On Grassmannian Description of the Constrained KP Hierarchy

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Abstract

This note develops an explicit construction of the constrained KP hierarchy within the Sato Grassmannian framework. Useful relations are established between the kernel elements of the underlying ordinary differential operator and the eigenfunctions of the associated KP hierarchy as well as between the related bilinear concomitant and the squared eigenfunction potential.

1 Introduction

The purpose of this note is to present construction of the constrained KP (cKP) hierarchy within the Sato Grassmannian context using elements of the kernels of the underlying differential operators. The fundamental concept is the canonical pairing (the bilinear concomitant) introduced here on the space of elements of the kernels of the underlying differential operator and its conjugated counterpart. The formalism is simplified by relations between the bilinear concomitant and the squared eigenfunction potential (SEP) which emerged before in the setting of the KP hierarchy \[1, 2\]. The claim is that use of SEP makes construction of the cKP hierarchy within the Sato Grassmannian theory of the KP hierarchy more transparent. The cKP hierarchy has recently been discussed in \[3\] and \[4, 5\] using Segal-Wilson modification of the Sato Grassmannian. This note provides the link between these works and the current formalism based on the Sato Grassmannian and the SEP method.

2 KP Hierarchy

We first briefly review the KP hierarchy of nonlinear evolution equations in the approach based on the calculus of the pseudodifferential operators. The main object here is the pseudo-differential Lax operator \(L\):

\[
L = D^r + \sum_{j=0}^{r-2} v_j D^j + \sum_{i\geq 1} u_i D^{-i}
\]

The operator \(D\) satisfies the generalized Leibniz rule so for instance \([D, f] = f'\) with \(f' = \partial f = \partial f/\partial x\).
The associated isospectral flows are described by the Lax equations:

\[
\frac{\partial}{\partial t_n} L = \left[ L^{n/r}, L \right] \quad n = 1, 2, \ldots
\]

(2.2)

with \( x \equiv t_1 \). In (2.2) and below, the subscripts \((\pm)\) of pseudo-differential operators indicate projections on purely differential/pseudo-differential parts. Commutativity of the isospectral flows \( \partial/\partial t_n \) (2.2) is then assured by the Zakharov-Shabat equations.

For a given Lax operator \( L \), which satisfies Sato’s flow equation (2.2), we call the function \( \Phi (\Psi) \), whose flows are given by the expression:

\[
\frac{\partial \Phi}{\partial t_l} = L^l_r (\Phi) \quad ; \quad \frac{\partial \Psi}{\partial t_l} = -(L^*_r)^l_r (\Psi) \quad l = 1, 2, \ldots
\]

(2.3)

an (adjoint) eigenfunction of \( L \). In (2.3) we have introduced an operation of conjugation, defined by simple rules \( D^* = -D \) and \((AB)^* = B^*A^*\). Throughout this paper we will follow the convention that for any (pseudo-)differential operator \( A \) and a function \( f \), the symbol \( Af \) will be just a product of \( A \) with the zero-order (multiplication) operator \( f \).

One can also represent the Lax operator in terms of the dressing operator \( W = 1 + \sum_{i=1}^{\infty} w_i D^{-i} \) through \( L = W D^r W^{-1} \). In this framework equation (2.2) is equivalent to the so called Wilson-Sato equation:

\[
\frac{\partial_n W}{\partial n} = - \left( WD^n W^{-1} \right)_- W
\]

(2.4)

where \( \partial_n = \partial/\partial t_n \). Next, we define corresponding wave-eigenfunction via:

\[
\psi_W(t, \lambda) = W(e^{\xi(t,\lambda)}) = \left( 1 + \sum_{i=1}^{\infty} w_i(t) \lambda^{-i} \right) e^{\xi(t,\lambda)}
\]

(2.5)

where

\[
\xi(t, \lambda) \equiv \sum_{n=1}^{\infty} t_n \lambda^n \quad ; \quad t_1 = x
\]

(2.6)

Similarly, there is also an adjoint wave-eigenfunction:

\[
\psi^*_W = W^{-1}(e^{-\xi(t,\lambda)}) = \left( 1 + \sum_{i=1}^{\infty} w^*_i(t) \lambda^{-i} \right) e^{-\xi(t,\lambda)}
\]

(2.7)

As seen from (2.4) and (2.3) the wave-eigenfunction, is an eigenfunction which in addition to eqs. (2.3) also satisfies the spectral equations \( L \psi_W(\lambda, t) = \lambda^r \psi_W(\lambda, t) \). The wave-eigenfunction and its adjoint enter the fundamental Hirota’s bilinear identity:

\[
\int d\lambda \psi^*_W(t, \lambda) \psi_W(t', \lambda) = 0
\]

(2.8)

which generates the entire KP hierarchy via Hirota’s equations for the underlying tau-functions (see f.i. [6]). In (2.8) and in what follows integrals over spectral parameters are understood as: \( \int d\lambda \equiv \oint_{0} d\lambda \frac{d\lambda}{2\pi i} = \text{Res}_{\lambda=0} \). The proper understanding of (2.8) requires, following
f.i. \[8\] expanding of \(\psi_W(t',\lambda)\) in \((2.8)\) as formal power series w.r.t. \(t'_n - t_n, n = 1, 2, \ldots\) according to

\[
\psi_W(t') = \sum \frac{(t'_1 - t_1)^{k_1} \cdots (t'_n - t_n)^{k_n}}{k_1! \cdots k_n!} \partial_t^{k_1} \cdots \partial_t^{k_n} \psi_W(t) \tag{2.9}
\]

The wave function is an oscillatory function of order 0. Generally, the oscillatory function of order \(l\) is of the form:

\[
f(t, \lambda)e^{\xi(t,\lambda)} = \left(\lambda^l + \sum_{j<l} a_j(t)\lambda^j\right)e^{\xi(t,\lambda)} \tag{2.10}
\]

It will be of importance for us that the action of the differential operator \(D\) can be uniquely inverted on the space of oscillatory functions according to

\[
D^{-1}f(t,\lambda)e^{\xi(t,\lambda)} = \sum_{\alpha=0}^{\infty} (-1)^{\alpha} f^{(\alpha)}(t,\lambda)\lambda^{-1-\alpha}e^{\xi(t,\lambda)} \tag{2.11}
\]

Consider now the one-form: \(\omega = \sum_n \text{Res} \left( D^{-1}\Psi L_+^{n/r} \Phi D^{-1} \right) dt_n \) defined for the couple of (adjoint) eigenfunctions \(\Phi, \Psi\). One shows \[1\] using the Zakharov-Shabat equations that \(\omega\) is a closed form with respect to the exterior derivative \(d \equiv \sum_n \partial_n dt_n\). By the usual argument one concludes from \(d\omega = 0\) that the one form \(\omega\) can be rewritten as \(\omega = dS(\Phi, \Psi)\). This procedure defines (up to a constant) a squared eigenfunction potential (SEP) \(S(\Phi, \Psi)\). In particular, the flows of \(S(\Phi, \Psi)\) are given by

\[
\partial_n S(\Phi, \Psi) = \text{Res} \left( D^{-1}\Psi L_+^{n/r} \Phi D^{-1} \right) \tag{2.12}
\]

