HAUSDORFF OPERATORS ON MODULATION AND WIENER AMALGAM SPACES

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Abstract. We give the sharp conditions for boundedness of Hausdorff operators on certain modulation and Wiener amalgam spaces.

1. Introduction and Preliminary

Hausdorff operator, originated from some classical summation methods, has a long history in the study of real and complex analysis. We refer the reader to [1] and [12] for a survey of some historic background and recent developments about Hausdorff operator.

For a suitable function \( \Phi \), one of the corresponding Hausdorff operator \( H_\Phi \) can be defined by

\[
H_\Phi f(x) = \int_{\mathbb{R}^n} \Phi(y) f\left(\frac{x}{|y|}\right) dy.
\] (1.1)

Although there is a general definition where \( f(A(y)x) \), with matrix \( A \), stays in place of \( f(x/|y|) \) in (1.1), we only consider the special case in this paper. However we do not exclude that the general case will prove to be of interest as well and we are keep interested in the general case.

There are many known results about the boundedness of Hausdorff operators on various function spaces, such as [10, 11, 13, 14]. Unfortunately, the sharp conditions on boundedness of Hausdorff operator can be characterized in only few cases. One can see [21] for the sharp characterization for the boundedness of Hausdorff operators on \( L^p \), and see [3, 17] for the sharp characterization for the boundedness of Hausdorff operators on Hardy spaces \( H^1 \) and \( h^1 \). We observe that the characterizations of boundedness of Hausdorff operator were also established in some other function spaces (see [1, 6]). However, we find that these spaces have some similar properties as the \( L^p \) spaces. Let us briefly describe this fact in the following.

In order to prove the necessity of boundedness of Hausdorff operator on \( L^p \), we must choose a suitable function \( f \) and estimate \( \|H_\Phi f\|_{L^p} \) from below by some integral regarding \( \Phi \). The space \( L^p \) is fit for this lower estimates, since for a function \( f \), the norm \( \|f\|_{L^p} \) only depends on the absolute value of \( f \) and the \( L^p \) norm has the scaling property \( \|f(s\cdot)\|_{L^p} = s^{-n/p}\|f\|_{L^p} \). We observe that the function spaces, for which the characterizations of boundedness of Hausdorff...
operator are established so far, all have the above two properties as the \( L^p \) spaces so that the proof of the necessity follows the same line as that on \( L^p \). However, in the case of frequency decomposition spaces, such as the modulation spaces or Wiener amalgam spaces, the situation becomes quite different and complicated.

The modulation spaces \( M_{p,q} \) were first introduced by Feichtinger [4] in 1983. As function spaces associated with the uniform decomposition (see [19]), modulation spaces have a close relationship to the topic of time-frequency analysis (see [7]), and they have been regarded as appropriate function spaces for the study of partial differential equations (see [20]). We refer the reader to [5] for some motivations and historical remarks. One can also refer our recent paper [8, 9] for the properties of modulation spaces and Wiener amalgam spaces.

As a frequency decomposition space, the norm of \( f \) in a modulation space cannot be completely determined by the absolute value of the function. On the other hand, the scaling property of modulation spaces is not as simple as that of \( L^p \) (see [15]). Thus, it is interesting to find out the sharp conditions for the boundedness of Hausdorff operator on modulation spaces, since in this case the method used in the \( L^p \) case is not adoptable.

We also consider the boundedness of Hausdorff operator on Wiener amalgam spaces \( W_{p,q}^s \). In general, a Wiener amalgam space can be represented by \( W(B, C) \), where \( B \) and \( C \) are served as the local and global component respectively. In this paper, we consider a special case \( W(\mathcal{F}^{-1}L_q^s, L_p) \), which is closely related to modulation spaces. For simplicity in the notation, we also use \( W_{p,q}^s \) to denote this function space. Before stating the main theorems, we make some preparations as follows.

We need to add some suitable assumptions on \( \Phi \). Firstly, in order to establish the sharp conditions for the boundedness of Hausdorff operator, we assume \( \Phi \geq 0 \). In the proof of the necessity part, we must make some (pointwise) estimates from below. That is why the assumption \( \Phi \geq 0 \) is necessary in most of the known characterizations for the boundedness of Hausdorff operator on function spaces (see [3, 17, 21]).

Secondly, we make another assumption for \( \Phi \) as following:

\[
\int_{B(0,1)} |y|^n \Phi(y) dy < \infty, \quad \text{and} \quad \int_{B(0,1)^c} \Phi(y) dy < \infty. \tag{1.2}
\]

We would like to give following remarks not only for explaining the reasonability of the assumption (1.2), but also to give some important properties of Hausdorff operator under the assumption (1.2).

**Remark 1.1 (Assumption (1.2) is weakest).** In fact, (1.2) is the weakest assumption to ensure that the Schwartz function can be mapped into tempered distribution by Hausdorff operator \( H_{\Phi} \).
On one hand, if $H_{\Phi} f \in \mathcal{S}'$, it must be locally integrable, and since $\Phi \geq 0$, we have

$$\int_{B(0,1)} |H_{\Phi} f(x)| \, dx = \int_{B(0,1)} \int_{\mathbb{R}^n} \Phi(y) f(x/|y|) \, dy \, dx$$

$$= \int_{\mathbb{R}^n} \Phi(y) \int_{B(0,1)} f(x/|y|) \, dx \, dy = \int_{\mathbb{R}^n} \Phi(y) |y|^n \int_{B(0,1)} f(x) \, dx \, dy$$

$$= \int_{B(0,1)} \Phi(y) |y|^n \int_{B(0,1/|y|)} f(x) \, dx \, dy + \int_{B^c(0,1)} \Phi(y) |y|^n \int_{B(0,1/|y|)} f(x) \, dx \, dy.$$

