BRAUER GROUPS AND TATE-SHAFAREVICH GROUPS

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Abstract. Let $X_K$ be a proper, smooth and geometrically connected curve over a global field $K$. In this paper we generalize a formula of Milne relating the order of the Tate-Shafarevich group of the Jacobian of $X_K$ to the order of the Brauer group of a proper regular model of $X_K$. We thereby partially answer a question of Grothendieck.

0. Introduction

Let $K$ be a global field, i.e. $K$ is a finite extension of $\mathbb{Q}$ (the “number field case”) or is finitely generated and of transcendence degree 1 over a finite field (the “function field case”). In the number field case we let $U$ denote a nonempty open subscheme of the spectrum of the ring of integers of $K$, and when $K$ is a function field in one variable with finite field of constants $k$, we let $U$ denote a nonempty open subscheme of the unique smooth complete curve over $k$ whose function field is $K$. We will write $S$ for the set of primes of $K$ not corresponding to a point of $U$. Further, we will write $\bar{K}$ for the separable algebraic closure of $K$ and $\Gamma$ for the Galois group of $\bar{K}$ over $K$. The completion of $K$ at a prime $v$ will be denoted by $K_v$.

Assume now that a connected, regular, 2-dimensional scheme $X$ is given together with a proper morphism $\pi: X \to U$ whose generic fiber $X_K = X \times_U \text{Spec} K$ is a smooth geometrically connected curve over $K$. Let $\delta$ (resp. $\delta'$) denote the index (resp. period) of $X_K$. These integers may be defined as the least positive degree of a divisor on $X_K$ and the least positive degree of a divisor class in $\text{Pic}(X_{\bar{K}})$, respectively, where $X_{\bar{K}} = X_K \otimes_K \bar{K}$. For each prime $v$ of $K$, we will write $\delta_v$ and $\delta'_v$ for the analogous quantities associated to the curve $X_{K_v}$. It is known that there are only finitely many primes $v$ for which $\delta_v \neq 1$. Further, Lichtenbaum [21] has shown that $\delta_v$ equals either $\delta'_v$ or $2\delta'_v$ for each prime $v$. We will write $d$ for the number of primes $v$ for which $\delta_v = 2\delta'_v$. These primes were called “deficient” in [35], and we sometimes refer to $d$ as the number of deficient primes of $X_K$. Now let $A$ be the Jacobian variety of $X_K$. It has long been known (see [46], §3) that there exist close connections between the Brauer group $\text{Br}(X)$ of $X$ and the Tate-Shafarevich group $\text{Ш}(A)$ of $A$. These connections were explored at length by Grothendieck in his
paper [15], in a more general setting than the one considered here. The following theorem can be extracted from [15], p. 121.

**Theorem** (Grothendieck). *In the function field case, suppose that \( \delta_v = 1 \) for all primes \( v \) of \( K \). Then there exist a finite group \( T_2 \subset \text{III}(A) \) of order \( \delta' \), a finite group \( T_3 \) of order dividing \( \delta \), and an exact sequence*

\[
0 \to \text{Br}(X) \to \text{III}(A)/T_2 \to T_3 \to 0.
\]

Regarding this result, Grothendieck (op. cit., p. 122) asked for the exact order of \( T_3 \). This problem was solved by Milne in [28] (see also [29], III.9.6), who used Cassels-Tate duality to compute the order of \( T_3 \). In order to state Milne’s result, which covers both the function field and number field cases, we need the following definition. Let

\[
\text{Br}(X)' = \text{Ker}\left[ \text{Br}(X) \to \bigoplus_{v \in S} \text{Br}(X_{K_v}) \right].
\]

Thus \( \text{Br}(X)' \) is the group of elements in the Brauer group of \( X \) becoming trivial on \( X_{K_v} \) for all \( v \in S \). Then one has the

**Theorem** (Milne). *Assume that \( \delta_v = 1 \) for all primes \( v \) of \( K \) and that \( \text{III}(A) \) contains no nonzero infinitely divisible elements. Then the period of \( X_K \) equals its index, i.e. \( \delta' = \delta \), and there exist finite groups \( T_2 \) and \( T_3 \) of order \( \delta \) and an exact sequence*

\[
0 \to \text{Br}(X)' \to \text{III}(A)/T_2 \to T_3 \to 0.
\]

*In particular, if one of III(A) or Br(X)' is finite, then so is the other, and their orders are related by*

\[
\delta^2[\text{Br}(X)'] = [\text{III}(A)].
\]

Grothendieck (loc. cit.) went on to pose the problem of making explicit the relations between \( \text{Br}(X) \) and \( \text{III}(A) \) when the integers \( \delta_v \) are no longer assumed to be equal to one\(^1\). In this paper we generalize the methods developed by Milne in [28] to prove the above theorem and obtain the following stronger result, which may be viewed as a partial solution to Grothendieck’s problem\(^2\).

**Main Theorem.** *Assume that the integers \( \delta_v' \) are relatively prime in pairs (i.e., \( (\delta_v', \delta_v) = 1 \) for all \( v \neq \bar{v} \)) and that \( \text{III}(A) \) contains no nonzero infinitely divisible elements. Then there is an exact sequence*

\[
0 \to T_0 \to T_1 \to \text{Br}(X)' \to \text{III}(A)/T_2 \to T_3 \to 0,
\]

*in which \( T_0, T_1, T_2 \) and \( T_3 \) are finite groups of orders*

\[
[T_0] = \delta/\delta',
\]
\[
[T_1] = 2^c,
\]
\[
[T_2] = \delta'/\prod \delta_v',
\]
\[
[T_3] = \frac{\delta'/\prod \delta_v'}{2^f},
\]

\(^1\)To be precise, Grothendieck stated an equivalent form of this problem.

