Circularly Symmetric Locally Univalent Functions

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Abstract Let $D \subset \mathbb{C}$ and $0 \in D$. A set $D$ is circularly symmetric if, for each $\varrho \in \mathbb{R}^+$, a set $D \cap \{\zeta \in \mathbb{C} : |\zeta| = \varrho\}$ is one of three forms: an empty set, a whole circle, a curve symmetric with respect to the real axis containing $\varrho$. A function $f$ analytic in the unit disk $\Delta \equiv \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ and satisfying the normalization condition $f(0) = f'(0) - 1 = 0$ is circularly symmetric, if $f(\Delta)$ is a circularly symmetric set. The class of all such functions is denoted by $X$. In this paper, we focus on the subclass $X'$ consisting of functions in $X$ which are locally univalent. We obtain the results concerned with omitted values of $f \in X'$ and some covering and distortion theorems. For functions in $X'$ we also find the upper estimate of the $n$-th coefficient, as well as the region of variability of the second and the third coefficients. Furthermore, we derive the radii of starlikeness, convexity and univalence for $X'$.

Keywords Locally univalent functions · Radius of univalence · Radius of starlikeness · Coefficients’ estimates

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1 Introduction

Let $\mathcal{A}$ denote the class of all functions analytic in the unit disk $\Delta \equiv \{ \zeta \in \mathbb{C} : |\zeta| < 1 \}$ which satisfy the condition $f(0) = f'(0) - 1 = 0$. A function $f$ is said to be typically real if the inequality $(\text{Im } z)(\text{Im } f(z)) \geq 0$ holds for all $z \in \Delta$. The class of functions which are typically real is denoted by $\tilde{T}$ and the class of typically real functions which belong to $\mathcal{A}$ is denoted by $T$. For a typically real function $f$, $z \in \Delta^+ \iff f(z) \in \mathbb{C}^+$ and $z \in \Delta^- \iff f(z) \in \mathbb{C}^-$. The symbols $\Delta^+$, $\Delta^-$, $\mathbb{C}^+$, $\mathbb{C}^-$ stand for the following sets: the upper and the lower halves of the disk $\Delta$, the upper halfplane and the lower halfplane, respectively.

Jenkins \cite{3} established the following definitions.

**Definition 1** Let $D \subset \mathbb{C}$, $0 \in D$. A set $D$ is circularly symmetric if, for each $\varrho \in \mathbb{R}^+$, a set $D \cap \{ \zeta \in \mathbb{C} : |\zeta| = \varrho \}$ is one of three forms: an empty set, a whole circle, a curve symmetric with respect to the real axis containing $\varrho$.

**Definition 2** A function $f \in \mathcal{A}$ is circularly symmetric if $f(\Delta)$ is a circularly symmetric set. The class of all such functions is denoted by $X$.

In fact Jenkins considered only those circularly symmetric functions which are univalent. This assumption is rather restrictive. A number of interesting problems appear while discussing non-univalent circularly symmetric functions. For these reasons, we decided to define a circularly symmetric function as in Definition 2. In this paper, we focus on the set $X'$ consisting of locally univalent circularly symmetric functions.

According to Jenkins (see, \cite{3}), if $f \in X$ is univalent then $zf'(z)f(z)$ is a typically real function. Additionally, he observed that the property $zf'(z)f(z) \in \tilde{T}$ does not guarantee the univalence of $f$. In fact, we have

$$f \in X \iff \frac{zf'(z)}{f(z)} \in \tilde{T}. \quad (1)$$

Jenkins also gave a nice geometric property of $f$ in $X$. He proved that $f \in X$ if and only if, for a fixed $r \in (0, 1)$, a function $|f(re^{i\varphi})|$ is nonincreasing for $\varphi \in (0, \pi)$ and nondecreasing for $\varphi \in (\pi, 2\pi)$. From (1), all coefficients of the Taylor series expansion of $f \in X$ are real.

In \cite{9} the following relation between $X'$ and $T$ was proved:

$$f \in X' \iff \frac{zf'(z)}{f(z)} = (1 + z)^2 \frac{h(z)}{z}, \quad h \in T. \quad (2)$$

It is known that each function of the class $T$ can be represented by the formula

$$h(z) = \int_{-1}^{1} \frac{z}{1 - 2rz + z^2} d\mu(r), \quad (3)$$

where $\mu$ is a probability measure on $[-1, 1]$ (see, \cite{7}). Applying (3) in (2) and integrating it, one can write a function $f \in X'$ in the form
\[ f(z) = z \exp \left( \int_0^z \int_{-1}^1 \frac{2(1 + t)}{1 - 2t\zeta + \zeta^2} \, d\mu(t) \, d\zeta \right). \]  

(4)

Putting \( \cos \psi \) instead of \( t \) in (4) and integrating it with respect to \( \zeta \), we get the integral representation of a function in \( X' \):

\[ f \in X' \iff f(z) = z \exp \left( \int_0^\pi i \cot \frac{\psi}{2} \log \frac{1 - ze^{i\psi}}{1 - ze^{-i\psi}} \, d\mu(\psi) \right), \]  

(5)

where \( \mu \) is a probability measure on \([0, \pi]\).

Applying the well-known equivalence

\[ f \in T \iff \frac{1 - z^2}{z} f(z) \in P_\mathbb{R}, \]  

(6)

we obtain the relation between \( X' \) and the set \( P_\mathbb{R} \) of functions with positive real part which have real coefficients:

\[ f \in X' \iff \frac{zf'(z)}{f(z)} = \frac{1 + z}{1 - z} p(z), \quad p \in P_\mathbb{R}. \]  

(7)

It is known (Robertson, [6]) that the set of extreme points for \( T \) has the form

\[ \left\{ \frac{z}{1 - 2zt + z^2} : t \in [-1, 1] \right\}. \]

Putting these functions into formula (2) as \( h \), we get

\[ \frac{zf'(z)}{f(z)} = \frac{(1 + z)^2}{1 - 2zt + z^2}. \]  

(8)

It is easy to check that the functions \( f \), which satisfy (8), are of the form

\[ f_t(z) = z \exp \left( i \cot \frac{\psi}{2} \log \frac{1 - ze^{i\psi}}{1 - ze^{-i\psi}} \right), \quad t \in [-1, 1], \]  

(9)

where \( t = \cos \psi \). Obviously, \( t \in [-1, 1] \iff \psi \in (0, \pi] \). Furthermore,

\[ \lim_{\psi \to 0^+} \left( i \cot \frac{\psi}{2} \log \frac{1 - ze^{i\psi}}{1 - ze^{-i\psi}} \right) = \frac{4z}{1 - z}, \]

so we can write

\[ f_1(z) = z \exp \left( \frac{4z}{1 - z} \right). \]  

(10)

Besides \( f_t \), we need another family of functions belonging to \( X' \). Since the set \( T \) is convex, every linear combination of any two functions from \( T \) also belongs to \( T \). Hence, taking

\[ \frac{1 + t}{2} \frac{z}{(1 + z)^2} + \frac{1 - t}{2} \frac{z}{(1 - z)^2}, \quad t \in [-1, 1] \]  

as \( h \) in (2), we obtain

\[ \frac{zf'(z)}{f(z)} = \frac{1 + 2zt + z^2}{(1 - z)^2}. \]  

(11)
Let us denote by $g_t$ the solutions of Eq. (11). From this equation
\[
g_t(z) = z \exp \left( \frac{2(1+t)z}{1-z} \right), \quad t \in [-1, 1]. \tag{12}
\]
In particular, we have
\[
g_{-1}(z) = f_{-1}(z) = z \quad \text{and} \quad g_1(z) = f_1(z) = z \exp \left( \frac{4z}{1-z} \right).
\]

2 Properties of $f_t$ and $g_t$

Firstly, we shall describe the sets $f_t(\Delta)$ and $g_t(\Delta)$, where $f_t$, $g_t$ are defined by (9) and (12), respectively.

