Kleene stars of the plane, polylogarithms and symmetries

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Abstract

We extend the definition and construct several bases for polylogarithms \( \text{Li}_T \), where \( T \) are some series, recognizable by a finite state (multiplicity) automaton of alphabet \( X = \{ x_0, x_1 \} \). The kernel of this new “polylogarithmic map” \( \text{Li}_\bullet \), is also characterized and provides a rewriting process which terminates to a normal form. We concentrate on algebraic and analytic aspects of this extension allowing index polylogarithms at non positive multi-indices, by rational series and regularize polyzetas at non positive multi-indices.\(^5\)

Keywords: Rational power series, Kleene stars, algebraic independence, polylogarithms, transcendence basis, compact convergence, multiplicity automaton.

\(^4\) The space of rational series considered here is \( \left( \mathbb{C}(X) \sqcup \mathbb{C}^{\text{rat}}(\langle x_0 \rangle) \sqcup \mathbb{C}^{\text{rat}}(\langle x_1 \rangle) \right)^* \).

\(^5\) This research of Ngo Quoc Hoan is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.04-2017.320
1 Introduction

As a matter of fact, the interest of rational series, over the alphabets $Y_0 = \{y_n\}_{n \in \mathbb{N}}, Y = Y_0 \setminus \{y_0\}$ and $X = \{x_0,x_1\}$, is twofold: algebraic and analytic.

Firstly, (from the algebraic point of view) these series are closed under shuffle products and the shuffle exponential of letters (and their linear combinations) is precisely their Kleene star \(^6\). Secondly, the growth of their coefficients is tame \([11,25,26]\) and as such their associated polylogarithms can be rightfully computed \([20,23,24]\).

Doing this, we recover many functions (as simple polynomials, for instance) forgotten in the straight algebra of polylogarithms at positive indices, which can be viewed as the image of the following isomorphism of algebras \([21]\):

\[
\text{Li}_\bullet : (\mathbb{C}\langle X \rangle, + , 1_X^*) \longrightarrow (\mathbb{C}\{\text{Li}_w\}_{w \in X^*}, \times, 1),
\]

\[
x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1 \longrightarrow \text{Li}_{s_1 \ldots ,s_r},
\]

\[
\forall n \geq 0, x_0^n \longrightarrow \log^n(z) \cdot \frac{1}{n!}.
\]

To study multi-indexed polylogarithms, one relies on the one-to-one correspondence between the multi-indices \((-s_1, \ldots , -s_r)\), in $\mathbb{Z}_{\leq 0}^r$ (or \((s_1, \ldots , s_r) \in \mathbb{N}_+^r\)), and the words $y_{s_1} \ldots y_{s_r}$, in the monoid $Y_0^*$, indexing polylogarithms by $y_{s_1} \ldots y_{s_r}$ as follows \([10,11,21,22]\):

\[
\text{Li}_{y_{s_1} \ldots y_{s_r}} = \text{Li}_{s_1 \ldots ,s_r} \quad \text{and} \quad \text{Li}_{-y_{s_1} \ldots y_{s_r}} = \text{Li}_{-s_1 \ldots ,-s_r}.
\]

We will explain the whole project to extend $\text{Li}_\bullet$ over a sub algebra of rational power series. In particular, we study here various aspects of $\mathcal{C}\{\text{Li}_w\}_{w \in X^*}$, where $\mathcal{C}$ denotes the ring of polynomials in $z, z^{-1}$ and $(1-z)^{-1}$, with coefficients in $\mathbb{C}$, and we will express polylogarithms (resp. harmonic sums) at negative multi-indices as polynomials in $(1-z)^{-1}$ (resp. $N \in \mathbb{N}$), with coefficients in $\mathbb{Z}$ (resp. $\mathbb{Q}$).

We will concentrate, in particular, on algebraic and analytic aspects of this extension allowing index polylogarithms at non positive multi-indices by rational series and regularize (divergent) polyzetas at non positive multi-indices.

The paper is structured as follows:

(i) In Section 2, we will provide some background consisting in the algebraic

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\(^6\) i.e. for any $S \in \mathbb{C}\langle X \rangle$ such that $\langle S | 1_X^* \rangle = 0$, $S^*$ denotes the sum $1_X^* + S + S^2 + S^3 + \ldots$ and is called its Kleene star (see [1]).

\(^7\) i.e. for such a rational series $S$ over $X$, there exists real numbers $K, R > 0$ such that, for any $w \in X^*$, the coefficient $|\langle S | w \rangle|$ is bounded from above by $K \cdot R^{|w|}$. 
combinatorial framework on which the first structures of polylogarithms and
harmonic sums rely, namely their indexation by words and then by (non com-
mutative) polynomials.

(ii) In Section 3, we observe that, as such, the derivations, \( z \frac{d}{dz} \) and \( (1-z) \frac{d}{dz} \) are
continuous (for the standard topology) and so are their sections \( \int_{z_0}^{z} \frac{dz}{s} \) and \( \int_{z_0}^{z} \frac{dz}{1-s}\) (for \( z_0 \) in a suitable non-void open domain) but, in order to satisfy
the standard asymptotic condition (see eq. 14), we need other sections which
we develop in this paragraph. We will then study the bi-integro-differential
algebra and some functional analysis aspects of polylogarithms, here defined
on the cleft complex plane \( \mathbb{C} \setminus \{ -\infty,0 \} \cup \{1, +\infty \} \). We will give also a linear
basis of the algebra of polylogarithms with respect to coefficients being \( \mathbb{C} \)-polynomials of \( \{z,1/z,1/(1-z)\} \).

(iii) In Section 4, in order to extend, to classes of rational series, the indexation
of \( \text{Li}_* \), we review properties of rational series and the notation of rational
expressions.

(iv) In Section 5, in order to study Kummer functional equations on polyloga-
rithms, via their non commutative generating series satisfying noncommuta-
tive differential equation, we define polylogarithms on the universal covering
of \( \mathbb{C} \setminus \{0,1\} \).

(v) Finally, in Section 6, the extended double regularization of divergent polyze-
tas, on \( \mathbb{C}\langle X \rangle \cup \mathbb{C}[x^*_0] \cup \mathbb{C}(-x^*_0) \cup \mathbb{C}[x^*_1] \) and \( \mathbb{C}\langle Y \rangle \cup \mathbb{C}[y^*_0] \cup \mathbb{C}[y^*_1] \), is obtained.
These studies will be applied to obtain solutions of \( KZ_3 \) and examples of associators
with rational coefficients [9,27].

2 Polylogarithms and algebraic combinatorial framework

Let us, now, go into details, using the notations of [1,33],

(i) We construct the bialgebras.

\[
(\mathbb{C}\langle X \rangle, \text{conc}, \Delta_{\cup}, 1_{X^*}, \epsilon) \text{ and } (\mathbb{C}\langle Y_0 \rangle, \text{conc}, \Delta_{\cup}, 1_{Y^*_0}, \epsilon)
\]

in which, for any \( i = 0,1 \) and \( j \geq 0 \), one has

\[
\Delta_{\cup}(x_i) = x_i \otimes 1_{X^*} + 1_{X^*} \otimes x_i,
\]

\[
\Delta_{\cup}(y_j) = y_j \otimes 1_{Y^*} + 1_{Y^*} \otimes y_j + \sum_{k+l=j} y_k \otimes y_l.
\]

and \( \text{conc} \) is the usual concatenation product between noncommutative polynomials. Out of these two, only the first one is Hopf, the last one contains
$1 + y_0$ which is group-like and has no inverse (see infiltration product phenomenon in [2]) and therefore has no antipode.

(ii) Let $\mathbb{C}^{\text{rat}}(\langle X \rangle)$ denote the closure of $\mathbb{C}(\langle X \rangle)$ by rational operations $\{+,,,*\}$ [1,8] (it is closed by shuffle products). By Kleene-Schützenberger’s theorem (see below paragraph 4.1), any power series $S$ belongs to $\mathbb{C}^{\text{rat}}(\langle X \rangle)$ if and only if it is recognizable by an automaton admitting a linear representation $((\beta, \mu, \eta))$ of dimension $n$, with

$$\beta \in \mathcal{M}_{1,n}(\mathbb{C}), \mu : X^* \longrightarrow \mathcal{M}_{n,n}(\mathbb{C}), \eta \in \mathcal{M}_{n,1}(\mathbb{C})$$

such that, for any $w \in X^*$, one has (see [1,8] and paragraph 4.1)

$$\langle S \mid w \rangle = \beta \mu(w) \eta.$$

(iii) Let us consider the following morphism of algebras, defined by

$$\pi_X : (\mathbb{C}(\langle Y \rangle, \text{conc}, 1_{Y*}) \longrightarrow (\mathbb{C}(\langle X \rangle, \text{conc}, 1_{X*}),$$

$$y_1 \cdots y_s \mapsto x_0^{s_1-1} x_1 \cdots x_0^{s_r-1} x_1.$$

It admits an adjoint $\pi_Y$ for the two standard scalar products, i.e.

$$\forall p \in \mathbb{C}(\langle X \rangle), \forall q \in \mathbb{C}(\langle Y \rangle), \langle \pi_Y(p) \mid q \rangle_Y = \langle p \mid \pi_X(q) \rangle_X.$$

One checks immediately that $\pi_Y(x_0^{s_1-1} x_1) = y_s$, $\ker(\pi_Y) = \mathbb{C}(\langle X \rangle x_0$ and $\pi_Y$ restricted to the subalgebra $(\mathbb{C} 1_{X*} \oplus \mathbb{C}(\langle X \rangle x_1,)$ is an isomorphism, inverse of $\pi_X$.

In this work, unless symmetries are involved (i.e. until section 5) $\Omega$ denotes the cleft plane $\mathbb{C} \setminus (\mathbb{C} \cup (1,1,\infty])$ and $\mathcal{H}(\Omega)$, the set of holomorphic functions over the simply connected domain $\Omega$.

