Existence of long time solutions and validity of the Nonlinear Schrödinger approximation for a quasilinear dispersive equation

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Abstract

We consider a nonlinear dispersive equation with a quasilinear quadratic term. We establish two results. First, we show that solutions to this equation with initial data of order $O(\varepsilon)$ in Sobolev norms exist for a time span of order $O(\varepsilon^{-2})$ for sufficiently small $\varepsilon$. Secondly, we derive the Nonlinear Schrödinger (NLS) equation as a formal approximation equation describing slow spatial and temporal modulations of the envelope of an underlying carrier wave, and justify this approximation with the help of error estimates in Sobolev norms between exact solutions of the quasilinear equation and the formal approximation obtained via the NLS equation.

The proofs of both results rely on estimates of appropriate energies whose constructions are inspired by the method of normal-form transforms. To justify the NLS approximation, we have to overcome additional difficulties caused by the occurrence of resonances. We expect that the method developed in the present paper will also allow to prove the validity of the NLS approximation for a larger class of quasilinear dispersive systems with resonances.

1 Introduction

In this paper, we consider the quasilinear dispersive equation

$$\partial_t u = K_0 u - u \partial_x u ,$$

(1)

where $x,t,u(x,t) \in \mathbb{R}$ and the linear operator $K_0$ is defined by its symbol

$$\hat{K}_0(k) = -i \tanh(k) .$$

(2)

First, we show that solutions of (1) with initial data of order $O(\varepsilon)$ in Sobolev norms exist for a time span of order $O(\varepsilon^{-2})$ for sufficiently small $\varepsilon$, although equation (1) has a quadratic nonlinearity. More precisely, we prove

**Theorem 1.1.** Let $s \geq 2$. There are constants $a, \varepsilon_0 > 0$ and $C \geq 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ and $u_0 \in H^s$ with

$$\|u_0\|_{H^s} \leq \varepsilon ,$$

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there exists a solution \( u \in C(I_\varepsilon, H^s) \cap C^1(I_\varepsilon, H^{s-1}) \), where \( I_\varepsilon = [-a/\varepsilon^2, a/\varepsilon^2] \), of (1) with \( u(x,0) = u_0(x) \) for all \( x \in \mathbb{R} \), which satisfies

\[
\sup_{t \in I_\varepsilon} \| u(t) \|_{H^s} \leq C \| u_0 \|_{H^s}.
\]

Secondly, we derive the Nonlinear Schrödinger (NLS) approximation for equation (1) and prove its validity. The NLS equation plays an important role in describing approximately slow modulations in time and space of an underlying spatially and temporarily oscillating wave packet in dispersive systems, for example, the water wave equations, see [1]. In order to derive the NLS approximation, we make the ansatz \( u = \varepsilon \psi = \varepsilon \psi_{NLS} + O(\varepsilon^2) \), with

\[
\varepsilon \psi_{NLS}(x,t) = \varepsilon A(\varepsilon(x - c_g t), \varepsilon^2 t, \varepsilon^2) e^{i(k_0 x - \omega_0 t)} + \text{c.c.}.
\]

Here \( 0 < \varepsilon \ll 1 \) is a small perturbation parameter, \( \omega_0 > 0 \) the basic temporal wave number associated to the basic spatial wave number \( k_0 > 0 \) of the underlying carrier wave \( e^{i(k_0 x - \omega_0 t)} \), \( c_g \) the group velocity, \( A \) the complex-valued amplitude, and c.c. the complex conjugate. With the help of (3) we describe slow spatial and temporal modulations of the envelope of the underlying carrier wave. Inserting the above ansatz into (1) we find that \( A \) satisfies at leading order in \( \varepsilon \) the NLS equation

\[
\partial_x A = i \nu_1 \partial^2_x A + i \nu_2 |A|^2 A, \quad (4)
\]

where \( X = \varepsilon(x - c_g t), T = \varepsilon^2 t, \) and \( \nu_j = \nu_j(k_0) \in \mathbb{R} \). \( T \) is the slow time scale and \( X \) is the slow spatial scale, that means, the time scale of the modulations is \( O(1/\varepsilon^2) \) and the spatial scale of the modulations is \( O(1/\varepsilon) \). See Figure 1. The basic spatial wave number \( k = k_0 \) and the basic temporal wave number \( \omega = \omega_0 \) are related via the linear dispersion relation of equation (1), namely

\[
\omega(k) = \tanh(k). \quad (5)
\]

Then the group velocity \( c_g \) of the wave packet is given by \( c_g = \partial_k \omega |_{k=k_0} \). Our ansatz leads to waves moving to the right. To obtain waves moving to the left, \( -\omega_0 \) and \( c_g \) have to be replaced by \( \omega_0 \) and \( -c_g \).

To justify the NLS approximation for (1), we prove

**Theorem 1.2.** Fix \( s_A \geq 7 \). Then for all \( k_0 > 0 \) and for all \( C_1, T_0 > 0 \) there exist \( C_2 > 0, \varepsilon_0 > 0 \) such that for all solutions \( A \in C([0, T_0], H^{s_A}(\mathbb{R}, \mathbb{C})) \) of the NLS equation (4) with

\[
\sup_{T \in [0, T_0]} \| A(\cdot, T) \|_{H^s_A(\mathbb{R}, \mathbb{C})} \leq C_1
\]

the following holds. For all \( \varepsilon \in (0, \varepsilon_0) \) there are solutions \( u \in C([0, T_0/\varepsilon^2], H^{s_A}(\mathbb{R}, \mathbb{R})) \) of equation (1) which satisfy

\[
\sup_{T \in [0, T_0/\varepsilon^2]} \| u(\cdot, t) - \varepsilon \psi_{NLS}(\cdot, t) \|_{H^s_A(\mathbb{R}, \mathbb{R})} \leq C_2 \varepsilon^{3/2}.
\]
Figure 1: The envelope (advancing with the group velocity $c_g$) of the oscillating wave packet (advancing with the phase velocity $c_p = \omega_0/k_0$) is described by the amplitude $A$ which solves the NLS equation (4).

The error of order $O(\varepsilon^{3/2})$ is small compared with the solution $u$ and the approximation $\varepsilon\psi_{NLS}$, which are both of order $O(\varepsilon)$ in $L^\infty$ such that the dynamics of the NLS equation can be found in equation (1), too. The NLS equation is a completely integrable Hamiltonian system, which can be solved explicitly with the help of some inverse scattering scheme, see, for example, [1]. We remark that such an approximation theorem should not be taken for granted. There are various counterexamples, where approximation equations derived by reasonable formal arguments make wrong predictions about the dynamics of the original systems, see, for example, [15, 17]. For an introduction into theory and applications of the NLS approximation we refer to [16].

Now, we explain the main ideas for the proofs of our theorems. The main difficulty in the proof of Theorem 1.1 is to show that $I_\varepsilon$ is of order $O(\varepsilon^{-2})$. If $u_0$ is of order $O(\varepsilon)$, then, due to the fact that the nonlinear term $-u\partial_x u$ is quadratic, direct energy estimates only guarantee an existence interval of order $O(\varepsilon^{-1})$ for $u$. A standard strategy to address this problem is to try to eliminate the quadratic term and transfer it into a cubic term with the help of a normal-form transform of the form

$$\tilde{u} := u + N(u, u),$$

where $N$ is an appropriately constructed bilinear mapping, see [21, 11]. In the case of equation (1), a direct computation of the evolution equation for $\tilde{u}$ with the help of equation (1) yields that $\tilde{u}$ solves an evolution equation of the form

$$\partial_t \tilde{u} = K_0 \tilde{u} + h(u, \partial_x u),$$

where $K_0$ is a constant linear operator and $h$ is a bilinear mapping.
where \( h(u, \partial_x u) \) is a cubic term if \( N \) satisfies
\[
- K_0 N(u, u) + N(K_0 u, u) + N(u, K_0 u) = \frac{1}{2} \partial_x (u^2). \tag{8}
\]
Since \( K_0 \) satisfies the identity
\[
K_0 (fg) - K_0 (f) g - f K_0 (g) = K_0 (K_0 (f) K_0 (g)), \tag{9}
\]
see Lemma 2.1 below, it follows
\[
N(u, u) = - \frac{1}{2} K_0^{-1} \partial_x (K_0^{-1} u)^2. \tag{10}
\]
However, this condition for \( N \) causes two problems. The first problem is that \( K_0^{-1} u \) may not exist, and the second one is that \( N(u, u) \) loses one derivative, that means, \( u \mapsto N(u, u) \) maps \( H^{m+1}(\mathbb{R}, \mathbb{C}) \) into \( H^m(\mathbb{R}, \mathbb{C}) \) or \( C^{n+1}(\mathbb{R}, \mathbb{C}) \) into \( C^n(\mathbb{R}, \mathbb{C}) \). Even if it was possible to invert the normal-form transform (6), the cubic term \( h \) expressed in terms of \( \tilde{u} \) would lose two derivatives such that it would not be possible to use equation (7) to derive closed energy estimates for \( \tilde{u} \).

To overcome these problems, we do not perform the normal-form transform (6) explicitly, but only use the term \( N \) to construct an energy of the form
\[
\mathcal{E}_s = \sum_{\ell=0}^{s} E_\ell, \tag{11}
\]
where the summands \( E_\ell \) are defined by a slight, \( \ell \)-dependent modification of the equation
\[
E_\ell = \frac{1}{2} \| \partial_\ell^x u \|_{L^2}^2 + \int_\mathbb{R} \partial_\ell^x u \partial_\ell^x N(u, u) \, dx \tag{12}
\]
to get around the problem that \( K_0^{-1} u \) may not exist. More precisely, since
\[
\frac{d}{dt} \| u \|_{L^2} = 0 \tag{13}
\]
for any sufficiently regular solution \( u \) of (1), see Lemma 2.6 below, we define
\[
E_0 := \frac{1}{2} \| u \|_{L^2}^2. \tag{14}
\]
Moreover, due to
\[
\int_\mathbb{R} \partial_\ell^x f \partial_\ell^x (f \partial_x f) \, dx = \sum_{a=1}^{\ell-1} \binom{\ell}{a} \int_\mathbb{R} \partial_\ell^x f \partial_a^x f \partial_\ell^{\ell-a+1} f \, dx + \frac{1}{2} \int_\mathbb{R} \partial_\ell^x f \partial_\ell^x \partial_x f \, dx \tag{15}
\]
for sufficiently regular functions \( f \), which follows with the help of Leibniz’ rule and partial integration, and because of the facts that \( K_0^{-1} \) is skew symmetric and \( K_0^{-1} \partial_x g \) exists for any \( g \in H^1 \), we define

\[
E_\ell := \frac{1}{2} \| \partial_\ell^x u \|_{L^2}^2 + \sum_{a=1}^{\ell-1} \left( \frac{\ell}{a} \right) \int_{\mathbb{R}} K_0^{-1} \partial_\ell^x u K_0^{-1} \partial_\ell^a u K_0^{-1} \partial_{\ell-a+1}^u dx
\]

\[
+ \frac{1}{2} \int_{\mathbb{R}} K_0^{-1} \partial_\ell^x u K_0^{-1} \partial_\ell^x u K_0^{-1} \partial_x u dx
\]

for \( \ell > 0 \).

