Dynamical Features of Maggiore’s Generalised Commutation Relations

S. Kalyana Rama

Institute of Mathematical Sciences, C. I. T. Campus,
Taramani, CHENNAI 600 113, India.
email: krama@imsc.ernet.in

ABSTRACT

We study the dynamical features of Maggiore’s generalised commutation relations. We focus on their generality and, in particular, their dependence on the Hamiltonian $H$. We derive the generalisation of the Planck’s law for black body spectrum, study the statistical mechanics of free particles, and study the early universe evolution which now exhibits non trivial features. We find that the dynamical features, found here and in our earlier work, are all generic and vary systematically with respect to the asymptotic growth of the Hamiltonian $H$.

PACS numbers: 11.25.-w, 05.90.+m, 98.80.Cq, 04.70.-s
The generalised commutation relations (GCRs) are the generalisation of the standard Heisenberg commutation relations. Many generalisations are possible [1, 2] and are currently of wide interest [1]−[7]. It is important to explore the physical consequences of the GCRs, which turn out to be interesting and non trivial. The GCRs, however, are kinematical only. The dynamics is governed by the Hamiltonian $H$ which, to the best of our knowledge, is not determined uniquely by any physical principle. In [7], we studied Maggiore’s GCRs [1], determined completely under certain assumptions and, considering two specific choices of $H$, found that these GCRs lead naturally to varying speed of light, modified dispersion relations, and reduction in the thermodynamical degrees of freedom at high temperatures, features which are all non trivial and are interesting in their own rights [8, 9].

In this paper, we study further the dynamical features of Maggiore’s GCRs [1]. Throughout in the following, we focus on the generality of these features and, in particular, their dependence on the choice of the Hamiltonian $H$. We derive the generalisation of the Planck’s law for black body spectrum, study the statistical mechanics of free particles with different statistics and study the early universe evolution which now exhibits non trivial features.

Since $H$ itself is not determined uniquely by any physical principle, we impose two physically reasonable requirements on $H$. We then consider a few generic choices of $H$ and, thereby, illustrate the dynamical features of Maggiore’s GCRs and their dependence on $H$. We find that the dynamical features, found here and in [7], are all generic and vary systematically with respect to the asymptotic growth of the Hamiltonian $H$.

The plan of the paper is as follows. In section 2, we present the relevant details. In section 3, we derive the generalisation of Planck’s law for black body spectrum, and discuss a few general features which are independent of the choice of $H$. In section 4, we impose requirements on $H$, present a few generic choices, and study the dynamical features of Maggiore’s GCRs and their dependence on $H$. In section 5, we mention briefly the resulting consequences for the early universe evolution. In section 6, we conclude with a brief summary, mentioning a few issues for further study.

2. The standard Heisenberg commutation relations between the position operators $X_i$ and the momentum operators $P_j$, $i, j = 1, 2, \cdots, d$, in $d$−dimensional space can be generalised in many ways [1, 2]. Maggiore has derived in [1] a set of generalised commutation relations (GCRs) between $X_i$ and $P_j$ for $d = 3$, determined completely under the following assumptions: (i)
The spatial rotation group and, hence, the commutators $[J_{ij}, J_{kl}]$, $[J_{ij}, X_k]$, and $[J_{ij}, P_k]$ are undeformed where $J_{ij}$ is the angular momentum operator in the $(i, j)$-plane. (ii) The translation group and, hence, the commutators $[P_i, P_j]$ are undeformed. (iii) The commutators $[X_i, X_j]$ and $[X_i, P_j]$ depend on a deformation parameter $\lambda$, with dimension of length, and reduce to the undeformed ones in the limit $\lambda \to 0$. Extended to the arbitrary $d$ case, Maggiore’s GCRs are given by

$$
[X_i, X_j] = -i \tilde{f} J_{ij}, \quad \tilde{f} = \frac{\lambda^2}{4\pi^2} \\
[X_i, P_j] = i \hbar f \delta_{ij}, \quad f = \sqrt{1 + \frac{\lambda^2}{\hbar^2}(P^2 + m^2c_0^2)}.
$$

(1)

The functions $\tilde{f}$ and $f$, and thus the GCRs, are determined completely up to the sign of the $\lambda^2$-term, chosen to be positive here. In the above equations, $m$ is the particle mass, $\hbar = 2\pi\hbar$ is the Planck’s constant, and $c_0 \simeq 3 \times 10^8 \text{ m/sec}$ is the standard speed of light in vacuum. In the following, we set $\hbar = c_0 = 1$ unless indicated otherwise.

