Global analysis of an SAIS model

M.R. Razvan\(^a\) and S. Yasaman\(^b\)

\(^a\)Department of Mathematical Sciences, Sharif University of Technology, P.O. Box 11155-9415, Tehran, Iran; \(^b\)School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395-5746, Tehran, Iran

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This paper is concerned with global analysis of an SAIS epidemiological model in a population of varying size introduced by Busenberg and van den Driessche. In this model the population is divided into three subgroups of susceptible, asymptomatic and infective individuals. It has been shown that this system has no periodic solutions and all its trajectories tend to the equilibria of the system. We use the Poincaré Index theorem to determine the number of the equilibria and their stability properties. We have shown that bistability occurs for suitable values of parameters and found a set of examples of all possible dynamics of the system.

Keywords: epidemiological model; SAIS model; disease-free equilibrium; endemic equilibrium; Poincaré Index; bifurcation

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1. Introduction

In 1991 Busenberg and van den Driessche introduced an SAIS model in which the population is divided into three subgroups susceptibles, asymptomatics and infectives [2]. They proved that this system has no periodic solutions and gave a sufficient condition for the stability of disease-free equilibrium (DFE). This model is a general model including differential infectivity (DI), SIRS, SEIS and SIAS models as its special cases. A complete global analysis of an SIRS model with no vertical transmission has been given in [1]. An SIRS model with standard incidence rate and treatment rate of infectious individuals was proposed in [6] to understand the effect of the capacity for treatment of infectives on the disease spread and it was also found that a backward bifurcation occurs. Busenberg and van den Driessche in [1] showed that SEIS models do not have periodic solutions and gave local analysis of the model. In [10] Mena Lorca analysed an SEI model, in which there is no cycling back into the susceptible group (SEI model can be obtained from the SEIS model by choosing some parameters to be equal to zero). An SIAS model in a population of constant size considered by Cooke and the sharp threshold was found in [4]. Hyman
et al. [9] introduced a DI model with dissimilar groups of infective individuals. There are some models which combine DI and susceptibility [8]. In [7] a couple of SIS and SIR models with DI and differential susceptibility have been considered and the equilibria, their stabilities and the reproductive number have been determined for these models. A disease transmission model with two groups of infectives has been analysed in [14]. Multiple equilibria for a DI model with two dissimilar groups of infective individuals have been reported in [16]. In this paper we show that an SAIS model may exhibit bistability for suitable values of parameters too.

We first explain in Section 2 the model equations which is a homogeneous system of degree one. For such a system, it is more convenient to consider the proportions system. In Section 3, we state a result concerning the nonexistence of certain types of the solutions for the proportions system [2,3]. This result helps us to show that every solution in the feasibility region tends to a rest point. The remainder of this paper is mainly concerned with the discussion of existence and stability of equilibria. The technique used here is the Poincaré Index theorem which is based on a careful choice of Jordan curves and counting the number of the rest points inside them. In Section 4, we reduce our system to a planar one to examine its local analysis. In Section 5, we first prove that the degenerate cases occur for a zero measure set of parameter values and hence degenerate cases determine the bifurcations of the system. Then we state the main theorems of this paper which explore the possible behaviours of the system and then we give a set of examples for each of these behaviours. In Section 6, a numerical example of the bifurcations of the planar system is given and we elaborate upon how we can control the disease by varying the parameters of the system.

2. The model

The model introduced by Busenberg and van den Driessche in [2] is obtained by dividing the population into three groups: susceptible individuals, asymptomatic (infective without symptoms) individuals and infective individuals (infective with symptoms), the size of each of these groups at time $t$ is denoted by $S(t)$, $A(t)$ and $I(t)$, respectively. Let $N = S + A + I$ be the total size of the population which is assumed to be varying in this paper. The following parameters appear in their model with the same notations as in [2]:

$b_{SS}, b_{AS}, b_{IS}, b_{AA}, b_{IA}, b_{AI}, b_{II}$: per capita birth rates, the first subscript denotes the parent class and the second one the class of the offspring;

d: per capita disease-free death rate;
$p$: per capita rate of passing from $A$ to $I$;
$q$: per capita rate of passing from $I$ to $A$;
$\delta, \varepsilon$: excess per capita death rate of $A$ and $I$, respectively;
$\lambda_A, \lambda_I$: effective per capita contact rate of infective individuals in groups $A$ and $I$, respectively;
$c_A, c_I$: per capita recovery rate of $A$ and $I$ entering $S$, respectively;
$\alpha, \beta$: per capita rate of passing of those infected by asymptomatics and infectives into $I$, respectively;

where $\alpha, \beta \in [0, 1]$. All the parameters are assumed to be positive unless otherwise specified. These hypotheses lead to the following system of differential equations in $\mathbb{R}_+^3$, where $'$ denotes the derivatives with respect to time, $t$:

$$S' = b_{SS}S + b_{AS}A + b_{IS}I - dS - \lambda_A \frac{SA}{N} - \lambda_I \frac{SI}{N} + c_AA + c_I I,$$

$$A' = b_{AA}A + b_{IA}I + (1 - \alpha)\lambda_A \frac{SA}{N} + (1 - \beta)\lambda_I \frac{SI}{N} - (d + \delta + c_A + p)A + qI,$$
\[ I' = b_{AI}A + b_{II}I + \alpha \lambda_A \frac{SA}{N} + \beta \lambda_I \frac{SI}{N} - (d + \varepsilon + c_I + q)I + pA, \]

where \( \lambda_A(SA/N) \) and \( \lambda_I(SI/N) \) are of the proportionate (or random) mixing type [5,11].

