1.3.1 Definition. a) A function $\phi : \mathcal{K}(V) \to \mathbb{C}$ is called a valuation if for any $K_1, K_2 \in \mathcal{K}(V)$ such that their union is also convex one has

$$\phi(K_1 \cup K_2) = \phi(K_1) + \phi(K_2) - \phi(K_1 \cap K_2).$$

b) A valuation $\phi$ is called continuous if it is continuous with respect to the Hausdorff metric on $\mathcal{K}(V)$.

Let us denote by $\text{Val}(V)$ the space of translation invariant continuous valuations on $\mathcal{K}(V)$. Equipped with the topology of uniform convergence on compact subsets of $\mathcal{K}(V)$ the space $\text{Val}(V)$ becomes a Banach space (see e.g. Lemma A.4 in [2]).

1.3.2 Definition. Let $k$ be a real number. A valuation $\phi$ is called $k$-homogeneous if for every convex compact set $K$ and for every scalar $\lambda > 0$

$$\phi(\lambda K) = \lambda^k \phi(K).$$

Let us denote by $\text{Val}_k(V)$ the space of $k$-homogeneous translation invariant continuous valuations.

1.3.3 Theorem (McMullen [12]).

$$\text{Val}(V) = \bigoplus_{k=0}^{n} \text{Val}_k(V),$$

where $n = \dim V$.

Note in particular that the degree of homogeneity is an integer between 0 and $n = \dim V$. It is known that $\text{Val}_0(V)$ is one-dimensional and is spanned by the Euler characteristic $\chi$, and $\text{Val}_n(V)$ is also one-dimensional and it is spanned by a Lebesgue measure [10]. The space $\text{Val}_n(V)$ is also denoted by $| \wedge V^*|$ (the space of complex valued Lebesgue measures on $V$). One has further decomposition with respect to parity:

$$\text{Val}_k(V) = \text{Val}_k^{\text{ev}}(V) \oplus \text{Val}_k^{\text{odd}}(V),$$

where $\text{Val}_k^{\text{ev}}(V)$ is the subspace of even valuations ($\phi$ is called even if $\phi(-K) = \phi(K)$ for every $K \in \mathcal{K}(V)$), and $\text{Val}_k^{\text{odd}}(V)$ is the subspace of odd valuations ($\phi$ is called odd if $\phi(-K) = -\phi(K)$ for every $K \in \mathcal{K}(V)$). The Irreducibility Theorem is as follows.

1.3.4 Theorem (Irreducibility Theorem [1]). The natural representation of the group $\text{GL}(V)$ on each space $\text{Val}_k^{\text{ev}}(V)$ and $\text{Val}_k^{\text{odd}}(V)$ is irreducible.
In this theorem, by the natural representation one means the action of \( g \in GL(V) \) on \( \phi \in Val(V) \) as \( (g\phi)(K) = \phi(g^{-1}K) \) for every \( K \in K(V) \). The subspace of smooth valuations with respect to this action in sense of Definition 1.1.1 is denoted by \( Val^{sm}(V) \).

In [3] we have introduced the notion of a smooth valuation on a linear space \( V \). Let us remind this notion. Let us denote by \( CV(V) \) the space of continuous valuations on \( V \). Equipped with the topology of uniform convergence on compact subsets of \( K(V) \), \( CV(V) \) becomes a Fréchet space. Let \( QV(V) \) denote the space of continuous valuations on \( V \) which satisfy the following additional property: the map given by \( K \mapsto \phi(tK + x) \) is a continuous map \( K(V) \to C^m([0, 1] \times V) \). We call such valuations quasi-smooth.

In the space \( QV(V) \) we have the natural linear topology defined as follows. Fix a compact subset \( G \subset V \). Define a seminorm on \( QV(V) \)

\[
||\phi||_G := \sup \{ ||\phi(tK + x)||_{C^m([0, 1] \times G)} \mid K \in K(V), K \subset G \}.
\]

Note that the seminorm \( ||\cdot||_G \) is finite. One easily checks the following claim.

1.3.5 Claim. Equipped with the topology defined by this family of seminorms the space \( QV(V) \) is a Fréchet space.

Note also that the natural representation of the group \( \text{Aff}(V) \) of affine transformations of \( V \) in the space \( QV(V) \) is continuous. We will denote by \( SV(V) \) the subspace of \( \text{Aff}(V) \)-smooth vectors in \( QV(V) \). It is a Fréchet space.

1.3.6 Definition. Elements of \( SV(V) \) are called smooth valuations on the linear space \( V \).

Let us remind notions of characteristic and normal cycle of a convex compact set \( K \in K(V) \). Clearly \( T^*V = V \times V^* \). Let \( K \in K(V) \). Let \( x \in K \).

1.3.7 Definition. A tangent cone to \( K \) at \( x \) is a set denoted by \( T_xK \) which is equal to the closure of the set \( \{ y \in V \mid \exists \varepsilon > 0 \ t + \varepsilon y \in K \} \).

It is easy to see that \( T_xK \) is a closed convex cone.

1.3.8 Definition. A normal cone to \( K \) at \( x \) is the set

\[
\text{Nor}_xK := \{ y \in V^* \mid y(x) \geq 0 \forall x \in T_xK \}.
\]

Thus \( \text{Nor}_xK \) is also a closed convex cone.

1.3.9 Definition. Let \( K \in K(V) \). The characteristic cycle of \( K \) is the set

\[
CC(K) := \bigcup_{x \in K} \text{Nor}_x(K).
\]

1.3.10 Remark. The notion of the characteristic cycle is not new. First an almost equivalent notion of normal cycle (see below) was introduced by Wintgen [18], and then studied further by Zähle [19] by the tools of geometric measure theory. Characteristic cycles of subanalytic sets of real analytic manifolds were introduced by Kashiwara (see [11], Chapter 9) using the tools of the sheaf theory, and independently by J. Fu [8] using rather different tools of geometric measure theory. Below we will remind an elementary definition of characteristic cycle of a differentiable subpolyhedron of a smooth manifold. This elementary approach will be sufficient for the purposes of this article.
For a linear space $W$ let us denote by $\mathbb{P}_+(W)$ the manifold of oriented lines in $W$ passing through the origin. Similarly for a vector bundle $E$ over a manifold $X$ let us denote by $\mathbb{P}_+(E)$ the vector bundle over $X$ whose fiber over a point $x \in X$ is equal to $\mathbb{P}_+(E_x)$ where $E_x$ is the fiber of $E$ over $X$.

It is easy to see that $CC(K)$ is a closed $n$-dimensional subset of $T^*V = V \times V^*$ invariant with respect to the multiplication by non-negative numbers acting on the second factor. Sometimes we will also use the following notation. Let $0$ denote the zero section of $T^*V$, i.e. $0 = V \times \{0\}$. Set

$$CC(K) := CC(K) \setminus 0,$$

$$\tilde{CC}(K) := \frac{CC}{\mathbb{R}_{>0}}.$$

Thus $\tilde{CC}(K) \subset \mathbb{P}_+(T^*V)$. Let us denote by $N(K)$ the image of $\tilde{CC}(K)$ under the involution on $\mathbb{P}_+(T^*V)$ of the change of an orientation of a line. $N(K)$ is called the normal cycle of $K$.

In this article for a manifold $Y$ we denote by $\Omega^k := \wedge^k T^*Y$ the vector bundle of $k$-forms over $Y$. Usually it will be clear from the context which manifold is meant.

Let us denote by

$$p : T^*V \to V$$

the canonical projection. Let us denote by $o$ the orientation bundle of $V$. Note that a choice of orientation on $V$ induces canonically an orientation on $CC(K)$ and $N(K)$ for any $K \in \mathcal{K}(V)$. Let us denote by $\tilde{C}^1(T^*V, \Omega^n \otimes p^*o)$ the space of $C^1$-smooth sections of $\Omega^n \otimes p^*o$ over $T^*V$ such that the restriction of $p$ to the support of this section is proper.

1.3.11 Theorem [4]. For any $\omega \in \tilde{C}^1(T^*V, \Omega^n \otimes p^*o)$ the map $\mathcal{K}(V) \to \mathbb{C}$ given by $K \mapsto \int_{\tilde{CC}(K)} \omega$ defines a continuous valuation on $\mathcal{K}(V)$.

1.3.12 Corollary. For any $\eta \in C^1(\mathbb{P}_+(T^*V), \Omega^{n-1} \otimes p^*o)$ the map $\mathcal{K}(V) \to \mathbb{C}$ given by $K \mapsto \int_{N(K)} \eta$ defines a continuous valuation on $\mathcal{K}(V)$.

We will also need the following statement.

1.3.13 Theorem [4]. The map $\mathcal{K}(V) \times (C^1(V, |\omega|) \oplus C^1(\mathbb{P}_+(V^*), \Omega^{n-1} \otimes p^*o)) \to \mathbb{C}$ given by

$$(K, (\omega, \eta)) \mapsto \int_K \omega + \int_{N(K)} \eta$$

is continuous.

1.3.14 Corollary [3, Corollary 5.1.7]. (i) The map $C^1(V, |\omega|) \oplus C^1(\mathbb{P}_+(V^*), \Omega^{n-1} \otimes p^*o) \to CV(V)$ given by $(\omega, \eta) \mapsto [K \mapsto \int_K \omega + \int_{N(K)} \eta]$ is continuous.

(ii) For any compact set $G \subset V$ the exists a larger compact set $\tilde{G} \subset V$ and a constant $C = C(G)$ such that for any $(\omega, \eta) \in C^1(V, |\omega|) \oplus C^1(\mathbb{P}_+(V^*), \Omega^{n-1} \otimes p^*o)$ one has

$$\sup_{K \subset G, K \in \mathcal{K}(V)} |\int_K \omega + \int_{N(K)} \eta| \leq C(||\omega||_{C^1(\tilde{G})} + ||\eta||_{C^1(\tilde{G})}).$$
1.3.15 Proposition ([3], Proposition 5.1.8). (i) For any 

\[(\omega, \eta) \in C^\infty(V, |\omega_V|) \oplus C^\infty(\mathcal{P}_+(V^*), \Omega^{n-1} \otimes p^*o)\]

the valuation \([K \mapsto \int_K \omega + \int_{N(K)} \eta]\) is smooth, i.e. belongs to \(SV(V)\).

(ii) The induced map 

\[C^\infty(V, |\omega_V|) \oplus C^\infty(\mathcal{P}_+(V^*), \Omega^{n-1} \otimes p^*o) \to SV(V)\]

is continuous.

1.3.16 Theorem ([3], Theorem 5.2.2). The map 

\[C^\infty(V, |\omega_V|) \oplus C^\infty(\mathcal{P}_+(T^*V), \Omega^{n-1} \otimes p^*o) \to SV(V)\]

is onto.

In [3] we have defined a decreasing filtration \(W_\bullet\) by closed subspaces on \(SV(V)\):

\[SV(V) = W_0 \supset W_1 \supset \cdots \supset W_n.\]

Here 

\[W_i := \{\phi \in SV(V) | \frac{d^k}{dt^k}\phi(tK + x)\}|_{t=0} = 0 \forall k < i, \forall K \in \mathcal{K}(V), \forall x \in V\].

It is clear that \(W_i\) are \(Aff(V)\)-invariant closed subspaces of \(SV(V)\). Obviously \(SV(V) = W_0 \supset W_1 \supset \cdots\).

