Monotonicity properties and bounds involving the two-parameter generalized Grötzsch ring function

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Abstract
In the article, we present several new monotonicity properties and bounds involving the generalized Grötzsch ring functions \( \mu_{a,b} \) in the theory of Ramanujan’s generalized modular equation for \( 0 < a, b < 1 \). Our results are the variants and extensions of some previously known results.

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1 Introduction
Given \( x \in (-1,1) \) and real numbers \( a, b, \) and \( c \) with \( c \neq 0, -1, -2, \ldots \), the Gaussian hypergeometric function \( F(a, b; c; x) [1–18] \) is defined by

\[
F(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!},
\]

where \( (a)_0 = 1 \) for \( a \neq 0 \) and \( (a)_n = a(a+1)(a+2) \cdots (a+n-1) \) for \( n = 1, 2, \ldots \). \( F(a, b; c; x) \) is said to be zero-balanced if \( c = a + b \). If \( x \to 1 \), then the following asymptotic formulas

\[
\begin{align*}
F(a, b; c; 1) &= \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)}, \quad a + b < c, \\
B(a, b) F(a, b; c; x) + \log(1-x) &= R(a, b) + O((1-x) \log(1-x)), \quad a + b = c, \\
F(a, b; c; x) &= (1-x)^{c-a-b} F(c-a, c-b; c; x), \quad a + b > c,
\end{align*}
\]

can be found in the literature [19, Theorems 1.19 and 1.48], where \( \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt \) [20–26] and \( B(p, q) = [\Gamma(p) \Gamma(q)] / \Gamma(p + q) \) [27–30] are respectively the classical Euler gamma and beta functions, and

\[
R(a, b) = -\psi(a) - \psi(b) - 2 \gamma, \quad R\left(\frac{1}{2}, \frac{1}{2}\right) = \log 16, \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}.
\]
and
\[ \gamma = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log n \right) = 0.5772156649 \ldots \]
is the Euler–Mascheroni constant [31–33].

Ramanujan’s generalized modular equation with order (or degree) \( p > 0 \) is given by
\[ \frac{F(a, b; c; 1 - s^2)}{F(a, b; c; s^2)} = p \frac{F(a, b; c; 1 - r^2)}{F(a, b; c; r^2)}, \quad 0 < r < 1. \tag{1.3} \]

It is well known that equation (1.3) has a unique solution for \( s \) if \( a, b, c > 0 \) with \( a + b \geq c \) [34, Lemma 4.5].

The two-parameter generalized Grötzsch ring function is defined by
\[ \mu_{a, b}(r) = \frac{B(a, b)}{2} \frac{F(a, b; (a + b + 1)/2; 1 - r^2)}{F(a, b; (a + b + 1)/2; 1 - s^2)}, \quad r \in (0, 1) \tag{1.4} \]
if \( a + b \geq 1 \).

Our interest is to focus on \( c = (a + b + 1)/2 \), which makes the derivative formula of the two-parameter generalized Grötzsch ring function defined by (1.4) simpler.

Let \( 0 < a, b < 1 \) with \( a + b \geq 1 \) and \( r \in (0, 1) \). Then the two-parameter generalized elliptic integrals of first and second kinds [34, (1.6)–(1.8)] are defined by
\[ K = K_{a, b} = K_{a, b}(r) = \frac{B(a, b)}{2} F \left( a, b; \frac{a + b + 1}{2}; r^2 \right), \tag{1.5} \]
\[ E = E_{a, b} = E_{a, b}(r) = \frac{B(a, b)}{2} F \left( a - 1, b; \frac{a + b + 1}{2}; r^2 \right), \tag{1.6} \]
\[ K' = K'_{a, b} = K'_{a, b}(r'), \quad E' = E'_{a, b} = E'_{a, b}(r'), \tag{1.7} \]
where and in what follows \( r' = \sqrt{1 - r^2} \). Moreover, it follows from (1.2) that
\[ K_{a, b}(0^+) = E_{a, b}(0^+) = \frac{B(a, b)}{2}, \]
\[ K_{a, b}(1^-) = \infty, \quad E_{a, b}(1^-) = \frac{B(a, b)B((a + b + 1)/2, (3 - a - b)/2)}{2B((b - a + 3)/2, (a - b + 1)/2)}. \]

In this paper, we study the two-parameter generalized Grötzsch ring function \( \mu_{a, b}(r) \) for \( a, b \in (0, 1) \), as well as the related functions \( K_{a, b}, E_{a, b}, \) and
\[ m_{a, b}(r) = \frac{2}{B(a, b)} r^2 K_{a, b} K'_{a, b} \quad r \in (0, 1). \tag{1.8} \]

The so-called Legendre \( M \)-function introduced in [35] can be used to study the derivative of \( m_{a, b}(r) \) and satisfies the formula
\[ \left[ \frac{B(a, b)}{2} \right]^2 M(r^2) = \frac{a + b - 1}{2} K K' + \frac{b - a + 1}{2} (K E' + K' E - K K'). \tag{1.9} \]
for \( r \in (0, 1) \). Furthermore, \( \mathcal{M}(r) \) can be rewritten as

\[
\mathcal{M}(r) = \frac{\Gamma\left((a + b + 1)/2\right) [r(1 - r)]^{1-a-b}/2}{\Gamma(a) \Gamma(b)},
\]

\hspace{1cm} (1.10)

and \( \mathcal{M}(r) \) becomes a constant if and only if \( a + b = 1 \), in which case \( \mathcal{M}(r^2) \) degenerates to be the generalized Legendre relation.

In the case of \( a + b = 1 \), these functions coincide with the special functions \( \mu_a(r), \kappa_a(r), \mathcal{E}_a(r) \), and \( m_a(r) \), respectively, which were studied in [36–49]. In particular, if \( a = b = 1/2 \), then these functions reduce to the classical cases denoted by \( \mu(r), \kappa(r), \mathcal{E}(r) \), and \( m(r) \), which appeared frequently in the geometric function theory and number theory [50–69].

The main purpose of the article is to find the sub-regions of \( (a, b) \in \mathbb{R}^2 | 0 < a, b < 1, a + b > 1 \) such that certain quotient functions involving \( \mu_{a,b}(r), \kappa_{a,b}(r), \mathcal{E}_{a,b}(r) \), and \( m_{a,b}(r) \) are monotonic on their corresponding sub-regions. As a consequence, several new bounds for \( \mu_{a,b}(r) \) and \( m_{a,b}(r) \) are discovered, which are the variants and extensions of the results given in [42, Theorems 1.1 and 1.2] for the case of zero-balanced.

