Points with Large Quadrant-Depth

[Extended Abstract] †

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ABSTRACT

Given a set $P$ of points in the plane we are interested in points that are ‘deep’ in the set in the sense that they have two opposite quadrants both containing many points of $P$. We deal with the extremal version of this problem. A pair $(a,b)$ of numbers is admissible if every point set $P$ contains a point $p \in P$ that determines a pair $(Q,Q^o)$ of opposite quadrants, such that $Q$ contains at least an $a$-fraction and $Q^o$ contains at least a $b$-fraction of the points of $P$. We provide a complete description of the set $\mathcal{F}$ of all admissible pairs $(a,b)$. This amounts to identifying three line segments and a point on the boundary of $\mathcal{F}$.

In higher dimensions we study the maximal $a$, such that $(a,a)$ is opposite orthant admissible. We show that $1/(2\gamma) \leq a \leq 1/\gamma$ for $\gamma = 2^{d-1}2^{d-1}$. Finally we deal with a variant of the problem where the opposite pairs of orthants need not be determined by a point in $P$. Again we are interested in values $a$, such that all subsets $P \subseteq \mathbb{R}^d$ admit a pair $(O, O^o)$ of opposite orthants both containing at least an $a$-fraction of the points. The maximal such value is $a = 1/2^d$. Generalizations of the problem are also discussed.

Categories and Subject Descriptors
G.2.1 [Discrete Mathematics]: Combinatorics

General Terms
Theory

1. INTRODUCTION

A point $p = (p_1, p_2)$ in the plane defines four quadrants $Q_1(p), Q_2(p), Q_3(p)$ and $Q_4(p)$ centered at $p$. They are the connected regions left after removing the horizontal and the vertical lines through $p$ from the plane. As usual the quadrants are numbered in counterclockwise order starting from $Q_1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > p_1 \text{ and } x_2 > p_2\}$. There are two pairs of opposite quadrants ($Q_1, Q_3$) and ($Q_2, Q_4$). We write $Q^{op}$ to denote the quadrant opposite to $Q$, e.g., $Q^{op} = Q_1$. Throughout the paper we only consider sets of points in general position; in our context this means that the intersection of any horizontal or vertical line with $P$ consists of at most one point. Actually this assumption is necessary for our results; if the points are not in general position and quadrants are open, then all quadrants of all points in $P$ may have empty intersection with $P$.

For a given set $P$ of points we ask for points $p \in P$ that are ‘deep’ in the set in the sense that $p$ has two opposite quadrants both containing many points of $P$. We deal with
the extremal version of this problem. That is, we ask for pairs \((a, b)\), such that for every set \(P\) of \(n\) points in the plane there is a point \(p \in P\) and a pair \((Q, Q^\emptyset)\) of opposite quadrants centered at \(p\), such that \(|Q \cap P| \geq an - c\) and \(|Q^\emptyset \cap P| \geq bn - c\), where \(c\) is a small constant. In this case we call the pair \((a, b)\) admissible and the pair \((Q, Q^\emptyset)\) of quadrants \((a, b)\)-admissible.

Brönnimann, Lenchner, and Pach [BLP] define the notion of opposite-quadrant depth for point sets in the plane as the maximum \(s\), such that \((a, a)\) is admissible. They prove that every set of points in the plane has opposite-quadrant depth at least \(1/3\). We give a new and simpler proof of this result below, Theorem 1.1.

In Section 2 we provide a complete description of the set \(\mathcal{F}\) of all admissible pairs. The shape of \(\mathcal{F}\) turns out to be surprisingly complicated (see Figure 3).

In Section 3 we ask for the maximum \(s\), such that \((a, a)\) is admissible in higher dimensions. In dimension \(d\) we obtain upper and lower bounds that differ by a factor of \(2\). In Section 4 we discuss further generalizations.

The notion of opposite quadrant-depth, resp. opposite orthant-depth is related to centerpoints and some measures of statistical depth, like hyperplane depth. We refer to [Ede] for information on centerpoints and to [LPS] for statistical depth. Brönnimann et al. [BLP] also mention a connection with conflict-free colorings. Related notions of depth have been studied e.g. in [BPZ1] and [BPZ2].

As a warm up and for the purpose of introducing some convenient notations we now reprove the main result from [BLP].

