The background information about perturbative quantum gravity

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Abstract The purpose of this Chapter is to give a general introduction and status review on the perturbative approach to quantum gravity (QG). This text is a modified version of the corresponding chapters of Part II of the recent textbook on quantum field theory (QFT) and QG, co-authored with I.L. Buchbinder and published in Oxford University Press. We discuss the choice of the starting action in the QG models, degrees of freedom and propagator of metric perturbations, power counting and renormalizability of these models, the problems related to higher derivative theories and ghosts, such as quantum unitarity and the stability of classical solutions in general relativity; and the perspective to overcome these problems. The gauge fixing and parametrization dependencies are discussed in detail using the corresponding general QFT theorems developed in gauge theories. On top of that, we present a basic example of deriving the one-loop divergences and discuss an important example of the renormalization group in QG. The gauge invariant renormalizability of QG is considered in another Chapter of the Handbook, written together with P.M. Lavrov.

Keywords

Quantum gravity; gravitational action; gauge fixing; propagator; power counting; renormalizability; higher derivatives; ghosts; stability.

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1 Introduction

Perturbative quantum gravity (QG) aims to construct quantum gravitational theory in a manner close to how QFT describes other fundamental forces, such as electroweak and high-energy strong interactions. One of the main purposes of perturbative QG is to get the quantum corrections to the classical (tree-level) action of gravity and the corresponding effective equations of motion for the metric.

The importance of loop corrections to the gravitational equations, coming from QG or from the quantum effects of matter fields (semiclassical approach), is partially owing to the fact that, at some point, we hope to be able to compare these corrections with experimental or observational data. Another reason to study QG is that the consistency of quantum theory may be useful to establish the restrictions on the modifications of general relativity (GR). It is assumed we may be able to select those gravity theories that are consistent at the quantum level and discard other models. Independent of all these reasons, in the last six decades the perturbative QG became an important part of the general QFT scenario and, especially, of the gauge fields theories.

When initiating the QG, we meet two main choices representing kinds of the points of bifurcation, where one can choose the direction of how to construct the theory. The first question is to decide what should be the object of quantization. In the traditional perturbative quantum gravity, we choose to quantize the spacetime metric, and apply to it the rules of quantization that are common to all gauge theories, such as the Yang-Mills. It is worth mentioning that one can choose other quantum variables to describe gravity, e.g., the tetrad or consider the connection independent of the metric (first order formalism). Owing to the size and contents limitations, we will not discuss these possibilities in what follows.

On the other hand, it is possible to extend the set of the quantum fields, that is include other fields along with the metric, or even replacing the metric. One of the most interesting approaches to make such extension is to provide an extended symmetry, as it is done in supergravity. We can refer the reader to the corresponding Section of the present Handbook for the reviews of supergravity, but will not discuss it here. Another possibility is to add quantum matter fields. Let us note that there is a reduced, semiclassical approach, when only the matter fields are assumed quantum and metric is regarded as a classical background. This kind of theories share many technical and conceptual features with perturbative QG, so the reader may find useful to study the semiclassical approach first. There are many useful monographs on this subject, let us mention just a few of them, [1, 2, 3, 4, 5, 6] and also recent textbook [7], where the reader can also find an introduction to QG². In what follows, we shall address the same subjects which are discussed in sections 18-21 of this textbook. Most of the present Chapter can be seen as a modified and more review-style extract from this book.

² For the interested reader, there are good books and reviews of semiclassical and quantum gravity theories, e.g., [8, 9, 10, 11]. One can find many other useful sources starting from these references.
The second important choice appears after we agree to quantize the metric. One has to choose the classical theory which should serve as a basis for the perturbative quantum gravity. Obviously, such initial model may be the Einstein’s GR, however there are strong reasons to try also other models of gravity. The choice of a model defines many important features of the theory, including its particle contents, renormalizability, unitarity, stability of classical solutions, high energy (UV) and low-energy (IR) behaviors, as we shall discuss below.

In what follows, we review the main elements of the perturbative QG, referring to the recent textbook [7] for most of the technical details. First of all, we consider the choice of the action and the corresponding gauge-fixing conditions, mostly, in the framework of the background field method [1, 12] (the reader can find more references starting from these two). After that, we review the structure of propagator of the metric perturbations in various models and the corresponding analysis of the degrees of freedom in QG.

The proof of the gauge-invariant renormalizability in QG is left for the second chapter of this Section [13]. This analysis possesses higher level of complexity compared to the mentioned textbook, but, as a result, the main statements are obtained in a more general and more concrete way. The main two outputs of this consideration, based on the BRST symmetry and the Batalin-Vilkovisky technique, are the following two statements:

i) The counterterms in the generally covariant theory of QG may have the same symmetry, i.e., the diffeomorphism invariance, as the initial classical action. This symmetry holds at the quantum level if we use a regularization preserving this symmetry, such that the general covariance is not violated by quantum corrections. For instance, this is the case for the perturbative QG in even dimension $n = 2m + 2$ (where $m = 1, 2, \ldots$), including $n = 4$, the last will be used by default in the rest of this Chapter.

ii) The dependence on the choice of the gauge fixing and on the parametrization of quantum field, at any order of loop expansion, is proportional to the effective equations of motion. As an important consequence of this rule, the respective dependencies in the one-loop divergences vanish on the classical mass shell.

The first feature implies that the general structure of divergences in any loop order may be defined on the basis of power counting of the Feynman diagrams, that means it can be reduced to the use of the dimensional arguments. In this way we can, in most of the cases, say that the given model of QG is renormalizable, non-renormalizable, or super-renormalizable, and even establish the structure of possible divergences in any loop order – without making explicit calculations. The remaining questions concern only the coefficients of the given terms in the divergences.

The second feature is also important for practical purposes, for the following two reasons. From one side, one can choose the gauge fixing condition in such a way that makes the calculations technically simpler. And, on another hand, we can extract that part of the quantum corrections which are gauge-fixing and parametrization invariant and, in principle, consider this as a physical output of the loop calculations.

The Chapter is organized as follows. In Sec. 2 we formulate the most relevant models of QG that may constitute a sound basis of the perturbative treatment. Let us
note that we only slightly touch the nonlocal models which represent, nowadays, a popular object of studies. The reason is that there is a special Section devoted to the nonlocal theories and we do not like to have repetitions. Thus, we mainly consider the QG based on GR and on the different kinds of local polynomial models. In Sec. 3, one can find the bilinear expansions of all metric-dependent terms that define the propagator of the quantum metric. In Sec. 5 we describe these propagators different models of QG. The next Sec. 6 is briefly formulating the the main statements about gauge-invariant renormalization in QG, while the detailed discussion is postponed to the separate Chapter of this Section. Sec. 7 describes the power counting and classification of the QG models to renormalizable, non-renormalizable and super-renormalizable ones. Sec. 8 discusses the problem of ghosts in higher derivative QG. This problem certainly represents the main difficulty of all the QG program. We give a basic introduction and a brief report on the existing results in this area. In Sec. 9 we describe the gauge-fixing and parametrization dependencies of the one-loop effective action and Sec. 10 shows the detailed derivation of the one-loop divergences in quantum GR, in the simplest parametrization and gauge fixing. Sec. 11 discusses an interesting example of the renormalization group applied to the quantum GR in a manner that fits the effective approach to QG. There is a special Section of the present Handbook, devoted to the subject of effective QG, so we do not go too deep into the subject and only show this particular example. Sec. 12 gives a short review of the one-loop calculations in other models of QG. Owing to the size limitations, this part is very much incomplete and does not cover the extensive literature on the subject, so it is included just as a starting point for the interested readership. Finally, in Sec. 13 we draw our conclusions and present final discussions of the current situation in perturbative QG.

In the rest of the Chapter, we use DeWitt notations, when the covariant integral \( \int d^4x \sqrt{-g} = \int \) may be assumed but not written explicitly when this is obvious. We denote functional trace Tr and determinant Det, that includes the same covariant integration over spacetime coordinates, and use pseudoeuclidean notations such as \( \sqrt{-g} \), even in case of using heat kernel technique, that requires Euclidean metric. In these cases, we assume the Wick rotation, regardless it may be a nontrivial issue in some models of QG. On top of this, the notations include the signature \( \eta_{\alpha\beta} = \text{diag}(+ - - -) \), the definition of the Riemann tensor

\[
R^\lambda_{\tau\alpha\beta} = \partial_\alpha \Gamma^\lambda_{\tau\beta} - \partial_\beta \Gamma^\lambda_{\tau\alpha} + \Gamma^\lambda_{\gamma\alpha} \Gamma^\gamma_{\tau\beta} - \Gamma^\lambda_{\gamma\beta} \Gamma^\gamma_{\tau\alpha},
\]

(1)

Ricci tensor \( R^a_{\mu\alpha\nu} = R_{\mu\nu} \), and its trace \( R = R_{\mu\nu} g^{\mu\nu} \), that is Ricci scalar.

2 Models of QG. General classification and gauge fixing

Our purpose is to construct the quantum theory of the metric field \( g_{\mu\nu} \). In GR, the metric is subject to the gauge transformation, also called diffeomorphism, corresponding to the infinitesimal coordinate transformation
The background information about perturbative quantum gravity

\[ x^\mu \to x'^\mu = x^\mu + \xi^\mu, \quad \xi^\mu = \xi^\mu(x). \quad (2) \]

Let us start by deriving the diffeomorphism transformation for the metric. Keeping only the terms of the first order in \( \xi^\mu \) and their derivatives, we get

\[ g'_{\alpha \beta}(x') = g'_{\alpha \beta}(x') - \partial_{\lambda} g_{\alpha \beta} \xi^\lambda). \]

Using the tensor transformation rule,

\[ g'_{\alpha \beta}(x') = \partial_{\rho} \partial_{\sigma} g_{\alpha \beta} (x') = \left( \delta_{\rho}^\alpha - \partial_{\alpha} \xi^\rho \right) \left( \delta_{\sigma}^\beta - \partial_{\beta} \xi^\sigma \right) \]

\[ = g_{\alpha \beta} - \partial_{\rho} g_{\alpha \beta} \partial_{\sigma} \xi^\rho - g_{\alpha \rho} \partial_{\beta} \xi^\sigma - \partial_{\sigma} \xi^\rho \partial_{\rho} \xi^\sigma \quad (3) \]

and taking the two expressions together, we get

\[ \delta g_{\alpha \beta}(x) = g'_{\alpha \beta}(x) - g_{\alpha \beta}(x) = -g_{\lambda \beta}(x) \partial_{\alpha} \xi^\lambda(x) - g_{\alpha \lambda}(x) \partial_{\beta} \xi^\lambda(x) \]

\[ \delta g_{\alpha \beta}(x) = -\nabla_\alpha \xi^\lambda - \nabla_\beta \xi^\lambda, \quad (4) \]

that defines the generators of the gauge transformations

\[ \delta S = R_{\mu \nu \lambda}(g) \xi^\lambda, \quad \delta S_{\mu \nu \lambda}(g) = -g_{\mu \lambda} \nabla_\nu - g_{\nu \lambda} \nabla_\mu \quad (5) \]

We assume that any sort of the QG candidate action \( S = S(g) \) possesses the symmetry under (5), i.e., satisfies the Noether identity,

\[ \frac{\delta S}{\delta g_{\mu \nu}} R_{\mu \nu \lambda}(g) = 0. \quad (6) \]

In the next Chapter [13] (see also [7]), it is shown that QG is the gauge theory of the Yang-Mills type, that means the algebra of the generators is closed off shell. This feature represents the basis of the two statements \( i \) and \( ii \), mentioned in the Introduction.

In many situations, it is useful to parameterize the metric as a perturbation over the Minkowski spacetime

\[ g_{\mu \nu} = \eta_{\mu \nu} + h_{\mu \nu}, \quad (7) \]

and use the following representation of \( h_{\mu \nu} \) (see [11] and Sec. 5 for more details):

\[ h_{\mu \nu} = \tilde{h}_{\mu \nu}^+, \quad \tilde{\mu} \nu \mu + \partial_\mu \epsilon_\nu^+ + \partial_\nu \epsilon_\mu^+ + \partial_\mu \partial_\nu \epsilon + \frac{1}{4} h \eta_{\mu \nu}. \quad (8) \]

In this expression, the tensor component (spin-2 mode) is traceless and transverse, i.e., \( \tilde{h}_{\mu \nu}^+ \eta^{\mu \nu} = 0 \) and \( \partial_\mu \tilde{h}_{\mu \nu}^+ = 0 \). The irreducible vector component (spin-1 mode) satisfies the condition \( \partial_\mu \epsilon^{\perp \mu} = 0 \). There are also two scalar fields (or modes) \( \epsilon \) and \( h \). The indices are raised and lowered with the flat metric.
In the rest of this section, we describe the most popular actions which are used for constructing the QG models. This description includes the proper action and the details of the DeWitt-Faddeev-Popov (or Faddeev-Popov) procedure required for the Lagrangian quantization of the model. For the sake of generality, we perform the consideration of these procedures using the background field method. The last means that, instead of \( g_{\mu\nu} \), the expansion is performed around an arbitrary background metric,

\[
g_{\mu\nu} \longrightarrow g'_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}.
\]

The interested reader can find more details of the expansion (9) and useful exercises in the book \[7\].

\[2.1\] **DeWitt-Faddeev-Popov method for QG**

Let us briefly sketch the Faddeev-Popov (or DeWitt-Faddeev-Popov) method for QG. There is no critical difference with other gauge models, and we shall use the notations close to the general ones. Namely, we shall denote \( g' = g_{\mu\nu} \) and keep in mind that, depending on our intentions, \( g' \) may be also used for \( h_{\mu\nu} \), as defined in (9). Then the transformation rule and the generator from Eq. (5) will be denoted as

\[
\delta g' = R_{\alpha}^{i} \xi_{\alpha} \quad \text{and} \quad R_{\alpha}^{i} = R_{\mu\nu\alpha}.
\]

The last note about notations is that \( g' \) may be also used for other parameterizations of the quantum metric\(^3\), and the formulas can be modified accordingly.

The starting point is the naive expression for the functional integral over the quantum metric

\[
Z = \int dg e^{iS(g)},
\]

where we use \( g = g' \) for the arguments of a functional \( S \) and the integration variable, to make formulas more readable. We can generalize the last integral to the generating functional of Green functions by replacing \( S \rightarrow S + gJ \) and assuming that the source term \( gJ = \int_{x} g'J_{\mu}\delta^{\mu}_{\nu} \) is diffeomorphism invariant. As this replacement does not cause real changes, we shall work with the formula (11).

The space of integration \( g' \) includes the *orbits*, i.e. the subspaces defined by the gauge transformations of the metric field (10). Since the action remains constant over any such subspace, it is clear that each of these subspaces contributes infinite to the integral, which is, therefore, badly defined. Such a divergence is similar to taking logarithm of determinant of a degenerate matrix. This divergence is a direct consequence of the gauge invariance and represents the problem solved by the

\(^3\) An example is Eq. (132) below.
The background information about perturbative quantum gravity

DeWitt-Faddeev-Popov method [14, 15] in gravity and in the Yang-Mills theory (see also, e.g., [16] for QG and further references in the next Chapter [13]).

The first step is to insert the factor of unity in the integrand of (11), in the form

\[ 1 = \Delta(g) \int d\xi^\alpha \delta(\chi^\alpha(g) - l^\alpha). \] (12)

Here \( l^\alpha \) is an arbitrary vector field and \( \chi^\alpha(g) \) is the gauge fixing condition. There can be different choices of this condition but, in principle, one can think that the surface \( \chi^\alpha(g) = l^\alpha \) (or, simply, \( \chi^\alpha(g) = 0 \)) crosses each orbit in a unique “point”, such that the degeneracy is removed. Indeed, this is not the way Faddeev-Popov method works, as we will see.

The integral (12) can be easily taken by making the change of integration variables to \( \chi^\alpha \),

\[ \Delta^{-1}(g) = \int d\chi^\beta \operatorname{Det} \left( \frac{\delta \xi^\alpha}{\delta \chi^\beta} \right) \delta(\chi^\alpha(g) - l^\alpha). \] (13)

The Jacobian of this transformation is inverse to the matrix

\[ \left( \frac{\delta \xi^\alpha}{\delta \chi^\beta} \right)^{-1} = \frac{\delta \chi_a}{\delta \xi^\beta} = \frac{\delta \chi_a}{\delta g^{i}} R^i_{\beta} = \frac{\delta \chi_a}{\delta g_{\mu \nu}} R_{\mu \nu}^\beta = M_{\alpha}^\beta. \] (14)

It is important that the determinant of the matrix \( M_{\alpha}^\beta \) does not depend on \( \xi \), but only on \( g^{i} \) and the form of the gauge fixing condition \( \chi(g) \). In this way, we arrive at the equivalent, albeit already non-degenerate, form of (11),

\[ Z = \int dg \delta(\chi^\alpha(g) - l^\alpha) \operatorname{Det} (M_{\alpha}^\beta) e^{iS(g)}. \] (15)

To achieve a more useful form of this expression, we insert in the integrand a unit in the form

\[ 1 = (\operatorname{Det} Y_{\alpha \beta})^{1/2} \int dl^\alpha \exp \left\{ \frac{i}{2} l^\alpha Y_{\alpha \beta} l^\beta \right\}. \] (16)

Here \( Y_{\alpha \beta} \) is a new object identified as weight operator. We will discuss several forms of this operator, adapted to different models of QG, in what follows.

Taking the integral with the delta function, we get

\[ Z = \int dg (\operatorname{Det} Y_{\alpha \beta})^{1/2} (\operatorname{Det} M_{\alpha}^\beta) e^{iS + iS_{gf}}, \] (17)

where

\[ S_{gf} = \frac{1}{2} \chi^\alpha Y_{\alpha \beta} \chi^\beta. \] (18)

In some situations, it is useful to deal with the modified form of the operator,
\[ \tilde{M}_{\alpha\beta} = Y_{\alpha\lambda} M_{\lambda\beta}^{\beta}, \]  

providing an alternative form of (17), i.e.,

\[
Z = \int dg \left( \text{Det} Y_{\alpha\beta} \right)^{-1/2} \left( \text{Det} \tilde{M}_{\alpha\lambda}^{\beta} \right) e^{iS + is_{gf}}
= \int dg d\bar{C} dC \left( \text{Det} Y_{\alpha\beta} \right)^{-1/2} e^{iS + is_{gf} + is_{gh}},
\]  

where the action of the Faddeev-Popov ghosts has the form

\[ S_{gh} = \bar{C}^{\alpha} \tilde{M}_{\alpha\beta}^{\beta} C_{\beta}. \]  

It is clear that the ghost fields \( \bar{C}^{\alpha} \) and \( C_{\beta} \) should be fermions (i.e., have odd Grassmann parity) to provide the positive power of \( \text{Det} \tilde{M}_{\alpha\beta}^{\beta} \). One can trade the functional determinant \( \left( \text{Det} Y_{\alpha\beta} \right)^{-1/2} \) for another (third) ghost. This ghost may have even or odd Grassmann parity, respectively, for the versions with \( \tilde{M}_{\alpha\beta} \) or \( M_{\beta}^{\lambda} \).

Thus, the total Faddeev-Popov action \( S_t = S + S_{gf} + S_{gh} \) depends on the choice of the gauge fixing condition \( \chi^\alpha \) and the weight operator (sometimes called weight function) \( Y_{\alpha\beta} \). In what follows, we consider a few important examples of the choice of these two objects in different models of QG. As with the most of this review, the reader can find more detailed discussion in \( [7] \).

### 2.2 Quantum general relativity

The first and most obvious candidate for the starting action of QG is the Einstein-Hilbert functional

\[
S_{EH} = -\frac{1}{\kappa^2} \int d^4x \sqrt{-g} \left( R + 2\Lambda \right).
\]  

Here we denoted \( \kappa^2 = 16\pi G \), as it proves useful when we consider the perturbative expansion. The reason is that, modifying the expansion (9) to

\[ g_{\mu\nu} \rightarrow g'_{\mu\nu} = g_{\mu\nu} + \kappa h_{\mu\nu}, \]  

the bilinear in \( h_{\mu\nu} \) terms are free from the parameter \( \kappa \), while the higher orders in \( h_{\mu\nu} \) terms have positive powers \( \kappa \). One of the consequences is that \( \kappa \) plays the role of the interaction constant in this QG model. In quantum theory, \( \kappa \) turns out the loop expansion parameter. Let us stress that this is, typically, not so for other QG models, which may have other parameters of the loop expansion. Other two relevant observations is that, since gravity is a non-polynomial theory, the parametrization (23) results in that the action (22) has unbounded powers of \( \kappa \). Furthermore, things
get more complicated in more general parameterizations but, typically, $\kappa$ remains the parameter of the loop expansion.

The flat limit requires a vanishing cosmological constant and we get

$$\quad g_{\mu \nu} = \eta_{\mu \nu} + \kappa h_{\mu \nu}. \quad (24)$$

As we shall see below, this means that the propagator of quantum metric is free of $\kappa$ by construction.

Instead of the expansion of the metric $g_{\mu \nu}$, one can start from the expansion of the inverse metric $g^{\mu \nu}$, or even a more general parametrization of the quantum metric. Later on, we consider the version which is the most general for the one-loop calculations and has various arbitrary parameters.

The Faddeev-Popov procedure requires introducing the gauge fixing term $S_{gf}$.

Such a term is called to make the highest derivative in $h_{\mu \nu}$ terms of the action $S + S_{gf}$ non-degenerate. Then one can obtain the propagator of this field or, e.g., apply the heat-kernel method and Schwinger-DeWitt technique for calculating the divergences in a covariant form. Since the theory (22) has at most second derivatives, we can choose the weight operator proportional to the background metric and arrive at the gauge-fixing term in the form

$$S_{gf} = \frac{1}{\alpha} \int d^4x \sqrt{-g} \chi^\mu \chi^\mu, \quad \text{where} \quad \chi^\mu = \nabla^\nu h^\nu_\mu - \beta \nabla^\mu h, \quad (25)$$

and $\alpha$ and $\beta$ are the gauge-fixing parameters. The dependence on the choice of the gauge fixing and on the parametrization of quantum metric, represents an important part of the QG development.

