Fixed points and stability of A class of Stochastic dynamical system driven by Brownian motion

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Abstract-In real life, many models and systems are affected by random phenomena. For this reason, experts and scholars propose to describe these stochastic processes with Brownian motion respectively. In this paper we consider a kind of stochastic Vollterra dynamical systems of nonconvolution type and give some new conditions to ensure that the zero solution is asymptotically stable in mean square by means of fixed point method. The theorems of asymptotically stability in mean square with a necessary conditions are proved. Some results of related papers are improved.

1. Introduction

The Lyapunov direct method is very effective in establishing stability results for various dynamical systems. However, there are a host of problems which can not be solved. Recently, Burton and other authors have studied stability\textsuperscript{(1-8)} by using the fixed point theory. Research results show that many of the problems can be solved by the fixed point theory when we use Lyapunov direct method in stability studies. The Lyapunov direct method usually requires a point state condition, while the stability result of the fixed point theory requires an average nature condition. Following it, Zhang\textsuperscript{(9)} also use the fixed point methods to study the stability.

When we study the stability of stochastic dynamical systems, some methods were applied\textsuperscript{(10-11)}. Recently, Luo\textsuperscript{(12)} firstly used fixed point method to study the stability of stochastic dynamical systems. However, while we study the stability by means of fixed point method, the operator we define is very important. In this paper, we consider the asymptotically stability in mean square of a kind of stochastic dynamical systems of nonconvolution type by means of fixed point method too.

According to the authors, one of the innovations of this paper is that few experts have used the fixed point method to study the destabilizing Volterra dynamical system of nonconvolution. In addition, the result of Burton\textsuperscript{(11)} is improved and generalized by defining different operator. It is shown that the fixed point method is more flexible and efficient than other methods such as Lyapunov's direct method in the study of the null solution stability of the logistic volatile terra system of nonconvolution type.

The rest of this paper is arranged as following. First, we state the main theorems and relative proofs in section 1. Secondly, , an example shows that our stability results are indeed better than those in [1] in section 2.

2. Main results

Define \{Ω, F, P\} as a complete probability space equipped with some filtration \{F_t\}_{t\geq0} which
satisfies the usual conditions. For example, the filtration is $\mathcal{F}_t$ contains all $P$-null sets and right continuous. Let $\{\omega(t), t \geq 0\}$ be a standard one-dimension Wiener process defined on $\{\Omega, \mathcal{F}, P\}$. The mapping $a(t) \in C(\mathbb{R}^+, \mathbb{R})$.

Consider the following one-dimensional stochastic Volterra dynamical system of nonconvolution type

$$dx(t) = f(t, x(t))dt + \int_{r(t)}^{t} b(t, s)g(x(s))ds + c(t)x(t)d\omega(t), \quad t \geq 0. \quad (1)$$

With the initial condition $x(0) = \phi(t) \in C([\pi(0), R])$. Where $x(t) \in R$, $\pi(0)$ is lower bound of the function $r(t)$ ($t \geq 0$), and $r(t) \in C(R^+, R^+)$ satisfy $t - r(t) \to \infty$ as $t \to \infty$, $c(t) \in C(R^+, R^+)$ and $b : R^+ \times R^+ \to R$. Assume that there exists a unique solution in system (1) which is denoting by $x(t)$.

The function $g$ satisfy the Lipschitz condition and positive constant $M$ and $N$ exists, such that for any $t \geq 0$ and $x, y \in R$, we have

$$|f(x) - f(y)| \leq N|x - y|, \quad |g(x) - g(y)| \leq M|x - y|$$

As a special case, while $c(t) = 0$, then

$$dx(t) = -a(t)x(t)dt + \int_{r(t)}^{t} b(t, s)g(x(s))ds, \quad t \geq 0. \quad (2)$$

Burton[1] has studied the asymptotically stability of this system. And have the result as following:

**Theorem A[Burton[1]]** Assume there exist $\alpha < 1$, such that when $t \geq 0$,

(i) $\int_{0}^{t} e^{-\int_{s}^{t} a(\mu)d\mu} \int_{s-r(s)}^{s} |b(s, \nu)| d\nu ds \leq \alpha$,

(ii) for each $\varepsilon > 0$, there exist $T_1 > 0$ and $T > 0$, such that $T_2 \geq T_1$ and $t \geq T_2 + T$ imply that

$$e^{-\int_{0}^{t} a(\mu)d\mu} < \varepsilon \text{ and } e^{-\int_{0}^{t} a(\mu)d\mu} \to 0 \text{ as } t \to \infty.$$ 

Then the zero solution of system (2) is mean square asymptotic stable at $t_0=0$.

