Statistical Mechanics of Non-stretching Elastica in Three Dimensional Space

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Abstract.
Recently I proposed a new calculation scheme of a partition function of an immersion object using path integral method and theory of soliton (J.Phys.A (1998) 31 Mar). I applied the scheme to problem of elastica in two-dimensional space and Willmore surface in three dimensional space. In this article, I will apply the scheme to elastica in three dimensional space as a more physical model in polymer science. Then orbit space of the nonlinear Schrödinger and complex modified Korteweg-de Vries equations can be regarded as the functional space of the partition function.

§1. Introduction

Elastica problem in two dimensional space $\mathbb{R}^2$ has long history [1,2]. It is known that by observing a shape of thin elastic beam, James Bernoulli named the shape elastica. It might be regarded as birth of the elastica problem and germination of the mathematical physics, including the elliptic function theory, mode analysis, nonlinear science, elliptic differential theory, algebraic analysis and so on. The elastica in $\mathbb{R}^2$ [1,2] is defined as a curve with the Bernoulli-Euler functional,

$$ E = \int ds \ k^2, \quad (1-1) $$

where $k$ is its curvature.

Recently I presented a new calculation scheme of a partition function of non-stretching elastics in $\mathbb{R}^2$ under the condition preserving its local length [3]. The partition function is formally defined as

$$ Z = \int DX \ e^{-\beta \int ds \ k^2}, \quad (1-2) $$

where $DX$ is the Feynman measure for an affine vector of a point of the elastica $X$ and $\beta$ is the inverse of temperature. Goldstein and Petrich discovered that the virtual motion of non-stretching curve obeys the modified Korteweg-de Vries (MKdV) equation,

$$ \partial_t k + \frac{3}{2} k^2 \partial_s k + \partial_s^3 k = 0, \quad (1-3) $$

and its hierarchy [5,6]. Using the Goldstein-Petrich scheme, I found that the functional space of the partition function (1-2) are completely represented by the MKdV equation (1-3). In other words, the MKdV flows conserves the energy functional (1-1). The functional space (1-2) is classified by the solutions of the MKdV equation (1-3).
After that, I applied this method to the Willmore surface in three dimensional space $\mathbb{R}^3$ [4]. Instead of the MKdV equation, there appears the modified Novikov-Veselov equation which classifies the functional space of the partition function.

In this article, I will investigate a partition function of an elastica in $\mathbb{R}^3$ with the energy functional

$$E = \int ds |\kappa|^2,$$  

where $\kappa$ is a complex curvature of the elastica in $\mathbb{R}^3$. I will also require that the elastica does not stretch.

Then the partition function of an elastica in $\mathbb{R}^3$ with the energy (1-4) can be also evaluated. Due to the non-stretching condition, instead of Goldstein-Petrich scheme of the MKdV hierarchy [3,5,6], the Langer-Perline scheme of the nonlinear Schrödinger (NLS) hierarchy and the complex MKdV (CMKdV) hierarchy appears in the calculation of the partition function [7,8].

Whereas the NLS equation is well known as the integrable equation and investigated well, the properties of the CMKdV equation is not sufficiently studied. According to the result of Mohammad and Can [10], the different version of the CMKdV equation does not pass the Painlevé test [10]. In this article, I will also argue the properties of the CMKdV equation and the relation between the CMKdV and the NLS equations.

On the other hand, the study of elastic chain model of a large polymer is current [11]. According to recent review of a large polymer [11], statistical mechanics of a polymer model is closely connected with the mathematical science. Due to the complexity, investigation of its properties is not simple in general. However it sometimes can be exactly performed owing to deep symmetry [11]. In fact an exact partition function of elastic chain with the energy functional (1-4) was obtained by Saitô et al. using the path integral [12]. However they paid no attention upon isometry condition as thermal fluctuation of the path integration even though they required isometry condition after all computations; they summed all over configuration space without isometry condition rather than over restricted functional space. It should be noted that the constraint does not commute with such evaluation of the partition function in general.

Thus as another limit, it is of interest to investigate the partition function with the energy (1-4) under the isometry condition. One of purposes of this article is to investigate the partition function of a non-stretching space curve with the energy functional (1-4) as a polymer model.

Furthermore, a space curve in $\mathbb{R}^3$ also interests us from the viewpoint of the string theory [15]. Grinevich and Schmidt investigated closed condition of a space curve obeying the NLS equation because a kind of its complexification becomes a surface with Kähler metric [14]. Thus the problem is associated with the string theory [15]. (However as I mentioned in ref.[3], it should be noted that the elastica absolutely differs from a string in the string theory, even though it influences the theory [15].) Thus although it is not main purpose, another hidden purpose of this article is to investigate the moduli of non-stretching curve in $\mathbb{R}^3$ by taking into the consideration of such relation as a generalization to the surface problem [4,14].

