Spanning Analysis of Stock Market Anomalies under Prospect Stochastic Dominance

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Abstract

We develop and implement methods for determining whether introducing new securities or relaxing investment constraints improves the investment opportunity set for prospect investors. We formulate a new testing procedure for prospect spanning for two nested portfolio sets based on subsampling and Linear Programming. In an application, we use the prospect spanning framework to evaluate whether well-known anomalies are spanned by standard factors. We find that of the strategies considered, a few expand the opportunity set of the prospect type investors, thus have real economic value for them. In-sample and out-of-sample results prove remarkably consistent in identifying genuine anomalies for prospect investors.

Keywords and phrases: Nonparametric test, prospect stochastic dominance efficiency, prospect spanning, market anomaly, Linear Programming.

JEL Classification: C12, C14, C44, C58, D81, G11, G40.

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1 Introduction

Traditional models in economics and finance assume that investors evaluate portfolios according to the expected utility framework. The theoretical motivation for this goes back to Von Neumann and Morgenstern (1944). Nevertheless, experimental and empirical work has shown that people systematically violate expected utility theory when choosing among risky assets. Prospect theory, first described by Kahneman and Tversky (1979) (see also Tversky and Kahneman (1992)), is widely viewed as a better description of how people evaluate risk in experimental settings. While the theory contains many remarkable insights, it has proven challenging to apply these insights in asset pricing, and it is only recently that there has been real progress in doing so (Barberis et al. (2021)). Barberis and Thaler (2003) and Barberis (2013) are excellent reviews on behavioral finance and prospect theory.

Stock market anomalies are key drivers of innovation in asset pricing. These are tradable portfolio strategies, usually constructed as long-short portfolios based on the top and bottom deciles of sorted stocks, according to specific characteristics (anomalies). Under the standard Mean-Variance (M-V) paradigm, establishing a cross-sectional return pattern as an anomaly involves testing for pricing based on a factor model. If factors are traded, spanning regressions relate to M-V criterion. Arbitrage pricing stipulates that a portfolio of factors is M-V efficient and no other portfolio can achieve a higher Sharpe Ratio (SR). In that sense, an anomaly is a strategy that exhibits higher SR and should be traded away. However, we can question M-V spanning for portfolio selection if returns do not follow elliptical distributions, or investor preferences depend on more than the first two moments of the return distribution. Chalamandaris et al. (2020) compare the M-V spanning and second-order stochastic spanning of 13 standard empirical asset pricing anomalies under various factor models using the Huberman-Kandel M-V spanning test (Huberman and Kandel (1987)). They show that the M-V spanning tests do not reconcile in-sample and out-of-sample. In the in-sample tests, almost all these anomalies reject the M-V spanning, but out-of-sample only a few
of them are genuine anomalies. On the contrary, the results under second-order stochastic spanning match in-sample and out-of-sample. Our paper shows that it holds as well for prospect stochastic spanning on those 13 anomalies and 5 additional ones used by Barberis et al. (2021). Besides, experimental evidence (Baucells and Heukamp (2006)) suggests that investors do not always act as risk averters. Instead, they behave in a much more complex fashion, exhibiting characteristics of both risk-loving and risk-averting. They behave differently on gains and losses, and they are more sensitive to losses than to gains (loss aversion). The relevant utility function could be concave for gains and convex for losses (S-Shaped).

The present study contributes to this literature by introducing, operationalizing and applying new prospect spanning tests for portfolio analysis. The general research question is whether a given investment possibility set $\mathbb{K}$, namely the benchmark set, contains portfolios which prospect dominate all alternatives in an expanded investment possibility set $\mathbb{L}$. Imposing less restrictive assumptions and allowing for risk-seeking preferences, prospect spanning tests may include portfolios in the efficient set that the M-V criterion may exclude. On the other hand, the less informationally demanding M-V criterion may include portfolios that the stochastic spanning criterion, i.e., the more informationally demanding one, may exclude from the efficient set. Therefore, efficient portfolio sets under the stochastic spanning and M-V criterion may be nonnested, in the sense that neither of them is a subset of the other. The use of a M-V criterion by a prospect investor might induce an opportunity cost coming from an expected utility loss caused by wrong decision making in terms of investments.

Stochastic spanning (Arvanitis et al. (2019)) is a model-free alternative to M-V spanning of Huberman and Kandel (1987) (see also Jobson and Korkie (1989), De Roon et al. (2001)). Spanning occurs if introducing new securities or relaxing investment constraints does not improve the investment possibility set for a given class of investors. M-V spanning checks if the M-V frontier of a set of assets is identical to the M-V frontier of a larger set made of those assets plus additional assets (Kan and Zhou (2012), Penaranda and Sentana (2012)). Here, we investigate such a problem for investors with prospect type preferences which are
interested in the whole return distributions generated by two sets of assets, namely we study stochastic dominance. First, we introduce the concept of prospect spanning, which is consistent with prospect type investors. We propose a theoretical measure for prospect spanning based on stochastic dominance and derive the exact limit distribution for the associated test statistic, its empirical counterpart, for a general class of dynamic processes. To check prospect spanning on data, we develop consistent and feasible testing procedures based on subsampling and Linear Programming (LP).

Similarly to Arvanitis et al. (2019), it is easy to see that if the prospect efficient set is non-empty, a prospect spanning set is essentially any superset of the former. As such, we can use a prospect spanning set to provide an outer approximation of the efficient set. It is useful in at least two ways. First, if the spanning set is small enough, the problem of optimal choice is reduced to a potentially simpler problem. Indeed, a spanning set is a reduction of the original portfolio set without loss of investment opportunities for any investor with S-shaped preferences. Second, if an algorithm for the choice of non-trivial candidate spanning sets is available, we can use it to construct decreasing sequences of prospect spanning sets that ensure to converge to the efficient set. Given the complexity of the prospect efficient set (see for example Ingersoll (2016)), such an approach can be useful for the determination of its properties.

The second contribution of the paper is to examine if well-known stock market anomalies expand the investment opportunity set for prospect investors. To do so, we test if trading strategies are genuine violations of standard factor models. More precisely, in the in-sample analysis, we use the prospect spanning test in order to check whether a portfolio set originating from a standard factor model, $\mathbb{K}$, spans the same set augmented with a market anomaly, $\mathbb{L}$. This check is relevant to the empirical analysis of financial markets. If the hypothesis of prospect spanning holds, the particular market anomaly can be explained by the factor model. Then, the trading strategy that is identified in the literature as market anomaly may not be an attractive investment opportunity for prospect investors. On the contrary, if the
hypothesis is not true, the anomaly expands the opportunity set for prospect investors, and is useful to that extent. It explains the reason of the focus of our analysis on the notion of spanning, instead of the stronger notion of stochastic bounding of Arvanitis et al. (2021). Rejection of bounding does not necessarily identify a fixed prospect investment opportunity inside $L - K$. We also examine whether the cross-sectional patterns that found to expand the set of factors in-sample, maintain their abnormal returns out-of-sample. Therefore, we use out-of-sample backtesting experiments as an independent criterion for robustness of in-sample test results (Harvey et al. (2016)). It turns out that prospect spanning tests produce remarkably consistent results both in- and out-of-sample in identifying trading strategies as genuine market anomalies for prospect investors. Thus, our framework helps validating stock market anomalies for prospect preferences.

The third contribution of the paper is to compare prospect spanning with M-V spanning both in- as well as out-of-sample. That comparison reveals several differences, for example, in terms of portfolio performance and weight allocations. Here, we opt for the portfolio perspective, and we contribute to the anomalies literature by asking whether the so-called anomalies are a good investment opportunity for prospect investors. Prospect theory is a valid alternative to M-V for building portfolios (and judging anomalies as investment opportunities). Andrew (2014) as well as Cochrane’s NBER keynote speech (2021), criticise M-V optimizers as too sensitive on mean and covariance estimates. The portfolio optimization using average return and covariance matrix estimates could be devilishly unstable. They both argue for using alternative utility functions when building portfolios. In our paper, we test whether well-known factor models span the augmented universe with a prominent stock market anomaly, and if not, whether the result is supported out-of-sample.

Let us briefly review applications of prospect theory in finance. Benartzi and Thaler (1995) utilize prospect theory to present an approach called myopic loss aversion which consists of two behavioural concepts, namely loss aversion and mental accounting. Barberis et al. (2001) study asset prices in an economy where investors derive direct utility not only
from consumption but also from fluctuations in the value of their financial wealth. They are loss averse over these fluctuations and how loss averse they are depends on their prior investment performance. The design of their model is influenced by prospect theory. Barberis and Huang (2008) study the pricing of financial securities when investors make decisions according to cumulative prospect theory. Several other papers confirm that positively skewed stocks have lower average returns (Boyer et al. (2010), Bali et al. (2011), Kumar (2009), Conrad et al. (2013)). Barberis and Xiong (2009, 2012) and Ingersoll and Jin (2013) show that theoretical investment models based on S-Shape utility maximisers help to understand the disposition effect found empirically in many studies (see, e.g., Odean (1988), Grinblatt and Han (2005), Frazzini (2006), Calvet et al. (2009)). Kyle et al. (2006) provide a formal framework to analyze the liquidation decisions of economic agents under prospect theory. He and Zhou (2011) study the impact of prospect theory on optimal risky exposures in portfolio choice through an analytical treatment. Ebert and Strack (2015) set up a general version of prospect theory and prove that probability weighting implies skewness preference in the small. Barberis et al. (2016) test the hypothesis that, when thinking about allocating money to a stock, investors mentally represent the stock by the distribution of its past returns and then evaluate this distribution in the way described by prospect theory. Moreover, Barberis et al. (2021) present a model of asset prices in which investors evaluate risk according to prospect theory and examine its ability to explain prominent stock market anomalies.

The paper is organised as follows. In Section 2, we review the definition of prospect stochastic dominance relation and we define the relevant concept of prospect spanning. We provide with a new representation of the relation, based on a class of S-shaped utility functions constructed as convex mixtures of appropriate “ramp functions”, in the spirit of Russel and Seo (1989). It avoids the differentiability assumption in the representation of Levy and Levy (2002) and constitutes a convenient setting for numerical analysis as well as economic interpretation in terms of expected utility. Using an empirical approximation of the latter, we construct a test for the null hypothesis of spanning based on subsampling. The con-
struction is based on the limiting null distribution of the test statistic which has the form of a saddle-type point of a zero mean Gaussian process. We specify a weak condition on the structure of the parameter contact sets, under which we show that the test is asymptotically exact and consistent. It avoids the tenuous comparison of the extreme points of the parameter sets used in Arvanitis et al. (2020) to obtain exactness in large samples.

In Section 3, we provide with a numerical approximation of the statistic that is based on the utility representation derived before. The utility functions are univariate, and normalized. We use a finite set of increasing piecewise-linear functions, restricted to the bounded empirical supports, that are compatible with the aforementioned representation. For every such utility function, we solve two embedded linear maximization problems. It is an improvement over the implementation in Arvanitis and Topaloglou (2017) and Arvanitis et al. (2020) where they formulate tests in terms of Mixed-Integer Programming (MIP) problems. MIP problems are NP-complete, and far more difficult to solve. Our numerical approximations are simple and fast since they are based on standard LP. They suit better resampling methods, which otherwise become quickly computationally demanding in empirical work. We also show that the numerical approximation converges to the test statistic for each sample size, when the number of piecewise linear components of the utilities approaches infinity.

In Section 4, we perform an empirical application where we use the prospect spanning tests to evaluate stock market anomalies using standard factor models. We consider three such models that build on the pioneer three-factor model of Fama and French (1993): the four-factor model of Hou et al. (2015), the five-factor model of Fama and French (2015), and the four-factor model of Stambaugh and Yuan (2017). Given the extensive set of results produced under alternative spanning criteria, the analysis is confined to 11 well-known strategies used to construct Stambaugh-Yuan factors, along with 7 extra (18 overall) that attracted significant attention, namely Betting against Beta, Quality minus Junk, Size, Growth Option, Value (Book to Market), Idiosyncratic Volatility and Profitability. The 11 anomalies used in Stambaugh and Yuan (2017) are realigned appropriately to yield positive average returns. In
particular, anomaly variables that relate to investment activity (Asset Growth, Investment to Assets, Net Stock Issues, Composite Equity Issue, Accruals) are defined low-minus-high decile portfolio returns, rather than high-minus-low. All the other anomalies are constructed as high-minus-low decile portfolio returns. These 18 trading strategies constitute our playing field for comparing spanning test results. Yet, we emphasize that this paper is not intended to compare factor models in terms of their ability to capture the cross-section of expected returns under prospect preferences. Instead, we use alternative factor models as a robustness check for testing the consistency of in- and out-of-sample results under the prospect spanning framework. Each factor model is our initial system of investment coordinates which we take as a granted opportunity set, without questioning its asset pricing validity. We view here the factors solely as investable assets (since they correspond to tradable strategies based on asset portfolios), and similarly for the anomalies. The anomalies might be labelled by other authors as factors if indeed priced in the cross-section, but we do not address such a research question in this paper. In the empirical analysis under our portfolio perspective, we also investigate the post-publication period for each anomaly as in Chinco et al. (2021) to test which anomalies survive after publication. Finally, we conduct M-V spanning tests and compare the results both in-sample and out-of-sample.

Finally, Section 5 concludes the paper. In Appendix A, we provide a short description of the stock market anomalies used in the empirical application. In Appendix B, we also provide a short description of the performance measure used in the out-of-sample analysis. We give in a separate Online Appendix: i) the limiting properties of the testing procedures under sequences of local alternatives, ii) a Monte Carlo study of the finite sample properties of the test, iii) the proofs of the main results, as well as several auxiliary lemmata and their proofs, iv) summary statistics of the factor and anomaly returns over our sample period from January 1974 to December 2016, and v) additional empirical results on out-of-sample analysis of market anomalies.
2 Prospect Stochastic Dominance and Stochastic Spanning

The theory of stochastic dominance (SD) gives a systematic framework for analyzing investor behavior under uncertainty (see Chapter 4 of Danthine and Donaldson (2014) for an introduction oriented towards finance). Stochastic dominance ranks portfolios based on general regularity conditions for decision making under risk (see Hadar and Russell (1969), Hanoch and Levy (1969), and Rothschild and Stiglitz (1970)). SD uses a distribution-free assumption framework which allows for nonparametric statistical estimation and inference methods. We can see SD as a flexible model-free alternative to M-V dominance of Modern Portfolio Theory (Markowitz (1952)). The M-V criterion is consistent with expected utility for elliptical distributions such as the normal distribution (Chamberlain (1983), Owen and Rabinovitch (1983), Berk (1997)), but has limited economic meaning when we cannot completely characterize the probability distribution by its location and scale. Simaan (1993), Athayde and Flores (2004), and Mencia and Sentana (2009) develop a mean-variance-skewness framework based on generalizations of elliptical distributions that are fully characterized by their first three moments. SD presents a further generalization that accounts for all moments of the return distributions without necessarily assuming a particular family of distributions.

Inspired by previous work, Levy and Levy (2002) formulate the notions of prospect stochastic dominance (PSD) (see also Levy and Wiener (1998), Levy and Levy (2004)) and Markowitz stochastic dominance (MSD). Those notions extend the well-known first-degree stochastic dominance (FSD) and second-degree stochastic dominance (SSD). PSD and MSD investigates choices by investors who have S-shaped utility functions and reverse S-shaped utility functions. Arvanitis and Topaloglou (2017) develop consistent tests for PSD and MSD efficiency which is an extension to the case where full diversification is allowed. Arvanitis et al. (2020) investigate MSD spanning. This paper extends those works to prospect spanning, which is consistent with prospect preferences.
2.1 Stochastic Spanning for Prospect Dominance and Analytical Representation

Given a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), suppose that \(F\) denotes the cdf of some probability measure on \(\mathbb{R}^n\). Let \(G(z, \lambda, F)\) be \(\int_{\mathbb{R}^n} 1_{\{\lambda^T u \leq z\}} dF(u)\), i.e., the cdf of the linear transformation \(x \in \mathbb{R}^n \rightarrow \lambda^T x\) where \(\lambda\) assumes its values in \(\mathbb{L}\), which denotes the portfolio space. We suppose that the portfolio space is a closed non-empty subset of \(\mathbb{S} = \{\lambda \in \mathbb{R}^n_+ : 1^T \lambda = 1\}\), possibly formulated by further economic, legal restrictions, etc. In many applications, we have that \(\mathbb{L} = \mathbb{S}\). We denote a distinguished sub-collection of \(\mathbb{L}\) by \(\mathbb{K}\) and generic elements of \(\mathbb{L}\) by \(\lambda, \kappa\), etc. In order to define the concepts of PSD and subsequently of stochastic spanning, we consider \(\mathcal{J}(z_1, z_2, \lambda, F) := \int_{z_1}^{z_2} G(u, \lambda, F) \, du\).

**Definition 1.** \(\kappa\) weakly Prospect-dominates \(\lambda\), written as \(\kappa \succeq_P \lambda\), iff we have the system of inequalities \(P_1(z, \lambda, \kappa, F) := \mathcal{J}(z, 0, \kappa, F) - \mathcal{J}(z, 0, \lambda, F) \leq 0, \forall z \in \mathbb{R}_-\) and \(P_2(z, \lambda, \kappa, F) := \mathcal{J}(0, \kappa, F) - \mathcal{J}(0, z, \lambda, F) \leq 0, \forall z \in \mathbb{R}_{++}\).

The expected utility representation of the relation in Levy and Levy (2002), as well as the Russel and Seo (1989) ramp function construction, show that the conditions on \(P_2\) are consistent with concave preferences on the positive domain, while the conditions on \(P_1\) are consistent with convex preferences on the negative domain. PSD is thus associated with non-global dispositions towards risk: risk loving preferences in the negative domain and risk aversion in the positive domain. As such, it is fundamentally different from SSD which involves the class of utilities that represent global risk aversion. The following simple example demonstrates the discrepancy between the two forms of stochastic dominance.

Consider the pair \(\kappa, \lambda\), with

\[ G(u, \kappa, F) := \begin{cases} 0, & \text{for } u < -1, \\ \frac{1}{8}, & \text{for } -1 \leq u < 0, \\ \Phi(u), & \text{for } 0 \leq u, \end{cases} \quad G(u, \lambda, F) := \begin{cases} 0, & \text{for } u < -1, \\ \frac{1}{4}(u + 1), & \text{for } -1 \leq u < 0, \\ \Phi(u), & \text{for } 0 \leq u, \end{cases} \]
where $\Phi$ denotes the standard Normal cdf. Here, while $\lambda$ dominates $\kappa$ w.r.t. SSD, we have on the contrary that $\kappa \succ_p \lambda$. On the negative domain, every risk loving agent chooses $\kappa$ over $\lambda$ due to the behavior of the two cdf on the interval $[-1, 0)$. That behavior also implies that $\kappa$ is not chosen over $\lambda$ by any risk averse agent on that interval. However, whenever the base assets vector is entirely supported on the positive real line, then Prospect dominance is equivalent to second-order dominance. If the joint distribution is further elliptical, it is then also equivalent to M-V dominance (see Arvanitis et al. (2018)).

Given the stochastic dominance relation above, stochastic spanning occurs when augmentation of the portfolio space does not enhance investment opportunities, or equivalently, investment opportunities are not lost when the portfolio space is reduced. The following definition clarifies the concept w.r.t. the Prospect dominance relation.

**Definition 2.** $K$ Prospect-spans $L$ ($K \succ_p L$) iff for any $\lambda \in L$, $\exists \kappa \in K : \kappa \succ_p \lambda$. If $K = \{\kappa\}$, the element $\kappa$ of the singleton $K$ is termed as Prospect super-efficient.

The efficient set of the dominance relation is the subset of $L$ that contains the maximal elements. The efficient set is a spanning subset of the portfolio space. Thereby, any superset of the efficient set is also a spanning subset of $L$. We can consider a spanning set as an outer approximation of the efficient set. Given a candidate spanning set exists, the question is whether it actually spans the portfolio space. If a method for answering such a question also exists, we can accurately approximate the efficient set via the choice of finer spanning subsets of the portfolio space. It helps in understanding decision making and investment choice.

Hence, the question we address here is: given a candidate $K$, is $K \succ_p L$? The following lemma provides an analytical characterization by means of nested optimizations, which, along with the economic representation of the relation in terms of expected utility in the following section, is key for a numerical implementation on real data and statistical inference. Its proof (see the Online Appendix) is based on parameter continuity arguments for the functionals involved. It is of similar form to the analogous functional employed in the SSD spanning by
Arvanitis et al. (2017), yet containing an additional layer of optimization, a supplementary complexity to be handled in the proof. It is due to the two component system of inequalities that appears in the definition of the relation and it is analogous to the representation of spanning w.r.t. the Markowitz dominance relation in Arvanitis et al. (2020).

**Lemma 3.** Suppose that $\mathbb{K}$ is closed. Then $\mathbb{K} \succeq_{P} \mathbb{L}$ iff we get the condition

$$\rho(F) := \max_{i=1,2} \sup_{\lambda \in L} \inf_{z \in A_i} \inf_{\kappa \in \mathbb{K}} P_i(z, \lambda, \kappa, F) = 0,$$

where $A_1 = \mathbb{R}_-$, $A_2 = \mathbb{R}_+$. 

### 2.2 Representation By Utility Functions

Here, we provide an expected utility characterization of spanning. First, it generalizes the utility characterization of PSD in Levy and Levy (2002), in that it does not require two-sided differentiability of the utilities involved. Local representations of the convex/concave components of the utilities involved imply that they have locally integrable one sided derivatives (see Appendix C.3 of Pollard (2002)), but need not have integrable derivatives. Hence, our representation enriches the class of functions involved. It allows us to analyze markets that contain investors with non-smooth preferences. Such preferences can be associated with situations in which information about the optimality of the equilibrium allocation is not fully characterized by equilibrium prices (see Ohtaki (2019)). Hence, our extension permits market conditions potentially involving ambiguity about asset equilibrium allocations. Second and foremost, our approach is in the spirit of the Russel and Seo (1989) representations for SSD. We rely on utilities represented as unions of graphs of convex mixtures of appropriate “ramp functions” on each half-line. Aside its economic interpretation, and given Lemma 3, it is key to the numerical LP implementation of the inferential procedures. In the next section, it enables a finite dimensional approximation of the utility class by functions with piecewise linear components (see Arvanitis et al. (2017) for a simpler construction for SSD).

To this end, we denote with $\mathcal{W}_-, \mathcal{W}_+$, the sets of Borel probability measures on the real line with supports that are closed subsets of $\mathbb{R}_-$ and $\mathbb{R}_+$, respectively, with existing first moments and uniformly integrable. The latter requirement is convenient yet harm-
less since orderings are invariant to utility rescalings. Those sets are convex, and closed w.r.t. the topology of weak convergence and their union contains the set of degenerate measures. Define $V_- := \left\{ v_w : \mathbb{R}_- \to \mathbb{R}, v_w(u) = \int_{\mathbb{R}_-} [z1_{u \leq z} + u1_{z \leq u \leq 0}] dw(z), w \in \mathcal{W}_- \right\}$, and $V_+ := \left\{ v_w : \mathbb{R}_+ \to \mathbb{R}, v_w(u) = \int_{\mathbb{R}_+} [u1_{0 \leq u \leq z} + z1_{z \leq u < +\infty}] dw(z), w \in \mathcal{W}_+ \right\}$. Every element of $V_+$ is increasing and concave, and dually every element of $V_-$ is increasing and convex. Furthermore, any function defined by the union of the graph of an arbitrary element of $V_+$ with the graph of an arbitrary element of $V_-$ is the graph of an S-shaped utility function as defined by Levy and Levy (2002). Such a utility function is concave for gains and convex for losses. Denote the set of S-shaped utility functions obtained by such graph unions as $V$.

