The Eisenstein and winding elements of modular symbols for odd square-free level

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Abstract We explicitly write down the Eisenstein elements inside the space of modular symbols for Eisenstein series with integer coefficients for the congruence subgroups \( \Gamma_0(N) \) with \( N \) odd square-free. We also compute the winding elements explicitly for these congruence subgroups. Our results are explicit versions of the Manin-Drinfeld Theorem [Thm. 6]. These results are the generalization of the paper [1] results to odd square-free level.

Keywords Eisenstein series · Modular symbols · Special values of \( L \)-functions

Mathematics Subject Classification Primary: 11F67 · Secondary: 11F11 · 11F20 · 11F30

1 Introduction

Mazur gave a list of all possible torsion subgroups of elliptic curves [cf. Thm. 8, [2]]. Merel wrote down explicit expression of the Eisenstein elements for the congruence subgroups \( \Gamma_0(p) \) for any odd prime \( p \) [3]. He used this to give an uniform upper bound of the torsion points of elliptic curves over any number fields in terms of extension degrees of these number fields [4]. The explicit expressions of winding elements for \( \Gamma_0(p) \) [3] are used by Calegari and Emerton to study the ramifications of Hecke algebras at the Eisenstein primes [5]. Several authors investigated the arithmetic invariants of the elliptic curves over number fields using modular symbols.

In this paper, we give an “explicit version” of the proof of the Manin-Drinfeld theorem [Thm. 6] for the special case of the image in \( H_1(X_0(N), \mathbb{R}) \) of the path in \( H_1(X_0(N), \partial(X_0(N)), \mathbb{Z}) \) joining 0 and \( i \infty \).

For \( m \mid N \), consider the basis \( \{ E_m \} \) of \( E_2(\Gamma_0(N)) \) [§ 5] for which the constant term \( a_0(E_m) \in \mathbb{Z} \).

Definition 1 (Eisenstein elements) The intersection pairing \( \circ \) induces a perfect, bilinear pairing \( H_1(X_0(N), \partial(X_0(N)), \mathbb{Z}) \times H_1(Y_0(N), \mathbb{Z}) \to \mathbb{Z} \). Let \( \pi_{E_m} : H_1(Y_0(N), \mathbb{Z}) \to \mathbb{Z} \) be the “period” homomorphism of \( E_m \) [§ 2.7]. Since \( \circ \) is a non-degenerate, there is an unique modular symbol \( \mathcal{E}_{E_m} \in H_1(X_0(N), \partial(X_0(N)), \mathbb{Z}) \) such that \( \mathcal{E}_{E_m} \circ c = \pi_{E_m}(c) \). This unique element \( \mathcal{E}_{E_m} \) is the Eisenstein element corresponding to the Eisenstein series \( E_m \).

Definition 2 (Winding elements) Let \( [0, \infty) \) denote the projection of the path from 0 to \( \infty \) in \( \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}) \) to \( X_0(N) \). We have an isomorphism \( H_1(X_0(N), \mathbb{Z}) \otimes \mathbb{R} \simeq \text{Hom}_\mathbb{C}(H^0(X_0(N), \Omega^1), \mathbb{C}) \). Let \( e_N : H^0(X_0(N), \Omega^1) \to \mathbb{C} \) be given by the homomorphism \( \omega \mapsto -f_0^\infty \omega \). The modular symbol \( e_N \in H_1(X_0(N), \mathbb{R}) \) is called the winding element.
The winding elements are the elements of the space of modular symbols whose annihilators define ideals of the Hecke algebras with the $L$-functions of the corresponding quotients of the Jacobian non-zero. Since the algebraic part of the special values of $L$-function are obtained by integrating differential forms on these modular symbols, our explicit expression of the winding elements are useful to understand the algebraic parts of the special values of the $L$-functions of the quotient $J_0(N)$ [7]. In this paper, we find an explicit expression of the winding elements and the Eisenstein elements [Theorem 3].

Let $B_1(x)$ be the first Bernoulli’s polynomial of period 1 defined by $B_1(0) = 0$, $B_1(x) = x - \frac{1}{2}$, if $x \in (0, 1)$. Let $u, v \in \mathbb{Z}$ such that $(u, v) = 1$ and $v \geq 1$, we define the Dedekind sum by the formula:

$$S(u, v) = \sum_{t=1}^{v-1} B_1\left(\frac{tu}{v}\right)B_1\left(\frac{u}{v}\right).$$

For any $k \in \mathbb{Z}$, let $\delta_k \in \{0, 1\}$ such that $\delta_k \equiv k$ (mod 2). Let $s_k = (k + (\delta_k - 1)N)$. Then $s_k$ is always an odd integer. Let $x$ be one of the prime divisor of $N$. Any coset representative $g \in \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$ can be written as either $(1, \pm t)$ or $(\pm t, 1)$. We observe that $(r - 1, r + 1) \sim (1, \pm t)$ or $(\pm t, 1)$ for some $r \leftrightarrow t = \pm 1 \pm kx$ for some $k$. Choose integers $s$, $s'$ and $l$, $l'$ such that $l(s_kx + 2) - 2sN = 1$ and $l's_kx - 2s'N = 1$. Let $\gamma_{y, k}^1 = \left(\frac{1 + 4rN - 22}{-4s(s_kx + 2)N + 4sN}\right)$ and $\gamma_{y, k}^2 = \left(\frac{1 + 4s'N - 22}{-4v(s_kx + 2)N + 4s'N}\right)$ for $y = 1, 2$, consider the integers

$$P_m(y_{y, k}^x) = \text{sgn}(r(y_{y, k}^x))(2(S(s(y_{y, k}^x), |r(y_{y, k}^x)|) - S(s(y_{y, k}^x), |r(y_{y, k}^x)|))$$

$$-S(s(y_{y, k}^x), \frac{|r(y_{y, k}^x)|}{2}) + S(s(y_{y, k}^x), \frac{|r(y_{y, k}^x)|}{2})$$

where

$$s(y_{y, k}^x) = 1 - 4sN(1 + s_kx), \quad r(y_{y, k}^x) = -2l(2s(s_kx + 2)N), \quad s(y_{y, k}^y) = 1 - 4s'N(s_k - \frac{1}{x}), \quad r(y_{y, k}^y) = -2l' - 2s'N.$$

Let $\xi : \text{SL}_2(\mathbb{Z}) \to H_1(X_0(N), \text{cusp}, \mathbb{Z})$ be the Manin map [5] and $F_m : \mathbb{P}^1(\mathbb{Z}/m\mathbb{Z}) \to \mathbb{Z}$ be defined by

$$F_m(g) = \begin{cases} 2(S(r, m) - 2S(r, 2m)) & \text{if } g = (r - 1, r + 1), \\ P_m(y_{1, k}^x) - P_m(y_{2, k}^x) & \text{if } g = (1 + kx, 1) \text{ or } g = (-1 - kx, 1), \\ -P_m(y_{1, k}^x) + P_m(y_{2, k}^x) & \text{if } g = (1, 1 + kx) \text{ or } g = (1, -1 - kx), \\ 0 & \text{if } g = (\pm 1, 1). \end{cases}$$

**Theorem 3** Let $N$ be odd square-free level. The modular symbol

$$\mathcal{E}_{E_m} = \sum_{g \in \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})} F_m(g)\xi(g) \in H_1(X_0(N), \partial(X_0(N), \mathbb{Z})$$

corresponding to the Eisenstein series $E_m \in E_2(\Gamma_0(N))$.

The element $e_N = \frac{1}{(1 - N)} \sum_{v \in (\mathbb{Z}/N\mathbb{Z})^*} F_N((1, v))(0, 1/v)$ is the winding element for $\Gamma_0(N)$.

