Anomaly polynomial of general 6D SCFTs

Kantaro Ohmori\(^1\), Hiroyuki Shimizu\(^1\), Yuji Tachikawa,\(^1,2\,*\) and Kazuya Yonekura\(^3\)

\(^1\)Department of Physics, Faculty of Science, University of Tokyo, Bunkyo-ku, Tokyo 133-0022, Japan
\(^2\)Institute for the Physics and Mathematics of the Universe, University of Tokyo, Kashiwa, Chiba 277-8583, Japan
\(^3\)School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540, USA

\(^*\)E-mail: yuji.tachikawa@ipmu.jp

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We describe a method to determine the anomaly polynomials of general 6D \(\mathcal{N} = (2, 0)\) and \(\mathcal{N} = (1, 0)\) superconformal field theories (SCFTs), in terms of the anomaly matching on their tensor branches. This method is almost purely field theoretical, and can be applied to all known 6D SCFTs. We demonstrate our method in many concrete examples, including \(\mathcal{N} = (2, 0)\) theories of arbitrary type and the theories on M5 branes on asymptotically locally Euclidean (ALE) singularities, reproducing the \(N^3\) behavior. We check the results against the anomaly polynomials computed M-theoretically via the anomaly inflow.

1. Introduction

In the past few years, 6D \(\mathcal{N} = (2, 0)\) superconformal theories have been used effectively as a way to organize and understand various features of lower-dimensional supersymmetric dynamics. We might hope that similar development with 6D \(\mathcal{N} = (1, 0)\) theories is not entirely out of reach. To orient ourselves, we would like to start by understanding better the properties of 6D theories themselves.

Let us quickly recall known 6D \(\mathcal{N} = (1, 0)\) theories in the literature: The \(\mathcal{N} = (2, 0)\) theories, with the ADE classification, were introduced in Refs. [1,2]: they are of course \(\mathcal{N} = (1, 0)\) theories. The E-string theories are obtained by putting M5 branes within the end-of-the-world \(E_8\) brane [3,4]. In Refs. [5,6] theories were found that become gauge theories on their tensor branch. M5 branes can also be put on the asymptotically locally Euclidean (ALE) singularity, with or without the end-of-the-world \(E_8\) brane. Another method is to consider coincident D5 branes in type IIB or type I theory on top of the ALE singularity [7,8]. The theories discussed so far can be uniformly analyzed in terms of F-theory [9]; the brane construction with D6 branes and NS5 branes can also be used [10,11]. F-theory gives a uniform perspective to discuss these theories: the classification was started in Ref. [12] and the details are being worked out, e.g. Refs. [13,14].\(^1\)

One feature of these 6D superconformal theories is that they have the tensor branch, i.e. the moduli space of vacua parameterized by the scalars in the tensor multiplets. On the tensor branch, the infrared theory is simpler and described by a system of almost free tensor multiplets, gauge fields, and matter contents that can either be free hypermultiplets or other superconformal field theories. The scalars

\(^1\)There are also approaches to study 6D \(\mathcal{N} = (1, 0)\) superconformal theories using Lagrangian descriptions; see e.g. Refs. [15–17] and [18,19].
in the tensor multiplets often control the coupling constant of the non-Abelian gauge multiplets. The objective of this paper is to show how this feature can be used to determine the anomaly polynomial of the original ultraviolet theory, providing us with at least one additional physical observable for each 6D superconformal theory.

The essential idea is that going to the tensor branch does not break any symmetry other than the conformal symmetry. Therefore, the whole anomaly of the ultraviolet theory can be found on the tensor branch by the anomaly matching. The anomaly there has two sources: the one-loop anomaly and the Green–Schwarz contribution. The one-loop anomaly follows from the standard formulas, and therefore all we need to do is to determine the Green–Schwarz contribution, which can be found by either of the following two methods:

1. If there is no gauge group whose coupling is controlled by the tensor multiplet scalar, we compactify the system on $S^1$ and determine the Chern–Simons term in 5D, which can be lifted back to 6D.
2. If there is a gauge group whose coupling is controlled by the tensor multiplet scalar, the requirement of the cancellation of the gauge anomaly uniquely fixes the Green–Schwarz term.

These methods allow us, in particular, to derive the characteristic $N^3$ behavior of the number of the degrees of freedom on 6D superconformal theories in an almost purely field theoretical manner. We think it best to demonstrate our methods using a few concrete examples here.

1.1. Without 6D gauge group

Let us consider the anomaly of the low-energy theory on $Q$ M5 branes, which is the 6D $\mathcal{N} = (2, 0)$ theories of type $A_{Q-1}$. There is a long history behind the computation of the anomaly polynomials of these theories, using M-theoretic techniques. For a single M5 brane, it was first discussed in Ref. [21]. The anomaly inflow analysis for a single M5 brane was done in Ref. [22], where a subtlety concerning the normal bundle anomaly was found. This subtlety was successfully resolved in Ref. [23], which led to the determination of the anomaly for general number $Q$ of the M5 branes in Ref. [24].

Our trick is to go to its $\mathcal{N} = (1, 0)$ tensor branch. On generic points on the tensor branch, we just have $\mathcal{Q} \mathcal{N} = (1, 0)$ tensor multiplets and $Q$ hypermultiplets, whose contribution to the $c_2(R)^2$ term in the anomaly polynomial is just

$$I^\text{one-loop} = \frac{Q}{24} c_2(R)^2. \quad (1.1)$$

Going to the $\mathcal{N} = (1, 0)$ branch, however, does not break the $SU(2)_R$ symmetry. Therefore, we should be able to see the full $SU(2)_R$ anomaly of the interacting theory on the tensor branch: it

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2 This essential idea, of the anomaly matching on the tensor branch, was independently found earlier by Ken Intriligator, and it appeared on the arXiv as Ref. [20].
should have an additional contribution from the Green–Schwarz term. Namely, if the \(i\)th tensor field has the modification of the Bianchi identity as

\[
d H_i = I_i = \rho_i c_2(R)
\]

(1.2)

the Green–Schwarz contribution to the anomaly is \([25–27]\)

\[
I^{\text{GS}} = \frac{1}{2} \delta^{ij} I_i I_j = \frac{1}{2} |\rho|^2 c_2(R)^2.
\]

(1.3)

We just need to determine \(\rho_i\).

To do this, we perform a dimensional reduction on \(S^1\). We have maximally supersymmetric \(U(Q)\) theory on 5D. The \(N = (1, 0)\) tensor branch in 6D corresponds to giving the vacuum expectation value (vev) to only one direction of the scalars out of \(\mathbb{R}^5\), breaking \(U(Q)\) down to the \(U(1)^Q\) gauge group. Let us say that the vev is

\[
\phi_1 < \phi_2 < \cdots < \phi_Q.
\]

(1.4)

We have corresponding \(U(1)\) gauge fields \(A_i = 1, \ldots, Q\). For each pair \((i, j)\) with \(i \neq j\), we have massive vector multiplets with real mass \(\phi_i - \phi_j\), with charge +1 for \(A_i\) and charge -1 for \(A_j\). In five dimensions, integrating out fermions in these massive multiplets generates Chern–Simons interactions. We are interested in \(A_i - F_R - F_R\) Chern–Simons terms, where \(F_R\) is the background gauge field for the \(SU(2)_R\) symmetry. A multiplet with \(U(1)\) charge \(q\) and real mass \(m\) gives a contribution proportional to \(q\) sign \(m\). Under the \(i\)th \(U(1)\) field \(F_i\), the pairs \((i, j)\) all have charge +1, but those with \(j > i\) have positive real mass, whereas those with \(j < i\) have negative real mass. Therefore, we see

\[
d \star F_i \propto [(Q - i) - (i - 1)] \text{Tr} F_R^2.
\]

(1.5)

Lifting it and carefully fixing the coefficients, which we will do in Sect. 2, we find

\[
d H_i = \frac{1}{2} (Q + 1 - 2i)c_2(R).
\]

(1.6)

This determines the Green–Schwarz contribution \(I^{\text{GS}}\) of (1.3), and we get

\[
I^{\text{tot}} = I^{\text{one--loop}} + I^{\text{GS}} = \frac{Q^3}{24}c_2(R)^2.
\]

(1.7)

This correctly reproduces the \(Q^3\) behavior of the anomaly polynomial including the coefficients. In addition, this procedure applies equally well to 6D \(\mathcal{N} = (2, 0)\) theories of any type \(G = A_n, D_n,\) and \(E_n\). The general formula was conjectured in Ref. [28], and the anomaly polynomial of the \(\mathcal{N} = (2, 0)\) theory of \(D_n\) was obtained by inflow analysis in Ref. [29].\(^3\) There has been, however, no derivation for the theory of type \(E\). Our method gives the first derivation of the anomaly polynomial of the 6D theory of type \(E\).

We will present the details of the procedure described in this subsection in Sect. 2. We will treat the \(\mathcal{N} = (2, 0)\) theories and E-string theories there.

\(^3\) In Ref. [30], compactification on \(T^2\) and comparison with \(\mathcal{N} = 4\) super Yang–Mills in 4D were used to deduce the \(N^3\) behavior of the anomaly of the 6D \(\mathcal{N} = (2, 0)\) theory of type \(A\) and \(D\) in the large \(N\) limit.
1.2. With 6D gauge group

1.2.1. Rank-1 examples. Let us next consider the class of 6D \( \mathcal{N} = (1, 0) \) theories whose tensor branch is 1D, such that on its generic point we just have pure gauge theory with gauge group \( G = \text{SU}(3), \text{SO}(8), F_4, E_{6,7,8} \). These theories were first introduced in Refs. [5,6].

The anomaly polynomial of the gauge multiplet is

\[
I^{\text{vec}} = -\frac{1}{24} \left( \frac{3}{4} w_G (\text{Tr} F^2)^2 + 6 h_{\Sigma}^\vee \text{Tr} F^2 c_2(R) + d_G c_2(R)^2 \right),
\]

where \( 3w_G/4 \) is the coefficient converting \( \text{tr}_{\text{adj}} F^4 \) to \( (\text{Tr} F^2)^2 \), \( h_{\Sigma}^\vee \) and \( d_G \) are the dual Coxeter number and the dimension of \( G \), respectively. These and other data and also our conventions are collected in Appendix A. For simplicity we only showed the terms that only involve the gauge fields and the \( R \) symmetry. The one-loop anomaly on the tensor branch is then

\[
I^{\text{one-loop}} = I^{\text{vec}} + I^{\text{tensor}}
\]

where \( I^{\text{tensor}} = \frac{1}{24} c_2(R)^2 \) is the contribution from the tensor multiplet. The self-dual tensor field can have a deformation of the Bianchi identity \( dH = I \) where \( I \) is a linear combination of \( \text{Tr} F^2 \) and \( c_2(R) \). Depending on the normalization of \( H \), it contributes to the anomaly by \( I^{\text{GS}} = a I^2 \) where \( a \) is a positive number. To cancel the pure and mixed gauge anomalies in (1.8), the essentially unique choice is to take

\[
I^{\text{GS}} = \frac{w_G}{2} \left( \frac{1}{4} \text{Tr} F^2 + \frac{h_{\Sigma}^\vee}{w_G} c_2(R) \right)^2.
\]

We conclude that the total anomaly polynomial is

\[
I^{\text{tot}} = I^{\text{one-loop}} + I^{\text{GS}} = \left( \frac{(h_{\Sigma}^\vee)^2}{2w_G} - \frac{d_G - 1}{24} \right) c_2(R)^2.
\]

Note that this is the anomaly polynomial of the ultraviolet fixed point. This is explicit and concrete, but not very illuminating. Let us move on to another class of examples.