The non vanishing commutator $[X_i, X_j]$ implies the non commutativity of space which shows up at length scales of $\mathcal{O}(\lambda)$ or smaller [1]. Other generalisations with different properties are also possible. Typically, they contain one or more arbitrary function(s). For example, in Kempf’s generalisation [2], the commutator $[X_i, X_j]$ vanishes and, thus, space is commutative. This is achieved by adding a term $F(P^2)P_iP_j$ to $\delta_{ij}f(P^2)$ in equation (1) with $F$ and $f$ constrained to satisfy a relation [2], thus leaving one function arbitrary. In the following, we consider Maggiore’s generalisation only where the GCRs are determined completely.

The energy and momentum scales set by $\lambda$, and assumed to be much larger than those set by $m$, are given by

$$
E_* = p_* c_0 = \frac{\hbar c_0}{\lambda} \gg m c_0^2.
$$

(2)

The low energy, low temperature limit and the high energy, high temperature limit are then given by

$$
E \ll E_* \quad \iff \quad p \ll p_* \quad \iff \quad \beta \gg \lambda \quad (3) \\
E \gg E_* \quad \iff \quad p \gg p_* \quad \iff \quad \beta \ll \lambda \quad (4)
$$

[1] Choosing the negative sign implies an upper bound on (the eigenvalues of) $P$, whose physical significance is not clear to us.
where $\beta = T^{-1}$ is the inverse temperature. Note that the limit $\lambda \to 0$ is equivalent to the low energy limit (3).

The GCRs (1) are kinematical only. The dynamics is governed by the Hamiltonian $H$. Once $H$ is specified, the velocity operator $V_i$ for a particle, given by (1)

$$V_i = dX_i/dt = \frac{i}{\hbar} [H, X_i], \quad (5)$$

can be calculated. We assume rotational invariance. Hence, $H$ is a function of $P^2$ or equivalently $\sqrt{P^2 + m^2}$. For free particles, $H$ is independent of $X_i$ also. The speed $v$ for free particles is then given by (7)

$$v(E) = \left( \sum_{i=1}^{d} v_i^2 \right)^{\frac{1}{2}} = \frac{pfE'}{\sqrt{p^2 + m^2}} \quad (6)$$

where $v_i$, $p$, and the energy $E$ are the eigenvalues of the operators $V_i$, $P$, and the Hamiltonian $H$ respectively, and $E'$ is the derivative of $E$ with respect to $\sqrt{P^2 + m^2}$. The speed of light, denoted by $c_\lambda(E)$, can be identified naturally with the speed of a particle with mass $m = 0$. Equation (6) then gives

$$c_\lambda(E) = fE' \quad \text{and} \quad v(E) \leq c_\lambda(E). \quad (7)$$

Once $H(P)$, equivalently $E(p)$, is specified, the statistical mechanics of a system of free particles in a $d$-dimensional volume $V$ obeying the GCRs (1) can also be studied (7). Various thermodynamical quantities, calculable easily in the grand canonical ensemble approach, are given by standard expressions (10). For example, we have, in standard notation,

$$-\beta F = \beta PV = \ln Z = \frac{1}{a} \int_0^\infty dE \ g(E) \ \ln(1 + ae^{-\beta(E-\mu)})$$

$$U = -\frac{\partial \ln Z}{\partial \beta} \quad (8)$$

where $a = -1, 0,$ or $+1$ if the particles obey, respectively, Bose-Einstein, Maxwell-Boltzmann, or Fermi-Dirac statistics, $F$ is the free energy, $P$ is the pressure, $U$ is the internal energy, etc.