The organization of this article is as follows. In §2, I will evaluate the partition function of non-stretching elastica in $\mathbb{R}^3$. Section 3 gives a discussion of the results.

§2. Partition Function of Non-stretching Elastica in $\mathbb{R}^3$

I will denote by $\mathcal{C}$ a shape of an elastica (a real one-dimensional curve) immersed in three
dimensional space $\mathbb{R}^3$ and by $X(s) = (X^1, X^2, X^3)$ its affine vector,

$$S^1 \ni s \mapsto X(s) \in C \subset \mathbb{R}^3, \quad \partial_s^n X(s + L) = \partial_s^n X(s), \quad (n \in \mathbb{N} + \{0\})$$

(2-1)

where $L$ is the length of the elastica $s$ is a parameter of the curve and $N$ is natural number. I consider a closed polymer in $\mathbb{R}^3$; its center axis is a space curve $C$. Here I will fix the metric of the curve $C$ induced from the natural metric of $\mathbb{R}^3$:

$$ds = \sqrt{dXdX}.$$  (2-2)

As I stated in ref.[3], a reader should not confuse an elastica with a "string" in a string theory; they are absolutely different.

There is the orthonormal system along $C$, $(n_0, n_1, n_2)$ with fixing $n_0$ as the tangent unit vector; $n_0 = \partial_s X$, where $\partial_s := \partial/\partial s$. We make them, first, satisfy the Frenet-Serret relation [16],

$$\partial_s \begin{pmatrix} n_0 \\ n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} n_0 \\ n_1 \\ n_2 \end{pmatrix}.$$  (2-3)

Here $k$ is the curvature, $\tau$ is the Frenet-Serret torsion and they are functions of only $s$. We rotate the orthonormal frame $\text{SO}(2)$ fixing $a_0 := n_0$ so that we obtain $(a_0, a_1, a_2)$ [17-19],

$$\partial_s \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1 & \kappa_2 \\ -\kappa_1 & 0 & 0 \\ -\kappa_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix},$$  (2-4)

where $\kappa_1 := k \cos \theta$, $\kappa_2 := k \sin \theta$ and

$$\theta(s) := \int_{s_0}^s \tau(s')ds'.$$  (2-5)

For convenience, we introduce a complex curvature as

$$\kappa := \kappa_1 + i\kappa_2 = ke^{i\theta}.$$  (2-6)

In this article, I will deal with a non-stretching elastica in $\mathbb{R}^3$ with the energy functional

$$E = \int_0^L ds \ |\kappa|^2,$$  (2-7)

which I will also call Bernoulli-Euler functional [3].

It is worth while noting that in general, there appear other potential terms in the energy functional for a general elastic rod. For example, there might appear elastic torsion term, stretching term and so on. An elastica is usually defined as a curve realized as a stationary point of an energy functional related to an elastic rod, at least, in the meaning of the classical mechanics. Hence the word "elastica" sometimes has ambiguity. Depending upon the potential term, its shape might belong to individual class. Thus reader should not confuse the word "elastica" with another one in another context. In this article, the word of "elastica" is meaning of a curve with the Bernoulli-Euler functional (2-7).
The elastica I deal with here is a model of a polymer which can freely rotate around its center axis but does not stretch and is forced by the potential (2-7). In other words, I assume that the force from the elastic torsion can be negligible but stretching can not. Furthermore, I will neglect the kinetic term of the elastica. Physically speaking, I will consider the polymers in the liquid whose temperature is determined and viscosity is very large. I also suppose that each polymer behaves independently and interaction among them are neglected.

Let the elastica closed and preserve its local infinitesimal length for even thermal fluctuation; it does not stretch. Under the conditions, I will consider this partition function of the elastica given as [3],

\[ Z = \int D\mathbf{X} \exp \left( -\beta \int_0^L ds |\kappa|^2 \right). \]  

(2-8)

Following the calculation scheme which I proposed in refs.[3,4], I will evaluate the partition function (2-8) under the non-stretching condition.

