Entanglement dynamics of two qubits under the influence of external kicks and Gaussian pulses

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Abstract
We have investigated the dynamics of entanglement between two spin-1/2 qubits that are subject to independent kick and Gaussian pulse-type external magnetic fields analytically and numerically. We show that 'almost-steady' high entanglement can be created between two initially unentangled qubits by using carefully designed kick or pulse sequences.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction
Control and manipulation of entanglement which is a quantifiable resource for quantum information tasks such as quantum computing [1], communication [2] and cryptography [3] have been studied along many directions in the last decade [4–13]. Among these studies, systems that are modelled as a 1D Heisenberg chain with qubits of spin-1/2 particles as the main unit are one of the prototypical examples [14]. For such systems, the control of entanglement between the qubits can be manipulated with the help of various types of external magnetic fields [4–13]. In particular, Heule et al investigated the feasibility of a local operator control in arrays of interacting qubits modelled as isotropic Heisenberg spin chains [4]. Along similar lines, Caneva et al explored optimal quantum control by appropriate pulses to affect the required transformations by the numerical Krotov algorithm [5]. Wu et al showed that one qubit gate can be constructed with global magnetic fields and controllable Heisenberg exchange interactions [6]. Levy demonstrated a scheme that uses pairs of spin-1/2 particles to form logic qubits and Heisenberg exchange only to produce all gate operations [7]. Malinovsky and Sola studied phase control of entanglement in two qubit systems and showed that by changing the relative phase of control pulses, one can control entanglement at will [8]. Sadiek et al studied the control and manipulation of entanglement evolution for a two-qubit system coupled through the XYZ Heisenberg interaction influenced by a time-varying external field [9]. Wang et al demonstrated that near perfect entanglement can be obtained by applying
a magnetic field on a single spin of an isotropic Heisenberg chain of length \( N \) [10]. Abliz et al studied the entanglement dynamics for a two-qubit Heisenberg XXZ model effected by population relaxation in the presence of various types of magnetic fields and showed that it is possible to produce, control and modulate high entanglement with the help of time-dependent external fields despite the existence of dissipation [11].

For a general time-dependent external field, time-ordering effects might be important and the dynamics cannot be found analytically. Most of the aforementioned studies employ numerical methods to investigate the entanglement control [4, 5, 8–10]. Although the numerical methods are fast and reliable, analytic solutions provide a clearer picture of the physics behind the dynamics. For a single qubit, the time evolution of populations and coherence under the influence of an external field in the form of a Gaussian pulse or a delta function kick was investigated by Kaplan et al and Shakov et al [15, 16]. The fast pulse or kick is defined based on the relation between the energy splitting of the qubit \( \Delta E \) and the duration of the pulse \( \tau \); if \( \Delta E \tau \ll 1 \), then the pulse is called a kick [15, 16]. Many experimental and theoretical studies have been carried out to investigate the dynamics of two-level quantum systems under the influence of a single or a series of kicks for quantum gates [17, 18], NMR [19], excitation of electronic states in molecules [20], chemical reactions [21] and quantum computing [22]. However, there have been no studies on the control of entanglement between two qubits by using fast pulses, to the best of our knowledge.

Fast pulses provide an efficient way of full population transfer in a qubit [15, 16] and because of that are expected to be an important way of controlling the entanglement. From this point, in this study, we consider two qubits with a Heisenberg XXX-type interaction. Each qubit is under the influence of a local time-dependent magnetic field that acts in the \( z \)-direction. We consider one, two, three and four kicks as well as Gaussian pulse sequences and their effect on the dynamics of entanglement between the qubits. We show that entanglement can be controlled by a careful design of the sequence of kicks.

The organization of this paper is as follows. In section 2, we introduce the model and basic formulation necessary to solve time evolution exactly. In section 3, we discuss the time-ordering effect on the dynamics. In section 4, the Wootters concurrence as an entanglement measure is briefly introduced. In section 5, the analytic entanglement dynamics of kicked qubits in the presence of time ordering is discussed by choosing single and multiple (up to four) kicks. In section 6, the effect of finite pulse width on the entanglement dynamics is studied numerically by choosing a Gaussian pulse or pulse sequence as an external field. We conclude with a summary of the important results in section 7.

2. The model and basic formulation

In this paper, we consider two Heisenberg XXX-coupled qubits in a time-dependent external magnetic field acting in the \( z \)-direction. The typical time-dependent Hamiltonian for this system may be expressed as [9] (we set \( \hbar = 1 \))

\[
\hat{H}(t) = \hat{H}_0 + \hat{H}_{\text{int}}(t),
\]

where

\[
\hat{H}_0 = J \sum_{i=x, y, z} \hat{\sigma}_i^1 \hat{\sigma}_i^2,
\]

\[
\hat{H}_{\text{int}}(t) = - \sum_{i=1}^{2} B_i(t) \hat{\sigma}_i^z.
\]
where $\sigma^{1,2,3}(i = x, y, z)$ are the usual Pauli spin matrices, $J$ is the qubit–qubit interaction strength and $B^1(t)$ and $B^2(t)$ are the time-dependent magnetic fields acting on qubits 1 and 2, respectively. It should be noted that the qubit–qubit interaction term in equation (1) is given by $H_0$ which is constant in time, and the time-dependent part of the total Hamiltonian is called $H_{int}(t)$ which describes the qubit–magnetic field interaction and is assumed to be a single real function of $t$.

The most general form of an initial pure state of the two-qubit system is $|\Psi(0)\rangle = a_1(0)|11\rangle + a_2(0)|10\rangle + a_3(0)|01\rangle + a_4(0)|00\rangle$, where $a_i(0)(i = 1, 2, 3, 4)$ are complex numbers with $\sum_{i=1}^{4}|a_i(0)|^2 = 1$; then the probability amplitudes evolve in time under Hamiltonian (1) according to the Schrödinger equation as

$$\frac{d}{dt} \begin{bmatrix} a_1(t) \\ a_2(t) \\ a_3(t) \\ a_4(t) \end{bmatrix} = \begin{bmatrix} J - B_T(t) & 0 & 0 & 0 \\ 0 & -J + \Delta B(t) & 2J & 0 \\ 0 & 2J & -J - \Delta B(t) & 0 \\ 0 & 0 & 0 & J + B_T(t) \end{bmatrix} \begin{bmatrix} a_1(t) \\ a_2(t) \\ a_3(t) \\ a_4(t) \end{bmatrix}. \quad (3)$$

where $\Delta B(t) = B^2(t) - B^1(t)$ and $B_T(t) = B^1(t) + B^2(t)$. The formal solution of equation (3) may be written in terms of the time-evolution matrix $\hat{U}(t)$ as

$$\begin{bmatrix} a_1(t) \\ a_2(t) \\ a_3(t) \\ a_4(t) \end{bmatrix} = \hat{U}(t) \begin{bmatrix} a_1(0) \\ a_2(0) \\ a_3(0) \\ a_4(0) \end{bmatrix}. \quad (4)$$

