Causal diffusions, causal Zeno effect and collision number

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We consider diffusion processes with the help of Markov random walk models. Especially the process of diffusion of a relativistic particle in a relativistic equilibrium system is considered. We interpret one of the results as causal Zeno effect for its similarity to quantum Zeno effect. Another problem we considered is about collision number. Basing on our numerical results, we propose that in the considered situation the probability density distribution among different collision numbers is a lognormal distribution.

I. INTRODUCTION

Diffusion process is very common in physics, chemistry, biology and many other fields. It can be described by a diffusion equation with or without a finite maximum velocity [1, 2, 3, 4]. Another approach toward diffusion process is numerical simulation [5, 6, 7]. Yet, it can also be considered with the help of random walk models (RWM). Many literatures take this way, such as [8, 9, 10].

In this letter, we consider the diffusion of a particle in an equilibrium system in the framework of Markov random walk model. We begin our study with the one-particle velocity distribution of a non-relativistic/relativistic equilibrium system.

As it is well-known, the one-particle velocity distribution of a near-free non-relativistic equilibrium gas is the Maxwell’s probability density function (PDF):

$$ f_M(v; m, \beta) = \frac{(\beta m)^{d/2}}{2 \pi^{d/2}} e^{-\beta m v^2/2}. \quad (1) $$

The first relativistic generalization of Maxwell’s function was proposed by F. Jüttner from a consideration of maximum entropy principle [11, 12, 13]:

$$ f_J(p; m, \beta) = \frac{1}{Z_J} e^{-\beta E}, \quad (2) $$

where $E = \sqrt{m^2 + p^2} = m \gamma(v)$, $p = mv \gamma(v)$, and $\gamma(v) = 1/\sqrt{1 - v^2}$. $Z_J$ is the normalization factor. Written in velocity PDF, Jüttner’s distribution is

$$ f_J(v; m, \beta) = \frac{m^d}{Z_J} \gamma(v)^{d/2} e^{-\beta m \gamma(v)}. \quad (3) $$

A lot of work has been done basing on Jüttner’s distribution since its proposal. However, in the 80’s of the last century, doubt about Jüttner’s function was expressed and “modified Jüttner’s function” was proposed [13, 14, 15]. But recent numerical simulations favored Jüttner distribution [9] as the correct one-particle velocity distribution of a relativistic equilibrium system [3, 6].

So, in this letter Maxwell’s function (1) and Jüttner’s function (3) are used. Numerical analysis is done. By comparing the results from the two distributions, we interpret one of the results as causal Zeno effect for its similarity with the Quantum Zeno Effect [16, 17].

Another problem we consider is that how many times a particle will collide with other particles during the process of diffusion from a initial point to a final point in a given time interval. As one can conceive, there will be a probability distribution among different collision numbers. One of our results is that if the final point is not specified, the probabilities to collide whatever N times will be the same. For the case of that the final position is also fixed, we make numerical analysis and propose that the probability distribution is the lognormal distribution [18, 19, 20].

II. RANDOM WALK MODELS

In this section, we consider diffusions as random walk processes. At first we give some general descriptions, then we apply the model to the non-relativistic Markov diffusions, finally we consider the relativistic Markov diffusions. Our discussions will be confined to the one-dimensional case.

A. General descriptions

When a particle is diffusing in an environment, one wants to know the probability for it to reach some position from a given position within a given time interval. The formal kinematics formula is

$$ \xi(t) = x_0 + \int_0^t ds v(s), \quad (4) $$

where $v(s)$ is its velocity.

For a free particle, its velocity is a constant. While if it interacts with other matters, its velocity will change. Here, we assume that the particle interacts with other particles only by point collision. Thus the above formula...
transforms into
\[ \xi(t) = x_0 + \sum_{i=1}^{N} v_i(t_i - t_{i-1}), \]  
(5)
where \( t_i, i = 1, 2, ..., N - 1 \) are the times of collisions and \( t_N \) is the given ending time. We write \( \tau_i = t_i - t_{i-1} \) for simplicity. Although the actual collision number is not \( N \) but \( N - 1 \), one can roughly say it is or it is indicated by \( N \).

