

SU(3)-structures on hypersurfaces of manifolds with $G_2$-structure

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Abstract. We study $SU(3)$-structures induced on orientable hypersurfaces of seven-dimensional manifolds with $G_2$-structure. Taking Gray-Hervella types for both structures into account, we relate the type of $SU(3)$-structure and the type of $G_2$-structure with the shape tensor of the hypersurface. Additionally, we show how to compute the intrinsic $SU(3)$-torsion and the intrinsic $G_2$-torsion by means of exterior algebra.

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1 Introduction

The exceptional Lie group $G_2$ is the group of automorphisms of the Cayley numbers $\mathbb{O}$: the non-associative normed algebra over $\mathbb{R}$ of dimension eight having an orthonormal basis $\{1, e_0, \ldots, e_6\}$ with multiplication determined by

\[
e_i^2 = -1, \quad e_i e_j = -e_j e_i, \quad (i \neq j),
\]

\[
e_i e_{i+1} = e_{i+3}, \quad e_{i+3} e_i = e_{i+1}, \quad e_{i+1} e_{i+3} = e_i, \quad (i, j \in \mathbb{Z}_7).
\]

The set $\text{Im}\mathbb{O}$ of pure imaginary Cayley numbers is the span of $\{e_0, \ldots, e_6\}$. The group $G_2$ can be equivalently defined as the subgroup of $SO(7)$ acting on $\text{Im}\mathbb{O}$ consisting of those elements which preserve the three-form given by

\[(1.1) \quad \varphi = \sum_{i \in \mathbb{Z}_7} e_i \wedge e_{i+1} \wedge e_{i+3},\]
where we have also denoted by $e_i$ the dual one-form of $e_i$. Furthermore, it can be shown that $G_2$ acts trivially on $\mathbb{R}$ as a subalgebra of $\mathfrak{O}$, so $G_2$ acts preserving the splitting $\mathfrak{O} = \mathbb{R} \oplus \text{Im} \, \mathfrak{O}$.

The subgroup of $G_2$ consisting of the automorphisms on $\mathfrak{O}$ that leave invariant an imaginary unit, for instance, $e_0$, is isomorphic to $SU(3)$. On this fact it is essentially based the possibility of defining an $SU(3)$-structure on an orientable hypersurface of a manifold equipped with a $G_2$-structure. Thus, Calabi in [1] and Gray in [9] considered orientable hypersurfaces $M$ of $\text{Im} \, \mathfrak{O}$, studying the induced $SU(3)$-structure on $M$ as an almost Hermitian structure ($U(3)$-structure). We recall that manifolds with $SU(n)$-structures, i.e., special almost Hermitian manifolds, are defined as almost Hermitian manifolds $(M, I, \langle \cdot, \cdot \rangle)$ equipped with a complex volume form $\Psi = \psi_+ + i\psi_-$. Likewise, a $G_2$-structure on a seven-dimensional manifold $\overline{M}$ is a reduction of the structure group of the tangent bundle to $G_2$ and is determined by a global three-form $\varphi$ which may locally written as in (1.1).

In the present work, we consider the structure on $M$ in its full condition as $SU(3)$-structure. So, we complete the information of the particular situations studied by Calabi and Gray with information coming from the complex volume form $\Psi$. On the another hand, we extend the results of Calabi and Gray to orientable hypersurfaces $M$ of manifolds $\overline{M}$ with $G_2$-structures in general. Thus, in a first instance, we prove results relating tensors involving the almost complex structure $I$, the complex volume form $\Psi$, the shape tensor of $M$ and the three form $\varphi$. Then, fixing the type of $G_2$-structure on $\overline{M}$, we apply such results in deriving geometrical conditions to be satisfied by $M$ to characterize types of $SU(3)$-structure.

The paper is organised as follows. In §2 we present some preliminary material (definitions, results, notation, etc.) about special almost Hermitian six-manifolds and give a new alternative way to characterize the diverse types of such structures (Proposition 2.7). Furthermore, we explicitly describe the intrinsic $SU(3)$-torsion in terms of the exterior derivatives of the Kähler form $\omega$ and the complex volume form $\Psi$. Then in §3, we recall some definitions, results and notations about $G_2$-structures. Likewise, we express the intrinsic $G_2$-torsion in terms of the exterior derivatives $d\varphi$ and $d\ast \varphi$.

In §4 we consider orientable hypersurfaces $M$ of seven-dimensional manifolds $\overline{M}$ with $G_2$-structure. Thus we start with the description of the induced $SU(3)$-structure on $M$. Then we prove the key results of the present work which are contained in Proposition 4.2. Such results are repeatedly applied in the proves of the successive theorems. In each one of these theorems,
we fix a type of $G_2$-structure on $M$. To avoid a too long exposition, we only consider types of $G_2$-structure corresponding to the irreducible modules $X_1, \ldots, X_4$ which occurs as summands in the $G_2$-decomposition of the space $X$ of covariant derivatives of $\varphi$ [8]. Finally, in §5, we give some examples.

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## 2 $SU(3)$-structures

In this section we give a brief summary of properties and results relative with special almost Hermitian six-manifolds. For more detailed and exhaustive information see [2, 15]. In the final part of this section, we present a new alternative way to characterize the diverse types of $SU(3)$-structure (Proposition 2.7) and express the intrinsic $SU(3)$-torsion by means of exterior algebra.

An almost Hermitian manifold is a $2n$-dimensional manifold $M$ with a $U(n)$-structure. This means that $M$ is equipped with a Riemannian metric $\langle \cdot, \cdot \rangle$ and an orthogonal almost complex structure $I$. Each fibre $T_m M$ of the tangent bundle can be consider as complex vector space by defining $ix = Ix$. We will write $T_m M_C$ when we are regarding $T_m M$ as such a space.

We define a Hermitian scalar product $\langle \cdot, \cdot \rangle_C = \langle \cdot, \cdot \rangle + i\omega(\cdot, \cdot)$, where $\omega$ is the Kähler form given by $\omega(x, y) = \langle x, Iy \rangle$. The real tangent bundle $TM$ is identified with the cotangent bundle $T^*M$ by the map $x \rightarrow \langle \cdot, x \rangle = x$. Analogously, the conjugate complex vector space $T_m M_C$ is identified with the dual complex space $T_m^* M_C$ by the map $x \rightarrow \langle \cdot, x \rangle_C = x_C$. It follows immediately that $x_C = x + iIx$.

If we consider the spaces $\Lambda^p T_m^* M_C$ of skew-symmetric complex forms, one can check $x_C \wedge y_C = (x + iIx) \wedge (y + iIy)$. There are natural extensions of scalar products to $\Lambda^p T_m^* M$ and $\Lambda^p T_m^* M_C$, defined respectively by

$$\langle a, b \rangle = \frac{1}{p!} \sum_{i_1, \ldots, i_p = 1}^{2n} a(e_{i_1}, \ldots, e_{i_p})b(e_{i_1}, \ldots, e_{i_p}),$$

$$\langle a_C, b_C \rangle_C = \frac{1}{p!} \sum_{i_1, \ldots, i_p = 1}^{n} a_C(u_{i_1}, \ldots, u_{i_p})b_C(u_{i_1}, \ldots, u_{i_p}),$$

where $e_1, \ldots, e_{2n}$ is an orthonormal basis for real vectors and $u_1, \ldots, u_n$ is a unitary basis for complex vectors.
The following conventions will be used in this paper. If \( b \) is a \((0, s)\)-tensor, we write
\[
I_{(i)} b(X_1, \ldots, X_i, \ldots, X_s) = -b(X_1, \ldots, I X_i, \ldots, X_s),
\]
\[
I b(X_1, \ldots, X_s) = (-1)^s b(I X_1, \ldots, I X_s),
\]

A special almost Hermitian manifold is a \( 2n \)-dimensional manifold \( M \) with an \( SU(n) \)-structure. This means that \( (M, \langle \cdot, \cdot \rangle, I) \) is an almost Hermitian manifold equipped with a complex volume form \( \Psi = \psi_+ + i \psi_- \) such that \( \langle \Psi, \Psi \rangle_C = 1 \). Note that \( I_{(i)} \psi_+ = \psi_- \).

In the following, we will only consider special almost Hermitian six-manifold. If \( e_1, e_2, e_3 \) is a unitary basis for complex vectors such that \( \Psi(e_1, e_2, e_3) = 1 \), i.e., \( \psi_+(e_1, e_2, e_3) = 1 \) and \( \psi_-(e_1, e_2, e_3) = 0 \), then \( e_1, e_2, e_3, I e_1, I e_2, I e_3 \) is an orthonormal basis for real vectors adapted to the \( SU(3) \)-structure. The three forms \( \psi_+ \) and \( \psi_- \) can be respectively expressed by
\[
\psi_+ = e_1 \wedge e_2 \wedge e_3 - I e_1 \wedge I e_2 \wedge e_3 - I e_1 \wedge e_2 \wedge I e_3 - e_1 \wedge I e_2 \wedge I e_3,
\]
\[
\psi_- = -I e_1 \wedge I e_2 \wedge I e_3 + I e_1 \wedge e_2 \wedge e_3 + e_1 \wedge I e_2 \wedge e_3 + e_1 \wedge e_2 \wedge I e_3.
\]
Moreover, it is straightforward to check \( \omega^3 = 6 e_1 \wedge e_2 \wedge e_3 \wedge I e_1 \wedge I e_2 \wedge I e_3 \), where \( \omega^3 = \omega \wedge \omega \wedge \omega \). If we fix the form \( Vol \) such that \( 6 Vol = \omega^3 \) as real volume form, it follows next lemma.

**Lemma 2.1 ([15]).** Let \( M \) be a special almost Hermitian 6-manifold, then

(i) \( \psi_+ \wedge \omega = \psi_- \wedge \omega = 0 \);

(ii) \( \psi_+ \wedge \psi_- = -4 Vol \) and \( \psi_+ \wedge \psi_+ = \psi_- \wedge \psi_- = 0 \);

(iii) for \( 1 \leq i < j \leq 3 \), \( I_{(i)} I_{(j)} \psi_+ = -\psi_+ \) and \( I_{(i)} I_{(j)} \psi_- = -\psi_- \); and

(iv) \( x \wedge \psi_+ = I x \wedge \psi_- = -(I x \wedge \psi_+ \wedge \omega \wedge \omega \wedge x \wedge \psi_+ = I x \wedge \psi_- \), for all vector \( x \), where \( \cdot \) denotes the interior product.

Furthermore, let us note that there are two Hodge star operators, defined on \( M \), respectively associated with the volume forms \( Vol \) and \( \Psi \). Relative to the real Hodge star operator \( * \), we have the equations
\[
* (\mu \wedge \psi_+) = * (\mu \wedge \psi_-) = 2 \mu,
\]
\[
* (\mu \wedge \psi_-) = * (\mu \wedge \psi_+) = 2 I \mu.
\]
We are dealing with $G$-structures where $G$ is a subgroup of the linear group $GL(m, \mathbb{R})$. If $M$ possesses a $G$-structure, then there always exists a $G$-connection defined on $M$. Moreover, if $(M^m, \langle \cdot, \cdot \rangle)$ is an orientable $m$-dimensional Riemannian manifold and $G$ a closed and connected subgroup of $SO(m)$, then there exists a unique metric $G$-connection $\tilde{\nabla}$ such that $\xi_x = \tilde{\nabla}_x - \nabla_x$ takes its values in $\mathfrak{g}^\perp$, where $\mathfrak{g}^\perp$ denotes the orthogonal complement in $\mathfrak{so}(m)$ of the Lie algebra $\mathfrak{g}$ of $G$ and $\nabla$ denotes the Levi-Civita connection [17, 4]. The tensor $\xi$ is the intrinsic torsion of the $G$-structure and $\tilde{\nabla}$ is called the minimal $G$-connection.