On the other hand, for any nonnegative Schwartz function $f$ satisfying $f = 1$ on $B(0,1)$, we have

$$\int_{B(0,1)} |H_{\Phi} f(x)| \, dx \geq \int_{B(0,1)} \Phi(y) |y|^n \int_{B(0,1)} f(x) \, dx \, dy + \int_{B^c(0,1)} \Phi(y) |y|^n \int_{B(0,1/|y|)} f(x) \, dx \, dy$$

$$\sim \int_{B(0,1)} \Phi(y) |y|^n \, dy + \int_{B^c(0,1)} \Phi(y) \, dy.$$

This implies that

$$\int_{B(0,1)} \Phi(y) |y|^n \, dy + \int_{B^c(0,1)} \Phi(y) \, dy < \infty.$$

**Remark 1.2** ($H_{\Phi} f$ is well defined as a tempered distribution). If $\Phi$ satisfies (1.2), $H_{\Phi} f$ makes sense for all $f \in \mathcal{S}(\mathbb{R}^n)$ for the reason that for $x \neq 0$,

$$|H_{\Phi} f(x)| \leq \left( \int_{B(0,1)} + \int_{B^c(0,1)} \right) \Phi(y) f(x/|y|) \, dy$$

$$\leq |x|^{-n} \int_{B(0,1)} |y|^n \Phi(y) (|x/|y||^n f(x/|y|)) + \|f\|_{L^\infty} \int_{B^c(0,1)} \Phi(y) \, dy$$

$$\lesssim C_f (1 + |x|^{-n}) \left( \int_{B(0,1)} |y|^n \Phi(y) \, dy + \int_{B^c(0,1)} \Phi(y) \, dy \right) < \infty,$$

and

$$\int_{B(0,1)} |H_{\Phi} f(x)| \, dx \leq \int_{B(0,1)} \int_{\mathbb{R}^n} \Phi(y) f(x/|y|) \, dy \, dx$$

$$= \int_{B(0,1)} \int_{B(0,1)} \Phi(y) f(x/|y|) \, dy \, dx + \int_{B(0,1)} \int_{B^c(0,1)} \Phi(y) f(x/|y|) \, dy \, dx$$

$$\leq \int_{B(0,1)} \Phi(y) \int_{\mathbb{R}^n} |f(x/|y|)| \, dx \, dy + |B(0,1)| \cdot \|f\|_{L^\infty} \cdot \int_{B^c(0,1)} \Phi(y) \, dy$$

$$\leq \int_{B(0,1)} |y|^n \Phi(y) \, dy \|f\|_{L^1} + |B(0,1)| \cdot \|f\|_{L^\infty} \cdot \int_{B^c(0,1)} \Phi(y) \, dy < \infty.$$

Thus, for $f \in \mathcal{S}(\mathbb{R}^n)$, $H_{\Phi} f$ is a locally integrable function, which has polynomial growth at infinity. It implies that $H_{\Phi} f$ is a tempered distribution for $f \in \mathcal{S}(\mathbb{R}^n)$. Write

$$\langle H_{\Phi} f, g \rangle = \int_{\mathbb{R}^n} H_{\Phi} f(x) g(x) \, dx,$$
where \( \langle u, f \rangle \) means the action of a tempered distribution \( u \) on a Schwartz function \( f \).

**Remark 1.3 (\( H_\Phi : \mathcal{S} \to \mathcal{S}' \) is continuous).** For \( f, g \in \mathcal{S}(\mathbb{R}^n) \), we have that

\[
\int_{\mathbb{R}^n} |f(x/|y|)g(x)| dx \leq \|f\|_{L^\infty} \|g\|_{L^1}
\]

and

\[
\int_{\mathbb{R}^n} |f(x/|y|)g(x)| dx \leq \|g\|_{L^\infty} \|f(\cdot/|y|)\|_{L^1} \leq |y|^n \|g\|_{L^\infty} \|f\|_{L^1}.
\]

It follows that

\[
\int_{\mathbb{R}^n} |\Phi(y)| \int_{\mathbb{R}^n} |f(x/|y|)g(x)| dx dy \\
\lesssim (\|f\|_{L^1} + \|f\|_{L^\infty})(\|g\|_{L^1} + \|g\|_{L^\infty}) \int_{\mathbb{R}^n} |\Phi(y)| \min\{1, |y|^n\} dy.
\]

Thus,

\[
|\langle H_\Phi f, g \rangle| = |\int_{\mathbb{R}^n} H_\Phi f(x)g(x)dx| \\
\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi(y)|f(x/|y|)| dy g(x) dx \\
\leq \int_{\mathbb{R}^n} \Phi(y) \int_{\mathbb{R}^n} |f(x/|y|)| \cdot |g(x)| dx dy \\
\lesssim (\|f\|_{L^1} + \|f\|_{L^\infty})(\|g\|_{L^1} + \|g\|_{L^\infty}) \int_{\mathbb{R}^n} |\Phi(y)| \min\{1, |y|^n\} dy.
\]

Using the definition of Schwartz function space, we have \( |\langle H_\Phi f, g_l \rangle| \to 0 \), for \( f, g_l \in \mathcal{S}(\mathbb{R}^n) \) satisfying that \( g_l \to 0 \) as \( l \to \infty \) in the topology of \( \mathcal{S} \).

**Remark 1.4 (Fourier transform of \( H_\Phi f \)).** Define

\[
\tilde{H}_\Phi f(x) = \int_{\mathbb{R}^n} \Phi(y)|y|^n f(|y|x) dy.
\]

By a similar method used before, we can verify that \( \tilde{H}_\Phi f \) is a tempered distribution and that the map \( \tilde{H}_\Phi : \mathcal{S} \to \mathcal{S}' \) is continuous.

