Abstract. A black hole solution in three spacetime dimensions, endowed with an \( SU(2) \) charge, is presented. The construction is based on two main features of three dimensions: i) AdS\(_3\) spacetime is locally Lorentz-flat, that is, it can be covered with a congruence of local inertial observers, just like flat Minkowski space; ii) The \( SO(2, 1) \) and \( SU(2) \) groups are isomorphic, so that a flat connection of the first can be mapped to a flat connection of the second. The global nontrivial nature of the solution is a consequence of the topology produced by the identification in the covering space that gives rise to the 2+1 black hole.

It can be seen that this solution belongs to the vacuum (matter-free) sector of a supersymmetric theory based on the Chern–Simons action for the \( su(1,2|2) \) superalgebra. The \( SU(2) \) gauge symmetry is interpreted as the freedom to choose locally the definition of spin quantization axis for the electrons.

1. The Lorentz flatness of AdS\(_3\) spacetime

Three-dimensional gravity is a source of surprises. In the standard theory, all classical solutions have constant Riemannian curvature, with a radius determined by the cosmological constant in the action. Hence, given the action, the solutions have a uniquely fixed curvature and the geometry is completely determined by the global topology. The geometry of each solution is determined by two parameters that characterize the particular identification applied to the covering AdS space. If closed timelike curves and naked singularities are excluded, a family of black holes labeled by a mass \( M \) and angular momentum \( J \) are obtained [1, 2]. A few other solutions dual to the black holes can also be found [3] and, if moderate naked singularities like conical defects are allowed, also spinning point particles can be included in the spectrum [4, 5].

Due to its simple dynamical structure 2+1 gravity is a remarkable test ground to study gravitational phenomena, analogous to those in 3+1 dimensions, in a simplified setting.

This simplicity can be pushed one step further by the following little

**Theorem:** A 2+1 locally AdS spacetime is Lorentz flat [6].

**Proof:** Consider a three-dimensional manifold \( M \) endowed with a Lorentz connection \( \omega^{ab} = -\omega^{ba} \), and an open simply connected region of \( M \) where the connection is flat,

\[
R^{ab} = d\omega^{ab} + \omega^{a}_{\ c}\omega^{cb} = 0, \quad (1)
\]
where $R^{ab}$ is the Lorentz curvature two-form. The metric structure of the space is determined by the one-form fields defining a local orthonormal frame $e^a$ (also called the soldering form, vielbein or coframe field). The exterior covariant derivative of $e^a$ defines the torsion two-form $T^a = De^a = de^a + \omega^{ac}e^c$. From (25) it is straightforward to infer that the torsion is covariantly constant,

$$0 = R^{ab}e^b = DD e^a = DT^a.$$  

We therefore learn that the Lorentz flat geometry must have covariantly constant torsion $DT^a = 0$. In three dimensions, this equation is integrated as

$$T^a = \tau \varepsilon^a_{\ b}e^b e^c,$$

where $\tau$ is an integration constant. It is always possible to split $\omega^{ab}$ as

$$\omega^{ab} = \tilde{\omega}^{ab} + \kappa^{ab},$$

where $\tilde{\omega}^{ab}$ is the torsion-free (Riemannian) part,

$$de^a + \tilde{\omega}^{ac}e^c = 0,$$

and $\kappa^{ab}$ is the contorsion. Therefore, $T^a = De^a = de^a + \omega^{ac}e^c \equiv \kappa^{ac}e^c$, and

$$R^{ab} = \tilde{R}^{ab} + \tilde{D}\kappa^{ac}e^c + \kappa^{ac}\kappa^{cb},$$

where $\tilde{D}$ is the covariant exterior derivative in the connection $\tilde{\omega}^{ab}$. Then, in view of (3) and (4), (6) becomes

$$R^{ab} = \tilde{R}^{ab} + \tau^2 e^a e^b,$$

which implies that anti-de Sitter spaces

$$\tilde{R}^{ab} = -\tau^2 e^a e^b,$$

are Lorentz-flat manifolds. (Q.E.D.)

This means that any simply connected open set of 2+1 AdS space can be covered with a family of local frames related with each other by Lorentz transformations. In particular, this implies that by parallel transport of a vector with a Lorentz connection, a globally defined vector field can be constructed that completely covers any simply connected open region. This can be done with a basis of three vectors at one point, thus covering the entire region with three independent vector fields.

