Abstract

The paper considers the problem of finding the metric of space time around a rotating, weakly gravitating body. Both external and internal metric tensors are consistently found, together with an appropriate source tensor. All tensors are calculated at the lowest meaningful approximation in a power series. The two physical parameters entering the equations (the mass and the angular momentum per unit mass) are assumed to be such that the mass effects are negligible with respect to the rotation effects. A non zero Riemann tensor is obtained. The order of magnitude of the effects at the laboratory scale is such as to allow for experimental verification of the theory.
Space time and rotations

A. Tartaglia

Gravity Probe B, Hansen Experimental Physics Labs, Stanford University,
Stanford (CA) and INFN, Torino, Italy*
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*Permanent address: Dip. Fisica, Politecnico, Torino, Italy.

**Permanent address: Dip. Fisica, Politecnico, Torino, Italy.
I. INTRODUCTION

Rotational motion has a peculiarity on its own since it appears to be absolute, unlike translational motion, which is purely relative. This absoluteness of rotation posed a principle problem since the very time of Newton with his rotating bucket example, which led precisely to the conclusion that rotational motion was absolute [1]. A couple of decades after Newton’s Principia, George Berkeley questioned his notion of an absolute space [2] and successively Ernst Mach, looking for the origin of inertia, stuck to the idea that even rotations are relative [3]. Mach’s approach was one of the ideas that inspired Einstein in developing the general theory of relativity [4], although the incorporation of Mach’s principle into the theory is not entirely satisfactory. In general, rotation is rather poorly treated in general relativity. Its effect is essentially reduced to affecting the dynamical mass of the rotating body, exactly as the translational motion does, and the space time geometry through gravitomagnetic effects [5], mediated again by the mass of the source. This is the case of the Lense-Thirring effect [6] and of the gravitomagnetic clock effects [7]. However the treatment of the translational motion incorporates the very notion of inertial reference frames and inertial observers, whereas rotating systems do not identify any class of equivalent observers. The rotation state itself is usually identified with respect to the asymptotic flatness of space time, or more specifically to the constraint that the metric tensor far away from the source assumes the Minkowski form. The rotation is confronted with a non rotating (Minkowskian) space time, which serves as an (absolute) reference frame.

The study of exact vacuum solutions of the Einstein equations has produced many metric tensors, which are stationary and endowed with axial symmetry [8]. These solutions include situations which correspond to rotating sources in an asymptotically flat space time. The most renowned is of course the Kerr metric [9], which contains two independent parameters characterizing the source: the asymptotic mass and the asymptotic angular momentum per unit mass. The latter quantity is interesting because it actually does not depend on the mass itself, rather expresses a sort of purely rotational property. Unfortunately however up to now it proved impossible to link the vacuum Kerr solution to a satisfying internal metric of a given matter distribution.

From general relativity we know that a mass curves space time around. If the mass rotates, the peculiar motion introduces further warps in space time. Suppose now that the influence of the very mass becomes weak (as it is normally the case within the Solar system, for instance), but the rotation stays important: would there be a residual effect on space time? In fact the condition in which mass is negligible and rotation is not is rather easy to obtain [10].

This is the problem this paper will address. The first step will be to identify the physical parameters describing the body and its rotation state. Then the metric tensor outside the spinning body will be found, in the form of an inverse powers of the distance development. Next will come the metric tensor inside the rotating matter distribution and finally the source tensor corresponding to the internal metric tensor and generating the external one. As we shall see, a complete consistent solution will be found, producing expectedly measurable effects on space time.
II. GENERAL FORM OF THE APPROXIMATED EXTERNAL METRIC TENSOR

We start from the empty space-time line element in polar coordinates:

\[ ds^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 \]

Let us assume that the presence of an axially symmetric steadily rotating mass at the origin introduces a perturbation such that the metric becomes

\[ ds^2 = c^2 (1 + h_{00}) dt^2 - (1 + h_{rr}) dr^2 - r^2 (1 + h_{\theta\theta}) d\theta^2 - r^2 \sin^2 \theta (1 + h_{\phi\phi}) d\phi^2 + 2h_{t\phi} r \sin \theta c dt d\phi \]

with the condition

\[ h_{\mu\nu} << 1 \]

The elements of the perturbation tensor, because of the assumed symmetries, depend geometrically on \( r \) and \( \theta \) only. Physically the perturbation must depend on two quantities, which characterize the body and its motion: the mass \( M \) and the angular velocity \( \Omega \), or its combination with the mass into the angular momentum \( J \). Both \( \Omega \) and \( J \) are defined from the viewpoint of an inertial observer at rest with respect to the rotation axis of the body, thus coinciding with the components of the corresponding three-vectors on that axis. Expressing the mass and the angular momentum as lengths it is possible to introduce the quantities \( \mu = GM/c^2 \) and \( a = J/Mc \); \( a \), apart from a factor depending on the shape of the body and the matter distribution inside it, is proportional to \( \Omega \) and does not depend on \( M \) any more.