\(^2\)In the last section of the paper we show that if a certain plausible conjecture holds, then we can give a complete answer to Grothendieck’s question.
where
\[ e = \max(0, d - 1) \]
and
\[ f = \begin{cases} 
1 & \text{if } \delta' / \prod \delta'_v \text{ is even and } d \geq 1 \\
0 & \text{otherwise.} 
\end{cases} \]

Here \( d \) is the number of deficient primes of \( X_K \) defined previously. In particular, if one of \( \Sha(A) \) or \( \Br(X)' \) is finite, then so is the other, and their orders are related by
\[ \delta \delta' \Br(X)' = 2^{e + f} \prod (\delta'_v)^2 \Sha(A). \]

Some immediate corollaries are

**Corollary 1.** Suppose that one of \( \Sha(A) \) or \( \Br(X)' \) is finite. Assume also that \( \delta_v = \delta'_v \) for all \( v \) (which holds for instance if \( X_K \) has genus 1\(^3\)), and that these integers are relatively prime in pairs. Then \( \delta = \delta' \), and
\[ \delta^2 \Br(X)' = \prod \delta_v^2 \Sha(A). \]
In particular if \( \delta_v = 1 \) for all \( v \), then
\[ \delta^2 \Br(X)' = \Sha(A). \]

Note that the last formula in the statement of Corollary 1 is precisely the formula of Milne stated before.

**Corollary 2.** Assume that one of \( \Sha(A) \) or \( \Br(X)' \) is finite and that \( \delta'_v = 1 \) for all \( v \) (the latter holds for instance if \( X_K \) has genus 2). Then
\[ \delta \delta' \Br(X)' = 2^{e + f} \Sha(A), \]
where \( e \) and \( f \) are as in the statement of the theorem.

In the function field case our main result, when combined with a formula of Gordon [10, p. 196], should imply the expected equivalence of the conjectures of Artin and Tate for \( X \) [46, Conj. C] and Birch and Swinnerton-Dyer for \( A \) [46, Conj. B] (this is Tate’s “elementary” conjecture (d) of [46]), at least under the additional assumption that the structural morphism \( \pi: X \to U \) is cohomologically flat in dimension 0. We expect to address this issue in a separate publication.

**Acknowledgements**

I am grateful to the people of Fondecyt for their financial support and their patience. I also thank the library staff at IMPA (Rio de Janeiro, Brasil) for their bibliographical assistance.

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\(^3\)Regarding the parenthetical remarks in the statements of both corollaries, see [21]
1. Preliminaries

We keep the notations introduced in the previous section. Thus in particular $X$ is a regular connected scheme of dimension 2 equipped with a proper morphism $\pi: X \to U$ whose generic fiber $X_K$ is a smooth geometrically connected curve over $K$.

Remark. The $U$-scheme $X$ is a proper regular model of its generic fiber. Conversely, if we start with a (geometrically connected) proper and smooth curve $X_K$ over $K$, then there is a closed immersion $X_K \to \mathbb{P}^n_K$ for some $n$ and we can obtain a $U$-scheme $X$ as above, with generic fiber $X_K$, by applying Lipman’s desingularization process [2], [22] to the schematic image of $X_K$ in $\mathbb{P}^n_K$.

The Picard scheme of $X_K/K$, $\text{Pic}_{X_K/K}$, is a smooth group scheme over $K$ whose identity component, $\text{Pic}^0_{X_K/K}$, is an abelian variety, the Jacobian variety of $X_K$. Henceforth, we will write $P$ for $\text{Pic}_{X_K/K}$ and (in accordance with previous notations) $A$ for $\text{Pic}^0_{X_K/K}$. The following holds. If $L$ is any field containing $K$ such that $X_K(L)$ is nonempty, then $P(L) = \text{Pic}(X_L)$ (for the basic facts on the relative Picard functor, see [4, §8.1] or [12]). Further, there is an exact sequence of $\Gamma$-modules

$$0 \to A(\bar{K}) \to P(\bar{K}) \xrightarrow{\text{deg}} \mathbb{Z} \to 0,$$

where $\text{deg}$ is the degree map on $P(\bar{K}) = \text{Pic}(X_{\bar{K}})$. In particular $A(\bar{K})$ may be identified with $\text{Pic}^0(X_{\bar{K}})$, the subgroup of $\text{Pic}(X_{\bar{K}})$ consisting of divisor classes of degree zero (in this paper we shall regard the elements of $\text{Pic}(X_{\bar{K}})$ mainly as classes of divisors. See below). Now we observe that $P(K) = P(\bar{K})^\Gamma = \text{Pic}(X_{\bar{K}})^\Gamma$. Further, there is an exact sequence (deduced from the preceding one by taking $\Gamma$-invariants)

$$0 \to A(K) \to P(K) \xrightarrow{\text{deg}} \delta' \mathbb{Z} \to 0,$$

where $\delta'$ is the period of $X_K$ as defined previously.