2 Notations, formulas, and lemmas

In order to prove our main results, we need several derivative formulas and lemmas, which we present in this section.

2.1 Notations

Throughout the article, we denote \( B(a, b) \) by \( B \) if no risk for confusion. Let

\[
D = \frac{B\left(\frac{a+b+1}{2}, \frac{a+b-1}{2}\right)}{2},
\]

\[
E = \frac{B(a, b)B\left(\frac{a+b+1}{2}, \frac{1-a-b}{2}\right)}{2B\left(\frac{a+b+1}{2}, \frac{b-a+1}{2}\right)},
\]

\[
\kappa_1(a, b) = a + b + 1 - 2ab(4 - a - b),
\]

\[
\kappa_2(a, b) = 3 + 7(a + b) + 2(a - b)^2 - 6(a + b)^2 - 5(a + b)^3 - (a + b)^5
\]

\[
+ 8ab\left[(a + b)^2 + (a + b)^3 - (a + b) + 8ab\right],
\]

\[
\kappa_3(a, b) = 5 + 7(a + b) - 3(a + b)^2 - 7(a + b)^3 - 2(a + b)^4
\]

\[
+ 4ab\left[3(a + b)^2 + 4(a + b) - 3\right],
\]

\[
\kappa_4(a, b) = 9 + 5(a + b) - 9(a + b)^2 - 5(a + b)^3 + 16ab(a + b),
\]

\[
\kappa_5(a, b) = (a + b + 1)^2 - 12ab.
\]

For the convenience of readers, we also introduce three sub-regions \( \Omega_1, \Omega_2, \) and \( \Omega_3 \) of \( (a, b) \in \mathbb{R}^2 | 0 < a, b < 1 \), which are illustrated in Fig. 1.

\[
\Omega_1 = \{(a, b) | 0 < a, b < 1, a + b > 1, \kappa_1(a, b) \geq 0\},
\]

\[
\Omega_2 = \{(a, b) | 0 < a, b < 1, a + b > 1, \kappa_2(a, b) \leq 0, \kappa_3(a, b) \leq 0\},
\]

\[
\Omega_3 = \{(a, b) | 0 < a, b < 1, a + b > 1, \kappa_5(a, b) \geq 0\}.
\]
2.2 Formulas

Let \( r \in (0, 1) \) and \( 0 < a, b < 1 \) with \( a + b > 1 \). Then the following derivative formulas

\[
\frac{dK}{dr} = \frac{1}{r^{p^2}} \left[ 2bE - r^2 K \right] + (a + b - 1)(K - E),
\] (2.1)

\[
\frac{dE}{dr} = \frac{2(a - 1)}{r} (K - E),
\] (2.2)

\[
\frac{d(K - E)}{dr} = \frac{1}{r^{p^2}} \left[ 2br^2 K - (a + b - 1 + 2(1 - a)r^2)(K - E) \right],
\] (2.3)

\[
\frac{d(E - r^2 K)}{dr} = \frac{1}{r} \left[ 2(1 - b)r^2 K + (a + b - 1)(K - E) \right]
\] (2.4)

can be found in [34, Theorem 4.15].

Note that Theorem 1.19(9) of [19] gives the derivative formula

\[
\frac{d\mu_{a,b}(r)}{dr} = \frac{(a + b - 1)B^2 D}{4r^{p^2} + b^{p^2} + 1 + K^2}
\] (2.5)

for \( \mu_{a,b}(r) \) if \( d = c = (a + b + 1)/2 \).
From (1.7), (1.9), (1.10), and (2.1) we clearly see that
\[
\frac{dm_{a,b}(r)}{dr} = \frac{4}{Br} \left[ (2b-1)r^2 K - (b-a+1)(K - E) \right].
\] (2.6)

2.3 Lemmas

Lemma 2.1 ([70, Theorem 2.1]) Suppose that the power series \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) and \( g(x) = \sum_{n=0}^{\infty} b_n x^n \) have the radius of convergence \( r > 0 \) with \( b_n > 0 \) for all \( n \in \{0,1,2,\ldots\} \). Let \( h(x) = f(x)/g(x) \) and \( Hf_g = (f'/g')g - f \). Then the following statements hold true:

(i) If the non-constant sequence \( \{a_n/b_n\}_{n=0}^{\infty} \) is increasing (decreasing) for all \( n \geq 0 \), then \( h(x) \) is strictly increasing (decreasing) on \((0,r)\);

(ii) If there exists \( n_0 > 0 \) such that the non-constant sequence \( \{a_n/b_n\}_{n=0}^{\infty} \) is increasing (decreasing) for \( 0 \leq n \leq n_0 \) and decreasing (increasing) for \( n \geq n_0 \), then \( h(x) \) is strictly increasing (decreasing) on \((0,r)\) if and only if \( Hf_g(r^+) \geq (\leq) 0 \). Moreover, if \( Hf_g(r^+) < (>) 0 \), then there exists \( x_0 \in (0,r) \) such that \( h(x) \) is strictly increasing (decreasing) on \((0,x_0)\) and strictly decreasing (increasing) on \((x_0,r)\).

Lemma 2.2 ([19, Theorem 1.25]) Suppose that \(-\infty < a < b < \infty, f, g : [a, b] \to \mathbb{R} \) are continuous on \([a, b]\) and differentiable on \((a, b)\), and \( g'(x) \neq 0 \) on \((a, b)\). If \( f'/g' \) is increasing (decreasing) on \((a, b)\), then so are the functions
\[
\frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)}.
\]
If \( f'/g' \) is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2.3 Let \( 0 < a, b < 1 \) with \( a + b > 1 \). Then the following assertions are valid:

(i) The function \( (K - E)/(r^2 K) \) is strictly increasing from \((0,1)\) onto \( (2b/(a + b + 1), 1)\);

(ii) The function \( r^{2p-1} K \) has positive Maclaurin coefficients and maps \((0,1)\) onto \((B/2,D)\);

(iii) The function \( r^p K \) is strictly decreasing from \((0,1)\) onto \((0,B/2)\) if \( p \geq 4ab/(a + b + 1) \).

Proof Items (i) and (ii) follow directly from [34, Lemma 4.22]. We only need to prove item (iii).

