INCIDENCE ESTIMATES FOR TUBES IN COMPLEX SPACE

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Abstract. In this paper, we prove a complex version of the incidence estimate of Guth, Solomon and Wang [GSW19] for tubes obeying certain strong spacing conditions, and we use one of our new estimates to resolve a discretized variant of Falconer's distance set problem in \( \mathbb{C}^2 \).

1. Introduction

It has long been known that in Fourier analysis and related fields, only some of the results true in the real space hold in the complex space as well. For instance,

- the Kakeya problem in three dimension under the Wolff axioms, with the bound \( \frac{5}{2} + \varepsilon \), is true over \( \mathbb{R} \) [KZ19] but false over \( \mathbb{C} \) because of the Heisenberg example [KLT19];
- the Falconer distance problem and its cousins, the Erdős ring problem and the Furstenberg problem, have counterexamples over \( \mathbb{C} \) because of the existence of the quadratic subfield \( \mathbb{R} \) [Wol03], but are still open over \( \mathbb{R} \);
- the Szemerédi-Trotter theorem is true over both \( \mathbb{R} \) [ST83] and \( \mathbb{C} \) [Tót15];
- the Erdős distinct distances problem, with an \( \varepsilon \)-loss, is true over both \( \mathbb{R} \) [GK15] and \( \mathbb{C} \) [SZ21].

The results in this paper are analogous to the bounds obtained in Theorem 1.2, Theorem 1.1 and Theorem 1.3, and Theorem 1.4 of [GSW19], giving some more examples of problems that have positive answers over both \( \mathbb{R} \) and \( \mathbb{C} \).

Theorem 1.1. Let \( \Theta \) be a partition of \( \mathbb{C}^{n-1} \) into “almost caps”. For each \( \theta \in \Theta \), let \( \{T_{\theta,j} : 1 \leq j \leq W^{2(n-1)}\} \) be a family of essentially distinct complex tubes of radius \( \delta \) and length 1 essentially contained in \( B_{\mathbb{C}^n}(0,1) \) with the property that every complex tube of radius \( W^{-1} \) and length 1 essentially contained in \( B_{\mathbb{C}^n}(0,1) \) with direction in \( \theta \) contains, essentially, \( \sim N \) of the complex tubes \( T_{\theta,j} \). Let \( T \) denote the set of all the tubes \( T_{\theta,j} \). Then, for \( r \gtrsim \varepsilon^{-1/2} |T| \),

\[ |P_r(T)| \lesssim \varepsilon^{-\varepsilon} W^{-2(n-1)} r^{-2} |T|^2. \]

Theorem 1.2. Let \( n = 2 \) or \( 3 \), \( 1 \leq W \leq \delta^{-1} \), and \( T \) be a collection of complex \( \delta \)-tubes essentially contained in \( B_{\mathbb{C}^n}(0,1) \). Suppose each distinct complex \( W^{-1} \)-tube essentially contained in \( B_{\mathbb{C}^n}(0,1) \) contains \( C_0 \) complex \( \delta \)-tube \( T \in T \). Then, for \( r \geq \max\{\delta^{2(n-1)-\varepsilon}|T|, 1 + C_0\} \),

\[ |P_r(T)| \lesssim_{\varepsilon,n,C_0} \delta^{-\varepsilon} |T|^\frac{\varepsilon+1}{n} r^{-\frac{n+1}{2}}. \]

Corollary 1.3. Fix \( 1 < s < 2 \). Let \( E \) be a set of \( \sim \delta^{-s} \) many \( \delta \)-balls in \( B_{\mathbb{C}^2}(0,1) \) with at most \( C \) \( \delta \)-balls in each ball of radius \( \delta^{s/4} \). Then, the number of disjoint
\[ \Delta(E) := \{(x_1 - x_2)^2 + (y_1 - y_2)^2 \in \mathbb{C} : \vec{p}_1, \vec{p}_2 \in \bigcup E \} \]

is \( \gtrsim_{\varepsilon,s} \delta^{-s+\varepsilon} \) for all \( \varepsilon > 0 \).

In Section 2, we review some definitions and conventions that will be used in the rest of the paper. In Section 3, we prove a lemma which allows us to capture the behavior of a large number of small balls using a few slightly larger balls, and thus lays the foundation for induction on scale. In Section 4, we consider two strong spacing conditions on complex tubes, and prove an upper bound on the number of rich balls in each case. And lastly, in Section 5, we derive a lower bound on the number of distinct complex distances for certain sparse sets in \( \mathbb{C}^2 \) by reducing the problem to one about incidences between complex tubes.

2. Definitions, Notations and Conventions

By “complex line”, we mean the translation of some 1-dimensional vector subspace of \( \mathbb{C}^n \) (over \( \mathbb{C} \)). There are two equivalent\(^1\) ways to define the angle between two complex lines, say

\[ \ell_1 = \{z \mathbf{u}_1 + \mathbf{t}_1 \in \mathbb{C}^n : z \in \mathbb{C} \} \quad \text{and} \quad \ell_2 = \{z \mathbf{u}_2 + \mathbf{t}_2 \in \mathbb{C}^n : z \in \mathbb{C} \}, \]

with \( \|\mathbf{u}_1\| = \|\mathbf{u}_2\| = 1 \):

1. Treat the complex lines as 2-planes in \( \mathbb{R}^{2n} \), and define the angle between the complex lines to be the first principal angle between the planes:

\[ \min_{z_1, z_2 \in \mathbb{C} : |z_1| = |z_2| = 1} \arccos \left( \iota(z_1 \mathbf{u}_1) \cdot \iota(z_2 \mathbf{u}_2) \right), \]

where \( \iota(\mathbf{u}) \) denotes the usual embedding of \( \mathbf{u} \in \mathbb{C}^n \) into \( \mathbb{R}^{2n} \).

2. Define the angle numerically to be \( \arccos |\langle \mathbf{u}_1, \mathbf{u}_2 \rangle| \), where \( \langle \cdot, \cdot \rangle \) is the usual inner product on \( \mathbb{C}^n \); this is also the Fubini-Study distance, up to a scaling constant, between the equivalent classes of \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \) in \( \mathbb{CP}^{n-1} \).

Note that for 2-planes in \( \mathbb{R}^{2n} \) corresponding to complex tubes, the two principal angles are equal\(^2\). We will use the latter definition, when partitioning the set of

\(^1\)To see the equivalence, write \( \mathbf{u}_1 = (a_1 + b_1 i, c_1 + d_1 i) \) and \( \mathbf{u}_2 = (a_2 + b_2 i, c_2 + d_2 i) \), with \( a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \in \mathbb{R} \). Then, for any \( z = e^{i \zeta} \) with \( \zeta \in \mathbb{R} \),

\[ \iota(\mathbf{u}_1) \cdot \iota(\mathbf{u}_2) = (a_1 a_2 + b_1 b_2 + c_1 c_2 + d_1 d_2) \cos \zeta + (-a_1 b_2 + a_2 b_1 - c_1 d_2 + c_2 d_1) \sin \zeta. \]

Hence,

\[ \min_{z_1, z_2 \in \mathbb{C} : |z_1| = |z_2| = 1} \arccos \left( \iota(z_1 \mathbf{u}_1) \cdot \iota(z_2 \mathbf{u}_2) \right) \]

\[ = \arccos \left( \max_{z \in \mathbb{C} : |z| = 1} \iota(\mathbf{u}_1) \cdot \iota(\mathbf{u}_2) \right) \]

\[ = \arccos \sqrt{(a_1 a_2 + b_1 b_2 + c_1 c_2 + d_1 d_2)^2 + (-a_1 b_2 + a_2 b_1 - c_1 d_2 + c_2 d_1)^2} \]

\[ = \arccos |\langle \mathbf{u}_1, \mathbf{u}_2 \rangle|. \]

\(^2\)Let’s assume WLOG that \( \mathbf{t}_1 = \mathbf{t}_2 = \mathbf{0} \). Suppose the first principal angle is attained for the pair of unit vectors \( (\mathbf{v}_1, \mathbf{v}_2) \) in \( e^{i \theta} \mathbf{u}_1 \times e^{i \theta} \mathbf{u}_2 \). Since the orthogonal complement of \( \iota(\mathbf{v}_1) \) in \( \iota(\ell_j) \) is \( \iota(iv_j) \mathbb{R} \) for \( j \in \{1, 2\} \), the second principal angle between \( \iota(\ell_1) \) and \( \iota(\ell_2) \) is

\[ \arccos (\iota(\mathbf{v}_1) \cdot \iota(\mathbf{v}_2)) = \arccos (\iota(\mathbf{v}_1) \cdot \iota(\mathbf{v}_2)). \]
A complex tube of length $l$ and radius $r$, is the $r$-neighbourhood of some line segment $\{zu + t : |z| < l/2\}$ with $u, t \in \mathbb{C}^n$ and $\|u\| = 1$. We will require $l > 10r$ so as to guarantee the existence of certain implicit constants. By “the angle between the tubes”, we will mean the angle between the line segments generating the tubes.

We say two solids $A_1, A_2 \subseteq \mathbb{C}^n \simeq \mathbb{R}^{2n}$ (e.g. balls, complex tubes, etc.) essentially intersect each other if their intersection is at least half as large as the maximum intersection between rigid transformations of the objects, i.e.

$$|A_1 \cap A_2| \geq \frac{1}{2} \max_{\sigma_1, \sigma_2 \in \text{SE}(2n)} |\sigma_1(A_1) \cap \sigma_2(A_2)|$$

And two geometric objects of the same kind are said to be essentially distinct if they do not essentially intersect. For example, two $\delta$-tubes $T_1$ and $T_2$ are essentially distinct if the volume of $T_1 \cap T_2$ is no more than half the volume of $T_1$. A solid $A_1$ is said to essentially contain another solid $A_2$ if $A_1$ essentially intersects $A_2$ and at least one essentially distinct translation of $A_2$.

The following proposition implies that two tubes with the same center are essentially distinct only if the angle between them is $\gtrsim \delta$. More generally, if tubes $T_1$ and $T_2$ intersect with

$$|T_1 \cap T_2| > \frac{1}{2} \max_{t \in \mathbb{C}^n} \left( |T_1 \cap (T_2 + t)| \right),$$

then $T_1$ and $T_2$ are essentially distinct only if the angle between $T_1$ and $T_2$ is $\gtrsim \delta$.

**Proposition 2.1.** Consider $\delta$-neighborhoods of the lines

$$\ell_1 = \{zu_1 : z \in \mathbb{C}\} \quad \text{and} \quad \ell_2 = \{zu_2 : z \in \mathbb{C}\}$$

for some direction vectors $u_1$ and $u_2$ that form angle $\theta > 0$ to each other, according to the definition above. Let $\iota : \mathbb{C}^n \to \mathbb{R}^{2n}$ be the usual embedding. Then

$$|\iota(N_\delta(\ell_1)) \cap \iota(N_\delta(\ell_2))| \sim \frac{\delta^{2n}}{\sin^2 \theta}. \quad (1)$$

We prove this proposition in the appendix.

Given any shape $A \subseteq \mathbb{C}^n \simeq \mathbb{R}^{2n}$ and any scalar $b \in [0, \infty)$, we denote by $bA$ the uniformly scaled shape $\{b(\alpha - c_A) + c_A : \alpha \in A\}$, with $c_A$ being the centre of mass of $A$.

By fixing a bump function for the unit ball, and then pre-composing the bump with translation and isotropic scaling (and rotation if we wish), we can construct smooth bump functions for balls in $\mathbb{C}^n$ in a uniform way. In a similar manner, we can uniformly construct bump functions for complex tubes as well. These uniformly constructed bump functions will be called “the bump functions” for the balls and the tubes.

