Separation and symmetry on two dimensional manifolds

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Abstract

We introduce notions of a separated solution and of a simple symmetry that generates a differential operator on a smooth manifold. We prove that a differential operator on a two dimensional manifold has a separated solution if it has a homogeneous simple symmetry of degree one which does not generate the operator.

Keywords: Symmetry operators; separation of variables; separation on manifolds; Bessel functions

Classification: 58J70, 35A30, 94A11

1 Introduction

We investigate separable solutions of partial differential operators on smooth manifolds. Our aim is a precise formulation and a proof of a separation theorem. It is well known that the problem of variables’ separations is connected to symmetries. See Miller [7], Kostant [5] and Eastwood [2] for varied approaches. Let us remark that the notion of a separable operator is not precisely defined in the book of Miller [7], although assertions concerning this notion are contained there. In Koornwinder [4], we find a broad discussion on what an appropriate definition of a separable operator shall be and a history of this problem. According to [4], the history of the ‘separation problem’ goes back to Stäckel [10]. Koornwinder writes in the ibid. citation that he is motivated by a missing definition of a separable operator in Miller [7]. This was one of the motivations for authors of this paper as well. Miller gives a definition of a separable operator, which is subsequently, used and investigated. See, e.g., Hainzl [3].

In particular we are interested in smooth manifolds of dimension two. Let us remark that we define neither a separable differential operator, nor a separable equation. We define the notion of a separable solution with respect to a

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chosen coordinate system instead. Our definition of a symmetry operator of a
differential operator is slightly different from the definition in Miller [7], who
set it in the case of the Helmholtz and Laplace operators. Namely, we do not
bound the order of the symmetry operator. The definition is also different from
the definitions of Eastwood [2] and Kostant [5]. These definitions are compared
in the paper briefly.

We also define the notions of the invariance of an operator with respect to a
smooth function, and give conditions under which a symmetry operator is said
to generate a given differential operator. In the case of the Helmholtz operator
on the Euclidean spaces $\mathbb{R}^2$ and $\mathbb{R}^3$, Miller in [7] does not treat the problem
explicitly. We prove that a differential operator $D$ defined on a two dimensional
manifold has a non-constant separated solution if there is a homogeneous degree
one symmetry which is so called simple and does not generate $D$ (Theorem 3).
Let us remark that a homogeneous degree one symmetry is a vector field. If we
omitted in the formulation of the theorem that the solution is non-constant, the
assertion would be almost trivial because the zero solution is always separated.
At the end of the paper, we apply our results in the well known case of the
Helmholtz operator on the Euclidean plane.

2 Separation theorem

Let $M$ be a smooth manifold of dimension $n$ and let us denote the filtered
ring of complex linear differential operators on $M$ by $DO(M)$. The filtration
is given by the order of the operator. See, e.g., Seeley [9] for a definition of
a differential operator on a manifold. The ring $DO(M)$ is non-commutative if
$n \geq 1$. The word linear is usually omitted in differential geometry. We follow
this convention. The multiplication in the ring is the composition and the
addition is the addition of operators. The ring $DO(M)$ is a left module over the
ring $C^\infty(M)$ of smooth complex valued functions on $M$. The module structure is
defined by setting $(f \cdot D)(g) = fD(g)$ for $f, g \in C^\infty(M)$ and $D \in DO(M)$, where
at the right hand side the point-wise multiplication of functions is considered.
In the paper, we omit writing the dot in $f \cdot D$.

Let $R$ denote the subring of the ring $C^\infty(M)$ consisting of constant functions.
The ring $DO(M)$ is also an $R$-algebra, i.e., it is a ring and an $R$-module satisfying
the compatibility condition $r(D_1D_2) = (rD_1)D_2 = D_1(rD_2)$ for each $D_1, D_2 \in
DO(M)$ and $r \in R$. We identify $R$ with the ring $\mathbb{C}$ of complex numbers.

Notation: Let us recall that an ordered $n$-tuple with entries in $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
is called a multiindex of length $n$. For a multiindex $I = (i_1, \ldots, i_n)$, we set $|I| = \sum_{j=1}^n i_j$. We consider the component-wise addition and subtraction of
multiindices.