Identifying group A with a recovered group R, the SIRS model is obtained by setting \( b_{SS} = b_{AS} = b_{IS} = b > 0, b_{AA} = b_{AI} = b_{IA} = b_{II} = p = c_I = \lambda_A = 0 \) and \( \beta = 1 \). We can obtain an SEIS model by identifying A with an exposed (but not yet infectious) group E, and setting \( q = c_A = \lambda_A = \beta = 0 \). If we identify A with an infective group (with different parameters of infectivity in comparison with I) and let \( b_{AA} = b_{AI} = b_{IA} = b_{II} = p = q = 0 \) and \( b_{SS} = b_{AS} = b_{IS} = b \), then we arrive at DI model. Now, consider the above model with no vertical transmission, no possibility of moving directly between groups A and I, i.e. \( b_{AA} = b_{AI} = b_{IA} = b_{II} = p = q = 0 \) and \( \alpha = \beta \). The resulting model corresponds to the SIAS model.

The total population equation is obtained by the sum of the above three equations:

\[ N' = b_{SS}S + (b_{AS} + b_{AA} + b_{AI} - \delta)A + (b_{IS} + b_{IA} + b_{II} - \varepsilon)I - dN. \]

Since the above system is homogeneous of degree one, it is more convenient to consider the proportions of the population, \( s = S/N, a = A/N \) and \( i = I/N \). The dynamics of \( s, a \) and \( i \) are governed by the following system of equations:

\[
\begin{align*}
    s' &= b_{SS}s + b_{AS}a + b_{IS}i - \lambda_{AS}a - \lambda_{AI}s + c_AA + c_{II}i \\
    &- s[b_{SS}s + (b_{AS} + b_{AA} + b_{AI} - \delta)a + (b_{IS} + b_{IA} + b_{II} - \varepsilon)i], \\
    a' &= b_{AA}a + b_{IA}i + (1 - \alpha)\lambda_{AS}a + (1 - \beta)\lambda_{AI}s - (\delta + c_A)a - pa + qi \\
    &- a[b_{SS}s + (b_{AS} + b_{AA} + b_{AI} - \delta)a + (b_{IS} + b_{IA} + b_{II} - \varepsilon)i], \\
    i' &= b_{AI}a + b_{II}i + \alpha \lambda_{AS}a + \beta \lambda_{AI}s - (\varepsilon + c_I)i + pa - qi \\
    &- [b_{SS}s + (b_{AS} + b_{AA} + b_{AI} - \delta)a + (b_{IS} + b_{IA} + b_{II} - \varepsilon)i].
\end{align*}
\]

In this paper we investigate the dynamics of the proportions system in the feasibility region:

\[ D = \{(s, a, i) : s + a + i = 1, s \geq 0, a \geq 0, i \geq 0 \}. \]

### 3. Nonexistence of periodic solutions

Given an autonomous system of ordinary differential equations in \( \mathbb{R}^n \)

\[
\frac{dx}{dt} = f(x),
\]

we denote by \( x(t) \) the value of the solution of this system at time \( t \) that is \( x \) initially. For \( V \subseteq \mathbb{R}^n \) and \( J \subseteq \mathbb{R} \), we let \( V \cdot J = \{ x \cdot t : x \in V, t \in J \} \). The set \( V \) is called invariant if \( V \cdot \mathbb{R} = V \) and it is called positively invariant if \( V \cdot \mathbb{R}^+ \subset V \). For \( Y \subseteq \mathbb{R}^n \), the \( \omega \)-limit (respectively, the \( \alpha \)-limit) set of \( Y \) is defined to be the maximal invariant set in the closure of \( Y \cdot [0, \infty) \) \((Y \cdot (-\infty, 0]) \). An orbit \( y(t) \) is a heteroclinic orbit if \( \lim_{t \to -\infty} y(t) = x \) and \( \lim_{t \to +\infty} y(t) = y \), where \( x \) and \( y \) are the rest points and it is called a homoclinic orbit when \( x \) coincides with \( y \). A closed curve connecting several rest points whose segments between successive rest points are heteroclinic orbits is called a phase polygon. By a sink we mean a rest point at which all the eigenvalues of the linearized system have negative real parts. A rest point is called a source point if these eigenvalues have positive real parts and it is called a saddle point if some of these eigenvalues have positive real parts and the others have negative real parts. A rest point is called nondegenerate if all of
these eigenvalues are nonzero and it is called hyperbolic if all of its eigenvalues have nonzero real parts. The following theorem is a special case of the results of [3] concerning the nonexistence of certain types of solutions.

**Theorem 3.1** Let in (4), \( f \) be a smooth vector field in \( \mathbb{R}^3 \) and \( \gamma(t) \) be a closed piecewise smooth curve which is the boundary of an orientable smooth surface \( S \subset \mathbb{R}^3 \). Suppose \( g: U \rightarrow \mathbb{R}^3 \) is defined and is smooth in a neighbourhood \( U \) of \( S \). Moreover, it satisfies \( g(\gamma(t)) \cdot f(\gamma(t)) = 0 \) and \( \text{curl} \ g \cdot n < 0 \), where \( n \) is the unit normal to \( S \). Then \( \gamma \) is not a finite union of the orbits of the system (4).