1.2.2. Q M5 branes on \( \mathbb{C}^2/\mathbb{Z}_k \). Consider \( Q \) M5 branes on the singularity \( \mathbb{C}^2/\mathbb{Z}_k \), without the center-of-mass mode. The tensor branch is \( (Q - 1) \)-dimensional, and on its generic point, the theory is a linear quiver theory \( [\text{SU}(k_0) \times \cdots \times \text{SU}(k_{Q-1}) \times [\text{SU}(k_Q)]^Q] \) with \( (Q - 1) \) gauge factors \( \text{SU}(k_1), \ldots, \text{SU}(k_{Q-1}) \) and flavor symmetry \( \text{SU}(k_0) \times \text{SU}(k_Q) \), with bifundamental hypermultiplets under \( \text{SU}(k_i) \times \text{SU}(k_{i+1}) \). These theories were first considered in Ref. [7] and studied using various stringy constructions in Refs. [8,10,11].

Let us determine the anomaly polynomial of this strongly coupled theory. The anomaly polynomial of the gauge multiplet for \( \text{SU}(k_i) \) is

\[
I^{\text{vec}}_i = -\frac{1}{24} \left( 2k \text{tr}_{\text{fund}} F_i^4 + \left( \frac{3}{2} \right) (\text{Tr} F_i^2)^2 + 6k \text{Tr} F_i^2 c_2(R) + (k^2 - 1)c_2(R)^2 \right).
\]

Similarly, the anomaly of the bifundamental charged under \( \text{SU}(k_i) \times \text{SU}(k_{i+1}) \) is

\[
I^{\text{bif}}_{i,i+1} = \frac{1}{24} \left( k \text{tr}_{\text{fund}} F_i^4 + k \text{tr}_{\text{fund}} F_{i+1}^4 + \left( \frac{3}{2} \right) \text{Tr} F_i^2 \text{Tr} F_{i+1}^2 \right)
\]

and that of one tensor multiplet is

\[
I^{\text{tensor}} = \frac{1}{24} c_2(R)^2.
\]
The contribution so far sums up to
\[ I_{\text{one-loop}} = -\frac{1}{32} \eta^{ij} \text{Tr} F_i^2 \text{Tr} F_j^2 - \frac{k}{4} \text{Tr} F_i^2 \rho^j c_2(R) - \frac{1}{24} (Q - 1)(k^2 - 2)c_2(R)^2 \]  
(1.15)
where \( \eta^{ij} \) for \( i, j = 1, \ldots, Q - 1 \) is given by the Cartan matrix of \( A_{Q - 1} \), i.e.,
\[
\eta^{ij} = \begin{pmatrix}
2 & -1 & -1 \\
-1 & 2 & -1 \\
-1 & -1 & 2 \\
\end{pmatrix},
\]
(1.16)
the vector \( \rho \) is
\[
\rho^i = (1, 1, \ldots, 1),
\]
(1.17)
and for simplicity we set the flavor background to be zero: \( F_0 = F_Q = 0 \).

This gauge theory is consistent only because there are \( (Q - 1) \) self-dual tensor fields whose Green–Schwarz interaction cancels the purely and mixed gauge anomalies. In general, the Green–Schwarz contribution from the self-dual tensor fields to the anomaly polynomial is
\[ I^{\text{GS}} = \frac{1}{2} \Omega^{ij} I_i I_j \]
(1.18)
where \( \Omega^{ij} \) is a positive-definite matrix and \( I_i \) is the modification to the Bianchi identity for the \( i \)th self-dual field
\[ dH_i = I_i. \]
(1.19)
Here, the essentially unique choice to cancel the gauge anomaly in (1.15) is to take
\[
\Omega^{ij} = \eta^{ij}, \quad I_i = \frac{1}{4} \text{Tr} F_i^2 + k(\eta^{-1})_{ij} \rho^j c_2(R).
\]
(1.20)
Then we conclude that
\[
I^{\text{tot}} = I_{\text{one-loop}} + I^{\text{GS}} = \left( \frac{k^2}{2} \rho^j (\eta^{-1})_{ij} \rho^j - \frac{1}{24} (Q - 1)(k^2 - 2) \right) c_2(R)^2
\]
\[
= \frac{1}{24} \left( (Q^3 - Q)k^2 - (Q - 1)(k^2 - 2) \right) c_2(R)^2
\]
(1.21)
where we used \( \rho^j (\eta^{-1})_{ij} \rho^j = (Q^3 - Q)/12 \).

We already see that this purely field theoretical method already gives the leading cubic behavior \( Q^3 \). The coefficient is exactly what is expected from AdS/\( CFT \); the whole structure can also be obtained from the anomaly inflow in M-theory; see Appendix B.

### 1.3. Organization of the paper

The rest of the paper is organized as follows. In Sect. 2, we give the details of the determination of the Green–Schwarz term when the tensor multiplet scalar does not control the coupling of any gauge field. It happens that, in this particular case, we have good control over the field theoretical behavior of its \( S^1 \) compactification to 5D, which allows us to determine the 5D Chern–Simons terms appearing from integrating out massive fermions. Concretely, we treat 6D \( \mathcal{N} = (2, 0) \) theories of arbitrary type \( G = A_n, D_n, \) and \( E_n \), and also the E-string theory of arbitrary rank. Our results reproduce the known
anomaly polynomials computed using the anomaly inflow in M-theory. Note that this is the first time that the anomaly of $\mathcal{N} = (2, 0)$ theory of type $E$ has been successfully computed.

In Sect. 3, we describe the methods to find the Green–Schwarz term when the scalar in the tensor multiplet determines the coupling of a gauge field. Here, the varieties of 6D $\mathcal{N} = (1, 0)$ theories that we can treat is vast. We will treat M5 branes on $\mathbb{C}^2 / \Gamma$ for arbitrary $\Gamma$ as the main examples. Most of the results we obtain in this section are new. At various steps, one needs to use the anomaly polynomials of the E-string theories as inputs. We end the section by rephrasing our results in terms of the F-theory geometry, used in the classification in Ref. [12].

We conclude with a discussion of future directions in Sect. 4.

We have two appendices: in Appendix A, we gather various standard formulas, such as the anomaly polynomials of various $\mathcal{N} = (1, 0)$ multiplets and the group theoretical constants. In Appendix B, we generalize the anomaly inflow analysis of Refs. [23,24] to determine the anomaly polynomials of M5 branes on $\mathbb{C}^2 / \Gamma$. This gives an independent confirmation of our methods in Sect. 3.

2. Tensor branches without gauge multiplets

In this section we determine the anomaly polynomials of $\mathcal{N} = (2, 0)$ theories of arbitrary type and of E-string theories of arbitrary rank, by going to their tensor branches. If the reader accepts the anomaly polynomials of these theories as known from the M-theoretic anomaly inflow, the content of this section is not necessary and the reader can directly go to the next section.

2.1. Generalities of self-dual tensor fields in 6D

We start by recalling various properties of self-dual tensor fields in 6D. Let us first introduce the charge pairing in 6D. Before that, it is useful to recall the situation in 4D. We normalize the 4D Dirac–Zwanziger pairing of particles with dyonic charges $q = (e, m)$ and $q' = (e', m')$ to be given by

$$\langle q, q' \rangle_{4D} = em' - e'm \in \mathbb{Z}$$

(2.1)

so that $\langle q, q' \rangle h/2$ gives the angular momentum carried by the electromagnetic fields. This pairing is antisymmetric.

In 6D with $n$ self-dual tensor fields, there are self-dual strings with charges taking values in an $n$-dimensional lattice $\Lambda$. The pairing is symmetric: for $q, q' \in \Lambda$, $\langle q, q' \rangle_{6D} = \langle q', q \rangle_{6D}$, and we normalize $\langle q, q' \rangle_{6D}$ using the compactification on $T^2$. Namely, the self-dual string of charge $q$ wound on the cycle $mA + nB$ of $T^2$ can be said to have 4D charge $q(mA + nB)$, and we require

$$\langle qA, q'B \rangle_{4D} = \langle q, q' \rangle_{6D} \langle A, B \rangle_{T^2}$$

(2.2)

where $\langle A, B \rangle_{T^2}$ is the intersection number of $A$ and $B$ on $T^2$.

Let us explicitly introduce $q = (q_i)_{i=1,\ldots,n} \in \Lambda$, and express the pairing using a symmetric matrix $\Omega^{ij}$ as

$$\langle q, q' \rangle_{6D} = \Omega^{ij} q_i q'_j.$$  

(2.3)

Accordingly, introduce the self-dual three-form field strengths $H_i$ normalized such that

$$dH_i = q_i \prod_{a=2,3,4,5} \delta(x_a) dx_a$$

(2.4)

when a self-dual string of charge $q$ exists at $x_{a=2,3,4,5} = 0$. 
At this point, suppose we have a modification of the Bianchi identity
\[ dH_i = I_i \] (2.5)
where \( I_i \) is a four-form constructed out of the metric and the gauge fields, which can either be dynamical or non-dynamical. The Green–Schwarz contribution to the anomaly is [25–27]
\[ I_{GS} = \frac{1}{2} \Omega^{ij} I_i I_j. \] (2.6)

2.2. 6D Green–Schwarz and 5D Chern–Simons

Next, we recall the relation of the 6D Green–Schwarz terms and the 5D Chern–Simons terms, and also how Chern–Simons terms are induced in 5D. The \( S^1 \) reduction of \( n \) self-dual fields \( H_i \) gives rise to \( n \) Abelian gauge fields \( A_i \). The field strengths are related as \( F_{\mu\nu} = 2\pi R \cdot H_{\mu\nu5} \), where \( R \) is the radius of \( S^1 \) and “5” is the direction of \( S^1 \). The 5D kinetic term is \( \frac{1}{2} \Omega^{ij} F_i \wedge \ast F_j \), and the reduction of (2.5) is
\[ d \left( \frac{1}{2\pi R} \ast F_i \right) = I_i, \] (2.7)
meaning that there is a 5D Chern–Simons term
\[ \frac{1}{2\pi} S^{CS} = \Omega^{ij} A_i I_j = A_i I^i, \] (2.8)
where indices are raised and lowered by \( \Omega^{ij} \) and \( (\Omega^{-1})_{ij} \), e.g. \( I^i = \Omega^{ij} I_j \).

Now, let us consider a 5D fermion \( \psi \) with mass term \( m \psi \bar{\psi} \); note that the sign of \( m \) is meaningful. Suppose it has charge \( q \) under a U(1) field, and furthermore, it couples to an additional non-Abelian background gauge field \( F_G \) in a representation \( p \) and of course the metric. By a careful computation of the triangle diagrams [31], the induced Chern–Simons term from integrating out \( \psi \) is
\[ \frac{1}{2} (\text{sign } m) q A \left( \frac{1}{2} \text{tr}_\rho F^2_G + \frac{1}{24} d_\rho p_1(T) \right). \] (2.9)
This result can also be obtained as follows. Consider we have a time-translation invariant situation where we have nontrivial \( F_G \) and/or nontrivial metric on the spatial slice. In this background, the fermion \( \psi \) has \( \nu = -\text{tr}_\rho F^2/2 - d_\rho p_1(T)/24 \) zero modes. By quantizing the fermionic zero modes in the instanton background, we can see that each zero mode shifts the electric charge by \( \pm q/2 \), depending on the sign of \( m \). Then the worldline Lagrangian for \( \psi \) has an additional coupling \( \pm(q\nu/2)A \), which indicates the 5D Chern–Simons term (2.9).