Thereby,

$$V := \left\{ v : \mathbb{R} \to \mathbb{R}, v(u) = \begin{cases} v_{w_1}(u), & u \leq 0 \\ v_{w_2}(u), & u \geq 0 \end{cases}, \text{where } v_{w_1} \in V_-, v_{w_2} \in V_+ \right\}.$$ 

**Lemma 4.** We have $\rho(F) = \max_{i=1,2} \sup_{v \in V_i} \sup_{\lambda \in \mathcal{L}} \mathbb{E}_\lambda [1_{u \in A}, v_w(u)] - \sup_{\kappa \in \mathcal{K}} \mathbb{E}_\kappa [1_{u \in A}, v_w(u)]$, where $\mathbb{E}_\lambda$ denotes expectation w.r.t. $G(z, \lambda, F)$. If the hypotheses of Lemma 3 hold and $\mathcal{K}$ is convex, then $\mathcal{K} \supseteq \mathcal{L}$ iff, $\sup_{v \in V} \sup_{\lambda \in \mathcal{L}} \mathbb{E}_\lambda [v] - \sup_{\kappa \in \mathcal{K}} \mathbb{E}_\kappa [v] = 0$.

The first part of the lemma connects the functional that represents spanning to the aforementioned classes of utilities. It is exploited below in order to obtain feasible numerical formulations based on LP. Those formulations are reminiscent of the LP programs developed in the early papers of testing for SSD efficiency of a given portfolio by Post (2003) and Kuosmanen (2004). The second part of Lemma 4 crystalizes the economic characterization of spanning w.r.t. investment opportunities. It states that spanning holds if and only if the reduction of investment opportunities from $\mathcal{L}$ to $\mathcal{K}$ does not reduce optimal choices uniformly w.r.t. this class of preferences. Equivalently, it essentially states that when spanning does not hold, the restriction of the portfolio possibilities from $\mathcal{L}$ to $\mathcal{K}$, results to the maximal expected
utility loss given by $\sup_{v \in V} [\sup_{\lambda \in L} E_\lambda [v] - \sup_{\kappa \in K} E_\kappa [v]] > 0$. The reason is that $K$ misses some PSD efficient elements. The proof (see the Online Appendix) depends on standard integration by arguments for Lebesgue-Stieljes integrals, and a recursive application of a min-max theorem.

2.3 An Asymptotically Exact and Consistent Test for Spanning

We cannot directly rely on Lemma 3 for empirical work if $F$ is unknown and/or the optimizations are infeasible. We construct a feasible statistical test for the null hypothesis of $K \succcurlyeq_P L$ by utilizing an empirical approximation of $F$ and by building feasible and fast optimisations with LP. The null and alternative hypotheses take the following forms: $H_0 : \rho (F) = 0$, and $H_a : \rho (F) > 0$. In the special case of singleton $K$, the hypotheses write as in Arvanitis and Topaloglou (2017).

We consider a process $(Y_t)_{t \in \mathbb{Z}}$ taking values in $\mathbb{R}^n$. $Y_{i,t}$ denotes the $i^{th}$ element of $Y_i$. The sample path of size $T$ is the random element $(Y_t)_{t=1,..,T}$. In our empirical finance framework, it represents returns of $n$ financial assets upon which we can construct portfolios via convex combinations. $F$ is the cdf of $Y_0$ and $F_T$ is the empirical cdf associated with the random element $(Y_t)_{t=1,..,T}$. Under our assumptions below, $F_T$ is a consistent estimator of $F$, so we consider the following Kolmogorov-Smirnov type test statistic $\rho_T := \sqrt{T} \rho (F_T) = \sqrt{T} \max_{i=1,2} \sup_{\lambda \in L} \sup_{z \in A_i} \inf_{\kappa \in K} P_i (z, \lambda, \kappa, F_T)$, which is the scaled empirical analog of $\rho (F)$. The consideration of empirical analogues of the functionals that represent SD spanning properties, instead of traditional M-V spanning tests (Huberman and Kandel (1987), Jobson and Korkie (1989), De Roon et al. (2001)) in the context of SSD, is explained in Section 4 of Arvanitis et al. (2017) upon deviations from ellipticity. A fortiori, in our context of S-shaped preferences, traditional M-V spanning tests are generally non-admissible due to the fundamental differences between the M-V and Prospect dominance relations when the supports include components of the negative domain as explained and shown with a counterexample in Section 2.1. In addition to the economic interpretation of
an estimated maximal expected utility loss from the discussion after Lemma 4, we prefer to use a Kolmogorov-Smirnov type statistic, instead of, say, a Cramer-von Mises type statistic, because of the availability and tractability of the numerical approximation through LP that is exemplified in the next section.

The following assumption enables the derivation of the limit distribution of $\rho_T$ under $H_0$ and is weaker than Assumption 2 in Arvanitis et al. (2020).

**Assumption 5.** $F$ is absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{R}^n$ with convex support that is bounded from below, and for some $0 < \delta$, $E\left[\|Y_0\|^{2+\delta}\right] < +\infty$. $(Y_t)_{t \in \mathbb{Z}}$ is $\alpha$-mixing with mixing coefficients $a_T = O(T^{-a})$ for some $a > 1 + \frac{2}{q}$, $0 < \eta < 2$, as $T \to \infty$.

The lower bound hypothesis is harmless in our empirical finance framework since we use financial returns that are naturally bounded from below by -1. The mixing part is readily implied by concepts such as geometric ergodicity which holds for many stationary models used in the context of financial econometrics under parameter restrictions and restrictions on the properties of the underlying innovation processes. Examples are the strictly stationary versions of (possibly multivariate) ARMA or several GARCH and stochastic volatility type of models (see Francq and Zakoian (2011) for several examples). Counter-examples are models that exhibit long memory, etc. The moment condition is established in the aforementioned models via restrictions on the properties of building blocks and the parameters of the processes involved.

For the derivation of the limit theory of $\rho_T$ under the null hypothesis, we consider the contact sets $\Gamma_i = \{\lambda \in \mathbb{L}, \kappa \in \mathbb{K}_\lambda^c, z \in A_i : P_i(z, \lambda, \kappa, F) = 0\}$, where $\mathbb{K}_\lambda^c := \{\kappa \in \mathbb{K} : \kappa \succcurlyeq P \lambda\}$ which under the null contains elements different from $\lambda$ for any element of $\mathbb{L} - \mathbb{K}$. For any $i$, the set $\Gamma_i$ is non empty since $\Gamma_i^* := \{(\kappa, \kappa, z) : \kappa \in \mathbb{K}, z \in A_i\} \subseteq \Gamma_i$. Furthermore, $(\lambda, \kappa, 0) \in \Gamma_1$, $\forall \lambda, \kappa$. Since $\hat{z} := \inf_{\lambda Y_0} \lambda Y_0$ exists from Assumption 5, we have that for all $z \leq \hat{z}$, $(\lambda, \kappa, z) \in \Gamma_i$, $\forall \lambda \in \mathbb{L}, \kappa \in \mathbb{K}_\lambda^c$ for the $i$ that corresponds to the sign of $\hat{z}$. In what follows, we denote convergence in distribution by $\Rightarrow$. 

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Proposition 6. Suppose that $K$ is closed, Assumption 5 holds and that $H_0$ is true. Then as $T \to \infty$, $\rho_T \rightsquigarrow \rho_\infty$, where $\rho_\infty := \max_{i=1,2} \sup_{\lambda} \sup_z \inf_{\kappa} P_i(z, \lambda, \kappa, G_F), (\lambda, z, \kappa) \in \Gamma_i$, and $G_F$ is a centered Gaussian process with covariance kernel given by $\text{Cov}(G_F(x), G_F(y)) = \sum_{t \in \mathbb{Z}} \text{Cov}(1_{\{y_0 \leq x\}}, 1_{\{y_t \leq y\}})$ and $\mathbb{P}$ almost surely uniformly continuous sample paths defined on $\mathbb{R}^n$.

The limiting random variable $\rho_\infty$ obtained by optimisation on a functional of a Gaussian process is well defined since the following inequalities hold

$$\int_0^{+\infty} \sum_{t \in \mathbb{Z}} \text{Cov}(1_{\{\lambda^T Y_0 \leq u\}}, 1_{\{\lambda^T Y_t \leq u\}}) du \leq 2 \sum_{t=0}^{+\infty} \sqrt{a_T} \int_0^{+\infty} \sqrt{1 - G(u, \lambda, F)} du < +\infty,$$

and

$$\int_{-\infty}^0 \sum_{t \in \mathbb{Z}} \text{Cov}(1_{\{\lambda^T Y_0 \leq u\}}, 1_{\{\lambda^T Y_t \leq u\}}) du \leq 2 \sum_{t=0}^{\infty} \sqrt{a_T} \int_{-\infty}^0 \sqrt{G(u, \lambda, F)} du < +\infty,$$

where the first inequalities in each of the previous expressions follow from inequality 1.12b in Rio (2000), and the second ones follow from Assumption 5 (see also p. 196 of Horvath et al. (2006)).

Since $F$ and $\Gamma_i$ are unknown in practice, we use the results of the previous lemma to construct a decision procedure based on subsampling, in the spirit of Linton et al. (2014) (see also Linton et al. (2005))\footnote{The partitioning used to get the results in Proposition 6 directly leads to the consideration of subsampling as a resampling procedure. A testing procedure based on (block) bootstrap as in Scaillet and Topaloglou (2010), can, due to the form of the recentering, be consistent, but can be too conservative asymptotically, and thereby suffer from a lack of power compared to the subsampling under particular local alternatives (see also the relevant discussion in Arvanitis et al. (2019)). The potential of asymptotic exactness for the subsampling test justifies the particular resampling choice for inference.}

Algorithm 7. It consists of the following steps:

1. Evaluate $\rho_T$ at the original sample value.

2. For $0 < b_T \leq T$, generate subsample values from the original observations $(Y_t)_{t=t-b_T+1}^{t+T-1}$ for all $t = 1, 2, \ldots, T - b_T + 1$.

3. Evaluate the test statistic on each subsample value thereby obtaining $\rho_{T,b_T,t}$ for all $t = 1, 2, \ldots, T - b_T + 1$. 
4. Approximate the cdf of the asymptotic distribution under the null of \( \rho_T \) by

\[
s_{T,b}(y) = \frac{1}{T-b_T+1} \sum_{t=1}^{T-b_T+1} 1 (\rho_{T,b_T,t} \leq y)
\]
and calculate its \( 1 - \alpha \) quantile

\[
q_{T,b_T} (1 - \alpha) = \inf_y \{ s_{T,b}(y) \geq 1 - \alpha \}
\]
for the significance level \( 0 < \alpha < .5 \).

5. Reject the null hypothesis \( H_0 \) if \( \rho_T > q_{T,b_T} (1 - \alpha) \).

In order to derive the limit theory for the testing procedure, namely its asymptotic exactness and consistency stated in the next theorem, we first use the following standard assumption that restricts the asymptotic behaviour of \( b_T \) governing the size \( b_T + 1 \) of each subsample.

**Assumption 8.** Suppose that \((b_T),\) possibly depending on \((Y_t)_{t=1,...,T},\) satisfies the condition

\[
P (l_T \leq b_T \leq u_T) \to 1,
\]
where \((l_T)\) and \((u_T)\) are real sequences such that \(1 \leq l_T \leq u_T\) for all \(T,\) \(l_T \to \infty\) and \(u_T \to 0\) as \(T \to \infty\).

**Theorem 9.** Suppose Assumptions 5 and 8 hold. For the testing procedure described in Algorithm 7, we have that

1. If \( H_0 \) is true, and for \( \lambda \in \mathbb{L} - \mathbb{K}, \inf Y_0 \lambda^T Y_0 \leq 0 \) there exists \( (\kappa, z) \in \mathbb{K}_\lambda^\mathbb{R} \times \mathbb{R}_{++} \) with

\[
(\lambda, \kappa, z) \in \Gamma_2 \text{ and that if } (\lambda, \kappa^*, z^*) \in \Gamma_2 \text{ for } \kappa^* \neq \kappa \text{ then } z^* \neq z\text{, then for all } \alpha \in (0, .5)
\]

\[
\lim_{T \to \infty} P (\rho_T > q_{T,b_T} (1 - \alpha)) = \alpha.
\]

2. If \( H_a \) is true then \( \lim_{T \to \infty} P (\rho_T > q_{T,b_T} (1 - \alpha)) = 1 \).

When for \( \lambda \in \mathbb{L} - \mathbb{K}, \inf Y_0 \lambda^T Y_0 \leq 0 \) then due to Assumption 5 for any contact triple \((\lambda, \kappa, z) \in \Gamma_2\) we have that \( P_2 (z, \lambda, \kappa, G_F) \) must be non-degenerate. Whenever \( z \) corresponds solely to the particular \( \kappa \), we obtain that \( \rho_\infty \) is non-degenerate and if its cdf jumps at the infimum of its support, then the jump magnitude is bounded above by .5. Hence, in this case, the test is asymptotically exact for all the usual choices of the significance level since the probability of rejection under the null hypothesis, i.e., the size of the test, reaches \( \alpha \) in large samples. We combine Proposition 6 above and Theorem 3.5.1 of Politis et al. (1999) in the proof of the exactness statement, namely point 1 of Theorem 9. To get exactness,
the condition imposed on $L - K$ is significantly weaker than the assumption on the relation between the extreme points of $L$ and $K$ adopted by Arvanitis, et al. (2020). It amounts to the existence of a spanned portfolio whose support is not strictly positive and so that, in the event of positive returns, there exists an elementary increasing and concave utility for positive returns and a unique portfolio such that the latter dominates the former and we are indifferent between the two portfolios with this particular utility. Besides, the test is also consistent since the probability of rejection under the alternative hypothesis, i.e., the power of the test, reaches 1 in large samples. We show in the proof of the consistency statement, namely point 2 of Theorem 9, that the test statistic diverges to $+\infty$ under the alternative hypothesis when $T$ goes to $+\infty$.

We opt for the “bias correction” regression analysis of Arvanitis et al. (2019) to reduce the sensitivity of the quantile estimates $q_{T,b_T}(1 - \alpha)$ on the choice of $b_T$ in empirically realistic dimensions for $n$ and $T$ (see also Arvanitis et al. (2020) for further evidence on its better finite sample properties). Specifically, given $\alpha$, we compute the quantiles $q_{T,b_T}(1 - \alpha)$ for a “reasonable” range of $b_T$. In the empirical section, we use $b_T \in \{T^{0.6}, T^{0.7}, T^{0.8}, T^{0.9}\}$. Next, we estimate the intercept and slope of the following regression line by OLS: $q_{T,b_T}(1 - \alpha) = \gamma_{0,T,1-\alpha} + \gamma_{1,T,1-\alpha}(b_T)^{-1} + \nu_{T,1-\alpha,b_T}$. Finally, we estimate the bias-corrected $(1 - \alpha)$-quantile as the OLS predicted value for $b_T = T$: $q_{T}^{BC}(1 - \alpha) := \hat{\gamma}_{0,T,1-\alpha} + \hat{\gamma}_{1,T,1-\alpha}(T)^{-1}$. Since $q_{T,b_T}(1 - \alpha)$ converges in probability to $q(\rho_\infty,1 - \alpha)$ and $(b_T)^{-1}$ converges to zero as $T \to 0$, $\hat{\gamma}_{0,T,1-\alpha}$ converges in probability to $q(\rho_\infty,1 - \alpha)$ and the asymptotic properties are not affected.

In the Online Appendix, we also show that under further assumptions, the test is asymptotically locally unbiased under given sequences of local alternatives. Besides, the Monte Carlo analysis reported in the Online Appendix shows that the test performs well with an empirical size close to 5% and an empirical power above 90% for a significance level $\alpha = 5\%$.  

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3 Numerical Implementation

In this section, we exploit the results of Lemma 4 in order to provide with a finitary approximation of the test statistic. We rely on this approximation to provide with a numerical implementation based on LP below. We construct the S-shaped utility functions as follows. We approximate the utility function in the risk-seeking area using a finite set of increasing and convex piecewise-linear functions, while we approximate the risk-averse part by increasing and concave piecewise-linear functions. The latter is also used in Arvanitis et al. (2019) for SSD spanning. We denote expectation w.r.t. the empirical measure by $E_{F_T}$. Let $\mathcal{R}^-$ denote $\max_{i=1,\ldots,n} \text{Range} \left( Y_{i,t} 1_{Y_{i,t} \leq 0} \right)_{t=1,\ldots,T} = [\bar{x}, 0]$. Partition $\mathcal{R}^-$ into $n_1$ equally spaced values as $\bar{x} = z_1 < \cdots < z_{n_1} = 0$, where $z_n := \bar{x} - \frac{n-1}{n_1-1}\bar{x}$, $n = 1, \ldots, n_1$; $n_1 \geq 2$. Furthermore, partition the interval $[0, 1]$, as $0 < \frac{1}{n_2-1} < \cdots < \frac{n_2-2}{n_2-1} < 1$, $n_2 \geq 2$. Similarly, $\mathcal{R}^+ := \max_{i=1,\ldots,n} \text{Range} \left( Y_{i,t} 1_{Y_{i,t} \geq 0} \right)_{t=1,\ldots,T} = [0, \bar{x}]$. Partition $\mathcal{R}^+$ into $p_1$ equally spaced values as $0 = z_1 < \cdots < z_{p_1} = \bar{x}$, where $z_p := \frac{p-1}{p_1-1}\bar{x}$, $n = 1, \ldots, p_1$; $p_1 \geq 2$, and again partition the interval $[0, 1]$, as $0 < \frac{1}{p_2-1} < \cdots < \frac{p_2-2}{p_2-1} < 1$, $p_2 \geq 2$. Using the above, we consider the test statistic:

$$\rho_T^* := \sqrt{T} \max_{i=1,2} \sup_{v \in V^*_i} \left[ \sup_{\lambda \in \mathcal{L}} E_{F_T}[v(\lambda^T Y)] - \sup_{\kappa \in \mathcal{K}} E_{F_T}[v(\kappa^T Y)] \right],$$

where the set of utility functions for negative returns is:

$$V^*_- := \left\{ v : v(u) = \sum_{n=1}^{n_1} w_n \left[ z_n 1_{\bar{x} \leq u \leq z_n} + u 1_{z_n \leq u \leq 0} \right], (w_1, \ldots, w_{n_1}) \in \mathcal{W}^- \right\},$$

$$\mathcal{W}^- := \left\{ (w_1, \ldots, w_{n_1}) \in \left\{ 0, \frac{1}{n_2-1}, \cdots, \frac{n_2-2}{n_2-1}, 1 \right\}^{n_1} : \sum_{n=1}^{n_1} w_n = 1 \right\},$$

and the set of utility functions for positive returns is:

$$V^*_+ := \left\{ v : v(u) = \sum_{p=1}^{p_1} w_p \left[ u 1_{0 \leq u \leq z_p} + z_p 1_{z_p \leq u \leq \bar{x}} \right], (w_1, \ldots, w_{p_1}) \in \mathcal{W}^+ \right\},$$
\[ W^+ := \left\{ (w_1, \ldots, w_{p_1}) \in \left\{ 0, \frac{1}{p_2 - 1}, \ldots, \frac{p_2 - 2}{p_2 - 1}, 1 \right\}^{p_1} : \sum_{p=1}^{p_1} w_p = 1 \right\}. \]

We obtain the following result on the approximation of \( \rho_T \) by \( \rho_T^* \).

**Proposition 10.** As \( n_1, n_2, p_1, p_2 \to \infty \), we have \( \rho_T^* \to \rho_T \), \( \mathbb{P} \) a.s.

Our feasible computational strategy builds on LP formulations for the numerical evaluation using the previous finitary approximation of the test statistic.

We have a set of increasing and convex utility functions: \( v(u) = \sum_{n=1}^{n_1} w_n \max(u, z_n) \) for the negative part, namely the risk-seeking area. For every \( v \in V^* \), we have at most \( n_2 \) line segments with knots at \( n_1 \) possible outcome levels. Then, we can enumerate all \( n_3 = \frac{1}{(n_1 - 1)!} \prod_{i=1}^{n_1-1} (n_2 + i - 1) \) elements of \( V^* \). Our application in Section 4 uses \( n_1 = 10 \), and \( n_2 = 5 \), which gives \( n_3 = 715 \) distinct utility functions, and a total of 1430 small LP problems for the two embedded maximisation problems in (1). Solving (1) yields simultaneously the optimal factor portfolio \( \kappa \), and the optimal augmented portfolio \( \lambda \) that maximize the expected utility. Below, we give the mathematical formulation for the first optimization problem \( \sup_{\lambda \in \Lambda} \mathbb{E}_{F_N} [u(\lambda^T Y)] \), that yields the optimal augmented portfolio \( \lambda \). The same formulation is used for the second optimization \( \sup_{\kappa \in \kappa} \mathbb{E}_{F_N} [u(\kappa^T Y)] \).

Let us define: \( c_{0,n} := \sum_{m=n}^{n_1} (c_{1,m} - c_{1,m+1}) z_m \), \( c_{1,n} := \sum_{m=n}^{n_1} w_m \), and \( \mathcal{N} := \{ n = 1, \ldots, n_1 : w_n > 0 \} \cup \{ n_1 \} \). For any given \( u \in V_- \), \( \sup_{\lambda \in \Lambda} \mathbb{E}_{F_N} [u(\lambda^T Y)] \) is the optimal value of the objective function of the following LP problem in canonical form:

\[
\begin{align*}
\max & \quad T^{-1} \sum_{t=1}^{T} y_t \\
\text{s.t.}, & \quad \text{for } t = 1, \ldots, T, \quad n \in \mathcal{N}, \quad i = 1, \ldots, M, \\
& \quad y_t \leq \lambda^T Y_t c_{1,n} + Q_t^- + Q_t^+, \quad y_t \leq c_{0,n} + Q_t^- + Q_t^+, \\
& \quad Q_t^- \geq c_{0,n} - \lambda^T Y_t c_{1,n}, \quad Q_t^+ \geq \lambda^T Y_t c_{1,n} - c_{0,n}, \quad Q_t^- \geq 0, \quad Q_t^+, \geq 0, \\
& \quad \sum_{i=1}^{M} \lambda_i = 1, \quad \lambda_i \geq 0, \quad \text{and } y_t \text{ being free}. 
\end{align*}
\]
For the positive part, namely the risk-averse area, we take a set of concave utility functions: \( v(u) = \sum_{p=1}^{p_1} w_p \min(u, z_p) \). Again, for every \( v \in V^*_+ \), we have at most \( p_2 \) line segments with knots at \( p_1 \) possible outcome levels. As before, the number of elements of \( V^*_+ \) is \( p_3 = \frac{1}{(p_1-1)!} \prod_{i=1}^{p_1-1} (p_2 + i - 1) = 1430 \), for \( p_1 = 10 \) and \( p_2 = 5 \).

Let us define: \( c_{0,p} := \sum_{m=p}^{p_1} (c_{1,m} - c_{1,m+1}) z_m, \quad c_{1,p} := \sum_{m=p}^{p_1} w_m, \) and \( \mathcal{P} := \{ p = 1, \ldots, p_1 : w_p > 0 \} \cup \{ p_1 \} \). For any given \( u \in V_+ \), \( \sup_{\lambda \in \Lambda} \mathbb{E}_{F_N}[u(\lambda^T Y)] \) is the optimal value of the objective function of the following LP problem in canonical form:

\[
\max \sum_{t=1}^{T} y_t (3) \\
\text{s.t., for } t = 1, \ldots, T, \quad i = 1, \ldots, M, \\
y_t \leq \lambda^T Y_t c_{1,p}, \quad y_t \leq c_{0,p}, \quad \sum_{i=1}^{M} \lambda_i = 1 \quad \lambda_i \geq 0, \quad \text{and } y_t \text{ being free.}
\]

The total run time for each computation does not exceed one minute when we use a desktop PC with a 3.6 GHz, 6-core Intel i7 processor, with 16 GB of RAM, using MATLAB and GAMS with the Gurobi optimization solver.