We will try to study the asymptotically stability in mean square of system (1) by using fixed point method in section 2. The result of Burton[1] is improved and generalized by defining different operator.

**Theorem 1** Assume there exist a continuous function $h(t) : [0, \infty) \to R^+$ and $\eta \in (0, 1)$, such that when $t \geq 0$,

(i) $\liminf_{t \to \infty} \int_{0}^{t} h(s)ds > -\infty$,

(ii) $\int_{0}^{t} |h(s) + N| e^{-\int_{s}^{t} h(\mu)d\mu} ds + M \int_{0}^{t} e^{-\int_{s}^{t} h(\mu)d\mu} (\int_{s-r(s)}^{s} b(s, \nu)d\nu)ds + \left(\int_{0}^{t} e^{-\int_{s}^{t} h(\mu)d\mu} \left|c(s)\right|^2 ds\right)^{1/2} \leq \eta < 1$;

If and only if

(iii) $\int_{0}^{t} h(s)ds \to \infty \text{ as } t \to \infty.$

Then, the zero solution of system (1) is mean square asymptotic stable.

**Proof** Define $S$ as the Banach space of all $F$-adapted processes $\varphi(t, \omega) : [\pi(0), +\infty) \times \Omega \to R$, which is almost surely continuous in $t$ for fixed $\omega \in \Omega$. In addition, $\varphi(s, \omega) = \varphi(s)$ for $s \in [\pi(0), 0]$ and $E[\varphi(t, \omega)]^2 \to 0$ as $t \to \infty$.

Define an operator $\Psi : S \to S$ by $(\Psi \varphi)(0) = \varphi(t)$ for $t \in [\pi(0), 0]$ and when $t \geq 0$,
\[(\Psi \varphi)(t) = \phi(0)e^{-\int_{t}^{0} \frac{h(l)\,dl}{\mu}} + \int_{0}^{t} e^{-\int_{s}^{t} \frac{h(l)\,dl}{\mu}} h(s)\varphi(s)\,ds + \int_{0}^{t} e^{-\int_{s}^{t} \frac{h(l)\,dl}{\mu}} f(s, \varphi(s))\,ds + \int_{0}^{t} e^{-\int_{s}^{t} \frac{h(l)\,dl}{\mu}} \int_{s-r(s)}^{r(s)} b(s, r)\varphi(r)\,dr\,ds + \int_{0}^{t} e^{-\int_{s}^{t} \frac{h(l)\,dl}{\mu}} c(s)\varphi(s)\,d\omega(s) := \sum_{i=1}^{5} J_{i}. \] (3)

Firstly, it is obvious that \(\Psi\) is continuous. Secondly, we prove \(\Psi(S) \subset S\), that is \(E(\Psi \varphi)(t)\rightarrow 0\) as \(t \to \infty\). As \(t \to \infty\), \(t-r(t) \to \infty\). For any \(\sigma > 0\), there exists \(t_{1} > 0\) such that \(t \geq t_{1}\), we get the following results: \(E|\varphi(t)|^{2} < \sigma\) and \(E|\varphi(t-r(t))|^{2} < \sigma\). Therefore

