LIE SUPERALGEBRAS ARISING FROM BOSONIC REPRESENTATION

Naihuan Jing\(^1\),\(^2\) and Chongbin Xu\(^1\),\(^3\)

\(^1\)School of Sciences, South China University of Technology, Guangzhou, China
\(^2\)Department of Mathematics, North Carolina State University, Raleigh, North Carolina, USA
\(^3\)College of Mathematics and Information Science, Wenzhou University, Zhejiang, China

A 2-toroidal Lie superalgebra is constructed using bosonic fields and a ghost field. The superalgebra contains \(osp(1|2n)^{(1)}\) as a distinguished subalgebra and behaves similarly to the toroidal Lie superalgebra of type \(B(0,n)\). Furthermore, this algebra is a central extension of the algebra \(osp(1|2n) \otimes \mathbb{C}[s, s^{-1}, t, t^{-1}]\).

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1. INTRODUCTION

Realization of Lie (super)algebras via bosonic or fermionic fields has been a successful approach in solving various problems in mathematical physics. From mathematical perspectives, representing an algebra in terms of known classical algebraic structures (Weyl or Clifford algebras) amounts to constructing a homomorphism from the initial algebra to the target algebra. In this way properties of the initial algebra can be studied by techniques of linear algebra. On one hand representations of the target algebra give rise to representations of the initial algebra which then enable one to gain information about eigenvalues of the operators or the expectation values of various physical quantities. On the other hand, physical intuition usually suggests how this homomorphism can be established, and this may happen in several different contexts or via different realizations. Mathematically, it may be proved that all these realizations are actually representing the same algebra.

Lie (super)algebras play an important role in both mathematics and physics. The realization problem, in particular for infinite dimensional cases, is one of the first questions to be studied. It was well-known in physics literature that finite-dimensional simple Lie (super)algebras of classical types can be realized by fermionic or bosonic operators, i.e., within Clifford or Weyl algebras. However, the
problem for affine Lie (super)algebras is more involved and requires sophisticated generalization. By 1980’s the problem of realizing affine Lie (super)algebras had been solved partly by several groups [3, 13]. In the famous work [2], A. Feingold and I. Frenkel realized classical affine Lie algebras using fermionic fields and bosonic fields respectively. Since then, their method has been generalized to other algebras such as extended affine Lie algebras, affine Lie superalgebras, Tits–Kantor–Köcher algebras, toroidal Lie algebras, Lie algebras with central extensions, two-parameter quantum affine algebras etc. (see [5, 8, 10, 14, 15, 17]).

More recently, in [7, 9] 2-toroidal Lie algebras of classical types were realized uniformly using bosonic fields or fermionic fields with help of a ghost field based on the Moody–Rao–Yokonuma presentation of toroidal Lie algebras [16]. The method used in these two papers not only generalizes Feingold–Frenkel construction to toroidal algebras but also fills up missing bosonic/fermionic realizations for orthogonal/symplectic types.

One can ask a similar question on how to realize 2-toroidal Lie superalgebra of classical types. In this paper, we first define a central extension of the superalgebra $\mathfrak{g} \otimes \mathbb{C}[s, s^{-1}, t, t^{-1}]$ which we will call the loop-like toroidal Lie superalgebra $\mathcal{Z}(B(0, n))$ of type $B(0, n)$ in light of [16] presentation, and then construct its representation using mixed bosons and fermions as well as a ghost field. Since the kernel of the homomorphism is contained in fields corresponding to imaginary roots, our representation can be lifted to that of the universal central extension of the 2-toroidal Lie superalgebra. Thus our loop-like toroidal algebra $\mathcal{Z}(B(0, n))$ and the universal toroidal Lie algebra $T(B(0, n))$ are mutual homomorphic images. It would be interesting to show that the loop-like toroidal superalgebra is indeed the universal central extension of the superalgebra $\mathfrak{g} \otimes \mathbb{C}[s, s^{-1}, t, t^{-1}]$. Our result may be viewed as a testing example for this, and we hope further computations can be made to reveal the structure of the kernel of the map from the loop-like toroidal superalgebra to the universal central extension.