Once the reference frame has been fixed, the sign of the parameter \( a \) varies according to the two directions of rotation: \( a \) is odd for reversal of time. The line element (1) must of course be even in time; this implies that the diagonal terms of the metric tensor must be even in time too, i.e. must contain even powers of \( a \) only. In order to have the mixed term in the line element being even in time, since it contains \( dt \), we must impose to the off diagonal term of the metric tensor to be odd, which means to contain odd powers of \( a \) only and no pure powers of \( \mu \).

In practice, introducing the dimensionless variables \( \varepsilon = \mu/r \) and \( \alpha = a/r \) and expressing the \( r \) dependence in the form of an inverse powers development leads to:

\[ h_{00} = A_0 \varepsilon + B_0 \alpha^2 + ... \]
\[ h_{rr} = A_1 \varepsilon + B_1 \alpha^2 + ... \]
\[ h_{\theta\theta} = A_2 \varepsilon + B_2 \alpha^2 + ... \]
\[ h_{\phi\phi} = A_3 \varepsilon + B_3 \alpha^2 + ... \]
\[ h_{t\phi} = A_4 \alpha + B_4 \varepsilon \alpha + ... \]

The \( A \) and \( B \) coefficients are functions of \( \theta \) only, which in turn is the physical angle between the position three-vector and the angular velocity axial three-vector.

Considering \( h_{t\phi} \) we see that the linear term (in \( \alpha \)), when introduced in (1), produces a constant, which means that the metric would not be flat at infinity. To avoid this it must be
A\_4 = 0. Finally, considering and comparing the orders of magnitude of the different terms, we assume that, in general, the mass contributions (\(\varepsilon\) terms) are negligible with respect to the rest; as said in the introduction this condition is rather simple to obtain \[10\]. Under this assumption the approximated line element will be

\[
ds^2 = c^2 \left(1 + B_0 \alpha^2\right) dt^2 - \left(1 + B_1 \alpha^2\right) dr^2 - r^2 \left(1 + B_2 \alpha^2\right) d\theta^2 - r^2 \sin^2 \theta \left(1 + B_3 \alpha^2\right) d\phi^2
\]
or, more explicitly,

\[
ds^2 = c^2 \left(1 + B_0 \frac{a^2}{r^2}\right) dt^2 - \left(1 + B_1 \frac{a^2}{r^2}\right) dr^2 - r^2 \left(1 + B_2 \frac{a^2}{r^2}\right) d\theta^2 - r^2 \sin^2 \theta \left(1 + B_3 \frac{a^2}{r^2}\right) d\phi^2
\]

\[\text{(2)}\]

**A. Conditions to be imposed on the metric tensor**

From the metric tensor corresponding to \(\text{(2)}\) one can calculate the Ricci tensor up to terms in \(a^2\). Since, by hypothesis, we are in empty space time, all the elements of the Ricci tensor must vanish. This condition corresponds to the following equations for the \(B\)’s (\(a\) \(\theta\) denotes differentiation with respect to \(\theta\)):

\[
(2B_0'' + 4B_0) \sin \theta + 2B_0' \cos \theta = 0
\]

\[
(4B_3' + 4B_2' + 2B_1' + 6B_0') \sin \theta + 4 (B_3 - B_2) \cos \theta = 0
\]

\[
(4B_3 + 4B_2 + 2B_1' + 6B_0') \sin \theta + 4 (B_3 - B_2) \cos \theta = 0
\]

\[
(2B_3'' - 4B_3 + 4B_2 + 2B_1' - 4B_0) \sin \theta + 2 (2B_3' - B_2') \cos \theta = 0
\]

\[
(2B_3'' - 4B_3) \sin \theta + (4B_3' - 2B_2' + 2B_1' + 2B_0') \cos \theta = 0
\]

