Spin Related Effects in Transport Properties of “Open” Quantum Dots

Y. Ahmadian, G. Catelani and I.L. Aleiner

Physics Department, Columbia University, New York, NY 10027

(Dated: July 3, 2018)

We study the interaction corrections to the transport coefficients in open quantum dots (i.e. dots connected to leads of large conductance \( G \gg e^2/\pi \hbar \)), via a quantum kinetic equation approach. The effects of all the channels of the universal (in the Random Matrix Theory sense) interaction Hamiltonian are accounted for at one loop approximation. For the electrical conductance we find that even though the magnitude of the triplet channel interaction is smaller than the charging energy, the differential conductance at small bias is greatly affected by this interaction. Furthermore, the application of a magnetic field can significantly change the conductance due to the Zeeman splitting, producing finite bias anomalies. For the thermal conductance we find that the Wiedemann-Franz law is violated by the interaction corrections, and we investigated the effect of magnetic field on the Lorentz ratio for contacts of finite reflection. The charge and triplet channel corrections to the electrical and thermal conductance vanish for reflectionless contacts. In the latter case the temperature and magnetic field dependence of the conductance is determined by the Maki-Thompson correction in the Cooper channel.

PACS numbers: 73.23.-b, 73.21.La, 73.63.Kv, 71.70.Gm

I. INTRODUCTION

Transport properties of quantum dots are strongly affected by electron-electron interactions, the most studied example being the so-called Coulomb blockade phenomenon (see Ref. [4] for a review) in dots connected to external electrodes by contacts of low conductance, \( G \ll e^2/\pi \hbar \equiv G_q \); if the temperature \( T \) is smaller than the charging energy \( E_c \) associated with the addition of one electron to the dot, transport through the dot is exponentially suppressed (except at charge degeneracy points). As the contacts’ conductance increases, quantum fluctuations of the dot’s charge will eventually lift the Coulomb blockade; even so, interactions in such “open” \( (G \gg G_q) \) or weakly blockaded dots do influence transport phenomena. In this paper we consider the interaction corrections to transport coefficients of first order in \( 1/g \equiv G_q/G \), similar to the well-known corrections in the bulk.

It is known that transport properties of dots with a large number of electrons can be described within the framework of Random Matrix Theory (RMT) (see e.g. Refs. [4,10] for reviews). We consider a system separated into clean leads without interaction, a closed dot and the coupling between the dot and the leads as shown in Fig. 1. All interference and interaction effects are associated with the closed dot Hamiltonian \( H_D \). In the RMT approach, the orbital dynamics of non-interacting spinless electrons in the closed dot is described by the \( M \to \infty \) limit of an \( M \times M \) Hamiltonian matrix with random entries \( H_{mn}^0 \) belonging to the Gaussian ensemble:

\[
(H_{mn}^0 H_{jn}^0)_{\text{RM}} = M \frac{\delta_{ij}}{\pi^2} \left[ \delta_{mj} \delta_{ni} + \left( 1 - \frac{g_h}{4M} \right) \delta_{mi} \delta_{nj} \right],
\]

where \( \delta_1 \) is the one-particle mean level spacing, \( \langle \cdots \rangle_{\text{RM}} \) denotes averaging over the random matrix ensemble, and

\[
g_h \equiv \frac{\delta_e}{\delta_1},
\]

with \( \delta_e \approx hT \) and \( \delta_1 \approx \hbar \omega \), representing the Thouless energy and mean level spacing, respectively.

In the absence of spin-orbit interactions (we neglect spin-orbit interactions in this paper; the interplay of exchange and spin-orbit interactions was studied in Ref. [11]), the spin of the electrons can be accounted for by defining the following non-interacting Hamiltonian operator (in the second quantized notation):

\[
H^0 = \psi_m \sum_{m} H_{mn}^0 \psi_n,
\]

where \( \psi_m^\dagger \) (\( \psi_m \)) is a two component spinor operator, whose components create (annihilate) electrons in the
For interacting electrons, the closed dot Hamiltonian is given by

$$H_D = H^0 + H_{\text{int}},$$  \hspace{1cm} (1.4)$$

where the dominant part of the interaction Hamiltonian has the universal form:

$$H_{\text{int}} = E_c N^2 + J_s S^2 + J_c T \mathcal{T}.$$  \hspace{1cm} (1.5)$$

Here

$$N = \sum_{n=1}^{M} \psi_n^\dagger \psi_n,$$

$$\tilde{S} = \frac{1}{2} \sum_{n=1}^{M} \psi_n^\dagger \sigma \psi_n,$$

$$\mathcal{T} = \frac{1}{2} \sum_{n=1}^{M} \psi_n \sigma \psi_n,$$  \hspace{1cm} (1.6a-c)$$

are respectively the number of electrons in the dot, the total spin of the dot electrons, and the pairing operator. Here, $\sigma$ and $\sigma^y$ are Pauli matrices acting in the spinor space.

The last term in the right-hand side of Eq. (1.5) describes pairing between electrons (interaction in the Cooper channel) and for $J_c < 0$ it drives the dot towards the superconducting state; studies of such superconducting grains have been reviewed e.g. in Ref. 13. Here we will assume that either $J_c > 0$ and hence no superconducting transition at any temperature, or $J_c < 0$ but $T - T_c \gtrsim T_c$, so that the dot is in the normal phase, and furthermore, the superconducting fluctuations are small.

The second term (triplet channel) gives the dependence of the dot’s energy on the total spin in the dot – the effects of this term on the tunneling density of states and on the spin susceptibility and on the peak spacing in the Coulomb blockade regime have been recently considered. Finally, the first term (singlet channel) describes the charging energy and it is responsible for the Coulomb blockade. In the weakly blockaded regime, only this term has been considered previously in the literature as its contribution is expected to be the dominant one for the repulsive Coulomb interaction. However this term is not affected by an external magnetic field, whereas the two remaining terms are. Our goal is to calculate the interaction corrections to the transport coefficients with the full universal Hamiltonian $H_D$ taken into account and to examine the dependence of these corrections on the applied magnetic field. In particular, we consider the non-linear conductance for voltage-biased dots and their (linear response) thermal conductance. To evaluate these transport coefficients we construct, starting from the RMT description, a quantum kinetic equation analogous to the one developed for the description of disordered metals.

The remainder of the paper is organized as follows: in the next section we summarize our results for the conductance of metallic quantum dots. In Sec. III we present the derivation of the 0-dimensional Usadel equation in RMT. This equation is the starting point for the derivation of the kinetic equation as outlined in Sec. IV. In Sec. V we give the explicit calculation of the transport coefficients.

II. SUMMARY OF THE RESULTS

Here we present our results for the interaction corrections to the differential electrical conductance and the linear thermal conductance of quantum dots in the presence of a magnetic field. These results are derived in Sec. V.

A. Electrical Conductance

The total ensemble averaged differential conductance $G$ is

$$G = \frac{dI}{dV} = G_0 + \Delta G,$$  \hspace{1cm} (2.1)$$

where

$$G_0 = \frac{e^2}{\pi \hbar} \frac{g_L g_R}{g_L + g_R}$$  \hspace{1cm} (2.2)$$

is the classical conductance and $\Delta G$ is the interaction correction. We do not include the weak localization correction for non-interacting electrons which can be found e.g. in Ref. 9. We also do not study the contribution of $\Delta G$ to the mesoscopic fluctuations of the conductance, which are smaller than $\Delta G$ by the additional factor $G_q/G_0$. (For the charge channel this effect was studied in Ref. 10).

In Eq. (2.2) $g_L$ and $g_R$ are the dimensionless conductances of the left and right contacts respectively:

$$g_L = \sum_{n=1}^{N_L} T_n, \quad g_R = \sum_{n=N_L+1}^{N_R} T_n,$$  \hspace{1cm} (2.3)$$

where $T_n$ is the transmission coefficient of the $n$-th channel.

The interaction correction $\Delta G$ has distinct contributions from each term in the interaction Hamiltonian:

$$\Delta G = \Delta G_c + \Delta G_s + \Delta G_{\text{Cooper}}.$$  \hspace{1cm} (2.4)$$

We postpone the discussion of the Cooper channel correction, $\Delta G_{\text{Cooper}}$, until the end of this section. For the charge and triplet channel contributions we have found

$$\Delta G_c = \frac{e^2}{\pi \hbar} (g_L + g_R)^2 \Xi (\Gamma_0, 4E_c, T; V),$$  \hspace{1cm} (2.5)$$

where
and
\[
\Delta G_s = \frac{e^2}{\pi \hbar} \frac{h_L g_L^2 + h_R g_R^2}{(g_L + g_R)^2} \times \sum_{m=0, \pm 1} \Xi (\Gamma_0 + imE'_Z, J_s, T; V).
\]  
(2.6)

The form factors \( h_L \) and \( h_R \) are given by
\[
h_{L(R)} = \sum_{n \in L(R)} T_n (1 - T_n).
\]  
(2.7)

Factors of this form were first obtained in Ref. 18 and reproduced in Refs. 6,7. This structure was originally missed in the formalism of Ref. 5 but was recovered in Ref. 5.

In Eqs. 2.5 and 2.6 the dimensionless function \( \Xi \) is defined by
\[
\Xi (\Gamma, \varepsilon, T; V) \equiv \text{Re} \frac{\Gamma_0}{T} \left[ \Psi \left( \frac{\Gamma - ieV}{2\pi T} \right) - \Psi \left( \frac{(1 + \varepsilon)\Gamma - ieV}{2\pi T} \right) \right],
\]  
(2.8)

where
\[
\Psi(z) \equiv \psi^{(0)}(z) + z\psi^{(1)}(z),
\]  
(2.9)

and \( \psi^{(i)}(z) \) is the \( i \)-th derivative of the digamma function. The form of this function agrees with the result derived in Refs. 4,5 for the charge channel. We denote with \( V \) and \( T \) the bias voltage and the temperature respectively, \( \Gamma_0/\hbar \) is the escape rate:
\[
\Gamma_0 = \frac{\hbar}{\tau} = \frac{\delta_1}{2\pi} \sum_{n=1}^{N_{ch}} T_n,
\]  
(2.10)

and \( E'_Z \) is the Zeeman energy renormalized by the exchange interaction:
\[
E'_Z = \frac{E_Z}{1 + J_s/\delta_1} = \frac{g_L \mu_B B}{1 + J_s/\delta_1}.
\]  
(2.11)

Here \( B \) is the magnetic field, \( g_L \) is the Lande g-factor, and \( \mu_B \) is the Bohr magneton.

In Fig. 2 we plot \( \Delta G = \Delta G_c + \Delta G_s \) in units of \( G_0 = e^2/\pi \hbar \), for different values of the triplet channel interaction constant \( J_s \) in the absence of magnetic field. We have taken \( N_L = N_R \) and \( T_n = 1/2 \) for all channels, so that the prefactors in Eqs. 2.5 and 2.6 are equal to \( 1/2 \). The interaction constant takes values ranging from \( J_s = 0 \) to \( J_s = -0.75\delta_1 \), which apply to most metals and quantum dots (the values of \( J_s/\delta_1 \) are reviewed e.g. in Ref. 11). The charging energy \( E_c \) on the other hand is much larger than the mean level spacing. A theoretical estimate for two dimensional dots yields \( E_c/\delta_1 \approx r_s k_F L \), where \( r_s \) is the gas parameter, \( k_F \) is the Fermi wavelength, and \( L \) is the lateral dimension of the dot. This is large because of the large factor \( k_F L \gg 1 \). As shown in Fig. 2 the charge channel correction generates a dip in the conductance at zero bias with a characteristic width of order \( (1 + 4E_c/\delta_1)\Gamma_0 \), analogous to the zero bias anomaly in bulk systems. This width can also be estimated as \( \hbar/\tau_c \), where \( \tau_c < \tau \) is the classical recharging time governing the charge dynamics. On the other hand, the attractive triplet interaction produces a peak with a much smaller width of order \( (1 + J_s/\delta_1)\Gamma_0 \equiv \hbar/\tau_s \), corresponding to the slow spin dynamics: \( \tau_s > \tau \). Therefore we have a competition between the two corrections at zero bias as seen in Fig. 2.

In Fig. 3 we plot the magneto-conductance, \( G(E_Z) - G(E_Z = 0) \), for different values of the Zeeman energy

![FIG. 2: Interaction correction to the dimensionless differential conductance for varying strength of the triplet interaction in the absence of Zeeman splitting. Here \( T = 0.1\delta_1, \Gamma_0 = 5\delta_1 \) and \( E_c = 100\delta_1 \) where \( \delta_1 \) is the mean level spacing, and we have taken \( N_L = N_R \) and \( T_n = 1/2 \) for all channels. The value of \( J_s \) is shown above each graph. For clarity, graphs are shifted upwards by 0.05 at each step.](image1)

![FIG. 3: The magneto-conductance of the starred curve of Fig. 2 vs. bias voltage, for different values of Zeeman splitting energy (shown above each graph). The graphs are shifted upwards by 0.03 at each step.](image2)
(and with the same assumptions about the form factors made for Fig. 2). In the presence of a magnetic field, the triplet channel contribution decomposes into three terms due to the Zeeman splitting. These terms produce peaks at $eV \simeq \pm E_Z^* \pm 0$. As the value of the Zeeman energy increases so does the width of the displaced peaks (for $m = \pm 1$); in the limit $E_Z \gg \Gamma_0$ this width is given by the bare Zeeman energy $E_Z$. In Fig. 4 we plot $\Delta G$ vs. $E_Z^*$ for different bias voltages.

In the case of reflectionless contacts, ($T_n = 1$ for all $n$), the corrections in Eqs. (2.5) and (2.6) vanish due to the vanishing of the form factors $h_L$ and $h_R$ of Eq. (2.7); a non-zero contribution to $G$ is given by the zero-dimensional analog of the Maki-Thompson correction to the conductivity:

$$\Delta G_{\text{Cooper}} = \Delta G_{\text{MT}} = \left( \frac{1}{\varepsilon^2} \right) \frac{e^2}{\pi \hbar} \frac{g_L g_R}{(g_L + g_R)^2} \times \sum_{\alpha, \beta = L, R} g_{\alpha} g_{\beta} \Upsilon_{\alpha \beta} (\Gamma_*; E_R^*, E_Z^*; T; V),$$

where $\Upsilon_{LL} = \Upsilon_{RR}$ and $\Upsilon_{LR} = \Upsilon_{RL}$ are dimensionless functions given below, and $\Gamma_*/h$ is the escape rate modified by the effect of the magnetic field on the orbital motion of the electrons [see the discussion after Eq. (1.12)]:

$$\Gamma_* \equiv \frac{h}{\tau_*} = \frac{\delta_1}{2\pi} \sum_{n=1}^{N_{ch}} T_n + g_h.$$  

Here

$$\varepsilon \simeq \ln \frac{\Gamma_*}{\max(T, eV, \Gamma_*)}.$$  

and $T_c$ is defined in Eq. (1.102). In the attractive case, $T_c$ is the critical temperature of the superconducting transition. In this case we only consider the normal state at $E_T \gg T \gg T_c$, and hence $|\varepsilon| \gg 1$. In the repulsive case, $T_c \gg E_T$, and since by assumption $E_T$ is larger than all the other energy scales in the problem, $\varepsilon \gg 1$. This means that the correction in Eq. (2.12) is logarithmically suppressed for normal dots and is in general much smaller than the correction due to charge and triplet channels. For this reason we report here the Cooper channel correction only for reflectionless contacts, where it is the only non-vanishing correction. The results for the general transmission coefficient can be found in Sec. V C.

The functions $\Upsilon_{\alpha \beta}$ cannot be calculated in a compact form and we therefore consider approximate expressions valid in certain regions of parameters. In the low temperature, low voltage regime (i.e. $T, eV \ll \Gamma_*$) we have:

$$\Upsilon_{\alpha \beta} = \frac{4}{[1 + (E_Z^*/\Gamma_*)^2]} \left[ A_{\alpha \beta} \left( \frac{eV}{\Gamma_*} \right)^2 + \frac{\pi^2}{3} \left( \frac{T}{\Gamma_*} \right)^2 \right],$$

where $A_{LR} = A_{RL} = 1$, and $A_{LL} = A_{RR} = 1/4$. The $T^2$ and $V^2$ dependences are due to the inelastic nature of the processes at the origin of the Maki-Thompson correction.

At finite bias and low temperature ($T \ll eV, \Gamma_*$), we have:

$$\Upsilon_{LR} \approx \frac{1}{2} \sum_{m=\pm 1} \frac{h}{\Gamma_*} \left( \frac{eV}{\Gamma_*} - mE_Z^* \Gamma_* \right),$$

$$\Upsilon_{LL} \approx \frac{1}{2} \Upsilon_{LR} - \frac{1}{4} \left( \sum_{m=\pm 1} \arctan \frac{eV - mE_Z^*}{\Gamma_*} \right)^2,$$

where the full expression for function $h$ is given in Eq. (5.04). This function is used in Fig. 5 to plot
\( \varepsilon^2 \Delta G_{MT} \) (in units of \( \varepsilon^2/\pi h \)) for symmetric leads at different values of the Zeeman energy. An approximate expression for \( h \) is:

\[
h \approx \sum_{m,n=\pm 1} \arctan \left( \frac{2eV + 2(m - n)E_Z^*}{\Gamma_0} \right) \quad (2.17)
\]

\[
\times \left[ \arctan \left( \frac{eV + mE_Z^*}{\Gamma_0} \right) - \arctan \left( \frac{mE_Z^*}{\Gamma_0} \right) \right].
\]

This formula qualitatively renders the shape of the correction, although it overestimates it in the central plateau [cf. Fig. 5]; on the other hand, Eq. (2.17) is a good approximation when \( |V| \gtrsim E_Z^* \) and \( E_Z^* \gg 1/\tau_s \), i.e. it gives a quantitative description of both the steps at \( V \sim \pm E_Z^* \) and the asymptotic regions at large bias for sufficiently large Zeeman energy. At small bias \( (V \ll 1/\tau_s) \), the correction is better represented by Eqs. (2.18).