However there is trivial affine symmetry of the centroid and direction of the elastica and the partition function naturally diverges [3]. For an affine transformation (translation and rotation \( g \in \text{SO}(3) \)), \( \mathbf{X}(s) \rightarrow \mathbf{X}_0 + g \mathbf{X}(s) \), (\( \mathbf{X}_0 \) and \( g \) are constants of \( s \)), the curvature \( \kappa \) and the Bernoulli-Euler functional (2-7) does not change; this is a gauge freedom and the energy functional (2-7) has infinitely degenerate states. In the path integral method, I must sum over all possible states, \( Z \) includes the integration over \( \mathbb{R}^3 \) and naturally diverges. As well as the arguments in refs.[3,4], I will regularize it,

\[ Z_{\text{reg}} = \frac{Z}{\text{Vol(Aff)}}, \]  

(2-9)

where \( \text{Vol(Aff)} \) is the volume of the space related to the affine transformation. By this regularization, I can concentrate the classification of shapes of elastica.

Next I will investigate the condition preserving local length even for the thermal fluctuation. I will expand the affine vector around the point which is an extremum point of the Bernoulli-Euler functional (2-7). I will call the point quasi-classical point according to the semi-classical method in path integral [3]. In the path integral, I must pay attention to the higher perturbations of \( \epsilon \) in order to obtain an exact result. Hence I will assume that \( X \) is parameterized by a parameter \( t \). I will express a perturbed affine vector \( \mathbf{X} \) around an extremum point \( \mathbf{X}_{\text{qcl}} \) in the partition function (2-9) as [3,4,7,8],

\[ \mathbf{X}(s, t) := e^{\epsilon \partial_t} \mathbf{X}_{\text{qcl}}(s, t), \quad \epsilon \partial_t \mathbf{X}_{\text{qcl}} = \mathbf{X}_{\text{qcl}} - \mathbf{X} + \mathcal{O}(\epsilon^2). \]  

(2-10)

with the relation

\[ \partial_t \mathbf{X}_{\text{qcl}} = u_0 a_0 + u_1 a_1 + u_2 a_2, \quad u_a(L) = u_a(0), \quad (a = 0, 1, 2), \]  

(2-11)

where \( u_a \)'s are real function of \( s \) and \( t \). I will regard (2-11) as virtual dynamics of the curve describing the thermal fluctuation [3]. As in refs.[3,7,8], due to the isometry condition, I require \( [\partial_t, \partial_s] = 0 \) for \( \mathbf{X} \). Since \( ds_{\text{qcl}} := \sqrt{\partial_s \mathbf{X}_{\text{qcl}} \cdot \partial_s \mathbf{X}_{\text{qcl}}} ds \), the isometry condition exactly preserves, \( ds = ds_{\text{qcl}} \). Here I will note that the deformation (2-10) generally contains non-trivial ones through \( u_a(s) \) and the ”equation of motion” (2-11).

Let us compute the non-stretching condition \( [\partial_t, \partial_s] \mathbf{X}_{\text{qcl}} = 0 \). I will introduce ”velocities” \( (\partial_t \phi_1, \partial_t \phi_2) \) as

\[ \partial_t a_0 \equiv \partial_t \partial_s \mathbf{X}_{\text{qcl}} := (\partial_t \phi_2) a_1 - (\partial_t \phi_1) a_2. \]  

(2-12)
and

\[ \partial_s \partial_t X_{\text{qcl}} = (\partial_s u_0 - \kappa_1 u_1 - \kappa_2 u_2) a_0 + (\partial_s u_1 + \kappa_1 u_0) a_1 + (\partial_s u_2 + \kappa_2 u_0) a_2. \] (2-13)

From the condition, I have the relation between \( \partial_t \phi_c \) (\( \phi_c := \phi_1 + i \phi_2 \)) and a complex "velocity", \( u_c := u_1 + i u_2 \),

\[ \partial_t \phi_c = i(\kappa_{\text{qcl}} u_0 + \partial_s u_c) = i(\kappa_{\text{qcl}} \partial_s^{-1} \text{Re}(\overline{\kappa_{\text{qcl}} u_c}) + \partial_s u_c) =: Q(u_c). \] (2-14)

Here I use the notation \( \kappa_{\text{qcl}} := \kappa_1 + i \kappa_2 \) and I introduce the pseudo-differential operator \( \partial_s \) in the meaning of

\[ \partial_s u_0 = \text{Re}(\overline{\kappa_{\text{qcl}} u_c}) = (\kappa_{\text{qcl}} \overline{u_c} + \overline{\kappa_{\text{qcl}} u_c})/2, \]

\[ u_0 = \partial_s^{-1} \text{Re}(\overline{\kappa_{\text{qcl}} u_c}) = \int_s^t ds' \text{Re}(\overline{\kappa_{\text{qcl}}(s')u_c(s')}). \] (2-15)