If $(U, \phi)$ is a map in the atlas of the manifold $M$, we define functions $x^i : U \to \mathbb{R}$, $i = 1, \ldots, n$ (the coordinate functions) by
$\phi(m) = (x^1(m), \ldots, x^n(m))$, where $m \in U$. We write $\partial_{x^i}$ for the appropriate local vector field $\frac{\partial}{\partial x^i}$ which is
For each $i$, vector fields. We define a function that does not vanish. These are actually vector fields on the manifold. We refer also to Björk [1] for algebraic properties of the subalgebra of $DO(\mathbb{R}^n)$ which consists of operators which are of the form $\sum p^I \partial_I$ (finite sum), where $p^I$ are polynomials with respect to the Cartesian coordinates on $\mathbb{R}^n$, the so called Weyl algebra.

**Remark:** Notice that when writing $D = \sum f^I \partial_I$ (finite sum), even locally on an open set $U$, the coefficients $f^I$ are uniquely defined by a choice of the manifold map. We write the functions first and then the composition of the vector fields.

For a chosen map, we can decompose $D$ of order $d$ as $\sum_{i=0}^d \sum_{|I|=i} f^I \partial_I$. For each $i$, the sum $\sum_{|I|=i} f^I \partial_I$ is called the homogeneous component of $D$ of degree $i$. A differential operator is called a homogeneous operator of degree $i$ if it equals to a single homogeneous component of degree $i$ only. Note that the notion of the homogeneous operator depends on coordinates (see the example of the Laplace operator below). However, the zero degree component is independent on coordinates, since it is given by the evaluation of the operator on the constant function $1$. Independently on maps, we may speak about first order homogeneous operators as those first order operators whose homogeneous component of degree zero vanishes. These are actually vector fields on the manifold.

We refer also to Björk [1] for algebraic properties of the subalgebra of $DO(\mathbb{R}^n)$ which consists of operators which are of the form $\sum f^I \partial_I$ (finite sum), where $p^I$ are polynomials with respect to the Cartesian coordinates on $\mathbb{R}^n$, the so called Weyl algebra.

**Definition 1:** Let $(U, \phi)$ be a map on $M$ and $1 \leq k \leq n$. We say that $f \in C^\infty(U)$ does not $\phi$-depend on the first $k$ variables, if

$$(f \circ \phi^{-1})(x'_1, \ldots, x'_k, x_{k+1}, \ldots, x_n) = (f \circ \phi^{-1})(x''_1, \ldots, x''_k, x_{k+1}, \ldots, x_n)$$

for any $(x'_1, \ldots, x'_k, x_{k+1}, \ldots, x_n), (x''_1, \ldots, x''_k, x_{k+1}, \ldots, x_n) \in \phi(U)$. Similarly we define a function that does not $\phi$-depend on the last $k$ variables.

**Remark:**

1. Degenerate cases. If $f$ does not $\phi$-depend on the first $n$ or on the last $n$ variables, then $f$ is locally constant. Naturally, we are interested in cases when $f$ is not locally constant, which leads to $k < n$.

2. If $f$ does not $\phi$-depend on the first $k$ variables, we have $\partial_{x^i} f = \ldots = \partial_{x^j} f = 0$ where $\phi = (x^1, \ldots, x^n)$. The opposite implication holds locally. Moreover, if $M \subseteq \mathbb{R}^n$ and $U$ is a convex set, the implication can be reversed.
**Definition 2:** Let \((U, \phi)\) be a map on \(M\) and \(g\) be a smooth function on \(U\) which does not \(\phi\)-depend on the first or on the last \(k\) variables. We say that a differential operator \(D\) is \(g\)-invariant with respect to \((U, \phi)\) if there exists a differential operator \(D'\) such that \(D(gh) = gD'(h)\) for any smooth function \(h\) on \(U\) which does not \(\phi\)-depend on the last or on the first \(n - k\) variables, respectively.

**Remark:** If \(g\) is locally constant, then any differential operator is \(g\)-invariant (with respect to any manifold map) since one can take \(k = n\) and \(D' = D\).

**Notation:** If \((U, \phi)\) is a map, \(f : U \to \mathbb{C}\) is a function on \(U\) and \((x_0^1, \ldots, x_0^n) \in \phi(U)\) is a point in \(\mathbb{R}^n\), we write \(f(x_0^1, \ldots, x_0^n)\) briefly instead of \((f \circ \phi^{-1})(x_0^1, \ldots, x_0^n)\).

