DERIVATIONS OF GROUP ALGEBRAS

A. A. Arutyunov, A. S. Mishchenko, and A. I. Shtern

UDC 517.986.3+514.169+512.544.42

Abstract. In the paper, a method of describing the outer derivations of the group algebra of a finitely presentable group is given. The description of derivations is given in terms of characters of the groupoid of the adjoint action of the group.

Dedicated to the memory of Yu. P. Solovyov

1. Introduction

Consider an algebra $\mathcal{A}$ and some bimodule $E$ over the algebra $\mathcal{A}$. Denote by $\text{Der}(\mathcal{A}, E)$ the space of all derivations from the algebra $\mathcal{A}$ to the bimodule $E$, i.e., the set of mappings $D : \mathcal{A} \rightarrow E$, satisfying the condition

$$D(ab) = D(a)b + aD(b), \quad a, b \in \mathcal{A}$$

(see [5,7]). Among the derivations $\text{Der}(\mathcal{A}, E)$, we can single out the so-called inner derivations $\text{Int}(\mathcal{A}, E) \subset \text{Der}(\mathcal{A}, E)$ that are defined by the adjoint representation,

$$\text{ad}_x(a) \overset{\text{def}}{=} xa - ax, \quad x \in E, \quad a \in \mathcal{A}.$$

The derivation problem is formulated as follows: Are all derivations inner? This problem was considered for group algebras $\mathcal{A} = C[G]$ of some group $G$ rather than for all algebras. To be more precise, the group algebra $\tilde{\mathcal{A}} = L^1(G)$ and the bimodule $E = M(G)$ are considered, where $M(G)$ stands for the algebra of all bounded measures on $G$ with the multiplication operation defined by the convolution of measures.

A question in [3] (Question 5.6.B, p. 746) is formulated as follows: Let $G$ be a locally compact group. Is every derivation from the algebra $\mathcal{A} = L^1(G)$ to the bimodule $E = M(G)$ an inner derivation? An affirmative answer is supported by the following consideration.

For the case in which the group $G$ is a finitely generated discrete free Abelian group, i.e., $G \cong \mathbb{Z}^n$, the algebra $\tilde{\mathcal{A}} = L^1(G)$ can also be identified with the Fourier algebra $A(\mathbb{T}^n)$ of continuous functions on the $n$-dimensional torus $\mathbb{T}^n$ whose Fourier coefficients form an absolutely convergent multiple series, $\mathcal{A} = A(\mathbb{T}^n) \subset C(\mathbb{T}^n)$, (the Fourier algebra is smaller than the algebra of continuous functions). There are no derivations on the algebra $A(\mathbb{T}^n)$ because it has sufficiently many nonsmooth functions; certainly, there are no inner derivations as well, because the algebra $\tilde{\mathcal{A}} = L^1(G)$ is commutative.

We are interested however in a dense subalgebra $\mathcal{A} = C[G] \subset \tilde{\mathcal{A}}$ of the whole Banach algebra $\tilde{\mathcal{A}} = L^1(G)$ (rather than in the algebra $\tilde{\mathcal{A}} = L^1(G)$ itself); this subalgebra consists of the so-called smooth elements of the algebra $\tilde{\mathcal{A}} = L^1(G)$, in the terminology of Connes [1, p. 247]. For the group algebra $\mathcal{A} = C[G]$, one can also formulate a similar problem: Describe the algebra of all outer derivations of the group algebra $\mathcal{A} = C[G]$.

Translated from Fundamentalnaya i Prikladnaya Matematika, Vol. 21, No. 6, pp. 65–78, 2016.
2. Group Algebra $C[G]$

Consider the group algebra $A = C[G]$. Assume that $G$ is a finitely presentable discrete group. An arbitrary element $u \in A$ is a finite linear combination

$$u = \sum_{g \in G} \lambda^g \cdot g.$$

Consider an arbitrary linear operator

$$X : A \to A.$$

A linear operator $X$ has the following matrix form:

$$X(u) = \sum_{h \in G} \left( \sum_{g \in G} x^h_g \lambda^g \right) \cdot h,$$

where $x^h_g$ is defined by the equation

$$X(g) = \sum_{h} x^h_g \cdot h \in A.$$

Since the sum in (2) must be finite, this means that the matrix $X = \| x^h_g \|_{g,h \in G}$ must satisfy the following natural condition:

(F1) for every subscript $g \in G$, the set of superscripts $h \in G$ for which $x^h_g$ is nonzero is finite.

