A MATHEMATICAL MODEL FOR VALUE ESTIMATION WITH PUBLIC INFORMATION AND HERDING

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Abstract. This paper deals with a class of integro-differential equations modeling the dynamics of a market where agents estimate the value of a given traded good. Two basic mechanisms are assumed to concur in value estimation: interactions between agents and sources of public information and herding phenomena. A general well-posedness result is established for the initial value problem linked to the model and the asymptotic behavior in time of the related solution is characterized for some general parameter settings, which mimic different economic scenarios. Analytical results are illustrated by means of numerical simulations and lead us to conclude that, in spite of its oversimplified nature, this model is able to reproduce some emerging behaviors proper of the system under consideration. In particular, consistently with experimental evidence, the obtained results suggest that if agents are highly confident in the product, imitative and scarcely rational behaviors may lead to an over-exponential rise of the value estimated by the market, paving the way to the formation of economic bubbles.

1. Introduction. Experimental evidence supports the idea that the actual human behavior differs, in systematic ways, from the behavior of *homo economicus* as it is proposed by the standard economic models of unbounded rationality [7, 8, 24]. Human strategies are widely heterogeneous and often follow mental shortcuts and rules of thumbs, as it has been shown once again by the economic crisis that has recently shaken the modern society. Such a crisis has emphasized the need for new tools to envisage the occurrence of sudden changes in the dynamics of the World Economy. In particular, together with Economists, scientists traditionally focused on the study of inert matter, such as Physicists and Mathematicians, are now challenged to develop new structures and methods able to deal with complex socio-economic systems, namely local and international markets, composed of a large number of interacting agents, who express heterogeneous sets of strategies and try to maximize their pay-off under the constrains imposed by the limited available information about the global economic milieu. This is namely testified by the recent and interesting book by P. Ball, [2], where a view of our society as a complex system is proposed and emerging behaviors are considered as consequences, in a bottom-up

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fashion, of the interactions among individuals and agents of the society. The same
book highlights how ideas and methods typical of the science of complexity can
be usefully applied to model and manage different socio-economic phenomena and
scenarios.

This work takes advantage of the considerations drawn in [1, 4, 5, 10, 11, 13,
17, 20, 22, 25] and presents a mathematical model for the dynamics of a market
or a large pool of agents that estimate the value of a given good. Such a system
is assumed to be embedded in an outer environment defined by external sources
of public information and the value estimated by the pool as a whole emerges
from interactions between agents and public information as well as among agents
themselves [12]. As external sources of public information we may refer, for instance,
to newspapers, web pages, reliability systems, networks of credit rating agencies and
reputation systems, as the ones recently developed to help the users in determining
who to trust while doing transactions over the internet (e.g. the ones used by eBay
or similar websites).

The model is developed under the assumption that each agent assigns a given
value to the product, while each one of the external information sources suggests a
value for the product. Agents are assumed to update the estimated value through in-
teractions with both information sources and other agents. Proper modeling strate-
gies are developed to translate into mathematical terms these phenomena and lead
to define an initial value problem for a class of integro-differential equations.

From an economic perspective, we are interested in collective behaviors emerging
in those economic scenarios where most of the agents trust each other and give little
importance to the strategy suggested by public information. The definition of a
model well suited for quantitative forecasts is beyond our present scope. In fact,
this paper is rather meant to show how even a simple model, such as the one here
introduced, is able to predict a possible overestimation of the values in a market
where everybody does what everyone else is doing, although public information
suggests doing something else, that is, a market where naïve herding takes place [15].
In agreement with experimental evidence [16, 19], the outputs of the model suggest
that, if agents are highly confident in the product, such an imitative and scarcely
rational behavior may lead to an over-exponential rise of the value estimated by the
pool of agents, paving the way to the formation of economic bubbles. Of course,
being not an exact science, there is not an univocal evidence. In fact, the opposite
may happen in some cases; that is, herding behavior may occur even in those cases
where the agents do not have a precise idea about the value of a given product and
thus they pay more attention to what is done by the others.

From a mathematical standpoint, we are interested in studying the asymptotic
behavior of the mathematical problem’s solution under some suitable parameter
settings, which mimic different economic scenarios, such as an efficient market where
an estimated value that incorporates all public information can emerge [3, 7, 21], a
market ruled by naïve herding and a highly confident market where naïve herding
takes place.

The contents of the paper are organized as follows:

- Section 2 aims at outlining the essential features of the socio-economic phe-
nomena under consideration and to present the mathematical model.
- Section 3 summarizes some results about the well-posedness of the Cauchy
  Problem linked to the model and the asymptotic behavior of the related so-
  lution in the limit of large times.
- Section 4 illustrates analytical results by means of numerical simulations and discusses their economic meaning.
- Section 5 contains the conclusions of this work.