It follows from (2.1) that
\[
\frac{d(r^p K)}{dr} = \frac{r^{p-2}}{r} \left[ (2b-p)r^2 K - (b-a+1)(K - E) \right]
= (b-a+1)r^{p-2}K \left[ \frac{2b-p}{b-a+1} - \frac{K-E}{r^2 K} \right].
\] (2.7)

Lemma 2.3(ii) and (2.7) enable us to know that \( r^p K \) is strictly decreasing on \((0,1)\) if \((2b-p)/(b-a+1) \leq 2b/(a + b + 1)\), that is, \( p \geq 4ab/(a + b + 1) \).

Note that
\[
r^p K = \frac{B(a,b)}{2} r^{p+1-a-b} F \left( \frac{b-a+1}{2}, \frac{a-b+1}{2}; \frac{a+b+1}{2}; r^2 \right).
\] (2.8)

If \( p \geq 4ab/(a + b + 1) \), then \( p + 1 - a - b \geq (a - b + 1)(b - a + 1)/(a + b + 1) > 0 \). This in conjunction with (1.2) and (2.8) gives \( \lim_{r \to 1^{-}} r^p K = 0 \). \( \square \)
In the following Lemma 2.4 we provide an asymptotic formula for $K$ as $r \to 1$ in the case of $a + b > 1$, which is the analog for the zero-balanced hypergeometric function (1.2).

**Lemma 2.4** Let $0 < a, b < 1$ with $a + b > 1$. Then one has

$$K(\sqrt{r}) = D(1 - r)^{(1-a-b)/2} + E + o((1 - r)^{(a+b+1)/4} \log(1 - r))$$

as $r \to 1$.

**Proof** It follows from $F(a, b; (a + b + 1)/2; r)$ is asymptotically to $2D(1 - r)^{(1-a-b)/2}/B$ [19, Theorem 1.19(5)] as $r \to 1$ for $a + b > 1$ and the derivative formula

$$\frac{dF(a, b; c; r)}{dr} = \frac{ab}{c} F(a + 1, b + 1; c + 1; r)$$

(2.9)

given in [19, (1.16)] for the hypergeometric function together with (1.2), and L’Hôpital’s rule that

$$\lim_{r \to 1} \frac{K(\sqrt{r}) - D(1 - r)^{(1-a-b)/2} - E}{(1 - r)^{(a+b-1)/2} \log(1 - r)} = \lim_{r \to 1} \frac{BF(\frac{b-a+1}{2}, \frac{a-b+1}{2}, \frac{a+b+1}{2}; r) - 2D - 2E(1 - r)^{(a+b-1)/2}}{2(1 - r)^{(a+b-1)} \log(1 - r)}$$

$$= \lim_{r \to 1} \frac{[(b-a)^2 - 1]BF(\frac{b-a+3}{2}, \frac{a-b+3}{2}, \frac{a+b+3}{2}; r) - 2[(a+b)^2 - 1]E(1 - r)^{(a+b-3)/2}}{4(a + b + 1)(1 - r)^{(a+b-1)/2}[(a + b - 1) \log(1 - r) + 1]}$$

$$= \lim_{r \to 1} \frac{ab[1 - (b-a)^2]B(1 - r)^{(a+b)^2 - 1}F(\frac{b-a+3}{2}, \frac{a-b+3}{2}, \frac{a+b+5}{2}; r)}{(a + b + 3)[(a + b)^2 - 1][(a + b - 1) \log(1 - r) + 3]}$$

$$= 0.$$

This completes the proof. □

Lemma 2.4 leads to Corollary 2.5 immediately.

**Corollary 2.5** Let $0 < a, b < 1$ and $a + b > 1$. Then

$$Dr^{1-a-b} + E - m_{a,b}(r) \to 0 \quad \text{and} \quad Dr^{1-a-b} + E - m_{a,b}(r) \to 0$$

as $r \to 0$.

**Proof** By replacing $r$ with $1 - r^2$ in Lemma 2.4, we clearly see that

$$K' = Dr^{1-a-b} + E + o(r^{a+b-1} \log r^2).$$

(2.10)
By definition, it is easy to know that \((K - B/2)/r \to 0\) as \(r \to 0\). This in conjunction with (2.10) and \(a + b < 2\) yields

\[
Dr^{1-a-b} + E - \mu_{a,b}(r) = \frac{B}{2K}(Dr^{1-a-b} + E - K') + \frac{1}{K}(K - B/2)(Dr^{1-a-b} + E) \to 0
\]
as \(r \to 0\). The second asymptotic formula can be proved by similar arguments. \(\square\)

**Lemma 2.6** Let \(0 < a, b < 1\) with \(a + b > 1\). Then the following assertions are valid:

(i) If \(\kappa_1(a, b) \geq 0\), then \(\kappa_5(a, b) > 0\) and \(a + b < 3/2\);

(ii) \(\kappa_4(a, b) < \kappa_3(a, b)\);

(iii) If \(\kappa_5(a, b) \geq 0\) and \(a \leq b\), then \(3 - 3a - b > 0\).

**Proof** (i) We only need to prove that it is not possible for \(\kappa_1(a, b) \geq 0\) and \(\kappa_5(a, b) \leq 0\). By calculations, the inequality \(\kappa_1(a, b) \geq 0\) is equivalent to \(0 < a \leq 1/2\) and \(-1 < b < 1\) or \(1/2 < a < 1\) and \(-1 < b \leq b_1(a)\), where \(b_1(a) = \frac{1}{4d}[-1 + 8a - 2a^2 - \sqrt{1 - 24a + 60a^2 - 32a^3 + 4a^4}]\) and \(\kappa_5(a, b) \leq 0\) is equivalent to \(1/2 < 2(2 - \sqrt{3}) < a < 1\) and \(b_2(a, b) \leq b_1 < 1\), where \(b_2(a) = -1 + 5a - 2\sqrt{3(2a^2 - a)}\).