The measure $g(E)$ in equation (8) describes the one-particle density of states. In the present case where the free particles obey the GCRs (1), $g(E)$
is calculated easily \cite{7} and can be written as
\[
g(E) = \frac{\Omega_{d-1}V}{h} \left( \frac{p}{hf} \right)^{d-1} \frac{1}{v(E)} \quad (9)
\]
where \( \Omega_n \) is the area of a unit \( n \)-dimensional sphere and \( v(E) \) is given by \( (6) \).

3. We now derive the generalisation of the Planck’s law for spectral radiation density of a perfect black body in thermal equilibrium at temperature \( T = \beta^{-1} \). Let \( R(\omega, T) = \frac{d^3N}{dt d\omega} \) and \( Q(\omega, T) = \hbar \omega R(\omega, T) \) be, respectively, the number of photons and radiative energy per unit area per unit time per unit frequency interval at frequency \( \omega \), as observed through a small hole of area \( A \) in the wall of a cavity containing the photons in thermal equilibrium. It then follows from a standard derivation \cite{10} that
\[
Q(\omega, T) = \hbar \omega c \lambda(E) \left( \frac{g_s \Omega_{d-2}}{2 \Omega_{d-1}} \right) \left( \frac{\hbar g(E)}{V(e^{\beta \hbar \omega} - 1)} \right) \quad (10)
\]
where \( E = \hbar \omega \), \( g_s = (d - 1) \) is the number of polarisation degrees of freedom for the photons, and \( g(E) \) is given by equation \( (9) \) with \( m = 0 \). In the expression \( (10) \) for \( Q(\omega, T) \), the \( c_\lambda(E) \) factor arises from the speed of photons coming out of the hole of the cavity, the factors in the first parenthesis from kinematics, and those in the second parenthesis from the average number density of photons at energy \( E \) inside the cavity. Note that the explicit \( c_\lambda(E) \) factor in \( (10) \) is cancelled by the \( c_\lambda(E) \) factor coming from \( g(E) \). Hence, using equation \( (9) \) only, with \( m = 0 \), we obtain
\[
Q(\omega, T) = \frac{g_s \Omega_{d-2}}{4\pi} \left( \frac{p}{hf} \right)^{d-1} \frac{\hbar \omega}{(e^{\beta \hbar \omega} - 1)} . \quad (11)
\]
Equation \( (11) \) is the generalisation of Planck’s law for the black body spectrum when the system obeys the GCRs \( (1) \).

We now discuss a few general features of \( c_\lambda(E) \), \( g(E) \), and \( Q(\omega, T) \).

- Consider the low energy limit \( (3) \). We require
\[
E(p) \to \sqrt{p^2c_0^2 + m^2c_0^4} \quad \text{as} \quad \lambda \to 0 . \quad (12)
\]
so as to be consistent with the known results in this limit. In this limit, \( f \to 1 \), and \( g(E) \) and \( Q(\omega, T) \) reduce to the standard expressions as in the \( \lambda = 0 \) case. For example, for \( d = 3 \), we have

\[
Q(\omega, T) \to \frac{1}{4\pi^2 c_0^2} \frac{\hbar \omega^3}{(e^{\beta \hbar \omega} - 1)}
\]

which is the standard Planck’s law [10]. Now, \( c_\lambda(E) \) is given in this limit by

\[
c_\lambda(E) = 1 + a_1 \lambda^2 E^2 + \mathcal{O}(\lambda^4 E^4)
\]

where the constant \( a_1 \) depends on the choice of \( E(p) \) but is, generically, of \( \mathcal{O}(1) \). Equation (14) leads to a bound on \( \lambda \). For example, upon using the result for the \( \gamma \)-ray velocity [11], we obtain

\[
\lambda^{-1} > \mathcal{O}(30 \text{ GeV})
\]

This bound can likely be improved further perhaps by analyses similar to those of [12]. Such a study, however, is beyond the scope of the present paper and will not be pursued here.

