Momentum conservation and correlation analyses in heavy-ion collisions at ultrarelativistic energies

Nicolas Borghini

Fakultät für Physik, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld, Germany

(Dated: March 31, 2022)

Global transverse-momentum conservation induces correlations between any number of particles, which contribute in particular to the two- and three-particle correlations measured in heavy-ion collisions. These correlations are examined in detail, and their importance for studies of jets and their interaction with the medium is discussed.

PACS numbers: 25.75.Gz

I. INTRODUCTION

The high-statistics data collected at the Brookhaven Relativistic Heavy-Ion Collider (RHIC) have opened the possibility of novel precision measurements aiming at characterizing the medium created in ultrarelativistic heavy-ion collisions. Among these new developments, significant effort is being devoted to extracting correlations between pairs or triplets of high-momentum particles from the data, to perform jet physics in the high-multiplicity environment of a high-energy nucleus–nucleus collision. Thus, the initial result from a study of azimuthal correlations between particles with transverse momenta $2 \text{ GeV} / c < p_T < 6 \text{ GeV} / c$ was the convincing demonstration that the back jet, at $180^\circ$ from a high-$p_T$ reference particle, is suppressed, in the sense that it does not emerge above the background $^1$.

More recently, the focus has been to obtain a more quantitative description of the behaviour in the recoil region away from a high-$p_T$ “trigger” particle. On the one hand, various physical mechanisms have been proposed, involving the interaction of the high-momentum parton initially emitted back-to-back to the trigger with the medium through which it propagates, that predict non-trivial structures in this away-side region: shock-waves along a Mach cone $^2$ generated by the energy deposited by the fast-moving parton $^3$; gluon bremsstrahlung $^5$ or Cherenkov-like radiation $^6, 7, 8$ by the parton; the deflection of the back jet by the collective movement (“flow”) of the expanding medium $^9$; or a path-length-based selection of the particles emerging a random walk through the medium $^{10}$.

In parallel, several analysis techniques were developed and are being applied to the experimental data, using two- $^{11}$ or three-particle $^{12, 13, 14}$ correlations, to distinguish between the different scenarios. A common need in these approaches is the necessity to deal with the background properly. Thus, all measurements take into account the modulation of azimuthal correlations induced by anisotropic collective flow. However, the methods differ with respect to the other sources of correlations. The philosophy of the cumulant study advocated in Ref. $^{12}$ is to perform a measurement with minimal assumptions on the nature of the correlations, and later to compute the contributions (assumed to be additive) of the different sources of correlation to the three-particle cumulant. In contrast, all other methods rely on the assumption, be it explicitly stated or not, that the measured correlation consists of a jet signal, which vanishes away enough from the jet, and an uncorrelated background.

The purpose of this paper is to emphasize the role of the unavoidable contribution of global transverse momentum conservation to the measured correlation, either between two or three particles. The realization that global momentum conservation can significantly affect measurements in heavy-ion physics is not new. Thus early measurements of collective anisotropic flow in nuclear collisions at intermediate and relativistic energies were corrected $^{15}$ for that effect. Later, it was shown that momentum conservation biases the standard measurement of the first anisotropic-flow harmonic $v_1$ at SPS $^{16}$ and an explicit correction to the analysis was devised $^{17}$ and applied to the NA49 data $^{18}$. Recently, the possibility that the conservation of momentum and energy could significantly affect the correlation functions measured in femtoscopy analyses was addressed in Ref. $^{19}$.

Here I shall argue that correlations at high transverse momentum are especially sensitive to the conservation of total transverse momentum. The latter has several effects. On the one hand, it contributes to the cumulant measured in the technique of Ref. $^{12}$. The specific dependence on the particle momenta of the three-particle cumulant from global transverse momentum conservation, which was already computed in Ref. $^{20}$, will thus be investigated in Sec. IIII. On the other hand, global momentum conservation correlates a jet to the remainder of the event, thereby invalidating the assumption that jet and background are uncorrelated. Some implications of this observation will be discussed in Sec. IIII.