Moreover, we have

\[
\tilde{H}_\Phi f = \tilde{H}_\Phi \tilde{f} \quad \text{in the distribution sense.}
\]
Indeed, for \( f, g \in \mathcal{S}(\mathbb{R}^n) \), we have
\[
\langle \widehat{H_\Phi f}, g \rangle = \langle H_\Phi f, \overline{g} \rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi(y) f(x/|y|) dy \overline{g}(x) dx
\]
\[
= \int_{\mathbb{R}^n} \Phi(y) \int_{\mathbb{R}^n} f(x/|y|) \overline{g}(x) dx dy
\]
\[
= \int_{\mathbb{R}^n} \Phi(y) \int_{\mathbb{R}^n} f(-|y|)(x) g(x) dx dy
\]
\[
= \int_{\mathbb{R}^n} \Phi(y) \int_{\mathbb{R}^n} |y|^n \hat{f}(|y|x) g(x) dx dy
\]
\[
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |y|^n \Phi(y) \hat{f}(|y|x) dy g(x) dx = \langle \widehat{H_\Phi f}, g \rangle.
\]

**Remark 1.5 (Adjoint operator of \( H_\Phi f \)).** We define the complex inner product
\[
\langle f | g \rangle = \int_{\mathbb{R}^n} f(x) \overline{g}(x) dx.
\]
The adjoint operator of \( H_\Phi f \) is defined by
\[
\langle H_\Phi f | g \rangle = \langle f | H_\Phi^* g \rangle
\]
for \( f, g \in \mathcal{S}(\mathbb{R}^n) \). By a direct calculation, we have
\[
\langle H_\Phi f | g \rangle = \int_{\mathbb{R}^n} H_\Phi f(x) \overline{g}(x) dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi(y) f(x/|y|) dy \overline{g}(x) dx
\]
\[
= \int_{\mathbb{R}^n} \Phi(y) \int_{\mathbb{R}^n} f(x/|y|) \overline{g}(x) dx dy = \int_{\mathbb{R}^n} |y|^n \Phi(y) \int_{\mathbb{R}^n} f(x) \overline{g}(|y|x) dx dy
\]
\[
= \int_{\mathbb{R}^n} f(x) \int_{\mathbb{R}^n} \Phi(y) |y|^n \overline{g}(|y|x) dy dx = \langle f | \widehat{H_\Phi} g \rangle.
\]
It follows that
\[
H_\Phi^* g = \widehat{H_\Phi} g \quad \text{in the distribution sense.}
\]

We turn to give definitions of modulation and Wiener amalgam spaces.

Let \( \mathcal{S} := \mathcal{S}(\mathbb{R}^n) \) be the Schwartz space and \( \mathcal{S}' := \mathcal{S}'(\mathbb{R}^n) \) be the space of tempered distributions. We define the Fourier transform \( \mathcal{F} f \) and the inverse Fourier transform \( \mathcal{F}^{-1} f \) of \( f \in \mathcal{S}(\mathbb{R}^n) \) by
\[
\mathcal{F} f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx, \quad \mathcal{F}^{-1} f(x) = f^\vee(x) = \int_{\mathbb{R}^n} f(\xi) e^{2\pi i x \cdot \xi} d\xi.
\]

The translation operator is defined as \( T_{x_0} f(x) = f(x - x_0) \) and the modulation operator is defined as \( M_\xi f(x) = e^{2\pi i \xi \cdot x} f(x) \), for \( x, x_0, \xi \in \mathbb{R}^n \). Fixed a nonzero function \( \phi \in \mathcal{S} \), the short-time Fourier transform of \( f \in \mathcal{S}' \) with respect to the window \( \phi \) is given by
\[
V_\phi f(x, \xi) = \langle f, M_\xi T_x \phi \rangle,
\]
and that can be written as
\[
V_\phi f(x, \xi) = \int_{\mathbb{R}^n} f(y) \overline{\phi(y - x)} e^{-2\pi i y \cdot \xi} dy.
\]
if \( f \in \mathcal{S} \). We give the (continuous) definition of modulation space \( \mathcal{M}_{p,q}^s \) as follows.

**Definition 1.6.** Let \( s \in \mathbb{R} \), \( 0 < p, q \leq \infty \). The (weighted) modulation space \( \mathcal{M}_{p,q}^s \) consists of all \( f \in \mathcal{S}'(\mathbb{R}^n) \) such that the (weighted) modulation space norm

\[
\|f\|_{\mathcal{M}_{p,q}^s} = \left\| \|V_{\phi} f(x, \xi)\|_{L_{x,p}} \right\|_{L_{\xi,q}^s}^{s/q}
\]

is finite, with the usual modifications when \( p = \infty \) or \( q = \infty \). This definition is independent of the choice of the window \( \phi \in \mathcal{S} \).

Applying the frequency-uniform localization techniques, one can give an alternative definition of modulation spaces (see [19] for details).

We denote by \( Q_k \) the unit cube with the center at \( k \). Then the family \( \{Q_k\}_{k \in \mathbb{Z}^n} \) constitutes a decomposition of \( \mathbb{R}^n \). Let \( \eta \in \mathcal{S}(\mathbb{R}^n) \), \( \eta : \mathbb{R}^n \to [0, 1] \) be a smooth function satisfying that \( \eta(\xi) = 1 \) for \( |\xi| \leq 1/2 \) and \( \eta(\xi) = 0 \) for \( |\xi| \geq 3/4 \). Let

\[
\eta_k(\xi) = \eta(\xi - k), k \in \mathbb{Z}^n
\]

be a translation of \( \eta \). Since \( \eta_k(\xi) = 1 \) in \( Q_k \), we have that \( \sum_{k \in \mathbb{Z}^n} \eta_k(\xi) \geq 1 \) for all \( \xi \in \mathbb{R}^n \). Denote

\[
\sigma_k(\xi) = \eta_k(\xi) \left( \sum_{l \in \mathbb{Z}^n} \eta_l(\xi) \right)^{-1}, k \in \mathbb{Z}^n.
\]

It is easy to know that \( \{\sigma_k\}_{k \in \mathbb{Z}^n} \) constitutes a smooth partition of the unity, and \( \sigma_k(\xi) = \sigma(\xi - k) \). The frequency-uniform decomposition operators can be defined by

\[
\Box_k := \mathcal{F}^{-1}\sigma_k \mathcal{F}
\]

for \( k \in \mathbb{Z}^n \). Now, we give the (discrete) definition of modulation space \( M_{p,q}^s \).