**Example 1:** Operator \(D'\) from Definition 2 (the case of first \(k\) variables) may contain partial derivatives with respect to some of the last \(n - k\) variables and still fulfil the definition. Namely, let \(M = \mathbb{R}^2 \setminus (-\infty, 0] \times \{0\}\) be equipped with the polar patch \((r, \theta) \rightarrow (r \cos \theta, r \sin \theta) \in M\), where \((r, \theta) \in (0, \infty) \times (-\pi, \pi)\). We denote the appropriate inverse of the patch by \(\phi\) and set \(f(r, \theta) = e^{\lambda \theta}, \lambda \in \mathbb{C}\). The function does not \(\phi\)-depend on the first variable. Since the Euclidean Laplace operator \(\Delta = \partial_r^2 + \partial_\theta^2\), restricted to smooth functions on \(M\), equals to \(\partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2\), we see that for any function \(h : r \mapsto h(r), \ r \in (0, \infty)\), which does not \(\phi\)-depend on the last coordinate

\[
(\partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2)(e^{\lambda \theta} h) = e^{\lambda \theta}(\partial_r^2 + \frac{1}{r} \partial_r + \frac{\lambda^2}{r^2}) h = e^{\lambda \theta} D'(h),
\]

where \(D' = \partial_r^2 + \frac{\lambda^2}{r^2}\). Thus for any \(\lambda \in \mathbb{C}, \Delta\) is \(e^{\lambda \theta}\)-invariant. We see that \(D'\) contains only partial derivatives in the variable \(r\). Nevertheless, we can take \(D' = \partial_r^2 + \frac{1}{r} \partial_r + \frac{\lambda^2}{r^2} + D'',\) where \(D''\) is a differential operator containing partial derivatives only with respect to \(\theta\) and whose zero order term is zero. In particular, we see that \(D'\) may depend on the last variable \(\theta\), and also that it is not unique.

Any associative algebra \(A\) over a field gives rise to the Lie algebra \((A, [\cdot, \cdot])\) with a Lie bracket that equals to the commutator, which is defined by \([C, D] = CD - DC\) for elements \(C, D \in A\).

**Definition 3:** Let \(D\) be a differential operator. We call a differential operator \(L\) a symmetry operator of \(D\) if there exists a differential operator \(G\) such that \([D, L] = GD\). We call \(L\) a simple symmetry operator of \(D\) if \([D, L] = 0\). We say that a differential operator \(L\) generates a differential operator \(D\) if there exist an open set \(U\) and smooth functions \(f_0, \ldots, f_m\) defined on \(U\) such that the restriction of \(D\) to \(C^\infty(U)\) satisfies \(D = \sum_{i=0}^m f_i L^i\), where \(L^0\) denotes the identity operator on \(C^\infty(U)\).
Remark:

1. The definition of a symmetry operator in Eastwood [2] is more general than our definition. The above definition of a symmetry operator generalizes the definition of Miller [7], pp. 2 and 7. The symmetry operator as defined in Kostant [5], p. 101, is less general than the symmetry operator of Miller in [7] if the formula in the Miller’s definition were considered for a general partial differential operator $D$.

2. For an open set $U$, we consider the set of differential operators $DO(U)$ as a left $C^\infty(U)$-module as explained above. For a differential operator $L$ let us consider the left $C^\infty(M)$-hull (finite sums) of the set $\{L^i, i \in \mathbb{N} \cup \{0\}\}$ and denote it by $\langle L \rangle$. Then “$L$ generates $D$” means precisely that there is an open set $U$ such that $D \in \langle L \rangle$.

Let us mention the following basic property of the symmetry operators though we shall not need it in the proof of the main result (Theorem 3).

In the next lemma, we generalize the invariance which we studied in the case of the Euclidean Laplace operator in Example 1.

**Lemma 1**: If $L$ is a symmetry operator of a differential operator $D$, then $L$ preserves the kernel of $D$, i.e., $L(\text{Ker } D) \subseteq \text{Ker } D$.