2.3. \( \mathcal{N} = (2, 0) \) theories

Now we have all the tools to compute the anomaly polynomial of 6D \( \mathcal{N} = (2, 0) \) theory of arbitrary type \( G = A_r, D_r, \) and \( E_r \). Its \( T^2 \) reduction is 4D \( \mathcal{N} = 4 \) theory with gauge group \( G \). Therefore, the 6D charge lattice of the self-dual strings is the root lattice of \( G \), and the matrix \( \Omega^{ij} \) in (2.3) is the Cartan matrix \( \eta^{ij} \) of type \( G \).

\[ \text{Our normalization of } p\text{-form fields (including gauge fields) is such that their field strengths take values in integer cohomology } H^{p+1}(M, \mathbb{Z}) \text{ for a smooth manifold } M. \text{ This makes the normalization of gauge fields different by a factor of } 2\pi \text{ from the usual one.} \]
The \( R \) symmetry of the \( \mathcal{N} = (2, 0) \) theory is \( \text{SO}(5)_R \). As an \( \mathcal{N} = (1, 0) \) theory, we can see the symmetry \( \text{SU}(2)_R \times \text{SU}(2)_L \cong \text{SO}(4)_R \subset \text{SO}(5)_R \). \( \text{SU}(2)_L \) is a flavor symmetry from the \( \mathcal{N} = (1, 0) \) viewpoint. Going to the \( \mathcal{N} = (1, 0) \) tensor branch does not break \( \text{SO}(4)_R \).

Now, reduce the system on \( S^1 \). The \( \mathcal{N} = (1, 0) \) tensor branch corresponds to giving vevs to only one direction out of \( \mathbb{R}^5 \) worth of scalars. We consider a generic vev \( v \in \mathfrak{h} \), where \( \mathfrak{h} \) is the Cartan subalgebra of \( g \), such that the low-energy system is just \( U(1)' \). For each root \( \alpha \in \mathfrak{h} \), we have a massive charged \( \mathcal{N} = 2 \) vector multiplet of mass \( |v \cdot \alpha| \), i.e. a pair of a massive \( \mathcal{N} = 1 \) vector multiplet and a massive \( \mathcal{N} = 1 \) hypermultiplet. Note that the fermion mass term is of the form

\[
\psi \Gamma^I \phi^I \psi
\]

where \( \psi \) is in the spinor representation of \( \text{SO}(5)_R \) and \( I = 1, \ldots, 5 \) is the index for the vector representation of \( \text{SO}(5)_R \). We are giving the vev to only \( \phi^{I=5} \). Therefore, the \( \mathcal{N} = 1 \) vector multiplet is charged only under \( \text{SU}(2)_R \) and has the real mass \(-v \cdot \alpha\), whereas the \( \mathcal{N} = 1 \) hypermultiplet is charged only under \( \text{SU}(2)_L \) and has the real mass \(+v \cdot \alpha\).

We can use \( v \) to determine the positive and negative sides of the Cartan subalgebra; accordingly, the roots \( \alpha \) can be separated into positive and negative roots. The induced Chern–Simons terms for the \( U(1)' \) field \( A \) valued in \( \mathfrak{h} \) is then

\[
\frac{1}{2} \sum_{\alpha > 0} (\alpha \cdot A) \left[ \left( c_2(L) + \frac{2}{24} p_1(T) \right) - \left( c_2(R) + \frac{2}{24} p_1(T) \right) \right] = \rho \cdot A (c_2(L) - c_2(R))
\]

where \( \rho \) is the Weyl vector. Lifting it back to 6D, the Bianchi identity of the self-dual tensor fields is given by

\[
dH = \rho (c_2(L) - c_2(R)),
\]

and therefore, the Green–Schwarz contribution to the anomaly of the 6D theory is

\[
\frac{1}{2} \langle \rho, \rho \rangle (c_2(L) - c_2(R))^2 = \frac{h_{G \ell} dG}{24} (c_2(L) - c_2(R))^2
\]

where we used the strange formula of Freudenthal and de Vries.

We conclude that the anomaly polynomial of the 6D \( \mathcal{N} = (2, 0) \) theory of type \( G \) is given by

\[
I_G^{(2, 0)} = \frac{h_{G \ell} dG}{24} p_2(N) + r_G I^{(2, 0) \text{ tensor}},
\]

where we added the contribution from \( r_G \) free \( \mathcal{N} = (2, 0) \) tensor multiplets on the tensor branch, and used the fact that \( \chi_4(N) = c_2(L) - c_2(R) \), \( p_2(N) = \chi_4(N)^2 \) when the \( \text{SO}(5)_R \) bundle is in fact an \( \text{SO}(4)_R \cong \text{SU}(2)_L \times \text{SU}(2)_R \) bundle.

The anomaly of \( Q \) M5 branes is obtained by adding one additional free \( \mathcal{N} = (2, 0) \) tensor to the 6D \( \mathcal{N} = (2, 0) \) theory of type \( A_{Q-1} \), and has the form

\[
I_Q^{\text{M5s}} = \frac{Q^3}{24} p_2(N) - Q I_8
\]

where

\[
I_8 = \frac{1}{48} (p_2(N) + p_2(T) - \frac{1}{4} (p_1(N) - p_1(T))^2).
\]
2.4. E-string theory of arbitrary rank

Next, let us consider the E-string theory of rank $Q$. This is the low-energy theory on $Q$ M5 branes on top of the end-of-the-world $E_8$ brane. For now, let us include the free hypermultiplet corresponding to the motion parallel to the $E_8$ brane.

We use the fact that when it is put on $S^1$ with a holonomy of $E_8$ chosen so that it breaks $E_8$ to SO(16), the 5D theory is given by $\mathcal{N} = 1$ USp(2$Q$) theory with an antisymmetric hypermultiplet and 8 hypermultiplets in the fundamental representation. This allows us to reconstruct the full anomaly polynomial, since SO(16) is a maximal rank subgroup of $E_8$.

We can go to the generic point on the tensor branch and repeat the analysis as in the case of $\mathcal{N} = (2,0)$ theories. Instead, let us consider a rather special point on the tensor branch such that $Q$ M5 branes are still coincident but are separated from the end-of-the-world brane. There is only one tensor multiplet scalar activated, which is a diagonal sum of $Q$ free tensor multiplets on the generic points. Therefore, the matrix $\Omega^{ij}$ in (2.3) is $Q$.

In 5D, this point on the Coulomb branch corresponds to giving a vev to the adjoint scalar of the vector multiplet that breaks USp(2$Q$) to U($Q$); we would like to determine the Chern–Simons term involving the U(1) part. We have $Q$ massive hypermultiplets in the vector representation of SO(16) with U(1) charge 1, and $Q^2 + Q$ massive vector multiplets and $Q^2 - Q$ hypermultiplets, both with U(1) charge 2. Recalling that the vector is charged under the SU(2)$_R$ and the hypers in the antisymmetric representation under the SU(2)$_L$, we find the induced Chern–Simons term to be

$$
\frac{1}{2} A \wedge \left[ Q \left( \frac{\text{Tr} F^2}{2} + \frac{16p_1(T)}{24} \right) \right] + 2(Q^2 - Q) \frac{1}{2} \left( c_2(L) + \frac{2p_1(T)}{24} \right) - 2(Q^2 + Q) \frac{1}{2} \left( c_2(R) + \frac{2p_1(T)}{24} \right) \right] = \Omega A \wedge \left( \frac{Q}{2} \chi_4(N) + I_4 \right)
$$

(2.17)

where $\Omega = Q$ and

$$
I_4 = \frac{1}{4} \left( \text{Tr} F^2 + p_1(T) + p_1(N) \right)
$$

(2.18)

and we again used

$$
\chi_4(N) = c_2(L) - c_2(R), \quad p_1(N) = -2(c_2(L) + c_2(R)).
$$

(2.19)

From this, we find that

$$
I_{\text{E-string, rank } Q\text{-free hyper}} = I_0^M + \frac{Q}{2} \left( \frac{Q}{2} \chi_4(N) + I_4 \right)^2
$$

(2.20)

$$
= \frac{Q^3}{6} \chi_4(N)^2 + \frac{Q^2}{2} \chi_4(N)I_4 + Q \left( \frac{1}{2} I_4^2 - I_8 \right),
$$

(2.21)

where $I_8$ was given above. This reproduces the result in Ref. [32] obtained via the anomaly inflow.

Note that this contains the contribution of a free hypermultiplet with the anomaly

$$
I_{\text{free}} = \frac{7p_1(T)^2 - 4p_2(T)}{5760} + \frac{c_2(L)p_1(T)}{48} + \frac{c_2(L)^2}{24}.
$$

(2.22)

When the E-string theory is used as a matter content, we always need to subtract this contribution (2.22) from (2.21).
3. Tensor branches with gauge multiplets

In this section we develop the method to determine the anomaly polynomials of 6D $\mathcal{N} = (1, 0)$ theories when we have non-Abelian gauge fields on the tensor branch. As briefly explained in the introduction, we can uniquely determine the Green–Schwarz terms by requiring that the gauge anomalies vanish. After explaining the basic ideas, we focus on the case of the worldvolume theories on multiple coincident M5 branes on ALE singularities of arbitrary type. We end the section by explaining the relation between the Green–Schwarz terms and the F-theory construction of arbitrary 6D $\mathcal{N} = (1, 0)$ theories.

3.1. Basic ideas

A 6D $\mathcal{N} = (1, 0)$ superconformal theory can have the tensor branch such that the infrared theory at a point on the tensor branch consists of $t$ vector multiplets in gauge group $G_A$, $A = 1, \ldots, t$, and $t$ free tensor multiplets whose scalars give the coupling constants of $G_A$, together with a number of charged “bifundamental matter contents”. This “bifundamental matter” can either be Lagrangian hypermultiplets or another 6D SCFT whose flavor symmetries are gauged by $G_A$. We assume that anomalies of the “bifundamental matter” are already known. This is indeed the case for all the theories discussed in Ref. [12], where we can have E-string theories of rank one or two as the “matter contents”.

Note that although the full tensor branch of the theory may have a dimension larger than the number of the gauge groups $t$, we always stay on the subspace of the full tensor branch where the number of almost free tensor multiplets is the same as that of the gauge groups. In particular, this means that we do not give vevs to the tensor modes of E-string theories.

Now, the “one-loop” anomaly (i.e. the anomaly without Green–Schwarz contribution) on the tensor branch is given by

$$I_{\text{one-loop}} = \sum_A I^\text{vec}_{FA} + \sum_{A,B} I^\text{matter}_{FA,FB} + t I^\text{tensor}. \quad (3.1)$$

It contains pure gauge and mixed gauge-background terms,

$$I_{\text{one-loop}} \supset -\frac{1}{32} c^{AB} \text{Tr} F_A^2 \text{Tr} F_B^2 - \frac{1}{4} X^A \text{Tr} F_A^2. \quad (3.2)$$

where $X^A$ consists of only background flavor and gravity fields. One needs to cancel these gauge anomalies by the Green–Schwarz contribution,

$$\frac{1}{2} \Omega^{ij} I_i I_j. \quad (3.3)$$

Here $\Omega^{ij}$ is the symmetric matrix introduced in (2.3), which, roughly speaking, is the matrix appearing in the kinetic term of the tensor multiplets. The anomaly cancellation requires

$$I_i = \frac{1}{4} d_i^A \text{Tr} F_A^2 + (\Omega^{-1})_{ij} (d^{-1})_A^i X^A, \quad d_i^A d_j^B \Omega^{ij} = c^{AB}, \quad (3.4)$$

where we have assumed that the matrix $c^{AB}$ has the maximal rank $t$, which implies that the matrix $d_i^A$ is invertible. This is the point at which we need the number of free tensor multiplets and the gauge groups $G_A$ to be the same.
Although the matrix $d^A_i$ is not completely determined, the Green–Schwarz contribution is uniquely determined in terms of $c^{AB}$ and $X^A$ as

$$\frac{1}{2} I_i I_j = \frac{1}{32} c^{AB} \text{Tr} F^2_A \text{Tr} F^2_B + \frac{1}{4} X^A \text{Tr} F^2_A + \frac{1}{2} (c^{-1})_{AB} X^A X^B. \quad (3.5)$$

The first two terms cancel the gauge anomalies, and the third term gives the Green–Schwarz contribution to the anomaly of background fields.