In [8] and [1], Eisenstein elements are described for $N = p^2$, $N = pq$ respectively. In this article, we give an explicit description in terms of two matrices $\gamma_{y, k}^1$ and $\gamma_{y, k}^2$ for $N$ odd square-free level. Note that if $h = \text{gcd}(N - 1, 12)$ and $n = \frac{N - 1}{h}$, then a multiple of winding element $ne_N$ belongs to $H_1(X_0(N), \mathbb{Z})$. Main and Drinfeld proved that the modular symbol $[0, \infty] \in H_1(X_0(N), \mathbb{Q})$. For the congruence subgroup $\Gamma_0(N)$ with $N$ square-free, we use the relative homology group $H_1(X_0(N), R \cup I, \mathbb{Z})$ and $H_1(X_{\Gamma} - P_+, P_+, \mathbb{Z})$. We intersect with the congruence subgroup $\Gamma(2)$ to ensure that the Manin maps become bijective (rather than only surjective). These relative homology groups are used in the study of modular symbol by the discovery of Merel. We follow his approach [cf. [3], Prop. 11] to prove our results.

This paper is arranged as follows. In Section 2, we write down some preliminaries. In Section 3, we compute the coset representatives and cusps. In Section 4, we explicitly write down the even Eisenstein elements [Theorem 21]. In Sections 5 and 6, we write the boundary of the Eisenstein series and Eisenstein elements. In Section 7, we prove our main theorem.
2 Preliminaries

For any natural number $M > 4$, the congruence subgroup $\{ \Gamma_0(M) = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{SL}_2(\mathbb{Z})/M \mid c \}$ acts on $\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}) = \mathbb{H}$. Let $Y_0(M)$ be the quotient space $\Gamma_0(M) \backslash \mathbb{H}$. These are non-compact Riemann surfaces and hence algebraic curves defined over $\mathbb{C}$. There are models of these algebraic curve defined over $\mathbb{Q}$ and they parametrize elliptic curves with cyclic subgroups of order $M$. Let $X_0(M)$ be the compactification of $Y_0(M)$. The set of cusps of $X_0(M)$ is given by $\Gamma_0(M) \backslash \mathbb{P}^1(\mathbb{Q})$. The modular curve $X_0(M) = Y_0(M) \cup \partial(X_0(M))$ is a smooth projective curve $X_0(N)$ defined over $\mathbb{Q}$, we have $\Gamma_0(N) \backslash \mathbb{H} \simeq X_0(N)$. We are interested to understand the $\mathbb{Q}$-structure of $X_0(N)$. Let $\Gamma \subset \text{SL}_2(\mathbb{Z})$ be a congruence subgroup. The topological space $X_\Gamma(\mathbb{C}) = \Gamma \backslash \mathbb{H}$ has a natural structure of a smooth compact Riemann surface. The projection map $\pi : \mathbb{H} \rightarrow X_\Gamma(\mathbb{C})$ is unramified outside the elliptic points and the set of cusps $\partial(X_\Gamma)$. Both these sets are finite.

2.1 Classical modular symbols

Recall the following fundamental theorem of Manin.

**Theorem 4** For $\alpha \in \mathbb{H}$, consider the map $c : \Gamma \rightarrow H_1(X_0(N), \mathbb{Z})$ defined by $c(g) = [\alpha, g\alpha]$. The map $c$ is a surjective group homomorphism and does not depend on $\alpha$. In fact, the kernel of this homomorphism is generated by the commutator, the elliptic elements, and the parabolic elements of the congruence subgroup $\Gamma$. In particular, $[\alpha, g\alpha] = 0$ for all $\alpha \in \mathbb{P}^1(\mathbb{Q})$ and $g \in \Gamma$.

2.2 The Manin map

Let $T, S$ be the matrices $\left( \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right), \left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right)$ and $R = ST$ be the matrix $\left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$. The modular group $\text{SL}_2(\mathbb{Z})$ is generated by $S$ and $T$.

**Theorem 5** *(Manin)*\[9] Let $\xi : \text{SL}_2(\mathbb{Z}) \rightarrow H_1(X_0(N), \partial(X_0(N)), \mathbb{Z})$ be the map that takes $g \in \text{SL}_2(\mathbb{Z})$ to the class in $H_1(X_0(N), \partial(X_0(N)), \mathbb{Z})$ of the image in $X_0(N)$ of the geodesic in $\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$ joining $g0 \text{ and } g\infty$. Then $\xi$ is surjective and $\forall g \in \Gamma_0(N) \backslash \text{SL}_2(\mathbb{Z})$, $\xi(g) + \xi(gS) + \xi(gR) + \xi(gR^2) = 0$.

2.3 Manin-Drinfeld theorem

We state the Manin-Drinfeld theorem.

**Theorem 6** *(Manin-Drinfeld)* \[10] Let $\Gamma$ be a congruence subgroup and $\alpha, \beta \in \mathbb{P}^1(\mathbb{Q})$ be any two cusps.

The path $[\alpha, \beta] \in H_1(X_\Gamma, \mathbb{Q})$.

As a corollary, we observe that $\{x, y\} \in H_1(X_\Gamma, \mathbb{Q})$ if and if $m(\pi_\Gamma(x) - \pi_\Gamma(y))$ is a divisor of a function for some positive integer $m$. That is, the degree zero divisors supported on the cusps are of finite order in the divisor class group. Manin-Drinfeld proved it using the extended action of the usual Hecke operators. In particular, it says that $\{0, \infty\} \in H_1(X_\Gamma, \mathbb{Q})$. We have a short exact sequence, $0 \rightarrow H_1(X_0(N), \mathbb{Z}) \rightarrow \delta' : H_1(X_0(N), \partial(X_0(N)), \mathbb{Z}) \rightarrow \mathbb{Z}^{\partial(X_0(N))} \rightarrow \mathbb{Z} \rightarrow 0$. The first map is a canonical injection. The second map $\delta : H_1(X_0(N), \partial(X_0(N)), \mathbb{Z}) \rightarrow \mathbb{Z}^{\partial(X_0(N))}$ is a boundary map which takes a geodesic, joining the cusps $r$ and $s$ to the formal symbol $[r] - [s]$ and the third map is the sum of the coefficients.

2.4 Relative homology group $H_1(X_0(N) - R \cup I, \partial(X_0(N)), \mathbb{Z})$ and almost Eisenstein series

Let $v$ be the geodesic joining the elliptic points $i$ and $\rho = \frac{1+\sqrt{-3}}{2}$ of $X_0(N)$. Set $R = \pi(\text{SL}_2(\mathbb{Z}) \rho)$ and $I = \pi(\text{SL}_2(\mathbb{Z}) i)$). A small checks shows that these two sets are disjoint.

For $g \in \text{SL}_2(\mathbb{Z})$, let $g|_N$ be the class of $\pi(gv)$ in the relative homology group $H_1(Y_0(N), R \cup I, \mathbb{Z})$. Let $\rho^* = -\rho$ be another point on the boundary of the fundamental domain. The homology groups $H_1(Y_0(N), \mathbb{Z})$ are subgroups of $H_1(Y_0(N), R \cup I, \mathbb{Z})$. Suppose $z_0 \in \mathbb{H}$ be such that $|z_0| = 1$ and $\frac{1}{1-i} < Re(z_0) < 1$. Let $\gamma$ be
the union of the geodesic in \( \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}) \) joining 0 and \( z_0 \) and \( z_0 + i \infty \). For \( g \in \Gamma_0(N) \backslash \text{SL}_2(\mathbb{Z}) \), let \( [g]^{\ast} \) be the class of \( \pi(g\gamma) \) in \( H_1(X_0(N) - R \cup I, \partial(X_0(N)), \mathbb{Z}) \). We have a short exact sequence,

\[
0 \to H_1(X_0(N) - R \cup I, \partial(X_0(N)), \mathbb{Z}) \to H_1(Y_0(N), R \cup I, \mathbb{Z}) \to \mathbb{Z} \to 0.
\]

The first map is a canonical injection. The second map \( \delta \) is the boundary map which takes a geodesic, joining the cusps \( r \) and \( s \) to the formal symbol \([r] - [s]\) and the third map is the sum of the coefficients. Note that \( \delta'(\xi(g)) = \delta([g]^{\ast}) \) for all \( g \in \text{SL}_2(\mathbb{Z}) \).