In particular, it follows from condition (F1) that, in the matrix representation (1), the outer sum is also finite.

Certainly, the converse also holds: if a matrix $X = \| x^h_g \|_{g,h \in G}$ satisfies condition (F1), then it well defines a linear operator $X : A \to A$ by formula (1). All this justifies the fact that the operator $X$ and its matrix $X = \| x^h_g \|_{g,h \in G}$ are denoted by the same symbol $X$.

Consider now the so-called derivation in the algebra $A$, i.e., an operator $X$ satisfying the condition

(F2) $X(u \cdot v) = X(u) \cdot v + u \cdot X(v)$, $u, v \in A$.

The set of all derivations of the algebra $A$ is denoted by $\text{Der}(A)$ and forms a Lie algebra with respect to the commutator of operators.

A natural problem is to describe all derivations of the algebra $A$. To this end, one should satisfy the conditions (F1) and (F2). The verification of each of these conditions separately is a more or less simple task. The simultaneous validity of these conditions is the content of the present paper.

There is a class of the so-called inner derivations, i.e., operators of the form

$$X = \text{ad}(u), \quad X(v) = \text{ad}(u)(v) = [u, v] = u \cdot v - v \cdot u, \quad u, v \in A.$$  

All inner derivations satisfy both conditions (F1) and (F2) automatically. The set of these derivations is denoted by $\text{Int}(A)$; it forms a Lie subalgebra in the Lie algebra $\text{Der}(A)$,

$$\text{Int}(A) \subseteq \text{Der}(A).$$

Proposition 1. The Lie subalgebra $\text{Int}(A) \subseteq \text{Der}(A)$ is an ideal.

Indeed, we are to verify the validity of the condition

$$[\text{Int}(A), \text{Der}(A)] \subseteq \text{Int}(A).$$

If $\text{ad}(u) \in \text{Int}(A)$, $X \in \text{Der}(A)$, then the commutator $[\text{ad}(u), X]$ is evaluated by the formula

$$[\text{ad}(u), X](v) = \text{ad}(u)(X(v)) - X(\text{ad}(u)(v))$$

$$= [u, X(v)] - X([u, v]) = [u, X(v)] - [X(u), v] - [u, X(v)] = - \text{ad}(X(u))(v),$$

which implies that $[\text{ad}(u), X] \in \text{Int}(A)$. 710
3. Description of Derivations as Functions on the Groupoid $G$

Denote by $G$ the groupoid associated with the adjoint action of the group $G$ (or the corresponding action groupoid; see, e.g., [4, p. 18, Example j]).

The groupoid $G$ consists of the objects $\text{Obj}(G) = G$ and the morphisms

$$\text{Mor}(a, b) = \{ g \in G: ga = bg \text{ or } b = \text{Ad}(g)(a), \ a, b \in \text{Obj}(G) \}.$$

It is convenient to denote elements of the set of all morphisms

$$\text{Mor}(G) = \coprod_{a, b \in \text{Obj}(G)} \text{Mor}(a, b)$$

in the form of columns

$$\xi = \left( \begin{array}{c} a \\ g \\ b \end{array} \right) \in \text{Mor}(a, b), \ b = gag^{-1} = \text{Ad}(g)(a).$$