In order to apply the above theorem for the surface \( D \), we define \( g = g_1 + g_2 + g_3 \) where

\[
g_1(a, i) = \begin{bmatrix} 0, & -\frac{f_3(1-a-i, a, i)}{ai} \\ \frac{f_2(1-a-i, a, i)}{ai} \end{bmatrix},
\]
\[
g_2(s, i) = \begin{bmatrix} \frac{f_3(s, 1-s-i, i)}{si}, & 0 \\ -\frac{f_1(s, 1-s-i, i)}{si} \end{bmatrix},
\]
\[
g_3(s, a) = \begin{bmatrix} -\frac{f_2(s, a, 1-s-a)}{sa}, & \frac{f_1(s, a, 1-s-a)}{sa} \\ 0 \end{bmatrix},
\]

and \( f_j \)'s denote the right-hand sides of the proportions system reduced to the functions of two variables by using \( s_1 + s_2 + i = 1 \). For the system \((1')–(3')\), we have \( g \cdot f = 0 \) and

\[
\text{curl} \ g \cdot (1, 1, 1) = -\left( \frac{b_{IS} + c_I}{s^2 a} + \frac{b_{IA} + q}{sa^2} + \frac{b_{AI} + p}{si^2} + \frac{b_{AS} + c_A}{is^2} + \frac{\alpha \lambda_A}{i^2} + \frac{(1-\beta) \lambda_I}{a^2} \right).
\]

**Corollary 3.2** The system \((1')–(3')\) has no periodic orbits, homoclinic orbits or phase polygons in \( \text{int}(D) \) (the interior of \( D \)).

**Proof** Use Theorem 3.1 for \( f = (f_1, f_2, f_3) \) and \( g = g_1 + g_2 + g_3 \) in \( \text{int}(D) \).

We modify the arguments of [2] to show that every trajectory of the system tends to an equilibrium. This result reduces our analysis to the study of the equilibria and their stability.

**Lemma 3.3** The \( \omega \)-limit set of any orbit of the system \((1')–(3')\) with initial point in \( D \) is an equilibrium.

**Proof** Since the \( \omega \)-limit set of an orbit is invariant, all of its regular points are in \( \text{int}(D) \). We know that the \( \omega \)-limit set is connected and the system has finitely many equilibria. Hence, if the \( \omega \)-limit set is not a single point, then it must have a regular point in \( \text{int}(D) \). Let \( x \) be such a point and \( h \) be its first return map. For a point \( y \) near \( x \) on the transversal, let \( V \) be the region bounded by the orbit \( \gamma \) from \( y \) to \( h(y) \) and the segment between them. This region is known as the Bendixson sack (see Figure 1). Now by Stokes’ theorem

\[
\int_V (\text{curl} \ g) \cdot (1, 1, 1) \ d\sigma = \int_{\gamma} g \cdot f \ dt + \int_{0}^{1} g(ty + (1-t)h(y)) \cdot (y - h(y)) \ dt.
\]

Since \( g \cdot f = 0 \) and \( h(x) = x \), the right-hand side of the above equality tends to zero when \( y \) tends to \( x \). On the other hand, the left-hand side tends to the integral over the region bounded by the \( \omega \)-limit set. This is a contradiction since \( (\text{curl} \ g) \cdot (1, 1, 1) < 0 \) in \( \text{int}(D) \).
**Remark 3.4** When the $\omega$-limit set lies in $\text{int}(D)$, the above result is easily concluded by the generalized Poincaré–Bendixson theorem [12,13] and Corollary 3.2. Similarly if the $\alpha$-limit set of an orbit of the system (1′)–(3′) lies in $\text{int}(D)$, it must be a single point.

The remainder of this paper is concerned with the number of the equilibria and their stability properties. We start with local stability of the equilibria of the system in the next section. In order to determine the equilibria of the system, we reduce it to a two-dimensional system which is more appropriate.

### 4. Local analysis

In this section, we desire to study the proportions system in its feasibility region. Let $\Sigma = s + a + i$, then

$$\Sigma' = (1 - \Sigma)(sb_{SS} + (b_{AS} + b_{AA} + b_{AI} - \delta)a + (b_{IS} + b_{IA} + b_{II} - \varepsilon)i).$$

Therefore, the plane $\Sigma = 1$ is invariant. Furthermore on the boundary of $D (\partial D)$, we have

- $s = 0 \Rightarrow s' = b_{AS}a + b_{IS}i + cAa + cI i > 0,$
- $a = 0 \Rightarrow a' = b_{IA}i + (1 - \beta)\lambda_I si + qi \geq 0,$
- $i = 0 \Rightarrow i' = b_{AI}a + \alpha\lambda_As + pa \geq 0.$

Since all the parameters are positive, the vector field points inward on $\partial D$ except at $(1, 0, 0)$ which is an equilibrium of the system. Therefore, $D$ is positively invariant with a DFE on its boundary.

The proportions system on $D$ is essentially two-dimensional. Indeed, one can use the relation $s + a + i = 1$ to reduce the system (1′)–(3′) to the following planar system:

$$a' = a[b_{AA} + (1 - \alpha)\lambda_A - \delta - c_A - p - b_{SS}] - a^2[(1 - \alpha)\lambda_A - b_{SS} + b_{AS} + b_{AA} + b_{AI} - \delta]$$
$$- ai[(1 - \alpha)\lambda_A + (1 - \beta)\lambda_I - b_{IS} + b_{IS} + b_{IA} + b_{II} - \varepsilon] - i^2[(1 - \beta)\lambda_I]$$
$$+ i[b_{IA} + (1 - \beta)\lambda_I + q],$$

$$i' = i[b_{II} + \beta\lambda_I - \varepsilon - c_I - q - b_{SS}] - i^2[\beta\lambda_I - b_{SS} + b_{IS} + b_{IA} + b_{II} - \varepsilon]$$
$$- ai[\alpha\lambda_A + \beta\lambda_I - b_{SS} + b_{AS} + b_{AA} + b_{AI} - \delta] - a^2[\alpha\lambda_A] + a[b_{AI} + \alpha\lambda_A + p].$$

It can be easily verified that the system (1′)–(3′) on $D$ is equivalent to the system (5)–(6) in its feasibility region:

$$D_1 = \{(a, i) : a + i \leq 1, a \geq 0, i \geq 0\}.$$
So, instead of analysing the system \((1')- (3')\), we can analyse the system \((5)-(6)\). The following lemma has two immediate consequences which are useful in the analysis of the system.

