Bunching and anti-bunching in electronic transport

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In quantum optics the $g^{(2)}$-function is a standard tool to investigate photon emission statistics. We define a $g^{(2)}$-function for electronic transport and use it to investigate the bunching and anti-bunching of electron currents. Importantly, we show that super-Poissonian electron statistics do not necessarily imply electron bunching, and that sub-Poissonian statistics do not imply anti-bunching. We discuss the information contained in $g^{(2)}(\tau)$ for several typical examples of transport through nano-structures such as few-level quantum dots.

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Current noise has long-since been established as an important tool for studying the physics of transport through mesoscopic and nano-scale conductors. The character of the noise is typically assessed by considering the Fano factor, the ratio of the zero-frequency noise to the current, and comparing with a Poisson process for which the Fano factor is equal to one. Systems with $F < 1$ are described as sub-Poissonian (non-interacting systems fall in this class) and systems which have $F > 1$ are called super-Poissonian. A common interpretation of this comparison is that a super-Poissonian Fano factor indicates a bunching of the current’s constituent electrons, whereas sub-Poissonian values indicates anti-bunching (Fig. 1).

In this paper we directly investigate bunching and anti-bunching in electronic transport as a phenomenon in the time domain through the introduction of a second-order correlation function $g^{(2)}(\tau)$, analogous to that used in quantum optics [1]. Within a quantum master equation (QME) framework in the appropriate limit, the $g^{(2)}$-function is seen to be proportional to the conditional probability that, given an electron is emitted into the collector at time $t = 0$, a further such jump is observed a time $\tau$ later. Following quantum optics, we identify

$$g^{(2)}(0) > g^{(2)}(\tau) \quad \text{bunching}$$
$$g^{(2)}(0) < g^{(2)}(\tau) \quad \text{anti-bunching}$$

(1)

since bunching means that particles are more likely to be emitted together than apart, and conversely for anti-bunching. By relating our $g^{(2)}$-function to the correlation function between the current at two different times, we clarify the relationship between the $g^{(2)}$-function, (anti-)bunching and the Fano factor.

We then investigate bunching and anti-bunching in several widely-discussed transport models in the Coulomb blockade (CB) regime (see Fig. 2). This analysis shows that the simple picture relating super-Poissonian Fano factors to bunching and sub-Poissonian ones to anti-bunching is often an oversimplification, and can even be outright wrong. In particular we discuss a simple quantum-dot (QD) model which has a Fano factor less than one, and is thus sub-Poissonian, and yet has $g^{(2)}(0) > g^{(2)}(\tau)$ for all $\tau > 0$ such that, according to Eq. (1), the electron flow is completely bunched. We also give a model for which the converse is true, i.e. we find a super-Poissonian Fano factor in conjunction with electron anti-bunching. These results mirror the work of Singh and Zou and Mandel, who have made similar points for quantum-optical systems.

This paper proceeds by first reviewing the situation in quantum optics. We then define by analogy our $g^{(2)}$-function for transport systems and examine its properties, including its relationship with current fluctuations and the zero-frequency Fano factor. We then consider our concrete examples and conclude.

I. THE SECOND-ORDER CORRELATION FUNCTION IN QUANTUM OPTICS

The second-order degree of coherence of the electric field at position $r$ and times $t$ and $t + \tau$; $\tau \geq 0$ is defined as

$$g^{(2)}(r, t, \tau) \equiv \frac{\langle E^{(-)}(r, t)E^{(-)}(r, t + \tau)E^{(+)}(r, t + \tau)E^{(+)}(r, t) \rangle}{\langle E^{(-)}(r, t)E^{(+)}(r, t) \rangle \langle E^{(-)}(r, t + \tau)E^{(+)}(r, t + \tau) \rangle}. \quad (2)$$

Here, linear polarisation in direction $e$ is assumed such that $E^{(\pm)} \equiv e \cdot E^{(\pm)}$. In the stationary limit and sup-
pressing the position dependence, we have
\[ g^{(2)}(\tau) = \frac{\langle E^{(-)}(\tau) E^{(+)}(\tau) E^{(-)}(\tau) E^{(+)}(\tau) \rangle}{\langle E^{(-)}(\tau) E^{(+)}(\tau) \rangle^2}, \]  
(3)

with \( E^{(\pm)} = E^{(\pm)}(r, t = 0) \).

