Higher order Riesz transforms for Hermite expansions

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Abstract
In this paper, we consider the Riesz transform of higher order associated with the harmonic oscillator $L = -\Delta + |x|^2$, where $\Delta$ is the Laplacian on $\mathbb{R}^d$. Moreover, the boundedness of Riesz transforms of higher order associated with Hermite functions on the Hardy space is proved.

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1 Introduction
Let $H_k(x)$ denote the Hermite polynomials on $\mathbb{R}$, which can be defined as

$$H_k(x) = (-1)^k \frac{d^k}{dx^k} (e^{-x^2}) e^{x^2}, \quad k = 0, 1, 2, \ldots$$

The normalized Hermite functions are defined by

$$h_k(x) = (\pi^{1/2} 2^k k!)^{1/2} H_k(x) \exp(-x^2/2), \quad k = 0, 1, \ldots$$

The high dimensional Hermite functions on $\mathbb{R}^d$ can be defined in the following way. For $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha_i \in \{0, 1, \ldots\}$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$,

$$h_\alpha(x) = \prod_{j=1}^d h_{\alpha_j}(x_j).$$

$\{h_\alpha\}$ forms a complete orthonormal basis of $L^2(\mathbb{R}^d)$. Let $|\alpha| = \alpha_1 + \cdots + \alpha_d$, then we have

$$L h_\alpha = (2|\alpha| + d) h_\alpha.$$

A very famous reference for Hermite functions is [1].

The operator $L$ is positive and symmetric on $L^2(\mathbb{R}^d)$. Let $\{T_t^L\}_{t \geq 0}$ be the heat kernel defined by

$$T_t^L f = e^{-tL} f = \sum_{n=0}^{\infty} e^{-t(2n+d)} P_n f$$
for \( f \in L^2(\mathbb{R}^d) \) and

\[
\mathcal{P}_n f = \sum_{|\alpha|=n} (f, h_\alpha)h_\alpha.
\]

Then the Poisson semigroup is defined as

\[
p_t^L f = e^{-tL^{1/2}} f = \sum_{n=0}^{\infty} e^{-t(2n+d)/2} \mathcal{P}_n f, \quad f \in L^2(\mathbb{R}^d).
\]

The relation between the heat kernel and the Poisson kernel is

\[
p_t^L f(x) = \frac{t}{\sqrt{4\pi}} \int_0^\infty s^{-3/2} \exp(-t^2/4s) T_t^L s f(x) ds. \tag{1}
\]

Let \( A_j = \frac{\partial}{\partial x_j} + x_j \) and \( A_{-j} = A_j^* = -\frac{\partial}{\partial x_j} + x_j, \ j = 1, 2, \ldots, d \). Then we can denote \( L \) as

\[
L = -\frac{1}{2} \left[ (\nabla + x) \cdot (\nabla - x) + (\nabla - x) \cdot (\nabla + x) \right] = \frac{1}{2} \sum_{j=1}^d A_j A_{-j} + A_{-j} A_j.
\]

We define operators \( R^L_j, j = 1, 2, \ldots, d \)

\[
R^L_j = A_j L^{-1/2}, \quad R^L_{-j} = A_{-j} L^{-1/2}.
\]

\( R_j \) and \( R_{-j} \) are called the Riesz transforms associated with \( L \). The definition was first suggested by Thangavelu in [2].

Let \( e_j \) be the coordinate vectors in \( \mathbb{R}^d \), then

\[
A_j h_\alpha = (2\alpha_j + 2)^{1/2} h_{\alpha+e_j}, \quad A_{-j} h_\alpha = (2\alpha_j)^{1/2} h_{\alpha-e_j}.
\]

Therefore, for \( f \in L^2(\mathbb{R}^d) \),

\[
R^L_j f = \sum_\alpha \left( \frac{2\alpha_j}{2|\alpha| + d} \right)^{1/2} (f, h_\alpha)h_{\alpha-e_j}
\]

\[
= \sum_{n=0}^{\infty} \sum_{|\alpha|=n} \left( \frac{2\alpha_j}{2n + d} \right)^{1/2} (f, h_\alpha)h_{\alpha-e_j}, \tag{2}
\]

and

\[
R^L_{-j} f = \sum_\alpha \left( \frac{2(\alpha_j + 1)}{2|\alpha| + d} \right)^{1/2} (f, h_\alpha)h_{\alpha+e_j}
\]

\[
= \sum_{n=0}^{\infty} \sum_{|\alpha|=n} \left( \frac{2(\alpha_j + 1)}{2n + d} \right)^{1/2} (f, h_\alpha)h_{\alpha+e_j}. \tag{3}
\]

In [1], the author proved that \( R^L_j \) were bounded on the local Hardy spaces \( \mathcal{H}^1(\mathbb{R}^d) \) which were defined by Goldberg in [3]. Thangavelu asked one question: whether it was possible
to characterize $h^1(\mathbb{R}^d)$ by $R_j^d$, i.e., whether the equality

$$h^1(\mathbb{R}^d) = \{ f \in L^1(\mathbb{R}^d) : R_j^d f \in L^1(\mathbb{R}^d), j = 1, 2, \ldots, d \}$$

is true. In [4], the author proved the boundedness of $R_j^d$ on Hardy spaces $H^1_2(\mathbb{R}^d)$, $d \geq 3$, where $H^1_2(\mathbb{R}^d)$ are the Hardy spaces for $L$ (cf. [5]).