Finally, in the high temperature regime \((T \gg \Gamma_s)\) we have:

\[
\Upsilon_{LR} \approx \frac{\pi^2}{2} - \frac{\pi^2}{4} \coth \left( \frac{E_Z^*/2T}{\Gamma_0} \right) \sum_{m=\pm} c_m \left( \frac{E_Z^* - meV}{2T} \right),
\]

\[
\Upsilon_{LL} \approx \frac{\pi^2}{8} \sum_{m=\pm} c_0 \left( \frac{eV - mE_Z^*/2T}{\Gamma_0} \right),
\]

where

\[
c_n(x) \equiv \frac{d^n}{dx^n} (x \coth x).
\]

B. Thermal Conductance

For the thermal conductance we calculate the linear response in singlet and triplet channels only, and we do not report the contribution of Cooper channel, as it is smaller by the factor \( 1/\varepsilon^2 \) [cf. Eq. (2.14)].

The thermal conductance is a combination of two parts

\[
\kappa = \frac{J_T^0}{T_L - T_R} = \kappa_{WF} + \Delta \kappa.
\]

The first term \( \kappa_{WF} \) respects the Wiedemann-Franz law:

\[
\kappa_{WF} = \frac{\pi^2 T}{3e^2} G(V = 0) = \frac{\pi^2 T}{3e^2} \left[ G_0 + \Delta G(V = 0) \right],
\]

with \( \Delta G \) given by Eqs. (2.20) - (2.26), whereas the correction \( \Delta \kappa \) violates this law:

\[
\Delta \kappa = \Delta \kappa_c + \Delta \kappa_s,
\]

\[
\Delta \kappa_c = \frac{g_R^2 h_L + g_L^2 h_R}{(g_L + g_R)^3} \pi T \frac{\Gamma_0}{9h}
\]

\[
\times \left[ g_1 \left( \frac{2\pi T}{(1 + \frac{4E_Z^*}{\Gamma_0})\Gamma_0} \right) - g_1 \left( \frac{2\pi T}{\Gamma_0} \right) \right].
\]

FIG. 6: The relative change in the Lorentz number vs. temperature for different values of Zeeman splitting energy (shown above each graph). Here, \( E_c = 100\delta_1 \) and \( J_s = -0.45\delta_1 \), \( \Gamma_0 = 5\delta_1 \), and we have taken \( N_L = N_R = 1 \) and \( T_n = 1/2 \) for all channels. The graphs are shifted upwards by 0.03 at each step.

\[
\Delta \kappa_c = \frac{g_R^2 h_L + g_L^2 h_R}{(g_L + g_R)^3} \pi T \frac{\Gamma_0}{9h}
\]

\[
\times \left[ g_1 \left( \frac{2\pi T}{(1 + \frac{4E_Z^*}{\Gamma_0})\Gamma_0} \right) - g_1 \left( \frac{2\pi T}{\Gamma_0} \right) \right].
\]

In Eqs. (2.22b) we use the notation

\[
\Gamma_m = \Gamma_0 + imE_Z^*,
\]

and

\[
g_1(x) = \frac{6}{x^3} \psi^{(1)} \left( \frac{1}{x} \right) - \frac{6}{x^2} - 3, \quad (2.24)
\]

where \( \psi^{(1)} \) is the derivative of the digamma function. The deviation from the Wiedemann-Franz law can be quantified by defining a generalized Lorentz number according to

\[
L \equiv \frac{\kappa}{TG(V = 0)}. \quad (2.25)
\]

such that it would yield the usual \( L_0 = \pi^2/3e^2 \) in the absence of interactions. In Fig. 6 we plot the relative change in the Lorentz number, \( L/L_0 - 1 \), as a function of temperature for different values of Zeeman splitting energy (and with the same assumptions about the form factors made for Fig. 4).

We note that the interaction corrections to thermal conductance Eqs. (2.22b) - (2.22c) vanish for reflectionless contacts, and so the Wiedemann-Franz law is satisfied in this case. That this law is not violated by inelastic processes in this case can be explained by the following qualitative picture. Consider an electron with energy \( \epsilon \) and charge \( e \) entering the dot from the right lead with an influx proportional to the number of channels \( N_R \) in this
lead \((T_n = 1\) for all channels). The electron will subsequently leave the dot through the right or the left lead with probability \(N_R/N_{ch}\) and \(N_L/N_{ch}\) respectively, thus contributing an outflux in the right lead proportional to \(N_{ch}^2/N_{ch}\). Therefore, the total ingoing electric current in this lead will be

\[
I_R \propto eN_R - eN_R^2/N_{ch} = eN_LN_R/N_L + N_R, \tag{2.26}
\]
i.e. the electric conductance is given by the classical conductance Eq. \(2.22\). If in the process the electron loses an energy \(\omega\) to the collective excitations this will still hold within the linear spectrum approximation as the bosonic excitations are neutral and do not contribute to electric current. This is apparently not the case for the energy current as the electron going out of the right lead now has energy \(\epsilon - \omega\). However, the collective excitation with energy \(\omega\) will eventually decay into an electron hole pair and these will enter the leads with the same probabilities as before, and due to energy conservation will carry the lost energy back to the right lead with the same rate, i.e.

\[
I_R \propto eN_R - (\epsilon - \omega)N_{ch}^2/N_{ch}^2 = eN_LN_R/N_L + N_R. \tag{2.27}
\]

We see that the relation between the energy current and the electric one Eq. \(2.20\) is not altered and we recover the Wiedemann-Franz law. It is crucial in this reasoning that the contacts be reflectionless, since only in this case the transmission coefficients will not be renormalized by interaction and the \(T_n\)’s remain equal to one independent of energy. This is not the case for \(T_n \neq 1\) and the renormalized transmission coefficients depend on energy, thus violating Eq. \(2.27\) and therefore the Wiedemann-Franz law.

### III. DERIVATION OF THE 0D USADEL EQUATION

In this section we derive the zero dimensional Usadel equation\textsuperscript{25} that describes electrons in open quantum dots, taking into account the electron-electron interaction but neglecting weak localization and mesoscopic fluctuation effects. Our treatment here is based on the Keldysh technique\textsuperscript{23} for non-equilibrium systems, and is analogous to Ref. 17 with appropriate modifications for the 0-d case. First we start with the description of coupling to the leads in subsection IIIA. In the rest of the subsections we derive equations for Green functions averaged over the appropriate random matrix ensemble. To overcome the difficulties involving ensemble averaging in the presence of the quartic interactions of the universal Hamiltonian Eq. \(1.12\), we employ a Hubbard-Stratonovich transformation for each channel of interaction, rendering the fermion Hamiltonian quadratic. This is done in subsections IIIB to IIID.

#### A. Description of the Leads

The open dot [see Fig. 1] is described by the Hamiltonian\textsuperscript{8}

\[
H = H_D + H_L + H_{LD}, \tag{3.1}
\]

where \(H_D = H^0 + H_{int}\) is the (RMT) Hamiltonian for the interacting electrons in the dot [see Eqs. \(1.1\)–\(1.5\)], \(H_F\) is the Hamiltonian for the free electrons in the leads and \(H_{LD}\) describes the coupling between the dot and the leads. The electron spectrum in the leads near the Fermi surface can be linearized:

\[
H_L = v_F \sum_{\alpha} \int \frac{dk}{2n} k\bar{n}_\alpha(k)\psi_\alpha(k), \tag{3.2}
\]

where \(v_F\) is the Fermi velocity, and \(\alpha\) labels different channels in the leads: \(1 \leq \alpha \leq N_L\) for the left, and \(N_L + 1 \leq \alpha \leq N_{ch}\) for the right lead channels, and \(N_{ch} = N_L + N_R\). Here, the field \(\psi_\alpha\) is understood to be a two component spinor and we suppress the spin indices unless stated otherwise.

The leads-dot coupling Hamiltonian is

\[
H_{LD} = \sum_{\alpha, n, k} e^{-\eta/2} (W_{\alpha n} \psi_\alpha^\dagger(k)\psi_n + h.c.), \tag{3.3}
\]

where the coupling constants are defined as

\[
W_{\alpha n} = t_n \sqrt{\frac{M_d}{\pi^2}} \delta_{\alpha n}, \tag{3.4}
\]

for \(n = \alpha \leq N_{ch}\) and zero otherwise, and the exponential at \(\eta \to 0^+\), is used to regularize the coupling at large \(|k|\). Here \(\nu = 1/(2\pi v_F)\) is the density of states per spin at the Fermi level. We can always write the matrix \(W\) in the above diagonal form by choosing the appropriate basis for the random matrix.

We introduce the exact Green functions of the electrons in the dot \(\hat{G}\) and in the leads \(\hat{F}\). As usual they are \(2 \times 2\) matrices in the Keldysh and spin spaces

\[
\hat{G}_{nm} = \begin{pmatrix} G^R_{nm}(t_1; t_2) & G^K_{nm}(t_1; t_2) \\ G^Z_{nm}(t_1; t_2) & G^K_{nm}(t_1; t_2) \end{pmatrix},
\]

\[
\hat{F}_{\alpha\beta} = \begin{pmatrix} F^R_{\alpha\beta}(t_1, k_1; t_2, k_2) & F^K_{\alpha\beta}(t_1, k_1; t_2, k_2) \\ F^Z_{\alpha\beta}(t_1, k_1; t_2, k_2) & F^K_{\alpha\beta}(t_1, k_1; t_2, k_2) \end{pmatrix}, \tag{3.5}
\]

where \(1 \leq n, m \leq M\) (\(M \to \infty\) being the size of the random matrix), and \(\alpha, \beta\) label different channels in each lead. The entries of the matrix in Eq. \(3.5\) are given by

\[
F^R = -i\theta_{12} \langle \psi_\alpha(k_1, t_1)\psi_\beta^\dagger(k_2, t_2) + \psi_\beta(k_2, t_2)\psi_\alpha(k_1, t_1) \rangle,
\]

\[
F^K = -i\theta_{21} \langle \psi_\alpha(k_1, t_1)\psi_\beta^\dagger(k_2, t_2) - \psi_\beta(k_2, t_2)\psi_\alpha(k_1, t_1) \rangle,
\]

\[
F^Z = 0. \tag{3.6}
\]

The expression for \(G\) is obtained from Eq. \(3.6\) by replacing \(\alpha, \beta\) with \(n, m\) and removing the \(k\) variables.
Here $\theta_{12} \equiv \theta(t_1 - t_2)$, where $\theta(t)$ is the step function, the fermionic spinor operators are in the Heisenberg representat-
where \( \hat{1} = \hat{1}_R \delta(t_1 - t_2) \), and

\[
\Pi^K_{\phi}(t_1, t_2) = \Pi^K_{\phi}(t_2, t_1) = \frac{1}{2i \theta} \sum_{n=1}^{\infty} \text{Tr}_s(G^K_{nm}(t_1, t_1|\phi)) \\
\Pi^A_{\phi}(t_1, t_2) = \frac{1}{2i \theta} \sum_{n=1}^{\infty} \text{Tr}_s(G^A_{nm}(t_1, t_1|\phi))
\]

Here we have introduced the Green function of the dot electrons as a functional of the field \( \phi \) in its matrix form in Keldysh space:

\[
\hat{G}_{nm}(t_1, t_2|\phi) = \left( G^R_{nm}(t_1, t_2|\phi) G^K_{nm}(t_1, t_2|\phi) G^A_{nm}(t_1, t_2|\phi) \right)
\]

such that its average over the fluctuating field \( \phi \) gives the usual expressions as in Eq. (3.18), with the averaged \( G^Z \) vanishing. We will suppress the argument \( \phi \) in the subsequent formulas.

We construct perturbation theory for the RM averaged Green function \( \langle \hat{G}_{nm} \rangle_{RM} = G_{nm} \delta_{nm} \), as an expansion in powers of the random matrix Hamiltonian \( H_0 \), and the Hubbard-Stratonovitch field \( \phi \) (see Fig. 7 for the diagrammatic form of the Dyson equation). The first term in the self energy expansion (Fig. 7.b) is due to the effect of the Hubbard-Stratonovitch field \( \phi \), which only changes the phase of the Green function and does not lead to scattering between different orbitals. The second term is due to scattering and to and from the leads. The third term (Fig. 7.c) gives the contribution of the non-interacting closed dot Hamiltonian \( H^0 \) after averaging over the random matrix ensemble, where the random matrix correlation in panel (e) is given by (3.11).

The crossing diagrams in Fig. 7.c are smaller than the non-crossing diagram only when \( m = n \) (see Fig. 7.c), and is therefore small by a factor of \( 1/M \). This means that the orthogonal and unitary ensembles are equivalent for charge and triplet channel (the Cooper channel case is considered in Sec. 1111). Thus, in the \( M \to \infty \) limit, we obtain the following Dyson equation for \( \hat{G}_n \)

\[
i \frac{\partial \hat{G}_n}{\partial t_1} = \hat{1} + \Sigma_n \hat{G}_n,
\]

\[
\Sigma_n = \phi - i \pi \nu W_{n\alpha} f_n W^\dagger_{n\alpha} + M \left( \frac{\delta \pi}{\pi} \right)^2 \sum_{m=1}^{M} \hat{G}_m
\]

Here, as in subsection 1111A, the products are understood as operator (matrix) multiplication in time as well as in Keldysh and spin spaces. In particular for the unit operator we have \( \hat{1} = \delta(t_1 - t_2) \Pi_{\phi} \otimes 1_s \), and \( \phi \) is understood as the operator \( \Pi_{\phi}(t_1) \Pi_{\phi}(t_1 - t_2) \Pi_{\phi} \). We have also defined

\[
f_\alpha = \frac{i}{\pi \nu} \int \frac{dk}{2\pi} e^{-|k|} f_\alpha^{(0)}(k)
\]

where \( f_\alpha^{(0)}(k) \) was defined in Eq. (3.17). Using Eq. (3.17) we integrate over \( k \) obtaining

\[
f_\alpha(t_1 - t_2) = \int \frac{dk}{2\pi} e^{-|k|} f_\alpha^{(0)}(k)
\]

and the function \( f_\alpha^{(0)}(k) \) was defined in Eq. (3.21). Substituting Eq. (3.21) into Eq. (3.20) and defining

\[
\hat{g}_n = i M \frac{\delta \pi}{\pi} \hat{G}_n, \quad \hat{g} = \sum_{n=1}^{M} \frac{\hat{g}_n}{M}
\]

we obtain

\[
\frac{1}{M} \frac{\partial \hat{g}_n}{\partial t_1} + \frac{i}{\delta \pi} \hat{g}_n + |t_n|^2 \hat{f}_n \hat{g}_n + \hat{g} \hat{g}_n = \hat{1},
\]

where we have adopted units such that \( \delta \pi = 1 \):

\[
t \to \frac{t}{\delta \pi}, \quad \phi \to \phi \frac{\delta \pi}{\pi}.
\]

As we are interested in the limit \( M \to \infty \) of the RMT, we can neglect the first two terms on the left side of Eq. (3.25), obtaining

\[
|t_n|^2 \hat{f}_n \hat{g}_n + \hat{g} \hat{g}_n = \hat{1}, \quad (n \leq N_{ch}),
\]

\[
\hat{g} \hat{g}_n = \hat{1}, \quad (n > N_{ch}).
\]
Summing Eq. (3.26) over all $n$ and neglecting terms of order $N_{ch}/M$ we obtain the following constraint for $\hat{g}$:

$$\hat{g} \cdot \hat{g} = \hat{1}. \quad (3.27)$$

Using Eqs. (3.25) (3.27), we solve for $\hat{g}_n$'s, obtaining

$$\hat{g}_n = \frac{\hat{1}}{\hat{g} + |t_n|^2 \hat{f}_n} = \hat{g}(\hat{1} + |t_n|^2 \hat{f}_n)^{-1} \hat{g} = (\hat{1} + |t_n|^2 \hat{f}_n)^{-1} \hat{g}. \quad (3.28)$$

Thus, given $\hat{g}$ and the lead Green functions Eq. (3.23), we can completely determine the $\hat{g}_n$'s using Eq. (3.28). As for $\hat{g}$, Eq. (3.21) in general only constrains $\hat{g}$ to a certain manifold, and does not determine it further. The non-equilibrium evolution of $\hat{g}$ on this manifold is given by the terms of order $1/M$ in Eq. (3.26), which were neglected in reducing that equation to Eq. (3.26). In order to separate those $1/M$ terms, we subtract from Eq. (3.25) its transpose, obtaining

$$\frac{1}{M} \frac{\partial \hat{g}}{\partial t} + i \frac{1}{M} [\hat{\phi} : \hat{g}_n] = -|t_n|^2 [\hat{f}_n : \hat{g}_n] + [\hat{g}_n : \hat{g}],$$

which after summation over all $n$ gives

$$\frac{\partial \hat{g}}{\partial t} + i [\hat{\phi} : \hat{g}] = -\sum_{n=1}^{N_A} |t_n|^2 [\hat{f}_n : \hat{g}_n]. \quad (3.29)$$

Here $\partial/\partial t \equiv \partial/\partial t_1 + \partial/\partial t_2$ is the derivative with respect to the “center of mass” time, and $[a; b] = ab - ba$ is the commutator. Using Eq. (3.26), we can rewrite Eq. (3.29) as

$$\frac{\partial \hat{g}}{\partial t} + i [\hat{\phi} : \hat{g}] = \sum_{n=1}^{N_A} [\hat{g}_n : \hat{g}_n], \quad (3.30)$$

which we call the 0D Usadel equation. It is easy to see that this equation is consistent with the constraint Eq. (3.27).

