Entanglement manipulation and distillability beyond LOCC

Eric Chitambar, Julio I. de Vicente, Mark W. Girard, and Gilad Gour

Department of Physics and Astronomy, Southern Illinois University, Carbondale, Illinois 62901, USA
Department of Mathematics and Statistics, University of Calgary, Alberta T2N 1N4, Canada
Institute for Quantum Computing, University of Waterloo, Ontario N2L 3G1, Canada

September 2, 2018

Abstract

When a quantum system is distributed to spatially separated parties, it is natural to consider how the system evolves when the parties perform local quantum operations with classical communication (LOCC). However, the structure of LOCC operations is exceedingly complex leaving many important physical problems unsolved. In this paper we consider generalized resource theories of entanglement based on different relaxations to the class of LOCC. The behavior of various entanglement measures are studied under non-entangling operations, as well as the newly introduced dually non-entangling and PPT-preserving operational classes. In an effort to better understand the nature of LOCC bound entanglement, we study the problem of entanglement distillation in these generalized resource theories. We show that all NPT entangled states can be distilled using operations that are both PPT and dually non-entangling. Furthermore, for any entangled state $\rho$ and any $\epsilon > 0$, we prove the existence of a non-entangling map that is $\epsilon$ close to the set of LOCC and which is able to distill pure-state entanglement from $\rho$. This finding reveals a type of fragility to the phenomenon of bound entanglement in LOCC processing. We then turn to the stochastic convertibility of multipartite pure states and show that any two states can be interconverted by any polytope approximation to the set of separable operations. Finally, as an analog to $k$-positive maps, we introduce and analyze the set of $k$-non-entangling operations.

1 Introduction

In the distant laboratory paradigm of quantum information theory, a multipartite quantum system is distributed to spatially separated parties. The individual parties are restricted to performing only local quantum operations on their respective subsystems, but they are permitted to coordinate their actions through global classical communication. Quantum operations arising from this scenario are known as LOCC (local operations and classical communication), and they emerge as the natural operational class to consider for distributed quantum information processing tasks. Entanglement becomes a resource under the restriction of LOCC; it is consumed when performing information processing tasks (such as quantum teleportation), and entangled states cannot be generated using LOCC alone. It is therefore natural to study entanglement in the framework of a quantum resource
theory, which characterizes the convertibility of entangled states under LOCC and investigates their applicability in various information-theoretic tasks (see the reviews [1] and [2]).

Despite the conceptually intuitive description of LOCC in the distant-lab setting, mathematically characterizing the set of LOCC maps is notoriously complex [3]. One way to overcome this difficulty is by relaxing the constraints of LOCC to encompass a more mathematically simple class of operations. Any task found to be impossible by these more general operations is therefore also impossible by LOCC. In fact, LOCC is not the largest class of operations that cannot create entanglement from unentangled, or separable, states. There exists a plethora of operational classes “beyond” LOCC that act invariantly on the set of separable states, and each of these defines a different quantum resource theory of entanglement manipulation.

It is interesting to study what differences arise in these generalized resource theories of entanglement compared to the standard LOCC scenario. Doing so has already proven successful in the asymptotic scenario where, under LOCC, certain mixed states demonstrate irreversibility in terms of pure-state entanglement distillation and the reverse process of entanglement dilution [4, 5]. The most dramatic example of this irreversibility is the phenomenon of bound entanglement [6], which refers to entangled mixed states that require the consumption of pure-state entanglement for their preparation, and yet no pure-state entanglement can be distilled back from them. Remarkably, the theory of entanglement distillation and dilution becomes much more amenable and elegant after enlarging the class of operations to include all asymptotically non-entangling operations. The work in [7, 8, 9] shows that bound entanglement no longer exists under this larger class of operations, and more importantly, the entanglement in any mixed state can be reversibly distilled and diluted at a rate given by the regularized relative entropy of entanglement.

In this paper we analyze in detail entanglement manipulation using operations beyond LOCC. Specifically, we study which state transformations are possible and which entanglement measures retain their validity within generalized resource theories of entanglement. One of our main motivations is to better understand the nature of bound entanglement within the standard framework of LOCC. While it is known that no pure-state entanglement can be distilled from any state having a positive partial transpose (PPT), one of the major open problems in quantum information theory is to determine if the converse is also true [10]; namely, does there exists non-PPT (NPT) bound entanglement? A strategy that would allow us to answer this question in the affirmative would be to identify a class of operations that contains LOCC and which lacks the ability to distill certain NPT states.

The first comprehensive attempt in this direction was carried out in [11], where it was shown that NPT bound entanglement does not exist when the class of operations is enlarged to contain all so-called PPT operations, as originally introduced in [12, 13] (see also Sect. 2 for definitions). Even stronger, the results of [7, 8, 9] imply that no bound entanglement (not even PPT bound entanglement) exists for the class of all strictly non-entangling maps. Here, our primary focus will be on classes of operations that are larger than LOCC (for which bound entanglement exists) but smaller the the class of non-entangling maps (for which there is none). We perform a systematic study of operations laying between these two extremes, and we analyze their distillation power.

In short, the operational classes we explore are as follows. Separable maps are those whose Choi operator is separable [14], while non-entangling maps are precisely those that do not produce an entangled output state for any separable input. The class of dually non-entangling maps, which we introduce here, consists of channels $\Lambda$ such that both $\Lambda$ and its dual map $\Lambda^*$ are non-entangling. We also consider two classes of operations that estimate LOCC and separable maps in terms of
Table 1: A summary of the entanglement distillation powers for different operational classes studied in this paper.

| Type of map                                      | PPT Bound Entanglement | NPT Bound Entanglement |
|-------------------------------------------------|------------------------|------------------------|
| LOCC                                            | Yes                    | ??                     |
| Separable                                       | Yes                    | ??                     |
| Dually non-entangling and PPT                    | Yes                    | No                     |
| Dually non-entangling and $\epsilon$-LOCC       | No                     | No                     |
| Entangling undetected                           | No                     | No                     |
| $k$-non-entangling (for $1 < k < d$)             | ??                     | ??                     |

geometrical proximity. The first consists of $\epsilon$-LOCC maps which are maps that are $\epsilon$-close to the set of LOCC channels. The second class is based on the idea of entanglement witnesses applied at the level of Choi operators. For every finite collection of such entanglement witness the channels whose Choi matrices are not detected by any one of the witnesses constitutes a class that we call entangling undetected operations. A summary of our findings for the problem of entanglement distillation is contained in Figure 1. Our work shows that bound entanglement is a rather delicate phenomenon that vanishes as one moves ever so slightly beyond separable or PPT operations.

Beyond the question of distillability, we also examine the difference between the class of channels whose Choi operator is PPT and the class of PPT-preserving channels. While the former refers to the well-known PPT operations originally defined by Rains [12, 13], to our knowledge the latter has not been previously studied. Finally, analogous to the relation between completely positive and $k$-positive maps, we also introduce here the classes of $k$-non-entangling and completely non-entangling maps. The former refers to maps that are non-entangling when tensored with the identity on a $k$-dimensional ancilla, while the later refers to maps that are $k$-non-entangling for all $k$.

The rest of the paper is structured as follows. Section 2 introduces the necessary background and notation, including precise definitions for the classes of maps that we study. Section 3 investigates how certain entanglement measures behave under different classes of non-entangling operations. The entanglement measures studied in this section include the robustness of entanglement, Schmidt rank, the Rényi $\alpha$-entropies of entanglement (as well as the relative Rényi entropies of entanglement), and the negativity. Although the robustness remains monotonic under non-entangling operations, we use the robustness to derive conditions for the convertibility of pure states under non-entangling operations that are independent of the celebrated majorization criterion for LOCC operations [15]. We also show that the Rényi $\alpha$-entropies of entanglement (which are entanglement measures for all $\alpha \in [0, +\infty)$) can be increased arbitrarily under non-entangling operations for $\alpha \in [0, 1/2]$. On the other hand, we show that the $\alpha$-entropies of entanglement in the range $\alpha \in [1/2, +\infty]$ coincide with a related relative Rényi entropy of entanglement measure, and therefore they cannot increase under non-entangling operations. In particular, this gives a closed-form expression for the relative $\alpha$-entropies of entanglement in certain cases. We also demonstrate that the Schmidt rank can be increased under the more restrictive class of dually non-entangling maps. Finally, we show that the negativity can be increased by an arbitrarily large fractional amount under PPT-preserving maps, thus highlighting the difference between the classes of PPT and PPT-preserving maps.

Entanglement distillation by classes of operations larger than LOCC is investigated in Section
4. We show that all entangled states are distillable under dually non-entangling operations, as well as under non-entangling $\epsilon$-LOCC operations for any $\epsilon > 0$. While it is known that all non-PPT entangled states are distillable under PPT operations, we show that such states are still distillable under the more restrictive class of dually non-entangling PPT operations as well. We go further and demonstrate the distillability of any state (even separable ones) under $\epsilon$-LOCC operations as well as any class of class of entangling undetected operations.

In Section 5, we relax the constraint of trace-preserving maps and turn to the problem of stochastic convertibility between pure states. For multipartite state spaces, a natural way to categorize entanglement is in terms of stochastic convertibility under LOCC. That is, two states $|\psi\rangle$ and $|\tilde{\psi}\rangle$ belong to the same entanglement class if there is invertible LOCC transformation from $|\psi\rangle$ to $|\tilde{\psi}\rangle$ that succeeds with some nonzero probability [16]. We show that this entanglement structure completely collapses under any class of entangling undetected operations. Connections to the problem of tensor rank calculation are also discussed.

The structure of $k$-non-entangling maps and $k$-PPT-preserving maps are analyzed in Section 6. We show that complete non-entangling is equivalent to $d$-non-entangling when the local systems have dimension $d$, and we show that the structure of $k$-non-entangling maps is nontrivial. That is, for every $k < d$ there exists $k$-non-entangling maps that are not $(k + 1)$-non-entangling. We also investigate distillability under this class of maps but we have not been able to answer whether or not they allow for bound entanglement. Lastly, a detailed proof of Theorem 4 is presented in Section 7 while a concluding discussion is given in Section 8.

2 Notation and definitions

Let $L(H)$ denote the space of linear operators on a Hilbert space $H$. An operator $\rho$ is in the set $D(H)$ of density operators (or states) on $H$ if $\rho \geq 0$ and $\text{Tr}(\rho) = 1$. The tensor product of two Hilbert spaces $H_A$ and $H_B$ is denoted by $H_A \otimes H_B$ or $H_{AB}$. The identity operator on a Hilbert space $H_A$ is denoted $1_A$, while the identity map on $L(H_A)$ is denoted $\text{id}_A$ and the identity operator on $C^d \otimes C^d$ is denoted by $1_{C^d}$.

We write $\psi$ to denote the density operator $\psi = |\psi\rangle\langle\psi|$ associated with a pure state vector $|\psi\rangle \in H$. For bipartite pure states $|\psi\rangle \in H_{AB}$ we typically assume, without loss of generality, that $|\psi\rangle$ is in Schmidt form,

$$|\psi\rangle = \sum_{i=1}^{d} \sqrt{\lambda_i} |ii\rangle,$$

where $\lambda = (\lambda_1, \ldots, \lambda_d)$ are the Schmidt coefficients of $\psi$ (in decreasing order) such that $\sum_{i=1}^{d} \lambda_i = 1$ and $d \leq \min(\text{dim} H_A, \text{dim} H_B)$. The Schmidt rank of the state is defined to be the number of non-zero entries in $\lambda$, which we denote by $\text{rank}(\lambda)$. A positive operator $\sigma \in L(H_{AB})$ is said to be separable with respect to $A : B$ if it can be written as $\sigma = \sum_i \tau_i \otimes \omega_i$ for some positive operators $\tau_i \in L(H_A)$ and $\omega_i \in L(H_B)$. The set of separable density operators on $H_{AB}$ will be denoted $S(H_{AB})$, or simply $S$ if the spaces $A$ and $B$ are clear from context.

The partial transpose of an operator $X \in L(H_{AB})$, denoted $X^{\Gamma_A}$, is the operator that results from applying the transposition map $A \mapsto A^T$ to system $A$. We simply write $X^{\Gamma}$ if the system being transposed is clear from context. A positive operator $\rho \in L(H_{AB})$ is positive under partial transpose (PPT) with respect to $A:B$ if $\rho^{\Gamma} \geq 0$, and is otherwise said to be non-PPT (or NPT). It is
well-known that every separable state is PPT, with the converse holding true only in dimensions $2 \otimes 2$ and $2 \otimes 3$ [17].

