EFFECTS OF DISPERSAL FOR A PREDATOR-PREY MODEL IN A HETEROGENEOUS ENVIRONMENT

Yaying Dong
School of Science, Xi’an Polytechnic University
Xi’an, Shaanxi, 710048, China

Shanbing Li
School of Mathematics and Statistics, Xidian University
Xi’an, Shaanxi 710071, China

Yanling Li
College of Mathematics and Information Science
Shaanxi Normal University, Xi’an, Shaanxi 710062, China

(Communicated by Hermen Jan Hupkes)

Abstract. In this paper, we study the stationary problem of a predator-prey cross-diffusion system with a protection zone for the prey. We first apply the bifurcation theory to establish the existence of positive stationary solutions. Furthermore, as the cross-diffusion coefficient goes to infinity, the limiting behavior of positive stationary solutions is discussed. These results implies that the large cross-diffusion has beneficial effects on the coexistence of two species. Finally, we analyze the limiting behavior of positive stationary solutions as the intrinsic growth rate of the predator species goes to infinity.

1. Introduction. For most predator-prey systems, the prey would become extinct due to the large growth rate of the predator or the high predation rate. Hence, in order to protect certain species, human interference is necessary. From this viewpoint, Du and Shi [3] studied the following predator-prey model with a protection zone:

\begin{align*}
  &u_t = \Delta u + u \left( \frac{\lambda - u - b(x)v}{1 + mu} \right), \quad x \in \Omega, \quad t > 0, \\
  &v_t = \Delta v + v \left( \mu - v + \frac{cu}{1 + mu} \right), \quad x \in \Omega \setminus \bar{\Omega}_0, \quad t > 0, \\
  &\partial_n u = 0, \quad x \in \partial \Omega, \quad t > 0, \\
  &\partial_n v = 0, \quad x \in \partial \Omega, \quad t > 0, \\
  &u(x, 0) = u_0(x) \geq 0, \quad x \in \Omega, \\
  &v(x, 0) = v_0(x) \geq 0, \quad x \in \Omega \setminus \bar{\Omega}_0.
\end{align*}

(1)

2000 Mathematics Subject Classification. Primary: 35J55, 92D40; Secondary: 35B30, 35B32.

Key words and phrases. Predator-prey model, cross-diffusion, protection zone, positive stationary solutions, limiting behavior.

The work is supported by the Natural Science Foundation of China (11801431, 61672021), the Postdoctoral Science Foundation of China (2018T111014, 2018M631133), the Natural Science Foundation of Shaanxi Province (2018JQ1004, 2018JQ1017), Scientific Research Program Funded by Shaanxi Provincial Education Department (Program No. 18JK0049).
From the results obtained in [3], we find that the protection zone has great influence on the dynamic behavior of the model (1). To be more precise, there is a critical size for the protection zone such that when the protection zone is below the critical size, the dynamical behavior is similar to the no-protection zone case: for small $\mu$, the prey survives but the predator becomes extinct; for large $\mu$, an opposite result occurs, that is, the predator survives but the prey is extinct; for intermediate $\mu$, positive stationary solutions exist, and the system is permanent. However the dynamical behavior is changed once the protection zone is beyond the critical size: the two species can coexist even if $\mu$ is large, moreover, for sufficiently large $\mu$, the two species stabilize at a unique positive stationary solution. Besides, the effect of a protection zone has been studied for predator-prey model with strong Allee effect [2], Leslie predator-prey model [6], ratio-dependent predator-prey model [23], Beddington-DeAngelis predator-prey model [7,21], and Lotka-Volterra competition model [5]. Finally, we point out that a related but different situation for the Holling type II predator-prey model, called a degeneracy, is studied in [4,12–14] and references therein.

The effect of cross-diffusion is not considered in [3]. Therefore, in this paper, we introduce the cross-diffusion for the prey in (1) and study the effect of cross-diffusion on the dynamical behavior. That is, we will study the following predator-prey cross-diffusion system with a protection zone

\[
\begin{aligned}
&u_t = \Delta[(1 + k \rho(x)v)u] + u \left( \lambda - u - \frac{b(x)v}{1 + mu} \right), & x \in \Omega, & t > 0, \\
v_t = \Delta v + v \left( \mu - v + \frac{c(x)u}{1 + mu} \right), & x \in \Omega \setminus \Omega_0, & t > 0, \\
\partial_n u = 0, & x \in \partial \Omega, & t > 0, \\
\partial_n v = 0, & x \in \partial \Omega \cup \partial \Omega_0, & t > 0, \\
u(x, 0) = u_0(x) \geq 0, & x \in \Omega, \\
v(x, 0) = v_0(x) \geq 0, & x \in \Omega \setminus \Omega_0, 
\end{aligned}
\]  

(2)

in a bounded domain $\Omega \subset \mathbb{R}^N (N \leq 3)$ with smooth boundary. Here $k$, $\lambda$ are positive constants and $\mu$ is constant which may be negative values; $\Omega_0$ is a subdomain of $\Omega$ with smooth boundary and is regarded as a protection zone for the prey where the predator cannot freely enter; $\rho(x) = b(x) = 1$ in $\overline{\Omega} \setminus \Omega_0$, whereas $\rho(x) = b(x) = 0$ in $\Omega_0$; $c(x) > 0$ is a Hölder continuous function in $\overline{\Omega} \setminus \Omega_0$. One can refer to [3,17] for more detailed biological meaning of the parameters in (2). About the effect of cross-diffusion and protection zone, one can find more results for different reaction-diffusion models in [8–11,17,18,20] and references therein.

Denote $\Omega_1 = \Omega \setminus \Omega_0$. The stationary solutions of (2) satisfy

\[
\begin{aligned}
&\Delta[(1 + k \rho(x)v)u] + u \left( \lambda - u - \frac{b(x)v}{1 + mu} \right) = 0, & x \in \Omega, \\
&\Delta v + v \left( \mu - v + \frac{c(x)u}{1 + mu} \right) = 0, & x \in \Omega_1, \\
&\partial_n u = 0, & x \in \partial \Omega, \\
&\partial_n v = 0, & x \in \partial \Omega_1.
\end{aligned}
\]  

(3)

In this paper, we are mainly concerned with the stationary problem (3), that is, we mainly study the positive stationary solutions of (3). Since the set of positive stationary solutions may play an important role in the stationary patterns, it is extremely important to obtain more information on the set of positive stationary solutions. The main purpose of this present paper has three aspects:
1. Provide the sufficient condition for the existence of positive stationary solutions.

2. Discuss the effect of cross-diffusion on (3) and analyze the limiting behavior of positive stationary solutions as $k \to \infty$.

3. Analyze the limiting behavior of positive stationary solutions as $\mu \to \infty$.

Focusing on these questions, we will state our main results (note that: some biological explanations related to these results will be given in last section). For this, we first introduce some notations. Let $\lambda^N_1(q(x), O)$ and $\lambda^D_1(q(x), O)$ be the principal eigenvalues of $-\Delta + q(x)$ over a region $O$, where $O$ is any bounded domain in $\mathbb{R}^N$, with Neumann or Dirichlet boundary conditions respectively, in particular, one can think that $q(x) = 0$ in $\overline{O}$ when $q(x)$ is omitted in these notations. For any $\phi(x) \in L^p(O)$, we denote the norm of $L^p(O)$ by $\|\phi\|_{p,O} = \left(\int_O |\phi(x)|^p dx\right)^{1/p}$, where $p \in [1, \infty]$. Let $|O|$ represent the measure of $O$. By the results obtained in [3, 17, 18], we can summarize the following lemma.

**Lemma 1.1.** Suppose that $\Omega_0 \neq \emptyset$ and $k > 0$.

(i) For $\mu < 0$, there is a continuous and strictly decreasing function $\lambda_\ast(\mu)$ such that $\lim_{\mu \to 0} \lambda_\ast(\mu) = 0$ and

$$\left\{ (\lambda, \mu) \in [0, \infty) \times (-\infty, 0) : \lambda^N_1 \left( -\mu - \frac{c(x)\lambda}{1 + m\lambda}, \Omega_1 \right) = 0 \right\} = \left\{ (\lambda_\ast(\mu), \mu) : \mu < 0 \right\};$$

(ii) For $\mu \geq 0$, there is a continuous function $\lambda_\ast(\mu, k)$ such that

$$\left\{ (\lambda, \mu) \in [0, \infty)^2 : \lambda^N_1 \left( \frac{b(x)\mu - \lambda}{1 + k\rho(x)\mu}, \Omega_1 \right) = 0 \right\} = \left\{ (\lambda_\ast(\mu, k), \mu) : \mu \geq 0 \right\},$$

moreover,

(a) $\lambda_\ast(\mu, k)$ is strictly increasing with $\mu \geq 0$ such that $\lambda_\ast(0, k) = 0$, $\lambda_\ast(\mu, k) < \mu$ for any $\mu > 0$ and $\lim_{\mu \to \infty} \lambda_\ast(\mu, k) = \lambda^\ast_{\infty}(k) \leq \lambda^D_1($O$)$.

(b) $\lambda^\ast(\mu, k)$ and $\lambda^\ast_{\infty}(k)$ are strictly decreasing with respect to $k$, and $\lambda^\ast_{\infty}(k)$ satisfies $\lim_{k \to \infty} k\lambda^\ast_{\infty}(k) = |\Omega_1|/|\Omega_0|$.

By the global bifurcation theorem, together with Lemma 1.1, we obtain the sufficient condition for the existence of positive solutions of (3).

**Theorem 1.2.** For any given $\Omega_0 \neq \emptyset$ and $k > 0$.

(i) If $\mu \geq 0$, then (3) admits a positive solution for each $\lambda > \lambda^\ast(\mu, k)$.

(ii) If $\lambda^N_1(-c(x)/m, \Omega_1) < \mu < 0$, then (3) admits a positive solution for each $\lambda > \lambda_\ast(\mu)$. By using the similar argument to Theorem 1.2, we can obtain the following result.