It remains to show that \(b_2(a) > b_1(a)\) for \(2(2 - \sqrt{3}) < a < 1\). A simple calculation leads to

\[
b_2(a) - b_1(a) = \frac{1}{4d\sqrt{3a(2a - 1)}} > 0
\]

if and only if

\[
(1 - 12a + 22a^2 + \sqrt{1 - 24a + 60a^2 - 32a^3 + 4a^4})^2 - (8a\sqrt{3a(2a - 1)})^2
\]
\[
= 2[(22a^2 - 12a + 1)\sqrt{1 - 24a + 60a^2 - 32a^3 + 4a^4}
+ 1 - 24a + 124a^2 - 184a^3 + 52a^4] > 0,
\]

which is also equivalent to

\[
(22a^2 - 12a + 1)^2(1 - 24a + 60a^2 - 32a^3 + 4a^4)
- (1 - 24a + 124a^2 - 184a^3 + 52a^4)^2
= 64a^3(a + 1)(3 - 2a)(2a - 1)(3a^2 - 3a + 1) > 0
\]

for \(1/2 < a < 1\). On the other hand, as we know, \(\kappa_1(a, b)\) can be thought of as a quadratic function of \(b\) and the parabola opens up. It is easy to verify that \(\kappa_1(a, 1 - a) = 2(1 - 3a + 3a^2) > 0\) and \(\kappa_1(1, a) = -(2 - a)(2a - 1) < 0\) for \(1/2 < a < 1\). Combining this with \(\kappa_1(a, 3/2 - a) = -5(1 - a)(a - 1/2) < 0\) for \(1/2 < a < 1\), we conclude that \(3/2 - a < b < 1\) makes \(\kappa_1(a, b)\) negative. This completes the first assertion.

(ii) Observe that \(\kappa_4(a, b) - \kappa_5(a, b) = 2(a + b + 1)(a + b - 1)Q(b)\), where

\[
Q(b) = a^2 + a - 2 - (4a - 1)b + b^2
\]

(2.11)
is a quadratic function in terms of $b$. Since the parabola of $Q(b)$ opens up, it follows from $Q(1-a) = -6a(1-a) < 0$ and $Q(1) = -a(3-a) < 0$ that $Q(b) < 0$ for $0 < a < 1$ and $1 - a < b < 1$. This in conjunction with (2.11) yields $\kappa_5(a, b) < \kappa_3(a, b)$.

(iii) If the conclusion is not true, that is, $3 - 3a - b \leq 0$, it follows that $b \geq \max(a, 3(1-a))$. As we know, $\kappa_5(a, b) = b^2 - 2(5a - 1)b + (a + 1)^2$ is a quadratic function of $b$. We divide the proof into two cases.

**Case 1**: $a \geq 3(1-a)$. Then we clearly see that $a \leq b < 1$ and $3/4 \leq a < 1$. Since the symmetric axis $5a - 1 > 1$, $\kappa_5(a, b)$ is strictly decreasing for $a < b$. This gives $\kappa_5(a, b) \leq \kappa_5(a, a) = -[8(a - 3/4)^2 + 8(a - 3/4) + 1/2] < 0$, which is a contradiction.

**Case 2**: $a < 3(1-a)$. In other words, $3(1-a) < b < 1$ and $2/3 < a < 3/4$. Similarly, the monotonicity of $\kappa_5(a, b)$ gives rise to $\kappa_5(a, b) \leq \kappa_5(a, 3(1-a)) = 4(2a - 1)(5a - 4) < 0$, which is also a contradiction. □

**Lemma 2.7** Let $0 < a, b < 1$ with $a + b > 1$ and $a + b + 1 \geq 4ab$, and $\varphi(r)$ be defined by

$$\varphi(r) = \frac{1}{r^a b^{-1} - 1 + B^2/(4r^a b^1 K^2)} - 1.$$  

Then $\varphi(r)$ is strictly decreasing from $(0, 1)$ onto $(0, (a b^{-1} + 1)/(2a b^{-1} b + b^2 - 6ab))$.

**Proof** Let $\varphi_1(r) = 1/(r^a b^{-1} - 1$ and $\varphi_2(r) = B^2/(4r^a b^1 K^2) - 1$. Then $\varphi(r) = \varphi_1(r)/\varphi_2(r)$ and $\varphi_1(0) = \varphi_2(0) = 0$. Combining this with Lemma 2.2, we clearly see that the monotonicity of $\varphi(r)$ depends on $\varphi'(r)/\varphi_2(r)$, that is,

$$\frac{\varphi_1'(r)}{\varphi_2'(r)} = \frac{4(a + b - 1)}{B^2} \cdot (r')^2 \cdot \frac{r^2 K}{(a + 1 - 3b)r^2 K + 2(b - a + 1)(K - E)}.$$  

(2.12)

It follows from Lemma 2.3(i) that $(a + 1 - 3b) + 2(b - a + 1)(K - E)/(r^2 K)$ is strictly increasing from $(0, 1)$ onto $(1 + 2a^2 + 2 + 2a + 2 + b^2 - 6ab, 3 - a - b)$. Since $a + b + 1 \geq 4ab$, Lemma 2.3(iii) leads to the conclusion that $r' K$ is strictly decreasing from $(0, 1)$ onto $(0, B/2)$. This in conjunction with (2.12) implies that $\varphi_1'(r)/\varphi_2'(r)$ is strictly decreasing on $(0, 1)$.

On the other hand, it follows from L'Hôpital's rule and (2.12) that

$$\varphi(0^+) = \lim_{r \to +0} \varphi_1'(r)/\varphi_2'(r) = \frac{(a + b - 1)(a + b + 1)}{1 + 2a + 2b + a^2 + b^2 - 6ab}, \quad \varphi(1^-) = 0.$$  

□

**Lemma 2.8** Let $(a, b) \in \Omega_1$ and $f(r)$ be defined by

$$f(r) = \frac{(1 - 2b)r^3 K + (b - a + 1)(K - E)}{B^2/4r^a b^{-1} K^2} - 1.$$  

Then $f(r)$ is strictly decreasing from $(0, 1)$ onto $(0, (a b^{-1} - 4ab)B/(1 + 2a^2 + 2b + 2a + 2 + b^2 - 6ab))$.

**Proof** Let $f_1(r) = (1 - 2b)r^3 K + (b - a + 1)(K - E)$ and $f_2(r) = B^2/(4r^a b^1 K^2) - 1$. Then we clearly see that $f(r) = f_1(r)/f_2(r)$ and $f_1(0) = f_2(0) = 0$.

