ON THE DIOPHANTINE EQUATION IN THE FORM THAT A SUM OF CUBES EQUALS A SUM OF QUINTICS

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Abstract. In this paper, the elliptic curves theory is used for solving the Diophantine equations

\[ a(X_1^5 + X_2^5) + \sum_{i=0}^{n} a_iX_i^5 = b(Y_1^3 + Y_2^3) + \sum_{i=0}^{m} b_iY_i^3, \]

where \( n, m \in \mathbb{N} \cup \{0\} \), and, \( a, b, a_i, b_i \), are fixed arbitrary rational numbers. We solve the Diophantine equation for some values of \( n, m, a, b, a_i, b_i \), and obtain nontrivial integer solutions for each case. By our method, we may find infinitely many nontrivial integer solutions for the Diophantine equation for every \( n, m, a, b, a_i, b_i \), and show among the other things that how sums of some 5th powers can be written as sums of some cubics.

1. Introduction

Euler conjectured in 1969 that the Diophantine equation \( A^4 + B^4 + C^4 = D^4 \), or more generally \( A_1^N + A_2^N + \cdots + A_{N-1}^N = A_N^N, (N \geq 4) \), has no solution in positive integers (see [1]). Nearly two centuries later, a computer search (see [6]) found the first counterexample to the general conjecture (for \( N = 5 \)):

\[ 27^5 + 84^5 + 110^5 + 133^5 = 144^5. \]

In 1986, Noam Elkies, by elliptic curves, found counterexamples for the \( N = 4 \) case (see [2]). His smallest counterexample was:

\[ 2682440^4 + 15365639^4 + 18796760^4 = 20615673^4. \]

The authors in three different papers, used elliptic curves to solve three Diophantine equations

\[ \sum_{i=1}^{n} a_i x_i^4 = \sum_{j=1}^{n} a_j y_j^4, \]

where \( a_i, n \geq 3 \), are fixed arbitrary integers.
\begin{equation}
X^4 + Y^4 = 2U^4 + \sum_{i=1}^{n} T_i U_i^{\alpha_i},
\end{equation}

where \(n, \alpha_i \in \mathbb{N}\), and \(T_i\), are appropriate fixed arbitrary rational numbers, and,

\begin{equation}
\sum_{i=1}^{n} a_i x_i^6 + \sum_{i=1}^{m} b_i y_i^3 = \sum_{i=1}^{n} a_i X_i^6 \pm \sum_{i=1}^{m} b_i Y_i^3,
\end{equation}

where \(n, m \geq 1\) and \(a_i, b_i\), are fixed arbitrary nonzero integers. (see [3], [4], [5])

In this paper, we are interested in the study of the Diophantine equation:

\begin{equation}
a(X_1^{r_5} + X_2^{r_5}) + \sum_{i=0}^{n} a_i X_i^{5} = b(Y_1^{r_3} + Y_2^{r_3}) + \sum_{i=0}^{m} b_i Y_i^{3},
\end{equation}

where \(n, m \in \mathbb{N} \cup \{0\}\), and, \(a, b \neq 0, a_i, b_i\), are fixed arbitrary rational numbers.

2. OUR MAIN THEOREM

Our main result is the following:

Main Theorem 2.1. Consider the Diophantine equation (1.4).

Let \(Y^2 = X^3 + FX^2 + GX + H\), be an elliptic curve in which the coefficients \(F, G,\) and \(H\), are all functions of \(a, b, a_i, b_i\) and the other rational parameters \(\alpha_i, \beta_i, t, x_1, v\), yet to be found later. If the elliptic curve has positive rank, depending on the values of \(\alpha_i, \beta_i, x_1\), (This is done by choosing appropriate arbitrary values for \(\alpha_i(0 \leq i \leq n), \beta_i(0 \leq i \leq m), x_1\)), the Diophantine equation has infinitely many integer solutions.

Proof. We solve the Diophantine equation (1.4), if we find rational solutions for the above Diophantine equation, then by canceling the denominators of \(X_1', X_2', X_i', Y_1', Y_2', Y_i\), and by multiplying the both sides of Diophantine equation by the appropriate value of \(M\), we may obtain integer solutions.
for the Diophantine equation.

