Classification of Frobenius, two-step solvable Lie poset algebras

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Abstract

We show that the isomorphism class of a two-step solvable Lie poset subalgebra of a semisimple Lie algebra is determined by its dimension. We further establish that all such algebras are absolutely rigid.

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1 Introduction

Notation: Throughout, we assume that \( k \) is an an algebraically closed field of characteristic zero – which we may assume is \( \mathbb{C} \).

If \( P \) is a finite poset with partial order \( \preceq \), then the associative poset (or incidence) algebra \( A = A(P, k) \) is the span over \( k \) of elements \( e_{ij}, \ i \preceq j \), with multiplication given by setting \( e_{ij}e_{j'k} = e_{ik} \) if \( j = j' \) and 0 otherwise. One may define the commutator bracket \( [a, b] = ab - ba \) on \( A = (A, k) \) to yield the “Lie poset algebra” \( g(P) = g(P, k) \). If \( |P| = N \), in which case we may assume that \( P = \{1, \ldots, N\} \) with partial order compatible with the linear order, then \( A \) and \( g \) may be viewed as subalgebras of the algebra of all upper-triangular matrices in \( \mathfrak{gl}(N) \). See Figure 1, where the left panel illustrates the Hasse diagram of \( P = \{1, 2, 3, 4\} \) with \( 1 \preceq 2 \preceq 3, 4 \). In the right panel, one has the matrix form defining both \( A(P) \) and \( g(P) \) – each “generated” as above by \( P \). The possible non-zero entries from \( k \) are marked by *’s.

![Figure 1: Hasse diagram of \( P = \{1, 2, 3, 4\} \), 1 \preceq 2 \preceq 3, 4 and matrix representation](image)

In [1], Coll and Gerstenhaber focus on the restriction of \( g(P) \) to \( A_{|P|-1} = \mathfrak{sl}(|P|) \) and denote these algebras by \( g_A(P) \). They show that both \( g(P) \) and \( g_A(P) \) are precisely the subalgebras lying between the...
Borel subalgebra of upper-triangular matrices and the Cartan subalgebra of diagonal matrices in \( \mathfrak{gl}(|P|) \) and \( \mathfrak{sl}(|P|) \), respectively. The authors further suggest that this model may be used to define Lie poset algebras more generally; that is, a Lie poset algebra is one that lies between a Cartan and associated Borel subalgebra of a simple Lie algebra.

Following the suggestion of [1], Coll and Mayers [3] extend the notion of Lie poset algebra to the other classical families: \( B_n = \mathfrak{so}(2n + 1) \), \( C_n = \mathfrak{sp}(2n) \), and \( D_n = \mathfrak{so}(2n) \), by providing the definitions of posets of types B, C, and D which encode the standard matrix forms of Lie poset algebras in types B, C, and D, respectively. There are no conditions on the “generating” posets in either \( \mathfrak{gl}(n) \) or \( \mathfrak{sl}(n) \). This is not true for types B, C, and D (see Definition 4). For the exceptional cases, the Lie poset algebra definition still applies – even if a generating poset may not be present.

Here, we go one step further and define Lie poset algebras to be subalgebras of a semisimple Lie algebra lying between a Borel and corresponding Cartan subalgebra. In particular, we are concerned with such Lie poset algebras which are Frobenius and two-step solvable. These two-step solvable algebras can exhibit remarkable complexity (see [9]), but when they are Frobenius (index zero) they are determined up to isomorphism by their dimension (see Theorem 2 – Classification Theorem). Moreover, we find that all such Frobenius, height-one Lie poset algebras are absolutely rigid (see Theorem 3 - Rigidity Theorem). The Classification and Rigidity theorems constitute the main results of this article. The latter result may be regarded as an extension, to the other classical types, of a recent result of [2] which establishes that a type-A Frobenius, Lie poset algebra of derived length less than three is absolutely rigid.

It is important to note that the rigidity theorem of [2] relies heavily on a recent cohomological result of Coll and Gerstenhaber [1], where they establish, in particular, that if \( g \) is a type-A Lie poset algebra then the space of infinitesimal deformations \( H^2(g, g) \), arising from the Chevalley-Eilenberg cochain complex, is a direct sum of three components,

\[
H^2(g, g) = \left( \bigwedge^2 \mathfrak{h}^* \otimes \mathfrak{c} \right) \bigoplus \left( \mathfrak{h}^* \otimes H^1(\Sigma, k) \right) \bigoplus H^2(\Sigma, k).
\]

(1)

Here, \( \mathfrak{h}^* \) is the linear dual of a Cartan subalgebra of \( g \), \( \mathfrak{c} \) is the center of \( g \), and \( \Sigma \) is the nerve of \( P \), now viewed as a category. Accordingly, the cohomology groups on the right side of (1) are simplicial in nature.