The principal object of the present paper, as in [10,11], is the polylogarithm well defined, for any $(s_1, \ldots, s_r) \in \mathbb{C}^r, r \in \mathbb{N}_+$ and for any $z \in \mathbb{C}$ such that $|z| < 1$, by

$$\text{Li}_{s_1,\ldots,s_r}(z) := \sum_{n_1 > \ldots > n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \cdots n_r^{s_r}}.$$

So is the following Taylor expansion

$$\frac{\text{Li}_{s_1,\ldots,s_r}(z)}{1-z} = \sum_{N \geq 0} H_{s_1,\ldots,s_r}(N) z^N,$$

where the arithmetic function

$$H_{s_1,\ldots,s_r} : \mathbb{N} \longrightarrow \mathbb{Q}$$

is expressed by

$$H_{s_1,\ldots,s_r}(N) := \sum_{N \geq n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}}.$$
Here, $\text{Li}_{s_1,\ldots,s_r}$ can also be obtained by iterated integrals (which provide for free their analytic continuations), along paths in $\Omega$ and with respect to the differential forms

$$\omega_0(z) = \frac{dz}{z} \quad \text{and} \quad \omega_1(z) = \frac{dz}{1-z}.$$ 

Let $\mathcal{H}_r[17,36]$ denote the following domain

$$\mathcal{H}_r = \{ (s_1,\ldots,s_r) \in C^r \mid \forall m = 1,\ldots,r; \Re(s_1) + \ldots + \Re(s_m) > m \}.$$ 

After a theorem by Abel, for any $r \geq 1$, if $(s_1,\ldots,s_r) \in \mathcal{H}_r$, we have

$$\zeta(s_1,\ldots,s_r) := \lim_{z \to 1} \text{Li}_{s_1,\ldots,s_r}(z) = \lim_{N \to \infty} H_{s_1,\ldots,s_r}(N).$$

These limits are no longer valid in the divergent cases and requires the renormalization of the corresponding divergent polyzetas. This has been done for the case of polyzetas at positive multi-indices $[3,5,25]$ and in $[15,18,30]$ and completed in $[10,11]$ for the case of negative multi-indices.

The technique used in $[10,11]$ (based on encoding polylogarithms, harmonic sums at positive multi-indices by words in $Y^*$) allows renormalize globally polyzetas at non positive multi-indices (via their noncommutative generating series) but the regularization is not achieved yet. To do that, in the present work, we introduce the rational series as a new and extended encoding suitable to regularize these functions (see [27] for the analytical justification of such algebraic process). This technique is already presented in [12], as a preprint, but never was published before.

### 3 Bi-integro-differential algebra of polylogarithms

#### 3.1 Differential rings

Let us consider the following group of transformations of $B = \mathbb{C} \setminus \{0,1\}$ which permutes the singularities in $\{0,1,\infty\}$

$$\mathcal{G} := \{ z \mapsto z, z \mapsto 1-z, z \mapsto z^{-1}, z \mapsto (1-z)^{-1}, z \mapsto 1-z^{-1}, z \mapsto z(z-1)^{-1} \}$$

and let us also consider the following rings :

$$\mathcal{G}_0 := \mathbb{C}[z^{-1}], \mathcal{G}_1 := \mathbb{C}[(1-z)^{-1}], \mathcal{G}_0 := \mathbb{C}[z,z^{-1}], \mathcal{G}_1 := \mathbb{C}[z,(1-z)^{-1}],$$

$$\mathcal{G}' := \mathbb{C}[z^{-1},(1-z)^{-1}], \mathcal{G} := \mathbb{C}[z,z^{-1},(1-z)^{-1}],$$

which are differential rings, endowed with the differential operator $\partial_z := d/dz$ and with the neutral element $1_\Omega : \Omega \to \mathbb{C}$, mapping $z$ to $1_\Omega(z) = 1$. It follows that

**Lemma 3.1** One has the following properties

(i) For $i = 0$ or $1$, 

\[ C'_i \subseteq C_i \subseteq C \text{ and } C'_i \subseteq C' \subseteq C. \]

(ii) The differential ring \( C \) is closed under action of \( \mathcal{G} \):
\[
\forall G(z) \in C, \forall g \in \mathcal{G}, G(g(z)) \in C.
\]

(iii) The subrings \( C_0, C_1 \) are closed by the involutions \( \{ z \mapsto z^{-1}, z \mapsto 1 - z \} \) and are exchanged by \( \{ z \mapsto 1 - z^{-1}, z \mapsto z(z - 1)^{-1} \} \), respectively.

Proof.

(i) It is immediate from the definitions of \( C, C_i, C', i = \{0, 1\} \) in (1).

(ii) This is an easy consequence of the fact that the element \( G \in C \) can be represented in the form:
\[
G(z) = \sum_{n=1}^{N_1} \frac{1}{z^n} + \sum_{m=-N_2}^{N_3} (1 - z)^m, N_1, N_2, N_3 \in \mathbb{N}.
\]

(iii) It is immediate from the definitions.

3.2 Differential and integration operators

Now, let us consider also the differential operators, acting on \( \mathcal{H}(\Omega) [26] \):
\[
\theta_0 = z \frac{d}{dz} \text{ and } \theta_1 = (1 - z) \frac{d}{dz}
\]
and integration operators
\[
\iota_0(f) = \int_{x_0}^{z} f(s) \omega_0(s) \text{ and } \iota_1(f) = \int_{x_0}^{z} f(s) \omega_1(s).
\]

One has \( \theta_i \iota_0 = \text{Id } \mathcal{H}(\Omega) \) (sections of the \( \theta_i \)).

One has other sections of \( \theta_i \), defined on \( C \{ \text{Li}_w \}_{w \in X^*} \) named \( I_i \) (without superscripts).

They are in fact, much more interesting (and adapted to the explicit computation of associators), these operators (\( I_i \) without superscripts), mentioned in the introduction are (more rigorously) defined by means of a \( C \)-basis of \( C \{ \text{Li}_w \}_{w \in X^*} = C \otimes_C C \{ \text{Li}_w \}_{w \in X^*} \).

(The family \( \{ \text{Li}_w \}_{w \in X^*} \) can be shown to be free free w.r.t. to all rational functions, see [6].)

Now, we recall that a word is Lyndon if it is always less (for the lexicographic ordering defined by \( x_0 < x_1 \)) than its proper right factors. Their set, noted \( \mathcal{Lyn}(X) \), is a transcendance basis of the shuffle algebra \( (C \langle X \rangle, \shuffle, 1_X) \) (Radford’s theorem [31]). Then
\[C\{L_i\}_{w \in X^*} \cong C[\mathcal{L}yn(X)],\]

one can partition the alphabet of this polynomial algebra in

\[(\mathcal{L}yn(X) \cap X^* \cup \{x_0\})\]

and then get the decomposition

\[\mathcal{C}\{L_i\}_{w \in X^*} \cong \mathcal{C} \otimes C \mathcal{C}\{L_i\}_{w \in X^*} \otimes C \mathcal{C}\{L_i\}_{w \in x_0^*}.\]

Using the following identity [20],

\[u x_1 x_0^n = u x_1 x_0^n - \sum_{k=1}^{n} (u x_0^k) x_1 x_0^{n-k},\]

we get

\[u x_1 x_0^n = \sum_{m=0}^{n} P_m x_1 x_0^m,\]

where \(P_m \in \mathbb{C}\langle X \rangle\) is uniquely defined by the above. Thus

\[L_i u x_1 x_0^n(z) = \sum_{m \leq n} L_i P_m x_1(z) \frac{\log^m(z)}{m!}.\]

This means that

\[\mathcal{B} := (z^k L_i u x_1(z) L_i x_0^n(z))_{(k, n, u) \in \mathbb{Z} \times \mathbb{N} \times X^*}\]

satisfies

\[\Box ((1 - z)^{-l} L_i u x_1(z) L_i x_0^n(z))_{(l, n, u) \in \mathbb{N} \times \mathbb{N} \times X^*}\]

\[\Box (z^k L_i x_0^n(z))_{(k, n) \in \mathbb{Z} \times \mathbb{N}}\]

\[\Box ((1 - z)^{-l} L_i x_0^n(z))_{(l, n) \in \mathbb{N} \times \mathbb{N}},\]

is a \(\mathbb{C}\)-basis of \(\mathcal{C}\{L_i\}_{w \in X^*}\). With this basis, we can define the operator \(t_0\) as follows

**Definition 3.2** Define the index map \(\text{ind} : \mathcal{B} \rightarrow \mathbb{Z}\) by

\[\text{ind}(z^k (1 - z)^{-l} L_i x_0^n(z)) = k \text{ and } \text{ind}(z^k (1 - z)^{-l} L_i u x_1(z) \log^n(z)) = k + |ux_1|.\]

Then \(t_0\) is computed as follows

\[t_0(b) = \begin{cases} \int_0^z b(s) \omega_0(s), & \text{if } \text{ind}(b) \geq 1, \\ \int_1^z b(s) \omega_0(s), & \text{if } \text{ind}(b) \leq 0. \end{cases}\]

and, as \(z \neq 1\), \(t_1\) is defined by

\[t_1(f) = \int_0^z f(s) \omega_1(s)\]
We will see in section 3.3 that \( t_0 \) is discontinuous. Nevertheless the pair \( \{t_0, t_1\} \) is adapted to computation of the special solution \( L_i \). One can check easily the following properties.

