GLOBAL DIFFUSION ON A TIGHT THREE-SPHERE

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Abstract. We consider an integrable Hamiltonian system weakly coupled with a pendulum-type system. On some fixed energy level, the uncoupled system is assumed to possess a normally hyperbolic invariant manifold diffeomorphic to a three-sphere, on which the Hamiltonian satisfies a strict convexity condition, and whose stable and unstable invariant manifolds coincide. The Hamiltonian flow on the three-sphere is equivalent to the Reeb flow for the induced contact form. The strict convexity condition implies that the contact structure on the three-sphere is tight. When a small, generic coupling is added to the system, the normally hyperbolic invariant manifold is preserved, and its stable and unstable manifolds split, yielding transverse intersections. We show that there exist trajectories that follow prescribed collections of frequencies of motion on the three-sphere. In this sense, the perturbed system exhibits global diffusion on the tight three-sphere.

1. Introduction

In this paper we are concerned with nearly integrable systems that can be locally described in terms of action-angle and hyperbolic variables; these are referred to as a priori unstable systems (see [10]). The diffusion problem in Hamiltonian dynamics asserts that, generically, such systems exhibit trajectories along which the action variable changes by some positive distance that is independent of the size of the perturbation (see [3]). A related phenomenon is the existence of trajectories that exhibit symbolic dynamics, i.e., they visit prescribed open sets in the action variable domain in a given order, or they follow prescribed sequence of invariant objects (KAM tori, Aubry-Mather sets) for given time intervals. The diffusion phenomenon in a priori unstable systems has been extensively studied, for instance in [10, 39, 40, 13, 11, 3].

A typical approach to the diffusion problem is to start with a region of the phase space of the unperturbed system which is foliated by Liouville tori, apply the KAM theorem to the perturbed system restricted to that region, and use KAM tori in combination with other geometric objects created by the perturbation to construct diffusing trajectories. Such trajectories lie within the region that was originally populated by Liouville tori. In this sense, this type of diffusion is local.

The global geometry of an integrable system can in general be quite complicated, with the phase space divided out by the singular leaves of the Liouville foliation into distinct regions that are populated by different families of Liouville tori. In this paper

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we formulate a question on whether, under generic conditions, there exist diffusing orbits that travel arbitrarily far across these distinct regions, and follow prescribed collections of frequencies of motion within the frequency range. We shall refer to this phenomenon as global diffusion.

We investigate the question on global diffusion for a class of a priori unstable Hamiltonian systems of three degrees of freedom. The unperturbed Hamiltonian system is a product of a two-degree of freedom integrable Hamiltonian and a one-degree of freedom pendulum. We assume that, on some fixed energy level, there exists a normally hyperbolic invariant manifold diffeomorphic to a three-sphere, on which the Hamiltonian satisfies a strict convexity condition. The sphere is divided out by the singular leaves into disjoint, open domains, each completely described by one action and two angle coordinates. We assume that the stable and unstable invariant manifolds of the three-sphere coincide. We apply a small perturbation. The normally hyperbolic invariant manifold survives if the perturbation is small enough. Assuming some non-degeneracy conditions that are generic, the stable and unstable invariant manifolds intersect transversally along homoclinic manifolds. To each homoclinic manifold, one can associate a scattering map that describes the asymptotic behavior of the homoclinic trajectories. The scattering map associated to a homoclinic manifold is defined on some open subset of the normally hyperbolic invariant manifold. Typically, there are many geometrically distinct homoclinic manifolds, with corresponding scattering maps. We assume that there exists a family of homoclinic manifolds with the property that the domains of the corresponding scattering maps sweep all possible action coordinate values on each action-angle domain on the three-sphere. Under these assumptions, we show that there exist trajectories that follow any prescribed sequence of frequencies of motion within the frequency range on the sphere. In this sense, we say that the perturbed system exhibits global diffusion relative to the three-sphere.

There are two key ingredients that are essential for us to establish this global diffusion type of result. The first ingredient is the assumption that the domains of the scattering maps cover all values of the action coordinate on each action-angle domain on the three-sphere. This condition is satisfied in very general situations. The second ingredient is the strict convexity condition of the unperturbed Hamiltonian on the three-sphere. This can be replaced by a weaker condition in terms of the contact structure induced on the three-sphere. More precisely, one can replace it by the following conditions: (i) the three-sphere is of contact type, (ii) the contact structure is tight, (iii) the Reeb flow associated to the contact structure is equivalent to the Hamiltonian flow restricted to the sphere, (iv) every periodic orbit of the Reeb flow has Conley-Zehnder index greater than or equal to three. Under these conditions, a deep result from [24] says that the Reeb flow has a disk-like global surface of section, on which the Poincaré return map is equivalent to an area preserving map. Here we can view the three-sphere, which represents an energy hypersurface of the two-degree of freedom integrable Hamiltonian, embedded in $\mathbb{R}^4$. We show that, when this Hamiltonian is coupled with the pendulum via a small perturbation, as the three-sphere survives as a normally hyperbolic invariant manifold in $\mathbb{R}^6$, there still exist a global surface of section of the flow restricted to the sphere. We use the return map to this global surface of section to capture the existence of trajectories that exhibit global diffusion. More specifically,
each action-angle domain determines a rotation set for the return map. We show that for any sequence of rotation numbers (frequencies) from the rotation set, there exists a trajectory whose motion follows every frequency for some prescribed time, and in the prescribed order.

A motivation for the question on global diffusion is furnished by the dynamics of the spatial circular restricted three body problem (SCRTBP) near one of its equilibrium points. The SCRTBP describes the spatial motion of an infinitesimal particle (e.g., a satellite) under the gravity of two heavy masses, referred to as primaries (e.g., Earth and Sun), moving on circular orbits about their common center of mass. (It is assumed that the infinitesimal particle exerts no influence on the motion of the primaries). The motion of the infinitesimal particle relative to a co-rotating system of coordinates can be described by a Hamiltonian system of three-degrees of freedom. One of the equilibrium points is located between the primaries, and we denote it $L_1$. The linear stability of $L_1$ is of (center $\times$ center $\times$ saddle)-type, hence, by the Lyapunov Center Theorem, there exists a normally hyperbolic invariant manifold diffeomorphic to a four-sphere near $L_1$. There is a natural action-angle coordinate system on the four-sphere induced by the linearized dynamics near $L_1$. Fixing an energy level above, but sufficiently close to that of $L_1$, the Hamiltonian flow restricted to that energy level possesses a normally hyperbolic invariant manifold diffeomorphic to a three-sphere. This sphere can be parametrized by one action and two angle coordinates, with the action variable corresponding to the out-of-plane amplitude of the motion. Provided that the mass ratio of the primaries is sufficiently small, and the energy level is sufficiently close to that of $L_1$, one can use analytical results combined with rigorous computer experiments to show that there exist trajectories that visit closely any prescribed sequence of action level sets on the three-sphere. Each of the action level sets is characterized by some frequency of motion. This argument is carried in [38], and will be further detailed in an upcoming work. The same mechanism of global diffusion can be established via rigorous numerics for arbitrary mass ratios and energy levels; see [11, 12]. The practical application of this global diffusion mechanism is that it yields a zero cost procedure to design a satellite trajectory that changes its out-of-plane amplitude, relative to the ecliptic, from nearly zero to the maximum possible that can be achieved for the given energy level.

Similarities and differences between the Hamiltonian model considered in this paper and the three-body problem described above are discussed in Remark 3.1.

Besides proposing the study of global diffusion, in this paper we hope to further explore the integration of symplectic dynamics ideas into problems in Hamiltonian instability, celestial mechanics, and dynamical systems in general. See, e.g., [1, 7, 25].

2. SET-UP AND MAIN RESULT

We start by describing certain structures that are needed to formulate the main result of this paper. Then we describe a class of nearly integrable Hamiltonian systems, and we formulate a global diffusion-type of result.
2.1. Preliminaries.

2.1.1. Strictly convex energy hypersurfaces for two-degrees of freedom Hamiltonian systems. Let $H : \mathbb{R}^4 \to \mathbb{R}$ be a smooth Hamiltonian, where $\mathbb{R}^4$ is endowed with the canonical symplectic form $\omega = \sum_{i=1}^{2} dq_i \wedge dp_i$. Let $\lambda = \frac{1}{2} \sum_{i=1}^{2} (q_i dp_i - p_i dq_i)$ be the Liouville form.

Suppose that $c$ is a regular value for $H$ and let $\Lambda = \{(p_1, p_2, q_1, q_2) \mid H(p_1, p_2, q_1, q_2) = c\}$ be the corresponding energy hypersurface of $H$. We say that $\Lambda$ is star-shaped relative to the origin if

(i) for every $x \in \Lambda$, the ray $t(x - x_0)$ intersects $\Lambda$ only once, at $x$, and
(ii) $x \notin T_x \Lambda$.

Then $\Lambda$ is diffeomorphic to the 3-dimensional unit sphere $S^3$ in $\mathbb{R}^4$, and there exists a positive smooth function $g : S^3 \to \mathbb{R}^+$, such that $\Lambda = \{z \sqrt{g(z)} \mid z \in S^3\}$ (see [16]).

The restriction of $\lambda$ to $\Lambda$ is a contact form. The contact form $\lambda$ determines a contact structure, which is the plane bundle on $\Lambda$ given by $\xi = \ker(\lambda)$.

Contact forms can be overtwisted, if there exists an overtwisted disk, i.e., an embedded 2-dimensional open disk $D$ with $T \partial D \subseteq \xi$ and $T_p D \neq \xi_p$ for all $p \in \partial D$, and tight, otherwise. Since $\Lambda$ is star-shaped, the contact form $\lambda|_{\Lambda}$ is equivalent to the contact form $g\lambda|_{S^3}$. This contact form is tight.

The Reeb vector field $X_\lambda$ associated to $\lambda$ is defined by $i_{X_\lambda} (\lambda) = 1$ and $i_{X_\lambda} (d\lambda) = 0$, and the flow generated by the Reeb vector field is called the Reeb flow. A sufficient condition for the Reeb flow to be equivalent to the Hamiltonian flow restricted to the energy hypersurface is that the Liouville vector field $\eta$, defined by $L_\eta \omega = \omega$, is transverse to $\Lambda$ (here $L$ denotes the Lie derivative).

The Liouville vector field is transverse to the star-shaped energy hypersurface, hence the Hamiltonian flow restricted to $\Lambda$ is equivalent (up to the reparametrization of orbits) to the Reeb flow on $\Lambda$ defined by the tight contact form $g\lambda|_{S^3}$.

An energy hypersurface $\Lambda$ is said to bound a strictly convex domain if there exists $\delta > 0$ such that $D^2 H_{00} - \delta \cdot \text{id}$ is positive definite at all points on $\Lambda$. If an energy hypersurface is strictly convex, then it is star-shaped.

An energy hypersurface $\Lambda$ is said to be dynamically convex if every periodic solution $x$ of period $T$ of the Reeb flow on $\Lambda$ has Conley-Zehnder index $\tilde{\mu}(x, T) \geq 3$. Dynamical convexity is a symplectic invariant. A strictly convex energy hypersurface is always dynamically convex.

See Appendix for further details.

2.1.2. Integrable two-degrees of freedom Hamiltonian systems. Consider a Hamiltonian system $H : \mathbb{R}^4 \to \mathbb{R}$ and an energy hypersurface $\Lambda$ as above.

The Hamiltonian system $H$ is said to be Liouville integrable if there exists a smooth function $K : \mathbb{R}^4 \to \mathbb{R}$ such that $H$ and $K$ are functionally independent a.e. and in involution, i.e., $\{H, K\} = 0$, where $\{\cdot, \cdot\}$ denotes the Poisson bracket. The function $K$ is said to be a first integral of the Hamiltonian system. The mapping $H : \mathbb{R}^4 \to \mathbb{R}^2$, $H = (H, K)$ is called the momentum mapping.

The momentum mapping determines a foliation of the phase space, called the Liouville foliation, given by the connected components of $\mathcal{L} = \{H^{-1}(\bar{a}) \mid \bar{a} \in \mathbb{R}^2\}$. By the
Liouville-Arnold Theorem [2], each regular leaf of $\mathcal{L}$ is a two-torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ that has a neighborhood $U$ diffeomorphic to $T^2 \times D^2$, where $D^2$ is an open disk in $\mathbb{R}^2$, and so that there exists a canonical system of action-angle coordinates $(I_1, I_2, \phi_1, \phi_2)$ on $U$ with $\sum_{i=1}^2 dq_i \wedge dp_i = \sum_{i=1}^2 d\phi_i \wedge dI_i$, $(I_1, I_2) \in \mathbb{R}^2$, $(\phi_1, \phi_2) \in T^2$, such that the Hamiltonian $H$ on $U$ depends only on $I_1, I_2$. The domain $U$ is called an action-angle domain.

We say that $K$ is a Bott function if the set of critical points of $K$ on $\Lambda$ is a disjoint union of smooth submanifolds, each of which being non-degenerate in the following sense: $D^2K$ is non-degenerate on the transversals to the submanifold (at each point). The class of Bott systems was introduced in [17] in the study of the topology of integrable systems (see also [6]). It is known that the set of Bott systems on a given energy level is a set of first category in the weak $C^r$ metric, $r > 2$.

In general, a connected critical submanifold of a Bott function on a compact, regular level set of a Hamiltonian with two degrees of freedom is diffeomorphic either to a circle, or to a torus, or to the Klein bottle. It is known that, by an arbitrary small perturbation, one can turn an integrable Hamiltonian system with a Bott integral into an integrable Hamiltonian system whose only critical submanifolds are circles. See [28, 29]. If these circles are either elliptic or hyperbolic orbits for the Hamiltonian flow, the system is said to be coherent.

In [31] it is shown that the each connected component $U$ of the complement of the set of the singular leaves of the Liouville foliation is a maximal action-angle domain.

2.1.3. Iso-energetic KAM theorem for two-degrees of freedom Hamiltonian systems.

Consider an integrable Hamiltonian $H : \mathbb{R}^4 \to \mathbb{R}$ and an action-angle domain $T^2 \times D^2$ as before. Define the frequency mapping $I = (I_1, I_2) \in D^2 \mapsto W(I) = [\omega_1(I) : \omega_2(I)] \in P^1(\mathbb{R})$, where $\omega_1 = \partial H/\partial I_1$, $\omega_2 = \partial H/\partial I_2$. For each $I \in D^2$, $W(I)$ represents the frequency ratio of the torus determined by $I$. We denote $\omega(I) = (\omega_1(I), \omega_2(I))$.

