APPROXIMATION BY SMOOTH FUNCTIONS WITH NO CRITICAL POINTS ON SEPARABLE BANACH SPACES

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Abstract. We characterize the class of separable Banach spaces $X$ such that for every continuous function $f : X \to \mathbb{R}$ and for every continuous function $\varepsilon : X \to (0, +\infty)$ there exists a $C^1$ smooth function $g : X \to \mathbb{R}$ for which $|f(x) - g(x)| \leq \varepsilon(x)$ and $g'(x) \neq 0$ for all $x \in X$ (that is, $g$ has no critical points), as those Banach spaces $X$ with separable dual $X^*$. We also state sufficient conditions on a separable Banach space so that the function $g$ can be taken to be of class $C^p$, for $p = 1, 2, \ldots, +\infty$. In particular, we obtain the optimal order of smoothness of the approximating functions with no critical points on the classical spaces $\ell^p(\mathbb{N})$ and $L^p(\mathbb{R}^n)$. Some important consequences of the above results are (1) the existence of a non-linear Hahn-Banach theorem and (2) the smooth approximation of closed sets, on the classes of spaces considered above.

1. Introduction and main results

The Morse-Sard theorem \cite{MS1, MS2} states that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is a $C^r$ smooth function, with $r > \max\{n - m, 0\}$, and $C_f$ is the set of critical points of $f$, then the set of critical values $f(C_f)$ is of Lebesgue measure zero in $\mathbb{R}^m$. This result has proven to be very valuable in a large number of areas, especially in differential topology and analysis (see for instance \cite{MS1, MS2} and the references therein). Additional geometric and analytical properties of the set of critical values in different versions of the Morse-Sard theorem, together with a study on the sharpness of the hypothesis of the Morse-Sard theorem, have been obtained in \cite{MS1, MS2, MS3, MS4, MS5, MS6}.

For many important applications of the Morse-Sard theorem, it is enough to know that any given continuous function can be uniformly approximated by a smooth map whose set of critical values has empty interior \cite{MS1, MS2}. We refer to this as an approximate Morse-Sard theorem. The same type of approximation could prove key to the study of related problems in the infinite-dimensional domain.

In this paper, we will prove the strongest version of an approximate Morse-Sard theorem that one can expect to be true for a general infinite-dimensional separable Banach space, namely that every continuous function $f : X \to \mathbb{R}$, where $X$ is an infinite-dimensional Banach space $X$ with separable dual $X^*$, can be uniformly approximated by a $C^1$ smooth function $g : X \to \mathbb{R}$ which does not have any critical point. In some
cases where more information about the structure of the Banach space $X$ is known, we will extend our result to higher order of differentiability, $C^p$ ($p > 1$).

Our result will also allow us to demonstrate two important corollaries. The first one is the existence of a non-linear Hahn-Banach theorem which shows that two disjoint closed subsets in $X$ can be separated by a 1-codimensional $C^p$ smooth manifold of $X$ (which is the set of zeros of a $C^p$ smooth function with no critical points on $X$). The second one states that every closed subset of $X$ can be approximated by $C^p$ smooth open subsets of $X$.

To put our work in context, let us briefly review some of the work established for the infinite-dimensional version of the Morse-Sard theorem. Smale [26] proved that if $X$ and $Y$ are separable connected smooth manifolds modelled on Banach spaces and $f : X \to Y$ is a $C^r$ Fredholm map then $f(C_f)$ is of first Baire category and, in particular, $f(C_f)$ has no interior points provided that $r > \max\{\text{index}(df(x)), 0\}$ for all $x \in X$. Here, $\text{index}(df(x))$ stands for the index of the Fredholm operator $df(x)$, that is, the difference between the dimension of the kernel of $df(x)$ and the codimension of the image of $df(x)$, which are both finite. These assumptions are very strong as they impose that when $X$ is infinite-dimensional then $Y$ is necessarily infinite-dimensional too (in other words, there is no Fredholm map $f : X \to \mathbb{R}$). In fact, as Kupka proved in [19], there are $C^\infty$ smooth functions $f : \ell_2 \to \mathbb{R}$ (where $\ell_2$ is the separable Hilbert space) such that their sets of critical values $f(C_f)$ contain intervals and hence have non-empty interiors and positive Lebesgue measure. Bates and Moreira [19] showed that this function $f$ can even be taken to be a polynomial of degree three. Azagra and Cepedello-Boiso [2] have shown that every continuous mapping from the separable Hilbert space into $\mathbb{R}^m$ can be uniformly approximated by $C^\infty$ smooth mappings with no critical points. Unfortunately, since the core of their proof requires the use of the special properties of the Hilbertian norm, this cannot be extended to non-Hilbertian Banach spaces. P. Hájek and M. Johanis [17] established the same kind of result in the case when $X$ is a separable Banach space which contains $c_0$ and admits a $C^p$-smooth bump function. In this case, the approximating functions are of class $C^p$, $p = 1, 2, \ldots, \infty$. This method is based on the result that the range of the derivative of a $C^2$ smooth function from $c_0$ to $\mathbb{R}$ is a countable union of compact sets [16]. However, as the authors noted, their method is not applicable when the space $X$ has the Radon-Nikod´ym property (e.g., when $X$ is reflexive), which leaves out all the classical Banach spaces $\ell_p$ and $L_p(\mathbb{R}^n)$ for $1 < p < \infty$.

As stated above, we prove that for any infinite-dimensional Banach space $X$ with a separable dual $X^*$, the set of $C^1$ smooth, real-valued functions with no critical points is uniformly dense in the space of all continuous, real-valued functions on $X$. This solves completely the problem of the approximation on separable Banach spaces by smooth, real-valued functions with no critical points when the order of smoothness of the approximating functions is one. Hence, we obtain the following characterization. For a separable Banach space $X$, the following are equivalent: (i) $X^*$ is separable, and (ii) the set of $C^1$ smooth, real-valued functions on $X$ with no critical points is uniformly dense in the space of all continuous, real-valued functions on $X$.

This result can be included in our main theorem which also applies to higher order of differentiability. Before stating our main theorem, recall that a norm $|| \cdot ||$ in a Banach space $X$ is LUR (locally uniformly rotund [11]) if $\lim_n ||x_n - x|| = 0$ whenever the sequence $\{x_n\}_n$
and the point $x$ are included in the unit sphere of the norm $\| \cdot \|$ and $\lim_n \| x_n + x \| = 2$. A norm $\| \cdot \|$ in $X$ is $C^p$ smooth if it is $C^p$ smooth in $X \setminus \{0\}$.

**Theorem 1.1.** Let $X$ be an infinite dimensional separable Banach space $X$ with a LUR and $C^p$ smooth norm $\| \cdot \|$, where $p \in \mathbb{N} \cup \{\infty\}$. Then, for every continuous mapping $f : X \to \mathbb{R}$ and for every continuous function $\varepsilon : X \to (0, \infty)$, there exists a $C^p$ smooth mapping $g : X \to \mathbb{R}$ such that $|f(x) - g(x)| \leq \varepsilon(x)$ for all $x \in X$ and $g$ has no critical points.

Our proof involves: i) a special construction of carefully perturbed partitions of unity in an open subset of the unit sphere of the Banach space $Y = X \oplus \mathbb{R}$ by means of a sequence of linear functionals in $Y^*$, ii) the study and use of the properties of the range of the derivative of the norm in $Y$, $Y^*$ and their finite dimensional subspaces (Lemmas 2.1 and 2.2 below), and iii) the use of $C^p$ deleting diffeomorphisms from $X$ onto $X \setminus O$, where $O$ is a bounded, closed, convex subset of $X$.

The following example gives the optimal order of smoothness of the approximation functions with no critical points for $\ell_p(\mathbb{N})$ and $L_p(\mathbb{R}^n)$.

**Example 1.2.** It follows immediately from Theorem 1.1 that one can approximate every continuous, real-valued function on $\ell_p(\mathbb{N})$ and $L_p(\mathbb{R}^n)$ $(1 < p < \infty)$ with $C^\infty$ smooth, real-valued functions with no critical points, where $\overline{p} = [p]$ if $p$ is not an integer, $\overline{p} = p - 1$ if $p$ is an odd integer, and $\overline{p} = \infty$ if $p$ is an even integer. Indeed, the standard norms of the classical separable Banach spaces $\ell_p(\mathbb{N})$ and $L_p(\mathbb{R}^n)$ are LUR and $C^\infty$ smooth $\|\|$.

Since every Banach space with separable dual admits an equivalent LUR and $C^1$ smooth norm $\|\|$, we immediately deduce from Theorem 1.1 the announced characterization of the property of approximation by $C^1$ smooth functions with no critical points.

**Corollary 1.3.** Let $X$ be a separable Banach space. The following are equivalent:

1. The dual space $X^*$ is separable,
2. for every continuous mapping $f : X \to \mathbb{R}$ and for every continuous function $\varepsilon : X \to (0, \infty)$, there exists a $C^1$ smooth mapping $g : X \to \mathbb{R}$ such that $|f(x) - g(x)| \leq \varepsilon(x)$ and $g$ has no critical points.

Next, we establish a similar statement for higher order smoothness on separable Banach spaces with a $C^p$ smooth bump function $(p \geq 2)$ and unconditional basis. We combine Theorem 1.1 and the results on fine approximation given in [13], to obtain the optimal order of smoothness of the approximating functions with no critical points on a large class within the Banach spaces with separable dual. In particular the following Corollary applies even when the space $X$ lacks a norm which is simultaneously LUR and $C^2$ smooth.

**Corollary 1.4.** Let $X$ be a separable Banach space with unconditional basis. Assume that $X$ has a $C^p$ smooth Lipschitz bump function, where $p \in \mathbb{N} \cup \{\infty\}$. Then, for every continuous mapping $f : X \to \mathbb{R}$ and for every continuous function $\varepsilon : X \to (0, \infty)$, there exists a $C^p$ smooth mapping $g : X \to \mathbb{R}$ such that $|f(x) - g(x)| \leq \varepsilon(x)$ and $g$ has no critical points.
Proof. Since $X$ is separable and admits a $C^p$ smooth bump function, the dual space $X^*$ is separable. Thus we obtain, from Corollary 1.3 a $C^1$ smooth function $h : X \to \mathbb{R}$ such that $h'(x) \neq 0$ and $|f(x) - h(x)| < \frac{\varepsilon(x)}{2}$ for every $x \in X$. Let us denote by $\| \cdot \|$ the dual norm on $X^*$. Define the continuous function $\overline{\varepsilon} : X \to (0, \infty)$, $\overline{\varepsilon}(x) = \frac{1}{2} \min \{\varepsilon(x), ||h'(x)||\}$, for $x \in X$. Now, by the main result of Remark 1.3, there is a $C^p$ smooth function $g : X \to \mathbb{R}$ such that $|h(x) - g(x)| < \overline{\varepsilon}(x)$ and $||h'(x)| - ||g'(x)||| < \overline{\varepsilon}(x)$, for every $x \in X$. The latter implies that $|h'(x)| < ||g'(x)|| < \frac{1}{2}||h'(x)||$, and therefore $0 < \frac{1}{2}||h'(x)|| < ||g'(x)||$ for every $x \in X$. Hence, $g$ is a $C^p$ smooth function with no critical points and $|f(x) - g(x)| < |f(x) - h(x)| + |h(x) - g(x)| < \varepsilon(x)$ for every $x \in X$. □

The proof of the above corollary yields to the following remark.

Remark 1.5. Assume that a separable Banach space $X$ satisfies the $C^1$-fine approximation property by $C^p$ smooth, real-valued functions, i.e., for every $C^1$ smooth function $f : X \to \mathbb{R}$ and every continuous function $\varepsilon : X \to (0, \infty)$ there is a $C^p$ smooth function $h : X \to \mathbb{R}$ such that $|f(x) - h(x)| \leq \varepsilon(x)$ and $|f'(x) - h'(x)| \leq \varepsilon(x)$, for every $x \in X$. Then, the conclusion of Corollary 1.4 holds.

Furthermore, our results allow us to make the following conclusions.

Remark 1.6.

(1) All of the results presented above hold in the case when one replaces $X$ with an open subset $U$ of $X$. Actually, the same proof given in the section to follow (with obvious modifications) can be used.

(2) Whenever $X$ has the property that every continuous, real-valued function on $X$ can be approximated by $C^p$ smooth, real-valued functions with no critical points, one can deduce the following Corollaries.

Corollary 1.7 (A nonlinear Hahn-Banach theorem). Let $X$ be any of the Banach spaces considered in the above results. Then, for every two disjoint closed subsets $C_1$, $C_2$ of $X$, there exists a $C^p$ smooth function $\varphi : X \to \mathbb{R}$ with no critical points, such that the level set $M = \varphi^{-1}(0)$ is a 1-dimensional $C^p$ smooth submanifold of $X$ that separates $C_1$ and $C_2$, in the following sense: define $U_1 = \{x \in X : \varphi(x) < 0\}$ and $U_2 = \{x \in X : \varphi(x) > 0\}$, then $U_1$ and $U_2$ are disjoint $C^p$ smooth open sets of $M$ with common boundary $\partial U_1 = \partial U_2 = M$, and such that $C_i \subseteq U_i$ for $i = 1, 2$.

Recall that an open subset $U$ of $X$ is said to be $C^p$ smooth provided its boundary $\partial U$ is a $C^p$ smooth one-dimensional submanifold of $X$.

Corollary 1.8 (Smooth approximation of closed sets). Every closed subset of any of the Banach spaces $X$ considered above can be approximated by $C^p$ smooth open subsets of $X$ in the following sense: for every closed set $C \subseteq X$ and every open set $W$ containing $C$ there is a $C^p$ smooth open set $U$ so that $C \subseteq U \subseteq W$.

Corollary 1.9 (Failure of Rolle’s Theorem). For every open subset $U$ of any of the above Banach spaces $X$ there is a continuous function $f$ on $X$ whose support is the closure of $U$, and such that $f$ is $C^p$ smooth on $U$ and yet $f$ has no critical point in $U$. 

2. Proof of Theorem 1.1

Recall that a norm $\|\cdot\|$ in a Banach space $E$, is (1) strictly convex if the unit sphere of the norm $\|\cdot\|$ does not include any segment line. Equivalently, $N(\frac{x+y}{2}) < 1$ for every $x, y$ in the unit sphere with $x \neq y$; (2) WUR (weakly uniformly rotund) if $\lim_n(x_n - y_n) = 0$ in the weak topology whenever the sequences $\{x_n\}_n$ and $\{y_n\}_n$ are included in the unit sphere and $\lim_n N(x_n + y_n) = 2$.

We denote by $\mathbb{R}^*$ the set of non zero real numbers. We also let $[u_1, ..., u_n]$ stand for the linear span of the vectors $u_1, ..., u_n$. Let us denote by $S_{\|\cdot\|}$ and $S_{\|\cdot\|}^*$ the unit sphere of a Banach space $(Z, \|\cdot\|)$ and its dual $(Z^*, \|\cdot\|*)$, respectively.

The following two geometrical lemmas will be essential to the proof of Theorem 1.1.

**Lemma 2.1.** Let $Z = [u_1, ..., u_n]$ be a $n$-dimensional space ($n > 1$) with a differentiable norm $\|\cdot\|$ (Gâteaux or Fréchet differentiable, since both notions coincide for convex functions defined on finite dimensional spaces). Let us consider real numbers $0 < \alpha_i < 1$, for $i = 1, ..., n - 1$ and define $\mathcal{R}$ as the subset of numbers $\alpha \in \mathbb{R}^*$ satisfying that

$$\{ T \in S_{\|\cdot\|}^*: T(u_i) = \alpha_i, \ i = 1, ..., n - 1, \ T(u_n) = \alpha \} = \emptyset.$$  

Then, the cardinal of $\mathbb{R}^* \setminus \mathcal{R}$ is at most two.

**Proof.** First, assume that the set

$$F = \{ T \in S_{\|\cdot\|}^*: T(u_1) = \alpha_1, ..., T(u_{n-1}) = \alpha_{n-1} \}$$

is non-empty (otherwise we have finished).

As the above pictures for the cases $n = 2$ and $n = 3$ suggest, there are at most two different tangent affine hyperplanes to the unit sphere $S_{\|\cdot\|}$ containing the affine subspace passing through the points $\frac{u_1}{\alpha_1}, ..., \frac{u_{n-1}}{\alpha_{n-1}}$, that is there are at most two different linear mappings $T_1, T_2$ in $F$. Indeed, assume first that the cardinal of $F$ is one and let us denote by $T_1$ the element of $F$. Then any real number $\alpha \notin \{0, T_1(u_n)\}$ satisfies condition (2.1) and $\mathcal{R} = \mathbb{R}^* \setminus \{T_1(u_n)\}$. Now, if there are two elements on $F$, $T_1 \neq T_2$, we claim that any other different element of $F$, say $T_3$, can be written as $T_3 = \gamma T_1 + (1 - \gamma)T_2$ for some
\( \gamma \in \mathbb{R} \setminus \{0,1\} \). Indeed, since \( T_3 - T_2 \neq 0 \) and \( T_1 - T_2 \neq 0 \) belong to the one dimensional subspace \( u_1^* \cap \ldots \cap u_{n-1}^* \cap \{u_1, \ldots, u_n\}^* \), we have that there is \( \gamma \in \mathbb{R} \) with \( T_3 - T_2 = \gamma (T_1 - T_2) \). Since \( T_3 \neq T_2 \), the constant \( \gamma \neq 0 \). In addition, since \( T_3 \neq T_1 \), we have that \( \gamma \neq 1 \), which proves our claim.

Furthermore, this implies that the three different points \( T_1, T_2 \) and \( T_3 \) of the dual unit sphere lie in a common line, and, because \( \| \cdot \|^* \) is convex, the segment line which passes through \( T_1, T_2 \) and \( T_3 \), and whose end points are two of these three points, is included in \( S_{\| \cdot \|^*} \). But this is in contradiction to the fact that the dual norm \( \| \cdot \|^* \) is strictly convex (because the norm \( \| \cdot \| \) is differentiable and \( Z \) is finite dimensional, see [11]). Finally, any real number \( \alpha \notin \{ 0, T_1(u_n), T_2(u_n) \} \) satisfies condition (2.1) and \( R = \mathbb{R}^* \setminus \{ T_1(u_n), T_2(u_n) \} \).

From the proof of Lemma 2.1 we obtain the following.

**Lemma 2.2.** Let \( Z = [u_1, \ldots, u_n] \) be a \( n \)-dimensional space \( (n \in \mathbb{N}) \) with a differentiable norm \( \| \cdot \| \). Consider real numbers \( 0 < \alpha_i < 1 \), for \( i = 1, \ldots, n - 1 \). Then, the cardinal of the set

\[
\{ T \in S_{\| \cdot \|^*} : T(u_i) = \alpha_i, \ i = 1, \ldots, n - 1 \}
\]

is at most two.