2. **The model.** This section presents a mathematical model for value estimation in goods trading processes under the effect of public information and herding phenomena. Related contents are organized into two subsections:

Subsection 2.1 briefly states the economic problem in view of the mathematical formalization.

Subsection 2.2 describes the strategies developed to translate into mathematical terms the socio-economic phenomena under consideration and presents the model.

2.1. **Brief phenomenological overview.** Local and international markets can be seen as complex systems composed of large numbers of interacting agents that try to estimate the value of traded goods on the basis of public and private information as well as of personal feelings.

Agents can be very different from each other and can use the same information in several different ways; thus, the set of all possible estimation strategies is highly heterogeneous.

In principle, if all agents would be fully rational, the value of goods estimated by the market would faithfully reflect all known information about the traded products. Loosely speaking, this is the basic feature that makes a market to be efficient [3, 21]. However, rationality of agents is bounded [24]. This implies that the available information is often not incorporated into those values that emerge from interactions among agents [7].

As a result, the complexity of market dynamics is highly increased by the non rationality that often pervades human behaviors, which can lead individuals to emulate others (herding) or even to do what everyone else is doing regardless what their sources of information suggest to do (naïve herding) [15, 19]. Herding mechanisms can be particularly dangerous in those markets where agents are highly confident in the product. In fact, they can reinforce positive feelings leading the estimated value to grow over-exponentially fast in time; this can be a prelude for the formation of economic bubbles [16].

2.2. **The model.** Taking advantage of the considerations drawn in [1, 17, 20], the mathematical model here proposed refers to a large market where agents estimate the value of a given product. The state of each agent is characterized by means of a continuous variable $u$, which stands for the value assigned to the product normalized with respect to a suitable reference value. Normalization implies that the domain of $u$ can be identified with the compact interval $\Omega := \{0, 1\} \subset \mathbb{R}^+$ and the number of agents in state $u$ at time $t$, normalized with respect to the total number of agents in the market, is represented by function $f(t, u)$

$$f : \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R}^+.\]$$

Assuming the independent variable $u$ to be continuous is consistent if the number of agents is sufficiently high. On the other hand, if the number of agents is not large enough, the present formalism can be easily adapted by considering suitable
discretization, as in the standard discrete kinetic theory for dilute gases, see e.g. [14]. The total density of agents at time \( t \) can be computed as:

\[
\rho(t) = \int_{\Omega} f(t,u)du
\]

and the value estimated by the market is modeled by the function

\[
U(t) = \frac{\int_{\Omega} uf(t,u)du}{\rho(t)}.
\]  

(1)

It should be noted that the time variable \( t \) is assumed to be normalized with respect to the duration of a reference trading period.

As previously noted, agents in the market are exposed to the action of some sources of public information, which can influence the evaluation process by suggesting a value for the product under consideration. Information sources are modeled as external agents whose state is identified by a suitable normalization of the proposed value, which is identified by a continuous variable \( w \in \Omega \). As in the case of the agents in the market, a function \( g(t,w) \) is introduced

\[
g : \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R}^+,
\]

which stands for the number of information sources in state \( w \) at time \( t \), normalized with respect to the total number of information sources, and it is supposed to be a given non-negative function of its argument such that

\[
g(t,w) \in C(\mathbb{R}^+; L^1(\Omega)), \quad \int_{\Omega} g(t,w)dw = 1.
\]  

(2)

For instance, function \( g \) can be defined evaluating the average rating of the product provided by a certain reputation system.

The estimated value \( U(t) \) results from a dynamical equilibrium and it can be computed if the value of \( f(t,u) \) is known. Function \( f \) evolves according to the following initial value problem:

\[
\begin{cases}
\frac{\partial}{\partial t} f(t,u) = F[f] := F^I[f] + F^H[f], & u \in \Omega, \quad t > 0 \\
f(0,u) = f^0(u) \in L^1(\Omega), & f^0(u) \geq 0 \text{ a.e. on } \Omega, \quad \int_{\Omega} f^0(u)du = 1,
\end{cases}
\]  

(3)

where, in analogy with a kinetic formalism, see [20] and references therein, \( F^I \) and \( F^H \) are evolution terms modeling, respectively, the net inflow of agents in the infinitesimal volume element \( du \) centered at \( u \) due to interactions among agents and the sources of public information or among agents themselves. More in detail:

**Public information based estimation (\( F^I[f] \)).** The number of agents assigning the \( u \) value may change over time due to decision phenomena based on public information, thus mediated by interactions among agents and some information sources. These phenomena are included in the model through a kernel \( K(u|u_*,w;\alpha) \) such that

\[
\int_{\Omega} K(u|u_*,w;\alpha)du = 1, \quad \forall u_*, w \in \Omega, \quad \forall \alpha \in (0,1) \subset \mathbb{R}^+,
\]  

(4)

which models the probability that agents in state \( u_* \) move into the state \( u \) at time \( t \) because of interactions with information sources in state \( w \), and parameter \( \alpha \) measures, on average, the attitude of agents in the market to change their mind.
because of interactions with public information sources. Kernel $K$ can be defined as follows:

$$K(u|u_*, w; \alpha) := \delta (u - (u_* + \alpha(w - u_*)))$$,

where $\delta$ is the Dirac delta distribution. The above definition relies on the idea that if agents in the market trust an information source, then they update their value to make it closer the one suggested by the source. This choice is analogous to the one used in models of opinion formation where interacting agents average their current opinions, see e.g. [18].