Why the action (22) requires the simplest weight functional in the case of QG? The reason is that the Lagrangian of GR (22) has at most two derivatives of the quantum metric $h_{\mu \nu}$ in the action. Thus, a “correct” (or, better to say, appropriate) way of breaking down the degeneracy in the total action requires that the gauge fixing term $S_{gf}$ also has two derivatives. Since each of $\chi^\mu$ in Eq. (25) is linear in derivatives, we are forced to implement the choice $Y_{\mu \nu} = \text{const} \times g_{\mu \nu}$ in this case. With some adjustments, the same logic will be used in all models of QG which will be discussed below (and even those which will not be).

The action of ghosts is constructed in a standard way, as

$$S_{gh} = \int d^4x \sqrt{-g} \bar{C} M^{\beta}_\alpha C_\beta, \quad (26)$$

where the operator $M$ is the variation of gauge fixing condition with respect to the transformation function,

$$M^{\beta}_\alpha = \frac{\delta \chi^\alpha}{\delta \bar{C}^\beta} = \frac{\delta \chi^\alpha}{\delta h_{\mu \nu}} R^{\beta}_{\mu \nu}. \quad (27)$$

It is important that the ghost fields in (26) satisfy the second-order equations. This means, the propagators of both gravitational field $h_{\mu \nu}$ and ghosts have the same
type of the UV behavior $1/k^2$. In the next models, we shall see that providing the homogeneity of the propagators of different modes of the metric (8) and of the ghosts may require some extra efforts. And such a homogeneity worth these efforts, as without it, the quantum theory gains a lot of artificial complications.

2.3 Fourth derivative gravity

The next model of common interest is based on the fourth-derivatives action. If the previous choice, namely, the Einstein-Hilbert action of GR, is strongly motivated by the success of Einstein’s classical gravitational theory, the fourth-derivative model is motivated by the consistency conditions of semiclassical gravity. It is known from the early paper by Utiyama and DeWitt [17] (see also the aforementioned books [2, 3, 5, 7]) that if the matter fields are quantum, the action of vacuum (i.e., the gravitational action) of renormalizable theory has to include both Einstein-Hilbert action (22) and the covariant local fourth-derivative terms

$$S_{HD} = \int d^4 x \sqrt{-g} \left\{ a_1 R^2_{\mu \nu \alpha \beta} + a_2 R^2_{\mu \nu} + a_3 R^2 + a_4 \Box R \right\}. \quad (28)$$

In QG, it is more useful to write this action in another basis, including the square of the Weyl tensor and the integrand of the Gauss-Bonnet topological term

$$C^2 = C_{\mu \nu \alpha \beta} C^{\mu \nu \alpha \beta} = R^2_{\mu \nu \alpha \beta} - 2R^2_{\mu \nu} + \frac{1}{3} R^2,$$
$$E_4 = R^2_{\mu \nu \alpha \beta} - 4R^2_{\mu \nu} + R^2. \quad (29)$$

In this basis, the fourth-derivative action has the form

$$S_{HD} = -\int d^4 x \sqrt{-g} \left\{ \frac{1}{2 \lambda} C^2 - \frac{1}{\rho} E_4 + \frac{1}{\xi} R^2 + \frac{1}{\kappa^2} \Box R + \frac{1}{\kappa^2} (R - 2 \Lambda) \right\}, \quad (30)$$

Sometimes other notations for the couplings $\rho$ and $\xi$ are used, e.g., $\theta = \lambda / \rho$ and $\omega = -3\lambda / \xi$, or the ones used in the special chapter [18] about the one-loop calculations and renormalization group flows in the theory (28). The important part is the positive sign of the coupling $\lambda$, as it is required by the positivity of the energy of the massless tensor mode (graviton) in the high energy region [19, 20] (see also an alternative treatment of the same problem in [7]).

Introducing the gauge-fixing term in the fourth-derivative theory of QG is a more complicated task compared to the quantum GR. Since we are interested to maintain homogeneity of the propagator, the gauge-fixing term should have four derivatives of the quantum metric and hence we introduce the expression for the gauge-fixing action

$$S_{gf} = \frac{1}{2} \int d^4 x \sqrt{-g} \chi^\mu Y_{\mu \nu} \chi^\nu, \quad (31)$$
where the gauge condition $\chi^\mu$ is still defined by the formula (25), but the new weight function $Y_{\mu\nu}$ should be a non-degenerate operator of the second order in derivatives. In the framework of background field method, its most general form is

$$Y_{\mu\nu} = \frac{1}{\alpha} \left( g_{\mu\nu} \Box + \gamma \nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu + p_1 R_{\mu\nu} + p_2 R g_{\mu\nu} \right),$$

(32)

where the non-degeneracy requires $\gamma \neq 0$. Compared to the quantum GR, the new gauge-fixing condition depends on a larger number of arbitrary gauge-fixing parameters, i.e., on $\alpha = (\alpha, \beta, \gamma, p_1, p_2)$, where $\beta$ comes from Eq. (25). In the case of the flat background metric, $Y_{\mu\nu}$ has only two arbitrary parameters $\alpha$ and $\gamma$.

With the definition (31) and (32), all the modes of the quantum metric in (8) have the same leading UV behaviour of the propagator, $G^{-1}_i(k) \propto k^4$. As we shall see in the forthcoming sections, the power counting is greatly simplified if the ghost action has the same number of derivatives as the action of the $h_{\mu\nu}$ field. It is clear that the problem can be solved by introducing a modified ghost action (21), namely

$$S_{gh} = \int d^4x \sqrt{-g} C^\alpha \tilde{M}_\alpha^\beta C_\beta, \quad \text{where} \quad \tilde{M}_\alpha^\beta = M_\alpha^\beta(g) Y_\alpha^\beta(g).$$

(33)

The functional integral is defined by the general expression (20) with the specific choice (32) of the weight operator. Since both $M_\alpha^\beta$ and $Y_\alpha^\beta$ are second order operators, the propagator of the Faddeev-Popov ghosts behaves like $k^{-4}$ in the UV, exactly as the propagator of gravitational perturbations. This feature proves useful for evaluating the power counting of the Feynman diagrams in this theory.

### 2.4 Quantum gravity models polynomial in derivatives

The previous two examples of QG models are minimal versions. In particular, GR fits all observational and experimental tests for a classical gravity [21] and, in this sense, can be regarded as a reference theory. On the contrary, there is not a single experimental test for the fourth-derivative model of gravity but, from another side, fourth derivative terms are required for the renormalizability of semiclassical gravity. And the same model guarantees also the renormalizability of QG [19]. On another hand, the fourth derivative theory has serious problems related to nonphysical ghosts and stability, as we shall discuss below. In this situation, one may look beyond the minimal theories and it is natural to try the models with more than four derivatives. Then, the number of derivatives may be finite or infinite. In the first case we meet the polynomial in derivatives models of QG, suggested in [22].

To construct the polynomial models, we impose the condition that the highest-derivative terms in the action should be homogeneous in the derivatives. In the next section, we shall see that this homogeneity may provide the superrenormalizability

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4 Which have nothing to do with the Faddeev-Popov ghosts, the traditional use of the same word here is a mere coincidence.
of the theory. The action of the theory has the form

\[ S_N = \int d^4x \sqrt{-g} \left\{ \partial_{N,R} R \Box^N R + \partial_{N,C} C \Box^N C + \partial_{N,GB} GB_N \\
+ \partial_{N-1,R} R \Box^{N-1} R + \partial_{N-1,C} C \Box^{N-1} C + \partial_{N-1,GB} GB_{N-1} + \ldots \\
+ \partial_0 R^2 + \partial_0 C^2 + \partial_0 GB_0 + \partial_{EH} R + \partial_{cc} + \mathcal{O}(R^3) \right\}, \]  

(34)

where \( N = 1, 2, \ldots \) and all \( \vartheta \)'s are arbitrary parameters of the action. It is assumed that both \( \vartheta_{N,R} \) and \( \vartheta_{N,C} \) are non-zero and that the maximal power of metric derivatives in the terms \( \mathcal{O}(R^3) \) is \( 2N + 4 \), i.e., is not higher than of the terms of the second order in curvatures, i.e., \( \mathcal{O}(R^2) \).

Furthermore, there are the squares of the Weyl tensor with extra factors of \( \Box \),

\[ C \Box^n C = C_{\mu \nu \alpha \beta} \Box^n C^{\mu \nu \alpha \beta} = R_{\mu \nu \alpha \beta} \Box^n R^{\mu \nu \alpha \beta} - 2R_{\mu \nu} \Box^n R^{\mu \nu} + \frac{1}{3} R \Box^n R. \]  

(35)

Similarly, using integrations by parts and the Bianchi identities, the generalized Gauss-Bonnet invariants can be shown to have the property

\[ GB_n = R_{\mu \nu \alpha \beta} \Box^n R^{\mu \nu \alpha \beta} - 4R_{\mu \nu} \Box^n R^{\mu \nu} + R \Box^n R = \mathcal{O}(R^3). \]  

(36)

This term is not topological for \( n \geq 1 \), but it contributes only to the third- and higher-order terms in the curvature tensor, and to the total derivatives. Thus, it may affect the vertices but not the propagators of QG, as we shall explicitly check out in what follows. On a flat background, the \( \mathcal{O}(R^2) \) terms may contribute to the propagator of the gravitational perturbation, while \( \mathcal{O}(R^3) \) terms affect only the vertices. In the action (34), the \( \mathcal{O}(R^2) \)'s are given in the basis of Weyl-squared and \( R \)-squared terms. As will be shown below, the Weyl-squared terms affect the propagation of the tensor mode \( \bar{h}_{\mu \nu} \), and the \( R \)-squared \( R \cdot R \) terms affect the propagation of the scalar modes. Using the higher-derivative actions in the form (34), we separate the propagators of the tensor and scalar modes at the level of the action.

To provide the homogeneity in derivatives for the propagator of all modes (8), the gauge-fixing terms should have the same highest power in derivatives as the main action. Since the gauge fixing conditions \( \chi_\mu \) are always chosen in the form (25), the \( \mathcal{O}(p^{-4-2N}) \) propagator of the quantum metric requires the weight function to be

\[ Y_{\mu \nu} = -\frac{1}{\alpha} (g_{\mu \nu} \Box + \gamma \nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) \Box^{N+1}. \]  

(37)

One can add here many terms with lower order of derivatives, but this does not change the UV behaviour of the propagator.

To provide the same power of derivatives in the ghost sector, as in the quadratic in curvature action, one can redefine the ghost action as (33), this time with the weight operator (37).

Furthermore, one can write the action (34) in an alternative form,
The background information about perturbative quantum gravity

\[ S_N = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} C_{\mu
u\alpha\beta} P_1(\Box) C^{\mu\nu\alpha\beta} + \frac{1}{2} R P_2(\Box) R + \vartheta_{\text{EH}} R + \vartheta_{\text{cc}} + \mathcal{O}(R^3) \right\}, \tag{38} \]

where \( P_{1,2}(x) \) are polynomials of the same order \( N \) and the terms \( \mathcal{O}(R^3) \) have at most \( 4 + 2N \) derivatives of the metric. One can make further generalization of (38), by trading the polynomials to the infinite series of \( \Box \). The discussion of these theories can be found in the corresponding Section of the present Handbook. Let us just quote the expression for the general (i.e., polynomial or non-polynomial) action

\[ S_{\text{gen}} = \int d^4x \sqrt{-g} \left\{ -\frac{1}{\kappa^2} (R + 2\Lambda) + \frac{1}{2} C_{\mu
u\alpha\beta} \Phi(\Box) C^{\mu\nu\alpha\beta} + \frac{1}{2} R \Psi(\Box) R + \mathcal{O}(R^3) \right\}. \tag{39} \]

To complete the bilinear in curvature part of the action we can add the third higher-derivative term, which boils down to the Gauss-Bonnet topological term for a constant form factor \( \Omega \),

\[ S_{\text{GB}} = \frac{1}{2} \int d^4x \sqrt{-g} \left\{ R_{\mu\nu\alpha\beta} \Omega(\Box) R^{\mu\nu\alpha\beta} - 4 R_{\mu\nu} \Omega(\Box) R_{\mu\nu} + R \Omega(\Box) R \right\}. \tag{40} \]

Despite this term is equivalent to \( \mathcal{O}(R^3) \) in (38), it makes sense to verify this feature, at least at the level of the propagator. The homogeneity of the propagator of all modes of the metric perturbations (8), requires the functions \( \Phi(x) \) and \( \Psi(x) \) to have analogous behavior in the UV. This can be achieved by requiring that

\[ \lim_{x \to \infty} \frac{\Psi(x)}{\Phi(x)} = C, \tag{41} \]

with \( C \) being a non-zero constant. For the polynomial functions \( P_{1,2}(x) \) of the same order, in (38), the last condition is guaranteed. In the case of the non-polynomial functions, the simplest useful choice is

\[ \Phi(x) = -\frac{1}{\kappa^2 x} \left( e^{\alpha_1 x} - 1 \right) \quad \text{and} \quad \Psi(x) = -\frac{C}{\kappa^2 x} e^{\alpha_2 x}, \tag{42} \]

such that condition (41) reduces to \( \alpha_1 = \alpha_2 \).

The gauge-fixing term in theory (39) that provides the homogeneity of the propagators, is of the standard form (31), but requires the special weight operator

\[ Y_{\mu\nu} = (g_{\mu\nu} \Box - \gamma^\mu \gamma_\nu \gamma_v + p_1 R_{\mu\nu} + p_2 R g_{\mu\nu}) W(\Box). \tag{43} \]

In many cases, it is sufficient to consider \( W(\Box) \propto \Phi(\Box) \), but we shall keep this function arbitrary for generality, until some point.

Finally, the homogeneity in the momentum at the UV for the quantum metric and for the Faddeev-Popov ghosts, can be achieved by the standard replacement (33).
3 Bilinear forms and linear approximation

The analysis of propagators in different models of quantum gravity requires the bilinear expansions of the relevant quantities depending on the curvature tensor, on a flat metric background. On the other hand, similar expressions with an arbitrary background metric are useful for the one-loop calculations in the background field method. Thus, we consider a general case and assume the expansion (9), i.e. $g_{\mu\nu} \rightarrow g'_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$.

Let us refer the interested reader to the book [7] for technical details and only give the following list of basic expansions:

$$ g^{\mu\nu} = h^{\mu\nu} + h^{\mu\lambda} h_{\lambda\nu} - h^{\mu\lambda} h^{\nu\kappa} h_{\lambda\kappa} + \ldots $$

$$ \sqrt{-g'} = \sqrt{-g} \left( 1 + \frac{1}{2} h + \frac{1}{8} h^2 - \frac{1}{4} h_{\mu\nu} h^{\mu\nu} + \ldots \right), $$

$$ \Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} + \delta \Gamma^\alpha_{\beta\gamma}, \quad (44) $$

where

$$ \delta \Gamma^\alpha_{\beta\gamma} = \frac{1}{2} \left( \delta^{\alpha\lambda} - h^{\alpha\lambda} + h^{\alpha\lambda} h^\kappa_{\lambda k} - h^{\alpha\lambda} h^\tau_{\lambda \tau \lambda} + \ldots \right) \left( \nabla^\mu h^\mu_{\lambda \lambda} + \nabla^\nu h^\nu_{\beta \lambda} - \nabla^\lambda h^\lambda_{\mu \lambda} \right). $$

In these formulas, the Greek indices are lowered and raised with the background metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$. One has to remember that the variation $\delta \Gamma^\alpha_{\mu\nu}$ is a tensor, and, therefore, can be a subject to covariant differentiation.

The first two orders of expansion of the Riemann tensor have the form

$$ R^{\alpha}_{\beta\mu\nu} = R^{\alpha}_{\beta\mu\nu} + \delta R^{\alpha}_{\beta\mu\nu}, \quad \text{where} \quad \delta R^{\alpha}_{\beta\mu\nu} = R^{(1)\alpha}_{\beta\mu\nu} + R^{(2)\alpha}_{\beta\mu\nu}, \quad (45) $$

$$ R^{(1)\alpha}_{\beta\mu\nu} = \frac{1}{2} \left( \nabla^\mu \nabla^\nu h^\alpha_{\mu\nu} - \nabla^\nu \nabla^\mu h^\alpha_{\mu\nu} + \nabla^\mu \nabla^\nu h^\alpha_{\mu\nu} - \nabla^\mu \nabla^\nu h^\alpha_{\mu\nu} \right), $$

$$ R^{(2)\alpha}_{\beta\mu\nu} = \frac{1}{2} \left( h^{\alpha\lambda} \left( \nabla^\mu h^\lambda_{\mu\nu} - \nabla^\nu h^\lambda_{\mu\nu} \right) + \nabla^\mu \nabla^\nu h^\lambda_{\mu\nu} - \nabla^\mu \nabla^\nu h^\lambda_{\mu\nu} \right) $$

$$ + \left[ \nabla^\nu, \nabla^\mu \right] h^\lambda_{\mu\nu} + \frac{1}{4} \left( \nabla^\mu h^\alpha_{\mu\nu} \right) \left( \nabla^\nu h^\alpha_{\mu\nu} - \nabla^\mu h^\nu_{\mu\nu} - \nabla^\nu h^\lambda_{\mu\nu} \right) $$

$$ - \left( \nabla^\mu h^\lambda_{\mu\nu} + \nabla^\mu h^\mu_{\nu\nu} \right) \left( \nabla^\nu h^\alpha_{\mu\nu} - \nabla^\nu h^\alpha_{\mu\nu} \right) + \left( \nabla^\mu h^\nu_{\mu\nu} \right) \left( \nabla^\nu h^\lambda_{\mu\nu} - \nabla^\nu h^\lambda_{\mu\nu} \right) $$

$$ - \left( \nabla^\nu h^\mu_{\mu\nu} \right) \left( \nabla^\nu h^\alpha_{\mu\nu} - \nabla^\nu h^\alpha_{\mu\nu} \right) - \left( \nabla^\nu h^\lambda_{\mu\nu} \right) \left( \nabla^\mu h^\alpha_{\mu\nu} - \nabla^\mu h^\alpha_{\mu\nu} \right) $$

$$ + \left( \nabla^\mu h^\nu_{\mu\nu} + \nabla^\nu h^\lambda_{\mu\nu} \right) \left( \nabla^\mu h^\alpha_{\mu\nu} - \nabla^\nu h^\lambda_{\mu\nu} \right). $$

Here and in what follows, points indicate the positions of the raised indices. Similar formulas for the Ricci tensor and scalar curvature are
where the background information about perturbative quantum gravity.

\[ R_{\beta \nu} = R_{\mu \nu} + \delta R_{\mu \nu}, \quad \text{where} \quad \delta R_{\mu \nu} = R_{\mu \nu}^{(1)} + R_{\mu \nu}^{(2)} \]

\[ R_{\mu \nu}^{(1)} = \frac{1}{2} \left( \nabla_{\lambda} \nabla_{\beta} h_{\mu}^{\lambda} + \nabla_{\nu} h_{\mu}^{\lambda} - \nabla_{\mu} \nabla_{\nu} h - \Box h_{\mu \nu} \right) \]

\[ R_{\mu \nu}^{(2)} = \frac{1}{2} h^{\alpha \beta} \left( \nabla_{\alpha} \nabla_{\beta} h_{\mu \nu} + \nabla_{\mu} \nabla_{\nu} h_{\alpha \beta} - \nabla_{\alpha} \nabla_{\mu} h_{\beta \nu} - \nabla_{\alpha} \nabla_{\nu} h_{\beta \mu} \right) + \frac{1}{4} \left( \nabla_{\mu} h_{\alpha \beta} \right) \left( \nabla_{\nu} h_{\alpha \beta} \right) + \frac{1}{4} \left( 2 \nabla_{\beta} h_{\alpha \beta} - \nabla_{\alpha} h \right) \left( \nabla_{\alpha} h_{\mu \nu} - \nabla_{\mu} h_{\alpha \nu} - \nabla_{\nu} h_{\alpha \mu} \right) \]

and

\[ R' = R + \delta R, \quad \text{where} \quad \delta R = R^{(1)} + R^{(2)} \]

\[ R^{(1)} = \nabla_{\mu} \nabla_{\nu} h^{\mu \nu} - \Box h - R_{\mu \nu} h^{\mu \nu} \]

\[ R^{(2)} = h^{\alpha \beta} \left( \nabla_{\alpha} \nabla_{\beta} h + \Box h_{\alpha \beta} - \nabla_{\alpha} \nabla_{\mu} h_{\beta \mu} - \nabla_{\mu} \nabla_{\alpha} h_{\beta \mu} \right) - \frac{1}{4} \left( \nabla_{\alpha} h \right) \left( \nabla_{\alpha} h \right) + \frac{1}{4} \left( \nabla_{\mu} h_{\alpha \beta} \right) \left( 3 \nabla_{\mu} h_{\alpha \beta} - 2 \nabla_{\alpha} h_{\mu \beta} \right) + \left( \nabla_{\beta} h_{\alpha \beta} \right) \left( \nabla_{\beta} h - \nabla_{\mu} h_{\beta \mu} + R_{\mu \nu} h_{\alpha \nu} h^{\mu \nu} \right) . \]

Using the expressions listed above, one can easily get the expansions of the terms in the action, up the second order in \( h_{\mu \nu} \). The results can be written for the terms in the four derivative action \( (30) \), but it is easy to show how, in the particular case of a flat metric, these expansions can be mapped to the more general action \( (39) \) with an arbitrary (finite or even infinite) number of derivatives.

