A CLASSIFICATION OF COMPLETE 3-DIMENSIONAL SELF-SHRINKERS IN EUCLIDEAN SPACE $\mathbb{R}^4$

QING-MING CHENG, ZHI LI AND GUOXIN WEI

Abstract. In this paper, we completely classify 3-dimensional complete self-shrinkers with constant norm $S$ of the second fundamental form and constant $f_3$ in Euclidean space $\mathbb{R}^4$, where $h_{ij}$ are components of the second fundamental form, $S = \sum_i h_{ij}^2$ and $f_3 = \sum_{i,j,k} h_{ij} h_{jk} h_{ki}$.

1. INTRODUCTION

It is well-known that mean curvature flow has been used to model various things in material sciences and physics such as cell, bubble growth and so on. Hence, study on the mean curvature flow is a very important subject in the differential geometry. One of the most important problems in the mean curvature flow is to understand the possible singularities that the flow goes through. In order to describe singularities of the mean curvature flow, self-shrinkers of the mean curvature flow play a very key role. A hypersurface $X : M^n \to \mathbb{R}^{n+1}$ of Euclidean space $\mathbb{R}^{n+1}$ is called a self-shrinker of the mean curvature flow if it satisfies

\begin{equation}
H + \langle X, e_{n+1} \rangle = 0,
\end{equation}

where $e_{n+1}$ and $H$ denote the normal vector and the mean curvature of $X : M^n \to \mathbb{R}^{n+1}$, respectively.

For classifications of complete self-shrinkers, many nice works have been done. Abresch and Langer [11] classified closed self-shrinker curves completely. These curves are so-called Abresch-Langer curves. In [18], Huisken proved that sphere $S^n(\sqrt{n})$ is the only $n$-dimensional compact self-shrinkers in $\mathbb{R}^{n+1}$ with non-negative mean curvature. Furthermore, in [19] and [14], Huisken and Colding and Minicozzi completely classified complete self-shrinkers with non-negative mean curvature and polynomial volume growth. According to the results of Halldorsson [17], Ding and Xin [15], Cheng and Zhou [13], one knows that $\gamma \times \mathbb{R}^{n-1}$ is a complete self-shrinker without polynomial volume growth in $\mathbb{R}^{n+1}$, where $\gamma$ is a complete self-shrinking curve of Halldorsson [17]. Hence, the condition of polynomial volume growth in [14] is essential. On the other hand, Colding and Minicozzi [14] (cf. Andrews, Li and Wei [2], Arezzo and Sun [3] for higher co-dimensions) gave a classification of

2020 Mathematics Subject Classification: 53E10, 53C40.

Key words and phrases: mean curvature flow, self-shrinker, the generalized maximum principle.

The first author was partially supported by JSPS Grant-in-Aid for Scientific Research: No.16H03937 and No. 22K03303 and the fund of Fukuoka University: No. 225001. The second author was partially supported by China Postdoctoral Science Foundation Grant No. 2022M711074. The third author was partly supported by grant No. 12171164 of NSFC, GDUPS (2018), Guangdong Natural Science Foundation Grant No.2023A1515010510.
n-dimensional $\mathcal{F}$-stable complete self-shrinker with polynomial volume growth in $\mathbb{R}^{n+1}$. Furthermore, Cao and Li \cite{Cao-Li} (cf. Le and Sesum \cite{Le-Sesum}) gave a classification for complete self-shrinkers of the mean curvature flow with polynomial volume growth and $S \leq 1$, where $S$ denotes the squared norm of the second fundamental form. By making use of the generalized maximum principle, Cheng and Peng \cite{Cheng-Peng} studied complete self-shrinkers of the mean curvature flow without the assumption of polynomial volume growth. For study on the rigidity of complete self-shrinkers, many works have been done (cf. \cite{Cheng}, \cite{Cheng-Peng}, \cite{Cheng-Li}, \cite{Chen-Huang}, \cite{Chen-Lin}, \cite{Cheng-Lin}, \cite{Cheng-Lin-Zhang}, \cite{Cheng-Lin-Zhang}, \cite{Cheng-Lin-Zhang} and so on).

For complete self-shrinkers with constant squared norm of the second fundamental form, the following conjecture is well-known and very important (cf. \cite{Chen}, \cite{Cheng}).

**Conjecture.** An $n$-dimensional complete self-shrinker $X : M \to \mathbb{R}^{n+1}$ with constant squared norm of the second fundamental form is isometric to one of $S^n(\sqrt{n})$, $\mathbb{R}^n$, $S^k(\sqrt{k}) \times \mathbb{R}^{n-k}$, $1 \leq k \leq n-1$.

In \cite{Cheng}, by estimating the first eigenvalue of the Dirichlet eigenvalue problem, Ding and Xin proved that a 2-dimensional complete self-shrinker $X : M^2 \to \mathbb{R}^3$ with polynomial volume growth and with constant squared norm $S$ of the second fundamental form is isometric to one of $S^2(\sqrt{2})$, $\mathbb{R}^2$, $S^1(1) \times \mathbb{R}$. Recently, Cheng and Ogata \cite{Cheng-Ogata} have solved the above conjecture for $n = 2$ affirmatively, that is, they have proved the following:

**Theorem CO.** A 2-dimensional complete self-shrinker $X : M \to \mathbb{R}^3$ with constant squared norm of the second fundamental form is isometric to one of $S^2(\sqrt{2})$, $\mathbb{R}^2$, $S^1(1) \times \mathbb{R}$.

Cheng and Peng \cite{Cheng-Peng} proved that an $n$-dimensional complete self-shrinker $X : M \to \mathbb{R}^{n+1}$ with constant squared norm of the second fundamental form is isometric to one of $S^k(\sqrt{k}) \times \mathbb{R}^{n-k}$, $1 \leq k \leq n-1$, and $S^n(\sqrt{n})$ if $\inf H^2 > 0$. Hence, in order to solve the above conjecture, one only need to prove $H \equiv 0$ if $\inf H^2 = 0$.