We allow agents in the market, that are in different states, to be influenced in different ways by information sources in a given state. In particular, we assume agents to rely only on those sources suggesting a value not too much different from the one that they assign. As a result, a threshold value $R^f \in (0, 1] \subset \mathbb{R}^+$ is introduced and the rate of interactions between agents and sources of public information $\eta^f(u_*, w)$ is defined as

$$\eta^f : \Omega \times \Omega \to \{0, 1\}, \quad \eta^f(u_*, w) := 1_{\{|u_* - w| \leq R^f\}},$$

where $1$ is the indicator function. The above definition is analogous to the one used in models for opinion dynamics of Deffuant-Weisbuch type [10], where $R^f$ is usually called bound of confidence, and implies that $\eta^f(u_*, w) = \eta^f(w, u_*)$ for all $u_*, w \in \Omega$.

**Herding based estimation ($F^H[f]$).**

The number of agents that assign the $u$ value to the product may change over time also because of interactions between agents themselves. Here we are interested in the effects of herding and imitation phenomena, which are modeled, similarly to the bounded confidence models very popular in opinion formation modelling, see e.g. [6], by assuming that, due to interactions with agents in state $u^*$, an agent in state $u_*$ can move into a state $u$, that is closer to $u^*$, with a probability described by a kernel $H(u|u_*, u^*; \beta)$ such that

$$\int_{\Omega} H(u|u_*, u^*; \beta) du = 1, \quad \forall u_*, u^* \in \Omega, \forall \beta \in (0, 1) \subset \mathbb{R}^+,$$

with:

$$H(u|u_*, u^*; \beta) := \delta (u - (u_* + \beta(u^* - u_*))), \quad \beta \in (0, 1) \subset \mathbb{R}^+,$$

where parameter $\beta$ measures, on average, the attitude of agents to change their mind because of herding mechanisms and $\delta$ indicates the Dirac delta distribution.

Two possible definitions for the rate of interactions among agents $\eta^H(u_*, u^*)$ are:

$$\eta^H : \Omega \times \Omega \to \{0, 1\}, \quad \eta^H(u_*, u^*) := 1_{\{|u_* - u^*| \leq R^H\}},$$

where function $\gamma$ is defined as follows:

$$\gamma(u_*, u^*):= \begin{cases} 
\frac{u^*(1 - u_*)}{R^H(u^* - u_*)}, & \text{if } u^* > u_* \\
0, & \text{otherwise.}
\end{cases}$$

Definition (9) is analogous to the one previously introduced for function $\eta^f$. It relies on the idea that interactions occur, at time $t$, only among agents whose states are at a distance smaller than a threshold value $R^H \in (0, 1] \subset \mathbb{R}^+$ and it ensures that $\eta^H(u_*, u^*) = \eta^H(u^*, u_*)$ for all $u_*, u^* \in \Omega$. On the other hand, definition (10)
mimics a socio-economic scenario where agents are highly confident in the product. In fact:

- agents in state $u_*$ trust only other agents that assign a higher value $u^*$ (i.e. $\eta^H(u_*, u^*) \neq 0$ if $u^* > u_*$);
- higher is the value assigned by an agent, higher is the probability that other agents interact with her (i.e. $\eta^H(u_*, u^*) \propto u^*$);
- agents that assign a small value tend to herd with a higher frequency (i.e. $\eta^H(u_*, u^*) \propto (1 - u_*)$);
- the frequency of herding phenomena increases as the agents’ bound of confidence with respect to the sources of public information decreases (i.e. $\eta^H(u_*, u^*) \propto \frac{1}{u^*}$);
- agents estimating similar values interact with a higher rate (i.e. $\eta^H(u_*, u^*) \propto \frac{1}{u^* - u_*}$).

It is worth noting that the underling assumptions of the present model are similar to the ones of adaptative agent-based models in Economic Theory where the agents adapt to the information but they do not influence the existing information.