In order to find the connection between \( \phi_c \) and \( \kappa \), I will also investigate the fluctuation of \( a_a \) \((a = 1, 2)\). By the virtual dynamics of \( a_0 \), differentiation of \( a_a \) \((a = 1, 2)\) by \( t \) must have the form,

\[ \partial_t a_1 = -\partial_t \phi_2 a_0 - v a_2, \quad \partial_t a_2 = \partial_t \phi_1 a_0 + v a_2, \] (2-16)

where \( v \) means the rotation in the plane spanned by \( a_a \) \((a = 1, 2)\). By requirement of the isometry, the virtual dynamics of \( a_a \) is constrained as \([\partial_t, \partial_s] a_a = 0 \) \((a = 1, 2)\),

\[ -\partial_s \partial_t a_1 = (\partial_s \partial_t \phi_2 - \kappa_2 v) a_0 + (\partial_t \phi_2 \kappa_1) a_1 + (\partial_t \phi_2 \kappa_1 + \partial_s v) a_2, \]

\[ -\partial_s \partial_t a_2 = -(\partial_s \partial_t \phi_1 - \kappa_2 v) a_0 - (\partial_t \phi_1 \kappa_1) a_1 - (\partial_t \phi_1 \kappa_2 + \partial_s v) a_2, \]

\[ -\partial_t \partial_s a_1 = \partial_t \kappa_1 a_0 + (\kappa_1 \partial_t \phi_2) a_1 - (\kappa_1 \partial_t \phi_2) a_2, \]

\[ -\partial_t \partial_s a_2 = \partial_t \kappa_2 a_0 + (\kappa_2 \partial_t \phi_2) a_1 - (\kappa_2 \partial_t \phi_2) a_2. \] (2-17)

Hence I have the relation [7,8],

\[ \partial_t \kappa_{\text{qcl}} = -Q(\partial_t \phi). \] (2-18)

Accordingly I have the relation between \( \partial_t \kappa \) and complex velocity \( u_c \) as the "equation of motion" of the deformation satisfied with the isometry condition [7,8],

\[ \partial_t \kappa_{\text{qcl}} = -Q^2(u_c). \] (2-19)

I will remark that \( Q^2 \) is known as the recursion operator of the NLS and CMKdV equations.

For this non-stretching deformation, the Bernoulli-Euler functional (2-7) changes as

\[
\int |\kappa|^2 ds = \int (|\kappa_{\text{qcl}}|^2 + \epsilon(\overline{\kappa_{\text{qcl}}} \partial_t \kappa_{\text{qcl}} + \kappa_{\text{qcl}} \partial_t \overline{\kappa_{\text{qcl}}}) \\
+ \epsilon^2 (|\partial_t \kappa_{\text{qcl}}|^2 + \overline{\kappa_{\text{qcl}}} \partial_t^2 \kappa_{\text{qcl}} + \kappa_{\text{qcl}} \partial_t^2 \overline{\kappa_{\text{qcl}}}) + \cdots) ds \\
= \int (\kappa_{\text{qcl}}^2 - \epsilon(\overline{\kappa_{\text{qcl}}} Q^2(u_0) + \kappa_{\text{qcl}} Q^2(u_0)) \\
+ \epsilon^2 (|\partial_t \kappa_{\text{qcl}}|^2 + \overline{\kappa_{\text{qcl}}} \partial_t^2 \kappa_{\text{qcl}} + \kappa_{\text{qcl}} \partial_t^2 \overline{\kappa_{\text{qcl}}}) + \cdots) ds \\
=: E_{\text{qcl}} + \delta^{(1)} E_{\text{qcl}} + \delta^{(2)} E_{\text{qcl}} + \cdots.
\] (2-20)
Since I wish to expand the complex curvature $\kappa$ around the extremum point in the functional space, I will require the extremum condition [3],

$$\delta^{(1)} E_{qcl} = 0. \quad (2-21)$$

In this method, I will sum the weight function over all extremum points. Since they are extremum rather than stationary points, they need not be realized in zero temperature.

Noting the relation $\partial_s u_0 = (\kappa_{qcl} u_c + \kappa_{qcl} u_c) / 2$ and above notices, supposed that $\kappa_{qcl} Q^2(u_c) + \kappa_{qcl} Q^2(u_c)$ could be regarded as another function $\kappa_{qcl} u_c + \kappa_{qcl} u_c$ of the variation of the normal direction in (2-15), I might find the relation

$$\int ds Re(\kappa_{qcl} Q^2(u_c)) \sim \int ds Re(\kappa_{qcl} u_c') = \int ds (\partial_s u_0') = 0. \quad (2-22)$$

I supposed that the deformation is described by one parameter $t$. However there is no requirement that I must go along with only one parameter $t$ to characterize this system. In the calculation of the partition function, one must sum up the weight function over events if the possibility of occurrence of the events can be considerable. I will search for all possible extremum points.

