A REMARK ON THE $sl_2$ APPROXIMATION OF THE KONTSEVICH INTEGRAL OF THE UNKNOT

SVETLANA TYURINA* AND ALEXANDER VARCHENKO**.1

*MPIM, Bonn, Germany
tyurina@mpim-bonn.mpg.de

**Department of Mathematics, University of North Carolina at Chapel Hill, Chapel Hill, NC 27599-3250, USA
av@math.unc.edu

Abstract. The Kontsevich integral of a knot $K$ is a sum $I(K) = 1 + \sum_{n=1}^{\infty} h^n \sum_{D \in A_n} a_D D$ over all chord diagrams with suitable coefficients. Here $A_n$ is the space of chord diagrams with $n$ chords. A simple explicit formula for coefficients $a_D$ is not known even for the unknot. Let $E_1, E_2, \ldots$ be elements of $A = \oplus_n A_n$. Say that a sum $I'(K) = 1 + \sum_{n=1}^{\infty} h^n E_n$ is an $sl_2$ approximation of the Kontsevich integral if the values of the $sl_2$ weight system $W_{sl_2}$ on both sums are equal, $W_{sl_2}(I(K)) = W_{sl_2}(I'(K))$.

For any natural $n$ fix points $a_1, \ldots, a_{2n}$ on a circle. For any permutation $\sigma \in S_{2n}$ of $2n$ elements define the chord diagram $D(\sigma)$ with $n$ chords as the diagram with chords formed by pairs $a_{\sigma(2i-1)}$ and $a_{\sigma(2i)}$, $i = 1, \ldots, n$. We show that

$$1 + \sum_{n=1}^{\infty} \frac{h^{2n}}{2^n (2n)! (2n+1)!} \sum_{\sigma \in S_{2n}} D(\sigma)$$

is an $sl_2$ approximation of the Kontsevich integral of the unknot.

1. $sl_2$ APPROXIMATIONS

The Kontsevich integral of a knot $K$ is a sum $I(K) = 1 + \sum_{n=1}^{\infty} h^n \sum_{D \in A_n} a_D D$ over all chord diagrams with suitable coefficients $[K]$. Here $A_n$ is the space of chord diagrams with $n$ chords, $h$ is formal parameter. A simple explicit formula for coefficients $a_D$ is not known even for the unknot. Let $E_1, E_2, \ldots$ be elements of $A = \oplus_n A_n$. Say that a sum $I'(K) = 1 + \sum_{n=1}^{\infty} h^n E_n$ is an $sl_2$ approximation of the Kontsevich integral if the values of the $sl_2$ weight system $W_{sl_2}$ on both sums are equal, $W_{sl_2}(I(K)) = W_{sl_2}(I'(K))$.

For any natural $n$ fix points $a_1, \ldots, a_{2n}$ on a circle. For any permutation $\sigma \in S_{2n}$ of $2n$ elements define the chord diagram $D(\sigma)$ with $n$ chords as the diagram with chords formed

Date: June 2001.

1Supported in part by NSF grant DMS-9801582.
by pairs $a_{\sigma(2i-1)}$ and $a_{\sigma(2i)}$, $i = 1, \ldots, n$. Introduce an element $\Sigma_n = \sum_{\sigma \in S_{2n}} D(\sigma)$ in $A_n$.

**Theorem 1.1.** The sum

$$1 + \sum_{n=1}^{\infty} \frac{h^{2n}}{2^n (2n)! (2n+1)!} \Sigma_n$$

is an $sl_2$ approximation of the Kontsevich integral of the unknot.

The theorem is proved in Section 6.

We thank S. Chmutov for useful discussions.

2. **Three algebras**

The Kontsevich integral takes values in the graded completion of the chord diagram algebra $A = \oplus_n A_n$. The $\mathbb{Q}$ vector space $A$ is generated by the usual chord diagrams modulo the four-term relation

$$\begin{array}{c}
\includegraphics[width=2cm]{four_term_relation}\n\end{array} = 0.
$$

The product of chord diagrams is their connected sum, see [B].

We also consider the algebra $T$ of trivalent diagrams. A trivalent diagram is a connected graph with only trivalent vertices and a distinguished oriented circle, such that at each vertex, which does not lie on the circle, one of two possible cyclic orderings of the three edges meeting at this vertex is chosen. The $\mathbb{Q}$ vector space $T$ is generated by trivalent diagrams modulo the STU relation,

$$\begin{array}{c}
\includegraphics[width=2cm]{stu_relation}\n\end{array}.
$$

These three trivalent diagrams are identical outside the corresponding fragment on the picture. Pieces of the circle are pictured by thick lines. The product of trivalent diagrams is defined as their connected sum with respect to the distinguished circles.

