Second Order Derivative Supersymmetry and Scattering Problem

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Abstract

Extensions of standard one-dimensional supersymmetric quantum mechanics are discussed. Supercharges involving higher order derivatives are introduced leading to an algebra which incorporates a higher order polynomial in the Hamiltonian. We study scattering amplitudes for that problem. We also study the role of a dilatation of the spatial coordinate leading to a $q$-deformed supersymmetric algebra. An explicit model for the scattering amplitude is constructed in terms of a hypergeometric function which corresponds to a reflectionless potential with infinitely many bound states.
1 Introduction

The search for dynamical connections between different quantum systems (or subsystems) is one of the most interesting and fundamental problems in modern quantum mechanics. Such connections are essential to establish in order to investigate the spectral properties of certain quantum models as well as to generate new systems with given spectral characteristics.

The first step in developing explicit relations between different spectral problems was realized in the remarkable papers by Moutard and by Darboux on the spectral properties of the Sturm-Liouville differential problem, more than hundred years ago. The corresponding approach in non relativistic quantum theory was introduced in the forties by Schrödinger for the one dimensional harmonic oscillator problem and is known as the factorization method. It was extensively developed by Infeld and Hull who classified one-dimensional solvable problems.

Another useful look at the spectral equivalence of Hamiltonians intertwined by Darboux transformations originated from the supersymmetrical approach to the above problem. The supersymmetric quantum mechanics (SSQM) as such has been introduced by Witten in 1981 as a toy model to illustrate the problem of supersymmetry breaking in the general framework of supersymmetric quantum field theories. In fact, the supersymmetric quantum mechanics allows to combine two isospectral Hamiltonians, which differ at most by a single bound state, into a single Schrödinger equation at the price of introducing additional fermionic degrees of freedom. The two supersymmetric partners are related by a particular Darboux transformation. This last approach appears to be fruitful in providing possibilities of generalizing to multidimensional problems the one dimensional Darboux transformations or the traditional factorization method. The operator approach based on the factorization method allows to unravel the detailed structure of supercharges, superhamiltonians and Hilbert superspaces. SSQM has now developed in its own right as a source for establishing relations between the wave functions and various observables of different physical systems with identical energy spectra but for a few states (see for example). There are actually many different ways of constructing families of isospectral Hamiltonians, generated for instance by iterative applications of basic procedures like the Darboux one, or such as the Abraham and Moses one based upon the Gel’fand-Levitan equation and the Pursey one, to quote some.

Recently extensions of SSQM have been elaborated using different realizations of the intertwining (super)generators. In the first one, supercharges are constructed in terms of higher-derivative operators (HSSQM) and the corresponding superalgebra (HSUSY) becomes polynomial in the Hamiltonian. In the second extension a dilatation of the coordinates is involved in building supercharges. The corresponding algebra obeys $q$-deformed SUSY relations. Both extensions open new ways for exploring quantum mechanical systems with related energy spectra and wave functions.

The aim of our paper is to analyze links between these new approaches and to examine their features in the scattering regime. Attention is paid to the Witten criterion of spontaneous SUSY breaking which is no longer applicable to the case of HSUSY. In Section 2 we review the basics of one-dimensional SSQM. The Witten criterion is formulated in its conventional form. In Section 3 SSQM with supercharges of second-order in derivatives is
built in the most general form and it is found that there exist cases where the superpartners cannot be constructed by iterations of two ordinary Darboux transformations. Thus we find new irreducible elements, not discussed in [10], for building the HSUSY systems with polynomial superalgebras. We also discuss within the one dimensional problem the radial case. In Section 4 the consequences for scattering properties of the partner systems in one dimensional SUSY and HSUSY are described. In the radial problem, the necessary restrictions which provide the correct properties of potentials and wave functions are investigated. The peculiarities of the Witten index in second-derivative SSQM are discussed. In Section 5 the \( q \)-deformation of SUSY and HSUSY induced by the dilatation of the coordinate is considered. The interrelation between \( q \)-deformed SUSY algebra with the true Hamiltonian and the ordinary SUSY algebra with \( q \)-deformed Hamiltonian is established and further generalized onto the higher-derivative SUSY. The relation between scattering amplitudes of two \( q \)-superpartners is displayed as well. In conclusion, the condition of self-similarity for potentials is briefly summarized and the dual condition of self-similarity for scattering amplitudes is introduced. It is solved, in the reflectionless case, within \( q \)-deformed SSQM, in terms of a hypergeometric function.

2 SUSY Quantum Mechanics in one dimension

SSQM is generated \([1],[2]\) by supercharge operators \( Q^+ \) and \( Q^- = (Q^+)^\dagger \) which together with the Hamiltonian \( H \) of the system fulfil the algebra

\[
(Q^\pm)^2 = 0, \quad [H, Q^\pm] = 0, \quad \{Q^+, Q^-\} = H = Q^2
\]

where we have introduced the hermitian supercharge operator, \( Q = Q^+ + Q^- \).

The one-dimensional representation is realized by the \( 2 \times 2 \) supercharges

\[
Q^- = \begin{pmatrix} 0 & 0 \\ a^- & 0 \end{pmatrix}, \quad \text{and} \quad Q^+ = \begin{pmatrix} 0 & a^+ \\ 0 & 0 \end{pmatrix},
\]

where

\[
a^\pm = \pm \partial + W(x)
\]

and the superhamiltonian is assembled from two ordinary Schrödinger Hamiltonians,

\[
H = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} = \begin{pmatrix} a^+a^- & 0 \\ 0 & a^-a^+ \end{pmatrix} = -\partial^2 + \begin{pmatrix} W^2 + W' & 0 \\ 0 & W^2 - W' \end{pmatrix} = (-\partial^2 + W^2)1 + \sigma_3 W'
\]

\[
h_i \equiv -\partial^2 + V_i(x),
\]

where \( \sigma_3 \) is a Pauli matrix.