**Proposition 1** Let $j = 1, 2, \ldots, d$. Then the operators $R_j^d$ are bounded on $H^1_2(\mathbb{R}^d)$, that is, there exists $C > 0$ satisfying

$$\| R_j^d f \|_{H^1_2} \leq C \| f \|_{H^1_2}.$$

Moreover, he characterized $H^1_2(\mathbb{R}^d)$ by $R_j^d$, $j = 1, 2, \ldots, d$. Therefore, we cannot characterize $h^1(\mathbb{R}^d)$ by $R_j^d$.

**Remark 1** When we consider the boundedness of Riesz transforms for $L$ on Hardy spaces, the main tool is Littlewood-Paley characterizations of Hardy spaces. In fact, we have the following equality (cf. [4]):

$$t_0 e^{-\frac{t_0^2}{2t}L_{L+2}^{1/2}} \left( R_j^d f \right) = -t \left( \pm \frac{\partial}{\partial x_j} + x_j \right) e^{-\frac{t}{2}L_{L+2}^{1/2}} f$$

for all $j = 1, 2, \ldots, d$ and $f \in L^2(\mathbb{R}^d)$. If we prove the boundedness of Riesz transforms $R_j^d$ on Hardy spaces, we need to consider the operator $L - 2$. Since the Hardy spaces $H^1_2(\mathbb{R}^d)$, $d \geq 3$, associated with $L$ defined in [5] are for nonnegative potentials, it is maybe natural to just consider $R_j^d$. In [6], the authors proved the boundedness of $R_j^d$ on $L^p(\mathbb{R}^d)$, where they considered the semigroup generated by $L + b$ for $b < 0$ on $L^p(\mathbb{R}^d)$.

In this paper, we prove that the higher ordered Riesz transforms are bounded on the Hardy spaces associated with Hermite functions. More precisely, let

$$L^{-m/2}h_x = (2|\alpha| + d)^{-m/2}h_x,$$

and define the $m$-ordered Riesz transforms as

$$R_{i_1 \cdots i_m} = A_{i_1} A_{i_2} \cdots A_{i_m} L^{-m/2},$$

where $1 \leq i_j \leq d$ and $1 \leq j \leq m$.

We define Hardy space $H^1_2(\mathbb{R}^d)$ for $d \geq 3$ as follows (cf. [5]):

$$H^1_2(\mathbb{R}^d) = \left\{ f \in L^1(\mathbb{R}^d) : \mathcal{M}_f f \in L^1(\mathbb{R}^d) \right\},$$

where $\mathcal{M}_f f(x) = \sup_{r > 0} |T^r f(x)|$.

Define

$$\rho(x) = \frac{1}{1 + |x|},$$

we say $a(x)$ is an atom for the space $H^1_2(\mathbb{R}^d)$ if there exists a ball $B(x_0, r)$ such that

1. $\text{supp } a \subset B(x_0, r)$,
(2) \( \|a\|_{L^\infty} \leq |B(x_0, r)|^{-1} \),
(3) if \( r < \rho(x_0) \), then \( \int a(x) \, dx = 0 \).

The atomic quasi-norm in \( H^1_1(\mathbb{R}^d) \) can be defined as
\[
\|f\|_{L\text{-atom}} = \inf \left\{ \sum |c_j| \right\}.
\]

In [5], the authors proved the following result.

**Proposition 2**  There exists \( C > 0 \) satisfying
\[
C^{-1} \|f\|_{H^1_1} \leq \|f\|_{L\text{-atom}} \leq C \|f\|_{H^1_1}.
\]

Let \( b \in \mathbb{R}^d \). We define
\[
G^b_t(x, y) = e^{-bt} G^b_t(x, y).
\]

Then
\[
G^b_t(f)(x) = \int_{\mathbb{R}^d} G^b_t(x, y)f(y) \, dy
\]
is a semigroup for the spaces \( L^p(\mathbb{R}^d) \), \( 1 \leq p < \infty \), and \( \|G^b_t(f)\|_{L^p(\mathbb{R}^d)} \leq e^{-bt} \|f\|_{L^p(\mathbb{R}^d)} \). This semigroup is generated by the operator \(-L + b\).

The subordination formula is
\[
P^b_t(x, y) = \frac{t}{\sqrt{4\pi}} \int_0^\infty G^b_s(x, y)s^{-3/2}e^{-t^2/4s} \, ds.
\]  (5)

The Poisson integral of \( f(x) \) can be defined as
\[
u_b(x, t) = P^b_t(f)(x) = \int_{\mathbb{R}^d} P^b_t(x, y)f(y) \, dy,
\]
\[
= \frac{t}{\sqrt{4\pi}} \int_{\mathbb{R}^d} \int_0^\infty G^b_s(x, y)f(y)s^{-3/2}e^{-t^2/4s} \, ds \, dy.
\]

Let
\[
G_b(f)(x) = \left( \int_0^\infty \sum_{j=0}^d |tA_ju_b(x, t)|^2 \frac{dt}{t} \right)^{1/2}
\]
and
\[
G^1_b(f)(x) = \left( \int_0^\infty \left| t\partial_t u_b(x, t) \right|^2 \frac{dt}{t} \right)^{1/2},
\]
where \( A_0 = \partial_t \).