The first expansion has the form

\[ \left( \int d^{4}x \sqrt{-g} \left[ R' + 2\Lambda \right] \right)^{(2)} = \frac{1}{4} \int d^{4}x \sqrt{-g} h^{\mu \nu} \left[ \delta_{\mu \nu, \alpha \beta} \Box - g_{\mu \nu} g_{\alpha \beta} \Box 

- 2 g_{\mu \nu} \nabla_{\nu} \nabla_{\beta} + \left( g_{\mu \nu} \nabla_{\alpha} g_{\beta \nu} - g_{\alpha \beta} \nabla_{\nu} g_{\mu \nu} \right) - \left( g_{\mu \nu} R_{\alpha \beta} - g_{\alpha \beta} R_{\mu \nu} \right) \right.

+ 2 R_{\mu \alpha \nu \beta} - \left( R + 2\Lambda \right) \left( \delta_{\mu \nu, \alpha \beta} - \frac{1}{2} g_{\mu \nu} g_{\alpha \beta} \right) \right] h_{\alpha \beta} , \]

where we used the DeWitt notation for the unit matrix in the symmetric tensors space

\[ \delta_{\mu \nu, \alpha \beta} = \frac{1}{2} \left( g_{\mu \alpha} g_{\nu \beta} + g_{\nu \alpha} g_{\mu \beta} \right) \]

and the short notations that assume the symmetrization, e.g.,

\[ g_{\mu \alpha} \nabla_{\nu} \nabla_{\beta} \rightarrow \frac{1}{4} \left( g_{\mu \alpha} \nabla_{\nu} \nabla_{\beta} + g_{\nu \alpha} \nabla_{\mu} \nabla_{\beta} + g_{\mu \beta} \nabla_{\nu} \nabla_{\alpha} + g_{\nu \beta} \nabla_{\mu} \nabla_{\alpha} \right) . \]

For the sake of brevity, the remaining expansions will be given only for a flat background. The complete expressions can be found, e.g., in [7]. The simplest of the remaining relevant bilinear expansions is...
\[
\left( \int d^4 x \sqrt{-g} R^2 \right)_{\text{flat}}^{(2)} = \int d^4 x \ h^{\mu \nu} \left[ \eta_{\alpha \beta} \eta_{\mu \nu} \Box^2 \eta_{\mu \nu} \partial_\alpha \partial_\beta - \eta_{\alpha \beta} \partial_\mu \partial_\nu \partial_\alpha \partial_\beta \right] h^{\alpha \beta}.
\]

(50)

One can note the absence of the term \(\delta_{\mu \nu, \alpha \beta} \Box^2\) in this expression. As a result, the \(R^2\) term does not affect the propagation of the spin-2 mode \(\bar{h}_{\mu \nu}^{\perp \perp}\) of the metric perturbation (8).

The next expansions are the one for the square of the Riemann tensor,

\[
\left( \int d^4 x \sqrt{-g} R^2_{\mu \nu \alpha \beta} \right)_{\text{flat}}^{(2)} = \int d^4 x \ h^{\mu \nu} \left[ \delta_{\mu \nu, \alpha \beta} \Box^2 + \partial_\alpha \partial_\beta \partial_\mu \partial_\nu - 2 \eta_{\nu \beta} \partial_\alpha \partial_\beta \right] h^{\alpha \beta}
\]

(51)

and for the square of the Ricci tensor,

\[
\left( \int d^4 x \sqrt{-g} R^2_{\mu \nu} \right)_{\text{flat}}^{(2)} = \frac{1}{2} \int d^4 x \ h^{\mu \nu} \left[ \frac{1}{2} \left( \delta_{\mu \nu, \alpha \beta} + \eta_{\mu \nu} \eta_{\alpha \beta} \right) \Box^2 - \eta_{\nu \beta} \Box \nabla_\alpha \nabla_\mu + \nabla_\alpha \nabla_\mu \nabla_\beta \nabla_\nu - \frac{1}{2} \eta_{\mu \nu} \Box \nabla_\alpha \nabla_\beta - \frac{1}{2} \eta_{\alpha \beta} \Box \nabla_\mu \nabla_\nu \right] \ h^{\alpha \beta}.
\]

(52)

The last two expansions (51) and (52) possess the \(\delta_{\mu \nu, \alpha \beta} \Box^2\) terms. This means, these two terms contribute to the flat-space propagator of the transverse and traceless mode of the gravitational perturbation \(\bar{h}_{\mu \nu}^{\perp \perp}\) in the representation (8).

One can note that it would be impossible to have only one of the terms (51) and (52) contributing to the propagation of the spin-2 mode, because (50) does not contribute to this mode and the linear combination of the three terms (29) form a topological invariant \(E_4\).

4 Gravitational waves, quantization, and gravitons

The gravitational wave is a dynamical classical solution of Einstein’s GR without matter sources. This term can be also used for the solutions of the same sort in modified gravity models. However, since in these models the additional modes are typically massive and, therefore, do not propagate for a long distances, it is most common to attribute the notion of a gravitational wave to the solutions in GR\(^5\).

Nowadays, gravitational waves represent one of the most successful parts of gravitational physics, both experimental and theoretical. However, in this short section we present only a brief survey of the basic facts concerning the gravitational waves on a flat background in GR.

\(5\) The topics related to the models of massive gravity are left beyond the present Handbook. The main reason is that, in these models, quantum aspects do not play an important role.
4.1 Gravitational waves in a weak-gravity regime

We start with an action of gravity with \( \Lambda = 0 \) and use the bilinear expansion \((48)\) on the flat background metric \( g_{\mu \nu} = \eta_{\mu \nu} \). To discuss the emission of the gravitational wave (in the simplest case), we also add the action of matter and consider it approximation that provides a linear equation for \( h_{\mu \nu} \). In this way, we obtain the action of GR in the linearized regime,

\[
S_{\text{total}}^{(\text{lin})} = -\frac{1}{32\pi G} \int d^4x \ h^{\mu \nu} \left\{ \frac{1}{2} \delta_{\mu \nu, \alpha \beta} \Box - \frac{1}{2} \eta_{\mu \nu} \eta_{\alpha \beta} \Box - \eta_{\mu \alpha} \partial_\nu \partial_\beta \right\} h^{\alpha \beta} - \frac{1}{16} \int d^4x h^{\mu \nu} T_{\mu \nu},
\]

(53)

where \( \Box = \eta^{\mu \nu} \partial_\mu \partial_\nu \) and \( T_{\mu \nu} \) is the energy-momentum tensor of matter in flat space-time background. The equation for metric perturbations has the form

\[
\left\{ \delta_{\mu \nu, \alpha \beta} \Box - \eta_{\mu \nu} \eta_{\alpha \beta} \Box - 2 \eta_{\mu \alpha} \partial_\nu \partial_\beta + (\eta_{\mu \nu} \partial_\alpha \partial_\beta - \eta_{\alpha \beta} \partial_\mu \partial_\nu) \right\} h^{\alpha \beta} = 16\pi G T_{\mu \nu}.
\]

(54)

Multiplying both sides of Eq. (54) by the matrix

\[
K^{\mu \nu, \rho \sigma} = \delta^{\mu \nu, \rho \sigma} - \frac{1}{2} \eta^{\mu \nu} \eta^{\rho \sigma},
\]

(55)

we arrive at the equation for the modified stress tensor, \( S_{\mu \nu} = T_{\mu \nu} - \frac{1}{2} T \gamma^\lambda g_{\mu \nu} \),

\[
\partial_\lambda h^\lambda_\mu + \partial_\lambda h^\lambda_\mu - \Box h_{\mu \nu} - \partial_\mu \partial_\nu h = 16\pi G S_{\mu \nu}.
\]

(56)

Here \( h = h_{\mu \nu} \eta^{\mu \nu} \). The last equation describes both propagation and emission of the gravitational waves in the linear approximation. This equation has to be supplemented by the gauge transformation (4) \( \delta h_{\mu \nu} = - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu \), and requires imposing the gauge-fixing condition, e.g., using the de Donder (also called Fock-de Donder) condition

\[
\partial_\mu h^\mu_\nu - \frac{1}{2} \partial_\nu h = 0.
\]

(57)

Using condition (57) in Eq. (56), the last is cast in the form (see, e.g., [23])

\[
\Box h_{\mu \nu} = -16\pi G S_{\mu \nu}.
\]

(58)

For the plane wave propagating along an arbitrary axis, the components of the metric perturbation (8) which are gauge invariant and can be called physical, are transverse ones, \( \tilde{h}_{\mu \nu} \). Thus, the gravitational wave in GR is a propagation of the spin-2 state.

The emission of the gravitational wave in the linear regime corresponds to the solution in the standard form of retarded potential,
\[ h_{\mu \nu}(x,t) = \frac{4}{M_P^2} \int d^3x' \frac{S_{\mu \nu}(x', t - |x - x'|)}{|x - x'|}. \] (59)

The factor \(1/M_P^2 = G\) in this expression shows that the emission of the gravitational waves is suppressed by the square of the Planck mass. And after the wave travels a very long distance, there is a similar Planck suppression at the moment of its detection, that explains the difficulty of detecting the gravitational wave. The remarkable detection by LIGO is explained by the incredible quantity of energy emitted in the merger of two black holes or other extremely compact and massive objects, such as neutron stars.

### 4.2 Quantization and the notion of graviton

At quantum level, the physical degrees of freedom corresponding to the state of a free gravitational field on a flat background correspond to the degrees of freedom of the linear gravitational wave described above. The corresponding particle with zero mass and spin-2 is called a graviton.

To derive the spin-2 part of the propagator of the gravitational perturbation \(h_{\mu \nu}\), we can use the tensor part of the propagator [see, e.g., Eq. (87) below] and setting \(\Phi = 0\). For the sake of simplicity, we omit the coefficient \(\kappa^2\) and obtain the spin-2 part of the Euclidean propagator in the form

\[ \langle h_{\mu \nu} h_{\alpha \beta} \rangle^{(2)} = G^{(2)}_{\mu \nu, \alpha \beta}(k) = \frac{P^{(2)}_{\mu \nu, \alpha \beta}(k)}{k^2}, \] (60)

where the projector \(P^{(2)}_{\mu \nu, \alpha \beta}(k)\) to the spin-2 states will be defined below. This equation describes the propagation of the massless degrees of freedom associated with the spin-2 states in GR, which is the graviton.

In other models of QG, the propagator can be more complicated because \(\Phi\) (and also \(\Psi\), because there may be a relevant dynamics in the scalar sector) are typically nonzero. For instance, including the fourth-derivative terms, there may be the following two changes:

i) Instead of the unique massless pole in the propagator (60), there may be additional massive poles.

ii) The scalar components of the metric perturbation (8) gain a massive, gauge-independent sector in the propagator.

Adding more derivatives, which means using polynomial or even nonlocal models, the modifications always concern the same two points. Namely, there will be (in the polynomial models) a growing number of poles in the spin-2 sector and the scalar sector. In contrast, some choices of nonlocal action may provide that there would not be any massive poles in the tree-level propagator, in both spin-2 and spin-0 sectors.
The background information about perturbative quantum gravity

It is worth noting that the count of degrees of freedom, based on the simple analysis of the gravitational propagator, was confirmed by the canonical quantization of the gravitational theory in the cases of quantum GR and fourth-derivative quantum gravity (see, e.g., [3]).

5 Propagator of metric and the Barnes-Rivers projectors

At this point we note a common aspect of all mentioned models of QG, namely the ones based on the actions (22), (30), (39). Since the propagator of $h_{\mu \nu}$ in flat background is defined by the quadratic in curvatures (Riemann, Ricci or scalar $R$) terms in the action, the form factors depending on $\Box$ do not influence the tensor structure of the bilinear form.

The propagator $G$ of the quantum metric, in any QG model, obeys the equation

$$H_{\mu \nu, \alpha \beta}(x) G_{\alpha \beta, \rho \sigma}(x, y) = \delta^4(x - y) \delta_{\rho \sigma} \delta_{\mu \nu}, \quad (61)$$

where

$$H_{\mu \nu, \alpha \beta}(x) = \frac{1}{2\sqrt{-g}} \frac{\delta^2 S}{\delta g_{\mu \nu}(x) \delta g_{\alpha \beta}(y)}, \quad (62)$$

is the bilinear in quantum fields form of the classical action $S$ of a model of quantum gravity. The action in (62) should include the gauge-fixing term, as otherwise we meet a degeneracy. For the sake of definiteness, we start from the gauge-invariant form and denote the corresponding degenerate bilinear form coming from the initial action, as $H_{(0), \mu \nu, \alpha \beta}$, while the non-degenerate version, after the Faddeev-Popov procedure, will be denoted as $H_{\mu \nu, \alpha \beta}$.

In this part of the Chapter, all spacetime indices are raised and lowered with the flat background metric. Then, independently of the model and the choice of the gauge fixing, the operator $H_{\mu \nu, \alpha \beta}(x)$ has the following tensor structure:

$$H_{\mu \nu, \alpha \beta}(x; g) = a_1 \delta_{\mu \nu, \alpha \beta} \Box + a_2 \eta_{\mu \nu} \eta_{\alpha \beta} \Box + a_3 (\eta_{\mu \nu} \partial_{\alpha} \partial_{\beta} + \eta_{\alpha \beta} \partial_{\mu} \partial_{\nu}) + a_4 (\eta_{\mu \alpha} \partial_{\beta} \partial_{\nu} + \eta_{\nu \alpha} \partial_{\beta} \partial_{\mu} + \eta_{\mu \beta} \partial_{\alpha} \partial_{\nu} + \eta_{\nu \beta} \partial_{\alpha} \partial_{\mu}) - a_5 \partial_{\alpha} \partial_{\beta} \partial_{\mu \nu}, \quad (63)$$

where $a_k = a_k(\Box)$ are five model-dependent functions of the d’Alembert operator.

In the higher-derivative cases, all of these functions are proportional to the linear combinations of $\Phi(\Box)$ and $\Psi(\Box)$ in Eq. (39). In particular, for the fourth-derivative model (30), $a_{1,2,3,4}$ are linear functions of $\Box$ and $a_5 = const$. In the case of quantum GR, there are the constant functions $a_{1,2,3,4}$ and $a_5 = 0$. In what follows, we consider the general analysis of the propagator, which is valid for all types of models.

Making a Fourier transform, we can rewrite the bilinear form in the momentum representation,
\[ H_{\mu\nu,\alpha\beta}(k; \eta) = -[a_1(k^2)\delta_{\mu\nu,\alpha\beta}k^2 + a_2(k^2)\eta_{\alpha\nu}\eta_{\alpha\beta}k^2 \\
+ a_3(k^2)(\eta_{\mu\nu}k_\alpha k_\beta + \eta_{\alpha\beta}k_\mu k_\nu) \\
+ a_4(k^2)(\eta_{\mu\nu}k_\beta k_\nu + \eta_{\nu\alpha}k_\beta k_\alpha + \eta_{\mu\nu}k_\alpha k_\beta + \eta_{\nu\beta}k_\alpha k_\mu) + a_5(k^2)k_\beta k_\nu k_\mu k_\nu] \tag{64} \]

where \( k^2 = k_\mu k^\mu \) is the square of the four-dimensional momentum and \( \delta_{\mu\nu,\alpha\beta} \) is similar to (48), but this time it is constructed from the flat metric \( \eta_{\mu\nu} \).

It is useful to present (64) in a slightly different form, providing more generality by using the \( n \)-dimensional versions of the formulas,

\[ \hat{H} = s_1 \mathcal{T}_1 + s_2 \mathcal{T}_2 + s_3 \mathcal{T}_3 + s_4 \mathcal{T}_4 + s_5 \mathcal{T}_5, \tag{65} \]

where \( \mathcal{T}_n = T_{\mu\nu,\alpha\beta}^{(n)} \) and

\[ \mathcal{T}_1 = \delta_{\mu\nu,\alpha\beta}, \quad \mathcal{T}_2 = \eta_{\mu\nu}\eta_{\alpha\beta}, \quad \mathcal{T}_3 = \frac{1}{k^2}(\eta_{\mu\nu}k_\alpha k_\beta + \eta_{\alpha\beta}k_\mu k_\nu), \quad \mathcal{T}_4 = \frac{1}{4k^2}(\eta_{\mu\nu}k_\beta k_\nu + \eta_{\nu\alpha}k_\beta k_\alpha + \eta_{\mu\beta}k_\alpha k_\nu + \eta_{\nu\beta}k_\alpha k_\mu), \quad \mathcal{T}_5 = \frac{1}{k^4}k_\alpha k_\beta k_\mu k_\nu. \tag{66} \]

The coefficients depend on momentum, \( s_j = s_j(k^2) \), and these dependencies may be nontrivial, e.g., in the polynomial or nonlocal models. However, the tensor structure of the expressions (65) is the same for all QG models, i.e., for quantum GR, or for a higher-derivative polynomial, or nonlocal models (39).

To invert the operator (64) and take care about its possible degeneracy, consider the operators called Barnes-Rivers projectors [24, 25]. The starting point is to formulate the projectors to the transverse and longitudinal subspaces of the vector space. In the momentum representation we have

\[ \omega_{\mu\nu} = \frac{k_\mu k_\nu}{k^2}, \quad \theta_{\mu\nu} = \eta_{\mu\nu} - \omega_{\mu\nu}, \tag{67} \]

with the standard properties

\[ \omega_{\mu\nu} \omega_{\lambda}^\nu = \omega_{\mu\lambda}, \quad \theta_{\mu\nu} \theta_{\lambda}^\nu = \theta_{\mu\lambda}, \quad \omega_{\mu\nu} \theta_{\lambda}^\nu = 0. \tag{68} \]

Then, the projectors to the spin-2, spin-1 and spin-0 states in the symmetric tensors space are written in the form

\[ \hat{P}^{(2)}_{\mu\nu,\alpha\beta} = P^{(2)}_{\mu\nu,\alpha\beta} = \frac{1}{2}(\theta_{\mu\alpha}\theta_{\nu\beta} + \theta_{\mu\beta}\theta_{\nu\alpha}) - \frac{1}{n-1} \theta_{\mu\nu}\theta_{\alpha\beta}, \]

\[ \hat{P}^{(1)}_{\mu\nu,\alpha\beta} = P^{(1)}_{\mu\nu,\alpha\beta} = \frac{1}{2}(\theta_{\mu\alpha}\omega_{\nu\beta} + \theta_{\nu\alpha}\omega_{\mu\beta} + \theta_{\mu\beta}\omega_{\nu\alpha} + \theta_{\nu\beta}\omega_{\mu\alpha}), \]

\[ \hat{P}^{(0-)}_{\mu\nu,\alpha\beta} = P^{(0-)}_{\mu\nu,\alpha\beta} = \frac{1}{n-1} \theta_{\mu\nu}\theta_{\alpha\beta}, \quad \hat{P}^{(0-)}_{\mu\nu,\alpha\beta} = P^{(0-)}_{\mu\nu,\alpha\beta} = \omega_{\mu\nu}\omega_{\alpha\beta}. \tag{69} \]
The background information about perturbative quantum gravity

\[ \hat{P}^{(ws)} = P^{(ws)}_{\mu\nu,\alpha\beta} = \frac{1}{\sqrt{n-1}} \theta_{\mu\nu} \omega_{\alpha\beta}, \quad \hat{P}^{(sw)} = P^{(sw)}_{\mu\nu,\alpha\beta} = \frac{1}{\sqrt{n-1}} \omega_{\mu\nu} \theta_{\alpha\beta}. \tag{70} \]

The algebra for the vector and tensor projectors is simple,

\[ \hat{P}^{(2)} \hat{P}^{(i)} = \hat{P}^{(2)} \delta^i_2 \quad \text{and} \quad \hat{P}^{(1)} \hat{P}^{(i)} = \hat{P}^{(1)} \delta^i_1, \tag{71} \]

where \( i = (2, 1, 0 - w, 0 - s, sw, ws) \). In the scalar sector, one has to construct the matrix projector operator

\[ \hat{P}_0 = \frac{1}{2} \begin{pmatrix} P^{(0-s)} & P^{(sw)} \\ P^{(ws)} & P^{(0-w)} \end{pmatrix}, \tag{72} \]

satisfying the relation \( \hat{P}_0^2 = \hat{P}_0 \). To end this part, the last two terms, which represent the scalar sector of (8), can be written, in momentum representation, as

\[ h_{\mu\nu}^{\text{scalar}} = \frac{1}{4} h \theta_{\mu\nu} + \left( \frac{1}{4} h - k^2 \right) \omega_{\mu\nu}, \tag{73} \]

such that acting by each of the two projectors \( P^{(0-s)} \) and \( P^{(0-w)} \), one of these terms remains invariant and another vanish.

Now we are in a position to find the propagator. Solving Eq. (61) requires the inversion of the expression

\[ \hat{H} = b_2 \hat{P}^{(2)} + b_1 \hat{P}^{(1)} + b_{ow} P^{(0-w)} + b_{sw} \left[ P^{(sw)} + \hat{P}^{(sw)} \right], \tag{74} \]

that means one has to find such an operator

\[ \hat{G} = c_2 \hat{P}^{(2)} + c_1 \hat{P}^{(1)} + c_{ow} P^{(0-w)} + c_{sw} \left[ P^{(sw)} + \hat{P}^{(sw)} \right], \tag{75} \]

that the product with \( \hat{B} \) is unity,

\[ \hat{H} \hat{G} = \hat{1} = \hat{P}^{(2)} + \hat{P}^{(1)} + P^{(0-s)} + \hat{P}^{(0-w)}. \tag{76} \]

Using the aforementioned algebra of projectors, we get the solution to this problem,

\[ c_2 = \frac{1}{b_2}, \quad c_1 = \frac{1}{b_1}, \quad c_{ow} = -\frac{b_{ow}}{\Delta}, \quad c_{sw} = -\frac{b_{sw}}{\Delta}, \quad c_{sw} = \frac{b_{sw}}{\Delta}, \tag{77} \]

where \( \Delta = b_{sw}^2 - b_{ow} b_{sw} \).