### 3. A BRIDGE BETWEEN DIFFERENT SCALES

**Fact 3.1.** For any $0 < \delta \ll \sigma < 1$, the maximum number of dual complex slabs of $\delta$-separated complex tubes of length $\sim \delta^{-1}$ and radius $\sim 1$ centred at the origin passing through a point $\omega \in \partial B_\mathbb{C}^n(0, \sigma)$ is

$$\sim \sigma^{-2}\delta^{-2(n-2)}.$$
Proof. It’s not hard to see that the maximum is no more than a constant multiple of the average number of dual slabs (centered at the origin) in a maximal collection that pass through a point on the sphere $\partial B_{\mathbb{C}^n}(0, \sigma)$. A double-counting argument then gives

$$\max \text{ # of dual slabs through a point on } \partial B_{\mathbb{C}^n}(0, \sigma) \sim \frac{\delta^{-2(n-1)} \cdot \sigma^{2(n-2)} \delta^2}{\delta^{-2(n-1)}} = \sigma^{-2} \delta^{-2(n-2)}.$$  

Lemma 3.2. Suppose $P$ is a set of essentially distinct unit balls in $B_{\mathbb{C}^n}(0, D)$, and $T$ is a set of essentially distinct tubes of length $D$ and radius 1 in $B_{\mathbb{C}^n}(0, D)$ such that each ball of $P$ lies in $\sim E$ (more specifically, at least $E$ and less than $2E$)$\rho$ tubes of $\mathbb{C}^n$, $T$. Then, for any $0 < \varepsilon < 1$ and $1 \ll \rho^{-1} \ll \lambda \ll D$ (say $\lambda := D^{\pm \delta}$ and $\rho := D^\varepsilon \lambda^{-1}$), at least one of the following is true:

1. (Thin case) $|P| \lesssim \lambda^2 \rho^{-2} E^{-2} D^{2(n-1)} |\mathbb{C}^n|$. 
2. (Thick case) There is a set of finitely overlapping $2\lambda$-balls $Q_j$ such that
   a) $\bigcup_j Q_j$ contains $\gtrsim \lambda^{-2(n+1)} \rho^{-2n}$ fraction of the balls $q \in P$;
   b) Each $Q_j$ intersects $\gtrsim E \lambda^2 \rho^{-2n}$ tubes $T \in \mathbb{C}^n$.

Proof. For each unit ball $q \in P$, let $\psi_q$ be “the” smooth function with $\psi_q^I = 1$ and supp $\psi_q \subseteq 2q$; and for each tube $T \in \mathbb{T}$, let $\psi_T$ be “the” smooth function with $\psi_T^I = 1$ and supp $\psi_T \subseteq 2T$. Define

$$f := \sum_{q \in P} \psi_q, \quad g := \sum_{T \in \mathbb{T}} \psi_T,$$

and

$$I(P, T) := \{(q, T) \in P \times \mathbb{T} : |q \cap T| \geq \frac{1}{2} |q|\}.$$  

Then,

$$I(P, T) \lesssim \int fg = \int \hat{f} \hat{g} = \int \eta \hat{f} \hat{g} + \int (1 - \eta) \hat{f} \hat{g},$$

where $\eta$ is “the” smooth bump function approximating $\chi_{B(0, \rho)}$.

In the low frequency case,

$$I(P, T) \lesssim \int \eta \hat{f} \hat{g} = \int \hat{f}(g * \tilde{\eta}) = \sum_{q \in P} \sum_{T \in \mathbb{T}} \int \psi_q (\psi_T * \tilde{\eta}).$$

Since $\tilde{\eta}$ decays rapidly outside of $B(0, \rho^{-1})$, $\lambda \gg \rho^{-1} \gg 1$ and supp $\psi_T \subseteq 2T$,

$$|\psi_T * \tilde{\eta}| \lesssim \sup_{\omega \in \mathbb{C}^n} |B(\omega, \lambda) \cap 2T| \cdot ||\tilde{\eta}||_{L^\infty} \sim \lambda^2 \cdot \rho^{2n}.$$  

Hence, for each $q \in P$,

$$\sum_{T \in \mathbb{T}} \int \psi_q (\psi_T * \tilde{\eta}) \lesssim \lambda^2 \rho^{2n} \cdot \# \{T \in \mathbb{T} : q \cap N_\lambda(T) \neq \emptyset\}.$$  

It follows that

$$E|P| \sim I(P, T) \lesssim \sum_{q \in P} \lambda^2 \rho^{2n} \cdot \# \{T \in \mathbb{T} : q \cap N_\lambda(T) \neq \emptyset\}. $$

Thus, for a $\gtrsim \lambda^{-2(n+1)} \rho^{-2n}$ fraction of the unit balls $q \in P$, we have

$$\# \{T \in \mathbb{T} : q \cap N_\lambda(T) \neq \emptyset\} \gtrsim E \lambda^2 \rho^{-2n}. $$
Note that
\[ \{ T \in \mathbb{T} : q \cap n_\lambda(T) \neq \emptyset \} = \{ T \in \mathbb{T} : T \cap n_\lambda(q) \neq \emptyset \}, \]
and each neighbourhood \( n_\lambda(q) \) can be covered by a \( 2\lambda \)-ball.

In the high frequency case,
\[
I(P, \mathbb{T}) \lesssim \int (1 - \eta) \| \hat{g} \| \leq \left( \int (1 - \eta) |\hat{f}|^2 \right)^{\frac{1}{2}} \left( \int (1 - \eta) |\hat{g}|^2 \right)^{\frac{1}{2}}
\]
where
\[
\left( \int (1 - \eta) |\hat{f}|^2 \right)^{\frac{1}{2}} \leq \| \hat{f} \|_{L^2} = \| f \|_{L^2} \sim |P|^\frac{1}{2}.
\]

Group the tubes in \( \mathbb{T} \) according to their directions: Let \( \Theta \) be a partition of \( \mathbb{CP}^{n-1} \) into “almost caps” \( \theta \) of radius \( \sim D^{-1} \). (To be precise, an “almost cap \( \theta \) of radius \( D^{-1} \theta \) is a connected subset of \( \mathbb{CP}^{n-1} \) for which there exists a point \( x \in \mathbb{CP}^{n-1} \) so that \( B_{\mathbb{CP}^{n-1}}(x, D^{-1}) \leq \theta \leq B_{\mathbb{CP}^{n-1}}(x, D^{-1}) \), where the distance is the one induced from Fubini-Study metric.) Define \( \Theta_\theta \) to be the set of tubes in \( \mathbb{T} \) with direction in \( \theta \), and set
\[
g_\theta := \sum_{T \in \Theta_\theta} \psi_T.
\]
For each \( \theta \in \Theta \), fix a direction \( \nu_\theta \in \theta \) and a tube \( T_\theta \) centred at the origin of length \( \sim 1 \) and radius \( \sim D^{-1} \) in direction \( \nu_\theta \) which is contained in all tubes centred at the origin of length \( \sim 1 \) and radius \( \sim D^{-1} \) with direction in \( \theta \). Then, for each \( \theta \in \Theta \) and \( T \in T_\theta \), \( \psi_T \) decays rapidly outside of \( T_\theta \). So,
\[
\int (1 - \eta) |\hat{g}|^2
\]
\[
= \int (1 - \eta) \left| \sum_{\theta \in \Theta} \hat{g}_\theta \right|^2
\]
\[
\sim \int (1 - \eta) \left( \sum_{\theta : \lambda T_\theta \ni \omega} \hat{g}_\theta(\omega) \right) d\omega + \int (1 - \eta) \left( \sum_{\theta : \lambda T_\theta \ni \omega} |\hat{g}_\theta(\omega)|^2 d\omega \right)
\]
\[
\sim \int_{\mathbb{C}^n \setminus B_\infty(0, \rho)} (1 - \eta(\omega)) \cdot \# \{ \theta \in \Theta : \lambda \Theta_\theta \ni \omega \} \cdot \sum_{\theta \in \Theta} |\hat{g}_\theta(\omega)|^2 d\omega \quad \text{(by Fact 3.1)}
\]
\[
\lesssim (\lambda^{-1} \rho)^{-2} D^{2(n-2)} \sum_{\theta \in \Theta} \int |\hat{g}_\theta|^2.
\]
where
\[
\sum_{\theta \in \Theta} \int |\hat{g}_\theta|^2 = \sum_{\theta \in \Theta} \int |g_\theta|^2 \lesssim \sum_{T \in \mathbb{T}} \int |\psi_T|^2 \sim |T| \cdot D^2.
\]
Putting things together, we get
\[
E|P| \sim I(P, \mathbb{T}) \lesssim |P|^\frac{1}{2} \cdot \lambda \rho^{-1} D^{n-2} \cdot |T| \cdot D^2,
\]
which reduces to
\[
|P| \lesssim E^{-2} (\lambda \rho^{-1})^2 D^{2(n-1)} |T|.
\]
4. Proof of the main theorems

**Theorem 4.1.** Let $\Theta$ be a partition of $\mathbb{CP}^{n-1}$ into almost-caps $\theta$ of radius $\sim \delta$. Suppose $1 \leq W \leq \delta^{-1}$ and $1 \leq N \leq W^{-1} \delta^{-1}$. Suppose $1 \leq W \leq \delta^{-1}$ and $1 \leq N \leq W^{-1} \delta^{-1}$. For each $\theta \in \Theta$, let $\{T_{\theta,j} : 1 \leq j \leq W^{2(n-1)}\}$ be a family of essentially distinct complex tubes of radius $\delta$ and length $1$ essentially contained in $B_{\mathbb{C}}(0,1)$ with the property that, for each direction $\theta$ and for a fixed maximal collection of essentially distinct complex tube of radius $W^{-1}$ and length $1$ essentially contained in $B_{\mathbb{C}}(0,1)$ with direction in $\theta$, each of the complex $W^{-1}$-tube contains, essentially, $\sim N$ of the tubes $T_{\theta,j}$. Let $T$ denote the set of all the tubes $T_{\theta,j}$. Then, for each $\varepsilon > 0$, there exist constants $c_1(\varepsilon)$ and $c_2(\varepsilon)$ such that, for $r \geq c_1(\varepsilon) \delta^{2(n-1)-\varepsilon}|T|$, 

$$|P_r(T)| \leq c_2(\varepsilon) \delta^{-\varepsilon} W^{-2(n-1)} r^{-2}|T|^2.$$  

**Proof.** We will prove the theorem using induction on scale. Let’s start with two base cases:

(I) If $r \geq \alpha_n \delta^{-2(n-1)}$, where $\alpha_n$ is a sufficiently large dimensional constant, then $P_r(T) = \emptyset$.

(II) Suppose $W \geq \beta_{n,\varepsilon} \delta^{-1+\frac{\varepsilon}{2(n-1)}}$, where $\beta_{n,\varepsilon}$ is a constant to be determined later. Then, 

$$\delta^{2(n-1)-\varepsilon}|T| \geq \gamma_n \delta^{2(n-1)-\varepsilon} W^{2(n-1)} \delta^{-2(n-1)} \geq \gamma_n \beta^{2(n-1)} \delta^{-2(n-1)},$$

where $\gamma_n$ is some dimensional constant. So, if we take $\beta_{n,\varepsilon}$ sufficiently large that 

$$c_1(\varepsilon) \gamma_n \beta^{2(n-1)} \geq \alpha_n,$$

then it follows from the earlier case that $P_r(T) = \emptyset$ for all $r \geq c_1(\varepsilon) \delta^{2-\varepsilon}|T|$.

Now, assume that the following estimate holds when $\bar{r} > r$ or $\tilde{\delta} > \delta$:

$$|P_{\bar{r}}(T)| \leq c_2(\varepsilon) \delta^{-\varepsilon} W^{-2(n-1)} r^{-2}|T|^2.$$  

Let $P \subseteq P_r(T)$ be the set of $\delta$-balls that lie in $< 2r$ tubes of $T$. If $|P_r(T)| \geq 10|P|$, then

$$|P_r(T)| \leq \frac{10}{9} |P_{2r}(T)|$$

$$\leq \frac{10}{9} c_2(\varepsilon) \delta^{-\varepsilon} W^{-2(n-1)} (2r)^{-2}|T|^2 \quad \text{(by the induction hypothesis)}$$

$$\leq c_2(\varepsilon) \delta^{-\varepsilon} W^{-2(n-1)} r^{-2}|T|^2.$$  

Thus, we may assume WLOG that $|P_r(T)| < 10|P|$.