The next step is a re-formulation of the traditional Heisenberg operator definition Eq. (5) in the (somewhat more flexible) Schrödinger (master equation) picture. For example, in the theory of resonance fluorescence\(^\text{11}\), one can express the electric field in the far-field limit in terms of the atomic (two-level) operators \( \sigma_{\pm}(t) \) via
\[ \mathbf{E}^{(+)}(r, t) = C(r - r_0) e \sigma_{-}(t - t_0), \]  
(4)

where \( C(r) \) is some position-dependent constant, \( e \) the appropriate polarisation vector and \( t_0 = |r - r_0|/c \) the time-of-flight from atom to detector\(^2\). With this substitution, the stationary \( g^{(2)} \) function of the field can be written purely in terms of atomic degrees of freedom as
\[ g^{(2)}(\tau) = \frac{\langle \sigma_{+}(\tau) \sigma_{-}(\tau) \sigma_{+}(\tau) \sigma_{-}(\tau) \rangle}{\langle \sigma_{+}(\tau) \sigma_{-}(\tau) \rangle^2}, \]  
(5)

where all the constants \( C \) cancel. The main technical step is now a formulation of the dissipative quantum dynamics in terms of a quantum master equation with Liouvillian \( \mathcal{W} \), \( \dot{\rho} = \mathcal{W} \rho \), for the state \( \rho \) of the atom in Born-Markov approximation. The quantum regression theorem then allows one to write\(^3\)
\[ g^{(2)}(\tau) = \frac{\text{Tr} \left\{ \sigma_{-} \mathcal{W}_{\gamma} \sigma_{-} \rho_{\text{stat}} \sigma_{+} \sigma_{+} \right\}}{\text{Tr} \left\{ \sigma_{-} \rho_{\text{stat}} \sigma_{+} \right\}^2}, \]  
(6)

where \( \rho_{\text{stat}} = \rho(t \rightarrow \infty) \) is the stationary state of the atom at large times \( t \). One then introduces the downwards-jump super-operator \( \mathcal{J}_\gamma \) through its action on an arbitrary density matrix \( \rho \)
\[ \mathcal{J}_\gamma \rho = \gamma \sigma_{-} \rho \sigma_{+}, \]  
(7)

where \( \gamma \) is the spontaneous emission rate, leading to
\[ g^{(2)}(\tau) = \frac{\text{Tr} \left\{ \mathcal{J}_\gamma e^{\mathcal{W}_\tau} \mathcal{J}_\gamma \rho_{\text{stat}} \right\}}{\text{Tr} \left\{ \mathcal{J}_\gamma \rho_{\text{stat}} \right\}^2}. \]  
(8)

This equation makes it clear that, at least from a resonance fluorescence perspective, the \( g^{(2)} \) function measures the correlation between two system jumps separated by a time \( \tau \).}

## II. A SECOND-ORDER CORRELATION FUNCTION FOR QUANTUM TRANSPORT

We consider here transport systems in the infinite-bias limit such that the time-evolution of the system density matrix can be described by a Markovian master equation \( \dot{\rho} = \mathcal{W} \rho \). We assume that we are only interested in the current in a single lead, assumed to be a collector, and decompose the Liouvillian as \( \mathcal{W} = \mathcal{W}_0 + \mathcal{J} \) where, similarly to Eq. (7), the jump super-operator \( \mathcal{J} \) describes incoherent transitions of the system. Here \( \mathcal{J} \) describes the emission of an electron into the lead in question. The remaining part of the Liouvillian, \( \mathcal{W}_0 \), describes the evolution of the system without such jumps (jumps to and from other leads are included in \( \mathcal{W}_0 \)). When we discuss our examples in the next section, we will give the kernels in the full-counting-statistics form \( \mathcal{W}(\chi) = \mathcal{W}_0 + \mathcal{J} e^{\chi} \) with \( \chi \) the counting field such that \( \mathcal{W} = \mathcal{W}(\chi = 0) \) and \( \mathcal{J} = -i \hbar \mathcal{A} \mathcal{W}(\chi) |\chi = 0\rangle \langle \chi = 0| \).

We then define the \( g^{(2)} \)-function for transport in analogy with Eq. (5), replacing the optical jump super-operators with their transport counterparts:
\[ g^{(2)}(\tau) = \frac{\langle \langle \mathcal{J} \mathcal{\Omega}(\tau) \mathcal{J} \rangle \rangle}{\langle \langle \mathcal{J} \rangle \rangle^2}, \]  
(9)

where we have explicitly included the master equation propagator \( \mathcal{\Omega}(\tau) = e^{\mathcal{W}_\tau} \), and \( \langle \langle \cdot \rangle \rangle \) denotes the stationary expectation value, \( \langle \langle A \rangle \rangle = \text{Tr} \{ A \rho_{\text{stat}} \} \) with \( \mathcal{W}_\rho_{\text{stat}} = 0, \rho_{\text{stat}} \) assumed unique. The interpretation of \( g^{(2)}(\tau) \) is analogous to the quantum optical case, and in particular the designation of bunching and anti-bunching follows Eq. (11).

