Stability and collapse of localized solutions of the controlled three-dimensional Gross-Pitaevskii equation

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On the basis of recent investigations, a newly developed analytical procedure is used for constructing a wide class of localized solutions of the controlled three-dimensional (3D) Gross-Pitaevskii equation (GPE) that governs the dynamics of Bose-Einstein condensates (BECs). The controlled 3D GPE is decomposed into a two-dimensional (2D) linear Schrödinger equation and a one-dimensional (1D) nonlinear Schrödinger equation, constrained by a variational condition for the controlling potential. Then, the above class of localized solutions are constructed as the product of the solutions of the transverse and longitudinal equations. On the basis of these exact 3D analytical solutions, a stability analysis is carried out, focusing our attention on the physical conditions for having collapsing or non-collapsing solutions.

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I. INTRODUCTION

About 85 years ago, the seminal work of Bose [1] opened up the study of the statistical properties of bosons in ultra-cold quantum systems. Bose’s idea was further developed by Einstein [2], leading to the theoretical prediction of the condensation of atoms in the lowest quantum state below a certain temperature. The idea of Bose-Einstein of atom condensation in the ground state has been experimentally verified in a dilute gas composed of atoms [3, 4, 5, 6]. The dynamics of nonlinearly interacting bosons in ultra-low temperature gases is governed by the Gross-Pitaevskii equation (GPE), which is an extension of the nonlinear Schrödinger equation (NLSE) by including the confining potential and inter-atomic interactions [7]. The GPE, without the external potential, admits localized solutions in the form of one-dimensional dark and bright solitons, as well as radially symmetric vortex structures. Nonlinear localized excitations involving BECs arise due to the balance between the spatial dispersion of matter waves and nonlinearities caused by repulsive or attractive inter-atomic interactions in BECs. Recent observations [8, 9, 10, 11, 12, 13, 14, 15] conclusively demonstrated the existence of bright [12, 13, 14, 15] and dark/grey [8, 9, 10, 11] matter wave solitons and quantum vortices [10].

Although the area of investigations of localized solutions of the GPE is quite fascinating, most of the theoretical results deals with approximate solutions in 3D or in reduced geometries [17]. They are well supported by suitable numerical evaluations [18] and adequately compared with a very broad spectrum of experimental observations (for a review, see Ref.s [19, 20].) Nevertheless, this testifies that finding exact localized solutions of the 3D GPE in a trapping external potential well, and preserving their stability for a long time, is still a challenging task. In particular, one encounters serious difficulties in attempting to find soliton solutions in one or more spatial dimensions, although several kinds of solitons have been found using certain approximations [21, 22]. This leads us to arrive at the conclusion that, in order to have exact soliton solutions in BECs, some sort of "control of the system" may be necessary.

The very large body of experience suggests that interacting bosons constitute a nonlinear and nonautonomous system [23], for which coherent stationary structures (i.e. solutions of the 3D GPE) exist only if suitable time-dependent external potentials are taken "ad hoc" [24]. Therefore, the correct analysis of the system should include...
a 'controlling potential' term in the GPE, to be determined self-consistently with the desired solutions (e.g. the localized solutions). This procedure may be, in principle, extended to an arbitrary 'controlled solution' with the appropriate choice of the controlling potential \[25\]. The controlling potential method (CPM) has been proposed in the literature, and used to find the multi-dimensional controlled localized solutions of the GPE \[25\]. Preliminary investigations \[25\] have suggested that control operations introduced by this method ensures the stability of coherent solutions against relatively small errors in experimental realizations of the prescribed controlling potential. This idea could be realized by techniques that involve lithographically designed circuit patterns, providing the electromagnetic guides and microtraps for ultracold systems of atoms in BEC experiments [27], and by the optically induced 'exotic' potentials \[28\].

Another important aspect of solitons in BECs is the phenomenon of collective collapse/explosion, that has been predicted \[22\] and observed experimentally \[29, 30\]. In particular, this phenomenon seems to be dependent on the parameters of the BECs and on the confining or repulsive potential \[22\].

The stabilization and control of BECs in asymmetric traps have been investigated via time-dependent solutions of the GPE \[31\]. Stable condensates, with the limited number of \(^7\)Li atoms with attractive interaction, have been observed in a magnetically trapped gas \[32\].

Recently, a mathematical investigation oriented towards the construction of 3D analytical solutions of the controlled GPE has been carried out \[33\] and applied to the construction of 3D exact localized solutions \[34\]. In Ref. \[33\], it has been proven that, under the assumption of the separability of the external trapping potential well \(V_{\text{trap}}\) in the spatial coordinates [viz. \(V_{\text{trap}}(x, y, z, t) = V_x(x, y, t) + V_z(z, t)\), where \(V_x\) and \(V_z\) are referred to as the 'transverse' and the 'longitudinal' potentials, respectively], and a suitable constrained variational condition for the controlling potential \(V_{\text{contr}}\) (i.e. the average over the 'transverse' \(x - y\) plane of \(V_{\text{contr}}\) is required to be a stationary functional of the BEC’s transverse profile), the factorized form of the solution of the 3D controlled GPE, in the form \(\psi(x, y, z, t) = \psi_\perp(x, y, t)\psi_z(z, t)\), can be constructed, so that \(\psi_\perp\) and \(\psi_z\) satisfy a 2D linear Schrödinger equation and a 1D nonlinear Schrödinger equation, respectively.

In this paper, we apply the results of the above investigation \[33\] to develop a new analytical procedure for constructing a broad class of exact localized solutions of the controlled 3D GPE, with a parabolic external potential well. In particular, we extend our recent investigation \[31\] to a wider family of exact localized solutions of the controlled GPE and perform an analysis that establishes the physical conditions and parameter regimes for having collapsing and non-collapsing localized solutions. In the next section, we formulate our problem and present the controlled GPE, and we briefly summarize the results found in Ref. \[33\]. In section II we apply these results to obtain localized solutions of the controlled GPE in the form of bright, dark and grey solitons for the longitudinal profile \(|\psi_z|^2\), and in the form of the Hermite-Gauss functions for the transverse profile \(|\psi_\perp|^2\). We use the decomposition procedure of the controlled GPE suggested in Ref. \[33\] and solve the 2D transverse linear Schrödinger equation to obtain \(\psi_\perp\) in terms of Hermite-Gauss modes. Then, the 1D longitudinal controlled NLSE is solved using a method based on the Madelung’s fluid representation \[35, 36\], which separates the NLSE into a pair of equations, composed of one continuity equation and one Korteweg-de Vries (KdV)-type equation. It is shown that the phase of the longitudinal wavefunction \(\psi_z\) has a parabolic dependence on the variable \(z\). As the transverse and longitudinal equations are coupled both through the coefficient of the nonlinear term (in the longitudinal equation) and through the controlling potential, the consistency condition between the transverse and longitudinal solutions set up a relationship between the transverse and longitudinal restoring forces of the external trapping potential well. From the latter, the explicit spatio-temporal dependence of the controlling potential is self-consistently determined in each particular case. In section IV a detailed analysis of the dynamics of the above exact 3D solutions is developed both analytically and numerically. In particular, we study the properties of the controlled BEC states in the 2D case, showing that they feature breathing (oscillations of the amplitude and position) due to the oscillations of the perpendicular solution, as well as the oscillations in the parallel direction, arising from the initial displacement of the structure from the bottom of the potential well in the parallel direction. A stability analysis of controlled GPE structures is carried out in terms of a set of six parameters related through four equations. Therefore, two of them can be assumed arbitrarily. We discuss some examples of parameters corresponding to collapsing and non-collapsing 3D solutions. Finally, the conclusions are summarized in section V.