II. MOMENTUM CONSERVATION AND CUMULANTS

In a nucleus–nucleus collision, all $N$ emitted particles are correlated together by the requirement that
their transverse momenta $p_T$, add up to 0. As a consequence, the joint $M$-particle probability distribution $f(p_{T1}, \ldots, p_{TM})$ differs from the product of the single-particle probability distributions $f(p_{Tj1}) \cdots f(p_{TM})$, where $1 \leq j_1 < \ldots < j_M \leq N$ and $2 \leq M \leq N$. In particular, the cumulant of the $M$-particle distribution, which will be denoted by $f_c(p_{T1}, \ldots, p_{TM})$ and corresponds to the part of the joint probability distribution that cannot be expressed in terms of distributions involving less than $M$ particles [21], is finite. Using a generating function of the multiparticle probability distributions and a saddle-point calculation, one can compute the successive cumulants to leading order in $1/N$ [20]. Thus, it was shown that the $M$-particle cumulant scales like $1/N^{M-1}$. Assuming that the single-particle $p_T$ distribution is isotropic, one in particular finds that the two- and three-particle cumulants due to global transverse-momentum conservation read

$$f_c(p_{T1}, p_{T2}) = \frac{-2p_{T1} \cdot p_{T2}}{N(p_T^2)},$$

$$f_c(p_{T1}, p_{T2}, p_{T3}) = \frac{-2}{N^2(p_T^2)}(p_{T1} \cdot p_{T2} + p_{T1} \cdot p_{T3} + p_{T2} \cdot p_{T3})$$

$$+ \frac{2}{N^2(p_T^2)^2}[12(p_{T1} \cdot p_{T2})(p_{T1} \cdot p_{T3}) + (p_{T1} \cdot p_{T2})(p_{T2} \cdot p_{T3}) + (p_{T1} \cdot p_{T3})(p_{T2} \cdot p_{T3})],$$(2)

where $\langle p_T^2 \rangle$ denotes the average over many particles and events of the squared transverse momentum.

The meaning of the two-particle cumulant [1] is clear and intuitive: the correlation is back-to-back and stronger between particles with higher transverse momenta. In plain words, given a high-$p_T$ trigger-particle, there is a larger probability to find an “associated” particle away from it in azimuth than close to it, just because of global transverse-momentum conservation.

The interpretation of the three-particle cumulant arising from global transverse-momentum conservation, Eq. (2), is slightly more involved, as it implies two terms with opposite signs. Yet, if one considers three particles with transverse momenta significantly larger than the rms transverse momentum $(p_T^2)^{1/2}$, one sees that the “attractive” term, in the second line, dominates over the “repulsive” one. To illustrate the behaviour of the three-particle cumulant, I shall now study Eq. (2) by promoting particle 1 to the role of “trigger particle,” with respect to which the azimuths of the other two are measured: $\Delta \varphi_{12} \equiv \varphi_2 - \varphi_1$, $\Delta \varphi_{13} \equiv \varphi_3 - \varphi_1$. I shall assume for simplicity that the “associated particles” 2 and 3 have equal transverse momenta $p_{T2} = p_{T3} \leq p_{T1}$, and use the notation $f_c(p_{T1}, p_{T2}, p_{T3}; \Delta \varphi_{12}, \Delta \varphi_{13})$ for the cumulant.

Consider first the symmetric case $\Delta \varphi_{13} = 2\pi - \Delta \varphi_{12}$. For symmetry reasons (parity and $2\pi$-periodicity), the three-particle cumulant $f_c(p_{T1}, p_{T2}, p_{T3}, \Delta \varphi_{12}, -\Delta \varphi_{12})$ has trivial extrema at $\Delta \varphi_{12} = 0$ and $\Delta \varphi_{12} = \pi$. The former is a maximum for values of $p_{T2} = p_{T3}$ larger than a minimal value

$$p_{T_{\text{min}}} = \langle p_T^2 \rangle / 10 p_{T1} \left( \frac{-p_{T1}^2}{\langle p_T^2 \rangle} + 1 + \sqrt{\frac{p_{T1}^4}{\langle p_T^2 \rangle^2} + \frac{8p_{T1}^2}{\langle p_T^2 \rangle}} + 1 \right),$$