We are now prepared to take the $M \rightarrow \infty$ limit in the formula for the current Eq. (3.13). Using the definition Eq. (3.24) and constraint Eq. (3.26) in Eq. (3.13), we obtain

$$I_a = \frac{e}{2} \sum_{n,a} \text{Tr}_s \left( -|t_n|^2 [\hat{f}_n : \hat{g}_n]^K \right),$$

$$= \frac{e}{2} \sum_{n,a} \text{Tr}_s [\hat{g}_n : \hat{g}_n]^K. \quad (3.31)$$

Thus the total outgoing current is proportional to the Keldysh part of the right hand side of the Usadel equation, Eq. (3.29), which states nothing but the conservation of number of particles.

We can rewrite the right hand side of Eq. (3.30) in a form that is more convenient for later use. Using Eq. (3.28), we write this commutator in terms of $\hat{g}$ and $\hat{f}_n$ only:

$$[\hat{g}_n : \hat{g}_n] = (\hat{1} + |t_n|^2 \hat{f}_n)^{-1} \hat{g}(\hat{1} + |t_n|^2 \hat{f}_n)^{-1} - (\hat{1} + |t_n|^2 \hat{f}_n)^{-1} \hat{g}(\hat{1} + |t_n|^2 \hat{f}_n)^{-1} \hat{g}$$

$$= |t_n|^2 \hat{g}_n \cdot \hat{g}_n \cdot \hat{g}_n \cdot \hat{g}_n = \frac{1}{4} T_n \hat{g}_n \cdot \hat{g}_n \cdot \hat{g}_n \cdot \hat{g}_n (\{\hat{g}_n, \hat{f}_n \} - \hat{2})^{-1}$$

where $\{a; b\} = ab + ba$ is the anti-commutator, and in obtaining the third line from the second, we have used Eq. (3.28) and Eq. (3.27) to write

$$\hat{g}_n \cdot \hat{g}_n = (\hat{1} + |t_n|^2 \hat{f}_n \hat{g}_n \cdot (\hat{1} + |t_n|^2 \hat{f}_n)^{-1} \hat{g}_n$$

$$= (\hat{1} + |t_n|^2)^2 \hat{f}_n + |t_n|^2 \{\hat{g}_n, \hat{f}_n \} - \hat{2})^{-1}.$$
C. Triplet Channel

The triplet part of the interaction Eq. (1.5) is given by

\[ H_{\text{int}} = J_s \vec{S}^2, \]  

(3.36)

where the total spin operator \( \vec{S} \) was defined in Eq. (1.6b). This can similarly be decoupled using a vector Hubbard-Stratonovitch field \( \hat{\Delta} \), with each component having the same Keldysh structure as in Eq. (3.6). The presence of this field leads to the replacement \( \phi \rightarrow \phi + \hat{h} \),

\[ \hat{h} \equiv \sum_i \hat{h}_i \otimes \sigma^i, \]  

(3.37)

in the self-energy Eq. (3.20). Here, \( i, j = x, y, z \), and \( \sigma^i \) are the Pauli matrices in spin space. The propagators for this field are defined as in Eq. (3.16) by replacing the \( \psi \)'s with different components of the vector field \( \hat{h} \), such that now each Keldysh component of the propagator acquires a 3 \( \times \) 3 tensor structure. In the saddle point approximation we have

\[ \hat{D}_h = \frac{J_s}{2} \left( \hat{1} + \hat{\Pi}_h \hat{D}_h \right), \]  

(3.38)

where \( \hat{1} = \hat{1}_K \delta \sigma^i \delta(t_1 - t_2) \), and the polarization operator tensors are given similarly as in Eq. (3.19) by variational derivative of the Green functions with respect to components of the field \( \hat{h} \). Repeating the steps leading to Eq. (3.34), we obtain

\[ [\Pi_h^f]_{ij}(t_1, t_2) = -\frac{2\delta_{ij}}{\pi} \delta(t_1 - t_2) - \frac{\delta \text{Tr}_s(\sigma^j g^K(t_1, t_2))}{2\delta h^z_j(t_2)}, \]  

(3.39)

where \( i, j = x, y, z \). Furthermore, in the presence of a magnetic field in the z-direction, we must add to the dot’s Hamiltonian the Zeeman energy term

\[ H_Z = E_Z \hat{z} \cdot \vec{S}, \quad (E_Z = g_L \mu_B B), \]  

(3.40)

where \( g_L \) is the Lande g-factor and \( \mu_B \) is the Bohr magneton. This causes the z-component of the field \( \hat{h} \) to acquire a non-zero average at the saddle point, given by

\[ \langle h^z_+ \rangle = \int dt_2 [\mathcal{D}_h^\dagger \Pi_h^f]_{zz}(t_1, t_2) \frac{E_Z}{2}, \]  

(3.41)

where the other components of the tensor vanish due to the symmetry of spin rotations around the z-axis. Here, the factor 1/2 arises from the same factor accompanying the Pauli matrix in the definition of the operator \( \vec{S} \), Eq. (1.6b). Separating this average from the fluctuating part, and redefining \( \hat{h} \) to stand for the latter, we obtain the modified Usadel equation:

\[ \frac{\partial \hat{g}}{\partial t} + \frac{1}{2} i E_Z^2 [\sigma_z \hat{g}] + i \hat{h} \hat{g} = \sum_{n=1}^{N, s} [\hat{g} \hat{g}_n], \]  

(3.42)

where \( E_Z^2 \) is the renormalized Zeeman energy

\[ E_Z^2 = E_Z^2 + 2 \langle h^z_+ \rangle. \]  

(3.43)

Equation (3.42) together with Eqs. (3.38)–(3.39), and the constraint (3.16) and identity Eq. (3.32) which are still valid, completely determine the kinetics of the quantum dot under the effect of spin fluctuations.

D. Cooper Channel

Interaction in the Cooper channel is given by

\[ H_{\text{int}} = J_c T^\dagger T, \]  

(3.44)

where the pairing operator \( T \) was defined in Eq. (1.6b). This interaction can be decoupled using a complex Hubbard-Stratonovitch field \( \hat{\Delta} \) which also has the structure Eq. (3.19) in Keldysh space. This pairing field gives rise to anomalous Green functions which can be taken into account by introducing the standard Gor’kov-Nambu (GN) spinor

\[ \Psi = \left( \begin{array}{c} \psi \\ i \sigma \tilde{\psi} \end{array} \right), \]  

(3.45)

where \( \sigma^y \) acts in the spin space, and we denote Pauli matrices in the GN space by \( \tau^i \). We also redefine the dot and lead Green functions \( \hat{G}_{nm} \) and \( \hat{F}_{nm} \) to be 2 \( \times \) 2 matrices in the GN, as well as in the Keldysh and spin spaces (we will use bold symbols to represent matrices in GN space), such that after averaging over the Hubbard-Stratonovitch field different Keldysh components will have the same structure as in Eq. (3.6), with \( \psi \) and \( \tilde{\psi} \) replaced by \( \tau^i \Psi \) and \( \Psi^\dagger \) respectively, and the products understood as direct products in GN as well as in spin spaces. Because of the standard convention to include \( \tau^z \) in the definition of GN Green functions, the time derivatives in the properly modified Eqs. (3.38) and (3.39) will be accompanied by this Pauli matrix [see Eq. (3.53) below]. We introduce the notation

\[ \tilde{\hat{g}} = \left( \begin{array}{c} \hat{g} \\ \tilde{\hat{F}} \end{array} \right)_N, \]  

(3.46)

to represent the GN components of the Green function \( \hat{g} \) (related to \( \hat{G}_{nm} \)'s as in Eq. (3.24)). The upper left GN component of this matrix gives the Green function used in the singlet and triplet channels, and the off-diagonal components are the anomalous Green functions. Since the components of the GN spinor are not independent
For repulsive interaction \( (J_c > 0) \) or for attractive interaction \( (J_c < 0) \) at \( T > T_c \), the field \( \Delta \) can be considered as Gaussian with propagators
\[
\langle \Delta_+ (1) \Delta_+ (2) \rangle_\Delta = \frac{i}{2} D_\Delta^R (1, 2),
\]
\[
\langle \Delta_+ (1) \Delta_- (2) \rangle_\Delta = \frac{i}{2} D_\Delta^A (1, 2),
\]
\[
\langle \Delta_- (1) \Delta_+ (2) \rangle_\Delta = \frac{i}{2} D_\Delta^A (1, 2),
\]
\[
\langle \Delta_- (1) \Delta_- (2) \rangle_\Delta = 0.
\]
In the saddle point approximation, the propagators satisfy the matrix (in Keldysh space) Dyson equation
\[
\hat{D}_\Delta = J_c \left( \hat{1} + \Pi_\Delta \hat{D}_\Delta \right),
\]
where \( \hat{1} = \hat{1}_K \delta(t_1 - t_2) \), and the polarization operators are given by
\[
\Pi_\Delta^R (t_1, t_2) = \Pi_\Delta^A (t_2, t_1) = \frac{1}{4} \frac{\delta \text{Tr}_s (\hat{F}^K (t_1, t_1))}{\delta \Delta_+ (t_2)},
\]
\[
\Pi_\Delta^K (t_1, t_2) = \frac{1}{4} \frac{\delta \text{Tr}_s (\hat{F}^K (t_1, t_1))}{\delta \Delta^- (t_2)}.
\]

Equation (3.34) together with Eqs. (3.35), (3.36), and the constraint (3.37) and identity Eq. (3.32) (which are still valid with the additional GN structure understood), constitute a complete description of the kinetics of the quantum dot under the effect of pairing fluctuations.

**IV. DERIVATION OF THE KINETIC EQUATION**

Even though the Usadel equation (3.29) gives a complete description of the kinetics of the open quantum dot, its solution for a general form of the Hubbard-Stratonovich fields is not tractable due to its non-linearity and non-locality in the time domain. The purpose of this section is to reduce the Usadel equation to the kinetic equation for all three channels of the interaction. The kinetic equation describes processes characterized by the time \( \Delta t \) which are slow. In the particular case of the quantum dots, slowness means \( \Delta t \gg 1/\omega^* \), where \( \omega^* \) is the characteristic energy transfer in the electron-electron interaction, (the scale \( \omega^* \) is determined by the shape of the distribution function \( f^R (c) \), e.g. for thermal equilibrium \( \omega^* \approx T \)). The relation between \( \Delta t \) and the relaxation scale determined by the kinetic equation itself (i.e. \( 1/\tau \) defined in Eq. (2.10) may be arbitrary (see e.g. Sec. II B of Ref. [16] for more detailed discussion).

In the first loop approximation for the interactions, the different channels are decoupled and we will treat them separately.
A. Charge Channel

1. Gauge Transformation

The condition that justifies the first loop approximation in spite of the large charging energy \( E_c \gg \delta_1 \), is the largeness of the leads’ conductance which suppresses the effect of fluctuations of the Hubbard-Stratonovich field for small frequency \( \omega \simeq 1/\tau \gg \delta_1 \). On the other hand, if the transmitted frequency is larger than \( 1/\tau \), the dot can be considered as closed. For the closed dot, the field \( \phi \) is coupled only with the total number of electrons, \( N \), which commutes with the Hamiltonian. This interaction causes the motion of the one-particle energy levels without any redistribution of particles between them, which even though produces a large, singular effect in the one-particle Green function (known as Coulomb blockade), has nothing to do with relaxation inside the dot. To eliminate the effect of the trivial motion of the levels from the very beginning, we invoke a gauge transformation (first proposed in Ref. [27]) in the Usadel equation (3.30):

\[
\hat{g}(t_1, t_2) = e^{-i\hat{K}(t_1)\tilde{\phi}}(t_1, t_2)e^{i\hat{K}(t_2)},
\]

where the matrix field

\[
\hat{K} = \begin{pmatrix} K_+ & K_- \\ K_- & K_+ \end{pmatrix},
\]

will be chosen such that almost all of the contributions of \( \phi \) from frequencies \( \omega \gtrsim 1/\tau \), are eliminated. The corrections to \( \hat{g} \) can then be found by perturbation theory.

We substitute Eqs. (3.32) and (4.1) into Eq. (3.30) and obtain

\[
\frac{\partial \hat{g}}{\partial t} + i [\hat{\phi} - \partial_t \hat{K}; \hat{g}] = T_n \left[ \hat{g}; e^{i\hat{K}}\hat{f}_n e^{-i\hat{K}} \right] \left[ 1 + \frac{T_n}{4} \left( \left[ \hat{g}; e^{i\hat{K}}\hat{f}_n e^{-i\hat{K}} \right] - 2 \right) \right]^{-1}.
\]

We will look for a perturbative solution to Eq. (4.3) in the first loop approximation; to do so it suffices to retain terms that are at most quadratic in the field \( \hat{K} \) on the right hand side. At zeroth order in \( \hat{\phi} \), the Green function has the form

\[
\hat{g} = \begin{pmatrix} \delta(t_1 - t_2) & g^K(t_1, t_2) \\ 0 & -\delta(t_1 - t_2) \end{pmatrix} K^-.
\]

so that

\[
K^- = -\mathcal{L}^g \phi_-.
\]

Here the bar indicates complex conjugation, and the 0-dimensional diffuson \( \mathcal{L}^g \) is given by

\[
\mathcal{L}^g(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \mathcal{L}^g(\omega)
\]

\[
\mathcal{L}^g(\omega) = \frac{1}{-i\omega + 1/\tau},
\]

with \( 1/\tau \) the escape rate Eq. (2.10) in units where \( \delta_1 = \pi \)

\[
\frac{1}{\tau} = \sum_n T_n = \frac{g_L + g_R}{2},
\]

and we introduced the operator notation

\[
[\mathcal{L}^g \phi_-](t_1) = \int dt_3 \mathcal{L}^g(t_1 - t_3)\phi_-(t_3).
\]

The diffuson \( \mathcal{L}^g(t) \) gives the classical probability for an electron introduced to the dot at \( t = 0 \) to remain in it at time \( t \). Note that in the present case the diffuson is much simpler than in higher dimensional systems, where it depends also on position in space and direction of momentum on the Fermi surface. The same is true for the fields \( \phi \) and \( \hat{K} \).

To solve the Keldysh component of the gauge transformed Usadel equation (4.3), we will look for \( K^+ \) in the form

\[
[\mathcal{L}^g]^{-1} K^+ = \phi^+ - 2\hat{K}^-, \quad \hat{K}^- \equiv (i\partial_t)^{-1}MK^-,
\]

where the operator \( M(t_1, t_2) \), will be chosen to simplify further expansion of \( g^K \). Substituting Eq. (4.10) into the Keldysh component of Eq. (4.3), and expanding the right hand side to second order in \( \hat{K} \) we obtain:

\[
\frac{\partial g^K}{\partial t} = \sum_n T_n \left[ f_n^+ - g^K \right]
\]

\[
+i \sum_n T_n \left[ K_+; f_n^+ - g^K \right] + iQK_- - 2i \left[ K_-; g^K \right]
\]

\[
- \sum_n T_n \left\{ [K_+; [K_+; f_n^+]]ight.\}

\[
+ (1 - T_n) \left[ K_+ f_n^+ K_- g^K - g^K f_n^+ K_+ \right] + T_n \left[ K_+ f_n^+ K_- f_n^+ K_+ - K_- f_n^+ K_+ \right] \}

Here we do not display second order terms that vanish after averaging over the field \( \phi \), e.g. \( K_- K_- \) (see Eq. (4.18)) and \( K_+(t_3)K_-(t_3) \). The latter product vanishes due to the analytic properties of the retarded propagator (see Eqs. (4.18) and (4.30)). Defining \( \delta g^K \) to be the linear
order correction to $g^K$, we find:

$$\delta g^K = \delta g^K + \delta g^K,$$

\[
\left(\frac{\partial}{\partial t} + \frac{i}{\tau} \right) \delta g^K = i \sum_n T_n \left( K^+; f_n^K - g^K \right), \quad (4.12a) \]

\[
\left(\frac{\partial}{\partial t} + \frac{i}{\tau} \right) \delta g^K = i Q K^+ - 2i \left[ \delta g^K \right], \quad (4.12b) \]

where the $K$’s are understood as operators: $K(t_1, t_2) = K(t_1)\delta(t_1 - t_2)$. The operator $Q(t_1, t_2; t_3)$ acts on $K$ as

$$[Q K_-] (t_1, t_2) = \int dt_3 \, Q(t_1, t_2; t_3) K_- (t_3), \quad (4.13)$$

and is related to products of the Keldysh Green function of the dot and the leads:

$$Q(t_1, t_2; t_3) = \frac{1}{4} \sum_n \left\{ T_n \left( f_n^K (t_1, t_3) f_n^K (t_3, t_2) + g^K (t_1, t_3) g^K (t_3, t_2) \right) \right.\]

\[
+ T_n (1 - T_n) \left[ (g^K - f_n^K) (t_1, t_3) (f_n^K - g^K) (t_3, t_2) \right] \right\}. \quad (4.14)
\]

The Keldysh Green functions are singular at coinciding times:

$$g^K (t_1, t_2)|_{t_2 \to t_1} = -\frac{2i}{\pi (t_1 - t_2)} + \text{regular}, \quad (4.15)$$

(similarly for $f_n^K$) and therefore the products in the second line of Eq. (4.12) are understood as

$$g^K (t_1, t_3) g^K (t_3, t_2) \equiv \frac{1}{2} \sum_{\sigma = \pm} g^K (t_1, t_3 + \sigma i 0) g^K (t_3 + \sigma i 0, t_2), \quad (4.16)$$

and similarly for $f_n^K \cdot f_n^K$. We will see later that the operator $Q$ has the meaning of the fluctuations of the charge going into and out of the dot. The third line in Eq. (4.12) vanishes in equilibrium and represents the non-equilibrium shot-noise, whereas the second line is present even in equilibrium and has the meaning of Nyquist noise.