It is interesting to observe that the minus sign in the 0-0 component of the Lorentzian metric is responsible for the negative sign in the curvature. Had the metric been Euclidean, the resulting geometry would have been that of a three sphere instead of AdS3. The little theorem presented here can then be seen as the Lorentzian version of Adams’ theorem, which states that a Euclidean three sphere is parallelizable, namely, it can be globally covered by a family of three independent vector fields [7, 8]. Adams’ theorem goes even further, stating that also the seven sphere is parallelizable, from which we can conjecture that a Minkowskian AdS7 is also Lorentz flat.

1 We implicitly assume exterior (wedge) products for forms except where confusion may arise.
2. The 3D black hole as a Lorentz flat geometry

Since a 2+1 black hole is locally an AdS geometry with radius of curvature $\ell$, it can be identified with a Lorentz flat spacetime with constant torsion given by (3) with $\tau = \eta \ell^{-1}$, where $\eta = \pm 1$. How can that be?

In order to illustrate the point, consider the geometry of a static BTZ black hole [9] (the rotating case is left as an exercise to the reader),

$$ds^2 = f^2 dt^2 + f^2 dr^2 + r^2 d\phi^2,$$

with $f^2 = M + r^2/\ell^2$, and $M$ is the mass. The vielbein are given by

$$e^0 = f(r) dt, \quad e^1 = dr/f(r), \quad e^2 = r d\phi.$$  \hspace{1cm} (10)

Then, the torsion-free (Riemannian) connection, satisfying (5), is given by

$$\tilde{\omega}^0_1 = \frac{r}{\ell^2} dt, \quad \tilde{\omega}^1_2 = -f d\phi, \quad \tilde{\omega}^2_0 = 0,$$  \hspace{1cm} (11)

and can be directly confirmed that $\tilde{R}^{ab} = -\ell^{-2} e^a e^b$. It is also straightforward to see that the torsion given by (3) is

$$T^0 = -\eta \frac{r}{\ell^2} dr \wedge d\phi, \quad T^1 = \eta \frac{f}{\ell} d\phi \wedge dt, \quad T^2 = \frac{\eta}{\ell} dt \wedge dr.$$  \hspace{1cm} (12)

The Lorentz connection, on the other hand, is given by (4), where

$$\kappa^0_1 = \eta \frac{r}{\ell} d\phi, \quad \kappa^1_2 = -\eta \frac{f}{\ell} dt, \quad \kappa^2_0 = -\eta \frac{dr}{\ell f}.$$  \hspace{1cm} (13)

It is then simple algebra to show that

$$\omega^0_1 = \eta \frac{r}{\ell} \left[ \frac{dt}{\ell} + d\phi \right], \quad \omega^1_2 = -f \left[ \frac{dt}{\ell} + d\phi \right], \quad \omega^2_0 = -\eta \frac{dr}{\ell f}.$$  \hspace{1cm} (14)

is Lorentz flat, $R^{ab} = d\omega^a_b + \omega^a_c \omega^c_b \equiv 0$. Finally, direct computation shows that the torsion is indeed covariantly constant in the Lorentz connection, $dT^a + \omega^a_b T^b \equiv 0$.

It should be noted that the flatness of the Lorentz connection does not imply that parallel-transporting a vector round an arbitrary closed path will take it back to itself. That is only so for a path contained in a simply connected open patch of AdS$_3$ which, in the case of a black hole –or a point particle–, means a region that does not include the central singularity. In particular, a path that winds around the line $r = 0$ would yield a nontrivial holonomy because such a path is not contractible and the region where it lies is not simply connected.

Another important point is the fact that the flatness of the Lorentz connection (14) does not depend on the value of $\ell$, so long as $\ell \neq 0$. In fact, $\ell$ can be set equal to 1 by the re-scaling of $r \rightarrow r' = r/\ell$, $t \rightarrow t' = t/\ell$.