A direct consequence of Eqs. \((1),(2)\) is that all eigenvalues of \( H \) are non-negative. In terms of components, the algebra defined by Eqs. \((1),(2)\) means that the Hamiltonians \( h_1 \)
and \( h_2 \) in Eqs. (5) are factorized. To express it differently, we can say that to a given factorizable Hamiltonian \( h_1 \), one can associate a supersymmetric partner \( h_2 \) such that both partners are linked by the intertwining relations

\[
h_1 a^+ = a^+ h_2 \quad \text{and} \quad a^- h_1 = h_2 a^-.
\] (6)

These relations lead to the double degeneracy of all positive energy levels of \( H \) belonging to the "bosonic" or "fermionic" sectors specified by the grading operator \( \tau = (-)^{\hat{N}_F} = \sigma_3 \) where \( \hat{N}_F \) is the fermion number operator. The grading operator commutes with the Hamiltonian \( H \) and anticommutes with the supercharge \( Q \). Thus the supercharge operator transforms eigenstates with \( \tau = +1 \) (bosons) into eigenstates with \( \tau = -1 \) (fermions) and vice versa. Boson and fermion wave functions are eigenfunctions of \( h_1 \) and \( h_2 \) respectively. They are connected via Eqs. (6) by the operators \( a^\pm \)

\[
\sqrt{E} \Psi_E^{(2)} = a^- \Psi_E^{(1)} \quad \text{and} \quad \sqrt{E} \Psi_E^{(1)} = a^+ \Psi_E^{(2)}.
\] (7)

The existence of zero energy states depends \([4,5]\) on the asymptotics of the superpotential \( W(x) \) which appears in Eq.(4). For appropriate \( W(x) \), that is for particular topologies, they can arise either in the bosonic sector, i.e.,

\[
a^-(x) = 0,
\]

or in the fermionic one, i.e.,

\[
a^+(x) = 0.
\]

The explicit form of the solutions for the wave functions reads

\[
\Psi^{(1,2)}(x) = \exp(\pm \int x \, dy \, W(y))
\] (8)

and shows that depending on the asymptotics of the superpotential there may be three different types of vacuum characterized by the number of bosons \( N_B \) and of fermions \( N_F \):

\[
N_B = 0, N_F = 1 \quad \text{or} \quad N_B = 1, N_F = 0
\]

for which supersymmetry is an exact symmetry and the vacuum wave functions are non degenerate. The third case, where the equations \( a^\mp \Psi^{(1,2)} = 0 \) have no normalizable solutions, corresponds to

\[
N_B = N_F = 0.
\]

Supersymmetry is then spontaneously broken and the vacuum states which have positive energies are degenerate. The above difference between unbroken and spontaneously broken supersymmetry can be formulated in terms of an order parameter, the Witten index \([4]\), which may be defined by

\[
\Delta_W = \dim \ker a^+ - \dim \ker a^- = \text{ind} \ a^+ = N_B - N_F
\] (9)

and takes the values 0 and ±1. Obviously, in the case of ordinary SSQM, \( \Delta_W = 0 \) unambiguously characterizes the models with spontaneously broken supersymmetry.

\[4\]The factorization that follows from Eqs. (5) defines the Hamiltonians up to an overall constant which is related to the arbitrariness of the energy scale. We have chosen here this constant to be zero.
3 Higher-derivative SSQM

3.1 One-dimensional problem

We now explore other realizations where we keep relations (1) but allow for modifications of the relation (2). An example of a non-standard realization has been introduced in [10]; it relies on the use of higher order derivative operators in the definition of the supercharges. Instead of the linear operators of Eq. (4), let us define the second order differential operators

\[ A^+ = (A^-)^\dagger = \partial^2 - 2f(x)\partial + b(x). \]  

(10)

Then, with \( Q = Q^+ + Q^- \), Eq. (2) is transformed to

\[ \{Q^+, Q^-\} = Q^2 = K. \]

The quasihamiltonian \( K \) is thus given by the conventional superalgebra but it is now a fourth order differential operator, hence not of the Schrödinger form.

Let us assume that there exists a diagonal Hamiltonian \( H \) of Schrödinger type

\[ H = \begin{pmatrix} h^{(1)} & 0 \\ 0 & h^{(2)} \end{pmatrix}, \]  

(11)

which commutes with supercharges \( Q^\pm \) constructed from \( A^\pm \) given in \([10]\). Then it follows, from intertwining relations similar to (6),

\[ h^{(1)}A^+ = A^+h^{(2)} \quad \text{and} \quad A^-h^{(1)} = h^{(2)}A^-, \]  

(12)

that the quasihamiltonian \( K \) commutes with \( H \) and is, furthermore, given in terms of \( H \) by

\[ K = H^2 - 2\alpha H + \beta. \]  

(13)

where \( \alpha \) and \( \beta \) are constants because of the nondegeneracy of spectra of \( h^{(1)} \) and \( h^{(2)} \) in the one-dimensional problem and because \([K, Q] = 0\). The intertwining relations \([12]\) for \( H \) require that

\[ b(x) = f(x)^2 - f'(x) - \frac{f''(x)}{2f(x)} + \left( \frac{f'(x)}{2f(x)} \right)^2 + \frac{d}{4f(x)^2} \]  

(14)

where

\[ d = \beta - \alpha^2. \]  

(15)

This condition is necessary and sufficient for the existence of a generalized, polynomial, HSUSY algebra defined by

\[ (Q^\pm)^2 = 0, \quad [H, Q^\pm] = 0, \quad \{Q^+, Q^-\} = Q^2 = (H - \alpha)^2 + d. \]  

(16)

Respectively, because of Eq. (14), the potentials of the superpartner Hamiltonians are expressed solely in terms of \( f(x) \) and its derivatives,

\[ V_{1,2} = \mp 2f'(x) + f(x)^2 + \frac{f''(x)}{2f(x)} - \left( \frac{f'(x)}{2f(x)} \right)^2 - \frac{d}{4f(x)^2} + \alpha. \]  

(17)
The eigenfunctions of $h^{(1)}$ and $h^{(2)}$ are obtained from one another through the second order differential operators $A^\pm$ according to Eqs. (7).