It is clear that the action of a generally covariant theory before adding the gauge fixing term has either \( b_1 = 0 \) or \( \Delta = 0 \). From this perspective, the purpose of the Faddeev-Popov procedure is to remove this degeneracy.

It remains to present the projectors (69) and the transfer operators (70) as the linear combinations of the expressions (66), and v.v. The first set of formulas is
\[ \hat{\rho}^{(2)} = \hat{T}_1 - \frac{1}{n-1} \hat{T}_2 + \frac{1}{n-1} \hat{T}_3 = 2\hat{T}_4 + \frac{n-2}{n-1} \hat{T}_5, \quad \hat{\rho}^{(1)} = 2(\hat{T}_4 - \hat{T}_5), \]

\[ \rho^{(0-w)} = \hat{T}_5, \quad \rho^{(0-s)} = \frac{1}{n-1} (\hat{T}_2 - \hat{T}_3 + \hat{T}_5), \]

\[ \rho^{(ws)} + \rho^{(sw)} = \frac{1}{\sqrt{n-1}} (\hat{T}_3 - 2\hat{T}_5). \] (78)

Finally, by inverting these relations, one can express the matrices in (66) as

\[ \hat{T}_1 = \hat{\rho}^{(2)} + \hat{\rho}^{(1)} + \rho^{(0-s)} + \rho^{(0-w)}, \]

\[ \hat{T}_2 = (n-1)\rho^{(0-s)} + \sqrt{n-1}[\rho^{(ws)} + \rho^{(sw)}] + \rho^{(0-w)}, \]

\[ \hat{T}_3 = \sqrt{n-1} \rho^{(ws)} + \sqrt{n-1} \rho^{(sw)} + 2\rho^{(0-w)}, \]

\[ \hat{T}_4 = \frac{1}{2} \hat{\rho}^{(1)} + \rho^{(0-w)}, \quad \hat{T}_5 = \rho^{(0-w)}. \] (79)

Correspondingly, the transformations between the coefficients of (65) and (74) are given by the inverse relations to the ones of “basic vectors”, i.e.,

\[ b_2 = s_1 + s_2 + s_3 + s_4, \quad b_1 = (n-1)s_3 + \sqrt{n-1}s_4, \]

\[ b_{0s} = \sqrt{n-1}s_5 + 2s_4, \quad b_{0w} = \frac{1}{2}s_2 + s_4, \quad b_{sw} = s_4. \] (80)

and

\[ s_1 = b_2 - \frac{1}{n-1} b_1 + \frac{1}{n-1} b_{0s} - 2b_{0w} + \frac{n-2}{n-1} b_{sw}, \quad s_2 = (b_{0w} - b_{sw}), \]

\[ s_3 = \frac{1}{n-1} (b_1 - b_{0s} + b_{sw}), \quad s_4 = b_{sw}, \quad s_5 = \frac{1}{\sqrt{n-1}} (b_{0s} - 2b_{sw}). \] (81)

The solution to Eq. (61) consists of casting the bilinear form in the standard form (66), using the relations between \( \hat{T}_i \) and projectors (79), inverting the result using (77) and, finally, using the inverse relations (78). In principle, this procedure works for the bilinear form of the total action (63) for an arbitrary model of quantum gravity if the chosen gauge-fixing term makes the bilinear form non-degenerate. Usually, the Faddeev-Popov procedure is sufficient in this respect, but if the original theory had an extra symmetry (e.g., the conformal one), one needs to apply an additional gauge fixing, e.g., setting \( h = h_{\mu}^{\mu} = 0 \) [26].

To put the described procedure in practice, one has to use definitions (35), with \( \Box^n \rightarrow \Phi \), and (40), and insert the form factors \( \Phi, \Psi \) and \( \Omega \) into expansions (51), (52), (50), and (48). The bilinear form of the general gauge-fixing term (31) with the weight (43), can be easily calculated,

\[ H_{\mu \nu \sigma \rho}^{(GF)} (k; \eta) = \hat{H}_{GF} = W(-k^2)k^4 \left\{ \beta^2 (\gamma - 1) \hat{T}_2 + \beta (1 - \gamma) \hat{T}_3 - \frac{1}{4} \hat{T}_4 + \gamma \hat{T}_5 \right\}. \] (82)
Summing up all the terms, including the bilinear form of the original action $H_{(0)\mu\nu,\alpha\beta}(k;\eta)$, contribution of (40) and of the gauge-fixing term, we arrive at the expression (65)

$$\hat{H} = \hat{H}_{(0)} + \hat{H}_{GB} + \hat{H}_{GF},$$

with the coefficients

\begin{align*}
s_1 &= \frac{1}{2} \Phi k^4 + \frac{1}{2\kappa^2} k^2 + \frac{\Lambda}{\kappa^2}, \\
s_2 &= \left(\Psi - \frac{1}{6} \Phi\right) k^4 - \frac{1}{2\kappa^2} k^2 - \frac{\Lambda}{2\kappa^2} + \beta^2(\gamma - 1)W k^4, \\
s_3 &= \left(\frac{1}{6} \Phi - \Psi\right) k^4 + \frac{1}{2\kappa^2} k^2 + \beta(1 - \gamma)W k^4, \\
s_4 &= -\Phi k^4 - \frac{1}{\kappa^2} k^2 - W k^4, \\
s_5 &= \left(\frac{1}{3} \Phi + \Psi\right) k^4 + \gamma W k^4.
\end{align*}

The remarkable detail is that there is no $\Omega(-k^2)$ in these expressions. This is certainly an expected result because this function comes from the “generalized” topological invariant (40), but we observe that this feature holds for any $\Omega(-k^2)$, not only for a constant, when this term in the action is really topological.

Another expected characteristic of expressions (84) is that all the coefficients except $s_1$ depend on the gauge fixing parameters. Since (83) has three linearly independent coefficients $\frac{1}{4\pi}$, $\frac{\beta}{2\pi}$, and $\frac{1-\gamma}{2\pi}$, by using the choice of the three parameters $\alpha$, $\beta$ and $\gamma$, one can eliminate terms with the coefficients $a_{3,4,5}$ in the bilinear form (83). As a result, in any QG model, one can provide the minimal form of the total bilinear operator,

$$H_{\mu\nu,\alpha\beta}^{total,\ minimal}(k;\eta) = -\left[a_1(k^2)\delta_{\mu\nu,\alpha\beta} + a_2'(k^2)\eta_{\mu\nu}\eta_{\alpha\beta}\right]k^2,$$

where $a_2'$ differs from $a_2$ in Eq. (64) because of the contribution of the gauge-fixing term. This property of the bilinear form holds also for an arbitrary background metric and is important for the one- or higher-loop calculations in QG.

The main advantage of the minimal form (85) compared to the general one, is that the minimal operators are directly suited for the use of the heat-kernel technique. Indeed, it is possible to work with the nonminimal operators in quantum gravity, e.g., using the generalized Schwinger-DeWitt technique [27], but it is always simpler to work with the minimal bilinear forms.

In some QG models, the choice of parameters is more restricted. E.g., in quantum GR, we meet only two gauge-fixing parameters, i.e., $\alpha$ and $\beta$, but there are only four nonminimal terms because of $a_5 = 0$. As a result, one can choose $\alpha$ and $\beta$ to achieve the minimal form (85). One technical observation is that, if the initial action includes, simultaneously, higher derivative ($\Phi$- and $\Psi$-terms) and the Einstein-Hilbert action, one can provide minimality only in the higher order terms (typically,
in all of them, greatly simplifying calculations in the superrenormalizable models),
but not in the second derivative sector of the operator.

Let us now analyse the situation from another perspective. Expressions (84) remain the same in any dimension $n$. Thus, we can use (80) to transform the operator $\hat{H}$ into the form (74). The result of this transformation is

$$b_2 = \frac{1}{2} \, \Phi k^4 + \frac{1}{2 \kappa^2} k^2 + \frac{\Lambda}{\kappa^2},$$

$$b_1 = -\frac{1}{2} \, W k^4 + \frac{\Lambda}{\kappa^2},$$

$$b_{0s} = -\frac{n-4}{6} \, \Phi k^4 + (n-1) \, [\Psi + \beta^2 (\gamma-1) W] k^4 - \frac{n-2}{2 \kappa^2} k^2 - \frac{(n-3) \Lambda}{2 \kappa^2},$$

$$b_{0v} = (1-\beta)^2 (\gamma-1) W k^4 + \frac{k^2}{\kappa^2} + \frac{\Lambda}{2 \kappa^2},$$

$$b_{sv} = \sqrt{n-1} \left[ \beta (\beta-1) (\gamma-1) k^4 W - \frac{\Lambda}{2 \kappa^2} \right]. \quad (86)$$

Once again, only the spin-2 coefficient is gauge-fixing independent.

The inversion formulas (77) are trivial in the spin-2 and spin-1 sectors. In the tensor sector, we get, for the $\Lambda = 0$ case, independent on the dimension $n$ and on the gauge fixing,

$$c_2(\Lambda = 0) = \frac{2 \kappa^2}{k^2 (1 + \kappa^2 k^2 \Phi)}. \quad (87)$$

The vector part depends on the gauge fixing and has no direct physical interpretation. Thus, we concentrate on the results in the scalar sector. The reader can easily obtain the complete formulas, but since these formulas are cumbersome, we shall present only the qualitative results and the most interesting expression. In the cases of $\Lambda \neq 0$ or $n \neq 4$, all the scalar coefficients are gauge-fixing dependent. However, in case $\Lambda = 0$ and $n = 4$, one important coefficient is invariant$^6$,

$$c_{os}(\Lambda = 0, n = 4) = -\frac{\kappa^2}{k^2 (1 - 3 \kappa^2 \Psi k^2)}. \quad (88)$$

It is easy to see from these formulas, that the propagator (87) of the spin-2 mode depends only on the function $\Phi$, while the spin-0 propagator (88) depends only on the function $\Psi$. Let us note that this output was anticipated already at the level of bilinear expansions of the classical action. Furthermore, both spin-2 and spin-0 propagators are gauge-fixing independent and contribute to the tree-level $S$-matrix of gravitational perturbations. It is interesting that these features, which are looking quite special, hold independent on the form of the form factors $\Phi(\Box)$ and $\Psi(\Box)$ in the action (39). The same concerns the irrelevance of the third form factor $\Omega$ for

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$^6$ I am grateful to Dr. Leslaw Rachwał for indicating to me this feature.
the propagator. Of course, this property is not valid for the vertices, if \( \Omega(x) \) is not a constant function.

A peculiar situation occurs in the usual second-derivative gravity. Even if one starts from the pure GR, the result is the same as if setting \( \Psi \to 0 \) in (88), i.e.,

\[
c_{\alpha\alpha}(GR \text{ with } \Lambda = 0) = -\frac{\kappa^2}{k^2}.
\]  

(89)

This formula is in apparent contradiction with what we saw in the analysis of the gravitational waves in GR, where the unique sort of the propagating degrees of freedom are the tensor (transverse and traceless) modes. The explanation of this apparent discrepancy is that the smooth \( \Psi \to 0 \) limit in (89) corresponds to the theory after the Faddeev-Popov procedure, which extends the space of the propagating modes and makes the whole propagator non-degenerate. On the contrary, the result for the gravitational waves is based on another procedure, i.e., removing all degrees of freedom by using gauge transformation and its remnant (see, e.g., [23]).

From the QG perspective, the smooth limit (89) of the general expression (88) for the scalar sector is important, as it provides universal IR limit of the propagator (for \( n = 4 \) only!) in any QG model (39) at the tree level, i.e., in both relevant sectors of the propagator.

6 Gauge-invariant renormalization in quantum gravity

In the next Chapter of this Section there is a detailed proof of the two main statements concerning the gauge-invariant renormalization in quantum gravity [28] (see also pioneer work [19], [29] and more recent and [30, 31]).

Both theorems were already mentioned in the Introduction. However, since these two statements are relevant for the rest of this Chapter, let us formulate them here in more detail.

1. The renormalization preserves the diffeomorphism invariance (general covariance) of the model of QG in four spacetime dimensions (i.e., in \( n = 4 \)), if the initial classical theory possesses this symmetry. This means, one can remove the divergences in all loop orders by adding covariant counterterms. This statement applies literally only to the divergences that take place in the framework of the background field method, which is especially designed to avoid non-covariant counterterms. Let us stress that the background field method is not necessary for the gauge-invariant renormalization in QG. However, using the non-covariant parametrization of the metric, such as (24), one has to go through a relatively complicated procedure or additional renormalization transformations, as described in [19]. The final output is always the same in the sense of the same essential covariant counterterms. For this reason, in what follows we shall assume (24) when evaluating the power counting in

\footnote{The factor of \( \kappa^2 \) in this formula appears because we used the expansion (9) with a flat background \( g_{\mu \nu} = \eta_{\mu \nu} \). If using the expansion (24), there is no such coefficient.}
different models of QG. However, we shall switch to the more general parametrization with the general background metric (9) when making the practical calculations.

2. The dependence on the choice of the gauge condition (31), (e.g., on the parameters $\beta$, $\gamma$ and the function $W(\Box)$ in the weight operator (43) is proportional to the effective equations of motion. The same concerns the dependence of the parametrization of the quantum metric. In particular, both ambiguities vanish on the classical mass shell for the one-loop divergences of the effective action. In what follows (see more details in [7]) we shall demonstrate how this feature works in the practical calculations in QG.

On top of that, there is the third statement, representing the general feature of QFT and valid, in particular, for QG. The counterterms required to remove UV divergences, in all loop orders, are local functionals of the fields. The mathematically rigid proof of this statement (usually called Weinberg’s theorem [32]) is complicated and can be found, e.g., in [33]. In what follows, we shall apply these three statement to describe the renormalization in QG.

7 Power counting, and classification of quantum gravity models

To estimate the power counting for the Feynman diagrams in QG is somehow simpler than in the quantum theories of other fields. The reason is that the metric is a dimensionless field. For this reason, the dimensions of the counterterm that emerge for diagram $G$, with $L$ loops, is defined only by the number of derivatives of the background metric or, in case of a non-covariant parametrization such as (24), by the number of derivatives of $h_{\mu\nu}$. The power counting of a diagram is essentially equivalent to the count of dimensions and, therefore, it does not depend on the number of external lines of $h_{\mu\nu}$. Let us use the last version of the expansion, but with $\kappa \to 1$ for generality, as this enables us to include the higher derivative theories into consideration.

The superficial degree of divergences will be denoted $\omega(G)$ (sometimes it is also called the index of divergence) and $d(G)$ is the number of partial derivatives of the external lines of the field $h_{\mu\nu}$ in the diagram. Taking into account the powers of momenta in all elements of the diagrams (see, e.g., [7] for a detailed general treatment), the general expression is

$$\omega(G) + d(G) = \sum_{l=1}^{R} (4 - r_l) - 4V + 4 + \sum_{V} K_V,$$

where the first sum is over all $I$ internal lines of the diagram, $r_l$ is the inverse power of momentum in the propagator of an internal line and $V$ is the number of vertices. The last sum is taken over all the vertices, where $K_V$ is the power of momenta, (or number of derivatives, in the coordinate representation) of all the lines coming to the given vertex.
It is easy to see that formula (90) is insufficient to evaluate the renormalizability of the given QFT or QG model. In addition to this formula, there is the simple topological relation

\[ L = I - V + 1 \]  

(91)

valid for all the relevant diagrams.

Before going on to consider concrete models of QG, let us make the following observation. The diagrams in quantum gravity, which we intend to analyze, have external lines of the field \( h_{\mu \nu} \) only, but there are internal lines of both \( h_{\mu \nu} \) and the Faddeev-Popov ghosts. However, with the modified definitions of the ghost actions (33), the values of \( r_I \) are the same for both kinds of quantum fields. E.g., in quantum GR, in both cases we have \( r_I = 2 \), in fourth-derivative gravity in both cases \( r_I = 4 \), in the polynomial models \( r_I = 2N + 4 \). Finally, in the nonlocal QG models (39), both \( r_I \) and \( K_V \) are infinite for both metric and ghosts. Then the use of the combination of (90) and (91) is not possible. However, we shall see how to deal with this special case using the topological relation (91) alone. For a while, we assume that in all models of interest, \( r_I \) are identical for the quantum metric and the ghosts.

### 7.1 Power counting in quantum gravity based on GR

As the first step, consider power counting in quantum GR, where with \( r_I = 2 \) for all internal lines. The vertices coming from the Einstein-Hilbert term have \( K_{EH} = 2 \). If we include the cosmological constant term, there are also vertices \( K_A = 0 \). However, looking only for strongest divergences, at first we consider only the diagrams with \( K_V = 2 \) vertices. Then (90), together with the topological relation (91), yields

\[ \omega(G) + d(G) = 2I - 4V + 4 + 2V = 2I - 2V + 4 = 2 + 2L. \]  

(92)

The last result clearly shows that the QG based on GR is non-renormalizable. At one-loop \( L = 1 \) and the logarithmic divergences with \( \omega(G) = 0 \) have \( d(G) = 4 \). Taking into account the diffeomorphism invariance, this indicates towards the counterterms repeating the covariant structures included in the fourth-derivative action (30). Indeed, at the one-loop order, there are counterterms of the Einstein-Hilbert form \( \sim \int \sqrt{-g} R \) with \( d(G) = 2 \) but with quadratic divergences only, since \( \omega(G) = 2 \).

The logarithmic divergences of this type are also possible, but only if we introduce cosmological constant term. If there is one vertex with \( K_A = 0 \), the diagram produces the logarithmic divergence with two derivatives. Assuming the covariance, this means an Einstein-Hilbert type counterterm. With two such vertices, we meet a logarithmic divergence without derivatives, i.e., with \( d(G) = 0 \). In one of the next sections, we confirm these conclusions by direct calculations and also analyze the gauge-fixing and parametrization dependence of the one-loop counterterms.

One can rewrite the one-loop divergences using the relations
\[ C^2 = E_4 + 2W \quad \text{(where} \quad W = R^2_{\mu \nu} - \frac{1}{3} R^2 \text{),} \quad E_4, \quad R^2, \quad \Box R. \quad (93) \]

We know that \( E_4 \) and \( \Box R \) are surface terms, which do not affect the dynamics of the theory, and the other two terms vanish on the classical equations of motion, when \( R_{\mu \nu} = 0 \). Thus, the one-loop \( S \)-matrix in the pure quantum GR (pure means without matter contents) is finite. In the presence of matter this feature doesn’t hold \footnote{34, 35}, but let us concern only pure QG.

In the two-loop order \( L = 2 \). According to Eq. (92), the logarithmic divergences without \( K_\lambda \) vertices have dimension six. A complete list of the corresponding terms has been elaborated in the works on the conformal anomaly in six spacetime dimensions \footnote{36}. This list includes

\[
\begin{align*}
\Sigma_1 &= R_{\mu \nu} R^{\alpha \beta} R_{\alpha}^{\nu} \\
\Sigma_4 &= R_{\mu \nu} R^{\mu \lambda \alpha \beta} R^{\nu \lambda \alpha \beta} \\
\Sigma_5 &= R^{\mu \nu \alpha} R^{\alpha \beta} \tau R^{\lambda \tau \mu \nu} \\
\Sigma_6 &= R^{\mu \nu \alpha} R^{\alpha \beta} \tau R^{\lambda \mu \nu} \\
\Sigma_7 &= (\nabla_\lambda R_{\mu \nu})^2 \\
\Sigma_8 &= R_{\mu \nu} \Box R^{\mu \nu} \\
\Sigma_{10} &= R \Box R \\
\Sigma_{11} &= (\nabla_\alpha R_{\mu \nu}) \nabla^\mu R^{\nu \alpha} \\
\Sigma_{12} &= R^{\mu \nu} \nabla_\mu \nabla_\nu R \\
\Sigma_{13} &= R_{\mu \nu} R^{\alpha \beta} R^{\mu \nu \alpha \beta} \\
\end{align*}
\] (94)

as well as the set of surface terms,

\[
\begin{align*}
\Xi_1 &= \Box^2 R \\
\Xi_2 &= \Box R_{\mu \nu \alpha \beta} \\
\Xi_3 &= \Box R^2_{\mu \nu} \\
\Xi_4 &= \Box R^2 \\
\Xi_5 &= \nabla_\mu \nabla_\nu (R_{\mu \lambda \alpha \beta} R^{\nu \lambda \alpha \beta}) \\
\Xi_6 &= \nabla_\mu \nabla_\nu (R_{\alpha \beta} R^{\mu \alpha \beta}) \\
\Xi_7 &= \nabla_\mu \nabla_\nu (R_{\alpha} R^{\nu \alpha}) \\
\Xi_8 &= \nabla_\mu \nabla_\nu (R R^{\mu \nu}) \\
\end{align*}
\] (95)

satisfying the identity

\[
\Xi_2 - 4 \Xi_3 + 4 \Xi_4 - 8 \Xi_5 + 8 \Xi_7 - 4 \Xi_8 = 0. \quad (96)
\]

All these structures can show up in the two-loop divergences, but only two of these terms, namely, \( \Sigma_5 \) and \( \Sigma_6 \) are critically important because they do not vanish on shell. The two-loop calculation were done in \footnote{37} and confirmed in \footnote{38} by using another calculational approach. The results confirmed the non-zero coefficient of \( \Sigma_5 \). The conclusion is that there are no miracles and the theory of QG based on GR is non-renormalizable.