The general strategy of the proof of Theorem 1.1 is as follows. We consider the space \( Y = X \oplus \mathbb{R} \) and define the following norm on \( Y \):

- If \( p > 1 \), for every \( y = (x, r) \in Y \), put \( N(y) = N(x, r) = (\|x\|^2 + r^2)^{1/2} \), where \( \| \cdot \| \) is a LUR and \( C^p \) smooth norm on \( X \). Then, clearly the norm \( N \) is LUR and \( C^1 \) smooth on \( Y \). Moreover, \( N \) is \( C^p \) smooth on the open set \( Y \setminus \{(0, \lambda) : \lambda \in \mathbb{R}\} \). Define \( \nu = (0, 1) \) and take \( \beta \in Y^* \setminus \{0\} \) such that \( X = \ker \beta \). Select \( \beta_1 \in Y^* \setminus \beta \) such that \( \beta_1(\nu) = 0 \) and \( \omega \in \ker \beta \setminus \ker \beta_1 \). Consider the closed hyperplane of \( Y \), \( X_1 = \ker \beta_1 \). Then, the restriction of the norm \( N \) to \( X_1 \) is a \( C^p \) smooth and LUR norm on \( X_1 \). Now, the (equivalent) norm considered in \( Y = X_1 \oplus [\omega] \), defined as \( \|z + \lambda \omega\| = (N(z)^2 + \lambda^2)^{1/2} \), where \( z \in X_1 \) and \( \lambda \in \mathbb{R} \), is LUR and \( C^1 \) smooth on \( Y \) and \( C^p \) smooth on \( Y \setminus [\omega] \). In particular, the norm \( \| \cdot \| \) is \( C^p \) smooth on the open set \( U = Y \setminus \ker \beta = \{(x, r) : x \in X, r \neq 0\} \).

It could also be proved that the Banach space \( Y \) admits an equivalent LUR and \( C^p \) smooth norm on \( Y \) with bounded derivatives up to the order \( p \). Nevertheless, a LUR and \( C^1 \) smooth norm on \( Y \) and \( C^p \) smooth on \( U \), is sufficient to prove our result. Recall that if \( X \) has a LUR and \( C^p \) smooth norm and \( p > 1 \), then \( X \) is superreflexive [11].

- If \( p = 1 \), since the dual space \( Y^* \) is separable, there is a norm \( \| \cdot \| \) on \( Y \) which is LUR, \( C^1 \) smooth and WUR whose dual is strictly convex [11]. Recall that if the norm \( \| \cdot \| \) is WUR, then the dual norm \( \| \cdot \|^* \) is uniformly Gâteaux smooth, and thus, Gâteaux smooth.

Therefore, if \( X \) is reflexive, the dual norm \( \| \cdot \|^* \) is LUR and \( C^1 \) smooth [11]. If \( X \) is not reflexive, the dual norm \( \| \cdot \|^* \) is strictly convex and Gâteaux smooth.
Let us denote $S := S_{1, |\cdot|}$, the unit sphere of $(Y, |\cdot|)$ and $S^* := S_{1, |\cdot|^*}$, the unit sphere of $(Y^*, |\cdot|^*)$. Let us consider, the duality mapping of the norm $|\cdot|$ defined as

$$D : S \rightarrow S^*$$

$$D(x) = |\cdot|^*(x),$$

which is $|\cdot| - |\cdot|^*$ continuous because the norm $|\cdot|$ is of class $C^1$.

We establish a $C^p$ diffeomorphism $\Phi$ between $X$ and half unit sphere in $Y$, $S^+ := \{y = (x, r) \in Y : r > 0\}$, as follows: $\Phi : X \rightarrow S^+$ is the composition $\Phi = \Pi \circ i$, where $i$ is the inclusion $i : X \rightarrow Y$, $i(x) = (x, 1)$ and $\Pi$ is defined by $\Pi : Y \setminus \{0\} \rightarrow S$, $\Pi(y) = \frac{y}{|y|}$.

In order to simplify the notation, we will make the proof for the case of a constant $\varepsilon > 0$. By taking some standard technical precautions the same proof will work in the case of a positive continuous function $\varepsilon : X \rightarrow (0, +\infty)$ (at the end of the proof we will explain what small changes should be made).

Now, given a continuous function $f : X \rightarrow \mathbb{R}$, we consider the composition $F := f \circ \Phi^{-1} : S^+ \rightarrow \mathbb{R}$, which is continuous as well. For any given $\varepsilon > 0$ we will $3\varepsilon$-approximate $F$ by a $C^p$ smooth function $H : S^+ \rightarrow \mathbb{R}$ with the properties that:

- the set of critical points of $H$ is the countable union of a family of disjoint sets $\{K_n\}_n$,
- there are countable families of open slices $\{O_n\}_n$ and $\{B_n\}_n$ in $S^+$, such that $\bigcup_n \overline{B_n}$ is relatively closed in $S^+$, $\text{dist}(B_n, X \setminus \{0\}) > 0$, $K_n \subset O_n \subset B_n$, $\text{dist}(O_n, S^+ \setminus B_n) > 0$ and $\text{dist}(B_n, \cup_{m \neq n} B_m) > 0$, for every $n \in \mathbb{N}$,
- the oscillation of $F$ in every $B_n$ is less than $\varepsilon$.

(We will consider slices of $S^+$ of the form $\{x \in S : f(x) > r\}$, where $f$ is a continuous linear functional of norm one and $0 < r < 1$. Recall also that the distance between two sets $A$ and $A'$ in a Banach space $(M, |\cdot|_M)$ is defined as the real number $\text{dist}(A, A') := \inf\{|a - a'| : a \in A, a' \in A'\}$.)

Then we will prove that the function $h := H \circ \Phi$ is a $C^p$ smooth function on $X$, which $3\varepsilon$-approximates $f$, and the set of critical points of $h$, $C := \{x \in X : h'(x) = 0\}$, can be written as $C = \bigcup_{n=1}^{\infty} K_n$, where, for every $n \in \mathbb{N}$, the set $K_n := \Phi^{-1}(K_n)$ is contained in the open, convex, bounded and $C^p$ smooth body $O_n := \Phi^{-1}(O_n)$, which in turn is contained in the open, convex, bounded and $C^p$ smooth body $B_n := \Phi^{-1}(B_n)$, in such a way that $\text{dist}(O_n, X \setminus B_n) > 0$, the oscillation of $f$ in $B_n$ is less than $\varepsilon$, $\cup_n \overline{B_n}$ is closed and $\text{dist}(B_n, \cup_{m \neq n} B_m) > 0$. Once we have done this, we will compose the function $h$ with a sequence of deleting diffeomorphisms which will eliminate the critical points of $h$. More precisely, for each set $O_n$ we will find a $C^p$ diffeomorphism $\Psi_n$ from $X$ onto $X \setminus \overline{O_n}$ so that $\Psi_n$ is the identity off $B_n$. Then, by defining $g := h \circ \bigcap_{n=1}^{\infty} \Psi_n$, we will get a $C^p$ smooth function which $4\varepsilon$-approximates $f$ and which has no critical points.

The most difficult part in this scheme is the construction of the function $H$. We will inductively define linearly independent functionals $g_k \in Y^*$, open subsets $U_k$ of $S^+$, points $x_k \in U_k$, real numbers $a_k \neq 0$ satisfying $|a_k - F(x_k)| < \varepsilon$, real numbers $\gamma_k$ and $\gamma_{i,j}$ in the interval $(0, 1)$ (with $i + j = k$), functions $h_k$ of the form $h_k = \varphi_k(g_k) \phi_{k-1,1}(g_{k-1}) \cdots \phi_{1,k-1}(g_1)$, where the $\varphi_k$, $\phi_{k-1,1}, \ldots, \phi_{1,k-1}$ are suitably chosen.
Notice that the interior of the support of $h$ where $\varphi_S \diamond \cdot$ Recall that the oscillation of $F$, $\text{dist}(\cdot)$, $\leq 1 = 1 = \cdots = 1$, so that the oscillation of $F$ will be less than $\varepsilon$. Denote by $T_x$ the (vectorial) tangent hyperplane to $S^+$ at the point $x$, that is $T_x := \ker D(x)$. The derivative of $H_k$ at every point $x \in U_1 \cup \ldots \cup U_k$ will be shown to be the restriction to $T_x$ of a nontrivial linear combination of the linear functionals $g_1, \ldots, g_k$. Then, by making use of Lemmas 2.1 and 2.2 and choosing the $\gamma_{i,j}$ close enough to $\gamma_i$, we will prove that the set of critical points of $H_k$ is a finite union of pairwise disjoint sets which are contained in a finite union of pairwise disjoint slices, with positive distance between any two slices (see Figure 2 below). These slices will be determined by functionals in finite sets $N_k \subset Y^*$ defined by a repeated application of Lemmas 2.1 and 2.2. The function $H$ will be then defined as

$$H = \sum_{k=1}^{\infty} \frac{a_k r_k h_k}{\sum_{k=1}^{\infty} h_k}.$$ 

Let us begin with the formal construction of the functions $H_k$. We will use the notation $H_k$ and $H'_k$ when $H_k$ and its derivative $H'_k$ are thought to be defined on an open subset of $Y$ and reserve the symbols $H_k$ and $H'_k(x)$ for the restriction of $H_k$ and $H'_k(x)$ to a subset of $S$ and to the tangent space $T_x$ of $S$ at $x$, respectively.

Since the norm $| \cdot |$ is LUR we can find, for every $x \in S^+$, open slices $R_x = \{ y \in S : f_x(y) > \delta_x \} \subset S^+$ and $P_x = \{ y \in S : f_x(y) > \delta^1_x \} \subset S^+$, where $0 < \delta_x < 1$ and $|f_x| = 1 = f_x(x)$, so that the oscillation of $F$ in every $P_x$ is less than $\varepsilon$. We also assume, for technical reasons, and with no loss of generality, that $\text{dist}(P_x, X \times \{0\}) > 0$.

Since $Y$ is separable we can subdivide a countable subfamily of $\{ R_x \}_{x \in S^+}$, which covers $S^+$. Let us denote this countable subfamily by $\{S_n\}_n$, where $S_n := \{ y \in S : f_n(y) > \delta_n \}$. Recall that the oscillation of $F$ in every $P_n := \{ y \in S : f_n(y) > \delta^1_n \}$ is less than $\varepsilon$ and $\text{dist}(P_n, X \times \{0\}) > 0$.

$\diamond$ For $k = 1$, define

$$h_1 : S^+ \longrightarrow \mathbb{R}$$

$$h_1 = \varphi_1(f_1),$$

where $\varphi_1$ is a $C^\infty$ function on $\mathbb{R}$ satisfying

$$\varphi_1(t) = 0 \text{ if } t \leq \delta_1$$

$$\varphi_1(1) = 1$$

$$\varphi'_1(t) > 0 \text{ if } t > \delta_1.$$ 

Notice that the interior of the support of $h_1$ is the open set $S_1$. Denote by $x_1$ the point of $S^+$ satisfying $f_1(x_1) = 1$. Now select $a_1 \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$ with $|a_1 - F(x_1)| < \varepsilon$ and define
the auxiliary function
\[ r_1 : S^+ \to \mathbb{R}, \quad r_1 = s_1f_1 + (1 - s_1f_1(x_1)), \]
where we have selected \( s_1 \) so that \( a_1s_1 > 0 \) and \( |s_1| \) small enough so that the oscillation of \( r_1 \) on \( S_1 \) is less than \( \frac{\varepsilon}{|a_1|} \). Notice that \( r_1(x_1) = 1 \). Define
\[
H_1 : S_1 \to \mathbb{R} \quad \text{and} \quad H_1 = \frac{a_1r_1h_1}{h_1} = a_1r_1.
\]
The function \( H_1 \) is \( C^\mu \) smooth in \( S_1 \) and the set of critical points of \( H_1 \),
\[ Z_1 = \{ x \in S_1 : H_1'(x) = 0 \text{ on } T_x \} \]
consists of the unique point \( x_1 \). Indeed, \( H_1'(x) = H_1'(x)|_{T_x} = a_1s_1f_1|_{T_x} \equiv 0 \) iff \( D(x) = f_1 \).
This implies that \( Z_1 = \{ x_1 \} \). Now select real numbers \( \gamma_{1,1}, t_1 \) and \( l_1 \) such that \( \delta_1 < \gamma_{1,1} < t_1 < l_1 < 1 \) and define the open slices
\[ O_{f_1} = \{ x \in S : f_1(x) > l_1 \} \quad \text{and} \quad B_{f_1} = \{ x \in S : f_1(x) > t_1 \}. \]
Clearly the above sets satisfy that \( Z_1 \subset O_{f_1} \subset B_{f_1} \subset S_1 \), \( \text{dist}(O_{f_1}, S \setminus B_{f_1}) > 0 \) and \( \text{dist}(B_{f_1}, \{ x \in S : f_1(x) \leq \gamma_{1,1} \}) > 0 \).

In order to simplify the notation in the rest of the proof, let us denote by \( \gamma_1 = \delta_1, U_1 = R_1 = S_1, g_1 = f_1, z_1 = x_1 \) and \( \Gamma_1 = N_1 = \{ g_1 \} \). Let us define \( \sigma_{1,1} = a_1s_1 \) and write \( H_1' = \sigma_{1,1}g_1 \) on \( U_1 \) where \( g_1 \) is the restriction of \( g_1 \) to \( T_x \) whenever we evaluate \( H_1'(x) \). In addition, if \( A \subset S \), we denote by \( A^c = S \setminus A \).

\( \diamond \) For \( k = 2 \). Let us denote by \( y_2 \in S^+ \) the point satisfying \( f_2(y_2) = 1 \). If either \( \{ g_1, D(y_2) = f_2 \} \) are lineally dependent (this only occurs when \( g_1 = f_2 \) or \( g_1(y_2) = 1 \)), we use the density of the norm attaining functionals (Bishop-Phelps Theorem) and the continuity of \( D \) to modify \( y_2 \) and find \( z_2 \in S^+ \) so that \( \{ g_1, D(z_2) := g_2 \} \) are l.i., \( g_1(z_2) \neq \gamma_1 \) and
\[ \{ x \in S : f_2(x) > \delta_2 \} \subset \{ x \in S : g_2(x) > \nu_2 \} \subset \{ x \in S : f_2(x) > \delta_3 \}, \]
for some \( \nu_2 \in (0, 1) \). If \( g_1(y_2) \neq \gamma_1 \) and \( \{ g_1, f_2 \} \) are l.i., define \( g_2 = f_2 \) and \( z_2 = y_2 \). Then, apply Lemma 2.1 to the 2-dimensional space \([g_1, g_2]\) with the norm \( | \cdot |^* \) (the restriction to \([g_1, g_2]\) of the dual norm \( | \cdot |^* \) considered in \( Y^* \)) and the real number \( \gamma_1 \in (0, 1) \) to obtain \( \gamma_2 \in (0, 1) \) close enough to \( \nu_2 \) so that
\[ S_2 = \{ x \in S : f_2(x) > \delta_2 \} \subset \{ x \in S : g_2(x) > \nu_2 \} \subset \{ x \in S : f_2(x) > \delta_3 \} = P_2 \]
and
\[ (2.3) \quad \{ T \in [g_1, g_2] : |T| = 1, T(g_1) = \gamma_1 \text{ and } T(g_2) = \gamma_2 \} = \emptyset \]
Recall that the norm \( | \cdot |^* \) is Gâteaux differentiable on \( Y^* \) and therefore the restriction of this norm to \([g_1, g_2]\), which we shall denote by \( | \cdot |^* \) as well, is a differentiable norm on the space \([g_1, g_2]\) (Gâteaux and Fréchet notions of differentiability are equivalent in the case of convex functions defined on finite-dimensional spaces). Therefore, we can apply Lemma 2.1 to the norm \( | \cdot |^* \) in the space \([g_1, g_2]\). In fact, the same argument works for any finite
dimensional subspace of $Y^*$ and we will apply Lemma 2.1 in the next steps to larger finite dimensional subspaces of $Y^*$. Define the sets

$$R_2 = \{ x \in S : g_2(x) > \gamma_2 \}, \quad \text{and}$$

$$U'_2 = \{ x \in S : g_1(x) < \gamma'_{1,1}, \ g_2(x) > \gamma_2 \}.$$ 

Assume that $U_1 \cap R_2 \neq \emptyset$ and consider the set

$$M_2 = D^{-1}([g_1, g_2]) \cap U'_2 \cap U_1.$$ 

In the case that $M_2 = \emptyset$, we select as $\gamma_{1,1}$ any point in $(\gamma_1, \gamma'_{1,1})$. In the case that $M_2 \neq \emptyset$ and $\gamma_1 < \inf\{g_1(x) : x \in M_2 \}$, we select $\gamma_{1,1}$ so that

$$\gamma_1 < \gamma_{1,1} < \inf\{g_1(x) : x \in M_2 \}.$$ 

In the case that $\gamma_1 = \inf\{g_1(x) : x \in M_2 \}$ and in order to obtain an appropriate $\gamma_{1,1}$ we need to study the limits of the sequences $\{x_n\} \subset M_2$ such that $\lim_n g_1(x_n) = \gamma_1$. Define

$$F'_2 = \{ T \in [g_1, g_2]^\ast : |T| = 1 \quad \text{and} \quad T(g_1) = \gamma_1 \}.$$ 

From Lemma 2.2 we deduce that the cardinal of the set $F'_2$ is at most two. Furthermore, since $|\cdot|^\ast$ is strictly convex, the cardinal of the set

$$N'_2 = \{ g \in S^\ast \cap [g_1, g_2] : T(g) = 1 \quad \text{for some} \quad T \in F'_2 \}$$

is at most two.

Let us take any sequence $\{x_n\} \subset M_2$ with $\lim_n g_1(x_n) = \gamma_1$. Consider every $x_n$ as an element of $X^{**}$ and denote by $x_n^\ast$ its restriction to $[g_1, g_2]$. Recall that if $x_n \in M_2$, then $D(x_n) \in S^\ast \cap [g_1, g_2]$, for every $n \in \mathbb{N}$. Moreover, the sequence of restrictions $\{x_n^\ast\} \subset [g_1, g_2]^\ast$ satisfies that

$$1 = |x_n| \geq |x_n^\ast| = \max\{x_n(h) : h \in S^\ast \cap [g_1, g_2] \} \geq x_n^\ast(D(x_n)) = D(x_n)(x_n) = 1,$$

for every $n \in \mathbb{N}$. Thus, there is a subsequence $\{x_{n_j}\}$ converging to an element $T \in [g_1, g_2]^\ast$ with $|T| = 1$. Since $\lim_j g_1(x_{n_j}) = \gamma_2$ and $\lim_j x_{n_j}(g_1) = \gamma_1$, then $T(g_1) = \gamma_1$ and this implies that $T \in F'_2$. Furthermore, if $g \in N'_2$ and $T(g) = 1$, then $\lim_j x_{n_j}(g) = 1$. In addition, $T(g_2) = \lim_j x_{n_j}(g_2) = \lim_j g_2(x_{n_j}) \geq \gamma_2$. Then, from condition (2.3), we deduce that $T(g_2) > \gamma_2$. Let us define

$$F_2 = \{ T \in F'_2 : \text{there is a sequence} \quad \{x_n\} \subset M_2 \quad \text{with} \quad \lim_n x_n = T \quad \text{and} \quad \lim_n x_n(g_1) = \gamma_1 \},$$

$$N_2 = \{ g \in N'_2 : T(g) = 1 \quad \text{for some} \quad T \in F_2 \}.$$ 

Select a real number $\gamma'_2$ satisfying $\gamma_2 < \gamma'_2 < \min\{T(g_2) : T \in F_2\}$ (recall that $F_2$ is finite). Let us prove the following Fact.