Note that if \( (X_1', X_2', X_0, \cdots, X_n, Y_1', Y_2', Y_0, \cdots, Y_m), \)
is a rational solution for the Diophantine equation \( (1.1) \), then for every arbitrary rational number \( \mu \),
\( (\mu^3 X_1', \mu^3 X_2', \mu^3 X_0, \cdots, \mu^3 X_n, \mu^5 Y_1', \mu^5 Y_2', \mu^5 Y_0, \cdots, \mu^5 Y_m) \)
is a solution for \( (1.1) \), too.

Let:
\[
X_1' = t + x_1, \quad X_2' = t - x_1, \quad Y_1' = t + v, \quad Y_2' = t - v, \quad X_i = \alpha_i t, \quad Y_i = \beta_i t,
\]
where all variables are rational numbers. By substituting these variables in the above Diophantine equation, we get:

\[
(2.1) \quad a(2t^5 + 10x_1^4 t + 20x_1^2 t^3) + \sum_{i=0}^n a_i \alpha_i^5 t^5 = b(2t^3 + 6tv^2) + \sum_{i=0}^m b_i \beta_i^3 t^3.
\]

Then after some simplifications and clearing the case of \( t = 0 \), we obtain:

\[
(2.2) \quad v^2 = \left( \frac{2a + \sum_{i=0}^n a_i \alpha_i^5}{6b} \right) t^4 + \left( \frac{20ax_1^2 - 2b - \sum_{i=0}^m b_i \beta_i^3}{6b} \right) t^2 + \left( \frac{5a}{3b} \right) x_1^4.
\]

Now by choosing appropriate arbitrary values for \( \alpha_i \) (0 ≤ i ≤ n), \( \beta_i \) (0 ≤ i ≤ m), \( x_1 \), such that the rank of the quartic elliptic curve (1.4) to be positive, and by calculating \( X_1', X_2', X_i, Y_1', Y_2', Y_i \), from the relations
\[
X_1' = t + x_1, \quad X_2' = t - x_1, \quad X_i = \alpha_i t, \quad Y_1' = t + v, \quad Y_2' = t - v, \quad Y_i = \beta_i t,
\]
some simplifications and canceling the denominators of \( X_1', X_2', X_i, Y_1', Y_2', Y_i \), we obtain infinitely many integer solutions for the Diophantine equation.

The proof of the theorem is complete.

\( \square \)

**Remark 2.1.** If in the quartic elliptic curve (2.2),
\[
(2.3) \quad Q := \left( \frac{5a}{3b} \right) x_1^4,
\]
to be square (It is right for appropriate values of \( a, b \)), say \( q^2 \), we may use the following lemma for transforming this quartic to a cubic elliptic curve of the form
\[ y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6, \text{ where } a_i \in \mathbb{Q}. \]

Then we solve the cubic elliptic curve just obtained of the rank \( \geq 1 \), and get infinitely many solutions for the Diophantine equation (1.4).

Generally, it is not necessary for \( \mathbb{Q} \) to be square as we may transform the quartic (2.2) to a new quartic in which the constant number is square if the rank of the quartic (2.2) is positive. The only important thing is that the rank of the quartic elliptic curve (2.2), be positive for getting infinitely many solutions for (1.4). See the example 1.

**Lemma 2.2.** Let \( K \) be a field of characteristic not equal to 2. Consider the equation
\[ v^2 = au^4 + bu^3 + cu^2 + du + q^2, \text{ with } a, b, c, d \in K. \]

Let \( x = \frac{2q(v+q)+du}{u^2}, \ y = \frac{4q^2(v+q)+2q(du+cu^2)-(d^2u^2)}{u^3}. \)

Define \( a_1 = \frac{d}{q}, \ a_2 = c - \left( \frac{d^2}{4q^2} \right), \ a_3 = 2qb, \ a_4 = -4q^2a, \ a_6 = a_2a_4. \)