**Definition 1.7.** Let \( s \in \mathbb{R} \), \( 0 < p, q \leq \infty \). The modulation space \( \mathcal{M}_{p,q}^s \) consists of all \( f \in \mathcal{S}'(\mathbb{R}^n) \) such that the (quasi-)norm

\[
\|f\|_{\mathcal{M}_{p,q}^s} := \left( \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{sq} \|\Box_k f\|_p^q \right)^{1/q}
\]

is finite. We write \( M_{p,q} := M_{p,q}^0 \) for short. We also recall that this definition is independent of the choice of \( \{\sigma_k\}_{k \in \mathbb{Z}^n} \) and the definitions of \( \mathcal{M}_{p,q}^s \) and \( M_{p,q}^s \) are equivalent [20].

**Definition 1.8.** Let \( 0 < p, q \leq \infty \), \( s \in \mathbb{R} \). Given a window function \( \phi \in \mathcal{S}\setminus\{0\} \), the Wiener amalgam space \( \mathcal{W}_{p,q}^s \) consists of all \( f \in \mathcal{S}'(\mathbb{R}^n) \) such that the norm

\[
\|f\|_{\mathcal{W}_{p,q}^s} = \left\| \|V_{\phi} f(x, \xi)\|_{L_{x,p}^s} \right\|_{L_{\xi,q}^s}^{s/q}
\]

is finite. We write \( \mathcal{W}_{p,q} := \mathcal{W}_{p,q}^0 \) for short. We also recall that this definition is independent of the choice of \( \sigma_k \) and the definitions of \( \mathcal{M}_{p,q}^s \) and \( M_{p,q}^s \) are equivalent [20].
is finite, with the usual modifications when \( p = \infty \) or \( q = \infty \). We write \( W^0_{p,q} \) for short.

Now, we state our main results as follows.

**Theorem 1.9.** Let \( 1 \leq p, q \leq \infty \), \((1/p - 1/2)(1/q - 1/p) \geq 0\), \( \Phi \) be a nonnegative function satisfying the basic assumption (1.2). Then \( H_\Phi \) is bounded on \( M_{p,q} \) if and only if

\[
\int_{\mathbb{R}^n} \left( |y|^{n/p} + |y|^{n/q'} \right) \Phi(y) dy < \infty.
\]

**Theorem 1.10.** Let \( 1 \leq p, q \leq \infty \), \((1/q - 1/2)(1/q - 1/p) \leq 0\), \( \Phi \) be a nonnegative function satisfying the basic assumption (1.2). Then \( H_\Phi \) is bounded on \( W_{p,q} \) if and only if

\[
\int_{\mathbb{R}^n} \left( |y|^{n/p} + |y|^{n/q'} \right) \Phi(y) dy < \infty.
\]

Our paper is organized as follows. In Section 2, we collect some basic properties of modulation and Wiener amalgam spaces and give the proof of Theorem 1.9 and 1.10.

Throughout this paper, we will adopt the following notations. We use \( X \lesssim Y \) to denote the statement that \( X \leq CY \), with a positive constant \( C \) that may depend on \( n, p \), but it might be different from line to line. The notation \( X \sim Y \) means the statement \( X \lesssim Y \lesssim X \). We use \( X \lesssim_{\lambda} Y \) to denote \( X \leq C_\lambda Y \), meaning that the implied constant \( C_\lambda \) depends on the parameter \( \lambda \). For a multi-index \( k = (k_1, k_2, ..., k_n) \in \mathbb{Z}^n \), we denote \( |k|_{\infty} := \max_{i=1,2,...,n} |k_i| \) and \( \langle k \rangle := (1 + |k|^2)^{1/2} \).

2. **Proof of main theorem**

First, we list some basic properties about modulation spaces as follows.

**Lemma 2.1** (Symmetry of time and frequency). \( \| \mathcal{F}^{-1} f \|_{M_{p,q}} \sim \| \mathcal{F} f \|_{M_{q,p}} \).

**Proof.** By the fact that \( |V_\phi f(x, \xi)| = |V_{\hat{\phi}} \hat{f}(\xi, -x)| \), the conclusion follows by the definition of modulation and Wiener amalgam space. \(\square\)

**Lemma 2.2** ([18] Dilation property of modulation space). Let \( 1 \leq p, q \leq \infty \), \((1/p - 1/2)(1/q - 1/p) \geq 0\). Set \( f_\lambda(x) = f(\lambda x) \). Then

\[
\| f_\lambda \|_{M_{p,q}} \lesssim \max \{ \lambda^{-n/p}, \lambda^{-n/q'} \} \| f \|_{M_{p,q}}.
\]

**Lemma 2.3** ([16] Embedding relations between modulation and Lebesgue spaces). The following embedding relations are right:

1. \( M_{p,q} \hookrightarrow L^p \) for \( 1/q \geq 1/p \geq 1/2 \);
2. \( L^p \hookrightarrow M_{p,q} \) for \( 1/q \leq 1/p \leq 1/2 \).

**Lemma 2.4** ([2] Embedding relations between Wiener amalgam and Lebesgue spaces). The following embedding relations are right:
Lemma 2.5. Let \( 1 \leq p, q \leq \infty \). We have

\[
(1) \quad \left| \int \frac{f(x)g(x)dx}{R^n} \right| \leq ||f||_{M_{p',q'}} ||g||_{M_{p,q}}
\]

\[
(2) \quad \left| \int \frac{f(x)g(x)dx}{R^n} \right| \leq ||f||_{W_{p',q'}} ||g||_{W_{p,q}}.
\]