\[\begin{align*}
E(\Psi \varphi)(t) & \leq 5\sum_{i=1}^{5} E(J_{i}(t))^{2}. \\
& \leq E(\sup_{s \in [0,t]} |J_{1}(s)|^{2}) + E(\sup_{s \in [0,t]} |J_{2}(s)|^{2})
& \leq E(\sup_{s \in [0,t]} |\varphi(s)|^{2}) + E(\sup_{s \in [0,t]} |\varphi(r)|^{2})
& \leq E(\sup_{s \in [0,t]} |\varphi(s)|^{2}) + \int_{0}^{t} e^{-\int_{s}^{t} \frac{h(l)\,dl}{\mu}} |\varphi(r)|^{2}\,dr + \int_{0}^{t} e^{-\int_{s}^{t} \frac{h(l)\,dl}{\mu}} |\varphi(r)|^{2}\,dr \\
& \leq E(\sup_{s \in [0,t]} |\varphi(s)|^{2}) + \int_{0}^{t} e^{-\int_{s}^{t} \frac{h(l)\,dl}{\mu}} |\varphi(s)|^{2}\,dr + \int_{0}^{t} e^{-\int_{s}^{t} \frac{h(l)\,dl}{\mu}} |\varphi(r)|^{2}\,dr.
\end{align*}\] (4)

From condition (ii) and (iii), there is \(t_{2} \geq t_{1}\) that such when \(s \geq t_{2}\) we have

\[E(\sup_{s \in [0,t]} |x(s)|^{2}) \leq \int_{0}^{t} e^{-\int_{s}^{t} \frac{h(l)\,dl}{\mu}} |x(r)|^{2}\,dr < \varepsilon\] (5)

From condition (ii), we know \(E(\sup_{s \in [0,t]} |J_{5}(s)|^{2}) < \sigma + \eta \sigma < 2\sigma\). Thus \(E(\sup_{s \in [0,t]} |J_{5}(s)|^{2}) \to 0\).

With the similar method, from the conditions (ii) and (iii), we have \(E(\sup_{s \in [0,t]} |J_{i}(s)|^{2}) \to 0, i = 1, ..., 4\). So, \(\Psi(S) \subset S\).

Thirdly, we will prove \(\Psi\) is contractive mapping in \(S\). From the condition (ii), we could find a constant \(K > 0\), such that

\[\begin{align*}
(1 + \frac{1}{K})\left|\int_{0}^{t} e^{-\int_{s}^{t} \frac{h(l)\,dl}{\mu}} |h(s) + N|\,ds + M \int_{0}^{t} e^{-\int_{s}^{t} \frac{h(l)\,dl}{\mu}} \left(\int_{s-r(s)}^{r(s)} b(s, r)\,dr\right)\,ds\right|^{2} +
(1 + K)\left|\int_{0}^{t} e^{-\int_{s}^{t} \frac{h(l)\,dl}{\mu}} |c(s)|^{2}\,ds\right|^{2} \leq \eta^{2} < 1
\end{align*}\] (6)

For any \(\varphi \in S\) and \(\psi \in S\), we have
E^\sup_{\sigma \in (0,1)} \left| (\Psi \varphi)(s) - (\Psi \psi)(s) \right|^2 = \\
E^\sup_{\sigma \in (0,1)} \left\{ s - \int_0^s \int_0^r h(\mu) \frac{d\mu}{\mu} \right\} + \int_0^r e^{-\int_0^r h(\mu) \frac{d\mu}{\mu}} \left[ f(v, \varphi(v)) - f(v, \psi(v)) \right] dv + \\
\int_0^r e^{-\int_0^r h(\mu) \frac{d\mu}{\mu}} \left( \int_{r-r(v)}^r b(v, r) (g(\varphi(r)) - g(\psi(r))) dr \right) dv + \\
\int_0^r e^{-\int_0^r h(\mu) \frac{d\mu}{\mu}} c(v) (\varphi(v) - \psi(v)) d\omega(v) = \\
E^\sup_{\sigma \in (0,1)} \left( \sup_{s \in [0,1]} \left| \varphi(s) - \psi(s) \right|^2 \right) \left( \sup_{s \in [0,1]} \left( 1 + \frac{1}{K} \right) \int_0^r e^{-\int_0^r h(\mu) \frac{d\mu}{\mu}} \left| h(v) + N \right| dv + \\
M \int_0^r e^{-\int_0^r h(\mu) \frac{d\mu}{\mu}} \left( \int_{r-r(v)}^r b(v, r) \left| dr \right| \right) dv + (1 + K) \int_0^r e^{-\int_0^r h(\mu) \frac{d\mu}{\mu}} \left| c(v) \right|^2 dv \right). (7)

Thus we get that \( \Psi \) is contractive by (6). According to the contraction mapping principle, \( \Psi \) has a unique fixed point \( x(t) \in S \), which is a solution of system (1) with \( x(r) = \phi(r) \) on \( r \in [\pi(0), 0] \) and \( E|x(r)|^2 \to 0 \) as \( t \to \infty \).