For $U(3)$-structures, the minimal $U(3)$-connection is given by $\tilde{\nabla} = \nabla + \xi$, with

\[
\xi_x Y = -\frac{1}{2}I(\nabla X)Y.
\]

see [6]. Since $U(3)$ stabilises the Kähler form $\omega$, it follows that $\tilde{\nabla} \omega = 0$. Then $\nabla \omega = -\xi \omega \in T^*M \otimes \mathfrak{u}(3)^\perp$. Thus, one can identify the $U(3)$-components of $\xi$ with the $U(3)$-components of $\nabla \omega$.

For $SU(3)$-structures, we have the decomposition $\mathfrak{so}(6) = \mathfrak{su}(3) + \mathbb{R} + \mathfrak{u}(3)^\perp$, i.e., $\mathfrak{su}(3)^\perp = \mathbb{R} + \mathfrak{u}(3)^\perp$. Therefore, the intrinsic $SU(3)$-torsion $\eta + \xi$ is such that $\eta \in T^*M \otimes \mathbb{R} \cong T^*M$ and $\xi$ is still determined by Equation (2.3). The tensors $\omega$, $\psi_+$ and $\psi_-$ are stabilised by the $SU(3)$-action, and $\tilde{\nabla}^{SU(3)} \omega = 0$, $\tilde{\nabla}^{SU(3)} \psi_+ = 0$ and $\tilde{\nabla}^{SU(3)} \psi_- = 0$, where $\tilde{\nabla}^{SU(3)} = \nabla + \eta + \xi$ is the minimal $SU(3)$-connection. Since $\tilde{\nabla}^{SU(3)}$ is metric and $\eta \in T^*M \otimes \mathbb{R}$, we have $\langle Y, \eta X Z \rangle = \langle I \eta(X) \omega(Y, Z) \rangle$, where $\eta$ on the right side is a one-form. Hence

\[
\eta_X Y = I\eta(X)Y.
\]

One can check $\eta \omega = 0$, then from $\tilde{\nabla}^{SU(3)} \omega = 0$ it is obtained

$$\nabla \omega = -\xi \omega \in T^*M \otimes \mathfrak{u}(3)^\perp = \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3 + \mathcal{W}_4,$$

where the summands $\mathcal{W}_i$ are the irreducible $U(3)$-modules given by Gray and Hervella [11] and $+$ denotes direct sum. The spaces $\mathcal{W}_3$ and $\mathcal{W}_4$ are also irreducible as $SU(3)$-modules. However, $\mathcal{W}_1$ and $\mathcal{W}_2$ admit the decompositions $\mathcal{W}_i = \mathfrak{W}_i^+ + \mathfrak{W}_i^-$, $i = 1, 2$, into irreducible $SU(3)$-components, where $\mathfrak{W}_i^+$ ($\mathfrak{W}_i^-$) includes those $a \in \mathcal{W}_i \subseteq T^*M \otimes \Lambda^2 T^*M$ such that the bilinear form $r(a)$, defined by $2r(a)(x, y) = \langle x, a \rangle_y y \psi_+$, is symmetric (skew-symmetric).
On the other hand, since $\nabla^{SU(3)}\psi_+ = 0$ and $\nabla^{SU(3)}\psi_- = 0$, we have $\nabla\psi_+ = -\eta\psi_+ - \xi\psi_+$ and $\nabla\psi_- = -\eta\psi_- - \xi\psi_-$. Therefore, from Equations (2.3) and (2.4) we obtain the following expressions

$$-\eta_X\psi_+ = -3I\eta(X)\psi_-, \quad -\xi_X\psi_+ = \frac{1}{2}(e_i \nabla_X\omega) \wedge (e_i \psi_-),$$

$$-\eta_X\psi_- = 3I\eta(X)\psi_+, \quad -\xi_X\psi_- = -\frac{1}{2}(e_i \nabla_X\omega) \wedge (e_i \psi_+),$$

where the summation convention is used. We use this convention along the present paper.

It is obvious that $-\eta\psi_+ \in W_5^- = T^*M \otimes \psi_-$ and $-\eta\psi_- \in W_5^+ = T^*M \otimes \psi_+$. The tensors $-\xi\psi_+$ and $-\xi\psi_-$ are described in the following proposition, where we need to consider the two $SU(3)$-maps

$$(2.5) \quad \Xi_+, \Xi_- : T^*M \otimes u(3)\perp \rightarrow T^*M \otimes \Lambda^3T^*M$$

respectively defined by $\nabla_\omega \rightarrow 1/2(e_i \nabla_\omega) \wedge (e_i \psi_-)$ and $\nabla_\omega \rightarrow -1/2(e_i \nabla_\omega) \wedge (e_i \psi_+)$. 

**Proposition 2.2 ([15]).** The $SU(3)$-maps $\Xi_+$ and $\Xi_-$ are injective and

$$\Xi_+ (T^*M \otimes u(3)\perp) = \Xi_- (T^*M \otimes u(3)\perp) = T^*M \otimes T^*M \wedge \omega.$$ 

Last result and above considerations give rise to the following theorem which describes the tensors $\nabla\psi_+$ and $\nabla\psi_-$. 

**Theorem 2.3 ([15]).** Let $M$ be a special almost Hermitian 6-manifold with Kähler form $\omega$ and complex volume form $\Psi = \psi_+ + i\psi_-$. Then

$$\nabla\psi_+ \in W_1^{\Xi_+} + W_2^{\Xi_+} + W_3^{\Xi_+} + W_4^{\Xi_+} + W_5^{\Xi_+},$$

$$\nabla\psi_- \in W_1^{\Xi_-} + W_2^{\Xi_-} + W_3^{\Xi_-} + W_4^{\Xi_-} + W_5^{\Xi_-},$$

where $W_i^{\Xi_+} = \Xi_+(W_i^+) = \Xi_-(W_i^-)$, $W_i^{\Xi_-} = \Xi_+(W_i^-) = \Xi_-(W_i^+)$, $i = 1, 2$; $W_3^{\Xi_+} = \Xi_+(W_3^-)$, $W_3^{\Xi_-} = \Xi_-(W_3^+)$, $j = 3, 4$; $W_5^+ = T^*M \otimes \psi_+$ and $W_5^- = T^*M \otimes \psi_-$. 

Further details of the $SU(3)$-components of $\nabla\psi_+$ and $\nabla\psi_-$ can be found in [15]. Moreover, if we consider the alternation maps $\tilde{a}_\pm : W_\Xi + W_5^{\Xi} \rightarrow \Lambda^3T^*M$, we get the following consequences of Theorem 2.3.
Corollary 2.4 ([15]). For $SU(3)$-structures, the exterior derivatives of $\psi_+$ and $\psi_-$ are such that
\[ d\psi_+, d\psi_- \in \Lambda^4 T^* M = W_1^a + W_2^a + W_{4,5}^a, \]
where $\tilde{a}_\pm(W_1^\pm) = W_1^a$, $\tilde{a}_\pm(W_2^\pm) = W_2^a$ and $\tilde{a}_\pm(W_5^\mp) = W_{4,5}^a$. Moreover, $\text{Ker}(\tilde{a}_\pm) = W_1^\pm = \mathbb{R}\omega \wedge \omega$, $W_2^a = \mathfrak{su}(3) \wedge \omega$ and $W_{4,5}^a = T^* M \wedge \psi_+ = T^* M \wedge \psi_- = \mathfrak{u}(3) \perp \wedge \omega$.

Now, if we compute the images by the maps $\tilde{a}_\pm \circ \Xi_\pm$ of the $W_4$-part of $\nabla\omega$, we obtain the $W_{4,5}$-parts of $d\psi_+$ and $d\psi_-$, i.e.,
\[ (d\psi_\pm)_{4,5} = - \left( 3\eta + \frac{1}{2} Id^*\omega \right) \wedge \psi_\pm, \]
where $d^*\omega$ means the coderivative of $\omega$. Then, making use of Equations (2.1) and (2.2) in Equation (2.6), one can explicitly describe the one-form $\eta$. This will complete the definition of the $SU(3)$-connection $\tilde{\nabla}^{SU(3)}$.

Theorem 2.5 ([15]). For an $SU(3)$-structure, the $W_5$-part $\eta$ of the torsion can be identified with $-\eta\psi_+ = -3I\eta \otimes \psi_-$ or $-\eta\psi_- = 3I\eta \otimes \psi_+$, where $\eta$ is a one-form such that
\[ \ast(*d\psi_+ \wedge \psi_+) = 6\eta + I d^*\omega = -I \ast(*d\psi_+ \wedge \psi_-) = I \ast(*d\psi_- \wedge \psi_+). \]

Finally, we are going to show an alternative way to describe the summand $\xi$ of the intrinsic torsion of an $SU(3)$-structure. In fact, if $e_1, e_2, e_3, e_4 = Ie_1, e_5 = Ie_2, e_6 = Ie_3$ is an adapted basis to an $SU(3)$-structure, we have
\[ \nabla\omega = \sum_{i,j=1}^{6} a_{ij}e_i \otimes e_j \wedge \psi_. \]

Now, we consider the $SU(3)$-map $r : T^* M \otimes \mathfrak{u}(3)^\perp \rightarrow \otimes^2 T^* M$ defined by
\[ r(\alpha)(x, y) = \frac{1}{2} \langle x \wedge \alpha, y \wedge \psi_+ \rangle. \]

It is straightforward to check that, for $\alpha$ given by Equation (2.7), $r(\alpha) = \sum_{i,j=1}^{6} a_{ij}e_i \otimes e_j$ and the following results follow.
Proposition 2.6. If \( r \) is the map defined by (2.8), then

(i) \( r \) is an SU(3)-isomorphism;

(ii) \( r(W^+_1) = \mathbb{R} \langle \cdot, \cdot \rangle, \ r(W^-_1) = \mathbb{R} \omega; \)

(iii) \( r(W^+_2) = \{ b \in \otimes^2 T^*M \mid b \text{ is trace free symmetric and } Ib = b \}; \)

(iv) \( r(W^-_2) = \{ b \in \otimes^2 T^*M \mid b \text{ is skew-symmetric, } Ib = b \text{ and } \langle \omega, b \rangle = 0 \}; \)

(v) \( r(W_3) = \{ b \in \otimes^2 T^*M \mid b \text{ is symmetric and } Ib = -b \}; \) and

(vi) \( r(W_4) = \{ b \in \otimes^2 T^*M \mid b \text{ is skew-symmetric and } Ib = -b \}. \)

Finally, we point out that the coderivative of \( \omega \) is given by

\[
(2.9) \quad d^*\omega = \sum_{i=1}^{6} \sum_{\{j,k \mid \psi_+(e_i, e_j, e_k) = 1\}} (r(\omega)(e_j, e_k) - r(\omega)(e_k, e_j)) e_i,
\]

where we denote \( r(\omega) = r(\nabla \omega) \).

In [15] it is proved that the intrinsic SU(3)-torsion \( \eta + \xi \) can be computed from \( d\omega, d\psi_+ \) and \( d\psi_- \). An explicit description of this fact is given in next Theorem, where we will write \( (d\psi_\pm)_{\xi} = d\psi_\pm + 3\eta \wedge \psi_\pm, \) and \( (X \wedge Y)_{\omega} d\psi_\pm = d\psi_\pm(X, Y, \cdot, \cdot, \cdot) \). We recall that \( \eta \) is given in Theorem 2.5, it only remains to describe \( \xi \).