**Definition 9** (Almost Eisenstein elements) For \( m \mid N \), the differential form \( E_m(z)dz \) is of first kind on the Riemann surface \( Y_0(N) \). Since \( \phi \) is a non-degenerate bilinear pairing, there is an unique element \( \mathcal{E}_m' \) \( \in H_1(X_0(N) - R \cup I, \partial(X_0(N)), \mathbb{Z}) \) such that \( \mathcal{E}_m' \circ c = \pi_{E_m}(c) \) for all \( c \in H_1(Y_0(N), R \cup I, \mathbb{Z}) \). We call \( \mathcal{E}_m' \) the almost Eisenstein element corresponding to the Eisenstein series \( E_m \).

### 2.5 Modular Curves with bijective manin maps

The Riemann sphere or the projective complex plane \( \mathbb{P}^1(\mathbb{C}) \) is the only one simply connected (genus zero) compact Riemann surface up to conformal bijections. A theorem of Belyi states that every (compact, connected, non-singular) algebraic curve \( X \) has a model defined over \( \mathbb{Q} \) if and only if it admits a map to \( \mathbb{P}^1(\mathbb{C}) \) branched over three points. Let \( \Gamma(2) = \left\{ \left( \begin{array}{cc} a & b \\ d & c \end{array} \right) \in \text{SL}_2(\mathbb{Z}) \mid (a, b, c, d) \equiv \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \right) \pmod{2} \right\} \) and \( X(2) = \Gamma(2) \mod \mathbb{H} \). Then \( \partial(X(2)) = \Gamma(2)0, \Gamma(2)1, \Gamma(2)\infty \). Hence, \( X(2) \) is the simply connected Riemann surface \( \mathbb{P}^1(\mathbb{C}) \) with the three marked points \( \Gamma(2)0, \Gamma(2)1 \) and \( \Gamma(2)\infty \).

Let \( \Gamma = \Gamma_0(N) \cap \Gamma(2) \). Let \( \pi_0 : \Gamma \backslash \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}) \to \Gamma(2) \backslash \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}) \) be the map \( \pi_0(\Gamma z) = \Gamma(2)z \). Let \( P_- = \pi_0^{-1}(\Gamma(2)1), P_+ = [\pi_0^{-1}(\Gamma(2)0) \cup [\pi_0^{-1}(\Gamma(2)\infty)] \). The Riemann surface \( X_\Gamma \) has boundary \( P_+ \cup P_- \).

### 2.6 Relative homology group \( H_1(X_\Gamma - P_-, P_+, \mathbb{Z}) \) and even Eisenstein elements

We study the relative homology groups \( H_1(X_\Gamma - P_-, P_+, \mathbb{Z}) \) and \( H_1(X_\Gamma - P_+, P_-, \mathbb{Z}) \). The intersection pairing is non-degenerate bilinear pairing \( \circ : H_1(X_\Gamma - P_+, P_- \cup P_+ , \mathbb{Z}) \times H_1(X_\Gamma - P_-, P_+ , \mathbb{Z}) \to \mathbb{Z} \). For \( g \in \Gamma \backslash \Gamma(2) \), let \( [g]^{\ast} \) (respectively \( [g_0]^{\ast} \)) be the image in \( X_\Gamma \) of the geodesic in \( \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}) \) joining \( g\gamma \) (respectively \( g\gamma(0) \)) and \( g\gamma(\infty) \) (respectively \( g\gamma(\infty) \)). We state two fundamental theorems.

**Theorem 10 ([3])** Let \( \xi_0 : \mathbb{Z}^{\Gamma \backslash \Gamma(2)} \to H_1(X_\Gamma - P_+, P_- , \mathbb{Z}), \xi_0 : \mathbb{Z}^{\Gamma \backslash \Gamma(2)} \to H_1(X_\Gamma - P_-, P_+ , \mathbb{Z}) \) be the maps which take \( g \in \Gamma \backslash \Gamma(2) \) to the element \( [g]_0 \) and \( g \in \Gamma \backslash \Gamma(2) \) to the element \( [g]^{\ast} \) respectively. The homomorphisms \( \xi_0 \) and \( \xi_0 \) are isomorphisms.

**Theorem 11 ([3])** For \( g, g' \in \Gamma(2) \), we have \( [g]_0 \circ [g']^{\ast} = \begin{cases} 1 & \text{if } \Gamma g = \Gamma g' \\ 0 & \text{otherwise} \end{cases} \).
Definition 12 (Even Eisenstein elements). For \( E_m \in \mathbb{E}_N \), let \( \lambda_{E_m} : X_0(N) \to \mathbb{C} \) be the rational function whose logarithmic differential is \( 2\pi i E_m(z) dz = 2\pi i \omega_{E_m} \). Consider the rational function \( \lambda_{E_{m,2}} = \left[ \lambda_{E_m} \omega_m \right]^2 \) on \( X_{\Gamma} \).

By Lemma 19, this function has no zeros and poles in \( P_- \). Let \( \kappa^*(\omega_{E_m}) \) be the logarithmic differential of the function. Let \( \varphi_{E_m}(c) = \int_c \kappa^*(\omega_{E_m}) \) be the corresponding period homomorphism form \( H_1(X_{\Gamma - P_+}, P_-, \mathbb{Z}) \) to \( \mathbb{Z} \).

By the non-degeneracy of the intersection pairing, there is a unique element \( E_{m}^c \in H_1(X_{\Gamma - P_+}, P_+, \mathbb{Z}) \) such that \( \int E_{m}^c \circ c = \varphi_{E_m}(c) \) for all \( c \in H_1(X_{\Gamma - P_+}, P_-, \mathbb{Z}) \). The modular symbol \( E_{m}^c \) is the even Eisenstein element corresponding to the Eisenstein series \( E_m \).

2.7 Period Homomorphisms

We state the period homomorphisms for the differential forms of third kind. Refer [[12], p. 10, [3], p. 14] for some properties of the following period map \( \pi_{E_m} \).

Definition 13 (Period homomorphism) For \( E_m \in \mathbb{E}_N \), the differential forms \( E_m(z) dz \) are of third kind on the Riemann surface \( X_0(N) \) but of first kind on the non-compact Riemann surface \( Y_0(N) \). For any \( z_0 \in \mathbb{H} \) and \( \gamma \in \Gamma_0(N) \), let \( c(\gamma) \) be the class in \( H_1(Y_0(N), \mathbb{Z}) \) of the image of \( Y_0(N) \) of the geodesic in \( \mathbb{H} \) joining \( z_0 \) and \( \gamma(z_0) \). This class is non-zero [Thm. 4] and is independent of the choice of \( z_0 \in \mathbb{H} \). Let \( \pi_{E_m}(\gamma) = \int_{c(\gamma)} E_m(z) dz \).

This map \( \pi_{E_m} : \Gamma_0(N) \to \mathbb{Z} \) is the “period” homomorphism of \( E_m \).

Proposition 14 Let \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) be an element of \( \Gamma_0(N) \) and \( \mu = \gcd(m - 1, 12) \).

1. \( \pi_{E_m} \) is a homomorphism \( \Gamma_0(N) \to \mathbb{Z} \) and \( \pi_{E_m}(\gamma) = \pi_{E_m}(\alpha = \begin{pmatrix} \frac{d}{m} \\ \frac{c}{a} \end{pmatrix}) \).

2. The image of \( \pi_{E_m} \) lies in \( \mu \mathbb{Z} \) and \( \pi_{E_m}(\gamma) = \begin{pmatrix} \frac{ad - bc}{\mu} \alpha \end{pmatrix}(m - 1) + 12\sgn(c)(S(d, |c|) - S(d, |c|/\mu)) \) if \( c \neq 0 \), \( \pi_{E_m}(\gamma) = \begin{pmatrix} \frac{ad}{\mu} \alpha \end{pmatrix}(m - 1) \) if \( c = 0 \).

3 Coset representatives and Cusps

We have a canonical bijection \( \Gamma_0(N) \setminus \mathrm{SL}_2(\mathbb{Z}) \cong \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) \) given by \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to (c, d) \).