**Theorem 1.1 ([BLP]).**

1. Any set \(P\) of \(n\) points in general position in the plane has opposite-quadrant depth \(\geq 1/3\).
2. If \(P\) is in convex position, then it has opposite-quadrant depth \(\geq 1/4\).

Before starting with the proof let us introduce the following convenient notation. The weight of a set of points \(A\) with \(|A| = n\) is \(\omega(A) = \frac{|A|}{n - 1}\). For a point \(p \in P\) and a quadrant \(Q\) centered at \(p\) we define the weight of \(Q\) as

\[
\pi(Q) = \omega(Q \cap P) = \frac{|Q \cap P|}{n - 1}.
\]

Note that by the general position assumption the quadrants of any point \(p\) fulfill \(\pi(Q_1(p)) + \pi(Q_2(p)) + \pi(Q_3(p)) + \pi(Q_4(p)) = 1\). In many cases we will need a subset \(A\) of \(P\) of some specified weight. E.g. when we choose a set \(A\) of weight \(\tau\) from \(P\) we mean that \(\tau = \frac{1}{n - 1} < \omega(A) \leq \tau\). We extend the loose interpretation to general inequalities: let \(\omega(A) \leq \tau\) and \(\omega(A) \geq \tau\) be abbreviations for \(\omega(A) \leq \tau + \frac{1}{n - 1}\) and \(\omega(A) \geq \tau - \frac{1}{n - 1}\) respectively, here \(c\) is some constant. The reason for allowing such small errors in inequalities is that our results talk about fractions of the point set, about asymptotic and not about single points. Because the sets considered in our proofs are obtained by at most four iterations of a directive of the form “choose \(A\) of weight \(w\) from \(A\)” we can assure that the constants \(c\) for all inequalities in this paper can be chosen to be \(\leq 4\).

The theorem is an immediate consequence of the following lemma.

**Lemma 1.2.** Every set \(P\) in the plane contains a point \(p\), such that

\[
\min(\pi(Q_1(p)), \pi(Q_3(p))) + \min(\pi(Q_2(p)), \pi(Q_4(p))) \geq \frac{1}{4}.
\]

**Proof.** Consider the sets \(P_L, P_R, P_T\) and \(P_B\) of weight \(1/4\) each consisting of the first points in \(P\) from the left, right, top and bottom, see Figure 1. From the definition of weights it follows that \(P' = P - (P_L \cup P_R \cup P_T \cup P_B) \neq \emptyset\). We claim that every point in \(P'\) has the desired property. Let \(p\) be such a point and assume that \(\min(\pi(Q_1(p)), \pi(Q_3(p))) = s = \pi(Q_1(p))\). Since \(P_T\) is contained in \(Q_1(p) \cup Q_2(p)\) it follows that \(\pi(Q_2(p)) \geq \frac{1}{4} - s\). Considering \(P_B\) we obtain \(\pi(Q_4(p)) \geq \frac{1}{4} - s\). Consequently \(\min(\pi(Q_1(p)), \pi(Q_3(p))) = s\) and \(\min(\pi(Q_2(p)), \pi(Q_4(p))) \geq \frac{1}{4} - s\), this proves the lemma.

![Figure 1: An illustration of the proof of Lemma 1.2.](image1)

**Figure 1: An illustration of the proof of Lemma 1.2.**

**Proof.** (of Theorem 1.1) For part one of the theorem it is enough to observe that either \(s\) or \(1/4 - s\) is at least \(1/4\). For the second part note that if \(P\) is in convex position one of the four quadrants of \(p\) is empty. Therefore, one of the two minima in the lemma is zero and the other minimum is at least \(1/4\).

![Figure 2: Opposite-quadrant depth \(\leq 1/4\).](image2)

**Figure 2: Opposite-quadrant depth \(\leq 1/4\).**

It is easy to see that the second part of Theorem 1.1 is best possible by taking \(P\) to be the set of vertices of a regular \(n\)-gon. In [BLP] it is shown that the first part of Theorem 1.1 is best possible for arbitrarily large values of \(n\) of the form \(n = 4 \cdot 3^k\). The example in Figure 2 shows a simple construction that proves that the first part of Theorem 1.1 cannot be improved. To describe the example we identify a point set with the induced dominance order, i.e., we say \(p \leq q\) if \(p_x \leq q_x\) and \(p_y \leq q_y\). Based on this order we can talk about chains and antichains of a point set. The example of Figure 2 consists of eight chains each of weight \(\leq 1/4\). In the figure the chains are represented by gray segments.