If we are in the thin case, then, taking $E := r$ in Lemma 3.2, we get 

$$|P| \leq \delta^{-\frac{\varepsilon}{2(n-1)}+2\varepsilon^3} W^{-2(n-1)} |T|$$

$$\leq \delta^{-\frac{\varepsilon}{2(n-1)}+2\varepsilon^3} W^{-2(n-1)} |T|^2 \quad \text{(because } |T| \sim W^{2(n-1)} \delta^{-2(n-1)} N)$$

$$= \delta^{(1-\frac{\varepsilon}{2(n-1)})+2\varepsilon^3} \cdot \delta^{-\varepsilon r^{-2} W^{-2(n-1)}|T|^2}.$$  

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3We call a connected set $\theta \subset \mathbb{CP}^{n-1}$ a $\delta$-almost-cap if there exists a point $u \in \mathbb{CP}^{n-1}$ such that $B_{\mathbb{CP}^{n-1}}(u, \frac{1}{10} \delta) \subseteq \theta \subseteq B_{\mathbb{CP}^{n-1}}(u, 10\delta)$. 
It follows that for $0 < \delta \ll_{\varepsilon} 1^4$,
\[
|P_r(T)| < 10|P| \leq c_2(\varepsilon) \delta^{-\varepsilon} W^{-2(n-1)} r^{-2} |T|^2.
\]

If we are in the thick case, then there exists a collection $\tilde{P}$ of essentially distinct $2\delta$-balls $Q_j$ such that $\bigcup_j Q_j$ contains a $\geq \delta^{2n}\varepsilon$ fraction of the $\delta$-balls in $P$, and each $Q_j$ intersects $\geq r \delta^{-\varepsilon} \frac{1}{2} + 2n\varepsilon^3$ tubes of $T$. Thicken each $\delta$-tube in $T$ to a $\lambda\delta$-tube, and call the new collection $\tilde{T}$. For each $1 \leq M \leq \lambda^{(n-1)}$, let $\tilde{T}_M$ be the set of $2\lambda\delta$-tubes in $\tilde{T}$ which contain $M$ tubes of $T$. By dyadic pigeonholing, we can find a particular $M_0$ such that $\tilde{T}_{M_0}$ contains a $\geq (n-1) \log \lambda$ fraction of the incidences between $\tilde{T}$ and $\tilde{P}$.

Observe that for each $2\lambda\delta$-tube $\tilde{T}$, every $2\lambda\delta$-tube that is not essentially distinct from $\tilde{T}$ lies completely in the tube $10\tilde{T}$. Let $\tilde{T}_{M_0,\max} \subseteq \tilde{T}_{M_0}$ be a maximal subset of essentially distinct $2\lambda\delta$-tubes. Then,
\[
I(10\tilde{T}_{M_0,\max}, \tilde{P}) \geq I(\tilde{T}_{M_0}, \tilde{P}),
\]
where $10\tilde{T}_{M_0,\max} := \{10\tilde{T} : \tilde{T} \in \tilde{T}_{M_0,\max} \}$. Further, since each $\delta$-tube is contained in $\lesssim n$ essentially distinct $2\lambda\delta$-tubes,
\[
|\tilde{T}| \lesssim n M_0 \big|10\tilde{T}_{M_0,\max}\big| = M_0 \big|\tilde{T}_{M_0,\max}\big|.
\]
At this point, we redefine $\tilde{T} := 10\tilde{T}_{M_0,\max}$.

It’s not hard to see that for $0 < \delta \ll_{\varepsilon} 1$, a $\gtrsim \delta^{2n}$ fraction of the $2\lambda\delta$-balls $Q_j \in \tilde{P}$ intersects $\gtrsim \delta^{2n} \cdot r \delta^{-\varepsilon} \frac{1}{2} + 2n\varepsilon^3 M_0^{-1}$ tubes of $\tilde{T}$, with the implicit constant in the latter inequality being, say, $\tilde{C} = C(n, \varepsilon)$. Since each $2\lambda\delta$-ball contains $\lesssim_n \lambda^{2n}$ essentially distinct $\delta$-balls,
\[
|P_r(T)| \lesssim |P| \lesssim \delta^{-\varepsilon} \lambda^{2n} \big|\bigcup_{Q_j \in \tilde{P}} Q_j\big| \lesssim \delta^{-2n - \varepsilon/2} (2\lambda\delta)^2 n \tilde{P} \sim \delta^{-\varepsilon} \lambda^{2n} \big|\tilde{P}\big| \lesssim \lambda^{2n} \delta^{-\varepsilon} \tilde{P},
\]
where $\tilde{r} := \tilde{C} r \delta^{-\varepsilon} \frac{1}{2} + 2n\varepsilon^3 M_0^{-1} \gtrsim r \delta^{-\varepsilon} \frac{1}{2} + 2n\varepsilon^3 M_0^{-1} \gtrsim (n-1) \log \lambda = r \delta^{2n\varepsilon^3} + \varepsilon^5$.

It follows from the given properties of $T$ that $\tilde{T}$, together with $\tilde{\delta} := 2\lambda\delta$ and $\tilde{W} := W$, meets the premise of our induction hypothesis. Moreover,
\[
\tilde{r} \gtrsim r \delta^{2n\varepsilon^3} + \varepsilon^5 \geq c_1(\varepsilon) \delta^{2(1-\varepsilon)} |T| \delta^{2n\varepsilon^3} + \varepsilon^5
\]
\[
\gtrsim \varepsilon (\lambda\delta)^{2(1-\varepsilon)} \lambda^{-2(n-1)+\varepsilon} M_0 \big|\tilde{T}\big| \delta^{2n\varepsilon^3} + \varepsilon^5
\]
\[
\gtrsim (\lambda\delta)^{2(n-1)-\varepsilon} \big|\tilde{T}\big| \lambda^{\varepsilon} \delta^{2n\varepsilon^3} + \varepsilon^5,
\]

\(^{4}\)The precise bound is
\[
\delta \lesssim \min \left\{ c_2(\varepsilon)^{(1-\varepsilon)} (\tilde{C}(\varepsilon)^{2} (2 \lambda \delta)^{2(n-1)-\varepsilon} (\lambda \delta)^{2(n-1)+\varepsilon} M_0 \big|\tilde{T}\big| (\delta^{2n\varepsilon^3} + \varepsilon^5) \right\}
\]
See below for the definition of $\tilde{C}$. 

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with $\lambda^2 \delta^{2n^2+\varepsilon^5} = \delta^{-\frac{3n+5}{4n+1}} + 2n \varepsilon^5 + \varepsilon^5 \gg 1$ for $\varepsilon$ sufficiently small. Thus, by the induction hypothesis,
\[
\left| P_r \left( \frac{T}{2} \right) \right| \leq c_2(\varepsilon) \left( 2 \lambda \delta \right)^{-\varepsilon} W^{-2(n-1)} T^{-2} \left| \frac{T}{2} \right|^2 \leq c_2(\varepsilon) \delta^{-\varepsilon} W^{-2(n-1)} r^{-2} \left| \frac{T}{2} \right|^2 \cdot \widetilde{C}^{-2} \lambda^{-\varepsilon} \delta^{\frac{2(n-1)}{4n+1}} \varepsilon^{-4n \varepsilon^3 - 2\varepsilon^5}.
\]
Hence,
\[
\left| P_r \left( \frac{T}{2} \right) \right| \leq c_2(\varepsilon) \delta^{-\varepsilon} W^{-2(n-1)} r^{-2} \left| \frac{T}{2} \right|^2 \cdot \widetilde{C}^{-2} \delta^{\frac{3n+5}{4n+1}} \varepsilon^{-4n \varepsilon^3 - 2\varepsilon^5 - \varepsilon^7},
\]
where $\widetilde{C}^{-2} \delta^{\frac{3n+5}{4n+1}} \varepsilon^{-4n \varepsilon^3 - 2\varepsilon^5 - \varepsilon^7} < 1$ when $0 < \delta \ll 1$. This closes the induction. \hfill \Box

**Theorem 4.2.** Let $n = 2$ or $3$, $1 \leq W \leq \delta^{-1}$, and $T$ be a collection of complex $\delta$-tubes essentially contained in $B_{C^2}(0,1)$. Suppose, for a fixed maximal family of essentially distinct complex $W^{-1}$-tubes essentially contained in $B_{C^2}(0,1)$, each of the $W^{-1}$-tubes contains exactly $H_0$ complex $\delta$-tube $T \in T$. Then, for richness $r \geq \max\{\delta^{2(n-1)-\varepsilon}|T|, 1 + H_0\}$,
\[
\left| P_r(T) \right| \lesssim_{\varepsilon, n, H_0} \delta^{-\varepsilon} W^{4n} r^{-\frac{n+1}{n}}.
\]

**Remark 4.3.** Suppose each distinct complex $W^{-1}$-tube contains at most $H_0$ complex $\delta$-tube $T \in T$, and $r \geq \max\{\delta^{2(n-1)-\varepsilon} W^{4(n-1)}, 1 + H_0\}$, then we still have
\[
\left| P_r(T) \right| \lesssim_{\varepsilon, n, H_0} \delta^{-\varepsilon} W^{4n} r^{-\frac{n+1}{n}}.
\]

**Proof.** We will again use induction on scale. For the base cases,

(I) Suppose $W \leq \delta^{-\varepsilon^4}$. Then, the set $P_r(T)$ is nonempty only if
\[
r \leq W^{2n-2} \leq \delta^{-(n-1)\varepsilon^4}.
\]
Thus, we may assume without loss of generality that $r \lesssim \delta^{-2(n-1)\varepsilon^4}$. Since
\[
|T| \lesssim W^{2(n-2)} \leq \delta^{-4(n-1)\varepsilon^4},
\]
it follows that
\[
\left| P_r(T) \right| \lesssim |P_2(T)| \lesssim |T|^2 \lesssim \delta^{-\varepsilon^4} |T|^\frac{1}{n+1} \cdot \delta^{-4(n-2)\varepsilon^4 + \varepsilon} \lesssim \delta^{-\varepsilon^4} |T|^\frac{1}{n+1} r^{-\frac{n+1}{n}}.
\]

(II) Suppose $W > \delta^{-1 + \frac{1}{n}}$. Then,
\[
r \geq \delta^{2(n-1)-\varepsilon} W^{4(n-1)} + 1 > \delta^{-2(n-1) - \frac{2}{n}},
\]
and therefore $|P_r(T)| = 0$.

We will assume that the bound $|P_r(T)| \leq c_0 \delta^{-\varepsilon^4} |T|^\frac{1}{n+1} r^{-\frac{n+1}{n}}$ holds when $\delta > \delta$, $\widetilde{r} > r$, and $\widetilde{r} \geq \max\{\delta^{2(n-1)-\varepsilon} W^{4(n-1)}, 1 + H_0\}$. By dyadic pigeonholing, we can find an $\alpha = \alpha(r) > W^{-1}$ such that the subcollection
\[
P_{r,\alpha}(T) := \left\{ p \in P_r(T) : \text{The maximal angle between complex tubes of } T \text{ passing through the } \delta\text{-ball } p \text{ is } \sim \alpha \right\}
\]
has size $\gtrsim (\log \delta)^{-1} |P_r(T)|$. The lower bound $W^{-1}$ comes from the fact that $r \geq 1 + H_0$, which is to say, the complex $\delta$-tubes passing through any $\delta$-ball $p$ cannot all lie in the same $W^{-1}$-tube.

Let’s first consider the case(s) where $r < \delta^{-3^4}$. 

(a) Suppose $\alpha \leq \delta^{\frac{2}{3}}$. Let $\{\tau\} \subseteq \mathbb{CP}^{n-1}$ be a maximal set of $\alpha$-separated directions. For each $\tau$, cover $B_{C^n}(0,1)$ with $\sim \alpha^{-2(n-1)}$ complex $\alpha$-tubes $\square_\tau$ in the direction $\tau$. Let $\mathbb{T}_{\square}$ denote the collection of complex tubes $T \in T$ that are essentially contained in $\square_\tau$. Then, our assumption about the spacing of the complex tubes implies that $|\mathbb{T}_{\square_\tau}| \lesssim (\alpha W)^{4(n-1)}$.