2. TOROIDAL LIE SUPERALGEBRA OF TYPE $B(0, n)$

A Lie superalgebra $\mathfrak{g} = \mathfrak{g}_\mathfrak{e} \oplus \mathfrak{g}_\mathfrak{r}$ is a $\mathbb{Z}_2$-graded vector space equipped with a bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that the following equation hold:

1) $[\mathfrak{g}_\mathfrak{e}, \mathfrak{g}_\mathfrak{e}] \subseteq \mathfrak{g}_\mathfrak{e}$;
2) $[a, b] = (-1)^{\deg(a)\deg(b)}[b, a]$;
3) $[a, [b, c]] = [[a, b], c] + (-1)^{\deg(a)\deg(b)}[b, [a, c]],$

where $a, b, c \in \mathbb{Z}_2$ and $a, b, c$ are homogenous elements.

According to Kac [11] simple Lie superalgebras are classified into two families—the classical types and the Cartan types. Among the classical superalgebras, one usually separates the strange series $P(n)$ and $Q(n)$ from the list of basic superalgebras: the series $A(m, n), B(m, n), C(n + 1), D(m, n)$ and the exceptional types $F(4), G(3)$ and $D(2, 1; \alpha)$. The other symplectic series $B(m, n)$ can be further divided into two classes: $m > 0$ and $m = 0$. In this paper, we consider the simplest case $B(0, n)$.

Let $R = \mathbb{C}[s, t]$ be the ring of Laurent polynomials in $s, t$. Let $\Omega_R$ be the $R$-module of differentials spanned by $da, a \in R$, and $d\Omega_R$ is the space of exact forms. Then $\Omega_R/d\Omega_R$ has a basis consisting of $s\partial s, t\partial t, dt, ds$. Let $\mathfrak{g}$ be a simple
Lie superalgebra, the toroidal Lie superalgebra $T(\mathfrak{g})$ is the central extension of the loop superalgebra $\mathfrak{g} \otimes R$:

$$T(\mathfrak{g}) = \mathfrak{g} \otimes R \oplus \Omega_R/d\Omega_R$$

under the Lie (super)bracket: $(x, y \in \mathfrak{g}, a, b \in R)$

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab + (x \mid y)(da)b.$$

with parities defined by $\deg(x \otimes a) = \deg(x)$, $\deg(\Omega_R/d\Omega_R) = 0$.

In practice, one can define a Lie superalgebra by generators and relations with appropriate parities for generators. In what follows, we will define the so-called toroidal Lie superalgebra of type $B(0, n)$ this way.

For $n \geq 1$, let $A = (a_{ij})$ be the extended distinguished Cartan matrix of the affine Lie superalgebra $B(0, n)$, i.e.,

$$A = \begin{pmatrix}
2 & -1 & 0 & \cdots & 0 & 0 & 0 \\
-2 & 2 & -1 & \cdots & 0 & 0 & 0 \\
0 & -1 & 2 & \cdots & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 2 & -1 & 0 \\
0 & 0 & 0 & \cdots & -1 & 2 & -1 \\
0 & 0 & 0 & \cdots & 0 & -2 & 2
\end{pmatrix},$$

and let $Q = \mathbb{Z}x_0 \oplus \mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_n$ be the root lattice, where $x_0, \ldots, x_{n-1}$ are even roots and $x_n$ is odd. The standard invariant form is given by

$$(x_i, x_j) = d_i a_{ij},$$

where $(d_0, d_1, \ldots, d_{n-1}, d_n) = (2, 1, \ldots, 1, 1/2)$, and thus $2(x_i, x_j)/(x_i, x_i) = a_{ij}$. Let $\delta = x_0 + 2 \sum_{i=1}^n x_i$.

**Definition 2.1.** The (loop-like) toroidal Lie superalgebra $\mathcal{X}$ of type $B(0, n)$ is the Lie superalgebra generated by

$$\{\mathcal{X}, x_i(k), x_i^+(k) \mid 0 \leq i \leq n, k \in \mathbb{Z}\}$$

with parities given as

$$\deg(\mathcal{X}) = \deg(x_i(k)) = 0, \quad 0 \leq i \leq n, \quad k \in \mathbb{Z},$$

$$\deg(x_i^+(k)) = \begin{cases}
\bar{0} & \text{if } 0 \leq i \leq n-1, \quad k \in \mathbb{Z}, \\
1 & \text{if } i = n, \quad k \in \mathbb{Z},
\end{cases}$$

and defining the following relations:

1) $[\mathcal{X}, x_i(k)] = [\mathcal{X}, x_i^+(k)] = 0$;