**Corollary 1.** For any given $\Omega_0 \neq \emptyset$ and $k > 0$.

(i) If $0 < \lambda < \lambda^\ast_{\infty}(k)$, then (3) admits a positive solution for each $\mu \in \left( \lambda^N_1 \left( -\frac{c(x)\lambda}{1 + m\lambda}, \Omega_1 \right), \mu^* \right)$, where $\mu^*$ is uniquely determined by $\lambda^N_1 \left( \frac{b(x)\mu^* - \lambda}{1 + k\rho(x)\mu^*}, \Omega_1 \right) = 0$.

(ii) If $\lambda \geq \lambda^\ast_{\infty}(k)$, then (3) admits a positive solution for each $\mu > \lambda^N_1 \left( -\frac{c(x)\lambda}{1 + m\lambda}, \Omega_1 \right)$.

Lemma 1.1 implies that $\lim_{k \to \infty} \lambda^\ast_{\infty}(k) = 0$. Hence it follows from Theorem 1.2 that when $k \to \infty$, (3) has a positive solution for each $\lambda > 0$ and $\mu \geq 0$. Our next theorem gives the limiting behavior of positive solutions of (3) as $k \to \infty$.

**Theorem 1.3.** Suppose that $(u_k, v_k)$ is any positive solution of (3).
(i) For any fixed $\lambda > 0$ and $\mu \geq 0$,
\[
\lim_{k \to \infty} (u_k, u_k, v_k) = (\lambda, 0, \mu) \text{ in } C^1(\Omega_0) \times C^1(\Omega_1) \times C^1(\Omega_1),
\]
\[
\lim_{k \to \infty} kv_k = \infty \text{ uniformly in } \Omega_1,
\]
passing to a subsequence.

(ii) For any fixed $\lambda > \lambda^*_\infty(k)$ and $\mu < 0$,
\[
\lim_{k \to \infty} u_k = \overline{u} \text{ uniformly in } \Omega,
\]
\[
\lim_{k \to \infty} (v_k, kv_k) = (0, \overline{w}) \text{ in } C^1(\Omega_1) \times C^1(\Omega_1),
\]
passing to a subsequence, where $(\overline{u}, \overline{w})$ satisfies
\[
\begin{cases}
\Delta [(1 + \rho(x)\overline{w})\overline{u}] + \overline{u}(\lambda - \overline{u}) = 0, & x \in \Omega, \\
\Delta \overline{w} + \overline{w} \left( \mu + \frac{c(x)\overline{u}}{1 + m\overline{u}} \right) = 0, & x \in \Omega_1, \\
\partial_n \overline{u} = 0, & x \in \partial \Omega, \\
\partial_n \overline{w} = 0, & x \in \partial \Omega_1.
\end{cases}
\]

For the limiting system (4), we establish the bifurcation structure of positive solutions in the following theorem.

**Theorem 1.4.** Suppose that $\lambda > 0$ and $\lambda^*_\infty(k) < \mu < 0$. Then the set of positive solutions of (4) forms an unbounded connected set of $\Gamma$ in $\mathbb{R} \times L^\infty(\Omega) \times C^1(\Omega_1)$ which joins the semitrivial solution branch $\{ (\mu, \overline{u}, \overline{w}) = (\mu, \lambda, 0) : \mu \in \mathbb{R} \}$ at $(\lambda^*_\infty(k), \lambda, 0)$ and remains bounded until $\mu$ approaches 0, where it blows up. Moreover, passing to a subsequence,
\[
\begin{cases}
\lim_{\mu \to 0} \overline{u}_\mu = \lambda \text{ in } C^1(\Omega_0), \\
\lim_{\mu \to 0} (\overline{u}_\mu, \overline{w}_\mu) = (0, \infty) \text{ uniformly in } \Omega_1.
\end{cases}
\]

It follows from Corollary 1 that when $\lambda > \lambda^*_\infty(k)$, (3) has a positive solution even for large $\mu$. Therefore, our final theorem gives the limiting behavior of positive solutions of (3) as $\mu \to \infty$.

**Theorem 1.5.** Let $\lambda > \lambda^*_\infty(k)$ and $k > 0$ be fixed. Suppose that $(u_\mu, v_\mu)$ is any positive solution of (3). Then, passing to a subsequence,
\[
\begin{cases}
\lim_{\mu \to \infty} u_\mu = U_{\lambda,k} \text{ in } C^1(\Omega_0), \\
\lim_{\mu \to \infty} (u_\mu, v_\mu) = (0, \infty) \text{ uniformly in } \Omega_1,
\end{cases}
\]
where $U_{\lambda,k}$ is the unique positive solution of
\[
\Delta U + U \left\{ (\lambda - U)\chi_{\Omega_0} - \left(1/k\right)\chi_{\Omega_1} \right\} = 0 \text{ in } \Omega, \ \partial \n U = 0 \text{ on } \partial \Omega.
\]

This paper is organized as follows. In Section 2, we prove some preliminary results which are used to show our main results. In Section 3, we complete the proof of our main results. In Section 4, we summary our results and discuss their biological implications.
2. Preliminary results. In this section, we show some preliminary results which will be used to prove our main results, including the non-existence of positive solutions, a priori estimates of any positive solution and the local bifurcation from semitrivial solutions.

Denote

\[ U = (1 + k \rho(x)v)u. \]

Then (3) is written as

\[
\begin{cases}
\Delta U + \frac{U}{1 + k \rho(x)v} \left( \frac{U}{1 + k \rho(x)v} - \frac{b(x)v(1 + k \rho(x)v)}{c(x)U} \right) = 0, & x \in \Omega, \\
\Delta v + v \left( \mu - v + \frac{1 + k \rho(x)v + mU}{1 + k \rho(x)v + mU} \right) = 0, & x \in \Omega_1, \\
\partial_n U = 0, & x \in \partial \Omega, \\
\partial_n v = 0, & x \in \partial \Omega_1.
\end{cases}
\]

To derive a priori estimates of any positive solution of (7), we cite Harnack inequality and maximum principle for weak solutions (see Lemma 2.2 and Lemma 2.3 in [15]).

Lemma 2.1. Assume that \( O \subset \mathbb{R}^N \) is a bounded Lipschitz domain, and \( m(x) \in L^q(O) \) for some \( q > N/2 \). If \( z \in W^{1,2}(O) \) satisfies

\[ \Delta z + m(x)z = 0 \text{ in } O, \quad \partial_n z = 0 \text{ on } \partial O, \]

then there exists a constant \( C > 0 \) such that

\[ \sup_O z \leq C \inf_O z, \]

where \( C \) depends on only \( \|m(x)\|_q, \ q \text{ and } O. \)

Lemma 2.2. Assume that \( O \subset \mathbb{R}^N \) is a bounded Lipschitz domain, and \( f \in C(\overline{O} \times \mathbb{R}) \). If there exists a positive constant \( M \) such that \( f(x,z) < (>0) \) for \( z > (>M), \) and \( z \in W^{1,2}(O) \) satisfies

\[ \Delta z + f(x,z) \geq (\leq)0 \text{ in } O, \quad \partial_n z \leq (\geq)0 \text{ on } \partial O, \]

then \( z \leq (\geq)M \) a.e. in \( O. \)

By means of Lemmas 2.1 and 2.2, we obtain a priori estimates of any positive solution of (7).

Proposition 1. (i) Suppose that \( (U,v) \) is any positive solution of (7). Then for each given \( k > 0, \) there is a positive constant \( C = C(\lambda, k, |\Omega|, |\Omega_0|) \) independent of \( \mu \) such that \((U,v)\) satisfies

\[ 0 < u \leq U \leq C \text{ in } \Omega, \quad \max_{\Omega_1} \{\mu, 0\} \leq v \leq \mu + \max_{\Omega_1} c(x)/m \text{ in } \Omega_1. \]

(ii) Suppose that \( (U,v) \) is any positive solution of (7). Then for large \( k(> M), \) there is a positive constant \( C' = C'(\lambda, |\Omega|, |\Omega_0|) \) independent of \( \mu \) and \( k \) such that \((U,v)\) satisfies

\[ 0 < u \leq U \leq C' \text{ in } \Omega, \quad \max_{\Omega_1} \{\mu, 0\} \leq v \leq \mu + \max_{\Omega_1} c(x)/m \text{ in } \Omega_1. \]

Proof. (i) Suppose that \((U,v)\) is any positive solution of (7). Then from the equation for \( v \) of (7), it follows that

\[ -\Delta v \leq v \left( \mu + \max_{\Omega_1} c(x)/m - v \right), \quad -\Delta v \geq v (\mu - v). \]
Thus, we apply Lemma 2.2 to get $\max\{\mu,0\} \leq v \leq \mu + \max_{\Omega_1} c(x)/m$ in $\Omega_1$.

We integrate the equation for $U$ of (7) and apply the Schwarz inequality to obtain

$$
\int_{\Omega} u^2 dx = \int_{\Omega} u \left( \lambda - \frac{b(x)v}{1 + mu} \right) dx \leq \lambda \int_{\Omega} u dx \leq \lambda |\Omega|^{1/2} ||u||_{2, \Omega}.
$$

This implies that

$$
||u||_{2, \Omega} \leq \lambda |\Omega|^{1/2},
$$

and thus

$$
|\Omega_0|^{1/2} \inf_{\Omega_0} u \leq ||u||_{2, \Omega_0} \leq ||u||_{2, \Omega} \leq \lambda |\Omega|^{1/2}.
$$

As a result, we derive

$$
\inf_{\Omega_0} u \leq \lambda (|\Omega|/|\Omega_0|)^{1/2}.
$$

Denote

$$
m(x) = \frac{1}{1 + k\rho(x)v} \left( \lambda - \frac{U}{1 + k\rho(x)v} - \frac{b(x)v(1 + k\rho(x)v)}{1 + k\rho(x)v + mu} \right).
$$

The equation for $U$ of (7) is written as

$$
\Delta U + m(x)U = 0, \; x \in \Omega, \; \partial_{\nu} U = 0, \; x \in \partial\Omega.
$$

Note that

$$
|m(x)| \leq \frac{\lambda}{1 + k\rho(x)v} + \frac{u}{1 + k\rho(x)v} + \frac{b(x)v}{(1 + k\rho(x)v)(1 + mu)} \leq \lambda + u + \frac{1}{k}.
$$

Thus

$$
||m(x)||_{2, \Omega} \leq C_1 + ||u||_{2, \Omega} \leq C_2.
$$

Then we apply Lemma 2.1 to obtain

$$
\sup_{\Omega} U \leq C_3 \inf_{\Omega} U \leq C_3 \inf_{\Omega_0} U \leq C_3 \lambda (|\Omega|/|\Omega_0|)^{1/2} \leq C,
$$

where $C$ is independent of $\mu$.

