Weighted least squares estimators for the Parzen tail index

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Abstract

Estimation of the tail index of heavy-tailed distributions and its applications are essential in many research areas. We propose a class of weighted least squares (WLS) estimators for the Parzen tail index. Our approach is based on the method developed by Holan and McElroy (2010). We investigate consistency and asymptotic normality of the WLS estimators. Through a simulation study, we make a comparison with the Hill, Pickands, DEdH (Dekkers, Einmahl and de Haan) and ordinary least squares (OLS) estimators using the mean square error as criterion. The results show that in a restricted model some members of the WLS estimators are competitive with the Pickands, DEdH and OLS estimators.

Keywords: density-quantile, tail exponent, weighted least squares estimators.

1. Introduction and main results

The problem of estimating the tail characteristics of probability distributions has received enormous attention in the last decades. Let \( F \) be an absolutely continuous probability distribution function with density function \( f \) and let \( Q \) denote the corresponding quantile function defined as

\[
Q(s) := \inf\{x : F(x) \geq s\}, \quad 0 < s \leq 1, \quad Q(0) := Q(0+).
\]
Parzen (1979) used the density-quantile function $f_Q(\cdot) = f(Q(\cdot))$ to classify probability distributions. Parzen assumed that the limit

$$\nu_1 := \lim_{u \to 1} \frac{(1 - u)J(u)}{f_Q(u)}$$

exists, where $J$ is the score function defined as $J(u) = -(f_Q)'(u)$. Assumption (1) yields the following approximation for $u$ values near 1:

$$f_Q(u) \approx C(1 - u)^{\nu_1},$$

for some positive constant $C$. Based on the parameter $\nu_1$, Parzen classified the probability distributions. Heavy tailed distributions correspond to $\nu_1 > 1$.

Parzen (2004) assumed that $f_Q(\cdot)$ is regularly varying at 0 and 1:

$$f_Q(u) = u^{\nu_0}L_0(u), \quad u \in [0, 1/2),$$

$$f_Q(u) = (1 - u)^{\nu_1}L_1(1 - u), \quad u \in (1/2, 1],$$

where $\nu_0, \nu_1 > 0$ are finite constants and $L_0$ and $L_1$ are slowly varying at zero. The parameters $\nu_0$ and $\nu_1$ are called the left and right tail exponents of the density-quantile function.

Holan and McElroy (2010) considered the following orthogonal series expansion for $L_i$:

$$L_i(u) = \exp \left\{ \theta_{i,0} + 2 \sum_{k=1}^{\infty} \theta_{i,k} \cos(2\pi ku) \right\}, \quad i = 0, 1.$$  

(4)

In order to estimate the tail exponents, they assumed that $L_i$ admits the representation

$$L_i(u) = L_i^{(p_i)}(u) = \exp \left\{ \theta_{i,0} + 2 \sum_{k=1}^{p_i} \theta_{i,k} \cos(2\pi ku) \right\}, \quad i = 0, 1,$$

where $p_i$ is fixed and unknown. In the representation (2) and (3) they considered $f_Q(u)$ for $u \in (0, u_t]$ and $u \in [u_r, 1)$, where $u_t \leq 1/2$ and $u_r \geq 1/2$ are chosen by the statistician. Using these representations, they assumed the equations

$$\log f_Q(u) = \nu_0 \log u + \theta_{0,0} + 2 \sum_{k=1}^{p_0} \theta_{0,k} \cos(2\pi ku), \quad u \in (0, u_t],$$

$$\log f_Q(u) = \nu_1 \log(1 - u) + \theta_{1,0} + 2 \sum_{k=1}^{p_1} \theta_{1,k} \cos(2\pi k(1 - u)), \quad u \in [u_r, 1).$$
Based on some estimator \( \hat{f}_Q(u) \) of the density-quantile \( f_Q(u) \), this leads to the regression equations

\[
\log \hat{f}_Q(u_j) = \nu_0 \log u_j + \theta_{0,0} + 2 \sum_{k=1}^{\tilde{p}_0} \theta_{0,k} \cos(2\pi k u_j) + \varepsilon(u_j),
\]

\[
\log \hat{f}_Q(1 - u_j) = \nu_1 \log u_j + \theta_{1,0} + 2 \sum_{k=1}^{\tilde{p}_1} \theta_{1,k} \cos(2\pi k u_j) + \varepsilon(1 - u_j),
\]

where \( \varepsilon(u) = \log \left( \frac{\hat{f}_Q(u)}{f_Q(u)} \right) \) is the residual process, \( u_j = j/n, j = u_{[na]}, \ldots, u_{[nb]} \) and \( 0 < a < b < 1 \), so the percentiles \( u_j \) are chosen from a subset \( U = [a, b] \) of the interval \((0, 1)\). Holan and McElroy (2010) obtained some estimators \( \hat{\nu}_0 \) and \( \hat{\nu}_1 \) for the tail exponents \( \nu_0 \) and \( \nu_1 \) using ordinary least squares regression.