For any integer $d \geq 2$, the maximally entangled pure state on $\mathbb{C}^d \otimes \mathbb{C}^d$ is denoted

$$|\phi^+_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle,$$

and we write $\phi^+_d = |\phi^+_d\rangle\langle\phi^+_d|$ to denote the corresponding density operator. We denote the so-called ‘flip’ operator on $\mathbb{C}^d \otimes \mathbb{C}^d$ by

$$F_d = \sum_{i,j=1}^{d} |ij\rangle\langle ji|,$$

which is precisely the partial transpose of $\phi^+_d$ scaled by $d$. Analogously, we write $\phi^+_A$ to denote the (unnormalized) maximally entangled operator on the Hilbert space $H_A \otimes H_A$,

$$\phi^+_A = \sum_{i,j=1}^{\dim(H_A)} |ii\rangle\langle jj|,$$

where $|ii\rangle$ is an orthonormal basis for $H_A$. Note that we can write the maximally entangled operator of the four-partite space $\phi^+_A \in L(H_A^2 \otimes H_B^2)$ as

$$\phi^+_A = \phi^+_A \otimes \phi^+_B.$$

Given a linear map $\Lambda : L(H_A^1) \to L(H_A^2)$, its Choi operator is the linear operator $J(\Lambda) \in L(H_A^2 \otimes H_A^1)$ defined by

$$J(\Lambda) = \Lambda \otimes \text{id}_{A^1}(\phi^+_A).$$

For any $X \in L(H_A^1)$, the action of the map $\Lambda$ on $X$ can be written in its Choi representation by $\Lambda(X) = \text{Tr}_{A_1}(J(\Lambda)(\mathbb{1}_{A_2} \otimes X^T))$. Recall that $\Lambda$ is completely positive (CP) if and only if its Choi representation is a positive semidefinite operator, and $\Lambda$ is trace preserving (TP) if and only if it holds that $\text{Tr}_{A_2}(J(\Lambda)) = \mathbb{1}_{A_2}$. Given two density operators $\rho_1, \rho_2 \in D(H')$ and an operator $A \in L(H)$, the linear map $\Lambda : L(H) \to L(H')$ defined by

$$\Lambda(X) = \text{Tr}(AX)\rho_1 + \text{Tr}((\mathbb{1} - A)X)\rho_2$$

is CPTP if $0 \leq A \leq \mathbb{1}$. Many of the maps constructed in this work take this form. Given a linear map $\Lambda : L(H) \to L(H')$, its dual map $\Lambda^* : L(H') \to L(H)$ is the unique linear map such that $\text{Tr}(\Lambda(X)Y) = \text{Tr}(X\Lambda^*(Y))$ for all operators $X \in L(H)$ and $Y \in L(H')$. It follows readily from the definition that the dual of a map of the form in (1) is given by

$$\Lambda^*(Y) = \text{Tr}(\rho_1 Y)A + \text{Tr}(\rho_2 Y)(\mathbb{1} - A),$$

while the Choi representation of a map of the form in (1) is given by

$$J(\Lambda) = \rho_1 \otimes A^T + \rho_2 \otimes (\mathbb{1} - A)^T.$$
2.1 Definitions of classes of operations

We now define all of the classes of channels that will be explored in this paper. We first recall the definitions of separable and PPT channels. Let \( \Lambda : \mathcal{L}(\mathcal{H}_{A_1:B_1}) \rightarrow \mathcal{L}(\mathcal{H}_{A_2:B_2}) \) be a completely positive linear map. Then \( \Lambda \) is said to be separable if its Choi representation \( J(\Lambda) \in \mathcal{L}(\mathcal{H}_{A_2:B_2} \otimes \mathcal{H}_{A_1:B_1}) \) is separable with respect to \( A_2:A_1:B_2:B_1 \). To keep the bipartition clear, we will often denote \( \mathcal{H}_{A_2:A_1:B_2:B_1} \) as \( \mathcal{H}_{A_2:A_1:B_2:B_1} \). Note that \( \Lambda \) is separable if and only if it can be written as

\[
\Lambda = \sum_j \Phi_j \otimes \Psi_j
\]

for some CP maps \( \Phi_j : \mathcal{L}(\mathcal{H}_{A_1}) \rightarrow \mathcal{L}(\mathcal{H}_{A_2}) \) and \( \Psi_j : \mathcal{L}(\mathcal{H}_{B_1}) \rightarrow \mathcal{L}(\mathcal{H}_{B_2}) \). Analogously, the map \( \Lambda \) is said to be PPT if the partial transpose of its Choi representation is positive semi-definite, i.e., if \( J(\Lambda)^T = J(\Lambda)^{\Gamma_{A_2:A_1}} \geq 0 \). It holds that \( \Lambda \) is PPT if and only if the map \( \Lambda^\Gamma \) defined by

\[
\Lambda^\Gamma (X) = (\Lambda(X^{\Gamma_{A_1}}))^{\Gamma_{A_2}}
\]

is also completely positive. In particular, note that \( J(\Lambda^\Gamma) = J(\Lambda)^T \). This definition of PPT channels is due to Rains [12].

In this paper, we are primarily concerned with the non-entangling operations, which are defined explicitly as follows. We provide here an analogous definition for PPT preserving maps.

**Definition 2.1.** Let \( \Lambda : \mathcal{L}(\mathcal{H}_{A_1:B_1}) \rightarrow \mathcal{L}(\mathcal{H}_{A_2:B_2}) \) be a completely positive map. Then \( \Lambda \) is separable preserving (or non-entangling) if \( \Lambda(\sigma) \) is separable (with respect to \( A_2:A_1:B_2:B_1 \)) for any operator \( \sigma \) that is separable (with respect to \( A_1:B_1 \)). Analogously, \( \Lambda \) is PPT-preserving if \( \Lambda(\sigma) \) is PPT for any PPT operator \( \sigma \).

By definition, the non-entangling maps comprise the largest class of operations which cannot create entanglement, while the PPT-preserving maps are exactly those that cannot generate an NPT state from PPT inputs. It is clear that every separable (resp. PPT) map is separable preserving (resp. PPT-preserving).

We will also consider the following set of operations that is even more restricted than the non-entangling operations which nonetheless contains the set of LOCC operations.

**Definition 2.2.** A CPTP map \( \Lambda : \mathcal{L}(\mathcal{H}_{A_1:B_1}) \rightarrow \mathcal{L}(\mathcal{H}_{A_2:B_2}) \) is said to be dually non-entangling if \( \Lambda \) is non-entangling and the output of its dual map \( \Lambda^\Gamma(\rho) \) is separable for any separable density operator \( \rho \in \mathcal{S} \).

Note that a map is completely positive if and only if its dual map is, but the dual of a trace-preserving map is not necessarily trace-preserving. Dually non-entangling operations form a subset of the non-entangling operations which still contains all of LOCC. Indeed, the dual of any separable operation is clearly separable, hence separable operations are dually non-entangling (and thus LOCC operations are as well).

Any quantum channel is a linear map that must not only preserve positivity, but it must also preserve positivity when it acts on the systems tensored with any ancilla space for linear map to represent a physical manipulation of quantum states. Motivated by this requirement of complete positivity for a quantum channel, in the following we define a notion of complete non-entangling. The analogous definition is provided for PPT preserving maps.
Definition 2.3. Let \( \Lambda : \mathcal{L}(\mathcal{H}_{A:B}) \rightarrow \mathcal{L}(\mathcal{H}_{A:B}) \) be a completely positive map and let \( k \geq 1 \) be an integer. Then \( \Lambda \) is \( k \)-separable-preserving (or \( k \)-non-entangling) if, for all \( k \)-dimensional systems \( \mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^k \), the map \( \Lambda \otimes \text{id}_{AB} \) is separable preserving (with respect to \( A_1A : B_1B \) and \( A_2A : B_2B \)). If \( \Lambda \) is \( k \)-non-entangling for all integers \( k \), then \( \Lambda \) is said to be completely non-entangling.

Definition 2.4. Let \( \Lambda : \mathcal{L}(\mathcal{H}_{A:B}) \rightarrow \mathcal{L}(\mathcal{H}_{A:B}) \) be a completely positive map and let \( k \geq 1 \) be an integer. Then \( \Lambda \) is \( k \)-PPT-preserving if, for all \( k \)-dimensional systems \( \mathcal{H}_A \) and \( \mathcal{H}_B \), the map \( \Lambda \otimes \text{id}_{AB} \) is PPT preserving. If \( \Lambda \) is \( k \)-PPT-preserving for all integers \( k \), then it is said to be completely PPT-preserving.

Lastly, we present two families of operations that are based on approximating the set of LOCC or separable operations in the topological sense. We capture the distance between two operators \( A, B \in \mathcal{L}(\mathcal{H}_{AB}) \) in terms of the trace norm \( \|A - B\|_1 \), where \( \|X\|_1 = \text{Tr} \sqrt{X^\dagger X} \).

Definition 2.5. For any \( \epsilon \geq 0 \), a completely positive map \( \Lambda : \mathcal{L}(\mathcal{H}_{A:B}) \rightarrow \mathcal{L}(\mathcal{H}_{A:B}) \) is called \( \epsilon \)-LOCC if there exists an LOCC map \( \Lambda_0 : \mathcal{L}(\mathcal{H}_{A:B}) \rightarrow \mathcal{L}(\mathcal{H}_{A:B}) \) such that \( \|\Lambda - \Lambda_0\|_1 \leq \epsilon \).

For our purposes, the choice of trace norm in this definition is not crucial. If \( \epsilon = 0 \), then this is simply the class of LOCC maps.

For the final family of operations, we recall the notion of entanglement witnesses. A hermitian operator \( W \in \mathcal{L}(\mathcal{H}_{AB}) \) is an entanglement witness if \( \text{Tr}(W \rho) \geq 0 \) for all separable states \( \rho \in S(\mathcal{H}_{AB}) \). Since the Choi operator for any separable map is separable, we can use entanglement witnesses on the level of Choi operators to detect non-separable maps. Even stronger, it is known that entanglement witnesses provide a full characterization of separability in the sense that an operator is separable if and only if \( \text{Tr}(WX) \geq 0 \) for all entanglement witnesses \( W \) [17]. Geometrically, every entanglement witness can be seen as a hyperplane in the space of hermitian operators that separates the positive convex cone of separable operators (see Fig. 1). Furthermore, the set of separable operators is not a polytope [18, 19], and therefore, any finite set of entanglement witnesses is insufficient to determine separability. Therefore, one obtains a strict relaxation of the separability constraint by requiring that \( \text{Tr}(WX) \geq 0 \) hold for not all witnesses, but instead for just some finite subset of witnesses. This idea motivates the following type of operations.

Definition 2.6. Let \( \{W_i\}_{i=1}^n \) be a finite collection of entanglement witnesses on \( \mathcal{H}_{A:A_i:B_i} \). A completely positive map \( \Lambda : \mathcal{L}(\mathcal{H}_{A:B}) \rightarrow \mathcal{L}(\mathcal{H}_{A:B}) \) is said to be entangling undetected by \( \{W_i\}_{i=1}^n \) if \( \text{Tr}(W_iJ(\Lambda)) \geq 0 \) for all \( i = 1, \cdots, n \).

Note that for any convex set in \( \mathbb{R}^d \), a containing polytope can be constructed that approximates the set to arbitrary precision [20]. Thus for any \( \epsilon > 0 \), there exists a collection of entanglement witnesses \( \{W_i\}_{i=1}^n \) such that a non-separable \( \Lambda \) is entangling undetected by \( \{W_i\}_{i=1}^n \) only if \( \|J(\Lambda) - \Omega\|_1 < \epsilon \) for all separable \( \Omega \in \mathcal{L}(\mathcal{H}_{A:A_i:B_i}) \).

3 Entanglement measures under maps beyond LOCC

In this section we explore what kinds of transformations among states are possible under non-entangling operations but not possible under LOCC. This is done by finding transformations under non-entangling operations that increase some entanglement measure. We examine which entanglement measures retain their monotonicity under non-entangling operations, dually non-entangling operations, PPT-preserving operations, and 2-symmetric extendible operations.
3.1 Entanglement measures under non-entangling maps

We first explore certain entanglement measures that can be increased with non-entangling operations, and show how certain other measures can yield conditions for conversion under such maps. Consider the relative entropy of entanglement [21, 22],

\[ E_R(\rho) := \min_{\sigma \in S} S(\rho\|\sigma), \]

where \( S(\rho\|\sigma) := \text{Tr}(\rho \log \rho) - \text{Tr}(\rho \log \sigma) \) is the relative entropy of \( \rho \) and \( \sigma \). This is an entanglement measure that is clearly also monotonic under non-entangling operations, a fact that follows directly from the joint monotonicity of the relative entropy, i.e. \( S(\Lambda(\rho)\|\Lambda(\sigma)) \leq S(\rho\|\sigma) \) for every CPTP map \( \Lambda \) [23]. Another entanglement measure that retains its monotonicity under non-entangling operations is the robustness of entanglement [24], which is defined as

\[ R(\rho) := \min_{\sigma \in S} R(\rho\|\sigma), \]

where, for any separable state \( \sigma \in S \),

\[ R(\rho\|\sigma) := \min\{s : \rho + s\sigma \text{ is separable}\}. \]

Indeed, if \( \Lambda \) is a non-entangling map, it is clearly seen that \( R(\Lambda(\rho)\|\Lambda(\sigma)) \leq R(\rho\|\sigma) \) for any separable \( \sigma \). For pure states, the robustness reduces to [24]

\[ R(\psi) = \left( \sum_i \sqrt{\lambda_i} \right)^2 - 1. \]

In particular, it holds that \( R(\phi_k^+) = k - 1 \) for any integer \( k \geq 2 \).

Here we show that the robustness yields necessary and sufficient conditions for certain transformations among pure states under non-entangling operations. Whereas pure state transformation under LOCC is governed strictly by majorization of Schmidt coefficients [15], certain transformations among pure states under non-entangling operations are possible if and only if the robustness of one is greater than the robustness of the other. As Theorem 1 shows, this holds when the input state is a maximally entangled pure state.
**Theorem 1.** Let \( \psi \) be a pure state. There exists a non-entangling map \( \Lambda \) such that \( \Lambda(\phi_k^+) = \psi \) if and only if \( R(\psi) \leq R(\phi_k^+) \).