II. DECOMPOSITION OF THE CONTROLLED GROSS-PITAEVSKII EQUATION

It is well known that the spatio-temporal evolution of an ultracold system of identical atoms forming a Bose Einstein condensate (BEC) in the presence of an external potential \(U_{\text{ext}}(\mathbf{r}, t)\), within the mean field approximation, is governed by the 3D GPE, viz.

\[
i \hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar}{2m_a} \nabla^2 \Psi + NQ|\Psi|^2 \Psi + U_{\text{ext}}(\mathbf{r}, t) \Psi,
\]
where \( \Psi(\mathbf{r}, t) \) is the wavefunction describing the BEC state, \( m_a \) is the atom mass, \( Q \) is a coupling coefficient related to the short range scattering \((s\text{-wave})\) length \( a \) representing the interactions between atomic particles, i.e. \( Q = 4\pi \hbar^2 a/m_a \), and \( N \) is the number of atoms. Note that the short range scattering length can be either positive or negative. Solitons have been observed in BECs of \(^7\text{Li}\) atoms with a small scattering length \( a \approx -0.2 \text{ nm} \), in correspondence of the following typical values of the parameters: \( N = 10^2 - 10^5 \) at a temperature of \( 1-10 \mu \text{K} \) and a magnetic field \( \sim 400-600 \mu \text{T} \). 

We assume that \( U_{\text{ext}} \) is the sum of a 3D trapping potential well, say \( V_{\text{trap}} \), to confine the particles of a BEC, and a controlling potential, say \( V_{\text{contr}} \), to be determined self-consistently. Furthermore, under this assumptions, we introduce the variable \( s = c t \) (\( c \) being the speed of light) and divide both sides of Eq. (1) by \( m_a c^2 \), in such a way that

\[
\frac{U_{\text{ext}}(\mathbf{r}, t)}{m_a c^2} = V_{\text{trap}}(\mathbf{r}, s) + V_{\text{contr}}(\mathbf{r}, s),
\]

and Eq. (1) can be cast into the form

\[
i\lambda_c \frac{\partial \psi}{\partial s} = -\frac{\lambda_c^2}{2} \nabla^2 \psi + [V_{\text{trap}}(\mathbf{r}, s) + V_{\text{contr}}(\mathbf{r}, s) + q|\psi|^2] \psi,
\]

where \( \psi(\mathbf{r}, s) = \Psi(\mathbf{r}, t = s/c) \), \( \lambda_c = \hbar/m_a c^2 \) is the Compton wavelength of the single atom of the BEC and \( q = N Q / m c^2 \).

In order to decompose Eq. (3), we briefly summarize the results of Ref. [33]. To this end, we first assume that

\[
V_{\text{trap}}(\mathbf{r}, s) = V_\perp(\mathbf{r}_\perp, s) + V_\parallel(z, s),
\]

where, in Cartesian coordinates, \( \mathbf{r} \equiv (x, y, z) \) and \( \mathbf{r}_\perp \equiv (x, y) \) denotes, by definition, the 'transverse' part of the particle’s vector position \( \mathbf{r} \) and \( z \) the 'longitudinal' coordinate. Additionally, let us denote, in Cartesian coordinates, \( \nabla_\perp \equiv \hat{x} \partial/\partial x + \hat{y} \partial/\partial y \).

Let us suppose that \( \psi_\perp(\mathbf{r}_\perp, s) \) and \( \psi_\parallel(z, s) \) are two complex functions satisfying the following 2D linear Schrödinger equation

\[
\left(i\lambda_c \frac{\partial}{\partial s} - \tilde{H}_\perp\right) \psi_\perp = 0,
\]

where

\[
\tilde{H}_\perp = -\frac{\lambda_c^2}{2} \nabla_\perp^2 + V_\perp(\mathbf{r}_\perp, s)
\]

and the following 1D NLSE

\[
\left(i\lambda_c \frac{\partial}{\partial s} - \tilde{H}_\parallel\right) \psi_\parallel = 0,
\]

where

\[
\tilde{H}_\parallel = -\frac{\lambda_c^2}{2} \frac{\partial^2}{\partial z^2} + V_\parallel(z, s) + q_{1D}(s) |\psi_\parallel(z, s)|^2 + V_0,
\]

respectively. In the latter, \( V_0 \) is an arbitrary real constant and the function \( q_{1D}(s) \) is defined as

\[
q_{1D}(s) = q \int d^2 \mathbf{r}_\perp |\psi_\perp|^4.
\]

Hereafter \( q_{1D} \) is referred to as the 'controlling parameter'.

Furthermore, let us assume that the controlling potential depends in principle on \( \mathbf{r}_\perp \) through \( |\psi_\perp|^2 \), viz. \( V_{\text{contr}} = V_{\text{contr}}(\rho_\perp(\mathbf{r}_\perp, s), z, s) \), where \( \rho_\perp(\mathbf{r}_\perp, s) \equiv |\psi_\perp(\mathbf{r}_\perp, s)|^2 \).

Provided that

\[
V_{\text{contr}}(\mathbf{r}_\perp, z, s) = [q_{1D}(s) - q |\psi_\perp(\mathbf{r}_\perp, s)|^2] |\psi_\parallel(z, s)|^2 + V_0,
\]

which makes stationary the functional (with respect to variation \( \delta \rho_\perp \) of \( \rho_\perp; z \) and \( s \) play here a role of parameters)

\[
\mathcal{V}[\rho_\perp; z, s] = \int \rho_\perp(\mathbf{r}_\perp, s) V_{\text{contr}}(\rho_\perp(\mathbf{r}_\perp, s), z, s) d^2 \mathbf{r}_\perp,
\]
under suitable constraints provided by the normalization condition for $\psi$, viz. $\int \rho (r, s) \, dr \, dz = 1$, and by Eq. (9), where $q_{1D}$ is thought as a given function of $s$, it can be shown that the complex function

$$
\psi(r, s) = \psi_{\perp}(r, s) \psi_{z}(z, s)
$$

(12)
is a 3D solution of the controlled Gross-Pitaevskii equation (3).

In this way, among all possible choices of $V_{\text{contr}}$, we adopt the one which does not change the mean energy of the system (the average of the Hamiltonian operator in Eq. (3) is the same with or without $V_{\text{contr}}$) and therefore minimizes the effects introduced by our control operation.

In the next section, we will apply the results obtained here to the case of parabolic potentials, $V_{\perp}$ and $V_{z}$, to give exact 3D controlled localized solutions of Eq. (3).

III. EXACT LOCALIZED SOLUTIONS OF THE CONTROLLED 3D GPE WITH A 3D PARABOLIC POTENTIAL WELL

Let us assume that $V_{\perp}(r, s)$ and $V_{z}(z, s)$ are the usual parabolic potential wells to confine the particle of a BEC, viz.

$$
V_{\perp}(r, s) = \frac{1}{2} \left[ \omega_{x}^{2}(s) x^2 + \omega_{y}^{2}(s) y^2 \right],
$$

(14) and

$$
V_{z}(z, s) = \frac{1}{2} \omega_{z}^{2}(s) z^2,
$$

(15)

where, in general, the frequency $\omega_{j}$, $j = x, y, z$, are supposed functions of time. The standard confining potential wells (along each direction) corresponds to the assumption that they are real quantity (positivity of their squares). However, our analysis can be extended to the case in which they are assumed imaginary (negativity of their squares).

