Order of Contact and Ruled Submanifolds

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Abstract

We prove a generalization of the Monge-Cayley-Salmon theorem on osculation and ruled submanifolds using geometric measure theory.

1 Introduction

Analytic surfaces in $\mathbb{R}^3$ have the following remarkable property, that played a key role in the proof of the Erdős distinct distances problem in dimension two by Guth and Katz [2].

Theorem 1.1 (Monge, Cayley, Salmon). Let $M \subset \mathbb{R}^3$ be a proper 3-dimensional analytic surface. Assume there exists a smooth family $\ell_x, x \in M$ of lines in $\mathbb{R}^3$ such that, for all $x \in M$, $\ell_x$ and $M$ have a contact of order 3 at $x$. Then, $\ell_x \subset M$ for all $x \in M$.

A proof of Theorem 1.1 can be found in [8]. In [3], Guth and Zahl proved a version of Theorem 1.1 for an arbitrary field instead of $\mathbb{R}$. For a detailed exposition on the Monge-Cayley-Salmon theorem and its relation to the Erdős distinct distances problem, we refer to [5] and [9]. The aim of the present paper is to prove the following generalization of Theorem 1.1. Along the line, we present a novel elementary proof of the Monge-Cayley-Salmon theorem.
A curve $\Gamma \subset \mathbb{R}^n$ is said to be of class $k \in \mathbb{N}$ if it can be parametrized by a map $\mathbb{R} \to \mathbb{R}^n$ whose coordinates are polynomial functions of degree at most $k$.

In the terminology of Definition 1.2, the lines are curves of class 1.

Theorem 1.3. Let $M \subset \mathbb{R}^n$ be a proper $m$-dimensional analytic submanifold. Assume there exists a smooth family $\Gamma_x, x \in M$ of class-$k$ curves in $\mathbb{R}^n$ such that, for all $x \in M$, $\Gamma_x$ and $M$ have a contact of order $k(m + 1)$ at $x$. Then, $\Gamma_x \subset M$ for all $x \in M$.

The proof of Theorem 1.3 for $k = 1$ and for analytic submanifolds of $\mathbb{C}^n$ or $\mathbb{CP}^n$ can be found in [7]. That proof uses techniques of algebraic geometry.

The main idea of our proof is to consider the $(m + 1)$-dimensional volume swept by $M$ as each point $x$ of $M$ moves along $\Gamma_x$. It turns out (see Proposition 3.2 on page 8) that this volume is equal to 0 precisely when $\Gamma_x \subset M$ for all $x \in M$. The order-of-contact condition, on the other hand, implies that the rate at which the volume is swept is sufficiently slow (see Proposition 3.3 on page 9). What bridges these two facts (the vanishing volume and the volume being swept at a sufficiently slow rate) is a result of a Weyl-tube-formula type (Proposition 3.1 on page 6). Now, we sketch this step in the case of a hypersurface in $\mathbb{R}^n$. The volume swept by the hypersurface is a polynomial in the time-variable $t$ of degree at most $k(m + 1) = kn$. If a polynomial of degree at most $kn$ grows slower than $t^{kn}$ as we approach 0, then it vanishes identically.

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2 Preliminaries

2.1 The nearest point map

Definition 2.1. Let $M \subset \mathbb{R}^n$ be a smooth submanifold, and let $N$ be a normal tubular neighbourhood of $M$. The nearest point map $r : N \to M$ is the map that sends each point of $N$ to the unique nearest point in $M$. 


The nearest point map of a smooth submanifold is smooth [4, page 109]. If the submanifold is analytic, then the nearest point map is analytic as well [1, page 240].

### 2.2 Order of contact

**Definition 2.2.** Two smooth curves \( \gamma_1, \gamma_2 : \mathbb{R} \rightarrow \mathbb{R}^n \) are said to have a contact of order \( k \in \mathbb{N} \cup \{0\} \) at a point \( t_0 \in \mathbb{R} \) if

\[
(\forall j \in \{0, 1, \ldots, k\}) \quad \gamma_1^{(j)}(t_0) = \gamma_2^{(j)}(t_0).
\]

**Definition 2.3.** A smooth curve \( \gamma_1 : \mathbb{R} \rightarrow \mathbb{R}^n \) is said to have a contact of order \( k \in \mathbb{N} \cup \{0\} \) with a smooth submanifold \( M \subset \mathbb{R}^n \) at a point \( t_0 \in \mathbb{R} \) if there exists a smooth curve \( \gamma_2 : \mathbb{R} \rightarrow M \) such that the curves \( \gamma_1 \) and \( \gamma_2 \) have a contact of order \( k \) at \( t_0 \).