**Theorem 2.7.** The \( u(3) \)-part \( \xi \) of the torsion of the minimal SU(3)-connection \( \tilde{\nabla}^{SU(3)} = \nabla + \eta + \xi \) is given by

\[
(2.10) \quad \xi_X Y = -\frac{1}{2} r(\omega)(X, e_i) \psi_+(e_i, e_j, e_k) Ie_j,
\]

for all vectors \( X, Y \). Furthermore, the \( (0,2) \)-tensor field \( r(\omega) \) can be alternatively given by

\[
(2.11) \quad 2r(\omega)(X, Y) = \langle X, d\omega, Y, \psi_+ \rangle - \langle (X \wedge Y)_{\omega} (d\psi_+)(\xi, \omega) \rangle + \langle (IX \wedge Y)_{\omega} (d\psi_-)(\xi, \omega) \rangle.
\]

**Proof.** From Equation (2.7) it follows

\[
(2.12) \quad d\omega = a_{ij} e_i \wedge (e_j \wedge \psi_+).
\]
Furthermore, computing $\Xi_+(\nabla \omega)$ and $\Xi_-(\nabla \omega)$, defined by Equation (2.5), and then alternating the obtained result, we will get

\[
\begin{align*}
(d\psi_+)_\xi &= a_{ij} e_i \wedge e_j \wedge \omega, \\
(d\psi_-)_\xi &= -a_{ij} e_i \wedge Ie_j \wedge \omega.
\end{align*}
\]

From Equations (2.12), (2.13) and (2.14), one can check Equation (2.11). Finally, from Equation (2.3) we can obtain

\[
\xi_X Y = -\frac{1}{2}(\nabla_X \omega)(e_j, Y)Ie_j.
\]

Now using Equation (2.7), we will obtain Equation (2.10).

\[\square\]

### 3 $G_2$-structures

We recall briefly some facts about $G_2$-structures and derive some results we will need in subsequent sections.

A $G_2$-structure on a Riemannian seven-manifold $(\overline{M}, \langle \cdot, \cdot \rangle)$ is by definition a reduction of the structure group of the tangent bundle to $G_2$. This is equivalent to the existence of a global three-form $\varphi$ which may be locally written as in (1.1). For all $m \in \overline{M}$, the tangent space $T_{\overline{M}}$ is then associated to the representation $\text{Im} \otimes$ of $G_2$. Since $G_2$ preserves $\langle \cdot, \cdot \rangle$, one can define a two-fold vector cross product $P$ given by $\langle P(x, y), z \rangle = \varphi(x, y, z)$. A local orthonormal frame $\{e_0, \ldots, e_6\}$ for vectors is a Cayley frame, if $P(e_i, e_{i+1}) = e_{i+3}$ for all $i \in \mathbb{Z}_7 [8, 13]$. The four-form $*\varphi$ defined by $*\varphi(x, y, z, u) = \mathcal{S}_{yzu}(P(x, y), P(z, u))$, $\mathcal{S}$ denotes a cyclic sum, is another tensor which plays a key role in $G_2$-structures. In terms of a Cayley frame, $*\varphi$ is locally given by

\[
*\varphi = -\sum_{i \in \mathbb{Z}_7} e_{i+2} \wedge e_{i+4} \wedge e_{i+5} \wedge e_{i+6}.
\]

As volume form we fix the form $Vol$ such that $\varphi \wedge *\varphi = 7Vol = 7e_0 \wedge \cdots \wedge e_6$.

If $\nabla^{G_2} = \nabla + \xi^{G_2}$ is the minimal $G_2$-connection, then the Levi-Civita covariant derivative $\nabla_\varphi = -\xi^{G_2} \varphi \in T^*\overline{M} \otimes g_2^\perp \subseteq T^*\overline{M} \otimes \Lambda^3 T^*\overline{M}$. Since $g_2^\perp = \{x, * \varphi \mid x \in T\overline{M}\}$, the space $\mathcal{X}$ of covariant derivatives of $\varphi$ can be described in the way contained in the following lemma.
Lemma 3.1 ([13]). If \( \{e_0, \ldots, e_6\} \) is a Cayley frame, then \( \alpha \in \mathcal{X} \) if and only if

\[
\alpha = \sum_{i,j \in \mathbb{Z}_7} a_{ij} e_i \otimes e_j \ast \varphi = - \sum_{i,j \in \mathbb{Z}_7} a_{ij} e_i \otimes (e_j \wedge \varphi). \tag{3.1}
\]

Therefore, if \( \nabla \) denotes the Levi-Civita connection of the metric tensor field \( \langle \cdot, \cdot \rangle \) on \( M \), the tensor \( \mathbf{\nabla}_\varphi \) can be expressed as \( \alpha \) in Equation (3.1). Moreover, the exterior derivative \( d\varphi \) and the coderivative \( d^*\varphi \) are given by

\[
d\varphi = -\sum_{i \in \mathbb{Z}_7} (a_{i+2,i+2} + a_{i+4,i+4} + a_{i+5,i+5} + a_{i+6,i+6}) e_{i+2} \wedge e_{i+4} \\
+ \sum_{i \in \mathbb{Z}_7} (a_{i+4,i+5} + a_{i+1,i+3} + a_{i+2,i+6}) e_i \wedge e_{i+1} \wedge e_{i+2} \wedge e_{i+4} \\
+ \sum_{i \in \mathbb{Z}_7} (a_{i+1,i+1} - a_{i+3,i+1} + a_{i+2,i+6}) e_i \wedge e_{i+2} \wedge e_{i+3} \wedge e_{i+5} \\
+ \sum_{i \in \mathbb{Z}_7} (a_{i+4,i+5} - a_{i+3,i+1} - a_{i+6,i+2}) e_i \wedge e_{i+3} \wedge e_{i+4} \wedge e_{i+6} \\
+ \sum_{i \in \mathbb{Z}_7} (-a_{i+5,i+4} + a_{i+1,i+3} - a_{i+6,i+2}) e_i \wedge e_{i+5} \wedge e_{i+6} \wedge e_{i+1}, \tag{3.2}
\]

\[
d^*\varphi = \sum_{i \in \mathbb{Z}_7} (a_{i+5,i+4} - a_{i+4,i+5} + a_{i+6,i+2} - a_{i+2,i+6}) e_{i+1} \wedge e_{i+3} \\
+ \sum_{i \in \mathbb{Z}_7} (a_{i+4,i+5} - a_{i+1,i+3} + a_{i+6,i+2} - a_{i+2,i+6}) e_{i+4} \wedge e_{i+5} \\
+ \sum_{i \in \mathbb{Z}_7} (a_{i+3,i+1} + a_{i+5,i+4} - a_{i+4,i+5} + a_{i+1,i+3}) e_{i+2} \wedge e_{i+6}. \tag{3.3}
\]

Note that one can compute \( a_{ij} \) from \( d\varphi \) and \( d^*\varphi \), for all \( i, j \in \mathbb{Z}_7 \). As a consequence, the expression for \( \mathbf{\nabla}_\varphi \) can be rebuilt from \( d\varphi \) and \( d^*\varphi \).

Fernández and Gray [8] proved that \( \mathcal{X} \), under the action of \( G_2 \), has four irreducible components, \( \mathcal{X} = \mathcal{X}^{(1)}_1 + \mathcal{X}^{(14)}_2 + \mathcal{X}^{(27)}_3 + \mathcal{X}^{(7)}_4 \), where the upper index indicates the corresponding dimension. Thus \( \mathbf{\nabla}_\varphi \in \mathcal{X} \) has four components giving rise to sixteen types of \( G_2 \)-structure. Conditions in terms of \( d\varphi \) and \( d^*\varphi \) can be given to characterize each type of \( G_2 \)-structure [13]. In [12], it was showed that the irreducible \( G_2 \)-summands of \( \mathcal{X} \) can be alternatively described by the following \( G_2 \)-equivariant map

\[
\mathcal{X} \to T^*M \otimes T^*M \\
\alpha \to \mathcal{F}(\alpha), \tag{3.4}
\]
| $\mathcal{P}$ | $\tau(\varphi) = 0$ |
|------------|-------------------|
| $\mathcal{X}_1 = \mathcal{N} \mathcal{P}$ | $\tau(\varphi) = \frac{k}{4} \langle \cdot , \cdot \rangle$, $k$ constant |
| $\mathcal{X}_2 = \mathcal{A} \mathcal{P}$ | $\tau(\varphi) \in \mathfrak{g}_2$ |
| $\mathcal{X}_3$ | $\tau(\varphi) \in S_{0}^{2} T^*M$ |
| $\mathcal{X}_4 = \mathcal{L} \mathcal{C} \mathcal{P}$ | $\tau(\varphi) = -\frac{1}{12} p_{\varphi} \varphi$ |
| $\mathcal{X}_1 + \mathcal{X}_2$ | $\tau(\varphi) + \tau(\varphi)^t$ is scalar and $\tau(\varphi) - \tau(\varphi)^t \in \mathfrak{g}_2$ |
| $\mathcal{X}_1 + \mathcal{X}_3 = \mathcal{S} \mathcal{P}$ | $\tau(\varphi)$ is symmetric |
| $\mathcal{X}_2 + \mathcal{X}_3$ | $\tau(\varphi) + \tau(\varphi)^t \in S_{0}^{2} T^*M$ and $\tau(\varphi) - \tau(\varphi)^t \in \mathfrak{g}_2$ |
| $\mathcal{X}_1 + \mathcal{X}_4 = \mathcal{L} \mathcal{C} \mathcal{N} \mathcal{P}$ | $\tau(\varphi) + \tau(\varphi)^t$ is scalar and $\tau(\varphi) - \tau(\varphi)^t = -\frac{1}{6} p_{\varphi} \varphi$ |
| $\mathcal{X}_2 + \mathcal{X}_4 = \mathcal{L} \mathcal{C} \mathcal{A} \mathcal{P}$ | $\tau(\varphi)$ is skew-symmetric |
| $\mathcal{X}_3 + \mathcal{X}_4$ | $\tau(\varphi) + \tau(\varphi)^t \in S_{0}^{2} T^*M$ and $\tau(\varphi) - \tau(\varphi)^t = -\frac{1}{6} p_{\varphi} \varphi$ |
| $\mathcal{X}_1 + \mathcal{X}_2 + \mathcal{X}_3$ | $\tau(\varphi) - \tau(\varphi)^t \in \mathfrak{g}_2$ |
| $\mathcal{X}_1 + \mathcal{X}_2 + \mathcal{X}_4$ | $\tau(\varphi) + \tau(\varphi)^t$ is scalar |
| $\mathcal{X}_1 + \mathcal{X}_3 + \mathcal{X}_4$ | $\tau(\varphi) - \tau(\varphi)^t = -\frac{1}{6} p_{\varphi} \varphi$ |
| $\mathcal{X}_2 + \mathcal{X}_3 + \mathcal{X}_4$ | $\tau(\varphi) + \tau(\varphi)^t \in S_{0}^{2} T^*M$ |
| $\mathcal{X}$ | no relation |

Table 1: Types of $G_2$-structures
where $\mathfrak{r}(\alpha)(x,y) = \frac{1}{4} \langle x \cdot \alpha, y \cdot \ast \varphi \rangle$. For all $\alpha \in \mathcal{X}$ given by Equation (3.1), we have

\[ (3.5) \quad \mathfrak{r}(\alpha) = \sum_{i,j \in \mathbb{Z}_7} a_{ij} e_i \otimes e_j. \]

Hence $\mathfrak{r}$ is a $G_2$-isomorphism. For the covariant two-tensors on $\overline{M}$, we have the following decomposition into $G_2$-irreducible components, $\otimes^2 T^* \overline{M} = \mathbb{R} + S^2_0 T^* \overline{M} + g_2 + g_2^\perp$, where $S^2_0 T^* \overline{M}$ is the space of trace free symmetric two-tensors, $g_2$ is the Lie algebra of $G_2$ and $g_2^\perp$ is the orthogonal complement of $g_2$ in $\Lambda^2 T^* \overline{M}$. Now by the $G_2$-isomorphism $\mathfrak{r}$, using the Schur’s Lemma, we get

\[ (3.6) \quad \sum_{\{i,k \mid \varphi(e_i,e_j,e_k)=1\}} b(e_j,e_k) = 0, \]

for all $i \in \mathbb{Z}_7$, where $\{e_0,\ldots,e_6\}$ is a Cayley frame.

For sake of simplicity we will write $\mathfrak{r}(\varphi) = \mathfrak{r}(\nabla \varphi)$. Now, by the map $\mathfrak{r}$, using Schur’s Lemma, one can characterize each type of $G_2$-structure ([12]). Such characterizations are shown in Table 1.