The chains come in two groups of four chains each. One group of weight \(1/7\) consists of the first points in \(P\) from the
top (and the right) and the other group of weight $\frac{1}{4}$ consists of the first points in $P$ from the bottom (and the left).

Since the example is symmetric it is enough to consider points in the first group to prove that there is no point in $P$ with a pair of opposite quadrants each of weight strictly more than $\frac{1}{4}$. We consider two cases: First for $p \in P$ look at the pair $(Q_2(p), Q_4(p))$. The chains in each group are arranged in such a way that $Q_2(p)$ and $Q_4(p)$ both contain an integral number of chains. Therefore in order to have $\pi(Q_2(p)) > \frac{1}{4}$ and $\pi(Q_4(p)) > \frac{1}{4}$ the second and fourth quadrant of $p$ both have to contain at least two chains. But there is no point $p$ with this property.

To complete the argument, we observe that $\pi(Q_4(p)) \leq \frac{1}{4}$ holds for all $p$ in the first group. Hence every point in $P$ has two adjacent quadrants with weight $\leq \frac{1}{4}$.

2. THE SET OF ADMISSIBLE PAIRS

Recall that a pair $(a, b) \in [0, 1]^2$ is called admissible if every set $P$ of $n$ points in the plane contains a point $p \in P$, such that there is a quadrant $Q$ centered at $p$ with $\pi(Q) \geq a$ and $\pi(Q^{op}) \geq b$. For pairs $(a, b) \neq (a', b')$ in $[0, 1]^2$ we write $(a', b') \prec (a, b)$ if $a' \leq a$ and $b' \leq b$.

In this section we provide a full characterization of the set $F$ of all admissible pairs. We begin with some easy observations.

- $F$ is symmetric in the sense that if $(a, b) \in F$, then also $(b, a) \in F$.
- $F$ is also monotone decreasing, that is, if $(a, b) \in F$ and $(0, 0) \prec (a', b') \prec (a, b)$ then $(a', b') \in F$.

Figure 3 includes the half of the set $F$, where $a \geq b$, the other half is obtained by reflection about the diagonal line $a = b$. In our analysis, we will determine the pair of the boundary of $F$ shown in the figure. That is, we will always assume that $a \geq b$.

Theorem 1.1 shows that $(\frac{1}{4}, \frac{1}{4}) \in F$ and the example from Figure 2 implies that if both $a$ and $b$ are greater than $\frac{1}{4}$, then $(a, b) \notin F$.

To see that no pair $(a, b)$ beyond $(\frac{1}{4}, 0)$ is in $F$ it is enough to consider three independent chains each of weight $\leq \frac{1}{4}$. In this example $\pi(Q(p)) > \frac{1}{4}$ implies $\pi(Q^{op}(p)) \leq 0$. Therefore:

**Observation 2.1.** Pairs $(a, b)$ with $a > \frac{1}{4}$ and $b > 0$ are not admissible, i.e., they are not in $F$.

Though the next proposition does not really contribute to the boundary of $F$ it provides a good first approximation to the set.

**Proposition 2.2.** Every pair $(a, b)$ with $3a + 5b \leq 1$ is in $F$.

**Proof.** First note that we only have to prove that every pair $(a, b)$, with $3a + 5b = 1$, is in $F$. The proposition then follows for all pairs $(a', b') \prec (a, b)$.

So given a set $P$ of points and a pair $(a, b)$ with $3a + 5b = 1$. Choose a vertical line, such that the set $A$ of points left of the line has weight $2a$. Choose another vertical line, such that the set $Z$ of points right of this line has weight $a+b$. Consider the horizontal median point $p$ in the strip $S$ between the vertical lines. From the assumption it follows that $\omega(S) \geq 4b$ (see Figure 4).