For any regular connected scheme $Y$, $R(Y)$ will denote the field of rational functions on $Y$. There is an exact sequence

$$0 \to R(X_{\bar{K}})^*/\bar{K}^* \to \text{Div}^0(X_{\bar{K}}) \to \text{Pic}^0(X_{\bar{K}}) \to 0$$

which induces an exact sequence

$$0 \to R(X_K)^*/K^* \to \text{Div}^0(X_K) \to A(K) \to A(K)/\text{Pic}^0(X_K) \to 0. \quad (1)$$

Note that $A(K)/\text{Pic}^0(X_K)$ is a finite abelian group since it is finitely generated (by the Mordell-Weil theorem) and isomorphic to a subgroup of the torsion group $H^1(\Gamma, R(X_K)^*/\bar{K}^*)$. Similarly, $P(K)/\text{Pic}(X_K)$ is finite and there is an exact sequence

$$0 \to R(X_K)^*/K^* \to \text{Div}(X_K) \to P(K) \to P(K)/\text{Pic}(X_K) \to 0. \quad (2)$$
We also have an exact sequence
\[ 0 \to \text{Div}^0(X_K) \to \text{Div}(X_K) \xrightarrow{\text{deg}} \delta Z \to 0, \]
where \( \delta \) is the index of \( X_K \) as defined previously. We note that \( \delta \) may also be defined as the greatest common divisor of the degrees of the fields \( L \) over \( K \) such that \( X_K(L) \neq \emptyset \).

**Lemma 1.1.** The period of \( X_K \) divides its index, i.e. \( \delta' | \delta \), and
\[ [P(K) : \text{Pic}(X_K)] = (\delta/\delta') \cdot [A(K) : \text{Pic}^0(X_K)]. \]

**Proof.** The first assertion of the lemma follows from the definitions. Regarding the second, an application of the snake lemma to the commutative diagram
\[
\begin{array}{cccccc}
0 & \to & \text{Div}^0(X_K) & \to & \text{Div}(X_K) & \to & \delta Z & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A(K) & \to & P(K) & \to & \delta' Z & \to & 0
\end{array}
\]
yields, using (1) and (2) above, an exact sequence
\[ 0 \to A(K)/\text{Pic}^0(X_K) \to P(K)/\text{Pic}(X_K) \to \delta' Z/\delta Z \to 0. \]
The lemma is now immediate. \( \square \)

It is clear from the definitions that if \( L \) is any field containing \( K \), then the index (resp. period) of \( X_L \) divides the index (resp. period) of \( X_K \). Thus for any prime \( v \) of \( K \), \( \delta_v | \delta \) and \( \delta'_v | \delta' \), where \( \delta_v \) and \( \delta'_v \) are, respectively, the index and period of \( X_{K_v} \). Further, since \( X_K(K_v) \neq \emptyset \) for all but finitely many primes \( v \) (see for example [18, p. 249, Remark 1.6]), we conclude that there are only finitely many primes \( v \) for which \( \delta_v \neq 1 \). The analogous statement with \( \delta_v \) replaced by \( \delta'_v \) holds true as well, since \( \delta'_v | \delta_v \) for each \( v \). We now write \( \Delta \) (resp. \( \Delta' \)) for the least common multiple of the integers \( \delta_v \) (resp. \( \delta'_v \)). Clearly, \( \Delta | \delta \) and \( \Delta' | \delta' \). Further, since \( \delta_v = \delta'_v \) or \( 2\delta'_v \) for each \( v \) as mentioned earlier, we have
\[ \frac{\Delta}{\Delta'} = \begin{cases} 1 & \text{if } \delta_v = \delta'_v \text{ for all } v \\ 2 & \text{otherwise.} \end{cases} \]

We now consider the map
\[ \Sigma : \bigoplus_v \delta^{-1}_v Z/Z \to Q/Z, \quad (x_v) \mapsto \sum x_v. \]

**Lemma 1.2.** We have
\[ [\text{Ker } \Sigma] = \prod \delta_v / \Delta \]
and
\[ \text{Im } \Sigma = \Delta^{-1} Z/Z. \]

**Proof.** That the image of \( \Sigma \) is exactly \( \Delta^{-1} Z/Z \) follows easily from the fact that \( \gcd(\Delta) = 1 \). The rest of the lemma is clear. \( \square \)

Next we consider the map
\[ D : Z/\delta' Z \to \bigoplus_v Z/\delta'_v Z \]
given by \( x \mod \delta' \mapsto (x \mod \delta'_v) \). Since the kernel of \( D \) is \( \Delta' Z/\delta' Z \), the following lemma is clear.
Lemma 1.3. We have
\[ \text{[Ker } D\text{]} = \delta' \frac{v}{\Delta} \]
and
\[ \text{[Coker } D\text{]} = \prod \delta'_v \frac{v}{\Delta}. \]

For each prime \( v \) of \( K \) we will write \( \Gamma_v \) for the Galois group of \( \overline{K}_v \) over \( K_v \).

From the cohomology sequence associated to the exact sequence of \( \Gamma_v \)-modules
\[ 0 \to A(\overline{K}_v) \to P(\overline{K}_v) \overset{\text{deg}}{\to} Z \to 0 \]
we get an exact sequence
\[ 0 \to \mathbb{Z}/\delta'_v \mathbb{Z} \to H^1(\Gamma_v, A) \to H^1(\Gamma_v, P) \to 0. \] (3)

Now by work of Lichtenbaum [21] and Milne [26] (see also [28, Remark I.3.7]), there exists a perfect pairing
\[ H^0(\Gamma_v, A) \times H^1(\Gamma_v, A) \to \mathbb{Q}/\mathbb{Z}, \]
where \( H^0(\Gamma_v, A) \) denotes \( A(K_v)/N_{\overline{K}_v/K_v}A(\overline{K}_v) \) if \( v \) is archimedean and \( A(K_v) \) otherwise. Relative to this pairing, the annihilator of the image of \( \text{Pic}^0(X_{K_v}) \) in \( H^0(\Gamma_v, A) \) under the canonical map \( A(K_v) \to H^0(\Gamma_v, A) \) is exactly the image of \( \mathbb{Z}/\delta'_v \mathbb{Z} \) in \( H^1(\Gamma_v, A) \) under the map in (3). Consequently the following holds