\( \sqrt{E_s} \) is equivalent to \( \| u \|_{H^s} \) for \( s \geq 2 \) and \( \| u \|_{H^1} = \mathcal{O}(\varepsilon) \), see Lemma 2.5. Due to the skew symmetry of \( K_0 \) and \( \mathbb{S} \), the right-hand side of the evolution equation for \( E_s \) contains neither quadratic nor cubic terms. Moreover, the right-hand side of the evolution equation for \( E_s \) can be written as a sum of integral terms containing at most one factor \( \partial^{n+1} u \) and not two. Consequently, using partial integration and estimates for the commutator \( [K_0^{-1}, u] \partial_x u \), we obtain

\[
\frac{d}{dt} E_s \lesssim \varepsilon^2 E_s
\]

as long as \( \| u \|_{H^2} = \mathcal{O}(\varepsilon) \) such that Gronwall’s inequality yields the \( \mathcal{O}(1) \)-boundedness of \( E_s \) and hence of \( u \) for all \( t \in I_\varepsilon \). For further details, see Section 2.

There is an equation which is related to (1), namely

\[
\partial_t u = H u - u \partial_x u ,
\]

where \( x, t, u(x, t) \in \mathbb{R} \) and \( H \) is the Hilbert transform. For this equation, the analogon of Theorem 1.1 was proven in [9]. The proof also relies on energy estimates inspired by a normal-form transform of the form (1), but the details of the proof are simpler in the following sense. Since the Hilbert transform also satisfies the identity (11) (with \( K_0 \) replaced by \( H \)), one obtains for the bilinear mapping \( N \) the condition (10) with \( K_0^{-1} \) replaced by \( H^{-1} \). Because \( H^{-1} = -H \) is well-defined in \( L^2 \), an appropriate energy can be defined directly by (11) and (12).

In [8] and [10], the techniques from [9] were further developed to apply them to the water wave equations with infinite depth of water. Since the water wave equations with finite depth of water can be obtained from the water wave equations with infinite depth of water by replacing \( H \) by \( K_0 \), we think that the techniques developed in the present paper can also be generalized to apply them to the water wave equations with finite depth of water.

In order to prove Theorem 1.2 we estimate the error

\[
\varepsilon^3 R := u - \varepsilon \psi
\]
for all $t \in [0, T_0/\varepsilon^2]$ to be of order $O(\varepsilon^\beta)$ for a $\beta \geq 3/2$, that means, we prove that $R$ is of order $O(1)$ for all $t \in [0, T_0/\varepsilon^2]$. The error $R$ satisfies the equation

$$
\partial_t R = K_0 R + 2\varepsilon B(\psi, R) + \varepsilon^\beta B(R, R) + \varepsilon^{-\beta} \text{Res}(\varepsilon \psi) \tag{20}
$$

with

$$
B(f, g) = -\partial_x(fg). \tag{21}
$$

If $A$ is a sufficiently regular solution of the NLS equation $[4]$, it is possible to construct an approximation function $\varepsilon \psi$ with $\varepsilon \psi = \varepsilon \psi_{NLS} + O(\varepsilon^2)$ and to choose $\beta$ such that

$$
\partial_t R = K_0 R + 2\varepsilon B(\psi, R) + O(\varepsilon^2). \tag{22}
$$

Since the Fourier transform $\hat{\varepsilon \psi}_{NLS}$ is strongly concentrated around the wave numbers $\pm k_0$, the approximation $\varepsilon \psi$ can be split into

$$
\varepsilon \psi = \varepsilon \psi_c + \varepsilon^2 \psi_s \tag{23}
$$

with

$$
\text{supp} \hat{\psi}_c = \{k \in \mathbb{R} : |k \mp k_0| \leq \delta\}, \tag{24}
$$

where $\delta \in (0, k_0)$ is small, but independent of $\varepsilon$. Hence, we have

$$
\partial_t R = K_0 R + 2\varepsilon B(\psi_c, R) + O(\varepsilon^2) \tag{25}
$$

such that the main difficulty is to control the quadratic term $2\varepsilon B(\psi_c, R)$ for $t \in [0, T_0/\varepsilon^2]$.

Constructing again a normal-form transform of the form

$$
\tilde{R} := R + \varepsilon N(\psi_c, R), \tag{26}
$$

with an appropriate bilinear mapping $N$, to obtain

$$
\partial_t \tilde{R} = K_0 \tilde{R} + O(\varepsilon^2) \tag{27}
$$

yields

$$
-K_0 N(\psi_c, R) + N(K_0 \psi_c, R) + N(\psi_c, K_0 R) = -2B(\psi_c, R) \tag{28}
$$

such that because of $[39]$ and $[21]$ it follows

$$
N(\psi_c, R) = -2K_0^{-1} \partial_x(K_0^{-1} \psi_c K_0^{-1} R). \tag{29}
$$

$K_0^{-1} \psi_c$ exists due to $[24]$, but we have the problems that $K_0^{-1} R$ may not exist and that $R \mapsto N(\psi_c, R)$ loses one derivative. Since the $L^2$-norm of $R$ is not a conserved quantity and $N$ depends on the two different functions $\psi_c$ and $R$, we cannot define an energy in an analogous way as in the proof of Theorem $[14]$ to overcome these problems. Nevertheless,
it is still possible to use the method of normal-form transforms for constructing an appropriate energy to control the error, but it takes some additional effort.

The problem that \( K_0^{-1} R \) may not exist is related to the occurrence of so-called resonances. In Fourier space, we have

\[
\hat{N}(\psi_c, R)(k) = \int_R \hat{n}(k, k - m, m) \hat{\psi}_c(k - m) \hat{R}(m) \, dm
\]

with

\[
\hat{n}(k, k - m, m) = -2ik \hat{K}_0^{-1}(k) \hat{K}_0^{-1}(k - m) \hat{K}_0^{-1}(m) .
\]

Because of (24), it is instructive to analyze the behavior of \( \hat{n} \) for \( |k - m| \approx k_0 \). We have

\[
\hat{n}(k, k - m, m) \approx \begin{cases} 
- \frac{2ik}{K_0(k) \hat{K}_0(k_0) \hat{K}_0(k - k_0)} & \text{for } k - m \approx k_0 , \\
- \frac{2ik}{K_0(k) \hat{K}_0(-k_0) \hat{K}_0(k + k_0)} & \text{for } k - m \approx -k_0 .
\end{cases}
\]

The denominators of the fractions in (32) have the following zeros, which are called resonances. Both denominators have a zero at \( k = 0 \). Since the numerators also vanish at \( k = 0 \) and \( \lim_{|k| \to 0} k / \tanh(k) = 1 \), the singularity at \( k = 0 \) is removable. Such a resonance is called a trivial resonance. The fact that the resonance at \( k = 0 \) is trivial correlates with the fact that \( K_0^{-1} \partial_x g \) exists for any \( g \in H^1 \). Moreover, both denominators have one more zero - the first denominator at \( k = k_0 \), the second one at \( k = -k_0 \). At these resonances, the respective numerators do not vanish. Such a resonance is called a non-trivial resonance. The fact that the resonances at \( k = \pm k_0 \) are non-trivial correlates with the fact that \( K_0^{-1} R \) may not exist.

In the situation of a trivial resonance at \( k = 0 \) and non-trivial resonances at \( k = \pm k_0 \), it is possible to apply a technique from [6] for constructing a modified normal-form transform. The construction principle is as follows.

Since \( \hat{B}(\psi_c, R)(k) \) vanishes at \( k = 0 \), one can expect that \( \hat{R}(k) \) will grow for \( k \) near 0 more slowly than for \( k \) further away from 0. Hence, it makes sense to rescale the error with the help of the weight function

\[
\hat{\vartheta}(k) = \begin{cases} 
1 & \text{for } |k| > \delta , \\
\varepsilon + (1 - \varepsilon)|k|/\delta & \text{for } |k| \leq \delta ,
\end{cases}
\]

where \( \delta \) is chosen as above. More precisely, by writing

\[
u = \varepsilon \psi_c + \varepsilon^2 \psi_s + \varepsilon^{5/2} \partial R ,
\]

where \( \partial R \) is defined by \( \hat{\partial} \hat{R} \), one obtains for the rescaled error \( R \) an evolution equation of the form

\[
\partial_t R = K_0 R - 2 \varepsilon \partial^{-1} \partial_x (\psi_c \partial P_{\psi, \infty} R) + O(\varepsilon^2) .
\]
Here, $P_{\varepsilon, \infty}$ is a linear operator with the symbol
\[ \hat{P}_{\varepsilon, \infty}(k) = (1 - \chi_{[-\varepsilon, \varepsilon]})(k), \]
where $\chi_{[-\varepsilon, \varepsilon]}$ is the characteristic function on $[-\varepsilon, \varepsilon]$. Now, constructing a normal-form transform of the form (26) yields
\[ N(\psi, R) = -2\vartheta^{-1} K_0^{-1} \partial_x (K_0^{-1} \psi \partial_P \varepsilon, R), \tag{36} \]
where $K_0^{-1} \partial_P \varepsilon, R$ exists for any $R \in L^2$. However, since $(\hat{\vartheta}(k))^{-1} = O(\varepsilon^{-1})$ for $|k| < \delta$, the transformed error $\tilde{R}$ satisfies an evolution equation of the form
\[ \partial_t \tilde{R} = K_0 \tilde{R} - \varepsilon \sum_{j=\pm 1} (1 - P_{\delta, \infty}) N(\psi_j, 2\varepsilon \vartheta^{-1} \partial_x (\psi_j \partial_P \varepsilon, R)) + O(\varepsilon^2), \tag{37} \]
with $\hat{\psi}_j = \hat{\psi}_c|_{jk_0 - \delta, jk_0 + \delta}$ and $(1 - P_{\delta, \infty}) N(\psi_j, 2\varepsilon \vartheta^{-1} \partial_x (\psi_j \partial_P \varepsilon, R)) = O(1)$. But the term of order $O(\varepsilon)$ on the right-hand side of (37) can be eliminated with the help of a second normal-form transform of the form
\[ \hat{\tilde{R}} = \tilde{R} + \varepsilon^2 \sum_{j=\pm 1} T_j(\psi_j, \psi_j, R) \tag{38} \]
with appropriate trilinear mappings $T_j$. The construction of the trilinear mappings is similar to the construction of bilinear mappings for normal-form transforms. In the case of equation (37), no resonances occur in the context of the construction of the trilinear mappings such that straightforward calculations yield
\[ \hat{T}_j(\psi_j, \psi_j, R)(k) = \int_R \int_R \hat{t}_j(k) \hat{\psi}_j(k - m) \hat{\psi}_j(m - n) \hat{R}(n) dn \, dm \tag{39} \]
with
\[ \hat{t}_j(k) = -\frac{k(k - jk_0 \vartheta(k - 2jk_0) \chi_{[-\delta, \delta]}(k)}{\vartheta(k) \tanh(k) \tanh(jk_0) \tanh(k - jk_0)} \times (\tanh(k) - 2 \tanh(jk_0) - \tanh(k - 2jk_0))^{-1}. \tag{40} \]
After these two normal-form transforms we have
\[ \partial_t \hat{R} = K_0 \hat{R} + O(\varepsilon^2). \tag{41} \]
For further details about the two normal-form transforms discussed just now, we refer to [6].

However, since the error equation (20) is quasilinear, also the modified normal-form transform $R \mapsto \hat{R}(R)$ loses one derivative. It can be shown that this normal-form transform is nevertheless invertible, but the term of order $O(\varepsilon^2)$ in the transformed error equation (41) loses two derivatives if it is expressed in terms of $\hat{R}$.