- Consider the high energy limit (4). It then follows easily, upon using equation (2), that \( g(E) \simeq g_*(E) \) and \( Q(\omega, T) \simeq Q_*(\omega, T) \) where

\[
g_*(E) = \frac{\Omega_{d-1} V}{\hbar \lambda^{d-1}} \frac{1}{c_\lambda(E)}
\]

\[
Q_*(\omega, T) = \frac{g_* \Omega_{d-2}}{4\pi \lambda^{d-1}} \frac{\hbar \omega}{(e^{\beta \hbar \omega} - 1)}
\]

The limiting forms \( g_*(E) \) and \( Q_*(\omega, T) \) are, upto numerical factors, independent of the number of spatial dimensions \( d \). The speed of light \( c_\lambda(E) \) and, hence, \( g_*(E) \) depend on the choice of \( E(p) \). The limiting form of the black body spectrum \( Q_*(\omega, T) \), however, is independent of the choice of \( E(p) \) also. Moreover, upto numerical factors, it is formally same as the standard black body spectrum but with \( d = 1 \).

4. To proceed further, \( p \) and \( f \) in equations (8), (7), and (6) are to be expressed in terms of the energy \( E \). This requires knowing the Hamiltonian \( H(P) \) or, equivalently, the energy \( E(p) \) explicitly. However, to the best of
our knowledge, there is no physical principle that determines the energy \( E(p) \) uniquely in the present context. In the absence of such a principle, we impose the conditions that the energy \( E(p) \) satisfy (12), and that

\[
E(p) \rightarrow \infty \quad \text{as} \quad p \rightarrow \infty .
\] (18)

These requirements are physically reasonable but are insufficient to determine \( E(p) \) uniquely. Nevertheless, it turns out that the general dynamical features of the GCRs (1) and their dependence on \( E(p) \) can be illustrated by studying a few generic choices of \( E(p) \) satisfying (12) and (18).

The effects of the GCRs (1) in the low energy, low temperature limit (3) will be negligible since, by construction, \( E(p) \) satisfies (12). Therefore, throughout in the following, we consider the high energy high temperature limit (4) only where the dynamical features of the GCRs (1) can be seen clearly, and study them for three generic choices of \( E(p) \) which satisfy (18).

It turns out that the leading terms of various quantities up to numerical factors are sufficient to illustrate the dynamical features. Hence, in the following, we calculate the leading terms up to numerical factors only, and present them in a tabular form indicating the choices of \( E(p) \) as 1, 2, and 3.

For the sake of comparison, we present the results for the standard case also, i.e. for the \( \lambda = 0 \), \( f = 1 \) case, in the high energy high temperature limit where \( E \approx p \). We indicate this case as 0.

The three generic choices of \( E(p) \) we consider and the corresponding \( c_\lambda(E) \) are given in Table I. Note that \( g(E) \approx g_*(E) \propto c_\lambda^{-1} \), see equation (16).

|       | 0   | 1             | 2             | 3             |
|-------|-----|---------------|---------------|---------------|
| \( E \) | \( p \) | \( \lambda^{-1} (\ln \lambda p)^n \) | \( \lambda^{-1} (\lambda p)^n \) | \( \lambda^{-1} e^{\lambda p} \) |
| \( c_\lambda(E) \) | \( 1 \) | \( n (\lambda E)^{\frac{n}{n-1}} \) | \( n (\lambda E) \) | \( \lambda E \ln(\lambda E) \) |

**Table I:** Choices of \( E(p) \) and the corresponding \( c_\lambda(E) \). \( n > 0 \).

The above choices of \( E(p) \) indicate a few generic ways in which the requirement (18) can be satisfied. Some of these choices may perhaps have a
natural origin. For example, the choice 1 with \( n = 1 \) can be obtained from the first Casimir operator \[1, 7\]; the choice 2 with \( n = 1 \) can be obtained by simply assuming that the standard Hamiltonian remains valid when \( \lambda \neq 0 \) also \[7\]; the choice 2 with an integer \( n > 1 \) may perhaps be thought of as arising from higher derivative terms in an effective action. A detailed analysis of naturalness and origins of \( E(p) \) is, however, beyond the scope of the present letter. The choices of \( E(p) \) in Table I are chosen mainly to illustrate the general dynamical features of the GCRs \( \{1\} \) and their dependence on \( E(p) \).