**Definition 2.4.** Two submanifolds \( M_1, M_2 \subset \mathbb{R}^n \) are said to have a contact of order \( k \in \mathbb{N} \cup \{0\} \) at a point \( p \in M_1 \cap M_2 \) if for every smooth curve \( \gamma_1 : \mathbb{R} \rightarrow M_1 \) such that \( \gamma_1(0) = p \) there exists a smooth curve \( \gamma_2 : \mathbb{R} \rightarrow M_2 \) such that \( \gamma_1 \) and \( \gamma_2 \) have a contact of order \( k \) at 0.

In the following lemma (and in the rest of the paper), we denote by \( d(x, M) \) the distance between a point \( x \in \mathbb{R}^n \) and a subset \( M \subset \mathbb{R}^n \), i.e.

\[
d(x, M) := \inf \{\|x - y\| \mid y \in M\}.
\]

**Lemma 2.5.** Let \( M \) be a submanifold of \( \mathbb{R}^n \), let \( I \) be an open interval containing 0, and let \( \phi_t : M \rightarrow \mathbb{R}^n \), \( t \in I \) be a smooth family of embeddings. Assume, for all \( x \in M \), the curve \( t \mapsto \phi_t(x) \) has a contact of order \( k \in \mathbb{N} \cup \{0\} \) with \( M \) at \( t = 0 \). Then,

\[
\lim_{t \rightarrow 0} \frac{d(\phi_t(x), M)}{t^k} = 0
\]

uniformly on compact subsets of \( M \).

**Proof.** Let \( N \) be a normal tubular neighbourhood of \( M \), and let \( r : N \rightarrow M \) be the nearest point map. Let \( K \subset M \) be an arbitrary compact subset, and let \( \delta > 0 \) be such that \([-\delta, \delta] \subset I \) and such that \( \phi_t(x) \in N \) for \( x \in K \) and \( t \in [-\delta, \delta] \). Denote

\[
C := \max_{x \in K, t \in [-\delta, \delta]} \left\| \partial_t^{k+1}(\phi_t(x) - r \circ \phi_t(x)) \right\|.
\]

Fix \( x \in M \), and denote by \( \gamma = (\gamma_1, \ldots, \gamma_n) : I \rightarrow \mathbb{R}^n \) the curve defined by \( \gamma(t) = \phi_t(x) - r(\phi_t(x)) \). Since \( \phi_t(x) \) has a contact of order \( k \) with \( M \), the
derivatives of $\gamma$ up to order $k$ are equal to 0. The Taylor approximation implies

$$|\gamma_i(t)| \leq \frac{t^{k+1}}{(k+1)!} \cdot \max_{|t| \leq \delta} |\partial^{k+1}_t \gamma_i(t)| \leq t^{k+1} \cdot \frac{C}{(k+1)!},$$

for all $i \in \{1, \ldots, n\}$ and $t \in (-\delta, \delta)$. Hence

$$\frac{d(\phi_t(x), M)}{t^k} = \frac{\|\gamma(t)\|}{t^k} \leq t \cdot \sqrt{n} \cdot \frac{C}{(k+1)!},$$

for all $x \in K$ and $t \in (-\delta, \delta)$. This finishes the proof.

### 2.3 Volume of a $C^1$ map

The purpose of this section is to introduce the notion of a volume $\text{Vol}(\phi)$ of a $C^1$ map $\phi : M \to N$ from a smooth manifold $M$ to a Riemannian manifold $(N, g)$. Intuitively, the volume of a $C^1$ map $\phi : M \to N$ is the $(\dim M)$–dimensional volume swept by $\phi$ in $N$. We will, actually, formally define only the volume of a $C^1$ map from an open subset of $\mathbb{R}^m$ to a Riemannian manifold. This definition extends to the general case via the standard trick which uses a collection of charts and a subordinate partition of unity.