**Fact 2.3.**

1. There are numbers $0 < t_2 < l_2 < 1$ such that for every $g \in N_2$, the slices

$$O_g := \{ x \in S : g(x) > l_2 \} \quad \text{and} \quad B_g := \{ x \in S : g(x) > t_2 \}$$

satisfy that

$$O_g \subset B_g \subset \{ x \in S : g_1(x) < \gamma'_{1,1}, \ g_2(x) > \gamma'_2 \} \quad \text{and}$$

$$\text{dist}(B_g, B_{g'}) > 0, \quad \text{whenever} \quad g, g' \in N_2, \ g \neq g'.$$
(2) There is \( \gamma_{1,1} \in (\gamma_1, \gamma_{1,1}) \), such that if \(|x| \in M_2 \) and \( g_1(x) < \gamma_{1,1} \), then \( x \in O_g \), for some \( g \in N_2 \).

**Proof of Fact 2.3.**

(1) First, if \( X \) is reflexive, we know that for every \( g \in N_2 \) there is \( x_g \in S \) such that \( D(x_g) = g \). Since \( x_g(g) = 1 \) and \( |\cdot|^* \) is Gâteaux smooth, then \( x_g \in F_2 \). This implies that \( x_g(g_1) = \gamma_1 < \gamma_{1,1} \) and \( x_g(g_2) > \gamma_2' \). Hence, \( x_g \in \{ x \in S : g_1(x) < \gamma_{1,1}, g_2(x) > \gamma_2' \} \). Now, since the norm \( |\cdot| \) is LUR and \( D(x_g) = g \), the functional \( g \) strongly exposes \( S \) at the point \( x_g \). Taking into account that \( N_2 \) is finite we can hence obtain real numbers \( 0 < t_2 < l_2 < 1 \) and slices \( O_g \) and \( B_g \) satisfying conditions \( \mathcal{L}_1 \) and \( \mathcal{L}_5 \), for every \( g \in N_2 \).

Now consider a non reflexive Banach space \( X \). Let us first prove \( \mathcal{L}_4 \). Assume, on the contrary, that there is a point \( g \in N_2 \) and there is a sequence \( \{ y_n \} \subset S \) satisfying \( g(y_n) > 1 - \frac{1}{n} \) with either \( g_1(y_n) \geq \gamma_{1,1} \) or \( g_2(y_n) \leq \gamma_2' \) for every \( n \in \mathbb{N} \). Since \( g \in N_2 \) there is a sequence \( \{ x_n \} \subset M_2 \) with \( \lim_n g_1(x_n) = \gamma_1 \), \( \lim_n g_2(x_n) > \gamma_2' \) and \( \lim_n g(x_n) = 1 \). In particular,

\[
g(x_n) + 1 - \frac{1}{n} < g \left( \frac{x_n + y_n}{2} \right) \leq \left| \frac{x_n + y_n}{2} \right| \leq 1,
\]

and thus \( \lim_n \left| \frac{x_n + y_n}{2} \right| = 1 \). Recall that, in this case the norm \( |\cdot| \) is WUR, and hence \( x_n \to y_n \) weakly converges to zero. This last assertion gives a contradiction since either \( \limsup_n g_1(x_n - y_n) \leq \gamma_1 - \gamma_{1,1} < 0 \) or \( \liminf_n g_2(x_n - y_n) \geq \lim_n g_2(x_n) - \gamma_2' > 0 \). Therefore we can find real numbers \( 0 < t_2 < l_2 < 1 \) and slices \( O_g \) and \( B_g \) for every \( g \in N_2 \), satisfying condition \( \mathcal{L}_4 \). In order to obtain \( \mathcal{L}_6 \) we just need to modify \( t_2 \) and \( l_2 \) and select them close enough to 1. Indeed, assume on the contrary, that there are sequences \( \{ y_n \} \subset S \) and \( \{ z_n \} \subset S \) and \( g, g' \in N_2 \), \( g \neq g' \), such that \( \lim_n g(y_n) = 1 \), \( \lim_n g'(z_n) = 1 \) and \( \lim_n |y_n - z_n| = 0 \). Then,

\[
g(y_n) + g'(z_n) \leq \left( g + g' \right)(y_n) + g'(z_n - y_n) \leq \left( g + g' \right)(y_n) + |z_n - y_n| \leq \frac{|g + g'|^*}{2} + \frac{|z_n - y_n|}{2} \leq 1 + \frac{|z_n - y_n|}{2}.
\]

Since the limit of the first and last terms in the above chain of inequalities is 1, we deduce that \( |g + g'|^* = 2 \). Since the norm \( |\cdot|^* \) is strictly convex, we deduce that \( g = g' \), a contradiction.

(2) Assume, on the contrary, that for every \( n \in \mathbb{N} \), there is \( x_n \in M_2 \) with \( g_1(x_n) \leq \gamma_1 + \frac{1}{n} \) and \( \{ x_n : n \in \mathbb{N} \} \cap \left( \bigcup_{g \in N_2} O_g \right) = \emptyset \). Since \( \lim_n g_1(x_n) = \gamma_1 \) and \( \{ x_n \} \subset M_2 \), from the comments preceding Fact \( \mathcal{L}_3 \) we know that there is a subsequence \( \{ x_{n_i} \} \) and \( g \in N_2 \) satisfying that \( \lim_j g(x_{n_j}) = 1 \), which is a contradiction. This finishes the proof of Fact \( \mathcal{L}_3 \). \( \square \)

If \( U_1 \cap R_2 = \emptyset \), we can select as \( \gamma_{1,1} \) any number in \( (\gamma_1, \gamma_{1,1}) \). Now, we define,

\[
h_2 : S^+ \to \mathbb{R},
\]

\[
h_2 = \varphi_2(g_2) \varphi_{1,1}(g_1)
\]
where $\varphi_2$ and $\phi_{1,1}$ are $C^\infty$ functions on $\mathbb{R}$ satisfying:

\[
\begin{align*}
\varphi_2(t) &= 0 \quad \text{if} \quad t \leq \gamma_2 \\
\varphi_2(1) &= 1 \\
\varphi_2'(t) &= 0 \quad \text{if} \quad t > \gamma_2,
\end{align*}
\]

and

\[
\begin{align*}
\phi_{1,1}(t) &= 1 \quad \text{if} \quad t \leq \frac{\gamma_1+\gamma_1}{2} \\
\phi_{1,1}(t) &= 0 \quad \text{if} \quad t \geq \gamma_1, \\
\phi_{1,1}'(t) &= 0 \quad \text{if} \quad t \in (\frac{\gamma_1+\gamma_1}{2}, \gamma_1,1).
\end{align*}
\]

Notice that the interior of the support of $h_2$ is the open set

\[U_2 = \{ x \in S : g_1(x) < \gamma_1, g_2(x) > \gamma_2 \}.
\]

Select one point $x_2 \in U_2$, a real number $a_2 \in \mathbb{R}^+$ with $|a_2 - F(x_2)| < \varepsilon$ and define the auxiliary function

\[
\begin{align*}
r_2 : S^+ &\longrightarrow \mathbb{R}, \\
r_2 &= s_2g_2 + (1 - s_2g_2(x_2)),
\end{align*}
\]

where we have selected $s_2$ so that $s_2a_2 > 0$ and $|s_2|$ is small enough so that the oscillation of $r_2$ on $U_2$ is less than $\frac{\varepsilon}{|a_2|}$. Notice that $r_2(x_2) = 1$.

Let us study the critical points $Z_2$ of the function

\[
\begin{align*}
H_2 : U_1 \cup U_2 &\longrightarrow \mathbb{R}, \\
H_2 &= \frac{a_1r_1h_1 + a_2r_2h_2}{h_1 + h_2}.
\end{align*}
\]

Let us prove that $Z_2 = \{ x \in U_1 \cup U_2 : H_2'(z) = 0 \}$ on $T_x$ can be included in a finite number of pairwise disjoint slices within $U_1 \cup U_2$ by splitting it conveniently into up to four sets.

First, if $x \in U_1 \setminus U_2$, we have that $H_2(x) = a_1r_1(x)$ and $H_2'(x) = H_2'(x)|_{T_x} = a_1s_1g_1|_{T_x} \equiv 0$ iff $D(x) = g_1$. Thus, $Z_2 \cap (U_1 \setminus U_2) \subseteq \{ z_1 \}$. Second, if $x \in U_2 \setminus U_1$, we have $H_2(x) = a_2r_2(x)$ and $H_2'(x) = H_2'(x)|_{T_x} = a_2s_2g_2|_{T_x} \equiv 0$ iff $D(x) = g_2$. Then, if $z_2 \in U_2 \setminus U_1$, $H_2$ has one critical point in $U_2 \setminus U_1$, namely $z_2$; in this case, since $g_1(z_2) \neq \gamma_1$, the point $z_2$ actually belongs to $U_2 \setminus U_1$.

Now, let us study the critical points of $H_2$ in $U_1 \cap U_2$. In order to simplify the notation, let us put $A_1 = \frac{b_1}{h_1 + h_2}$, and denote by $g_1'$ and $g_2'$ the restrictions $g_1|_{T_x}$ and $g_2|_{T_x}$, respectively, whenever we consider $H_2'(x)$ and $A_1'(x)$. Then, $H_2 = a_1r_1A_1 + a_2r_2(1 - A_1)$ and

\[
\begin{align*}
H_2' &= a_1s_1A_1g_1 + a_2s_2(1 - A_1)g_2 + (a_1r_1 - a_2r_2)A_1' \\
&= \sigma_{1,1}A_1g_1 + a_2s_2(1 - A_1)g_2 + (H_1 - a_2r_2)A_1'.
\end{align*}
\]
By computing $\Lambda'_1$, we obtain $\Lambda'_1 = \xi_{1,1} g_1 + \xi_{1,2} g_2$, where the coefficients $\xi_{1,1}$ and $\xi_{1,2}$ are continuous functions on $U_1 \cup U_2$ and have the following form,

$$\xi_{1,1} = \frac{\varphi'_1(g_1) h_2 - h_1 \varphi_2(g_2) \phi_{1,1}(g_1)}{(h_1 + h_2)^2},$$

$$\xi_{1,2} = \frac{-h_1 \varphi'_2(g_2) \phi_{1,1}(g_1)}{(h_1 + h_2)^2}.$$

Thus $H'_2 = \sigma_{2,1} g_1 + \sigma_{2,2} g_2$, where $\sigma_{2,1}$ and $\sigma_{2,2}$ are continuous functions on $U_1 \cup U_2$ and have the following form

$$(2.6) \quad \sigma_{2,1} = \sigma_{1,1} \Lambda_1 + (H_1 - a_2 r_2) \xi_{1,1},$$

$$\sigma_{2,2} = a_2 s_2 (1 - \Lambda_1) + (H_1 - a_2 r_2) \xi_{1,2}.$$ Notice that if $x \in U_1 \cap U_2$, then $\sigma_{2,1} > 0$, $a_2 s_2 > 0$, $\Lambda_1 > 0$, $1 - \Lambda_1 > 0$, $\xi_{1,1} > 0$ and $\xi_{1,2} < 0$. Therefore, on $U_1 \cap U_2$, the coefficient $\sigma_{2,1}$ is strictly positive whenever $H_1 - a_2 r_2 \geq 0$, and the coefficient $\sigma_{2,2}$ is strictly positive whenever $H_1 - a_2 r_2 \leq 0$. Since the vectors $g_1$ and $g_2$ are l.i., if $x \in U_1 \cap U_2$ and $H'_2(x) : T_x \to \mathbb{R}$ is identically zero, there is necessarily $g \neq 0$ with $D(x) = g(\sigma_{2,1}(x) g_1 + \sigma_{2,2}(x) g_2)$. Thus, $D(x) \in [g_1, g_2]$.

The set $Z_2$ can be split into the disjoint sets $Z_2 = Z_1 \cup Z_{2,1} \cup Z_{2,2}$, where

$$Z_{2,1} = \begin{cases} \{z_2\}, & \text{if } z_2 \in U_2 \setminus U_1 \\ \emptyset, & \text{otherwise} \end{cases}$$

and $Z_{2,2}$ is a subset (possibly empty) within $U_1 \cap U_2 \cap D^{-1}([g_1, g_2])$. Now, let us check that $Z_{2,2} \subset \bigcup_{g \in N_2} O_g$. Indeed, if $x \in Z_{2,2}$, then $x \in U_1 \cap U_2 \subset U_1 \cup U'_2$, $D(x) \in [g_1, g_2]$ and $g_1(x) < \gamma_{1,1}$. This implies, according to Fact 2.3, that $x \in \bigcup_{g \in N_2} O_g$.

In the case when $Z_{2,1} = \{z_2\}$ and $z_2 \not\in \bigcup_{g \in N_2} O_g$, we select, if necessary, a larger $t_2$ with $t_2 < 1$, so that $z_2 \not\in \bigcup_{g \in N_2} \overline{O}_g$. Since the norm $|\cdot|$ is LUR and $D(z_2) = g_2$, the functional $g_2$ strongly exposes $S$ at the point $z_2$ and we may select numbers $0 < t'_2 < l'_2 < 1$ and open slices, which are neighborhoods of $z_2$, defined by

$$O_{g_2} := \{x \in S : g_2(x) > l'_2\} \quad \text{and} \quad B_{g_2} := \{x \in S : g_2(x) > t'_2\},$$

satisfying $O_{g_2} \subset B_{g_2} \subset \{x \in S : g_1(x) < \gamma'_{1,1}, g_2(x) > \gamma'_2\}$ and $\text{dist}(B_{g_2}, B_g) > 0$ for every $g \in N_2$. In this case, we define $\Gamma_2 = N_2 \cup \{g_2\}$.

Now, if $Z_{2,1} = \{z_2\} \subset \bigcup_{g \in N_2} \overline{O}_g$, we select, if necessary, a smaller constant $l_2$, with $0 < t_2 < l_2 < 1$, so that $Z_{2,1} = \{z_2\} \subset \bigcup_{g \in N_2} O_g$. In this case, and also when $Z_{2,1} = \emptyset$, we define $\Gamma_2 = N_2$.

Notice that, in any of the cases mentioned above, Fact 2.3 clearly holds for the (possibly) newly selected real numbers $t_2$ and $l_2$.

Notice that the distance between any two sets $B_g$, $B_{g'}$, $g, g' \in \Gamma_1 \cup \Gamma_2$, $g \neq g'$, is positive. Moreover, $Z_1 \subset O_{g_1} \subset B_{g_1} \subset U_1 = R_1$, and $Z_{2,1} \cup Z_{2,2} \subset \bigcup_{g \in \Gamma_2} O_g \subset \bigcup_{g \in \Gamma_2} B_g \subset U'_2 \subset R_2$. Therefore, $Z_2 = Z_1 \cup Z_{2,1} \cup Z_{2,2} \subset \bigcup_{g \in \Gamma_1 \cup \Gamma_2} O_g \subset \bigcup_{g \in \Gamma_1 \cup \Gamma_2} B_g \subset U_1 \cup U_2 = R_1 \cup R_2$. In addition, we have $\text{dist}(\bigcup_{g \in \Gamma_1 \cup \Gamma_2} B_g, (U_1 \cup U'_2)^o) > 0$.

It is worth remarking that $H'_2 = \sigma_{2,1} g_1 + \sigma_{2,2} g_2$ in $U_1 \cup U_2$, where $\sigma_{2,1}$ and $\sigma_{2,2}$ are continuous functions and at least one of the coefficients $\sigma_{2,1}(x)$, $\sigma_{2,2}(x)$ is strictly positive,
for every \( x \in U_1 \cup U_2 \). Moreover, \( \sigma_{2,1}(x) = 0 \) whenever \( x \not\in U_1 \), and \( \sigma_{2,2}(x) = 0 \) whenever \( x \not\in U_2 \).

In order to clarify the construction in the general case, let us also explain in detail the construction of the function \( h_3 \) and locate the critical points of the function \( H_3' \).

\( \diamond \) For \( j = 3 \), let us denote by \( y_3 \in S \) the point satisfying \( f_3(y_3) = 1 \). If \( \{g_1, g_2, f_3\} \) are linearly dependent, or if \( g_1(y_3) = \gamma_1 \), or if \( g_2(y_3) = \gamma_2 \), we can use the density of the norm attaining functionals (Bishop-Phelps Theorem) and the continuity of \( \gamma \) and \( M \) lineally dependent, or if \( g_3 = f_3 \) and \( g_3(y_3) = y_3 \). Then, we apply Lemma 2.1 to the l.i. vectors, \( \{g_1, g_2, g_3\} \) and the real numbers \( \gamma_1 \in (0, 1) \), \( \gamma_2 \in (0, 1) \) and obtain \( \gamma_3 \in (0, 1) \) close enough to \( \nu_3 \) so that

\[
S_3 = \{ x \in S : f_3(x) > \delta_3 \} \subset \{ x \in S : g_3(x) > \gamma_3 \} \subset \{ x \in S : f_3(x) > \delta_3 \} = P_3.
\]

(2.7) \( \{ T \in [g_1, g_2, g_3]^* : T(g_1) = \gamma_1, T(g_2) = \gamma_2, T(g_3) = \gamma_3 \) and \( |T| = 1 \} = \emptyset, \)

(2.8) \( \{ T \in [g_1, g_3]^* : T(g_1) = \gamma_1, T(g_3) = \gamma_3 \) and \( |T| = 1 \} = \emptyset, \)

(2.9) \( \{ T \in [g_2, g_3]^* : T(g_2) = \gamma_2, T(g_3) = \gamma_3 \) and \( |T| = 1 \} = \emptyset. \)

Select \( \gamma_{2,1} \in (\gamma_2, \gamma_2') \) and define

\[
R_3 = \{ x \in S : g_3(x) > \gamma_3 \} \quad \text{and} \quad U'_3 = \{ s \in S : g_1(x) < \gamma_{1,2}', g_2(x) < \gamma_{2,1}' \} \quad \text{and} \quad g_3(x) > \gamma_3 \},
\]

where \( \gamma_{1,2}' \) is a number in \( (\gamma_1, \frac{\gamma_1 + \gamma_{1,2}}{2}) \).

Notice that \( \text{dist}(B_g, U'_3) > 0 \) for every \( g \in \Gamma_1 \cup \Gamma_2 \). Assume that \( R_3 \cap (U_1 \cup U_2) \neq \emptyset, \)

and consider the sets

\[
M_{3,1} = \{ x \in (U_1 \cap U'_3) \setminus U_2 : D(x) \in [g_1, g_3] \},
\]

\[
M_{3,2} = \{ x \in (U_2 \cap U'_3) \setminus U_1 : D(x) \in [g_2, g_3] \},
\]

\[
M_{3,1,2} = \{ x \in U_1 \cap U_2 \cap U'_3 : D(x) \in [g_1, g_2, g_3] \},
\]

and \( M_3 = M_{3,1} \cup M_{3,2} \cup M_{3,1,2} \).