Proof. By Lemma 2.1, we only give the proof of the first inequality. Denote \( \eta_k = \sum_{l \in \mathbb{Z}^n: \eta_k \eta_l \neq 0} \eta_l \) and \( \Box_k = \mathcal{F}^{-1} \eta_k \mathcal{F} \). By the definition of modulation spaces and Plancherel’s equality we get that

\[
\left| \int \frac{f(x)g(x)dx}{R^n} \right| = \left| \int \frac{\hat{f} \hat{g}d\xi}{R^n} \right| = \left| \int \sum_{k \in \mathbb{Z}^n} \sigma_k \hat{f} \cdot \sum_{l \in \mathbb{Z}^n} \sigma_l \hat{g}dx \right|
\]

\[
= \left| \int \sum_{k \in \mathbb{Z}^n} \sigma_k \hat{f} \cdot \sigma_k^* \hat{g}dx \right| = \left| \int \sum_{k \in \mathbb{Z}^n} \Box_k f \cdot \Box_k^* gdx \right|
\]

\[
\leq \sum_{k \in \mathbb{Z}^n} \int \left| \Box_k f \cdot \Box_k^* g \right| dx \leq \sum_{k \in \mathbb{Z}^n} ||\Box_k f||_{L^{p'}} ||\Box_k^* g||_{L^p}
\]

\[
\leq \left( \sum_{k \in \mathbb{Z}^n} ||\Box_k f||_{L^{p'}}^q \right)^{1/q'} \left( \sum_{k \in \mathbb{Z}^n} ||\Box_k^* g||_{L^p}^q \right)^{1/q} \leq ||f||_{M_{p',q'}} ||g||_{M_{p,q}},
\]

where we use Hölder’s inequality in the last two and third inequality and the fact that the definition of modulation space is independent of the decomposition function. \( \square \)

In order to make the proof more clear, we give the following technical Proposition.

Proposition 2.6 (For technique). Let \( 1/2 \leq 1/p \leq 1/q \leq 1 \), and \( \Phi \) be a non-negative function satisfying the basic assumption \((1.2)\). Then

(1) if \( H_\Phi : M_{p,q} \rightarrow L^p \) is bounded, we have

\[
\int_{R^n} |y|^{n/p} \Phi(y)dy < \infty;
\]

(2) if \( H_\Phi^* : W_{q,p} \rightarrow L^q \) is bounded, we have

\[
\int_{R^n} |y|^{n/q'} \Phi(y)dy < \infty;
\]

(3) if \( H_\Phi^* : M_{p,q} \rightarrow L^p \) is bounded, we have

\[
\int_{R^n} |y|^{n/p'} \Phi(y)dy < \infty;
\]
(4) if $H_{\Phi} : W_{q,p} \to L^q$ is bounded, we have

$$\int_{\mathbb{R}^n} |y|^{n/q} \Phi(y) dy < \infty.$$ 

**Proof.** We only give the proof of statements (1) and (2), since the other cases can be handled similarly.

Suppose $H_{\Phi} : M_{p,q} \to L^p$ is bounded. Let $\psi : \mathbb{R}^n \to [0, 1]$ be a smooth bump function supported in the ball $\{ \xi : |\xi| < \frac{3}{2} \}$ and be equal to 1 on the ball $\{ \xi : |\xi| < \frac{3}{4} \}$. Let $\rho(\xi) = \psi(\xi) - \psi(2\xi)$. Then $\rho$ is a positive smooth function supported in the annulus $\{ \xi : \frac{3}{4} < |\xi| < \frac{3}{2} \}$, satisfying $\rho(\xi) = 1$ on a smaller annulus $\{ \xi : \frac{3}{4} \leq |\xi| \leq \frac{3}{2} \}$. Denote $\rho_j(\xi) := \rho(\xi/2^j)$. We have $\text{supp} \rho_j \subset \{ \xi : \frac{3}{4} \cdot 2^j \leq |\xi| \leq \frac{3}{2} \}$. Thus, we have $\text{supp} \sum_{j=1}^{N} \rho_j(\xi) \subset \{ \xi : \frac{3}{4} \leq |\xi| \leq \frac{3}{2} \cdot 2^N \}$ and $\sum_{j=1}^{N} \rho_j(\xi) = 1$ on $\{ \xi : \frac{3}{4} \leq |\xi| \leq \frac{3}{2} \cdot 2^N \}$.

Take $\varphi$ to be a nonnegative smooth function satisfying that $\text{supp} \hat{\varphi} \subset B(0, 1/2)$, $\varphi(0) = 1$. Choose $f_N(x) = \left( \sum_{j=0}^{N+1} \rho_j(x) \cdot |x|^{-n/p} \right) \ast \varphi$. So we have

$$\text{supp} \hat{f}_N \subset B(0, 1/2) \text{ and } f_N(x) \gtrsim \sum_{j=1}^{N} \rho_j(x) \cdot |x|^{-n/p}. \quad (2.1)$$

In above, the previous inclusion relation follows from the support condition of $\hat{\varphi}$. We interpret the latter inequality. We only need to prove it when the right hand is nonzero, i.e. $x \in \{ \frac{4}{3} \cdot 2^N \leq |x| \leq \frac{3}{2} \cdot 2^N \}$. For the nonnegative function $\varphi$ satisfying $\varphi(0) = 1$, there exists a positive constant $\delta < \min\{4/3, 1/12\}$, such that $\varphi(x) > 1/2$ when $|x| < \delta$. By the triangle inequality and the properties of $\varphi$ we have that

$$f_N(x) = \left( \sum_{j=0}^{N+1} \rho_j(x) \cdot |x|^{-n/p} \right) \ast \varphi = \int_{\mathbb{R}^n} \left( \sum_{j=0}^{N+1} \rho_j(x - y) \cdot |x - y|^{-n/p} \right) \varphi(y) dy$$

$$\gtrsim \int_{\frac{4}{3} \cdot 2^N \leq |x - y| \leq \frac{3}{2} \cdot 2^N} \sum_{j=0}^{N+1} \rho_j(x - y) \cdot |x - y|^{-n/p} dy \gtrsim \sum_{j=1}^{N} \rho_j(x) \cdot |x|^{-n/p},$$