**Definition 2.6.** Let $U \subset \mathbb{R}^m$ be an open subset, let $(N, g)$ be a Riemannian manifold, and let $\phi : U \to N$ be a $C^1$ map. The **volume** of the map $\phi$ is defined by

$$\text{Vol}(\phi) := \int_U \|\partial_1 \phi(x) \wedge \cdots \wedge \partial_m \phi(x)\| \, dx.$$ 

**Remark 2.7.** We found it convenient to use exterior algebra $\bigwedge^m T_{\phi(x)} N$ to express the volume element. In our conventions, if $e_1, \ldots, e_n$ is an orthonormal basis of $T_{\phi(x)} N$, then $e_{i_1} \wedge \cdots \wedge e_{i_m}$, $1 \leq i_1 < \cdots < i_m \leq n$ is an orthonormal basis of $\bigwedge^m T_{\phi(x)} N$. Alternatively, $\|\partial_1 \phi(x) \wedge \cdots \wedge \partial_m \phi(x)\|$ can be written as

$$\|\partial_1 \phi(x) \wedge \cdots \wedge \partial_m \phi(x)\| = \sqrt{\det [g(\partial_i \phi(x), \partial_j \phi(x))]_{i,j}}.$$ 

**Lemma 2.8.** Let $\psi : V \to U$ be a $C^1$ diffeomorphism between two $m$–dimensional manifolds, let $(N, g)$ be a Riemannian manifold, and let $\phi : U \to N$ be a $C^1$ map. Then, $\text{Vol}(\phi \circ \psi) = \text{Vol}(\phi)$. 

4
Proof. Without loss of generality, assume \( U \) and \( V \) are two open subsets of \( \mathbb{R}^m \). For a linear map \( A : W_1 \to W_2 \) (from a vector space \( W_1 \) to a vector space \( W_2 \)), denote by \( \bigwedge^k A : \bigwedge^k W_1 \to \bigwedge^k W_2 \) the linear map defined by

\[
(\forall v_1, \ldots, v_k \in W_1) \quad \left( \bigwedge^k A \right) ^*(v_1 \wedge \cdots \wedge v_m) := (Av_1) \wedge \cdots \wedge (Av_k).
\]

The lemma follows from the following sequence of equalities

\[
\begin{align*}
\Vol(\phi \circ \psi) &= \int_V \| \partial_1 (\phi \circ \psi) \wedge \cdots \wedge \partial_m (\phi \circ \psi) \| \, dx \\
&= \int_V \| (D\phi(\psi(x))\partial_1 \psi(x)) \wedge \cdots \wedge (D\phi(\psi(x))\partial_m \psi(x)) \| \, dx \\
&= \int_V \left\| \bigwedge^k (D\phi(\psi(x))) \partial_1 \psi(x) \wedge \cdots \wedge \partial_m \psi(x) \right\| \, dx \\
&= \int_V \left\| \bigwedge^k (D\phi(\psi(x))) \det D\psi(x) e_1 \wedge \cdots \wedge e_m \right\| \, dx \\
&= \int_V \left\| (D\phi(\psi(x))) e_1 \wedge \cdots \wedge e_m \right\| \cdot |\det D\psi(x)| \, dx \\
&= \int_U \| \partial_1 \phi(\psi(y)) \wedge \cdots \wedge \partial_m \phi(\psi(y)) \| \cdot |\det D\psi(y)| \, dy \\
&= \Vol(\phi).
\end{align*}
\]

Here, \( e_1, \ldots, e_m \) stands for the standard basis of \( \mathbb{R}^m \).

