Presenting parabolic subgroups
Françoise Dahmani and Vincent Guirardel

Abstract

Consider a relatively hyperbolic group $G$. We prove that if $G$ is finitely presented, so are its parabolic subgroups. Moreover, a presentation of the parabolic subgroups can be found algorithmically from a presentation of $G$, a solution of its word problem, and generating sets of the parabolic subgroups. We also give an algorithm that finds parabolic subgroups in a given recursively enumerable class of groups.

Consider a relatively hyperbolic group $G$ with parabolic subgroups $H_1, \ldots, H_n$. It is well known that if each $H_i$ is finitely generated (or finitely presented), then so is $G$. Osin showed conversely that if $G$ is finitely generated, then so are $H_1, \ldots, H_n$ \cite[Prop. 2.27]{Osi06}. Whether finite presentation of $G$ implies finite presentation of $H_1, \ldots, H_n$ is an important question raised by Osin in \cite[Problem 5.1]{Osi06}.

On the algorithmic side, given a finite presentation of a relatively hyperbolic group $G$ and a generating set of the parabolic subgroups, can one find a presentation of the parabolic subgroups?

We give a positive answer to these two questions.

**Theorem 1.** Let $G$ be a finitely presented group. Assume that $G$ is hyperbolic relative to $H_1, \ldots, H_n$. Then each $H_i$ is finitely presented.

**Theorem 2.** There exists an algorithm that takes as input a finite presentation of a group $G$, a solution to its word problem, and a collection of finite subsets $S_1, \ldots, S_n \subset G$, and that terminates if and only if $G$ is hyperbolic relative to $\langle S_1 \rangle, \ldots, \langle S_n \rangle$.

In this case, the algorithm outputs a linear isoperimetry constant $K$ for the corresponding relative presentation, a finite presentation for each of the parabolic subgroups $\langle S_i \rangle$, and says whether $G$ is properly relative hyperbolic relative to $\langle S_1 \rangle, \ldots, \langle S_n \rangle$ (i.e. $\langle S_i \rangle \subsetneq G$ for all $i$).

In this statement, the linear isoperimetry constant $K$ is for the relative presentation $X_\infty$ as defined in Section 1.2.

If one is not given generating sets of the parabolic subgroups, one can search for them, and require that they lie in some recursively enumerable class of groups.

**Theorem 3.** There exists an algorithm as follows. It takes as input a finite presentation of a group $G$, a solution for its word problem, and a recursive
class of finitely presented groups $C$ (given by a Turing machine enumerating
presentations of these groups).

It terminates if and only if $G$ is properly hyperbolic relative to subgroups that
are in the class $C$.

In this case, the algorithm outputs an isoperimetry constant $K$, a generating
set and a finite presentation for each of the parabolic subgroups.

The Turing machine enumerating $C$ is a machine that enumerates some finite
presentations, each of which represents a group in $C$, and such that every group
in $C$ has at least one presentation that is enumerated.

This paper can be seen as a continuation, extension, and precision, on the
form and the substance of [Dah08]. It is based on the analysis of some Van
Kampen diagrams in different truncated relative presentations. The main tool
is Proposition 2.9 which says that if some relative presentation does not satisfy
a linear isoperimetric inequality, then this shows up on some diagram of small
area and small complexity.

Section 1 recalls definitions about isometric inequalities, introduces trun-
cated relative presentations, and defines the complexity of a diagram. Section
2 contains the main technical results. Section 3 is devoted to corollaries. Theo-
rems 1, 2 and 3 follow from Corollaries 3.3, 3.5 and 3.6.

1 Context

1.1 Linear isoperimetric inequalities

Consider a finitely generated group $G$, with an arbitrary (non necessarily finite)
generating set $S$. A presentation of $G$ over $S$ is a set $R \subset \mathbb{F}_S$ that normally
generates the kernel of the natural map from the free group $\mathbb{F}_S$ to $G$. The
elements of $R$ are called defining relations, and we usually write $G = \langle S | R \rangle$.

We say that a presentation is triangular if every defining relation has length
2 or 3 as word over the alphabet $S^\pm$. If one allows to increase the generating
set, it is not restrictive to consider triangular presentations: from an arbitrary
finite presentation, one can construct effectively a triangular one.

Consider $w \in \mathbb{F}_S$, viewed as a reduced word over the alphabet $S^\pm$. If $w$
represents the trivial element of $G$ (we write $w \overset{G}{=} 1$), the area of $w$ for the
presentation $G = \langle S | R \rangle$, denoted by $\text{Area}(w)$, is the minimal number $n$
such that $w$ is the product in $\mathbb{F}_S$ of $n$ conjugates of elements of $R$.

Given a word $w$ such that $w \overset{G}{=} 1$, a Van Kampen diagram for $w$ over the
presentation $G = \langle S | R \rangle$, is a simply connected planar 2-complex such that
oriented edges are labeled by elements of $S^\pm$, such that reversing the orientation
changes the label to its inverse, and such that every 2-cell has its boundary
labeled by a cyclically reduced word conjugate to an element of $R \cup R^{-1}$, and
such that the boundary of the diagram itself is labeled by $w$. Sometimes, we
just say cell instead of 2-cell. It is well known that $\text{Area}(w)$ is the minimal
number of 2-cells of Van Kampen diagrams for $w$. See [LS01, Section 5.1] for
more details.
An isoperimetric function of a presentation \( \langle S \mid R \rangle \) is a function \( f : \mathbb{N} \to \mathbb{N} \) such that for all \( w \in F_S \), \( \text{Area}(w) \leq f(\text{length}(w)) \). Note that if \( S \) is infinite, there are infinitely many words of a given length, and it may happen that no such function (with finite values) exists.