The main results of this paper are as follows.
Theorem 1 \(f \in H^1_L(\mathbb{R}^d)\) is equivalent to \(G_b(f) \in L^1(\mathbb{R}^d)\) and \(f \in L^1(\mathbb{R}^d)\). Moreover,
\[
\|f\|_{H^1_L} \sim \|G_b(f)\|_{L^1} + \|f\|_{L^1}.
\]

Theorem 2 The operators \(R_{i_1j_1} \cdots R_{i_mj_m} = A_{i_1}A_{j_2} \cdots A_{i_m}L^{-m/2}\) are bounded on \(H^1_L(\mathbb{R}^d)\) for all \(1 \leq i_j \leq d\) for every \(1 \leq j \leq m\), that is, there exists \(C > 0\) satisfying
\[
\|R_{i_1j_1} \cdots R_{i_mj_m}f\|_{H^1_L} \leq C\|f\|_{H^1_L}.
\]

The organization of this paper is as follows. In Section 2, we give some estimations of the heat kernel and the Poisson kernel associated with \(L + b\). In Section 3, Theorem 1 is proved. In Section 4, we prove Theorem 2.

Throughout the article, we use \(A\) and \(C\) to denote the positive constants, which are independent of the main parameters and may be different at each occurrence. By \(B_1 \sim B_2\), we mean that there exists a constant \(C > 1\) such that \(C_B_1 \leq B_2 \leq C_B_1\).

2 Estimations of the kernels
Let \(G^b_t(x,y)\) be the heat kernel of \(\{T^L_t + b\}\). Then the following inequality can be proved by the Feynman-Kac formula:
\[
G^b_t(x,y) \leq W_t(x-y),
\]
where
\[
W_t(x) = (4\pi t)^{-d/2} \exp(-|x|^2/(4t))
\]
is the heat kernel on \(\mathbb{R}^d\).

Since \(G^b_t(x,y) \leq G^L_t(x,y)\), we have (cf. [7]) the following lemma.

Lemma 1
(a) For \(N \in \mathbb{N}\), there exists \(C_N > 0\)
\[
0 \leq G^L_t(x,y) \leq C_N t^{-d/2} e^{-|x-y|^2/(2t)} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^N. \tag{6}
\]

(b) There are constants \(0 < \delta < 1\) and \(C > 0\), for \(N > 0\), there is \(C_N > 0\) which satisfies for all \(|h| \leq \frac{|x-y|}{x}\),
\[
|G^L_t(x+h,y) - G^L_t(x,y)| \leq C_N \left(\frac{|h|}{\sqrt{t}}\right)^\delta t^{-d/2} e^{-|x-y|^2/(2t)} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^N. \tag{7}
\]

By the subordination formula, we get the following.

Lemma 2
(a) For \(N \in \mathbb{N}\), there is \(C_N > 0\) satisfying
\[
0 \leq P^b_t(x,y) \leq C_N \frac{t}{(t^2 + A|x-y|^2)^d+2/2} \left(1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)}\right)^N. \tag{8}
\]
(b) Let $0 < \delta < 1$ and $|h| < \frac{|x-y|}{2}$. Then, for $N \in \mathbb{N}$, there are $C > 0$, $C_N > 0$ satisfying

$$
|P^h_t(x+h,y) - P^h_t(x,y)|
\leq C_N \left( \frac{|h|}{t} \right)^\delta \frac{t}{(t^2 + A|x-y|^2)^{(d+1)/2}} \left( 1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)} \right)^{-N}.
$$

Proof (a) By subordination formula and Lemma 1, we have

$$
0 \leq P^h_t(x,y) \leq \frac{1}{\sqrt{\pi}} \int_0^\infty G^h_{t^2/4\mu}(x,y) e^{-\mu} \mu^{-1/2} \, d\mu
$$

By (10) and

$$
P^h_t(x,y) \leq \frac{t}{(t^2 + A|x-y|^2)^{(d+1)/2}},
$$

we get

$$
0 \leq P^h_t(x,y) \leq C_N \frac{t}{(t^2 + A|x-y|^2)^{(d+1)/2}} \left( 1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)} \right)^{-N}.
$$

(b) By subordination formula again, we know

$$
|P^h_t(x+h,y) - P^h_t(x,y)|
\leq \frac{1}{\sqrt{\pi}} \int_0^\infty \left| G^h_{t^2/4\mu}(x+h,y) - G^h_{t^2/4\mu}(x,y) \right| e^{-\mu} \mu^{-1/2} \, d\mu
$$

By (10) and

$$
P^h_t(x,y) \leq \frac{t}{(t^2 + A|x-y|^2)^{(d+1)/2}},
$$

we get

$$
0 \leq P^h_t(x,y) \leq C_N \frac{t}{(t^2 + A|x-y|^2)^{(d+1)/2}} \left( 1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)} \right)^{-N}.
$$
We also have

\[
\left| P_t^b(x+h,y) - P_t^b(x,y) \right| \\
\leq C_N \int_0^\infty \left( \frac{t^2}{4\mu} \right)^{-\frac{d}{2}} e^{-C_t r^2(4\mu|x-y|^2)} \left( \frac{\sqrt{4\mu}|h|}{t} \right)^{d'} e^{-\mu t^{-1/2}} d\mu \\
= C_N \left( \frac{|h|}{t} \right)^{d'} \int_0^\infty \left( \frac{t^2}{4\mu} \right)^{-\frac{d}{2}} e^{-C_t r^2(4\mu|x-y|^2)} e^{-\mu t^{-1/2}} d\mu \\
\leq C_N \left( \frac{|h|}{t} \right)^{d'} \int_0^\infty \left( \frac{t^2}{4\mu} \right)^{-\frac{d}{2}} e^{-C_t r^2(4\mu|x-y|^2)} e^{-\mu t^{-1/2}} d\mu \\
= C_N \left( \frac{|h|}{t} \right)^{d'} \left( \frac{t}{(t^2 + A|x-y|^2)^{d+1}/2} \right)^N. 
\] (12)

Then (b) follows from (11) and (12). \hfill \Box

Let \( D_t^b(x,y) = t^k \partial_t^k P_t^b(x,y) \). Then, by Lemma 2, we can prove (cf. [8] or [9]) the following.