The vector field $p_{\varphi}$ which occurs in some conditions contained in Table 1 is such that $pd^* \varphi = \ast(*)d \varphi \wedge \varphi = \langle p_{\varphi}, \cdot \rangle$. In [14], it is showed that

\[ (3.7) \quad pd^* \varphi = -2 \sum_{i \in \mathbb{Z}_7} \sum_{\{j,k \mid \varphi(e_i,e_j,e_k)=1\}} (\mathfrak{r}(\varphi)(e_j,e_k) - \mathfrak{r}(\varphi)(e_k,e_j)) e_i. \]

It is well known that the intrinsic torsion $\xi^{G_2}$ of a $G_2$-structure can be computed from $d \varphi$ and $d \ast \varphi$. Here we will give an explicit description of this fact.

**Theorem 3.2.** The minimal $G_2$-connection is given by $
abla^{G_2} = \nabla + \xi^{G_2}$, where $\xi^{G_2}$ is defined by

\[ \xi^{G_2}_X Y = -\frac{1}{3} \sum_{i \in \mathbb{Z}_7} \mathfrak{r}(\varphi)(X,e_i) P(e_i,Y), \]

for all vectors $X,Y$, and $\mathfrak{r}(\varphi) = 1/4 \langle \nabla \varphi, \cdot \ast \varphi \rangle$. Moreover, the bilinear form $\mathfrak{r}(\varphi)$ is expressed in terms of $d \varphi$ and $\ast \varphi$ by

\[ (3.8) \quad 4\mathfrak{r}(\varphi)(X,Y) = \langle X \ast d \varphi, Y \ast \varphi, \cdot \rangle - \langle Y \ast (X \wedge \ast \varphi), d \varphi \rangle + 2d^\ast \varphi(X,Y). \]
Proof. Equation (3.8) can be checked for $\nabla G^2(\varphi)(e_i,e_j)$, using the expressions (3.2) and (3.3) for $d\varphi$ and $d^*\varphi$, respectively.

It is also immediate that $\xi G^2 \in \mathfrak{g}_2$. Finally, it is straightforward to check that $\nabla G^2 \varphi = 0$. Hence $\nabla G^2$ is the minimal $G_2$-connection. $\square$

Remark 3.3. As a direct consequence of last Theorem, we obtain an expression for $\nabla \varphi$ in terms of $d\varphi$ and $d^*\varphi$, i.e., $\nabla \varphi = -\xi G^2 \varphi$. Also note that the $G_2$-connection $\nabla G^2$, particularised to $G_2$-structures of type $\mathcal{X}_2$, coincides with the one given by Cleyton and Ivanov in [3].

4 Orientable hypersurfaces of manifolds with $G_2$-structure

From now on, $M$ will be an orientable hypersurface of a seven-dimensional Riemannian manifold $\overline{M}$ with a $G_2$-structure and $\iota : M \to \overline{M}$ will denote the inclusion map. Associated with the $G_2$-structure, we have the metric $\langle \cdot, \cdot \rangle$, the fundamental three-form $\varphi$ and the two-fold vector cross product $P$.

Proposition 4.1. Let $\overline{M}$ be a Riemannian manifold with a $G_2$-structure. If $M$ is an orientable hypersurface and $n$ is a unit normal vector field on $M$, then there is a special almost Hermitian structure on $M$ defined by the almost complex structure

$$(4.1) \quad Ix = P(n,x),$$

and the complex volume form given by

$$(4.2) \quad \Psi = \cos \theta^*\varphi - \sin \theta^*(n \lrcorner * \varphi) + i (\sin \theta^*\varphi + \cos \theta^*(n \lrcorner * \varphi)),$$

where $\theta$ is a smooth function on $M$.

The almost complex structure defined by (4.1) has been already considered by Calabi in [1] and by Gray in [10]. On each point of $M$ there is a Cayley basis with $n$ as first element, i.e., $\{n,e_1,\ldots,e_6\}$. The local frame of $M$ given by $\{e_1,e_2,e_4,e_3,e_6,e_5\}$ is an adapted local frame for the special almost Hermitian structure defined in Proposition 4.1 and the special unitary group $SU(3)$ can be considered as included in $G_2$ in the following way

$$SU(3) = \{g \in G_2 \subset SO(7) \mid g.n = n \}.$$ 

In next lemma we relate the bilinear form $r(\omega) = r(\nabla \omega)$, defined by Equation (2.8), with the bilinear form $\overline{\nabla}(\varphi)$ and the shape tensor $II$. 

Proposition 4.2. For the SU(3)-structure and the $G_2$-structure considered in Proposition 4.1, if $B(x, y) = \langle II(x, y), n \rangle$ denotes the second fundamental form, $\text{Tr}$ denotes trace, $c_\omega$ means the contraction by $\omega$ and $h$ is the length of the mean curvature $H$, i.e., $h = \langle H, n \rangle$, then

\begin{align*}
(4.3) \quad r(\omega) &= \cos \theta (-I_2^*) \tau(\varphi) + B) - \sin \theta (I^* \tau(\varphi) + I_2^*) B, \\
(4.4) \quad 2Id^* \omega &= I^* pd^* \varphi - 2\tau(\varphi)(n, \iota^* I \cdot) + 2\tau(\varphi)(\iota^* I \cdot, n), \\
(4.5) \quad pd^* \varphi(n) &= -2 \cos \theta \text{Tr} (r(\omega)) - 2 \sin \theta c_\omega (r(\omega)) + 12h, \\
(4.6) \quad Tr (I^* \tau(\varphi)) &= -\sin \theta \text{Tr} (r(\omega)) + \cos \theta c_\omega (r(\omega)), \\
(4.7) \quad 3I\eta &= d\theta - \tau(\varphi)(\iota^* I \cdot, n).
\end{align*}

Proof. On each point of $M$, we consider a Cayley frame $\{n, e_1, \ldots, e_6\}$ and, using Equation (3.5) and Lemma 3.1, obtain

\[
\tau(\varphi)(e_i, e_j) = \bar{a}_{ij} = (\nabla_{e_i} \varphi) (e_4, e_6, n) = \langle (\nabla_{e_i} P) (n, e_4), e_6 \rangle \\
= \langle (\nabla_{e_i} I) e_4, e_6 \rangle - \langle \nabla_{e_i} n, P(e_4, e_6) \rangle \\
= -\langle \nabla_{e_i} \omega \rangle (e_4, e_6) + B(e_i, Ie_1).
\]

Since we have

\[
2 (\nabla_{e_i} (\omega) (e_4, e_6) = \langle \nabla_{e_i} \omega, Ie_1 \iota^* (n \cdot \varphi) \rangle = -\langle \nabla_{e_i} \omega, e_1 \iota^* (n \cdot \varphi) \rangle,
\]

it is not hard to show

\[
\tau(\varphi)(X, IY) = \frac{1}{2} \langle \nabla_X \omega, Y \iota^* (\varphi) \rangle - B(X, Y), \\
\tau(\varphi)(X, Y) = \frac{1}{2} \langle \nabla_X \omega, Y \iota^* (n \cdot \varphi) \rangle + B(X, IY).
\]

for all vectors $X, Y \in TM$. From these two identities, Equation (4.3) follows.

Equation (4.4) is deduced from Equation (3.7), taking Equations (4.3) and (2.9) into account. Equations (4.5) and (4.6) are derived by computing $Tr (r(\omega))$ and $c_\omega (r(\omega))$, taking Equations (3.7) and (4.3) into account.

For Equation (4.7), we firstly consider that $\theta$ is constant and equal to 0. In such a case, we have $\psi_+ = i^* \varphi$ and $\psi_- = i^* (n \cdot \varphi)$. Noting that $di^* \varphi = i^* d\varphi$ and making use of the explicit expression for $d\varphi$ given by Equation (3.2), we obtain the following identities

\[
*(\ast d i^* \varphi \wedge i^* \varphi) = Id^* \omega - 2\tau(\varphi)(\iota^* I \cdot, n), \\
*(\ast d i^* \varphi \wedge i^* (n \cdot \varphi)) = -d^* \omega - 2\tau(\varphi)(\iota^* I \cdot, n).
\]
From these identities, taking Theorem 2.5 into account, it follows

\begin{align}
(4.8) \quad * (\ast d\iota (\varphi) \wedge \iota^*(\varphi)) &= \ast (\ast d\varphi \wedge \iota^*(\varphi)) \\
&= \Id^* \omega - 2\mathcal{T}(\varphi)(\iota_* I, n),
\end{align}

\begin{align}
(4.9) \quad -\ast (\ast d\iota (\varphi) \wedge \iota^*(\varphi)) &= \ast (\ast d\varphi \wedge \iota^*(\varphi)) \\
&= -d^* \omega - 2\mathcal{T}(\varphi)(\iota_* I, n).
\end{align}

On the other hand, making use of Equations (2.1) and (2.2), for all \( \mu \in T^* M \), we have

\begin{align}
(4.10) \quad * (\ast (\mu \wedge \iota^* \varphi) \wedge \iota^* \varphi) &= \ast (\ast (\mu \wedge \iota^*(\varphi)) \wedge \iota^*(\varphi)) = -2\mu, \\
(4.11) \quad (\ast (\mu \wedge \iota^*(\varphi)) \wedge \iota^* \varphi) &= \ast (\ast (\mu \wedge \iota^* \varphi) \wedge \iota^*(\varphi)) = 2\mu.
\end{align}

For sake of simplicity we have deduced Equations (4.8), (4.9), (4.10) and (4.11) fixing \( \Psi = \iota^* \varphi + i\iota^*(n_* \varphi) \). But really, these equations mainly depend of the Hodge star operator \( \ast \) which is determined by the metric, and not of what complex volume form \( \Psi \) we have previously fixed. Thus Equations (4.8), (4.9), (4.10) and (4.11) are still true when \( \Psi \) is given by (4.2). In such a case, we would have \( \psi_+ = \cos \theta \iota^* \varphi - \sin \theta \iota^*(n_* \varphi) \) and need to compute \( \ast (\ast d\psi_+ \wedge \psi_+) \) to determine \( \eta \) (see Theorem 2.5). In fact, taking Equations (4.8), (4.9), (4.10) and (4.11) into account, we would get

\[ 6\eta + \Id^* \omega = \ast (\ast d\psi_+ \wedge \psi_+) = -2\Id \theta + \Id^* \omega - 2\mathcal{T}(\varphi)(\iota_* I, n). \]

From these identities Equation (4.7) follows.

Proposition 4.2, Table 1 and Proposition 2.6 give place to results relating the type of \( G_2 \)-structures on the ambient manifold, type of the \( SU(3) \)-structure on the hypersurface and the shape tensor. In the following theorem we only mention the more relevant consequences in such a direction.

**Theorem 4.3.** Let \( \overline{M} \) be a seven-dimensional Riemannian manifold with a \( G_2 \)-structure of type \( \mathcal{X}_1 \) such that \( d\varphi = k \ast \varphi \). Let \( M \) be an orientable hypersurface with unitary normal vector field \( n \). We consider an \( SU(3) \)-structure on \( M \) defined as in Proposition 4.1. Then \( M \) is of type \( \mathcal{W}^+_1 + \mathcal{W}^-_1 + \mathcal{W}^+_2 + \mathcal{W}^-_2 + \mathcal{W}_3 + \mathcal{W}_5 \) and the conditions displayed in Table 2 characterise types of \( SU(3) \)-structure on \( M \).
Proof. Taking Proposition 4.2 into account, since \( \overline{\mathfrak{r}}(\varphi) = k/4\langle \cdot, \cdot \rangle \) and \( pd^*\varphi = 0 \) (see Table 1), we have

\[
r(\omega) = \cos \theta \left( \frac{k}{4} \omega + B \right) - \sin \theta \left( \frac{k}{4} \langle \cdot, \cdot \rangle + I(2)B \right),
\]

\[
Id^*\omega = 0, \quad 3I\eta = d\theta.
\]

Now, all parts of Theorem are direct consequences of these equations and Proposition 2.6. \( \square \)

Remark 4.4. (i) Conditions given in 10th, 11th and 12th lines of Table 2 has been already shown by Calabi [1] for the particular case \( \overline{\mathfrak{M}} = \mathbb{R}^7 \) considered as the pure imaginary Cayley numbers. Also for such a case, Gray [10] proved that any orientable hypersurface is of type \( \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3 \) as almost Hermitian manifold, and deduced conditions with correspond with those contained in 2nd and 9th lines of Table 2. Here the context is more general and we give more detailed information with respect to the \( SU(3) \)-structure defined on the hypersurface.