The composition $*$ of two morphisms is defined by the formula

$$\left( \begin{array}{c} a \\ g_2g_1 \end{array} \right) \rightarrow \left( \begin{array}{c} b \\ g_2 \end{array} \right) \ast \left( \begin{array}{c} a \\ g_1 \end{array} \right),$$

$$b = \text{Ad}(g_1)(a), \ c = \text{Ad}(g_2)(b) = \text{Ad}(g_2)(\text{Ad}(g_1)(a)) = \text{Ad}(g_2\text{Ad}(g_1)(a)),$$

which corresponds to the diagram

$$\xymatrix{ a \ar[r]^{g_1} & b \ar[r]^{g_2} \\ a \ar[r]_{g_2g_1} & c }.$$

There is another symbol for the morphism:

$$\xi = \left( \begin{array}{c} a \\ g \\ b \end{array} \right)$$

and for the composition of two morphisms:

$$\xymatrix{ a \\ g_2g_1 \\ a \ar[r]_{g_1a=bg_1} & b \ar[r]_{g_2b=cg_2} & c }.$$

**Operators as functions on the groupoid.** A linear operator $X: A \rightarrow A$ is described by a matrix $X = \|x^h_g\|_{g, h \in G}$ satisfying the following condition:

(F1) for every subscript $g \in G$, the set of all superscripts $h \in G$ for which $x^h_g$ is nonzero is finite.

The matrix $X = \|x^h_g\|_{g, h \in G}$ defines a function on the groupoid $G$,

$$T^X: \text{Mor}(G) \rightarrow R,$$

associated with the operator $X$; this function is defined by the following formula: if

$$\xi = \left( \begin{array}{c} a \\ g \\ b \end{array} \right) \in \text{Mor}(G),$$

then we set

$$T^X(\xi) = T^X \left( \begin{array}{c} a \\ g \\ b \end{array} \right) = x^g_{a=b}. 711$$
The condition (F1) imposed on the coefficients of the matrix $X$ can be reformulated in terms of the function $T$:

(T1) for every element $g \in G$, the set of morphisms of the form

$$\xi = \left( \frac{a \rightarrow b}{g} \right)$$

for which $T^X(\xi) \neq 0$ is finite.

The set of all morphisms $\text{Mor}(G)$ can be represented in the form of a disjoint union

$$\text{Mor}(G) = \coprod_{g \in G} \mathcal{H}_g,$$

where

$$\mathcal{H}_g = \left\{ \xi = \left( \frac{a \rightarrow b}{g} \right) : a \in G, \ b = gag^{-1} \in G \right\}.$$

Then condition (T1) imposed on the function $T$ can be reformulated in an equivalent way as follows.

**Proposition 2.** A function $T^X: \text{Mor}(G) \rightarrow \mathbb{C}$ is defined by a linear operator $X: A \rightarrow A$ if and only if, for every element $g \in G$, the restriction $(T^X)|_{\mathcal{H}_g}: \mathcal{H}_g \rightarrow \mathbb{C}$ is a finitely supported function.

**Theorem 1.** An operator $X: A \rightarrow A$ is a derivation if and only if the function $T^X$ on the groupoid $G$ associated with the operator $X$ satisfies the condition

(T2) $T^X(\eta \ast \xi) = T^X(\eta) + T^X(\xi)$

for every pair of morphisms $\xi$ and $\eta$ admitting the composition $\eta \ast \xi$.

**Proof.** Let the matrix of the operator $X$ be of the form $X = \|x_{gh}\|_{g,h \in G}$. Thus, the function $T^X$ takes the value

$$T^X(\xi) = T^X\left( \frac{a \rightarrow b}{g} \right) = x_{g g^{-1} b g^{-1}}^g.$$ 

Let

$$\xi = \left( \frac{a \rightarrow b}{g_1} \right), \ \eta = \left( \frac{b \rightarrow c}{g_2} \right), \ \eta \ast \xi = \left( \frac{a \rightarrow c}{g_1 g_2} \right).$$

Then

$$T^X(\eta \ast \xi) = x_{g_2 g_1 a b g_2} = x_{g_2 g_1}^h, \quad T^X(\xi) = x_{g_1 b g_2} = x_{g_1}^{g_2}, \quad T^X(\eta) = x_{g_2 g_1} = x_{g_1}^{g_2}.$$ 

On the other hand,

$$X(g_2 g_1) = X(g_2) g_1 + g_2 X(g_1).$$

In other words,

$$X(g_2 g_1) = \sum_{h \in G} x_{g_2 g_1}^h \cdot h = \sum_{h \in G} x_{g_2}^h \cdot g_1 + g_2 \cdot \sum_{h \in G} x_{g_1}^h \cdot h = \sum_{h \in G} x_{g_2}^{g_1} \cdot h + \sum_{h \in G} x_{g_1}^{g_2} \cdot h.$$
Therefore,

\[ x^h_{g_2g_1} = x^h_{g_2} + x^g_{g_1} \]