Within the standard perturbative approach, the non-renormalizability means the theory has no predictive power. With every new order of the loop expansion, there are new types of local covariant divergences, with the growing number of derivatives of the metric. And every time when a new type of counterterm is introduced, it is necessary to fix the renormalization condition. Each of these conditions requires making a measurement and using its result to fix the value of the corresponding parameter. In quantum GR, this sequence of operations is formally infinite. Thus, before making a single prediction, it is necessary to use an infinite amount of experimental data.

What are the possible ways out of this situation? The main options are as follows.
1. Change standard perturbative approach to something else. The reader can consult other Sections of our Handbook, to see how the problem is solved in the framework of non-perturbative approaches, superstring theory, etc. Let us say that there are many interesting options, but their consistency and the relation to the QG program are not completely clear, in all cases.

2. Restrict the area of application of QG to the low-energy domain. The reader can read about this possibility in the Section about effective QG. The main problem with this approach is that the QG is initially supposed to be a concept describing extreme high-energy Physics, with the typical energy scale of the Planck order of magnitude. It is certainly important to know what remains from the QG effects in the IR, but this does not reduce the importance of formulating QG that would be applicable at high energies.

3. Change the theory, i.e., start from the model different from GR, to construct QG. This option is the mainstream direction in perturbative QG. It is important that the problems we meet in this way, such as the problem of nonphysical ghosts coming from higher derivative terms, persist in many non-traditional approaches which we mentioned in the first point (see, e.g., the discussion in [39]).

7.2 Power counting in fourth-derivative gravity models

The next example is the power counting in the fourth-derivative quantum gravity (30). We assume the Faddeev-Popov procedure with the second-order weight operator (32) and the modified action of ghosts (33). In this case, for all modes of the gravitational perturbation $h_{\mu\nu}$ and ghosts, we have $r_l = 4$, while the vertices $K_V$ include $K_{4d}$, $K_{EH}$, and $K_{A}$.

Let us denote $n_{4d}$ the number of vertices with fourth power of momenta, $n_{EH}$ the one with two, and $n_A$ with zero power of momenta. Then

$$n_{4d} + n_{EH} + n_A = V \quad \text{and} \quad n_{4d}K_{4d} + n_{EH}K_{EH} + n_AK_A = \sum V K_V.$$ \hspace{1cm} (97)

The general expression (90), together with the topological relation (91), give the following result:

$$\omega(G) + d(G) = 4 - 2n_{EH} - 4n_A.$$ \hspace{1cm} (98)

As a starting point, consider the diagrams with the strongest divergences, where all of the vertices are of the $K_{4d}$ type, i.e., $V = n_{4d}$. In this case, (98) means that the logarithmic divergences have $d(G) = 4$. Taking into account locality and covariance arguments, the possible counterterms are of the $C^2$, $R^2$, $E_4$ and $\Box R$ types. This means, in all loop orders the divergences have the same form as the fourth-derivative terms in the classical action (30). Then, for $n_{4d} = V - 1$ and $n_{EH} = 1$, we obtain $d(G) = 2$, corresponding to the counterterm linear in $R$. Finally, for $n_{EH} = 2$ and $V - 2 = n_{4d}$, or for $n_A = 1$ and $V - 1 = n_{4d}$, there is a counterterm with $d(G) = 0$. 

which is the cosmological constant. Thus, the theory under consideration is multiplicatively renormalizable. In the next sections, we shall see that this does not mean that the theory is completely consistent, as there is a massive nonphysical ghost in the spectrum and the subsequent problems with quantum unitarity and even with the stability of classical solutions.

For the particular case of the general model (30) without dimensional parameters, the classical action has a global conformal symmetry under the transformation $g_{\mu \nu} \rightarrow g_{\mu \nu} e^{2\lambda}$, with $\lambda = \text{const}$. The power counting (98) can be perfectly well applied in this case, yielding $\omega(G) + d(G) = 4$. This means that the theory is also multiplicatively renormalizable. The disadvantage of this model of gravity is that there is no automatic Einstein limit in the low-energy domain. Let us remember that such a limit is one of the main conditions of consistency of any model which generalizes or modifies GR, so this situation should be seen as a problem of the model.

We note, by passing, that one can start from such a theory of globally conformal theory, coupled to a massless scalar field. At the quantum level, the loop corrections to the scalar potential may produce such effective potential that the global conformal symmetry is dynamically broken, producing GR in the low-energy limit. This idea resurrected several times in the literature (see, e.g., [40, 41, 42]) in different QG frameworks and, in general, looks attractive and promising. Unfortunately, the real deal is that, in the IR one has to break the global conformal symmetry. And then, in the broken phase, we come back to the massive ghost problems and to the related issue of instabilities, as it will be discussed in the subsequent section 8.

Another particular case, which is instructive to consider, is the $R + R^2$-gravity, which is model (30) without the $C^2$-term. As we saw in the previous sections, in this model the traceless component of the metric $\bar{h}_{\mu \nu}$ has $1/k^2$ propagator, while the scalar mode has a propagator behaving as $1/k^4$ in the UV. Furthermore, there are vertices $K_{dd}$ connecting all these modes. It is easy to check that the power counting in this model is dramatically different from that in the general fourth-derivative model. The theory is non-renormalizable and the power counting is even much worse than in quantum GR.

The next example is the model (30) without the $R^2$ term. This particular model may be interesting since the fourth-derivative part of the action possesses local conformal symmetry. This symmetry is softly broken by the Einstein-Hilbert and cosmological terms. The expression “soft breaking” means that the symmetry does not hold in the terms with dimensional parameters. Can it be that the softly broken conformal symmetry “saves” the power counting in this model? The answer is certainly negative. The propagator of the traceless mode of the metric, $\bar{h}_{\mu \nu}$, in this case, has the UV behavior $1/k^4$, and that of the scalar mode the different UV behavior, $\propto 1/k^2$, due to the presence of the $R$-term. At the same time, there are $K_{dd}$ vertices that link all of the modes, and hence the power counting is qualitatively the same as in the previous $R + R^2$ case. The theory is not renormalizable.

The situation is much more complicated if the original theory has the pure Weyl-squared term in the Lagrangian, i.e., possesses local conformal symmetry. The theoretical proof of the renormalizability in conformal theories exists only for the quan-
tum theory in curved space (semiclassical gravity) [43] and, in the literature, there is no proof for conformal QG. On the other hand, at the one-loop order this theory is renormalizable, as was shown by direct calculations [26, 44] and confirmed in [45]. However, it is expected that this model is not renormalizable at higher loops because of the conformal anomaly. But, in the situation when this expectation is not supported by direct higher-loop calculations or the analysis similar to [43], the question should be regarded as open.

### 7.3 Power counting in the polynomial theory

Consider power counting in the polynomial model (34). As before, we assume that the Faddeev-Popov quantization is done with the weight operator (37) and the correspondingly modified ghost term (33). In the general case, both coefficients of the highest derivative terms $\omega_{N,R}$ and $\omega_{N,C}$ are non-zero. Then the propagators of both metric perturbations $h_{\mu\nu}$ and ghosts have the UV behavior $\propto k^{-4-2N}$, i.e., we have $r_l \equiv 4 + 2N$. For the vertices, the generalization of Eq. (97) is

$$ \sum_V K_V = n_{4+2N}K_{4+2N} + n_{2+2N}K_{2+2N} + n_{2N}K_{2N} + \ldots + n_{EH}K_{EH} + n_AK_A, $$

$$ V = n_{4+2N} + n_{2+2N} + n_{2N} + \ldots + n_{EH} + n_A, $\tag{99}$$

where $K_{4+2N} = 4 + 2N$, $K_{2+2N} = 2 + 2N$, $\ldots$ $K_{EH} = 2$, $K_A = 0$, and $n_{4+2N}$, $n_{2+2N}$, $n_{2N}$, $\ldots$, $n_{EH}$, and $n_A$ are the numbers of the respective vertices.

Consider the diagrams with the strongest divergences. This means $V = n_{4+2N}$, such that the other types of vertices are absent. Then the power counting becomes quite simple, because of $\sum_V K_V = V(4 + 2N)$. The expression (100) becomes

$$ \omega(G) + d(G) = (4 - 4 - 2N)I - 4V + 4 + V(4 + 2N) $$

$$ = 4 + 2N(V - I) = 4 + 2N(1 - L), $$\tag{100}$$

where we used the topological relation (91) in the form $V - I = 1 - L$. It is easy to see that the power counting in the four-derivative model (98) is a particular case, corresponding to $N = 0$. Thus, we assume $N \geq 1$.

According to (100), the sum $\omega(G) + d(G)$ decreases with a growing number of loops $L$. The strongest divergences occur for $L = 1$, when the aforementioned sum equals 4 and the logarithmic divergences correspond to the one-loop counterterms with, at most, four derivatives. Taking the covariance and locality, this means that the one-loop counterterms are of the $C^2$, $R^2$, $E_4$ and $\Box R$ types. In other words, all the terms with six and more derivatives do not need to be renormalized at the one-loop order. But, if there just one vertex with two less derivatives, i.e., $n_{2+2N} = 1$ and $n_{4+2N} = V - 1$, then we meet only the Einstein-Hilbert type divergence. And finally, in the case $n_{2+2N} = 2$ or $n_{2N} = 1$, the unique divergence is that of the cosmological constant.
The features of the $L = 1$ approximation that we listed above, do not depend on $N \geq 1$. Starting from $L \geq 2$, the structure of divergences starts to depend on the value of $N$. In particular, for $N \geq 3$, according to (100), the second- and higher-loop diagrams are all finite. This creates a situation when the one-loop beta functions are the exact ones. In the case of $N = 2$, there are two-loop divergences, but only of the cosmological constant type. Finally, for $N = 1$, there are two-loop divergences of the Einstein-Hilbert and the cosmological constant type and also three-loop divergences, but only of the cosmological constant type.

All in all, a theory with both $\omega_{N,R} \neq 0$ and $\omega_{N,C} \neq 0$, is superrenormalizable. In contrast, in the degenerate case, when only one of these coefficients is zero, the theory is non-renormalizable. The situation is essentially similar to what we discussed above for two similar four-derivative models of QG.

Finally, the power counting in the QG models (39) with non-polynomial (typically exponential) functions $\Phi$ and $\Psi$ cannot be performed on the basis of the formula (90), because both the number of derivatives in the vertices and the parameter $r_l$ are infinite. However, if the condition (41) is satisfied, the evaluation can be done using only the topological relation [46]. Let us assume, for simplicity, that the functions are those from (42). Then, in Euclidean signature, each propagator brings the exponential of $-\alpha k^2$ and each vertex contributes with the exponential of $+\alpha k^2$. This means, if the number of vertices is different from the number of propagators, the last integral in the given diagram will be either strongly divergent, or completely convergent. Taking this into account, from the relation (91) we learn that the divergences are present only in the one-loop diagrams and that these divergences have fourth powers of momenta. Thus, the power counting in the theories of this class is the same as in the polynomial models with $N \geq 3$.

8 Massive ghosts in higher-derivative models

Previously we saw that quantum GR, based on the Einstein-Hilbert action, is a non-renormalizable theory. On the other hand, by adding the general fourth-derivative terms, we arrive at the model providing multiplicative renormalizability. Another strong argument for including fourth-derivative terms is that they are required for the renormalizability of the semiclassical theory, when gravity is an external field [2, 3, 5, 7]. And this is something one cannot disregard. The point is that the concepts of QFT from one side and of the curved space from another, are well-established and, to a great extent, verified by experiments and observations. Thus, if QFT in curved space produces the four-derivative terms in the action, we have to admit that these terms are there. The problem concerns not exactly UV divergences and formal renormalizability. Together with the logarithmic divergences there are always logarithmic nonlocal form factors. In the IR such a form factor behaves, effectively, as a constant [55, 7]. This means, if we do not include fourth derivatives into the action, they will come there anyway as legitimate corrections coming from quantum matter fields. Therefore, independent on which approach to QG we choose,
it makes sense to include fourth derivatives terms into the gravitational action and check out the physical consequences of this inclusion.

At the classical level and in the low-energy domain, the fourth derivative terms look irrelevant because the coefficients of these terms in the action (30) are just a numbers while the coefficient of the $R$-term is $1/G$, that is $M_P^2 \approx 10^{38}$GeV. As a consequence, until the metric derivatives, in the momentum representation, do not have Planck-order frequencies, the fourth-derivative terms cannot compete with the Einstein-Hilbert term. This situation is usually called a Planck suppression. So, from the first sight the deal is perfect: including the fourth derivatives terms we get a renormalizable QG (and semiclassical too!), while all classical solutions remain the same as in GR and one can enjoy the well-verified gravitational theory.

Unfortunately, even if theory (30) is, formally, multiplicatively renormalizable, it does not make it consistent at either the quantum or even classical levels. As we shall see in brief, the spectrum of this model includes states that have negative kinetic energy. These states, or particles, are called massive nonphysical ghosts. The presence of ghosts violates the unitarity of the theory at the quantum level. Worse than that, in the presence of massive ghosts or, more generally, ghost-like states, classical solutions of the theory can be unstable with respect to the metric perturbations. Qualitatively, the same situation takes place not only in the fourth-derivative theory but in all polynomial models of quantum gravity.

The problem of ghosts is certainly the main obstacle for building a consistent QG theory. For this reason, we consider this problem here. We shall closely follow [7]. The interested reader can address this book for more details, or go directly to the original works, such as the reviews [47] on the Ostrogradsky instabilities, or the papers [48] and [49, 50, 51] for the approach we follow to explore and interpret the instabilities caused by massive ghosts.

### 8.1 What means a massive ghost

Consider the propagator of the transverse and traceless part of the metric (87) or the one of the scalar, gauge-invariant mode (88). For simplicity, we can set the cosmological constant to be zero, then the fourth-derivative action (30) becomes

$$ S = - \int d^4x \sqrt{-g} \left\{ \frac{1}{\lambda} R_{\mu\nu}^2 + \frac{\omega - 1}{3\lambda} R^2 + \frac{1}{\kappa^2} R \right\}. \tag{101} $$

Formulas (87) and (88) change accordingly, with $\Phi$ and $\Psi$ becoming constants.

We can consider at once both scalar and tensor modes, because the formulas are similar. For the definiteness sake, consider the tensor mode $h = \hat{h}_{\mu\nu}^{\perp\perp}$, which is not affected by the $R^2$ term. Using the expansions (50) and (48), the action of this mode becomes

$$ S^{(2)}_{\text{tensor}} = \int d^4x \left\{ - \frac{1}{4\lambda} (\Box h)^2 - \frac{1}{4\kappa^2} h \Box h \right\} = - \frac{1}{4\lambda} \int d^4x h (\Box + m_2^2) \Box h. \tag{102} $$
where $m_2^2 = \lambda / \kappa^2$ is the mass of the mode that is called a tensor ghost, a massive tensor ghost or higher-derivative ghost. The reason for this exotic name is that the Euclidean propagator of the spin-2 mode in this theory can be cast in the form

$$G_2(k) \propto \frac{1}{m_2^2} \left( \frac{1}{k^2} - \frac{1}{k^2 + m_2^2} \right) \hat{p}^{(2)}.$$  

(103)

The negative sign of the second term indicates that the corresponding mode is not a usual particle. In fact, we have not one but two degrees of freedom of the tensor field. One of these degrees of freedom has positive kinetic energy and zero mass, and it corresponds to the first term in Eq. (103). The second degree of freedom has the mass $m_2$ and corresponds to the second term in (103). As we shall see in what follows, its kinetic energy is negative, and, for this reason, it is called a ghost.

The separation of the two degrees of freedom can be most simply explored by using an auxiliary field $\Phi$ (see, e.g., [52], a more detailed discussion in [53] and more general formulations in [54]). Consider the Lagrangian density

$$\mathcal{L}' = -\frac{m_2^2}{4\lambda} h \Box h + \lambda \phi^2 - \phi \Box h.$$  

(104)

The Lagrange equation for $\phi$ can be solved as $\phi = \frac{1}{\lambda} h \Box h$. Substituting this expression back into (104), we arrive at Eq. (102), which shows the dynamical equivalence of the models (102) and (104).

The two fields $\Psi$ and $h$ in (104) are not factorized. To improve on this issue, we change to the new variables $\theta$ and $\psi$,

$$h = \frac{\sqrt{2\lambda}}{m_2} (a_1 \theta + a_2 \psi), \quad \phi = \frac{\sqrt{2\lambda}}{m_2} a_3 \psi.$$  

(105)

where the unknown coefficients $a_{1,2,3}$ should provide the separation of the modes and also the standard coefficients $\frac{1}{2}$ or $-\frac{1}{2}$ in the kinetic terms. A small algebra shows that the condition $a_2 + a_3 = 0$ is necessary to separate the variables and, also, the condition $a_1 = a_2 = 1$ is required to provide standard normalization of the kinetic terms. Then, in the new variables, the Lagrangian (104) becomes

$$\mathcal{L}' = \frac{1}{2} \eta^{\mu\nu} \partial_\mu \theta \partial_\nu \theta - \frac{1}{2} \left( \eta^{\mu\nu} \partial_\mu \psi \partial_\nu \psi - m_2^2 \psi^2 \right).$$  

(106)

We can conclude that the theory (30) has healthy tensor massless degrees of freedom $\theta$ and, on top of this, tensor massive degrees of freedom $\psi$ with negative kinetic energy, called nonphysical massive ghost.
8.2 Classification of ghosts and tachyons

Consider a basic classification of ghosts and tachyons following [55, 7]. The general action of a free second-order field \( h(x) = h(t, r) \) can be written as

\[
S(h) = \frac{s_1}{2} \int d^4x \left\{ \eta^{\mu\nu} \partial_\mu h \partial_\nu h - s_2 m^2 h^2 \right\} = \frac{s_1}{2} \int d^4x \left\{ \dot{h}^2 - (\nabla h)^2 - s_2 m^2 h^2 \right\}.
\]  

(107)

Here \( s_1 \) and \( s_2 \) are sign factors \( \pm 1 \) for different types of fields. In what follows, we consider all four combinations of these signs.

It is useful to perform the Fourier transform in the space variables,

\[
h(t, r) = \frac{1}{(2\pi)^3} \int d^3k e^{ik \cdot r} h(t, k).
\]  

(108)

In a free theory, one can consider the dynamics of each component \( h \equiv h(t, k) \) separately. Such a dynamics is defined by the action

\[
S_k(h) = \frac{s_1}{2} \int dt \left\{ \dot{h}^2 - k^2 h^2 - s_2 m^2 h^2 \right\} = \frac{s_1}{2} \int dt \left\{ \dot{h}^2 - m_k^2 h^2 \right\},
\]  

(109)

where \( k^2 = k \cdot k \) and \( m_k^2 = s_2 m^2 + k^2 \).

(110)

The properties of the field are defined by the signs of \( s_1 \) and \( s_2 \). The possible options can be classified as follows:

i) **Normal healthy field** corresponds to \( s_1 = s_2 = 1 \). The kinetic energy of the field is positive and the equation of motion has the oscillatory form,

\[
\ddot{h} + m_k^2 h = 0,
\]  

(111)

with the usual periodic solutions.

ii) **A tachyon** has \( s_1 = 1 \) and \( s_2 = -1 \). The classical dynamics of tachyons is described in the literature, e.g., in [56, 57], but, for our purposes, it is sufficient to give only a basic survey. For a relatively small momentum \( k = |k| \), there is \( m_k^2 < 0 \) in Eq. (110), and the equation of motion is

\[
\ddot{h} - \omega^2 h = 0, \quad \omega^2 = |m_k^2|,
\]  

(112)

with exponential solutions

\[
h = h_1 e^{\omega t} + h_2 e^{-\omega t}.
\]  

(113)

However, if such a particle moves faster than light, the solution is of the normal oscillatory kind, indicating that such a motion is “natural” for this kind of particle.

iii) **A massive ghost** has \( s_1 = -1 \) and \( s_2 = 1 \). It is not a tachyon, because \( m_k^2 \geq 0 \). In this case, the kinetic energy of the field is negative, but the Lagrange equation
(leaving aside its derivation from the least action principle) has a normal oscillatory equation (111).

iv) A tachyonic ghost has $s_1 = s_2 = -1$. For relatively small $k^2$, one meets $m_k^2 < 0$. The kinetic energy is negative and the mass is imaginary. So, along with the problems typical for the ghosts, the free wave solutions are exponential, as in (113). One can find a discussion of the implication of tachyonic ghosts in [55].

**8.3 Massive ghosts in the fourth-order model**

Let us come back to the fourth-order gravity model (101). Consider the free tensor modes in the theory (102) and change to the momentum representation,

$$\Box h = \dddot{h} - \Delta h \rightarrow \dddot{h} + k^2 h.$$  \hspace{1cm} (114)

Here $k$ is the wave vector of an individual mode. It is important that, owing to the presence of both massless and massive modes, the standard massless dispersion relation between the frequency and the wave vector does not hold in this case. The Lagrange function of the wave with fixed $k$ can be obtained from (102):

$$L = -\frac{1}{4\lambda} (\dddot{h} + k^2 h)^2 - \frac{1}{4\kappa^2} h(\dddot{h} + k^2 h).$$  \hspace{1cm} (115)

The Lagrange equation for $L = L(q, \dot{q}, \ddot{q})$ has the form

$$\frac{\partial L}{\partial \dot{q}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{q}} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \dddot{q}} = 0$$  \hspace{1cm} (116)

and the energy can be easily obtained in the form

$$E = \dot{q} \left( \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{q}} \right) + \ddot{q} \frac{\partial L}{\partial \ddot{q}} - L.$$  \hspace{1cm} (117)

In our case (115), this formula gives the energy of the individual wave with momentum $k$,

$$E = \frac{1}{4\lambda} \left( 2h^{(m)} \dddot{h} - \dddot{h}^2 \right) + \left( \frac{1}{4\kappa^2} + \frac{k^2}{2\lambda} \right) \dddot{h}^2 + \left( \frac{k^2}{4\kappa^2} + \frac{k^4}{4\lambda} \right) h^2.$$  \hspace{1cm} (118)

This formula provides some information about the fourth-derivative theory. We can separate it into the following points:

i) In the limit $\lambda \rightarrow \infty$, the remaining expression for the energy is positively defined, as it should be for Einstein’s gravity.

ii) The fourth time derivative terms are given by the first summand in (118). It is easy to see that this term is not positively defined. This sign indefiniteness should be expected, as a direct consequence of the presence of the massive nonphysical ghost.
iii) In the model under discussion, the low-energy limit (IR) means

\[ \dot{\hbar}^2 \ll k^2 \hbar^2 \quad \text{and} \quad |\hbar h^{(\text{in})}| \ll k^2 \hbar^2. \]  

(119)

In this case, the first indefinite term (with fourth derivatives) in (118) is small, and the sign of the energy is defined by the second term, providing a relevant constraint on the action (30). The positivity of the theory in this limit does not depend on fourth time derivatives. However, the kinetic energy can be still unbounded from below for the negative coupling \( \lambda < 0 \). Owing to the violated dispersion relation between the wave vector \( k \) and the time derivatives, it is possible to have a large \( k^2 \) with the conditions (119) satisfied. Thus, the sign of the coupling \( \lambda \) in the action (30) should be positive, as it was always assumed in the literature, e.g., in the classical works \([19, 20]\) and \([26]\).