Whilst we have defined our transport \( g^{(2)} \)-function here by analogy between master-equation formalisms (i.e. in terms of jump operators), result Eq. (9) can also be derived from a microscopic Heisenberg-picture expression similar to Eq. (2). This derivation is discussed in the appendix.

### A. Properties

We use a Liouville space ket \(|0\rangle\rangle = \rho_{\text{stat}}\rangle\rangle and bra \langle\langle 0\rangle\rangle\rangle such that the trace over the stationary state is written \( \langle\langle A \rangle\rangle = \langle\langle 0| A |0\rangle\rangle\rangle \). It is then useful to define the projector onto the stationary state \( \mathcal{P} = |0\rangle\rangle \langle\langle 0| \) via \( \mathcal{W} \mathcal{P} = \mathcal{P} \mathcal{W} = 0 \), along with its compliment \( \mathcal{Q} = \mathcal{I} - \mathcal{P} \). This allows us to decompose the propagator as \( \mathcal{\Omega}(\tau) = \mathcal{P} + \mathcal{R}(\tau) \) with irreducible part \( \mathcal{R}(\tau) = \mathcal{Q} \mathcal{\Omega}(\tau) \mathcal{Q} \). Using \( \langle\langle \mathcal{J} \mathcal{P} \mathcal{J} \rangle\rangle = \langle\langle 0| \mathcal{J} \mathcal{P} \mathcal{J} |0\rangle\rangle\rangle = \langle\langle \mathcal{J} \rangle\rangle^2 \), we find
\[ g^{(2)}(\tau) = 1 + \frac{\langle\langle \mathcal{J} \mathcal{R}(\tau) \mathcal{J} \rangle\rangle}{\langle\langle \mathcal{J} \rangle\rangle^2} \; \text{for} \; \tau > 0. \]  
(10)

In this long-time limit, all non-zero eigenvalues of \( \mathcal{W} \) have negative real-parts and \( \mathcal{R}(\tau) \) decays with \( \tau \rightarrow \infty \) such that \( g^{(2)}(\tau \rightarrow \infty) = 1 \).

In the limit of strong Coulomb blockade such that at most one excess electron occupies the system at any time, we have \( \mathcal{J}^2 = 0 \), since the exit of an electron from the system leaves it empty and a second jump can not take place immediately. In this case
\[ g^{(2)}(\tau) = 0. \]  
(11)

This leads to the conclusion that, according to the definition Eq. (11), a system in the strong CB regime is always
anti-bunched, since \( g^{(2)}(\tau = 0) < g^{(2)}(\tau) \) as \( g^{(2)}(\tau) \) is always positive.

The current noise at finite frequency is defined as
\[
S(\omega) = \frac{1}{2} \int dt e^{i\omega t} \langle [\delta I(t), \delta I(0)] \rangle.
\]
In the QME approach with unidirectional transport, this can be written in terms of system quantities as
\[
S(\omega) = \langle J \rangle + \langle J \mathcal{R}(-i\omega) + \mathcal{R}(i\omega) J \rangle,
\]
with Laplace-transformed irreducible propagator \( \mathcal{R}(z) = [z - \mathcal{W} \mathcal{Q}]^{-1} \). The mean current reads \( \langle I \rangle = \langle J \rangle \).

Transformation back into the time-domain yields,\(^{12,13}\)
\[
S(\tau) = \langle I \rangle \delta(\tau) + \langle J \mathcal{R}(|\tau|) J \rangle; \quad \forall \tau.
\]
Thus providing we set \( \tau > 0 \), we can equate
\[
g^{(2)}(\tau) = \frac{S(\tau)}{\langle J \rangle^2} + 1; \quad \tau > 0.
\]
Or, in the frequency domain, we have
\[
S(\omega) = \int_{-\infty}^{+\infty} d\tau e^{i\omega \tau} S(\tau) = \langle I \rangle + 2\langle I \rangle^2 \int_0^{+\infty} d\tau e^{i\omega \tau} \left( g^{(2)}(\tau) - 1 \right).
\]
These results show that, in the infinite-bias limit, the \( g^{(2)} \)-function can be directly related to the current-current correlation function either in frequency or time domain.

