A suboptimal algorithm for synthesis of a movable object control

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Abstract. In this paper a suboptimal algorithm for synthesis of a movable object control, based on the known approach, according to which a control is formed depending on a control error – the difference between current and required values of a regulated variables vector is considered. The distinctive peculiarity of the algorithm was based on the vector of regulated parameters estimation approach. This vector is proposed to be estimated by solving a mobile left endpoint problem derived from the original one. It is shown by examples that the proposed approach allows to simplify the control synthesis problem solution and obtain effective control algorithms, providing a suboptimal control and, in some cases, presented in this paper, an optimal control. The suggested algorithm can be used to determine continuous and pulse controls of aircrafts (including spacecrafts) in problems of their approaching (guidance) and spacecraft docking, carrying out of correcting maneuvers, descent and soft landing of spacecrafts at al.

1. Introduction

The problem of effective movable objects control algorithms development is actual. Its solution, as a rule, is formed as follows:

1) firstly a required (basis) object trajectory is determined by a solution of a control (optimum control) problem with the fixed initial conditions [1, 2]. At the same time the required trajectory may be rebuilt [3], if the current object state is selected as initial conditions;

2) secondly a control, provides for the movement along the required path with using the control error – the difference between required (which corresponds to required trajectory) and current values of a regulated variables vector is generated.

The main problem is to solve the first task in the above list. This paper deals with the mentioned above approach, according to which a control is formed depending on a control error – the difference between current and required values of a regulated variables vector.

The main peculiarity and the novelty of the considered algorithm related with the required value of regulated variables vector determining method (see the task no 1 in the list above). We suggest to estimate this vector by solving of a mobile left endpoint problem instead of the fixed left endpoint problem [3], as it is generally accepted.

This approach allows to: 1) substantially simplify the optimal control problem solution; 2) synthesize the control which either coincides with the optimal one or it is close to the optimal control and this fact is confirmed by examples shown.

Moreover, at approximate solution of the optimal control problem with the mobile left endpoint one can use the widely applicable parametric representation of a desirable trajectory in the form of a given...
function of one of co-ordinates and a set of selected parameters even in cases when it cannot be made
in the original optimum control problem with the fixed left endpoint.

When described problems solving, the widely known approaches – the Pontryagin's maximum
principle, direct variational methods and others can be applied [4, 5].

In this paper we’ll show that the proposed approach makes it possible to simplify the control synthesis
problem solution and obtain effective control algorithms, providing a suboptimal control and, in some
cases, presented in this paper, an optimal control.

2. Materials and research methods

The optimum control of a movable object problem, which we should call the "original", is considered:
\[
\min_{u} I
\]
with connections (motion equations)
\[
\dot{x} = f(x, u, t),
\]
under constraints
\[
x \in X, \; \bar{u} \in U,
\]
initial
\[
x(0) = \bar{x}_0
\]
and terminal conditions
\[
x(T) \in M_T.
\]

In formulas (1)–(5) the following designations are accepted: \( I \) – criterion; \( x \) – \( n \)-dimensional state
vector; \( u \) – control vector; \( t \) – current time; \( X, U, M_T \) – given sets; \( \bar{x}_0 \) – given vector of initial conditions;
\( T \) – time of movement.

For the approximate solution of "original" task let’s use the approach, according to which the control
\( \bar{u} \) is set depending on the control error vector \( E \), estimating the deviation of \( m \)-dimensional (\( m < n \))
vector of adjustable parameters \( \bar{x}_m \) from its given (required) value \( \bar{x}_{mR} \). Thus, a vector \( \bar{x}_m \) during the
movement should be modified so that to ensure \( E \to 0 \) or, that is the same, \( \bar{x}_m \to \bar{x}_{mR} \). This approach
is known as well as control algorithms, depending on an error \( E = g(E) \),
g – an operator), denoted by A A Krasovskiy [4] and widely used in practice [5, 6]. Thereby, the
implementation of a considered approach and the control problem solution optimization is related with
the estimation of \( \bar{x}_{mR} \).

A vector \( \bar{x}_m \) is a function of a state vector \( \bar{x} \).