**Proof.** If $Df = 0$ for a smooth function $f$, we have $D(Lf) = L(Df) + [D, L]f = GDf = 0$ for a convenient operator $G$. □

**Lemma 2**: Let $M$ be a manifold, $(U, \phi = (x^1, \ldots, x^n))$ be a map on this manifold, $D = \sum_{I \in K} f^I \partial_I \in DO(U)$ be a differential operator and $g(x^1, \ldots, x^n) = e^{\lambda x^1}$ for all $(x^1, \ldots, x^n) \in \phi(U)$. If for each $I \in K$, $f^I$ does not $\phi$-depend on the first variable $x^1$, then for every $\lambda \in \mathbb{C}$ the operator $D$ is $g$-invariant with respect to $(U, \phi)$.

**Proof.** Let $h \in C^\infty(U)$ be a function that does not $\phi$-depend on the first
coordinate. We have

\[
\left( \sum_{I \in K} f^I \partial_I \right) (e^{\lambda x^1} h) = \sum_{I=(i_1, \ldots, i_n) \in K} f^I \partial_{x_1} \ldots \partial_{x_{i_1}} \partial_I - (i_1, 0, \ldots, 0) (e^{\lambda x^1} h) = \sum_{I=(i_1, \ldots, i_n) \in K} f^I \partial_{x_1} \ldots \partial_{x_{i_1}} (e^{\lambda x^1} \partial_I - (i_1, 0, \ldots, 0) h) = \sum_{I=(i_1, \ldots, i_n) \in K} f^I \lambda^{i_1} e^{\lambda x^1} \partial_I - (i_1, 0, \ldots, 0) h = e^{\lambda x^1} \sum_{I=(i_1, \ldots, i_n) \in K} \lambda^{i_1} f^I \partial_{(0, i_2, \ldots, i_n)} h.
\]

Setting \( D' = \sum_{I=(i_1, \ldots, i_n) \in K} \lambda^{i_1} f^I \partial_{(0, i_2, \ldots, i_n)} \), we see that \( D \) is \( e^{\lambda x^1} \)-invariant with respect to \((U, \phi)\).

**Definition 4:** Let \((U, \phi)\) be a map on a manifold \(M\). We say that a solution \(f \in C^\infty(U)\) of \(Df = 0\) is \(\phi\)-separated (or a \(\phi\)-separated solution) if there exists an integer \(1 \leq k < n\) and functions \(g, h \in C^\infty(U)\) such that \(f = gh\) where \(g\) does not \(\phi\)-depend on the first \(k\) variables and \(h\) does not \(\phi\)-depend on the last \(n-k\) variables.

**Remark:** Let us suppose that the dimension of the manifold \(n > 1\). Then \(f = 0\) is always a \(\phi\)-separated solution. If \(D\) does not contain the zero order term (in some and consequently in any coordinates), any constant function is a separated solution of \(D\).

Notice also that allowing \(k = n\) in the above definition would lead to the conclusion that any solution \(f\) is a \(\phi\)-separated solution since we might take \(g\) to be an arbitrary non-zero constant function and set \(h = f/g\).

If \(n = 1\), we agree that no separable solution exists.

In the next theorem, we prove that the existence of a specific first order homogeneous simple symmetry operator implies the existence of a separated solution, which is non-constant. (The operator is actually a vector field commuting with \(D\).)

**Theorem 3:** Let \(M\) be a two dimensional manifold, \(D \in DO(M)\) be a differential operator and \(L\) be a first order homogeneous simple symmetry operator of \(D\) which does not generate \(D\). Then there is a point \(m \in M\) and a map \((U, \psi)\) around \(m\) such that there exists a non-constant \(\psi\)-separated solution of \(Df = 0\).

**Proof:** Since \(L\) is a first order homogeneous operator, it is a non-zero vector field in particular. Let us pick a point \(m_0 \in M\) such that \((Lf)(m_0) \neq 0\) for a smooth function \(f\) on \(M\). By the theory of ordinary differential equations, there
exists a map \((U_0, \phi = (x^1, x^2))\) around \(m_0\) such that \(L = \partial_{x^1}\) on \(U_0\). (The map can be defined using the local flow of \(L\) and the flow of a vector field \(L'\) which may be taken perpendicular to \(L\) with respect to a Riemannian metric defined around \(m_0\).)