We propose a more general class of estimators using weighted least squares regression. We choose some nonnegative weights of the form \( w_{j,n} = R(j/n) \) with some weight function \( R \). Set \( y_j := \log \hat{f}_Q(u_j) \),

\[
y := (y_{[na]}, \ldots, y_{[nb]})',
\]

\[
W := \text{diag}(w_{[na],n}, \ldots, w_{[nb],n}),
\]

and let \( X := [G^*, G_0, 2G_1, \ldots, 2G_{\tilde{p}_0}] \), where

\[
G^* = (\log(u_{[na]}), \ldots, \log(u_{[nb]}))',
\]

\[
G_k = (\cos(2\pi k u_{[na]}), \ldots, \cos(2\pi k u_{[nb]}))', \quad k = 0, \ldots, \tilde{p}_0,
\]

and \( \tilde{p}_0 > p_0 \) is chosen by the statistician. Set \( \beta_{\tilde{p}_0} := (\nu_0, \theta_{0,0}, \theta_{0,1}, \ldots, \theta_{0,\tilde{p}_0})' \), where \( \theta_{0,j} = 0 \) if \( j > p_0 \). By minimizing the weighted sum of squares

\[
\sum_{j=|na|}^{[nb]} w_{j,n}(y_j - \nu_0 \log u_j - \theta_{0,0} - 2 \sum_{k=1}^{\tilde{p}_0} \theta_{0,k} \cos(2\pi k u_j))^2,
\]

we obtain the following estimator of \( \beta_{\tilde{p}_0} \):

\[
\hat{\beta}_{\tilde{p}_0} = (X'WX)^{-1}X'Wy.
\]

Then the weighted least squares estimator of \( \nu_0 \) can be written in the form

\[
\hat{\nu}_0 = e_1'(\hat{\beta}_{\tilde{p}_0}) = e_1'(X'WX)^{-1}X'Wy,
\]

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where $\epsilon_1$ is the $\tilde{p}_0 + 2$ dimensional vector defined as $\epsilon_1 = (1, 0, 0, \ldots, 0)'$. The right tail exponent $\nu_1$ can be estimated similarly.

A crucial point of this method is to choose a good estimator for the density-quantile $fQ(u)$. Letting $q(u) := Q'(u)$ denote the quantile density function, and using the identity $fQ(u)Q'(u) = 1$, one wish to estimate $q(u)$ instead of $fQ(u)$. Given a sample $X_1, \ldots, X_n$ with distribution function $F$, let $F_n$ denote its empirical distribution function and define $Q_n := F_n^{-1}$ to be the empirical quantile function. Holan and McElroy (2010) used the kernel quantile estimator of the empirical quantile function. Holan and McElroy (2010) used the kernel quantile estimator of $Q(u)$:

$$\hat{q}_n(u) = \frac{d}{du} \int_0^1 Q_n(t)K_n(u,t)\,d\mu_n(t), \quad u \in (0, 1),$$

(5)

where the kernel function $K_n(u,t)$ and the measure $\mu_n$ satisfy the following conditions of Cheng (1995):

(K₁) For every $n$, $0 < \mu_n([0, 1]) < \infty$, and $\mu_n(\{0, 1\}) = 0$.

(K₂) For every $n$ and each $(u, t)$, $K_n(u, t) \geq 0$, and for every $u \in U$,

$$\int_0^1 K_n(u,t)\,d\mu_n(t) = 1.$$

(K₃) For every $n$, $\int_0^1 tK_n(u,t)\,d\mu_n(t) = u$, $u \in U$.

(K₄) There is a sequence $\delta_n \downarrow 0$ such that $\sup_{u \in U} |\int_{u-\delta_n}^{u+\delta_n} K_n(u,t)\,d\mu_n(t) - 1| \downarrow 0$ as $n \uparrow \infty$.