For the dimension 3, by studying the infimum of $H^2$, Cheng, Li and Wei \cite{Cheng-Li} have proved the above conjecture is true under the assumption that $f_3$ is constant. In this paper, by considering the supremum and the infimum of $H$, we prove the above conjecture is true if $f_3$ is constant. In fact, we prove the following:

**Theorem 1.1.** Let $X : M^3 \to \mathbb{R}^4$ be a 3-dimensional complete self-shrinker in $\mathbb{R}^4$.

If the squared norm $S$ of the second fundamental form and $f_3$ are constant, then $X : M^3 \to \mathbb{R}^4$ is isometric to one of

1. $S^3(\sqrt{3})$,
2. $\mathbb{R}^3$,
3. $S^1(1) \times \mathbb{R}^2$,
4. $S^2(\sqrt{2}) \times \mathbb{R}^1$.

In particular, $S$ must be 0 and 1; $f_3$ must be 0, 1, $\frac{\sqrt{3}}{2}$ and $\frac{\sqrt{3}}{3}$, where $h_{ij}$ are the components of the second fundamental form, $S = \sum_{i,j} h_{ij}^2$ and $f_3 = \sum_{i,j,k} h_{ij} h_{jk} h_{ki}$.

**Remark 1.1.** Recently, Cheng, Wei and Yano \cite{Cheng-Wei-Yano} proved that, for an $n$-dimensional complete self-shrinker $X : M \to \mathbb{R}^{n+1}$ in $\mathbb{R}^{n+1}$, if the squared norm $S$ of the second
fundamental form and $f_3$ are constant and $S$ satisfies

$$S \leq 1.83379,$$

then $M$ is isometric to one of $S^n\sqrt{n}$, $\mathbb{R}^n$, $S^k\sqrt{k} \times \mathbb{R}^{n-k}$, $1 \leq k \leq n - 1$.

2. Preliminaries

For an $n$-dimensional hypersurface $X : M^n \to \mathbb{R}^{n+1}$ of $(n + 1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$, we choose a local orthonormal frame field $\{e_A\}_{A=1}^{n+1}$ in $\mathbb{R}^{n+1}$ with dual co-frame field $\{\omega_A\}_{A=1}^{n+1}$, such that, restricted to $M^n$, $e_1, \cdots, e_n$ are tangent to $M^n$. We make use of the following conventions on the ranges of indices,

$$1 \leq i, j, k, l \leq n.$$

$\sum_i$ means taking summation from 1 to $n$ for $i$. Then we have

$$dX = \sum_i \omega_i e_i$$

and the Levi-Civita connection $\omega_{ij}$ of the hypersurface satisfies

$$de_i = \sum_j \omega_{ij} e_j + \omega_{in+1} e_{n+1},$$

$$de_{n+1} = \omega_{n+1i} e_i, \quad \omega_{n+1i} = -\omega_{in+1}.$$

By restricting these forms to $M$, we get

$$(2.1) \quad \omega_{n+1} = 0.$$

Thus, we obtain, from Cartan lemma,

$$(2.2) \quad \omega_{in+1} = \sum_j h_{ij} \omega_j, \quad h_{ij} = h_{ji},$$

$$h = \sum_{i,j} h_{ij} \omega_i \otimes \omega_j; \quad H = \sum_i h_{ii}$$

are called second fundamental form and mean curvature of $X : M \to \mathbb{R}^{n+1}$, respectively. Let $S = \sum (h_{ij})^2$ be the squared norm of the second fundamental form of $X : M \to \mathbb{R}^{n+1}$. The Gauss equations of hypersurface are given by

$$(2.3) \quad R_{ijkl} = h_{ik} h_{jl} - h_{il} h_{jk}.$$

Defining covariant derivative of $h_{ij}$ by

$$(2.4) \quad \sum_k h_{ijk} \omega_k = dh_{ij} + \sum_k h_{ik} \omega_{kj} + \sum_k h_{kj} \omega_{ki},$$

we obtain the Codazzi equations

$$(2.5) \quad h_{ijk} = h_{ikj}.$$

By defining

$$(2.6) \quad \sum_l h_{ijk} \omega_l = dh_{ijk} + \sum_l h_{ijl} \omega_{kl} + \sum_l h_{ilk} \omega_{lj} + \sum_l h_{ilj} \omega_{kl},$$

and the mean curvature $H$ of the hypersurface $X : M \to \mathbb{R}^{n+1}$ is given by

$$H = \sum_i h_{ii}.$$
and
\[ \sum_m h_{ijklm} \omega_m = dh_{ijkl} + \sum_m h_{mjk} \omega_{mi} + \sum_m h_{imk} \omega_{mj} \]
(2.7)
\[ + \sum_m h_{ijml} \omega_{mk} + \sum_m h_{ijkl} \omega_{ml} \]
we have the following Ricci identities
(2.8)
\[ h_{ijkl} - h_{ijlk} = \sum_m h_{mjk} R_{mikl} + \sum_m h_{imk} R_{mjkl} \]
(2.9)
\[ h_{ijklq} - h_{ijkql} = \sum_m h_{mjk} R_{milq} + \sum_m h_{imk} R_{mjql} + \sum_m h_{ijm} R_{klim} \]

For a smooth function \( f \), we define
\[ df := \sum_i f, i \omega_i, \quad df, j := df, i + \sum_j f, j \omega_{ji} \]
Therefore, we know that the norm of gradient \( \nabla f \) and Laplacian of \( f \) are given by
\[ |\nabla f|^2 = \sum_i (f, i)^2, \quad \Delta f = \sum_i f, ii \]