The KAM theorem states that, assuming that $H$ is smooth enough and that $\det(\partial \omega/\partial I) \neq 0$ for all $I \in D^2$, if a small Hamiltonian perturbation is added to the system, then, for the perturbed Hamiltonian $H_\varepsilon = H + \varepsilon H'$, most of the tori $I = \text{const.}$ from the unperturbed case survive, slightly deformed, to the perturbed case. The non-degeneracy condition is equivalent to the fact that the frequencies $\omega_1(I), \omega_2(I)$ are functionally independent at each $I \in D^2$.

The motion on each perturbed torus is still quasi-periodic with the same collection of frequencies $\omega_1, \omega_2$ as those on the corresponding unperturbed torus. The perturbed tori generically form a nowhere dense set, and the Lebesgue measure of the complement to this set is $O(\sqrt{\varepsilon})$. More precisely, assuming that $H$ and $H'$ are $C^r$ with $r > 4$, the tori that survive the perturbation are those satisfying the Diophantine condition

$$|k_1 \omega_1(I) + k_2 \omega_2(I)| > C(|k_1| + |k_2|)^r,$$

where $1 < \tau < \frac{1}{2}(r - 2)$ and $C > 0$. Note that the KAM tori lie on different energy levels.

The iso-energetic KAM theorem is concerned with the persistence of tori on the same energy level. For this, one replaces the above non-degeneracy condition with the
Arnold non-degeneracy condition
\[ \det \begin{pmatrix} \partial \omega / \partial I & \omega \\ \omega^T & 0 \end{pmatrix} \neq 0, \]
for all \( I \in D^2 \), where \( (\cdot)^T \) denotes the transpose of a matrix. This condition means that \( \partial W / \partial I \neq 0 \), when the derivative is taken at constant energy, hence the frequency ratio map \( I \mapsto W(I) \) from the energy hypersurface to the projective line \( P^1(\mathbb{R}) \) has maximal rank. The iso-energetic non-degeneracy condition implies that the rotation number changes between neighboring tori. This condition can also be reformulated in terms of transversality, see [15].

2.1.4. Aubry-Mather theory. Assume that \( H \) satisfies the iso-energetic non-degeneracy condition on a domain \( T^2 \times D^2 \) as above. Assume that \( \Sigma \) is a local surface of section for the Hamiltonian flow on \( \Lambda \). For some fixed \( \varepsilon \), let \( T_1 \) and \( T_2 \) be two two-dimensional tori in \( \Lambda \) that are invariant under the flow. Assume that the region in \( \Lambda \) between \( T_1 \) and \( T_2 \) intersects \( \Sigma \) in an annulus \( A \simeq T^1 \times [0,1] \), bounded by two circles \( \hat{T}_1 \) and \( \hat{T}_2 \), and that the Poincaré first return map \( f \) to \( \Sigma \) is well defined on \( A \).

Let \( \mathbb{R} \times [0,1] \) be the universal cover, let \( \pi_x \) be the projection onto the first component, and \( \pi_y \) be the projection onto the second component. We fix a lift of \( f \) to \( \mathbb{R} \times [0,1] \). With an abuse of notation, we also denote the universal cover by \( A \) and the lift by \( f \).

The iso-energetic non-degeneracy condition implies that \( f \) satisfies a monotone twist-condition on \( A \), that is, \( |\partial (\pi_x \circ f) / \partial y| > 0 \) at all points in the annulus \( A \). Let us assume that \( f \) is a positive twist, meaning that \( \partial (\pi_x \circ f) / \partial y > 0 \) at all points. The map \( f \) restricted to the boundary components \( \hat{T}_1, \hat{T}_2 \) of the annulus has well defined rotation numbers \( \rho_1, \rho_2 \), respectively, with \( \rho_1 < \rho_2 \). Here we recall that for a point \( z \in A \) the rotation number is defined by
\[ \rho(z) = \lim_{n \to \infty} \frac{\pi_x(f^n(z))}{n}, \]
provided that the limit exists.

By an essential invariant circle in \( A \) we mean a circle invariant under \( f \) that cannot be homotopically deformed into a point inside the annulus. A region in \( A \) between two essential invariant circles \( \hat{T}_1 \) and \( \hat{T}_2 \) is called a Birkhoff Zone of Instability (BZI) provided that there is no essential invariant circle in the interior of the region.

A invariant subset \( M \subseteq A \) is said to be monotone (cyclically ordered) if \( \pi_x(z_1) < \pi_x(z_2) \) implies \( \pi_x(f(z_1)) < \pi_x(f(z_2)) \) for all \( z_1, z_2 \in M \). For \( z \in A \) the extended orbit of \( z \) is the set \( EO(z) = \{ f^n(z) + (j,0) : n, j \in \mathbb{Z} \} \). The orbit of \( z \) is said to be monotone if the set \( EO(z) \) is monotone. If the orbit of \( z \in A \) is monotone, then its rotation number \( \rho(z) \) is well defined. All points in the same monotone set have the same rotation number.

An Aubry-Mather set is a minimal, monotone, \( f \)-invariant subset of given rotation number \( \rho \). Here by a minimal set we mean a closed invariant set that does not contain any proper closed invariant subsets. (Equivalently, the orbit of every point in the set is dense in the set.) This should not be confused with action-minimizing sets.
For every $\rho \in [\rho_1, \rho_2]$, there exists a non-empty Aubry-Mather set of rotation number $\rho$. Aubry-Mather sets defined as such can be obtained as limits of monotone Birkhoff periodic orbits [30].

In [34], Mather has proved that given a bi-infinite sequence of action-minimizing Aubry-Mather sets inside a BZI, there exists a trajectory that visits arbitrarily close each of the Aubry-Mather set in the sequence, in the prescribed order.

There is a topological version of this result, due to Hall [22], providing shadowing orbits of Aubry-Mather sets that are not necessarily action minimizing. We say that an orbit $(f^t(\zeta)) \in \mathbb{Z}$ of $f$ moves with the frequency $\rho$ for $m$ iterations if there exists a monotone orbit $(f^i(z)) \in \mathbb{Z}$, with $\rho(z) = \rho$, and pair of points $z_l, z_r \in EO(z)$, and some $t_0$ such that

$$\pi_x(f^t(z_l)) \leq \pi_x(f^t(\zeta)) \leq \pi_x(f^t(z_r))$$

for all $t \in \{t_0, t_0 + 1, \ldots, t_0 + m - 1\}$. Hall’s version of Mather’s shadowing theorem states that, given a bi-infinite sequence of Aubry-Mather sets inside a BZI, there exists a trajectory that moves, for some prescribed length of time, with the frequencies of one the Aubry-Mather sets in the sequence, then goes on and moves, for some time, with the next prescribed frequency, and so on.

2.1.5. Linearization of normally hyperbolic flows. Let $M$ be a $C^r$-smooth, $m$-dimensional manifold (without boundary), with $r \geq 1$, and $\phi : M \times \mathbb{R} \to M$ a $C^r$-smooth flow on $M$. A submanifold (possibly with boundary) $\Lambda$ of $M$ is said to be a normally hyperbolic invariant manifold for $\phi$ if $\Lambda$ is invariant under $\phi_t$, there exists a splitting of the tangent bundle of $TM$ into sub-bundles

$$TM = E^u \oplus E^s \oplus T\Lambda,$$

that are invariant under $D\phi^t$ for all $t \in \mathbb{R}$, and there exist a constant $C > 0$ and rates $0 < \beta < \alpha$, such that for all $x \in \Lambda$ we have

$$||D\phi^t(x)(v)|| \leq Ce^{-\alpha t}\|v\| \text{ for all } t \geq 0, \text{ if and only if } v \in E^s_x;$$

$$||D\phi^t(x)(v)|| \leq Ce^{\alpha t}\|v\| \text{ for all } t \leq 0, \text{ if and only if } v \in E^u_x;$$

$$||D\phi^t(x)(v)|| \leq Ce^{\beta|t|}\|v\| \text{ for all } t \in \mathbb{R}, \text{ if and only if } v \in T_x\Lambda.$$

In [23] it is proved that in some neighborhood of $\Lambda$ the flow $\phi^t$ is conjugate with its linearization. That is, there exists a neighborhood $\mathcal{U}$ of $\Lambda$ and a homeomorphism $h$ from $\mathcal{U}$ to some neighborhood $\mathcal{V}$ of the zero section of the normal bundle to $\Lambda$ such that

$$D\phi^t \circ h = h \circ \phi^t.$$

The homeomorphism $h$ defines a system of coordinates $(x_c, x_s, x_u) \in \Lambda \oplus E^s \oplus E^u$. The flow written in these coordinates takes the form

$$h(\phi^t(x)) = (\phi^t(x_c), D\phi^t(x_s), D\phi^t(x_u)),$$

for $x \in \mathcal{U}$ and $h(x) = (x_c, x_s, x_u) \in \mathcal{V}$.
2.2. Main result. We consider a $C^r$-differentiable Hamiltonian system, with $r$ sufficiently large, given by a smooth function $H_\varepsilon : \mathbb{R}^6 \to \mathbb{R}$ of the form
\begin{equation}
H_\varepsilon(p, q) = H_0(p, q) + \varepsilon H_1(p, q),
\end{equation}
where $p = (p_1, p_2, p_3)$, $q = (q_1, q_2, q_3)$, and the unperturbed Hamiltonian $H_0$ is of the form
\begin{equation}
H_0(p, q) = H_{00}(p_1, q_1, p_2, q_2) + H_{01}(p_3, q_3).
\end{equation}
On $\mathbb{R}^6$ we consider the canonical symplectic form $\omega = \sum_{i=1}^3 dq_i \wedge dp_i$.

Suppose that $c$ is a regular value for $H_{00}$ and that
$\Lambda_0 = \{(p_1, q_1, p_2, q_2) \mid H_{00}(p_1, q_1, p_2, q_2) = c\}$
is the corresponding energy hypersurface of $H_{00}$.

We now make some assumptions on the Hamiltonian $H_\varepsilon$ which will be used in formulating the main result.

(A1) The Hamiltonian $H_{00}$ satisfies the following properties:

(i) The Hamiltonian $H_{00}$ is Liouville integrable: there exists a first integral $K$ such that \{H_{00}, K\} are functionally independent almost everywhere and in involution. We assume that $K$ is a Bott function. Moreover, we assume that the critical submanifolds of $K$ on $\Lambda_0$ are only circles of either elliptic or hyperbolic type.

(ii) On each maximal action-angle domain $U$, with action-angle coordinates $(I_1, I_2, \phi_1, \phi_2)$, we assume that the Hamiltonian $H_{00}$ satisfies the isoenergetic non-degeneracy condition
\begin{equation}
\det \begin{pmatrix}
\frac{\partial^2 H_{00}}{\partial I_i \partial I_j} & \frac{\partial H_{00}}{\partial I_i} \\
\frac{\partial H_{00}}{\partial I_i} & 0
\end{pmatrix} \neq 0.
\end{equation}

(iii) For some $c > 0$, the energy hypersurface
$\Lambda_0 = \{(p_1, q_1, p_2, q_2) \mid H_{00}(p_1, q_1, p_2, q_2) = c\}$
bounds a strictly convex domain.

(A2) The Hamiltonian $H_{01}$ satisfies the following properties:

(i) The Hamiltonian $H_{01}$ has a non-degenerate saddle point at $(0, 0)$, and the set
$\gamma = \{(p_3, q_3) \mid H_{01}(p_3, q_3) = H_{01}(0, 0)\}$
is the union of two smooth curves $\gamma^-, \gamma^+$ (separatrices) that meet only at $(0, 0)$, and contain no other critical point of $H_{01}$. Each $\gamma^\pm$ represents the stable and unstable manifold of $(0, 0)$; the two manifolds coincide.

(ii) The absolute values of the Lyapunov exponents of the flow of $H_{00}$ on $\Lambda_0$ are less than $\mu$, where $\pm\mu$ denote the Lyapunov exponents of the saddle point $(0, 0)$ for the flow of $H_{01}$.

Now we discuss the above assumptions.

The assumption (A1-i) says that the Hamiltonian system $H_{00}$ is Bott integrable. Hence, $\Lambda_0$ is the union of the critical leaves of the Liouville foliation, which, by assumption, are only circles of elliptic or hyperbolic type, and maximal action-angle
domains. Let $\Omega^j_0$, $j = 1, \ldots, k$, be the components in $\Lambda_0$ of the maximal action-angle domains. These are mutually disjoint open sets, and the complement of $\bigcup_{j=1,\ldots,k} \Omega^j_0$ is a nowhere dense set in $\Lambda_0$. Each domain $\Omega^j_0$ can be described by a system $(I_1, \phi_1, \phi_2)$ of one action and two angle coordinates.

Assumption (A1-ii) says that $H_{00}$ satisfies the iso-energetic non-degeneracy condition on each maximal action-angle domain.

Assumption (A1-iii) implies that the energy hypersurface $\Lambda_0$ is star-shaped, so that the Hamiltonian flow on it is equivalent to the Reeb flow on a tight three-sphere.

Assumption (A2) implies that $\Lambda_0$ is a normally hyperbolic invariant manifold for the Hamiltonian flow, and $W^u(\Lambda_0) = W^s(\Lambda_0)$. The theory of normal hyperbolicity implies that, if the perturbation $\epsilon$ is sufficiently small, then $\Lambda_0$ is survived by a normally hyperbolic invariant manifold $\Lambda_\epsilon$, for which there exist $W^u(\Lambda_\epsilon)$ and $W^s(\Lambda_\epsilon)$ (see [23]). From [13], there exists a smooth parametrization $k_\epsilon : \Lambda_0 \to \Lambda_\epsilon$ of $\Lambda_\epsilon$ that depends smoothly on $\epsilon$, and with $k_0 = \text{id}_{\Lambda_0}$. Let $\Omega^j_\epsilon = k_\epsilon(\Omega^j_0)$, $j = 1, \ldots, k$. We note that the boundaries of the domains $\Omega^j_\epsilon$, $j = 1, \ldots, k$, are no longer invariant sets, as they may get destroyed by the perturbation. Using the parametrization $k_\epsilon$ we can define a system of one action and two angle coordinates on each domain $\Omega^j_\epsilon$, $j = 1, \ldots, k$, which, with an abuse of notation, we still denote by $(I_1, \phi_1, \phi_2)$.

The iso-energetic KAM theorem implies that in $\Lambda_\epsilon$ there exist a Cantor family of 2-dimensional tori $\{\mathcal{T}_\alpha\}_{\alpha \in \Delta}$ that survive the perturbation, with the Lebesgue measure of the KAM tori approaching full measure when the size of the perturbation approaches 0.

For a perturbation $H_1$ of a ‘generic type’, and for all $\epsilon$ sufficiently small, $W^u(\Lambda_\epsilon)$ and $W^s(\Lambda_\epsilon)$ intersect transversally. More precisely, we will show that there exists an open and dense set of perturbations $H_1$ for which there exists a collection of homoclinic channels $\Gamma^i_\epsilon$, $i = 1, \ldots, n$, and corresponding scattering maps $S^i_\epsilon : U^i_\epsilon \to V^i_\epsilon$ associated to $\Gamma^i_\epsilon$, with $U^i_\epsilon \subseteq \Lambda_\epsilon$, such that, for every $j \in \{1, \ldots, k\}$, each level set of $I_1$ in $\Omega^j_\epsilon$ intersects some $U^i_\epsilon$. The notion of a scattering map is recalled in Appendix 5.3.