(ii) In Theorem 4.3, if we consider the \( G_2 \)-structure on \( \overline{\mathfrak{M}} \) of type \( \mathcal{P} \), we would obtain the same conclusions but for \( k = 0 \). So that we will not show a specific table for such a situation.

(iii) In Table 2, it is not given an exhaustive list of conditions which characterise all possible types. We only has written conditions for those more relevant types. However, it is straightforward to deduce the remaining characterizations from conditions contained in the table.

For most of the following theorems, their proves are deduced using analog arguments as in the proof of Theorem 4.3. For such a reason, we give some of those theorems without an explicit proof.

Two special particular cases of Theorem 4.3 for \( \theta \) constant are respectively when \( \theta = 0 \) and \( \theta = \pi/2 \).

Theorem 4.5. Let \( \overline{\mathfrak{M}} \) be a seven-dimensional Riemannian manifold with a \( G_2 \)-structure of type \( \mathcal{X}_1 \) such that \( d\varphi = k \ast \varphi \). Let \( \mathfrak{M} \) be an orientable hypersurface with unitary normal vector field \( n \). We consider an \( SU(3) \)-structure on \( \mathfrak{M} \) defined as in Proposition 4.1, taking \( \theta = 0 (\theta = \pi/2) \) constant. Then \( \mathfrak{M} \) is of type \( \mathcal{W}_1^{+(-)} + \mathcal{W}_1^{-(+)} + \mathcal{W}_2^{-(-)} + \mathcal{W}_3 \) and the conditions displayed in Table 3 characterise types of \( SU(3) \)-structure on \( \mathfrak{M} \).
\begin{tabular}{ll}
$W_1^+ + W_1^- + W_2^+ + W_2^- + W_3$ & $\theta$ is constant \\
$W_1^+ + W_1^- + W_2^+ + W_2^- + W_5$ & $IB = B$ \\
$W_1^+ + W_1^- + W_2^+ + W_3 + W_5$ & $\sin \theta (1 + I)B = 2h \sin \theta \langle \cdot, \cdot \rangle$ \\
$W_1^+ + W_2^- + W_3 + W_5$ & $\cos \theta (1 + I)B = 2h \cos \theta \langle \cdot, \cdot \rangle$ \\
$W_1^+ + W_2^+ + W_2^- + W_3 + W_5$ & $-4h \sin \theta = k \cos \theta$ \\
$W_1^- + W_2^+ + W_3 + W_5$ & $4h \cos \theta = k \sin \theta$ \\
$W_1^+ + W_2^- + W_3 + W_5$ & $(1 + I)B = 2h \langle \cdot, \cdot \rangle$ \\
$W_2^+ + W_2^- + W_3 + W_5$ & $M$ is a minimal variety and $\overline{M}$ is of type $\mathcal{P}$ \\
$W_1^+ + W_1^- + W_3 + W_5$ & $M$ is totally umbilic \\
$W_2^+ + W_3 + W_5$ & $IB = B$, $M$ is a minimal variety and $\overline{M}$ is of type $\mathcal{P}$ \\
$W_3 + W_5$ & $IB = -B$ and $\overline{M}$ is of type $\mathcal{P}$ \\
$W_5$ & $M$ is totally geodesic and $\overline{M}$ is of type $\mathcal{P}$ \\
$\{0\}$ & $M$ is totally geodesic, $\theta$ is constant and $\overline{M}$ is of type $\mathcal{P}$ \\
\end{tabular}

Table 2: $\overline{M}$ of type $\mathcal{X}_1$
\[
\begin{array}{ll}
W_i^{+(-)} + W_j^{-(+)} + W_k^{+(-)} & IB = B \\
W_i^{+(-)} + W_j^{-(+)} + W_3 & (1 + I)B = 2h\langle \cdot, \cdot \rangle \\
W_i^{+(-)} + W_j^{-(+)} + W_3 & \overline{M} \text{ is of type } \mathcal{P} \\
W_i^{+(+)} + W_j^{+(+)} + W_3 & M \text{ is a minimal variety} \\
W_i^{+(+)} + W_j^{+(+)} & M \text{ is totally umbilic} \\
W_i^{+(+)} + W_j^{+(+)} & IB = B \text{ and } \overline{M} \text{ is of type } \mathcal{P} \\
W_i^{+(+)} + W_j^{+(+)} & IB = B \text{ and } M \text{ is a minimal variety} \\
W_i^{+(+)} + W_3 & (1 + I)B = 2h\langle \cdot, \cdot \rangle \text{ and } \overline{M} \text{ is of type } \mathcal{P} \\
W_i^{+(+)} + W_3 & IB = -B \\
W_i^{+(+)} + W_3 & M \text{ is a minimal variety and } \overline{M} \text{ is of type } \mathcal{P} \\
W_3 & IB = -B \text{ and } \overline{M} \text{ is of type } \mathcal{P} \\
W_3 & IB = B, M \text{ is a minimal variety and } \overline{M} \text{ is of type } \mathcal{P} \\
W_i^{+(+)} & M \text{ is totally geodesic} \\
W_i^{+(+)} & M \text{ is totally umbilic variety and } \overline{M} \text{ is of type } \mathcal{P} \\
\{0\} & M \text{ is totally geodesic and } \overline{M} \text{ is of type } \mathcal{P}
\end{array}
\]

Table 3: $\overline{M}$ of type $\mathcal{X}_1$ and $\theta = 0(\pi/2)$ constant

A particular case of Theorem 4.5 is when the $G_2$-structure on $\overline{M}$ is of type $\mathcal{P}$. In such a situation we have the following result.

**Theorem 4.6.** Let $\overline{M}$ be a seven-dimensional Riemannian manifold with a $G_2$-structure of type $\mathcal{P}$. Let $M$ be an orientable hypersurface with unitary normal vector field $n$. We consider an $SU(3)$-structure on $M$ defined as in Proposition 4.1, taking $\theta = 0(\theta = \pi/2)$ constant. Then $M$ is of type $W_i^{+(-)} + W_j^{-(+)} + W_3$ and the conditions displayed in Table 4 characterise types of $SU(3)$-structure on $M$.

Next, we describe the situation when the $G_2$-structure is almost parallel.

**Theorem 4.7.** Let $\overline{M}$ be a seven-dimensional Riemannian manifold with a $G_2$-structure of type $\mathcal{X}_2$. Let $M$ be an orientable hypersurface with unitary normal vector field $n$. We consider an $SU(3)$-structure on $M$ defined as in
\[
\mathcal{W}_1^+(-) + \mathcal{W}_2^+(-) \quad IB = B \\
\mathcal{W}_1^+(-) + \mathcal{W}_3 \quad (1 + I)B = 2\mathfrak{h}(\cdot, \cdot) \\
\mathcal{W}_2^+(-) + \mathcal{W}_3 \quad M \text{ is a minimal variety} \\
\mathcal{W}_3 \quad IB = -B \\
\mathcal{W}_2^+(-) \quad IB = B \text{ and } M \text{ is a minimal variety} \\
\mathcal{W}_1^+(-) \quad M \text{ is totally umbilic} \\
\{0\} \quad M \text{ is totally geodesic}
\]

Table 4: \( M \) of type \( \mathcal{P} \) and \( \theta = 0(\theta = \pi/2) \) constant

**Proposition 4.1.** Then we have \( d^*\omega = 6I\eta - 2d\theta = -2\mathfrak{r}(\varphi)(\iota_*\cdot, n) \) and the conditions displayed in Table 5 characterise types of SU(3)-structure on \( M \).

**Proof.** Taking conditions of Table 1 into account, it follows that, for a \( G_2 \)-structure of type \( \mathcal{X}_2 \), we have

\[
c_\omega (\iota^*\mathfrak{r}(\varphi)) = 0, \quad pd^*\varphi = 0, \quad \mathfrak{r}(\varphi)(X,Y) = -\mathfrak{r}(\varphi)(Y,X), \\
\mathfrak{r}(\varphi)(X,Y) - \mathfrak{r}(\varphi)(IX,IY) = \mathfrak{r}(\varphi)(n, P(n, P(X,Y))),
\]

for all vector fields \( X, Y \) tangent to \( M \). Now, Theorem follows from these equations, Proposition 4.2 and Proposition 2.6.

Note that Table 5 only contains conditions for some types of SU(3)-structure on \( M \). However, conditions for remaining types can be easily derived from those which are given in the mentioned table. Next we give the result corresponding to Theorem 4.7 when \( \theta = 0(\theta = \pi/2) \) constant.

**Theorem 4.8.** Let \( \overline{M} \) be a seven-dimensional Riemannian manifold with a \( G_2 \)-structure of type \( \mathcal{X}_2 \). Let \( M \) be an orientable hypersurface with unitary normal vector field \( n \). We consider an SU(3)-structure on \( M \) defined as in Proposition 4.1, taking \( \theta = 0(\theta = \pi/2) \) constant. Then \( M \) is of type \( \mathcal{W}_1^+(-) + \mathcal{W}_2^+(-) + \mathcal{W}_3 + \mathcal{W}_4 + \mathcal{W}_5 \), \( d^*\omega = 6I\eta = -2\mathfrak{r}(\varphi)(\iota_*\cdot, n) \) and the conditions displayed in Table 6 characterise types of SU(3)-structure on \( M \).

Now, we will see the corresponding theorem to \( G_2 \)-structures of type \( \mathcal{X}_3 \).
\[ W^+_1 + W^-_1 + W^+_2 + W^-_2 + W_3 + W_4 \quad d\theta = 2\tau(\varphi)(\iota^\ast, n) \]
\[ W^+_1 + W^-_1 + W^+_2 + W^-_2 + W_3 + W_5 \quad \tau(\varphi)(\iota^\ast, n) = 0 \]
\[ W^+_1 + W^-_1 + W^+_2 + W^-_2 + W_4 + W_5 \quad IB = B \]
\[ W^+_1 + W^-_1 + W^+_2 + W^-_2 + W_3 + W_4 + W_5 \quad \sin \theta \left( (I(1) - I(2)) \iota^\ast \tau(\varphi) + (1 + I)B \right) = 2h \sin \theta \langle \cdot, \cdot \rangle \]
\[ W^+_1 + W^-_1 + W^+_2 + W^-_2 + W_3 + W_4 + W_5 \quad \cos \theta \left( (I(1) - I(2)) \iota^\ast \tau(\varphi) + (1 + I)B \right) = 2h \cos \theta \langle \cdot, \cdot \rangle \]
\[ W^+_1 + W^-_1 + W^+_2 + W^-_2 + W_3 + W_4 + W_5 \quad h \sin \theta = 0 \]
\[ W^+_1 + W^-_1 + W^+_2 + W^-_2 + W_3 + W_4 + W_5 \quad h \cos \theta = 0 \]
\[ W^+_1 + W^-_1 + W^+_2 + W^-_2 + W_3 \quad \tau(\varphi)(\iota^\ast, n) = 0 \text{ and } \theta \text{ is constant} \]
\[ W^+_1 + W^-_1 + W_3 + W_4 + W_5 \quad (I(1) - I(2)) \iota^\ast \tau(\varphi) + (1 + I)B = 2h \langle \cdot, \cdot \rangle \]
\[ W^+_2 + W^-_2 + W_3 + W_4 + W_5 \quad M \text{ is a minimal variety} \]

Table 5: $\overline{M}$ of type $X_2$

\[ W^{+(-)}_1 + W^{-(-)}_2 + W_3 \quad \tau(\varphi)(\iota^\ast, n) = 0 \]
\[ W^{+(-)}_1 + W^{-(-)}_2 + W_4 + W_5 \quad IB = B \]
\[ W^{+(-)}_1 + W_3 + W_4 + W_5 \quad (I(1) - I(2)) \iota^\ast \tau(\varphi) + (1 + I)B = 2h \langle \cdot, \cdot \rangle \]
\[ W^{+(-)}_2 + W_3 + W_4 + W_5 \quad h \text{ is a minimal variety} \]

Table 6: $\overline{M}$ of type $X_2$ and $\theta = 0(\theta = \pi/2)$ constant
Theorem 4.9. Let $\overline{M}$ be a seven-dimensional Riemannian manifold with a $G_2$-structure of type $\mathcal{X}_3$. Let $M$ be an orientable hypersurface with unitary normal vector field $n$. We consider an $SU(3)$-structure on $M$ defined as in Proposition 4.1. Then $M$ is of type $\mathcal{W}_{1}^+ + \mathcal{W}_{1}^- + \mathcal{W}_{2}^+ + \mathcal{W}_{2}^- + \mathcal{W}_{3} + \mathcal{W}_{5}$ and the conditions displayed in Table 7 characterise others types of $SU(3)$-structure on $M$.

