Finding the set of global minimizers of a piecewise affine function

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Abstract
Coexhausters are families of convex compact sets that allow one to represent the approximation of the increment of a function at a given point in the form of minmax or maxmin of affine functions. We demonstrate that this representation can be used to define a piecewise affine function and therefore coexhausters are a natural technique for studying the problem of finding a global minimum of piecewise affine functions. All the conditions and methods in the current study were obtained by means of coexhausters theory. Firstly, we apply coexhauster based conditions to state and prove necessary and sufficient conditions for a piecewise affine function to be bounded from below. Secondly, we use coexhausters to construct a simple method which allows one to get the minimum value of the studied function and the corresponding set of all its global minimizers. Illustrative numerical examples are provided throughout the paper.

Keywords Global minimum · Piecewise affine function · Coexhausters · Optimality conditions · Optimization

1 Introduction
Piecewise affine functions play important roles for different areas of mathematics and applications. For example they are used in DC programming [1], optimal control [2], global optimization [3, 4], approximation problems [5, 6] and so on. We will consider piecewise affine functions from the coexhausters theory point of view. Coexhausters are an effective tool for the study of nondifferentiable functions and are important objects of the constructive nonsmooth analysis [7].

Local behavior of any smooth function at a given point is provided by the linear function which is described by means of the gradient. Therefore optimality conditions for the smooth functions as well as techniques for building descent and ascent directions can be expressed...
via gradient. Obviously one needs a different form of approximation of the increment in the nonsmooth case. One of the ways is to consider this approximation in the form of minmax or maxmin of affine functions. Coexhausters are families of convex compact sets that provide this type of representation for the approximation of the increment. This notion was proposed by Demyanov in [8, 9]. The well-developed calculus of coexhausters enables one to build these families for a wide class of functions. Optimality conditions in terms of coexhausters as well as techniques for obtaining descent and ascent directions, in cases when these optimality conditions are not satisfied, were derived in [10–12]. This led to the emergence of the coexhausters based optimization algorithms.

Thus piecewise affine functions are naturally built in the coexhausters theory. Therefore, it is not surprising that this theory is a suitable tool to study for such functions.

The paper is organized as follows. In Sect. 1 we discuss the notion of coexhausters and codifferentials as well as the definition of piecewise affine function. In Sect. 2 we describe optimality conditions in terms of coexhausters and use these results to derive the condition which is equivalent to the lower boundedness of the studied function. In Sect. 3 we describe a simple technique for finding the minimum value of the function. The method for obtaining the set of all global minimizers is developed in Sect. 4. Throughout the paper we provide numerical examples and illustrations.

2 Coexhausters and piecewise affine functions

2.1 Codifferentiable functions

Let $X \subset \mathbb{R}^n$ be an open set. We say that a function $f : X \rightarrow \mathbb{R}$ is codifferentiable at a point $x$, if and only if there exist convex compact sets $d f (x) \subset \mathbb{R}^{n+1}$ and $d f (x) \subset \mathbb{R}^{n+1}$ such that

$$f(x + \Delta) = f(x) + \max_{[a,v] \in d f (x)} [a + \langle v, \Delta \rangle] + \min_{[b,w] \in d f (x)} [b + \langle w, \Delta \rangle] + o_x(\Delta),$$  \hspace{1cm} (1)

where

$$\lim_{\alpha \downarrow 0} \frac{o_x(\alpha \Delta)}{\alpha} = 0, \quad \forall \Delta \in \mathbb{R}^n.$$  \hspace{1cm} (2)

Here $a, b \in \mathbb{R}$; $v, w \in \mathbb{R}^n$.

The pair $D f (x) = [d f (x), d f (x)]$ is called a codifferential of the function $f$ at the point $x$. A codifferential is a pair of sets in the space $\mathbb{R}^{n+1}$.