### 3.1.1. A consistency condition on the theory

Before going to the applications of the above method, we would like to make an interesting digression here. The self-dual three-form field strengths $H_i$ in the tensor multiplets satisfy $d H_i = I_i$ as discussed in Sect. 2.1. From the above expression for $I_i$, a point-like instanton of the gauge field $\frac{1}{4} \text{Tr} F^2_A$ gives a string in 6D with charge $q_i = d^A_i$. Then, each element of $c^{AB} = \Omega^{ij} d^A_i d^B_j = \langle d^A, d^B \rangle_{6D}$ must be an integer precisely by the 6D charge quantization condition discussed around (2.3). This imposes a strong constraint on the theory. For example, the theories that are excluded based on global anomalies in Ref. [6] can already be excluded by this constraint alone, because the one-loop anomaly polynomial yields non-integer $c^{AB}$ in those theories.

### 3.2. M5 branes on ALE singularities

As an example of the method outlined in Sect. 3.1, we determine the anomalies of $Q$ M5 branes on an ALE singularity $\mathbb{C}^2/\Gamma$. When $\Gamma$ is of type $A_k$, there is a $\text{U}(1)$ symmetry acting on $\mathbb{C}^2/\Gamma$, but we ignore this symmetry for simplicity.

In M-theory, the singular locus extends along seven dimensions, on which lives a 7D dynamical gauge multiplet in the gauge group $G$ determined by $\Gamma$. M5 branes are 6D, and therefore we consider the singular locus to form a line of singularities transverse to the worldvolume.

We first separate $Q$ M5 branes along this line of singularities. The system can be described as a linear quiver theory

$$[G_0] \times G_1 \times \cdots \times G_{Q-1} \times [G_Q] \quad (3.6)$$

with $(Q-1)$ gauge factors $G_1, \ldots, G_{Q-1}$ and flavor symmetry $G_0 \times G_Q$, and “conformal matters” charged under $G_i \times G_{i+1}$. The “conformal matter” is a theory that is realized on a single M5 brane on the singularity. So, the computation of the anomaly of $Q$ M5 branes involves two steps. The first is to compute the anomalies of each “conformal matter”. The second is to compute the anomalies of the linear quiver theory.

### 3.2.1. Anomalies of “conformal matters”

The “conformal matter” is a Lagrangian hypermultiplet when $G$ is of type $A$, and another nontrivial 6D SCFT when $G$ is of type $D$ or $E$. Since we already know anomalies of Lagrangian hypermultiplets, we concentrate on the latter.

The tensor branch of these superconformal field theories (SCFTs) is investigated in Ref. [13]. What happens is that a single M5 brane can split to several fractional M5 branes along the line of singularities. On a generic point on the tensor branch, the low-energy theory consists only of tensor multiplets, hypermultiplets, and vector multiplets. The tensor multiplet scalars are the distances between two adjacent fractional M5 branes. However, it is not always the case that there is a non-trivial gauge group on a segment between two fractional M5 branes. If this happens, we make these fractional M5 branes coalesce. Then we have the situation where there are an equal number of tensor and vector multiplets, coupled to Lagrangian matter fields and/or E-string theories of rank 1 and 2. Then, we can just apply the method in Sect. 3.1.
Let us list the structure of the “conformal matters” at the point on the tensor branch that we use to identify the anomaly polynomial, following Ref. [13]. We will use the $E_6$ case to illustrate the detailed steps of the computation.

$(A_{k-1}, A_{k-1})$ conformal matter. This is just a hypermultiplet in the bifundamental of $SU(k) \times SU(k)$.

$(D_k, D_k)$ conformal matter. The tensor branch is 1D, or equivalently, the M5 brane can fractionate into two. The first fractional M5 brane changes the gauge group from $SO(2k)$ to $USp(2k - 8)$. The second fractional M5 brane changes it back to $SO(2k)$. We can depict the setup

$$SO(2k)|USp(2k - 8)|SO(2k)$$

where $|$ stands for a fractional M5 brane, and the groups displayed are the gauge groups on the particular half-line or segment of the $D_k$ singularities. One can also regard it as describing the linear quiver gauge theory, where the two $SO(2k)$ at the ends are flavor symmetries, and $USp(2k - 8)$ is a gauge symmetry. The fractional M5 brane between $SO(2k)$ and $USp(2k - 8)$ provides a half-hypermultiplet in the bifundamental.

In the case $k = 4$, there is no $USp$ gauge group between two fractional M5 branes, so our method cannot be applied. The conformal matter realized on one full M5 brane on $D_4$ is actually the rank-1 E-string theory. However, the anomaly polynomial of this theory is also given by putting $k = 4$ in the general formula that we will present later.

$(E_6, E_6)$ conformal matter. The tensor branch is 3D, and the M5 brane can fractionate into four. The gauge groups that occur between the fractional M5 branes are

$$E_6|empty|SU(3)|empty|E_6.$$}

To compute the anomaly, we make pairs of fractional M5 branes coalesce:

$$E_6||SU(3)||E_6.$$}

Now we have an $SU(3)$ vector multiplet plus one $(1, 0)$ tensor multiplet, and the matter content between $E_6$ and $SU(3)$ is in fact the rank-1 E-string theory, via the embedding

$$E_6 \times SU(3) \subset E_8.$$}

The anomalies of two rank-1 E-string theories and an $SU(3)$ vector multiplet plus one $(1, 0)$ tensor multiplet are given by

$$I^{\text{one-loop}} = I^{\text{rank-1 E-string}}_E \left( Tr F_L^2 + Tr F_{SU(3)}^2 \right) + I^{\text{vector}}_{SU(3)} \left( Tr F_{SU(3)}^2 \right) + I^{\text{tensor}} + I^{\text{rank-1 E-string}}_E \left( Tr F_{SU(3)}^2 + Tr F_R^2 \right)$$

$$= \frac{1}{32} \left( Tr F_L^2 \right)^2 + \frac{1}{32} \left( Tr F_R^2 \right)^2 + \left( Tr F_L^2 + Tr F_R^2 \right) \left( \frac{1}{16} p_1(T) - \frac{1}{4} c_2(R) \right)$$

$$+ \frac{19}{24} c_2^2(R) - \frac{29}{48} c_2(R) p_1(T) + \frac{373}{5760} p_1^2(T) - \frac{79}{1440} p_2(T)$$

$$- \frac{1}{32} \left( Tr F_{SU(3)}^2 \right)^2 + Tr F_{SU(3)}^2 \left( -\frac{5}{4} c_2(R) + \frac{1}{16} p_1(T) + \frac{1}{16} Tr F_L^2 + \frac{1}{16} Tr F_R^2 \right)$$

where $F_L$, $F_{SU(3)}$, and $F_R$ are the background field strength of $E_6^L$, $SU(3)$, and $E_6^R$, respectively. The anomaly of the rank-1 E-string $I^{\text{rank-1 E-string}}_E$ is given in (2.21), but note that one needs to subtract the contribution from a free hypermultiplet given in (2.22). Also note that we call these contributions the “one-loop” contribution from a lack of better terminology, although there is no concept of loop computations in the E-string theory.
The Green–Schwarz term that cancels the SU(3) part of the anomalies is found to be
\[
I^{GS} = \frac{1}{2} \left( \frac{1}{4} \text{Tr} F_{SU(3)}^2 + 5c_2(R) - \frac{1}{4} p_1(T) - \frac{1}{4} \text{Tr} F_L^2 - \frac{1}{4} \text{Tr} F_R^2 \right)^2. \tag{3.11}
\]

Therefore, the total anomaly is
\[
I_{E_6,E_7}^{\text{bif}}(F_L,F_R) = I^{\text{one-loop}} + I^{GS} = \frac{1}{16} (\text{Tr} F_L^2)^2 + \frac{1}{16} \text{Tr} F_L^2 \text{Tr} F_R^2
+ \frac{1}{16} \left( \text{Tr} F_L^2 \right)^2 + \left( \text{Tr} F_L^2 + \text{Tr} F_R^2 \right) \left( \frac{1}{8} p_1(T) - \frac{3}{2} c_2(R) \right)
+ \frac{319}{24} c_2(R) - \frac{89}{48} c_2(R) p_1(T) + \frac{553}{5760} p_1^2(T) - \frac{79}{1440} p_2(T). \tag{3.12}
\]

\((E_7, E_7)\) conformal matter. The tensor branch is 5D, and the M5 brane fractionates into six. The structure is given by
\[
E_7|\text{empty}|SU(2)|SO(7)|SU(2)|\text{empty}|E_7. \tag{3.13}
\]

To compute the anomaly, we make two pairs coalesce to the situation
\[
E_7||SU(2)|SO(7)|SU(2)||E_7. \tag{3.14}
\]

Now we have vector multiplets in \(SU(2) \times SO(7) \times SU(2)\) plus three \((1,0)\) tensor multiplets. The matter content between \(E_7\) and \(SU(2)\) is again the rank-1 E-string theory, via the embedding\n\[
E_7 \times SU(2) \subset E_8, \tag{3.15}
\]

and that between \(SU(2)\) and \(SO(7)\) is the half-hypermultiplet in the fundamental of \(SU(2)\) and the spinor of \(SO(7)\).

