Unimodal Bandits without Smoothness

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Abstract

We consider stochastic bandit problems with a continuum set of arms and where the expected reward is a continuous and unimodal function of the arm. No further assumption is made regarding the smoothness and the structure of the expected reward function. We propose Stochastic Pentachotomy (SP), an algorithm for which we derive finite-time regret upper bounds. In particular, we show that, for any expected reward function \( \mu \) that behaves as \( \mu(x) = \mu(x^*) - C|x - x^*|^{\xi} \) locally around its maximizer \( x^* \) for some \( \xi, C > 0 \), the SP algorithm is order-optimal, i.e., its regret scales as \( O(\sqrt{T \log(T)}) \) when the time horizon \( T \) grows large. This regret scaling is achieved without the knowledge of \( \xi \) and \( C \). Our algorithm is based on asymptotically optimal sequential statistical tests used to successively prune an interval that contains the best arm with high probability. To our knowledge, the SP algorithm constitutes the first sequential arm selection rule that achieves a regret scaling as \( O(\sqrt{T}) \) up to a logarithmic factor for non-smooth expected reward functions, as well as for smooth functions with unknown smoothness.

1 Introduction

This paper considers the problem of stochastic unimodal optimization with bandit feedback which is a generalization of the classical multi-armed bandit problem solved by Lai and Robbins [19]. The problem is defined by a continuous and unimodal expected reward function \( \mu \) defined on the interval \([0, 1]\). The algorithm repeatedly selects an arm \( x \in [0, 1] \), and gets a noisy reward of mean \( \mu(x) \). The performance of the algorithm is characterized by its regret up to time horizon \( T \) (the number of observed noisy rewards) defined as the difference between the average cumulative reward one would obtain if the function \( \mu \) was known, i.e., \( T \sup_{x \in [0, 1]} \mu(x) \), and the actual average cumulative reward achieved under the algorithm. It has been proved that for linear reward functions, one can only expect to achieve a regret scaling as \( \Omega(\sqrt{T}) \). Our objective is to devise an algorithm whose regret scales \( O(\sqrt{T}) \) up to a logarithmic factor for a large class of unimodal and continuous reward functions. Such an algorithm would hence be order-optimal. Importantly we merely make any assumption on the smoothness of the reward function – the latter can even be non-differentiable. This contrasts with all existing work investigating similar continuum-armed bandit problems, and where strong assumptions are made on the structure and smoothness of the reward function. These structure and smoothness are known to the decision maker, and are explicitly used in the design of efficient algorithms.

We propose Stochastic Pentachotomy (SP), an algorithm for which we derive finite-time regret upper bounds. In particular, we show that its regret scales as \( O(\sqrt{T \log(T)}) \) for any unimodal and continuous reward function \( \mu \) that behaves as \( \mu(x) = \mu(x^*) - C|x - x^*|^{\xi} \) locally around its maximizer \( x^* \) for some \( \xi, C > 0 \). This regret scaling is achieved without the knowledge of \( \xi \) or \( C \). The SP algorithm

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consists in successively narrowing an interval in $[0, 1]$ while ensuring that the arm with highest mean reward remains in this interval with high probability. The narrowing subroutine is a sequential test that takes as input an interval and samples a few arms in the interior of this interval until it gathers enough information to actually reduce the interval. We investigate a general class of such sequential tests. In particular, we provide a (finite time) lower bound of their expected sampling complexity, and design a sequential test that matches this lower bound. This is the test used in the SP algorithm. Interestingly, we show that to be efficient, a sequential test needs to sample at least three arms in the interior of the interval to reduce. This implies that a bandit version of the celebrated Golden section search algorithm cannot achieve a reasonably low regret over a large class of reward functions (indeed such an algorithm would sample only two arms in the interval to reduce). We illustrate the performance of our algorithms using numerical experiments and compare its regret to that of existing algorithms that leverage the smoothness and structure of the reward function.

To our knowledge, Stochastic Pentachotomy is the first order-optimal algorithm for continuous unimodal bandit problems for a large class of expected reward functions: Its regret scales as $O\left(\sqrt{T \log(T)}\right)$ for non-smooth reward functions, as well as for smooth functions with unknown smoothness.

**Related work.** Stochastic bandit problems with a continuum set of arms have recently received a lot of attention. Various kinds of structured reward function have been explored, i.e., linear [10], Lipschitz [2, 17, 5], and convex [1, 22]. In these papers, the knowledge of the structure greatly helps the design of efficient algorithms (for Lipschitz bandits, except for [6], the Lipschitz constant is assumed to be known). More importantly, the smoothness or regularity of the reward function near its maximizer is also assumed to be known and leveraged in the algorithms. Indeed, most existing algorithms use a discretization of the set of arms that depends on this smoothness, and this is crucial to guarantee a regret scaling as $\sqrt{T}$. As discussed in [5, 4], without the knowledge of the smoothness, these algorithms would yield a much higher regret (e.g. scaling as $T^{2/3}$ for the algorithm proposed in [4]).

Unimodal bandits have been addressed in [9, 24]. In [9], the author shows that Kiefer-Wolfowitz (KW) stochastic approximation algorithm achieves a regret of the order of $O\left(\sqrt{T}\right)$ under some strong regularity assumptions on the reward function (strong convexity). LSE, the algorithm proposed in [24], has a regret that scales as $O\left(\sqrt{T \log(T)}\right)$, but requires the knowledge of the smoothness of the reward function. LSE is a bandit version of the Golden section search algorithm, and iteratively eliminates subsets of arms based on PAC-bounds derived after appropriate sampling. By design, under LSE, the sequence of parameters used for the PAC bounds is pre-defined, and in particular does not depend of the observed rewards. As a consequence, LSE may explore too much sub-optimal parts of the set of arms. Our algorithm exploits more adaptive sequential statistical tests to remove subsets of arms, and yields a lower regret even without the need to know the smoothness of the reward function. A naive way to address continuum bandit problems consists in discretizing the set of arms, and in applying efficient discrete bandit algorithms. This method was introduced in [18], and revisited in [7] in the case of unimodal rewards. To get a regret scaling as $O\left(\sqrt{T \log(T)}\right)$ using this method, the reward function needs to be smooth and the discretization should depend on the smoothness of the function near its maximizer.

Our problem is related to stochastic derivative-free optimization problems where the objective is to get close to the maximizer of the reward function as quickly as possible, see e.g. [23, 14], and references therein. However, as explained in [11], minimizing regret constitutes a different and more subtle objective. Finally, it is worth mentioning papers investigating the design of sampling strategies to identify the best arm in multi-armed bandit problems, see e.g. [21, 11, 3, 15, 13]. These strategies apply to discrete finite sets of arms, but resemble our sequential statistical tests to reduce the interval containing the best arm. We believe that our analysis (e.g. we derive finite-time lower bounds for the expected sampling complexity of a set of tests), and our proof techniques are novel.
2 Problem Formulation and Notation

We consider continuous bandit problems where the set of arms is the interval $[0, 1]$, and where the expected reward $\mu(x)$ is a continuous and unimodal function of the arm. More precisely, there exists $x^*$ such that $x \mapsto \mu(x)$ is strictly increasing (resp. decreasing) in $[0, x^*]$ (resp. in $[x^*, 1]$). We denote by $\mathcal{U}$ the set of such functions. Define $\mu^* = \mu(x^*)$.

Time proceeds in rounds indexed by $n = 1, 2, \ldots$. When arm $x$ is selected in round $n$, the observed reward $X_n(x)$ is a random variable whose mean is $\mu(x)$ and distribution is $\nu(\mu(x))$, where $\nu$ refers to an exponential family of distributions with one parameter (for example, Bernoulli, exponential, normal, ...). We assume that the rewards $(X_n(x), n \geq 1)$ are i.i.d., and are independent across arms. At each round, a decision rule or algorithm selects an arm depending on the arms chosen in earlier rounds, and the corresponding observed rewards. Let $x^n(n)$ denote the arm selected in round $n$ under the algorithm $\pi$. The set $\Pi$ of all possible algorithms consists of sequential decision rules $\pi$ such that for any $n \geq 2$, $x^n(n)$ is $\mathcal{F}_{n-1}$-measurable where $\mathcal{F}_n$ is the $\sigma$-algebra generated by $(x^s(s), X_s(x^s(s)), s = 1, \ldots, n)$.

