A UNIFIED FRAMEWORK FOR GRAPH ALGEBRAS AND QUANTUM CAUSAL HISTORIES

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Abstract. We present a mathematical framework that unifies the quantum causal history formalism from theoretical high energy physics and the directed graph operator framework from the theory of operator algebras. The approach involves completely positive maps and directed graphs and leads naturally to a new class of operator algebras.

1. Introduction

In this paper we present a new operator theoretic framework that provides a unified approach for recent studies in theoretical high energy physics and contemporary operator algebra theory. More specifically, this approach involves completely positive maps and directed graphs, and includes the quantum causal history formalism from recent work towards a quantum theory of gravity on the one hand and the graph-operator framework from the theory of directed graph operator algebras on the other. We also define a new class of operator algebras that is naturally defined through this approach.

The basic physical properties that a quantum theory of gravity must satisfy motivated F. Markopoulou to invent a formalism called “Quantum Causal Histories” [1, 2, 3, 4, 5]. A secondary goal of this paper is to introduce this formalism to the broader mathematical community. Fundamental examples include causal evolutions of spin networks [6, 8] and quantum computers [3]. The basic definitions have been refined through a series of papers and now a clean mathematical definition is emerging [1]. Mathematically, and somewhat roughly speaking at this point, a quantum causal history (QCH) is given by a directed graph with a finite-dimensional Hilbert space for each vertex and a quantum operation associated with each directed edge. The vertex spaces correspond to events, or observables, within a local history and the quantum operations indicate causal links between pairs of related events.

2000 Mathematics Subject Classification. 46L05, 47L75, 81P68, 83C45.

Key words and phrases. directed graph, completely positive map, partial isometry, quantum operation, quantum causal history.
As described below, the QCH framework incorporates further structure motivated by the characterization of evolution within quantum systems.

On the other hand, Cuntz and Krieger [9] were motivated by a problem in dynamical systems to initiate the study of operator algebras associated with directed graphs. Over the past two decades the study of directed graph operator algebras and related topics has taken on a life of its own and now, it is fair to say, plays a central role in the modern theory of operator algebras. We mention the articles [10] – [37] as entrance points into the extensive literature on the subject.

In § 2 we recall some basic properties of completely positive maps and quantum operations. We define the “CP – directed graph framework” and associated C*-algebras in § 3, and show how graph algebras fit into the framework. In § 4 we discuss in some detail the QCH formalism, draw a connection with quantum computing, and prove a theorem for the QCH C*-algebras.

2. Completely Positive Maps

Given a Hilbert space \( H \) we let \( \mathcal{B}(H) \) be the set of bounded operators that act on \( H \). A completely positive (CP) map is a linear map \( \Phi : \mathcal{B}(H_1) \to \mathcal{B}(H_2) \) such that the “ampliated” maps

\[
\mathbb{1}_k \otimes \Phi : \mathcal{M}_k \otimes \mathcal{B}(H_1) \longrightarrow \mathcal{M}_k \otimes \mathcal{B}(H_2)
\]

are positive for \( k \geq 1 \). (Here \( \mathcal{M}_k \) denotes the set of \( k \times k \) complex matrices and \( \mathbb{1}_k \) denotes the identity operator, the “maximally mixed state”, inside \( \mathcal{M}_k \).) For basic mathematical properties of CP maps see [38] and physical motivations see [39].

A fundamental technical device in the study of CP maps is the operator-sum representation given by the theorem of Choi [40] and Kraus [41]. For every CP map \( \Phi \) on finite-dimensional space, there is a set of noise operators (or errors) \( \{A_i\} \subseteq \mathcal{B}(H_1, H_2) \) such that

\[
\Phi(\rho) = \sum_{i} A_i \rho A_i^* \quad \forall \rho \in \mathcal{B}(H_1).
\]

The map is unital if also \( \sum_i A_i A_i^* = \mathbb{1}_{H_2} \). It is always possible to choose a family of noise operators with cardinality at most \( \text{dim}(H_1) \text{dim}(H_2) \). On infinite dimensional space not all CP maps have such a form, and when they do the sum in (1) converges in the strong operator topology. For brevity, we assume all the CP maps considered here have a representation as in (1).