In the case that \( M_3 = \emptyset \), we select as \( \gamma_{2,1} \) any point in \( (\gamma_2, \gamma_{2,1}') \) and \( \gamma_{1,2} \) any point in \( (\gamma_1, \gamma_{1,2}') \).

In the case that \( M_3 \neq \emptyset \) and \( \text{dist}(M_3, (U_1 \cup U_2)^c) > 0 \) we can easily find \( \gamma_{2,1} \in (\gamma_2, \gamma_{2,1}') \) and \( \gamma_{1,2} \in (\gamma_1, \gamma_{1,2}') \) with \( M_3 \subset \{ x \in S : g_1(x) > \gamma_{1,2} \} \cup \{ x \in S : g_2(x) > \gamma_{2,1} \} \).

In the case that \( \text{dist}(M_3, (U_1 \cup U_2)^c) = 0 \) and in order to obtain suitable constants \( \gamma_{2,1} \) and \( \gamma_{1,2} \), we need to study the limits of the sequences \( \{x_n\} \subset M_3 \) such that
\[ \lim_n \text{dist}(x_n, (U_1 \cup U_2)^c) = 0. \]

Define the sets
\[
\begin{align*}
F'_{3,i} &= \{ T \in [g_i, g_3]^* : T(g_i) = \gamma_i \text{ and } |T| = 1 \} \quad \text{for } i = 1, 2, \\
F'_{3,1,2} &= \{ T \in [g_1, g_2, g_3]^* : T(g_1) = \gamma_1, T(g_2) = \gamma_2 \text{ and } |T| = 1 \},
\end{align*}
\]
and
\[
\begin{align*}
N'_{3,i} &= \{ g \in S^* \cap [g_i, g_3] : T(g) = 1 \text{ for some } T \in F'_{3,i} \} \quad \text{for } i = 1, 2, \\
N'_{3,1,2} &= \{ g \in S^* \cap [g_1, g_2, g_3] : T(g) = 1 \text{ for some } T \in F'_{3,1,2} \}.
\end{align*}
\]

Since the norm \(| \cdot |^*\) is Gâteaux smooth, we apply Lemma 2.2 to the finite dimensional space \([g_1, g_2, g_3]\) and the restriction of the norm \(| \cdot |^*\) to \([g_1, g_2, g_3]\) (which is a differentiable norm on the space \([g_1, g_2, g_3]\)), and deduce that the cardinal of any of the sets \(F'_{3,i}, F'_{3,1,2}\) is at most two. Furthermore, from the strict convexity of the norm \(| \cdot |^*\) we obtain that the cardinal of any of the sets \(N'_{3,i}\) and \(N'_{3,1,2}\) is at most two. Let us consider, for \(i = 1, 2\), the norm-one extensions to \([g_1, g_2, g_3]\) of the functionals \(F'_{3,i}\), that is,
\[
F''_{3,i} = \{ T \in [g_1, g_2, g_3]^* : T|_{[g_i, g_3]} \in F'_{3,i} \text{ and } |T| = 1 \}.
\]

Since the norm \(| \cdot |^*\) is Gâteaux smooth, for every \(G \in F''_{3,i}\) there is exactly one norm-one extension \(T\) to \([g_1, g_2, g_3]\). Therefore the cardinal of the set \(F''_{3,i}\) is at most two. Hence the sets \(F''_3 := F''_{3,1} \cup F''_{3,2} \cup F''_{3,1,2}\) and \(N''_3 := N'_{3,1} \cup N'_{3,2} \cup N'_{3,1,2}\) are finite. In addition, as a consequence of the equalities (2.8), (2.9), and (2.10), we deduce that \(T(g_3) \neq \gamma_3\) for every \(T \in F''_3\). Indeed, if \(T \in F''_{3,1,2}\) the assertion follows immediately from (2.8). If \(T \in F''_{3,i}\) for some \(i \in \{1, 2\}\), then \(T|_{[g_i, g_3]} \in F'_{3,i}\), that is, \(|T|_{[g_i, g_3]}| = 1\) and \(T(g_i) = \gamma_i\). From (2.8) for \(i = 1\), and (2.9) for \(i = 2\), we obtain that \(T(g_3) \neq \gamma_3\). We can restrict our study to one of the following kind of sequences:

1. Fix \(i \in \{1, 2\}\). Consider any sequence \(\{x_n\} \subset M_{3,i}\) such that \(\lim_n \text{dist}(x_n, (U_1 \cup U_2)^c) = 0\). Then, it easily follows that \(\lim_n g_i(x_n) = \gamma_i\). Indeed,
   - if \(\{x_n\} \subset M_{3,1}\), then in particular \(\{x_n\} \subset U_1 = R_1\). Therefore, \(\text{dist}(x_n, (U_1 \cup U_2)^c) \geq \text{dist}(x_n, R_1^c)\). Thus, \(\lim_n \text{dist}(x_n, R_1^c) = 0\) and this implies that \(\lim_n g_1(x_n) = \gamma_1\);
   - if \(\{x_n\} \subset M_{3,2}\), then in particular \(\{x_n\} \subset U_2 = R_2\). Recall that \(U_1 \cup U_2 = R_1 \cup R_2\). Therefore, \(\text{dist}(x_n, (U_1 \cup U_2)^c) \geq \text{dist}(x_n, R_2^c)\). Thus, \(\lim_n \text{dist}(x_n, R_2^c) = 0\) and this implies that \(\lim_n g_2(x_n) = \gamma_2\).

Now, let us take any sequence \(\{x_n\} \subset M_{3,i}\) such that \(\lim_n g_i(x_n) = \gamma_i\). Consider every \(x_n\) as an element of \(X^*\) and denote by \(x_n\) its restriction to \([g_1, g_2, g_3]\). Recall that \(D(x_n) \in S^* \cap [g_i, g_3]\) for every \(n \in \mathbb{N}\). Then, the sequence of restrictions \(\{x_n\} \subset [g_1, g_2, g_3]^*\) satisfies that
\[
1 = |x_n| \geq |x_n| \geq |x_n|_{[g_i, g_3]} = \max \{ x_n(h) : h \in S^* \cap [g_i, g_3] \}
\]
\[
\geq x_n(D(x_n)) = D(x_n)(x_n) = 1,
\]
for every \(n \in \mathbb{N}\). Thus, there is a subsequence \(\{x_{n_j}\}\) converging to an element \(T \in [g_1, g_2, g_3]^*\) with \(|T| = |T|_{[g_i, g_3]} = 1\). Since \(\lim_{j} g_i(x_{n_j}) = \lim_j x_{n_j}(g_i) = \gamma_i\), we have that \(T(g_i) = \gamma_i\) and this implies that \(T|_{[g_i, g_3]} \in F'_{3,i}\) and \(T \in F''_{3,i}\). Furthermore, if \(g \in N'_{3,i}\) and \(T(g) = 1\), then \(\lim_j x_{n_j}(g) = 1\). In addition, \(T(g_3) = \lim_j x_{n_j}(g_3) = \lim_j g_3(x_{n_j}) \geq \gamma_3\) because \(\{x_{n_j}\} \subset U_3\). Then, from condition (2.9)
if $i = 1$ and condition (2.7), if $i = 2$, we deduce that $T(g_3) > \gamma_3$. Finally, let us check that $T(g_s) = \lim_n x_n(g_s) \leq \gamma_s$, where $s \in \{1, 2\}$ and $s \neq i$:

- if $i = 1$, the sequence $\{x_n\} \subset M_{3,1}$ and thus $\{x_n\} \subset (U_1 \cap U_3') \setminus U_2$. In particular $\{x_n\} \subset U_3'$ and $g_1(x_n) < \gamma_{1,2} < \gamma_{1,1}$ for every $j \in \mathbb{N}$. Therefore, if $x_{n_j} \notin U_2$ for all $j$, we must have $x_{n_j}(g_2) = g_2(x_{n_j}) \leq \gamma_2$ for every $j \in \mathbb{N}$.
- if $i = 2$, the sequence $\{x_n\} \subset M_{3,2}$ and thus $\{x_n\} \subset (U_2 \cap U_3') \setminus U_1$. In particular $x_{n_j} \notin U_1 = R_1$, for every $j \in \mathbb{N}$ and this implies $x_{n_j}(g_1) = g_1(x_{n_j}) \leq \gamma_1$ for every $j \in \mathbb{N}$.

(2) Consider a sequence $\{x_n\} \subset M_{3,1,2}$, such that $\lim_n \text{dist}(x_n,(U_1 \cup U_2)^c) = 0$. Then, it easily follows that $\lim_n g_i(x_n) = \gamma_i$ for $i = 1, 2$. Indeed, $U_1 \cup U_2 = R_1 \cup R_2$ and then $\text{dist}(x_n,(R_1 \cup R_2)^c) \geq \text{dist}(x_n,R_i^c)$ for every $n \in \mathbb{N}$ and $i = 1, 2$. Hence $\lim_n \text{dist}(x_n,R_i^c) = 0$. Since $\{x_n\} \subset R_i$, for $i = 1, 2$, we obtain that $\lim_n g_i(x_n) = \gamma_i$, for $i = 1, 2$.

Now, let us take any sequence $\{x_n\} \subset M_{3,1,2}$ such that $\lim_n g_i(x_n) = \gamma_i$, for every $i = 1, 2$. Consider every $x_n$ as an element of $X^*$ and denote by $x_n$ its restriction to $[g_1, g_2, g_3]$. Then, the sequence of restrictions $\{x_n\} \subset [g_1, g_2, g_3]^*$ satisfies that

\[ 1 = |x_n| \geq |x_n| = \max\{x_n(h) : h \in S^* \cap [g_1, g_2, g_3]\} \geq x_n(D(x_n)) = D(x_n)(x_n) = 1, \]

for every $n \in \mathbb{N}$. Thus, there is a subsequence $\{x_{n_i}\}$ converging to an element $T \in [g_1, g_2, g_3]^*$ with $|T| = 1$. Since $\lim_j g_i(x_{n_j}) = \lim_j x_{n_j}(g_i) = \gamma_i$ for $i = 1, 2$, then $T(g_i) = \gamma_i$ for $i = 1, 2$, and this implies that $T \in F_{3,1,2}$. Furthermore, if $g \in N_{3,1,2}'$ and $T(g) = 1$, then $\lim_j x_{n_j}(g) = 1$. In addition, $T(g_3) = \lim_j x_{n_j}(g_3) = \lim_j g_3(x_{n_j}) \geq \gamma_3$ because $\{x_{n_j}\} \subset U_3$. Then, from condition (2.7), we deduce that $T(g_3) > \gamma_3$.

Let us define, for $i = 1, 2$,

\[ F_{3,i} = \{ T \in F_{3,i}' : \text{there is } \{x_n\} \subset M_{3,i} \text{ with } \lim_n x_n(g_i) = \gamma_i, \text{ and } \lim_n x_n = T \}, \]

\[ F_{3,1,2} = \{ T \in F_{3,1,2}' : \text{there is } \{x_n\} \subset M_{3,1,2} \text{ with } \lim_n x_n(g_1) = \gamma_1, \lim_n x_n(g_2) = \gamma_2 \]

and

\[ F_3 = F_{3,1} \cup F_{3,2} \cup F_{3,1,2}. \]

Select a real number $\gamma'_3$ satisfying $\gamma_3 < \gamma'_3 < \min\{T(g_3) : T \in F_3\}$ (recall that $F_3$ is finite), and define,

\[ N_{3,i} = \{ g \in N_{3,i}' : \text{there is } T \in F_{3,i} \text{ with } T(g) = 1 \}, \quad \text{for } i = 1, 2, \]

\[ N_{3,1,2} = \{ g \in N_{3,1,2}' : \text{there is } T \in F_{3,1,2} \text{ with } T(g) = 1 \}, \]

and $N_3 = N_{3,1} \cup N_{3,2} \cup N_{3,1,2}$. Let us prove the following Fact.

**Fact 2.4.** (1) There are numbers $0 < t_3 < l_3 < 1$ such that for every $g \in N_3$, the slices

\[ O_g := \{ x \in S : g(x) > l_3 \} \quad \text{and} \quad B_g := \{ x \in S : g(x) > t_3 \} \]
satisfy that
\begin{align}
(2.11) \quad & O_g \subset B_g \subset \{ x \in S : g_1(x) < \gamma_{1,2}, g_2(x) < \gamma_{2,1}, g_3(x) > \gamma_3' \} \quad \text{and} \\
(2.12) \quad & \text{dist}(B_g, B_{g'}) > 0, \quad \text{whenever } g, g' \in N_3, \ g \neq g'.
\end{align}

(2) There are numbers \( \gamma_{1,2} \in (\gamma_1, \gamma_{1,2}') \) and \( \gamma_{2,1} \in (\gamma_2, \gamma_{2,1}') \) such that if \( x \in M_3 \), \( g_1(x) < \gamma_{1,2} \) and \( g_2(x) < \gamma_{2,1} \), then \( x \in O_g \), for some \( g \in N_3 \).

**Proof of Fact 2.24**  
(1) First, if \( X \) is reflexive, we know that for every \( g \in N_3 \) there is \( x_g \in S \) such that \( D(x_g) = g \). Let us study the three possible cases:

- If \( g \in F_{3,1} \), denote by \( x_g \) the restriction of \( x_g \) to \([g_1, g_2, g_3]\). Since \( x_g(g) = 1 \) and \( |\cdot|^* \) is Gâteaux smooth, then \( x_g = T \) for some \( T \in F_{3,1} \). This implies that \( x_g(g_1) = \gamma_1 < \gamma_{1,2}' \), \( x_g(g_3) > \gamma_3' \) and \( x_g(g_2) < \gamma_{2,1}' \). Hence, \( x_g \in \{ x \in S : g_1(x) < \gamma_{1,2}' \), \( g_2(x) > \gamma_{2,1}' \) and \( g_3(x) > \gamma_3' \} \).

- If \( g \in F_{3,2} \), denote by \( x_g \) the restriction of \( x_g \) to \([g_1, g_2, g_3]\). Since \( x_g(g) = 1 \) and \( |\cdot|^* \) is Gâteaux smooth, then \( x_g = T \) for some \( T \in F_{3,2} \). This implies that \( x_g(g_2) = \gamma_2 < \gamma_{2,1}' \), \( x_g(g_3) > \gamma_3' \) and \( x_g(g_1) < \gamma_{1,2}' \). Hence, \( x_g \in \{ x \in S : g_1(x) < \gamma_{1,2}' \), \( g_2(x) > \gamma_{2,1}' \) and \( g_3(x) > \gamma_3' \} \).

- If \( g \in F_{3,1,2} \), denote by \( x_g \) the restriction of \( x_g \) to \([g_1, g_2, g_3]\). Since \( x_g(g) = 1 \) and \( |\cdot|^* \) is Gâteaux smooth, then \( x_g = T \) for some \( T \in F_{3,1,2} \). This implies that \( x_g(g_1) = \gamma_1 < \gamma_{1,2}' \), \( x_g(g_2) = \gamma_2 < \gamma_{2,1}' \) and \( x_g(g_3) > \gamma_3' \). Hence, \( x_g \in \{ x \in S : g_1(x) < \gamma_{1,2}' \), \( g_2(x) > \gamma_{2,1}' \) and \( g_3(x) > \gamma_3' \} \).

Now, since the norm \( |\cdot| \) is LUR and \( D(x_g) = g \), the functional \( g \) strongly exposes \( S \) at the point \( x_g \) for every \( g \in N_3 \). Since \( N_3 \) is finite, we can hence obtain real numbers \( 0 < t_3 < l_3 < 1 \) and slices \( O_g \) and \( B_g \) for every \( g \in N_3 \), satisfying conditions (2.11) and (2.12).

Now consider a non reflexive Banach space \( X \). Let us first prove (2.11). Assume, on the contrary, that there is a point \( g \in N_3 \) and there is a sequence \( \{ y_n \} \subset S \) satisfying \( g(y_n) > 1 - \frac{1}{n} \) and such that \( g_1(y_n) \geq \gamma_{1,2}' \), or \( g_2(y_n) \geq \gamma_{2,1}' \), or \( g_3(y_n) \leq \gamma_3' \) for every \( n \in \mathbb{N} \). If \( g \in N_3 \) there is a sequence \( \{ x_n \} \subset M_3 \) with \( \lim_n g_i(x_n) \leq \gamma_i \), for \( i = 1, 2 \), \( \lim_n g_3(x_n) > \gamma_3' \) and \( \lim_n g(x_n) = 1 \). In particular,
\[
\frac{g(x_n) + \frac{1}{n}}{2} \leq g \left( \frac{x_n + y_n}{2} \right) \leq \frac{|x_n + y_n|}{2} \leq 1,
\]
and thus \( \lim_n \frac{|x_n + y_n|}{2} = 1 \). Recall that in the non reflexive case, the norm \( |\cdot| \) is WUR, and then \( x_n - y_n \to 0 \) (weakly converges to zero). This last assertion gives a contradiction since we have \( \limsup_n g_1(x_n - y_n) \leq \gamma_1 - \gamma_{1,2}' < 0 \) or \( \limsup_n g_2(x_n - y_n) \leq \gamma_2 - \gamma_{2,1}' < 0 \) or \( \liminf_n g_3(x_n - y_n) \geq \lim_n g_3(x_n - \gamma'_3) > 0 \). Therefore we can find real numbers \( 0 < t_2 < t_3 < 1 \) and slices \( O_g \) and \( B_g \) for every \( g \in N_3 \), satisfying condition (2.11). The proof of (2.12) is the same as the one given in Fact 2.24 where the only property we need is the strict convexity of \( |\cdot|^* \).

(2) Assume, on the contrary, that for every \( n \in \mathbb{N} \), there is \( x_n \in M_3 \) with \( g_i(x_n) \leq \gamma_i + \frac{1}{n} \), for \( i = 1, 2 \) and \( \{ x_n : n \in \mathbb{N} \} \cap (\cup_{g \in N_3} O_g) = \emptyset \). Then, there is a subsequence of \( \{ x_n \} \), which we keep denoting by \( \{ x_n \} \), such that \( \{ x_n \} \subset M_{3,1} \), or \( \{ x_n \} \subset M_{3,2} \), or
\( \{x_n\} \subset M_{3,1.2} \). In the first case, \( \lim_{n} g_1(x_n) = \gamma_1 \). In the second case, \( \lim_{n} g_2(x_n) = \gamma_2 \). In the third case, \( \lim_{n} g_i(x_n) = \gamma_i \), for every \( i = 1, 2 \). From the definition of \( F_3 \) and \( N_3 \) and the comments preceding Fact 2.4, we know that there is a subsequence \( \{x_{n_j}\} \) and \( g \in N_3 \) satisfying that \( \lim_j g(x_{n_j}) = 1 \), which is a contradiction. This finishes the proof of Fact 2.4. \( \Box \)

If \( R_3 \cap (U_1 \cup U_2) = \emptyset \) we may select as \( \gamma_{1,2} \) any number in \( (\gamma_1, \gamma'_{1,2}) \) and \( \gamma_{2,1} \) any number in \( (\gamma_2, \gamma'_{2,1}) \).