Assuming the notations given hereafter to hold throughout the paper

$$f(t) = f(t, u), \quad f_*(t) = f(t, u_*), \quad f^*(t) = f(t, u^*), \quad g(t, w) = g(t),$$

and recalling that $F^I[f](t, u)$ and $F^H[f](t, u)$ model, respectively, the net inflow of agents at time $t$ in the infinitesimal volume element $du$ centered at $u$ due to interactions among agents and the sources of public information or among agents themselves, the following definitions can be introduced:

$$F^I[f](t, u) = \int_{\Omega} \int_{\Omega} \eta^I(u_*, w) K(u|u_*, w; \alpha) f_*(t) g(t) du_* dw +$$

$$\quad - \int_{\Omega} \int_{\Omega} \eta^I(u, w) K(u_*|u, w; \alpha) f(t) g(t) du_* dw$$

$$= \int_{\Omega} \int_{\Omega} \eta^I(u_*, w) K(u|u_*, w; \alpha) f_*(t) g(t) du_* dw +$$

$$\quad - f(t) \int_{\Omega} \eta^I(u, w) g(t) dw$$

$$F^H[f](t, u) = \int_{\Omega} \int_{\Omega} \eta^H(u_*, u^*) H(u|u_*, u^*; \beta) f^*(t) f^*(t) du_* du^* +$$

$$\quad - \int_{\Omega} \int_{\Omega} \eta^H(u, u^*) H(u_*|u, u^*; \beta) f(t) f^*(t) du_* du^*$$

$$= \int_{\Omega} \int_{\Omega} \eta^H(u_*, u^*) H(u|u_*, u^*; \beta) f^*(t) f^*(t) du_* du^* +$$

$$\quad - f(t) \int_{\Omega} \eta^H(u, u^*) f^*(t) du^*,$$

(12)

which rely on assumptions (4) and (7).

To sum up, the above model is characterized by means of four parameters only:

- parameter $R^I$ is the agents’ bound of confidence with respect to the sources of public information;
- parameter $\alpha$ measures, on average, the attitude of agents to change the assigned value because of interactions with public information sources;
- parameter $\beta$ measures, on average, the attitude of agents to change the assigned value because of interactions with public information sources;
- parameter $R^H$ is the agents’ bound of confidence with respect to other agents;
- parameter $\beta$ measures, on average, the attitude of agents to change the assigned value because of interactions with other agents in the market.

As a consequence, this model is not enough refined to make some quantitative forecasts; however, it is well suited for studying the emergence of collective behaviors resulting from the interactions under consideration.

3. Analytical results. This section is devoted to present some analytical results related to the properties of the solution of Cauchy Problem (3). In more detail:

Subsection 3.1 proves, by means of standard fixed point arguments, that the initial value problem (3) is well-posed in the sense of Hadamard, i.e. the solution exists, it is unique and depends continuously on the initial data.

Subsection 3.2 studies the behavior of $f(t,u)$ for $t \to \infty$ in three different asymptotic scenarios of particular interest from the economic perspective. Demonstration strategies rely, case by case, on a systematic characterization of the average opinion and the related variance.

3.1. A general well-posedness result. The existence of a unique solution for the Cauchy Problem (3) is stated by the following:

**Theorem 3.1.** Assume conditions (2), (4), (6), (7) and (9) or (10) to hold. Then, the mathematical problem (3) admits a unique non-negative solution $f \in C(\mathbb{R}^+; L^1(\Omega))$ that satisfies the identity

$$\varrho(t) = \varrho(0) = 1, \quad \forall t \geq 0,$$

(13)

for any value of parameters $\alpha$, $\beta$, $R^I$ and $R^H$.

**Proof.** Due to assumptions (4), (6), (7) and (9) or (10), the a priori estimate (13) can be derived by integrating over $u$ both the sides of the equation for the time derivative of $f$ established by (3). Both definitions (9) and (10) ensure that $\eta^H$ is bounded from above by a positive real constant $\bar{\eta}^H$.

Functional $F[f]$ is a Lipschitz continuous map from $L^1(\Omega)$ into itself. In fact, for any $f_1, f_2 \in L^1(\Omega)$ at time $t$, straightforward estimations show that:

$$\|F[f_1] - F[f_2]\|_{L^1(\Omega)} \leq \kappa \|f_1 - f_2\|_{L^1(\Omega)}, \quad \kappa \in \mathbb{R}^+,$$

where the explicit definition of $\kappa$ is not reported for the sake of conciseness. Let us define $T = \frac{1}{2\kappa}$; consider the Banach space $C([0,T], L^1(\Omega))$ and introduce the application $I[f]$ defined as the formal primitive of $F$:

$$I[f](t,u) := f(t,u) = f^0(u) + \int_0^t F[f](s,u)ds = f^0(u) + \Xi[f](t,u).$$

(14)