Then \( y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6. \)

The inverse transformation is \( u = \frac{2q(x+c)-(\frac{d^2}{2q})}{y}, \ v = -q + \frac{u(xz-d)}{2q}. \)

The point \((u, v) = (0, q)\) corresponds to the point \((x, y) = \infty\) and \((u, v) = (0, q)\) corresponds to \((x, y) = (a_2, a_1a_2 - a_3). \) (see [8])

**Remark 2.3.** If in the Diophantine equation (1.4), \( n, m \) to be odd, \( a_{2k} = a_{2k+1}, \ (0 \leq k \leq \frac{n-1}{2}), \) and \( b_{2k} = b_{2k+1}, \ (0 \leq k \leq \frac{m-1}{2}), \) we may by second beautiful method transform the Diophantine equation (1.4) to another quadrant elliptic curve and then solve it for getting infinitely many nontrivial solutions for the Diophantine equation.

Now we have
\[
a(X_1^5 + X_2^5) + a_0(X_0^5 + X_1^5) + a_2(X_2^5 + X_3^5) + \cdots + a_{n-1}(X_{n-1}^5 + X_n^5) = b(Y_1^3 + Y_2^3) + b_0(Y_0^3 + Y_1^3) + b_2(Y_2^3 + Y_3^3) + \cdots + b_{m-1}(Y_{m-1}^3 + Y_m^3). \]
Then (after renaming the coefficients, the number of terms, and the variables) the above Diophantine equation (or (1.4)) is in the form

\[(2.4) \quad \sum_{i=0}^{N} A_i (Z_i^5 + Z_{i+1}^5) = \sum_{i=0}^{M} B_i (W_i^3 + W_{i+1}^3).\]

Let \(Z_i = t + x_i\), \(Z_{i+1}^\prime = t - x_i\), \(W_i = t + y_i\), \(W_{i+1}^\prime = t - y_i\) \((i \geq 0)\).

By substituting these variables in the above Diophantine equation, we get:

\[(2.5) \quad \sum_{i=0}^{N} A_i (2t^5 + 10x_i^4t + 20x_i^2t^3) = \sum_{i=0}^{M} B_i (2t^3 + 6t^2y_i^2).\]

Then after some simplifications and clearing the case of \(t = 0\), and letting \(y_0 = v\), we obtain:

\[(2.6) \quad v^2 = (\frac{\sum_{i=0}^{N} A_i}{3B_0})t^4 + (\frac{10\sum_{i=0}^{N} A_i x_i^2 - \sum_{i=0}^{M} B_i}{3B_0})t^2 + (\frac{5\sum_{i=0}^{N} A_i x_i^4 - 3\sum_{i=1}^{M} B_i y_i^2}{3B_0}).\]

Now by choosing appropriate values for \(x_i\) \((0 \leq i \leq N)\), \(y_i\) \((1 \leq i \leq M)\), such that the rank of the quartic elliptic curve \((2.6)\) to be positive, we obtain infinitely many solutions for the Diophantine equation \((1.4)\). The proof of the theorem is complete.

### 3. Application to Examples

Now we are going to solve some couple of examples:

**3.1. Example:** \(X_1^5 + X_2^5 + X_3^5 = Y_1^3 + Y_2^3 + Y_3^3\).

i.e., the sum of 3 fifth powers can be written as the sum of 3 cubics.

Let: \(X_1 = t + x_1\), \(X_2 = t - x_1\), \(X_3 = at\), \(Y_1 = t + v\), \(Y_2 = t - v\), \(Y_3 = \beta t\).
Then we get:

\[
(3.1) \quad v^2 = \frac{2 + \alpha^5}{6} t^4 + \frac{20x_1^2 - 2 - \beta^3}{6} t^2 + \left(\frac{5}{3}\right)x_1^4.
\]