Our approach is based on the fact that a group is relatively hyperbolic if and only if it has a presentation of a particular kind with a linear isoperimetric function [Osi06], see Theorem 1.2 below. Another important fact is that the failure of a specific linear isoperimetric inequality can be observed in a set of words of controlled area (Gromov [Gro87], Bowditch [Bow95], Papasoglu [Pap95]).

**Theorem 1.1** ([Pap95]). Let \( G = \langle S \mid R \rangle \) be an arbitrary (non necessarily finite) triangular presentation of an arbitrary group.

Assume that there is a word \( w \) over the alphabet \( S^\pm \) such that \( w \in G = 1 \) and \( \text{Area}(w) > K \text{length}(w) \). Then there exists a word \( w' \) over the alphabet \( S^\pm \) such that \( w' \in G = 1 \), and such that

- \( \text{Area}(w') \in \left[ \frac{K}{240K}, 240K \right] \)
- \( \text{Area}(w') > \frac{1}{22^{100}} \text{length}(w')^2 \).

### 1.2 Truncated and exact relative presentations

Since finite generation of a relatively hyperbolic group implies finite generation of its maximal parabolic subgroups [Osi06 Prop. 2.27], we always assume that relatively hyperbolic groups and their maximal parabolic subgroups are finitely generated.

Let \( G \) be a finitely presented group, and \( H_1, \ldots, H_n \) be finitely generated subgroups of \( G \). For each \( i \), let \( S_i \) be a finite symmetric generating set of \( H_i \). Consider a finite triangular presentation \( G = \langle S \mid R \rangle \) where \( S \) is a finite symmetric generating set of \( G \) containing each \( S_i \), and \( R \) is a finite set of triangular relations over \( S \).

To introduce truncated relative presentations, we need auxiliary groups \( \hat{H}_1, \ldots, \hat{H}_n \), with generating sets \( \hat{S}_1, \ldots, \hat{S}_n \), and with epimorphisms \( p_i : \hat{H}_i \to H_i \) that map \( \hat{S}_i \) bijectively to \( S_i \). Informally, \( \hat{H}_i \) is a group obtained from a presentation of \( H_i \) over \( S_i \) by removing some relations. Exact relative presentations will correspond to the case where each \( p_i \) is an isomorphism.

Let \( T(\hat{H}_i) \subset \hat{H}_i^2 \) be the multiplication table of \( \hat{H}_i \), i.e. the set tuples of at most 3 elements of \( F_{S_i} \), whose product is trivial in \( \hat{H}_i \). Thus, we have \( (a, b, c) \in T(\hat{H}_i) \) if and only if \( abc = 1 \) in \( \hat{H}_i \).

Let \( \hat{S} = S \sqcup \hat{H}_1 \sqcup \ldots \sqcup \hat{H}_n \). To each element of \( \hat{S} \) corresponds naturally an element of \( G \) via the inclusion \( S \subset G \) or via \( p_i \). These elements of \( G \) form a generating set, in general infinite.

Given the initial presentation \( G = \langle S \mid R \rangle \), \( H_i \) and its generating set \( S_i \), the auxiliary groups \( \hat{H}_i \) and the epimorphisms \( p_i : \hat{H}_i \to H_i \), we associate the truncated relative presentation of \( G \) as follows:

\[
G = \left\langle \hat{S} \mid \mathcal{R}', (T(\hat{H}_i))_{i=1 \ldots n} \right\rangle
\]  

(1)
where $\mathcal{R}'$ consists of $\mathcal{R}$ together with all two-letter relators of the form $\tilde{\sigma}^{-1} p_i(\tilde{\sigma})$ for $\tilde{\sigma} \in \tilde{S}_i$, ($p_i(\tilde{\sigma})$ being an element of $S$). Obviously, this infinite presentation is indeed a triangular presentation of $G$.

We say that this presentation is truncated because only the multiplication table of $\tilde{H}_i$ is included, and not the one of $H_i$ (although all relations of $H_i$ are consequences of $\mathcal{R}'$). We say that a truncated relative presentation as above is exact if for all $i$, $p_i : \tilde{H}_i \to H_i$ is an isomorphism.

We will be particularly interested in the following one-parameter family of truncated relative presentations $X_{\rho}$. Given $G$, $H_1, \ldots, H_n$, $S_i$ as above, and $\rho \in \mathbb{N} \cup \{\infty\}$, we define $R_{\rho}(S_i)$ be the set of all words of length $\leq \rho$ on $S_i$ that are trivial in $H_i$, $\tilde{H}_i = \langle S_i | R_{\rho}(S_i) \rangle$, and $p_i : \tilde{H}_i \to H_i$ the obvious epimorphism. We define $X_{\rho}$ the truncated relative presentation corresponding to this data. In particular, $X_{\infty}$ is an exact relative presentation, and if all $H_i$ are finitely presented, then $X_{\rho}$ and $X_{\infty}$ coincide (as presentations) for $\rho$ large enough.

**Theorem 1.2 (Osi06 Th. 1.7, Def. 2.29).** $G$ is hyperbolic relative to $H_1, \ldots, H_n$ if and only if the exact presentation $X_{\infty}$ satisfies a linear isoperimetric inequality.

The subgroups $H_1, \ldots, H_n$ of $G$ are called the maximal parabolic subgroups. Since there is no risk of confusion, we will simply call them parabolic subgroups.

**Remark 1.3.** Osin includes all words of any length in the multiplication table. One easily checks that this does not change the result.