Since $\Xi[f]$ is a continuous mapping from $C([0,T], L^1(\Omega))$ into $C([0,T], L^1(\Omega))$, we can state that also $I[f]$ maps $C([0,T], L^1(\Omega))$ into itself. Moreover, the Lipschitz continuity of $F[f]$ implies that $I[f]$ is contractive. In fact, for any $f_1, f_2 \in C([0,T], L^1(\Omega))$, with $f_1(t = 0, u) = f_2(t = 0, u) = f^0(u)$, the following inequality holds true:

$$\sup_t \|I[f_1] - I[f_2]\|_{L^1(\Omega)} \leq \kappa T \|f_1 - f_2\|_{L^1(\Omega)}.$$

As a result, the Banach fixed point theorem ensures that $I[f]$ admits a unique fixed point on $[0,T]$. 
Since the explicit solution of Problem (3) reads as
\[ f(t, u) = e^{-A(t,u)}f^0(u) + \int_0^t e^{A(s,u) - A(t,u)}B(s,u)ds, \] (15)
see e.g. [23] for similar systems, where:
\[
A[f](t, u) = \int_0^t \int_\Omega \eta^I(t, w)g(s)dwds + \int_0^t \int_\Omega \eta^H(u, u^*)f^*(s)du^*ds,
\]
\[
B[f](t, u) = \int_\Omega \int_\Omega \eta^I(u^*, u)K(u|u^*, w; \alpha)f_*(t)g(t)du_*dw
+ \int_\Omega \int_\Omega \eta^H(u^*, u^*)H(u|u^*, u^*; \beta)f_*(t)f^*(t)du_*du^*,
\]
the non-negativity of \( f \) is verified in view of the positivity of the exponential, the non–negativity of \( f^0 \) and \( g \) and the positivity property of the integral.

Finally, the non-negativity of function \( f \) together with identity (13) allow us to iterate the above reasoning to conclude that the Problem (3) admits a unique solution \( f \in C(\mathbb{R}^+, L^1(\Omega)) \). \( \square \)

3.2. Asymptotic analysis. Three different asymptotic scenarios are particularly interesting from the economic point of view:

- **Case 1. efficient market** (\( \beta \to 0 \), \( R^I = 1 \) and \( g(t, w) = \delta(w - W) \)).
  Agents tend to weakly modify the estimated value because of interactions among themselves, but they are highly confident in public sources of information, which suggest the same value for the product under consideration.

- **Case 2. na"ive herding** (\( \alpha \to 0 \) and \( \eta^H \) defined by (9) with \( R^H \geq \max(U(0), 1 - U(0)) \)). Each agent pays few attention to public information and imitates the other agents, although they assign a value even quite different from her own one.

- **Case 3. bubble formation** (\( \alpha \to 0 \) and \( \eta^H \) defined by (10)).
  Agents are scarcely influenced by public information and tend to imitate those agents that assign a value higher than their own ones. The market as a whole is highly confident in the product.

The behavior of \( f(t, u) \) for \( t \to \infty \) is characterized, in these cases, by Theorem 3.3, whose proof relies on the following preliminary results:

**Lemma 3.2.** Let \( F^I \) and \( F^H \) be defined by (12). Then, the following identities hold for all test functions \( \varphi(u) \) belonging to the completion of \( C^\infty_c(\Omega) \) in \( W^{1,\infty}(\Omega) \):
\[
\lim_{\alpha \to 0} \int_\Omega \varphi(u)F^I(f)(t, u)du = 0, \quad \lim_{\beta \to 0} \int_\Omega \varphi(u)F^H(f)(t, u)du = 0, \quad \forall t \geq 0. \] (16)

**Proof.** Definition of kernel \( K \) implies that
\[
\int_\Omega \varphi(u)F^I[f](t, u)du = \int_\Omega \int_\Omega \eta^I(u^*, w)g_*(t)(\varphi(u^* + \alpha(w - u_*)) - \varphi(u_*))du_*dw
= \alpha \int_\Omega \int_\Omega \eta^I(u^*, w)g_*(t)(w - u_*) \left( \frac{\varphi(u^* + \alpha(w - u_*)) - \varphi(u_*)}{\alpha(w - u_*)} \right) du_*dw,
\]
where the right hand side tends to zero under the limit \( \alpha \to 0 \), since functions \( \eta^I \), \( f \) and \( g \) are bounded and in the limit \( \alpha \to 0 \) the difference quotient approximates...
the first derivative of \( \varphi \) with respect to \( u \). By approximation this holds for the completion of \( C_\infty^\ast(\Omega) \) in \( W^{1,\infty}(\Omega) \). Finally, the second identity can be proved in an analogous way due to the definition of kernel \( H \) and the definitions for function \( \eta^H \) under consideration.