Note that \(\left(\frac{5}{3}\right)x_1^4\), is not a square. Then we may not use from the above theorem for transforming the above quartic to a cubic elliptic curve, but we do this work by another beautiful method. Let us take \(x_1 = 1, \alpha = \beta = 2\). Then the quartic \((3.1)\) becomes

\[
(3.2) \quad v^2 = \frac{17}{3} t^4 + \frac{5}{3} t^2 + \frac{5}{3}.
\]

By searching, we see that the above quartic has two rational points \(P_1 = (1, 3)\), and \(P_2 = (7, 117)\), among others. Let us put \(T = t - 1\). Then we get

\[
(3.3) \quad v^2 = \frac{17}{3} T^4 + \frac{68}{3} T^3 + \frac{107}{3} T^2 + 26T + 9.
\]

Now with the inverse transformation

\[
(3.4) \quad T = \frac{6(X + \frac{107}{3}) - \frac{26^2}{Y}}{Y},
\]

and

\[
(3.5) \quad v = -3 + \frac{T(XT - 26)}{6},
\]

the quartic \((3.3)\), maps to the cubic elliptic curve

\[
(3.6) \quad Y^2 + \frac{26}{3} XY + 136Y = X^3 + \frac{152}{9} X^2 - 204X - \frac{10336}{3}.
\]

The rank of this elliptic curve is 2 and its generator are the points \(P_1 = (X', Y') = (-\frac{152}{9}, \frac{269}{27})\), and \(P_2 = (X'', Y'') = (-\frac{44}{3}, \frac{20}{9})\).
To square the left hand of (3.6), let us put $M = Y + \frac{13}{3}X + 68$. Then the cubic elliptic curve (3.6) transforms to the Weierstrass form

$$M^2 = X^3 + \frac{107}{3}X^2 + \frac{1156}{3}X + \frac{3536}{3}.$$  

The generators for this new cubic elliptic curve are the two points $G_1 = (X', M') = (\frac{-44}{3}, \frac{20}{3})$, and $G_2 = (X'', M'') = (\frac{-110}{9}, \frac{140}{27})$. Thus we conclude that we could transform the main quartic (3.2) to the cubic elliptic curve (3.7) of rank equal to 2.

Because of this, the above cubic elliptic curve has infinitely many rational points and we may obtain infinitely many solutions for the Diophantine equation too.

Since $G_1 = (X', M') = (\frac{-44}{3}, \frac{20}{3})$, we get $(t, v) = (7, -117)$, that is on the (3.2), by calculating $X_i, Y_i$, from the above relations and after some simplifications and canceling the denominators of $X_i, Y_i$, we obtain a solution for the Diophantine equation

$$X_1^5 + X_2^5 + X_3^5 = Y_1^3 + Y_2^3 + Y_3^3$$

as

$$8^5 + 6^5 + 14^5 = (-110)^3 + 124^3 + 14^3.$$ 

It is interesting that we see, $8 + 6 + 14 = (-110) + 124 + 14$, too.

Also we have $2G_2 = (\frac{373}{36}, \frac{-21721}{216})$.

By using this new point $2G_2 = (\frac{373}{36}, \frac{-21721}{216})$, we get $(t, v) = (\frac{11}{17}, \frac{2943}{2209})$, on the (3.2), by calculating $X_i, Y_i$, from the above relations and after some simplifications and canceling the denominators of $X_i, Y_i$, we obtain another solution for the Diophantine equation as

$$128122^5 + (-79524)^5 + 48598^5 = 359227580^3 + (-251874598)^3 + 107352982^3.$$ 

By choosing the other points on the elliptic curve such as $nG_1, nG_2$ ($n = 3, 4, \cdots$) we obtain infinitely many solutions for the Diophantine equation.
3.2. Example: \(5(X_1^5 + X_2^5) = 3(Y_1^3 + Y_2^3)\).

Let: \(X_1' = t + x_1, X_2' = t - x_1, Y_1' = t + v, Y_2' = t - v\).