It follows that Eq. (3) becomes

$$
i \lambda c \frac{\partial \psi}{\partial s} = - \frac{\lambda c^2}{2} \nabla^2 \psi + \frac{1}{2} \left[ \omega_{x}^{2}(s) x^2 + \omega_{y}^{2}(s) y^2 + \omega_{z}^{2}(s) z^2 \right] \psi + q |\psi|^2 \psi + V_{\text{contr}}(x, y, z, s) \psi.
$$

(16)

According to the results and the assumptions of the previous sections, if we seek a solution in the factorized form

$$
\psi(x, y, z, s) = \psi_{\perp}(x, y, s) \psi_{z}(z, s),
$$

(17)

Eq. (16) can be decomposed into the following set of equations:

$$
i \lambda c \frac{\partial \psi_{\perp}}{\partial s} + \frac{\lambda c^2}{2} \nabla^2_{\perp} \psi_{\perp} - \frac{1}{2} \left[ \omega_{x}^{2}(s) x^2 + \omega_{y}^{2}(s) y^2 \right] \psi_{\perp} = 0,
$$

(18) and

$$
i \lambda c \frac{\partial \psi_{z}(z, s)}{\partial s} + \frac{\lambda c^2}{2} \frac{\partial^2 \psi_{z}(z, s)}{\partial z^2} - \frac{1}{2} \omega_{z}^{2}(s) z^2 \psi_{z}(z, s) - q_{1D}(s) |\psi_{z}(z, s)|^2 \psi_{z}(z, s) = 0,
$$

(19)

$$
V_{\text{contr}}(x, y, z, s) = \left[q_{1D}(s) - q |\psi_{\perp}(r, s)|^2 \right] |\psi_{z}(z, s)|^2.
$$

(20)
A. Solution of the transverse equation with a 2D parabolic potential well

Equation (18) is readily solved, and its solution can be found in the standard literature, but we present it here for completeness. The general solution \( \psi_\perp(x, y, t) \) can be expressed as the superposition of Hermite-Gauss modes, viz..

\[
\psi_\perp (x, y, s) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \alpha_{n,l} \psi_{x,n} (x, s) \psi_{y,l} (y, s)
\]

where \( \alpha_{n,l} \) are arbitrary constants and

\[
\psi_{j,k} (j, s) = \left[ \pi 2^{2k+1} (k!)^2 \sigma_j^2 (s) \right]^{-\frac{1}{4}} \exp \left[ \frac{i \gamma_j (s) j^2}{2 \lambda_c} - \frac{j^2}{4 \sigma_j^2 (s)} + i \phi_{j,k} (s) \right] H_k \left[ \frac{j}{\sqrt{2} \sigma_j (s)} \right],
\]

where \( j = x, y \). The perpendicular spatial \( \sigma_j \) (i.e., the root of mean square) of the Hermite-Gauss functions satisfies the Ermakov-Pinney equation

\[
\frac{d^2 \sigma_j (s)}{ds^2} + \omega_j^2 (s) \sigma_j (s) - \frac{\lambda_c^2}{4 \sigma_j^2 (s)} = 0,
\]

and the phase functions \( \gamma_j (s) \) and \( \phi_{j,k} (s) \) are given by

\[
\gamma_j (s) = \frac{\sigma_j' (s)}{\sigma_j (s)},
\]

\[
\phi_{j,k} (s) = \phi_{j,k,0} - \frac{\lambda_c}{4} (2k + 1) \int_0^s \frac{ds'}{\sigma_j^2 (s')},
\]

where \( \phi_{j,k,0} \) is an arbitrary constant.

B. Exact solution of the longitudinal NLSE with a 1D parabolic trapping potential well

In order to solve Eq. (19), we first observe that, according to Eqs. (9), (21) and (22), the controlling parameter \( q_{1D} \) can be expressed in terms of the features of \( \Psi_\perp \), viz.

\[
q_{1D} (s) = q \frac{\mathcal{F} [\sigma_x (s), \sigma_y (s)]}{\sigma_x (s) \sigma_y (s)},
\]

where \( \mathcal{F} [\sigma_x (s), \sigma_y (s), s] \) is a (relatively complicated) positive definite functional of \( \sigma_x \) and \( \sigma_y \) given by

\[
\mathcal{F} [\sigma_x (s), \sigma_y (s)] = \int \frac{e^{-2u^2}}{\sqrt{2 \pi}} \int \frac{e^{-2v^2}}{\sqrt{2 \pi}} \left| \sum_{n,m,l,p} \mathcal{F}_{n,m,l,p} [\sigma_x (s), \sigma_y (s)] H_n (u) H_m (v) H_l (u) H_p (v) \right|^2,
\]

with \( u = x/\sqrt{2 \sigma_x}, \ v = y/\sqrt{2 \sigma_y} \) and

\[
\mathcal{F}_{n,m,l,p} [\sigma_x (s), \sigma_y (s)] = \frac{\alpha_{n,l} \alpha_{m,p}}{\sqrt{2 \pi} n! m! l! p!} e^{i \phi_{n,m,l,p} (0)} \frac{\sigma_x (s)}{\sigma_y (s)}
\]

\[
\times \left[ \int \frac{d\phi_{n,m,l,p}}{2 \pi} \left| \sum_{n,m,l,p} \mathcal{F}_{n,m,l,p} [\sigma_x (s), \sigma_y (s)] H_n (u) H_m (v) H_l (u) H_p (v) \right|^2 \right]^{\frac{1}{2}}
\]

From the above equation, it is clear that, in general, \( q_{1D} \) depends in a non-trivial manner on \( \sigma_x (s) \) and \( \sigma_y (s) \). However, in the simple case when the perpendicular solution contains only one Gaussian-Hermite mode, say the \( (n, l) \)-mode, Eqs. (21) and (22), \( \mathcal{F} \) becomes a real positive constant, viz.,

\[
\mathcal{F} [\sigma_x (s), \sigma_y (s)] = \int \frac{e^{-2u^2}}{\sqrt{2 \pi}} \int \frac{e^{-2v^2}}{\sqrt{2 \pi}} \equiv \delta_n \delta_l = \text{constant}
\]

and, therefore, \( q_{1D} \) becomes

\[
q_{1D} (s) = q \frac{\delta_n \delta_l}{\sigma_x (s) \sigma_y (s)} \equiv q_{1D}^{nl} (s).
\]
1. Reduction of the 1D NLSE to a KdV-like equation by means of the Madelung’s fluid approach

Equation (19) is a 1D GPE with a time dependent parabolic potential. Its approximate solutions are well known in the literature, but we attempt here to find an exact solution which is also compatible with the ones of the transverse part, in such a way to give a solution of the full controlled 3D GPE (16). We seek \( \psi(\rho, z, s) \) using the following standard Madelung’s fluid representation

\[
\psi(\rho, z, s) = \sqrt{\rho(z, s)} \exp \left[ \frac{i \Theta(\rho, z, s)}{\lambda_c} \right], \tag{31}
\]

which, after the substitution into Eq. (19) and the separation of real and imaginary parts, yields

\[
\frac{\partial \Theta}{\partial s} + \frac{1}{2} \left( \frac{\partial \Theta}{\partial z} \right)^2 + U(z, s) - \frac{\lambda_c^2}{2} \frac{1}{\rho^{1/2}} \frac{\partial^2 \rho^{1/2}}{\partial z^2} = 0, \tag{32}
\]

and

\[
\frac{\partial \rho}{\partial s} + \frac{\partial}{\partial z} \left( \rho \frac{\partial \Theta}{\partial z} \right) = 0, \tag{33}
\]

where

\[
U(z, s) = \frac{1}{2} \omega(z,s)^2 z^2 + q_{1D}(s) \rho(z, s)
\]

is the ‘longitudinal potential energy’ which is a functional of \( \rho \).