Furthermore in a microcanonical system at energy $E_0$, the entropy $S$ of the system is defined as $S := \log Z|_{E=E_0}$ and can be regarded as the logarithm of the volume of the functional space. From primitive consideration, the dimension of the functional space in the statistical physics is related to the degrees of freedom corresponding to $E_0$ and the degrees of freedom of the elastica are not finite and its dimension need not one.

Along the line of the arguments of ref.[3], I will give up to express the thermal fluctuation using only one parameter $t$ and I will introduce the sequence for mathematical times $t := (t_1, t_3, t_5, \ldots, t_{2n+1}, \ldots)$ in this system so that (2-22) is satisfied. I will redefine the fluctuation (2-10) and introduce infinite parameters family, which can sometimes become finite set as I will show later,

$$X_{\delta t} = e^{(1/\sqrt{\beta}) \sum_{n=0}^{\delta t_{2n+1}} t_{2n+1} X_{qcl}}$$

$$= X_{qcl} + (1/\sqrt{\beta}) \sum_{n=0}^{\delta t_{2n+1}} t_{2n+1} X_{qcl} + O(1/\beta), \quad (2-23)$$

where $\epsilon$ was replaced with $(1/\sqrt{\beta}) \delta t_{2n+1}$ and $t_{2n+1}$ $X_{qcl}$ is expressed as

$$\partial_{t_{2n+1}} X_{\epsilon} = u_0^{(n)} a_0 + u_1^{(n)} a_1 + u_2^{(n)} a_2, \quad u_0^{(n)} = \partial_s^{-1} Re(\kappa_{qcl} u_c^{(n)}), \quad u_c^{(n)} = Q^{2n}(u_c^{(0)}). \quad (2-24)$$

The virtual equations of motion for the deformation are expressed as

$$\partial_{t_{2n+1}} \kappa = Q^{2n}(u_c^{(0)}). \quad (2-25)$$

Thus (2-25) represents the thermal fluctuation which conserves the local length.

However it should be noted that there are two manifest symmetries in this system; one exhibits the symmetry of choice of the origin $s$ and another is for the symmetry of $U(1)$ phase of $\kappa$; the later one is the same as the choice of the $s_0$ at the integration (2-5). For the transformation $\kappa(s) \rightarrow e^{it} \kappa(s - t)$, the partition function is invariant.
I require that the virtual motions must include such manifest symmetries

\[ \partial_{\bar{t}_1} \kappa_{\text{qcl}} = \partial_s \kappa_{\text{qcl}} \]  

(2-26)

and

\[ \partial_{t_1} \kappa_{\text{qcl}} = i \kappa_{\text{qcl}}. \]  

(2-27)

As in refs. [3,4], instead of the single deformation parameter, I will assign the infinite dimensional parameters in (2-23) to those which fulfill this requirement; \( \tilde{t} := (t, \bar{t}) = (t_1, t_3, \cdots, \tilde{t}_1, \tilde{t}_3, \cdots) \).

In terms of these, I will investigate the moduli space of the partition function (2-9). In other words I will give a minimal set of the virtual equations of motion, which is satisfied with this physical requirement that the deformation contains the manifest symmetries (2-26) and (2-27),

\[ \partial_{\bar{t}_{2n+1}} \kappa_{\text{qcl}} = (-Q^2)^n (\partial_s \kappa_{\text{qcl}}), \quad \partial_{t_{2n+1}} \kappa_{\text{qcl}} = -Q^2 (\partial_{\bar{t}_{2n-1}} \kappa_{\text{qcl}}), \quad (n = 1, 2, \cdots). \]  

(2-28)

\[ \partial_{t_{2n+1}} \kappa_{\text{qcl}} = (-Q^2)^n (i \kappa_{\text{qcl}}), \quad \partial_{\bar{t}_{2n+1}} \kappa_{\text{qcl}} = -Q^2 (\partial_{t_{2n-1}} \kappa_{\text{qcl}}), \quad (n = 1, 2, \cdots). \]  

(2-29)

They are the CMKdV and the NLS hierarchies respectively.

As stated in the introduction, the properties of the CMKdV equation is not well-known as far as I know. It has not been concluded that it is soliton equation yet. However even though it might not be integrable, properties of the CMKdV hierarchy and the CMKdV equation are very regular as I show as follows.