To overcome the regularity problems, we pursue again the strategy from the proof of Theorem 1.1 that we do not perform the normal-form transform explicitly, but only use it to construct an energy of the form
\[ \hat{E}_s = \sum_{\ell=0}^{s} \hat{E}_\ell, \tag{42} \]
where the summands $\tilde{E}_\ell$ are defined by a slight, $\ell$-dependent modification of the equation

$$\tilde{E}_\ell = \frac{1}{2} \| \partial^\ell_x R \|_{L^2}^2 + \varepsilon \int_\mathbb{R} \partial^\ell_x R \partial^\ell_x N(\psi_c, R) \, dx + \varepsilon^2 \sum_{j=\pm 1} \int_\mathbb{R} \partial^\ell_x R \partial^\ell_x T_j(\psi_j, \psi_j, R) \, dx, \quad (43)$$

where $R$ is defined by (34), $N$ is defined by (36), and the mappings $T_j$ are as in (38).

Since $k^\ell(\hat{\vartheta}(k))^{-1} = \mathcal{O}(1)$ for $|k| < \delta$ if $\ell \geq 1$, we do not need to include the second normal-form transform in our energy for $\ell \geq 1$. Hence, we define

$$\tilde{E}_\ell := \frac{1}{2} \| \partial^\ell_x R \|_{L^2}^2 + \varepsilon \int_\mathbb{R} \partial^\ell_x R \partial^\ell_x N(\psi_c, R) \, dx \quad (44)$$

for $\ell \geq 1$. Then partial integration yields

$$\tilde{E}_\ell = \frac{1}{2} \| \partial^\ell_x R \|_{L^2}^2 + \varepsilon \mathcal{O}(\| R \|_{H^s}^2) \quad (45)$$

for $\ell \geq 1$. Because the mapping

$$R \mapsto \int_\mathbb{R} R \hat{R}(R) \, dx$$

is in general not positive definite, we have to perform the full normal-form transform in the case of $\ell = 0$ and define

$$\tilde{E}_0 := \| \hat{R} \|_{L^2}^2. \quad (46)$$

The resulting loss of regularity does not mind here because it can be compensated with the help of the other components of our energy such that we obtain the equivalence of $\sqrt{\tilde{E}_s}$ and $\| R \|_{H^s}$ for $s \geq 1$ and sufficiently small $\varepsilon$, see Corollary 4.7. Consequently, the right-hand side of the evolution equation of $\tilde{E}_s$ can be written as a sum of integral terms containing at most one factor $\partial^{s+1}_x R$ and not two. Moreover, since $\| \hat{R} \|_{H^s}$ differs from $\tilde{E}_s$ only by terms of order $\mathcal{O}(\varepsilon^2)$, the evolution equations of $\tilde{E}_s$ and $\| \hat{R} \|_{H^s}$ share the property that their right-hand sides are of order $\mathcal{O}(\varepsilon^2)$. Therefore, by using partial integration, we obtain

$$\partial_t \tilde{E}_s \lesssim \varepsilon^2 (\tilde{E}_s + 1) \quad (47)$$

as long as $\| R \|_{H^s} = \mathcal{O}(1)$ such that Gronwall’s inequality yields the $\mathcal{O}(1)$-boundedness of $\tilde{E}_s$ and hence of $R$ for all $t \in [0, T_0/\varepsilon^2]$.

For the reasons discussed above, the justification of the NLS approximation for dispersive systems with quasilinear quadratic terms is a highly nontrivial problem, which has been remained unsolved in general for more than four decades. The first and very general NLS approximation theorem for quasilinear dispersive wave systems was shown in [11]. However, the occurrence of quasilinear quadratic terms was excluded explicitly. In the case of quasilinear quadratic terms, an NLS approximation theorem was proven for dispersive wave systems where the right-hand sides lose only half a derivative. The 2D water wave problem without surface tension and finite depth of water in
Lagrangian coordinates falls into this class. In this case the elimination of the quadratic terms is possible with the help of normal-form transforms. The right-hand sides of the transformed systems then lose one derivative and can be handled with the help of the Cauchy-Kowalevskaya theorem \cite{18, 7}. Furthermore, the NLS approximation was justified for the 2D and 3D water wave problem without surface tension and infinite depth of water \cite{23, 22} by finding a different transform adapted to the special structure of that problem. Similarly, for the quasilinear Korteweg-de Vries equation the result can be obtained by simply applying a Miura transform \cite{20}. In \cite{2}, the NLS approximation of time oscillatory long waves for equations with quasilinear quadratic terms was proven for analytic data without using a normal-form transform. Moreover, another approach to address the problem of the validity of the NLS approximation can be found in \cite{13}. Finally, some numerical evidence that the NLS approximation is also valid for quasilinear equations was given in \cite{3}.

Very recently, the first validity proof of the NLS approximation of a nonlinear wave equation with a quasilinear quadratic term in Sobolev spaces was given in \cite{5}. The proof also relies on estimates of an appropriate energy which is constructed with the help of a normal-form transform. The construction of the energy is easier in the sense that no problems with resonances occur, but more difficult in the sense that the energy has to allow to control a system of two coupled error equations.

In forthcoming papers, we intend to combine the methods of the present paper with the methods from \cite{5, 6} to prove the validity of the NLS approximation for a larger class of quasilinear dispersive systems with resonances.

The plan of the paper is as follows. In Section \ref{2} we prove Theorem \ref{1.1}. In Section \ref{3} we derive the NLS approximation. In Section \ref{4} we perform the error estimates to prove Theorem \ref{1.2}.

**Notation.** We denote the Fourier transform of a function $u \in L^2(\mathbb{R}, \mathbb{K})$, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ by
\[
F(u)(k) = \hat{u}(k) = \frac{1}{2\pi} \int_{\mathbb{R}} u(x)e^{-ikx}dx.
\]
Let $H^s(\mathbb{R}, \mathbb{K})$ be the space of functions mapping from $\mathbb{R}$ into $\mathbb{K}$ for which the norm
\[
\|u\|_{H^s(\mathbb{R}, \mathbb{K})} = \left( \int_{\mathbb{R}} |\hat{u}(k)|^2(1+|k|^2)^s dk \right)^{1/2}
\]
is finite. We also write $L^2$ and $H^s$ instead of $L^2(\mathbb{R}, \mathbb{R})$ and $H^s(\mathbb{R}, \mathbb{R})$. Moreover, we use the space $L^p(m)(\mathbb{R}, \mathbb{K})$ defined by $u \in L^p(m)(\mathbb{R}, \mathbb{K}) \iff u\sigma^m \in L^p(\mathbb{R}, \mathbb{K})$, where $\sigma(x) = (1 + x^2)^{1/2}$.

Furthermore, we write $A \lesssim B$, if $A \leq CB$ for a constant $C > 0$, and $A = O(B)$, if $|A| \lesssim B$. 

10
2 Long time solutions

In this section, we prove Theorem 1.1. To address this issue, we will need the following properties of the operator $K_0$.

**Lemma 2.1.** Let $f, g \in L^2$ and $fg \in L^2$. Then we have

$$K_0(fg) - K_0(f)g - fK_0(g) = K_0(K_0(f)K_0(g)).$$

(48)

**Proof.** Considering the symbol of $K_0$, we obtain the assertion of the Lemma due to

$$\tanh(k) - \tanh(m) - \tanh(k-m) = -\tanh(k)\tanh(m)\tanh(k-m)$$

for all $m, k \in \mathbb{R}$, which can be directly verified.

**Lemma 2.2.** $K_0^{-1}\partial_x$ is a continuous linear operator from $H^{s+1}$ into $H^s$ for any $s \geq 0$ and satisfies

$$\|K_0^{-1}\partial_x f\|_{H^s} \leq \|f\|_{H^{s+1}}$$

(49)

**Proof.** The assertion of the Lemma is a consequence of

$$|k| \leq |\tanh(k)|(1 + k^2)^{1/2}$$

(50)

for all $k \in \mathbb{R}$, which can be directly verified.

**Lemma 2.3.** Let $j \geq 0$, $q > \frac{1}{2}$, $r \geq \max\{1 + q, j\}$ and $u \in H^r$. Then we have the commutator estimate

$$\|[K_0^{-1}, u]\partial_x u\|_{H^j} \lesssim \|u\|_{H^{1+q}}\|u\|_{H^j}.$$  

(51)

**Proof.** Since $u\partial_x u = \frac{1}{2}\partial_x (u^2)$, the Fourier transform of $[K_0^{-1}, u]\partial_x u$ has the two representations

$$\mathcal{F}([K_0^{-1}, u]\partial_x u)(k) = -\int_{\mathbb{R}} \left( \frac{m}{\tanh(k)} - \frac{m}{\tanh(m)} \right) \hat{u}(k-m)\hat{u}(m) \, dm$$

$$= -\int_{\mathbb{R}} \left( \frac{k}{2\tanh(k)} - \frac{m}{\tanh(m)} \right) \hat{u}(k-m)\hat{u}(m) \, dm.$$  

Hence, using Young’s inequality for convolutions and the Cauchy-Schwarz inequality, we obtain

$$\|[K_0^{-1}, u]\partial_x u\|_{H^j} \lesssim \left( \sup_{k,m \in \mathbb{R}} G(k,m) \right) \|\hat{u}\|_{L^2(1+q)}\|\hat{u}\|_{L^2(j)}$$

$$\lesssim \left( \sup_{k,m \in \mathbb{R}} G(k,m) \right) \|u\|_{H^{1+q}}\|u\|_{H^j}.$$
with

\[
G(k, m) = \begin{cases} 
\frac{k}{2 \tanh(k)} - \frac{m}{\tanh(m)} \frac{(1 + k^2)^{j/2}}{(1 + (k - m)^2)^{j/2}(1 + m^2)^{1/2}} & \text{for } |k| \leq 1, \\
\frac{m}{\tanh(k)} - \frac{m}{\tanh(m)} \frac{(1 + k^2)^{j/2}}{(1 + (k - m)^2)^{j/2}(1 + m^2)^{1/2}} & \text{for } |k| > 1, 
\end{cases}
\]

where \( y \mapsto y / \tanh(y) \) is continued by 1 for \( y = 0 \).

In order to show the boundedness of the supremum we distinguish three cases. \( G(k, m) \) is obviously uniformly bounded for all \((k, m) \in \mathbb{R}^2\) with \(|k| \leq 1\) or \(|m| \leq 1\). If \(|k|, |m| \geq 1\) and \(\text{sign}(m) = \text{sign}(k)\), we have

\[
\left| \frac{1}{\tanh(k)} - \frac{1}{\tanh(m)} \right| \lesssim e^{-2|k|} + e^{-2|m|}
\]

such that

\[
G(k, m) \lesssim \frac{e^{-2|k|(1 + k^2)^{j/2}m|}}{(1 + (k - m)^2)^{j/2}(1 + m^2)^{1/2}} + \frac{e^{-2|m|(1 + 2(k - m)^2 + 2m^2)^{j/2}}|m|}{(1 + (k - m)^2)^{j/2}(1 + m^2)^{1/2}}.
\]

Consequently, \( G(k, m) \) is also uniformly bounded in this case. Finally, if \(|k|, |m| \geq 1\) and \(\text{sign}(m) = -\text{sign}(k)\), we have

\[
\left| \frac{1}{\tanh(k)} - \frac{1}{\tanh(m)} \right| \lesssim 1
\]

such that

\[
G(k, m) \lesssim \frac{(1 + k^2)^{j/2}|m|}{(1 + (|k| + |m|)^2)^{j/2}(1 + m^2)^{1/2}}
\]

and \( G(k, m) \) is uniformly bounded in this case as well. Hence, the supremum is bounded, which implies the assertion of the lemma.