For $f_t$, $t \in (-1, 1)$ (i.e. for $\psi \in (0, \pi)$), from (9) we obtain
\[
\left| f_t(e^{i\varphi}) \right| = \exp \left( \cot \frac{\psi}{2} \arg \frac{1 - e^{i(\varphi - \psi)}}{1 - e^{i(\varphi + \psi)}} \right).
\]

We shall derive the argument which appears in the above expression. To do this, we need the following identity:
\[
\arg \left( 1 + e^{i\alpha} \right) = \frac{\alpha}{2} - \pi \cdot \left[ \frac{\alpha + \pi}{2\pi} \right] \quad \text{for} \quad \alpha \neq (1 + 2k)\pi, \quad k \in \mathbb{Z}.
\]
Hence
\[
\left| f_t(e^{i\varphi}) \right| = \exp \left( -\psi \cot \frac{\psi}{2} \right) \quad \text{for} \quad \varphi \in (\psi, 2\pi - \psi) \tag{13}
\]
and
\[
\left| f_t(e^{i\varphi}) \right| = \exp \left( (\pi - \psi) \cot \frac{\psi}{2} \right) \quad \text{for} \quad \varphi \in [0, \psi) \cup (2\pi - \psi, 2\pi]. \tag{14}
\]

From the above expressions, it follows that a function $\left| f_t(e^{i\varphi}) \right|$, with fixed $t \in (-1, 1)$, does not depend on the variable $\varphi$. Moreover, $\left| f_t(e^{i\varphi}) \right|$ in (13) is less than 1 and $\left| f_t(e^{i\varphi}) \right|$ in (14) is greater than 1. Additionally,
\[
\arg \left( f_t(e^{i\varphi}) \right) = \varphi + \cot \frac{\psi}{2} \log \left| \frac{\sin \frac{\varphi + \psi}{2}}{\sin \frac{\varphi - \psi}{2}} \right|,
\]
which means that the curves $\{f_t(e^{i\varphi}), \ \varphi \in [0, \phi]\}$ and $\{f_t(e^{i\varphi}), \ \varphi \in (\phi, \pi]\}$ wind around circles given by (13) and (14) infinitely many times. Hence, for $t \in (-1, 1)$,
\[
f_t(\Delta) \subset \left\{ w \in \mathbb{C} : |w| < \exp \left( (\pi - \psi) \cot \frac{\psi}{2} \right) \right\} \tag{16}.
\]
It is easily seen that $f_{-1}(\Delta) = \Delta$.

Now, we shall show that $f_1$ omits only one point on the real axis.

**Theorem 1**  The condition $f_1(z) \neq -e^{-2}$ holds for all $z \in \Delta$.

**Proof**  On the contrary, suppose that there exists $z \in \Delta$ such that $f_1(z) = -e^{-2}$. In fact, we can assume that $\arg z \in [0, \pi]$ because the coefficients of $f_1$ are real. For this reason,

$$z \exp \left( \frac{2(1+z)}{1-z} \right) = -1,$$

or equivalently,

$$\frac{1+z}{1-z} = \frac{1}{2} \left( -\log |z| + i \arg \frac{-1}{z} \right).$$

Let $z = re^{i\varphi}$, $\varphi \in [0, \pi]$. Comparing the arguments of both sides of this equality, we get

$$\frac{2r \sin \varphi}{1 - r^2} = \frac{\pi - \varphi}{-\log r}.$$

For a fixed $r \in (0, 1)$, let us consider a function

$$h(\varphi) = \frac{2r}{1-r^2} \sin \varphi + \frac{1}{\log r} (\pi - \varphi), \quad \varphi \in [0, \pi].$$

For $\varphi \in [0, \pi]$ the function $h'(\varphi)$ decreases and

$$h'(\varphi) \geq -\frac{2r}{(1-r^2) \log r} \cdot g(r),$$

where

$$g(r) = \log r + \frac{1-r^2}{2r}.$$

Since $g'(r) = -(1-r)^2/2r^2$, the function $g(r)$ decreases for $r \in (0, 1)$ and

$$g(r) \geq g(1) = 0.$$

Both factors on the right-hand side of (20) are positive. Consequently, $h(\varphi)$ increases for $\varphi \in (0, \pi)$, so $h(\varphi) \leq h(\pi) = 0$.

Taking into account the last inequality, we can see that $\varphi = \pi$ is the only solution of (19). It means that Eq. (18) is satisfied only for $z = re^{i\pi} = -r$; so

$$\frac{1-r}{1+r} + \frac{1}{2} \log r = 0.$$

Let us denote the left-hand side of (21) by $k(r)$. Since $k$ is increasing for $r \in (0, 1)$,

$$\sup \{k(r) : r \in (0, 1)\} = k(1) = 0.$$
Therefore, (21) has no solutions in the open set \((0, 1)\), which contradicts (18), and, consequently we obtain the desired result. □

Applying a similar argument, one can prove the following more general theorem.

**Theorem 2** For \(g_t, t \in (-1, 1)\) given by (12),

(i) \(g_t(z) \neq -e^{-1-t} \) for \(z \in \Delta\),

(ii) the equation \(g_t(z) = -\varphi e^{-1-t}\) has a solution for any \(\varphi > 1\),

(iii) the equation \(g_t(z) = -e^{-1-t} e^{i\theta}\) has a solution for any \(\theta \in (-\pi, \pi)\).

