SCALING OF VOIDS IN
THE LARGE SCALE DISTRIBUTION OF MATTER

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Abstract

Voids are a prominent feature of the galaxy distribution but their quantitative study is hindered by the lack of a precise definition of what constitutes a void. Here we propose a definition of voids in point distributions that uses methods of discrete stochastic geometry, in particular, Delaunay and Voronoi tessellations, and we construct a new void-finder. We then apply the void-finder to scaling point distributions. First, we find the voids of pure fractals with a transition to homogeneity and show that the rank ordering of the voids also scales (Zipf’s law) and, in addition, shows the transition to homogeneity. However, a pure fractal is arguably not a good model of the galaxy distribution, so we construct from a cosmological N-body simulation a bifractal mock galaxy sample representing two galaxy populations, which we identify as “wall” and “field” galaxies. The wall galaxy distribution fits a pure fractal with a transition to homogeneity and, furthermore, the rank ordering of its voids show a scaling range with the right slope plus a transition to homogeneity.

Introduction

The standard statistical analysis of the distribution of galaxies consists in the derivation of the correlation functions, chiefly, the two-point correlation function $\xi$. The large value of this correlation function ($\xi \gg 1$) on small scales is interpreted as the tendency of galaxies to cluster strongly. The reverse side of strong clustering is the appearance of void regions, with essentially no galaxies. Voids have an intrinsic appeal, since our sensory system is well suited to perceive patterns, so we immediately perceive void shapes while looking at a point set. The observation of voids in the distribution of galaxies has led to the compilation of void catalogues and to their statistical analysis (Einasto, Einasto & Gramann, 1989; Vogeley, Geller & Huchra, 1991; El-Ad, Piran & da Costa, 1997, Mueller et al., 2000; Hoyle & Vogeley, 2002). A precise definition of void is however a delicate matter, and it is a moot point to determine to what
extent the conclusions drawn from statistics of voids depend on the particular definition adopted.

Analytic models of structure formation such as the adhesion model as well as numerical simulations show that the distribution of matter and, in particular, the distribution of galaxies constitutes an interpenetrating network of clusters and voids, which has been dubbed “the cosmic web”. This network has a self-similar nature. On the other hand, the hierarchical structure of clusters of galaxies and the analysis of the two-point correlation function provides evidence for a self-similar fractal structure (Mandelbrot, 1977; Coleman & Pietronero, 1992; Sylos Labini, Montuori & Pietronero, 1998) or multifractal structure (Martinez et al, 1990; Jones et al, 1992). It was noted years ago that the cosmic web seems to exhibit a hierarchy of voids, but the scaling properties of voids in the distribution of galaxies were hardly studied (as an exception, see Einasto, Einasto & Gramann, 1989).

Here, we begin by recalling the properties of voids in fractal distributions and its application to the galaxy distribution derived in previous work (Gaite & Manrubia, 2002). Self-similarity is the most obvious property and manifests itself in the scaling of void sizes, which can be conveniently described by the Zipf law (a rank-ordering power law), as shown by Gaite & Manrubia (2002) using regular shape voids. We also showed how to obtain the fractal dimension from this law. However, the lists of voids in galaxy catalogues that we examined fail to show any scaling. We attributed this to shortcomings of current void searching algorithms, regarding their definition of void. Therefore, the analysis of void sizes and their scaling properties is a promising tool but it cannot be really effective until having a suitable definition of void. Here we propose a new definition of void and a new void-finder (using that definition), and explore its application to the scaling properties of cosmological matter distributions.

Existing definitions of void and void-finders differ. We must mention: (i) the empty sphere method (Einasto, Einasto & Gramann, 1989); (ii) the improvement with elliptical regions (Ryden & Melott, 1996); (iii) the progressive construction of voids with cubes + rectangular prisms (Kaufmann & Fairall, 1991); (iv) the related method that uses connected spheres (El-Ad & Piran, 1997); (v) the method of distance field maxima (Aikio & Mahonen, 1998); (vi) the use of the smoothed density field (Shandarin, Sheth & Sahni, 2004). For us it is natural to rely on the methods of discrete stochastic geometry, namely, Delaunay and Voronoi tessellations, the use of which in Cosmology has been pioneered by Rien van de Weygaert and collaborators (Schaap & van de Weygaert, 2000).

