INEQUALITIES VIA $\varphi_{h,m}$-CONVEXITY

M.E. ÖZDEMIR ♡ AND MERVE AVCİ ♡ ♠

Abstract. In this paper, we define $\varphi_{h,m}$-convex functions and prove some inequalities for this class.

1. INTRODUCTION

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval $I$ of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2} \tag{1.1}$$

is known as Hermite-Hadamard’s inequality for convex functions, [2].

In [1], Toader defined $m$-convexity as the following.

**Definition 1.** The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be $m$-convex where $m \in [0,1]$, if we have

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0,1]$. We say that $f$ is $m$-concave if $(-f)$ is $m$-convex.

In [4], Varošanec defined the following class of functions. $I$ and $J$ are intervals in $\mathbb{R}$, $(0,1) \subseteq J$ and functions $h$ and $f$ are real non-negative functions defined on $J$ and $I$, respectively.

**Definition 2.** Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $f : I \to \mathbb{R}$ is an $h$-convex function, or that $f$ belongs to the class $SX(h, I)$, if $f$ is non-negative and for all $x, y \in I, \alpha \in (0,1)$ we have

$$f (\alpha x + (1-\alpha) y) \leq h(\alpha)f(x) + h(1-\alpha)f(y) \tag{1.2}$$

If inequality (1.2) is reversed, then $f$ is said to be $h$-concave, i.e. $f \in SV(h, I)$.

In [5], Sarıkaya et al. proved a variant of Hadamard inequality which holds for $h$-convex functions.

**Theorem 1.** Let $f \in SX(h, I)$, $a, b \in I$, with $a < b$ and $f \in L_1 ([a, b])$. Then

$$\frac{1}{2h \left( \frac{1}{2} \right)} f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq [f(a) + f(b)] \int_0^1 h(\alpha) \, d\alpha. \tag{1.3}$$

In [3], Özdemir et al. defined $(h, m)$-convexity and obtained Hermite-Hadamard-type inequalities as following.

*Corresponding Author.*
Definition 3. Let \( h : J \subset \mathbb{R} \to \mathbb{R} \) be a non-negative function. We say that \( f : [0, b] \to \mathbb{R} \) is a \((h, m)\)–convex function, if \( f \) is non-negative and for all \( x, y \in [0, b] \), \( m \in [0, 1] \) and \( \alpha \in (0, 1) \), we have

\[
f(\alpha x + m(1 - \alpha)y) \leq h(\alpha)f(x) + mh(1 - \alpha)f(y).
\]

If the inequality is reversed, then \( f \) is said to be \((h, m)\)–concave function on \([0, b]\).

Theorem 2. Let \( f : [0, \infty) \to \mathbb{R} \) be an \((h, m)\)–convex function with \( m \in (0, 1] \), \( t \in [0, 1] \). If \( 0 \leq a < b < \infty \) and \( f \in L_1 [a, b] \), then the following inequality holds:

\[
\frac{1}{m + 1} \left[ \frac{1}{mb - a} \int_a^b f(x) \, dx + \frac{1}{b - ma} \int_{ma}^b f(x) \, dx \right] \leq [f(a) + f(b)] \int_0^1 h(t) \, dt.
\]

Let us consider a function \( \varphi : [a, b] \to [a, b] \) where \([a, b] \subset \mathbb{R}\). In [7], Youness defined the \( \varphi \)–convex functions as the following:

Definition 4. A function \( f : [a, b] \to \mathbb{R} \) is said to be \( \varphi \)–convex on \([a, b]\) if for every two points \( x \in [a, b], \ y \in [a, b] \) and \( t \in [0, 1] \), the following inequality holds:

\[
f(t\varphi(x) + (1 - t)\varphi(y)) \leq tf(\varphi(x)) + (1 - t)f(\varphi(y)).
\]

In [6], M.Z. Sarıkaya defined \( \varphi_h \)–convex functions and obtained the following inequalities for this class.

Definition 5. Let \( I \) be an interval in \( \mathbb{R} \) and \( h : (0, 1) \to (0, \infty) \) be a given function. We say that a function \( f : I \to [0, \infty) \) is \( \varphi_h \)–convex if

\[
(1.4) \quad f(t\varphi(x) + (1 - t)\varphi(y)) \leq h(t)f(\varphi(x)) + h(1 - t)f(\varphi(y))
\]

for all \( x, y \in I \) and \( t \in (0, 1) \). If inequality (1.4) is reversed, then \( f \) is said to be \( \varphi_h \)–concave.