**Proposition 3.3** ([10,11,23,24]) One has the following properties

(i) The operators \( \{\theta_0, \theta_1, t_0, t_1\} \) satisfy in particular,

\[
\theta_1 + \theta_0 = [\theta_1, \theta_0] = \partial_z \quad \text{and} \quad \forall k = 0, 1, \theta_k t_k = \text{Id},
\]

\[
[\theta_0 t_1, \theta_1 t_0] = 0 \quad \text{and} \quad (\theta_0 t_1)(\theta_1 t_0) = (\theta_1 t_0)(\theta_0 t_1) = \text{Id}.
\]

(ii) The subspace \( \mathcal{C}_w \{L_i\}_{w \in X^*} \) is closed under the action of \( \{\theta_0, \theta_1\} \) and \( \{t_0, t_1\} \).

This means that, for any \( w = x_1 \ldots x_r \in Y^* \) (then \( \pi_X(w) = x_0^{r-1} x_1 \ldots x_r^{r-1} x_1 \)) and \( u = y_1 \ldots y_s \in Y^* \), the functions \( L_i \) and \( \tilde{L}_u \) satisfy

\[
L_i = (t_0^{r_i-1} t_1 \ldots t_0^{r_r-1} t_1) l_\Omega \quad \text{and} \quad \tilde{L}_u = (\theta_0^{r_i+1} t_1 \ldots \theta_0^{r_r+1} t_1) l_\Omega,
\]

\[
t_0 L_i \pi_X(w) = L_{x_0 \pi_X(w)} \quad \text{and} \quad t_1 L_i \pi_X(w) = L_{x_1 \pi_X(w)},
\]

\[
\theta_0 L_i x_0 \pi_X(w) = L_i \pi_X(w) \quad \text{and} \quad \theta_1 L_i x_1 \pi_X(w) = L_i \pi_X(w).
\]

(iii) The bi-integro differential ring \( \mathcal{C}_w \{L_i\}_{w \in X^*}, \theta_0, t_0, \theta_1, t_1 \) is stable under the action of \( \mathcal{D} \)

\[
\forall h \in \mathcal{C}_w \{L_i\}_{w \in X^*}, \forall g \in \mathcal{D}, h(g(z)) \in \mathcal{C}_w \{L_i\}_{w \in X^*},
\]

(iv) \( \theta_0 t_1 \) and \( \theta_1 t_0 \) are scalar operators within \( \mathcal{C}_w \{L_i\}_{w \in X^*} \), respectively with eigenvalues \( \lambda := z \rightarrow z(1 - z) \) and \( 1/\lambda \), i.e.

\[
\forall f \in \mathcal{C}_w \{L_i\}_{w \in X^*}, \quad (\theta_0 t_1) f = \lambda f \quad \text{and} \quad (\theta_1 t_0) f = (1/\lambda) f.
\]

**Proof.** The three first points can be checked by (more or less) straightforward computations. The last point needs on the one hand identification of each \( L_i \) with its unique lifting to \( C \setminus \{0, 1\} \) which coincides with \( L_i \) on \( \Omega \) and on the other hand to lift the elements of \( \mathcal{D} \) so that this group acts on \( C \setminus \{0, 1\} \).

\[
3.3 \quad \text{Topology on } \mathcal{H}(\Omega) \quad \text{and continuity or discontinuity of the sections } t_i
\]

The algebra \( \mathcal{H}(\Omega) \) is that of analytic functions defined over \( \Omega \). We quickly describe the standard topology on it (see also [32]), namely that of compact convergence whose seminorms are indexed by compact subsets of \( \Omega \), and defined by

\[
p_K(f) := ||f||_K = \sup_{s \in K} |f(s)|.
\]

Of course,

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8 When the functions \( L_i \) and \( \mathcal{C} \) are extended to \( \tilde{B} \).

9 For details, see section 5.1.
\[ P_{K_1 \cup K_2} = \sup(P_{K_1}, P_{K_2}), \]
and therefore the same topology is defined by extracting a fundamental subset of seminorms, which can be chosen denumerable. As \( \mathcal{H}(\Omega) \) is complete with this topology it is a Fréchet space (see [34]) 10.

With the standard topology above, an operator \( \phi \in \text{End}(\mathcal{H}(\Omega)) \) is continuous iff, with \( K_i \) compacts of \( \Omega \),

\[
(\forall K_2)(\exists K_1)(\exists M_{21} > 0)(\forall f \in \mathcal{H}(\Omega))(||\phi(f)||_{K_2} \leq M_{21}||f||_{K_1}),
\]
the algebra \( \mathcal{C}\{L_i^w\}_{w \in X^*} \) (and \( \mathcal{H}(\Omega) \)) is closed under the operators \( \theta_i, i = 0, 1 \). We have build sections of them \( t_i^{z_0}, t_i \), which are continuous and, \( t_0 \) which is discontinuous and adapted to renormalisation and the computation of associators.

For \( z_0 \in \Omega \), let us define \( t_i^{z_0} \in \text{End}(\mathcal{H}(\Omega)) \) by

\[
t_i^{z_0}(f) = \int_{z_0}^{z} f(s) \omega_i(s) \text{ and } t_i^{z_0}(f) = \int_{z_0}^{z} f(s) \omega_1(s).
\]
It is easy to check that \( \theta_i t_i^{z_0} = \text{Id}_{\mathcal{H}(\Omega)} \) and that they are continuous on \( \mathcal{H}(\Omega) \) (for the topology of compact convergence) because for all \( K \subset \text{compact } \Omega \), we have

\[
|p_K(t_i^{z_0}(f))| \leq p_K(f)[\sup_{z \in K} \int_{z_0}^{z} \omega_i(s)]],
\]
and this is sufficient to prove continuity. The operators \( t_i^{z_0} \) are also well defined on \( \mathcal{C}\{L_i^w\}_{w \in X^*} \) and it is easy to check that

\[
t_i^{z_0}(\mathcal{C}\{L_i^w\}_{w \in X^*}) \subset \mathcal{C}\{L_i^w\}_{w \in X^*}.
\]
Due to the decomposition of \( \mathcal{H}(\Omega) \) into a direct sum of closed subspaces

\[
\mathcal{H}(\Omega) = \mathcal{H}_{z_0 \to 0}(\Omega) \oplus \mathbb{C}1_\Omega,
\]
it is not hard to see that the graphs of \( \theta_i \) are closed, thus, the \( \theta_i \) are also continuous.

To show discontinuity of \( t_0 \), one of the possibilities consists in exhibiting two sequences \( f_n, g_n \in \mathcal{C}\{L_i^w\}_{w \in X^*} \) converging to the same limit but such that

\[
\lim t_0(f_n) \neq \lim t_0(g_n).
\]
Here, we choose the function \( z \) for being approached in a twofold way and if \( t_0 \) were continuous, we would have equality of the limits of the image-sequences (and this is not the case). We first remark that

\[
z = \sum_{n \geq 0} \frac{\log^n(z)}{n!} = \sum_{n \geq 1} (-1)^{n+1} \frac{\log^n((1-z)^{-1})}{n!}
\]
Set

\[
10 \text{ It is even a Fréchet algebra with unit, but we will not use the multiplicative structure here.}
\]
\[
f_n = \sum_{0 \leq m \leq n} \frac{\log^m(z)}{m!} \quad \text{and} \quad g_n = \sum_{1 \leq m \leq n} \frac{(-1)^{m+1} \log^m((1-z)^{-1})}{m!}
\]

(These two sequences are in \( \mathbb{C} \{ \text{Li}_w \}_{w \in X^*} \). It is easily seen that
\[
t_0(f_n) = f_{n+1} - 1
\]
and then
\[
\lim_{n \to +\infty} t_0(f_n)(z) = z - 1.
\]

Now, for any \( s \in [0, z] \) with \( z \in ]0, 1[ \), one has
\[
|g(s)| = \left| \sum_{m=1}^{n} (-1)^{m+1} \frac{\log^m(1-s)}{m!} \right|
\]
\[
\leq \sum_{m=1}^{n} \left| \frac{\log^m(1-s)}{m!} \right|
\]
\[
\leq \exp(-\log(1-s)) - 1
\]
\[
= \frac{s}{1-s}.
\]

In order to exchange limits, we apply Lebesgue’s dominated convergence theorem to the measure space \((]0, z], \mathcal{B}, dz/z\) (\( \mathcal{B} \) is the usual Borel \( \sigma \)-algebra) and the function \( p(x) = s(1-s)^{x-1} \) which is - as are the functions \( g_n \) - integrable on \( ]0, z] \) (for every \( z \in ]0, 1[ \)). Then
\[
\lim(t_0(g_n)) = \lim_{n \to +\infty} \int_{0}^{z} g_n(s) \frac{ds}{s} = \int_{0}^{z} \lim_{n \to +\infty} g_n(s) \frac{ds}{s} = \int_{0}^{z} \frac{ds}{s} = z.
\]
Hence, for \( z \in ]0, 1[ \), we obtain,
\[
\lim(t_0(f_n)) = z - 1 \neq z = \lim(t_0(g_n)).
\]
This completes the proof.

4 Extension of \( \text{Li}_* \) to its rational domain.

4.1 Rational series

Rational series arise from an extension of finite state (boolean) automata to graphs with costs or weights \([14,35]\). They have many connections with computer science \([1,35]\) but also with operator and Hopf algebras \([8,13]\). In short a weighted graph is a finite directed graph with edges marked by weights (taken in a semiring, ring or field) and letters (taken in an alphabet \( X \)) as follows
\[
< \text{tail} > \xrightarrow{s|x|\alpha} < \text{head} >
\]
this amounts to giving a map \( \mu : X \rightarrow \mathcal{M}_{n,n}(\mathbb{C}) \) which is extended to words as
\[
\mu : X^* \rightarrow \mathcal{M}_{n,n}(\mathbb{C})
\]
by morphism. Along a graph path, weights multiply and letters concatenate, this gives the \textit{behaviour} of the automaton, which has an initial vector as input \( \beta \in \mathcal{M}_{1,n}(\mathbb{C}) \) and a final vector \( \eta \in \mathcal{M}_{n,1}(\mathbb{C}) \) as output; this \textit{behaviour} is a series \( S \in \mathbb{C}\langle\langle X\rangle\rangle \).