The evolution operator for the general time-dependent Hamiltonian of two qubits is not easy to obtain analytically; a number of systematic procedures are obtained in [23] and [24] based on dynamical groups of the system when time ordering is not important. Here, the time-evolution operator $\hat{U}(t)$ may be expressed as

$$\hat{U}(t) = \hat{T} e^{-i\int_0^t \hat{H}(t') dt'} = \hat{T} e^{-i\int_0^t (\hat{H}_0 + \hat{H}_{int}(t')) dt'}$$

$$= \hat{T} \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_0^t \hat{H}(t_n) \, dt_n \cdots \int_0^t \hat{H}(t_2) \, dt_2 \int_0^t \hat{H}(t_1) \, dt_1. \quad (5)$$

The only non-trivial time dependence on $\hat{U}(t)$ arises from time dependent $\hat{H}(t)$ and time ordering $\hat{T}$. The Dyson time-ordering operator $\hat{T}$ specifies that $\hat{H}(t_j)\hat{H}(t_j)$ is properly ordered [15, 16, 25]:

$$\hat{T}\hat{H}(t_i)\hat{H}(t_j) = \hat{H}(t_i)\hat{H}(t_j) + \hat{\theta}(t_j - t_i)[\hat{H}(t_j), \hat{H}(t_i)], \quad (6)$$

where $\hat{\theta}(t_j - t_i)$ is the Heaviside step function whose value is zero if $(t_j - t_i)$ is negative and 1 if $(t_j - t_i)$ is positive. It should be noted that time ordering imposes a connection between the effects of $\hat{H}(t_i)$ and $\hat{H}(t_j)$ and gives rise to observable, non-local, time-ordering effects when $[\hat{H}(t_j), \hat{H}(t_i)] \neq 0$ [26, 27].

### 3. Time ordering

If one takes $\hat{T} = 1$ in equation (5) to obtain $\hat{U}(t)$, the time evolution is said to contain no time ordering. So the difference between the result obtained by an exact treatment of $\hat{T}$ in equation (5) and $\hat{T} \rightarrow 1$ is called the effect of time ordering on the dynamics [15, 16]. One should note that removing time ordering as $\hat{T} \rightarrow 1$ corresponds to the zeroth order term in an eikonal-like Magnus expansion in the commutator terms [28].
3.1. Limit of no time ordering

Replacing $\hat{T}$ with 1 in equation (5), in the Schrödinger picture we have

$$
\hat{U}(t) = \hat{T} e^{-i\int_0^t \hat{H}(t') dt'} \to \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \left[ \int_0^t \hat{H}(t') dt' \right]^n
$$

$$
= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \left[ \hat{H}_0 t + \int_0^t \hat{H}_{\text{int}}(t') dt' \right]^n = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} [(\hat{H}_0 + \hat{H}_{\text{int}}) t]^n
$$

$$
= e^{-i\hat{H}t} = \hat{U}^0(t),
$$

(7)

where $\hat{H}_{\text{int}} = \int_0^t \hat{H}_{\text{int}}(t') dt'$ is the averaged interaction field, and $\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}$ and $[\hat{H}_0, \hat{H}_{\text{int}}]$ terms are non-zero. By expanding in powers of $[\hat{H}(t''), \hat{H}(t')]$, it is straightforward to show that to leading order in $\hat{H}_{\text{int}}(t)$ and $\hat{H}_0$, the time-ordering effect is given by

$$
\hat{U} - \hat{U}^0 \simeq -\frac{1}{2} \int_0^t dt'' \int_0^{t''} dt' [\hat{H}(t''), \hat{H}(t')] = -\frac{1}{2} [\hat{H}_0, \hat{H}_{\text{int}}] \int_0^t dt' (t - 2t') f(t'),
$$

(8)

where $\hat{H}_{\text{int}}(t') = \hat{H}_{\text{int}}^0 f(t')$. This leading term disappears if the pulse centroid $T_k = t/2$ and $f(t')$ is symmetric about $T_k$. Furthermore, $\hat{U} - \hat{U}^0$ vanishes identically in the special cases of $H_{\text{int}}(t') = 0$, $\hat{H}_{\text{int}}(t') = \hat{H}_{\text{int}}$ [15, 16]. Also the commutator $[\hat{H}(t''), \hat{H}(t')]$ (i.e. the time-ordering effect) vanishes for $B_1^z(t) = B_2^z(t)$ or $J = 0$ because by using the total Hamiltonian (1), we have

$$
[\hat{H}(t''), \hat{H}(t')] = 2J \{(B_1^z(t') - B_2^z(t')) - (B_1^z(t'') - B_2^z(t''))\} (\hat{\sigma}_+ \hat{\sigma}_- - \hat{\sigma}_- \hat{\sigma}_+),
$$

(9)

which vanishes when either the time-dependent magnetic fields on qubits 1 and 2 are equal or the qubit–qubit interaction is neglected.

In general, there is no simple analytic form for the exact result $\hat{U}(t)$, except for special cases [9, 15, 16]. For the result without time ordering with averaged magnetic fields $\bar{B}_1^z(t) = \int_0^t B_1^z(t') dt' = \alpha$ and $\bar{B}_2^z(t) = \int_0^t B_2^z(t') dt' = \beta$, the time-evolution matrix $\hat{U}^0(t)$ in equation (7) can be easily calculated as

$$
\hat{U}^0(t) = e^{-i(\hat{H}_0 t + \hat{H}_{\text{int}} t)}
$$

$$
= \begin{bmatrix}
y y^* & 0 & 0 & 0 \\
0 & y(u + iv) & y(-w + iz) & 0 \\
0 & y(w + iz) & y(u - iv) & 0 \\
0 & 0 & 0 & y^*_y y^*
\end{bmatrix},
$$

(10)

where

$$
y = e^{i\theta'},
$$

$$
y_1 = e^{i\omega + \beta},
$$

$$
u = \cos (\Gamma),
$$

$$
v = \frac{(\alpha - \beta)}{\Gamma} \sin (\Gamma),
$$

$$
w = 0,
$$

$$
z = -\frac{2Jt}{\Gamma} \sin (\Gamma),
$$

(11)

where $\Gamma = \sqrt{4J^2 t^2 + (\alpha - \beta)^2}$ and $\alpha$ and $\beta$ are called the integrated magnetic strengths associated with the magnetic fields acting on qubits 1 and 2, respectively.