As the right hand side of Eq. (4.12a) vanishes for $t_1 = t_2$, we see that $\delta g^K (t_1, t_1) = 0$. This means that $\delta g^K$ does not contribute to physical quantities of the dot itself (e.g. total charge or total energy in the dot). Furthermore, taking the limit $t_1 \to t_2$ in the right hand side of Eq. (4.12a) and using Eq. (4.11b), it is seen that the same condition can be imposed on $\delta g^K$ if we choose the operator $M$ to be

$$M (t_1, t_2) = -\frac{\pi}{4} Q(t_1, t_3; t_2). \quad (4.17)$$

We note how the structure of the $Q$ and $M$ operators is similar to that of Eq. (5.30) of Ref. 16 for the case of disordered metals, the difference being that in that reference the Green functions are characterized by different directions instead of different channels.

The next step is to average Eq. (4.11) over the Hubbard-Stratonovitch fields. For this we need the propagators for the fields $K_\pm$ defined as

$$\langle K_+ (t_1) K_+ (t_2) \rangle = \frac{i}{2} K^K (t_1, t_2),$$

$$\langle K_+ (t_1) K_- (t_2) \rangle = \frac{i}{2} K^R (t_1, t_2),$$

$$\langle K_- (t_1) K_+ (t_2) \rangle = \frac{i}{2} K^A (t_1, t_2),$$

$$\langle K_- (t_1) K_- (t_2) \rangle = 0. \quad (4.18)$$

Their relations to the propagators (5.10) and their explicit expressions will be given in subsection (4.18).

After averaging over the fluctuating fields and dropping from now on the superscript $K$ in the averaged Green functions ($g = (g^K)_\phi$), Eq. (4.11) takes the form

$$\frac{\partial \delta g}{\partial t} = \sum_n \frac{T_n}{2} (f_n - g) + \tilde{S}^\phi + \tilde{S}^\phi, \quad (4.19)$$

where the collision integrals $\tilde{S}^\phi$ and $\tilde{S}^\phi$ arise from averaging terms of first and second order in $K_\pm$ respectively. To average the second order terms (given by the last three lines of Eq. (4.11)), we replace $g^K$ with its average $g$ – keeping the deviation from average will result in higher order terms beyond our approximation– and average the $K_\pm$ pairs using the propagators (4.18), obtaining

$$\tilde{S}^\phi (t_1, t_2) = \frac{i}{8} \sum_n T_n \tilde{K}^K (t_1, t_2) f_n (t_1, t_2), \quad (4.20a)$$

$$-\frac{1}{2} T_n (1 - T_n) \int dt_3 \left( K^R_{1,3} - K^A_{1,3} \right) \left( f^R_{1,3} g_{3,2} + g_{1,3} f^R_{3,2} \right), \quad (4.20b)$$

$$- T_n^2 \int dt_3 \left( K^R_{1,3} - K^A_{1,3} \right) f^R_{1,3} f^R_{3,2}, \quad (4.20c)$$

$$\times \int dt_3 \left( f^R_{1,3} g_{3,2} - g_{1,3} f^R_{3,2} \right) \left( K^R_{1,3} + K^A_{1,3} \right), \quad (4.20d)$$

where

$$\tilde{K}^K (t_1, t_2) = 2 K^K (t_1, t_2) - K^K (t_1, t_1) - K^K (t_2, t_2). \quad (4.21)$$

As for the first order terms (given by the second line of Eq. (4.11)), we write $g^K = g + \delta g^K$, where $\delta g^K$ given by Eqs. (4.12a)–(4.12b) is the correction linear in $K_\pm$. Keeping terms of second order in $K_\pm$, we obtain

$$\tilde{S}^\phi = \tilde{S}^\phi + \tilde{S}^\phi, \quad (4.22a)$$

$$\tilde{S}^\phi = i [\delta Q] K_- - 2i \left[ \tilde{K}^\phi - \delta g^K \right], \quad (4.22b)$$

$$\tilde{S}^\phi = -\frac{i}{\tau} \left[ K^\phi + \delta g^K \right], \quad (4.22c)$$

for the case of disordered metals, the difference being that in that reference the Green functions are characterized by different directions instead of different channels.
where $\delta Q$ is the first variation of $Q$ [Eq. (4.14)] with respect to $g^K$:

$$\delta Q = \frac{\delta Q}{\delta g^K} \cdot \delta g^K. \quad (4.22d)$$

Using the solutions of Eqs. (4.12a)–(4.12b) for $\delta g^K$, one can explicitly average $\bar{S}_{\pm \pm}$ using Eq. (4.13), but as it will be shown in the next subsection the collision integral $\bar{S}_1$ does not contribute to physical quantities and we will not need its explicit form.

This formally concludes the derivation of the equation for the Keldysh Green function in the time domain. However we need to calculate explicitly the propagators introduced in Eq. (4.15); this is done in the next subsection.

2. Propagators and Collective Excitations

The propagators defined in Eq. (4.15) can be expressed in terms of the propagators $D$, Eq. (4.10), thanks to the relations between the fields $K_\pm$ and $\phi_\pm$; see Eqs. (4.6) and (4.10). We obtain

$$K^R = -L^g D^R_\phi L^g, \quad (4.23)$$
$$K^K = L^g D^K_\phi L^g + 2i \left[ L^g \partial t^{-1} M K^A - K^R M \partial t^{-1} L^g \right]. \quad (4.24)$$

Therefore we need to evaluate the $D$ propagators, given by Eq. (3.17) and Eq. (3.34). We note that the Green functions appearing in Eq. (4.24) are the ones before the gauge transformation Eq. (4.1), so that to linear order in $\phi$, the $\delta \hat{g}$'s obtained from Eqs. (4.12a)–(4.12b) have to be modified according to

$$\delta \hat{g}^K \to \delta \hat{g}^K - i[K_+; \hat{g}^K] - 2K_\phi \delta(t_1 - t_2),$$
$$\delta \hat{g}^Z \to 2K_\phi \delta(t_1 - t_2). \quad (4.25)$$

We also note that, thanks to Eq. (4.15):

$$\lim_{t_1 \to t_2} -i[K_+; \hat{g}^K] = -\frac{2}{\pi} \partial t K^+. \quad (4.26)$$

Using this result together with Eqs. (4.6)–(4.10) and the property

$$\delta \hat{g}^K(t_1, t_1) = 0, \quad (4.27)$$

from Eq. (3.17) we find

$$\Pi^K_\phi(t_1, t_2) = -\frac{2}{\pi} \left[ \delta(t_1 - t_2) - \partial t, L^g(t_1, t_2) \right], \quad (4.28)$$

or after the Fourier transformation (4.7)

$$\Pi^K_\phi(\omega) = -\frac{2}{\pi} \frac{1/\tau}{\omega + 1/\tau}. \quad (4.29)$$

The result for the Keldysh component is

$$\Pi^K_\phi = \frac{4i}{\pi} \bar{L}^g M L^g. \quad (4.30)$$

Substituting Eq. (4.29) into the retarded component of Eq. (3.17) we obtain

$$D^R_\phi(\omega) = 2E_e \frac{-i\omega + \frac{1}{\tau}}{-i\omega + \left(1 + \frac{4E_e}{\delta_1}\right)^{1/\tau}}, \quad (4.31)$$

where we restored dimensionful units (the difference being the mean level spacing appearing in the denominator instead of $\pi$). Then we can calculate the $K^R$ propagator in Eq. (4.28); it can be written in terms of the diffusion (“ghosts”) propagator $L^g$ and the following propagator $L^p$ for the collective excitations in the charge channel

$$L^p(\omega) = \frac{1}{-i\omega + \left(1 + \frac{4E_e}{\delta_1}\right)^{1/\tau}}, \quad (4.32)$$

as

$$K^R(\omega) = \frac{\delta_1}{2} \frac{1}{-i\omega} \left[ L^g(\omega) - \left(1 + \frac{4E_e}{\delta_1}\right) L^p(\omega) \right]. \quad (4.33)$$

To find $K^K$ we use the Keldysh part of Eq. (3.17):

$$D^K_\phi = D^R_\phi D^K_\phi D^A_\phi, \quad (4.34)$$

together with Eqs. (4.21) and (4.30) and obtain

$$K^K = -\frac{\delta_1}{2} \frac{1}{-i\omega} \left[ L^g N^g + N^g L^g \right]$$

$$- \left(1 + \frac{4E_e}{\delta_1}\right) \left[ L^p N^p + N^p L^p \right] \frac{1}{\partial t}, \quad (4.35)$$

written in the time domain and in the operator notation. Here the bosonic “density matrices” $N^K, \alpha = g, \rho$, are defined through

$$(L^g)^{-1} N^g + N^g (L^g)^{-1} = 2M,$$

$$(L^p)^{-1} N^p + N^p (L^p)^{-1} = 2(1 + F^p) M, \quad (4.36)$$

i.e. they are required to satisfy the kinetic equations

$$\frac{\partial}{\partial t} N^K = -2 (1 + F^K) \left(\frac{1}{\tau} N^K - M\right), \quad (4.37)$$

with

$$F^g = 0, \quad F^p = \frac{4E_e}{\delta_1}. \quad (4.38)$$

While in general we will not solve these kinetic equations, their existence is needed to obtain the conservation law for the energy. To convince the reader of the bosonic nature of the collective excitations, let us briefly consider the thermodynamic equilibrium $g(\epsilon) = f_\alpha(\epsilon) = 2 \text{tanh}(\epsilon/2T)$. In this case, after Fourier transforming Eq. (4.17) we find

$$M_{cg}(\omega) = \frac{1}{\tau} \omega \coth \frac{\omega}{2T}.$$
and introducing the distribution functions \( N^\alpha(t, \omega) \) by
\[
N^\alpha(t_1, t_2) = \int \frac{d\omega}{2\pi} e^{-i\omega(t_1-t_2)} \omega \left[ 2N^\alpha \left( \frac{t_1+t_2}{2}, \omega \right) + 1 \right]
\]
we arrive at the following solution of Eq. (4.37):
\[
N_{eq}^\alpha(\omega) = N_P(\omega) \equiv \frac{1}{e^{\omega/T} - 1},
\]
i.e. the Planck distribution.

### 3. Conservation Laws and Currents

The validity of the conservation laws in the kinetic equation approach is related to certain properties of the collision integral. For example in a closed system the charge conservation law follows from the vanishing of the collision integral in the limit \( t_2 \to t_1 \). In the present case however we are dealing with an open 0-dimensional system and therefore the conservation law for any physical quantity like the charge \( q(t) \) in the dot should have the form:
\[
\frac{\partial q}{\partial t} + I = 0,
\]
where \( I = I_L + I_R \) represents the total charge flux leaving the dot.

To obtain the charge conservation law, we rely on the following general properties of the collision integrals (4.20)-(4.22):
\[
\lim_{t_2 \to t_1} \hat{S}^{in} = 0,
\]
\[
\lim_{t_2 \to t_1} \hat{S}^+ = 0.
\]
The last property is a direct consequence of the definition (4.22b) together with Eq. (1.27). The proof of the first property can be obtained by following the steps described in Appendix E of Ref. 12. We also notice that for the stationary state [see Appendix A]:
\[
\lim_{t_2 \to t_1} \hat{S}^- = 0.
\]
The charge in the dot is given by:
\[
q(t_1) = \frac{e}{2} \lim_{t_2 \to t_1} \text{Tr}_s g(t_1, t_2),
\]
and taking the same limit of both sides of the kinetic equation (4.19) we obtain Eq. (4.41) with:
\[
I = I_L + I_R + I_-, \quad (4.45a)
\]
\[
I_\alpha = e \sum_{n=\alpha} \int \frac{d\omega}{2\pi} \left[ \frac{T}{2} (f_n(\epsilon) - g(\epsilon)) + I_n^\alpha(\epsilon) \right], \quad (4.45b)
\]
\[
I_- = e \lim_{t_2 \to t_1} \hat{S}^-, \quad (4.45c)
\]
where \( \alpha = L, R \) and
\[
I_n^\ell(\epsilon) = -\frac{i}{8} \int \frac{d\omega}{2\pi} T_n (1 - T_n) K^R(\omega) \times [f_n(\epsilon - \omega)g(\epsilon) - g(\epsilon - \omega)f_n(\epsilon)], \quad (4.45d)
\]
The contribution \( I_- \) vanishes in the steady state, thanks to Eq. (4.43), and will not be given any further consideration. We mention that the above expressions for the electric current can also be obtained by applying the gauge transformation Eq. (1.1) to the current formula Eq. (3.31).

We now turn to the energy conservation law, whose validity is based on the properties
\[
\lim_{t_2 \to t_1} \left( \frac{\partial I_1 - \partial I_2}{2} \right) \hat{S}^{in} = \partial_t u_b(t_1) + \sum_{n=1}^{N_{ch}} I_{n}^{in}(t_1), \quad (4.46a)
\]
\[
\lim_{t_2 \to t_1} \left( \frac{\partial I_1 - \partial I_2}{2} \right) \hat{S}^+ = 0, \quad (4.46b)
\]
where
\[
I_{n}^{in}(t_1) = T_n \left[ \frac{i}{4\pi} \partial_t K^R - \frac{1}{8} \int dt_3 \left[ \partial_t K^R \right]_{1,3} g_{1,3} g_{3,1} \right] - \frac{1}{16} T_n (1 - T_n) \int dt_3 \left[ \partial_t K^R \right]_{1,3} (g(f_n - g) + (f_n - g)g), \quad (4.47)
\]
or after the Wigner transform
\[
I_{n}^{in} = T_n \left[ \frac{i}{4\pi} \int d\omega \omega^2 K^R(\omega) \right. \left. - \frac{1}{8} \int d\omega \int \frac{d\omega}{2\pi} \omega \text{Im} K^R(\omega) g(\epsilon - \omega)g(\epsilon) \right] - \frac{T_n (1 - T_n)}{8} \int d\omega \omega \text{Im} K^R(\omega) (\epsilon - \omega) (f_n(\epsilon) - g(\epsilon)).
\]

As before, the last property can be proved straightforwardly, while the derivation of the first one is delineated in Appendix E of Ref. 12. In order to write it in the above form we used the kinetic equations (4.37) and defined \( u_b \) as
\[
u_b = u_\rho - u_g, \quad u_\alpha = \frac{1}{2} \mathcal{L}^\alpha N^\alpha, \quad \alpha = \rho, g. \quad (4.49)
\]
Below we will identify it as the contribution of collective excitations to the dot’s energy. The separation in Eq. (4.46a) of a energy contribution from a energy current one may seem arbitrary; however there are two independent tests of the validity of Eqs. (4.48)-(4.49). First, since the leads’ electrons are non-interacting, we can proceed similarly to the derivation of Eq. (3.31) for the electric current and find that the energy current is obtained by replacing \( e \to i(\partial_{\ell_1} - \partial_{\ell_2})/2 \) in that equation; substituting in the resulting formula Eqs. (3.32) and (4.1), expanding to second order in \( K \) and averaging over the
fluctuating field, we again arrive at the results presented below, Eqs. 4.151. Second, the definition 4.49) of the collective excitations’ energy is in agreement with the result of the thermodynamic calculation for such quantity, see Appendix E. Finally, we note that the property similar to Eq. 4.47) holds in the steady state:

$$\lim_{t_2 \to t_1} (\partial_{t_1} - \partial_{t_2}) \hat{S}_t = 0. \quad (4.50)$$

The electrons’ contribution to the dot’s energy \( u \) is

$$u_e(t_1) = -\frac{i}{4} \lim_{t_2 \to t_1} (\partial_{t_1} - \partial_{t_2}) \text{Tr}_x g(t_1, t_2), \quad (4.51)$$

and from Eq. 4.19) we arrive at

$$\partial_t u_{tot} + I^e = 0, \quad (4.52)$$

where the total energy \( u_{tot} \) is the sum of the contributions of the electrons and collective excitations:

$$u_{tot} = u_e + u_b. \quad (4.53)$$

The energy current \( I^e \) is:

$$I^e = I^e_L + I^e_R + I^e_e, \quad (4.54a)$$

$$I^e_\alpha = \sum_{n \in \alpha} \left[ \int \frac{d\epsilon}{2\pi} \frac{\epsilon}{T_n} \left( f_n(\epsilon) - g(\epsilon) \right) + I^e_{n, in} \right], \quad (4.54b)$$

$$I^e = \frac{i}{2} \lim_{t_2 \to t_1} (\partial_{t_1} - \partial_{t_2}) \hat{S}_t, \quad (4.54c)$$

where \( \alpha = L, R, I^e_{n, in} \) is defined in Eq. 4.53d) and \( I^e_{n, in} \) in Eq. 4.58. Again, since \( I^e \) vanishes in the steady state – see Eq. 4.10 – we will not discuss it anymore.