Then we get:

\[
(3.8) \quad v^2 = \frac{5}{9} t^4 + \frac{50}{9} x_1^2 - \frac{3}{9} t^2 + \left(\frac{25}{9}\right)x_1^4.
\]

Note that for every arbitrary rational number \(x_1\), \(\left(\frac{25}{9}\right)x_1^4\), is a square, then by using the above theorem, we may transform the above quartic to a cubic elliptic curve. Now, we must choose appropriate value for \(x_1\), such that the rank of the above quartic to be positive. Let \(x_1 = 1\).

Then the quartic [3.8] becomes

\[
(3.9) \quad v^2 = \frac{5}{9} t^4 + \frac{47}{9} t^2 + \left(\frac{25}{9}\right)x_1^4.
\]

With the inverse transformation

\[
(3.10) \quad t = \frac{10}{3} \left(\frac{x + \frac{47}{9}}{Y}\right),
\]

and

\[
(3.11) \quad v = -\frac{5}{3} + \frac{t^2 X}{\frac{10}{3}},
\]

the quartic [3.9], maps to the cubic elliptic curve

\[
(3.12) \quad Y^2 = X^3 + \frac{47}{9} X^2 - \frac{500}{81} X - \frac{23500}{729}.
\]

The rank of this elliptic curve is 1 and its generator is the point

\(P = (X, Y) = \left(\frac{-609566}{164025}, \frac{-225298052}{66430125}\right)\). Because of this, the above elliptic curve has infinitely many rational points and we may obtain infinitely many solutions for the Diophantine equation too.

Since \(P = (X, Y) = \left(\frac{-609566}{164025}, \frac{-225298052}{66430125}\right)\), we get \((t, v) = \left(\frac{-335475}{226658}, \frac{-633289965055}{154121546892}\right)\),
by calculating $X'_1$, $X'_2$, $Y'_1$, $Y'_2$, from the above relations and after some simplifications and canceling the denominators of $X'_1$, $X'_2$, $Y'_1$, $Y'_2$, we obtain a solution for the Diophantine equation

$$5.(X'_1^5 + X'_2^5) = 3.(Y'_1^3 + Y'_2^3)$$

as

$$X'_1 = -1509393993133320876,$$

$$X'_2 = -779731267671365724,$$

$$Y'_1 = -812465974773035511128800250760,$$

$$Y'_2 = 382156766244967634925328109160.$$

By choosing the other points on the elliptic curve such as $nP$, ($n = 2, 3, \cdots$) we obtain infinitely many solutions for the Diophantine equation.

3.3. Example: $n.(X_1^5 + X_2^5 + X_3^5 + X_4^5) = m.(Y_1^3 + Y_2^3 + Y_3^3 + Y_4^3)$.

By letting, $X_1 = t + x_1$, $X_2 = t - x_1$, $X_3 = t + x_2$, $X_4 = t - x_2$, $Y_1 = t + v$, $Y_2 = t - v$, $Y_3 = t + y_2$, $Y_4 = t - y_2$,

we get:

$$v^2 = \left(\frac{2n}{3m}\right)t^4 + \left(\frac{10nx_1^2 + 10nx_2^2 - 2m}{3m}\right)t^2 + \left(\frac{5nx_1^4 + 5nx_2^4 - 3my_2^2}{3m}\right).$$

We see that if

$$\left(\frac{5nx_1^4 + 5nx_2^4 - 3my_2^2}{3m}\right)$$

is a square, say $q^2$, then we may transform the above quartic to a cubic elliptic curve. For every arbitrary values of $n, m$, this is done by choosing appropriate values of $x_1, x_2, y_2$. 
As an example, if \( m = 85, n = 6 \), we may set \( x_1 = 1, x_2 = 2, y_2 = 1, (q = 1) \).

Then (3.14) becomes

\[
(3.15) \quad v^2 = \frac{4}{85} t^4 + \frac{26}{51} t^2 + 1.
\]

With the inverse transformation

\[
(3.16) \quad t = \frac{2(X + \frac{26}{51})}{Y},
\]

and

\[
(3.17) \quad v = -1 + \frac{t^2 X}{2},
\]

the corresponding cubic elliptic curve is

\[
(3.18) \quad Y^2 = X^3 + \frac{26}{51} X^2 - \frac{16}{85} X - \frac{416}{4335}.
\]

The rank of this elliptic curve is 2 and its generators are the points \( P_1 = (X', Y') = (\frac{2}{3}, \frac{28}{51}) \), and \( P_2 = (X'', Y'') = (\frac{20777}{21075}, -\frac{1908281}{1842375}) \).