Let us choose the connected component of \(U_0\) containing \(m_0\), keeping denoted it by \(U_0\). Write the restriction of \(D\) to \(C^\infty(U_0)\) as \(D = \sum_{|\ell| \leq d} f^\ell \partial^\ell\) (with respect to the map \(\phi\)), where \(f^\ell\) are smooth functions on \(U_0\) and \(d\) is a non-negative integer. Since \(L\) is a simple symmetry, we have \(0 = [L, D] = [\partial_{x^1}, D] = \sum_{|\ell| \leq d} (\partial_{x^1} f^\ell) \partial^\ell\) because the coordinate vector fields commute with each other. In particular, \(\partial_{x^1} f^\ell = 0\) and thus, there exists an open set \(U_1\) such that \(f^\ell\) does not \(\phi\)-depend on \(x^1\). By Lemma 2, \(D\) is \(g\)-invariant for \(g(x^1, x^2) = e^{\lambda x^1}\) with respect to \((U_1, \phi)\).

Because \(D\) is \(g\)-invariant, we have the operator
\[
D' = \sum_{j=0}^{d_2} \left( \sum_{i=0}^{d_1} \lambda^{i} f^{(i,j)} \right) \partial_{x^2}^{j},
\]
at our disposal. We take it in this form, that is derived in the proof of Lemma 2. By assumption, \(L = \partial_{x^1}\) does not generate \(D = \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} f^{(i,j)} \partial_{x^1}^i \partial_{x^2}^j\).

Thus \(D\) has to contain at least one non-zero function coefficient \(f^{(i,j)}\) in front of \(\partial_{x^1}^i \partial_{x^2}^j\) for \(0 \leq i \leq d_1\) and \(0 < j \leq d_2\). Let us denote its indices by \(i_0\) and \(j_0\).

Let us consider a point \(m_1 \in U_1\) such that \(f^{(i_0,j_0)}(m_1) \neq 0\), and the polynomial \(P(\lambda) = \sum_{i=0}^{d_1} f^{(i,j_0)}(m_1) \lambda^i\), which has to be non-zero. As a consequence of the continuity of \(P\), there is an infinite set of complex numbers \(\lambda\) for which this polynomial attains a non-zero value. Especially, it is possible to choose \(\lambda_0 \neq 0\) from this set. Since \(j_0 \geq 1\), \(D'\) is a differential operator of order at least one. From now, we consider \(D'\) only for \(\lambda = \lambda_0\).

Since the chosen operator \(D'\) contains derivatives only in the second variable and its function coefficients do not \(\phi\)-depend on \(x^1\), \(D'\) is a differential operator in \(x^2\). By the basic theory of ordinary differential equations, \(D' h = 0\) has a non-zero solution in an open neighbourhood \(U \subseteq U_1\) of \(m_1\) since the degree of \(D'\) is at least 1. We set \(\psi = \phi|_U\), choose such a non-zero solution that does not \(\psi\)-depend on \(x^1\), and denote it by \(\tilde{h}\). The function \(f = g\tilde{h} = e^{\lambda_0 x^1} \tilde{h}\) is non-constant. It is \(\psi\)-separated and it is a solution of \(Df = 0\) since
\[
D(e^{\lambda_0 x^1} \tilde{h}) = e^{\lambda_0 x^1} D'(\tilde{h}) = 0.
\]

It follows, that \((U, \psi = \phi|_U)\) is a map around \(m = m_1\) and \(f = e^{\lambda_0 x^1} \tilde{h}\) is a non-constant \(\psi\)-separated solution of \(Df = 0\) defined on \(U\).

\[\square\]

**Remark:**

1. Instead of the continuity of \(P\) in the above proof, we could use the fact that non-zero polynomials in one variable have only a finite number of roots.
2. From the proof of Theorem 3, we see that if we can take even $j \geq 1$, there would be a non-constant solution $\tilde{h}$ of $D'h = 0$, which does not $\phi$-depend on $x^1$. Thus $f(x^1, x^2) = e^{Ax^1} \tilde{h}(x^2)$ would be a separated solution, whose first factor depends on $x^1$ only and the second one only on $x^2$. More precisely, it is not true that the first factor does not $\phi$-depend on $x^1$ and it is not true that the second one does not $\phi$-depend on $x^2$.

