Bounding quantum contextuality with lack of third-order interference

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Abstract

Recently much interest has been generated by the search for simple principles that can explain the quantum limitations on possible sets of experimental probabilities in non-locality and contextuality experiments, as compared to more general theories. Approaching from considerations of quantum gravity, Sorkin has proposed lack of irreducible third-order interference as the most fundamental property of quantum mechanics. Here it is shown that this principle implies the principle known as consistent exclusivity or local orthogonality. This explains the previous result that lack of third-order interference (along with the condition that several independent copies of any realisable behaviour should again be realisable) rules out the existence of the Popescu Rorlich box and the Wright pentagon, and implies new results such as bounds on violations of the CHSH inequality, and lack of quantum advantage in the Guess Your Neighbour’s Input game.

1 Introduction

It is of great interest to formulate simple principles obeyed by quantum mechanics (QM), from which other interesting features of the theory can be derived in a straightforward way. Such principles can provide simple derivations of otherwise mysterious or difficult results. They also clarify the options when we consider what properties of quantum mechanics are likely to be most fundamental, i.e. most likely to persist in more developed physical theories. This question has relevance for quantum gravity, where many have considered going beyond standard quantum mechanics in the light of such issues as black hole evaporation and the problems of time (see e.g. [1, 2, 3]). This has led to a convergence of interests from the study of quantum information and quantum gravity, in which both sides stand to gain new understanding.

In nonlocality experiments, QM allows only a specific set of experimental probabilities or “behaviours” [4, 5, 6], and a ban on superluminal signalling is far from being enough to explain these limitations [7]. It is interesting to look for an explanation of this in terms of simple

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principles; the same question can be asked for broader classes of “contextuality scenarios”. As such principles proliferate in the literature [8, 9, 10, 11, 12], it becomes increasingly important to ask whether there are logical relations between them [13]. The principle of local orthogonality or consistent exclusivity [14, 15] (also called the “E principle,” and closely related to orthomodularity and orthocoherence in the earlier quantum logic literature [16]) is of particular interest. This very simple operational principle does indeed explain many important features of quantum nonlocality and contextuality. Coming from the quantum gravity perspective, Sorkin has suggested that “lack of (irreducible) third order interference” be viewed as the most fundamental feature of QM [17]. This principle has recently been applied in the context of generalised probabilistic theories as one of a number of postulates from which quantum mechanics can be reconstructed [18, 19]. As we will see, superficially the two principles have some similarities, and Sorkin’s principle has been shown to imply some of the same restrictions as has consistent exclusivity [20, 21]. Here, we explain this by showing that the former implies the latter, for a broad set of contextuality scenarios that is natural from the perspective of Sorkin’s original definition. Some relations between a strengthening of this condition and others are to be found in [9, 22]; here the weakest form of the principle is investigated in a more general (and more simply defined) class of experimental scenarios.

**Partition scenarios.** Consider a hypothetical experiment in which a number of measurements can be made, which may be incompatible in the sense that carrying out one may affect the results of others. We will allow the identification of particular outcomes of different measurements (a concrete example being the identification of outcomes in QM experiments when they correspond to the same Hilbert space subspace). Now consider a “sample space” \( \Xi \), and let us identify every measurement with a partition \( \mathcal{M} \) of \( \Xi \), and every “fine-grained” outcome of that measurement with a set \( A \in \mathcal{M} \). In this way an element of \( \Xi \) specifies an outcomes for every experiment, although this carries little physical meaning so far. The set of all measurements will be called \( \mathcal{M} \). The term coarse-grained outcomes for a measurement \( M \) will refer to unions of the fine-grained outcomes in \( M \). For each \( M \in \mathcal{M} \) the set of all coarse-grained outcomes together with the empty set form a Boolean algebra, \( \mathfrak{a}_M \), generated by the fine-grained outcomes. The set of all coarse-grained outcomes across all measurements will be denoted \( \mathcal{C} := \bigcup_{M \in \mathcal{M}} \mathfrak{a}_M \). Note that, as desired, the same outcome may appear in two different measurements. The space \( \Xi \) together with the set \( \mathcal{M} \) of all measurements specifies a partition scenario \( \mathcal{S} = \{ \Xi, \mathcal{M} \} \).

An experimental behaviour is defined by a partition scenario along with a consistent probability function \( P(\cdot) \) whose domain is the set of all outcomes \( \mathcal{C} \). The function \( P \) is only required to be a probability measure when restricted to the Boolean algebra of outcomes \( \mathfrak{a}_M \) for a measurement \( M \); thus it provides all experimental probabilities while making sure that outcomes that are identified in different measurements have the same probability. This is a broad class of contextuality scenarios, which is more general than the “joint measurement scenarios” of [22] and thus includes all nonlocality scenarios. Other frameworks can be defined which treat cases not covered by partition scenarios, however [23].