3 Family of embeddings and the swept volume

In this section, we consider the maps of the form

\[
\phi : M \times \mathbb{R} \to \mathbb{R}^n, \\
\phi(x, t) := x + t \cdot v_1(x) + \cdots + t^k \cdot v_k(x),
\]

where \( M \) is an \( m \)-dimensional submanifold, and \( v_i : M \to \mathbb{R}^n \) is a smooth map for \( i \in [1, \ldots, k] \). Proposition 3.1 proves that the rate of growth of \( \Vol \left( \phi|_{M \times (-t, t)} \right) \) at 0 cannot be arbitrary. More precisely, it shows that
Proposition 3.1. Let $M \subset \mathbb{R}^n$ be an $m$-dimensional submanifold, and let $v_i : M \to \mathbb{R}^n$, $i \in \{1, \ldots, k\}$ be smooth maps. Denote by $\phi : M \times \mathbb{R} \to \mathbb{R}^n$ the map defined by

$$\phi(x, t) := x + t \cdot v_1(x) + \cdots + t^k \cdot v_k(x).$$

If

$$\lim_{t \to 0} \frac{\text{Vol} \left( \phi|_{M \times (-t, t)} \right)}{t^{k(m+1)}} = 0,$$

then $\text{Vol}(\phi) = 0$.

Proof. Denote $d := k(m + 1) - 1$. Let $\alpha : \mathbb{U} \to \mathbb{R}^m$, where $\mathbb{U} \subset \mathbb{R}^m$ is open, be a parametrization of a subset of $M$. Denote by $\psi : \mathbb{U} \times \mathbb{R} \to \mathbb{R}^n$ the map defined by $\psi(x, t) := \phi(\alpha(x), t)$. The map $\psi$ is equal to the composition of the restriction $\phi|_{\alpha(\mathbb{U}) \times \mathbb{R}}$ with the diffeomorphism

$$\mathbb{U} \times \mathbb{R} \to \alpha(\mathbb{U}) \times \mathbb{R} : (x, t) \mapsto (\alpha(x), t).$$

Hence

$$\text{Vol} \left( \psi|_{\mathbb{U} \times (-t, t)} \right) = \text{Vol} \left( \phi|_{\alpha(\mathbb{U}) \times (-t, t)} \right) \leq \text{Vol} \left( \phi|_{M \times (-t, t)} \right),$$

and, consequently,

$$\lim_{t \to 0} \frac{\text{Vol} \left( \psi|_{\mathbb{U} \times (-t, t)} \right)}{t^{d+1}} = 0.$$

The volume element $\|\partial_1 \psi(x, t) \wedge \cdots \wedge \partial_m \psi(x, t) \wedge \partial_t \psi(x, t)\|$ is equal to

$$\left\| \bigwedge_{i=1}^m \left( \partial_1 \alpha(x) + \sum_{j=1}^k t^j \cdot \partial_1 [v_j \circ \alpha](x) \right) \wedge \left( \sum_{j=1}^k j \cdot t^{j-1} \cdot [v_j \circ \alpha](x) \right) \right\|.$$

After developing the expression above by distributive law and after applying the Pythagorean theorem, the volume element transforms into the form

$$\sqrt{A_1(x, t)^2 + \cdots + A_\ell(x, t)^2}.$$
where $A_1, \ldots, A_\ell$ are polynomials in $t$ of degree at most $d$ whose coefficients are smooth functions $U \to \mathbb{R}$ in $x$-variable, and $\ell := \binom{n}{m+1}$. Assume there exists $j \in \{1, \ldots, \ell\}$ such that $A_j(x, t)$ is not identically equal to 0. Then,

$$A_j(x, t) = a_0(x) + a_1(x) \cdot t + \cdots + a_d(x) \cdot t^d,$$

where $a_1, \ldots, a_d : U \to \mathbb{R}$ are smooth functions that are not all identically equal to 0. Let $b \in \{0, \ldots, d\}$ be the smallest index such that $a_b$ is not identically equal to 0. There exists $\varepsilon > 0$ such that

$$\frac{1}{2} \int_U |a_b(x)| \, dx \cdot |t|^b \geq \sum_{i=b+1}^d \int_U |a_i(x)| \, dx \cdot |t|^i,$$

for $|t| < \varepsilon$. Hence, for $|t| < \varepsilon$,

$$\int_U \sqrt{A_1(x, t)^2 + \cdots + A_\ell(x, t)^2} \, dx \geq \int_U |A_j(x, t)| \, dx \geq \int_U |a_b(x)| \, dx \cdot |t|^b - \sum_{i=b+1}^d \int_U |a_i(x)| \, dx \cdot |t|^i.$$