In section 3, we are going to prove that if $X_{\infty}$ satisfies a linear isoperimetric inequality, so does $X_{\rho}$ for $\rho$ large enough. This will easily imply that parabolic subgroups are finitely presented.

### 1.3 Complexities

Since $X_{\rho}$ is an infinite presentation, it is convenient to have a measure of complexity for letters and words on $\hat{S}$. Recall that $\hat{S} = S \cup \tilde{H}_1 \cup \cdots \cup \tilde{H}_n$. For $a \in \hat{H}_i$, we denote by $|a|_{\hat{S}_i}$ the word length of $a$ relative to the generating set $\hat{S}_i$. We define the complexity $||a||$ of $a \in \hat{S}$ as 1 if $a \in S$, and as $|a|_{\hat{S}_i}$ if $a \in \tilde{H}_i$.

Given a word $w = a_1 \cdots a_n$ over $\hat{S}$, we define

- $\text{length}(w) = n$
- $||w||_1 = \sum_{i=1}^{n} ||a_i||$
- $||w||_\infty = \max_{i=1}^{n} ||a_i||$

Note that if $w$ is a one-letter word, then $||w||_1 = ||w||_\infty = ||w||$.

Similarly, if $D$ is a diagram (or a path) whose edges are labeled by elements of $\hat{S}$, we define $||D||_1$ and $||D||_\infty$ as the sum and the maximum of the complexities of the labels of its edges. For a labeled path $p$, $\text{length}(p)$ denotes its number of edges, and $\text{Area}(D)$ denotes the number of 2-cells of a diagram $D$. 
2 Diagrams

The goal of this section is to prove that if \( X_\rho \) does not satisfy a linear isoperimetric inequality, this shows up on diagrams of small area and small complexity (Proposition 2.9).

2.1 Vocabulary

**Thickness.** Let \( D \) be a Van Kampen diagram over the presentation \( X_\rho \) (\( \rho \) being fixed in \( \mathbb{N} \cup \{\infty\} \)). We denote by \( D_{\text{thick}} \subset D \) the union of all 2-cells, and of all vertices and edges that are contained in the boundary of a 2-cell. We say that \( D \) is **thick** if \( D = D_{\text{thick}} \) i. e. if every edge lies in the boundary of a 2-cell.

**Clusters.** We define cells of type \( \mathcal{R}' \) (resp. of type \( \tilde{H}_i \)) as those labeled by a word of \( \mathcal{R}' \) (resp. by a word in \( \mathcal{T}_{S_i}(\tilde{H}_i) \)). Note that two cells of type \( \tilde{H}_i \) and \( \tilde{H}_j \) cannot share an edge if \( i \neq j \). Two cells of the same type \( \tilde{H}_i \) and sharing an edge are said **cluster-adjacent**. A **cluster** is an equivalence class for the transitive closure of this relation. All 2-cells of a cluster have the same type \( \tilde{H}_i \), which we define as the type of the cluster. We identify a cluster with the closure \( C \) of the 2-cells it is made of. Note that clusters are contained in \( D_{\text{thick}} \).

If \( C \) is a cluster, we denote by \( \partial C \) (its boundary) the union of closed edges of \( C \) that are in only one 2-cell of \( C \).

*Remark 2.1.* Note that for any cluster \( C \), any edge in \( \partial C \setminus \partial D \) has complexity 1. Indeed, the 2-cell of \( D \setminus C \) containing this edge is labeled by a relator \( \tilde{s}^{-1}p_i(\tilde{s}) \) for some \( \tilde{s} \in S_i \).

2.2 Simply connected clusters, standard filling

![Diagram](image)

**Figure 1:** Standard filling.

Note that a cluster \( C \) (as a subset of the plane) is simply connected if and only if \( C \) is a disk and \( \partial C \) is an embedded circle in the plane. We will mostly deal with diagrams whose clusters are simply connected.

Consider a simply connected cluster \( C \), with \( \partial C \) labeled by the cyclic word \( a_1, \ldots, a_n \) (where each \( a_j \in \tilde{H}_i \)). A **standard filling** of \( \partial C \) is a diagram with
boundary $\partial C$, with $n - 2$ triangles as in figure\textup{[1]} all whose vertices are in $\partial C$, and whose interior edges are labeled by $a_1 \ldots a_j$ for $j \leq n - 2$, where $a_1 \ldots a_j$ is viewed as an element of $\hat{H}_i$.

**Lemma 2.2.** If $C$ is an arbitrary simply connected cluster, then $\|\partial C\|_1 \leq 3\text{Area}(D) + \|\partial D\|_1$.

If $C$ is standardly filled, then $\text{Area}(C) = \text{length}(\partial C) - 2$, and $\|C\|_\infty \leq \|\partial C\|_1$.

**Proof.** Let us partition $\partial C$ into edges that are in $\partial D$ and inner edges. There are at most $\text{length}(\partial C) \leq 3\text{Area}(D)$ inner edges, each of which is of complexity 1, by Remark 2.1. The sum of complexities of the edges in $\partial D$ is bounded by $\|\partial D\|_1$. This proves the first assertion. The second assertion is clear from the definition. \qed

**Remark 2.3.** If $C$ is any cluster, then $\text{Area}(C) \geq \text{length}(\partial C) - 2$. Indeed, Denoting by $F$, $E_{\text{int}}$, $E_{\text{ext}}$ the number of 2-cells, interior edges and boundary edges, connectedness of the dual graph implies $F - 1 \leq E_{\text{int}}$. Since cells of $C$ have at most 3 sides, $2E_{\text{int}} + E_{\text{ext}} \leq 3F$. It follows that $E_{\text{ext}} \leq F + 2$ as required.

The following lemma shows that in many situations, clusters are simply connected.