**Proof** ad (i) Suppose that there exists \(z \in \Delta\) such that \(g_t(z) = -e^{-1-t}\). Then

\[
z \exp \left( (1 + t) \frac{1 + z}{1 - z} \right) = -1,
\]

and putting \(z = re^{i\varphi}\), we have

\[
\frac{1 + re^{i\varphi}}{1 - re^{i\varphi}} = \frac{1}{1 + t} \left(-\log r + i(\pi - \varphi)\right).
\]

Comparing the arguments on both sides, we obtain the same function \(h\) as in the proof for Theorem 4; consequently \(h(\varphi) \leq 0\). Therefore, Eq. (23) holds only if \(z = -r\), but in this case

\[
\frac{1 - r}{1 + r} + \frac{1}{1 + t} \log r = 0.
\]

Let the left-hand side of (24) be denoted by \(k(r)\), which is an increasing and nonpositive function of \(r \in (0, 1)\). It yields that (24) has no solutions in the open set \((0, 1)\), which contradicts the assumption.

ad (ii) Consider an equation \(g_t(z) = -\varphi e^{-1-t}\) with fixed \(\varphi > 1\). It takes the following form

\[
z \exp \left( (1 + t) \frac{1 + z}{1 - z} \right) = -\varphi.
\]

Putting \(z = re^{i\varphi}\) into (25), we have

\[
\frac{1 + re^{i\varphi}}{1 - re^{i\varphi}} = \frac{1}{1 + t} \left(\log \frac{\varphi}{r} + i(\pi - \varphi)\right).
\]

Hence

\[
\frac{2r \sin \varphi}{1 - r^2} - \frac{\pi - \varphi}{\log \frac{\varphi}{r}} = 0.
\]

Let \(h(\varphi)\) denote the left-hand side of this equality. The function \(h'(\varphi)\) is decreasing; \(h'(0) = \frac{2r}{1-r^2} + \frac{1}{\log \frac{\varphi}{r}} > 0\) and \(h'(\pi) = \frac{-2r}{1-r^2} + \frac{1}{\log \frac{\varphi}{r}}\). One can easily prove that there exists only one number \(r_0 \in (0, 1)\) such that \(h'(\pi) > 0\) for \(r \in (0, r_0)\) and \(h'(\pi) < 0\) for \(r \in (r_0, 1)\).

Hence, for suitably taken \(r\), the function \(h\) first increases and then decreases. Furthermore, \(h(0) = -\pi/\log \frac{\varphi}{r} < 0\) and \(h(\pi) = 0\). Consequently, there exists \(\varphi_0 \in (0, \pi)\)
such that \( h(\varphi_0) = 0 \). It means that (26) is satisfied by \( z_0 = re^{i\varphi_0} \); so (25) holds for \( z = z_0 \).

**ad (iii)** The proof of this part is similar to the proof of (ii). \( \square \)

**Corollary 1** \( g_t(\Delta) = \mathbb{C}\setminus\{e^{-1-t}\} \) for all \( t \in (-1, 1] \).

**Proof** Let \( t \in (-1, 1] \) be fixed. For \( r \in (0, 1) \), we have \( g_t(-r) = -r \exp(-2(1 + t)r/(1 + r)) \). Observe that \( |g_t(-r)| \) is a continuous and increasing function of \( r \in [0, 1) \). For this reason, \( g_t(-r) \) achieves all values in \( (-e^{-1-t}, 0] \). From the definition of circularly symmetric function it follows that if \( c \) belongs to the negative real axis and \( c \in g_t(\Delta) \), then the whole circle with radius \(|c|\) centered at the origin is also contained in this set. Hence, for each \( \varphi \) in \([0, 1)\) we have \( \{w \in \mathbb{C} : |w| = \varphi e^{-1-t}\} \subset g_t(\Delta) \).

By Theorem 2, part (ii), \((−\infty, −e^{-1-t})\) is contained in \( g_t(\Delta) \). Let \(-\varphi e^{-1-t}, \varphi > 1\) be an arbitrary point of this ray. Applying the same argument as above, we conclude that for any \( \varphi > 1 \), \( \{w \in \mathbb{C} : |w| = \varphi e^{-1-t}\} \subset g_t(\Delta) \).

Combining these facts with points (i) and (iii) of Theorem 2 completes the proof. \( \square \)

Let us consider a function \( b(\varphi) = \arg g_t(re^{i\varphi}) \), \( \varphi \in (0, \pi) \) where \( t \in [-1, 1] \) and \( r \in (0, 1) \) are fixed. Analyzing the derivative of this function it can be observed that if \( t - 3 + 4\sigma^2 \leq 0 \), where \( \sigma = \frac{2r}{1+r} \), then \( b(\varphi) \) increases in \((0, \pi)\). On the other hand, if \( t - 3 + 4\sigma^2 > 0 \) then, for \( \varphi \in (0, \pi) \), the function \( b(\varphi) \) increases at the beginning, then it decreases, only to increase again at the end. From this observation we conclude that for small \( r \), a set \( g_t(\{z \in \mathbb{C} : |z| < r, \text{Im} z \geq 0\}) \) is contained in the upper halfplane. If \( r \) is greater than \( r_t = \frac{\sqrt{3-t}}{2+\sqrt{1+t}} \), then this set is not contained in the upper halfplane; its boundary is wound around the origin.

If \( r = 1 \), then

\[
g_t(e^{i\varphi}) = \exp\left(-1 - t + i \left( \varphi + (1 + t) \cot \frac{\varphi}{2} \right) \right). \tag{28}
\]

Hence

\[
|g_t(e^{i\varphi})| = \exp(-1 - t) \tag{29}
\]

and

\[
\arg\left(g_t(e^{i\varphi})\right) = \varphi + (1 + t) \cot \frac{\varphi}{2}. \tag{30}
\]

Therefore, \( g_t(\Delta) \) is wound around the origin infinitely many times, or, more precisely, it is wound around the circle with radius \( e^{-1-t} \).

### 3 Coefficients of Functions in \( X' \)

To start with, let us look into the coefficients of \( f_1 \) given by (10). Although it is complicated to find an explicit formula for the \( n \)-th coefficient of this function, the formula of the logarithmic coefficients \( \gamma_n \) of \( f_1 \) can be easily derived. Indeed,

\[
\frac{1}{2} \log \frac{f_1(z)}{z} = \sum_{k=1}^{\infty} 2e^{k},
\]

\( \square \) Springer
thus
\[
\gamma_n = 2 \quad \text{for all } n \in \mathbb{N}.
\]

The Taylor series expansion of \( f_1 \) is given by
\[
f_1(z) = z + \sum_{k=1}^{\infty} \frac{4^k}{k!} z^{k+1} (1 - z)^{-k} = z + \sum_{k=1}^{\infty} \frac{4^k}{k!} z^{k+1} \sum_{j=0}^{\infty} \binom{j+k-1}{k-1} z^j
\]
\[
= z + \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} b_{j,k} z^{k+j+1},
\]
where
\[
b_{j,k} = \frac{4^k}{k!} \binom{j+k-1}{k-1}, \quad k \geq 1, \quad j \geq 0.
\]

Denoting the \( n \)-th coefficient of \( f_1 \) by \( A_n \), we can write
\[
A_n = \sum_{s=0}^{n-2} b_{s,n-1-s} = \sum_{s=0}^{n-2} \frac{4^{n-1-s}}{(n-1-s)!} \binom{n-2}{n-2-s},
\]
and consequently,
\[
A_n = \sum_{j=1}^{n-1} \frac{4^j}{j!} \binom{n-2}{j-1}.
\]

The first four values of \( A_n \) are
\[
A_2 = 4, \quad A_3 = 12, \quad A_4 = \frac{92}{3}, \quad A_5 = \frac{212}{3}.
\]

On the other hand, for \( A_n \) the formula
\[
nA_{n+1} = (2n + 2)A_n - (n - 2)A_{n-1}
\]
holds for \( n \geq 2 \). Indeed, expanding both sides of the equality
\[
z f_1'(z) = f_1(z) \left( \frac{1 + z}{1 - z} \right)^2,
\]
we get
\[
\sum_{n=1}^{\infty} n A_n z^n = \sum_{n=1}^{\infty} A_n z^n \cdot \left( 1 + \sum_{k=1}^{\infty} 2z^k \right)^2.
\]
Comparing the coefficients at $z^n$, we obtain (32).