The expansion (1) implies

$$f(x + \Delta) = f(x) + \max_{[a,v] \in d f (x)} \min_{[b,w] \in d f (x)} [a + b + \langle v + w, \Delta \rangle] + o_x(\Delta)$$

$$= f(x) + \max_{C \in E(x)} \min_{[b,w] \in C} [b + \langle w, \Delta \rangle] + o_x(\Delta),$$  \hspace{1cm} (3)

where

$$E(x) = \{ C \subset \mathbb{R}^{n+1} | C = [a, v] + d f (x), \ [a, v] \in d f (x) \}.$$  

Analogously one gets

$$f(x + \Delta) = f(x) + \min_{[b,w] \in d f (x)} \max_{[a,v] \in d f (x)} [a + b + \langle v + w, \Delta \rangle] + o_x(\Delta)$$

$$= f(x) + \min_{C \in E(x)} \max_{[a,v] \in C} [a + \langle v, \Delta \rangle] + o_x(\Delta),$$  \hspace{1cm} (4)
where 
\[ \overline{E}(x) = \{ C \subset \mathbb{R}^{n+1} | C = [b, w] + \overline{d}f(x), [b, w] \in \overline{d}f(x) \}. \]

The functions
\[ h_1(x, \Delta) = \max_{C \in \overline{E}(x)} \min_{[b, w] \in C} [b + \langle w, \Delta \rangle], \]
\[ h_2(x, \Delta) = \min_{C \in \overline{E}(x)} \max_{[a, v] \in C} [a + \langle v, \Delta \rangle], \]
represent approximations of the increment of the function \( f \) in the neighbourhood of \( x \).

The notion of codifferential was introduced in [13] where necessary optimality conditions as well as codifferential calculus formulas were derived. The expansions (3) and (4) give way to the following generalization of the notion of codifferential.

### 2.2 Coexhausters

Let a function \( f : X \to \mathbb{R} \) be continuous at a point \( x \in X \). We say that at the point \( x \) the function \( f \) has an upper coexhauster if and only if the following expansion holds for all \( \Delta \in \mathbb{R}^n \setminus \{0_n\} \):
\[ f(x + \Delta) = f(x) + \min_{C \in \overline{E}(x)} \max_{[a, v] \in C} [a + \langle v, \Delta \rangle] + o_x(\Delta), \tag{5} \]
where \( \overline{E}(x) \) is a family of convex compact sets in \( \mathbb{R}^{n+1} \), and \( o_x(\Delta) \) satisfies (2).

The set \( \overline{E}(x) \) is called an upper coexhauster of \( f \) at the point \( x \).

We say that at the point \( x \) the function \( f \) has a lower coexhauster if and only if the following expansion holds for all \( \Delta \in \mathbb{R}^n \setminus \{0_n\} \):
\[ f(x + \Delta) = f(x) + \max_{C \in \overline{E}(x)} \min_{[b, w] \in C} [b + \langle w, \Delta \rangle] + o_x(\Delta), \tag{6} \]
where \( \overline{E}(x) \) is a family of convex compact sets in \( \mathbb{R}^{n+1} \), and \( o_x(\Delta) \) satisfies (2).

The set \( \overline{E}(x) \) is called a lower coexhauster of the function \( f \) at the point \( x \).

In what follows we assume that the families \( \overline{E}(x) \) and \( \overline{E}(x) \) are totally bounded, then from the continuity of the function \( f \) and expansions (5) and (6) it follows that
\[ \min_{C \in \overline{E}(x)} \max_{[a, v] \in C} a = \max_{C \in \overline{E}(x)} \min_{[b, w] \in C} b = 0. \tag{7} \]

The notion of coexhauster was introduced in [8, 9]. The calculus of coexhausters that allows one to built these families for a wide class of nonsmooth functions as well as optimality conditions in terms of these objects were developed in [10–12].

### 2.3 Piecewise affine functions

Recall that if a set can be represented as the intersection of a finite family of closed halfspaces it is called polyhedral set [14]. A function \( \phi : \mathbb{R}^n \to \mathbb{R} \) of the form \( \phi(\Delta) = a + \langle v, \Delta \rangle \), where \( a \in \mathbb{R}, v \in \mathbb{R}^n \) is called affine.