By calculations, one has

$$\frac{f_1'(r)}{f_2'(r)} = \frac{4}{B^2} \cdot [r^a b^{-1/3} K] \cdot \bar{f}(r),$$  

(2.13)
where

\[
\hat{f}(r) = \left[ \sigma_1(a, b) + \sigma_2(a, b)r^2 \right] r^2 K - \left[ \sigma_3(a, b) + \sigma_4(a, b)r^2 \right] (K - \mathcal{E}) \frac{\hat{f}_1(r)}{\hat{f}_2(r)} \quad (2.14)
\]

and

\[
\sigma_1(a, b) = 2(1 - b - ab + b^2), \quad \sigma_2(a, b) = 2(1 - b)(2b - 1),
\]

\[
\sigma_3(a, b) = (a + b - 1)(b - a + 1), \quad \sigma_4(a, b) = (b - a + 1)(3 - 2a - 2b).
\]

Let

\[
\hat{f}_{11}(r) = \sigma_1(a, b) - \sigma_3(a, b) \frac{K - \mathcal{E}}{r^2 K}, \quad \hat{f}_{12}(r) = -r^2 \left[ \sigma_4(a, b) \frac{K - \mathcal{E}}{r^2 K} - \sigma_2(a, b) \right].
\]

Then \( \hat{f}_1(r)/(r^2 K) = \hat{f}_{11}(r) + \hat{f}_{12}(r) \).

It follows from Lemma 2.3(i) and \( \sigma_3(a, b) > 0 \) that \( \hat{f}_{11}(r) \) is strictly decreasing on \((0, 1)\). For \((a, b) \in \Omega_1\), namely \(0 < a, b < 1, a + b > 1\) and \( \kappa_1(a, b) > 0 \), we clearly see from Lemma 2.6(i) that \( \kappa_3(a, b) = (a + b + 1)^2 - 12ab > 0, a + b < 3/2, \) and then \( \sigma_4(a, b) > 0 \). This in conjunction with Lemma 2.3(i), (iii) implies that \( \nu^{(a + b + 1)/3 K} \) is strictly decreasing on \((0, 1)\) and

\[
\sigma_2(a, b) \frac{K - \mathcal{E}}{r^2 K} - \sigma_3(a, b) > \sigma_4(a, b) \cdot \frac{2b}{a + b + 1} - \sigma_2(a, b) = \frac{2\kappa_1(a, b)}{a + b + 1} > 0. \quad (2.15)
\]

Lemma 2.3(i) and (2.15) enable us to know that \( \hat{f}_{12}(r) \) is strictly decreasing on \((0, 1)\). This gives the monotonicity of \( \hat{f}_1(r)/(r^2 K) \). So \( \hat{f}_1(r)/(r^2 K) > \sigma_1(a, b) + \sigma_2(a, b) - \sigma_3(a, b) - \sigma_4(a, b) = (2 - a - b)(a + b - 1) > 0 \). Moreover, it is easy to verify from Lemma 2.3(i) that \( \hat{f}_2(r)/(r^2 K) \) is strictly increasing from \((0, 1)\) onto \((1 + 2a + 2b + a^2 + b^2 - 6ab, 3 - a - b)\). Combining with (2.14), the monotonicity of \( \hat{f}_1(r)/(r^2 K) \) and \( \hat{f}_2(r)/(r^2 K) \) leads to the conclusion that \( \hat{f}(r) \) is strictly decreasing on \((0, 1)\).

Therefore, the monotonicity of \( f(r) \) follows from Lemma 2.2 and (2.13) together with the monotonicity of \( \nu^{(a + b + 1)/3 K} \) and \( \hat{f}(r) \).

To this end, by L'Hôpital's rule and (2.13), (2.14),

\[
f(0^+) = \lim_{r \to 0^+} \frac{f_1(r)}{\hat{f}_2(r)} = \frac{(a + b + 1 - 4ab) B}{1 + 2a + 2b + a^2 + b^2 - 6ab}, \quad f(1^-) = 0. \quad \square
\]

**Lemma 2.9** Let \((a, b) \in \Omega_2\) and \( g(r) \) be defined by

\[
g(r) = \frac{B^2/(4\nu^{(a + b + 1)/3 K}) - 1}{[2b^2 K + (a - b - 1)(K - \mathcal{E})]/r^2}.
\]

Then \( g(r) \) is strictly decreasing from \((0, 1)\) onto \((0, \frac{1 + 2a + 2b + a^2 + b^2 - 6ab}{4ab B})\).

**Proof** Let \( g_1(r) = B^2/(4\nu^{(a + b + 1)/3 K}) - 1 \) and \( g_2(r) = [2b^2 K + (a - b - 1)(K - \mathcal{E})]/r^2 \). Then \( g(r) = g_1(r)/g_2(r) \) and \( g_1(0) = g_2(0) = 0 \).

By calculations, one has

\[
\frac{g_1'(r)}{g_2'(r)} = \frac{B^2}{4[\nu^{(a + b + 1)/3 K}]^2} \frac{\hat{g}_1(r)}{\hat{g}_2(r)}. \quad (2.16)
\]
where
\[
\hat{g}_1(r) = \frac{r^{a+b-1}[(a - 3b + 1)r^2K_c + 2(b - a + 1)(K_c - E)]}{r^2K_c},
\]  
(2.17)
\[
\hat{g}_2(r) = \frac{[\lambda_1(a, b) + \lambda_2(a, b)r^2]r^2K_c + [\lambda_3(a, b) + \lambda_4(a, b)r^2](K_c - E)}{r^2}
\]  
(2.18)
and
\[
\lambda_1(a, b) = 2b(a - b + 1), \quad \lambda_2(a, b) = 4b^2,
\]
\[
\lambda_3(a, b) = (b - a + 1)(a + b - 1), \quad \lambda_4(a, b) = -2(a + b)(b - a + 1).
\]