By differentiating Eq. (32) with respect to \( z \) and introducing the ‘current velocity’ \( V \equiv \partial \Theta/\partial z \), a non-trivial series of transformations given in Refs. [35, 36] allows us to obtain the following generalized KdV equation

\[
-\rho \frac{\partial V}{\partial s} + V \frac{\partial \rho}{\partial s} + 2 \left[ c_0(s) - \int z \frac{\partial V}{\partial s} dz \right] \frac{\partial \rho}{\partial z} - \left( \frac{\partial U}{\partial z} + 2U \frac{\partial \rho}{\partial z} \right) + \frac{\lambda_c^2}{4} \frac{\partial^3 \rho}{\partial z^3} = 0, \tag{35}
\]

where \( c_0(s) \) is an arbitrary function of \( s \). Making use of definition (34), Eq. (35) becomes

\[
-\rho \frac{\partial V}{\partial s} + V \frac{\partial \rho}{\partial s} + 2 \left[ c_0(s) - \int z \frac{\partial V}{\partial s} dz \right] \frac{\partial \rho}{\partial z} - \omega(z,s)^2 z \rho - \omega(z,s)^2 z_2 \frac{\partial \rho}{\partial z} - 3q_{1D}(s) \rho \frac{\partial \rho}{\partial z} + \frac{\lambda_c^2}{4} \frac{\partial^3 \rho}{\partial z^3} = 0. \tag{36}
\]

Therefore, the system of equations (32) and (33) is now replaced by the system of equations (36) and (33).

We look for a solution for \( \rho \), assuming that \( V \) is a linear function of \( z \), viz.

\[
V(z, s) = g(s) z + \kappa(s), \tag{37}
\]

and this corresponds to seek a quadratic form of the solution for the phase \( \theta(z, s) \), viz.

\[
\Theta(z, s) = \Theta_0(s) + \kappa(s) (z + \frac{1}{2} g(s) z^2), \tag{38}
\]

where the ‘initial phase’, \( \Theta_0(s) \), the ‘wavenumber’, \( \kappa(s) \), and the ‘dispersive’ term, \( g(s) \), are real functions of the time-like variable \( s \). It is easy to see that the arbitrary function of \( s \), \( c_0(s) \), appearing in Eq. (35), is proportional to \( \Theta_0' \) (hereafter the prime stands for the first-order derivative with respect to \( s \)), i.e. \( c_0(s) = -\Theta_0'(s) \).

After substituting Eq. (37) into Eq. (36), the system of equations (33) and (36) can be cast into the form

\[
\frac{\partial \rho}{\partial s} + (g z + \kappa) \frac{\partial \rho}{\partial z} + g \rho = 0, \tag{39}
\]

and

\[
-\rho \left( g' z + \kappa' \right) + (g z + \kappa) \frac{\partial \rho}{\partial s} + (-2\Theta_0' - g' z^2 - 2\kappa' z) \frac{\partial \rho}{\partial z} - \omega(z,s)^2 z \rho - \omega(z,s)^2 z_2 \frac{\partial \rho}{\partial z} - 3q_{1D} \rho \frac{\partial \rho}{\partial z} + \frac{\lambda_c^2}{4} \frac{\partial^3 \rho}{\partial z^3} = 0. \tag{40}
\]
Then, substituting Eq. (39) into Eq. (40) we obtain
\[-(g' + g^2 + \omega_z^2) (\rho + z \frac{\partial \rho}{\partial z}) z - (\kappa' + g \kappa) (\rho + 2 z \frac{\partial \rho}{\partial z}) - 2\theta_0' \frac{\partial \rho}{\partial z} - 3 q_{1D} \frac{\partial \rho}{\partial z} + \frac{\lambda^2}{4} \frac{\partial^3 \rho}{\partial z^3} = 0 , \] (41)
where \( \theta_0'(s) \equiv \Theta_0'(s) + \kappa^2(s)/2 \). To reduce Eq. (41) to the following KdV-like equation
\[-2\theta_0'(s) \frac{\partial \rho}{\partial z} - 3 q_{1D} \frac{\partial \rho}{\partial z} + \frac{\lambda^2}{4} \frac{\partial^3 \rho}{\partial z^3} = 0 , \] (42)
we have to impose that the coefficients of \( (\rho + z \frac{\partial \rho}{\partial z}) z \) and \( (\rho + 2 z \frac{\partial \rho}{\partial z}) \) are zero, namely we automatically find that \( g(s) \) satisfies the following Riccati’s equation, viz.
\[ g' + g^2 + \omega_z^2 = 0 , \] (43)
while \( \kappa(s) \) is related with it through
\[ \kappa' + g \kappa = 0 , \] (44)
which is readily integrated as
\[ \kappa(s) = \kappa_0 e^{-\int_0^s g(\tau) d\tau} , \] (45)
where \( \kappa_0 \) is an arbitrary constant.

We look for functions \( \rho(z, s) \) which satisfies simultaneously the KdV-like equation (42) and the continuity equation (39). By using Eqs. (31), (38) and (43)-(45), they allow us to construct also solutions of the longitudinal equation (19).

To this end, under the coordinate transformation
\[ \xi = \xi(z, s) = q_{1D}(s) z + R(s) \] (46)
\[ s' = s'(z, s) = s , \]
where \( R(s) \) is a real function, the system of Eqs. (39) and (42) becomes
\[ \left[ \left( q_{1D}' + g \right) (\xi - R) + R' + \kappa q_{1D} \right] \frac{\partial \rho}{\partial \xi} + \frac{\partial \rho}{\partial s'} + g \rho = 0 , \] (47)
and
\[-2\theta_0' \frac{\partial \rho}{\partial \xi} - 3 q_{1D} \frac{\partial \rho}{\partial \xi} + \frac{\lambda^2}{4} q_{1D} \frac{\partial^3 \rho}{\partial \xi^3} = 0 , \] (48)
where the prime denotes differentiation with respect to \( s \).

To find solutions in the factorized form
\[ \rho(\xi, s') = A(s') F(\xi) , \] (49)
satisfying simultaneously (47) and (48), the following conditions have to be imposed
\[ q_{1D}' + g q_{1D} = 0 \] (50)
\[ R' + \kappa q_{1D} = 0 . \]
Consequently, Eqs. (47) and (48) become the following ordinary differential equations, respectively,
\[ A' + g A = 0 , \] (51)
and
\[-2\theta_0' \frac{dF}{d\xi} - 3 q_{1D} A F \frac{dF}{d\xi} + \frac{\lambda^2}{4} q_{1D}^2 \frac{d^3 F}{d\xi^3} = 0 . \] (52)
We note that the first condition \ref{eq:50} and continuity equation \ref{eq:51} imply, respectively
\[ q_{1D}(s') = q_0 e^{-\int_0^{s'} g(\tau) d\tau}, \]  \[ A(s') = A_0 e^{-\int_0^{s'} g(\tau) d\tau}, \] where \( q_0 \) and \( A_0 \) are integration constants, i.e. \( q_0 = q_{1D}(s' = 0), A_0 = A(s' = 0) \). Additionally, by using solutions \ref{eq:53} and \ref{eq:45}, the second condition \ref{eq:50} can be easily solved for \( R(s') \), viz.
\[ R(s') = R_0 - \kappa_0 q_0 \int_0^{s'} ds'' e^{-2 \int_0^{s''} g(\tau) d\tau}, \] where \( R_0 = R(s' = 0) \).