The performance of an algorithm $\pi \in \Pi$ is characterized by its regret up to time horizon $T$ defined by: $R^\pi(T) = T\mu^* - \sum_{n=1}^T \mu(x^n(n))$. Our objective is to devise an algorithm minimizing regret. Importantly, the only information available to the decision maker about the reward function $\mu$ is that $\mu \in \mathcal{U}$. In particular, the smoothness of $\mu$ around $x^*$ remains unknown – actually $\mu$ could well not be differentiable, e.g. $\mu(x) = \mu^* - |x - x^*|^\xi$ for $\xi \in (0, 1)$.

**Notation.** In what follows, for any $\alpha, \beta$, we denote by $\text{KL}(\alpha, \beta)$ the Kullback-Leibler divergence between distributions $\nu(\alpha)$ and $\nu(\beta)$. When $\alpha, \beta \in [0, \beta]$, and when $\nu(\cdot)$ is the family of Bernoulli distributions, this KL divergence is denoted by $\text{KL}_2(\alpha, \beta) = \text{KL}(\alpha, \beta) = \alpha \log(\frac{\alpha}{\beta}) + (1 - \alpha) \log(\frac{1 - \alpha}{1 - \beta})$.

3 Stochastic Polychotomy Algorithms

We present here a family of sequential arm selection rules, referred to as Stochastic Polychotomy (SP). These algorithms consist in successively narrowing an interval in unimodal functions – as it samples only two arms in the interior of the interval that needs to be trimmed. We discuss the design of the narrowing subroutine in the next section. In particular, we show that to around its maximizer $x^*$ for some $\xi > 0$, the regret scales as $O(\sqrt{T \log(T)})$ as the time horizon $T$ grows large. Hence this algorithm is order-optimal and does not require any assumption or knowledge about the smoothness of the reward function $\mu$. 


The next theorem provides a lower bound on the expected number of times each arm terminates.

**Theorem 4.1** Let \( \chi \) be a sequential test with minimax risk \( \alpha \). Let \( \mu \in \mathcal{U} \), and \( s \in \{1, 2\} \). Denote by \( \beta = \mathbb{P}_\mu[S^\chi = s] \), the probability that the final outcome \( S^\chi \) is \( s \) and consider \( \alpha \leq \beta \). Then:

\[
\inf_{\lambda \in B_s} \sum_{k=1}^{K} \mathbb{E}_\mu[t_k^\chi(T) | KL(\mu(x_k), \lambda(x_k)) \geq KL_2(\beta, \alpha)].
\]

From the above result, we deduce Corollary 4.2 stating that any sequential test with time horizon \( T \) and with minimax risk \( T^{-\gamma} \), for \( \gamma \in (0, 1] \), has a length that scales at least as \( \gamma \log(T) \) as \( T \) grows large. Later on, we propose a sequential test whose length matches this lower bound.
Corollary 4.2 Let $\gamma \in (0, 1]$, $s \in \{1, 2\}$, and $\mu \in \mathcal{U}$. Consider a sequence (indexed by $T$) of sequential tests $\chi_T$ with time horizon $T$ and minimax risk $\alpha^T \Delta_T = T^{-\gamma}$, such that $\lim_{T \to \infty} \mathbb{P}_\mu[\mathcal{S}_X = s] = \beta > 0$. Then: $\lim \inf_{T \to \infty} \inf_{\lambda \in \mathcal{B}_2} \sum_{k=1}^{K} \frac{\mathbb{E}[(\chi_T(T))] \log(T)}{\mu_k(\lambda(x_k))} \geq \gamma \beta$.

Another consequence of Theorem 4.1 is presented in Corollary 4.3. The latter states that it is impossible to construct a sequential test that samples at most two arms in the interior of $I$, that terminates before the time horizon $T$ with probability larger than $1/2$ and that has a minimax risk strictly less than $1/4$. Note that if a test terminates before $T$ with probability less than $1/2$, its expected length is at least $T/2$. Such a test would be useless in bandit problems since running it would incur a regret linearly growing with $T$.

Corollary 4.3 Consider the family of sequential tests running on the interval $I = [\underline{x}, \overline{x}]$, and arms $\underline{x} = x_1 < x_2 < x_3 < x_4 = \overline{x}$. There exists $\mu \in \mathcal{U}$, such that for any sequential test $\chi$ of this family with arbitrary finite time horizon $T$ and minimax risk $\alpha < 1/4$, we have $\mathbb{P}_\mu[\mathcal{S}_X \not= \emptyset] \leq 1/2$ (i.e., the test does not terminate before $T$ with probability $1/2$).

Recall that Kiefer’s Golden section search algorithm uses two points in the interior of the interval to reduce. Hence, the above corollary implies that it is impossible to construct a bandit version of this algorithm that performs well without additional assumptions on the smoothness and structure of the reward function. Actually, LSE, proposed in [24], is a bandit version of the Golden section search algorithm, but to analyze its regret, additional assumptions on the structure of the reward function are made (its minimal slope and smoothness).

Corollary 4.3 is a direct consequence of Theorem 4.1: the choice of the reward function $\mu$ used in Corollary 4.3 is illustrated in Figure 1 (left), and the result is obtained by considering a sequence (indexed by $\epsilon > 0$) of unimodal functions $\lambda_\epsilon \in B_1$. An efficient test must distinguish between $\mu$ and $\lambda_\epsilon$ based on the reward samples at $x_1, x_2, x_3, x_4$. By letting $\epsilon \to 0$, we see that under such a test, the number of samples from $x_3$ must be arbitrary large.

### 4.2 IT$_3$: An Asymptotically Optimal Sequential Test

We propose IT$_K$, a set, indexed by $K \geq 3$, of sequential tests with time horizon $T$ and minimax risk $O(T^{-\gamma} \log(T)^{3K})$. Those tests are asymptotically optimal, in the sense that the number of times arms are sampled before the test terminates matches the lower bound derived in Theorem 4.1. $\chi$ samples $K$ arms in the interior of $I = [\underline{x}, \overline{x}]$, i.e., $\underline{x} = x_1 < x_2 < \ldots < x_K < \overline{x}$. To simplify the presentation, we assume that for $k = 1, \ldots, K$, $x_k = \underline{x} + k(\overline{x} - \underline{x})/(K + 1)$. This assumption is not crucial, and our analysis remains valid for any choice of arms provided that they lie in the interior of $I$. Interestingly, IT$_3$ samples only 3 arms to take a decision. Recall that in contrast, in the Golden section search algorithm (an algorithm that does not perform well in view of Corollary 4.3), four arms are sampled.

To describe the sequential test $\chi = \Gamma_K$, we introduce for any $s \in \{1, 2\}$, the function $i_s : \mathbb{R}_+^K \to \mathbb{R}$ with

$$i_s(\mu_1, \ldots, \mu_K) = \inf_{\lambda \in \mathcal{B}_2} \sum_{k=1}^{K} \text{KL}(\mu_k, \lambda(x_k)).$$

We also define the empirical average reward of arm $x_k$ up to round $n \leq \bar{L}_\chi$ as:

$$\hat{\mu}_k(n) = \frac{1}{t^\chi(n)} \sum_{n'=1}^{n} X_{x_k}(n') 1\{x^\chi(n') = x_k\},$$

if $t^\chi(n) > 0$ and $\hat{\mu}_k(n) = 0$ otherwise. Let $\bar{\mu}(n) = (\hat{\mu}_1(n), \ldots, \hat{\mu}_K(n))$ and $\bar{L}(n) = \min_{1 \leq k \leq K} t^\chi_k(n)$. The sequential test $\chi$ is defined as follows: for any $n \leq T$:  

(i) If there exists $s \in \{1, 2\}$ such that $\ell(x(n))_{s}(\hat{\mu}(n)) \geq \gamma \log(T)$, then $S^{\chi}(n) = s$, i.e., $\chi$ terminates and its final output is $S^{\chi} = s$ (ties are broken arbitrarily if both conditions $\ell(x(n))_{s}(\hat{\mu}(n)) \geq \gamma \log(T)$ for $s = 1, 2$ hold).

(ii) Otherwise $S^{\chi}(n) = 0$, and $\chi$ samples arm $x^{\chi}(n) = x_{1 + (n \mod K)}$.

The sequential test $\chi$ outputs the interval $I_{S^{\chi}}$ where $I_{1} = [x_{1}, x_{K}]$ and $I_{2} = [x_{1}, x]$, and its length $L^{\chi}$.