Thus,

\[ T^X(\eta \ast \xi) = T^X(\eta) + T^X(\xi). \]

We refer to a function \( T : \operatorname{Mor}(G) \to R \) on the groupoid \( G \) satisfying the additivity condition (T2) as a character; denote the set of all characters on the groupoid \( G \) by \( T(G) \). Denote the space of all locally finitely supported characters of the groupoid \( G \) by \( T_f(G) \subset T(G) \).

Thus, there is a mapping

\[ \operatorname{Der}(A) \xrightarrow{T} T_f(G), \]

which is one-to-one.

4. Inner Derivations

There are works (see, e.g., [7]) related to the so-called inner derivations of the group algebra. The commutator in the algebra is a derivation, which is called an inner derivation.

A natural question arises: How are the inner derivations described in terms of the matrix of the operator of the derivation?

The answer can be formulated as follows. Let \( a \in G \), and let \( \text{ad}(a) \) be the commutator,

\[ \text{ad}(a)(x) = [a, x], \quad x \in C^\infty(G). \]

This is an inner derivation. Denote by \( \|A^h_g\| \) the matrix of the derivation \( \text{ad}(a) \). Then

\[ \text{ad}(a)(g) = \sum_{h \in G} A^h_g \cdot h. \]

Since \( \text{ad}(a)(g) = ag - ga \), it follows that

\[ A^h_g = \delta^ag_h - \delta^ga_h. \]

The matrix of the operator \( \text{ad}(a) \) defines the function \( T^{\text{ad}(a)} \) on the set of all morphisms \( \operatorname{Mor}(G) \) of the category \( G \). Let

\[ \xi = \left( \begin{array}{c} \alpha \\ g \end{array} \right) \]

be a morphism in the category \( G \), and let \( g\alpha = \beta g = g \). Then

\[ T^{\text{ad}(a)}(\xi) = T^{\text{ad}(a)} \left( \begin{array}{c} \alpha \\ g \end{array} \right) = A^g_{\alpha=\beta} = \delta^ag_{\alpha=\beta} - \delta^ga_{\alpha=\beta}. \]

The first summand in the function \( T^{\text{ad}(a)}(\xi) \) is equal to 1 if and only if \( \beta = a \), i.e., if and only if \( \xi \in \operatorname{Mor}(g^{-1}ag, a) \). Similarly, the second summand in the function \( T^{\text{ad}(a)}(\xi) \) is equal to \(-1\) if and only if \( \alpha = a \), i.e., if and only if \( \xi \in \operatorname{Mor}(a, gag^{-1}) \).

In other words, the matrix \( \text{ad}(a) \) is equal to 1 on the morphisms \( \operatorname{Mor}(g^{-1}ag, a) \), to \(-1\) on the morphisms \( \operatorname{Mor}(a, gag^{-1}) \), and is equal to 0 on the morphisms \( \operatorname{Mor}(u, u) \) and on the morphisms \( \operatorname{Mor}(a, a) \) and \( \operatorname{Mor}(v, v) \) in the following diagram:

\[
\begin{array}{ccc}
T=0 & T=0 & T=0 \\
\text{u} & \text{a} & \text{v} \\
T=+1 & T=-1 & \\
\end{array}
\]

This proves the following theorem.
Theorem 2 (on the inner derivations). The characters of the inner derivations are trivial on $\text{Mor}(a, a)$:

$\begin{align*}
\text{Int}(A) & \longrightarrow \text{Der}(A) \\
\cap \quad \cong \quad T \\
\ker p_a & \longrightarrow T_f(G) \quad \longrightarrow_{p_a} T_f(\text{Mor}(a, a))
\end{align*}$

The set $T_f(\text{Mor}(a, a))$ coincides with the group of all characters

$T_f(\text{Mor}(a, a)) = T(\text{Mor}(a, a))$.