![Figure 4: An illustration of the proof of Proposition 2.2.](image)

From $\omega(A) = 2a$ it follows that one of the left quadrants of $p$ has weight $\geq a$. If $\pi(Q_4(p)) > b$ we are done. Hence, we may assume that $\pi(Q_4(p)) < b$. From the weight of $Z$ it then follows that $\pi(Q_4(p)) \geq a$. The weight of points below $p$ in $S$ is $\geq 2b$, therefore, $\pi(Q_4(p)) \leq b$ implies $\pi(Q_4(p)) \geq b$. We have thus found an appropriate pair of opposite quadrants.

The next observation implies that $(\frac{1}{4}, 0)$ belongs to the boundary of $F$.

**Observation 2.3.** The pair $(\frac{1}{4}, 0)$ belongs to $F$. Moreover, for every $a > \frac{1}{4}$ the pair $(a, 0)$ does not belong to $F$.

**Proof.** Let $P$ be a set of $n$ points in the plane and let $p \in P$ be the point of $P$ with the largest $x$-coordinate. All points of $P \setminus \{p\}$ are contained in the second and third quadrant of $p$. Hence, one of these two quadrants has weight $\geq \frac{1}{2}$. This shows that $(\frac{1}{4}, 0)$ belongs to $F$.

To see that $(a, 0)$ does not belong to $F$ for any $a > \frac{1}{4}$, consider a set $P$ of $n$ points evenly distributed on a circle, or equivalently the vertices of regular $n$-gon. It is left to the reader to verify that no point in $P$ has a quadrant of weight $> \frac{1}{4}$.

We now get to the concavity at $(\frac{1}{5}, \frac{1}{10})$ on the boundary of $F$ and the two segments bounding $F$ that meet in this point.

**Theorem 2.4.** Every pair $(a, b)$ with $4a + 2b = 1$ and $\frac{3}{4} \leq a \leq \frac{1}{3}$ is on the boundary of $F$.

**Proof.** In the proof we will need that $2a \geq 3b$, this is where the restriction $\frac{3}{4} \leq a$ comes from.

Given a set $P$ of points and a pair $(a, b)$. Choose two vertical lines, such that the set $A$ of points left of both lines has weight $2a - b$, the weight of the set $S$ in the strip between the lines has weight $4b$ and the set $Z$ of points on the right of the two lines has the remaining weight $2a - b$. Consider the horizontal median point $p$ of the middle set $S$ (see Figure 5).

One of the quadrants of $p$ has weight at least $\frac{3}{4}$. Without loss of generality we assume that this is true for $Q_1(p)$, i.e., $\pi(Q_1(p)) \geq \frac{1}{4} \geq a$. If $\pi(Q_2(p)) \geq b$ we are done. Hence, we may assume that $\pi(Q_4(p)) < b$. From the weight of points below $p$ it then follows that $\pi(Q_4(p)) < b$. Since the weight of $A$ is $2a - b \geq 2b$ it follows that $\pi(Q_4(p)) \geq b$. If one of the quadrants $Q_2(p)$ and $Q_4(p)$ has weight at least $a$ we are done. But if $\pi(Q_2(p)) < a$ and $\pi(Q_4(p)) < a$ the points in the union of the second, third and fourth quadrant of $p$...
The example given in Figure 6.

and follow the very same argumentation as in Theorem 2.5.

Suppose one of the quadrants of \( p \) or \( q \) has weight at least \( a \) we sum up the weights of the first and second quadrant of \( p \) and the third and fourth quadrant of \( q: \pi(Q_1(p)) + \pi(Q_2(p)) + \pi(Q_3(q)) + \pi(Q_4(q)) = 4a. \)

To see that at least one of the quadrants of \( p \) or \( q \) has weight at least \( a \) we sum up the weights of the first and second quadrant of \( p \) and the third and fourth quadrant of \( q: \pi(Q_1(p)) + \pi(Q_2(p)) + \pi(Q_3(q)) + \pi(Q_4(q)) = 4a. \)

The proof that there are no admissible pairs with \( a \) beyond the segment \( 3a + 3b = 1 \) consists in analysing the example given in Figure 6.

**Theorem 2.5.** Every pair \((a, b)\) with \( 2a + 6b = 1 \) and \( \frac{1}{3} \leq a \leq \frac{1}{2} \) is on the boundary of \( \mathcal{F} \).

The proof of this theorem is given in the full paper.