3 The Helmholtz operator

As an application of Theorem 3, we derive a family (depending on a continuous parameter $\lambda$) of classical separated solutions for the Helmholtz operator in two variables, i.e., for

$$D = \partial_x^2 + \partial_y^2 + \omega^2, \text{ where } \omega \in (0, \infty)$$

acting on smooth functions on $M = \mathbb{R}^2$. Notice that the result is well known (see, e.g., [7]). The operator $L = x\partial_y - y\partial_x$ is a first order homogeneous operator, which is a simple symmetry as the following computation shows

$$\begin{align*}
[x\partial_y - y\partial_x, D] &= [x\partial_y - y\partial_x, \partial_x^2 + \partial_y^2] \\
&= (x\partial_y - y\partial_x)(\partial_x^2 + \partial_y^2) - (\partial_x^2 + \partial_y^2)(x\partial_y - y\partial_x) \\
&= x\partial_y \partial_x^2 + x\partial_y \partial_y^2 - y\partial_x \partial_x^2 - y\partial_x \partial_y^2 \\
&\quad - \partial_x (x\partial_y + x\partial_x \partial_y) + y\partial_x^3 - x\partial_y^3 + \partial_x (x\partial_y + y\partial_y \partial_x) \\
&= x\partial_y \partial_x^2 + x\partial_y \partial_y^2 - y\partial_x \partial_x^2 - y\partial_x \partial_y^2 - \partial_x \partial_y - \\
&\quad - x\partial_x^2 \partial_y + y\partial_y^3 - x\partial_y^3 + \partial_y \partial_x + \partial_y \partial_x + y\partial_x \partial_y \partial_x = 0.
\end{align*}$$

We shall find the map $\phi$ and also a $\phi$-separated solution. For simplicity, let us consider only such points which do not lie on the non-positive part of the $x$-axis. (This limitation can be removed by translating the found map by a constant vector.) By the proof of Theorem 3, the symmetry operator $L$ has to be locally equal to a derivative with respect to a coordinate function of the map $\phi$. It is easy to realize that $L = \partial_\theta$, where we use the polar patch $(r, \theta)$ in the form introduced in Example 1. (Obviously, $L$ does not determine the map $\phi$ uniquely.) In the considered polar patch, the Helmholtz operator has the form

$$D = \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2 + \omega^2.$$

Function coefficients of $D$ in the polar patch in front of the partial derivatives do not $\phi$-depend on the second variable, i.e., on $\theta$. From this expression, we see that $L$ does not generate $D$ which contains a differentiation with respect to $r$. According to Theorem 3, the Helmholtz operator has a non-constant separated solution with respect to the map $\phi$ around any point in $M$. 

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Using the construction from the proof of Theorem 3, we get
\[(\partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2 + \omega^2)(e^{\lambda \theta} h) = e^{\lambda \theta} (\partial_r^2 + \frac{1}{r} \partial_r + \frac{\lambda^2}{r^2} + \omega^2) h\]
where \(h : r \mapsto h(r)\) does not \(\phi\)-depend on the last variable. We choose \(D' = \partial_r^2 + \frac{1}{r} \partial_r + \frac{\lambda^2}{r^2} + \omega^2\). This operator is of degree 2 independently on \(\lambda\). By the theory
of ordinary differential equations, we know that there exists not only a non-zero but also a non-constant solution of \(D'h = 0\), i.e., of \((\partial_r^2 + \frac{1}{r} \partial_r + \frac{\lambda^2}{r^2} + \omega^2) h = 0\).
Some of the solutions of this equation are the well known Bessel functions of the first kind \(\tilde{h}(r) = J_l(\omega r), r > 0\), where \(l = \pm i \lambda\). (We omit a description of the whole fundamental system of solutions for \(D'h = 0\), i.e., for the so called Bessel equation. See references below.) All of these functions are non-constant, the function \(J_0(\omega r)\) inclusively. (See, e.g., [6], [8] or [11].) Hence for every complex \(\lambda \neq 0\), the functions \((0, \infty) \times (-\pi, \pi) \ni (r, \theta) \mapsto f_{\pm \lambda}(r, \theta) := e^{\lambda \theta} J_{\pm \lambda}(\omega r)\) are 
\((r, \theta)\)-separated solutions for the Helmholtz equation \(Df = 0\) on \(M\).

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