Note that, if a character $T \in T(\text{Mor}(a, a))$ vanishes on $\text{Mor}(a, a)$, then it vanishes on $\text{Mor}(u, u)$ for every conjugate element $u \in [a]$, $u = gag^{-1}$. Thus, the diagram has the following form:

$\begin{align*}
\text{Int}(A) & \longrightarrow \text{Der}(A) \\
\cap \quad \cong \quad T \\
\ker p_a & \longrightarrow T_f(G) \quad \longrightarrow_{p_a} T(\text{Mor}(a, a))
\end{align*}$

From the viewpoint of the Johnson derivation problem [6], it is natural to denote by $\text{Out}(A)$ the quotient group $\text{Out}(A) = \text{Der}(A)/\text{Int}(A)$ and refer to it as the algebra of outer derivations of the algebra $A$. Thus, the previous diagram is completed to the diagram

$\begin{align*}
0 & \longrightarrow \text{Int}(A) \longrightarrow \text{Der}(A) \longrightarrow \text{Out}(A) \longrightarrow 0 \\
\cap \quad \cong \quad T \\
0 & \longrightarrow \ker p_a \longrightarrow T_f(G) \quad \longrightarrow_{p_a} T(\text{Mor}(a, a))
\end{align*}$

Description of Inner Derivations. First, let us note that the set of morphisms $\text{Mor}(G)$ of the groupoid $G$ is decomposed into a disjoint union of morphisms over the conjugacy classes of $G$, which are the objects of the groupoid $G$ by definition. The group $G$ is decomposed into the disjoint union of the conjugacy classes

$G = \bigsqcup_{g \in G} [g], \quad [g] = \{h : \exists a \in G, h = aga^{-1}\}$.

The set of morphisms is also represented as the disjoint union

$\text{Mor}(G) = \bigsqcup_{[g]} \text{Mor}(G_{[g]})$.

This means that the construction of every derivation can be carried out in the form of derivations $\text{Der}_{[g]}(A)$ independently in every subcategory $G_{[g]}$ as locally finitely supported characters on each of these subcategories.

A natural problem is to find out whether or not the set of all derivations trivial on all $\text{Mor}(u, u)$ coincides with the set of inner derivations, in other words, whether or not the embedding

$\text{Int}_{[u]}(A) \subseteq \ker p_u$

is an isomorphism:

$\begin{align*}
\text{Int}_{[u]}(A) & \subseteq \ker p_u \subseteq T_f(G) \quad \subseteq_{p_u} T(\text{Mor}(u, u))
\end{align*}$

The investigation of this problem enables us to formulate specific conditions on a locally finitely supported character $T : \text{Mor}(G) \rightarrow R$ that realizes a given inner derivation $X \in \text{Int}(G)$,

$T = T^X$. 

714
Case of the identity element \([e]\). In particular, one of the subcategories corresponds to the identity element \(e \in G\) for which \([e] = \{e\}\). In this special case, the subcategory \(G_{[e]}\) consists of a single object \(e \in G\) and the set of morphisms is isomorphic to the group \(G\), \(\text{Mor}(e, e) \approx G\). In particular, the set of locally finitely supported characters \(T_f(G_{[e]}) \approx T(G_{[e]}) \approx T(G)\) is isomorphic to the group of all characters on the group \(G\). Every character on the group, \(T \in T(G)\), is realized as a derivation \(X \in \text{Der}(G)\), \(TX = T\). Indeed, the character \(T \in T(G)\) is a character on the category \(G\) which is equal to \(T\) on \(\text{Mor}(e, e)\) and to zero on the other summands \(\text{Mor}(G_{[g]}), g \neq e\). Therefore, the corresponding matrix \(\|X_g^h\|\) of the operator \(X\) is given by the formula

\[
X_g^h = T(g)\delta_g^h.
\]

All derivations corresponding to the characters on the subcategory \(G_{[e]}\) are not inner derivations.

Similar considerations fit for the other conjugacy classes that consist of finitely many elements, i.e., when \(#[g] < +\infty\). In particular, this holds for the elements in the center \(g \in Z(G)\).