We define functions \( f_3 \) and \( f_4 \) as follows:
\[ f_3 = \sum_{i,j,k} h_{ij} h_{jk} h_{ki}, \quad f_4 = \sum_{i,j,k,l} h_{ij} h_{jk} h_{kl} h_{li} \]

As in [14], we define the \( \mathcal{L} \)-operator by
\[ \mathcal{L} f = \Delta f - \langle X, \nabla f \rangle \]

A hypersurface \( X : M \to \mathbb{R}^{n+1} \) is called a self-shrinker of mean curvature flow if
\[ H + \langle X, e_{n+1} \rangle = 0 \]

By a simple calculation, we have the following basic formulas.
(2.10)
\[ H, i = \sum_k h_{ik} \langle X, e_k \rangle, \]
\[ H, ij = \sum_k h_{ijk} \langle X, e_k \rangle + h_{ij} - H \sum_k h_{ik} h_{kj} \]

Using the above formulas and the Ricci identities, we can get the following Lemma (cf. [9], [6]):

**Lemma 2.1.** Let \( X : M^n \to \mathbb{R}^{n+1} \) be an \( n \)-dimensional self-shrinker in \( \mathbb{R}^{n+1} \). We have
(2.11)
\[ \mathcal{L} H = H (1 - S), \]
(2.12)
\[ \frac{1}{2} \mathcal{L} H^2 = |\nabla H|^2 + H^2 (1 - S), \]
(2.13) \[ \frac{1}{2} \mathcal{L} S = \sum_{i,j,k} h^2_{ijk} + S(1 - S), \]

(2.14) \[ \frac{1}{3} \mathcal{L} f_3 = 2 \sum_{i,j,k,l} h_{ijl}h_{jkl}h_{ki} + (1 - S)f_3. \]

**Lemma 2.2.** Let \( X : M^n \to \mathbb{R}^{n+1} \) be an \( n \)-dimensional self-shrinker in \( \mathbb{R}^{n+1} \). If \( S \) is constant, we have

\[ \sum_{i,j,k,l} h^2_{ijkl} = (S - 2) \sum_{i,j,k} h^2_{ijk} - 6 \sum_{i,j,k,l,p} h_{ijk}h_{ijl}h_{jklp} + 3 \sum_{i,j,k,l,p} h_{ijk}h_{ijkl}h_{kjp}h_{lp}. \]

**Proof.** Since \( S \) is constant, we know from the lemma 2.1,

\[ \sum_{i,j,k} h^2_{ijk} = S(S - 1). \]

By making use of the Ricci identities (2.8), (2.9) and a direct calculation, we have

\[ \frac{1}{2} \mathcal{L} \sum_{i,j,k} h^2_{ijk} = \sum_{i,j,k,l} h^2_{ijkl} - (S - 2) \sum_{i,j,k} (h_{ijk})^2 + 6 \sum_{i,j,k,l,p} h_{ijk}h_{ijl}h_{jklp} - 3 \sum_{i,j,k,l,p} h_{ijk}h_{ijkl}h_{kjp}h_{lp}. \]

Thus we obtain (2.15) since \( \sum_{i,j,k} h^2_{ijk} \) is constant. \( \square \)

In order to prove our theorem, we need the following generalized maximum principle which is proved by Cheng and Peng [8].

**Lemma 2.3.** Let \( X : M^n \to \mathbb{R}^{n+p} \) be a complete self-shrinker with Ricci curvature bounded from below. Let \( f \) be any \( C^2 \)-function bounded from above on this self-shrinker. Then, there exists a sequence of points \( \{p_t\} \in M^n \), such that

\[ \lim_{t \to \infty} f(X(p_t)) = \sup f, \quad \lim_{t \to \infty} |\nabla f|(X(p_t)) = 0, \quad \limsup_{t \to \infty} \mathcal{L} f(X(p_t)) \leq 0. \]

### 3. Proof of Theorem 1.1

From (2.13), we know that \( S = 0 \) or \( S = 1 \) or \( S > 1 \). If \( S = 0 \), we know that \( X : M^3 \to \mathbb{R}^4 \) is \( \mathbb{R}^3 \). Next, we assume that \( S \geq 1 \). We will prove the following

**Proposition 3.1.** For a 3-dimensional complete self-shrinker \( X : M^3 \to \mathbb{R}^4 \) with nonzero constant squared norm \( S \) of the second fundamental form, if \( f_3 \) is constant, we have that either \( S = 1 \) or \( \sup H = \frac{3f_3}{2S} \).
Proof. Since $S$ is constant, if $S \leq 1$, according to the results of Cheng and Peng [8], we have $S = 1$. Thus, we only need to consider $S > 1$. We choose $e_1$, $e_2$ and $e_3$, at each point $p \in M^3$, such that

$$h_{ij} = \lambda_i \delta_{ij}. $$

From the definitions of $S$ and $H$, we obtain

$$H^2 \leq 3S.$$

Since $S$ is constant, from the Gauss equations, we know that the Ricci curvature of $X : M^3 \to \mathbb{R}^4$ is bounded from below. We can apply the generalized maximum principle for $\mathcal{L}$-operator to $H$. Thus, there exists a sequence $\{p_t\}$ in $M^3$ such that

$$\lim_{t \to \infty} H(X(p_t)) = \sup H, \quad \lim_{t \to \infty} |\nabla H|(X(p_t)) = 0, \quad \lim_{t \to \infty} \sup \mathcal{L}H(X(p_t)) \leq 0.$$

Since $S$ and $f_i$ are constant, by (2.13) and (2.15), we know that $\{h_{ij}(p_t)\}, \{h_{ijkl}(p_t)\}$ and $\{h_{ijk}(p_t)\}$ are bounded sequences for $i, j, k, l = 1, 2, 3$. Hence, we can assume that they convergence if necessary, taking a subsequence.