As the NLS hierarchy, a solution of the \( n \)-th CMKdV equation,

\[ \partial_{\bar{t}_{2n+1}} \kappa_{\text{qcl}} - (-Q^2)^n (\partial_s \kappa_{\text{qcl}}) = 0, \]  

(2-30)

is satisfied with the simultaneous equations by introducing unknown parameter \( \bar{t}_{2n-1} \),

\[ \partial_{t_{2n+1}} \kappa_{\text{qcl}} - Q^2 (\partial_{\bar{t}_{2n-1}} \kappa_{\text{qcl}}) = 0, \quad \partial_{\bar{t}_{2n-1}} \kappa_{\text{qcl}} = (-Q^2)^{n-1} (\partial_s \kappa_{\text{qcl}}). \]  

(2-31)

This is a kind of Bäcklund transformation. Thus by ladder calculations, it can be proved that the solutions of the higher order equations belonging to the CMKdV hierarchy are also satisfied with the CMKdV equation,

\[ \partial_t \kappa + \frac{3}{2} |\kappa|^2 \partial_s \kappa + \partial_s^3 \kappa = 0. \]  

(2-32)

In other words, the nontrivial deformation obeys the CMKdV equation as the soliton hierarchy does. (One might have a question why the ladder relation terminates at \( \bar{t} = \tilde{t}_3 \) rather than \( \bar{t} = \tilde{t}_1 \). From (2-26) \( \tilde{t}_1 \) is determined as \( \tilde{t}_1 \equiv s + s_0 \) and thus \( \tilde{t}_1 \) is not an unknown parameter in the sense of (2-31). Thus (2-32) is a minimal non-trivial equation.) Further it is worth while noting that for intrinsically real initial condition

\[ \kappa \in e^{i\alpha_0} \mathbb{R} \] for all \( s \) and constant \( \alpha_0 \) for \( s \) and \( t \), the CMKdV equation is reduced to the MKdV equation. Thus the solution space of the CMKdV equation has the completely integrable region in the meaning of the soliton theory.

Similarly, I have the NLS equation as nontrivial deformation of the NLS hierarchy,

\[ i \partial_t \kappa + \frac{1}{2} |\kappa|^2 \kappa + \partial_s^2 \kappa = 0. \]  

(2-33)
For the NLS equation, this reduction can be naturally justified in the Jacobi variety of the hyperelliptic curve as a solution space [20].

Furthermore it is a very remarkable fact that for the variation of \(\bar{t}\) obeying the CMKdV equation, the Bernoulli-Euler functional (2-7) is invariant,

\[-\partial_\bar{t} \int ds |\kappa(s, t, \bar{t})|^2 = \int ds \partial_s \left( \frac{3}{4} |\kappa|^4 + (\partial_s^2 \bar{\kappa}) \kappa + \bar{\kappa} \partial_s^2 \kappa - |\partial_s \kappa|^2 \right) = 0, \quad (2-34)\]
as the NLS flows conserves the first integral,

\[\partial_t \int ds |\kappa(s, t, \bar{t})|^2 = -i \int ds \partial_s ((\partial_s \bar{\kappa}) \kappa - \bar{\kappa} \partial_s \kappa) = 0. \quad (2-35)\]

Thus regardless of integrability of the CMKdV equation, it was clarified that its properties are very regular.

Here I will comment upon the result of Mohammad and Can [10]. They investigated the “complex MKdV” equation and concluded that it is not a soliton equation. However their “complex MKdV equation” is expressed as

\[\partial_t \kappa + \frac{1}{2} \partial_s (|\kappa|^2 \kappa) + \partial_s^2 \kappa = 0, \quad (2-36)\]

which is a kind of “complexification” of the MKdV equation but differs from (2-32). Thus their result does not directly affect the studies on the integrability of our CMKdV equation (2-32).

As I obtained the isometry deformation which includes the manifest symmetries (2-26) and (2-27), I will consider, here, the partition function (2-9).

Since the CMKdV and NLS problems are initial value problems, for any regular shape of elastica satisfied with the boundary conditions, the ”time” \(t\) and \(\bar{t}\) developments of the curvature are uniquely determined. Furthermore noting that if one gives the real value \(\kappa\), \(\kappa\) goes on real in the ”time” \(\bar{t}\) development of the CMKdV equation (2-32) whereas for the NLS equation (2-33) its ”time” \(t\) development includes the complex value due to the pure imaginary in the first term in (2-33). Thus the ”time” \(dt\) and \(d\bar{t}\) are expected orthogonal in the moduli of the CMKdV and NLS equations. The ”time” developments of both equations differ each other. In other words, for a given regular curve, there exist individual families of the solutions of the CMKdV (2-32) and NLS (2-33) equations which contain the given curve as an initial condition. Due to relations (2-34) and (2-35), during the motion of \(t\) and \(\bar{t}\), the Bernoulli-Euler functional (2-7) does not change its value. Hence the deformation parameter \(t\) and \(\bar{t}\) draw the trajectories of the functional space which have the same value of the Bernoulli-Euler functional (2-7).