**Lemma 4.1** The trace of the linearization of the system \((5)-(6)\) at a rest point in \(\text{int}(D_1)\) is negative.

**Proof** If we take the derivatives of the right-hand sides of the system \((5)-(6)\), then we obtain:

\[
\frac{\partial a'}{\partial a} = [b_{AA} + (1-\alpha)\lambda_A - \delta - c_A - p - b_{SS}] - 2a[(1-\alpha)\lambda_A - b_{SS} + b_{AS} + b_{AA} + b_{AI} - \delta]
- i[(1-\alpha)\lambda_A + (1-\beta)\lambda_I - b_{SS} + b_{IS} + b_{IA} + b_{II} - \varepsilon],
\]

\[
\frac{\partial i'}{\partial i} = [b_{II} + \beta\lambda_I - \varepsilon - c_I - q - b_{SS}] - 2i[\beta\lambda_I - b_{SS} + b_{IS} + b_{IA} + b_{II} - \varepsilon]
- a[\alpha\lambda_A + \beta\lambda_I - b_{SS} + b_{AS} + b_{AA} + b_{AI} - \delta].
\]

Since \(a' = i' = 0\) at every equilibrium, by some calculations it is obtained that

\[
\frac{\partial a'}{\partial a} = -a[(1-\alpha)\lambda_A - b_{SS} + b_{AS} + b_{AA} + b_{AI} - \delta] + \frac{i^2}{a}[(1-\beta)\lambda_I]
- i\frac{a}{b_{II} + (1-\beta)\lambda_I + q],
\]

\[
\frac{\partial i'}{\partial i} = -i[\beta\lambda_I - b_{SS} + b_{IS} + b_{IA} + b_{II} - \varepsilon] + \frac{a^2}{i}[\alpha\lambda_A] - a\frac{b_{AI} + \alpha\lambda_A + p].
\]

Now by using the relation \(s + a + i = 1\), it yields that

\[
\text{trace} = \frac{\partial a'}{\partial a} + \frac{\partial i'}{\partial i} = -a[\lambda_A + b_{AS} + b_{AA} + b_{AI}] - i[\lambda_I + b_{IS} + b_{IA} + b_{II}]
- \frac{i}{a}[b_{II} + s(1-\beta)\lambda_I + q] - \frac{a}{i}[b_{AI} + \alpha\lambda_A + p] + (1-s)b_{SS} + a\delta + i\varepsilon.
\]

Since \(s' = 0\) at every equilibrium, we conclude that

\[(1-s)b_{SS} + a\delta + i\varepsilon < a[\lambda_A + b_{AS} + b_{AA} + b_{AI}] + i[\lambda_I + b_{IS} + b_{IA} + b_{II}],\]

hence \(\text{trace} < 0\). □

**Corollary 4.2** The system \((5)-(6)\) has no source points in \(\text{int}(D_1)\).

**Corollary 4.3** Every nondegenerate equilibrium of the system \((5)-(6)\) in \(\text{int}(D_1)\) is hyperbolic.

We close this section by investigating the stability of the origin that is the unique DFE of the system \((5)-(6)\) in \(D_1\). Corollary 4.3 is true for DFE as well. Indeed DFE cannot have two pure imaginary eigenvalues, since \(D_1\) is positively invariant. The Jacobian matrix at DFE is

\[
J = \begin{bmatrix}
    b_{AA} + (1-\alpha)\lambda_A - \delta - c_A - p - b_{SS} & b_{II} + (1-\beta)\lambda_I + q \\
    b_{AI} + \alpha\lambda_A + p & b_{II} + \beta\lambda_I - \varepsilon - c_I - q - b_{SS}
\end{bmatrix},
\]

\[
\iff \det J = (b_{AA} + (1-\alpha)\lambda_A - \delta - c_A - p - b_{SS})(b_{II} + \beta\lambda_I - \varepsilon - c_I - q - b_{SS})
- (b_{II} + (1-\beta)\lambda_I + q)(b_{II} + \alpha\lambda_A + p).
\]

The necessary and sufficient condition for the degeneracy of DFE is \(\det J = 0\). Busenberg and van den Driessche in [2] proved the following theorem which gives a sufficient condition for the
stability of DFE in terms of

\[ R_{0A} = \frac{(1 - \alpha) \lambda_A}{c_A + p + \delta + b_{SS} - b_{AA}} \quad \text{and} \quad R_{0I} = \frac{\beta \lambda_I}{c_I + q + \varepsilon + b_{SS} - b_{II}}. \]

**Theorem 4.4** DFE is locally stable if \( R_{0A} < 1 \), \( R_{0I} < 1 \) and

\[
(1 - R_{0A})(1 - R_{0I}) > \frac{(b_{IA} + (1 - \beta) \lambda_I + q)(b_{AI} + \alpha \lambda_A + p)}{(\delta + c_A + p + b_{SS} - b_{AA})(\varepsilon + c_I + q + b_{SS} - b_{II})}.
\]

5. **Global analysis**

Let \( \Omega \) be the parameter space of the system (5)–(6) which is an open subset of \( R_+^{17} \) and \( \Omega_0 \) be the set of all possible values of the parameters for which this system has a nonhyperbolic (or equivalently degenerate) equilibrium in \( D_1 \). The following proposition shows that the bifurcation points of the system are the degenerate cases.