**Theorem 2.6.** Every pair \((a, b)\) with \( 3a + 3b = 1 \) and \( \frac{1}{4} \leq a \leq \frac{1}{3} \) is on the boundary of \( \mathcal{F} \).

Proof. Given a set \( P \) of points and a pair \((a, b)\). Choose a vertical line, such that the set \( A \) of points left of the line has weight \( a + b \). Choose another vertical line, such that the set \( Z \) of points right of this line has weight \( a + b \). The weight of the set \( S \) in the strip between the lines also has weight \( a + b \). This set \( S \) is divided vertically into a top part \( T \), a middle part \( M \) and a bottom part \( B \), such that each of \( T \) and \( B \) has weight \( 2b \) (see Figure 8). The vertical lines are chosen, such that the weight of \( M \) is \((a + b) - 4b = a - 3b \geq 0\), this is where the restriction \( a \geq \frac{1}{3} \) comes from.

Suppose one of the quadrants of \( p \) or \( q \) has weight at least \( a \). In this case we can simply disregard the middle part \( M \) and follow the very same argumentation as in Theorem 2.5 to find an appropriate pair of opposite quadrants.

3. **Higher Dimensions**

A point \( p \) in \( \mathbb{R}^d \) defines \( 2^d \) orthants centered at \( p \). Again there is an obvious notion of an opposite orthant \( O^\pi \) of a given orthant \( O \). The weight \( \pi(O) \) of an orthant \( O \) with respect to a point set \( P \) is the fraction of points of \( P \) contained in \( O \). For a more formal definition of the weight we refer to the introduction.

Define the opposite-orthant depth \( \alpha_d \) for point sets in \( \mathbb{R}^d \) as the maximum \( a \), such that every point set \( P \subset \mathbb{R}^d \) contains a point \( p \) that determines a pair \((O, O^\pi)\) of opposite orthants with \( \pi(O) \geq a \) and \( \pi(O^\pi) \geq a \). Brönnimann et al. [BLP] have asked for \( \alpha_3 \). They also claim that \( \alpha_3 \geq n/2016 \), this however is based on the false assumption that every set of 9
points in $\mathbb{R}^3$ has a point with two opposite nonempty orthants. Indeed, the least $n$ such that this holds is $n = 17$. With this correction their proof only yields $\alpha_3 \geq n/16320$. The case $d = 3$ of the theorem below gives $\alpha_3 \geq n/32$.

For a point $x \in \mathbb{R}^d$ and $i = 1, \ldots, d$ define the open half-spaces $H_i^+(x) = \{ y : y_i > x_i \}$ and $H_i^-(x) = \{ y : y_i < x_i \}$. A sign pattern is a vector $\sigma = (\sigma_1, \ldots, \sigma_d)$ with $\sigma_i \in \{+, -\}$. For every point $x$ and sign pattern $\sigma$ we define the orthant $O^\sigma(x) = \bigcap_i H_i^{\sigma_i}(x)$.

**Theorem 3.1 (Lower Bound).** The opposite-orthant depth $\alpha_d$ of point sets in $\mathbb{R}^d$ satisfies $\alpha_d \geq 2^{-(2^d-1+d)}$.

**Proof.** A set of points is $t$-good if it contains no point determining a pair of opposite orthants such that both contain $t+1$ or more points from $P$.

We will prove that $|P| > 2^{2^d-1}(t2^d)$ implies that $P$ is not $t$-good. Hence $\alpha_d(t2^{d-1}+d+1) \geq t+1$ which yields the bound stated in the theorem.

Let $P$ be a $t$-good set. One of the orthants from each pair of opposite orthants determined by $p \in P$ is small in the sense that it contains at most $t$ points from $P$.

The pattern assigned to $p$ is a collection $\phi(p)$ of $2^d-1$ sign vectors, such that

- $O^\sigma(p)$ is small for each $\sigma \in \phi(p)$ and for each pair $(O^\sigma, O^\bar{\sigma})$ of opposite orthants determined by $p$ either $\sigma$ or $\bar{\sigma}$ is in $\phi(p)$.

For a given pattern $\phi$ we collect all points $p \in P$ with $\phi(p) = \phi$ in a set $P_\phi$. Figure 10 shows an example.