In the next theorem, we show that asymptotically, $\chi = IT_{K}$ has a minimax risk $O(T^{-\gamma} \log(T)^{3K})$, and that for any $s \in \{1, 2\}$, if $\mu \notin B_{s}$, the expected number of times arm $k$ is sampled is less than $\gamma \log(T)/i_{s}(\mu(x_{1}), \ldots, \mu(x_{K}))$. This establishes the asymptotical optimality of IT$_{K}$.

**Theorem 4.4** Let $K \geq 3$.

(i) There exists a function of $K$, $C_{K} \geq 0$ such that the sequential test IT$_{K}$ has minimax risk less than $C_{K}T^{-\gamma} \log(T)^{3K}$ for all $T \geq 1$.

(ii) Under the sequential test $\chi = IT_{K}$, for all $\mu \in \mathcal{U}$, if $\mu \notin B_{s}$, then we have for all $k = 1, \ldots, K$:

$$
\lim_{T \to \infty} \sup_{T} \frac{\mathbb{E}_{\mu}[L_{s}^{\chi}(T)\mathbb{I}_{i_{s}}]}{\log(T)} \leq \frac{\gamma}{i_{s}(\mu(x_{1}), \ldots, \mu(x_{K}))}.
$$

4.3 IT$'$_{3}$: A Computationally Efficient Sequential Test

Next we present IT$'$_{3}, a sequential test which is computationally simpler than IT$_{3}$. IT$'$_{3} is not asymptotically optimal, but its implementation is much simpler than that of IT$_{3}$. Its rationale involves calculating a fully explicit lower bound for function $i_{s}$.

For $\epsilon \geq 0$, we define the function KL$^{*, \epsilon} : \mathbb{R}^{2} \to \mathbb{R}$ as:

$$
\text{KL}^{*, \epsilon}(\mu_{1}, \mu_{2}) = \mathbb{1}_{\{\mu_{1} < \mu_{2}\}} \left[ \text{KL} \left( \mu_{1} + \epsilon, \frac{\mu_{1} + \mu_{2}}{2} - \epsilon \right) + \text{KL} \left( \mu_{2} - \epsilon, \frac{\mu_{1} + \mu_{2}}{2} + \epsilon \right) \right].
$$

and KL$^{*}(\mu_{1}, \mu_{2}) = \text{KL}^{*, 0}(\mu_{1}, \mu_{2})$.

The sequential test $\chi' = IT_{3}'$ is defined by, for any $n \leq T$,

(i) If $\ell(x(n))\text{KL}^{*}(\hat{\mu}_{1}(n), \hat{\mu}_{2}(n)) \geq \gamma \log(T)$, then $S^{\chi'}(n) = 1$, i.e., $\chi'$ terminates and its final output is $S^{\chi'} = 1$. Similarly if $\ell(x(n))\text{KL}^{*}(\hat{\mu}_{3}(n), \hat{\mu}_{2}(n)) \geq \gamma \log(T)$, then $S^{\chi'}(n) = 2$.

(ii) Otherwise $S^{\chi'}(n) = 0$, and $\chi'$ samples arm $x^{\chi'}(n) = x_{1 + (n \mod 3)}$. 

Figure 1: Illustrations of Corollary 4.3 (left) and Theorem 4.3 (right).
4.4 Finite time analysis and risk of IT₃ and IT₃'

The next theorem provides an explicit lower bound of the expected length of IT₃ and IT₃'. To this aim, we derive a lower bound of \( i_s(\mu_1, \mu_2, \mu_3). \) Theorem 4.5 will be instrumental in the regret analysis of the Stochastic Pentachotomy algorithm. We restrict the analysis to Bernoulli rewards. This is mainly for simplicity, and the proof techniques can be extended to sub-Gaussian rewards with straightforward modifications.

**Theorem 4.5** Let \( \chi \in \{IT₃, IT₃'\}, \)
(i) There exists a constant \( C_{\chi} \geq 0 \) such that the sequential test \( \chi \) has minimax risk less than \( C_{\chi} T^{-\gamma} \log(T)^9 \) for all \( T \geq 1. \)
(ii) Define \( m = 1 \) if \( x^* \in [x_2, \bar{x}] \) and \( m = 3 \) otherwise. Define \( \delta = (\mu(x_2) - \mu(x_m))/2. \) Then, we have that for all \( 0 < \epsilon < \delta/2, \) for all \( k = 1, 2, 3 \) and all \( T \geq 1: \)

\[
E_\mu[t_{\chi}(T)] \leq \frac{\gamma \log(T)}{KL^*(\mu(x_m), \mu(x_2))} + 2\epsilon^{-2}.
\]

As a consequence we have the following inequalities:

\[
E_\mu[t_{\chi}(T)] \leq \frac{\gamma \log(T) + 32}{\delta^2} \quad \text{and} \quad \limsup_{T \to \infty} \frac{E_\mu[t_{\chi}(T)]}{\log(T)} \leq \frac{\gamma}{KL^*(\mu(x_m), \mu(x_2))}.
\]

**Theorem 4.5** is illustrated by Figure 1 (right). Let \( \mu \) such that \( x^* \in [x_2, \bar{x}], \) then any unimodal function which attains its maximum in \([\bar{x}, x_1]\) cannot be at a distance lesser than \( \delta \) from \( \mu \) at points \( x_1 \) and \( x_2 \) simultaneously.

5 Regret analysis

In this section, we analyze the regret of Stochastic Pentachotomy algorithms. We refer to as SP* the algorithm using the narrowing subroutines IT₃' (instead of IT₃ for SP). We first derive an upper bound valid for all \( \mu \in \mathcal{U} \) and all time horizon \( T. \) We then specify the bound when \( \mu \) behaves as \( \mu(x) = \mu(x^*) - C|x - x^*|^\xi \) locally around its maximizer \( x^* \) for some \( \xi, C > 0. \) To simplify the presentation, our bounds are stated and proved for Bernoulli rewards, but the analysis can be extended to other exponential families of distributions.

Let \( \mu \in \mathcal{U}. \) Define for any \( \Delta > 0: \)

\[
g_\mu(\Delta) = \mu^* - \max(\mu(x^* - \Delta), \mu(x^* + \Delta)) \]

\[
h_\mu(\Delta) = \min \left\{ \min_{x \in [x^* - \Delta/4, x^* + \Delta/4]} (\mu(x) - \mu(x + \Delta/4)), \min_{x \in [x^* - \Delta, x^* + \Delta]} (\mu(x) - \mu(x - \Delta/4)) \right\}
\]

**Theorem 5.1** Let \( \psi = 3/4. \) Under Algorithm \( \pi = SP \) or \( \pi = SP*, \) the expected regret satisfies: for all \( T \geq 1, \) and all \( N \geq 1, \)

\[
R^\pi(T) \leq CNT^{1-\gamma} \log(T)^9 + Tg_\mu(\psi^N) + 3(\gamma \log(T) + 32) \sum_{N=0}^{N-1} g_\mu(\psi^N)h_\mu(\psi^N)^{-2}.
\]

for some constant \( C > 0. \)

Next we restrict our attention to the expected reward functions that satisfy Assumption 1.
therefore, \( R_{\pi}(T) \) scales as \( O(\sqrt{T} \log(T)) \) without the knowledge of the behavior of the reward function around its maximizer.

6 Numerical Experiments

In this section, we briefly explore the performance of SP' (using parameter \( \gamma = 0.6 \)), and compare it to that of two other algorithms, namely KL-UCB(\( \delta \)) and KW. KL-UCB(\( \delta \)) consists in applying the KL-UCB algorithm \([12]\) to the discrete set of arms \( \{0, \delta, 2\delta, \ldots, 1\} \). KW is the algorithm proposed in \([9]\). The performance of LSE \([24]\) is not reported here, since it is generally outperformed by KL-UCB(\( \delta \)), as shown in \([7]\).

We consider two reward functions satisfying Assumption \([1]\) with \( \xi = 1/2 \) and \( \xi = 2 \), respectively. More precisely, \( \mu(x) = 1 - (2(1/2 - x))^\xi \) for \( x \in [0, 1] \). The first function is not differentiable at its maximizer, whereas the second function is just quadratic. Note that KW should then perform well for the quadratic rewards (there the regret scales as \( O(\sqrt{T}) \) \([9]\)), but there is not guarantee that it would do well for the non-differentiable reward functions. For KL-UCB(\( \delta \)), the optimal discretization step \( \delta \) depends on the smoothness of the reward function, and is set to \( \log(T) / \sqrt{T} \) \(^{1/\xi}\).