4 Empirical Application

Here, we examine if we can explain well-known stock market anomalies by standard factors within a new breed of asset pricing models, for prospect type investor preferences. For this purpose, we use the prospect spanning tests, both in-sample and out-of-sample.

4.1 Factor Models and Anomalies

We start with a benchmark factor model from a set of models that have generated support in the recent finance literature, and we ask whether a characteristic identified in the literature as stock market anomaly, is a market anomaly for prospect investors. To answer this question,
we consider three models that build on the pioneer three-factor model of Fama and French (1993): the four-factor model of Hou et al. (2015), the five-factor model of Fama and French (2015), and the four-factor model of Stambaugh and Yuan (2017). Fama and French (1993) aim to capture the part of average stock returns left unexplained in CAPM of Sharpe (1964) and Lintner (1965) by including, in addition to the market factor, two extra risk factors relating to size (measured by market equity) and the ratio of book-to-market equity. In addition to the market excess return, the influential three-factor model of Fama and French (1993) includes a book-to-market or "value" factor, HML, and a size factor, SMB, based on market capitalization. Motivated by Miller and Modigliani (1961), Fama and French (2015) five-factor model (henceforth, FF-5) augments the original Fama-French three-factor model by two extra factors, one for profitability and another for investment. Hou et al. (2015) consider a four-factor model (dubbed the q factor model) that includes the original market and size factors of Fama and French (1993) augmented by a profitability and investment factor. Stambaugh and Yuan (2017) consider a four-factor model (henceforth, M-4) including the standard market and size factors along with two composite factors for investment and profitability. To construct the composite factors, they combine information from 11 market anomalies relating to investment and profitability measures. We use alternative factor models as a robustness check, namely for testing the consistency of in- and out-of-sample results under the prospect preferences, and not for a horse race in cross-sectional asset pricing.

The stock market anomalies we examine in this paper have a long history in the relevant literature. A common theme in the original papers that first highlighted these patterns, is that they all challenge the rational asset pricing paradigm as they exhibit returns that are not in line with the risks taken. However, notwithstanding whether they are caused by sentiment (a catch-all term that stand for all kinds of irrational decision-making) or by market frictions (e.g., margin requirements), it is also acknowledged that most of them persist because they cannot be “arbitraged” away. From the perspective of the Arbitrage Pricing Theory, it implies that arbitrageurs cannot trade against them without exposing themselves
to significant risks. In this paper, we test the 11 strategies used to construct Stambaugh-Yuan factors, along with Betting against Beta, Quality minus Junk, Size, Growth Option, Value (Book to Market), Idiosyncratic volatility and Profitability. The 11 anomalies used in Stambaugh and Yuan (2017) are Accruals, Asset Growth, Composite Equity Issue, Distress, Growth Profitability Premium, Investment to Assets, Momentum, Net Operating Assets, Net Stock Issues, O-Score, and Return on Assets. They are realigned appropriately to yield positive average returns. In particular, anomaly variables that relate to investment activity (Asset Growth, Investment to Assets, Net Stock Issues, Composite Equity Issues, Accruals) are defined low-minus-high decile portfolio returns, rather than high-minus-low, as in Hou et al. (2015). All the other anomalies are constructed as high-minus-low decile portfolio returns. A short description of the 18 market anomalies that we study in the paper is given in Appendix A (see Stambaugh and Yuan (2017) for further details). 12 of these anomalies are used in Barberis et al. (2021). Returns of the Fama and French 5 factors were downloaded from Kenneth French’s site. The dataset consists of all monthly observations from January 1974 until December 2016. M-4 factor returns and anomaly spread return series were downloaded from the websites of Robert Stambaugh and AQR. In the Online Appendix, we report summary statistics of the factor and anomaly returns over our sample period.

4.2 In-Sample Analysis

In this section, we test in-sample the null hypothesis that the set of standard factors prospect spans the set enlarged with a particular market anomaly. We test separately for the Fama and French 5 factors, the Stambaugh-Yuan 4 factors as well as Hou-Xue-Zhang 4 factors, with respect to each one of the 18 additional anomalies.

The hypothesized portfolio manager with prospect preferences forms optimal portfolios from two separate asset universes: the first universe consists only of factors from a factor model (FF-5, M-4, q), the set $\mathcal{K}$. The second universe is the respective set of factors aug-
mented by a single trading (spread) strategy, the set \( \mathbb{L} \). Portfolio managers are assumed to solve portfolio optimization problems, motivated by the prospect spanning framework, effectively looking for a portfolio picked from the augmented universe \( \mathbb{L} \) that prospect stochastically dominates all portfolios of the respective factor universe \( \mathbb{K} \).

The rejection of the prospect spanning hypothesis implies that there exists at least one portfolio in \( \mathbb{L} \) built from the factors (of each particular factor model) and one market anomaly, which is weakly prefered to every factor portfolio in \( \mathbb{K} \) by at least one S-shaped utility function (see Definition 2). Such a portfolio is by construction efficient w.r.t. \( \mathbb{K} \) (see Definition 2.1 in Linton et al. (2014) for the SSD case which we can easily generalize to our PSD case). The empirical version of such a portfolio is the optimal portfolio \( \lambda \) that maximizes \( \rho_T \) for the particular sample value.

Many of the anomalies employed were published post 1974 and since they build on CRSP/Compustat data, they typically work at least from the mid 60s until the publication year. To account for that, we repeat the analysis for each anomaly only post publication as in Chinco et al. (2021), to test which anomalies survive after publication. In order to have enough observations to perform the tests, we use weekly data in this experiment.

We additionally test for M-V spanning and compare the results. For the M-V spanning we are looking for the portfolio augmented with the anomaly so that we get the highest quadratic utility improvement over the optimal (in the expected quadratic utility terms) M-V factor portfolio. We solve the embedded expected-utility optimization problems (for every given quadratic utility function) using quadratic programming.

### 4.2.1 Prospect spanning

We compute the subsampling distribution of the test statistic for subsample size \( b_T \in \{T^{0.6}, T^{0.7}, T^{0.8}, T^{0.9}\} \). Using OLS regression on the empirical quantiles \( q_{T,b_T} (1 - \alpha) \) for a significance level \( \alpha = 5\% \), we get the estimate \( q_T^{BC} \) for the bias-corrected critical value. We reject spanning if the test statistic \( \rho_T^* \) is higher than the regression estimate \( q_T^{BC} \). Tables 13.
report the test statistics $\rho^*_T$ as well as the regression estimates $q^{BC}_T$ when we test for spanning of the alternative factor models w.r.t. each one of the 18 market anomalies, both in the full period as well as in the post publication period.

For the full sample, we observe that the FF-5 model spans 6 out of 18 market anomalies, that is, Accruals, Asset Growth, Return on Assets, Size, Growth Option, and Profitability. The M-4 model spans the same 6 market anomalies, while the q model spans Return on Assets, Betting against Beta, Size, and Profitability. Thus, in most cases, optimal portfolios based on the investment opportunity set that includes a market anomaly is not spanned by the corresponding optimal portfolio strategies based on the original factors. We also observe that Return on Assets, Size, and Profitability are spanned by all the factor models, indicating the robustness of these characteristics being not considered as genuine market anomalies by prospect investors.

In contrast to the full period, only 5 out of the 18 characteristics are not spanned by any factor model in the post publication period. These genuine anomalies are the Composite equity issue, Momentum, Betting against Beta, Value and Idiosyncratic volatility. Although Net stock issues is not spanned by the first two factor models, it is spanned by the q model.
| Variable                          | Test statistic $\rho^*_T$ | Regression estimates $q_{BC}^T$ | Result         |
|----------------------------------|-----------------------------|---------------------------------|----------------|
| **Panel (a): Full period**       |                             |                                 |                |
| Accruals                         | 0.0016                      | 0.0025                          | Spanning       |
| Asset Growth                     | 0.0                         | 0.0                             | Spanning       |
| Composite Equity Issue           | 0.0015                      | 0.0003                          | Reject Spanning|
| Distress                         | 0.0045                      | 0.0005                          | Reject Spanning|
| Growth Profitability Premium     | 0.0015                      | 0.0012                          | Reject Spanning|
| Investment to Assets             | 0.0014                      | 0.0001                          | Reject Spanning|
| Momentum                         | 0.0696                      | 0.0204                          | Reject Spanning|
| Net Operating Assets             | 0.0268                      | 0.0009                          | Reject Spanning|
| Net Stock Issues                 | 0.0011                      | 0.0003                          | Reject Spanning|
| O-Score                          | 0.0129                      | 0.0092                          | Reject Spanning|
| Return on Assets                 | 0.0024                      | 0.0047                          |                |
| Betting against Beta             | 0.0235                      | 0.0176                          | Reject Spanning|
| Quality minus Junk               | 0.0088                      | 0.0061                          | Reject Spanning|
| Size                             | 0.0                         | 0.0                             | Spanning       |
| Growth Option                    | 0.0                         | 0.0                             | Spanning       |
| Value (Book to Market)           | 0.1921                      | 0.1878                          | Reject Spanning|
| Idiosyncratic Volatility         | 0.0195                      | 0.0100                          | Reject Spanning|
| Profitability                    | 0.0                         | 0.0                             | Spanning       |
| **Panel (b): Post publication period** |                             |                                 |                |
| Accruals                         | 0.1340                      | 0.6832                          | Spanning       |
| Asset Growth                     | 0.3619                      | 0.5102                          | Spanning       |
| Composite Equity Issue           | 0.4399                      | 0.3464                          | Reject Spanning|
| Distress                         | 0.4984                      | 0.7748                          | Spanning       |
| Growth Profitability Premium     | 0.4225                      | 0.7096                          | Spanning       |
| Investment to Assets             | 0.0006                      | 0.0035                          | Spanning       |
| Momentum                         | 0.6965                      | 0.6286                          | Reject Spanning|
| Net Operating Assets             | 0.6697                      | 0.8177                          | Spanning       |
| Net Stock Issues                 | 0.3246                      | 0.2661                          | Reject Spanning|
| O-Score                          | 0.0029                      | 0.0192                          | Spanning       |
| Return on Assets                 | 0.4975                      | 0.5484                          | Spanning       |
| Betting against Beta             | 0.0165                      | 0.0113                          | Reject Spanning|
| Quality minus Junk               | 0.0083                      | 0.0086                          | Spanning       |
| Size                             | 0.0                         | 0.0                             | Spanning       |
| Growth Option                    | 0.0                         | 0.0                             | Spanning       |
| Value (Book to Market)           | 0.1921                      | 0.1878                          | Reject Spanning|
| Idiosyncratic Volatility         | 0.0195                      | 0.0100                          | Reject Spanning|
| Profitability                    | 0.0                         | 0.0                             | Spanning       |

Entries report the test statistics $\rho^*_T$ and the regression estimates $q_{BC}^T$ for spanning of the Fama and French (FF-5) model with respect to each one of the 18 market anomalies. We reject spanning at significance level $\alpha = 5\%$ if $\rho^*_T > q_{BC}^T$. Panel (a) uses the full period while Panel (b) uses the post publication period.
Table 2: Test statistics: Stambaugh-Yuan (M-4) Factors

| Variable                        | Test statistic $\rho^*_T$ | Regression estimates $q^{BC}_T$ | Result          |
|--------------------------------|---------------------------|-------------------------------|-----------------|
| **Panel (a): Full period**     |                           |                               |                 |
| Accruals                       | 0.0081                    | 0.0083                        | Spanning        |
| Asset Growth                   | 0.0057                    | 0.0069                        | Spanning        |
| Composite Equity Issue         | 0.0143                    | 0.078                         | Reject Spanning |
| Distress                       | 0.0533                    | 0.0020                        | Reject Spanning |
| Growth Profitability Premium   | 0.0113                    | 0.0049                        | Reject Spanning |
| Investment to Assets           | 0.0116                    | 0.0164                        | Reject Spanning |
| Momentum                       | 0.1189                    | 0.1143                        | Reject Spanning |
| Net Operating Assets           | 0.0653                    | 0.0071                        | Reject Spanning |
| Net Stock Issues               | 0.0145                    | 0.0073                        | Reject Spanning |
| O-Score                        | 0.0133                    | 0.0122                        | Reject Spanning |
| Return on Assets               | 0.0012                    | 0.0015                        | Spanning        |
| Betting against Beta          | 0.0755                    | 0.0703                        | Reject Spanning |
| Quality minus Junk             | 0.0374                    | 0.0099                        | Reject Spanning |
| Size                           | 0.0000                    | 0.0000                        | Spanning        |
| Growth Option                  | 0.0000                    | 0.0000                        | Spanning        |
| Value (Book to Market)         | 0.2939                    | 0.2817                        | Reject Spanning |
| Idiosyncratic Volatility       | 0.2593                    | 0.1039                        | Reject Spanning |
| Profitability                  | 0.0000                    | 0.0000                        | Spanning        |
| **Panel (b): Post publication period** |                   |                               |                 |
| Accruals                       | 0.1565                    | 0.8790                        | Spanning        |
| Asset Growth                   | 0.0642                    | 0.0884                        | Spanning        |
| Composite Equity Issue         | 0.5912                    | 0.3786                        | Reject Spanning |
| Distress                       | 0.1973                    | 0.2189                        | Spanning        |
| Growth Profitability Premium   | 0.3651                    | 0.5844                        | Spanning        |
| Investment to Assets           | 0.0006                    | 0.0012                        | Spanning        |
| Momentum                       | 0.6482                    | 0.5534                        | Reject Spanning |
| Net Operating Assets           | 0.6511                    | 0.6719                        | Spanning        |
| Net Stock Issues               | 0.4188                    | 0.2852                        | Reject Spanning |
| O-Score                        | 0.0063                    | 0.0084                        | Spanning        |
| Return on Assets               | 0.5590                    | 0.8730                        | Spanning        |
| Betting against Beta          | 0.0189                    | 0.0091                        | Reject Spanning |
| Quality minus Junk             | 0.0054                    | 0.0079                        | Spanning        |
| Size                           | 0.0000                    | 0.0000                        | Spanning        |
| Growth Option                  | 0.0000                    | 0.0000                        | Spanning        |
| Value (Book to Market)         | 0.2939                    | 0.2817                        | Reject Spanning |
| Idiosyncratic Volatility       | 0.2593                    | 0.1039                        | Reject Spanning |
| Profitability                  | 0.0000                    | 0.0000                        | Spanning        |

Entries report the test statistics $\rho^*_T$ and the regression estimates $q^{BC}_T$ for spanning of the Stambaugh-Yuan (M-4) model with respect to each one of the 18 market anomalies. We reject spanning at significance level $\alpha = 5\%$ if $\rho^*_T > q^{BC}_T$. Panel (a) uses the full period while Panel (b) uses the post publication period.
Table 3: Test statistics: Hou-Xue-Zhang (q) Factors

| Variable                      | Test statistic $\rho^*_T$ | Regression estimates $q^{BC}_T$ | Result         |
|-------------------------------|---------------------------|---------------------------------|----------------|
| **Panel (a): Full period**    |                           |                                 |                |
| Accruals                      | 0.0106                    | 0.0039                          | Reject Spanning|
| Asset Growth                  | 0.0176                    | 0.0101                          | Reject Spanning|
| Composite Equity Issue        | 0.0163                    | 0.0159                          | Reject Spanning|
| Distress                      | 0.0386                    | 0.0133                          | Reject Spanning|
| Growth Profitability Premium  | 0.0084                    | 0.0038                          | Reject Spanning|
| Investment to Assets          | 0.0157                    | 0.0123                          | Reject Spanning|
| Momentum                      | 0.0835                    | 0.0305                          | Reject Spanning|
| Net Operating Assets          | 0.0449                    | 0.0059                          | Reject Spanning|
| Net Stock Issues              | 0.0178                    | 0.0170                          | Reject Spanning|
| O-Score                       | 0.0140                    | 0.0109                          | Reject Spanning|
| Return on Assets              | 0.0235                    | 0.0321                          | Spanning       |
| Betting against Beta         | 0.0404                    | 0.0384                          | Reject Spanning|
| Quality minus Junk            | 0.0304                    | 0.0177                          | Reject Spanning|
| Size                          | 0.0                       | 0.0                             | Spanning       |
| Growth Option                 | 0.0029                    | 0.0                             | Reject Spanning|
| Value (Book to Market)        | 0.2045                    | 0.1878                          | Reject Spanning|
| Idiosyncratic Volatility      | 0.2386                    | 0.0101                          | Reject Spanning|
| Profitability                 | 0.0                       | 0.0                             | Spanning       |
| **Panel (b): Post publication period** |                           |                                 |                |
| Accruals                      | 0.2140                    | 0.5378                          | Spanning       |
| Asset Growth                  | 0.3811                    | 0.4887                          | Spanning       |
| Composite Equity Issue        | 0.5266                    | 0.3172                          | Reject Spanning|
| Distress                      | 0.4850                    | 0.7257                          | Spanning       |
| Growth Profitability Premium  | 0.1936                    | 0.4851                          | Spanning       |
| Investment to Assets          | 0.0027                    | 0.0064                          | Spanning       |
| Momentum                      | 0.6734                    | 0.5891                          | Reject Spanning|
| Net Operating Assets          | 0.4855                    | 0.7969                          | Spanning       |
| Net Stock Issues              | 0.1405                    | 0.1723                          | Spanning       |
| O-Score                       | 0.0035                    | 0.0075                          | Spanning       |
| Return on Assets              | 0.3212                    | 0.5415                          | Spanning       |
| Betting against Beta         | 0.0175                    | 0.0122                          | Reject Spanning|
| Quality minus Junk            | 0.0021                    | 0.0034                          | Spanning       |
| Size                          | 0.0                       | 0.0                             | Spanning       |
| Growth Option                 | 0.0                       | 0.0                             | Spanning       |
| Value (Book to Market)        | 0.2939                    | 0.2817                          | Reject Spanning|
| Idiosyncratic Volatility      | 0.2593                    | 0.1039                          | Reject Spanning|
| Profitability                 | 0.0                       | 0.0                             | Spanning       |

Entries report the test statistics $\rho^*_T$ and the regression estimates $q^{BC}_T$ for spanning of the Hou-Xue-Zhang (q) model with respect to each one of the 18 market anomalies. We reject spanning at significance level $\alpha = 5\%$ if $\rho^*_T > q^{BC}_T$. Panel (a) uses the full period while Panel (b) uses the post publication period.
4.2.2 M-V spanning tests

Tables 4-6 report M-V spanning test results of each market anomaly relative to FF-5, M-4 and q factors, respectively, using three test statistics. The first and second column in each table display the value and heteroskedasticity-adjusted t-statistic of alpha coefficient. The third and fourth columns relate to the regression-based test of Huberman and Kandel (1987) showing, respectively, the likelihood ratio (LR) and Lagrange multiplier (LM) statistic, both of which are suitable for asymptotic tests (Kan and Zhou (2008)).

M-V spanning tests broadly agree that most strategies are market anomalies, irrespective of factor set. It differs markedly from prospect spanning. Table 4 shows that M-V spanning tests agree that 17 out of 18 strategies represent anomalies (reject spanning) with respect to FF-5 factors in the full sample, and 12 out of 18 in the post publication period. Table 5 presents qualitatively similar results when assessing trading strategies against M-4 factors. In this case, test statistics agree that 15 out of 18 strategies represent market anomalies in the full sample, and again 12 out of 18 post publication. Similar results obtain with respect to q factors, where 15 out of 18 strategies represent market anomalies in the full sample, and 12 out of 18 post publication.

Although in the post publication period the number of anomalies is reduced, still many strategies are found to be anomalies. The strategies that seem to be explained by the factor models after publication are the Composite Issue, Gross Profitability, Investment/Assets, Net Operating Assets, Return on Assets and Profitability. In-sample prospect spanning tests agree less often that strategies are market anomalies in the post publication period than M-V spanning.
Table 4: M-V spanning tests of market anomalies with respect to Fama-French (FF-5) factors

| Anomaly                  | Alpha (bps) | t-stat | LM     | LR     |
|--------------------------|-------------|--------|--------|--------|
| **Panel (a): Full period** |             |        |        |        |
| Accruals                 | 42          | 3.04** | 111.58** | 123.18** |
| Asset Growth             | 7           | 0.71   | 14.01** | 14.17** |
| Composite Issue          | 34          | 3.01** | 14.72** | 14.89** |
| Distress                 | 93          | 4.14** | 33.48** | 34.60** |
| Gross Profitability      | 36          | 2.79** | 67.66** | 71.50** |
| Investment/Assets        | 342         | 3.69** | 35.43** | 36.44** |
| Momentum                 | 128         | 4.28** | 37.48** | 38.62** |
| Net Operating Assets     | 50          | 3.65** | 78.71** | 83.97** |
| O-Score                  | 47          | 3.90** | 266.04** | 343.54** |
| Return On Assets         | 52          | 3.97** | 20.47** | 20.86** |
| Net Stock Issues         | 34          | 3.53** | 27.78** | 28.40** |
| Betting against Beta     | 43          | 2.99** | 25.76** | 26.29** |
| Quality minus Junk       | 77          | 6.84** | 118.66** | 131.20** |
| Size                     | -198        | -7.79** | 232.58** | 264.37** |
| Growth Option            | -159        | -8.55** | 187.23** | 199.14** |
| Value (Book to Market)   | 143         | 5.94** | 123.87** | 130.27** |
| Idiosyncratic Volatility | 110         | 3.82** | 93.12** | 98.27** |
| Profitability            | -67         | -3.74** | 46.65** | 49.28** |
| **Panel (b): Post publication period** |             |        |        |        |
| Accruals                 | 20          | 0.93   | 22.57** | 24.11** |
| Asset Growth             | 13          | 0.63   | 65.47** | 69.93** |
| Composite Issue          | 10          | 0.52   | 7.53    | 7.69    |
| Distress                 | 114         | 2.96** | 134.54** | 142.11** |
| Gross Profitability      | 0           | 0.00   | 11.17   | 11.43   |
| Investment/Assets        | 3           | 0.13   | 25.75** | 25.97** |
| Momentum                 | 112         | 2.52** | 26.17** | 27.04** |
| Net Operating Assets     | 50          | 2.32** | 53.22** | 55.17** |
| O-Score                  | 40          | 2.14*  | 85.14** | 86.11** |
| Return On Assets         | 6           | 0.22   | 3.11    | 3.14    |
| Net Stock Issues         | 34          | 1.36   | 7.43    | 7.46    |
| Betting against Beta     | 109         | 4.93** | 22.36** | 23.11** |
| Quality minus Junk       | 111         | 1.80   | 18.34** | 19.18** |
| Size                     | -213        | -7.59** | 138.34** | 143.98** |
| Growth Option            | -128        | 3.18** | 45.11** | 47.17** |
| Value (Book to Market)   | 129         | 3.71** | 87.45** | 91.23** |
| Idiosyncratic Volatility | 113         | 2.19*  | 54.93** | 56.87** |
| Profitability            | 35          | 0.69   | 7.86    | 8.33    |