A quantum operation (or quantum evolution, or quantum channel) is a CP map \( \Phi : \mathcal{B}(H_1) \to \mathcal{B}(H_2) \) that is also trace preserving. When \( \Phi \)
is represented as in (1), trace preservation is equivalent to the identity
\[ \sum_i A_i^*A_i = \mathbb{1}_{H_1}. \]

Thus, a quantum operation \( \Phi \) is a map that satisfies (1) and (2) for some set of operators \( \{A_i\} \). Equivalently, a quantum operation is a CP map such that its associated dual map, denoted by \( \Phi^\dagger : \mathcal{B}(H_2) \rightarrow \mathcal{B}(H_1) \), is unital. (Recall that the dual map for a map \( \Phi \) is defined via the equation \( \text{trace}(\rho \Phi^\dagger(\sigma)) = \text{trace}(\Phi(\rho) \sigma) \).)

The ideal physical examples of quantum operations are unitary maps as they characterize evolution of states within a closed quantum system. Such a map is of the form \( \Phi(\rho) = U \rho U^* \) for some unitary operator \( U \).

When evolution occurs in an open system (i.e., when the system of interest is exposed to an external environment) quantum operations have the more general form given by (1) and (2). See [42] for further discussions and references.

### 3. The CP – Directed Graph Framework

Let \( E = (E^0, E^1, r, s) \) be a (countable) directed graph with vertices \( x \in E^0 \), directed edges \( e \in E^1 \) and range and source maps \( r, s : E^1 \rightarrow E^0 \) giving the initial \( s(e) \) and final \( r(e) \) vertices of an edge \( e \). When \( e \in E^1 \) satisfies \( s(e) = x \) and \( r(e) = y \), we shall write \( e = (x, y) \).

Suppose we have a Hilbert space \( \{H(x) : x \in E^0\} \) for each vertex and a family of CP maps \( \Psi = \{\Phi_e : e \in E^1\} \) with domains and ranges that satisfy
\[
\begin{align*}
(i) & \quad \text{Dom}(\Phi_e) = \mathcal{B}(H(s(e))) \quad \forall e \in E^1 \\
(ii) & \quad \text{Ran}(\Phi_e) \subseteq \mathcal{B}(H(r(e))) \quad \forall e \in E^1
\end{align*}
\]

Given such a family of spaces and maps, define the Hilbert space \( \mathcal{H} = \oplus_{x \in E^0} \mathcal{H}(x) \) and let \( P_x \) be the projection of \( \mathcal{H} \) onto \( \mathcal{H}(x) \).

**Definition 3.1.** Given a directed graph \( E \), let \( \{\mathcal{H}(x) : x \in E^0\} \) be Hilbert spaces and let \( \Psi = \{\Phi_e : e \in E^1\} \) be a family of CP maps that satisfy (†). Suppose \( \Phi_e = \{A_{e,i} : i \in \mathcal{I}_e\} \) is an operator-sum representation of \( \Phi_e \) for each \( e \in E^1 \). We can naturally regard the \( A_{e,i} \) as operators acting on \( \mathcal{H} \). Define \( \mathfrak{A}_\Psi \) to be the \( C^* \)-algebra generated by all operators \( A_{e,i} \) and vertex projections \( P_x \); so that,
\[ \mathfrak{A}_\Psi = C^*(\{P_x, A_{e,i} : x \in E^0, e \in E^1, i \in \mathcal{I}_e\}). \]

The choice of noise operators that represent a given CP map in (1) is of course not unique. However, as our notation suggests the algebras \( \mathfrak{A}_\Psi \) are independent of these choices.
Proposition 3.2. Let $E$ be a directed graph and let $\Psi = \{\Phi_e : e \in E^1\}$ be a family of CP maps that satisfy (†). Then the algebra $\mathfrak{A}_\Psi$ is independent of the choice of operators $\{A_{e,i}\}$ that represent the maps $\Phi_e$ as in (†).

Proof. Suppose that $\{A_{e,i}\}$ and $\{A'_{e,j}\}$ represent $\Phi_e$ via equation (†). By possibly including zero operators we may assume the cardinality of these two sets is the same. Then from the structure theory for CP maps, there is a scalar unitary matrix $U = (u_{ij})$ such that

$$A_{e,i} = \sum_j u_{ij} A'_{e,j} \quad \forall i.$$ 

It follows that the algebras generated by $\{P_x, A_{e,i}\}_{x,e,i}$ and $\{P_x, A'_{e,j}\}_{x,e,j}$ coincide, and the result follows. ■

3.1. Graph Algebras. We now discuss one of the motivating special cases for this framework. Let $E = (E^0, E^1, r, s)$ be a directed graph. Consider families of operators $\{P_x, S_e : x \in E^0, e \in E^1\}$, where the $P_x$ are projections and the $S_e$ are partial isometries (or equivalently, unitary operators restricted to a subspace), that act on the same Hilbert space and satisfy:

$$\tag{‡} \begin{align*}
(i) & \quad S_e^* S_e = P_{s(e)} \quad \forall e \in E^1 \\
(ii) & \quad S_e S_e^* \leq P_{r(e)} \quad \forall e \in E^1
\end{align*}$$

Then the structure of $E$ determines the relations satisfied by $\{P_x, S_e\}$ in the sense that the initial projection for each $S_e$ is equal to the projection for the source vertex of $e$ and the range projection for each $S_e$ is supported on the projection for the range vertex of $e$.