Let \( \Omega = \cup_{m \mid N} M_m \), where for any \( m \mid N \) with \( 1 < m < N \),
\[
M_m = \{ \beta_1 = \begin{pmatrix} -1 \\ M_m \end{pmatrix}, \, 0 \leq l \leq \frac{N}{m} - 1 \}, \, M_N = \{ \alpha_N = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \}, \, M_1 = \{ \alpha_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \}, \, 0 \leq k \leq N - 1 \}.
\]

Lemma 15 The set \( \Omega \) is a complete set of coset representatives of \( \Gamma_0(N) \setminus \mathrm{SL}_2(\mathbb{Z}) \equiv \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) \).

Proof Let \( m_1, m_2 \) be two divisors of \( N \) such that \( \beta_1^{m_1}(\beta_2^{m_2})^{-1} \in \Gamma_0(N) \) for some \( 0 \leq l_1 \leq \frac{N}{m_1} - 1 \) and \( 0 \leq l_2 \leq \frac{N}{m_2} - 1 \). This implies that \( m_1 l_2 - l_1 - m_1 + m_2 \equiv 0 \) (mod \( N \)). Since \( m_1, m_2 \) are divisors of \( N \), the above expression implies that \( m_1 = m_2 \). Therefore the above expression reduces to \( m_1^2(l_2 - l_1) \equiv 0 \) (mod \( N \)), which further implies that \( m_1(l_2 - l_1) \equiv 0 \) (mod \( \frac{N}{m_1} \)). Since \( m_1 \equiv \frac{N}{m_1} \equiv 1 \), we get \( l_1 = l_2 \). Since \( ab^{-1} \notin \Gamma_0(N) \) for \( a \in M_1 \cup M_N \) and \( b \in M_m \) and \#\( \Omega = |\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})| = \sum_{m \mid N} \frac{N}{m} \), the result follows.

There are \( 2r \) cusps of \( X_0(N) \), where \( r \) is the number of primes dividing the level \( N \). They are explicitly given in the following lemma.

Lemma 16 The cusps \( \partial(X_0(N)) \) can be identified with the set \( \{ 0, \infty \} \cup \{ \frac{1}{m}, m \mid N \} \).

Proof If \( \frac{a}{x} \) and \( \frac{a'}{x'} \) are in \( \mathbb{P}^1(\mathbb{Q}) \), then \( \Gamma_0(N) \frac{a}{x} = \Gamma_0(N) \frac{a'}{x'} \iff \begin{pmatrix} ay \\ cx \end{pmatrix} \equiv \begin{pmatrix} a'y + cx' \end{pmatrix} \) (mod \( N \)), for some \( x \) and \( y \) such that \( \gcd(x, N) = 1 \) [[13], p. 99]. This implies that \( \Gamma_0(N)0, \, \Gamma_0(N)\infty \) and \( \Gamma_0(N) \frac{1}{m}, (m \mid N) \) are disjoint.
We list the rational numbers $\alpha(0)$ and $\alpha(\infty)$ with $\alpha \in \Omega$ as equivalence classes of cusps as follows:

| $0$ | $1$  |
|------|------|
| $\frac{1}{m}$, $m \mid N$ | $\frac{1}{x}$, $(k, m) > 1$ |
| $\frac{1}{m}$, $(t \frac{N}{m} - 1, m) = 1$ | $\frac{1}{x^{a-1}}, (a \frac{N}{m} - 1, m) > 1$ |

Choose $a$ and $b$ to be two unique integers such that $a \frac{N}{m} + bm \equiv 1 \pmod{N}$ with $1 \leq a \leq (\frac{N}{m} - 1)$ and $1 \leq b \leq (m - 1)$. Let $\tilde{\Omega} = \cup_{m \mid N} \tilde{M}_m$, where for any $m \mid N$ with $1 < m < N$,

$$
\tilde{M}_m = \{ \tilde{\rho}_m^i \} = \left\{ \left( \begin{array}{ccc} -1 & -l_m + \delta_m \frac{N}{m} & 0 \\ (N+m)(-1) & 1 & \frac{N}{m} \\ 0 & 0 & 1 \end{array} \right); 0 \leq l_m \leq \frac{N}{m} - 1 \right\},
$$

$$
\tilde{M}_N = \{ \tilde{\alpha}_N = (\frac{N}{N+1} \frac{N-1}{N}) \} \tilde{M}_1 = \{ \tilde{\alpha}_k = \left( \begin{array}{ccc} s_k \frac{N^2}{s_k} & s_k \frac{N-1}{s_k} & 0 \\ s_k & s_k \end{array} \right); 0 \leq k \leq N - 1 \}.
$$

**Lemma 17** The set $\tilde{\Omega}$ is a complete set of coset representatives of $\Gamma_0(N) \backslash \text{SL}_2(\mathbb{Z}) \cong \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$ and $\Gamma \backslash \Gamma(2) \cong \Gamma_0(N) \backslash \text{SL}_2(\mathbb{Z})$.

**Proof** The orbits $\Gamma_0(N) A_1$, $\Gamma_0(N) A_2$ are not equal, for distinct $A_1$, $A_2 \in \tilde{\Omega}$. The lemma follows from $|\tilde{\Omega}| = |\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})| = \sum_{m \mid N} \phi(m)$. The set $\tilde{\Omega} \subset \Gamma(2)$ and $\Gamma_0(N) \tilde{\alpha}_i = \Gamma_0(N) \tilde{\alpha}_i$, $\Gamma_0(N) \tilde{\beta}_i = \Gamma_0(N) \tilde{\beta}_i$. Hence there is a canonical bijection $\Gamma_0(N) \backslash \text{SL}_2(\mathbb{Z}) \cong \Gamma \backslash \Gamma(2)$. \hfill $\square$

**Lemma 18** We can explicitly write the set $P_-$ is of the form $\Gamma \frac{x}{y}$ with $x$ and $y$ both odd. We can also write $P_- = \{ \Gamma \frac{a}{m}, m \mid N \}$.

**Proof** We can write any element $a$ in $P_- = \pi_0^{-1}(\Gamma(2)1)$ as $\pi \theta$ for some $\theta \in \tilde{\Omega}$. (Lemma 17). Let $\delta \mid N$, then $a = \Gamma \frac{a}{m}$ with $\text{gcd}(u, v\delta) = 1$ and $\text{gcd}(uv\delta, \frac{N}{\delta}) = 1$. Choose an odd integer $m$ and an even integer $l$ such that $lu - m\delta = 1$. We have $(\frac{1}{l} \frac{c}{l}) (\frac{1}{c} \frac{-c}{l}) 1 = \frac{1}{a}$ and $(\frac{-c}{l} \frac{1}{c} \frac{1}{l} - \frac{-c}{l} \frac{1}{c} \frac{1}{l} = \frac{u}{m} \delta$. Hence $A(\frac{a}{m}) = \frac{1}{a}$, where $A = (\frac{1}{l} \frac{0}{l}) (\frac{1}{c} \frac{-c}{l}) (\frac{1}{l} \frac{1}{l} - \frac{-c}{l} \frac{1}{l} - \frac{-c}{l} \frac{1}{l} = \frac{u}{m} \delta$. This matrix $A \in \Gamma$ if and only if $u \delta \equiv l \pmod{\frac{N}{\delta}}$. Since $\text{gcd}(v\delta, \frac{N}{\delta}) = 1$, there is always such an $c$. Hence, the set $P_-$ consists of $2^r$ elements $\{ \Gamma \frac{a}{m}, m \mid N \}$. \hfill $\square$

### 4 Even Eisenstein elements

We construct differential forms of first kind on the ambient Riemann surface $X_\pi - P_+$ by using the following lemma.

**Lemma 19** Let $f : X_0(N) \to \mathbb{C}$ be a rational function. The divisors of $\kappa(f)$ are supported on $P_+$.

**Proof** Let $f$ be a meromorphic function on $X_0(N)$. Then $f = \frac{g}{h}$ with $g$ and $h$ holomorphic function on $X_0(N)$. Every element of $P_-$ is of the form $\Gamma \frac{1}{m}$ with $m \mid N$ [Lemma 18]. Every holomorphic map on Riemann surface locally looks like $z \to z^i$ [p. 44, [14]].