$$\lim_{t \to \infty} H(p_t) = \bar{H}, \quad \lim_{t \to \infty} h_{ij}(p_t) = \bar{h}_{ij} = \bar{\lambda}_i \delta_{ij},$$

$$\lim_{t \to \infty} h_{ijk}(p_t) = \bar{h}_{ijk}, \quad \lim_{t \to \infty} h_{ijkl}(p_t) = \bar{h}_{ijkl}, \quad i, j, k, l = 1, 2, 3.$$

By taking exterior derivative of $\bar{H}$, from (3.1) and by taking limits, we know

$$\bar{H}_i = \bar{h}_{11i} + \bar{h}_{22i} + \bar{h}_{33i} = 0, \quad i = 1, 2, 3.$$

According to the definition of the self-shrinker, we have

$$H_i = \sum_k h_{ik} \langle X, e_k \rangle, \quad i = 1, 2, 3,$$

$$H_{ij} = \sum_k h_{ijk} \langle X, e_k \rangle + h_{ij} - H \sum_k h_{ik} h_{kj}, \quad i, j = 1, 2, 3.$$

Thus, we get

$$\bar{H}_i = \bar{h}_{11i} + \bar{h}_{22i} + \bar{h}_{33i} = \bar{\lambda}_i \lim_{t \to \infty} \langle X, e_i \rangle(p_t), \quad i = 1, 2, 3,$$

and

$$\bar{H}_{ij} = \sum_k \bar{h}_{ijk} \lim_{t \to \infty} \langle X, e_k \rangle(p_t) + \bar{\lambda}_i - \bar{H} \bar{\lambda}_i^2, \quad i = 1, 2, 3,$$

$$\bar{H}_{ij} = \sum_k \bar{h}_{ijk} \lim_{t \to \infty} \langle X, e_k \rangle(p_t), \quad i \neq j, \quad i, j = 1, 2, 3.$$

Since $S$ is constant, we obtain

$$\sum_{i,j} h_{ij} h_{ijk} = 0, \quad k = 1, 2, 3,$$

$$\sum_{i,j} h_{ij} h_{ijkl} + \sum_{i,j} h_{ijkl} h_{ijl} = 0, \quad k, l = 1, 2, 3.$$

Under the processing by taking limits, we have

$$\bar{\lambda}_1 \bar{h}_{11k} + \bar{\lambda}_2 \bar{h}_{22k} + \bar{\lambda}_3 \bar{h}_{33k} = 0, \quad k = 1, 2, 3,$$

$$\bar{\lambda}_i h_{ijklm} = 0, \quad i = 1, 2, 3.$$
and

\[
\left\{ \begin{align*}
\sum_i \lambda_i \bar{h}_{ikk} &= - \sum_{i,j} \bar{h}_{ijk}^2, \quad k = 1, 2, 3, \\
\sum_i \tilde{\lambda}_i \bar{h}_{ikl} &= - \sum_{i,j} \bar{h}_{ijk} \bar{h}_{ijl}, \quad k \neq l, \quad k, l = 1, 2, 3.
\end{align*} \right.
\] (3.6)

It follows from Ricci identities (2.8) that

\[
\bar{h}_{ijkl} - \bar{h}_{ijlk} = \bar{\lambda}_i \bar{\lambda}_j \bar{\lambda}_k \delta_i \delta_j - \bar{\lambda}_i \bar{\lambda}_j \delta_i \delta_k + \bar{\lambda}_i \bar{\lambda}_j \delta_k \delta_j - \bar{\lambda}_i \bar{\lambda}_j \delta_i \delta_k \delta_j,
\] that is,

\[
\left\{ \begin{align*}
\bar{h}_{1212} - \bar{h}_{1221} &= \bar{\lambda}_1 \bar{\lambda}_2 (\bar{\lambda}_1 - \bar{\lambda}_2), \\
\bar{h}_{1313} - \bar{h}_{1331} &= \bar{\lambda}_1 \bar{\lambda}_3 (\bar{\lambda}_1 - \bar{\lambda}_3), \\
\bar{h}_{2323} - \bar{h}_{2332} &= \bar{\lambda}_2 \bar{\lambda}_3 (\bar{\lambda}_2 - \bar{\lambda}_3), \\
\bar{h}_{iikl} - \bar{h}_{iilk} &= 0, \quad i, k, l = 1, 2, 3.
\end{align*} \right.
\] (3.7)

Since \(f_3\) is constant, by (2.10), we know that

\[
\bar{\lambda}_1^2 \bar{h}_{11k} + \bar{\lambda}_2^2 \bar{h}_{22k} + \bar{\lambda}_3^2 \bar{h}_{33k} = 0, \quad k = 1, 2, 3,
\] (3.8)

and

\[
\left\{ \begin{align*}
\sum_i \bar{\lambda}_i^2 \bar{h}_{ikk} &= -2 \sum_{i,j} \bar{\lambda}_i \bar{h}_{ijk}^2, \quad k = 1, 2, 3, \\
\sum_i \bar{\lambda}_i^2 \bar{h}_{ikl} &= -2 \sum_{i,j} \bar{\lambda}_i \bar{h}_{ijk} \bar{h}_{ijl}, \quad k \neq l, \quad k, l = 1, 2, 3.
\end{align*} \right.
\] (3.9)

From the above equations (3.2) to (3.9), we can prove the following claim: Claim. If \(S > 1\), we have

\[
\bar{H} = \frac{3f_3}{2S}
\]

holds.