The equation for tensor perturbations can be derived from (116),

\[ h^{(iv)} + 2k^2 \ddot{h} + k^4 h + \frac{\lambda}{16\pi \kappa^2} (\dot{h} + k^2 h) = 0. \]  

(120)

One can introduce the new notation,

\[ \frac{\lambda}{16\pi \kappa^2} = s^2 m^2, \]  

(121)

where \( s^2 = \text{sign} \lambda \) and \( m^2 > 0 \). Then Eq. (120) becomes

\[ \left( \frac{\partial^2}{\partial t^2} + k^2 \right) \left( \frac{\partial^2}{\partial t^2} + m^2 \right) h = 0, \]  

(122)

where \( m^2 \) was defined in (110). The solutions of the last equation can be different, depending on the sign of \( \lambda \) and, hence, that of \( s^2 \). The general formula for the frequencies in \( h \sim \exp \{ \pm \omega t \} \) has the form

\[ \omega_{1,2} \approx \pm i (k^2)^{1/2} \quad \text{and} \quad \omega_{3,4} \approx \pm (\mp m^2)^{-1/2}. \]  

(123)

For a positive \( \lambda \), there are only imaginary \( \omega^2 \)'s and, hence, oscillator-type solutions. In contrast, for \( \lambda < 0 \), we have \( s^2 = -1 \) and the roots \( \omega_{3,4} \) are real, since, in this case, \(- m^2 > 0 \) for sufficiently small \( k^2 \). Indeed, the first couple of roots corresponds to the massless graviton, and the second couple to the massive particle. According to the classification presented above, this particle is a ghost for \( \lambda > 0 \) and it is a tachyonic ghost for \( \lambda < 0 \).

Finally, we conclude that the model (30) has ghosts (and maybe tachyonic ghosts) owing to the presence of fourth derivatives.

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8 The more general equation describing the dynamics of tensor perturbations on the cosmological background, will be discussed below; see Eq. (128).
8.4 Massive ghosts in the six and higher-order models

If the number of derivatives in the polynomial model of six or greater, the structure of the propagator is more complicated than in the fourth-order theory. First of all, the massive poles can be either real or complex. In the last case, they emerge in the complex conjugate pairs. There is an important theorem about the structure of the propagator in the case of real poles \[22\]. In this case, instead of \[103\], we meet

\[
G_2(k) = \frac{A_0}{k^2} + \frac{A_1}{k^2 + m_1^2} + \frac{A_2}{k^2 + m_2^2} + \ldots + \frac{A_{N+1}}{k^2 + m_{N+1}^2},
\]

(124)

where the squares of the masses \(m_j^2\) are real. Assuming that there is the following hierarchy of the masses:

\[
0 < m_1^2 < m_2^2 < m_3^2 < \ldots < m_{N+1}^2,
\]

(125)

one can prove that the signs of the coefficients \(A_j\) alternate, i.e., \(\text{sign } [A_j] = - \text{sign } [A_{j+1}]\). This feature means one cannot choose the theory in such a way that all degrees of freedom instead of the heaviest one are normal particles and the mass of the ghost is infinitely large.

The complex poles are also possible, but their detailed discussion lies beyond the scope of this review. Interesting hints about the role of real and complex poles come from the analysis of the Newtonian limit and the bending of light in the polynomial models \[58, 59, 39\].

8.5 On the quantum consistency and stability of classical solutions

We have seen that the theory with fourth derivatives, typically, has a massive ghost or a tachyonic ghost, and this conclusion can be extended to the polynomial models with more than four derivatives. So, it is interesting to understand what the ghost means, from the physical viewpoint. Let us give just a brief qualitative description of the situation.

A particle with negative kinetic energy tends to the minimum of the action and therefore tends to achieve a maximal speed. If such a particle is free, it cannot accelerate, as this would violate energy conservation. Hence, a free ghost does not produce any harm to the environment, being isolated from it. However, in the case when there is an interaction of a ghost with healthy fields, the argument about energy conservation in a closed system does not work. Since any physical system tends to the state with minimal action, a ghost tends to accelerate, transmitting the extra positive energy to the healthy fields interacting with it, in the form of the quantum or classical emission of the corresponding particles. A systematic study of this situation at the quantum level has been given by Veltman in \[60\].
Since gravitons are massless and the metric-dependent gravitational theories have non-polynomial interactions, a massive ghost always couples to an infinite amount of gravitons. This fact may lead to dramatic consequences. The energy conservation does not forbid a spontaneous creation of a massive ghost from the vacuum, even in the flat Minkowski space. It is clear that such a spontaneous creation of a ghost also implies that the corresponding amount of positive energy should be released with the creation of massless gravitons. As the mass of the ghost has the Planck magnitude, these gravitons have to accumulate with the Planck energy density.

Assuming the existence of even a single real ghost, such a particle should accelerate, emitting and scattering gravitons. The magnitude of the energy of the ghost would increase, and hence the energy of the created and scattered gravitons would increase too, without an upper bound for the emitted gravitational energy. After a while, the ghost would acquire an infinite amount of negative energy and start to emit an infinite amount of positive energy. It is clear that if some objects of this sort would be around, we would certainly know about it or, rather, we would feel it.

Thus, the main theoretical problem is to explain why this dramatic scenario does not work. We have to say that, at the moment, there is no solution to this important problem. The solution could be, e.g., an explanation of why gravitons cannot agglomerate with the Planck density, but no mechanism for this has been formulated. Obviously, we have to assume that some kind of solution exists. Probably, it is related to the large mass of the massive ghost in higher-derivative gravity.

The simplest way to preserve the unitarity is to admit the existence of a ghost. But, as we described above, this leads to the physically inconsistent output. Thus, one avoids ghosts by forming the in states only with gravitons. Owing to the interactions, ghost wakes up from the vacuum and emerge in the out states – that means the scattering matrix is not unitary and we arrive at the contradiction.

Historically, the main efforts in solving the problem of ghosts was related to the quantum aspects. In this respect we can start the list from the mentioned paper by Veltman, regardless it has nothing specific about quantum gravity. Soon after the seminal work of Stelle with the proof of renormalizability of the fourth-derivative QG, there were first works about solving the problem of ghosts. The main common idea of these works was that the loop corrections to the propagator transform the unique massive pole into a pair of the complex conjugate poles, with the positions of these poles being gauge fixing dependent. In this case, one can prove the unitarity of the S-matrix, violated by the presence of massive ghosts. Unfortunately, it was shown, that definite conclusion on this issue can be taken only on the basis of exact knowledge of the dressed propagator of $h_{\mu\nu}$. E.g., the one-loop corrections or the $1/N$ approximation are insufficient for solving the problem. It is interesting that starting from the six-derivative theory, one can provide the desirable features (pair of complex conjugate poles, for instance) already at the tree-level and, also, prove that the loop corrections do not change this structure. In this situation, one can use the optical theorem and prove the unitarity of the S-matrix in the Lee-Wick approach.

The problem with this solution of the ghost problem is that the Lee-Wick approach assumes that the scattering occurs between the asymptotical states, where
and *out* states describe the free particles. However, in gravity (and especially in its quantum version) the notion of free massive particle is not perfectly well defined, because any such particle produces a gravitational field, starts to interact with it and, therefore, is not completely free. Therefore, the definite resolution of the ghost problem by means of the $S$-matrix does not look really promising, at least at the fundamental level.

What we can learn from all the quantum considerations is that the issues with ghosts described above would become impossible in the presence of a natural cut-off on the energy density of gravitons, such that this density never achieves the Planck order of magnitude. Then the gravitons cannot agglomerate to create a ghost and the $S$-matrix remains unitary. The problem is that there is no theoretical mechanism for such a cut-off. One can say that this is the main reason of why the problem of ghosts does not have a solution.

### 8.6 Stable solutions in the presence of massive ghosts

Another aspect of the problem of ghosts is related to the stability of classical solutions. The most important issue related to massive ghosts is whether their presence can be compatible with the stability of the classical solutions of GR. As we mentioned above, in higher derivative models of gravity one typically meets the Planck suppression. As a result, classical solutions of GR represent high-quality approximations to the corresponding solutions in the presence of the higher-derivative terms. This logic can be successfully applied to the fourth-derivative theory and extended to the polynomial theories ([34](#)), if we assume that all massive parameters are of the Planck order of magnitude [39].

The excellence in the approximation of the solution of GR does not necessarily mean the stability of this solution. In general, providing the stability of a gravitational solution under arbitrary small perturbations (that do not have the symmetry of solutions themselves) may be not simple even in GR, for some gravitational backgrounds. Let us mention, for instance, the study of the stability of the Schwarzschild solution in GR [66, 67] (see also [68]). In the presence of $C^2$ term, one can expect that the same solution will not be stable. Regardless of existing contradictions in the literature, in general, this expectation is confirmed [69, 70]. Owing to the high level of technical difficulty, this case will not be discussed here.

From the technical side, it is much simpler to consider the stability of classical cosmological solutions in the presence of fourth-order terms. The advantage of this simpler case is that the physical interpretation of the results is relatively explicit. The analysis of [48] and after that in [49, 50, 51], was done for the fourth-derivative action ([30](#)) with anomaly-induced semiclassical corrections (see, e.g., [7](#) for the review). It turns out that these corrections do not change the main result. We shall explain this result omitting all technical details except the basic formulas.

The stability we need to explore is related to the presence of massive spin-2 ghost degrees of freedom, which means the transverse and traceless modes of the
metric perturbation on the homogeneous and isotropic, cosmological background. According to the theory of cosmological perturbations (see, e.g., [71, 72, 73]), the background cosmological metric with tensor perturbations is

\[ ds^2 = a^2(\eta)[d\eta^2 - (\gamma_{ij} + h_{ij})dx^idx^j], \]  

(126)

where \( \eta \) is the conformal time, \( a(\eta) \) corresponds to a background cosmological solution, and we imposed the synchronous coordinate condition \( \gamma_{ij} \equiv \eta_{ij} \) and set the cosmological constant to vanish, \( \Lambda = 0 \).

Since we are interested in the gravitational wave dynamics, it is sufficient to retain only the traceless and transverse parts of \( h_{ij} \), which are the purely tensor modes, by imposing

\[ \partial_j h^{ij} = 0, \quad h_{kk} = 0. \]  

(127)

As before, we do not need to write indices and set \( h = h^{ij} \).

The Lagrange equation for \( h(t) \), in terms of physical time, has the form [74]

\[
\frac{1}{3} h^{(iv)} + 2H h^{(iii)} + \left( H^2 - \frac{\lambda M_p^2}{16\pi} \right) h + \frac{2}{3} \left( \frac{1}{4} \frac{\nabla^4 h}{a^4} - H \frac{\nabla^2 h}{a^2} - \frac{\nabla^2 \dot{h}}{a^2} \right) \\
- \left( \dot{H} + 6H^3 + \frac{3\lambda M_p^2 H}{16\pi} \right) h + \left[ \frac{\lambda M_p^2}{16\pi} + \frac{4}{3} (H + 2H^2) \right] \frac{\nabla^2 h}{a^2} \\
- \left[ 24\dot{H}H^2 + 12H^2 + 16\dot{H} \right] \frac{1}{3} H^{(iii)} + \frac{\lambda M_p^2}{8\pi} (2H + 3H^2) \right] h = 0. \]  

(128)

The contribution of the fourth-derivative terms depends on the unique parameter \( \lambda \) from the action (30). The reason is that the Gauss-Bonnet combinations do not affect the equations of motion and another invariant is \( R^2 \), which contributes to the equation for the conformal factor \( a(t) \), but does not affect the propagation of the tensor mode.

The next step is to make the Fourier transformation for the space coordinates

\[ h_{\mu\nu}(r, t) = \int \frac{d^3k}{(2\pi)^3} h_{\mu\nu}(k, t) e^{ikr}. \]  

(129)

One can treat the wave vector \( k \) as a constant and hence will be interested only in the time evolution of the perturbation \( h_{\mu\nu}(k, t) \). The validity of such a treatment is restricted to the linear perturbations, but this is what we need now. In this way, the complicated partial differential equation (128) are reduced to the much simpler ordinary differential equation for each of the individual modes.

Using the notation \( h = h(t, k) = h(t, k) \), the equation has the form
\[ h^{(iv)} + 6Hh^{(iii)} + \left( 3H^2 - \frac{3\lambda M_p^2}{16\pi} \right) \ddot{h} + \left( \frac{1}{2} \frac{k^4}{a^4} + \frac{2k^2}{a^2} \frac{\ddot{h}}{a^2} + \frac{2k^2}{a^2} H \ddot{h} \right) \]

\[ - 3 \left( H \dot{H} + \dot{H} + 6H^3 + \frac{3\lambda M_p^2 H}{16\pi} \right) \dot{h} - \left[ \frac{3\lambda M_p^2}{16\pi} + 4 \left( \dot{H} + 2H^2 \right) \right] \frac{k^2}{a^2} \dot{h} \]

\[ - 72HH^2 + 36H^2 + 48H \dot{H} + 8H^{(iii)} + \frac{3\lambda M_p^2}{8\pi} \left( 2H + 3H^2 \right) \right] h = 0, \quad (130) \]

where \( k = |k| \) is the frequency of the massless field. Finally, the initial conditions for the perturbations will be chosen according to the quantum fluctuations of free fields. The spectrum is identical to that of a scalar quantum field in Minkowski space \[ \eta \]

\[ h(x, \eta) = h(\eta) e^{\pm ikx}, \quad h(\eta) \propto e^{\pm i\kappa \eta} \sqrt{2\kappa}, \quad (131) \]

As before, \( \eta \) is conformal time. The transition to the physical time is \( a(\eta)d\eta = dt \).

The possible instabilities can be explored using Eq. (130). According to the known mathematical theorems about the stability of the fixed points of differential equations, linear stability guarantees non-linear (at least perturbative) stability for sufficiently small perturbations. In the present case, this point was confirmed in [50, 51] for the Bianchi-I metric, that is a reduced form of the perturbation for \( k = 0 \).

The main qualitative result of the numerical analysis of Eq. (130) performed in [48, 49] is that the linear stability in the fourth-derivative model is possible, but the two conditions should be fulfilled. First of all, the frequency \( k \) should be essentially smaller than the Planck mass. The threshold value for \( k \) slightly depends on the type of the cosmological solution (dominated by radiation, dust or cosmological constant), but, in all cases for \( k < 0.1 M_P \), there is no growth of \( h(t, k) \), while such a growth is evident starting from \( k \approx 0.6 M_P \). This requirement exactly corresponds to our expectation that one needs a Planck density of gravitons to wake up the ghost from the vacuum. For a small frequency \( k \), the ghost remains as a virtual mode and cannot be created from vacuum to become a real particle. Let us remember that the creation of a ghost from the vacuum requires positive-energy gravitons with the Planck energy density (Planck energy in the space volume of a cube of the Planck-scale Compton wavelength). If the frequency of the gravitational wave is insufficient, the ghost is not created, and there is no instability.

Second, the signs of both parameters \( \lambda \) and \( \kappa^2 \) should be positive. For a negative sign of the calibrated Newton constant \( \kappa^2 \), the ghost becomes also a tachyon and there is no stability for any frequency. The negative sign of \( \lambda \) means the graviton becomes a ghost and the massive particle is normal. Then there is no threshold for creating a ghost and we observe instabilities at all frequencies. And this is exactly what was observed in the numerical analysis in [48]. It worth noting that, in the previous subsection, we saw that \( \lambda > 0 \) is a condition of stability in the flat spacetime.
Now we can see that this is confirmed by the analysis of stability on the cosmological background. Thus, in what follows we assume that $\lambda$ and $\kappa^2$ are both positive.

As we mentioned already a few times, the solution of the ghost problem (and, consequently, of the QG problem in general) requires an explanation of why gravitons cannot accumulate with the Planck energy density. Intuitively, it is easier to accept that such accumulation may occur only when the background metric describes an intensive gravitational field, as in the early Universe.

In this respect, an interesting thing happens when the frequency $k$ is greater than the energy threshold but the external cosmological background is described by a strong gravity. The last means that the Hubble parameter has a large value and the Universe is rapidly expanding. It turns out that there is a very fast growth of $h(t,k)$ but such an explosion of the perturbations does not last for a long period of time. To understand why this happens, one can take a look at the main equation, (130). The frequency $k$ enters this equation in the combination $q = k/a(t)$. For a sufficiently fast expansion of the Universe, the explosive growth of the perturbations lasts only until the magnitude of $q$ becomes smaller than the energy threshold. After that, the amplitude of the perturbations vanish exponentially. Thus, the perturbations do not violate the cosmological principle, i.e., the Universe remains homogeneous and isotropic at the large scale and the effect of ghosts does not contradict the observations.

To end this subsection, let us stress that the result of [48, 49] and [50, 51] cannot be interpreted as a solution of the problem of massive ghosts. In our opinion, it should be regarded as a hint to the direction where such a solution can be found.

### 8.7 Effective approach to the problem of ghosts

The effective approach to QG is a subject of a special Section of our Handbook, so it makes no sense to go into details of this approach here. Let us just briefly explain what is conventionally understood as an effective solution to the problem of massive ghosts, as introduced by Simon in [76] and elaborated further in [77].

The proposal in [76] is to consider the Einstein equations as the basic gravitational theory, regarding all higher-derivative terms in the gravitational action and the respective dynamical equations as a small perturbation. According to this treatment, the gravitational theory should be described by the two physical degrees of freedom of GR by definition – independent of what the action of the theory is and the form of the quantum corrections to this action. The propagator of the quantum metric $h_{\mu\nu}$ is derived from Einstein’s gravity, and no corrections can produce additional poles in this propagator. By construction, there cannot be any kind of massive ghosts, and hence there are no problems with unitarity and instabilities at the classical or quantum levels.

This solution certainly looks mathematically correct and efficient. On the other hand, there are serious problems with its consistency, especially at the high energy

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9 One can find the discussions of other implications of this unknown physical principle in [75].
scale, where the fourth-derivative terms gain the magnitude comparable to the GR action. On top of that, there is a problem with the uniqueness of the procedure. E.g., one can modify this effective approach to include an $R^2$ term, or any $f(R)$ term, into the main part of the action, because these terms do not produce a ghost. The same concerns many other terms, e.g., all $\mathcal{O}(R^3)$’s. At the same time, it should be strictly forbidden to do the same with the $R \Box R$-term, which produces a scalar ghost. As it was discussed in [39] and also in [7], the analogy with QED and the standard resolution of the problem of “run-away solutions” is not convincing. So, all the scheme looks as an ad hoc procedure, without the physical background. It is as saying that we do not like ghosts and will therefore forbid them. If we follow the same approach in other branches of QFT, it is perfectly well possible to modify any theory in a way we like and provide the predictions we like. Despite this may look a universal solution of all the problems, the theories created in this way would not be reliable or, better say, would not provide reliable predictions.

However, if we restrict the area of application of QG to the energies essentially lower that the Planck threshold, the described effective approach becomes a normal feature of the theory that can be fixed only by the observations and/or experiments. In this case, the approach of [76, 77] becomes equivalent to the one described in the previous subsection. The results on the stability of the cosmological background of [48] show that, in the IR, one can trade using GR as a basic theory “by definition” to the restrictions on the initial seeds of the tensor mode of cosmological perturbations.

To conclude this section, we have to say that the problem of ghosts is unsolved, at least if we do not restrict the applicability of QG to the low-energy (IR) domain. However, there are certain clues about the directions in which the solution may be found. It seems that we need a new physical principle forbidding the concentration of gravitons with the Planck densities. This means, at the Planck frequencies gravity should dramatically change. Such a change may be because of the nonlocalities in the action, but the problem may require a more complicated solution. At the UV, ghosts may be generated from the vacuum. In the preprint [78], Hawking made a hypothesis, that in this situation, the QFT approach should be modified, taking into account that the ghost is not an independent particle, but appears paired with the graviton. Thinking along this line, one can expect to find a solution by working with bound states or condensates, including ghosts and maybe some normal degrees of freedom, e.g., in the framework of the superrenormalizable QG.

9 Gauge-fixing dependence using general formalism

Starting from this point, in this and the next sections, we discuss the loop corrections in the models of QG. This is an extensive subject and it is traditionally one of the most worked out parts of QG. Obviously, all of it cannot be settled into a short review, so we shall discuss only two particular aspects. Namely, we perform an important general analysis of the gauge- and parametrization dependence of the loop corrections in QG, and show the derivation of loop corrections in the simplest case.
of quantum GR in the simplest gauge and parametrization of quantum metric field, i.e., repeat the main part of the paper by 't Hooft and Veltman [34]. Another Chapter of this Section is devoted to the divergences in the fourth-derivative QG model.