Consider the morphism $\pi'$ and the point $\Gamma \frac{1}{m}$ with $m \mid N$. The local coordinates at the points $\Gamma_0(N)\theta$, $\Gamma_0(N)\infty$ and $\Gamma_0(N)\frac{1}{m}$, $1 < m < N$, $m \mid N$ are given by $q_0(z) = e^{2\pi i \frac{1}{m}}, q_\infty(z) = e^{2\pi i \frac{1}{m}}$ and $q_\frac{1}{m}(z) = e^{2\pi i \frac{1}{m} (\frac{a+c}{m}+1)}$ respectively. For $X_\pi$, the local coordinates around the points of $P_-$ are given by $q_1(z) = e^{2\pi i \frac{1}{m} (\frac{a+c}{m}+1)}, q_\frac{1}{m}(z) = e^{2\pi i \frac{1}{m} (\frac{a+c}{m}+1)}, 1 < m < N, m \mid N$. Around the point $\Gamma \frac{1}{N}$, we have $q_0 \circ \pi = q_1^2, q_0 \circ \pi' = q_1^4, q_\frac{1}{N} \circ \pi = q_\frac{1}{N}^2$ and $q_\frac{1}{m} \circ \pi' = q_\frac{1}{N}^2$. Let $y = \frac{1}{m}$ with $1 < m < N, m \mid N$.

Consider the map $\pi$ and $t = (\begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix})$ is a matrix such that $t(y) = i \infty$ and $e = \frac{N}{m}$. The local coordinate around the point $\Gamma \frac{1}{m}$ is $z \to e^{2\pi i \frac{1}{m} \frac{t(z)}{N}}$ and the map $\pi$ takes it to $e^{2\pi i \frac{t(z)}{N}}$. In this coordinate chart, the map $\pi$ is given by $z \to z^2$ and $\pi'$ is given by $z \to z^4$. Thus the function $\frac{(f \circ \pi)^2}{f \circ \pi}^2$ has no zero or pole on $P_-$. \hfill $\square$
For $E_m \in \mathbb{E}_N$, define a function $F_m : \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) \to \mathbb{Z}$ by

$$F_m(g) = \varphi_{E_m}(\xi_0(g)) = \int_{g(1)}^{g(\infty)} [2E_m(z) - E_m(\frac{z + 1}{2})]dz.$$ 

For any $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma(2)$, $h \gamma h^{-1} = \left( \begin{array}{cc} \frac{a+c}{2c} & \frac{b+ad-c}{2c} \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z})$, where $h = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$ and for any matrix $F_m(g) = \frac{2\pi E_m(\gamma) - \pi E_m(h \gamma h^{-1})}{12}$ is given by

$$P_m(\gamma) = \text{sgn}(t(\gamma)) [2(S(s(\gamma), |t(\gamma)|N) - S(s(\gamma), |t(\gamma)|)) - S(s(\gamma), |t(\gamma)|) - \frac{t(\gamma)}{2} |N| + S(s(\gamma), |t(\gamma)|) - \frac{t(\gamma)}{2}],$$

where $t(\gamma) = b + d - a - c$ and $s(\gamma) = a + c$. In particular, $P_m(\gamma) \in \mathbb{Z}$ [[1], Remark 27, Lemma 28].

**Proposition 20**

$$F_m(g) = \begin{cases} 12(S(r, m) - 2S(r, 2m)) & \text{if } g = (r - 1, r + 1), \\ 6(P_m(\gamma_{1,k}^{x,k}) - P_m(\gamma_{2,k}^{x,k})) & \text{if } g = (1 + kx, 1) \text{ or } g = (-1 - kx, 1), \\ -6(P_m(\gamma_{1,k}^{x,k}) - P_m(\gamma_{2,k}^{x,k})) & \text{if } g = (1, -1 + kx) \text{ or } g = (1, 1 + kx), \\ 0 & \text{if } g = (\pm 1, 1). \end{cases}$$

**Proof** If $g = (r - 1, r + 1)$ and $E_m \in \mathbb{E}_N$, we get [[3], p. 18]

$$F_m(g) = \varphi_{E_m}(\xi_0(g)) = 12(S(r, m) - 2S(r, 2m)).$$

The differential form $k^\ast(\omega_{E_m})$ is of first kind on the Riemann surface $X_\Gamma - P_+$. Since all the Fourier coefficients of the Eisenstein series are real valued, so an argument similar to [[3], p. 19] shows that $F_m(s_k x + 1, 1) = F_m(-\frac{1}{s_k x} + 1, 1)$, the path $\{ \frac{1}{s_k x} + \frac{1}{s_k x + 2} \} = \{ \frac{1}{s_k x} + \frac{1}{s_k x + 1} \} + \{ \frac{1}{s_k x + 1} + \frac{1}{s_k x + 2} \}$. The rational number $\frac{1}{s_k x}$ correspond to a point of $P_-$ in the Riemann surface $X_\Gamma$. The differential form $k^\ast(\omega_{E_m})$ has no zeros and poles on $P_-$. We conclude that

$$\int_{-\frac{1}{s_k x + 2}}^{-\frac{1}{s_k x}} k^\ast(\omega_{E_m}) = \int_{-\frac{1}{s_k x}}^{-\frac{1}{s_k x + 2}} k^\ast(\omega_{E_m}) + \int_{\frac{1}{s_k x}}^{\frac{1}{s_k x + 2}} k^\ast(\omega_{E_m}) = 2F_m(s_k x + 1, 1) + \int_{\frac{1}{s_k x}}^{\frac{1}{s_k x + 2}} k^\ast(\omega_{E_m}).$$

Let $\gamma_{1,k}^{x,k}$ and $\gamma_{2,k}^{x,k}$ be two matrices in $\Gamma$ such that $\gamma_{1,k}^{x,k}(\frac{1}{s_k x + 1}) = -\frac{1}{s_k x + 1}$ and $\gamma_{2,k}^{x,k}(\frac{1}{s_k x}) = -\frac{1}{s_k x}$. Hence $2F_m(s_k x + 1, 1) = \int_{\frac{1}{s_k x}}^{\frac{1}{s_k x + 1}} k^\ast(\omega_{E_m}) - \int_{\frac{1}{s_k x}}^{\frac{1}{s_k x + 1}} k^\ast(\omega_{E_m}).$

We now prove that the $\int_{\frac{1}{s_k x}}^{\gamma_{1,k}^{x,k}(\frac{1}{s_k x})} k^\ast(\omega_{E_m})$ is independent of the choice of the matrices $\gamma_{2,k}^{x,k} \in \Gamma$ that take $\frac{1}{s_k x}$ to $-\frac{1}{s_k x}$. For, $\gamma_{2,k}^{x,k}$ and $\gamma_{2,k}^{x,k}$ are two matrices such that $\gamma_{2,k}^{x,k}(\frac{1}{s_k x}) = \gamma_{2,k}^{x,k}(\frac{1}{s_k x}) = -\frac{1}{s_k x}$. Since $\gamma_{2,k}^{x,k} \in \Gamma$, the integral $\varphi_{E_m}(\gamma_{2,k}^{x,k}) = \int_{\frac{1}{s_k x}}^{\gamma_{2,k}^{x,k}(\frac{1}{s_k x})} k^\ast(\omega_{E_m})$ is independent of the choice of any point in $\mathbb{H} \cup \{-1\}$, hence by replacing $\frac{1}{s_k x}$ with $(\gamma_{2,k}^{x,k})^{-1}(\gamma_{2,k}^{x,k}) \frac{1}{s_k x}$, we get the above integral is same as $\int_{\frac{1}{s_k x}}^{\gamma_{2,k}^{x,k}(\frac{1}{s_k x})} k^\ast(\omega_{E_m})$ and the integral is independent of the choice of the matrices. Similarly, we can prove that $\int_{\frac{1}{s_k x + 2}}^{\gamma_{2,k}^{x,k}(\frac{1}{s_k x + 2})} k^\ast(\omega_{E_m})$ is also independent of the choice of the matrices that take $\frac{1}{s_k x + 2}$ to $-\frac{1}{s_k x + 2}$. The above calculation shows that

$$2\pi E_m(\gamma_{1,k}^{x,k}) - \pi E_m(h \gamma_{1,k}^{x,k} h^{-1}) = 2F_m(s_k x + 1, 1) + 2\pi E_m(\gamma_{2,k}^{x,k}) - \pi E_m(h \gamma_{2,k}^{x,k} h^{-1}).$$