The same discussion can be repeated for a rotating black hole, or for a point particle spinning or not (naked conical singularity). All of these are locally AdS$_3$ spacetimes, the only difference being the type of identification that is made to produce them and therefore they belong to different topological classes but with the same local geometry.
3. Adding SU(2) hair on a 2+1 BH

Black holes do no suffer hair gladly. The possibility of adding hair to a black hole is always a challenging problem as there are not many standard black hole solutions with minimally coupled nontrivial gauge fields. The previous discussion however, hints at a possibility of including an SU(2) hair in 2+1 dimensions, based on the isomorphism of this gauge group with SO(2, 1). The point is that since these two groups are isomorphic, having a nontrivial locally flat configuration of the latter suggests the possibility of a nontrivial flat configuration for the former.

The isomorphism SO(2, 1) ≅ SU(2) means that there exists a one-to-one correspondence between the elements of each group and their corresponding algebras,

\[ [J_a, J_b] = \varepsilon_{abc} J_c \quad \leftrightarrow \quad [G_I, G_J] = i \varepsilon_{IJK} G_K, \]

where \( \varepsilon_{abc} = \eta^{ad} \varepsilon_{abd} \), and both \( \varepsilon_{abc} \) and \( \varepsilon_{IJK} \) are the three-dimensional Levi-Civita symbols, with \( \varepsilon_{012} = \varepsilon_{123} = 1 \). An explicit representation for the SO(2, 1) and SU(2) algebras is given by \( J_a = (\Gamma_a)^{\alpha} \beta \), and \( G_I = (\sigma_I)^{\ell} \), where \( \gamma_a \) and \( \sigma_I \) are the 2 × 2 representations of the Dirac and Pauli matrices respectively.

In particular, starting from the connection (14) one can produce a flat SU(2) connection given by

\[ A^3 = \eta' r \left[ \eta' \frac{dt}{\zeta} + d\phi \right] \sim \omega^0_1, \quad A^2 = i h \left[ \eta' \frac{dt}{\zeta} + d\phi \right] \sim \omega^1_2, \quad A^1 = \eta' \frac{dr}{\zeta h} \sim \omega^2_0, \]

where \( h^2(r) = r^2/\zeta^2 - Q \) and \( \eta' = \pm 1 \). The global nontriviality is reflected by the fact that \( d\phi \) is not an exact form in an open region containing \( r = 0 \).

It is straightforward to check that \( A^I \) defines a flat SU(2) connection,

\[ F^I = dA^I + \epsilon^{IJK} A^J A^K \equiv 0, \]

that holds for all values of \( \zeta \) and \( \eta' \). Here \( \zeta \) plays the role of the AdS radius, although there is no need to identify \( \zeta \) with \( \ell \). Similarly, the charge \( Q \) is an integration constant of the solution, similar to the mass of the black hole. However, since there is no analogue for the metric in the SU(2) sector, there is no restriction for the charge corresponding to the cosmic censorship that restricts \( M \) to non negative values in order to avoid naked singularities.

The analysis can be extended to the case of rotating solutions as reported in the forthcoming paper [9], where the BPS configurations are also discussed. The physical meaning of these solutions requires the identification of the physical observables of the system, the conserved charges and this cannot be discussed without knowing the action principle.

4. Action principle

The construction above would be a rather artificial exercise if the equations \( R^{\alpha \beta} = 0 \) and \( F^I = 0 \) did not originate in some action principle. The natural three-dimensional Lagrangians that give rise to these equations are the Chern-Simons (CS) forms for the connections \( \omega^{a\beta} \) and \( A^I \), respectively, so one might tentatively consider

\[ I_0[\omega, A] = k_1 \int \left( \frac{1}{2} A^I dA^I + \frac{1}{3} \varepsilon_{IJK} A^I A^J A^K \right) + k_2 \int \left[ \frac{1}{2} \omega^{a\beta} d\omega^{b\alpha} + \frac{1}{3} \omega^{a\beta} \omega^{b\gamma} \omega^{c\alpha} \right]. \]

However this action is still somewhat uninteresting since there is no coupling between the fields and there seem to be no reason to put them together. In addition, the identification of the second term as something related to the spacetime geometry and gravity could only be justified if the metric properties of spacetime are to play a role, otherwise the earlier discussion about
the AdS spacetime would be completely unjustified. The link between the spacetime symmetry represented by the Lorentz connection and the internal SU(2) gauge symmetry is naturally provided by a spin-1/2 field. Fermions are intrinsically sensitive to the Lorentz symmetry because spinors are fundamental representations of the Lorentz group. In addition, fermions can be charged with respect to an SU(N) gauge group as we know from the Standard Model of particle physics. So, the presence of spinning matter can provide a natural link between the two free fields in (18).