Exact sequence. If we get rid of the condition that the characters are locally finitely supported, then one can establish that some sequence is exact, as is formulated in the following theorem.

**Theorem 3.** The following sequence is exact:

\[
0 \to \ker p_a \to T(G_{[a]}) \xrightarrow{p_a} T(\text{Mor}(a, a)) \to 0, \quad a \in U.
\]

**Proof.** One should prove only that the mapping \(p(a)\) is epimorphic.

Let \(\chi \in X(G^a)\). Choose an element \(a \in U\) and arbitrary elements \(g_{a,b} \in \text{Mor}(a, b)\) for all \(b \in U\) such that \(g_{a,a} = e\). Elements of this kind exist indeed, because \(\text{Mor}(a, b)\) is nonempty. Write further

\[
g_{b,c} = g_{a,c}g_{a,b}^{-1} \in \text{Mor}(b, c).
\]

Note that the condition \(g_{b,c} \in \text{Mor}(b, c)\) means that

\[
g_{b,c}b = cg_{b,c}.
\]

Let us prove this condition. First,

\[
g_{a,b}a = bg_{a,b}
\]

by construction. Further, \(g_{b,a} = g_{a,a}g_{a,b}^{-1} = g_{a,b}^{-1} \in \text{Mor}(b, a)\), i.e.,

\[
g_{b,c} = g_{a,c}g_{a,b}.
\]

Thus,

\[
g_{b,c}b = g_{a,c}g_{b,a}b = g_{a,c}ag_{b,a} = cg_{a,c}g_{b,a} = cg_{b,c},
\]

i.e.,

\[
g_{b,c} \in \text{Mor}(b, c).
\]

Hence, the following relation holds:

\[
g_{c,d}g_{b,c} = g_{b,d} \in \text{Mor}(b, d).
\]

Indeed,

\[
g_{c,d}g_{b,c} = g_{a,d}g_{c,a}g_{a,c}g_{b,a} = g_{a,d}g_{b,a} = g_{b,d} \in \text{Mor}(b, d).
\]

Let us now construct a character \(X\) on the groupoid \(G_U\), \(X : \text{Mor}(U) \to \mathbb{R}\). Let \(x_{b,c} \in \text{Mor}(b, c)\) be an arbitrary morphism. Then

\[
g_{a,x_{b,c}g_{a,b}} \in \text{Mor}(a, a) = G^a.
\]

Write

\[
X(x_{b,c}) \overset{\text{def}}{=} \chi(g_{c,a}x_{b,c}g_{a,b}).
\]

It can readily be seen that the mapping \(X : \text{Mor}(U) \to \mathbb{R}\) is additive,

\[
X(x_{c,d}x_{b,c}) = \chi(g_{d,a}(x_{c,d}x_{b,c})g_{a,b}) = \chi(g_{d,a}x_{c,d}x_{b,c}g_{a,c}x_{b,c}g_{a,b}) = \chi(g_{d,a}x_{c,d}g_{a,c}x_{b,c}g_{a,b})X(x_{c,d})X(x_{b,c}).
\]

715
The restriction of the character \( X \) to \( \text{Mor}(a,a) = G^a \) coincides with \( \chi \):
\[
X(x_{a,a}) = \chi(g_{a,a}x_{a,a}g_{a,a}) = \chi(x_{a,a}).
\]

**Reduction to groups of cochains.** Let us return to the study of characters \( T \) that are trivial on the subspace \( \text{Mor}(a,a) \),
\[
\textbf{Int}_a(A) \xrightarrow{\subset} \ker p^f_a \xrightarrow{\subset} T_f(G_{[a]}) \xrightarrow{\rho^f} T(\text{Mor}(a,a)).
\]
Denote by \( \Delta(G_{[a]}) \) the simplex whose vertices are the elements of the conjugacy class \( [a] \subset G = \text{Obj}(G_{[a]}). \)