so we prove (2.1).
\[
\|H_{\Phi} f_N\|_{L^p} = \left\| \int_{\mathbb{R}^n} \Phi(y) f_N(x/|y|) dy \right\|_{L^p}
\]
\[
\geq \left\| \int_{B(0, \frac{2}{3}2^M) \setminus B(0, \frac{2}{3}2^{-M})} \Phi(y) |y|^{\frac{n}{p}} \cdot \sum_{j=1}^{N} \rho_j(x/|y|) \cdot |x|^{-\frac{n}{p}} dy \right\|_{L^p}
\]
\[
\geq \left\| \int_{B(0, \frac{2}{3}2^M) \setminus B(0, \frac{2}{3}2^{-M})} \Phi(y) |y|^{\frac{n}{p}} \cdot \sum_{j=1}^{N} \rho_j(x/|y|) \cdot |x|^{-\frac{n}{p}} dy \right\|_{L^p}
\]
\[
= \int_{B(0, \frac{2}{3}2^M) \setminus B(0, \frac{2}{3}2^{-M})} \Phi(y) |y|^{\frac{n}{p}} dy \cdot \left( \frac{\log 2^{N-2M}}{\log 2^N} \right)^{\frac{1}{p}},
\]
where we use the fact that \( \sum_{j=1}^{N} \rho_j(x/|y|) = 1 \) for \( y \in B(0, \frac{2}{3}2^M) \setminus B(0, \frac{2}{3}2^{-M}) \) and \( x \in B(0, 2^M) \setminus B(0, 2^{N-M}) \). On the other hand, observing that \( \text{supp} \widetilde{f}_N \subset B(0, 1/2) \), we have that
\[
\|f_N\|_{M_{p,q}} = \left( \sum_{\sigma_k \widetilde{f}_N \neq 0} \left( \mathcal{F}^{-1}(\sigma_k \widetilde{f}_N) \right)^\frac{q}{p} \right)^{\frac{1}{q}} \geq \left( \sum_{\sigma_k \widetilde{f}_N \neq 0} \|f_N\|_{L^p} \right)^{\frac{1}{q}}
\]
\[
\leq \|f_N\|_{L^p} \leq \left\| \sum_{j=0}^{N+1} \rho_j(x) \cdot |x|^{-\frac{n}{p}} \right\|_{L^p} \sim (\ln 2^N)^{1/p}.
\]
Using the boundedness of \( H_{\Phi} \) and the above estimates for \( H_{\Phi} f_N \) and \( f_N \), we have that
\[
\|H_{\Phi}\|_{M_{p,q} \rightarrow L^p} \geq \frac{\|H_{\Phi} f_N\|_{L^p}}{\|f_N\|_{M_{p,q}}} \geq \int_{B(0, \frac{2}{3}2^M) \setminus B(0, \frac{2}{3}2^{-M})} \Phi(y) |y|^{\frac{n}{p}} dy \left( \frac{\log 2^{N-2M}}{\log 2^N} \right)^{\frac{1}{p}}.
\]
Letting \( N \to \infty \), we have
\[
\int_{B(0, \frac{2}{3}2^M) \setminus B(0, \frac{2}{3}2^{-M})} \Phi(y) |y|^{\frac{n}{p}} dy \leq \|H_{\Phi}\|_{M_{p,q} \rightarrow L^p}.
\]
By the arbitrariness of \( M \), we let \( M \to \infty \) and obtain that \( \int_{\mathbb{R}^n} \Phi(y) |y|^{\frac{n}{p}} dy \leq \|H_{\Phi}\|_{M_{p,q} \rightarrow L^p} \).

Now we turn to give the proof for the second conclusion. Suppose \( H_{\Phi}^* : W^{q,p} \to L^q \) is bounded. As in the proof of conclusion (1), we take \( g_N(x) = \sum_{j=1}^{N} \rho_j(x) |x|^{-n/q} \).
A direction calculation yields that

\[ \|H_\Phi^* g_N\|_{L^q} = \left\| \int_{\mathbb{R}^n} \Phi(y) |y|^n g_N(|y|x) dy \right\|_{L^q} \]

\[ = \left\| \int_{\mathbb{R}^n} \Phi(y) |y|^{n/q'} \sum_{j=1}^{N} \rho_j(|y|x) \cdot |x|^{-n/q} dy \right\|_{L^p} \]

\[ \geq \left\| \int_{B(0,4/3-M) \setminus B(0,3/2-M)} \Phi(y) |y|^{n/q'} \sum_{j=1}^{N} \rho_j(|y|x) \cdot |x|^{-n/q} dy \right\|_{L^q} \]

\[ \geq \int_{B(0,4/3-M) \setminus B(0,3/2-M)} \Phi(y) |y|^{n/q'} dy \cdot \| |x|^{-n/q} \chi_{\{2M < |x| < 2N-M\}}(x) \|_{L^q} \]

\[ \succeq \int_{B(0,4/3-M) \setminus B(0,3/2-M)} \Phi(y) |y|^{n/q'} dy \cdot (\lg 2^{N-2M})^{1/q}. \]

On the other hand,

\[ \| \mathcal{F}^{-1}(\sigma_k g_N) \|_{L^p} = \left\| \mathcal{F}^{-1}(\sigma_k \sum_{j=1}^{N} \rho_j(x) \cdot |x|^{-n/q}) \right\|_{L^p} \lesssim \langle k \rangle^{-n/q} \| \mathcal{F}^{-1}(\sigma_k \sum_{j=1}^{N} \rho_j(x)) \|_{L^p} \]

\[ \lesssim \langle k \rangle^{-n/q}. \]

Using Lemma 2.1, we obtain that

\[ \|g_N\|_{W_{q,p}} = \left\| \mathcal{F}^{-1} g_N \right\|_{M_{p,q}} = \left( \sum_{k \in \mathbb{Z}^n} \| \mathcal{F}^{-1}(\sigma_k g_N) \|_{L^p}^q \right)^{1/q} \]

\[ \lesssim \left( \sum_{|k| < 2^{n+1}} \langle k \rangle^{-n} \right)^{1/q} \sim (\lg 2^N)^{1/q}. \]

We deduce that

\[ \|H_\Phi^*\|_{W_{q,p} \to L^q} \geq \frac{\|H_\Phi^* f_N\|_{L^q}}{\|f_N\|_{W_{q,p}}} \gtrsim \int_{B(0,4/3-M) \setminus B(0,3/2-M)} \Phi(y) |y|^{n/q'} dy \left( \frac{\lg 2^{N-2M}}{\lg 2^N} \right)^{1/q}. \]

Letting \( N \to \infty \), we have

\[ \int_{B(0,4/3-M) \setminus B(0,3/2-M)} \Phi(y) |y|^{n/q'} dy \lesssim \|H_\Phi^*\|_{W_{q,p} \to L^q}. \]

By the arbitrariness of \( M \), we let \( M \to \infty \) and obtain that \( \int_{\mathbb{R}^n} \Phi(y) |y|^{n/q'} dy \lesssim \|H_\Phi^*\|_{W_{q,p} \to L^q}. \)

Next, we establish the following two propositions for reduction.
Proposition 2.7 (For reduction of modulation space). Let $1/2 \leq 1/p \leq 1/q \leq 1$, $\Phi$ be a nonnegative function satisfying (1.2). If the Hausdorff operator $H_\Phi$ is bounded on $M_{p,q}$, we have

1. $H_\Phi : M_{p,q} \to L^p$ is bounded,
2. $H_\Phi^* : W_{q,p} \to L^q$ is bounded.