In this section, we show how the general statement about the on-shell gauge-fixing and parametrization independence of the effective action can be used in different models of QG. The practical applications described below, were introduced in [26] and later on, formulated in a more explicit form in [79] and [80].

9.1 Gauge-fixing dependence in quantum GR

Consider the gauge-fixing and parametrization dependence in QG based on GR. We consider the non-zero cosmological constant for generality and also because an interesting application to the on-shell renormalization group [26].

Let us start from some historic note and references. The subject was pioneered in the paper [81], where the calculations for the two-parameters gauge were performed using Feynman diagrams. Even more general diagrams-based calculation in the non-minimal gauge were done in [82]. The use of the heat-kernel methods required the generalized Schwinger-DeWitt technique [27], the results were applied to quantum GR in [83]. On the other hand, in [84] it was noted that one can simplify things by exploring the parametrization ambiguity instead of the minimal gauge fixing. The most general version of such a calculation [85], used the background field method with the parametrization

\[
g_{\alpha \beta} \rightarrow g'_{\alpha \beta} = e^{2\kappa r \sigma} \left[g_{\alpha \beta} + \kappa (\gamma_1 \phi_{\alpha \beta} + \gamma_2 \phi \phi_{\alpha \beta}) + \kappa^2 (\gamma_3 \phi_{\alpha \rho} \phi_{\rho \beta} + \gamma_4 \phi_{\alpha \theta} \phi_{\theta \beta} + \gamma_5 \phi \phi_{\alpha \beta} + \gamma_6 \phi^2 g_{\alpha \beta}) \right],
\]

where \( g_{\alpha \beta} \) is the background metric and \( \phi_{\alpha \beta} \) and \( \sigma \) are the quantum fields. Furthermore, the trace is defined as \( \phi = \phi^\mu_\mu \) and the indexes are lowered and raised with the background metric \( g_{\alpha \beta} \) and with its inverse \( g^{\alpha \beta} \). For the one loop effects, Eq. (132) is generalized version of the simplest parametrization (9), as it includes arbitrary coefficients \( \gamma_1, \ldots, \gamma_6 \) and \( r \), which parameterize the choice of the quantum variables. An important detail is that, since the one-loop divergences are defined only by the bilinear in quantum fields part of the action, (132) is the most general parametrization at the one-loop order. The next point is that, for any choice of \( \gamma_1, \ldots, \gamma_6 \) and \( r \), one can choose the gauge fixing parameters \( \alpha \) and \( \beta \) such that the bilinear form of the total action with the \( S_{ef} \)-term (18), be a minimal operator. This approach enables one to verify the general statements about gauge-fixing and parametrization dependence in a relatively economic way, avoiding working with the nonminimal operators.

In this review, we will not go into details of the practical calculations, which can be seen in the mentioned original works, but follow [80, 85] and [7] to explore all the mentioned dependencies in the general framework.
The classical equations of motion corresponding to the Einstein-Hilbert action with the cosmological constant term (22), are (we ignore the irrelevant factor of $\kappa^2$)

$$\varepsilon^{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} = R^{\mu\nu} - \frac{1}{2} (R + 2\Lambda) g^{\mu\nu}. \quad (133)$$

The general statement about gauge-fixing and parametrization independence on-shell can be used together with the locality of the divergent part of the effective action. The power counting tells us that this divergence has the form

$$\Gamma^{(1)}_{div} = \frac{1}{\varepsilon} \int d^4x \sqrt{-g} \{ c_1 R^2 + c_2 R^2 + c_3 R^2 + c_4 \Box R + c_5 R + c_6 \}, \quad (134)$$

where $\varepsilon = (4\pi)^2(n-4)$ is the regularization parameter and $c_{1,2,\ldots,6}$ are some coefficients. Our purpose is to explore how these coefficients depend on the parametrization and gauge fixing choices.

We denote $\alpha$, the full set of arbitrary parameters characterizing the gauge fixing and parametrization of the quantum metric. The special values $\alpha^0$ of these parameters correspond to some fixed choice, e.g., to those in [34]. The $\alpha$,-related ambiguities in $\Gamma^{(1)}_{div}$ do not violate the locality of this expression. Taking this into account, the on-shell universality tells us that

$$\delta \Gamma^{(1)}_{div} = \Gamma^{(1)}_{div}(\alpha) - \Gamma^{(1)}_{div}(\alpha^0) = \frac{1}{\varepsilon} \int d^4x \sqrt{-g} \left( b_1 R_{\mu\nu} + b_2 R_{g\mu\nu} + b_3 \Lambda g_{\mu\nu} + b_4 g_{\mu\nu} \Box + b_5 \nabla_{\mu} \nabla_{\nu} \right) \varepsilon^{\mu\nu}, \quad (135)$$

where the new parameters $b_{1,2,\ldots,5}$ in (135) depend on $\alpha$, and the explicit form of the dependence can be seen only from the real calculations. However, one can draw relevant conclusions directly from (135). In the simplest case of $\Lambda = 0$, this formula tells us that only the Gauss-Bonnet counterterm $\int E_4$ cannot be set to zero by choosing $\alpha$. This is exactly the result that was discovered by direct calculation in [31]. The $S$-matrix for the gravitational perturbations corresponds to the on-shell limit of the effective action and thus, it is finite.

In the general theory, with $\Lambda \neq 0$, we note that the parameter $b_5$ has no effect on divergences because of the third Bianchi identity $\nabla_{\mu} G_{\nu}^{\mu} = 0$ and that $\nabla_{\nu} \Lambda = 0$. Thus, there is a four-parameter $b_{1,2,3,4}$ ambiguity for the six existing coefficients $c_{1,2,\ldots,6}$. Therefore, only two combinations of these six coefficients can be expected to be gauge-fixing and parametrization independent. Obviously, one of these combinations is the coefficient of $\int E_4$ defined in (29). This directly follows from the fact that the $\Lambda$-term cannot affect the four-derivative divergences, as we know from the power counting.

Let us find the second combination of the parameters. A simple calculation using (135) shows that the coefficients in the expression (134) vary according to

$$c_1 \rightarrow c_1, \quad c_2 \rightarrow c_2 + b_1, \quad c_3 \rightarrow c_3 - \left( b_2 + \frac{1}{2} b_1 \right),$$

$$c_4 \rightarrow c_4 - b_4, \quad c_5 \rightarrow c_5 - \left( b_1 + 4 b_2 + b_3 \right) \Lambda, \quad c_6 \rightarrow c_6 - 4 b_3 \Lambda^2. \quad (136)$$
It is an easy exercise to show that the two gauge-fixing and parametrization invariants which do not change under the transformations of $c_1, c_2, \ldots, c_5$ in (136), are

$$c_1 \quad \text{and} \quad c_{\text{inv}} = c_6 - 4\Lambda c_5 + 4\Lambda^2 c_2 + 16\Lambda^2 c_3. \quad (137)$$

The last observation is that the on-shell expressions for the classical action and divergences have the forms

$$S \bigg|_{\text{on shell}} = \frac{2\Lambda}{k^2} \int d^4x \sqrt{-g},$$

$$\Gamma^{(1)}_{\text{div}} \bigg|_{\text{on shell}} = \frac{1}{\varepsilon} \int d^4x \sqrt{-g} \{c_1 E_4 + c_{\text{inv}}\}. \quad (138)$$

In these two functionals there are only invariant quantities. This feature forms the basis of the so-called on-shell renormalization group equation, to be discussed below. An additional small detail is that $c_{\text{inv}}$ does not change if we replace $E_4$ in (138) by the square of the Riemann tensor.

### 9.2 Gauge-fixing dependence in higher-derivative models

Adding more derivatives in the classical action, the gauge-fixing and parametrization dependence in the divergent part of the effective action becomes smaller. Among all higher-derivative models of QG, the unique non-trivial example is the fourth-derivative model (30). Let us first consider this model following [80].

In the four-derivative theory, the formula analogous to (135) has the form

$$\Gamma^{(1)}_{\text{div}}(\alpha_i) - \Gamma^{(1)}_{\text{div}}(\alpha^0_i) = \frac{1}{\varepsilon} \int d^4x \sqrt{-g} f_{\mu\nu} \varepsilon^{\mu\nu}_{(4)}, \quad (139)$$

where $f_{\mu\nu} = f_{\mu\nu}(\alpha_i)$ is an unknown tensor depending on $\alpha_i$ and

$$\varepsilon^{\mu\nu}_{(4)} = \frac{1}{\sqrt{-g}} \frac{\delta S_{\text{HD}}}{\delta g_{\mu\nu}} \quad (140)$$

are the equations of motion for the fourth-derivative gravity.

To find $f_{\mu\nu}$, let us remember that the fourth-derivative quantum gravity is a renormalizable theory. Therefore, all three of the expressions $\Gamma^{(1)}_{\text{div}}(\alpha^0_i)$, $\Gamma^{(1)}_{\text{div}}(\alpha_i)$ and $\varepsilon^{\mu\nu}_{(4)}$ have dimension 4, as the classical action. Since the divergencies in (139) are local functionals, $f_{\mu\nu}$ is a dimensionless tensor. Then the only possible choice is

$$f_{\mu\nu}(\alpha_i) = g_{\mu\nu} f(\alpha_i), \quad (141)$$

where $f(\alpha_i)$ is an arbitrary (can be defined only by explicit calculations) dimensionless function of the set of parameters of gauge fixing and metric parametrization. Thus, the gauge/parametrization dependence of the divergent part of effective
action is controlled by the “conformal shift” of the classical action

\[ \Gamma^{(1)}_{\text{div}}(\alpha_i) - \Gamma^{(1)}_{\text{div}}(\alpha^0_i) = f(\alpha_i) \int d^4x \ g_{\mu\nu} \frac{\delta S}{\delta g_{\mu\nu}}. \] (142)

In the case of the conformal model, the r.h.s. of this equation simply vanishes, i.e., in purely conformal, Weyl-squared, gravity theory, the divergences of the effective action do not depend on \( \alpha_i \) because the classical action satisfies the Noether identity for the conformal invariance. For the general model (30), the \( C^2, E_4 \) and \( R \) terms in the action do not contribute to the r.h.s. of (142). Then, the gauge- and parametrization dependencies are defined by the Einstein-Hilbert, cosmological and \( R^2 \) terms. It is easy to get

\[ \Gamma^{(1)}_{\text{div}}(\alpha_i) - \Gamma^{(1)}_{\text{div}}(\alpha^0_i) = f(\alpha_i) \int d^4x \sqrt{\tilde{g}} \left\{ \frac{2\omega}{\kappa} \Box R - \frac{1}{\kappa^2} (R + 4\Lambda) \right\}. \] (143)

The divergent coefficient of the \( \Box R \) term depends on the gauge fixing, the same is true for the coefficients of the Einstein-Hilbert and cosmological terms. At the same time, there are two gauge-invariant combinations of these coefficients.

The easiest part is the gauge and parametrization dependence of the counterterms in superrenormalizable models with more than four derivatives, both polynomial or nonlocal. It is easy to see that these models do not have such dependencies. To get this result, we note that formula (139) is valid for all such models, both polynomial and nonlocal, with trading of \( \varepsilon^{\mu\nu}_{(4)} \) to the variational derivative of the corresponding action \( \varepsilon^{\mu\nu} \). As we have seen above, according to the power counting arguments, the divergences are given by local expressions with four, two or zero derivatives of the metric. On the other hand, in all superrenormalizable models, the equations of motion \( \varepsilon^{\mu\nu} \) have more than four derivatives of the metric. Thus, the non-zero r.h.s. of (139) is incompatible with the locality of the function \( f_{\mu\nu} \), proving the statement about the universality of renormalization [22, 86, 46].

10 One-loop divergences in quantum GR

The derivation of one-loop divergences in quantum GR has great historical [34] and practical importance. This calculation is a starting point for many other developments, in many different models, including pure QG, models of more and more complicated versions of pure QG, gravity coupled to quantum matter, etc. For these reasons, the review on perturbative QG should include this calculation and the list of the most important extensions and corresponding references.

In the rest of this section, we repeat the original derivation of divergences in pure QG from the classical paper [34]. Since this is not complicated, we shall also include the nonzero cosmological constant, as it was done in [87]. The standard calculation uses the background field method based on (9), the heat-kernel expansion and the Schwinger-DeWitt technique [1] (see also [7] for a detailed introduction).
The bilinear expansion of the action \((22)\) is given in Eq. \((48)\), and the gauge-fixing term with the two gauge fixing parameters \(\alpha\) and \(\beta\) is given by \((25)\). The ghost action can be easily obtained from \((27)\), but we postpone this part until fixing the values of \(\alpha\) and \(\beta\). For this, we rewrite \((25)\) as

\[
S_{gf} = \frac{1}{\alpha} \int d^4 x \sqrt{-g} \left[ g_{\mu \alpha} \nabla_\nu \nabla_\beta - \beta (g_{\mu \nu} \nabla_\alpha \nabla_\beta - g_{\alpha \beta} \nabla_\mu \nabla_\nu) \right] + \beta^2 g_{\mu \nu} g_{\alpha \beta} \Box h_{\alpha \beta}.
\] (144)

Adding this expression to \((48)\), we require that the sum includes the minimal operator \(H_{\mu \nu, \alpha \beta}\):

\[
S_{EH}^{(2)} + S_{gf} = \frac{1}{2} \int d^4 x \sqrt{-g} h_{\mu \nu} H_{\mu \nu, \alpha \beta} h_{\alpha \beta},
\]

\[
H_{\mu \nu, \alpha \beta} = K_{\mu \nu, \alpha \beta} \Box + M_{\mu \nu, \alpha \beta},
\] (145)

where \(K_{\mu \nu, \alpha \beta}\) and \(M_{\mu \nu, \alpha \beta}\) are \(c\)-number operators. This is achieved for \(\alpha = 2\) and \(\beta = 1/2\). After that, we arrive at the expression \((145)\) with

\[
K_{\mu \nu, \alpha \beta} = \frac{1}{2} \left( \delta_{\mu \nu, \alpha \beta} - \frac{1}{2} g_{\mu \nu} g_{\alpha \beta} \right),
\] (146)

\[
M_{\mu \nu, \alpha \beta} = R_{\mu \nu \rho \sigma} + g_{\nu \beta} R_{\mu \alpha} - \frac{1}{2} (g_{\mu \nu} R_{\alpha \beta} + g_{\alpha \beta} R_{\mu \nu}) - \frac{1}{2} R (\delta_{\mu \nu, \alpha \beta} - \frac{1}{2} g_{\alpha \beta} g_{\mu \nu}).
\]

It is easy to see that the matrix \(2K_{\rho \sigma, \alpha \beta}\) is equal to its own inverse,

\[
\left( \delta_{\mu \nu, \alpha \beta} - \frac{1}{2} g_{\mu \nu} g_{\alpha \beta} \right) \left( \delta_{\alpha \beta, \rho \sigma} - \frac{1}{2} g_{\alpha \beta} g_{\rho \sigma} \right) = \delta_{\mu \nu, \rho \sigma}.
\] (147)

On the other hand, \(\text{Tr} \ln (2K_{\rho \sigma, \alpha \beta})\) does not contribute to the divergences (e.g., in the dimensional regularization) since this operator has no derivatives. Thus, regarding the divergences,

\[
\text{Tr} \ln \left( H_{\rho \sigma, \alpha \beta} \right) = \text{Tr} \ln \left( 2K_{\mu \nu, \rho \sigma} H_{\rho \sigma, \alpha \beta} \right) = \text{Tr} \ln \left( H'_{\rho \sigma, \alpha \beta} \right) = \text{Tr} \ln \left( \delta_{\mu \nu, \alpha \beta} \Box + \Pi_{\rho \sigma, \alpha \beta} \right).
\] (148)

A small calculation gives

\[
\hat{\Pi} = \Pi_{\rho \sigma, \alpha \beta} = 2K_{\mu \nu, \rho \sigma} M_{\rho \sigma, \alpha \beta} = M_{\mu \nu, \alpha \beta}.
\] (149)

It is evident that Eq. \((148)\) enables one to use the standard Schwinger-DeWitt formula for the operator

\[
\hat{H} = \hat{\Pi} + 2 \hat{h}^\mu \nabla_\mu + \hat{\Pi}.
\] (150)

For this, we need to define
The divergent part of the one-loop effective action is an integral over the "magic" multiplication table:

\[ S_{\mu\nu} = [\nabla_\nu, \nabla_\mu] \hat{I} + (\nabla_\nu \hat{h}_\mu - \nabla_\mu \hat{h}_\nu) + (\hat{h}_\nu \hat{h}_\mu - \hat{h}_\mu \hat{h}_\nu) \]  \hspace{1cm} (151)

and

\[ \hat{P} = \hat{\Pi} + \frac{i}{6} R - \nabla_\mu \hat{h}_\mu - \hat{h}_\mu \hat{h}_\mu. \]  \hspace{1cm} (152)

The divergent part of the one-loop effective action is an integral over the "magic" \( a_2 \) coefficient,

\[ \Gamma^{(1)}_{d_{iv}} = - \frac{\mu^{n-4}}{\xi} \int d^n x \sqrt{-g} \, \text{tr} \hat{a}_2, \]  \hspace{1cm} (153)

where

\[ \hat{a}_2 = \hat{a}_2(x, x) = \frac{i}{180} (R^2_{\mu\nu\alpha\beta} - R^2_{\alpha\beta} + \Box R) + \frac{1}{2} \hat{P}^2 + \frac{1}{6} (\Box \hat{P}) + \frac{1}{12} \hat{S}^2_{\mu\nu}. \]  \hspace{1cm} (154)

In our case, we have a simple situation because \( \hat{h}_\mu = 0 \). Thus,

\[ \hat{P} = \hat{\Pi} + \frac{i}{6} R \quad \text{and} \quad \hat{S}_{\mu\nu} = [\nabla_\nu, \nabla_\mu]. \]  \hspace{1cm} (155)

A simple calculation gives

\[ \hat{P} = P_{\mu\nu,\alpha\beta} = \hat{K}_1 + \hat{K}_2 - \frac{5}{12} \hat{K}_3 + \frac{1}{4} \hat{K}_5, \]  \hspace{1cm} where

\[ \hat{K}_1 = R_{\mu\nu\alpha\beta}, \quad \hat{K}_2 = g_{\nu\beta} R_{\mu\alpha}, \quad \hat{K}_3 = g_{\mu\nu} R_{\alpha\beta} + g_{\alpha\beta} R_{\mu\nu}, \]

\[ \hat{K}_4 = \delta_{\mu\nu,\alpha\beta} R, \quad \hat{K}_5 = R g_{\alpha\beta} g_{\mu\nu}, \]

and

\[ \hat{S}_{\lambda\tau} = [S_{\lambda\tau}]_{\mu\nu,\alpha\beta} = -2 R_{\mu\alpha\lambda\tau} g_{\nu\beta}. \]  \hspace{1cm} (156)

For the sake of compactness, in all of these expressions, we assume automatic symmetrization over the couples of indices \((\mu\nu)\) and \((\alpha\beta)\). It is easy to get the following multiplication table:

\[ \text{tr} \hat{K}_1 \cdot \hat{K}_1 = \frac{1}{2} R^2_{\mu\nu,\alpha\beta}, \quad \text{tr} \hat{K}_2 \cdot \hat{K}_2 = \frac{3}{2} R^2_{\mu\nu} + \frac{1}{4} R^2, \quad \text{tr} \hat{K}_3 \cdot \hat{K}_3 = 8 R^2_{\mu\nu} + 2 R^2, \]

\[ \text{tr} \hat{K}_4 \cdot \hat{K}_4 = 10 R^2, \quad \text{tr} \hat{K}_5 \cdot \hat{K}_5 = 16 R^2, \quad \text{tr} \hat{K}_1 \cdot \hat{K}_2 = -\frac{1}{2} R^2_{\mu\nu}, \]

\[ \text{tr} \hat{K}_1 \cdot \hat{K}_3 = 2 R^2_{\mu\nu}, \quad \text{tr} \hat{K}_1 \cdot \hat{K}_4 = -\frac{1}{2} R^2, \quad \text{tr} \hat{K}_1 \cdot \hat{K}_5 = R^2, \quad \text{tr} \hat{K}_2 \cdot \hat{K}_3 = 2 R^2_{\mu\nu}, \]

\[ \text{tr} \hat{K}_2 \cdot \hat{K}_4 = \frac{5}{2} R^2, \quad \text{tr} \hat{K}_2 \cdot \hat{K}_5 = R^2, \quad \text{tr} \hat{K}_3 \cdot \hat{K}_4 = 2 R^2, \quad \text{tr} \hat{K}_3 \cdot \hat{K}_5 = 8 R^2, \]

\[ \text{tr} \hat{K}_4 \cdot \hat{K}_5 = 4 R^2, \quad \text{and} \quad \hat{S}_{\lambda\tau} \cdot \hat{S}_{\lambda\tau} = -6 R^2_{\mu\nu,\alpha\beta}. \]  \hspace{1cm} (157)

Substituting these values into Eq. (154), we get...
\[
\text{tr} \left( \frac{1}{2} \hat{P} \cdot \hat{P} + \frac{1}{12} \hat{S}_{\lambda \tau} \cdot \hat{S}_{\lambda \tau} \right) = R_{\mu \nu \alpha \beta}^2 - 3 R_{\mu \nu \alpha \beta}^2 + \frac{59}{36} R^2 + \frac{26}{3} R \Lambda + 20 \Lambda^2 .
\]

(158)

In this and subsequent expressions, we ignore the total derivative term \( \Box R \). The complete expressions can be found, e.g., in the original publication [85], including for the general parametrization of quantum metric (132).