When forming a state vector \( \bar{x} \) in appropriate way, one can achieve, that \( \bar{x}_m \) includes last \( m \) components of the state vector \( \bar{x} \). Then
\[
\bar{x} = (\bar{x}_{n-m}, \bar{x}_m),
\]
where \( \bar{x}_{n-m} \) – the vector which includes first \( n - m \) components of \( \bar{x} \).

The required value of \( \bar{x}_{mR} \) we suggest to determine by solving the optimal control problem which we
should call “auxiliary”. It differs from the "original" one in that the condition (4) is substituted to
\[
\bar{x}_{n-m,0} (0) = \bar{x}_{n-m,0,0}, \; \bar{x}_m (0) \in X_m.
\]

Consequently, the "original" problem with the fixed left endpoint transforms into the “auxiliary”
problem with the mobile left endpoint.

The vector \( \bar{x}_m \) and the set \( X_m \) are chosen from practical reasons according to control object’s
properties and received information. Often the value of \( \bar{x}_m (0) \) is not restricted by the constraint
\( \bar{x}_m (0) \in X_m \).
The value of \( \bar{x}_m(0) \), obtained when solving the “auxiliary” problem, also is the required \( \bar{x}_{mk} \).

To obtain the approximate solution of the “auxiliary” problem and, therefore, to estimate \( \bar{x}_{mk} \) we can apply different parametric methods (such as direct variational methods and others).

It should be noted that, that constraints imposed on state vectors and controls (3) and also motion equations (2) in the “auxiliary” problem may be more simple than in the "original" problem, because the exact accounting of the "original" problem constraints may be provided by forming the control which depends on the error \( E \).

Let’s now consider solutions of some control problems with usage of the described approach.

3. Results and discussion

Example 1. The "original" optimal control problem:

\[
\begin{align*}
\min_u & \int_0^T dt, \\
\dot{x}_1 &= x_2, \quad \dot{x}_2 = u, \quad |u| \leq u_m, \\
x_1(0) &= x_{10}, \\
x_2(0) &= x_{20}, \\
x_1(T) &= 0, \quad x_2(T) &= 0,
\end{align*}
\]

(8)–(12)

where \( \bar{x} = (x_1, x_2) \), \( x_1, x_2 \) – position and velocity; \( u \) – control; \( u_m, \bar{x}_0 = (x_{10}, x_{20}) \) – given constraints on control and initial conditions.

The decision of this problem is known. Below it is written in the form of synthesis:

\[
\begin{align*}
u &= \begin{cases} -u_m \cdot \text{sign}(P), & \text{if } P \neq 0; \\ u_m \cdot \text{sign}(x_i), & \text{if } P = 0; \end{cases} \\
P &= 2 \cdot u_m \cdot x_1 + x_2 \cdot |x_2|.
\end{align*}
\]

(13)–(14)

Let’s take the velocity \( x_2 \) as the adjustable parameter and the error of control will be estimated by formula:

\[
E = x_{2R} - x_2,
\]

(15)

where \( x_{2R} \) – is the required value of the adjustable parameter \( x_2 \).

When finding \( x_{2R} \) we shall not restricted \( x_2(0) \). Then, according to the described approach, for to estimate \( x_{2R} \) the "auxiliary" optimum control problem which differs from the "original" problem (8)–(12) in that, the initial condition is absent, is to be solved. As the result, we should get (in the synthesis form):

\[
x_{2R} = -\sqrt{2 \cdot u_m \cdot |x_1| \cdot \text{sign}(x_i)}.
\]

(16)

The control by the error \( E \) (15) we may construct, having used an obvious relation:

\[
u = \begin{cases} u_m \cdot \text{sign}(E), & \text{if } E \neq 0; \\ u_m \cdot \text{sign}(x_i), & \text{if } E = 0. \end{cases}
\]

(17)

Let’s note that expressions (13), (16) are easy to obtain when using the maximum principle of L S Pontryagin.