Especially, \(\partial_x S(\Phi(t), \Psi(t)) = \Phi(t)\Psi(t)\). As shown in \[2\] the squared eigenfunction potential defines a spectral representation of (adjoint) eigenfunctions. The statement is as follows. Any (adjoint) eigenfunction of the general KP hierarchy possesses a spectral representation:

\[
\Phi(t) = -\int d\lambda \psi_W(t,\lambda) S(\Phi(t'),\psi_W^*(t',\lambda)) \tag{2.13}
\]

\[
\Psi(t) = \int d\lambda \psi_W^*(t,\lambda) S(\psi_W(t',\lambda), \Psi(t')) \tag{2.14}
\]

with spectral densities given by the squared eigenfunction potentials at some multi-time \(t' = (t'_1, t'_2, \ldots)\) taken at some arbitrary fixed value. The r.h.s. of \((2.13)\) and \((2.14)\) do not depend on \(t'\). Furthermore, the closed expressions have been found in \[4\] for those squared eigenfunction potentials which have as argument at least one oscillating wave-eigenfunction:

\[
S(\Phi(t), \psi_W^*(t)) = -\frac{1}{\lambda} \psi_W^*(t) \Phi \left( t + [\lambda^{-1}] \right) \tag{2.15}
\]

\[
S(\psi_W(t), \Psi(t)) = \frac{1}{\lambda} \psi_W(t) \Psi \left( t - [\lambda^{-1}] \right) \tag{2.16}
\]

In the above equation \(S(\psi_W(t,\lambda), \Psi(t))\) is the squared eigenfunction potential (SEP) associated with a pair of eigenfunctions \(\psi_W(t,\lambda)\) and \(\Psi(t)\). It is an oscillatory function of order \(-1\):

\[
S(\psi_W(t,\lambda), \Psi(t)) = \sum_{j=1}^{\infty} s_j(t)\lambda^{-j}e^{\xi(t,\lambda)} = \left[ \Psi(t)\lambda^{-1} + O(\lambda^{-2}) \right] e^{\xi(t,\lambda)} \tag{2.17}
\]
We will now make connection to the language of universal Sato Grassmannian $\mathcal{G}_r$. Consider the hyperplane $W \in \mathcal{G}_r$ defined through a linear basis of Laurent series $\{f_k(\lambda)\}$ in $\lambda$ in terms of the wave eigenfunction as a generating function:

$$W \equiv \text{span}\{f_0(\lambda), f_1(\lambda), f_2(\lambda), \ldots\}$$

$$f_k(\lambda) = \frac{\partial^k}{\partial x^k} \psi_W(t, \lambda) \bigg|_{x=t_2=t_3=\ldots=0} \quad (2.18)$$

Obviously, $W$ is closed under the differentiation $\partial/\partial x$. From the fact that $\psi_W(t, \lambda)$ satisfies eq. $(2.3)$ we obtain an alternative definition of $W$:

$$W = \text{span}\{\psi_W(t, \lambda), \text{ all } t \in \mathbb{C}^\infty\} \quad (2.19)$$

A typical element of $W$: $f_k(\lambda) = \left(\lambda^k + O(\lambda^{k-1})\right)\exp \xi(t, \lambda)$, has an order $k \geq 0$. Consequently, the set of orders of all of elements of $W$ is given by the set of non-negative integers.

In case of the standard $r$-th KdV reduction, where the corresponding Lax operator $L = D + \sum_1^\infty u_i D^{-i}$ satisfies $\mathcal{L}^r = \mathcal{L}_{+}^r$, the latter constraint translates to the Grassmannian language as $\lambda^r W \subset W$.

It is clear that $\int d\lambda \psi_W^*(t, \lambda) \chi(t', \lambda) = 0$ for any $\chi(t, \lambda) \in W$. We will make here a plausible assumption that the inverse holds as well. More precisely, the statement is as follows. Let $F(\psi_W(t', \lambda))$ be a linear functional of $\psi_W(t, \lambda)$ of a positive order for which the following bilinear equation $\int d\lambda \psi_W^*(t, \lambda)F(\psi_W(t', \lambda)) = 0$ holds for all $t'$, then $F(\psi_W(t', \lambda)) \in W$.

### 3 Differential Operators and the Canonical Pairing Structure

Consider a differential operator of order $m$

$$L_m = D^m + u_{m-1} D^{m-1} + \ldots + u_1 D + u_0 \quad (3.1)$$

The differential operator of order $m$ is called monic if its leading term is $D^m$. A monic differential operator of order $m$ is fully characterized by $m$ elements of its kernel. For instance, let functions $\phi_i$, $i = 1, \ldots, m$ constitute a basis for $\text{Ker} L_m = \{\phi_1, \ldots, \phi_m\}$ then

$$L_m(f) = \frac{W_m[\phi_1, \ldots, \phi_m, f]}{W_m[\phi_1, \ldots, \phi_m]} \quad (3.2)$$

The elements of $\text{Ker} L_m$ are assumed to be linearly independent so that the Wronskian matrix:

$$(W_{m \times m})_{1 \leq i, j \leq m} = \partial^{j-1}_x \phi_i \quad (3.3)$$

is invertible. In different words the Wronskian determinant $W_m[\phi_1, \ldots, \phi_m] = \text{det} ||W_{m \times m}||$ must be different from zero. We define $\left(W_{m \times m}^{-1}\right)_{ij}$ for $i, j = 0, \ldots, m-1$ to be the matrix elements of the matrix of the inverse of the Wronskian matrix $W_{m \times m}$. The following relations are then satisfied:

$$\sum_{j=1}^m \left(\left(W_{m \times m}^{-1}\right)_{ij}\right) \phi_k^{(j-1)} = \delta_{i,k} \quad ; \quad \sum_{k=1}^m \phi_k^{(j-1)} \left(\left(W_{m \times m}^{-1}\right)_{kl}\right) = \delta_{j,l} \quad (3.4)$$
It is easy to verify that

\[
(W_{m \times m})^{-1}_{ij} = (-1)^{i+j} \frac{\det_{(j,i)} \| \mathcal{W}_{m \times m} \|}{W_m[\phi_1, \ldots, \phi_m]} \tag{3.5}
\]

where the determinant on the right hand side is the minor determinant obtained by extracting the \(j\)'th row and \(i\)'th column from the Wronski matrix \(\mathcal{W}_{m \times m}\) given in eq. (3.3).