**Proof.** Since we have already pointed out that \( R \) cannot increase under non-entangling operations, it remains to construct an explicit map achieving the transformation whenever \( R(\psi) \leq R(\phi_k^+) \). This is done by the map defined by

\[
\Lambda(X) = \text{Tr}(\phi_k^+ X)\psi + \text{Tr}((1 - \psi^+)X)\sigma,
\]

where \( \sigma \) is the separable state for which \( R(\psi) = R(\psi||\sigma) \). To see that this map is non-entangling, note that \( \text{Tr}(\phi_k^+ \rho) \leq 1/k \) holds for all \( \rho \in S \). Therefore, for all \( \rho \in S \), it holds that \( \Lambda(\rho) = p\psi + (1 - p)\sigma \) for some \( p \leq 1/k \). Note that \( 1/k = 1/(1 + R(\phi_k^+)) \) and that \( p\psi + (1 - p)\sigma \in S \) for any \( p \leq 1/(1 + R(\psi)) \) by construction. Thus the above map is non-entangling whenever \( R(\psi) \leq R(\phi_k^+) \). □

Theorem 1 indicates that the Schmidt rank of a pure state (which cannot be increased under LOCC or separable operations [25]) can in certain cases be increased under non-entangling operations. In fact, for any integers \( k \) and \( d \) with \( k < d \), there exists a pure state with Schmidt rank \( k \) that can be transformed to a state with Schmidt rank \( d \). This fact is formalized in the following corollary, which follows directly from Theorem 1.

**Corollary 2.** For all integers \( k \) and \( d \) with \( 2 \leq k < d \), there exists a non-entangling map \( \Lambda \) such that \( \Lambda(\phi_k^+) \) is a pure state with Schmidt rank \( d \).

The above result demonstrates that the Schmidt rank is not monotonic under non-entangling operations. It is natural to ask then which other entanglement measures remain monotonic under non-entangling operations and which ones do not. We now investigate other entanglement measures that can be increased under non-entangling operations. The Rényi entropy of entanglement of order \( \alpha \) is defined in terms of the Schmidt coefficients of pure states as

\[
E_\alpha(\psi) = H_\alpha(\lambda)
\]

for any \( \alpha \in [0, +\infty) \), where \( H_\alpha \) are the Rényi entropies

\[
H_\alpha(\lambda) = \begin{cases} 
\frac{1}{\alpha} \log \sum_i \lambda_i^\alpha & \alpha \neq 0, 1, +\infty \\
-\sum_i \lambda_i \log \lambda_i & \alpha = 1 \\
-\log \text{rank}(\lambda) & \alpha = 0 \\
-\log \max_i \{\lambda_i\} & \alpha = +\infty.
\end{cases}
\]

These quantities are Schur concave as functions of the Schmidt coefficients, hence \( E_\alpha \) is an entanglement measure for pure states. Notice that the Schmidt rank of a pure state is in direct correspondence with the Rényi entropy of order zero. Since we have seen that \( E_0 \) can be increased under non-entangling operations, it is natural to consider whether \( E_\alpha \) is not monotonic under non-entangling operations for other values of \( \alpha \) as well. The Rényi entropy of order 1 reduces to the standard entropy of entanglement,

\[
E_1(\psi) = E(\psi) = -\sum_i \lambda_i \log(\lambda_i),
\]
which is known to be monotonic under non-entangling operations since it coincides with the relative entropy of entanglement $E_R$ for pure states [22]. Furthermore, note from (4) that the Rényi entropy of entanglement of order $1/2$ is a monotonic function of the robustness, since

$$E_{1/2}(\psi) = \log(R(\psi) + 1),$$

so $E_{1/2}$ is also monotonic under non-entangling operations by monotonicity of $R$.

In the following, we show that $E_\alpha$ can be increased under non-entangling operations for any $\alpha \in [0, 1/2)$, while it cannot be increased under non-entangling operations for any $\alpha \geq 1/2$. The proof of the former is actually a simple corollary of Theorem 1. To show this, it suffices to find a pure state $\psi$ with $E_{1/2}(\psi) = E_{1/2}(\phi_2^+)$ but that $E_\alpha(\psi) > E_\alpha(\phi_2^+)$ for $\alpha < 1/2$. Since $E_{1/2}(\phi_2^+) = 1$, Theorem 1 indicates that the transformation $\phi_2^+ \to \psi$ is possible under non-entangling operations as long as $E_{1/2}(\psi) \leq 1$.

**Corollary 3.** The measures $E_\alpha$ can be increased by non-entangling operations for any $\alpha \in [0, 1/2)$.

**Proof.** Let $\psi$ be a pure state with Schmidt rank greater than 2 such that $E_{1/2}(\psi) = 1$. States with this property certainly exist, since we may take for instance the state with Schmidt coefficients $\lambda = (1 - 2\epsilon, \epsilon, \epsilon)$ and $\epsilon = 1/18$. Note that $E_\alpha(\psi)$ is a continuous and non-increasing function of $\alpha$, and that

$$\frac{dE_\alpha(\psi)}{d\alpha} \bigg|_{\alpha = 1/2} < 0$$

for this choice of $\psi$. It follows that $E_\alpha(\psi) > 1$ for all $\alpha < 1/2$. Note however that $E_\alpha(\phi_2^+) = 1$ for all $\alpha$. In particular, $E_{1/2}(\phi_2^+) = 1$ so the transformation $\phi_2^+ \to \psi$ is possible with non-entangling operations by Theorem 9. This completes the proof. $\square$

To prove that $E_\alpha$ is monotonic under non-entangling operations for all $\alpha \geq 1/2$, we will show that the value of $E_\alpha$ coincides with that of another entanglement measure that is known to be monotonic under non-entangling operations. For every $\alpha \in [0, +\infty)$ and density operators $\rho$ and $\sigma$, the Rényi $\alpha$-relative entropy of $\rho$ with respect to $\sigma$ is defined as

$$S_\alpha(\rho||\sigma) = \begin{cases} \frac{1}{\alpha-1} \operatorname{Tr}(\rho^\alpha \sigma^{1-\alpha}) & \text{supp } \rho \subseteq \text{supp } \sigma \text{ or } \alpha \in [0, 1) \\ \frac{1}{\alpha-1} \operatorname{Tr}(\rho (\log \rho - \log \sigma)) & \text{supp } \rho \subseteq \text{supp } \sigma \text{ and } \alpha = 1 \\ +\infty & \text{otherwise}, \end{cases}$$

where supp denotes the support. Note that $S_1 = S$ is the standard relative entropy. The Rényi $\alpha$-relative entropy of entanglement of a state $\rho$ is

$$E_{R,\alpha}(\rho) := \inf_{\sigma \in S} S_\alpha(\rho||\sigma),$$

where the minimization is taken over all separable states $\sigma$. The Rényi $\alpha$-relative entropies are known to be **contractive** under CPTP maps for all $\alpha \in [0, 2]$, i.e.,

$$S_\alpha(\Lambda(\rho)||\Lambda(\sigma)) \leq S_\alpha(\rho||\sigma)$$

for all CPTP maps $\Lambda$ [26]. It readily follows that $E_{R,\alpha}$ is monotonic under non-entangling operations for $\alpha \in [0, 2]$. In the case $\alpha = 1$, this reduces to the well-known relative entropy of entanglement $E_R$. It is known that $E_1(\psi) = E(\psi) = E_R(\psi)$ for pure states [22], which establishes that $E_1$ is monotonic under non-entangling operations. Here, we show that $E_{R,\alpha}(\psi) = E_{1/\alpha}(\psi)$ for all $\alpha \in [0, 2]$. 

10
Theorem 4. For pure states \( \psi \), it holds that \( E_{R,\alpha}(\psi) = E_{1/\alpha}(\psi) \) for all \( \alpha \in [0, 2] \).

The proof of Theorem 4, which can be found in Section 7, uses similar methods to those in Ref. [22] which prove that \( E(\psi) = E_R(\psi) \). That \( E_\alpha \) on pure states is monotonic under non-entangling operations for any \( \alpha \geq 1/2 \) now follows directly from Theorem 4.

Corollary 5. For all \( \alpha \in [1/2, +\infty] \), the measures \( E_\alpha \) are monotonic for pure states under non-entangling operations.

While majorization provides necessary and sufficient conditions for conversions of pure states under LOCC, whether one can find similar necessary and sufficient conditions for convertibility of arbitrary pure states under non-entangling operations remain unknown. In the following, we provide a sufficient condition for convertibility under non-entangling operations of arbitrary pure states that is independent of the majorization criterion. We first provide the following Lemma.

Lemma 6. For every pure state \( \psi \), it holds that \( \max_{\sigma \in S} \text{Tr}(\psi \sigma) = \lambda_1 \) where \( \lambda_1 \) is the largest Schmidt coefficient of \( \psi \).

Proof. We may suppose without loss of generality that \( \psi \) is in Schmidt form \( |\psi\rangle = \sum_{i=1}^n \sqrt{\lambda_i} |ii\rangle \). We first show that \( \text{Tr}(\psi \sigma) \leq \lambda_1 \) holds for all \( \sigma \in S \). It will suffice to show that \( \text{Tr}(\psi \phi) \leq \lambda_1 \) for all separable pure states \( |\phi\rangle = \sum_{ij} u_i |i\rangle |v_j\rangle \) with \( \sum_i |u_i|^2 = \sum_j |v_j|^2 = 1 \). Note that \( \text{Tr}(\psi \phi) = |\langle \phi | \phi \rangle|^2 \) and

\[
|\langle \psi | \phi \rangle|^2 = \sum_i |\sqrt{\lambda_i} u_i v_i|^2 \\
\leq \lambda_1 (\sum_i |u_i|^2 |v_i|^2)^2 \\
\leq \lambda_1,
\]

as desired. Since \( |11\rangle \) is separable and \( |\langle \psi | 11 \rangle|^2 = \lambda_1 \), the maximum is achieved. \( \square \)

Theorem 7. Let \( \psi \) and \( \phi \) be pure states with Schmidt coefficients \( \lambda \) and \( \mu \) respectively. If it holds that \( R(\phi) \leq 1/\lambda_1 \), then there exists a non-entangling operation \( \Lambda \) such that \( \Lambda(\psi) = \phi \).

Proof. The proof is very similar to that of Theorem 1. The desired operation is given by

\[
\Lambda(X) = \text{Tr}(\psi X) \phi + \text{Tr}((1 - \psi) X) \sigma
\]

where \( \sigma \in S \) is the separable state such that \( R(\sigma) = R(\phi) \). It is evident that \( \Lambda(\psi) = \phi \), so it remains to show that \( \Lambda \) is non-entangling. By construction, it holds that \( p \phi + (1 - p) \sigma \in S \) whenever \( p \leq 1/R(\phi) \). By Lemma 6, it holds that \( \text{Tr}(\psi \rho) \leq \lambda_1 \) for any separable state \( \rho \). If the condition that \( R(\phi) \leq 1/\lambda_1 \) is met, it follows that \( \Lambda(\rho) \in S \) whenever \( \rho \) is separable and thus \( \Lambda \) is non-entangling. \( \square \)

3.2 Entanglement measures under dually non-entangling maps

We now explore what transformations are possible under the more restrictive class of dually non-entangling operations. Recall that these are the maps such that both \( \Lambda \) and its dual \( \Lambda^* \) do not generate entanglement. In particular, we show that the Schmidt rank can be increased by this more restrictive class of transformations.
We first recall some conditions for certain operators to be separable. For a pure state \( \psi \) with Schmidt coefficients \( \lambda = (\lambda_1, \ldots, \lambda_d) \) (in decreasing order), it holds that \( R(\psi\|\frac{1}{d^2} \mathbf{1}_{d^2}) = d^2 \sqrt{\lambda_1 \lambda_2} \) (see Appendix B of [24]). Hence operators of the form \( \psi + s \mathbf{1}_{d^2} \) for \( s \geq 0 \) are separable if and only if \( s \geq \sqrt{\lambda_1 \lambda_2} \). Additionally, for any hermitian operator \( \Lambda \) on a bipartite space with local dimensions \( d_1 \) and \( d_2 \), the operator \( \mathbf{1}_{d_1 d_2} + \Lambda \) is always separable whenever \( \|\Lambda\| \leq 1 \), where \( \|\cdot\| \) is the Frobenius norm (see Theorem 1 of Ref. [27]). We now state a useful condition for certain types of maps to be dually non-entangling.

**Lemma 8.** Let \( \psi \) and \( \phi \) be pure states with Schmidt coefficients (in decreasing order) \( \lambda \) and \( \mu \) respectively, and consider the CPTP map \( \Lambda : \mathcal{L}(\mathbb{C}^k \otimes \mathbb{C}^d) \rightarrow \mathcal{L}(\mathbb{C}^d \otimes \mathbb{C}^d) \) defined by

\[
\Lambda(X) = \text{Tr}(\psi X) \phi + \text{Tr}((\mathbb{1}_{k^2} - \psi) X) \frac{1}{d^2} \mathbb{1}_{d^2}.
\]  
(7)

If it holds that

\[
\lambda_1 \leq \frac{1}{d^2 \sqrt{\mu_1 \mu_2}}
\]  
(8)

then \( \Lambda \) is non-entangling. If it furthermore holds that

\[
d^2 \mu_1 \leq 1 + \frac{1}{\sqrt{\lambda_1 \lambda_2}}
\]  
(9)

then \( \Lambda \) is also dually non-entangling.