4. Final Form of the Kinetic Equation

Closing this section, we present for completeness the final form of the kinetic equation for the dot’s distribution function \( n(\epsilon) = \frac{1}{\delta} - \frac{1}{\delta} g(\epsilon) \). The collision integral is a functional of \( n(\epsilon) \), the leads’ distribution functions \( \tilde{n}_n(\epsilon) = \frac{1}{\delta} - \frac{1}{\delta} f_n(\epsilon) \) and the bosonic distribution functions \( N^\alpha(\omega) \) defined in Eq. 4.39):

$$\partial n(\epsilon) \quad (4.55)$$

$$\hat{S}_t \equiv \hat{S}_t \{ n, \tilde{n}_n, N^\alpha \}, \quad (4.55)$$

$$\hat{S}_t = \hat{S}_t \{ n, \tilde{n}_n \} + \hat{S}_t \{ n, \tilde{n}_n, N^\rho \} - \hat{S}_t \{ n, \tilde{n}_n, N^\sigma \}. \quad (4.55)$$

The last in the second line, we separated in \( \hat{S}_t \) three physically distinct contributions. The first term on the right hand side describes the relaxation of \( n \) due to tunneling into and out of the contacts – this mechanism is present even for non-interacting electrons. The two other terms describe the interaction effects. Explicitly, they are given by

$$\hat{S}_t(\epsilon) = -\sum_n \frac{T_n}{2} \left[ n(\epsilon) - \tilde{n}_n(\epsilon) \right], \quad (4.56)$$

and

$$\hat{S}_t^\alpha(\epsilon) = -\frac{\delta_1}{\delta} \left( 1 + F^\alpha \right) \sum_n \int d\omega \frac{1}{2\pi} \frac{1}{\omega} \left( \frac{T_n}{4} \left[ L^\alpha(\omega) \tilde{Y}_n^\alpha(\epsilon, \omega) + \tilde{Y}_n^\alpha(\epsilon, \omega) L^\alpha(\omega) \right] + \frac{T_n - T_n}{4} \left[ \text{Re} L^\alpha(\omega) \tilde{n}_n(\epsilon - \omega) \left( \tilde{n}_n(\epsilon) - n(\epsilon) \right) \right] \right), \quad (4.57)$$

where the propagators \( L^\alpha \), \( \alpha = g, \rho \), are defined in Eqs. 4.47) and 4.32), and we introduced the combination of distribution functions:

$$\tilde{Y}_n^\alpha(\epsilon, \omega) = \left( N^\alpha(\omega) + 1 \right) \tilde{n}_n(\epsilon) \left( 1 - \tilde{n}_n(\epsilon - \omega) \right) - N^\alpha(\omega) \left( 1 - \tilde{n}_n(\epsilon) \right) \tilde{n}_n(\epsilon - \omega). \quad (4.58)$$

The bosonic distribution functions \( N^\alpha(\omega) \), \( \alpha = g, \rho \), were defined in Eq. 4.39) and they satisfy the kinetic equation

$$\partial n^*(\omega) = \frac{1}{\omega} \sum_n \frac{T_n}{2} \int d\epsilon \left\{ \tilde{Y}_n^g(\epsilon, \omega) + \tilde{Y}_n^\rho(\epsilon, \omega) \right\} + (1 - T_n) [n(\epsilon) - n(\epsilon)] \left[ n(\epsilon - \omega) - \tilde{n}_n(\epsilon - \omega) \right] \}, \quad (4.59)$$

which follows from Eq. 4.39). Here, \( \tilde{Y}(\epsilon, \omega) \) is given by Eq. 4.55) after the replacement \( \tilde{n}_n \to n \). The combination \( \tilde{Y}(\epsilon, \omega) \) can be obtained by the standard argument for the creation and annihilation of the one-particle excitations in the dot-lead system. On the other hand, the last line in Eq. 4.57) may be understood as the renormalization of the scattering coefficients \( T_n \) of the non-interacting collision integral Eq. 4.60, due to interaction with the self-consistent potential in the dot.

B. Triplet Channel

The case of interaction in the triplet channel can be treated similarly to the singlet channel, the main difference being that the Hubbard-Stratonovich field, Eq. 3.37), is now a vector. Therefore the phase factors in the gauge transformation must also possess this structure and in Eq. 4.11 we substitute

$$\tilde{K}(t_1) \to \tilde{K}(t_1) \otimes \tilde{\sigma}. \quad (4.60)$$

This transformation does not commute with the Zeeman energy term in Eq. 4.42), after the gauge transformation and expanding up to second order in \( \tilde{K} \) this term becomes

$$i \frac{\delta \hat{S}_t}{\delta \sigma} \{ \sigma^z; \tilde{\sigma} \} = i \frac{\delta \hat{S}_t}{\delta \sigma} \{ \sigma^z; \tilde{\sigma} \} + i \frac{\delta \hat{S}_t}{\delta \sigma} \{ \sigma^z; \tilde{\sigma} \} \), \quad (4.61)$$

where we sum over the repeated index \( \alpha = x, y, z \). Note that the second order terms vanish identically.
The linear order equations for $K^a_\alpha$ can be decoupled by using the following basis for matrices in the spin space:

$$A = \frac{1}{2} \text{Tr}_s(A_1 s + A_m \sigma^m), \quad \left( A_m = \frac{1}{2} \text{Tr}_s(\sigma^{-m} A) \right),$$

$$\sigma^0 \equiv \sigma^z, \quad \sigma^{\pm 1} = \frac{1}{\sqrt{2}} (\sigma^x \pm i \sigma^y). \quad (4.62)$$

These matrices obey the following commutation relations:

$$[\sigma^0, \sigma^{\pm 1}] = \pm 2 \sigma^{\pm 1}, \quad [\sigma^{+1}, \sigma^{-1}] = 2 \sigma^0,$$  \hspace{1cm} (4.63)

and all other commutators vanish. Then the solution for $K^m_\alpha$, $m = 0, \pm 1$ is

$$K^m_\alpha = -\mathcal{L}^g_m h^m_\alpha \quad (4.64)$$

(no summation over $m$), where $\mathcal{L}^g_m$ is obtained by shifting the frequency of $\mathcal{L}^g_\alpha$ in Eq. (4.42):

$$\mathcal{L}^g_\alpha(\omega) = \mathcal{L}^g_\alpha(\omega - m E^g_Z). \quad (4.65)$$

We can similarly obtain the expressions for $K^{\alpha}_m$. In the new basis, the formula for the polarization operators $\Pi_{m,-n}$ is given by Eq. (4.38), with $(i,j)$ replaced by $(m,n)$, and we similarly obtain

$$\left[ \hat{\Pi}^R_m \right]_{mn} (\omega) = \frac{-2}{\pi} \frac{1}{\omega - m E^g_Z} + \frac{i}{\pi} \delta_{m,-n}, \quad (4.66)$$

and using Eq. (4.39)

$$\left[ D^R_m \right]_{mn} (\omega) = \frac{J_s}{2} \frac{-i (\omega - E^g_Z) + \frac{1}{\tau}}{-i \omega + (1 + \frac{1}{\delta_1}) \left( \frac{1}{\tau} + i m E^g_Z \right)} \delta_{m,-n}, \quad (4.67)$$

for the interaction propagators. Using these expressions in the zero frequency limit, together with Eqs. (4.31), (4.38) we obtain the expression Eq. (2.41) for $E^g_Z$. Finally for the retarded propagator we obtain $K^R_m = K^R_m \delta_{m,-n},$ with

$$K^R_m(\omega) = \frac{\delta_1}{2} \frac{1}{-i \omega} \left[ \mathcal{L}^g_m(\omega) - (1 + F^s) \mathcal{L}^s_m(\omega) \right], \quad (4.68)$$

where

$$\mathcal{L}^s_m(\omega) = \frac{1}{-i (\omega - m E^g_Z) + (1 + F^s) \frac{1}{\tau}}, \quad (4.69)$$

and

$$F^s = \frac{J_s}{\delta_1}. \quad (4.70)$$

Similarly, $K^{K}_m = K^{K}_m \delta_{m,-n}$ where

$$K^{K}_m = -\frac{\delta_1}{2} \frac{1}{\partial_i} \left[ \mathcal{L}^g_m N^g_{m} + N^g_m \mathcal{L}^g_m \right]$$

$$\left. - (1 + F^s) \left[ \mathcal{L}^s_m N^s_{m} + N^s_m \mathcal{L}^s_m \right] \right\} \frac{1}{\partial_i}, \quad (4.71)$$

and the density matrices $N^\alpha_m, \alpha = g, s,$ are defined similarly to Eqs. (4.36).

We could also repeat (with necessary modifications) the calculations of Sec. IV A and find the final form of the fermionic collision integral. The kinetic equation Eq. (4.36), is modified by adding the following collision integrals to the right hand side:

$$\sum_{m=0,\pm 1} \left( \tilde{\mathcal{S}}^g_{m} - \tilde{\mathcal{S}}^g_{m} \right), \quad (4.72)$$

where $\tilde{\mathcal{S}}^\alpha_{m}$ are given by Eq. (4.37), after replacing $\mathcal{L}^\alpha$ with $\mathcal{L}^g_{m}$, and $\mathcal{N}^\alpha$ with $\mathcal{N}^g_{m}$. The $\mathcal{N}^\alpha$, $\alpha = g, s,$ are distribution functions for bosons with unit spin and are defined in terms of the density matrices $N^\alpha_{m}$ as in Eq. (4.39). They satisfy the kinetic equation Eq. (4.59), after the replacement $\mathcal{N}^\alpha \rightarrow \mathcal{N}^g_{m}$. It also follows that the expressions for the curvets are obtained by replacing $K^R \to \sum_{m=0,\pm 1} K^R_m \quad (4.73)$

in Eqs. (4.40) and (4.41).

C. Cooper Channel

For the interaction in the Cooper channel, we resort to a perturbative approach which, in contrast to the charge and triplet channels, does not start with a gauge transformation. Instead, we use the constraint (3.24), to write the retarded and advanced Green functions as:

$$g^R = \left( \hat{1} - \frac{1}{2} \mathcal{F}^R, \mathcal{F}^R \right), \quad -\hat{1} + \frac{1}{2} \mathcal{F}^R, \mathcal{F}^R \right) \quad (4.74)$$

$$g^A = \left( -\hat{1} + \frac{1}{2} \mathcal{F}^A, \mathcal{F}^A \right), \quad \hat{1} - \frac{1}{2} \mathcal{F}^A, \mathcal{F}^A \right) \quad (4.75)$$

and the Keldysh anomalous Green functions as:

$$\mathcal{F}^K = -\frac{1}{2} \left( \mathcal{F}^R g^K + g^K \mathcal{F}^A \right),$$

$$\bar{\mathcal{F}}^K = -\frac{1}{2} \left( \mathcal{F}^R \bar{g}^K + \bar{g}^K \mathcal{F}^A \right), \quad (4.76)$$

where we use the notation Eq. (4.40). In the normal phase the anomalous Green functions have only fluctuating parts proportional to $\Delta$, and hence the above expressions are valid up to second order in the fluctuating field. Using these formulas together with Eq. (4.32), and defining

$$\hat{\mathcal{S}}_{m} = -i [\Delta : \hat{g}], \quad \hat{\mathcal{S}}_{m} = \sum_{n=1}^{N_B} [\hat{g} : \hat{g}_n], \quad (4.77)$$

we can write the upper left GN component of the Keldysh part of the collision integral $\mathcal{S}_{m}$ as

$$\left( \hat{\mathcal{S}}_{1} \right)_{11} = \sum_{n=1}^{N_B} \left[ T_n (\bar{g}^K - g^K) + \hat{\mathcal{S}}_{11}^{MT} + \hat{\mathcal{S}}_{11}^{MT} \right] \quad (4.78)$$
where we have identified the Maki-Thompson and elastic parts as
\[
\widehat{S}_n^{MT} = \frac{1}{8} T_n^2 R f_n^K - \bar{g}^K \widehat{F}^A
\] (4.79)
\[
\widehat{S}_n^{el} = \frac{1}{8} T_n (T_n - 1) \left[ \mathcal{F}^R \mathcal{F}^R (f_n^K - \bar{g}^K) + (f_n^K - \bar{g}^K) \mathcal{F}^A \mathcal{F}^A \right].
\] (4.80)

We note that there is no term analogous to the Aslamazov-Larkin contribution – this is due to the independence of the field \( \Delta \) from the spatial coordinate. Similarly, the substitution of Eqs. (4.4) - (4.7) into \( \tilde{S}_1 \) gives:
\[
(\tilde{S}_1)^{K}_{11} = -i \Delta_+ \mathcal{F}^K - i \mathcal{F}^K \Delta_+ - i \Delta_- \mathcal{F}^A - i \mathcal{F}^R \Delta_+^*
\]
\[
= -\frac{i}{2} \Delta_- \mathcal{F}^R \mathcal{F}^R - \frac{i}{2} \Delta_+ \mathcal{F}^K - \frac{i}{2} \Delta_+ \mathcal{F}^A + \frac{i}{2} \mathcal{F}^R \mathcal{F}^K \Delta_+^*
\]
\[
+ \frac{i}{2} \mathcal{F}^A \mathcal{F} A \Delta_+^* - i \Delta_- \mathcal{F}^A - i \mathcal{F}^R \Delta_+^*
\] (4.81)

In order to average over the fluctuating field \( \Delta \), we need to solve the equations of motion for the \( \mathcal{F} \)'s. Using Eq. (4.89) for the \( \mathcal{F}^R \) component, we obtain
\[
\left( \partial_t - \partial_{t_2} + \frac{1}{\tau_e} \right) \mathcal{F}^R = \frac{i}{2} E Z \{ \sigma^z ; \mathcal{F}^R \} = \frac{2i \Delta_+ \delta(t_1 - t_2) + i g K \Delta_+}{\tau_e}
\] (4.82)

where
\[
\frac{1}{\tau_e} = \frac{1}{\tau} + \frac{g_h}{2},
\] (4.83)

and \( g_h \) was defined in Eq. (1.2). This modification of the escape rate is due to the last term on the left hand side of Eq. (3.83), which describes the breaking of time reversal invariance for the anomalous Green functions by the orbital magnetic field. It is clear from this equation that the \( \mathcal{F} \)'s are diagonal in spin space, and if we decompose them according to
\[
\mathcal{F} = \mathcal{F}_+ \frac{1}{2} + \sigma^z \mathcal{F}_- \frac{1}{2} + \sigma^z \mathcal{F}_- \frac{1}{2},
\] (4.84)

the solution to Eq. (4.82) can be expressed in terms of the cooperon \( C \) as:
\[
\mathcal{F}^R(t_1 - t_2) = i C^R \left( \frac{t_1 - t_2}{2} \right) \Delta_+ \left( \frac{t_1 + t_2}{2} \right)
\]
\[
+ i \int dt_3 C^R \left( \frac{t_1 - t_2}{2} - t_3 \right) \Delta_- \left( \frac{t_1 + t_2}{2} - t_3 \right) g^K (2t_3),
\] (4.85)

where
\[
C^R(t) = \int \frac{d\epsilon}{2\pi} \frac{e^{-i\epsilon t}}{\epsilon + \epsilon^2 + 1}.
\] (4.86)

The \( \mathcal{F}^R \) is obtained similarly and is given by Eq. (4.85), after replacing \( \Delta_+ \) by \( \mp \Delta_+ \), and \( g^K \) by \( g^K \bar{g} \). For the advanced components we can use Eq. (3.47) to write:
\[
\mathcal{F}^A(t_1 - t_2) = \mathcal{F}^R(t_2 - t_1);
\] (4.87)

the same relation holds for the \( \mathcal{F} \)'s. From Eq. (3.47) we also have
\[
g^K(t_1, t_2) = g^K(t_2, t_1),
\] (4.88)

and similarly for \( f_n \) and \( f_n \) (here and for the rest of this section we drop the superscript \( K \) from the dot and leads Keldysh Green functions).

Using these solutions for \( \mathcal{F} \)'s and the propagators \( \mathcal{C} \) to average over the fluctuating fields, we arrive at the following expressions for the singlet part of the Fourier transformed collision integral \( \tilde{S}_2 \):
\[
\frac{1}{2} \text{Tr}_s (\tilde{S}_n^{MT} (\epsilon)) = \frac{1}{8} T_n^2 \int \frac{d\omega}{2\pi} (f_n (\omega - \epsilon) - g (\omega - \epsilon))
\]
\[
\times \sum_{m = \pm 1} |C_m|^2 (2\epsilon - \omega) J(\omega, \epsilon) |D^R_m(\omega)|^2,\] (4.89)

\[
\frac{1}{2} \text{Tr}_s (\tilde{S}_n^{el} (\epsilon)) = \frac{1}{8} T_n (1 - T_n) \int \frac{d\omega}{2\pi} (f_n (\epsilon) - g (\epsilon))
\]
\[
\times \left\{ -2 \sum_{m = \pm 1} \text{Re} C_m^2 (2\epsilon - \omega) J(\omega, \epsilon) |D_m^R(\omega)|^2ight.
\]
\[
+ g(\omega - \epsilon) \sum_{m = \pm 1} \text{Im} \left[ C_m^d (2\epsilon - 2\epsilon) D_m^R(\omega) \right],\] (4.90)

with the kernel \( J(\epsilon, \omega) \) given by
\[
J(\epsilon, \omega) = i \Pi^K_\omega(\omega) + g(\epsilon) \text{Im} \Pi^R(\omega).
\] (4.91)

To write the collision integral in this form we used the identity
\[
[i \Pi^K_\omega(\omega) + g(\epsilon) \text{Im} \Pi^R(\omega)] |D^R_\omega(\omega)|^2
\]
\[
= i D^R_\omega(\omega) + g(\epsilon) \text{Im} D^R_\omega(\omega),
\] (4.92)

which follows from the Dyson equation (3.55) for the propagators. At \( \omega \) much larger than \( T \) or applied voltage \( eV \), \( i D^R_\omega(\omega) = -2 \text{sgn}(\omega) \text{Im} D^R_\omega(\omega) \), and the contribution from such large \( \omega \)’s in Eq. (4.89) vanishes. This is a direct manifestation of the inelastic nature of Maki-Thompson processes.