The following general features of \( c_\lambda(E) \) can be seen clearly from Table I:

- Faster the asymptotic growth of \( E(p) \), larger is the speed of light \( c_\lambda(E) \) and, hence, smaller is \( g_*(E) \).

- In units where \( c_0 = 1 \), \( c_\lambda(E) \ll 1 \) for the choice 1 with \( n < 1 \); \( c_\lambda(E) = 1 \) for the choice 1 with \( n = 1 \); and \( c_\lambda(E) \gg 1 \) for the choice 1 with \( n > 1 \), and for the choices 2 and 3. The physical implications of \( c_\lambda(E) \neq 1 \) and its energy dependences are discussed in \[7, 8\].

Consider now various thermodynamical quantities given in equations \( \{8\} \). Their behaviour in the low temperature limit \( \{3\} \), namely in the limit \( \beta \gg \lambda \), is unaffected since \( E(p) \) is required to satisfy equation \( \{12\} \). Therefore, we study the high temperature limit \( \{4\} \), namely the limit \( \beta \ll \lambda \), only. With no loss of generality, we set \( m = 0 \) and, hence, \( \mu = 0 \) in equations \( \{8\} \) and study the \( a = -1 \) and \( a = 0 \) cases i.e., cases where the particles obey Bose-Einstein and Maxwell-Boltzmann statistics respectively. The results for \( a = 1 \) case are formally similar to those for \( a = 0 \) case in this limit.

In general, explicit evaluation of the partition function \( \ln Z \) is difficult, if possible at all. However, in the limit \( \beta \ll \lambda \), the leading order behaviour of \( \ln Z \) can be obtained easily upto numerical factors, which suffices for our purposes. The method we follow is to split the integral in \( \{8\} \) as follows:\[2\]

\[
\int_0^\infty dE (*) = \int_0^{\lambda^{-1}} dE (*) + \int_{\lambda^{-1}}^{\beta^{-1}} dE (*) + \int_{\beta^{-1}}^\infty dE (*). \]

We then obtain, upto numerical factors, the \( \beta \) dependence of each term and, in the limit \( \beta \ll \lambda \), use the leading order contribution to calculate the quantities of interest, namely \(-\beta F\) and \(\beta U\), given in equations \( \{8\} \).

\[2\] We thank J. Magueijo for a helpful correspondence on this point.
The calculations involved are straightforward and are not particularly illuminating. Hence, we omit the details and present only the results. The results for $-\beta F$ and $\beta U$ are presented in Table II for the $a = -1$ case and in Table III for the $a = 0$ case, where $C_0 \equiv \frac{\Omega_{d=1} V}{\hbar^2}$ and $C \equiv \frac{\Omega_{d=1} V}{\hbar^2 \lambda^2}$.

|       | 0   | 1                | 2                | 3                |
|-------|-----|------------------|------------------|------------------|
| $E$   | $p$ | $\lambda^{-1} (\ln \lambda p)^n$ | $\lambda^{-1} (\lambda p)^n$ | $\lambda^{-1} e^{\lambda p}$ |
| $-\beta F$ | $C_0 \left( \frac{1}{\beta} \right)^d$ | $C \left( \frac{1}{\beta} \right)^{\frac{1}{n}}$ | $C \left( \ln \frac{1}{\beta} \right)^2$ | $C \left( \ln \frac{1}{\beta} \right) \ln \left( \ln \frac{1}{\beta} \right)$ |
| $\beta U$ | $dC_0 \left( \frac{1}{\beta} \right)^d$ | $\frac{c}{n} \left( \frac{1}{\beta} \right)^{\frac{1}{n}}$ | $C \left( \ln \frac{1}{\beta} \right)$ | $C \ln \left( \ln \frac{1}{\beta} \right)$ |