In the case that I immersed an elastica in \(\mathbb{R}^2\), the thermal fluctuation obeys the MKdV equation and there appears only one sort of hierarchy or the MKdV hierarchy [3]. In this article, the codimension of the immersion of the elastica in \(\mathbb{R}^3\) is two while the former problem is one [3]. Accordingly it is natural that there appear twice degrees of freedom of the elastica in \(\mathbb{R}^2\), \(t\) and \(\bar{t}\) for the elastica in \(\mathbb{R}^3\).

Thus I can formally estimate the functional space for each functional value. By the ”time” development of \(t_3\) and \(\bar{t}_3\), I can classify the functional space of the partition function (2-9). In other words by investigating the moduli of the CMKdV and NLS equations which are satisfied with the boundary conditions,

\[\kappa(0) = \kappa(L), \quad X_{qcl}(0) = X_{qcl}(L), \quad (2-37)\]
the measure of the functional integral $d\mu$ can be decomposed as,

$$d\mu = \sum_E d\mu_E.$$  \hfill (2-38)

So I denote by $\Xi_E$ the set of these trajectories which occupy the same energy $E$.

Hence the partition function can be represented as

$$Z_{\text{reg}} = \int d\mu \exp(-\beta E) = \sum_E \exp(-\beta E) \int_{\Xi_E} d\mu_E = \sum_E \exp(-\beta E) \text{Vol}(\Xi_E),$$

where

$$\text{Vol}(\Xi_E) = \int_{\Xi_E} d\mu_E$$

is the volume of the trajectories $\Xi_E$.

It is known that any solutions of the NLS equation (2-33) can be also expressed by the hyperelliptic function and its modulus agrees with the modulus of the hyperelliptic curves [13,20]. Grinevich and Schmidt studied the moduli of the NLS equation (2-33) whose corresponding space curve is satisfied with the boundary condition (2-37). For the NLS equation (2-33), there are infinite Jacobi varieties who have the same energy $E$ in general. Thus it is expected that the CMKdV equation is connected among these Jacobi varieties induced from the NLS equation (2-33).

As well as the arguments in ref.[3], even though I introduced the infinite dimensional coordinates $t$ in (2-23), they are reduced to finite dimensional space, as the Jacobi variety of a hyperelliptic curve with finite dimension is embedded in the universal grassmannian manifold. Using the genus $g$ of the hyperelliptic curves, I will evaluate the subspace and submesure of $(\Xi_E, d\mu_E)$ as NLS part. The NLS part $(\Xi_{E}^{\text{NLS}}, d\mu_{E}^{\text{NLS}})$ can be decomposed as $(\Xi_{E}^{\text{NLS}}, d\mu_{E}^{\text{NLS}}) = \prod_g (\Xi_{E}^{\text{NLS}(g)}, d\mu_{E}^{\text{NLS}(g)})$.

For the case of a solution represented by the hyperelliptic function of genus $g$ which is satisfied with (2-37), $d\mu_{E}^{\text{NLS}(g)}$ is expressed as $dt_3 \wedge dt_5 \wedge \cdots \wedge dt_{2g-1}$. For each point, there is CMKdV flows. Even though it has not been confirmed that the trajectories of the CMKdV equation are linear and regarded as the vector space, it is clear that their cotangent space is flat and can be regarded as the vector space locally. Thus I can locally express the measure of $d\mu_{E}^{(g)}$ as

$$d\mu_{E}^{(g)} = dt_3 \wedge dt_5 \wedge \cdots \wedge dt_{2g-1} \wedge \tilde{t}_3 \wedge \tilde{t}_5 \wedge \cdots \wedge \tilde{t}_{2g-1}.$$ \hfill (2-41)

Here I remove $dt_1 \wedge \tilde{t}_1$ in the measure because it exhibits trivial symmetries [3]. (2-41) is a subset of the infinite dimensional deformation parameters $t$ in (2-23). Hence (2-40) becomes

$$\text{Vol}(\Xi_E) = \sum_g \text{Vol}(\Xi_E^{(g)}) = \sum_g \int_{\Xi_E^{(g)}} d\mu_{E}^{(g)}. \hfill (2-42)$$

By exchanging the coordinate $dt_i$ and $d\tilde{t}_j$ of multi-times $t$, the volume of $\Xi_E^{(g)}$ is estimated by the unit of the elastica length $L$. Since the dimension of the Bernoulli-Euler functional $E$ is the inverse of length and $\beta/|\text{length}|$ is order unit, the multiple of the length can be interpreted as the multiple of the inverse temperature $\beta^{-1}$. Hence the sum of terms with different dimensional volume which appear in (2-39) can be regarded as expansion of power of $\beta$. 