We shall now prove that the upper bound of $n$-th coefficient of a function $f \in X'$ is achieved when $f$ is equal to $f_1$. To do this, we apply the relation (7).

Suppose that functions $f \in X'$ and $p \in \mathcal{P}_\mathbb{R}$ are of the form $f(z) = \sum_{n=1}^\infty a_n z^n$ and $p(z) = \sum_{n=0}^\infty p_n z^n$ with $a_1 = 1$, $p_0 = 1$. Equation (7) yields

$$z + \sum_{n=2}^\infty n a_n z^n = \left( z + \sum_{n=2}^\infty a_n z^n \right) \cdot \left( 1 + \sum_{n=1}^\infty c_n z^n \right),$$

where

$$c_n = p_n + 2 \sum_{k=0}^{n-1} p_k.$$

Comparing the coefficients at $z^n$, $n \geq 2$, we obtain

$$(n - 1)a_n = \sum_{j=1}^{n-1} a_j c_{n-j} = \sum_{j=1}^{n-1} a_j \left( p_{n-j} + 2 \sum_{k=0}^{n-j-1} p_k \right). \tag{33}$$

Taking into account (33) and the coefficient estimates of a function in $\mathcal{P}_\mathbb{R}$, we conclude

$$(n - 1)a_n \leq 4 \sum_{j=1}^{n-1} |a_j| (n - j), \quad n \geq 2. \tag{34}$$

Equality in (34) holds only if all $p_j$ in (33) are equal to 2, which means that $p(z) = \frac{1+z}{1-z}$.

From (34), when $n = 2$, there is $a_2 \leq 4|a_1| = 4$. Equality in this estimate holds for $f_1$ only. Now, it is sufficient to apply mathematical induction in order to prove that successive coefficients $a_n$ of any $f \in X'$ are bounded by corresponding coefficients $A_n$ of $f_1$. Hence

**Theorem 3** Let $f \in X'$ have the form $f(z) = z + \sum_{n=2}^\infty a_n z^n$ and let $A_n$ be given by (31). Then, for $n \geq 2$,

$$a_n \leq A_n.$$  

Our next problem is to find the set of variability of $(a_2, a_3)$ for a function in $X'$. For a given class of analytic functions $A$, let $A_{i,j}(A)$ denote a set $\{(a_i(f), a_j(f)) : f \in A\}$.

For the class $\mathcal{P}_\mathbb{R}$ of functions with positive real part and having real coefficients, the following result is known:

$$A_{1,2}(\mathcal{P}_\mathbb{R}) = \{(x, y) : -2 \leq x \leq 2, x^2 - 2 \leq y \leq 2\}. \tag{35}$$

Based on this result, we can prove
Theorem 4

\[ A_{2,3}(X') = \left\{ (x, y) : 0 \leq x \leq 4, x^2 - x \leq y \leq \frac{1}{2}x^2 + x \right\} \]

and

**Corollary 2** Let \( f \in X' \) have the form \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \). Then \( a_3 \geq -\frac{1}{4} \).

**Proof of Theorem 4** Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in X' \) and \( p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in P_\mathbb{R} \). It follows from (33) that

\[
\begin{align*}
a_2 &= p_1 + 2, \\
2a_3 &= p_2 + 2p_1 + 2 + a_2(p_1 + 2),
\end{align*}
\]

or equivalently,

\[
\begin{align*}
p_1 &= a_2 - 2, \\
p_2 &= 2a_3 - a_2^2 - 2a_2 + 2.
\end{align*}
\]

Combining these relations with the estimates given in (35) completes the proof.

The points of intersection of two parabolas described in Theorem 4, i.e.: \((0, 0)\) and \((4, 12)\) correspond to the functions \( f_{-1}(z) = z \) and \( f_1(z) = z \exp \left( \frac{4z}{1-z} \right) = z + 4z^2 + 12z^3 + \cdots \), respectively.

Observe that the class \( X' \) is not convex. Indeed, if \( X' \) is a convex set, then, for every fixed \( \alpha \in (0, 1) \), a function \( \alpha f_1(z) + (1 - \alpha) f_{-1}(z) = \alpha z \exp \left( \frac{4z}{1-z} \right) + (1 - \alpha)z = z + 4\alpha z^2 + 12\alpha z^3 + \cdots \), would be in \( X' \). This will imply that \((4\alpha, 12\alpha) \in A_{2,3}(X')\), a contradiction with Theorem 4.

**4 Distortion Theorems**

Directly from the definition of a circularly symmetric function, it follows that

\[
|f(-r)| \leq |f(re^{i\varphi})| \leq |f(r)|
\]

(36)

for every function \( f \in X' \) and for all \( \varphi \in [0, 2\pi] \) and \( r \in (0, 1) \).

From (4), for any function \( f \in X' \) and any number \( r \in (0, 1) \),

\[
\begin{align*}
f(r) &= r \exp \left( \int_0^r \int_{-1}^1 \frac{2(1+t)}{1-2tx+x^2} d\mu(t) dx \right) \\
&\leq r \exp \left( \int_0^r \frac{4}{(1-x)^2} dx \right) = r \exp \left( \frac{4r}{1-r} \right) = f_1(r).
\end{align*}
\]

\( \square \) Springer
Similarly,
\[
|f(-r)| = r \exp \left( \int_0^{-r} \int_{-1}^{1} \frac{2(1+t)}{1-2tx+x^2} d\mu(t) dx \right)
= r \exp \left( -\int_0^{r} \int_{-1}^{1} \frac{2(1+t)}{1+2ty+y^2} d\mu(t) dy \right)
\geq r \exp \left( -\int_0^{r} \frac{4}{(1+y)^2} dy \right) = r \exp \left( \frac{-4r}{1+r} \right) = |f_1(-r)|.  \tag{38}
\]

Equalities in the above estimates hold only if \(\mu\) is a measure concentrated in point 1; it means that \(h(z) = \frac{z}{(1-z)^2}\). We have proved

**Theorem 5** For any \(f \in X'\) and \(r = |z| \in (0, 1)\),
\[
r \exp \left( \frac{-4r}{1+r} \right) \leq |f(z)| \leq r \exp \left( \frac{4r}{1-r} \right),  \tag{39}
\]

Equalities in the above estimates hold only for \(f_1\) and points \(z = -r\) and \(z = r\).

**Corollary 3** For any \(f \in X'\), we have \(f(\Delta) \supset \Delta_{e^{-2}}\).

The estimates of \(|f'(z)|\) for \(f \in X'\) can be obtained from (2) and Theorem 5.