The existence of a relevant Schrödinger operator $H$ in Eq. (13) is assured, for a given quasihamiltonian $K$

$$K = \partial^4 + \{P(x), \partial^2\} + R(x), \quad (18)$$

if and only if

$$R(x) - P^2(x) = d, \quad (19)$$

with $d$ a constant defined in (15).

In particular cases one can factorize the elementary operators $A^\pm$ in terms of ordinary superpotentials $W_1$ and $W_2$, as in Eq. (4),

$$A^+ = a_1^+ a_2^- = (\partial + W_1(x)) (\partial + W_2(x)). \quad (20)$$

When $\alpha = 0$ and $\beta = 0$ they are connected by the ladder equation

$$a_1^- a_1^+ = a_2^- a_2^+ \quad \text{or} \quad -W_1' + W_1^2 = W_2' + W_2^2. \quad (21)$$

The superpotentials $W_{1,2}(x)$ are determined by

$$W_{1,2}(x) = \pm \frac{f'(x)}{2f(x)} - f(x) \quad (22)$$

The factorization Eq. (20) arises from two successive standard SSQM transformations

$$\left( \begin{array}{cc} h^{(1)} & 0 \\ 0 & h \end{array} \right) = \left( \begin{array}{cc} a_1^+ a_1^- & 0 \\ 0 & a_1^- a_1^+ \end{array} \right) \quad (23)$$

and

$$\left( \begin{array}{cc} h & 0 \\ 0 & h^{(2)} \end{array} \right) = \left( \begin{array}{cc} a_2^+ a_2^- & 0 \\ 0 & a_2^- a_2^+ \end{array} \right) \quad (24)$$

together with the ladder condition (21). One can then define $H$ as

$$H = \left( \begin{array}{ccc} a_1^+ a_1^- & 0 & 0 \\ 0 & a_2^- a_2^+ \end{array} \right). \quad (25)$$

This polynomial, higher-derivative algebra for SSQM (HSSQM) obviously yields the double degeneracy of the positive energy part of the spectrum of $H$.

Nontrivial effects can be found in the lowest energy subspace of both $h^{(1)}$ and $h^{(2)}$. In particular, in this case, the Witten criterion (9) of spontaneous breaking of supersymmetry is not effective. To illustrate this point, let us write the most general normalizable solution of the zero-mode equations, i.e. $A^\pm \Psi_{B,F}(x) = 0$,

$$\Psi_{B,F}(x) = A_{B,F} \sqrt{f(x)} \exp(\pm \int_{x_0}^{x} dy f(y)) \quad (26)$$

where $A_{B,F}$ are constants. The asymptotic behaviour of $f(x)$ determines the number of zero modes of $A^\pm$ and their properties. The Witten index of operators $K$ and $H$ in the
present case varies in general from $-1$ to $+1$ but the most interesting situation arises for configurations such that

$$f(x) \rightarrow 0 \quad x \rightarrow \pm \infty$$

and

$$\int_{-\infty}^{+\infty} dy f(y) < \infty$$

(28)

One has then zero modes of operator $H$ with both fermionic and bosonic components,

$$N_B = N_F = 1$$

We can conclude thus that the Witten criterion is not applicable to HSSQM. Namely, this configuration has $\Delta_W = 0$ though it does not reveal spontaneous breaking of supersymmetry due to the existence of the doubly degenerate zero modes of $A^\pm$.

One can proceed to a similar analysis if one imposes a constant shift $c$ to the ladder equation

$$a_2^+ a_2^- + c = a_1^+ a_1^-$$

(29)

where we consider $c > 0$ without loss of generality. This equation implies the following relation

$$\{Q^+, Q^-\} = \tilde{H}(\tilde{H} - c)$$

(30)

where to compare with Eq. (13)

$$\tilde{H} = H + \frac{1}{2}(c - 2\alpha) = \begin{pmatrix} a_1^+ a_1^- & 0 \\ 0 & a_2^- a_2^+ + c \end{pmatrix}$$

(31)

The constant $c$ is actually related to $d$ defined in Eq. (15) by

$$c^2 = -4d$$

(32)

and the shifted ladder condition is only meaningful for $d < 0$.

We consider now the case $c = 2\alpha$ which implies $\beta = 0$. The superpotentials $W_{1,2}$ can then be parametrized in terms of a function $f(x)$ as follows

$$W_{1,2} = \pm \frac{2f'(x) + c}{4f(x)} - f(x).$$

(33)

If there exist zero-modes of both supercharges $Q^\pm$ then they can generally expressed in terms of $f(x)$ and $c$ as follows,

$$\psi_{B,F} = A_{B,F} \exp\left\{ \int_a^x \left( \mp f(y) + \frac{2f'(y) + c}{4f(y)} dy \right) \right\}$$

$$+ D_{B,F} \exp\left\{ \int_a^x \left( \mp f(y) + \frac{2f'(y) - c}{4f(y)} dy \right) \right\}.$$

(34)

The Witten index for $c \neq 0$ is defined according to Eq. (9) where $N_{B,F}$ characterize the number of zero modes of supercharges $Q^\pm$ which may not be the ground states of $H$. For $c > 0 (c < 0)$ it can take the values between $-1$ and $+2 (-2 \text{ and } +1)$. 

7
The Witten’s criterion is not working again for functions \( f(x) \) such that
\[
f(x) \to f_\pm; \quad x \to \pm \infty;
\]
\[
\text{sign} f_+ = -\text{sign} f_- = +1, \quad 16 f_\pm^4 < c^2,
\] (35)
and
\[
f(x) \big|_{x \to x_i} = -\frac{1}{2}c(x-x_i) + o(x-x_i).
\] (36)
The limiting case \( c = 0 \) is reproduced only if simultaneously \( c \to 0 \) and \( x_l \to \infty \).