Furthermore, we also observe that, given the set of Eqs. \ref{eq:43}, \ref{eq:45}, \ref{eq:53} - \ref{eq:55}, we can conveniently express the functions \( A(s), k(s) \) and \( \omega_z(s) \) in terms of the controlling parameter \( q_{1D}(s) \) (which is directly connected with the transverse part of the GPE solution), i.e.
\[ A(s) = \frac{A_0}{q_0} q_{1D}(s), \]  \[ \kappa(s) = \frac{\kappa_0}{q_0} q_{1D}(s), \]  \[ R(s) = -\frac{\kappa_0}{q_0} \int_0^s q_{1D}^2(\tau) d\tau + R_0, \]  \[ \omega_z^2(s) = -q_{1D}(s) \frac{d^2}{ds'^2} \left[ \frac{1}{q_{1D}(s)} \right]. \]

In particular, Eq. \ref{eq:59} has been obtained by substituting the first of equations \ref{eq:50} into Riccati’s equation \ref{eq:43}. It establishes a 'control condition' by the transverse part of the GPE solution on the longitudinal parabolic potential. In fact, it indicates which time dependence of \( \omega_z = \omega_z(s) \) has to be taken, provided that \( q_{1D}(s) \), given by Eq. \ref{eq:26}, is solution of the first of Eqs. \ref{eq:50}. Note that, according to Eq. \ref{eq:26}, it results that
\[ q_0 = q \frac{\mathcal{F}[\sigma_x(s), \sigma_y(s)]|_{s=0}}{\sigma_x(s = 0) \sigma_y(s = 0)}. \] (Note that \( \mathcal{F}[\sigma_x(s), \sigma_y(s)]|_{s=0} \) does not coincides with \( \mathcal{F}[\sigma_x(s = 0), \sigma_y(s = 0)] \).) To cast Eq. \ref{eq:52} as an equation with constant coefficients, we can choose the arbitrary function \( \Theta'_0(s') \) proportional to \( e^{-2 \int_0^{s'} g(\tau) d\tau} \) and therefore
\[ \Theta'_0(s') = \frac{\bar{\Theta}'_0}{q_0^2} q_{1D}(s') \] where \( \bar{\Theta}'_0 \) is an arbitrary constant, in such a way that, according to Eq. \ref{eq:45} and the definition of \( \theta'_0(s') \) given above, we have
\[ \theta'_0(s') = \frac{\bar{\Theta}'_0 + \kappa_0^2/2}{q_0^2} q_{1D}(s') = \frac{\bar{\Theta}'_0}{q_0^2} q_{1D}(s'). \]

By substituting Eqs. \ref{eq:56}, \ref{eq:57} and \ref{eq:62} in Eq. \ref{eq:52}, we finally obtain the following ordinary differential equations with constant coefficients (stationary KdV equation)
\[ -2\bar{\Theta}'_0 \frac{dF}{d\xi} - 3q_0 A_0 F \frac{dF}{d\xi} + \frac{\lambda^2 q_0^2}{4} \frac{d^3 F}{d\xi^3} = 0. \]
Let us now determine the phase $\Theta(z, s)$. To this end, we first observe that $\Theta_0(s)$ can be obtained by integrating Eq. (61), i.e.

$$\Theta_0(s) = \varphi_0 + \frac{\bar{\Theta}_0}{q_0} \int_0^s q_1^2(\tau) d\tau,$$

where $\varphi_0$ is an integration constant. Without loss of generality, we can put: $\varphi_0 = 0$. Then, by using the first of conditions (50) and Eqs. (57) and (64), Eq. (38) can be easily cast into the form

$$\Theta(z, s) = -\frac{1}{2} q_1^2(s) \left[ z - \zeta(s) \right]^2 + \Delta(s),$$

where

$$\zeta(s) = \frac{\kappa_0 q_1^2(s)}{q_0 q'_1(s)},$$

$$\Delta(s) = \frac{\bar{\Theta}_0}{q_0} \int_0^s q_1^2(\tau) d\tau + \frac{\kappa_0^2 q_1^3(s)}{q_0^2 q'_1(s)}.$$

2. Soliton solutions

As it is well known, Eq. (63) admits both localized and periodic solutions [39, 40, 41]. Typically, the latter are expressed in terms of Jacobian elliptic functions, whose suitable asymptotic limits of their parameters recover the localized solutions in the form of bright, dark and grey solitons. However, a very useful integration approach of Eq. (63) that has a simple physical meaning of the solutions is the well-known Sagdeev’s pseudo-potential method [42, 43].

(i). Bright solitons

If $F$ (and consequently $\psi_z$) is a normalizable wave function, we can look for a bright soliton solution of Eq. (63), which satisfies the following boundary conditions: $F \to 0$, for $\xi \to \pm \infty$. This solution exists for $\bar{\Theta}_0 > 0$, i.e.

$$F(\xi) = -\frac{2\bar{\Theta}_0}{q_0 A_0} \text{sech}^2 \left( \frac{\sqrt{2\bar{\Theta}_0}}{\lambda_c|q_0|} \xi \right).$$

Then, going back to the old variables, $z$ and $s$ and using Eqs. (49), (56) and (58), we finally get the following bright soliton solution of Eq. (42)

$$\rho(z, s) = -\frac{2\bar{\Theta}_0}{q_0} q_1^2(s) \text{sech}^2 \left\{ \left[ \sqrt{2\bar{\Theta}_0} \frac{\kappa_0}{\lambda_c|q_0|} q_1^2(s) [z - z_0(s)] \right] \right\},$$

where

$$z_0(s) = \frac{1}{q_1^2(s)} \left[ \kappa_0 \int_0^s q_1^2(\tau) d\tau - R_0 \right].$$

The positivity of $\rho(z, s)$ implies that $q_1 < 0$ and therefore $q < 0$ and $q_0 < 0$. On the other hand, if we require that $\psi_z$ is normalized, i.e.

$$\int dz |\psi_z|^2 = 1,$$

then

$$\bar{\Theta}_0 = \frac{q_0^2}{8} = \frac{q^2}{8} \left[ \frac{\mathcal{F} \left[ \sigma_x(s), \sigma_y(s) \right]_{s=0}}{\sigma_x(s=0) \sigma_y(s=0)} \right]^2.$$


It follows that, in the case of bright solitons, Eq. (67) becomes
\[
\Delta(s) = \frac{1}{2q_0^2} \left( \frac{q_0^2}{4} - \kappa_0^2 \right) \int_0^s q_1^2(q_0^2) d\tau + \frac{\kappa_0^2 q_1^2(s)}{q_0^2 q_1^2(s)}.
\] (73)

(ii). Grey and dark solitons

If \( F \) (and consequently \( \psi_z \)) is a non-normalizable solution, we can look for dark or grey soliton solutions of Eq. (63), which satisfy the following boundary conditions: \( F \to F_0 \), for \( \xi \to \pm \infty \), where \( |F_0| < \infty \). These solutions are given by the general form
\[
F(\xi) = F_0 \left[ 1 - \epsilon^2 \text{sech}^2 \left( \frac{\xi}{\Delta_0} \right) \right],
\] (74)
where \( \epsilon \) is a real parameter,
\[
F_0 = \frac{2\tilde{\theta}_0^2}{q_0 A_0 (\epsilon^2 - 3)},
\] (75)
and
\[
\Delta_0 = \frac{\lambda_c^2 q_0^2}{2\tilde{\theta}_0^2} \left( 1 - \frac{3}{\epsilon^2} \right).
\] (76)