**Lemma 2.4.** Let $w$ be a word over $\hat{S}$ defining the trivial element in $G$. Let $D$ be a minimal Van Kampen diagram for $w$ over the presentation $X_\rho$. Assume that $\rho \geq 3\text{Area}(D)$.

If $D$ is chosen among diagrams for $w$ over $X_\rho$ to minimize successively the area, and the number of 2-cells of type $\mathcal{R}'$, then every cluster of $D$ is simply connected.

Assume either that $D$ is as above and that all its clusters are standardly filled, or that $D$ minimizes successively the area, the number of 2-cells of type $\mathcal{R}'$ and $\|D\|_\infty$. Then

$$\|D\|_\infty \leq 3\text{Area}(D) + \|w\|_1.$$  

**Proof.** Assume by contradiction that there exists a cluster $C$ of type $\hat{H}_i$ that is not simply connected. Then there is a simply connected subdiagram $D' \subset D$ such that edges of $\partial D'$ are all in $\partial C \setminus \partial D$. Since edges of $\partial D'$ lie in a 2-cell, $\text{length}(\partial D') \leq 3\text{Area}(D)$. Moreover $\|\partial D'\|_\infty = 1$, since by Remark 2.1 every edge in $\partial C \setminus \partial D$ has complexity 1. Thus, $\|\partial D'\|_1 \leq 3\text{Area}(D)$. Since $\rho \geq 3\text{Area}(D)$, the definition of $X_\rho$ says that the word labeled by $\partial D'$ is trivial in $\hat{H}_i$. One can then replace the subdiagram bounded by $c$ by a diagram with same combinatorics, and with cells of type $\hat{H}_i$. This contradicts the minimality of $D$ for the number of 2-cells of type $\mathcal{R}'$. It follows that all clusters of $D$ are simply connected.

Assume now that all clusters are standardly filled. By Lemma 2.2 for each cluster $C$, $\|C\|_\infty \leq \|\partial C\|_1 \leq 3\text{Area}(D) + \|w\|_1$. Since each edge of $D_{\text{thick}}$ of
complexity at least 2 is contained in a cluster, this implies that $\|D_{\text{thick}}\|_\infty \leq 3\text{Area}(D) + \|w\|_1$.

Finally, assume that $D$ minimizes successively the area, the number of 2-cells of type $R'$ and $|D|_\infty$. Since clusters of $D$ are simply connected, we can modify $D$ to a diagram $D'$ whose clusters are standardly filled, and having the same area and the same number 2-cells of type $R'$ as $D$. In particular, $|D|_\infty \leq |D'|_\infty$. By the argument above, $|D'|_\infty \leq 3\text{Area}(D) + \|w\|_1$ which concludes the proof. \hfill \Box

2.3 Complicated clusters

A cluster $C$ is said to be complicated if $\partial C \cap \partial D$ contains at least two edges.

Lemma 2.5. Assume that $D$ is a Van Kampen diagram, and $C \subset D$ is a simply connected cluster.

If $C$ is not complicated, then $\|\partial C\|_\infty \leq \text{length}(\partial C)$, $\|\partial C\|_1 \leq 2\text{length}(\partial C)$.

Proof. Denote by $\tilde{H}_i$ the type of the cluster $C$, so that edges of $C$ are labeled by elements of $\tilde{H}_i$. If $C$ is not complicated, all edges of $\partial C$ but one have complexity 1. The cluster being simply connected, the label of the remaining edge has the same image in $\tilde{H}_i$ as a product of $\text{length}(\partial C) - 1$ elements of $S_i$. Therefore, this edge has complexity at most $\text{length}(\partial C) - 1$. It follows that $\|\partial C\|_\infty \leq \text{length}(\partial C)$, and $\|\partial C\|_1 \leq (\text{length}(\partial C) - 1) + \sum_{e \in \partial C} 1$. This proves the lemma. \hfill \Box

Lemma 2.6 (See also [Osi06, Lemma 2.27]). Let $D$ be a Van Kampen diagram whose clusters are simply connected, non complicated, and standardly filled.

Then $\|D_{\text{thick}}\|_\infty \leq 6\text{Area}(D)$.

Proof. Any edge of $D_{\text{thick}}$ is either contained in a cell of type $R'$ (it has complexity 1) or in a cluster $C$. Since the number of edges of $D$ that lie in the boundary of a 2-cell is bounded by $3 \times \text{Area}(D)$, we have $\text{length}(\partial C) \leq 3 \times \text{Area}(D)$. Since $C$ is not complicated, $\|C\|_\infty \leq 6 \times \text{Area}(D)$ by Lemma 2.5. The lemma follows. \hfill \Box

2.4 Arcs-of-clusters and pieces

![Diagram](image)

Figure 2: 3 complicated clusters, 4 regular pieces, and 6 arcs-of-clusters
Consider a diagram $D$ whose clusters are simply connected. An arc-of-cluster is a maximal subpath $c \subset \partial C$ for some complicated cluster $C$ that does not contain any edge of $\partial D$ (see Figure 2). Since $\partial C$ is an embedded circle, each arc-of-circle $c$ is an embedded arc with endpoints in $\partial D$, and $c \cap \partial D$ contains no edge, but it may contain vertices distinct from its endpoints.

We define regular pieces of $D$ as the connected components of $D \setminus \mathcal{C}$ where $\mathcal{C}$ denotes the interior in $D$ of the union of all complicated clusters in $D$ (edges in $\partial D \cap \partial C$ for some complicated cluster are in $\mathcal{C}$), see Figure 2. Regular pieces and complicated clusters are called pieces.