In the case \( c > 0 \), the regularized Witten index \([4],[12]\) reads
\[
\Delta_{\text{reg}}^W = \text{Tr}[(-)^{NF}e^{-\gamma H}] = 1 - e^{-\gamma c}.
\] (37)
and takes any values between 0 and 1 for positive \( \gamma \). Hence, we have seen that for HSSQM systems the Witten index does not characterize the spontaneous breaking of supersymmetry.

As we have seen, the shifted ladder equation (29) is only meaningful if the discriminant \( d \) is negative; it is this property that allows to introduce the intermediate hermitian Hamiltonian in Eqs. (23), (24). If \( d > 0 \), no such relation as (29) can be defined; we therefore refer to this class of second order derivative SSQM as irreducible.

Such a class represents a new primitive element in building of supersymmetrical ladders and respectively of polynomial SSQM. Clearly, the corresponding supercharge cannot possess any zero modes since it is bounded from below by the constant \( \sqrt{d} \), as shown by Eqs (16). Therefore, for such a supersymmetric Hamiltonian the Witten criterion is valid. In the case under discussion the function \( f(x) \) should be assumed to be nodeless in order to avoid singular supercharges and potentials, Eq. (17). For instance, one can make the following ansatz, \( f(x) = \exp(x) + \exp(-2x) \) and obtain two equivalent Hamiltonians with wave functions connected by operators \( A^\pm \), Eq. (10). The generalization of second order derivative SUSY algebra to differential operators \( A^\pm \) of higher order is straightforward and leads to polynomials of \( H \) in the right hand side of Eq. (16). The ladder construction (21), (29) is applicable when it is made of the primitive elements of the first and second order in derivatives. Thus one expects, in general, the following polynomial superalgebra,
\[
Q^2 = \prod_{i+j=n} (H - c_i)((H - \alpha_j)^2 + d_j); \quad d_j > 0.
\] (38)

### 3.2 Radial problem

Sofar we have considered one-dimensional SSQM on the line, \( x \in (\infty, +\infty) \). In problems with rotational symmetry one introduces the radial Schrödinger operator on the half line, \( r \in [0, +\infty) \), and one can similarly discuss features of second-derivative SSQM; for definiteness, we restrict ourselves to the 3-dimensional case.

The radial problem differs by the boundary conditions. After reduction of the radial part of Schrödinger operator to the conventional form, Eq. (3), one finds in standard SSQM with nonsingular potentials that in the \( l + m \) partial wave the behavior of the wave function is regular at the origin \( \Psi \sim r^{l+1} \). Furthermore the corresponding superpotential behaves as \( W(r) \sim (l+1)/r \) or \( \sim -l/r \) for \( r \sim 0 \). As a consequence the superpartners describe different partial waves: \( l_2 = l_1 \pm 1 \).
Here we take a slightly more general approach allowing for different behavior of the wave function reflecting singularities of centrifugal type and also possible existence of bound states at zero energy (generating as we will see in the next section anomalies in the scattering problem). Therefore we only require that the reduced radial wave function vanishes at the origin and (for the discrete spectrum) decreases at infinity faster than $1/\sqrt{r}$.

Thus in order to construct the HSSQM Hamiltonian we assume that

$$f(r) \mid_{r \to 0} \sim f_0 \cdot r^\lambda$$

(39)

Acceptable values of $\lambda$ are constrained by the combination of two requirements:

a) the potentials $V_{1,2}$ from Eq.(17) with centrifugal terms at $r \sim 0$ for the angular momentum $l$ and with a core at short range,

$$V_{1,2} \sim \mp 2 f_0 \lambda r^{\lambda-1} + f_0^2 r^{2\lambda} + \frac{\lambda(\lambda - 2)}{4r^2} - \frac{d}{4f_0^2 r^{2\lambda}} \sim \frac{l(l+1) + \gamma_{1,2}}{r^2},$$

(40)

should be nonsingular, i.e., $\gamma_{1,2} > -1/4$; we consider the core coupling constants $\gamma_{1,2}$ as independent of $l$;

b) the mapping of wave functions realized by the operators $A^\pm$ of Eq. (11)

$$\sqrt{E} \Psi_E^{(2)} = A^- \Psi_E^{(1)} = \left(V_1(r) - E + 2f(r)\partial_r + 2f'(r) + b(r)\right) \Psi_E^{(1)}$$

$$= \left(2f^2(r) - f'(r) - E + 2f(r)\partial_r\right) \Psi_E^{(1)}$$

(41)

(where the basic Schrödinger equation has been applied) should reproduce the correct physical behavior when $r \to 0$ for a particular angular momentum $l$.

We obtain two possible behaviors for $f(r)$ in the vicinity of the origin:

1) For $\lambda = -1$ the consistency requirements can be satisfied for all angular momenta $l_1$ only if $\gamma = 0$. The related parameter $f_0$ takes the following values,

$$f_0 = -l_1 - 3/2; \quad \text{for} \quad l_2 = l_1 + 2;$$

$$f_0 = l_1 - 1/2; \quad \text{for} \quad l_2 = l_1 - 2 \quad \text{if} \quad l_1 \geq 2.$$  

(42)

(43)

If $d < 0$ the HSSQM system can be embedded into a ladder of two standard SSQM problems (see Eq. (21)) but the case $d > 0$ is again irreducible.

2) For $\lambda = +1$ the solution exists for $d < 0$ only which can be always embedded into the ladder of standard SUSY systems with raising and lowering of angular momentum so that as a result,

$$l_1 = l_2 = l, \quad \text{and} \quad f_0^2 = \frac{-d}{(2l+1)^2 + 4\gamma}; \quad \gamma_{1,2} = \gamma.$$  

(44)

For other values of $\lambda$ and $f_0$ either the operators $A^\pm$ map from the physical space into an unphysical one or a potential is singular. In both cases the formal intertwining relations between $h_1$ and $h_2$ do not result in the equivalence of energy spectra and in the connection of physical wave functions.