Controls evaluated by (13) and (17) coincide. Thus, control (17) is optimal for the “original” problem (8)–(13). Besides, to solve an "auxiliary" problem is easier, than an "original", because an "original" problem may have a switch of control, but control does not vary in an "auxiliary" problem.
In the use of parametric methods, when a trajectory or a control vector are determined by a set of unknown parameters, with reference to an "auxiliary" problem a trajectory \( x(t) \) in the first case may describe by the second order polynomial of time \( t \), and control \( u(t) \) – in the second case – by the zero order polynomial, because it is necessary to satisfy for three boundary conditions (10), (12). At the same time it is not difficult to get formula (16) in both cases by minimizing the motion time \( T \), and then to synthesize the optimum control for an "original" problem by formula (17).

Therefore, with reference to an "original" problem polynomials orders, which describe a trajectory \( x(t) \) or control \( u(t) \), are to be increased, because it is necessary to fulfill not to three, but to four boundary conditions (10)–(12), so the obtained control in both cases should not be optimal by the criterion (8).

Example 2. The "original" optimal control problem: criterion (8), movement equations, constraints, initial and terminal conditions are

\[
\begin{align*}
\dot{x}_1 &= V \cdot \cos(x_1), \quad \dot{x}_2 = V \cdot \sin(x_1), \quad \dot{x}_3 = u, \quad |u| \leq u_m, \quad (18) \\
x_1(0) &= x_{10}, \quad x_2(0) = x_{20}, \quad (19) \\
x_3(0) &= x_{30}, \quad (20) \\
x_1(T) &= x_{1f}, \quad x_2(T) = x_{2f}, \quad x_3(T) = x_{3f}, \quad (21)
\end{align*}
\]

where \( x = (x_1, x_2, x_3) \), \( x_1, x_2 \) – coordinates, \( x_3 \) – trajectory inclination; \( V \) – velocity; \( u \) – control; \( u_m \), \( x_{10}, x_{20}, x_{30} \), \( x_{1f}, x_{2f}, x_{3f} \) – given constraints on control, initial and terminal conditions.

The solution of the given problem is known and the optimal programmed control is a piecewise constant function of time, which takes at time intervals one of following values: \( u = u_m \), \( u = -u_m \), \( u = 0 \). Appropriate optimal paths are: "turn – straight line – turn", "turn – turn – turn".

Let’s take the path inclination \( x_3 \) as the adjustable parameter and the error of control \( \tilde{E} \) we should estimate by the equation

\[
\tilde{E} = x_{3R} - x_3, \quad (22)
\]

where \( x_{3R} \) – the required value of the adjustable parameter \( x_3 \).

We should not restrict \( x_3(0) \) during the search of \( x_{3R} \). Then, to evaluate the value of \( x_{3R} \) it is necessary to solve the "auxiliary" optimal control problem, which differs from the "original" problem (8), (18)–(21) in that, the initial condition (20) is absent. As a result we get (in a synthesis form):

\[
x_{3R} = \begin{cases} 
\alpha_1 + \pi - \alpha_2, & \text{if } \rho \geq R; \\
\beta_1 + \beta_2, & \text{if } \rho < R;
\end{cases} \quad (23)
\]

where

\[
R = \frac{V}{u_m}; \quad \rho = \sqrt{x_1^2 + (x_2 - R)^2}; \quad (24)
\]

\[
\alpha_2 = \arcsin\left(\frac{R}{\rho}\right); \quad \alpha_1 = \begin{cases} 
\alpha_{10}, & \text{if } x_1 \geq 0; \\
\pi - \alpha_{10}, & \text{if } x_1 < 0;
\end{cases} \quad \alpha_{10} = \arcsin\left(\frac{x_2 - R}{\rho}\right); \quad (25)
\]

\[
\beta_1 = \beta_{12} - \beta_{11}; \quad \beta_{12} = \arcsin\left(\frac{x_{1e}}{\rho_1}\right); \quad \beta_{11} = \arcsin\left(\frac{x_1}{\rho_1}\right); \quad (26)
\]

\[
x_{1e} = \sqrt{R^2 + (x_{2e} - R)^2}; \quad x_{2e} = \frac{\rho_1^2 - R^2}{4R}; \quad \rho_1 = \sqrt{x_1^2 + (x_2 + R)^2}; \quad (27)
\]

\[
\beta_2 = 2 \cdot \arcsin\left(\frac{x_{2e}}{\rho_2}\right); \quad \rho_2 = \sqrt{x_{1e}^2 + x_{2e}^2}. \quad (28)
\]
Relations (23)–(28) are obtained under conditions: \( x_2 \geq 0, x_{1T} = 0, x_{2T} = 0, x_{3T} = 0 \). These conditions are always satisfied at the expense of the transition to the new coordinate system, which origin is to be set in a point with coordinates \( x_{1T}, x_{2T} \) and its axes are oriented in appropriate way.