The following technical identity, which is valid for an arbitrary function \(\chi\), follows directly from (3.3)-(3.5):

\[
\sum_{j=1}^{m} (W_{m \times m})^{-1}_{ij} \chi^{(j-1)} = (-1)^{m+i} \frac{W_m[\phi_1, \ldots, \hat{\phi}_i, \ldots, \phi_m, \chi]}{W_m[\phi_1, \ldots, \phi_m]} ; \quad i = 1, \ldots, m \tag{3.6}
\]

In addition, we also need to consider an adjoint operator \(L_m^*\) obtained from (3.1) by a process of conjugation described below eq.(2.3). Let \(\psi_1, \ldots, \psi_m\) be elements of the kernel of an adjoint operator \(L_m^*\):

\[
\text{Ker } L_m^* = \{ \psi_1, \ldots, \psi_m \} \tag{3.7}
\]

They are given in terms of elements of Ker \(L_m\) as follows [7, 8]:

\[
\psi_i = (-1)^{m+i} \frac{W_{m-1}[\phi_1, \ldots, \hat{\phi}_i, \ldots, \phi_m]}{W_m[\phi_1, \ldots, \phi_m]} ; \quad i = 1, \ldots, m \tag{3.8}
\]

Comparing with (3.5) we see that the relation (3.8) expresses the fact that \((\psi_1, \ldots, \psi_m)^T\) is the last column in the inverse \(\mathcal{W}^{-1}\) of the Wronskian matrix \(\mathcal{W}\) of \((\phi_1, \ldots, \phi_m)\). In particular, we see that the functions \(\{\psi_1, \ldots, \psi_m\}\) are also linearly independent.

Some of the obvious consequences of definition (3.8) and connection between \(\psi_i\) and the matrix \(\mathcal{W}^{-1}\) are:

\[
\sum_{i=1}^{m} \phi_i^{(k)}(t) \psi_i(t) = \delta_{k,m-1} \quad \text{for} \quad k = 0, 1, \ldots, m - 1 \tag{3.9}
\]

For completeness let us list the extension of (3.9) to \(k = m\):

\[
\sum_{i=1}^{m} \phi_i^{(m)}(t) \psi_i(t) = \sum_{i=1}^{m} (-1)^{m-i} \frac{W_{m-1}[\phi_1, \ldots, \hat{\phi}_i, \ldots, \phi_m] \phi_i^{(m)}}{W_m[\phi_1, \ldots, \phi_m]} = \partial_x \ln W_m[\phi_1, \ldots, \phi_m] \tag{3.10}
\]

Consider the quantity \(N = \sum_{i=1}^{m} \phi_i D^{-1} \psi_i\). It follows easily that

\[
(L_m N)_{-} = \sum_{i=1}^{m} L_m(\phi_i) D^{-1} \psi_i = 0 \tag{3.11}
\]

Moreover, using the Leibniz rule we obtain from (3.9) and (3.10)

\[
(L_m N)_{+} = \left( L_m \left( \sum_{i=1}^{m} \sum_{\alpha=1}^{\infty} D^{-1-\alpha} \phi_i^{(\alpha)} \psi_i \right) \right)_{+} \tag{3.12}
\]

\[
= \left( L_m(D^{-m} + D^{-1-m} \partial_x \ln W_m[\phi_1, \ldots, \phi_m] + O(D^{-2-m})) \right)_{+} = 1
\]
Hence, as in [4] we obtain from (3.11) and (3.12)

\[ L_m^{-1} = \sum_{i=1}^{m} \phi_i D^{-1} \psi_i \]  

(3.13)

Consider, now Res \( D^{-1} \psi_j L_m \sum_{i=1}^{m} \phi_i D^{-1} \psi_i \). In view of (3.13) we find:

\[ \text{Res} \left( D^{-1} \psi_j L_m \sum_{i=1}^{m} \phi_i D^{-1} \psi_i \right) = \psi_j = \sum_{i=1}^{m} \text{Res} \left( D^{-1} \psi_j L_m \phi_i D^{-1} \right) \psi_i \]  

(3.14)

One notices that

\[ \partial_x \text{Res} \left( D^{-1} \psi_j L_m \phi_i D^{-1} \right) = \psi_j L_m (\phi_i) + L_m^* (\psi_j) \phi_i = 0 \]  

(3.15)

and therefore \( \text{Res} \left( D^{-1} \psi_j L_m \phi_i D^{-1} \right) \) is a constant in \( x \). Since, functions \( \psi_i \) are linearly independent, we conclude in view of equation (3.14) and (3.15) that:

\[ \text{Res} \left( D^{-1} \psi_j L_m \phi_i D^{-1} \right) = \delta_{i,j} \]  

(3.16)

It appears, therefore, that \( \{ \psi_1, \ldots, \psi_m \} \) can be viewed as the dual basis of \( \{ \phi_1, \ldots, \phi_m \} \) with respect to a canonical pairing defined in terms of the so-called bilinear concomitant (see [4, 10, 11]):

\[ \langle \phi | \psi \rangle_{L_m} \equiv \text{Res} \left( D^{-1} \psi L_m \phi D^{-1} \right) = \sum_{i=1}^{m} \sum_{j=0}^{i-1} (-1)^j \phi^{(i-j-1)} (u_i \psi)^{(j)} \]  

(3.17)

(3.18)

with \( u_m = 1 \). In this setting the bases \( \{ \phi_1, \ldots, \phi_m \} \) and \( \{ \psi_1, \ldots, \psi_m \} \) related through (3.8) are dual to each other in the sense of satisfying \( \langle \phi_i | \psi_j \rangle_{L_m} = \delta_{ij} \) for \( i, j = 1, \ldots, m \) due to (3.16). The following technical Lemma provides a useful characterization of the products \( \langle \chi | \psi_i \rangle_{L_m} \) for an arbitrary function \( \chi \) and \( \psi_i \in \text{Ker} L_m^* \).

**Lemma 3.1** The following identity:

\[ \langle \chi | \psi_i \rangle_{L_m} = \sum_{j=1}^{m} \left( W^{-1}_{m \times m} \right)_{ij} \chi^{(j-1)} \; ; \; i = 1, \ldots, m \]  

(3.19)

holds for an arbitrary function \( \chi \) and \( \psi_i \in \text{Ker} L_m^* \).

