Weak Essential Fuzzy Submodules Of Fuzzy Modules

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Abstract

Throughout this paper, we introduce the notion of weak essential F-submodules of F-modules as a generalization of weak essential submodules. Also, we study the homomorphic image and inverse image of weak essential F-submodules.

Keywords: Semi-prime F-submodules, essential F-submodules.

1.Introduction

Let $S \neq \emptyset$. Zadeh [1] defined F-subset $X$ of $S$ as a mapping $X: S \rightarrow [0,1]$. Negoita and Ralescu [2] introduced the concept of F-modules. Mashinchi and Zahedi [3] introduced the notion of F-submodules.

Mona [4] introduced and studied the concept of weak essential submodules, where a submodule $H$ of $\mathcal{M}$ is called a weak essential, if $H \cap L \neq (0)$, for each non-zero semiprime submodule $L$ of $\mathcal{M}$. In this paper, we introduce the notion weak essential F-submodule of F-module. We investigate some basic results about weak essential submodules.

Next, throughout this paper $\mathcal{R}$ is a commutative ring with identity, $\mathcal{M}$ is an $\mathcal{R}$-module and $X$ is a F-module of an $\mathcal{R}$-module $\mathcal{M}$.

Finally, (shortly fuzzy set, fuzzy submodule and fuzzy module is F-set, F-submodule and F-module).

S.1 Preliminaries

In this section, we shall give the concepts of F-sets and operations on F-sets, with some important properties of them, which are used in this paper.
Definition 1.1 [1]:

Let $S$ be a non-empty set and let $I$ be a closed interval $[0,1]$ of the real line (real number). A $F$-set $X$ in $S$ (a fuzzy subset $X$ of $S$) is characterized by a membership function $X : S \rightarrow I$.

Definition 1.2 [2]:

Let $x_t : S \rightarrow I$, be a F-set in $S$, where $x \in S$, $t \in I$, defined by:

$$x_t = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

Then $x_t$ a said $F$-singleton.

If $x = 0$ and $t = 1$ then:

$$0_t(y) = \begin{cases} 1 & \text{if } y = 0 \\ 0 & \text{if } y \neq 0 \end{cases}$$

We shall call such F-singleton the $F$-zero singleton.

Proposition 1.3 [3]:

Let $a_t, b_k$ be two F-singletons of a set $S$. If $a_t = b_k$, then $a = b$ and $t = k$, where $t, k \in I$.

Definition 1.4 [5]:

Let $A_1, A_2$ are F-sets in $S$, then:

1. $A_1 = A_2$ if and only if $A_1(x) = A_2(x)$, $\forall x \in S$.
2. $A_1 \subseteq A_2$ if and only if $A_1(x) \leq A_2(x)$, $\forall x \in S$.

If $A_1 \subseteq A_2$ and there exists $x \in S$ such that $A_1(x) < A_2(x)$, then $A_1$ is called a proper F-subset of $A_2$.

3. $x_t \subseteq A$ if and only $x_t(y) \leq A(y)$, $\forall y \in S$ and if $t > 0$ then $A(x) \geq t$. Thus $x_t \subseteq A(x \in A_t)$, (that is $x \in A_t$ if and only if $x_t \subseteq A$)

Definition 1.5 [5]:

Let $A_1, A_2$ are F-sets in $S$, then:

1. $(A_1 \cup A_2)(x) = \max\{A_1(x), A_2(x)\}$, $\forall x \in S$.
2. $(A_1 \cap A_2)(x) = \min\{A_1(x), A_2(x)\}$, $\forall x \in S$.

$A_1 \cup A_2$ and $A_1 \cap A_2$ are F-sets in $S$.

In general if $\{A_\alpha, \alpha \in \Lambda \}$, is a family of F-sets in $S$, then:

$$\bigcap_{\alpha \in \Lambda} A_\alpha (x) = \inf\{A_\alpha(x), \alpha \in \Lambda\}, \text{for all } x \in S.$$  

$$\bigcup_{\alpha \in \Lambda} A_\alpha (x) = \sup\{A_\alpha(x), \alpha \in \Lambda\}, \text{for all } x \in S.$$  

Now, we give the definition of level subset, which is a set between F-set and ordinary set.

Definition 1.6 [6]:

Let $A$ be a F-set in $S$. For $t \in I$, the set $A_t = \{x \in S, A(x) \geq t\}$ is called level subset of $X$.