By (2.2) and (2.9), we clearly see that
\[
\frac{K_c - E}{r^2} = \frac{B(a, b)}{4(a - 1)r} \frac{dF(a - 1, b; (a + b + 1)/2; r^2)}{dr}
\]
\[
= \frac{bB(a, b)}{a + b + 1} F(a, b + 1; (a + b + 3)/2; r^2).
\]  
(2.19)
It follows from (1.2), (1.5), and (2.19) that
\[
r^{a+b-1}K_c = \frac{B(a, b)}{2} F \left( \frac{b - a + 1}{2}, \frac{a - b + 1}{2}; \frac{a + b + 1}{2}; r^2 \right),
\]  
(2.20)
\[
r^{a+b-1}\frac{K_c - E}{r^2} = \frac{B(a, b)b}{a + b + 1} F \left( \frac{b - a + 3}{2}, \frac{a - b + 1}{2}; \frac{a + b + 3}{2}; r^2 \right).
\]  
(2.21)
Combining with (2.17), (2.18), (2.20), and (2.21), we rewrite \(\hat{g}_1(r)\) and \(\hat{g}_2(r)\) in terms of power series:
\[
\hat{g}_1(r) = \sum_{n=0}^{\infty} \frac{(\frac{b-1}{2}, b-b)}{(\frac{a-1}{2}, b-b)} \xi_{a,b}(n)r^{2n},
\]  
(2.22)
\[
\hat{g}_2(r) = \frac{B(a, b)}{2} \sum_{n=0}^{\infty} \frac{(a, n-1)(b, n-1)}{(a+b+1, n+1)} \xi_{a,b}(n)r^{2n},
\]  
(2.23)
where
\[
\xi_{a,b}(n) = \frac{1 + 2a + 2b + a^2 + b^2 - 6ab + 2(a + b + 1)n}{1 + a + b + 2n},
\]
\[
\zeta_{a,b}(n) = \left[ (a + b)^2 - 1 \right] n^3 + 2(a + b - 1)(a + b + 2ab - 1)n^2
\]
\[
+ \left[ 4ab(a + b + ab - 1) - 3(a + b)^2 + 4(a + b) - 1 \right] n
\]
\[
+ 4ab(1 - a)(1 - b).
\]
We now claim that \(\hat{g}_1(r)\) is strictly decreasing on \((0, 1)\) and \(\hat{g}_2(r)\) is strictly increasing on \((0, 1)\); furthermore, \(\hat{g}_2(r)\) has positive Maclaurin coefficients.
Lemma 2.3(ii), we conclude that

\[ g^n(\alpha, b, n) = \frac{(\frac{a}{2} - b + 1, n)(\frac{b - a + 1}{2}, n)}{(a, n)(b, n)} \xi_{a,b}(n) \]  

\[ \frac{\alpha_{n+1}}{\alpha_n} = \frac{1 + b - a + 2n(1 + a - b + 2n)\xi_{a,b}(n + 1)}{4(b + n)(a + n)\xi_{a,b}(n)} \leq 1 \]

A simple calculation yields

\[ \frac{\Delta_{a,b}(n)}{(1 + a + b + 2n)(3 + a + b + 2n)} \leq 0, \]

\[ \Delta_{a,b}(n) = (1 + b - a + 2n)(1 + a - b + 2n)\xi_{a,b}(n + 1) - 4(a + n)(b + n)\xi_{a,b}(n) \]

\[ = \frac{\Delta_{a,b}(n)}{(1 + a + b + 2n)(3 + a + b + 2n)} \leq 0, \]

where

\[ \Delta_{a,b}(n) = \kappa_2(a, b) + 4\kappa_3(a, b)n + 4\kappa_4(a, b)n^2 - 16[(a + b)^2 - 1]n^2. \]

For \((a, b) \in \Omega_2,\) namely \(0 < a, b < 1, a + b > 1, \kappa_2(a, b) \leq 0, \kappa_3(a, b) \leq 0,\) and then \(\kappa_4(a, b) \leq 0\) by Lemma 2.6(ii). This in conjunction with (2.24) and (2.25) implies that the sequence \(\{\alpha_n\}_{n=0}^{\infty}\) is decreasing. So the first assertion is valid.

- We mention that the Poisson symbol \((a, -1)(b, 1) = \frac{1}{(a-1)(b-1)} > 0\) for \(0 < a, b < 1.\) It only needs to prove \(\zeta_{a,b}(n) > 0\) for \(n \geq 0\) with \(0 < a, b < 1\) and \(a + b > 1.\)

Clearly, \(\zeta_{a,b}(0) = 4ab(1 - a)(1 - b) > 0\) and \(\zeta_{a,b}(1) = 4ab(a + b + 2ab - 1) > 0.\)
Moreover, \(\zeta_{a,b}(n)\) is strictly increasing for \(n \geq 0.\) This gives \(\zeta_{a,b}'(n) \geq \zeta_{a,b}'(1) = 4q(b)\) for \(n \geq 1,\) where \(q(b) = (a^2 + 3a + 1)b^2 + (3a^2 - a - 1)b + a(a - 1)\) is regarded as a quadratic function in terms of \(b\) and its parabola opens up.

Observe that

\[ \frac{3a^2 - a - 1}{2(a^2 + 3a + 1)} = (1 - a) = \frac{a(1 - a)(2a + 3) + 1}{2(a^2 + 3a + 1)} < 0, \]

that is to say, the symmetric axis of \(q(b)\) lies on the left side of the interval \([1 - a, 1].\)

This in conjunction with \(q(1 - a) = a^2(a - 1)^2 > 0\) implies that \(q(b) > 0\) for \(1 - a < b < 1.\)
So \(\zeta_{a,b}(n)\) is strictly increasing for \(n \geq 1\) and \(\zeta_{a,b}(n) \geq \zeta_{a,b}(1) > 0\) for \(n \geq 1.\) This completes the second assertion.

Therefore, \(\tilde{g}_1(r) / \tilde{g}_2(r)\) is strictly decreasing on \((0, 1)\) follows from the above assertions together with \(\tilde{g}_1(r) > 0\) and \(\tilde{g}_2(r) > 0.\) Combining this with (2.16), Lemma 2.2 and Lemma 2.3(ii), we conclude that \(g(r)\) is strictly decreasing on \((0, 1).\)

It remains to compute two end values of \(g(r).\) By L'Hôpital's rule and (2.16) together with Lemma 2.3(i), (ii),

\[ g(0^+) = \lim_{r \to 0^+} \frac{g_1'(r)}{g_2'(r)} = \frac{1 + 2a + 2b + a^2 + b^2 - 6ab}{4abB}, \quad g(1^-) = 0. \]
Lemma 2.10 Let $0 < a \leq b < 1$ with $a + b > 1$ and $\kappa_3(a, b) \geq 0$, and $h(r)$ be defined by
\[
h(r) = \frac{2br^2\kappa - [2(a + b - 1) + (3 - 3a - b)r^2](\kappa - \mathcal{E})}{(a - 3b + 1)r^2\kappa + 2(b - a + 1)(\kappa - \mathcal{E})}.
\]
Then $h(r)$ is strictly decreasing from $(0, 1)$ onto $(\frac{a + b - 1}{3 - a - b}, \frac{2b(3 - a - b)}{1 + 2a + 2b + a^2 + b^2 - 6ab})$.