This further implies

$$0 = \lim_{t \to 0} \frac{\text{Vol} \left( \psi|_{U \times (-t, t)} \right)}{t^{d+1}}$$

$$\geq \lim_{t \to 0} t^{-(d+1)} \cdot \int_{-t}^t \frac{1}{2} \int_U |a_b(x)| \, dx \cdot |s|^b \, ds$$

$$= \lim_{t \to 0} t^{-(d+1)} \cdot \int_0^t \int_U |a_b(x)| \, dx \cdot |s|^b \, ds$$

$$= \lim_{t \to 0} t^{-(d+1)} \cdot \frac{t^{d+1}}{d+1} \cdot \int_U |a_d(x)| \, dx$$

$$= \frac{1}{d+1} \int_U |a_b(x)| \, dx.$$
The continuity of $a_b$ now implies $a_b(x) = 0$ for all $x \in U$. Contradiction! Therefore

$$A_1(x, t) = \cdots = A_{\ell}(x, t) = 0,$$

for all $x \in U$ and $t \in \mathbb{R}$, and, consequently,

$$\text{Vol} \left( \phi|_{\alpha(U) \times \mathbb{R}} \right) = \text{Vol}(\psi) = 0.$$

This holds for all charts $\alpha$ of $M$. Hence $\text{Vol}(\phi) = 0$. \hfill $\blacksquare$

**Proposition 3.2.** Let $M$ be an $m$-dimensional manifold, let $I$ be an open interval containing 0, and let $\phi_t : M \to \mathbb{R}^n$ be a smooth family of embeddings. Assume $\text{Vol}(\phi : M \times I \to \mathbb{R}^n) = 0$. Then, for all $x \in M$, there exists $\epsilon = \epsilon(x) > 0$ such that

$$(\forall t \in (-\epsilon, \epsilon)) \quad \phi_t(x) \in \phi_0(M).$$

**Proof.** Suppose there exists $(x_0, t_0) \in M \times I$ such that $D\phi(x_0, t_0)$ is of rank $m + 1$. Then, there exists a neighbourhood $U \subset M \times I$ of $(x_0, t_0)$ such that $\phi_t|_U$ is an embedding. The volume of an embedding is positive. Hence

$$\text{Vol}(\phi) \geq \text{Vol}(\phi|_U) > 0.$$

This contradicts $\text{Vol} \phi = 0$. Therefore

$$(\forall (x, t) \in M \times I) \quad \text{rank} \ D\phi(x, t) \leq m.$$}

Since $\phi_t$ is a family of embeddings, the rank of $D\phi(x, t)$ is less than $m + 1$ if, and only if,

$$\partial_t \phi_t(x) \in \text{im} \ D\phi_t(x).$$

Denote by $Y_t$ the smooth vector field on $M$ defined by

$$\partial_t \phi_t(x) = D\phi_t(x)Y_t(x).$$

Denote by $\psi$ the (locally defined) flow of the vector field $-Y_t$. Fix $x \in M$. Let $O$ be a neighbourhood of $x$ and let $\epsilon > 0$ be such that $\psi_t(y)$ is well defined for $y \in O$ and $t \in (-\epsilon, \epsilon)$. Let $\delta > 0$ be such that $\psi^{-1}_t(x) \in O$ for all $t \in (-\delta, \delta)$. Since

$$\frac{\partial}{\partial t} (\phi_t(\psi_t(x))) = D\phi_t(\psi_t(x))Y_t(\psi_t(x)) + D\phi_t(\psi_t(x))\partial_t \psi_t(x)$$

$$= D\phi_t(\psi_t(x))Y_t(\psi_t(x)) + D\phi_t(\psi_t(x))(-Y_t(\psi_t(x)))$$

$$= 0,$$
for all \( y \in O \) and \( t \in (-\varepsilon, \varepsilon) \), we have \( \phi_t(\psi_t(y)) = \phi_0(y) \). By substituting \( y = \psi_t^{-1}(x) \), one gets

\[
(\forall t \in (-\delta, \delta)) \quad \phi_t(x) = \phi_0(\psi_t^{-1}(x)) \in \phi_0(M).
\]

\[\blacksquare\]

**Proposition 3.3.** Let \( M \subset \mathbb{R}^n \) be an analytic submanifold, and let \( \phi : M \times \mathbb{R} \to \mathbb{R}^n \) be a smooth map such that

- \( \phi(x,0) = x \), for all \( x \in M \),
- \( \phi(x,t) = x \), for \( t \in \mathbb{R} \) and for \( x \) outside of a compact set,
- the curve \( t \mapsto \phi(x,t) \) is analytic and has a contact of order \( k \in \mathbb{N} \) with \( M \) at \( t = 0 \) for all \( x \in M \).