The following are some properties of the level subset:

Remark 1.7 [1]:

Let $A, B$ are F-subsets of $S$, $t \in I$, then:

1. $(A \cap B)_t = A_t \cap B_t$.
2. $(A \cup B)_t = A_t \cup B_t$.
3. $A = B$ if and only if $A_t = B_t$, for all $t [0,1]$. 
Definition 1.8 [7]:
Let \( f \) be a mapping from a set \( \mathcal{M}_1 \) into a set \( \mathcal{M}_2 \), let \( A \) be a F-set in \( \mathcal{M}_1 \) and \( B \) be a F-set in \( \mathcal{M}_2 \). The image of \( A \) denoted by \( f(A) \) is the F-set in \( \mathcal{M}_2 \) defined by:
\[
f(A)(y) = \begin{cases} \sup \{ A(z) | z \in f^{-1}(y) \} & \text{if } f^{-1}(y) \neq \emptyset, \text{ for each } y \in \mathcal{M}_2 \\ 0 & \text{o.w.} \end{cases}
\]
where \( f^{-1}(y) = \{ x : f(x) = y \} \).
And the inverse of \( B(x) \), denoted by \( f^{-1}(B) \) is the F-set in \( \mathcal{M}_1 \) defined by:
\[
f^{-1}(B) = B \left( f \left( \mathcal{M}_1 \right) \right), \text{ for all } x \in \mathcal{M}_1.
\]

Definition 1.9 [8]:
Let \( f \) be a function from a set \( \mathcal{M}_1 \) into a set \( \mathcal{M}_2 \). A F-subset \( A \) of \( \mathcal{M}_1 \) is said \( f \)-invariant if \( A(x) = A(y) \), whenever \( f(x) = f(y) \), where \( x, y \in \mathcal{M}_1 \).

Proposition 1.10 [8]:
If \( f \) is a function defined on a set \( \mathcal{M} \), \( A_1 \) and \( A_2 \) are F-subsets of \( \mathcal{M} \), \( B_1 \) and \( B_2 \) are F-subset of \( f(\mathcal{M}) \). The followings are true:
1. \( A_1 \subseteq f^{-1}(f(A_1)) \).
2. \( A_1 = f^{-1}(f(A_1)) \), whenever \( A_1 \) is \( f \)-invariant.
3. \( f(f^{-1}(B_1)) = B_1 \).
4. If \( A_1 \subseteq A_2 \), then \( f(A_1) \subseteq f(A_2) \).
5. If \( B_1 \subseteq B_2 \), then \( f^{-1}(B_1) \subseteq f^{-1}(B_2) \).
6. Let \( f \) be a function from a set \( \mathcal{M} \) into \( \mathcal{N} \). If \( B_1 \) and \( B_2 \) are F-subsets of \( \mathcal{N} \), then \( f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2) \) [9].

Definition 1.11 [2]:
A said F-set \( X \) is F-module of an \( \mathcal{R} \)-module \( \mathcal{M} \) if:
1. \( X(v - \mu) \geq \min \{ X(v), X(\mu) \} \), \( \forall v, \mu \in \mathcal{M} \).
2. \( X(rv) \geq X(v) \), \( \forall v \in \mathcal{M} \) and \( r \in \mathcal{R} \).
3. \( X(0) = 1 \) (0 is the zero element of \( \mathcal{M} \)).

Definition 1.12 [3]:
Let \( X_1, X_2 \) are F-modules of an \( \mathcal{R} \)-module \( \mathcal{M} \). \( X_2 \) is a said F-submodule of \( X_1 \) if \( X_2 \subseteq X_1 \)."

Definition 1.15 [3]:
Let \( A \) be a F-set in \( \mathcal{M}_1 \), then we define:
1. \( A^* = \{ x \in \mathcal{M}_1 : A(x) > 0 \} \) is called support of \( A \), also \( A^* = \bigcup A_t, t \in (0,1] \).
2. \( A_* = \{ x \in \mathcal{M}_1 : A(x) = 1 = A(0_{\mathcal{M}_1}) \} \).
Definition 1.16 [12]:
A F-submodule $A$ of a F-module $X$ is called an essential (briefly $A \leq_e X$), if $A \cap B \neq 0$, for any non-trivial F-submodule $B$ of $X$.