**Figure 7:** An example showing that the pairs $(a, b)$ from Theorem 2.5 are on the boundary of $\mathcal{F}$.

\[
\begin{array}{c|c|c|c}
A & T & Z \\
a + b & q & a + b \\
M & a - 2b & 2b \\
B & p & 2b \\
\end{array}
\]

\(a - 2b \geq 0 \iff b \leq \frac{a}{14}\) because

\[2a + 6b = 1\]

**Figure 8:** An illustration of the proof of Theorem 2.6.

We have partitioned the points of $P$ according to their pattern. The upper bound on the size of any $t$-good set $P$ follows from counting the possible patterns and bounding the number of points in each class $P_\phi$.

There are $2^d$ sign vectors paired up in $2^{d-1}$ pairs $\sigma, \bar{\sigma}$ belonging to pairs of opposite orthants. A pattern is a selection of one sign vector from each such pair, therefore:

- There are at most $2^{2^d-1}$ different patterns.

For any $p$ and $\sigma$ let $v_\sigma(p) = |O^\sigma(p) \cap P|$. Define the score of $p$ as $s(p) = \sum_{\sigma \in \phi(p)} v_\sigma(p)$. From the definition of $\phi(p)$ it follows that $v_\sigma(p) \leq t$ for all $\sigma \in \phi(p)$, hence, $s(p) \leq t2^{d-1}$. Note that the score of $p$ is the number of points in the small orthants $O^\sigma(p)$ with $\sigma \in \phi(p)$.

Consider two points $p, q$ and note that $q \in O^\sigma(p)$ if and only if $p \in O^\bar{\sigma}(q)$. Suppose $p$ and $q$ both belong to $P_\phi$, since one of $\sigma$ and $\bar{\sigma}$ is in $\phi$ we note that either $p$ is counted in the score of $q$ or $q$ is counted in the score of $p$. From this we obtain:

\[
\frac{|P_\phi||P_\phi| - 1}{2} = \left(\frac{|P_\phi|}{2}\right)^2 \leq \sum_{p \in P_\phi} s(p) \leq \sum_{p \in P_\phi} t2^{d-1} = |P_\phi|t2^{d-1}.
\]

To reduce the upper bound on $\sum s(p)$ by one observe that
for each \( \sigma \in \sigma \) there are points in \( P_\sigma \) with \( v_\sigma(p) < t \). This yields the following bound on \( |P_\sigma| \):

- For each class \( P_\sigma \) we have \( |P_\sigma| \leq 2t \, 2^{d-1} \).

Combining the bounds for the number of patterns and the size of the classes we find that a \( t \)-good set \( P \) has at most \( 2^{d-1} (t \, 2^{d-1}) \) points.

The upper bound on the opposite-orthant depth \( \alpha_d \) shown with the following theorem is only a factor of two apart from the lower bound of Theorem 3.1. It is evident that the lower bound is not tight. In dimension 2 the upper and lower bounds yield \( \frac{1}{16} \leq \alpha_2 \leq \frac{1}{4} \). From Theorem 1.1 and the example of Figure 2 we know that \( \alpha_2 = \frac{1}{4} \). Indeed we suspect that in all dimensions the upper bound gives the true value of \( \alpha_d \).

**Theorem 3.2 (Upper bound).** The opposite-orthant depth \( \alpha_d \) of point sets in \( \mathbb{R}^d \) satisfies \( \alpha_d \leq 2^{-(d-1)+d-1} \).

**Proof.** We have to construct large point sets with small opposite-orthant depth. The construction is in two steps. In the first step we build a set \( P_0 \) of \( 2^{d-1} \) points with opposite-orthant depth zero. In the second step we replace each point of \( P_0 \) with a carefully chosen set of \( t \, 2^{d-1} \) points, such that the depth remains bounded by \( t \). Hence \( t + 1 \geq \alpha_d (2^{d-1} - t \, 2^{d-1}) \) which yields the bound stated in the theorem.

Let \( \sigma \) be a sign vector and \( \bar{\sigma} \) be the sign vector of the orthant opposite to \( \sigma \). Based on \( \sigma \) we define a binary relation on \( P \), let \( p \sim_\sigma q \) if \( p \in O^\sigma(q) \) or \( p \in O^{\bar{\sigma}}(q) \), i.e., \( q \in O^\sigma(p) \).