Then,

\[
\lim_{t \to 0} \frac{\text{Vol}(\phi|_{M \times (-t,t)})}{t^k} = 0.
\]

**Proof.** Without loss of generality, assume that \( M \) is covered by a single chart \( \alpha : U \subset \mathbb{R}^m \to M \). Let \( N \) be a normal tubular neighbourhood of \( M \), and let \( r : N \to M \) be the nearest point map. Since \( \phi_t(x) \) is \( t \)-independent for \( x \) outside of a compact set (and since \( \phi_0(x) \in M \)), there exists \( \delta > 0 \) such that \( \phi(M \times (-\delta, \delta)) \subset N \). Let \( \delta_1 \in (0, \delta) \) be such that \( \phi_t : M \to \mathbb{R}^n \) is an embedding for all \( t \in [-\delta_1, \delta_1] \). Such \( \delta_1 \) exists because the set of embeddings \( M \to \mathbb{R}^n \) is open [4, Theorem 1.4]. Let \( Y_t, t \in [-\delta_1, \delta_1] \) be the vector field on \( M \) defined by

\[
D\phi_t(x)Y_t(x) = -dr(\partial_t \phi_t(x)).
\]

Denote by \( \theta_t : M \to M \) the flow of the vector field \( Y_t \). Let \( \psi_t : M \to \mathbb{R}^n \) be the smooth family of embeddings defined by \( \psi_t(x) := \phi_t(\theta_t(x)) \). For \( t \in [-\delta_1, \delta_1] \), the following holds

\[
\phi_t(M) = \psi_t(M), \quad \text{Vol}(\phi|_{M \times (-t,t)}) = \text{Vol}(\psi|M \times (-t,t)).
\]

There exists a compact set \( K \subset U \) such that \( \psi_t(x) = x \) for \( x \in \alpha(U \setminus K) \) and \( t \in [-\delta_1, \delta_1] \). Denote

\[
C := \max_{x \in K, |t| \leq \delta_1} \| \partial_1(\psi_t \circ \alpha)(x) \wedge \cdots \wedge \partial_m(\psi_t \circ \alpha)(x) \|.
\]
Since
\[ \|v_1 \wedge \cdots \wedge v_\ell\| \leq \|v_1\| \cdot \|v_2 \wedge \cdots \wedge v_\ell\|, \]
we get
\[ \text{Vol}(\phi|_{M \times (-t, t)}) = \text{Vol}(\psi|_{M \times (-t, t)}) = \int_\mathbb{U} \int_{-t}^t \|\partial_1(\psi_s \circ \alpha)(x) \wedge \cdots \wedge \partial_m(\psi_s \circ \alpha)(x)\| \, ds \, dx \]
\[ \leq \int_\mathbb{U} \int_{-t}^t \|\partial_s \psi_s \circ \alpha(x)\| \cdot \|\partial_1(\psi_s \circ \alpha)(x) \wedge \cdots \wedge \partial_m(\psi_s \circ \alpha)(x)\| \, ds \, dx \]
\[ = \int_K \int_{-t}^t \|\partial_s \psi_s \circ \alpha(x)\| \cdot \|\partial_1(\psi_s \circ \alpha)(x) \wedge \cdots \wedge \partial_m(\psi_s \circ \alpha)(x)\| \, ds \, dx \]
\[ \leq C \cdot \int_K \int_{-t}^t \|\partial_s \psi_s \circ \alpha(x)\| \, ds, \]
for \( t \in [0, \delta_1) \). It is enough to prove
\[ \lim_{t \to 0} \frac{1}{t^k} \int_{-t}^t \|\partial_s \psi_s(x)\| \, ds = 0. \tag{1} \]