\((\{v_i\}; \{\tau_i\})\) defines a path in the space. If the particle has a probability density \( f(\{v_i\}; \{\tau_i\}) \) to follow this path. Then the transition probability density function of order \( N \) can be written as
\[ p_N(t, x|0, x_0) = \int dv_1...dv_N \int d\tau_1...d\tau_N f(\{v_i\}; \{\tau_i\})\delta(x - \xi(t)). \]  
(6)

If the collision number is not concerned about, the total transition PDF is
\[ p(t, x|0, x_0) = \sum_{N=1}^{\infty} p_N(t, x|0, x_0). \]  
(7)

In the following, we set \( \tau_i \) be equal (= \( \tau = t/N \)). And we consider Markov process in which case \( f(\{v_i\}; \{\tau_i\}) \) can be written as \( \prod_{i=1}^{N} f(v_i; \tau_i) \).

Thus the \( N \)-order transition PDF is
\[ p_N(t, x|0, x_0) = \left[ \prod_{i=1}^{N} \int dv_i f(v_i; \tau_i) \right] \delta(x - \xi(t)) \]  
(8)

Using
\[ \delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx} \]  
(9)
together with (6), one can get
\[ p_N(t, x|0, x_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ik(x-x_0)} \left[ \prod_{i=1}^{N} \int dv_i f(v_i; \tau)e^{-ikv_i\tau} \right] \]  
\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ik(x-x_0)} \varphi(-k\tau)^N \]  
(10)
where \( \varphi(u) \) is the characteristic function of \( f(v) \), i.e. its Fourier transformation.

One can also get an expression for \( p_N(t, x|0, x_0) \) by invoking the mean velocity. Define
\[ V_{SN} = \sum_{i=1}^{N} v_i, \]  
(11)
then the mean velocity
\[ \overline{V}_N = \frac{1}{t} \sum_{i=1}^{N} v_i \tau_i = \frac{1}{N} \sum_{i=1}^{N} v_i = \frac{1}{N} V_{SN}. \]  
(12)

If the velocity probability density function of \( v_i \) is \( f_i(v) \), then the PDF of \( V_{SN} \) is
\[ f_{SN}(V) = f_1(v) * f_2(v) * ... * f_N(v), \]  
(13)
here \( * \) means convolution.

This leads to the probability density function of \( \overline{V}_N \):
\[ f_{MN}(V) = f_{SN}(NV)N. \]  
(14)
Finally the \( N \)-order transition PDF is
\[ p_N(t, x|0, x_0) = f_{MN}(\frac{x-x_0}{t})^{1/t}. \]  
(15)

We will follow this way when we make numerical analysis.

### B. Non-relativistic Markov diffusions

Firstly we apply this model to the case of non-relativistic Markov diffusions. The equilibrium velocity PDF of a non-relativistic system is the Maxwell distribution.

Substituting (11) into (10), and assuming
\[ \beta = \frac{1}{k_B T} = \frac{\tau}{2mD}, \]  
(16)
one gets
\[ p_N(t, x|0, x_0) = \frac{1}{\sqrt{4\pi NDt}} \frac{1}{\tau} e^{-\frac{(x-x_0)^2}{4ND\tau}} \]  
\[ = \frac{1}{\sqrt{4\pi DT}} e^{-\frac{(x-x_0)^2}{4DT}}, \]  
(17)
which is the solution to the ordinary diffusion equation with a proper initial condition.

One can easily see that this expression of transition PDF is in conflict with the special relativity, for when \( |x-x_0| > ct \), there is still a small but non-vanishing probability for the particle to be found.

Another property of non-relativistic Markov diffusions is that for different values of \( N \), the \( p_N(t, x|0, x_0) \)'s are the same, i.e. no matter how much the collision number is, even when it is infinite, \( p_N(t, x|0, x_0) \) is the same as above. So, the particle can always diffuse. As we will see, this is not the case in relativistic Markov diffusions.

### C. Relativistic Markov diffusions

#### 1. Formula development

For relativistic Markov diffusions, one should use Jüttner’s distribution instead of Maxwell’s distribu-
tion. Then the N-order transition PDF will be

$$p_N(t, x, 0|x_0, x_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dke^{ik(x-x_0)} \left[ \int dv \{J(v; \tau)e^{-ik\tau v} \}^N \right]$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dke^{ik(x-x_0)} \varphi_f(-k\tau)^N.$$  \hspace{1cm} (18)

This is very difficult, or perhaps, even impossible to carry out the integral analytically. For a compensation, we consider its large-N asymptotic behavior by invoking its N-order mean velocity $\nabla_V$.

