Generalisation of Hill’s Yield Function for Planar Plastic Anisotropy

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Abstract. In this paper we present a new generalised quadratic yield function for plane stress analysis \cite{7} for the description of the plastic anisotropy of metals and also for the study of the asymmetric behaviour in tension-compression typical of the Hexagonal Closed-Pack (HCP) materials. The new yield function has a quadratic form in the stress tensor and it simultaneously predicts accurately the r-values and directional flow stresses. It also describes very well the biaxial symmetric stress state which is fundamental for the accurate modelling of aluminium alloys. The new quadratic yield function represents the non-symmetric biaxial stress state by performing a linear interpolation from pure uni-axial loading to a biaxial symmetric stress state. The most advanced yield functions for plastic planar anisotropy characterise very well the bi-axial symmetric stress region by using experimental flow stresses for the symmetric bi-axial stress state. However, the behaviour of the alloy between the uni-axial stress state and the symmetric bi-axial stress state is still not very well characterised. In this new yield function that behaviour can be assessed from interpolation from the uni-axial stress state into the symmetric bi-axial stress state until experimental yield locus fitting is achieved within a reasonable tolerance in that region. The main advantages of this new yield function is that it can be used for the modelling of metals with any crystallographic structure (BCC, FCC or HCP), it only has five anisotropic coefficients and also that it is a simple quadratic yield criterion that is able to accurately reproduce the plastic anisotropy of metals whilst using an associated flow rule. In the results and discussion we validate the yield locus for FCC and HCP alloys and we apply the new yield function in a cup drawing simulation for the assessment of the cup earing profile.

1. Introduction

One of the first yield functions for plastic anisotropy was developed in the pioneer work by Hill (1948) \cite{10}. Hill developed a quadratic yield function with anisotropic coefficients that could either predict the r-values or directional flow stresses, but never both simultaneously. Moreover, Hill’s original yield function does not include the effect of the biaxial symmetric stress and so it is not accurate in the modelling of aluminium alloys. Many posteriori yield functions \cite{1}, \cite{2}, \cite{3}, \cite{4}, \cite{5}) were developed after Hill’s and the coefficients of these yield functions were designed to include the biaxial symmetric stress effect. Whilst the equi-biaxial flow stress has been defined in these functions, none of them have characterised the coefficients for the unsymmetric biaxial stress state between pure uniaxial loading and equi-biaxial loading.

Hexagonal Close-Packed (HCP) materials, such as magnesium and titanium alloys, have less active slip systems at low/room temperatures but they have additional twinning systems that accommodate plastic deformation by a different mechanism known as twinning or distortion of the lattice. Twinning is a polar deformation mechanism (it only develops in one direction)
and this is the main reason for the asymmetric behaviour observed on HCP alloys in tension-compression.

Cazacu et al. [8] proposed a criterion based on a linear transformation that accounts for the strength-differential effect, particularly prominent in Hexagonal Closed-Pack (HCP) materials, with the work being extended in Plunkett et al. [12] and Plunkett et al. [11] by including the effect of texture development in the yield function. Barlat et al. [4], [5] later introduced two linear transformations which were applied on the sum of two yield functions in the case of plane and general stress states, in order to improve the accuracy of the functions by Cazacu et al. [8], Plunkett et al. [12] and Plunkett et al. [11] in the modelling of the anisotropic behaviour of aluminium sheets. Bron and Besson [6] further extended Karafillis and Boyce’s approach to two linear transformations. These recently proposed yield functions include more anisotropy coefficients and therefore give a better description of the anisotropic properties of a material. Although the mathematical formulations are complex and very heavy from a computational point of view.

Most of the early developed phenomenological yield potentials (e.g. von-Mises [14] and Hill’s [10]) are quadratic in the stress tensor. These yield potentials were mainly designed from distortion energy balance equations, and they were developed primarily for steel alloys, with Hill’s 1948 [10] going a step further by including plastic anisotropy in the potential. It is widely accepted that these potentials fit the yield locus very well for steel, but are unable to accurately predicting the anomalous behaviour of aluminium alloys (Dodd and Caddell [9]), especially in reproducing the yield locus on the vicinity of the symmetric biaxial stress state. There are two ways of accurately model aluminium alloys: i) by using non-quadratic yield functions with associated flow rules; ii) by using quadratic yield functions with non-associated flow rules (Stoughton and Yoon [13]), where in this case a plastic potential needs to be defined for the plastic flow. The use of non-associated flow rules allows for the use of simpler yield potentials, such as the quadratic potential of Hill 1948 [10], but a second plastic potential needs to be used for the plastic flow. The use of two different potentials in the non-associated flow rule can however lead to difficulties during return mapping procedures, especially if the loci of the two potentials (yield and plastic potential) are of considerably different shapes. Considering however the flexibility in phenomenological modelling, it must be possible to develop a quadratic generalised yield function for simultaneously predicting r-values and directional flow stresses accurately as opposed to the individual treatment that has been adopted so far. It also follows to say that it must be possible to simultaneously match the r-values and directional flow stresses for any stress state, as for example under planar anisotropy assumption. This generalised yield function must be able to accurately predict the anomalous aluminium behaviour [9] and the symmetric biaxial stress state. Certainly, the yield function must be accurate for a wide range of cases and valid if, and only if, it is proven to be convex in the principal stress space. Therefore, the main ideas for the new yield function proposed in this paper are as follows:

- A new quadratic yield function is developed for the simultaneous prediction of r-values and directional flow stresses;
- This new model can simultaneously predict the r-values and directional flow stress accurately for any given angle from the rolling direction;
- The biaxial symmetric flow stress is incorporated in this new quadratic yield function. However, the biaxial r-value is not included in this formulation;
- It is postulated that the stress tensor changes in a linear manner between symmetric biaxial stress state and uniaxial stress state, hence it is included in the new quadratic yield function in an interpolatory manner;
- Consequently due to this new quadratic yield function, it is possible to simultaneous predict r-values and directional flow stress from the use of an associated flow rule;
The main objective of this research work is therefore to develop the yield function for plane stress analysis as general as possible so that it can work with associated flow rules for the modelling of planar anisotropy for both FCC and HCP materials and also that it is able to describe the asymmetric behaviour in tension-compression typical of HCP materials.

2. Model Formulation

In this formulation, the well-established Hill’s [10] yield potential is used and extra flexibility is introduced in some coefficients in order to achieve a fully generalised function. Hill [10] proposed a yield function that can be used for the study of the planar anisotropy of metals and for which the equivalent stress is defined as:

$$\bar{\sigma} = \sqrt{\frac{3}{2} \left( (G + H) \sigma_{xx}^2 + (F + H) \sigma_{yy}^2 - 2H \sigma_{xx} \sigma_{yy} + 2N \tau_{xy}^2 \right)}$$

The coefficients $F$, $G$, $H$ and $N$ are designed to fit the r-values or, alternatively, the directional flow stresses but never both in simultaneous. This is a major limitation in Hill’s yield potential because both fitting is required for the accurate modelling of planar plastic anisotropic metals. Hill’s yield potential has however some great advantages which are its quadratic form and the simplicity of the model for the description of plastic anisotropy.

Therefore, a new yield function (YldParam) is defined as:

$$\bar{\sigma} = \frac{1}{C_a(u)} \sqrt{\frac{3}{2} \left( (G + H) \sigma_{xx}^2 + [F(u) + H] \sigma_{yy}^2 - 2H \sigma_{xx} \sigma_{yy} + 2N \tau_{xy}^2 \right)}$$

where $C_a(u)$ is a new coefficient which defines the anisotropy in the yield stresses and the anisotropy for the r-values is conserved from the adaptation of coefficient $F$ from the original Hill’s model by making it variable. Both coefficients are a function of a parametric coordinate $u$, which represents the orientation of the loading direction when measured from the rolling direction. This parametric variable $u$ exists within the limits $0 \leq u \leq 1$, with $u = 0$ defining the rolling direction, $u = 0.125$ being $45^\circ$ from the rolling direction, $u = 0.25$ defining $90^\circ$ with the rolling direction and finally $u = 1.0$ again defining the rolling direction. The coefficient $C_a(u)$ is designed to fit the yield stresses while the coefficient $F(u)$ is designed to fit the r-values. The Hill’s coefficients $H$, $G$ and $N$ are obtained from the experimental r-values at $0^\circ$, $45^\circ$ and $90^\circ$ from the rolling direction [10]:

$$H = \frac{2r_0}{1 + r_0}$$

$$G = \frac{2}{1 + r_0}$$

$$N = \frac{2 \left( r_0 + r_{90} \right) \left( 2r_{45} + 1 \right)}{2r_{90} \left( 1 + r_0 \right)}$$

The r-value anisotropic coefficient $F(u)$ will be calculated from the use of a Non-Uniform Rational B-Spline (NURBS) approximation on the parametric coordinate $u$. For that purpose, it will be necessary to calculate first the r-value anisotropic coefficient $F$ at every $15^\circ$ from the rolling direction and then use the NURBS approximation to build a function $F(u)$ that can generate the F-coefficient for any loading orientation $\theta$. The Hill’s 1948 [10] formulae for this coefficient can be used as presented in the following equation:

$$F(\theta) = \frac{H \left( 1 - 4 \sin^2 \theta \cos^2 \theta \right) - G \left( \sin^2 \theta \cos^2 \theta + r_\theta \cos^2 \theta \right) + 2N \sin^2 \theta \cos^2 \theta}{\sin^2 \theta \cos^2 \theta + r_\theta \sin^2 \theta}$$
This function for $F(\theta)$ is singular at $0^\circ$ and so the following alternative function was used for the calculation of the coefficient at $0^\circ$:

$$F(0^\circ) = \frac{2r_0}{r_90(1 + r_0)}$$

(5)