Now we define \( h_3 \) as follows:

\[
h_3 : S^+ \rightarrow \mathbb{R}
\]

\[
h_3 = \varphi_3(g_3) \phi_{2,1}(g_2) \phi_{1,2}(g_1),
\]

where \( \varphi_3, \phi_{2,1} \) and \( \phi_{1,2} \) are \( C^\infty \) functions on \( \mathbb{R} \) satisfying that

\[
\varphi_3(t) = 0 \quad \text{if} \quad t \leq \gamma_3
\]

\[
\varphi_3(1) = 1
\]

\[
\varphi'_3(t) > 0 \quad \text{if} \quad t > \gamma_3
\]

and

\[
\phi_{1,2}(t) = 1 \quad \text{if} \quad t \leq \frac{\gamma_1 + \gamma_{1,2}}{2}
\]

\[
\phi_{1,2}(t) = 0 \quad \text{if} \quad t \geq \gamma_{1,2}
\]

\[
\phi_{1,2}(t) < 0 \quad \text{if} \quad t \in \left(\frac{\gamma_1 + \gamma_{1,2}}{2}, \gamma_{1,2}\right)
\]

\[
\phi_{2,1}(t) = 1 \quad \text{if} \quad t \leq \frac{\gamma_1 + \gamma_{2,1}}{2}
\]

\[
\phi_{2,1}(t) = 0 \quad \text{if} \quad t \geq \gamma_{2,1}
\]

\[
\phi_{2,1}(t) < 0 \quad \text{if} \quad t \in \left(\frac{\gamma_1 + \gamma_{2,1}}{2}, \gamma_{2,1}\right)
\]

Clearly the interior of the support of \( h_3 \) is the set

\[
U_3 = \{ x \in S^+ : g_1(x) < \gamma_{1,2}, g_2(x) < \gamma_{2,1} \quad \text{and} \quad g_3(x) \geq \gamma_3 \}.
\]

Select one point \( x_3 \in U_3 \), a real number \( a_3 \in \mathbb{R}^* \) with \( |a_3 - F(x_3)| < \varepsilon \) and define the auxiliary function

\[
r_3 : S^+ \rightarrow \mathbb{R},
\]

\[
r_3 = s_3 g_3 + (1 - s_3 g_3(x_3)),
\]

where we have selected \( s_3 \) so that \( s_3 a_3 > 0 \) and \( |s_3| \) is small enough so that the oscillation of \( r_3 \) on \( U_3 \) is less than \( \frac{\varepsilon}{|a_3|} \). Notice that \( r_3(x_3) = 1 \).

Let us study the critical points \( Z_3 \) of the \( C^p \) smooth function

\[
H_3 : U_1 \cup U_2 \cup U_3 \rightarrow \mathbb{R},
\]

\[
H_3 = \frac{a_1 r_1 h_1 + a_2 r_2 h_2 + a_3 r_3 h_3}{h_1 + h_2 + h_3}.
\]

Let us prove that \( Z_3 := \{ x \in U_1 \cup U_2 \cup U_3 : H_3'(x) = 0 \ \text{on} \ T_x \} \) can be included in a finite number of disjoint slices within \( U_1 \cup U_2 \cup U_3 \) by splitting it conveniently into the (already defined) \( Z_1, Z_2 \) and up to four more disjoint sets within \( U_3 \), as the Figure 2 suggests.

The function \( H_3 \) can be written as \( H_3' = \sigma_{3,1} g_1 + \sigma_{3,2} g_2 + \sigma_{3,3} g_3 \), where \( \sigma_{3,i} \) are continuous and real functions on \( U_1 \cup U_2 \cup U_3 \) and \( g_i \) denotes the restriction \( g_i|_{T_x} \), \( i = 1, 2, 3 \), whenever we evaluate \( H_3(x) \) on \( T_x \).
Clearly $H_3$ and $H_3'$ restricted to $(U_1 \cup U_2) \setminus U_3$ coincide with $H_2$ and $H_2'$ respectively. Then, $Z_3 \setminus U_3 = Z_3' \setminus U_3 = Z_2$. Let us study the set $Z_3 \cap U_3$. First, if $x \in U_3 \setminus (U_1 \cup U_2)$, then $H_3(x) = a_3r_3(x)$ and $H_3'(x) = a_3r_3'(x) = a_3s_3g_3$. Therefore $H_3'(x) = a_3s_3g_3|_{U_3} \equiv 0$ if $D(x) = g_3$. If the point $z_3 \in U_3 \setminus (U_1 \cup U_2)$ then, $H_3$ has exactly one critical point in $U_3 \setminus (U_1 \cup U_2)$; in this case, $g_1(z_3) \neq \gamma_1$ and $g_2(z_3) \neq \gamma_2$, the point $z_3$ actually belongs to $U_3 \setminus (U_1 \cup U_2)$.

Now, let us study the critical points of $H_3$ in $U_3 \cap (U_1 \cup U_2)$. If we define $\Lambda_2 = \frac{h_1 + h_3}{h_1 + h_2 + h_3}$, then we can rewrite $H_3$ in $U_3 \cap (U_1 \cup U_2)$ as

$$H_3 = \frac{a_1r_1h_1 + a_2r_2h_2}{h_1 + h_2} + \frac{a_3r_3h_3}{h_1 + h_2 + h_3} = H_2 \Lambda_2 + a_3r_3(1 - \Lambda_2),$$

and

$$H_3' = H_2' \Lambda_2 + a_3s_3(1 - \Lambda_2)g_3 + (H_2 - a_3r_3)\Lambda_2'.$$

By computing $\Lambda_2$, we obtain $\Lambda_2 = \xi_{2,1}g_1 + \xi_{2,2}g_2 + \xi_{2,3}g_3$, where the coefficients $\xi_{2,1}$, $\xi_{2,2}$, and $\xi_{2,3}$ are continuous functions of the following form:

$$(2.13) \quad \xi_{2,1} = \frac{-\varphi_3(g_3)\phi_{2,1}(g_2)\phi_{1,2}(g_1)(h_1 + h_2) + h_3\varphi_1'(g_1) + h_3\varphi_2(g_2)\phi_{1,1}'(g_1)}{(h_1 + h_2 + h_3)^2};$$

$$\xi_{2,2} = \frac{-\varphi_3(g_3)\phi_{2,1}'(g_2)\phi_{1,2}(g_1)(h_1 + h_2) + h_3\varphi_2'(g_2)\phi_{1,1}(g_1)}{(h_1 + h_2 + h_3)^2};$$

$$\xi_{2,3} = \frac{-\varphi'_3(g_3)\phi_{2,1}(g_2)\phi_{1,2}(g_1)(h_1 + h_2)}{(h_1 + h_2 + h_3)^2}.$$

Since $g_1(x) < \gamma_1 < \frac{\gamma_1 + \gamma_1}{2}$ for every $x \in U_3$, we have that $\phi_{1,1}'(g_1(x)) = 0$ for every $x \in U_3$, and we can drop the term $h_3\varphi_2(g_2)\phi_{1,1}'(g_1)$ in the above expression of $\xi_{2,1}$. Thus, if $x \in U_3 \cap (U_1 \cup U_2)$, the coefficients $\sigma_{3,1}$, $\sigma_{3,2}$, $\sigma_{3,3}$ for $H_3'$ have the following form:

$$\sigma_{3,1} = \xi_{2,1} = \frac{a_3s_3(1 - \Lambda_2) + (H_2 - a_3r_3)\xi_{2,1}}{a_3s_3 > 0, \quad \Lambda_2 > 0, \quad 1 - \Lambda_2 > 0, \quad \xi_{2,1} \geq 0, \quad \xi_{2,2} \geq 0, \quad \xi_{2,1} + \xi_{2,2} > 0 \quad \text{and} \quad \xi_{2,3} < 0 \quad \text{on} \quad U_3 \cap (U_1 \cup U_2).$$

Therefore, if $H_2 - a_3r_3 \leq 0$, the coefficient $\sigma_{3,3} > 0$. When $H_2 - a_3r_3 \geq 0$ and $\sigma_{3,2} > 0$, we have that $\sigma_{3,2} > 0$. Finally, when $H_2 - a_3r_3 \geq 0$ and $\sigma_{3,2} > 0$, we have $\sigma_{3,1} > 0$ (recall that for every $x \in U_1 \cup U_2$, there is $j \in \{1, 2\}$ such that $\sigma_{2,j}(x) > 0$). Since the vectors $\{g_1, g_2, g_3\}$ are lineally independent we get that, if $H_3'(x) = 0$ for some $x \in U_3 \cap (U_1 \cup U_2)$, then there necessarily exists $\rho \neq 0$ such that $D(x) = \rho(\sigma_{3,1}(x)g_1 + \sigma_{3,2}(x)g_2 + \sigma_{3,3}(x)g_3)$, that is $D(x) \in [g_1, g_2, g_3]$.

In fact we can be more accurate and obtain that if $x \in (U_3 \cap U_2) \setminus U_1$ and $H_3'(x) = 0$ then $D(x) \in [g_2, g_3]$. Indeed, in step 2 we proved that $\sigma_{2,1} = 0$ in $U_2 \setminus U_1$. Moreover, the functions $\varphi_1(g_1), \phi_{1,1}(g_1)$ and $\phi_{1,2}(g_1)$ are constant outside $U_1$, thus their derivatives vanish outside $U_1$. This implies $\xi_{2,1} = 0$ and consequently $\sigma_{3,1} = 0$ in $(U_3 \cap U_2) \setminus U_1$. Similarly, if $x \in (U_3 \cap U_1) \setminus U_2$ and $H_3'(x) = 0$, then $D(x) \in [g_1, g_3]$. Indeed, from step 2 we know that $\sigma_{2,2} = 0$ on $U_1 \setminus U_2$. Moreover, the function $\varphi_2'(g_2)\phi_{1,1}(g_1)$ vanishes outside...
$U_2$. In addition, if $x \in (U_3 \cap U_1) \setminus U_2$ then $g_1(x) < \gamma_{1,2} < \gamma_{1,1}$ and hence $g_2(x) \leq \gamma_2$. Thus $\phi_{2,1}(g_2(x)) = 0$, which implies $\xi_{2,2}(x) = 0$. Consequently $\sigma_{3,2}(x) = 0$ if $x \in (U_3 \cap U_1) \setminus U_2$.

Define the sets

$$Z_{3,1} = \begin{cases} \{z_3\}, & \text{if } z_3 \in U_3 \setminus (\overline{U_1 \cup U_2}) \\ \emptyset, & \text{otherwise} \end{cases}$$

$$Z_{3,2} = Z_3 \cap U_3 \setminus (U_1 \cup U_2).$$

Now, let us check that $Z_{3,2} \subset \cup_{g \in N_g} O_g$. Indeed, if $x \in Z_{3,2}$, then $x \in (U_1 \cup U_2) \cap U_3$. Now,

- if $x \in (U_1 \cap U_3) \setminus U_2$, then $D(x) \in [g_{1,3}].$ Since $(U_1 \cap U_3) \setminus U_2 \subset (U_1 \cap U_3') \setminus U_2$ we can deduce that $x \in M_{3,1} \subset M_3$.
- If $x \in (U_2 \cap U_3) \setminus U_1$, then $D(x) \in [g_{2,3}].$ Since $(U_2 \cap U_3) \setminus U_1 \subset (U_2 \cap U_3') \setminus U_1$ we can deduce that $x \in M_{3,2} \subset M_3$.
- If $x \in U_1 \cap U_2 \cap U_3$, then $D(x) \in [g_{1,2}, g_{2,3}].$ Since $U_1 \cap U_2 \cap U_3 \in U_1 \cup U_2 \cup U_3'$ we can deduce that $x \in M_{3,1,2} \subset M_3$.

Finally, since $x \in U_3$, we have that $g_1(x) < \gamma_{1,2}$ and $g_2(x) < \gamma_{2,1}$. We apply Fact 2.4 (2) to conclude that there is $g \in N_3$ such that $x \in O_g$.

In the case when $Z_{3,1} = \{z_3\} \notin \cup_{g \in N_g} \overline{O_g}$, we select if necessary, a larger $t_3$, with $t_3 < t_3'$, so that $z_3 \notin \cup_{g \in N_3} \overline{B_g}$. Since the norm is LUR and $D(z_3) = g_3$ we may select numbers $0 < t_3' < l_3 < 1$ and open slices, which are neighborhoods of $z_3$ defined by

$$O_{g_3} := \{x \in S : g_3(x) > t_3'\} \quad \text{and} \quad B_{g_3} := \{x \in S : g_3(x) > t_3'\},$$

satisfying $O_{g_3} \subset B_{g_3} \subset \{x \in S : g_1(x) < \gamma_{1,2}', \quad g_2(x) < \gamma_{2,1}', \quad g_3(x) > \gamma_3'\}$ and $\text{dist}(B_{g_3}, B_g) > 0$ for every $g \in N_3$. In this case, we define $\Gamma_3 = N_3 \cup \{g_3\}$.

Now, if $Z_{3,1} = \{z_3\} \in \cup_{g \in N_3} \overline{O_g}$, we select, if necessary, a smaller constant $l_3$, with $0 < t_3 < l_3 < 1$, so that $Z_{3,1} = \{z_3\} \in \cup_{g \in N_3} O_g$. In this case, and also when $Z_{3,1} = \emptyset$, we define $\Gamma_3 = N_3$.

Notice that, in any of the cases mentioned above, Fact 2.4 clearly holds for the (possibly) newly selected real numbers $t_3$ and $t_3'$.

Then, the distance between any two sets $B_g, \quad B_g', \quad g, g' \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \quad g \neq g'$, is strictly positive. Moreover $Z_{3,1} \cup Z_{3,2} \subset \cup_{g \in \Gamma_3} O_g \subset \cup_{g \in \Gamma_3} B_g \subset U_3' \subset R_3$. Therefore, $Z_3 = Z_1 \cup Z_2 \cup Z_{3,1} \cup Z_{3,2} \subset \cup_{g \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3} O_g \subset \cup_{g \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3} B_g \subset U_1 \cup U_2 \cup U_3 = R_1 \cup R_2 \cup R_3$. Finally, recall that $\text{dist}(B_g, R_3) > 0$, for every $g \in \Gamma_3$ and $\text{dist}(B_g, (U_1 \cup U_2 \cup U_3^c)) > 0$ for every $g \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$.

It is worth mentioning that, by combining all the results obtained in the step $n = 3$, we deduce that $H_3' = \sigma_{3,1} g_1 + \sigma_{3,2} g_2 + \sigma_{3,3} g_3$, where $\sigma_{3,i}$ are continuous functions on $U_1 \cup U_2 \cup U_3$, $\sigma_{3,i}(x) = 0$ whenever $x \notin U_i$, and for every $x \in U_1 \cup U_2 \cup U_3$ there is at least one coefficient $\sigma_{3,j}(x) > 0$.

$\diamond$ Assume that, in the steps $j = 2, \ldots, k$, with $k \geq 2$, we have selected points $z_j \in S^+$ and constants $\gamma_j \in (0, 1)$, with $g_1(z_j) \neq \gamma_1, \ldots, g_{j-1}(z_j) \neq \gamma_{j-1}$, $\{g_1, \ldots, g_k := D(z_k)\}$ linearly
Assume we have also defined the functions \( h_j = \phi_j(g_j) \phi_{j-1,1}(g_{j-1}) \cdots \phi_{1, j-1}(g_1) \), where \( \phi_j, \phi_{j-1,1}, \ldots, \phi_{1, j-1} \) are \( C^\infty \) functions on \( \mathbb{R} \) satisfying

\[
\begin{align*}
\varphi_j(t) &= 0 & \text{if } t &\leq \gamma_j \\
\varphi_j(1) &= 1 \\
\varphi_j'(t) &> 0 & \text{if } t &> \gamma_j
\end{align*}
\]

and

\[
\begin{align*}
\phi_{1, j-1}(t) &= 1 & \text{if } t &\leq \frac{\gamma_1 \cdots \gamma_{j-1}}{2}, \\
\phi_{1, j-1}(t) &= 0 & \text{if } t &> \gamma_{j-1} \\
\phi_{1, j-1}'(t) &< 0 & \text{if } t &\in \left( \frac{\gamma_1 \cdots \gamma_{j-1}}{2}, \gamma_{j-1} \right) \\
\phi_{1, j-1}'(t) &> 0 & \text{if } t &\in \left( \frac{\gamma_1 \cdots \gamma_{j-1}}{2}, \gamma_{j-1} \right)
\end{align*}
\]

where \( \gamma_1 < \gamma_{j-1}, \ldots, \gamma_{j-1} < \gamma_{j-1}, \) and \( 2 \leq j \leq k \).

The interior of the support of \( h_j \) is the set

\[
U_j = \{ x \in S : g_1(x) < \gamma_{1, j-1}, \ldots, g_{j-1}(x) < \gamma_{j-1,1} \text{ and } g_j(x) > \gamma_j \}.
\]

Assume we have also defined the \( C^p \) smooth functions \( r_j \) and \( H_j \):

\[
\begin{align*}
r_j : S^+ &\longrightarrow \mathbb{R}, \\
H_j : U_1 \cup U_2 \cup \ldots \cup U_j &\longrightarrow \mathbb{R}, \\
\quad r_j = s_j g_j + (1 - s_j g_j(x_j)), \\
\quad H_j = \frac{\sum_{i=1}^{j} a_i r_i h_i}{\sum_{i=1}^{j} h_i}
\end{align*}
\]

for \( 2 \leq j \leq k \), where \( x_j \in U_j \) the numbers \( a_j, s_j \in \mathbb{R}^* \) satisfy that \( |a_j - F(x_j)| < \varepsilon \), \( s_j a_j > 0 \), and the oscillation of \( r_j \) on \( U_j \) is less than \( \frac{1}{|a_j|} \).