Now we can give a definition of a piecewise affine function (see [15–17]).

**Definition 1** Let \( \{M_j \mid j \in J\} \) and \( \{\phi_j(x) \mid j \in J\} \) be finite sets of polyhedral sets and affine functions respectively, where \( J = \{1, \ldots, k\} \). We say that the function \( \Phi(x) \) is a piecewise affine on \( \mathbb{R}^n \) if the following conditions hold:
1. \( \text{int} M_j \neq \emptyset, \forall j \in J \),
2. \( \bigcup_{j \in J} M_j = \mathbb{R}^n \),
3. \( \text{int} M_j \cap \text{int} M_i = \emptyset, i \neq j \),
4. \( \Phi(x) = \phi_j(x), \forall x \in M_j, \forall j \in J \),

where \( \text{int} M \) means the interior of the set \( M \).

As it was shown in [16] the function \( \Phi(x) \) is piecewise affine if and only if it can be presented in any of the following forms

\[
\Phi(x) = \min_{1 \leq i \leq k} \max_{1 \leq j \leq m_i} a_{ij}(x),
\]

or

\[
\Phi(x) = \max_{1 \leq i \leq k} \min_{1 \leq j \leq m_i} b_{ij}(x),
\]

where \( a_{ij} \) and \( b_{ij} \) are affine functions for any \( j = 1, \ldots, m_i \) and \( i = 1, \ldots, k \). Obviously, these representations are equivalent to the expressions for the main part of the increment in (5) and (6) when coexhausters consist of finite numbers of sets which are convex hulls of finite number of points. Therefore coexhausters provide a piecewise affine approximation for the increment of the studied function at the considered point.

### 3 Bounded below piecewise affine function

First we describe optimality conditions in terms of coexhausters. The advantage of the optimality conditions of this form is disclosed subsequently, as all the proposed results and methods are based on these conditions. Since we consider the problem of finding a global minimum, it is natural to deal with functions which are bounded below. In this section we derive the condition which is equivalent to the lower boundedness.

Optimality conditions for the minimum of a piecewise affine function most organically are stated in terms of upper coexhauster. Therefore we deal with the representation of a piecewise affine function of the form

\[
h(x) = \min_{C \in \mathcal{E}} \max_{[a, v] \in C} [a + \langle v, x \rangle],
\]

where \( \mathcal{E} \) is the family that consists of a finite number of convex polytopes. Let \( h(0_n) = 0 \). This equation holds when \( h \) is the coexhauster approximation of the increment of the studied function at the considered point (see (7)). The following theorem gives us an optimality condition for the global minimum of the function \( h \) at the origin.

**Theorem 1** For the condition \( h(x) \geq 0 \) to hold, for any \( x \in \mathbb{R}^n \), it is necessary and sufficient that

\[
C \cap L_0^+ \neq \emptyset, \quad \forall C \in \mathcal{E},
\]

where \( L_0^+ = \{ [a, 0_n] \in \mathbb{R}^{n+1} \mid a \geq 0 \} \).

This theorem was derived in [11]. Based on this result we can formulate necessary and sufficient conditions for the function \( h \) to be bounded from below.
**Theorem 2** A function \( h(x) \) is bounded from below if and only if the following condition holds
\[
C \bigcap L_0 \neq \emptyset, \quad \forall C \in \overline{E},
\] (8)
where \( L_0 = \{[a, 0_n] \in \mathbb{R}^{n+1}\} \).