Theorem 3. Let \( h : (0, 1) \to (0, \infty) \) be a given function. If \( f : I \to [0, \infty) \) is Lebesgue integrable and \( \varphi_h \)–convex for continuous function \( \varphi : [a, b] \to [a, b] \), then the following inequality holds:

\[
\frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x) f(\varphi(a) + \varphi(b) - x) \, dx \leq \left[ f'^2(\varphi(x)) + f'^2(\varphi(y)) \right] \int_0^1 h(t)h(1 - t) \, dt + 2f(\varphi(x)) f(\varphi(y)) \int_0^1 h^2(t) \, dt.
\]

Theorem 4. Let \( h : (0, 1) \to (0, \infty) \) be a given function. If \( f, g : I \to [0, \infty) \) is Lebesgue integrable and \( \varphi_h \)–convex for continuous function \( \varphi : [a, b] \to [a, b] \), then the following inequality holds:

\[
\frac{1}{\varphi(b) - \varphi(a)} - \int_{\varphi(a)}^{\varphi(b)} f(x) g(x) \, dx \leq M(a, b) \int_0^1 h^2(t) \, dt + N(a, b) \int_0^1 h(t)h(1 - t) \, dt
\]

where

\[
M(a, b) = f(\varphi(x)) g(\varphi(x)) + f(\varphi(y)) g(\varphi(y))
\]
and
\[ N(a, b) = f(\varphi(x))g(\varphi(y)) + f(\varphi(y))g(\varphi(x)). \]

The aim of this paper is to define a new class of convex function and then establish new Hermite-Hadamard-type inequalities.

2. MAIN RESULTS

In the beginning we give a new definition $\varphi_{h,m}$-convex function.

$I$ and $J$ are intervals on $\mathbb{R}$, $(0, 1) \subseteq J$ and functions $h$ and $f$ are real non-negative functions defined on $J$ and $I$, respectively.

**Definition 6.** Let $h : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $f : [0, b] \subseteq [0, \infty) \rightarrow \mathbb{R}$ is a $\varphi_{h,m}$-convex function, if $f$ is non-negative and satisfies the inequality
\[(2.1) \quad f(t\varphi(x) + m(1-t)\varphi(y)) \leq h(t)f(\varphi(x)) + mh(1-t)f(\varphi(y))\]
for all $x, y \in [0, b]$, $t \in (0, 1)$.

If the inequality (2.1) is reversed, then $f$ is said to be $\varphi_{h,m}$-concave function on $[0, b]$.

Obviously, if we choose $h(t) = t$ and $m = 1$ we have non-negative $\varphi$-convex functions. If we choose $m = 1$, then we have $\varphi$-convex functions. If we choose $m = 1$ and $\varphi(x) = x$ the two definitions $\varphi_{h,m}$-convex and $h$-convex functions become identical.

The following results were obtained for $\varphi_{h,m}$-convex functions.

**Proposition 1.** If $f, g$ are $\varphi_{h,m}$-convex functions and $\lambda > 0$, then $f + g$ and $\lambda f$ are $\varphi_{h,m}$-convex functions.

**Proof.** From the definition of $\varphi_{h,m}$-convex functions we can write
\[ f(t\varphi(x) + m(1-t)\varphi(y)) \leq h(t)f(\varphi(x)) + mh(1-t)f(\varphi(y)) \]
and
\[ g(t\varphi(x) + m(1-t)\varphi(y)) \leq h(t)g(\varphi(x)) + mh(1-t)g(\varphi(y)) \]
for all $x, y \in [0, b]$, $m \in (0, 1]$ and $t \in [0, 1]$. If we add the above inequalities we get
\[ (f + g)(t\varphi(x) + m(1-t)\varphi(y)) \leq h(t)(f + g)(\varphi(x)) + mh(1-t)(f + g)(\varphi(y)). \]

And also we have
\[ \lambda f(t\varphi(x) + m(1-t)\varphi(y)) \leq h(t)\lambda f(\varphi(x)) + mh(1-t)\lambda f(\varphi(y)) \]
which completes the proof.

