Nonessential Functionals in Multiobjective Optimal Control Problems

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Abstract

We address the problem of obtaining well-defined criteria for multiobjective optimal control systems. Necessary and sufficient conditions for an optimal control functional to be nonessential are proved. The results provide effective tools for determining nonessential objectives in vector-valued optimal control problems.

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1 Introduction

Multiobjective optimal control attracts more and more attention and is source of strong current research (see e.g. [2, 7, 18] and references therein). We consider multiobjective problems of optimal control governed by ordinary differential dynamical systems. This comprises an important class of problems which naturally appear on practical applications to Economic [13, Chap. 8] and Engineering modelling [17]. Our main goal is to extend the results found in the literature on nonessential functions of mathematical static optimization programming [5, 12] to functionals of optimal control theory.

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It is well known that the concept of Pareto optimality or efficiency play a crucial role on optimal control [8, 17]. The question of obtaining well-defined criteria for multiple criteria decision making problems seems, however, being considered in the literature only for static multiobjective optimization problems (cf. [5, 12] and references therein). In this work we investigate the problem of obtaining well-defined criteria for multicriteria optimal control dynamical systems.

One of the approaches dealing with the problem of obtaining well-defined criteria for multiple criteria static decision making problems is the concept of nonessential objective functions. A certain objective function is called nonessential if it does not influence the set of efficient solutions of the vector-valued optimization problem, that is, the set of efficient solutions is the same both with or without that objective function. Information about nonessential objectives helps a decision maker to know and to understand better the problem and this might be a good starting point for further investigation or revision of the mathematical model. Dropping nonessential functions leads to a problem with a smaller number of objectives, which can be solved more easily. For this reason, the issue of nonessential objectives is a substantial feature for multiple criteria decision making [5, 6, 11]. To the best of the authors knowledge, no study has been done in this field for optimal control problems. We are interested in generalizing the previous results on nonessential objectives found in the literature to cover optimal control problems with a vector-valued functional to minimize. More precisely, we generalize the concept of nonessential objective to multicriteria functionals of optimal control systems and we give the first steps on the corresponding theory. Main results provide methods for identifying nonessential objectives in nonlinear and optimal control vector-valued optimization problems.

2 Optimal control with a vector-valued cost

We consider a dynamical control system described by \( n \) state variables \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) and \( r \) control variables \( u = (u_1, \ldots, u_r) \in \mathbb{R}^r, \ r \leq n \). Both state and control variables vary with respect to the scalar variable \( t \in \mathbb{R} \). Given a control vector function \( u : [a, b] \rightarrow \mathbb{R}^r \), the state evolution over \([a, b] \), namely \( x : [a, b] \rightarrow \mathbb{R}^n \), must satisfy the control system

\[
\dot{x}(t) = h(t, x(t), u(t)),
\]

the boundary conditions

\[
x(a) = \alpha, \quad x(b) = \beta,
\]

and \( m \) inequality constraints

\[
g_i(t, x(t), u(t)) \leq 0, \quad i = 1, \ldots, m.
\]

We would like to find a piecewise-continuous control function \( u(\cdot) \) and the corresponding state trajectory \( x(\cdot) \), satisfying (2.1), (2.2) and (2.3), which minimizes
a finite number $N$ of cost functionals, called the optimal control multiobjective criteria or an optimal control multiobjective performance-index:

$$\min \int_{a}^{b} f(t, x(t), u(t))dt = \min \left( \int_{a}^{b} f_1(t, x(t), u(t))dt, \cdots, \int_{a}^{b} f_N(t, x(t), u(t))dt \right).$$

All functions $f(t, x, u)$, $g(t, x, u)$ and $h(t, x, u)$ are assumed to be continuously differentiable with respect to $t$ and $x$ variables. To simplify notation, we write

$$I_N[x, u] = \int_{a}^{b} f(t, x(t), u(t))dt$$

and

$$I_i[x, u] = \int_{a}^{b} f_i(t, x(t), u(t))dt, \quad i = 1, \ldots, N.$$ 

In general does not exist a pair of functions $(x, u)$ that renders the minimum value to each cost functional $I_i$, $i = 1, \ldots, N$, simultaneously, and one uses the concept of Pareto-optimality. Let us denote by $S$ the set of feasible solutions, i.e. the set of all admissible functions $(x, u)$. The multiobjective control problem consists to find all feasible solutions that are efficient in the sense of Definition 2.1. This problem is denoted in the sequel by $(P)$. We remark that many practical applications that appear in engineering and economics can be written in the form of problem $(P)$ [17].