Let $S_n$ be the unique closed subset of $(0, 1)$ such that $\mu_n((0, 1) \setminus S_n) = 0$ and $\mu_n((0, 1) \setminus S_n') > 0$ for any $S_n' \subset S_n$. For the sequence $\delta_n$ in (K₁), let $I_n(u) = [u - \delta_n, u + \delta_n]$, $I_n^c(u) = (0, 1) \setminus I_n(u)$, for $u \in U$. Define $\Lambda(u; K_n) = \int_{I_n(u)} |K_n(u,t)|\,d\mu_n(t)$, $u \in U$, and for a well-defined function $g$ on $(0, 1)$, let $\Psi(g; K_n) = \sup_{u \in U} \int_{I_n(u)} |g(t)K_n'(u,t)|\,d\mu_n(t)$. It is also assumed that the derivative $K_n'(u,t) = \partial K_n(u,t)/\partial u$ satisfies the conditions (K₅)–(K₇) below:

(K₅) For every $n$, $\sup_{u \in U} \int_0^1 |K_n'(u,t)|\,d\mu_n(t) < \infty$.

(K₆) For every $n$ and each $u \in U$, $K_n(u,t) \equiv 0$, $t \in I_n^c(u)$; or $S_n \subseteq [\varepsilon, 1 - \varepsilon] \subset (0, 1)$, with $U \subset [\varepsilon, 1 - \varepsilon]$ for some $0 < \varepsilon < 1/2$.

(K₇) For the sequence $\delta_n$ in (K₄), $\sup_{u \in U} \Lambda(u; K_n) \to 0$ and $\Psi(1; K_n) \to 0$ as $n \uparrow \infty$.

Similarly as in Holan and McElroy (2010), in some cases we assume that the kernel function has the form $K_n(u,t) = K(h_n^{-1}(t-u))h_n^{-1}$ and satisfies
the condition
\[(K_8) \quad \sup_{u \in U} \left| h_n^{-1} K \left( \frac{s - u}{h_n} \right) - h_n^{-1} K \left( \frac{t - u}{h_n} \right) \right| \leq C_n |t - s|^\beta \quad \text{and} \quad |K''(x)| \leq C/|x| \]
for some constants $C, \beta > 0$ and $|x|$ sufficiently large, and $C_n$ are positive constants such that $\sup_{n \geq 1} C_n < \infty$.

Moreover, Holan and McElroy (2010) used the following assumptions of Cheng (1995) on $q(u)$:

\begin{enumerate}[(Q_1)]
  \item The quantile density function is twice differentiable on (0,1).
  \item There exists a positive constant $\gamma$ such that $\sup_{u \in (0,1)} u(1-u) \left| J(u) \right| / fQ(u) \leq \gamma$, where $J$ is the score function in (1).
  \item Either $q(0) < \infty$ or $q(u)$ is nonincreasing in some interval $(0, u_*)$, and either $q(1) < \infty$ or $q(u)$ is nondecreasing in some interval $(u^*, 1)$.
\end{enumerate}

We will show that the limit matrix $M(a, b, R) := \lim_{n \to \infty} n^{-1} X'WX$ exists (see the proof of Theorem 1). Let $(v^*, v_0, \ldots, v_{\tilde{p}_i})$ be the first row of $M(a, b, R)^{-1}$, and set $G_R(u) := R(u) \left( v^* \log u + v_0 + 2 \sum_{k=1}^{\tilde{p}_i} v_k \cos(2\pi ku) \right)$, $i = 0, 1$.

Finally, we assume that the weight function $R$ satisfies the following condition:

\begin{enumerate}[(R)]
  \item $R$ is nonnegative and Riemann integrable on $[a, b]$.
\end{enumerate}

Let $\overset{P}{\to}$ denote convergence in probability, $\overset{D}{\to}$ denote convergence in distribution, and let $N(\mu, \sigma^2)$ stand for the normal distribution with mean $\mu$ and variance $\sigma^2$. Limiting and order relations are always meant as $n \to \infty$ if not specified otherwise. Our main results are contained in the following two theorems:

**Theorem 1.** Suppose that the conditions $(Q_1) - (Q_3)$ are satisfied for the quantile density $q(u)$, and $\hat{q}(u)$ is a kernel smoothed estimator with kernel function satisfying $(K_1) - (K_7)$, the weight function $R$ satisfies the condition $(R)$, and the matrix $M(a, b, R)$ is invertible. Moreover, assume that the percentiles $u_j$ are chosen from a closed set $U = [a, b]$ such that $u_j = j/n$, $j = \lceil na \rceil, \ldots, \lfloor nb \rfloor$, and $\tilde{p}_i > p_i$, $i = 0, 1$. Then $\hat{\nu}_i \overset{P}{\to} \nu_i$, $i = 0, 1$. 