M-V spanning tests of each market anomaly relative to FF-5 factors. The first column displays the alpha coefficient, and the second the heteroskedasticity-adjusted t-statistic. The third and fourth columns display the likelihood ratio (LR) and the Lagrange multiplier (LM) statistics of the regression-based test of Huberman and Kandel (1987). Panel (a) uses the full period while Panel (b) the post publication period.
Table 5: M-V spanning tests of market anomalies with respect to Stambaugh-Yuan (M-4) factors

| Anomaly                          | Alpha(bps) | t-stat | LM      | LR      |
|----------------------------------|------------|--------|---------|---------|
| **Panel (a): Full period**       |            |        |         |         |
| Accruals                         | 18         | 1.24   | 71.27** | 75.77** |
| Asset Growth                     | -19        | -1.73  | 385.75**| 586.19**|
| Composite Issue                  | 3          | 0.29   | 16.34** | 16.54** |
| Distress                         | -6         | -0.37  | 195.91**| 245.99**|
| Gross Profitability              | 12         | 0.98   | 118.05**| 130.32**|
| Investment/Assets                | 10         | 0.82   | 2.53    | 2.53    |
| Momentum                         | -5         | 0.85   | -0.20   | 30.92** |
| Net Operating Assets             | 18         | 1.33   | 24.40** | 24.88** |
| O-Score                          | 32         | 2.46** | 236.37**| 294.04**|
| Return on Assets                 | 30         | 2.23** | 36.02** | 37.28** |
| Net Stock Issues                 | 3          | 0.34   | 24.38** | 24.85** |
| Betting against Beta             | 32         | 2.01*  | 5.80    | 5.83    |
| Quality minus Junk               | 47         | 3.52** | 109.33**| 119.85**|
| Size                             | -229       | -8.74**| 287.34**| 311.76**|
| Growth Option                    | -167       | -7.13**| 213.89**| 237.44**|
| Value (Book to Market)           | 201        | 8.03** | 144.56**| 157.11**|
| Idiosyncratic Volatility         | 144        | 4.47** | 88.23** | 89.36** |
| Profitability                    | -67        | -3.03**| 47.87** | 49.53** |
| **Panel (b): Post publication period** |        |        |         |         |
| Accruals                         | 2          | 0.10   | 21.32** | 23.14** |
| Asset Growth                     | -1         | -0.02  | 25.28** | 25.91** |
| Composite Issue                  | 13         | 0.73   | 14.48   | 15.11   |
| Distress                         | 47         | 1.89   | 31.19** | 32.18** |
| Gross Profitability              | -5         | -0.19  | 11.02   | 11.92   |
| Investment/Assets                | -13        | -0.69  | 5.03    | 5.77    |
| Momentum                         | -28        | -0.84  | 44.87** | 45.11** |
| Net Operating Assets             | 30         | 1.58   | 1.18    | 1.37    |
| O-Score                          | 30         | 1.51   | 127.25**| 128.36**|
| Return On Assets                 | -15        | -0.54  | 7.11    | 7.87    |
| Net Stock Issues                 | 23         | 1.06   | 24.87** | 25.11** |
| Betting against Beta             | 90         | 4.05** | 29.44** | 32.86** |
| Quality minus Junk               | 50         | 1.05   | 16.27** | 20.33** |
| Size                             | -256       | -8.86**| 187.34**| 201.88**|
| Growth Option                    | -122       | -2.45**| 39.23** | 41.12** |
| Value (Book to Market)           | 216        | 6.25** | 84.49** | 97.12** |
| Idiosyncratic Volatility         | 166        | 3.81** | 23.11** | 25.87** |
| Profitability                    | -13        | -0.24  | 8.13    | 9.17    |

M-V spanning tests of each market anomaly relative to M-4 factors. The first column displays the alpha coefficient, and the second the heteroskedasticity-adjusted t-statistic. The third and fourth columns display the likelihood ratio (LR) and the Lagrange multiplier (LM) statistics of the regression-based test of Huberman and Kandel (1987). Panel (a) uses the full period while Panel (b) the post publication period.
Table 6: M-V spanning tests of market anomalies with respect to Hou-Xue-Zhang \((q)\) factors

| Anomaly                          | Alpha (bps) | t-stat | LM   | LR   |
|----------------------------------|-------------|--------|------|------|
| **Panel (a): Full period**       |             |        |      |      |
| Accruals                         | 57          | 4.19** | 81.81** | 88.83** |
| Asset Growth                     | 13          | 1.16   | 3.68  | 3.69  |
| Composite Issue                  | 52          | 4.07** | 57.81** | 60.79** |
| Distress                         | 44          | 2.07*  | 13.35** | 13.53** |
| Gross Profitability              | 38          | 2.33** | 98.07** | 107.09** |
| Investment/Assets                | 36          | 2.92** | 15.72** | 15.93** |
| Momentum                         | 42          | 1.51   | 13.00** | 13.15** |
| Net Operating Assets             | 38          | 2.73** | 45.44** | 47.26** |
| O-Score                          | 46          | 3.60** | 255.63** | 333.12** |
| Return on Assets                 | 11          | 1.00   | 4.67  | 4.69  |
| Net Stock Issues                 | 38          | 3.60** | 53.84** | 56.41** |
| Betting against Beta             | 31          | 2.07*  | 16.78** | 17.02** |
| Quality minus Junk               | 65          | 5.00** | 128.61** | 144.74** |
| Size                             | -226        | -9.30** | 311.18** | 343.73** |
| Growth Option                    | -148        | -7.06** | 202.36** | 224.12** |
| Value (Book to Market)           | 209         | 8.56** | 187.11** | 196.54** |
| Idiosyncratic Volatility         | 167         | 5.96** | 103.87** | 112.40** |
| Profitability                    | -91         | -4.92** | 76.12** | 79.24** |
| **Panel (b): Post publication period** |             |        |      |      |
| Accruals                         | 39          | 1.98   | 84.88** | 93.29** |
| Asset Growth                     | 5           | 0.27   | 22.73** | 23.84** |
| Composite Issue                  | 29          | 1.48   | 2.34  | 2.45  |
| Distress                         | 91          | 2.64** | 20.17** | 22.11** |
| Gross Profitability              | 17          | 0.45   | 11.17 | 12.03 |
| Investment/Assets                | -10         | -0.50  | 7.87  | 8.14  |
| Momentum                         | 40          | 1.01   | 37.92** | 38.45** |
| Net Operating Assets             | 28          | 1.31   | 7.29  | 7.44  |
| O-Score                          | 42          | 2.28*  | 34.28** | 34.59** |
| Return On Assets                 | -20         | -0.99  | 11.07 | 12.27 |
| Net Stock Issues                 | 55          | 2.19*  | 28.18 | 29.60** |
| Betting against Beta             | 95          | 4.38** | 46.34** | 51.17** |
| Quality minus Junk               | 56          | 1.34   | 23.18** | 25.11** |
| Size                             | -2.18       | -7.90** | 155.38** | 158.33** |
| Growth Option                    | -138        | -2.80** | 41.44** | 48.17** |
| Value (Book to Market)           | 190         | 5.68** | 77.50** | 79.40** |
| Idiosyncratic Volatility         | 128         | 2.94** | 22.38** | 23.90** |
| Profitability                    | 6           | 0.12   | 6.89  | 7.13  |

M-V spanning tests of each market anomaly relative to \(q\) factors. The first column displays the alpha coefficient, and the second the heteroskedasticity-adjusted t-statistic. The third and fourth columns display the likelihood ratio (LR) and the Lagrange multiplier (LM) statistics of the regression-based test of Huberman and Kandel (1987). Panel (a) uses the full period while Panel (b) the post publication period.
4.3 Out-of-Sample Analysis

In this section, we conduct out-of-sample backtesting experiments using both prospect and M-V spanning in the post publication period, to examine whether the inclusion of a market anomaly in the investment opportunity set benefits to investors out-of-sample. Those experiments allows us to check the robustness of in-sample results. Although we reject the null hypothesis of prospect spanning in some cases for in-sample tests, it is not known a priori whether an optimal augmented portfolio also outperforms an optimal portfolio made of factors only in an out-of-sample analysis. It is because by construction we form these portfolios at time \( t \), based on the information prevailing at time \( t \), while we reap the portfolio returns over \([t, t+1]\) (next week). The out-of-sample test is a real-time exercise avoiding a potential look-ahead bias and mimicking the way that a real-time investor acts in practice.

We resort to backtesting experiments on a rolling horizon basis. The rolling windows cover the period from 01/1974 to 12/2016. First, we specify the publication date of each anomaly. At each week, we use the data from the previous 4 years (208 weekly observations) to calibrate the procedure. We solve the resulting optimization problem for the prospect stochastic spanning test and record the optimal portfolios. The clock is advanced and we determine the realized returns of the optimal portfolios from the actual returns of the various assets. Then, we repeat the same procedure for the next time period and we compute the ex post realized returns over the entire period for both portfolios.

We compute a number of commonly used performance measures: the average return (Mean), the standard deviation (SD) of returns, the Sharpe ratio, the downside Sharpe ratio (D. Sharpe ratio) of Ziemba (2005), the upside potential and downside risk (UP) ratio of Sortino and van der Meer (1991), the opportunity cost of Simaan (2013), and a measure of the portfolio risk-adjusted returns net of transaction costs (Return Loss) of DeMiguel et al. (2009). The downside Sharpe and UP ratios are considered to be more appropriate measures of performance than the typical Sharpe ratio given the asymmetric return distribution of the anomalies. For the calculation of the opportunity cost, we use the following utility function
which satisfies the curvature of prospect theory (S-shaped): \( U(R) = R^\alpha \) if \( R \geq 0 \) or \(-\gamma(-R)^\beta \) if \( R < 0 \), where \( \gamma \) is the coefficient of loss aversion (usually \( \gamma = 2.25 \)) and \( \alpha, \beta < 1 \). We provide a short description of those performance measures in Appendix B. In the next lines, we only detail the results of the out-of-sample tests for the Momentum market anomaly. The latter is well documented on diverse markets and asset classes (Asness et al. (2013)). In the Online Appendix, we report the performance measures for the 5 Fama and French, the 4 Stambaugh and Yuan and the 4 Hou-Xue-Zhang optimal factor portfolios, and the optimal augmented portfolios for all the other market anomalies that we test.

Table 7 reports the performance measures for the Momentum anomaly under each factor model (Panels A, B and C, respectively). These performance measures supplement the evidence obtained from the in-sample analysis.

For prospect spanning, we observe that the Mean, the Sharpe ratio, downside Sharpe ratio and UP ratio of the optimal augmented portfolio are improved with respect to the optimal factor portfolio. Although these measures are based on the first two moments, they support the in-sample result that the set enlarged with the momentum anomaly is not spanned by any factor model. The same is true when we take into account transaction costs. The Return Loss is always positive. We observe that augmenting the factors by Momentum increases the performance of the optimal portfolio with respect to each factor model. These results are consistent with the in-sample tests.

In contrast, the out-of-sample results for M-V spanning are not consistent with in-sample tests. Although in-sample the Momentum is found to be anomaly in the M-V framework, out-of-sample we observe that the augmented portfolio is not improved compared to the factors optimal portfolio. Thus, we confirm the robustness of in-sample and out-of sample results for prospect spanning but not for M-V spanning. One possible reason for this inconsistency is that M-V optimization average return and covariance matrix estimates over each rolling window could be unstable.

Moreover, we compare the M-V optimal portfolio with the prospect portfolio in terms of
prospect utility loss. We measure it, by the opportunity cost $\theta$, which is the return that needs to be added to the M-V portfolio so that the investor is indifferent in prospect utility terms. In Table 7 we see that the opportunity cost is always positive, indicating that the prospect investor is better off compared to choices under an M-V criterion. The opportunity cost measure takes into account the entire distribution of returns under a given characterization of preferences. The Momentum characteristic is negatively skewed and leptokurtic, which makes it attractive for prospect investors.

In the Online Appendix, we present analogous Tables for the other market anomalies. The analysis indicates that the Composite Equity Issue, Momentum, Betting against Beta, Value, and Idiosyncratic Volatility emerge as unambiguously genuine market anomalies under all factor sets, both in-sample and out-of-sample.

Prospect investors would benefit from including these characteristics in their portfolios, expanding the investment opportunity set offered by factor portfolios. They can also prefer strategies that can produce opportunities with low skewed returns, which are attractive for prospect investors. All these anomalies have high Sharpe ratios, and the skewness is low as expected for a portfolio built for investors with preferences towards risk that are associated with risk aversion for gains and risk loving for losses. We stress that the prospect spanning approach is particularly robust in- and out-of-sample. This remarkable consistency offers good incentives for adopting such an approach when exploring instances of apparent market inefficiency.

In the M-V framework, the Composite Issue, Distress, Gross Profitability, Net Operating Assets, Return on Assets, Net Stock Issues, Betting against Beta, Value and Idiosyncratic Volatility improve the opportunity set of M-V investors out-of-sample. These characteristics exhibit high Sharpe ratio, which makes them attractive for investors that take into account the first two moments only. Finally, we observe that the prospect utility loss is almost in all cases positive, indicating the superiority of the prospect portfolios in prospect utility terms compared to the M-V portfolios. It means that one needs to give a positive return equal to
θ to a M-V investor so that she becomes as happy as a prospect investor in expected utility terms. The computation of the opportunity cost requires the computation of the expected utility and hence the use of the whole probability distribution of portfolio returns. Thus, the calculated opportunity cost has taken into account the higher order moments.

Tables 8-10 report the average weight allocation of the optimal augmented portfolios under the Prospect and the M-V Spanning, to understand better how the choices of a prospect investor and a M-V investor differ. Those choices vary substantially most of the time. More specifically, prospect investors often include a much higher proportion of their wealth in the anomaly than the M-V investors, specially in the anomalies that provide clear portfolio improvements. Prospect investors take concentrated positions in characteristics with joined low skewness and high kurtosis. The optimal weight of Momentum is between 72.6% and 80.3% for the prospect investors, indicating the superior performance of this anomaly for the prospect investor. For M-V investors, the weight of Momentum is only between 0.2% and 18.3%.
Table 7: Performance measures of the optimal portfolios spanning portfolios. The case of the Momentum anomaly.

Prospect Spanning

|                | Panel A | Panel B | Panel C |
|----------------|---------|---------|---------|
|                | FF-5    | M-4     | q       |
| **Mean**       | 0.0020  | 0.0021  | 0.0028  |
| **SD**         | 0.0231  | 0.1938  | 0.0166  |
| **Sharpe ratio** | 0.0760  | 0.1087  | 0.0895  |
| **D. Sharpe ratio** | 0.1408  | 0.4433  | 0.0535  |
| **UP ratio**   | 0.7435  | 1.2393  | 0.9736  |
| **Return Loss** | 0.085%  | 0.024%  | 0.008%  |

M-V spanning

|                | Panel A | Panel B | Panel C |
|----------------|---------|---------|---------|
|                | FF-5    | M-4     | q       |
| **Mean**       | 0.0016  | 0.0019  | 0.0020  |
| **SD**         | 0.0134  | 0.0130  | 0.0141  |
| **Sharpe ratio** | 0.1042  | 0.1275  | 0.1595  |
| **D. Sharpe ratio** | 0.1102  | 0.1653  | 0.1595  |
| **UP ratio**   | 0.6120  | 0.6650  | 0.8010  |
| **Return Loss** | -0.003% | -0.004% | -0.020% |

Prospect utility loss

(Opportunity Cost)

- $\alpha = \beta = 0.2$
  - 0.162% 1.139% 0.287% 0.491% 0.159% 0.839%
- $\alpha = \beta = 0.4$
  - 0.146% 1.025% 0.228% 0.442% 0.143% 0.775%
- $\alpha = \beta = 0.6$
  - 0.131% 0.923% 0.175% 0.398% 0.129% 0.617%

Entries report the performance measures (Mean, Standard Deviation, Sharpe ratio, Downside Sharpe ratio, UP ratio and Returns Loss) for the factor optimal portfolios, as well as the augmented with the Momentum optimal portfolio under prospect spanning and M-V spanning. Panel A report measures for the case of the FF-5 factors. Panel B for the case of the M-4 factors, while panel C for the case of the q factors. Finally the Table yields the prospect utility loss of the M-V portfolios over the prospect portfolios.
Table 8: Optimal Prospect and M-Vspanning portfolios. The case of Fama-French (FF-5) factors

| Anomaly                        | Prospect | Variance |
|-------------------------------|----------|----------|
|                               | MKt      | SMB      | HML      | RMW      | CMA      | Anom     | MKt      | SMB      | HML      | RMW      | CMA      | Anom     |
| Accruals                      | 15.9     | 8.2      | 6.5      | 15.1     | 11.5     | 42.8     | 52.6     | 0.2      | 8.2      | 18.2     | 12.8     | 7.9      |
| Asset Growth                  | 6.3      | 5.1      | 3.6      | 11.4     | 6.2      | 67.5     | 52.1     | 0.1      | 7.9      | 17.7     | 10.2     | 11.8     |
| Composite Issue               | 3.9      | 2.5      | 3.2      | 14.9     | 7.9      | 67.6     | 48.4     | 0.7      | 7.2      | 9.7      | 12.5     | 21.5     |
| Distress                      | 6.0      | 6.3      | 4.9      | 28.4     | 14.6     | 39.8     | 43.1     | 0.6      | 13.2     | 8.8      | 17.4     | 16.8     |
| Gross Profitability           | 20.3     | 7.2      | 7.3      | 28.5     | 20.5     | 16.2     | 53.4     | 0.0      | 12.9     | 12.7     | 14.0     | 7.0      |
| Investment/Assets             | 5.5      | 3.9      | 4.2      | 9.5      | 18.9     | 58.0     | 50.5     | 0.2      | 6.7      | 21.5     | 8.2      | 12.9     |
| Momentum                      | 15.9     | 1.4      | 2.2      | 5.4      | 2.5      | 72.6     | 34.0     | 0.8      | 11.0     | 22.9     | 17.3     | 14.0     |
| Net Operating Assets          | 6.5      | 3.7      | 1.9      | 5.3      | 1.9      | 80.6     | 28.6     | 1.4      | 8.0      | 21.4     | 14.0     | 26.6     |
| O-Score                       | 14.9     | 10.3     | 9.9      | 32.4     | 20.2     | 12.4     | 53.3     | 0.0      | 10.9     | 15.5     | 16.5     | 3.9      |
| Return On Assets              | 8.9      | 0.2      | 0.2      | 1.0      | 0.4      | 89.2     | 39.7     | 1.9      | 10.4     | 0.6      | 20.3     | 27.2     |
| Net Stock Issues              | 6.4      | 1.3      | 0.7      | 9.9      | 4.5      | 77.2     | 46.0     | 1.6      | 8.9      | 5.4      | 8.0      | 30.0     |
| Betting against Beta          | 19.9     | 2.8      | 3.6      | 12.1     | 8.7      | 52.8     | 33.4     | 0.7      | 3.7      | 17.7     | 18.3     | 26.2     |
| Quality minus Junk            | 36.7     | 2.0      | 2.5      | 4.9      | 4.2      | 49.7     | 38.5     | 6.9      | 11.2     | 0.1      | 15.2     | 28.1     |
| Size                          | 22.8     | 8.4      | 8.3      | 34.5     | 25.9     | 0.0      | 53.9     | 0.1      | 10.7     | 18.5     | 16.9     | 0.0      |
| Growth Option                 | 21.8     | 8.5      | 9.6      | 34.5     | 25.7     | 0.0      | 53.9     | 0.1      | 10.5     | 18.6     | 17.0     | 0.0      |
| Value (Book/Market)           | 40.6     | 0.0      | 0.0      | 0.0      | 0.0      | 59.4     | 24.6     | 10.9     | 1.0      | 26.0     | 23.4     | 14.2     |
| Idiosyncratic Volatility      | 36.4     | 5.6      | 9.3      | 27.1     | 17.1     | 4.6      | 26.7     | 0.1      | 10.2     | 35.5     | 18.3     | 9.2      |
| Profitability                 | 16.6     | 9.8      | 10.3     | 36.1     | 26.4     | 0.8      | 53.9     | 0.1      | 10.7     | 18.4     | 16.9     | 0.0      |

The Table exhibits the average weight allocation of the optimal augmented portfolios under the Prospect and the M-V Spanning. The FF-5 model is used, for the post-publication period.
Table 9: Optimal Prospect and M-V spanning portfolios. The case of Stambaugh-Yuan (M-4) factors

| Anomaly          | Prospect | Variance |         |         |         |         |         |         |
|------------------|----------|----------|---------|---------|---------|---------|---------|---------|
|                  | MKt      | SMB      | MGMT    | PERF    | Anom    | MKt     | SMB     | MGMT    | PERF    | Anom    |
| Accruals         | 13.9     | 13.1     | 19.8    | 8.7     | 44.5    | 37.4    | 0.0     | 8.0     | 49.0    | 5.5     |
| Asset Growth     | 7.0      | 9.6      | 12.4    | 8.3     | 62.6    | 38.0    | 0.1     | 4.7     | 48.4    | 8.8     |
| Composite Issue  | 3.9      | 2.3      | 10.1    | 6.0     | 77.7    | 29.7    | 0.4     | 7.7     | 47.3    | 14.9    |
| Distress         | 7.8      | 5.9      | 21.5    | 17.7    | 47.1    | 22.1    | 0.0     | 11.7    | 40.1    | 26.0    |
| Gross Profitability | 16.0    | 26.4     | 38.2    | 16.3    | 3.2     | 37.9    | 0.0     | 11.8    | 50.3    | 0.0     |
| Investment/Assets| 10.5     | 12.2     | 23.1    | 6.5     | 47.7    | 37.6    | 0.3     | 6.2     | 44.2    | 11.7    |
| Momentum         | 10.4     | 3.7      | 4.7     | 4.8     | 76.5    | 27.0    | 0.3     | 24.3    | 30.1    | 18.3    |
| Net Operating Assets | 6.2      | 4.7      | 6.4     | 2.1     | 80.6    | 14.3    | 1.7     | 13.3    | 38.4    | 32.3    |
| O-Score          | 18.4     | 15.7     | 39.1    | 16.1    | 10.8    | 37.9    | 0.0     | 11.8    | 50.3    | 0.0     |
| Return On Assets | 2.6      | 0.7      | 0.3     | 0.8     | 95.5    | 23.8    | 0.3     | 12.0    | 32.8    | 31.2    |
| Net Stock Issues | 5.5      | 0.7      | 2.3     | 6.3     | 85.2    | 28.7    | 0.3     | 8.3     | 45.9    | 16.7    |
| Betting against Beta | 18.7    | 12.5     | 19.1    | 6.1     | 43.6    | 15.5    | 1.3     | 17.6    | 23.9    | 41.7    |
| Quality minus Junk | 29.0    | 12.3     | 19.5    | 4.0     | 35.1    | 20.8    | 0.5     | 12.4    | 29.2    | 37.1    |
| Size             | 25.8     | 14.0     | 42.1    | 18.1    | 0.0     | 37.9    | 0.0     | 11.8    | 50.3    | 0.0     |
| Growth Option    | 25.7     | 14.5     | 42.1    | 17.7    | 0.0     | 37.9    | 0.0     | 11.8    | 50.3    | 0.0     |
| Value (Book/Market) | 33.0    | 0.0      | 0.0     | 0.0     | 67.0    | 15.8    | 16.1    | 20.1    | 33.3    | 14.7    |
| Idiosyncratic Volatility | 30.8    | 2.4      | 43.3    | 20.1    | 3.3     | 9.6     | 0.0     | 24.2    | 55.6    | 10.6    |
| Profitability    | 18.5     | 19.6     | 40.7    | 17.9    | 3.3     | 37.9    | 0.0     | 11.8    | 50.3    | 0.0     |

The Table exhibits the average weight allocation of the optimal augmented portfolios under the Prospect and the M-V Spanning. The M-4 model is used, for the post-publication period.
Table 10: Optimal Prospect and M-V spanning portfolios. The case of Hou-Xue-Zhang (q) factors

| Anomaly                  | MKt | ME  | IA  | ROE | Anom | MKt | ME  | IA  | ROE | Anom |
|--------------------------|-----|-----|-----|-----|------|-----|-----|-----|-----|------|
| Accruals                 | 9.9 | 8.9 | 14.8| 4.3 | 62.1 | 28.1| 12.1| 37.6| 18.7| 3.6  |
| Asset Growth             | 7.2 | 4.9 | 6.4 | 1.6 | 79.9 | 28.4| 12.8| 40.1| 18.6| 0.1  |
| Composite Issue          | 5.0 | 6.5 | 10.9| 1.9 | 75.8 | 28.4| 12.7| 38.1| 18.4| 2.3  |
| Distress                 | 8.3 | 11.6| 15.9| 4.3 | 60.0 | 28.7| 12.9| 40.0| 17.7| 0.8  |
| Gross Profitability      | 8.7 | 15.4| 26.5| 4.9 | 44.5 | 29.0| 10.1| 40.1| 15.0| 5.8  |
| Investment/Assets        | 9.7 | 8.4 | 12.3| 3.5 | 66.1 | 28.2| 12.1| 38.8| 18.9| 2.0  |
| Momentum                 | 11.0| 2.6 | 3.3 | 2.8 | 80.3 | 28.3| 12.9| 40.3| 18.3| 0.2  |
| Net Operating Assets     | 5.4 | 3.8 | 2.2 | 0.3 | 88.3 | 25.3| 13.6| 35.7| 17.3| 8.1  |
| O-Score                  | 17.8| 14.0| 20.9| 5.8 | 41.5 | 28.6| 11.0| 38.5| 15.9| 6.0  |
| Return On Assets         | 5.2 | 0.8 | 1.9 | 0.4 | 91.8 | 28.0| 13.9| 36.8| 10.5| 10.8 |
| Net Stock Issues         | 7.7 | 2.6 | 5.9 | 0.9 | 83.0 | 28.3| 12.5| 36.7| 17.7| 4.8  |
| Betting against Beta     | 19.6| 5.7 | 7.9 | 1.5 | 65.2 | 27.4| 13.3| 37.6| 17.5| 4.3  |
| Quality minus Junk       | 17.7| 4.9 | 16.2| 11.5| 49.7 | 29.4| 14.9| 32.3| 10.7| 12.7 |
| Size                     | 11.0| 9.8 | 38.4| 40.8| 0.0  | 28.3| 12.8| 40.1| 18.5| 0.3  |
| Growth Option            | 12.2| 8.8 | 38.7| 40.3| 0.0  | 28.5| 11.6| 40.4| 18.8| 0.6  |
| Value (Book/Market)      | 0.0 | 0.0 | 0.0 | 0.0 | 1.0  | 21.5| 20.9| 32.6| 18.0| 7.0  |
| Idiosyncratic Volatility | 27.5| 0.5 | 22.6| 20.1| 29.3 | 23.0| 14.9| 42.4| 16.5| 3.1  |
| Profitability            | 10.3| 12.8| 41.4| 33.3| 2.2  | 28.6| 12.2| 39.9| 17.8| 1.5  |

The Table exhibits the average weight allocation of the optimal augmented portfolios under the Prospect and the M-V Spanning. The q model is used, for the post-publication period.
5 Conclusions

In this paper, we develop and implement methods for determining whether introducing new securities or relaxing investment constraints improves the investment opportunity set for prospect investors. We develop a testing procedure for prospect spanning for two nested portfolio sets based on subsampling and standard LP.