In order to formulate the main result of this paper, we need to discuss the existence of a disk-like global surfaces of section for the flow of $H_\epsilon$ restricted to $\Lambda_\epsilon$. See Appendix 5.3.

**Proposition 2.1.** There exists $\epsilon_0 > 0$ such that for $\epsilon \in [0, \epsilon_0)$ there exists a family of disk-like global surfaces of sections $\mathcal{D}_\epsilon$, smoothly depending on $\epsilon$, for the Hamiltonian flow of $H_\epsilon$ restricted to $\Lambda_\epsilon$. The return map $f_\epsilon$ to $\mathcal{D}_\epsilon$ is conjugate to an area preserving map.

The proof of this proposition is given in Section 6.

Each domain $\Omega^j_\epsilon$, $j = 1, \ldots, k$, determines an open domain $\hat{\Omega}^j_\epsilon = \Omega^j_\epsilon \cap \mathcal{D}_\epsilon$ in $\mathcal{D}_\epsilon$. Each $\hat{\Omega}^j_\epsilon$ is diffeomorphic to an annulus (punctured disk) and can be described in terms of a system of action-angle coordinates $(J, \theta)$ depending on $j$. We denote the rotation set $\{\rho(z) : z \in \hat{\Omega}^j_\epsilon\}$ of $f_\epsilon$ on $\hat{\Omega}^j_\epsilon$ by $\mathcal{R}^j_\epsilon$. We also denote $\mathcal{R}_\epsilon = \bigcup_{j=1}^k \mathcal{R}^j_\epsilon$.

Let $\mathcal{U}_\epsilon$ be a neighborhood of $\Lambda_\epsilon$ in the energy hypersurface where the Hamiltonian flow can be linearized. That is, $\phi^t_\epsilon$ is conjugate with the linearized flow $D\phi^t_0$ on $\mathcal{U}_\epsilon$, in the sense that there exists a homeomorphism $h_\epsilon$ from $\mathcal{U}_\epsilon$ to a neighborhood $\mathcal{V}_\epsilon$ of the
zero section of the normal bundle to $\Lambda_{\varepsilon}$ to such that

$$D\phi^t_{\varepsilon} \circ h_{\varepsilon} = h_{\varepsilon} \circ \phi^t_{\varepsilon}.$$ 

Thus, in the $(x_c, x_s, x_u)$-coordinates given by $h_{\varepsilon}$, the flow $\phi^t_{\varepsilon}$ is given by

$$\phi^t_{\varepsilon}(x) = (\phi^t_{\varepsilon}(x_c), D\phi^t_{\varepsilon}(x_s), D\phi^t_{\varepsilon}(x_u)),$$

where $h_{\varepsilon}(x) = (x_c, x_s, x_u)$, $x_c \in \Lambda_{\varepsilon}$, $x_s \in B^g_\delta(0) \subseteq E^c_{\varepsilon}$, $x_u \in B^g_\delta(0) \subseteq E^u_{\varepsilon}$, for some $\delta > 0$. For as long as a trajectory $\phi^t_{\varepsilon}(x)$ stays in the neighborhood $U_{\varepsilon}$ of $\Lambda_{\varepsilon}$, there exists a `matching’ trajectory $\phi^t_{\varepsilon}(x_c)$ on $\Lambda_{\varepsilon}$ defined by the above conjugacy. We can always construct trajectories of the Hamiltonian flow $\phi^t_{\varepsilon}(x)$ of $H_\varepsilon$ that spend any prescribed length of time in $U_{\varepsilon}$. Hence we can consider the intersections of the matching trajectory $\phi^t_{\varepsilon}(x_c)$ with the surface of section $D_{\varepsilon}$, and characterize the dynamics via the Poincaré map $f_\varepsilon$.

Given a frequency $\rho \in R_{\varepsilon}$ and a time interval $[t', t'']$ with $t'' - t' > 0$, we say that an orbit $\phi^t_{\varepsilon}(x)$ of $H_{\varepsilon}$ moves with the frequency $\rho$ for the time interval $[t', t'']$, if $\phi^t_{\varepsilon}(x) \in U_{\varepsilon}$ for all $t \in [t', t'']$, and the matching trajectory $\phi^t_{\varepsilon}(x_c)$ in $\Lambda_{\varepsilon}$ determines an orbit $(f^t_{\varepsilon}(\zeta))_{n=1, \ldots, m}$ of $f_\varepsilon$, for some $\zeta \in D_{\varepsilon}$, which moves with the frequency $\rho$ for $m$ iterations, where $m$ is determined by the interval $[t', t'']$, by the surface of section $D_{\varepsilon}$, and by the point $\zeta$.

**Theorem 2.2.** Assume that $H_{\varepsilon}$ satisfies the assumptions (A1)-(A2) from above. Then, there exist an open and dense set of smooth perturbations $H_1$ and $\varepsilon_2 > 0$ sufficiently small, such that, for every $\varepsilon \in (0, \varepsilon_2)$, every sequence of frequencies $(\rho_s)_{s \in \mathbb{Z}}$ in $R_{\varepsilon}$, and every sequence of time lengths $(T_s)_{s \in \mathbb{Z}}$ with $T_s > 0$ for all $s$, there exists a sequence of time intervals $[t'_s, t''_s]_{s \in \mathbb{Z}}$, with $t''_s - t'_s = T_s$ and $t''_s < t'_{s+1}$ for all $s$, and a trajectory $\phi^t_{\varepsilon}(x)$ of the Hamiltonian flow, such that the trajectory $\phi^t_{\varepsilon}(x)$ moves with the frequency $\rho_s$ for the time interval $[t'_s, t''_s]$ for each $s \in \mathbb{Z}$.

The proof of Theorem 2.2 is given in Section 3. The main idea of the proof is to consider a collection of Aubry-Mather sets of rotation numbers $\rho_s \in R_{\varepsilon}$, $s \in \mathbb{Z}$, each of them lying in some annular region $D^j_{\varepsilon}$, $j = 1, \ldots, k$. These sets can be invariant tori or Cantor type sets. We discretize the dynamics. Then we use the inner dynamics, given by $f_{\varepsilon}$ on $D_{\varepsilon}$, and the outer dynamics, given by the scattering maps, as in [20], to construct trajectories that follow these Aubry-Mather sets for prescribed lengths of times.

**Remark 2.3.** The assumption in (A1)-(i) that the critical submanifolds of $K$ on $\Lambda_0$ are only circles does not seem to be essential; we make it mainly to simplify the geometric picture. If a critical submanifold is a torus, than it is shown in [31] that there is a neighborhood of the torus which is an action-angle domain. If a critical submanifold is a Klein bottle, then there is neighborhood of the Klein bottle where the dynamics can be reduced, via a two-sheet covering, to the dynamics near a critical torus.

**Remark 2.4.** The global diffusion problem may be formulated in some other ways as well, for example, we can show that given any sequence of invariant tori, chosen from different action-angle domains, there exists a trajectory of the Hamiltonian flow that visits these tori in the prescribed order. The argument for Theorem 2.2 can be
easily adapted to prove this formulation of the global diffusion problem. For the latter formulation, we may not need to use a global surface of section for the flow restricted to $\Lambda_{\varepsilon}$. Hence, the strict convexity condition on the energy hypersurface may not be needed in that case. However, the statement of Theorem 2.2 is stronger, as one captures the existence of orbits that explore a larger portion of the energy hypersurface. Some of the prescribed frequencies may correspond to invariant tori while some others to Cantori. The existence of a global surface of section for the flow restricted to $\Lambda_{\varepsilon}$, which is a necessary ingredient in the current approach, allows us to appeal to the Aubry-Mather theory for twist maps of the annulus.

**Remark 2.5.** For most of the argument, the differentiability class $C^r$ of the Hamiltonian can be chosen with $r$ fairly low, e.g. $r \geq 5$. The existence of a global surface of sections for Hamiltonian flows on three-dimensional strictly convex energy surfaces is proved in [24] for $C^\infty$-differentiable systems. In a private conversation, H. Hofer communicated to us that the result remains valid for $C^r$-differentiable systems, with $r$ sufficiently large.

### 3. Example

We consider a Hamiltonian system consisting of the coupling of two non-harmonic oscillators and one pendulum subject to a small perturbation. This is described by the following Hamiltonian

$$H_\varepsilon(p, q) = H_0(p, q) + \varepsilon H_1(p, q),$$

where

$$H_0(p, q) = a_1 p_1^2 + a_2 p_2^2 + b_1 \left(\frac{p_1^2 + q_1^2}{2}\right)^2 + b_2 \left(\frac{p_2^2 + q_2^2}{2}\right)^2 + \left[\frac{1}{2} p_3^2 + \lambda^2 (\cos q_3 - 1)\right].$$

The corresponding Hamilton equations are:

$$\dot{q}_1 = a_1 p_1 + b_1 p_1 \frac{p_1^2 + q_1^2}{2} + \varepsilon \frac{\partial H_1}{\partial p_1},$$

$$\dot{p}_1 = -a_1 q_1 - b_1 q_1 \frac{p_1^2 + q_1^2}{2} - \varepsilon \frac{\partial H_1}{\partial q_1},$$

$$\dot{q}_2 = a_2 p_2 + b_2 p_2 \frac{p_2^2 + q_2^2}{2} + \varepsilon \frac{\partial H_1}{\partial p_2},$$

$$\dot{p}_2 = -a_2 q_2 - b_2 q_2 \frac{p_2^2 + q_2^2}{2} - \varepsilon \frac{\partial H_1}{\partial q_2},$$

$$\dot{q}_3 = p_3 + \varepsilon \frac{\partial H_1}{\partial p_3},$$

$$\dot{p}_3 = \lambda^2 \sin q_3 - \varepsilon \frac{\partial H_1}{\partial q_3},$$

where $a_1, a_2, b_1, b_2 > 0$, $a_1 \neq a_2$, $b_1 \neq b_2$. 

First, we let \( \varepsilon = 0 \). The point \((p, q) = 0_{\mathbb{R}^6}\) is an equilibrium point of the Hamiltonian flow, of linearized type center-saddle. The eigenvalues of the linearized system at this point are \( \pm a_1, \pm a_2, \pm \lambda \). Assume that \( \lambda > 0 \).

The Hamiltonian

\[
H_{00}(p_1, p_2, q_1, q_2) = a_1 \frac{p_1^2}{2} + q_1^2 + a_2 \frac{p_2^2 + q_2^2}{2} + \frac{b_1}{2} \left( \frac{p_1^2 + q_1^2}{2} \right)^2 + \frac{b_2}{2} \left( \frac{p_2^2 + q_2^2}{2} \right)^2
\]

is integrable, with two first integrals \( K_1(p, q) = a_1 \frac{p_1^2 + q_1^2}{2} + \frac{b_1}{2} \left( \frac{p_1^2 + q_1^2}{2} \right)^2 \) and \( K_2(p, q) = a_2 \frac{p_2^2 + q_2^2}{2} + \frac{b_2}{2} \left( \frac{p_2^2 + q_2^2}{2} \right)^2 \). Of course \( H_{00} = K_1 + K_2 \). The value \( c = 0 \) is the only critical value of \( H_{00} \). We note that the Hamiltonian \( H_0 \) is also integrable.

The set \( \Lambda_0 = \{(p, q) \in H_0^{-1}(c) \mid p_3 = q_3 = 0\} \) is invariant under the Hamiltonian flow of \( H_0 \), and \( \Lambda_0 \approx S^3 \). If \( c = 0 \) then the invariant set coincides with \( \mathbb{R}^6 \). If \( c > 0 \), then \( \Lambda_0 \) is diffeomorphic to \( S^3 \).

We fix \( c > 0 \) sufficiently small. The critical submanifolds of \( \Lambda_0 \) are the circles

\[
p_1^2 + q_1^2 = \frac{-a_1 + \sqrt{a_1^2 + 2a_2 c}}{b_1}, \quad p_2 = q_2 = 0, \quad \text{and} \quad p_1^2 + q_1^2 = \frac{-a_2 + \sqrt{a_2^2 + 2b_2 c}}{b_2}, \quad p_1 = q_1 = 0.
\]

Both circles represent closed orbits of elliptic type. Note that these two circles form a Hopf link.

We make a canonical change of coordinates to action-angle coordinates

\[
I_1 = \frac{1}{2}(q_1^2 + p_1^2), \quad \phi_1 = \tan^{-1}(p_1/q_1),
\]

\[
I_2 = \frac{1}{2}(q_2^2 + p_2^2), \quad \phi_2 = \tan^{-1}(p_2/q_2),
\]

so that \( \sum_{i=1,2} dq_i \wedge dp_i = \sum_{i=1,2} dI_i \wedge d\phi_i \).

The Hamiltonian \( H_{00} \) in action-angle coordinates is

\[
H_{00}(I_1, \phi_1, I_2, \phi_2) = a_1 I_1 + a_2 I_2 + \frac{b_1}{2} I_1^2 + \frac{b_2}{2} I_2^2.
\]

The equations of motion on \( \Lambda_0 \) in action-angle coordinates are

\[
\dot{I}_1 = 0,
\]

\[
\dot{\phi}_1 = a_1 + b_1 I_1,
\]

\[
\dot{I}_2 = 0,
\]

\[
\dot{\phi}_2 = a_2 + b_2 I_2.
\]

We have

\[
\det \begin{pmatrix}
\frac{\partial^2 H_{00}}{\partial^2 I_1} & \frac{\partial H_{00}}{\partial I_1} \\
\frac{\partial H_{00}}{\partial I_2} & 0
\end{pmatrix}_T = -b_2(a_1 + b_1 I_1)^2 - b_1(a_2 + b_2 I_2)^2.
\]

Since \( a_1, a_2, b_1, b_2 > 0 \), this never vanishes, so the iso-energetic non-degeneracy condition is satisfied.