The N-order mean velocity during the time interval from 0 to $t$ is

$$\nabla_V = \frac{\xi(t) - x_0}{t} = \frac{1}{N} \sum_{i=1}^{N} \tau_i v_i = \frac{1}{N} \sum_{i=1}^{N} v_i.$$

(19)

Since $M(v_i) = 0, \text{Var}(v_i) = \sigma^2 < \infty, \text{Central Limit Theorem (CLT)}$ \hspace{1cm} (18) asserts that the distribution of $\sum_{i=1}^{N} v_i/\sqrt{N}$ converges to Gauss’s normal distribution with parameters $(0, 1)$:

$$N(x; 0, 1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$ \hspace{1cm} (20)

Then, the probability density about $\nabla_V$ will be

$$\frac{1}{\sqrt{N\pi\sigma^2}} e^{-\frac{(x-x_0)^2}{2\sigma^2}}.$$ \hspace{1cm} (21)

And the N-order transition PDF will be

$$p_N(t, x|0, x_0) = \frac{1}{\sqrt{2\pi\tau\sigma^2}} e^{-\frac{(x-x_0)^2}{2\tau\sigma^2}}.$$ \hspace{1cm} (22)

Here $\sigma^2$ is the variance of Jüttner’s distribution \hspace{1cm} (A11); it is a function of $\chi(=m\beta)$, and $\tau = t/N$.

At a given reciprocal temperature $\beta, \sigma^2$ is fixed. When $N \to \infty, \tau \to 0$. Then

$$p_N(t, x|0, x_0) \to \delta(x - x_0).$$ \hspace{1cm} (23)

This result means that the particle will not diffuse at all in this situation. By comparing the results in non-relativistic and relativistic situations, one can see that they are contrary to each other. In the non-relativistic situation, we have seen that $p_N(t, x|0, x_0)$ is independent of $N$, and always a Gaussian distribution. But here, in the relativistic case, we have the result \hspace{1cm} (23).

2. Causal Zeno effect

We venture to interpret the result \hspace{1cm} (23) as causal Zeno effect for its similarity with the quantum Zeno effect. Quantum Zeno effect is a name coined by George Sudarshan and Baidyanath Mishra in 1977 in their analysis of the situation in which an unstable particle, if observed continuously, will never decay \hspace{1cm} (10). One can nearly “freeze” the evolution of the system by measuring it frequently enough in its (known) initial state \hspace{1cm} (17).

In the random walk model, $N \to \infty$ means that the particle is continuously collided by other particles. In other words, it is continuously observed. So the result \hspace{1cm} (23) means that the particle will never depart from its initial position if it is continuously observed. However, as we have seen, in the non-relativistic situation, the particle can always diffuse.

This difference between the two situations arises from the different velocity distribution functions. Furthermore, as one can see, when $\chi$ becomes large, Jüttner’s distribution \hspace{1cm} (3) tends to Maxwell’s distribution, save for its finite support. Thus the boundedness of velocity is indispensable for the convergence to a delta function.

One can also perceive that at a given temperature, there will be a probability distribution among different values of $N$, and the behavior of $p_N(t, x|0, x_0)$ with the most possible value of $N$ will dominate in the diffusion process of the particle.

3. $\lambda_N = 1$

As for the probability distribution among different values of $N$, we first consider the following expression:

$$\lambda_N \equiv \int_{-\infty}^{\infty} p_N(t, x|0, x_0) dx.$$ \hspace{1cm} (24)

This expression quantifies the probability of the particle colliding $N - 1$ times with other particles in a given time interval $t - 0$ from the initial position $x_0$ to non-specified final positions.

Substituting \hspace{1cm} (10) into the above expression, the outcome is

$$\lambda_N = \int_{-\infty}^{\infty} dx \frac{1}{2\pi} \int_{-\infty}^{\infty} dke^{ik(x-x_0)} \varphi_f(-k\tau)^N.$$ \hspace{1cm} (25)

As

$$\delta(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{ik(x-x_0)},$$

one gets

$$\lambda_N = \varphi_f(0)^N = 1,$$

which means that the probabilities for the particle to collide different $N - 1$ times with other particles are the same. And it is right even for cases that $\tau_i (i=1,2,...,N)$ are different and the process is non-Markovian.

It is surprising as we have thought that there would be a non-trivial probability distribution. However, in the case with a specified final position, one may anticipate the probabilities will be different. We will check this by numerical analysis.
III. NUMERICAL ANALYSIS

A. $P_N(t, x|0, x_0)$ against $x$

The formulae we use for numerical calculation have been presented in Section II. In the following, we use J"uttner’s function and we set $x_0=0$.