In Figure \([2]\) we present the regret of the various algorithms (averaged over 10 independent runs). Observe that without the knowledge of the smoothness of the function, SP' is able to significantly outperform the two other algorithms. As expected, KW does not perform well when \( \xi = 1/2 \), but outperforms KL-UCB(\( \delta \)) for \( \xi = 2 \). Additional numerical experiments are presented in Appendix.
7 Conclusion

In this paper, we have presented the first order-optimal algorithms for one-dimensional continuous unimodal bandit problems that do not explicitly take into account the structure or the smoothness of the expected reward function. In some sense, the proposed algorithm learns and adapts its sequential decisions to the smoothness of the function. Future work will be devoted to applying the techniques used to devise our algorithms to other structured bandits with continuum set of arms (i.e., Lipschitz or convex bandits). We also would like to extend our analysis to the case where the set of arms lies in a space of higher dimension.
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A Additional numerical experiments

Figure 3 presents a graphical illustration of a typical run of SP’. We consider a run of algorithm SP’ with reward function \( \mu(x) = 1 - (2|1/2 - x|)^\xi, \) \( \xi = 0.5 \) (left), and \( \xi = 2 \) (right), time horizon \( T = 10^6 \) and \( \gamma = 0.6 \). We represent the shape of \( \mu \) and the successive intervals returned by IT’3, starting at the bottom of the y-axis. The thickness of the segments is an increasing function of the length of IT’3. In both cases, we observe that the successive intervals contain the optimal arm \( x^\star \). When the search interval gets narrower (we are closer to the peak), the intervals get thicker since the duration of the test increases when the separation between arms \( \{x_1, x_2, x_3\} \) decreases. Also remark that when the expected reward function is flatter (here \( \xi = 2 \)), the algorithm tends to spend more time on each given interval. The evolution of the regret over time for these two reward functions is presented in the core of the paper.

Finally, Figure 4 compares the regret of the various algorithm for a triangular reward function, and illustrates a typical run of the SP’ algorithm.

Figure 3: Illustration of a run of SP’ with reward function \( \mu(x) = 1 - (2|1/2 - x|)^\xi, \) \( \xi = 0.5 \) (left), and \( \xi = 2 \) (right) and time horizon \( T = 10^6 \).

Figure 4: Reward function: \( \mu(x) = 1 - (2|1/2 - x|)^\xi \). (Left) Regret vs time of various algorithms. (Right) Illustration of a run of SP’ with time horizon \( T = 10^6 \).
B Proofs

B.1 Proof of Theorem 4.1

We work with a given sequential test $\chi$ throughout the proof and we omit the superscript $\chi$ for clarity. Without loss of generality, let $s = 1$. We work with a fixed parameter $\lambda \in B_1$. We denote by $Y(T) = (X_1(x(1)), \ldots, X_T(x(T)))$ the observed rewards from round 1 to round $T$. We denote by $P_T$ and $Q_T$ the probability distribution of $Y(T)$ under $\mu$ and $\lambda$ respectively. From Lemma B.3 (stated and proved at the end of the appendix), we have:

$$\text{KL} (P_T || Q_T) = \sum_{k=1}^{K} \mathbb{E}[t_k(T)] \text{KL} (\mu(x_k), \lambda(x_k)).$$

Consider the event $S(T) = 1$. Since the sequential test $\chi$ has minimax risk smaller than $\alpha$, and $\lambda \in B_1$, we have $P_\lambda[S(T) = 1] \leq \alpha$. Recall that by assumption $P_\mu[S(T) = 1] = \beta$ and $\alpha \leq \beta$. Now $S(T)$ is a function of $Y(T)$. Using Lemma B.2 (stated at the end of the appendix):

$$\text{KL} (P_T || Q_T) \geq \text{KL}_2(P_\mu[S(T) = 1], P_\lambda[S(T) = 1]) \geq \text{KL}_2(\beta, \alpha).$$

where we have used the fact that $\alpha \mapsto \text{KL}_2(\beta, \alpha)$ is decreasing for $\alpha \leq \beta$. Putting (1) and (2) together, we obtain:

$$\sum_{k=1}^{K} \mathbb{E}[t_k(T)] \text{KL} (\mu(x_k), \lambda(x_k)) \geq \text{KL}_2(\beta, \alpha).$$

Taking the infimum over $\lambda \in B_1$, we obtain the claimed result:

$$\inf_{\lambda \in B_1} \sum_{k=1}^{K} \mathbb{E}[t_k(T)] \text{KL} (\mu(x_k), \lambda(x_k)) \geq \text{KL}_2(\beta, \alpha).$$

B.2 Proof of Corollary 4.2

Let us denote $\beta_T = P_\mu[S(T) = 1]$. Since $\beta_T \to_{T \to \infty} \beta > 0$ there exists $T_0$ such that for all $T \geq T_0$ we have $\beta_T \geq T^{-\gamma}$. Since $\chi_T$ has minimax risk $\alpha = T^{-\gamma}$, for all $T \geq T_0$, applying Theorem 4.1, we obtain:

$$\inf_{\lambda \in B_1} \sum_{k=1}^{K} \mathbb{E}[t_k(T)] \text{KL} (\mu(x_k), \lambda(x_k)) = \text{KL}_2(\beta_T, \alpha_T) = \text{KL}_2(\beta_T, T^{-\gamma}).$$

Now by definition of $\text{KL}_2$, we have that:

$$\text{KL}_2(\beta_T, T^{-\gamma}) = \beta_T \log(\beta_T) + \beta_T^{-\gamma} \log(T) + (1 - \beta_T) \log(1 - \beta_T) + (1 - \beta_T) \log(1 - T^{-\gamma}).$$

Since $\beta_T \to_{T \to \infty} \beta > 0$, we have that $\text{KL}_2(\beta_T, T^{-\gamma}) \sim_{T \to \infty} \gamma \beta \log(T)$. Letting $T \to \infty$ in (3) we have:

$$\lim_{T \to \infty} \inf_{\lambda \in B_1} \sum_{k=1}^{K} \mathbb{E}[t_k(T)] \text{KL} (\mu(x_k), \lambda(x_k)) \geq \gamma \beta,$$

which concludes the proof.
We upper-bound the risk of IT for any \( \mu \). Without loss of generality we consider interval \( I = [0, 1] \). Consider the function \( \mu(x) = 1 - 2|x|/2 \). \( \mu \) is clearly unimodal, with \( x^\ast = 1/2 \) and \( \mu^\ast = 1 \).

We proceed by contradiction. Consider a test \( \chi \) such that \( \mathbb{P}_\mu[S^\chi(T) \neq 0] \geq 1/2 \). Without loss of generality we consider \( s = 1 \). Let \( \epsilon > 0 \), and define the function \( \lambda^\epsilon \) which is linear on intervals \( [(x_1, x_2), [x_2, x_3], x_2, (x_3 + x_4)/2], [(x_3 + x_4)/2, x_4] \) with \( \lambda(x_k) = \mu(x_k), k \neq 3 \) and \( \lambda(x_3) = \mu(x_2) + \epsilon \), and \( \lambda((x_3 + x_4)/2) = 1 \). One can check that \( \lambda^\epsilon \) is unimodal, and attains its maximum in \([x_3, x_4]\). We recall that \( \alpha < 1/4 \) and applying Theorem 4.1, we obtain the following inequality:

\[
\sum_{k=1}^K \mathbb{E}_\mu[t_k(T)]KL(\mu(x_k), \lambda(x_k)) \geq KL(1/4, \alpha).
\]

Since \( KL(\mu(x_k), \lambda(x_k)) = KL(\mu(x_k), \mu(x_k)) = 0 \), for \( k \neq 3 \), and \( t_3(T) \leq T \) we obtain:

\[
T KL(\mu(x_3), \mu(x_3) + \epsilon) \geq KL(1/4, \alpha).
\]

Therefore inequality \((4)\) cannot hold for all \( \epsilon > 0 \). This is a contradiction and proves that a test \( \chi \) as considered here cannot exist, which concludes the proof.