**Proof.** Outputs of the map in (7) are of the form \( \Lambda(\rho) = p \phi + (1 - p) \frac{\mathbf{1}_{d^2}}{d^2} \) for any state \( \rho \), where \( p = \text{Tr}(\psi \rho) \). From the observation above, this is separable if and only if \( p \leq 1/(1 + d^2 \sqrt{\mu_1 \mu_2}) \) since \( \|\phi\| = 1 \). Since \( \text{Tr}(\psi \rho) \leq \lambda_1 \) for any separable state \( \rho \) (see Lemma 6), we obtain the desired result that \( \Lambda \) is non-entangling whenever the condition in (8) holds. On the other hand, for any density operator \( \rho \), its output under the dual map of \( \Lambda \) is given by

\[
\Lambda^*(\rho) = \left( \text{Tr}(\phi \rho) - \frac{1}{d^2} \right) \psi + \frac{1}{d^2} \mathbf{1}_{d^2}.
\]

Let \( \rho \) be a separable state. If it holds that \( \text{Tr}(\phi \rho) \leq 1/d^2 \) then the operator \( (d^2 \text{Tr}(\phi \rho) - 1) \psi + \mathbf{1}_{d^2} \) is separable by Theorem 1 of [27], since \( \|\psi\| = 1 \) and \( |d^2 \text{Tr}(\phi \rho) - 1| \leq 1 \). We may therefore assume that \( \text{Tr}(\phi \rho) > 1/d^2 \), in which case \( \Lambda^*(\rho) \) is separable if and only if \( d^2 \text{Tr}(\phi \rho) \leq 1 + 1/\sqrt{\lambda_1 \lambda_2} \). Since \( \text{Tr}(\phi \rho) \leq \mu_1 \) holds for any separable state \( \rho \), it follows that \( \Lambda^*(\rho) \) is separable for all separable states \( \rho \), as long as the condition in (9) holds.

Now that we have conditions for a map of the form in (7) to be dually non-entangling, we may construct such dually non-entangling maps that increase the Schmidt rank of pure states arbitrarily. The trick is to construct pure states \( \psi \) and \( \phi \) with Schmidt coefficients \( \lambda \) and \( \mu \) that satisfy both (8) and (9). This is done in the proof of Theorem 9.

**Theorem 9.** For all integers \( k \) and \( d \) with \( 2 \leq k < d \), there exists pure states \( \psi \) and \( \phi \) with Schmidt ranks equal to \( k \) and \( d \), respectively, and a dually non-entangling map \( \Lambda \) such that \( \Lambda(\psi) = \phi \).
Proof. Let \( k \) and \( d \) be integers with \( 2 \leq k < d \), and choose positive real numbers \( \delta \) and \( \epsilon \) small enough so that

\[
1 - \delta \leq \frac{1}{1 + \frac{d^2}{\sqrt{k-1}} \sqrt{(1-\epsilon)\epsilon}}
\]

and

\[
d^2(1-\epsilon) \leq 1 + \frac{\sqrt{k-1}}{\sqrt{(1-\delta)\delta}}
\]

are both satisfied. This can be done, since for example the values \( \delta = d^{-4} \) and \( \epsilon = d^{-12} \) satisfy both (10) and (11) (see Appendix A for proof). Let \( \psi \) and \( \phi \) be pure states with Schmidt coefficients \( \lambda \) and \( \mu \) given by

\[
\lambda = \left(1 - \delta, \frac{\delta}{k-1}, \ldots, \frac{\delta}{k-1}\right)
\]

and

\[
\mu = \left(1 - \epsilon, \frac{\epsilon}{d-1}, \ldots, \frac{\epsilon}{d-1}\right)
\]

such that \( \psi \) and \( \phi \) have Schmidt rank \( k \) and \( d \), respectively. The desired operation is given by

\[
\Lambda(X) = \text{Tr}(\psi X)\phi + \text{Tr}(\mathbb{1}_{k^2} - \psi)X\frac{1}{d^2}\mathbb{1}_{d^2},
\]

which is dually non-entangling by Lemma 8 and performs the transformation \( \Lambda(\psi) = \phi \). □

We remark that it does not seem possible to strengthen the above results by considering maps whose outputs are mixtures of \( \phi \) and some separable state \( \sigma \) other than the identity, as we did in Theorems 1 and 7 for the case of non-entangling operations. The reason is that \( \sigma \) can be rank-deficient [27] so that there exists some other separable state \( \rho \in \mathcal{S} \) such that \( \text{Tr}(\sigma \rho) = 0 \). With this state, the output of the dual map \( \Lambda(\rho) \) is entangled and thus \( \Lambda \) cannot be dually non-entangling.

3.3 Negativity under PPT-preserving operations

We now investigate the difference between the class of PPT maps defined by Rains [12] and the class of PPT-preserving maps defined in this paper. Despite some confusion in the literature [2], these classes are not equivalent. (This distinction goes away, however, if we require that a map be PPT-preserving when tensored with the identity on any ancilla space.) We will show that these classes of maps are distinct by presenting a transformation under PPT-preserving maps that is not possible under PPT maps. This will be done by investigating the negativity, a quantity defined as

\[
N(\rho) = \frac{\text{Tr}[\rho^T] - 1}{2}
\]
for bipartite states $\rho$. It is well-known that the negativity is monotonic under PPT maps (and thus under LOCC maps as well), so the negativity is an entanglement measure. Here, however, we show that the negativity can be increased by PPT-preserving maps, and conclude that this set of maps is strictly larger than the class of PPT maps.

Before proceeding, we first note that states of the form $a \frac{d^2}{d+1} (a - \frac{d}{d+1})^\Gamma \phi^+_{d^2} + (1 - a) \frac{d}{d+1} \phi^+_{d}$ are PPT if and only if $a \geq \frac{d}{d+1}$. Indeed, note that

$$\left( a \frac{d^2}{d+1} (a - \frac{d}{d+1})^\Gamma \phi^+_{d^2} + (1 - a) \frac{d}{d+1} \phi^+_{d^2} \right) = a \frac{d^2}{d+1} \left( \phi^+_{d^2} + \frac{d}{a} \frac{d}{d+1} \phi^+_{d} \right),$$

which is positive exactly when $a \geq \frac{d}{d+1}$, since the eigenvalues of $\phi^+_{d}$ are $\pm 1$. We now show that the negativity can be increased by PPT-preserving maps.

**Theorem 10.** The negativity is not a monotone under PPT-preserving maps.

**Proof.** We construct a PPT-preserving map $\Lambda$ such that $N(\Lambda(\rho)) > N(\rho)$ for some bipartite state $\rho$. Let $d > 3$ be an integer. Consider the operator

$$A = \frac{1}{d+1} (d \frac{1}{d+1} \phi^+_{d^2} + \phi^+_{d}),$$

and the corresponding CPTP map $\Lambda : L(\mathbb{C}^d \otimes \mathbb{C}^d) \to L(\mathbb{C}^d \otimes \mathbb{C}^d)$ defined by

$$\Lambda(X) = \text{Tr}(AX) \frac{1}{d+1} \phi^+_{d^2} + \text{Tr}((d \frac{1}{d+1} \phi^+_{d^2} - A)X) \phi^+_{d}.$$

By the observation above, $\Lambda(\rho)$ is PPT whenever $\text{Tr}(A \rho) \geq \frac{d}{d+1}$, which certainly holds for any PPT state $\rho$. We conclude that $\Lambda$ is PPT-preserving. Now consider the state

$$\rho = \frac{1}{d(d+1)} (\frac{1}{d+1} \phi^+_{d^2} - \phi^+_{d}).$$

which is the most entangled Werner state and has negativity equal to $N(\rho) = 1/d$. However

$$\Lambda(\rho) = \frac{d-1}{d+1} \frac{1}{d} \phi^+_{d^2} + \frac{2}{d+1} \phi^+_{d},$$

which has negativity

$$N(\Lambda(\rho)) = \frac{d-1}{2d}.$$

Then $N(\Lambda(\rho)) > N(\rho)$ for this state, since $d > 3$. \qed

Note that for $\rho$ and $\Lambda$ as in the proof above, we have $N(\Lambda(\rho))/N(\rho) = (d-1)/2$. Therefore, this ratio can be made arbitrarily big by taking a large enough $d$. Moreover, a PPT-preserving channel that is not PPT can be constructed from any NPT state $\rho$ and a corresponding pure state vector $|\psi\rangle$ corresponding to a negative eigenvalue of $\rho^\Gamma$. Indeed, for such a $\rho$ and $|\psi\rangle$, define the operator

$$A = \frac{1}{d+1} (d \frac{1}{d+1} \phi^+_{d^2} + |\psi\rangle \langle \psi|).$$
The corresponding CPTP map $\Lambda$ of the form in (15) is clearly PPT-preserving since it maps every state to a PPT state, which results from the fact that $\text{Tr}(A\sigma) \geq \frac{d}{(d+1)}$ for every state $\sigma$. Nonetheless, this map is not PPT, since the operator $J(\Lambda)^T$ is not positive semidefinite. Indeed, note that

$$\text{Tr}(J(\Lambda)^T((\mathbb{1}_d - F_d) \otimes \rho^T)) = \left(1 - \frac{1}{d}\right)\text{Tr}(A\rho^T) + (1 - d)(1 - \text{Tr}(A\rho^T))$$

$$< 0$$

since $\text{Tr}(A\rho^T) < \frac{d}{(d+1)}$.

### 4 Distillability beyond LOCC

In order to explore the possible existence of NPT bound entanglement under LOCC, we address the question of distillable entanglement under classes of operations that are strictly larger than LOCC but nevertheless non-entangling. As mentioned in the introduction, it has been shown that all NPT states are distillable by PPT operations [11], but non-entangling operations are independent from this class of maps. Reference [8] has shown that that all entangled states are not only distillable under asymptotically non-entangling operations but also under non-entangling operations. Here, we will prove that the later is still the case under the more restrictive class of dually non-entangling operations. Furthermore, while all NPT states are known to be distillable under PPT maps, we also show that such states are also distillable under the more restrictive classes of maps that are both dually non-entangling and PPT. Next we turn to $\epsilon$-LOCC operations and demonstrate that all entangled states are distillable under the subclass of non-entangling $\epsilon$-LOCC channels. Finally, we show that every family of entangling undetected operations is able to distill entanglement from all states. These results are summarized in Table 1.

A state $\rho$ is said to be distillable under some class of operations if there exists a sequence of maps $\{\Lambda_n\}$ in this class such that $\lim_{n \to \infty} ||\Lambda_n(\rho^{\otimes n}) - \phi_2^+||_1 = 0$, where $||\cdot||_1$ denotes the trace norm. It has been shown that every entangled two-qubit state is distillable by LOCC operations [28]. Thus, to prove that some entangled state $\rho \in \mathcal{L}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2})$ is distillable under some class of operations that contains LOCC, it suffices to show that there exist $n \in \mathbb{N}$ and a map in this class $\Lambda : \mathcal{L}(\mathbb{C}^{nd_1} \otimes \mathbb{C}^{nd_2}) \rightarrow \mathcal{L}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ such that $\Lambda(\rho^{\otimes n})$ is entangled. In the cases we study it turns out that it suffices to consider $n = 1$.

Our techniques will involve maps whose outputs are always mixtures of some operator with the maximally entangled state of two qubits. Note that the state $p\mathbb{1}_4 + (1-p)\phi_2^+$ is separable iff $p \geq 2/3$ [29]. We first show how any entangled state can be distilled using dually non-entangling operations.

**Lemma 11.** Let $W$ be an entanglement witness $W$ such that $||W||_2 \leq 1$ (where $||\cdot||_2$ is the Frobenious norm), and let $A = (W + 2\mathbb{1})/3$. Then the map $\Lambda$ defined by

$$\Lambda(X) = \text{Tr}(AX)\frac{1}{4}\mathbb{1}_4 + \text{Tr}((\mathbb{1} - A)X)\phi_2^+$$

(16)

is dually non-entangling.
Proof. First note that \( ||W||_2 \leq 1 \) implies that \(-I \leq W \leq I\) and thus \( 0 \leq A \leq 1\), so \( \Lambda \) is CPTP. Since \( W \) is an entanglement witness, it holds that \( \text{Tr}(W \sigma) \geq 0 \) for all separable states \( \sigma \). Hence \( \text{Tr}(A \sigma) \geq 2/3 \) holds for any separable state \( \sigma \) and therefore, as discussed above, \( \Lambda(\sigma) \) is separable whenever \( \sigma \) is separable. Thus \( \Lambda \) is non-entangling. To conclude that \( \Lambda \) is also dually non-entangling, we show that \( \Lambda^*(\sigma) \) is separable for all two-qubit states \( \sigma \). Note that

\[
\Lambda^*(\sigma) \propto I + \frac{1}{2} - \frac{1}{2} \, \text{Tr}(\phi_2^+ \sigma) \frac{W}{W} + \frac{1}{2} \, \text{Tr}(\phi_2^+ \sigma)
\]

and that \( 0 \leq \text{Tr}(\phi_2^+ \sigma) \) holds for any two-qubit state \( \sigma \). Since

\[
\frac{1}{2} \leq \frac{1}{2} - \frac{1}{2} \, \text{Tr}(\phi_2^+ \sigma) \leq \frac{1}{2}
\]

and \( ||W||_2 \leq 1 \), it holds that \( \Lambda^*(\sigma) \) is separable by Theorem 1 of Ref. [27]. \( \square \)

**Theorem 12.** All entangled states are distillable under dually non-entangling operations.

Proof. Let \( \rho \) be an arbitrary entangled state. Then there exists an entanglement witness \( W \) such that \( \text{Tr}(W \rho) < 0 \) but that \( \text{Tr}(W \sigma) \geq 0 \) for all separable states \( \sigma \in S \). We may suppose without loss of generality \( ||W||_2 \leq 1 \). Indeed, we may otherwise take the witness \( W' = W/||W||_2 \). Consider the map \( \Lambda \) in (16), which is dually non-entangling by Lemma 11. However, it holds that \( \text{Tr}(\Lambda(\rho) \phi_2^+) > 1/2 \) so by Lemma 6 \( \Lambda(\rho) \) is an entangled state of two qubits and is therefore distillable under LOCC (and thus also distillable under dually non-entangling operations). \( \square \)

Lastly, while it is known that all non-PPT entangled states are distillable under PPT operations, we next show that all such states are distillable under the more restrictive set of operations that are both PPT and dually non-entangling.