Now we show that the sphere \( \Lambda_0 \) bounds a strictly convex domain. We compute

\[
D^2 H_{00} = \begin{pmatrix}
D_1 & 0 \\
0 & D_2
\end{pmatrix},
\]
where
\[ D_i = \begin{pmatrix} a_i + \frac{b_i}{2}(3p_i^2 + q_i^2) & b_ip_iq_i \\ b_ip_iq_i & a_i + \frac{b_i}{2}(p_i^2 + 3q_i^2) \end{pmatrix}, \]
with \( i = 1, 2. \)

We have
\[
\langle D^2 H_{00}(u_1, v_1, u_2, v_2), (u_1, v_1, u_2, v_2) \rangle = \\
= \sum_{i=1,2} a_i(u_i^2 + v_i^2) + \frac{b_i}{2}(3p_i^2 + q_i^2)u_i^2 + 4p_iq_i u_i v_i + (p_i^2 + 3q_i^2)v_i^2 \\
\geq \sum_{i=1,2} a_iu_i^2 + a_iv_i^2 \\
\geq \sum_{i=1,2} \delta u_i^2 + \delta v_i^2 \\
= \delta \langle (u_1, v_1, u_2, v_2), (u_1, v_1, u_2, v_2) \rangle,
\]
provided we choose \( \delta < \min\{a_1, a_2\}. \) Consequently \( \Lambda_0 \) is strictly convex, hence dynamically convex, i.e., every periodic solution \( x \) of period \( T \) has Conley-Zehnder index \( \tilde{\mu}(x, T) \geq 3. \) Theorem 5.1 tells us that there exists a disk-like global surface of section \( D_0 \) of the flow of \( H_{00}. \) In this example, such a global surface of section is given by \( p_2 = q_2 = 0, \) which is a disk bounded by the closed orbit given by \( p_1^2 + q_1^2 = \frac{-a_1 + \sqrt{a_1^2 + 2b_1c}}{b_1}, \) \( p_2 = q_2 = 0. \) See Fig. 1. This disk is foliated by invariant circles of the type \( \{I_1 = \text{const}\}, \) with \( I_2 \) implicitly given by the energy condition \( H_{00}(I_1, I_2) = c. \)

There exists \( c_0 > 0 \) sufficiently small such that, for each \( 0 < c < c_0, \) \( \Lambda_0 \) is a normally hyperbolic invariant manifold (NHIM) for the Hamiltonian flow of \( H_0. \) The condition of smallness depends on \( \lambda. \)
Now let $0 < \varepsilon < \varepsilon_0$, with $\varepsilon_0$ sufficiently small. The standard theory of normal hyperbolicity \cite{23} implies that there exists $\Lambda_\varepsilon$ surviving $\Lambda_0$ as a NHIM for the Hamiltonian flow of $H_\varepsilon$.

Since the dynamical condition is stable under a small perturbation, then $\Lambda_\varepsilon$ bounds a dynamically convex region provided $\varepsilon_0$ is sufficiently small. Then, by Theorem \ref{thm:5.1} there exists a disk-like global surface of section $D_\varepsilon \subseteq \Lambda_\varepsilon$. The boundary $\partial D_\varepsilon$ is a periodic orbit $\chi_{\varepsilon,1}$. There exists another periodic orbit $\chi_{\varepsilon,0}$ in $\Lambda_\varepsilon$ transverse to $\text{int}(D_\varepsilon)$, forming a Hopf link with $\chi_{\varepsilon,1}$. Let $p_{\varepsilon,0}$ be the intersection point between $\chi_{\varepsilon,0}$ with $D_\varepsilon$. Let $f_\varepsilon : \text{int}(D_\varepsilon) \rightarrow \text{int}(D_\varepsilon)$ be the Poincaré return map to $D_\varepsilon$ for the flow restricted to $\Lambda_\varepsilon$. The return map restricted to the open annulus $A_\varepsilon = D_\varepsilon \setminus \{p_{\varepsilon,0}\}$ is equivalent to an area preserving map. It also satisfies a monotone twist condition.

Now, since the unperturbed Hamiltonian satisfies the iso-energetic non-degeneracy condition, we can apply KAM theorem and obtain a Cantor family of tori $\{T_\varepsilon\}_{\varepsilon \in A}$ in $\Lambda_\varepsilon$, that survives the perturbation, provided $0 < \varepsilon < \varepsilon_1$ for some $\varepsilon_1$ sufficiently small. The family of tori $\{T_\varepsilon\}_{\varepsilon \in A}$ contains gaps of size $O(\sqrt{\varepsilon})$, corresponding to the resonances of the system (see Subsection 2.1.3).

Assume that $H_1$ satisfies the non-degeneracy condition (B) from Section 4.2.3. Under these assumptions, the stable and unstable manifold $W^u(\Lambda_\varepsilon)$ and $W^s(\Lambda_\varepsilon)$ intersect transversally, and there exists a finite collection of homoclinic channels $\{\Gamma_\varepsilon^i\}$, and corresponding scattering maps $S^i_\varepsilon : U^i_\varepsilon \rightarrow V^i_\varepsilon$, $i = 1, \ldots, n$, such that $\bigcup_{i=1,\ldots,n} U^i_\varepsilon$ covers the whole range of the action variable $I_1$ on $\Lambda_\varepsilon$. That is, $\Pi_{I_1}(\bigcup_{i=1,\ldots,n} U^i_\varepsilon) \supseteq \Pi_{I_1}(\Lambda_\varepsilon \setminus (\chi_{\varepsilon,0} \cup \chi_{\varepsilon,1}))$.

Under these conditions, Theorem \ref{thm:2.2} implies that the Hamiltonian flow exhibits global diffusion on $\Lambda_\varepsilon$.

Remark 3.1. We compare this example with the spatial circular restricted three-body problem. In a co-rotating coordinate system, with the primaries of masses $1 - \mu$, $\mu$ placed at $-\mu$, $1 - \mu$, respectively, the motion of the infinitesimal mass is described by the following Hamiltonian

$$H(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \omega(x, y, z),$$

where the ‘effective potential’ $\omega$ is given by

$$\omega(x, y, z) = \frac{1}{2}(x^2 + y^2) + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2},$$

with $r_1 = ((x+\mu)^2+y^2+z^2)^{1/2}$ and $r_2 = ((x-1+\mu)^2+y^2+z^2)^{1/2}$. This problem has five equilibria, $L_1, L_2, L_3$ of (center\times center\times saddle)-type, and $L_4, L_5$ of (center\times center\times center)-type. We denote by $L_1$ the equilibrium between the primaries.

Let us assume that $\mu$ is very small. We fix an energy level $\{H = c\}$ above the energy level to that of $L_1$, but very close to it. On this energy level, near $L_1$ there exists a normally hyperbolic invariant manifold $\Lambda$ diffeomorphic to $S^3$.

Following \cite{38}, using a normal form about $L_1$, the Hamiltonian can be written in a neighborhood of $L_1$ in the form

$$H(p, q) = H_{00}(p_1, q_1, p_2, q_2) + H_{01}(p_3, q_3) + H_1(p, q),$$
with \( H_{00} \) (in action-angle coordinates) and \( H_{01} \) of the form
\[
H_{00}(I_1, I_2) = a_1 I_1 + a_2 I_2 - \frac{b_1}{2} I_1^2 - \frac{b_2}{2} I_2^2 - b_3 I_1 I_2,
\]
\[
H_{01}(p_3, q_3) = \lambda p_3 q_3,
\]
and \( H_1 \) sufficiently small.

A significant difference from the previous example is that \( \Lambda \subseteq \{ H = c \} \) does not satisfy the strict convexity condition. However, the problem is symmetric in \( z \), so the hyperplane \( \{ z = 0 \} \) is a global surface of section for the Hamiltonian flow. In particular, it is a global surface of section for the flow restricted to \( \Lambda \). Thus, the lack of the strict convexity condition is not an impediment for the above geometric mechanism.

Another significant difference is that the term \( H_1 \) is intrinsic to the problem, and cannot be regarded, in general, as a small perturbation of a generic type. In order to establish the existence of orbits that exhibit global diffusion relative to \( \Lambda \), in this case, one would have to explicitly find a collection of homoclinic intersections and corresponding scattering maps that can be used to achieve global diffusion.

4. Proofs of the results

4.1. Proof of Proposition 2.1. As it was mentioned in Subsection 2.2, we have that \( \Lambda_\varepsilon \) can be parametrized via some smooth map \( k_\varepsilon : \Lambda_0 \to \Lambda_\varepsilon \) with \( \lambda_0 = \text{id} \). Choose a parametrization of \( \Lambda_0 \) given by \( z \in S^3 \mapsto \kappa(z) = \sqrt{g(z)} z \in \Lambda_0 \), with \( g : S^3 \to \mathbb{R} \) smooth. Let \( \tilde{k}_\varepsilon = k_\varepsilon \circ \kappa \). Let \( \lambda_\varepsilon \) be the contact form on \( \Lambda_\varepsilon \) and let \( \tilde{k}_\varepsilon^* \lambda_\varepsilon \) be the contact form on \( S^3 \) given by the pull back of \( \lambda_\varepsilon \). Since \( \lambda \) is tight, then \( \tilde{k}_\varepsilon^* \lambda_\varepsilon \) is tight for all \( \varepsilon \) sufficiently small.

At this point on \( S^3 \) we have the standard contact form \( \lambda \), and the family of tight contact forms \( \tilde{k}_\varepsilon^* \lambda_\varepsilon \), with \( \varepsilon > 0 \) small. By Gray’s Stability Theorem, there exists a smooth family of positive functions \( \Psi_\varepsilon \) such that \( \tilde{k}_\varepsilon^* \lambda_\varepsilon = \sqrt{\Psi_\varepsilon} \lambda \). We have that \( \Psi_\varepsilon \) is \( C^r \)-close to the identity for \( \varepsilon \) sufficiently small, and so the Reeb flow of \( \tilde{k}_\varepsilon^* \lambda_\varepsilon \) is \( C^r \)-close to the Reeb flow of \( \lambda \). Since \( \lambda \) is strictly convex, it is dynamically convex. It then follows that \( \tilde{k}_\varepsilon^* \lambda_\varepsilon \) is dynamically convex for all \( \varepsilon \) sufficiently small.

By virtue of Theorem 5.1, we can choose \( \varepsilon_0 \) sufficiently small such that, for all \( 0 < \varepsilon < \varepsilon_0 \), there exists a disk-like surface of section \( D_\varepsilon \) for the flow of \( H_\varepsilon \) restricted to \( \Lambda_\varepsilon \). Moreover, since the property of being a global surface of section is an open condition, then we can choose the family of disks \( D_\varepsilon \) to be smoothly depending on \( \varepsilon \).

In addition, the boundary \( \partial D_\varepsilon \) is a periodic orbit \( \chi_{\varepsilon,1} \). There exists another periodic orbit \( \chi_{\varepsilon,0} \) in \( \Lambda_\varepsilon \) transverse to \( \text{int}(D_\varepsilon) \), forming a Hopf link with \( \chi_{\varepsilon,1} \).

4.2. Proof of Theorem 2.2

4.2.1. The geometry of the unperturbed system. The assumption (A1)-(i) is that all critical submanifolds of the first integral mapping \( K \) are circles that are non-degenerate in the sense of Bott. These circles are necessarily periodic orbits of the Hamiltonian flow. The index of a critical circle is the number of negative eigenvalues of the restriction of \( D^2 K \) on a subpace transverse to the circle. These circles are also assumed to be
of hyperbolic or elliptic type. In particular, the circles of index 0 or 2 are elliptic, and the circles of index 1 are hyperbolic.

The connected components of the complement of the circles are action-angle domains, i.e., on each component $U$ there exist a system of action-angle coordinates $(I_1, I_2, \phi_1, \phi_2)$ as in (A1)-(ii) such that $H_0 = h(I_1, I_2)$ for some function $h$ on this domain. For a critical circle $\chi$ we have the following possibilities (see [3, 8, 32, 5, 31]):

(i) If the index of $\chi$ is 0 or 2, then there exists a neighborhood of $\chi$ that is foliated by regular Liouville tori of the type $I_1 = \text{const.}$, $I_2 = \text{const.}$, where $(I_1, I_2, \phi_1, \phi_2)$ is a system of action-angle coordinates near $\chi$. The intersections of these tori with the surface of section $D_0$ are circles surrounding $\chi \cap D_0$.

(ii) If the index of $\chi$ is 1, then there exists a neighborhood of $\chi$ that is locally the product between $\chi$ and a ‘cross’ composed of separatrices of the orbit. The trajectory has an orientable separatrix diagram or a non-orientable separatrix diagram (see [17]). The whole connected component of the critical level of $K$ containing $\chi$ is a finite union of critical circles and cylinders whose boundary is either made of one or two critical circles; all these critical circles have index 1. The intersection between this neighborhood and $D_0$ determines separatrices for the dynamics of the Poincaré map $f_0$.

Consequently, the Liouville foliation determines a partition of $D_0$ into a finite collection of annuli (including punctured disks) $\Omega^j$, $j = 1, \ldots, k$, on each of which the dynamics can be described by a system of action-angle coordinates $(J, \theta)$. (Here $J$ can be chosen $I_1$, and $\theta$ the symplectic conjugate of $J$ relative to $d\lambda|_{D_0}$. ) The boundaries of these annuli consist of points and circles (including the separatrices). Each annulus is foliated by invariant circles of the type $J = \text{const.}$ The iso-energetic non-degeneracy condition (2.3) ensures that the Poincaré map $f_0$ satisfies a monotone twist condition near each circle $J = \text{const.}$ An example illustrating a possible topology of the intersection between the leaves of the Liouville foliation with the disk is shown in Fig. 2.

For each $j = 1, \ldots, k$, we denote by $\Pi_J : \Omega^j \to \mathbb{R}$ the projection mapping onto the $J$-coordinate. The image of each domain $\Omega^j$ by $\Pi_J$ is an interval in $\mathbb{R}$.

4.2.2. The geometry of the perturbed system. We now look into the effect of a small perturbation on the Liouville foliation. The main idea is that, if the perturbation is small enough, then, by the KAM theorem, a positive measure set of invariant tori from the unperturbed case will survive the perturbation, although slightly deformed, with the measure of the set of the KAM tori approaching full measure as the size of the perturbation tends to zero. The destroyed tori will lead to ‘gaps’ between KAM tori, including ‘large gaps’ of the order of the square root of the size of the perturbation.

For $0 < \varepsilon < \varepsilon_0$, the domains $\Omega^j_0$ in $\Lambda_0$ can be continued to domains $\Omega^j_\varepsilon$ in $\Lambda_\varepsilon$, $j = 1, \ldots, k$, which can still be parametrized by one action and two angle coordinates. The corresponding domains $\tilde{\Omega}^j_\varepsilon$ in $D_\varepsilon$ can be parametrized by one action and one angle coordinate. With an abuse of notation, we will still denote these coordinates by $(J, \theta)$.

Note that $\tilde{\Omega}^j_\varepsilon$ are no longer bounded by separatrices, which may be destroyed by the perturbation.
We make the following additional assumptions on the perturbation $H_1$ for each $0 < \varepsilon < \varepsilon_0$, for some $\varepsilon_0$ sufficiently small:

(B) (i) For each component of the boundary $\partial \Omega_0^j$ of an action-angle domain which is of the type $W^u(\chi_0) = W^s(\chi_0)$ for the unperturbed system, with $\chi_0$ a critical circle of hyperbolic type, for the perturbed system we have $W^u(\chi_\varepsilon) \cap W^s(\chi_\varepsilon)$, where $\chi_\varepsilon$ is the hyperbolic circle corresponding to $\chi_0$ after perturbation.