As a consequence of last Theorem, we analyse the situation when $\theta = 0(\theta = \pi/2)$ constant.

Theorem 4.10. Let $\overline{M}$ be a seven-dimensional Riemannian manifold with a $G_2$-structure of type $\mathcal{X}_3$. Let $M$ be an orientable hypersurface with unitary normal vector field $n$. We consider an $SU(3)$-structure on $M$ defined as in Proposition 4.1, taking $\theta = 0(\theta = \pi/2)$ constant. Then $M$ is of type $\mathcal{W}_{1}^+ + \mathcal{W}_{1}^- + \mathcal{W}_{2}^+ + \mathcal{W}_{2}^- + \mathcal{W}_{3} + \mathcal{W}_{5}$ and the conditions displayed in Table 8 characterise types of $SU(3)$-structure on $M$.

Finally, we pay attention to locally conformal parallel $G_2$-structures.

Theorem 4.11. Let $\overline{M}$ be a seven-dimensional Riemannian manifold with a $G_2$-structure of type $\mathcal{X}_4$. Let $M$ be an orientable hypersurface with unitary normal vector field $n$. We consider an $SU(3)$-structure on $M$ defined as in Proposition 4.1. Then $3Id^*\omega = \iota^*pd^*\varphi = 12Id\theta + 36\eta$ and the diverse types of $SU(3)$-structure on $M$ are characterised by the conditions displayed in Table 9.

Proof. Taking conditions of Table 1 into account, it follows that, for a $G_2$-structure of type $\mathcal{X}_4$, we have

$$2c_\omega(r(\omega)) = \sin(\theta)(12h - pd^*\varphi(n)),$$
$$2\text{Tr}(r(\omega)) = \cos(\theta)(12h - pd^*\varphi(n)),$$
$$pd^*\varphi(\iota_*I*) = 12\overline{\varphi}(\iota_*n) = \varphi(p\varphi, n, \cdot),$$
$$\overline{\varphi}(X, Y) = -\frac{1}{12}pd^*\varphi(P(X, Y)),$$
$$pd^*\varphi(P(X, IY)) - pd^*\varphi(P(IX, Y)) = 2pd^*\varphi(n)\langle X, Y \rangle,$$

for all vector fields $X, Y$ tangent to $M$. Now, Theorem follows from these equations, Proposition 4.2 and Proposition 2.6.

Theorem 4.12. Let $\overline{M}$ be a seven-dimensional Riemannian manifold with a $G_2$-structure of type $\mathcal{X}_4$. Let $M$ be an orientable hypersurface with unitary
\[ W^+_1 + W^-_1 + W^+_2 + W^-_2 + W_3 + W_5 \]
\[ d\theta = \tau(\varphi)(\iota^* \cdot, n) \]
\[ W^+_1 + W^-_1 + W^+_2 + W^-_2 + W_5 \]
\[ (I_{(1)} + I_{(2)})\tau(\varphi) = (1 - I)B \]
\[ W^+_1 + W^-_1 + W^+_2 + W^-_2 + W_3 + W_5 \]
\[ 3 \cos \theta(1 + I)\tau(\varphi) + 3 \sin \theta(1 + I)B \]
\[ = (6h \sin \theta - \cos \theta \tau(\varphi)(n, n))\langle \cdot, \cdot \rangle \]
\[ W^+_1 + W^-_1 + W^+_2 + W^-_2 + W_3 + W_5 \]
\[ -3 \sin \theta(1 + I)\tau(\varphi) + 3 \cos \theta(1 + I)B \]
\[ = (6h \cos \theta + \sin \theta \tau(\varphi)(n, n))\langle \cdot, \cdot \rangle \]
\[ W^+_1 + W^-_2 + W^+_3 + W^-_3 + W_5 \]
\[ \cos \theta \tau(\varphi)(n, n) = 6h \sin \theta \]
\[ W^-_1 + W^+_2 + W^-_3 + W^+_3 + W_5 \]
\[ \sin \theta \tau(\varphi)(n, n) = -6h \cos \theta \]
\[ W^+_1 + W^-_1 + W_3 + W_5 \]
\[ 3(1 + I)\iota^* \tau(\varphi) = -\tau(\varphi)(n, n)\langle \cdot, \cdot \rangle \]
\[ (1 + I)B = 2h\langle \cdot, \cdot \rangle \]
\[ W^+_2 + W^-_3 + W_3 + W_5 \]
\[ \tau(\varphi)(n, n) = 0 \text{ and } M \text{ is a minimal variety} \]

Table 7: $\overline{M}$ of type $X_3$

normal vector field $n$. We consider an $SU(3)$-structure on $M$ defined as in Proposition 4.1, taking $\theta = 0 (\theta = \pi/2)$ constant. Then $3ID^*\omega = \iota^*pd^*\varphi = 36\eta$, $M$ is of type $W^+_1(-) + W^+_2(-) + W_3 + W_4 + W_5$ and the diverse types of $SU(3)$-structure on $M$ are characterised by the conditions displayed in Table 10.

Finally, it is of some interest to consider the particular situation such that the vector field $p_\varphi$ is tangent to $M$.

**Corollary 4.13.** In the same conditions as in Theorem 4.11, but with $p_\varphi$ tangent to $M$. Then the diverse types of $SU(3)$-structure on $M$ are characterized by the conditions given in Table 11.

Fernández & Gray showed that there are at most sixteen types of seven-dimensional manifolds with $G_2$-structure [8]. 'At most', because topological conditions can do impossible the existence of some particular type. In fact, the type $W_1 + W_2 - (W_1 \cup W_2)$ can not be found on a connected manifold [13]. Examples for the remaining types were shown in [8, 7, 13, 16]. Thus, there are really fifteen types of $G_2$-structure. For each type of $G_2$-structure, theorems and results of the sort here exposed can be given. Their proves would make reiterated use of Proposition 4.2, Table 1 and Proposition 2.7. For sake of brevity, we only give the results presented until this point and the following one that we consider of some special interest.
\[ \mathcal{W}_1^+ + \mathcal{W}_1^- + \mathcal{W}_2^+ + \mathcal{W}_2^- + \mathcal{W}_3 \quad \mathcal{F} (\varphi)(\iota_+, n) = 0 \]

(idem)

\[ \mathcal{W}_1^+ + \mathcal{W}_1^- + \mathcal{W}_2^+ + \mathcal{W}_2^- + \mathcal{W}_5 \quad (I(1) + I(2)) \mathcal{F} (\varphi) = (1 - I)B \]

(idem)

\[ \mathcal{W}_1^+ + \mathcal{W}_1^- + \mathcal{W}_2^+ + \mathcal{W}_3 + \mathcal{W}_5 \quad 3(1 + I) \mathcal{F} (\varphi) = -\mathcal{F} (\varphi)(n, n) \langle \cdot, \cdot \rangle \]

((1 + I)B = 2h \langle \cdot, \cdot \rangle)

\[ \mathcal{W}_1^+ + \mathcal{W}_1^- + \mathcal{W}_2^- + \mathcal{W}_3 + \mathcal{W}_5 \quad (1 + I)B = 2h \langle \cdot, \cdot \rangle \]

(3(1 + I) \mathcal{F} (\varphi) = -\mathcal{F} (\varphi)(n, n) \langle \cdot, \cdot \rangle)

\[ \mathcal{W}_1^+ + \mathcal{W}_2^+ + \mathcal{W}_2^- + \mathcal{W}_3 + \mathcal{W}_5 \quad \mathcal{F} (\varphi)(n, n) = 0 \]

(M is a minimal variety)

\[ \mathcal{W}_1^- + \mathcal{W}_2^+ + \mathcal{W}_2^- + \mathcal{W}_3 + \mathcal{W}_5 \quad M \text{ is a minimal variety} \]

(\mathcal{F} (\varphi)(n, n) = 0)

Table 8: $\overline{M}$ of type $\mathcal{X}_3$ and $\theta = 0 (\theta = \pi / 2)$ constant

\[ \mathcal{W}_1^+ + \mathcal{W}_1^- + \mathcal{W}_2^- + \mathcal{W}_2^+ + \mathcal{W}_3 + \mathcal{W}_4 \quad \nu^* \rho^* \varphi = 12I d\theta \]

\[ \mathcal{W}_1^+ + \mathcal{W}_1^- + \mathcal{W}_2^- + \mathcal{W}_2^+ + \mathcal{W}_3 + \mathcal{W}_5 \quad \rho_\varphi \text{ is normal to } M \]

\[ \mathcal{W}_1^+ + \mathcal{W}_1^- + \mathcal{W}_2^- + \mathcal{W}_2^+ + \mathcal{W}_3 + \mathcal{W}_5 \quad IB = B \]

\[ \mathcal{W}_1^+ + \mathcal{W}_1^- + \mathcal{W}_2^- + \mathcal{W}_2^+ + \mathcal{W}_3 + \mathcal{W}_4 + \mathcal{W}_5 \quad \sin \theta \ 2h \langle \cdot, \cdot \rangle = \sin \theta (1 + I)B \]

\[ \mathcal{W}_1^+ + \mathcal{W}_1^- + \mathcal{W}_2^- + \mathcal{W}_2^+ + \mathcal{W}_3 + \mathcal{W}_4 + \mathcal{W}_5 \quad \cos \theta \ 2h \langle \cdot, \cdot \rangle = \cos \theta (1 + I)B \]

\[ \mathcal{W}_1^+ + \mathcal{W}_2^+ + \mathcal{W}_2^- + \mathcal{W}_3 + \mathcal{W}_4 + \mathcal{W}_5 \quad \sin \theta (\rho^* \varphi(n) - 12h) = 0 \]

\[ \mathcal{W}_1^+ + \mathcal{W}_2^+ + \mathcal{W}_2^- + \mathcal{W}_3 + \mathcal{W}_4 + \mathcal{W}_5 \quad \cos \theta (\rho^* \varphi(n) - 12h) = 0 \]

\[ \mathcal{W}_1^+ + \mathcal{W}_1^- + \mathcal{W}_3 + \mathcal{W}_4 + \mathcal{W}_5 \quad (1 + I)B = 2h \langle \cdot, \cdot \rangle \]

\[ \mathcal{W}_2^+ + \mathcal{W}_2^- + \mathcal{W}_3 + \mathcal{W}_4 + \mathcal{W}_5 \quad \rho^* \varphi(n) = 12h \]

\[ \mathcal{W}_1^+ + \mathcal{W}_1^- + \mathcal{W}_4 + \mathcal{W}_5 \quad M \text{ is totally umbilic} \]