**Theorem 13.** All NPT states are distillable under maps that are both dually non-entangling and PPT.

Proof. Let \( \rho \) denote an arbitrary \( d_1 \times d_2 \) NPT state and let \( |\eta\rangle \) denote a normalized eigenvector of \( \rho^T \) corresponding to a negative eigenvalue. Then \( W = |\eta\rangle \langle \eta|^T \) is an entanglement witness that detects \( \rho \), since \( \text{Tr}(W \rho) < 0 \) but \( \text{Tr}(W \sigma) \geq 0 \) for all separable states \( \sigma \). Furthermore, note that the Frobenious norm of this witness is \( ||W||_2 = |||\eta\rangle \langle \eta|||^T = 1 \). It follows from Lemma 11 that the map \( \Lambda \) in (16) is dually non-entangling. To show that this map is also PPT, note that the Choi representation of this map is

\[
J(\Lambda) = \frac{1}{4} \mathbb{1}_4 \otimes A^T + \phi_2^+ \otimes (\mathbb{1}_{d_1d_2} - A^T),
\]

where \( A = (W + 2\mathbb{1}_{d_1d_2})/3 \). The partial transpose of this operator is

\[
J(\Lambda)^T = \frac{1}{4} \mathbb{1}_4 \otimes |\bar{\eta}\rangle \langle \bar{\eta}| + \frac{1}{6} (\mathbb{1}_4 + F_2) \otimes (\mathbb{1}_{d_1d_2} - |\bar{\eta}\rangle \langle \bar{\eta}|),
\]

where \( F_2 = 2(\phi_2^+)^T \) is the flip operator. This is clearly positive since \(-\mathbb{1}_4 \leq F_2 \). Lastly, note that \( \Lambda(\rho) \) is an entangled two-qubit state, so it is distillable under LOCC (and thus distillable under dually non-entangling and PPT maps as well). \( \square \)
We now investigate distillability under operations that closely approximate LOCC and separable operations. We first turn to \( \epsilon \)-LOCC maps, and even stronger we consider such maps that are also non-entangling. Hence, separable states remain as free states under these operations. Interestingly, it turns out that all entangled states are distillable under such operations for any value \( \epsilon > 0 \).

**Theorem 14.** All entangled states are distillable under non-entangling \( \epsilon \)-LOCC operations for any \( \epsilon > 0 \).

**Proof.** Let \( \rho \) be an entangled state. Let \( \Lambda \) be the same non-entangling map from (16) in the proof of Theorem 12 such that \( \Lambda(\rho) \) is an entangled two-qubit state. Let

\[
\sigma = \frac{1}{6} \mathbb{1}_4 + \frac{1}{3} \phi^+_{2},
\]

which is a state on the boundary of the separable states, and define the map \( \Lambda'(X) = \text{Tr}(X)\sigma \). Since \( \sigma \) is separable, the map \( \Lambda' \) is certainly LOCC. Finally, consider the maps \( \Omega_\delta = \delta \Lambda + (1 - \delta)\Lambda' \) for \( \delta \in [0,1] \). Then \( \Omega_\delta \) is non-entangling since the set of non-entangling maps is convex. Moreover, for all \( \delta > 0 \) it holds that \( \Omega_\delta(\rho) = \delta \Lambda(\rho) + (1 - \delta)\sigma \) is an entangled two-qubit state. Hence it is distillable under any set of operations containing LOCC. Taking \( \delta \) sufficiently small, we can make \( \Omega_\delta \) to be \( \epsilon \)-LOCC for any \( \epsilon > 0 \). \( \square \)

Finally, we show that all states (even separable ones) are distillable under any class of entangling undetected operations.

**Theorem 15.** Let \( \{W_i\}_{i=1}^n \) be a finite number of entanglement witnesses for operators on \( \mathcal{H}_{A_2A_1B_2B_1} \). Then every state is distillable using operations that are entangling undetected by \( \{W_i\}_{i=1}^n \).

**Proof.** Let \( \Pi = \sum_{i,j=0}^{1} |ij\rangle \langle ij| \) be the projector onto an arbitrary two-qubit subspace of \( \mathcal{H}_{A_2B_2} \). We first note that the projected partial traces of the witnesses \( V_i := \Pi [\text{Tr}_{A_1B_1}(W_i)] \Pi \) are again entanglement witnesses for two-qubit states with support on the span of \( |ij\rangle_{i,j=0}^1 \subset \mathcal{H}_{A_2B_2} \). Indeed, let \( \sigma \) be a separable state with support on span\( |ij\rangle_{i,j=0}^1 \). Then \( \sigma \otimes \mathbb{1}_{A_2B_2} \) is separable with respect to \( A_2A_1 : B_2B_1 \) and thus

\[
\text{Tr}(V_i \sigma) = \text{Tr}(W_i (\Pi \sigma \Pi \otimes \mathbb{1}_{A_1B_1})) = \text{Tr}(W_i (\sigma \otimes \mathbb{1}_{A_1B_1})) \geq 0.
\]

Next consider the set of two-qubit density operators which are not detected by the \( V_i \). This is the intersection of a convex set \( \mathcal{D} \) (the set of all two-qubit density operators) with a polyhedron \( \mathcal{P} \), the set of all hermitian operators satisfying \( \text{Tr}(V_i X) \geq 0 \) for all \( i = 1, \ldots, n \). If \( \mathcal{D} \) is contained entirely in the polyhedron, then there obviously exists an entangled two-qubit state \( \rho \) not detected by any of the \( V_i \). If \( \mathcal{D} \) is not contained entirely in the polyhedron, then \( \mathcal{D} \cap \mathcal{P} \) has a facet [30]. However, the set of two-qubit separable states does not have a facet [19], and thus an entangled state \( \rho \) exists in \( \mathcal{D} \cap \mathcal{P} \). We then define the map \( \Lambda : \mathcal{L}(\mathcal{H}_{A_1B_1}) \to \mathcal{L}(\mathcal{H}_{A_2B_2}) \) by \( \Lambda(X) = \text{Tr}(X)\rho \). This map has Choi representation \( J(\Lambda) = \rho \otimes \mathbb{1}_{A_2B_2} \), which clearly satisfies \( \text{Tr}(W_i J(\Lambda)) \geq 0 \) for all \( i \). The desired result follows since any two-qubit entangled state \( \rho \) is distillable under separable operations. \( \square \)

## 5 Stochastic Convertibility of Pure States

In this section we relax the condition that the quantum operation is trace-preserving. Let \( \Lambda_1 \) be an arbitrary non-trace-preserving map, which by appropriate scaling, we can assume is trace non-increasing; i.e., \( \text{Tr}(X) \geq \text{Tr}(\Lambda_1(X)) \geq 0 \) for all \( X \geq 0 \). The rescaled map \( \Lambda_1 \) can always be “completed”
by another CP map \( \Lambda_2 \) so that \( \Lambda(\cdot) = \Lambda_1(\cdot) \otimes |1\langle 1| + \Lambda_2(\cdot) \otimes |2\langle 2| \) is CPTP map. After performing the channel \( \Lambda \) on \( \rho \), the classical register \( |i\rangle \langle i| \) can then be measured so that the post-measurement state \( \Lambda_i(\rho)/\text{Tr}[\Lambda_i(\rho)] \) is obtained with probability \( \text{Tr}[\Lambda_i(\rho)] \). Thus, any non-trace-preserving CP map \( \Lambda_1 \) can be stochastically implemented in this way.

While all the discussion thus far has focused on bipartite entanglement, here we consider multipartite systems. To fix notation, let \( S_1 \cdots S_N \) be \( N \) systems with joint state space \( \mathcal{H}_{S_1 \cdots S_N} \). Convertibility between any two states by stochastic LOCC (SLOCC) provides a natural way to classify entanglement. Under this partitioning of states, two states \( \rho \) and \( \hat{\rho} \) are said to be equivalent in terms of their entanglement iff \( \rho \rightarrow \hat{\rho} \) and \( \hat{\rho} \rightarrow \rho \) by SLOCC [16]. When considering pure states, it is well-known that two states \( |\psi\rangle \) and \( |\hat{\psi}\rangle \) belong to the same entanglement class iff there exists invertible linear operators \( A_i \) such that \( |\hat{\psi}\rangle = A_1 \otimes A_2 \otimes \cdots A_N |\psi\rangle \). A paradigmatic example of two inequivalent states is the three-qubit GHZ state, \(|GHZ\rangle = \sqrt{1/2}(|000\rangle + |111\rangle)\), and W state, \(|W\rangle = \sqrt{1/3}(|000\rangle + |010\rangle + |001\rangle)\), which cannot be converted from one to the other using SLOCC [16]. In fact, for systems with more than three parties, or for tripartite systems with dimensions greater than \( 2 \otimes 3 \otimes 6 \), there are an infinite number of inequivalent SLOCC entanglement classes [31].

We will now show that the situation is dramatically different if any family of entangling undetected operations is considered. This is similar in spirit to Theorem 15, except our proof is simpler in that it does not rely on the convex structure of the set of separable states. For a CP map \( \Lambda : \mathcal{L}(\mathcal{H}_{S_1 \cdots S_N}) \rightarrow \mathcal{L}(\mathcal{H}_{S_1 \cdots S_N}) \), its Choi operator is an \( N \)-partite positive semi-definite operator

\[
J(\Lambda) = \Lambda \otimes \text{id}_{S_1 \cdots S_N}(\phi^+_{S_1 \cdots S_N} \in \mathcal{L}(\mathcal{H}_{S_1 \cdots S_N}))
\]

where \( \phi^+_{S_1 \cdots S_N} = \phi^+_{S_1} \otimes \cdots \otimes \phi^+_{S_N} \). Conversely, for any positive operator \( \Omega \in \mathcal{L}(\mathcal{H}_{S_1 \cdots S_N}) \), the map \( \Lambda_\Omega(X) = \text{Tr}_{S_1 \cdots S_N}(\Omega[I_{S_1 \cdots S_N} \otimes X^T]) \) is CP. Direct calculation shows that \( J(\Omega) \) is a fully separable operator whenever \( \Omega \) is a separable operator, and conversely \( \Lambda_\Omega \) is a separable map whenever \( \Omega \) is fully separable. By fully separable, it is meant that \( \Omega \) can be expressed as a convex combination of product operators on \( \mathcal{H}_{S_1 S_2 \cdots S_N} \). Hence, we have a one-to-one correspondence between separable operators and separable maps.

Like in the bipartite case, we say that a hermitian operator \( W \) on \( \mathcal{H}_{S_1 S_2 \cdots S_N} \) is an entanglement witness if \( \text{Tr}(W \rho) \geq 0 \) for all fully separable \( N \)-partite states \( \rho \in \mathcal{S}(\mathcal{H}_{S_1 S_2 \cdots S_N}) \).

**Theorem 16.** Let \( [W_i]_i^n \) be a finite number of entanglement witnesses for operators on \( \mathcal{H}_{S_1 S_2 \cdots S_N} \) and let \( |\psi\rangle \in \mathcal{H}_{S_1 S_2 \cdots S_N} \) and \( |\hat{\psi}\rangle \in \mathcal{H}_{S_1 S_2 \cdots S_N} \) be any two states. Then there exists a CP map \( \Lambda \) that is entangling undetected by \( [W_i]_i^n \) and such that \( \Lambda(|\psi\rangle\langle\psi|) = |\hat{\psi}\rangle\langle\hat{\psi}| \).