**Proposition 5.1** \( \Omega_0 \) is a closed nonempty subset of \( \Omega \) with zero measure.

**Proof** It is easy to find some values of parameters for which DFE is a degenerate point, hence \( \Omega_0 \) is nonempty. Since the eigenvalues of the linearization of the system (5)–(6) are continuous with respect to the entries of the Jacobian matrix and the elements of Jacobian matrix are continuous with respect to the parameters, we can easily conclude that \( \Omega_0 \) is a closed subset of \( \Omega \).

In order to show that \( \Omega_0 \) is of zero measure, we use Sard’s theorem. Since \( s' = 0 \) at every equilibrium, by substituting the relation \( s + a + i = 1 \) in Equation (1') we obtain:

\[
a = \frac{[b_{IS} + c_I + (b_{SS} - b_{IA} - b_{II} - \lambda_I + \varepsilon)s](1 - s)}{c_I - c_A + b_{IS} - b_{AS} + (b_{AS} + b_{AA} + b_{AI} - b_{IS} - b_{IA} - b_{II} + \lambda_A - \lambda_I - \delta + \varepsilon)s}, \quad (7)
\]

\[
i = \frac{[b_{AS} + c_A + (b_{SS} - b_{AS} - b_{AA} - b_{AI} - \lambda_A + \delta)s](1 - s)}{c_A - c_I + b_{AS} - b_{IS} + (b_{IS} + b_{IA} + b_{II} - b_{AS} - b_{AA} - b_{AI} + \lambda_I - \lambda_A - \varepsilon + \delta)s}. \quad (8)
\]

If at an equilibrium in \( \text{int}(D_1) \) we have

\[
c_A - c_I + b_{AS} - b_{IS} + (b_{IS} + b_{IA} + b_{II} - b_{AS} - b_{AA} - b_{AI} + \lambda_I - \lambda_A - \varepsilon + \delta)s = 0,
\]

then

\[
(c_I + b_{IS})(b_{SS} - b_{AS} - b_{AA} - b_{AI} - \lambda_A + \delta) = (c_A + b_{AS})(b_{SS} - b_{IS} - b_{IA} - b_{II} - \lambda_I + \varepsilon),
\]

which is the equation of an algebraic hyperplane in \( \Omega \). Hence its measure is zero and Equations (7) and (8) make sense almost everywhere. Moreover, by considering Equation (2') at an equilibrium it yields that

\[
b_{AA} = \frac{(b_{AS} + B_{AA} - \delta)a^2 + (b_{IS} + b_{IA} + b_{II} - \varepsilon)i + ((1 - \alpha)\lambda_A - b_{SS})sa - (1 - \beta)\lambda_I si}{(\delta + c_A + p)a}
\]

\[
- \frac{(b_{IA} + q)i}{a(1 - a)}.
\]

By substituting the statements (7) and (8) in the above equation, \( b_{AA} \) can be written as a function in term of variable \( s \), namely \( b_{AA} = h(s) \). Notice that \( a(1 - a) > 0 \) for an equilibrium in \( \text{int}(D_1) \).
If this equilibrium is degenerate, then $b_{AA}$ is a critical value of $h$. Furthermore, Sard’s theorem asserts that the set of critical values of $h$ is of zero measure (this set is finite since $h$ is a rational function). Hence $\Omega_0$ intersects almost all of the lines which are parallel to $b_{AA}$ axis in a set of zero measure. Now Fubini’s theorem completes the proof for the critical point in $\text{int}(D_1)$. For $DFE$ the degeneracy occurs in a zero measure subset of $\Omega$. Hence $\Omega_0$ is contained in the union of these subsets of $\Omega$ with zero measure.

5.1. **Global analysis when $DFE$ is stable**

**Theorem 5.2** If $DFE$ is stable, then one of the following statements holds:

1. $DFE$ is globally stable in $D_1$ (monostability).
2. There are two endemic equilibria, one of them stable and the other one a saddle point (bistability).
3. There is a unique endemic equilibrium and it is degenerate (degeneracy).

In addition, all of these statements occur for suitable values of parameters.

**Proof** We first assume that all the rest points of the system are nondegenerate. In order to construct a Jordan curve, we choose a sufficiently small ellipse-like neighborhood $U_1$ around DFE such that our vector field is inward on the boundary of $U_1$. Now, let $\Gamma = \partial(D_1 \cup U_1)$ (Figure 2).

Thus, our vector field is tangent or inward on $\Gamma$. It is known that the Poincaré Index of $\Gamma$, $I(\Gamma)$, is 1 [15]. Suppose that $\mu_0(\Gamma), \mu_1(\Gamma)$ and $\mu_2(\Gamma)$ are the number of stable, saddle and source points in the interior of $\Gamma$. Since all the rest points in $\Gamma$ are hyperbolic, we have

$$\mu_0(\Gamma) - \mu_1(\Gamma) + \mu_2(\Gamma) = 1, \quad \mu_0(\Gamma) + \mu_1(\Gamma) + \mu_2(\Gamma) \leq 4, \quad \mu_2(\Gamma) = 0,$$

by the Poincaré Index theorem [13], Bézout’s theorem and Corollary 4.2 , respectively. Thus, $\mu_0(\Gamma) = 1, \mu_1(\Gamma) = \mu_2(\Gamma) = 0$ or $\mu_0(\Gamma) = 2, \mu_1(\Gamma) = 1, \mu_2(\Gamma) = 0$. The first one implies (1) and the second one implies (2).