Lemma 1.4. We have
\[ [A(K_v) : \text{Pic}^0(X_{K_v})] = \delta'_v. \]

We now combine the previous lemma with an analogue of Lemma 1.1 to obtain

Lemma 1.5. We have
\[ [P(K_v) : \text{Pic}(X_{K_v})] = \delta_v. \]

Remark. The archimedean case of Lemma 1.5 was originally established by Witt in 1935 [50]. For an interesting review of this and other related classical results in terms of étale cohomology, see [38, §20.1].

We now derive a slight variant of the snake lemma (Proposition 1.6 below). It is one of the basic ingredients of the proof of the Main Theorem.

Consider the following exact commutative diagram in the category of abelian groups
\[
\begin{array}{ccccccc}
0 & \to & A_1 & \to & A_2 & \overset{f}{\to} & A_3 & \to & A_4 & \to & A_5 & \to & 0 \\
& & \downarrow & & \downarrow \eta & & \downarrow \lambda & & \downarrow \mu \\
0 & \to & B_1 & \to & B_2 & \overset{g}{\to} & B_3 & \to & B_4 & \to & B_5 & \to & 0
\end{array}
\] (4)

(we have labeled only those maps which are relevant to our purposes). We have an induced exact commutative diagram
\[
\begin{array}{ccccccc}
A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 & \longrightarrow & 0 \\
\downarrow \eta & & \downarrow \lambda & & \downarrow \mu \\
0 & \longrightarrow & B_3/\text{Im } g & \longrightarrow & B_4 & \longrightarrow & B_5 & \longrightarrow & 0
\end{array}
\]

where \( \eta \) is the composition of \( \eta \) with the canonical map \( B_3 \to B_3/\text{Im } g \). An application of the snake lemma to the above diagram using the fact that \( \text{Im } f \subset \text{Ker } \eta \) yields
\textbf{Proposition 1.6.} To any exact commutative diagram of the form (4) there is associated an exact sequence

$$0 \to\text{Im } f \to \text{Ker } \bar{\eta} \to \text{Ker } \lambda \to \text{Ker } \mu \to \text{Coker } \bar{\eta},$$

where $\bar{\eta}$ is as defined above.

The following result supplements Proposition 1.6.

\textbf{Lemma 1.7.} With the above notations, there is an exact sequence

$$0 \to \text{Ker } \eta \to \text{Ker } \bar{\eta} \to \text{Im } g \to \text{Coker } \eta \to \text{Coker } \bar{\eta} \to 0.$$

\textbf{Proof.} This is nothing more than the kernel-cokernel sequence for the pair of maps $A_3 \stackrel{\eta}{\to} B_3 \to B_3/\text{Im } g$. The maps in the exact sequence of the lemma are the natural ones, e.g. $\text{Im } g \to \text{Coker } \eta$ is the composite $\text{Im } g \to B_3 \to B_3/\text{Im } \eta = \text{Coker } \eta$.

\section{Proof of the Main Theorem}

All cohomology groups below will be either Galois cohomology groups or \'{e}tale cohomology groups. We will view $\Gamma_v$ as a subgroup of $\Gamma$ in the standard way, i.e., by identifying it with the decomposition group of some fixed prime of $\bar{K}$ lying above $v$. For each $v$, $\text{inv}_v : \text{Br}(K_v) \to \mathbb{Q}/\mathbb{Z}$ will denote the usual invariant map of local class field theory. The $n$-torsion subgroup of an abelian group $M$ will be denoted by $M_{n-\text{tor}}$.

We begin by recalling a fundamental exact sequence. Since $H^q(X_K, \mathbb{G}_m) = 0$ for all $q \geq 2$ [27, Ex. 2.23(b), p. 110], the Hochschild-Serre spectral sequence

$$H^p(\Gamma, H^q(X_K, \mathbb{G}_m)) \Rightarrow H^{p+q}(X_K, \mathbb{G}_m)$$

yields (see [7, XV.5.11]) an exact sequence

$$0 \to \text{Pic}(X_K) \to P(K) \to \text{Br}(K) \to \text{Br}(X_K) \to H^1(\Gamma, P) \to 0,$$

where the zero at the right-hand end comes from the fact that $H^3(\Gamma, \bar{K}^*) = 0$ [29, I.4.21]. (We have used here the well-known facts that $\text{Pic}(X_K) = H^1(X_K, \mathbb{G}_m)$ and that the Brauer group of a regular scheme of dimension $\leq 2$ agrees with the cohomological Brauer group of the scheme.) Similarly, for each prime $v$ of $K$ there is an exact sequence

$$0 \to \text{Pic}(X_{K_v}) \to P(K_v) \stackrel{g_v}{\to} \text{Br}(K_v) \to \text{Br}(X_{K_v}) \to H^1(\Gamma_v, P) \to 0.$$