\((E_8, E_8)\) conformal matter. The tensor branch is 11D, and the M5 brane fractionates into twelve. The structure is given by
\[
E_8|\text{empty}|\text{empty}|SU(2)|G_2|\text{empty}|F_4|\text{empty}|G_2|SU(2)|\text{empty}|\text{empty}|E_8. \tag{3.16}
\]

The matter content between \(SU(2)\) and \(G_2\) is the half-hypermultiplet in the bifundamental. Each \(SU(2)\) also has a half-hypermultiplet in the fundamental. After coalescing, the matter between \(G_2 \times F_4\) is again the rank-1 E-string theory, via the embedding
\[
G_2 \times F_4 \subset E_8. \tag{3.17}
\]

To compute the anomaly, we go to the point where we have
\[
E_8|||SU(2)|G_2||F_4||G_2|SU(2)|||E_8. \tag{3.18}
\]

From the F-theory description given in Ref. [12], we see that the matter between \(E_8\) and \(SU(2)\) is now the rank-2 E-string theory, whose anomaly was given as (2.21) minus (2.22). Here the \(SU(2)\) gauge group is coupled to the \(SU(2)_L\) symmetry explained in Sect. 2.4. This interpretation can be supported as follows: on a generic point on the tensor branch of the rank-2 E-string, there is one free hypermultiplet, which describes the relative position of 2 M5 branes parallel to the end-of-the-world \(E_8\) brane. This counts as one half-hypermultiplet in the fundamental of \(SU(2)\), which should be identified as the half-hypermultiplet of \(SU(2)\) mentioned just below (3.16).
By doing the same exercise that we did in the $E_6$ case for all $(G, G)$ conformal matters, where $G$ is an ADE group, we get the following anomaly polynomial:

$$I_{G,G}^{\text{bif}}(F_L, F_R) = \frac{\alpha}{24} c_2(R)^2 - \frac{\beta}{48} c_2(R) p_1(T) + \frac{\gamma}{2} p_1(T)^2 - 4 p_2(T)$$

$$+ \left( -\frac{x}{8} c_2(R) + \frac{y}{96} p_1(T) \right) (\text{Tr} F_L^2 + \text{Tr} F_R^2)$$

$$+ \frac{1}{48} (\text{Tr}_G F_L^4 + \text{Tr}_G F_R^4) \right) - \frac{1}{2} \left( \frac{1}{4} \text{Tr} F_L^2 - \frac{1}{4} \text{Tr} F_R^2 \right)^2$$

where coefficients are listed in Table 1. From this table, we can easily read off that $\gamma = \dim_G + 1$, $x = |\Gamma_G| - h_G^2$, and $y = h_G^2 - \alpha$ and $\beta$ are more complicated combinations of group theoretical data, which we will display as part of the formula for a general number $Q$ of M5 branes on the ALE singularity below.

### 3.2.2. Anomaly polynomial

Now let us determine the anomaly polynomial of $Q$ full M5 branes on the ALE singularity $\mathbb{C}^2/\Gamma$. We go to a point on the tensor branch where it is a quiver gauge theory with flavor and gauge groups $[G_0] \times G_1 \times \cdots \times G_{Q-1} \times [G_Q]$. We have just computed the anomaly of the “conformal matters” of $G_i \times G_{i+1}$. We also have $Q - 1$ free tensor multiplets, describing the relative positions of the M5 branes. In this section we are going to compute the total anomaly. We include the center-of-mass motion of $Q$ M5 branes just for convenience of computation, but this does not affect the final result as long as we subtract the contribution of the center-of-mass mode (both one-loop and Green–Schwarz) at the end of the computation.\(^5\)

The one-loop anomaly is then given by

$$I^{\text{one-loop}} = \sum_{i=0}^{Q-1} I_{G,G}^{\text{bif}}(F_i, F_{i+1}) + \sum_{i=1}^{Q-1} I_G^{\text{vec}}(F_i) + Q I^{\text{tensor}}.$$  \hspace{1cm} (3.20)

We find that the gauge anomalies can be canceled by the Green–Schwarz term

$$I^{\text{GS}} = \frac{1}{2} \sum_{i=0}^{Q-1} \mathcal{H}_i \mathcal{H}_i$$

for the self-dual tensor fields with the Bianchi identity

$$d \mathcal{H}_i = \mathcal{I}_i = \frac{1}{4} \text{Tr} F_i^2 - \frac{1}{4} \text{Tr} F_{i+1}^2 + \frac{1}{2} (2i - Q + 1) |\Gamma| c_2(R),$$

where $\mathcal{H}_i$ ($i = 0, 1, \ldots, Q - 1$) are the three-form fields of the tensor multiplets whose scalars represent the positions of $Q$ M5 branes. Combining all of them, we get the anomaly polynomial of $Q$

\(^5\) If we compute the anomaly by the inflow argument as in Appendix B, the center-of-mass mode is automatically included there.

---

### Table 1. Table of anomaly coefficients for $(G, G)$ conformal matters.

| $G$ | SU($k$) | SO(2$k$) | $E_6$ | $E_7$ | $E_8$ |
|-----|---------|---------|-------|-------|-------|
| $\alpha$ | 0 | $10k^2 - 57k + 81$ | 319 | 1670 | 12489 |
| $\beta$ | 0 | $2k^2 - 3k - 9$ | 89 | 250 | 831 |
| $\gamma$ | $k^2$ | $k(2k - 1) + 1$ | 79 | 134 | 249 |
| $\chi$ | 0 | $2k - 6$ | 12 | 30 | 90 |
| $\psi$ | $k$ | $2k - 2$ | 12 | 18 | 30 |
M5 branes at the ALE singularity $\mathbb{C}^2/\Gamma_G$:

$$I_G^{\text{tot}} = I^{\text{GS}} + I^{\text{one-loop}} = |\Gamma|^2 Q \left( \frac{c_2^2(R)}{24} - \frac{Q}{48} c_2(R) \left( |\Gamma|^2(r_G + 1) - 1 \right) \right) \left( 4c_2(R) + p_1(T) \right)
- \frac{Q}{8} |\Gamma| c_2(R) \left( \text{Tr} F_0^2 + \text{Tr} F_Q^2 \right) + \frac{Q}{8} \left( \frac{1}{4} c_2(R) p_1(T) - \frac{1}{6} p_2(T) + \frac{1}{24} p_1^2(T) \right)
- \frac{1}{2} I^{\text{vec}}(F_0) - \frac{1}{2} I^{\text{vec}}(F_Q).$$

(3.23)

Here we give two comments about the result (3.23). The first comment is about the center-of-mass tensor multiplet. The anomaly polynomial of the UV SCFT is determined by subtracting the contributions of the center-of-mass tensor multiplet,

$$I_{\text{tot}} = I^{\text{SCFT}} + I^{\text{ten}} + \frac{1}{2} \left( \frac{1}{4} \text{Tr} F_0^2 - \frac{1}{4} \text{Tr} F_Q^2 \right)^2.$$

(3.24)

Here the third term is a Green–Schwarz term for the center-of-mass tensor multiplet: it has the Bianchi identity

$$d \left( \frac{1}{Q} \sum_i \mathcal{H}_i \right) = \frac{1}{Q} \sum_i I_i = \frac{1}{Q} \left( \frac{1}{4} \text{Tr} F_0^2 - \frac{1}{4} \text{Tr} F_Q^2 \right),$$

(3.25)

and the additional factor $Q$ comes from the factor $Q$ in front of the kinetic term of the center-of-mass tensor multiplet, i.e. $\Omega_{\text{center-of-mass}} = Q$.

The second comment is about the leading behavior. This field theoretical method gives the cubic behavior $\frac{1}{24} Q^3 |\Gamma|^2 c_2(R)^2$. The coefficient is exactly what is expected from AdS$_7$/CFT$_6$. In fact, the whole structure of (3.23), including its coefficients, can be reproduced from an anomaly inflow, as will be explained in Appendix B.

3.3. Green–Schwarz terms for F-theory constructions

In this subsection we investigate the Green–Schwarz terms for general 6D $\mathcal{N} = (1, 0)$ theories constructed in Ref. [12]. Although the Green–Schwarz contribution for such theories can be computed by the method we have developed so far, here we want to investigate a more direct way to relate the Green–Schwarz terms and F-theory constructions.

3.3.1. On generic points on the tensor branch. First, we recall how we can determine the Green–Schwarz terms associated with metric, gauge, and flavor background starting from Type IIB supergravity on $\mathbb{R}^{1.5} \times B$ where $B$ is a noncompact (possibly singular) manifold that contains compact or noncompact rational curves $C_a$ possibly wrapped by 7-branes. We let the index $a, b, \ldots$ run through all curves in $B$, while $i, j, \ldots$ are only for compact ones.

The 5-form field strength $F_5$ in 10D spacetime decomposes into 6D self-dual 3-form field strengths $H_i$ as

$$F_5 = H_i \wedge \omega^i$$

(3.26)

where $\omega^i$ is the Poincaré dual of $C_i$. If we have the Bianchi identity for the 5-form field strength $F_5$ written as

$$d F_5 = Z,$$

(3.27)

for some 6-form $Z$ consisting of background field strengths, then the Bianchi identities for the 3-form strengths $H^i$ become

$$d H_i = I_i, \quad \eta^{ij} I_j = - \int_B Z \wedge \omega^j,$$

(3.28)
where the matrix $\eta_{ij} = -\int_B \omega_i \wedge \omega_j = -C_i \cdot C_j$ is $-1$ times the intersection form of compact cycles in $B$. We extend this intersection form to $\eta^{ia}$, which includes intersections between compact and noncompact cycles. The contributions for the anomaly polynomial from these Green–Schwarz terms are\(^6\)

$$I^{\text{GS}} = -\frac{1}{2} \int_B Z^2 = \frac{1}{2} \eta_{ij} I_i I_j.$$  \hspace{1cm} \text{(3.29)}

The matrix $\Omega^{ij}$ introduced in (2.3) is given by $\Omega^{ij} = \eta^{ij}$ in this class of theories.

As described in Ref. [25], the 10D Green–Schwarz term $Z$ is

$$Z = \frac{1}{4} c_1(B) \wedge p_1(T) + \frac{1}{4} \sum_a \omega^a \text{Tr} F_a^2$$  \hspace{1cm} \text{(3.30)}

where $F_a$ is the field strength on the 7-branes wrapping $C_a$. So we get

$$\eta^{ij} I_j = \frac{1}{4} (\eta^{ia} \text{Tr} F_a^2 - K^i p_1(T)), \quad K^i := \int_B c_1(B) \wedge \omega^i = 2 - \eta^{ii},$$  \hspace{1cm} \text{(3.31)}

up to the term proportional to $c_2(R)$. This expression is supposed to be true even for non-perturbative F-theory background, and can be checked for concrete 6D $\mathcal{N} = (1, 0)$ theories by the method developed throughout this paper.

Next, we consider the terms associated with the $\text{SU}(2)_R$ $R$ symmetry, which are not directly visible by the geometry of F-theory construction. Each Green–Schwarz term $I_i$ should contain contributions proportional to $c_2(R)$ to cancel mixed gauge-$\text{SU}(2)_R$ anomalies, so we write

$$\eta^{ij} I_j = \frac{1}{4} (\eta^{ia} \text{Tr} F_a^2 - K^i p_1(T)) + y^i c_2(R).$$  \hspace{1cm} \text{(3.32)}

Our next task is to determine the coefficients $y^i$. The contribution to the mixed anomalies between gauge and $R$ symmetries from the Green–Schwarz terms are

$$I^{\text{GS}} \supset \frac{1}{4} y^i \text{Tr} F_i^2 c_2(R).$$  \hspace{1cm} \text{(3.33)}

Consider a generic point of the tensor branch. There we have only Lagrangian degrees of freedom, that are vector multiplets, hyper multiplets, and tensor multiplets, and only the vector multiplets have mixed anomaly between the $R$ and gauge symmetries described in (1.8). Then, if a cycle $C_i$ has a nontrivial gauge group $G_i$, we can immediately conclude that $y^i = h_i^{1\gamma}$ for that cycle. This agrees with what we saw in (1.10).

For $-1$ and $-2$ curves without gauge groups, we cannot determine $y^i$ with this method. One can circumvent this problem by going to the points of the tensor branch where such curves are shrunk giving rank-1 or 2 E-string theories, as we have done in previous subsections. We will discuss this process of shrinking curves in the next subsection. It will turn out that $y^i = 1$ gives consistent results in the process.

Alternatively, $y^i$ should be fixed so that $I^{\text{GS}}$ reproduces the correct $Q^3$ dependence of the anomaly polynomials of rank $Q \mathcal{N} = (2, 0)$ or E-string theories. This requires $y^i$ to be 1 for those curves of self-intersection $-1$ and $-2$, assuming that $y^i$ is independent of the information for any other curves $C_j$ for $j \neq i$. Then the subleading terms of $Q$ are also correctly reproduced.