**Theorem 6** For any \(f \in X'\) and \(|z| = r \in (0, 1)\),
\[
\left( \frac{1-r}{1+r} \right)^2 \exp \left( \frac{-4r}{1+r} \right) \leq |f'(z)| \leq \left( \frac{1-r}{1+r} \right)^2 \exp \left( \frac{4r}{1-r} \right).  \tag{40}
\]

Equalities in the above estimates hold only for \(f_1\) and points \(z = -r\) and \(z = r\).

**Proof** Let \(f \in X'\). From (2),
\[
f'(z) = (1+z)^2 \frac{h(z) f(z)}{z},
\]
where \(h \in T\). Therefore, if \(|z| = r \in (0, 1)\) then
\[
|f'(z)| \leq (1+r)^2 \frac{1}{(1-r)^2} \frac{f_1(r)}{r}
\]
and
\[
|f'(z)| \geq (1-r)^2 \frac{1}{(1+r)^2} \frac{|f_1(-r)|}{r},
\]
which is equivalent to (40). Moreover, equalities in both estimates appear when \(h\) is equal to \(\frac{z}{(1-z)^2}\) and \(z\) is equal to \(r\) and \(-r\), respectively. It means that \(f_1\) is the extremal function for (40). \(\square\)
Finally, we shall prove two lemmas which will be useful in our research on the convexity of functions in $X'$.

**Lemma 1** For a fixed point $z \in \Delta^+$, the set $\Omega(z)$ of variability of the expression $\frac{zf'(z)}{f(z)}$, while $f$ varies in $X'$, is of the form

$$\Omega(z) = \text{conv } \gamma(z),$$

where $\gamma(z)$ is an upper halfplane located arc of a circle containing three nonlinear points: $z_0 = 0$, $z_1 = 1$, $z_2 = \left(\frac{1+z}{1-z}\right)^2$, with endpoints $z_1$ and $z_2$.

**Lemma 2** For any $f \in X'$ and $z \in \Delta$,

$$\text{Re} \frac{zf'(z)}{f(z)} \geq \begin{cases} \text{Re} \left(\frac{1+z}{1-z}\right)^2 & \text{for } \text{Re}(z + 1/z) \leq 2 \\ 1 & \text{for } \text{Re}(z + 1/z) \geq 2. \end{cases} \quad (41)$$

**Proof of Lemma 1** Let $z \in \Delta^+$. Applying (2) and the representation formula for a function in $T$, we have

$$\frac{zf'(z)}{f(z)} = \int_{-1}^{1} \frac{(1+z)^2}{1-2zt + z^2} d\mu(t), \quad (42)$$

where $\mu$ is a probability measure on $[-1, 1]$.

With a fixed $z \in \Delta$, we denote by $q_z(t)$ an integrand in (42). The image set $\{q_z(t) : t \in \mathbb{R}\}$ coincides with a circle going through the origin. Furthermore, $q_z(-1) = 1$ and $q_z(1) = (\frac{1+z}{1-z})^2$.

For $z$ such that $\text{Im } z > 0$,

$$\text{Im} \left(\frac{1+z}{1-z}\right)^2 = \text{Im} \left(1 + \frac{4}{w-2}\right) = -4 \frac{\text{Im } w}{|w-2|^2} = \frac{4(1/|z|^2 - 1) \text{Im } z}{|w-2|^2} > 0,$$

where $w = z + 1/z$.

Hence, the set $\{q_z(t) : t \in [-1, 1]\}$ is an arc of the circle with endpoints $q_z(-1)$ and $q_z(1)$, which does not contain the origin. For this reason, this set coincides with $\gamma(z)$ and one endpoint of this arc is always 1, independent of $z$. Finally, $\Omega(z)$ is a section of the disk bounded by $\gamma(z)$ and the line segment with endpoints $q_z(-1)$ and $q_z(1)$. \hfill \Box

**Proof of Lemma 2** Every function $f$ in $X'$ has real coefficients, so $f(\Delta)$ is symmetric with respect to the real axis. Hence, it is sufficient to prove (41) only for $z \in \Delta^+$. But Lemma 1 leads to

$$\inf \left\{ \text{Re} \frac{zf'(z)}{f(z)} : f \in X' \right\} = \begin{cases} \text{Re } q_z(1) & \text{for } \text{Re}(z + 1/z) \leq 2 \\ \text{Re } q_z(-1) & \text{for } \text{Re}(z + 1/z) \geq 2. \end{cases} \quad (43)$$

\hfill \Box
It is easy to check that for \( z \in \Delta \),
\[
\Re \left( \frac{1+z}{1-z} \right)^2 \leq 1 \quad \Leftrightarrow \quad \Re(z + 1/z) \leq 2. \tag{44}
\]

Consequently, (41) can be written as follows:
\[
\Re \frac{zf'(z)}{f(z)} \geq \min \left\{ \Re \left( \frac{1+z}{1-z} \right)^2, 1 \right\}.
\]

## 5 Starlikeness and Convexity

The relation (2) and the estimates of the argument for typically real functions imply that for \( z \in \Delta^+ \),
\[
\arg \frac{zf'(z)}{f(z)} = 2 \arg(1 + z) + \arg \frac{g(z)}{z} \leq 2 \arg(1 + z) + \arg \frac{1}{(1-z)^2} = 2 \arg \frac{1+z}{1-z}. \tag{45}
\]

Furthermore,
\[
\left| \arg \frac{1+z}{1-z} \right| \leq \arctan \frac{2r}{1-r^2}. \tag{46}
\]

The condition for starlikeness \( |\arg \frac{zf'(z)}{f(z)}| \leq \frac{\pi}{2} \) and the bounds given above result in
\[
r_{S^*}(X') = \sqrt{2} - 1. \tag{47}
\]

Equality in (45) holds for \( g(z) = \frac{z}{(1-z)^2} \), and, consequently, for \( f = f_1 \). This result will be generalized in two ways.

First, we estimate \( \Re \frac{zf'(z)}{f(z)} \) for \( z \in H = \{ z \in \Delta : |1 + z^2| > 2|z| \} \). This set appears in the research on typically real functions. It is the domain of univalence and local univalence in \( T \) (see, [2]). The set \( H \), called the Golusin lens, is the common part of two disks with radii \( \sqrt{2} \) which have the centers in points \( i \) and \( -i \). Moreover,
\[
H = \left\{ z \in \mathbb{C} : \Re \left( \frac{1+z}{1-z} \right)^2 > 0 \right\}.
\]

From Lemma 2, we obtain

**Theorem 7** For each \( f \in X' \) and \( z \in H \),
\[
\Re \frac{zf'(z)}{f(z)} \geq 0.
\]
It is worth noticing that this theorem is still true even if $X'$ is replaced by $T$. This property is very interesting because the classes $X'$ and $T$ have a non-empty intersection, but one is not included in the other.

As a corollary, from Theorem 7 we obtain (47).

Another generalization of (47) refers to the radius of starlikeness of order $\alpha$ and the radius of strong starlikeness of order $\alpha$ (for definitions and other details the reader is referred to [1, 5, 8]).