The following relations follow from the STU relation,

(AS):

$$\begin{array}{c}
\includegraphics[width=2cm]{as_relation}\n\end{array} = - \includegraphics[width=2cm]{as_relation}.
$$

(IHX):

$$\begin{array}{c}
\includegraphics[width=2cm]{ihx_relation}\n\end{array} = \includegraphics[width=2cm]{ihx_relation} + \includegraphics[width=2cm]{ihx_relation}.
$$

Applying the STU relation one can express a given trivalent diagram as a linear combination of chord diagrams. This gives a natural mapping $T \to A$ which is an isomorphism of algebras, see [B].
The third algebra is the algebra $\mathcal{U}$ of uni-trivalent diagrams. A uni-trivalent diagram is a graph whose vertices either univalent or trivalent, at each trivalent vertex a cyclic ordering of the three edges is chosen. The vertices of valency 1 of a uni-trivalent diagram are called “legs” of the diagram. We consider the $\mathbb{Q}$ vector space $\mathcal{U}$ generated by uni-trivalent graphs modulo the AS and IHX relations. The product in the algebra $\mathcal{U}$ is the disjoint union $\sqcup$. There is a natural isomorphism $\mathcal{U} \rightarrow \mathcal{T}$ as vector spaces, but not as algebras. The isomorphism maps every uni-trivalent diagram to the average of all possible ways of placing its univalent vertices along the circle, see [B].

Introduce uni-trivalent graphs $w_{2n}$, the “wheels with $2n$ legs”,

\[ w_2 = \bigcirc, \quad w_4 = \bigcirc, \quad \ldots \]

Under the isomorphism $\mathcal{U} \rightarrow \mathcal{T}$, for instance, we have

\[ \bigcirc \mapsto \frac{1}{2} \left( \bigcirc + \bigcirc \right). \]

3. The $\mathfrak{sl}_2$-weight system

A weight system on $A$ (resp. on $\mathcal{T}$, $\mathcal{U}$) with values in a vector space is a linear homomorphism of $A$ (resp. of $\mathcal{T}$, $\mathcal{U}$) to the vector space. The composition of the Kontsevich integral of a knot with the linear homomorphism defines a knot invariant of the knot. Here we recall the construction of the weight system associated to a Lie algebra $\mathfrak{g}$ with an $ad$-invariant nondegenerate bilinear form $\Phi$.

Let $a_1, \ldots, a_m$ and $b_1, \ldots, b_m$ be two dual bases of $\mathfrak{g}$: $\Phi(a_i, b_j) = \delta_{i,j}$. Fix a chord diagram $D$ and a base point on its circle which is different from the endpoints of the chords. Label each chord by a number $i$ such that $1 \leq i \leq m$. Attach to one endpoint of the chord labelled by $i$ the element $a_i$ and to another endpoint the element $b_i$. For example, for the chord diagram $D = \bigcirc$ with the base point $\ast$ and labels $i$ and $j$ we have

\[ a_i \quad b_j \quad a_j \quad b_i \]

Walk around the circle starting from the base point in the direction of the orientation of the circle and write in one word the elements associated to the endpoints. The constructed word is an element of the universal enveloping algebra $U(\mathfrak{g})$. Define $W(D)$ to be the sum of such words where the sum is over all labels of the chords. In the example,

\[ W(D) = \sum_{i,j} a_i b_j b_i a_j \in U(\mathfrak{g}). \]

In our pictures we always assume that the circle is oriented counterclockwise.
The element $W(D)$ does not depend on the base point, does not depend on the choice
of dual bases in $\mathfrak{g}$, belongs to the center $Z(\mathfrak{g})$ of the universal enveloping algebra, satisfies
the four-term relation. The mapping $W : A \to Z(\mathfrak{g})$ is an algebra homomorphism,
$W(D_1 \cdot D_2) = W(D_1) \cdot W(D_2)$, see [K].

**Example.** $W(\square) = c$ is the quadratic Casimir element of $Z(\mathfrak{g})$ associated to the
chosen invariant form.