Only four out of these equations can be independent since the Ricci tensor is symmetric and consequently it can be diagonalized at any moment and place in space time. The number of independent equations is further reduced because of the rotation symmetry of the body and steadiness of the motion. Solving for the \(B\) functions one obtains:

\[
B_0 = C_0 \cos \theta + D_0 \left(1 + \frac{1}{2} \cos \theta \ln \left(\frac{1 - \cos \theta}{1 + \cos \theta}\right)\right)
\]

\[
B_1 = -2f \cos^2 \theta - 3C_0 \cos \theta + C_1 \cos^2 \theta
\]

\[
B_2 = 2f \sin^2 \theta - C_1 \sin^2 \theta - 4 (\sin \theta \cos \theta) f' + (\cos^2 \theta) f''
\]

\[
B_3 = \frac{\cos^3 \theta}{\sin \theta} f'
\]

where \(f = f (\theta)\) is an arbitrary function. \(C_0, C_1\) and \(D_0\) are constants. Actually only finite solutions can be accepted (in order the development to be consistent all the \(B\) functions must be \(\sim 1\)), consequently it must be \(D_0 = 0\).

\[
\begin{align*}
B_0 & = C_0 \cos \theta \\
B_1 & = -2f \cos^2 \theta - 3C_0 \cos \theta + C_1 \cos^2 \theta \\
B_2 & = 2f \sin^2 \theta - C_1 \sin^2 \theta - 4 (\sin \theta \cos \theta) f' + (\cos^2 \theta) f'' \\
B_3 & = \frac{\cos^3 \theta}{\sin \theta} f' \\
\end{align*}
\]

\[\text{(3)}\]
If it is $C_0 \neq 0$ the correction to the $g_{00}$ term of the metric has the form of a dipolar potential, consistent with the axial character of the angular velocity three-vector.

Now we can calculate the non-zero terms of the Riemann tensor, which are

\[
\begin{align*}
R^t_{\text{tr}} &= -3C_0 \frac{a^2}{r^2} \cos \theta \\
R^t_{\phi t} &= \frac{3}{2} C_0 \frac{a^2}{r^2} \cos \theta \\
R^t_{\theta t} &= \frac{3}{2} C_0 \frac{a^2}{r^2} \cos \theta \\
R^t_{\phi \phi} &= \frac{3}{2} C_0 \frac{a^2}{r^2} \cos \theta \\
R^t_{\phi \theta} &= -3C_0 \frac{a^2}{r^2} \cos \theta \sin^2 \theta
\end{align*}
\]

The presence of these terms indicates the existence of real physical effects depending solely on the rotation of the body. As it is seen they are there only when $C_0 \neq 0$.

Though in the form of a power series the metric corresponding to (3) belongs to Weyl’s class of axially symmetric stationary vacuum solutions of the Einstein field equations [5].

**III. DETERMINING THE SOURCE TENSOR**

In order to find the source tensor for the metric corresponding to (3) it is convenient to proceed as it is usually done in the linearized theory of gravity, writing the metric tensor as the sum of the Minkowski metric tensor (written in polar coordinates, in our case) $g_{\text{flat}}$ and a small correcting tensor $h_{\mu \nu}$.

\[
g_{\mu \nu} = g_{\text{flat}} + h_{\mu \nu}
\]

The explicit expressions for the perturbations are

\[
h_{\mu \nu} = \begin{pmatrix}
B_0 \frac{a^2}{r^2} & 0 & 0 & 0 \\
0 & -B_1 \frac{a^2}{r^2} & 0 & 0 \\
0 & 0 & -B_2 a^2 & 0 \\
0 & 0 & 0 & -B_3 a^2 \sin^2 \theta
\end{pmatrix}
\]

The trace $h = h^\mu_\mu$ is

\[
h = \frac{a^2}{r^2} (B_0 + B_1 + B_2 + B_3)
\]

Following the standard method, let us define the auxiliary tensor $\overline{h}_{\mu \nu} = h_{\mu \nu} - \frac{1}{2} g_{\text{flat}} h$. It is

\[
\begin{align*}
\overline{h}_0 &= \frac{1}{2} \frac{a^2}{r^2} (B_0 - B_1 - B_2 - B_3) \\
\overline{h}_r &= -\frac{1}{2} \frac{a^2}{r^2} (B_0 - B_1 + B_2 + B_3) \\
\overline{h}_\theta &= -\frac{1}{2} \frac{a^2}{r^2} (B_0 + B_1 - B_2 + B_3) \\
\overline{h}_\phi &= -\frac{1}{2} \frac{a^2}{r^2} (B_0 + B_1 + B_2 - B_3)
\end{align*}
\]
The determinant of the metric tensor will be assumed to coincide with the one of flat space time, since all corrections are of order $a^4$: $g = g_{flat} = -c^2 r^4 \sin^2 \theta$.