**Proof** Let us start from necessity. Suppose that there exists \( M \) such that \( h(x) - M \geq 0 \) for all \( x \in \mathbb{R}^n \). This means validity of the inequality
\[
\min_{C \in \overline{E}} \max_{[a, v] \in C} [a + \langle v, x \rangle] \geq 0,
\]
where
\[
\widetilde{E} = \{ \tilde{C} = C - [M, 0_n] | C \in E \}, \quad C - [M, 0_n] = \{[a - M, v] | [a, v] \in C\}.
\]
Theorem 1 implies that
\[
\tilde{C} \bigcap L_0^+ \neq \emptyset, \quad \forall \tilde{C} \in \tilde{E}.
\] (9)
Choose and fix an arbitrary set \( \tilde{C} \in \tilde{E} \). Condition (9) yields that there exists \( y \in L_0^+ \) and \( z \in C \), such that \( y = z - [M, 0_n] \). Hence \( z \in L_0^+ + [M, 0_n] \subset L_0 \) wherefrom we conclude
\[
C \bigcap L_0 \neq \emptyset, \quad \forall C \in E.
\]
Now proceed to the sufficiency. Suppose that the condition \( C \bigcap L_0 \neq \emptyset \) is true for all \( C \in E \). Choose an arbitrary set \( \hat{C} \) from \( E \) and denote \( \hat{a} = \max_{[a, 0_n] \in \hat{C}} a \) and \( a_* = \min_{C \in E} \max_{[a, 0_n] \in C} a \).
The inequality
\[
a_* = \min_{C \in E} \max_{[a, 0_n] \in C} a \leq \max_{[a, 0_n] \in \hat{C}} a = \hat{a},
\]
holds, therefore \( [\hat{a} - a_*, 0_n] \in L_0^+ \), whence it is obvious that
\[
(\hat{C} - [a_*, 0_n]) \bigcap L_0^+ \neq \emptyset.
\] (10)
Since \( \hat{C} \) was chosen arbitrarily, condition (10) is valid for all \( C \in E \)
\[
(C - [a_*, 0_n]) \bigcap L_0^+ \neq \emptyset, \quad \forall C \in E.
\]
Theorem 1 implies the inequality
\[
\min_{C \in E} \max_{[a, v] \in C} [a - a_* + \langle v, x \rangle] \geq 0, \quad \forall x \in \mathbb{R}^n,
\]
which yields that \( h(x) \geq a_* \) for all \( x \in \mathbb{R}^n \).

Note that a similar result was obtained in terms of codifferentials in [18]. The following example demonstrates how Theorem 2 works.

**Example 1** Let the function \( h_1 : \mathbb{R} \to \mathbb{R} \) be given by
\[
h_1(x) = \min \left\{ \max \left\{ 2 + x, 1 + \frac{1}{2} x \right\}, \max \{-2 + x, -x\} \right\}.
\]
The family \( \overline{E} = \{C_1, C_2\} \), where
\[
C_1 = \text{conv} \left\{ [2, 1], \left[ 1, \frac{1}{2} \right] \right\}, \quad C_2 = \text{conv} \{[-2, 1], [0, -1]\},
\]
where \( \text{conv} \) denotes the convex hull.
Fig. 1 The upper coexhauster and the graph of the function $h_1$

is an upper coexhauster of the function $h_1$. Since $C_1 \cap L_0 = \emptyset$ (see Fig. 1a) condition (8) is not fulfilled here. Thus $h_1$ is not bounded below (see Fig. 1b).

4 Finding the minimum value of a piecewise affine function

Based on Theorems 1 and 2, we can derive a technique of finding the minimum value of a piecewise affine function.

**Theorem 3** Let a piecewise affine function $h: \mathbb{R}^n \to \mathbb{R}$ be given by

$$h(x) = \min_{C \in \mathcal{E}} \max_{[a,v] \in C} \{a + \langle v, x \rangle\},$$

where $\mathcal{E}$ is a finite family of convex polytopes. If there exists $C \in \mathcal{E}$ such that $C \cap L_0 = \emptyset$, then $h$ is not bounded from below on $\mathbb{R}^n$. Otherwise

$$\min_{x \in \mathbb{R}^n} h(x) = a_* = \min_{C \in \mathcal{E}} \max_{[a,0_n] \in C} a,$$

i.e., $a_*$ is the minimum value of the function $h$ on $\mathbb{R}^n$.