Hence, we get

$$F_m(s_k x + 1, 1) = \frac{2\pi E_m(\gamma_{1,k}^{x,k}) - \pi E_m(h \gamma_{1,k}^{x,k} h^{-1}) - 2\pi E(\gamma_{2,k}^{x,k}) + \pi E(h \gamma_{2,k}^{x,k} h^{-1})}{2} = 6(P_m(\gamma_{1,k}^{x,k}) - P_m(\gamma_{2,k}^{x,k})).$$

We also have $F_m((1 + s_k x, 1)) = -F_m((1, -1 - s_k x))$, the proposition follows. □
Theorem 21 For $E_m \in E_2(\Gamma_0(N))$, the even Eisenstein elements $E_{E_m}^0$ of $H_1(X_\Gamma - P_-, P_+, \mathbb{Z})$ is given by $E_{E_m}^0 = \sum_{g \in \Pi^1(\mathbb{Z}/N\mathbb{Z})} H_E_{E_m}(g) e^0(g)$. 

Proof Let the even Eisenstein element be $E_{E_m}^0 = \sum_{g \in \Pi^1(\mathbb{Z}/N\mathbb{Z})} H_E_{E_m}(g) e_0^0(g)$ for some $H_{E_m}(g)$. Since by theorem 11, $H_{E_m}(g) = \sum_{g \in \Pi^1(\mathbb{Z}/N\mathbb{Z})} H_{E_m}(g) e_0^0(g) \circ \xi_0^0(g) = E_{E_m}^0 \circ \xi_0^0(g) = \varphi_{E_m}(\xi_0^0(g)) = F_m(g)$. 

5 Boundary of the Eisenstein series for $\Gamma_0(N)$

Let $\sigma_{1}(n)$ denote the sum of the positive divisors of $n$. Let $E_{E_m}^0(z) = 1 - 24(\sum_{n} \sigma_{1}(n)) e^{2\pi i n z} \Delta$ and $\Delta$ be the Ramanujan’s cusp form of weight 12. For all $N \in \mathbb{N}$, the function $z \rightarrow \frac{\Delta(Nz^2)}{\Delta(z)}$ is a function on $\mathbb{H}$ invariant under $\Gamma_0(N)$. The logarithmic differential of this function is $2\pi i E_{N}(z) dz$ and $E_{N}$ is a modular form of weight two for $\Gamma_0(N)$ with constant term $N - 1$. The differential form $E_{N}(z) dz$ is a differential form of third kind on $X_0(N)$. The periods $[\mathbb{Z}, \mathbb{Z}]$ of these differential forms are in $\mathbb{Z}$. By [[13], Thm. 4.6.2], the set $\mathbb{E}_{N} = \{ E_{m}, m > 1, m \mid N \}$ is a basis of $E_{2}(\Gamma_0(N))$. Let $\text{Div}^{0}(X_0(N), \partial(X_0(N)), \mathbb{Z})$ be the group of degree zero divisors supported on cusps. For all cusps $y$, let $e_{\Gamma_0(N)}(y)$ denote the ramification index of $y$ over $SL_{2}(\mathbb{Z}) \backslash \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$ and $r_{\Gamma_0(N)}(y) = e_{\Gamma_0(N)}(y)a_0(E[y])$. By [[15], p. 23], there is a canonical isomorphism $\delta : E_{2}(\Gamma_0(N)) \rightarrow \text{Div}^{0}(X_0(N), \partial(X_0(N)), \mathbb{Z})$ given by

$$\delta(E) = \sum_{y \in \Gamma_0(N) \backslash \mathbb{P}^1(\mathbb{Q})} r_{\Gamma_0(N)}(y)[y].$$

By [16], we see that $e_{\Gamma_0(N)}(y) = \begin{cases} \frac{N}{m} & \text{if } x = \frac{1}{m} \\ 1 & \text{if } y = \infty \\ N & \text{if } y = 0. \end{cases}$

Since $\sum_{s \in \partial(X_0(N))} e_{\Gamma_0(N)}(s)a_0(E[x]) = 0$, we get $\delta(E) = a_0(E)(\infty - 0) + \sum_{1 < m < N, m \mid N} \frac{N}{m} a_0(E[\frac{1}{m}]).$

6 Boundaries of the Eisenstein elements

The level $N$ is square-free. Hence for $m \mid N$, $N$ with $1 < m < N$, there exists $a(m), b(m)$ are two unique integers such that $a(m) \frac{N}{m} + b(m)m \equiv 1 \pmod{N}$ with $1 \leq a(m) \leq m - 1$ and $1 \leq b(m) \leq \frac{N}{m} - 1$.

Lemma 22 For all $k$ with $1 \leq k \leq \frac{N}{m} - 1$, we can choose an integer $s(k) \in (\mathbb{Z}/\frac{N}{m}\mathbb{Z})$ such that $(km, -1) = (m, s(k)m - 1)$ in $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$. The map $k \rightarrow s(k)$ is a bijection $(\mathbb{Z}/\frac{N}{m}\mathbb{Z})^{*} \rightarrow \phi(\prod_{p\mid N}(\mathbb{Z}/p\mathbb{Z}) - \{b(m)p\})$, where $\phi : \prod_{p\mid N}(\mathbb{Z}/p\mathbb{Z}) \rightarrow (\mathbb{Z}/\frac{N}{m}\mathbb{Z})$ is the standard isomorphism and $b(p)$ are chosen such that $a_{m}(p)p + b_{m}(p)m = 1 \forall p \mid \frac{N}{m}$.

Proof For all $k$ with $1 \leq k \leq (\frac{N}{m})^{*}$, let $k'$ be the inverse of $k$ in $(\mathbb{Z}/\frac{N}{m}\mathbb{Z})^{*}$.

By Chinese remainder theorem, we choose an unique $x$ such that $x \equiv -1 \pmod{m}$ and $x \equiv -k' \pmod{\frac{N}{m}}$. It is possible to find such an $x$. Observe $x$ is coprime to both $m$ and $\frac{N}{m}$. We write $x = s(k)m - 1$ for an unique $s(k)$ with $0 \leq s(k) \leq \frac{N}{m} - 1$. Since $\Gamma_0(N), SL_2(\mathbb{Z}) \cong \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$, we deduce that $(km, -1) = (xkm, -x) = (m, x) = (m, s(k)m - 1)$ in $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$.

Consider the map $\psi : (\mathbb{Z}/\frac{N}{m}\mathbb{Z})^{*} \rightarrow (\mathbb{Z}/\frac{N}{m}\mathbb{Z})$ given by $k \rightarrow s(k)$. This map is one-one since if $s(y) = s(h)$ then $y \equiv h \pmod{\frac{N}{m}}$. We have $\psi(\mathbb{Z}/\frac{N}{m}\mathbb{Z})^{*} \leq \phi(\mathbb{P}^1(\mathbb{Z}/p\mathbb{Z}) - \{b(p)m\})$, where $b(p)$ are chosen such that $a_{m}(p)p + b_{m}(p)m = 1p \mid \frac{N}{m}$. For suppose $x \in (\mathbb{Z}/\frac{N}{m}\mathbb{Z}) - \phi(\mathbb{P}^1(\mathbb{Z}/p\mathbb{Z}) - \{b(p)m\})$, then there exists a prime $p$ which divides $\frac{N}{m}$ such that $x \equiv \phi(b(p)m)$. Suppose $x \in \text{Image of } \psi$. We have $x = s(k)$ for some $k$. Hence $xm - 1$ is a unit mod $\frac{N}{m}$. Thus $xm - 1$ is a unit mod $p$. Now look at $xm - 1 (mod p) = b(m)p - 1 = 0 (mod p)$. Which is a contradiction. Hence the map $\psi$ is onto. Hence, the map $(\mathbb{Z}/\frac{N}{m}\mathbb{Z})^{*} \rightarrow \phi(\mathbb{P}^1(\mathbb{Z}/p\mathbb{Z}) - \{b(p)m\})$, $k \rightarrow s(k)$ is a bijection. \qed
Proposition 23 The boundary of \( X = \sum_{g \in \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})} F(g)[g]^* \in H_1(X_0(N) - R \cup I, \partial(X_0(N)), \mathbb{Z}) \) is
\[
\delta(X) = \sum_{m \mid N, 1 < m < N} A_m(X)([1/m] - [0]) + C(X)([\infty] - [0]),
\]
with
\[
A_m(X) = \sum_{l=0}^{N/m-1} [F(\beta_l^m) - F(\beta_l^m S)] \text{ and } C(X) = [F(0, 1) - F(1, 0)].
\]