2) $[x_i(k), x_j(l)] = k(x_i \mid x_j)\delta_{m-n, \mathcal{X}}$;
3) \([x_i(k), x_j^+(l)] = \pm (x_i, x_j)x_i^+(k + l);\)
4) \([x_i^-(k), x_j^-(l)] = -\delta_{ij}\frac{k}{(k + l)}[x_i(k + l) + k\delta_{k,-l}];\)
5) \([x_i^+(k), x_j^+(l)] = [x_i^-(k), x_j^-(l)] = 0; (\text{ad}x_i^+(0))^{1-a_{ij}}x_i^+(k) = 0, \text{if } i \neq j; (\text{ad}x_i^-(0))^{1-a_{ij}}x_i^+(k) = 0, \text{if } i \neq j,

where the super-bracket is adopted: \([X, Y] = XY - (-1)^{\text{deg}(X)\text{deg}(Y)}YX.\)

The algebra \(\mathcal{Z}\) is a \(Q \times \mathbb{Z}\)-graded Lie superalgebra under the grading \(\text{deg} \mathcal{Z} = (0, 0), \text{deg} x_i(k) = (0, k), \text{deg} x_i^+(k) = (\pm x_i, k).\) We denote the subspace of degree \((x, k)\) by \(\mathcal{Z}^x_k.\)

\[\mathcal{Z} = \bigoplus_{(x, k) \in Q \times \mathbb{Z}} \mathcal{Z}^x_k.\]  

(2.1)

We remark that the center of this algebra is contained in the subalgebra generated by \(\mathcal{Z}_0^{0}, k \in \mathbb{Z}.

Let \(e_i, f_i, h_i (i = 1, 2, \ldots, n)\) be the Chevalley generators of Lie superalgebra \(\mathcal{G}\) of type \(B(0, n)\) corresponding to the distinguished simple root system \(\Pi = \{x_1, \ldots, x_n\}\) and \(\theta\) be the longest root relative to \(\Pi.\) We also choose \(e_0 \in \mathcal{G}_0, f_0 \in \mathcal{G}_{-\theta}\) and \(h_0\) as in the affine Lie algebra, then we have the following proposition.

**Proposition 2.2.** The following map defines a surjective homomorphism from loop-like toroidal superalgebra \(\mathcal{Z}\) to the algebra \(T(B(0, n)).\)

\[\mathcal{Z} \mapsto s^{-1}ds\]
\[x_i(k) \mapsto d_i(h_i \otimes s^k + \delta_{i0}s^k dt), \quad i = 0, \ldots, n\]
\[x_i^+(k) \mapsto e_i \otimes s^k, \quad i = 1, \ldots, n\]
\[x_i^-(k) \mapsto -f_i \otimes s^k, \quad i = 1, \ldots, n\]
\[x_0^+(k) \mapsto e_0 \otimes s^k t^{-1},\]
\[x_0^-(k) \mapsto -f_0 \otimes s^k t\]

**Proof.** It is straightforward to check that the elements on the right satisfy the relations in Definition 2.1. For example,

\[\left[ h_i \otimes s^k, h_j \otimes s^l \right] = [h_i, h_j] \otimes s^{k+l} + (h_i, h_j) s^k d(s^l)\]
\[= d_i^{-1}d_j^{-1}k(x_i, x_j)\delta_{k,-l}s^{-1}ds.\]

In fact the loop-like toroidal Lie superalgebra \(\mathcal{Z}\) is a central extension of the algebra \(\mathcal{G} \otimes \mathbb{C}[t, t^{-1}, s, s^{-1}],\) as the kernel is contained in the subspace \(\bigoplus_{n,k} \mathcal{Z}^{n,k}_0,\) which is clearly central by the commutation relations. General theory of central extension of Lie superalgebra has been studied as the usual Lie algebras [6]. According to [6] \(T(B(0, n))\) is the universal central extension of the algebra \(\mathcal{G} \otimes \mathbb{C}[t, t^{-1}, s, s^{-1}],\) thus the loop-like toroidal algebra \(\mathcal{Z}\) is also a homomorphic image of \(T(B(0, n)).\) It would be interesting to show that the algebra \(\mathcal{Z}\) is indeed the universal central extension.
For convenience, we will present the structure of \( \mathcal{E} \) in terms of generating series. To this end, we define the generating series with coefficients from \( \mathcal{E} \):

\[
\alpha_i(z) = \sum_{k \in \mathbb{Z}} \alpha_i(k) z^{-k-1}, \quad x_i^+(z) = \sum_{k \in \mathbb{Z}} x_i^+(k) z^{-k-1}.
\]