**Proof** First part of the theorem is a straight corollary of Theorem 2. Let us show that $a_*$ is the minimum value of the function $h$ on $\mathbb{R}^n$. We will consider the case $a_* \geq 0$ as the opposite case can be treated similarly. Let $C_*$ be a set in $\mathcal{E}$ such that

$$a_* = \min_{C \in \mathcal{E}} \max_{[a,0_n] \in C} a = \max_{[a,0_n] \in C_*} a.$$

Repeating the same arguments as in the proof of sufficiency of Theorem 2, we can obtain the following chain of inequalities

$$h(x) \geq h(x_*) \geq a_*,$$

where $x_*$ is a minimum point of $h$. To show that $h(x_*) = a_*$ we assume the contrary, i.e., $h(x_*) > a_*$. The piecewise affine function $h(x) - h(x_*)$ is nonnegative for any $x$ in $\mathbb{R}^n$ therefore applying Theorem 2 we get the condition
From the other side, for any element of the set \( C_\ast \in \overline{E} \) of the form \([a, 0_n]\) the inequality
\[ a - h(x_\ast) \leq a_\ast - h(x_\ast) < 0, \]
holds. Hence it is valid the condition
\[ (C_\ast - [h(x_\ast), 0]) \cap L_0^+ = \emptyset, \]
which contradicts (11).

To find a minimum of a piecewise affine functions, one has to calculate \( \max_{[a, 0_n] \in C} a \) for any \( C \in \overline{E} \). This can be done via solving a linear programming problem. Consider an arbitrary \( C \in \overline{E} \). Since it is convex polytope it can be described in the form \( C = \text{conv} \{ [a_i, v_i] \mid i \in I_C \} \), where \( I_C \) is a finite index set. Then the solution of the following linear programming problem

\[
\begin{align*}
\max & \quad \sum_{i \in I_C} \lambda_i a_i \\
\text{s.t.} & \quad \sum_{i \in I_C} \lambda_i v_i = 0_n \\
& \quad \sum_{i \in I_C} \lambda_i = 1 \\
& \quad \lambda_i \geq 0, \ i \in I_C,
\end{align*}
\]

(12)
gives us needed value \( \max_{[a, 0_n] \in C} a \).

5 Finding a global minimizer of a piecewise affine function

Now we can identify sets from the family \( \overline{E} \) that determine the global minimizer.

Lemma 1 Let \( h: \mathbb{R}^n \to \mathbb{R} \) be a bounded below piecewise affine function
\[ h(x) = \min_{C \in \overline{E}} \max_{[a, v] \in C} \{ a + \langle v, x \rangle \} , \]
where \( \overline{E} \) is a finite family of convex polytopes. Then for any global minimizer \( x_\ast \) of the function \( h \) on \( \mathbb{R}^n \), the following condition holds
\[ h(x_\ast) = \min_{C \in \overline{E}} \max_{[a, v] \in C} \{ a + \langle v, x_\ast \rangle \} = \min_{C \in \overline{E}_\ast} \max_{[a, v] \in C} \{ a + \langle v, x_\ast \rangle \} , \]
where \( \overline{E}_\ast = \left\{ C \in \overline{E} \mid \max_{[a, 0_n] \in C} a = a_\ast \right\} \) and \( a_\ast = \max_{C \in \overline{E}[a, 0_n] \in C} a \), i.e., global minimum can be attained on sets \( C \in \overline{E}_\ast \) only.

Proof Let the point \( x_\ast \) be a global minimizer of the function \( h \). Choose an arbitrary \( C \in \overline{E} \) such that \( C \notin \overline{E}_\ast \). We have the chain of inequalities
\[ \max_{[a, v] \in C} \{ a + \langle v, x_\ast \rangle \} \geq \max_{[a, 0_n] \in C} \{ a + \langle 0_n, x_\ast \rangle \} = \max_{[a, 0_n] \in C} a > a_\ast , \]
whence
\[ h(x_*) = a_* < \min_{C \in \mathcal{E} \setminus \mathcal{E}_s} \max_{[a, v] \in C} [a + \langle v, x_* \rangle]. \]