Defining the Fano factor as the ratio of zero-frequency noise to the mean current, we have
\[
F(0) \equiv \frac{S(\omega = 0)}{\langle I \rangle} = 1 + 2\langle I \rangle \int_0^{+\infty} d\tau \left( g^{(2)}(\tau) - 1 \right).
\]
This result for the Fano factor coincides with the optical Mandel Q factor \( Q = F(0) - 1 \) in the limit of long counting times \( T \to \infty \).\(^{15,16}\)

We also note that the \( g^{(2)} \)-function defined here bears some resemblance to the waiting-time distribution.\(^{15,16}\)
\[
w(\tau) = \frac{\langle J e^{W_0 \tau} J \rangle}{\langle J \rangle}, \quad \tau > 0,
\]
but there are two differences: (i) the normalisation is different, and (ii) the waiting-time propagator is defined with Liouvillian \( W_0 \) which excludes additional jumps within the interval \( \tau \). The \( g^{(2)} \)-function, in contrast, contains the full propagator \( W_0 \).

III. EXAMPLES

We now discuss the features of the transport \( g^{(2)} \)-function for several master-equation models, widely used to describe the electronic transport through CB systems such as quantum dots (QDs) or molecules. In each case, we consider unidirectional transport in a two lead set-up, where electrons tunnel into the system from the left lead with the rate \( \Gamma_L \), and out of the system into the right with rate \( \Gamma_R \).

![FIG. 2. Sketch of four of the five transport models discussed here: (a) The dynamical channel blockade model; (b) the double quantum dot; (c) the Anderson model \((j = 2)\); and (d) the Anderson model with only one singly-occupied level participating in transport \((j = 1)\). In each case, electrons tunnel with the rate \( \Gamma_L (\Gamma_R) \) into (out of) the quantum dot systems except when the outgoing rate is modified by the factor \( x \).](image)

A. Single resonant level

Our first example, the single resonant level (SRL), describes the transport through a single level (in a QD, for example) in the strong CB regime. Written in a basis such that the relevant part of the density matrix is collected into the vector \( \rho = \{ \rho_0, \rho_1 \} \), where 0 and 1 denote an empty or occupied level, the Liouvillian reads
\[
\mathcal{W}^{SRL}(\chi) = \begin{pmatrix}
-\Gamma_L & \Gamma_R e^{ix} \\
\Gamma_L & -\Gamma_R
\end{pmatrix},
\]
These steady state is current given by \( \langle I_{SRL} \rangle = \frac{\Gamma_R \Gamma_L}{\Gamma_R + \Gamma_L} \) and zero frequency Fano factor is \( \frac{\Gamma_R^2 + \Gamma_L^2}{(\Gamma_R + \Gamma_L)^2} \).
\[
F_{SRL}(0) = \frac{\Gamma_R^2 + \Gamma_L^2}{(\Gamma_R + \Gamma_L)^2}.
\]
Since the rates \( \Gamma_L \) and \( \Gamma_R \) are (positive) real numbers the Fano factor is always sub-Poissonian, \( F(0) < 1 \). The corresponding \( g^{(2)}_{SRL}(\tau) \)-function is given by
\[
g^{(2)}_{SRL}(\tau) = 1 - e^{-\left(\Gamma_R + \Gamma_L\right)\tau} \quad \text{and} \quad g^{(2)}_{SRL}(0) = 0.
\]
Starting by zero at \( \tau = 0 \) the \( g^{(2)}_{SRL} \)-function increases monotonously to one (not plotted), which indicates strictly anti-bunched electrons. Since, here, the Fano-factor is sub-Poissonian and the electron flow anti-bunched, the SRL model reflects the naive interpretation that sub-Poissonian statistics corresponds to anti-bunching. Our second example illustrates that this is not necessarily the case, however.
and super-Poissonian (line) Fano factor. Inset: The dimensionless factor $x$ changes the rate for outgoing spin-down electrons relative to spin-up ones. The steady-state current of the DCB model is $\langle J_{\text{DCB}} \rangle = \frac{\Gamma_R \Gamma_L}{\Gamma_L + \Gamma_R + x \Gamma_R}$, and the zero-frequency Fano factor is

$$F_{\text{DCB}}(0) = \frac{\Gamma^2_R x + \Gamma^2_L}{(x \Gamma_R + x \Gamma_L + \Gamma_L)^2}.$$ (22)