It can be proved the following theorem

\textbf{Theorem 4.1 ([1,8,35])} Let \( X \) be a finite alphabet and \( S \in \mathbb{C}\langle\langle X\rangle\rangle \). The following are equivalent

\begin{enumerate}
  \item \( S \) admits a linear representation \((\beta, \mu, \eta)\) of dimension \( n \) i.e. it exists
    \[
    \beta \in \mathcal{M}_{1,n}(\mathbb{C}), \mu : X^* \rightarrow \mathcal{M}_{n,n}(\mathbb{C}), \eta \in \mathcal{M}_{n,1}(\mathbb{C})
    \]
    such that, for all \( w \in X^* \)
    \[
    \langle S \mid w \rangle = \beta \mu(w) \eta \tag{2}
    \]
  \item \( S \) belongs to the smallest (concatenation) subalgebra of \( \mathbb{C}\langle\langle X\rangle\rangle \), containing \( \mathbb{C}\langle X\rangle \) and closed by \( S \rightarrow S^{-1} \) (rational closure\footnote{Of course \( S \rightarrow S^{-1} \) is only partially defined, its domain is the set of series such that \( \langle S \mid 1_{X^*} \rangle \neq 0 \).} of \( \mathbb{C}\langle X\rangle \)).
\end{enumerate}

\section{Domain of \( \text{Li}_\bullet \)}

Under suitable conditions of convergence (see below), the extension of \( \text{Li}_\bullet \) in general and to some subdomain of \( \mathbb{C}\text{rat}\langle\langle x_0 \rangle\rangle \) can be done as follows: call \( \text{Dom}(\text{Li}_\bullet) \) the set of series
\[
S = \sum_{n \geq 0} S_n \text{ with } S_n := \sum_{|w|=n} \langle S \mid w \rangle w
\]
such that \( \sum_{n \geq 0} \text{Li}_{S_n} \) converges uniformly any compact of \( \Omega \). Then

\textbf{Proposition 4.2} One has

\begin{enumerate}
  \item The set \( \text{Dom}(\text{Li}_\bullet) \) is closed by shuffle products.
  \item For any \( S, T \in \text{Dom}(\text{Li}_\bullet) \), one has \( \text{Li}_{S \shuffle T} = \text{Li}_S \text{Li}_T \).
  \item One has \( \mathbb{C}\langle X \rangle \shuffle \mathbb{C}\text{rat}\langle\langle x_0 \rangle\rangle \shuffle \mathbb{C}\text{rat}\langle\langle x_1 \rangle\rangle \subset \text{Dom}(\text{Li}_\bullet) \).
\end{enumerate}

\textbf{Proof.} (i) and (ii): Suppose \( S, T \in \text{Dom}(\text{Li}_\bullet) \) and \( S = \sum_{p \geq 0} S_p \) (resp. \( T = \sum_{q \geq 0} T_q \)) their decomposition in homogeneous components, then the family \( \text{Li}_{S_p} \) (resp. \( \text{Li}_{T_q} \)) is summable in \( \mathcal{H}(\Omega) \). This implies ([4] Ch III §6) that the families
\((\text{Li}_p \text{Li}_q)_{p,q \geq 0}\) and then \((\sum_{p+q=n} \text{Li}_p \text{Li}_q)_{n \geq 0}\) are summable in \(\mathcal{H}(\Omega)\). As
\[
\sum_{p+q=n} \text{Li}_p \text{Li}_q = \sum_{p+q=n} \text{Li}_{p+q} = \text{Li}_{p+q}\]
one gets that \(S \cup T \in \text{Dom}(\text{Li} \ast)\) and \(\text{Li}_{S \cup T} = \text{Li}_S \text{Li}_T\).

(iii): In view of (i,ii) it suffices to check that each of \(\mathbb{C} \langle X \rangle, \mathbb{C}_{\text{rat}} \langle \langle x_0 \rangle \rangle, \mathbb{C}_{\text{rat}} \langle \langle x_1 \rangle \rangle\) is in \(\text{Dom}(\text{Li} \ast)\). The first being given, the property for the last two is a consequence of Kronecker’s theorem [37] i.e. the fact that
\[
\mathbb{C}_{\text{rat}} \langle \langle x \rangle \rangle = \{P/Q\} \text{ where } P,Q \in \mathbb{C}[x] \text{ and } Q(0) \neq 0
\]
and the partial fraction decomposition. \(\Box\)

This extension is compatible with identities between rational series as Lazard’s elimination \(^{12}\), for instance, for all \(S \in \mathbb{C}_{\text{rat}} \langle \langle S \rangle \rangle \cap \text{Dom}(\text{Li} \ast)\):
\[
\text{Li}_S(z) = \sum_{n \geq 0} \langle S \mid x_0^n \rangle \frac{\log^n(z)}{n!} + \sum_{k \geq 1} \sum_{w \in (x_0^k x_1)^k x_0^k} \langle S \mid w \rangle \text{Li}_w(z),
\]

**Remark 4.3** Here we will be mostly interested by rational series within \(\text{Dom}(\text{Li} \ast)\). But there are other series as the following (infinite sum of rational series).
\[
T = \sum_{n \geq 0} \frac{(nx_0)^*}{n!} = \sum_{n \geq 0} \frac{1}{n!(1-nx_0)} \text{Treves} = e \sum_{k \geq 0} B_k x_0^k
\]
where the Treves topology is just the product topology and therefore limits, for it, are computed term by term \(^{13}\).

Now, it is easy to see that we have compact convergence because on \(\Omega\) (or, below, \(\tilde{\Omega}\)) as for all \(\phi \in \mathcal{H}(\Omega)\), and \(K \subset \text{compact} \Omega\) (or \(\tilde{\Omega}\)) one gets
\[
||e \sum_{k=0}^N B_k \phi^k_k ||_K \leq e \sum_{k=0}^N B_k \frac{||\phi||_K^k}{k!} \leq e \sum_{k=0}^\infty B_k \frac{||\phi||_K^k}{k!} = ee^{\phi\|\phi\|_K} = e^{e^{\phi\|\phi\|_K}}.
\]
Now, remarking that \(\text{Li}_{x_0^k}(z) = \log^k(z)/k!\), this proves that

\(T \in \text{Dom}(\text{Li} \ast)\) and \(\text{Li}_T(z) = e^{e^z}\).

\(^{12}\text{i.e. } X = (x_0^* x_1)^* x_0^*\). In other words, in \(\mathbb{C} \langle X \rangle\), \((1 - (x_0 + x_1))^{-1} = (1 - (1 - x_0^{-1} x_1))^{-1} (1 - x_0)^{-1}\).

\(^{13}\text{i.e. } S_n \rightarrow S \text{ iff } (\forall w \in X^*) (\lim n \langle S_n \mid w \rangle = \langle S \mid w \rangle)\).
The morphism $L_i$ is no longer injective on its domain but the family $(L_w)_{w \in X^*}$ is still $\mathcal{C}$-linearly independent [23]. We will use several times the following lemma which is characteristic-free.

4.3 Stars of the plane

**Lemma 4.4** Let $(\mathcal{A}, d)$ be a commutative differential ring without zero divisors, and $R = \ker(d)$ be its subring of constants. Let $z \in \mathcal{A}$ such that $d(z) = 1$ and $S = \{e_\alpha\}_{\alpha \in I}$ be a set of eigenfunctions of $d$, with all different eigenvalues (for example, take $I \subset R$) i.e., $e_\alpha \neq 0$ and $d(e_\alpha) = \alpha e_\alpha, \forall \alpha \in I$. Then the family $(e_\alpha)_{\alpha \in I}$ is $R[z]$-linearly free.

**Proof.** If there is no non-trivial $R[z]$-linear relation, we are done. Otherwise let us consider relations

$$\sum_{j=1}^{N} P_j(z) e_{\alpha_j} = 0, \tag{3}$$

with $P_j \in R[t]_{\text{pol}}, P_j(z) \neq 0$ for all $j$ (we will, in the sequel refer to these expressions as packed linear relations). We choose one of them minimal with respect to the triplet $[N, \deg(P_N), \sum_{j<N} \deg(P_j)]$, lexicographically ordered from left to right.

Remarking that $d(P(z)) = P'(z)$ (because $d(z) = 1$), we apply the operator $d - \alpha_N Id$ to both sides of (3) and get

$$\sum_{j=1}^{N} (P'_j(z) + (\alpha_j - \alpha_N) P_j(z)) e_{\alpha_j} = 0. \tag{4}$$

Minimality of (3) implies that (4) is trivial i.e.

$$P'_N(z) = 0 \text{ and } (\forall j = 1..N-1) \ (P'_j(z) + (\alpha_j - \alpha_N) P_j(z) = 0). \tag{5}$$

Now (3) implies

$$\prod_{j=1}^{N-1} (\alpha_N - \alpha_j) \sum_{j=1}^{N} P_j(z) e_{\alpha_j} = 0,$$

which, because $\mathcal{A}$ has no zero divisors, is a packed linear relation and has the same associated triplet as (3). From (5), we see that for any $k < N$, one has

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14 Here $R[z]$ is understood as ring adjunction i.e. the smallest subring generated by $R \cup \{z\}$.

15 Here, $R[t]_{\text{pol}}$ means the formal univariate polynomial ring (the subscript is here to avoid confusion).

16 i.e. consider the ones with $N$ minimal and among these, we choose one with $\deg(P_N)$ minimal and among these we choose one with $\sum_{j<N} \deg(P_j)$ minimal.
\[
\prod_{j=1}^{N-1} (\alpha_N - \alpha_j)P_k(z) = \prod_{j=1, j \neq k}^{N-1} (\alpha_N - \alpha_j)P_k'(z),
\]
so, if \( N \geq 2 \), we would get a relation of lower triplet. This being impossible, we get \( N = 1 \) and (3) boils down to \( P_N(z)e_N = 0 \) which, as \( \mathcal{A} \) has no zero divisors, implies \( P_N \equiv 0 \), a contradiction. Then the \( (e_\alpha)_{\alpha \in I} \) are \( R[z] \)-linearly independent. \( \square \)

**Remark 4.5** If \( \mathcal{A} \) is of characteristic zero, \( d(z) = 1 \) implies that \( z \) is transcendent over \( R \) and the two notations \( R[z] \) and \( R[z]_{pol} \) coincide.