(ii) By replacing the upper bound of $|m(x)|$ by $\lambda + u + 1/M$, we can use the similar argument as above to derive the assertion (ii). \qed

For any $\lambda, \mu > 0$, (7) has two semi-trivial solutions: $(\lambda, 0)$ and $(0, \mu)$. Thus, (7) exists two curves of these solutions in the space of $(\lambda, U, v)$:

$$
\Gamma_U = \{ (\lambda, U, v) = (\lambda, \lambda, 0) : \lambda > 0 \}, \; \Gamma_v = \{ (\lambda, U, v) = (\lambda, 0, \mu) : \lambda > 0 \}.
$$

Define

$$
X_1 = W_n^{2,p}(\Omega) \times W_n^{2,p}(\Omega_1), \; X_2 = L^p(\Omega) \times L^p(\Omega_1),
$$

where $W_n^{2,p}(O) = \{ w \in W^{2,p}(O) : \partial_{\nu} w = 0 \text{ on } \partial O \}$ and $p > N$. Define

$$
E = C_n^1(\Omega) \times C_n^1(\Omega_1),
$$

where $C_n^1(\Omega) = \{ u \in C^1(\Omega) : \partial_{\nu} u = 0 \text{ on } \partial O \}$.

Let $\phi^*$ be a positive solution of

$$
-\Delta \phi^* + \frac{b(x)\mu - \lambda^*}{1 + k\rho(x)\mu} \phi^* = 0 \text{ in } \Omega, \; \partial_{\nu} \phi^* = 0 \text{ on } \partial\Omega, \quad (8)
$$

where $\lambda^* = \lambda^*(\mu, k)$ is defined in Lemma 1.1(ii). Define

$$
\psi^* = ( -\Delta + \mu I )_{\Omega_1}^{-1} \left[ \frac{c(x)\mu}{1 + k\mu} \phi^* \right]. \quad (9)
$$

Thus we apply the local bifurcation theorem [1] to derive the following bifurcation result from $\Gamma_v$. 

Proposition 2. For each given $\mu > 0$, $(\lambda^*, 0, \mu)$ is a bifurcation point of (7) where positive solutions bifurcate from $\Gamma_0$, moreover, positive solutions of (7) near $(\lambda^*, 0, \mu) \in \mathbb{R} \times X_1$ are expressed by

$$\Gamma_{local} = \{\lambda, U, v \} = (\lambda(s), s(\phi^* + U(s)), \mu + s(\psi^* + v(s)) : s \in (0, \delta)\},$$

where $\delta > 0$ is a small constant, $(\lambda(s), U(s), v(s))$ is a smooth function with respect to $s$ which satisfies $(\lambda(0), U(0), v(0)) = (\lambda^*, 0, 0)$ and $\int_{\Omega} U(s) \phi^* dx = 0$.

Proof. Denote $V = v - \mu$ in (7) and define an operator $F : \mathbb{R} \times X_1 \to X_2$ by

$$F(\lambda, U, V) = \left( \begin{array}{c} \Delta U + f_1(\lambda, U, V + \mu) \\ \Delta V + f_2(U, V + \mu) \end{array} \right),$$

where

$$\begin{cases} f_1(\lambda, U, v) = \frac{U}{1 + k\rho(x)v} \left( \frac{\lambda - b(x)v^0}{1 + k\rho(x)v} - \frac{b(x)v(1 + k\rho(x)v)}{1 + k\rho(x)v + mU} \right), \\ f_2(U, v) = v \left( \mu - \frac{b(x)v}{1 + k\rho(x)v + mU} \right). \end{cases}$$

(10)

It is clear that $F(\lambda, 0, 0) = 0$ for any $\lambda$, and the Fréchet derivative of $F$ at $(U, V) = (0, 0)$ is defined by

$$F_{(U, V)}(\lambda, 0, 0)[\phi, \psi] = \left( \begin{array}{c} \Delta \phi + \frac{\lambda - b(x)\mu}{1 + k\rho(x)\mu} \phi \\ \Delta \psi - \mu \psi + \frac{c(x)\mu}{1 + k\rho(x)\mu} \phi \end{array} \right).$$

(11)

It follows from Krein-Rutman theorem [22] and Lemma 1.1 that $F_{(U, V)}(\lambda, 0, 0)[\phi, \psi] = (0, 0)$ has a solution with $\phi > 0$ iff $\lambda = \lambda^*$. By (8), (9) and (11), a simple calculation yields that

$$\text{Ker } F_{(U, V)}(\lambda^*, 0, 0) = \text{span}\{ (\phi^*, \psi^*) \}.$$

This implies that $\dim \text{Ker } F_{(U, V)}(\lambda^*, 0, 0) = 1$.

The adjoint operator of $F_{(U, V)}(\lambda^*, 0, 0)$, denoted by $F^*_n(U, V)(\lambda^*, 0, 0)$, is given by

$$F^*_n(U, V)(\lambda^*, 0, 0)[\phi, \psi] = \left( \begin{array}{c} \Delta \phi^* + \frac{\lambda^* - b(x)\mu}{1 + k\rho(x)\mu} \phi^* \\ \Delta \psi^* - \mu \psi^* + \frac{c(x)\mu}{1 + k\rho(x)\mu} \phi^* \end{array} \right).$$

Then $\text{Ker } F^*_n(U, V)(\lambda^*, 0, 0) = \text{span}\{ (\phi^*, 0) \}$. Hence, we apply Fredholm alternative theorem to obtain

$$\text{Range } F_{(U, V)}(\lambda^*, 0, 0) = \left\{ (\phi, \psi) \in X_2 : \int_{\Omega} \phi \cdot \phi^* dx = 0 \right\}. \quad (12)$$

Thus $\text{codim } \text{Range } F_{(U, V)}(\lambda^*, 0, 0) = 1$. Moreover, by (12), we have

$$F_{\lambda(U, V)}(\lambda^*, 0, 0)[\phi^*, \psi^*] = \left( \begin{array}{c} \phi^* \\ 0 \end{array} \right) \notin \text{Range } F_{(U, V)}(\lambda^*, 0, 0).$$

Therefore, we complete the proof by using the local bifurcation theorem [1]. \qed

Let $\psi_*$ be a positive solution of

$$-\Delta \psi_* - \frac{c(x)\lambda_*}{1 + m\lambda_*} \psi_* = \mu \psi_* \text{ in } \Omega_1, \quad \partial_n \psi_* = 0 \text{ on } \partial \Omega_1,$$
where \(\lambda_\ast = \lambda_\ast(\mu)\) is defined in Lemma 1.1(i). Define
\[
\phi_\ast = -(\Delta - \lambda_\ast I)^{-1} \left[ \lambda_\ast \left( k\rho(x)\lambda_\ast - \frac{b(x)}{1 + m\lambda_\ast} \right) \psi_\ast \right].
\]

By the similar argument to Proposition 2, we can derive the following bifurcation result.

**Proposition 3.** For each given \(\lambda_1^N = \lambda_\ast^N < 0, (\lambda_\ast, \lambda_\ast, 0)\) is a bifurcation point of (7) where positive solutions bifurcate from \(\Gamma_U\), moreover, positive solutions of (7) near \((\lambda_\ast, \lambda_\ast, 0)\) \(\in \mathbb{R} \times X_1\) are expressed by
\[
\{(\lambda, U, v) = ((\tilde{\lambda}(s), \lambda_\ast + s(\phi_\ast + \tilde{U}(s)), s(\psi_\ast + \tilde{v}(s))): s \in (0, \tilde{\delta})\},
\]
where \(\tilde{\delta} > 0\) is a small constant, and \((\tilde{\lambda}(s), \tilde{U}(s), \tilde{v}(s))\) is a smooth function with respect to \(s\) which satisfies \((\tilde{\lambda}(0), \tilde{U}(0), \tilde{v}(0)) = (\lambda_\ast, 0, 0)\) and \(\int_{\Omega} \tilde{v}(s)\psi_\ast dx = 0\).