### B.4 Proof of Theorem 4.4

**Proof of (i) (Minimax risk).** We upper-bound the risk of IT for any \( \mu \). Recall the definition of the risk:

\[
\alpha^\lambda(\mu) = \sum_{s=1}^2 1\{\mu \in B_s\} \mathbb{P}_\mu[S^\lambda(T) = s].
\]

Let \( \mu \in \mathcal{U} \). If \( \mu \notin B_1 \cup B_2 \), then \( \alpha^\lambda(\mu) = 0 \) so that the risk is indeed \( O(T^{-\gamma} \log(T)^3K) \). Now we assume that \( \mu \in B_s \) and we derive an upper bound of \( \mathbb{P}_\mu[S^\lambda(T) = s] \). By definition of \( IT_K \), the event \( S^\lambda(T) = s \) implies that there exists \( n \leq T \) such that \( \ell(n)i_s(\hat{\mu}(n)) \geq \gamma \log(T) \). Using the two facts: (a) \( \mu \in B_s \) and (b) \( t_k(n) \geq \ell(n) \), we have:

\[
\gamma \log(T) \leq \ell(n)i_s(\hat{\mu}(n)) = \ell(n) \inf_{\lambda \in B_s} \sum_{k=1}^K KL(\hat{\mu}_k(n), \lambda(x_k)) \leq (a) \ell(n) \sum_{k=1}^K KL(\hat{\mu}_k(n), \mu(x_k)) \leq (b) \ell(n) \sum_{k=1}^K t_k(n) KL(\hat{\mu}_k(n), \mu(x_k)).
\]

Therefore we have proven that:

\[
\alpha^\lambda(\mu) \leq \mathbb{P} \left[ \sum_{k=1}^K t_k(n) KL(\hat{\mu}_k(n), \mu(x_k)) \geq \gamma \log(T) \right]
\]

And using Theorem \ref{thm:B.3} (presented at the end of the appendix) with \( \delta = \gamma \log(T) \), we obtain that:

\[
\alpha^\lambda(\mu) \leq e^{K+1} K^{-K} T^{-\gamma} (\log(T))^{3K}.
\]
This proves that the minimax risk $\alpha^\chi$ is $O(T^{-\gamma} \log(T)^{3K})$ and concludes the proof of (i).

Proof of (ii) (Expected duration of the test).

We now consider $1 \leq k \leq K$ and we derive an upper bound of $E_{\hat{t}_k}(T)$. Fix $\epsilon > 0$, and define $t_0 = (1 + \epsilon) \gamma \log(T)/i_s(\mu(x_1), \ldots, \mu(x_K))$. Introduce the two sets of instants:

$$
A = \{1 \leq n \leq T : x(n) = x_k, \ell(n) \leq t_0\},
$$

$$
B = \{1 \leq n \leq T : x(n) = x_k, \ell(n) \geq t_0\}.
$$

Hence we have $t_k(T) \leq |A| + |B|$. Furthermore, at each instant $n \in A$, $t_k(n)$ is incremented, therefore $|A| \leq t_0$. Let us bound the expected size of $B$. Let $n \in B$. By design of IT$_K$, this implies that: $t_0 \leq \ell(n)$ and $\ell(n)i_s(\mu(n)) \leq \gamma \log(T)$. Therefore:

$$
t_0i_s(\mu(n)) \leq \gamma \log(T),
$$

and thus:

$$
i_s(\mu(n)) \leq i_s(\mu(x_1), \ldots, \mu(x_K))/(1 + \epsilon).
$$

Now one can easily verify that the function $(\lambda_1, \ldots, \lambda_K) \mapsto \sum_{k=1}^K KL(\mu(x_k), \lambda_k)$ attains its infimum on $B_s$. By continuity of KL in its second argument, there must exist $\lambda^* \in B_s$ such that:

$$
i_s(\mu(x_1), \ldots, \mu(x_K)) = \sum_{k=1}^K KL(\mu(x_k), \lambda^*(x_k)).
$$

Let $\eta > 0$ such that we have $|\hat{\mu}_k(n) - \mu(x_k)| \leq \eta$ for all $k$. Since $\lambda^* \in B_s$, this implies that:

$$
i_s(\hat{\mu}(n)) = \inf_{\lambda \in B_s} \sum_{k=1}^K KL(\hat{\mu}_k(n), \lambda(x_k)) \leq \sum_{k=1}^K KL(\hat{\mu}_k(n), \lambda^*(x_k)).
$$

Since $|\hat{\mu}_k(n) - \mu(x_k)| \leq \eta$ for all $k$, the r.h.s. of (6) tends to $i_s(\mu(x_1), \ldots, \mu(x_K)) < i_s(\mu(x_1), \ldots, \mu(x_K))/(1 + \epsilon)$ as $\eta \to 0$. Hence the inequality (5) cannot hold for arbitrary small $\eta$.

Hence, there exists $\eta_0$ such that $n \in B$ implies $\max_k |\hat{\mu}_k(n) - \mu(x_k)| \geq \eta_0$. Note that $\eta_0$ might depend on $\epsilon$ and $\mu(x_1), \ldots, \mu(x_K)$. Using Lemma B.5 we get $E[|B|] = o(\log(T))$.

Therefore we have:

$$
E[t_k(T)] \leq \frac{(1 + \epsilon) \gamma \log(T)}{i_s(\mu(x_1), \ldots, \mu(x_K))} + o(\log(T)).
$$

and:

$$
\limsup_{T \to \infty} \frac{E[t_k(T)]}{\log(T)} \leq \frac{(1 + \epsilon) \gamma}{i_s(\mu(x_1), \ldots, \mu(x_K))}.
$$

Since the above inequality holds for all $\epsilon > 0$, we obtain the announced result:

$$
\limsup_{T \to \infty} \frac{E[t_k(T)]}{\log(T)} \leq \frac{\gamma}{i_s(\mu(x_1), \ldots, \mu(x_K))},
$$

which concludes the proof of (ii).

B.5 Proof of Theorem 4.5

We start by proving lemma B.1 which shows that $i_s$ can be bounded using the KL$^*$ function.
Lemma B.1 Consider Bernoulli rewards. Define \( m = 1 \) if \( s = 1 \) and \( m = 3 \) otherwise. Then we have for all \( s \in \{1, 2\} \):

\[
i_s(\mu(x_1), \mu(x_2), \mu(x_3)) \geq \text{KL}^*(\mu(x_0), \mu(x_2)).
\]

Proof. We prove only the statement for \( s = 1 \), as the case \( s = 2 \) follows by symmetry. By a slight abuse of notation we denote \( \mu(x_k) \) and \( \lambda(x_k) \) by \( \mu_k \) and \( \lambda_k \) respectively.

First note that if \( \mu_2 < \mu_1 \), we have \( \text{KL}^*(\mu_1, \mu_2) = 0 \) and the statement holds because \( i_s(\mu(x_1), \mu(x_2), \mu(x_3)) \geq 0 \), since the KL divergence is positive.

Now consider the case \( \mu_2 \geq \mu_1 \). We have the inequality:

\[
i_1(\mu_1, \mu_2, \mu_3) = \inf_{\lambda \in B} \sum_{k=1}^3 \text{KL}(\mu_k, \lambda_k) \geq \inf_{\lambda \in B} \sum_{k=1}^2 \text{KL}(\mu_k, \lambda_k).
\]

Define function \( f : [0, 1]^2 \to \mathbb{R} \) by \( f(\lambda_1, \lambda_2) = \sum_{k=1}^2 \text{KL}(\mu_k, \lambda_k) \). Define the set \( \Lambda = \{ (\lambda_1, \lambda_2) : \lambda_1 \geq \lambda_2 \} \). Consider \( \lambda \in B_1 \), then \( x \mapsto \lambda(x) \) attains its maximum in \([x_0, x_1]\), and since \( \lambda \) is unimodal we must have \( \lambda_1 \geq \lambda_2 \). Therefore:

\[
i_1(\mu_1, \mu_2, \mu_3) \geq \min_{(\lambda_1, \lambda_2) \in \Lambda} f(\lambda_1, \lambda_2).
\]

Consider \((\lambda_1^*, \lambda_2^*) \in \arg \min_{(\lambda_1, \lambda_2) \in \Lambda} f(\lambda_1, \lambda_2) \). We are going to prove that we must have \( \lambda_1^* = \lambda_2^* \). Consider two subcases (a) \( 0 \leq \lambda_2^* \leq \mu_2 \) and (b) \( \mu_1 \leq \lambda_2^* \leq 1 \). In case (a) we must have \( \lambda_1^* = \mu_1 \) since \( \lambda \mapsto \text{KL}(\mu_k, \lambda_1) \) attains its minimum at \( \mu_1 \). In turn we must have \( \lambda_2^* = \mu_1 = \lambda_1^* \) since \( \lambda \mapsto \text{KL}(\mu_1, \lambda_1) \) is decreasing for \( \lambda_1 \leq \mu_1 \leq \mu_2 \). In case (b), we must have \( \lambda_1^* = \lambda_2^* \) because \( \lambda \mapsto \text{KL}(\mu_1, \lambda_1) \) is increasing for \( \lambda_1 \geq \lambda_2^* \geq \mu_1 \). In both cases we have proven that \( \lambda_1^* = \lambda_2^* \).