Proof We denote by $h_1(r) = 2b - [2(a + b - 1) + (3 - 3a - b)r^2](\kappa - \mathcal{E})/(r^2\kappa)$ and $h_2(r) = (a - 3b + 1) + 2(b - a + 1)(\kappa - \mathcal{E})/(r^2\kappa)$.

If $0 < a \leq b < 1$, $a + b > 1$, and $\kappa_3(a, b) \geq 0$, then $3 - 3a - b > 0$ follows from Lemma 2.6(iii).

Combining this with Lemma 2.3(i), we conclude that $h_1(r)$ is strictly decreasing from $(0, 1)$ onto $(a - 1, 2\frac{b(3 - a - b)}{a + b - 1})$ and $h_2(r)$ is strictly increasing from $(0, 1)$ onto $(\frac{1 + 2a + 2b + a^2 + b^2 - 6ab}{a + b - 1}, 3 - a - b)$. This gives the monotonicity of $h(r) = h_1(r)/h_2(r)$ together with two limiting values $h(0^+)$ and $h(1^-)$.

3 Main results

Theorem 3.1 Let $(a, b) \in \Omega_1$ and $F(r)$ be defined on $(0, 1)$ by
\[
F(r) = \frac{Dr^{1-a-b} + E - m_{a,b}(r)}{Dr^{1-a-b} + E - \mu_{a,b}(r)}
\]
Then $F(r)$ is strictly decreasing from $(0, 1)$ onto $(1, L_0)$, where
\[
L_0 = \frac{(a + b + 1)[(a + b - 1)^2 + 4] - 16ab}{(a + b - 1)(1 + 2a + 2b + a^2 + b^2 - 6ab)}.
\]
In particular, the double inequality
\[
m_{a,b}(r) < \mu_{a,b}(r) < \frac{1}{L_0}m_{a,b}(r) + \left(1 - \frac{1}{L_0}\right)(Dr^{1-a-b} + E)
\]
holds for $r \in (0, 1)$.

Proof Let $F_1(r) = Dr^{1-a-b} + E - m_{a,b}(r)$ and $F_2(r) = Dr^{1-a-b} + E - \mu_{a,b}(r)$. Clearly, $F(r) = F_1(r)/F_2(r)$ and $F_1(0^+) = F_2(0^+) = 0$ follow from Corollary 2.5.

By calculations, one has
\[
F_1(r) = \left(\frac{1}{\rho r^{a+b-1} - 1} + \frac{4r^{a+b-1}\kappa'}{(a + b - 1)BD}[1 - 2br^2\kappa + (b - a + 1)(\kappa - \mathcal{E})] \right) \frac{B^2}{4r^{a+b-1}\kappa}\rho^{a+b-1} = 1
\]
\[
= \varphi(r) + \frac{4}{(a + b - 1)BD} \cdot \rho^{a+b-1}\kappa' \cdot f(r),
\]
where $\varphi(r)$ and $f(r)$ are defined as in Lemma 2.7 and Lemma 2.8, respectively.

Since $\rho^{a+b-1}\kappa'$ can be regarded as the composition of $x^{a+b-1}\kappa(x)$ and $x = r' = \sqrt{1 - r^2}$, Lemma 2.3(ii) enables us to know that $\rho^{a+b-1}\kappa'$ is strictly decreasing from $(0, 1)$ onto $(B/2, D)$. This in conjunction with (3.1) together with Lemma 2.2, Lemma 2.7, and Lemma 2.8 gives rise to the monotonicity of $F(r)$ and also, by L'Hôpital's rule.
and (3.1),
\[ F(0^+) = \lim_{r \to 0^+} \frac{F'_r(r)}{F''_r(r)} = \varphi(0^+) + \frac{4}{(a + b - 1)BD} \cdot D \cdot f(0^+) = L_0, \]
and \( F(1^-) = 1 \) follows directly from \( m_{a,b}(1^-) = \mu_{a,b}(1^-) = 0. \)

**Corollary 3.2** Let \((a, b) \in \Omega_1 \) and \( \hat{F}(r) \) be defined on \((0, 1)\) by
\[ \hat{F}(r) = \frac{m_{a,b}(r) - D(r^{1-a-b} - 1)}{\mu_{a,b}(r) - D(r^{1-a-b} - 1)}. \]
Then \( \hat{F}(r) \) is strictly decreasing from \((0, 1)\) onto \((0, 1)\).

**Proof** Let \( \hat{F}_1(r) = m_{a,b}(r) - D(r^{1-a-b} - 1) \) and \( \hat{F}_2(r) = \mu_{a,b}(r) - D(r^{1-a-b} - 1) \). Then \( \hat{F}(r) = \hat{F}_1(r)/\hat{F}_2(r) \) and \( \hat{F}_1(1^-) = \hat{F}_2(1^-) = 0 \).

Since \( \hat{F}_1(r)/\hat{F}_2(r) = F_1(r)/F_2(r), \) Lemma 2.2 enables us to know the monotonicity of \( \hat{F}(r) \) depends on that of \( F_1(r)/F_2(r), \) which follows from Theorem 3.1. It only remains to compute two limiting values \( \hat{F}(0^+) \) and \( \hat{F}(1^-) \).

By Corollary 2.5, it is easy to see that \( \hat{F}(0^+) = (D + E)/(D + E) = 1 \). By L'Hôpital's rule and (3.1) together with Lemma 2.7, Lemma 2.8,
\[ \hat{F}(1^-) = \lim_{r \to 1^-} \frac{\hat{F}_1(r)}{\hat{F}_2(r)} = \varphi(1^-) + \frac{4}{(a + b - 1)BD} \cdot B \cdot f(1^-) = 0. \]

**Theorem 3.3** Let \((a, b) \in \Omega_2 \) and \( G(r) \) be defined on \((0, 1)\) by
\[ G(r) = \frac{Dr^{1-a-b} + E - \mu_{a,b}(r)}{K - B/2}. \]
Then \( G(r) \) is strictly decreasing from \((0, 1)\) onto \((0, \infty)\).

**Proof** We denote \( G_1(r) = Dr^{1-a-b} + E - \mu_{a,b}(r) \) and \( G_2(r) = K - B/2 \). Then we clearly see that \( G(r) = G_1(r)/G_2(r) \) and \( G_1(0^+) = G_2(0^+) = 0 \).