\(^6\) We use a convention that $F_5$ is anti-self-dual so that the 6D fields $H_i$ become self-dual, because $\omega^i$, which have a negative definite intersection matrix, are anti-self-dual. The minus sign in front of $\frac{1}{2} \int Z^2$ comes from this anti-self-dual (instead of self-dual) property of $F_5$. 

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Therefore, we claim that \( y^i = 1 \) for the cycles without gauge groups when none of the cycles are shrunk. Then we can calculate the anomaly polynomial for any of 6D \( \mathcal{N} = (1, 0) \) theories constructed in Ref. [12]. At a generic point of the tensor branch, we have only fields described by Lagrangians, so the calculation of one-loop anomaly polynomial is straightforward. Then all we have to do is just add the Green–Schwarz contribution (3.29).

3.3.2. Shrinking \(-1\) curves. Now let us describe a convenient algorithm for calculating Green–Schwarz terms for a non-generic point of the tensor branch where some of the \(-1\) curves are shrunk. This will justify the above claim that \( y^i = 1 \) for cycles without gauge groups, and can also be used to obtain (3.22). For simplicity, we consider blowing down a certain \(-1\) curve \( C_A \) in \( B \) that intersects with curves \( C_{A-1} \) and \( C_{A+1} \). We can obtain the result for the case in which multiple \(-1\) are shrunk by recursion.

Let \( \hat{B} \) be the manifold obtained by shrinking (i.e., blowing down) the \(-1\) curve \( C_A \) in \( B \), and \( p : B \to \hat{B} \) be the blow-down map. The homology classes of cycles \( \hat{C}_i = p(C_i) \), \( i \neq A \) in \( \hat{B} \) and \( C_i \) in \( B \) are related by

\[
p^*\{\hat{C}_i\} = \begin{cases} [C_i] + [C_A] & i = A - 1, A + 1 \\ [C_i] & i \neq A - 1, A, A - 1. \end{cases} \tag{3.34}
\]

where \([C_i]\) means the homology class of \( C_i \). In the following, we demand that indices \( i, j \) do not take the value \( A \). Their intersection form becomes

\[
\hat{\eta}^{ij} = -\hat{C}^i \cdot \hat{C}^j = \begin{cases} \eta^{ii} - 1 & i = j = A \pm 1 \\ -1 & (i, j) = (A - 1, A + 1), (A + 1, A - 1), \\ \eta^{ij} & \text{otherwise} \end{cases} \tag{3.35}
\]

There are 3-form field strengths \( \hat{H}_i \) each associated with cycles \( \hat{C}_i \) that satisfy

\[
d\hat{H}_i = \hat{I}_i, \quad \hat{I}_i = \begin{cases} I^i + I^A & i = A + 1, A - 1 \\ I^i & \text{otherwise} \end{cases} \tag{3.36}
\]

where \( I^i = \eta^{ij} I_j \). This means that the \( \text{Tr} F_i^2 \) and \( p_1(T) \) dependence of the Green–Schwarz terms \( \hat{I}_i \) is again given by (3.31), as it should be. The coefficients of \( c_2(R) \) in \( \hat{I}_i \) can be easily calculated from (3.36).

Then, the new Green–Schwarz contribution to the anomalies after the blow-down is

\[
\hat{I}^{\text{GS}} = \frac{1}{2} \hat{\eta}^{ij} \hat{I}_i \hat{I}_j = \frac{1}{2} (\hat{\eta}^{-1})_{ij} \hat{I}_i \hat{I}_j. \tag{3.37}
\]

The anomaly of the total system is

\[
I^{\text{tot}} = I^{\text{one–loop}} + I^{\text{GS}} = \hat{I}^{\text{one–loop}} + \hat{I}^{\text{GS}}. \tag{3.38}
\]

This equality must hold because of the anomaly matching. We also have a mathematical relation\(^7\)

\[
I^{\text{GS}} = \hat{I}^{\text{GS}} + \frac{1}{2} (I^A)_2. \tag{3.39}
\]

\(^7\) The relation is shown as follows. Let us change the basis of two-forms of \( B \) from \( \omega^i \) to \( \omega'^i = p^* \omega^i \) (\( i \neq A \)), \( \omega'^A = \omega^A \). Then, \( I^i = - \int \omega'^i \wedge \omega'^i \) is given by \( I^i = \hat{I}^i \) (\( i \neq A \)) and \( I^A = \hat{I}^A \). The “intersection form” in the new basis \( \eta'^{ij} = - \int \omega'^i \wedge \omega'^j \) is given by a block diagonal form, \( \eta'^{ij} = \hat{\eta}^{ij} \) (\( i, j \neq A \)), \( \eta'^{AA} = 0 \) (\( i \neq A \)), and \( \eta'^{AA} = 1 \). Using these, \( \text{(3.39)} \) immediately follows.
Thus the one-loop factor before and after the blow-up must be related by

\[ \hat{I}_{\text{one-loop}} = I_{\text{one-loop}} + \frac{1}{2} (I^A)^2. \] (3.40)

Let us apply the result (3.40) to the case where the curve \( C_A \) does not have any gauge group. Then a copy of the rank-1 E-string theory appears by the blow-down. So the difference between the one-loop anomalies before and after the blow-down must be the difference between the rank-1 E-string and a free tensor multiplet,

\[ \hat{I}_{\text{one-loop}} - I_{\text{one-loop}} = I_{\text{rank-1 E-string}} - I_{\text{tensor}} = \frac{1}{2} \left( c_2(R) - \frac{1}{4} p_1(T) - \frac{1}{4} \text{Tr} F_{E_8}^2 \right)^2. \] (3.41)

On the other hand, from (3.32) we get

\[ \frac{1}{2} (I^A)^2 = \frac{1}{2} \left( \eta_A c_2(R) - \frac{1}{4} p_1(T) - \frac{1}{4} \text{Tr} F_{A-1}^2 - \frac{1}{4} \text{Tr} F_{A+1}^2 \right)^2. \] (3.42)

For these results to be consistent, we must have \( \eta_A = 1 \), justifying our claim in the previous subsection. We can also see that the gauge groups \( G_{A-1} \times G_{A+1} \) are embedded in the \( E_8 \) of the E-string theory such that \( \text{Tr} F_{E_8}^2 = \text{Tr} F_{A-1}^2 + \text{Tr} F_{A+1}^2 \).

As an example, let us calculate the Green–Schwarz terms associated with the tensor branch mode that represents the distance between two full M5 branes on an \( E_6 \) type zsingularity using (3.36). In the dual F-theory description, the intersections of cycles and the coefficients \( y_i \) at a generic point of the tensor branch (corresponding to fractionated M5 branes) are given by

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
E_6 & | & empty & | & SU(3) & | & empty & | & E_6 & | & empty & | & SU(3) & | & empty & | & E_6 \\
\hline
\eta^{ii} & 1 & 3 & 1 & 6 & 1 & 3 & 1 \\
\eta^i & 1 & 3 & 1 & 12 & 1 & 3 & 1 \\
\hline
\end{array}
\]

Here the spaces between two adjacent | and | represent compact cycles and we explicitly write the corresponding gauge symmetries as \( E_6, SU(3) \), or empty. The leftmost and rightmost \( E_6 \) represent the noncompact cycles that provide \( E_6 \times E_6 \) flavor symmetries. The numbers in the second and third rows are \( \eta^{ii} \) (i.e. \(-1\) times the self-intersection number of the corresponding compact cycle) and the coefficients \( y_i \), respectively.

When we shrink all of the \(-1\) curves in the above figure, we get

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
E_6 & || & SU(3) & || & E_6 & || & SU(3) & || & E_6 \\
\hline
\eta^{ii} & 1 & 4 & 1 \\
y^i & 5 & 14 & 5 \\
\hline
\end{array}
\]

where || represents that the curve in between has shrunk, and finally this goes to

\[
\begin{array}{|c|c|c|c|}
\hline
E_6 & |||| & E_6 & |||| & E_6 \\
\hline
\eta^{ii} & 2 \\
y^i & 24 \\
\hline
\end{array}
\]

The symbol |||| now corresponds to a full M5 brane in M-theory, and therefore the tensor mode between two |||| is what we wanted. Note that 24 is equal to their order \( |\Gamma_{E_6}| \) of the binary tetrahedral group \( \Gamma_{E_6} \).

In general, the F-theory description of \( Q \) separated (but not fractionated) full M5 branes on the \( \mathbb{C}^2/\Gamma \) singularity is a sequence of \( Q - 1 \) curves of self-intersection number \(-2\) that are decorated by the gauge group \( G \). Let \( H_i \) be the tensor multiplet for the \( i \)th 2-cycle, with the Bianchi identity
\[ dH_i = I_i. \] This \( I_i \) can be easily determined in the same way as the above computation for \( E_6 \), and we obtain
\[ I^i = \eta^{ij} I_j = \frac{1}{4} (2 \text{Tr} F_i^2 - \text{Tr} F_{i-1}^2 - \text{Tr} F_{i+1}^2) + |\Gamma_1| c_2(R). \] (3.43)
Note that the tensor multiplet containing \( H^i = \eta^{ij} H_j \) corresponds to the distance of two M5 branes.\(^8\)
On the other hand, the positions of the M5 branes are denoted by \( \mathcal{H}_i \) in Sect. 3.2.2, and hence we have
\[ H^i = \mathcal{H}_i - \mathcal{H}_{i-1}. \] (3.44)
This means that \( I^i \) above should be given by \( I^i - I_{i-1} \), where \( d\mathcal{H}_i = I_i \) was given in (3.22), and this is indeed the case.

4. Conclusions and discussions
In this paper, we have described methods that allow us to determine the anomaly polynomials of very general 6D \( \mathcal{N} = (1, 0) \) superconformal theories. The essential idea was that the tensor branch vevs do not break any symmetry other than the conformal symmetry, and therefore the anomaly polynomial at the origin of the tensor branch can be obtained by the anomaly matching on the tensor branch. For this, we need to determine the Green–Schwarz term carried by the self-dual tensor fields on the tensor branch. We described two methods to do so.

The first was applicable when there was no gauge field on generic points on the tensor branch. In this case, we had sufficient control of the behavior of the \( S^1 \) compactification. Then we can determine the induced Chern–Simons terms in 5D, that can then be lifted to 6D to fix the Green–Schwarz term.

The second was applicable when the number of the gauge fields and the number of the tensor fields are equal on some points on the tensor branch. In this case, we can determine the Green–Schwarz term just by requiring that there is no gauge anomaly.

We used the first method to derive the anomaly polynomials of \( \mathcal{N} = (2, 0) \) theories of arbitrary type, and of the E-string theory of arbitrary rank. In most of the cases, the results were known from the analysis of the M-theory anomaly inflow, and our method gives an independent confirmation. For \( \mathcal{N} = (2, 0) \) theory of type \( E \), ours is the first derivation.

We then used the second method to derive the anomaly polynomials of the worldvolume theories on \( Q \) M5 branes on the ALE singularities. We found a general formula, which can be successfully checked against the anomaly inflow computation reported in Appendix B. We also gave a general procedure to determine the Green–Schwarz contribution for \( \mathcal{N} = (1, 0) \) theories recently discussed using F-theory in Refs. [12,13].

Our methods can definitely be used to study various other \( \mathcal{N} = (1, 0) \) theories already known in the literature. Hopefully, the anomaly polynomials we determined here and the methods themselves will be useful in the study of the compactifications of \( \mathcal{N} = (1, 0) \) theories to various lower dimensions.

