PROOF OF A CONJECTURE OF WIEGOLD

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1. Introduction

In this short note we confirm a conjecture of James Wiegold [1 4.69]. The breadth $b(x)$ of an element $x$ of a finite $p$-group $G$ is defined by the equation $|G : C_G(x)| = p^{b(x)}$, where $C_G(x)$ is the centralizer of $x$ in $G$. We prove the following:

**Theorem 1.1.** Let $G$ be a finite $p$-group and let $|G'| > p^{n(n-1)/2}$ for some non-negative integer $n$. Then the group $G$ can be generated by the elements of breadth at least $n$.

An overview of this problem can be found in [2]. Also M.R.Vaughan-Lee in [3] proved that if in a $p$-group $G$ we have $|G'| > p^{n(n-1)/2}$, then there exists an element in $G$ of breadth at least $n$.

In this article we prove that in the case $p \neq 2$ a more general result is true:

**Theorem 1.2.** Let $p \neq 2$ be a prime number. Let $G$ be a finite $p$-group and let $|G'| > p^{n(n-1)/2}$ for some non-negative integer $n$. Then the set of elements of breadth at least $n$ cannot be covered by two proper subgroups in $G$.

In the particular case $p = 2$ we also get a more general result:

**Theorem 1.3.** Let $G$ be a finite 2-group and let $|G'| > 2^{n(n-1)/2}$ for some non-negative integer $n$. Then the set of elements of breadth at least $n$ cannot be covered by two proper subgroups in $G$, one of which has index at least 4 in $G$.

So Theorem 1.1 is a consequence of the two Theorems 1.2 and 1.3.

2. Proofs of Theorems 1.2 and 1.3

The breadth $b_H(g)$ of an element $g$ of a finite $p$-group $G$ with respect to a subgroup $H \subseteq G$ is defined by the equation $|H : C_H(g)| = p^{b_H(g)}$, where $C_H(g) = \{h \in H | hg = gh\}$ is the centralizer of $g$ in $H$. By definition, $b(g) = b_G(g)$.

First we formulate Lemmas 2.1, 2.2, 2.3, which will be useful in the proofs of the Theorems 1.2 and 1.3.

**Lemma 2.1.** Let $G$ be a finite $p$-group and let $C$ be a finite subgroup of index $p$. Then for any element $g$ from the set $G \setminus C$ we get $\log_p |G'| \leq b(g) + \log_p |C'|$.

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Proof. The cardinality of the set $X = \{ [g, c] | c \in C \}$ is not bigger than $p^{k(n)}$. So it is enough to prove that $G' = XC'$. This follows from the following properties of the set $XC'$:

1. $XC'$ is a subgroup in $G : [g, c_1][g, c_2]C' = [g, c_1c_2]C'$;
2. $XC'$ is a normal subgroup;
3. Group $G/XC'$ is abelian.

These facts imply Lemma 2.1. □

The next two lemmas are the well-known facts in theory of $p$-groups, so we state them without proof.

**Lemma 2.2.** Let $G$ be a finite $p$-group and let $|G : Z(G)| \leq p^2$. Then $\log_p |G'| \leq 1$.

**Lemma 2.3.** Let $G$ be a finite $p$-group and let $G = H_1 \cup H_2 \cup H_3$ for some three proper subgroups of $G$. Then $p = 2$ and $H_i$ has index 2 in $G$ for $i = 1, 2, 3$.

### 2.1. **Proof of Theorem 1.2.** Assume the converse. Let the proper subgroups $H_1$ and $H_2$ cover all the elements of breadth at least $n$. We will prove that then $|G'| \leq p^{n(n-1)/2}$. The proof is by induction on $|G|$. We can assume that $|G : H_i| = p$ (because every proper subgroup is contained in a subgroup of index $p$) and $H_1 \neq H_2$ (because in every non-cyclic $p$-group, there are at least two different maximal proper subgroups).