**Theorem 3.3.** Assume conditions (2), (4), (6), (7) and (9) or (10) to hold. Then, there exists a subsequence of \( f \), denoted again as \( f \), such that:

i) (Establishing convergence)

\[
 f \to f^\infty, \text{ in the weak sense of measure, as } t \to \infty,
\]

where \( f^\infty \) is a bounded non-negative measure.

ii) (Identifying the limit \( f^\infty \))

**Case 1 (efficient market).** If \( R^I = 1 \), \( g(t,w) = g(w) = \delta(w-W) \) with \( W \in \Omega \) for any \( t \geq 0 \) and \( \beta \to 0 \), then:

\[
 f^\infty(u) = \delta(u-W). \tag{17}
\]

**Case 2 (naïve herding).** If \( \eta^H \) is defined by (9) with \( R^H \geq \max(U(0),1-U(0)) \) and \( \alpha \to 0 \), then:

\[
 f^\infty(u) = \delta(u-U(0)). \tag{18}
\]

**Case 3 (bubble formation).** If \( \eta^H \) is defined by (10) and \( \alpha \to 0 \), then

\[
 f^\infty(u) = \delta(u-1) \quad \text{and} \quad \frac{d}{dt}U(t) \geq CU(t)(1-U(t)), \quad C \in \mathbb{R}^+. \tag{19}
\]

**Proof.** We split the proof into two parts.

i) (Establishing convergence)

The result established by Theorem 3.1 implies that \( f(t,u) \) is a bounded non-negative measure for any \( t \geq 0 \) and for all values of \( R^I, \alpha, R^H \) and \( \beta \). As a result, the Banach Alaoglu theorem allows us to conclude that, up to extraction,

\[
 f(t,u) \rightharpoonup f^\infty(u), \quad \text{as } t \to \infty,
\]

where \( f^\infty \) is a bounded non-negative measure such that

\[
 \varrho^\infty = \int_\Omega f^\infty(u)du = 1.
\]

ii) (Identifying the limit \( f^\infty \))

**Case 1.** Using the result established by Lemma 3.2 with \( \varphi(u) = u \), we can conclude that

\[
 \lim_{\beta \to 0} \frac{d}{dt}U(t) = \int_\Omega uF_I[f](t,u)du.
\]

Then, it is easy to verify that, under hypothesis \( R^I = 1 \),

\[
 \lim_{\beta \to 0} \frac{d}{dt}U(t) = \alpha \int_\Omega wg(w)dw - \alpha U(t),
\]
that is, under the limit $\beta \to 0$:

$$U(t) \to \int_{\Omega} w g(w) dw, \quad \text{as} \quad t \to \infty.$$  

Furthermore, choosing $\varphi(u) = (u - w)^2$ and using Lemma 3.2 we are led to the following conclusion:

$$\lim_{\beta \to 0} \frac{d}{dt} \int_{\Omega} \int_{\Omega} (u - w)^2 f(t) g(t) dw du = \int_{\Omega} \int_{\Omega} (u - w)^2 g(t) F^f[f](t,u) dw du.$$  

Due to the above identity, straightforward calculations show that, in the limit $\beta \to 0$,

$$\frac{d}{dt} \int_{\Omega} \int_{\Omega} (u - w)^2 f(t) g(w) dw du = (\alpha^2 - 2\alpha) \int_{\Omega} \int_{\Omega} (u - w)^2 f(t) g(w) dw du + \int_{\Omega} (w - \int_{\Omega} w g(w) dw)^2 g(w) dw.$$  

As a result, if $g(w) = \delta(w - W)$,

$$\frac{d}{dt} \int_{\Omega} \int_{\Omega} (u - w)^2 f(t) g(w) dw du = (\alpha^2 - 2\alpha) \int_{\Omega} \int_{\Omega} (u - w)^2 f(t) g(w) dw du.$$  

The above identity implies that the average distance between the points belonging to the support of $f$ and the points belonging to the support of $g$ monotonically decreases across time. As a result, identity (17) follows in the limit $t \to \infty$.

**Case 2.** Lemma 3.2 with $\varphi(u) = u$ implies that

$$\lim_{\alpha \to 0} \frac{d}{dt} \int_{\Omega} u f(t) du = \int_{\Omega} u F^H[f](t,u) du = 0,$$  

where the last identity follows from assumptions (7) and (9). Thus, we are led to conclude that

$$U(t) = U(0), \quad \forall t > 0, \quad \text{in the limit} \quad \alpha \to 0.$$  

Using the result established by Lemma 3.2 with $\varphi(u) = (u - U(0))^2$, it is possible to prove that, whether $R^H \geq \max(U(0), 1 - U(0))$, there exists a positive real constant $C$ such that

$$\lim_{\alpha \to 0} \frac{d}{dt} \int_{\Omega} (u - U(0))^2 f(t,u) du \leq -C \int_{\Omega} (u - U(0))^2 f(t,u) du.$$  