**Proof.** Let \( |\psi^\perp\rangle = |a_1\rangle \cdots |a_N\rangle \) be any product state that is orthogonal to \( |\psi^\perp\rangle \). Define the hermitian operators

\[
R_i = \text{Tr}_{S_1 \cdots S_N}(W_i[I_{S_1} \cdots I_{S_N} \otimes |\psi^\perp\rangle\langle\psi^\perp|]), \quad S_i = \text{Tr}_{S_1 \cdots S_N}(W_i[I_{S_1} \cdots I_{S_N} \otimes (I - |\psi^\perp\rangle\langle\psi^\perp|)])
\]

which by construction are entanglement witnesses on \( \mathcal{H}_{S_1 \cdots S_N} \). There must exist a separable state \( \omega \) such that \( \text{Tr}(\omega R_i) > 0 \) for all \( i = 1, \cdots, n \). Indeed such a state can be constructed as follows. For each \( R_i \), let \( \rho_i \) be a separable state such that \( \text{Tr}(R_i \rho_i) > 0 \); since the set of separable states span the space of hermitian operators, we are assured that such a \( \rho_i \) exists. Then take \( \omega = \sum_{i=1}^n \rho_i \), and the fact that each \( R_i \) is an entanglement witness guarantees that \( \text{Tr}(R_i \omega) > 0 \) for all \( i \). Define

\[
a = \min_{i=1,\cdots,n} \text{Tr}(R_i \omega) > 0, \quad b = \min_{i=1,\cdots,n} \langle \hat{\psi} | S_i | \hat{\psi} \rangle,
\]

(18)
and the positive semi-definite operator

$$\Omega = \frac{|b|}{d} \omega \otimes |\psi^+\rangle\langle \psi^+| + |\tilde{\psi}\rangle\langle \tilde{\psi}| \otimes (|\psi^+\rangle\langle \psi^+| - |\psi^+\rangle\langle \psi^+|).$$  \hfill (19)

It can be directly seen that \(\text{Tr}(W_i \Omega) \geq 0\) for all \(i\), and the CP map given by

$$\Lambda(\rho) = \text{Tr}_{S_1 \cdots S_N} \left( \Omega [\mathbb{I}_{S_1 \cdots S_N} \otimes \rho^T] \right)$$

satisfies \(\Lambda(|\psi\rangle\langle \psi|) = |\tilde{\psi}\rangle\langle \tilde{\psi}|.\) \hfill \(\square\)

Theorem 16 has interesting consequences for the problem of tensor rank calculation. The tensor rank of an \(N\)-partite state \(|\psi\rangle\) is the minimum number of product states whose linear span contains \(|\psi\rangle\); i.e.

$$\text{Tensor rank}(|\psi\rangle) = \min \left\{ r : |\psi\rangle = \sum_{i=1}^{r} |\phi_i^{(S_1)}\rangle \otimes \cdots \otimes |\phi_i^{(S_N)}\rangle \right\}. \hfill (20)$$

It is easy to see that the tensor rank of a state can be equivalently characterized as an SLOCC convertibility problem,

$$\text{Tensor rank}(|\psi\rangle) = \min \left\{ r : |\text{GHZ}_r^{(N)}\rangle \rightarrow |\psi\rangle \text{ by SLOCC} \right\}, \hfill (21)$$

where \(|\text{GHZ}_r^{(N)}\rangle = \sqrt{1/r} \sum_{i=1}^{r} |i \cdots i\rangle_{S_1 \cdots S_N}\). Given the one-to-one correspondence between separable maps and separable operators, Eq. (21) can be expressed in terms of entanglement witnesses as

$$\text{Tensor rank}(|\psi\rangle) = \min \left\{ r : \text{Tr}_{S_1 \cdots S_N} \left( \Omega [\mathbb{I}_{S_1 \cdots S_N} \otimes \Phi_r^{(N)}] \right) = |\psi\rangle\langle \psi|, \quad \Omega \geq 0, \quad \text{Tr}[\Omega W] \geq 0 \quad \forall W \in \mathbb{W}_{S_1 \cdots S_N} \right\}, \hfill (22)$$

where \(\Phi_r^{(N)} = |\text{GHZ}_r^{(N)}\rangle \langle \text{GHZ}_r^{(N)}|\) and \(\mathbb{W}_{S_1 \cdots S_N}\) is the collection of all entanglement witnesses on \(\mathcal{H}_{S_1 \cdots S_N}\).

From (20), the tensor rank can be seen as a multipartite generalization of the Schmidt rank. However, unlike the Schmidt rank, the tensor rank is in general very difficult to compute; in fact already in tripartite systems the problem is NP-Complete [33]. One way to tackle the tensor rank problem is to start from the characterization (22) and move “beyond” SLOCC. Specifically, instead of considering all entanglement witnesses in the minimization of (22), one could relax the problem and work with some finite subset. A lower bound on this relaxed problem would also be a lower bound on the tensor rank of \(|\psi\rangle\). However, Theorem 16 implies that such a strategy will fail since the minimum for any finite set of witnesses will always be one. This finding reflects the general difficulty even in computing non-trivial lower bounds for the tensor rank.

### 6 \(k\)-resource-non-generating maps

We now examine the classes of \(k\)-non-entangling maps and \(k\)-PPT-preserving maps. As Proposition 17 below indicates, the class of completely non-entangling maps coincides with the class of separable maps. In particular, a map \(\Lambda\) is completely non-entangling if and only if it is \(d\) non-entangling,
where \( d \) is the dimension of Alice’s and Bob’s input spaces. Since a map is separable if and only if its Choi representation is a separable operator, this parallels the notion that complete positivity of a map is equivalent to positivity of its Choi representation. The same holds for maps that are PPT and completely PPT-preserving.

**Proposition 17.** Let \( \mathcal{H}_A \) and \( \mathcal{H}_B \) be \( d \)-dimensional systems and consider a completely positive map \( \Lambda : \mathcal{L}(\mathcal{H}_{AB}) \rightarrow \mathcal{L}(\mathcal{H}_{AB}) \). The following are equivalent:

(i) \( \Lambda \) is separable (respectively PPT);

(ii) \( \Lambda \) is \( d \)-non-entangling (respectively \( d \)-PPT-preserving);

(iii) \( \Lambda \) is completely non-entangling (respectively completely PPT-preserving).

**Proof.** The implications (iii) \( \Rightarrow \) (ii) \( \Rightarrow \) (i) follow from the definitions. It remains to prove the implication (i) \( \Rightarrow \) (iii), which follows from the following observations. Note that the tensor product of separable maps is separable, since if \( \Lambda : \mathcal{L}(\mathcal{H}_{AB}) \rightarrow \mathcal{L}(\mathcal{H}_{AB}) \) and \( \Lambda' : \mathcal{L}(\mathcal{H}_{A'B'}) \rightarrow \mathcal{L}(\mathcal{H}_{A'B'}) \) are both separable, then \( \Lambda \otimes \Lambda' : \mathcal{L}(\mathcal{H}_{A_1A'_1} \otimes \mathcal{H}_{B_1B'_1}) \rightarrow \mathcal{L}(\mathcal{H}_{A_2A'_2} \otimes \mathcal{H}_{B_2B'_2}) \) can be written as

\[
\sum_{i,j} \Phi_i \otimes \Phi'_j \otimes \Psi_i \otimes \Psi'_j
\]

for some CP maps

\[
\Phi_i : \mathcal{L}(\mathcal{H}_{A_1}) \rightarrow \mathcal{L}(\mathcal{H}_{A_2}) \quad \quad \quad \Psi_i : \mathcal{L}(\mathcal{H}_{B_1}) \rightarrow \mathcal{L}(\mathcal{H}_{B_2})
\]

\[
\Phi'_i : \mathcal{L}(\mathcal{H}_{A'_1}) \rightarrow \mathcal{L}(\mathcal{H}_{A'_2}) \quad \quad \quad \Psi'_i : \mathcal{L}(\mathcal{H}_{B'_1}) \rightarrow \mathcal{L}(\mathcal{H}_{B'_2})
\]

such that \( \Lambda = \sum_i \Phi_i \otimes \Psi_i \) and \( \Lambda' = \sum_j \Phi'_j \otimes \Psi'_j \), and thus \( \Lambda \otimes \Lambda' \) is separable. Analogously, the tensor product of any PPT maps is PPT, which can be clearly seen from the fact that \( (\Lambda \otimes \Lambda')^T = \Lambda^T \otimes \Lambda'^T \). In particular, the identity map \( \text{id}_{AB} = \text{id}_A \otimes \text{id}_B \) is separable (resp. PPT) for any systems \( A \) and \( B \), hence the map \( \Lambda \otimes \text{id}_{AB} \) is separable (resp. PPT) whenever \( \Lambda \) is separable (resp. PPT). The desired result follows from the fact that every separable map is non-entangling (and every PPT map is PPT-preserving). \( \square \)

Note that 1-non-entangling is the same as non-entangling. (Similarly, 1-PPT-preserving is the same as PPT-preserving). While it is evident that non-entangling is a strictly larger set than completely non-entangling operations, it is not evident from the definitions that there must exist maps that are \( k \)-non-entangling for some \( k < d \) that are not also completely non-entangling. In the following, we show that there exist \( k \)-non-entangling maps which are not \((k+1)\)-non-entangling for every \( k < d \). This shows that the structure of these maps is as rich as it could be.

We first examine a useful condition for determining when a map on \( \mathcal{L}(\mathbb{C}^d \otimes \mathbb{C}^d) \) is \( k \)-non-entangling. Note that any pure state \( |u \rangle \in \mathbb{C}^d \otimes \mathbb{C}^k \) can be obtained from \( |\phi^+_d \rangle \) by some rank-\( k \) operator \( X \) such that

\[
|u \rangle = 1_d \otimes X |\phi^+_d \rangle.
\]

A map \( \Lambda \) on \( \mathcal{L}(\mathbb{C}^d \otimes \mathbb{C}^d) \) is \( k \)-non-entangling if and only if

\[
\Lambda \otimes \text{id}_{d'}(|u \rangle \langle u| \otimes |v \rangle \langle v|)
\]

is completely non-entangling for some \( k \).
is separable for all pure states $|u\rangle, |v\rangle \in \mathbb{C}^d \otimes \mathbb{C}^k$. To show that $\Lambda$ is $k$-non-entangling, it suffices to check only when $X$ is a $k$-dimensional projection. A $k$-dimensional projection is a linear operator $P : \mathbb{C}^d \to \mathbb{C}^k$ such that $P^TP = 1_k$. In particular, for such an operator it holds that $\|P\| = \sqrt{\text{Tr}(P^TP)} = \sqrt{k}$, where $\|\cdot\|_2$ is the Frobenius norm. The above observations allow us to state the following characterization of $k$-non-entangling maps in Lemma 18.

**Lemma 18.** A map $\Lambda$ on $\mathcal{L}(\mathbb{C}^d \otimes \mathbb{C}^d)$ is $k$-non-entangling if and only if the operator

$$
(1_d \otimes P \otimes Q)\Lambda(1_d \otimes P \otimes Q)^\dagger
$$

is separable for all $k$-dimensional projections $P$ and $Q$.

The construction of $k$-non-entangling maps that are not $(k+1)$-non-entangling is based on the well-known Werner states. The Werner states are a family of symmetric states on $\mathbb{C}^d \otimes \mathbb{C}^d$ that are defined by

$$
\rho_d(\beta) = \frac{1}{d^2 - (\beta + 1)} \left( 1_d \otimes \frac{\beta + 1}{d} F_d \right)
$$

for $-(d + 1) \leq \beta \leq d - 1$, where $F_d = d(\phi_d^+)^\dagger$. These states are entangled for $\beta > 0$, and furthermore they are PPT if and only if they are separable. The following lemma presents a fact about Werner states on $\mathbb{C}^d \otimes \mathbb{C}^d$ that will be employed in our construction of $k$-non-entangling maps in Theorem 20.

**Lemma 19.** Let $k$ and $d$ be integers with $2 \leq k < d$ and let $-(d + 1) \leq \beta \leq d - 1$. The operator $(P \otimes Q)\rho_d(\beta)(P \otimes Q)^\dagger$ is separable for all $k$-dimensional projections $P$ and $Q$ if and only if it holds that $\beta \leq (d - k)/k$.

**Proof.** First assume that $\beta \leq (d - k)/k$ and let $P$ and $Q$ be $k$-dimensional projections. Let $\overline{P}$ denote the matrix whose entries are complex conjugate of those of $P$, and let $|\psi\rangle = (\overline{P} \otimes Q|\phi^+_d\rangle) / \sqrt{c}$ denote the normalized pure state with Schmidt rank $k$ and normalization constant $c > 0$ given by

$$
c = \text{Tr}(\overline{P} \otimes Q \phi^+_d P^T \otimes Q^T)
= \frac{1}{d} \text{Tr}(QP^T PQ^T)
\leq \frac{k}{d},
$$

where the inequality follows from the fact that $QP^T PQ^T \leq 1_k$ with equality if and only if $P = Q$. Hence $c \leq k/d$ with equality if and only if $P = Q$. Since $\|\psi\|_2 = 1$, note that

$$(P \otimes Q)\rho_d(\beta)(P \otimes Q)^\dagger \propto 1_k - c(\beta + 1)\psi^\dagger
$$

is separable as long as $c(\beta + 1) \leq 1$ (by Theorem 1 of [27]). Then it is clearly separable since $\beta \leq (d - k)/k$ by assumption. On the other hand, if $\beta > (d - k)/k$ we may choose $P = Q$ such that

$$
\left((P \otimes Q)\rho_d(\beta)(P \otimes Q)^\dagger\right)^\dagger \propto 1_k - \frac{k}{d}(\beta + 1)\phi^+_k \not\propto 0.
$$

We conclude that $(P \otimes Q)\rho_d(\beta)(P \otimes Q)^\dagger$ is NPT and therefore entangled. \qed
We are now ready to present the construction of \( k \)-non-entangling maps in Theorem 20. The maps presented here are trivially non-entangling since they map every state to a separable state. Nonetheless, the maps defined in Theorem 20 are not completely non-entangling and showcase the hierarchy of \( k \)-non-entangling maps.

**Theorem 20.** For all integers \( k \) and \( d \) with \( 2 \leq k < d \), there exists a \( k \)-non-entangling map \( \Lambda : \mathcal{L}(\mathbb{C}^d \otimes \mathbb{C}^d) \to \mathcal{L}(\mathbb{C}^2 \otimes \mathbb{C}^2) \) that is not \((k + 1)\)-non-entangling.