(ii) There exists a collection of homoclinic channels $\Gamma_\varepsilon^i$, $i = 1, \ldots, n$, and corresponding scattering maps $S_\varepsilon^i : U_\varepsilon^i \to V_\varepsilon^i$, associated to $\Gamma_\varepsilon^i$, with $U_\varepsilon^i, V_\varepsilon^i \subseteq \Lambda_\varepsilon$, such that, for every $j \in \{1, \ldots, k\}$, each level set of $I_1$ in $\Omega_\varepsilon^j$ intersects some $U_\varepsilon^i$. Moreover, for every $j$, and every action level set $\{I_1 = J_0\}$, there exist $i$ and point $x_1 \in U_\varepsilon^i \cap \{I_1 = J_0\}$ such that $I_1(S_\varepsilon^i(x_1)) = J_1 < J_0$, and there exist a point $x_2 \in U_\varepsilon^i \cap \{I_1 = J_0\}$ such that $I_1(S_\varepsilon^i(x_2)) = J_2 > J_0$.

(iii) For every critical circle $\chi_\varepsilon$ which is of hyperbolic type and has an orientable separatrix, there exists $i$ such that $U_i$ intersects $W^{u,s}(\chi_\varepsilon)$. Moreover, there exist scattering maps $S_\varepsilon^i$ that move points from either side of $W^{u,s}(\chi_\varepsilon)$ to the opposite side.

We will show in Subsection 4.2.4 that the assumption (B) is satisfied by an open and dense set of perturbations $H_1$. Assumption (B)-(i) implies that the separatrices of the action-angle domains that correspond to stable and unstable manifolds of hyperbolic circles are destroyed by the perturbation, yielding transverse homoclinic intersections. Assumption (B)-(ii) implies that there exists a collection of scattering maps that is rich enough so that it covers the whole action range inside each maximal action-angle domain in $\Lambda_\varepsilon$, as well as the stable/unstable manifolds of hyperbolic invariant circles that separate these domains. Moreover, it says that inside such a domain one can use one of the scattering maps to increase/decrease the action coordinate $I_1$. Assumption (B)-(iii) also says that the scattering maps can be used to cross the stable/unstable
manifolds of the hyperbolic circles from one side to the other. Of course, when $\chi_0$ has a non-orientable separatrix, the scattering maps are no longer needed to cross the stable/unstable manifolds.

The iso-energetic non-degeneracy condition (A)-(ii) allows us to apply the KAM theorem inside each domain $\Omega^j_2$, and infer that there exist a family of 2-dimensional tori $\{\tau_0\}_{\alpha \in A_j}$ that survives the perturbation. These tori are primary tori, in the sense that they can be written as graphs over the $(\phi_1, \phi_2)$ variables, where $(I_1, \phi_1, \phi_2)$ are the coordinates on $\Omega^j_2$. The scattering map $S^j_ε$ defined on $U^j_ε$ allows us to construct finitely many transition chains of tori alternating with 'large gaps', where the large gaps are regions between $\phi^j_ε$-invariant tori in $\Lambda_ε$ that do not contain other primary invariant tori in between. The tori at the ends of these large gaps are not necessarily KAM tori, and they are not necessarily smooth. More precisely, we can form a sequence of invariant 2-dimensional tori of the type $\{\tau_i\}_{i=1, \ldots, \tau_j}$ in $\Omega^j_2$, for some integer $\tau_j > 0$, with the following properties:

- Each torus $\tau_i$ is the limit of some other invariant tori, i.e., there exists a sequence of invariant tori $(\tau_i)(i \in \mathbb{N})$ that approaches $\tau_i$ in the $C^0$-topology.
- Each torus $\tau_i$ is a part of a transition chain $\{\tau_i, \tau_{i+1}, \ldots, \tau_{i+l}\} \subseteq \{\tau_i\}_{i=1, \ldots, \tau_j}$, where for each $t = 0, \ldots, l - 1$ there is an $i \in \{1, \ldots, n\}$ such that $S^j_ε(\tau_{i+t})$ topologically crosses $\tau_{i+t+1}$ - for the definition of topological crossing see, e.g., [21]; if $\tau_i$ is an end torus in the transition chain, then it is Lipschitz continuous; if $\tau_i$ is not an end torus in the transition chain, then it is a KAM torus, and so it is smooth;
- If $\{\tau_i, \tau_{i+1}, \ldots, \tau_{i+l}\}$ is a transition chain, then the region in $\Omega^j_2$ between $\tau_{i+t}$ and $\tau_{i+t+1}$ does not contain any invariant primary torus in its interior; the torus $\tau_{i+t+1}$ is an end torus in a subsequent transition chain.

Relative to the surface of section $\mathcal{D}_ε$, assumption (B) implies that the separatrices that bound different action-angle domains $\hat{\Omega}_0^j$ are destroyed by the perturbation, and they give rise to hyperbolic periodic orbits whose stable and unstable manifolds intersect transversally. Assume that $O(p) = \{p_1, \ldots, p_m\}$ is a hyperbolic periodic orbit for $f_ε$, with $W^u(O(p))$ transverse to $W^s(O(p))$ in $\mathcal{D}_ε$. These invariant manifolds separate different regions $\hat{\Omega}_0^j$, which correspond to maximal action-angle domains in the unperturbed case.

Since the Poincaré map $f_0$ satisfies a monotone twist condition on $\hat{\Omega}_0^j$, the perturbed map $f_ε$ also satisfies a monotone twist condition on $\hat{\Omega}_ε^j$, provided that $ε$ is sufficiently small. Along the hyperbolic invariant manifolds, the twist is infinite.

Inside each region $\hat{\Omega}_0^j$ there exist a family of invariant tori, as described above. There exists a last invariant torus that is the closest to the hyperbolic invariant manifolds. The region between the last invariant torus and the hyperbolic invariant manifolds is a BZI. Thus, there exist trajectories that move any prescribed neighborhood of the torus at one boundary of the BZI to any prescribed neighborhood of the hyperbolic invariant manifolds (see [20]).

Therefore, we can form transition chains of tori interspersed with gaps, as above, and move across different action-angle domains $\Omega^j_2$ using the hyperbolic invariant manifolds.
that separate them and the scattering maps that intersect these invariant hyperbolic manifolds. These transition chains include, in addition to invariant tori, stable and unstable invariant manifolds of hyperbolic periodic orbits. Similar constructions appear in [13].

In the next section we will use these transition chains of tori interspersed with gaps as in [20] to construct diffusing orbits with the desired features.

4.2.3. The existence of diffusing trajectories. We now describe how to combine the outer dynamics, given by the family of scattering maps, the inner dynamics, given by the restriction of the flow to $\Lambda_\varepsilon$, and the reduced dynamics, given by the return map $f_\varepsilon$ to the surface of section $D_\varepsilon$, in order to construct orbits with the desired characteristics. The main part of the argument relies heavily on the constructions from [20]; hence we will not repeat some of the technical details, and focus on how to utilize these constructions in order to prove the main result. A description of the ingredients of [20] is given in Appendix S.

First, we show how to produce trajectories that stay close enough to $\Lambda_\varepsilon$ for an arbitrarily long time. Let $U_\varepsilon$ be a neighborhood of $\Lambda_\varepsilon$ in the energy hypersurface where the flow can be linearized, as in Subsection 2.1.5. Given any $T > 0$, each point in the set $U_\varepsilon^T = \bigcap_{t \in [0,T]} \phi_\varepsilon^{-t}(U_\varepsilon)$ stays in $U_\varepsilon$ for at least a time $T > 0$. The trajectory can be chosen so that, after it spends at least a time $T \geq 0$ in $U_\varepsilon$, it will follow one’s choice of a scattering map $S_\varepsilon$ and re-enter a neighborhood $U_\varepsilon^{T'}$ corresponding to some other prescribed $T' > 0$.

We discretize the dynamics by considering the time-1 map $F_\varepsilon$ of the flow $\phi_\varepsilon$, which is defined on the energy hypersurface. The above considerations for the flow dynamics relative to $U_\varepsilon$ remain valid for the time-1 map dynamics (we can choose $T \in \mathbb{Z}^+$. We also remark that the scattering map for the time-1 map coincides with the scattering map for the flow [14]. Hence, we will use the same notation $S_\varepsilon$ to refer to the scattering map for the flow $\phi_\varepsilon$ and the scattering map for the time discretization of the flow $F_\varepsilon$.

We can also define a scattering map for the return map $f_\varepsilon$ to $D_\varepsilon$. For each scattering map $S_\varepsilon$ for the flow/time discretization, there exists a scattering map $\hat{S}_\varepsilon$ that enjoys similar properties. See [12].

We describe a procedure to combine the dynamics of the time-1 map $F_\varepsilon$ restricted to $\Lambda_\varepsilon$ with the dynamics of the return map $f_\varepsilon$ on $D_\varepsilon$. Consider a point $z \in D_\varepsilon$ and its image $f^n_\varepsilon(z)$ under some power of $f_\varepsilon$. Then $f^n_\varepsilon(z) = \phi_{\varepsilon t(z)}^{t(z,n)}(z)$ for some time $t(z) \in \mathbb{R}$. Consider the orbit of $z$ under the time-discretization map $F_\varepsilon$. To each $n \in \mathbb{Z}$ there exists a unique $n'(z,n) \in \mathbb{Z}$ such that $n'(z,n) \leq t(z,n) < n'(z,n) + 1$. The points $f^n_\varepsilon(z), F^{n'}(z,n)(z)$ lie on the same trajectory $\phi_{\varepsilon t(z)}(z)$.

We now describe the geometric objects that organize the dynamics of $f_\varepsilon$. Each domain $\Omega_\varepsilon$ determines a maximal action-angle domain $\hat{\Omega}_\varepsilon \subseteq D_\varepsilon$ with corresponding action-angle variables $(J,\theta)$. The tori $T_\tau$ define $f_\varepsilon$-invariant circles $\hat{T}_\tau, \tau = 1, \ldots, \tau_j$ in $\hat{\Omega}_\varepsilon$.

Each ‘large gap’ in $\Omega_\varepsilon$ between a pair of successive tori $T_\tau$ and $T_{\tau+1}$ defines a BZI in $\Omega_\varepsilon$, i.e., an annular region in $\hat{\Omega}_\varepsilon$ bounded by $\hat{T}_\tau$ and $\hat{T}_{\tau+1}$ which does not contain any topologically non-trivial, $f_\varepsilon$-invariant circle in its interior.
For each such BZI, assumption (B)-(ii) says that there exist scattering maps whose domain/codomain intersect the boundaries of the BZI, so that one can go from either side to such a boundary to the other, in \( D_\varepsilon \).

Consider the collection of frequencies \( \mathcal{R}_{\text{seq}} = \{ \rho_s \}_{s \in \mathbb{Z}} \cap \mathcal{R}^j \) corresponding to the prescribed frequencies that correspond to the dynamics of \( f_\varepsilon \) restricted to \( \hat{\Omega}_\varepsilon \). Some of the frequencies \( \rho_s \in \mathcal{R}_{\text{seq}} \) yield invariant circles, and some others yield Aubry-Mather sets of Cantor type. Consider the subset of these frequencies that are between \( \rho \) and \( z \).

Let us also denote by \( \hat{\varepsilon} \) the scattering maps \( S_{\varepsilon}^{(\tau')} \) to \( \hat{\varepsilon}(\tau'+1) \) for the return map \( f_\varepsilon \). Assume that:

- the codomain \( \hat{V}_i(\tau') \) of the scattering map \( \hat{S}_{\varepsilon}^{(\tau')} \) intersects \( \mathcal{T}_{\tau'} \), and \( \hat{S}_{\varepsilon}^{(\tau')} \) maps points from \( D_\varepsilon \setminus \hat{A}_{\tau'} \) to \( \hat{A}_{\tau'} \);
- the domain \( \hat{U}_i(\tau'+1) \) of the scattering map \( \hat{S}_{\varepsilon}(\tau'+1) \) intersects \( \mathcal{T}_{\tau'+1} \), and \( \hat{S}_{\varepsilon}(\tau'+1) \) maps points from \( \hat{A}_{\tau'} \) to \( D_\varepsilon \setminus \hat{A}_{\tau'} \).

The next step repeats the construction of correctly aligned windows from [20]. The purpose of this step is to provide a topological argument for the existence of orbits of \( f_\varepsilon \) that start near one boundary of the BZI, visit a prescribed sequence of Aubry-Mather sets inside the BZI (that is, it move with prescribed frequencies for prescribed time intervals), and end up near the other boundary of the BZI. The method of correctly aligned windows is summarized in Appendix 7.

The procedure described in [20] yields a sequence of 2-dimensional windows in \( D_\varepsilon \)

\[
\hat{R}_0, \hat{R}_1, \ldots, \hat{R}_p, \hat{R}_{p+1},
\]

such that:

- \( \hat{R}_0 \subseteq \hat{V}_i(\tau') \);
- \( \hat{R}_{p+1} \subseteq \hat{U}_i(\tau'+1) \);
- \( \hat{R}_1, \ldots, \hat{R}_p \) are contained in \( \hat{A}_{\tau'} \);
- \( \hat{R}_q \) is correctly aligned with \( \hat{R}_{q+1} \) under some power \( f_\varepsilon^{m_{s_q}} \) of \( f_\varepsilon \) for each \( q = 0, \ldots, p \);
- Every orbit of \( f_\varepsilon \) that visits these windows in the prescribed order moves with a frequency \( \rho_{s_q} \) for a prescribed time \( m_{s_q} \), for all \( q = 1, \ldots, p \).

Each window \( \hat{R}_q \) is a rectangle of the type

\[
\hat{R}_q = \{(J, \theta) \in A_{\tau'} \mid \theta(z^q_l) \leq \theta \leq \theta(z^q_r)\},
\]

where \( z^q_l, z^q_r \) are two points on the Aubry-Mather set of rotation number \( \rho_q \). This rectangle is bounded by \( \hat{T}_{\tau'} \) and \( \hat{T}_{\tau'+1} \), and by the lines \( \theta = \text{const.} \) passing through \( z^q_l \)
and $z^j$. The exit set of this window is defined by

$$\hat{R}^e_q = \{ (J, \theta) \in \hat{R}_q \mid \theta = \theta(z^j_\epsilon), J \leq J(z^j_\epsilon) \}$$

$$\cup \{ (J, \theta) \in \hat{R}_q \mid \theta = \theta(z^j_\epsilon), J \geq J(z^j_\epsilon) \}$$

$$\cup (\hat{R}_q \cap \hat{T}_\tau)$$

$$\cup (\hat{R}_q \cap \hat{T}_{\tau+1}),$$

and the entry set is $\hat{R}^e_\infty = \text{cl}(\partial \hat{R}_q \setminus \hat{R}^e_q)$.