**Proposition 2.** Let $h_1, h_2 : (0, 1) \rightarrow (0, \infty)$ be functions such that $h_2(t) \leq h_1(t)$ for all $t \in (0, 1)$. If $f$ is $\varphi_{h_2,m}$-convex on $[0, b]$, then for all $x, y \in [0, b]$ $f$ is $\varphi_{h_1,m}$-convex on $[0, b]$.

**Proof.** Since $f$ is $\varphi_{h_2,m}$-convex on $[0, b]$, for all $x, y \in [0, b]$ and $t \in (0, 1)$, we have
\[ f(t\varphi(x) + m(1-t)\varphi(y)) \leq h_2(t)f(\varphi(x)) + mh_2(1-t)f(\varphi(y)) \leq h_1(t)f(\varphi(x)) + mh_1(1-t)f(\varphi(y)) \]
which completes the proof.
Theorem 5. Let \( f \) be \( \varphi_{h,m} \)-convex function. Then i) if \( \varphi \) is linear, then \( f \circ \varphi \) is \( (h-m) \)-convex and ii) if \( f \) is increasing and \( \varphi \) is \( m \)-convex, then \( f \circ \varphi \) is \( (h-m) \)-convex.

Proof. i) From \( \varphi_{h,m} \)-convexity of \( f \) and linearity of \( \varphi \), we have
\[
  f \circ \varphi [tx + m(1-t)y] = f [\varphi (tx + m(1-t)y)]
\]
\[
= f [t \varphi(x) + m(1-t) \varphi(y)]
\]
\[
\leq h(t) f \circ \varphi(x) + mh (1-t) f \circ \varphi(y)
\]
which completes the proof for first case.

ii) From \( m \)-convexity of \( \varphi \), we have
\[
\varphi [tx + m(1-t)y] \leq t \varphi(x) + m(1-t) \varphi(y).
\]
Since \( f \) is increasing we can write
\[
f \circ \varphi [tx + m(1-t)y] \leq f [t \varphi(x) + m(1-t) \varphi(y)]
\]
\[
\leq h(t) f \circ \varphi(x) + mh (1-t) f \circ \varphi(y).
\]
This completes the proof for this case. \( \square \)

Theorem 6. Let \( h : J \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a non-negative function, \( h \neq 0 \) and \( f : [0, b] \subseteq [0, \infty) \rightarrow \mathbb{R} \) be an \( \varphi_{h,m} \)-convex function with \( m \in (0, 1] \) and \( t \in (0, 1) \). Then for all \( x, y \in [0, b] \), the function \( g : [0, 1] \rightarrow \mathbb{R} \), \( g(t) = f(t \varphi(x) + m(1-t) \varphi(y)) \) is \( (h-m) \)-convex on \([0, b] \).

Proof. Since \( f \) is \( \varphi_{h,m} \)-convex function, for \( x, y \in [0, b] \), \( \lambda_1, \lambda_2 \in (0, 1) \) with \( \lambda_1 + \lambda_2 = 1 \) and \( t_1, t_2 \in (0, 1) \) we obtain
\[
g(\lambda_1 t_1 + m \lambda_2 t_2)\]
\[
= f [(\lambda_1 t_1 + m \lambda_2 t_2) \varphi(x) + m (1 - \lambda_1 t_1 - m \lambda_2 t_2) \varphi(y)]
\]
\[
= f [\lambda_1 (t_1 \varphi(x) + m (1 - t_1) \varphi(y)) + m \lambda_2 (t_2 \varphi(x) + m (1 - t_2) \varphi(y))]
\]
\[
\leq h (\lambda_1) f (t_1 \varphi(x) + m (1 - t_1) \varphi(y)) + mh (\lambda_2) f (t_2 \varphi(x) + m (1 - t_2) \varphi(y))
\]
\[
= h (\lambda_1) g (t_1) + mh (\lambda_2) g (t_2)
\]
which shows the \( (h-m) \)-convexity of \( g \). \( \square \)