Now we consider the degenerate cases. Suppose that the system has a degenerate equilibrium in $\text{int}(D_1)$. We know that there are at most four equilibria (counted with multiplicity) and every degenerate point is of multiplicity more than one. Hence, if there is another equilibrium in $\text{int}(D_1)$, then it must be nondegenerate. Since $\Omega_0$ has zero measure, by applying a suitable perturbation the number of saddle points of the planar system is odd. One can see that this is a contradiction by choosing a Jordan curve as in Figure 2. Hence if there is a degenerate endemic equilibrium, then it is the unique equilibrium in $\text{int}(D_1)$.

![Figure 2. The Jordan curve when DFE is stable.](image-url)
In the following, we show that the cases 1, 2 and 3 of the above theorem occur for some suitable values of parameters by analysing a special case of the planar system. Indeed, we find examples of monostability and bistability for this special case and through a perturbation, we show that cases (1) and (2) occur for general system.

**Proposition 5.3** If \( b_{SS} + \delta > b_{AA} + b_{AI} + \lambda_A \) and \( b_{SS} + \varepsilon > b_{IA} + b_{II} + \lambda_I \), then DFE is globally stable.

**Proof** By applying the statement \( s_1 + s_2 + i = 1 \), we can rewrite Equation (1') in the following form:

\[
s' = (b_{AS}a + b_{SI}i)(1 - s) + c_Aa + c_Ii + s(a(b_{SS} + \delta - b_{AA} - b_{AI} - \lambda_A) + i(b_{SS} + \varepsilon - b_{IA} - b_{II} - \lambda_I)).
\]

If \( b_{SS} + \delta > b_{AA} + b_{AI} + \lambda_A \) and \( b_{SS} + \varepsilon > b_{IA} + b_{II} + \lambda_I \), then \( s' > 0 \) or equivalently \( (a + i)' < 0 \). Thus, the function \( (a + i) \) is a Lyapunov function which decreases along the orbits in \( \text{int}(D_1) \). By LaSalle’s Invariance Principle, DFE is globally asymptotically stable. \( \blacksquare \)

**Proposition 5.4** If \( b_{SS} = b_{AA} = b_{II} = b \), \( \alpha = \beta \), \( (1 - \alpha)\lambda_A/\delta + \beta \lambda_I/\varepsilon < 1 \), \( \delta < \varepsilon \), \( \delta < \lambda_A \) and \( \varepsilon > \lambda_I \), then for sufficiently small values of \( b_{AS}, b_{IS}, b_{AI}, b_{IA}, c_A, c_I, p \) and \( q \), the system is bistable, i.e., there is a stable endemic equilibrium besides the stable DFE.

**Proof** Let \( b_{AS} = b_{IS} = b_{AI} = b_{IA} = c_A = c_I = p = q = 0, \alpha = \beta \) and \( b_{SS} = b_{AA} = b_{II} = b \). Then we arrive at:

\[
a' = a[(1 - \alpha)\lambda_A - \delta] - a^2[(1 - \alpha)\lambda_A - \delta] - ai[(1 - \alpha)\lambda_A + (1 - \beta)\lambda_I - \varepsilon] - i^2[(1 - \beta)\lambda_I] + i((1 - \beta)\lambda_I),
\]

\[
i' = i[\beta \lambda_I - \varepsilon] - i^2[\beta \lambda_I - \varepsilon] - ai[\alpha \lambda_A + \beta \lambda_I - \delta] - a^2[\alpha \lambda_A] + a[\alpha \lambda_A].
\]

This system has three equilibria \((0, 0)\), \((1, 0)\) and \((0, 1)\). The Jacobian matrix at DFE is

\[
J(0, 0) = \begin{bmatrix}
(1 - \alpha)\lambda_A - \delta & (1 - \beta)\lambda_I \\
\alpha \lambda_A & \beta \lambda_I - \varepsilon
\end{bmatrix}.
\]

If \( (1 - \alpha)\lambda_A/\delta + \beta \lambda_I/\varepsilon < 1 \), then det \( J(0, 0) > 0 \) and trace \( J(0, 0) < 0 \) and hence DFE is stable. Furthermore, the Jacobian matrix at \((1, 0)\) is

\[
J(1, 0) = \begin{bmatrix}
\delta - (1 - \alpha)\lambda_A & \varepsilon - (1 - \alpha)\lambda_A \\
-\alpha \lambda_A & \delta - \varepsilon - \alpha \lambda_A
\end{bmatrix}.
\]

One can easily check that the eigenvalues of the Jacobian matrix at \((1, 0)\) are \( \delta - \lambda_A \) and \( \delta - \varepsilon \) and the eigenvalues at \((0, 1)\) are \( \varepsilon - \delta \) and \( \varepsilon - \lambda_I \). Thus, if \( (1 - \alpha)\lambda_A/\delta + \beta \lambda_I/\varepsilon < 1 \), \( \delta < \varepsilon \), \( \delta < \lambda_A \) and \( \varepsilon > \lambda_I \), then DFE and \((0, 1)\) are stable and \((0, 0)\) is a source point. Since this system is quadratic, there is at most one equilibrium in \( \text{int}(D_1) \) which must be nondegenerate. Now, choose some sufficiently small ellipse-like neighborhoods \( B_1, B_2 \) and \( B_3 \) around \((0, 0), (1, 0) \) and \((0, 1)\), respectively, and let \( \Gamma = \partial((D_1 \cup B_1 \cup B_2) \setminus B_3) \) (Figure 3).