Moreover, we will use the well-known interpolation inequalities

\[
\|\partial_x^j f \partial_x^\ell g\|_{L^2} \lesssim \|g\|_{L^\infty} \|\partial_x^{j+\ell} f\|_{L^2} + \|\partial_x^{j+\ell} g\|_{L^2} \|f\|_{L^\infty}, \tag{52}
\]

\[
\|\partial_x^j f \partial_x^\ell g\|_{L^2} \lesssim \|g\|_{H^q} \|\partial_x^{j+\ell} f\|_{L^2} + \|\partial_x^{j+\ell} g\|_{L^2} \|f\|_{H^q}, \tag{53}
\]

for \( j, \ell \in \mathbb{N}, k \geq 1, j + \ell \leq k, \frac{1}{2} < q \leq k \), and \( f, g \in H^k \) as well as the identity

\[
\int_{\mathbb{R}} fg \partial_x f \, dx = -\frac{1}{2} \int_{\mathbb{R}} f^2 \partial_x g \, dx \tag{54}
\]

for \( f, g \in H^1 \).
As motivated in Section 1, we define the energy

$$E_s := \sum_{\ell=0}^{s} E_{\ell},$$

(55)

with

$$E_0 := \frac{1}{2} \|u\|_{L^2}^2$$

(56)

and

$$E_\ell := \frac{1}{2} \|\partial_\ell^\nu u\|_{L^2}^2 + \sum_{a=2}^{\ell-1} \left( \begin{array}{c} \ell \\ a \end{array} \right) \int_{\mathbb{R}} K_0^{-1} \partial_\ell^\nu u K_0^{-1} \partial_x^a u K_0^{-1} \partial_x^{\ell-a+1} u \, dx$$

(57)

$$+ \frac{1}{2} \int_{\mathbb{R}} K_0^{-1} \partial_\ell^\nu u K_0^{-1} \partial_x^a u K_0^{-1} \partial_x u \, dx$$

for $\ell > 0$.

**Remark 2.4.** One may wonder why we do not write $E_\ell$ for $\ell > 0$ in the equivalent form

$$E_\ell = \frac{1}{2} \|\partial_\ell^\nu u\|_{L^2}^2 + \sum_{a=2}^{\ell-1} \left( \begin{array}{c} \ell \\ a \end{array} \right) \int_{\mathbb{R}} K_0^{-1} \partial_\ell^\nu u K_0^{-1} \partial_x^a u K_0^{-1} \partial_x^{\ell-a+1} u \, dx$$

$$+ \frac{2\ell + 1}{2} \int_{\mathbb{R}} K_0^{-1} \partial_\ell^\nu u K_0^{-1} \partial_x^a u K_0^{-1} \partial_x u \, dx,$$

but it turns out that the form (16) is more convenient for our calculations below.

The Cauchy-Schwarz inequality and (52) directly imply

**Lemma 2.5.** For $\ell \geq 1$, we have

$$E_\ell = \frac{1}{2} \|\partial_\ell^\nu u\|_{L^2}^2 + \mathcal{O}(\|K_0^{-1} \partial_x u\|_{L^\infty}) \|K_0^{-1} \partial_\ell^\nu u\|_{L^2}^2.$$ 

(58)

Now, we would like to estimate the time derivative of $E_s$ for any sufficiently regular solution of (1). We obtain

**Lemma 2.6.** For $\ell = 0$, we have

$$\frac{d}{dt} E_0 = 0.$$ 

(59)

**Proof.** Due to (1), the skew symmetry of $K_0$ and (54), we get

$$\frac{d}{dt} E_0 = \int_{\mathbb{R}} u K_0 \partial_x u \, dx - \int_{\mathbb{R}} u^2 \partial_x u \, dx = 0.$$
Lemma 2.7. For \( \ell \geq 1 \), we have
\[
\frac{d}{dt} E_\ell \lesssim \| u \|_{H^2}^2 \| u \|_{H^\ell}^2.
\]  

Proof. Using (1), we get
\[
\frac{d}{dt} E_\ell = \int \partial^\ell_x u \partial^\ell_x K_0 u \, dx - \int \partial^\ell_x u \partial^\ell_x (u \partial_x u) \, dx
\]
\[
+ \sum_{a=1}^{\ell-1} \left( \binom{\ell}{a} \int \partial^\ell_x u K_0^{-1} \partial^a_x u K_0^{-1} \partial^{\ell-a+1}_x u \, dx + \frac{1}{2} \int \partial^\ell_x u K_0^{-1} \partial^a_x u K_0^{-1} \partial_x u \, dx \right)
\]
\[
+ \sum_{a=1}^{\ell-1} \left( \binom{\ell}{a} \int K_0^{-1} \partial^\ell_x u \partial^a_x u K_0^{-1} \partial^{\ell-a+1}_x u \, dx + \frac{1}{2} \int K_0^{-1} \partial^\ell_x u K_0^{-1} \partial^a_x u \partial_x u \, dx \right)
\]
\[
- \sum_{a=1}^{\ell-1} \left( \binom{\ell}{a} \int K_0^{-1} \partial^\ell_x (u \partial_x u) K_0^{-1} \partial^a_x u K_0^{-1} \partial^{\ell-a+1}_x u \, dx \right)
\]
\[
- \sum_{a=1}^{\ell-1} \left( \binom{\ell}{a} \int K_0^{-1} \partial^\ell_x (u \partial_x u) K_0^{-1} \partial^a_x u K_0^{-1} \partial^{\ell-a+1}_x (u \partial_x u) \, dx \right)
\]
\[
- \frac{1}{2} \int K_0^{-1} \partial^\ell_x u K_0^{-1} \partial^\ell_x u K_0^{-1} \partial_x (u \partial_x u) \, dx.
\]

Due to the skew symmetry of \( K_0 \), the first integral equals zero. By construction all cubic terms cancel. This can be seen by using Leibniz’s rule and (52) for the second integral as well as (18) and the skew symmetry of \( K_0 \) for the other cubic terms. Hence, we have
\[
\frac{d}{dt} E_\ell = - \sum_{a=1}^{\ell} \left( \binom{\ell}{a} \int K_0^{-1} \partial^\ell_x (u \partial_x u) K_0^{-1} \partial^a_x u K_0^{-1} \partial^{\ell-a+1}_x u \, dx \right)
\]
\[
- \frac{1}{2} \int K_0^{-1} \partial^\ell_x u K_0^{-1} \partial^\ell_x u K_0^{-1} \partial_x (u \partial_x u) \, dx.
\]
Since $\binom{\ell}{a} = \binom{\ell}{\ell-a}$, we obtain by partial integration
\[
\frac{d}{dt} E_{\ell} = \sum_{a=1}^{\ell} \binom{\ell}{a} I_{a},
\]
with
\[
I_{a} = - \int_{\mathbb{R}} K_{0}^{i} \partial_{x}^{\ell-a} (u \partial_{x} u) K_{0}^{-1} \partial_{x}^{\ell-a+1} u \, dx
+ \int_{\mathbb{R}} K_{0}^{-1} \partial_{x}^{\ell-a} (u \partial_{x} u) K_{0}^{i} \partial_{x}^{\ell-a+1} u \, dx
= - \int_{\mathbb{R}} \partial_{x}^{\ell-a} (u K_{0}^{i} \partial_{x} u) K_{0}^{i} \partial_{x}^{\ell-a+1} u \, dx
+ \int_{\mathbb{R}} K_{0}^{-1} \partial_{x}^{\ell-a} (u K_{0}^{i} \partial_{x} u) K_{0}^{i} \partial_{x}^{\ell-a+1} u \, dx
- \int_{\mathbb{R}} \partial_{x}^{\ell-a} [K_{0}^{-1}, u] \partial_{x} u K_{0}^{i} \partial_{x}^{\ell-a+1} u \, dx
+ \int_{\mathbb{R}} K_{0}^{-1} \partial_{x}^{\ell-a} [K_{0}^{-1}, u] \partial_{x} u K_{0}^{i} \partial_{x}^{\ell-a+1} u \, dx
= : \sum_{k=1}^{4} I_{a,k}.
\]
For $I_{a,1} + I_{a,2}$, Leibniz’s rule, multiple integration by parts, the Cauchy-Schwarz inequality and (53) yield
\[
I_{a,1} + I_{a,2} = - \int_{\mathbb{R}} u K_{0}^{i} \partial_{x}^{\ell+1} u K_{0}^{i} \partial_{x}^{\ell-a+1} u \, dx
- \sum_{i=1}^{\ell} \binom{\ell}{i} \int_{\mathbb{R}} \partial_{x}^{i} u K_{0}^{-1} \partial_{x}^{\ell-i+1} u K_{0}^{i} \partial_{x}^{\ell-a+1} u \, dx
+ \int_{\mathbb{R}} K_{0}^{-1} \partial_{x}^{\ell-a} u K_{0}^{i} \partial_{x}^{\ell-a+1} u \, dx
+ \sum_{j=1}^{\ell-a} \binom{\ell-a}{j} \int_{\mathbb{R}} K_{0}^{-1} \partial_{x}^{\ell-a+1} u \partial_{x}^{j} u K_{0}^{-1} \partial_{x}^{\ell-a-j+1} u \, dx
= - \sum_{i=1}^{\ell} \binom{\ell}{i} \int_{\mathbb{R}} \partial_{x}^{i} u K_{0}^{-1} \partial_{x}^{\ell-i+1} u K_{0}^{i} \partial_{x}^{\ell-a+1} u \, dx
+ \sum_{j=1}^{\ell-a} \binom{\ell-a}{j} \int_{\mathbb{R}} K_{0}^{-1} \partial_{x}^{\ell-a+1} u \partial_{x}^{j} u K_{0}^{-1} \partial_{x}^{\ell-a-j+1} u \, dx
\lesssim \|u\|_{H^{2}}^{2} \|u\|_{H^{\ell}}^{2}.
\]
Finally, using the Cauchy-Schwarz inequality, (51) and (53), we get
\[
I_{a,3} \lesssim \|u\|_{H^{2}}^{2} \|u\|_{H^{\ell}}^{2},
\]
for
and with the aid of partial integration, the Cauchy-Schwarz inequality, (51), (53) and (54), we obtain
\[ I_{a,4} \lesssim \|u\|_{H^2}^2 \cdot \|u\|_{H^\ell}^2. \]

Now, combining the estimates (59), (60) and (58), we get
\[ \frac{d}{dt} E_s \lesssim \varepsilon^2 E_s \]
for any solution \( u \in C(I, H^s) \cap C^1(I, H^{s-1}) \), where \( I \subseteq \mathbb{R} \) and \( s \geq 2 \), of (1) with \( \|u\|_{H^2} \leq \varepsilon \). Because of the local existence results for quasi-linear symmetric hyperbolic systems from [12] and Gronwall’s inequality, we obtain the \( O(1) \)-boundedness of \( E_s \) and therefore of \( \|u\|_{H^s} \) for all \( t \in I_\varepsilon \), which proves Theorem 1.1.