1.3.17 Proposition ([3], Proposition 3.1.1.).

\[W_{n+1} = 0.\]

1.3.18 Proposition ([3], Proposition 3.1.2). \(W_n\) coincides with the space of smooth densities on \(V\).

Let us denote by \(Val(TV)\) the (infinite dimensional) vector bundle over \(V\) whose fiber over \(x \in V\) is equal to the space of translation invariant \(GL(V)\)-smooth valuations on the tangent space \(T_xV = V\). Similarly we can define the vector bundle \(Val_k(TV)\) of \(k\)-homogeneous smooth translation invariant valuations. Clearly \(C^\infty(V, Val_k(TV)) = C^\infty(V, Val^{sm}_k(V))\) where the last space denotes the space of infinitely smooth functions on \(V\) with values in the Fréchet space \(Val^{sm}_k(V)\).

1.3.19 Theorem ([3], Proposition 3.1.5.). There exists a canonical isomorphism of Fréchet spaces of the associated graded space \(\text{gr}_W SV(V) := \bigoplus_{i=0}^n W_i/W_{i+1}\) and \(C^\infty(V, Val(TV))\).

Remind also the construction of this isomorphism. More precisely there is an isomorphism 

\[I_i : W_i/W_{i+1} \to C^\infty(V, Val^{sm}_i(V)).\]

The map \(I_i\) is defined as follows. For \(\phi \in W_i, x \in V, K \in \mathcal{K}(V)\)

\[(I_i\phi)(x, K) := \lim_{r \to +0} \frac{\phi(rK + x)}{r^i}.\] (1)
Now let us describe the filtration $W_\bullet$ in terms of integration with respect to the characteristic cycle following [3]. Let us start with some general remarks.

Let $X$ be a smooth manifold. Let $p : P \to X$ be a smooth bundle. Let $\Omega^N(P)$ be the vector bundle over $P$ of $N$-forms. Let us introduce a filtration of $\Omega^N(P)$ by vector subbundles $W_i(P)$ as follows. For every $y \in P$ set

$$(W_i(P))_y := \{ \omega \in \wedge^N T^*_y P \mid \omega|_F \equiv 0 \text{ for all } F \in \text{Gr}_N(T_y P) \text{ with } \dim(F \cap T_y(p^{-1}p(y))) > N - i \}.$$ 

Clearly we have

$$\Omega^N(P) = W_0(P) \supset W_1(P) \supset \cdots \supset W_N(P) \supset W_{N+1}(P) = 0.$$ 

Let us discuss this filtration in greater detail following [3].

Let us make some elementary observations from linear algebra. Let $L$ be a finite dimensional vector space. Let $E \subset L$ be a linear subspace. For a non-negative integer $i$ set

$$W_i(L, E) := \{ \omega \in \wedge^N L^* \mid \omega|_F \equiv 0 \text{ for all } F \subset L \text{ with } \dim(F \cap E) > N - i \}.$$ 

Clearly

$$\wedge^N L^* = W_0(L, E) \supset W_1(L, E) \supset \cdots \supset W_N(L, E) \supset W_{N+1}(L, E) = 0.$$ 

1.3.20 Lemma ([3], Lemma 5.2.3). There exists canonical isomorphism of vector spaces

$$W(L, E)_i / W(L, E)_{i+1} = \wedge^{N-i} E^* \otimes \wedge^i (L/E)^*.$$ 

Let us apply this construction in the context of integration with respect to the characteristic cycle. Let $X$ be a smooth manifold of dimension $n$. Let $P := T^*X$ be the cotangent bundle. Let $p : P \to X$ be the canonical projection. Let us denote by $o$ the orientation bundle on $X$. The above construction gives a filtration of $\Omega^n(P)$ by subbundles

$$\Omega^n(P) = W_0(\Omega^n(P)) \supset \cdots \supset W_n(\Omega^n(P)).$$ 

Twisting this filtration by $p^*o$ we get a filtration of $\Omega^n(P) \otimes p^*o$ by subbundles denoted by $W_i(\Omega^n(P) \otimes p^*o)$.

Let us denote by $\tilde{C}^\infty(P, W_i(\Omega^n \otimes p^*o))$ the space of infinitely smooth sections of the bundle $W_i(\Omega^n \otimes p^*o)$ such that the restriction of the projection $p$ to the support of these sections is proper. The next result is a trivial reformulation of Proposition 5.2.5 from [3].

1.3.21 Theorem. Consider the map $\Xi : \tilde{C}^\infty(P, \Omega^n \otimes p^*o) \to SV(V)$ given by

$$(\Xi(\omega))(K) = \int_{CC(K)} \omega.$$ 

This map is surjective, and moreover for every $i = 0, 1, \ldots, n$ the map $\Xi$ maps $\tilde{C}^\infty(P, W_i(\Omega^n) \otimes p^*o)$ onto $W_i$ surjectively.
2 Differentiable polyhedra, finitely additive measures, and smooth valuations.

In Subsection 2.1 we discuss the notion of the differentiable polyhedron. In Subsection 2.2 we discuss a combinatorial notion of finitely additive measure on a family of sets which is not necessarily closed under finite intersections and unions but satisfies some other assumptions. In Subsection 2.3 we introduce a notion of finitely additive measure on the class of simple differentiable subpolyhedra of a smooth manifold. Finally in Subsection 2.4 we introduce the main object of this article, namely the notion of a smooth valuation on a manifold.

2.1 Differentiable polyhedra.

We remind the definition and basic properties of differentiable polyhedra. The exposition in the beginning this subsection (up to Lemma 2.1.6) follows very closely [5].

A convex angle in \( \mathbb{R}^n \) is a set defined by finitely many inequalities \( \{ x | < \xi_\nu, x > \geq 0, 0 \leq \nu \leq N \} \). Note that a convex angle is a convex cone in particular. We say that a convex angle \( C \) is of type \( r \) if it contains an \( r \)-dimensional linear subspace and does not contain linear subspaces of larger dimensions.

Let \( P^n \) be a compact connected topological space for which there has been given a covering by open subsets \( \Omega_i \) and a homeomorphic mapping \( \phi_i \) of each \( \Omega_i \) onto an \( n \)-dimensional convex angle \( C_i \) (which may be \( \mathbb{R}^n \)). \( P^n \) is called an \( n \)-dimensional differentiable polyhedron if the maps \( \phi_i \phi_k^{-1} \) are of class \( C^\infty \) on the domain of their definition.

A differentiable cell is, by definition, a differentiable polyhedron which is diffeomorphic of class \( C^\infty \) with a convex compact polyhedron in \( \mathbb{R}^n \).

Let \( P^n \) be a differentiable polyhedron. For any point \( z \in P^n \) one defines the tangent space \( \tilde{T}_z P^n \) to \( P^n \) at \( z \) in the obvious way. \( \tilde{T}_z P^n \) is a linear space. The (tangent) angle of \( P^n \) at \( z \) is the subset of \( \tilde{T}_z P^n \) consisting of those \( v \in \tilde{T}_z P^n \) such that there exists a \( C^\infty \)-smooth map \( \gamma : [0, 1] \rightarrow P^n \) such that \( \gamma(0) = z, \gamma'(0) = v \). We will denote it by \( T_z P^n \). It is clear that \( T_z P^n \) is a convex angle in \( \tilde{T}_z P^n \).

If \( C \) is the tangent angle of \( P^n \) at a point \( z \) then \( z \) has a neighborhood homeomorphic to \( C \). If \( C \) is of type \( r \) then we say that \( z \) is of type \( r \) in \( P^n \). Points of type \( n \) in \( P^n \) are called inner points of \( P^n \). Points of type at most \( r \) (where \( 0 \leq r \leq n \)) form a compact subset of \( P^n \). The set of inner points of \( P^n \) will be called (relative) interior of \( P^n \) and will be denoted by \( \text{int} P^n \).

2.1.1 Definition. A regular differentiable subpolyhedron \( Q^p \) in \( P^n \), is the one-to-one image of a differentiable polyhedron \( Q_0^p \) in \( P^n \) provided that this map is of class \( C^\infty \) and its differential is injective at every point.

2.1.2 Definition. A finite set of distinct regular subpolyhedra \( Q^r _\rho \) of \( P^n \) form a subdivision \( D \) of \( P^n \) if the following conditions are satisfied:

1. each point of \( P^n \) is an inner point of at least one \( Q^r _\rho \) in \( D \);
2. if \( Q^r _\rho \) and \( Q^r _\sigma \) in \( D \) are such that there is an inner point of \( Q^r _\rho \) contained in \( Q^r _\sigma \), then \( Q^r _\rho \subset Q^r _\sigma \).
From condition (2) it follows that no two differentiable polyhedra in $\mathcal{D}$ can have an inner point in common unless they coincide.

The following result was proved in [5], Lemma 7.

2.1.3 Lemma. If $Q^r$ is a differentiable polyhedron in a subdivision $\mathcal{D}$ of $P^n$, all inner points of $Q^r$ have the same type in $P^n$.

2.1.4 Lemma. Let $\mathcal{D}$ be a subdivision of $P^n$. Let $A$ and $B$ be two subsets of $P^n$ which are unions of finitely many elements of the subdivision $\mathcal{D}$. Then $A \cap B$ is also a union of finitely many elements of $\mathcal{D}$.

**Proof.** It is enough to prove the lemma under the assumption that $A$ and $B$ are elements of $\mathcal{D}$. Assume that $z \in A \cap B$. Then there is a unique cell $P^r_\lambda \in \mathcal{D}$ such that $z$ belongs to its interior. Then by part (2) of Definition 2.1.2 $P^r_\lambda \subset A$ and similarly $P^r_\lambda \subset B$. Hence $P^r_\lambda \subset A \cap B$. The result follows. Q.E.D.

2.1.5 Definition. A subdivision $\mathcal{D}'$ of $P^n$ is called a refinement of a subdivision $\mathcal{D}$ of $P^n$ if for any differentiable polyhedron $P^r_\lambda$ in $\mathcal{D}$ all differentiable polyhedra of $\mathcal{D}'$ contained in $P^r_\lambda$ form a subdivision of $P^r_\lambda$.

Lemma 2.1.3 implies that if a differentiable polyhedron $Q^r$, in a subdivision $\mathcal{D}$ of $P^n$, contains at least one inner point of $P^n$ then all inner points of $Q^r$ are inner points of $P^n$; $Q^r$ is called an inner polyhedron of the subdivision $\mathcal{D}$.

2.1.6 Lemma ([5], Lemma 8). Let $\mathcal{D}$ be a subdivision of $P^n$ and let $z$ be any point of $P^n$. Then the tangent angles at $z$ of those differentiable polyhedra in $\mathcal{D}$ which contain $z$ form a subdivision of the tangent angle of $P^n$ at $z$. Moreover the inner angles (i.e. those of maximal dimension) in the latter subdivision are the angles of the inner polyhedra in $\mathcal{D}$ which contain $z$.

A differentiable cell is a differentiable polyhedron diffeomorphic to a convex compact polytope. We now define a cellular subdivision of a differentiable polyhedron $P^n$ as a subdivision $\mathcal{D}$, every polyhedron $Z^r_\nu$ in which is a differentiable cell.

2.1.7 Definition. (1) A differentiable polyhedron $P^n$ is called simple if every point $z \in P^n$ has a neighborhood diffeomorphic to $\mathbb{R}^r \times \mathbb{R}^{n-r}_{\geq 0}$ for some $0 \leq r \leq n$.

(2) A subdivision $\mathcal{D}$ of a differentiable polyhedron $P^n$ is called simple if any element of $\mathcal{D}$ is simple.

(3) A triangulation of a differentiable polyhedron $P^n$ is a subdivision every element of which is diffeomorphic to a simplex.

The following result is well known.