**Proposition 3** There are \( C > 0, \ 0 < \delta' < \delta \), for \( N \in \mathbb{N} \), there is \( C_N \) such that

(a) \( \left| D_t^{jk}(x,y) \right| \leq C_N \frac{t}{(t^2 + C|x-y|^2)^{d+1}/2} \left( 1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)} \right)^N \); \\
(b) \( \left| D_t^{jk}(x+h,y) - D_t^{jk}(x,y) \right| \leq C_N \left( \frac{|h|}{t} \right)^{d'} \frac{t}{(t^2 + C|x-y|^2)^{d+1}/2} \left( 1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)} \right)^N \)

for all \( |h| \leq \frac{|x-y|}{2} \).

Let \( t = \frac{1}{2} \ln \frac{x^2}{|x-y|^2}, \ s \in (0,1). \) Then

\[
G_t(x,y) = \left( \frac{1-s^2}{4\pi ts} \right)^{d/2} \exp \left( -\frac{1}{4} \left( s|x+y|^2 + \frac{1}{s}|x-y|^2 \right) \right) = K_s(x,y). \quad (13)
\]

The proof of the following proposition is motivated by [10].

**Proposition 4** There is \( A > 0 \), for \( N \in \mathbb{N} \) and \( |x-x'| \leq \frac{|x-y|}{2} \), we can find \( C_N > 0 \) such that

(a) \( \left| tA_j G_t^b(x,y) \right| \leq C_N t^{-2} \exp \left( -\frac{|x-y|^2}{At} \right) \left( 1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)} \right)^N \); \\
(b) \( \left| tA_j G_t^b(x,y) - tA_j G_t^b(x',y) \right| \leq C_N \frac{|x-x'|}{t} t^{-2} \exp \left( -\frac{|x-y|^2}{At} \right) \left( 1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)} \right)^N. \)
Proof By

\[ |A_j G_t(x, y)| = \left| \frac{\partial}{\partial x_j} G_t(x, y) + x_j G_t(x, y) \right| \]

\[
\leq \left| \frac{\partial}{\partial x_j} G_t(x, y) \right| + |x_j G_t(x, y)| \approx I_1 + I_2,
\]

and \( t = \frac{1}{2} \ln \frac{1+s}{1-s} \sim s, s \to 0^+ \), for \( s \in (0, \frac{1}{2}] \), we have

\[ I_2 \leq C|x_j|s^{-\frac{d}{2}} \exp \left( -\frac{1}{4} s|x + y|^2 \right) \exp \left( -\frac{1}{4} \frac{|x - y|^2}{s} \right) \]

\[ \leq C|x|s^{-\frac{d-1}{2}} \exp \left( -\frac{1}{4} \frac{|x - y|^2}{8s} \right). \]

If \( x \cdot y \leq 0 \), then \( |x| \leq |x - y| \). So

\[ I_2 \leq Cs^{-\frac{d}{2}} |x - y| \exp \left( -\frac{1}{4} s|x + y|^2 \right) \exp \left( -\frac{1}{4} \frac{|x - y|^2}{s} \right) \]

\[ \leq Ct^{-\frac{d-1}{2}} \exp \left( -\frac{|x - y|^2}{4s} \right). \]

If \( x \cdot y \geq 0 \), then \( |x| \leq |x + y| \). So

\[ I_2 \leq Cs^{-\frac{d}{2}} |x + y| \exp \left( -\frac{1}{4} s|x + y|^2 \right) \exp \left( -\frac{1}{4} \frac{|x - y|^2}{s} \right) \]

\[ \leq Cs^{-\frac{d+1}{2}} \exp \left( -\frac{|x - y|^2}{4s} \right) \leq Ct^{-\frac{d+1}{2}} \exp \left( -\frac{|x - y|^2}{4t} \right). \]

Therefore,

\[ |I_2| \leq C(t^{\frac{d}{2}} + t^{\frac{d+1}{4}}) e^{-bt} t^{-\frac{d}{4}} \exp \left( -\frac{|x - y|^2}{8t} \right) \leq Ct^{-\frac{d}{2}} \exp \left( -\frac{|x - y|^2}{8t} \right). \] (14)

When \( s \in [\frac{1}{2}, 1) \),

\[ I_2 \leq C|x| \exp \left( -\frac{1}{4} \left( s|x + y|^2 + \frac{|x - y|^2}{s} \right) \right) \]

\[ \leq C|x|s^{-\frac{d}{2}} \exp \left( -\frac{1}{4} \left( s|x + y|^2 + \frac{|x - y|^2}{s} \right) \right) \]

\[ \leq C(|x + y| + |x - y|)s^{-\frac{d}{2}} \exp \left( -\frac{1}{4} \left( s|x + y|^2 + \frac{|x - y|^2}{s} \right) \right) \]

\[ \leq C \exp \left( -\frac{|x - y|^2}{8s} \right). \]

Since \( t = \frac{1}{2} \ln \frac{1+s}{1-s} > s \) for \( s \in [\frac{1}{2}, 1) \), we get

\[ I_2 \leq C \exp \left( -\frac{|x - y|^2}{8t} \right). \]
Therefore,
\[ |tI_2| \leq Cte^{-bt} \exp\left(-\frac{|x-y|^2}{8t}\right) \leq Ct^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{8t}\right). \] (15)