We will first describe our void-finder and its geometrical basis. The problem of definition of void is similar in two or higher dimensions, so we will begin with two-dimensional fractal point sets (with transition to homogeneity), whose voids are easy to visualize. This is useful since the notion of void
arises as an intuitive visual notion. The conclusions can be extrapolated to three-dimensional fractals. Next, we show how to apply the method to the galaxy distribution, by means of mock distributions obtained from cosmological $N$-body simulations. Finally, we discuss the results.

1. **Discrete geometry methods and void finder**

   The Delaunay tessellation of a three-dimensional point set consists of a set of links between points forming simplices such that their respective circumscribing spheres do not contain any other point of the set. This tessellation is unique and, moreover, is particularly suited for the search for voids because each circumscribing sphere is void. The Delaunay tessellation can be generalized to any dimension and, in particular, in two dimensions is called the Delaunay triangulation. There is a dual tessellation formed by the centers of the circumscribing (hyper)spheres, called the Voronoi tessellation. Each Voronoi cell is the neighbourhood of a point of the set, in the sense that the points of the interior of the cell are closer to that point than to any other point of the set.

   The Delaunay and Voronoi tessellations are fundamental constructions associated to a point set. In the search for voids, the defining property of the Delaunay tessellation is obviously adequate for the definition of voids: we can consider each simplex as an elementary void. Then we grow a given elementary void by joining adjacent simplices if appropriate, according to some criteria. Consequently, we define a void as set of adjacent in pairs simplices with a boundary given by the separation criteria.

   The natural separation criterium between elementary spherical voids is given by the fractional overlap being below a predetermined threshold, as used by Hoyle & Vogeley (2002). Naturally, our elementary spherical voids are the circumscribing spheres corresponding to adjacent simplices. However, this criterium is not sufficient and we need an additional condition on the relative diameter of the added simplices to prevent the merging of elementary voids of very different size, similar to the prescription of El-Ad & Piran (1997). So our procedure relies on preceding void-finders. Its advantage is that the elementary spheres are given by the Delaunay tessellation.

   To begin the search for voids we need to estimate where we may find the largest one, but we cannot measure the size of a void until it has been found; so we look for the largest Delaunay simplex. The algorithm consists of the following steps:

   1. Construct the Delaunay and Voronoi tessellations for the given point set.
   2. Sort the simplices of the Delaunay triangulation by size and select the largest one to begin to build the first void.
   3. Grow the void by adding adjacent simplices (in this process, the Voronoi tessellation is useful). A simplex is added if the overlap criterium is
met and the ratio of the diameter of its touching face to the diameter of the initial simplex is above a given value ("link ratio"). The set of all simplices found constitutes the void.

4. Iterate by finding the largest simplex among the remaining ones until they are exhausted.

The algorithm is valid in any dimension but we have only applied it to two and three dimensional point sets.

2. Properties of voids in fractal distributions

Our void finding algorithm produces a list of voids roughly ordered by their size. Actually, one must measure the sizes of voids and reorder them accordingly. Thus one gets the voids listed by decreasing size, which is suitable for a study with rank-ordering techniques, that is, for studying how the size of voids decreases with rank. If there is a hierarchy of voids, one must expect a regular decreasing; in particular, a power-law decreasing is the Zipf law (that is, a simple linear decrease in the log-log plot). The interest of this law is that it does not mark any scale and it is, therefore, associated with scale invariance (Gaite & Manrubia, 2002). Concretely, a scaling fractal set must have a scaling void hierarchy. This is simple to prove in one dimension, in which the voids are well defined as intervals, but a proof in higher dimensions requires before a definition of void, such as the one used in our void-finder. So we will apply it to scaling fractal sets to test the Zipf law.

Firstly, let us consider, in one dimension, how the transition to homogeneity is achieved and its effect on the Zipf law. A transition to homogeneity in a one-dimensional random Cantor-like fractal can be achieved by joining by the ends several realizations of it. For the sake of the argument, let us assume that a single realization follows a perfect Zipf law $\Lambda_R = \Lambda_1 R^{-\zeta}$. Then we have in 10 copies of the fractal, say, 10 voids with size $\Lambda_1$ and ranks $R = 1, \ldots, 10$, 10 voids with size $\Lambda_2$ and ranks $R = 11, \ldots, 20$, etc. So the sizes follow the law $\log \Lambda_{10n} = \log \Lambda_1 + \zeta \log 10 - \zeta \log (10n)$, $\Lambda$ being constant between ranks $10n - 9$ and $10n$. This is a stepcase with steps of exponentially decreasing width and linearly descending ends with slope $\zeta$. Relating the condition of an initial perfect Zipf law we smooth the steps, so we conclude that the effect of the transition to homogeneity is the flattening of the Zipf law for small ranks and that the width of the flattened portion measures the scale of homogeneity.