2.1.8 Proposition. Every simple polyhedron admits a triangulation.

2.1.9 Definition. Let $\mathcal{D} = \{P_\lambda\}$ and $\mathcal{D}' = \{P'_\nu\}$ be two subdivisions of a differentiable polyhedron $P$. We say that $\mathcal{D}$ and $\mathcal{D}'$ are transversal to each other if for any $P_\lambda \in \mathcal{D}$, any $P'_\nu \in \mathcal{D}'$, and any $x \in P_\lambda \cap P'_\nu$ the maximal linear subspaces contained in the cones $T_x P_\lambda$ and $T_x P'_\nu$ intersect transversally.
2.1.10 Lemma. Let $X^{(1)}$ and $X^{(2)}$ be two regular differentiable subpolyhedra of a smooth $n$-manifold $M$. Assume that for any $x \in X^{(1)} \cap X^{(2)}$ the maximal linear subspaces contained in the cones $T_x X^{(1)}$ and $T_x X^{(2)}$ intersect transversally. Then $X^{(1)} \cap X^{(2)}$ is a differentiable polyhedron. Moreover if $X^{(1)}$ and $X^{(2)}$ are simple then $X^{(1)} \cap X^{(2)}$ is also simple.

Proof. Fix $x \in X^{(1)} \cap X^{(2)}$. Let $x$ has type $p_1$ in $X^{(1)}$ and type $p_2$ in $X^{(2)}$. Then there exist $C^\infty$-smooth real valued functions $f_1^{(k)}, \ldots, f_{N_k}^{(k)}; k = 1, 2$, such that

(1) for $k = 1, 2$ for each $j > n - p_k$ the function $f_j^{(k)}$ is a linear combination with constant coefficients of $f_l^{(k)}$'s with $l \leq n - p_k$;

(2) in a neighborhood of $x$ $X^{(k)} = \{f_j^{(k)} \geq 0| 1 \leq j \leq N_k\}$;

(3) $|f_1^{(1)}|_x, \ldots, |f_{n-p_1}^{(1)}|_x; |f_1^{(2)}|_x, \ldots, |f_{n-p_2}^{(2)}|_x$ are linearly independent.

Let $q := n - ((n - p_1) + (n - p_2))$. Let us choose $C^\infty$-smooth functions $g_1, \ldots, g_q$ such that

$$df_1^{(1)}|_x, \ldots, df_{n-p_1}^{(1)}|_x; df_1^{(2)}|_x, \ldots, df_{n-p_2}^{(2)}|_x; dg_1|_x, \ldots, dg_q|_x$$

form a basis of $T_x^*M$. Then the sequence of functions $f_1^{(1)}, \ldots, f_{n-p_1}^{(1)}, f_1^{(2)}, \ldots, f_{n-p_2}^{(2)}, g_1, \ldots, g_q$

form a coordinate system in a neighborhood of $x$. It is clear that in this coordinate system $X^{(1)}, X^{(2)}, X^{(1)} \cap X^{(2)}$ are defined by finite systems of linear inequalities, and hence they are convex angles. The last part of the lemma also follows. Q.E.D.

2.1.11 Proposition. Let $\mathcal{D} = \{P_\lambda\}$ and $\mathcal{D}' = \{P'_\nu\}$ be two transversal subdivisions of a polyhedron $P$. Let $\mathcal{D} \cap \mathcal{D}' := \{P_\lambda \cap P'_\nu | P_\lambda \in \mathcal{D}, P'_\nu \in \mathcal{D}'\}$. Then $\mathcal{D} \cap \mathcal{D}'$ is also a subdivision of $P$. Moreover it refines both $\mathcal{D}$ and $\mathcal{D}'$.

To prove this proposition we need first of all the following lemma.

2.1.12 Lemma. Let $M$ be a smooth manifold. Let $P$ and $Q$ be two regular differentiable subpolyhedra of $M$. Assume that for any $x \in P \cap Q$ the maximal linear subspaces contained in the cones $T_x P$ and $T_x Q$ intersect transversally. Then $\text{int}(P \cap Q) = \text{int}P \cap \text{int}Q$.

Proof. By Lemma 2.1.10 $P \cap Q$ is a differentiable polyhedron. Fix $z \in P \cap Q$. Let $z$ has type $p$ in $P$ and type $q$ in $Q$. Consider the tangent space $T_z M$ to $M$ at $z$ and tangent angles $P_1$ and $Q_1$ to $P$ and $Q$ respectively at $z$. Then $P_1, Q_1 \subset T_z M$ are convex angles. $P_1$ contains a $p$-dimensional linear subspace $A$ such that the image of $P_1$ in $T_z M/A$ is a cornered convex angle (i.e. it does not contain any non-zero linear subspace). Similarly $Q_1$ contains a $q$-dimensional linear subspace $B$ such that the image of $Q_1$ in $T_z M/B$ is a cornered convex angle. It follows from the assumptions of the lemma that $A$ and $B$ are transversal to each other. Put $C := A \cap B$. Choose $A'$ a complement of $C$ in $A$, and $B'$ a complement of $C$ in $B$. Then there exist cornered convex angles $R \subset B'$ and $S \subset A'$ such that

$$P_1 = A \times R = C \times A' \times R, Q_1 = B \times S = C \times S \times B'.$$
Then $P_1 \cap Q_1 = C \times S \times R$. This is also a convex angle and $\text{int}(P_1 \cap Q_1) = C \times \text{int}S \times \text{int}R = \text{int}P_1 \cap \text{int}Q_1$. It is easy to see that in a small neighborhood of $z$ the polyhedron $P \cap Q$ is diffeomorphic to $P_1 \cap Q_1$. This implies the lemma. Q.E.D.

Proof of Proposition 2.1.11 Now let us check that $D \cap D'$ is indeed a subdivision of $P$. Fix any $z \in P$. Then by Lemma 2.1.12 there exits $P_\lambda \in D$ and $P'_\nu \in D'$ such that $z \in \text{int}P_\lambda$ and $z \in \text{int}P'_\nu$. Hence $z \in \text{int}P_\lambda \cap \text{int}P'_\nu = \text{int}(P_\lambda \cap P'_\nu)$.

It remains to check condition (2) of Definition 2.1.2. Assume that $z \in \text{int}(P_\lambda \cap P'_\nu) = \text{int}P_\lambda \cap \text{int}P'_\nu$ and $z \in P_s \cap P'_t$. Then it follows that $P_\lambda \subset P_s$ and $P'_\nu \subset P'_t$. Hence $P_\lambda \cap P'_\nu \subset P_s \cap P'_t$. Q.E.D.

2.1.13 Definition. Let $D$ be a subdivision of a differentiable polyhedron $P$. Let $\{U_\alpha\}$ be an open covering of $P$. We say that $D$ is subordinate to $\{U_\alpha\}$ if any element of $D$ is contained in at least one element of the covering $\{U_\alpha\}$.

2.1.14 Lemma. Let $D$ be a cellular subdivision of a differentiable polyhedron $P$. Let $\{U_\alpha\}$ be an open covering of $P$. Then there exists a refinement $D'$ of $D$ which is a triangulation of $P$ and subordinate to $\{U_\alpha\}$.

Proof. Assume that we have constructed a triangulation of each element of $D$ of dimension less than $r$ subordinate to $\{U_\alpha\}$. Let us fix a cell $Q \in D$ of dimension $r$ and let us construct its subdivision which extends the subdivision of the boundary of $Q$ obtained previously and which is subordinate to $\{U_\alpha\}$. Let us fix a point $x \in \text{int}Q$. For any cell $R$ contained in the boundary of $Q$ and belonging to the subdivision constructed previously, let us consider the cone over $R$ with vertex at $x$. All such cones form a subdivision of $Q$.

Now we are reduced to the following situation. Given a convex compact polytope $S$ of dimension $r$ and given its $(r-1)$-dimensional face $F \subset S$ which is a simplex. Given an open covering $\{U_\alpha\}$ of $S$ such that $F$ is contained in at least one of the elements of the covering. We have to find a triangulation of $S$ subordinate to $\{U_\alpha\}$ such that $F$ is one of the elements of this subdivision. But this problem in the affine space can be solved easily. Q.E.D.

2.1.15 Lemma. Every differentiable polyhedron can be regularly imbedded into a smooth compact manifold.

Proof. We can choose a finite open covering $\{U_\alpha\}_{\alpha=1}^N$ of $P$, open sets $\{V_\alpha\}_{\alpha=1}^N$ such that $U_\alpha \subset V_\alpha$, and there exist diffeomorphisms $\phi_\alpha$ of $V_\alpha$ onto a convex angle in $\mathbb{R}^n$. Let $l_\alpha^1, \ldots, l_\alpha^n$ be the corresponding coordinate functions on $V_\alpha$. Let us choose a smooth partition of unity $\{\phi_\alpha\}$ such that $\phi_\alpha \equiv 1$ on $U_\alpha$, $\text{supp}(\phi_\alpha) \subset V_\alpha$, and $\sum_\alpha \phi_\alpha \equiv 1$. Then the collection of functions $\{\phi_\alpha^1\}$ defines an immersion of $P$ into $\mathbb{R}^{nN}$. Indeed let us fix $x_0 \in P$. There exists $\alpha_0$ such that $x_0 \in U_{\alpha_0}$. Then for any $x \in U_{\alpha_0}$

$$(\phi_{\alpha_0}^1 l_{\alpha_0})(x) = l_{\alpha_0}^1(x).$$

hence the functions $\{\phi_{\alpha_0}^1 l_{\alpha_0}\}$ define an imbedding of $U_{\alpha_0}$ to $\mathbb{R}^N$. Hence all the functions $\{\phi_\alpha^1 l_\alpha\}$ define an imbedding of $U_{\alpha_0}$ to $\mathbb{R}^{nN}$. Thus the functions $\{\phi_\alpha^1 l_\alpha\}$ define an immersion of $P$ to $\mathbb{R}^{nN}$.

Now let us assume that we have an immersion $f : P \to \mathbb{R}^M$. Let us construct an imbedding $g : P \to \mathbb{R}^{M'}$. Note that the fibers of $f$ are discrete sets. Since the space $P$ is compact
the cardinality of fibers of $f$ is uniformly bounded. Fix a point $y_0 \in \mathbb{R}^M$ lying in the image of $f$. Let $f^{-1}(y_0) = \{x_1, \ldots, x_k\} \subset P$. One can choose a smooth function $g_{y_0} : P \to \mathbb{R}$ such that

$$g_{y_0}(x_i) \neq g_{y_0}(x_j) \text{ for } i \neq j.$$ 

It is clear that there exists a neighborhood $\mathcal{O}$ of $y_0$ such that for any $y \in \mathcal{O}$ the function $g_{y_0}$ takes different values on points from $f^{-1}(y)$. Choosing a finite covering $\{\mathcal{O}_\beta\}$ of $\text{Im}f$ by such subsets we construct smooth functions $\{g_\beta\}_{\beta=1}^k$ such that for each $\beta = 1, \ldots, k$, $g_\beta : P \to \mathbb{R}$, and for any $y \in \text{Im}f$ there exists $\gamma = 1, \ldots, k$ such that $g_\gamma$ takes different values on points from $f^{-1}(y)$.

Consider the map

$$g := (f, g_1, \ldots, g_k) : P \to \mathbb{R}^M \times \mathbb{R}^k = \mathbb{R}^{M+k}.$$ 

Obviously this map is an imbedding. Since $\mathbb{R}^{M+k}$ can be imbedded as an open subset into the sphere $S^{M+k}$ the result follows. Q.E.D.

**2.1.16 Proposition.** Let $M$ be a compact smooth manifold. Let $P \subset M$ be a differentiable polyhedron. Let $\mathcal{D}$ be a subdivision of $P$. Let $T$ be a subdivision of $M$. Then the set of $C^\infty$-diffeomorphisms $f$ of $M$ such that for each $T_\lambda \in T$ its image $f(T_\lambda)$ is transversal to each $P_\nu \in \mathcal{D}$, is open and dense in the group $\text{Diff}(M)$ of all $C^\infty$-diffeomorphisms of $M$.