First of all, let us prove

**Lemma 4.6** Let \( \mathcal{A} \) be a \( \mathbb{Q} \)-algebra (associative, unital, commutative) and \( z \) an indeterminate, then \( e^z \in \mathcal{A}[z] \) is transcendent over \( \mathcal{A}[z] \).

**Proof.** It is a straightforward consequence of Remark 4.5. Note that this can be rephrased as \([z, e^z]\) are algebraically independent over \( \mathcal{A} \).

**Proposition 4.7** Let \( Z = \{z_n\}_{n \in \mathbb{N}} \) be an alphabet, then \([z_0^*, z_1^*]\) is algebraically independent on \( \mathbb{C}[Z] \) within \((\mathbb{C}[[Z]], \omega, 1_{Z^*})\).

**Proof.** Recall that, for \( x \) a letter, one has

\[
x^* := \sum_{n=0}^{+\infty} x^n = \sum_{n=0}^{+\infty} \frac{x^{\perp n}}{n!} = e^x.
\]

By using Lemma 4.6, one can prove by induction that \([e_{i_1}^{(0)}, e_{i_2}^{(0)}, \cdots, e_{i_k}^{(0)}, z_0, z_1, \cdots, z_k]\), are algebraically independent. This implies that \( Z \cup \{e_{i_1}^{(0)}\}_{Z \in Z} \) is an algebraically independent set and, by restriction \( Z \cup \{e_{i_1}^{(0)}, e_{i_2}^{(0)}\} \) whence the proposition. \( \square \)

Using Lemma 4.7, we obtain the following proposition.

**Proposition 4.8** One has

(i) The family \( \{x_0^*, x_1^*\} \) is algebraically independent over \((\mathbb{C}[X], \omega, 1_{X^*})\) within \((\mathbb{C}[\langle X \rangle]^\text{rad}, \omega, 1_{X^*})\).

(ii) The module \((\mathbb{C}[X], \omega, 1_{X^*})[x_0^*, x_1^*, (-x_0)^*]\) is free over \( \mathbb{C}[X] \) and the family \( \{(x_0^*)^{(k)} \perp \omega (x_1^*)^{(l)}\}_{(k,l) \in \mathbb{Z} \times \mathbb{N}} \) is a \( \mathbb{C}[X] \)-basis of it.

(iii) As a consequence, \( \{w^{(k)}(x_0^*)^{(k)} \perp \omega (x_1^*)^{(l)}\}_{w \in X^*, \quad (k,l) \in \mathbb{Z} \times \mathbb{N}} \) is a \( \mathbb{C} \)-basis of it.

Now, we can construct the following morphism

**Definition 4.9** The following morphism

\[
\text{Li}^{(1)} : (\mathbb{C}[X], \omega, 1_{X^*})[x_0^*, (-x_0)^*, x_1^*] \longrightarrow \mathcal{H}(\Omega)
\]
can be defined, for any $w \in X^*$ and $L_i^{(1)} = L_i^w$, by

$$L_0^w = z, \quad L_{(-x_0)}^w = z^{-1}, \quad L_i^w = (1-z)^{-1}.$$

In fact existence and uniqueness of this morphism obtained as a consequence of Proposition 4.8. Moreover, its kernel and image are given by the following result 17:

**Theorem 4.10** We have $\text{Im}(L_i^{(1)}) = \mathcal{C}\{L_i^w\}_{w \in X^*}$ and $\ker(L_i^{(1)})$ is the ideal generated by $x_0^* \uplus x_i^* - x_i^* + 1_{X^*}$.

**Proof.** As $\mathbb{C}(X)[x_0^*, x_i^*, (-x_0)^*]$ admits $\{(x_0^*)^{\uplus k} \uplus (x_i^*)^{\uplus l}\}_{k \in \mathbb{Z}, l \in \mathbb{N}}$ as a basis for its structure of $\mathbb{C}(X)$-module, it suffices to remark that

$$L_i^{(1)}(x_0^*)^{\uplus k} \uplus (x_i^*)^{\uplus l}(z) = \frac{z^k}{(1-z)^l},$$

is a generating system of $\mathcal{C}$. As regards the second assertion, let us prove the following lemma (in this lemma and its proof, all sums are supposed finitely supported)

**Lemma 4.11** Let $M_1$ and $M_2$ be $K$-modules ($K$ is a unitary commutative ring). Suppose the following map is linear

$$\phi : M_1 \rightarrow M_2$$

Let $N \subset \ker(\phi)$ be a submodule. If there is a system of generators in $M_1$, namely $\{g_i\}_{i \in I}$ and $J \subset I$, such that

(i) For any $i \in I \setminus J$, $g_i \equiv \sum_{j \in J \setminus I} c_i^j g_j \pmod{N}$, $(c_i^j \in K; \forall j \in J)$;

(ii) $\{\phi(g_j)\}_{j \in J}$ is $K$-free in $M_2$;

then $N = \ker(\phi)$.

**Proof.** Suppose $P \in \ker(\phi)$. Then

$$P \equiv \sum_{j \in J} p_j g_j \pmod{N}$$

with $\{p_j\}_{j \in J} \subset K$. Thus

$$0 = \phi(P) = \sum_{j \in J} p_j \phi(g_j).$$

From the fact that $\{\phi(g_j)\}_{j \in J}$ is $K$-free in $M_2$, we obtain $p_j = 0$ for any $j \in J$. This means that $P \in N$. Thus $\ker(\phi) \subset N$ and, finally, $N = \ker(\phi)$. □

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17 This result is already presented in [12], as a preprint, but never was published before.
4.4 Examples of polylogarithms indexed by rational series

Let now \( \mathcal{J} \) be the ideal generated by \( x_0^* \cup x_1^* - x_1^* + 1 X^* \). It is easily checked, from the following formulas\(^{18}\), for \( l \geq 1 \),
\[
\begin{align*}
\{ w \cup (x_0^*)^\pm l \cup (x_1^*)^\pm l \} & = w \cup (x_1^*)^\pm l - w \cup (x_1^*)^\pm l - 1 [\mathcal{J}], \\
\{ w \cup (x_0^*)^\pm l \cup (x_1^*)^\pm l \} & = w \cup (x_1^*)^\pm l - 1 + w \cup (x_1^*)^\pm l [\mathcal{J}],
\end{align*}
\]
that one can rewrite [mod \( \mathcal{J} \)] any monomial \( w \cup (x_0^*)^\pm l \cup (x_1^*)^\pm l \) as linear combination of such monomials with \( kl = 0 \). Applying Lemma 4.11 with the following data:

- the morphism \( \phi = \text{Li}_i^{(1)} \),
- the modules \( M_1 = \mathbb{C}(X)[x_0^*, x_1^*], (-x_0)^*], M_2 = \mathcal{H}(\Omega), N = \mathcal{J} \),
- the families \( \{ g_i \} = \{ w \cup (x_1^*)^\pm l \cup (x_0^*)^\pm l \} \}_{(w,n,m) \in I} \),
- and the indices \( I = X^* \times \mathbb{N} \times \mathbb{Z}, J = (X^* \times \mathbb{N} \times \{ 0 \}) \sqcup (X^* \times \{ 0 \} \times \mathbb{Z}) \),

we have the second point of Theorem 4.10. \( \square \)

4.4 Examples of polylogarithms indexed by rational series

Proposition 4.12 ([19,20]) One has

(i) For \( x \in X, i \in \mathbb{N}_+, a \in \mathbb{C}, |a| < 1 \),
\[
\text{Li}^{(1)}_{\{2a\}_i} (z) = a^{i-1} \left( \frac{i-1}{k} \right) \frac{(a \log(z))^k}{k!},
\]
\[
\text{Li}^{(1)}_{\{1a\}_i} (z) = \frac{1}{(1-z)^a} \sum_{k=0}^{i-1} \left( \frac{i-1}{k} \right) \frac{(a \log((1-z)^{-1}))^k}{k!}.
\]

(ii) For any \( (s_1, \ldots, s_r) \in \mathbb{N}_+^r \) and \( |t_1| < 1, \ldots, |t_r| < 1 \),
\[
\text{Li}^{(1)}_{\{t_{1}x_0^*, x_1^* \}_i} (z) = \sum_{n_1 > \ldots > n_r > 0} \frac{z^{n_1}}{(n_1 - t_1)^{s_1} \ldots (n_r - t_r)^{s_r}}.
\]

\(^{18}\) In Figure 1, \( (w,l,k) \) codes the element \( w^{\perp \cup} (x_0^*)^\pm l \cup (x_1^*)^\pm l \).
In particular,  
\[
\operatorname{Li}^{(1)}_{(t_1, x_0) \cdots (t_r, x_0) \cdot x_1} (z) = \sum_{n_1, \ldots, n_r > 0} \operatorname{Li}_{x_0 \cdots x_{n_1-1} x_0 \cdots x_{n_r-1} x_1} (z) t_0^{n_1-1} \cdots t_r^{n_r-1}.
\]

To prove this proposition, we use the following easy lemma:

**Lemma 4.13** For any \(i, n \in \mathbb{N}^*\), we have

\[
\binom{n + i - 1}{n} = \sum_{k=0}^{n} \binom{i - 1}{k} \binom{n}{n-k} (ax)^n,
\]

Now, we give the proof of Proposition 4.12.