Therefore
\[ h(x_*) = \min_{C \in \mathcal{E}_s} \max_{[a, v] \in C} [a + \langle v, x_* \rangle]. \]

Let us proceed to define the set of global minimizers of the function \( h \). Consider an arbitrary \( C_* \in \mathcal{E}_s, C_* = \text{conv} \{ [a_i, v_i] \mid i \in I_{C_*} \} \), where \( I_{C_*} \) is a finite index set. Build the set
\[ X(C_*) = \{ x \in \mathbb{R}^n \mid | a_i + \langle v_i, x \rangle \leq a_*, \forall i \in I_{C_*} \}. \]

For any point \( x \in X(C_*) \) we have
\[ a_* \leq h(x) = \min_{C \in \mathcal{E}_s} \max_{[a, v] \in C} [a + \langle v, x \rangle] \leq \max_{[a, v] \in C_*} [a + \langle v, x \rangle] \leq a_* , \]
whence \( h(x) = a_* \), i.e., \( x \) is a global minimizer of the function \( h \).

So \( X = \bigcup_{C \in \mathcal{E}_s} X(C) \) is a set of global minimizers of the function \( h \).

Let us show that any global minimizer of the function \( h \) belongs to the set \( X \). Indeed, assume that for some point \( \tilde{x} \in \mathbb{R}^n \) the equality \( h(\tilde{x}) = a_* \) is true. Then Lemma 1 implies that there exists \( C_* \in \mathcal{E}_s \) such that
\[ a_* = \min_{C \in \mathcal{E}_s} \max_{[a, v] \in C} [a + \langle v, \tilde{x} \rangle] = \min_{C \in \mathcal{E}_s} \max_{[a, v] \in C} [a + \langle v, \tilde{x} \rangle] = \max_{[a, v] \in C_*} [a + \langle v, \tilde{x} \rangle], \]
which means that \( \tilde{x} \in X(C_*) \).

By these reasonings we proved the following result.

**Theorem 4** Let \( h : \mathbb{R}^n \to \mathbb{R} \) be a bounded below piecewise affine function
\[ h(x) = \min_{C \in \mathcal{E}} [a + \langle v, x \rangle], \]
where \( \mathcal{E} \) is a finite family of convex polytopes. Then for a point \( x_* \) to be a global minimizer of the function \( h \) on \( \mathbb{R}^n \) it is necessary and sufficient that \( x_* \in X = \bigcup_{C \in \mathcal{E}_s} X(C) \).

Theorem 4 gives important corollaries.

**Corollary 1** Let for some \( C_* \in \mathcal{E}_s, C_* = \text{conv} \{ [a_i, v_i] \mid i \in I_{C_*} \} \) the inequalities \( a_i \leq a_* \) are valid for any \( [a_i, v_i] \in I_{C_*} \). Then \( 0_n \in X, i.e., 0_n \) is a global minimizer of the function \( h \) on \( \mathbb{R}^n \).

**Proof** Since
\[ a_* \leq h(0_n) = \min_{C \in \mathcal{E}_s} \max_{[a, v] \in C} a \leq \max_{[a, v] \in C_*} a \leq a_* , \]
we have \( h(0_n) = a_* \). Hence \( 0_n \) is a global minimizer of \( h \). \( \square \)
Corollary 2 The function $h$ has a unique global minimizer $x_*$ if and only if the set $X = \{x_*\}$ is a singleton, i.e., for any $C \in \mathcal{E}_*$ where $C = \text{conv}\{[a_i, v_i] \mid i \in I_C\}$, the system of inequalities

$$a_i + \langle v_i, x \rangle \leq a_* \quad i \in I_C,$$

is trivial in the sense that the point $x_*$ is its only solution.