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Theorem 2. Assume that the conditions of Theorem 1 are satisfied, and suppose that the kernel function is symmetric and differentiable on $[-1, 1]$, and satisfies the condition $(K_8)$. Suppose that the derivative $g_R(u) := G_R'(u)$ exists, and $g_R$ and $G_R$ are uniformly bounded on $U$. Let $h_n$ be a sequence such that $nh_n^2 \to \infty$, $nh_n^4 \to 0$ and $h_n \to 0$, and assume that $\tilde{p}_i > p_i$, $i = 0, 1$. Then
\[
\sqrt{n}(\tilde{p}_i - \nu_i) \overset{D}{\to} N(0, V), \quad i = 0, 1,
\]
where
\[
V = \int_a^b G_R^2(u) du + \int_a^b \int_a^b G_R(u)G_R(v) \left( 1 + [(u \wedge v) - uv] \frac{q'(u)q'(v)}{q(u)q(v)} \right) du dv.
\]

In the special case when the weight function $R$ is identically 1, the two theorems above reduces to Theorems 1 and 2 of Holan and McElroy (2010).

2. Classical tail index estimation

A distribution function $F$ has right heavy tail with tail index $\alpha_1 > 0$ if $1 - F(x)$ is regularly varying at infinity with index $-1/\alpha_1$, i.e.,
\[
1 - F(x) = x^{-1/\alpha_1} \ell_1(x), \quad 0 < x < \infty,
\]
where $\ell_1$ is a function slowly varying at infinity. Similarly, $F$ has left heavy tail with tail index $\alpha_0 > 0$ if $F(-x)$ is regularly varying at infinity with index $-1/\alpha_0$. Let $X_1, X_2, \ldots$ be independent random variables with a common distribution function $F$ having right heavy tail with tail index $\alpha_1$, and for each $n \in \mathbb{N}$, let $X_{1:n} \leq \cdots \leq X_{n:n}$ denote the order statistics pertaining to the sample $X_1, \ldots, X_n$. Several estimators exist for $\alpha_1$ among which Hill’s estimator is the most classical. Hill (1975) proposed the following estimator for the tail index $\alpha_1$:
\[
\hat{\alpha}_1 = \frac{1}{k_n} \sum_{j=1}^{k_n} \log X_{n-j+1:n} - \log X_{n-k_n:n} = \frac{1}{k_n} \sum_{j=1}^{k_n} \log \frac{X_{n-j+1:n}}{X_{n-k_n:n}},
\]
where the $k_n$ are some integers satisfying
\[
1 \leq k_n < n, \quad k_n \to \infty \quad \text{and} \quad k_n/n \to 0 \quad \text{as} \quad n \to \infty.
\]
The left tail analogue of the Hill estimator is the following:

\[ \hat{\alpha}_0 = \frac{1}{k_n} \sum_{j=1}^{k_n} \log \frac{X_{j,n}}{X_{k_n+1,n}}. \]

Another estimators were proposed by [Pickands (1975), Dekkers et al. (1989)], to name a few. As it was pointed out by [Holan and McElroy (2010)], there is the following relationship between the Parzen and classical tail indices:

\[ \nu_i = 1 + \alpha_i, \quad i = 0, 1. \]

Thus the classical tail index estimators also can be used to estimate the Parzen tail index.

### 3. Comparison of tail index estimators

#### 3.1. Asymptotic variances

We evaluate the limiting variance \([\tilde{\alpha}_0 = 1]\) for different weight functions and tail indices to compare the WLS and the unweighted (ordinary least squares) estimators in the following submodel of \((\text{4})\):

\[ L_0(u) = \exp \left\{ 2 \cos(2\pi u) \right\}, \quad u \in [a, b]. \]

The limiting variances are contained in Table \([\text{1}]\) in the Appendix. For the calculations we used numerical integration performed by the Wolfram Mathematica software. We see that in some cases the use of the weights makes the asymptotic variance smaller.