Under the hypothesis of a weak field and considering that the metric tensor is stationary, one can write

$$\nabla^2 \mathcal{T}^\nu = 16\pi S^\nu$$

whence

$$\mathcal{T}^\nu = -4 \int \frac{S^\nu}{|r - r'|} \sqrt{\gamma} d^3 x'$$

The square root of the determinant of the space part of the Minkowski metric tensor is written as $\sqrt{\gamma}$. Developing in multipolar components and keeping the lowest order contributions, one has

$$\mathcal{T}^\nu = -\frac{4}{r} \int S^\nu \sqrt{\gamma} d^3 x' - \frac{4 \cos \theta}{r^2} \int S^\nu r' \cos \theta' \sqrt{\gamma} d^3 x'$$

Looking at (4) we see that there is no $1/r$ term. This means that $S^\nu$ must be odd with respect to an odd number of integration variables. Considering the independence from $\phi$ (invariance for rotation) and the nature of $r$, only $\theta$ is left. Let us assume that

$$S^\nu = S^\nu w(\theta)$$

with $S^\nu w$ depending at most on $r$ and $w(\theta)$ an odd function of $\theta$ (odd with respect to the equatorial plane of the body), so that in general

$$\mathcal{T}^\nu = -\frac{4 \cos \theta}{r^2} \int_0^\pi \int_0^R \int_0^{2\pi} S^\nu r^3 w(\theta') \cos \theta' \sin \theta' d\theta' d\phi'$$

$$= -8\pi \frac{\cos \theta}{r^2} \int_0^\pi \int_0^R S^\nu r^3 w(\theta') \cos \theta' \sin \theta' d\theta' d\phi'$$

(5)

The result of the integration depends of course also on the shape of the body and in particular on the shape of the meridian section of the body expressed through the function $r' = R(\theta')$ representing the border of that section. A further assumption may be that the body is homogeneous so that $S^\nu$ is not depending on $r'$ either (or we can refer to the average value of $S^\nu$ along the radius). So one has

$$\mathcal{T}^\nu = -2\pi \frac{\cos \theta}{r^2} S^\nu \int_0^\pi R^4(\theta') w(\theta') \cos \theta' \sin \theta' d\theta' = -2\pi F \frac{\cos \theta}{r^2} S^\nu$$

(6)

where $F$ is a constant depending on the shape of the section of the body.

Comparing this result with (4) we see that it must be

$$S^0 = -\frac{1}{4\pi F} \frac{a^2 (B_0 - B_1 - B_2 - B_3)}{\cos \theta}$$

$$S^r = \frac{1}{4\pi F} \frac{a^2 (B_0 - B_1 + B_2 + B_3)}{\cos \theta}$$

$$S^\theta = \frac{1}{4\pi F} \frac{a^2 (B_0 + B_1 - B_2 + B_3)}{\cos \theta}$$

$$S^\phi = \frac{1}{4\pi F} \frac{a^2 (B_0 + B_1 + B_2 - B_3)}{\cos \theta}$$

(7)
The left hand side does not depend on \( \theta \), so of course the same should happen to the right hand side. This fact poses constraints on the \( f \) function appearing in (3). Equating all (7) to constants one obtains that it must be

\[
f = \frac{W}{\cos \theta} + \frac{C_1}{2}
\]

Now \( W \) is a constant. Consequently the \( B \) functions are

\[
\begin{align*}
B_0 &= C_0 \cos \theta \\
B_1 &= -(2W + 3C_0) \cos \theta \\
B_2 &= W \cos \theta \\
B_3 &= W \cos \theta
\end{align*}
\]

and the source tensor is

\[
\begin{align*}
S_0^0 &= -\frac{1}{\pi F} a^2 C_0 w(\theta) \\
S_r^r &= \frac{1}{\pi F} (C_0 + W) w(\theta) \\
S_\theta^\theta &= -\frac{1}{2\pi F} (C_0 + W) w(\theta) \\
S_\phi^\phi &= -\frac{1}{2\pi F} (C_0 + W) w(\theta)
\end{align*}
\]