which is $\approx p_{T1}/5$ when the transverse momentum $p_{T1}$ of the trigger particle is large. Then there is a second local maximum, beyond that at $\Delta \varphi_{12} = 0$, for

$$p_{T_{\text{min}}} = \langle p_T^2 \rangle / 10 p_{T1} \left( \frac{-p_{T1}^2}{\langle p_T^2 \rangle} + 1 + \sqrt{\frac{p_{T1}^4}{\langle p_T^2 \rangle^2} + \frac{8p_{T1}^2}{\langle p_T^2 \rangle}} + 1 \right),$$

which is $\approx p_{T1}/5$ when the transverse momentum $p_{T1}$ of the trigger particle is large. Then there is a second local maximum, beyond that at $\Delta \varphi_{12} = 0$, for

$$\cos \Delta \varphi_{12} = \frac{\langle p_T^2 \rangle}{12 p_{T1} p_{T2}} \left[ 1 - \frac{p_{T1}^2}{\langle p_T^2 \rangle} + \sqrt{\frac{p_{T1}^4}{\langle p_T^2 \rangle^2} - 1} + \frac{12p_{T1}^2}{\langle p_T^2 \rangle} \left( 1 + \frac{2p_{T1}^2}{\langle p_T^2 \rangle} \right) \right].$$

One can check that the $\Delta \varphi_{12}$ position of this minimum is an increasing function of $p_{T2}$, which reaches a maximal value $\Delta \varphi_{12} = \arccos(1/3 + (p_{T1}^2)/6p_{T2}^2)$ $\approx 1.23$ rad (70.5°) for $p_{T2} = p_{T1}$. The next extremum of the three-particle cumulant in the range $0 \leq \Delta \varphi_{12} \leq \pi$ only exists if the transverse momentum of the associated particles $p_{T2}$ is larger than

$$p_{T_{\text{min}}} = \langle p_T^2 \rangle / 10 p_{T1} \left( \frac{-p_{T1}^2}{\langle p_T^2 \rangle} + 1 + \sqrt{\frac{p_{T1}^4}{\langle p_T^2 \rangle^2} + \frac{8p_{T1}^2}{\langle p_T^2 \rangle}} + 1 \right),$$

which is $\approx p_{T1}/5$ when the transverse momentum $p_{T1}$ of the trigger particle is large. Then there is a second local maximum, beyond that at $\Delta \varphi_{12} = 0$, for

$$\cos \Delta \varphi_{12} = \frac{\langle p_T^2 \rangle}{12 p_{T1} p_{T2}} \left[ 1 - \frac{p_{T1}^2}{\langle p_T^2 \rangle} + \sqrt{\frac{p_{T1}^4}{\langle p_T^2 \rangle^2} - 1} + \frac{12p_{T1}^2}{\langle p_T^2 \rangle} \left( 1 + \frac{2p_{T1}^2}{\langle p_T^2 \rangle} \right) \right].$$

The position of this maximum decreases with increasing $p_{T2}$, reaching a minimal $\Delta \varphi_{12} = 2\pi/3$ when $p_{T2} = p_{T1}$. This is actually a rather intuitive result: when the three particles have equal transverse momenta, the cumulant
has a maximum at the symmetric configuration $\Delta \varphi_{12} = \Delta \varphi_{23} = \Delta \varphi_{13} = 120^\circ$. Eventually, the three-particle cumulant $f_c(p_{T1}, p_{T2}, p_{T3}, \Delta \varphi_{12}, \Delta \varphi_{13})$ has the already-mentioned extremum at $\Delta \varphi_{12} = \pi$, which is a minimum if $p_{T2} > p_{T2}^{T\text{min}}$, a maximum otherwise.