Moreover, introducing fermions turns out to be a very economical solution because, as Weyl noted long ago [10], spinors also naturally couple to the metric of the manifold on which they live. One way of seeing this is that in order to write down the Dirac equation in an arbitrary spacetime background it is necessary to know the metric structure of the spacetime geometry. The need can be seen in the fundamental structure for spin-1/2 representation, the Clifford algebra,

\[ \{ \gamma_a, \gamma_b \} = 2\eta_{ab}, \]  

where \( \eta_{ab} \) is the Lorentz invariant metric of Minkowski space. Then, in order to define the Dirac gamma matrices, it is necessary to transmute the tangent space index of the Dirac gamma matrix into a spacetime index. The instrument that performs this is the local Lorentz (co-)frame \( e^a = e^a_\mu dx^\mu \) and its inverse, the local vector frame

\[ E^a = E^\mu_a \partial_\mu, \]  

and its inverse, the local vector frame

\[ E^a = E^\mu_a \partial_\mu. \]  

This means that the manifold where spinors propagate must be endowed with a metric structure that projects the tangent space metric down to the spacetime manifold,

\[ g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu. \]  

Thus, a simple and natural way to link the SU(2) and SO(2,1) connections, bringing a metric structure at the same time is to add to (18) a minimally coupled spin-1/2 SU(2)-charged Dirac field

\[ I[\omega, A, e, \psi] = I_0[\omega, A] + \int L_F, \]  

where

\[ L_F = \overline{\psi} \left[ \hat{\psi} + iA - \frac{1}{4} \gamma_a \gamma^a \gamma_b \right] \psi |d^3 x - \overline{\psi} \psi e^a T_a. \]  

The first term in brackets on the right is just the usual Dirac Lagrangian for a complex spinor minimally coupled to the the SU(2) gauge field and to the standard spin connection coupling required by covariance in a curved background. The last term results from the requirement of hermiticity,

\[ 1/2 \left[ \overline{\psi} \hat{\psi} + h.c. \right] = \overline{\psi} \hat{\psi} + \overline{\psi} \psi e^a T_a, \]  

and the torsion is produced by the presence of the frame \( E^\mu_a \) in \( \hat{\psi} \), as noted by Weyl [10]. The field equations for this action are:

\[
\delta A^\mu : \quad F^I = \frac{i}{2} \varepsilon^{IJK} \overline{\psi}^J \gamma_a \gamma^b \psi^K e^a e^b \\
\delta \omega : \quad R^{ab} = -\frac{1}{2} \overline{\psi} \gamma^a e^b \\
\delta \overline{\psi} : \quad \left[ \hat{\psi} + iA - \frac{1}{4} \gamma_a \gamma^a \gamma_b + e^a_\mu \epsilon T^b_{\nu\lambda} \eta_{ab} \epsilon^{\mu\nu\lambda} \right] \psi = 0 \\
\delta e^a : \quad \overline{\psi} \epsilon_{abc} \gamma^b \left[ \overline{d} e^c - \overline{d} e^c + 2ie^c A \right] \psi + 2 \overline{\psi} \psi T^a = 0.
\]
It is immediately obvious that in the limit of vanishing fermionic field, the $SU(2)$ and $SO(2, 1)$ connections are flat, $F = 0 = R^{ab}$. This means that the discussion of Sect. 1 refers to the matter-free sector of this system. It is interesting, however, that even if $\psi \neq 0$, the Lorentz curvature is such that $R^{ab} e^{b} = 0$, so that the torsion is still covariantly constant. Therefore (3) would still be valid and (8) would be replaced by

$$\tilde{R}^{ab} = - \left[ \tau^2 + \overline{\psi} \gamma^I \psi \right] e^a e^b.$$  

This in turn means that the fermions can be included perturbatively as corrections to the curvature around the locally AdS spacetime background. The fact that even in the presence of fermions the torsion is covariantly constant and therefore of the form $T^a = \tau \epsilon^{abc} e_b e_c$, implies that the last term in the Dirac equation is always a mass term,

$$e^a_\mu T^b_{\nu \lambda} \eta_{ab} \epsilon^{\mu \nu \lambda} = -6\tau \equiv m.$$  

Thus, in this model although there is no mass parameter and no cosmological constant in the action, the fermion mass and the cosmological constants result from the integration constant related to the torsion of the spacetime background.