Proof The proof follows along the same line as the proof of [Proposition 32, [1]]. \( \square \)

Proposition 24 The boundary of any element \( X = \sum_{g \in \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})} F(g)[g]^* \in H_1(X_\Gamma - P_-, P_+, \mathbb{Z}) \) is
\[
\delta^0(X) = \sum_{m \mid N, 1 < m < N} \tilde{A}_m(X)([1/m] - [0]) + \tilde{C}(X)([\infty] - [0]),
\]
where \( \tilde{A}_m(X) = \sum_{l=0}^{N/m-1} [F(\tilde{\beta}_l^m) - \sum_{i=0}^{N/m-1} F(\tilde{\alpha}_{im})] - F(\tilde{\beta}_l^m), \) and \( \tilde{C}(X) = [F(0, 1) - F(\tilde{\alpha}_N)]. \)

Proof This is a straightforward calculation using the coset representatives of \( \Gamma \setminus \Gamma(2) \) [cf. Lemma 17]. \( \square \)

Proposition 25 For \( E \in \mathbb{E}_N \), the boundary of the almost Eisenstein elements \( E' \in H_1(X_0(N) - R \cup I, \partial(X_0(N)), \mathbb{Z}) \) corresponding to the Eisenstein series \( E \) is \( -\delta(E) \) [§ 5].

Proof For \( E \in \mathbb{E}_N \), let \( E'_E = \sum_{g \in \mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})} G_E(g)[g]^* \) be the almost Eisenstein element. According to Proposition 23, we need to calculate \( A_m(E'_E) \). For all \( 0 \leq l < (\frac{N}{m} - 1), \beta_l^m T = \beta_l^m + 1 \) and \( \beta_0^N T = \gamma_0 \) with \( \gamma = \left( \frac{1 + N}{m - N}, \frac{N}{m - N}, 1 - N \right) \). We have an inclusion \( H_1(Y_0(N), \mathbb{Z}) \to H_1(Y_0(N), R \cup I, \mathbb{Z}) \). Since \( \{\rho^*, \gamma \rho^*\} = \{\beta_0 \rho^*, \gamma \beta_0 \rho^*\} = -\sum_{k=0}^{N/m-1} \{\beta_k^m \rho, \beta_k^m \rho^*\} \), we deduce that
\[
\pi_E(\gamma) = \int_{z_0}^{N \gamma z_0} E(z) dz = E'_E \circ \{z_0, \gamma \gamma_0\} = -E'_E \circ \left( \sum_{k=0}^{N/m-1} \{\beta_k^m \rho, \beta_k^m \rho^*\} \right) = -\sum_{k=0}^{N/m-1} E'_E \circ \{\beta_k^m \rho, \beta_k^m \rho^*\}.
\]
Applying Cor. 8, we have \( \sum_{k=0}^{N/m-1} E'_E \circ \{\beta_k^m \rho, \beta_k^m \rho^*\} = \sum_{k=0}^{N/m-1} [G_E(\beta_k^m) - G_E(\beta_k^m S)] = -A_m(E'_E) \). Hence, we prove that \( A_m(E'_E) = -\pi_E(\gamma) \). We now calculate \( \pi_E(\gamma) \) by using [16]. Recall, \( \frac{1}{m} \) is a cusp with \( \epsilon_{\Gamma(2)}(\frac{1}{m}) = \frac{N}{m} \). Consider the matrix \( x = \left( \begin{array}{cc} 1 & -N \\ -m & 1 + N \end{array} \right) \). We have \( x \left( \frac{1}{m} \right) x^{-1} = \gamma. \) Notice that \( x(i \infty) = \Gamma_0(N) \frac{1}{m} \). By [16], p. 524), we deduce that \( \pi_E(\gamma) = \epsilon_{\Gamma_0(N)}(\frac{N}{m}) a_0(E(\frac{1}{m})). \) By Proposition 23, the boundary of the almost Eisenstein element corresponding to an Eisenstein series \( E \) is
\[
\delta(E'_E) = \sum_{m \mid N, 1 < N} A(E'_E)([1/m] + + C(E'_E)([\infty] - (A(E'_E) + C(E'_E))[0])
\]
with \( A_m(E'_E) = \frac{N}{m} a_0(E(\frac{1}{m})) \) and \( C(E'_E) = -[F(I) - F(S)]. \) By Cor. 8 again, we deduce that \( F(I) - F(S) = \int_{\rho}^{\rho'} E(z) dz = -a_0(E). \) Hence \( 5 \delta(E) = \delta(E'_E) \). \( \square \)

Let \( \beta \) and \( h \) be the matrices \( \left( \begin{array}{cc} 1 & \frac{1}{2} \\ 0 & 1 \end{array} \right) \) and \( \left( \begin{array}{cc} 1 & \frac{1}{2} \\ 0 & 1 \end{array} \right) \) respectively. The modular curve \( X_0(N) \) has no obvious morphism to \( X(2) \). Hence, we consider the modular curve \( X_\Gamma \). There are two natural maps \( \pi, \pi' : \Gamma \setminus \mathbb{H} \to \Gamma_0(N) \setminus \mathbb{H} \) be the maps \( \pi(\Gamma z) = \Gamma_0(N)z \) and \( \pi'(\Gamma z) = \Gamma_0(N)z \). For the modular curve \( X_\Gamma \), we have a similar short exact sequence
\[
0 \to H_1(X_\Gamma - P_-, P_+, \mathbb{Z}) \to H_1(X_\Gamma - R \cup I, \partial(X_0(N)), \mathbb{Z}) \to \mathbb{Z}^{P_+} \to \mathbb{Z} \to 0.
\]
The boundary map \( \delta^0 \) takes a geodesic, joining the top \( \rho \) and \( s \) of \( P_+ \) to the formal symbol \( [r] - [s] \). Let \( \pi_* : H_1(X_\Gamma - P_-, P_+, \mathbb{Z}) \to H_1(X_0(N) - R \cup I, \partial(X_0(N)), \mathbb{Z}) \) be the isomorphism defined by \( \pi_*(\xi_0(g)) = [g]^* \) [11, Cor. 1]. It is easy to see that \( \delta(\pi_*(X)) = \delta^0(X) \) for all \( X \in H_1(X_\Gamma - P_-, P_+, \mathbb{Z}) \).
Proposition 26  For all $E \in \mathbb{E}_N$, let $\mathcal{E}^0_E$ be the even Eisenstein element in $H_1(X_\Gamma - P_-, P_+, \mathbb{Z})$ [§ 4]. The boundary of the modular symbol $\pi_a(\mathcal{E}^0_E)$ is $-6\delta(E)$.