Let me now study the three-particle cumulant $f_c$ in the specific case $p_{T2} = p_{T3}$ and $\Delta \varphi_{13} = \Delta \varphi_{12}$. Due to parity and $2\pi$-periodicity, it has two extrema at $\Delta \varphi_{12} = 0$ and $\Delta \varphi_{12} = \pi$. The latter is always a maximum, irrespective of the value of $p_{T2}$. That means that the point $\Delta \varphi_{12} = \Delta \varphi_{13} = \pi$ is either a saddle point for the cumulant $f_c(p_{T1}, p_{T2}, p_{T2}, \Delta \varphi_{12}, \Delta \varphi_{13})$, if $p_{T2} > p_{T2}^{T\text{min}}$, or a local maximum if $p_{T2} \leq p_{T2}^{T\text{min}}$. The nature of the extremum at $\Delta \varphi_{12} = 0$ depends on the value of the transverse momentum of the associated particle. If $p_{T2} \leq \sqrt{(p_{T1}^2 - 2(p_{T1}^2 - p_{T1})/2)$ — which is $\approx (p_{T1}^2)/2p_{T1} \ll \langle p_{T1}^2 \rangle^{1/2}$ in the regime $p_{T1} \gg \langle p_{T1}^2 \rangle^{1/2}$, so that the case is most likely irrelevant for studies of correlations at high transverse momentum — then the cumulant has a minimum at $\Delta \varphi_{12} = 0$. On the contrary, when $p_{T2} > \sqrt{(p_{T1}^2 - 2(p_{T1}^2 - p_{T1})/2$ the cumulant has a maximum at $\Delta \varphi_{12} = 0$, and a minimum for

$$\cos \Delta \varphi_{12} = -\frac{2p_{T2}^2 - \langle p_{T1}^2 \rangle}{2p_{T1}p_{T2}}.$$  

The position of this minimum grows with $p_{T2}$, reaching $\Delta \varphi_{12} = \arccos(-1 + \langle p_{T2}^2 \rangle/2p_{T1})$ for $p_{T2} = p_{T1}$. In the case of a large transverse momentum $p_{T1}$ of the trigger particle, this minimum sits at $\Delta \varphi_{12} \approx \pi - \sqrt{\langle p_{T1}^2 \rangle}/\sqrt{2}p_{T1}$, close to the maximum at $\pi$.

Figure 1 shows the profile of the three-particle cumulant due to global transverse-momentum conservation $f_c(p_{T1}, p_{T2}, p_{T3}, \Delta \varphi_{12}, \Delta \varphi_{13})$, in the case of equal transverse momenta of the associated particles, $p_{T2} = p_{T3}$, larger than the rms transverse momentum $\langle p_{T1}^2 \rangle^{1/2}$. Both choices $p_{T2} > p_{T2}^{T\text{min}} \sim p_{T1}/5$ (left) and $p_{T2} < p_{T2}^{T\text{min}}$ (right) are displayed to illustrate the different behaviours discussed above. As anticipated, in the first case, the cumulant has a saddle point at $\Delta \varphi_{12} = \Delta \varphi_{13} = \pi$ and two clearly separated maxima away from $\Delta \varphi_{12} = \Delta \varphi_{13} = 0$ (modulo $2\pi$) along the line $\Delta \varphi_{12} = 2\pi - \Delta \varphi_{13}$; whereas in the second case, there are two maxima for values of $\Delta \varphi_{12} = \Delta \varphi_{13}$ equal to 0 or $\pi$, and no further structure.

The values of the total number of particles $N = 8000$ and the rms transverse momentum $\langle p_{T1}^2 \rangle^{1/2} = 450$ MeV/c adopted in Fig. 1 were chosen so as to mimic a central Au–Au collision at RHIC. Such numbers result in a three-particle cumulant from transverse-momentum conservation of order $10^{-4}$ for transverse momenta of the trigger and associated particles of 3 and 1 GeV/c respectively. This is admitted a small correlation, yet it is of the same size as the contribution of anisotropic collective flow to the cumulant $\Delta \varphi_{12}^{\text{flow}}$. In addition, the correlation will be stronger in more peripheral collisions, since it scales as the inverse squared multiplicity $1/N^2$, see Eq. (2). Similarly, the value of the correlation will be larger in smaller systems and at lower collision energies. In fact, the multiplicity $N$ that enters Eqs. (12) might not be the total event multiplicity, but one could argue that different rapidity slices are decorrelated, so that the constraint from transverse momentum conservation would actually be driven by a smaller multiplicity. That would also give values of the cumulant of the correlation from momentum conservation higher than those plotted in Fig. 1.