Here is an alternative definition. For each complicated cluster $C$, consider properly embedded arcs with endpoints in $\partial D$, that are very close and parallel to each arc-of-cluster, obtained by pushing inside $C$ the arcs-of-clusters. Let $\mathcal{A}$ be the union of such embedded arcs when $C$ ranges over all complicated clusters. Then connected components of $S \setminus \mathcal{A}$ are in one-to-one correspondence with pieces. On figure 2, $\mathcal{A}$ is represented by dotted lines.

Clearly, the set of pieces induces a partition of the set of 2-cells of $D$. There is a natural incidence graph $\mathcal{G}$ for this partition, whose vertices are the pieces, whose edges are the arcs-of-clusters, the two endpoints of an edge being the cluster and the regular piece on both sides of the corresponding arc-of-cluster. 

**Lemma 2.7.** Let $D$ be a Van Kampen diagram, and assume that any cluster of $D$ is simply connected.

The incidence graph $\mathcal{G}$ is a bipartite tree and the degree of a vertex $v$ associated to a complicated cluster $C$ is at most the number of edges in $\partial D \cap \partial C$, with strict inequality when the vertex is $v$ is a leaf of the tree $\mathcal{G}$.

**Proof.** The graph is bipartite by definition. It is connected because $D$ is. Since every arc-of-cluster separates $D$, every edge of the incidence graph disconnects it. This proves that $\mathcal{G}$ is a tree.

Consider a vertex $v$ associated to a complicated cluster $C$. The degree of $v$ is, by definition, the number of arcs-of-clusters on $\partial C$. Since $C$ is simply connected, $\partial C$ is an embedded circle, and since $C$ is complicated, $\partial C$ contains an edge of $\partial D$. By maximality in the definition of arc-of-clusters, each such arc is followed in $\partial C$ (with a chosen fixed orientation) by an edge of $\partial C \cap \partial D$. This association, which is clearly one-to-one, ensures the bound on the degree.

Finally, if $v$ is a leaf of $\mathcal{G}$, its degree is 1 and $\partial D \cap \partial C$ contains at least 2 edges because $C$ is complicated.

The following result of [Dah08] was, to some extend, left to the reader. We include a proof.

**Lemma 2.8.** Let $D$ be a Van Kampen diagram. If every cluster is simply connected, then the number of pieces, and the number of arc-of-clusters are both bounded by $\operatorname{length}(\partial D)$.

**Proof.** The number $N$ of pieces is the number of vertices of the incidence graph $\mathcal{G}$. Since $\mathcal{G}$ is a tree, $N = E + 1$ where $E$ is the number of edges of $\mathcal{G}$, i.e. the number of arcs-of-clusters. Denote by $v_C$ the vertex corresponding to a
cluster $C$, by $d(v_C)$ its degree, and by $V_{cl}$ the set of all vertices of $G$ corresponding to clusters. Since $G$ is bipartite, $E = \sum_{v_C \in V_{cl}} d(v_C)$. By lemma 2.7, $d(v_C)$ is bounded by the number $e(C)$ of edges of $\partial C \cap \partial D$. Therefore $E \leq \sum_{v_C \in V_{cl}} e(C) \leq \text{length}(\partial D)$.

Finally, if some $v_C$ is a leaf of $G$, this last inequality is a strict inequality, which yields $N = E + 1 \leq \text{length}(\partial D)$. There remains the case where some leaf of $G$ is a regular piece $B$. This means that $\partial B = \alpha \cup \beta$ where $\alpha$ is an arc-of-cluster, and $\beta$ is a path in $\partial D$. Since clusters are simply connected, the endpoints of $\alpha$ are distinct, so $\beta$ contains at least an edge. This implies that $\sum_{v_C \in V_{cl}} e(C) < \text{length}(\partial D)$, and concludes the lemma. 

2.5 Reduction to diagrams of small complexity

We are now ready to state and prove the main statement of this section. It claims that if $X_\rho$ does not satisfy a linear isoperimetric inequality, this shows up on diagrams of small area (this is Papasoglu’s theorem) and small complexity.

**Proposition 2.9** ([Dah08, Prop. 1.5]). Let $K \geq 10^6$ and $\rho \in \mathbb{N} \cup \{\infty\}$, $\rho \geq 3 \times 10^4 K$.

Assume that $X_\rho$ fails to satisfy a linear isoperimetric inequality of constant $K$ (that is, there exists a word $w$ over the alphabet $\hat{S}$ such that $\text{Area}(w) > K \text{Length}(w)$).

Then, there exists a word $w''$ over the alphabet $\hat{S}$, and a minimal Van Kampen diagram $D''$ (over $X_\rho$) for $w''$, such that

1. $\text{Area}(D'') \leq 240 K$
2. $\|D''\|_{\infty} \leq 2 \times 10^6 K^2$
3. $\text{Area}(D'') > \sqrt{K} \times \text{length}(\partial D'')$.

*Proof.* The first step is to apply Papasoglu’s Theorem 1.1 to the presentation $X_\rho$ to obtain a word $w'$ over $\hat{S}$ for which $K/2 \leq \text{Area}(w') \leq 240 K$, and $\text{Area}(w') > \sqrt{K} \times \text{length}(w')$.