5. Superconnection theory

The fact that fermionic fields couple naturally to the Lorentz connection $\omega^{ab}$ and to an internal gauge connection $\lambda$ suggests that there could be a way to combine the two symmetries $SO(2,1)$ and $SU(2)$ into a larger group $G$. However, some well-known no-go theorems make this possibility unlikely for a Lie group, except for the trivial case $G \sim SO(2,1) \times SU(2)$ [11].

As noted long ago by Haag, Lopuszanski and Sohnius [12] a way to circumvent this difficulty is to consider a graded Lie group, also referred to as a supergroup. The idea is that, together with the generators of the Lie groups $\mathfrak{J}_a$ and $\mathfrak{G}_I$ (bosonic sector), fermionic generators $Q_i^\alpha$, $\overline{Q}_\alpha$ that transform in a spin-1/2 representation of $SO(2,1)$ and a fundamental representation of $SU(2)$ are introduced. These supercharges obey

$$[\mathfrak{J}_a, Q] = -\frac{1}{2} \gamma_a Q, \quad [\mathfrak{J}_a, \overline{Q}] = \frac{1}{2} \overline{Q} \gamma_a$$

$$[\mathfrak{G}_I, Q] = \frac{1}{2} Q \sigma_I, \quad [\mathfrak{G}_I, \overline{Q}] = -\frac{1}{2} \sigma_I \overline{Q},$$

where we have suppressed all but the relevant indices. The simplest $4 \times 4$ matrix representation for this is

$$\mathfrak{J}_a = \begin{bmatrix} \frac{1}{2} (\gamma_a)^\alpha_{\beta} & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathfrak{G}_I = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} (\sigma_I)^j_i \end{bmatrix}; \quad Q_i^\alpha = \begin{bmatrix} 0 & \delta_i^\alpha \delta_i^j \\ \delta_i^\alpha \delta_i^j & 0 \end{bmatrix}, \quad \overline{Q}_\alpha = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$  

The superalgebra is completed by the anticommutator,

$$\{Q_i^\alpha, \overline{Q}_\beta\} = i \delta_i^\alpha \delta_j^\beta Z + \delta_i^\beta (\gamma_a)^\alpha_{\beta} \mathfrak{J}_a + \delta_i^\alpha (\sigma_I)^j_i \mathfrak{G}_I.$$  

Here $Z$ is an abelian central extension that can be identified with the $U(1)$ generator

$$Z = -i \begin{bmatrix} \delta_i^\alpha & 0 \\ 0 & \delta_i^j \end{bmatrix}.$$  

Note that all these generators have vanishing supertrace, and the superalgebra can be identified as $su(2,1|2) \oplus u(1)$. 

6
At this point a legitimate question would be, what is the meaning of this enlarged symmetry? It turns out that the action (22) can be written in a very compact way, as a Chern-Simons form,
\[
str \left[ \frac{1}{2} AdA \frac{1}{3} AAA \right] = I[\omega, A, e, \psi] + AdA,
\]
where \(\text{str}\) stands for the graded trace (supertrace) in the superalgebra and \(A\) is the connection
\[
A = \omega^a J_a + A^I G_I + \overline{Q}^j \gamma_a e^a \psi_j - \overline{\psi} e^a \gamma_a Q_i + AZ,
\]
(here \(\omega^a \equiv \varepsilon^a_{bc} \omega^{bc}\)). Since the \(U(1)\) field \(A\) is decoupled from the rest, the theory can be consistently truncated dropping the \(u(1)\) generator from the superalgebra. Alternatively, one could include a nontrivial coupling between the fermion and the \(U(1)\) connection \(A\).

