Chapter 7
Embeddings of Four-valent Framed Graphs into 2-surfaces

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Abstract  It is well known that the problem of detecting the least (highest) genus of a surface where a given graph can be embedded is closely connected to the problem of embedding special four-valent framed graphs, i.e. 4-valent graphs with opposite edge structure at vertices specified. This problem has been studied, and some cases (e.g., recognizing planarity) are known to have a polynomial solution.

The aim of the present survey is to connect the problem above to several problems which arise in knot theory and combinatorics: Vassiliev invariants and weight systems coming from Lie algebras, Boolean matrices etc., and to give both partial solutions to the problem above and new formulations of it in the language of knot theory.

7.1 Introduction

Assume 4-valent graph $\Gamma$ with each vertex endowed with opposite half-edge structure, that is, at each vertex the four half-edges are split into two pairs of formally opposite edges. Classify the surfaces $S$ where $\Gamma$ can be embedded in a way such that the formal opposite half-edge structure coincides with the opposite half-edge structure induced by the embedding.

A natural question is to study the highest (least) genus of the surface the graph can be embedded into. We restrict ourselves only to the case of embeddings which decompose the surface into 2-cells. We shall address this general question later in this paper. We shall start with the following partial cases of it. One of them, more general, deals with embedded graphs whose first $\mathbb{Z}_2$-homology class is orienting. As a partial case of this, we address the following

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Fig. 7.1 Any embedded graph generates a 4-valent framed graph

Problem 1 Which is the least possible (highest possible) genus of a 2-surface $S$ (closed, but not necessarily orientable) this graph can be embedded into in such a way that the embedding represents the zero homology class in the surface (alternatively, the complement to the graph is checkerboard colourable).

Embeddings of such graphs representing the $\mathbb{Z}_2$-homology class are well studied for the case of the plane (see e.g., [Ros99, LM76, RR78, Man05b]) and in the general case (see e.g., [LRS87, CR01]).

In fact, any embedding of a 4-graph in $\mathbb{R}^2$ defines a checkerboard colouring on the set of faces (we consider the infinite domain as a face of $S^2$, the latter being a one-point compactification of $\mathbb{R}^2$) because the plane has trivial first homology. On the other hand, any graph $\Gamma$ embedded into a 2-surface $S$ (orientable or not) can be transformed into a 4-graph by taking the medial graph $\Gamma'$: the vertices of $\Gamma'$ are the middle points of the edges of $\Gamma$, the edges of $\Gamma'$ connect adjacent edges (sharing the same angle), and faces of $\Gamma'$ correspond to faces (white) and vertices (black) of $\Gamma$, see Fig. 7.1.

Such 4-valent graphs appeared with many names in different problems of low-dimensional topology: as atoms (see rigorous definition ahead), originally due to Fomenko [Fom91], see the connection between atoms and knots in [MU05a], they are connected to Grothendieck’s dessins d’enfant, see [LZ03] and [DFK+06].

There is a nice connection between combinatorics of Vassiliev invariants and other invariants of knots and virtual knots and many well-known functions on graphs, see [CDM] and references therein.

Finally, the genus of the atom (the genus of the checkerboard surface we are interested in) is closely connected to the estimates of the thickness for Khovanov and Ozsváth-Szabó homology for classical and virtual knots, see [Man] and [Low07].

In [CR01] there was a reformulation of the problem stated above in terms of ranks of some matrices.