Since every character \( T \in \ker p^f_a \) takes equal values on the set of all morphisms \( \text{Mor}(b,c) \), \( b, c \in [a] \subset G \), it follows that there is a natural embedding
\[
\varphi: \ker p^f_a \hookrightarrow C^1(\Delta(G_{[a]}))
\]
in the group of cochains of the simplex \( \Delta(G_{[a]}), \) and every character \( T \in \ker p^f_a \) is a cocycle:
\[
\begin{array}{c}
C^0(\Delta(G_{[a]})) \xrightarrow{\varphi_0} C^0(\Delta(G_{[a]})) \xrightarrow{\delta} C^1(\Delta(G_{[a]})) \xrightarrow{\delta} C^2(\Delta(G_{[a]}))
\end{array}
\]

The embedding \( \varphi_0 \) is an isomorphism. The image
\[
\delta\left(\varphi_0(\textbf{Int}_a(A))\right) \subset \delta\left(C^0_f(\Delta(G_{[a]}))\right) \subset \ker p^f_a \subset C^1(\Delta(G_{[a]}))
\]
can be described as some set of cocycles satisfying certain conditions. Consider the conjugacy class \( [a] \) on which the group \( G \) acts by the adjoint action
\[
G \times [a] \xrightarrow{\text{ad}} [a], \quad \text{ad}_g(b) = gbg^{-1}, \quad b \in [a].
\]
For an arbitrary element \( g \in G \), consider the graph \( \Gamma_{g,a} \subset \Delta(G_{[a]}) \), formed by directed edges whose beginnings are the elements \( b \in [a] \) and the ends are \( gbg^{-1} \in [a] \). Thus, every edge is of the form of a directed segment \([b, gbg^{-1}]\). The graph \( \Gamma_g \) is decomposed into a disjoint union of directed paths formed by the directed edges
\[
\Gamma_{g,a} = \bigcup_{\alpha} \Gamma_{g,a}^\alpha.
\]
Every directed path \( \Gamma_{g,a}^\alpha \) can be infinite in both directions or finite, in which case this path is cyclic.

**Theorem 4.** Let \( X \in \textbf{Int}_{[a]}(A) \). Then the cochain \( \varphi(TX) \in C^1(\Delta(G_{[a]})) \) satisfies the following condition:

(FF) the cochain \( \varphi(TX) \) is finitely supported on the graph \( \Gamma_{g,a} \), and the sum of values of the cochain \( \varphi(TX) \) on every directed path \( \Gamma_{g,a}^\alpha \) vanishes.

Conversely, if an operator \( X \in \text{Der}(A) \), \( TX \in \ker p^f_a \subset T_f(G_{[a]}), \) satisfies condition (FF), then \( X \in \textbf{Int}_{[a]}(A) \).

The kernel \( \ker p^f_a \) differs from the set of inner derivations. There are examples of groups for which there are locally finitely supported characters \( T \in \ker p^f_a \) that do not satisfy condition (FF). For the simplest example, consider the free group with two generators \( G = \langle x_1, x_2 \rangle \). For the conjugacy class we take the class \([x_1] \subset G\).

Consider the character \( T: \text{Mor}(G_{[x_1]}) \to R \), defining the values of the character on the generators of the groupoid \( \text{Mor}(G_{[x_1]}) \) independently of one another. The set of generators of the groupoid \( \text{Mor}(G_{[x_1]}) \) consists of the morphisms of the form
\[
\xi = \left( \frac{\alpha \rightarrow \beta}{g} \right), \quad g = x_1, x_2; \quad \beta = g\alpha g^{-1}, \quad \alpha \in [x_1],
\]

\[716\]
Set
\[ T\left(x_2x_1x_2^{-1}x_1x_2x_1x_2^{-1}x_1^{-1}\right) = 1, \]
and let the character \( T \) vanish on the other generators. Since in the free group there are no relations except for natural reductions in words, it follows that the function \( T \) can be extended by additivity to some character on the groupoid \( G_{x_1} \).

Hence, the character \( T \) takes only one value equal to 1 on one of the directed paths of the form \( \Gamma_1^\alpha \), and this character is identically equal to 0 on the other directed paths. This means that condition (T1) holds, and condition (FF) fails to hold.