**Proof.** Let \( M_i = \sum_{j=1}^{m} \left( W^{-1}_{m \times m} \right)_{ij} D^j \) be \( (m-1) \)-order differential operator for the fixed \( i \). We know its \( m-1 \) null-functions \( \phi_k \) such that \( M_i (\phi_k) = 0 \) for \( k \neq i \). We also have \( M_i (\phi_i) = 1 \). This characterizes \( M_i \) completely. Note, that the \( (m-1) \)-order differential operator \( \langle \cdot | \psi_i \rangle_{L_m} \) agrees with \( M_i \) on \( \phi_i, \; i = 1, \ldots, m \), which completes the proof. □

Recalling identity (3.6) we find an alternative way of writing (3.19) as

\[ \langle \chi | \psi_i \rangle_{L_m} = (-1)^{m+i} \frac{W_m [\phi_1, \ldots, \hat{\phi_i}, \ldots, \phi_m, \chi]}{W_m [\phi_1, \ldots, \phi_m]} \]  

(3.20)

from which it follows:
\[ \langle \chi | \psi \rangle_{L_m} = \psi_i L_{m,i}(\chi) \] (3.21)

where \( L_{m,i} \) are the ordinary differential operators of order \( m - 1 \), whose kernels are given by \( \text{Ker} (L_{m,i}) = \{ \phi_1, \ldots, \hat{\phi}_i, \ldots, \phi_m \} \). Correspondingly, the action of \( L_{m,i} \) is defined through

\[ L_{m,i}(\chi) \equiv \frac{W_{m+1}[\phi_1, \ldots, \hat{\phi}_i, \ldots, \phi_m, \chi]}{W_m[\phi_1, \ldots, \phi_i, \ldots, \phi_m]} \] (3.22)

We will now introduce isospectral deformations of the differential operator \( L_m \) of the form:

\[ \partial_n L_m = \tilde{B}_n L_m - L_m B_n \quad ; \quad n = 1, 2, \ldots \] (3.23)

In this setting we will show that the product \( \langle \cdot | \cdot \rangle_{L_m} \) defines a canonical pairing \( \text{Ker} L_m \times \text{Ker} L_m^* \to \mathbb{C} \). As discussed in [10, 11] this pairing is nonsingular.

The two families of differential operators \( \tilde{B}_n, B_n \) are both assumed to satisfy Zakharov-Shabat equations:

\[ 0 = \partial_k \tilde{B}_n - \partial_n \tilde{B}_k + \left[ \tilde{B}_n, \tilde{B}_k \right] \]
\[ 0 = \partial_k B_n - \partial_n B_k + \left[ B_n, B_k \right] \quad ; \quad k, n = 1, 2, \ldots \] (3.24)

to ensure commutativity of flows defined in (3.23). From (3.23) we find that

\[ \partial_n L_m^{-1} = B_n L_m^{-1} - L_m^{-1} \tilde{B}_n \quad ; \quad n = 1, 2, \ldots \] (3.25)

The following result applies to this case [8]:

**Lemma 3.2** Equations (3.23) imply that \( \phi_i \in \text{Ker} L_m \) and \( \psi_i \in \text{Ker} L_m^* \) are “up to a gauge rotation” (adjoint) eigenfunctions satisfying:

\[ \partial_n \phi_i = B_n(\phi_i) \quad i = 1, \ldots, m \] (3.26)
\[ \partial_n \psi_i = -\tilde{B}_n^*(\psi_i) \quad i = 1, \ldots, m \] (3.27)

**Proof.** From \( \partial_n L_m(\phi_i) = 0 \) and (3.23) we find that \( B_n(\phi_i) - \partial_n \phi_i \in \text{Ker} L_m \). Hence we can write

\[ B_n(\phi_i) - \partial_n \phi_i = -\sum_{j=1}^{m} \phi_j c_{ji}^{(n)}(\bar{t}) \] (3.28)

where \( \bar{t} = (t_2, t_3, \ldots) \). We now proceed in a way similar to the one used, in a slightly different setting, in [12]. Define \( (\Delta_n)_{jk} \equiv \partial_j \delta_{jk} - c_{kj}^{(n)} \) so that we can compactly rewrite (3.28) as \( (\Delta_n)_{jk} \phi_k = B_n(\phi_j) \). The Zakharov-Shabat equations (3.24), ensure the zero curvature equation \( ([\Delta_n, \Delta_{\bar{t}}])_{ik} \phi_k = 0 \). Thus the “connection” \( c_{ij}^{(n)} \) is a pure gauge and can be cast in a form

\[ c_{ij}^{(n)}(\bar{t}) = (c^{-1})_{ik}(\bar{t}) \partial_n c_{kj}(\bar{t}) \quad ; \quad n \geq 2 \] (3.29)
Define accordingly
\[ \tilde{\phi}_j \equiv \phi_k (c^{-1})_{kj} \]  \hspace{1cm} (3.30)

It is easy to verify that \( \tilde{\phi}_j \) satisfy
\[ \partial_n \tilde{\phi}_j = (\Delta_n \phi)_k (c^{-1})_{kj} = B_n(\tilde{\phi}_j) \]  \hspace{1cm} (3.31)

Similarly, from \( \partial_n L^*_m(\psi_i) = 0 \) we arrive at
\[ \tilde{B}^*_n(\psi_i) + \partial_n \psi_i = \sum_{j=1}^{m} \tilde{c}_{ij}^{(n)} (\tilde{t}) \psi_j \]  \hspace{1cm} (3.32)

We will now establish a relation between coefficients \( c_{ij}^{(n)} \) and \( \tilde{c}_{ij}^{(n)} \). We need at this point and the technical identity:
\[ [K, f D^{-1} g]_l = K(f)D^{-1}g - f D^{-1} K^*(g) \]  \hspace{1cm} (3.33)

valid for a purely differential operator \( K \) and arbitrary functions \( f, g \). We find from (3.25) and the above equation that
\[ \left( \partial_n L_m^{-1} \right)_l = \sum_{i=1}^{m} (B_n(\phi_i)) D^{-1} \psi_i - \sum_{i=1}^{m} \phi_i D^{-1} \tilde{B}^*_n(\psi_i) \]  \hspace{1cm} (3.34)

Equations (3.28) and (3.32) agree with (3.34) provided
\[ \sum_{i,j=1}^{m} \left( c_{ij}^{(n)} + \tilde{c}_{ij}^{(n)} \right) \phi_i D^{-1} \psi_j = 0 \]  \hspace{1cm} (3.35)

Define a differential operator of \( m - 1 \) order
\[ K[\phi] \equiv \sum_{s=1}^{m} \sum_{l=0}^{s-1} u_l D^l(\phi)^{(s-l-1)} \]  \hspace{1cm} (3.36)
such that \( K^*[\phi](\psi) = \langle \phi | \psi \rangle_{L_m} \). From (3.35) and (3.16) we find
\[ \left( \sum_{i,j=1}^{m} \left( c_{ij}^{(n)} + \tilde{c}_{ij}^{(n)} \right) \phi_i D^{-1} \psi_j K[\phi_k] \right)_l = 0 \]  \hspace{1cm} (3.37)
or
\[ \sum_{i=1}^{m} \left( c_{ik}^{(n)} + \tilde{c}_{ik}^{(n)} \right) \phi_i = 0 \hspace{1cm} ; \hspace{1cm} k = 1, \ldots, m \]  \hspace{1cm} (3.38)