**Theorem 8** The radius of starlikeness of order $\alpha$, $\alpha \in [0, 1)$, in $X'$ is equal to

$$r_{S^*(\alpha)}(X') = \begin{cases} \sqrt{\frac{2}{1-2\alpha}} - \sqrt{\frac{1+2\alpha}{1-2\alpha}} & \text{for } \alpha \in [0, 1/3], \\ \frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}} & \text{for } \alpha \in [1/3, 1). \end{cases}$$

**Corollary 4** $r_{S^*(1/2)}(X') = (\sqrt{2} - 1)^2 = 0.171\ldots$

**Theorem 9** The radius of strong starlikeness of order $\alpha$, $\alpha \in (0, 1]$, in $X'$ is equal to

$$r_{SS^*(\alpha)}(X') = \tan\left(\frac{\pi}{8}\alpha\right).$$

**Corollary 5** $r_{SS^*(2/3)}(X') = 2 - \sqrt{3}$.

**Proof of Theorem 8** By Lemma 2,

$$\Re \frac{zf'(z)}{f(z)} \geq \begin{cases} 1 + 4 \Re \frac{z}{(1-z)^2} & \text{for } \Re(z + 1/z) \leq 2, \\ 1 & \text{for } \Re(z + 1/z) \geq 2. \end{cases} \tag{48}$$

Let $r = |z|$ be a fixed number, $0 < r \leq 2 - \sqrt{3}$. It is known that $h(z) = \frac{z}{(1-z)^2}$ is convex for $|z| \leq 2 - \sqrt{3}$. Hence

$$\Re \frac{z}{(1-z)^2} \geq \frac{-r}{(1+r)^2} \tag{49}$$

with equality for $z = -r$.

From (48) and (49) it follows that

$$\Re \frac{zf'(z)}{f(z)} \geq \begin{cases} 1 - \frac{4r}{(1+r)^2} & \text{for } \Re(z + 1/z) \leq 2, \\ 1 & \text{for } \Re(z + 1/z) \geq 2, \end{cases}$$

and so

$$\Re \frac{zf'(z)}{f(z)} \geq \left(\frac{1-r}{1+r}\right)^2, \tag{50}$$

with equality for $z = -r$. For this $z$, the condition $\Re(z + 1/z) \leq 2$ is satisfied.
Now, suppose that \( r \in (2 - \sqrt{3}, 1) \). The real part of \( \frac{z}{1-z} \) for \( z = re^{i\phi} \) can be written as a function \( h(\cos \phi) \), \( \phi \in [0, 2\pi] \), where \( h(x) = \frac{r(1+x^2)x-2r^2}{(1-2rx+r^2)^2} \). If \( r \in (2 - \sqrt{3}, 1) \), one can check that

\[
\min \{h(x) : x \in [-1, 1]\} = h(x_0) = \frac{(1 + r^2)^2}{8(1 - r^2)^2},
\]

where

\[
x_0 = \frac{1 - 6r^2 + r^4}{2r(1 + r^2)}.
\]

Thus

\[
\text{Re} \frac{zf'(z)}{f(z)} \geq \begin{cases} 1 - \frac{(1+r^2)^2}{2(1-r^2)^2} & \text{for } \text{Re}(z + 1/z) \leq 2, \\ 1 & \text{for } \text{Re}(z + 1/z) \geq 2. \end{cases}
\]

Consequently

\[
\text{Re} \frac{zf'(z)}{f(z)} \geq \frac{1 - 6r^2 + r^4}{2(1 - r^2)^2},
\]

with equalities for points \( z_0 = re^{i\phi_0} \) and \( \overline{z_0} \), where \( \phi_0 = \arccos x_0 \). Furthermore,

\[
\text{Re}(z_0 + 1/z_0) - 2 = (1/r + r) \cos \phi_0 - 2 = (1/r + r)x_0 - 2 = -\frac{(1 - r^2)^2}{2r^2}.
\]

The condition \( \text{Re}(z_0 + 1/z_0) \leq 2 \) is satisfied in this case also.

Combining (50) and (51), we get

\[
\text{Re} \frac{zf'(z)}{f(z)} \geq \begin{cases} \frac{1}{1-i\alpha} & \text{for } r \in (0, 2 - \sqrt{3}], \\ \frac{1-6r^2+r^4}{2(1-r^2)^2} & \text{for } r \in [2 - \sqrt{3}, 1). \end{cases}
\]

In the first case, substituting \( \left( \frac{1-r}{1+r} \right)^2 \) by \( \alpha \), we obtain \( r = \frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}} \). The condition \( r \in (0, 2 - \sqrt{3}] \) is equivalent to \( \alpha \in [1/3, 1) \).

While discussing the second possibility in (52), we should remember that the radius of starlikeness in \( X' \) is equal to \( \sqrt{2} - 1 \). For this reason, we substitute \( \frac{1-6r^2+r^4}{2(1-r^2)^2} = \alpha \) only for \( r \in [2 - \sqrt{3}, \sqrt{2} - 1] \). This results in \( r = \sqrt{\frac{2}{1-2\alpha}} - \sqrt{\frac{1+2\alpha}{1-2\alpha}} \) and \( \alpha \in [0, 1/3] \).

The bound in (52) is sharp; equality holds for \( f \) satisfying \( \frac{zf'(z)}{f(z)} = \frac{z}{(1-z)^2} \), so for \( f_1 \).
The proof of Theorem 9 is easier. In fact, we need the condition for strong starlikeness and inequality (46). Thus we obtain

\[ 2 \arctan \frac{2r}{1-r^2} \leq \frac{\pi}{2} \alpha, \]

and hence

\[ r^2 + 2r \cot \left( \frac{\pi}{4} \alpha \right) - 1 \leq 0. \]

Solving this inequality with respect to \( r \), the assertion of Theorem 9 follows.

The next theorem is concerned with the problem of convexity of a function in \( X' \).

**Theorem 10** The radius of convexity for \( X' \) is equal to \( r_{CV}(X') = r_0 \), where \( r_0 = 0.139 \ldots \) is the only solution of equation \( 1 - 7r - r^2 - r^3 = 0 \). The extremal function is \( f_1 \).

In the proof of this theorem, we need the following result of Todorov for \( h \in T \) (see [10]):

\[
\text{Re} \frac{zh'(z)}{h(z)} \geq \begin{cases} 
\frac{1-r}{1+r}, & 0 \leq r \leq 2 - \sqrt{3}, \\
\frac{1-6r^2+r^4}{1-r^4}, & 2 - \sqrt{3} \leq r < 1.
\end{cases} \tag{53}
\]

**Proof** From (2), if \( f \in X' \) then \( \frac{zf''(z)}{f'(z)} = (1+z)^2 \frac{h(z)}{z} \), where \( h \in T \). Hence

\[
1 + \frac{zf''(z)}{f'(z)} = \frac{zf'(z)}{f(z)} + \frac{zh'(z)}{h(z)} - \frac{1-z}{1+z}. \tag{54}
\]

In further calculation, we shall apply Lemma 2.