**Proof.** The openness is obvious. Let us prove the density. Clearly it is enough to prove that in any neighborhood of the identity diffeomorphism of $M$ there is a transformation we need. Let $n := \dim M$. We can choose a finite open covering $\{U_i\}_{i=1}^N$ of $M$ such that there exist open subsets $\{V_i\}_{i=1}^N$ such that $\bar{U}_i \subset V_i$, and there exist diffeomorphisms $f_i : V_i \to D^n$ where $D^n$ denotes the unit ball in $\mathbb{R}^n$. Let $l^1_i, \ldots, l^n_i$ be the corresponding coordinate functionals on $V_i$. Let us fix a partition of unity $\{\phi_i\}_{i=1}^N$ such that $\phi_i \equiv 1$ on $\bar{U}_i$, $\text{supp}(\phi_i) \subset V_i$, and $\sum_i \phi_i \equiv 1$. Then for small enough real numbers $a_{ij}$ the map $x \mapsto x + \sum_j a_{ij} \phi_i l^j$ is a globally defined diffeomorphism of $M$. Let $A \subset \mathbb{R}^{nN}$ be a small neighborhood of $0$ in the space of parameters $\{a_{ij}\}$. Thus we get a map

$$\Xi : A \times M \to M.$$ 

Let us fix $T_\lambda \in T$ and $P_\nu \in \mathcal{D}$. It is clear that if we restrict $\Xi$ to $A \times T_\lambda$ we get a submersion

$$\Xi' : A \times T_\lambda \to M.$$ 

In particular $\Xi'$ is transversal to $P_\nu$. Then by Theorem 10.3.3 of [6] for $a$ from a dense subset of $A$ the map

$$\Xi'_a := \Xi'(a, \cdot) : T_\lambda \to M$$ 

is transversal to $P_\nu$. (Though in [6] this is proved under assumption that $T_\lambda$, $P_\nu$ are closed submanifolds, but the same proof works when $T_\lambda$ and $P_\nu$ are differentiable subpolyhedra.) Q.E.D.

**2.1.17 Proposition.** Let $P$ be a differentiable polyhedron. Let $\{U_\alpha\}$ be an open covering of $P$. Let $\mathcal{D}$ be a simple subdivision of $P$. Then there exists a refinement $\mathcal{D}'$ of $\mathcal{D}$ which is simple and subordinate to $\{U_\alpha\}$. 
Proof. Using Lemma 2.1.5 let us imbed $P$ into a smooth compact manifold $M$. Let $\tilde{U}_\alpha$ be an open subset of $M$ such that $\tilde{U}_\alpha \cap P = U_\alpha$. Consider the open covering of $M$ by \{\tilde{U}_\alpha\} \cup \{M \setminus P\}$. Since any smooth manifold admits a triangulation, by Lemma 2.1.14 we can choose a triangulation $T$ of $M$ subordinate to this covering. By Proposition 2.1.16 we can choose a generic diffeomorphism of $M$ close to the identity so that the image of $T$ is transversal to $D$. We may assume that it is $T$ itself. Choosing $D' := D \cap T$ and applying Lemma 2.1.10 and Proposition 2.1.11 we prove the proposition. Q.E.D.

2.1.18 Proposition. Let $P$ be a differentiable polyhedron. Let $D_1$ and $D_2$ be two subdivisions of $P$. Then there exist subdivisions $D_3$, $D'$, $D''$ of $P$ such that

1. $D'$ is a refinement of $D_1$ and $D_3$;
2. $D''$ is a refinement of $D_2$ and $D_3$.

Moreover if $D_1$ and $D_2$ are simple then $D_3$, $D'$, and $D''$ can also be chosen simple.

Proof. Using Lemma 2.1.15 let us fix an imbedding of $P$ into a compact smooth manifold $M$. Fix any triangulation $T$ of $M$. Let $D_3$ be the image of $T$ under a generic diffeomorphism of $M$ (we use Proposition 2.1.16). Then $D_3$ is transversal to $D_1$ and $D_2$. Now let us define $D' := D_1 \cap D_3$, $D' := D_2 \cap D_3$. The result now follows from Lemma 2.1.10 and Proposition 2.1.11. Q.E.D.

2.1.19 Definition. Let $D$ be a subdivision of $P^n$. Assume that a subset $X \subset P^n$ admits a presentation as a union $X = \bigcup_{j=1}^s P_{\lambda_j}$ where $P_{\lambda_j} \in D$. We say that this presentation of $X$ is reduced if no one of the polyhedra in this union is contained in another, i.e. $P_{\lambda_i} \nsubseteq P_{\lambda_j}$ for $i \neq j$.

2.1.20 Lemma. Let us assume that a subset $X \subset P^n$ has two reduced decompositions

$$X = A \cup (\bigcup_j P_{\lambda_j}) = A \cup (\bigcup_l Q_{\nu_l})$$

where $A \in D$ is a polytope of type $r$ and $P_{\lambda_j}, Q_{\nu_l}$ are polytopes of type at most $r$. Then

$$\bigcup_j P_{\lambda_j} = \bigcup_l Q_{\nu_l}.$$

Proof. Set $B := \bigcup_j P_{\lambda_j}$, $C := \bigcup_l Q_{\nu_l}$. By symmetry it is enough to prove that $B \subset C$. Let $z \in B$, say $z \in P_{\lambda_i}$. If $z \notin A$ then $z \in C$. Let us assume now that $z \in A$. By assumption $P_{\lambda_i} \nsubseteq A$. Fix any point $w$ from the interior of $P_{\lambda_i}$. Then $w \notin A$. Hence $w \in C$. In particular $z \in C$. Q.E.D.

2.1.21 Corollary. Let $D$ be a subdivision of $P^n$. Let $X \subset P^n$ be a subset presentable as a union of some elements of $D$. Then $X$ admits a reduced decomposition, and it is unique.

Proof. The existence of a reduced decomposition is obvious. Let us prove the uniqueness. Let us denote by $r := \dim X$. Assume that we have two reduced decompositions of $X$:

$$X = \bigcup_{j=1}^s A_{\lambda_j} = \bigcup_{l=1}^r B_{\nu_l}. \quad (2)$$

Take some $B_{\nu_p}$ of dimension $r$. Fix any interior point $z$ of $B_{\nu_p}$. Then $z \in A_{\lambda_q}$ for some $\lambda_q$. Hence $B_{\nu_p} \subset A_{\lambda_q}$. Since $A_{\lambda_q}$ has dimension at most $r$ we conclude that $B_{\nu_p} = A_{\lambda_q}$. By Lemma 2.1.20 we can omit $B_{\nu_p} = A_{\lambda_q}$ from the second equality in (2). Continuing this process we prove the statement. Q.E.D.
2.2 Finitely additive measures.

In this subsection we will discuss a combinatorial notion of finitely additive measure on a family of sets which is not necessarily closed under finite intersections and unions but satisfies some other assumptions.

2.2.1 Definition. Let $S$ be a family of sets which is closed under finite unions and finite intersections. A functional $\mu : S \rightarrow \mathbb{C}$ is called a finitely additive measure if for any $A, B \in S$ one has

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B).$$

It is easy to see by induction that finitely additive measures satisfy a stronger inclusion-exclusion property. Namely for any $A_1, \ldots, A_s \in S$ one has

$$\mu(\bigcup_{i=1}^{s} A_i) = \sum_{I \subset \{1, \ldots, s\}, I \neq \emptyset} (-1)^{|I|+1} \mu(\bigcap_{i \in I} A_i).$$

Let $D = \{A_\lambda\}_{\lambda \in \Lambda}$ be a finite family of subsets of some set. Assume that we are given a decomposition of the set of indices $\Lambda$ into a disjoint union

$$\Lambda = \Lambda_0 \coprod \Lambda_1 \coprod \cdots \coprod \Lambda_n.$$

For $r = 0, 1, \ldots, n$ let us call sets $A_\lambda$ with $\lambda \in \Lambda_r$ the sets of type $r$. Set $\Lambda_\leq r := \cup_{i=0}^{r} \Lambda_i$. Let $X$ be a finite union of some of elements of $D$. Let us call a presentation $X = \bigcup_{j=1}^{s} A_{\lambda_j}$, $\lambda_j \in \Lambda$ reduced if no set $A_{\lambda_j}$ in this presentation is contained in the other one.

Let us make the following assumptions on $D$:

1. for any sets $A_{\lambda_1}$ and $A_{\lambda_2}$ from $D$ of types $r_1$ and $r_2$ respectively, their intersection $A_{\lambda_1} \cap A_{\lambda_2}$ is a finite union of sets from $D$ of types at most $\min\{r_1, r_2\}$;
2. if for $\lambda_1 \neq \lambda_2$ the sets $A_{\lambda_1}$ and $A_{\lambda_2}$ are of the same type $r$ then $A_{\lambda_1} \cap A_{\lambda_2}$ is a finite union of sets from $D$ of types strictly less than $r$;
3. for every set $X$ as above, a reduced decomposition is unique.

Let us denote by $T$ the family of all finite unions of subsets from $D$. Then clearly under the above assumptions $T$ is closed under finite unions and finite intersections.

Assume we are given a function $m : \Lambda \rightarrow \mathbb{C}$. Then we have

2.2.2 Lemma. Under the above assumptions there exists unique finitely additive measure $\mu$ on $T$ (in the sense of Definition 2.2.1) such that for any $\lambda \in \Lambda$

$$\mu(A_\lambda) = m(\lambda).$$

Proof. Let us denote by $T_r$ the family of all finite unions of subsets from $D$ of types at most $r$. Then clearly $T_r$ is closed under finite unions and finite intersections. Moreover we have:

$$T_0 \subset T_1 \subset \cdots \subset T_n = T.$$

The construction of the measure $\mu$ on $T_r$ will be by induction on $r$. First let $r = 0$. For any $\lambda_1, \lambda_2 \in \Lambda_0$, $\lambda_1 \neq \lambda_2$ we have $A_{\lambda_1} \cap A_{\lambda_2} = \emptyset$. Hence any set $X$ from $T_0$ has the form $X = A_{\lambda_1} \coprod \cdots \coprod A_{\lambda_s}$ where all $\lambda_j \in \Lambda_0$ and are distinct. Then there is only one way to
define \( \mu \) on \( \mathcal{T}_0 \), namely \( \mu(X) = \sum_{j=1}^{s} m(A_{\lambda_j}) \). Clearly we get a well defined measure \( \mu \) on \( \mathcal{T}_0 \).

Assume we have constructed uniquely defined finitely additive measure \( \mu \) on \( \mathcal{T}_{r-1} \). Let us extend it to \( \mathcal{T}_r \) and prove uniqueness of this extension. Let \( X \in \mathcal{T}_r \). Then \( X \) has a (non-unique) presentation \( X = \bigcup_{j=1}^{s} A_{\lambda_j} \) where \( \lambda_j \in \Lambda_{\leq r} \) and all \( A_{\lambda_j} \) are pairwise distinct. The only way to define \( \mu(X) \) is

\[
\mu(X) := \sum_{j=1}^{s} m(A_{\lambda_j}) + \sum_{I \subset \{1,\ldots,s\}, |I| > 1} (-1)^{|I|+1} \mu(\cap_{i \in I} A_{\lambda_i}).
\]

Note that in this formula the second sum is defined by the assumption of induction. We have to prove that \( \mu \) is well defined on \( \mathcal{T}_r \) and that it is indeed a finitely additive measure.