Assume that for \( 2 \leq j \leq k \) the set of critical points \( Z_j \) of \( H_j \) is a union of the form \( Z_j = Z_{j-1} \cup Z_{j,1} \cup Z_{j,2} \), where \( Z_{j-1} \) is the set of critical points of \( H_{j-1} \), the sets

**Figure 2.** Case \( n = 3 \): the decomposition of \( Z_3 \).
$Z_{j-1}, Z_{j,1}, Z_{j,2}$ are pairwise disjoint, $Z_j \subset D^{-1}([g_1, ..., g_j])$, $Z_{j-1} \subset (U_1 \cup ... \cup U_{j-1}) \setminus U_j$ and $Z_{j,1} \cup Z_{j,2} \subset U_j$. Furthermore, assume that there is an open subset $U'_j$ such that

$$U_j \subset U'_j \subset R_j := \{x \in S : g_j(x) > \gamma_j\}$$

and $\text{dist}(B_g, U'_j) > 0$, for every $g \in \Gamma_j \cup ... \cup \Gamma_{j-1}$, there is a finite subset $\Gamma_j \subset S^*$ and open slices of $S$,

$$B_g := \{x \in S : g(x) > t_j\} \quad \text{and} \quad O_g := \{x \in S : g(x) > l_j\}, \quad 0 < t_j < l_j < 1,$$

satisfying $B_g \subset U'_j$ for every $g \in \Gamma_j$, $\text{dist}(B_g, B_{g'}) > 0$ whenever $g, g' \in \Gamma_j$, $g \neq g'$ and

there is $\gamma'_j \in (\gamma_j, 1)$ such that

$$Z_{j,1} \cup Z_{j,2} \subset \bigcup_{g \in \Gamma_j} O_g \subset \bigcup_{g \in \Gamma_j} B_g \subset U'_j \cap \{x \in S : g_j(x) > \gamma'_j\} \subset U_j.$$  

Finally, assume that for $2 \leq j \leq k$, $H'_j = \sigma_{j,1} g_1 + ... + \sigma_{j,j} g_j$ on $U_1 \cup ... \cup U_j$, where $\sigma_{j,i}$ are continuous functions on $U_1 \cup ... \cup U_j$, for every $i = 1, ..., j$, and assume that for every $x \in U_1 \cup ... \cup U_j$, there is at least one strictly positive coefficient $\sigma_{j,m}(x)$, and that if $x \in (U_1 \cup ... \cup U_j) \setminus U_m$, with $m \in \{1, ..., j\}$, then $\sigma_{j,m}(x) = 0$.

Now, let us denote by $y_{k+1} \in S$ the point satisfying $f_{k+1}(y_{k+1}) = 1$. If either $\{g_1, ..., g_k, f_{k+1}\}$ are lineally dependent or $g_i(y_{k+1}) = \gamma_i$ for some $i \in \{1, ..., k\}$, we can use the density of the norm attaining functional (Bishop-Phelps Theorem) and the continuity of $D$ to slightly modify $y_{k+1}$ and find $z_{k+1} \in S$ so that: $g_i(z_{k+1}) \neq \gamma_i$, for every $i = 1, ..., k$, 

$$\{x \in S : f_{k+1}(x) > \delta_{k+1}^2\} \subset \{x \in S : g_{k+1}(x) > \nu_{k+1}\} \subset \{x \in S : f_{k+1}(x) > \delta_{k+1}^3\},$$

for some $\nu_{k+1} \in (0, 1)$. If $g_i(y_{k+1}) \neq \gamma_i$ for every $i \in \{1, ..., k\}$ and $\{g_1, ..., g_k, f_{k+1}\}$ are lineally independent, we define $z_{k+1} = y_{k+1}$ and $g_{k+1} = f_{k+1}$. Then we apply Lemma 2.1 to the l.i. vectors, $\{g_1, ..., g_{k+1}\}$ and the real numbers $\gamma_1, ..., \gamma_k$ and obtain $\gamma_{k+1} \in (0, 1)$ close enough to $\nu_{k+1}$ so that

$$S_{k+1} = \{x \in S : f_{k+1}(x) > \delta_{k+1}\}$$

$$\subset \{x \in S : g_{k+1}(x) > \gamma_{k+1}\} \subset \{x \in S : f_{k+1}(x) > \delta_{k+1}^4\} = P_{k+1}$$

and

(2.15)

$$\{T \in [g_1, ..., g_{i_1}, g_{i_2}, g_{k+1}]^*: T(g_{i_1}) = \gamma_{i_1}, ..., T(g_{i_s}) = \gamma_{i_s}, T(g_{k+1}) = \gamma_{k+1}\ \text{and} \ \|T\| = 1\} = \emptyset$$

for every $1 \leq i_1 < ... < i_s \leq k$ and $1 \leq s \leq k$.

Define

$$R_{k+1} = \{x \in S : g_{k+1}(x) > \gamma_{k+1}\}.$$  

Recall that $\bigcup_{g \in \Gamma_k} B_g \subset U'_k \cap \{x \in S : g_k(x) > \gamma'_k\}$ and select $\gamma'_{k,1} \in (\gamma_k, \gamma'_k)$. In addition, we select numbers

(2.16)  

$$\gamma'_{k-1,2} \in (\gamma_{k-1}, \frac{\gamma_{k-1} + \gamma_{k-1,1}}{2}), ..., \gamma'_{k,k} \in (\gamma_k, \frac{\gamma_k + \gamma_{1,k-1}}{2}),$$

and define the open set

(2.17)  

$$U'_{k+1} = \{x \in S : g_{i}(x) < \gamma'_{1,k}, ..., g_{k}(x) < \gamma'_{k,1} \text{ and } g_{k+1}(x) > \gamma_{k+1}\}.$$
Notice that $\text{dist}(B_g, U_{k+1}') > 0$ for every $g \in \Gamma_1 \cup ... \cup \Gamma_k$.

Assume that $R_{k+1} \cap (U_1 \cup ... \cup U_k) \neq \emptyset$ and define, for every $1 \leq i_1 < ... < i_s \leq k$ and $1 \leq s \leq k$, the set

$$M_{k+1,i_1,...,i_s} = \{ x \in U_{i_1} \cap ... \cap U_{i_s} \cap U'_{k+1}: x \notin U_j \text{ for every } j \in \{1,...,k\} \setminus \{i_1,...,i_s\},$$

and $D(x) \in [g_{i_1},...,g_{i_s},g_{k+1}]$. In the case when $M_{k+1} = \emptyset$ we select as $\gamma_{1,k}$ any point in $(\gamma_1,\gamma_{1,k}',...)$, and $\gamma_{k,1}$ any point in $(\gamma_k,\gamma_{k,1}')$.

Notice that $U_1 \cup ... \cup U_k = R_1 \cup ... \cup R_k$. In the case when $M_{k+1} \neq \emptyset$ and $\text{dist}(M_{k+1}, (U_1 \cup ... \cup U_k)') = \text{dist}(M_{k+1}, (R_1 \cup ... \cup R_k)') > 0$, we can immediately find $\gamma_{1,k} \in (\gamma_1,\gamma_{1,k}')$, $...$, $\gamma_{k,1} \in (\gamma_k,\gamma_{k,1}')$ with $M_{k+1} \subset \{ x \in S: g_1(x) > \gamma_{1,k} \} \cup ... \cup \{ x \in S: g_k(x) > \gamma_{k,1} \}$. Indeed, if $\text{dist}(x_n, (U_1 \cup ... \cup U_k)') = 0$, Let us prove that for every $1 \leq i_1 < ... < i_s \leq k$ and $1 \leq s \leq k$, the sets

$$F_{k+1,i_1,...,i_s}' = \{ T \in [g_{i_1},...,g_{i_s},g_{k+1}]: T(g_i) = \gamma_i \text{ for every } i \in \{i_1,...,i_s\} \text{ and } |T| = 1 \},$$

$$N_{k+1,i_1,...,i_s}' = \{ g \in S' \cap [g_{i_1},...,g_{i_s},g_{k+1}]: T(g) = 1 \text{ for some } T \in F_{k+1,i_1,...,i_s}' \}.$$

Since the norm $\| \cdot \|$ is Gâteaux smooth, we can apply Lemma [2.2] to the finite dimensional space $[g_{i_1},...,g_{i_s},g_{k+1}]$ with the norm $\| \cdot \|$ restricted to this finite dimensional space, and deduce that the cardinal of any of the sets $F_{k+1,i_1,...,i_s}'$ is at most two. Moreover, since the norm is strictly convex, the cardinal of each set $N_{k+1,i_1,...,i_s}'$ is at most two. Let us consider the norm-one extensions to $[g_{i_1},...,g_{i_s},g_{k+1}']$ of the elements of $F_{k+1,i_1,...,i_s}'$, that is,

$$F_{k+1,i_1,...,i_s}' = \{ T \in [g_{i_1},...,g_{i_s},g_{k+1}]: T|_{[g_{i_1},...,g_{i_s},g_{k+1}]} \in F_{k+1,i_1,...,i_s}' \text{ and } |T| = 1 \}.$$

Since the norm $\| \cdot \|$ is Gâteaux smooth, for every $G \in F_{k+1,i_1,...,i_s}'$ there is a unique norm-one extension $T$ defined on $[g_{i_1},...,g_{k+1}]$. Thus, the cardinal of each set $N_{k+1,i_1,...,i_s}'$ is at most two. Therefore the sets

$$F_{k+1}' = \bigcup \{ F_{k+1,i_1,...,i_s}': 1 \leq i_1 < ... < i_s \leq k \text{ and } 1 \leq s \leq k \}$$

$$N_{k+1}' = \bigcup \{ N_{k+1,i_1,...,i_s}': 1 \leq i_1 < ... < i_s \leq k \text{ and } 1 \leq s \leq k \}$$

are finite. As a consequence of equality [2.15], we deduce that $T(g_{k+1}) \neq \gamma_{k+1}$ for every $T \in F_{k+1}'$. We can restrict our study to the following kind of sequences: Fix $1 \leq s \leq k$ and $1 \leq i_1 < ... < i_s \leq k$ and consider a sequence $\{ x_n \} \subset M_{k+1,i_1,...,i_s}$ such that $\lim_n \text{dist}(x_n, (U_1 \cup ... \cup U_k)') = 0$. Let us prove that for every $i \in \{i_1,...,i_s\}$, $\lim_n g_i(x_n) = \gamma_i$. Indeed, if $\{ x_n \} \subset M_{k+1,i_1,...,i_s}$, then in particular $\{ x_n \} \subset U_i \subset R_i$ for every $i \in \{i_1,...,i_s\}$. Recall that $U_1 \cup ... \cup U_k = R_1 \cup ... \cup R_k$. Therefore, $\text{dist}(x_n, (U_1 \cup ... \cup U_k)') \geq \text{dist}(x_n, R_i)$ for every $i \in \{i_1,...,i_s\}$. Thus, $\lim_n \text{dist}(x_n, R_i) = 0$ for every $i \in \{i_1,...,i_s\}$. Since $\{ x_n \} \subset R_i$, this implies that $\lim_n g_i(x_n) = \gamma_i$, for every $i \in \{i_1,...,i_s\}$. 


Now, let us take any sequence \( \{x_n\} \subset M_{k+1,i_1,...,i_s} \) such that \( \lim_n g_i(x_n) = \gamma_i \), for \( i \in \{i_1,...,i_s\} \). Consider every \( x_n \) as an element of \( X^{**} \) and denote by \( x_n \) its restriction to \([g_1,...,g_{k+1}]\). Recall that \( D(x_n) \in S^* \cap [g_1,...,g_{i_s},g_{k+1}] \) for every \( n \in \mathbb{N} \). Then, the sequence of restrictions \( \{x_n\} \subset [g_1,...,g_{k+1}]^* \) satisfies that

\[
1 = |x_n| \geq |x_n| \geq |x_n|[g_{i_1}...,g_{i_s}g_{k+1}] = \max\{x_n(h) : h \in S^* \cap [g_1,...,g_{i_s},g_{k+1}]\}
\]

\[
\geq x_n(D(x_n)) = D(x_n)(x_n) = 1,
\]

for every \( n \in \mathbb{N} \). Thus, there is a subsequence \( \{x_{n_j}\} \) converging to an element \( T \in [g_1,...,g_{k+1}]^* \) with \( |T| = 1 \) and \( |T|[g_1,...,g_{i_s},g_{k+1}] = 1 \). Since \( \lim_j g_i(x_{n_j}) = \gamma_i \) for every \( i \in \{i_1,...,i_s\} \), we have that \( T(g_i) = \gamma_i \) for every \( i \in \{i_1,...,i_s\} \). This implies that \( T|[g_1,...,g_{i_s},g_{k+1}] \in F_{k+1,i_1,...,i_s} \) and \( T \in F_{k+1,i_1,...,i_s} \). Furthermore, if \( g \in N'_{k+1,i_1,...,i_s} \) and \( T(g) = 1 \), then \( \lim_j x_{n_j}(g) = 1 \). In addition, \( T(g_{k+1}) = \lim_j x_{n_j}(g_{k+1}) = \lim_j g_{k+1}(x_{n_j}) \geq \gamma_{k+1} \) because \( \{x_{n_j}\} \subset U'_{k+1} \). Then, from condition \( \Box \), we deduce that \( T(g_{k+1}) > \gamma_{k+1} \). Finally, let us check that \( g_i(x_n) \leq \gamma_i \) for every \( i \in \{1,...,k\} \\setminus \{i_1,...,i_s\} \) and \( n \in \mathbb{N} \). Indeed, since \( \{x_n\} \subset U'_{k+1} \), we have \( g_i(x) \leq \gamma_i \). \( g_i(g_i g_{i-1}...,g_{i-1}g_i...) \leq \gamma_i \). Now, from the definition of \( U_i \) and the fact that \( x_n : n \in \mathbb{N} \cap U_i = \emptyset \), we deduce that \( g_i(x_n) \leq \gamma_i \), for every \( n \in \mathbb{N} \). Finally, if \( T = \lim_j x_{n_j} \) in \([g_1,...,g_{k+1}]\), then \( T(g_i) = \lim_j x_{n_j}(g_i) \leq \gamma_i \), for every \( i \in \{1,...,k\} \\setminus \{i_1,...,i_s\} \).

Let us define, for every \( 1 \leq s \leq k \) and \( 1 \leq i_1 < ... < i_s \leq k \), the sets

\[
F_{k+1,i_1,...,i_s} = \{T \in F_{k+1,i_1,...,i_s} : \text{there is } \{x_n\} \subset M_{k+1,i_1,...,i_s}, \text{ with } \lim_n x_n(g_i) = \gamma_i, \text{ for } i \in \{i_1,...,i_s\} \text{ and } \lim_n x_n = T\},
\]

\[
N_{k+1,i_1,...,i_s} = \{g \in N_{k+1,i_1,...,i_s} : \text{there is } T \in F_{k+1,i_1,...,i_s} \text{ with } T(g) = 1\},
\]

and

\[
F_{k+1} = \bigcup\{F_{k+1,i_1,...,i_s} : 1 \leq s \leq k \text{ and } 1 \leq i_1 < ... < i_s \leq k\},
\]

\[
N_{k+1} = \bigcup\{N_{k+1,i_1,...,i_s} : 1 \leq s \leq k \text{ and } 1 \leq i_1 < ... < i_s \leq k\},
\]

which are all finite. Select a real number \( \gamma'_{k+1} \) satisfying \( \gamma_{k+1} < \gamma'_{k+1} < \min\{T(g_{k+1}) : T \in F_{k+1}\} \).

**Fact 2.5.**

1. There are numbers \( 0 < t_{k+1} < l_{k+1} < 1 \) such that for every \( g \in N_{k+1} \), the slices

\[
O_g := \{x \in S : g(x) > l_{k+1}\} \quad \text{and} \quad B_g := \{x \in S : g(x) > t_{k+1}\}
\]

satisfy that

\[
O_g \subset B_g \subset \{x \in S : g_1(x) < \gamma'_{1,k} , ..., g_k(x) < \gamma'_{k,1} , g_{k+1}(x) > \gamma'_{k+1} \} \quad \text{and}\quad \text{dist}(B_g, B_{g'}) > 0, \text{ whenever } g, g' \in N_{k+1}, g \neq g'.
\]

(2.18)

(2.19)

2. There are numbers \( \gamma_{1,k} \in (\gamma_1, \gamma'_{1,k}), ..., \gamma_{k,1} \in (\gamma_k, \gamma'_{k,1}) \) such that if \( x \in M_{k+1} \), \( g_1(x) < \gamma_{1,k} , ..., g_k(x) < \gamma_{k,1} \), then \( x \in O_g \), for some \( g \in N_{k+1} \).
Proof of Fact 2.5. (1) First, if $X$ is reflexive, we know that for every $g \in N_{k+1}$ there is $x_g \in S$ such that $D(x_g) = g$. There is $1 \leq s \leq k$ and $1 \leq i_1 < \ldots < i_s \leq k$ such that $g \in F_{k+1,i_1,\ldots,i_s}$. Denote by $x_g$ the restriction of $x_g$ to $[g_1, \ldots, g_{k+1}]$. Since $x_g \in T$ for some $T \in F_{k+1,i_1,\ldots,i_s}$. This implies that $x_g(g_i) = \gamma_i \leq \gamma_{i,k+1}$, and $x_g(g_k) > \gamma_{k+1}$ for every $i \in \{i_1, \ldots, i_s\}$, $x_g(g_k) > \gamma_{k+1}$ and $x_g(g_k') \leq \gamma_i < \gamma_{i,k+1}$, whenever $i \in \{i_1, \ldots, i_s\}$. Hence, $x_g \in \{x \in S : g_1(x) < g_1(x'), g_k(x) < g_k(x') \text{ and } g_{k+1}(x) > g_{k+1}(x')\}$.

Now, since the norm $\| \cdot \|$ is LUR and $D(x_g) = g$, the functional $g$ strongly exposes $S$ at the point $x_g$ for every $g \in N_{k+1}$. Since $N_{k+1}$ is finite, we can obtain real numbers $0 < t_{k+1} < l_{k+1} < 1$ and slices $O_g$ and $B_g$, for every $g \in N_{k+1}$, satisfying conditions (2.18) and (2.19).

Now consider a non reflexive Banach space $X$. Let us first prove (2.18). Assume, on the contrary, that there is a point $g \in N_{k+1}$ and there is a sequence $\{y_n\} \subset S$ satisfying $g(y_n) > 1 - \frac{1}{n}$ with $g_1(y_n) \geq \gamma_{i,k+1}$, $\ldots$, or $g_k(y_n) \geq \gamma_{i,k+1}$, or $g_{k+1}(y_n) \leq \gamma_{k+1}$, for every $n \in \mathbb{N}$. If $g \in N_{k+1}$ there is a sequence $\{x_n\} \subset M_{k+1}$ with $\lim_n g_i(x_n) = \gamma_i$ for every $i \in \{1, \ldots, k\}$, if $\lim_n g_k(x_n) \geq \gamma_{k+1}$ and $\lim_n g(x_n) = 1$. In particular,

$$\frac{g(x_n) + 1 - \frac{1}{n}}{2} \leq g \left( \frac{x_n + y_n}{2} \right) \leq \left| \frac{x_n + y_n}{2} \right| \leq 1,$$

and thus $\lim_n \frac{x_n + y_n}{2} = 1$. Since in this case the norm $\| \cdot \|$ is WUR, we have that $x_n - y_n \xrightarrow{w} 0$ (weakly converges to zero). This last assertion gives a contradiction since either $\limsup_n g_i(x_n - y_n) \leq \gamma_i - \gamma_{i,k+1}$ or $\liminf_n g_{k+1}(x_n - y_n) \geq \liminf_n g_{k+1}(x_n) - \gamma_{k+1} > 0$. Therefore, we can find real numbers $0 < t_{k+1} < l_{k+1} < 1$ and slices $O_g$ and $B_g$ for every $g \in N_{k+1}$, satisfying condition (2.18). The proof of (2.18) is the same as the one given in Fact 2.5, where the only property we need is the strict convexity of $\| \cdot \|$.