Unfortunately, the spectral sequence argument used to establish (1) does not extend to the type-B, C, and D cases as the nerves of the generating posets do not provide an appropriate simplicial complex (see Example 1). Broadening the context – but restricting the focus – we offer an alternative to (1). In particular, we establish, via direct calculation, that the second cohomology group of a Frobenius, two-step solvable Lie poset algebra with coefficients in its adjoint representation is trivial (see Theorem 3).

The structure of the paper is as follows. In Section 2 we set the notation and review some background definitions. In Sections 3 and 4, we establish the Classification and Rigidity theorems, respectively. In the Epilogue we discuss motivations and pose several questions for further study.

2 Preliminaries

We begin by formally stating the definition of Lie poset algebras.

**Definition 1.** Given a semisimple Lie algebra \( g \), let \( b \subset g \) denote a Borel subalgebra and \( \mathfrak{h} \) its associated Cartan subalgebra. A Lie subalgebra \( p \subset g \) satisfying \( \mathfrak{h} \subset p \subset b \) is called a Lie poset algebra.

The next definition is fundamental to our study.

**Definition 2.** The index of a Lie algebra \( g \) is defined as

\[
\text{ind } g = \min_{F \in g^*} \dim(\ker(B_F)),
\]
where $B_F$ is the skew-symmetric Kirillov form defined by $B_F(x, y) = F([x, y])$, for all $x, y \in g$. The Lie algebra $g$ is Frobenius if it has index zero. An index-realizing functional $F$ is one for which the natural map $g \to g^*$ is an isomorphism. In such a case $F$ is called a Frobenius functional.

**Remark 1.** Frobenius Lie algebras are of interest to those studying deformation and quantum group theory stemming from their relation to the classical Yang-Baxter equation (see [7, 8]).

For our work here, we make use of the following, more computational, characterization of the index. Let $g$ be an arbitrary Lie algebra with basis $\{x_1, \ldots, x_n\}$. The index of $g$ can be expressed using the commutator matrix, $([x_i, x_j])_{1 \leq i, j \leq n}$, over the quotient field $R(g)$ of the symmetric algebra $\text{Sym}(g)$ as follows (cf [4]).

**Theorem 1.** $\text{ind } g = n - \text{rank}_{R(g)}([x_i, x_j])_{1 \leq i, j \leq n}$.

**Proof.** Let $M = ([x_i, x_j])_{1 \leq i, j \leq n}$ and $M_f = (f([x_i, x_j]))_{1 \leq i, j \leq n}$, for $f \in g^*$. Take any $f \in g^*$ and $y = \sum_{k=1}^n a_k x_k \in g$. We have that $y \in \ker(B_f)$ if and only if $f([x_i, y]) = 0$, for all $i = 1, \ldots, n$, if and only if $\sum_{k=1}^n a_k f([x_i, x_k]) = 0$, for all $i = 1, \ldots, n$. Thus, $y \in \ker(B_f)$ if and only if the sequence of values $a_1, \ldots, a_n$ define an element in the nullspace of $M_f$. Thus, $\text{dim}(\ker(B_f)) = \text{dim}(\ker(M_f)) = n - \text{rank}(M_f)$. Therefore, showing

$$\text{rank}(M) = \max_{f \in g^*} \text{rank}(M_f)$$

will establish the result.

If $M$ is of rank $r$, then there exists an $r \times r$ submatrix $A$ of $M$ such that $\det(A) \neq 0$. Since $\det(A)$ is a polynomial in $\{x_1, \ldots, x_n\}$ and we can extend any $f \in g^*$ to an algebra homomorphism into $k$, there exists $f_A \in g^*$ such that $\det(f_A(A)) = f_A(\det(A)) \neq 0$; that is, $M_{f_A}$ has an $r \times r$ submatrix with non-zero determinant. Thus,

$$\text{rank}(M) \leq \max_{f \in g^*} \text{rank}(M_f).$$

Conversely, assume

$$\text{rank}(M_{f'}) = r = \max_{f \in g^*} \text{rank}(M_f),$$

for $f' \in g^*$. Further, let $A$ be an $r \times r$ submatrix of $M_{f'}$ with non-zero determinant, and which corresponds to the submatrix $A_M$ of $M$. Then $\det(f'(A_M)) = \det(f'(A_M)) = \det(A) \neq 0$; that is, $\det(A_M) \neq 0$ and

$$\text{rank}(M) \geq \max_{f \in g^*} \text{rank}(M_f).$$

The claim follows. $\square$

Since Lie poset algebras are evidently solvable, we lastly recall the following basic definition, since we require certain associated terminology for the Classification theorem.