Proof  By Theorem 21, suppose the even Eisenstein element $\mathcal{E}^0_E$ in the relative homology group $H_1(X_\Gamma - P_-, P_+, \mathbb{Z})$ is $\mathcal{E}^0_E = \sum_{g \in \Gamma(\mathbb{Z}/N\mathbb{Z})} F_E(g)\mathbb{E}_0(g)$. According to Proposition 24, we need to calculate $\tilde{A}_m(\mathcal{E}^0_E), \tilde{C}(\mathcal{E}^0_E)$. For $0 \leq l_m < (\frac{N}{m} - 2)$, we have $\tilde{\beta}_l^m \beta = \tilde{\beta}_{l+2}^m$. A small check shows that $\tilde{\beta}_{l-1}^m \beta = \tilde{\beta}_l^m$ and $\tilde{\beta}_{l-2}^m \beta = \gamma' \tilde{\beta}_0$ with

$$\gamma' = \left(1 + 2N(1 + \frac{N}{m}) - 2\frac{N}{m}, 1 - 2N(1 + \frac{N}{m})\right) \in \Gamma.$$ 

In $H_1(X_\Gamma - P_+, P_-, \mathbb{Z})$, we have

$$\{-1, \gamma'(-1)\} = \{\tilde{\beta}_0(-1), \gamma'\tilde{\beta}_0(-1)\} = - \sum_{l=0}^{\frac{N}{m}-1} \{\tilde{\beta}_l^m(1), \tilde{\beta}_l^m(-1)\} = - \sum_{l=0}^{\frac{N}{m}-1} \{\tilde{\beta}_{l-1}^m(1), \tilde{\beta}_{l-1}^m(-1)\}.$$ 

By the definition of the even Eisenstein elements, we conclude that

$$\int_{\mathcal{E}^0_E} k^*(\omega_E) = \mathcal{E}^0_E \circ \{z_0, \gamma'z_0\} = -\mathcal{E}^0_E \circ \{\sum_{l=0}^{\frac{N}{m}-1} (\tilde{\beta}_l^m(1), \tilde{\beta}_l^m(-1))\} = - \int_{\mathcal{E}^0_E} \{\tilde{\beta}_l^m(1), \tilde{\beta}_l^m(-1)\}.$$ 

It is easy to see that $hASBh^{-1} \in \text{SL}_2(\mathbb{Z})$ for all $A, B \in \Gamma(2)$. Since $[\tilde{\alpha}_y S] = [\tilde{\beta}_{s(y)}]$ in $\mathbb{F}^1(\mathbb{Z}/N\mathbb{Z})$, so $\kappa' = \tilde{\alpha}_y \tilde{S} \tilde{\beta}_{s(y)}^{-1} \in \Gamma_0(N)$ and $h\kappa'h^{-1} \in \Gamma_0(N)$. We deduce that the differential form

$$k^*(\omega_E) = f(z)dz = [2E(z) - \frac{1}{2}E(z + \frac{1}{2})]dz$$

is invariant under $\kappa'$. Hence

$$F_E(\tilde{\alpha}_y \tilde{S}) = \int_{\tilde{\alpha}_y \tilde{S}} f(z)dz = \int_{\tilde{\alpha}_y \tilde{S}} f(z)dz = - \int_{\tilde{\alpha}_y \tilde{S}} f(z)dz = \int_{\tilde{\alpha}_y \tilde{S}} f(z)dz.$$ 

By Theorem 10, we have

$$\sum_{k=0}^{\frac{N}{m}-1} F_E(\tilde{\beta}_k^m) = \sum_{l=0}^{\frac{N}{m}-1} \mathcal{E}^0_E \circ \{\tilde{\beta}_l^m(1), \tilde{\beta}_l^m(-1)\} = - \int_{\mathcal{E}^0_E} k^*(\omega_E).$$

By the definition of the period $\pi_E$ of the Eisenstein series $E(z)$, we get

$$\int_{\mathcal{E}^0_E} k^*(\omega_E) = \int_{\mathcal{E}^0_E} [2E(z) - \frac{1}{2}E(z + \frac{1}{2})]dz = 2\pi_E(\gamma') - \pi_E(h\gamma'h^{-1}).$$

As in the proof of Proposition 35, p.no 281, [1] replacing $p$ by $m$ and $q$ by $\frac{N}{m}$, we have $\pi_E(h\gamma'h^{-1}) = \frac{N}{m}a_0(E[\frac{1}{m}])$ and $\pi_E(\gamma') = 2\frac{N}{m}a_0(E[\frac{1}{m}])$ and $\int_{\mathcal{E}^0_E} k^*(\omega_E) = 3a_0(E[\frac{1}{m}]).$

$$F_E(I) = -F_E(\alpha_N) = \int_{\frac{1}{2}}^{1} [2E(z) - \frac{1}{2}E(z + \frac{1}{2})]dz = - \int_{-1}^{\beta(-1)} [2E(z) - \frac{1}{2}E(z + \frac{1}{2})]dz = - 3a_0(E),$$

we conclude that $\tilde{C}(\mathcal{E}^0_E) = [F_E(I) - F_E(\alpha_N)] = -6a_0(E)$ and hence $\delta^0(\mathcal{E}^0_E) = \delta(\mathcal{E}^0_E) = -6\delta(E)$. □

The inclusion map $i : (X_0(N) - R \cup I, \partial(X_0(N))) \to (X_0(N), \partial(X_0(N)))$ induces an onto map $i_0 : H_1(X_0(N) - R \cup I, \partial(X_0(N), Z) \to H_1(X_0(N), \partial(X_0(N)), Z)$ with $i_0([g]^\ast) = \xi(g)$. Note that $\delta([g]^\ast) = [g.0] - [g.\infty] = \delta'(\xi(g)) = \delta'(i_0([g]^\ast))$. From [§ 2.4], we have $\delta(c) = \delta'(i_0(c))$ for all homology class $c \in H_1(X_0(N) - R \cup I, \partial(X_0(N), Z)$.}
7 Proof of Theorem 3

Proof By [[11], Cor. 3], we obtain \( i_\ast (E'_E) \circ c = E'_E \circ i^\ast c = \int f_i (E(z)dz) \). Hence, \( i_\ast (E'_E) \) is the Eisenstein element inside the space of modular symbols corresponding to \( E \). By Proposition 25 and 26, the boundary of \( \pi_\ast (E'_E) \) is same as the boundary of \( \delta_\ast (E'_E) \). There is a non-degenerate bilinear pairing \( S_2(\Gamma_0(N)) \times H_1(X_0(N), \mathbb{R}) \rightarrow \mathbb{C} \) given by \( (f, c) = \int f(z)dz \). Hence, the integrals of the holomorphic differentials over \( H_1(X_0(N), \mathbb{Z}) \) are not always zero. By [[3], Lemma 5], the integrals of every holomorphic differentials over \( i_\ast (E'_E) \) and \( i_\ast (\pi_\ast (E'_E)) \) are always zero.

We deduce that \( E_E = i_\ast (E'_E) = \frac{1}{6} i_\ast \pi_\ast (E'_E) = \frac{1}{6} \sum_{g \in \mathbb{P}(\mathbb{Z}/N\mathbb{Z})} F_E(g) \xi(g) \), for \( E \in \mathbb{E}_N \).

Let \( e_N \in H_1(X_0(N), \mathbb{Z}) \otimes \mathbb{R} \) be the winding element. Let \( 1 < m < N, m | N \). The constant Fourier coefficients of \( E_N \) at cusps 0 and \( \frac{1}{m} \), and \( \infty \) are \( \frac{N}{2m^2} \), 0, 0 and \( \frac{N}{2m} \) respectively [as in the proof of Lemma 38, [1] (replacing \( p \) by \( m \) and \( q \) by \( \frac{N}{m} \)]. Hence, we obtain

\[
(1 - N)e_N = \sum_{v \in (\mathbb{Z}/N\mathbb{Z})^*} F_N((1, v))(0, \frac{1}{N}).
\]

\[ \square \]

8 Concluding Remarks

Generalization of the results in this paper to any arbitrary level \( N \) is an interesting question. The methods in this paper works only for squarefree level.

Remark 27 For the Eisenstein series \( E_m \in E_2(\Gamma_0(m)) \), \( \frac{1}{m} \) represents the cusp \( \infty \) and \( \frac{m}{N} \) represents the cusp 0. We deduce that \( a_0(E_m[\beta_0]) = \frac{m-1}{2m} \) and \( a_0(E_m[\gamma_0]) = \frac{1-m}{2m} \).

Remark 28 For the Eisenstein series \( E_N \), by [Lemma 4, [3]] the Eisenstein elements can be written explicitly if \( g = (r - 1, r + 1) \) as follows.

\[
F_N((r - 1, r + 1)) = \sum_{h=0}^{N-1} B_H(\frac{hr}{2N}).
\]

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