Consider any subgroup $C$ of $G$ such that $C$ has index $p$ in $G$ and $C \cap H_1 = C \cap H_2 = H_1 \cap H_2$. Notice that the set $\{ c \in C | b_C(c) = b(c) \}$ is contained in the subgroup $Y = \{ c \in C | [c, g] \in C', \forall g \in G \}$. This is because if $b_C(c) = b(c)$, then for any $g \in G$ there exists $c' \in C$ such that $[c, g] = [c, c']$. Consider the case when $Y = C$. In this case the central subgroup $C/C'$ has a prime index in $G/C'$, so $G/C'$ is abelian and $G' = C'$. The rest follows from the induction hypothesis: the group $C$ has smaller order and all its elements of breadth at least $n$ are contained in the subgroup $C \cap H_1 = C \cap H_2$ (because $b_C(c) \leq b(c)$). Now consider the case $Y \neq C$. Notice that the set $\{ c \in C | b_C(c) \geq n - 1 \}$ is contained in $(C \cap H_1) \cup Y$ (because $b_C(c) < b(c)$, when $c \notin Y$). Apply the induction hypothesis to the group $C$ and its proper subgroups $C \cap H_1$ and $Y$. We conclude that $\log_p |C'| \leq \frac{(n-2)(n-1)}{2}$. Consider any element $g$ not lying in $H_1 \cup H_2 \cup C$ (this is possible because of Lemma 2.3 and $p > 2$). Its breadth is less than $n$ (because $g \notin H_1 \cup H_2$), so from Lemma 2.1 we get that $\log_p |G'| \leq b(g) + \log_p |C'| \leq \frac{n(n-1)}{2}$. □

**Theorem 2.1.** Let $G$ be a finite $p$-group. Let given that for some integers $n \leq k + 1$

1. The set of all elements of the breadth at least $n$ can be covered by two proper subgroups of $G$.
2. The set of elements of the breadth at most $k$ generates $G$.

Then $\log_p |G| \leq \frac{(n-1)(n-2)}{2} + k$.

**Proof.** The proof is by induction on $|G|$. Let the subgroups $H_1$ and $H_2$ cover all the elements of the breadth at least $n$. We can assume that $|G : H_i| = p$ (because any proper subgroup is contained in the subgroup of index $p$) and $H_1 \neq H_2$ (because in
every non-cyclic $p$-group, there are at least two different maximal proper subgroups). Consider any subgroup $C$ of $G$ such that $C$ has index $p$ in $G$ and $C \cap H_1 = C \cap H_2 = H_1 \cap H_2$. Notice that the set $\{c \in C | b_C(c) = b(c)\}$ is contained in the subgroup $Y = \{c \in C | [c, g] \in C', \forall g \in G\}$. It is because if $b_C(c) = b(c)$, then for any $g \in G$ there exists $c' \in C$ such that $[c, g] = [c, c']$. So we can conclude that the set $\{c \in C | b_C(c) \geq n - 1\}$ is contained in $(C \cap H_1) \cup Y$ (because $b_C(c) < b(c)$ if $c \notin Y$). Consider the case $Y = C$. In this case the central subgroup $C/C'$ has a prime index in $G/C'$, so $G/C'$ is abelian and $C' = C'$. The rest follows from the induction hypothesis: group $C$ has smaller order and all its elements of the breadth at least $n$ are contained in the subgroup $C \cap H_1 = H_1 \cap H_2$ (because $b_C(c) \leq b(c)$), also the set $C \setminus H_1$ is contained in the set $\{c \in C | b_C(c) \leq k\}$ (because $b_C(c) \leq b(c) \leq n - 1 \leq k$ for $c \in C \setminus H_1$) and so $C$ is generated by the elements of the breadth at most $k$ in subgroup $C$.

So we can assume that $Y$ is a proper subgroup of $C$. Consider the case $|C : Y| = p$ and consider a homomorphism $\pi : G \to G/C'$. It is clear that $\pi(Y)$ is the central subgroup of $\pi(G)$ and $|\pi(G) : \pi(Y)| \leq p^2$. Apply Lemma 2.2 to $\pi(G)$, so we conclude that $\log_p |\pi(G)'| \leq 1$ and $\log_p |G'| \leq \log_p |C'| + 1$. Apply the induction hypothesis to the group $C$ and to its two proper subgroups $H_1 \cap H_2, Y$, so we get $\log_p |C'| \leq \frac{(n - 3)(n - 2)}{2} + n - 1$ (every element from the set $C \setminus H_1$ is of the breadth at most $n - 1$ in $C$ and this set generates $C$, also for any element $g$ from the set $C \setminus (H_1 \cup Y)$, we have $b_C(g) \leq n - 2$). So if $k \geq 2$ we get $\log_p |G'| \leq \log_p |C'| + 1 \leq \frac{(n - 3)(n - 2)}{2} + \frac{(n - 2)(n - 1)}{2} + 2 \leq \frac{(n - 2)(n - 1)}{2} + k$ and the induction step is clear. In the case $k \leq 1$, we get $n = 2$ (the case $n \leq 1$ is trivial) and the set $C \setminus (H_1 \cup Y)$ is contained in the center of $C$ (because $b_C(g) \leq b(g) - 1 \leq 0$ for $g \in C \setminus (H_1 \cup Y)$). It is clear that the set $C \setminus (H_1 \cup Y)$ generates the central subgroup of index at most $p$ in $C$, so $C$ is abelian. From the conditions of Theorem 2.1 there exists an element $g \notin C$ such that $b(g) \leq k \leq 1$. So the group $C_G(g)$ has index at most $p$ in $G$. Also the subgroup $C_G(g) \cap C$ has index at most $p^2$ in $G$ and is central in $G$. Apply Lemma 2.2 to the group $G$, so we conclude that $\log_p |G'| \leq 1 \leq \frac{(n - 1)(n - 2)}{2} + k$.