For the ghost action, we obtain

\[
\frac{\delta \chi^\mu}{\delta h_{\rho \sigma}} = \delta^{\mu \rho \sigma} \nabla_{\lambda} - \frac{1}{2} g^{\rho \sigma} \nabla_{\mu} , \quad R^{\nu}_{\rho \sigma} = - \delta^{\nu}_{\rho} \nabla_{\sigma} - \delta^{\nu}_{\sigma} \nabla_{\rho} .
\]

This is a minimal vector operator, that enables one to use the standard Schwinger-DeWitt formula (153). For the commutator, we get

\[
\hat{S}_{\lambda \tau} = [\hat{S}_{\lambda \tau}, \alpha, \beta] = - R_{\alpha \beta \lambda \tau} .
\]

(160)

Thus,

\[
\text{tr} \left( \frac{1}{2} \hat{P} \cdot \hat{P} + \frac{1}{12} \hat{S}_{\lambda \tau} \cdot \hat{S}_{\lambda \tau} \right)_{\text{ghost}} = - \frac{1}{12} R_{\mu \nu \alpha \beta}^2 + \frac{3}{2} R_{\mu \nu \alpha \beta}^2 + \frac{2}{9} R^2 .
\]

(161)

Finally, replacing all the expressions above into (154) and taking into account that, in the \( h_{\mu \nu} \) sector, \( \text{tr} \hat{1} = \delta_{\mu \nu \alpha \beta} \delta_{\mu \nu \alpha \beta} = 10 \) and, in the ghost sector, \( \text{tr} \hat{1}_{\text{ghost}} = 4 \), we arrive at the famous result of [34],

\[
R_{\text{div, total}}^{I} = \frac{i}{2} \text{Tr} \ln \hat{H} - i \text{Tr} \ln \hat{H}_{\text{ghost}} \]

\[
= - \frac{\mu^{n-4}}{\epsilon} \int d^n x \sqrt{-g} \left\{ \frac{53}{45} R_{\mu \nu \alpha \beta}^2 - \frac{361}{90} R_{\alpha \beta}^2 + \frac{43}{36} R^2 + \frac{26}{3} R \Lambda + 20 \Lambda^2 \right\}
\]

\[
= - \frac{\mu^{n-4}}{\epsilon} \int d^n x \sqrt{-g} \left\{ \frac{53}{45} E_4 + \frac{7}{10} R_{\alpha \beta}^2 + \frac{1}{60} R^2 + \frac{26}{3} R \Lambda + 20 \Lambda^2 \right\},
\]

where, as usual, \( \epsilon = (4 \pi)^2 (n - 4) \), and \( \mu \) is the renormalization parameter.

As we already know from section 9.1, the coefficient \( \frac{53}{45} \) of the Gauss-Bonnet term is invariant, while other coefficients can be modified by changing the gauge-fixing conditions or the parametrization of the quantum metric. The second invariant coefficient in (137) can be easily found by using the on shell conditions \( R_{\mu \nu} = - \Lambda g_{\mu \nu} \) and \( R = - 4 \Lambda \). The result of this operation is

\[
c_{\text{inv}} = - \frac{58}{5} \Lambda^2 .
\]

(163)

This means, the invariant, on shell, version of (162) is
\begin{equation}
\Gamma_{\text{div.}}^{1, \text{total}} = -\frac{\mu^{n-4}}{\varepsilon} \int d^n x \sqrt{-g} \left\{ \frac{53}{45} E_4 - \frac{58}{5} \Lambda^2 \right\}, \tag{164}
\end{equation}

To conclude this section, let us make a few observations.

i) The reader could note that in the formulas in the previous section, e.g., in (143), we used \( \int d^4 x \sqrt{-g} \), while in this section, the more complicated integration rule \( \int d^n x \mu^{n-4} \sqrt{-g} \) was used. The point is that the two formulas are equivalent when it concerns the divergences. The expression with \( \int d^4 x \) is just shorter and the one with \( \int d^n x \) may be more useful, e.g., for deriving the beta function in the Minimal Subtraction Scheme of renormalization.

ii) One may be curious why we identify the square of the Riemann tensor in the formula for invariant part of divergences in Eq. (138) with the Gauss-Bonnet topological invariant \( E_4 \) and not with the square of the Weyl tensor. This is an important question and we shall give an extensive answer to it.

This issue requires addressing the nonlocal finite contributions corresponding to the logarithmic UV divergences. One can calculate these nonlocal terms directly (see, e.g., [79, 88] for the heat-kernel approach that can be used for this purpose), or using Feynman diagrams, including in the theory of massive fields [89] (see also [90, 91, 7]). In the massless limit, the nonlocal logarithmic terms correspond the higher-derivative divergences (134) and have the form [7]

\begin{equation}
\Gamma_{\text{fin.}, \text{HD}}^{(1)} = \int d^4 x \sqrt{-g} \left\{ c_1 R_{\mu \nu \alpha \beta} \ln \left( -\frac{\Box}{\mu^2} \right) R^{\mu \nu \alpha \beta} \right. \\
+ \left. c_2 R_{\alpha \beta} \ln \left( -\frac{\Box}{\mu^2} \right) R^{\alpha \beta} + c_3 R \ln \left( -\frac{\Box}{\mu^2} \right) R \right\}. \tag{165}
\end{equation}

Let us note that one can also derive the nonlocal form factor for the Einstein-Hilbert term [90, 91], regardless there is an ambiguity even in the nonlocal representation of the proper action of GR [92]. However, what we need is just the expression (165), which is the physical reflection of the divergences.

The correspondence between divergences and nonlocal terms holds independent on the choice of gauge fixing and parametrization, such that the transformations (136) apply to the coefficients of the logarithmic terms in (165). As we know from section 5, all three parts of this expression affect the propagation of the spin-2 or spin-0 modes of the metric. The ambiguity (136) affects this propagation and, in particular, can eliminate all of it. As we know from the discussion of propagator in the general models (39) with an extra term (40), the unique part of the expression which does not affect the propagation of the spin-2 or spin-0 modes of the metric, is the generalized Gauss-Bonnet term

\begin{equation}
\Gamma_{\text{GB.}, \text{nonloc}}^{(1)} = -c_1 \int d^4 x \sqrt{-g} \left\{ R_{\mu \nu \alpha \beta} \ln \left( -\frac{\Box}{\mu^2} \right) R^{\mu \nu \alpha \beta} \right. \\
- \left. 4 R_{\alpha \beta} \ln \left( -\frac{\Box}{\mu^2} \right) R^{\alpha \beta} + R \ln \left( -\frac{\Box}{\mu^2} \right) R \right\}. \tag{166}
\end{equation}
This means, the invariant coefficient $c_1$ in (137) should be attributed to the $E_4$ and not to the Weyl-squared term. In case of QG based on GR, this coefficient is $53/45$. There may be further contributions from matter fields, which do not depend on the gauge fixing in the QG sector of the theory.

iii) As one can see already from the power counting arguments, the renormalization of the Einstein-Hilbert term in QG based on GR is possible only owing to the nonzero cosmological constant $\Lambda$. This feature holds beyond the one-loop order and can be seen as a general result. However, this statement corresponds to the most universal, logarithmic, divergences. The quadratic divergences of the Einstein-Hilbert type are possible even for $\Lambda = 0$. However, we know that these divergences depend on the choice of regularization and, therefore, the physical results which one can obtain from quadratic divergences always have uncertain physical sense. This issue was discussed in more detail, e.g., in [7].

11 On shell renormalization group in quantum GR

The formulation of renormalization group running in QG is a complicated problem because renormalizable and superrenormalizable models of QG always have massive ghosts and other massive degrees of freedom, such that the physical sense of the running (and quantum effects, in general) in these models is not clear. The reason is that, loop contributions in these models come from the functional integral over massless mode of the gravitational field and, also, over the massive modes. It is a standard assumption that the contributions of massive fields vanish at the energy scale below their masses. In the fourth derivative QG this expectation was formulated, for the first time, in [26]. Nowadays, it is partially supported by the calculations in semiclassical theories [89, 91] and in the toy model of QG [93]. On the other hand, in QG all massive modes have masses of the Planck order of magnitude (see the discussion of this issue in [39]). Thus, the physical applicability of the renormalization group running in fourth- and higher-order models of QG is restricted by the UV domain, with energies above the Planck scale.

Thus, despite the running of parameters of the actions in all renormalizable QG models might have great importance from the theoretical perspective, its application to any kind of physics is unclear. Since the universal QG theory in the IR is quantum GR [94], we can try to explore the running using this model. However, the use of renormalization group to explore the running in the quantum GR is a non-trivial issue because the theory is not renormalizable. Assuming that all massive degrees of freedom decouple in the IR (below the Planck scale) and the physically interesting running is in the region below the Planck scale of energies, we arrive at the subject of effective approach to QG, which is treated in a separate Section of this Handbook.

Let us consider a version of renormalization group running in QG which enables one to avoid the mentioned difficulties. This version of renormalization group is not perfectly well defined, but it gives an idea of what we can expect from the running. The on shell version of renormalization group uses the expressions (138) for the
classical action and one-loop divergences on the classical equations of motion. The on shell divergences are universal, i.e., do not depend on the parametrization of quantum metric and on the gauge-fixing choice. Ignoring the topological term $c_1E_4$, the aforementioned expressions can be used to perform the renormalization on shell and, consequently, to achieve the renormalization group running.

Let us define the dimensionless parameter

$$\gamma = \kappa^2 \Lambda.$$  \hskip 1.0cm (167)

In terms of this parameter, the classical action and the one-loop counterterm (both on-shell) have the form

$$S_{EH}\big|_{\text{on-shell}} = -\frac{1}{\kappa^4} \int d^n x \sqrt{-g} \mu^{n-4} (-2 \gamma),$$

$$\Delta S^{(1)}\big|_{\text{on-shell}} = \frac{1}{\epsilon} \frac{1}{\kappa^4} \int d^n x \sqrt{-g} \mu^{n-4} \left( -\frac{58}{5} \gamma^2 \right).$$  \hskip 1.0cm (168)

As these two expressions have an identical dependence of the metric, we can remove the divergence by making the renormalization transformation,

$$\gamma_0 = \mu^{n-4} \left( \gamma - \frac{29}{5\epsilon} \gamma^2 \right),$$  \hskip 1.0cm (169)

from what immediately follows the general $\beta$-function in $n$ spacetime dimensions,

$$\mu \frac{d \gamma}{d \mu} = -(n-4) \gamma - \frac{29}{5(4\pi)^2} \gamma^2.$$  \hskip 1.0cm (170)

The standard $\beta$-function for $\gamma$ can be obtained in the limit $n \to 4$ and we arrive at the renormalization group equation that corresponds to an asymptotically free theory,

$$\mu \frac{d \gamma}{d \mu} = \beta_\gamma = -a^2 \gamma^2, \quad a^2 = \frac{29}{5(4\pi)^2}.$$  \hskip 1.0cm (171)

The solution of this equation can be easily found if we set the initial value at some scale, $\gamma(\mu_0) = \gamma_0$,

$$\gamma(\mu) = \frac{\gamma_0}{1 + a^2 \gamma_0 \ln(\mu/\mu_0)}.$$  \hskip 1.0cm (172)

The physical interpretation of this solution meets several difficulties, so let us present some points discussing this subject.

1. It is possible to identify $\mu$ with one or another physical parameter, in different physical situations. As far as $\gamma$ is the dimensionless cosmological constant, the standard application of the solution should be in cosmology and then the natural choice of the scale is the Hubble parameter [95, 96, 97].

2. The cosmological constant that enters the definition (167) is not the one which is responsible for the observed accelerated expansion of the Universe. The observ-
The background information about perturbative quantum gravity 55

able density of the cosmological constant $\rho_{\Lambda}^{\text{obs}} = \Lambda / (8\pi G)$ is a sum of the vacuum quantity $\rho_{\Lambda}^{\text{vac}} = \rho_{\Lambda}^{\text{vac}}$ and the induced density $\rho_{\Lambda}^{\text{ind}}$ coming from, e.g., the electroweak phase transition. Both summands are many orders of magnitude greater than the observed quantity $\rho_{\Lambda}^{\text{obs}}$. For this reason, the numerical value of $\gamma_0$ may be small, but it is not that small as some reader might think. This issue was discussed in detail in [98], with the consideration based on the unique effective action formalism [99, 100, 101]. This interesting approach is discussed in a separate Chapter of this Section and does not fit the present review. However, we shall briefly discuss the differences between the two universal equations for $\gamma(\mu)$ below.

3. Another interesting point is that, if $\gamma_0$ is sufficiently small are negligible and this equation can be regarded as non-perturbative. To see this, it is sufficient to pick up the power counting in the quantum GR, as it was presented above.

To end this section, let us compare the running (172) with the one in the effective approach to quantum GR with cosmological constant [98] (see also earlier work [101] with the same result but without effective interpretation). In the effective approach, the higher derivative terms should be neglected, but without these terms, the invariants (137) cannot be constructed. The way out is to use the Vilkovisky-DeWitt scheme of unique effective action, that does not depend of parametrization or gauge fixing by construction [99, 100]. The calculations in this case give [101]

$$\Gamma_{\text{div}}^{1, \text{total}} = -\frac{\mu^{n-4}}{\varepsilon} \int d^n x \sqrt{-g} \left\{ \frac{53}{45} E_4 + \frac{121}{60} R_2^{\alpha \beta} - \frac{29}{60} R^2 + 8 R A + 12 A^2 \right\}. \tag{173}$$

This is different from (162), but it is easy to check that the on shell expression (164) is the same. But in this case, we do not need to rely on the on shell version. Neglecting all fourth derivative terms in (173) and using the standard formalism, one gets the renormalization group equations for the constants in the action (22),

$$\frac{d}{d\mu} \frac{1}{\kappa^2} = \frac{8A}{(4\pi)^2}, \tag{174}$$

$$\frac{dA}{d\mu} = -\frac{2A^2 \kappa^2}{(4\pi)^2}. \tag{175}$$

In this case, the equation for the dimensionless combination (167) has the same general form (171), but this time with the coefficient

$$a^2 = \frac{10}{(4\pi)^2}. \tag{176}$$

This change of the beta function illustrates the role of the physical interpretation in QG. The difference between the two cases is that in (171) we do not ignore the terms with four derivatives and get the invariant beta function by using the on

\footnote{I am grateful to Breno Giacchini and Tibério de Paula Netto for the stimulating discussions of this issue.}
shell condition. On the contrary, in \((176)\) we follow the effective approach, ignore higher derivative terms and use the Vilkovisky’s unique effective action to provide an invariant result.

Both approaches look satisfactory and both are not perfect. Which one is better? In a “normal” physical model the answer would be probably given by experiment, but in QG this is not an option.

12 One-loop divergences in other models of QG

There is a vast number of publications on the one-loop calculations and we do not pretend to make a complete review or even list the most relevant of these works (including the ones of the present author) here. In what follows, I will separate those papers which were first of the kind for the most relevant models.

The first of these works concerned the one loop divergences in quantum GR coupled to quantum matter. Already in the first paper with one-loop calculations, ’t Hooft and Veltman derived also the divergences for QG coupled to the minimal scalar field \([34]\). It is interesting that the case of quantum GR coupled to the nonminimal quantum scalar \(\phi\) (nonminimal means the presence of the term \(\xi R\phi^2\)) can be reduced to the calculations with a minimal scalar by the change of variables, i.e., the conformal transformation \([102]\). However, the correspondence with the direct calculation \([103]\) required much greater effort and was achieved only two decades later in \([104]\). This example shows importance and complications related to the choice of parametrization. The calculations in gravity-vector (Abelian and nonabelian) and gravity-fermion models were done almost at the same time as \([34]\) by Deser, van Nieuwenhuisen et al in \([35]\). Let us note that more general combinations of quantum fields including QG were explored in supergravity, but since there is a special Section of this Handbook about supergravity, we do not need to discuss this part here. The important general result of the mentioned calculations is that, for matter-gravity systems, the one-loop divergences do not vanish on shell \([34, 35]\).

The one-loop divergences were calculated in other models of QG, including fourth-derivative QG with the action \((30)\) and the polynomial superrenormalizable model \((34)\). The case of the theory \((30)\) is treated in full detail in another Chapter of this Section of our Handbook and we will not discuss the technical details here. From the general perspective, this calculation was relevant for several reasons. There is a growing understanding that the construction of QG which is not restricted to the low-energy domain, cannot be successful without higher derivative and the same is true in the semiclassical theory. Then, one of the “traditional” expectations is to advance in the problem of ghosts and instabilities by exploring quantum corrections and, at the first place, the logarithmic contributions \([61, 62]\). As we already know, these contributions are directly related to the UV divergences.

Historically, the calculation of divergences in the fourth-derivative QG started in the work \([105]\), using Feynman diagrams and separating massless and massive internal lines. This way to make calculations was never used after that (up to our
knowledge) as all subsequent calculations were performed by technically more simple heat-kernel (or Schwinger-DeWitt) technique, adapted for the four-derivative operators in [26] and later on, in a more systematic way, in [27]. However, the diagrammatic approach of the pioneering work of Julve and Tonin [105] may be, by the end of the day, rather important. The reason is that, using diagrams enable one to separate the graphs with only massless internal lines from the ones which have the lines of the massive degrees of freedom (i.e., ghosts, in the present case). This method can be used to explore the IR limit of the quantum corrections in higher derivative theories. As we already mentioned above, in this area we have only the general statement about the universality of quantum GR as an effective theory of QG in the IR [26, 94] and the calculations of decoupling in the models of massive matter fields [89, 93] (see further references therein) that can be used as toy models for the problem of decoupling massive degrees of freedom in QG. Let is stress that completing the program of decoupling in the higher derivative models would be important for improved understanding of QG as a whole.

The next important step was done in the seminal paper [26] by Fradkin and Tseytlin. This was the first publication were the relevance of the weight operator (32) for quantum contributions was noted, there was also the first heat-kernel treatment of the fourth derivative operator and of the non-minimal vector operator, the first discussion of the IR decoupling in higher derivative theories, and the first observation concerning the difference between general and conformal QG models. We can say, that [26] paved the way for the development of a large part of QG in the subsequent decades. Furthermore, the correct result for the general and conformal versions of the fourth-derivative QG models was obtained in [106] and [107], respectively. Finally, there was a confirmation of the correctness of these results in [45, 108], where another interesting aspect of QG was addressed. Much earlier, in the paper [109], it was noted that the Gauss-Bonnet term \( \int E_4 \) may play a nontrivial role in QG. The point is that the topological nature of this term becomes badly seen in the perturbative approach, especially if one uses the dimensional regularization. In the previous sections we saw that the topological term does not contribute to the propagator, in any dimension \( n \). However, for \( n \neq 4 \) it does affect the vertices. The question posed in [109] was whether it is true that the contributions of these vertices cancel in the divergent part of effective action, or even in the finite part of it. The answer is that the divergences really do not depend on the \( \int E_4 \) term, and this dependence takes place only in the local finite terms.

Finally, let us mention the derivation of the one-loop divergences in the polynomial, superrenormalizable models of QG [86]. From the technical side, these calculations require the generalized Schwinger-DeWitt technique developed by Barvinsky and Vilkovisky [27].
13 Concluding discussion

Quantum gravity is intended as a theory to be most relevant in the vicinity of the singularities, i.e., at extremely high energies. The idea to formulate QG in a way of perturbative QFT, that proved efficient in other theories, is the first that comes to mind when one decides how to quantize gravity. In this sense, the perturbative approach is the most basic part of the whole QG program.

To a great extent, the current situation in perturbative QG is similar to the one in QG, in general. We have many approaches to incorporating the quantum effects of gravity within the well-established perturbative formalism of quantum field theory. On the other hand, there is no perfect model of quantum gravity. The theory based on GR is a promising candidate to describe low energy effects (see the corresponding Section on effective QG), but it has serious problems with non-renormalizability in the UV. From another side, fourth derivative model is renormalizable and enables one to make controllable calculations at any energy scale. However, the spectrum of particles in this theory includes nonphysical massive ghosts. The instabilities of classical solutions, generated by these ghosts, look non-avoidable. However, our experience in cosmology shows that these instabilities show up only in the regions with the typical energies are of the Planck order of magnitude. The role of the ghosts and the emergence of instabilities in the superrenormalizable QG models were not explored at the same level of quality, but it is expected that the output may be, qualitatively, the same.

We can draw two main conclusions about the main problem of perturbative QG. First, this problem is the conflict between renormalizability and the lack of physical unitarity (which does not reduce to the unitarity of the S-matrix) or instabilities of classical solutions owing to the presence of ghosts. Second, the resolution of this conflict does not look impossible, but it will (most likely, at least) require new ideas and new approaches. According to DeWitt and Molina-Paris, long ago W. Pauli remarked “It will take somebody really smart” to construct a quantum theory of gravity. After many decades that passed after this prediction, we are in a position to say that it will take somebody really smart to find modifications of gravity explaining how Nature prevents the ghosts to emerge. The opinion of the present author is that the problem of ghosts shows up in the perturbative approach, but its solution will, most likely, be found only beyond this framework.

From the historical perspective, one of the main impacts of QG to quantum field theory was the development of the classical and quantum theory of gauge fields, that helped to develop the Yang-Mills theory and gave a theoretical basis to the Standard Model of Particle Physics. In this respect, it is sufficient to mention the work of DeWitt. It is quite possible that the further progress in QG will be based on the flux of ideas in the opposite direction, that is using methods which already exist in field theory, but are not sufficiently familiar to the QG community.

As one can learn from the variety of approaches presented in this Handbook, we have many theories of QG. The main problem is perhaps not the shortage of the theories, but that none of these theories can be currently tested in experiments and/or observations. However, the increasing volume and improving quality of the obser-
The background information about perturbative quantum gravity may help to close this gap, someday. Thus, at least part of the nowadays theoretical developments in QG may find their use in the future.

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