Geometrically nonlinear analysis of a flexible wire (chain)

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Abstract. In this work the deformation analysis of a flexible cable is considered in detail. Nonlinear finite element analysis is used for solving the problem. In deriving the final form of equations, geometrically nonlinear analysis is used and validity of simple Hooke’s law is assumed. The resulting nonlinear algebraic equations are solved by using simple iteration method. Numerical examples are considered.

1. General Formulation of the Problem

Cable-stayed structures are widely spread in construction. Such a structure consists of flexible cables or wires that can carry only a tensile load [1, 2]. Cables are used in cable-stayed bridges, suspension bridges, including simple suspension bridges, transmission lines [2–4]. Cable nets can be used for covering long span spaces of stadiums, exhibition halls, etc [2]. Such structures are typically flexible and can experience large displacements even though deformations remain small. In these circumstances, the application of geometrically nonlinear analysis is necessary for proper design of these structures [5].

The analysis of a single cable as a constituent of a more complex cable-stayed structure is important and is considered in this paper. The analysis of a cable based on solving differential equations is presented in [6]. Instead of solving differential equations, the present analysis follows the approach presented in [1] and is strictly discrete. That means that the cable is subdivided into a large number of linear segments or elements and the load is applied at the nodes of the discretized structure. Geometrically non-linear analysis is applied as is required for flexible structures. The equilibrium equations are derived for the deformed configuration of the structure but certain simplifications are made to obtain the final form of the governing equations. Such simplifications include the assumption of validity of simple Hooke’s law even when the structure experiences large displacements. The resulting non-linear algebraic equations are solved numerically using simple iteration method. The specific numerical examples considered serve as a benchmark for more realistic problems. Among the numerical examples chosen are the problem of a hanging chain under its own weight and the problem of a chain under a point load.

Let us consider a wire of length \( L \) fixed at two endpoints. Distance between the supports (endpoints) is denoted by \( L_0 \) and \( L_0 < L \) (figure 1). To discretize configuration of the wire, we divide it into \( n \) segments each of which has the same horizontal projection equal to \( l_0 = L_0 / n \). In the course of division, we obtain \( n + 1 \) nodes, which are the endpoints of two adjacent elements. The number of
internal nodes is denoted by \( m = n - 1 \). Each element \( i \) is bounded by two endpoints \( i - 1 \) and \( i \). The length of the \( i \)-th element in the undeformed state is denoted by \( l_i \). The angle of inclination of the \( i \)-th element with respect to the horizontal axis is denoted by \( \alpha_i \) and it is assumed positive when the rotation from the horizontal axis to the element is seen clockwise.

\[
\begin{align*}
\text{Figure 1. Geometry of the wire and the load applied to its nodes.}
\end{align*}
\]

The distributed load is applied to the wire and we are able to lump this load to the nodes. Thus, at each node \( i \), in general, we can have two force components: vertical, in the direction of the \( y \)-axis, \( F_{yi} \) and horizontal, in the direction of the \( x \)-axis, \( F_{xi} \). The unknown displacements for each node: vertical \( u_{yi} \) and horizontal \( u_{xi} \). The endpoints have zero displacements and thus can be excluded from the analysis. We can assemble nodal forces and displacements into the vectors \( \{ F \} \) and \( \{ u \} \), respectively,

\[
\{ F \} = \{ F_{y1}, F_{x1}, F_{y2}, F_{x2}, \ldots, F_{yn}, F_{xn} \}, \quad \{ u \} = \{ u_{y1}, u_{x1}, u_{y2}, u_{x2}, \ldots, u_{yn}, u_{xn} \} \tag{1.1}
\]

Similarly, we can define a vector of internal forces and the vector of elongations (change in length) in members (elements) of the wire as

\[
\{ N \} = \{ N_1, N_2, \ldots, N_n \}, \quad \{ \Delta \} = \{ \Delta_1, \Delta_2, \ldots, \Delta_n \} \tag{1.2}
\]

The considered system has the property of having large displacements even when the deformation of individual segments (elements) of the wire is small. Thus, there is no linear dependence between deformation and displacements. Such systems are called geometrically nonlinear. For such systems the equilibrium equations are derived for the deformed state of the system and consequently the equilibrium matrix will depend on the displacement vector.