Since
\[ \partial_t \psi_t(x) = (\partial_t \phi_t)(\theta_t(x)) + D\phi_t(\theta_t(x))\partial_t \theta_t(x) \]
\[ = (\partial_t \phi_t)(\theta_t(x)) + D\phi_t(\theta_t(x))Y_t(\theta_t(x)) \]
\[ = (\partial_t \phi_t)(\theta_t(x)) - dr(\partial_t \phi_t(\theta_t(x))) \]
\[ \in \ker dr, \]
we have
\[ \partial_t r(\psi_t(x)) = dr(\psi_t(x))\partial_t \psi_t(x) = 0, \]
and, consequently, \( r(\psi_t(x)) = x \) for \( t \in (-\delta_1, \delta_1) \). By Lemma 3.4 below, for \( x \in M \), there exists \( \epsilon_x > 0 \) such that the coordinates of \( r(\psi_t(x)) - \psi_t(x) = x - \psi_t(x) \) are monotone (with respect to \( t \)) for \( t \in [-\epsilon_x, 0] \) and for \( t \in [0, \epsilon_x] \).
Lemma 3.5 below implies

\[ \int_{-t}^{t} \left\| \partial_s \psi_s (x) \right\| ds = \int_{-t}^{t} \left\| \partial_s (x - \psi_s (x)) \right\| ds \]

\[ = \int_{-t}^{0} \left\| \partial_s (x - \psi_s (x)) \right\| ds + \int_{0}^{t} \left\| \partial_s (x - \psi_s (x)) \right\| ds \]

\[ \leq n \cdot \left( \left\| \psi_{-t} (x) - x \right\| + \left\| \psi_t (x) - x \right\| \right) \]

\[ = n \cdot \left( \sup_{y \in M} d(\psi_{-t} (y), M) + \sup_{y \in M} d(\psi_t (y), M) \right) \]

\[ \leq n \cdot \left( \sup_{y \in M} d(\phi_{-t} (y), M) + \sup_{y \in M} d(\phi_t (y), M) \right) \]

for \( t \in [0, \varepsilon]. \) Hence (by Lemma 2.5) (11) holds, and the proof is finished. \( \blacksquare \)

**Lemma 3.4.** Let \( M \subset \mathbb{R}^n \) be an analytic submanifold, and let \( \gamma : \mathbb{R} \to \mathbb{R}^n \) be an analytic curve such that \( \gamma (0) \in M. \) Denote by \( r : N \to M \) the nearest point map defined in a normal tubular neighbourhood \( N \) of \( M. \) Then, there exists \( \varepsilon > 0 \) such that the coordinates of the function

\[ [-\varepsilon, \varepsilon] \to \mathbb{R}^n : t \mapsto r(\gamma (t)) - \gamma (t) \]

are monotone (not necessarily strictly) on \([-\varepsilon, 0]\) and \([0, \varepsilon].\)

**Proof.** Since \( M \) is an analytic submanifold of \( \mathbb{R}^n, \) the nearest point map is analytic [1, page 240]. The set \( \gamma^{-1} (N) \) is open. Hence there exists \( \delta > 0 \) such that \( \gamma (t) \in N \) for all \( t \in (-\delta, \delta). \) Denote by

\[ f = (f_1, \ldots, f_n) : (-\delta, \delta) \to \mathbb{R}^n \]

the analytic map defined by \( f(t) = r(\gamma (t)) - \gamma (t). \) Fix \( j \in \{1, \ldots, n\}. \) If \( f_j^{(k)} (0) = 0 \) for all \( k \in \mathbb{N}, \) then (since \( f_j \) is analytic) there exists \( \varepsilon > 0 \) such that \( f_j (t) = 0 \) for \( t \in (-\varepsilon, \varepsilon). \) Consequently, \( f_j \) is monotone on \((-\varepsilon, \varepsilon).\)