**Proof.**

(i) Let’s choose \(i \in \mathbb{N}^*, a \in \mathbb{C}\) and \(x \in X\). Note that

\[
(ax)^i = \sum_{n=0}^{\infty} (ax)^n = \frac{1}{1 - ax},
\]

\[
((ax)^*)^i = \sum_{n=0}^{\infty} \binom{n + i - 1}{n} (ax)^n,
\]

\[
(ax)^i \cdot (1 + ax)^i-1 = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{i - 1}{k} \binom{n}{n-k} (ax)^n,
\]

and use Lemma 4.13, we obtain, for \(i \in \mathbb{N}^*, a \in \mathbb{C}, x \in X\),

\[
((ax)^*)^i = (ax)^i \cdot (1 + ax)^i-1.
\]

Thus, for \(i \in \mathbb{N}^*, |a| < 1, x \in X\),

\[
\operatorname{Li}^{(1)}_{((ax)^*)^i} = \operatorname{Li}^{(1)}_{(ax)^*} \operatorname{Li}^{(1)}_{(1+ax)^i-1} = \operatorname{Li}^{(1)}_{(ax)^*} \sum_{k=0}^{i-1} \binom{i - 1}{k} a^k \operatorname{Li}^{(1)}_{x^k}.
\]

It follows then the expected results.

(ii) Using Lemma 4.13, we obtain this statement by direct calculation.

\[\square\]

**Corollary 4.14** One has

\[
\{ \operatorname{Li} \}_{S \in \mathbb{C}(X) \cup \mathbb{C}[\langle x_0 \rangle] \cup \mathbb{C}[\langle -x_0 \rangle] \cup \mathbb{C}[x_1]} = \operatorname{span}_\mathbb{C} \left\{ \frac{z^a}{(1 - z)^b} \operatorname{Li}_w(z) \right\}_{w \in X^*} \\
\subset \operatorname{span}_\mathbb{C} \left\{ \operatorname{Li}_{s_1, \ldots, s_r} \right\}_{s_1, \ldots, s_r \in \mathbb{Z}^r} \\
\oplus \operatorname{span}_\mathbb{C} \{ z^a | a \in \mathbb{Z} \},
\]

\[
\{ \operatorname{Li} \}_{S \in \mathbb{C}(X) \cup \mathbb{C}^{\text{rat}}(\langle x_0 \rangle) \cup \mathbb{C}^{\text{rat}}(\langle x_1 \rangle)} = \operatorname{span}_\mathbb{C} \left\{ \frac{z^a}{(1 - z)^b} \operatorname{Li}_w(z) \right\}_{w \in X^*}.
\]
\[ \subset \text{span}_\mathbb{C}\{\text{Li}_{s_1,\ldots,s_r}\}_{s_1,\ldots,s_r\in\mathbb{C}}^r \oplus \text{span}_\mathbb{C}\{z^a|a\in\mathbb{C}\}. \]

5 Symmetries and transition elements

5.1 Framework

Up to now (and historically [29]), the polylogarithms are computed in \( \Omega = \mathbb{C} \setminus (-\infty,0]\cup[1,\infty) \), cleft in order to cope with the two singularities \{0,1\}. But \( B = \mathbb{C} \setminus \{0,1\} \) is acted on by the following group of symmetries (which permutes, in fact, \{0,1,\infty\}).

\[ \mathcal{G} := \{ z \mapsto z, z \mapsto 1 - z, z \mapsto z^{-1}, z \mapsto (1 - z)^{-1}, z \mapsto 1 - z^{-1}, z \mapsto z(z - 1)^{-1} \} \]

To this end and because \( \mathbb{C} \setminus (-\infty,0]\cup[1,\infty) \) is not stable by this group we have now to work on \( \Omega = \tilde{B} \).

5.2 Monodromy Principle

For convenience, we consider the following situation

\[
\begin{array}{ccc}
  \tilde{B} & \xrightarrow{s} & \tilde{B} \\
  j & & \downarrow p \\
 \Omega & \xrightarrow{\tilde{B}} & B \\
\end{array}
\]

and recall the monodromy principle (see [7] 16.28.8)

**Theorem 5.1 (Monodromy Principle)** Let \((Y,X,p)\) be a covering of a differential manifold \(X\) and let \(f : X' \to X\) be a \(C^\infty\)-mapping of a simply-connected, differential manifold. Let \(a' \in X'\) and \(b \in p^{-1}(f(a'))\). Then there exists a unique \(C^\infty\)-mapping \(g : X' \to Y\) such that \(g(a') = b\) and \(p \circ g = f\) (the mapping \(g\) is said to be a lifting of \(f\)).

Here we specialize this to \(B = \mathbb{C} \setminus \{0,1\}\) choose a universal covering \((B,\tilde{B},p)\) and a section \(s : \Omega \to \tilde{B}\) of \(p\), lifted from the canonical embedding \(j : \Omega \hookrightarrow B\)

---

19 Nowadays simply-connected implies path-connected.
We first remark that any \( g \in \mathcal{G} \) maps in fact \( B = \mathbb{C} \setminus \{0, 1\} \) to itself and apply the Monodromy Principle to the following situation

\[
\begin{array}{ccc}
\tilde{B} & \xrightarrow{\tilde{g}} & \tilde{B} \\
\downarrow p & & \downarrow p \\
B & \xrightarrow{g} & B
\end{array}
\]

where \((\tilde{B}, B, p)\) is any universal covering of \( B \).

First remark that \( \mathcal{G} \) is a copy of \( \mathcal{G}_3 \) as it permutes the three singularities and

\[
\begin{array}{c}
10\infty \\
\downarrow 1/z \\
01\infty & 1\infty 0 \\
\downarrow 1/z & \downarrow 1/z \\
\infty 10 & 0\infty 1 \\
\downarrow 1/z & \downarrow 1/z \\
\infty 01
\end{array}
\]

Fig. 2. Orbit of the singularities

choose an orbit in \( \Omega \) (as the orbit of \( i \)), for instance now, we can pinpoint the lifting

\[
\begin{array}{c}
i \\
\downarrow 1/z \\
1-i & -i \\
\downarrow 1/z & \downarrow 1/z \\
\frac{1+i}{2} & \frac{1+i}{2}
\end{array}
\]

Fig. 3. Orbit of \( i \)

(i.e. find points like \( a', b \) in Theorem 5.1).

Let us note \( g_{1-z}(z) = 1 - z, g_{1/z}(z) = z^{-1} \in \mathcal{G}_B \) and set \( \tilde{g}_\phi \in \mathcal{G}_{\tilde{B}} \) such that

\[
p \circ \tilde{g}_\phi = g_\phi \circ p \text{ with } p \circ \tilde{g}_\phi(s(z_0)) = g_\phi(z_0).
\]
where \( z_0 = i \).

**Remark 5.2** An involutive bi-homomorphism cannot always be lifted as a permutation of finite order as shows the example of \( \mathbb{Z}/2\mathbb{Z} \) acting on \( \mathbb{C}^* = \mathbb{C} \setminus \{0\} \) by \( a(z) = -z \). It can be shown (exercise left to the reader) that every lifting \( \tilde{a} \) of \( a \) i.e.

\[
\begin{array}{c}
\tilde{\mathbb{C}}^*
\xrightarrow{\tilde{a}}
\tilde{\mathbb{C}}^*
\end{array}
\]

\[
\begin{array}{c}
\mathbb{C}^*
\xrightarrow{a}
\mathbb{C}^*
\end{array}
\]

is of infinite order.

Before computing the transition maps for \( \tilde{L} \) (extension of \( L \) to \( \tilde{B} \)) under \( \tilde{g}_1 - z \) and \( \tilde{g}_1/z \), we must take an excursion to noncommutative differential equations.

### 5.3 Noncommutative differential equations

Let \((V, d)\) be a one-dimensional \( C^0 \) connected complex manifold with its derivation \( d = d/dz \). We endow \( \mathcal{H}(V)\langle\langle X\rangle\rangle \) with \( d \)

\[
d(S) = \sum_{w \in X^*} d(\langle S | w \rangle)w.
\] (7)

It can be easily checked that \( d \) is a derivation of the \( \mathbb{C} \)-algebra \( \mathcal{H}(V)\langle\langle X\rangle\rangle \).

We are now able to define noncommutative differential equations (left multiplier case).

**Definition 5.3** A noncommutative differential equation, on \( V \), with left multiplier \( M \in \mathcal{H}(V)_+\langle\langle X\rangle\rangle \) is an equality

\[
d(S) = MS.
\] (8)

An initial condition can be pinpointed (with \( z_0 \in V \) and \( S_0 \in \mathbb{C}\langle\langle X\rangle\rangle \))

\[
\begin{cases}
  d(S) = MS, \\
  S(z_0) = S_0.
\end{cases}
\] (9)

It can be asymptotic \(^2\text{1}\)

\(^2\text{0}\) A Riemann surface in short.

\(^2\text{1}\) Below and in general \( \mathfrak{F} \) is a filter, the reader who is not familiar with these objects can replace \( \mathfrak{F} \) by any mode of convergence with respect to a subset (e.g. a cone, cluster point, to infinity - full or with restrictions - etc.).
\[
\begin{align*}
\text{d}(S) &= MS, \\
\lim\limits_{S} S(z) &= S_0.
\end{align*}
\] (10)

We gather here the needed results

**Proposition 5.4 ([6,9])** We have the following properties:

(i) If \( V \) is simply connected, equation (9) has a unique solution:

- with \( S_0 = 1_X \), it can be computed, through Picard’s process, by iterated integrals with lower bond \( z_0 \), this solution will be noted \( S_{\text{Pic}}^{\text{0}} \).
- in the general case (initial condition \( S_0 \)) the solution is \( S_{\text{Pic}}^{\text{0}} S_0 \).

(ii) If \( V \) is connected, solutions to equation (10) may not exist, but if it does, the solution is unique.