3 The derivation of the NLS approximation

In this section, we derive the NLS equation as an approximation equation for the quasi-linear dispersive equation (1). In doing so, we make the ansatz
\[ u = \varepsilon \tilde{\psi} = \varepsilon \tilde{\psi}_1 + \varepsilon^2 \tilde{\psi}_0 + \varepsilon^2 \tilde{\psi}_2 + \varepsilon^2 \tilde{\psi}_-2, \]
with \( \tilde{\psi}_j(x, t) = \tilde{A}_j(\varepsilon(x - c_g t), \varepsilon^2 t) E^j \) and \( \tilde{A}_j = \tilde{A}_j \) for \( j \in \{0, 1, 2\} \), where \( 0 < \varepsilon \ll 1 \), \( k_0 > 0 \), \( \omega_0 = \tanh(k_0) \), \( c_g = \tanh'(k_0) = \sech^2(k_0) \), and \( E = e^{i(4k_0x - \omega_0 t)} \).

Remark 3.1. Our ansatz leads to waves moving to the right. For waves moving to the left one has to replace in the above ansatz \( \omega_0 \) by \( -\omega_0 \) and \( c_g \) by \( -c_g \).

We insert our ansatz (62) in equation (1). Then we expand all terms of the form \( K_0 \tilde{\psi}_j \) by using the Taylor series of the hyperbolic tangent around \( k = jk_0 \). (For more details compare Lemma 25 in [13], for example.) After that we equate the coefficients in front of the \( \varepsilon^m E^j \) to zero. In detail, we get for

\[(m, j) = (1, 1) : \quad i\omega_0 \tilde{A}_1 = i \tanh(k_0) \tilde{A}_1,\]
\[(m, j) = (2, 1) : \quad c_g \partial_X \tilde{A}_1 = \sech^2(k_0) \partial_X \tilde{A}_1,\]
\[(m, j) = (2, 2) : \quad i(-2\omega_0 + \tanh(2k_0)) \tilde{A}_2 = i k_0(\tilde{A}_1)^2,\]
\[(m, j) = (3, 0) : \quad (-c_g + \sech^2(0)) \partial_X \tilde{A}_0 = \partial_X (\tilde{A}_1 \tilde{A}_{-1}),\]
\[(m, j) = (3, 1) : \quad \partial_T \tilde{A}_1 = -i \tanh(k_0) \sech^2(k_0) \partial_X^2 \tilde{A}_1 + ik_0 (\tilde{A}_0 \tilde{A}_1 + \tilde{A}_{-1} \tilde{A}_2),\]
where \( X = \varepsilon(x - c_g t) \) and \( T = \varepsilon^2 t \).
The equations for \((m,j) = (1,1)\) and \((m,j) = (2,1)\) are satisfied due to the definitions of \(\omega_0\) and \(c_g\). Since for \(k_0 \neq 0\) and all integers \(j \geq 2\) the non-resonance conditions
\[
\tanh(jk_0) \neq j \tanh(k_0), \quad (63)
\]
\[
\tanh'(k_0) \neq \tanh'(0) \quad (64)
\]
hold, we can choose \(\tilde{A}_0\) and \(\tilde{A}_2\) depending on \(\tilde{A}_1\), such that the equations for \((m,j) = (2,2)\) and \((m,j) = (3,0)\) are satisfied and the equation for \((j,m) = (3,1)\) becomes the NLS equation
\[
\partial_T \tilde{A}_1 = i\nu_1 \partial_x^2 \tilde{A}_1 + i\nu_2 |\tilde{A}_1|^2 \tilde{A}_1, \quad (65)
\]
with
\[
\nu_1 = \frac{1}{2} \tanh''(k_0) = -\tanh(k_0) \text{sech}^2(k_0),
\]
\[
\nu_2 = k_0 \left( \frac{k_0}{\tanh(2k_0) - 2 \tanh(k_0)} + \frac{1}{\tanh^2(k_0)} \right).
\]

To prove the approximation property of the NLS equation \((65)\) it will be helpful to make the residual
\[
\text{Res}(\varepsilon \tilde{\psi}) = -\partial_t (\varepsilon \tilde{\psi}) + K_0 (\varepsilon \tilde{\psi}) - \varepsilon \tilde{\psi} \partial_x (\varepsilon \tilde{\psi}) \quad (66)
\]
which contains all terms that do not cancel after inserting ansatz \((62)\) into system \((1)\), smaller by modifying \(\varepsilon \tilde{\psi}\) in the following way. First, the above approximation \(\varepsilon \tilde{\psi}\) is extended by higher order terms. Secondly, by some cut-off function the support of the modified approximation in Fourier space is restricted to small neighborhoods of a finite number of integer multiples of the basic wave number \(k_0 > 0\). Since the approximation in Fourier space is strongly concentrated around these wave numbers, the approximation is only changed slightly by this modification, but this second step will give us a simpler control of the error and makes the approximation an analytic function.

Since non-resonance conditions \((63)-(64)\) hold, we can proceed analogously as in \([7]\) to replace \(\varepsilon \tilde{\psi}\) by a new approximation \(\varepsilon \psi\) of the form
\[
\varepsilon \psi = \sum_{|j| \leq 5} \sum_{\beta(j,n) \leq 5} \varepsilon^{\beta(j,n)} \psi_j^n, \quad (67)
\]
where \(j \in \mathbb{Z}, n \in \mathbb{N}_0\),
\[
\beta(j,n) = 1 + |j| - 1 + n,
\]
\[
\psi_j^n(x,t) = A_j^n (\varepsilon (x + c_g t), \varepsilon^2 t) E_j^n,
\]
\(A_{-j} = A_j\), and the functions \(\psi_j^n\) have the compact support
\[
\{k \in \mathbb{R}: |k - jk_0| \leq \delta < k_0/20\}
\]
in Fourier space for sufficiently small \(\varepsilon > 0\). For later purposes we fix \(\delta \in (0,k_0/20)\) such that
\[
|\tanh(k) - 2 \tanh(jk_0) - \tanh(k - 2jk_0)| \geq C > 0 \quad (68)
\]
for a constant \( C = C(k_0) \), which is possible due to (63). Furthermore, we define

\[
\psi_{\pm 1} := \psi_{\pm 1}^0, \\
\psi_c := \psi_{-1} + \psi_1, \\
\psi_s := \varepsilon^{-1}(\psi - \psi_c).
\]

(69) (70) (71)

As in Section 2 of [7], we get the following estimates for the modified residual.

**Lemma 3.2.** Let \( s_A \geq 7, \hat{A}_1 \in C([0, T_0], H^{s_A}(\mathbb{R}, \mathbb{C})) \) be a solution of the NLS equation (65) with

\[
\sup_{T \in [0, T_0]} \| \hat{A}_1(T) \|_{H^{s_A}} \leq C_A,
\]

and \( \delta \) be chosen as above. Then for all \( s \geq 0 \) there exist \( C_{\text{Res}}, C_\psi, \varepsilon_0 > 0 \) depending on \( C_A \), where \( \varepsilon_0 < \delta \), such that for all \( \varepsilon \in (0, \varepsilon_0) \) the approximation \( \varepsilon \psi \) satisfies

\[
\sup_{T \in [0, T_0/\varepsilon^2]} \| \text{Res}(\varepsilon \psi) \|_{H^s} \leq C_{\text{Res}} \varepsilon^{11/2},
\]

(72)

\[
\sup_{T \in [0, T_0/\varepsilon^2]} \| \varepsilon \psi - (\varepsilon \hat{\psi}_1 + \varepsilon \hat{\psi}_{-1}) \|_{H^{s_A}} \leq C_\psi \varepsilon^{3/2},
\]

(73)

\[
\sup_{T \in [0, T_0/\varepsilon^2]} (\| \hat{\psi}_{\pm 1} \|_{L^1(s+1)} + \| \hat{\psi}_s \|_{L^1(s+1)}) \leq C_\psi.
\]

(74)

**Remark 3.3.** The bound (74) will be used for instance to estimate

\[
\| \psi_{jn}^n f \|_{H^s} \leq C \| \psi_{jn}^n \|_{C^s} \| f \|_{H^s} \leq C \| \hat{\psi}_{jn} \|_{L^1(s)} \| f \|_{H^s}
\]

without loss of powers in \( \varepsilon \) as it would be the case with \( \| \hat{\psi}_{jn} \|_{L^2(s)} \).

Moreover, by an analogous argumentation as in the proof of Lemma 3.3 in [7] we obtain the fact that \( \partial_t \hat{\psi}_{\pm 1} \) can be approximated by \( K_0 \hat{\psi}_{\pm 1} \). More precisely, we get

**Lemma 3.4.** For all \( s > 0 \) there exists a constant \( C > 0 \) such that

\[
\| \partial_t \hat{\psi}_{\pm 1} - K_0 \hat{\psi}_{\pm 1} \|_{L^1(s)} \leq C \varepsilon^2.
\]

(75)

4 **The error estimates**

Now, we write a solution \( u \) of (11) as the sum of approximation and error. To avoid problems arising from the resonances at \( k = \pm k_0 \), we rescale the error with the help of the weight function

\[
\hat{\vartheta}(k) = \begin{cases} 
1 & \text{for } |k| > \delta, \\
\varepsilon + (1 - \varepsilon)|k|/\delta & \text{for } |k| \leq \delta,
\end{cases}
\]

(76)
where $\delta$ is chosen as above and $\varepsilon \in (0, \varepsilon_0)$, with $\varepsilon_0$ as in Lemma 3.2. That means, we write
\[ u = \varepsilon \psi + \varepsilon^{5/2} \vartheta R, \tag{77} \]
where $\vartheta R$ is defined by $\hat{\vartheta}R = \hat{\vartheta}R$. By this choice $\vartheta R(k)$ is small at the wave numbers close to zero reflecting the fact that the nonlinearity of (11) vanishes at $k = 0$.

Inserting this ansatz into (11) leads to
\[ \partial_t R = K_0 R - \varepsilon \vartheta^{-1} \partial_x (\psi \vartheta R) - \frac{1}{2} \varepsilon^{5/2} \vartheta^{-1} (\vartheta R)^2 + \varepsilon^{-5/2} \vartheta^{-1} \text{Res}(\varepsilon \psi), \tag{78} \]
where the operator $\vartheta^{-1}$ is defined by its symbol $\hat{\vartheta}^{-1}(k) = \hat{\vartheta}^{-1}(k) = (\hat{\vartheta}(k))^{-1}$.

Due to the structure of the nonlinear terms in the error equation (78), the size of the Fourier transform of these terms depends on whether $k$ is close to zero or not. In order to separate the behavior in these two regions more clearly, we define projection operators $P_{0,\alpha}$ and $P_{\alpha,\infty}$ for $\alpha > 0$ by the Fourier multipliers $\hat{P}_{0,\alpha}(k) = \chi_{[-\alpha,\alpha]}(k)$, \tag{79} 
and $\hat{P}_{\alpha,\infty}(k) = (1 - \chi_{[-\alpha,\alpha]})(k)$, \tag{80}
where $\chi_{[-\alpha,\alpha]}$ is the characteristic function on $[-\alpha, \alpha]$.