**Proof.** Let \(z := U - \lambda\) in (7) and define an operator \(\Phi: \mathbb{R} \times X_1 \rightarrow X_2\) by
\[
\Phi(\lambda, z, v) = \left( \begin{array}{c} \Delta z + f_1(\lambda, z + \lambda, v) \\ \Delta V + f_2(z + \lambda, v) \end{array} \right),
\]
where \(f_1\) and \(f_2\) are functions defined by (10). By a simple calculation, the Fréchet derivative of \(\Phi\) at \((z, v) = (0, 0)\) is given by
\[
\Phi_{(z,v)}(\lambda, 0, 0)[\phi, \psi] = \begin{pmatrix} \Delta \phi - \lambda \phi + \lambda \left( k\rho(x)\lambda_\ast - \frac{b(x)}{1 + m\lambda_\ast} \right) \psi \\ \Delta \psi + \left( \mu + \frac{c(x)\lambda_\ast}{1 + m\lambda_\ast} \right) \psi \end{pmatrix}, \tag{13}
\]
By the Krein-Rutman theorem [22] and Lemma 1.1, \(\Phi_{(z,v)}(\lambda, 0, 0)[\phi, \psi] = (0, 0)\) has a solution with \(\phi > 0\) if and only if \(\lambda = \lambda_\ast\); thus \(\lambda_\ast\) is the only possible bifurcation point where positive solutions of (7) bifurcate from \(\Gamma_U\). Similar to the calculation of Proposition 2, we can obtain
\[
\ker \Phi_{(z,v)}(\lambda_\ast, 0, 0) = \text{span}\{(\phi_\ast, \psi_\ast)\},
\]
and
\[
\text{range} \Phi_{(z,v)}(\lambda_\ast, 0, 0) = \left\{(\phi, \psi) \in X_2: \int_{\Omega} \psi \cdot \psi_\ast dx = 0 \right\},
\]
Thus it holds that \(\dim \ker \Phi_{(z,v)}(\lambda_\ast, 0, 0) = \text{codim range} \Phi_{(z,v)}(\lambda_\ast, 0, 0) = 1\). Moreover, it is easy to check that
\[
\Phi_{(z,v)}(\lambda_\ast, 0, 0)[\phi_\ast, \psi_\ast] \notin \text{range} \Phi_{(z,v)}(\lambda_\ast, 0, 0),
\]
Consequently, we can apply the local bifurcation theorem [1] to \(\Phi\) at \((\lambda_\ast, 0, 0)\). Therefore, the proof of Proposition 3 is complete.

3. **Proof of main results.** In this section, by combining the results obtained in Section 2, we apply the global bifurcation theorem, elliptic regularity theory and various elliptic estimates to complete the proof of our main results.
3.1. **Proof of Theorem 1.2.** In this subsection, we apply the global bifurcation theorem (see Theorem 6.4.3 of López-Gómez [16] based on the global bifurcation theory of Rabinowitz [19]) to prove Theorem 1.2.

**Proof of Theorem 1.2.** For the case \( \mu > 0 \). We define an operator:

\[
H(\lambda, U, v) = \begin{pmatrix} U \\ v \end{pmatrix} - \begin{pmatrix} (-\Delta + I)_{\Omega}^{-1}[U + f_1(\lambda, U, v)] \\ (-\Delta + I)_{\Omega}^{-1}[v + f_2(U, v)] \end{pmatrix}.
\]

For each given \( \lambda > 0 \), it follows from the elliptic regularity theory that the second term of \( H \) is a compact operator. It is clear that (7) is equivalent to \( H(\lambda, U, v) = 0 \).

By the standard argument, it is not hard to check that the conditions of Theorem 6.4.3 in [16] hold true (for details, one can see Theorem 3.2 in [10]). Therefore, by Proposition 2 and Theorem 6.4.3 in [16], the local bifurcation branch \( \Gamma_{local} \) can be extended a maximal connected set \( \Gamma_{global} \subset \mathbb{R} \times E \), which satisfies

\[
\Gamma_{local} \subset \Gamma_{global} \subset \{ (\lambda, U, v) \in \mathbb{R} \times E \setminus \{(\lambda^*, 0, \mu)\} : H(\lambda, U, v) = 0 \}.
\]

Moreover, the maximal connected set \( \Gamma_{global} \) satisfies one of the following:

1. \( \Gamma_{global} \) is unbounded in \( \mathbb{R} \times E \);
2. \( \Gamma_{global} \) contains a certain point \((\lambda^{**}, 0, \mu)\) with \( \lambda^{**} \neq \lambda^* \);
3. \( \Gamma_{global} \) contains a certain point \((\lambda', \phi', \psi') \in \mathbb{R} \times (Y \setminus \{(0, \mu)\}) \), where \( Y \) is the supplement of span \( \{(\phi^*, \psi^*)\} \) which is given by

\[
Y = \left\{ (\phi, \psi) \in E : \int_{\Omega} \phi \cdot \phi^* = 0 \right\}.
\]

In the following, we will show only case (1) can occur. For this, we prove

\[
\Gamma_{global} \subset \mathbb{R} \times P_{\Omega} \times P_{\Omega_1},
\]

where \( P_{\Omega} = \{ w \in C^1(\Omega) : w > 0 \text{ in } \Omega \} \). If this is not true, then there is a sequence \( \{(\lambda_i, U_i, v_i)\}_{i=1}^\infty \subset \Gamma_{global} \cap (\mathbb{R} \times P_{\Omega} \times P_{\Omega_1}) \) such that

\[
\lim_{i \to \infty} (\lambda_i, U_i, v_i) = (\lambda_\infty, U_\infty, v_\infty) \text{ in } \mathbb{R} \times E,
\]

where

\[
(\lambda_\infty, U_\infty, v_\infty) \in \Gamma_{global} \cap (\mathbb{R} \times \partial(P_{\Omega} \times P_{\Omega_1})).
\]

Furthermore, by the maximum principle, \((U_\infty, v_\infty)\) satisfies one of the following:

(a) \( U_\infty \equiv 0 \) in \( \overline{\Omega} \), \( v_\infty \equiv 0 \) in \( \overline{\Omega} \);
(b) \( U_\infty > 0 \) in \( \overline{\Omega} \), \( v_\infty \equiv 0 \) in \( \overline{\Omega} \);
(c) \( U_\infty \equiv 0 \) in \( \overline{\Omega} \), \( v_\infty > 0 \) in \( \overline{\Omega} \).

By the boundary condition of \( v_i \), we integrate the second equation of (7) with \((U, v) = (U_i, v_i)\) to obtain

\[
\int_{\Omega_1} v_i \left( \mu - v_i + \frac{c(x)U_i}{1 + kv_i + mU_i} \right) dx = 0.
\]

Suppose that (a) or (b) occurs. Then for sufficiently large \( i \in N \),

\[
\mu - v_i + \frac{c(x)U_i}{1 + kv_i + mU_i} > 0 \text{ in } \Omega_1
\]

because of \( \mu > 0 \). This shows that for sufficiently large \( i \in N \), the left-hand side of (18) is positive, a contradiction. Suppose that (c) occurs. Then \( v_\infty \) satisfies

\[
\Delta v_\infty + v_\infty (\mu - v_\infty) = 0 \text{ in } \Omega_1, \quad \partial_\nu v_\infty = 0 \text{ on } \partial\Omega_1.
\]

Hence we must have \( v_\infty = \mu \) in \( \overline{\Omega_1} \) due to \( v_\infty > 0 \) in \( \overline{\Omega_1} \). By Proposition 2, we derive

\[
(\lambda_\infty, U_\infty, v_\infty) = (\lambda^*, 0, \mu).
\]

This is impossible due to (14) and (17). Consequently, (16) is true.
Thanks to (16), it is clear that case (2) is impossible. In view of (15), (16) and \( \phi^* > 0 \) in \( \overline{\Omega}_1 \), case (3) also cannot occur. As a result, the only possibility is case (1). By means of Proposition 1 and (16), system (7) admit a positive solution if \( \lambda > \lambda^*(\mu, k) \). Therefore, we complete the proof for the case \( \mu > 0 \).

By a similar manner, we can show that (7) admit a positive solution if \( \lambda > \lambda_*(\mu) \) for the case \( \mu < 0 \), and so we omit the details here.

As a last step, we discuss the case \( \mu = 0 \). Since we have proved the existence of positive solutions for \( \mu > 0 \) and \( \mu < 0 \) respectively, we choose a sequence \( \{(\mu_i, U_i, v_i)\}_{i=1}^{\infty} \) satisfying \( \lim_{i \to \infty} \mu_i = 0 \) such that \( (U_i, v_i) \) is a positive solution of (7) with \( \mu = \mu_i \). By Proposition 1 and the boundedness of \( \{\mu_i\}_{i=1}^{\infty} \), we can further choose a subsequence such that

\[
\lim_{i \to \infty} (U_i, v_i) = (U_\infty, v_\infty) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}_1),
\]

where \( (U_\infty, v_\infty) \) is a non-negative solution of (7) with \( \mu = 0 \). Thus it follows from the maximum principle that \( (U_\infty, v_\infty) \) satisfies either one of (a)-(c) or \( U_\infty > 0 \) and \( v_\infty > 0 \). From the first equation of (7) with \( (U, v) = (U_i, v_i) \), we derive

\[
\int_{\Omega_i} \frac{U_i}{1+k\rho(x)v_i} \left( \lambda - \frac{U_i}{1+k\rho(x)v_i} - \frac{b(x)v_i(1+k\rho(x)v_i)}{1+k\rho(x)v_i+mU_i} \right) \, dx = 0
\]

for any \( i \in N \), thus (a) cannot occur because of \( \lambda > 0 \). We integrate the second equation of (7) with \( (U, v) = (U_i, v_i) \) to obtain

\[
\int_{\Omega_i} v_i \left( \mu_i - v_i + \frac{c(x)U_i}{1+kv_i+mU_i} \right) \, dx = 0
\]

for any \( i \in N \), both (b) and (c) cannot occur due to \( \lim_{i \to \infty} \mu_i = 0 \). Hence, the only possibility is that \( U_\infty > 0 \) in \( \overline{\Omega} \) and \( v_\infty > 0 \) in \( \overline{\Omega}_1 \). This shows that for any fixed \( \lambda > 0 \), (7) with \( \mu = 0 \) has a positive solution.

3.2. Proof of Theorem 1.3. In this subsection, we will complete the proof of Theorem 1.3. For this, we establish the following lemmas.

**Lemma 3.1.** Suppose that \( (u_{k_i}, v_{k_i}) \) is any positive solution of (3) with \( k = k_i \) satisfying \( \lim_{i \to \infty} k_i = \infty \). Then, passing to a subsequence,

\[
\lim_{i \to \infty} (U_{k_i}, v_{k_i}) = (\overline{U}, \max\{\mu, 0\}) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}_1),
\]

where \( U_{k_i} = (1+k_i\rho(x)v_{k_i})u_{k_i} \), and \( \overline{U} \in C^1(\overline{\Omega}) \) is a non-negative function.