An even simpler superalgebra can be constructed by accommodating the generators \(J_a\) of \(SO(2,1)\) together with a complex two-component spinor supercharge \(Q\) and a \(U(1)\) generator \(K\) dropping the \(SU(2)\) part [13], so that instead of (36) one has
\[
A = \omega^a J_a + A K + \overline{Q} \gamma_a e^a \psi - \overline{\psi} e^a \gamma_a Q.
\]
(37)
Here the generators are in a \(3 \times 3\) matrix representation,
\[
J_a = -\frac{1}{2} \begin{bmatrix} \gamma_a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad K = i \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad Q^a = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right\},
\]
and \(\overline{Q}_a = (Q^a)^\top\). The resulting superalgebra is \(osp(2|2)\) and the corresponding CS action describes a complex Dirac field minimally coupled to a \(U(1)\) gauge field in a curved geometry described by the Lagrangian
\[
L(A, \omega, \psi) = AdA + \omega^a_b d\omega_a + \frac{2}{3} \omega^a_b \omega^b_c e^c - \overline{\psi} \left[ \phi + i A - \frac{1}{4} \gamma_a \gamma^a \gamma_b \right] \psi e^3 d^2 x - \overline{\psi} \psi e^a T_a.
\]
Clearly, the recipe can be extended to dimensions \(D = 2n + 1\): take your favorite gauge group \(G\), then define a supercharge \(Q_K^a\), in a vector representation of \(G\) \((K = 1, 2, ..., s)\) and a spin-1/2 representation of the Lorentz group \((\alpha = 1, 2, ..., 2^n)\). The graded Lie algebra spanned by the supercharges together with Lorentz and \(G\)-generators can always be made close by including additional bosonic generators. With this superalgebra, a connection one-form analogous to (36) can be constructed and with it the Chern-Simons form is uniquely defined. This CS Lagrangian is guaranteed to be quadratic and first order in fermions only for dimension three. In higher dimensions the CS form may include quartic or higher powers of the fermions.

The lack of CS forms in even dimensions, forces the next best option, the Yang-Mills Lagrangian for the superconnection. The resulting Lagrangian in four dimensions is the Dirac equation, minimally coupled to standard electromagnetism and Einstein gravity with positive cosmological constant, plus a four fermion Nambu–Jona-Lasinio term. The resulting theory has only a residual \(SO(3,2) \times U(1)\) gauge symmetry and supersymmetry is explicitly broken [14].

6. Relation with graphene
Graphene is a remarkable system consisting of a one-atom thick layer of carbon atoms in a two-dimensional hexagonal array. This crystal is best described as a superposition of two dislocated triangular sublattices with an average of one free electron per atom that can hop between nearest neighbors [15].
Since nearest neighbor sites cannot be in the same triangular sublattice, hopping between nearest neighbors implies jumping between the two sublattices, which can be pictured as the destruction of an electron in one sublattice and simultaneous creation of one in the other. The electron field can be represented by a two-component spinor,

$$\psi = \begin{pmatrix} a \\ b \end{pmatrix}, \quad \psi^\dagger = \begin{pmatrix} a^\dagger \\ b^\dagger \end{pmatrix},$$

where $a, b$ destroy electrons on each sublattice, and $a^\dagger, b^\dagger$ create them.

It can be shown that the dispersion relation for conduction electrons near the Fermi surface has the form of a massless relativistic particle, $E \simeq v_F |k|$ [15, 16]. Hence, the action that describes the propagation of electrons in graphene is the Dirac Lagrangian for a massless spin-1/2 field, where the speed of light corresponds to the Fermi velocity, $v_F \sim c/300 = 1,000 \text{ km/sec}$. (For comprehensive reviews on graphene see, e.g., [17, 18].)

Interestingly enough, a good description of graphene in the presence of electromagnetism is given by (39). If the geometry is fixed, one can ignore the dynamic features of the field $\omega$. In that case, the Lorentz connection is a classical external field and therefore its variation in the action is irrelevant. Similarly, if the electromagnetic interaction comes from an external classical field, one can ignore the CS term $A \wedge A$, assuming $A$ to represent an external electromagnetic field in which the graphene sample is immersed. The fact that the CS term is intrinsically three-dimensional may seem incorrect from the point of view of the experimental setup, where the graphene sheet defines a 2+1 space embedded in a 3+1 laboratory and the electromagnetic field should therefore be described by the usual Maxwell action. However, in the presence of a weak magnetic field, the electrons in a graphene crystal behave as in the presence of a $U(1)$ field described by the CS Lagrangian like in (39) [19].