For $\mathfrak{g} = \mathfrak{sl}_2$, we have $Z(\mathfrak{sl}_2) \cong \mathbb{C}[c]$, and for a chord diagram $D$ with $n$ chords,

$$W_{\mathfrak{sl}_2}(D) = c^n + \lambda_1 c^{n-1} + \lambda_2 c^{n-2} + \ldots + \lambda_{n-1} c.$$

We choose $Tr$ as an $\text{ad}$-invariant form on $\mathfrak{sl}_2$ where $Tr$ is the trace of matrices in
the standard two dimensional representation of $\mathfrak{sl}_2$. A recurrent formula for $W_{\mathfrak{sl}_2}$ is constructed in [CV].

**Theorem 3.1.** Let $W = W_{\mathfrak{sl}_2}$ be the weight system associated to $\mathfrak{sl}_2$ and the $\text{ad}$-invariant
form $Tr$. Then

$$W\left(\begin{array}{c}
\square
\end{array}\right) - W\left(\begin{array}{c}
\square
\end{array}\right) - W\left(\begin{array}{c}
\square
\end{array}\right) + W\left(\begin{array}{c}
\square
\end{array}\right) =$$

$$= 2W\left(\begin{array}{c}
\square
\end{array}\right) - 2W\left(\begin{array}{c}
\square
\end{array}\right);$$

$$W\left(\begin{array}{c}
\square
\end{array}\right) - W\left(\begin{array}{c}
\square
\end{array}\right) - W\left(\begin{array}{c}
\square
\end{array}\right) + W\left(\begin{array}{c}
\square
\end{array}\right) =$$

$$= 2W\left(\begin{array}{c}
\square
\end{array}\right) - 2W\left(\begin{array}{c}
\square
\end{array}\right);$$

$$W\left(\begin{array}{c}
\square
\end{array}\right) - W\left(\begin{array}{c}
\square
\end{array}\right) - W\left(\begin{array}{c}
\square
\end{array}\right) + W\left(\begin{array}{c}
\square
\end{array}\right) =$$

$$= 2W\left(\begin{array}{c}
\square
\end{array}\right) - 2W\left(\begin{array}{c}
\square
\end{array}\right);$$
This theorem allows one to compute $W(D)$ since the two chord diagrams of the right hand side have one chord less than the diagrams of the left hand side, and the last three diagrams of the left hand side are simpler than the first one since they have less intersections between their chords.

The theorem indicates six-term elements of the kernel of the $sl_2$ weight system. The subspace $I$ of the algebra $A$ generated by the six-term elements forms an ideal. The quotient algebra $A/I$ is generated by two elements $\bigotimes$ and $\bigotimes$. The ideal generated by the six term elements and the element $\bigotimes + 2\bigotimes - \bigotimes \cdot \bigotimes$ is the kernel of the $sl_2$ weight system.

The linear isomorphisms $U \rightarrow T$ and $T \rightarrow A$ induce weight systems $W_{sl_2}: T \rightarrow Z(sl_2)$, $W_{sl_2}: U \rightarrow Z(sl_2)$.

**Theorem 3.2.** [CV] The weight system $W_{sl_2}$ satisfies the following three term relation

$$W_{sl_2}\left(\begin{array}{c} \bigotimes \\ \bigotimes \end{array}\right) = 2W_{sl_2}\left(\begin{array}{c} \bigotimes \\ \bigotimes \end{array}\right) - 2W_{sl_2}\left(\begin{array}{c} \bigotimes \\ \bigotimes \end{array}\right)$$

for any uni-trivalent diagrams differed only by the pictured fragments.

**Corollary 3.3.** [CV]

$$W_{sl_2}\left(\begin{array}{c} \bigotimes \\ \bigotimes \end{array}\right) = 4W_{sl_2}\left(\begin{array}{c} \bigotimes \\ \bigotimes \end{array}\right).$$

4. **Bernoulli numbers and Bernoulli polynomials**

The modified Bernoulli numbers are defined by the series

$$\sum_{n=0}^{\infty} b_{2n} x^{2n} = \frac{1}{2} \ln \frac{e^{x/2} - e^{-x/2}}{x/2}.$$ 

The Bernoulli polynomials $B_n(x)$ are defined by the series

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!}.$$ 

The polynomial $B_n(x)$ has degree $n$. Its top coefficient equals 1.
Following \cite{LM} introduce the shifted Bernoulli polynomials \( q_n(x) \) by the condition

\[
q_n\left(\frac{x^2 - 1}{2}\right) = \frac{2}{(2n + 1)!} \frac{B_{2n+1}(\frac{1+x}{2})}{x}.
\]

The polynomial \( q_n(x) \) has degree \( n \). Its top coefficient equals \( \frac{1}{2^n(2n+1)!} \).