The relations (‡) provide the fundamental base case for investigations into operator algebras associated with directed graphs. In the most general context, a graph algebra is an operator algebra generated by a family $\{S_e, P_x\}$. There are a number of refinements and generalizations of the formulation (‡). In many instances the $S_e$ are assumed to have mutually orthogonal ranges. The projections $P_x$ are typically assumed to have mutually orthogonal ranges as well, or sometimes just mutually commuting ranges. There are also topological graph generalizations wherein the vertices and edges are locally compact spaces and the range and source maps are continuous maps. However, in every setting the motivating case is the same: A Hilbert space $\mathcal{H}(x)$ associated with every vertex $x$ in $E$ and for every directed edge $e = (x, y)$ a partial isometry $S_e$ that maps from $\mathcal{H}(x)$ to $\mathcal{H}(y)$.

If $\{S_e, P_x\}$ satisfy (‡), observe that for each $e = (x, y)$ the operator $S_e$ defines a unitary from $\mathcal{H}(x) = P_x \mathcal{H}$ into $\mathcal{H}(y) = P_y \mathcal{H}$ and a unitary
CP map $\Phi_e : \mathcal{B}(\mathcal{H}(x)) \to \mathcal{B}(\mathcal{H}(y))$ via
$$\Phi_e(\rho) = S_e' \rho (S_e')^* \text{ where } S_e' = S_e|_{\mathcal{H}(x)}.$$ 
Thus the corresponding algebra $\mathfrak{A}_\Psi$ defines a graph algebra, and so graph algebras form a subclass of the algebras $\mathfrak{A}_\Psi$.

4. THE QUANTUM CAUSAL HISTORY FORMALISM

The mathematical formalism for QCH’s has undergone a series of refinements since being introduced in [4]. The presentation below is most closely related to the recent formulation of Hawkins, Markopoulou and Sahlmann [1]. The nomenclature we use is slightly different than [1], we do this to mesh with the graph algebra terminology. We shall focus on the mathematical aspects and touch on the physical motivations for various constraints.

To define a QCH then, we begin with a graph $E = (E^0, E^1, r, s)$, which may also be interpreted as a partial order when there are no loops. This graph represents a causal set wherein the vertices correspond to a set of local events in the universe and vertices linked by directed edges indicate causal relations between events. From the postulates of quantum mechanics, events are represented by density operators on Hilbert space. Recent work in string theory and loop quantum gravity (see [2]) suggests that any finite region of space should contain a finite amount of information. Thus, each of the event spaces is assumed to be finite-dimensional. Since causality can be interpreted as transferring information from one event to another and because, by definition, a QCH describes local causality at the quantum level, a causal relation given by a directed edge $e = (x, y) \in E^1$ corresponds to a quantum operation $\Phi(x, y) : \mathcal{B}(\mathcal{H}(x)) \to \mathcal{B}(\mathcal{H}(y))$ between event spaces.

Thus, at its mathematical core, a QCH consists of a directed graph, with a finite-dimensional Hilbert space for each vertex, and a quantum operation for each directed edge. There are further constraints within a QCH and we discuss them now briefly.

First some terminology. Given $x, y \in E^0$ write $x \leq y$ when $x$ precedes $y$ as an event. In this case there is a future-directed curve from $x$ to $y$ and this is represented by a directed edge $e = (x, y) \in E^1$. If $x \leq y$ or $y \leq x$ then $x$ and $y$ are related and otherwise they are spacelike separated and we use $x \sim y$ to denote this. A path in $E$ corresponds to a future-directed path through the events in the history. Such a path is future (past) inextendible if there is no event in $E$ which is in the future (past) of the entire path. Loops in $E$ correspond to closed timelike curves. From the finiteness assumption discussed above, $E$
is locally finite in the sense that for any $x, y \in E^0$ there are at most finitely many $z \in E^0$ such that $x \leq z \leq y$. Given $x, y \in E^0$, there is also no generality lost in assuming there is at most one edge $e = (x, y)$ in $E$ from $x$ to $y$. (If $s(e) = x = s(f)$ and $r(e) = y = r(f)$ then the operations associated with these edges could be combined to form a single operation that encodes the relevant causal structure from event $x$ to event $y$.)