Since \( \{\phi_i\} \) are linearly independent we find from (3.38) that \( c_{ij}^{(n)} = -\tilde{c}_{ij}^{(n)} \) for all \( i, j = 1, \ldots, m \). Accordingly, \( (\Delta_n^*)_{jk} \psi_k = -\left( L_{m+1}^* \right)^{n/r}_+ (\psi_j) \), with \( (\Delta_n^*)_{jk} \equiv \partial_n \delta_{jk} + c_{jk}^{(n)} \). Define, next
\[ \tilde{\psi}_j \equiv c_{jk} \psi_k \]  \hspace{1cm} (3.39)

It follows that
\[ \partial_n \tilde{\psi}_j = c_{jk} (\Delta_n^*)_{jk} \psi_k = -\tilde{B}^*_n(\tilde{\psi}_j) \]  \hspace{1cm} (3.40)

Hence we succeeded to find a mutually inverse gauge rotations taking \( \phi_i \in \text{Ker} L_m \) and \( \psi_i \in \text{Ker} L_m^* \) into (adjoint) eigenfunctions satisfying (3.28) and (3.27). \( \square \)
Lemma 3.3 Let $\phi$ and $\psi$ satisfy (3.26) and (3.27) with respect to flows from (3.23), then
\[
\partial_n \langle \phi \mid \psi \rangle_{L_m} = \text{Res} \left( D^{-1} \psi B_n(L_m(\phi) - D^{-1}) - \text{Res} \left( D^{-1} L_m^*(\psi) B_n \phi D^{-1} \right) \right) \\
= \langle L_m(\phi) \mid \psi \rangle_{B_n} - \langle \phi \mid L_m^*(\psi) \rangle_{B_n} \quad n = 1, 2, \ldots
\] (3.41)

Proof. Proof follows from the technical Lemma [1]:
\[
\text{Res} \left( D^{-1} L_1 L_2 D^{-1} \right) = \text{Res} \left( D^{-1} (L_1)_0 L_2 D^{-1} \right) + \text{Res} \left( D^{-1} L_1 (L_2)_0 D^{-1} \right)
\] (3.42)
Where $L_1, L_2$ are arbitrary differential operators and $(\cdot)_0$ denotes projection on the zero-order term. With the help of relation (3.42) we can rewrite $\text{Res} \left( D^{-1} \psi L_m B_n \phi D^{-1} \right)$ as a sum of $\text{Res} \left( D^{-1} L_1^* (L_2)_0 \right)$ and $\text{Res} \left( D^{-1} L_2^* (L_1)_0 \right)\). □

Corollary 3.2 For $\phi \in \text{Ker} L_m$ and $\psi \in \text{Ker} L_m^*$ and $L_m$ satisfying eq. (3.23) it holds that $\partial_n \langle \phi \mid \psi \rangle_{L_m} = 0$ for $n = 1, 2, \ldots$. Accordingly, $(\cdot \mid \cdot)_{L_m}$ defines a canonical pairing $\text{Ker} L_m \times \text{Ker} L_m^* \to \mathbb{C}$.

As a special case ($n = 1$) of Lemma 3.3 we have equation
\[
\partial_x \langle \phi \mid \psi \rangle_{L_m} = L_m (\phi) \psi - \phi L_m^* (\psi)
\] (3.43)
Note, that the result (3.43) is valid this time for an arbitrary $\phi, \psi$ as follows by verification.

Another consequence of Lemma 3.3 reads :

Corollary 3.3 For $\phi, \psi$ satisfying condition of Lemma 3.3 and $L_m$ whose isospectral flows are given in (3.23) the following relation holds:
\[
\langle \phi \mid \psi \rangle_{L_m} = S(L_m(\phi), \psi) - S(\phi, L_m^*(\psi))
\] (3.44)
up to a constant (in the multi-time $t$).

Eq. (3.44) follows from eq. (3.41) and relations:
\[
\partial_n S(L_m(\phi), \psi) = \langle L_m(\phi) \mid \psi \rangle_{B_n} \quad ; \quad \partial_n S(\phi, L_m^*(\psi)) = \langle \phi \mid L_m^*(\psi) \rangle_{B_n}
\] (3.45)

4 Lax operator representation of the CKP hierarchy

We are studying here the class of constrained cKP hierarchies for which we have the Lax representation:
\[
L = D^r + \sum_{i=0}^{r-2} u_i D^i + \sum_{i=1}^{m} \Phi_i D^{-1} \Psi_i = B_r + \sum_{i=1}^{m} \Phi_i D^{-1} \Psi_i
\] (4.1)
with $\Phi_i, \Psi_i$ being (adjoint) eigenfunctions of the Lax operator $L$ as in (2.3). As shown in [8, 4, 5] the cKP$_{r,m}$ hierarchy can be expressed in terms of two normalized differential
operators $L_m$, $L_{m+r}$ of order $m$ and $r + m$, respectively. The Lax operator (4.1) of the $cKP_{r,m}$ hierarchy is in this representation being rewritten as a ratio:

$$L \equiv L_m^{-1}L_{m+r} = \sum_{i=1}^{m} \phi_i D^{-1}L^*_{m+r}(\psi_i) + B_r$$

(4.2)

The wave eigenfunction $\psi_W(t, \lambda)$ of (4.1) is an eigenfunction (as in eq.(2.3)) which additionally satisfies the following spectral equation:

$$L\psi_W(t, \lambda) = B_r(\psi_W(t, \lambda)) + \sum_{i=1}^{m} \Phi_i(t) S(\psi_W(t, \lambda), \Psi_i(t)) = \lambda^r \psi_W(t, \lambda)$$

(4.3)

In eq.(4.3) $S(\psi_W(t, \lambda), \Psi_i(t))$ is the squared eigenfunction potential (SEP) associated with a pair of eigenfunctions $\psi_W(t, \lambda)$ and $\Psi_i(t)$.

5 Universal Sato’s Grassmannian construction of the CKP Hierarchy

Let us first introduce the following basic definition:

**Definition 5.1** For the wave-eigenfunction of the KP hierarchy and $\psi_i \in \text{Ker} L^*_m$ we define $m$ objects:

$$S_i(t, \lambda) \equiv \lambda^r \langle \psi_W | \psi_i \rangle_{L_m}; \quad i = 1, \ldots, m$$

(5.1)

As seen from eq. (3.43) the $m$ objects $S_i(t, \lambda)$ defined in (5.1) satisfy:

$$\partial_x S_i(t, \lambda) = \lambda^r \psi_i L_m(\psi_W); \quad i = 1, \ldots, m$$

(5.2)

Note, also that the expressions (5.1) and (3.21) lead to

$$S_i(t, \lambda) = \lambda^r \psi_i L_{m,i}(\psi_W)$$

(5.3)

where $L_{m,i}$ are the ordinary differential operators of order $m - 1$, whose kernels are given by

$$\text{Ker}(L_{m,i}) = \{ \phi_1, \ldots, \hat{\phi}_i, \ldots, \phi_m \}.$$  The action of $L_{m,i}$ is defined in eq.(3.22)

The following Lemma establishes a connection between $cKP_{r,m}$ reduction of the KP hierarchy and the Grassmannian formulation.