**Theorem 4.1.** For any natural \( n \), we have

\[
W_{sl_2}(w_{2n}) = 2^{n+1} (2n + 1)! q_n(c), \quad \sum_{\sigma \in S^{2n}} W_{sl_2}(D(\sigma)) = 2^n (2n)! (2n + 1)! q_n(c).
\]

The theorem is proved in Section 6.

5. THE KONTSEVICH INTEGRAL OF THEUNKNOT

A formula for the logarithm of the Kontsevich integral \( I \) of the unknot in terms of wheels is given in \cite{BGRT, T}.

**Theorem 5.1.**

\[
I = 1 + \exp\left(\sum_{n=1}^{\infty} b_{2n} h^{2n} w_{2n}\right) = 1 + \left(\sum_{n=1}^{\infty} b_{2n} h^{2n} w_{2n}\right) + \frac{1}{2}\left(\sum_{n=1}^{\infty} b_{2n} h^{2n} w_{2n}\right)^2 + \ldots
\]

The value of the \( sl_2 \) weight system on the Kontsevich integral of the unknot is calculated in \cite{LM}.

**Theorem 5.2.**

\[
W_{sl_2}(I) = \sum_{n=0}^{\infty} q_n(c) h^{2n}.
\]

6. PROOFS

**Lemma 6.1.** For \( n_1 + n_2 + \ldots + n_k = n \), we have

\[
W_{sl_2}(w_{2n_1} \sqcup w_{2n_2} \sqcup \ldots \sqcup w_{2n_k}) = 2^{n_1+n_2+\ldots+n_k+k} \frac{W_{sl_2}(\Sigma_n)}{(2n)!}.
\]

**Proof of the lemma.** We prove the lemma for \( k = 1 \), general case is similar. The three term relation applied to a vertex of \( w_{2n} \) gives

\[
W_{sl_2}(w_{2n}) = 2W_{sl_2}(\{ | \} \sqcup w_{2n-2}) - 2W_{sl_2}(t_{2n-2})
\]

where \( \{ | \} \) is the uni-trivalent graph with one edge and two univalent vertices and \( t_{2n-2} = \underbrace{\ldots \ldots}_{(2n-2) \text{ legs}} \).

Application of the three term relation to the first two vertices of \( t_{2n-2} \) gives

\[
W_{sl_2}(t_{2n-2}) = 2W_{sl_2}(\{ | \} \sqcup t_{2n-4}) - 2W_{sl_2}(\{|\} \sqcup t_{2n-4}) = 0.
\]

Then Corollary 3.3 implies \( W_{sl_2}(w_{2n}) = 2^{n+1} W_{sl_2}(\sqcup_n \{ | \}) \) where \( \sqcup_n \{ | \} \) is the diagram with \( n \) edges and \( 2n \) univalent vertices. Glueing legs of \( \sqcup_n \{ | \} \) to the circle in all possible...
ways and dividing the sum of the resulting chord diagrams by \((2n)!\) gives the lemma for \(k = 1\),

\[
W_{sl_2}(w_{2n}) = \frac{2^{n+1}}{(2n)!} \sum_{\sigma \in S_{2n}} W_{sl_2}(D_n(\sigma)).
\]

**Proof of Theorems 1.1 and 4.1.** Theorems 5.2 and Lemma 6.1 imply that \(W_{sl_2}(\Sigma_n) = \text{const } q_n(c)\). Both sides are polynomials in \(c\) of degree \(n\). Comparing the coefficients of \(c^n\) one gets Theorem 1.1 and the second equality of Theorem 4.1. Equation (1) implies the first equality of Theorem 4.1.

**REFERENCES**

[B] D.Bar-Natan, On the Vassiliev knot invariants, – Topology. 34, 1995, 423-472.

[BGRT] D. Bar-Natan, S.Garoufalidis, L.Rozansky, D.Thurston, Wheels, wheeling, and the Kontsevich integral of the unknot, arXiv:q-alg/9703025.

[CV] S.Chmutov, A.Varchenko, Remarks on the Vassiliev Knot Invariants Coming from sl_2. – Topology. 36, No.1, 1996, 153-178.

[K] M.Kontsevich, Vassiliev’s knot invariants. – Adv. in Sov. Math. 16, part 2, 1993, 137-150.

[LM] T.Q.T. LE and J. MURAKAMI, Parallel version of the universal Vassiliev-Kontsevich invariant. – J. Pure and Appl. Alg. 121, 1997, 271-291.

[T] D. Thurston, Wheeling: a diagrammatic analogue of the Duflo isomorphism, Ph.D.thesis, arXiv:math.QA/0006083.