By (13), we get
\[ \frac{\partial}{\partial x_j} K_s(x, y) = -\frac{1}{2} \left(s(x_j + y) + \frac{1}{s}(x_j - y)\right) K_s(x, y), \]
and
\[ I_1 \leq C \left(s|x_j + y| + \frac{1}{s}|x_j - y|\right) K_s(x, y) \leq C \left(s|x + y| + \frac{1}{s}|x - y|\right) K_s(x, y). \]

Therefore, when \( s \in (0, \frac{1}{2}] \), we have
\[ I_1 \leq Cs^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{8s}\right) \leq Ct^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{8t}\right). \]

When \( s \in [\frac{1}{2}, 1) \), we have
\[ I_1 \leq C \exp\left(-\frac{|x-y|^2}{8s}\right) \leq C \exp\left(-\frac{|x-y|^2}{8t}\right). \]

Then
\[ \left| t \frac{\partial}{\partial x_j} G_t^b(x, y) \right| \leq Ct(1 + t^{-\frac{d}{2}}) e^{-bt} \exp\left(-\frac{|x-y|^2}{8t}\right) \leq Ct^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{8t}\right). \] (16)

By (14)-(16), we get
\[ |tA_j G_t^b(x, y)| \leq Ct^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{8t}\right). \] (17)

Similar to the proof of (17), for any \( N > 0 \), we can prove
\[ \left(\sqrt{t} |x|\right)^N |tA_j G_t^b(x, y)| \leq C_N t^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{8t}\right) \]
and
\[ t^N |tA_j G_t^b(x, y)| \leq C_N t^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{8t}\right). \]

Since \( \rho(x) = \frac{1}{\sqrt{1 + |x|}} \), we get \( \frac{\sqrt{t}}{\rho(x)} = \sqrt{t}(1 + |x|) \). Then, for \( N > 0 \),
\[ \left(\frac{\sqrt{t}}{\rho(x)}\right)^N |tA_j G_t^b(x, y)| \leq C_N t^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{8t}\right). \] (18)
Since $x$ and $y$ are symmetric, we also have
\[
\left( \frac{\sqrt{t}}{\rho(y)} \right)^N |tA_jG^H_t(x,y)| \leq CN t^{-\frac{d}{2}} \exp \left( -\frac{|x-y|^2}{8t} \right). \tag{19}
\]

Then (a) follows from (17)-(19).

(b) Note that
\[
|tA_jG^H_t(x',y) - tA_jG^H_t(x,y)| \\
\leq \left| t \frac{\partial}{\partial x_j} G^H_t(x',y) - t \frac{\partial}{\partial x_j} G^H_t(x,y) \right| + \left| t x'_j G^H_t(x',y) - t x_j G^H_t(x,y) \right| \\
= J_1 + J_2.
\]

For $J_2$, let
\[
\varphi(z) = \varphi_{y,s}(z) = z_j \exp \left( -\frac{1}{4} \alpha(s,z,y) \right),
\]
where $\alpha(s,z,y) = s|z+y|^2 + \frac{1}{s}|z-y|^2$.

Then
\[
\frac{\partial \varphi}{\partial z_k}(z) = \left( \delta_k - \frac{s}{2} z_j(z_k + y_k) - \frac{1}{2s} z_j(z_k - y_k) \right) \exp \left( -\frac{1}{4} \alpha(s,z,y) \right).
\]

Therefore
\[
\left| \frac{\partial \varphi}{\partial z_k}(z) \right| \leq C \left( 1 + s|z+y| + \frac{1}{s}|z-y| \right) \exp \left( -\frac{1}{4} \alpha(s,z,y) \right) \\
\leq C \left( 1 + s^{1/2}|z| + \frac{1}{s^{1/2}}|z| \right) \exp \left( -\frac{1}{8} \alpha(s,z,y) \right) \\
\leq C \left( 1 + s^{1/2}(|z-y| + |z+y|) + \frac{1}{s^{1/2}}(|z-y| + |z+y|) \right) \exp \left( -\frac{1}{8} \alpha(s,z,y) \right) \\
\leq C \left( 1 + s + \frac{1}{s} \right) \exp \left( -\frac{1}{16s} |z-y|^2 \right) \\
\leq Cs^{-1} \exp \left( -\frac{1}{16s} |z-y|^2 \right). \tag{20}
\]

Let $\theta = \lambda x + (1-\lambda)x'$, $0 < \lambda < 1$. Then
\[
J_2 = te^{-bt}|x'_jK_s(x',y) - x_jK_s(x,y)| \\
\leq Ct^{-d/2} |x-x'| \sup_{\theta} |\nabla \varphi(\theta)| \\
\leq Ct^{-d/2} \frac{|x-x'|}{s} \sup_{\theta} \exp \left( -\frac{|\theta-y|^2}{16s} \right) \\
\leq Ct^{-d/2} \frac{|x-x'|}{t} \sup_{\theta} \exp \left( -\frac{|\theta-y|^2}{16t} \right).
\]
When $|x - x'| \leq \frac{|x - y|}{2}$, we can get $|\theta - y| \sim |x - y|$. Therefore, there exists $A > 0$ such that

$$J_2 \leq Ct^{-d/2} \frac{|x - x'|}{t} \exp \left(-\frac{|x-y|^2}{At} \right). \tag{21}$$

For $J_1$,

$$J_1 = \left| t \frac{\partial}{\partial x_i} G_t^b(x', y) - t \frac{\partial}{\partial x_i} G_t^b(x, y) \right|$$

$$= te^{-bt} \left| \frac{\partial}{\partial x_j} K(x', y) - \frac{\partial}{\partial x_j} K(x, y) \right|$$

$$= te^{-bt} \left[ s(x_j + y_j) + \frac{1}{s}(x_j - y_j) \right] \exp \left(-\frac{1}{4} \alpha(s, x, y) \right)$$

$$- \left[ s(x'_j + y_j) + \frac{1}{s}(x'_j - y_j) \right] \exp \left(-\frac{1}{4} \alpha(s, x', y) \right) \right|.$$ 