The higher-order polynomial superalgebra for the radial problem can be constructed in the same form as Eq.(38). However in this case one obtains generally the spectral equivalence between different partial waves when keeping in mind the preceding analysis.
4 SSQM and scattering problem

4.1 One-dimensional scattering

Consider the one dimensional scattering problem on the line, i.e. \( x \in (-\infty, +\infty) \), for the Hamiltonian

\[
h_1 = -\partial^2 + V_1(x) = a^+ a^-
\]

(45)

with a potential that reaches its constant asymptotic value (fast enough)

\[
V_1(x) \to C \quad x \to \pm \infty
\]

(46)

where the bound state energy spectrum of \( h_1 \) is bounded by the constant \( C \) \( (0 \leq E_n \leq C) \)

\[
h_1 \psi^{(1)}_n(x) = E_n \psi^{(1)}_n(x); \quad n = 0, 1, ..
\]

(47)

while the continuous spectrum is given by, \( E(k) \geq C \),

\[
h_1 \psi^{(1)}(k) = E(k) \psi^{(1)}(k); \quad E(k) = k^2 + C.
\]

(48)

The scattering wave function fulfils the asymptotic conditions (see, for example [14])

\[
\psi^{(1)}_{k, -\infty} = e^{ikx} + R^{(1)}(k) e^{-ikx}
\]

(49)

and

\[
\psi^{(1)}_{k, +\infty} = T^{(1)}(k) e^{ikx}
\]

(50)

where \( R^{(1)}(k), T^{(1)}(k) \) are the reflection and transmission coefficients respectively.

The ladder operators \( a^\pm \) are asymptotically expressed as

\[
a^{-\infty}_{\pm} = -\partial + W_\pm; \quad a^{+\infty}_{\pm} = \partial + W_\pm
\]

(51)

with

\[
W_\pm = \lim_{x \to \pm \infty} W(x); \quad W^2_\pm = C
\]

(52)

The asymptotic scattering wave function of the partner Hamiltonian \( h_2 \) is then proportional to

\[
a^{-\infty}_{\psi^{(1)}_{k, -\infty}} = (-ik + W_-) \left[ e^{ikx} - R^{(1)}(k) \frac{k - iW_-}{k + iW_-} e^{-ikx} \right]
\]

(53)

while

\[
a^{+\infty}_{\psi^{(1)}_{k, +\infty}} = (-ik + W_-) \left[ T^{(1)}(k) \frac{k + iW_-}{k + iW_-} e^{ikx} \right].
\]

(54)

Hence the transmission and reflection coefficients associated to \( h_2 \) are respectively [8], [15]

\[
T^{(2)}(k) = T^{(1)}(k) \frac{k + iW_+}{k + iW_-}
\]

(55)

and

\[
R^{(2)}(k) = -R^{(1)}(k) \frac{k - iW_-}{k + iW_-}
\]

(56)
One can recognize, as is well known \[14\], that the transmission coefficient contains physical
poles in the upper half of the complex \( k \)-plane, their positions corresponding to the energies
of bound states \( E_j = -\kappa_j^2 + C, \ k_j = i\kappa_j \). Thus the difference in physical pole structure of
\( T^{(1)} \) and \( T^{(2)} \) depends on the signs of \( W_\pm \). Unitarity always holds for \( T^{(2)}, R^{(2)} \) whenever it
holds for \( T^{(1)}, R^{(1)} \).

We now want to relate the above properties of transmission coefficient to the Witten
index introduced in the previous section. Consider the equations

\[
a^\mp \Psi_E^{(1),(2)}(x) = 0 \tag{57}
\]

whose solutions are given in Eq.(8). Three different cases arise

1. \( W_- > 0 \quad W_+ < 0 \quad \Delta_W = 1 - 0 = 1 \quad E_0^{(1)} = 0 \) and \( E_n^{(2)} > 0 \)
2. \( W_- < 0 \quad W_+ > 0 \quad \Delta_W = 0 - 1 = -1 \quad E_0^{(2)} = 0 \) and \( E_n^{(1)} > 0 \)
3. \( W_+ W_- > 0 \quad \Delta_W = 0 \quad E_n^{(1)} > 0 \) and \( E_n^{(2)} > 0 \)

This establishes the following connection

\[
T^{(2)}(k = 0) = T^{(1)}(k = 0) \exp(i\pi \Delta_W) \tag{58}
\]
between the relative phase of the supersymmetric partner amplitudes and the Witten index
(see also D.Boyanovsky and R.Blankenbecler \[15\]).

The scattering problem for HSSQM, as defined in Eqs. (16), (17) can be deduced from
Eqs. (55),(56) by iteration:

\[
T^{(2)}(k) = T^{(1)}(k) \frac{(k + iW_2')(k + iW_1')}{(k + iW_2')(k + iW_1')} \tag{59}
\]

where, assuming \( f_\pm = \lim f(x) \) for \( x \to \pm\infty \) are constant so that the potentials \( V_{1,2} \) at
\( x \to \pm\infty \) are finite and following Eq. (33), the asymptotic values of the superpotentials are
given by

\[
W_{1\pm} = \frac{c}{4f_\pm} - f_\pm; \quad W_{2\pm} = -\frac{c}{4f_\pm} - f_\pm. \tag{60}
\]

In order to guarantee that the asymptotic values of the potentials \( V_i \) both at \(+\infty\) and at
\(-\infty\) are equal, one must have either \( f_+^2 = f_-^2 \) or \( 16f_+^2 f_-^2 = c^2 \). The Witten index is still
determined by Eq. (9) and obviously can take any integer value between 2 and -1 for \( c > 0 \).
As can be seen from Eq. (33) the breaking of the Witten criterion occurs if \( f_+ = -f_- \) and
\( 16f_\pm^4 < c^2 \).
The peculiar situation in which the Witten criterion is not valid appears to be described by \( c = 0 \) and \( W_{1,2} = 0 \).\(^5\) Although the wave functions, connected by the intertwining relations, are different as well as the potentials, the spectrum and the phase shifts of the two supersymmetric partners are identical. This illustrates explicitly the fact that the knowledge of the phase shifts and of the position of the bound states do not identify uniquely a local potential.