In the empirics, we apply the prospect spanning framework to asset prices in which investors evaluate risk according to prospect theory and examine its ability to explain 18 well-known stock market anomalies. The setting deploys prospect theory in a fully nonparametric way. We find that of the strategies considered, a few expand the opportunity set of prospect investors, thus have real economic value for them.

Most importantly, we show that the prospect spanning approach is particularly robust between in- and out-of-sample analysis. We also compare the prospect spanning with M-V spanning both in- as well as out-of-sample. We observe that M-V spanning results are not that robust in- and out-of-sample. Moreover, in most cases, the prospect investors are better off compared to choices under an M-V criterion, as measured by the prospect utility loss.

The paper contributes to a current strand of literature aiming to reevaluate published anomalies and discern those with real economic content for prospect investors. From a practitioner perspective, this robust framework for establishing investment opportunities for prospect investors can be of real value, especially in the case of quantitative investment funds that combine talent, capital and computational power to the purpose of exploiting the existing anomalies and discovering new ones.

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APPENDIX A: Description of Stock Market Anomalies
Below we provide the origin and a short description of the 18 market anomalies used in the empirical application.

1. Accruals: Sloan (1996) argues that investors tend to overestimate in their earnings expectations the persistence of the earnings' component that is due to accruals. As a result, firms with low accruals earn on average abnormally higher returns than firms with high accruals.

2. Asset Growth: Cooper, Gulen, and Schill (2008) maintain that investors tend to overreact positively right after asset expansions. According to the authors, this behavior causes firms with high growth in their total assets to exhibit relatively lower returns over the subsequent fiscal years.

3. Composite Equity Issues: Daniel and Titman (2006) base their analysis on a measure of equity issuance that they devised finding that equity issuers tend to underperform non-issuer firms.

4. Distress: Campbell, Hilscher, and Szilagyi (2008) find that firms with high default probability tend to exhibit lower subsequent returns. This pattern is counter-intuitive in the context of rational asset pricing, given that according to the standard models high risk entails high expected return and vice versa.

5. Gross Profitability Premium: Novy-Marx (2013) argues that gross profit is the most objective profitability metric. As a result, firms with the strongest gross profit have on average higher returns than the less profitable ones.

6. Investment to Assets: Titman, Wei, and Xie (2004) argue that investors are put off by empire-building managers who over-invest. For this reason, firms showing a significant increase in gross property, plant, equipment or inventories tend to underperform the market.

7. Momentum: Momentum (Jegadeesh and Titman (1993)) is perhaps the most cited anomaly in asset pricing. Since Carhart factor model (1997), it has been included in various reduced-form models of the SDF as a factor. The momentum effect is attributed to sentiment and describes the pattern of “winner” stocks gaining higher subsequent returns and “loser”
stocks relatively lower.

8. Net Operating Assets: Hirshleifer et al. (2004) suggest that investors often neglect information about cash profitability and focus instead on accounting profitability. Because of this bias, firms with high net operating assets (measured as the cumulative difference between operating income and free cash flow) get to have negative long-run stock returns.

9. Net Stock Issues: Ritter (1991) and Loughran and Ritter (1995) indicate that equity issuers underperform non-issuers with similar characteristics. Fama and French (2008) demonstrate that net stock issues are negatively correlated with subsequent returns.

10. O-Score: This anomaly coincides with the distress anomaly we mentioned earlier. In this case, the spread portfolios are constructed from stock ranking based on the O-score (Ohlson (1980)) to measure distress likelihood.

11. Return on Assets: Chen, Novy-Marx, and Zhang (2010) associate high past return on assets with abnormally high subsequent returns. Return on assets is measured as the ratio of quarterly earnings to last quarter’s assets.

12. Betting against Beta: Black, Jensen and Scholes (1972) showed that low (high) beta stocks have consistently positive (negative) risk-adjusted returns. Frazzini and Pedersen (2014) propose an investment strategy ("betting-against-beta" (BAB)) that exploits this anomaly by buying low-beta stocks and shorting high-beta stocks. Because of its robustness, this anomaly is currently one of the most widely examined APT violations.

13. Quality minus Junk: Asness, Frazzini and Pedersen (2013) show that high-quality stocks (safe, profitable, growing, and well managed) exhibit high risk-adjusted returns. The authors attribute this pattern to mispricing.

14. Size: The market capitalization is computed as the log of the product of price per share and number of shares outstanding, computed at the end of the previous month.

15. Growth Option: Growth Option measure represents the residual future-oriented firm growth potential. This future (yet-to-be exercised) growth option measure is calculated as the % of a firm’s market value (V) arising from future-oriented growth opportunities.
(PVGO/V). It is inferred by subtracting from the current market value of the firm (V) the perpetual discounted stream of expected operating cash flows under a no-further growth policy (see, e.g., Kester (1984), Anderson and Garcia-Feijoo (2006), Berk, Green, and Naik (1999)).

16. Value (Book to market): The log of book value of equity scaled by market value of equity, computed following Fama and French (1992) and Fama and French (2008); firms with negative book value are excluded from the analysis.

17. Idiosyncratic Volatility: Standard deviation of the residuals from a firm-level regression of daily stock returns on the daily Fama-French three factors using data from the past month. See Ang et al. (2006).

18. Profitability.: It is measured as revenue minus cost of goods sold at time t, divided by assets at time t-1. Stocks with high profitability ratios tend to outperform on a risk-adjusted basis (Novy-Marx (2013), Novy-Marx and Velikov (2015)). Recent research suggests that profitability is one of the stock return anomalies that has the largest economic significance (see Novy-Marx (2013)).
APPENDIX B: Description of Performance Measures

For the downside Sharpe ratio, first we need to calculate the downside variance (or more precisely the downside risk), $\sigma^2_{P_-} = \frac{\sum_{t=1}^{T}(x_t - \bar{x})^2}{T-1}$, where the benchmark $\bar{x}$ is zero, and the $x_t$ taken are those returns of portfolio $P$ at month $t$ below $\bar{x}$, i.e., those $t$ of the $T$ months with losses. To get the total variance, we use twice the downside variance namely $2\sigma^2_{P_-}$ so that the downside Sharpe ratio is, $S_P = \frac{\bar{R}_p - \bar{R}_f}{\sqrt{2}\sigma_{P_-}}$, where $\bar{R}_p$ is the average period return of portfolio $P$ and $\bar{R}_f$ is the average risk free rate. The UP ratio compares the upside potential to the shortfall risk over a specific target (benchmark) and is computed as follows. Let $R_t$ be the realized monthly return of portfolio $P$ for $t = 1, ..., T$ of the backtesting period, where $T = 216$ is the number of experiments performed and let $\rho_t$ be respectively the return of the benchmark (risk free rate) for the same period. Then, we have, UP ratio $= \frac{\frac{1}{K} \sum_{t=1}^{K} \max[0, R_t - \rho_t]}{\sqrt{\frac{1}{K} \sum_{t=1}^{K} (\max[0, \rho_t - R_t])^2}}$. It is obvious that the numerator of the above ratio is the average excess return over the benchmark and so reflects upside potential. In the same way, the denominator measures downside risk, i.e., shortfall risk over the benchmark.

Next, we use the concept of opportunity cost presented in Simaan (2013) to analyse the economic significance of the performance difference of the two optimal portfolios. Let $R_{Aug}$ and $R_F$ be the realized returns of the optimal augmented and the optimal factors portfolios, respectively. Then, the opportunity cost $\theta$ is defined as the return that needs to be added to (or subtracted from) the optimal factors portfolio return $R_F$, so that the investor is indifferent (in utility terms) between the strategies imposed by the two different investment opportunity sets, i.e., $E[U(1 + R_F + \theta)] = E[U(1 + R_{Aug})]$.

A positive (negative) opportunity cost implies that the investor is better (worse) off if the investment opportunity set allows for the market anomaly factor prospect type investing. The opportunity cost takes into account the entire probability distribution of asset returns and hence it is suitable to evaluate strategies even when the asset return distribution is not normal. For the calculation of the opportunity cost, we use the following utility function
which satisfies the curvature of prospect theory (S-shaped): \( U(R) = R^\alpha \) if \( R \geq 0 \) or \(-\gamma(-R)^\beta \) if \( R < 0 \), where \( \gamma \) is the coefficient of loss aversion (usually \( \gamma = 2.25 \)) and \( \alpha, \beta < 1 \).

Finally, we evaluate the performance of the two portfolios under the risk-adjusted (net of transaction costs) returns measure, proposed by DeMiguel et al. (2009) which indicates the way that the proportional transaction cost, generated by the portfolio turnover, affects the portfolio returns. Let \( trc \) be the proportional transaction cost, and \( R_{P,t+1} \) the realized return of portfolio \( P \) at time \( t+1 \). The change in the net of transaction cost wealth \( NW_{P,t} \) of portfolio \( P \) through time is, \( NW_{P,t+1} = NW_{P,t}(1 + R_{P,t+1})[1 - trc \times \sum_{i=1}^{N}(|w_{P,i,t+1} - w_{P,i,t}|)]. \) The portfolio return, net of transaction costs is defined as \( RTC_{P,t+1} = \frac{NW_{P,t+1}}{NW_{P,t}} - 1 \). Let \( \mu_F \) and \( \mu_{Aug} \) be the out-of-sample mean of monthly \( RTC \) factors and the Augmented optimal portfolio, respectively, and \( \sigma_F \) and \( \sigma_{Aug} \) be the corresponding standard deviations. Then, the return-loss measure is, \( R_{Loss} = \frac{\mu_{Aug}}{\sigma_{Aug}} \times \sigma_F - \mu_F \), i.e., the additional return needed so that the factors performs equally well with the optimal augmented with the market anomaly portfolio. We follow the literature and use 35 bps for the transaction costs.
Spanning analysis of stock market anomalies under Prospect Stochastic Dominance

Stelios Arvanitis, Olivier Scaillet, Nikolas Topaloglou

Online Appendix

This online Appendix contains: i) the limiting properties of the testing procedures under sequences of local alternatives, ii) a Monte Carlo study of the finite sample properties of the test, iii) the proofs of the main results, as well as auxiliary lemmata and their proofs, iv) summary statistics of the factor and anomaly returns over our sample period from January 1974 to December 2016 and v) additional empirical results on out-of-sample analysis of market anomalies. We keep the numbering of assumptions and results as in the main text. We introduce a local numbering for assumptions and results that only appear here.

1 Local Alternatives

We enhance the consistency results of Theorem 9 by considering the limiting behavior of the testing procedure under a sequence of local to spanning alternatives. In this respect, \( \rightsquigarrow \) denotes weak convergence under the measure with cdf \( G \). Furthermore, \( L^0_2(F) \) denotes the space of random variables with zero mean and finite second moment.

**Assumption 1 (LOCAL).** There exists a sequence of cdf \( (F^*_t)_{t \in \mathbb{N}} \) such that for some \( h \in \mathbb{N} \)
\[L_2^0(F)\]
\[
\int_{\mathbb{R}^n} \left[ \sqrt{T} \left( \sqrt{dF_T^*} - \sqrt{dF} \right) - h \sqrt{dF} \right] \to 0, \text{ as } T \to \infty.
\]

Assumptions 5 and LOCAL along with Theorem 7.3 of Rio (2000) and the analogous extension of Theorem 1 of Wellner (1992) imply that \[
\sqrt{T} (F_T - F) \Rightarrow G_F + \delta_h \text{ where } \delta_h(x) := \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \ldots \int_{-\infty}^{x_n} h dF \text{ for any } x \in \mathbb{R}^n.
\]

**Proposition 2 (LOCLIM).** Under Assumptions 5 and LOCAL, as \(T \to \infty\),

\[
\rho_T \Rightarrow \rho^*_\infty := \max_{i=+,-} \sup_{\lambda} \sup_{z} \inf_{\kappa} \left[ P_i(z, \lambda, \kappa, G_F) + P_{ih}(z, \lambda, \kappa, F) \right], \quad (\lambda, z, \kappa) \in \Gamma_i,
\]

where

\[
P_{ih}(z, \lambda, \kappa, F) := J_{h}(z, 0, \kappa, F) - J_{h}(z, 0, \lambda, F), \quad z \in \mathbb{R}_-,
\]

\[
P_{2h}(z, \lambda, \kappa, F) := J_{h}(0, z, \kappa, F) - J_{h}(0, z, \lambda, F), \quad z \in \mathbb{R}_{++},
\]

\[
J_{h}(z_1, z_2, \lambda; F) := \int_{z_1}^{z_2} G_{h}(u, \lambda, F) \, du,
\]

and

\[
G_{h}(z, \lambda, F) := \int_{\mathbb{R}^n} \mathbb{I}\{\lambda^T u \leq z\} h(u) \, dF(u).
\]

Using the premises of the previous proposition and if \(P_{ih} > C > 0\) for all \(i\) then a result such as the C2 one in Theorem 1 of Wellner (1992), and arguments analogous to the proof of Proposition 6 imply that

\[
\rho(F_T^*) = \frac{1}{\sqrt{T}} \max_{i=+,-} \sup_{\lambda} \sup_{z} \inf_{\kappa} \left[ P_{ih}(z, \lambda, \kappa, F) \right], \quad (\lambda, z, \kappa) \in \Gamma_i.
\]

Hence, in such a case, we have that \(K \not\subseteq P \subseteq L\), and we can construct the following sequence
of local alternative hypotheses:

\[ H^*_T : \rho (F^*_T) = \frac{c}{\sqrt{T}}, \]

where \( c := \max_{i=+,-} \sup_{\lambda} \sup_z \inf_\kappa [P_{ih}(\lambda, \kappa, F)] > 0 \) and \((\lambda, z, \kappa) \in \Gamma_i\). Obviously, \( H^*_T \) approximates the null hypothesis as \( T \to \infty \).

**Proposition 3** (LOCUN). Suppose that Assumptions 5 and LOCAL hold, \( P_{ih} > C > 0 \), while, for some \( \lambda \in L-\mathbb{K} \), \( \inf_{Y_0} \lambda^T Y_0 \leq 0 \), there exists \((\kappa, z) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 \) with \((\lambda, \kappa, z) \in \Gamma_2\), and that if \((\lambda^*, \kappa^*, z^*) \in \Gamma_2 \) for some \( \kappa^* \neq \kappa \) then \( z^* \neq z \). Under \( H^*_T \) and as \( T \to \infty \),

\[
\lim_{T \to \infty} \mathbb{P} (\rho_T > q_{T,b_T} (1 - \alpha)) = \mathbb{P} (\rho^*_\infty > q (\rho^*_\infty, \alpha)) > \alpha.
\]

Hence the test is asymptotically locally unbiased under the chosen sequence of local alternatives.

## 2 Monte Carlo Study

We design and perform a set of Monte Carlo experiments to evaluate the size and power of the proposed tests in finite samples. We do so in a framework of conditional heteroskedasticity that is partially consistent with empirical findings on returns of financial data and relevant to the empirical application that we develop in the main text. We construct \((Y_t)_{t \in \mathbb{Z}}\) as a vector GARCH(1,1) process that also contains an appropriately transformed element. Under the relevant restrictions, this allows for both temporal as well as cross sectional dependence between the random variables that constitute the vector process.

Suppose that \( z_t \overset{\text{iid}}{\sim} N(0,1) \), \( t \in \mathbb{Z} \). Furthermore for all \( t \in \mathbb{Z} \), for \( i = 1, 2, 3, \ omega_i, \alpha_i, \beta_i \in \mathbb{R}_+^2 \), \( \mu_i \in \mathbb{R}_+ \) define \( y_{i,t} = \mu_i + z_i h_i^{1/2} \), with \( h_i = \omega_i + (\alpha_i z_{i-1}^2 + \beta_i) h_{i-1} \), such that \( \mathbb{E} (\alpha_i z_0^2 + \beta_i)^{1+\epsilon} < 1 \), for some \( \epsilon > 0 \), while, for \( i = 4 \) and \( v_1, v_2 \in \mathbb{R} \), define \( y_{4,t} = v_1 (z_i h_{3,t}^{1/2})_+ + v_2 (z_i h_{3,t}^{1/2})_- \). Suppose that \( Y_t = (y_{1,t}, y_{2,t}, y_{3,t}, y_{4,t})' \). Arvanitis and Topaloglou
(2017) establish that the vector process above satisfies our assumption framework. Let \( \tau = (0, 0, 1, 0) \), \( \tau^* = (0, 0, 0, 1) \) and \( \mathbb{L} = \{ (\lambda, 1 - \lambda, 0, 0) \, \lambda \in [0, 1], \tau, \tau^* \} \). Using this portfolio space we obtain the following result on Prospect-spanning. Its proof follows directly from Proposition 4 of Arvanitis and Topaloglou (2017) and it essentially depends on the fact that \( \tau^* \) is a Prospect super-efficient portfolio w.r.t. the portfolio space.

**Proposition 4 (MC).** If \( \mu_i = 0 \) for \( i = 1, 2, 3 \), \( |v_1| > \sqrt{\frac{\max\{|\omega_i, \alpha_i, \beta_i, i = 1, 2, 3\}|}{\min\{|\omega_i, \alpha_i, \beta_i, i = 1, 2, 3\}} \) and \( |v_2| < \sqrt{\frac{\min\{|\omega_i, \alpha_i, \beta_i, i = 1, 2, 3\}|}{\max\{|\omega_i, \alpha_i, \beta_i, i = 1, 2, 3\}} \) then \( \mathbb{K} := \{ (\lambda, 1 - \lambda, 0, 0) \, \lambda \in [0, 1] \} \cup \{ \tau^* \} \) Prospect-spans \( \mathbb{L} \), while \( \mathbb{K} - \{ \tau^* \} \) does not Prospect-span \( \mathbb{L} \).

**Scenarios** We use as DGPs instances of the GARCH processes conforming to the previous Proposition 4 in order to evaluate the size and power under a fixed \( T \).

**Size Evaluation Scenario-Parameters Selection:** To approximate the fixed \( T \) size, we test for PSD spanning by setting \( \mu_i = 0 \) for \( i = 1, 2, 3 \), \( \omega_1 = 0.5 \), \( \omega_2 = 0.5 \), and \( \omega_3 = 0.5 \), \( a_1 = 0.4 \), \( a_2 = 0.45 \), and \( a_3 = 0.5 \) and \( \beta_1 = 0.5 \), \( \beta_2 = 0.45 \), \( \beta_3 = 0.4 \), \( v_1 = 1.5 \) and \( v_2 = 0.5 \). In this case, we have that \( |v_1| > \sqrt{\frac{\max\{|\omega_i, \alpha_i, \beta_i, i = 1, 2, 3\}|}{\min\{|\omega_i, \alpha_i, \beta_i, i = 1, 2, 3\}} \) and \( |v_2| < \sqrt{\frac{\min\{|\omega_i, \alpha_i, \beta_i, i = 1, 2, 3\}|}{\max\{|\omega_i, \alpha_i, \beta_i, i = 1, 2, 3\}} \).

**Power evaluation Scenario-Parameters Selection:** To approximate the fixed \( T \) power, we test for PSD spanning by setting \( \mu_i = 0 \) for \( i = 1, 2, 3 \), \( \omega_1 = 0.5 \), \( \omega_2 = 0.5 \), and \( \omega_3 = 0.8 \), \( a_1 = 0.3 \), \( a_2 = 0.4 \), and \( a_3 = 0.45 \) and \( \beta_1 = 0.3 \), \( \beta_2 = 0.4 \), \( \beta_3 = 0.45 \), \( v_1 = 2 \) and \( v_2 = 0.2 \). In this case, we have that \( \omega_1 < \omega_3 \), \( a_1 < a_3 \) and \( \beta_1 < \beta_3 \).

**Results** We present our Monte Carlo results in Table 1. We use three cases. In the first case, \( T = 300 \) and we get the subsampling distribution of the test statistic for subsample size \( b_T \in \{50, 100, 150, 200\} \). In the second case, \( T = 500 \) and \( b_T \in \{100, 200, 300, 400\} \). Finally, in the third case, \( T = 1000 \) and \( b_T \in \{120, 240, 360, 480\} \). We present the results using the original subsampling critical values (without bias correction) as well as the ones obtained using the bias correction method. We observe that for small samples (\( T=300 \) and
T=500) the bias correction method is more efficient and more powerful. The test with the bias correction method seems to perform well in all cases with an empirical size close to 5% and an empirical power above 90% for a nominal size $\alpha = 5\%$.

We observe that the computational time is not increasing with the number of assets, it is only increasing with the number of observations.

| Monte Carlo Results |
|---------------------|
|                     |
| Without bias correction |
| Cases | $T=300$ | $T=500$ | $T=1000$ |
| Empirical size | 12.9% | 11.8% | 9.5% |
| Empirical power  | 82.6% | 85.8% | 90.2% |

| With bias correction |
|----------------------|
| Cases | $T=300$ | $T=500$ | $T=1000$ |
| Empirical size | 3.9% | 4.6% | 4.1% |
| Empirical power  | 91.5% | 92.3% | 94.3% |

Table 1: Monte Carlo Results. Entries report the empirical size and empirical power based on 1000 replications and a nominal size $\alpha = 5\%$. The rejection probabilities are calculated both without and with the bias correction method.