An acausal set $\xi \subseteq E^0$ is defined by the property that $x \sim y$ whenever $x, y \in \xi$. Such a set is a complete future for an event $x$ if $\xi$ intersects any future inextendible future-directed path that starts at $x$. A complete past is defined analogously. The composite state space for $x \sim y$ (the physical existence of which is guaranteed by quantum mechanics) is $\mathcal{H}(\{x, y\}) = \mathcal{H}(x) \otimes \mathcal{H}(y)$ and more generally $\mathcal{H}(\xi) = \bigotimes_{x \in \xi} \mathcal{H}(x)$. For $x \in E^0$ write $\mathcal{A}(x)$ for the matrix algebra $\mathcal{B}(\mathcal{H}(x))$ and similarly define $\mathcal{A}(\xi) = \bigotimes_{x \in \xi} \mathcal{A}(x)$ for a set $\xi \subseteq E^0$. Given an acausal set $\xi$ and an event $x \in \xi$, there is a natural unital embedding $\iota_x : \mathcal{A}(x) \to \mathcal{A}(\xi)$ and we shall write $\mathcal{A}(x) \subseteq \mathcal{A}(\xi)$.

If $\xi$ and $\zeta$ are acausal sets such that $\xi$ is a complete past for $\zeta$ and $\zeta$ is a complete future for $\xi$, then we write $\xi \preceq \zeta$ and say that $(\xi, \zeta)$ form a complete pair. Such a pair represents an evolution in a closed quantum system, hence woven into the fabric of the QCH there should be a unitary operator $U(\xi, \zeta) : \mathcal{H}(\xi) \to \mathcal{H}(\zeta)$. Such an operator determines a unitary map (an isomorphism) $\Phi(\xi, \zeta) : \mathcal{A}(\xi) \to \mathcal{A}(\zeta)$ via

$$\Phi(\xi, \zeta)(\rho) = U(\xi, \zeta) \rho U(\xi, \zeta)^* \quad \forall \rho \in \mathcal{A}(\xi).$$

Note the restriction of $\Phi(\xi, \zeta)$ (respectively $\Phi(\xi, \zeta)^\dagger$) to $\mathcal{A}(x) \subseteq \mathcal{A}(\xi)$ for $x \in \xi$ (respectively $\mathcal{A}(z) \subseteq \mathcal{A}(\zeta)$ for $z \in \zeta$) is a $*$-homomorphism. This gives the structure of a QCH at the global level, but does not indicate how the isomorphisms $\Phi(\xi, \zeta)$ should depend on the individual causal relations between events in $\xi$ and $\zeta$. This is the role played by the operations $\Phi(x, y)$ on individual edges.

We now give a precise mathematical definition of a QCH. We note that the maps in the definition below have directions reversed from the presentation in [11]. Here we take the dual approach so the “directions” of the maps are in line with the graph structure. Recall that if $i_A : \mathcal{A}_1 \to \mathcal{A}_2$ and $i_B : \mathcal{B}_1 \to \mathcal{B}_2$ are $*$-monomorphisms and $\Psi : \mathcal{A}_2 \to \mathcal{B}_2$ is a map, then the reduction of $\Psi$ to $\mathcal{A}_1 \leftrightarrow \mathcal{B}_1$ is the map $\Phi = i_B^\dagger \circ \Psi \circ i_A$.

**Definition 4.1.** A quantum causal history consists of a directed graph $E$ with a Hilbert space $\{\mathcal{H}(x) : x \in E^0\}$ for each event and a quantum operation $\Phi(x, y) : \mathcal{A}(x) \to \mathcal{A}(y)$ for each pair of related events $x \leq y$ such that the following axioms are satisfied:
(i) (Extension) For all \( y \in E^0 \) and \( \zeta \subseteq E^0 \) a complete future of \( y \), there is a homomorphism \( \Phi_F(y, \zeta) : A(y) \to A(\zeta) \) such that \( \Phi_F(y, \zeta) \dagger \) is a quantum operation and for all \( z \in \zeta \), the reduction of \( \Phi_F(y, \zeta) \dagger \) to \( A(z) \) is \( \Phi(y, z) \dagger \). Likewise, for all \( y \in E^0 \) and \( \xi \subseteq E^0 \) a complete past of \( y \), there is a quantum operation \( \Phi_P(\xi, y) : A(\xi) \to A(y) \) such that \( \Phi_P(\xi, y) \dagger \) is a homomorphism and for all \( x \in \xi \) the reduction of \( \Phi_P(\xi, y) \dagger \) to \( A(x) \) is \( \Phi(x, y) \dagger \).