Now, fix a complex tube $\square_\tau$, and rescale $\square_\tau$ to essentially a unit ball. Then, the complex $\delta$-tubes in $\mathbb{T}_{\square_\tau}$ become complex $\hat{\delta}$-tubes, the collection of which we call $\mathbb{T}_{\hat{\square_\tau}}$. Let $\hat{\delta} := \frac{\delta}{\alpha}$. The collection $\mathbb{T}_{\hat{\square_\tau}}$ satisfies the spacing condition with $\hat{W} := \alpha W$ and $\delta := \frac{\delta}{\alpha} \geq \delta^{-\hat{\delta}^3} \delta$. Thus, we have the trivial upper bound,

$$\left| P_2 \left( \mathbb{T}_{\hat{\square_\tau}} \right) \right| \lesssim \hat{\delta}^{-\varepsilon} \left| \mathbb{T}_{\hat{\square_\tau}} \right|^{\frac{n}{\alpha} \cdot \hat{\delta}^{-\varepsilon}} \cdot \alpha^{-2} \quad (2)$$

Rescaling the unit ball back to $\square_\tau$, we see that each 2-rich $\hat{\delta}$-balls becomes a complex tube of radius $\delta$ and length $\frac{\delta}{\alpha}$, which can be viewed as an almost disjoint union (or, more precisely, a finitely overlapping union) of $\sim \alpha^{-2}$ of the $\delta$-balls in $P_{2,\alpha}(T)$. As a result,

$$|P_{r,\alpha}(r)(T)| \leq |P_{2,\alpha}(r)(T)| \leq \sum_r \sum_{\square_\tau} \left| P_2 \left( \mathbb{T}_{\hat{\square_\tau}} \right) \right|^{\frac{n}{\alpha} \cdot \hat{\delta}^{-\varepsilon}} \cdot \alpha^{-2}$$

$$\lesssim \alpha^{-4(n-1)} \cdot \left( \frac{\delta}{\alpha} \right)^{-\varepsilon} (\alpha W)^{4(n-1)} \frac{n}{\alpha} \cdot \alpha^{-2}$$

$$\sim \delta^{-\varepsilon} |T|^{\frac{n}{\alpha} \cdot \hat{\delta}^{-\varepsilon}} \cdot \alpha^{-4(n-1)+\varepsilon+4n-2},$$

where

$$\alpha^{-4(n-1)+\varepsilon+4n-2} \leq \delta^{\frac{3}{2} \cdot \varepsilon^{2}(2+\varepsilon)} = \delta^{3\varepsilon} \cdot \delta^{\frac{3}{2} \cdot \varepsilon^{2}} \leq \delta^{\frac{n+1}{n} \cdot \varepsilon^{2}} \cdot \delta^{\frac{n+1}{n} \cdot \varepsilon^{2}} < r \cdot \frac{n+1}{n} \cdot \delta^{\frac{3}{2} \cdot \varepsilon^{2}}.$$

So, we can close the induction in the case where $r < \delta^{-\varepsilon}$ and $\alpha \leq \delta^{\frac{3}{2} \cdot \varepsilon}$.  

(b) Next, suppose $\alpha > \delta^{\frac{3}{2} \cdot \varepsilon}$. Clearly,

$$\delta^{-\varepsilon} |T|^{\frac{n}{\alpha} \cdot \hat{\delta}^{-\varepsilon}} \cdot \alpha^{-4(n-1)+\varepsilon+4n-2} \geq \delta^{-\varepsilon} W^{4n} \delta^{\frac{n+1}{n} \cdot \varepsilon^{2}}.$$

Thus, when $W > \delta^{-\frac{1}{2} \cdot \varepsilon^{2}}$, we have

$$\delta^{-\varepsilon} |T|^{\frac{n}{\alpha} \cdot \hat{\delta}^{-\varepsilon}} \cdot \alpha^{-4(n-1)+\varepsilon+4n-2} \geq \delta^{-2n-\frac{1}{2} \cdot \varepsilon^{2}+\frac{n+1}{n} \cdot \varepsilon^{2}} \geq \delta^{-2n} \gtrsim |P_{r}(T)|.$$

For this reason, we may as well assume that $W \leq \delta^{-\frac{1}{2} \cdot \varepsilon^{2}}$ in the rest of part (b). Then,

$$W^{-2} = \delta^{-\frac{1}{2} \cdot \varepsilon^{2}} \gg \delta.$$

Let $\mathbb{T}$ be the collection of complex $W^{-2}$-tubes obtained by thickening each complex $\delta$-tube in $T$. As $W^{-2} < W^{-1}$, the spacing condition of $T$ implies that the complex tubes in $\mathbb{T}$ are essentially distinct. Let $Q$ be a minimal covering of $B_{C^n}(0,1)$ consisting of $W^{-2}$-balls, and let $Q_{X,M}$ be the collection of $W^{-2}$-balls in $Q$ that contains $\sim X$ of the $\delta$-balls in $P_{r,\alpha}(T)$ and intersects $\sim M$ of the complex $\delta$-tubes in $T$. It’s not hard to see that the set $Q_{X,M}$ is nonempty only if $X \lesssim \left( \frac{W^{-1}}{\delta} \right)^{2n}$ and $M \lesssim W^{4(n-1)}$. Then, by dyadic pigeonholing twice, we can find some particular $X_0$ and $M_0$ such that $\bigcup_{Q \in Q_{X_0,M_0}} q$ covers a $\gtrsim_n (- \log \delta)^{-1}$ fraction of the $\delta$-balls in $P_{r,\alpha}(T)$. By definition, each $\delta$-ball in $P_{r,\alpha}(T)$ is intersected by two complex tubes
in \( \mathbb{T} \) with directions differing by \( \sim \alpha \). Hence, the number of \( \delta \)-balls from \( P_{r,\alpha}(T) \) contained in each \( q \) is \( \lesssim \alpha^{-2}M_0^2 \), and consequently we have

\[
X_0 \lesssim \alpha^{-2}M_0^2 \lesssim \delta^{-3\varepsilon^3}M_0^2.
\]

By the induction hypothesis,

\[
|P_{M_0} \left( \overline{I} \right)| \lesssim W^{2\varepsilon} \left| \overline{I} \right|^{\frac{n-\varepsilon}{n}} M_0^{-\frac{\alpha+1}{n-\varepsilon}}.
\]

Therefore,

\[
|P_r(T)| \lesssim (-\log \delta) \cdot X_0 \cdot |P_{M_0} \left( \overline{I} \right)|
\]

\[
\lesssim (-\log \delta) \delta^{-3\varepsilon^3} W^{2\varepsilon} \left| \overline{I} \right|^{\frac{n-\varepsilon}{n}} M_0^{-\frac{\alpha+1}{n-\varepsilon}}
\]

\[
\leq (-\log \delta) \delta^{-3\varepsilon^3} + 2\varepsilon \left( -\frac{\varepsilon}{2} + \frac{\varepsilon}{2} \right) |\overline{I}|^{\frac{n-\varepsilon}{n}} M_0^{-\frac{\alpha+1}{n-\varepsilon}}
\]

\[
= (-\log \delta) \delta^{\frac{\varepsilon^2}{2} - 3\varepsilon^3} \cdot \delta^{-\varepsilon |\overline{I}|^{\frac{n-\varepsilon}{n}}} M_0^{-\frac{\alpha+1}{n-\varepsilon}},
\]

with \( M_0^{2-\frac{\alpha+1}{n-\varepsilon}} \leq 1 \) when \( n = 2 \) or \( 3 \), and

\[
(-\log \delta) \delta^{\frac{\varepsilon^2}{2} - 3\varepsilon^3} = (-\log \delta) \delta^{\frac{\varepsilon^2}{2} - \left( 3 + \frac{\alpha+1}{n-\varepsilon} \right) \varepsilon^3} \cdot \delta^{\frac{\varepsilon^2}{n-\varepsilon} + 3} \ll r^{-\frac{\varepsilon^2}{2}},
\]

assuming that \( \varepsilon \) is sufficiently small. This closes the induction in the case where \( r < \delta^{-\varepsilon^3} \) and \( \alpha > \delta^{\frac{\varepsilon^2}{2}} \).

Hereinafter, suppose \( r \geq \delta^{-\varepsilon^3} \). Take \( D := \delta^{-\varepsilon^4} < W \), and let \( \Omega \) be a minimal covering of \( B_{C^n}(0,2) \) consisting of \( D\delta \)-balls \( q \). By our induction hypothesis, we may assume without loss of generality that \( |P| \sim |P_r(T)| \), where \( P := P_r(T) \setminus P_{2r}(T) \).

For any two intersecting \( T \) and \( q \), the intersection \( T \cap q \) is contained in a complex tube of length \( 2D\delta \) and radius \( \delta \), which we call \( T \). Let \( T_{q,N} \) be the set of complex tubes \( T \) which are essentially contained in \( q \), and \( \sim N \) of the complex tubes \( T \in \mathbb{T} \).

Note that \( T_{q,N} \neq \emptyset \) only if \( N \lesssim W^{2(n-1)} \) because of the spacing condition. As in the proof of Theorem 4.1, we first use dyadic pigeonholing to pick a particular \( N_0 \) such that

\[
\sum_{q \in \Omega} N_0 \cdot \left| I \left( P_q, T_{q,N_0} \right) \right| \gtrsim n (-\log \delta)^{-1} |I(P,T)|,
\]

where \( P_q \) is the collection of \( \delta \)-balls \( p \in P \) that are essentially contained in \( q \). Then, for each \( q \in \Omega \), we fix a maximal subset \( T_{q,N_0,\max} \subseteq T_{q,N_0} \) consisting of essentially distinct complex tubes. We note that any chopped tube \( \tilde{T} \) essentially the same as a chopped tube \( T \) would lie completely in \( 10\tilde{T} \). Let \( P_{q,E} \) be the set of \( \delta \)-balls in \( P \) which are essentially contained in \( q \) and \( \sim E \) of the complex tubes \( 10\tilde{T} \in 10T_{q,N_0,\max} \). Again, by dyadic pigeonholing, we can find a particular \( E_0 \) such that

\[
\sum_{q \in \Omega} N_0 \cdot \left| I \left( P_{q,E_0}, 10\tilde{T}_{q,N_0,\max} \right) \right| \gtrsim n (-\log \delta)^{-2} |I(P,T)|.
\]

By the definition of \( P_{q,E} \), we have \( \left| I \left( P_{q,E_0}, 10\tilde{T}_{q,N_0,\max} \right) \right| \sim E_0 |P_{q,E_0}| \); and by the definition of \( P \), \( |I(p,T)| \sim r|P| \). It then follows that

\[
\sum_q |P_{q,E_0}| \gtrsim (\log \delta)^{-2} \frac{r}{N_0 E_0} |P|. \tag{3}
\]
On the other hand,
\[ \sum_q |P_{q,E_0}| \leq \sum_q |P_q| \leq 2|P|, \]
Hence, \( |P| \gtrsim (\log \delta)^{-2} \frac{r}{N_0 E_0} |P| \), which implies
\[ N_0 E_0 \gtrsim (\log \delta)^{-2} r. \] (4)

Next, let’s establish a dichotomy which states that either our problem reduces to the assumed bound on \( P_{C_n,r} \), where \( C_n \) is some large dimensional constant, or we have an estimate matching the one right above:

Recall that the tubes \( \mathcal{T} \in T_{q, N_0, \text{max}} \) are, roughly speaking, chopped out of \( q \).

Thus, when studying the incidences of chopped tubes and \( \delta \)-balls essentially contained in a certain \( q \), we may as well consider the fattened version, which we will denote \( 10\mathcal{T} \), in place of the proportionally enlarged \( 10\mathcal{T} \). For any fattened chopped tube \( 10\mathcal{T} \) (of width \( 20\delta \)) essentially containing a \( \delta \)-ball \( p \), the original chopped tube \( \mathcal{T} \) passes through the middle of the ball \( 50p \) in the sense that the distance between the centre of \( p \) and the axis of \( \mathcal{T} \) is no more than \( 11\delta \) (i.e. \( 11 \cdot 50 \cdot \delta \)), we have that
\[ \left| \mathcal{T} \cap 50p \right| \sim n_1, \] (5)
with the implicit constants dependent only on the dimension \( n \). Consequently, for each \( p \in P_{q,E_0} \),
\[ \int_{50p} \sum_{\mathcal{T} \in \mathcal{T}, N_0, \text{max}: 10\mathcal{T} \supseteq q} \chi_{\mathcal{T}} = \sum_{\mathcal{T} \in \mathcal{T}, N_0, \text{max}: 10\mathcal{T} \supseteq q} \left| \mathcal{T} \cap 50p \right| \sim_n E_0. \]
Thus, there exists some point \( x_p \in 50p \) at which
\[ \sum_{\mathcal{T} \in \mathcal{T}, N_0, \text{max}: 10\mathcal{T} \supseteq q} \chi_{\mathcal{T}}(x_p) \sim E_0. \]
A \( \gtrsim_n 1 \) fraction of the \( \delta \)-ball \( B(x_p, \delta) \) is contained in each of the \( \mathcal{T} \) containing \( x_p \).