**Lemma 5.1** The $cKP_{r,m}$ reduction within the KP hierarchy is equivalent to the following system defined in terms of the Grassmannian:

1) Let $\{ \Phi_1, \ldots, \Phi_m \}$ be $m$ linearly independent functions

2) Let $m$ objects $S_i(t, \lambda)$ be defined as in Def.5.1 in terms of $\psi_i$ dual to $\Phi_i$ according to (3.3). Let $S_i(t, \lambda)$, furthermore, satisfy the following two conditions:

$$\partial_x S_i(t, \lambda) \in W$$

(5.4)

$$\sum_{i=0}^{m} c_i S_i(t, \lambda) \in W \quad \text{implies} \quad c_i = 0$$

(5.5)
Proof. We start the proof with the cKP\(_{r,m}\) system as given in (1.1) with both \(\{\Phi_i\}\) and \(\{\Psi_i\}\) being linearly independent set of functions. It follows from (1.3), (2.18) and \(\partial_x S (\psi_W(t, \lambda), \Psi_i(t)) = \psi_W(t, \lambda)\Psi_i(t)\) that

\[
\lambda^r \psi_W^{(j)}(t, \lambda) - \sum_{i=1}^m \Phi_i^{(j)}(t) S(\psi_W(t, \lambda), \Psi_i(t)) \in W \quad ; \quad j = 0, \ldots, m - 1
\]  

(5.6)
or in the matrix notation:

\[
\lambda^r \left( \begin{array}{c}
\psi_W(t, \lambda) \\
\psi_W^{(1)}(t, \lambda) \\
\vdots \\
\psi_W^{(m-1)}(t, \lambda)
\end{array} \right) - W_{m \times m}^{-1} \left( \begin{array}{c}
S(\psi_W(t, \lambda), \Psi_1(t)) \\
S(\psi_W(t, \lambda), \Psi_2(t)) \\
\vdots \\
S(\psi_W(t, \lambda), \Psi_m(t))
\end{array} \right) \in W
\]

(5.7)

meaning that each element of the above combination of columns belongs to the Grassmannian \(W\). One finds easily from (5.7) and \(\partial_x S (\psi_W(t, \lambda), \Psi_i(t)) = \psi_W(t, \lambda)\Psi_i(t)\) that

\[
\lambda^r \partial_x W_{m \times m}^{-1} \left( \begin{array}{c}
\psi_W(t, \lambda) \\
\psi_W^{(1)}(t, \lambda) \\
\vdots \\
\psi_W^{(m-1)}(t, \lambda)
\end{array} \right) \in W
\]

(5.8)

In terms of the matrix elements \((W_{m \times m}^{-1})_{ij}\) from (3.4) the relation (5.8) takes a form

\[
\lambda^r \partial_x \sum_{j=0}^{m-1} (W_{m \times m}^{-1})_{ij} \psi^{(j)}(t, \lambda) \in W
\]

(5.9)

for each \(i = 1, \ldots, m\). This yields condition (5.4) due to the Lemma 3.1.

Recalling Lemma 3.1 we see that (5.7) implies

\[
S(\psi_W(t, \lambda), \Psi_i(t)) - S_i(t, \lambda) \in W \quad ; \quad i = 1, \ldots, m
\]

(5.10)

If now it holds that

\[
\sum_{i=1}^m c_i S_i(t, \lambda) \in W
\]

(5.11)
then because of relation (5.10) eq. (5.11) implies that

\[
\sum_{i=1}^m c_i S(\psi_W(t, \lambda), \Psi_i(t)) \in W
\]

(5.12)
and therefore due to eq. (2.17) \(\sum_{i=1}^m c_i \Psi_i \lambda^{-1} \exp \xi(t, \lambda) = 0\). We note that from the linear independence of \(\{\Psi_1, \ldots, \Psi_m\}\) it follows that \(c_i = 0\), which is the desired result.

From now on, we will assume that given is the system of the linearly independent functions \(\{\Phi_1, \ldots, \Phi_m\}\) together with conditions (5.4)-(5.5). We are going to show that the KP hierarchy associated with the wave-eigenfunction \(\psi_W(t, \lambda)\) satisfying constraints (5.4) and (5.5) belongs to the cKP\(_{r,m}\) class.
We start by defining $m$ functions:

$$
\Psi_i(t, t_0) \equiv \int d\lambda \, \psi^*_W(t, \lambda) \, \mathcal{S}_i(t_0, \lambda) ; \quad i = 1, \ldots, m
$$

(5.13)

with $\mathcal{S}_i(t_0, \lambda)$ defined as in (5.14). First, it follows clearly from the definition (5.13) that $\Psi_i$ is an adjoint eigenfunction in the multi-time $t$. Secondly, $\Psi_i$ is non-zero only for $\mathcal{S}_i(t_0, \lambda)$ not in $\mathcal{W}$ due to the Hirota’s identity. According to the condition (5.3) the $m$ functions $\Psi_i$ are linearly independent.

What remains to be proven in order to establish that $\Psi_i$ from eq. (5.13) are adjoint eigenfunctions is that the functions $\Psi_i$ do not depend on the second multi-parameter $t_0$. Indeed, from the condition (5.4) it follows immediately that $\partial_{x_0} \Psi_i(t, t_0) = 0$ and accordingly $\Psi_i$ does not depend on $x_0 = (t_0)_1$. To complete the proof it remains to show that indeed $\partial \Psi_i / \partial (t_0)_n = 0$ for $n > 1$.