We first observe that since \( \Delta_0 > 0 \), the following condition holds:
\[
\tilde{\theta}_0^2 (\epsilon^2 - 3) > 0.
\] (77)

Taking into account Eqs. (49) and (74)-(76), we easily get:
\[
\rho(\xi, s') = \frac{2\tilde{\theta}_0^2 q_1^2(s')}{q_0^2 (\epsilon^2 - 3)} \left[ 1 - \epsilon^2 \text{sech}^2 \left( \sqrt{\frac{2\tilde{\theta}_0^2}{1 - 3/\epsilon^2}} \frac{\xi}{\lambda_c |q_0|} \right) \right],
\] (78)
which, going back to the variables \( z \) and \( s \), can be cast into the form
\[
\rho(z, s) = \frac{2\tilde{\theta}_0^2 q_1^2(s)}{q_0^2 (\epsilon^2 - 3)} \left[ 1 - \epsilon^2 \text{sech}^2 \left( \sqrt{\frac{2\tilde{\theta}_0^2}{1 - 3/\epsilon^2}} \frac{q_1^2(s)}{\lambda_c |q_0|} (z - z_0(s)) \right) \right].
\] (79)

Taking into account condition (77), from non-negativity of \( \rho(z, s) \) it follows that \( q_1^2 > 0 \). By virtue of Eq. (26), this condition implies, in turn, that \( q > 0 \) and, consequently, due to Eq. (60), that \( q_0 > 0 \).

- If we choose \( \epsilon = \pm 1 \), then Eq. (77) implies that \( \tilde{\theta}_0' < 0 \), while Eqs. (74) and (79) take the form of standard 'dark solitons', i.e.
\[
F(\xi) = -\frac{\tilde{\theta}_0'}{A_0} \tanh^2 \left( \sqrt{-\frac{\tilde{\theta}_0'}{\lambda_c |q_0|}} \xi \right),
\] (80)
and
\[
\rho(z, s) = -\frac{\tilde{\theta}_0'}{q_0^2} \tanh^2 \left( \sqrt{-\frac{\tilde{\theta}_0'}{\lambda_c |q_0|}} \frac{q_1^2(s)}{\lambda_c |q_0|} (z - z_0(s)) \right).
\] (81)

- For \( \epsilon \neq \pm 1 \), condition (77) and Eq. (79) take the form of standard 'grey solitons', in the following range of the parameters \( \epsilon \) and \( \tilde{\theta}_0' \), i.e.
\[
-1 < \epsilon < 1 \quad \text{and} \quad \tilde{\theta}_0' < 0.
\] (82)
IV. THE EVOLUTION AND COLLAPSE OF THE CONTROLLED GPE STRUCTURES

In this section, we carry out a stability analysis of our system, taking into account the control condition (59) and the explicit dependence of the controlling parameter $q_{1D}$ on the transverse scale lengths $\sigma_x$ and $\sigma_y$ through Eq. (26). For simplicity, we present the results for the simple case when the perpendicular solution is the product of only two single Hermite-Gauss modes, which implies for $q_{1D}$ the expression (60) for the transverse mode $(n, l)$. The generalization to the multiple modes’ solutions may be somewhat lengthy, but is straightforward.

We note that the characteristic spatial scales $\sigma_x$, $\sigma_y$, and $q_{1D}(s)$ are related with the coefficients of the restoring force $\omega_x$, $\omega_y$, and $\omega_z$, through four equations. These are the Ermakov-Pinney equations (23), the minimization condition for the control, Eq. (30), and the consistency condition of the amplitude and the phase of the parallel solution, Eq. (31).

\[
\frac{d^2\sigma_j(s)}{ds^2} + \omega^2_j(s)\sigma_j(s) - \frac{\lambda^2_c}{4\sigma_j(s)^3} = 0, \quad j = x, y, \quad \text{(83)}
\]

\[
q_{1D}^{n,l}(s) = q \frac{\delta_n \delta_l}{\sigma_x(s) \sigma_y(s)}, \quad \text{(84)}
\]

\[
\omega^2_z(s) = -q_{1D} \frac{d^2}{ds^2} \left( \frac{1}{\sigma_x(s) \sigma_y(s)} \right). \quad \text{(85)}
\]

Obviously, if we wish to produce a controlled state in a BEC experiment, only two coefficients of the restoring force can be adopted arbitrarily, while the third is determined by the above constraints. Alternatively, we may adopt two arbitrary dependencies in the form $F_1(\omega_x, \omega_y, \omega_z, s) = 0$, $F_2(\omega_x, \omega_y, \omega_z, t) = 0$, from which the quantities $\omega_j(s)$, $j = x, y, z$ are uniquely determined with the use of the constraints (83)-(85). The temporal evolution and the stability of the resulting coherent nonlinear state depends strongly on our choice of the temporal dependence of the confining potential $V_{\text{trap}}$. Both stable and collapsing solutions can be obtained under different conditions, which is studied in more details below.

A. Stable solutions

First we study the important particular case when the perpendicular restoring force is stationary. For this case, we demonstrate the existence of stable (non-collapsing) nonlinear modes when the perpendicular solution contains only one Gauss-Hermite mode in each direction, see Eq. (21).

The Ermakov-Pinney equation (23) is easily solved if $\omega_j = \text{constant}$, where $j = x, y$. The general solution for the perpendicular scale and for the phase functions, Eqs. (24) and (25), can be written as

\[
\sigma_j(s) = \frac{\sqrt{4\gamma_{j,0}^2\sigma_{j,0}^4 + 4\omega_j^2\sigma_{j,0}^4 + 8\gamma_{j,0}\omega_j\sigma_{j,0}^4 \sin(2\omega_j s) + \lambda^2_c - \left[4\left(\gamma_{j,0}^2 - \omega_j^2\right)\sigma_{j,0}^4 + \lambda_c^2\right] \cos(2\omega_j s)}}{2\sqrt{2}\sigma_{j,0}\omega_j}, \quad \text{(86)}
\]

\[
\gamma_j(s) = \frac{\omega_j \left\{8\gamma_{j,0}\omega_j\sigma_{j,0}^4 \cos(2\omega_j s) + \left[4\left(\gamma_{j,0}^2 - \omega_j^2\right)\sigma_{j,0}^4 + \lambda_c^2\right]\sin(2\omega_j s)\right\}}{4\gamma_{j,0}^2\sigma_{j,0}^4 + 4\omega_j^2\sigma_{j,0}^4 + 8\gamma_{j,0}\omega_j\sigma_{j,0}^4 \sin(2\omega_j s) + \lambda_c^2 - \left[4\left(\gamma_{j,0}^2 - \omega_j^2\right)\sigma_{j,0}^4 + \lambda_c^2\right] \cos(2\omega_j s)}, \quad \text{(87)}
\]

and

\[
\phi_{j,k}(s) = \phi_{j,k,0} - \frac{2k + 1}{2} \left\{\tan^{-1} \left[\frac{2\gamma_{j,0}\sigma_{j,0}^2}{\lambda_c} + \frac{4\gamma_{j,0}^2\sigma_{j,0}^4 + \lambda_c^2}{2\lambda_c^2\sigma_{j,0}\omega_j} \tan(\omega_j s)\right] - \tan^{-1} \left[\frac{2\gamma_{j,0}\sigma_{j,0}^2}{\lambda_c} \right]\right\}, \quad \text{(88)}
\]

where $\sigma_{j,0} = \sigma_j(0)$, $\gamma_{j,0} = \gamma_j(0)$, and $\phi_{j,k,0} = \phi_{j,k}(0)$ are arbitrary initial conditions.