Let \( \Gamma_n \) denote the maximal number of essentially distinct unit balls \( B \subseteq \mathbb{C}^n \) whose neighbourhood \( 50B \) contains the origin. If for every \( q \in \Omega \) and \( p \in P_{q,E_0} \), there exists no point \( x \in 50p \) at which
\[ \sum_{\mathcal{T} \in \mathcal{T}, N_0, \text{max}: 10\mathcal{T} \supseteq q} \chi_{\mathcal{T}}(x) \sim E_0 \text{ and } B(x, \delta) \not\subseteq P_{C_n,r}, \]
where \( C_n := \frac{n-1}{n} \), then,
\[ \sum_Q |P_{Q,E_0}| \leq \Gamma_n |P_{C_n,r}| \]
\[ \leq \Gamma_n \cdot c_n \delta^{-\varepsilon} |T|^{\frac{n}{n-1}} (C_n r)^{-\frac{n-1}{n}} \] (by the induction hypothesis)
\[ = c_n \delta^{-\varepsilon} |T|^{\frac{n}{n-1}} - \frac{n-1}{n}, \]
as desired. Otherwise, there exists some \( q \in \Omega \), \( p \in P_{q,E_0} \), and \( x \in 50p \) such that
\[ \sum_{\mathcal{T} \in \mathcal{T}, N_0, \text{max}: 10\mathcal{T} \supseteq q} \chi_{\mathcal{T}}(x) \sim E_0 \text{ and } B(x, \delta) \not\subseteq P_{C_n,r}. \]
Since $B(x, \delta)$ is essentially contained in $\approx E_0$ of the chopped tubes $\hat{T} \in T_{q, N_0, \text{max}}$, by the definition of $T_{q, N_0}$, the $\delta$-ball $B(x, \delta)$ is essentially contained in $\approx N_0 E_0$ of the complex tubes $T \in \mathbb{T}$. Then,

$$N_0 E_0 \lesssim C_n r \sim r.$$  \hfill (6)

Thus, by eq. (3),

$$|P| \lesssim_n (\log \delta)^2 \sum_{q \in \mathbb{D}} |P_{q, E_0}| \lesssim (\log \delta)^2 \left( \sum_{q \text{ thin}} |P_{q, E_0}| + \sum_{q \text{ thick}} |P_{q, E_0}| \right).$$

Now we apply Lemma 3.2 to each pair of $P_{q, E_0}$ and $\hat{T}_q := \hat{T}_{q, N_0, \text{max}}$.

(c) For each $q$ falling into the thin case,

$$|P_{q, E_0}| \lesssim D_{\hat{T}_q}^{n-2\varepsilon} E_0^{-2} D^{2(n-1)} |\hat{T}_q|.$$  

So, if the thin case dominates, then

$$|P| \lesssim_n (\log \delta)^2 \sum_{q \text{ thin}} |P_{q, E_0}| \lesssim (\log \delta)^2 \cdot D_{\hat{T}_q}^{n-2\varepsilon} E_0^{-2} D^{2(n-1)} \sum_{q} |\hat{T}_q|.$$  

To estimate $\sum_q |\hat{T}_q|$, let $\{\sigma\} \subseteq \mathbb{CP}^{n-1}$ be a maximal set of $D^{-1}$-separated directions, and for each $\sigma$, let $\{\square_\sigma\}$ be a minimal covering of $B_{\mathbb{C}^n}(0, 1)$ consisting of complex $D^{-1}$-tubes $\square_\sigma$ in the direction $\sigma$. Let $\tilde{T}_\square_\sigma$ denote the collection of complex tubes $T \in \mathbb{T}$ that are essentially contained in $\square_\sigma$. Each fattened chopped complex tube $\hat{T} \in \hat{T}_q$ intersects essentially $\approx N_0$ complex tubes of $\mathbb{T}$, all of which lie essentially in the same $\square_\sigma$. Rescale each $\square_\sigma$ to essentially a unit ball, and let $\tilde{T}_\square_\sigma$ be the resulting set of complex $\tilde{\delta}$-tubes, where $\tilde{\delta} := D \delta$. Then, $\tilde{T}_\square_\sigma$ meets the spacing condition with $\tilde{W} := WD^{-1}$. Moreover,

$$\sum_q |\hat{T}_q| \lesssim \sum_{\sigma} \sum_{\square_\sigma} |P_{\hat{T}}(\tilde{T}_\square_\sigma)|,$$

where $\tilde{r} \sim N_0$ (or, more precisely, $\tilde{r}$ is a constant fraction of $N_0$). Since $\tilde{\delta} = D \delta = \delta^{1-\varepsilon^2} \gg \delta$,

we can apply our induction hypothesis and get

$$|\tilde{T}_\square_\sigma| \lesssim \tilde{\delta}^{-\varepsilon} |\tilde{T}_\square_\sigma|^{\frac{n}{2n-4} \tilde{r} - \frac{n+1}{n-1}} \sim (D \delta)^{-\varepsilon} (WD^{-1})^{4(n-1)} \frac{n}{2n-4} N_0^{-\frac{n+1}{n-1}}$$

$$\sim (D \delta)^{-\varepsilon} D^{-4n} |T|^{\frac{n}{2n-4}} N_0^{-\frac{n+1}{n-1}},$$

which implies

$$\sum_q |\hat{T}_q| \lesssim D^{4(n-1)} \cdot (D \delta)^{-\varepsilon} D^{-4n} |T|^{\frac{n}{2n-4}} N_0^{-\frac{n+1}{n-1}} = D^{-4}(D \delta)^{-\varepsilon} |T|^{\frac{n}{2n-4}} N_0^{-\frac{n+1}{n-1}},$$

and consequently

$$|P_r(T)| \lesssim (\log \delta)^2 \cdot D^{2n-6+\frac{n}{2n-4} \delta^{-\varepsilon}} (D \delta)^{-\varepsilon} E_0^{-2} N_0^{-\frac{n+1}{n-1} |T|^{\frac{n}{2n-4}}}$$

$$\lesssim (\log \delta)^2 \cdot D^{2n-6+\frac{n+1}{n-1} \delta^{-\varepsilon}} D^{2n-6+\frac{n}{2n-4} \delta^{-\varepsilon}} E_0^{-2} \tilde{r}^{-\frac{n+1}{n-1} |T|^{\frac{n}{2n-4}}},$$

(by eq. (4))
When \(-2 + \frac{n+1}{n-1} \leq 0\), i.e. when \(n \geq 2\) or \(3\), we have

\[
E_0^{-2 + \frac{n+1}{n-1}} \lesssim \left( D^{2(n-1)} \right)^{-2 + \frac{n+1}{n-1}},
\]
and hence

\[
|P_r(\mathbb{T})| \lesssim (\log \delta)^{\frac{dn}{D(n-1)}} D(n^{-1})^{-2 + \frac{n+1}{n-1}} \delta^{-\varepsilon} r^{-\frac{n+1}{n-1}} |\mathbb{T}|^{\frac{n}{n-1}},
\]

where \((\log \delta)^{\frac{dn}{D(n-1)}} D(n^{-1})^{-2 + \frac{n+1}{n-1}} \ll 1\).

(d) For each \(q\) falling into the thick case, there exists a collection \(\tilde{P}_q\) of finitely overlapping balls \(\tilde{p}\) of width \(\sim D \frac{1}{n^{m-1}} \delta\) such that the union \(\bigcup_{\tilde{p} \in \tilde{P}_q} \tilde{p}\) contains a \(\gtrsim D^{-\frac{2n+m}{2} + \frac{2m+1}{2}}\) fraction of the balls \(p \in P_{q,E_0}\), and each ball \(\tilde{p}\) intersects \(\gtrsim E_0 D \frac{1}{n^{m-1}} \delta^{-2 + \frac{n+1}{n-1}}\) of the complex tubes \(\tilde{T} \in \tilde{\mathbb{T}}\). Set

\[
\tilde{P} := \bigcup_{q \text{ thick}} \tilde{P}_q.
\]

Then, each \(\tilde{p} \in \tilde{P}\) intersects \(\gtrsim M_0 E_0 D \frac{1}{n^{m-1}} \delta^{-2 + \frac{n+1}{n-1}}\) of the complex tubes \(T \in \mathbb{T}\). To apply the induction hypothesis, we fatten each complex \(\delta\)-tube \(T \in \mathbb{T}\) into a complex tube \(\tilde{T}\) of width \(\sim D \frac{1}{n^{m-1}} \delta\), and call the collection of all these complex tubes \(\tilde{\mathbb{T}}\). Then, \(\tilde{P} \subseteq P_r(\tilde{\mathbb{T}})\), where

\[
\tilde{T} \sim M_0 E_0 D \frac{1}{n^{m-1}} \delta^{-2 + \frac{n+1}{n-1}} \gtrsim (\log \delta)^{-2} r D \frac{1}{n^{m-1}} \delta^{-2 + \frac{n+1}{n-1}} \gg r.
\]

The collection \(\tilde{\mathbb{T}}\) meets the spacing condition with \(\bar{\delta} \sim D \frac{1}{n^{m-1}} \delta\), \(\tilde{W} = W\) and

\[
\tilde{W} = W \lesssim \delta^{-\frac{1}{2} - \frac{1}{n(n-1)}} \sim \delta^{-1 + \varepsilon} (1 + \frac{1}{n^{m-1}}) \sim \bar{\delta}^{-1}.
\]

Hence,

\[
\sum_{Q \text{ thick}} |\tilde{P}_Q| \leq |P_r(\tilde{\mathbb{T}})| \lesssim \tilde{\delta}^{-\varepsilon} r^{-\frac{n+1}{n-1}} |\tilde{\mathbb{T}}|^{\frac{n}{n-1}}
\]

\[
\lesssim \delta^{-\varepsilon} (1 + \frac{1}{n^{m-1}}) \log \delta \frac{1}{\tilde{n}^{-1}} \cdot \delta^{-\varepsilon} r^{\frac{n+1}{n-1}} |\mathbb{T}|^{\frac{n}{n-1}},
\]

and

\[
|P_r(\mathbb{T})| \lesssim (\log \delta)^2 \sum_{Q \text{ thick}} |P_{Q,E_0}| \lesssim \delta^{-\varepsilon} (1 + \frac{1}{n^{m-1}}) \log \delta \frac{1}{\tilde{n}^{-1}} \cdot \delta^{-\varepsilon} r^{\frac{n+1}{n-1}} |\mathbb{T}|^{\frac{n}{n-1}}
\]

\[
\lesssim \delta^{-\varepsilon} r^{\frac{n+1}{n-1}} |\mathbb{T}|^{\frac{n}{n-1}}.
\]

The induction is now complete.

\[\square\]

5. Application to a variant of Falconer’s distance set problem

Given any two points \(\tilde{p}_1 = (x_1, y_1)\) and \(\tilde{p}_2 = (x_2, y_2)\) in \(\mathbb{C}^2\), we define the auxiliary line

\[
l_{\tilde{p}_1, \tilde{p}_2} := \left\{ (\frac{x_1 + x_2}{2} + \frac{y_1 - y_2}{2} z, \frac{y_1 + y_2}{2} - \frac{x_1 - x_2}{2} z, z) : z \in \mathbb{C} \right\}.
\]
Theorem 5.1. Fix $1 < s < 2$. Let $E$ be a set of $\sim N \delta^{-s}$ many $\delta$-balls in $B_{C^2}(0, 1)$ with at most $N$ $\delta$-balls in each ball of radius $\delta^{s/4}$. Then, the number of disjoint $\delta$-balls needed to cover the difference set

$$\Delta(E) := \left\{(x_1 - x_2)^2 + (y_1 - y_2)^2 \in \mathbb{C} : \vec{p}_1, \vec{p}_2 \in \bigcup E\right\}$$

is $\gtrsim \varepsilon, \delta^{-s+\varepsilon}$ for all $\varepsilon > 0$.