**Proposition 2.3.** The relations of \( \mathcal{E} \) can be written as follows:

1') \( [\mathcal{E}, \alpha_i(z)] = [\mathcal{E}, x_i^+(z)] = 0; \)
2') \( [\alpha_i(z), x_j(w)] = (\alpha_i | x_j) \hat{e}_w \delta(z - w) \mathcal{E}; \)
3') \( [\alpha_i(z), x_i^+(w)] = \pm (\alpha_i | x_i) x_i^+(w) \delta(z - w); \)
4') \( [x_i^+(z), x_j^+(w)] = -\delta_{ij} \frac{z^{a_{ij}}}{(z-w)\hat{e}_i \mathcal{E}} (\alpha_i(w) \delta(z - w) + \hat{e}_i \delta(z - w) \mathcal{E}); \)
5') \( [x_i^+(z), x_j^-(w)] = [x_i^-(z), x_j^-(w)] = 0; \quad (\prod_{k=1}^{1-a_{ij}} \text{ad} x_i^+(z_k)) x_j^+(w) = 0, \text{if } i \neq j; \quad (\prod_{k=1}^{1-a_{ij}} \text{ad} x_i^-(z_k)) x_j^-(w) = 0, \text{if } i \neq j. \)

**Proof.** The proposition follows from Definition 2.1 and the two useful expansions of formal delta function \( \delta(z - w) = \sum_{k \in \mathbb{Z}} w^k z^{-k-1} \) (cf. [12]):

\[
\delta(z - w) = i_{z,w} \frac{1}{z-w} + i_{w,z} \frac{1}{z-w}, \quad \hat{e}_w \delta(z - w) = i_{z,w} \frac{1}{(z-w)z} - i_{w,z} \frac{1}{(z-w)^2},
\]

where the symbol \( i_{z,w} \) means expansion in the domain \( |z| > |w| \). For simplicity, \( i_{z,w} \) is omitted when there is no confusion in expansion direction. \( \square \)

**3. REPRESENTATION OF \( \mathcal{E} \)**

In this section, we realize the loop-like toroidal Lie superalgebra defined in Section 2 using bosonic fields and a ghost field.

Let \( \{ \varepsilon_i | 0 \leq i \leq n + 1 \} \) be an orthonormal basis of the vector space \( \mathbb{C}^{n+2} \) with inner product \( (\varepsilon_i, \varepsilon_j) = \delta_{ij} \). Then the distinguished simple and positive roots of simple Lie superalgebra of type \( B(0, n) \) can be realized as follows:

\[
\Pi : \varepsilon_1 = \varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} = \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n = \varepsilon_n, \\
\Delta_+ = \{ 2\varepsilon_i, \varepsilon_i \pm \varepsilon_j, 1 \leq i \leq n, 1 \leq i < j \leq n \},
\]

and the longest root is

\[
\theta = 2\varepsilon_1 + \cdots + 2\varepsilon_n = 2\varepsilon_1.
\]

Introduce \( \tau = \varepsilon_0 + \sqrt{-1} \varepsilon_{n+1} \), and define

\[
\alpha_0 = \tau - \theta \quad \text{and} \quad \beta = \varepsilon_1 - \frac{1}{2} \tau.
\]

Then

\[
P = \mathbb{Z}\bar{\tau} \oplus \mathbb{Z}\varepsilon_1 \oplus \cdots \oplus \mathbb{Z}\varepsilon_n
\]

is the weight lattice of \( B(0, n)^{(1)} \). Note that \( (\beta | \beta) = 1, (\beta | \varepsilon_i) = \delta_{1i}, \alpha_0 = -2\beta \).
Let $P_\mathbb{C} = P \otimes \mathbb{C}$ be the vector space spanned by $\mathbb{C}$, $P_\mathbb{C}$, $P_\mathbb{C}$ are both maximal isotropic subspaces under the antisymmetric bilinear form on $\mathbb{C}$ given by

$$\langle b^*, a \rangle = -\langle a, b^* \rangle = \langle a, b \rangle, \quad \langle a, b \rangle = \langle a^*, b^* \rangle = 0, \quad a, b \in \mathbb{C}_0.$$

Let $\Lambda(\infty)$ be the associative algebra generated by

$$\{u(k), e(k) \mid u \in \mathbb{C}, k \in \mathbb{Z}\}$$

with the defining relations

$$u(k)v(l) - v(l)u(k) = \langle u, v \rangle \delta_{k,-l},$$

and

$$u(k)e(l) - e(l)u(k) = 0, \quad e(k)e(l) + e(l)e(k) = \delta_{k,-l},$$

where $u, v \in \mathbb{C}, k, l \in \mathbb{Z}$.