If our aim is to find one arbitrary global minimizer of $h$, we can consider any $C_* \in \mathcal{E}_*$, $C_* = \text{conv}\{[a_i, v_i] \mid i \in I_{C_*}\}$ and choose at least one point satisfying the system of inequalities

$$a_i + \langle v_i, x \rangle \leq a_* \quad i \in I_{C_*}.$$

This problem can be reduced to LP (linear programming) problem or to unconstrained minimization of the following continuously differentiable function

$$\sum_{i \in I_{C_*}} (\max\{0, a_i - a_* + \langle v_i, x \rangle\})^2,$$

which its gradient obviously equals

$$\sum_{i \in I_{C_*}} 2v_i \max\{0, a_i - a_* + \langle v_i, x \rangle\}.$$

Consider some examples to demonstrate obtained results. The following one shows that the developed approach omits local minimums and bring us to the global solution.

Example 2 Let the function $h_2: \mathbb{R} \rightarrow \mathbb{R}$ be given by

$$h_2(x) = \min \left\{ \max\{-9 + 2x, 9 - 4x\}, \max\{4 + x, -2 - \frac{1}{2}x\} \right\}.$$

The family $\mathcal{E} = \{C_1, C_2\}$, where

$$C_1 = \text{conv}\{[-9, 2], [9, -4]\}, \quad C_2 = \text{conv}\{[4, 1], \left[-2, -\frac{1}{2}\right]\},$$

is an upper coexhauster of the function $h_2$. Since $C_i \cap L_0 \neq \emptyset$ for all $i = 1, 2$, (see Fig. 2a) condition (8) is fulfilled here. Thus $h_2$ is bounded below (see Fig. 2b).

It is obvious that $a_* = -3$ and $\mathcal{E}_* = \{C_1\}$ here. To find a set of all global minimizers we have to solve the system of inequalities

$$\begin{cases} -9 + 2x \leq -3 \\ 9 - 4x \leq -3, \end{cases}$$

which is trivial since point $x_* = 3$ is the only solution of the system. Hence the conditions of Corollary 2 are satisfied and the function $h_2$ has only one global minimizer $x_*$.
which has solution \((x^*, y^*) = (3, -3)\). Or we could construct and minimize the function (13)

\[
(\max\{0, 12 - 4x\})^2 + (\max\{0, -6 + 2x\})^2.
\]

The gradient of this function

\[
-8 \max\{0, 12 - 4x\} + 4 \max\{0, -6 + 2x\},
\]

equals zero at the point \(x^*\).

Below is the example from [19], where quadratic programming is employed to solve the problem. Our results permits to find the solution easier.

**Example 3** (see [19]) Consider the function \(h_3: \mathbb{R}^2 \rightarrow \mathbb{R}\) which is given by

\[
h_3(x) = \min\{\max\{-x_1, x_1, -x_2, x_2\}, \\
\quad \max\{-3 + 2x_1, 5 - 2x_1, -1 + x_2, 3 - x_2\}\}.
\]

The family \(\overline{E} = \{C_1, C_2\}\), where

\[
C_1 = \text{conv} \{(0, -1, 0), [0, 1, 0], [0, 0, -1], [0, 0, 1]\}
\]

\[
C_2 = \text{conv} \{[-3, 2, 0], [5, -2, 0], [-1, 0, 1], [3, 0, -1]\}.
\]

is an upper coexhauster of the function \(h_3\). Omitting calculations for the problems (12) for each of sets \(C_1\) and \(C_2\), we write final results

\[
\overline{E}_* = \{C_1\}, \quad a_* = 0.
\]
As we see the condition from Corollary 1
\[ a \leq 0, \quad \forall [a, v] \in C_1, \]
is fulfilled here, whence \( x_a = 0_n \) is a global minimizer of \( h_3 \). Moreover, since the system of inequalities
\[
\begin{cases}
-x_1 \leq 0 \\
x_1 \leq 0 \\
-x_2 \leq 0 \\
x_2 \leq 0,
\end{cases}
\]
is trivial this point is unique global minimizer of \( h_3 \), i.e., \( X = \{x_a\} \).

Next example demonstrates that the developed method allows us to describe the whole set of global minimizers even in cases when this set is infinite.