**Conditions on experimental behaviours.** Non-contextuality in this framework is a condition on experimental behaviours. It requires that there exists a joint probability distribution \( P_J \) on \( \Xi \) such that \( P_J(A) = P(A) \ \forall A \in \mathcal{C} \), that is, the behaviour can be derived from a
probability distribution over the whole sample space. It is well-known that this principle is incompatible with quantum mechanics. Consistent Exclusivity (CE) represents a weakening of non-contextuality \([15, 14, 23]\). We will say that outcomes \(A\) and \(B\) are exclusive if there exists a measurement \(M \in \mathcal{M}\) such that \(A, B \in M\).

**Definition 1.** Consider a behaviour \(\{P, S\}\) and a set of fine-grained outcomes \(S\) such that \(A\) and \(B\) are exclusive for all pairs \(\{A, B\} \subset S\). The behaviour obeys Consistent Exclusivity if, for all such sets,

\[
\sum_{A \in S} P(A) \leq 1. \tag{1}
\]

Note that this condition implies that the same holds for sets of coarse-grained outcomes.

Sorkin’s “lack of third order interference” has a similar flavour, only here, instead of retreating from noncontextuality by only assigning probabilities to outcomes, the approach is to carry on considering all subsets of \(\Xi\) on a similar footing, but to attribute a joint “generalised measure” to these sets instead of a joint probability measure. This may allow “interference of probabilities” in the sense that the Kolmogorov rule need no longer apply. In particular a quantum measure is one that bans tripartite interference, in the following sense.

**Definition 2.** A behaviour \(\{P, S\}\) admits a joint quantum measure, if there exists a function \(\mu : 2^{\Xi} \rightarrow \mathbb{R}\) such that

(a) \[\mu(A) \geq 0 \quad \forall A \subset \Xi; \tag{2}\]

(b) For any three disjoint sets \(A \subset \Xi, B \subset \Xi\) and \(C \subset \Xi,\)

\[
\mu(A) + \mu(B) + \mu(C) - \mu(A \cup B) - \mu(B \cup C) - \mu(C \cup A) + \mu(A \cup B \cup C) = 0; \tag{3}\]

(c) \[\mu(A) = P(A) \quad \forall A \in \mathcal{C}. \tag{4}\]

Equation (3) is known as the Sorkin sum rule (or “quantum sum rule”). Substituting the Kolmogorov sum rule here would give the definition of a joint probability distribution. The last condition, (4), states that the quantum measure reduces to the experimental probabilities when restricted to the measurement outcomes. This is true of the quantum measure as usually defined in standard quantum mechanics (and, though Sorkin intended the definition of the quantum measure to be a starting point for a realistic, histories-based interpretation of QM, this is all we need in the present context). It is not hard to check that both consistent exclusivity and lack of tripartite interference are implied if there is a standard quantum model for the behaviour [23, 22].

Having defined both of these principles, some similarities are apparent. Having given up on non-contextuality, both principles aim to partially restore it, and both state that if pairs of events in some set behave classically in some sense, then the whole set will as well. The main difference is that the first principle refers only to the experimental probabilities while the
second makes its statements in terms of a generalised measure on the same “non-contextual” sample space. This connection can be formalised. To do so, the following lemma will be useful, which essentially shows that the Sorkin sum rule implies that there is no higher-order interference that is not a consequence of pairwise interference.

**Lemma 3.** Consider a behaviour \( \{P, S\} \) that admits a joint quantum measure, and consider a partition \( Q \) of a set \( X \subseteq \Xi \). If \( \mu(A) + \mu(B) = \mu(A \cup B) \) for all \( A, B \in Q \) then

\[
\mu(X) = \sum_{A \in Q} \mu(A) \tag{5}
\]

**Proof.** For \( |Q| = 2 \) the statement is trivially true. Assume the inductive hypothesis that the lemma holds in all cases with \( |Q| \leq n \) for some \( n \geq 2 \), and consider a partition \( Q \) of a set \( X \) for which \( |Q| = n + 1 \), obeying the condition \( \mu(A) + \mu(B) = \mu(A \cup B) \) for all \( A, B \in Q \). Let \( A, B \in Q \) be two events in the partition and let \( Y = X \setminus (A \cup B) \). Applying the Sorkin sum rule \( (3) \) to \( \{Y, A, B\} \), the following is obtained:

\[
\mu(Y) + \mu(A) + \mu(B) - \mu(Y \cup A) - \mu(A \cup B) - \mu(Y \cup B) + \mu(X) = 0. \tag{6}
\]

The inductive hypothesis implies that \( \mu(Y \cup B) = \mu(Y) + \mu(B) \), and by assumption \( \mu(A \cup B) - \mu(A) = \mu(B) \), and so this implies

\[
\mu(X) = \mu(Y \cup A) + \mu(B) = \sum_{C \in Q} \mu(C), \tag{7}
\]

where the inductive hypothesis has been applied to the set \( Y \cup A \) in the last step. \( \square \)

With this we can prove the following theorem.