In order to acquire the mean square asymptotic stability, we should prove that it is mean square stable in the zero solution of system (1). Let \( \sigma > 0 \) be given and choose \( \sigma > \delta > 0 \) satisfying

\[(1 + K) \delta e^{-\int_0^r h(\mu) \frac{d\mu}{\mu}} + \eta \sigma < \sigma \]

Where \( K \) is defined in (6). If \( x(t) = x(t, 0, \pi(0)) \) is a unique solution of system (1) with \( E|\phi|^2 < \delta \), then \( x(t) = (\Psi x)(t) \) defined in (2). We obtain that \( E|x(t)|^2 < \varepsilon \) for all \( t \geq 0 \). If there exists \( t^* > 0 \) such that \( E|x(t^*)|^2 = \varepsilon \) and \( E|x(s)|^2 < \varepsilon \) for \( 0 \leq s < t^* \), then it follows from (3) and (6) that

\[ E|x(t^*)|^2 \leq (1 + K) E|\phi(0)|^2 e^{-\int_0^{t^*} h(\mu) \frac{d\mu}{\mu}} + \varepsilon (1 + \frac{1}{K}) \int_0^{t^*} e^{-\int_0^r h(\mu) \frac{d\mu}{\mu}} \left| h(s) + N \right| ds + M \int_0^{t^*} e^{-\int_0^r h(\mu) \frac{d\mu}{\mu}} \left| c(s) \right|^2 ds \leq (1 + K) \delta e^{-\int_0^r h(\mu) \frac{d\mu}{\mu}} + \eta \varepsilon < \varepsilon \]

Which is contradictory to the definition of \( t^* \). This explains that if the condition (iii) holds, the zero solution of system (1) is mean square asymptotic stable.

In the contrary, if the condition (iii) fails, there exists a sequence \( \{ t_n \mid t_n \to \infty \text{ as } n \to \infty \} \) by the condition (i). And we can find a positive constant \( G \) satisfying \( -G \leq \int_0^{t_n} h(s) ds = \theta \) for some \( \theta \in R \) by the condition (i). We give a definition

\[ F(s) := M \int_{r-r(s)}^{t_n} b(s, r) g(\varphi(r)) dr \]

For all \( s \geq 0 \). By the condition (ii), we have, This yields \( \int_0^{t_n} e^{-\int_0^r h(\mu) \frac{d\mu}{\mu}} F(s) ds \leq \eta e^{\int_0^{t_n} h(\mu) \frac{d\mu}{\mu}} \leq e^G \).

As the sequence \( \{ \int_0^{t_n} e^{-\int_0^r h(\mu) \frac{d\mu}{\mu}} F(s) ds \} \) is bounded, there exists a convergent subsequence. We may
assume $\int_0^\infty e^{\int^{d\mu}_{h\mu}} F(s) ds = \lambda$ for a positive constant $\lambda$. Then we choose a positive integer $k$ so large such that
\[
\lim_{n \to \infty} \int_0^\infty e^{-\int^{d\mu}_{h\mu}} F(s) ds \leq \frac{\delta_0}{8L}
\]
for all $n \geq k$, where $L = \sup_{t \geq 0} e^{-\int^{d\mu}_{h\mu}}, \delta_0 > 0$ satisfies $8\delta_0 Le^G + \eta < 1$.