Similarly, the time-evolution matrix without time ordering in the interaction picture can be studied [15, 16]. However, in [15] and [16], it was shown that the occupation probabilities
for a kicked qubit for the dynamics in the limit $\hat{T} \to 1$ depend on the chosen picture. On the other hand, the exact results (the results including time ordering) are independent of the chosen picture, as expected. Thus in the following sections, we will discuss the entanglement dynamics of kicked qubits by using exact time-ordered results, and we will work in the Schrödinger representation. The matrix in equation (10) will be used to check the correctness of our results, because as mentioned before, when $B_2(t) = B_0(t)$ or $J = 0$, the time-ordering effect defined as $\hat{U}^B(t) - \hat{U}^0(t)$ vanishes.

4. Measure of entanglement

For a pair of qubits, the Wootters concurrence can be used as a measure of entanglement [29]. The concurrence function varies from $C = 0$ for a separable state to $C = 1$ for a maximally entangled state. To calculate the concurrence function, one needs to evaluate the matrix

$$
\hat{R} = \hat{\rho}(t)(\hat{\sigma}_y \otimes \hat{\sigma}_y)\hat{\rho}^*(t)(\hat{\sigma}_y \otimes \hat{\sigma}_y),
$$

(12)

where $\hat{\rho}(t)$ is the density matrix of the system and $\hat{\rho}^*(t)$ is its complex conjugate. Then the concurrence is defined as

$$
C(\hat{\rho}) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\},
$$

(13)

where $\lambda_1, \lambda_2, \lambda_3$ and $\lambda_4$ are the positive square roots of the eigenvalues of $\hat{R}$ in a descending order.

It should be noted that due to the discrete symmetry (conservation of parity under flipping of the $\hat{\sigma}_i$, $i = x, y, z$, and $j = 1, 2$, i.e. when $\hat{\sigma}_i \to -\hat{\sigma}_i$) of the total Hamiltonian (1), the states $|\Phi\rangle = a_2|10\rangle + a_3|01\rangle$ and $|\Psi\rangle = a_1|11\rangle + a_4|00\rangle$ can never get mixed in time due to that symmetry [9], as can be seen from equation (3). Thus, we consider the time evolution of the concurrence of these states individually. The concurrence function for a pure state $|\Phi(t)\rangle = a_2(t)|10\rangle + a_3(t)|01\rangle$ with the density matrix $\hat{\rho}(t) = |\Phi(t)\rangle\langle\Phi(t)|$ is given by

$$
C(\hat{\rho}) = \max\{0, 2|a_2(t)a_3(t)|\}.
$$

(14)

Similarly, for the pure state $|\Psi(t)\rangle = a_1(t)|11\rangle + a_4(t)|00\rangle$ with the density matrix $\hat{\rho}(t) = |\Psi(t)\rangle\langle\Psi(t)|$, the concurrence function reads

$$
C(\hat{\rho}) = \max\{0, 2|a_1(t)a_4(t)|\},
$$

(15)

where according to equation (4), the time-dependent coefficients read

$$
a_4(t) = \sum_{j=1}^{4} U_{ij}(t)a_j(0),
$$

(16)

where $U_{ij}(t)(i, j = 1, 2, 3, 4)$ are the matrix elements of $\hat{U}(t)$.

5. Entanglement dynamics of kicked qubits

In this section, we will examine the entanglement dynamics of kicked qubits by taking into account the time-ordering effects for the initially pure separable $|\Psi(0)\rangle = |01\rangle$ and maximally entangled $|\Phi(0)\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$ Bell states. We will work in the Schrödinger picture and present analytic expressions for the time-evolution operator of the two-qubit system for a single kick as well as a positive followed by a negative kick and a sequence of two, three and four equally distanced kicks. For all kick sequences, we consider two integrated magnetic strength regimes: $\alpha = 2\beta$ and $\alpha = 3\beta$ and for convenience we shall set $J = 1$ and $\beta = 1$. From
those results, we will use equations (14) and (16) to analyse and discuss the time evolution of the entanglement between the qubits.

We have also considered the other Bell states $\frac{1}{\sqrt{2}}(|11\rangle \pm |00\rangle)$ and $\frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$ as well as separable states of two qubits as $|11\rangle$, $|00\rangle$, $|10\rangle$. Under the influence of $\hat{H}(t)$ of equation (1), the states of the type $|00\rangle$ and $|11\rangle$ remain separable, while the entanglement of $\frac{1}{\sqrt{2}}(|11\rangle \pm |00\rangle)$ does not change with time. These may be checked using equation (15) and the solution of the expansion coefficients $a_1(t)$ and $a_2(t)$ in equation (3) for the considered initial states. The dynamics of $\frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$ is the same as that of $\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$ and that of $|01\rangle$ is the same as $|10\rangle$ that are noted after specifying the propagators for kicked qubits and by using equations (14) and (16). So, we consider only $|\Psi(0)\rangle = |01\rangle$ and $|\Phi(0)\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$ as initial states.

One point we want to emphasize is that before the field is active, the propagator is equal to $e^{-iH_0}$ and given by equation (23). Based on equations (14), (16) and (23), the concurrence of the initial state $|\Phi(0)\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$ is equal to 1, while the concurrence of $|\Psi(0)\rangle = |01\rangle$ is $|\sin(4Jt)|$ at time $t$ before the kick. Note that the entanglement dynamics for the initial state $|\Phi(0)\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$ is unperturbed by the qubit–qubit interaction in the absence of the external field, while the qubit–qubit interaction creates a high degree of entanglement that oscillates between 0 and 1 if the qubits are initially prepared in $|\Psi(0)\rangle = |01\rangle$ state.

5.1. Single kick

Here, we consider two qubits whose states are coupled with an interaction field which can be expressed as a sudden ‘kick’ at $t = T_1$, namely $B_1(t) = a\delta(t - T_1)$, $B_2(t) = \beta\delta(t - T_1)$. For such a kick the integration over time is trivial and the time-evolution matrix in equation (5) becomes [15]

$$\hat{U}^K(t) = e^{-i\hat{H}_0(t-T_1)} e^{-i\int_{T_1}^{t} H_{\alpha}(t') dt'} e^{-i\hat{H}_0T_1},$$

(17)

in the same form as equation (10) with elements

$$y = e^{i\beta},$$
$$y_1 = e^{i(\alpha+\beta)},$$
$$u = \cos(2Jt) \cos(\alpha - \beta),$$
$$v = \cos(2J(t-T_1)) \sin(\alpha - \beta),$$
$$w = \sin(2J(t-T_1)) \sin(\alpha - \beta),$$
$$z = -\sin(2Jt) \cos(\alpha - \beta),$$

(18)

for $t > T_1$. The propagator without time ordering is given in equation (10) and as explained before when $\alpha = \beta$ or $J = 0$, the time-ordering effect, $\hat{U}^K(t) - \hat{U}^0(t)$, vanishes after the field is active.