For $x < \frac{2 \Gamma_L \Gamma_R - \sqrt{4 \Gamma^2_L \Gamma^2_R + 4 \Gamma^2_R \Gamma^2_L}}{4(\Gamma_L - \Gamma_R)}$ ($x < \frac{3}{\gamma}$ in the limit $\Gamma_L = \Gamma_R$), the Fano factor of this model is super-Poissonian, see Fig. 3 (Inset), corresponding to DCB. Our $g_{\text{DCB}}^{(2)}(\tau)$-function for this model reads

$$g_{\text{DCB}}^{(2)}(\tau) = 1 - \frac{1}{2 \gamma} \left[ (\gamma_L + x \gamma) e^{-\frac{1}{2} \tau (2 \Gamma_L + x \Gamma_R + \Gamma_R + \gamma)} - (\gamma_L - x \gamma) e^{-\frac{1}{2} \tau (2 \Gamma_L + x \Gamma_R + \Gamma_R - \gamma)} \right].$$ (23)

where $\gamma = \sqrt{4 \Gamma^2_L + (x - 1)^2 \Gamma^2_R}$ and $\gamma_L = \Gamma_L (x^2 + 1)$. Fig. 3 shows $g_{\text{DCB}}^{(2)}(\tau) - 1$ plotted for $x = 1, 1/3$, and $1/5$ for which the corresponding Fano factors are $F_{\text{DCB}}(0) = 0.55$, $F_{\text{DCB}}(0) = 1$, and $F_{\text{DCB}}(0) = 1.41$, respectively.

The sub-Poissonian case (represented by $x = 1$ here) shows a monotonously increasing $g_{\text{DCB}}^{(2)}(\tau)$-function with interpretation identical to that of the SRL. In the $x = 1/5$ case, the Fano factor is super-Poissonian and the function $g_{\text{DCB}}^{(2)}(\tau) - 1$ starts off negative for $\tau = 0$, but rises rapidly through zero and remains positive for large $\tau$. For all times, $g_{\text{DCB}}^{(2)}(\tau) > g_{\text{DCB}}^{(0)}(0)$, and the electron flow is anti-bunched. Thus, at least from the perspective of definition Eq. (11), we find contradiction in associating a super-Poissonian Fano factor with anti-bunching, as the flow here is anti-bunched independent of the Fano factor. Nevertheless, there is a clear difference between super- and sub-Poissonian cases, since the $g^{(2)}$-function of the former exhibits a clear maximum and a negative slope for large times. The area of the $g^{(2)}$ curve then integrates to a positive number and the Fano factor is greater than one.

This discrepancy can be arises because there are two competing processes occurring. At short times the electrons must naturally be anti-bunched due to the strong CB. At longer times, the DCB effect means however, that after one electron has tunnled through the device, there is an increased probability that further electrons will tunnel once the system has had a chance to ”recharge”, and the electrons are indeed bunched, but only over timescales larger than is required for the system recharge. This analysis mirrors that of Kießlich et al. who studied the current-current correlation function for this model and obtained a similar result to Fig. 3.

The dotted-curve in Fig. 3 shows the crossover case (here at $x = 1/3$) where the Fano factor is exactly $F_{\text{DCB}}(0) = 1$. Here the $g_{\text{DCB}}^{(2)}(\tau)$-function profile is qualitatively and sub-Poissonian case, but the “negative-area” exactly cancels the “positive area” resulting in $F_{\text{DCB}}(0) = 1$. From the point-of-view of the $g^{(2)}$-correlation function nothing particularly special occurs at a value $x = 1/3$, although from the Fano factor, this may appear to be the case. Note that, although at this point $F_{\text{DCB}}(0) = 1$, the complete statistics of the charge transfer are clearly not Poissonian since $g_{\text{DCB}}^{(2)}(\tau) \neq 1$ is not equal to one at all times $\tau$. This distinction can also be shown by calculating the higher-order Fano factors, which for this model deviate from the Poissonian value of one.

C. Double quantum dot

Our third example is the double quantum dot (DQD) in the strong CB, which allows us to study the influence of internal coherence on the $g^{(2)}$-function. A sketch of the DQD is shown in Fig. 2(b) and the system can either
ing increases, the Fano factor becomes super-Poissonian, but as detuning increases, the Fano factor becomes sub-Poissonian, and convolution between the two dots imprints damped oscillations onto the waiting time distribution for a double quantum dot, or in resonance fluorescence in quantum optics. This is in contrast to the other systems without quantum coherences discussed in this paper, where the Fano factor either monotonously increases towards one or has a single peak and then decays towards one for \( \tau \to \infty \). We therefore expect oscillations in the Fano factor to be an interesting tool for an experimental detection of quantum coherences.