As motivated in Section 1, we define the energy
\[ \tilde{E}_s = \sum_{\ell=0}^{s} \tilde{E}_\ell, \tag{81} \]
\[ \tilde{E}_\ell = \begin{cases} \| \hat{R} \|^2_{L^2} & \text{for } \ell = 0, \\ \frac{1}{2} \| \partial_x^\ell R \|^2_{L^2} + \varepsilon \int_{\mathbb{R}} \partial_x^\ell R \partial_x^\ell N(\psi_c, R) \, dx & \text{for } \ell > 0 \end{cases} \tag{82} \]
with
\[ \hat{R} = R + \varepsilon N(\psi_c, R) + \varepsilon^2 \hat{T}(\psi_c, \psi_c, R), \tag{83} \]
\[ N(\psi_c, R) = -\vartheta^{-1} K_0 \partial_x (K_0^{-1} \psi_c K_0 \vartheta P_{\varepsilon,\infty} R), \tag{84} \]
\[ \hat{T}(\psi_c, \psi_c, R)(k) = \sum_{j=\pm 1} \hat{T}_j(\psi_j, \psi_j, R)(k), \tag{85} \]
\[ \hat{T}_j(\psi_j, \psi_j, R)(k) = \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{\psi}_j(k-m) \hat{\psi}_j(m-n) \hat{R}(n) \, dn \, dm, \tag{86} \]
\[ \hat{\psi}_j(k) = -\frac{k(k-jk_0) \hat{\vartheta}(k-2jk_0) \chi_{[-\delta,\delta]}(k)}{\vartheta(k) \tanh(k) \tanh(jk_0) \tanh(k-jk_0) \times (\tanh(k) - 2 \tanh(jk_0) - \tanh(k-2jk_0))^{-1}}, \tag{87} \]
where \( s = s_A \geq 7 \), in order to control the error.

To perform our energy estimates we will need the following lemmas.

**Lemma 4.1.** The operator \( N \) has the following properties:

a) \( f \mapsto N(\psi_c, f) \) defines a continuous linear map from \( H^1(\mathbb{R}, \mathbb{R}) \) into \( L^2(\mathbb{R}, \mathbb{R}) \), and there exists a constant \( C = C(\psi_c) > 0 \), such that for all \( f \in H^1(\mathbb{R}, \mathbb{R}) \) and all \( g \in H^2(\mathbb{R}, \mathbb{R}) \) we have

\[
\|N(\psi_c, f)\|_{L^2} \leq C\varepsilon^{-1}\|f\|_{H^1},
\]

(88)

\[
\|P_{s,\infty}N(\psi_c, f)\|_{L^2} \leq C\|f\|_{H^1},
\]

(89)

\[
\|\partial_x N(\psi_c, g)\|_{L^2} \leq C\|g\|_{H^2}.
\]

(90)

b) For all \( f \in H^1(\mathbb{R}, \mathbb{R}) \) we have

\[
\partial N(\psi_c, f) = \partial_x(\mathbf{K}_0^{-1} \psi_c f) + Q(\psi_c, f)
\]

(91)

with

\[
\|Q(\psi_c, f)\|_{H^s} = O(\|f\|_{L^2})
\]

(92)

for all \( s \geq 0 \).

c) For all \( f \in H^1(\mathbb{R}, \mathbb{R}) \) we have

\[
-\mathbf{K}_0 N(\psi_c, R) + N(\psi_c, R) + N(\psi_c, \mathbf{K}_0 R) = \partial^{-1}_x(\psi_c \partial_x P_{\delta,\infty} R).
\]

(93)

d) For all \( f \in L^2(\mathbb{R}, \mathbb{R}) \) we have

\[
P_{0,\delta} N(\psi_c, P_{0,\delta} f) = 0.
\]

(94)

e) For all \( f, g \in H^1(\mathbb{R}, \mathbb{R}) \) we have

\[
\int_{\mathbb{R}} f \partial N(\psi_c, g) \, dx = -\int_{\mathbb{R}} g \partial N(\psi_c, f) \, dx + \int_{\mathbb{R}} S(\partial_x \psi_c, f) g \, dx + \int_{\mathbb{R}} Z(\psi_c, f, g) \, dx,
\]

(95)

where

\[
S(\partial_x \psi_c, f) = \mathbf{K}_0^{-1} \partial_x \psi_c f,
\]

\[
Z(\psi_c, f, g) = f Q(\psi_c, g) + g Q(\psi_c, f).
\]

Proof. In Fourier space, we have}

\[
\hat{N}(\psi_c, f)(k) = \int_{\mathbb{R}} \hat{n}(k, k - m, m) \hat{\psi}_c(k - m) \hat{f}(m) \, dm
\]

(96)
with
\[ \hat{n}(k, k - m, m) = -\hat{v}^{-1}(k) \hat{K}_0^{-1}(k) ik \hat{K}_0^{-1}(k - m) \chi_e(k - m) \hat{K}_0^{-1}(m) \hat{v}(m) \hat{P}_{\epsilon,\infty}(m), \]
where $\chi_e = \chi_{{\text{supp}}(\psi_c)}$. Now, we estimate the kernel $\hat{n}$. We have

\[
|\hat{K}_0^{-1}(m) \hat{v}(m) \hat{P}_{\epsilon,\infty}(m)| = \begin{cases} 
0 & \text{for } 0 < |m| \leq \epsilon, \\
\frac{\epsilon}{|\tanh(m)|} + \frac{(1 - \epsilon)|m|}{\delta |\tanh(m)|} & \text{for } \epsilon \leq |m| \leq \delta, \\
\frac{1}{|\tanh(m)|} & \text{for } |m| \geq \delta.
\end{cases}
\]

Exploiting the monotonicity properties of $m \mapsto \frac{1}{|\tanh(m)|}$ and $m \mapsto \frac{|m|}{|\tanh(m)|}$, we obtain

\[
\frac{\epsilon}{|\tanh(m)|} + \frac{(1 - \epsilon)|m|}{\delta |\tanh(m)|} \leq \frac{\epsilon}{|\tanh(\epsilon)|} + \frac{(1 - \epsilon)|\delta|}{\delta |\tanh(\delta)|} \leq 1 + \frac{\delta}{\tanh(\delta)}
\]

for $\epsilon \leq |m| \leq \delta$. This yields

\[
\sup_{m \in \mathbb{R}} |\hat{K}_0^{-1}(m) \hat{v}(m) \hat{P}_{\epsilon,\infty}(m)| \leq \frac{1 + \delta}{\tanh(\delta)}. \quad (97)
\]

Furthermore, we have

\[
\sup_{k-m \in \mathbb{R}} |\hat{K}_0^{-1}(k - m) \chi_e(k - m)| \leq \frac{1}{\tanh(k_0 - \delta)}. \quad (98)
\]

The definitions of $\hat{v}$ and $\hat{P}_{\delta,\infty}$ directly imply

\[
\sup_{k \in \mathbb{R}} |\hat{v}^{-1}(k)| = \epsilon^{-1}, \quad (99)
\]
\[
\sup_{k \in \mathbb{R}} |\hat{P}_{\delta,\infty}(k)\hat{v}^{-1}(k)| = 1. \quad (100)
\]

Moreover, we have

\[
|k \hat{v}^{-1}(k)| = \begin{cases} 
|k| & \text{for } |k| > \delta, \\
|k| & \text{for } |k| \leq \delta.
\end{cases}
\]

Since

\[
\frac{|k|}{\epsilon + (1 - \epsilon)\frac{|k|}{\delta}} = \frac{1}{\frac{|k|}{\delta} + \frac{(1 - \epsilon)|k|}{\delta}} \leq \frac{1}{\frac{\delta}{\delta} + \frac{(1 - \epsilon)|k|}{\delta}} = \delta
\]
for $0 \neq |k| \leq \delta$, we get
\[
\sup_{k \in \mathbb{R}} |k \hat{a}^{-1}(k)| = \max\{\delta, |k|\}.
\] (101)

Now, using (96)-(101), (50), (74), Young’s inequality for convolutions,
\[
\hat{n}(k, k - m, m) = \hat{n}(-k, -(k - m), -m) \in \mathbb{R}
\]
and the fact that $\psi$ is real-valued, we obtain the validity of all statements of a).

Let $k_1 > 0$ be a constant such that $|k| \geq k_1$ and $|k - m - k_0| \leq \delta$ imply $|m| \geq \delta$ and $\text{sign}(k) = \text{sign}(m)$. Then, by using
\[
\tanh(k) = \text{sign}(k) \left(1 - \frac{2}{1 + e^{2|k|}}\right)
\]
we get
\[
\hat{v}(k) \hat{n}(k, k - m, m) = \frac{k \chi_c(k - m)}{\tanh(k - m)(1 + \mathcal{O}(e^{-2|k|}))} (1 + \mathcal{O}(e^{-2|k - m|}))
\]
\[
= \frac{k \chi_c(k - m)}{\tanh(k - m)(1 + \mathcal{O}(e^{-|k|}))}
\]
for $|k| \geq k_1$ provided that $k_1$ is chosen large enough. This yields statement b).

(93) follows by construction of $N$ due to (48). (94) is a direct consequence of
\[
\chi_{[-\delta, \delta]}(k) \chi_{[-\delta, \delta]}(m) \chi_c(k - m) = 0.
\]
Finally, (95) follows from a) and b) by partial integration. \qed

Lemma 4.2. Fix $p \in \mathbb{R}$. Assume that $\kappa \in C(\mathbb{R}^3, \mathbb{C})$, that $g \in C^2(\mathbb{R}, \mathbb{C})$ has a finitely supported Fourier transform and that $f \in H^s(\mathbb{R}, \mathbb{C})$ for $s \geq 0$.

a) If $\kappa$ is Lipschitz continuous with respect to its second argument in some neighborhood of $p$, then there exist $C_{g,\kappa,p} > 0$, $\varepsilon_0 > 0$ such that
\[
\left\| \int (\kappa(\cdot, \cdot - \ell, p) - \kappa(\cdot, p, \ell)) e^{-1} \hat{g} \left(\frac{\cdot - \ell - p}{\varepsilon}\right) \hat{f}(\ell) d\ell \right\|_{L^2(s)} \leq C_{g,\kappa,p} \varepsilon \|f\|_{H^s} \quad (102)
\]
for all $\varepsilon \in (0, \varepsilon_0)$.

b) If $\kappa$ is globally Lipschitz continuous with respect to its third argument, then there exist $D_{g,\kappa} > 0$, $\varepsilon_0 > 0$ such that
\[
\left\| \int (\kappa(\cdot - \ell, \cdot - \ell, \cdot - p) - \kappa(\cdot, \cdot, \cdot - p)) e^{-1} \hat{g} \left(\frac{\cdot - \ell - p}{\varepsilon}\right) \hat{f}(\ell) d\ell \right\|_{L^2(s)} \leq D_{g,\kappa} \varepsilon \|f\|_{H^s} \quad (103)
\]
for all $\varepsilon \in (0, \varepsilon_0)$.