**Proof.** By Proposition 1(ii) and elliptic regularity theory, we may assume that

\[
\lim_{i \to \infty} (U_{k_i}, v_{k_i}) = (\overline{U}, \overline{v}) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}_1).
\]

By Proposition 1(ii) again, it is clear that

\[
\overline{v} \geq \max\{\mu, 0\} \text{ in } \overline{\Omega}_1.
\]  

(19)

Since \( \rho(x) = 1 \) in \( \overline{\Omega}_1 \), we have

\[
\lim_{i \to \infty} \frac{u_{k_i}(x)v_{k_i}(x)}{1+mu_{k_i}(x)} = \lim_{i \to \infty} \frac{U_{k_i}(x)}{1+k_i v_{k_i}(x)} \cdot \frac{v_{k_i}(x)}{1+mu_{k_i}(x)} = 0 \text{ in } \overline{\Omega}_1.
\]

Thus, we apply Lebesgue dominated convergence theorem to obtain

\[
0 = \lim_{i \to \infty} \int_{\Omega_1} v_{k_i} \left( \mu - v_{k_i} + \frac{c(x)u_{k_i}}{1+mu_{k_i}} \right) \, dx = \int_{\Omega_1} \overline{v}(\mu - \overline{v}) \, dx.
\]  

(20)

It follows from (19) and (20) that \( \overline{v} \equiv \max\{\mu, 0\} \) in \( \overline{\Omega}_1 \). \( \square \)
Lemma 3.2. For any \( \lambda > \lambda_+(\mu) \) with \( \mu \leq 0 \). Suppose that \( (u_k, v_k) \) is any positive solution of (3) with \( k = k_i \) satisfying \( \lim_{i \to \infty} k_i = \infty \), and suppose that \( \{\max_{1 \leq i \leq k} u_{k_i}, v_{k_i}\} \) is bounded. Then \( \mu < 0 \) and

\[
\lim_{i \to \infty} u_{k_i} = \pi \text{ uniformly in } \Omega, \quad \lim_{i \to \infty} k_i v_{k_i} = w \text{ in } C^1(\Omega_1),
\]

passing to a subsequence. Here \((\pi, w)\) is a positive solution of (4).

Proof. Denote \( w_{k_i} = k_i v_{k_i} \). Then \( (u_k, w_k) \) is a positive solution of

\[
\begin{align*}
\Delta[(1 + \rho(x)w_{k_i})u_{k_i}] + u_{k_i} \left( \lambda - u_{k_i} - \frac{b(x)v_{k_i}}{1 + mu_{k_i}} \right) &= 0, \quad x \in \Omega, \\
\Delta w_{k_i} + w_{k_i} \left( \mu - v_{k_i} + \frac{c(x)u_{k_i}}{1 + mu_{k_i}} \right) &= 0, \quad x \in \Omega_1, \quad (21) \\
\partial_n u_{k_i} &= 0, \quad x \in \partial \Omega, \\
\partial_n w_{k_i} &= 0, \quad x \in \partial \Omega_1.
\end{align*}
\]

If \( \{\max_{1 \leq i \leq k} u_{k_i}, v_{k_i}\} \) is bounded, then it is clear that \( v_{k_i} \to 0 \) uniformly in \( \Omega_1 \). It follows from the elliptic regularity theory and Lemma 3.1 that

\[
\lim_{i \to \infty} \left( (1 + \rho(x)w_{k_i})u_{k_i}, v_{k_i}, w_{k_i} \right) = (U, 0, \pi) \text{ in } C^1(\Omega) \times C^1(\Omega_1)<C^1(\Omega_1). \quad (22)
\]

Thus,

\[
\lim_{i \to \infty} u_{k_i} = \lim_{i \to \infty} \frac{U_{k_i}}{1 + \rho(x)w_{k_i}} = \frac{U}{1 + \rho(x)\pi} =: \pi \geq 0 \text{ uniformly in } \Omega. \quad (23)
\]

Letting \( i \to \infty \) in (21), together with (22) and (23), we find that \((\pi, w)\) satisfies (4).

We next prove that \( \pi > 0 \) in \( \Omega \) and \( w > 0 \) in \( \Omega_1 \). It follows from (4) and (23) that \( U \) is a non-negative solution of

\[
\Delta U + \frac{U}{1 + \rho(x)\pi} \left( \lambda - \frac{U}{1 + \rho(x)\pi} \right) = 0 \text{ in } \Omega, \quad \partial_n U = 0 \text{ on } \partial \Omega.
\]

Thus either \( U > 0 \) or \( U \equiv 0 \) in \( \Omega \) by the maximum principle. If \( U \equiv 0 \) in \( \Omega \), then, in view of (23), \( \lim_{i \to \infty} u_{k_i} = 0 \) uniformly in \( \Omega_1 \). Due to \( \lambda > 0 \) and (22), we obtain

\[
\int_{\Omega} u_{k_i} \left( \lambda - u_{k_i} - \frac{b(x)v_{k_i}}{1 + mu_{k_i}} \right) dx > 0 \text{ for large } i.
\]

This is a contradiction. Hence we must have \( U > 0 \) in \( \Omega \), this implies that \( \pi > 0 \) in \( \Omega \). Similarly, it follows from the maximum principle that either \( w > 0 \) or \( w \equiv 0 \) in \( \Omega_1 \). If \( w \equiv 0 \) in \( \Omega_1 \), then

\[
\Delta \pi + \pi(\lambda - \pi) = 0 \text{ in } \Omega, \quad \partial_n \pi = 0 \text{ on } \partial \Omega.
\]

The positivity of \( \pi \) implies that \( \pi \equiv \lambda \in \Omega \). Then from the equation of \( v_{k_i} \), it follows that

\[
0 = \lambda^N_1 \left( -\mu - \frac{c(x)v_{k_i}}{1 + mu_{k_i}} + v_{k_i}, \Omega_1 \right)
\]

This is a contradiction, which means that \( \pi \) must be positive in \( \Omega_1 \). Therefore, \((\pi, w)\) is a positive solution of (4).
It remains to show $\mu < 0$. We have shown that $(\overline{\Omega}, \overline{\Omega})$ is a positive solution of (4). Then

$$\mu = \lambda_1^N \left( - \frac{c(x)\overline{\Omega}}{1 + m\overline{\Omega}}, \Omega_1 \right) - \frac{c(x)\overline{\Omega}}{1 + m\overline{\Omega}} < 0 \text{ in } \Omega_1.$$ 

Hence, it follows from the monotonicity of the principal eigenvalue that $\mu < 0$. \hfill \Box

**Lemma 3.3.** Let $\mu = 0$. Suppose that $(u_{k_i}, v_{k_i})$ is any positive solution of (3) with $k = k_i$ satisfying $\lim_{i \to \infty} k_i = \infty$. Then $\{\min_{\Omega_i} k_i v_{k_i}\}_{i=1}^\infty$ is unbounded.

**Proof.** Denote $w_{k_i} = k_i v_{k_i}$. Then we observe that $\{\max_{\Omega_i} w_{k_i}\}_{i=1}^\infty$ is unbounded. Otherwise, we assume that $\{\max_{\Omega_i} w_{k_i}\}_{i=1}^\infty$ is bounded. Thus, Lemma 3.2 implies $\mu < 0$, which contradicts the assumption $\mu = 0$. Note that $w_{k_i}$ satisfies

$$\Delta w_{k_i} + w_{k_i} \left( \frac{c(x)u_{k_i}}{1 + m k_i} - v_{k_i} \right) = 0 \text{ in } \Omega_1, \quad \partial_n w_{k_i} = 0 \text{ on } \partial \Omega_1.$$ 

By Harnack inequality, there exists some positive constant $C$ independent of $i$ such that

$$\max_{\Omega_i} w_{k_i} \leq C \min_{\Omega_i} w_{k_i}.$$ 

This shows that $\{\min_{\Omega_i} k_i v_{k_i}\}_{i=1}^\infty$ is unbounded. \hfill \Box

By means of Lemmas 3.1-3.3, we can complete the proof of Theorem 1.3.

**Proof of Theorem 1.3.** (i) Suppose that $(u_{k_i}, v_{k_i})$ is any positive solution of (3) with $k = k_i$ satisfying $\lim_{i \to \infty} k_i = \infty$. Then by Lemma 3.1 and Lemma 3.3, passing to a subsequence, we see that

$$\lim_{i \to \infty} (U_{k_i}, v_{k_i}) = (\overline{U}, \mu) \text{ in } C^1(\overline{\Omega}) \times C^1(\overline{\Omega}).$$

and

$$\lim_{i \to \infty} k_i v_{k_i} = \infty \text{ uniformly in } \overline{\Omega}.$$ \hfill (24)

Hence, it follows from (24) and $\rho(x) = 1$ in $\overline{\Omega}$ that

$$\lim_{i \to \infty} u_{k_i} = \lim_{i \to \infty} \frac{U_{k_i}}{1 + k_i \rho(x) v_{k_i}} = 0 \text{ in } C^1(\overline{\Omega}).$$ \hfill (25)

For this part, it remains to prove that $\lim_{i \to \infty} u_{k_i} = \lambda$ in $C^1(\Omega_0)$. By integrating the $U$-equation in (7) with $(U, v) = (U_{k_i}, v_{k_i})$, we get

$$\int_{\Omega_0} u_{k_i} (\lambda - u_{k_i}) dx + \int_{\Omega \setminus \Omega_0} u_{k_i} \left( \lambda - u_{k_i} - \frac{v_{k_i}}{1 + m u_{k_i}} \right) dx = 0.$$ 

Here we use the fact that $b(x) = 1$ in $\overline{\Omega}$ and $b(x) = 0$ in $\Omega_0$. Furthermore, it follows from (25) that

$$\int_{\Omega_0} \pi (\lambda - \pi) dx = 0.$$ \hfill (26)

For $U$-equation in (7) with $(U, v) = (U_{k_i}, v_{k_i})$, we divide it by $U_{k_i}$ and integrate the obtained equation to derive

$$\int_{\Omega} \frac{\lambda - u_{k_i} - \frac{b(x) v_{k_i}}{1 + m u_{k_i}}}{1 + k_i \rho(x) v_{k_i}} dx = - \int_{\Omega} \frac{|\nabla U_{k_i}|^2}{U_{k_i}^2} dx \leq 0.$$ 

After some arrangement we have

$$\int_{\Omega_0} (\lambda - u_{k_i}) dx + \int_{\Omega \setminus \Omega_0} \frac{\lambda - u_{k_i} - \frac{v_{k_i}}{1 + m u_{k_i}}}{1 + k_i v_{k_i}} dx \leq 0.$$
By (24), the above inequality implies that
\[ \int_{\Omega_0} (\lambda - \pi) dx \leq 0. \]  
(27)

It follows from (26) and (27) that
\[ \int_{\Omega_0} (\lambda - \pi)^2 dx = \lambda \int_{\Omega_0} (\lambda - \pi) dx - \int_{\Omega_0} \pi (\lambda - \pi) dx \leq 0. \]

Thus we must have \( \pi \equiv \lambda \) in \( \Omega_0 \). Therefore, we complete the proof of part (i).