**Definition 2.1 (Pareto-optimality).** A pair of functions $(\tilde{x}, \tilde{u}) \in S$ is said to be an efficient (Pareto-optimal) solution of the problem $(P)$ if, and only if, there exists no $(x, u) \in S$ such that $I_N[x, u] \leq I_N[\tilde{x}, \tilde{u}]$, where

$$I_N[x, u] \leq I_N[\tilde{x}, \tilde{u}] \Leftrightarrow \forall i \in \{1, \ldots, N\} : I_i[x, u] \leq I_i[\tilde{x}, \tilde{u}] \land \exists j \in \{1, \ldots, N\} : I_j[x, u] < I_j[\tilde{x}, \tilde{u}].$$

The set of efficient solutions of $(P)$ is denoted by $S^N_P$.

The central result in optimal control theory is given by the celebrated Pontryagin maximum principle [15], which is a necessary optimality condition. A version of the Pontryagin maximum principle for Pareto-solutions of control systems with multiple criteria was proved already in the sixties [3]. Roughly speaking, one can say that the necessary and sufficient conditions for Pareto-optimality are obtained converting the vector performance optimal control problem to a family of scalar-index optimal control problems by forming an auxiliary scalar integral functional as a function of the vector-index and a vector of weighting parameters [9, 16]. For a gentle introduction to optimal control, including necessary and sufficient conditions and the question of existence, we refer the readers to [10, 14] (scalar case) and [8, 17] (Pareto optimal control). Here we just recall three basic lemmas (cf. [8, Chap. 17]) that relate the Pareto-solution of a multiobjective control problem with the solutions of an appropriate scalar-valued cost problem.
Lemma 2.2. If the feasible pair \((\tilde{x}, \tilde{u}) \in S\) is efficient for \((P)\), then it is optimal for the scalar-valued cost

\[ I_i(x, u), \quad i \in \{1, \ldots, N\} \]

subject to the constraints \((x, u) \in S\) and

\[ I_j(x, u) - I_j(\tilde{x}, \tilde{u}) \leq 0, \quad j = 1, \ldots, N \text{ and } j \neq i. \]

Lemma 2.2 is very useful because it implies that the necessary conditions [4, 15] are also necessary for Pareto-optimality in the optimal control problem with a vector-valued cost. As with the necessary conditions, next two lemmas reduce the sufficient conditions for Pareto-optimality to sufficient conditions with a scalar-valued cost functional.

Lemma 2.3. A feasible pair \((\tilde{x}, \tilde{u}) \in S\) is efficient for \((P)\) if there exists a constant \(\gamma \in \mathbb{R}^N\), with \(\gamma_i > 0\) for \(i = 1, \ldots, N\) and \(\sum_{i=1}^{N} \gamma_i = 1\), such that

\[ \sum_{i=1}^{N} \gamma_i I_i(x, u) \geq \sum_{i=1}^{N} \gamma_i I_i(\tilde{x}, \tilde{u}) \]

for every \((x, u) \in S\).

Unlike Lemma 2.3, not all components of \(\gamma\) in the next Lemma 2.4 need to be nonzero. However, in Lemma 2.4 the minimum of \(\sum_{i=1}^{N} \gamma_i I_i(x, u)\) must be achieved by a unique \((\tilde{x}, \tilde{u}) \in S\).