3 Proofs

3.1 Proofs of Main Results

Proof of Lemma 3. i. ($\Leftarrow$) If $\mathbb{K} \succeq_P \mathbb{L}$, we have from Definition 1 that, for any $\lambda$, there exists some $\kappa$ such that $\sup_{z \in A_1} P_1(z, \lambda, \kappa, F) \leq 0$ and $\sup_{z \in A_2} P_2(z, \lambda, \kappa, F) \leq 0$. It implies that $\max_{i=1,2} \sup_{z \in A_i} \inf_{\kappa \in \mathbb{K}} P_i(z, \lambda, \kappa, F) \leq 0$, which in turn implies that $\rho(F) \leq 0$. Since $\mathbb{K}$ is closed and thereby compact, the Dominated Convergence Theorem implies that $\mathcal{J}(z, 0, \kappa, F)$ is continuous w.r.t. $\kappa$. This continuity along with the compactness of $\mathbb{K}$ imply that $\arg \min_{\kappa \in \mathbb{K}} \mathcal{J}(z, 0, \kappa, F)$ is non empty. Let $\kappa^*$ be an element of the latter. Then, the first equality follows from $\rho(F) \geq \inf_{\kappa \in \mathbb{K}} \mathcal{J}(z, 0, \kappa, F) - \mathcal{J}(z, 0, \kappa^*, F) = 0$. If $\mathbb{K} \not\succeq_P \mathbb{L}$ then for some $\lambda^* \in \mathbb{L}$, and any $\kappa \in \mathbb{K}$, there exists some $i^*$ and $z^* \in A_{i^*}$ such that $P_{i^*}(z^*, \lambda^*, \kappa, F) > 0$. Then the continuity of $\mathcal{J}(z, 0, \kappa, F)$ and $\mathcal{J}(0, \lambda, \kappa, F)$ w.r.t. $\kappa$ and the compactness of $\mathbb{K}$,
implies that for any \( z \in A_i \), \( \exists \kappa_{\lambda,z,i} \in \mathbb{K} \) such that \( \inf_{\kappa \in \mathbb{K}} P_i(z, \lambda, \kappa, F) = P_i(z, \lambda, \kappa_{\lambda,z,i}, F) \), and thereby \( \rho(F) \geq P_i^*(z^*, \lambda^*, \kappa_{\lambda^*,z^*,i^*}, F) > 0 \).

ii. \( (\Rightarrow) \) If \( \rho(F) = 0 \), then for any \( \lambda \in \mathbb{L} \) we get that \( \max_{i=1,2} \sup_{z \in A_i} \inf_{\kappa \in \mathbb{K}} P_i(z, \lambda, \kappa, F) \leq 0 \). Hence, there exists an element of \( \mathbb{K} \) for which \( P_i(z, \lambda, \kappa, F) \leq 0 \), for every \( z \in A_i, i = 1, 2 \).

Proof of Lemma 4. Integrating by parts, for any \(-\infty < \alpha < \beta < +\infty\), we have that

\[
\int_\alpha^\beta G(u, \lambda, F) \, du = (u - \beta) G(u, \lambda, F)|^\beta_\alpha - \int_\alpha^\beta (u - \beta) \, dG(u, \lambda, F)
\]

\[
= (\beta - \alpha) G(\alpha, \lambda, F) - \int \alpha^\beta (u - \beta) 1_{\alpha \leq u \leq \beta} \, dG(u, \lambda, F)
\]

\[
= \int \int_\mathbb{R} [(\beta - \alpha) 1_{\alpha \leq u} + (\beta - u) 1_{\alpha \leq u \leq \beta}] \, dG(u, \lambda, F).
\]

Hence, for \( \alpha = z \in \mathbb{R}_- \), \( \beta = 0 \), we obtain from Definition 1 that

\[
P_1(z, \lambda, \kappa, F) = \int_{\mathbb{R}} [-z 1_{u \leq z} - u 1_{z \leq u \leq 0}] \, d[G(u, \kappa, F) - G(u, \lambda, F)]
\]

\[
= \int_{\mathbb{R}} [z 1_{u \leq z} + u 1_{z \leq u \leq 0}] \, d[G(u, \lambda, F) - G(u, \kappa, F)].
\]

Analogously, for \( \alpha = 0 \), \( \beta = z \in \mathbb{R}_+ \), we obtain from Definition 1 that

\[
P_2(z, \lambda, \kappa, F) = \int_{\mathbb{R}} [z 1_{u \leq 0} + (z - u) 1_{0 \leq u \leq z}] \, d[G(u, \kappa, F) - G(u, \lambda, F)]
\]

\[
= \int_{\mathbb{R}} [-u 1_{0 \leq u \leq z} + z 1_{-\infty < u \leq z}] \, d[G(u, \kappa, F) - G(u, \lambda, F)]
\]

\[
= \int_{\mathbb{R}} [u 1_{0 \leq u \leq z} - z (1 - 1_{z \leq +\infty})] \, d[G(u, \lambda, F) - G(u, \kappa, F)]
\]

\[
= \int_{\mathbb{R}} [u 1_{0 \leq u \leq z} + z 1_{z \leq u < +\infty}] \, d[G(u, \lambda, F) - G(u, \kappa, F)].
\]

The previous along with Fubini Theorem, enabled by the existence of the first moment for
the elements of $\mathcal{W}_-, \mathcal{W}_+$, and supposing that the analogous suprema and infima exist, imply that for any $\lambda, \kappa$,

$$
\sup_{z \leq 0} \inf_{\kappa \in \mathbb{K}} P_1 (z, \lambda, \kappa, F) = \sup_{w \in \mathcal{W}_-} \inf_{\kappa \in \mathbb{K}} \int_{\mathbb{R}_-} P_1 (z, \lambda, \kappa, F) \, dw (z)
$$

$$
= \sup_{w \in \mathcal{W}_-} \inf_{\kappa \in \mathbb{K}} \int_{\mathbb{R}_-} \int_{\mathbb{R}} [z 1_{u \leq z} + u 1_{z \leq u \leq 0}] \, d [G (u, \lambda, F) - G (u, \kappa, F)] \, dw (z)
$$

$$
= \sup_{w \in \mathcal{W}_-} \inf_{\kappa \in \mathbb{K}} \int_{\mathbb{R}_-} \int_{\mathbb{R}} [z 1_{u \leq z} + u 1_{z \leq u \leq 0}] \, dw (z) \, d [G (u, \lambda, F) - G (u, \kappa, F)]
$$

$$
= \sup_{v_w \in V_-} \left[ E_\lambda [1_{u \leq 0} v_w (u)] - \sup_{\kappa \in \mathbb{K}} E_\kappa [1_{u \leq 0} v_w (u)] \right],
$$

and analogously,

$$
\sup_{z \geq 0} \inf_{\kappa \in \mathbb{K}} P_2 (z, \lambda, \kappa, F) = \sup_{w \in \mathcal{W}_+} \inf_{\kappa \in \mathbb{K}} \int_{\mathbb{R}_+} P_2 (z, \lambda, \kappa, F) \, dw (x)
$$

$$
= \sup_{v_w \in V_+} \left[ E_\lambda [1_{u \geq 0} v_w (u)] - \sup_{\kappa \in \mathbb{K}} E_\kappa [1_{u \geq 0} v_w (u)] \right].
$$

These representations and the commutativity of suprema imply that

$$
\rho (F) = \max_{i=1,2} \sup_{v_w \in V_i} \left[ \sup_{\lambda \in \mathbb{L}} E_\lambda [1_{u \in \mathbb{A}_i} v_w (u)] - \sup_{\kappa \in \mathbb{K}} E_\kappa [1_{u \in \mathbb{A}_i} v_w (u)] \right],
$$

and the first result follows. For the second one, due to Lemma 3, the previous implies that

$\mathbb{K} \succeq_P \mathbb{L}$ iff, $\max_{i=1,2} \sup_{v_w \in V_i} \left[ \sup_{\lambda \in \mathbb{L}} E_\lambda [1_{u \in \mathbb{A}_i} v_w (u)] - \sup_{\kappa \in \mathbb{K}} E_\kappa [1_{u \in \mathbb{A}_i} v_w (u)] \right] = 0$. If the latter holds then $\sup_{v_w \in V_i} \left[ \sup_{\lambda \in \mathbb{L}} E_\lambda [1_{u \in \mathbb{A}_i} v_w (u)] - \sup_{\kappa \in \mathbb{K}} E_\kappa [1_{u \in \mathbb{A}_i} v_w (u)] \right] \leq 0$, $\forall i = 1, 2$, which due to the convexity of $\mathbb{K}$ and a double application of the Sion (1958) Minimax Theorem implies that $\sup_{v \in V} \left[ \sup_{\lambda \in \mathbb{L}} E_\lambda [v] - \sup_{\kappa \in \mathbb{K}} E_\kappa [v] \right] \leq 0$, and the result follows from $\mathbb{K} \subseteq \mathbb{L}$.

Now suppose that $\sup_{v \in V} \left[ \sup_{\lambda \in \mathbb{L}} E_\lambda [v] - \sup_{\kappa \in \mathbb{K}} E_\kappa [v] \right] = 0$, which implies from $\mathbb{K} \subseteq \mathbb{L}$ that $\sup_{v \in V} \inf_{\kappa \in \mathbb{K}} [E_\lambda [v] - E_\kappa [v]] \leq 0$, $\sup_{v \in V} \sup_{\lambda \in \mathbb{L}} \inf_{\kappa \in \mathbb{K}} [E_\lambda [v] - E_\kappa [v]] \leq 0$, where
\[ V_* = \begin{cases} v : \mathbb{R} \to \mathbb{R}, \quad v(u) = \begin{cases} v_w(u), & u \leq 0 \\ 0, & u \geq 0 \end{cases}, \text{where } v_w \in V_- \end{cases} \]

\[ V^* = \begin{cases} v : \mathbb{R} \to \mathbb{R}, \quad v(u) = \begin{cases} 0, & u \leq 0 \\ v_w(u), & u \geq 0 \end{cases}, \text{where } v_w \in V_+ \setminus \mathbb{R} \end{cases} \]

Using the obvious identification of \( V_*, V^* \) with \( V_-, V_+ \), the latter display implies that

\[
\sup_{v_w \in V_i} \left[ \sup_{\lambda \in \mathbb{L}} \mathbb{E}_\lambda [1_{u \in A_i} v_w(u)] - \sup_{\kappa \in \mathbb{K}} \mathbb{E}_\kappa [1_{u \in A_i} v_w(u)] \right] \leq 0, \quad \forall i = 1, 2,
\]

and the result follows from \( \mathbb{K} \subseteq \mathbb{L} \).

\[ \square \]

**Proof of Proposition 6.** The results in the auxiliary Lemma 15 imply that

\[
\begin{pmatrix} P_1(z_1, \lambda, \kappa, \sqrt{T} (F - F)) \\ P_2(z_2, \lambda, \kappa, \sqrt{T} (F - F)) \end{pmatrix}
\]

weakly converges to

\[
\begin{pmatrix} P_1^*(z_1, \lambda, \kappa, \mathbb{G}_F) \\ P_2^*(z_2, \lambda, \kappa, \mathbb{G}_F) \end{pmatrix}
\]

w.r.t. the product topology of lower semi-continuous (lsc) real valued functions (see e.g. Knight (1999) for the dual notion of epi-convergence). This product space is metrizable as complete and separable (see again Knight (1999)). Hence, Skorokhod representations are applicable (as above, see for example Theorem 1 in Cortissoz (2007)) and thereby for any \((z_1, z_2, \lambda)\) and any sequence \((z_{1,T}, z_{2,T}, \lambda_T) \to (z_1, z_2, \lambda)\), there exist an enhanced probability space and processes

\[
\begin{pmatrix} P_{1,T}(z_1, \lambda, \kappa) \\ P_{2,T}(z_2, \lambda, \kappa) \end{pmatrix} \overset{d}{\longrightarrow} \begin{pmatrix} P_1^*(z_1, \lambda, \kappa, \sqrt{T} (F - F)) \\ P_2^*(z_2, \lambda, \kappa, \sqrt{T} (F - F)) \end{pmatrix},
\]

defined on it such that \( \begin{pmatrix} P_{1,T} \\ P_{2,T} \end{pmatrix} \to \begin{pmatrix} P_1^* \\ P_2^* \end{pmatrix} \) almost surely, w.r.t. to the product topology of epi-convergence, where \( \overset{d}{=} \) denotes equality in distribution. Notice that,

\[
\begin{pmatrix} P_1(z_{1,T}, \lambda_T, \kappa, \sqrt{TF_T}) \\ P_2(z_{2,T}, \lambda_T, \kappa, \sqrt{TF_T}) \end{pmatrix} \overset{d}{=} \begin{pmatrix} P_{1,T} \\ P_{2,T} \end{pmatrix} + \sqrt{T} \begin{pmatrix} P_1(z_{1,T}, \lambda_T, \kappa, F) \\ P_2(z_{2,T}, \lambda_T, \kappa, F) \end{pmatrix},
\]

\[ 8 \]
and for each $i$ consider the function

$$
P_i^\infty (z_i, \lambda, \kappa) := \begin{cases} 
P_i^* (z_i, \lambda, \kappa), & P_i (z_i, \lambda, \kappa, F_T) = 0 \\
+\infty, & P_i (z_i, \lambda, \kappa, F_T) > 0 \\
-\infty, & P_i (z_i, \lambda, \kappa, F_T) < 0
\end{cases}
$$

Notice that for $P_i^\infty (z_i, \lambda, \kappa) = P_i^* (z_i, \lambda, \kappa)$ for each $(z_i, \lambda, \kappa) \in \Gamma_i$. Suppose that, $H_0$ holds. Then, for each $i$ and for any compact $K$ that contains $\kappa \in K$ such that $(z_{i,T}, \lambda_T, \kappa)$ converges on the boundary of $\Gamma_i$ we have that almost surely,

$$
\lim \inf_{T \to \infty} \inf_K P_i \left( z_{i,T}, \lambda_T, \kappa, \sqrt{T} F_T \right) \geq \inf_K P_i (z_i, \lambda, \kappa, G_F) \right) + \lim \inf_{T \to \infty} \sqrt{T} \inf_K P_i (z_{i,T}, \lambda_T, \kappa, F_T) 
\geq \inf_K P_i^\infty (z_i, \lambda, \kappa).
$$

Hence, due to Proposition 3.2.(ii)-(iii) (ch. 5, p. 337) of Molchanov (2006), $P_i \left( z_i, \lambda, \kappa, \sqrt{T} F_T \right)$ almost surely epi-converges w.r.t. $\kappa$, continuously w.r.t. $(z_i, \lambda)$ to $P_i^\infty (z_i, \lambda, \kappa)$. The compactness of $K$ and Theorem 3.4 (ch. 5, p. 338) of Molchanov (2006) imply that, almost surely

$$
\inf_\kappa P_i \left( z_{i,T}, \lambda_T, \kappa, \sqrt{T} F_T \right) \to \inf_\kappa P_i^\infty (z_i, \lambda, \kappa) = \begin{cases} 
\inf_\kappa P_i^* (z_i, \lambda, \kappa), & \forall \kappa \in K_\lambda^+, (z_i, \lambda, \kappa) \in \Gamma_i \\
-\infty & \exists \kappa \in K_\lambda^+, (z_i, \lambda, \kappa) \notin \Gamma_i
\end{cases}
$$

The existence of $\mathbf{z}$, the compactness of $L$, the fact that $P_i \left( z_i, \lambda, \kappa, \sqrt{T} F_T \right)$ is a monotone transformation of $P_i (z_i, \lambda, \kappa, F_T)$, the fact that continuous convergence implies hypoconvergence by Theorem 7.11 of Rockafellar and Wets (2009), the dual version of Theorem 3.4 of Molchanov (2006), the fact that $\Gamma_i \neq \emptyset$ for all $i$, imply the result by reverting to the original probability space. \qed
Proof of Theorem 9. The first result follows by a direct application of Theorem 3.5.1.i of Politis et al. (1999) due to the results of Proposition 6, since the limiting cdf is continuous at any $q_{1-\alpha}$ for all $\alpha \in (0, \frac{1}{2})$ due to the auxiliary Lemma 16. Notice that if $H_a$ is true then for $\lambda^* \in \mathbb{L} - \mathbb{K}$, and any $\kappa \in \mathbb{K}$ there exists $i^*, z^* \in A_i$, such that $P_{i^*}(z^*, \lambda^*, \kappa, F) > 0$. Then we have that $\rho_T \geq \inf_{\kappa \in \mathbb{K}} P_{i^*}(z^*, \lambda^*, \kappa, \sqrt{T}(F_T - F)) + \sqrt{T}\inf_{\kappa \in \mathbb{K}} P_{i^*}(z^*, \lambda^*, \kappa, F)$, and due to arguments analogous to the ones used in the proof of Proposition 6, we have that the first term in the rhs of the last display is asymptotically tight, while due to the arguments used in the proof of Proposition 3, the second term in the rhs of the last display diverges to $+\infty$. The result follows from the properties of $b_T$.

Proof of Proposition LOCLIM. Analogous to the proof of Proposition 6.

Proof of Proposition LOCUN. Follows directly by Proposition LOCLIM and Theorem 3.5.1.iii of Politis et al. (1999).

Proof of Proposition 10. First notice that the integration by parts formula and the proof of Lemma 4 imply that

$$
\rho_T = \sqrt{T}\max_{i=1,2} \sup_{v_w \in V_i} \left[ \sup_{\lambda \in \mathbb{L}} \mathbb{E}_{F_T}\left[ 1_{\lambda^T Y \in A_i} v_w(\lambda^T Y) \right] - \sup_{\kappa \in \mathbb{K}} \mathbb{E}_{F_T}\left[ 1_{\kappa^T Y \in A_i} v_w(\kappa^T Y) \right] \right].
$$

Since $V_i^* \subset V_i$, $i = -, +$ (as a matter of fact, we have that $\mathbb{P}$ a.s. $\rho_T \geq \rho_T^*$), the result obtains if for any $v_w \in V_i$, there exists some $v_w^* \in V_i^*$, such that $\mathbb{E}_{F_T}\left[ 1_{\lambda^T Y \in A_i} v_w^*(\lambda^T Y) \right]$ converges uniformly $\mathbb{P}$ a.s. to $\mathbb{E}_{F_T}\left[ 1_{\lambda^T Y \in A_i} v_w(\lambda^T Y) \right]$ on $\mathbb{L} \times \mathcal{R}^i$, $i = -, +$. It follows from the $\mathbb{P}$ a.s. compactness of $\mathcal{R}^i \times [0, 1]$, the Density Theorem (see Theorem 15.10 of Aliprantis and Border (2006)), and Theorem 15.11 of Aliprantis and Border (2006).

Proof of Proposition MC. From Proposition 4 of Arvanitis and Topaloglou (2017), we have that $\tau^*$ strictly Prospect dominates every portfolio in $\mathbb{L}$. Hence $\mathbb{K} \succ_P \mathbb{L}$ and thereby $\mathbb{K} \supset \{\tau^*\} \succ_P \mathbb{L}$. Due to the same reasoning there is no element in $\mathbb{K} - \{\tau^*\}$ that Prospect dominates $\tau^*$. Hence $\mathbb{K} - \{\tau^*\}$ cannot Prospect-span $\mathbb{L}$.

10
3.2 Auxiliary Lemmata

The following are auxiliary results used in the proofs above.

**Lemma 5.** *Under Assumption 5*

\[
\begin{pmatrix}
P_1 \left(z_1, \lambda, \kappa, \sqrt{T} (F_T - F) \right) \\
\sqrt{T} (F_T - F) \\
F_T - F
\end{pmatrix} \sim \begin{pmatrix}
P_1 \left(z_1, \lambda, \kappa, G_F \right) \\
P_2 \left(z_2, \lambda, \kappa, G_F \right)
\end{pmatrix}
\]

as random elements with values on the space of \(\mathbb{R}^2\)-valued bounded functions on \(L \times K \times \mathbb{R}_- \times \mathbb{R}_{++}\) equipped with the sup-norm. The limiting process has continuous sample paths.

*Proof.* See Lemma 2 in Arvanitis and Topaloglou (2017).

**Lemma 6.** *Under Assumptions 5, \(P(\rho_\infty \geq 0) = 1\) its cdf is absolutely continuous on \((0, +\infty)\) and it may have a jump discontinuity at zero. Suppose moreover that for \(\lambda \in L - K\), \(\inf_{Y_0} \lambda^T Y_0 \leq 0\) and there exists \((\kappa, z) \in K^+ \times \mathbb{R}_{++}\) with \((\lambda, \kappa, z) \in \Gamma_2\) and that if \((\lambda, \kappa^*, z^*) \in \Gamma_2\) for \(\kappa^* \neq \kappa\) then \(z^* \neq z\). Then \(P(\rho_\infty > 0) \geq \frac{1}{2}\).