Table II: $-\beta F$ and $\beta U$ for the $a = -1$ case.

|       | 0   | 1                | 2                | 3                |
|-------|-----|------------------|------------------|------------------|
| $E$   | $p$ | $\lambda^{-1} (\ln \lambda p)^n$ | $\lambda^{-1} (\lambda p)^n$ | $\lambda^{-1} e^{\lambda p}$ |
| $-\beta F$ | $C_0 \left( \frac{1}{\beta} \right)^d$ | $C \left( \frac{1}{\beta} \right)^{\frac{1}{n}}$ | $C \left( \ln \frac{1}{\beta} \right)$ | $C \ln \left( \ln \frac{1}{\beta} \right)$ |
| $\beta U$ | $dC_0 \left( \frac{1}{\beta} \right)^d$ | $\frac{c}{n} \left( \frac{1}{\beta} \right)^{\frac{1}{n}}$ | $C$ | $\frac{c}{\ln \lambda}$ |

Table III: $-\beta F$ and $\beta U$ for the $a = 0$ case.

The temperature dependence of $-\beta F$ indicates the effective thermodynamic degrees of freedom (d.o.f) in the system, see [9] for example. The following general features can then be seen clearly from Tables II and III:

- Faster the asymptotic growth of $E(p)$, larger is the reduction in the d.o.f.

- The d.o.f in the case of choice 1 are formally equivalent to that of the standard case but with an effective dimension $d_{eff} = \frac{1}{n}$. 

9
• Compared to the standard case, the d.o.f are reduced in the case of choices 2 and 3, and in the case of choice 1 with $n > \frac{1}{d}$, whereas they are increased in the case of choice 1 with $n < \frac{1}{d}$.

• The reduction in the d.o.f in the case of choice 1 with $n = 1$ is of the type found in the case of strings at temperatures much higher than the Hagedorn temperature (if $\lambda \simeq \text{String length}$) [9].

• The reduction in the d.o.f in the $a = 0$ case for the choice 2 with any $n$ is of the type found in a lattice theory with a finite number of bose oscillators at each site [13], or in certain topological field theories [13] with general coordinate invariance restored at short distances [9]. It will be of interest to investigate whether the types of reduction in the d.o.f seen in the remaining cases also arise in other contexts.

• Let $w = \frac{P}{\rho}$, where $\rho = \frac{U}{V}$ is the energy density. For perfect fluids in the standard case, $w$ is constant and must be $< 1$ since the speed of sound $v_s < 1$ in units where $c_0 = 1$. In the present case, $w$ can be $> 1$, see Tables II and III. It can then be checked easily that $v_s$ can also be $> 1$ but remains $< c_\lambda$ always.

5. High temperatures where $\beta \ll \lambda$ arise naturally in the early universe, dominated by radiation for which $a = -1$ and which we assume obeys the GCRs [1]. We mention briefly the resulting consequences for the early universe evolution, assumed to be determined by the standard equations with radiation pressure $P$ and its energy density $\rho = \frac{U}{V}$ given in Table II in the limit $\beta \ll \lambda$. See [14] for similar studies.

   The relevent line element is given by $ds^2 = -c_\lambda^2 dt^2 + A^2(t) \sum_{i=1}^d dX^i dX^i$. The comoving horizon radius $r_h$ at time $t_0 > 0$ is given by $r_h = \int_{t_0}^{t_0} dt \left( \frac{c_\lambda}{A} \right)$ where $t = 0$ is the time of big bang singularity. Taking the temperature $T$ to be the independent variable and using the results given in Tables I and II, it is straightforward to calculate $t$, $A$, and $r_h$ to the leading order in the limit $\beta \ll \lambda$. The results for $t$, $A$, and $r_h$ are presented in Table IV, where $K \equiv \left( 1 - \frac{2}{d(n+1)} - \frac{2(n-1)}{n} \right)$. 