Let \( r \leq 2 - \sqrt{3} \). For \( z \) such that \( \text{Re}(z + 1/z) \geq 2 \),

\[
\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \geq 1 + \text{Re} \left( \frac{zh'(z)}{h(z)} - \frac{1-z}{1+z} \right) = \text{Re} \left( \frac{zh'(z)}{h(z)} + \frac{2z}{1+z} \right).
\]

Estimate (53) and the inequality \( \text{Re} \frac{2z}{1+z} \geq -\frac{2r}{1-r} \) result in

\[
\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \geq \frac{1 - 4r - r^2}{1 - r^2}.
\]

This estimate is not sharp because equalities in the two previous inequalities appear only if \( z = -r \), but in this case \( \text{Re}(z + 1/z) < 2 \). From the above, we conclude that if \( \text{Re}(z + 1/z) \geq 2 \) and \( r \in [0, \sqrt{5} - 2) \) then \( \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0 \).
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Assume now that \( \text{Re}(z + 1/z) \leq 2 \). In this case

\[
\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \geq \text{Re} \left( \frac{(1 + z)}{(1 - z)} + \frac{zh'(z)}{h(z)} - \frac{1 - z}{1 + z} \right)
= \text{Re} \frac{zh'(z)}{h(z)} + \text{Re} \frac{2z}{1 + z} + \text{Re} \frac{4z}{(1 - z)^2}.
\]

(55)

The first two components can be estimated as above. Based on (49), the third one is greater than or equal to \( -\frac{4r}{(1 + r)^2} \). Since each estimate is sharp (with equality for \( z = -r \)), the estimate of the expression \( \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \) is also sharp. Consequently,

\[
\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \geq \frac{1 - 7r - r^2 - r^3}{(1 + r)^2(1 - r)}.
\]

The function in the numerator of the right-hand side of this inequality is decreasing for \( t \in \mathbb{R} \). For this reason, it has in \((0, 1)\) the only solution \( r_0 \). We have proven that if \( \text{Re}(z + 1/z) \leq 2 \) and \( r \in [0, r_0] \) then \( \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \geq 0 \). But \( r_0 < \sqrt{3} - 2 \).

Taking into account both parts of the proof, we obtain the assertion. Equality in (53) holds for \( h(z) = \frac{z}{(1 - z)^2} \) and \( z = -r \). It means that (55) is sharp, with equality for \( f_1 \) and \( z = -r \).

\[\square\]

6 Univalence

The problems of the univalence of functions in \( X' \) are more complicated. Based on the already proved results, we know that the radius of univalence \( r_S(X') \) is greater than or equal to \( \sqrt{2} - 1 \). On the other hand, one can easily find the upper estimate of \( r_S(X') \). Namely, discuss a function \( F(z) = \frac{1}{r_*} f_1(r_* z) \), where \( f_1 \) is given by (10) and \( r_* \) is equal to \( r_S(X') \) which we want to derive. The function \( F \) is univalent in \( \Delta \) and it has normalization \( F(0) = F'(0) - 1 = 0 \). From (31) it follows that \( F(z) = z + 4r_* z^2 + \ldots \).

The estimate of the second coefficient of functions in \( S \) results in \( r_* \leq 1/2 \).

The main theorem of this section is as follows.

**Theorem 11** The radius of univalence in \( X' \) is equal to \( r_S(X') = r_1 \), where \( r_1 = 0.454 \ldots \) is the only solution of equation

\[
\arcsin \frac{1 - r^2}{2r} + \frac{2(1 - r^2)}{1 + r^2 - \sqrt{-1 + 6r^2 - r^4}} = \pi
\]

in \((\sqrt{2} - 1, 1)\). The extremal function is \( f_1 \).

In the proof of this theorem we need two lemmas.

**Lemma 3** For each \( f_t, t \in [-1, 1] \) given by (9) and for each \( r \in (0, \sqrt{3}/3) \) and \( \varphi \in [0, \pi] \) the following inequality is true:
\[
\arg f_1(r e^{i\varphi}) \leq \arg f_1(r e^{i\psi}).
\] (57)

**Lemma 4** The function \( f_1 \) is univalent in the disk \(|z| < r_1\), where \( r_1 \) is the only solution of (56).

**Proof of Lemma 3** Let \( t \in [-1, 1] \) and \( r \in (0, \sqrt{3}/3) \) be fixed. Let us denote by \( g(\psi) \) the argument of \( f_t(r e^{i\varphi}) \) with a fixed \( \varphi \in [0, \pi] \), where \( \psi \) and \( t \) are connected by \( t = \cos \psi \). Applying (9), \( g \) can be written as

\[
g(\psi) = \varphi + \cot \frac{\psi}{2} k(\psi),
\]

where

\[
k(\psi) = \log \left| \frac{1 - r e^{i(\varphi+\psi)}}{1 - r e^{i(\varphi-\psi)}} \right|.
\]

Since

\[
\left| \frac{1 - r e^{i(\varphi+\psi)}}{1 - r e^{i(\varphi-\psi)}} \right| \geq 1,
\]

for \( \varphi \) and \( \psi \) in \([0, \pi]\), we conclude that \( k(\psi) \geq 0 \) for all \( \psi \in [0, \pi] \).

Now, we shall show that \( g(\psi) \) is a decreasing function of the variable \( \psi \). We have

\[
g'(\psi) = \frac{-1}{2 \sin^2 \frac{\psi}{2}} \cdot h(\psi),
\]

where

\[
h(\psi) = k(\psi) - k'(\psi) \sin \psi.
\]

A long and tedious calculation shows that

\[
h'(\psi) = \frac{\sin \varphi(1 - \cos \psi)}{[(q \cos \psi - \cos \varphi)^2 + (q^2 - 1) \sin^2 \psi]^2} \cdot W(\psi),
\]

with \( q = (1 + r^2)/2r, q > 1 \) and

\[
W(\psi) = [(1 - 2q^2) \cos \varphi - q] \cos^2 \psi + 2[q \cos^2 \varphi + \cos \varphi + q(q^2 - 1)] \cos \psi - \cos^3 \varphi - q \cos^2 \varphi - (q^2 - 1) \cos \varphi + q(q^2 - 1).
\]

Hence

\[
W(0) = (q - \cos \varphi)(3q^2 - 4 + \cos^2 \varphi) \quad \text{and} \quad W(\pi) = -(q + \cos \varphi)^3.
\]
It is obvious that $W(\pi) < 0$. On the other hand, $W(0) > 0$, providing that $r \in (0, \sqrt{3}/3)$, or equivalently, $q^2 > 4/3$. From these observations, taking into account that $W$ is a quadratic function of $\cos \psi$, we can see that $W(\psi)$ has exactly one solution in $[0, \pi]$; let us denote it by $\psi_0$. Hence, $h'(\psi)$ has only one solution $\psi_0$ in $(0, \pi)$. Thus, $h'(\psi)$ for $\psi \in (0, \psi_0)$ increases, and for $\psi \in (\psi_0, \pi)$ decreases. Combining it with $h(0) = h(\pi) = 0$, we obtain $h(\psi) \geq 0$ for $\psi \in [0, \pi]$. This implies $g'(\psi) \leq 0$ for $\psi \in (0, \pi)$; so $g(\psi)$ is decreasing in $(0, \pi)$.