Proof. The Lemma is a special case of Lemma 3.5 in [7]. \qed
Lemma 4.3. The operator $T$ has the following properties:

a) Fix functions $g, h$ with $\hat{g}_c := \chi_{\text{supp}(\psi_c)} \hat{g} \in L^1(\mathbb{R}, \mathbb{C})$ and $\hat{h}_c := \chi_{\text{supp}(\psi_c)} \hat{h} \in L^1(\mathbb{R}, \mathbb{C})$. Then $f \mapsto T(g_c, h_c, f)$ defines a continuous linear map from $L^2(\mathbb{R}, \mathbb{C})$ into $L^2(\mathbb{R}, \mathbb{C})$, and there exists a constant $C > 0$ such that for all $f \in L^2(\mathbb{R}, \mathbb{C})$ we have

$$\|T(g_c, h_c, f)\|_{L^2} \leq C\varepsilon^{-1}\|\hat{g}_c\|_{L^1}\|\hat{h}_c\|_{L^1}\|f\|_{L^2}. \quad (104)$$

b) For all $f \in H^2(\mathbb{R}, \mathbb{C})$ we have

$$-K_0T(\psi_c, \psi_c, f) + T(K_0\psi_c, \psi_c, f) + T(\psi_c, K_0\psi_c, f) + T(\psi_c, \psi_c, K_0f) \quad (105)$$

with

$$\|Y(\psi, f)\|_{L^2} = O(\|f\|_{H^2}) \quad (106)$$

for sufficiently small $\varepsilon > 0$.

c) For all $f \in L^2(\mathbb{R}, \mathbb{C})$ we have

$$P_{\delta, \infty}T(\psi_c, \psi_c, f) = 0. \quad (107)$$

**Proof.** To show a), we use the triangle inequality, Young’s inequality for convolutions, (50), (99) and (68) to get

$$\|T(g_c, h_c, f)\|_{L^2} \lesssim \sum_{j=\pm 1} \|\hat{f}_j\|_{L^\infty}\|\hat{g}_c\|_{L^1}\|\hat{h}_c\|_{L^1}\|f\|_{L^2} \lesssim \varepsilon^{-1}\|\hat{g}_c\|_{L^1}\|\hat{h}_c\|_{L^1}\|f\|_{L^2}. \quad (108)$$

To prove b), we first show that

$$N(\psi_c, \vartheta^1\partial_x(\psi \vartheta f)) = \sum_{j=\pm 1} P_{0, \delta}N(\psi_j, \vartheta^1\partial_x(\psi_j \vartheta f)) + O(\|f\|_{H^2})$$

such that it is sufficient to prove that the $L^2$-norm of

$$\tilde{Y} := \sum_{j=\pm 1} \left(-K_0T_j(\psi_j, \psi_j, f) + T_j(K_0\psi_j, \psi_j, f) + T_j(\psi_j, K_0\psi_j, f) + T_j(\psi_j, \psi_j, K_0f) - P_{0, \delta}N(\psi_j, \vartheta^1\partial_x(\psi_j \vartheta f)) \right)$$

is of order $O(\|f\|_{L^2})$, which we will obtain by construction of $T$ and because of Lemma 4.2.

To verify (108), we split $N$ into

$$N(\psi_c, \vartheta^1\partial_x(\psi \vartheta f)) = \sum_{j=\pm 1} P_{0, \delta}N(\psi_j, \vartheta^1\partial_x(\psi_j \vartheta f)) + \sum_{j=\pm 1} P_{0, \delta}N(\psi_j, \vartheta^1\partial_x(\psi_{-j} \vartheta f)) + P_{\delta, \infty}N(\psi_c, \vartheta^1\partial_x(\psi \vartheta f)) + \varepsilon P_{0, \delta}N(\psi_c, \vartheta^1\partial_x(\psi \vartheta f)).$$
Due to (88), (89) and (101), the $L^2$-norm of the sum of the last two summands is of order $O(\|f\|_{L^2})$. Furthermore, in Fourier space, we have
\[
\hat{P}_{0,\delta}(k) \int_{\mathbb{R}} \int_{\mathbb{R}} K(k, k - m, m, n) \hat{\psi}_j(k - m) \hat{\psi}_j(m - n) \hat{f}(n) \, dn \, dm
\]
with
\[
K(k, k - m, m, n) = -\frac{ikm \, \hat{\vartheta}(n)}{\vartheta(k) \tanh(k - m) \tanh(m)} ,
\]
where $y \mapsto y/\tanh(y)$ is continued by 1 for $y = 0$.

For $\ell = -j$ we can apply Fubini’s theorem, Young’s inequality for convolutions and Lemma 4.2 to obtain
\[
\| \hat{P}_{0,\delta}(\psi_j, \vartheta - 1 \partial_x (\psi \vartheta f)) \|_{L^2} = O(\|f\|_{L^2})
\]
such that we have verified (108).

To estimate $\| \hat{Y} \|_{L^2}$, we use
\[
\hat{Y}(k) = \sum_{j = \pm 1} \hat{P}_{0,\delta}(k) \int_{\mathbb{R}} \int_{\mathbb{R}} K_j(k, k - m, m - n, n) \hat{\psi}_j(k - m) \hat{\psi}_j(m - n) \hat{f}(n) \, dn \, dm
\]
\[
- \sum_{j = \pm 1} \hat{P}_{0,\delta}(k) \int_{\mathbb{R}} \int_{\mathbb{R}} K(k, k - m, m, n) \hat{\psi}_j(k - m) \hat{\psi}_j(m - n) \hat{f}(n) \, dn \, dm ,
\]
where $y \mapsto y/\tanh(y)$ is continued by 1 for $y = 0$. For $\ell = -j$ we can apply Fubini’s theorem, Young’s inequality for convolutions and Lemma 4.2 to obtain
\[
\| \hat{P}_{0,\delta}(\psi_j, \vartheta - 1 \partial_x (\psi \vartheta f)) \|_{L^2} = O(\|f\|_{L^2})
\]
such that we have verified (108).
where
\[
K_j(k, k - m, m - n, n) = -\frac{ik(k - jk_0) \hat{\vartheta}(k - 2jk_0)}{\hat{\vartheta}(k) \tanh(k) \tanh(jk_0) \tanh(k - jk_0)}
\times \frac{\tanh(k) - \tanh(k - m) - \tanh(m - n) - \tanh(n)}{\tanh(k) - 2 \tanh(jk_0) - \tanh(k - 2jk_0)}
\]
and \(K\) is as above. We can apply again Fubini’s theorem, Young’s inequality for convolutions and Lemma 4.2 to obtain
\[
\hat{\tilde{Y}}(k) = \sum_{j = \pm 1} \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{P}_{0, \delta}(k) K_j(k, jk_0, jk_0, k - 2jk_0) \hat{\psi}_j(k - m) \hat{P}_{\epsilon, \infty}(m) \hat{\psi}_j(m - n) f(n) dndm
\]
\[
- \sum_{j = \pm 1} \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{P}_{0, \delta}(k) K(k, jk_0, k - jk_0, k - 2jk_0) \hat{\psi}_j(k - m) \hat{P}_{\epsilon, \infty}(m) \hat{\psi}_j(m - n) f(n) dndm
\]
\[
+ \mathcal{O}(\|f\|_{L^2}),
\]
where we used that due to the support of \(\hat{\psi}_j\) the first integrand vanishes for \(|m| \leq \varepsilon\).

Since
\[
K_j(k, jk_0, jk_0, k - 2jk_0) = K(k, jk_0, k - jk_0, k - 2jk_0),
\]
the two integral kernels, which are both of order \(\mathcal{O}(\varepsilon^{-1})\), cancel each other out such that we get
\[
\|\tilde{Y}\|_{L^2} = \mathcal{O}(\|f\|_{L^2}).
\]

Hence, we have proven b).

Finally, c) follows directly by the definition of \(T\).

Lemma 4.4. Let \(f \in H^\ell(\mathbb{R}, \mathbb{R})\) and \(g \in H^m(\mathbb{R}, \mathbb{R})\) with \(\ell, m \geq 0\). Then we have
\[
\int_{\mathbb{R}} \partial_x^\ell f \partial_x^m g dx = \int_{\mathbb{R}} \partial_x^\ell f \partial_x^m g dx + \mathcal{O}(\|f\|_{L^2}\|g\|_{L^2}),
\]
\[
\int_{\mathbb{R}} \partial_x^\ell f \partial_x^{m+1} g dx = \int_{\mathbb{R}} \partial_x^\ell f \partial_x^{m+1} g dx + \mathcal{O}(\|f\|_{L^2}\|g\|_{L^2}).
\]

Proof. Using the definition of \(\vartheta\), we get
\[
\int_{\mathbb{R}} \partial_x^\ell f \partial_x^m g dx = \int_{\mathbb{R}} \partial_x^\ell f \partial_x^m g dx + (-1)^\ell \int_{\mathbb{R}} f \partial_x^{\ell+m} P_{0, \delta}(\vartheta - 1) g dx,
\]
\[
\int_{\mathbb{R}} \partial_x^\ell f \partial_x^{m+1} g dx = \int_{\mathbb{R}} \partial_x^\ell f \partial_x^{m+1} g dx + (-1)^\ell \int_{\mathbb{R}} f \partial_x^{\ell+m+1} P_{0, \delta}(\vartheta - 1) g dx,
\]
which yields \((109)\) and, due to \((101)\), also \((110)\).
Lemma 4.5. For sufficiently small $\varepsilon > 0$ there exist constants $C, \hat{C} > 0$ such that

$$\|\hat{R}\|_{L^2} \leq C\|R\|_{H^1}, \quad (111)$$

$$\|R\|_{L^2} \leq \hat{C}\|\hat{R}\|_{L^2}. \quad (112)$$

Proof. Estimate (111) is a direct consequence of the estimates (88) and (104).

To prove (112) we introduce $R_0 := P_{0,\delta}R, \hat{R}_0 := P_{0,\delta}\hat{R}, R_1 := P_{0,\infty}R, \hat{R}_1 := P_{0,\infty}\hat{R}$ and split $R, \hat{R}$ into $R = R_0 + R_1$ and $\hat{R} = \hat{R}_0 + \hat{R}_1$. Because of (94) and (107), $R_0$ satisfies

$$R_0 + \varepsilon^2 \mathcal{T}(\psi_c, \psi_c, R_0) = \hat{R}_0 - \varepsilon P_{0,\delta}N(\psi_c, R_1) - \varepsilon^2 \mathcal{T}(\psi_c, \psi_c, R_1). \quad (113)$$

Using (88) and (104) yields

$$\|R_0\|_{L^2} \lesssim \|\hat{R}_0\|_{L^2} + \|R_1\|_{L^2} \quad (114)$$

for sufficiently small $\varepsilon > 0$. Moreover, $R_1$ satisfies

$$R_1 + \varepsilon P_{0,\infty}N(\psi_c, R_1) = \hat{R}_1 - \varepsilon P_{0,\infty}N(\psi_c, R_0). \quad (115)$$

Multiplying this equation with $R_1$, integrating and using $P_{0,\infty}N = P_{0,\infty}\partial N$ as well as (89) yields

$$\|R_1\|_{L^2}^2 + \varepsilon \int R_1 \partial N(\psi_c, R_1) \, dx \lesssim (\|\hat{R}_1\|_{L^2} + \varepsilon \|R_0\|_{L^2}) \|R_1\|_{L^2}. \quad (116)$$

Because of (95) and (114), we get

$$\|R_1\|_{L^2} \lesssim \|\hat{R}_1\|_{L^2} + \varepsilon \|R_0\|_{L^2} \lesssim \|\hat{R}\|_{L^2} \quad (117)$$

and

$$\|R_0\|_{L^2} \lesssim \|\hat{R}\|_{L^2} \quad (118)$$

for sufficiently small $\varepsilon > 0$. Combining (116) and (117) yields (112). \hfill \Box

The assertions of Lemma 4.1 and Lemma 4.5 imply

Lemma 4.6. For $\ell \geq 1$, we have

$$\tilde{E}_\ell = \frac{1}{2} \|\partial_x^\ell R\|_{L^2}^2 + \varepsilon\mathcal{O}(\|R\|_{H^s}^2). \quad (118)$$

Corollary 4.7. $\sqrt{E_s}$ is equivalent to $\|R\|_{H^s}$ for all $s \geq 1$ if $\varepsilon > 0$ is sufficiently small.