In fact, straightforward computations show that, in the limit $\alpha \to 0$,

$$\frac{d}{dt} \int_{\Omega} (u - U(0))^2 f(t,u) du = (\beta^2 - \beta) \int_{\Omega} \int_{\Omega} \eta^L(U_*, U_*)(u_*, u^*)^2 f_*(t) f^*(t) du_* du^*,$$  

which implies, if $R^H \geq \max(U(0), 1 - U(0))$,

$$\frac{d}{dt} \int_{\Omega} (u - U(t))^2 f(t,u) du = \frac{d}{dt} \int_{\Omega} (u - U(0))^2 f(t,u) du \leq \beta(\beta - 1) C_1 \int_{\Omega} \eta^L(U_*, U(0))(u_*, U(0))^2 f_*(t) du_*$$

$$= -\beta(1 - \beta) C_1 \int_{\Omega} (u_*, U(0))^2 f_*(t) du_*,$$

where $C_1$ is a positive real constant smaller than one. As a result, defining $C = \beta(1 - \beta) C_1$, we get differential inequality (20), which allows us to conclude that $f(t,u)$ converges to a Dirac delta centered in $U(0)$, in the limit $\alpha \to 0$ and $t \to \infty$.  


Case 3. Using the result established by Lemma 3.2 with \( \phi(u) = u \), it is straightforward to check that, in the limit \( \alpha \to 0 \), definition (11) implies
\[
\frac{d}{dt} U(t) = \int_{\Omega} u F^H[f](t,u)du = \frac{\beta}{R^I} \int_{\Omega} \int_{u^* > u^*} u^*(1 - u^*)f^*(t)f^*(t)du^*du^*,
\]
which leads us to conclude that there exists a constant \( C \in \mathbb{R}^+ \) such that
\[
\frac{d}{dt} U(t) \geq CU(t)(1 - U(t)).
\]
The above differential inequality implies that, if \( U(0) \) is sufficiently smaller than 1, there exists a time interval where \( U(t) \) grows over-exponentially in \( t \) and then saturates at 1. Finally, under the limit \( \alpha \to 0 \), it is possible to use Lemma 3.2 with \( \phi(u) = (u - U(t))^2 \) to verify that
\[
\frac{d}{dt} \int_{\Omega} (u - U(t))^2 f(t,u)du = \left( \int_{\Omega} (u - U(t))^2 F^H[f](t,u)du \right)
= 2\beta(1 - \beta) \int_{\Omega} (u - U(t))^2 f(t)du,
\]
which implies that \( f \to \delta(u - 1) \) in the limit \( t \to \infty \) and \( \alpha \to 0 \). This concludes the proof. \( \square \)

4. Computational results and economic interpretation. This section presents the results of some numerical simulations of the Cauchy Problem (3), which are aimed at illustrating the above analytical results about the asymptotic behavior of function \( f \) under the parameters settings considered by Theorem 3.3. Some economic considerations are also drawn. In more detail:

Subsection 4.1 refers to Case 1, i.e. an efficient market.

Subsection 4.2 deals with Case 2, i.e. a market ruled by naïve herding.

Subsection 4.3 refers to Case 3, i.e. a highly confident market where naïve herding may lead to an over-exponential growth of the estimated value, so paving the way to economic bubbles.

Along all simulations, we assume \( f^0(u) = 1 \), i.e. the distribution of agents among all the possible values is uniform at the beginning of observations, which implies \( U(0) = 0 \). Simulations are developed using a spectral collocation method.

4.1. Case 1: Efficient market transition. We perform simulations setting \( \alpha = 0.5, R^I = 1, \beta = 0.001, \eta^H \) defined by (9) with \( R^H = 0.6 \) and
\[
g(t,w) = g(w) = \delta(w - 0.63).
\]
As previously noted, such a setting mimics an ideal economic scenario where agents interact with several other agents (i.e. \( R^H > 0.5 \)) but they tend to weakly modify the estimated value because of these interactions (i.e. \( \beta \to 0 \)). Agents are confident in sources of public information independently from the value that they suggest (i.e. \( R^I = 1 \) ), which is assumed to be the same for all sources (i.e. \( g(w) = \delta(w - W) \), with \( W = 0.63 \)), and tend to make their value closer to the one of the sources (i.e. 

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\( \alpha = 0.5 \). Loosely speaking, this setting reproduces an economic scenario where all agents are fully rational.