For \( d > 0 \) and nonsingular potentials the function \( f(x) \) is nodeless and in order to have equal asymptotics Eq. (46) one should impose \( f_+ = f_- = f_\infty \). Respectively the transmission and reflection coefficients are connected as follows:

\[
T^{(2)} = T^{(1)} \quad R^{(2)} = R^{(1)} \left( \frac{(k + i f_\infty)^2 - \frac{d}{4f_\infty}}{(k - i f_\infty)^2 - \frac{d}{4f_\infty}} \right) \tag{61}
\]

The equality of the transmission coefficients is accounted for by equal asymptotic values for each potential at \( \pm \infty \), as in (46).

### 4.2 Radial scattering problem

The radial problem can be treated in a very close manner except that the running variable \( r \) varies from 0 to \( \infty \) and the partner potentials \( V_{1,2} \) go asymptotically to the constant value \( C \). The value at the origin may be finite or infinite but the \( s \)-wave wavefunction must go to 0 linearly with \( r \) as \( r \) goes to 0. Asymptotically one has then

\[
\Psi^{(1),l}_{k,\infty} = e^{-ikr} + (-1)^{l+1} S^{(1)}_{l}(k)e^{ikr} \tag{62}
\]

and, in analogy with the preceding developments,

\[
a_{\infty}^{-} \Psi^{(1),l_1}_{k,\infty} = (ik + W_\infty) \left[ e^{-ikr} + (-1)^{l_1+1} S_{l_1}^{(1)}(k) \frac{k + i W_\infty}{k - i W_\infty} e^{ikr} \right] \tag{63}
\]

It is clear that the supersymmetric transformation introduces singularities in the potential. Hence, there are two different interpretations possible. One may consider a fixed \( l \) partial wave, each supersymmetric transformations adding a singular contribution, reminiscent of a centrifugal barrier, to the preceding potential [8]. The second interpretation is based on the recombination of the original potential with the dynamically generated centrifugal barriers and hence the superpartner is considered to be associated to a different orbital momentum \( l \) (see, for instance, F. Cooper et al. in ref. [15]). We will refer from now on to this last interpretation.

If \( W_\infty \neq 0 \) the \( S \)-matrix element \( S_{l_2}^{(2)}(k) \) associated to the partner Hamiltonian is given by

\[
S_{l_2}^{(2)}(k) = -S_{l_1}^{(1)}(k) \frac{k + i W_\infty}{k - i W_\infty}, \quad l_2 = l_1. \tag{64}
\]

\(^5\)This case can provide examples of zero energy bound states. There is the possibility that one partner has indeed a normalizable state at zero energy whereas the companion state is not normalizable. This is due to the fact that the relations between the norms involves the square of the energy (i.e.0). We thus would realize an example of partnership between a bound state and a continuum like state.
reflecting the presence of long-range forces $\sim r^{-3}$. In this case the behaviour of $S_{l_2}^{(2)}(k)$ for small $k$ is of $s$–wave type independently of the angular momentum. We remind that the correct behaviour of wave functions at the origin requires that $l_2 = l_1 \pm 1$. It can be made compatible with Eq. (64) only at the expense of introducing a centrifugal core which depends on angular momentum.

On the other side for $W_\infty = 0$ the two $S$-matrices coincide

$$S_{l_2}^{(2)}(k) = S_{l_1}^{(1)}(k), \quad l_2 = l_1 \pm 1.$$  \hspace{1cm} (65)

signaling an anomalous threshold behaviour related to the existence of zero energy pole which makes the effective range expansion invalid.

The radial scattering problem for the second order derivative SUSY system can be readily solved if $d = 0$, by repetition of the single SSQM relations. Evidently $S$-matrices will coincide belonging to partial waves $l_2 = l_1, l_1 \pm 2$.

Let us analyze the features of scattering problem for HSSQM Hamiltonians when $d \neq 0$. We assume the following asymptotics for $f(r)$:

$$f(r) = \bar{f}_0 + \bar{f}_1 r^\lambda + \bar{f}_2 r^{2\lambda} + \cdots; \quad r \to \infty,$$  \hspace{1cm} (66)

where $\bar{f}_{0,1} \neq 0$. Let us restrict ourselves to potentials decreasing at least as fast as its centrifugal part (or faster in the $s$-wave) that gives the bound $\lambda \leq -1$.

We have found four essentially different types of potentials depending on their asymptotics.

a) The case $\lambda = -1$ is consistent for

$$d = -4\bar{f}_0^4 < 0,$$  \hspace{1cm} (67)

if the Coulomb-type long range forces are absent. The relevant SUSY transformation gives rise to the following change in the angular momentum,

$$l_2 = l_1 - 1, \quad \bar{f}_1 = \frac{1}{2} l_1;$$

$$l_2 = l_1 + 1, \quad \bar{f}_1 = -\frac{1}{2} (l_1 + 1).$$  \hspace{1cm} (68)

When comparing with the requirements on the wave function at the origin, Eqs. (P3), (P4), one concludes that they can be satisfied if the related potentials have short range core of centrifugal type which depends on angular momentum.

b) In the case $-2 < \lambda < -1$ the consistency is again assured by Eq. (67) and the potentials fall off $\sim r^{\lambda-1}$ and display $s$-wave behavior.

c) For $\lambda = -2$, whatever is the sign of $d$, the centrifugal term is not changed under the HSUSY transformation and the coefficients $\bar{f}_1$ obey the relation

$$2\bar{f}_0 \cdot \bar{f}_1 \left(1 + \frac{d}{4\bar{f}_0^4}\right) = l(l+1).$$  \hspace{1cm} (69)

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6For a discussion see Amado et al.Int.J.Mod.Phys. quoted in Ref.8
7for a discussion see Amado et al.Phys.Rev. C43 quoted in Ref.8
d) For $\lambda < -2$, only $s$-wave is allowed.