(ii) According to the result of Lemma 3.2, we only need to verify \( \{ \max_{\Omega_1} k_i v_{k_i} \}_{i=1}^{\infty} \) is bounded. Suppose that the result is false, that is, \( \{ \max_{\Omega_1} k_i v_{k_i} \}_{i=1}^{\infty} \) is unbounded. Similar to Lemma 3.3, we can still apply Harnack inequality to prove that \( \{ \min_{\Omega_1} k_i v_{k_i} \}_{i=1}^{\infty} \) is unbounded. Then we may assume that
\[ \lim_{i \to \infty} \min_{\Omega_1} k_i v_{k_i} = \infty. \]  
(28)

From Proposition 1 and (28), it follows that
\[ \lim_{i \to \infty} u_{k_i} = \lim_{i \to \infty} \frac{U_{k_i}}{1 + k_i v_{k_i}} = 0 \text{ uniformly in } \overline{\Omega_1}. \]  
(29)

Denote \( \tilde{v}_{k_i} = v_{k_i} / \max_{\Omega_1} v_{k_i} \). Then
\[ \Delta \tilde{v}_{k_i} + \tilde{v}_{k_i} \left( \mu - v_{k_i} + \frac{c(x) u_{k_i}}{1 + m u_{k_i}} \right) = 0 \text{ in } \Omega_1, \quad \partial_n \tilde{v}_{k_i} = 0 \text{ on } \partial \Omega_1, \]  
(30)

where \( \max_{\Omega_1} \tilde{v}_{k_i} = 1 \). By the elliptic regularity theory, there exists some non-negative function \( \tilde{v} \in C^1(\overline{\Omega_1}) \) with \( \max_{\Omega_1} \tilde{v} = 1 \) such that
\[ \lim_{i \to \infty} \tilde{v}_{k_i} = \tilde{v} \text{ in } C^1(\overline{\Omega_1}). \]  
(31)

It follows from Lemma 3.1 and (29)-(31) that \( \tilde{v} \) is a non-negative solution of
\[ \Delta \tilde{v} + \mu \tilde{v} = 0 \text{ in } \Omega_1, \quad \partial_n \tilde{v} = 0 \text{ on } \partial \Omega_1. \]

By Harnack inequality, it is clear that \( \tilde{v} > 0 \) in \( \overline{\Omega_1} \) due to \( \max_{\Omega_1} \tilde{v} = 1 \). This leads to \( \mu = 0 \), a contradiction. Hence \( \{ \max_{\Omega_1} k_i v_{k_i} \}_{i=1}^{\infty} \) is bounded. We complete the proof of part (ii).

3.3. Proof of Theorem 1.4. Denote \( \overline{U} := (1 + \rho(x) \pi) \pi \). Then (4) can be written as
\[ \begin{cases} 
\Delta \overline{U} + g_1(\overline{U}, \overline{w}) = 0, & x \in \Omega, \\
\Delta \overline{w} + g_2(\mu, \overline{U}, \overline{w}) = 0, & x \in \Omega_1, \\
\partial_n \overline{U} = 0, & x \in \partial \Omega, \\
\partial_n \overline{w} = 0, & x \in \partial \Omega_1,
\end{cases} \]  
(32)

where
\[ \begin{align*}
g_1(\overline{U}, \overline{w}) &= \frac{\overline{U}}{1 + \rho(x) \pi} \left( \lambda - \frac{\overline{U}}{1 + \rho(x) \pi} \right), & x \in \Omega, \\
g_2(\mu, \overline{U}, \overline{w}) &= \overline{w} \left( \mu + \frac{c(x) \overline{U}}{1 + \overline{w} + m \overline{U}} \right), & x \in \Omega_1.
\end{align*} \]

Let
\[ \mu_1 \triangleq \lambda_1 ^N \left( - \frac{c(x) \lambda}{1 + m \lambda}, \Omega_1 \right). \]

The following lemma gives the local bifurcation result for (32).
Lemma 3.4. For any fixed \( \lambda > 0 \) and \( \mu \in (\lambda_1^N (-c(x)/m, \Omega_1), 0) \), \((\mu_1, \lambda, 0)\) is a bifurcation point of \((32)\) where positive solutions bifurcate from \(\{ (\mu, \lambda, 0) : \mu \in \mathbb{R} \}\), moreover, positive solutions of \((32)\) near \((\mu_1, \lambda, 0) \in \mathbb{R} \times X_1\) are expressed by

\[
\Gamma_{local} = \{(\mu, \overline{U}, \overline{w}) = (\mu(s), \lambda + s(\overline{\delta} + \overline{U}(s)), s(\psi_* + \overline{w}(s))) : s \in (0, \overline{\delta})\},
\]

where \(\overline{\delta} > 0\) is a small constant, \(\overline{\delta} = (-\Delta + \lambda I)_{\Omega}^{-1}[\rho(x)\lambda^2 \psi_*]\), and \((\mu(s), \overline{U}(s), \overline{w}(s))\) is a smooth function with respect to \(s\) which satisfies \((\lambda(0), \overline{U}(0), \overline{w}(0)) = (\lambda_*, 0, 0)\) and \(\int_{\Omega_1} \overline{\delta}(s) \psi_* dx = 0\).

Proof. The proof is similar to that of Proposition 2, so we omit it. \(\square\)

The following lemma gives further information on the bifurcation curve \(\Gamma_{local}\) obtained in Lemma 3.4.

Lemma 3.5. For any fixed \( \lambda > 0 \) and \( \mu \in (\lambda_1^N (-c(x)/m, \Omega_1), 0) \), there is an unbounded connected set \(\Gamma_{global}\) of positive solutions of \((32)\) in \(\mathbb{R} \times E\) which bifurcates from \(\{ (\mu, \overline{U}, \overline{w}) = (\mu, \lambda, 0) : \mu \in \mathbb{R} \}\) at \((\mu_1, \lambda, 0)\) and remains bounded until \(\mu\) approaches 0, where it blows up. Moreover, \((\mu_1, 0) \subset \text{Proj}_\mu \Gamma_{global} \subset (\overline{\mu}, 0)\) for some \(\overline{\mu} \in (\lambda_1^N (-c(x)/m, \Omega_1), \mu_1]\), \(\overline{U}_\mu\) is bounded in \(C^1(\overline{\Omega})\) and \(\lim_{\mu \to 0} \overline{w}_\mu = \infty\) uniformly in \(\overline{\Omega}\), where \((\mu, \overline{U}_\mu, \overline{w}_\mu) \in \Gamma_{global}\).

Proof. By the global bifurcation theorem, we use the similar argument to Theorem 1.2 to claim that the local bifurcation curve \(\Gamma_{local}\) can be extended into a global curve \(\Gamma_{global}\) which is contained in the set of positive solutions of \((32)\), moreover, \(\Gamma_{global}\) is unbounded in \(\mathbb{R} \times E\).

Let \((\mu, \overline{U}_\mu, \overline{w}_\mu) \in \Gamma_{global}\). Then \(\overline{w}_\mu\) is a positive solution of

\[
-\Delta \overline{w}_\mu - \frac{c(x)\overline{U}_\mu}{1 + \overline{w}_\mu + m\overline{U}_\mu} \overline{w}_\mu = \mu \overline{w}_\mu \text{ in } \Omega_1, \partial_\mu \overline{w}_\mu = 0 \text{ on } \partial \Omega_1. \tag{33}
\]

This implies that

\[
\lambda_1^N (-c(x)/m, \Omega_1) \leq \mu = \lambda_1^N \left( -\frac{c(x)\overline{U}_\mu}{1 + \overline{w}_\mu + m\overline{U}_\mu}, \Omega_1 \right) \leq 0.
\]

Therefore \(\text{Proj}_\mu \Gamma_{global} \subset (\overline{\mu}, 0)\) for some \(\overline{\mu} \in (\lambda_1^N (-c(x)/m, \Omega_1), \mu_1]\).