*Proof.* Notice first that for all \((\lambda, \kappa)\), \(P_i \left(\lambda, \kappa, \min_{i \leq N \leq T} (Y_i), \sqrt{T} F_T \right) = 0\), for the \(i\) that corresponds to the sign of \(\tilde{z}\) which then implies that \(P_i \left(\lambda, \kappa, \sqrt{T} F_T \right) \geq 0\) a.s., and then due to the Portmanteau Theorem and Proposition 6, we get \(0 = \lim_{T \to \infty} P(\rho_T < 0) \geq P(\rho_\infty < 0)\). Now, for \(\Lambda = L \times K \times \{1, 2\} \times \mathbb{R}_- \times \mathbb{R}_{++}\) where \(\{1, 2\}\) is equipped with the discrete metric, consider \(X_\mu := 1_{i=1} P_i \left(z_1, \lambda, \kappa, G_F \right) + 1_{i=2} P_2 \left(z_2, \lambda, \kappa, G_F \right)\), for \(\mu = (\lambda, \kappa, i, z_1, z_2)\), and \(1_j\) is the indicator of \(\{j\}\). \(X_\mu\) is zero mean Gaussian and has continuous sample paths due to the final assertion of Lemma 5. Since \(P_i \left(z, \kappa, \kappa, G_F \right) = 0\) almost surely for all \(z\) and \(i\), and due Lemma 18.15 of van der Vaart (2000), we have that for \(\mu^* = (\kappa, \kappa, i, z_1, z_2)\)

\[
0 \leq \sigma^2 := \sup_{\Lambda} E \left(X_\mu^2 \right) = \sup_{\Lambda} E \left((X_\mu - X_{\mu^*})^2 \right) \leq \sup_{\mu, v \in \Lambda} E \left((X_\mu - X_v)^2 \right) < +\infty.
\]
Hence due to the zero mean function of $X_\mu$, and Furnique inequality (see Relation (1,1) in Samorodnitsky (1991)), we have that for $0 < \varepsilon < 1$, there exists $\kappa(\varepsilon)$, such that

$$
\mathbb{E}\left(\sup_\Lambda X_\mu^2\right) = \int_0^{+\infty} \mathbb{P}\left(\sup_\Lambda |X_\mu| > \sqrt{y}\right) dy \leq 2\kappa(\varepsilon) \int_0^{+\infty} \exp\left(-\frac{(1-\varepsilon)}{2\sigma^2} y\right) dy < +\infty.
$$

Then Ch. 2 of Nualart (2006), (see the remark after the proof of Proposition 2.1.11 (p. 109)) implies the existence of the square integrable Malliavin derivative for $X_\mu$. The zero mean Gaussianity, via the exclusion of $\mathbb{P}$-negligible events, implies that $X_\mu$ is zero only when $\kappa = \lambda$ or $\kappa \neq \lambda$ and $z \leq \inf Y_0$, and at most only then that $X_\mu$ has degenerate variance. Hence, Nualart (2006) implies then that the Malliavin derivative of $X_\mu$ equals zero only then. The previous lines imply the validity of Assumption 1 of Arvanitis, Scaillet and Topaloglou (2020) for $\mathcal{T} = \{0\}$ in their notation, and the second assertion follows by Theorem 1 there. For the final assertion notice that since $\inf Y_0 \lambda^{Tr} Y_0 \leq 0$ and $\lambda \in \Lambda - K$, then $\text{Var}(P_2(z, \lambda, \kappa, G_F)) > 0$ due to Assumption 5. Then since for any $(\lambda, \kappa^*, z^*) \in \Gamma_2$ with $\kappa^* \neq \kappa$ then $z^* \neq z$, we have that $\mathbb{P}(\sup_z \inf_\kappa P_2(z, \lambda, \kappa, G_F) > 0, (\lambda, \kappa, z) \in \Gamma_2) = \mathbb{P}(\sup_z P_2(z, \lambda, \kappa, G_F) > 0, (\lambda, \kappa, z) \in \Gamma_2) \geq \mathbb{P}(P_2(z, \lambda, \kappa, G_F) > 0, (\lambda, \kappa, z) \in \Gamma_2) = \frac{1}{2}$, due to non-degeneracy and zero mean Gaussianity. We deduce the result since $\mathbb{P}(\rho_\infty > 0)$ is greater than or equal from the probability in the lhs of the inequality. \qed

4 Summary Statistics of the Factor and Anomaly Returns

Table 2 reports summary statistics of the factor and anomaly returns over our sample period.
Table 2: Descriptive Statistics of monthly returns

| Panel | Composite EquityA: Factor Models | Mean  | SD    | Skewness | Kurtosis | Sharpe ratio |
|-------|----------------------------------|-------|-------|----------|----------|--------------|
| Market|                                  | 0.0098| 0.0454| -0.5234  | 2.0989   | 0.1306       |
|       | FF-5 model                        |       |       |          |          |              |
| SMB   |                                  | 0.0028| 0.0299| 0.3853   | 4.2534   | -0.0363      |
| HML   |                                  | 0.0036| 0.0293| 0.0777   | 2.1566   | -0.0108      |
| RMW   |                                  | 0.0030| 0.0235| -0.3615  | 12.444   | -0.0387      |
| CMA   |                                  | 0.0034| 0.0198| 0.3913   | 1.9190   | -0.0251      |
| M-4 model |                                |       |       |          |          |              |
| SMB   |                                  | 0.0045| 0.0280| 0.2565   | 2.0487   | 0.0198       |
| MGMT1 |                                  | 0.0061| 0.0283| 0.1510   | 1.8210   | 0.0789       |
| PERF1 |                                  | 0.0065| 0.0393| -0.0486  | 3.8711   | 0.0650       |
| ME    |                                  | 0.0034| 0.0305| 0.6317   | 6.3636   | -0.0150      |
| IA    |                                  | 0.0040| 0.0184| 0.2052   | 1.8422   | 0.0041       |
| ROE   |                                  | 0.0055| 0.0261| -0.7203  | 4.8811   | 0.0624       |

| Panel B: Anomalies | Mean  | SD    | Skewness | Kurtosis | Sharpe ratio |
|-------------------|-------|-------|----------|----------|--------------|
| Accruals          | 0.0031| 0.0310| 0.0066   | 1.0947   | -0.0272      |
| Asset Growth      | 0.0052| 0.0328| 0.5986   | 3.6047   | 0.0390       |
| Composite Equity Issues | 0.0049| 0.0337| 0.0480   | 2.3218   | 0.0301       |
| Distress          | 0.0045| 0.0638| 0.0830   | 3.6412   | 0.0102       |
| Growth Profitability Premium | 0.0021| 0.0377| 0.2643   | 1.2312   | -0.0480      |
| Investment to Assets | 0.0053| 0.0291| 0.0804   | 0.1216   | 0.0478       |
| Momentum          | 0.0107| 0.0652| -0.8537  | 5.5629   | 0.1046       |
| Net Operating Assets | 0.0056| 0.0291| 0.1552   | 1.0253   | 0.0595       |
| O-Score           | 0.000 | 0.0362| 0.2574   | 1.1787   | -0.1070      |
| Return on Assets  | 0.0057| 0.0417| 0.3637   | 2.6874   | 0.0425       |
| Net Stock Issues  | 0.0051| 0.0271| 0.1013   | 2.5132   | 0.0433       |
| Betting against Beta | 0.0088| 0.0340| -0.6509  | 3.3393   | 0.1431       |
| Quality minus Junk | 0.0051| 0.0454| 0.0979   | 1.5959   | 0.0269       |
| Size              | -0.0186| 0.0637| -1.8285  | 8.4206   | -0.3533      |
| Growth Option     | -0.0218| 0.0535| 2.0057   | 10.1429  | -0.4819      |
| Value (Book to Market) | 0.0181| 0.0596| 0.0919   | 13.4067  | 0.2384       |
| Idiosyncratic Volatility | 0.0103| 0.0857| 1.8921   | 8.6201   | 0.0743       |
| Profitability     | -0.0037| 0.0515| -1.9601  | 8.9951   | -0.1481      |

Entries report the descriptive statistics of the factor and anomaly returns. The dataset spans the period from January, 1974 to December, 2016.

5 Out-of-Sample Analysis: Prospect Spanning

Tables 3-19 reports the performance measures for the 5 Fama and French, the 4 Stambaugh and Yuan and the 4 Hou-Xue-Zhang optimal factor portfolios, and the augmented portfolios.
with each of the market anomaly (Panels A, B and C respectively) under prospect and M-V spanning.

Table 3: Performance measures of the optimal spanning portfolios. The case of the Accruals anomaly.

|                      | Prospect Spanning |                     |                     | M-V spanning |                     |
|----------------------|-------------------|---------------------|---------------------|--------------|---------------------|
|                      | Panel A           | Panel B             | Panel C             | Panel A      | Panel B             | Panel C             |
|                      | FF-5 + anom.      | M-4 + anom.        | q + anom.           | FF-5 + anom. | M-4 + anom.        | q + anom.           |
| Mean                 | 0.0016            | 0.0014             | 0.0018              | 0.0015       | 0.0014             | 0.0018              |
| SD                   | 0.0161            | 0.0122             | 0.0119              | 0.0125       | 0.0119             | 0.0118              |
| Sharpe ratio         | 0.0855            | 0.0903             | 0.1265              | 0.0984       | 0.0903             | 0.1265              |
| D. Sharpe ratio      | 0.0857            | 0.1265             | 0.1218              | 0.0695       | 0.0517             | 0.1508              |
| UP ratio             | 0.5513            | 0.5415             | 0.8022              | 0.5292       | 0.5591             | 0.6703              |
| Return Loss          | -0.011%           | -0.005%            | -0.023%             | -0.010%      | -0.005%            | -0.023%             |
| Prospect utility loss|                   |                     |                     |              |                     |                     |
| Opportunity Cost     |                   |                     |                     |              |                     |                     |
| \(\alpha = \beta = 0.2\) | 0.038%            | 0.001%             | 0.152%              | 0.046%       | 0.052%             |
| \(\alpha = \beta = 0.4\) | 0.034%            | 0.001%             | 0.137%              | 0.042%       | 0.058%             |
| \(\alpha = \beta = 0.6\) | 0.031%            | 0.0005%            | 0.123%              | 0.037%       | 0.063%             |

Entries report the performance measures (Mean, Standard Deviation, Sharpe ratio, downside Sharpe ratio, UP ratio and Returns Loss) for the factor optimal portfolios, as well as the augmented with the Accruals optimal portfolio under prospect spanning and M-V spanning. Panel A report measures for the case of the FF-5 factors. Panel B for the case of the M-4 factors, while panel C for the case of the q factors. Finally the Table yields the prospect utility loss of the M-V portfolios over the prospect portfolios.
Table 4: Performance measures of the optimal spanning portfolios. The case of the Asset Growth anomaly.

| Prospect Spanning | Panel A | Panel B | Panel C |
|-------------------|---------|---------|---------|
|                   | FF-5    | M-4     | q       |
| Mean              | 0.0014  | 0.0009  | 0.0057  | 0.0127  | 0.0019  |
| SD                | 0.0238  | 0.0169  | 0.0384  | 0.0396  | 0.0918  |
| Sharpe ratio      | 0.0471  | 0.0357  | 0.0338  | 0.0364  | 0.0175  |
| D. Sharpe ratio   | 0.0296  | 0.0413  | 0.3144  | 0.0237  | 0.8442  |
| UP ratio          | 0.4674  | 0.0363  | 1.2032  | 0.5124  | 1.5398  |
| Return Loss       | -0.002% | -0.025% | -0.025% |

M-V spanning

| Panel A | Panel B | Panel C |
|---------|---------|---------|
| FF-5    | M-4     | q       |
| Mean    | 0.0016  | 0.0018  | 0.0018  | 0.0025  | 0.0024  |
| SD      | 0.0132  | 0.0128  | 0.0137  | 0.0149  | 0.0162  |
| Sharpe ratio | 0.1049  | 0.1234  | 0.1167  | 0.1516  | 0.1332  |
| D. Sharpe ratio | 0.1158  | 0.1572  | 0.1497  | 0.2700  | 0.2157  |
| UP ratio | 0.6312  | 0.6801  | 0.6797  | 0.8163  | 0.7578  |
| Return Loss | -0.002% | -0.010% | -0.029% |

Prospect utility loss (Opportunity Cost)

| α = β = 0.2 | -0.250% | -0.680% | 0.670% | -0.293% | 0.298% | -0.374% |
| α = β = 0.4 | -0.271% | -0.748% | 0.612% | -0.323% | 0.168% | -0.412% |
| α = β = 0.6 | -0.305% | -0.823% | 0.551% | -0.354% | 0.101% | -0.453% |

Entries report the performance measures (Mean, Standard Deviation, Sharpe ratio, Downside Sharpe ratio, UP ratio and Returns Loss) for the factor optimal portfolios, as well as the augmented with the Asset Growth optimal portfolio under prospect spanning and M-V spanning. Panel A report measures for the case of the FF-5 factors. Panel B for the case of the M-4 factors, while panel C for the case of the q factors. Finally the Table yields the prospect utility loss of the M-V portfolios over the prospect portfolios.
Table 5: Performance measures of the optimal spanning portfolios. The case of the Composite Issue anomaly.

| Prospect Spanning | Panel A | Panel B | Panel C |
|-------------------|---------|---------|---------|
| FF-5 + anom.      | 0.0014  | 0.0126  | 0.0017  |
| Mean              | 0.0091  | 0.0126  | 0.0017  |
| SD                | 0.0223  | 0.1127  | 0.0161  |
| Sharpe ratio      | 0.0538  | 0.1097  | 0.0933  |
| D. Sharpe ratio   | 0.3591  | 0.8554  | 0.1730  |
| UP ratio          | 0.7189  | 1.5753  | 0.8852  |
| Return Loss       | 0.025%  | 0.007%  | 0.009%  |
| M-V spanning      |         |         |         |
| FF-5 + anom.      | 0.0017  | 0.0019  | 0.0019  |
| Mean              | 0.0017  | 0.0019  | 0.0019  |
| SD                | 0.0130  | 0.0122  | 0.0123  |
| Sharpe ratio      | 0.1086  | 0.1344  | 0.1375  |
| D. Sharpe ratio   | 0.1204  | 0.1290  | 0.1813  |
| UP ratio          | 0.6184  | 0.6126  | 0.7051  |
| Return Loss       | 0.003%  | 0.004% -0.014% |

Prospect utility loss (Opportunity Cost)

| α = β = 0.2  | 0.339% | 0.689% | 0.169% | 0.743% | 0.085% | 0.475% |
| α = β = 0.4  | 0.305% | 0.529% | 0.152% | 0.668% | 0.077% | 0.428% |
| α = β = 0.6  | 0.275% | 0.368% | 0.137% | 0.602% | 0.069% | 0.385% |

Entries report the performance measures (Mean, Standard Deviation, Sharpe ratio, Downside Sharpe ratio, UP ratio and Returns Loss) for the factor optimal portfolios, as well as the augmented with the Composite Issue optimal portfolio under prospect spanning and M-V spanning. Panel A report measures for the case of the FF-5 factors. Panel B for the case of the M-4 factors, while panel C for the case of the q factors. Finally the Table yields the prospect utility loss of the M-V portfolios over the prospect portfolios.
Table 6: Performance measures of the optimal spanning portfolios. The case of the Distress anomaly.

| Prospect Spanning | Panel A | Panel B | Panel C |
|-------------------|--------|---------|---------|
|                   | FF-5 + anom. | M-4 + anom. | q + anom. |
| Mean              | 0.0015 | 0.0019 | 0.0035 | 0.0028 | 0.0027 |
| SD                | 0.0173 | 0.0127 | 0.1472 | 0.0143 | 0.0249 |
| Sharpe ratio      | 0.0752 | 0.1335 | 0.0218 | 0.1817 | 0.0981 |
| D. Sharpe ratio   | 0.1217 | 0.1876 | 0.1838 | 0.3892 | 0.2184 |
| UP ratio          | 0.7127 | 0.6949 | 0.7688 | 0.8728 | 0.9486 |
| Return Loss       | -0.010% | -0.010% | -0.009% |

| M-V spanning      | Panel A | Panel B | Panel C |
|-------------------|--------|---------|---------|
|                   | FF-5 + anom. | M-4 + anom. | q + anom. |
| Mean              | 0.0017 | 0.0010 | 0.0021 | 0.0026 | 0.0027 |
| SD                | 0.0135 | 0.0082 | 0.0134 | 0.0142 | 0.0148 |
| Sharpe ratio      | 0.1089 | 0.0940 | 0.1356 | 0.1688 | 0.1645 |
| D. Sharpe ratio   | 0.1315 | 0.0298 | 0.2105 | 0.3186 | 0.3128 |
| UP ratio          | 0.6271 | 0.4591 | 0.7093 | 0.8495 | 0.8242 |
| Return Loss       | 0.002% | 0.001% | -0.007% |

Prospect utility loss
( Opportunity Cost)

\[
\begin{align*}
\alpha = \beta = 0.2 & \quad 0.076\% \quad -0.083\% \quad 0.404\% \quad 0.033\% \quad 0.094\% \quad 0.030\% \\
\alpha = \beta = 0.4 & \quad 0.068\% \quad -0.091\% \quad 0.363\% \quad 0.030\% \quad 0.085\% \quad 0.026\% \\
\alpha = \beta = 0.6 & \quad 0.061\% \quad -0.100\% \quad 0.327\% \quad 0.025\% \quad 0.076\% \quad 0.024\%
\end{align*}
\]

Entries report the performance measures (Mean, Standard Deviation, Sharpe ratio, Downside Sharpe ratio, UP ratio and Returns Loss) for the factor optimal portfolios, as well as the augmented with the Distress optimal portfolio under prospect spanning and M-V spanning. Panel A report measures for the case of the FF-5 factors. Panel B for the case of the M-4 factors, while panel C for the case of the q factors. Finally the Table yields the prospect utility loss of the M-V portfolios over the prospect portfolios.
Table 7: Performance measures of the optimal spanning portfolios. The case of the Gross Profitability anomaly.

|                  | Prospect Spanning | M-V spanning |
|------------------|-------------------|--------------|
|                  | Panel A | Panel B | Panel C | Panel A | Panel B | Panel C | Panel A | Panel B | Panel C | Panel A | Panel B | Panel C |
|                  | FF-5 + anom. | M-4 + anom. | q + anom. | FF-5 + anom. | M-4 + anom. | q + anom. | FF-5 + anom. | M-4 + anom. | q + anom. | FF-5 + anom. | M-4 + anom. | q + anom. |
| Mean             | 0.0007 | 0.0014 | 0.0017 | 0.0012 | 0.0018 | 0.0016 | 0.0018 | 0.0018 | 0.0025 | 0.0024 | 0.0016 | 0.0015 | 0.0018 |
| SD               | 0.0147 | 0.0113 | 0.0270 | 0.0096 | 0.0371 | 0.0124 | 0.0114 | 0.0109 | 0.0129 | 0.0122 | 0.0124 | 0.0120 | 0.0114 |
| Sharpe ratio     | 0.0305 | 0.1022 | 0.0539 | 0.0969 | 0.0243 | 0.1071 | 0.1373 | 0.1442 | 0.1750 | 0.1765 | 0.1071 | 0.1082 | 0.1373 |
| D. Sharpe ratio  | 0.5067 | 0.0595 | 0.0270 | 0.0012 | 0.1092 | 0.1095 | 0.1783 | 0.1919 | 0.3276 | 0.3307 | 0.1095 | 0.1072 | 0.1783 |
| UP ratio         | 0.5480 | 0.6371 | 0.6114 | 0.4770 | 0.4832 | 0.6189 | 0.7031 | 0.7256 | 0.8888 | 0.9092 | 0.6189 | 0.6137 | 0.7031 |
| Return Loss      | -0.015% | -0.015% | -0.014% | -0.002% | 0.009% | 0.002% | 0.009% | 0.003% | 0.002% | 0.009% | 0.002% | 0.009% | 0.003% |

Prospect utility loss (Opportunity Cost)

\[ \alpha = \beta = 0.2 \]
\[ \alpha = \beta = 0.4 \]
\[ \alpha = \beta = 0.6 \]

Entries report the performance measures (Mean, Standard Deviation, Sharpe ratio, Downside Sharpe ratio, UP ratio and Returns Loss) for the factor optimal portfolios, as well as the augmented with the Gross Profitability optimal portfolio under prospect spanning and M-V spanning. Panel A report measures for the case of the FF-5 factors. Panel B for the case of the M-4 factors, while panel C for the case of the q factors. Finally the Table yields the prospect utility loss of the M-V portfolios over the prospect portfolios.
Table 8: Performance measures of the optimal spanning portfolios. The case of the Investment/Asset anomaly.

| Prospect Spanning | Panel A   | Panel B   | Panel C   |
|-------------------|-----------|-----------|-----------|
|                   | FF-5 + anom. | M-4 + anom. | q + anom. |
| Mean              | 0.0016   | 0.0019   | 0.0016   | 0.0029 | 0.0026 |
| SD                | 0.0143   | 0.0162   | 0.0282   | 0.0278 | 0.0128 |
| Sharpe ratio      | 0.0953   | 0.0474   | 0.0997   | 0.0472 | 0.0956 | 0.1856 |
| D. Sharpe ratio   | 0.1851   | 0.1351   | 0.1273   | 0.1042 | 0.2407 | 0.2221 |
| UP ratio          | 0.9935   | 0.6526   | 0.7537   | 0.7173 | 0.8842 | 0.5787 |
| Return Loss       | -0.016%  | -0.015%  | -0.001%  |

| M-V spanning      | Panel A   | Panel B   | Panel C   |
|-------------------|-----------|-----------|-----------|
|                   | FF-5 + anom. | M-4 + anom. | q + anom. |
| Mean              | 0.0016   | 0.0016   | 0.0018   | 0.0018 | 0.0025 | 0.0024 |
| SD                | 0.0126   | 0.0123   | 0.0123   | 0.0126 | 0.0144 | 0.0153 |
| Sharpe ratio      | 0.1075   | 0.1090   | 0.1283   | 0.1257 | 0.1573 | 0.1430 |
| D. Sharpe ratio   | 0.1111   | 0.1105   | 0.1617   | 0.1587 | 0.2758 | 0.2272 |
| UP ratio          | 0.6238   | 0.6234   | 0.6921   | 0.7037 | 0.8190 | 0.7632 |
| Return Loss       | 0.002%   | -0.004%  | -0.022%  |

Prospect utility loss (Opportunity Cost)

| α = β = 0.2 | 0.444% | 0.054% | 0.027% | -0.041% | 0.030% | 0.020% |
| α = β = 0.4 | 0.399% | 0.048% | 0.024% | -0.045% | 0.027% | 0.019% |
| α = β = 0.6 | 0.359% | 0.044% | 0.020% | -0.049% | 0.025% | 0.012% |

Entries report the performance measures (Mean, Standard Deviation, Sharpe ratio, Downside Sharpe ratio, UP ratio and Returns Loss) for the factor optimal portfolios, as well as the augmented with the Investment/Asset optimal portfolio under prospect spanning and M-V spanning. Panel A report measures for the case of the FF-5 factors. Panel B for the case of the M-4 factors, while panel C for the case of the q factors. Finally the Table yields the prospect utility loss of the M-V portfolios over the prospect portfolios.
Table 9: Performance measures of the optimal portfolios spanning portfolios. The case of the Net Operating Assets anomaly.

| Prospect Spanning | Panel A | Panel B | Panel C |
|-------------------|---------|---------|---------|
| **Mean** FF-5 + anom. | 0.0017 | 0.0095 | 0.0015 |
| **SD** | 0.0217 | 0.2451 | 0.0176 |
| **Sharpe ratio** | 0.0692 | 0.0379 | 0.0717 |
| **D. Sharpe ratio** | 0.1136 | 0.0284 | 0.6779 |
| **UP ratio** | 0.9044 | 0.4920 | 1.9576 |
| **Return Loss** | -0.025% | -0.024% | -0.026% |

| M-V spanning | Panel A | Panel B | Panel C |
|--------------|---------|---------|---------|
| **Mean** FF-5 + anom. | 0.0017 | 0.0018 | 0.0020 |
| **SD** | 0.0126 | 0.0124 | 0.0122 |
| **Sharpe ratio** | 0.1143 | 0.1282 | 0.1411 |
| **D. Sharpe ratio** | 0.1305 | 0.1491 | 0.1951 |
| **UP ratio** | 0.6392 | 0.6543 | 0.7177 |
| **Return Loss** | 0.011% | 0.011% | -0.008% |

Prospect utility loss (Opportunity Cost)

\[ \alpha = \beta = 0.2 \]
\[ \alpha = \beta = 0.4 \]
\[ \alpha = \beta = 0.6 \]

Entries report the performance measures (Mean, Standard Deviation, Sharpe ratio, Downside Sharpe ratio, UP ratio and Returns Loss) for the factor optimal portfolios, as well as the augmented with the Net Operating Assets optimal portfolio under prospect spanning and M-V spanning. Panel A report measures for the case of the FF-5 factors. Panel B for the case of the M-4 factors, while panel C for the case of the q factors. Finally the Table yields the prospect utility loss of the M-V portfolios over the prospect portfolios.
Table 10: Performance measures of the optimal portfolios spanning portfolios. The case of the O-Score anomaly.

| Prospect Spanning | Panel A | Panel B | Panel C |
|------------------|---------|---------|---------|
|                  | FF-5 + anom. | M-4 + anom. | q + anom. |
| Mean             | 0.0018   | 0.0015   | 0.0167   | 0.0184   | 0.0022   |
| SD               | 0.0211   | 0.0176   | 0.3489   | 0.3198   | 0.0169   |
| Sharpe ratio     | 0.0743   | 0.0720   | 0.0571   | 0.0568   | 0.1134   |
| D. Sharpe ratio  | 1.0841   | 1.1319   | 0.0677   | 1.0613   | 0.0628   |
| UP ratio         | 2.0375   | 1.8480   | 0.3996   | 1.9382   | 0.6457   |
| Return Loss      | -0.024%  | -0.020%  | -0.019%  |

| M-V spanning     | Panel A | Panel B | Panel C |
|------------------|---------|---------|---------|
|                  | FF-5 + anom. | M-4 + anom. | q + anom. |
| Mean             | 0.0015   | 0.0018   | 0.0018   | 0.0025   | 0.0023   |
| SD               | 0.0122   | 0.0122   | 0.0105   | 0.0128   | 0.0121   |
| Sharpe ratio     | 0.1037   | 0.1381   | 0.1461   | 0.1718   | 0.1698   |
| D. Sharpe ratio  | 0.0941   | 0.1751   | 0.1856   | 0.3042   | 0.2840   |
| UP ratio         | 0.5809   | 0.6955   | 0.7104   | 0.8453   | 0.8158   |
| Return Loss      | -0.002%  | 0.011%   | -0.001%  |

Prospect utility loss
(Opportunity Cost)

- $\alpha = \beta = 0.2$: 0.447% 0.047% 0.109% -0.429% 0.850% -0.391%
- $\alpha = \beta = 0.4$: 0.202% 0.045% 0.089% -0.472% 0.665% -0.430%
- $\alpha = \beta = 0.6$: 0.098% 0.038% 0.071% -0.519% 0.499% -0.473%