10
\[
\begin{array}{|c|c|c|c|c|}
\hline
E & p & \lambda^{-1} (\ln p)^n & \lambda^{-1} (\lambda p)^n & \lambda^{-1} e^{\lambda p} \\
\hline
\lambda t & (\lambda T)^{-d+\frac{1}{2}} & (\lambda T)^{-d+\frac{1}{2}} & (\lambda T)^{-d+\frac{1}{2}} & (\lambda T)^{-d+\frac{1}{2}} \\
A & (\lambda T)^{-1} & (\lambda T)^{-\frac{1}{2}} & (\lambda T)^{-\frac{1}{2}} & (\lambda T)^{-\frac{1}{2}} \\
\frac{r_h}{\lambda} & (\lambda T)^{-d+\frac{1}{2}} & (\lambda T)^{-d+\frac{1}{2}} & (\lambda T)^{-d+\frac{1}{2}} & (\lambda T)^{-d+\frac{1}{2}} \\
\hline
\end{array}
\]

Table IV: \(t(T), A(T), \) and \(r_h(T)\) for the early universe.

The following general features can be seen clearly from Table IV:

- In all the cases, \(T \to \infty\) and \(A \to 0\) in the limit \(t \to 0\). The curvature invariants diverge at \(t = 0\), resulting in a big bang singularity.

- For the choice 1 with \(n > \frac{1}{d}\), and for the choices 2 and 3, \(A(t) \to 0\) more slowly than in the standard case.

- Let \(T \to \infty\). Then, for the choices 2 and 3, \(r_h \to \infty\). This is true for the choice 1 also if \(K < 0\), equivalently if \(n > n_*\) where \(n_* = \frac{d-2+\sqrt{(d-2)^2+8d^2}}{2d}\). Note that \(1 \leq n_* \leq 2\) for \(1 \leq d \leq \infty\). Hence, \(r_h \to \infty\) for the choice 1 also if \(n \geq 2\). It can then be inferred from table I that \(r_h \to \infty\) if \(c_\lambda(E)\) grows at least as fast as \(\sqrt{\lambda E}\) in the limit \(\lambda E \gg 1\).

Note that \(r_h\) is finite for the choice 1 with \(1 < n < n_*\) despite the speed of light \(c_\lambda(E)\) increasing with energy, see Table I. This implies that \(c_\lambda\) increasing with energy alone is not sufficient to cause \(r_h\) to diverge but the scale factor \(A\) must also vanish at a sufficiently slow pace. Thereby, photons with speed \(c_\lambda(E) \gg 1\), whose number is non negligible as can be seen from the black body spectrum [17], have sufficient time to establish causal contact within the horizon \(r_h\) before encountering the big bang singularity.
6. We have studied the dynamical features of the GCRs (1) and their
dependence on the energy $E(p)$ by considering three generic choices of $E(p)$.
The dynamical features can be seen clearly in the high energy high temper-
ature limit (4), and are summarised briefly as follows. In the limit (4):

- The dynamical quantities are, upto numerical factors, independent of
  the number of spatial dimensions $d$.

- The black body spectrum approaches the limiting form (17), which is
  independent of the choice of $E(p)$ also.

- Generically, the speed of light depends on energy and it is larger, faster
  the asymptotic growth of $E(p)$; the one particle density of states is
  correspondingly smaller.

- The thermodynamical relations are highly modified. The effective ther-
  modynamical degrees of freedom depend on energy, and their reduction
  is larger, faster the asymptotic growth of $E(p)$.

- In the early universe, the scale factor evolves more slowly; the hori-
  zon size $r_h$ increases faster, faster the asymptotic growth of $E(p)$ and,
  generically, $r_h \to \infty$.