We define the normal ordering of a quadratic expression to be

$$: u(k)v(l) ::= \begin{cases} u(k)v(l), & k < 0 \\ 1/2(u(0)v(l) + v(l)u(0)), & k = 0 \\ v(l)u(k), & k > 0, \end{cases}$$

and

$$: e(k)e(l) ::= \begin{cases} e(k)e(l), & k < 0 \\ 1/2(e(0)e(l) - e(l)e(0)), & k = 0 \\ -e(l)e(k), & k > 0, \end{cases}$$

and

$$: u(k)e(l) ::= e(l)u(k) ::= u(k)e(l), \quad k, l \in \mathbb{Z}.$$

Introduce the $\Lambda(\infty)$-module

$$V = \Lambda(\infty) \left/ \left[ \sum_{k \in \mathbb{Z}^+} (\Lambda(\infty)u(k) + \Lambda(\infty)e(k)) \right] \right.,$$

which is isomorphic to the associative algebra $\Lambda^-(\infty)$ generated by $u(k), e(k), u \in \mathbb{C}$, $k \in \mathbb{Z}_-$ as vector spaces. Denote the image of $x$ by $x \mid 0$. Then we have

$$V = \Lambda^-(\infty) \mid 0.$$
For any two bosonic(fermionic) fields, we let
\[ : x(z)y(w) := \sum_{k,l \in \mathbb{Z}} : x(k)y(l) : z^{-k-1/2}w^{-l-1/2}. \]
Then it follows that
\[ : u(z)v(w) := v(w)u(z) : \quad \text{and} \quad : e(z)e(w) := - : e(w)e(z) : \]
and
\[ : u(z)e(w) := u(z)e(w) = : e(w)u(z) : . \]
Based on the normal ordering of two fields, one can define the normal product of \( n \) fields inductively.
We further define the contraction of two fields by
\[ x(z)y(w) = x(z)y(w) - : x(z)y(w) : \]
Since the element \( e \) is central, we have that for any \( u \in \mathbb{C} \)
\[ [c(z), u(w)] = 0. \]  

**Definition 3.1.** Let \( x_1, \ldots, x_n \) be generators in an algebra of operators having bosonic relations and a notions of normal ordering. Define
\[ : x_1 \cdots x_i \cdots x_j \cdots x_n : = x_i x_j (: x_1 \cdots x_{i-1} x_{i+1} \cdots x_{j-1} x_{j+1} \cdots x_n :). \]

**Remark.** In what follows, we understand that \( \langle e, e \rangle = 1, \langle e, \mathbb{C} \rangle = 0 \), so the above definition is also well defined when one or two of \( x_i \)'s is \( e \).

The following Wick's theorem is well known.

**Theorem 3.2 ([2, 12]).** For elements \( x_i, y_j, 1 \leq i \leq r, 1 \leq j \leq t \), we have
\[ : x_1 \cdots x_r : y_1 \cdots y_t := x_1 \cdots x_r y_1 \cdots y_t : + \sum \pm : x_1 \cdots x_r y_1 \cdots y_s :, \]
where the summation is taken over all possible combinations of contraction of some \( x \)'s and some \( y \)'s, and the sign is the sign of permutation of fermionic operators.

**Corollary 3.3.** In an algebra with both fermionic and bosonic generators, we have for any permutation
\[ : x_1 \cdots x_r : = (-1)^N : x_{\sigma(1)} \cdots x_{\sigma(r)} :, \]
where \( N \) is the number of fermionic-fermionic transposition in a decomposition of \( \sigma \).
The following statement is proved by the standard technique of OPE (cf. [9, 18]). The only difference occurs when one or two factors are odd operators, but this is taken care of by super-brackets from the construction of the field operators.

**Corollary 3.4.** For $r_1, s_1 \in \mathcal{C}, r_2, s_2 \in \mathcal{C} \cup \{e\}$ and $|z| > |w|$, we have

\[
\langle : r_1(z)s_1(z) : \rangle = : r_1(z)s_1(z)r_2(w)s_2(w) : + \langle \langle r_1, r_2 \rangle : s_1(z)s_2(w) : + \langle r_1, s_2 \rangle \\
\cdot : s_1(z)r_2(w) : + \langle s_1, r_2 \rangle : r_1(z)s_2(w) : + \langle s_1, s_2 \rangle : r_1(z)r_2(w) : \\
\cdot \cdot \frac{1}{z - w} + \langle \langle r_1, r_2 \rangle \langle s_1, s_2 \rangle + \langle r_1, s_2 \rangle \langle s_1, r_2 \rangle \rangle \cdot \frac{1}{(z - w)^2}.
\]

The following result is an immediate consequence.