In this paper, we only considered the part of the anomalies of the \( \mathcal{N} = (1, 0) \) theories that can be captured at the level of differential forms. It would be interesting to study the global anomalies of these theories, following Refs. [33,34]. Also, the partition functions of \( \mathcal{N} = (2, 0) \) theories are known to behave rather like conformal blocks of 2D chiral CFTs [35–38], and it would be interesting to understand what happens in \( \mathcal{N} = (1, 0) \) cases.

\(^8\) The scalars \( \phi_i \) in the tensor multiplets may be contained in the Kähler form \( J \) of the base \( B \) as \( J = -\phi_i \omega^i + \cdots \). The areas of the cycles \( C_i \) are given by \( \int_{C_i} J = \int_B \omega^i \wedge J = \eta^{ij} \phi_j = \phi^i \), and they correspond to the distances between adjacent (full or fractional) M5 branes.
Another possible application of our results is the following. In the case of 4D SCFTs, there are relations between the coefficients of anomaly polynomials and central charges $a$, $c$ and other flavor central charges [39,40]. If there are similar relations also in 6D SCFTs, the anomaly polynomials may be used to calculate the central charges. In particular, it might be useful for checking whether the $a$-theorem in 6D is valid or not (see Refs. [41–45] for some evidence for or against the $a$-theorem in 6D). One observation is that the Green–Schwarz contribution $\Omega^{ij} I_i I_j$ has a certain positivity property because the matrix $\Omega^{ij}$ is positive-definite. If a UV SCFT flows to an IR SCFT and some free fields including tensor multiplets, the coefficient of e.g. $c_2(R)^2$ always decreases between the UV and IR SCFT, assuming that the UV and IR $SU(2)_R$ are the same. Therefore, by relating the coefficients of the anomaly polynomials to $a$, we may be able to have evidence for the 6D $a$-theorem.

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Appendix A. Tables of anomalies and group theoretic constants
In this appendix we summarize the anomaly polynomials for multiplets of 6D $\mathcal{N} = (1, 0)$ supersymmetry, and other group theoretic notations. In this paper we do not concern ourselves with subtleties arising from global structures of gauge groups and are careless about whether we are talking about groups or algebras.

In this paper we use the notation in which the anomaly polynomial of Weyl fermions in a representation $\rho$ becomes
\[
\hat{A}(T) \text{tr}_\rho e^{iF}.
\]
where $\hat{A}(T)$ is the A-roof genus. In particular, $F$ is anti-Hermitian and includes a $(2\pi)^{-1}$ factor in its definition compared to the usual one. The anomaly polynomials for $\mathcal{N} = (1, 0)$ multiplets are the following:

- Hypermultiplet with representation $\rho$
\[
\frac{\text{tr}_\rho F^4}{24} + \frac{\text{tr}_\rho F^2 p_1(T)}{48} + d_\rho \frac{7p_1^2(T) - 4p_2(T)}{5760}
\]
(A2)

- Vector multiplet with group $G$
\[
\frac{-\text{tr}_{\text{adj}} F^4 + 6c_2(R)\text{tr}_{\text{adj}} F^2 + d_G c_2(R)^2}{24} - \frac{(\text{tr}_{\text{adj}} F^2 + d_G c_2(R))p_1(T)}{48} - \frac{7p_1^2(T) - 4p_2(T)}{5760}
\]
(A3)

- Tensor multiplet
\[
\frac{c_2(R)^2}{24} + \frac{c_2(R) p_1(T)}{48} + \frac{23p_1(T)^2 - 116p_2(T)}{5760}
\]
(A4)
Table A1. Group theoretical constants defined for all $G$. These constants are also listed in the appendix of Ref. [46]

| $G$ | SU(k) | SO(k) | USp(2k) | $G_2$ | $F_4$ | $E_6$ | $E_7$ | $E_8$ |
|-----|-------|-------|---------|-------|-------|-------|-------|-------|
| $r_G$ | $k - 1$ | $[k/2]$ | $k$ | 2 | 4 | 6 | 7 | 8 |
| $h_G^\vee$ | $k$ | $k - 2$ | $k + 1$ | 4 | 9 | 12 | 18 | 30 |
| $d_G$ | $k^2 - 1$ | $k(k - 1)/2$ | $k(2k + 1)$ | 14 | 52 | 78 | 133 | 248 |
| $d_{	ext{fund}}$ | $k$ | $k$ | $2k$ | 7 | 26 | 27 | 56 | 248 |
| $s_G$ | $1/2$ | 1 | $1/2$ | 1 | 3 | 3 | 6 | 30 |
| $t_G$ | $2k$ | $k - 8$ | $2k + 8$ | 0 | 0 | 0 | 0 | 0 |
| $u_G$ | 2 | 4 | 1 | $10/3$ | 5 | 6 | 8 | 12 |

Table A2. Group theoretical constants defined only for $G$ without an independent quartic Casimir operator.

| $G$ | SU(2) | SU(3) | $G_2$ | $F_4$ | $E_6$ | $E_7$ | $E_8$ |
|-----|-------|-------|-------|-------|-------|-------|-------|
| $w_G$ | $8/7$ | 3 | $10/7$ | 5 | 6 | 8 | 12 |
| $x_G$ | $1/6$ | $1/6$ | $1/5$ | 1 | 1 | 2 | 12 |

where $d_\rho$ and $d_G$ are the dimensions of representation $\rho$ and group $G$, respectively.

It is convenient to define the symbol $\text{Tr}_G$ as the trace in the adjoint representation divided by the dual Coxeter number $h_G^\vee$ of the gauge group $G$, listed in Table A1. One of the properties of $\text{Tr}$ is that $\int \frac{1}{4} \text{Tr} F^2$ is one when there is one instanton on a four-manifold. Moreover, if we have subgroup $G'$ in a group $G$ with Dynkin index of embedding 1, for an element $f$ of universal enveloping algebra of Lie algebra of $G'$, the following equation holds:

$$\text{Tr}_{G'} f = \text{Tr}_G f.$$  \hfill (A5)

All of the embeddings we consider in this paper have index 1, so we often omit the subscript $G$ in $\text{Tr}_G$.

To convert the above anomaly polynomials to a convenient form, we define some constants and write those values in Table A1. We define the constant $s_G$, which relates the trace of $F^2$ in the fundamental representation, and $\text{Tr} F^2$ as $\text{tr}_{\text{fund}} F^2 = s_G \text{Tr} F^2$. Then we have

$$\text{tr}_{\text{adj}} F^2 = h_G^\vee \text{Tr} F^2, \quad \text{tr}_{\text{fund}} F^2 = s_G \text{Tr} F^2,$$ \hfill (A6)

where the first equation is just the definition of $\text{Tr}$. For the trace of $F^4$, we define $t_G$ and $u_G$ by

$$\text{tr}_{\text{adj}} F^4 = t_G \text{tr}_{\text{fund}} F^4 + \frac{3}{4} u_G (\text{Tr} F^2)^2.$$ \hfill (A7)

For gauge groups $G = \text{SU}(2)$, $\text{SU}(3)$, and all exceptional groups, there are no independent quadratic Casimir operators, so we can relate $\text{tr}_G F^4$ and $(\text{Tr} F^2)^2$ by

$$\text{tr}_{\text{adj}} F^4 = \frac{3}{4} w_G (\text{Tr} F^2)^2, \quad \text{tr}_{\text{fund}} F^4 = \frac{3}{4} s_G (\text{Tr} F^2)^2.$$ \hfill (A8)

These constants are tabulated in Table A2. Note that because $t_{\text{SO}(8)} = 0$, we can also relate $\text{tr}_{\text{adj}} F^4$ to $(\text{Tr} F^2)^2$ for $G = \text{SO}(8)$.
All representations that we use in this paper are fundamental or adjoint, except for the spin representation 8 of SO(7). The conversion constant for this representation is

\[
\text{tr}_8 F^2 = \text{Tr} F^2, \\
\text{tr}_8 F^4 = -\frac{1}{2} \text{tr}_{\text{fund}} F^4 + \frac{3}{8} (\text{Tr} F^2)^2.
\]

Finally, let us note that the finite subgroup \( \Gamma_G \) of SU(2) of type \( G = A_n, \ D_n, \) and \( E_n \) has the following order:

\[
|\Gamma_{\text{SU}(k)}| = k, \quad |\Gamma_{\text{SO}(2k)}| = 4k - 8, \quad |\Gamma_{E_6}| = 24, \quad |\Gamma_{E_7}| = 48, \quad |\Gamma_{E_8}| = 120.
\]

Appendix B. Anomaly of M5s on ALE singularity via inflow

Here we derive the anomaly polynomials of the theory realized by M5 branes put on the ALE singularities by using the anomaly inflow. What we will compute includes contributions not only from the genuine SCFT part, but also from the center-of-mass tensor multiplet and its Green–Schwarz contribution.

B.1. Chern–Simons terms in M-theory

For the anomaly inflow, Chern–Simons terms involving the M-theory three-form \( C \) are important. In the 11D spacetime \( X_{11} \), if there is no magnetic source for the field strength four-form \( G = dC \), we have

\[
S_{CGG} = \frac{2\pi}{6} \int_{X_{11}} C \wedge G \wedge G = \frac{2\pi}{6} \int_{Y_{12}} G \wedge G \wedge G, \\
S_{CI8} = -2\pi \int_{X_{11}} C \wedge I_8 = -2\pi \int_{Y_{12}} G \wedge I_8,
\]

where, in the following, \( Y_{p+1} \) means a \( p + 1 \)-dimensional manifold whose boundary is \( X_p \), i.e., \( \partial Y_{p+1} = X_p \), and

\[
I_8 = \frac{1}{48} \left[ p_2(T X_{11}) - \frac{1}{4} p_1^2(T X_{11}) \right].
\]

When there is an orbifold singularity \( X_{11} = X_7 \times \mathbb{C}^2 / \Gamma \), we also have two types of Chern–Simons terms localized on the singularity. The first one can be determined in the following way. When \( X_{11} = X_7 \times \mathbb{C}^2 / \Gamma \), the structure group of the tangent bundle is decomposed as \( \text{SO}(11) \rightarrow \text{SO}(7) \times \text{SU}(2)_L \times \text{SU}(2)_R \). The orbifold \( \Gamma \) acts on \( \text{SU}(2)_L \). There is an \( \text{SU}(2) \) symmetry acting on \( \mathbb{C}^2 / \Gamma \), and by a slight abuse of notation, we denote this symmetry as \( \text{SU}(2)_R \). When \( \Gamma \) is of type \( A_k \), there is a \( \text{U}(1) \) symmetry acting on \( \mathbb{C}^2 / \Gamma \), but we ignore this symmetry for simplicity. Let \( c_2(L) \) and \( c_2(R) \) be the Chern classes of \( \text{SU}(2)_L \) and \( \text{SU}(2)_R \) respectively. This \( c_2(R) \) gives the Chern class of the connection field associated with the rotational symmetry \( \text{SU}(2)_R \). Then \( I_8 \) becomes

\[
I_8 = -\frac{1}{48} c_2(L)(4c_2(R) + p_1(T X_7)) + \frac{1}{48} \left[ p_2(T) - p_1(T)c_2(R) - \frac{1}{4} p_1(T)^2 \right].
\]

The singularity may be regarded as a gravitational instanton, and has some nontrivial curvature \( c_2(L) \) localized at the singularity, with

\[
\int_{\mathbb{C}^2 / \Gamma} c_2(L) =: \chi_{\Gamma}.
\]
where \( \chi_\Gamma \) can be thought of as a version of the “Euler number” of the singularity. Then, we get a Chern–Simons term on the singularity as

\[
S_{CI_8} = S_{CI_8}^{bulk} + 2\pi \int_{X_7 \times [0]} \frac{\chi_\Gamma}{48} C \wedge (4c_2(R) + p_1(T)),
\]

where \( S_{CI_8}^{bulk} \) is the contribution that is not localized on the singularity.