Lemma 2.4. A feasible pair \((\tilde{x}, \tilde{u}) \in S\) is efficient for \((P)\) if there exists a constant \(\gamma \in \mathbb{R}^N\), with \(\gamma_i \geq 0\) for \(i = 1, \ldots, N\) and \(\sum_{i=1}^{N} \gamma_i = 1\), such that

\[ \sum_{i=1}^{N} \gamma_i I_i(x, u) > \sum_{i=1}^{N} \gamma_i I_i(\tilde{x}, \tilde{u}) \]

for every \((x, u) \in S, (x, u) \neq (\tilde{x}, \tilde{u})\).

Together with the Pontryagin maximum principle [4, 15], Lemmas 2.2, 2.3 and 2.4 provide expedient tools to study concrete multiobjective problems of optimal control (cf. [4]).

3 Nonessential functionals: main results

We form a new multiobjective control problem \((\bar{P})\) from \((P)\) by adding a new functional \(I_{N+1}[x, u] = \int_{a}^{b} f_{N+1}(t, x(t), u(t))dt\) to problem \((P)\). Let \(S_E^{N+1}\) denote the set of efficient solutions of the problem \((\bar{P})\). With this notation we introduce the definition of nonessential functional.
**Definition 3.1.** The functional $I_{N+1}$ is said to be nonessential in $(\hat{P})$ if, and only if, $S_E^N = S_E^{N+1}$. A functional which is not nonessential will be called essential.

We are interested in characterizing the functionals which do not change the set of efficient solutions (nonessential objective functionals). Along the text we denote by $S_i, i = 1, 2, \ldots, N, N+1$, the set of solutions of the scalar optimal control problem

$$\min I_i[x, u]$$

subject to $S$. We start with a simple example.

**Example 3.2.** Consider a system characterized by a single state and control variable ($n = r = 1$) that evolves according to the state equation

$$\dot{x}(t) = u(t)$$

with control constraint set

$$U = \{u : [a, b] \to \mathbb{R} : |u(t)| \leq 1\}.$$  

The system is to be transferred from a given initial state $x(0) = \xi \neq 0$ to a given terminal state $x(T) = 0$ within an unspecified bounded interval $[0, T]$. Functionals to be minimized are

$$I_1 = \int_0^T dt, \quad I_2 = \int_0^T |u(t)|dt.$$  

Applying the Pontryagin maximum principle [15] we obtain:

$$S_1 = \{(x(t), u(t)) : u(t) = -\text{sgn}\{\xi}\}, \quad \min \int_0^T dt = |\xi|.$$  

and

$$S_2 = \{(x(t), u(t)) : u(t) = -\text{sgn}\{\xi\}v(t)\},$$

where

$$v(t) \in V = \{v(t) : 0 \leq v(t) \leq 1, t \in [0, T], v(t) \neq 0\}, \quad \min \int_0^T |-\text{sgn}\{\xi\}v(t)|dt = |\xi|.$$  

Details can be found in [1]. It is easy to see that $S_1 \cap S_2 = S_1$ (we can take $v(t) = 1, t \in [0, T]$). In this problem we have: $S_1 = S_E^1 = S_E^2 \subset S_2$. Hence $I_2$ is nonessential, but $I_1$ is essential (in order to see this we need only to change indices).

**Lemma 3.3.** One has $S_E^N \subset S_E^{N+1}$ if, and only if, for every $(x, u) \in S_E^N$ the following condition holds:

$$\exists (x', u') \in S : I_N[x', u'] = I_N[x, u] \Rightarrow I_{N+1}[x', u'] = I_{N+1}[x, u].$$
Proof. Let $S^N_E \subset S^{N+1}_E$ and assume, on the contrary, that exists $(\hat{x}, \hat{u}) \in S^N_E$ such that
\[ \exists (x', u') \in S : I^N[x', u'] = I^N[\hat{x}, \hat{u}] \] (3.1)
and
\[ I_{N+1}[x', u'] \neq I_{N+1}[\hat{x}, \hat{u}]. \] (3.2)
We conclude from (3.1) that $(x', u') \in S^N_E$. Therefore $(x', u')$ is not in $S^{N+1}_E$ by (3.2). This contradicts the fact that $S^N_E \subset S^{N+1}_E$. Let us prove now the second implication. If $S^N_E = \emptyset$, then $S^N_E \subset S^{N+1}_E$. Let $S^N_E \neq \emptyset$. Suppose that for every $(x, u) \in S^N_E$ holds:
\[ \exists (x', u') \in S : I^N[x', u'] = I^N[x, u] \Rightarrow I_{N+1}[x', u'] = I_{N+1}[x, u] \] (3.3)
and $S^N_E$ is not contained in $S^{N+1}_E$. In this case there exists $(\hat{x}, \hat{u}) \in S^N_E$ which is not in $S^{N+1}_E$. Hence
\[ \exists (\hat{x}, \hat{u}) \in S : I^{N+1}[\hat{x}, \hat{u}] \leq I^{N+1}[\hat{x}, \hat{u}]. \] (3.4)
This gives $I^N[\hat{x}, \hat{u}] = I^N[\hat{x}, \hat{u}]$ and from (3.3) we have $I_{N+1}[\hat{x}, \hat{u}] = I_{N+1}[\hat{x}, \hat{u}]$. Consequently $I^{N+1}[\hat{x}, \hat{u}] = I^{N+1}[\hat{x}, \hat{u}]$, contrary to (3.4).
\[ \square \]

Remark 3.4. Notice that in Example 3.2 the scalar optimal control problem
\[ \min I_1[x, u] \]
since $S$ has a unique solution. Therefore, Lemma 3.3 holds true for the example.

Definition 3.5. A function $f : \mathbb{R}^N \to \mathbb{R}$ is nondecreasing if for $y^1$ and $y^2 \in \mathbb{R}^N$: $y^1 \leq y^2$ imply $f(y^1) \leq f(y^2)$.

Theorem 3.6. If $I_{N+1}[x, u] = f(I^N[x, u])$, where $f : \mathbb{R}^N \to \mathbb{R}$, then $S^N_E \subset S^{N+1}_E$. Furthermore, $S^N_E = S^{N+1}_E$ if function $f$ is nondecreasing on the set $I^N(S)$.

Proof. Let $(x, u) \in S^N_E$. If there exists $(x', u') \in S$ such that $I^N[x', u'] = I^N[x, u]$, then $f(I^N[x', u']) = f(I^N[x, u])$ and so $I_{N+1}[x', u'] = I_{N+1}[x, u]$.
Therefore $S^N_E \subset S^{N+1}_E$ by Lemma 3.3.

We will now show the inclusion $S^{N+1}_E \subset S^N_E$. Let $(x, u) \in S$, $(x, u)$ being not an element of the set $S^N_E$. In this case there exists $(x', u') \in S$ such that $I^N[x', u'] \leq I^N[x, u]$. If $f$ is nondecreasing on $I^N(S)$, we know that $I_{N+1}[x', u'] = f(I^N[x', u']) \leq f(I^N[x, u]) = I_{N+1}[x, u]$. Hence $(x, u)$ is not an element of the set $S^{N+1}_E$.
\[ \square \]

Remark 3.7. Example 3.2 shows that sufficient condition in Theorem 3.6 for an optimal control functional to be nonessential, is not necessary.

Theorem 3.8. Let $S_{N+1} = \{(x^0, u^0)\}$. If the functional $I_{N+1}$ is nonessential, then $(x^0, u^0) \in S^N_E$. 

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Proof. Let $S_E^N = S_E^{N+1}$. If $(x^0, u^0)$ is not an element of the set $S_E^N$, then $(x^0, u^0)$ is also not an element of the set $S_E^{N+1}$. In this case, there exists $(x', u') \in S$ such that $I^{N+1}[x', u'] \leq I^{N+1}[x^0, u^0]$. So $I_{N+1}[x', u'] \leq I_{N+1}[x^0, u^0]$. This is a contradiction to the assumption that $S_{N+1} = \{(x^0, u^0)\}$. \hfill \Box

**Theorem 3.9.** Let the set $S$ be compact. If the functional $I_{N+1}$ is nonessential, then $S_{N+1} \cap S_E^N \neq \emptyset$.