2. Weak Essential Fuzzy Submodules

Mona in [4] introduced the concept of weak essential submodule, where a submodule $H$ of $\mathcal{M}$ is a said weak essential, if $H \cap L \neq (0)$, for each non-zero semiprime submodule $L$ of $\mathcal{M}$, where a submodule $N$ of an $\mathcal{R}$-module $\mathcal{M}$ is called semiprime if for each $r \in \mathcal{R}$ and $m \in \mathcal{M}$, if $r^2x \in N$, then $rx \in N$ [13]. We shall fuzzify this concept.

Definition 2.1 [14]:
Let $A$ be F-submodule of F-module $X$ is a said a semiprime $\mathcal{F}$-submodule if $r_t^k a_s \subseteq A$, for F-singleton $r_t$ of $\mathcal{R}$, $a_s \subseteq X$, $k \in \mathbb{Z}_+$, then $r_t a_s \subseteq A$. Equivalently, $A$ is semiprime F-submodule if $r_t^2 a_s \subseteq A$ for $a_s \subseteq X$ and $r_t$ a F-singleton of $\mathcal{R}$, then $r_t a_s \subseteq A$.

Definition 2.2:
Let $A_1$ be F-submodule of F-module $X$. $A_1$ is a said weak essential F-submodule if $A_1 \cap S \neq 0$, for each non-trivial semiprime F-submodules of $X$. Equivalently F-submodule $A$ of a F-module $X$ is called weak essential F-submodule if $A \cap S = 0$, then $S = 0$, for every semiprime F-submodule of $X$.

Next, proposition is a characterization of a weak essential F-submodule.

Proposition 2.3:
Let $X$ be a F-module and $A$ a non-trivial F-submodule of $X$ is a weak essential F-submodule if and only if for each non-trivial semiprime F-submodule $S$ of $X$, there exists $x_t \subseteq S$ and $r_t$ of $\mathcal{R}$, such that $x_t r_t \subseteq A$, $\forall t \in (0,1]$.

Proof:
Suppose that non-trivial semiprime F-submodule $S$ of $X$, there exists $x_t \subseteq S$ and $r_t$ of $\mathcal{R}$, such that $0_1 \neq x_t r_t \subseteq A$. Note that $x_t r_t \subseteq S$.

Conversely, $A$ is weak essential F-submodule, then $A \cap S \neq 0$, for each non-trivial semiprime F-submodule $S$ of $X$. Thus, there exists $0_1 \neq x_t \subseteq A \cap S$, implying that $x_t \subseteq A$ and hence $0_1 \neq x_t r_t \subseteq A$, $\forall t \in (0,1]$.

Now, we give the following Lemma, which we will need in proving the next result.

Lemma 2.4:
Let $A$ be a F-submodule of a F-module $X$ if $A_t$ weak essential submodule of $X_t$, $\forall t \in I$. Then $A$ is weak essential F-submodule in $X$.

Proof:
Assume $B$ a semiprime F-submodule of $X$ such that $B \neq 0_1$, since $B$ semiprime F-submodule of $X$, hence $B_t$ semiprime submodule of $X_t$, $\forall t \in (0,1]$, see [14, Theorem(2.4)], which implies $A_t \cap B_t \neq (0)$, since $A_t$ is weak essential submodule and $A_t \cap B_t = (A \cap B)_t \neq (0)$, hence $A \cap B \neq 0_1$ by Remark (1.7)(3). Thus, $A$ is a weak essential F-submodule of $X$.

Remark 2.5:
Every essential F-submodule is weak essential F-submodule. But the converse is not true in general, for example:

Example:
Let $\mathcal{M} = Z_{36}$ as Z-module. Define $X : \mathcal{M} \rightarrow I$, by:
X(a) = 1, for all \( a \in Z_{36} \)

Let \( A: \mathcal{M} \to I \), define by: 
\[
A(x) = \begin{cases} 
1 & \text{if } x = 0 \\
1/2 & \text{if } x \in (\bar{9}) - (0) \\
0 & \text{otherwise}
\end{cases}
\]

It is clear that \( A \) F-submodule of \( X \), \( A_1 = (\bar{9}) \) is weak essential by [4, Remarks(1.5)], then \( A \) is weak essential F-submodule by Lemma(2.4). Let

\[
B: \mathcal{M} \to I, \text{ as defined by: } B(x) = \begin{cases} 
1 & \text{if } x = 0 \\
1/2 & \text{if } x \in (\bar{4}) - (0) \\
0 & \text{otherwise}
\end{cases}
\]

It is clear that \( B \) F-submodule of \( X \). \( A \) is not essential, since

\[
A \cap B(x) = \begin{cases} 
1 & \text{if } x = 0 \\
0 & \text{otherwise}
\end{cases}
\]

\( A \cap B = 0_1 \) and \( B \neq 0_1 \); therefore \( A \) is not essential F-submodule.