The results summarized in Figure 1 illustrate the analytical ones presented in the previous section. In fact, \( f \) concentrates, across time, around point \( W \). This implies that all the agents concentrate, in the limit of large times, around the same value, which is the one suggested by information sources. Therefore, the value estimated by the market reflects all public information about the product, that is, the market is efficient.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Large time behavior (left) and evolution in time (right) of \( f(t,u) \) in Case 1 considered by Theorem 3.3. Function \( f \) concentrates, across time, around the point \( W = 0.63 \) where \( g \) is highly concentrated. Thus, all the agents concentrate, in the limit of large times, around the same value, which is the one suggested by information sources. Within the framework of our model, this result provides a mathematical formalization for the efficient market transition.}
\end{figure}

4.2. Case 2: Effects of naïve herding. We develop simulations with \( \alpha = 0.001 \), \( R^I = 1 \), \( \beta = 0.5 \), \( \eta^H \) defined by (9) with \( R^H = 0.6 \) and assuming definition (21) for \( g \) to hold. Such a parameter choice reproduces an economic scenario where agents interact with all sources (i.e. \( R^I = 1 \)), but they give little weight to public information (i.e. \( \alpha \to 0 \)). Each agent tends to imitate other agents (i.e. \( \beta = 0.5 \)) although they assign a value quite different from her own one (i.e. \( R^H > 0.5 \)). This means that naïve herding is assumed to occur within the market, that is, agents follow each other independently from public information.

In agreement with the analytical results established by Theorem 3.3, Figure 2 shows that \( f \) concentrates, across time, around point \( U(0) \). As a result, since interacting agents tend to average their current values (i.e. \( \beta = 0.5 \)), a consensus is reached for large times. In fact, at the end of observations, all the agents estimate the same value, which is equal to the one defined by the market as a whole at the beginning of observations and is independent from the one suggested by public information. In one word, naïve herding invalidates the efficient market axiom.
4.3. Case 3: Formation of economic bubbles. We develop simulations with $\alpha = 0.001$, $R^I = 0.1$, $\beta = 0.5$, $\eta^H$ defined by (10) and assuming definition (21) for $g$ to hold. Such a parameter choice reproduces an economic scenario where agents interact with very few sources (i.e. $R^I = 0.1$), give little weight to public information (i.e. $\alpha \to 0$), highly trust the value of the product and tend to naively imitate those other agents (i.e. $\beta = 0.5$) that assign a higher value (i.e. see considerations drawn about definition (10) in Section 2). Therefore, naïve herding takes place within a scarcely rational market, which is overconfident in the product under consideration.

Figure 3 highlights the dynamics of $f$, while Figure 4 shows the trend of the value estimated by the market and makes also a comparison between $U(t)$ and an exponential function of $t$ (i.e. $U(0)e^t$) over an interval close to the initial observation time. These numerical results are consistent with the ones presented by Theorem 3.3 in Case 3. In fact, as time goes by, function $f$ tends to be highly concentrated in 1 and so function $U$ tends to 1. This implies that, in a weakly rational and highly confident market, such as the one here considered, naïve herding can lead agents to estimate a very high value for the product under consideration. Furthermore, Figure 4 points out that, when $U$ is not yet affected by saturation effects (i.e. on small intervals close to the origin), the model predicts an over-exponential growth of this function. This is in agreement with experimental evidence supporting the idea that naïve herding may be one of the principal mechanisms fueling the growth of economic bubbles [12, 16], whose first symptom is a super-exponential increase in the value that the market assigns to a traded good.
Figure 3. Large time behavior (left) and evolution in time (right) of $f(t,u)$ in Case 3 considered by Theorem 3.3. Function $f$ concentrates, across time, around the point 1. Thus, as time goes by, all the agents tend to estimate the same high value for the product. This result provides a mathematical formalization for the idea that naïve herding can lead agents to estimate a very high value for traded products in a weakly rational and highly confident market.

5. Conclusions. This work stems from the idea, proposed in several previous papers [7, 8, 24], that local and international markets can be seen as complex systems composed of a large number of heterogeneous agents interacting within the external environment defined by public information sources. Focusing on a given traded good, the value estimated by the market is seen as an emerging property resulting from interactions between agents and public information, as well as from herding phenomena among agents.

Proper strategies to mathematically formalize such interactions have been developed and an integro-differential model relying on four parameters has been defined. Due to the small number of parameters, such a model is not enough refined to make quantitative forecasts, but it is well suited for studying the emergence of collective behaviors, which is the aim of the present paper.

A general well-posedness result has been established for the initial value problem linked to the model. Moreover, the asymptotic behavior in time of the related solution has been characterized for some general parameter settings, which imitate three different economic scenarios: an efficient market, a market ruled by naïve herding and a market where economic bubbles are created.

Analytical results have been illustrated by means of numerical simulations and lead us to conclude that, in spite of its oversimplified nature, this model is able to reproduce some emerging behaviors proper of the system under consideration. In particular, in agreement with experimental evidence [12, 16], the outputs of the model suggest that, if agents are highly confident in the product, imitative and scarcely rational behaviors may lead to an over-exponential rise of the value estimated by the market, paving the way to the formation of economic bubbles.
Several extensions and perspectives can be developed. Among others, it may be interesting to include in the model the idea that interacting agents can be characterized by different levels of influence, namely according to their reputation. For instance, opinion leaders may have a stronger influence on the overall estimated value, as in the case of models for opinion dynamics [18].

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