By comparing Eqs. (4.31) and (4.78) with the definition of \( \tilde{S}_2 \) [Eq. (4.77)] we find that the current in each channel is given by
\[
I_n = \frac{e}{2} \text{Tr}_s \int \frac{d\epsilon}{2\pi} \left[ \frac{T_n}{2} (f_n - g) + \tilde{S}_n^{el} + \tilde{S}_n^{MT} \right],
\] (4.93)

with the expressions (4.39) - (4.40) for the collision integrals. We will use this result in Sec. V C to calculate the corresponding interaction correction to the conductance.

By repeating the above steps for the collision integral \( \tilde{S}_1 \) we find:
\[
\frac{1}{2} \text{Tr}_s (\tilde{S}_1 (\epsilon)) = \frac{1}{2} \sum_{m = \pm 1} \int \frac{d\omega}{2\pi} \text{Re} (\mathcal{C}_m (2\epsilon - \omega)) K(\epsilon, \omega),
\] (4.94)
where the kernel $K(\epsilon, \omega)$ is
\[
K(\epsilon, \omega) = i \Delta^2(\omega) \left[ g(\epsilon) + g(\omega - \epsilon) \right] + \text{Im} \Delta^2(\omega) \left[ g(\epsilon)g(\omega - \epsilon) + 4 \right]. \tag{4.95}
\]

Having derived the collision integrals, we now find the explicit form of the propagators Eq. (3.54) and polarization operators Eq. (8.35). Using Eqs. (4.76) and (4.83) we obtain
\[
\Pi^R_\Delta(\omega) = -\frac{i}{2} \int \frac{d\epsilon}{2\pi} \sum_{m=\pm 1} C_m(2\epsilon)g\left(\frac{\omega}{2} - \epsilon\right), \tag{4.96a}
\]
\[
\Pi^A_\Delta(\omega) = \frac{i}{2} \int \frac{d\epsilon}{2\pi} \sum_{m=\pm 1} C_m(2\epsilon)g\left(\frac{\omega}{2} + \epsilon\right), \tag{4.96b}
\]
\[
\Pi^K_\Delta(\omega) = -\frac{i}{3} \int \frac{d\epsilon}{2\pi} \sum_{m=\pm 1} C_m(2\epsilon) \left[ g\left(\frac{\omega}{2} - \epsilon\right)g\left(\frac{\omega}{2} + \epsilon\right) + 4 \right]. \tag{4.97}
\]

In Eq. (4.97), the last term in the brackets does not follow from the Usadel equation. It arises to compensate for the incorrect order of summation over orbital states $m$ and integration over $\epsilon$ in Eq. (4.35), similarly to the anomaly in charge channel [cf. the discussion after Eq. (8.34)]. This ultraviolet factor can be obtained e.g. by explicit calculation in the original model. A more compact way, however, is to restore the correct term by requiring the fluctuation-dissipation theorem:
\[
\Pi^K_\Delta(\omega) = \coth\left(\frac{\omega}{2T}\right) \left[ \Pi^A_\Delta(\omega) - \Pi^R_\Delta(\omega) \right], \tag{4.98}
\]

to hold for the equilibrium “distribution function” $g(\epsilon) = 2 \tanh(\epsilon/2T)$. Furthermore the logarithmic divergence in Eq. (4.96) is cut off by $|\epsilon| \lesssim E_T$ as it determines the validity of RMT.

In equilibrium, with $g(\epsilon)$ given by the right hand side of Eq. (4.35) at temperature $T$ and $\mu = 0$, we can calculate $\Pi^K_\Delta$ explicitly by closing the integration contour in the upper half plane, obtaining a sum over the residues at the poles of $\tanh(\epsilon/2T)$. The result is
\[
\Pi^K_\Delta(\omega) = \sum_{m=\pm 1} \sum_{n=0}^{N} \frac{1}{n + \frac{1}{2} + z_m}, \tag{4.99}
\]
where
\[
z_m = -\frac{i(\omega - mE_T)}{4\pi T} + \frac{1}{\omega}. \tag{4.100}
\]
The upper cutoff $N$ is approximately given by $E_T/2\pi T$. We note that the coupling constant $J_c$ is also defined at the scale of $E_T$. We can express this sum in terms of the digamma function $\psi^{(0)}(x)$
\[
-2\pi\Pi^K_R(\omega) = \psi^{(0)}\left(\frac{1}{2} + \frac{E_T}{2\pi T} + z_m\right) - \psi^{(0)}\left(\frac{1}{2} + z_m\right) \approx \ln \left(\frac{E_T}{2\pi T}\right) - \psi^{(0)}\left(\frac{1}{2} + z_m\right). \tag{4.101}
\]

Defining the “critical temperature”
\[
\gamma_T \equiv \frac{2\gamma}{\pi} e^{\frac{\pi J_c^{-1}}{4}}, \tag{4.102}
\]
we can rewrite $\Pi^K_R$ as
\[
2\pi\Pi^K_R(\omega) = \pi J_c^{-1} + t + \psi^{(0)}\left(\frac{1}{2} + z_m\right) - \psi^{(0)}\left(\frac{1}{2}\right), \tag{4.103}
\]
where $t$ is the reduced temperature
\[
t \equiv \ln \frac{T}{T_c}. \tag{4.104}
\]

Therefore, in equilibrium we find
\[
D^K_{eq}(\omega) = -\frac{\pi}{t + \frac{1}{2} \sum_{m=\pm 1} \psi^{(0)}\left(\frac{1}{2} + z_m\right) - \psi^{(0)}\left(\frac{1}{2}\right)} \tag{4.105}
\]
Furthermore, in equilibrium we have the fluctuation-dissipation relation:
\[
D^K_{eq}(\omega) = \coth\left(\frac{\omega}{2T}\right) \left[ D^K_R(\omega) - D^K_{eq}(\omega) \right], \tag{4.106}
\]
as can be verified using Eqs. (4.105), (4.103) and (4.104). We can define the bosonic distribution function of the fluctuating cooper pairs by generalizing this relation to the non-equilibrium case:
\[
D^K_R(\omega) = (2N(\omega) + 1) \left[ D^K_R(\omega) - D^K_{eq}(\omega) \right], \tag{4.107}
\]
such that in equilibrium $N(\omega)$ is given by the Planck distribution Eq. (4.109).

The polarization operators must be recalculated in the non-equilibrium case, as they depend on the distribution functions of the electrons. We do this for the DC case in Sec. V C.

In closing this section we give the final form of the kinetic equation for interaction in the Cooper channel. Introducing $n(\epsilon) = \frac{1}{2} - \frac{1}{2} g(\epsilon)$ and $\bar{n}(\epsilon) = \frac{1}{2} - \frac{1}{2} f(\epsilon)$ for distribution functions of the electrons in the dot and the leads respectively, and using Eqs. (4.83), (4.84) and (4.108) we arrive at
\[
\frac{\partial n(\epsilon)}{\partial t} = -\sum_n \frac{T_n}{2} \left[ n(\epsilon) - \bar{n}(\epsilon) \right] + \tilde{S}_{el}\{n, \bar{n}, N_c\} + \tilde{S}_{MT}\{n, \bar{n}, N_c\} + \tilde{S}_1\{n, \bar{n}, N_c\}. \tag{4.108}
\]

The collision integrals are given by
\[
\tilde{S}_{el} = -\sum_n T_n (1 - T_n) \int \frac{d\omega}{2\pi} \mathcal{M}(\epsilon, \omega), \tag{4.109}
\]
where
\[
\mathcal{M}(\epsilon, \omega) = [N_c(\omega) + n(\epsilon)] \sum_{m=\pm 1} \text{Re} C^2_m(2\epsilon - \omega) \text{Im} \Delta^2(\omega)
\]
\[
-\frac{1}{2} n(\omega - \epsilon) \sum_{m=\pm 1} \text{Im} \left[ C^2_m(\omega - 2\epsilon) \Delta^2(\omega) \right]. \tag{4.110}
\]
and
\[
\hat{S}_{1MT} = \sum_n \frac{T_n^2}{2} \int \frac{d\omega}{2\pi} \sum_{m=\pm 1} |C_{m}^{\prime 2}(2\epsilon - \omega)| \text{Im}\mathcal{D}_{\Delta}^R(\omega) \\
\times \left[ \tilde{\Psi}_n(\epsilon, \omega) - \Psi(\epsilon, \omega) \right], \quad (4.111)
\]
\[
\hat{S}_{1} = -2 \int \frac{d\omega}{2\pi} \sum_{m=\pm 1} \text{Re} \mathcal{C}_m(2\epsilon - \omega) \text{Im}\mathcal{D}_{\Delta}^R(\omega) \Psi(\epsilon, \omega). \quad (4.112)
\]
Here we defined
\[
\Psi(\epsilon, \omega) = - (N^c(\omega) + 1) n(\epsilon)n(\omega - \epsilon) \\
+ N^c(\omega)(1 - n(\epsilon))(1 - n(\omega - \epsilon)), \quad (4.113)
\]
and \(\Psi_n\) is obtained by replacing \(n(\omega - \epsilon)\) with \(\tilde{n}_n(\omega - \epsilon)\) in Eq. (4.113). The bosonic distribution function \(N^c(\omega)\) is in turn given in terms of the fermionic distribution function \(n(\epsilon)\) through
\[
2N^c(\omega) + 1 = \frac{\Pi_{\Delta}^R(\omega)}{\Pi_{\Delta}^L(\omega) - \Pi_{\Delta}^R(\omega)}, \quad (4.114)
\]
and the expressions (4.90)–4.97 for the polarization operators.

V. CALCULATIONS OF THE TRANSPORT COEFFICIENTS

In this section we explicitly calculate the interaction corrections to the electrical and thermal conductances by solving the kinetic equations. We first consider the singlet and triplet channel corrections to the electrical conductance and then to the thermal conductance. Finally, we calculate the Cooper channel correction to the electrical conductance.

A. Electrical conductance: singlet and triplet channel

In the presence of a bias voltage, the energy levels in the leads are shifted by the voltage times the electron’s charge, so that the nonequilibrium “distribution functions” in the leads, Eq. (3.8), become
\[
f_{R(L)}(\epsilon) = 2 \text{tanh} \left( \frac{\epsilon - eV_{R(L)}}{2T} \right), \quad (5.1)
\]
with \(V_{R(L)} = \pm V/2\). For the future use we introduce
\[
\Delta f(\epsilon) \equiv f_L(\epsilon) - f_R(\epsilon). \quad (5.2)
\]
In the steady state we have \(I_L = -I_R \equiv I\), so we can write
\[
I = xI_L + (x - 1)I_R, \quad (5.3)
\]
for any constant \(x\). In terms of the current in each channel
\[
I = \sum_{n=1}^{N_{ch}} \Lambda_n I_n, \quad (5.4)
\]
where we choose the constants to be
\[
\Lambda_n \equiv \begin{cases} \frac{g_L}{g_L + g_R} 1 \leq n \leq N_L, \\
- \frac{g_L}{g_L + g_R} N_L + 1 \leq n \leq N_{ch}, \end{cases} \quad (5.5)
\]
with \(g_L, g_R\) defined in Eq. (4.45d). This choice will simplify calculations as we make use of the identity \(\sum_n \Lambda_n T_n = 0\).

Let us calculate the singlet channel correction first. Using Eq. (4.45b) we have
\[
I = e \sum_{n=1}^{N_{ch}} \Lambda_n \int \frac{d\epsilon}{2\pi} \left[ \frac{T_n}{2} (f_n(\epsilon) - g(\epsilon)) + I_n^l(\epsilon) \right], \quad (5.6)
\]
with \(I_n^l\) given in Eq. (4.45d). By shifting the integration variable \(\epsilon\) by \(\omega\) in the first term in square brackets in Eq. (4.45d), we can rewrite \(I\) as:
\[
I = e \int \frac{d\epsilon}{2\pi} \sum_{n=1}^{N_{ch}} \Lambda_n \left\{ \frac{T_n}{2} (f_n(\epsilon) - g(\epsilon)) \\
+ \frac{1}{4} T_n (1 - T_n) f_n(\epsilon) K(\omega) g(\epsilon - \omega) \right\}, \quad (5.7)
\]
where
\[
K(\omega) \equiv -\text{Im} K^R(\omega) = \text{Im} \left[ (\mathcal{L}^g)^2(\omega) \mathcal{D}^R(\omega) \right], \quad (5.8)
\]
and we used Eq. (4.29) in the last identity; in this form we can recognize that our expression for the current has the structure similar to the one found in Ref. [8] for higher dimensional systems. However, for practical purposes, we will use the following identity:
\[
K(\omega) = \frac{\pi T}{2} \text{Im} [\mathcal{L}^g(\omega) - \mathcal{L}^o(\omega)], \quad (5.9)
\]
where \(\mathcal{L}^g\) and \(\mathcal{L}^o\) were defined in Eqs. (4.7) and (4.32) respectively.

We note that since \(\sum_n \Lambda_n T_n g = 0\), the interaction correction to \(g\) does not enter the expression for the current to first order in the interaction propagator. So for the purpose of calculating the DC current, we can calculate \(g\) to zeroth order by equating to zero the first term on the right hand side of Eq. (4.19) and we find:
\[
g(\epsilon) = \frac{g_L f_L(\epsilon) + g_R f_R(\epsilon)}{g_L + g_R}. \quad (5.10)
\]
with \(g_L, g_R\) defined in Eq. (4.45d).

We write the differential conductance as
\[
G = \frac{dI}{dV} = G_0 + \Delta G, \quad (5.11)
\]
where $G_0$ and $\Delta G$ are the classical conductance and the interaction correction, originating from the first and second term in curly brackets in Eq. (5.1), respectively. Substituting Eq. (5.10) into Eq. (5.9) and using the definition (5.11) we obtain

$$G_0 = \frac{e^2}{2} \frac{g_L g_R}{g_L + g_R} \frac{\partial}{\partial \tau} \int \frac{d\epsilon}{2\pi} \Delta f(\epsilon) = \frac{e^2}{\pi} \frac{g_L g_R}{g_L + g_R}, \quad (5.12)$$

and

$$\Delta G = \frac{e^2}{2\pi} \frac{h_L h_R e^2}{(g_L + g_R)^2} \times \int \frac{d\epsilon}{2\pi} \int \frac{d\omega}{2\pi} \Delta f(\epsilon) K(\omega) g(\epsilon - \omega), \quad (5.13)$$

where the form factors $h_L$ and $h_R$ are defined in Eq. (5.4), and

$$\tilde{V} \equiv eV.$$

We define the shorthand notation

$$aKb \equiv \int \frac{d\epsilon}{2\pi} \int \frac{d\omega}{2\pi} a(\epsilon) K(\omega) b(\epsilon - \omega), \quad (5.14)$$

and notice that since $K$ is an odd function of $\omega$, we can further simplify Eq. (5.13) into

$$\Delta G = \frac{e^2}{\pi} \frac{h_L h_R}{(g_L + g_R)^2} \frac{\pi}{4} f_R^' K f_L \quad (5.15)$$

where here and below a prime denotes derivation with respect to the energy variable. The integration over $\epsilon$ can be readily performed and after a partial integration over $\omega$ we arrive at

$$\frac{\pi}{4} f_R^' K f_L = 2T \int \frac{d\omega}{2\pi} K'(\omega) \frac{\omega - \tilde{V}}{2T} \coth \frac{\omega - \tilde{V}}{2T}. \quad (5.16)$$

Using the identity (5.10) together with the distributions Eqs. (2.27) and (2.28), we integrate over $\omega$ by closing the contour in the upper half plane, obtaining a sum over the residues of the poles of $\coth (\omega/2T)$, which is expressible in terms of polygamma functions. The result is:

$$\frac{\pi}{4} f_R^' K f_L = \frac{\tau}{2} \text{Re} \left[ \Psi \left( \frac{1 - i\tilde{V}\tau}{2\pi\tau T} \right) - \Psi \left( \frac{1 + 4e\epsilon}{8\tau} - i\tilde{V}\tau \right) \right], \quad (5.17)$$

where

$$\Psi(z) \equiv z\psi^{(1)}(z) + \psi^{(0)}(z). \quad (5.18)$$

and $\psi^{(i)}(z)$ is the $i$-th derivative of the digamma function. Substitution into Eq. (5.10) yields Eq. (2.30) for the singlet channel correction to the electrical conductance (where we have also restored dimensionless units).

As for the triplet channel, the result is found by using the replacements $\epsilon \rightarrow 1/\tau + imE_2^*$ and $\epsilon \rightarrow \epsilon + 1/\tau + imE_2^*$ in Eqs. (5.19) and (5.12). Then Eq. (5.19) becomes:

$$K(\omega) \rightarrow \sum_{m=0, \pm 1} \frac{1}{2} \frac{1}{1/\tau + imE_2^*} \text{Im} \left[ \mathcal{L}_m^s(\omega) - \mathcal{L}_m^s(\omega) \right] \quad (5.19)$$

and repeating the above steps we obtain Eq. (2.30) for the triplet channel correction $\Delta G_s$.

B. Thermal conductance

The thermal conductance is found by assuming that there is a temperature difference $\delta T$ between the right and left leads, so that the distribution functions, cf. Eq. (5.9), are given by

$$f_R(L)(\epsilon) = 2 \tanh \left( \frac{\epsilon}{2T \pm \delta T} \right), \quad (5.20)$$

where $+ (-)$ signs corresponds to right (left) lead. At lowest order, $g$ is still given by the relation (5.10).