(2) Assume, on the contrary, that for every $n \in \mathbb{N}$, there is $x_n \in M_{k+1}$ with $g_i(x_n) \leq \gamma_i + \frac{1}{n}$, for every $i \in \{1, \ldots, k\}$ and $\{x_n : n \in \mathbb{N}\} \cap (\cup_{g \in N_{k+1}} O_g) = \emptyset$. Then there is a subsequence of $\{x_n\}$, which we denote by $\{x_n\}$ as well, and there are numbers $1 \leq s \leq k$ and $1 \leq i_1 < \ldots < i_s \leq k$ such that $\{x_n\} \subset M_{k+1,i_1,\ldots,i_s}$. In particular, $\{x_n\} \subset U_{i_1}$ and $\{x_n\} \subset U_{i_s}$. Then $g_i(x_n) > \gamma_i$ for every $i \in \{i_1, \ldots, i_s\}$ and $n \in \mathbb{N}$. Hence, $\lim_n g_i(x_n) = \gamma_i$ for every $i \in \{i_1, \ldots, i_s\}$. Since $\{x_n\} \subset M_{k+1,i_1,\ldots,i_s}$, from the comments preceding Fact 2.5, we know that there is a subsequence $\{x_{n_j}\}$ and $g \in N_{k+1,i_1,\ldots,i_s}$, satisfying that $\lim_j g(x_{n_j}) = 1$, which is a contradiction. This finishes the proof of Fact 2.5. □

If $R_{k+1} \cap (U_1 \cup \ldots \cup U_k) = \emptyset$ we may select any number in $(\gamma_1', \gamma_1'')$, and $\gamma_{k+1}$ any number in $(\gamma_{k+1}', \gamma_{k+1}'')$.

Now we define $h_{k+1}$,

$$h_{k+1} : S^+ \to \mathbb{R}$$

$$h_{k+1} = \varphi_{k+1}(g_{k+1}) \phi_{k,1}(g_k) \cdots \phi_{1,1}(g_1),$$
with \( \varphi_{k+1}, \phi_{k,1}, \ldots, \phi_{1,k} \) \( C^\infty \) functions on \( \mathbb{R} \) satisfying
\[
\varphi_{k+1}(t) = 0 \quad \text{if } t \leq \gamma_{k+1}
\]
\[
\varphi_{k+1}(1) = 1
\]
\[
\varphi_{k+1}'(t) > 0 \quad \text{if } t > \gamma_{k+1}
\]
and
\[
\phi_{1,k}(t) = 1 \quad \text{if } t \leq \frac{\gamma_1 + \gamma_{1,k}}{2}, \quad \ldots, \quad \phi_{k,1}(t) = 1 \quad \text{if } t \leq \frac{\gamma_k + \gamma_{k,1}}{2}
\]
\[
\phi_{1,k}(t) = 0 \quad \text{if } t \geq \gamma_{1,k}, \quad \ldots, \quad \phi_{k,1}(t) = 0 \quad \text{if } t \geq \gamma_{k,1}
\]
\[
\phi_{1,k}'(t) < 0 \quad \text{if } t \in \left( \frac{\gamma_1 + \gamma_{1,k}}{2}, \gamma_{1,k} \right), \quad \ldots, \quad \phi_{1,k}'(t) < 0 \quad \text{if } t \in \left( \frac{\gamma_k + \gamma_{k,1}}{2}, \gamma_{k,1} \right).
\]
Clearly the interior of the support of \( h_{k+1} \) is the set
\[
U_{k+1} = \{ x \in S : g_1(x) < \gamma_{1,k}, \ldots, g_k(x) < \gamma_{k,1} \text{ and } g_{k+1}(x) > \gamma_{k+1} \}.
\]
Select one point \( x_{k+1} \in U_{k+1} \), a real number \( a_{k+1} \in \mathbb{R}^* \) with \( |a_{k+1} - F(x_{k+1})| < \varepsilon \) and define the auxiliary function
\[
r_{k+1} : S^+ \rightarrow \mathbb{R},
\]
\[
r_{k+1} = s_{k+1}g_{k+1} + (1 - s_{k+1}g_{k+1}(x_{k+1})),
\]
where we have selected \( s_{k+1} \) so that \( s_{k+1}a_{k+1} > 0 \) and \( |s_{k+1}| \) is small enough so that the oscillation of \( r_{k+1} \) on \( U_{k+1} \) is less than \( \frac{\varepsilon}{|a_{k+1}|} \). Notice that \( r_{k+1}(x_{k+1}) = 1 \).
Let us study the set of critical points \( Z_{k+1} \) of the \( C^p \) smooth function
\[
H_{k+1} : U_1 \cup \ldots \cup U_{k+1} \rightarrow \mathbb{R}, \quad H_{k+1} = \sum_{i=1}^{k+1} a_i r_i h_i.
\]

Let us prove that \( Z_{k+1} := \{ x \in U_1 \cup \ldots \cup U_{k+1} : H_{k+1}'(x) = 0 \text{ on } T_x \} \) can be included in a finite union of disjoint slices within \( U_1 \cup \ldots \cup U_{k+1} \) by splitting \( Z_{k+1} \) conveniently into the (already defined) set \( Z_k \) and a finite number of disjoint sets within \( U_{k+1} \).

It is straightforward to verify that \( H_{k+1}' = \sigma_{k+1}g_1 + \ldots + \sigma_{k+1}g_{k+1} + \Lambda_k \), where \( \sigma_{k+1} \) are continuous functions on \( U_1 \cup \ldots \cup U_{k+1} \) and \( g_i \) denotes the restriction \( g_i|_{T_x}, i = 1, \ldots, k+1 \), whenever we evaluate \( H_{k+1}'(x) \).

Clearly the restrictions of \( H_{k+1} \) and \( H_{k+1}' \) to \( (U_1 \cup \ldots \cup U_k) \setminus U_{k+1} \) coincide with \( H_k \) and \( H_k' \) respectively. Then, \( Z_{k+1} \setminus U_{k+1} = Z_k = Z_{k+1} \setminus \overline{U}_{k+1} \). Let us study the set \( Z_{k+1} \cap U_{k+1} \). First, if \( x \in U_{k+1} \setminus (U_1 \cup \ldots \cup U_k) \), then \( H_{k+1}'(x) = a_{k+1}r_{k+1}(x) \) and \( H_{k+1}'(x) = a_{k+1}r_{k+1}(x) \). Therefore \( H_{k+1}'(x) = a_{k+1}r_{k+1}(x) \). If the point \( z_{k+1} \in U_{k+1} \setminus (U_1 \cup \ldots \cup U_k) \), then \( H_{k+1} \) has exactly one critical point in \( U_{k+1} \setminus (U_1 \cup \ldots \cup U_k) \); in this case, \( g_i(z_{k+1}) \neq \gamma_i \) for every \( i = 1, \ldots, k \), the point \( z_{k+1} \) actually belongs to \( U_{k+1} \setminus (U_1 \cup \ldots \cup U_k) \).

Now, let us study the critical points of \( H_{k+1} \) in \( U_{k+1} \cap (U_1 \cup \ldots \cup U_k) \). If we define \( \Lambda_k = \frac{\sum_{i=1}^{k+1} h_i}{\sum_{i=1}^{k+1} h_i} \), then we can rewrite \( H_{k+1} \) on \( U_{k+1} \cap (U_1 \cup \ldots \cup U_k) \) as
\[
H_{k+1} = \frac{\sum_{i=1}^{k+1} a_i r_i h_i}{\sum_{i=1}^{k+1} h_i} \cdot \frac{\sum_{i=1}^{k+1} h_i}{\sum_{i=1}^{k+1} h_i} + \frac{a_{k+1}r_{k+1}(h_{k+1})}{\sum_{i=1}^{k+1} h_i} = H_k \Lambda_k + a_{k+1}r_{k+1}(1 - \Lambda_k),
\]
and

\[ H'_{k+1} = H_k \Lambda_k + a_{k+1}s_{k+1}(1 - \Lambda_k)g_{k+1} + (H_k - a_{k+1}r_{k+1})\Lambda_k. \]

Notice that, on the open set \( U_{k+1} \), we have that \( \phi_{i,j}(g_i) \equiv 1 \), whenever \( i + j \leq k \). Indeed, on the one hand, if \( x \in U_{k+1} \), and \( i \in \{1, \ldots, k\} \), then \( g_i(x) < \gamma_{i,k+1} \leq \frac{\gamma_i + \gamma_{i+1}}{2} \), whenever \( i + j \leq k \). On the other hand, \( \phi_{i,j}(t) \equiv 1 \) if \( t \leq \frac{\gamma_i + \gamma_{i+1}}{2} \). Therefore \( h_i|_{U_{k+1}} = \varphi_i(g_i) \), for every \( i = 1, \ldots, k \), and

\[ \Lambda_k = \frac{\sum_{i=1}^{k} \varphi_i(g_i)}{\sum_{i=1}^{k} \varphi_i(g_i) + h_{k+1}}. \]

By computing \( \Lambda'_k \) in \( U_{k+1} \), we obtain \( \Lambda'_k = \xi_{k,1}g_1 + \cdots + \xi_{k,k+1}g_{k+1} \), where the coefficients \( \xi_{k,1}, \ldots, \xi_{k,k+1} \) are continuous functions of the following form:

\[ \xi_{k,j} = \frac{-\varphi_{k+1}(g_{k+1})}{(\prod_{i=1, i \neq j}^{k} \phi_{i,k+1-i}(g_i))} (\sum_{i=1}^{k} h_i) + h_{k+1} \varphi_j(g_j), \quad j = 1, \ldots, k \]

\[ \xi_{k,k+1} = \frac{-\varphi_{k+1}(g_{k+1})}{(\prod_{i=1}^{k} \phi_{i,k+1-i}(g_i))} (\sum_{i=1}^{k} h_i). \]

Thus, if \( x \in U_{k+1} \cap (U_1 \cup \ldots U_k) \), the coefficients \( \sigma_{k+1,1}, \ldots, \sigma_{k+1,k+1} \) for \( H'_{k+1} \) have the following form,

\[ \sigma_{k+1,j} = \sigma_{k,j}\Lambda_k + (H_k - a_{k+1}r_{k+1})\xi_{k,j}, \quad \text{for } j = 1, \ldots, k \]

\[ \sigma_{k+1,k+1} = a_{k+1}s_{k+1}(1 - \Lambda_k) + (H_k - a_{k+1}r_{k+1})\xi_{k,k+1}. \]

Notice that in \( U_{k+1} \cap (U_1 \cup \ldots U_k) \), \( a_{k+1}s_{k+1} > 0 \), \( \Lambda_k > 0 \), \( 1 - \Lambda_k > 0 \), \( \xi_{k,j} > 0 \), for every \( j = 1, \ldots, k \). \( \sum_{j=1}^{k} \xi_{k,j} > 0 \) and \( \xi_{k,k+1} < 0 \). Therefore, if \( H_k - a_{k+1}r_{k+1} \leq 0 \), the coefficient \( \sigma_{k+1,k+1} > 0 \). When \( H_k - a_{k+1}r_{k+1} > 0 \) and \( \sigma_{k+j} > 0 \), the coefficient \( \sigma_{k+1,j} > 0 \) (recall that, from the step \( k \) we know that, for every \( x \in U_1 \cup \ldots U_k \) there exists at least one \( j \in \{1, \ldots, k\} \) with \( \sigma_{k,j} > 0 \)). Hence, if \( H'_{k+1}(x) = 0 \) for some \( x \in U_{k+1} \cap (U_1 \cup \ldots U_k) \), there necessarily exists \( \varrho \neq 0 \) such that \( D(x) = \varrho(\sigma_{k+1,1}(x)g_1 + \cdots + \sigma_{k+1,k+1}(x)g_{k+1}) \), that is \( D(x) \in [g_1, \ldots, g_{k+1}] \).

In fact we can be more accurate and obtain that if \( H'_{k+1}(x) = 0 \), \( x \in U_{k+1} \cap (U_1 \cup \ldots U_k) \) and \( x \notin \cup_{j \in F}U_j \) for some proper subset \( F \subset \{1, \ldots, k\} \), then \( D(x) \in \text{span}\{g_j : j \in \{1, \ldots, k+1\} \setminus F\} \). Indeed, from step \( k \) we know that, if \( x \in (U_1 \cup \ldots U_k) \setminus U_j \), where \( j \in \{1, \ldots, k\} \), then \( \sigma_{k,j}(x) = 0 \). Now, if \( j \in F \) and \( j = 1 \), it is clear that the functions \( \varphi'_j(g_1) \) and \( \varphi'_{1,k}(g_1) \) vanish outside \( U_1 \). This implies \( \xi_{k,1}(x) = 0 \) and consequently \( \sigma_{k+1,1}(x) = 0 \). If \( j \in F \) and \( 2 \leq j \leq k \), since \( x \in U_{k+1} \) we know that

\[ g_1(x) < \gamma_{1,k} < \gamma_{1,j-1} \leq \cdots < g_{j-1} < g_{j-2} < \gamma_{j-1,1}, \]

and then necessarily \( g_j(x) \leq g_j \). Since the functions \( \varphi'_j(g_j) \) and \( \varphi'_{j,k+1-j}(g_j) \) vanish whenever \( g_j \leq g_j \), we deduce \( \xi_{k,j}(x) = 0 \) and thus \( \sigma_{k+1,j}(x) = 0 \).
Let us now define the sets
\[ Z_{k+1,1} = \begin{cases} \{z_{k+1}\}, & \text{if } z_{k+1} \in U_{k+1} \setminus (U_1 \cup \ldots \cup U_k) \\ \emptyset, & \text{otherwise} \end{cases} \]
\[ Z_{k+1,2} = Z_{k+1} \cap U_{k+1} \cap (U_1 \cup \ldots \cup U_k). \]

Now, let us check that \( Z_{k+1,2} \subset \bigcup_{g \in N_{k+1}} O_g \). Indeed, if \( x \in Z_{k+1,2} \), there are constants \( 1 \leq s \leq k \) and \( 1 \leq i_1 < \ldots < i_k \leq k \), such that \( x \in U_{k+1} \cap U_{i_1} \cap \ldots \cap U_{i_s} \) and \( x \not\in \bigcup_{j \in F} U_j \), where \( F = \{1, \ldots, k\} \setminus \{i_1, \ldots, i_s\} \). From the preceding assertion, \( D(x) \in \{g_{i_1}, \ldots, g_{i_s}, g_{k+1}\} \).

In the case when \( Z_{k+1,1} = \{z_{k+1}\} \not\subset \bigcup_{g \in N_{k+1}} O_g \), we define \( O_{g_{k+1}} := \{x \in S : g_{k+1}(x) > t'_{k+1}\} \) and \( B_{g_{k+1}} := \{x \in S : g_{k+1}(x) < t'_{k+1}\} \), satisfying \( O_{g_{k+1}} \subset B_{g_{k+1}} \subset \{x \in S : g_{k+1}(x) < t'_{k+1}\} \) and dist\( (B_{g_{k+1}}, O_g) > 0 \), for every \( g \in N_{k+1} \). In this case, we define \( \Gamma_{k+1} = N_{k+1} \cup \{g_{k+1}\} \).

Finally, let us notice that, by combining the results obtained in step \( k+1 \), we deduce that \( H_{k+1} = a_{k+1} R_{k+1} S_{k+1} \) in \( U_{k+1} \setminus (U_1 \cup \ldots \cup U_k) \) and \( H_{k+1} = H_k \) on \( (U_1 \cup \ldots \cup U_k) \setminus U_{k+1} \), and in general \( H_{k+1} = \sigma_{k+1,1} S_1 + \ldots + \sigma_{k+1,k+1} S_{k+1} \) on \( U_1 \cup \ldots \cup U_{k+1} \), where \( \sigma_{k+1,i} \) are continuous functions on \( U_1 \cup \ldots \cup U_{k+1} \), for \( i = 1, \ldots, k+1 \), and for every \( x \in U_1 \cup \ldots \cup U_{k+1} \) there is at least one \( i \in \{1, \ldots, k+1\} \) such that \( \sigma_{k+1,i}(x) > 0 \). Furthermore, \( \sigma_{k+1,j}(x) = 0 \) whenever \( x \in (U_1 \cup \ldots \cup U_{k+1}) \setminus U_j \), \( j \in \{1, \ldots, k+1\} \).

Once we have defined, by induction, the functions \( h_k \), \( r_k \) and the constants \( a_k \), for all \( k \in \mathbb{N} \), we define
\[
H : S^+ \longrightarrow \mathbb{R} \\
H = \frac{\sum_{k=1}^{\infty} a_k r_k h_k}{\sum_{k=1}^{\infty} h_k}.
\]
It is straightforward to verify that the family \( \{U_k\}_{k \in \mathbb{N}} \) of open sets of \( S^+ \) is a locally finite open covering of \( S^+ \). Thus, for every \( x \in S^+ \) there is \( k_x \in \mathbb{N} \) and a (relatively open
in $S^+$) neighborhood $V_x \subset S^+$ of $x$, such that $V_x \cap (\bigcup_{k>0} U'_k) = \emptyset$ and therefore $H|_{V_x} = H_{k_k}|_{V_x}$. Thus $H$ is $C^p$ smooth whenever the functions $\{h_k\}_{k \in \mathbb{N}}$ are $C^p$ smooth.