**Theorem 4.** Consider a behaviour \( \{P, S\} \). If the behaviour admits a joint quantum measure then it obeys Consistent Exclusivity.

**Proof.** Assume that the behaviour admits a joint quantum measure, and consider a set of fine-grained outcomes \( S \) such that \( A \) and \( B \) are exclusive for all pairs \( \{A, B\} \subseteq S \). Let us define the sets \( X = \bigcup_{A \in S} A \) and \( R = \Xi \setminus X \), and also the set \( Q = S \cup \{R\} \). The first step of the proof is to show that \( Q \) satisfies the conditions in lemma \( 3 \). Because \( A \) and \( B \) are exclusive for all pairs \( \{A, B\} \subseteq S \), all members of \( S \) are disjoint, and \( R \) is disjoint from these, so \( Q \) is a partition of \( \Xi \). Furthermore, using \( (4) \), we have that \( \mu(A) + \mu(B) = \mu(A \cup B) \) for all \( A, B \in S \). Now, for some \( B \in S \), let us apply the Sorkin sum rule \( (3) \) to the sets \( Y = X \setminus B \), \( B \) and \( R \). We obtain

\[
\mu(Y) + \mu(B) + \mu(R) - \mu(Y \cup B) - \mu(Y \cup R) - \mu(B \cup R) + \mu(\Xi) = 0 \tag{8}
\]

Applying lemma \( 3 \) gives \( \mu(Y) + \mu(B) = \mu(Y \cup B) \), and we have \( \mu(Y \cup R) + \mu(B) = \mu(\Xi) \) from \( (4) \) because \( B \) is a measurement outcome, giving the result

\[
\mu(B) + \mu(R) = \mu(B \cup R), \tag{9}
\]

and so because this applies for all \( B \in S \) we have \( \mu(A) + \mu(B) = \mu(A \cup B) \) for all \( A, B \in Q \). From this lemma \( 3 \) gives \( \sum_{A \in S} \mu(A) = \mu(\Xi) = 1 \). Subtracting \( \mu(R) \) and using \( (2) \), we have that \( \sum_{A \in S} \mu(A) \leq 1 \). Using \( (4) \), this establishes that consistent exclusivity holds for the behaviour, proving the theorem. \( \square \)
This allows a number of interesting results to be imported into quantum measure theory from the study of local orthogonality and Consistent Exclusivity, of which the following are of particular interest.

**Corollary 5.** The following are properties of all behaviours on partition scenarios that admit a joint quantum measure:

(i) they allow no advantage over classical (non-contextual) behaviours for the Guess Your Neighbour’s Input Game;

(ii) for the CHSH scenario, the existence of two independent copies of this behaviour with maximum violation of the CHSH inequality of more than 2.883 is banned;

(iii) They imply the quantum bound, $\sqrt{5}$, on the maximum violation of the KCBS inequality for two independent copies of the Wright pentagon scenario.

**Proof.** Nonlocality scenarios and copies thereof are partition scenarios [22], and so (i) and (ii) can be proved by combining theorem 4 with the first proof in the Methods section of [14] and the argument in section 4.3 of [24] respectively. The Wright pentagon can also be constructed as a partition scenario, either by noting that it is embedded in the CHSH nonlocality scenario [15], or directly from the following example: $\Xi = \{\{A, B, C\}, \{D, E, F\}, \{G, H, I\}\}$ and $\mathcal{M} = \{\{A, B, C\}, \{D, E, F\}, \{G, H, I\}\}, \{\{A, B, C\}, \{D, G, H\}, \{E, F, I\}\}, \{\{A, F, I\}, \{B, C, E\}, \{D, G, H\}\}, \{\{A, F, I\}, \{B, C, D\}, \{E, G, H\}\}, \{\{A, E, F\}, \{B, C, D\}, \{G, H, I\}\}\}$. Thus (iii) can be proved by combining theorem 4 with the arguments in [15].

There are many other facts known about Consistent Exclusivity and Local Orthogonality that are now relevant for the principle of lack of tripartite interference, which will not be listed here [23, 25, 26, 24]. Many interesting issue remain open. For example, it would be of interest to know if the converse of the above theorem also true, or whether the theorem be extended to other formalisms, e.g. those of [18, 23]. The stronger forms of joint quantum measure considered in [22] have been justified by appealing to composability, and so it would also be instructive to know if they can be derived from the above by adding some simple composability assumptions. Similarly, it has been asked whether local orthogonality can be strengthened by the addition of further strongly-motivated conditions. In the light of the results above, work on either one of these questions can now inform the other. Hopefully, answering some of these questions will help to clarify what needs to be added to these principles in order to characterise quantum non-locality and contextuality.

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