*Proof.* Consider the problem

$$\min \int_a^b f(t, x(t), u(t)) dt$$

subject to $S_{N+1}$. Let $\tilde{S}$ denote the set of efficient solutions of the problem $(3.5)$. By the compactness of the set $S$, the set $\tilde{S}$ is nonempty. Let $(x^0, u^0) \in S$. If $(x^0, u^0)$ is not an element of $S_E^N$, then by assumption $(x^0, u^0)$ is not an element of $S_E^{N+1}$. In this case, there exists $(x', u') \in S$ such that $I^{N+1}[x', u'] \leq I^{N+1}[x^0, u^0]$. Hence $(x', u') \in S_{N+1}$. This contradicts the fact that $(x^0, u^0)$ is an efficient solution of the problem $(3.5)$. \hfill \Box

**Remark 3.10.** Notice that $I_2$ is nonessential in Example 3.2 and we have $S_E^1 \cap S_2 \neq \emptyset$.

Next section provides an example of application of the obtained results to check whether a functional is nonessential.

## 4 An illustrative example

We illustrate the obtained results with a multiobjective control problem borrowed from [7, §4.3], where $N = 3$, $n = 2, r = 1, m = 4, a = 0, b = T$, with $T$ not fixed. We consider a mobile rocket car with mass one running on rails on a closed region $-3 \leq x_1 \leq 3$ (we denote the position of the center of the car at time $t$ by $x_1(t)$), whose movement we can control with its accelerator $u$, where the maximum allowable acceleration is 1 and the maximum break power is $-1$, i.e., $-1 \leq u \leq 1$ (negative force means break, positive force means acceleration). The dynamics of the system is given by Newton’s second law, force equals mass times acceleration, which in our setting reads as $u(t) = \dot{x}_1(t)$. The problem is to move the car from a given location to a pre-assigned destination. If the car is at a position $x_1 = 1$ at time $t = 0$, with no velocity, that is $\dot{x}_1(0) = 0$, we want to find a piecewise constant function $u(t)$ that drives the car to $x_1(T) = 0$ at some instant $T > 0$. The state of the system is given by the position $x_1(t)$ and the velocity $x_2(t) = \dot{x}_1(t)$ (where we are and how fast we are going at each instant of time $t$). Different cost criteria can be considered, for example, minimizing the time $T$ (functional $I_1$ below); maximizing the velocity at $T$ (maximizing $x_2(T)$, which corresponds to functional $I_2$ below); and a linear combination $I_3$ of these functionals: minimize

$$I_1 = \int_0^T 1 dt, \quad I_2 = \int_0^T -u(t) dt, \quad I_3 = I_1 + I_2.$$
subject to the control system
\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= u;
\end{align*}
\]  
(4.1)

to the boundary conditions
\[
x_1(0) = 1, \quad x_1(T) = 0, \quad x_2(0) = 0;
\]  
(4.2)

and inequality constraints
\[
|u| \leq 1, \quad |x_1| \leq 3.
\]  
(4.3)

Denoting by \( S_i, \ i = 1, 2, 3 \), the solution set of the scalar optimal control problem
\[
\min I_i[x, u] \text{ subject to (4.1)-(4.3)},
\]
we have:
\[
* S_1 = \{(x^1, u^1)\} \quad \text{with} \quad u^1 = -1,
\]
\[
x_1^1 = -\frac{t^2}{2} + 1, \quad x_2^1 = -t,
\]
\[
0 \leq t \leq T = \sqrt{2};
\]