Let

$$\psi(z) = \psi_{s,z}(z) = \left( s(z_j + y_j) + \frac{1}{s}(z_j - y_j) \right) \exp \left(-\frac{1}{4} \alpha(s, z, y) \right).$$

Then

$$\frac{\partial \psi}{\partial z_k}(z) = \left[ s + \frac{1}{s} \right] \delta_{jk} - \left[ s(z_j + y_j) + \frac{1}{s}(z_j - y_j) \right]$$

$$\times \left[ s(z_k + y_k) + \frac{1}{s}(z_k - y_k) \right] \exp \left(-\frac{1}{4} \alpha(s, z, y) \right).$$

Therefore, similar to the proofs of (20) and (21), we can prove

$$\left| \frac{\partial \psi}{\partial z_k}(z) \right| \leq Cs^{-1} \exp \left(-\frac{1}{4} \alpha(s, z, y) \right)$$

and

$$J_1 \leq Ce^{-bt} \sup_{\theta} \left| \nabla \psi(\theta) \right| \left| x - x' \right|$$

$$\leq Ct^{-d/2} \frac{|x - x'|}{t} \exp \left(-\frac{|x-y|^2}{At} \right). \tag{22}$$

Inequalities (21) and (22) show

$$\left| tA_j G_t^b(x, y) - tA_j G_t^b(x', y) \right| \leq C_N \frac{|x - x'|}{t} t^{-d/2} \exp \left(-\frac{|x-y|^2}{At} \right).$$

Then, similar to the proof of (a), we have

$$\left| tA_j G_t^b(x, y) - tA_j G_t^b(x', y) \right| \leq C_N \frac{|x - x'|}{t} t^{-d/2} \exp \left(-\frac{|x-y|^2}{At} \right) \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)} \right)^{N}.$$

This completes the proof of Proposition 4. \qed
The subordination formula gives the following lemma.

Lemma 3

(a) For \( N \in \mathbb{N} \), there is \( C_N > 0 \) satisfying

\[
|tA_tP^t(x,y)| \leq C_N \frac{t}{(t^2 + A|x-y|^2)^{(d+1)/2}} \left( 1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)} \right)^{-N}.
\]

(b) For any \( N > 0 \) and \( |x - x'| \leq \frac{|x-y|}{2} \), there are \( C_N > 0, C > 0 \), so that

\[
|tA_tP^t(x,y) - tA_tP^t(x',y)| \leq C_N \left( \frac{|x-x'|}{t} \right) \frac{t}{(t^2 + A|x-y|^2)^{(d+1)/2}} \left( 1 + \frac{t}{\rho(x)} + \frac{t}{\rho(y)} \right)^{-N}.
\]

3 Square function characterizations of \( H^1_L(\mathbb{R}^d) \)

We define square functions

\[
G^b_k f(x) = \left( \int_0^\infty |D^b_k f(x)|^2 \frac{dt}{t} \right)^{1/2}
\]

and

\[
S^b_k f(x) = \left( \int_0^\infty \int_{|x-y| < t} |D^b_k f(y)|^2 \frac{dy dt}{td^{d+1}} \right)^{1/2},
\]

where \( D^b_k f(x) = t^k (\partial^k_t P^b_t f)(x) \) for \( k = 1,2,\ldots \).

The proof of the following lemma can be found in [4].

Lemma 4 If \( f \in L^1(\mathbb{R}^d) \), we have \( f \in H^1_L(\mathbb{R}^d) \) is equivalent to \( f \in H^1_L(\mathbb{R}^d) \) for \( b > 0 \).

Then, by Lemma 4, we can prove (cf. Section 8 in [11] or [12]) the following.

Proposition 5 \( f \in H^1_L(\mathbb{R}^d) \) is equivalent to its area integral \( S^b_k f \in L^1(\mathbb{R}^d) \) and \( f \in L^1(\mathbb{R}^d) \). Moreover,

\[
\|f\|_{H^1_L} \sim \|f\|_{H^1_L} \sim \|S^b_k f\|_{L^1} + \|f\|_{L^1}.
\]

Motivated by [13], we can prove the following.

Lemma 5 There is \( C > 0 \) satisfying

\[
\|S^{b+k} f\|_{L^1} \leq C \|G^b_k f\|_{L^1}.
\]

Proof Let

\[
F(x)(t) = (\partial^k_t e^{-t\sqrt{L}F}) f(x), \quad V(x,s) = e^{-s\sqrt{L}F} F(x).
\]
Then

\[ V(x,s)(t) = e^{-\sqrt{s-t}b} (\partial_t^k e^{-\sqrt{s-t}b} f)(x) = (\partial_t^k e^{-(s+t)}\sqrt{s}b f)(x). \]

Therefore

\[
\int_0^{+\infty} |V(x,s)(t)|^2 t^{2k-1} dt = \int_0^{+\infty} |(\partial_t^k e^{-(s+t)}\sqrt{s}b f)(x) |^2 t^{2k-1} dt \\
= \int_s^{+\infty} |(\partial_t^k e^{-\sqrt{s-t}b} f)(x) |^2 (t-s)^{2k-1} dt.
\]

Hence

\[
\sup_{s>0} \int_0^{+\infty} |V(x,s)(t)|^2 t^{2k-1} dt \leq \int_0^{+\infty} |(t^k \partial_t^k e^{-\sqrt{s-t}b} f)(x) |^2 dt = (\gamma_L^h b f(x))^2.
\]