We note here that the windows $\hat{R}_1, \ldots, \hat{R}_p$ constructed in [20] correspond to Hall’s diagonal sets. The fact that diagonal sets can be described as windows is also discussed in [18].

As we noted in Subsection 4.2.2, we can construct orbits that cross from one action-angle domain $\Omega^j_\epsilon$ to another; moreover, we can obtain such orbits via the method of correctly aligned windows. More precisely, we do the following. Assume that $\chi_\epsilon$ is a critical circle of hyperbolic type with $W^u(\chi_\epsilon)$ intersecting transversally $W^s(\chi_\epsilon)$, as in (B)-(i). Let the intersection of $\chi_\epsilon$ with $\Omega_\epsilon$ be the hyperbolic periodic orbit $O(p)$ for $f_\epsilon$, with $W^u(p)$ and $W^s(p)$ intersecting transversally in $\Omega_\epsilon$. The domain $\hat{\Omega}^j_\epsilon = \Omega^j_\epsilon \cap \Omega_\epsilon$ is obtained from the domain $\hat{\Omega}^j_0 = \Omega^j_0 \cap \Omega_\epsilon$ through the perturbation. Let $T$ be the last invariant circle in $\hat{\Omega}^j_\epsilon$, defined by the property that there is no topologically non-trivial invariant circle between $T$ and $W^u(p) \cup W^s(p)$. The region between $T$ and $W^u(p) \cup W^s(p)$ is also a BZI. We can again call the arguments in [20] to construct a window $\hat{R}$ by $T$ which is correctly aligned under some power of $f_\epsilon$ with a window $\hat{R}^s$ by $W^s(p)$. By the standard construction of the correctly aligned windows by a hyperbolic periodic orbit, we can construct another window $\hat{R}^u$ by $W^u(p)$, such that $\hat{R}^s$ is correctly aligned with $\hat{R}^u$ under some power of $f_\epsilon$. In the case when the critical circle has a non-orientable separatrix, we can use the powers of $f_\epsilon$ to move from one side to the other of $W^u(p)$. In the case when the critical circle has an orientable separatrix, by (B)-(ii) we can use scattering maps to move from one side to the other of $W^u(p)$. The conclusion of this paragraph is that the construction of correctly aligned can also be extended across different action-angle domains.

At the next step, we use the above 2-dimensional windows in $\Omega_\epsilon$, which are correctly aligned under $f_\epsilon$, to construct 3-dimensional windows in $\Lambda_\epsilon$, that are correctly aligned under $F_\epsilon$.

For each 2-dimensional window $\hat{R}_q \subseteq \Omega_\epsilon$ we construct a 3-dimensional windows $R_q \subseteq \Lambda_\epsilon$, with the property that $R_q$ is correctly aligned with $R_{q+1}$ under some suitable power of $F_\epsilon$. Note that the flow lines of $\phi^t_\epsilon$ restricted to $\Lambda_\epsilon$ which are passing through $\hat{R}_q$ intersect $\hat{R}_{q+1}$ along $f^{t_n}_\epsilon(\hat{R}_q) \cap \hat{R}_{q+1}$.

We start with the construction of $\hat{R}_0$ and continue the construction inductively. We have that $\hat{R}_0$ is correctly aligned with $\hat{R}_1$ under $f^{t_{n_0}}_\epsilon$. Choose a point $z \in \hat{R}_0$ and let $n'(n_0) := n'_0$ be the unique integer with $n'_0 \leq t(z, n_0) < n'_0 + 1$. Since $F_\epsilon$ is a diffeomorphism, $F^{n'_0}(\Omega_\epsilon)$ is a diffeomorphic copy of $\Omega_\epsilon$ and it is itself a global surface of section. Define $\hat{R}_1$ as the the set of points of first intersection between the flow lines
through \( \hat{R}_1 \) and \( F^{n_0}(\mathcal{D}_\varepsilon) \). We define the window \( R_0 \) of the type
\[
R_0 = \bigcup_{z \in R_0} \phi_{\varepsilon}^{[t_1(0, z), t_2(0, z)]}(z),
\]
for some \( t_1(0, z), t_2(0, z) \in \mathbb{R} \), with the property that \( \hat{R}_0 = \bigcup_{z \in R_0} \phi_{\varepsilon}^{t_0(0, z)}(z) \) for some \( t_0(0, z) \in (t_1(0, z), t_2(0, z)) \). We let
\[
R_{0}^\text{ex} = \bigcup_{z \in R_{0}^\text{ex}} \phi_{\varepsilon}^{[t_1(0, z), t_2(0, z)]}(z) \cup \bigcup_{z \in R_0} \phi_{\varepsilon}^{[t_1(0, z), t_2(0, z)]}(z).
\]
Then we define
\[
R_1 = \bigcup_{z \in R_1} \phi_{\varepsilon}^{[t_1(1, z), t_2(1, z)]}(z),
\]
for some \( t_1(1, z), t_2(1, z) \in \mathbb{R} \), with the property that \( \hat{R}_1 = \bigcup_{z \in R_1} \phi_{\varepsilon}^{t_0(1, z)}(z) \) for some \( t_0(1, z) \in (t_1(1, z), t_2(1, z)) \). We let
\[
R_{1}^\text{ex} = \bigcup_{z \in R_{1}^\text{ex}} \phi_{\varepsilon}^{[t_1(1, z), t_2(1, z)]}(z) \cup \bigcup_{z \in R_1} \phi_{\varepsilon}^{[t_1(1, z), t_2(1, z)]}(z).
\]
It is immediate that for some suitable choices of the functions \( t_1(0, z), t_2(0, z), t_0(0, z), t_1(1, z), t_2(1, z), t_0(1, z) \), we get that \( R_0 \) is correctly aligned with \( R_1 \) under \( F^{n_0} \). More precisely, we require that \( t_1(0, z) + n_0' < t_1(1, z) < t_2(1, z) - t_2(0, z) + n_0' \). See Fig. 3.

The construction is continued inductively, providing a sequence of windows
\[
R_0, R_1, \ldots, R_p, R_{p+1},
\]
such that \( R_q \) is correctly aligned with \( R_{q+1} \) under some power \( F^{n_{q'}^{p}} \) of \( F_\varepsilon \).

We remark that we can also ‘thicken’ the windows \( \hat{R}_q \) to 3-dimensional windows \( \hat{\hat{R}}_q \) of the type
\[
\hat{\hat{R}}_q = \bigcup_{z \in \hat{R}_q} \phi_{\varepsilon}^{[t_1(q, z), t_2(q, z)]}(z),
\]
for some suitable \( t_1(q, z), t_2(q, z) \in \mathbb{R} \), with the property that \( \hat{\hat{R}}_q = \bigcup_{z \in \hat{R}_q} \phi_{\varepsilon}^{t_0(q, z)}(z) \) for some \( t_0(q, z) \in (t_1(q, z), t_2(q, z)) \). Then the relation between the windows \( R_q \) and the windows \( \hat{\hat{R}}_q \) is that each window \( R_q \) is correctly aligned with the corresponding \( \hat{\hat{R}}_q \) under some suitable translation mapping along the trajectories of the flow \( \phi_{\varepsilon}^{t} \) through \( \hat{\hat{R}}_q \).

The conclusion of this construction is that, by constructing the windows \( R_0, R_1, \ldots, R_p \) in a correctly aligned manner, we obtain orbits of \( F_\varepsilon \) that follow these windows. These \( F_\varepsilon \)-orbits correspond, via translations along the trajectories of the flow, to orbits of \( f_\varepsilon \).
that move with the frequency $\rho_s$ for a time $m_s$, for each $\rho_s \in \{\rho_s', \ldots, \rho_s''\}$. These orbits cross the large gaps between the invariant tori of the type $T_{\tau'}$ and $T_{\tau'+1}$ in $\Lambda_\varepsilon$.

For the next step of the construction, we construct 3-dimensional windows along the transition chains $\{T_{\tau'+t}\}_{t \in \{0, \ldots, l\}}$ in $\Lambda_\varepsilon$, that are correctly aligned under suitable powers of $F_\varepsilon$. By (B)-(i), for each $t \in \{0, \ldots, l-1\}$ there is a scattering map $S^i_\varepsilon$ such that $S^i_\varepsilon(T_{\tau'+t})$ topologically crosses $T_{\tau'+t+1}$. Thus we have that $W^u(T_{\tau'+t})$ is topologically crossing $W^s(T_{\tau'+t+1})$ along some homoclinic manifold $\Gamma^i_\varepsilon$, for $t \in \{0, \ldots, l-1\}$.

From the previous construction we have obtained 3-dimensional windows $R_{\tau'}$ and $R_{\tau'+l}$ at each end of the transition chain of tori. Using the scattering maps, we construct windows $R_{\tau'}^b, \{R_{\tau'+t}, R_{\tau'+t}^b\}_{t \in \{1, \ldots, l-1\}}$, such that, for each $t \in \{0, \ldots, l-1\}$ we have:

- $R_{\tau'+t}$ is correctly aligned with $R_{\tau'+t}^b$ under some power of $F_\varepsilon$, for $t \in \{0, \ldots, l-1\}$;
- $R_{\tau'+t}^b$ is correctly aligned with $R_{\tau'+t+1}$ under some scattering map $S^i_\varepsilon$ for some $i$.

We apply Lemma 7.4 from Appendix 7. This says that correctly aligned windows as such yield orbits that follow the windows in the prescribed order. The main argument of this Lemma is that, by using the Lambda Lemma, the 3-dimensional windows $R_{\tau'}$ can be used to construct corresponding 5-dimensional windows $W_{\tau'}$ that are correctly aligned under some powers of $F_\varepsilon$. Each window $W_{\tau'}$ is constructed by ‘thickening’ the

\[ \text{Figure 3. Windows correctly aligned by the return map and windows correctly aligned by the time-1 map.} \]
underlying window $R_\varepsilon$ in the hyperbolic directions normal to $\Lambda_\varepsilon$. The resulting shadowing orbits travel across different action-angle domains $\Omega^j_\varepsilon$, and move with frequency $\rho_s$ for a time length of $m_s$, for each $s$. This completes the argument.

**Remark 4.1.** Here we use the topological method of correctly aligned windows to show the existence of orbits that shadow Aubry-Mather sets lying in different maximal action-angle domains. The existence of orbits that shadow Aubry-Mather sets within a single action-angle domain has been established through variational methods in [34]. This approach has been applied to the diffusion problem in [8, 9]. The orbits found in these papers are minimal. As concatenations of minimal orbit does not necessarily yield minimal orbits, the variational approach does not seem to be suitable to find orbits traveling across different action-angle domains.

**Remark 4.2.** One of the main ingredients to achieve global diffusion in this problem is the combination of multiple dynamics, i.e., of the inner dynamics given by the restriction of the Hamiltonian flow to $\Lambda_\varepsilon$ and of the outer dynamics given by several scattering maps. This is closely related to the idea of polysystems, which are locally constant skew-products over a Bernoulli shift. Polysystems play a key role in the work of Marco on instability in a priori stable and in a priori unstable systems [33]. Polysystems, also known as iterated function systems, are further discussed in [37, 27, 36, 35].

4.2.4. **Genericity of the perturbation.** In this section we sketch out the argument that there exists an open and dense set of potentials $H_1$ for which condition (B) from Subsection 4.2.2 is satisfied. We follow the ideas in [13], where detailed arguments can be found.

We assume that a Hamiltonian $H_1$ is given, and we show that by an arbitrarily small perturbation of $H_1$ the resulting Hamiltonian $H_\varepsilon = H_0 + \varepsilon H_1$ satisfies the condition (B).

By assumption (A2) the Hamiltonian $H_{01}$ has two homoclinic orbits $\gamma^\pm$ to $(0, 0)$. We denote by $(p^3_\pm(\tau), q^3_\pm(\tau)), \tau \in \mathbb{R}$, some parametrizations of these homoclinic orbits with $(p^3_\pm(0), q^3_\pm(0)) = (0, 0)$, where $(0, 0)$ is the saddle equilibrium point of the pendulum-like system. These homoclinic orbits determine two corresponding branches of the stable and unstable manifolds of $\Lambda_0$, $W^{+,-}(\Lambda_0) = W^{+,-}(\Lambda_0) = \bigcup_{z \in \Lambda_0} \gamma^+$, and and $W^{+,u}(\Lambda_0) = W^{-,s}(\Lambda_0) = \bigcup_{z \in \Lambda_0} \gamma^-$. We want to measure the splitting of the stable and unstable manifolds of $\Lambda_\varepsilon$ when $\varepsilon \neq 0$ is sufficiently small. For this, we define the Melnikov potential for each of the homoclinic orbits $\gamma^\pm$ by

$$
\mathcal{M}^\pm(z, \tau) = \int_{-\infty}^{\infty} \left[ H_1(p_1(t), q_1(t), p_2(t), q_2(t), p^3_\pm(\tau + t), q^3_\pm(\tau + t)) \right.
$$

$$
\left. - H_1(p_1(t), q_1(t), p_2(t), q_2(t), 0, 0) \right] dt,
$$

where $z = (p_1, q_1, p_2, q_2) \in \Lambda_0$, and $(p_1(t), q_1(t), p_2(t), q_2(t))$ denotes the trajectory $\phi^t_{00}(z)$ on $\Lambda_0$ for the Hamiltonian $H_{00}$. 
If we restrict to a domain \( \Omega_0^j \subseteq \Lambda_0 \), with corresponding coordinates \((I_1, \phi_1, \phi_2)\), with respect to these coordinates we have

\[
\mathcal{M}^\pm(I_1, \phi_1, \phi_2, \tau) = \int_{-\infty}^{\infty} \left[ H_1(I_1, \phi_1 + tI_1, I_2, \phi_2 + tI_2, p_3^\pm(\tau + t), q_3^\pm(\tau + t)) - H_1(I_1, \phi_1 + tI_1, I_2, \phi_2 + tI_2, 0, 0) \right] \, dt,
\]

where the value \( I_2 \) is implicitly determined by \((I_1, \phi_1, \phi_2)\). To measure the splitting of the stable and unstable manifolds of \( \Lambda_\varepsilon \), we take a local section \( \Sigma \) to \( \gamma^\pm(\tau) \) and we measure the distance between the intersection points of \( W^{\pm,u}(k_\varepsilon(z)) \) and \( W^{\pm,s}(k_\varepsilon(z)) \) with \( \Sigma \), which turns out to be given by

\[
-\varepsilon \frac{d}{d\tau} \mathcal{M}^\pm(I_1, \phi_1, \phi_2, \tau) + O(\varepsilon^{1+\delta}),
\]

for some \( \delta > 0 \). The existence of a transverse intersection of the stable and unstable manifolds \( W^{\pm,s}(\Lambda_\varepsilon) \), \( W^{\pm,u}(\Lambda_\varepsilon) \) is guaranteed provided that the function \( \tau \mapsto \mathcal{M}^\pm(I_1, \phi_1, \phi_2, \tau) \) has a non-degenerate critical point \( \tau^*(I_1, \phi_1, \phi_2) \). Then \((I_1, \phi_1, \phi_2) \mapsto (I_1, \phi_1, \phi_2, \tau^*(I_1, \phi_1, \phi_2))\), with \((I_1, \phi_1, \phi_2)\) in some domain \( U_\varepsilon^0 \) in \( \Lambda_0 \) where \( \tau^* \) is a non-degenerate critical point of \( \tau \mapsto \mathcal{M}^\pm \), gives a parametrization of a homoclinic manifold \( \Gamma_\varepsilon^0 \subseteq W^{\pm,s}(\Lambda_\varepsilon) \cap W^{\pm,u}(\Lambda_\varepsilon) \). If \( \Gamma_\varepsilon^0 \) is chosen small enough so that it is a homoclinic channel, then there is an associated scattering map \( S_\varepsilon^0 : U_\varepsilon^0 \to V_\varepsilon^0 \). For each \( x \in \Gamma_\varepsilon^0 \), \( S_\varepsilon^0 \) assigns to the point \( \Omega^u(x) \in U_\varepsilon^0 \) the point \( \Omega^s(x) \in V_\varepsilon^0 \), where \( U_\varepsilon^0, V_\varepsilon^0 \subseteq \Lambda_\varepsilon \). See Appendix [6]

The change in the action \( I_1 \) by the corresponding scattering map \( S_\varepsilon^0 \) is given by

\[
I_1(\Omega^s(x)) - I_1(\Omega^u(x)) = -\varepsilon \frac{d}{d\tau} \{I_1, \mathcal{M}^\pm\}(I_1, \phi_1, \phi_2, \tau^*(I_1, \phi_1, \phi_2)),
\]

for \( x \in \Gamma_\varepsilon^0 \).