**Proof.** Let \( k \) and \( d \) be integers with \( 2 \leq k < d \), and let \( \beta > 0 \) such that

\[
\frac{d - (k + 1)}{k + 1} < \beta \leq \frac{d - k}{k}.
\]

Consider the map \( \Lambda_\beta \) defined by

\[
\Lambda_\beta(X) = \text{Tr}(\rho_d(\beta)X)\ket{00}\bra{00} + \text{Tr}((\mathbb{1}_d - \rho_d(\beta))X)\ket{11}\bra{11}.
\]  

(28)

Note that this map is trivially non-entangling since every output is separable. The corresponding Choi operator of this map is given by

\[
J(\Lambda_\beta) = \ket{00}\bra{00} \otimes \rho_d(\beta) + \ket{11}\bra{11} \otimes (\mathbb{1}_d - \rho_d(\beta)).
\]  

(29)

From Lemma 18, we have that \( \Lambda_\beta \) is \( k \)-non-entangling but not \((k + 1)\)-non-entangling iff

\[
\left( \mathbb{1}_4 \otimes (P \otimes Q) \right) J(\Lambda_\beta) \left( \mathbb{1}_4 \otimes (P \otimes Q)^\dagger \right)
\]

(30)

is separable for all \( k \)-dimensional projections \( P \) and \( Q \) on \( \mathbb{C}^d \) but entangled for some \((k + 1)\)-dimensional projections. Note that the operator \( \mathbb{1}_d - \rho_d(\beta) \) is always separable for any \( \beta \) (by [27]), and moreover, the operator in Eq. (29) is separable iff so is \((P \otimes Q)\rho_d(\beta)(P \otimes Q)^\dagger\). Thus, the use of Lemma 19 together with the assumption in Eq. (27) completes the proof. \( \square \)

Having seen that \( k \)-non-entangling maps for \( 1 < k < d \) is a strict superset of separable maps and a strict subset of non-entangling maps, we would like to see whether all entangled states are still distillable under these maps or whether some form of bound entanglement arises for \( k < d \). We have not been able to answer this question but we can show that the non-entangling distillation maps used in Sec. 4 are not \( k \)-non-entangling for every \( k < d \).

**Proposition 21.** The maps used to prove distillability under dually non-entangling and PPT operations (cf. Theorem 13) for the states \( \rho_d(\beta) \) with \((d - 3)/3 < \beta \leq (d - 2)/2\) are not 3-non-entangling.

**Proof.** Lemma 19 tells us that for the values of \( \beta \) mentioned above, there exist projections on a 3-dimensional subspace \( P \) and \( Q \) such that \((P \otimes Q)\rho_d(\beta)(P \otimes Q)^\dagger\) is entangled (and NPT). This immediately implies that there exists a pure state \( |\eta\rangle \) with Schmidt rank equal to 3 such that \( \text{Tr}(\rho_d(\beta)|\eta\rangle\langle\eta|)^\dagger < 0 \), where \( W = |\eta\rangle\langle\eta|^\dagger \) is an entanglement witness. Thus, as argued in Theorem 13, the map \( \Lambda \) of Eq. (16) with \( A = (W + \mathbb{1}_2)/3 \) has the property that \( \Lambda(\rho_d(\beta)) \) is an entangled 2-qubit state. We are now going to show that this map is not 3-non-entangling. Since for \((d - 3)/3 < \beta \leq (d - 2)/2\) the states \( \rho_d(\beta) \) are 1-undistillable [34], the above map cannot be separable and, therefore, its Choi operator

\[
J(\Lambda) = \left( \frac{2}{3} \mathbb{1}_4 + \frac{1}{3} \phi_2^\dagger \right) \otimes \mathbb{1}_d + \left( \frac{1}{4} \mathbb{1}_4 - \phi_2^\dagger \right) \otimes |\eta\rangle\langle\eta|^\dagger,
\]

(31)
even though PPT, must be entangled. Now, by Lemma 18 the map $\Lambda$ can be seen not to be 3-non-entangling if $J'(\Lambda) = (\mathbb{1}_4 \otimes (P \otimes Q))J(\Lambda)(\mathbb{1}_4 \otimes (P \otimes Q)^\dagger)$ is still entangled for some choice of projections on a 3-dimensional subspace $P$ and $Q$. Choosing $P$ and $Q$ to project on the Schmidt bases of $|\eta\rangle$ we obtain that
\[
J'(\Lambda) = \left(\frac{2}{3}\mathbb{1}_4 + \frac{1}{3}\phi_2^\perp\right) \otimes \mathbb{1}_9 + \left(\frac{1}{4}\mathbb{1}_4 - \phi_2^\perp\right) \otimes |\eta\rangle\langle\eta|^{\dagger},
\]
which amounts to
\[
J(\Lambda) = J'(\Lambda) + \left(\frac{2}{3}\mathbb{1}_4 + \frac{1}{3}\phi_2^\perp\right) \otimes \mathbb{1}_{(d-3)^2}. \tag{33}
\]
Now, since the operator corresponding to the second term in the right-hand-side of the above equation is separable, if $J'(\Lambda)$ were separable too, this would imply that $J(\Lambda)$ is separable, which is a contradiction. Thus, $J'(\Lambda)$ must be entangled. □

This is only modest evidence for the existence of NPT bound entanglement. There could still exist other 3-non-entangling maps allowing for distillation, particularly if we allow them to act on more copies of the input states. However, $k$-non-entangling maps constitute at this moment the only superset of LOCC operations with a well-defined mathematical structure for which the possibility of NPT bound entanglement has not been proved impossible. Thus, we believe that further investigation of the properties of this class, particularly in the context of distillation, deserve future attention.

7 Proof of Theorem 4

This section presents the proof of Theorem 4, which shows the equivalence of the $\alpha$-entropy of entanglement and the $(1/\alpha)$-relative entropy of entanglement for $\alpha \in [0, 2]$ on pure states. We first state the background necessary for proving this result.

Recall that the Rényi $\alpha$-relative entropies $S_\alpha$ are monotonic under CPTP maps for $\alpha \in [0, 2]$, and the corresponding $\alpha$-relative entropies of entanglement are defined by
\[
E_{R,\alpha}(\rho) = \inf_{\sigma \in S} S_\alpha(\rho\|\sigma).
\]
When $\rho = \psi$ is a pure state, the $\alpha$-relative entropy reduces to
\[
S_\alpha(\psi\|\sigma) = \frac{1}{\alpha - 1} \log \langle \psi | \sigma^{1-\alpha} | \psi \rangle.
\]
While $S_\alpha$ is jointly convex for all $\alpha \in [0, 1]$, it is also known to be convex in the second argument for $\alpha \in [0, 2]$ (see [35]). This allows us to state a useful minimization criterion for the $\alpha$-relative entropies of entanglement [36]. Indeed, for any state $\rho$ and any separable state $\sigma$, it holds that
\[
E_{R,\alpha}(\rho) = S_\alpha(\rho\|\sigma)
\]
if and only if it holds that
\[
\frac{d}{dt} S_\alpha(\rho\|(1-t)\sigma + t\sigma') \bigg|_{t=0} \geq 0 \tag{34}
\]
for all other separable states $\sigma' \in S$. Furthermore, we can limit our consideration only to pure separable states $\sigma' = |\phi\rangle\langle\phi|$ where $\phi$ is separable.
To show that \( E_{R,\alpha}(\psi) = E_{1/\alpha}(\psi) \) holds for all pure states \( \psi \) and all \( \alpha \in [0, 2] \), we must first find a separable state \( \sigma \) such that \( S_{\alpha}(\psi\|\sigma) = E_{1/\alpha}(\psi) \). We then show that (34) holds for all separable pure states \( \sigma' \). The necessary background for computing the derivatives in (34) is given in Section 7.1. Finally, the proof of Theorem 4 is given in Section 7.2.

7.1 Fréchet derivatives

Here we define the Fréchet derivative of matrix functions, following the formalism of [37, Section 3.4]. See also [36]. Given a differentiable function \( f : (0, \infty) \to \mathbb{R} \), its divided differences for \( x, y \in \mathbb{R} \) are defined as

\[
\begin{align*}
  f^{[1]}(x, y) = & \begin{cases} 
  \frac{f(x) - f(y)}{x - y} & x \neq y \\
  f'(x) & x = y.
  \end{cases}
\end{align*}
\]

Let \( X = \text{diag}(x_1, \ldots, x_n) \) be an \( n \times n \) positive diagonal matrix. Its matrix of divided differences is the \( n \times n \) matrix \( f^{[1]}(X) \) whose entries are given by

\[
(f^{[1]}(X))_{ij} = \begin{cases} 
  f^{[1]}(x_i, x_j) & x_i, x_j \neq 0 \\
  0 & x_i = 0 \text{ or } x_j = 0.
  \end{cases}
\]

The Fréchet differential operator of \( f \) at \( X \), denoted \( \partial f(X) \), is a linear map of matrices defined by

\[
\partial f(X)(Y) = f^{[1]}(X) \circ Y
\]

for arbitrary matrices \( Y \), where \( \circ \) denotes the entry-wise product of matrices such that the elements of \( \partial f(X)(Y) \) are given by

\[
(\partial f(X)(Y))_{ij} = \begin{cases} 
  f^{[1]}(x_i, x_j)Y_{ij} & x_i, x_j \neq 0 \\
  0 & x_i = 0 \text{ or } x_j = 0,
  \end{cases}
\]

and \( Y_{ij} \) are the entries of \( Y \). If \( A \) is a hermitian matrix such that \( \text{supp}(A) \subseteq \text{supp}(X) \), then

\[
\frac{d}{dt} \left|_{t=0} \text{Tr}(Af(X + tY)) \right. = \text{Tr}(A\partial f(X)(Y)).
\]

7.2 Proof of Theorem

We now present the proof that \( E_{R,\alpha}(\psi) = E_{1/\alpha}(\psi) \) for all pure states \( \psi \) and all \( \alpha \in [0, 2] \). This result is already known when \( \alpha = 1 \) [22], so here we prove it only for \( \alpha \neq 1 \). We will consider the case when \( \alpha = 0 \) separately.

Here, we will make use of the functions \( f_\alpha \) for \( \alpha \in [0, 1) \cup (1, 2] \) that are defined by \( f_\alpha(x) = x^{1-\alpha} \) and whose divided differences are given by

\[
f^{[1]}_\alpha(x, y) = \begin{cases} 
  \frac{x^{1-\alpha} - y^{1-\alpha}}{x - y} & x \neq y \\
  (1-\alpha)x^{-\alpha} & x = y.
  \end{cases}
\]

For these functions, it is straightforward to check that

\[
f^{[1]}_\alpha \left( \frac{x}{c}, \frac{y}{c} \right) = c^\alpha f^{[1]}_\alpha(x, y)
\]

holds for any real constant \( c > 0 \) and any \( x, y \geq 0 \). We now prove Theorem 4.
Proof (of Theorem 4). Let \( \psi = \sum_i \sqrt{\lambda_i} |ii\rangle \) be a pure state with Schmidt coefficients \( \lambda = (\lambda_1, \ldots, \lambda_d) \). First suppose that \( \alpha \neq 0 \) and define the following separable density operator

\[
\sigma = \frac{1}{\|\lambda\|_1^{1/\alpha}} \sum_i \lambda_i^{1/\alpha} |ii\rangle \langle ii|,
\]

where \( \|\lambda\|_1^{1/\alpha} = (\sum_i \lambda_i^{1/\alpha})^\alpha \). Note that the \( \alpha \)-relative entropy of \( \psi \) and \( \sigma \) is

\[
S_\alpha(\psi||\sigma) = \frac{1}{\alpha - 1} \log \langle \psi|\sigma^{1-\alpha}|\psi \rangle
= \frac{1}{\alpha - 1} \log \|\lambda\|_1^{1/\alpha}
= \frac{1}{1 - \alpha/\|\lambda\|_1^{1/\alpha}} \log \|\lambda\|_1^{1/\alpha}
\]

from which it follows that \( S_\alpha(\psi||\sigma) = E_{1/\alpha}(\psi) \). The matrix \( \sigma \) has eigenvalues \( \lambda_i^{1/\alpha}/\|\lambda\|_1^{1/\alpha} \) and its matrix of divided differences can be given by

\[
f^{(1)}_\alpha(\sigma) = \|\lambda\|_1^{1/\alpha} \sum_{ij} f^{(1)}_\alpha(\lambda_i^{1/\alpha}, \lambda_j^{1/\alpha}) |ii\rangle \langle jj|.
\]

For any other separable state \( \sigma' \), we may use the Fréchet derivative to compute

\[
\frac{d}{dt} S_\alpha(\psi|(1-t)\sigma + t\sigma') \bigg|_{t=0} = \frac{1}{\alpha - 1} \frac{d}{dt} \log \langle \psi|(1-t)\sigma + t\sigma')^{1-\alpha}|\psi \rangle \bigg|_{t=0}
= \frac{1}{\alpha - 1} \frac{1}{\langle \psi| \sigma^{1-\alpha} |\psi \rangle} \langle \psi| \partial f_\alpha(\sigma)(\sigma' - \sigma)|\psi \rangle
= 1 - \frac{1}{1 - \alpha/\|\lambda\|_1^{1/\alpha}} \langle \psi| \partial f_\alpha(\sigma)(\sigma')|\psi \rangle,
\]

where the final equality follows from the fact that \( \langle \psi| \sigma^{1-\alpha} |\psi \rangle = \|\lambda\|_1^{1/\alpha} \), the linearity of the map \( \partial f_\alpha(\sigma) \), and the fact that \( \partial f_\alpha(\sigma)(\sigma) = (1-\alpha)\sigma^{1-\alpha} \). When \( \sigma' = |\phi\rangle\langle \phi| \) is a separable pure state for some \( |\phi\rangle = \sum_{i,j} u_i |v_j i j\rangle \) with \( \sum_i |u_i|^2 = \sum_j |v_j|^2 = 1 \), it holds that

\[
\partial f_\alpha(\sigma)(\sigma') = f^{(1)}_\alpha(\sigma) \circ \sigma'
= \|\lambda\|_1^{1/\alpha} \sum_{i,j} u_i |v_j i j\rangle f^{(1)}_\alpha(\lambda_i^{1/\alpha}, \lambda_j^{1/\alpha}) |ii\rangle \langle jj|.
\]

Note that \( f^{(1)}_\alpha(x, y)/(1 - \alpha) \geq 0 \) for any \( x, y \geq 0 \). Hence

\[
\left| \frac{1}{1 - \alpha/\|\lambda\|_1^{1/\alpha}} \langle \psi| (\partial f_\alpha(\sigma)(\sigma')) |\psi \rangle \right| \leq \sum_{i,j} |u_i| |v_j||u_i||v_j| \sqrt{\lambda_i \lambda_j} \frac{f^{(1)}_\alpha(\lambda_i^{1/\alpha}, \lambda_j^{1/\alpha})}{1 - \alpha/\|\lambda\|_1^{1/\alpha}} \leq \sum_{i,j} |u_i| |v_j||u_i||v_j| \leq \sum_i |u_i|^2 \sum_i |v_i|^2 = 1,
\]

25
where the second inequality is due to Lemma 22 (see below) and the final inequality follows from the Cauchy-Schwarz inequality. This completes the proof, since we have found that

$$\frac{d}{dt}S_\alpha(\psi\|(1-t)\sigma + t\sigma')\bigg|_{t=0} \geq 0$$

for all pure separable states \(\sigma'\).