Using $\sqrt{\text{Area}(w')} > \frac{\text{length}(\partial w')}{\sqrt{2 \times 10^4}}$ and $\text{Area}(w') \geq K/2$, we get

$$\text{Area}(w') > \sqrt{\frac{\text{Area}(w')}{2 \times 10^4}} \times \text{length}(w') \geq \sqrt{\frac{K}{200}} \times \text{length}(w').$$

Choose a diagram $D'$ among minimal area diagrams over $X_\rho$ for $w'$ so that the number of 2-cells of type $R'$ is minimal. We claim that up to changing $w'$, we can assume that $D'$ is thick i. e. all edges lie in the boundary of a 2-cell. Indeed, if all connected components $A'_1, \ldots, A'_l$ of $D'_{\text{thick}}$ satisfy $\text{Area}(A'_i) \leq \sqrt{\frac{K}{200}} \times \text{length}(\partial A'_i)$, then

$$\text{Area}(D') = \sum_i \text{Area}(A'_i) \leq \sqrt{\frac{K}{200}} \sum_i \text{length}(\partial A'_i) \leq \sqrt{\frac{K}{200}} \times \text{length}(w').$$
a contradiction. It follows that some component \( A'_i \) satisfies \( \text{Area}(A'_i) > \frac{200}{2100} \times \text{length}(\partial A'_i) \). Obviously, \( \text{Area}(A'_i) \leq \text{Area}(D') \leq 240K \), and \( A'_i \) is a diagram for \( \partial A'_i \) that minimizes the area and the number of cells of type \( R' \) (if not, substituting a diagram of smaller area for \( \partial A'_i \) in \( D' \) contradicts minimality of \( D' \)). This proves that we can assume that \( D' \) is thick.

We do not have any control on the complexity of a diagram filling \( w' \) yet. By choice of \( \rho \), Lemma 2.4 shows that the clusters of \( D' \) are simply connected. We can modify \( D' \) and assume that all clusters are standardly filled. By Remark 2.3 \( D' \) still minimizes area and the number of cells of type \( R' \). By Lemma 2.8 the number of pieces in the decomposition into complicated clusters and regular pieces is at most length(\( \partial D' \)).

![Figure 3: Adding chords to the pieces of \( D' \), and regluing them together](image)

Let \( C'_1, \ldots, C'_s \) be the complicated clusters of \( D' \), and \( D'_1, \ldots, D'_r \), be the regular pieces. We construct new diagrams \( C''_1, D''_1 \), and \( C''_2, D''_2 \) from \( C'_1, D'_1 \) by first adding chords, then by changing the triangulation as follows (see Figure 3).

Fix a complicated cluster \( C'_k \) of \( D' \), and denote by \( \tilde{H}_i \) its type. Its boundary \( \partial C'_k \) is a union of pairwise disjoint arcs-of-clusters, together with arcs in \( \partial D' \). Consider an arc-of-cluster \( c \subset \partial C'_k \) whose edges are labeled by elements \( a_1, \ldots, a_n \) of \( \tilde{H}_i \), and let \( a_c = a_1 \ldots a_n \in \tilde{H}_i \) be their product. We glue along \( c \) a standardly filled disk with boundary labeled by \( a_1, \ldots, a_n, a_c^{-1} \). We name the new edge labeled by \( a_c^{-1} \) a chord. Performing this operation for each arc-of-cluster, we get a disk \( C''_k \) made of cells of type \( \tilde{H}_i \). Finally, we change the triangulation of this disk to a standard filling, and we call \( \tilde{C''}_k \) the obtained diagram. Note that \( \text{Area}(\tilde{C''}_k) \leq \text{length}(\partial \tilde{C''}_k) - 2 \).

Now, we perform a similar operation for each regular piece \( D'_j \). For each arc-of-cluster \( c \subset \partial D'_j \) labeled by \( a_1, \ldots, a_n \in \tilde{H}_i \) (now, the type \( \tilde{H}_i \) may depend on...
c), we define \( a_c = a_1 \ldots a_n \in \hat{H}_k \), and glue to \( C_k' \) along \( c \) a new cluster of type \( \hat{H}_k \), standardly filled, whose boundary is labeled by \( a_1, \ldots, a_n, a_c^{-1} \). Since the filling is standard, the area of the added cluster is \((n + 1) - 2 = \text{length}(c) - 1\). Performing this operation for each arc-of-cluster, we get the new diagram \( D_j'' \). Finally, we take for \( \tilde{D}_j'' \) a diagram with boundary \( \partial D_j'' \), and minimizing successively the area and the number of 2-cells of type \( \mathcal{R}' \).

We are going to bound \( \| \tilde{D}_j'' \|_\infty \) by first bounding \( \| D_j'' \|_\infty \). Since all complicated clusters of \( D' \) are \( C_1', \ldots, C_r', D_j'' \) has no complicated cluster coming from \( D' \). The newly created clusters in \( D_j'' \) have just one edge in \( \partial D_j'' \), so are not complicated. Therefore, clusters of \( D_j'' \) are not complicated, simply connected, and standardly filled. Since \( D' \) is thick, so is \( D_j'' \). Applying Lemma 2.6 to \( D_j'' \), we get \( \| D_j'' \|_\infty \leq 6 \times \text{Area}(D_j'') \leq 6 \times 240K \).

In particular, \( \| \partial \tilde{D}_j'' \|_\infty = \| \partial D_j'' \|_\infty \leq 6 \times 240K \), and since \( D_j'' \) is thick, \( \| \partial D_j'' \|_1 \leq 3 \text{Area}(D_j'') \| \partial D_j'' \|_\infty \leq 18 \times (240K)^2 \). Applying Lemma 2.4 to \( \tilde{D}_j'' \), we get

\[
\| \tilde{D}_j'' \|_\infty \leq 3 \text{Area}(D_j'') + \| \partial D_j'' \|_1 \leq 3 \times 240K + 18 \times (240K)^2 \leq 2.10^6 K^2.
\]

This proves that for all \( j \in \{1, \ldots, r\} \), \( \tilde{D}_j'' \) satisfies assertions (1) and (2) of the proposition.