#### 3.2. Simulation results

In order to make a comparison with existing proposals, simulations were done performed by the Matlab software. The samples were generated from the model \((\text{2})\) with \(L_0 \equiv 1\) using different tail indices \(\nu_0\). The Hill, Pickands, DEdH (Dekkers, Einmahl and de Haan) and the least squares estimators were included in the simulation study. Similarly as in [Holan and McElroy (2010)], for the simulations we used the Bernstein polynomial estimator of \(q(u)\). Let \(0 < \varepsilon < 1/2\) be a constant, and assume that \(U \subset [\varepsilon, 1 - \varepsilon]\). Set \(L_{\varepsilon} := 1 - 2\varepsilon\) and \(t_j := \varepsilon + (j/k)L_{\varepsilon}, \quad j = 0, 1, \ldots, k\). The Bernstein polynomial estimator is defined as

\[ \hat{q}_n^B(u) = \frac{1}{L_{\varepsilon}^k} \sum_{j=0}^{k-1} \frac{Q_n(t_{j+1}) - Q_n(t_j)}{1/k} \left( \begin{array}{c} k-1 \\ j \end{array} \right) (u - \varepsilon)^j (1 - \varepsilon - u)^{k-1-j}. \]
This estimator belongs to the class (5) and satisfies the conditions \( (K_1) - (K_7) \). We used the values \( k = n = 700, \varepsilon = 0.001, a = 0.001 \) and \( b = 0.4 \) for the regression estimators, and the weight function \( R(u) = u/300 \) for the WLS estimator. Tables 2 and 3 contain the average simulated estimates (mean) and the calculated empirical mean square errors (MSE). We used the sample fraction size \( k_n = 100 \) for the Hill, Pickands and DEdH estimators. All the simulations were repeated 200 times. We conclude that in the submodel

\[ L_0 \equiv 1 \]

for \( \alpha \) values between 0.8 and 1.5 the WLS estimator has better performance than the OLS estimator. Thus for thinner tails we propose the WLS estimator instead of the OLS estimator. The Hill estimator is the best among the examined estimators. This good performance is not surprising since the Hill estimator was obtained in the special case of (7) when the slowly varying function \( \ell_1(x) \) is constant for all \( x \geq x_{\alpha_1} \), for some threshold \( x_{\alpha_1} \). The Pickands estimator has also good performance. On the other hand, we emphasize that the WLS method can be applied not only for the estimation of the tail index but for the estimation of the slowly varying functions \( L_i \) in (2) and (3).

4. Proofs

The proof of Theorems 1 and 2 follows the general outline of the proof of Theorems 1 and 2 of Holan and McElroy (2010). We give a more detailed proof for Theorem 2.

**Proof of Theorem 1.** We deal only with the left tail exponent \( \nu_0 \), the proof for \( \nu_1 \) is similar. Set \( \gamma = (\gamma_{[na]}, \ldots, \gamma_{[nb]})' := \sqrt{W}X(X'WX)^{-1}e_1 \) and \( \xi := (\varepsilon(u_{[na]}), \ldots, \varepsilon(u_{[nb]}))' \). Then \( \tilde{\nu}_0 - \nu_0 = \gamma'\sqrt{W}\xi \), and hence, using the Cauchy-Schwarz inequality,

\[
|\tilde{\nu}_0 - \nu_0| = \left| \sum_{j=[na]}^{[nb]} \gamma_j \sqrt{w_{j,n}} \varepsilon(u_j) \right| \leq \left( \sum_{j=[na]}^{[nb]} \gamma_j^2 \right)^{1/2} \left( \sum_{j=[na]}^{[nb]} w_{j,n} \varepsilon^2(u_j) \right)^{1/2}.
\]
We have $\sum_{j=[na]}^{[nb]} \gamma_j^2 = \gamma' \gamma = c'_1(n^{-1}X'WX)^{-1} e_1 n^{-1}$ with the matrix

$$X'WX = \begin{bmatrix}
\sum_{j=[na]}^{[nb]} \log^2 u_j R(u_j) & \sum_{j=[na]}^{[nb]} \log u_j R(u_j) & 2 \sum_{j=[na]}^{[nb]} \log u_j \cos(2\pi u_j) R(u_j) \\
\sum_{j=[na]}^{[nb]} \log u_j R(u_j) & \sum_{j=[na]}^{[nb]} R(u_j) & 2 \sum_{j=[na]}^{[nb]} \cos(2\pi u_j) R(u_j) \\
\vdots & \vdots & \vdots
\end{bmatrix}.$$  