(iii) (\( V \) is connected) The set of solutions of (8) is a vector space. Two solutions which coincide at a point - actual (9) or asymptotic (10) - coincide everywhere.

(iv) (\( V \) is simply connected) The set of invertible solutions of (9) is the following orbit on the right

\[ \mathcal{P} = S_{\text{Pic}}^{\text{0}} (\mathbb{C}[X])^\times. \]

5.4 Equivariance of polylogs on \( \tilde{B} \).

Now, we will explain the property of polylogs expressed by formulas of the type

\[ \tilde{L}(g, z) = \mu_g(L(z))Z(g), \]

where \( \mu_g \) is a morphism of alphabets of the type

\[ \begin{pmatrix}
\mu_g(x_0) \\
\mu_g(x_1)
\end{pmatrix} = \begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
\begin{pmatrix}
x_0 \\
x_1
\end{pmatrix} \]

(11)

and \( Z(g) \in \mathbb{C}[\langle X \rangle] \).

We detail here the computation for \( \tilde{g}_{1-z} \).

(i) \( \tilde{g}_{1-z} \) has been chosen such that

\[ p \circ \tilde{g}_{1-z} = g_{1-z} \circ p \text{ and } p \circ \tilde{g}_{1-z}(s(i)) = 1 - i. \] (12)

(ii) From \( p : \tilde{B} \to B \), one has \( p^* : \mathcal{H}(B) \to \mathcal{H}(\tilde{B}) \) and define the functions \( \tilde{\phi}_z \) (resp. \( \tilde{\phi}_{1/z} \)) in \( \mathcal{H}(\tilde{B}) \) by

\[ \tilde{\phi}_z(u) = p(u) \in \mathbb{C} \text{ resp. } \tilde{\phi}_{1/z}(u) = \frac{1}{p(u)} \in \mathbb{C}. \] (13)
(iii) Solve, in $\mathcal{H}(\tilde{B})\langle\langle X\rangle\rangle$,

\[
\begin{align*}
\begin{cases}
    d(\tilde{S})(u) &= \left(\frac{x_0}{\phi_z(u)} + \frac{x_1}{\phi_{1-z}(u)}\right)\tilde{S}(u), \\
    \lim_{z \to 0, z \in \Omega} \tilde{S}(s(z))e^{-x_0\log(z)} &= 1_{X^*}.
\end{cases}
\end{align*}
\]

(14)

The uniqueness of such a solution is a result of Proposition 5.4 (iii). The existence can be obtained by lifting or remarking that the analog of (14) (first row) can be solved at the level of $\Omega$ and, for any choice of $z_0 \in \Omega$, one has

\[
\tilde{S}_{Pic}^{s(z_0)} \circ s = S_{Pic}^{s_0}
\]

then the solution of (14) is given by

\[
\tilde{L} = \tilde{S}_{Pic}^{s(z_0)} L(z_0)
\]

and satisfies

\[
\tilde{L} \circ s = L.
\]

(iv) Now

\[
d(\tilde{L} \circ \tilde{g}_{1-z}) \circ s = \frac{d}{dz}(\tilde{L} \circ \tilde{g}_{1-z}) \circ s
\]

and, in a neighbourhood of $i$, one has

\[
\tilde{L} \circ \tilde{g}_{1-z} \equiv L(1-z).
\]

(v) Setting $L_1(z) = L(1-z)$, one gets

\[
\frac{d}{dz}L_1 = \left(-\frac{x_0}{1-z} - \frac{x_1}{z}\right)L_1 = \mu\left(\frac{x_0}{z} + \frac{x_1}{1-z}\right)L_1,
\]

where $\mu$ is the morphism of alphabets

\[
\mu \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}
\]

(vi) as $\mu$ permutes with the derivation and is invertible, one has

\[
\frac{d}{dz}\mu^{-1}(L_1) = \left(\frac{x_0}{z} + \frac{x_1}{1-z}\right)\mu^{-1}(L_1).
\]

Hence,

\[
\mu^{-1}(L_1) = LZ_1 \text{ and } L_1 = \mu(L)\mu(Z_1).
\]

(vii) Finally, as they coincide on an open subset and are analytic, one gets

\[
\tilde{L}_1 = \mu(\tilde{L})\mu(Z_1).
\]
6 Applications

In this section, we will give some applications of this new presentation of polylogarithms at negative integer multi-indices.

6.1 A new presentation of harmonic sums with non-positive multi-indices

From Corollary 4.14, for any \( w \in Y^*_0 \), the corresponding polylogarithm \( \text{Li}_w^- \) is an element of the algebra \( \mathbb{Q}[ (1-z)^{-1}] \). Thus, for \( w \in Y^*_0 \), we suppose that \( \text{Li}_w^- \) can be expanded as follows

\[
\text{Li}_w^-(z) = \sum_{k=0}^{(w)+|w|} \frac{a^w_k}{(1-z)^k} \quad \text{with } a^w_i \in \mathbb{Q}.
\] (16)

We note that \( a^1_{y^0_0} = 1 \) and, for any \( n > 0 \), \( a^n_{y^0_0} = 0 \). Then, using Proposition 3.3, the sequences \( \{a^w_i\}_{w \in Y^*_0, n \in \mathbb{N}} \) are computed as follows:

Lemma 6.1 (Algorithm to compute \( \text{Li}_w^- \))

Let \( w = y_k u \in Y^*_0 = Y^*_0 \setminus \{y^0_0\} \). We have

(i) If \( k = 0 \) then

\[
a^y_0 u := \begin{cases} a^u_{i-1} & \text{for } i = (u) + |u| + 1, \\ a^u_{i-1} - a^u_i & \text{for } 1 \leq i \leq (u) + |u|, \\ -a^u_i & \text{for } i = 0, \\ 0 & \text{otherwise}. \end{cases}
\]

(ii) If \( k > 0 \) then

\[
a^{y_k u} := \begin{cases} (i-1)a^{y_{k-1} u}_{i-1} & \text{for } i = (u) + |u| + k + 1, \\ (i-1)a^{y_{k-1} u}_{i-1} - ia^{y_{k-1} u}_i & \text{for } 2 \leq i \leq (u) + |u| + k, \\ -a^{y_{k-1} u}_i & \text{for } i = 1, \\ 0 & \text{for otherwise}. \end{cases}
\]

Proof.

(i) If \( k = 0 \), we have \( \text{Li}_{y^0 u}^- = \text{Li}_{y^0_0}^- \text{Li}_u^- \). Hence, we obtain this condition.

(ii) This condition is a direct corollary of the identity \( \text{Li}_{y^k u}^- = \theta_0 \text{Li}_{y^{k-1} u}^- \).

Proposition 6.2 For any \( w \in Y^*_0 \), we have \( \text{Li}_w^- = \text{Li}_p^- \), where

\[\text{Such a (non-empty) word } w \text{ can always be written uniquely } w = y_k u \text{ where } y_k \in Y_0 \text{ and } u \in Y^*_0.\]
\[ P_{w} := \sum_{i=0}^{(w)+|w|} a_{i}^{w} (x_{i}^{*})^{i} \in (\mathbb{C}[x_{1}^{*}], \mathbb{1}, 1x_{1}^{*}) \]

and the coefficients \( \{a_{i}^{w}\}_{i \in \mathbb{N}} \) are defined as in Lemma 6.1.

| \( k \) \( i \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | ...
|------|---|---|---|---|---|---|---|---|------|
| 0    | 1 | 0 | -2| 1 | 0 | 0 | 0 | 0 | ...  |
| 1    | 0 | 2 | -4| 2 | 0 | 0 | 0 | 0 | ...  |
| 2    | 0 | -2| 10| -14| 6 | 0 | 0 | 0 | ...  |
| 3    | 0 | 2 | -22| 62| -66| 24| 0 | 0 | ...  |
| 4    | 0 | -2| 46| -230| 450| -384| 120| 0 | ...  |
| ...  | ...| ...| ...| ...| ...| ...| ...| ...| ...

Table 1
The value of \( a_{i}^{y_{0}} \)

Now, we note that for any \( k \in \mathbb{N} \),

\[
\frac{1}{(1-z)^{k+1}} = \frac{1}{k!} \sum_{n \geq k} \frac{n! \cdots (n-k+1)! z^{n-k}}{1-z} = \sum_{N \geq 0} \binom{N+k}{k} z^{N}.
\]

On the other hand, for any \( w \in Y_{0}^{*} \), denoting \( p = (w) + |w| \), we have

\[
\sum_{N \geq 0} H_{w}^{*}(N) z^{N} = L_{w}^{+}(z) = \frac{a_{p}^{w}}{(1-z)^{p+1}} + \cdots + \frac{a_{1}^{w}}{(1-z)^{2}} + \frac{a_{0}^{w}}{1-z}.
\]

Hence,

\[
\sum_{N \geq 0} H_{w}^{*}(N) z^{N} = a_{p}^{w} \sum_{N \geq 0} \binom{N+p}{p} z^{N} + \cdots + a_{0}^{w} \sum_{N \geq 0} \binom{N}{0} z^{N}.
\]

This means that,

**Proposition 6.3** For any \( w \in Y_{0}^{*} \),

\[ H_{w}^{*}(N) = \sum_{k=0}^{(w)+|w|} a_{k}^{w} \binom{N+k}{k}. \]

In fact, since the definition of the sequence \( \{a_{p}^{w}\}_{w \in Y_{0}^{*}, p \in \mathbb{N}} \), we can use the formula

\[
\binom{n+r}{r} - \binom{n+r-1}{r-1} = \binom{n+r-1}{r}
\]
and Proposition 6.3 to obtain an extension of Faulhaber’s result [28], i.e., the harmonic sum $H_w^-(N), w \in Y^*_0$ can be written like a finite linear combination of the elements in the family $\{(\binom{N+n}{m})_{n,m \in \mathbb{N}}\}$, where the number of terms is at most $\lfloor \frac{(w) + |w|}{2} \rfloor + 1$.