By inserting equation (18) into equations (14) and (16) for the initial states $|\Phi(0)\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$ and $|\Psi(0)\rangle = |01\rangle$, one can obtain the analytic expressions of the concurrence functions after the kick ($t > T_1$). For the maximally entangled state, the concurrence is given as

$$C(\hat{\rho}) = \max\{0, |\cos^2(\Delta) + e^{8J(t-T_1)} \sin^2(\Delta)|\},$$

(19)

while for $|\Psi(0)\rangle = |01\rangle$ the concurrence after $t = T_1$ can be obtained as

$$C(\hat{\rho}) = 2 \max\{0, |\Lambda|\},$$

(20)

where $\Lambda = (\cos(2Jt) \cos(\Delta) - i \cos(\zeta) \sin(\Delta))(i \cos(\Delta) \sin(2Jt) + \sin(\zeta) \sin(\Delta))$, $\Delta = \alpha - \beta$ and $\zeta = 2J(t - 2T_1)$. 
one may consider a sequence of two kicks of opposite sign at times
by the multiplication of several matrices of the form of equation (17) [15, 16]. For example,

The propagator for a sequence of either identical or non-identical kicks can be easily obtained
5.2. A positive followed by a negative kick

The dynamics of concurrence for the initial Bell state and the separable state for the
single kick which are given by equations (19) and (20) for \( t > T_1 \) are displayed in figures
1(a) and (b), respectively. The effect of the kick on the entanglement is pronounced for both
initial states; the concurrence of the Bell state starts oscillating with an amplitude that depends
on the ratio of the integrated magnetic strength of the external fields on qubits 1 and 2 (i.e.
\( \alpha \) and \( \beta \), respectively). The effect of the kick on the system initially in a separable state,
\( |01\rangle \), is similar with the exception that the concurrence variation amplitudes get lower after
the kick. One should also note that \( \mathcal{C}(\hat{\rho}) \) of the initial Bell state is independent of \( J \) before
the kick, while the frequency of its time dependence after the kick is proportional to the
qubit–qubit interaction strength \( J \), as can be seen from figure 1(a) and the analytic expression
equation (18).

Figure 1. Concurrence as a function of dimensionless time, \( J t \), for an ideal positive kick applied at
\( T_1 = 5 \) for the initial pure states \( |\Phi(0)\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) \) (a) and \( |\Psi(0)\rangle = |01\rangle \) (b). The dashed
lines correspond to \( \alpha = 2\beta \) and the solid lines to \( \alpha = 3\beta \).

5.2. A positive followed by a negative kick

The propagator for a sequence of either identical or non-identical kicks can be easily obtained
by the multiplication of several matrices of the form of equation (17) [15, 16]. For example,
one may consider a sequence of two kicks of opposite sign at times \( t = T_1 \) and \( t = T_2 \),
namely, \( B^\dagger_1(t) = \alpha (\delta(t - T_1) - \delta(t - T_2)) \), \( B^\dagger_2(t) = \beta (\delta(t - T_1) - \delta(t - T_2)) \). Following
the procedure given in equation (17), one obtains the time-evolution matrix for \( t > T_2 \) as [15]

\[
\hat{U}^K(t) = e^{-i T_2 H_0} e^{-i T_1 H_0} e^{-i \int_{T_2}^{T_1} dt' H_0} e^{-i \int_{T_1}^{T_2} dt' \Delta} \rho_{\nu_0(t-T_1)} \rho_{\nu_0(t-T_2)} e^{-i T_2 H_0} e^{-i T_1 H_0}, \tag{21}
\]

where the elements of equation (21) are of the same form as equation (10) with parameters
\[
\begin{align*}
y & = e^{i t}, \\
y_1 & = 1, \\
u & = \cos (2Jt) \cos(\Delta)^2 + \cos (2J(t - 2T_1)) \sin(\Delta)^2, \\
u & = (\cos(\xi_1) - \cos(\xi_2)) \sin(\Delta) \cos(\Delta), \\
u & = (\sin(\xi_1) - \sin(\xi_2)) \sin(\Delta) \cos(\Delta), \\
z & = - \sin (2Jt) \cos(\Delta)^2 - \sin (2J(t - 2T_1)) \sin(\Delta)^2, \tag{22}
\end{align*}
\]

where \( \xi_1 = 2J(t - 2T_1) \), \( \Delta = (\alpha - \beta) \) and \( T_i = T_2 - T_1 \). For this case, the time-evolution
matrix without time ordering is given by
because for a positive kick followed by a negative kick the averaged interaction Hamiltonian, $\hat{H}_{\text{int}} = 0$. For the cases $J = 0$, or $T_0 = 0$, or $\alpha = \beta$, the time-ordering effect defined as $\hat{U}^0(t) - \hat{U}^0(t)$ goes to zero, as expected.

The entanglement dynamics under the positive–negative kick sequence at times $t > T_2$ is obtained by using expression equation (22) in equations (14) and (16) and are displayed in figures 2(a) and (b) for the initial Bell and separable states, respectively. The effect of the negative kick at $T_2$ is found to be opposite for the $|\Phi(0)\rangle = |\varphi^+\rangle = (|10\rangle + |01\rangle)/\sqrt{2}$ and $|\Psi(0)\rangle = |01\rangle$ initial states; for the $|01\rangle$ state, the dependence of concurrence on the integrated magnetic strength vanishes while for the initial Bell state, amplitude of concurrence oscillations changes with $\alpha$ and $\beta$. One peculiar result from figure 2(a) is the observation that the negative kick has no influence on the dynamics of concurrence for $\alpha = 3\beta$ magnetic fields (solid line in figure 2(a)). The positive–negative kick sequence also demonstrates the strong effect of time ordering on the entanglement dynamics. As mentioned before, based on the propagator for $\hat{T} \rightarrow 1$ case given by equation (23), the concurrence for the Bell state is always equal to 1, while the concurrence for the initially separable state has oscillations between 0 and 1 for the times $t > T_2$. On the other hand, as can be seen from figures 2(a) and (b), the time-ordered propagator leads to different results in the concurrence for both initial states.