\[
S_2 = \{(x^2, u^2)\} \quad \text{with}
\]
\[
u^2 = \begin{cases} 
-1 & \text{if } 0 \leq t \leq 2, \\
+1 & \text{if } 2 \leq t \leq T = 4 + \sqrt{6}, 
\end{cases}
\]
\[
x_1^2 = \begin{cases} 
-\frac{t^2}{2} + 1 & \text{if } 0 \leq t \leq 2, \\
\frac{t^2}{2} - 4t + 5 & \text{if } 2 \leq t \leq 4 + \sqrt{6},
\end{cases} \quad x_2^2 = \begin{cases} 
-t & \text{if } 0 \leq t \leq 2, \\
t - 4 & \text{if } 2 \leq t \leq 4 + \sqrt{6};
\end{cases}
\]

\[
S_3 = \{(x^3, u^3)\} \quad \text{with}
\]
\[
u^3 = \begin{cases} 
-1 & \text{if } 0 \leq t \leq 1, \\
+1 & \text{if } 1 \leq t \leq T = 2;
\end{cases}
\]
\[
x_1^3 = \begin{cases} 
-\frac{t^2}{2} + 1 & \text{if } 0 \leq t \leq 1, \\
\frac{t^2}{2} - 2t + 2 & \text{if } 1 \leq t \leq 2,
\end{cases} \quad x_2^3 = \begin{cases} 
-t & \text{if } 0 \leq t \leq 1, \\
t - 2 & \text{if } 1 \leq t \leq 2.
\end{cases}
\]

Direct calculations show that
\[
I^2[x^1, u^1] = [\sqrt{2}, \sqrt{2}] = A,
\]
\[
I^2[x^2, u^2] = [4 + \sqrt{6}, \sqrt{6}] = B,
\]
\[
I^2[x^3, u^3] = [2, 0] = C.
\]

Let \( \zeta \) denote the set \( S^2_E \). It is the continuous, convex curve \( \overline{AB} \) (details can be found in [17, §4.3]). As \( C \in \zeta \) we have \( S^2_E \cap S_3 \neq \emptyset \). Moreover, let us notice that \( I_3 \) has a form \( I_3[x, u] = f(I^2[x, u]) \), where \( f : \mathbb{R}^2 \to \mathbb{R} \) is nondecreasing function. Therefore, the functional \( I_3 \) is nonessential by Theorem 3.6.

*The solutions to the scalar optimal control problems \( I_i[x, u] \to \min \) are found by application of the Pontryagin maximum principle [13]. Details can be found in [17, §4.3].
Remark 4.1. If we change the functional $I_3$ into
\[ I_3[x, u] = \gamma_1 I_1[x, u] + \gamma_2 I_2[x, u], \]
where $\gamma_i \in \mathbb{R}$ and $\gamma_i \geq 0, i = 1, 2$ or
\[ I_3[x, u] = [(I_1[x, u] - \sqrt{2})^p + (I_2[x, u] + \sqrt{6})^p]^{1/p}, \]
where $p \in [1, \infty]$, then again $I_3$ will be nonessential by Theorem 3.6. It is worth noting that functionals (4.4) and (4.5) can be used in order to find efficient solutions of the problem $\min I_2^2[x, u]$ subject to (4.1)-(4.3). We mentioned this in section 2, details can be found in [17] and [8].

5 Conclusions

The problem of optimizing a vector-valued criteria often arises in connection with the solution of problems in the areas of planning, organization of production, operational research and dynamical control systems. Currently, the problem of optimizing a vector-valued criteria is a central part of control theory and great attention is being given to it in the design and construction of modern automatic control systems, such as in concrete applications of seismology, energetic chemistry and metallurgy. In this work we use the notion of Pareto-optimality in control theory to define and investigate nonessential objective functionals of optimal control. For multicriteria optimal control systems this notion seems to be new and not used before. We claim the concept of nonessential objective functional to be an important issue in optimal control and we trust it will have an important role in the study of vector-valued optimization problems of control theory. For future work, it would be interesting to study the consequences of dropping nonessential objectives in multi-criteria optimal control systems.

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