For $P_N(t, x|0, 0)$ against $x$, there are three parameters: $\chi$ for the velocity distribution; $t$ for time of the diffusing process; $N$ for the kind of the diffusing pathes.

We give our results in Fig.(1) and Fig.(2). In Fig.(1), the value of $\chi$ is 0.1. The peak in the middle belongs to the curve of $P_N$ with $N=2$. This curve has another two peaks at $\pm 1$. The two second highest peaks which are symmetrically located belong to the curve of $N=3$. This curve has low sharp peaks at $\pm 1$ too. For $N=4$, there are 3 peaks symmetrically located besides the peaks at $\pm 1$. This case is not shown in the picture. For large $N$, the $P_N(t, x|0, 0)$ are Gaussian-like, which is predicted by CLT.

In Fig.(2), $\chi = 10$. This is the case of low-velocity. The peaks are more sharpened. The effective velocity of the diffusing particle is small and the particle is more confined in its neighborhood. And with $N$ increasing the peak sharpens. This is different with the result (17) which comes from Maxwell’s function for the PDF of velocities. J"uttner’s function with large $\chi$ applies to the low velocity situations as well as Maxwell’s function, but since the support of J"uttner’s function is finite, the causal structure is maintained.

B. $P_N(t, x|0, x_0)$ against $N$

For $P_N(t, x|0, 0)$ against $N$, the parameters are $\chi$, $t$ and $x$.

In the ultra-relativistic situation and in the region with $x$ far away from zero, the distribution among different values of $N$ will be characterized by sharp peaks. Because in the region with $x$ far away from zero, $P_N(t, x|0, 0)$ is characterized by sharp peaks (See Fig.(1)). If the parameter $x$ is fallen into a peak of some $P_N(t, x|0, 0)$, the probability distribution to this value of $N$ will be very large, and those to others will be very small.

As for the case with $x$ near zero, contributions to the total transition PDF (the sum of $P_N(t, x|0, 0)$’s with different values of $N$) come mostly from large $N$. However, for large $N$, $P_N(t, x|0, 0)$ is gaussian-like. So we just consider the large $\chi$ cases in which $P_N(t, x|0, 0)$’s are Gaussian-like.

We give the results in Fig.(3). From Fig.(3), one can see that the curves in this picture are much similar to the curves of the lognormal distributions in Fig.(4). Lognormal distribution has been widely used in chemistry, biology, ecology, social sciences and economics, and many other fields [19]. We propose that the probability distribution among different values of $N$ is a lognormal distribution.
IV. CONCLUSIONS

In this letter, we have considered random walk models of diffusion processes. One of our results is that there will be a causal Zeno effect in causal diffusions. As for the problem of collision number, we have found that the probabilities for the diffusing particle to collide whatever $N - 1$ time with other particles in a given time interval will be the same if the initial position is specified but the final position is not. For the case of that the final position is also fixed, we have made numerical analysis and proposed that the probability distribution among different values of $N$ is a lognormal distribution.

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APPENDIX A: JÜTTNER’S DISTRIBUTION

Jütter’s distribution in 1-dimension space (see Fig. 4 and Fig. 5) is

$$f_{J}(v; m, \beta) = \frac{1}{2K_{1}(\chi)} \frac{\gamma(v)}{e^{-\chi \gamma(v)}}$$

where $\chi = \beta m$, $K_{1}(\chi)$ is a modified Bessel function of the second kind.

The mean is 0 and the variance $\sigma^{2}$ is

$$\sigma^{2} = 1 - \frac{1}{2K_{1}(\chi)} \int_{-1}^{1} dv \gamma(v) e^{-\chi \gamma(v)}.$$  \hfill (A2)

APPENDIX B: LOGNORMAL DISTRIBUTION

Let $g_{\mu, \sigma^{2}}(x)$ be Gauss’s normal distribution,

$$g_{\mu, \sigma^{2}}(x) = \frac{1}{\sqrt{2\pi \sigma^{2}}} e^{-\frac{(x-\mu)^2}{2\sigma^{2}}}.$$  \hfill (B1)

then

$$f(x) = \begin{cases} \frac{1}{\sigma^{2}} g_{\mu, \sigma^{2}}(\log x), & \text{if } x > 0; \\ 0, & \text{if } x \leq 0. \end{cases}$$  \hfill (B2)

is called the Lognormal distribution with parameters $\mu, \sigma^{2}(-\infty < \mu < \infty, 0 < \sigma^{2} < \infty)$ [18, 19, 20] (see Fig. 6).

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