The value of \( \chi_\Gamma \) is given as \([47]\)

\[
\chi_\Gamma = r_\Gamma + 1 - \frac{1}{|\Gamma|},
\]

where \( r_\Gamma \) is the rank of the \( A_r, D_r, E_r \) group corresponding to \( \Gamma \), and \( |\Gamma| \) is the order of \( \Gamma \).

This formula can be understood as follows. Let \( M = \{ z \in \mathbb{C}^2 / \Gamma; |z|^2 \leq 1 \} \). The boundary is \( \partial M = S^3 / \Gamma \). The topological Euler number of this space is \( \chi(M) = r + 1 \), because \( \dim H_2(M) = r \), \( \dim H_0(M) = 1 \), and the others are zero. Now recall that the topological Euler number is also given as an integral of local quantities as \( \chi(M) = \int_M E_4 + \int_{\partial M} (\text{local term}) \), where we have denoted the Euler density as \( E_4 \). When \( \Gamma \) is trivial so that \( \partial M = S^3 \), the contribution from the boundary integral is 1 because \( r = 0 \) and \( E_4 = 0 \) in that case. Then this boundary contribution is \( 1/|\Gamma| \) when \( \partial M = S^3 / \Gamma \). Therefore we get \( \int_M E_4 = r + 1 - 1/|\Gamma| \).

The second type of Chern–Simons term involves gauge fields localized on the singularity. The gauge fields \( A_i (i = 1, \ldots, r_\Gamma) \) in the Cartan subalgebra of \( A_r, D_r, E_r \) gauge algebra localized on the singularity come from the three-form \( C \) as

\[
C = C^{bulk} + i \sum_{i=1}^{r_\Gamma} \omega^i \wedge A_i,
\]

where \( \omega^i \) are Poincaré duals to the two cycles that are collapsed at the singularity. The factor \( i = \sqrt{-1} \) was introduced to make \( A_i \) anti-Hermitian. Then we get

\[
S_{CGG} = S_{CGG}^{bulk} + \frac{2\pi}{2} \eta^{ij} \int_{X_7 \times [0]} C^{bulk} \wedge F_i \wedge F_j
\]

\[
= S_{CGG}^{bulk} + \frac{2\pi}{4} \int_{X_7 \times [0]} C^{bulk} \wedge \text{Tr} F^2,
\]

where \( \eta^{ij} = - \int \omega^i \wedge \omega^j \) is \(-1\) times the intersection matrix of the two-cycles given by the Cartan matrix of \( A_r, D_r, E_r \). Although it is obtained for gauge fields in the Cartan subalgebra, the last expression should be valid for more general non-Abelian fields.

Combining the above results, we get the Chern–Simons terms localized on the singularity as

\[
S_{CI_8} = 2\pi \int_{X_7 \times [0]} C^{bulk} \wedge J_4,
\]

\[
J_4 = \frac{\chi_\Gamma}{48} (4c_2(R) + p_1(T)) + \frac{1}{4} \text{Tr} F^2.
\]

### B.2. Effects of M5 on the theory on singularity

Before computing the inflow, let us explain the effects of inserting M5 branes on the singularity to the gauge fields living on the singularity. Consider the A-type singularity, and suppose that this singularity is realized in Taub–NUT space instead of ALE space. Then, by going to the type IIA description, the singularity becomes D6 branes and the M5 branes become NS5 branes. We get a system where NS5 branes are inserted into D6. In this case, NS5 branes have the effect that the
two sides of the NS5 become independent gauge theories, i.e., the gauge group is \( \text{SU}(r + 1)_L \times \text{SU}(r + 1)_R \) where \( \text{SU}(r + 1)_L \) is on one side of the NS5 brane and \( \text{SU}(r + 1)_R \) is on the other. Furthermore, the boundary condition of these gauge fields is such that a gauge theory between two NS5 becomes a 6D \( \mathcal{N} = (1, 0) \) vector multiplet instead of an \( \mathcal{N} = (1, 1) \) vector multiplet. That is, among the 7D \( \mathcal{N} = 1 \) fields, three scalars and a component of vector field normal to NS5 have Dirichlet boundary conditions, while vector fields tangent to NS5 have Neumann boundary conditions. The same things should happen when M5 branes are inserted in more general \( A, D, E \) singularities.

The above boundary condition makes a contribution to the anomaly, just as in the case of the end-of-the-world \( E_8 \) brane where the change of the gravitino boundary condition made contributions to the anomaly \([48,49]\). This contribution is given by

\[
- \frac{1}{2} I^\text{vec}_L - \frac{1}{2} I^\text{vec}_R, \tag{B12}
\]

where \( I^\text{vec}_L \) and \( I^\text{vec}_R \) are the anomalies of \( \mathcal{N} = (1, 0) \) vector multiplets with gauge groups \( G_L \) and \( G_R \) on the two sides of the M5 branes, respectively.

### B.3. Anomaly inflow on \( \mathbb{R} \times \mathbb{C}^2 / \Gamma \)

Now we calculate the anomaly inflow. Since the relevant calculations are almost the same as in Ref. \([32]\), we will be brief and neglect some of the subtleties.

Let us take the 11D space to be \( X_{11} = X_6 \times (\mathbb{R} \times \mathbb{C}^2 / \Gamma) \) and put \( Q \) M5 branes at the origin of \( \mathbb{R} \times \mathbb{C}^2 / \Gamma \). Let \( y^a (a = 1, 2, 3, 4, 5) \) be the coordinates of the covering space \( \mathbb{R}^5 = \mathbb{R} \times \mathbb{C}^2 \).

If \( \Gamma \) is trivial, the Bianchi equation for \( G \) is

\[
dG = Q \prod_{a=1}^{5} \delta(y^a) dy^a. \tag{B13}
\]

Its solution at \( y \neq 0 \) is given by

\[
G = \frac{Q}{2} e_4 + \text{(regular)}, \tag{B14}
\]

where \( \text{(regular)} \) represents terms that are not singular at \( y = 0 \). The four-form \( e_4 \), which is closed at \( y \neq 0 \), is given by

\[
e_4(y) = \frac{1}{32\pi^2} \epsilon_{a_1 \cdots a_5} \left[ (D \hat{y})^{a_1} (D \hat{y})^{a_2} (D \hat{y})^{a_3} (D \hat{y})^{a_4} \hat{y}^{a_5} - 2 F^{a_1 a_2} (D \hat{y})^{a_3} (D \hat{y})^{a_4} \hat{y}^{a_5} + F^{a_1 a_2} F^{a_3 a_4} \hat{y}^{a_5} \right], \tag{B15}
\]

where \( \hat{y}^a = y^a / |y| \), \( D \) is a covariant exterior derivative of \( \text{SO}(5)_R \) rotational symmetry around the origin of \( \mathbb{R}^5 \), and \( F^{a_1 a_2} \) is the field strength of \( \text{SO}(5)_R \). We restrict the \( \text{SO}(5)_R \) bundle to the subbundle \( \text{SU}(2)_R \subset \text{SO}(4)_R \subset \text{SO}(5)_R \), which will also be preserved when the space is divided by \( \Gamma \). Then, when we introduce the orbifold, the only change is that \( G = (|\Gamma|/Q/2) e_4 + \text{(regular)} \) as long as \( y^a \) are understood to be the coordinates of the covering space.

Some of the important properties of \( e_4 \) that we will use are

\[
\int_{S^4} e_4 = 2, \quad \int_{S^4} (e_4)^3 = 2 c_2(R)^2, \quad e_4|_{y=0} = - c_2(R) \text{ sign}(y^1), \tag{B16}
\]

where \( S^4 \) is a sphere around the origin of \( \mathbb{R} \times \mathbb{C}^2 \). When we divide by \( \Gamma \), there is an additional factor \( 1/|\Gamma| \) in the first two equations.
Now let us determine the contribution from the inflow from the Chern–Simons terms. Because $G$ is singular at the position of the M5 branes, we remove a small tubular neighborhood of the M5 branes in the integral of Chern–Simons terms. We denote a tubular neighborhood of a submanifold $M$ as $D_e(M)$. By an abuse of notation, we denote the submanifold where the M5 branes are located as $X_5$ (or $Y_7$ depending on whether we consider $X_{11}$ or $Y_{12}$).

Because $e_4$ is closed, it is (locally) written as $e_4 = de_3^{(0)}$. Now, the most singular part of the Chern–Simons term $S_{CGG}^{\text{bulk}}$ is given as

\[
S_{CGG}^{\text{bulk}} = \frac{2\pi}{6} \lim_{\epsilon \to 0} \int_{Y_7 \setminus D_e(Y_7)} G^{\text{bulk}} \wedge G^{\text{bulk}} \wedge G^{\text{bulk}}
\]

\[
\sim 2\pi \cdot \frac{Q^3|\Gamma|^3}{48} \lim_{\epsilon \to 0} \int_{Y_7 \setminus D_e(Y_7)} e_3 = -2\pi \cdot \frac{Q^3|\Gamma|^3}{48} \lim_{\epsilon \to 0} \int_{\partial D_e(Y_7)} e_3^{(0)} e_4^2
\]

\[
= -2\pi \cdot \frac{Q^3|\Gamma|^2}{24} \int_{Y_7} c_2(R)^{(0)} c_2(R),
\]

where $dc_2(R)^{(0)} = c_2(R)$ and we have used the second equation of (B16) in the last equation. Thus the contribution of this to the anomaly polynomial is $-(Q^3|\Gamma|^2/24)c_2(R)^2$.

In the same way, we get

\[
S_{CL_8}^{\text{bulk}} \sim 2\pi \cdot Q \int_{Y_7} I_7^{(0)},
\]

\[
S_{\Gamma} \sim 2\pi \cdot \frac{Q|\Gamma|}{2} \int_{Y_7} c_2(R)^{(0)} (J_{4,L} + J_{4,R}),
\]

where $dl_7^{(0)} = I_8$, the $J_{4,L}$ and $J_{4,R}$ are the $J_4$ defined in (B11) on the left and right of the M5 branes respectively, and we have used the first and third equations of (B16).

Combining these, the inflow of anomaly is given by

\[
-\frac{Q^3|\Gamma|^2}{24}c_2(R)^2 + QI_8 + \frac{Q|\Gamma|}{2}c_2(R)(J_{4,L} + J_{4,R}).
\]

This must be canceled by the anomaly of the theory living on the M5 branes. Taking into account the contribution (B12), we finally get the anomaly polynomial of the theory on M5 branes that are put on $\mathbb{R} \times \mathbb{C}^2/\Gamma$ as

\[
I^{\text{tot}}(Q \text{ M5}; \mathbb{R} \times \mathbb{C}^2/\Gamma)
\]

\[
= \frac{Q^3|\Gamma|^2}{24}c_2(R)^2 - QI_8 - \frac{Q|\Gamma|}{2}c_2(R)(J_{4,L} + J_{4,R}) - \frac{1}{2}I_{L}^{\text{vec}} - \frac{1}{2}I_{R}^{\text{vec}}.
\]

We can check that this formula is equal to (3.23).

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