Theorem 7. Let \( h : J \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a non-negative function, \( h \neq 0 \) and \( f : [0, b] \subseteq [0, \infty) \rightarrow \mathbb{R} \) be a \( \varphi_{h,m} \)-convex function with \( m \in (0, 1] \) and \( t \in (0, 1) \). If \( f \in L_1 [\varphi(a), m \varphi(b)], h \in L_1 [0,1], \) one has the following inequality:
\[
\frac{1}{m \varphi(y) - \varphi(x)} \int_{\varphi(x)}^{m \varphi(y)} f(u) f(\varphi(x) + m \varphi(y) - u) du
\]
\[
\leq f^2 (\varphi(x)) + m^2 f^2 (\varphi(y)) \int_0^1 h(t) h (1-t) dt + f (\varphi(x)) f (\varphi(y)) [m + 1] \int_0^1 h^2 (t) dt
\]
Proof. Since \( f \) is \( \varphi_{h,m} \)-convex function, \( t \in [0, 1] \) and \( m \in (0, 1) \), then
\[
f (t \varphi(x) + m(1-t) \varphi(y)) \leq h(t) f (\varphi(x)) + mh (1-t) f (\varphi(y))
\]
and
\[
f ((1-t) \varphi(x) + m t \varphi(y)) \leq h(1-t) f (\varphi(x)) + mh(t) f (\varphi(y))
\]
for all \( x, y \in [0, b] \).
If we change the variable $u = t\varphi(x) + m(1-t)\varphi(y)$, we obtain

\[\int_0^1 f(t\varphi(x) + m(1-t)\varphi(y)) f((1-t)\varphi(x) + mt\varphi(y)) dt \leq f^2(\varphi(x)) \int_0^1 h(t) h(1-t) dt + mf(\varphi(x)) f(\varphi(y)) \int_0^1 h^2(t) dt + m f(\varphi(x)) f(\varphi(y)) \int_0^1 h(t) h(1-t) dt + m^2 f^2(\varphi(y)) \int_0^1 h(t) h(1-t) dt\]

If we multiply these inequalities and integrate on $[0, 1]$ with respect to $t$, we obtain

\[\int_0^1 \left[ f^2(\varphi(x)) + m^2 f^2(\varphi(y)) \right] h(t) h(1-t) dt + f(\varphi(x)) f(\varphi(y)) [m + 1] \int_0^1 h^2(t) dt.\]

Remark 1. In Theorem 7, if we choose $m = 1$ Theorem 7 reduces to Theorem 5.

Theorem 8. Under the assumptions of Theorem 7 we have the following inequality

\[\frac{1}{m\varphi(y) - \varphi(x)} \int_{\varphi(x)}^{m\varphi(y)} f(u) du \leq [f(\varphi(x)) + f(\varphi(y))] \int_0^1 h(t) dt.\]

Proof. By definition of $\varphi_{h,m}$-convex function we can write

\[f(t\varphi(x) + m(1-t)\varphi(y)) \leq h(t)f(\varphi(x)) + mh(1-t)f(\varphi(y)).\]

If we integrate the above inequality on $[0, 1]$ with respect to $t$ and change the variable $u = t\varphi(x) + m(1-t)\varphi(y)$, we obtained the required inequality.

Remark 2. In Theorem 8, if we choose $m = 1$ and $\varphi : [a, b] \to [a, b], \varphi(x) = x$, we obtained the inequality which is the right hand side of (1.3).

Theorem 9. Under the assumptions of Theorem 8 we have the following inequality

\[\frac{1}{m + 1} \left[ \frac{1}{\varphi(y) - m\varphi(x)} \int_{m\varphi(x)}^{\varphi(y)} f(u) du + \frac{1}{m\varphi(y) - \varphi(x)} \int_{\varphi(x)}^{m\varphi(y)} f(u) du \right] \leq [f(\varphi(x)) + f(\varphi(y))] \int_0^1 h(t) dt\]

for all $0 \leq m\varphi(x) \leq \varphi(x) \leq m\varphi(y) < \varphi(y) < \infty$.

Proof. Since $f$ is $\varphi_{h,m}$-convex function, we can write

\[f(t\varphi(x) + m(1-t)\varphi(y)) \leq h(t)f(\varphi(x)) + mh(1-t)f(\varphi(y)),\]

\[f((1-t)\varphi(x) + mt\varphi(y)) \leq h(1-t)f(\varphi(x)) + mh(t)f(\varphi(y)),\]

\[f(t\varphi(y) + m(1-t)\varphi(x)) \leq h(t)f(\varphi(y)) + mh(1-t)f(\varphi(x)),\]