Optimal trajectories of the "auxiliary" task are: "straight line – turn", if \( \rho \geq R \) and “turn – turn” otherwise (if \( \rho < R \)).

To solve an "auxiliary" problem is easier, because the amount of control switches in it is less than in the “original”.

Let’s note that in situations of practical interest (the trajectory of the "original" problem is “turn – straight line – turn”) then, as a rule, the optimal controls of the "original" problem coincide with controls created by means of the above discussed approach.

Solutions of "original" and "auxiliary" problems, considered in the example 2, are easy to obtain by using of the maximum principle of L S Pontryagin.

Example 3. The "original" optimal control problem: (8), (18)–(21), where \[7\]:

\[
\begin{align*}
3\cos \nu x g & \Rightarrow V, \\
x_{10} = 0 \text{ km}, & x_{20} = 10 \text{ km}, x_{30} = 0 \text{ degrees}, \\
x_{1T} = 37 \text{ km}, & x_{2T} = 0 \text{ km}, x_{3T} = -80 \text{ degrees}, V = 300 \text{ m/s}.
\end{align*}
\]

In above formulas: \( x_1 \) – horizontal distance; \( x_2 \) – height of flight; \( n_y \) – normal load factor; \( g \) – acceleration of gravity.

In [7] the decision of this problem with the use of the parametric method is depicted, where the desirable trajectory was set in the form of a polynomial

\[
x_2(x_1) = A_0 + A_1 \cdot x_1 + A_2 \cdot x_1^2 + A_3 \cdot x_1^3,
\]

where unknown coefficients \( A_0, A_1, A_2, A_3 \) are determined by boundary conditions

\[
A_0 = x_{20}; \quad A_1 = \tan(x_{30});
\]

\[
A_2 = \frac{3(x_{2T} - x_{20})}{x_{1T} - x_{10}} - \frac{\tan(x_{1T})}{x_{1T} - x_{10}} - \frac{2\tan(x_{30})}{x_{1T} - x_{10}};
\]

\[
A_3 = -\frac{2(x_{2T} - x_{30})}{(x_{1T} - x_{10})^3} + \frac{\tan(x_{30})}{x_{1T} - x_{10}} + \frac{\tan(x_{3T})}{(x_{1T} - x_{10})^2}.
\]

While moving along the trajectory (32) the control \( u \) and the load factor \( n_y \) were determined by relations

\[
u = \left(2 \cdot A_2 + 6 \cdot A_3 \cdot (x_1 - x_{10})\right) \cdot V \cdot \cos^3(x_3),
\]

\[
n_y = \frac{V}{g} u + \cos(x_3).
\]

Estimated results are depicted in the table 1 below, where the following assumptions had been taken: the velocity \( V \) does not vary; at the desired trajectory (32) the constraint on the control \( u \) (and, consequently, on the normal load factor \( n_y \)) was not imposed; the control had been formed according the above considered approach (see example 2) and constraints on \( u_m \) were obtained by the formula

\[
u_m = \frac{g}{V} a, \quad a = \begin{cases} 2, \text{ when estimation of } R \text{ is used (see(24))}; \\ 3, \text{ when control is obtained depending on } E. \end{cases}
\]
Table 1. Results obtained in example 3.

| Method                          | $T$ (s) | $n_{y_{\text{max}}}$ | $W$ (s) |
|---------------------------------|---------|-----------------------|---------|
| Inverse dynamics problems concept| 242     | 3.4                   | 152     |
| Concept described in this article | 133     | 1.4                   | 119     |

According to table 1:

$$n_{y_{\text{max}}} = \max_{0 \leq t \leq T} n_y(t); \quad W = \int_0^T n_y^2 \, dt$$  \hspace{1cm} (39)

Note that, as a rule, constraints are imposed on $n_y$ instead of $u$. In order to ensure demands to $n_y$ according to considered methodic, we have to vary $u_m$ in appropriate way (see the relation (29)) during $x_{3R}$ estimation and the control error “elimination”.