\textbf{Lemma 2.1.} For each prime $v$ of $K$, we have

$$\text{Im } g_v = \text{Br}(K_v)_{\delta_v-\text{tor}}.$$

\textbf{Proof.} By Lemma 1.5, $\text{Im } g_v$ is a subgroup of $\text{Br}(K_v)$ of order $\delta_v$. On the other hand the invariant map $\text{inv}_v$ induces an isomorphism $\text{Br}(K_v)_{\delta_v-\text{tor}} \cong \delta_v^{-1} \mathbb{Z}/\mathbb{Z}$, whence the lemma follows. □
We now consider the commutative diagram

$$
\begin{array}{cccccccc}
0 & \rightarrow & \text{Pic}(X_K) & \rightarrow & P(K) & \rightarrow & \text{Br}(K) & \rightarrow & \text{Br}(X_K) & \rightarrow & H^1(\Gamma, P) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \eta & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \bigoplus_v \text{Pic}(X_{K_v}) & \rightarrow & \bigoplus_v P(K_v) & \rightarrow & \bigoplus_v \text{Br}(K_v) & \rightarrow & \bigoplus_v \text{Br}(X_{K_v}) & \rightarrow & \bigoplus_v H^1(\Gamma_v, P) & \rightarrow & 0
\end{array}
$$

where the direct sums extend over all primes of $K$, the vertical maps are the natural ones and $g = \bigoplus_v g_v$. This diagram is of the form considered at the end of Section 2, and we may therefore apply to it Proposition 1.6 above. Before doing so, however, we call upon

Lemma 2.2. (a) There is an exact sequence

$$
0 \rightarrow \text{Br}(K) \rightarrow \bigoplus_v \text{Br}(K_v) \rightarrow \bigoplus_v \text{Br}(X_{K_v}) \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0.
$$

(b) There is an exact sequence

$$
0 \rightarrow \text{Br}(X) \rightarrow \text{Br}(X_K) \rightarrow \bigoplus_{v \notin S} \text{Br}(X_{K_v}),
$$

where $S$ is the set of primes of $K$ not corresponding to a point of $U$.

Proof. Assertion (a) is one of the major theorems of class field theory. See [47, §11]. Assertion (b) is proved in [28, Lemma 2.6]. □

We now apply Proposition 1.6 to the diagram above using the preceding lemma. We get an exact sequence

$$
0 \rightarrow P(K)/\text{Pic}(X_K) \rightarrow \text{Ker} \tilde{\eta} \rightarrow \text{Br}(X)' \rightarrow \text{III}(P) \rightarrow \text{Coker} \tilde{\eta},
$$

where $\tilde{\eta}: \text{Br}(K) \rightarrow \bigoplus_v \text{Br}(K_v)/\text{Im} g$ is induced by $\eta$ and

$$
\text{Br}(X)' = \text{Ker} \left[ \text{Br}(X) \rightarrow \bigoplus_{v \in S} \text{Br}(X_{K_v}) \right].
$$

Now by Lemma 1.1, the order of $P(K)/\text{Pic}(X_K)$ equals $(\delta/\delta') \cdot [A(K) : \text{Pic}^0(X_K)]$. Regarding the kernel and cokernel of $\tilde{\eta}$, the following holds.

Proposition 2.3. We have

$$
[Ker \tilde{\eta}] = \prod \delta_v/\Delta,
$$

and the map $\sum \text{inv}_v : \bigoplus_v \text{Br}(K_v) \rightarrow \mathbb{Q}/\mathbb{Z}$ induces an isomorphism

$$
\text{Coker} \tilde{\eta} \simeq \mathbb{Q}/\Delta^{-1}\mathbb{Z}.
$$

Proof. By combining Lemmas 1.7, 2.1 and 2.2 we obtain an exact sequence

$$
0 \rightarrow \text{Ker} \tilde{\eta} \rightarrow \bigoplus_v \text{Br}(K_v)_{\delta_v - \text{tor}} \rightarrow \bigoplus_v \sum \text{inv}_v \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow \text{Coker} \tilde{\eta} \rightarrow 0.
$$

Now for each $v$ the invariant map $\text{inv}_v$ induces an isomorphism $\text{Br}(K_v)_{\delta_v - \text{tor}} \simeq \delta_v^{-1}\mathbb{Z}/\mathbb{Z}$, and it follows that the kernel and cokernel of the middle map in the above exact sequence may be identified with the kernel and cokernel of the map $\Sigma$ considered in Section 2. The proposition now follows from Lemma 1.2. □

We summarize the results obtained so far.
Corollary 2.4. There is an exact sequence

$$0 \to T_0 \to T_1 \to \text{Br}(X)' \to \mathbb{Q}/\Delta^{-1}\mathbb{Z},$$

where $T_0$ and $T_1$ are finite groups of orders

$$[T_0] = (\delta/\delta') \cdot [A(K) : \text{Pic}^0(X_K)]$$
$$[T_1] = \prod \delta_v/\Delta.$$

We now prove

Theorem 2.5. Assume that the integers $\delta_v'$ are relatively prime in pairs. Then

$$A(K) = \text{Pic}^0(X_K).$$

Proof. There is an exact commutative diagram

$$
\begin{array}{c}
0 \longrightarrow \ A(K)/\text{Pic}^0(X_K) \longrightarrow \ Br(K) \\
| \downarrow \quad \quad \quad \quad \downarrow \eta \\
0 \longrightarrow \bigoplus_{all \ v} A(K_v)/\text{Pic}^0(X_{K_v}) \longrightarrow \bigoplus_{all \ v} \text{Br}(K_v),
\end{array}
$$