Let \( [A] \) denote the equilibrium matrix of the system. The equilibrium equations for the geometrically nonlinear system can then be written in general form as \( [1] \)

\[
[A(\{u\})](\{N\}) = \{F\}, \tag{1.3}
\]

Let us consider derivation of the matrix \( [A] \). For that, consider displacements of node \( i \) and neighboring nodes \( i - 1 \) and \( i + 1 \). Considering equilibrium of the node \( i \) in the deformed configuration, we can write

\[
\begin{align*}
N_i \sin \alpha_i' - N_{i+1} \sin \alpha_{i+1}' &= F_{yi} \\
N_i \cos \alpha_i' - N_{i+1} \cos \alpha_{i+1}' &= F_{xi},
\end{align*}
\tag{1.4}
\]
where \( \alpha_i' \) are the angles of inclination of the elements after deformation. These angles can be expressed in terms of the initial coordinates of the nodes \((x_i, y_i)\) and the components of the displacement vector. Simple geometry gives the following relationships

\[
\begin{align*}
\sin \alpha_i' &= \frac{y_i - y_{i-1} + u_{y i} - u_{y i-1}}{l_i + \Delta_i} \\
\sin \alpha_{i+1}' &= \frac{y_{i+1} - y_i + u_{y i+1} - u_{y i}}{l_{i+1} + \Delta_{i+1}} \\
\cos \alpha_i' &= \frac{l_i + u_{x i} - u_{x i-1}}{l_i + \Delta_i} \\
\cos \alpha_{i+1}' &= \frac{l_{i+1} + u_{x i+1} - u_{x i}}{l_{i+1} + \Delta_{i+1}}
\end{align*}
\]

Here \( \Delta_i \) denotes the elongation of the element \( i \) and thus \( l_i + \Delta_i \) is the deformed length of the element. The above relationships can be simplified by omitting the term \( \Delta_i \) taking into account the fact that \( l_i \ll \Delta_i \).

Therefore, our equilibrium matrix will consists of the following blocks (written for each internal node \( i = 1, 2, \ldots, m \))

\[
[A_i] = \begin{bmatrix}
y_i - y_{i-1} + u_{x i} - u_{x i-1} & -y_{i+1} - y_i + u_{y i+1} - u_{y i} \\
l_i & l_{i+1} + \Delta_{i+1} \\
-l_0 + u_{x 1} - u_{x 0} & -l_0 + u_{y 2} - u_{y 1}
\end{bmatrix}.
\]

When evaluating this matrix for the nodes \( i = 1 \) and \( i = m \), we must prescribe zero displacements for the supports, i.e., \( u_{x 0} = u_{y 0} = 0 \) and \( u_{y m} = u_{y m} = 0 \). The whole matrix \([A]\) consists of individual blocks \([A_i]\) and for the case of, for example, 4 elements and 3 internal nodes it has the following structure

\[
\begin{bmatrix}
A_1^{11} & A_1^{12} \\
A_1^{21} & A_1^{22} \\
A_2^{11} & A_2^{12} \\
A_2^{21} & A_2^{22} \\
A_3^{11} & A_3^{12} \\
A_3^{21} & A_3^{22}
\end{bmatrix}
\]

We also need equations that relate displacements of the nodes to the elongations of the elements. These equations can be written in the form [1]

\[
[B(\{u\})]^T \{u\} = \{\Delta\},
\]
In geometrically linear analysis, the matrix $B$ is equal to the equilibrium matrix $A$ but in the nonlinear analysis that is not true. To derive the components of the matrix $B$, consider the square of the deformed length of the element

$$
(l_i + \Delta_i)^2 = \left[(y_i - y_{i-1}) + (u_{yi} - u_{yi-1})\right]^2 + \left[l_0 + (u_{xi} - u_{xi-1})\right]^2,
$$

(1.9)