Table 9: $\overline{M}$ of type $\mathcal{X}_4$
\( p_x \) is normal to \( M \)

\[
\begin{array}{ll}
W_1^+(-) + W_2^+(-) + W_3 & \
W_1^+(-) + W_2^+(-) + W_4 + W_5 & IB = B \\
W_1^+(-) + W_3 + W_4 + W_5 & (1 + I)B = 2h\langle \cdot, \cdot \rangle \\
W_2^+(-) + W_3 + W_4 + W_5 & pd^*\varphi(n) = 12h \\
W_1^+(-) + W_4 + W_5 & M \text{ is totally umbilic}
\end{array}
\]

Table 10: \( \overline{M} \) of type \( X_4 \) and \( \theta = 0 (\theta = \pi/2) \) constant

\[
\begin{array}{ll}
W_1^+ + W_2^- + W_3^+ + W_4^- + W_5 & \iota^*pd^*\varphi = 12Id\theta \\
W_1^+ + W_2^- + W_3^+ + W_4^- + W_5 & IB = B \\
W_1^+ + W_1^- + W_2^+ + W_3 + W_4 + W_5 & \sin \theta \ 2h\langle \cdot, \cdot \rangle = \sin \theta \ (1 + I)B \\
W_1^+ + W_1^- + W_2^+ + W_3 + W_4 + W_5 & \cos \theta \ 2h\langle \cdot, \cdot \rangle = \cos \theta \ (1 + I)B \\
W_1^+ + W_2^+ + W_2^- + W_3 + W_4 + W_5 & h \sin \theta = 0 \\
W_1^- + W_2^+ + W_2^- + W_3 + W_4 + W_5 & h \cos \theta = 0 \\
W_1^+ + W_1^- + W_3 + W_4 + W_5 & (1 + I)B = 2h\langle \cdot, \cdot \rangle \\
W_2^+ + W_2^- + W_3 + W_4 + W_5 & M \text{ is a minimal variety} \\
W_1^+ + W_1^- + W_4 + W_5 & M \text{ is totally umbilic} \\
W_4 + W_5 & M \text{ is totally geodesic}
\end{array}
\]

Table 11: \( \overline{M} \) of type \( X_4 \) and \( p_\varphi \) tangent to \( M \)
**Theorem 4.14.** Let us consider a seven-dimensional Riemannian manifold \( M \) equipped with a \( G_2 \)-structure, an orientable hypersurface \( M \) of \( M \) with unitary normal vector field \( n \), and an \( SU(3) \)-structure on \( M \) defined as in Proposition 4.1.

(i) If the \( G_2 \)-structure on \( M \) is of type \( X_1 + X_3 \), then the \( SU(3) \)-structure on \( M \) is of type \( W_1^+ + W_1^- + W_2^+ + W_2^- + W_3 + W_5 \).

(ii) If the \( G_2 \)-structure on \( M \) is of type \( X_2 + X_4 \) and \( p_\varphi \) is tangent to \( M \), then the \( SU(3) \)-structure on \( M \) is of type \( W_2^+ + W_2^- + W_3 + W_4 + W_5 \) if and only if \( M \) is a minimal variety.

(iii) If the \( G_2 \)-structure on \( M \) is of type \( X_2 + X_4 \), then the \( SU(3) \)-structure on \( M \) is of type \( W_1^+ + W_1^- + W_2^+ + W_2^- + W_4 + W_5 \) if and only if \( \varphi_B = B \).

5 **Examples**

1. **\( SU(3) \)-structures on \( S^6 \).**— Let us consider \( \mathbb{R}^7 \) as identified with \( Im \mathbb{O} \), the pure imaginary Cayley numbers. It is well known that \( \mathbb{R}^7 \), considered in this way, is equipped with a parallel (\( P \)) \( G_2 \)-structure defined by means of the product of Cayley numbers. Since the six-dimensional sphere \( S^6 \) is totally umbilic in \( \mathbb{R}^7 \), taking Theorem 4.3 into account, \( S^6 \) has \( SU(3) \)-structures of types \( W_1^+ + W_1^- + W_5 \) and \( W_1^+ + W_1^- \). Likewise, taking Theorem 4.5, \( S^6 \) has two \( SU(3) \)-structures of type \( W_1^+ + W_1^- \), respectively.

   Moreover, since \( B_{S^6} = \langle \cdot, \cdot \rangle \), by Proposition 4.2, we have \( r(\omega_{S^6}) = \cos \theta \langle \cdot, \cdot \rangle + \sin \theta \omega_{S^6} \). Finally, taking Equation (2.8) into account, we get

   \[ \nabla \omega_{S^6} = \cos \theta \psi_{S^6} + \sin \theta \psi_{S^6}. \tag{5.1} \]

2. **\( SU(3) \)-structures on \( S^6 \times S^1 \).**— Now we consider the \( SU(3) \)-structure on \( S^6 \) with \( \theta = 0 \) constant. On the product manifold \( S^6 \times S^1 \), we define a \( G_2 \)-structure by

   \[ \varphi = -\vartheta \wedge \omega_{S^6} + \psi_{S^6} + \psi_{S^6}, \quad *\varphi = -\frac{1}{2} \omega_{S^6} \wedge \omega_{S^6} + \vartheta \wedge \psi_{S^6}, \tag{5.2} \]

   where \( \vartheta \) is a Maurer-Cartan one-form on \( S^1 \). Since \( \nabla \omega_{S^6} = \psi_{S^6} + \eta_{S^6} = 0 \), taking Theorem 2.3 and Corollary 2.4 into account, we have

   \[ d\psi_{S^6} = 0, \quad d\psi_{S^6} = 2\omega_{S^6} \wedge \omega_{S^6}. \]
Therefore,
\[
d\varphi = 3\theta \wedge \psi_{S^6} = 3\theta \wedge \varphi;
\]
\[
d\ast \varphi = -2\theta \wedge \omega_{S^6} \wedge \omega_{S^6} = 4\theta \wedge \ast \varphi.
\]
Hence, \(S^6 \times S^1\) is equipped with a \(G_2\)-structure of type \(X_4\) with \(pd^\ast \varphi_1 = -12\theta\) (see [13]). This fact was pointed out in [14] and, moreover, we have \(\bar{\pi}(\varphi) = \varphi(\theta, \cdot, \cdot)\) (see Table 1).

The manifold \(S^5 \times S^1\) is contained in \(S^6 \times S^1\) as a totally geodesic orientable hypersurface. Thus, if \(n\) is the unit normal vector field on \(S^5 \times S^1\), we are in the conditions of Corollary 4.13 with \(Id^\ast \omega = 12\eta + 4Id\theta = -4\ast \theta \neq 0\). Therefore, the manifold \(S^5 \times S^1\) has \(SU(3)\)-structures of type
\[
(W_4 + W_5) - (W_4 \cup W_5).
\]

Remark 5.1. It is well known that \(S^6\) and \(S^5 \times S^1\) have almost Hermitian structures (\(U(3)\)-structures) of type \(W_1\) and \(W_4\), respectively. Here we give further information about \(S^6\) and \(S^5 \times S^1\) as special almost Hermitian manifolds.

3. Hypersurfaces of the manifold \(H(1,2)/\Gamma \times \mathbb{T}^2\).- Let \(H(1,2)\) the generalized Heisenberg group, i.e., the connected, simply connected and nilpotent Lie group consisting of matrices:
\[
a = \begin{pmatrix}
I_2 & X & Z \\
0 & 1 & y \\
0 & 0 & 1
\end{pmatrix},
\]
where \(X\) and \(Z\) are \(2 \times 1\) matrices of real numbers and \(y\) is a real number. If we write \(X' = (x_1, x_2)\) and \(Z' = (z_1, z_2)\), then a coordinate system on \(H(1,2)\) is given by:
\[
x_1(a) = x_1, \quad x_2(a) = x_2, \quad y(a) = y, \quad z_1(a) = z_1, \quad z_2(a) = z_2;
\]
and a basis of left invariant one-forms is:
\[
\{dx_1, \ dx_2, \ dy, \ dz_1 - x_1dy, \ dz_2 - x_2dy\}.
\]

Let \(\Gamma\) be the discrete subgroup of \(H(1,2)\) consisting of matrices with integer entries. We consider the quotient space \(H(1,2)/\Gamma\). Since the one-forms \(\{dx_1, dx_2, dy, dz_1 - x_1dy, dz_2 - x_2dy\}\) are left invariant, they descend to one-forms \(\{\beta_1, \beta_2, \lambda, \gamma_1, \gamma_2\}\) on \(H(1,2)/\Gamma\) such that:
\[
d\beta_1 = d\beta_2 = d\lambda = 0, \quad d\gamma_1 = \lambda \wedge \beta_1, \quad d\gamma_2 = \lambda \wedge \beta_2.
\]
Let us consider the product manifold $\overline{M} = H(1, 2)/\Gamma \times \mathbb{T}^2$, where $\mathbb{T}^2$ is a two-dimensional torus. If $\eta_1, \eta_2$ denote a basis of closed one-forms on $\mathbb{T}^2$, we consider on $\overline{M}$ the following basis for one-forms

$$e_0 = \beta_1, \quad e_1 = \gamma_2, \quad e_2 = \eta_2, \quad e_3 = \eta_1, \quad e_4 = \beta_2, \quad e_5 = \lambda, \quad e_6 = \gamma_1. $$

In [7] it is defined the $G_2$-structure such that $\{e_0, e_1, \ldots, e_6\}$ is a Cayley coframe for one-forms and found that such a $G_2$-structure is of type $\mathcal{X}_2$, i.e., $d\varphi = 0$. In fact, it was the first known example of compact calibrated manifold with $G_2$-structure. Moreover, one can compute

$$d^*\varphi = -\ast d\varphi = e_0 \wedge e_1 - e_4 \wedge e_6.$$ 

Now, using Equation (3.8), we obtain the bilinear form

$$2\tau(\varphi) = e_0 \wedge e_1 - e_4 \wedge e_6.$$ 

Let us recall that we have $de_i = -e_4 \wedge e_5$, $de_6 = -e_0 \wedge e_5$ and $de_i = 0$, for $i = 0, 2, 3, 4, 5$. From $e_i([e_j, e_k]) = -de_i(e_j, e_k)$, where $i, j, k \in \mathbb{Z}_7$ and $[,]$ denotes the Lie bracket, we get $[e_0, e_5] = e_6$, $[e_4, e_5] = e_1$ and $[e_i, e_j] = 0$ for $(i, j) \in \mathbb{Z}_7 \times \mathbb{Z}_7 - \{(0, 5), (5, 0), (4, 5), (5, 4)\}$. Now, using Koszul Formula, we get

$$\nabla_{e_0} e_5 = \frac{1}{2} e_6, \quad \nabla_{e_0} e_6 = -\frac{1}{2} e_5, \quad \nabla_{e_1} e_4 = -\frac{1}{2} e_5, \quad \nabla_{e_1} e_5 = \frac{1}{2} e_4 \quad \nabla_{e_4} e_1 = -\frac{1}{2} e_5, \quad \nabla_{e_4} e_5 = -\frac{1}{2} e_1, \quad \nabla_{e_5} e_0 = -\frac{1}{2} e_6, \quad \nabla_{e_5} e_1 = \frac{1}{2} e_4, \quad \nabla_{e_5} e_4 = -\frac{1}{2} e_1, \quad \nabla_{e_5} e_6 = \frac{1}{2} e_0, \quad \nabla_{e_6} e_0 = -\frac{1}{2} e_5, \quad \nabla_{e_6} e_5 = \frac{1}{2} e_0,$$ 

and $\nabla_{e_i} e_j = 0$, for all $(i, j) \in \mathbb{Z}_7 \times \mathbb{Z}_7 - \{(0, 5), (0, 6), (1, 4), (1, 5), (4, 1), (4, 5), (5, 0), (5, 1), (5, 4), (5, 6), (6, 0), (6, 5)\}$.