in which the vertical maps are the natural ones and the nontrivial horizontal maps are induced by the maps $P(K)/\text{Pic}(X_K) \to \text{Br}(K)$ and $P(K_v)/\text{Pic}(X_{K_v}) \to \text{Br}(K_v)$ coming from (5) and (6). Arguing as in the proof of Lemma 2.1 (using Lemma 1.4 in place of Lemma 1.5 there), we conclude that for each $v$ the image of $A(K_v)/\text{Pic}^0(X_{K_v})$ in $\text{Br}(K_v)$ equals $\text{Br}(K_v)\delta_v^{\text{tor}}$. It follows that $A(K)/\text{Pic}^0(X_K)$ injects into $\text{Ker} \ \bar{\eta}'$, where

$$\bar{\eta}' : \text{Br}(K) \to \bigoplus_{v} \text{Br}(X_{K_v})/\text{Br}(K_v)\delta_v^{\text{tor}}$$

is induced by $\eta$. Now arguing as in the proof of Proposition 2.3, we see that the order of $\text{Ker} \ \bar{\eta}'$ equals $\prod \delta_v'/\Delta'$. By hypothesis this number is 1, whence the theorem follows.$\square$

We now recall from the Introduction the integer $d$, which was defined to be the number of primes $v$ of $K$ for which $\delta_v = 2\delta_v'$. As noted just before the statement of Lemma 1.2, $\Delta/\Delta' = 1$ or $2$ according as $d = 0$ or $d \geq 1$. Now we observe that

$$\frac{\prod \delta_v}{\Delta} = \left(\frac{\Delta}{\Delta'}\right)^{-1} \prod (\delta_v/\delta_v') \cdot \frac{\prod \delta_v'}{\Delta'} = 2^e \frac{\prod \delta_v'}{\Delta'},$$

where

$$e = \max(0, d - 1).$$

Consequently if the integers $\delta_v'$ are relatively prime in pairs, then $\prod \delta_v/\Delta = 2^e$. Thus Corollary 2.4 and Theorem 2.5 together imply
**Corollary 2.6.** Assume that the integers $\delta'_v$ are relatively prime in pairs. Then there is an exact sequence

$$0 \to T_0 \to T_1 \to Br(X)' \to \mathbb{III}(P) \xrightarrow{\phi} \mathbb{Q}/\Delta^{-1}\mathbb{Z},$$

in which $T_0$ and $T_1$ are finite groups of orders

$$[T_0] = \frac{\delta}{\delta'},$$

$$[T_1] = 2^e,$$

where $e = \max(0, d - 1)$.

We turn now to the problem of relating $\mathbb{III}(P)$ to $\mathbb{III}(A)$. There is an exact commutative diagram

$$
\begin{array}{cccccc}
0 & \to & \mathbb{Z}/\delta'\mathbb{Z} & \to & H^1(\Gamma, A) & \to & H^1(\Gamma, P) & \to & 0 \\
 & & \downarrow D & & \downarrow & & \downarrow & & \\
0 & \to & \bigoplus \mathbb{Z}/\delta'_v\mathbb{Z} & \to & \bigoplus H^1(\Gamma_v, A) & \to & \bigoplus H^1(\Gamma_v, P) & \to & 0,
\end{array}
$$

in which $D$ is the diagonal map of Section 2 and the rows come from the cohomology sequences of

$$0 \to A \to P \xrightarrow{\deg} \mathbb{Z} \to 0$$

over $K$ and over $K_v$. Applying the snake lemma to the above diagram yields the exact sequence

$$0 \to \text{Ker } D \to \mathbb{III}(A) \to \mathbb{III}(P) \to \text{Coker } D. \quad (7)$$

Let $T_2 = \text{Ker } D$. Then, by Lemma 1.3,

$$[T_2] = \delta'/\Delta'. \quad (8)$$

Further, the order of $\text{Coker } D$ is $\prod \delta'_v/\Delta'$. Consequently the following holds

**Proposition 2.7.** Suppose that the integers $\delta'_v$ are relatively primes in pairs. Then there is an exact sequence

$$0 \to T_2 \to \mathbb{III}(A) \xrightarrow{\rho} \mathbb{III}(P) \to 0,$$

in which $T_2$ is a finite group of order

$$[T_2] = \delta'/\prod \delta'_v.$$
Corollary 2.8. Assume that the integers $\delta_v$ are relatively prime in pairs. Then there is an exact sequence

$$0 \to T_0 \to T_1 \to \text{Br}(X)' \to \text{III}(A)/T_2 \to \mathbb{Q}/\Delta^{-1}\mathbb{Z},$$

in which $T_0$, $T_1$ and $T_2$ are finite groups of orders

$$[T_0] = \delta / \delta'$$

$$[T_1] = 2^e$$

$$[T_2] = \delta' / \prod \delta_v,$$

where $e = \max(0, d - 1)$.

It remains only to compute $T_3$, the image of III$(A)/T_2$ in $\mathbb{Q}/\Delta^{-1}\mathbb{Z}$ under the map in Corollary 2.8, or equivalently, the image of the composite map III$(A) \xrightarrow{\rho} \text{III}(P) \xrightarrow{\phi} \mathbb{Q}/\Delta^{-1}\mathbb{Z}$, where $\phi$ and $\rho$ are the maps in Corollary 2.6 and Proposition 2.7, respectively. We will show that $T_3$ has the order stated in the Introduction by generalizing [28, Lemma 2.11].

We begin by recalling from [28, Remark 2.9] the explicit description of the map $\phi : \text{III}(P) \to \mathbb{Q}/\Delta^{-1}\mathbb{Z}$.