For the NURBS approximation for both $C_a(u)$ and $F(u)$ the following relation between the angle from the rolling direction and the parametric coordinate $u$ is necessary:

$$\theta = 2\pi u$$

(6)

and the angle $\theta$ can be easily obtained from the Mohr’s circle or from the equation for the principal directions from plane stress analysis:

$$\tan(2\theta) = \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}}$$

(7)

The coefficient $C_a(u)$ will be calculated from the new yield function from equation (2) and from the yield stresses $\sigma_0$ defined at every $15^\circ$ from the rolling direction. By doing so the following equation is obtained for coefficient $C_a(u)$:

$$C_a(u) = \frac{\sigma_0}{\bar{\sigma}} \sqrt{\frac{3}{2}} \sqrt{(G + H) \cos^4 \theta + [F(u) + H] \sin^4 \theta + 2(N - H) \cos^2 \theta \sin^2 \theta}$$

(8)

where $\frac{\sigma_0}{\bar{\sigma}}$ is the normalised flow stress at direction $\theta$ from the rolling direction obtained from experimental data.

2.1. Incorporation of the Biaxial Symmetric Flow Stress

The model defined so far does not consider the biaxial symmetric yield stress. If the stress tensor deviates considerably from the uniaxial stress state (defined by the principal stresses $\sigma_1 > 0$ and $\sigma_2 = 0$ for tension or $\sigma_2 < 0$ and $\sigma_1 = 0$ for compression), the differences in the accuracy can be substantial and this is more severe when the stress tensor is closer to the biaxial symmetric stress state. So, the incorporation of the biaxial symmetric flow stress in the model is important.

A generalised yield function in the normalised principal space, $\sigma_1/\bar{\sigma} - \sigma_2/\bar{\sigma}$ is shown in figure 1 and for each quadrant, the Mohr’s circle with the possible loading directions is depicted. The stress tensor as one transits from a symmetric biaxial state through an unsymmetric biaxial state to a uniaxial stress state remains unknown, however it is postulated in this work that a linear variation is valid. We can therefore interpolate between these two stress states by introducing a parameter $\beta$ defined in equations (10) and (11) that represents the deviation from a symmetric biaxial stress state. Therefore, we can define a coefficient $C_b(u)$ for the biaxial symmetric stress state and the coefficient $C_a(u)$ becomes:

$$C(u, \beta) = \beta \cdot C_a(u) + (1 - \beta) \cdot C_b(u)$$

(9)

where:

$$\beta = \frac{\sigma_1 - \sigma_2}{\sigma_1}$$

(10)

for biaxial tension and:

$$\beta = \frac{|\sigma_2| - |\sigma_1|}{|\sigma_2|}$$

(11)

for biaxial compression. For $\beta = 0$ we have symmetric biaxial stress state and for $\beta = 1$ we have uniaxial stress. For $0 < \beta < 1$ we have a stress state somewhere between uniaxial and symmetric biaxial.

Thus, the new quadratic yield function from equation (2) becomes:

$$\bar{\sigma} = \frac{1}{C(u, \beta)} \sqrt{\frac{3}{2}} \sqrt{(G + H) \sigma_{xx}^2 + [F(u) + H] \sigma_{yy}^2 - 2H\sigma_{xx}\sigma_{yy} + 2N\tau_{xy}^2}$$

(12)
3. Cup Drawing for Earing Prediction
The new yield function was tested for a cup drawing simulation for the prediction of the cup earing profile for an Al 2090-T3 aluminum alloy. The problem definition can be found in Cardoso and Adetoro [7].

The cup drawing simulation was carried out in the commercial software ABAQUS and by using a user material subroutine "VUMAT" for the new yield function. Figure 2 depicts the deformed configuration and the earing profile for the cup after the cup drawing operation.
In Figure 3 the cup earing profile is compared for the experimental results from Yoon et al. [15], Yld96 without translation [15] (or without consideration of the strength differential effects) and the new yield function. The earing magnitude is in good agreement with the simulations of Yoon et al. [15] for the Yld96 without translation and also in reasonable agreement with the measured result, but however, both Yld96 and the new yield function do not lead to the correct trend: the experimental cup height at 0 degrees is larger than that at 90 degrees, whereas the simulated results predict the reverse. The justification of this discrepancy is detailed in Cardoso and Adetoro [7].

4. Concluding Remarks

In this work, a new generalised quadratic yield function was developed for the description of planar plastic anisotropy in metallic alloys. The new yield function delivers a good prediction of both r-values and directional flow stresses and it also accurately describes the biaxial symmetric flow stress and the unsymmetric biaxial stress state. One coefficient $F(u)$ was made function of a directional parameter that represents the angle between the loading direction and the rolling direction. An additional coefficient $C(u)$ was added for the accurate prediction of directional flow stresses. Quadratic NURBS basis functions were used for the mathematical description of these two coefficients, making the method computationally effective.

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