Define an oscillatory function $\psi_V(t, \lambda)$ by

$$
L_m(\psi_W(t, \lambda)) \equiv \lambda^m \psi_V(t, \lambda).
$$

(5.14)

Since $\psi_W(t, \lambda) = W \exp \xi(t, \lambda)$ where $W$ is a dressing operator, we find that $\psi_V(t, \lambda) = V \exp \xi(t, \lambda)$ where $V = L_m W D^{-m}$ has like $W$ a form of the dressing operator $V = 1 + \sum_{i=1}^{\infty} x_i D^{-i}$. Alternatively, we can rewrite $L_m = V D^m W^{-1}$. Consider, $L_{m+r} \equiv V D^{m+r} W^{-1}$ such that $L_{m+r}(\psi_W(t, \lambda)) = \lambda^r L_m(\psi_W(t, \lambda))$. Since $\lambda^r L_m(\psi_W(t, \lambda)) \in \mathcal{W}$ the operator $L_{m+r}$ is an ordinary differential operator. Moreover, we find that the KP Lax operator $L = WD' W^{-1}$ can be written in terms of two ordinary differential operators as $L = L_{m+1} L_{m+r}$. From [13] and [14] we know that the KP hierarchy equations $\partial_n L = \left[ (L)^{n/r}, L \right]$ for $L = L_{m-1} L_{m+r}$ are equivalent to the following flows on the differential operators $L_m, L_{m+r}$:

$$
\partial_n L_m = \left( L_{m+r} L_m^{-1} \right)^{n/r}_{+} L_m - L_m \left( L_m^{-1} L_{m+r} \right)^{n/r}_{+}
$$

(5.15)

$$
\partial_n L_{m+r} = \left( L_{m+r} L_m^{-1} \right)^{n/r}_{+} L_{m+r} - L_{m+r} \left( L_m^{-1} L_{m+r} \right)^{n/r}_{+}
$$

(5.16)

It has been shown in [8] that equations (5.15) and (5.16) imply that $\phi_i \in \text{Ker} L_m$ and $\psi_i \in \text{Ker} L_{m+r}$ are “up to a gauge rotation” (adjoint) eigenfunctions satisfying:

$$
\partial_n \phi_i = \left( L_m^{-1} L_{m+r} \right)^{n/r}_{+} (\phi_i) \quad i = 1, \ldots, m
$$

(5.17)

$$
\partial_n \psi_i = - \left( (L_{m+r} L_m^{-1})^* \right)^{n/r}_{+} (\psi_i) \quad i = 1, \ldots, m
$$

(5.18)

We recognize in the above equations the setting of Lemma 3.2 with (5.17)-(5.18) appearing to be special cases of (3.26)-(3.27). Especially, we may use the results of Lemma 3.3 and equation (3.41) to find

$$
\partial_n \mathcal{S}_i = \lambda^r \langle \lambda^r L_m(\psi_W) \rangle \left( L_{m+r} L_m^{-1} \right)^{n/r}_{+} \psi_i = \lambda^r A_{n-1} \left( L_m(\psi_W) \right)
$$

(5.19)

with $A_{n-1}$ being $(n-1)$-th order differential operator. Since $\lambda^r A_{n-1} \left( L_m(\psi_W) \right) \in \mathcal{W}$ it follows immediately that $\partial \Psi_i / \partial (t_0)_n = 0$ and $\Psi$ is indeed a function of the multi-time $t$ only.
Note, that on basis of relation (3.19) the alternative form of the definition (5.13) appears to be:

\[ \Psi_{i}(t) \equiv \sum_{j=0}^{m-1} (\mathcal{W}^{-1})_{ij}(t_0) \int d\lambda \lambda^{r} \psi_{W}^{*}(t, \lambda) \psi_{W}^{(j)}(t_0, \lambda) \]  \hspace{1cm} (5.20)

Accordingly, using (3.4) we find (see also [15])

\[ \sum_{i=1}^{m} \Phi_{i}(t_0) \Psi_{i}(t) = \int d\lambda \lambda^{r} \psi_{W}^{*}(t, \lambda) \psi_{W}(t_0, \lambda) \]  \hspace{1cm} (5.21)

from which it follows that

\[ \sum_{i=1}^{m} \left( \partial_{n} \Phi_{i}(t_0) - B_{n} (\Phi_{i}(t_0)) \right) \Psi_{i}(t) = 0 \quad ; \quad n = 1, 2, \ldots \]  \hspace{1cm} (5.22)

or equivalently

\[ \sum_{i=1}^{m} \left( \partial_{n} \Phi_{i}(t_0) - B_{n} (\Phi_{i}(t_0)) \right) S_{i}(t, \lambda) \in \mathbb{W} \]  \hspace{1cm} (5.23)

From the last identity (5.23) and condition (5.3) we conclude that \( \Phi_{i} \) are eigenfunctions for \( i = 1, \ldots, m \).

Recall, that \( L_{W}(t, \lambda) = \lambda^{r} \psi_{W}(t, \lambda) = L_{+}(\psi_{W}(t, \lambda)) + L_{-}(\psi_{W}(t, \lambda)) \) with the pseudodifferential part \( L_{-}(\psi_{W}(t, \lambda)) \sim O(\lambda^{-1}) \exp \xi(t, \lambda) \). Inserting it back into eq.(5.21) we find

\[ \sum_{i=1}^{m} \Phi_{i}(t_0) \Psi_{i}(t) = \int d\lambda \lambda^{r} \psi_{W}^{*}(t, \lambda) L_{-}(\psi_{W}(t_0, \lambda)) \]  \hspace{1cm} (5.24)

From [2] we conclude that (5.24) implies

\[ L_{-}(\psi_{W}(t_0, \lambda)) = \sum_{i=1}^{m} \Phi_{i}(t_0) \frac{1}{\lambda} \psi_{W}(t_0, \lambda) \Psi_{i}(t_0 - [\lambda^{-1}]) \]  \hspace{1cm} (5.25)

up to terms in \( \mathbb{W} \). Equivalently, we can rewrite the last relation in the desired form

\[ L_{-}(\psi_{W}(t, \lambda)) = \sum_{i=1}^{m} \Phi_{i}(t) S(\psi_{W}(t, \lambda), \Psi_{i}(t)) \]  \hspace{1cm} (5.26)

from which eq.(4.1) follows due to the fact that the pseudodifferential operators act freely on the wavefunctions as seen from (2.11). \( \square \)

### 6 Truncated KP Hierarchy as cKP Hierarchy

Let us consider the truncated KP hierarchy defined by the dressing operator \( W \) containing only finite number of terms. Let \( K \) be a positive order differential operator of the order \( N \), such that \( N > m \) and such that \( W = K D^{-N} \). Accordingly, the corresponding Lax operator is

\[ L_{tr} = K D^{r} K^{-1} = W D^{r} W^{-1} \]  \hspace{1cm} (6.1)
Let $f_i, g_i$ with $i = 1, \ldots, N$ be elements of the kernels $\text{Ker} K$ and $\text{Ker} K^*$, respectively. As shown in [12] the Wilson-Sato equations (2.4) for the hierarchy defined by the Lax operator (6.1) take a simple form for the elements of $\text{Ker} K$:

$$\partial_n f_i = \partial_x^n f; \quad i = 1, \ldots, N \quad (6.2)$$

We have

$$K^{-1} = \sum_{i=1}^{N} f_i D^{-1} g_i \quad (6.3)$$

and consequently

$$\text{Res} \left( KD^r K^{-1} \right) = \sum_{i=1}^{N} KD^r(f_i) g_i = \sum_{i=1}^{N} (-1)^{N-i} \frac{W[f_1, \ldots, f_N, \hat{f}_i, \ldots, f_N]}{W^2[f_1, \ldots, f_N]} \quad (6.4)$$

where we used the Jacobi identity

$$W[W[f_1, \ldots, f_m, g], W[f_1, \ldots, f_m, h]] = W[f_1, \ldots, f_m]W[f_1, \ldots, f_m, g, h] \quad (6.5)$$

and (6.2).