---

1 The rms transverse momentum $\langle p_{T1}^2 \rangle^{1/2}$ may also decrease when going to more peripheral collisions, which contributes, although much less than the drop in multiplicity, to the growth of the cumulant.
FIG. 2: (Color online) Three-particle correlation due to global transverse-momentum conservation vs. the relative angles Δφ_{12}, Δφ_{13}, assuming N = 8000 particles per event with a rms transverse momentum \( (p_T^2)^{1/2} = 0.45 \text{ GeV}/c \), for particles of transverse momentum \( p_{T1} = 3.2 \text{ GeV}/c \) and \( p_{T2} = p_{T3} = 1.2 \text{ GeV}/c \), respectively.

III. DISCUSSION

Till now, I have mostly focussed on the cumulant \(^2\) of the three-particle correlation due to global transverse-momentum conservation. The three-particle correlation itself, which also involves two-body terms [controlled by Eq. (1)], is much larger, as displayed in Fig. 2 for the same RHIC-inspired values of the multiplicity, rms transverse momentum, and transverse momenta of the trigger and associated particles as in the left panel of Fig. 1. One sees that the main, quite intuitive effect of momentum conservation is to push the associated particles 2 and 3 away from the trigger particle 1. The resulting “bump” at Δφ_{12} = Δφ_{13} = π is broader along the Δφ_{12} = 2π − Δφ_{13} axis than along the perpendicular Δφ_{12} = Δφ_{13} axis. This is quite generic for the correlation due to momentum conservation — configurations with the two associated particles on each side of the direction defined by \( \vec{p}_T1 \) are more likely than those with both particles on the same side —, while the symmetric shape of the bump across the Δφ_{12} = Δφ_{13} axis in Fig. 2 reflects the specific choice \( p_{T2} = p_{T3} \).

A most prominent feature of the profile, shown in Fig. 2 of the three-particle correlation arising from the momentum-conservation constraint, is the “dip” around the origin Δφ_{12} = Δφ_{13} = 0. The meaning of this dip is transparent: if there is a high-\( p_T \) particle in the event, then its transverse momentum has to be balanced by the others. Therefore, the probability of finding another high-\( p_T \) particle pointing into the same direction is smaller than if transverse momentum were not conserved.\(^2\) In other words, a jet and the remainder of the “underlying” event are not uncorrelated, as is too often assumed, but global transverse-momentum conservation alone already constrains the momentum distribution of the particles “outside the jet.”

A consequence of the existence of this correlation is that the mere notion of there being “an underlying event over which the jet develops” is incorrect. The jet does distort the event to which it belongs, it is not merely embedded in it as is done in many simulations. This means that, at least in azimuth, there is no “region far from the jet” where its influence would vanish, thereby allowing one to determine a correctly normalized “background” to the jet: even in the over-simplified case of a one-particle jet and considering only two-particle correlations, the region where the event shape is not modified by the presence of the jet is restricted to two points only, at 90° away from it. Even in studies of correlations between “only” two particles, the approaches currently used to disentangle the “jet” from the harmonic modulation of azimuthal correlations due to anisotropic flow may thus be inaccurate, since they ignore the modulation induced by transverse-momentum conservation. The effect of the latter will be even more important in three-particle correlation studies.