We now restrict ourselves to a 2D geometry, $\partial / \partial y = 0$, where the analysis is particularly simple, but it can be easily generalized also to the 3D case. In a 2D case we have
and we also conveniently rewrite the expression for $\sigma_x$, Eq. (86), in the following form

$$\sigma^2_x(s) = D_x^2 \left\{ 2 \cos^2[\omega_x(s - S_x)] - 1 + \sqrt{1 + \frac{\lambda_c^2}{4\omega_x^2 D_x^2}} \right\},$$

(90)

where $D_x$ and $S_x$ are two arbitrary constants. It is obvious that $\sigma^2_x(s)$ is non-negative, and that it may have periodic zeros only in the absence of spatial dispersion, $\lambda_c = 0$. Thus, the amplitude and the characteristic wavenumber of the solution, $\propto q_{1D} \propto 1/\sigma_x(s)$ are regular functions, and the collapse does not occur. The corresponding variation of the coefficient of the parallel restoring force, $\omega_z$, in a 2D regime with a single Hermite-Gauss mode, using Eq. (59), $\omega_z$ is readily obtained as

$$\frac{\omega_z^2(s)}{\omega_z^2} = 1 - \frac{\lambda_c^2}{4\omega_x^2 D_x^2} \left\{ 2 \cos^2[\omega_x(s - S_x)] - 1 + \sqrt{1 + \frac{\lambda_c^2}{4\omega_x^2 D_x^2}} \right\}^{-2}$$

$$= \cos[2\omega_x(s - S_x)] + \sqrt{1 + \frac{\lambda_c^2}{(4\omega_x^2 D_x^2)}} \left\{ \cos[2\omega_x(s - S_x)] + \sqrt{1 + \frac{\lambda_c^2}{(4\omega_x^2 D_x^2)}} \right\}^2 \times \frac{1}{\sqrt{1 + \frac{\lambda_c^2}{(4\omega_x^2 D_x^2)}} + \lambda_c/(2\omega_x D_x^2)}.$$

(91)

The first term is positively definite, and the second is the sum of a harmonic function and a constant, of the form $\cos[2\omega_z(s - S_z)] + \zeta$, where $0 < \zeta < 1$, which obviously features the change of sign.

![Graph](image)

**FIG. 1: Left:** The temporal evolution ("breathing") of the perpendicular scale $\sigma_x^{-1}$ (solid line) and the parallel scale $q_{1D}(s)/q$ (dashed), found in a stable configuration, with $|q_0| = 1, \lambda_c = 1, \delta \equiv \lambda_c |q_0|/\sqrt{2|\theta_0^2|} = 1, \omega_x = 1, \sigma_{x,0} = 0.607107, \gamma_{x,0} = 0.15, \phi_{x,k,0} = 0$, and for the lowest order Hermite-Gauss mode, $k = 0$. **Right:** The corresponding solution for the parallel characteristic frequency $\omega_z^2(s)$ in the 2D case, given by Eq. (91). There is a periodic change of sign, from focussing to defocusing and vice versa.

The temporal evolution ("breathing") of the perpendicular scale $\sigma_x^{-1}$ and the corresponding evolution of $\omega_z^2(s)$ in the 2D case are shown in Fig. 1 respectively. Hereafter, we fix for all figures: $|q_0| = 1$.

The evolution of the 2D structures that in the parallel direction correspond to a bright, dark, and gray soliton, described by Eqs. (69), (80), and (79), respectively, is for the case of a ground Hermite-Gauss mode, $k = 0$ displayed in Figs. 21. The solution features "breathing" (the oscillations of the amplitude and of the position) due to the oscillations of the perpendicular solution, as well as oscillations in the parallel direction, described by the quantity $z_0(s)$, Eq. (70). The frequency of the latter oscillations is smaller than that of the "breathing", and they are not noticeable within the time span of Figs. 214.

The evolution of the stable, first order ($k=1$) mode, is displayed in Figs. 567. Its behavior is similar to that of the ground mode.
FIG. 2: The evolution of the lowest order \((k = 0)\), 2D controlled, stable BEC state whose parallel component is a bright soliton. The initial position is calculated with \(R_0 = 0.2\), and the other parameters of the solution are the same as in Fig. 1.

FIG. 3: The lowest order \((k = 0)\), 2D controlled, stable BEC state whose parallel component is a dark soliton. The parameters of the solution are the same as in Fig. 1.

### B. Collapsing solutions

Besides the stable solutions described in the preceding subsection, unstable controlled BEC states can also be generated, with a different choice of the confining potential \(V_{\text{trap}}\). This can be demonstrated in the simple case when the parallel restoring force is adopted to be stationary, \(\omega_z = \text{constant}\), when the equation (59) can be solved as

\[
q_{1D}(s) = \frac{1}{c_1 \cos(\omega_z s + \varphi_0)},
\]

where \(c_1\) and \(\varphi_0\) are arbitrary constants. Then, in the 2D case \((\partial/\partial y = 0)\) and using Eqs. (30), (23), (24), (25), and (70), we can readily write down the parameters of the perpendicular, Eq. (22), and parallel, Eqs. (69), (80), and...
FIG. 4: The lowest order \((k = 0)\), 2D controlled, stable BEC state that in the parallel direction behaves as a gray soliton with \(\epsilon = 0.7\). Other parameters are the same as in Fig. 1.

FIG. 5: The evolution of the first order \((k = 1)\), 2D controlled, stable BEC state whose parallel component is a bright soliton. The parameters of the solution are the same as in Figs. 2-4.

\[ \sigma_x(s) = \frac{\sigma_{x,0}}{\omega_z} \left[ \omega_z \cos(\omega_z s) + \gamma_{x,0} \sin(\omega_z s) \right], \quad (93) \]

\[ \gamma_x(s) = \omega_z \frac{\gamma_{x,0} \cos(\omega_z s) - \omega_z \sin(\omega_z s)}{\gamma_{x,0} \sin(\omega_z s) + \omega_z \cos(\omega_z s)}, \quad (94) \]

\[ \phi_{x,k}(s) = \phi_{x,k,0} - \frac{\lambda_c (1 + 2k)}{4 \sigma_{x,0}^2} \frac{\sin(\omega_z s)}{\omega_z \cos(\omega_z s) + \gamma_{x,0} \sin(\omega_z s)}. \quad (95) \]
FIG. 6: The first order \((k = 1)\), 2D controlled, stable BEC state whose parallel component is a dark soliton. The parameters of the solution are the same as in Figs. [24].

\[
\omega_x(s) = \omega_z \sqrt{1 + \frac{\lambda^2 \omega_z^2}{4 \sigma_x^2} \left[\omega_z \cos(\omega_z s) + \gamma_{x,0} \sin(\omega_z s)\right]}^{-1},
\]

(96)

\[
q_{1D}(s) = \frac{\delta_k \omega_z}{\sigma_{x,0}} \left[\omega_z \cos(\omega_z s) + \gamma_{x,0} \sin(\omega_z s)\right]^{-1},
\]

(97)

\[
z_0(s) = -\frac{R_0}{q_{1D}(s)} + \frac{\delta_k \kappa_0}{q_0 \sigma_{x,0} \omega_z} \sin(\omega_z s),
\]

(98)

where \(\delta_k\) is defined in Eq. [29], while \(\sigma_{x,0} = \sigma_x(0)\), \(\gamma_{x,0} = \gamma_x(0)\), \(\phi_{x,k,0} = \sigma_{x,k}(0)\), and \(R_0\) are arbitrary initial conditions.
We note that the characteristic wavenumbers and the amplitudes of both the perpendicular and the parallel components of the BEC controlled structure are periodic functions of time, with the frequency $\omega_z$, which take infinite values twice during each period. The displacement from the equilibrium position, $z_0(s)/q_{1D}(s)$, oscillates with the same frequency, but remains limited. Such behavior corresponds to a cyclic collapse and recovery of the soliton-like structure in both directions. However, the recovery does not occur in reality. Our basic GPE is no longer valid for large amplitudes, such as those associated with a collapsing solution, and the new physical effects that arise under such circumstances are likely to perturb the system in a way that prevents its recurrence.