Define function \( \tilde{f}(\lambda) = f(\lambda, \lambda) \), from the reasoning above we have that:

\[
\min_{(\lambda_1, \lambda_2) \in \Lambda} f(\lambda_1, \lambda_2) = \min_{\lambda \in [0, 1]} \tilde{f}(\lambda).
\]

- If \( \mu_1 = \mu_2 = 0 \), then \( \tilde{f}(0, 0) = 0 \) so that the optimum is \( \lambda^* = 0 \).
- If \( \mu_1 = \mu_2 = 1 \), then \( \tilde{f}(1, 1) = 0 \), so that the optimum is \( \lambda^* = 1 \).
- Otherwise, denote by \( \tilde{f}' \) the first derivative of \( \tilde{f} \). We have:

\[
\tilde{f}'(\lambda) = \frac{2 - (\mu_1 + \mu_2)}{1 - \lambda} - \frac{\mu_1 + \mu_2}{\lambda}.
\]

and \( \tilde{f}'(0^+) = -\infty \) and \( \tilde{f}'(1^-) = +\infty \) so that \( \tilde{f} \) attains its maximum in the interior of \([0, 1] \). Solving for \( \tilde{f}'(\lambda^*) = 0 \) we obtain the unique solution \( \lambda^* = (\mu_1 + \mu_2)/2 \).

We observe that in the three above cases, the optimum is \( \lambda^* = (\mu_1 + \mu_2)/2 \). We have proven the announced inequality:

\[
i_1(\mu(x_1), \ldots, \mu(x_k)) \geq \min_{(\lambda_1, \lambda_2) \in \Lambda} f(\lambda_1, \lambda_2) = \min_{\lambda \in [0, 1]} \tilde{f}(\lambda) = \tilde{f}((\mu_1 + \mu_2)/2) = \text{KL}^*(\mu_1, \mu_2).
\]

We are now ready to prove Theorem 4.5.

Proof of the theorem.

(i) Minimax risk of \( \text{IT}_3^* \)

The minimax risk of \( \text{IT}_3^* \) was already established to be \( O(T^{-\gamma} \log(T))^\theta \) by Theorem 4.4.

(ii) Minimax risk of \( \text{IT}_3 \)
Consider $\mu \in B_z$, we are going to prove that $S^{IT_s}(T) = s$ implies $S^{IT_s}(T) = s$ so that $\alpha^{IT_s} \leq \alpha^{IT_s}$ which is sufficient to prove claim (i).

Without loss of generality consider $s = 1$ and a time instant $n$ such that $S^{IT_s}(T) = 1$. By definition of IT, this implies that $\bar{l}(n)KL^*(\hat{\mu}_1(n), \hat{\mu}_2(n)) \geq \gamma \log(T)$. Using Lemma B.1

$$\bar{l}(n)\bar{l}(1)(\hat{\mu}_1(n), \ldots, \hat{\mu}_K(n)) \geq \bar{l}(n)KL^*(\hat{\mu}_1(n), \hat{\mu}_2(n)) \geq \gamma \log(T).$$

which proves that $S^{IT_s}(T) = 1$ and concludes the proof of (i).

(ii) Expected duration of IT:

By a slight abuse of notation we denote $\mu(x_k)$ by $\mu_k$. Without loss of generality, consider $\mu$ such that $x^* \in [x_2, x]$. Therefore we have that $\mu_2 > \mu_1$ since $\mu$ is unimodal. Fix $0 < \epsilon < \delta/2$, and define $t_0 = \gamma \log(T)/KL^*(\mu_1, \mu_2).

Introduce the two sets of instants:

$$A = \{1 \leq n \leq T : x(n) = x_k, \bar{l}(n) \leq t_0\} \quad B = \{n \geq 1 : x(n) = x_k, \max_{k' \in \{1, 2\}} |\hat{\mu}_{k'}(n) - \mu_{k'}| \geq \epsilon\}.$$

We are going to prove that $x(n) = x_k$ implies that $n \in A \cup B$. Consider $n$ such that $\bar{l}(n) \geq t_0$ and $|\hat{\mu}_{k'}(n) - \mu_{k'}| \leq \epsilon, k' \in \{1, 2\}$. Since $\epsilon < \delta/2 \leq (\mu_2 - \mu_1)/4$ we have:

$$\hat{\mu}_1(n) \leq \mu_1 + \epsilon \leq (\mu_1 + \mu_2)/2 - \epsilon \leq (\hat{\mu}_1(n) + \hat{\mu}_2(n))/2,$$

$$\hat{\mu}_2(n) \geq \mu_2 - \epsilon \geq (\mu_1 + \mu_2)/2 + \epsilon \geq (\hat{\mu}_1(n) + \hat{\mu}_2(n))/2,$$

so that $KL^*(\hat{\mu}_1(n), \hat{\mu}_2(n)) \geq KL^*(\mu_1, \mu_2)$. Applying Lemma B.1 we have:

$$\bar{l}(n)i(\hat{\mu}(n)) \geq \bar{l}(n)KL^*(\hat{\mu}_1(n), \hat{\mu}_2(n)) \geq t_0KL^*(\mu_1, \mu_2) = \gamma \log(T).$$

Therefore we cannot have $x(n) = x_k$.

We have proven that $t_k(T) \leq |A| + |B|$. Furthermore, at each instant $n \in A$, $\bar{l}(n)$ is incremented, therefore $|A| \leq t_0$. Let us upper bound the expected size of $B$. Decompose $B = B^1 \cup B^2$, with:

$$B^{k'} = \{n \geq 1 : x(n) = x_k, |\hat{\mu}_{k'}(n) - \mu_{k'}| \geq \epsilon\}.$$

Consider $n \in B^{k'}$ and define $a = \sum_{n' \leq n} 1[n' \in B^{k'}]$ so that $n$ is the $a$-th instant of $B^{k'}$. Then we have that $t_{k'}(n) \geq a$ and applying [8][Lemma 2.2] we have that for $k' \in \{1, 2\}$, $E||B^{k'}|| \leq \epsilon^2$. Therefore $E||B|| \leq 2\epsilon^2$. So the first statement of (ii) is proven:

$$\text{E}_{\mu}[t_k^*(T)] \leq t_0 + 2\epsilon^2 = \frac{\gamma \log(T)}{KL^*(\mu(x_1), \mu(x_2))} + 2\epsilon^2.$$

Using Pinsker’s inequality $KL^*(\alpha, \beta) \geq 2(\alpha - \beta)^2$, so that

$$KL^*(\mu_2, \mu_1) \geq 4((\mu_2 - \mu_1)/2 - 2\epsilon)^2 \geq 4(\delta - 2\epsilon)^2,$$

and setting $\epsilon = \delta/4$ we get $KL^*(\mu_2, \mu_1) \geq \delta^2$ and $2\epsilon^2 = 32\delta^{-2}$ so that we obtain the second claim:

$$\text{E}_{\mu}[t_k^*(T)] \leq \frac{\gamma \log(T) + 32}{\delta^2}.$$

The third statement of (ii) holds since for all $\epsilon$, we have:

$$\lim_{T \to \infty} \text{E}_{\mu}[t_k^*(T)] \leq \frac{\gamma}{KL^*(\mu(x_1), \mu(x_2))}.$$

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so that letting $\epsilon \to 0$ in the above expression yields:
\[
\lim_{T \to \infty} \sup_{\tau} \frac{\mathbb{E}_\mu[R^N(T)]}{\log(T)} \leq \frac{\gamma}{\text{KL}(\mu(x_1), \mu(x_2))},
\]
which concludes the proof of (ii).