We write $E$ for the canonical map $\text{Div}(X_K) \to \text{Pic}(X_K) = P(K)$. Represent $\alpha \in \text{III}(P)$ by a cocycle $a \in Z^1(\Gamma, P(K))$, and let $a \in C^1(\Gamma, \text{Div}(X_K))$ be such that $E(a) = a$. Then $\partial a \in Z^2(\Gamma, \text{Ker} E) = Z^2(\Gamma, R(X_K)^*/K^*)$ and, because $H^3(\Gamma, K^*) = 0$, it can be pulled back to an element $f \in Z^2(\Gamma, R(X_K)^*)$ (here $\partial$ is the boundary map). On the other hand, $\text{res}_v(a) = \partial a_v$ with $a_v \in C^0(\Gamma_v, P(K_v))$. Let $a_v \in C^0(\Gamma_v, \text{Div}(X_{K_v}))$ be such that $E(a_v) = a_v$. Then $\text{res}_v(a) = \partial a_v + \text{div}(f_v)$ with $f_v \in C^1(\Gamma_v, R(X_{K_v})^*)$, and $\text{res}_v f / \partial f_v \in Z^2(\Gamma_v, K_v^*)$. Let $\gamma_v$ be the class of $\text{res}_v f / \partial f_v$ in Br$(K_v)$. Then

$$\phi(\alpha) = q \left( \sum_v \text{inv}_v(\gamma_v) \right),$$

where $q$ is the canonical map $\mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\Delta^{-1}\mathbb{Z}$ induced by the identity map on $\mathbb{Q}$.

We note that, for any divisor $\varepsilon_v$ on $X_{K_v}$ such that neither $\text{res}_v f$ nor $\partial f_v$ has a zero or a pole in the support of $\varepsilon_v$,

$$(\text{res}_v f)(\varepsilon_v) / \partial f_v(\varepsilon_v) = (\deg \varepsilon_v) \text{res}_v f / \partial f_v$$

(see for example [21, §4]). Here $(\text{res}_v f)(\varepsilon_v) = f(\varepsilon_v) \in Z^2(\Gamma_v, K_v^*)$ is the value at $(\text{res}_v f, \varepsilon_v)$ of the cup-product pairing

$$Z^2(\Gamma_v, R(X_{K_v})^*) \times Z^0(\Gamma_v, \text{Div}(X_K)) \to Z^2(\Gamma_v, K_v^*)$$

which is induced by the evaluation pairing $R(X_{K_v})^* \times \text{Div}(X_{K_v}) \to K_v^*$, and similarly for $\partial f_v(\varepsilon_v)$. Since $\partial f_v(\varepsilon_v) = \partial (f_v(\varepsilon_v))$ with $f_v(\varepsilon_v) \in C^1(\Gamma_v, K_v^*)$, we see that $(\deg \varepsilon_v) \gamma_v$ is represented by $f(\varepsilon_v)$. Now choose a divisor $\varepsilon_v$ on $X_{K_v}$ of degree $\delta_v$ such that neither $f$ nor $\partial f_v$ has a zero or a pole in the support of $\varepsilon_v$ (this is always possible; see [19, App. 2]). Then $\delta_v \gamma_v$ is represented by $f(\varepsilon_v)$.
We now recall from [29, Remark I.6.12] the definition of a pairing
\[<, > : \text{III}(A) \times \text{III}(A) \rightarrow \mathbb{Q}/\mathbb{Z} \tag{9}\]
which annihilates only the divisible part of \(\text{III}(A)\).

Let \(\alpha \in \text{III}(A)\) be represented by \(a \in Z^1(\Gamma, A(K))\), and let \(\text{res}_v(a) = \partial a_v\) with \(a_v \in Z^0(\Gamma_v, A(K_v))\). Write
\[
\begin{align*}
a &= E(a), & a \in C^1(\Gamma, \text{Div}^0(X_{\bar{K}})) \\
a_v &= E(a_v), & a_v \in C^0(\Gamma_v, \text{Div}^0(X_{\bar{K}_v})).
\end{align*}
\]

Then \(\text{res}_v(a) = \partial a_v + \text{div}(f_v)\) in \(C^1(\Gamma_v, \text{Div}^0(X_{\bar{K}_v}))\) with \(f_v \in C^1(\Gamma_v, R(X_{\bar{K}_v})^*)\). Moreover, \(\partial a = \text{div}(f)\) with \(f \in Z^2(\Gamma, R(X_{\bar{K}})^*)\). Let \(\alpha'\) be a second element of \(\text{III}(A)\) and define \(a', a'_v, f'_v, f'\) as for \(\alpha\). Set
\[
< \alpha, \alpha' > = \sum_v \text{inv}_v(\gamma_v), \quad \gamma_v = \text{class of } (f'_v \cup \text{res}_v(a) - a'_v \cup \text{res}_v(f)),
\]
where \(\cup\) denotes the cup-product pairing induced by the evaluation pairing mentioned earlier. This definition is independent of the choices made, and \(f, a'_v, f'_v\) and \(a\) can be chosen so that \(f'_v(\text{res}_v(a))\) and \(f(a'_v)\) are defined. Moreover, \(< \alpha, \alpha > = 0\).

Now let \(\varepsilon : \mathbb{Q}/\mathbb{Z} \rightarrow \mathbb{Q}/\Delta^{-1}\mathbb{Z}\) be the natural isomorphism which is induced by multiplication by \(\Delta^{-1}\) on \(\mathbb{Q}\).