We can choose two balls $B_1$ and $B_2$ of radius $C_1 := 0.01$ with centres $C_2 := 1.2$ apart such that each $E_j := \{q \in E : q \subseteq B_j\}$ contains a $\frac{1}{2}$ fraction of $E$. Define $T$ to be the collection of complex $\delta$-tubes $\{T_{q_1, q_2} : (q_1, q_2) \in (E_1 \times E_2) \cup (E_2 \times E_1)\}$, where

$$T_{q_1, q_2} := \left(\bigcup_{\{\vec{p}_1, \vec{p}_2\} \in q_1 \times q_2} l_{\vec{p}_1, \vec{p}_2}\right) \cap B_{C^\infty}(0, 1);$$

and $Q$ to be the set of all quadruples $(q_1, q_2, q_3, q_4) \in E_1 \times E_2 \times E_2 \times E_1$ such that there exist some $(\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) \in q_1 \times q_2 \times q_3 \times q_4$ with

$$|\Delta(\vec{p}_1, \vec{p}_2) - \Delta(\vec{p}_3, \vec{p}_4)| < \delta.$$

Proposition 5.2. The collection $T$ meets the spacing condition of Theorem 4.2, with $W := \delta^{s/4}$ and $H_0 := N^2$.

Proof. To show that no $W^{-1}$-tube contains more than $N^2$-many tubes of $T$, it is sufficient to show the following claim:

Claim 5.3. Let $E', E'' \subset E$ be sets of $\delta$-balls so that no ball of radius $\delta^{s/4}$ contains two $\delta$-balls of $E'$ or of $E''$, and let $T' \subset T$ be the set

$$T' = \{T_{q_1, q_2} : q_1 \in E' \text{ and } q_2 \in E''\}.$$

Then any $W^{-1}$-tube in $B_{C^0}(0, 1)$ contains at most one tube of $T'$.

Assuming this claim, we can prove Proposition 5.2 in the following way:

Let $R$ be a $W^{-1}$-complex tube in $B_{C^0}(0, 1)$. To show that $R$ contains no more than $N^2$-many tubes of $T$, we will partition $T$ into $N^2$ parts and show that $R$ contains at most 1 tube from each of these parts. By our original choice of $E$, each $\delta^{s/4}$-ball contains at most $N$-many $\delta$-balls of $E$. Thus, we can color the $\delta$ balls with $\sim N$ colors in a way that guarantees that no two delta balls of the same color are in any $\delta^{s/4}$-almost-box. Suppose we used exactly $N'$ colors. For $j = 1, \ldots, N'$ let $E_j$ be the set of $\delta$-balls which were colored with color $j$. For $1 \leq j, k \leq N'$, let $T_{j, k} \subset T$ be the collection

$$T_{j, k} = \{T_{q_1, q_2} : q_1 \in E_j \text{ and } q_2 \in E_k\}.$$

By Claim 5.3, for any pair $(j, k)$ with $1 \leq j, k \leq N$, our $W^{-1}$-tube $R$ contains at most 1 tube in $T_{j, k}$. Since the sets $\{T_{j, k}\}_{1 \leq j, k \leq N}$ partition $T$, it follows that $R$ contains at most $(N')^2 \sim N^2$ tubes of $T$ altogether.

It remains to prove the Claim 5.3.

To prove Claim 5.3, let $j$ and $k$ be fixed. We will consider a pair of tubes in $T_{j, k}$ and show that no $W^{-1}$-tube in $B_{C^0}(0, 1)$ can contain both of these $\delta$-tubes. Let $\vec{p}_1 = (x_1, y_1)$ and $\vec{p}_3 = (x_3, y_3)$ be the respective centers of $\delta$-balls $q_1$ and $q_3$ in $E_j$, and let $\vec{p}_2 = (x_2, y_2)$ and $\vec{p}_4 = (x_4, y_4)$ be the centers of $q_2$ and $q_4$ in $E_k$. We let

$$\vec{u}_1 = (u_1, v_1) = \left(\frac{y_3 - y_4}{2}, -\frac{x_3 - x_4}{2}\right) \quad \text{and} \quad \vec{u}_2 = (u_2, v_2) = \left(\frac{y_2 - y_4}{2}, -\frac{x_2 - x_4}{2}\right)$$

say $\frac{1}{2} |B_{C^2}(0, 1)| N \delta^{-2} < |E| < 2 |B_{C^2}(0, 1)| N \delta^{-2}$.
so that \((\vec{u}_1, 1)\) is the direction vector for the line
\[
l_{\vec{p}_1, \vec{p}_0} = \left\{ \left( \frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}, 0 \right) + z \left( \frac{y_1-y_3}{2}, -\frac{x_1-x_3}{2}, 1 \right) : z \in \mathbb{C} \right\},
\]
and \((\vec{u}_2, 1)\) is the direction vector for the line
\[
l_{\vec{p}_2, \vec{p}_0} = \left\{ \left( \frac{x_2+x_4}{2}, \frac{y_2+y_4}{2}, 0 \right) + z \left( \frac{y_2-y_4}{2}, -\frac{x_2-x_4}{2}, 1 \right) : z \in \mathbb{C} \right\}.
\]
The tube \(T_{q_1, q_3}\) is the \(\delta\)-neighborhood of the complex line segment
\[
\overline{l_{\vec{p}_1, \vec{p}_3}} = \left\{ \left( \frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}, 0 \right) + z \left( \frac{y_1-y_3}{2}, -\frac{x_1-x_3}{2}, 1 \right) : z \in \mathbb{C}, |z| \leq \frac{1}{2} \right\},
\]
and \(T_{q_2, q_4}\) is the \(\delta\)-neighborhood of the complex line segment
\[
\overline{l_{\vec{p}_2, \vec{p}_4}} = \left\{ \left( \frac{x_2+x_4}{2}, \frac{y_2+y_4}{2}, 0 \right) + z \left( \frac{y_2-y_4}{2}, -\frac{x_2-x_4}{2}, 1 \right) : z \in \mathbb{C}, |z| \leq \frac{1}{2} \right\}.
\]
The complex lines \(l_{\vec{p}_1, \vec{p}_0}\) and \(l_{\vec{p}_2, \vec{p}_0}\) correspond to real two-planes \(P_{1,2}\) and \(P_{2,4}\).
If we let \(x_j = a_j + b_j i\) and \(y_j = c_j + d_j i\) for \(j = 1, \ldots, 4\), then we can define parameterizations \(G_{1,3}, G_{2,4} : \mathbb{R}^2 \to \mathbb{R}^6\) by
\[
G_{1,3}(\alpha, \beta) = \left( \frac{a_1+a_3}{2}, \frac{b_1+b_3}{2}, \frac{c_1+c_3}{2}, \frac{d_1+d_3}{2}, 0, 0 \right) + \alpha \left( \frac{c_1-c_3}{2}, \frac{d_1-d_3}{2}, -\frac{(a_1-a_3)}{2}, -\frac{(b_1-b_3)}{2}, 1, 0 \right) + \beta \left( -\frac{(d_1-d_3)}{2}, \frac{c_1-c_3}{2}, \frac{b_1-b_3}{2}, -\frac{(a_1-a_3)}{2}, 0, 1 \right)
\]
and
\[
G_{2,4}(\alpha, \beta) = \left( \frac{a_2+a_4}{2}, \frac{b_2+b_4}{2}, \frac{c_2+c_4}{2}, \frac{d_2+d_4}{2}, 0, 0 \right) + \alpha \left( \frac{c_2-c_4}{2}, \frac{d_2-d_4}{2}, -\frac{(a_2-a_4)}{2}, -\frac{(b_2-b_4)}{2}, 1, 0 \right) + \beta \left( -\frac{(d_2-d_4)}{2}, \frac{c_2-c_4}{2}, \frac{b_2-b_4}{2}, -\frac{(a_2-a_4)}{2}, 0, 1 \right).
\]
The complex line segments \(\overline{l_{\vec{p}_1, \vec{p}_0}}\) and \(\overline{l_{\vec{p}_2, \vec{p}_0}}\) correspond to the respective images under \(G_{1,3}\) and \(G_{2,4}\) of the disk \(B_{\mathbb{R}^2} \left( \vec{0}, \frac{1}{2} \right) = \{ (\alpha, \beta) \in \mathbb{R}^2 : (\alpha^2 + \beta^2)^{1/2} \leq \frac{1}{2} \} \).
To show that no \(W^{-1}\)-box (essentially) contains both \(T_{1,3}\) and \(T_{2,4}\), it suffices to show that there is some point \((\alpha_0, \beta_0) \in B_{\mathbb{R}^2} \left( \vec{0}, \frac{1}{2} \right)\) so that
\[
d(G_{1,3}(\alpha_0, \beta_0), P_{2,4}) \gtrsim W^{-1}.
\]
Here, \(d(G_{1,3}(\alpha_0, \beta_0), P_{2,4})\) is the distance from the point \(G_{1,3}(\alpha_0, \beta_0)\) to the plane \(P_{2,4}\), i.e.
\[
d(G_{1,3}(\alpha_0, \beta_0), P_{2,4}) = \min_{(\alpha, \beta) \in \mathbb{R}^2} \| G_{1,3}(\alpha_0, \beta_0) - G_{2,4}(\alpha, \beta) \|.
\]
We consider the subspace \(V = \operatorname{Span}(e_1, e_2, e_3, e_4) \subseteq \mathbb{R}^6\) and note that the principal angle between \(V\) and each of \(P_{1,3}\) and \(P_{2,4}\) is bounded away from 0. (This is because \(\|\vec{u}_1\|, \|\vec{u}_2\| \sim 1\); in particular, the size of each of \(\|\vec{u}_1\|\) and \(\|\vec{u}_2\|\) is bounded above.) Further note that each of \(P_{1,3}\) and \(P_{2,4}\) intersects any translate of \(V\) in a single point. This is because a basis for either, along with a basis for \(V\), forms a basis for \(\mathbb{R}^6\).
Fixing \((\alpha_0, \beta_0) \in B_{\mathbb{R}^2} \left( \vec{0}, \frac{1}{2} \right)\), we consider the affine subspace
\[
V_{\alpha_0, \beta_0} := V + (0, 0, 0, 0, \alpha_0, \beta_0).
\]
Since the principal angle between $V_{\alpha_0, \beta_0}$ and each of $P_{1,3}$ and $P_{2,4}$ is bounded away from 0, it follows that
\[ d(G_{1,3}(\alpha_0, \beta_0), P_{2,4}) \gtrsim \|G_{1,3}(\alpha_0, \beta_0) - G_{2,4}(\alpha_0, \beta_0)\|, \] (8)
or, in words, the line segment from $G_{1,3}(\alpha_0, \beta_0)$ to the point where $P_{2,4}$ intersects the slice $V_{\alpha_0, \beta_0}$ is not much longer than the path from $G_{1,3}(\alpha_0, \beta_0)$ to $P_{2,4}$ that is perpendicular to $P_{2,4}$. Specifically, the line segment in the slice $V_{\alpha_0, \beta_0}$ is at most $O(1)$ time as long as the perpendicular path.