(ii) Expected duration of IT$^3$:
The claim (ii)' can be proven using the same argument as that used to prove (ii).

\[\Box\]

B.6 Proof of Theorem 5.1

Fix $N$ throughout the proof. We introduce the following notations. The algorithm proceeds in rounds, each round corresponding to a call to IT$^3$ (or IT$^3_i$). We define $I^N$ the interval output by the $N'$-th call of IT$^3$, with $I^0 = [0, 1]$. We define $\tau^N$ the duration of the $N'$-th call of IT$^3$. Define the event:
\[A = \cap_{N'=0}^N \{x^* \in I^N\},\]
which corresponds to sample paths where the first $N$'-th calls of IT$^3$ have returned an interval containing the optimal arm $x^*$. We denote by $A^c$ the complement of $A$.

The regret due to sample paths in $A^c$ is upper bounded by $\mu^*TP[A^c]$. The regret due to the $N'$-th round of sample paths in $A$ is upper bounded by $\mathbb{E}[\tau^N \{A\}(\mu^* - \min_{x \in I^N} \mu(x))]$. This is true because the $N'$-th round has duration $\tau^N$, and during that round only arms in $I^N$ are sampled so that the regret of a sample in $I^N$ is upper bounded by $\mu^* - \min_{x \in I^N} \mu(x)$. Therefore the regret admits the following upper bound:
\[R^c(T) \leq \mu^*TP[A^c] + \sum_{N' \geq 0} \mathbb{E}[\tau^N \{A\}(\mu^* - \min_{x \in I^N} \mu(x))].\]

Consider a sample path in $A$, and $N' \leq N$, then we have $|I^{N'}| \leq \psi^{N'}$ and $x^* \in I^{N'}$. Therefore $\mu^* - \min_{x \in I^{N'}} \mu(x) \leq g_\mu(\psi^{N'})$ by definition of $g_\mu$. Similarly, consider a sample path in $A$, and $N' > N$. Then we have $I^N \subset I^{N'}$, $|I^N| \leq \psi^N$ and $x^* \in I^N$. Therefore:
\[\mu^* - \min_{x \in I^N} \mu(x) \leq g_\mu(\psi^N),\]
and the regret satisfies:
\[R^c(T) \leq \mu^*TP[A^c] + \sum_{N'=0}^N \psi^{N'} \mathbb{E}[\tau^N \{A\}] + g_\mu(\psi^N) \sum_{N' > N} \mathbb{E}[\tau^N \{A\}]\]
\[\leq \mu^*TP[A^c] + \sum_{N'=0}^N \psi^{N'} \mathbb{E}[\tau^N \{A\}] + g_\mu(\psi^N) \mathbb{E}[\sum_{N' > N} \tau^N],\]
\[\leq \mu^*TP[A^c] + \sum_{N'=0}^N \psi^{N'} \mathbb{E}[\tau^N \{A\}] + Tg_\mu(\psi^N),\]
where we have used the fact that $\sum_{N' > N} \tau^N \leq \sum_{N' \geq 0} \tau^N = T$.

We now upper bound the probability of $A^c$. Since $x^* \in I^0 = [0, 1]$, the occurrence of $A^c$ implies that there exists $N' < N$ such that $x^* \in I^{N'}$ and $x^* \notin I^{N'+1}$ so that we have the inclusion:
\[A^c \subset \cup_{N'=0}^{N-1} \{x^* \in I^{N'}, x^* \notin I^{N'+1}\}.\]
Since the event \( \{x^* \in I^N, x^* \notin I^{N+1}\} \) corresponds to an incorrect decision taken under IT3, we have \( \mathbb{P}[x^* \in I^N, x^* \notin I^{N+1}] \leq \alpha IT3 \leq CT^{-\gamma} \log(T)^9 \) for some \( C > 0 \) (because of the first statement of Theorem 4.4). Using a union bound we obtain the upper bound:

\[
P[A^c] \leq \sum_{N=0}^{N-1} \mathbb{P}[x^* \in I^N, x^* \notin I^{N+1}] \leq NCT^{-\gamma} \log(T)^9.
\]

The regret upper bound becomes:

\[
R^*(T) \leq \mu^* NCT^{1-\gamma} \log(T)^9 + T \log(T) + \sum_{N=0}^{N-1} g_\mu(\psi^N) \mathbb{E}[\tau^N 1\{A\}].
\]

Finally, from Theorem 4.5, we have that \( \mathbb{E}[\tau^N 1\{A\}] \leq 3(\gamma \log(T) + 32)h_\mu(\psi^N)^{-2} \) (we sample from 3 arms) where \( \delta(I^N) \) is the quantity \( \delta \) defined in the statement of Theorem 4.5, when the interval considered by IT3 is \( I^N \). Since we are considering a sample path in \( A \), and \( N' \leq N \) we have once again that \( |I^N| \leq \psi^N' \) and \( x^* \in I^N \) so that \( \delta(I^N) \geq h_\mu(\psi^N) \) by definition of \( h_\mu \). Therefore:

\[
\mathbb{E}[\tau^N 1\{A\}] \leq 3(\gamma \log(T) + 32)h_\mu(\psi^N)^{-2}.
\]

We obtain finally:

\[
R^*(T) \leq \mu^* NCT^{1-\gamma} \log(T)^9 + T \log(T) + 32 \sum_{N'=0}^{N-1} g_\mu(\psi^{N'})h_\mu(\psi^{N'})^{-2},
\]

which is the announced result and concludes the proof.

### B.7 Proof of Corollary 5.2

To prove Corollary 5.2, we use the following intermediate result.

**Proposition 1** Under Assumption 1, for all \( 0 < \Delta \leq \Delta_\mu \) we have:

(a) \( g_\mu(\Delta) \leq C_{\mu,1}\Delta^\xi; \)

(b) \( h_\mu(\Delta) \geq C_{\mu,2}\min(1, 2^\xi - 1)(\Delta/4)^\xi. \)

**Proof.** (a) By definition of \( g_\mu \) and Assumption 1 (statement (i)), we have:

\[
g_\mu(\Delta) = \mu^* - \min(\mu(x^* - \Delta), \mu(x^* + \Delta)) \leq C_{\mu,1}\Delta^\xi.
\]

(b) Let \( x \) such that \( x^* \leq x \leq x^* + \Delta/4 \). Using Assumption 1 (statement (ii)), we have:

\[
\mu(x) - \mu(x + \Delta/4) \geq C_{\mu,2}((x + \Delta/4 - x^*)^\xi - (x - x^*)^\xi).
\]

Fix \( \Delta \), and define the function \( l(x) = (x + \Delta/4 - x^*)^\xi - (x - x^*)^\xi \). Its first derivative is:

\[
l'(x) = \xi((x + \Delta/4 - x^*)^{\xi-1} - (x - x^*)^{\xi-1}).
\]

Therefore the function \( x \mapsto l(x) \) on interval \([x^*, x^* + \Delta/4]\) is increasing if \( \xi \geq 1 \) and decreasing if \( \xi < 1 \) so we get the lower bound:

\[
\min_{x \in [x^*, x^* + \Delta/4]} \mu(x) - \mu(x + \Delta/4) \geq \begin{cases} C_{\mu,2}(x^*) = C_{\mu,2}(\Delta/4)^\xi & \text{if } \xi \geq 1 \\
\mu_2(x^* + \Delta/4) = C_{\mu,2}(2^\xi - 1)(\Delta/4)^\xi & \text{if } \xi < 1 \end{cases}
\]

so that \( h_\mu(\Delta) \geq C_{\mu,2}\min(1, 2^\xi - 1)(\Delta/4)^\xi \) which proves the second statement of the proposition. □
Let us now prove Corollary 5.2. From Theorem 5.1, we can decompose the regret upper bound into three terms:

\[ R^\pi(T) \leq r_1(T) + r_2(T) + r_3(T), \]

with \( r_1(T) = \mu^* N CT^{1-\gamma} \log(T)^9, \) \( r_2(T) = T g_\mu(\psi^N) \) and:

\[ r_3(T) = 3(\gamma \log(T) + 32) \sum_{N'=0}^{N'} g_\mu(\psi^N')(h_\mu(\psi^N'))^{-2}. \]

Define \( N_0 = \lceil \log(\Delta, \log(\psi)) \rceil. \)

- For \( N' \leq N_0 \) we use the following upper bounds: \( g_\mu(\psi^N') \leq \mu^* \) and \( h_\mu(\psi^N') \geq C_{\mu,3} \) with \( C_{\mu,3} = \min_{N' \leq N_0} h_\mu(\psi^N') \). Note that \( C_{\mu,3} > 0 \) since \( h_\mu(\psi^N') > 0 \) for all \( N' \) by unimodality of \( \mu \).