Finally,

$$g(\psi) \leq g(0^+) \quad \text{for all } \psi \in [0, \pi],$$

where

$$g(0^+) = \lim_{\psi \to 0^+} g(\psi) = \varphi + \lim_{\psi \to 0^+} \frac{k(\psi)}{\tan \frac{\psi}{2}} = \varphi + 2k'(0) = \varphi + \frac{2\sin \varphi}{q - \cos \varphi}.$$

Moreover, if $\varphi = 0$ then $g(\psi) = 0$, and, if $\varphi = \pi$ then $g(\psi) = \pi$ for all $\psi \in [0, \pi]$. Consequently, (57) holds also for $\varphi = 0$ and $\varphi = \pi$.

Proof of Lemma 4 Consider a level curve $f_1(\{z \in \mathbb{C} : |z| = r\})$ with a fixed $r \in (0, 1)$. Since $f_1$ is a circularly symmetric function, $f_1(re^{i\varphi})$ decreases for $\varphi$ from 0 to $\pi$. Hence, $f_1$ is univalent when the level curves has no self-intersection points. It happens at small $r$, i.e. when $r < \sqrt{2} - 1$, because $f_1$ is starlike in this case. So it is enough to discuss for which $r \in [\sqrt{2} - 1, 1/2]$ the level curve $f_1(\{z \in \mathbb{C} : |z| = r\})$ is tangent to the real axis. Denoting the point of tangency by $w_0$, and denoting by $z_0 = re^{i\varphi_0}$ the corresponding point on circle $|z| = r$ for which $f(z_0) = w_0$, we obtain $\arg f(z_0) = \pi$.

The tangency of the level curve to the real axis in $w_0$ ensures that $\arg f_1(re^{i\varphi})$ is increasing for $\varphi \in (0, \varphi_0)$, decreasing for $\varphi \in (\varphi_0, \varphi_1)$, and once again increasing for $\varphi \in (\varphi_1, \pi)$, where $\varphi_1$ is a number from the interval $(\varphi_0, \pi)$. Hence, $\Re \frac{z_0 f_1'(z_0)}{f_1(z_0)} = 0$.

For this reason, we need to solve the system

$$\begin{align*}
\Re \frac{z_0 f_1'(z_0)}{f_1(z_0)} &= 0, \\
\arg f_1(z_0) &= \pi.
\end{align*} \tag{58}$$

The first equation can be written, using (8), as $\Re \left(1 + \frac{z_0}{1 - z_0}\right)^2 = 0$. Since $z_0 = re^{i\varphi}$, we obtain

$$\varphi_0 = \arcsin \frac{1 - r^2}{2r}. \tag{59}$$

But

$$f_1(re^{i\varphi_0}) = re^{i\varphi_0} \exp \left(\frac{4re^{i\varphi_0} - 4r^2}{1 - 2r \cos \varphi_0 + r^2}\right),$$
so
\[
\arg f_1(re^{i\varphi}) = \varphi_0 + \frac{4r \sin \varphi_0}{1 - 2r \cos \varphi_0 + r^2},
\]
which, together with the second equation of (58), proves that \( r \) is a solution of (56).

Finally, it can be observed that the right-hand side of the equality given above, let us denote it by \( b(r) \), satisfies
\[
b'(r) = \frac{(1 + r^2)(-1 + 6r^2 - r^4) + 8r^2 \sqrt{-1 + 6r^2 - r^4} + 4r^2(1 + r^2)}{r(1 - r^2)^2 \sqrt{-1 + 6r^2 - r^4}}.
\]
This means that \( b(r) \) increases from \( \pi/2 + \sqrt{2} \) to infinity, while \( r \in (\sqrt{2} - 1, 1) \). For this reason, (56) has only one solution. \( \square \)

Proof of Theorem 11 Let \( L = \{ \log f(z) : f \in X' \} \). In the paper [9], the authors showed that the extreme points of the class \( L \) are as follows:
\[
l_\psi(z) = i \cot \frac{\psi}{2} \log \frac{1 - ze^{i\psi}}{1 - ze^{-i\psi}}, \quad \psi \in [0, \pi].
\]
A functional \( L \ni l \to \text{Im}(l(z)) \) is linear, so for a fixed \( z \in \Delta \), there is
\[
\max \{ \text{Im} l(z) : l \in L \} = \max \{ \text{Im} l_\psi(z) : \psi \in [0, \pi] \}.
\]
But \( \text{Im} l(z) = \arg \frac{f(z)}{z} \) for \( l \in L \). Therefore
\[
\max \left\{ \arg \frac{f(z)}{z} : f \in X' \right\} = \max \left\{ \arg \frac{f_i(z)}{z} : t \in [-1, 1] \right\},
\]
where \( f_i \) is given by (9). Hence, for \( z \in \Delta \setminus \{0\} \),
\[
\max \{ \arg f(z) : f \in X' \} = \max \{ \arg f_i(z) : t \in [-1, 1] \}.
\]
Applying Lemma 3, we conclude that for every \( f \in X', r \in (0, \sqrt{3}/3) \) and \( \varphi \in [0, \pi] \), the following inequality holds:
\[
\arg f(re^{i\varphi}) \leq \arg f_1(re^{i\varphi}).
\]
Consequently, for every function \( f \in X' \), from \( |\arg f_1(re^{i\varphi})| \leq \pi \), it yields that \( |\arg f(re^{i\varphi})| \leq \pi \), which combined with Lemma 4 gives the assertion. \( \square \)

In the paper [4], the class \( T \) of semi-typically real functions was defined. Namely, \( f \in T \) if
\[
z \in (0, 1) \quad \text{if and only if} \quad f(z) > 0.
\]
This equivalence means that the values of \( f \) belonging to \( T \) are positive real numbers if and only if \( z \in \Delta \) is positive and real. According to this definition, \( T \subset \mathcal{T} \).

Based on the proof of Theorem 11, one can anticipate that functions \( f \in X' \) are semi-typically real at most in the disk with radius \( r_{\mathcal{T}} \). The number \( r_{\mathcal{T}} \) is chosen such that the level curves \( f(\{z \in \mathbb{C} : |z| = r\}) \) for \( r < r_{\mathcal{T}} \) and \( f \in X' \) may wind around the origin, yet they do not touch the positive real halfaxis. Moreover, one can anticipate that the extremal function is still \( f_1 \).

**Conjecture.** The radius of semi-typical reality in \( X' \) is equal to \( r_{\mathcal{T}}(X') = r_2 \), where \( r_2 = 0.718 \ldots \) is the only solution of equation

\[
\arcsin \frac{1 - r^2}{2r} + \frac{2(1 - r^2)}{1 + r^2 - \sqrt{1 + 6r^2 - r^4}} = 2\pi. \tag{64}
\]

It is worth emphasizing that in the proof of Lemma 3 we did apply the assumption \( r \in (0, \sqrt{3}/3) \), which is equivalent to \( q^2 > 4/3 \). The number \( \sqrt{3}/3 \) in this expression is not necessarily sharp. Hence, the argument given in the proof of Theorem 11 is not sufficient to prove this conjecture.

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