Let \((\mu, \overline{U}_\mu, \overline{w}_\mu) \in \Gamma_{global}\). We now show that \(\overline{U}_\mu\) is bounded in \(C^1(\overline{\Omega})\). It follows from the first equation of \((32)\) that

\[
\int_{\Omega} \left( \frac{\overline{U}_\mu}{1 + \rho(x)\overline{w}_\mu} \right)^2 dx = \lambda \int_{\Omega} \frac{\overline{U}_\mu}{1 + \rho(x)\overline{w}_\mu} dx \leq \lambda |\Omega|^{1/2} \left\| \frac{\overline{U}_\mu}{1 + \rho(x)\overline{w}_\mu} \right\|_{2, \Omega}.
\]

Thus we immediately obtain

\[
\left\| \frac{\overline{U}_\mu}{1 + \rho(x)\overline{w}_\mu} \right\|_{2, \Omega} \leq \lambda |\Omega|^{1/2}. \tag{34}
\]

Therefore, by Lemma 2.1 with \(p = 2\), there is some positive constant \(C^*\) independent of \(\mu\) such that

\[
\sup_{\Omega} \overline{U}_\mu \leq C^* \inf_{\Omega} \overline{U}_\mu. \tag{35}
\]

Moreover, by \((34)\), we have

\[
|\Omega|^{1/2} \inf_{\Omega} \overline{U}_\mu \leq \left\| \overline{U}_\mu \right\|_{2, \Omega_0} \leq \left\| \frac{\overline{U}_\mu}{1 + \rho(x)\overline{w}_\mu} \right\|_{2, \Omega} \leq \lambda |\Omega|^{1/2}.
\]
This shows that \( \inf_{\Omega} U_{\mu} \leq \lambda |\Omega|/|\Omega_0|^{1/2} \), and thus \( \sup_{\Omega} U_{\mu} \leq C^* \lambda (|\Omega|/|\Omega_0|)^{1/2} \) by (35). This means that \( U_{\mu} \) is bounded in \( \Omega \). Hence, by elliptic regularity theory and the Sobolev embedding theorem, we can obtain the boundedness of \( \|U_{\mu}\|_{C^1(\Omega)} \).

According to above results, we have known that \( \text{Proj}_{\mu} \Gamma_{\text{global}} \) and \( \{\|U_{\mu}\|_{C^1(\Omega)}\} \) are bounded, Hence \( \{\|\bar{\pi}_{\mu}\|_{C^1(\Omega)}\} \) is unbounded. This means that, by (33) and the elliptic regularity theory, \( \{\max_{\Omega} \bar{\pi}_{\mu}\} \) is unbounded. By Lemma 2.1, we can obtain that \( \{\min_{\Omega_1} \bar{\pi}_{\mu}\} \) is also unbounded. Therefore, we may choose a sequence \( \{\mu_i\}_{i=1}^{\infty} \) with \( \lim_{i \to \infty} \mu_i = \mu_{\infty} \) for some \( \mu_{\infty} \in [\bar{\pi}, 0] \) such that \( \lim_{i \to \infty} \min_{\Omega_1} \bar{\pi}_{\mu_i} = \infty \). Denote \( \bar{\omega}_{\mu_i} = \bar{\pi}_{\mu_i} / \max_{\Omega_1} \bar{\pi}_{\mu_i} \). It follows from the second equation of (32) that

\[
\Delta \bar{\omega}_{\mu_i} + \bar{\pi}_{\mu_i} \left( \mu_i + \frac{c(x) U_{\mu_i}}{1 + \bar{\pi}_{\mu_i} + m U_{\mu_i}} \right) = 0 \quad \text{in} \quad \Omega_1, \quad \partial_n \bar{\omega}_{\mu_i} = 0 \quad \text{on} \quad \partial \Omega_1
\]

with \( \max_{\Omega_1} \bar{\omega}_{\mu_i} = 1 \). Hence by elliptic regularity theory, we may assume that \( \lim_{i \to \infty} \bar{\omega}_{\mu_i} = \bar{w} \) in \( C^1(\Omega_1) \), where \( \bar{w} \) is a positive solution of

\[
\Delta \bar{w} + \mu_{\infty} \bar{w} = 0 \quad \text{in} \quad \Omega_1, \quad \partial_n \bar{w} = 0 \quad \text{on} \quad \partial \Omega_1
\]

with \( \max_{\Omega_1} \bar{w} = 1 \). Harnack inequality implies that \( \mu_{\infty} = 0 \), which shows that \( \text{Proj}_{\mu} \Gamma \supset (\mu_1, 0) \). Consequently, we show that

\[
(\mu_1, 0) \subset \text{Proj}_{\mu} \Gamma \subset (\bar{\pi}, 0)
\]

for some \( \bar{\pi} \in (\lambda_1^N (-c(x)/m, \Omega_1), \mu_1] \).

By view of Lemma 2.2, we derive from (32) that

\[
\min_{\Omega_1} \bar{\pi}_{\mu} \geq \lambda, \quad \min_{\Omega_1} \bar{\pi}_{\mu} \geq \frac{c(x_0) U_{\mu}(x_0)}{-\mu} - 1 - m U_{\mu}(x_0),
\]

where \( \bar{\pi}_{\mu}(x_0) = \min_{\Omega_1} \bar{\pi}_{\mu} \). Thus, it follows from the boundedness of \( \{\|U_{\mu}\|_{C^1(\Omega)}\} \) that

\[
\min_{\Omega_1} \bar{\pi}_{\mu} \geq \frac{c(x_0) U_{\mu}(x_0)}{-\mu} - 1 - m U_{\mu}(x_0) \geq \frac{c(x) \lambda}{-\mu} - 1 - m \max_{\Omega_1} U_{\mu} \to 0^- \quad \text{as} \quad \mu \to 0^-.
\]

Consequently, we show that \( \lim_{\mu \to 0^-} \bar{\pi}_{\mu} = \infty \) uniformly in \( \Omega_1 \). \( \square \)

By Lemmas 3.4 and 3.5, we can immediately prove Theorem 1.4.

**Proof of Theorem 1.4.** In order to complete the proof of Theorem 1.4, it suffices to obtain the convergence result of \( \bar{\pi}_{\mu} \) by Lemma 3.5. Note that the convergence result of \( \pi_{\mu} \) can be proved by the similar argument to the first part (i) of Theorem 1.3. We complete the proof of Theorem 1.4.

3.4. **Proof of Theorem 1.5.** In this subsection, we are devoted to the proof of Theorem 1.5. For this, we first show the existence and uniqueness of the single equation (5).

**Lemma 3.6.** Suppose that \( \lambda > \lambda_{\infty}^*(k) \) with \( k > 0 \). Then (5) has a unique positive solution.

**Proof.** Let \( \phi \) satisfy

\[
\begin{cases}
-\Delta \phi + (1/k) \chi_{\Omega_1} \phi - \lambda_{\infty}^*(k) \chi_{\Omega_0} \phi = 0 \quad \text{in} \quad \Omega, \\
\partial_n \phi = 0 \quad \text{on} \quad \partial \Omega, \\
\phi > 0 \quad \text{in} \quad \Omega, \quad \max_{\Omega_1} \phi = 1.
\end{cases}
\]
Define $\phi_\varepsilon = \varepsilon \phi$, where $\varepsilon > 0$ is any constant. By choosing enough small $\varepsilon > 0$, we can obtain
\[
\Delta \phi_\varepsilon + \phi_\varepsilon \left((\lambda - \phi_\varepsilon)\chi_{\Omega_0} - (1/k)\chi_{\Omega_1}\right) = \phi_\varepsilon (\lambda - \lambda_\infty^* (k) - \phi_\varepsilon)\chi_{\Omega_0} \geq 0 \text{ in } \Omega
\]
due to (36) and $\lambda > \lambda_\infty^* (k)$. Thus, $\phi_\varepsilon$ is a sub-solution of (5). On the other hand, by choosing large constant $M$, we see that $M$ is an super-solution of (5). By the standard super-sub solution method, the equation (5) has at least one positive solution. Moreover, (5) has a maximal solution $U^*$ and a minimal solution $U_*$, where $U^*$ and $U_*$ belong to the order interval $\{U \in C^1(\overline{\Omega}) : \phi_\varepsilon \leq U \leq M\}$, and $U_* \leq U^*$ in $\overline{\Omega}$. By multiplying (5) with $U_*$ and integrating over $\Omega$, we get
\[
\int_{\Omega} \left[-\nabla U^* \cdot \nabla U_* + U_* U^* \left((\lambda - U^*)\chi_{\Omega_0} - (1/k)\chi_{\Omega_1}\right)\right] dx = 0. \tag{37}
\]
Similarly, we multiply (5) with $U = U_*$ by $U^*$ and integrate over $\Omega$ to obtain
\[
\int_{\Omega} \left[-\nabla U_* \cdot \nabla U^* + U_* U^* \left((\lambda - U_*)\chi_{\Omega_0} - (1/k)\chi_{\Omega_1}\right)\right] dx = 0. \tag{38}
\]
It follows from (37) and (38) that
\[
\int_{\Omega} U_* U^*(U_* - U^*) dx = 0.
\]
This shows that
\[
U_* = U^* \text{ in } \Omega_0. \tag{39}
\]
On the other hand, we directly integrate (5) with $U = U^*$ or $U = U_*$ to derive
\[
0 = \int_{\Omega} U^* \left[(\lambda - U^*)\chi_{\Omega_0} - (1/k)\chi_{\Omega_1}\right] dx = \int_{\Omega} U^* (\lambda - U^*) dx - \frac{1}{k} \int_{\Pi_1} U^* dx \tag{40}
\]
and
\[
0 = \int_{\Omega} U_* \left[(\lambda - U_*)\chi_{\Omega_0} - (1/k)\chi_{\Omega_1}\right] dx = \int_{\Omega} U_* (\lambda - U_*) dx - \frac{1}{k} \int_{\Pi_1} U_* dx. \tag{41}
\]
Then it follows from (39)-(41) that
\[
\int_{\Pi_1} (U^* - U_*) dx = 0.
\]
Since $U_* \leq U^*$ in $\overline{\Omega}$, we get
\[
U^* = U_* \text{ in } \overline{\Pi_1}. \tag{42}
\]
(39) and (42) implies that $U^* = U_*$ in $\overline{\Pi_1}$, and hence (5) has a unique positive solution. \hfill \square