Thus, to show that (7) holds for some $(\alpha_0, \beta_0)$, it suffices to find some $(\alpha_0, \beta_0)$ with
\[ \|G_{1,3}(\alpha_0, \beta_0) - G_{2,4}(\alpha_0, \beta_0)\| \gtrsim W^{-1}. \] (9)

A natural candidate for $(\alpha_0, \beta_0)$ is the point $(0, 0)$. Using the parallelogram law, we can show that either $\|G_{1,3}(0, 0) - G_{2,4}(0, 0)\| \gtrsim W^{-1}$ or that there is a point $(\alpha_0, \beta_0)$ 'far away' from $(0, 0)$ but still in $B_{\delta/2} \left( \frac{1}{2}, \frac{1}{4} \right)$ for which (9) holds. Here are the details: Recall that the points $\bar{p}_1, \ldots, \bar{p}_4$ are at least $\delta^{\alpha/4}$-separated, and apply the parallelogram law to the vectors $\|\bar{p}_1 - \bar{p}_2\|$ and $\|\bar{p}_3 - \bar{p}_4\|$ to get
\[ 2\|\bar{p}_1 - \bar{p}_2\|^2 + 2\|\bar{p}_3 - \bar{p}_4\|^2 = \|(\bar{p}_1 - \bar{p}_2) + (\bar{p}_3 - \bar{p}_4)\|^2 + \|(\bar{p}_1 - \bar{p}_2) - (\bar{p}_3 - \bar{p}_4)\|^2 \]
\[ = \|(\bar{p}_1 + \bar{p}_3) - (\bar{p}_2 + \bar{p}_4)\|^2 + \|(\bar{p}_1 - \bar{p}_3) - (\bar{p}_2 - \bar{p}_4)\|^2. \] (10)

Note that
\[ \|(\bar{p}_1 + \bar{p}_3) - (\bar{p}_2 + \bar{p}_4)\| = 2\|G_{1,3}(0, 0) - G_{2,4}(0, 0)\|. \]

Meanwhile, to estimate $\|G_{1,3}(\alpha, \beta) - G_{2,4}(\alpha, \beta)\|$ for $(\alpha, \beta) \neq (0, 0)$, we begin by writing
\[ G_{1,3}(\alpha, \beta) - G_{2,4}(\alpha, \beta) = \left( \frac{(a_1 + a_2) - (a_2 + a_3)}{2}, \frac{(b_1 + b_3) - (b_2 + b_4)}{2}, \frac{(c_1 + c_3) - (c_2 + c_4)}{2}, \frac{(d_1 + d_3) - (d_2 + d_4)}{2}, 0, 0 \right) \]
\[ + \alpha \left( \frac{(c_1 - c_3) - (c_2 - c_4)}{2}, \frac{(d_1 - d_3) - (d_2 - d_4)}{2}, \frac{(a_1 - a_3) + (a_2 - a_4)}{2}, \frac{-(b_1 - b_3) + (b_2 - b_4)}{2}, 0, 0 \right) \]
\[ + \beta \left( \frac{-(d_1 - d_3) - (d_2 - d_4)}{2}, \frac{(c_1 - c_3) - (c_2 - c_4)}{2}, \frac{(b_1 - b_3) - (b_2 - b_4)}{2}, \frac{-(a_1 - a_3) + (a_2 - a_4)}{2}, 0, 0 \right). \]

We note that the latter two vectors (the ones with coefficients $\alpha$ and $\beta$) are orthogonal to each other and both have size $\|\left( \frac{y_1 - y_4}{2}, -\frac{x_1 - x_4}{2}, 1 \right) - \left( \frac{y_2 - y_4}{2}, -\frac{x_2 - x_4}{2}, 1 \right)\|$. Thus,
\[ \|G_{1,3}(\alpha, \beta) - G_{2,4}(\alpha, \beta)\| \]
\[ \geq \|(\alpha, \beta)\| \left( \|\left( \frac{y_1 - y_4}{2}, -\frac{x_1 - x_4}{2}, 1 \right) - \left( \frac{y_2 - y_4}{2}, -\frac{x_2 - x_4}{2}, 1 \right)\| + \|\left( \frac{x_1 + x_4}{2}, \frac{y_1 + y_4}{2}, 0 \right) - \left( \frac{x_2 + x_4}{2}, \frac{y_2 + y_4}{2}, 0 \right)\| \right) \]
\[ = \left( \frac{\|\bar{p}_1 - \bar{p}_3\| - \|\bar{p}_2 - \bar{p}_4\|}{2} \right) - \frac{1}{2} \|(\bar{p}_1 + \bar{p}_3) - (\bar{p}_2 + \bar{p}_4)\| \]
\[ \geq \|(\bar{p}_1 - \bar{p}_3) - (\bar{p}_2 - \bar{p}_4)\|. \] (11)

This motivates us to take cases on the relative sizes of $\|\bar{p}_1 + \bar{p}_3\| - \|\bar{p}_2 + \bar{p}_4\|$ and $\|\bar{p}_1 - \bar{p}_3\| - \|\bar{p}_2 - \bar{p}_4\|$.}

**Case 1:** First, suppose that
\[ \|\bar{p}_1 - \bar{p}_3\| - \|\bar{p}_2 - \bar{p}_4\| \leq 16 \|\bar{p}_1 + \bar{p}_3\| - \|\bar{p}_2 + \bar{p}_4\| \]. (12)
(Our choice of 16 above is rather arbitrary; we could replace 16 with any sufficiently large constant.) In this case, (10) becomes
\[
\|\vec{p}_1 - \vec{p}_2\|^2 + \|\vec{p}_3 - \vec{p}_4\|^2 \lesssim 257 \|\vec{p}_1 + \vec{p}_3\| \cdot \|\vec{p}_2 + \vec{p}_4\|^2.
\]
Since \(\|\vec{p}_1 - \vec{p}_2\|, \|\vec{p}_3 - \vec{p}_4\| \gtrsim \delta^{s/4}\), this implies that
\[
\|\vec{p}_1 + \vec{p}_3\| - \|\vec{p}_2 + \vec{p}_4\| \gtrsim \delta^{s/4},
\]
which is to say
\[
\|G_{1,3}(0,0) - G_{2,4}(0,0)\| \gtrsim \delta^{s/4}.
\]

**Case 2**: Now, suppose that
\[
\|\vec{p}_1 - \vec{p}_2\| - \|\vec{p}_3 - \vec{p}_4\| \gtrsim 16 \|\vec{p}_1 + \vec{p}_3\| - \|\vec{p}_2 + \vec{p}_4\|.
\]
In this case, (11) becomes
\[
\|G_{1,3}(\alpha, \beta) - G_{2,4}(\alpha, \beta)\| \gtrsim \frac{\delta}{16} \|\vec{p}_1 - \vec{p}_2\| - \|\vec{p}_3 - \vec{p}_4\| - \frac{1}{16} \|\vec{p}_1 - \vec{p}_2\| - \|\vec{p}_3 - \vec{p}_4\|.
\]
It follows that if \(\|\alpha, \beta\| \gtrsim \frac{1}{8}\), then
\[
\|G_{1,3}(\alpha, \beta) - G_{2,4}(\alpha, \beta)\| \gtrsim \left(\frac{1}{16} - \frac{1}{32}\right) \|\vec{p}_1 - \vec{p}_2\| - \|\vec{p}_3 - \vec{p}_4\|
\]
\[
\gtrsim \frac{1}{32} \|\vec{p}_1 - \vec{p}_2\| - \|\vec{p}_3 - \vec{p}_4\|.
\]
And again by (10), we have \(\|\vec{p}_1 - \vec{p}_2\| - \|\vec{p}_3 - \vec{p}_4\| \gtrsim \delta^{s/4}\). Thus, for \(\frac{1}{8} \leq \|\alpha, \beta\|\), we have
\[
\|G_{1,3}(\alpha, \beta) - G_{2,4}(\alpha, \beta)\| \gtrsim \delta^{s/4}.
\]
In conclusion, regardless of whether (12) holds, we can find a point \((\alpha, \beta) \in B_{\delta^2} (\vec{0}, \frac{1}{4})\) that satisfies (9), as claimed.

**Proposition 5.4.** Let \(q_1, q_4 \in E_1\) and \(q_2, q_3 \in E_2\) be distinct balls. If the quadruple \((q_1, q_2, q_3, q_4)\) is in \(Q\), then the complex \(\delta\)-tubes \(T_{q_1, q_3}\) and \(T_{q_2, q_4}\) intersect in a \(\delta\)-ball.

**Proof.** Fix a quadruple \((\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4) \in q_1 \times q_2 \times q_3 \times q_4\) with
\[
|\Delta(\vec{p}_1, \vec{p}_2) - \Delta(\vec{p}_3, \vec{p}_4)| < \delta.
\]
Then, the vectors
\[
\vec{v} := (\frac{\vec{p}_1 - \vec{p}_2}{2}, -\frac{\vec{p}_1 - \vec{p}_3}{2}, 1) \quad \text{and} \quad \vec{w} := (\frac{\vec{p}_1 - \vec{p}_4}{2}, -\frac{\vec{p}_2 - \vec{p}_3}{2}, 1)
\]
are parallel to the lines \(l_{\vec{p}_1, \vec{p}_3}\) and \(l_{\vec{p}_2, \vec{p}_4}\), respectively.

**Claim 5.5.** The angle between \(l_{\vec{p}_1, \vec{p}_3}\) and \(l_{\vec{p}_2, \vec{p}_4}\) is \(\gtrsim 1\).

It suffices to show that the angle between \(\vec{v}\) and \(z\vec{w}\) is \(\gtrsim 1\) for all \(z \in \mathbb{C}\) with \(|z| = 1\). Let \(\theta_z\) denote the angle between \(\vec{v}\) and \(z\vec{w}\). Then,
\[
\cos \theta_z = \frac{\|\vec{v}\|^2 + \|\vec{w}\|^2 - \|\vec{v} - z\vec{w}\|^2}{2 \|\vec{v}\| \|\vec{w}\|}.
\]
Since \(\vec{v}\) and \(\vec{w}\) are of size \(\asymp 1\), our claim reduces to showing that \(\|\vec{v} - z\vec{w}\|^2\) is bounded away from 0, say \(\gtrsim c_0\). Well, roughly speaking, in order for both
\[ |\frac{y_1 - y_3}{2} - z| = |\frac{y_1 - y_3}{2} - z| \] and \[ |\frac{x_1 - x_3}{2} - z| = |\frac{x_1 - x_3}{2} - z| \] to be small, \( z \) needs to be closer to \(-1\) than \( 1 \), in which case \(|1 - z| \) wouldn’t be small. To give a rigorous proof, let’s set
\[
m_0 := \frac{1}{2} \left( \sqrt{\left(\frac{1}{2}C + 1\right)^2 + 1^2} + \sqrt{\left(\frac{1}{2}C - 1\right)^2 + 1^2} \right) < 2
\]
and
\[
\varepsilon_0 := \frac{1}{2} \left( \sqrt{\left(\frac{1}{2}C + 1\right)^2 + 1^2} - \sqrt{\left(\frac{1}{2}C - 1\right)^2 + 1^2} \right) < \frac{1}{1000},
\]

so that \( m_0 - \varepsilon_0 \leq \|v\|, \|w\| \leq m_0 + \varepsilon_0 \). Let \( \varepsilon_1 > 0 \) be a tiny constant, say \( \varepsilon_1 := \frac{1}{10000}. \) Assume to the contrary that there exists a \( z \in C \) with \( |z| = 1 \) such that \( \cos \theta > 1 - \varepsilon_1 \). Then,
\[
\|v - zw\| < 2(m_0 + \varepsilon_0)^2 - 2(1 - \varepsilon_1)(m_0 - \varepsilon_0)^2 < 4\varepsilon_0m_0 + 2\varepsilon_1m_0^2 < \frac{1}{100},
\]
and as a result, \( \|\frac{1}{2}(\bar{p}_1 - \bar{p}_3) - \frac{1}{2}(\bar{p}_2 - \bar{p}_4)\| \) and \(|1 - z| \) are both less than \( \frac{1}{2} \). We may temporarily assume, without loss of generality, that the centres of the balls \( B_1 \) and \( B_2 \) lie in the same “vertical” complex line (i.e. the centres share the same first coordinate). After performing orthogonal projection onto the axis \( \{0\} \times \mathbb{C} \), the spacing between the small balls \( B_1 \) and \( B_2 \) tells us that
\[
|\text{Arg}(y_1 - y_3) - \text{Arg}(y_2 - y_4)| > \frac{\pi}{2},
\]

and
\[
C_2 - 2C_1 < |y_1 - y_3|, |y_2 - y_4| < 1 + 2C_2.
\]

Hence, by the triangle inequality,
\[
\left| \frac{1}{2}(C_2 - 2C_1) \frac{y_1 - y_3}{|y_1 - y_3|} - \frac{1}{2}(C_2 - 2C_1) \frac{y_2 - y_4}{|y_2 - y_4|} \right|
\]

\[
< \frac{1}{100} + 2C_2 + 2C_2 < \frac{1}{20} < \frac{1}{2}(C_2 - 2C_1),
\]

which happens only if \(|\text{Arg}(z)| > \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}\). On the other hand,
\[
|1 - z| < \frac{1}{100} < 2\sin \frac{\pi}{12}
\]

only if \(|\text{Arg}(z)| < \frac{\pi}{6}\). Thus we have arrived at a contradiction.