The fermionic character of the electrons in graphene is reflected in the Fermi-Dirac statistics of the creation and destruction operators for particles in each sublattice, $a, a^\dagger, b, b^\dagger$. The two components are not really related to the intrinsic angular momentum of the conduction electrons, but to the fact that the honeycomb array is not a single lattice but two interlocking sublattices.

The spin of the conduction electrons is an independent degree of freedom that requires a new label for the Dirac field. Its inclusion requires a field that can be denoted as $\psi^\alpha_a$, where $\alpha = 1, 2$ correspond to the components displayed in (40), and $a = 1, 2$ is a new spin index that represents the two possible orientations along the $z$-axis. Clearly, the orientation of the "$z$ axis" is conventional and nothing would change if one decides to modify its orientation arbitrarily at each lattice point. Thus, the inclusion of the spin degree of freedom requires introducing a local $SU(2)$ gauge freedom just as in the model described by (39)!

This idea of turning the freedom to choose locally the spin quantization axis and therefore inducing a coupling with an $SU(2)$ gauge field was examined in the context of the Hubbard model more than two decades ago [20]. The $SU(2)$-mediated interaction between conduction electrons is likely to produce a strong correlation between electrons with antiparallel spins, analogous to the formation of Cooper pairs in superconductivity.

For certain values of the parameters of these black hole solution and the $SU(2)$ charge, the geometry admits globally defined Killing spinors. These configurations are therefore candidates for stable supersymmetric vacua of the theory. It would be interesting to see if there are experimental counterparts of these special states predicted by the field theoretic model.

7. Summary

We started out by observing that Lorentz flatness does not necessarily imply flat metric geometry. In particular, in three dimensions Lorentz flatness can correspond to anti-de Sitter spaces of arbitrary radii, including infinite radius, that is, Minkowski space. As has been known for some
time now, locally AdS$_3$ spacetimes can take a number of different shapes, including black holes, static or rotating, so that Lorentz flat metrics may be quite nontrivial. This result can be seen as the Lorentzian counterpart of a classic theorem in geometry that states that a three sphere is parallelizable, i.e., it can be globally covered with a congruence of orthonormal frames. Another way of understanding a three-dimensional black hole then is as a geometry in which any simply connected patch can be covered by a congruence of “inertial observers”, so that the Lorentz connection can be consistently chosen to vanish everywhere in that patch.

Since the three-dimensional Lorentz group is isomorphic to SU(2), a black hole in 2+1 dimensions an also be endowed with a nontrivial SU(2) hair. This solution with vanishing both Lorentz and SU(2) curvatures, can be seen as a configuration in the matter-free sector of a system consisting of a Dirac field minimally coupled to a CS gauge field and CS gravity. This system can also be seen as a CS action constructed with a graded 1-form that combines bosonic and fermionic generators. The bosonic gauge fields are one-form connections, while the fermionic 1-forms are the product of a spin-1/2 zero-form and a dreibein.

Remarkably, the coupled CS SU(2) and Dirac system is exactly what one would expect of a graphene action if the spin degree of freedom of the conduction electron were also taken into account. So, in the end it may be that the vacuum sector of one of the simplest materials in nature, a two-dimensional array of identical atoms may have physical features that may resemble a 2+1 black hole with SU(2) hair.

Acknowledgements
I have benefited from many enlightening discussions with my collaborators, Pedro Alvarez, Gastón Giribet, Luis Huerta, Cristián Martínez, Pablo Pais, Eduardo Rodríguez, Patricio Salgado and Mauricio Valenzuela. Also, very fruitful discussions with Nicolás Grandi, Marcelo Loewe, Guillermo Silva and David Valenzuela, have helped me enormously to understand what I report here. This work is partially supported by Fondecyt grant 1140155. The Centro de Estudios Científicos is funded by the Chilean Government through the Centers of Excellence Base Financing Program of Conicyt.

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