We now prove that one of the diagrams \( \tilde{D}_j'' \), \( j = 1, \ldots, r \) must satisfy (3). Assume by contradiction that for all \( j \in \{1, \ldots, r\} \), \( \text{Area}(\tilde{D}_j'') \leq \frac{\sqrt{K}}{600} \text{length}(\partial \tilde{D}_j'') \).

Note that \( \check{C}_k'' \) satisfies this inequality as well. Indeed, \( \text{Area}(\check{C}_k'') \leq \text{length}(\partial \check{C}_k'') \), and by assumption, \( K \geq 10^6 \) so \( \frac{\sqrt{K}}{600} \geq 1 \).

Gluing together the diagrams \( \tilde{D}_1'', \ldots, \tilde{D}_r'' \) and \( \check{C}_1'', \ldots, \check{C}_r'' \) pairwise along the two chords corresponding to a given arc-of-cluster as shown on Figure \( \text{X} \), we get another (non necessarily minimal) Van Kampen diagram \( \tilde{D}' \) for \( w' \).

We have

\[
\text{Area}(D') \leq \text{Area}(\tilde{D}') = \sum_j \text{Area}(\tilde{D}_j'') + \sum_k \text{Area}(\check{C}_k'') \\
\leq \frac{\sqrt{K}}{600} \left( \sum_j \text{length}(\partial \tilde{D}_j'') + \sum_k \text{length}(\partial \check{C}_k'') \right) \\
\leq \frac{\sqrt{K}}{600} \left( \text{length}(\partial D') + 2n_a \right)
\]

where \( n_a \) is the number of arcs-of-clusters in \( D' \). By lemma 2.8, \( n_a \leq \text{length}(\partial D') \), so \( \text{Area}(D') \leq \frac{\sqrt{K}}{600} \times \text{length}(\partial D') \), thus contradicting the property of \( D' \) established at the beginning of the proof.

\section{3 Consequences}

\textbf{Corollary 3.1.} Assume that \( X_\infty \) satisfies a linear isoperimetric inequality of factor \( K \geq 10^6 \). Let \( K' = (600K)^2 \) and \( \rho(K) = 10^{26} K^5 \). Then for all \( \rho \geq \rho(K) \),
Proof of Corollary 3.1. Assume that $X_{\rho}$ fails to satisfy the predicted isoperimetric inequality (of factor $K'$), and argue towards a contradiction. By Proposition 2.9 there is a word $w''$ representing the trivial element, with a diagram $D''$, minimal over the presentation $X_{\rho}$, of area at most $240K'$, and complexity $\|D''\|_{\infty} \leq 2.10^6K'^2$ and such that Area$(D'') > K \times \text{length}(w'')$.

Consider the map $\rho : \hat{S}_{\rho} \to \hat{S}_\infty$ described above. Choose $D_0''$ among diagrams for $p(w'')$ in the presentation $X_\infty$, in order to minimize successively the area, the number of 2-cells of type $\mathcal{R}'$, and the complexity $\|D_0''\|_{\infty}$. Since $X_\infty$ satisfies a linear isoperimetric inequality of factor $K$, Area$(D_0'') < \text{Area}(D'') \leq 240K'$. By Lemma 2.4, $\|D_0''\|_{\infty} \leq 720K^2 + \|p(w'')\|_{\infty}$. On the other hand,

$$\|p(w'')\|_{\infty} \leq |w''| \leq \text{length}(w'') \|D''\|_{\infty} \leq \frac{1}{K} \text{Area}(D'') \times 2.10^6K'^2 \leq \frac{240K'}{K} \times 2.10^6K'^2 \leq 3.10^{25}K^5.$$

Since $K \geq 10^6$, $720K' \leq 10^9K^2 \leq K^5$. By hypothesis on $\rho$, we see that $\|D_0''\|_{\infty} \leq \rho/3$. It follows that $p_{w''}(D)$ is a diagram over $X_\rho$ for $w''$, of area $< \text{Area}(D'')$, a contradiction.

\begin{lemma}
Assume that $X_{\rho}$ satisfies a linear isoperimetric inequality of factor $K'$ with $\rho \geq \max(3K', 2)$.

Then $p_i : \hat{H}_i \to H_i$ is an isomorphism. In particular, $H_i$ is finitely presented, with a presentation whose defining relations are of length $\leq \rho$.

\end{lemma}

\begin{proof}
Assume by contradiction that $p_i : \hat{H}_i \to H_i$ is not injective, and consider $a \in \ker p_i \setminus \{1\}$. Then $a$ is a generator of the presentation $X_\rho$ that represents the trivial element of $G$. Note that since $\rho > 1$, $a \notin \hat{S}_i$. Therefore, there exists
a Van Kampen diagram $D$ over $X_\rho$ whose boundary consists of a single edge $e$ labeled $a$, and whose area is at most $K'$. We choose a diagram for $a$ over $X_\rho$ in order to minimize successively the area, the number of 2-cells of type $R'$, and $\|D\|_{\infty}$. Since $\rho \geq 3K'$, Lemma 2.4 implies that clusters of $D$ are simply connected. Since $a \notin \tilde{S}_i$, $e$ lies in a cluster $C$ of type $\tilde{H}_i$. But since $C$ is simply connected, and since a cluster of type $\tilde{H}_i$ involves only relations of $\tilde{H}_i$, we get that $a$ is trivial in $\tilde{H}_i$, a contradiction. 

Corollary 3.3. Assume that $X_\infty$ satisfies a linear isoperimetric inequality of factor $K$.