We write the linear thermal conductance as

$$\kappa \equiv \frac{I^e}{\delta T} = \kappa_0 + \Delta \kappa_{el} + \Delta \kappa_{in}. \quad (5.21)$$

where $\kappa_0$ is the classical thermal conductance and $\Delta \kappa_{el}$ and $\Delta \kappa_{in}$ are the contributions of the elastic and inelastic processes described respectively by the collision integrals in Eqs. (4.20c) and (4.20b). The calculation is similar to the one in the previous section: we can write the steady state energy current as $\sum_n \Lambda_n I_n^e$ and from Eqs. (4.48) and (4.49) we obtain

$$I^e = I_0^e + I_{cl}^e + I_{in}^e \quad (5.22a)$$

and

$$I_0^e = \frac{1}{2} \sum_{n=1}^{N_e} \Lambda_n T_n \int \frac{d\epsilon}{2\pi} \epsilon (f_n(\epsilon) - g(\epsilon)), \quad (5.22b)$$

$$I_{cl}^e = -\frac{i}{8} \sum_{n=1}^{N_e} \Lambda_n T_n (1 - T_n) \int \frac{d\epsilon}{2\pi} \int \frac{d\omega}{2\pi} K^R(\omega) \times \epsilon [f_n(\epsilon - \omega) g(\epsilon) - g(\epsilon) f_n(\epsilon - \omega)], \quad (5.22c)$$

$$I_{in}^e = -\frac{1}{16} \sum_{n=1}^{N_{ch}} \Lambda_n T_n (1 - T_n) \int \frac{d\epsilon}{2\pi} \int \frac{d\omega}{2\pi} \omega K^R(\omega) \times [g(\epsilon - \omega) (f_n - g) (\epsilon - \omega) f_n(\epsilon) + (f_n - g)(\epsilon - \omega) g(\epsilon)], \quad (5.22d)$$

We note that the part of $I_{in}^e$ proportional to $T_n$’s [see Eq. (4.48)], vanishes in the DC case after summing over the channels due to the identity $\sum_n \Lambda_n T_n = 0$, and was therefore omitted here. Summing over the channels explicitly and expanding the distribution functions to linear order in $\delta T$, we find

$$\kappa_0 = \frac{\pi G_0}{2e^2} \int \frac{d\epsilon}{2\pi} \frac{\epsilon^2}{2T} \frac{\partial f_0}{\partial \epsilon} = \frac{\pi^2}{3e^2} T G_0, \quad (5.23)$$
in agreement with the Wiedemann-Franz law, and

\[ \Delta \kappa_{el} = \frac{A}{4T} \int \frac{d\omega}{2\pi} \int \frac{d\epsilon}{2\pi} \mathcal{I}_-(\omega, \epsilon) \epsilon K_R^R(\omega), \]  
(5.24a)

\[ \Delta \kappa_{in} = -\frac{A}{8\pi} \int \frac{d\omega}{2\pi} \int \frac{d\epsilon}{2\pi} \mathcal{I}_+(\omega, \epsilon) \omega K_R^R(\omega), \]  
(5.24b)

where

\[ A = \frac{h_L g_L^2 + h_R g_R^2}{(g_L + g_R)^2}. \]  
(5.25)

is the same form factor that appears in equation (5.13), and \( \mathcal{I}_\pm \) are the following combinations of distribution functions:

\[ \mathcal{I}_\pm(\omega, \epsilon) = \frac{1}{\delta T} [f_0(\epsilon) \Delta f(\epsilon - \omega) \pm f_0(\epsilon - \omega) \Delta f(\epsilon)] . \]  
(5.26)

Note that at linear order

\[ \Delta f(\epsilon) \approx \frac{\delta T}{2T} \epsilon \frac{\partial f_0}{\partial \epsilon}. \]  
(5.27)

where \( f_0 \) is the equilibrium Green function given in Eq. (5.8) with temperature \( T \) and \( \mu = 0 \).

In Eqs. (5.24), we perform the integration over \( \epsilon \):

\[ \int \frac{d\epsilon}{2\pi} \mathcal{I}_-(\omega, \epsilon) = \frac{4}{2\pi T} \frac{\omega^3}{6} \frac{\partial}{\partial \omega} \coth \frac{\omega}{2T} \]  
(5.28a)

\[ + \frac{8T}{3} \frac{\epsilon}{\omega^2} \frac{\partial}{\partial \omega} \left( \omega \coth \frac{\omega}{2T} \right), \]  
(5.28b)

\[ \int \frac{d\epsilon}{2\pi} \mathcal{I}_+(\omega, \epsilon) = -\frac{4}{2\pi T} \frac{\omega^3}{3} \frac{\partial}{\partial \omega} \coth \frac{\omega}{2T}. \]  
(5.28c)

After substitution in Eq. (5.24), the last term in the right hand side of Eq. (5.28a) gives a term proportional \((after integrating \( \omega \) by parts)\) to the integral Eq. (5.10) appearing in the calculation of the electrical conductance, and so contributes to \( \Delta \kappa_{el} \) a term that obeys the Wiedemann-Franz law; the first term, on the other hand, gives a contribution proportional to the inelastic correction:

\[ \Delta \kappa_{el} = \frac{\pi^2}{3e^2} T \Delta G_e - \frac{A}{3} J_{in}, \]  
(5.29a)

\[ \Delta \kappa_{in} = -AJ_{in}, \]  
(5.29b)

where

\[ J_{in} = -\frac{1}{4\pi T} \int \frac{d\omega}{2\pi i} \frac{\epsilon^3}{\omega^3} \coth \frac{\omega}{2T} K_R^R(\omega). \]  
(5.30)

In this expression, the final integration over \( \omega \) can be done similar to Eq. (5.10) by closing the contour in the upper half plane, and the result can be expressed in terms of polygamma functions:

\[ J_{in} = \frac{\pi T}{12} \left[ g_1 \left( 2\pi T \right) - g_1 \left( \frac{2\pi T}{1 + \frac{4e^R}{3t}} \right) \right]. \]  
(5.31)

where

\[ g_1(x) = \frac{6}{x^3} \psi^{(1)} \left( \frac{1}{x} \right) - \frac{6}{x^2} - \frac{3}{x}. \]  
(5.32)

Equation (5.24) together with Eqs. (5.29) gives the singlet channel part of the result for the thermal conductance reported in Sec. 4.1. Evaluation of the triplet channel contribution is straightforward (as explained at the end of the previous subsection) and leads to Eq. (5.22c).

C. Electrical conductance: Cooper channel

The evaluation of the Cooper channel correction to the electrical conductance is based on the current formula (4.93). Writing the DC current as in Eq. (5.4) and summing over \( n \), the contribution of the first term in square brackets in Eq. (4.93) gives the classical conductance \( G_0 \), and the contribution of the other terms gives the Cooper channel correction which can be written as:

\[ \Delta G_{Cooper} = \Delta G_{el} + \Delta G_{MT}, \]  
(5.33)

where

\[ \Delta G_{el} = \frac{e^2}{\pi} A \left[ J_{DoS_1} + J_{DoS_2} \right], \]  
(5.34)

\[ \Delta G_{MT} = \frac{e^2}{\pi} BJ_{MT}, \]  
(5.35)

with the form factor \( A \) defined in Eqs. (5.29) and

\[ B = \frac{g_L g_R}{(g_L + g_R)^2} - A, \]  
(5.36)

which are characteristic of elastic and inelastic contributions respectively, and we introduced

\[ J_{MT} = \frac{2}{\pi} \frac{\partial}{\partial V} \int \frac{d\omega}{2\pi} \int \frac{d\epsilon}{2\pi} \sum_{m=\pm 1} |C_m|^2(2\epsilon) \]  
(5.37)

\[ \times \Delta f \left( \frac{\omega}{2} + \epsilon \right) \mathcal{J} \left( \frac{\omega}{2} - \epsilon, \omega \right) |D_R^{R}(\omega)|, \]

\[ J_{DoS_1} = -\frac{2}{\pi} \frac{\partial}{\partial V} \int \frac{d\omega}{2\pi} \int \frac{d\epsilon}{2\pi} \sum_{m=\pm 1} 2ReC_m^2(2\epsilon) \]  
(5.38)

\[ \times \Delta f \left( \frac{\omega}{2} + \epsilon \right) \mathcal{J} \left( \frac{\omega}{2} - \epsilon, \omega \right) |D_R^{R}(\omega)|, \]

\[ J_{DoS_2} = \frac{2}{\pi} \frac{\partial}{\partial V} \int \frac{d\omega}{2\pi} \int \frac{d\epsilon}{2\pi} \Delta f \left( \frac{\omega}{2} - \epsilon \right) \]  
(5.39)

\[ \times g \left( \frac{\omega}{2} + \epsilon \right) \text{Im} \left[ \sum_{m=\pm 1} \mathcal{C}_m^2(2\epsilon) D_R^{R}(\omega) \right]. \]

The kernel \( \mathcal{J}(\epsilon, \omega) \) and the function \( \Delta f(\epsilon) \) are defined in Eqs. (4.91) and (5.2) respectively.
To proceed further, we need the explicit form of the polarization operators in the presence of the bias voltage, as they determine the kernel $J(\epsilon, \omega)$. In this case to zeroth order in interaction we have [see Eq. (5.10)]:

$$\frac{1}{2} g K(\epsilon) = \sum_{\alpha = \pm 1} g_\alpha \tanh \left( \frac{\epsilon - \alpha V/2}{2T} \right),$$

(5.40)

where $g_{\pm 1}$ are defined as

$$g_{+1} \equiv \frac{g_R}{g_L + g_R}, \quad g_{-1} \equiv \frac{g_L}{g_L + g_R}.$$

(5.41)

Using the formula (obtained similarly to Eq. (4.100))

$$\int d\epsilon \text{Re} C_m(2\epsilon) \tanh \left( \frac{\epsilon - \omega/2}{2T} \right) = \text{Im} \psi^{(0)} \left( \frac{1}{2} + z_m \right),$$

(5.42)

with $z_m$ given in Eq. (4.100), and the definition

$$\chi^{m,\alpha}_i(\omega) \equiv \frac{1}{\pi} \text{Im} \psi^{(0)} \left( \frac{1}{2} + \frac{1}{2} + i(\omega - m E^*_Z - \alpha V) \right),$$

(5.43)

with $\psi^{(0)}$ the $i$-th polygamma function, from Eq. (4.107) we obtain

$$i \Pi_{\Delta}^R(\omega) = \sum_{m,\alpha,\beta} g_\alpha g_\beta \text{coth} \left( \frac{2\omega - (\alpha + \beta) V}{4T} \right) \chi^{m,\alpha}_0(\omega).$$

(5.44)

Here and below, the indices $m, \alpha, \beta$ are summed over $\pm 1$; the quantities $\tau_*$ and $E^*_Z$ are defined in Eqs. (2.41) and (2.42). Using again Eqs. (5.42)-5.43), Eqs. (4.96) give:

$$\text{Im} \Pi_{\Delta}^R(\omega) = -\frac{1}{2} \sum_{m,\alpha,\beta} g_\alpha \chi^{m,\alpha}_0(\omega),$$

(5.45)

Substituting the above results into Eq. (4.91) we find:

$$J(\epsilon, \omega) = \sum_{m,\alpha,\beta} g_\alpha g_\beta \chi^{m,\alpha}_0(\omega)$$

$$\times \left[ \text{coth} \left( \frac{2\omega - (\alpha + \beta) V}{4T} \right) - \tanh \left( \frac{2\epsilon - (\beta - \alpha) V}{4T} \right) \right].$$

(5.46)

What remains to be done, are the two integrals over $\omega$ and $\epsilon$ in Eqs. (5.39), (5.33); the latter can be evaluated exactly, while the former only approximately. Due to their similarity, we consider $J_{MT}$ and $J_{DoS}$ together in the next subsection, deferring the calculation of $J_{DoS}$ to a later subsection.

1. **Maki-Thompson and DoS** parts

Substituting Eq. (5.46) into Eqs. (5.37) and using Eq. (6.44) together with the identity:

$$\frac{1}{\tau_*} |C_m|^2(2\epsilon) = \text{Re} C_m(2\epsilon),$$

(5.47)

the result of the integration over $\epsilon$ is:

$$J_{MT} = \sum_{m,\alpha,\beta} g_\alpha g_\beta \frac{\pi}{4} \frac{\partial}{\partial V} \int \frac{d\omega}{2\pi} |D^R_\Delta|^2(\omega)$$

$$\times \left[ \text{coth} \left( \frac{2\omega - (\alpha + \beta) V}{4T} \right) - \text{coth} \left( \frac{2\omega - (\beta - \alpha) V}{4T} \right) \right],$$

(5.48)

where $g_\alpha$ and $\chi^{m,\alpha}_0(\omega)$ were defined in Eqs. (5.41) and (5.43), respectively, and the subscripts $m, n, \alpha$ and $\beta$ are summed over $\pm 1$. Moreover since

$$-\text{Re} C_m(2\epsilon) = \frac{\partial}{\partial (\tau_*^*)} \text{Re} C_m(2\epsilon),$$

(5.49)

the expression for $J_{DoS}$ is found by replacing the $\chi^{m,\alpha}_0$’s with $\chi^{m,\alpha}_1(\omega)$ in $2\pi \tau_*/(2\pi \tau_*)$ in Eq. (5.18). For convenience, in both $J_{MT}$ and $J_{DoS}$, we separate two contributions with different dependences on $g_L, g_R$ (and we drop a contribution odd in $\omega$ which vanishes upon integration):

$$J_a = \frac{2g_L g_R}{(g_L + g_R)^2} J^1_a + \frac{g_L^2 + g_R^2}{(g_L + g_R)^2} J^2_a,$$

(5.50)

where $a = MT, DoS$. For $a = MT$ we have

$$J^1_{MT} = \frac{\pi}{4} \frac{\partial}{\partial V} \int \frac{d\omega}{2\pi} |D^R_\Delta|^2(\omega) \sum_{m,\alpha,\beta} \chi^{m,\alpha}_0(\omega) \chi^{n,\alpha}_0(\omega)$$

$$\times \alpha \left[ \text{coth} \left( \frac{\omega - \alpha V}{2T} \right) - \text{coth} \left( \frac{\omega}{2T} \right) \right],$$

(5.51)

$$J^2_{MT} = \frac{\pi}{4} \frac{\partial}{\partial V} \int \frac{d\omega}{2\pi} |D^R_\Delta|^2(\omega) \sum_{m,\alpha,\beta} \chi^{m,\alpha}_0(\omega) \frac{1}{2} \sum_{n,\beta} \chi^{n,\beta}_0(\omega)$$

$$\times \alpha \left[ \text{coth} \left( \frac{\omega - \alpha V}{2T} \right) - \text{coth} \left( \frac{\omega}{2T} \right) \right],$$

(5.52)

and, as explained above, the formulas for $a = DoS$ are found by replacing $\chi^{m,\alpha}_0$ by $\chi^{m,\alpha}_1(\omega)$. Up to now, no approximation has been made. However the validity of the RMT for the metallic dots requires all the energy scales $|T, V, 1/\tau_*, E^*_Z|$ to be much smaller than $T_c$; we can then neglect the energy dependence of the propagator and approximate

$$D^R_\Delta(\omega) \approx D^R_\Delta(0) \approx \frac{\pi}{\varepsilon},$$

(5.53)

with $\varepsilon$ defined in Eq. (2.41). The last approximation is valid for logarithmic accuracy and can be verified by noticing that the dependence of the polarization operator on the relevant quantities is the same as in Eq. (5.45) (dropping “Im” on both sides of the equation) and using Eq. (4.100). Even with this approximation, the integrals cannot be calculated in closed form and we must resort to further approximations valid in different limits. In particular, we will consider the low and high temperature
regimes – the transition between the two occurs at \( T \sim \frac{1}{\tau_e} \), with \( \tau_e \) defined in Eq. (1.33).

In the low temperature limit \( T \ll \frac{1}{\tau_e} \), we can use in Eq. (5.43) the following asymptotics for the polygamma functions:

\[
\text{Im } \psi^{(0)} \left( \frac{1}{2} + \frac{1}{4\pi T} \right) \approx \arctan \tau_e \omega
\]

(5.54)

\[
\text{Im } \psi^{(1)} \left( \frac{1}{2} + \frac{1}{4\pi T} \right) \approx -4\pi T \tau_e \omega \left( \frac{\tau_e \omega}{1 + (\tau_e \omega)^2} \right)
\]

(5.55)

We further distinguish two cases: “low” and “high” voltage, when \( \tilde{V} \ll \frac{1}{\tau_e} \), and \( \tilde{V} \gg T \) respectively; it is evident that the two conditions are not mutually exclusive and the results derived below must agree with each other at intermediate voltages.

If the voltage is small, \( \tilde{V} \ll \frac{1}{\tau_e} \), we can expand \( \chi^{\alpha}_{\alpha}(\omega) \) in \( \tau_e \tilde{V} \) and \( \tau_e \omega \) as well (as \( |\omega| \) is itself limited by \( |\tilde{V}| \)) to obtain

\[
\sum_n \chi_0^{n,\alpha}(\omega) \approx \frac{2 \tau_e (\omega - \alpha \tilde{V})}{\pi (1 + (\tau_e \tilde{V})^2)}.
\]

(5.56)

Moreover in this approximation we have

\[
\sum_n \chi_1^{n,\alpha}(\omega) \approx -4\pi T \tau_e \sqrt{1 - (\tau_e \tilde{V})^2} \sum_n \chi_0^{n,\alpha}(\omega),
\]

(5.57)

so that in this regime

\[
J_{DOS1} = -2 \frac{1 - (\tau_e \tilde{V})^2}{1 + (\tau_e \tilde{V})^2} J_{MT},
\]

(5.58)

and we only need to calculate \( J_{MT} \).