Proof. The first conclusion can be deduced by the embedding relation $M_{p,q} \hookrightarrow L^p$ (see Lemma 2.3) directly. We turn to prove the second conclusion. For any Schwartz function $f$, by the property of $H_\Phi$ and Lemma 2.1, we have $\|f\|_{M_{p,q}} = \|\hat{f}\|_{W_{q,p}}$ and

$$\|H_\Phi f\|_{M_{p,q}} = \|\tilde{H}_\Phi f\|_{W_{q,p}} = \|\tilde{H}_\Phi \hat{f}\|_{W_{q,p}} = \|\tilde{H}_\Phi \hat{f}\|_{W_{q,p}}.$$ 

Thus, if $H_\Phi$ is bounded on $M_{p,q}$, we have

$$\|H_\Phi f\|_{M_{p,q}} \lesssim \|\hat{f}\|_{W_{q,p}}.$$ 

The embedding relation $W_{q,p} \hookrightarrow L^q$ then yields that

$$\|H_\Phi f\|_{L^q} \lesssim \|f\|_{W_{q,p}}$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$. □

Proposition 2.8 (For reduction of Wiener amalgam space). Let $1/2 \leq 1/q \leq 1/p \leq 1$, $\Phi$ be a nonnegative function satisfying (1.2). If the Hausdorff operator $H_\Phi$ is bounded on $W_{p,q}$, we have

1. $H_\Phi : W_{p,q} \to L^p$ is bounded,
2. $H_\Phi^* : M_{q,p} \to L^q$ is bounded.

Proof. The first conclusion can be deduced by the embedding relation $W_{p,q} \hookrightarrow L^p$ (see Lemma 2.4) directly. We turn to prove the second conclusion. For any Schwartz function $f$, by the property of $H_\Phi$ and Lemma 2.1, we have that $\|f\|_{W_{p,q}} = \|\hat{f}\|_{M_{q,p}}$ and

$$\|H_\Phi f\|_{W_{p,q}} = \|\tilde{H}_\Phi f\|_{M_{q,p}} = \|\tilde{H}_\Phi \hat{f}\|_{M_{q,p}} = \|\tilde{H}_\Phi \hat{f}\|_{M_{q,p}}.$$ 

Thus, if $H_\Phi$ is bounded on $W_{p,q}$, we have

$$\|H_\Phi f\|_{M_{q,p}} \lesssim \|\hat{f}\|_{M_{q,p}}.$$ 

The embedding relation $M_{q,p} \hookrightarrow L^q$ then yields that

$$\|H_\Phi f\|_{L^q} \lesssim \|f\|_{M_{q,p}}$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$. □

Now, we are ready to give the proof of Theorem 1.9.

Proof of Theorem 1.9. We divide this proof into two parts.

IF PART: Using the Minkowski inequality, we deduce that

$$\|H_\Phi f\|_{M_{p,q}} \lesssim \left\| \int_{\mathbb{R}^n} \Phi(y) f(x/|y|) dy \right\|_{M_{p,q}}$$

$$\lesssim \int_{\mathbb{R}^n} \Phi(y) \|f(x/|y|)\|_{M_{p,q}} dy.$$
Recalling the dilation properties of modulation space (see Lemma 2.2), we obtain that

\[ \|H_\Phi f\|_{M_{p,q}} \lesssim \int_{\mathbb{R}^n} \Phi(y) \max\{|y|^{n/p}, |y|^{n/q'}\} dy \|f\|_{M_{p,q}} \]

This implies the boundedness of \(H_\Phi\) on \(M_{p,q}\).

**ONLY IF PART:** Suppose \(H_\Phi\) is bounded on \(M_{p,q}\). If \(1/2 \leq 1/p \leq 1/q \leq 1\), the conclusion can be verified directly by Proposition 2.6 and 2.7.

We only need to deal with the case of \(1/q \leq 1/p \leq 1/2\). We use a dual argument to deal with this case. Recalling

\[ \langle H_\Phi^* f \mid g \rangle = \langle f \mid H_\Phi g \rangle \]

for all \(f, g \in \mathcal{S}(\mathbb{R}^n)\), by Lemma 2.5 and Lemma 2.3 we deduce that

\[ |\langle H_\Phi^* f \mid g \rangle| = |\langle f \mid H_\Phi g \rangle| \]

\[ \leq \|f\|_{M_{p',q'}} \|H_\Phi g\|_{M_{p,q}} \]

\[ \lesssim \|f\|_{M_{p',q'}} \|g\|_{M_{p,q}} \]

\[ \lesssim \|f\|_{M_{p',q'}} \|g\|_{L^p}, \]

which implies that

\[ \|H_\Phi^* f\|_{L^{p'}} \lesssim \|f\|_{M_{p',q'}} \] (2.2)

for all \(f \in \mathcal{S}(\mathbb{R}^n)\). In addition, by the boundedness of \(H_\Phi\) on \(M_{p,q}\), we use Lemma 2.1 to deduce that \(H_\Phi^*\) is also bounded on \(W_{q,p}\). Thus, by Lemma 2.5 and Lemma 2.4 we have

\[ |\langle H_\Phi f \mid g \rangle| = |\langle f \mid H_\Phi^* g \rangle| \]

\[ \leq \|f\|_{W_{q',p'}} \|H_\Phi^* g\|_{W_{q,p}} \]

\[ \lesssim \|f\|_{W_{q',p'}} \|g\|_{W_{q,p}} \]

\[ \lesssim \|f\|_{W_{q',p'}} \|g\|_{L^q}, \]

which implies that

\[ \|H_\Phi f\|_{L^{q'}} \lesssim \|f\|_{W_{q',p'}} \] (2.3)

for all \(f \in \mathcal{S}(\mathbb{R}^n)\).