Let us check first that \( \mu \) is well defined. Let \( X \in \mathcal{T}_r \). It is sufficient to show that for any decomposition of a set \( X \in \mathcal{T}_r \) the expression (3) gives the same value for \( \mu \) as for the reduced decomposition (which is unique by the assumptions on \( D \)). Assume that a decomposition \( X = \bigcup_{j=1}^{s} A_{\lambda_j} \) is not reduced, say \( A_{\lambda_1} \supset A_{\lambda_2} \). Then we have:

\[
\sum_{I \subset \{1,\ldots,s\}, I \neq \emptyset} (-1)^{|I|+1} \mu(\cap_{i \in I} A_{\lambda_i}) =
\sum_{I \subset \{3,\ldots,s\}, I \neq \emptyset} (-1)^{|I|+1} \mu(\cap_{i \in I} A_{\lambda_i}) + \sum_{I \subset \{3,\ldots,s\}} (-1)^{|I|+2} \mu(A_{\lambda_1} \cap (\cap_{i \in I} A_{\lambda_i})) + \sum_{I \subset \{3,\ldots,s\}} (-1)^{|I|+2} [\mu(A_{\lambda_2} \cap (\cap_{i \in I} A_{\lambda_i})) - \mu(A_{\lambda_1} \cap A_{\lambda_2} \cap (\cap_{i \in I} A_{\lambda_i}))].
\]

The last sum clearly vanishes. Hence we see that the set \( A_{\lambda_2} \) can be omitted. We can continue this procedure till we get a reduced decomposition of \( X \). This proves that \( \mu \) is well defined on \( \mathcal{T}_r \).

It remains to check that \( \mu \) is indeed a finitely additive measure on \( \mathcal{T}_r \). Since \( \mathcal{T}_r \) is closed under finite unions and finite intersections it is sufficient to check that for any two sets \( X, Y \in \mathcal{T}_r \) one has \( \mu(X \cup Y) = \mu(X) + \mu(Y) - \mu(X \cap Y) \). Let \( X = \bigcup_{j=1}^{s} A_{\lambda_j}, \ Y = \bigcup_{i=1}^{t} B_{\nu_i} \). Let us prove the statement by the induction in \( s \).

Let us assume that \( s = 1 \). Thus \( X = A_{\lambda_1} \). First let us check that

\[
\mu(X \cap Y) = \sum_{I \subset \{1,\ldots,t\}, I \neq \emptyset} (-1)^{|I|+1} \mu(X \cap (\cap_{i \in I} B_{\nu_i})).
\]

We have

\[
X \cap Y = \bigcup_{l=1}^{t} (A_{\lambda_1} \cap B_{\nu_l}),
X \cap (\cap_{i \in I} B_{\nu_i}) = \cap_{i \in I} (A_{\lambda_1} \cap B_{\nu_i}).
\]

If \( A_{\lambda_1} \neq B_{\nu_l} \) for any \( l \), then the type of \( A_{\lambda_1} \cap B_{\nu_l} \) is strictly less than \( r \), and (4) follows by the additivity of \( \mu \) on \( \mathcal{T}_{r-1} \). Assume now that \( A_{\lambda_1} = B_{\nu_l} \). Then \( \mu(X \cap Y) = \mu(B_{\nu_l}) \). Also the right hand side in (4) is equal to

\[
\mu(B_{\nu_l}) + \sum_{I \subset \{2,\ldots,t\}, I \neq \emptyset} (-1)^{|I|+1} (\mu(X \cap (\cap_{i \in I} B_{\nu_i})) - \mu(X \cap B_{\nu_l} \cap (\cap_{i \in I} B_{\nu_i}))) = \mu(B_{\nu_l}).
\]
This proves (4).

Next we have
\[
\mu(X \cup Y) = \mu(A_{\lambda_1} \cup (\cup_{i=1}^t B_{\nu_i})) = \\
\sum_{I \subset \{1, \ldots, t\}, I \neq \emptyset} (-1)^{|I|+1} \mu(\cap_{i \in I} B_{\nu_i}) + \mu(A_{\lambda_1}) + \\
\sum_{I \subset \{1, \ldots, t\}, I \neq \emptyset} (-1)^{|I|+2} \mu(A_{\lambda_1} \cap (\cap_{i \in I} B_{\nu_i}))) = \\
\mu(Y) + \mu(X) - \mu(X \cap Y).
\]

Let us assume that \(s > 1\). Then let us present \(X = F \cup G\) where \(F\) and \(G\) can be presented as a union of a smaller number than \(s\) of elements of \(D\). Then by the assumption of induction we have
\[
\mu(X \cup Y) = \mu(F \cup (G \cup Y)) = \mu(F) + \mu(G \cup Y) - \mu(F \cap (G \cup Y)) = \\
\mu(F) + (\mu(G) + \mu(Y) - \mu(G \cap Y)) - (\mu(F \cap G) + \mu(F \cap Y) - \mu(F \cap G \cap Y)) = \\
\mu(F \cup G) + \mu(Y) - \mu((F \cup G) \cap Y) = \mu(X) + \mu(Y) - \mu(X \cap Y).
\]
Thus \(\mu\) is indeed a finitely additive measure. Q.E.D.

### 2.3 The sheaf of finitely additive measures.

Let \(X\) be a smooth manifold (of class \(C^\infty\)). Let \(\mathcal{P}(X)\) denote the family of all simple regular subpolyhedra of \(X\) in sense of Definitions 2.1.1 and 2.1.7(1). Let \(n = \dim X\).

#### 2.3.1 Definition.

A finitely additive measure \(\mu\) on \(\mathcal{P}(X)\) is a functional
\[
\mu : \mathcal{P}(X) \rightarrow \mathbb{C}
\]
which satisfies the following property. Fix any \(P \in \mathcal{P}(X)\) and any simple subdivision \(\mathcal{D} = \{P_\lambda\}_{\lambda \in \Lambda}\) of \(P\). Define a function \(m : \Lambda \rightarrow \mathbb{C}\) by \(m(\lambda) := \mu(P_\lambda)\). For \(r = 0, \ldots, r\) let \(\Lambda_r := \{\lambda \in \Lambda | \dim P_\lambda = r\}\). Then clearly \(\Lambda = \Lambda_0 \bigsqcup \Lambda_1 \bigsqcup \cdots \bigsqcup \Lambda_r\). The assumptions (1)-(3) before Lemma 2.2.2 are satisfied. Let \(\mathcal{T}\) denote the family of all subsets representable as finite unions of elements of \(\mathcal{D}\). Clearly \(\mathcal{T}\) is a finite family closed under (finite) unions and intersections, and \(P \in \mathcal{T}\). Let \(\mu'\) denote the finitely additive measure on \(\mathcal{T}\) which is constructed from \(m\) as in Lemma 2.2.2. Then we call \(\mu\) to be a finitely additive measure on \(\mathcal{P}(X)\) if \(\mu(P) = \mu'(P)\) for any \(P\) and any subdivision \(\mathcal{D}\) of it.

The linear space of all finitely additive measures on \(\mathcal{P}(X)\) we will denote by \(\mathcal{M}(X)\). Now let us consider a presheaf \(\mathcal{M}_X\) of vector spaces on \(X\) defined as follows. For any open subset \(U \subset X\) set
\[
\mathcal{M}_X(U) := \mathcal{M}(U)
\]
with the obvious maps of restriction.

#### 2.3.2 Proposition.

The presheaf \(\mathcal{M}_X\) is a sheaf.
Proof. Let $U$ be any open subset of $X$. Let $\{U_\alpha\}$ be any open covering of $U$. We have to check the following two conditions:

1. if $\mu \in M_X(U)$ is such that $\mu|_{U_\alpha} = 0$ for any $\alpha$ then $\mu = 0$;
2. if we are given $\mu_\alpha \in M_X(U_\alpha)$ such that $\mu_\alpha|_{U_\alpha \cap U_\beta} = \mu_\beta|_{U_\alpha \cap U_\beta} \forall \alpha, \beta$

then there exists $\mu \in M_X(U)$ such that $\mu|_{U_\alpha} = \mu_\alpha$ for all $\alpha$.

First let us check the condition (1). Let $P \in P(U)$. By Proposition 2.1.17 we can choose a simple subdivision $D = \{P_\lambda\}$ of $P$ subordinate to $\{U_\alpha\}$. Then one has

$$\mu(P) = \sum_{I \subset \Lambda, I \neq \emptyset} (-1)^{|I|+1} \mu(\cap_{i \in I} P_i) = 0.$$ 

Let us check condition (2). Let $P \in P(U)$. Let us choose any subdivision $D = \{P_\lambda\}$ of $P$ subordinate to the covering $\{U_\alpha\}$. Let us define a function $m : \Lambda \rightarrow \mathbb{C}$ as follows. Let $\lambda \in \Lambda$. Choose $U_\alpha$ such that $P_\lambda \subset U_\alpha$. Define $m(\lambda) := \mu_\alpha(P_\lambda)$. Clearly $m$ is well defined. By Lemma 2.2.2 we can define a number $\mu_D(P)$ using this subdivision $D$.

2.3.3 Claim. The value $\mu_D(P)$ does not depend on the choice of a subdivision $D$ of $P$.

This value will be denoted by $\mu(P)$. Let us prove Claim 2.3.3. Let $D_1$ and $D_2$ be two simple subdivisions subordinate to the covering $\{U_\alpha\}$. By Proposition 2.1.18 we can choose simple subdivisions $D_3$, $D'$, $D''$ such that $D'$ is a refinement of $D_1$ and $D_3$, and $D''$ is a refinement of $D_2$ and $D_3$.

Thus in order to check that $\mu$ is well defined it remains to check that if $D'$ is a refinement of $D$ then

$$\mu_D(P) = \mu_{D'}(P).$$

But this statement follows immediately from the uniqueness in Lemma 2.2.2.

To finish the proof of Proposition 2.3.2 it remains to prove that $\mu$ is indeed a finitely additive measure. Let $P \in P(U)$. Let $D$ be any simple subdivision of $P$. Let $D'$ be a simple refinement of $D$ subordinate to $\{U_\alpha\}$. Then define $m : \Lambda \rightarrow \mathbb{C}$ by $m(\lambda) = \mu(P_\lambda)$. The result follows from Lemma 2.2.2. Q.E.D.

2.4 Smooth valuations.

In this subsection we introduce the main object of this article, namely smooth valuations.

Let $X$ be a smooth manifold of dimension $n$. Let $P \in P(X)$. For any point $x \in P$ let us define the tangent cone to $P$ at $x$, denoted by $T_xP$, the set

$$T_xP := \{\xi \in T_xX| \text{ there exists a } C^1 - \text{map } \gamma : [0, 1] \rightarrow P \text{ such that } \gamma(0) = x \text{ and } \gamma'(0) = \xi\}. \quad (5)$$

It is easy to see that $T_xP$ coincides with the usual tangent space if $x$ is an interior point of $P$. In general $T_xP \subset T_xX$ is a closed polyhedral cone.
2.4.1 Definition. The characteristic cycle of $P$ is defined by

$$CC(P) := \cup_{x \in P} (T_x P)^\circ$$

(6)

where for a convex cone $C$ in a linear space $W$ one denotes by $C^\circ$ the dual cone

$$C^\circ := \{ y \in W^* | y(x) \geq 0 \text{ for any } x \in C \}.$$

Then $CC(P)$ is an $n$-dimensional subset of $T^* X$. It is invariant under the group $R_{>0}$ of positive real numbers acting on $T^* X$ by the multiplication of cotangent vectors. Moreover it is a Lagrangian submanifold with singularities. Note that when $X$ is oriented the smooth part of $CC(P)$ carries an induced orientation; then it is a cycle, i.e. $\partial(CC(P)) = 0$.