In any dimension, we can generate periodic pure fractals (with transition to homogeneity) with a method based on the theory of fractional Brownian motion. We must check the fractal by computing its number-radius function, which has to be a power law, $N(r) \propto r^D$ (equivalent to $\xi \propto r^{D-3}$ and $\xi \gg 1$). This law must hold for a sufficient range of scales between the minimum inter-point distance and the scale of homogeneity.
In two dimensions, we have run our void-finder on various fractal point sets with different dimensions and various values of the overlap and link ratio thresholds. The Zipf law obtains, but to have the maximum scaling range, that is, similar to the scaling range of the number function, one has to tune the algorithm parameters. Fig. 1 shows the voids found in a fractal with dimension $D = 0.8$ and 3871 points (for both overlap and link ratio = 0.5). Fig. 2 shows its number-radius function and the Zipf’s law for voids. We observe that the transition to homogeneity has a similar aspect in both plots. The effect of the transition to homogeneity in the rank-ordering of voids is the flattening of the Zipf law for small ranks and the width of the flattened portion measures the scale of homogeneity. So an essential feature of both graphs is that there is a crossover between two different regions. The left flat region in the rank-ordering of voids corresponds to the transition to homogeneity whereas the right scaling region corresponds to the expected scaling with slope $-2/D = -2.5$. In contrast, in the graph of the number-radius function, the transition to homogeneity corresponds to the right region, where the slope tends to 3.

The rank-ordering of voids and the corresponding Zipf’s law convey no more information than the number-radius function and, in fact, one needs to tune somewhat the void-finder parameters to extract the same information. This is due to the fact that the morphology of voids depends on the type of fractal. The information on morphology (including features like lacunarity)
Figure 2 Number-radius function of the $D = 0.8$ fractal point set in $d = 2$ and rank ordering of voids (corresponding to the displayed void-finder setting).
Scaling of voids

has value of its own but the existence of various morphologies corresponding to the same scaling dimension poses difficulties for void-finders, which need to adapt to a particular morphology. Obviously, the void-finder parameters must only depend on relative magnitudes, like the overlap and link ratio which we have used, but one has some liberty in their choice.

3. **Bifractal distribution: field and wall galaxies**

Before looking for voids in a galaxy sample, it is usual to divide the sample between wall and field galaxies and remove the latter (El-Ad & Piran, 1997). The rationale for this is that the galaxy distribution is very inhomogeneous, in the sense that there are very rich clusters with high luminosity galaxies and there also are faint galaxies which are weakly clustered. In fact, this idea can be extended to the full dark matter distribution, in connection with the concept of bias: rich galaxy clusters are located in places with very high density of dark matter (massive haloes) and fainter galaxies in places with moderate density (normal haloes), whereas the regions of very low density of dark matter are depleted from galaxies. The definition and structure of voids have already been examined in this context (Benson et al, 2003; Gottloeber et al, 2003).

The pure fractals we have been using are not suitable to incorporate the idea of two different populations of galaxies or, more in general, the idea of biasing. This is because every particle of a pure fractal has the same properties in relation with the other particles. This uniformity characterizes monofractal distributions, as opposed to multifractal distributions, which can be pictured as a superposition of monofractals with various dimensions (Martinez et al, 1990; Jones, Coles & Martinez, 1992).

The general multifractal model of the dark matter distribution assumes a superposition of monofractals with a continuous range of dimensions (in principle, running from the lowest to the highest, namely, from zero to three). Therefore, it would be reasonable to associate to it a multifractal distribution of galaxies. This continuous superposition of monofractals would be impractical for void searching, so we use a cutoff to obtain a two-population distribution that we identify with the wall and field galaxy populations. A bifractal distribution has also been favoured on more fundamental grounds (Balian & Schaeffer, 1988).

We have applied this bifractal model to the distribution of dark matter halos in a simulation with the Hydra code, namely, the $z = 0$ positions of a simulation with $86^3$ particles (sufficient to test our method). We identify dark matter haloes with overdensities (by means of a spatial window) and we populate haloes with galaxies assuming linear bias. To illustrate the appearance of both populations, we show in Fig. 3 the result of this process on a slice of the Hydra
Figure 3 Selection of two populations in a slice: in red the more clustered population ($M = 3539$) and in blue the less clustered population ($M = 2255$).
simulation. Note that the more clustered population (with lower dimension) has larger total mass in spite of being sparser.