**Definition 3.** Let $g$ be a finite-dimensional Lie algebra. The derived series $g = g^0 \supset g^1 \supset g^2 \supset \ldots$ is defined by $g^i = [g^{i-1}, g^{i-1}]$, for $i > 0$. The least $k$ for which $g^k \neq 0$ and $g^{j} = 0$, for $j > k$, is called the derived length of $g$. The Lie algebra $g$ is solvable if its derived series terminates in the zero subalgebra. If a Lie algebra has finite derived length $k$, then it is $(k + 1)$-step solvable.

## 3 Classification

In this section, we show that any two Frobenius, two-step solvable Lie poset algebras of the same dimension are isomorphic.

We require the following lemma to show that a Lie poset algebra inherits a Cartan-Weyl basis from its parent semi-simple Lie algebra. Recall that a Cartan-Weyl basis of a Lie algebra, $g$, is defined to be a basis

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1 We are indebted to A. Ooms who provided the proof of Theorem 1 in a private communication. While the Theorem is widely quoted (and is due to Dixmier), we could not find a detailed proof in the literature.
we find that there are at least graphs generate Frobenius, type-A Lie poset algebras. Since any such connected, acyclic, bipartite graph can be paired with a tree in such a way that non-isomorphic trees get paired with non-isomorphic posets, we find that there are at least $\frac{\binom{n-2}{2}}{n}$ Frobenius, two-step solvable, $n$-dimensional type-A Lie poset algebras corresponding to non-isomorphic posets. Thus, Theorem 2 is not trivial.
4 Rigidity

In this section, we show $\Phi_n$ is absolutely rigid; that is, has no infinitesimal deformations. By Theorem 2, it will follow that all Frobenius, two-step solvable Lie poset algebras cannot be deformed.

To set the context for our rigidity result, we recall some basic facts from the (infinitesimal) deformation theory of Lie algebras. Recall the standard Chevalley-Eilenberg cochain complex $(C^\ast(g, g), C)$ of $g$ with coefficients in the $g$-module $g$; that is, $C^n(g, g)$ consists of forms $F^n : \bigwedge^n g \to g$ satisfying $\delta^2 F^n = 0$, where

$$\delta F^n(g_1, \ldots, g_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} [g_i, F^n(g_1, \ldots, \hat{g}_i, \ldots, g_{n+1})]$$

$$+ \sum_{1 \leq i \leq j \leq n+1} (-1)^{i+j} F^n([x_1, x_j], x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}).$$

In this setting, $Z^n(g, g) = \ker(\delta) \cap C^n(g, g)$, $B^n(g, g) = \text{Im}(\delta) \cap C^n(g, g)$, and $H^n(g, g) = Z^n(g, g)/B^n(g, g)$. These comprise, respectively, the $n$-cocycles, $n$-coboundaries, and $n^{th}$ cohomology group of $g$ with coefficients in $g$. Up to equivalence, the infinitesimal deformations of $g$ may be regarded as elements of $H^2(g, g)$ with the obstructions to their propagation to higher-order deformations lying in $H^3(g, g)$ (see [5] and [6]). If each element of $H^2(g, g)$ is obstructed, then $g$ is called rigid, and if $H^2(g, g) = 0$, then $g$ is said to be absolutely rigid.

**Theorem 3.** $H^2(\Phi_n, \Phi_n) = 0$.

**Proof.** Let $F^2 \in C^2(\Phi_n, \Phi_n)$ be defined by $F^2(x, y) = \sum_{k=1}^n (f_{d_k}^x d_k + f_{e_k}^x e_k)$, for $x, y \in \{d_1, e_1, \ldots, d_n, e_n\}$ such that $x \neq y$. Solving $\delta F^2 = 0$, we find that if $F^2 \in Z^2(\Phi_n, \Phi_n)$ and $i \neq j$, then

- $F^2(e_i, e_j) = f_{e_i, e_j}^{e_i} e_i + f_{e_j, e_i}^{e_i} e_j$;
- $F^2(d_i, d_j) = f_{d_i, d_j}^{d_i} d_i + f_{d_j, d_i}^{d_i} d_i$;
- $F^2(d_i, e_j) = f_{d_i, e_j}^{d_i} e_i + f_{e_j, d_i}^{d_i} e_j$; and
- $F^2(d_i, e_j) = f_{d_i, e_j}^{d_i} e_i + f_{e_j, d_i}^{d_i} d_i = \sum_{k \neq i} (f_{d_k}^{d_i, e_i} e_k + f_{e_k}^{e_i, e_k} d_k)$.