Entries report the performance measures (Mean, Standard Deviation, Sharpe ratio, Downside Sharpe ratio, UP ratio and Returns Loss) for the factor optimal portfolios, as well as the augmented with the O-Score optimal portfolio under prospect spanning and M-V spanning. Panel A report measures for the case of the FF-5 factors. Panel B for the case of the M-4 factors, while panel C for the case of the q factors. Finally the Table yields the prospect utility loss of the M-V portfolios over the prospect portfolios.
Table 11: Performance measures of the optimal portfolios spanning portfolios. The case of the Return on Assets anomaly.

| Prospect Spanning | Panel A | Panel B | Panel C |
|-------------------|---------|---------|---------|
|                    | FF-5 + anom. | M-4 + anom. | q + anom. |
| Mean               | 0.0010 | 0.0017 | 0.0042 | 0.0015 | 0.0006 |
| SD                 | 0.0182 | 0.0357 | 0.0118 | 0.0729 | 0.0137 | 0.0174 |
| Sharpe ratio       | 0.0406 | 0.0346 | 0.1188 | 0.0544 | 0.0920 | 0.0205 |
| D. Sharpe ratio    | 0.0223 | 0.0289 | 0.1894 | 0.0362 | 0.0232 | 0.0211 |
| UP ratio           | 0.5307 | 0.4575 | 0.7985 | 0.4235 | 0.6110 | 0.4448 |
| Return Loss        | -0.017% | -0.017% | -0.009% |

| M-V spanning | Panel A | Panel B | Panel C |
|--------------|---------|---------|---------|
|              | FF-5 + anom. | M-4 + anom. | q + anom. |
| Mean         | 0.0016 | 0.0019 | 0.0019 | 0.0025 | 0.0025 |
| SD           | 0.0128 | 0.0121 | 0.0122 | 0.0132 | 0.0129 |
| Sharpe ratio | 0.1082 | 0.1353 | 0.1393 | 0.1727 | 0.1718 |
| D. Sharpe ratio | 0.1178 | 0.1809 | 0.1972 | 0.3183 | 0.3122 |
| UP ratio     | 0.6139 | 0.6964 | 0.7123 | 0.8605 | 0.8463 |
| Return Loss  | 0.003% | 0.005% | -0.001% |

Prospect utility loss
(Opportunity Cost)

|          | 0.2 | 0.4 | 0.6 |
|----------|-----|-----|-----|
| α = β = 0.2 | -0.179% | -0.241% | 0.111% | -0.450% | -0.545% | -0.503% |
| α = β = 0.4 | -0.196% | -0.265% | 0.099% | -0.495% | -0.599% | -0.533% |
| α = β = 0.6 | -0.216% | -0.292% | 0.090% | -0.544% | -0.659% | -0.608% |

Entries report the performance measures (Mean, Standard Deviation, Sharpe ratio, Downside Sharpe ratio, UP ratio and Returns Loss) for the factor optimal portfolios, as well as the augmented with the Return on Assets optimal portfolio under prospect spanning and M-V spanning. Panel A report measures for the case of the FF-5 factors. Panel B for the case of the M-4 factors, while panel C for the case of the q factors. Finally the Table yields the prospect utility loss of the M-V portfolios over the prospect portfolios.
Table 12: Performance measures of the optimal portfolios spanning portfolios. The case of the Net Stock Issues anomaly.

| Prospect Spanning | Panel A | Panel B | Panel C |
|-------------------|---------|---------|---------|
| FF-5 + anom.       | 0.0011  | 0.0014  | 0.0014  |
| M-4 + anom.        | 0.0034  | 0.0014  | 0.0015  |
| q + anom.          | 0.0014  | 0.0015  |         |
| **Mean**           |         |         |         |
| SD                | 0.0128  | 0.0173  | 0.0125  |
| Sharpe ratio       | 0.0657  | 0.0770  | 0.0949  |
| D. Sharpe ratio    | 0.0103  | 0.1237  | 0.0267  |
| UP ratio           | 0.4677  | 0.8848  | 0.7151  |
| Return Loss        | 0.009%  | 0.010%  | -0.002% |

| M-V spanning       | Panel A | Panel B | Panel C |
|-------------------|---------|---------|---------|
| FF-5 + anom.       | 0.0017  | 0.0019  | 0.0020  |
| M-4 + anom.        | 0.0020  | 0.0121  | 0.0121  |
| q + anom.          | 0.0026  | 0.0139  | 0.0143  |
| **Mean**           |         |         |         |
| SD                | 0.0128  | 0.0128  | 0.1362  |
| Sharpe ratio       | 0.1099  | 0.1138  | 0.1362  |
| D. Sharpe ratio    | 0.1216  | 0.1315  | 0.1834  |
| UP ratio           | 0.6213  | 0.6185  | 0.7049  |
| Return Loss        | 0.005%  | 0.006%  | -0.009% |

Prospect utility loss (Opportunity Cost)

|         | α = β = 0.2 | α = β = 0.4 | α = β = 0.6 |
|---------|-------------|-------------|-------------|
| Mean    | -0.265%     | -0.292%     | -0.321%     |
| Standard Deviation | 0.259%     | 0.233%     | 0.209%      |
| Sharpe ratio         | -0.147%     | -0.161%     | -0.177%     |
| D. Sharpe ratio      | 0.055%      | 0.050%      | 0.046%      |
| UP ratio             | -0.497%     | -0.546%     | -0.601%     |
| Return Loss          | -0.394%     | -0.433%     | -0.477%     |

Entries report the performance measures (Mean, Standard Deviation, Sharpe ratio, Downside Sharpe ratio, UP ratio and Returns Loss) for the factor optimal portfolios, as well as the augmented with the Net Stock Issues optimal portfolio under prospect spanning and M-V spanning. Panel A report measures for the case of the FF-5 factors. Panel B for the case of the M-4 factors, while panel C for the case of the q factors. Finally the Table yields the prospect utility loss of the M-V portfolios over the prospect portfolios.
Table 13: Performance measures of the optimal portfolios spanning portfolios. The case of the Betting against Beta anomaly.

| Prospect Spanning | Panel A | Panel B | Panel C |
|-------------------|---------|---------|---------|
|                   | FF-5    | M-4     | q       |
| Mean              | 0.0016  | 0.0015  | 0.0018  | 0.0028  | 0.0033 |
| SD                | 0.0130  | 0.0130  | 0.0126  | 0.0089  | 0.0164 |
| Sharpe ratio      | 0.1021  | 0.0982  | 0.1230  | 0.1346  | 0.1896 |
| D. Sharpe ratio   | 0.1302  | 0.1260  | 0.1817  | 0.2786  | 0.2793 |
| UP ratio          | 0.7248  | 0.7058  | 0.7455  | 0.8970  | 0.9486 |
| Return Loss       | 0.004%  | 0.05%   | 0.009%  |

| M-V spanning      | Panel A | Panel B | Panel C |
|-------------------|---------|---------|---------|
|                   | FF-5    | M-4     | q       |
| Mean              | 0.0017  | 0.0019  | 0.0020  | 0.0026  | 0.0027 |
| SD                | 0.0130  | 0.0112  | 0.0125  | 0.0129  | 0.0138  | 0.0141 |
| Sharpe ratio      | 0.1145  | 0.1334  | 0.1355  | 0.1727  | 0.1719  |
| D. Sharpe ratio   | 0.1344  | 0.1751  | 0.1856  | 0.3142  | 0.3041  |
| UP ratio          | 0.6320  | 0.6892  | 0.6978  | 0.8518  | 0.8288  |
| Return Loss       | 0.010%  | 0.002%  | -0.002% |

Prospect utility loss
(Opportunity Cost)

\[
\alpha = \beta = 0.2 \\
\alpha = \beta = 0.4 \\
\alpha = \beta = 0.6
\]

Entries report the performance measures (Mean, Standard Deviation, Sharpe ratio, Downside Sharpe ratio, UP ratio and Returns Loss) for the factor optimal portfolios, as well as the augmented with the Betting against Beta optimal portfolio under prospect spanning and M-V spanning. Panel A report measures for the case of the FF-5 factors. Panel B for the case of the M-4 factors, while panel C for the case of the q factors. Finally the Table yields the prospect utility loss of the M-V portfolios over the prospect portfolios.
Table 14: Performance measures of the optimal portfolios spanning portfolios. The case of the Quality minus Junk anomaly.

|                     | Prospect Spanning | M-V spanning |                |
|---------------------|-------------------|--------------|---------------|
|                     | Panel A           | Panel B      | Panel C       |
|                     | FF-5 + anom.      | M-4 + anom.  | q + anom.     |
| Mean                | 0.0016            | 0.0016       | 0.0019        | 0.0020        | 0.0028        | 0.0029        |
| SD                  | 0.0157            | 0.0165       | 0.0125        | 0.0130        | 0.0159        | 0.0160        |
| Sharpe ratio        | 0.0861            | 0.0822       | 0.1338        | 0.1330        | 0.1611        | 0.1665        |
| D. Sharpe ratio     | 0.0988            | 0.0914       | 0.2099        | 0.2286        | 0.3165        | 0.3613        |
| UP ratio            | 0.6173            | 0.5699       | 0.7813        | 0.7532        | 0.9231        | 0.9054        |
| Return Loss         | -0.005%           | -0.001%      | -0.006%       |

Prospect utility loss (Opportunity Cost)

|                     | Panel A           | Panel B      | Panel C       |
|                     | FF-5 + anom.      | M-4 + anom.  | q + anom.     |
| α = β = 0.2         | -0.037%           | -0.189%      | 0.057%        | 0.040%        | 0.013%        | 0.052%        |
| α = β = 0.4         | -0.041%           | -0.208%      | 0.052%        | 0.036%        | 0.012%        | 0.047%        |
| α = β = 0.6         | -0.045%           | -0.228%      | 0.047%        | 0.035%        | 0.009%        | 0.043%        |

Entries report the performance measures (Mean, Standard Deviation, Sharpe ratio, Downside Sharpe ratio, UP ratio and Returns Loss) for the factor optimal portfolios, as well as the augmented with the Quality minus Junk optimal portfolio under prospect spanning and M-V spanning. Panel A report measures for the case of the FF-5 factors. Panel B for the case of the M-4 factors, while panel C for the case of the q factors. Finally the Table yields the prospect utility loss of the M-V portfolios over the prospect portfolios.
Table 15: Performance measures of the optimal portfolios spanning portfolios. The case of the Size anomaly.

Prospect Spanning

|                  | Panel A |          | Panel B |          | Panel C |          |
|------------------|---------|----------|---------|----------|---------|----------|
|                  | FF-5    | + anom.  | M-4     | + anom.  | q       | + anom.  |
| Mean             | 0.0013  | 0.0013   | 0.0028  | 0.0028   | 0.0019  | 0.0019   |
| SD               | 0.0151  | 0.0151   | 0.0108  | 0.0108   | 0.0142  | 0.0142   |
| Sharpe ratio     | 0.0706  | 0.0706   | 0.2350  | 0.2350   | 0.1190  | 0.1190   |
| D. Sharpe ratio  | 0.0895  | 0.0895   | 0.6194  | 0.6194   | 0.1799  | 0.1799   |
| UP ratio         | 0.6644  | 0.6644   | 1.1919  | 1.1919   | 0.7056  | 0.7056   |
| Return Loss      | 0.000%  | 0.000%   |         |          |         | 0.000%   |

M-V spanning

|                  | Panel A |          | Panel B |          | Panel C |          |
|------------------|---------|----------|---------|----------|---------|----------|
|                  | FF-5    | + anom.  | M-4     | + anom.  | q       | + anom.  |
| Mean             | 0.0012  | 0.0012   | 0.0016  | 0.0016   | 0.0019  | 0.0018   |
| SD               | 0.0125  | 0.0125   | 0.0120  | 0.0120   | 0.0136  | 0.0137   |
| Sharpe ratio     | 0.0928  | 0.0928   | 0.1335  | 0.1335   | 0.1395  | 0.1344   |
| D. Sharpe ratio  | 0.0247  | 0.0247   | 0.1182  | 0.1182   | 0.1588  | 0.1531   |
| UP ratio         | 0.5608  | 0.5608   | 0.6337  | 0.6337   | 0.6952  | 0.6856   |
| Return Loss      | 0.000%  | 0.000%   |         |          |         | -0.004%  |

Prospect utility loss
(Opportunity Cost)

|                  |         |         |         |         |         |
|------------------|---------|---------|---------|---------|---------|
| $\alpha = \beta = 0.2$ | 0.093%  | 0.093%  | 0.259%  | 0.259%  | 0.048%  | 0.048%   |
| $\alpha = \beta = 0.4$ | 0.084%  | 0.084%  | 0.153%  | 0.153%  | 0.043%  | 0.043%   |
| $\alpha = \beta = 0.6$ | 0.076%  | 0.076%  | 0.0123% | 0.123%  | 0.039%  | 0.039%   |

Entries report the performance measures (Mean, Standard Deviation, Sharpe ratio, Downside Sharpe ratio, UP ratio and Returns Loss) for the factor optimal portfolios, as well as the augmented with the Size optimal portfolio under prospect spanning and M-V spanning. Panel A report measures for the case of the FF-5 factors. Panel B for the case of the M-4 factors, while panel C for the case of the q factors. Finally the Table yields the prospect utility loss of the M-V portfolios over the prospect portfolios.
Table 16: Performance measures of the optimal portfolios spanning portfolios. The case of the Growth Option anomaly.

| Prospect Spanning | Panel A | Panel B | Panel C |
|-------------------|---------|---------|---------|
| Mean              | 0.0012  | 0.0026  | 0.0014  |
| SD                | 0.0147  | 0.0120  | 0.0109  |
| Sharpe ratio      | 0.0678  | 0.1949  | 0.1070  |
| D. Sharpe ratio   | 0.1164  | 0.3910  | 0.2219  |
| UP ratio          | 0.4845  | 0.8377  | 0.6745  |
| Return Loss       | 0.000%  | 0.000%  | 0.000%  |

| M-V spanning      | Panel A | Panel B | Panel C |
|-------------------|---------|---------|---------|
| Mean              | 0.0012  | 0.0014  | 0.0021  |
| SD                | 0.0130  | 0.0122  | 0.0133  |
| Sharpe ratio      | 0.0705  | 0.0921  | 0.1604  |
| D. Sharpe ratio   | 0.1139  | 0.0710  | 0.2208  |
| UP ratio          | 0.4308  | 0.5840  | 0.8119  |
| Return Loss       | 0.000%  | 0.000%  | 0.005%  |

Prospect utility loss (Opportunity Cost)

- $\alpha = \beta = 0.2$: 0.045% 0.045% 0.574% 0.574% -0.186% -0.170%
- $\alpha = \beta = 0.4$: 0.040% 0.040% 0.516% 0.516% -0.205% -0.177%
- $\alpha = \beta = 0.6$: 0.036% 0.036% 0.465% 0.465% -0.225% -0.184%

Entries report the performance measures (Mean, Standard Deviation, Sharpe ratio, Downside Sharpe ratio, UP ratio and Returns Loss) for the factor optimal portfolios, as well as the augmented with the Growth Option optimal portfolio under prospect spanning and M-V spanning. Panel A report measures for the case of the FF-5 factors. Panel B for the case of the M-4 factors, while panel C for the case of the q factors. Finally the Table yields the prospect utility loss of the M-V portfolios over the prospect portfolios.
Table 17: Performance measures of the optimal portfolios spanning portfolios. The case of the Value anomaly.

| Prospect Spanning | Panel A | Panel B | Panel C |
|-------------------|---------|---------|---------|
|                   | FF-5 + anom. | M-4 + anom. | q + anom. |
| **Mean**          | 0.0025  | 0.0064  | 0.0024  | 0.0066  | 0.0036  | 0.0039  |
| **SD**            | 0.0215  | 0.0303  | 0.0222  | 0.0329  | 0.0257  | 0.0265  |
| **Sharpe ratio**  | 0.1055  | 0.2043  | 0.0960  | 0.1925  | 0.1313  | 0.1381  |
| **D. Sharpe ratio** | 0.1896 | 0.4775  | 0.2168  | 0.4125  | 0.3498  | 0.4266  |
| **UP ratio**      | 0.7844  | 1.0555  | 0.7425  | 0.9596  | 1.0985  | 1.0872  |
| **Return Loss**   | 0.206%  | 0.206%  | 0.206%  | 0.206%  | 0.206%  | 0.206%  |

| M-V spanning | Panel A | Panel B | Panel C |
|--------------|---------|---------|---------|
|               | FF-5 + anom. | M-4 + anom. | q + anom. |
| **Mean**     | 0.0016  | 0.0016  | 0.0020  | 0.0022  | 0.0028  | 0.0030  |
| **SD**       | 0.0129  | 0.0129  | 0.0120  | 0.0121  | 0.0135  | 0.0135  |
| **Sharpe ratio** | 0.1072 | 0.1082  | 0.1458  | 0.1604  | 0.1889  | 0.2044  |
| **D. Sharpe ratio** | 0.1139 | 0.1161  | 0.2053  | 0.2459  | 0.3928  | 0.4613  |
| **UP ratio**  | 0.6279  | 0.6317  | 0.7195  | 0.7586  | 0.9560  | 1.0373  |
| **Return Loss** | 0.001%  | 0.017%  | 0.017%  | 0.017%  | 0.017%  | 0.021%  |

Prospect utility loss (Opportunity Cost)

\[
\alpha = \beta = 0.2: \quad 0.232\% \quad 0.785\% \quad 0.035\% \quad 0.368\% \quad 0.099\% \quad 0.095\%
\]

\[
\alpha = \beta = 0.4: \quad 0.209\% \quad 0.707\% \quad 0.031\% \quad 0.331\% \quad 0.090\% \quad 0.084\%
\]

\[
\alpha = \beta = 0.6: \quad 0.188\% \quad 0.636\% \quad 0.028\% \quad 0.298\% \quad 0.081\% \quad 0.072\%
\]

Entries report the performance measures (Mean, Standard Deviation, Sharpe ratio, Downside Sharpe ratio, UP ratio and Returns Loss) for the factor optimal portfolios, as well as the augmented with the Value (Book to Market) optimal portfolio under prospect spanning and M-V spanning. Panel A report measures for the case of the FF-5 factors. Panel B for the case of the M-4 factors, while panel C for the case of the q factors. Finally the Table yields the prospect utility loss of the M-V portfolios over the prospect portfolios.
Table 18: Performance measures of the optimal portfolios spanning portfolios. The case of the Idiosyncratic Volatility anomaly.

| Prospect Spanning | Panel A | Panel B | Panel C |
|-------------------|---------|---------|---------|
|                   | FF-5 + anom. | M-4 + anom. | q + anom. |
| Mean              | 0.0029  | 0.0029  | 0.0025  | 0.0033 |
| SD                | 0.0157  | 0.0104  | 0.0137  | 0.0168 |
| Sharpe ratio      | 0.1672  | 0.2466  | 0.1674  | 0.1798 |
| D. Sharpe ratio   | 0.3441  | 0.6196  | 0.7116  | 0.3294 |
| UP ratio          | 0.9076  | 1.1914  | 1.2843  | 0.9582 |
| Return Loss       | 0.004%  | 0.025%  | 0.012%  |

| M-V spanning      | Panel A | Panel B | Panel C |
|-------------------|---------|---------|---------|
|                   | FF-5 + anom. | M-4 + anom. | q + anom. |
| Mean              | 0.0017  | 0.0019  | 0.0020  | 0.0027  | 0.0028 |
| SD                | 0.0131  | 0.0123  | 0.0125  | 0.0135  | 0.0135 |
| Sharpe ratio      | 0.1108  | 0.1345  | 0.1798  | 0.1860  |
| D. Sharpe ratio   | 0.1246  | 0.1774  | 0.3456  | 0.3668  |
| UP ratio          | 0.6182  | 0.6813  | 0.8919  | 0.9090  |
| Return Loss       | 0.005%  | 0.003%  | 0.0208% |

Prospect utility loss (Opportunity Cost)

|                  | α = β = 0.2 | α = β = 0.4 | α = β = 0.6 |
|------------------|-------------|-------------|-------------|
| 0.509%           | 0.511%      | 0.952%      | 0.824%      | 0.824%      | 0.824%      | 0.824%      | 0.824%      | 0.824%      | 0.824%      | 0.824%      | 0.824%      |
| 0.511%           | 0.511%      | 0.952%      | 0.824%      | 0.824%      | 0.824%      | 0.824%      | 0.824%      | 0.824%      | 0.824%      | 0.824%      | 0.824%      |
| 0.458%           | 0.460%      | 0.857%      | 0.711%      | 0.455%      | 0.455%      | 0.824%      | 0.824%      | 0.824%      | 0.824%      | 0.824%      | 0.824%      |
| 0.412%           | 0.414%      | 0.771%      | 0.610%      | 0.041%      | 0.041%      | 0.041%      | 0.041%      | 0.041%      | 0.041%      | 0.041%      | 0.041%      |

Entries report the performance measures (Mean, Standard Deviation, Sharpe ratio, Downside Sharpe ratio, UP ratio and Returns Loss) for the factor optimal portfolios, as well as the augmented with the Idiosyncratic Volatility optimal portfolio under prospect spanning and M-V spanning. Panel A report measures for the case of the FF-5 factors. Panel B for the case of the M-4 factors, while panel C for the case of the q factors. Finally the Table yields the prospect utility loss of the M-V portfolios over the prospect portfolios.
Table 19: Performance measures of the optimal portfolios spanning portfolios. The case of the Profitability anomaly.

| Prospect Spanning | Panel A | Panel B | Panel C |
|-------------------|---------|---------|---------|
| **Prospect Spanning** | **Panel A** | **Panel B** | **Panel C** |
| FF-5 + anom. M-4 + anom. | Mean | 0.0021 | 0.0020 | 0.0027 | 0.0025 | 0.0015 | 0.0013 |
| SD | 0.0103 | 0.0104 | 0.0098 | 0.0099 | 0.0101 | 0.0105 |
| Sharpe ratio | 0.1813 | 0.1685 | 0.2501 | 0.2251 | 0.1261 | 0.1042 |
| D. Sharpe ratio | 0.4389 | 0.4218 | 0.6440 | 0.6242 | 0.2776 | 0.2238 |
| UP ratio | 0.9262 | 0.8704 | 1.1845 | 1.0883 | 0.7635 | 0.7014 |
| Return Loss | -0.004% | -0.025% | -0.023% |
| **M-V spanning** | **Panel A** | **Panel B** | **Panel C** |
| FF-5 + anom. M-4 + anom. | Mean | 0.0015 | 0.0015 | 0.0017 | 0.0017 | 0.0023 | 0.0021 |
| SD | 0.0130 | 0.0130 | 0.0121 | 0.0121 | 0.0137 | 0.0139 |
| Sharpe ratio | 0.0988 | 0.0988 | 0.1179 | 0.1179 | 0.1524 | 0.1317 |
| D. Sharpe ratio | 0.0935 | 0.0935 | 0.0966 | 0.0966 | 0.2607 | 0.2470 |
| UP ratio | 0.6181 | 0.6181 | 0.6060 | 0.6060 | 0.8039 | 0.7631 |
| Return Loss | 0.000% | 0.000% | -0.029% |
| **Prospect utility loss** | **(Opportunity Cost)** |
| α = β = 0.2 | 0.641% | 0.550% | 0.855% | 0.660% | -0.024% | -0.005% |
| α = β = 0.4 | 0.577% | 0.495% | 0.749% | 0.574% | -0.026% | -0.005% |
| α = β = 0.6 | 0.519% | 0.446% | 0.654% | 0.497% | -0.029% | -0.006% |

Entries report the performance measures (Mean, Standard Deviation, Sharpe ratio, Downside Sharpe ratio, UP ratio and Returns Loss) for the factor optimal portfolios, as well as the augmented with the Profitability optimal portfolio under prospect spanning and M-V spanning. Panel A report measures for the case of the FF-5 factors. Panel B for the case of the M-4 factors, while panel C for the case of the q factors. Finally the Table yields the prospect utility loss of the M-V portfolios over the prospect portfolios.

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