Combining (2.2) and (2.3), observing \(1/2 \leq 1/p' \leq 1/q'\), we use Proposition 2.6 to get the conclusion.

**Proof of Theorem 1.10.** We divide this proof into two parts.

**IF PART:** Using Lemma 2.1 and the Minkowski inequality, we deduce that

\[ \|H_\Phi f\|_{W_{p,q}} \sim \|\widehat{H_\Phi f}\|_{M_{q,p}} \sim \left\|\int_{\mathbb{R}^n} \Phi(y) |y|^n \hat{f}(|y| |x|) dy \right\|_{M_{q,p}} \]

\[ \lesssim \int_{\mathbb{R}^n} \Phi(y) |y|^n \left\|\hat{f}(|y| |x|)\right\|_{M_{q,p}} dy. \]
Recalling the dilation properties of modulation space (see Lemma 2.2), we obtain that

\[ \|H_\Phi f\|_{W_{p,q}} \lesssim \int_{\mathbb{R}^n} \Phi(y)|y|^n \max\{|y|^{-n/q}, |y|^{-n/p'}\} \, dy \|\hat{f}\|_{M_{q,p}} \]

\[ \lesssim \int_{\mathbb{R}^n} (|y|^{n/p} + |y|^{n/q'}) \Phi(y) \, dy \|f\|_{W_{p,q}}. \]

This implies the boundedness of $H_\Phi$ on $W_{p,q}$.

**ONLY IF PART:** Suppose $H_\Phi$ is bounded on $W_{p,q}$. If $1/2 \leq 1/q \leq 1/p \leq 1$, the conclusion can be verified directly by Proposition 2.6 and 2.8.

For the case $1/p \leq 1/q \leq 1/2$, the desired conclusion follows by a dual argument as in the proof of Theorem 1.9.

**Remark 2.9.** For some technical reasons, our main theorems only characterize the boundedness of Hausdorff operator on $M_{p,q}$ and $W_{p,q}$ in some special cases. Our theorems remain an open problem for the characterization of Hausdorff operator on the full range $1 \leq p, q \leq \infty$.

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**References**

1. J. Chen, D. Fan, S. Wang, Hausdorff operators on Euclidean spaces, Applied Mathematics-A Journal of Chinese Universities, 2013, 28(4), 548-564.
2. J. Cunanan, M. Kobayashi, M. Sugimoto, Inclusion relations between Lp-Sobolev and Wiener amalgam spaces, J. Funct. Anal. 268(1) (2015), 239-254.
3. D. Fan, X. Lin, Hausdorff operator on real Hardy spaces, Analysis, 2014, 34(4), 319-337.
4. H.G. Feichtinger, Modulation spaces on locally compact Abelian group, Technical Report, University of Vienna, 1983. Published in: “Proc. Internat. Conf. on Wavelet and Applications”, 99-140. New Delhi Allied Publishers, India, 2003.
5. H.G. Feichtinger, Modulation spaces: looking back and ahead, Sampling Theory in Signal and Image Processing, 5(2) (2006), 109-140.
6. G. Gao, F. Zhao, Sharp weak bounds for Hausdorff operators. Anal. Math. 41 (2015), no. 3, 163-173.
7. K. Gröchenig, Foundations of Time-Frequency Analysis, Birkhäuser, Boston, MA, 2001.
8. W. Guo, H. Wu, Q. Yang, G. Zhao, Characterization of inclusion relations between Wiener amalgam and some classical spaces. J. Funct. Anal., 2017, 273(1): 404-443.
9. W. Guo, H. Wu, G. Zhao, Inclusion relations between modulation and Triebel-Lizorkin spaces, Proc. Amer. Math. Soc, 2017.
10. A. Lerner, E. Liflyand, Multidimensional Hausdorff operator on the real Hardy space, J. Austr. Math. Soc. 83(2007), 65-72.
11. E. Liflyand, Boundedness of multidimensional Hausdorff operators on $H^1(\mathbb{R}^n)$, Acta Sci. Math. (Szeged). 74 (2008), 845-851.
12. E. Liflyand, Hausdorff Operators on Hardy Spaces, Eurasian Math. J. 4(2013), 101-141.
13. E. Liflyand, A. Miyachi, Boundedness of the Hausdorff operators in $H^p$ spaces, 0 < p < 1, Studia Math. 194(3) (2009), 279-292.
14. E. Liflyand, F. Móricz, The Hausdorff operator is bounded on the real Hardy space $H^1(\mathbb{R})$. Proc. Amer. Math. Soc., 2000, 128(5), 1391-1396.
15. M. Sugimoto, N. Tomita, The dilation property of modulation spaces and their inclusion relation with Besov spaces, J. Funct. Anal., 248(1) (2007), 79-106.
16. M. Kobayashi, M. Sugimoto, The inclusion relation between Sobolev and modulation spaces, J. Funct. Anal., 260 (2011) 11, 3189-3208.
17. J. Ruan, D. Fan, Hausdorff operators on the power weighted Hardy spaces, Journal of Mathematical Analysis and Applications, 2016, 433(1), 31-48.
18. M. Sugimoto, N. Tomita, The dilation property of modulation space and their inclusion relation with Besov spaces, J. Funct. Anal., 248 (2007), 79-106.
19. H. Triebel, Modulation spaces on the euclidean n-space, Z. Anal. Anwend., 2 (5) (1983), 443-457.
20. B. Wang, H. Hudzik, The global Cauchy problem for the NLS and NLKG with small rough data, J. Differential Equations, 232 (2007), 36-73.
21. X. Wu, J. Chen, Best constants for Hausdorff operators on n-dimensional product spaces, Science China Mathematics, 2014, 57(3), 569-578.

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