Hence we reproduced the well-known result that the tau function for the truncated KP hierarchy is the Wronskian $\tau_{\text{trun}} = W[f_1, \ldots, f_N]$.

Due to (6.2) we can rewrite $f_i$ as: $f_i = \int dz \tilde{f}_i(z) \exp(\xi(t, z))$. Notice, that

$$K \exp(\xi(t, \lambda)) = z^N \psi_W(t, \lambda) \quad (6.6)$$

due to the fact that $W = KD^{-N}$ is the dressing operator of the truncated hierarchy. It holds therefore that

$$0 = K(f_i) = \int dz z^N \tilde{f}_i(z) \psi_W(t, \lambda) \quad (6.7)$$

Accordingly, for any positive differential operator $B$ we find

$$\int dz z^N \tilde{f}_i(z) B(\psi_W(t, z)) = B \left( \int dz z^N \tilde{f}_i(z) \psi_W(t, z) \right) = 0 \quad (6.8)$$

We now investigate the condition for $L_{tr}$ to be within $c\text{KP}_{r,m}$ in the nontrivial case $N > m$. As shown above, the necessary condition for this to happen is that $\lambda^r L_m(\psi_W) \in W$ or that there exists a positive differential operator $B$ such that $\lambda^r L_m(\psi_W) = B(\psi_W)$. Comparing with (6.8) we find that $\lambda^r L_m(\psi_W) \in W$ translates into:

$$0 = \int dz z^N \tilde{f}_i(z) z^r L_m(\psi_W(t, z)) = L_m \left( \int dz \tilde{f}_i(z) z^{r+N} \psi_W(t, z) \right)$$

$$= L_m \left( \int dz \tilde{f}_i(z) z^r K e^{\xi(t, z)} \right) = L_m K D^r(f_i) \quad (6.9)$$

for all $i = 1, \ldots, N$. 
Hence $KD^r(f_i) \in \text{Ker}(L_m)$ for $i = 1, \ldots, N$. In [4] this condition was rewritten using a Jacobi identity for Wronskians as

$$W[f_1, \ldots, f_N, f_{i_1}^r, \ldots, f_{i_{m+1}}^r] = 0$$

\[ (6.10) \]

# 7 Concluding Remarks

We have seen that formulating the constrained KP hierarchy within the Sato Grassmannian becomes transparent when use is being made of the underlying ordinary differential operators with convenient parametrization of their kernels. Useful insight has been obtained by relating the notions of kernel elements of the underlying ordinary differential operators with that of the eigenfunctions of the KP hierarchy. The related connection of the bilinear concomitant introducing the canonical pairing structure on the space kernels to that of the squared eigenfunction potentials has then arisen naturally.

Let us complete our discussion by the following additional comments addressing fundamental questions of the formalism.

**Remark.** Due to relation (3.19) we have $\sum_{i=1}^{m} \Phi_i S_i = \lambda^r \psi_W$. Hence condition (5.5) can be understood as an obstruction to the usual KdV reduction with $\lambda^r \psi_W \in W$.

**Remark.** Alternatively to (5.3) we could have expressed the relevant assumption in terms of the integrals:

$$\sum_{i=0}^{m} c_i \int d\lambda \psi_W^* (t, \lambda) S_i(t_0, \lambda) = 0 \text{ implies } c_i = 0$$

\[ (7.1) \]

instead of involving the Sato Grassmannian $W$ in (5.3). The arguments used in the proof would then have worked with small adjustments but without any need of making an additional assumption that $\int d\lambda \psi_W^* (t, \lambda) F(\psi_W(t', \lambda)) = 0$ implies $F(\psi_W(t', \lambda)) \in W$ for $F(\psi_W(t', \lambda))$ of the positive order.

**Remark.** From the definition (5.14) and (5.15) we find that the flows of $\psi_V(t, \lambda)$ are given by:

$$\partial_t \psi_V(t, \lambda) = \left( L_m + L_m^{-1} \right)^{n/r} \psi_V(t, \lambda)$$

\[ (7.2) \]

Based on (3.27) and (7.3) it makes now sense to define the squared eigenfunction potential $S(\psi_V, \psi_i)$ for $\psi_V(t, \lambda)$ and $\psi_i$ with the following useful property:

$$S_i(t, \lambda) = \lambda^{r+m} S(\psi_V, \psi_i)$$

\[ (7.3) \]

Due to eq. (7.3) one can rewrite eq.(5.13) as

$$\Psi_i(t) = \int d\lambda \lambda^{r+m} \psi_W^* (t, \lambda) S(\psi_V(t_0, \lambda), \psi_i(t_0)) ; \quad i = 1, \ldots, m$$

\[ (7.4) \]

Since $\partial_{\rho_0} \Psi_i(t, t_0) = 0$ it holds that $\int d\lambda \lambda^{r+m} \psi_W^* \psi_V = 0$.

Furthermore it is easy to see that $\lambda^{r+m} \psi_W^* (t, \lambda) = L_{m+r}^* (\psi_V^*(t, \lambda))$ as follows from the definition (2.7) and $\psi_V^*(t, \lambda) = V^{-1} \exp -\xi(t, \lambda)$ together with $L_{m+r}^* = W^{-1} (-D)^{m+r} V^*$. Plugging it back in (7.4) we obtain:

$$\Psi_i(t) = \int d\lambda L_{m+r}^* (\psi_V^*(t, \lambda)) S(\psi_V(t_0, \lambda), \psi_i(t_0)) = L_{m+r}^* (\psi_i)$$

\[ (7.5) \]

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Remark. The inclusion $\partial_x S_i \in W$ can be rewritten as $\lambda^r L_m(\psi_W) \in W$, where $L_m$ is a $m$-order differential operator whose action on the wave function $\psi_W$ can be viewed as $m$ successive Darboux-Bäcklund transformations. With the kernel of $L_m$ being $\{\Phi_1, ..., \Phi_m\}$, let $w_j$ be such that $\Phi_j = \int d\lambda \lambda^{-1}(\psi_W w_j)$ for $j = 1, ..., m$. Accordingly, $w_j$ are orthogonal to the subspace $W' \equiv \text{span}\{L_m(\psi_W)\}$ of $W$ with respect to the inner product $\langle u|v \rangle \equiv \int d\lambda \lambda^{-1}uv$. Hence, the inclusion $\lambda^r W' \in W$ is a co-dimension $m$ inclusion. Also, the condition (5.3) in view of (5.3) expresses the assumption about the codimension $m$ being optimal. This establishes link to the formalism of [4, 5].

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