(ii) (Spacelike Commutativity) If \( x \sim y \in E^0 \) and \( \zeta \subseteq E^0 \) is a complete future of \( x \) and \( y \), then the images of \( \Phi_F(x, \zeta) \) and \( \Phi_F(y, \zeta) \) commute inside \( A(\zeta) \). Likewise, if \( y \sim z \in E^0 \) and \( \xi \subseteq E^0 \) is a complete past of \( y \) and \( z \), then the images of \( \Phi_F(\xi, y) \dagger \) and \( \Phi_P(\xi, z) \dagger \) commute inside \( A(\xi) \).

(iii) (Composition) If \( \zeta \subseteq E^0 \) is a complete future of \( x \) and a complete past of \( y \), then \( \Phi(x, y) = \Phi_P(\zeta, y) \circ \Phi_F(x, \zeta) \).

If \( \xi \preceq \zeta \) form a complete pair within a given QCH, it is proved in [1] that there is a unique unitary map \( \Phi(\xi, \zeta) : A(\xi) \to A(\zeta) \) such that the reduction of \( \Phi(\xi, \zeta) \) to \( A(x) \mapsto A(y) \) is \( \Phi(x, y) \) for all \( x \in \xi \) and \( y \in \zeta \). Thus the isomorphisms \( \Phi(\xi, \zeta) \) discussed above may be built up from the individual edge maps \( \Phi(x, y) \) and hence the edge maps are the fundamental building blocks for a QCH.

Example 4.2. (Quantum Computers) As discussed in [3], the basic model for a quantum computer fits into the QCH formalism. Specifically, each quantum algorithm may be interpreted as a QCH via its "circuit-gate" presentation. (See [42] for a brief mathematical introduction to quantum algorithms and references.) The QCH for a given algorithm has vertex spaces all equal to \( \mathbb{C}^2 \). The directed edges correspond to the choice of unitary gates within the algorithm and the vertex spaces encode the intermediate states of the quantum bits of information (the ‘qubits’). The structure of the associated directed graph is the same as the circuit-gate diagram, with the circuits labelled as vertices and the gates labelled as directed edges.

We finish by proving that the algebras \( \mathfrak{A}_\Psi \) associated with QCH’s are familiar objects from operator theory. For basic properties of AF-algebras we point the reader to the text [43].

**Theorem 4.3.** Let \( \mathfrak{A}_\Psi \) be the C*-algebra associated with a given quantum causal history. Then \( \mathfrak{A}_\Psi \) is an AF-algebra.

**Proof.** It is enough to prove that every finite set of elements of \( \mathfrak{A}_\Psi \) can be approximated by elements lying in a finite-dimensional subalgebra. But elements of the form \( A = A_1 \cdots A_n \), where each \( A_k = A_{e,i} \) or
$A_k = A^*_{e,i}$ for some $e$ and $i$, span a dense subspace of $\mathfrak{A}_\Psi$. (Note that the vertex projections are obtained via equation (2).) Hence it is enough to show that each finite set of such elements lies in a finite-dimensional subalgebra. Suppose $\mathcal{F}$ is such a set. Note that each $A^*_{e,i}$ belongs to $\mathcal{B}(\mathcal{H}(x), \mathcal{H}(y))$ for some $x, y \in E^0$, and so the same is true for every $A^*_{e,i}$ and all elements $A \in \mathcal{F}$. Thus, as $\dim \mathcal{H}(x) < \infty$ for all $x \in E^0$, it follows that the algebra generated by $\mathcal{F}$ is a finite-dimensional algebra which is also a subalgebra of $\mathfrak{A}_\Psi$, and the result follows.

**Remark 4.4.** While graph algebras and quantum causal histories provided the initial impetus for the CP–directed graph framework presented here, it is evident that this structure admits other possibilities. Indeed, there are many CP maps that are neither unitary maps nor quantum operations, and presumably the $C^*$-algebras $\mathfrak{A}_\Psi$ would go beyond the graph algebra and QCH subclasses in general. We have also not considered here the various possibilities for non-selfadjoint algebras defined by this framework. We plan to undertake investigations of these algebras elsewhere and hope this paper motivates others to do the same.

**Acknowledgements.** I would like to thank Fotini Markopoulou for enlightening conversations and Eli Hawkins for helpful comments on an early draft. I am also grateful to colleagues at the Institute for Quantum Computing and the Perimeter Institute for interesting discussions. This work was partially supported by an NSERC grant.

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