5.3. Two positive kicks

To show the difference between positive and negative kicks applied after the first positive kick on the entanglement dynamics of two qubits, one may consider a sequence of two positive kicks applied at times $t = T_1$ and $t = T_2$, namely, $B^2_1(t) = \alpha (\delta(t - T_1) + \delta(t - T_2))$ and $B^2_2(t) = \beta (\delta(t - T_1) + \delta(t - T_2))$. Following the procedure given in equation (17), one obtains the time-evolution matrix equation (21) for $t > T_2$ in the form as equation (10) with parameters

$$y = e^{i\delta t},$$
$$y_1 = e^{i(\alpha + \beta)} t,$$
$$u = \cos (2Jt) \cos(\Delta) - \cos (2J(t - 2T_1)) \sin(\Delta),$$
$$v = (\cos(\xi_1) + \cos(\xi_2)) \sin(\Delta) \cos(\Delta),$$
$$w = (\sin(\xi_1) + \sin(\xi_2)) \sin(\Delta) \cos(\Delta),$$
$$z = - \sin (2Jt) \cos (\Delta)^2 + \sin (2J(t - 2T_1)) \sin(\Delta)^2.$$  

(24)

where $\xi = 2J(t - 2T_1)$, $\Delta = (\alpha - \beta)$ and $T_0 = T_2 - T_1$. Here, the propagator without time ordering can be calculated by replacing $B^2_1(t) \rightarrow 2\alpha$ and $B^2_2(t) \rightarrow 2\beta$ in equation (10) and note for $\alpha = \beta$ or $J = 0$, $\hat{U}^0(t) - \hat{U}^0(t)$ vanishes, as expected according to equations (8) and (9).

The effect of two consecutive positive kicks on the dynamics of concurrence for two qubits is displayed in figures 3(a) and (b) for the initial Bell state, $|\varphi^+\rangle = (|10\rangle + |01\rangle)/\sqrt{2}$, and separable state, $|01\rangle$, respectively. Comparing the analytic expressions of the time-evolution operators for positive–negative and positive–positive kick sequences of equations (22) and (24), respectively, along with figures 2 and 3, the effect of the sign of the kicks in the sequence is to change the amplitude of the concurrence oscillations. The oscillation amplitude of $C(\tilde{\rho})$ for the Bell state as well as the separable state increases for the positive–positive sequence
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\(y\) in the same form as equation (10) with parameters compared to that of the positive–negative sequence of kicks. Also, the \(\alpha/\beta\) dependence of the amplitude is different, as can be seen from a comparison of figures 2 and 3.

5.4. Three and four positive kicks

One may consider a sequence of \(n\)-positive kicks applied at times \(t = T_1, t = T_2, \ldots, t = T_n\), namely \(B^i(t) = \sum_{i=1}^{n} \alpha \delta(t - T_i), B^j(t) = \sum_{j=1}^{n} \beta \delta(t - T_j)\). For example, following the procedure given in equation (17), one obtains the time-evolution matrix for three positive kicks at times \(t > T_3\) as

\[
\hat{U}^K(t) = e^{-i\hat{H}_0(T_3 - T_2)} e^{-i\int_{T_2-T_3}^T \hat{H}_\alpha(t') dt'} e^{-i\hat{H}_0(T_3 - T_2)} e^{-i\int_{T_2-T_3}^T \hat{H}_\alpha(t') dt'}
\times e^{-i\hat{H}_0(T_2 - T_1)} e^{-i\int_{T_2-T_1}^T \hat{H}_\alpha(t') dt'} e^{-i\hat{H}_0T_1},
\]

in the same form as equation (10) with parameters

\[
y = e^{i\beta},
\]

\[
y_1 = e^{i(\alpha + \beta)},
\]

Figure 2. Concurrence as a function of \(Jt\) for an ideal positive kick applied at \(T_1 = 5\) followed by an ideal negative kick at \(T_2 = 10\) for the initial pure states \(|\Phi(0)\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)\) (a) and \(|\Psi(0)\rangle = |01\rangle\) (b). The dashed lines correspond to \(\alpha = 2\beta\) and the solid lines to \(\alpha = 3\beta\).

Figure 3. Concurrence as a function of \(Jt\) for a sequence of two ideal positive kicks applied at \(T_1 = 5\) and \(T_2 = 10\) for the initial pure states \(|\Phi(0)\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)\) (a) and \(|\Psi(0)\rangle = |01\rangle\) (b). The dashed lines correspond to \(\alpha = 2\beta\) and the solid lines to \(\alpha = 3\beta\).
Similarly, the time-evolution matrix for four positive kicks at times $t > T_4$
\[
\hat{U}^K(t) = e^{-i\hat{H}_0(t-T_4)} e^{-i\int_{T_4}^{T} \hat{H}_{\alpha}(t') \, dt'} e^{-i\int_{T}^{T_1} \hat{H}_{\alpha}(t') \, dt'} e^{-i\int_{T_1}^{T_2} \hat{H}_{\alpha}(t') \, dt'} e^{-i\int_{T_2}^{T_3} \hat{H}_{\alpha}(t') \, dt'} e^{-i\int_{T_3}^{T_4} \hat{H}_{\alpha}(t') \, dt'} \times e^{-i\int_{T_4}^{T_5} \hat{H}_{\beta}(t') \, dt'} e^{-i\int_{T_5}^{T_6} \hat{H}_{\beta}(t') \, dt'} e^{-i\int_{T_6}^{T_7} \hat{H}_{\beta}(t') \, dt'} e^{-i\int_{T_7}^{T_8} \hat{H}_{\beta}(t') \, dt'},
\]
with parameters specified in equation (10)
\[
y = e^{i\theta},
\]
\[
y_1 = e^{i\theta_1},
\]
\[
u = \cos (2Jt) \cos (\Delta)^3 - \sum_{i,j=1}^{4} \cos (2J(t + 2(T_i - T_j))) \cos (\Delta)^2 \sin (\Delta)^2
\]
\[
+ \cos (2J(t + 2T_{1234})) \sin (\Delta)^4,
\]
\[
v = \sum_{j=1}^{4} \cos (\zeta_j) \cos (\Delta)^3 - \sum_{i,j,k=1}^{4} \cos (2J(t + 2(T_i - T_j + T_k))) \cos (\Delta) \sin (\Delta)^3,
\]
\[
w = \sum_{j=1}^{4} \sin (\zeta_j) \cos (\Delta)^3 - \sum_{i,j,k=1}^{4} \sin (2J(t + 2(T_i - T_j + T_k))) \cos (\Delta) \sin (\Delta)^3,
\]