Thus the consistent treatment of wave functions behaviour at the origin and at the infinity selects out the only possibility $l_1 = l_2, d < 0$. The irreducible transformation $d > 0$ induces the short range forces depending on angular momentum. The connection between $S$-matrices of superpartner systems is given by,

$$S^{(2)}_{l_2} = S^{(1)}_{l_1} \frac{k^2 - 2ik\hat{f}_0 - \hat{f}_0^2 - \frac{d}{4\hat{f}_0}}{k^2 + 2ik\hat{f}_0 - \hat{f}_0^2 - \frac{d}{4\hat{f}_0}},$$

(70)

and, generally, they differ by two poles.

The potentials belonging to the "a" or "b"-type only differ by one pole because of the constraint (67), so that

$$S^{(2)}_{l_2} = S^{(1)}_{l_1} \frac{k - 2i\hat{f}_0}{k + 2i\hat{f}_0},$$

(71)

5 $q$-deformed SUSY quantum mechanics

The concept of $q$-deformations has drawn recently much attention as a way to extend the description of symmetries based on Lie algebras [14]. In application to SSQM, an extension based on $q$-deformed SUSY algebra has been developed by V. Spiridonov [11] in a particular realization exploiting the dilatation of coordinates.

The main tool for a $q$-deformation is given by the dilatation operator,

$$T_q f(x) = \sqrt{q} f(qx), \quad T_q \partial_x = q^{-1} \partial_x T_q,$$

(72)

which can be represented by the following pseudodifferential operator

$$T_q = \sqrt{q} \exp(ln q x \partial_x), \quad T_q^\dagger = T_q^{-1}.$$  

(73)

Let us now introduce the $q$-deformed components of supercharges $Q^\pm$,

$$a^+_q = (\partial + W(x)) T_q, \quad a^-_q = T_q^\dagger (-\partial + W(x)),$$

(74)

in the notations of Section 2. The Hamiltonian components (see Eq. (5)) become now the $q$-deformed

$$h_1 = a^+_q a^-_q = -\partial^2 + W^2(x) + W'(x),$$

$$h_2 = q^{-2} a^-_q a^+_q = -\partial^2 + q^{-2} W^2(q^{-1} x) - q^{-1} W'(q^{-1} x),$$

(75)

if the kinetic terms are properly normalized. Evidently these Hamiltonians are not intertwined by means of the standard SUSY algebra (see Eq. (5)). Rather they obey the $q$-deformed intertwining relations,

$$h_1 a^+_q = q^2 a^+_q h_2, \quad a^-_q h_1 = q^2 h_2 a^-_q.$$  

(76)

The complete $q$-deformed SUSY algebra has the conventional form, Eq. (5) but with the $q$-(anti)commutators instead of the usual ones,

$$[X,Y]_q \equiv XY - q^{-2} YX, \quad \{X,Y\}_q \equiv XY + q^{-2} YX.$$  

(77)
In particular, one has

$$\{Q^+, Q^-\}_q = H; \quad [Q^+, H]_q = [H, Q^-]_q = 0.$$  \hfill (78)

Obviously the supercharges are not conserved now in the usual sense because they do not commute with the Hamiltonian. As a consequence the superpartner Hamiltonians are no more isospectral but their spectra are related by $q$-dilatation, $E^{(1)} = q^2 E^{(2)}$. We notice however that there exists a $q$-Hamiltonian, $H_q$, which commutes with supercharges and obeys the superalgebra,

$$\{Q^+, Q^-\} = H_q \equiv \begin{pmatrix} h_1 & 0 \\ 0 & q^2 h_2 \end{pmatrix}; \quad [H_q, Q^\pm] = 0.$$  \hfill (79)

The wave functions are unambiguously interrelated by the action of the operators $a_q^\pm$,

$$\psi^{(2)} \propto a_q^- \psi^{(1)}, \quad \psi^{(1)} \propto a_q^+ \psi^{(2)}.$$  \hfill (80)

Obviously the Witten-type analysis of spontaneous SUSY breaking can be extended for $q$-deformed SUSY, in terms of $q$-Hamiltonian $H_q$, in full analogy with the standard SSQM in Section 2.

The above realization of $q$-deformed SUSY is flexible enough to be easily combined with other extensions of SSQM such as the paraSUSY ([20]) and HSUSY (see Section 3) quantum mechanics. The particular (factorizable) construction of the second order derivative $q$-deformed SUSY algebra can be realized by means of a sequence of two $q$-deformations ([74]) with different dilatation parameters $q_1, q_2$. The relevant components of the $q$-supercharge are given by products,

$$A_q^+ = q_1 a_{q_1}^+ a_{q_2}^+ = q_1 (\partial + W_1(x)) T_{q_1} (\partial + \tilde{W}_2(x)) T_{q_2} = (A_q^-)^\dagger.$$  \hfill (81)

The supercharge components can be transformed to the form of second order derivative operator augmented with the single dilatation,

$$A_q^+ = (\partial + W_1(x)) (\partial + W_2(x)) T_q; \quad q = q_1 \cdot q_2; \quad W_2(x) \equiv q_1 \tilde{W}_2(q_1 x).$$  \hfill (82)

In such a form the generalization of second order derivative SUSY is straightforward and can be performed following the scheme of Section 3 with the dilatation at the last step both for the reducible ($d < 0$) and for the irreducible ($d > 0$) cases.