2.4.2 Definition. The normal cycle $N(P)$ of $P$ is defined by

$$N(P) := (a(CC(P)) \setminus \{0\}) / R_{>0}$$

(7)

where $a: T^* X \to T^* X$ is the natural involution of multiplication by $-1$ each cotangent vector, $\{0\}$ denotes the zero section of $T^* X$, and the quotient is taken with respect to the natural action of the group $R_{>0}$ mentioned above.

Thus $N(P) \subset \mathbb{P}_+(T^* X)$ is $(n-1)$-dimensional submanifold with singularities. An orientation of $X$ induces an orientation of $N(P)$; then it is a cycle. For some references on the notions of the normal and characteristic cycles see Remark 1.3.10.

Let $\mu$ be a finitely additive measure on $X$ in sense of Definition 2.3.1.

2.4.3 Definition. A measure $\mu$ is called continuous valuation if for any sequence of sets $\{P_N\} \subset P(X)$ which is contained in a compact subset of $X$ and such that $\sup\ vol(N(P_N)) < \infty$, and a subset $P \in P(X)$ such that

$$CC(P_N) \to CC(P)$$

in sense of currents, one has $\mu(P_N) \to \mu(P)$.

2.4.4 Remark. (1) Remind that the convergence in sense of currents means that for any $\omega \in C^\infty(T^* X, \Omega^n \otimes p^* o)$ such that the restriction of the projection $p$ to the support of $\omega$ is proper, one has

$$\int_{CC(P_N)} \omega \to \int_{CC(P)} \omega.$$

(2) The convergence used in Definition 2.4.3 is equivalent to the flat convergence of currents, see [7]. The equivalence is proved in [16], Theorem 31.2.

For any open subset $U \subset X$ let us denote by $\mathcal{C}(U)$ the space of continuous valuations on $U$. Clearly the correspondence $U \mapsto \mathcal{C}(U)$ is a sub-presheaf of $\mathcal{M}_X$. It will be denoted by $\mathcal{C}_X$.

We would like to formulate a conjecture.

2.4.5 Conjecture. The presheaf $\mathcal{C}_X$ is a sheaf.
Let us denote by $\mathcal{K}(\mathbb{R}^n)$ the family of convex compact subsets of $\mathbb{R}^n$.

### 2.4.6 Definition.
A measure $\mu$ is called smooth valuation if every point $x \in X$ has a neighborhood $U \ni x$ and a diffeomorphism $\phi : U \rightarrow \mathbb{R}^n$ such that the restriction of $\phi_*\mu$ to $\mathcal{P}(\mathbb{R}^n) \cap \mathcal{K}(\mathbb{R}^n)$ extends by continuity in the Hausdorff metric to $\mathcal{K}(\mathbb{R}^n)$ (clearly this extension is unique if it exists) and this extension belongs to $SV(\mathbb{R}^n)$ (see Subsection 1.3).

For any open subset $U \subset X$ let us denote by $V^\infty(U)$ the set of smooth valuations.

### 2.4.7 Lemma.
Let $V$ be an affine $n$-dimensional space. Let $\mu \in C(V)$. Assume that $\mu(P) = 0$ for any convex polytope $P$. Then $\mu = 0$.

**Proof.** Since by Proposition 2.4.8 any $P \in \mathcal{P}(V)$ admits a triangulation, it is enough to show that $\mu$ vanishes on any smoothly imbedded simplex $T$. Let $T = f(\Delta)$ where $\Delta$ be the standard $n$-dimensional simplex in $\mathbb{R}^n$, and $f$ is a diffeomorphism of a neighborhood of $\Delta$ onto an open subset in $V$. (The case of lower dimensional simplices in reduced to $n$-dimensional case by approximation.) Let $K_N$ be a sequence of convex compact subsets of $\mathbb{R}^n$ with smooth boundary which converges to $\Delta$ in the Hausdorff metric. Then $CC(K_N) \rightarrow CC(\Delta)$, and $\sup_N vol(N(K_N)) < \infty$ (this fact is known for a long time, see e.g. the end of Section 1 in M. Zähle [20] where this fact was stated without proof; for a proof we refer to [4] due to the lack of original reference). Set $A_N := f(K_N)$. Then $CC(A_N) \rightarrow CC(T)$ and $\sup_N vol(N(A_N)) < \infty$. Hence it is enough to show that $\mu(A_N) = 0$. Thus if one shows that for any compact domain $B$ with smooth boundary there exists a sequence of subsets $\{B_N\}$ presentable as a finite union of convex polytopes such that this sequence is contained in a compact subset, $CC(B_N)$ have uniformly bounded volume, and $CC(B_N) \rightarrow CC(B)$, then it follows that $\mu(A_N) = 0$. In this form this result is proved in [4]; however the main step in the proof showing convergence of the normal cycles (instead of the characteristic cycles) is due to M. Zähle [20]. The result follows. Q.E.D.

### 2.4.8 Lemma.
Let $X$ be a smooth manifold.

(i) Let $\nu \in C^\infty(X, |\omega_X|)$, $\eta \in C^\infty(\mathbb{P}_+(T^*X), \Omega^{n-1} \otimes p^*o)$. Then $P \mapsto \nu(P) + \int_{\mathcal{N}(P)} \eta$ defines a smooth valuation on $X$.

(ii) Let $\mu \in V^\infty(X)$. Let $x \in X$. Then there exists a neighborhood $U$ of $x$, $\nu \in C^\infty(U, |\omega_X|)$, $\eta \in C^\infty(\mathbb{P}_+(T^*U), \Omega^{n-1} \otimes o_U)$ such that for any $P \in \mathcal{P}(U)$ one has

$$\mu(P) = \nu(P) + \int_{\mathcal{N}(P)} \eta.$$ 

**Proof.** Part (i) follows from Proposition 1.3.15.

Part (ii) immediately follows from Lemma 2.4.7 and Theorem 1.3.16. Q.E.D.

### 2.4.9 Corollary.
For any open subset $U \subset X$ the set of smooth valuations $V^\infty(U)$ is a linear subspace of $\mathcal{M}_X(U)$.

**Proof.** This immediately follows from Lemma 2.4.8. Q.E.D.

### 2.4.10 Theorem.
The correspondence $U \mapsto V^\infty(U)$ is a subsheaf of $\mathbb{C}$-vector spaces of the sheaf $\mathcal{M}_X$.
This subsheaf will be denoted by $V_X^\infty$.

**Proof of Theorem 2.4.10.** By Corollary 2.4.9, $V^\infty(U)$ is a $C$-linear subspace of $M_X(U)$. It immediately follows from Lemma 2.4.8 that $V_X^\infty$ is a presheaf. Since the definition of $V^\infty(U)$ is local, the sheaf property of $V_X^\infty$ is satisfied automatically. Q.E.D.

Further properties of smooth valuations will be studied in the next section. Now we will remind the following well known lemma (see p.234 in [15]; compare with Theorem 1.8.8 of [15]).

**2.4.11 Lemma.** Let us fix a Euclidean metric on an affine space $V$. Let $\{K_N\}$ be a sequence in $K(V)$ converging in the Hausdorff metric to $K \in K(V)$. Let $A \in K(V)$. Then for almost all isometries $g$ of $V$ one has

$$K_N \cap (gA) \to K \cap (gA)$$

in the Hausdorff metric.

**2.4.12 Proposition.** Let $V$ be a linear space.

(i) The restriction map

$$C(V) \to CV(V)$$

is injective.

(ii) Under the above imbedding the image of $V^\infty(V)$ is equal to $SV(V)$.

**Proof.** Part (i) follows immediately from Lemma 2.4.7.

Let us prove part (ii). First observe that Theorem 1.3.16 and Lemma 2.4.8(1) imply immediately that $SV(V)$ is contained in the image of $V^\infty(V)$. To prove the opposite inclusion let us fix a Euclidean metric on $V$ and fix $\phi \in V^\infty(V)$. Let $\{U_\alpha\}$ be an open covering of $V$ such that, as in Lemma 2.4.8(ii), for any $\alpha$ and any $P \subset U_\alpha$

$$\phi(P) = \nu_\alpha(P) + \int_{N(P)} \eta_\alpha$$

where $\nu_\alpha \in C^\infty(U_\alpha, |\omega_\alpha|)$, $\eta_\alpha \in C^\infty(P_+ (T^* U_\alpha), \Omega^{n-1} \otimes p^* o)$.

Let $K_0 \in K(V)$, $t_0 \in [0, 1]$, $x_0 \in V$. Assume first that there exists $a_0$ such that $K_0 + t_0 + x_0 \subset U_{a_0}$. Then there exist neighborhoods $\mathcal{O}_1 \subset [0, 1]$ of $t_0$ and $\mathcal{O}_2 \subset V$ of $x_0$ such that for any $t \in \mathcal{O}_1$ and any $x \in \mathcal{O}_2$ one has

$$K_0 + t + x \subset U_{a_0}.$$  

Then the function $[(t, x) \mapsto \phi(tK_0 + x)]$ is infinitely smooth in $\mathcal{O}_1 \times \mathcal{O}_2$ by Proposition 1.3.15(i).

Now an arbitrary $K_0 \in K(V)$ can be represented as a finite union of convex compact sets $K_0 = \bigcup_{l=1}^s K_l$ such that for each $l = 1, \ldots, s$ the set $K_l + t_0 + x_0$ is contained in some element of the covering $\{U_\alpha\}$. The inclusion-exclusion property and the above case imply the smoothness of the function $[(t, x) \mapsto \phi(tK + x)]$ where $(t, x) \in [0, 1] \times V$.

Thus it remains to show that the map $K(V) \to C^\infty([0, 1] \times V)$ given by $K \mapsto [(t, x) \mapsto \phi(tK + x)]$ is continuous. Let us fix a lattice $L \subset V$. Let $Q$ be a unit parallelepiped for $L$. It is easy to see that if $\varepsilon > 0$ is small enough, then for any $x \in \varepsilon L$ the set $K \cap (x + \varepsilon Q)$ is
contains in one of the elements of the covering \( \{ U_\alpha \} \). Also we have \( K = \cup_{x \in L} (K \cap (x + \varepsilon Q)) \).

Replacing \( L \) by its image under generic isometry of \( V \) close to the identity and using Lemma 2.4.11 we may assume that for any \( x \in \varepsilon L \)

\[
K_N \cap (x + \varepsilon Q) \to K \cap (x + \varepsilon Q)
\]

and similar convergence holds for finite intersections of the above sets. Now the result follows from Proposition 1.3.15(i). Q.E.D.

3 Further properties of smooth valuations.

In Subsection 3.1 we introduce and study the filtration \( \mathcal{W}_i \) on smooth valuations; in Proposition 3.1.9 we obtain a description of smooth valuations in terms of the integration with respect to the characteristic cycle. In Subsection 3.2 we define the natural structure of Fréchet space on the space of smooth valuations. In Subsection 3.3 we introduce the Euler-Verdier involution on smooth valuations.

3.1 Filtration on smooth valuations.

3.1.1 Definition. Let \( 0 \leq i \leq n \). Let \( U \) be an open subset of a manifold \( X \). Let us denote by \( \mathcal{W}_i(U) \) the subset of \( \mathcal{V}_X^\infty(U) \) consisting of all elements \( \phi \in \mathcal{V}_X^\infty(U) \) such that for every point \( x \in U \) there exists a neighborhood \( V \) and a diffeomorphism \( f : V \to \mathbb{R}^n \) such that the image of \( f_*\phi \) in \( SV(\mathbb{R}^n) \) belongs to \( W_i \) (see Subsection 1.3).