Figure 4. Concurrence as a function of dimensionless time, $Jt$, for four successive ideal positive kicks for the initial pure states $|\Phi(0)\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$ (a) and $|\Phi(0)\rangle = |0\rangle$ (b). Here the dashed lines correspond to $\alpha = 2\beta$ and the solid lines to $\alpha = 3\beta$, and we take $T_1 = 5, T_2 = 10, T_3 = 15$ and $T_4 = 20$. 

\[
u = \sum_{j=1}^{3} \cos (\zeta_j) \sin (\Delta) \cos (\Delta)^2 - \cos (2J(t - 2(T_1 - T_2 + T_3))) \sin (\Delta)^3, 
\]

\[
w = \sum_{j=1}^{3} \sin (\zeta_j) \sin (\Delta) \cos (\Delta)^2 - \sin (2J(t - 2(T_1 - T_2 + T_3))) \sin (\Delta)^3,
\]
\[
z = -\sin (2Jt) \cos (\Delta)^3 + \sum_{i,j=1}^{3} \sin (2J(t + 2(T_i - T_j))) \cos (\Delta) \sin (\Delta)^2.
\]
Figure 5. The contour plot of concurrence versus $J_t$ and the ratio, $\alpha/\beta$, for the initial pure states $|\Phi(0)\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$ (a) and $|\Psi(0)\rangle = |01\rangle$ (b). Here the contour plots include four ideal positive kicks applied at $T_1 = 5, T_2 = 10, T_3 = 15$ and $T_4 = 20$. (There are ten equidistant contours of concurrence in the plots between 0 (black) and 1 (white).)

\[ z = -\sin (2Jt) \cos(\Delta)^4 + \sum_{i,j=1}^{4} \sin (2J(t + 2(T_i - T_j))) \cos(\Delta)^2 \sin(\Delta)^2 \]
\[ - \sin (2J(t + 2T_{1234})) \sin(\Delta)^4, \]  

(28)

where $\xi_i = 2J(t - 2T_i)$, $\Delta = (\alpha - \beta)$ and $T_{1234} = (T_1 - T_2 + T_3 - T_4)$. Here the propagator without time ordering can be calculated by replacing $\hat{B}^1_t \rightarrow n\alpha$ and $\hat{B}^2_t \rightarrow n\beta$ (here $n = 3$ for three positive kicks and $n = 4$ for four positive kicks) in equation (10) and note that for $\alpha = \beta$ or $J = 0$, the time-ordering effect defined as $\hat{U}^K(t) - \hat{U}^0(t)$ disappears after the field is active.

The dynamics of $C(\hat{\rho})$ under three and four positive kick sequence is shown in figures 4(a) and (b) for $\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$ and $|01\rangle$ initial states, respectively. The most important finding from these figures is that almost constant high entanglement can be obtained after the third kick for the $\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$ initial Bell state and after the first and fourth kicks for the $|01\rangle$ state at the external magnetic field ratio of $\alpha/\beta = 2$.

The integrated magnetic field strength dependence of the concurrence dynamics is shown in figure 5, where we display the contour plot of $C(\hat{\rho})$ as functions of $\alpha/\beta$ and $J_t$ for $\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$ and $|01\rangle$ initial states. For the $\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$ initial Bell state, the maximally entangled state is found to be unaffected by the external field for $\alpha/\beta = 1$ and $\alpha/\beta \geq 2.5, 4.25, 5.75, 7.25, 8.75$. The $\alpha/\beta$ periodicity of the maximum of $C(\hat{\rho})$ increases after each kick for the initial Bell state. The dependence of $C(\hat{\rho})$ on $\alpha/\beta$ for the initial separable state $|01\rangle$ is more complicated compared to the case of the initial Bell state. The almost periodic structures exist also in figure 5(b); their periodicity changes after each kick, but it is not easy to obtain an expression for that change. Most importantly, the high-entanglement regions, which are indicated in white in the contour plots, have long lifetimes for each positive kick for the
initial Bell state, while for the separable state they are distributed in a narrower area compared to the initial Bell state case and have long lifetimes only after the first, second and fourth kicks.

6. Entanglement dynamics of qubits perturbed by a sequence of Gaussian pulses

Depending on the physical implementation of the qubit, it might be difficult to obtain an external field that can be considered as a kick. Instead, a Gaussian pulse with finite width can be applied (for example, half-cycle electromagnetic pulses with width near \( \tau = 1 \) ps may be experimentally achievable \([30, 31]\)). Thus in this section, we will discuss the entanglement dynamics of two qubits under the influence of Gaussian pulses of the form \( B_{\pm}(t) = \frac{a_{\pm}}{\sqrt{\pi \tau}} e^{-\frac{(t-T_{\pm})^2}{\tau^2}} \) \((a_{1,2} = \alpha, \beta)\) centred at \( T_{\pm} \) with width \( \tau \). The dynamics of entanglement in the presence of time ordering for the initial pure states \(|\Phi(0)\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)\) and \(|\Psi(0)\rangle = |01\rangle\) are evaluated by numerically integrating the corresponding equations in equation (3) and using equation (14). Here, we will investigate how the entanglement depends on the pulse width \( \tau \) by choosing a single pulse, a positive pulse followed by a negative pulse, and multiple positive pulses up to four centred at times \( T_{1} = 5, T_{2} = 10, T_{3} = 15 \) and \( T_{4} = 20 \). For all pulse sequences we will consider two integrated magnetic strength regimes: \( \alpha = 2\beta \) and \( \alpha = 3\beta \) and for convenience we shall set \( J = 1 \) and \( \beta = 1 \). One should note that in the limit \( \tau \rightarrow 0 \), the results of entanglement dynamics of kicked qubits in the presence of time ordering should be the same as those that are analysed in the previous section.

6.1. Single pulse

In figure 6, we show the results of a calculation of the concurrence for the initial pure states \(|\Phi(0)\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)\) and \(|\Psi(0)\rangle = |01\rangle\) when strongly perturbed by a single Gaussian pulse centred at \( t = T_{1} \) with width \( \tau \). According to the Schrödinger equation (3) of the system considered here, the expansion coefficients \( a_{1}(t) \) and \( a_{2}(t) \) evolve independently, and \( a_{3}(t) \) and \( a_{4}(t) \) obey a first-order coupled differential equation set, for example for a single pulse, as

\[
\begin{align*}
\dot{a}_{2}(t) &= \left( -J - \frac{(\alpha - \beta)}{\sqrt{\pi \tau}} e^{-\frac{(t-T_{1})^2}{\tau^2}} \right) a_{2}(t) + 2Ja_{3}(t), \\
\dot{a}_{3}(t) &= \left( -J + \frac{(\alpha - \beta)}{\sqrt{\pi \tau}} e^{-\frac{(t-T_{1})^2}{\tau^2}} \right) a_{3}(t) + 2Ja_{2}(t),
\end{align*}
\]

(29)

which are solved numerically by using a fourth-order Runge–Kutta algorithm. The most important observation from figure 6 is the existence of almost constant high concurrence for the initially separable state at \( \alpha = 2\beta \) integrated magnetic strength and a high-width Gaussian pulse, while the entanglement continues to have high amplitude oscillations for \( \alpha = 3\beta \); its value for \( \alpha = 2\beta \) is almost constant at around 1 for \( J\tau \) and the dimensionless pulse width is greater than 0.15. In contrast, for the initial Bell state, the oscillation amplitude of \( C(h) \) increases with the \( J\tau \) of the pulse for each magnetic ratio \((\alpha/\beta = 2 \) and \( \alpha/\beta = 3)\).