In summary, I have shown that the conservation of the total transverse momentum yields a correlation between pairs or triplets of high-\( p_T \) particles that is sizable — in central Au–Au collisions at RHIC energies the resulting three-particle cumulant is of the same magnitude as that due to anisotropic collective flow — and therefore should be accounted for properly in experimental studies. This might be easily feasible in studies at the cumulant level \(^{12}\): there one only has to consider the three-particle cumulant \(^2\) — possibly generalized to include the known anisotropy of the \( \vec{p}_T \) distribution (see Ref. \(^{20}\)), especially in non-central collisions —, so that the non-measured multiplicity \( N \) and rms transverse momentum \( (p_T^2)^{1/2} \) enter the analysis only once. The three-particle cumulant from transverse-momentum conservation has a specific dependence on the values of the particle momenta, which was illustrated in special cases in Fig. 1. If the transverse momentum of the trigger particle is much larger (by a factor \( \sim 5 \)) than those of the associated particles, the cumulant has a simple shape with one peak on the trigger-particle side, and a broader back-to-back bump, more elongated along the Δφ_{12} = 2π − Δφ_{13} axis than perpendicular to it (see

\(^2\) Surprisingly, this mutual exclusion of high transverse-momentum particles seems to be entirely driven by the anticorrelation in the two-body term. On the contrary, unless the associated particles have transverse momenta smaller than \( (p_T^2)^{1/2} \), the three-particle cumulant is maximal at Δφ_{12} = Δφ_{13} = 0: at that level, all three high-\( p_T \) particles are grouped together by the attractive term in the second line of Eq. 2, rather than oriented away from each other.
However, if one decreases the trigger transverse momentum while keeping fixed the associated transverse momenta, then a two-bump structure develops along the $\Delta \varphi_{12} = 2\pi - \Delta \varphi_{13}$ axis on the side away from the trigger (Fig. 1 right). 

Taking properly into account the effect of total momentum conservation will be much more involved in analyses that rely on an assumed jet profile \[11, 12, 14, 22\]. In those, the effect of momentum conservation also has to be considered at the two-body level, in the correlations between “jet” and “background” or between two “background” particles (although the correlation is smaller between softer particles). In addition, the correlation induced by momentum conservation somehow blurs the distinction between jet and background, since the latter is unquestionably different in the presence of the jet from what it would be in its absence. Before trying to identify definite structures, which require high-precision measurements, in the recoil region of a high-$p_T$ particle, it is important to first determine precisely the reference pattern — including the effects of momentum conservation and anisotropic collective flow — over which they would develop.

[1] C. Adler et al. [STAR Collaboration], Phys. Rev. Lett. 90, 082302 (2003).
[2] H. G. Baumgardt et al., Z. Phys. A 273, 359 (1975).
[3] H. Stöcker, Nucl. Phys. A 750, 121 (2005).
[4] J. Casalderrey-Solana, E. V. Shuryak, and D. Teaney, J. Phys. Conf. Ser. 27, 22 (2005).
[5] I. Vitev, Phys. Lett. B 630, 78 (2005).
[6] I. M. Dremin, Nucl. Phys. A 767, 233 (2006).
[7] J. Ruppert and B. Müller, Phys. Lett. B 618, 123 (2005).
[8] V. Koch, A. Majumder, and X. N. Wang, Phys. Rev. Lett. 96, 172302 (2006).
[9] N. Armesto, C. A. Salgado, and U. A. Wiedemann, Phys. Rev. Lett. 93, 242301 (2004).
[10] C. B. Chiu and R. C. Hwa, Phys. Rev. C 74, 064909 (2006).
[11] N. N. Ajitanand et al., Phys. Rev. C 72, 011902 (2005).
[12] C. A. Pruneau, Phys. Rev. C 74, 064910 (2006).
[13] J. G. Ulery and F. Wang, nucl-ex/0609017.
[14] N. N. Ajitanand, nucl-ex/0609038.
[15] P. Danielewicz et al., Phys. Rev. C 38, 120 (1988).
[16] N. Borghini, P. M. Dinh, and J.-Y. Ollitrault, Phys. Rev. C 62, 034902 (2000).
[17] N. Borghini et al., Phys. Rev. C 66, 014901 (2002).
[18] C. Alt et al. [NA49 Collaboration], Phys. Rev. C 68, 034903 (2003).
[19] Z. Chajécki and M. Lisa, nucl-th/0612080.
[20] N. Borghini, Eur. Phys. J. C 30, 381 (2003).
[21] N. G. van Kampen, Stochastic Processes in Physics and Chemistry (North Holland, Amsterdam, 1981).
[22] S. Kniege, talk given at Quark Matter 2006.