Example 4. The "original" optimal control problem: (8), (18)–(21), (29)–(31) and additional constraint imposed on $x_3$:

$$x_3 \leq 0.$$ \hspace{1cm} (40)

Let’s take the path inclination $x_3$ as the adjustable parameter and the control error we should estimate by the formula (22).

In the "auxiliary" problem we should not set constraints on $(x_3(0), u)$ and eliminate the condition (40).

While the "auxiliary" problem (to find $x_{3R}$) approximate solving let’s use the inverse dynamics problems concept (see example 3) and set the desirable trajectory in the polynomial form (32) of:

- second order (not third order, as it is shown in the example 3);
- third order, thus, let’s demand, that the condition $n_y(T) = 0$ is to be satisfied.

As the result, we should get (in the synthesis form):

- for the second order polynomial

  $$x_{3R} = \arctg \left( \frac{2 (x_{3T} - x_i)}{x_{3T} - x_i} \cdot \tan(x_{3T}) \right);$$ \hspace{1cm} (41)

- for the third order polynomial

  $$x_{3R} = \arctg \left( -\frac{0.5 \cdot g \cdot (x_{3T} - x_i)}{v^2 \cdot \cos^2(x_{3T})} + \frac{3 (x_{3T} - x_i)}{x_{3T} - x_i} - 2 \cdot \tan(x_{3T}) \right).$$ \hspace{1cm} (42)

It is easy to observe, that (41) appears from conditions $A_1 = 0$ and $n_y (T) = 0$ (see (32), (35)–(37)), but parameters $x_{10}, x_{20}, x_{30}, x_i$ there should be replaced with $x_1, x_2, x_{3R}, x_{3T}$ respectively.

When using the control depending on $E$, the constraint on $u_m$ is determined by (38) and condition (40) is considered while the control error “elimination” by the correction of $x_{3R}$ as follows: if $x_{3R} > 0$, then it is set $x_{3R} = 0$. Results are shown in the table 2 below.

Table 2. Results of control estimation obtained in example 4.

| The order of the polynomial | $T$ (s) | $n_{y_{\text{max}}}$ | $W$ (s) |
|----------------------------|---------|-----------------------|---------|
| 2                          | 154     | 2.3                   | 195     |
| 3                          | 150     | 1.8                   | 152     |
Let's note that the representation of a desirable trajectory by polynomials in the form (32) in the “original” problem is possible, if \( |x_{00}| < \pi / 2 \), whereas in the “auxiliary” problem this demand is removed, because of \( |x_{00}| < \pi / 2 \).

The analyses of the direct methods applicability to design optimal short-term spatial maneuvers for an unmanned vehicle in a faster than real-time scale is performed in [8].

The task of the aerial vehicles adaptive control synthesis is considered in [9, 10]. Based on an advanced nonsingular fast terminal sliding mode control scheme and adaptive control, an adaptive nonsingular fast terminal sliding mode guidance law is there proposed in the presence of the target acceleration as an unknown bounded external disturbance.

4. Conclusion

The above considered examples demonstrate that:
- the proposed approach allows to simplify a solution of a control problem and to obtain effective control algorithms;
- at approximate solution of the optimal control problem with the mobile left endpoint one can use the widely applicable parametric representation of a desirable trajectory in the form of a given function of one of coordinates and a set of selected parameters even in cases when it cannot be made in the original optimum control problem with the fixed left endpoint (see examples 3 and 4 – the representation of a desirable trajectory by polynomials as (32) in the original problem is impossible at \( |x_{00}| \geq \pi / 2 \), but it is possible in the auxiliary problem because the value of \( x_{00} \) is selective).

In this paper we have shown that the proposed approach allows to simplify the control synthesis problem solution and obtain effective control algorithms, providing a suboptimal control and, in some cases, an optimal control.

The suggested algorithm can be used to determine continuous and pulse controls of aircrafts [10, 11] (including spacecrafts) in problems of their approaching (guidance) and spacecraft docking, carrying out of correcting maneuvers, descent and soft landing of spacecrafts at al. [12].

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