5. Appendix: Groups with Finitely Many Generators

Let a group \( G \) be finitely presentable, i.e., let it have finitely many generators \( \{g_1, g_2, \ldots, g_k\} \) and finitely many defining relations \( \{S_1, S_2, \ldots, S_l\} \):
\[ G = F\langle g_1, g_2, \ldots, g_k \rangle / \{S_1, S_2, \ldots, S_l\}. \]
Every relation \( S_i \) is a word of length \( s_i \) formed of generating elements \( g_1, g_2, \ldots, g_k \) or inverses of the generating elements \( g_1^{-1}, g_2^{-1}, \ldots, g_k^{-1} \). Thus, the relation \( S_i \) can be represented in the following form:
\[ S_i = g^\varepsilon_{j(i,1)} g^\varepsilon_{j(i,2)} g^\varepsilon_{j(i,3)} \cdots g^\varepsilon_{j(i,s_i)} = 1, \quad \varepsilon_{j(i,j)} = \pm 1. \]

Every relation \( S_i \) induces a series of relations on the groupoid \( G \):
\[ \xi_{j(i,1)} \xi_{j(i,2)} \xi_{j(i,3)} \cdots \xi_{j(i,s_i)} = \left( \frac{\alpha_{j(i,1)} \rightarrow \alpha_{j(i,1)}}{1} \right), \]
where the morphisms \( \xi_{j(i,j)} \) are defined by the rule
\[ \xi_{j(i,j)} = \left( \frac{\alpha_{j(i,j)} \rightarrow \beta_{j(i,j)}}{g^\varepsilon_{j(i,j)} \beta_{j(i,j)}} \right), \]
\[ \beta_{j(i,j)} = g^\varepsilon_{j(i,j)} \alpha_{j(i,j)} g^{-\varepsilon_{j(i,j)}}, \]
\[ \beta_{j(i,j)} = \alpha_{j(i,j+1)}, \quad \beta_{j(i,s_i)} = \alpha_{j(i,1)}. \]

Thus, to define a locally finitely supported character \( T \) on the groupoid \( G \), it suffices to define the values of the locally finitely supported character \( T \) on the set of generators \( \prod_{i=1}^k H_{g_i} \) in such a way that the additivity condition
\[ T(\xi_{j(i,1)}) + T(\xi_{j(i,1)}) + T(\xi_{j(i,3)}) + \cdots + T(\xi_{j(i,s_i)}) = 0 \]
holds on every relation of the form
\[ \xi_{j(i,1)} \xi_{j(i,2)} \xi_{j(i,3)} \cdots \xi_{j(i,s_i)} = \left( \frac{\alpha_{j(i,1)} \rightarrow \alpha_{j(i,1)}}{1} \right). \]

The first author was financially supported by Ministry of Education and Science of the Russian Federation (Agreement No. 02.a03.21.0008 of 24.06.2016). The second author was partially supported by the RFBR grant No. 14-01-00007. The third author was partially supported by the RFBR grant No. 14-01-00007.
REFERENCES

1. A. Connes, *Noncommutative Geometry*, Academic Press, New York (1994).
2. H. G. Dales, “Automatic continuity: a survey,” *Bull. London Math. Soc.*, **10**, No. 2, 129–183 (1978).
3. H. G. Dales, *Banach Algebras and Automatic Continuity*, Clarendon Press, Oxford Univ. Press, New York (2000).
4. A. V. Ershov, *Categories and Functors* [in Russian], Nauka, Saratov (2012).
5. F. Ghahramani, V. Runde, and G. Willis, “Derivations on group algebras,” *Proc. London Math. Soc.*, **80**, No. 2, 360–390 (2000).
6. B. E. Johnson, “The derivation problem for group algebras of connected locally compact groups,” *J. London Math. Soc.*, **63**, No. 2, 441–452 (2001).
7. V. Losert, “The derivation problem for group algebras,” *Ann. Math.*, **168**, No. 1, 221–246 (2008).

A. A. Arutyunov
Moscow Institute of Physics and Technology, Moscow, Russia
E-mail: andronick.arutyunov@gmail.com

A. S. Mishchenko
Moscow State University, Moscow, Russia
E-mail: asmish.prof@gmail.com

A. I. Shtern
Moscow State University, Moscow, Russia
E-mail: aishtern@mtu-net.ru