6.2. Positive–negative and positive–positive pulse sequence

In figures 7 and 8, we show the results of a calculation for the concurrence for the initial pure states \(|\Phi(0)\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)\) and \(|\Psi(0)\rangle = |01\rangle\) when strongly perturbed by a single Gaussian pulse centred at \( t = T_{1} \) followed by a negative or positive Gaussian pulse centred at \( t = T_{2} \) with the same width \( \tau \). For the double-pulse sequence, \( a_{2}(t) \) and \( a_{3}(t) \) obey the coupled
Figure 6. Concurrence as a function of $Jt$ for a single Gaussian pulse with width $\tau$ for the initial pure states $|\Phi(0)\rangle = \frac{1}{\sqrt{2}} (|10\rangle + |01\rangle)$ (a), (c), (e) and $|\Psi(0)\rangle = |01\rangle$ (b), (d), (f) and (h). The dashed lines correspond to $\alpha = 2\beta$ and the solid lines to $\alpha = 3\beta$. Here, we assume four dimensionless pulse widths as $J\tau = 0.05, 0.1, 0.15, 0.2$. 
Figure 7. Concurrence as a function of $Jt$ for a positive followed by a negative Gaussian pulse having the same width $\tau$ for the initial pure states $|\Psi(0)\rangle = |01\rangle$ (a), (c), (e) and (g) and $|\Phi(0)\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$ (b), (d), (f) and (h). The dashed lines correspond to $\alpha = 2\beta$ and the solid lines to $\alpha = 3\beta$. Here, we assume four dimensionless pulse widths as $Jt = 0.05, 0.1, 0.15, 0.2$. 
Figure 8. Concurrence as a function of $Jt$ for a sequence of two positive Gaussian pulses having the same width $\tau$ for the initial pure states $|\Phi(0)\rangle = \frac{1}{\sqrt{2}} (|10\rangle + |01\rangle)$ (a), (c), (e) and (g) and $|\Phi(0)\rangle = |01\rangle$ (b), (d), (f) and (h). The dashed lines correspond to $\alpha = 2\beta$ and the solid lines to $\alpha = 3\beta$. Here, we assume four dimensionless pulse widths as $J\tau = 0.05, 0.1, 0.15, 0.2$. 
equations:

\[
\begin{align*}
\dot{a}_2(t) &= \left( -J - \frac{(\alpha - \beta)}{\sqrt{\pi \tau}} \left( e^{-\frac{\alpha \cdot t^2}{\tau^2}} \pm e^{-\frac{\beta \cdot t^2}{\tau^2}} \right) \right) a_2(t) + 2Ja_3(t), \\
\dot{a}_3(t) &= \left( -J + \frac{(\alpha - \beta)}{\sqrt{\pi \tau}} \left( e^{-\frac{\alpha \cdot t^2}{\tau^2}} \pm e^{-\frac{\beta \cdot t^2}{\tau^2}} \right) \right) a_3(t) + 2Ja_2(t),
\end{align*}
\]

where + sign in the ± on the right-hand side is for the positive–positive pulse sequence, while − sign is for the positive–negative pulse sequence.

The concurrence dynamics for positive–negative and positive–positive pulse sequences obtained from the numerical solutions of equation (30) are displayed in figures 7 and 8, respectively, for the initial Bell and the separable states at different dimensionless pulse width values \( J\tau \approx 0.05, 0.10, 0.15, 0.20 \). For the initial Bell state under positive–negative sequence, the most important effect of the pulse width seems to be an increase in the \( C(\hat{\rho}) \) oscillation amplitudes for \( \alpha = 3\beta \) at times \( t > T_2 \). The almost constant high entanglement can be obtained for the (01) initial state for the positive–negative pulse sequence as can be seen from figure 7(h) for \( \alpha = 3\beta \) and \( t > T_2 \). One peculiarity of this figure is that high entanglement is obtained for \( \alpha = 2\beta \) after the first pulse, while it is obtained for \( \alpha = 3\beta \) after the second pulse. For the positive–positive Gaussian pulse sequence, the difference from positive–negative sequence becomes small as the width of the pulse gets larger as can be deduced from a comparison of figures 7 and 8. On the other hand, for a small pulse width, the difference is significant. For example for \( J\tau = 0.05 \) and \( \alpha = 3\beta \), the initial Bell state has nearly constant entanglement around 1 (see figure 7(a)) for the positive–negative pulse sequence after the negative pulse, while it oscillates between 1 and 0.25 for the positive–positive pulse sequence (see figure 8(a)).

6.3. A sequence of four positive pulses

The effect of integrated magnetic strength and the pulse width on the dynamics of concurrence for two qubits perturbed by a sequence of four positive Gaussian pulses is displayed in figures 9(a)–(h) for the initial Bell state, \(|\Phi(0)\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)\), and the separable state, \(|\Psi(0)\rangle = |01\rangle\). For this four positive pulse sequence the concurrence may be calculated by using the numerical solutions of the coupled equations:

\[
\begin{align*}
\dot{a}_2(t) &= \left( -J - \frac{(\alpha - \beta)}{\sqrt{\pi \tau}} \sum_{i=1}^{4} e^{-\frac{\alpha \cdot t^2}{\tau^2}} \right) a_2(t) + 2Ja_3(t), \\
\dot{a}_3(t) &= \left( -J + \frac{(\alpha - \beta)}{\sqrt{\pi \tau}} \sum_{i=1}^{4} e^{-\frac{\alpha \cdot t^2}{\tau^2}} \right) a_3(t) + 2Ja_2(t).
\end{align*}
\]