By expanding the last relationship, we obtain

$$
l_i^2 + \Delta_i^2 + 2l_i\Delta_i = \left(y_i - y_{i-1}\right)^2 + \left(u_{yi} - u_{yi-1}\right)^2 + 2\left(y_i - y_{i-1}\right)(u_{yi} - u_{yi-1})
+ \left[l_0 + (u_{xi} - u_{xi-1})\right]^2 + 2l_0(u_{xi} - u_{xi-1})\right).
$$

But considering the initial geometry of the wire, it is obvious that $l_i^2 = l_0^2 + (y_i - y_{i-1})^2$ and therefore the last relationship can be simplified as

$$
\Delta_i(2l_i + \Delta_i) = \left(2(y_i - y_{i-1}) + (u_{yi} - u_{yi-1})\right)(u_{yi} - u_{yi-1}) + \left(2l_0 + (u_{xi} - u_{xi-1})\right)(u_{xi} - u_{xi-1}).
$$

Taking into account that $\Delta_i << 2l_i$, the second term in the sum $2l_i + \Delta_i$ can be dropped and we obtain the approximate expression for the elongation

$$
\Delta_i = \frac{1}{2l_i}\left[2(y_i - y_{i-1}) + (u_{yi} - u_{yi-1})\right](u_{yi} - u_{yi-1}) + \left(2l_0 + (u_{xi} - u_{xi-1})\right)(u_{xi} - u_{xi-1})],
$$

(1.10)

The above equation for the elongation of the $i$-th element can be put in the matrix form as follows

$$
\Delta_i = \left[\begin{array}{c}
\frac{2(y_i - y_{i-1}) + (u_{yi} - u_{yi-1})}{2l_i} \\
-\frac{2l_0 + (u_{xi} - u_{xi-1})}{2l_i} \\
\frac{2l_0 + (u_{xi} - u_{xi-1})}{2l_i} \\
\end{array}\right],
$$

(1.11)

and these terms will be located in a particular row of the matrix $[B]^T$. Instead of defining the components of the matrix $[B]^T$, it is also possible to define the matrix $[B]$, which will have the same structure as the matrix $[A]$. Then for each internal node $i$ the 2x2 blocks of the matrix $[B]$ are given as follows (since the $i$-th node is between the elements $i$ and $i + 1$)

$$
[B_i] = \left[\begin{array}{c}
\frac{2(y_i - y_{i-1}) + (u_{yi} - u_{yi-1})}{2l_i} \\
\frac{2l_0 + (u_{xi} - u_{xi-1})}{2l_i} \\
\frac{2l_0 + (u_{xi} - u_{xi-1})}{2l_i} \\
\end{array}\right].
$$

(1.12)

Now it is evident that $[B] \neq [A]$. The representation (1.12) is perhaps more advantageous when it comes to computer implementation since the matrix $[B]$ can be implemented as the matrix $[A]$.

We also define the stiffness matrix $[K]$ that will be a diagonal matrix with the terms $E_i A_i/l_i$ standing on the main diagonal that represent the stiffness of each element. Here $E_i$ is the modulus of elasticity and $A_i$ is the area of the cross-section of each member of the wire.

Finally, we are ready to write down the resulting system of non-linear equations that we must solve for the unknown displacements. This system can be written as
\[ A(\{u\})[K][B(\{u\})]^{T}\{u\} = \{F\} \]

To solve the nonlinear system of equations (1.13) we use an iterative approach. The external force \( \{F\} \) is applied during one load increment. Approximation to the solution at the iteration \( j \) is denoted by \( \{u\}^{(j)} \). The increment is defined as \( \{\Delta u\}^{(j+1)} = \{u\}^{(j+1)} - \{u\}^{(j)} \). Using simple iteration method, this increment can be found by solving the following equation

\[ [D]\{\Delta u\}^{(j+1)} = \{F\} - [A(\{u\}^{(j)})][K][B(\{u\}^{(j)})]^{T}\{u\}^{(j)} \]  

(1.14)

Here \([D]\) is a diagonal matrix with the values chosen to be sufficiently large so that the convergence of the iteration procedure is ensured but, on the other hand, sufficiently small to reduce the number of iterations. If all elements have identical parameter \( EA \), then a good choice for the diagonal term is \( 2EA/l_0 \). The iteration process converges if the norm of the correction to the displacement vector \( \{\Delta u\}^{(j)} \) goes to zero as the number of iterations grows, i.e., when \( j \to \infty \).