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§3. Discussion

In this article, I gave a calculation scheme of the partition function of elastics in $\mathbb{R}^3$ in terms of solutions of the CMKdV equation (2-32) and NLS equation (2-33). Even though I could not give a concrete form of the partition function (2-9), I showed that its formal expansion is given by (2-39). As I thought that this scheme is based upon the soliton theory in refs.[3,4], I can not deny that it might be beyond the integrable system. In fact the CMKdV equation might be connected with the deformation of the Jacobi variety induced by the NLS equation (2-33). Hence I believe that this formulation might shed a new light upon the theories of the immersed object and its quantization (or evaluation of the partition function).

Here I will mention the knot configuration. Since the NLS and CMKdV equations are initial value problems, the solution space includes any configurations of a space curve in $\mathbb{R}^3$. In other words, they also include any knot configurations and so I need not pay any attention upon the ambient isotopy [21]. In fact the trajectories of NLS equation classify space curves immersed in $\mathbb{R}^3$ rather than ones embedded in $\mathbb{R}^3$; crossings are allowed and its topology disables us to distinguish such knot invariances or ambient isotopy. Since the knot configuration is physically discriminated by means of long range force such as the electromagnetic force and this theory in this article does not include such force, this notion can be physically interpreted. If one wishes to consider the knot configuration in this system, it might be related to the gauged NLS equation [22].

Next I will give two comments on the CMKdV equation. First, one might have a question why I need the CMKdV equation whereas the solution space of the NLS equation includes any configurations of a space curve in $\mathbb{R}^3$. I have been dealing with the measure of the functional space. An uncountable set of $\mathbb{R}$ becomes $\mathbb{R}^2$ if the elements are measurable and one can define $\mathbb{R}^2$ topology in the set. In the similar meaning, I need CMKdV equation in order to introduce the natural measure in the functional space.

The solutions of the NLS equation are described in terms of the hyperelliptic functions [13,20]. A hyperelliptic curve is embedded in a Jacobi variety. The trajectories of the NLS equation, $(t_1, t_3, \ldots, t_{2g-1})$, form the vector structure of the Jacobi variety. The NLS flow, which obeys the NLS equation (2-33), covers a subset of Jacobi variety. In each Jacobi variety, there exists compact subset as orbits of NLS flows. The individual Jacobi varieties are distinguished by points in the Siegel upper half space [20]. Since the CMKdV flows are perpendicular with the NLSE flows, the CMKdV flows might connect the different Jacobi varieties of solutions of the NLS equation. Thus I will conjecture, as the second comment upon the CMKdV equation, that the moduli of the CMKdV equation might be realized in the Siegel upper half space. It reminds me of the facts the theta function of the elliptic curve obeys the heat equation over the Siegel upper half plane, which is not integrable in the sense of the kinematic theory such as the soliton theory. The integrability of the soliton theory is associated with the time inversion symmetry and time translational symmetry, and the solutions are acted by a (continuous) group. On the other hand, the solution space of the heat equation is acted by only semi-group and thus it is not "integrable" in general. By complexification of the heat equation, imaginary time heat equation, or Schrödinger equation is kinematic equation and integrable in the sense of the kinematic theory. Thus even though the CMKdV equation includes integrable solutions as kinematic region [23], I have a question regarding role of the CMKdV equation in Jacobi varieties of the hyperelliptic curves; is it in the framework of the integrable system? However in this stage, I cannot explicitly express the role of the CMKdV equation because there are few studies on the CMKdV equation.
I state that the properties of the CMKdV should be investigated.

Finally I will comment upon the higher dimensional elastica problem, e.g., an elastica in n-dimensional space $C \subset \mathbb{R}^n$. The codimension of the elastica becomes $n-1$ and thus instead of $t = (t, \bar{t})$, there appear $(n-1)$ sets of infinite dimensional parameters $t = (t^{(1)}, t^{(2)}, \ldots, t^{(n-1)})$. As there appeared $U(1)$-bundle in this article, they represent the $(n-2)$-dimensional inner sphere of sphere bundle over the elastica $C$ and the normal radius direction of $C$. Thus there is naturally a principal bundle over $C$. In other words, one can add the group structure over the equations. Thus the generalized MKdV equation naturally appears [8,24] and it is expected that my computation scheme of the partition function can be extended.

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