Let \( X = L^2((0, \infty), t^{2k-1} dt) \). Then

\[
\sup_{s>0} \| e^{-\sqrt{s-t}b} F(x) \|_X = G_L^h b f(x) \in L^1(\mathbb{R}^d).
\]

Therefore \( F \in H_X^1(\mathbb{R}^d) \), here \( H_X^1(\mathbb{R}^d) \) is a vector-valued Hardy space. Therefore \( \tilde{S}_L \gamma^1 F(x) \in L^1(\mathbb{R}^d) \), where

\[
\tilde{S}_L^\gamma^1 F(x) = \left( \int_0^{+\infty} \int_{|y|<2t} \| D_t^{\gamma^1} F(y) \|_X^2 \frac{dy dt}{t^{d+1}} \right)^{1/2}.
\]

By

\[
(\tilde{S}_L^\gamma^1 F(x))^2 = \int_0^{+\infty} \int_{|x-y|<2t} \| D_t^{\gamma^1} (x) \|_X^2 \frac{dy dt}{t^{d+1}} \\
= \int_0^{+\infty} \int_{|x-y|<2t} \int_0^{+\infty} |(-t^\gamma L + b) e^{-\sqrt{s-t}b} F(y)(s) |^2 s^{2k-1} ds \frac{dy dt}{t^{d+1}} \\
= \int_0^{+\infty} \int_0^{+\infty} \int_{|x-y|<2t} |(-\sqrt{L} + b)^{k+1} e^{-(s+t)\sqrt{s}b} f(y) |^2 \\
\times t^{1-d} s^{2k-1} dy dt ds \\
= \int_0^{+\infty} \int_{|x-y|<2t} |(-\sqrt{L} + b)^{k+1} e^{-\sqrt{s-t}b} f(y) |^2 \\
\times (t-s)^{1-d} s^{2k-1} dy dt ds \\
\geq \int_0^{+\infty} \int_{|x-y|<2(t-s)} |(-\sqrt{L} + b)^{k+1} e^{-\sqrt{s-t}b} f(y) |^2 \\
\times (t-s)^{1-d} s^{2k-1} dy ds dt \\
\geq \int_0^{+\infty} \int_{|x-y|<2t} |(-\sqrt{L} + b)^{k+1} e^{-\sqrt{s-t}b} f(y) |^2 t^{1-d} s^{2k-1} dy ds dt.
\]
\[
\begin{align*}
&= \frac{1}{2k^{2k}} \int_0^{\infty} \int_{|x-y|<ct} \left| (-t\sqrt{L} + \mathcal{B})^{k+1} e^{-t\sqrt{L}+b} f(x) \right|^2 t^{-1-2\delta} dy dt \\
&= \frac{1}{2k^{2k}} \int_0^{\infty} \int_{|x-y|<ct} |D_t^{b,k+1} f(y)|^2 \frac{dy dt}{t^{d+1}} = \frac{1}{2k^{2k}} \left( S_b^{b,k} f(x) \right)^2,
\end{align*}
\]

we get \( \|S_b^{b,k+1} f\|_{L^1} \leq C \|G_{b,k}^L(f)\|_{L^1}. \) \( \Box \)

By Lemma 5, we can prove the following.

**Proposition 6** \( f \in H^1_L(\mathbb{R}^d) \) is equivalent to \( G_{b,k}^L f \in L^1(\mathbb{R}^d) \) and \( f \in L^1(\mathbb{R}^d) \). Moreover,

\[
\|f\|_{H^1_L} \sim \|G_{b,k}^L f\|_{L^1} + \|f\|_{L^1}.
\]

Similar to the proof of Lemma 14 in [9], we have the following.

**Lemma 6** Let \( a \) be an \( H^1_{L^\infty} \)-atom. Then we can find a constant \( C > 0 \) satisfying

\[
\|G_0(a)\|_{L^1} \leq C.
\]

As pointed out in [14], we cannot get that an operator is bounded on \( H^p_L(\mathbb{R}^d) \) by just proving that it is uniformly bounded on atoms. But we have the following lemma (cf. p.316, Theorem 7.3 in [15]).

**Lemma 7** Let \( T \) be an integral operator with the kernel in the Campanato space \( \Lambda_{d(1/p-1)} \) and satisfy \( \|Ta\|_{L^p} \leq C \) for all the \( H^0_{L^\infty} \)-atom \( a(x) \), then \( T \) is a bounded operator from \( H^p_L(\mathbb{R}^d) \) to \( L^p(\mathbb{R}^d) \).

In the following, we prove \( D_t^p(x,y) = tA_t^p(x,y) \) belongs to \( BMO_L \), which is defined in [8].