Of course the source tensor must satisfy also the zero divergence condition. Considering the symmetries and the fact that the weak field approximation holds supposedly also inside the body (use of the flat space time Christoffel symbols for the covariant derivatives), the null four-divergence condition reduces to

\[
\begin{align*}
&\frac{d}{d\theta} \left( S_\theta^\theta w(\theta) \right) + \frac{\cos \theta}{\sin \theta} w(\theta) \left( S_\theta^\theta - S_\phi^\phi \right) = 0 \\
&2S_r^r - S_\theta^\theta - S_\phi^\phi = 0
\end{align*}
\]

If \( w(\theta) \) must be odd through the equatorial plane, eq.s (10) can be satisfied only with \( S_r^r = S_\theta^\theta = S_\phi^\phi = 0 \). This fact in turn implies that

\[
W = -C_0
\]

The general forms of the \( B \)'s and of the source tensor are then

\[
\begin{align*}
B_0 &= C_0 \cos \theta \\
B_1 &= -C_0 \cos \theta \\
B_2 &= -C_0 \cos \theta \\
B_3 &= -C_0 \cos \theta
\end{align*}
\]

and
\[ S_0^0 = -\frac{1}{\pi} \frac{a^2}{F} C_0 w(\theta) \]
\[ S_r^r = 0 \]
\[ S_\theta^\theta = 0 \]
\[ S_\phi^\phi = 0 \]

(12)

IV. THE INTERNAL METRIC TENSOR

The next step is to determine a consistent metric tensor inside the matter distribution. As in the case of the external solution it is convenient to use the physical dimensionless variables

\[ \varepsilon = \frac{Gm}{c^2 r} \]
\[ \alpha = \frac{a}{r} \]

where now both \( m \) and \( a \) depend on \( r \). For an homogeneous, rigidly rotating sphere it would be \( m = \frac{4}{3} \pi \rho r^3 \) and \( a = \frac{2\Omega^2 c}{r} \), where \( \rho \) is the matter density and \( \Omega \) is the angular velocity. In general \( a \) would be expressed as a numerical factor depending on the shape, size and kind of matter distribution multiplying the angular velocity of the body and the square of some characteristic distance from the axis. On the basis of these considerations the dimensionless variables inside the body can be written as

\[ \varepsilon = \frac{G\rho}{c^2 r^2} \]
\[ \alpha = \frac{\Omega r}{c} \]

transferring all shape effects into the coefficients that will multiply the variables.

Here again we assume that

\[ \varepsilon \ll \alpha \]

Now let us write the internal line element as a power expansion in \( r \) where the same general symmetries as in the external case should hold. In practice the extra-diagonal term is negligible, as well as the mass terms.

The proposed line element is then

\[ ds^2 = c^2 (1 + \beta_0 \frac{\Omega^2}{c^2 r^2}) dt^2 - \left( 1 + \beta_1 \frac{\Omega^2}{c^2 r^2} \right) dr^2 - r^2 \left( 1 + \beta_2 \frac{\Omega^2}{c^2 r^2} \right) d\theta^2 - r^2 \sin^2 \theta \left( 1 + \beta_3 \frac{\Omega^2}{c^2 r^2} \right) d\phi^2 \]

(13)

where the \( \beta \)'s are functions of \( \theta \) only. From there one can straightforwardly calculate the Einstein tensor.
Introducing this, the border of the rotating body. These conditions reduce in practice to

\[ G'_t = -\frac{\Omega^2}{c^2} \frac{(\beta'' + 8\beta_3 + 10\beta_2 + \beta' + 6\beta_1)}{2 \sin \theta} \sin \theta + (2\beta'_3 - \beta'_2 + \beta'_1) \cos \theta \]

\[ G'_r = -\frac{\Omega^2}{c^2} \frac{(\beta'' + 2\beta_3 + 4\beta_2 - 2\beta_1 + \beta'_0 + 4\beta_0)}{2 \sin \theta} \sin \theta + (2\beta'_3 - \beta'_2 + \beta'_0) \cos \theta \]

\[ G'_\theta = -\frac{\Omega^2}{c^2} \frac{(\beta'' + 2\beta'_3 - \beta'_1 + \beta'_0)}{2 \sin \theta} \sin \theta + 2(\beta_3 - \beta_2) \cos \theta \]

\[ G'_\phi = -\frac{\Omega^2}{c^2} \frac{(\beta'_0 - \beta'_1 + \beta'_0 + 4\beta_0)}{2 \sin \theta} \sin \theta + (2\beta_3 - \beta_2) \cos \theta \]