In fact, if \(\bar{\lambda}_1 = \bar{\lambda}_2 = \bar{\lambda}_3\), we have

\[
\bar{H}^2 = 3S, \quad \bar{H} = 3\bar{\lambda}_1.
\] (3.10)

From (3.6) and (3.9), we have

\[
\left\{ \begin{align*}
\bar{\lambda}_1 \sum_i h_{i11} &= - (\bar{h}_{111}^2 + \bar{h}_{221}^2 + \bar{h}_{331}^2) - 2(\bar{h}_{112}^2 + \bar{h}_{113}^2 + \bar{h}_{123}^2), \\
\bar{\lambda}_1 \sum_i h_{i22} &= - (\bar{h}_{112}^2 + \bar{h}_{222}^2 + \bar{h}_{332}^2) - 2(\bar{h}_{221}^2 + \bar{h}_{223}^2 + \bar{h}_{123}^2), \\
\bar{\lambda}_1 \sum_i h_{i33} &= - (\bar{h}_{113}^2 + \bar{h}_{223}^2 + \bar{h}_{333}^2) - 2(\bar{h}_{331}^2 + \bar{h}_{332}^2 + \bar{h}_{123}^2),
\end{align*} \right.
\] and

\[
\left\{ \begin{align*}
\bar{\lambda}_1^2 \sum_i h_{i11} &= - 2\bar{\lambda}_1 (\bar{h}_{111}^2 + \bar{h}_{221}^2 + \bar{h}_{331}^2) - 4\bar{\lambda}_1 (\bar{h}_{112}^2 + \bar{h}_{113}^2 + \bar{h}_{123}^2), \\
\bar{\lambda}_1^2 \sum_i h_{i22} &= - 2\bar{\lambda}_1 (\bar{h}_{112}^2 + \bar{h}_{222}^2 + \bar{h}_{332}^2) - 4\bar{\lambda}_1 (\bar{h}_{221}^2 + \bar{h}_{223}^2 + \bar{h}_{123}^2), \\
\bar{\lambda}_1^2 \sum_i h_{i33} &= - 2\bar{\lambda}_1 (\bar{h}_{113}^2 + \bar{h}_{223}^2 + \bar{h}_{333}^2) - 4\bar{\lambda}_1 (\bar{h}_{331}^2 + \bar{h}_{332}^2 + \bar{h}_{123}^2).
\end{align*} \right.
\]
Thus, we infer
\begin{equation}
\bar{h}_{ijk} = 0, \quad i, j, k = 1, 2, 3.
\end{equation}

According to the lemma 2.1, we know $S = 1$. It is impossible because of $S > 1$.

If two of $\bar{\lambda}_1$, $\bar{\lambda}_2$ and $\bar{\lambda}_3$ are equal, without loss of generality, we assume that $\bar{\lambda}_1 \neq \bar{\lambda}_2 = \bar{\lambda}_3$. Then we get from (3.2) and (3.5)
\begin{equation}
\bar{h}_{11k} = 0, \quad \bar{h}_{22k} + \bar{h}_{33k} = 0, \quad k = 1, 2, 3.
\end{equation}

If $\bar{\lambda}_1 = 0$, then $\bar{\lambda}_2 = \bar{\lambda}_3 \neq 0$ because of $S > 1$. By making use of equations (3.6), (3.9) and (3.12), we know that
\begin{align*}
&\begin{cases}
\bar{\lambda}_2(\bar{h}_{2211} + \bar{h}_{3311}) = -2(\bar{h}_{221}^2 + \bar{h}_{123}^2), \\
\bar{\lambda}_2(\bar{h}_{2222} + \bar{h}_{3322}) = -2(\bar{h}_{222}^2 + \bar{h}_{223}^2) - 2(\bar{h}_{221}^2 + \bar{h}_{123}^2),
\end{cases} \\
\text{and} \\
&\begin{cases}
\bar{\lambda}_2^2(\bar{h}_{2211} + \bar{h}_{3311}) = -4\bar{\lambda}_2(\bar{h}_{221}^2 + \bar{h}_{123}^2), \\
\bar{\lambda}_2^2(\bar{h}_{2222} + \bar{h}_{3322}) = -4\bar{\lambda}_2(\bar{h}_{222}^2 + \bar{h}_{223}^2) - 2\bar{\lambda}_2^2(\bar{h}_{221}^2 + \bar{h}_{123}^2).
\end{cases}
\end{align*}

Hence, we get
\[\bar{h}_{221} = \bar{h}_{123} = 0, \quad \bar{h}_{222} = \bar{h}_{223} = 0.\]

Namely, we obtain
\begin{equation}
\bar{h}_{ijk} = 0, \quad i, j, k = 1, 2, 3.
\end{equation}

We have $S = 1$, which contradicts to $S > 1$.

If $\bar{\lambda}_1 \neq 0$, $\bar{\lambda}_2 = \bar{\lambda}_3 = 0$, combining it with (3.6), (3.9) and (3.12), we have
\begin{align*}
&\begin{cases}
\bar{\lambda}_1\bar{h}_{1111} = -2(\bar{h}_{221}^2 + \bar{h}_{123}^2), \\
\bar{\lambda}_1\bar{h}_{1122} = -2(\bar{h}_{222}^2 + \bar{h}_{223}^2) - 2(\bar{h}_{221}^2 + \bar{h}_{123}^2),
\end{cases} \\
\text{and} \\
&\begin{cases}
\bar{\lambda}_1^2\bar{h}_{1111} = 0, \\
\bar{\lambda}_1^2\bar{h}_{1122} = -2\bar{\lambda}_1(\bar{h}_{221}^2 + \bar{h}_{123}^2).
\end{cases}
\end{align*}

We infer
\[\bar{h}_{221} = \bar{h}_{123} = 0, \quad \bar{h}_{222} = \bar{h}_{223} = 0.\]

Thus, we conclude
\begin{equation}
\bar{h}_{ijk} = 0, \quad i, j, k = 1, 2, 3.
\end{equation}

Hence, we infer $S = 1$. It is also impossible because of $S > 1$.