Assume, now, there exists \( k \in \mathbb{N} \) such that \( f_j^{(k)} (0) \neq 0, \) and such that \( f_j^{(i)} (0) = 0 \) for all \( i \in \{1, \ldots, k - 1\}. \) The Taylor approximation for \( f_j \) implies

\[ f_j^{(i)} (t) = \frac{f_j^{(k)} (0)}{(k - 1)!} \cdot t^{k-1} + \frac{f_j^{(k+1)} (c_t)}{k!} \cdot t^k \]

\[ = \frac{t^{k-1}}{(k - 1)!} \left( f_j^{(k)} (0) + \frac{1}{k} \cdot f_j^{(k+1)} (c_t) \cdot t \right), \]
for \(t \in (-\delta, \delta)\) and for some \(c_t\) between 0 and \(t\). Since \(\frac{1}{k} f_j^{(k+1)}\) is a bounded function on \([-\frac{\delta}{2}, \frac{\delta}{2}]\), there exists \(\varepsilon \in (0, \delta)\) such that the function \(f_j'\) does not change the sign on intervals \((-\varepsilon, 0)\) and \((0, \varepsilon)\). Hence \(f_j\) is monotone on \([-\varepsilon, 0]\) and \([0, \varepsilon]\).

Lemma 3.5. Let \([a, b] \subset \mathbb{R}\) be a compact interval, and let \(\gamma = (\gamma_1, \ldots, \gamma_n) : [a, b] \to \mathbb{R}^n\) be a \(C^1\) curve such that \(\gamma_i : [a, b] \to \mathbb{R}\) is monotone for all \(i \in \{1, \ldots, n\}\). Then,

\[
\text{length}(\gamma) \leq n \cdot \|\gamma(a) - \gamma(b)\|.
\]

Proof. Since

\[
\sqrt{\sum_j (\gamma'_j(t))^2} \leq \sqrt{n} \cdot \max_j |\gamma'_j(t)| \leq \sqrt{n} \cdot \sum_j |\gamma'_j(t)|,
\]

the length of \(\gamma\) is bounded by

\[
\text{length}(\gamma) \leq \sqrt{n} \cdot \sum_j \int_a^b |\gamma'_j(t)| \, dt
\]

\[
= \sqrt{n} \cdot \sum_j \left| \int_a^b \gamma'_j(t) \, dt \right|
\]

\[
= \sqrt{n} \cdot \sum_j |\gamma_j(b) - \gamma_j(a)|
\]

\[
\leq n^{\frac{3}{2}} \cdot \sqrt{\frac{1}{n} \cdot \sum_j |\gamma_j(b) - \gamma_j(a)|^2}
\]

\[
= n \cdot \|\gamma(b) - \gamma(a)\|.
\]

In the sequence of inequalities above, we used

\[
\left| \int_a^b \gamma'_j(t) \, dt \right| = \int_a^b |\gamma'_j(t)| \, dt
\]

(which holds because \(\gamma_j\) is monotone) and the Cauchy-Schwarz inequality.

4 Proof of the main theorem

Proof of Theorem 1.3: Let \(p\) be an arbitrary point in \(M\), and let \(v_1, \ldots, v_k : M \to \mathbb{R}^n\) be smooth compactly supported maps such that

\[
x + t \cdot v_1(x) + \cdots + t^k \cdot v_k(x) \in \Gamma_x
\]
for all $x \in M$, and such that

$$ t \mapsto p + t \cdot v_1(p) + \cdots + t^k \cdot v_k(p) $$

is a parametrization of $\Gamma_p$. Denote by $\phi_t : M \to \mathbb{R}^n$, $t \in \mathbb{R}$ the family of smooth maps defined by

$$ \phi_t(x) := x + t \cdot v_1(x) + \cdots + t^k \cdot v_k(x). $$

Proposition 3.3 implies

$$ \lim_{t \to 0} \frac{\text{Vol} \left( \phi|_{M \times (-t,t)} \right)}{t^{k(m+1)}} = 0. \quad (2) $$

There exists $\varepsilon > 0$, such that $\phi_{t_1}, t_1 \in (-\varepsilon, \varepsilon)$ is a smooth family of embeddings (see [4, Theorem 1.4]). Therefore, due to Proposition 3.1, Proposition 3.2, and (2), $\phi_t(x) \in M$ for $|t|$ small enough. In particular, there exists an open segment $I_p$ of $\Gamma_p$ such that $p \in I_p \subset M$. Since $M$ is proper, and since $M$ and $\Gamma_p$ are analytic, the identity theorem for analytic functions [6, Corollary 1.2.7] implies $\Gamma_p \subset M$. ■

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