These figures can be compared with figures 5(a) and (b) to discern the effect of the pulse width. As the pulse gets wider, the \( \alpha/\beta \)-dependent oscillatory structures in the figure coalesce to produce non-periodic structures, especially after the third and fourth pulses. The high entanglement regions, which are shown in white, can still have long lifetimes, as indicated by white straight perpendicular sections in the contour plots of figure 9. In the case of an ideal kick, the maximally entangled state is found to be unaffected by the external field for \( \alpha/\beta = 1 \) and \( \alpha/\beta \geq 2.5, 4.25, 5.75, 7.25, 8.75 \) (see figure 5(a)). Comparing figure 5(a) with figures 9(a), (c), (e) and (g), the initial Bell state is found to be unperturbed by the highly wider Gaussian pulses if and only if \( \alpha/\beta = 1 \); especially seen obviously for the dimensionless pulse width greater than 0.15. One should note that this is one of the conditions in which time-ordering effects vanish.
Figure 9. The contour plot of concurrence versus $Jt$ and $\alpha/\beta$, for a sequence of four positive Gaussian pulses of width $\tau$ for the initial pure states $|\Phi(0)\rangle = \frac{1}{\sqrt{2}} (|10\rangle + |01\rangle)$ (a), (c), (e) and (g), and $|\Psi(0)\rangle = |01\rangle$ (b), (d), (f) and (h). Here we assume four dimensionless pulse widths as $J\tau = 0.05, 0.1, 0.15, 0.2$. (There are ten equidistant contours of concurrence in the plots between 0 (black) and 1 (white).)
The last point we want to emphasize is that what happens to the entanglement dynamics between two qubits if the time-ordering effect vanishes. As mentioned before the time-ordering effect vanishes for the special cases either $\alpha = \beta$ or $J = 0$. From the corresponding equations it can be noted that for the case $\alpha = \beta$, the concurrence function for the initial state $|\Phi(0)\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |10\rangle)$ is equal to 1, while for the initial state $|\Psi(0)\rangle = |01\rangle$ it is equal to $|\sin(4Jt)|$ and unaffected by the sequence of the kicks or Gaussian pulses. For the other case, $J = 0$, under the influence of the kick or Gaussian pulse sequences, the concurrence for the initial Bell state is always equal to 1, while the separable state remains separable at any time. These show an important fact that since the time ordering provides a connection between interactions at different times, it is responsible for the nonlocal correlations between the qubits in time.

7. Conclusion

We have investigated the dynamics of entanglement for two qubits that interact with each other via Heisenberg XXX-type interaction under a time-dependent external magnetic field. The initial state of the system is considered to be pure Bell or separable states. The main aim of the study was to investigate the controllability of the entanglement with a sequence of pulse or kick-type external fields.

The effect of time ordering in the dynamics of concurrence is found to be important; concurrence calculated when the time ordering is neglected is found to be completely different than when it is taken into account. Time-dependent concurrence obtained after one, two, three and four kicks at different magnetic field strengths indicates that one can employ carefully chosen kick or kick sequences to produce high entanglement between two initially non-entangled qubits.

We have also considered the effect of the pulse width of the external field on the entanglement dynamics by modelling the external field as a Gaussian pulse or a sequence of Gaussian pulses. Increasing the width of the pulse is found to enhance the control of high and steady entanglement.

One should note that the external control field considered in this study acts on both qubits at the same time. It might be possible to use pulse sequences acting on individual qubits at different times to obtain a better control of entanglement.

References

[1] Nielsen M A and Chuang I L 2000 Quantum Computation and Quantum Information (Cambridge: Cambridge University Press)
[2] Bennett C H, DiVincenzo D P, Smolin J A and Wootters W K 1996 Phys. Rev. A 54 3824
[3] Ekert A K 1991 Phys. Rev. Lett. 67 661
[4] Heule R, Bruder C, Burgarth D and Stojanovic V M 2010 arXiv:1007.2572 [quant-ph]
[5] Caneva T, Murphy M, Calarco T, Fazio F, Montangero S, Giovannetti V and Santoro G E 2009 Phys. Rev. Lett. 103 240501
[6] Wu L A, Lidar L A and Friesen M 2004 Phys. Rev. Lett. 93 030501
[7] Levy J 2002 Phys. Rev. Lett. 89 147902
[8] Malinovsky V S and Sola I R 2004 Phys. Rev. Lett. 93 190502
[9] Sadykov G, Lashin E I and Abdalla M S 2009 Physica B 404 1719
[10] Wang X, Bayat A, Schirmer S G and Bose S 2010 Phys. Rev. A 81 032312
[11] Abliz A, Gao H J, Xie X C, Wu Y S and Liu W M 2006 Phys. Rev. A 74 052105
[12] Wang X 2001 Phys. Rev. A 64 012313
[13] Sainz I, Burlak G and Klimov A B 2010 arXiv:1008.2784 [quant-ph]
[14] Imamoglu A, Awschalom D D, Burkard G, DiVicenzo D P, Loss D, Sherwin M and Small A 1999 Phys. Rev. Lett. 83 4204
[15] Kaplan L, Shakov K K, Chalastaras A, Maggio M, Burin A L and McGuire J H 2004 Phys. Rev. A 70 063401
[16] Shakov K K, McGuire J H, Kaplan L, Uskov D and Chalastaras A 2006 J. Phys. B: At. Mol. Opt. Phys. 39 1361
[17] Jones J A, Hansen R A and Mosca M 1998 J. Magn. Reson. 138 353
[18] Vandersypen L M K, Steffen M, Breyta G, Yannoni C S, Sherwood M H and Chuang I L 2001 Nature 414 883
[19] Slighcher C P 1996 Principles of Magnetic Resonance (Berlin: Springer)
[20] Kosloff R, Hammerich A D and Tannor D 1992 Phys. Rev. Lett. 69 2172
[21] Shi S, Woody A and Rabitz H 1988 J. Chem. Phys. 88 6870
[22] Palao J and Kosloff R 2002 Phys. Rev. Lett. 89 188301
[23] Rau A R P 1998 Phys. Rev. Lett. 81 4785
[24] Rau A R P, Selvaraj G and Uskov D 2005 Phys. Rev. A 71 062316
[25] Godunov A L and McGuire J H 2001 J. Phys. B: At. Mol. Opt. Phys. 34 223
[26] Zhao H Z, Lu Z H and Thomas J E 1997 Phys. Rev. Lett. 79 613
[27] Merabet H, Bruch R, Hanni J, Godunov A L and McGuire J H 2002 Phys. Rev. A 65 010703
[28] Magnus W 1954 Commun. Pure Appl. Math. 7 649
[29] Wootters W K 1998 Phys. Rev. Lett. 80 2245
[30] Jones R Y, You D and Buckhsbaum P H 1993 Phys. Rev. Lett. 70 1236
[31] Abique A M and Berakdar B 2004 Appl. Phys. Lett. 84 2346