Imposing the four-divergence of \( G'_\nu \) to be zero produces the two equations

\[
\frac{1}{2} \beta''_1 - \frac{1}{2} \beta''_0 - \beta''_3 + \beta_3 - \beta_2 - \frac{\cos \theta}{\sin \theta} \left( \frac{2\beta'_3 - \beta'_2}{2} - \frac{\beta'_0}{2} \right) = 0
\]

\[
\frac{1}{2} \beta' - \frac{1}{2} \beta' - \beta - \frac{\cos \theta}{\sin \theta} \beta_3 + \frac{\cos \theta}{\sin \theta} \beta_2 = 0
\]

These equations are not independent from each other since differentiating the second one, then subtracting it from the first one, the second equation is obtained again. A solution is found when \( \beta = \beta_0 = -\beta_1 = -\beta_2 = -\beta_3 \). This same solution brings all the \( G'_t \)'s to 0 too.

The Einstein tensor must be proportional to (12); let us call \( \chi \) the proportionality constant. The equality condition reduces to

\[
-\frac{\Omega^2}{c^2} \frac{(\beta'' + 8\beta_3 + 10\beta_2 + \beta' + 6\beta_1)}{2 \sin \theta} \sin \theta + (2\beta'_3 - \beta'_2 + \beta'_1) \cos \theta = -\frac{1}{\pi} \frac{a^2}{F} C_0 w(\theta)
\]

i.e.

\[
\left( \frac{\beta'' + 6\beta}{\sin \theta} \right) \frac{\sin \theta + \beta' \cos \theta}{\sin \theta} = \frac{1}{\pi} \frac{a^2}{F} \frac{c^2}{\Omega^2} C_0 w(\theta)
\]

Eq. (14) may be rewritten as \( (K = \frac{1}{\pi} \frac{a^2 c^2}{F \Omega^2} C_0) \)

\[
\frac{d^2 \beta}{d \theta^2} + \frac{d \beta \cos \theta}{d \theta \sin \theta} + 6\beta = K w(\theta)
\]

Further conditions to be imposed are related to the continuity of the metric tensor at the border of the rotating body. These conditions reduce in practice to

\[
\beta \frac{\Omega^2}{c^2} R^2 = B_0 \frac{a^2}{R^2}
\]

from where one has

\[
\beta = \frac{a^2 c^2}{\Omega^2 R^4} C_0 \cos \theta
\]

Introducing this \( \beta \) into (15) gives

\[
\frac{1}{R^4} \cos \theta + 2 \frac{R'}{R^5} \sin \theta + 5 \frac{R'}{R^6} \cos \theta - \frac{R''}{R^5} \cos \theta - \frac{R'}{R^5} \cos \theta \frac{\cos \theta}{\sin \theta} = \frac{1}{4\pi} \frac{\chi}{F} w(\theta)
\]

Once \( R \) is chosen (17) gives the expression for \( w(\theta) \).
A. The case of a rotating sphere

Let us assume our rotating body is a solid homogeneous sphere. In this case it is

\[ R = R = \text{constant} \]

and

\[ a = \frac{2 \Omega}{5 c} R^2 \]

From (17) we see that

\[ w(\theta) = \cos(\theta) \]

On the other hand one has from (13)

\[ F = \frac{2}{3} R^4 \]

Then, again from (17)

\[ \chi = \frac{8}{3} \pi \]  

Finally we can list the explicit expressions for the line elements and the source tensor.

Internal line element:

\[
d s^2 = c^2 (1 + \frac{4}{25} C_0 \frac{\Omega^2}{c^2} r^2 \cos \theta) dt^2 - \left( 1 - \frac{4}{25} C_0 \frac{\Omega^2}{c^2} r^2 \cos \theta \right) dr^2 - r^2 \left( 1 - \frac{4}{25} C_0 \frac{\Omega^2}{c^2} r^2 \cos \theta \right) d\theta^2 - r^2 \sin^2 \theta \left( 1 - \frac{4}{25} C_0 \frac{\Omega^2}{c^2} r^2 \cos \theta \right) d\phi^2 \]  

(19)