**Example 4** Let the function \( h_4 : \mathbb{R} \to \mathbb{R} \) be given by
\[
h_4(x) = \min \{ \max \{-4 - x, 0, 2 + x\}, \max \{2 - 2x, 0, -9 + 3x\} \}.
\]
The family \( \mathcal{E} = \{C_1, C_2\} \), where
\[
C_1 = \text{conv} \{[-4, -1], [0, 0], [2, 1]\}, \quad C_2 = \text{conv} \{[2, -2], [0, 0], [-9, 3]\},
\]
is an upper coexhauster of the function \( h_4 \).

As we see from Fig. 3a the function \( h_4 \) is lower bounded \( a_\ast = 0 \) and \( \mathcal{E}_\ast = \{C_1, C_2\} \). The set \( X(C_1) \) is defined by the system of inequalities
\[
\begin{cases}
-4 - x \leq 0 \\
2 + x \leq 0,
\end{cases}
\]
while the system
\[
\begin{cases}
2 - 2x \leq 0 \\
-9 + 3x \leq 0,
\end{cases}
\]
defines the set \( X(C_2) \). Consequently
\[
X = X(C_1) \bigcup X(C_2) = \{ x \in \mathbb{R} \mid -4 \leq x \leq -2 \} \bigcup \{ x \in \mathbb{R} \mid -9 \leq x \leq 3 \}.
\]
This corresponds to what we see on Fig. 3b.

The following example shows the advantages of the proposed method for some high-dimensional problems.

**Example 5** Consider the function \( h_5 : \mathbb{R}^{106} \to \mathbb{R} \) which is given by
\[
h_5(x) = \min \{ \max \{1 + \langle 2w, x \rangle, -1 + \langle -2w, x \rangle\}, \max \{3, 5 + \langle 2w, x \rangle, 5 + \langle -2w, x \rangle\} \},
\]
where \( w = 1 = (1, 1, \ldots, 1) \in \mathbb{R}^{106} \). The family \( \mathcal{E} = \{C_1, C_2\} \), where
\[
C_1 = \text{conv} \{[1, 2w], [-1, -2w]\}, \quad C_2 = \text{conv} \{[3, 0], [5, 2w], [5, -2w]\},
\]
is an upper coexhauster of the function \( h_5 \).
Theorem 2 implies that the function $h_5$ is bounded below. It is obvious that

$$\max_{[a,0_n] \in C_1} a = 0, \quad \max_{[a,0_n] \in C_2} a = 5.$$  

Hence $a^* = 0$, $E^* = \{C_1\}$. Consequently $X(C_1)$ is given by the system of inequalities

$$\begin{cases} 
1 + \langle 2w, x \rangle \leq 0 \\
-1 - \langle 2w, x \rangle \leq 0,
\end{cases}$$

whence $X = \left\{ x \in \mathbb{R}^{106} \mid \langle w, x \rangle = -\frac{1}{2} \right\}$. Based on this result we can take the point $x = -\frac{10^{-6}}{2} w$ as a global minimizer.

### 6 Conclusion

Obtained results allow one to get the minimum value of a piecewise affine function. In comparison with the algorithm in [19] this can be done by solving $m$ linear programming problems (12) instead of solving $m$ quadratic programming problems, where $m$ is a number of sets in an upper coexhauster of the studied function.

Note that for a set $C \in E$ where $C = \text{conv} \{ [a_i, v_i] \mid i \in I_C \}$, the problem of this kind is $|I_C|$-dimensional. So the minimum value can be obtained easily even for high-dimensional studied function, in cases when $|I_C|$ for any $C \in E$ are small (see Example 5). These advantages are due to the fact that instead of the function itself we deal with the geometric object that define it. This gives us a simple and transparent interpretation, which, in some cases, makes it significantly easy to solve the problem.

We can construct the set of all global minimizers of the studied piecewise affine function via its minimum value. In cases when an arbitrary global minimizer is needed we additionally solve a linear programming problem or the unconstrained optimization problem for the function (13).

The proposed approach is quite simple and can be applied in different branches of mathematics and various applications.
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Declarations

Data Availability  Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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