If $\bar{\lambda}_1 \neq 0$, $\bar{\lambda}_2 = \bar{\lambda}_3 \neq 0$, combining (3.2) and (3.3), we obtain
\begin{equation}
\lim_{t \to \infty} \langle X, e_k \rangle(p_t) = 0, \quad k = 1, 2, 3.
\end{equation}

It follows from (3.4), (3.6), (3.9), (3.12) and (3.15) that
Furthermore, (3.16) yields

\[
\begin{align*}
\bar{h}_{1111} + \bar{h}_{2211} + \bar{h}_{3311} &= \dot{\lambda}_1 - H\dot{\lambda}_1^2, \\
\bar{h}_{1122} + \bar{h}_{2222} + \bar{h}_{3322} &= \dot{\lambda}_2 - H\dot{\lambda}_2^2, \\
\bar{h}_{1133} + \bar{h}_{2233} + \bar{h}_{3333} &= \dot{\lambda}_3 - H\dot{\lambda}_3^2, \\
\bar{h}_{1112} + \bar{h}_{2212} + \bar{h}_{3312} &= 0, \\
\bar{h}_{1113} + \bar{h}_{2213} + \bar{h}_{3313} &= 0, \\
\bar{h}_{1123} + \bar{h}_{2223} + \bar{h}_{3323} &= 0,
\end{align*}
\]

(3.16)

From the lemma 2.1, From (3.20), we have \(\bar{h}_{111} = \bar{h}_{222} = \bar{h}_{333} = \dot{\lambda}_1 - H\dot{\lambda}_1^2\), \(\bar{h}_{112} = \bar{h}_{222} = \bar{h}_{333} = \dot{\lambda}_2 - H\dot{\lambda}_2^2\), \(\bar{h}_{113} = \bar{h}_{222} = \bar{h}_{333} = \dot{\lambda}_3 - H\dot{\lambda}_3^2\), \(\bar{h}_{1112} + \bar{h}_{2212} + \bar{h}_{3312} = 0\), \(\bar{h}_{1113} + \bar{h}_{2213} + \bar{h}_{3313} = 0\), \(\bar{h}_{1123} + \bar{h}_{2223} + \bar{h}_{3323} = 0\),

and

\[
\begin{align*}
\dot{\lambda}_1\bar{h}_{1111} + \dot{\lambda}_2(\bar{h}_{2211} + \bar{h}_{3311}) &= -2(\bar{h}_{221}^2 + \bar{h}_{123}^2), \\
\dot{\lambda}_1\bar{h}_{1122} + \dot{\lambda}_2(\bar{h}_{2222} + \bar{h}_{3322}) &= -2(\bar{h}_{222}^2 + \bar{h}_{233}^2) - 2(\bar{h}_{221} + \bar{h}_{123}^2), \\
\dot{\lambda}_1\bar{h}_{1133} + \dot{\lambda}_2(\bar{h}_{2233} + \bar{h}_{3333}) &= -2(h_{222}^2 + h_{233}^2) - 2(h_{221}^2 + h_{123}^2), \\
\dot{\lambda}_1\bar{h}_{1112} + \dot{\lambda}_2(\bar{h}_{2212} + \bar{h}_{3312}) &= -2(\bar{h}_{221}\bar{h}_{222} + \bar{h}_{223}\bar{h}_{123}), \\
\dot{\lambda}_1\bar{h}_{1113} + \dot{\lambda}_2(\bar{h}_{2213} + \bar{h}_{3313}) &= -2(\bar{h}_{221}\bar{h}_{223} - \bar{h}_{222}\bar{h}_{123}), \\
\dot{\lambda}_1\bar{h}_{1123} + \dot{\lambda}_2(\bar{h}_{2223} + \bar{h}_{3323}) &= 0,
\end{align*}
\]

(3.17)

Furthermore, (3.16) yields

\[
\bar{h}_{2212} + \bar{h}_{3312} = -\bar{h}_{1112}, \quad \bar{h}_{2213} + \bar{h}_{3313} = -\bar{h}_{1113}.
\]

(3.19)

Inserting (3.19) into (3.17) and (3.18), we derive to

\[
\bar{h}_{221}\bar{h}_{222} + \bar{h}_{223}\bar{h}_{123} = 0, \quad \bar{h}_{221}\bar{h}_{223} - \bar{h}_{222}\bar{h}_{123} = 0.
\]

(3.20)

It follows from (3.17) and (3.18) that

\[
\begin{align*}
\dot{\lambda}_1(\dot{\lambda}_2 - \dot{\lambda}_1)\bar{h}_{1111} &= 2\dot{\lambda}_2(h_{221}^2 + h_{123}^2), \\
\dot{\lambda}_1(\dot{\lambda}_2 - \dot{\lambda}_1)\bar{h}_{1112} &= 2\dot{\lambda}_2(h_{222}^2 + h_{233}^2) + 2\dot{\lambda}_1(h_{221}^2 + h_{123}^2), \\
\dot{\lambda}_1(\dot{\lambda}_2 - \dot{\lambda}_1)\bar{h}_{1113} &= 2\dot{\lambda}_2(h_{223}^2 + h_{233}^2) + 2\dot{\lambda}_1(h_{221}^2 + h_{123}^2).
\end{align*}
\]

(3.21)

From (3.21), we have \(h_{221}^2 + h_{123}^2 = 0\), or \(h_{222}^2 + h_{233}^2 = 0\).