**Fact 2.6.** The function $H$ $3\varepsilon$-approximates $F$ in $S^+$. 

**Proof.** Recall that the oscillation of $F$ in $U_k$ is less than $\varepsilon$, the oscillation of $r_k$ in $U_k$ is less than $\frac{\varepsilon}{|a_k|}$, $|a_k - F(x_k)| < \varepsilon$ and $r_k(x_k) = 1$, for every $k \in \mathbb{N}$. Now, if $h_k(x) \neq 0$, then $x \in U_k$ and

$$|a_k r_k(x) - F(x)| \leq |a_k r_k(x) - a_k r_k(x_k)| + |a_k r_k(x_k) - F(x)|$$

$$= |a_k||r_k(x) - r_k(x_k)| + |a_k - F(x)|$$

$$\leq |a_k| \frac{\varepsilon}{|a_k|} + |a_k - F(x_k)| + |F(x_k) - F(x)| \leq 3\varepsilon.$$

Hence,

$$|H(x) - F(x)| = \frac{\sum_{k=1}^{\infty} (a_k r_k(x) - F(x)) h_k(x)}{\sum_{k=1}^{\infty} h_k(x)} \leq \frac{\sum_{k=1}^{\infty} |a_k r_k(x) - F(x)| h_k(x)}{\sum_{k=1}^{\infty} h_k(x)} \leq 3\varepsilon.$$ \[\square\]

Let us denote by $C$ the critical points of $H$ in $S^+$. Since for every $x \in S^+$, there is $k_x \in \mathbb{N}$ and a (relatively open in $S^+$) neighborhood $V_x \subset S^+$ of $x$ such that $V_x \cap (\bigcup_{k>k_x} U_k) = \emptyset$, we have that $H|_{V_x} = H_{k_x}|_{V_x}$ and $C \subset \bigcup U_k Z_k$. Recall that $U_k Z_k \subset \bigcup \{O_g : g \in \bigcup \Gamma_k\} \subset \bigcup \{B_g : g \in \bigcup \Gamma_k\}$, the oscillation of $F$ on $B_g$ is less than $\varepsilon$ and dist($B_g, B_g'$) > 0, for every $g, g' \in \bigcup \Gamma_k$ with $g \neq g'$. Furthermore, from the inductive construction of the sets $\{B_g : g \in \bigcup \Gamma_k\}$, it is straightforward to verify that (i) for every $k > 1$, if $g \in \Gamma_k$ and $g' \in \bigcup_{m>1} \Gamma_m$, then dist($B_g, B_g'$) $\geq \gamma_k' - \gamma_k > 0$ and (ii) if $g \in \Gamma_1$ and $g' \in \bigcup_{m>1} \Gamma_m$, then dist($B_g, B_g'$) $\geq t_1 - \gamma_1' > 0$. Therefore, for every $g \in \bigcup \Gamma_k$,

$$\text{dist}(B_g, \bigcup \{B_{g'} : g' \in \bigcup \Gamma_k, \ g' \neq g\}) > 0.$$ \[2.21\]

We relabel the countable families of open slices $\{O_g\}_{g \in \bigcup \Gamma_k}$ and $\{B_g\}_{g \in \bigcup \Gamma_k}$ as $\{O_n\}$, $\{B_n\}$, respectively. Notice that the set $\bigcup B_n$ is a (relatively) closed set in $S^+$. Indeed, if $\{x_j\} \subset \bigcup B_n$ and lim$x_j = x \in S^+$, since $\bigcup U_n'$ is also a locally finite open covering of $S^+$, there is $n_x$ and a (relatively open in $S^+$) neighborhood $W_x \subset S^+$ of $x$, such that $W_x \cap (\bigcup_{n>n_x} U_n') = \emptyset$. In addition, from the construction of the family $\{B_n\}$, there is $N \in \mathbb{N}$ such that $\bigcup_{n>N} B_n \subset \bigcup_{n>n_x} U_n'$, and thus there is $j_0 \in \mathbb{N}$ with $\{x_j\}_{j>j_0} \subset \bigcup_{n=1}^{N} B_n$. Hence $x \in \bigcup_{n=1}^{N} B_n \subset \bigcup B_n$.

Let us denote $B_n = \Phi^{-1}(B_n)$ and $O_n = \Phi^{-1}(O_n)$, for every $n \in \mathbb{N}$.

**Fact 2.7.** $O_n$ and $B_n$ are open, convex and bounded subsets of $X$, for every $n \in \mathbb{N}$.

**Proof.** Since $\Phi$ is continuous, it is clear that $O_n$ and $B_n$ are open sets. The sets $O_n$ and $B_n$ are slices of the form $R = \{x \in S : b(x) > \delta\}$ for some $b \in S^*$ and $\delta > 0$ such that dist($R, X \times \{0\}$) > 0. Let us prove that $R := \Phi^{-1}(R)$ is convex and bounded in $X$. First, let us check that the cone in $Y$ generated by $R$ and defined by

$$\text{cone}(R) = \{\lambda x : x \in R, \ \lambda > 0\} = \{x \in Y : b(x) > \delta\}$$
is a convex set: consider $0 \leq \alpha \leq 1$ and $x, x' \in \text{cone}(R)$. Then,

$$b(\alpha x + (1 - \alpha)x') = \alpha b(x) + (1 - \alpha)b(x') > \alpha \delta |x| + (1 - \alpha)\delta|x'|$$

and this implies that $\alpha x + (1 - \alpha)x' \in \text{cone}(R)$. Therefore, the intersection of the two convex sets $\text{cone}(R) \cap (X \times \{1\}) = \Pi^{-1}(R)$ is convex. Now, it is clear that $R = \Phi^{-1}(R) = i^{-1}(\Pi^{-1}(R))$ is convex as well.

Let us prove that $\Pi^{-1}(R)$ is bounded in $Y$. Consider the linear bounded operator $\pi_2 : Y = X \oplus \mathbb{R} \rightarrow \mathbb{R}$, $\pi_2(x, r) = r$, for every $(x, r) \in X \oplus \mathbb{R}$. Then, $\Pi^{-1}(y) = \frac{y}{\pi_2(y)}$, for every $y \in S^+$. On the one hand, $d := \text{dist}(R, X \times \{0\}) > 0$ and then

$$\pi_2(x, r) = r = \frac{\|(x, r) - (x, 0)\|}{\|(0, 1)\|} \geq \frac{d}{\|(0, 1)\|} = s > 0, \quad \text{for every } (x, r) \in R.$$ 

On the other hand,

$$|\Pi^{-1}(y)| = \frac{|y|}{\pi_2(y)} = \frac{1}{\pi_2(y)} \leq \frac{1}{s}, \quad \text{for every } y \in R,$$

and thus $\Pi^{-1}(R)$ is bounded. Since the norm $\|\cdot\|$ considered on $X \times \{0\}$ (defined as $\|(x, 0)\| = |x|$) and the restriction of the norm $\|\cdot\|$ to $X \times \{0\}$ are equivalent norms on $X \times \{0\}$, there exist constants $m, M > 0$ such that $m\|x - x'\| \leq |i(x) - i(x')| = |(x, 1) - (x', 1)| = |(x - x', 0)| \leq M\|x - x'\|$, for every $x, x' \in X$. Hence,

$$|\Phi^{-1}(y)| = |i^{-1}(\Pi^{-1}(y)) - i^{-1}(0, 1)| \leq \frac{1}{m}|\Pi^{-1}(y) - (0, 1)|$$

$$\leq \frac{1}{m}(|\Pi^{-1}(y)| + |(0, 1)|) \leq \frac{1 + s\|(0, 1)\|}{s \cdot m},$$

for every $y \in R$, what shows that $R$ is bounded in $X$. □

**Fact 2.8.** $\overline{O}_n$ and $\overline{B}_n$ are (closed convex and bounded) $C^p$ smooth bodies, for every $n \in \mathbb{N}$.

**Proof.** We already know that these sets are closed, convex and bounded bodies, hence it is enough to prove that their boundaries $\partial O_n$ and $\partial B_n$ are $C^p$ smooth one-codimensional submanifolds of $X$. Since $\partial B_n = \Phi^{-1}(\partial B_n)$, $\partial O_n = \Phi^{-1}(\partial O_n)$, and $\Phi$ is a $C^p$ diffeomorphism, this is the same as showing that $\partial O_n$ and $\partial B_n$ are $C^p$ smooth one-codimensional submanifolds of $S$. But, if $O_n$ is defined by $O_n = \{y \in S : g_n(y) > \beta_n\}$, we have that $\partial O_n$ is the intersection of $S$ with the hyperplane $X_n = \{y \in Y : g_n(y) = \beta_n\}$ of $Y$, and $X_n$ is transversal to $S$ at every point of $\partial O_n$ (otherwise the hyperplane $X_n$ would be tangent to $S$ at some point of $\partial O_n$ and, by strict convexity of $S$, this implies that $\partial O_n = X_n \cap S$ is a singleton, which contradicts the fact that $O_n$ is a nonempty open slice of $S$), hence the intersection $\partial O_n = S \cap X_n$ is a one-codimensional submanifold of $S$. The same argument applies to $\partial B_n$. □

**Fact 2.9.** $\text{dist}(O_n, X \setminus B_n) > 0$ and $\text{dist}(B_n, \cup_{m \neq n} B_m) > 0$, for every $n \in \mathbb{N}$. 
Proof. This is a consequence of the fact that \( \text{dist}(O_n, S^+ \setminus B_n) > 0 \), \( \text{dist}(B_n, \cup_{m 
eq n} B_m) > 0 \), and \( \Phi \) is Lipschitz. Indeed, on the one hand, recall that \( |i(x) - i(x')| = |(x - x', 0)| \leq M |x - x'| \), for every \( x, x' \in X \). On the other hand,

\[
|\Pi(y) - \Pi(y')| = \frac{|y|}{|y'|} |y' - y| = \frac{|y|}{|y'|} \left( |y'| - |y| \right) \leq \frac{2|y|}{|y'|} \left( \frac{|y'|}{|y|} - 1 \right) \leq \frac{2\zeta}{|y'|},
\]

for every \( y, y' \in X \times \{1\} \), where \( \zeta = \text{dist}(0, X \times \{1\}) > 0 \). Therefore, \( |\Phi(x) - \Phi(x')| \leq \frac{2\zeta}{\zeta} x - x' \), for every \( x, x' \in X \). Now, if two sets \( A, A' \subset S^+ \) satisfy that \( \text{dist}(A, A') > 0 \), then \( \text{dist}(A, A') \leq |a - a'| \leq \frac{2M}{\zeta} |\Phi^{-1}(a) - \Phi^{-1}(a')| \), for every \( a \in A, a' \in A' \). Therefore, \( 0 < \text{dist}(A, A') \leq \frac{2M}{\zeta} \text{dist}(\Phi^{-1}(A), \Phi^{-1}(A')) \). \( \square \)

Fact 2.10. For every \( n \in \mathbb{N} \), there exists a \( C^p \) diffeomorphism \( \Psi_n \) from \( X \) onto \( X \setminus \overline{O_n} \) such that \( \Psi_n \) is the identity off \( B_n \).

Proof. Assume that \( 0 \in O_n \). Since \( \text{dist}(O_n, X \setminus B_n) > 0 \), there is \( \delta_n > 0 \) such that \( \text{dist}((1 + \delta_n)O_n, B_n) > 0 \). We can easily construct a \( C^p \) smooth radial diffeomorphism \( \Psi_{n,2} \) from \( X \setminus \{0\} \) onto \( X \setminus \overline{O_n} \) satisfying \( \Psi_{n,2}(x) = x \) if \( x \notin (1 + \delta_n)O_n \). Indeed, take a \( C^\infty \) smooth function \( \lambda_n : [0, \infty) \rightarrow [1, \infty) \) satisfying that \( \lambda_n(t) = t \) for \( t \geq 1 + \delta_n \), \( \lambda_n(0) = 1 \) and \( \lambda_n(t) > 0 \) for \( t > 0 \), and define

\[
\Psi_{n,2}(x) = \lambda_n(\mu_n(x)) \frac{x}{\mu_n(x)},
\]

for \( x \in X \setminus \{0\} \), where \( \mu_n \) is the Minkowski functional of \( \overline{O_n} \), which is \( C^p \) smooth on \( X \setminus \{0\} \).

Now, since \( 0 \in O_n \), there is \( \alpha_n > 0 \) such that \( \alpha_n B_{||\cdot||} \subset O_n \). According to [12, Proposition 3.1] and [13, Lemma 2] (see also [1]), there exists a \( C^p \) diffeomorphism \( \Psi_{n,1} \) from \( X \) onto \( X \setminus \{0\} \) such that \( \Psi_{n,1} \) is the identity off \( \alpha_n B_{||\cdot||} \) (this set may be regarded as the unit ball of a \( C^p \) smooth norm on \( X \)).

Then, the composition \( \Psi_n := \Psi_{n,2} \circ \Psi_{n,1} \) is a \( C^p \) diffeomorphism from \( X \) onto \( X \setminus \overline{O_n} \) such that \( \Psi_n \) is the identity off \( B_n \). If \( 0 \notin O_n \), select \( \omega_n \in O_n \) and repeat the above construction of the diffeomorphism with the sets \( O_n - \omega_n \) and \( B_n - \omega_n \). Then, \( \Psi_n := \tau_{\omega_n} \circ \Psi_{n,2} \circ \Psi_{n,1} \circ \tau_{-\omega_n} \) is the required \( C^p \) diffeomorphism, where \( \tau_{\omega_n}(x) = x + \omega_n \). \( \square \)

Now, the infinite composition \( \Psi = \bigcirc_{n=1}^\infty \Psi_n \) is a well-defined \( C^p \) diffeomorphism from \( X \) onto \( X \setminus \cup_n \overline{O_n} \) which is the identity outside \( \cup_n B_n \) and \( \Psi(B_n) \subset B_n \). This follows from the fact that, for every \( x \in X \), there is an open neighborhood \( V_x \) and \( n_x \in \mathbb{N} \) such that \( V_x \cap (\cup_{n \neq n_x} B_n) = \emptyset \), and therefore \( \Psi|_{V_x} = \Psi|_{V_x} \). Finally, let us check that the \( C^p \) smooth function

\[
g : X \longrightarrow \mathbb{R}
\]

4\( \varepsilon \)-approximates \( f \) on \( X \) and \( g \) does not have critical points. Indeed, for every \( x \in X \), if \( \Psi(x) \neq x \), then there is \( B_{n_x} \) such that \( x \in B_{n_x} \). Since the oscillation of \( f \) in \( B_{n_x} \) is less than \( \varepsilon \) and \( \Psi(x) \in B_{n_x} \), we can deduce that \( |f(\Psi(x)) - f(x)| < \varepsilon \), for every \( x \in X \). Recall
that \( F \circ \Phi = f \) and \(|H(x) - F(x)| < 3\varepsilon\), for every \( x \in S^+ \). Then,

\[
|g(x) - f(x)| = |H \circ \Phi(\Psi(x)) - F \circ \Phi(x)| \\
\leq |H(\Phi(\Psi(x))) - F(\Phi(\Psi(x)))| + |F \circ \Phi(\Psi(x)) - F \circ \Phi(x)| \\
\leq 3\varepsilon + \varepsilon = 4\varepsilon,
\]

for every \( x \in X \). Since \( \Phi \) and \( \Psi \) are \( C^p \) diffeomorphisms, we have that \( g'(x) = 0 \) if and only if \( H'(\Phi(\Psi(x))) = 0 \). For every \( x \in X \), \( \Psi(x) \notin \cup_n \overline{D}_n \) and thus \( \Phi(\Psi(x)) \notin \cup_n \overline{D}_n \).

It follows that \( H'(\Phi(\Psi(x))) \neq 0 \) and \( g \) does not have any critical point.

Before finishing the proof, let us say what additional precautions are required in the case when \( \varepsilon \) is a strictly positive continuous function:

- the slices \( S_k = \{ x \in S : f_k(x) > \delta_k \} \), \( (k \in \mathbb{N}) \) are selected with the additional property that the oscillation of the two functions \( F \) and \( \varepsilon = \varepsilon \circ \Phi^{-1} \) in \( S_k \) are less than \( \frac{\sigma(y_k)}{2} \), where \( y_k \) is the point of \( S^+ \) satisfying \( f_k(y_k) = 1 \); this implies, in particular, that \( \frac{3}{4} \sigma(y_k) < \varepsilon(x) < \frac{3}{2} \sigma(y_k) \), for every \( x \in S_k \);
- the real numbers \( a_k \in \mathbb{R}^* \) satisfy that \( |a_k - F(x_k)| < \frac{\sigma(y_k)}{2} \);
- the oscillation of \( r_k \) in \( S_k \) is less than \( \frac{\sigma(y_k)}{|a_k|} \).

From the above conditions and inequality \((2.20)\), it can be deduced that if \( x \in U_k \), then \( |a_k r_k(x) - F(x)| \leq 2\sigma(y_k) < 4\varepsilon(x) \). From this, it can be obtained that \(|H(x) - F(x)| \leq 4\varepsilon(x)\), for every \( x \in S^+ \). Equivalently, \(|H \circ \Phi(x) - F \circ \Phi(x)| = |H \circ \Phi(x) - f(x)| < 4\varepsilon(x)\), for every \( x \in X \). Now, if \( x \neq \Psi(x) \), then there is \( B_{nx} \) such that \( x, \Psi(x) \in B_{nx} \). Thus, \(|f(\Psi(x)) - f(x)| < \varepsilon \circ (\Phi^{-1}(y_k)) < \varepsilon \). Now, from inequality \((2.22)\), we obtain: (a) if \( x \in B_{nx} \) for some \( n_x \), then \(|g(x) - F(x)| \leq 4\varepsilon(\Psi(x)) + \varepsilon(x) \leq 6\varepsilon(\Phi^{-1}(y_k)) + \varepsilon(x) \leq 13\varepsilon(x) \), and (b) if \( x \notin \cup_n B_n \), then \(|g(x) - f(x)| \leq 4\varepsilon(\Psi(x)) = 4\varepsilon(x) \). This finishes the proof of Theorem \(1.1 \). \( \Box \)

Remark 2.11. The construction of the function \( g \) with no critical points that approximates \( f \) with a constant \( \varepsilon > 0 \), is considerably shorter in the case that either (i) \( X = \ell_2(\mathbb{N}) \) (and we use West Theorem \(28\)) or (ii) \( X \) is non-reflexive and the norm \(|\cdot|\) considered on \( Y \) can be constructed with the additional property that the set \( \{ f \in Y^* : f \) does not attain its norm\} contains a dense subspace (except the zero functional).

Indeed, in the first case, we can define as \(|\cdot|\) the standard norm on \( \ell_2(\mathbb{N}) \). In both cases, the use of the auxiliary functions \( r_n \) is not required, we can consider the slice \( R_n := S_n \) (that is, the additional construction of the sequence of slices \( \{ R_n \} \) is not required) and we can select for every \( n \in \mathbb{N} \), any strictly decreasing sequence \( \{ \gamma_{n,i} \}_i \) such that \( \lim_i \gamma_{n,i} = \delta_n \). Then, let us choose a non-zero functional \( w \in Y^* \setminus \{ f_n : n \in \mathbb{N} \} \) (where \( |f_n : n \in \mathbb{N} | \) denotes the space of all finite linear combinations of the set \( \{ f_n : n \in \mathbb{N} \} \) with \(|w|^* < \varepsilon \), and define \( H = \frac{\sum a_i h_i}{\sum_{i=1}^n h_i} + w \) and \( H_n = \frac{\sum a_i h_i}{\sum_{i=1}^n h_i} + w \), for every \( n \in \mathbb{N} \). We obtain in the case (i), that \( Z_n \) (the critical points of \( H_n \)), is included in the compact set \( D^{-1}([f_1,...,f_n,w] \cap S^*) \). Therefore, the set \( C \) of critical points of \( H \) and thus the set \( C \) of critical points of the composition \( H \circ \Phi \), are closed and locally compact sets of \( S^+ \) and \( \ell_2 \), respectively. Now, \( g \) is obtained, by applying West Theorem \(28\), considering a \( C^\infty \) deleting diffeomorphism \( \Psi \) from \( \ell_2(\mathbb{N}) \) onto \( \ell_2(\mathbb{N}) \setminus C \), with the additional property
that the family \( \{ (x, \Psi(x)) : x \in \ell_2(\mathbb{N}) \} \) refines the open covering \( \{ \Phi^{-1}(S_n) ; n \in \mathbb{N} \} \).

Finally, we can define \( g := H \circ \Phi \circ \Psi \).

In the case (ii), we can select the family \( G = \{ f_n : n \in \mathbb{N} \} \cup \{ w \} \) with the additional requirement that \( [G] \setminus \{ 0 \} \) is included in the set of non-norm attaining functionals. Thus, the sets of critical points of both \( H_n \) and \( H \) are empty. Therefore, the set of critical points of \( g := H \circ \Phi \) is empty and \( g \) approximates \( f \). Notice that this case is particularly interesting because the use of a deleting diffeomorphism is not required.

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