Then by Riemann sum approximation

$$\lim_{n \to \infty} n^{-1}X'WX = M(a, b, R)$$  

$$:= \begin{bmatrix}
\int_a^b \log^2 u R(u) du & \int_a^b \log u R(u) du & 2 \int_a^b \log u \cos(2\pi u) R(u) du \\
\int_a^b \log u R(u) du & \int_a^b R(u) du & 2 \int_a^b \cos(2\pi u) R(u) du \\
\vdots & \vdots & \vdots
\end{bmatrix}.$$  

It follows that for all $n$ large enough $c'_1(n^{-1}X'WX)^{-1} e_1 \leq C$ for some constant $C$, and hence

$$|\hat{\nu}_0 - \nu_0| \leq \sqrt{C} \left( n^{-1} \sum_{j=[na]}^{[nb]} R(u_j) \varepsilon^2(u_j) \right)^{1/2}.$$  

Let $C' > 0$ be a constant such that $R(u) \leq C'$, $0 \leq u \leq 1$. Then

$$|\hat{\nu}_0 - \nu_0| \leq \sqrt{CC'} \left( n^{-1} \sum_{j=[na]}^{[nb]} \varepsilon^2(u_j) \right)^{1/2}.$$  

Now, by Theorem 2.1 of [Cheng (1995)], $n^{-1} \sum_{j=[na]}^{[nb]} \varepsilon^2(u_j) = o_P(1)$ (cf. the proof of Theorem 1 of [Holan and McElroy (2010)]).  

**Proof of Theorem 2.** Write

$$\sqrt{n}(\hat{\nu}_0 - \nu_0) = \frac{1}{\sqrt{n}} c'_1(n^{-1}X'WX)^{-1} X'W \varepsilon$$

$$= \frac{1}{\sqrt{n}} c_1' M(a, b, R)^{-1} X'W \varepsilon + \frac{1}{\sqrt{n}} c'_1 ((n^{-1}X'WX)^{-1} - M(a, b, R)^{-1}) X'W \varepsilon.$$
By straightforward calculation,
\[
A_n := \frac{1}{\sqrt{n}} e_1'M(a, b, R)^{-1}X'W_\varepsilon = \frac{1}{\sqrt{n}} \sum_{j=\lceil na \rceil}^{\lfloor nb \rfloor} \varepsilon(u_j)G_R(u_j). \tag{9}
\]

It follows from Theorem 5 in Holan and McElroy (2010) that
\[
A_n \xrightarrow{D} G_R(b)W(b) - G_R(a)W(a) - \int_a^b W(u) \left( g_R(u) - G_R(u) \frac{q'(u)}{q(u)} \right) du,
\]
where \(W(u)\) is a Brownian bridge process. The limiting variance is given by (6). Next we show that
\[
B_n := \frac{1}{\sqrt{n}} e_1' \left( (n^{-1}X'WX)^{-1} - M(a, b, R)^{-1} \right) X'W_\varepsilon = \sigma_F(1).
\]

Let \((v_n^*, v_{0,n}, \ldots, v_{\tilde{p}_0,n})\) be the first row of \((n^{-1}X'WX)^{-1} - M(a, b, R)^{-1}\). By (8), \((v_n^*, v_{0,n}, \ldots, v_{\tilde{p}_0,n}) \to 0\). Set
\[
G^{(n)}(u) = R(u) \left( v_n^* \log u + v_{0,n} + 2 \sum_{k=1}^{\tilde{p}_0} v_{k,n} \cos(2\pi ku) \right).
\]

Similarly as in (9),
\[
B_n = \frac{1}{\sqrt{n}} \sum_{j=\lceil na \rceil}^{\lfloor nb \rfloor} \varepsilon(u_j)G^{(n)}(u_j)
\]
\[
= v_n^* \frac{1}{\sqrt{n}} \sum_{j=\lceil na \rceil}^{\lfloor nb \rfloor} \varepsilon(u_j)R(u_j) \log u_j + v_{0,n} \frac{1}{\sqrt{n}} \sum_{j=\lceil na \rceil}^{\lfloor nb \rfloor} \varepsilon(u_j)R(u_j)
\]
\[
+ 2 \sum_{k=1}^{\tilde{p}_0} v_{k,n} \frac{1}{\sqrt{n}} \sum_{j=\lceil na \rceil}^{\lfloor nb \rfloor} \varepsilon(u_j)R(u_j) \cos(2\pi ku_j).
\]