If both \(h_{221}^2 + h_{123}^2 = 0\) and \(h_{222}^2 + h_{233}^2 = 0\), we have

\[
\bar{h}_{ijk} = 0, \quad i, j, k = 1, 2, 3.
\]

From the lemma 2.1, \(S = 1\). It is a contradiction.
If \( \bar{h}_{221}^2 + \bar{h}_{123}^2 = 0 \) and \( \bar{h}_{222}^2 + \bar{h}_{223}^2 \neq 0 \), by making use of (3.21), we have

\[
\begin{align*}
\lambda_1 ( \lambda_2 - \lambda_1 ) \bar{h}_{1111} &= 0, \\
\lambda_1 ( \lambda_2 - \lambda_1 ) \bar{h}_{1122} &= 2 \lambda_2 ( \bar{h}_{222}^2 + \bar{h}_{223}^2 ), \\
\lambda_1 ( \lambda_2 - \lambda_1 ) \bar{h}_{1133} &= 2 \lambda_2 ( \bar{h}_{222}^2 + \bar{h}_{223}^2 ).
\end{align*}
\]

Thus,

\[
(3.22) \quad \bar{h}_{1111} = 0, \quad \bar{h}_{1122} = \bar{h}_{1133} = \frac{2 \lambda_2}{\lambda_1 ( \lambda_2 - \lambda_1 )} ( \bar{h}_{222}^2 + \bar{h}_{223}^2 ).
\]

Noting that \( \bar{h}_{1111} = 0 \) and by (3.17), we know

\[
\bar{h}_{2211} + \bar{h}_{3311} = 0, \quad \bar{H},11 = 0.
\]

Hence, by the first equation of (3.16), we have

\[
(3.23) \quad \lambda_1 \bar{H} = 1.
\]

Combining (3.16), (3.17) with (3.23), we know

\[
-2( \bar{h}_{222}^2 + \bar{h}_{223}^2 ) = \lambda_1 \bar{h}_{1122} + \lambda_2 \left( \bar{h}_{222} - \bar{H} \lambda_2^2 - \bar{h}_{1122} \right)
\]

\[
= \lambda_2 \left( \bar{h}_{222} - \bar{H} \lambda_2^2 \right) + ( \lambda_1 - \lambda_2 ) \bar{h}_{1122}
\]

\[
= \lambda_2 \left( \bar{h}_{222} - \bar{H} \lambda_2^2 \right) + ( \lambda_1 - \lambda_2 ) \cdot \frac{2 \lambda_2}{\lambda_1 ( \lambda_2 - \lambda_1 )} ( \bar{h}_{222}^2 + \bar{h}_{223}^2 )
\]

\[
= \lambda_2 \left( \bar{h}_{222} - \bar{H} \lambda_2^2 \right) - \frac{2 \lambda_2}{\lambda_1} ( \bar{h}_{222}^2 + \bar{h}_{223}^2 ).
\]

From (3.23), we obtain

\[
\bar{h}_{222}^2 + \bar{h}_{223}^2 = -\frac{\lambda_2^2}{2}.
\]

It is a contradiction.

If \( \bar{h}_{221}^2 + \bar{h}_{123}^2 \neq 0 \) and \( \bar{h}_{222}^2 + \bar{h}_{223}^2 = 0 \), according to (2.13) and (2.14) in Lemma 2.2, we have

\[
(3.24) \quad \sum_{i,j,k} \bar{h}_{ijk}^2 = S(S - 1), \quad 2 \sum_{i,j,k,l} h_{ijk} h_{jkl} h_{kli} = f_3(S - 1).
\]

Since \( \bar{h}_{222}^2 + \bar{h}_{223}^2 = 0 \), it follows from (3.12) that

\[
(3.25) \quad \sum_{i,j,k} \bar{h}_{ijk}^2 = 6( \bar{h}_{221}^2 + \bar{h}_{123}^2 ), \quad 2 \sum_{i,j,k,l} \bar{h}_{ijk} \bar{h}_{jkl} h_{kli} = 4 \bar{H}( \bar{h}_{221}^2 + \bar{h}_{123}^2 ).
\]

Therefore, by (3.24) and (3.25), we obtain

\[
6( \bar{h}_{221}^2 + \bar{h}_{123}^2 ) = S(S - 1), \quad 4 \bar{H}( \bar{h}_{221}^2 + \bar{h}_{123}^2 ) = (S - 1) f_3,
\]

that is,

\[
(2 \bar{H} S - 3 f_3)(S - 1) = 0.
\]

Therefore, we conclude

\[
2 \bar{H} S - 3 f_3 = 0, \quad \bar{H} = \frac{3 f_3}{2 S},
\]
because of $S > 1$.

If $\bar{\lambda}_1$, $\bar{\lambda}_2$ and $\bar{\lambda}_3$ are distinct, by using of (3.2), (3.5) and (3.8), we have that

$$h_{11k} = h_{22k} = h_{33k} = 0, \ k = 1, 2, 3,$$

then

$$\sum_{i,j,k} \bar{h}_{ijk} = 6\bar{h}_{123}^2, \ \sum_{i,j,k,l} \bar{h}_{ijl} \bar{h}_{jkl} \bar{h}_{kli} = 4\bar{H}\bar{h}_{123}^2.$$

It follows from the lemma 2.1

$$6\bar{h}_{123}^2 = S(S - 1), \ 4\bar{H}\bar{h}_{123}^2 = (S - 1)f_3,$$

namely,

$$(2\bar{H}S - 3f_3)(S - 1) = 0.$$

Hence, we get

$$2\bar{H}S - 3f_3 = 0, \ \bar{H} = \frac{3f_3}{2S}.$$

Thus, we complete the proof of the proposition 3.1. \qed

By applying the generalized maximum principle for $\mathcal{L}$-operator to the function $-\bar{H}$ and using the same proof as that of proving the proposition 3.1 we can obtain the following

**Proposition 3.2.** For a 3-dimensional complete self-shrinker $X : M^3 \to \mathbb{R}^4$ with nonzero constant squared norm $S$ of the second fundamental form. If $f_3$ is constant, then we have that either $S = 1$ or $\inf H = \frac{3f_3}{2S}$.