A set \( M \) of points in \( \mathbb{R}^d \) is monotone if there is a sign vector \( \sigma \), such that \( p \sim_\sigma q \) for all \( p, q \in M \). Equivalently \( M \) is monotone if there is an ordering of the points so that each coordinate is increasing or decreasing in this order. Repeated application of the Lemma of Erdős-Szekeres implies that any \( n \) points in \( \mathbb{R}^d \) contain a monotone subset of size at least \( \frac{n^{d-1}}{2} \). It is a widely known fact that this bound is best possible. A detailed construction of tight examples can be found e.g. in [Lit]. Due to this result there is a set \( P_0 \) of \( 2^{d-1} \) points that does not contain a monotone subset of size three. Hence, for every \( p \in P_0 \) and every pair of opposite orthants \((O, O^\sigma)\) determined by \( p \) at least one of the orthants contains no point of \( P_0 \).

An orthant \( O^\sigma(p) \) defined by \( p \in P_0 \) is small if \( O^\sigma(p) \cap P_0 = \emptyset \). As in the proof of the previous theorem we collect sign vectors of small orthants of \( p \in P_0 \) in a pattern \( \phi(p) \). Recall that \( \phi(p) \) contains the sign vector of one of each pair of opposite orthants. We construct the set \( P \) by replacing each point \( p \in P_0 \) by a set \( Q(p) \) of \( t \, 2^{d-1} \) points, such that \( q \in Q(p) \) and \( \sigma \in \phi(p) \) implies \(|O^\sigma(q) \cap P| = |O^\sigma(q) \cap Q(p)| \leq t \), i.e., the orthant \( O^\sigma(q) \) is \( t \)-small for all \( \sigma \in \phi(p) \).

For each \( p \in P_0 \) fix a "small" box \( B(p) \) containing \( p \), such that every choice of one point from each of these boxes yields a set of opposite orthant depth zero. Formally, for \( p, q \in P_0 \) with \( p \sim_\sigma q \) we require that \( p' \sim_\sigma q' \) for all \( p' \in B(p) \) and \( q' \in B(q) \). For the construction of \( Q(p) \) in the box \( B(p) \) it is convenient to think of \( B(p) \) as an open set. To begin with let \( \sigma_0, \sigma_1, \sigma_2, \ldots \) be an ordering of the \( 2^d \) sign vectors in \( \phi(p) \). Starting with \( S_0 = \emptyset \) we inductively define subboxes \( S_i \) for \( i = 1, \ldots, 2^{d-1} - 1 \) of \( B_0 = B(p) \) as follows: If \( S_{i-1} \) and \( B_{i-1} \) are defined choose a point \( s_i \) in \( B_{i-1} \) and let \( S_i = O^\sigma(s_i) \cap B_{i-1} \) and \( B_i = O^{\bar{\sigma}}(s_i) \cap B_{i-1} \). Finally let \( S_{2^{d-1}} = B_{2^{d-1}-1} \). An example is given in Figure 11.
A sequence of positive real numbers such that \( \pi \in \text{\textit{large}} \) measure (see Figure 12). This naturally generalizes to higher dimensions as well. The following theorem gives a partial answer to this problem in any dimension \( d \):

**Theorem 4.1.** Let \( \mu \) be a continuous probability measure in \([0, 1]^d\). Let \( n \) be a positive integer and let \( a_1, \ldots, a_n \) be a sequence of positive real numbers such that \( \sum_{i=1}^n a_i = 1 \). Then there exist numbers \( x_{i,j} \), where \( 1 \leq i \leq d \) and \( 0 \leq j \leq n \), and \( d \) permutations \( \pi_1, \ldots, \pi_d \) on \( \{1, \ldots, n\} \) with the following properties:

1. For every \( 1 \leq i \leq d \), \( x_{i,0} = 0 < x_{i,1} < \ldots < x_{i,n-1} < x_{i,n} = 1 \).
2. For every \( 1 \leq j \leq n \) we have \( \mu([x_{1,\pi_1(j)}-1, x_{1,\pi_1(j)}] \times \ldots \times [x_{d,\pi_d(j)}-1, x_{d,\pi_d(j)}]) \geq a_j \).