External line element

\[
d s^2 = c^2 \left( 1 + \frac{4}{25} C_0 \frac{\Omega^2 R^4}{c^2 r^2} \cos \theta \right) dt^2 - \left( 1 - \frac{4}{25} C_0 \frac{\Omega^2 R^4}{c^2 r^2} \cos \theta \right) dr^2 - r^2 \left( 1 - \frac{4}{25} C_0 \frac{\Omega^2 R^4}{c^2 r^2} \cos \theta \right) d\theta^2 - r^2 \sin^2 \theta \left( 1 - \frac{4}{25} C_0 \frac{\Omega^2 R^4}{c^2 r^2} \cos \theta \right) d\phi^2 \]  

(20)

Source tensor

\[
S^0_0 = - \frac{6}{25 \pi} \frac{\Omega^2}{c^2} C_0 \cos \theta \\
S^r_r = 0 \\
S^\theta_\theta = 0 \\
S^\phi_\phi = 0 \]  

(21)
V. INTERPRETATION OF THE SOURCE TENSOR

To interpret the tensor and to conjecture a value of $C_0$ let us consider what happens to the line element in a weak field approximation inside a homogeneous sphere when considering the pure effect of mass. In that case we expect the $g_{00}$ element of the metric tensor to be corrected by the local Newtonian potential in the form

$$c^2 h_{00} = -\frac{2}{r} \int G\rho \sqrt{\gamma} d^3 x'$$

Suppose now that the same role is played by the centrifugal potential inside the body. That potential at a given point is $\frac{1}{2} \Omega^2 r^2 \sin^2 \theta = \frac{1}{2} \Omega^2 \frac{r^3}{r} \sin^2 \theta$. Suppose you want to write it in the form of a volume integral up to a given $r$ and a given $\theta$; it could be:

$$\frac{1}{2} \Omega^2 r^2 \sin^2 \theta = \frac{3}{2\pi} \frac{\Omega^2}{r} \int_0^{2\pi} \int_0^r \int_0^\theta \cos \theta' \sqrt{\gamma} d^3 x' = \Omega^2 r^2 \int_0^\theta \cos \theta' \sin \theta' d\theta'$$

In this way, recalling also the 'coupling constant' (18), the same role as $G\rho/c^2$ before, is now played by

$$-4 \frac{\Omega^2}{c^2} \cos \theta$$

In the case of a pure mass effect the $T_{00}^0$ term of the stress energy tensor would contain precisely $G\rho/c^2$. Continuing on the same line of thought and looking at (21) we expect

$$\frac{6}{25\pi} \frac{\Omega^2}{c^2} C_0 \cos \theta = 4 \frac{\Omega^2}{c^2} \cos \theta$$

that implies

$$C_0 = \frac{50}{3} \pi \sim 10^2$$

VI. ZERO MASS LIMIT

Assuming a rigidly rotating body one must allow for some kind of force keeping the whole thing together against centrifugal forces. This force, when self gravitation is negligible, is provided by elasticity. To account for it the elastic energy momentum tensor should be added to the source tensor (12). A simpler approach is to think of an elastic membrane constraining the body and balancing the centrifugal push on the surface. In that case there would be a pure surface tension. If furthermore we assume that the membrane is indeed made of a big number of independent rings than the stress is purely directed along the 'parallels' of the membrane. Its value, if the interior is purely passive but nonetheless remains homogeneous, is

$$\sigma_{\phi\phi} = \frac{\rho}{3l} \Omega^2 R^3 \sin^3 \theta$$
The density of the inner material is \( \rho \), while \( l \) is the thickness of the membrane. There will be a contribution to the source tensor only at the membrane.

The presence of the elastic tensor, even in the simplified version of a surface tension, is necessary, from the mathematical point of view, to insure the continuity of the radial derivatives of the metric tensor at the boundary of the body. In fact from (20) and (19) the derivatives of the metric at the boundary turn out to be (considering the 00 and \( rr \) components) \( \frac{2}{R} \beta \) on the internal side and \(- \frac{2}{R} B_0\) on the external side. A \(- \frac{4}{R} \beta\) term is needed on the left to restore the equality. The additional surface term corresponds to the elastic force originated by the elastic stress in the membrane (per unit mass and dividing by \( c^2 \)).