Because of this, the above elliptic curve has infinitely many rational points and we may obtain infinitely many solutions for the Diophantine equation too.

By using the point \( P_1 \), we get \((t, v) = (\frac{30}{7}, \frac{251}{40})\), by calculating \( X_i, Y_i \), from the above relations and after some simplifications and canceling the denominators of \( X_i, Y_i \), we obtain a solution for the Diophantine equation as

\[
6.(1813^5 + 1127^5 + 2156^5 + 784^5) = 85.(158123^3 + (-14063)^3 + 88837^3 + 55223^3).
\]

By choosing the other points on the elliptic curve such as
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By taking $m = 17, n = 3$, in the (3.13), we may choose appropriate values $x_1 = 1, x_2 = 2, y_2 = 1, (q = 2)$.

Then (3.13) becomes

\[(3.19) \quad v^2 = \frac{2}{17} t^4 + \frac{116}{51} t^2 + 4.\]

With the inverse transformation

\[(3.20) \quad t = \frac{4(X + \frac{116}{51})}{Y},\]

and

\[(3.21) \quad v = -2 + \frac{t^2 X}{4},\]

the corresponding cubic elliptic curve is

\[(3.22) \quad Y^2 = X^3 + \frac{116}{51} X^2 - \frac{32}{17} X - \frac{3712}{867}.\]

The rank of this elliptic curve is $1$ and its generator is the point $P = (X, Y) = (4, \frac{160}{17})$. Because of this, the above elliptic curve has infinitely many rational points and we may obtain infinitely many solutions for the Diophantine equation too.

Since $P = (X, Y) = (4, \frac{160}{17})$, we get $(t, v) = (\frac{8}{3}, \frac{46}{9})$, by calculating $X_i, Y_i$ from the above relations and after some simplifications and canceling the denominators of $X_i, Y_i$, we obtain a solution for the Diophantine equation as

\[3.(99^5 + 45^5 + 126^5 + 18^5) = 17.(1890^3 + (-594)^3 + 891^3 + 405^3).\]

By choosing the other points on the elliptic curve such as $nP (n = 2, 3, \cdots)$ we obtain infinitely many solutions for the Diophantine equation.
The Sage software has been used for calculating the rank of the elliptic curves. (see [7])

References

[1] L. E. DICSON, History of the Theory of Numbers, Vol. II: Diophantine Analysis, G. E. STECHERT. Co., New York, (1934).
[2] N. ELKIES, “On $A^4 + B^4 + C^4 = D^4$”. Mathematics of Computation. 51(184) : 825835, 61(1987).
[3] F. IZADI and M. BAGHALAGHDAM, On the Diophantine equation $\sum_{i=1}^{n} a_i x_i^4 = \sum_{j=1}^{n} a_j y_j^4$, submitted, (2016).
[4] F. IZADI and M. BAGHALAGHDAM, On the Diophantine equation $X^4 + Y^4 = 2U^4 + \sum_{i=1}^{n} T_i U_i^{a_i}$, submitted, (2016).
[5] F. IZADI and M. BAGHALAGHDAM, On the Diophantine equations $\sum_{i=1}^{n} a_i x_i^6 + \sum_{i=1}^{m} b_i y_i^3 = \sum_{i=1}^{n} a_i X_i^6 \pm \sum_{i=1}^{m} b_i Y_i^3$, submitted, (2016).
[6] L. J. LANDER and T. R. PARKIN, “Counterexamples to Euler’s conjecture on sums of like powers,” Bull. Amer. Math. Soc, Vol.72, p.1079, (1966)
[7] SAGE software, available from http://sagemath.org
[8] L. C. WASHINGTON, Elliptic Curves: Number Theory and Cryptography, Chapman-Hall, (2008).

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