**Proposition 3.5.** The bosonic fields satisfy the following (super)commutation relations:

\[
[a(z), b(w)] = [a^*(z), b^*(w)] = 0,
\]

\[
[a(z), b^*(w)] = \langle a, b \rangle \delta(z - w),
\]

and the commutators among normal ordering products are given by

\[
\langle : a_1(z)a_2^*(z) : \rangle = : b_1(w)b_2^*(w) : \]

\[
= - (a_1, b_2) : a_2^*(z)b_1(z) : \delta(z - w) + (a_2, b_1) : a_1(z)b_2^*(z) : \delta(z - w)
\]

\[
- (a_1, b_2)(a_2, b_1) \delta(w) \delta(z - w),
\]

where $a, b, a_i, b_j \in P_E, i = 1, 2$.

The anti-symmetric inner product of the underlying Lie superalgebra can be extended to an inner product for the span of quadratic products:

\[
\langle : r_1r_2 : , : s_1s_2 : \rangle = - \langle r_1, s_1 \rangle \langle r_2, s_2 \rangle + \langle r_1, s_2 \rangle \langle r_2, s_1 \rangle.
\]

Now we can state and prove the main theorem.

**Theorem 3.6.** The correspondence

\[
x^+_i(z) = \begin{cases} \frac{1}{2} : \beta^*(z)\beta^*(z) : & i = 0 \\ : \epsilon_i(z)\epsilon_{i+1}^*(z) : & 1 \leq i \leq n - 1 \\ \sqrt{2} : \epsilon_i(z)\epsilon(z) : & i = n, \end{cases}
\]

\[
x^-_i(z) = \begin{cases} \frac{1}{2} : \beta(z)\beta(z) : & i = 0 \\ : \epsilon_{i+1}(z)\epsilon_i^*(z) : & 1 \leq i \leq n - 1 \\ -\sqrt{2} : \epsilon_i^*(z)\epsilon(z) : & i = n, \end{cases}
\]
Now we check the commutation relations involving root vectors:

\[ x_i(z) \begin{cases} -2 : \beta^*(z) \beta(z) : & \text{if } i = 0 \\ : e_i(z)e_i^*(z) : & \text{if } 1 \leq i \leq n - 1 \\ : e_n(z)e_n^*(z) : & \text{if } i = n \end{cases} \]

gives rise to a realization of toroidal Lie superalgebra \( \Xi \) of level \(-1\). Moreover, the correspondence also gives a representation of 2-toroidal superalgebra \( T(B(0, n)) \) through the map in Proposition 2.2.

**Proof.** First of all, we have for \( i \neq j \)

\[
\begin{align*}
\{ : e_i(z)e_j^*(z) : & : e_j(w)e_i^*(w) : \\
& = ( : e_i(z)e_j^*(z) : - : e_j(z)e_i^*(z) : ) \delta(z - w) - \hat{c}_w \delta(z - w), \\
\{ : e_i^*(z)e_j^*(z) : & : e_i(w)e_j^*(w) : \\
& = ( : e_j(z)e_i^*(z) : + : e_i(z)e_j^*(z) : ) \delta(z - w) + \hat{c}_w \delta(z - w),
\end{align*}
\]

which imply that all commutation relations for \( X(x_i, z) \), \( i = 1, \ldots, n - 1 \) are the same as those of the affine Lie algebra of type \( A_{n-1} \) (cf. [9]). The new commutation relations for the Heisenberg algebra are

\[
\begin{align*}
[x_{n-1}(z), x_n(w)] &= -\{ : e_n(z)e_n^*(z) : : e_n(w)e_n^*(w) : \\
&= \hat{c}_w \delta(z - w) = (x_{n-1}, x_n) \hat{c}_w \delta(z - w) \cdot (-1), \\
[x_n(z), x_n(w)] &= \{ : e_n(z)e_n^*(z) : : e_n(w)e_n^*(z) : \\
&= (x_n, x_n) \hat{c}_w \delta(z - w) \cdot (-1).
\end{align*}
\]