**Remark 2.6:**

The converse of Lemma (2.4) is not true in general.

**Example 2.7:**

Let \( \mathcal{M} = Z_6 \) as Z-module. Define \( X: \mathcal{M} \to I \), \( A: \mathcal{M} \to I \) by:
\[
X(a) = \begin{cases} 
1 & \text{if } a = 0 \\
1/2 & \text{if } a = 2, 4 \\
0 & \text{otherwise}
\end{cases}
\]

, \( A(a) = \begin{cases} 
1 & \text{if } a = 0 \\
1/3 & \text{if } a = 2, 4 \\
0 & \text{otherwise}
\end{cases}
\]

A is an essential F-submodule, then \( A \) is weak essential by Remark (2.5), but \( A_1 = (0) \) is not essential see [15, Remark (2.1)]. Also \( A_1 \) is not weak essential, since \( A_1 \cap S = (0) \), where \( S \) any semiprime submodule. Therefore \( A_1 \) is not weak essential of \( X_1 \).

**Proposition 2.8:**

Let \( A \) be a F-submodule of a F-module \( X \), then \( A \) is weak essential in \( X \) iff \( A_* \) is weak essential submodule in \( X_* \).

Proof:

Let \( A_* \) is a weak essential submodule in \( X_* \). To show \( A \) is weak essential F-submodule in \( X \).

Assume that \( S \) is semiprime F-submodule of \( X \) and \( A \cap S = 0_1 \), then \((A \cap S)_* = (0)\), implies that \( A_* \cap S_* = (0) \). But \( S \) is semiprime F-submodule, then \( S_* \) is semiprime see [14, Theorem (2.4)], so \( S_* \) is semiprime, hence \( S_* = (0) \), so \( S = 0_1 \). Thus, \( A \) is weak essential F-submodule in \( X \).

Conversely, let \( A \) is a weak essential F-submodule in \( X \), we have to show that \( A_* \) is weak essential submodule in \( X_* \).

Let \( N \) is semiprime submodule of \( X_* \), and \( A_* \cap N = (0) \), we must prove \( N = (0) \).

Define \( B: \mathcal{M} \to I \) by:
\[
B(x) = \begin{cases} 
1 & \text{if } x \in N \\
0 & \text{otherwise}
\end{cases}
\]

It is clear that \( B \) F-submodule of \( X \), \( B_* = N \), so \( A_* \cap B_* = (0) \), then \((A \cap B)_* = (0)\), hence by Remark(1.7)(3), \( A \cap B = 0_1 \) and \( B = 0_1 \), since \( A \) is weak essential F-submodule in \( X \), so \( B_* = (0) \); therefore
N = (0). Thus \( A \) is weak essential submodule in \( X \).

**Remarks 2.9:**

1. Let \( A, B \) are \( F \)-submodules of \( X \) such that \( A \subseteq B \) and \( B \) is weak essential \( F \)-submodule of \( X \), then \( A \) need not be weak essential \( F \)-submodule for example:

   Let \( \mathcal{M} \) be as \( Z \)-module \( Z_{36} \). Let \( X : \mathcal{M} \rightarrow I \), define by:
   \[ X(a) = 1, \quad \text{for all} \quad a \in Z_{36}. \]

   Define \( A : \mathcal{M} \rightarrow I \), \( B : \mathcal{M} \rightarrow I \) by:
   \[ A(x) = \begin{cases} 1 & \text{if } x \in \overline{18} \\ 0 & \text{otherwise} \end{cases}, \quad B(x) = \begin{cases} 1 & \text{if } x \in \overline{2} \\ 0 & \text{otherwise} \end{cases} \]

   It is clear that \( X_t = Z_{36} \) and \( A, B \) are \( F \)-submodules of \( X \).