In view of the results presented here, we believe that further detailed
studies of the GCRs (1), and those in 2, will be fruitful. It is of interest to
study, in particular, if and how the standard Lorentz invariance is modified
in the presence of the GCRs. Such a modification, if found, is likely to
determine the Hamiltonian uniquely. It is also likely to suggest a general
coordinate invariant formulation of the GCRs (1), which can then be used
to study rigorously the implications for cosmology and black hole physics.

Acknowledgement: We thank J. Magueijo for a helpful correspondence.

References

[1] M. Maggiore, Phys. Lett. B304 (1993) 65, hep-th/9301067; Phys. Rev.
D49 (1994) 5182, hep-th/9305163; Phys. Lett. B319 (1993) 83, hep-
th/9309034.
[2] A. Kempf, J. Math. Phys. 35 (1994) 4483, hep-th/9311147; J. Phys. A30 (1997) 2093, hep-th/9604045.

[3] J. Lukierski, H. Ruegg, and W. J. Zakrzewski, Ann. Phys. 243 (1995) 90, hep-th/9312153; H. Bacry, J. Phys. A: Math. Gen. 26 (1993) 5413.

[4] A. Kempf, G. Mangano, and R. B. Mann, Phys. Rev. D52 (1995) 1108, hep-th/9412167; J. Kowalski-Glikman, Mod. Phys. Lett. A17 (2002) 1, hep-th/0107054; hep-th/0111110.

[5] D. V. Ahluwalia, Phys. Lett. B339 (1994) 301, gr-qc/9308007; Phys. Lett. A275 (2000) 31, gr-qc/0002003.

[6] R. Brout, Cl. Gabriel, M. Lubo, and Ph. Spindel, Phys. Rev. D59 (1999) 044005, hep-th/9807063; M. Lubo, Phys. Rev. D61 (2000) 124009, hep-th/9911191.

[7] S. Kalyana Rama, Phys. Lett. B519 (2001) 103, hep-th/0107253.

[8] For studies related to varying speed of light, see for example, J. W. Moffat, Int. J. Mod. Phys. D2 (1993) 351, gr-qc/9211020; Found. of Phys. 23 (1993) 411, gr-qc/9209001; A. Albrecht, J. Magueijo, Phys. Rev. D59 (1999) 043516, astro-ph/9811018; J. D. Barrow, Phys. Rev. D59 (1999) 043515; J. Magueijo, Phys. Rev. D62 (2000) 103521, gr-qc/0007036; Phys. Rev. D63 (2001) 043502, astro-ph/0001059; J. D. Barrow and J. Magueijo, Phys. Lett. B443 (1998) 104, astro-ph/9811072; Phys. Lett. B447 (1999) 246, astro-ph/9811073; Class. Quant. Grav. 16 (1999) 1435, astro-ph/9901049; For studies related to modified dispersion relations, see for example, J. Martin, R. H. Brandenberger, astro-ph/0012031; astro-ph/0005432; Phys. Rev. D63 (2001) 123501, hep-th/0005209.

[9] J. J. Atick and E. Witten, Nucl. Phys. B310 (1988) 291.

[10] See, for example, W. Greiner, L. Neise, and H. Stocker, Thermodynamics and Statistical Mechanics, Springer Verlag (1995).

[11] T. Alväger et al, Arkiv Fysik 31 (1966) 145 as quoted in Z. Bay and J. A. White, Phys. Rev. D5 (1972) 796.
[12] S. Coleman and S. L. Glashow, Phys. Rev. D59 (1999) 116008, hep-ph/9812418; F. W. Stecker and S. L. Glashow, Astropart. Phys. 16 (2001) 97. See also D. Colladay and V. A. Kostelecky, Phys. Rev. D58 (1998) 116002, hep-ph/9809521.

[13] E. Witten, Comm. Math. Phys. 117 (1988) 353; Comm. Math. Phys. 118 (1988) 411; Phys. Lett. B206 (1988) 601.

[14] S. H. S. Alexander and J. Magueijo, hep-th/0104093; J. C. Niemeyer, Phys. Rev. D65 (2002) 083505, astro-ph/0111479; astro-ph/0201511.