After finding the displacement vector, the internal force vector can be found as follows

\[ \{N\} = [K][B(\{u\})]^{T}\{u\}, \]

(1.15)

2. Numerical Examples

In the first numerical example we consider a wire, suspended horizontally, subject to its own weight. Let the total weight of the wire be denoted by \( G \). Let us choose the initial (undeformed) configuration of the wire as two straight lines of length \( L/2 \) symmetric with respect to the center. The goal is to find the final (deformed) configuration of the wire. Since each element in the initial configuration has the same length, we can apply the identical vertical force to each node of the wire equal to \( G/n \), where \( n \) is the total number of elements. (The effect of the change of the elements’ lengths on the magnitude of the applied load is neglected.) The load vector will have the form

\[ \{F\} = \{G/n,0,G/n,0,\ldots,G/n,0\}, \]

(2.1)

Figure 2 shows the initial and deformed configurations for this wire. It can be verified that the deformed configuration resembles very closely a parabola, which is the shape in a theoretical solution of the present problem. For this example we chose the following parameters: \( G = 0.4 \), \( L_0 = 4 \), \( L = 4\sqrt{2} \), \( n = 20 \), \( E_iA_i = 2 \). Thus, the initial angle of inclination for each element is \( \pm 45^\circ \).
Figure 2. Initial and deformed geometries of the wire subject to its own weight.

Figure 3 shows initial and deformed configurations of the wire suspended vertically and subject to its own weight. For this problem the load vector is given by

$$\{F\} = \{0, G/n, 0, G/n, \ldots, 0, G/n\},$$

(2.2)

The theoretical solution for the deformed shape of the wire is a straight vertical line. This shape is predicted reasonably well by the numerical solution (see figure 3) but the convergence of the iteration procedure was much worse for this problem than that in the problem of horizontally suspended wire.

Figure 3. Initial and deformed geometries of the wire suspended vertically and subject to its own weight.
Figure 4 shows initial and deformed configurations of the wire suspended horizontally, as in figure 2, and subjected to the concentrated vertical force applied at a quarter span location from the left support. As we see, the deformed configuration in this case consists of two linear segments with a corner point coincident with the force application point.

When the concentrated load is applied closer to the support than at a quarter span location, the obtained solution turns out to be dependent upon the initial configuration of the wire and thus this solution cannot be considered as correct. To correct the situation, one must consider the action of wire’s self-weight and apply the concentrated load in combination with the self-weight.

![Initial and deformed geometries of the wire subjected to a vertical concentrated force applied at a quarter span location from the left support.](image)

**Figure 4.** Initial and deformed geometries of the wire subjected to a vertical concentrated force applied at a quarter span location from the left support.

3. Conclusions
In this paper we analyzed the problem of deformation of a flexible wire or cable subject to different kinds of loads. Among the loads considered are the wire’s own weight and the concentrated load. Two wire configurations were considered: wire suspended horizontally from two supports at equal elevation and wire hanging vertically.

The considered examples confirmed the necessity to use geometrically nonlinear analysis for the present problem. The analysis predicted the correct shape of the deformed wire in all cases. In case of the wire suspended horizontally, the parabolic shape of the deformed wire was obtained; in case of the wire hanging vertically, the deformed shape was a vertical line. When the concentrated load was applied to the wire, the resulting deformed shape consisted of two straight segments with a corner point at the point of load application, which can also be deduced from the overall equilibrium of the wire.

The results considered to be correct when they were independent of the initial configuration, which was chosen consisting of straight segments, for convenience. The iteration procedure proved to be reliable but for some problems the convergence was slow. Ways of improving the convergence of the method were not considered and will be the subject of future research.
References

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