**Claim 5.6.** *The distance between \( l_{\bar{p}_i, \bar{p}_i} \) and \( l_{\bar{p}_2, \bar{p}_4} \) is \( \leq \delta \), and the complex tubes \( T_{q_1, q_3} \) and \( T_{q_2, q_4} \) intersect in (essentially) a \( \delta \)-ball in \( B_{C_1} \).*

Let’s define
\[
A(s, t) := \left( \frac{\bar{p}_1 + \bar{p}_3}{2} - \frac{\bar{p}_2 + \bar{p}_4}{2}, 0 \right) + sv - tw.
\]

Observe that
\[
|\Delta(\bar{p}_1, \bar{p}_2) - \Delta(\bar{p}_3, \bar{p}_4)| \geq C_2 - 2C_1.
\]

If \(|x_1 - x_3 - x_2 + x_4| \geq |y_1 - y_3 - y_2 + y_4|\), then \(|x_1 - x_3 - x_2 + x_4| \geq \frac{1}{2}C_2 - C_1\), and, evaluating \( \|A\| \) at \( s = t = \frac{y_1 + y_3 - y_2 - y_4}{x_1 - x_3 - x_2 + x_4} \), we find that
\[
\frac{\|A(s, t)\|^2}{4|x_1 - x_3 - x_2 + x_4|} = \left( \frac{\delta}{C_2 - 2C_1} \right)^2
\]
with \( |s| = |t| \leq \frac{4C_1}{C_2 - C_1} < \frac{1}{2} \). Similarly, if \( |x_1 - x_3 - x_2 + x_4| \leq |y_1 - y_3 - y_2 + y_4| \), then, evaluating \( \|A\| \) at \( s = t = \frac{x_1 + x_3 - x_2 + x_4}{y_1 - y_3 - y_2 + y_4} \), we again have
\[
\|A(s, t)\| < \frac{\delta}{C_2 - 2C_1} < \delta,
\]
with \( |s| = |t| < \frac{1}{2} \). Thus, in either case, there exists a ball of radius \( \frac{\delta}{2} \) which contains a point from \( l_{\bar{r}_1, \bar{r}_3} \) and a point from \( l_{r_2, \bar{r}_4} \), and lies completely in \( B_{C_1}(0, \frac{\delta}{2}) \). The \( \delta \)-ball with the same centre witnesses the incidence between the complex tubes \( T_{r_1, q_3} \) and \( T_{r_2, q_4} \).

So we see that each quadruple in \( Q \) corresponds to a tube-tube incidence. As a result,
\[
|Q| \leq \sum_{r \text{ dyadic}} r^2 |P_r(T)|.
\]
For \( r \geq \delta^{4-\varepsilon}|T| \), we know from Theorem 4.2 that
\[
|P_r(T)| \lesssim \delta^{-\varepsilon}|T|^{\frac{3}{2}}r^{-2}.
\]
The same bound holds true for \( 2 \leq r < \delta^{4-\varepsilon}|T| \) since \( |T| \sim \delta^{-2s} \) and there are at most \( \sim \delta^{-6} \) essentially distinct \( \delta \)-balls in the unit ball \( B_{C_1}(0, 1) \). Hence,
\[
|Q| \lesssim \log_2 \delta^4 \cdot \delta^{-\varepsilon}|T|^{\frac{3}{2}} \sim |\log \delta| \cdot \delta^{-\varepsilon-3s}.
\]
Finally, by the Cauchy-Schwarz inequality,
\[
\#\Delta(E_1, E_2) \geq \left( \frac{|\#E_1 \cdot \#E_2|}{|Q|} \right)^2 \gtrsim |\log \delta|^{-1} \delta^{-s+\varepsilon},
\]
as desired.

6. APPENDIX

We return to the problem, introduced in Section 2, of estimating the volume of the intersection between the \( \delta \)-neighborhoods of two complex lines.

**Proposition 6.1.** Suppose that \( \ell_1, \ell_2 \subset \mathbb{C}^2 \) are two complex lines through the origin that make angle \( \theta > 0 \) to each other. If we let \( \iota : \mathbb{C}^2 \to \mathbb{R}^4 \) denote the embedding that sends \( (z_1, z_2) \) to \( (\Re(z_1), \Im(z_1), \Re(z_2), \Im(z_2)) \), then
\[
|\iota(N_\delta(\ell_1)) \cap \iota(N_\delta(\ell_2))| \sim \frac{\delta^4}{\sin^2 \theta}.
\]
More generally, if \( \ell_1, \ell_2 \) are lines through the origin in \( \mathbb{C}^n \) which make angle \( \theta > 0 \) to each other, then
\[
|\iota(N_\delta(\ell_1)) \cap \iota(N_\delta(\ell_2))| \sim \frac{\delta^{2n}}{\sin^2 \theta}.
\]
**Proof.** Let \( u_1 \) and \( u_2 \) be direction vectors for the tubes’ respective central axes. Suppose without loss of generality that \( u_1 = (1, 0) \). If \( u_2 \) is parallel to \( (0, 1) \), then \( \theta = \frac{\pi}{2} \), and \( |\iota(N_\delta(\ell_1)) \cap \iota(N_\delta(\ell_2))| = \delta^4 \). Henceforth, we may assume that \( u_2 = \frac{1}{\sqrt{1 + r^2}}(1, re^{i\zeta}) \) for some \( \zeta \in [0, 2\pi) \) and some real number \( r > 0 \). Note that in this case, we have that
\[
\sin^2 \theta = \frac{r^2}{1 + r^2}.
\]
Now, we set some notation. We define
\[ \mathbf{v}_1 = \iota(u_1) = (1, 0, 0, 0); \]
\[ \mathbf{v}_2 = \iota(iu_1) = (0, 1, 0, 0); \]
\[ \mathbf{w}_1 = \iota(u_2) = \frac{1}{\sqrt{1+r^2}}(1, 0, r \cos \zeta, r \sin \zeta); \]
\[ \mathbf{w}_2 = \iota(iu_2) = \frac{1}{\sqrt{1+r^2}}(0, 1, -r \sin \zeta, r \cos \zeta). \]

We let \( V = \text{Span}(\{ \mathbf{v}_1, \mathbf{v}_2 \}) \) and \( W = \text{Span}(\{ \mathbf{w}_1, \mathbf{w}_2 \}) \) so that \( \iota(N_\delta(\mathbf{f}_1)) = N_\delta(V) \) and \( \iota(N_\delta(\mathbf{f}_2)) = N_\delta(W) \). We write a point in \( \mathbb{R}^4 \) as \((\bar{x}, \bar{y}) \in \mathbb{R}^2 \times \mathbb{R}^2\). Then \( N_\delta(V) = \{(\bar{x}, \bar{y}) : \|\bar{y}\| \leq \delta\} \), and
\[
|\iota(T_1) \cap \iota(T_2)| = |N_\delta(V) \cap N_\delta(W)| = \int_{\mathbb{R}^4} 1_{N_\delta(V)} 1_{N_\delta(W)} \, dV
= \int_{B^2(\bar{0}, \delta)} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} 1_{N_\delta(W)}(\bar{x}, \bar{y}) \, dA_\bar{x} \right) \, dA_\bar{y},
\]
(15)

In the last line, we have used \( B^2(\bar{0}, \delta) \) to denote the ball of radius \( \delta \) about \( \bar{0} \) in \( \mathbb{R}^2 \). In the ensuing work, we will continue to use a superscript \( 2 \) when we wish to indicate a two-dimensional ball.

By Lemma 6.2, we have \((\bar{x}, \bar{y}) \in N_\delta(W)\) if and only if \( \bar{x} \) is in the ball of radius \( \sqrt{1+r^2} \) around the point \( \frac{1}{r} R_\zeta(\bar{y}) = \left( y_1 \cos \zeta + y_2 \sin \zeta, -y_1 \sin \zeta + y_2 \cos \zeta \right) \). (Here we have used \( R_\zeta \) to denote the standard matrix for rotation through an angle \( \zeta \) in the plane.) Thus, resuming from (15), we conclude that
\[
|\iota(T_1) \cap \iota(T_2)| = \int_{B^2(\bar{0}, \delta)} \left( \int_{B^2(\frac{1}{r} R_\zeta(\bar{y}), \sqrt{1+r^2})} 1 \, dA_\bar{x} \right) \, dA_\bar{y}
\sim \int_{B^2(\bar{0}, \delta)} \frac{\delta^2(1+r^2)}{r^2} \, dA_\bar{y} \sim \frac{\delta^4(1+r^2)}{\sin^2 \theta} = \frac{\delta^4}{\sin^2 \theta}.
\]

This completes the proof of (13). For \( n > 2 \), the orthogonal complements of \( \iota(\mathbf{f}_1) \) and \( \iota(\mathbf{f}_1) \) intersect in a subspace of dimension \( 2n - 4 \). Any slice of \( \iota(N_\delta(\mathbf{f}_1)) \cap \iota(N_\delta(\mathbf{f}_1)) \) by a translation of this subspace is a figure like that of the \( C^2 \) case. We can integrate the 4-dimensional cross section volume over the common short directions to arrive at (14).

**Lemma 6.2.** Let \( \mathbf{v}_1 = \iota(u_1) = (1, 0, 0, 0) \) and let \( \mathbf{v}_2 = \iota(iu_1) = (0, 1, 0, 0) \). Fixing \( r > 0 \) and \( \zeta \in [0, 2\pi) \), let \( \mathbf{w}_1 = \frac{1}{\sqrt{1+r^2}}(1, 0, r \cos \zeta, r \sin \zeta) \) and let \( \mathbf{w}_2 = \frac{1}{\sqrt{1+r^2}}(0, 1, -r \sin \zeta, r \cos \zeta) \). Let \( W = \text{Span}(\{ \mathbf{w}_1, \mathbf{w}_2 \}) \). Then we have
\[
(x_1, x_2, y_1, y_2) \in N_\delta(W) \iff (x_1, x_2) \in B^2 \left( \frac{1}{r} R_\zeta(y_1, y_2), \frac{\sqrt{1+r^2}}{r} \right),
\]
where \( R_\zeta(y_1, y_2) = (y_1 \cos \zeta + y_2 \sin \zeta, -y_1 \sin \zeta + y_2 \cos \zeta) \) is the rotation of \((y_1, y_2)\) through angle \( \zeta \).

**Proof.** Let \( \Pi_W \) denote orthogonal projection onto \( W \), and let \( \Pi_{W^\perp} \) denote orthogonal projection onto the orthogonal complement of \( W \). The point \((\bar{x}, \bar{y}) =
\((x_1, x_2, y_1, y_2)\) is in \(N_\delta(W)\) if and only if \(\|\Pi_{W^\perp}(\vec{x}, \vec{y})\| < \delta\). We compute that
\[
\|\Pi_{W^\perp}(\vec{x}, \vec{y})\|^2 = \|(\vec{x}, \vec{y}) - \Pi_{W^\perp}(\vec{x}, \vec{y})\|^2
= \|(\vec{x}, \vec{y}) - ((\vec{x}, \vec{y}) \cdot w_1)w_1 - ((\vec{x}, \vec{y}) \cdot w_2)w_2\|^2
= \frac{1}{1 + r^2} \left( (r^2 x_1^2 + r^2 x_2^2 + y_1^2 + y_2^2)
- 2r((x_1 y_1 + x_2 y_2) \cos \zeta + (x_1 y_2 - x_2 y_1) \sin \zeta) \right)
= \frac{1}{1 + r^2} \left( (rx_1 - (y_1 \cos \zeta + y_2 \sin \zeta))^2 + (rx_2 - (y_1 \sin \zeta + y_2 \cos \zeta))^2 \right)
= \frac{r}{1 + r^2} \left( (x_1 - \frac{1}{r}(y_1 \cos \zeta + y_2 \sin \zeta))^2 + (x_2 - \frac{1}{r}(y_1 \sin \zeta + y_2 \cos \zeta))^2 \right)
= \frac{r}{1 + r^2} \left\| \vec{x} - \frac{1}{r} R_\zeta(\vec{y}) \right\|^2.
\]
(Note that some steps are omitted between the second and third lines.)

We conclude that \(\|\Pi_{W^\perp}(\vec{x}, \vec{y})\| < \delta\) if and only if \((x_1, x_2)\) is within the ball of radius \(\delta \sqrt{1 + r^2}/r\) centered at the point
\[
\frac{1}{r} R_\zeta(\vec{y}) = \frac{1}{r}(y_1 \cos \zeta + y_2 \sin \zeta, -y_1 \sin \zeta + y_2 \cos \zeta),
\]
as claimed. \(\square\)

**Acknowledgement.** The first author is supported by an NSF GRFP fellowship. We would like to thank our advisor, Larry Guth, for suggesting the question and supporting us through the research. Also, we would like to thank Dominique Maldaque for discussions about the formulation of the problem.

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