This kind of collapse is well known for the bright solitons, Eq. (69), in the case of a focusing nonlinear term in the GPE, $q_{1D} < 0$ and $\theta_y^0 > 0$. Our analysis shows that the collapse occurs also for a defocusing nonlinearity, $q_{1D} > 0$ and $\theta_y^0 < 0$, when the nonlinear solution is periodic in $z$. The wavelength of such train of nonlinear structures is proportional to $1/\sqrt{|q_{1D}|}$, and becomes infinitesimally small in the singularities of $q_{1D}$, i.e. the train of coherent structures undergoes a collapse.

The temporal evolution of the perpendicular and parallel scales, $\sigma_x^{-1}$ and $q_{1D}(s)$, are displayed in Fig. 8 together with the coefficient of the perpendicular restoring force, $\omega_z^2(s)$. The collapse of the 2D structures, that in the parallel direction correspond to a bright, dark, and gray soliton is displayed in Figs. 9-11 in the case of a ground Hermite-Gauss mode, $k = 0$. The solution features an explosive collapse within a limited period of time. The evolution of the unstable (collapsing), first order ($k = 1$) mode, is displayed in Figs. 12-14. Its behavior is similar to that of the ground mode.

If there is an initial displacement in the parallel direction, $R_y \neq 0$, in the course of its collapse, the soliton-like structure will approach the equilibrium position $z = 0$. The collapse occurs roughly at the same rate in both directions, and that it needs to be supported by a rapid (explosive) growth of the restoring force.

The temporal evolution of the perpendicular and parallel scales, $\sigma_x^{-1}$ and $q_{1D}(s)$, is displayed in Fig. 8, found in a collapsing configuration, with $|q_0| = 1, \lambda_x = 1, \delta \equiv \lambda_y|q_0|/\sqrt{2|\theta_y^0|} = 1, \omega_z = 1, \sigma_{x,0} = 0.607107, \gamma_{x,0} = .15, \phi_{x,k,0} = 0$, and for the lowest order Hermite-Gauss mode, $k = 0$. The solution features an explosive collapse within a limited period of time. The evolution of the unstable (collapsing), first order ($k = 1$) mode, is displayed in Figs. 12-14. Its behavior is similar to that of the ground mode.

If there is an initial displacement in the parallel direction, $R_y \neq 0$, in the course of its collapse, the soliton-like structure will approach the equilibrium position $z = 0$. The collapse occurs roughly at the same rate in both directions, and that it needs to be supported by a rapid (explosive) growth of the restoring force.

The collapse of the controlled BEC structures, that occurs in the regime $\omega_z = \text{constant}$ and with an arbitrary dependence $F_1(\omega_x, \omega_y, s) = 0$, is essentially 2D. It is obvious from Eqs. (92) and (30) that the singularities of the effective parallel and perpendicular wavenumber scale as $q_{1D}(s) \propto 1/|\sigma_x(s)\sigma_y(s)| \propto 1/(s-c)$. Thus, if we adopt $\omega_y = \text{constant}$, we obtain a regular expression for $\sigma_y$, Eq. (80), i.e. the collapse would occur in the $x, y$ plane, producing a thin 1D string along $y$. Likewise, if we adopt the dependence $\sigma_x = \sigma_y$ (which is a stronger requirement than $\omega_x = \omega_y$), the collapse in the perpendicular, $x$ and $y$, directions will be the same, and much slower than in the parallel direction. In other words, the controlled BEC would collapse first into a pancake structure in the perpendicular plane, that would at a later stage and on a slower time scale, collapse also in the radial direction.

V. CONCLUSIONS

In this paper, we have used a newly developed analytical procedure to construct a wide class of localized solutions of the controlled 3D GPE governing the dynamics of BECs in the presence of a spatio-temporally varying external potential, where the latter is composed by a 3D parabolic time-varying potential trap plus a controlling potential to be determined self-consistently. This has been done on the basis of recent investigations [33, 34]. According to
these results, we have found a class of solutions in the factorized form $\psi(r, s) = \psi_{\perp}(r_{\perp}, s)\psi_{z}(z, s)$, which allows us to decompose the 3D GPE into a pair of coupled partial differential equations (i.e. a 2D linear Schrödinger equation, governing the evolution of $\psi_{\perp}$ and a 1D NLSE, governing the evolution of $\psi_{z}$), plus a constrained variational conditions which, among all possible choices of $V_{\text{contr}}$, does not change the mean energy of the system and therefore minimizes the effects introduced by our control operation. We have extended our previous investigation [34] to a wider family of localized 3D solutions of the controlled GPE. In particular, we have found a wide class of localized solutions whose transverse profile is expressed in terms of breathing Hermite-Gauss functions and the longitudinal one expressed in terms of breathing bright, dark or grey solitons. Furthermore, we have studied in details the properties of the controlled BEC states in the 2D case. It is demonstrated that they feature the breathing due to the oscillations of the perpendicular solution, as well as the oscillations in the parallel direction, arising from the initial displacement of the structure from the bottom of the potential well in the parallel direction. In the examples displayed in our Figs.
The stability of the controlled structures, both bright and dark/gray, is governed by the shape and the temporal dependence of the potential trap. We note that the minimization condition for the control, Eq. \( \text{(30)} \), introduces an additional constraint for the possible experimental realization of the trap. The parallel and perpendicular coefficients of the restoring force are related, and they can not be both adopted arbitrarily. We have demonstrated that a stable solution can be obtained if the perpendicular restoring force is stationary, while the parallel force periodically changes the sign. In other words, the external parallel force needs to be switched periodically from the confining to deconfining, in order to arrest the inherent collapse of the soliton due to the nonlinear interactions. Conversely, when the coefficient of the parallel restoring force is stationary, the controlled solution is unstable and undergoes a collapse within the finite period of time. To maintain the control throughout the collapse, the corresponding perpendicular restoring

![Diagram](image)

**FIG. 11**: The lowest order \((k = 0)\), 2D controlled, unstable BEC state that in the parallel direction behaves as a gray soliton with \( \epsilon = 0.7 \). Other parameters are the same as in Fig. 8.

![Diagram](image)

**FIG. 12**: The evolution of the first order \((k = 1)\), 2D controlled, unstable BEC state whose parallel component is a bright soliton. The parameters of the solution are the same as in Figs. 9-11.

The frequency of the latter oscillations was smaller than that of the "breathing", and the parallel oscillations were not noticeable on the "breathing" time scale.
force needs to have a simultaneous singularity in time. In a 3D geometry, the collapse remains predominantly 2D. Thus, in the regime when one of the perpendicular coefficients is constant, the controlled BEC collapses into a 1D string, while in a symmetric situation in the perpendicular plane, it collapses first into a pancake-like structure in the perpendicular plane, that will collapse also in the radial direction, at a later stage and on a slower time scale.

We want to point out that this stability analysis is not exhaustive, but a more general stability analysis will be given in a future work.

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