Eventually, we can assume that $|C : Y| \geq p^2$. So the set $C \setminus (H_1 \cup Y)$ generates $C$, because otherwise if it generates proper subgroup $H$ of $C$, then $C = (H_1 \cap C) \cup Y \cup H$ and from Lemma 2.3 we conclude that $|C : Y| \leq 2$, contradicting our assumption that $|C : Y| \geq p^2$. Also the set $C \setminus (H_1 \cup Y)$ is contained in the set $\{g \in C | b_C(g) \leq n - 2\}$, so $C$ is generated by the elements of the breadth at most $n - 2$. And we can apply the induction hypothesis to $C$ and its two subgroups $H_1 \cap H_2, Y$ and conclude that $\log_p |C'| \leq \frac{(n - 3)(n - 2)}{2} + n - 2 = \frac{(n - 2)(n - 1)}{2}$. Also from the conditions of the Theorem 2.1 there exists an element $a \notin C$ such that $b(a) \leq k$. Apply Lemma 2.1 to $C$ and $a$, so $\log_p |G'| \leq b(a) + \log_p |C'| \leq \frac{(n - 2)(n - 1)}{2} + k$. \qed
2.2. Proof of Theorem 1.3. Theorem 1.3 is a consequence of Theorem 2.1. Assume the converse. Let the proper subgroups $H_1$ and $H_2$ cover all the elements of breadth at least $n$ and $|G : H_2| \geq 4$. We will prove that then $|G'| \leq 2^{n(n-1)/2}$. The set $G \setminus (H_1 \cup H_2)$ generates $G$, because otherwise if it generates proper subgroup $H$ of $G$, then $G = H_1 \cap H_2 \cup H$ and from Lemma 2.3 we conclude that $|G : H_2| \leq 2$, contradicting our assumption that $|G : H_2| \geq 4$. Now, notice that all the elements from $G \setminus (H_1 \cup H_2)$ have breadth at most $n - 1$ in $G$. So denote $k = n - 1$ and apply Theorem 2.1 to $G, H_1, H_2$ and $n \leq k + 1$. We conclude that

$$
\log_2 |G'| \leq \frac{(n-1)(n-2)}{2} + k = \frac{n(n-1)}{2}.
$$

Remark 2.1. In fact, Theorem 1.2 is also a consequence of Theorem 2.1.

The following Theorems 2.2, 2.3 are generalizations of Theorems 1.2 and 1.3, respectively. Their proofs are similar to the proofs of Theorems 1.2 and 1.3.

Theorem 2.2. Let $p$ be a prime number. Let $G$ be a finite $p$-group and let $|G'| > p^{n(n-1)/2}$ for some non-negative integer $n$. Then the set of elements of breadth at least $n$ cannot be covered by $p - 1$ proper subgroups in $G$.

Theorem 2.3. Let $p$ be a prime number. Let $G$ be a finite $p$-group and let $|G'| > p^{n(n-1)/2}$ for some non-negative integer $n$. Then the set of elements of breadth at least $n$ cannot be covered by $p$ proper subgroups in $G$, one of which has index at least $p^2$ in $G$.

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References

[1] V.D. Mazurov and E.I. Khukhro (Editors), Unsolved problems in group theory - The Kourovka notebook, No. 18, arXiv:1401.0300v10, 8 Sep 2017.
[2] Wiegold. J, Commutator subgroups of finite $p$-groups, J. Australian Math, Soc., 10 (1969), 480-484.
[3] M.R. Vaughan-Lee, Breadth and commutator subgroups of $p$-groups, J. Algebra 32 (1974), 278-285.