\( \Gamma \) is a Jordan curve containing all of the fixed points in \( \text{int}(D_1) \) and the vector field is either tangent or inward on \( \Gamma \), hence \( I(\Gamma') = 1 \). It shows that there must be an equilibrium in \( \text{int}(D_1) \) whose Poincaré Index is \(-1\) and then it is a saddle point. Since hyperbolic points are structurally stable, by applying a suitable perturbation we obtain an example of the general model for which DFE is stable and there is a saddle point in \( \text{int}(D_1) \). If
we choose a Jordan curve as depicted in Figure 2 and apply the Poincaré Index theorem, then we conclude that there is a stable endemic equilibrium in \( \text{int}(D_1) \). Hence the second possible case of Theorem 5.3 occurs.

**Remark 5.5** Since cases (1) and (2) of Theorem 5.2 are structurally stable, the set of values of parameters for which cases (1) and (2) occur, are open subsets of \( \Omega \). Hence, it is enough to prove that the set of values of parameters for which DFE is stable, is a connected set in \( \Omega \). We apply a homotopy argument to show the connectedness of this set, that is we join two given values of parameters by a curve inside this subset. The Jacobian matrix at DFE is

\[
J = \begin{pmatrix}
b_{AA} + (1 - \alpha)\lambda_A - \delta - c_A - p - b_{SS} & b_{IA} + (1 - \beta)\lambda_I + q \\
b_{AI} + \alpha\lambda_A + p & b_{II} + \beta\lambda_I - \epsilon - c_I - q - b_{SS}
\end{pmatrix}.
\]

\( J \) does not have complex eigenvalues, since \( D_1 \) is positively invariant. If DFE is stable, then \( \det J > 0 \) and \( \text{trace} J < 0 \). Hence

\[
(b_{AA} + (1 - \alpha)\lambda_A - \delta - c_A - p - b_{SS})(b_{II} + \beta\lambda_I - \epsilon - c_I - q - b_{SS})
\]

\[
> (b_{IA} + (1 - \beta)\lambda_I + q)(b_{AI} + \alpha\lambda_A + p).
\]

Since the parameters are positive, the diagonal entries of \( J \) must be of the same sign.

If DFE is stable, then \( \det J > 0 \) and \( \text{trace} J < 0 \). Hence \( b_{AA} + (1 - \alpha)\lambda_A - \delta - c_A - p - b_{SS} < 0 \) and \( b_{II} + \beta\lambda_I - \epsilon - c_I - q - b_{SS} < 0 \). We first apply some suitable homotopies to make the parameters \( b_{AA}, b_{AI}, b_{IA}, b_{II}, \lambda_A \) and \( \lambda_I \) vanish. Then we arrive at the matrix whose determinant is

\[
(\delta + c_A + p + b_{SS})(\epsilon + c_I + q + b_{SS}) - pq > 0.
\]

Now, applying some homotopies to make \( c_A, c_I, \epsilon \) and \( \delta \) vanish, we arrive at the matrix:

\[
M = \begin{pmatrix}
-p - b_{SS} & q \\
p & -q - b_{SS}
\end{pmatrix} \implies \det M = b_{SS}(b_{SS} + p + q).
\]

Hence, \( \det M > 0 \) and \( \text{trace} M < 0 \) iff \( b_{SS} + p + q > 0 \) which is true in the connected set \( \Omega \).
5.2. **Global analysis when DFE is a saddle or a source point**

**Theorem 5.6** If DFE is a saddle or a source point, then one of the following statements holds:

1. There is a unique endemic equilibrium and it is globally asymptotically stable in $D_1$ (monostability).
2. There are three endemic equilibria, a saddle together with two stable points (bistability).
3. There is a degenerate endemic equilibrium (degeneracy).

In addition, all of these statements occur for suitable values of parameters.

**Proof** We first suppose that DFE is a saddle point. Since $D_1$ is positively invariant, the stable manifold of DFE cannot meet the interior of $D_1$ and one of the branches of the unstable manifold of DFE must lie in the interior of $D_1$ (Figure 4).

The local behaviours of DFE help us to find piecewise smooth Jordan curves $\Gamma$ containing all the endemic equilibria, on which the vector field of the planar system is either tangent or inward. In order to construct such a Jordan curve, we have used a trajectory which is sufficiently close to the stable manifold of DFE and continued it based on the local dynamics (Figure 5(a)).

Now assume that DFE is a source point, then we choose a sufficiently small ellipse-like neighbourhood $U_1$ around DFE on which the vector field is outward and let $\Gamma = \partial(D_1 \setminus U_1)$ (Figure 5(b)). Thus, our vector field is tangent or inward on $\Gamma$ and hence $I(\Gamma) = 1$. Furthermore, the Poincaré Index of $\Gamma$ is equal to the sum of the Poincaré Indices of the equilibria in the interior of $\Gamma$. Suppose that $\mu_0(\Gamma)$, $\mu_1(\Gamma)$ and $\mu_2(\Gamma)$ are the number of stable, saddle and source points in the interior of $\Gamma$. If all the rest points in $\Gamma$ are nondegenerate (or equivalently hyperbolic), then

![Figure 4](image1.png)  
**Figure 4.** Local behaviour of planar system near DFE when it is a saddle point.