**Proof of Theorem 1.1** If $S = 0$, we know that $X : M^3 \to \mathbb{R}^4$ is the plane $\mathbb{R}^3$. If $S \neq 0$, from the proposition 3.1 and the proposition 3.2 we have that either $S = 1$ or $\sup H = \inf H = \frac{3f_3}{2S}$. It follows from (2.13) that the mean curvature $H$ and the principal curvatures are constant. Then, $X : M^3 \to \mathbb{R}^4$ is an isoparametric hypersurface. $X : M^3 \to \mathbb{R}^4$ is either $S^1(1) \times \mathbb{R}^2$, or $S^2(\sqrt{2}) \times \mathbb{R}^1$ or $S^3(\sqrt{3})$. \qed

Acknowledgements. We would like to express our gratitude to referees for valuable comments and suggestions.

**References**

[1] U. Abresch and J. Langer, The normalized curve shortening flow and homothetic solutions, J. Differential Geom., 23(1986), 175-196.
[2] B. Andrews, H. Li and Y. Wei, $\mathcal{F}$-stability for self-shrinking solutions to mean curvature flow, Asian J. Math., 18 (2014), 757-778.
[3] C. Arezzo, J. Sun, Self-shrinkers for the mean curvature flow in arbitrary codimension, Math. Z., 274 (2013), no. 3-4, 993-1027.
[4] H.-D. Cao and H. Li, A gap theorem for self-shrinkers of the mean curvature flow in arbitrary codimension, Calc. Var. Partial Differential Equations, 46 (2013), 879-889.
[5] Q. -M. Cheng, H. Hori and G. Wei, Complete Lagrangian self-shrinkers in $\mathbb{R}^4$, Math. Z. 301 (2022), 3417-3468.
[6] Q. -M. Cheng, Z. Li, and G. Wei, Complete self-shrinkers with constant norm of the second fundamental form, Math. Z., 300 (2022), no. 1, 995-1018.
[7] Q. -M. Cheng and S. Ogata, 2-dimensional complete self-shrinkers in $\mathbb{R}^3$, Math. Z., 284 (2016), 537-542.
[8] Q. -M. Cheng and Y. Peng, Complete self-shrinkers of the mean curvature flow, Calc. Var. Partial Differential Equations, 52 (2015), no. 3-4, 497-506.
[9] Q. -M. Cheng and G. Wei, A gap theorem for self-shrinkers, Trans. Amer. Math. Soc., 367 (2015), 4895-4915.
[10] Q. -M. Cheng and G. Wei, Complete self-shrinkers of mean curvature flow, Proceedings of the International Consortium of Chinese Mathematicians 2018, pp.179-196, Int. Press, Boston, MA, 2020.
[11] Q. -M. Cheng and G. Wei, Complete $\lambda$-surfaces in $\mathbb{R}^3$, Cal Var. Partial Differential Equations, (2021), 60:46: 1-19.
[12] Q. -M. Cheng, G. Wei and W. Yano, The second gap on complete self-shrinkers, Proc. Amer. Math. Soc. 151 (2023), 339-348.
[13] X. Cheng and D. Zhou, Volume estimate about shrinkers, Proc. Amer. Math. Soc., 141 (2013), 687-696.
[14] T. H. Colding and W. P. Minicozzi II, Generic mean curvature flow I; Generic singularities, Ann. of Math., 175 (2012), 755-833.
[15] Q. Ding and Y. L. Xin, Volume growth, eigenvalue and compactness for self-shrinkers, Asian J. Math., 17 (2013), 443-456.
[16] Q. Ding and Y. L. Xin, The rigidity theorems of self shrinkers, Trans. Amer. Math. Soc., 366 (2014), 5067-5085.
[17] H. Halldorsson, Self-similar solutions to the curve shortening flow, Trans. Amer. Math. Soc., 364 (2012), 5285-5309.
[18] G. Huisken, Asymptotic behavior for singularities of the mean curvature flow, J. Differential Geom., 31 (1990), 285-299.
[19] G. Huisken, Local and global behaviour of hypersurfaces moving by mean curvature, Differential geometry: partial differential equations on manifolds (Los Angeles, CA, 1990), Proc. Sympos. Pure Math., 54, Part 1, Amer. Math. Soc., Providence, RI, (1993), 175-191.
[20] Nam Q. Le and N. Sesum, Blow-up rate of the mean curvature during the mean curvature flow and a gap theorem for self-shrinkers, Comm. Anal. Geom., 19 (2011), no.4, 1-27.
[21] Z. Li and G. Wei, Complete 3-dimensional $\lambda$-translators in the Minkowski space $\mathbb{R}^4_1$, J. Math. Soc. Japan, 75 (2023), no.1, 119-150.
[22] L. Lei, H. W. Xu and Z. Y. Xu, A new pinching theorem for complete self-shrinkers and its generalization, Sci. China Math., 63 (2020), no. 6, 1139-1152.
[23] H. Li and Y. Wei, Lower volume growth estimates for self-shrinkers of mean curvature flow, Proc. Amer. Math. Soc., 142 (2014), 3237-3248.
[24] H. Li and Y. Wei, Classification and rigidity of self-shrinkers in the mean curvature flow, J. Math. Soc. Japan, 66 (2014), 709-734.
[25] L. Wang, A Benstein type theorem for self-similar shrinkers, Geom. Dedicata, 15 (2011), 297-303.

QING-MING CHENG,
DEPARTMENT OF APPLIED MATHEMATICS, FACULTY OF SCIENCE,
FUKUOKA UNIVERSITY, FUKUOKA 814-0180, JAPAN.
Email address: cheng@fukuoka-u.ac.jp

ZHI LI,
COLLEGE OF MATHEMATICS AND INFORMATION SCIENCE, HENAN NORMAL UNIVERSITY,
453007, XINXIANG, HENAN, CHINA.
Email address: lizhihnsd@126.com
GUOXIN WEI,
SCHOOL OF MATHEMATICAL SCIENCES, SOUTH CHINA NORMAL UNIVERSITY,
510631, GUANGZHOU, CHINA.

Email address: weiguoxin@tsinghua.org.cn