**Remark.** Intuitively speaking, the numbers \( x_{i,j} \) in Theorem 4.1 define the \( d \)-dimensional array of \( n^d \) cells generated by the hyper-planes \( H_{i,j} = \{ (x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_i = x_{i,j} \} \), for all \( 1 \leq i \leq n \) and \( 1 \leq j \leq d \). The theorem then says that there exists such an array (while the measure \( \mu \) is given) that contains a generalized diagonal of \( n \) boxes with measures of at least \( a_1, \ldots, a_n \), respectively.

**Proof.** Assume to the contrary that there are numbers \( x_{i,j} \) and permutations \( \pi_i \) such that for each \( j \) with \( 0 \leq j \leq n \):

\[
\lambda([x_{1,\pi_1(j)}-1, x_{1,\pi_1(j)}] \times \ldots \times [x_{d,\pi_d(j)}-1, x_{d,\pi_d(j)}]) > \frac{1}{n^d}.
\]

We will use the means inequality, by which, for every \( d \) positive numbers \( a_1, \ldots, a_d \) we have \( d^{-1}(a_1 + \cdots + a_d) \geq (a_1a_2 \cdots a_d)^{\frac{1}{d}} \).

For every \( 1 \leq i \leq d \) and \( 1 \leq j \leq n \) put \( a_{i,j} = x_{i,\pi_i(j)} - x_{i,\pi_i(j)-1} \). For every \( 1 \leq i \leq d \) we may assume that \( x_{i,0} = 0 \) and \( x_{i,n} = 1 \), hence, \( \sum_{j=1}^n a_{i,j} = 1 \). On the other hand, our assumption on the measure and the cells of the generalized diagonal imply \( \prod_{j=1}^n a_{i,j} > \frac{1}{n^d} \) for every \( 1 \leq j \leq n \).

Therefore, by the means inequality, for every \( 1 \leq j \leq n \) we have:

\[
\sum_{i=1}^d a_{i,j} \geq d \left( \prod_{i=1}^d a_{i,j} \right)^{\frac{1}{d}} > \frac{d}{n} \quad \implies \quad d = \sum_{i=1}^d \sum_{j=1}^n a_{i,j} = \sum_{j=1}^n a_{i,j} > \frac{n^d}{n} = d.
\]

This contradiction completes the proof.

**5. CONCLUSION**

We gave a complete description of the set \( \mathcal{F} \) of all admissible pairs \((a, b)\) in the plane. This was done by identifying three line segments and the points \((\frac{1}{d}, \frac{1}{d})\) and \((0, \frac{1}{2})\) from the boundary of \( \mathcal{F} \).

In higher dimensions we were interested in the maximal number \( \alpha_d \) such that \((\alpha_d, \alpha_d)\) is admissible. We think that our upper bound on \( \alpha_d \) is tight, the lower bound leaves room for improvements. It would be interesting to get more information about the set \( \mathcal{F}_d \) of all admissible pairs \((a, b)\) for \( \mathbb{R}^d \).

In the relaxed setting, where the point determining a pair of opposite orthants need not belong to the point set we could determine the diagonal entry precisely, it is \((\frac{1}{d}, \frac{1}{d})\). This follows from more general bounds on generalized diagonals. In this case we have some additional results concerning admissible pairs \((a, b)\). There remain many questions to be asked and answered.

**6. REFERENCES**

[BPZ1] I. Ben-Dan, R. Pinchasi and R. Ziv *On a problem of Felsner about quadrant depth*, submitted

[BPZ2] I. Ben-Dan, R. Pinchasi and R. Ziv *Points with large \( \alpha \)-depth*, Journal of Combinatorial Theory A, 116 (2009) 747-â€”755.

[BLP] H. Brönnimann, J. Lenchner, and J. Pach, *Opposite-quadrant depth in the plane*. Graphs and Combinatorics 23, (2007), 145-152.

[Ed] H. Edelsbrunner. Algorithms in Combinatorial Geometry. Springer, 1987.

[Lit] L. Heinrich-Litan. *Monotone Subsequences in \( \mathbb{R}^d \).* Tech. Rep. B 00-19, FU Berlin, FB Math. u. Inf., 2000. http://heinrich-litan.de/PUB/MonSeq.pdf

[LPS] R. Liu, J. Parelius, and K. Singh. *Multivariate analysis by data depth: Descriptive statistics, graphics and inference*. Ann. Stat. 27 (1999) 783-858.