3.1.2 Proposition. (i) For any \( 0 \leq i \leq n \) and for any open subset \( U \subset X \), \( \mathcal{W}_i(U) \) is a vector subspace of \( \mathcal{V}_X^\infty(U) \).

(ii) The correspondence \( U \mapsto \mathcal{W}_i(U) \) is a subsheaf of \( \mathcal{V}_X^\infty \).

Proof. Let us fix an open subset \( V \subset U \) and a diffeomorphism \( f : V \to \mathbb{R}^n \). By Theorem 1.3.21 and Lemma 2.4.10 any valuation \( \phi \) on \( V \) such that \( f_*\phi \) lies in \( \mathcal{W}_i(V) \) has the following form: there exists \( \eta \in \mathcal{C}_\infty(\mathbb{T}^*U, \mathcal{W}_i(\mathbb{T}^*U) \otimes \mathbb{R}^n) \) such that

\[
\phi(P) = \int_{\mathbb{C}(P)} \eta.
\]

Obviously the set of valuations having the above form is a vector subspace of \( \mathcal{V}_X^\infty(V) \). This proves part (i) of the proposition.

The same reasoning implies that \( U \mapsto \mathcal{W}_i(U) \) is a sub-presheaf of \( \mathcal{C}_X \). Since the definition of \( \mathcal{W}_i \) is local, it is a sheaf. Q.E.D.

Remind that by Proposition 2.4.12 we have the identification \( \mathcal{V}_X^\infty(V) = SV(V) \). Using this identification we have the following proposition.

3.1.3 Proposition. Let \( V \) be a linear space. Then \( \mathcal{W}_i(V) = W_i \).

Proof. It is clear from the definition that \( W_i \subset \mathcal{W}_i(V) \). Let us prove that \( \mathcal{W}_i(V) \subset W_i \). Let \( \phi \in \mathcal{W}_i(V) \). Fix \( K \in \mathcal{K}(V), x \in V \). There exists a neighborhood \( U \) of \( x \) and a
diffeomorphism \( f : U \to \mathbb{R}^n \) such that \( f_* \phi \in W_i(\mathbb{R}^n) \). By Theorem 1.3.21 there exists \( \eta \in \tilde{C}^\infty(T^*\mathbb{R}^n, W_i(\Omega^n \otimes p^* o)) \) such that for any \( A \in \mathcal{K}(\mathbb{R}^n) \) one has

\[
(f_* \phi)(A) = \int_{CC(A)} \eta.
\]

By Proposition 2.3.12(i) the formula (8) still holds for any \( A \in \mathcal{P}(\mathbb{R}^n) \). Hence for \( 0 \leq t \ll 1 \) one has

\[
\phi(tK + x) = \int_{CC(tK + x)} \tilde{f}^* \eta
\]

where \( \tilde{f} \) is the natural lift of \( f \) to \( T^*U \). Set \( \omega := \tilde{f}^* \eta \in \tilde{C}^\infty(T^*U, W_i(\Omega^n \otimes p^* o)) \). Theorem 1.3.21 implies that

\[
\int_{CC(tK + x)} \omega = O(t^i).
\]

Hence \( \phi \in W_i \). Q.E.D.

Let us introduce more notation. Let us consider the following sheaf on \( X \):

\[
W'_i(U) := \tilde{C}^\infty(T^*U, W_i(\Omega^n \otimes p^* o)), \ i = 0, \ldots, n.
\]

Integration with respect to the normal cycle defines the following morphism of sheaves which, by Theorem 1.3.21 and Proposition 3.1.3, is an epimorphism:

\[
W'_i \to W_i.
\]

Clearly \( W'_i/W'_{i+1} \) is isomorphic to the sheaf \([U \mapsto \tilde{C}^\infty(U, W_i(\Omega^n \otimes p^* o)/W_{i+1}(\Omega^n \otimes p^* o))]\). Hence we have a continuous epimorphism

\[
\Xi_i : \tilde{C}^\infty(T^*U, W_i(\Omega^n \otimes p^* o)/W_{i+1}(\Omega^n \otimes p^* o)) \to W_i(U)/W_{i+1}(U).
\]

Next we have a continuous map

\[
\Psi_i : \tilde{C}^\infty(T^*U, \Omega^{n-i}_{T^*U/U} \otimes p^*(\wedge^i T^*U) \otimes p^* o) \to C^\infty(U, Val_i(TU)).
\]

This map \( \Psi_i \) is defined pointwise

\[
\Psi_i|_x : \tilde{C}^\infty(T^*_x X, \Omega^{n-i}_{T^*_x X} \otimes \wedge^i T^*_x X \otimes p^* o) \to Val^m_i(T_x X)
\]

using the integration with respect to the characteristic cycle of a subset of \( T^*_x X \).

By Lemma 1.3.20 there exists a canonical isomorphism of vector bundles

\[
J_i : W_i(\Omega^n \otimes p^* o)/W_{i+1}(\Omega^n \otimes p^* o) \to \Omega^{n-i}_{T^*U/U}(T^*U) \otimes p^*(\wedge^i T^*U) \otimes p^* o.
\]

Let us denote for brevity by \( Val_i(TX) \) the sheaf \([U \mapsto C^\infty(U, Val_i(TU))]\). Define sheaves \( \mathcal{R}_i, \mathcal{S}_i \) by

\[
\mathcal{R}_i(U) := \tilde{C}^\infty(T^*U, W_i(\Omega^n \otimes p^* o)/W_{i+1}(\Omega^n \otimes p^* o)),
\]

\[
\mathcal{S}_i(U) := \tilde{C}^\infty(T^*U, \Omega^{n-i}_{T^*U/U} \otimes p^*(\wedge^i T^*U) \otimes p^* o).
\]

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We have the canonical map
\[ \mathcal{W}_i(U)/\mathcal{W}_{i+1}(U) \to (\mathcal{W}_i/\mathcal{W}_{i+1})(U). \]
The composition of this map with \( \Xi_i \) from (10) gives a map
\[ \tilde{\Xi}_i(U) : R_i(U) \to (\mathcal{W}_i/\mathcal{W}_{i+1})(U). \]
(12)
This map is compatible with restrictions to open subsets. Hence we obtain a morphism of sheaves
\[ \tilde{\Xi}_i : R_i \to \mathcal{W}_i/\mathcal{W}_{i+1}. \]
(13)

3.1.4 Lemma. (i) There exists a natural isomorphism of sheaves
\[ \mathcal{W}'_i/\mathcal{W}'_{i+1} \simeq R_i. \]
(ii) For any open subset \( U \subset X \)
\[ R_i(U) = \mathcal{W}'_i(U)/\mathcal{W}'_{i+1}(U). \]

Proof. Part (i) is obvious. To prove part (ii) note that we have an exact sequence of \( \mathcal{O}_X \)-modules
\[ 0 \to \mathcal{W}'_i \to \mathcal{W}'_{i+1} \to R_i \to 0. \]
Hence from the long exact sequence we get
\[ 0 \to \mathcal{W}'_i(U) \to \mathcal{W}'_{i+1}(U) \to R_i(U) \to H^1(U, \mathcal{W}'_i). \]
But since by Example 1.2.11 \( \mathcal{O}_X \)-modules are acyclic we have \( H^1(U, \mathcal{W}'_i) = 0 \). The result follows. Q.E.D.

3.1.5 Lemma. The morphism \( \tilde{\Xi}_i : R_i \to \mathcal{W}_i/\mathcal{W}_{i+1} \) is an epimorphism of sheaves.

Proof. This follows immediately from the facts that \( \mathcal{W}'_i \to \mathcal{W}_i \) is an epimorphism, and \( R_i \simeq \mathcal{W}'_i/\mathcal{W}'_{i+1} \). Q.E.D.

We will need the following proposition.

3.1.6 Proposition. (i) There exists unique morphism of sheaves on \( X \)
\[ I_i : \mathcal{W}_i/\mathcal{W}_{i+1} \to \text{Val}(TX) \]
which makes the following diagram commutative:

\[ \begin{array}{ccc}
R_i & \xrightarrow{\tilde{\Xi}_i} & \mathcal{W}_i/\mathcal{W}_{i+1} \\
\downarrow{J_i} & & \downarrow{I_i} \\
S_i & \xrightarrow{\Psi_i} & \text{Val}(TX)
\end{array} \]

(ii) This morphism \( I_i \) is an isomorphism of sheaves.
The uniqueness of the morphism \( I_i \) follows immediately from the surjectivity of \( \tilde{\Xi}_i \).

Let us prove the existence. Observe first of all that for any open subset \( U \subset \mathbb{R}^n \) using Lemma 3.1.4 we have
\[
\mathcal{R}_i(U) = \mathcal{W}'_{i}(U)/\mathcal{W}'_{i+1}(U) \xrightarrow{\Xi} \mathcal{W}_i(U)/\mathcal{W}_{i+1}(U).
\]
This shows that
\[
\Xi_i(\mathcal{R}_i(U)) \hookrightarrow \mathcal{W}_i(U)/\mathcal{W}_{i+1}(U) \rightarrow (\mathcal{W}_i/\mathcal{W}_{i+1})(U).
\]

Let us fix an open subset \( U \subset X \) diffeomorphic to \( \mathbb{R}^n \), and fix a diffeomorphism \( f : U \xrightarrow{\sim} \mathbb{R}^n \). By Proposition 3.1.3 \( \mathcal{W}_j(\mathbb{R}^n) = \mathcal{W}_j \subset SV(\mathbb{R}^n) \). Theorem 1.3.21 and (15) imply that \( \Xi_i(\mathcal{R}_i(\mathbb{R}^n)) = \mathcal{W}_i/\mathcal{W}_{i+1} \). Let us construct a map denoted also \( I_i : \mathcal{W}_i/\mathcal{W}_{i+1} \to Val^i_{sm}(\mathbb{R}^n) \) which makes the following diagram commutative:

\[
\begin{array}{ccc}
\mathcal{R}_i(\mathbb{R}^n) & \xrightarrow{\Xi_i} & \mathcal{W}_i/\mathcal{W}_{i+1} \\
\downarrow J_i & & \downarrow I_i \\
\mathcal{S}_i(\mathbb{R}^n) \xrightarrow{\Psi_i} C^\infty(\mathbb{R}^n, Val_i(T\mathbb{R}^n)) = C^\infty(\mathbb{R}^n, Val^i_{sm}(\mathbb{R}^n))
\end{array}
\]

As in Subsection 1.3 for \( \phi \in \mathcal{W}_i/\mathcal{W}_{i+1} \) let us define
\[
(I_i \phi)(x, K) = \lim_{r \to 0} \frac{1}{r^i} \phi(rK + x)
\]
where \( K \in \mathcal{K}(\mathbb{R}^n), x \in \mathbb{R}^n \). Let us show that the diagram (16) is commutative. For \( \phi \in \mathcal{W}_i/\mathcal{W}_{i+1} \) and for all \( x \in V, K \in \mathcal{K}(V) \) one has
\[
(I_i \phi)(x, K) = \lim_{r \to 0} \frac{1}{r^i} \phi(rK + x).
\]

Let us fix \( \eta \in C^\infty(T^*\mathbb{R}^n, \mathcal{W}_i(\Omega^n \otimes p^*o)/\mathcal{W}_{i+1}(\Omega^n \otimes p^*o)) \). Let us fix a basis \( e_1^*, \ldots, e_n^* \) in \( V^* \). Then we can write
\[
J_i(\eta) = \sum_{j_1, \ldots, j_i} \eta_{j_1, \ldots, j_i} \otimes e_{j_1}^* \wedge \cdots \wedge e_{j_i}^*
\]
where \( \eta_{j_1, \ldots, j_i} \in C^\infty(T^*\mathbb{R}^n, \Omega^{n-i}_{T^*\mathbb{R}^n/X} \otimes p^*o) \). Then
\[
(I_i(\Xi_i \eta))(x, K) = \sum_{j_1, \ldots, j_i} \lim_{r \to 0} \frac{1}{r^i} \int_{CC(rK + r)} \eta_{j_1, \ldots, j_i} \otimes e_{j_1}^* \wedge \cdots \wedge e_{j_i}^*
\]
\[
= \sum_{j_1, \ldots, j_i} \int_{CC(K)} \eta_{j_1, \ldots, j_i} \big|_{p^{-1}(x)} \otimes e_{j_1}^* \wedge \cdots \wedge e_{j_i}^*
\]
\[
= (\Psi_i(J_i \eta))(x, K).
\]
Thus the diagram (16) is commutative. Pulling the diagram (16) back to $\mathcal{U}$ and using Proposition 3.1.3 we obtain commutative diagram of vector spaces

\[
\begin{array}{ccc}
\mathcal{R}_i(\mathcal{U}) & \xrightarrow{\Xi_i} & \mathcal{W}_i(\mathcal{U})/\mathcal{W}_{i+1}(\mathcal{U}) \\
\downarrow J_i & & \downarrow I_i \\
\mathcal{S}_i(\mathcal{U}) & \xrightarrow{\Psi_i} & C^\infty(\mathcal{U}, \text{Val}_i(T\mathcal{U})) \\
\end{array}
\] (17)

Note however that the map $I_i$ in (17) might depend on a choice of a diffeomorphism $f: \mathcal{U} \to \mathbb{R}^n$. This however does not happen due to the uniqueness of $I_i$ which has been proved.