Each term in the last sum tends to zero, e.g., in the first term \(v_n^* \to 0\) and using again Theorem 5 in Holan and McElroy (2010), the sequence \(\frac{1}{\sqrt{n}} \sum_{j=\lceil na \rceil}^{\lfloor nb \rfloor} \varepsilon(u_j)R(u_j) \log u_j\) has a weak limit. \(\square\)
Table 1: Limiting variances for different weight functions and tail indices.

| $\nu_0$ | $R(u)$ | unweighted |
|---------|--------|------------|
|         | $1 + \cos u$ | $e^{-u}$ | $-\log u$ | $1/u$ |         |
| $\nu_0 = 1.2$ |         |          |          |          |         |
| $a = 0.1, b = 0.4$ | 821.232 | 816.812 | 823.778 | 851.364 | 822.13 |
| $a = 0.1, b = 0.3$ | 1512.62 | 1513.46 | 1538.35 | 1600.46 | 1512.83 |
| $a = 0.2, b = 0.3$ | 269523 | 269655 | 270796 | 272081 | 269524 |
| $\nu_0 = 1.8$ |         |          |          |          |         |
| $a = 0.1, b = 0.4$ | 821.962 | 819.166 | 829.786 | 860.498 | 822.66 |
| $a = 0.1, b = 0.3$ | 1521.58 | 1523.69 | 1551.68 | 1617.04 | 1521.66 |
| $a = 0.2, b = 0.3$ | 267666 | 267807 | 268969 | 270267 | 267666 |
| $\nu_0 = 1.667$ |         |          |          |          |         |
| $a = 0.1, b = 0.4$ | 819.423 | 816.278 | 826.109 | 856.14  | 820.164 |
| $a = 0.1, b = 0.3$ | 1516.49 | 1518.31 | 1545.6  | 1610.22 | 1516.6  |
| $a = 0.2, b = 0.3$ | 268011 | 268151 | 269308 | 270604 | 268012 |
| $\nu_0 = 2.25$ |         |          |          |          |         |
| $a = 0.1, b = 0.4$ | 840.595 | 838.929 | 825.157 | 885.102 | 841.151 |
| $a = 0.1, b = 0.3$ | 1551.91 | 1555.02 | 1585.51 | 1653.45 | 1551.89 |
| $a = 0.2, b = 0.3$ | 266776 | 266924 | 268099 | 269406 | 266775 |
Table 2: Average simulated tail index estimates (Mean) for sample size $n = 700$ and for $L_0 \equiv 1$.

| $\nu (\alpha)$ | Mean | WLS | OLS | Hill | Pickands | DEdH |
|-----------------|------|-----|-----|------|-----------|------|
| 2.25(1.25)      | 2.3777 | 2.4571 | 2.5088 | 2.4271 | 2.4825 | 2.2936 | 2.2703 | 2.7346 |
| 2(1)            | 2.0741 | 2.1231 | 2.2432 | 2.0902 | 2.1162 | 2.1177 | 2.0038 | 1.9998 | 2.4988 |
| 1.833(0.833)    | 1.9119 | 1.9249 | 1.9405 | 1.9248 | 1.904 | 1.8959 | 1.8404 | 1.8471 | 2.3354 |
| 1.667(0.667)    | 1.7163 | 1.6915 | 1.7274 | 1.7217 | 1.7019 | 1.7058 | 1.6743 | 1.6902 | 2.1692 |
| 1.556(0.556)    | 1.5949 | 1.6294 | 1.5951 | 1.6017 | 1.5822 | 1.5637 | 1.5534 | 1.5567 | 2.0483 |
| 1.5(0.5)        | 1.5230 | 1.5448 | 1.5518 | 1.5222 | 1.5613 | 1.5668 | 1.5005 | 1.4942 | 1.9955 |
| 1.333(0.333)    | 1.3639 | 1.389 | 1.3874 | 1.3598 | 1.3355 | 1.3136 | 1.3347 | 1.3294 | 1.8296 |
| 1.25(0.25)      | 1.2956 | 1.2471 | 1.2424 | 1.2741 | 1.2585 | 1.2629 | 1.2476 | 1.2474 | 1.7426 |
| 1.2(0.2)        | 1.2281 | 1.2483 | 1.2189 | 1.1967 | 1.2204 | 1.2089 | 1.1993 | 1.2144 | 1.6942 |
| 1.182(0.182)    | 1.1742 | 1.1891 | 1.199 | 1.1776 | 1.1725 | 1.1677 | 1.1833 | 1.174 | 1.6783 |
| 1.167(0.167)    | 1.1628 | 1.1953 | 1.1826 | 1.162 | 1.158 | 1.1452 | 1.167 | 1.1624 | 1.662 |
| 1.1(0.1)        | 1.1116 | 1.0926 | 1.1538 | 1.0899 | 1.0755 | 1.0725 | 1.1006 | 1.0952 | 1.5955 |
| 1.067(0.067)    | 1.0761 | 1.106 | 1.0895 | 1.0456 | 1.0597 | 1.0431 | 1.0673 | 1.0562 | 1.5622 |
| 1.05(0.05)      | 1.0674 | 1.0607 | 1.0866 | 1.0527 | 1.0476 | 1.0438 | 1.0496 | 1.048 | 1.5445 |