We think over the zero solution $x(t) = x(t, t_i, \phi)$ of system (1) with $E|\phi(t_i)|^2 = \delta_0$ and $E|\phi(s)|^2 \leq \delta_0$ for every $s \leq t_i$. We know $E|x(t)|^2 \leq 1$ for every $t \geq t_i$. We can select $\phi$ such that $G(t_i) := \phi(t_i) \geq \frac{1}{2} \delta_0$. It follows from (3) and $x(t) = (\Psi x)(t)$ that for all $t \geq t_i$,
\[
E|x(t_n)|^2 \geq G(t_i) e^{-\int^{d\mu}_{h\mu}} \left\{ G(t_i) e^{-\int^{d\mu}_{h\mu}} - 2 \int_{t_i}^t e^{-\int^{d\mu}_{h\mu}} F(s) ds \right\}
\geq \frac{1}{2} \delta_0 e^{-\int^{d\mu}_{h\mu}} \left\{ \frac{1}{2} \delta_0 e^{-\int^{d\mu}_{h\mu}} - 2 \int_{t_i}^t e^{-\int^{d\mu}_{h\mu}} F(s) ds \right\}
\geq \frac{1}{2} \delta_0 e^{-\int^{d\mu}_{h\mu}} \left\{ \frac{1}{2} \delta_0 - 2K \int_{t_i}^t e^{-\int^{d\mu}_{h\mu}} F(s) ds \right\} \geq \frac{1}{8} \delta_0 e^{-2J} > 0
\]

However, if the zero solution of system (1) is mean square asymptotic stable, $E|x(t)|^2 = E|x(t, t_i, \phi)|^2 \to 0$. So we have $E|x(t_n)|^2 \to 0$ as $n \to \infty$. Which contradicts (10).

Hence the condition (iii) is necessary for the mean square asymptotic stability of the zero solution of (1). This completes the proof.

Remark 1: If we choose $h(s) \equiv a(s)$, the result of Theorem 1 is similar to the Theorem A. But, sometimes, if we define $h(s) \equiv a(s)$, the mean square asymptotic stability can not be obtained.

3. Examples

Example 1 Consider the following stochastic Volterra dynamical system
\[
dx(t) = -2x(t)dt + \int_{\frac{3}{2}}^{x(t)} e^{-\int^{d\mu}_{h\mu}} (x(s) + 5) ds + x(t) d\omega(t), \quad t \geq 0
\]
If we select $h(t) \equiv 2 > 0$ in Theorem 1, the zero solution of (10) is mean square asymptotic stability. The following example shows us that sometimes the results of this paper enjoy priority over Burton[1].

Example 2 Consider the following stochastic Volterra dynamical system
\[
dx(t) = -0.5x(t)dt - \int_{\frac{3}{2}}^{x(t)} e^{-\int^{d\mu}_{h\mu}} x(s) ds + x(t) d\omega(t), \quad t \geq 0
\]
If we select $h(t) \equiv t$ in Theorem 1, the zero solution of system (11) is mean square asymptotic stable provided that:
\[
\sup_{t \geq 0} e^{\frac{3}{2}} \left( \int_0^t (s + e^{-\frac{3}{2}} - 1.5) e^{\int_0^s ds} + c \right) \left( \int_0^t e^{\int_0^s ds} \right) < 1.
\]

Remark 2 But for $a(s) = \sin s$ in (12), so $e^{\int_0^\infty \sin ds} = e^{\cos 1} \leq 1$ can not be satisfy to the Theorem.
A in paper [1], and the stability can not be judged in paper [1].

**Remark 3** In this paper, we have studied the stability of stochastic Volterra dynamical systems of nonconvolution type by means of fixed point method. It enjoys priority over the results of Lyapunov theories. We introduced a new function in the study of stability by using fixed point theory, which makes the stability conditions be more feasible.

4. Conclusions

In this paper, we use the Banach fixed point method to discuss the mean square asymptotic stability of zero solution for a class of stochastic dynamical systems. The innovation points of this paper are as follows:

(1) this paper studies the complicated nonlinear stochastic integro-differential dynamical systems, the theory of the fixed point method in the study of complex nonlinear stochastic integro-differential dynamical systems is extended.

(2) the theorem obtained by the method of innovation in this paper improves and generalizes the relevant conclusions in reference [1].

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