Fixing a value \( I_1 = I_0^1 \), we can always ensure, by an arbitrarily small perturbation of \( H_1 \), if necessary, that there exists an open neighborhood \( U_\varepsilon^{i0} \) of a point \((I_0^1, \phi_0^1, \phi_0^2) \in \Lambda_\varepsilon \) for which the map \( \tau \mapsto \mathcal{M}^\pm(I_1, \phi_1, \phi_2, \tau) \) has a non-degenerate critical point \( \tau^*(I_1, \phi_1, \phi_2) \), for each \((I_1, \phi_1, \phi_2) \in U_\varepsilon^{i0} \). We can choose this perturbation to have compact support in some small tubular neighborhood of a point of \((I_1, \phi_1, \phi_2, \gamma^+(\tau))\) or \((I_1, \phi_1, \phi_2, \gamma^-(\tau))\). The homoclinic intersection given by \((I_1, \phi_1, \phi_2, \tau^*(I_1, \phi_1, \phi_2))\), with \((I_1, \phi_1, \phi_2) \in U_\varepsilon^{i0} \), is denoted by \( \Gamma_\varepsilon^i \).

Since in the unperturbed system we have available two homoclinic orbits \( \gamma^\pm \) that are geometrically different, we can produce arbitrarily small perturbations of \( H_1 \) that have mutually disjoint supports, and obtain scattering maps \( S_\varepsilon^i : U_\varepsilon^i \to V_\varepsilon^i \), with corresponding homoclinic manifolds \( \Gamma_\varepsilon^i \) such that the union of the domains \( U_\varepsilon^i \) covers the whole range of \( I_1 \). This ensures the genericity of condition (B)-(i) within each angle-action domain \( \Omega_\varepsilon^j \), \( j = 1, \ldots, k \).

Now we discuss the splitting of the hyperbolic invariant manifolds of the critical circles \( \chi_0 \). For the Hamiltonian flow of \( H_0 \), each such circle \( \chi_0 \) has two dimensional stable and unstable manifolds \( W^{\pm,s}(\chi_0), W^{\pm,u}(\chi_0) \), with one hyperbolic direction corresponding to the dynamics of \( \phi_0^1 \) on \( \Lambda_0 \), and another hyperbolic direction corresponding to the separatrix of the flow of \( H_{01} \). We have that \( W^{\pm,s}(\chi_0) = W^{\pm,u}(\chi_0) \). By an arbitrarily small perturbation of a given \( H_1 \), if necessary, we can make that \( W^{\pm,s}(\chi_0) \) intersects
transversally $W^{±,u}(\lambda_0)$. This ensures the genericity of the condition (B)-(ii), and also of the condition (B)-(i) at the boundaries of the angle-action domain $\Omega^j_\varepsilon$, $j = 1, \ldots, k$.

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5. APPENDIX A. BACKGROUND ON SYMPLECTIC DYNAMICS.

5.1. Contact geometry. Given a compact, connected, oriented, 3-dimensional manifold $M$, a contact form $\lambda$ on $M$ is a 1-form on $M$ such that $\lambda \wedge d\lambda$ is a volume form on $M$. The contact structure associated to $\lambda$ is the plane bundle in $TM$ given by $\xi = \ker(\lambda) = \{(x, h) \in TM | \lambda(x)(h) = 0\}$. The restriction $d\lambda|_{\xi ? \xi}$ defines a symplectic structure on each fiber of $\xi \to M$. The characteristic distribution of $M$ is the 1-dimensional distribution $L = \{(x, h) \in TM | x \in M, \omega(h, k) = 0 \text{ for all } k \in T_xM\}$. The corresponding 1-dimensional foliation is called the characteristic foliation.

The Reeb vector field $X$ associate to $\lambda$ is the vector field on $M$ uniquely defined by $i_X(d\lambda) = 0$ and $i_X(\lambda) = 1$. The Reeb vector field $X$ spans the characteristic distribution $L$ which has the canonical section $X$. The flow lines of $X$ are contained in the leaves of the characteristic foliation. We have that $TM$ naturally splits as $TM = L \oplus \xi = \mathbb{R}X \oplus \xi$.

A contact structure $\lambda$ is said to be tight provided that there are no overtwisted disks in $M$, that is, there is no embedded disk $D \subseteq M$ such that $T\partial D \subseteq \xi$ and $T_pD \neq \xi_p$ for all $p \in \partial D$. As an example, the 1-form on $\mathbb{R}^4$ of coordinates $(q_1, p_1, q_2, p_2)$,

$$\lambda_0 = \frac{1}{2}(q_1 dp_1 - p_1 dq_1 + q_2 dp_2 - p_2 dq_2),$$

gives a tight contact form on the 3-dimensional sphere $S^3 \subseteq \mathbb{R}^4$ when restricted to $S^3$. By a theorem of Eliashberg [16], every tight contact form on $S^3$ is diffeomorphic to $g\lambda_0$, for some $C^1$-differentiable $g : S^3 \to \mathbb{R} \setminus \{0\}$. The sphere $S^3$ equipped with this distinguished contact structure is called the tight three-sphere.

If we denote by $\phi^t$ the flow of $X$, we have that $(\phi^t)^*\lambda = \lambda$, and $(D\phi^t)_m : \xi_m \to \xi_{\phi^t(m)}$ is symplectic with respect to $d\lambda$.

5.2. The Conley-Zehnder index. To each contractible $T$-periodic solution $x(t)$ of the Reeb vector field, there is assigned the so called Conley-Zehnder index. The Conley-Zehnder index generalizes the usual Morse index for closed geodesics on a Riemannian manifold. Roughly speaking, the index measures how much neighboring trajectories of the same energy wind around the orbit.

We now recall the definition of the index. Assume that $x$ is a $T$-periodic solution, which is contractible. The derivative map $D\phi^T : T_{x(0)}M \to T_{x(T)}M$ maps the contact plane $\xi_{x(0)}$ to $\xi_{x(T)}$ and is symplectic with respect to $d\lambda$. We assume that $x(t)$ is non-degenerate, meaning that $D\phi^T$ does not contain 1 in the spectrum. Choose a smooth
disk \( u : D \to M \) s.t. \( u(e^{2\pi i t/T}) = x(t) \), where \( D = \{ z \in \mathbb{C} | |z| \leq 1 \} \). Then choose a symplectic trivialization \( \beta : u^*\xi \to D \times \mathbb{R}^2 \). We associate to \( x(t) \) an arc of symplectic matrices \( \Phi : [0,T] \to Sp(1) \), where \( Sp(1) = \{ A \in GL(2) | ATJA = J \} \), by
\[
\Phi(t) = \beta(e^{2\pi i t/T}) \circ (D\phi_t)|_{x(t)} \circ \beta(1)^{-1}.
\]
The arc starts at the identity \( \Phi(0) = Id \) and ends at \( \Phi(T) \), with \( \det(\Phi(T) - I) \neq 0 \), due to the non-degeneracy condition. Take \( z \in \mathbb{C} \) and compute the winding number of \( \Phi(t)z \),
\[
\Delta(z) = \theta(T) - \theta(0) \in \mathbb{R},
\]
where \( \theta(t) \) is a continuous argument of \( \Phi(t)z \), i.e., \( \Phi(t)z = r(t)e^{2\pi i \theta(t)} \). Then define the winding interval of the arc \( \Phi(t) \) by
\[
I(\Phi) = \{ \Delta(z) | z \in \mathbb{C} \setminus \{ 0 \} \}.
\]
Equivalently, we can put \( \Phi(t)e^{2\pi is} = r(t,s)e^{2\pi i \theta(t,s)} \) for all \( s \in [0,1] \), where \( \theta(0,s) = s \), and define \( \Delta(s) = \theta(T,s) - \theta(0,s) = \theta(T,s) - s \), and \( I(\Phi) = \{ \Delta(s) | s \in [0,1] \} \). The length of the winding interval is less than \( 1/2 \). Then the winding interval either lies between two consecutive integers or contains precisely one integer.

We define
\[
\mu(\Phi) = \begin{cases} 
2k, & \text{if } k \in I(\Phi); \\
2k + 1, & \text{if } I(\phi) \subset (k,k+1).
\end{cases}
\]

Then we define the Conley-Zehnder index \( \mu(x,T,[u]) \) of \( (x,T) \) by \( \mu(x,T,[u]) = \mu(\Phi) \). It depends on \( x \) and \( T \) and on the homotopy class of the choice of the disk map \( u : D \to M \) satisfying \( u(e^{2\pi i t/T}) = x(t) \). In the case when \( \pi_2(M) = 0 \) (e.g., if \( M = S^3 \)), the index is independent of the choice of the disk map \( u \).

5.3. Existence of global surfaces of section. Consider \( \mathbb{R}^4 = \{ x = (q_1,p_1,q_2,p_2) | q_1,p_1,q_2,p_2 \in \mathbb{R} \} \) endowed with the standard symplectic form \( \omega = \sum_{i=1}^2 dq_i \wedge dp_i \), and \( H : \mathbb{R}^4 \to \mathbb{R} \) a \( C^\infty \)-differentiable Hamiltonian function. If \( c \in \mathbb{R} \) is a regular value for \( H \), then \( S_c = \{ x | H(x) = c \} \) is a 3-dimensional manifold invariant under the Hamiltonian flow of \( H \). Assume that \( S_c \) is compact and connected.

The manifold \( S_c \) is said to bound a strictly convex domain provided that there exists \( \delta > 0 \) such that \( D^2H(x) - \delta \cdot \text{id} \) is positive definite for all \( x \in \mathbb{R}^4 \). This is equivalent with the conditions that \( W = \{ x | H(x) \leq c \} \) is bounded, and \( D^2H(x)(h,h) > 0 \) for each \( x \in W \) and each non-zero vector \( h \).

An energy manifold \( S_c \) of the Hamiltonian \( H \) is said to be of contact type if there exists a one-form \( \lambda \) on \( S_c \) such that \( d\lambda = -j^*\omega \) and \( i_{X_H}(\lambda) \neq 0 \) hold on \( S_c \), where \( j : S_c \to \mathbb{R}^4 \) is the inclusion map.

Assume now that \( S_c \) is diffeomorphic to \( S^3 \), that it is of contact type, and that the contact structure is tight. The manifold \( S_c \) is said to be dynamically convex if for every periodic solution \( (x,T) \) of the Reeb vector field, we have \( \mu(x,T) \geq 3 \).

If \( S_c \) is equipped with the contact form \( \lambda_{0|S_c} \), encloses \( 0_{\mathbb{R}^4} \) and is strictly convex, then it is dynamically convex. The converse is not true.

The following result provides sufficient conditions for the existence of a disk-like surface of section. Given a closed 3-dimensional manifold and a flow \( \phi \) with no rest points, we say that a topologically embedded 2-dimensional disk \( D \) is a disk-like global
surface of section provided that: (i) the boundary \(\text{bd}(D)\) is a periodic orbit (called spanning orbit), (ii) the interior of the disk \(\text{int}(D)\) is a smooth manifold transverse to the flow, and (iii) every orbit, other than the spanning orbit, intersects \(\text{int}(D)\) in forward and backward time.

**Theorem 5.1** \([24]\). Assume that \(S_c\) is diffeomorphic to \(S^3\), is equipped with a tight contact structure, and is dynamically convex. Then there exits a global disk-like surface of section \(D\) and an associated global return map \(f : \text{int}(D) \rightarrow \text{int}(D)\) that is smoothly conjugated to an area preserving mapping of the open unit disk in \(\mathbb{R}^2\). The spanning orbit \(\chi\) of prime period \(T\) has Conley-Zehnder index \(\tilde{\mu}(\chi, T) = 3\).

We note that a generalization of this result to non-dynamically convex tight contact forms on the three-sphere appears in \([26]\).

6. Appendix B. Background on the scattering map.

Consider a flow \(\Phi : M \times \mathbb{R} \rightarrow M\) defined on a manifold \(M\) that possesses a normally hyperbolic invariant manifold \(\Lambda \subseteq M\).

As the stable and unstable manifolds of \(\Lambda\) are foliated by stable and unstable manifolds of points, respectively, for each \(x \in W^u(\Lambda)\) there exists a unique \(x^u \in \Lambda\) such that \(x \in W^u(x^u)\), and for each \(x \in W^s(\Lambda)\) there exists a unique \(x^s \in \Lambda\) such that \(x \in W^s(x^s)\). We define the wave maps \(\Omega^s : W^s(\Lambda) \rightarrow \Lambda\) by \(\Omega^s(x) = x^u\), and \(\Omega^u : W^u(\Lambda) \rightarrow \Lambda\) by \(\Omega^u(x) = x^s\). The maps \(\Omega^s\) and \(\Omega^u\) are \(C^\ell\)-smooth.