   \( B_t \) a weak essential submodule in \( X_t \) see [4, Remarks(1.5)]. Thus \( B \) is weak essential \( F \)-submodule of \( X \) by Lemma (2.4). Let \( C : \mathcal{M} \rightarrow I \), as defined by:
   \[ C(x) = \begin{cases} 1 & \text{if } x \in \overline{12} \\ 0 & \text{otherwise} \end{cases}, \quad \text{where } C \text{ semiprime } F \text{-submodule} \]

   \( C_t = \overline{12} \), is semiprime submodule of \( X_t \) (\( \forall t > 0 \)). But \( A \cap C = 0_t \), therefore \( A \) is not weak essential \( F \)-submodule of \( X \).

2. Let \( A, B \) are \( F \)-submodule such that \( A \subseteq B \). If \( A \) is weak essential \( F \)-submodule in \( X \) implying \( B \) is a weak essential \( F \)-submodule of \( X \).

   Proof:
   Assume that \( B \cap S = 0_t \), for some semi-prime \( F \)-submodule \( S \) of \( X \), then \( A \cap S = 0_t \). But \( A \) is weak essential \( F \)-submodule, hence \( S = 0_t \). That is \( B \) is weak essential \( F \)-submodule of \( X \).

3. Let \( A, B \) be are \( F \)-submodules of \( F \)-module \( X \) if \( A \cap B \) a weak essential \( F \)-submodule of \( X \), then both of \( A \) and \( B \) are weak essential \( F \)-submodules of \( X \).

   Proof:
   It is clear by (2).

   Note that, the converse is not true in general, for example:

**Example:**

Let \( \mathcal{M} \) be \( Z_{36} \) as \( Z \)-module. Define \( X : \mathcal{M} \rightarrow I \) by:
\[ X(a) = 1, \quad \text{for all} \quad a \in Z_{36}. \]

Let \( A : \mathcal{M} \rightarrow I \), \( B : \mathcal{M} \rightarrow I \), define by:
\[ A(x) = \begin{cases} 1 & \text{if } x \in \overline{12} \\ 0 & \text{otherwise} \end{cases}, \quad B(x) = \begin{cases} 1 & \text{if } x \in \overline{18} \\ 0 & \text{otherwise} \end{cases} \]

Clearly \( A, B \) are \( F \)-submodules of \( X \), \( A_t = \overline{12} \), \( B_t = \overline{18} \), \( \forall t \in (0,1] \) are weak essential submodules of \( X_t \) by [4, Remark(1.5)]. Hence \( A, B \) are weak essential \( F \)-submodules of \( X \); see Lemma(2.4). But \( A \cap B = 0_t \); that is \( A \cap B \) is not weak essential \( F \)-submodule of \( X \).

Under some conditions the converse (3) will be true as in the following proposition.

**Proposition 2.10:**

Let \( A, B \) are \( F \)-submodules of \( F \)-module \( X \) such that \( A \) is an essential \( F \)-submodule, \( B \) weak essential \( F \)-submodule, then \( A \cap B \) is a weak essential \( F \)-submodule of \( X \).

Proof:
Suppose $S$ is a non-trivial semiprime $F$-submodule of $X$, but $B$ is weak essential $F$-submodule of $X$, hence $B \cap S \neq 0$. So $A$ is an essential $F$-submodule of $X$ and we have $A \cap (B \cap S) = (A \cap B) \cap S \neq 0_1$,
Hence, $A \cap B$ is weak essential $F$-submodule of $X$.

**Lemma 2.11:**
If $S$ is a semiprime $F$-submodule of $F$-module $X$, $B$ be a $F$-submodule of $X$ such that $B \not\subseteq S$, then $S \cap B$ is a semiprime $F$-submodule in $B$.

**Proof:**
Let $S$ be a semiprime $F$-submodule of $X$, then by \[14,\text{Theorem(2.4)}\], $S$ is a semiprime submodule and $B$ is a submodule of $X$; see Proposition (1.14) such that $B \not\subseteq X_t$, then by \[13,\text{Proposition(1.11)}\], $S_t \cap B_t = (S \cap B)_t$; see Proposition (1.7)(1) is a semiprime submodule in $B_t$, therefore $S \cap B$ is a semiprime $F$-submodule in $B$; see \[14, \text{Theorem(2.4)}\].