- For \( N' \geq N_0 \), we have \( \psi^N' \leq C_{\mu,1} \psi^N \) and \( h_\mu(\psi^N') \geq C_{\mu,2} \min(1, 2^\xi - 1)(\Delta/4)^\xi \).

Let \( N \geq N_0 \). From the above analysis, we have the upper bound:

\[ r_3(T) \leq 3(\gamma \log(T) + 32) \left[ \mu^* N_0 C_{\mu,3}^2 + C_{\mu,1} \left(C_{\mu,2} \min(1, 2^\xi - 1)/4\right)^2 \sum_{N'=N_0}^{N} \psi^{-\xi N'} \right]. \]

Using the fact that:

\[ \sum_{N'=N_0}^{N} \psi^{-\xi N'} \leq \sum_{N'=0}^{N} \psi^{-\xi N'} = \frac{\psi^{-\xi(N+1)}}{\psi^{-\xi} - 1}, \]

we obtain that \( r_3(T) = O(\log(T)\psi^{-\xi(N+1)}). \) Now in the regret upper bound, we set \( N = N(T) \) where:

\[ N(T) = \max \left( N_0, \left\lfloor \frac{\log(T/\log(T))}{2^\xi \log(1/\psi)} \right\rfloor \right). \]

We finally get:

- \( r_1(T) = O(\sqrt{T}) \), (since \( \gamma > 0.5 \))
- \( r_2(T) = O(T^{\psi^N(T)}) = O(\sqrt{T \log(T)}) \) and
- \( r_3(T) = O(\log(T)\psi^{-\xi(N(T)+1)}) = O(\sqrt{T \log(T)}) \)

Therefore

\[ R^\pi(T) = O(\sqrt{T \log(T)}), \]

which concludes the proof.

### B.8 Technical results

Lemma [B.2] gives a lower bound of the KL divergence of probability measures using the KL divergence between two Bernoulli distributions.

**Lemma B.2** Let \( P \) and \( Q \) be two probability measures on a probability space \((\Omega, \mathcal{F}, P)\). Assume that \( P \) and \( Q \) are both absolutely continuous with respect to measure \( m(dx) \). Then:

\[ KL(P\|Q) \geq \sup_{A \in \mathcal{F}} KL_2(P(A), Q(A)). \]
Proof. The proof is based on the log-sum inequality. We recall the derivation of the log-sum inequality here. Consider \( f(x) = x \log(x) \). We have that \( f''(x) = 1/x \), so that \( f \) is convex. We define \( p, q \) the densities of \( P, Q \) with respect to measure \( m \). Then for all \( A \in \mathcal{F} \):

\[
\int_A \log \left( \frac{p(x)}{q(x)} \right) p(x) m(dx) = \int_A f \left( \frac{p(x)}{q(x)} \right) q(x) m(dx) \\
= Q(A) \int_A f \left( \frac{p(x)}{q(x)} \right) \frac{q(x)}{Q(A)} m(dx) \\
\geq (a) Q(A) f \left( \int_A \frac{p(x)}{q(x)} \frac{q(x)}{Q(A)} m(dx) \right) \\
= Q(A) f \left( \frac{P(A)}{Q(A)} \right) = P(A) \log \left( \frac{P(A)}{Q(A)} \right).
\]

and (a) holds because of Jensen’s inequality. Applying the reasoning above to \( A \) and \( A^c = \Omega \setminus A \):

\[
\text{KL} (P||Q) = \int_\Omega \log \left( \frac{p(x)}{q(x)} \right) p(x) m(dx) \\
= \int_A \log \left( \frac{p(x)}{q(x)} \right) p(x) m(dx) + \int_{A^c} \log \left( \frac{p(x)}{q(x)} \right) p(x) m(dx) \\
\geq P(A) \log \left( \frac{P(A)}{Q(A)} \right) + P(A^c) \log \left( \frac{P(A^c)}{Q(A^c)} \right) \\
= P(A) \log \left( \frac{P(A)}{Q(A)} \right) + (1 - P(A)) \log \left( \frac{1 - P(A)}{1 - Q(A)} \right) \\
= \text{KL}_2(P(A), Q(A)).
\]

So for all \( A \) we have:

\[
\text{KL} (P||Q) \geq \text{KL}_2(P(A), Q(A)),
\]

and taking the supremum over \( A \in \mathcal{F} \) concludes the proof. \( \square \)

Lemma B.3 evaluates the KL divergence between sample paths of a given test under two different parameters. The proof follows from a straightforward conditioning argument and is omitted here.

Lemma B.3 We denote by \( Y(T) = (X_1(x(1)), \ldots, X_T(x(T))) \) the observed rewards from time 1 to \( T \). Consider \( \mu, \lambda \in \mathcal{U} \), and denote by \( P_T \) and \( Q_T \) the probability distribution of \( Y(T) \) under \( \mu \) and \( \lambda \) respectively. Then we have:

\[
\text{KL} (P_T||Q_T) = \sum_{k=1}^K \mathbb{E}[t_k(T)] \text{KL} (\mu(x_k), \lambda(x_k)).
\]

Theorem B.4 is a concentration inequality for sums of KL divergences. It was derived derived in [20], and is stated here for completeness.

Theorem B.4 [20] For all \( \delta \geq (K + 1) \) and \( n \in \mathbb{N} \) we have:

\[
\mathbb{P} \left[ \sum_{k=1}^K t_k(n) \text{KL} (\hat{\mu}_k(n), \mu(x_k)) \geq \delta \right] \leq e^{-\delta \left( \frac{[\delta \log(n)] \delta}{K} \right)} e^{K+1}.
\]

Lemma B.3 is a technical result showing that the expected number of times the empirical mean of i.i.d. variables deviates by more than \( \delta \) from its expectation is \( o(\log(n)) \), \( n \) being the time horizon.
Lemma B.5 Let \( \{X_n\}_{n \geq 1} \) be a family of i.i.d. random variables with common expectation \( \mu \) and finite second moment. Define \( \bar{\mu}(n) = (1/n) \sum_{n'=1}^{n} X_{n'} \). For \( \delta > 0 \) define
\[
D^\delta(n) = \sum_{n'=1}^{n} 1\{|\bar{\mu}(n') - \mu| \geq \delta\}.
\]
Then we have that for all \( \delta \):
\[
\frac{\mathbb{E}[D^\delta(n)]}{\log(n)} \to_{n \to \infty} 0.
\]

**Proof.** We define \( v^2 = \mathbb{E}[(X_1 - \mu)^2] \) the variance. Using the fact that \( \{X_n\}_{n \geq 1} \) is independent, we have that \( \mathbb{E}[(\bar{\mu}(n) - \mu)^2] = v^2/n \). Applying Chebychev’s inequality we have that:
\[
\mathbb{P}[|\bar{\mu}(n) - \mu| \geq \delta] \leq \frac{\mathbb{E}[(\bar{\mu}(n) - \mu)^2]}{\delta^2} = \frac{v^2}{n\delta^2}.
\]
Therefore, we recognize the harmonic series:
\[
\mathbb{E}[D^\delta(n)] = \sum_{n'=1}^{n} \mathbb{P}[|\bar{\mu}(n') - \mu| \geq \delta] \leq \frac{v^2}{\delta^2} \sum_{n'=1}^{n} \frac{1}{n'} \leq \frac{v^2(\log(n) + 1)}{\delta^2},
\]
so that \( \sup_n \mathbb{E}[D^\delta(n)]/\log(n) < \infty \).

Applying the law of large numbers, we have that \( \bar{\mu}(n) \to_{n \to \infty} \mu \) a.s., so that \( |\bar{\mu}(n) - \mu| \) occurs only finitely many times a.s. Hence \( \sup_n D^\delta(n) < \infty \) a.s and \( D^\delta(n)/\log(n) \to 0 \) a.s.

We have proven that \( \sup_n \mathbb{E}[D^\delta(n)]/\log(n) < \infty \) and \( D^\delta(n)/\log(n) \to 0 \) a.s. so applying Lebesgue’s dominated convergence theorem we get the announced result:
\[
\frac{\mathbb{E}[D^\delta(n)]}{\log(n)} \to_{n \to \infty} 0,
\]
which concludes the proof. \( \square \)