Using Eq. (3.1), we have that

\[
\{ : e_1(z)e_1^*(z) : : \beta(w)\beta^*(w) : \} = \{ : e_1(z)e_1^*(z) : : e_1(w)e_1(w) : \}
\]

\[ = -\hat{c}_w \delta(z - w). \]

It follows that

\[
\begin{align*}
[x_0(z), x_0(w)] &= 4\{ : \beta(z)\beta^*(z) : : \beta(w)\beta^*(w) : \\
&= 4\hat{c}_w \delta(z - w) = (x_0, x_0) \hat{c}_w \delta(z - w) \cdot (-1), \\
[x_0(z), x_1(w)] &= -2\{ : \beta(z)\beta^*(z) : : e_1(z)e_1^*(z) : \\
&= (x_0, x_1) \hat{c}_w \delta(z - w) \cdot (-1),
\end{align*}
\]

Now we check the commutation relations involving root vectors:

\[
\begin{align*}
[x_{n-1}(z), x_n^+(w)] &= -\sqrt{2} \{ : e_n(z)e_n^*(z) : : e_n(w)e(w) : \\
&= -\sqrt{2} : e_n(w)e(w) : \delta(z - w)
\end{align*}
\]

\[ = (x_{n-1}, x_n) x_n^+(w) \delta(z - w), \]

\[ [z_{n-1}(z), x_n^-(w)] = -\sqrt{2} \colon e_n(z)e_n^*(z) \colon e_n^*(w)e(w) ] \]
\[ = \sqrt{2} : e_n^*(z)e(w) : i\delta(z - w) \]
\[ = -(z_{n-1}, z_n)x_n^-(w)i\delta(z - w). \]

Similarly, we have
\[ [z_n(z), x_n^+(w)] = \pm(z_n, z_n)x_n^+(w)i\delta(z - w), \]
\[ [x_n^+(z), x_n^-(w)] = -2[ : e_n(z)e(z) : : e_n^*(w)e(w) ] \]
\[ = -2[ : e_n(w)e_n^*(w) : i\delta(z - w) - i\tilde{\epsilon}_w\delta(z - w)] \]
\[ = -2 \frac{1}{(z_n, z_n)} [z_n(z) + i\tilde{\epsilon}_w\delta(z - w) \cdot (-1)], \]
where we have used the facts that
\[ : e_n(z)e(z)e_n(w)e(w) := - : e_n(w)e(w)e_n(z)e(z) : \]
and \( e(w)e(w) := 0. \)

\[ [x_n^+(z), x_n^-(w)] = \frac{1}{4} [ : \beta^*(z)\beta^*(z) : : \beta(w)\beta(w) ] \]
\[ = \frac{1}{4} [ 4 : \beta(z)\beta^*(w) : i\delta(z - w) + 2i\tilde{\epsilon}_w\delta(z - w) ] \]
\[ = \frac{-2}{(z_n, z_n)} [z_n(z)\delta(z - w) + i\tilde{\epsilon}_w\delta(z - w) \cdot (-1)], \]
\[ [z_n(z), x_n^-(w)] = -[ : \beta(z)\beta^*(z) : : \beta^*(w)\beta^*(w) ] \]
\[ = 2 : \beta(z)\beta^*(w) : i\delta(z - w) = (z_n, z_n)x_n^-(w)i\delta(z - w), \]
and similarly, \( [z_n(z), x_n^+(w)] = -(z_n, z_n)x_n^-(w)i\delta(z - w). \)

Next we proceed to check Serre relations. In fact, we have that
\[ [x_i^+(z), x_i^+(w)] = 0, \quad 0 \leq i \leq n - 2, \]
and
\[ [x_{n-1}^+(z_2), [x_{n-1}^+(z_1), x_n^+(w)]] = \sqrt{2} [x_{n-1}^+(z_2), [ : e_{n-1}(z_1)e_n^*(z_1) : : e_n(w)e(w) ] ] \]
\[ = \sqrt{2} [ : e_{n-1}(z_2)e_n^*(z_1) : : e_{n-1}(w)e(w) ] i\delta(z_1/w) \]
\[ = 0, \]
\[ [x_{n-1}^+(z_3), [x_{n-1}^+(z_2), [x_n^+(z_1), x_{n-1}^+(w)]]] \]
\[ = \sqrt{2} [x_{n}^+(z_3), [x_n^+(z_2), [ : e_n(z_1)e(z_1) : : e_{n-1}(w)e_n^*(w) ] ] ] \]
\[ = -2 [x_n^+(z_3), [ : e_n(z_2)e(z_2) : : e_{n-1}(w)e(w) ] ] i\delta(z_1/w) \]
\[ -2\sqrt{2} \left[ : e_n(z_1) e(z_1) : , : e_n(w) e_{n-1}(w) : \right] \delta(z_1/w) \delta(z_2/w) = 0. \]