Let $F^1 \in C^1(\Phi_n, \Phi_n)$ be defined by $F^1(x) = \sum_{k=1}^n (f_{d_k}^x d_k + f_{e_k}^x e_k)$, for $x \in \{d_1, e_1, \ldots, d_n, e_n\}$. Setting

- $f_{d_i}^{e_i} = f_{e_i}^{e_i}$,
- $f_{e_i}^{d_i} = f_{e_i}^{d_i}$,
- $f_{e_i}^{e_j} = f_{d_i}^{e_j}$,
- $f_{d_i}^{e_i} = - f_{d_i}^{e_i}$,
- $f_{d_j}^{d_i} = f_{e_j}^{d_i}$, for all $i, j$,
- and all remaining terms equal to 0,

we get $\delta F^1 = F^2$. The result follows. 

\[\square\]
5 Epilogue

The original motivation for this article was to understand if the type-A cohomological result of equation (1) could be succinctly extended to the other classical types – this owing to the recent introduction of the definitions of Lie poset algebras in types B, C, and D (see [3] and Definition 4 below).

It is evident from (1) that type-A Lie poset algebras are simplicial in nature – the chains of the poset which generates a type-A Lie poset algebra conveniently furnishing the simplicial object \( \Sigma \). In the other classical cases, the analogously defined \( \Sigma \) fails to satisfy (1). To see this, we introduce the definitions of the posets of type C, D, and B. These define generating posets for Lie posets of type C, D, and B, respectively.

**Definition 4.** A type-C poset is a poset \( \mathcal{P} = \{-n, \ldots, -1, 1, \ldots, n\} \) such that

1. if \( i \preceq_p j \), then \( i \leq j \); and
2. if \( i \neq -j \), then \( i \preceq_p j \) if and only if \( -j \preceq_p -i \).

A type-D poset is a poset \( \mathcal{P} = \{-n, \ldots, -1, 1, \ldots, n\} \) satisfying 1 and 2 above as well as

3. \( -i \preceq_p i \).

A type-B poset is a poset \( \mathcal{P} = \{-n, \ldots, -1, 0, 1, \ldots, n\} \) satisfying 1 through 3 above.

**Example 1.** The poset \( \mathcal{P} \) on \( \{-3, -2, -1, 1, 2, 3\} \) defined by \(-1 \preceq 2, 3; -2 \preceq 1, 3; \) and \(-3 \preceq 1, 2\) forms a type-C poset. The Hasse diagram of \( \mathcal{P} \) is illustrated in Figure 3 (left). The matrix form defining the corresponding type-C Lie poset algebra, denoted \( g_C(\mathcal{P}) \), is illustrated in Figure 3 (right). Clearly, \( g_C(\mathcal{P}) \) is two-step solvable. Furthermore, using Theorem 1, it is straightforward to show that \( g_C(\mathcal{P}) \) is Frobenius. Thus, by Theorem 3, \( H^2(g_C(\mathcal{P}), g_C(\mathcal{P})) = 0 \). If equation (1) applied to \( g_C(\mathcal{P}) \), then \( H^2(g_C(\mathcal{P}), g_C(\mathcal{P})) \neq 0 \); this follows as the corresponding simplicial object would be the Hasse diagram of \( \mathcal{P} \) which is homotopy equivalent to a circle.

![Figure 3](image_url)
6 Appendix-Spectrum

**Theorem 4.** $\Phi_n$ has a binary spectrum.

*Proof.* Let $e_i^*$, for $i = 1, \ldots, n$, denote the functional on $\Phi_n$ which returns the coefficient of $e_i$. A straightforward exercise in linear algebra shows that $f = \sum_{i=1}^n e_i^*$ is a Frobenius functional on $\Phi_n$ with principal element $\hat{f} = \sum_{i=1}^n d_i$. Calculating $[\hat{f}, d_i]$ and $[\hat{f}, e_i]$, for $i = 1, \ldots, n$, establishes the result. \qed