Now, we are prepared to estimate the time derivative of $\tilde{E}_s$ for any sufficiently regular solution of (78). We obtain
Lemma 4.8. For sufficiently small $\varepsilon > 0$, we have
\[
\frac{d}{dt} \tilde{\mathcal{E}}_0 \lesssim \varepsilon^2 (\tilde{\mathcal{E}}_2 + \varepsilon^{1/2} \tilde{\mathcal{E}}_2^{3/2} + 1).
\] (119)

Proof. Because of (78) and (83) we get
\[
\frac{d}{dt} \tilde{\mathcal{E}}_0 = \int_{\mathbb{R}} \overline{\tilde{R}} \partial_t \tilde{R} \, dx + \int_{\mathbb{R}} \overline{\tilde{R}} \partial_t \tilde{R} \, dx
\]
with
\[
\partial_t \tilde{R} = K_0 \tilde{R} + \varepsilon^{-5/2} \partial^{-1} \text{Res}(\varepsilon \psi)
\]
\[
- \varepsilon (\partial^{-1} \partial_x (\psi_c \partial P_{\infty} R) + K_0 N(\psi_c, R) - N(K_0 \psi_c, R) - N(R, K_0 \psi_c))
\]
\[
+ \varepsilon (N(\partial_t \psi_c - K_0 \psi_c, R) + N(\psi_c, \varepsilon^{-5/2} \partial^{-1} \text{Res}(\varepsilon \psi)))
\]
\[
- \varepsilon \partial^{-1} \partial_x (\psi_c \partial P_{\infty} R) + \partial^{-1} \partial_x ((\tilde{\psi} - \psi_c) \partial R)
\]
\[
- \varepsilon^2 N(\psi_c, \partial^{-1} \partial_x (\psi \partial R))
\]
\[
- \varepsilon^2 (K_0 T(\psi_c, \psi_c, R) - T(K_0 \psi_c, \psi_c, R) - T(\psi_c, K_0 \psi_c, R) - T(\psi_c, \psi_c, K_0 R))
\]
\[
+ \varepsilon^2 (T(\partial_t \psi_c - K_0 \psi_c, \psi_c, R) + T(\psi_c, \partial_t \psi_c - K_0 \psi_c, R))
\]
\[
+ \varepsilon^2 T(\psi_c, \psi_c, \varepsilon^{-5/2} \partial^{-1} \text{Res}(\varepsilon \psi))
\]
\[
- \varepsilon^3 T(\psi_c, \psi_c, \partial^{-1} \partial_x (\tilde{\psi} \partial R)) - \frac{1}{2} \varepsilon^7/2 N(\psi_c, \partial^{-1} \partial_x (\partial R)^2),
\]
where $\tilde{\psi} = \psi + \frac{1}{2} \varepsilon^3/2 \partial R$.

Exploiting the skew symmetry of $K_0$ and the Cauchy-Schwarz inequality, we conclude
\[
\frac{d}{dt} \tilde{\mathcal{E}}_0 \leq 2 \|
\partial_t \tilde{R} - K_0 \tilde{R} \|_{L^2} \|
\tilde{R} \|_{L^2}.
\]

Using (67), the bounds (72), (74) and (75) for the approximation functions and the residual, the properties (88) and (93) of the operator $N$, the properties (104)-(106) of the operator $T$, the bounds (101) and
\[
\| \partial P_{\infty} f \|_{L^2} \lesssim \varepsilon \| f \|_{L^2}
\]
for $\partial$, the estimate (111) for $\tilde{R}$ as well as Corollary 4.7 we get
\[
\|
\partial_t \tilde{R} - K_0 \tilde{R} \|_{L^2} \lesssim \varepsilon^2 (\tilde{\mathcal{E}}_2^{1/2} + \varepsilon^{1/2} \tilde{\mathcal{E}}_2^{3/2} + 1),
\]
\[
\| \tilde{R} \|_{L^2} \lesssim \tilde{\mathcal{E}}_1^{1/2}
\]
and therefore
\[
\frac{d}{dt} \tilde{\mathcal{E}}_0 \lesssim \varepsilon^2 (\tilde{\mathcal{E}}_2 + \varepsilon^{1/2} \tilde{\mathcal{E}}_2^{3/2} + 1).
\]
\[\square\]
Lemma 4.9. For $\ell \geq 1$, $\theta \geq \max\{2, \ell\}$ and sufficiently small $\varepsilon > 0$, we have

$$\frac{d}{dt} \tilde{E}_\ell \lesssim \varepsilon^2 (\tilde{E}_\theta + \varepsilon^{1/2} \tilde{E}_\theta^{3/2} + 1).$$  \hfill (121)

Proof. We compute

\[
\frac{d}{dt} \tilde{E}_\ell = \int_\mathbb{R} \partial_x^\ell R \partial_0 \partial_x^\ell R \, dx + \varepsilon \left( \int_\mathbb{R} \partial_t \partial_x^\ell R \partial_x^\ell N(\psi_c, R) \, dx + \int_\mathbb{R} \partial_x^\ell R \partial_x N(\psi_c, \partial_t R) \, dx + \int_\mathbb{R} \partial_x^\ell R \partial_x N(\partial_t \psi_c, R) \, dx \right).
\]

Using the error equation (78), we get

\[
\frac{d}{dt} \tilde{E}_\ell = \int_\mathbb{R} \partial_x^\ell R K_0 \partial_x^\ell R \, dx + \int_\mathbb{R} \partial_x^\ell R \varepsilon^{-5/2} \partial_x^\ell \, \text{Res}(\varepsilon \psi) \, dx + \varepsilon \left( - \int_\mathbb{R} \partial_x^\ell R \varepsilon \partial_x^\ell N(\psi_c, \partial_t R) \, dx + \int_\mathbb{R} K_0 \partial_x^\ell R \partial_x^\ell N(\psi_c, R) \, dx + \int_\mathbb{R} \partial_x^\ell R \partial_x^\ell N(\psi_c, K_0 R) \, dx + \int_\mathbb{R} \partial_x^\ell R \partial_x^\ell N(K_0 \psi_c, R) \, dx + \int_\mathbb{R} \partial_x^\ell R \partial_x^\ell N(\partial_t \psi_c - K_0 \psi_c, R) \, dx + \int_\mathbb{R} \varepsilon^{-5/2} \partial_x^\ell \, \text{Res}(\varepsilon \psi) \partial_x^\ell N(\psi_c, R) \, dx + \int_\mathbb{R} \partial_x^\ell R \partial_x^\ell N(\psi_c, \varepsilon^{-5/2} \partial_x^\ell \, \text{Res}(\varepsilon \psi)) \, dx - \int_\mathbb{R} \partial_x^\ell R \varepsilon \partial_x^\ell (\tilde{\psi} - \psi_c) \, dx - \int_\mathbb{R} \partial_x^\ell R \varepsilon \partial_x^\ell \, dx \right) - 28
\]
\[ -\varepsilon^2 \left( \int_{\mathbb{R}} \partial^{-1}_x \partial^{\ell+1}_x (\tilde{\psi} \vartheta R) \partial^\ell_x N(\psi_c, R) \, dx \\
+ \int_{\mathbb{R}} \partial^\ell_x R \partial^\ell_x N(\psi_c, \partial^{-1}_x (\tilde{\psi} \vartheta R)) \, dx \right) \\
= \sum_{j=1}^{13} I_j \]

where \( \tilde{\psi} = \psi + \frac{1}{2} \varepsilon^{3/2} \vartheta R \).

Because of the skew symmetry of \( K_0 \), the integral \( I_1 \) equals zero. Since the operator \( N \) satisfies (93), we have

\[ I_3 + I_4 + I_5 + I_6 = 0. \]

Moreover, using (67), the bounds (72), (74) and (75) for the approximation functions and the residual, the properties of the operator \( N \) from Lemma 4.1, the bounds (101) and (120) for \( \vartheta \), identity (54), Corollary 4.7 and partial integration, the integrals \( I_2, I_7, I_8, I_9, \) and \( I_{10} \) can be bounded by \( C\varepsilon^2 (\hat{E}_\vartheta + 1) \) for a constant \( C > 0 \).

Next, we analyze \( I_{12} + I_{13} \). Due to (110), we have

\[ I_{12} + I_{13} = -\varepsilon^2 \left( \int_{\mathbb{R}} \partial^{-1}_x \partial^{\ell+1}_x (\tilde{\psi} \vartheta R) \partial^\ell_x N(\psi_c, R) \, dx \\
+ \int_{\mathbb{R}} \partial^\ell_x R \partial^\ell_x N(\psi_c, \partial^{-1}_x (\tilde{\psi} \vartheta R)) \, dx \right) \\
+ \varepsilon^2 O(\hat{E}_\vartheta + \varepsilon^{3/2} \hat{E}_\vartheta^{3/2}). \]

To extract all terms with more than \( \ell \) spatial derivatives falling on \( R \), we apply Leibniz’s rule and get

\[ I_{12} + I_{13} = -\varepsilon^2 \left( \int_{\mathbb{R}} \partial^{-1}_x \partial^{\ell+1}_x (\tilde{\psi} \vartheta R) \partial N(\psi_c, \partial^\ell_x R) \, dx \\
+ \varepsilon^2 \right) \]

To complete the analysis, further steps would be required to bound and simplify these integrals, ensuring that they contribute to the overall error term.
Because of (95), we obtain

\[ I_{12} + I_{13} = -\varepsilon^2 \left( \int_R \partial_x \partial_x^{\ell+1} (\tilde{\psi} \partial R) \left( \partial_x \psi_c, \partial_x^\ell R \right) \, dx + 2\ell \int_R \partial_x \partial_x^{\ell+1} (\tilde{\psi} \partial R) \partial N \left( \partial_x \psi_c, \partial_x^{\ell-1} R \right) \, dx \right) + \varepsilon^2 O(\tilde{E}_0 + \varepsilon^{3/2} \tilde{E}_0^{3/2}). \]

Using (91), (109) and (110) yields

\[ I_{12} + I_{13} = -(2\ell + 1) \varepsilon^2 \int_R K_0^{\ell} \partial_x \psi_c \tilde{\psi} \partial_x^{\ell+1} R \partial_x R \, dx + \varepsilon^2 O(\tilde{E}_0 + \varepsilon^{3/2} \tilde{E}_0^{3/2}) \]

where \( \tilde{\psi} = \psi + \varepsilon^{3/2} \partial R \). Finally, with the help of (54), we arrive at

\[ I_{12} + I_{13} = -\varepsilon^2 \frac{d}{dt} \tilde{E}_s + \varepsilon^2 O(\tilde{E}_0 + \varepsilon^{3/2} \tilde{E}_0^{3/2}). \]

Using again (54), (109) and (110) yields

\[ I_{11} = \varepsilon^2 \frac{d}{dt} \tilde{E}_s + \varepsilon^{1/2} \tilde{E}_s^{3/2}. \]

Hence, we obtain

\[ \frac{d}{dt} \tilde{E}_s \lesssim \varepsilon^2 (\tilde{E}_0 + \varepsilon^{1/2} \tilde{E}_0^{3/2} + 1). \]

Now, combining the estimates (119) and (121), we obtain

\[ \frac{d}{dt} \tilde{E}_s \lesssim \varepsilon^2 (\tilde{E}_s + \varepsilon^{1/2} \tilde{E}_s^{3/2} + 1) \quad (122) \]

for \( s = s_A > 7 \) and sufficiently small \( \varepsilon > 0 \). Consequently, Gronwall’s inequality yields the \( O(1) \)-boundedness of \( \tilde{E}_s \) for \( t \in [0, T_0/\varepsilon^2] \). Due to Corollary 4.7 and estimate (73), Theorem 1.2 follows.

\[ \Box \]

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