Inserting Eq. (5.56) into Eq. (5.51) and performing the integral over \( \omega \) we find

\[
J_{MT}^1 \approx \left[ \frac{2}{\varepsilon (1 + (\tau_e \tilde{V})^2)^2} \right] \left( \frac{\tau_e \tilde{V}^2 + \pi^2 (\tau_e T)^2}{3} \right),
\]

(5.59)

where we used the identity

\[
\frac{\partial}{\partial y} \int dx \left( \coth(x - y) - \coth(x + y) \right) (x^2 - y^2) = 8y^2 + \frac{2\pi^2}{3}.
\]

(5.60)

By repeating the above steps for Eq. (5.52) we find

\[
J_{MT}^2 \approx \left[ \frac{2}{\varepsilon (1 + (\tau_e \tilde{V})^2)^2} \right] \left[ \frac{1}{4} \left( \tau_e \tilde{V}^2 + \frac{\pi^2}{3} (\tau_e T)^2 \right) \right].
\]

(5.61)

Using the results (5.59) and (5.60) in Eq. (5.50) and then Eq. (5.55), we arrive at Eqs. (2.14).

At high voltages \( \tilde{V} \gg T \) (but still low temperatures \( T \ll \frac{1}{\tau_e} \)), we can approximate the hyperbolic cotangents in Eqs. (5.51), (5.52) with their zero-temperature limit: \( \coth \frac{\omega}{\tau_e} \to \text{sgn} \omega \). Performing the differentiation with respect to \( \tilde{V} \) and a partial integration, and using the approximation in Eq. (5.54), we obtain

\[
J_{MT}^1 = \frac{1}{2\pi^2} \sum_{n,m=\pm 1} \int \frac{d\omega}{\nu} \frac{\tau^* \arctan \tau^* (\omega + \tilde{V} - n \tilde{E}^*_\alpha)}{1 + \tau^2 (\omega - \tilde{V} - m \tilde{E}^*_\alpha)^2}
\]

(5.62)

The result of the \( \omega \)-integral can be expressed in terms of dilogarithms:

\[
J_{MT}^1 = \frac{1}{2\pi^2} \sum_{m=\pm 1} \left(\sum_{n} \arctan \tau^* (\tilde{V} - m \tilde{E}^*_\alpha) \right)^2,
\]

(5.63)

\[
J_{MT}^2 = \frac{1}{2} J_{MT}^1 - \frac{1}{4\pi^2} \left( \sum_{n} \arctan \tau^* (\tilde{V} - m \tilde{E}^*_\alpha) \right)^2,
\]

(5.64)

The second line is found in the approximation (5.54) by direct comparison of the definitions (5.51) - (5.52) and the function \( h \) is defined as:

\[
h(x, y) \equiv \frac{1}{2} \text{Re} \left\{ f(x, y) - f(-x, y) + f(y, x) - f(-y, x) \right\},
\]

(5.65)

with

\[
f(x, y) \equiv L_2 \left( \frac{x + y + i}{2} \right) - L_2 \left( \frac{x + y - i}{2} \right) + \ln \left( 1 + i x + y \right) \left[ \ln \frac{y - x + i}{2} - \ln \frac{y - x - i}{2} \right].
\]

(5.66)

The approximate expression (2.17) for \( h \) is found as follows: we first notice that in Eq. (5.62) we can restrict the integral to the region \((0, \tilde{V})\) if we multiply by 2; then we shift the integration variable: \( \omega \to \omega + \tilde{V} + m \tilde{E}^*_\alpha \) and we finally neglect the dependence of the numerator on \( \omega \). The resulting integral is straightforward and gives the formula (2.17).

We now turn to the high temperature regime \( T \gg \frac{1}{\tau_e} \). In this case, we can neglect \( 1/(4\pi \tau_e T) \) in the argument of the logarithm in the definition (5.43), then using the identity

\[
\frac{2}{\pi} \text{Im } \psi^{(0)} \left( \frac{1}{2} + \frac{i}{\pi} \right) = \tanh \omega, \quad (\text{Im } \omega = 0),
\]

(5.67)

we have

\[
\chi_0^{m,\alpha}(\omega) \approx \frac{1}{2} \tanh \left( \frac{\omega - \alpha \tilde{V} - m \tilde{E}^*_\alpha}{4T} \right).
\]

(5.68)

Substituting the above approximate expression into Eqs. (5.61) - (5.62) and performing both the differentiation with respect to \( \tilde{V} \) and the integration over \( \omega \) we
obtain
\[ J_{MT} \approx \frac{1}{2} \frac{\pi^2}{\epsilon^2} \left( 1 - \frac{1}{2} \coth \left( \frac{E_c^z}{2T} \right) \right) \]  
(5.68)
\[ J_{MT}^2 \approx \frac{\pi^2}{\epsilon^2} \sum_{m=\pm 1} \frac{1}{8} c_2 \left( \frac{\tilde{V} - mE_c^z}{2T} \right), \]  
(5.69)
where
\[ c_n(x) \equiv \frac{d^n}{dx^n} (x \coth x). \]  
(5.70)
Since \( J_{DoS_1} \) is smaller than \( J_{MT} \) by a factor of \((\tau_sT)^{-1}\) [see Eqs. (5.49)], it can always be neglected in this regime.

This concludes the calculation of the Maki-Thompson and “ReC\^2” contribution and hence the derivation of the results for \( \Delta G_{MT} \) reported in Sec. [IV]. In the next subsection we present the calculation of the remaining “DoS” correction for completeness.

2. DoS\(_2\) part

In this subsection we evaluate \( J_{DoS_2} \), Eq. (5.39), in the relevant limits. Using Eqs. (5.2), (5.40) and (4.86), integration over \( \epsilon \) can be done similarly to Eq. (5.11) by closing the integration contour in the upper half plane. The result is
\[ J_{DoS_2} = \frac{1}{16\tau_sT} \sum_{m,\alpha,\beta} \beta g_m \frac{\partial}{\partial V} \int \frac{d\omega}{2\pi} \coth \left( \frac{2\omega - (\alpha + \beta)\tilde{V}}{4T} \right) \]
\[ \times \text{Re} \left\{ \left[ \xi^{m,\beta}(\omega) - \xi^{m,\alpha}(\omega) \right] D^R(\omega) \right\}, \]  
(5.71)
where
\[ \xi^{m,\alpha}(\omega) \equiv \frac{1}{\pi} \psi^{(1)} \left( \frac{1}{2} + \frac{1}{2} i \frac{\omega - mE_c^z - \alpha\tilde{V}}{4\pi T} \right), \]  
(5.72)
and the subscripts \( m, \alpha \) and \( \beta \) are summed over \( \pm 1 \). In Eq. (5.71), the approximation amounts to neglecting the imaginary part of the propagator in comparison with its real part, so that
\[ J_{DoS_2} \approx \frac{1}{16\tau_sT} \sum_{m=\pm 1} \frac{\partial}{\partial V} \int \frac{d\omega}{2\pi} \coth \left( \frac{\omega}{2T} \right) \]
\[ \times \text{Re} \left\{ \left[ \xi^{m,+,\alpha}(\omega) - \xi^{m,-,\alpha}(\omega) \right] D^R(0) \right\}. \]  
(5.73)

In the low temperature regime \( T \ll \frac{1}{\tau_s} \), we can use the following asymptotic expansion:
\[ \psi^{(1)}(z) \approx \frac{1}{z}, \quad z \gg 1 \]  
(5.74)
so that
\[ J_{DoS_2} \approx \frac{1}{4\tau_s} \text{Re} D^R(0) \text{Re} \sum_{\alpha,\beta} \int \frac{d\omega}{2\pi} \coth \left( \frac{\omega}{2T} \right) \]
\[ \times \frac{\partial}{\partial V} \left[ \frac{\alpha}{\tau_s} + i(\omega - \alpha\tilde{V} - mE_c^z) \right]. \]  
(5.75)
The result of the integral can be expressed in terms of \( \psi^{(1)} \) and using again the expansion we find
\[ J_{DoS_2} \approx \frac{1}{2\epsilon} \sum_{m=\pm 1} \frac{1}{1 + \tau_s^2(\tilde{V} - mE_c^z)^2}. \]  
(5.76)
In the high temperature limit \( T \gg \frac{1}{\tau_s} \), we can use Eq. (5.67) to rewrite \( J_{DoS_2} \) as
\[ J_{DoS_2} \approx \frac{\text{Re} D^R(0)}{16\tau_sT} \sum_{m=\pm 1} \frac{\partial}{\partial V} \int \frac{d\omega}{2} \coth \frac{\omega}{2} \]
\[ \times \left[ \text{sech}^2(\tilde{V} - mE_c^z) - \text{sech}^2(\tilde{V} + mE_c^z) \right], \]  
(5.77)
and integrating over \( \omega \) we obtain
\[ J_{DoS_2} \approx \frac{\pi/\epsilon}{16\tau_sT} \sum_{m=\pm 1} c_2 \left( \frac{\tilde{V} - mE_c^z}{2T} \right), \]  
(5.78)
where \( c_2 \) was defined in Eq. (5.74). We see that for \( \tau_sT \gg 1 \) we have \( J_{DoS_2} \approx \epsilon J_{MT} \gg |J_{MT}| \). However the prefactor \( A \) in Eq. (5.71) vanishes for reflectionless contacts and for this reason we only report the MT contribution in Sec. [IV].

VI. CONCLUSIONS

In this paper we presented a quantum kinetic equation description of transport in open quantum dots with the inclusion of all the “universal” – in the Random Matrix Theory sense – interaction effects [see Eq. (1.5)]. While the effect of the charging energy was taken into account before, the interaction corrections in the triplet and Cooper channels are considered here for the first time.

The main result of the present work is that the triplet channel interaction can significantly affect the differential conductance of the quantum dot, see Fig. 2 and in contrast with the singlet channel contribution, it is sensitive to the magnetic field, cf. Eqs. (2.5) and (2.6) – see also Fig. 3. For transparent contacts both the singlet and triplet channel contributions to the electrical conductance vanish; the non-vanishing Cooper channel contribution is unfortunately expected to be a negligible one in metallic dots, as discussed after Eq. (2.3).

In addition to the electrical conductance, we were able to calculate the thermal conductance by applying the local kinetic equation approach developed in Ref. [14]. For the thermal conductance we find that the Wiedemann-Franz law is violated by the interaction corrections [se
Eq. (2.22), and we investigated the effect of magnetic field on the Lorentz ratio for contacts of finite reflection. The charge and triplet channel corrections to the thermal conductance also vanish for reflectionless contacts, and the Wiedemann-Franz law is not violated by the electron-hole channels in this case.

Acknowledgments

We would like to thank P.W. Brouwer, L.I. Glazman and A.I. Larkin for interesting discussions.

APPENDIX A: VANISHING OF $\hat{S}_t$ IN THE STEADY STATE

We want to show that the properties (4.13) and (4.15) hold in the steady state. To this end, we notice that the second term on the right hand side of the definition (4.22a) always satisfies these properties, as one can verify using the same arguments that prove the corresponding properties (4.12a) and (4.16a) of the collision integral $\hat{S}_{t+}$. Next we want to show that in the steady state

$$\langle \delta Q \rangle K_+ = 0, \quad \text{(A1)}$$

after the average over the fluctuating field but before any limit and/or time derivative is taken. Indeed in $\delta Q$ only the contributions proportional to $\delta g^K_+$ should be kept, since the ones proportional to $\delta g^K_-$ vanish after the average. We note that the right hand side of Eq. (4.12a) can be rewritten as

$$i \left[ K_+; \partial_t g^K_+ \right], \quad \text{(A2)}$$

by using the zeroth order (in the interaction) part of Eq. (4.19) for $g^K$, i.e. neglecting the collision integrals $\hat{S}_{t1}$ and $\hat{S}_{t2}$ – this is sufficient in the one-loop approximation. In the steady state, $\partial_t g_0 = 0$ and hence $\delta g_0 = 0$. This proves Eq. (A1) and therefore the properties (4.13) and (4.15).

APPENDIX B: THERMODYNAMICS

The effects of interaction on the thermodynamics can be obtained by calculating the leading contribution to the thermodynamic potential $\Omega$: this is given by the sum of the so-called ring diagrams of Fig. 8 [see e.g. Ref. 8]. This sum can be calculated in the Matsubara representation and after standard analytic continuation the correction $\delta \Omega$ is found to be:

$$\delta \Omega = \int d\omega \frac{1}{2\pi} \frac{1}{2} \coth \left( \frac{\omega}{2T} \right) \text{Im} \ln \left( 1 - 2E_c \Pi^K_0(\omega) \right). \quad \text{(B1)}$$

Using expression (4.29) for the polarization operator and the definitions (4.7) and (4.32), we can rewrite this as:

$$\delta \Omega = \int d\omega \frac{1}{2\pi} \frac{1}{2} \coth \left( \frac{\omega}{2T} \right) \text{Im} \left[ \ln \mathcal{L}^g - \ln \mathcal{L}^e \right]. \quad \text{(B2)}$$

We can now calculate the correction $\delta c_V$ to the specific heat and hence arrive at the expression for the energy density $u_b$; indeed

$$\delta c_V = -T \frac{\partial^2 \delta \Omega}{\partial T^2} = \frac{\partial u_b}{\partial T}. \quad \text{(B3)}$$

Using the first of the above relations and after an integration by parts we find

$$\delta c_V = \frac{\partial}{\partial T} \int_0^\infty d\omega \omega N_P(\omega) \left[ b^p(\omega) - b^q(\omega) \right], \quad \text{(B4)}$$

where we introduced the bosonic density of states:

$$b^\alpha(\omega) = \frac{1}{\pi} \text{Im} \partial_\omega \ln \mathcal{L}^\alpha. \quad \text{(B5)}$$

Comparing Eq. (B4) to the second relation in Eq. (B3) and generalizing this result to an arbitrary distribution function we have

$$u_b = \int_0^\infty d\omega \omega \left[ N^p(\omega) b^p(\omega) - N^q(\omega) b^q(\omega) \right]. \quad \text{(B6)}$$

It is straightforward to prove that this coincides with Eq. (4.39) [by substituing in the latter Eq. (4.38) and noticing the different limits of integration].

FIG. 8: Leading singular contribution to the thermodynamic potential. The shaded box corresponds to $2E_c$, defined through the two particle vertex [see e.g. Ref. 24]; the solid lines are coherent parts of the electron Green functions.
1. H.R. Zeller and I. Giaver, Phys. Rev. Lett. 20, 1504 (1968).
2. I.O. Kulik and R.I. Shekhter, Zh. Eksp. Teor. Fiz. 68, 623 (1975) [Sov. Phys. JETP 41, 308 (1975)].
3. D.V. Averin and K.K. Likharev, J. Low Temp. Phys. 62, 345 (1986).
4. I. Aleiner, P. Brouwer, and L. Glazman, Phys. Rep. 358, 309 (2002).
5. P. W. Brouwer and I. L. Aleiner, Phys. Rev. Lett. 82, 390 (1999).
6. D. S. Golubev and A. D. Zaikin, Phys. Rev. B 69, 0752318 (2004).
7. P. W. Brouwer, A. Lamackraft, and K. Flensberg, Phys. Rev. Lett. 94, 136801 (2005). P. W. Brouwer, A. Lamackraft, and K. Flensberg, Phys. Rev. B 72, 075316 (2005).
8. B.L. Altshuler and A.G. Aronov, in Electron-electron interactions in disordered systems, edited by A.L. Efros and M. Pollack (North-Holland, Amsterdam, 1985).
9. C. W. J. Beenakker, Rev. Mod. Phys. 69, 731 (1997).
10. Y. Alhassid, Rev. Mod. Phys. 72, 895 (2000).
11. D. A. Gorokhov and P. W. Brouwer, Phys. Rev. B 69, 155417 (2004).
12. I. Kurland, I. Aleiner, and B. Altshuler, Phys. Rev. B 62, 14886 (2000).
13. J. von Delft, Ann. Phys. 10, 219 (2001).
14. M.N. Kiselev and Y. Gefen, unpublished, cond-mat/0504751 (2005).
15. S. Vorotyntsev and H. Baranger, unpublished, cond-mat/0505569 (2005).
16. G. Catelani and I. Aleiner, JETP 100, 331 (2005).
17. G. Zala, B. Narozhny, and I. Aleiner, Phys. Rev. B 64, 214204 (2001).
18. K. Matveev, D. Yue, and L. Glazman, Phys. Rev. Lett. 71, 3351 (1993).
19. K. Maki, Prog. Theor. Phys. 39, 897 (1968).
20. R. Thompson, Phys. Rev. B 1, 327 (1970).
21. A.I. Larkin, Pis’ma Zh. Eksp. Teor. Fiz. 31, 239 (1980) [Sov. Phys. JETP Lett. 31, 219 (1980)].
22. A.I. Larkin and A.A. Varlamov, Theory of Fluctuations in Superconductors (Oxford University Press, Oxford, 2005).
23. L. Keldysh, Zh. Eksp. Teor. Fiz. 47, 1945 (1964) [Sov. Phys. JETP 20, 1018 (1964)].
24. A.A. Abrikosov, L.P. Gorkov, and I.E. Dzyaloshinski, Methods of Quantum Field Theory in Statistical Physics (Prentice-Hall, Englewood Cliffs, NJ, 1963).
25. K.D. Usadel, Phys. Rev. Lett. 25, 507 (1970).
26. L.P. Gor’kov, Zh. Eksp. Teor. Fiz. 34, 735 (1958) [Sov. Phys. JETP 7, 505 (1958)]; Y. Nambu, Phys. Rev. 117, 648 (1960).
27. A. Kamenev and A. Andreev, Phys. Rev. B 60, 2218 (1999).
28. L.G. Aslamazov and A.I. Larkin, Soviet Solid State 10, 875 (1968).
29. Ensemble averaging of the polarization operator separately from the Green function is allowed because of the large conductance $G_0 \gg G_q$.
30. To avoid rescaling the kinetic equation, we switch back to units $\delta_1/\pi = 1$.
31. The scalar product is now $a \cdot b = a^z b^z + a^+ b^- + a^- b^+$.