Note again, that, as for all systems in the ultra-strong CB regime, the electron flow is anti-bunched, according to definition Eq. (1). Once again though, this simple assignment misses the full complexity of the situation. Both sub- and super-Poissonian traces are qualitatively very similar, suggesting once again that the value of the Fano factor is itself, not particularly diagnostic of the transport mechanisms occurring in this model.

### D. Doubly-occupied Quantum dots

Away from the strong CB limit, the system can be occupied by more than just a single electron at a given time. This opens the path for the \( g^{(2)} \)-function at \( \tau = 0 \) to be non-zero, and the electron flow bunched.

As example of this class of model we consider a system with one empty state, \( j \in \{0,1\} \) single-occupied states (spin orbitals) and one double-occupied states within the transport window (and thus contributing to the current). In the basis \( \rho = \{\rho_0,\rho_1\} \), the kernel reads

\[
\mathcal{W}^{\text{QDD}}(\chi) = \begin{pmatrix}
-\Gamma_L & 0 & e^{i\chi} \Gamma_R & 0 \\
0 & 0 & i\Gamma_C & -i\Gamma_C \\
i\Gamma_C & -i\Gamma_C & -i\chi - \Gamma_R/2 & 0 \\
0 & i\Gamma_C & i\Gamma_C & 0 & i\chi - \Gamma_R/2
\end{pmatrix}.
\]

The stationary current and the Fano factor of this model are

\[
I_{\text{QDD}} = \frac{4T_C^2 \Gamma_L \Gamma_R}{4T_C^2 \Gamma_R + \Gamma_L (8T_C^2 + \Gamma_R^2)},
\]

\[
F_{\text{QDD}} = \frac{16 (4T_C^2 + \Gamma_R^2) T_C^2 + 8T_C^2 (12\epsilon^2 - \Gamma_R^2) T_C^2}{(4T_C^2 + \Gamma_L T_R + 4T_C^2 (2\Gamma_L + \Gamma_R))^2} + \frac{\Gamma_L^2 (4\epsilon^2 - 4\epsilon^2 + \Gamma_R^2)^2}{(4T_C^2 + \Gamma_L T_R + 4T_C^2 (2\Gamma_L + \Gamma_R))^2}.
\]

Once the right tunnel rate \( \Gamma_R \) is much smaller than the left tunnel rate \( \Gamma_L \) the Fano factor can become super-Poissonian for finite detunings, see Fig. 4 Inset. For zero detuning the Fano factor is sub-Poissonian, but as detuning increases, the Fano factor becomes super-Poissonian around \( \epsilon/\Gamma_L \approx 0.15 \) and reaches a maximum around \( \epsilon/\Gamma_L \approx 0.4 \), before decaying towards unity for large detunings.

In Fig. 4 we plot the \( g^{(2)} \) function for two sets of parameters: one that gives a sub-Poissonian Fano factor (dashed line) and one that gives a super-Poissonian one (continuous line). In both cases, coherent tunneling between the two dots imprints damped oscillations onto the waiting time distribution for a double quantum dot, or in resonance fluorescence in quantum optics. This is in contrast to the other systems without quantum coherences discussed in this paper, where the Fano factor either monotonously increases towards one or has a single peak and then decays towards one for \( \tau \to \infty \). We therefore expect oscillations in the Fano factor to be an interesting tool for an experimental detection of quantum coherences.

Note again, that, as for all systems in the ultra-strong CB regime, the electron flow is anti-bunched, according to definition Eq. (1). Once again though, this simple assignment misses the full complexity of the situation. Both sub- and super-Poissonian traces are qualitatively very similar, suggesting once again that the value of the Fano factor is itself, not particularly diagnostic of the transport mechanisms occurring in this model.
\[ G^{(2)}(\tau) = 1 + \frac{1}{8j\gamma_2^2} \left[ (a_1 + a_2)e^{-\frac{1}{2}(x_j + x_j - \gamma_2 + 2)\Gamma} + (a_1 - a_2)e^{-\frac{1}{2}(x_j + x_j + \gamma_2 + 2)\Gamma} \right], \]  

with \( \gamma_2 = \sqrt{j^2(x_1 - 1)^2 + 4}, \) \( a_1 = 2\gamma_2(1 + x - 3jx) \) and \( a_2 = -2(2(1 + x) + j^2x(1 + x) + j(4x - 3 - 3x^2)). \)

For \( \tau = 0 \), we have

\[ G^{(2)}_{2c}(0) = \frac{jx + x + 1}{4jx}, \]

which is non-zero and becomes \( G^{(2)}_{2c}(0) = 1 \) for \( x \to \frac{1}{3j\gamma_2^2} \).