The stress affordable by the membrane has an upper limit \( \sigma_m \), not depending on the density of the membrane by itself. This puts an upper limit to the angular velocity of the body too. The origin of the elastic resistance is in molecular interactions, i.e. in electromagnetic interaction. At the scale of the laboratory electromagnetic forces are far greater than the gravitational force. This is why gravitational effects can be neglected while rotation effects, whose limit is determined by elastic stresses, can not. Increasing the size of the body the gravitational effect keeps growing, while the molecular forces, which are short ranged, stay more or less the same: this is the reason why for big bodies gravity overtakes again and eventually rotational effects become secondary and more or less negligible.

However it is important to remark that no zero mass limit can produce any paradox since there are no electric charges without a mass and relativistically the electromagnetic field also contributes to the mass. So any zero mass limit is also a zero electromagnetic interaction limit. Sending the mass to zero implies turning the elastic interaction off and the weaker this is the smaller is the maximum possible angular velocity, then the limit for \( \rho \) going to zero is also an \( \Omega \) going to zero limit.

VII. EXPERIMENTAL VERIFICATION

Let us refer to (20) and suppose to send light along a closed optical fiber at constant \( r \) and constant \( \theta \). The corresponding line element would be

\[
0 = c^2 \left( 1 + \frac{8}{3} \frac{\Omega^2}{c^2} \frac{R^4}{r^2} \cos \theta \right) dt^2 - r^2 \sin^2 \theta \left( 1 - \frac{8}{3} \frac{\Omega^2}{c^2} \frac{R^4}{r^2} \cos \theta \right) d\phi^2
\]

The coordinate time of flight along the whole loop would be

\[
t_1 = \frac{2\pi}{c} r \sin \theta \sqrt{\frac{1 - \frac{8}{3} \frac{\Omega^2}{c^2} \frac{R^4}{r^2} \cos \theta}{1 + \frac{8}{3} \frac{\Omega^2}{c^2} \frac{R^4}{r^2} \cos \theta}} \\
\simeq \frac{2\pi}{c} r \sin \theta \left( 1 - \frac{8}{3} \frac{\Omega^2}{c^2} \frac{R^4}{r^2} \cos \theta \right)
\]

Let us now do the same using an identical loop at \( \theta' = \pi - \theta \). The time of flight would be

\[
t_2 \simeq \frac{2\pi}{c} r \sin \theta \left( 1 + \frac{8}{3} \frac{\Omega^2}{c^2} \frac{R^4}{r^2} \cos \theta \right)
\]
The difference between the two times is of course
\[ \Delta t = \frac{32}{3} \pi^2 \frac{\Omega^2 R^4}{c^3 r^2} \sin \theta \cos \theta \]

This difference would not be there if the body was not rotating. The maximum effect is produced when \( \theta = \frac{\pi}{4} \):
\[ \Delta t_M = \frac{16}{3} \pi^2 \frac{\Omega^2 R^4}{c^3 r^2} \simeq \frac{16}{3} \pi^2 \frac{\Omega^2 R^2 R}{c^2} \]

The value of \( \Delta t_M \) in laboratory conditions \( (R \sim 1 \text{ m}, \Omega R \sim 10^3 \text{ m/s}) \) is \( \sim 10^{-18} \text{ s} \). It could be measured by interferometric techniques.

VIII. CONCLUSION

Summarizing the results of this paper, we have shown that:

1) there exists a metric tensor in the vicinity of a weakly gravitating and rotating body which can be expressed, at the chosen level of approximation, as depending only on the square of the angular velocity of the body as viewed by a static (with respect to the rotation axis) inertial observer and on the shape of the body (here weakly gravitating means that the mass terms are negligible with respect to rotation terms);

2) the Riemann tensor calculated from the external metric has non zero terms, hence corresponding to physical and not only coordinate effects;

3) there exists an internal metric tensor matching the external one, provided an elastic force is allowed to resist the centrifugal force within the matter distribution;

4) the external metric tensor can be deduced from a source symmetric and conserved (in the sense of null four-divergence) tensor, which turns out to be proportional to the Einstein tensor calculated from the internal metric tensor;

5) the relevant term in the source tensor as well as in the metric tensors has the properties and appearance respectively of a dipolar density and dipolar potential, whose magnitude is given by the square of the angular velocity of the body;

6) considering the zero mass limit one sees that a vanishing mass corresponds to a vanishing elastic force, then vanishing allowed angular velocity too and eventually purely flat space time;

7) the numerical size of the various terms in "laboratory" conditions is such that physical effects can be measured by light interferometry;

8) studying the structure of space time, rotation effects should be accounted for adding to the energy momentum tensor a rotation source tensor.

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