In the following proposition, we prove the transitive property for non-trivial $F$-submodule.

**Proposition 2.12:**
Let $A, B$ be a non-trivial $F$-submodules of $F$-module $X$ such that $A \subseteq B$. If $A$ is a weak essential $F$-submodule in $B$ and $B$ is a weak essential $F$-submodule in $X$ implying $A$ is a weak essential $F$-submodule in $X$.

**Proof:**
Assume that $S$ is a semiprime $F$-submodule in $X$, such that $A \cap S = 0_1$. Note that $0_1 = A \cap S = (A \cap S) \cap B = A \cap (S \cap B)$. But $S$ is a semi-prime $F$-submodule of $X$, so we have two cases. If $B \subseteq S$, then $0_1 = A \cap (S \cap B) = A \cap B$. Hence, $A \cap B = 0_1$, but $A \subseteq B$ so $A \cap B = A$ implies $A = 0_1$ which is a contradiction with our assumption. Thus $B \not\subseteq S$ and by Lemma (2.11), $S \cap B$ is a semiprime $F$-submodule in $B$. Since $A$ is a weak essential $F$-submodule in $B$, therefore $S \cap B = 0_2$ and since $B$ is a weak essential $F$-submodule in $X$, then $S = 0_1$, then $A$ is a weak essential $F$-submodule in $X$.

Now, we study a homomorphic image of a weak essential $F$-submodule.

**Proposition 2.13:**
Let $X_1, X_2$ be $F$-modules of an $\mathcal{R}$-module $\mathcal{M}_1$ and $\mathcal{M}_2$ resp. and $f : X_1 \rightarrow X_2$ be $F$-epimorphism. If $A_1$ is a weak essential $F$-submodule of $X_1$ such that $A_1$ is $f$-invariant, then $f(A_1)$ is a weak essential $F$-submodule of $X_2$.

**Proof:**
To show $f(A_1)$ is a weak essential $F$-submodule of $X_2$, since $A_1$ is a $F$-submodule of $X_1$, then $f(A_1)$ is a $F$-submodule of $X_2$ by Proposition (1.13)(1). Now suppose that $S$ semiprime $F$-submodule of $X_2$ such that $f(A_1) \cap S = 0_1$; therefore $f^{-1}(f(A_1) \cap S) = f^{-1}(0_1)$, then $f^{-1}(f(A_1)) \cap f^{-1}(S) = 0_1$, see Proposition (1.10)(2). But $A_1$ is $f$-invariant implying that $A_1 \cap f^{-1}(S) = 0_1$, and $f^{-1}(S) = 0_1$, since $A_1$ is weak essential $F$-submodule and $f^{-1}(S)$ is $F$-submodule of $X_1$ by Proposition (1.13)(2). $f(f^{-1}(S)) = f(0_1)$, then $S = 0_1$, by Proposition (1.10)(3). That is $f(A_1)$ is a weak essential $F$-submodule.

Now, we consider the inverse image of a weak $F$-submodule.
Proposition 2.14:

Let $X_1, X_2$ are $F$-modules of an $\mathcal{R}$-module $\mathcal{M}_1$ and $\mathcal{M}_2$ resp. and $f : X_1 \rightarrow X_2$ be $F$-epimorphism. If $A_2$ is weak essential $F$-submodule of $X_2$, then $f^{-1}(A_2)$ is a weak essential $F$-submodule of $X_1$.

Proof:

Since $A_2$ $F$-submodule of $X_2$, then $f^{-1}(A_2)$ is $F$-submodule of $X$ see Proposition(1.13)(2). Now suppose $S$ is semiprime $F$-submodule of $X_1$, such that $f^{-1}(A_2) \cap S = 0_1$, hence $f(f^{-1}(A_2) \cap S) = f(0_1)$, implies that $f(f^{-1}(A_2)) \cap f(S) = f(0_1)$ see Proposition (1.10)(6). $A_2 \cap f(S) = 0_1$ (since $A_2$ is $f$-invariant and $f$ is epimorphism), then $f^{-1}(f(S)) = f^{-1}(0_1)$, implies that $S = 0_1$, since every $F$-submodule of $X_1$ is $f$-invariant, implies $f^{-1}(A_2)$ is weak essential $F$-submodule of $X_1$.

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