As in the ordinary $q$-deformed case the Hamiltonian $H$ does not commute with the supercharge but it satisfies Eqs. (78). The $q$-deformed SUSY algebra with the Hamiltonian $H$ reads

$$Q^+ Q^- + q^{-4} Q^- Q^+ = \{Q^+, Q^-\}_q^2 = (H - \alpha q^{\sigma_3 - 1})^2 + d \cdot q^{2(\sigma_3 - 1)},$$

$$[Q^+, H]_q = [H, Q^-]_q = 0,$$  \hfill (83)

where $\sigma_3$ is the Pauli matrix.

15
But again there exists a $q$-Hamiltonian $H_q$, Eq. (79), which commutes with the supercharges. Moreover in terms of $H_q$ the HSUSY algebra takes the usual form, Eq. (16).

Further steps in extension of the higher order derivative SUSY either lead to Eq. (38) with the $q$-Hamiltonian and the conventional SUSY or to its $q$-deformed version [10] with the true Hamiltonian $H$ and the primitive blocks defined in (83),

$$\{Q^+, Q^-\}_q = \prod_{i+j=n} (H - c_i \cdot q^{\sigma^1-1}) \left((H - \alpha_j q^{\sigma^3-1})^2 + d_j \cdot q^{2(\sigma^3-1)}\right);$$

$$[Q^+, H]_q = [H, Q^-]_q = 0; \quad d_j > 0. \quad (84)$$

Concerning the scattering problem for the above models, the asymptotic relations, Eqs. (53), (54), (55), (56) should be modified taking into account Eq. (74). As a result one finds the connection between transmission and reflection coefficients of $h_1$ and $h_2$ which now involves the dilatation parameter $q$,

$$T^{(2)}(k) = T^{(1)}(k) \frac{(kq + iW_+)}{(kq + iW_-)}$$

$$R^{(2)}(k) = -R^{(1)}(k) \frac{(kq - iW_-)}{(kq + iW_-)} \quad (85)$$

Thus both in the discrete and in the continuous parts of spectrum we remove the spectrum degeneracy typical for the ordinary SSQM and come to scaling relations.

In the framework of $q$-deformed SUSY mechanics the problem of the construction of self-similar potentials has recently been tackled [11] and a class of such potentials has been found [11], [21]. They are obtained from solutions of ladder equations with $q$-periodic closure conditions. For instance, in the case of the ordinary $q$-deformed SUSY mechanics (75) they lead to the following equation for the superpotential [21],

$$V_1(x) = W'(x) + W^2(x) = V_2(x) + c = -q^{-1}W'(q^{-1}x) + q^{-2}W^2(q^{-1}x) + c. \quad (86)$$

In the scattering regime the potentials are decreasing at the infinity and therefore $c = W^2_\pm(1 - q^{-2})$.

The $q$-deformation of scattering data allows us to formulate the dual self-similarity condition which in general does not imply the self-similarity of potentials. We formulate this condition again as a closure condition like [21] but for scattering amplitudes in momentum space,

$$T^{(2)}(k) = T^{(1)}(k); \quad R^{(2)}(k) = R^{(1)}(k), \quad (87)$$

which is combined with Eq. (85). Their solutions for $q > 1$ and $W_+ = -W_- > 0$ are

$$T^{(1)}_{q>1}(k) \equiv T(q, \frac{W_+}{k}) = \prod_{n=1}^{\infty} \frac{(k + iW_+/q^n)}{(k - iW_+/q^n)} t(ln k) \equiv \Phi_0(-1; iW_+/k)t(ln k);$$

$$R^{(1)}_{q>1}(k) \equiv R(q, \frac{W_+}{k}) = \prod_{n=1}^{\infty} \frac{(k + iW_-/q^n)}{(k - iW_-/q^n)} r(ln k) \equiv \Phi_0(-1; iW_+/k)r(ln k), \quad (88)$$
where \(|t|^2 + |r|^2 = 1\) and \(t(z)(r(z))\) are periodic (antiperiodic) functions,

\[
t(z + \ln q) = t(z), \quad r(z + \ln q) = -r(z)
\]

and the self-similar transmission and reflection coefficients are parametrized by the hyper-geometric function, \(1\Phi_0(a; z)\) \[22\]. If \(0 < q < 1\) and \(W_+ = -W_- < 0\) then the consistent solution is expressed in terms of the previous one, Eqs.(88),

\[
T_{q<1}^{(1)}(k) = T(1/q, -\frac{W_+}{k})\frac{k - iW_+}{k + iW_+}
\]  

(89)

and a similar equation takes place for \(R_{q<1}^{(1)}(k)\).

If one imposes the condition of vanishing of the reflection coefficient for high \(k\), which amounts to the validity of the Born expansion, \[14\], one is lead to assume \(r(\ln k) = 0\) and to put \(t(\ln k) = 1\), respectively. It is thus interesting to analyze the interconnections between self-similarities of reflectionless potentials and of scattering amplitudes \[23\]. It is clear that the underlying reflectionless potential has infinitely many bound states with an accumulation point around zero. As the Born expansion is valid it means that such a potential is slowly decreasing and oscillating. Our construction is connected to recent investigations concerning a class of reflectionless potentials with infinitely many bound states \[28\], we stress however that our starting point has been the exploration of the condition of self-similarity for the scattering coefficients leaving aside the detailed \(x\) dependence of the (super)potentials \[8\].

One can easily derive \(q\)-deformed HSUSY relations between the transmission and reflection coefficients guided by similar arguments as those leading to (59) and (61).

In conclusion we remark that both the polynomial SSQM and \(q\)-deformed SSQM may be generalized onto matrix potentials and partially extended onto higher space dimensions. Some of extensions are considered in \[24\] and \(q\)-deformations of polynomial SSQM will be described elsewhere.

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