By taking the derivative of \( G_1(r) \) and \( G_2(r) \), one has
\[ \frac{G'_1(r)}{G'_2(r)} = \frac{(a + b - 1)D}{r^{a+b-1}} \cdot \frac{B^2/(4r^{a+b+1}K^2) - 1}{[2br^2K + (a - b - 1)(K - E)]/r^2} = \frac{(a + b - 1)D}{r^{a+b-1}} \cdot g(r), \]
where \( g(r) \) is defined as in Lemma 2.9.

Therefore, the monotonicity of \( G(r) \) follows from Lemma 2.9 and that of \( 1/r^{a+b-1} \).

To this end, by L'Hôpital's rule and (3.2),
\[ G(0^+) = \lim_{r \to 0^+} \frac{G'_1(r)}{G'_2(r)} = \lim_{r \to 0^+} \frac{(a + b - 1)D}{r^{a+b-1}} \cdot g(0^+) = \infty, \quad G(1^-) = 0. \]

**Theorem 3.4** Let \((a, b) \in \Omega_3 \) and \( H(r) \) be defined on \((0, 1)\) by
\[ H(r) = \frac{r^{1-a-b}(B/2 - \varepsilon)}{Dr^{1-a-b} + E - \mu_{a,b}(r)}. \]
Then \( H(r) \) is strictly decreasing from \((0, 1)\) onto \((L_1, L_2)\), where

\[
L_1 = \frac{2b(1 - a)(3 - a - b)B}{(a + b - 1)(1 + 2a + 2b + a^2 + b^2 - 6ab)D}, \\
L_2 = \frac{(b - a + 1)B + 2(a + b - 1)E}{2(b - a + 1)(D + E)}.
\]

As a consequence, the double inequality

\[
\frac{\mu_{a,b}(r)}{r^{1-a-b}} \left( D - \frac{B}{2L_1} + \frac{E}{2L_2} \right) + E < \mu_{a,b}(r) < \frac{\mu_{a,b}(r)}{r^{1-a-b}} \left( D - \frac{B}{2L_1} + \frac{E}{2L_2} \right) + E
\]

holds for \( r \in (0, 1) \).

**Proof** Since \( H(r) \) is symmetric with respect to \( a, b \), we may assume that \( 0 < a \leq b < 1 \). Let \( H_1(r) = r^{1-a-b}(B/2 - E) \) and \( H_2(r) = Dr^{1-a-b} + E - \mu_{a,b}(r) \). Then we clearly see from Corollary 2.5 and \( a + b < 2 \) that \( H_1(r) = H_1(r)/H_2(r) \) and \( H_1(0^+) = H_2(0^+) = 0 \).

Moreover,

\[
\frac{H_1'(r)}{H_2'(r)} = \frac{r^{-(a+b)}[(1 - a - b)B/2 + (3a + b - 3)E + 2(1 - a)K]}{(a + b - 1)Dr^{-(a+b)}(\frac{B^2}{4a^2 + 3b^2} - 1)} = \frac{1}{(a + b - 1)D} \cdot \frac{(1 - a - b)B/2 + (3a + b - 3)E + 2(1 - a)K}{(\frac{B^2}{4a^2 + 3b^2} - 1)}
\]

(3.3)

\[
H_{11}(0^+) = H_{22}(0^+) = 0,
\]

and

\[
\frac{H_{11}'(r)}{H_{22}'(r)} = \frac{8(1 - a)}{(a + b - 1)B^2D} \cdot \left( r^{2a + b - 1}K \right)^3 \cdot h(r),
\]

(3.4)

where \( h(r) \) is defined as in Lemma 2.10.

If \((a, b) \in \Omega_3\), in other words, \( 0 < a \leq b < 1, a + b > 1 \), and \( \kappa_3(a, b) = (a + b + 1)^2 - 12ab \geq 0 \), then it follows from Lemma 2.3(iii) and Lemma 2.10 that \( r^{-(a+b)}K \) is strictly decreasing on \((0, 1)\) and \( h(r) \) is strictly decreasing on \((0, 1)\). This in conjunction with (3.3), (3.4), and Lemma 2.2 implies that \( H(r) \) is strictly decreasing on \((0, 1)\). By L'Hôpital's rule together with Lemma 2.10 and (3.3), (3.4),

\[
H(0^+) = \lim_{r \to 0^+} H(r) = \lim_{r \to 0^+} \frac{H_1'(r)}{H_2'(r)} = \lim_{r \to 0^+} \frac{H_{11}'(r)}{H_{22}'(r)} = \frac{8(1 - a)}{(a + b - 1)B^2D} \cdot \left( \frac{B}{2} \right)^3 \cdot h(0^+) = L_1,
\]

and \( H(1^-) = L_2 \) follows easily from \( \mu_{a,b}(1^-) = 0 \).

**Open Problem** What is the sub-region of \( \{(a, b) \in \mathbb{R}^2 | 0 < a, b < 1\} \) such that the function

\[
\mathcal{G}(r) = \frac{Dr^{1-a-b} + E - \mu_{a,b}(r)}{r^{1-a-b}(K - B/2)}
\]
is strictly decreasing from $(0, 1)$ onto $(0, L_3)$, where
\[
L_3 = \frac{(a + b - 1)(1 + 2a + 2b + a^2 + b^2 - 6ab)D}{2ab(3 - a - b)B}.
\]

4 Consequences and discussion
In the article, we study the monotonicity of the functions $F(r)$, $G(r)$, and $H(r)$ related to generalized Grötzsch ring function and generalized elliptic integrals, where $F(r)$, $G(r)$, and $H(r)$ are explicitly given by
\[
F(r) = \frac{Dr^{1-a-b} + E - m_{a,b}(r)}{Dr^{1-a-b} + E - \mu_{a,b}(r)},
\]
\[
G(r) = \frac{Dr^{1-a-b} + E - \mu_{a,b}(r)}{K - B/2},
\]
and
\[
H(r) = \frac{r^{1-a-b}(B/2 - E)}{Dr^{1-a-b} + E - \mu_{a,b}(r)}.
\]

5 Conclusion
In the article, we have found the sub-regions of $\{(a, b) \in \mathbb{R}^2 | 0 < a, b < 1, a + b > 1\}$ such that several quotient functions involving $\mu_{a,b}(r)$, $K_{a,b}(r)$, $E_{a,b}(r)$, and $m_{a,b}(r)$ are monotonic on their corresponding sub-regions, and established several inequalities for $\mu_{a,b}(r)$ and $m_{a,b}(r)$. Our results are the variants and extensions of the previous results of [42, Theorems 1.1 and 1.2] in the case of zero-balanced.

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