In Fig. 5 we plot \( G^{(2)}_{2c}(\tau) \) with this value of \( x \) for \( j = 1 \) (solid line) and \( j = 2 \) (dashed line). Although both \( G^{(2)}_{2c}(\tau) \) curves start at zero for \( \tau = 0 \), the behaviour at finite time \( \tau \) is completely different. In the \( j = 1 \) model, the \( G^{(2)}_{2c}(\tau) \)-function is negative and shows a minimum as a function of time \( \tau \). The electron flow is therefore bunched. In the \( j = 2 \) model, however, the \( G^{(2)}_{2c}(\tau) \)-function is everywhere positive, shows a maximum and indicates an anti-bunched electron flow. Looking at the zero-frequency Fano factor (see Fig. 5 Inset) we see that it is sub-Poissonian for the \( j = 1 \) case (negative area under \( G^{(2)}_{2c}(\tau) - 1 \)- function) and super-Poissonian for \( j = 2 \) (positive area under \( G^{(2)}_{2c}(\tau) - 1 \)- function). This is completely opposite to the intuitive understanding of the relationship between (anti-)bunching and the Fano factor.

### IV. CONCLUSIONS

The \( G^{(2)} \)- function is a well-known tool for investigating bunching and anti-bunching behaviour of photon emission, but can equally well be used to describe the statistics of electron emission in transport systems such as quantum dots. The unifying frame in both cases is the concept of quantum jumps in Markovian quantum master equations that govern the dissipative dynamics. The relation Eq. \( (10) \) between the \( G^{(2)} \)- function and the Fano factor clarifies that super-Poissonian statistics in electron transport do not necessarily correspond to bunching; sub-Poissonian statistics do not necessarily imply anti-bunching. As our examples show, single electron transport through nanostructures such as quantum dots offers the possibility to test these relations. An interesting further question would be the features of cross correlations and off-diagonal \( G_{kl}^{(2)} \)- functions defined with two different jump operators \( J_k \neq J_l \).

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### Appendix A: A microscopic derivation of Eq. \( (9) \)

We first specify the Hamiltonian of our transport system as composed of system, lead, and tunnel-coupling parts: \( H = H_S + H_{\text{leads}} + H_T \). The system Hamiltonian we write as \( H_S = \sum_{\alpha} E_\alpha |\alpha\rangle\langle\alpha| \), where \( |\alpha\rangle \) is a many-body system state of energy \( E_\alpha \). We assume that the leads can be described by the non-interacting Hamiltonian

\[ H_{\text{leads}} = \sum_{\alpha} H^\alpha_{\text{leads}} = \sum_{\alpha} \sum_k \omega_{\alpha k} c^\dagger_{\alpha k} c_{\alpha k}, \]

where \( c_{\alpha k} \) is the annihilation operator for an electron in lead \( \alpha \) with quantum numbers \( k \) and energy \( \omega_{\alpha k} \). We assume that system and the leads are coupled with the single-particle tunnel Hamiltonian

\[ H_T = \sum_{k,\alpha} V_{k\alpha} c^\dagger_{k\alpha} d_\alpha + \text{H.c.,} \]

where \( d_\alpha \) is the annihilation operator of an electron in a localised system state, which we assume to be unique for each lead, and where \( V_{k\alpha} \) is a tunnel amplitude.
To produce a correlation function analogous to Eq. (2), we first define for lead $\alpha$ the operator
\[
C^{(+)}_{\alpha}(t) = 2i \sum_k V^*_k c_{k\alpha}(t).
\] (A3)

This definition is analogous to the expansion of electric field operator $E^{(+)}$ in terms of the normal-mode annihilation operators. Here, the choice of coefficients is somewhat arbitrary, but $2iV^*_k$ is convenient. We then define our $g^{(2)}$-function for lead-$\alpha$ as
\[
g^{(2)}_{\alpha}(t, \tau) = \frac{\langle C^{(-)}_{\alpha}(t)C^{(-)}_{\alpha}(t+\tau)C^{(+)}_{\alpha}(t+\tau)C^{(+)}_{\alpha}(t) \rangle}{\langle C^{(-)}_{\alpha}(t)C^{(+)}_{\alpha}(t) \rangle \langle C^{(-)}_{\alpha}(t+\tau)C^{(+)}_{\alpha}(t+\tau) \rangle}.
\] (A4)