Thus we have constructed, by now, for every open subset $\mathcal{U} \subset X$ diffeomorphic to $\mathbb{R}^n$ the unique map $I_i: \mathcal{W}_i(\mathcal{U})/\mathcal{W}_{i+1}(\mathcal{U}) \to C^\infty(\mathcal{U}, \text{Val}_i(T\mathcal{U}))$ which makes the diagram (17) commutative. Since $\mathcal{W}_i/\mathcal{W}_{i+1}$ is the sheafification of the presheaf $[U \mapsto \mathcal{W}_i(U)/\mathcal{W}_{i+1}(U)]$ this defines in unique way the map of sheaves $I_i: \mathcal{W}_i/\mathcal{W}_{i+1} \to \text{Val}_i(TX)$ which makes the diagram (14) commutative. This proves part (i) of the proposition.

Part (ii) follows immediately from Theorem 1.3.19 and the description of $I_i$ after it.

Q.E.D.

We would like to state separately the following immediate corollary.

3.1.7 Corollary. The quotient sheaf $\mathcal{W}_i/\mathcal{W}_{i+1}$ is canonically isomorphic to the sheaf $\text{Val}_i(TX)$.

3.1.8 Proposition. The sheaves $\mathcal{W}_i$ are soft. In particular the sheaf $\mathcal{V}_X^\infty$ is soft.

Proof. Consider the filtration of $\mathcal{W}_i$ by subsheaves

\[ \mathcal{W}_i \supset \mathcal{W}_{i+1} \supset \cdots \supset \mathcal{W}_n. \]

By Corollary 3.1.7 $\mathcal{W}_j/\mathcal{W}_{j+1}$ is an $\mathcal{O}_X$-module for any $j$. Hence by Example 1.2.11 $\mathcal{W}_j/\mathcal{W}_{j+1}$ is soft. Hence $\mathcal{W}_i$ is also soft by Proposition 1.2.13. Q.E.D.

3.1.9 Proposition. For any section $\phi \in \Gamma(X, \mathcal{W}_i)$ there exists $\eta \in \tilde{C}^\infty(T^*X, \mathcal{W}_i(\Omega^n \otimes p^*o))$ such that for any $P \in \mathcal{P}(X)$

\[ \phi(P) = \int_{\mathcal{C}C(P)} \eta. \]

Proof. Remind that in (9) we have introduced the sheaves $\mathcal{W}_i'$. We have the canonical epimorphism

\[ \mathcal{W}_i' \twoheadrightarrow \mathcal{W}_i. \]

We have to show that the map

\[ \Gamma(X, \mathcal{W}_i') \to \Gamma(X, \mathcal{W}_i) \]

is an epimorphism. For $i = n$ this is obvious.
Since the sheaves $W'_j$ and $W_{j+1}$ are soft we have
\[
\Gamma(X, W_j'/W_{j+1}') = \Gamma(X, W_j')/\Gamma(X, W_{j+1}'),
\]
\[
\Gamma(X, W_j/W_{j+1}) = \Gamma(X, W_j)/\Gamma(X, W_{j+1}).
\]

By the descending induction in $i$ it is enough to show that the induced maps
\[
\Gamma(X, W_j'/W_{j+1}') \to \Gamma(X, W_j/W_{j+1})
\]
are epimorphisms for all $j$. We may assume that $j < n$. We have seen that the morphism $\tilde{\Xi}_i: W_j'/W_{j+1}' \to W_j/W_{j+1}$ is an epimorphism of sheaves. Moreover the sheaf $W_j'/W_{j+1}'$ is an $O_X$-module. But $\tilde{\Xi}_i = I^{-1}_i \circ \Psi_i \circ J_i$, and $\Psi_i$ and $J_i$ are morphisms of $O_X$-modules. Set $K := Ker\tilde{\Xi}_i$. Hence $K$ is isomorphic (via $I^{-1}_i$) to an $O_X$-module. Hence by Example 1.2.11 $H^i(X, K) = 0$ for $i > 0$. From the long exact sequence we have
\[
\Gamma(X, W_j'/W_{j+1}') \to \Gamma(X, W_j/W_{j+1}) \to H^1(X, K) = 0.
\]
Thus Proposition 3.1.9 follows. Q.E.D.

3.1.10 Corollary. For any $\phi \in W_n(X)$ there exists $\nu \in C^\infty(X, |\omega_X|)$ such that for any $P \in \mathcal{P}(X)$
\[
\phi(P) = \nu(P).
\]
Moreover for any $i = 0, 1, \ldots, n - 1$ and any $\phi \in W_i(X)$ there exist $\nu \in C^\infty(X, |\omega_X|)$ and $\omega \in C^\infty(\mathbb{P}_+(T^*X), W_i(\Omega^{n-1}) \otimes p'^*o)$ such that for any $P \in \mathcal{P}(X)$
\[
\phi(P) = \nu(P) + \int_{N(P)} \omega.
\]

3.2 Linear topology on smooth valuations.

Let us describe the canonical Fréchet space structure on the space of smooth valuations. By Corollary 3.1.10 we have an epimorphism of linear spaces
\[
\Theta: C^\infty(X, |\omega_X|) \bigoplus C^\infty(\mathbb{P}_+(T^*X), \Omega^{n-1} \otimes p'^*o) \to V_X^\infty(X).
\]
The source space has a canonical Fréchet space structure. It is easy to see that the kernel of $\Theta$ is closed. Let us define the topology on $V_X^\infty(X)$ as the quotient topology. This is a Fréchet topology. By the same argument we define a Fréchet topology on $V_X^\infty(U)$ for any open subset $U \subset X$. The following proposition is trivial.

3.2.1 Proposition. For any open subsets $U \subset V \subset X$ the restriction map $V_X^\infty(V) \to V_X^\infty(U)$ is continuous.

3.2.2 Proposition. Let $V$ be an $n$-dimensional linear space. Consider the isomorphism of linear spaces $V_X^\infty(V) \to SVV(V)$ from Proposition 2.4.12(ii).

Then this is an isomorphism of Fréchet spaces.
Proof. By the Banach inversion theorem it is enough to check that the map $V_\infty^\infty(V)\to SV(V)$ is continuous. This is clear from the definitions. Q.E.D.

3.2.3 Proposition. For any $i = 0, 1, \ldots, n$, $W_i(X)$ is a closed subspace of $V_X^\infty(X)$.

Proof. The definition of $W_i$ and Corollary 3.1.10 imply that a smooth valuation $\phi \in V_X^\infty(X)$ belongs to $W_i(X)$ if and only if for any open subset $U \subset X$ diffeomorphic to $\mathbb{R}^n$, any diffeomorphism $f: U \to \mathbb{R}^n$, any $K \in \mathcal{K}(\mathbb{R}^n)$, and any $x \in \mathbb{R}^n$ one has

$$\left.\frac{d^k}{dt^k}\right|_{t=0}(f_*\phi)(tK + x) = 0 \text{ for } k < i.$$

It is easy to see that $\phi \mapsto \left.\frac{d^k}{dt^k}\right|_{t=0}(f_*\phi)(tK + x)$ is a continuous linear functional on $V_X^\infty(X)$ for any $U, f, K, x, k$ as above. Hence $W_i(X)$ is a closed subspace of $V_X^\infty(X)$. Q.E.D.

3.3 The Euler-Verdier involution.

In this subsection we construct a canonical continuous involution on the sheaf of smooth valuations which we call the Euler-Verdier involution. Thus

$$\sigma: V_X^\infty \to V_X^\infty$$

satisfies $\sigma^2 = Id$. This involution preserves the filtration $W_i$.

Let us describe the construction of $\sigma$. Remind that we have the sheaf $W'_0$ on $X$ defined by

$$W'_0(U) = \tilde{C}^\infty(T^*U, \Omega^n \otimes p^*o)$$

where as previously the last space denotes the space of infinitely smooth sections of the bundle $\Omega^n \otimes p^*o$ such that the restriction of the projection $p$ to the support of these sections is proper. By Proposition 3.1.9 we have epimorphism of sheaves

$$\Theta: W'_0 \to V_X^\infty.$$

On the space $T^*X$ we have the involution $a$ of multiplication by -1 in each fiber of the projection $p: T^*X \to X$. It induces involution $a^*$ of the sheaf $W'_0$.

3.3.1 Proposition. The involution $(-1)^n a^*$ factorizes (uniquely) to involution of $V_X^\infty$ denoted by $\sigma$.

Proof. We have to show that if $\omega \in \tilde{C}^\infty(T^*U, \Omega^n \otimes p^*o)$ satisfies $\Theta(\omega) = 0$ then $\Theta(a^*(\omega)) = 0$.

It is easy to see that for any $\omega \in \tilde{C}^\infty(T^*X, \Omega^n \otimes p^*o)$ and any $P \in \mathcal{P}(X)$ one has

$$\int_{CC(P)} a^*\omega = (-1)^{n-\text{dim}P} \left( \int_{CC(P)} \omega - \int_{CC(\partial P)} \omega \right)$$

where $\partial P := P \setminus \text{int}P$, and $\text{int}P$ if the relative interior of $P$. The formula (18) immediately implies the proposition. Q.E.D.

The following result is clear from the discussion above.
3.3.2 Theorem. (i) The Euler-Verdier involution $\sigma$ preserves the filtration $W_\bullet$.

(ii) The induced involution on $\text{gr}_W V_X^\infty \simeq \text{Val}_\bullet(TX)$ comes from the involution on the bundle $\text{Val}(TX)$ defined as $\phi \mapsto [K \mapsto (-1)^{\deg \phi}(\phi(-K))]$ for any $\phi \in \text{Val}(T_xX)$ for any $x \in X$, and where $\deg \phi$ is the degree of homogeneity of $\phi$.

Thus the sheaf $V_X^\infty$ of smooth valuations decomposes under the action of the Euler-Verdier involution into two subsheaves $V_X^{\infty,+}$ and $V_X^{\infty,-}$ corresponding to eigenvalues 1 and -1 of $\sigma$ respectively. Thus

$$V_X^\infty = V_X^{\infty,+} \oplus V_X^{\infty,-}.$$ 

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