Example 6.4

\[
H_{y_0}^-(N) = \binom{N+1}{1} - \binom{N}{0} = \binom{N}{1},
\]
\[
H_{y_1}^-(N) = \binom{N+2}{2} - \binom{N+1}{1} = \binom{N+1}{2},
\]
\[
H_{y_2}^-(N) = 2 \binom{N+3}{3} - 3 \binom{N+2}{2} + \binom{N+1}{1} = 2 \binom{N+2}{3} - \binom{N+1}{1},
\]
\[
H_{y_3}^-(N) = 6 \binom{N+4}{4} - 12 \binom{N+3}{3} + 7 \binom{N+2}{2} - \binom{N+1}{1} = 6 \binom{N+2}{4} + \binom{N+1}{2},
\]
\[
H_{y_0}^-(N) = \binom{N+2}{2} - 2 \binom{N+1}{1} + \binom{N}{0} = \binom{N}{2},
\]
\[
H_{y_{1y_0}}^-(N) = 2 \binom{N+3}{3} - 4 \binom{N+2}{2} + 2 \binom{N+1}{1} = 2 \binom{N+1}{3},
\]
\[
H_{y_{2y_0}}^-(N) = 6 \binom{N+4}{4} - 14 \binom{N+3}{3} + 10 \binom{N+2}{2} - 2 \binom{N+1}{1} = 6 \binom{N+2}{4} + 2 \binom{N+1}{2}.
\]
6.2 Regularization of polyzetas at negative multi-indices

Let \( \{ t_i \}_{i \in \mathbb{N}_+} \) be a family of variables. The symmetric functions \( \{ \eta_k \}_{k \in \mathbb{N}} \) and the power sums \( \{ \psi_k \}_{k \in \mathbb{N}} \) are defined respectively by

\[
\eta_k(t) := \sum_{n_1 > \ldots > n_k > 0} t_{n_1} \ldots t_{n_k} \quad \text{and} \quad \psi_k(t) := \sum_{n > 0} t_n^k.
\]

Then the generating series of the family \( \{ \eta_k \}_{k \in \mathbb{N}} \) is defined by

\[
1 + \sum_{k \geq 1} \eta_k(t) z^k = \prod_{i > 0} (1 + t_i z) =: \eta(t \mid z).
\]

In the same way, we also define

\[
\sum_{k \geq 1} \psi_k(t) z^k = \sum_{i \geq 1} \frac{t_i z}{1 - t_i z} =: \psi(t \mid z).
\]

Note that the functions \( \eta(t \mid z) \) and \( \psi(t \mid z) \) satisfy Euler’s identity

\[
z \frac{d}{dz} \log(\eta(t \mid z)) = \psi(t \mid -z). \tag{17}
\]

For any \( w = y_{s_1} \cdots y_{s_r} \in \mathbb{Y}^* \), the quasi-symmetric function of depth \( |w| = r \) and of degree (or weight) \( (w) := s_1 + \ldots + s_r \), namely \( F_w \), is defined by

\[
F_w(t) := \sum_{n_1 > \ldots > n_r > 0} t_{n_1}^{s_1} \ldots t_{n_r}^{s_r}.
\]

Note that \( F_{y_1} = \eta_k \) and \( F_{y_k} = \psi_k \), and from equation (17), we obtain

\[
\sum_{k \geq 0} F_{y_1} z^k = \exp \left( - \sum_{k \geq 1} F_{y_k} \left( -z \right)^k \right). \tag{18}
\]

Remark that the harmonic sum \( H_w(N) \) can be obtained by specializing, in the quasi-symmetric function \( F_w \), the variables \( \{ t_i \}_{i \geq 1} \) as follows [24]:

\[
t_i = \frac{1}{i} \quad \text{if} \quad 0 < i \leq N \quad \text{else} \quad t_i = 0.
\]

On the other hand, we recall the morphisms of regularization [5]

\[
\gamma_* : (\mathbb{C}(Y), \mathbb{M}, 1_Y) \longrightarrow (\mathbb{C}, \cdot, 1),
\]

\[
\zeta_{\perp} : (\mathbb{C}(X), \mathbb{M}, 1_X) \longrightarrow (\mathbb{C}, \cdot, 1),
\]

which are defined by

\[
\gamma_{y_1} = \gamma, \quad \zeta_{\perp}(x_0) = \zeta_{\perp}(x_1) = 0
\]

and

\[
\forall l \in \mathcal{L}(Y)(X) - X, \quad \gamma_{y_l}(l) = \zeta(l).
\]

\[23\] We set \( \eta_0(t) = 1 \).
Then
\[ \forall t \in \mathbb{C}, \quad \zeta((tx_1)^*) = 1. \]
Note that, by a Newton-Girard like identity (see (18)), one has [5]
\[ \sum_{k \geq 0} H_{s_k} z^k = \exp \left( - \sum_{k \geq 1} H_{s_k} \frac{(-z)^k}{k} \right), \]
and by the properties of Gamma function, we obtain, for \(|t| < 1|,
\[ \gamma_{\mathcal{Y}}((tx_1)^*) = \exp \left( \gamma_{\mathcal{Y}} - \sum_{n \geq 2} \zeta(n) \frac{(-t)^n}{n} \right) = \frac{1}{\Gamma(1+t)}. \] (19)
For \(t \in \mathbb{N}\), we put
\[ \gamma_{\mathcal{Y}}((tx_1)^*) = \frac{1}{t!}. \]

**Proposition 6.5 ([27])** Let \( w = y_{s_1} \cdots y_{s_r} \in Y_0^\ast \). Then there exists a unique polynomial \( P_w \in (\mathbb{C}[x_1^\ast], \cup, 1x^\ast) \) such that
\[ \text{Li}_w = \text{Li}_{P_w}. \]
Setting \( \gamma_{-s_1, \ldots, -s_r} := \gamma_{\mathcal{Y}}(P_w) \), we get
\[ \gamma_{-s_1, \ldots, -s_r} = \sum_{k=0}^{s_1 + \cdots + s_r + r} a_k^w \frac{1}{k!}. \]

**Proof.** The first part of this proposition is the direct consequence of (16) and Proposition 6.2. See [27] for the analytical justification of such algebraic process and the proof of the uniqueness.

\[ \square \]

**Example 6.6**
\[ \gamma_0 = -1 + \frac{1}{1!} = 0, \]
\[ \gamma_{-1} = -\frac{1}{1!} + \frac{1}{2!} = -\frac{1}{2}, \]
\[ \gamma_{-2} = \frac{1}{1!} - \frac{3}{2!} + \frac{2}{3!} = -\frac{1}{6}, \]
\[ \gamma_{-3} = -\frac{1}{1!} + \frac{7}{2!} - \frac{12}{3!} + \frac{6}{4!} \]
\[
\gamma_{-4} = \frac{3}{4},
\gamma_{-4} = \frac{1}{1!} - \frac{15}{2!} + \frac{50}{3!} - \frac{60}{4!} + \frac{24}{5!}
= -\frac{15}{7},
\gamma_{0,0} = 1 - \frac{2}{1!} + \frac{1}{2!}
= -\frac{1}{2},
\gamma_{0,-1} = \frac{1}{1!} - \frac{2}{1!} + \frac{1}{3!}
= -\frac{6}{5},
\gamma_{-2,0} = -\frac{2}{1!} + \frac{10}{2!} - \frac{14}{3!} + \frac{6}{4!}
= \frac{11}{12},
\gamma_{0,-2} = -1 + \frac{4}{2!} - \frac{5}{3!} + \frac{2}{4!}
= \frac{1}{4},
\gamma_{-1,-1} = -1 + \frac{5}{2!} - \frac{7}{3!} + \frac{3}{4!}
= \frac{11}{24},
\gamma_{0,0,0} = -1 + \frac{3}{1!} - \frac{3}{2!} + \frac{1}{3!}
= \frac{2}{3},
\gamma_{0,-1,0} = -\frac{2}{1!} + \frac{6}{2!} - \frac{6}{3!} + \frac{2}{4!}
= \frac{1}{12},
\gamma_{0,-2,0} = -\frac{12}{2!} + \frac{6}{5!} + \frac{2}{1!} - \frac{20}{4!} + \frac{24}{3!}
= -\frac{47}{47},
\gamma_{-4,-4,-6} = 1 - \frac{16655}{2!} + \frac{5260444}{3!} - \frac{321370622}{4!} + \frac{7519806977}{5!}
- \frac{90280292235}{6!} + \frac{647428882810}{7!} - \frac{3028468246320}{8!}
\]
9748178974760 + 22298261594760 + 36869237126640
9! + 10! + 11!
44258208343200 + 38240776382400 + 23192869190400
12! + 13! + 14!
9376213916160 + 2270032128000 + 249080832000
15! + 16! + 17!
375 − 2270032128000
16!
17!
375
137837700
375
137837700.

7 Conclusion

In this work, we explained the whole project of extending \( \text{Li}_n \) over a shuffle subalgebra of rational power series.

In particular, we have studied different aspects of \( \mathcal{C}\{\text{Li}_w\}_{w \in X^*} \), where \( \mathcal{C} \) denotes the ring of polynomials in \( z, z^{-1} \) and \( (1 - z)^{-1} \), with coefficients in \( \mathbb{C} \).

On the other hand, we applied this new indexing of \( \text{Li}_n \) to express the polylogarithms (resp. harmonic sums) at negative multi-indices as polynomials in \( (1 - z)^{-1} \) (resp. \( N \)), with coefficients in \( \mathbb{Z} \) (resp. \( \mathbb{Q} \)).

We concentrated, particularly, on algebraic and analytic aspects of this extension allowing index polylogarithms, at non positive multi-indices, by rational series and to regularize divergent polyzetas, at non positive multi-indices.

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