Similarly,

\[ [ x_{n-1}^+(z_2), [ x_{n-1}^-(z_1), x_n^-(w) ] ] = 0, \]
\[ [ x_n^+(z_2), [ x_n^-(z_2), [ x_n^-(z_1), x_{n-1}^-(w) ] ] ] = 0. \]

Finally, this realization indeed gives a representation of the toroidal Lie superalgebra. In fact the imaginary field \( \delta(z) \) clearly commutes with all operators \( x^+_n(z) \), and the kernel of the map in Proposition 2.2 is contained in the subalgebra generated by \( \mathbb{Z}^n_k \), our representation can be lifted to the superalgebra \( T(B(0, n)) \). This completes the proof. \( \square \)

**NOTE ADDED IN PROOF**

We thank S. Eswara Rao for bringing our attention to his recent work [1] which has constructed a class of vertex representations for toroidal Lie superalgebras of classical types similar to the usual lattice vertex operator algebra realizations of affine Lie algebras [4]. It would be interesting to establish a super-analog of boson-boson correspondence between our construction and the vertex representations in [1].

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**REFERENCES**

[1] Eswara Rao, S. A new class of modules for toroidal Lie superalgebras. *Sao Paulo J. Math. Sci.* To appear (arXiv:1205.3604).
[2] Feingold, A. J., Frenkel, I. B. (1985). Classical affine algebras. *Adv. Math.* 56:117–172.
[3] Frenkel, I. B. (1980). Spinor representations of affine Lie algebras. *Proc. Natl. Acad. Sci. USA* 77(11):6303–6306.
[4] Frenkel, I. B., Lepowsky, J., Meurman, A. (1988). *Vertex Operator Algebras and the Monster*. New York: Academic Press.
[5] Gao, Y. (2002). Fermionic and bosonic representation of the extended affine Lie algebra of \( \mathfrak{gl}_N(C_q) \). *Canad. Math. Bull.* 45:623–633.
[6] Iohara, K., Koga, Y. (2001). Central extensions of Lie superalgebras. *Commun. Math. Helv.* 76:110–154.
[7] Jing, N., Misra, K. C. (2010). Fermionic realization of toroidal Lie algebras of classical types. *J. Alg.* 324:183–194.
[8] Jing, N., Misra, K. C., Tan, S. (2005). Bosonic realizations of higher level toroidal Lie algebras. *Pacific. J. Math.* 219:285–302.
[9] Jing, N., Misra, K. C., Xu, C. (2009). Bosonic realization of toroidal Lie algebras of classical types. *Proc. AMS.* 137:3609–3618.
[10] Jing, N., Zhang, H. (2009). Fermionic realization of two-parameter quantum affine algebra \( U_{c,0}(sl_n) \). *Lett. Math. Phys.* 89(2):159–170.
[11] Kac, V. G. (1977). Lie superalgebras. Adv. Math. 26(1):8–96.
[12] Kac, V. G. (1997). Vertex Algebras for Beginners. Univ. Lect. Ser. 10. Providence: Amer. Math. Soc.
[13] Kac, V. G., Peterson, D. (1981). Spin and wedge representations of infinite-dimensional Lie algebras and groups. Proc. Nat. Acad. Sci. U.S.A. 78(6):3308–3312.
[14] Kac, V. G., Wakimoto, M. (2001). Integrable highest weight modules over affine superalgebras and Appell’s function. Commun. Math. Phys. 215:631–682.
[15] Lau, M. (2005). Bosonic and Fermionic representation of Lie algebra with central extensions. Adv. Math. 194(2):225–245.
[16] Moody, R. V., Rao, S. E., Yokonuma, T. (1990). Toroidal Lie algebras and vertex representations. Geom. Ded. 35:283–307.
[17] Tan, S. (1999). Vertex operator representations for toroidal Lie algebra of type $B_c$. Comm. Algebra 27(8):3593–3618.
[18] Xu, X. (1998). Introduction to Vertex Operator Superalgebras and Their Modules. Dordrecht: Kluwer Academic Publishers.