We now describe the scattering map, following \([14]\). Assume that \(W^u(\Lambda)\) has a transverse intersection with \(W^s(\Lambda)\) along a \(l\)-dimensional homoclinic manifold \(\Gamma\). The manifold \(\Gamma\) consists of a \((l-1)\)-dimensional family of trajectories asymptotic to \(\Lambda\) in both forward and backwards time. The transverse intersection of the hyperbolic invariant manifolds along \(\Gamma\) means that \(\Gamma \subseteq W^u(\Lambda) \cap W^s(\Lambda)\) and, for each \(x \in \Gamma\), we have

\[
\begin{align*}
T_xM &= T_xW^u(\Lambda) + T_xW^s(\Lambda), \\
T_x\Gamma &= T_xW^u(\Lambda) \cap T_xW^s(\Lambda).
\end{align*}
\]

Let us assume the additional condition that for each \(x \in \Gamma\) we have

\[
\begin{align*}
T_xW^s(\Lambda) &= T_xW^s(x^s) \oplus T_x(\Gamma), \\
T_xW^u(\Lambda) &= T_xW^u(x^u) \oplus T_x(\Gamma),
\end{align*}
\]

where \(x^u, x^s\) are the uniquely defined points in \(\Lambda\) corresponding to \(x\).

The restrictions \(\Omega^s_\Gamma, \Omega^u_\Gamma\) of \(\Omega^s, \Omega^u\), respectively, to \(\Gamma\) are local \(C^{\ell-1}\) diffeomorphisms. By restricting \(\Gamma\) even further, if necessary, we can ensure that \(\Omega^s_\Gamma, \Omega^u_\Gamma\) are \(C^{\ell-1}\)-diffeomorphisms. A homoclinic manifold \(\Gamma\) for which the corresponding restrictions of the wave maps are \(C^{\ell-1}\)-diffeomorphisms will be referred as a homoclinic channel.

**Definition 6.1.** Given a homoclinic channel \(\Gamma\), the scattering map associated to \(\Gamma\) is the \(C^{\ell-1}\)-diffeomorphism \(S^\Gamma = \Omega^s_\Gamma \circ (\Omega^u_\Gamma)^{-1}\) defined on the open subset \(U^u := \Omega^u_\Gamma(\Gamma)\) in \(\Lambda\) to the open subset \(U^s := \Omega^s_\Gamma(\Gamma)\) in \(\Lambda\).
Theorem 7.3. Let \( \Lambda \) be a collection of continuous maps on \( M \). Then \( W^s(T_1) \) has a transverse intersection with \( W^s(T_2) \) in \( M \) if and only if \( S(T_1) \) has a transverse intersection with \( T_2 \) in \( \Lambda \).

7. Appendix C. Correctly aligned windows.

We follow [41, 21, 19, 20, 12].

Definition 7.1. An \( (m_1, m_2) \)-window in an \( m \)-dimensional manifold \( M \), where \( m_1 + m_2 = m \), is a compact subset \( R \) of \( M \) together with a \( C^0 \)-parametrization given by a homeomorphism \( \rho \) from some open neighborhood \( U_R \) of \([0, 1]^{m_1} \times [0, 1]^{m_2} \subseteq \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \) to an open subset of \( M \), with \( R = \rho([0, 1]^{m_1} \times [0, 1]^{m_2}) \), and with a choice of an 'exit set'

\[
R^m = \rho([0, 1]^{m_1} \times \partial[0, 1]^{m_2})
\]

and of an 'entry set'

\[
R^e = \rho([0, 1]^{m_1} \times \partial[0, 1]^{m_2} \times [0, 1]^{m_2})
\]

Let \( f \) be a continuous map on \( M \) with \( f(\text{im}(\rho_1)) \subseteq \text{im}(\rho_2) \). Denote \( f_\rho = \rho_2^{-1} \circ f \circ \rho_1 \).

Definition 7.2. Let \( R_1 \) and \( R_2 \) be \( (m_1, m_2) \)-windows, and let \( \rho_1 \) and \( \rho_2 \) be the corresponding local parametrizations. We say that \( R_1 \) is correctly aligned with \( R_2 \) under \( f \) if the following conditions are satisfied:

(i) There exists a continuous homotopy \( h : [0, 1] \times U_{R_1} \rightarrow \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \), such that the following conditions hold true

\[
h([0, 1], \partial[0, 1]^{m_1} \times [0, 1]^{m_2} \times [0, 1]^{m_2}) \cap \partial([0, 1]^{m_1} \times [0, 1]^{m_2}) = \emptyset,
\]

\[
h([0, 1], [0, 1]^{m_1} \times [0, 1]^{m_2} \times [0, 1]^{m_2}) \cap \partial([0, 1]^{m_1} \times [0, 1]^{m_2}) = \emptyset,
\]

and

(ii) There exists \( y_0 \in [0, 1]^{m_2} \) such that the map \( A_{y_0} : \mathbb{R}^{m_1} \rightarrow \mathbb{R}^{m_1} \) defined by

\[
A_{y_0} (x) = \pi_{m_1} (h_1(x, y_0))
\]

satisfies

\[
A_{y_0} (\partial([0, 1]^{m_1})) \subseteq \mathbb{R}^{m_1} \setminus [0, 1]^{m_1},
\]

\[
\text{deg}(A_{y_0}, 0) \neq 0,
\]

where \( \pi_{m_1} : \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \rightarrow \mathbb{R}^{m_1} \) is the projection onto the first component, and\n
Theorem 7.3. Let \( \{R_i\}_{i \in \mathbb{Z}} \) be a collection of \( (m_1, m_2) \)-windows in \( M \), and let \( f_i \) be a collection of continuous maps on \( M \). If for each \( i \in \mathbb{Z} \), \( R_i \) is correctly aligned with \( R_{i+1} \) under \( f_i \), then there exists a point \( p \in R_0 \) such that

\[
(f_i \circ \cdots \circ f_0)(p) \in R_{i+1}, \text{ for all } i \in \mathbb{Z}.
\]

Moreover, under the above conditions, and assuming that for some \( k > 0 \) we have \( R_i = R_{(i \mod k)} \) and \( f_i = f_{(i \mod k)} \) for all \( i \in \mathbb{Z} \), then there exists a point \( p \) as above that is periodic in the sense

\[
(f_{k-1} \circ \cdots \circ f_0)(p) = p.
\]
Assume that \( f : M \to M \) is a diffeomorphism on a manifold \( M \), \( \Lambda \subseteq M \) is an \( l \)-dimensional normally hyperbolic invariant manifold, and \( S : U \to V \) is a scattering map associated to some homoclinic channel \( \Gamma \).

**Lemma 7.4.** Let \( \{R_i, R'_i\}_{i \in \mathbb{Z}} \) be a bi-infinite sequence of \( l \)-dimensional windows contained in \( \Lambda \). Assume that the following properties hold for all \( i \in \mathbb{Z} \):

1. \( R_i \subseteq U \) and \( R'_i \subseteq V \).
2. \( R_i \) is correctly aligned with \( R'_{i+1} \) under the scattering map \( S \).
3. For each pair \( R'_{i+1}, R_{i+1} \) and for each \( L > 0 \) there exists \( L' > L \) such that \( R'_{i+1} \) is correctly aligned with \( R_{i+1} \) under the iterate \( f^L_{|\Lambda} \) of the restriction \( f_{|\Lambda} \) of \( f \) to \( \Lambda \).

Fix any bi-infinite sequence of positive real numbers \( \{\varepsilon_i\}_{i \in \mathbb{Z}} \). Then there exist an orbit \( (f^n(z))_{n \in \mathbb{Z}} \) of some point \( z \in M \), an increasing sequence of integers \( (n_i)_{i \in \mathbb{Z}} \), and some sequences of positive integers \( \{N_i\}_{i \in \mathbb{Z}}, \{K_i\}_{i \in \mathbb{Z}}, \{M_i\}_{i \in \mathbb{Z}} \), such that, for all \( i \in \mathbb{Z} \):

\[
\begin{align*}
&d(f^n(z), \Gamma) < \varepsilon_i, \\
&d(f^{n_i+N_i+1}(z), f_{|\Lambda}^{N_i+1}(R'_{i+1})) < \varepsilon_{i+1}, \\
&d(f^{n_i-M_i}(z), f_{|\Lambda}^{-M_i}(R_i)) < \varepsilon_i, \\
&n_{i+1} = n_i + N_{i+1} + K_{i+1} + M_{i+1}.
\end{align*}
\]

8. **Appendix D. Topological method for the diffusion problem.**

In this section we recall the main result from [20] Assume the following: (C1) \( M \) is a \( n \)-dimensional \( C^r \)-differentiable Riemannian manifold, and \( f : M \to M \) is a \( C^r \)-smooth map, for some \( r \geq 2 \).

(C2) There exists a submanifold \( \Lambda \) in \( M \), diffeomorphic to an annulus \( \Lambda \simeq T^1 \times [0, 1] \). We assume that \( f \) is normally hyperbolic to \( \Lambda \) in \( M \). Denote the dimensions of the stable and unstable manifolds of a point \( x \in \Lambda \) by \( \dim(W^s(x)) = n_s \) and \( \dim(W^u(x)) = n_u \). Then, \( n = 2 + n_s + n_u \).

(C3) On \( \Lambda \) there is a system of angle-action coordinates \( (\phi, I) \), with \( \phi \in T^1 \) and \( I \in [0, 1] \). The restriction \( f_{|\Lambda} \) of \( f \) to \( \Lambda \) is a boundary component preserving, area preserving, monotone twist map, with respect to the angle-action coordinates \( (\phi, I) \).

(C4) The stable and unstable manifolds of \( \Lambda \), \( W^s(\Lambda) \) and \( W^u(\Lambda) \), have a differentiability transverse intersection along a 2-dimensional homoclinic channel \( \Gamma \). We assume that the scattering map \( S : U^- \to U^+ \) associated to \( \Gamma \) is well defined.

(C5) There exists a bi-infinite sequence of Lipschitz primary invariant tori \( \{T_i\}_{i \in \mathbb{Z}} \) in \( \Lambda \), and a bi-infinite, increasing sequence of integers \( \{i_k\}_{k \in \mathbb{Z}} \) with the following properties:

1. Each torus \( T_i \) intersects the domain \( U^- \) and the range \( U^+ \) of the scattering map \( S \) associated to \( \Gamma \).
2. For each \( i \in \{i_k + 1, \ldots, i_{k+1} - 1\} \), the image of \( T_i \cap U^- \) under the scattering map \( S \) is topologically transverse to \( T_{i+1} \).
3. For each torus \( T_i \) with \( i \in \{i_k + 2, \ldots, i_{k+1} - 1\} \), the restriction of \( f \) to \( T_i \) is topologically transitive.
(iv) Each torus $T_i$ with $i \in \{i_k + 2, \ldots, i_{k+1} - 1\}$, can be $C^0$-approximated from both sides by other primary invariant tori from $\Lambda$

We will refer to a finite sequence $\{T_i\}_{i=i_k+1, \ldots, i_{k+1}}$ as above as a transition chain of tori.

(C6) The region in $\Lambda$ between $T_{i_k}$ and $T_{i_{k+1}}$ is a BZI.

(C7) Inside each region between $T_{i_k}$ and $T_{i_{k+1}}$ there is prescribed a finite collection of Aubry-Mather sets $\{\Sigma_{\rho_s^k}, \Sigma_{\rho_2^k}, \ldots, \Sigma_{\rho_s^k}\}$, where $s_k \geq 1$, and $\rho_s^k$ denotes the rotation number of $\Sigma_{\rho_s^k}$. These Aubry-Mather sets are assumed to be vertically ordered, relative to the $I$-coordinate on the annulus.

Instead of (C6) we can consider the following condition:

(C6′) The region $\Lambda_k$ in $\Lambda$ between $T_{i_k}$ and $T_{i_{k+1}}$ contains finitely many invariant primary tori $\{\Upsilon_{h_1^k}, \ldots, \Upsilon_{h_{l_k}^k}\}$, where $l_k \geq 1$, satisfying the following properties:

(i) Each $\Upsilon_{h_j^k}$ falls in one of the following two cases:

(a) $\Upsilon_{h_j^k}$ is an isolated invariant primary torus.

(b) There exists a hyperbolic periodic orbit in $\Lambda$ such that its stable and unstable manifolds coincide.

(ii) The invariant primary tori $\{\Upsilon_{h_1^k}, \ldots, \Upsilon_{h_{l_k}^k}\}$ are vertically ordered, relative to the $I$-coordinate on the annulus.

(iii) For each $\Upsilon_{h_j^k}$, $j = 1, \ldots, l_k$, the inverse image $S^{-1}(\Upsilon_{h_j^k} \cap U^+)$ forms with $\Upsilon_{h_j^k}$ a topological disk $D_{h_j^k} \subseteq U^+$ below $\Upsilon_{h_j^k}$, such that $S(D_{h_j^k}) \subseteq U^+$ is a topological disk above $\Upsilon_{h_j^k}$, which is bounded by $\Upsilon_{h_j^k}$ and $S(\Upsilon_{h_j^k} \cap U^-)$.

**Theorem 8.1.** Let $f : M \rightarrow M$ be a $C^r$-differentiable map, and let $(T_i)_{i \in \mathbb{Z}}$ be a sequence of invariant primary tori in $\Lambda$, satisfying the properties (C1)–(C6), or (C1)–(C5) and (C6′), from above. Then for each sequence $(\epsilon_i)_{i \in \mathbb{Z}}$ of positive real numbers, there exist a point $z \in M$ and a bi-infinite increasing sequence of integers $(N_i)_{i \in \mathbb{Z}}$ such that

\begin{equation}
\label{eq:8.1}
d(f^{N_i}(z), T_i) < \epsilon_i, \text{ for all } i \in \mathbb{Z}.
\end{equation}

In addition, if condition (C7) is assumed, and some positive integers $\{n_s^k\}_{s=1, \ldots, s_k}$, $k \in \mathbb{Z}$ are given, then there exist $z \in M$ and $(N_i)_{i \in \mathbb{Z}}$ as in (8.1), and positive integers $\{m_s^k\}_{s=1, \ldots, s_k}$, $k \in \mathbb{Z}$, such that, for each $k$ and each $s \in \{1, \ldots, s_k\}$, we have

\begin{equation}
\label{eq:8.2}
\pi_\phi(f^j(w_s^k)) < \pi_\phi(f^j(z)) < \pi_\phi(f^j(\bar{w}_s^k)),
\end{equation}

for some $w_s^k, \bar{w}_s^k \in \Sigma_{\omega_s^k}$ and for all $j$ with

\[ N_{i_k} + \sum_{t=0}^{s-1} n_t^k + \sum_{t=0}^{s-1} m_t^k \leq j \leq N_{i_k} + \sum_{t=0}^{s} n_t^k + \sum_{t=0}^{s-1} m_t^k. \]

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