and

\[f((1-t)\varphi(y) + mt\varphi(x)) \leq h(1-t)f(\varphi(y)) + mh(t)f(\varphi(x)).\]
By summing these inequalities and integrating on \([0,1]\) with respect to \(t\), we obtain
\[
\int_0^1 f(t\varphi(x) + m(1-t)\varphi(y))dt + \int_0^1 f((1-t)\varphi(x) + mt\varphi(y))dt \\
+ \int_0^1 f(t\varphi(y) + m(1-t)\varphi(x))dt + \int_0^1 f((1-t)\varphi(y) + mt\varphi(x))dt \\
\leq \left[ f(\varphi(x)) + f(\varphi(y)) \right] (m + 1) \left[ \int_0^1 h(t)dt + \int_0^1 h(1-t)dt \right].
\]

It is easy to see that
\[
\int_0^1 f(t\varphi(x) + m(1-t)\varphi(y))dt = \int_0^1 f((1-t)\varphi(y) + mt\varphi(x))dt = \frac{1}{m\varphi(y) - \varphi(x)} \int_{\varphi(x)}^{\varphi(y)} f(u)du,
\]
\[
\int_0^1 f(t\varphi(y) + m(1-t)\varphi(x))dt = \int_0^1 f((1-t)\varphi(x) + mt\varphi(y))dt = \frac{1}{\varphi(y) - m\varphi(x)} \int_{m\varphi(x)}^{\varphi(y)} f(u)du
\]
and
\[
\int_0^1 h(t)dt = \int_0^1 h(1-t)dt.
\]
If we write these equalities in the above inequality we obtain the required result. 

**Remark 3.** In Theorem 9 if we choose \(\varphi : [a, b] \to [a, b], \varphi(x) = x\) Theorem 9 reduces to Theorem 2.

**Theorem 10.** Let \(h : J \subseteq \mathbb{R} \to \mathbb{R}\) be a non-negative function, \(h \neq 0\) and \(f, g : [0, b] \subseteq [0, \infty) \to \mathbb{R}\) be \(\varphi_{h,m}\)-convex functions with \(m \in (0,1]\). If \(f\) and \(g\) are Lebesgue integrable, the following inequality holds:
\[
\int_{\varphi(x)}^{\varphi(y)} f(u)g(u)du \\
\leq M(a, b) \int_0^1 h^2(t)dt + mN(a, b) \int_0^1 h(t)h(1-t)dt
\]
where
\[
M(a, b) = f(\varphi(x))g(\varphi(x)) + m^2 f(\varphi(y))g(\varphi(y))
\]
and
\[
N(a, b) = f(\varphi(x))g(\varphi(y)) + f(\varphi(y))g(\varphi(x)).
\]

**Proof.** Since \(f\) and \(g\) are \(\varphi_{h,m}\)-convex functions, we can write
\[
f(t\varphi(x) + m(1-t)\varphi(y)) \leq h(t)f(\varphi(x)) + mh(1-t)f(\varphi(y))
\]
and
\[
g(t\varphi(x) + m(1-t)\varphi(y)) \leq h(t)g(\varphi(x)) + mh(1-t)g(\varphi(y)).
\]
If we multiply the above inequalities and integrate on $[0,1]$ with respect to $t$, we obtain
\[
\int_0^1 f(t\varphi(x) + m(1-t)\varphi(y))g(t\varphi(x) + m(1-t)\varphi(y))dt 
\leq f(\varphi(x))g(\varphi(x))\int_0^1 h^2(t)dt + m^2 f(\varphi(y))g(\varphi(y))\int_0^1 h^2(1-t)dt 
\]
\[
+ m[f(\varphi(x))g(\varphi(y)) + f(\varphi(y))g(\varphi(x))]\int_0^1 h(t)h(1-t)dt.
\]
If we change the variable $u = t\varphi(x) + m(1-t)\varphi(y)$, we obtain the inequality which is the required. □

**Remark 4.** In Theorem 11, if we choose $m = 1$ Theorem 11 reduces to Theorem 4.

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ATATURK UNIVERSITY, K. K. EDUCATION FACULTY, DEPARTMENT OF MATHEMATICS, 25240, ERZURUM, TURKEY

*E-mail address: emos@atauni.edu.tr*

ADIYAMAN UNIVERSITY, DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND ARTS, 02040, ADIYAMAN, TURKEY

*E-mail address: mavci@posta.adiyaman.edu.tr*