One can split the galaxies in two populations in several ways by assigning different halo mass cutoffs. We have found convenient to use a population of 1746 “wall galaxies” (corresponding to halos with total mass $M = 22,738$). In any division in two populations, for consistency, we must check that each population approximately corresponds to a monofractal. This is especially important for the “wall galaxy” population. We have calculated the number-radius relation for the population of 1746 “wall galaxies” and indeed found a power law in a scale range, corresponding to a surprisingly small dimension, namely, $D = 0.4$. Furthermore, the transition to homogeneity on the largest scales is clearly visible (see Fig. 4).

We now apply the void-finder to the population of 1746 “wall galaxies”. We find and rank order the voids, which are a juxtaposition of simplices, as shown in Fig. 5 (we only show from the third to the sixth rank void for ease of visualization). The rank-ordering plot, displayed in Fig. 6, has a wide left region corresponding to the transition to homogeneity and a scaling region with the right slope $-3/D = -7.5$. We have tried various void-finder parameters with similar results. The scaling region begins with the 300th void (approximately), with size $0.0001$ (corresponding to linear size $\sim 0.05$).

4. Discussion

We propose a new method for finding voids in the galaxy distribution based on discrete geometry constructions, namely, Delaunay and Voronoi tessella-
Figure 5 Voids with 2nd to 6th rank in the distribution of the 1746 Hydra “wall galaxies”.

Figure 6. Rank ordering of the voids in the 1746 Hydra “wall galaxies” (the slope $-3/0.4$ is marked by the straight line).
Scaling of voids

This method defines a void as a juxtaposition of Delaunay simplices separated from other voids by reasonable criteria, similar to separation criteria used in other void-finders. This definition assumes that the total space occupied by the sample is to be filled with voids of irregular shape. The size of voids in a scaling point distribution run from nearly the sample’s size to about the minimal interparticle distance. A convenient representation of the size distribution is the rank-ordering Zipf’s plot, which is a power-law for a scaling point distribution (Zipf’s law).

We have demonstrated that Zipf’s law holds for uniformly scaling point distributions, that is, pure fractals (or monofractals). However, it holds at the most over a scaling range as large as the scaling range defining the fractal (by the number-radius relation). This is shown by our study of fractals with transition to homogeneity.

On the other hand, we have argued, following previous work in the literature, that one must consider multifractal rather than monofractal models of the galaxy distribution. A multifractal distribution can be considered as a superposition of fractals with different dimensions. The simplest multifractal is a superposition of two fractals, which is in accord with the usual division of galaxies between wall and field galaxies. Therefore, we have assumed that a multifractal galaxy distribution can be approximately described as a superposition of just two fractal populations. So we have built two galaxy populations from the halo distribution found in an $N$-body simulation (by the Hydra Consortium). The strongly clustered population scales with a very low fractal dimension ($D = 0.4$), much smaller than the fractal dimensions of the full galaxy distribution reported in the literature, and it has a transition to homogeneity. We have used our method to find the voids in this strongly clustered population. Of course, these voids contain inside members of the weakly clustered population. The Zipf law holds for the voids, but in a limited range.

A relatively small range of the Zipf law means that most sizeable voids are outside the scaling region and are, therefore, strongly influenced by the transition to homogeneity. This can help to explain why no scaling is perceived in the void catalogues compiled in the literature (Gaite & Manrubia, 2002). Even if the void-finder employed is adequate, a poor selection of the sample and/or of its voids can spoil scaling. Regarding the sample, one has to select a sufficiently uniform population to be considered a monofractal. One should check that the “wall builder phase” of the usual void-finding procedure accomplishes it. Regarding the voids, the first hundreds of voids may only show the transition to homogeneity of a scaling distribution on smaller scales. However, it is usual to restrict the search for voids to a few hundreds. Hopefully, improvement along these lines will eventually show scaling in the void catalogues.
Acknowledgments

I thank Francisco Prada for a conversation and for pointing out the work of Gottloeber et al (2003). My work is supported by a “Ramon y Cajal” contract and by grant BFM2002-01014 of the Ministerio de Ciencia y Tecnologia.

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