**Proposition 2.9.** Let \(\beta\) be a generator of \(T_2 \subset \text{III}(A)\) and let \(\beta' = (\Delta/\Delta')\beta\). Then the composite
\[
\text{III}(A) \overset{\rho}{\rightarrow} \text{III}(P) \overset{\phi}{\rightarrow} \mathbb{Q}/\Delta^{-1}\mathbb{Z}
\]
is \(\alpha \mapsto \varepsilon(< \alpha, \beta'>)\).

**Proof.** Let \(\alpha \in \text{III}(A)\) and define \(a, a_v, f_v\) and \(f\) as above. Then \(\phi(\rho(\alpha)) = \sum_v \text{inv}_v(\gamma_v)\) where \(\delta_v\gamma_v\) is represented by \(f(\epsilon_v)\) for some divisor \(\epsilon_v\) of degree \(\delta_v\) on \(X_{\bar{K}_v}\).

On the other hand, we can choose \(\beta' \in \text{III}(A)\) to be represented by \(b' = E(b')\) where \(b' = \Delta \partial P\), \(P\) any closed point on \(X_{\bar{K}}\). Define
\[
b'_v = \Delta P - (\Delta/\delta_v)\epsilon_v \in C^0(\Gamma_v, \text{Div}^0(X_{\bar{K}_v})).
\]

Then \(b'_v = E(b'_v)\) satisfies
\[
\partial b'_v = E(\Delta \partial P) = E(\text{res}_v(b')) = \text{res}_v(b').
\]

Further, \(\text{res}_v(b') = \partial b'_v\), which means that we may choose \(f'_v = 1\). Thus \(< \alpha, \beta' > = -\sum_v \text{inv}_v(\gamma'_v)\), where \(\gamma'_v\) is represented by \(f(b'_v) = f(\Delta P - (\Delta/\delta_v)(\Delta')) = f(P)^{\Delta/\gamma'_v}\).

Consequently
\[
\varepsilon(< \alpha, \beta' >) = \varepsilon \left( \sum_v \text{inv}_v(\gamma_v) \right)
= q \left( \sum_v \text{inv}_v(\gamma_v) \right) = \phi(\rho(\alpha)),
\]
as claimed. \(\Box\)
Corollary 2.10. Assume that \( \mathfrak{III}(A) \) contains no nonzero infinitely divisible elements. Then

\[
[T_3] = \frac{\delta'/\Delta'}{(\delta'/\Delta')_2}.
\]

Proof. Under our hypothesis the pairing (9) is nondegenerate, and the proposition shows that \( T_3 \) is isomorphic to the dual of \( <\beta'> = (\Delta'/\Delta')T_2 \). The corollary now follows from (8). □

The Main Theorem of the Introduction may now be obtained by combining Corollaries 2.8 and 2.10.

3. Concluding Remarks

As shown in the proof of Theorem 2.5, the order of \( A(K)/\text{Pic}^0(X_K) \) divides \( \prod \delta'_v/\Delta'_v \). We believe that it is always equal to \( \prod \delta'_v/\Delta'_v \). This is equivalent to the following plausible statement. Consider the nondegenerate pairing

\[
\bigoplus_v \mathbb{Z}/\delta'_v\mathbb{Z} \times \bigoplus_v A(K_v)/\text{Pic}^0(X_{K_v}) \rightarrow \mathbb{Q}/\mathbb{Z},
\]

which is the sum of the local pairings

\[
\mathbb{Z}/\delta'_v\mathbb{Z} \times A(K_v)/\text{Pic}^0(X_{K_v}) \rightarrow \mathbb{Q}/\mathbb{Z}
\]

induced by Lichtenbaum duality (see the discussion preceding the statement of Lemma 1.4). Then, relative to this pairing, the image of the diagonal map

\[
D : \mathbb{Z}/\delta'\mathbb{Z} \rightarrow \bigoplus_v \mathbb{Z}/\delta'_v\mathbb{Z}
\]

is the exact annihilator of the image of the diagonal map

\[
\mathcal{D} : A(K)/\text{Pic}^0(X_K) \hookrightarrow \bigoplus_v A(K_v)/\text{Pic}^0(X_{K_v}).
\]

Assuming (for simplicity) that \( \mathfrak{III}(A) \) is finite, the above conjectural statement implies in addition that the exact sequence

\[
0 \rightarrow \text{Ker} \, D \rightarrow \mathfrak{III}(A) \rightarrow \mathfrak{III}(P) \rightarrow \text{Coker} \, D \rightarrow \text{Coker} \, \lambda \rightarrow \text{Coker} \, \mu \rightarrow 0
\]

(which is the continuation of the exact sequence (7)). Here \( \lambda \) and \( \mu \) are the natural maps \( H^1(\Gamma, A) \rightarrow \bigoplus H^1(\Gamma_v, A) \) and \( H^1(\Gamma, P) \rightarrow \bigoplus H^1(\Gamma_v, P) \) splits into two short exact sequences

\[
0 \rightarrow \text{Ker} \, D \rightarrow \mathfrak{III}(A) \rightarrow \mathfrak{III}(P) \rightarrow 0
\]

and

\[
0 \rightarrow \text{Coker} \, D \rightarrow \text{Coker} \, \lambda \rightarrow \text{Coker} \, \mu \rightarrow 0,
\]

the second of which is the dual of

\[
0 \rightarrow \text{Pic}^0(X_K) \rightarrow A(K) \rightarrow \text{Im} \, \mathcal{D} \rightarrow 0.
\]

Thus if the above conjecture is true, then we can give a complete answer to Grothendieck’s question stated in the Introduction.
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