Heisenberg’s equation of motion for the lead annihilators reads
\[
c_{k\alpha}(t) = -i\omega_{k\alpha} c_{k\alpha}(t) - iV_{k\alpha} d_{\alpha}(t).
\] (A5)

Introducing the Laplace transform $f(z) = \int_0^\infty dt e^{-zt} f(t)$ and solving gives
\[
c_{k\alpha}(z) = \frac{1}{z + i\omega_{k\alpha}} \left(c_{k\alpha}(t=0) - iV_{k\alpha} d_{\alpha}(z)\right).
\]

Summing over $k$ and regularizing we obtain
\[
C^{(+)}_{\alpha}(z) = \sum_k 2iV^*_k \left\{ \frac{\pi\delta(iz - \omega_{k\alpha}) + \mathbb{P}}{z + i\omega_{k\alpha}} \right\} \times \left\{ c_{k\alpha}(t=0) - iV_{k\alpha} d_{\alpha}(z) \right\},
\] (A6)

where $\mathbb{P}$ is the principal part. We choose the initial state of lead $\alpha$ such that $c_{k\alpha}(t=0)$ evaluates to zero in the expectation value (see below). Then
\[
C^{(+)}_{\alpha}(z) = \left\{ 2\pi \sum_k |V_{k\alpha}|^2 \delta(iz - \omega_{k\alpha}) + 2i\mathbb{P} \sum_k |V_{k\alpha}|^2 \frac{1}{iz - \omega_{k\alpha}} \right\} d_{\alpha}(z) = \{\Gamma_{\alpha}(iz) + ip_{\alpha}(iz)\} d_{\alpha}(z)
\] (A7)

which defines the frequency-dependent rate, $\Gamma_{\alpha}(\omega) = 2\pi \sum_k |V_{k\alpha}|^2 \delta(\omega - \omega_{k\alpha})$, and principal part $p_{\alpha}(\omega)$. If we assume that the rate is constant as a function of its argument, $\Gamma_{\alpha}(\omega) = \Gamma_{\alpha}$, and that the principal part vanishes, we obtain the result
\[
C^{(+)}(t) = \Gamma_{\alpha} d_{\alpha}(t)
\] (A8)

These assumptions can be justified by assuming that lead $\alpha$ starts in the equilibrium state $\rho^eq_{\alpha} = \exp{-\beta(H_{\text{leads}} - \mu_{\alpha} N_{\alpha})}$ in the infinite-bias limit $\mu_{\alpha} \to -\infty$. This also enforces the requirement $\Gamma_{k\alpha}(t=0)\rho^eq_{\alpha} = 0$.

Placing result Eq. (A8) in definition Eq. (A4) and taking the long-time limit yields
\[
g^{(2)}_{\alpha}(\tau) = \frac{\langle d^*_\alpha d^*_\alpha(\tau)d_{\alpha}(\tau)\rangle d_{\alpha}(\text{stat})}{\langle d^*_\alpha d_{\alpha}(\text{stat}) \rangle^2},
\] (A9)

analogous to Eq. (5).

If we assume that all leads start in this same infinite-bias limit, the time-evolution of the system density matrix can be described by the Markovian master equation:
\[
\dot{\rho} = -i[H_S, \rho] + \sum_{\alpha \in S} \Gamma_{\alpha} \left\{ d^*_\alpha \rho d_{\alpha} - \frac{1}{2} \rho d^*_\alpha d_{\alpha} - \frac{1}{2} d^*_\alpha d_{\alpha} \rho \right\}
\] (A10)

where the first sum is over source leads, the second over drains and where $\Gamma_{\alpha}$ are rates. Writing this as $\dot{\rho} = (\mathcal{W}_0 + J)\rho$ with jump super-operator $J\rho = \Gamma_{\alpha} d_{\alpha} d^*_\alpha$, we see that definitions Eq. (A3) and Eq. (A4) indeed reproduce Eq. 9 of the main text.

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identification of the counting rate $R = \langle \langle J \rangle \rangle$. With $\lambda(\tau)$ corresponding to our $S(\tau)/\langle \langle J \rangle \rangle^2 = g^{(2)}(\tau) - 1$ we obtain $F(0) = Q + 1 = 1 + 2\langle \langle J \rangle \rangle \int_0^\infty d\tau \lambda(\tau)$, thus Eq. (10).

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