**Proof of Theorem 1.5.** Let $\lambda > \lambda_\infty^* (k)$ and $k > 0$ be fixed. Suppose that $(u_{\mu_i}, v_{\mu_i})$ is any positive solution of (3) with $\mu = \mu_i$ and denote $U_{\mu_i} := (1 + k \rho (x) \varepsilon_{\mu_i}) u_{\mu_i}$, where $\lim_{i \to \infty} \mu_i = \infty$. From Proposition 1, it follows that
\[
\lim_{i \to \infty} v_{\mu_i} = \infty \text{ uniformly in } \overline{\Pi_1}. \tag{43}
\]
Thus, we have
\[
\lim_{i \to \infty} u_{\mu_i} = \lim_{i \to \infty} \frac{U_{\mu_i}}{1 + k v_{\mu_i}} = 0 \text{ uniformly in } \overline{\Pi_1}. \tag{44}
\]
By Proposition 1 and elliptic regularity theory, we may assume that \( \lim_{i \to \infty} U_{\mu_i} = U_\infty \) in \( C^1(\Omega) \) as \( \lim_{i \to \infty} \mu_i = \infty \), where \( U_\infty \in C^1(\Omega) \) is a non-negative function. Since \( u_{\mu_i} = U_{\mu_i} \) in \( \Omega_0 \), we find that
\[
\lim_{i \to \infty} u_{\mu_i} = U_\infty \quad \text{in} \quad C^1(\Omega_0). \tag{45}
\]
We multiply the first equation of (7) with \( (U, v) = (U_{\mu_i}, v_{\mu_i}) \) by \( \psi \) and integrate over \( \Omega \) to derive
\[
- \int_{\Omega} \nabla U_{\mu_i} \cdot \nabla \psi \, dx + \int_{\Omega_0} U_{\mu_i} (\lambda - U_{\mu_i}) \psi \, dx \\
+ \int_{\Omega_1} \frac{U_{\mu_i}}{1 + kv_{\mu_i}} \left( \lambda - \frac{U_{\mu_i}}{1 + kv_{\mu_i}} - \frac{v_{\mu_i}(1 + kv_{\mu_i})}{1 + kv_{\mu_i} + m U_{\mu_i}} \right) \psi \, dx = 0, \tag{46}
\]
where \( \psi \in H^1(\Omega) \) is a test function. Letting \( i \to \infty \) in above equation, together with (43) and (45), we have
\[
- \int_{\Omega} \nabla U_\infty \cdot \nabla \psi \, dx + \int_{\Omega_0} U_\infty (\lambda - U_\infty) \psi \, dx - \frac{1}{k} \int_{\Omega_1} U_\infty \psi \, dx = 0.
\]
This shows that \( U_\infty \) satisfies (5). By Hopf boundary lemma and maximum principle, we see that either \( U_\infty > 0 \) in \( \overline{\Omega} \) or \( U_\infty \equiv 0 \) in \( \overline{\Omega} \).

Multiplying (36) by \( U_{\mu_i} \), and integrating over \( \Omega \), we get
\[
\int_{\Omega} \nabla \phi \cdot \nabla U_{\mu_i} \, dx + \frac{1}{k} \int_{\Omega_1} \phi U_{\mu_i} \, dx - \lambda_\infty^*(k) \int_{\Omega_0} \phi U_{\mu_i} \, dx = 0. \tag{47}
\]
In (46), we set \( \psi = \phi \) and obtain
\[
- \int_{\Omega} \nabla U_{\mu_i} \cdot \nabla \phi \, dx + \int_{\Omega_0} U_{\mu_i} (\lambda - U_{\mu_i}) \phi \, dx \\
+ \int_{\Omega_1} \frac{U_{\mu_i}}{1 + kv_{\mu_i}} \left( \lambda - \frac{U_{\mu_i}}{1 + kv_{\mu_i}} - \frac{v_{\mu_i}(1 + kv_{\mu_i})}{1 + kv_{\mu_i} + m U_{\mu_i}} \right) \phi \, dx = 0. \tag{48}
\]
It follows from (47) and (48) that
\[
\int_{\Omega_0} (\lambda - \lambda_\infty^*(k) - U_{\mu_i}) U_{\mu_i} \phi \, dx \\
+ \int_{\Omega_1} \left\{ \frac{1}{k} + \frac{1}{1 + kv_{\mu_i}} \left( \lambda - \frac{U_{\mu_i}}{1 + kv_{\mu_i}} - \frac{v_{\mu_i}(1 + kv_{\mu_i})}{1 + kv_{\mu_i} + m U_{\mu_i}} \right) \right\} U_{\mu_i} \phi \, dx = 0.
\]
After some arrangement we have
\[
\int_{\Omega_0} (\lambda - \lambda_\infty^*(k) - U_{\mu_i}) U_{\mu_i} \phi \, dx \\
+ \int_{\Omega_1} \left\{ \frac{1}{k} - \frac{1}{1/k + 1/m U_{\mu_i}/v_{\mu_i}} + \frac{\lambda - U_{\mu_i}/(1 + kv_{\mu_i})}{1 + kv_{\mu_i}} \right\} U_{\mu_i} \phi = 0. \tag{49}
\]
Assume that \( U_\infty \equiv 0 \) in \( \overline{\Omega} \). Then it follows from the assumption \( \lambda > \lambda_\infty^*(k) \) and (43) that the left-hand side of (49) is positive for large \( i \in N \). Clearly, this is impossible due to (49). This implies that \( U_\infty \) is a positive solution of (5). Therefore, the proof of Theorem 1.5 is complete.

**Acknowledgments.** The authors thank the reviewers for some helpful comments and thank the anonymous referee for the helpful constructive comments of this paper.
REFERENCES

[1] M. G. Crandall and P. H. Rabinowitz, Bifurcation from simple eigenvalues, J. Funct. Anal., 8 (1971), 321–340.
[2] R. H. Cui, J. P. Shi and B. Y. Wu, Strong Allee effect in a diffusive predator-prey system with a protection zone, J. Differential Equations, 256 (2014), 108–129.
[3] Y. H. Du and J. P. Shi, A diffusive predator-prey model with a protection zone, J. Differential Equations, 229 (2006), 63–91.
[4] Y. H. Du and J. P. Shi, Allee effect and bistability in a spatially heterogeneous predator-prey model, Trans. Amer. Math. Soc., 359 (2007), 4557–4593.
[5] Y. H. Du and X. Liang, A diffusive competition model with a protection zone, J. Differential Equations, 244 (2008), 61–86.
[6] Y. H. Du, R. Peng and M. X. Wang, Effect of a protection zone in the diffusive Leslie predator-prey model, J. Differential Equations, 246 (2009), 3932–3956.
[7] X. He and S. N. Zheng, Protection zone in a diffusive predator-prey model with Beddington-DeAngelis functional response, J. Math. Biol., 75 (2017), 239–257.
[8] S. B. Li, S. Y. Liu, J. H. Wu and Y. Y. Dong, Positive solutions for Lotka-Volterra competition system with large cross-diffusion in a spatially heterogeneous environment, Nonlinear Anal. Real World Appl., 36 (2017), 1–19.
[9] S. B. Li and J. H. Wu, Effect of cross-diffusion in the diffusion prey-predator model with a protection zone, Discrete Contin. Dynam. Syst., 37 (2017), 411–430.
[10] S. B. Li, J. H. Wu and S. Y. Liu, Effect of cross-diffusion on the stationary problem of a Leslie prey-predator model with a protection zone, Calc. Var. Partial Differential Equations, 56 (2017), 56–82.
[11] S. B. Li and Y. Yamada, Effect of cross-diffusion in the diffusion prey-predator model with a protection zone II, J. Math. Anal. Appl., 461 (2018), 971–992.
[12] S. B. Li and J. H. Wu, Asymptotic behavior and stability of positive solutions to a spatially heterogeneous predator-prey system, J. Differential Equations, 265 (2018), 3754–3791.
[13] S. B. Li, J. H. Wu and Y. Y. Dong, Effects of a degeneracy in a diffusive predator-prey model with Holling II functional response, Nonlinear Anal. Real World Appl., 43 (2018), 78–95.
[14] S. B. Li, J. H. Wu and Y. Y. Dong, Effects of degeneracy and response function in a diffusion predator-prey model, Nonlinearity, 31 (2018), 1461–1483.
[15] G. M. Lieberman, Bounds for the steady-state Sel’kov model for arbitrary p in any number of dimensions, SIAM J. Math. Anal., 36 (2005), 1400–1406.
[16] J. López-Gómez, Spectral Theory and Nonlinear Functional Analysis, Research Notes in Mathematics, vol. 426, CRC Press, Boca Raton, FL, 2001.
[17] K. Oeda, Effect of cross-diffusion on the stationary problem of a prey-predator model with a protection zone, J. Differential Equations, 250 (2011), 3988–4009.
[18] K. Oeda, Coexistence states of a prey-predator model with cross-diffusion and a protection zone, Adv. Math. Sci. Appl., 22 (2012), 501–520.
[19] P. H. Rabinowitz, Some global results for nonlinear eigenvalue problem, J. Funct. Anal., 7 (1971) 487–513.
[20] Y. X. Wang and W. T. Li, Effects of cross-diffusion on the stationary problem of a diffusive competition model with a protection zone, Nonlinear Anal. Real World Appl., 14 (2013), 224–245.
[21] Y. X. Wang and W. T. Li, Uniqueness and global stability of positive stationary solution for a predator-prey system, J. Math. Anal. Appl., 462 (2018), 577–589.
[22] Q. X. Ye and Z. Y. Li, Introduction to Reaction-Diffusion Equations (in Chinese), Beijing: Science Press, 1990.
[23] X. Z. Zeng, W. T. Zeng and L. Y. Liu, Effect of the protection zone on coexistence of the species for a ratio-dependent predator-prey model, J. Math. Anal. Appl., 462 (2018), 1605–1626.

Received June 2018; revised November 2018.

E-mail address: dongyaying@xpu.edu.cn
E-mail address: lishanbing2006@163.com
E-mail address: yanlingl@snnu.edu.cn