Then the subgroups $P_i$ are finitely presented, with a presentation whose defining relations are of length $\leq \rho(\max(K, 10^6))$. 

Proof. Without loss of generality, we can assume $K \geq 10^6$. By Corollary 3.1, $X_{\rho(K)}$ satisfies a linear isoperimetric inequality of factor $K' = (600K)^2$. Lemma 3.2 concludes. 

Lemma 3.4 (see also [Osi06, Lemma 5.4]). Assume that $X_\infty$ satisfies a linear isoperimetric inequality of factor $K$.

If $s \in S$ represents an element $a$ of $H_i$, then $\|a\| \leq 12K$. 

Proof. The word $w = sa$ is a word of length 2 over $X_\infty$. If it represents the trivial element in $G$, then there is a Van Kampen diagram $D$ over $X_\rho$ whose boundary is a path of length 2 labeled $sa$, and whose area is at most $2K$. We choose $D$ among minimal area diagrams over $X_\infty$ for $w$ so that the number of 2-cells of type $R'$ is minimal. Since $\rho = \infty$, Lemma 2.4 implies that clusters of $D$ are simply connected, and we can assume that they are standardly filled.

Note that there is no complicated cluster as only the edge labeled $a$ of $\partial D$ can be in a cluster. By Lemma 2.6, this implies that $\|D\|_{\text{thickness}} \leq 12K$, so $\|a\| \leq 12K$. 

We obtain the following improvement of [Dah08]:

Corollary 3.5. There exists an algorithm that takes as input a finite presentation of a group $G$, a solution of its word problem, and a collection of finite subsets $S_1, \ldots, S_n \subset G$, and that terminates if and only if $G$ is hyperbolic relative to $\langle S_1 \rangle, \ldots, \langle S_n \rangle$.

In this case, produces an isoperimetry constant $K$ for the presentation $X_\infty$, a finite presentation for each of the parabolic subgroups, and says whether $G$ is parabolic (i.e. $G = \langle S_i \rangle$ for some $i$).

Proof. For a fixed $K \geq 10^6$, we consider all diagrams $D$ over $X_\infty$ such that $\|D\|_{\infty} \leq B = 2.10^6K^2$ and Area($D$) $\leq 240K$. There are only finitely many. The word problem in $G$ allows to list all relators of $\langle S_i \rangle$ of length at most $3B$, and hence to list these diagrams. Out of this list, we make the list $W(K)$ of words labeling the boundaries of these diagrams.

We claim that given $w \in W(K)$, we can compute Area($w$). Indeed, let $D'$ be a diagram for $w$ chosen to minimize area, the number of cells of type $R'$,
and $\|D\|_\infty$. By Lemma 2.4, $\|D\|_\infty \leq 3\text{Area}(D') + \|w\|_1 \leq 720K + \|w\|_1$. We can compute an upper bound $M \geq 720K + \|w\|_1$ for $\|D\|_\infty$, and we can list all diagrams $D'$ with $\text{Area}(D') \leq 240K$ and $\|D\|_\infty \leq M$ whose boundary is $w$. We can then compute $\text{Area}(w)$ as the minimal area of these diagrams.

Now we can check whether $\text{Area}(w) \leq \sqrt{\frac{K}{600}} \text{length}(w)$ for all $w \in W(K)$. If this is not the case, the algorithm increments $K$ and starts over.

If this is the case, then by Proposition 2.3, $X_\infty$ satisfies isoperimetric inequality of factor $K$, and the algorithm stops. It outputs $K$, and gives as set of relators for $\langle S_i \rangle$, the set of all words of length $\leq \rho(K)$ that are trivial in $G$; this can be done using the word problem in $G$, and this is indeed a presentation of $\langle S_i \rangle$ by Lemma 3.3. To check whether $G = \langle S_i \rangle$, one needs to check whether each $s \in S$ represents an element $a \in \langle S_i \rangle$. Lemma 3.3 bounds the complexity of $a$, and we can try all possibilities for $a$ using the word problem.

If $X_\infty$ does satisfy a linear isoperimetric inequality of factor $K_0$, then the process will obviously stop when $K$ will reach a value greater than $(600K_0)^2$.

**Corollary 3.6.** There exists an algorithm as follows. It takes as input a finite presentation of a group $G$, a solution for its word problem, and a recursive class of finitely presented groups $C$ (given by a Turing machine enumerating them). It terminates if and only if $G$ is properly hyperbolic relative to subgroups that are in the class $C$.

In this case, the algorithm produces an isoperimetry constant $K$, a generating set and a finite presentation for each of the parabolic subgroups.

**Proof.** First, enumerate all possible presentations of groups in $C$ using the Turing machine given as input, and Tietze transformations. In parallel, list all possible families of finite subsets $S = (S_1, \ldots, S_n)$ of $G$. For each of them, run in parallel the algorithm of Corollary 3.5 that stops if $G$ is hyperbolic relative to $\langle S_1 \rangle, \ldots, \langle S_n \rangle$ and outputs a presentation of $\langle S_i \rangle$ in this case, and says whether $G$ is parabolic. Get rid of those $S$ such that $G$ is parabolic.

Then stop if at some point, one sees that in some of the produced presentations, $\langle S_i \rangle$ lie in $C$. 

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François Dahmani
Institut Fourier,
Université Joseph Fourier (Grenoble 1)
BP 74,
F-38402 St Martin d’Hres cédex, France
francois.dahmani@ujf-grenoble.fr

Vincent Guirardel
Institut de Recherche en Mathématiques de Rennes (IRMAR)
Université de Rennes 1
263 avenue du Général Leclerc, CS 74205
F-35042 Rennes cédex, France
vincent.guirardel@univ-rennes1.fr