(i) Let us consider a hypersurface $M_1$ which is a maximal integral submanifold of the integrable distribution defined by the one-form $n = e_3$. Such hypersurface $M_1$ is diffeomorphic with $M \times \mathbb{T}^1$. Since $M_1$ is totally geodesic and $\tau(\varphi)(\iota_\ast n) = 0$, making use of Theorem 4.7, $M_1$ has induced an $SU(3)$-structure of type $\mathcal{W}_2^+ + \mathcal{W}_2^- + \mathcal{W}_5$. If we take $\theta = \pi/4$ constant, the $SU(3)$-structure is of type $\mathcal{W}_2^+ + \mathcal{W}_2^-$. Finally, making use of Theorem 4.8, we obtain two $SU(3)$-structures on $M_1$ of type $\mathcal{W}_2^+$ and $\mathcal{W}_2^-$, respectively.

(ii) Let us consider a hypersurface $M_2$ which is a maximal integral submanifold of the integrable distribution defined by the one-form $n = e_0$. The second fundamental form $M_2$ is given by $2B = e_5 \otimes e_6 + e_6 \otimes e_5$ and $2\iota^\ast \tau(\varphi) = -e_4 \wedge e_6$. Thus, $M_2$ is a minimal variety and it is satisfied $(I(1) - I(2))n^\ast \tau(\varphi) + (1 + I)B = 0$. Therefore, by making use of Theorem 4.7, the $SU(3)$-structure on $M_2$ is of type $\mathcal{W}_3 + \mathcal{W}_4 + \mathcal{W}_5$. Note that $IB \neq B$ and $-2\tau(\varphi)(\iota_\ast n) = e_1 \neq 0$. 

(iii) Let us consider a hypersurface $M_3$ which is a maximal integral sub-manifold of the integrable distribution defined by the one-form $n = e_5$. The second fundamental form of $M_3$ is given by $B = -e_0 \vee e_6 - e_1 \vee e_4$, where $2a \vee b = a \otimes b + b \otimes a$. Moreover, $2t^*\mathcal{F}(\varphi) = e_0 \wedge e_1 - e_4 \wedge e_6$. Thus, $M_3$ is a minimal variety, $IB = B$ and $\mathcal{F}(\varphi)(t^*, n) = 0$. Then, using Theorem 4.7, we have that $M_3$ is of type Kähler ($\mathcal{W}_5$). In particular, when $\theta$ is constant, the intrinsic $SU(3)$-torsion of $M_3$ vanishes, i.e., $M_3$ is $su(3)$-Kähler.

4. Hypersurfaces of a family of manifolds with $G_2$-structure of type $\mathcal{X}_3$. Let us consider the manifolds $M(k)$ described in [5] as follows. For a fixed $k \in \mathbb{R}$, $k \neq 0$, let $G(k)$ be the three-dimensional connected and solvable (non-nilpotent) Lie group consisting of the matrices

$$ a = \begin{pmatrix} e^{kz} & 0 & 0 & x \\ 0 & e^{-kz} & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}, $$

where $x, y, z \in \mathbb{R}$. Then, a global coordinate system $\{x, y, z\}$ for $G(k)$ is given by $x(a) = x$, $y(a) = y$, $z(a) = z$. A straightforward computation proves that a basis of right invariant one-forms on $G(k)$ is $\{dx - kxdz, dy + kydz, dz\}$.

The Lie group $G(k)$ can be also described as the semidirect product $\mathbb{R} \times \phi \mathbb{R}^2$, where $\phi : \mathbb{R} \to Aut(\mathbb{R}^2)$ is the representation defined by

$$ \phi(t) = \begin{pmatrix} e^{kz} & 0 \\ 0 & e^{-kz} \end{pmatrix}, \quad z \in \mathbb{R}. $$

Therefore $G(k)$ possesses a discrete subgroup $\Gamma(K)$ such that the quotient manifold $M(k) = G(k)/\Gamma(k)$ is compact. Moreover, the one-forms $dx - kxdz$, $dy + kydz$, $dz$ descend to $M(k)$. Let us denote by $\alpha, \beta, \gamma$, respectively, the induced one-forms on $M(k)$. Then we have $d\alpha = -k\alpha \wedge \gamma$, $d\beta = k\beta \wedge \gamma$, $d\gamma = 0$.

Let us consider the product manifold $M(k) \times T^4$ where $T^4$ is a four-dimensional torus. Let $\eta_1, \eta_2, \eta_3, \eta_4$ be a basis of closed one-forms on $T^4$. Then in $M(k) \times T^4$ we consider the following basis for one-forms

$$ e_0 = \alpha, \quad e_1 = \beta, \quad e_2 = \eta_2, \quad e_3 = \gamma, \quad e_4 = \eta_3, \quad e_5 = \eta_4, \quad e_6 = \eta_5. $$

Therefore, $de_0 = -ke_0 \wedge e_3$, $de_1 = ke_1 \wedge e_3$ and $de_i = 0$, for all $i \in \mathbb{Z} - \{0, 1\}$. 
We consider on $M(k) \times \mathbb{T}^4$ the $G_2$-structure determined by the four form $\varphi = \sum_{i \in \mathbb{Z}_7} e_i \wedge e_{i+1} \wedge e_{i+3}$. Since

\begin{align}
(5.3) \quad d\varphi &= k e_3 \wedge e_4 \wedge e_5 \wedge e_0 - k e_3 \wedge e_5 \wedge e_6 \wedge e_1 \\
&\quad + ke_3 \wedge e_6 \wedge e_0 \wedge e_2 - k e_3 \wedge e_1 \wedge e_2 \wedge e_4,
\end{align}

we have $d\varphi \wedge \varphi = 0$. Moreover, it is immediate to check that $d \ast \varphi = 0$. Therefore, the $G_2$-structure is of type $\mathcal{X}_3$. Moreover, with the indicated metric, these manifolds do not admit any $G_2$-structure of type $\mathcal{P}$ (see [14]).

Now, from $d^* \varphi = 0$ and the expression for $d\varphi$ given by (5.3), taking Equation (3.8) into account, we obtain

\begin{align}
(5.4) \quad r(\varphi) &= -2ke_0 \vee e_1.
\end{align}

From $e_i([e_j, e_k]) = -de_i(e_j, e_k)$, for $i, j, k \in \mathbb{Z}_7$, we get $[e_0, e_3] = ke_0$, $[e_1, e_3] = -ke_1$ and $[e_i, e_j] = 0$ for $(i, j) \in \mathbb{Z}_7 \times \mathbb{Z}_7 - \{(0, 3), (3, 0), (1, 3), (3, 1)\}$. Now, using Koszul Formula, we get $\nabla_{e_0} e_3 = ke_0$, $\nabla_{e_1} e_3 = -ke_1$, $\nabla_{e_0} e_0 = -ke_3$, $\nabla_{e_1} e_1 = ke_3$ and $\nabla_{e_i} e_j = 0$, for all $(i, j) \in \mathbb{Z}_7 \times \mathbb{Z}_7 - \{(0, 3), (1, 3), (0, 0), (1, 1)\}$.

(i) Let us consider a hypersurface $N_1$ which is a maximal integral submanifold of the integrable distribution defined by the one-form $n = e_2$. Such a hypersurface $N_1$ is diffeomorphic with $M(k) \times \mathbb{T}^3$. Since $N_1$ is totally geodesic, $r(\varphi)(\iota_\ast n) = 0$, $r(\overline{\varphi})(n, n) = 0$, and $\iota^* r(\varphi) = -2k e_0 \vee e_1$, taking Theorem 4.9 into account, $N_1$ has $SU(3)$-structure of types $\mathcal{W}_2^+ + \mathcal{W}_2^- + \mathcal{W}_3 + \mathcal{W}_5$. Taking $\theta$ constant, the $SU(3)$-structure on $N_1$ is of type $\mathcal{W}_2^+ + \mathcal{W}_2^- + \mathcal{W}_3$. In particular, when $\theta = 0(\theta = \pi/2)$ constant, making use of Theorem 4.10, we have that the $SU(3)$-structure is of type $\mathcal{W}_2^+ + \mathcal{W}_3$ ($\mathcal{W}_2^- + \mathcal{W}_3$).

(ii) Let us consider a hypersurface $N_2$ which is a maximal integral submanifold of the integrable distribution defined by the one-form $n = e_3$. The second fundamental tensor of $N_2$ is given by $B = ke_0 \otimes e_0 - ke_1 \otimes e_1$. Thus, $M_2$ is a minimal variety, $r(\varphi)(\iota_\ast n) = 0$, $r(\overline{\varphi})(n, n) = 0$, $IB = -B$ and $\iota^* r(\varphi) = -\iota^* r(\overline{\varphi})$. Then, using Theorem 4.9, $N_1$ has $SU(3)$-structures of type $\mathcal{W}_3 + \mathcal{W}_5$. In particular, when $\theta$ is constant, the $SU(3)$-structure on $N_2$ is of type $\mathcal{W}_3$.

(iii) Let us consider a hypersurface $N_3$ which is a maximal integral submanifold of the integrable distribution defined by the one-form $n = e_0$. It is
immediate that \( N_3 \) is totally geodesic, \( r(\varphi)(\iota_\ast n, n) = -ke_1, \iota^\ast r(\varphi) = 0 \) and \( r(\varphi)(n, n) = 0 \). Then, using Theorem 4.9, the \( SU(3) \)-structure on \( N_3 \) is of type \( \mathcal{W}_5 \). If we consider \( \theta \) constant, then \( N_3 \) is \( su(3) \)-Kähler.

References

[1] E. Calabi, *Construction and properties of some 6-dimensional almost complex manifolds*, Trans. Amer. Math. Soc. 87 (1958): 407–438.

[2] S. G. Chiossi and S. Salamon, *The intrinsic torsion of SU(3) and \( G_2 \) structures*, Differential geometry, Valencia 2001, World Sci. Publishing, River Edge, NJ (2002): 115–133.

[3] R. Cleyton and S. Ivanov, *On the geometry of closed \( G_2 \)-structures*, arXiv:math.DG/0306362, IMADA preprint PP-2003-11.

[4] R. Cleyton and A. Swann, *Einstein Metrics via Intrinsic or Parallel Torsion*, Math. Z. 247 (2004): 513–528.

[5] L. A. Cordero, M. Fernández and A. Gray, *Modelos minimales en geometría diferencial*, Proc. Recent Topics on Diff. Geom. Workshop, Ed. D. Chinea and J.M. Sierra, Univ. de La Laguna (Spain) (1990): 31–41.

[6] M. Falcitelli, A. Farinola and S. M. Salamon, *Almost-Hermitian geometry*, Differential Geom. Appl. 4 (1994): 259–282.

[7] M. Fernández, *An example of a compact calibrated manifold associated with the exceptional Lie Group \( G_2 \)*, J. Diff. Geom. 26 (1987): 367–370.

[8] M. Fernández and A. Gray, *Riemannian manifolds with structure group \( G_2 \)*, Ann. Mat. Pura Appl. (4) 32 (1982): 19–45.

[9] A. Gray, *Some examples of almost Hermitian manifolds*, Illinois J. Math. 10 (1969): 353–366.

[10] A. Gray, *Vector cross products on manifolds*, Trans. Amer. Soc. 148 (1969): 463–504.

[11] A. Gray and L. M. Hervella, *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. Pura Appl. (4) 123 (1980): 35–58.
[12] F. Martín Cabrera *Spin(7)-structures on principal bundles over Riemannian manifolds with $G_2$-structures*, Rend. Circ. Mat. Palermo (2) 44 (1995): 249–272.

[13] F. Martín Cabrera, *On Riemannian manifolds with $G_2$-structures*, Boll. Unione Mat. It. (7) 9-A (1996): 99–112.

[14] F. Martín Cabrera, *Orientable hypersurface of Riemannian manifolds with Spin(7)-structure*, Acta Math. Hungar. (3) 76 (1997): 235–247.

[15] F. Martín Cabrera, *Special almost Hermitian geometry*, J. Geom. Phys. (to appear) arXiv:math.DG/0409167.

[16] F. Martín Cabrera, M. D. Monar and A. Swann *Classification $G_2$-structures*, J. London Math. Soc. 53 (1996): 407–416.

[17] S. M. Salamon, *Riemannian geometry and holonomy groups*, Pitman Research Notes in Mathematics 201, Longman, Harlow (1989).

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