Lastly, to prove the claim when \(\alpha = 0\), set \(\sigma = |11\rangle\langle 11|\). Note that \(S_0(\psi||\sigma) = -\log \lambda_1\), where \(\lambda_1\) is the largest Schmidt coefficient of \(\psi\). For any other separable state \(\sigma' \in S\), it holds that \(\langle \psi|\sigma'|\psi \rangle \leq \lambda_1\) (by Lemma 6) and thus

\[
S_0(\psi||\sigma') = -\log \langle \psi|\sigma'|\psi \rangle \geq -\log \lambda_1 = S_0(\psi||\sigma),
\]

from which it follows that \(E_{R,0}(\psi) = -\log \lambda_1\). Since \(E_{+\infty}(\psi) = -\log \lambda_1\), the result follows. \(\square\)

The proof of Theorem 4 above in the case when \(\alpha \neq 0\) relies on the following lemma, which we prove below.

**Lemma 22.** Let \(\alpha \in (0,1) \cup (1,2]\). For all \(p, q \in (0,1]\), it holds that

\[
0 \leq \sqrt{pq} \frac{1}{1-\alpha} f^{(1)}_\alpha(p^{1/\alpha}, q^{1/\alpha}) \leq 1.
\]

The proof is trivial in the case when \(\alpha = 2\). Indeed, \(f_2(x) = x^{-1}\) and thus for \(p \neq q\) it holds that

\[
-\sqrt{pq} f^{(1)}_2(\sqrt{p}, \sqrt{q}) = -\sqrt{pq} \frac{1}{\sqrt{p}} - \frac{1}{\sqrt{q}} = 1.
\]

The proof in the case when \(\alpha \neq 2\) is a bit more technical and requires the following pair of results.

**Lemma 23.** For all \(x, y \geq 0\) with \(x \neq y\) and \(r \in (-1,0) \cup (0,1)\), it holds that

\[
\frac{1}{r} \frac{x^r - y^r}{x - y} \leq (\sqrt{xy})^{r-1}.
\]

**Proof.** For \(x, y \geq 0\) and \(r \in (-1,0) \cup (0,1)\), we have the following integral representations:

\[
rx^{r-1} = \frac{\pi}{\sin(r\pi)} \int_0^\infty \frac{t^r}{(x + t)^2} dt, \quad \frac{x^r - y^r}{x - y} = \frac{\pi}{\sin(r\pi)} \int_0^\infty \frac{t^r}{(x + t)(y + t)} dt.
\]

Note that \(x + y \geq 2 \sqrt{xy}\) for all \(x, y \geq 0\), and thus

\[
(x + t)(y + t) = xy + t(x + y) + t^2 \\
\geq xy + 2 \sqrt{xy} t + t^2 = (\sqrt{xy} + t)^2
\]
holds for all $t \geq 0$. Then

$$\frac{1}{r} x' - y' = \frac{1}{r} \pi \int_0^\infty \frac{t}{(x + t)(y + t)} dt$$

\begin{align*}
&\leq \frac{1}{r} \pi \int_0^\infty \frac{t}{(x + t)(y + t)} dt \\
&= \left( \sqrt{x} \right)^{-1},
\end{align*}

where we use the representations in (42) and (43).

\begin{lemma}
For all $p, q \in (0, 1)$ with $p \neq q$, and all $\alpha \in (0, 1) \cup (1, 2)$, it holds that

$$\frac{\sqrt{pq}}{1 - \alpha} \left( \frac{p^{(1-\alpha)/\alpha} - q^{(1-\alpha)/\alpha}}{p^{1/\alpha} - q^{1/\alpha}} \right) \leq 1.$$ (44)

\end{lemma}

\begin{proof}
Setting $r = 1 - \alpha$, $x = p^{1/\alpha}$, and $y = q^{1/\alpha}$, an application of Lemma 23 yields

$$\frac{\sqrt{pq}}{1 - \alpha} \left( \frac{p^{(1-\alpha)/\alpha} - q^{(1-\alpha)/\alpha}}{p^{1/\alpha} - q^{1/\alpha}} \right) \leq \sqrt{pq} \left( \sqrt{p^{1/\alpha} q^{1/\alpha}} \right)^{-\alpha}$$

$$= 1,$$

as desired. \qed

A proof of Lemma 22 in the case $\alpha \neq 2$ now follows.

\begin{proof}[of Lemma 22]
If $p \neq q$, we can apply Lemma 24. Otherwise we have

$$f^{(1)}_\alpha(p^{1/\alpha}, p^{1/\alpha}) = (1 - \alpha)(p^{1/\alpha})^{-\alpha} = \frac{1 - \alpha}{p}$$

and thus $f^{(1)}_\alpha(p^{1/\alpha}, p^{1/\alpha}) = 1$, which completes the proof. \qed

\section{Discussion}

Although LOCC maps provide the most physically meaningful choice of free operations for the resource theory of entanglement, from a resource-theoretic point of view, other choices of operations still provide consistent and well-defined theories. Furthermore, insight into LOCC is gained by studying more general resource theories since impossibility results for the latter imply impossibility results for the former. We have examined many different classes of non-entangling operations that are still larger than the class of LOCC (and separable) operations. While many entanglement measures are still monotonic under such non-entangling operations, we have presented examples of entanglement measures whose monotonicity fails to hold. This allowed us to find transformations among entangled states that are possible under non-entangling maps but not possible under LOCC (or separable maps). PPT operations have also been widely studied in the context of entanglement theory and, in a similar vein, here we have also shown that, perhaps surprisingly, the negativity is no longer a monotone when this class is extended to the set of PPT-preserving operations.
Although we believe that resource theories of entanglement under different classes of operations is an interesting research topic in itself, the main motivation behind this present work is to understand more deeply the phenomenon of bound entanglement. The question of whether NPT bound entanglement exists is one of the most challenging open problems in the field. If NPT bound entanglement does exist, this could be proven by finding some NPT entangled state that still remains undistillable by a larger class of operations than LOCC. So far, only PPT operations had been considered in this context [11]. Here, we have introduced and systematically studied entanglement distillation under hierarchies of operational classes that go beyond LOCC. Interestingly, we have proven that in most of these classes, the phenomenon of bound entanglement disappears. In particular, every entangled state is distillable by some non-entangling $\epsilon$-LOCC operation $\forall \epsilon > 0$. Hence if some superset of LOCC contains NPT bound entanglement, it must be a relatively constrained extension. Most importantly, the only class for which we have been unable to prove distillability of all NPT states is that of $k$-non-entangling maps. Thus, further investigation into the properties of this class could be a promising route to answer the long-standing question of NPT bound entanglement. In particular, even though we have proved that the class of $k$-non-entangling maps is a strict superset of LOCC and SEP maps, this alone does not necessarily imply that there exist pure-state transformations achievable by the former class of maps which are impossible by the latter. It would be interesting to study if such examples exist and, if so, which entanglement measures loose their monotonicity under $k$-non-entangling maps.

EC is supported by the National Science Foundation (NSF) Early CAREER Award No. 1352326. JdV acknowledges support from the Spanish MINECO through grants MTM2014-54692-P and MTM2014-54240-P and from the Comunidad de Madrid through grant QUITEMAD+CMS2013/ICE-2801. MG is acknowledges support from an Izaak Walton Killam Memorial Scholarship and an Alberta Innovates–Technology Futures (AITF) Graduate Student Scholarship. GG acknowledges support from the Natural Sciences and Engineering Research Council of Canada (NSERC).

References

[1] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki. Quantum entanglement. Rev. Mod. Phys., 81(2), 865–942 (2009). arXiv:quant-ph/0702225.

[2] M. B. Plenio and S. Virmani. An introduction to entanglement measures. Quantum Inf. Comput., 7(1), 1–51 (2007). arXiv:quant-ph/0504163.

[3] E. Chitambar, D. Leung, L. Mančinska, M. Ozols, and A. Winter. Everything You Always Wanted to Know About LOCC (But Were Afraid to Ask). Comm. Math. Phys., 328(1), 303–326 (2014). arXiv:1210.4583.

[4] G. Vidal and J. I. Cirac. Irreversibility in asymptotic manipulations of entanglement. Phys. Rev. Lett., 86(25), 5803–5806 (2001). arXiv:quant-ph/0102036.

[5] D. Yang, M. Horodecki, R. Horodecki, and B. Synak-Radtke. Irreversibility for all bound entangled states. Phys. Rev. Lett., 95(19), 1–4 (2005). arXiv:quant-ph/0506138.
[6] M. Horodecki, P. Horodecki, and R. Horodecki. Mixed-state entanglement and distillation: is there a “bound” entanglement in nature? *Phys. Rev. Lett.*, 80(24), 5239–5242 (1998). arXiv:quant-ph/9801069.

[7] F. G. S. L. Brandão and M. B. Plenio. Entanglement theory and the second law of thermodynamics. *Nat. Phys.*, 4(11), 873–877 (2008). arXiv:0810.2319.

[8] F. Brandão and M. B. Plenio. A Reversible Theory of Entanglement and its Relation to the Second Law. *Comm. Math. Phys.*, 295(3), 829–851 (2010). arXiv:0710.5827.

[9] F. G. S. L. Brandão, M. Christandl, and J. Yard. Faithful Squashed Entanglement. *Comm. Math. Phys.*, 306(3), 805–830 (2011). arXiv:1010.1750.

[10] L. Pankowski, M. Piani, M. Horodecki, and P. Horodecki. A Few Steps More Towards NPT Bound Entanglement. *IEEE Trans. Inf. Theory*, 56(8), 4085–4100 (2010). arXiv:0711.2613.

[11] T. Eggeling, K. G. H. Vollbrecht, R. F. Werner, and M. Wolf. Distillability via Protocols Respecting the Positivity of Partial Transpose. *Phys. Rev. Lett.*, 87(25), 257902 (2001). arXiv:quant-ph/0104095.

[12] E. M. Rains. Rigorous treatment of distillable entanglement. *Phys. Rev. A*, 60(1), 173–178 (1999). arXiv:quant-ph/9809078.

[13] E. M. Rains. Erratum: Bound on distillable entanglement [Phys. Rev. A 60, 179 (1999)]. *Phys. Rev. A*, 63(1), 019902 (2000).

[14] E. M. Rains. Entanglement purification via separable superoperators (1997). arXiv:quant-ph/9707002.

[15] M. A. Nielsen. Conditions for a Class of Entanglement Transformations. *Phys. Rev. Lett.*, 83(2), 436–439 (1999).

[16] W. Dürr, G. Vidal, and J. I. Cirac. Three qubits can be entangled in two inequivalent ways. *Phys. Rev. A*, 62(6), 062314 (2000). arXiv:quant-ph/0005115.

[17] M. Horodecki, P. Horodecki, and R. Horodecki. Separability of mixed states: necessary and sufficient conditions. *Phys. Lett. A*, 223(1-2), 1–8 (1996). arXiv:quant-ph/9605038.

[18] L. M. Ioannou and B. C. Travaglione. Quantum separability and entanglement detection via entanglement-witness search and global optimization. *Phys. Rev. A*, 73(5), 052314 (2006). arXiv:quant-ph/0602223.

[19] O. Gühne and N. Lütkenhaus. Nonlinear entanglement witnesses, covariance matrices and the geometry of separable states. *J. Phys. Conf. Ser.*, 67, 12004 (2007). arXiv:quant-ph/0612108.

[20] E. M. Bronstein. Approximation of convex sets by polytopes. *J. Math. Sci.*, 153(6), 727–762 (2008).

[21] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight. Quantifying Entanglement. *Phys. Rev. Lett.*, 78(12), 2275–2279 (1997). arXiv:quant-ph/9702027.
Here we show that the values $\delta = d^{-4}$ and $\epsilon = d^{-12}$ satisfy both (10) and (11). Indeed, for these values of $\epsilon$ and $\delta$, it is straightforward to check that

$$\frac{d^2}{\sqrt{d-1}} \sqrt{(1-\epsilon)\epsilon} \leq d^{-4}$$
and thus
\[
(1 - \delta) \left( 1 + \frac{d^2}{\sqrt{d - 1}} \sqrt{(1 - \epsilon)e} \right) \leq (1 - d^{-4})(1 + d^{-4}) \\
\leq 1,
\]
which implies that (10) holds. On the other hand, it clear that
\[
d^2(1 - \epsilon) \sqrt{(1 - \delta)\delta} \leq d^2 \sqrt{\delta} \\
= 1.
\]
Furthermore, it holds that $1 \leq \sqrt{k - 1}$ since $k \geq 2$. It follows that
\[
d^2(1 - \epsilon) \leq \frac{\sqrt{k - 1}}{\sqrt{(1 - \delta)\delta}}
\]
which proves that (11) holds as desired.