**Lemma 8** For every \( t > 0 \) and \( x \in \mathbb{R}^d \), we have \( D_t^p(x,y) \in BMO_L \).

**Proof** For any ball \( B(y_0,r) \), if \( r < \rho(y_0) \) and \( r < t \), then by Lemma 3(b) we have

\[
\begin{align*}
\frac{1}{|B|^{1/2}} \left( \int_B |D_t^p(x,y) - D_t^p(x,y_0)|^2 dy \right)^{1/2} &\leq Cr^{-d/2} \left( \int_B \frac{|y - y_0|^2}{t} \frac{t^{-2d}}{1 + t^{-2}|x-y_0|^2} \frac{1}{|x-y_0|^{d+1}} dy \right)^{1/2} \\
&\leq C r^{-d} \left( \frac{r}{t} \right) \leq C r^{-d}.
\end{align*}
\]

(25)

If \( t \leq r < \rho(y_0) \), then by Lemma 3(a)

\[
\begin{align*}
\frac{1}{|B|^{1/2}} \left( \int_B |D_t^p(x,y) - D_t^p(x,y_0)|^2 dy \right)^{1/2} &\leq C r^{-d}.
\end{align*}
\]

(26)
If \( r \geq \rho(y_0) \), then by Lemma 3(a) we have

\[
\frac{1}{|B|^{1/2}} \left( \int_B \left| D_t^b(x, y) \right|^2 \, dy \right)^{1/2} \leq C r^{-d/2} \left( \int_B \left( 1 + t^{-2} |x - y_0|^2 \right)^{d+1} \, dy \right)^{1/2} \leq C t^{-d} r^{-d/2} |B|^{1/2} \leq C t^{-d} \text{.} \tag{27}
\]

Then Lemma 8 follows from (25)-(27).

Now, let us prove Theorem 1.

\textbf{Proof of Theorem 1} When \( f \in L^1(\mathbb{R}^d) \) and \( \mathcal{G}_b(f) \in L^1(\mathbb{R}^d) \), by Proposition 5 and Lemma 5, we have

\[
\|f\|_{H^1_L} \leq \|f\|_{H^1_L} \leq C \left\{ \left\| \mathcal{S}_{L+b}^2(f) \right\|_{L^1} + \|f\|_{L^1} \right\} \leq C \left\{ \left\| \mathcal{G}_{L+b}^b(f) \right\|_{L^1} + \|f\|_{L^1} \right\}
\]

Therefore, \( f \in H^1_L(\mathbb{R}^d) \).

The reverse can be proved by Lemmas 6, 7 and 8.  

Theorem 1 is proved. \( \square \)

\section{4 Riesz transform associated with \( L \)}

We introduce the following version of Riesz transform:

\[
R_j^{L,b} = A_j(L + b)^{-1/2}, \quad j = 1, 2, \ldots, d, b > 0.
\]

If \( f \in L^2(\mathbb{R}^d) \), then

\[
R_j^{L,b} f = \sum_{\alpha} \left( \frac{2\alpha_j}{2|\alpha| + d + b} \right)^{1/2} \langle f, h_\alpha \rangle h_{\alpha - \alpha_j}
\]

\[
= \sum_{n=0}^{\infty} \sum_{|\alpha|=n} \left( \frac{2\alpha_j}{2n + d + b} \right)^{1/2} \langle f, h_\alpha \rangle h_{\alpha - \alpha_j} \text{.} \tag{28}
\]

We can prove the following.

\textbf{Theorem 3} Let \( j = 1, 2, \ldots, d \). Then \( R_j^{L,b} \) are bounded operators on \( H^1_L(\mathbb{R}^d) \), that is, there is \( C > 0 \) satisfying

\[
\|R_j^{L,b} f\|_{H^1_L} \leq C\|f\|_{H^1_L} \text{.}
\]

\textbf{Proof} When \( f \in L^2(\mathbb{R}^d) \), following (28), it is not difficult to check

\[
D_t^{b+2} R_j^{L,b} f = -tA_j h_b(x, t) \text{.} \tag{29}
\]
for \( j = 1, 2, \ldots, d \). As \( H^1_1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \) is dense in \( H^1_1(\mathbb{R}^d) \) (see [11]), we can assume \( f \in H^1_1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \). Then, by Lemma 5, Theorem 1, Proposition 6 and (29), we get

\[
\| R_j^{L,b} f \|_{H^1_2} \leq C \| R_j^{L,b} f \|_{H^1_{1+b/2}} \leq C \| G^{L+b}_{L,J}(R_j^{L,b} f) \|_{L^1} \\
= C \left( \int_0^\infty |t A_t H_b(x,t)|^2 \frac{dt}{t} \right)^{1/2} \leq C \| G_b(f) \|_{L^1} \leq C \| f \|_{H^1_1}.
\]

This proves Theorem 3. \( \square \)

The proof of the following lemma can be found in [16].

**Lemma 9** If \( \beta \in \mathbb{R} \) and \( f \in L^2(\mathbb{R}^d) \), then

\[
A_j L^\beta f = (L + 2)^\beta A_j f,
\]

for \( j = 1, 2, \ldots, d \).

Now, we can prove Theorem 2.

**Proof of Theorem 2** Since \( H^1_1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \) is dense in \( H^1_1(\mathbb{R}^d) \) (see [11]), we can assume \( f \in H^1_1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \). We prove Theorem 2 by an inductive argument.

When \( m = 1 \), Theorem 2 has been proved in [4]. We assume that Theorem 2 holds for \( m - 1 \), by Lemma 9 and Theorem 3,

\[
\| R_{i_1 i_2} \cdots i_m L^{-(m-2)} f \|_{H^1_2} = \| A_{i_1} (L + 2(m - 1))^{-1/2} A_{i_2} \cdots A_{i_m} L^{-(m-1)/2} f \|_{H^1_2} \leq \| A_{i_2} \cdots A_{i_m} L^{-(m-1)/2} f \|_{H^1_2} \leq \| f \|_{H^1_2}.
\]

Therefore Theorem 2 holds. \( \square \)

**5 Conclusions**

In this paper, we consider the Riesz transforms of higher order associated with a harmonic oscillator and prove the boundedness of them on the Hardy space. It is well known that the Riesz transforms play an important role in the study of harmonic analysis and partial differential equations. These results are very good progress on the harmonic analysis of Hermite operators.

**Competing interests**

The author declares that they have no competing interests.

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