Table 3: Empirical mean square errors (MSE) of tail index estimates for sample size $n = 700$ and for $L_0 \equiv 1$.

| $\nu (\alpha)$ | Mean | WLS | OLS | Hill | Pickands | DEdH |
|-----------------|------|-----|-----|------|-----------|------|
| 2.25(1.25)      | 0.0953 | 0.1565 | 0.2224 | 0.1540 | 0.2701 | 0.3855 | 0.017784 | 0.0592 | 0.2525 |
| 2(1)            | 0.0794 | 0.1121 | 0.1865 | 0.1029 | 0.1244 | 0.1942 | 0.011251 | 0.0491 | 0.2600 |
| 1.833(0.833)    | 0.0599 | 0.1134 | 0.1550 | 0.0714 | 0.1257 | 0.1673 | 0.0075016 | 0.0427 | 0.2598 |
| 1.667(0.667)    | 0.0594 | 0.0817 | 0.1164 | 0.0565 | 0.0832 | 0.1218 | 0.0062222 | 0.0412 | 0.2471 |
| 1.556(0.556)    | 0.0515 | 0.0935 | 0.0938 | 0.0404 | 0.0593 | 0.0845 | 0.0056131 | 0.0405 | 0.2482 |
| 1.5(0.5)        | 0.0465 | 0.1105 | 0.1352 | 0.0471 | 0.0640 | 0.0909 | 0.0036438 | 0.0395 | 0.2501 |
| 1.333(0.333)    | 0.0400 | 0.0679 | 0.1064 | 0.0292 | 0.0350 | 0.0627 | 0.003354 | 0.0397 | 0.2432 |
| 1.25(0.25)      | 0.0413 | 0.0754 | 0.0878 | 0.0229 | 0.0445 | 0.0580 | 0.0009903 | 0.0436 | 0.2447 |
| 1.2(0.2)        | 0.0388 | 0.0716 | 0.1090 | 0.0196 | 0.0301 | 0.0456 | 0.0007893 | 0.0358 | 0.2468 |
| 1.182(0.182)    | 0.0335 | 0.0620 | 0.0894 | 0.0216 | 0.0284 | 0.0365 | 0.0007318 | 0.0335 | 0.2453 |
| 1.167(0.167)    | 0.0304 | 0.0708 | 0.1008 | 0.0160 | 0.0341 | 0.0476 | 0.0005918 | 0.0372 | 0.2462 |
| 1.1(0.1)        | 0.0350 | 0.0788 | 0.1001 | 0.0191 | 0.0384 | 0.0489 | 0.0040866 | 0.0332 | 0.2454 |
| 1.067(0.067)    | 0.0358 | 0.0652 | 0.1013 | 0.0169 | 0.0318 | 0.0455 | 0.0002472 | 0.0313 | 0.2445 |
| 1.05(0.05)      | 0.0308 | 0.0625 | 0.0845 | 0.0149 | 0.0238 | 0.0315 | 0.0002247 | 0.0351 | 0.2443 |

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