![Figure 5](image2.png)  
**Figure 5.** The Jordan curve when DFE is a (a) saddle point, (b) source point.
we have:

\[ \mu_0(\Gamma) - \mu_1(\Gamma) + \mu_2(\Gamma) = 1, \quad \mu_0(\Gamma) + \mu_1(\Gamma) + \mu_2(\Gamma) \leq 3, \quad \mu_2(\Gamma) = 0. \]

Thus, \( \mu_0(\Gamma) = 1, \mu_1(\Gamma) = \mu_2(\Gamma) = 0 \) or \( \mu_0(\Gamma) = 2, \mu_1(\Gamma) = 1, \mu_2(\Gamma) = 0 \). The first case implies (1) and the second case implies (2). Furthermore, if one of the equilibria in \( \text{int}(D_1) \) is nonhyperbolic, then it must be degenerate, hence case (3) occurs.

We close this section by showing that the statements of the above theorem occur for suitable values of the parameters. At first we prove that the set of parameters values for which DFE is a saddle and source point are connected subsets of \( \Omega \):

The necessary and sufficient conditions for DFE to be a saddle point, is \( \det J < 0 \) where \( J \) is the Jacobian matrix at DFE or equivalently:

\[
\frac{(b_{AA} + (1 - \alpha)\lambda_A - \delta - c_A - p - b_{SS})(b_{II} + \beta\lambda_I - \varepsilon - c_I - q - b_{SS})}{-(\alpha\lambda_A + p)(b_{IA} + (1 - \beta)\lambda_I + q)}.
\]

Since the parameters are positive, the right-hand side of the above relation is a rational function. Hence, the set of values of parameters for which DFE is a saddle point, namely \( E \), is connected.

The necessary and sufficient conditions for DFE to be a source point, are \( \det J > 0 \) and \( \text{trace } J > 0 \). Suppose that the parameters have been chosen such that DFE is a source point, hence the diagonal entries of \( J \) are positive. If we use suitable homotopies to make the parameters \( \alpha, b_{IA}, b_{AI} \) and \( b_{SS} \) to be equal to zero and make \( \beta \) to be equal to 1, then we arrive at

\[ M = \begin{bmatrix} b_{AA} + \lambda_A - \delta - c_A - p & q \\ b_{II} + \lambda_I - \varepsilon - c_I - q \end{bmatrix}, \]

\[ \det M = (b_{AA} + \lambda_A - \delta - c_A)(b_{II} + \lambda_I - \varepsilon - c_I) - p(b_{II} + \lambda_I - \varepsilon - c_I) \]

\[ - q(b_{AA} + \lambda_A - \delta - c_A). \]

Moreover, \( \det M > 0 \) and \( \text{trace } M > 0 \) iff

\[
\frac{p}{b_{AA} + \lambda_A - \delta - c_A} + \frac{q}{b_{II} + \lambda_I - \varepsilon - c_I} < 1,
\]

which holds for a connected subset of \( \Omega \), namely \( F \). Since by some homotopies \( J \) changes to \( M \), the set of parameters for which DFE is a source point is connected. Therefore, if one shows that cases (1) and (2) occur for suitable values of parameters, then by connectedness of \( E \) and \( F \) and the structural stability of hyperbolic points it is concluded that the case (3) occurs too. In the following, we use a special type of the model that is a DI model to show that these cases occur for general system.

Let \( b_{AA} = b_{AI} = b_{IA} = b_{II} = p = q = 0 \) and \( b_{SS} = b_{AS} = b_{IS} = b \). By these assumptions, the system (5)–(6) is simplified as:

\[
a' = a[(1 - \alpha)\lambda_A - \delta - c_A - b] + a^2[\delta - (1 - \alpha)\lambda_A] - ai[(1 - \alpha)\lambda_A + (1 - \beta)\lambda_I - \varepsilon] \\
- i^2[(1 - \beta)\lambda_I] + i[(1 - \beta)\lambda_I],
\]

\[
i' = i[\beta\lambda_I - \varepsilon - c_I - b] - i^2[\beta\lambda_I - \varepsilon] - ai[\alpha\lambda_A + \beta\lambda_I - \delta] - a^2[\alpha\lambda_A] + a[\alpha\lambda_A].
\]
The Jacobian matrix at \((0, 0)\) is

\[
J(0, 0) = \begin{bmatrix}
(1 - \alpha)\lambda_A - \delta - c_A - b & (1 - \beta)\lambda_I \\
\alpha\lambda_A & \beta\lambda_I - \varepsilon - c_I - b
\end{bmatrix}.
\]

Now, we investigate two cases of the DI model:

1. \(\alpha = \beta\): A necessary condition for DFE to be a source point, is \(\det J(0, 0) > 0\). Some calculations show that if \((1 - \alpha)\lambda_A/(\delta + c_A + b) + \beta\lambda_I/(\varepsilon + c_I + b) < 1\), then \(\det J > 0\), but this relation implies that trace \(J(0, 0)\) < 0. Hence if \(\alpha = \beta\), then DFE cannot be a source point. If DFE is a saddle point, it has been proved that cases (1) and (2) occur [14].

2. \(\alpha = 0, \beta = 1\): One can easily find out that in this case the lines \(i = 0\) and \(a = 0\) is invariant with respect to the DI model. It is easy to see that \(WA = (1 - (b + c_A)/(\lambda_A - \delta), 0)\) lies on \(\{i = 0\} \cap D_1\) if \(b + c_A + \delta < \lambda_A\) and \(WI = (0, 1 - (b + c_I)/(\lambda_I - \varepsilon))\) lies on \(\{a = 0\} \cap D_1\) if \(b + c_I + \varepsilon < \lambda_I\) and both are true if DFE is a source point. Some calculations show that \(WA\) is stable iff \((\lambda_I - \delta)(1 - (b + c_A)/(\lambda_A - \delta)) > \lambda_I - \varepsilon - c_I - b\) and \(WI\) is stable iff \((\lambda_A - \varepsilon)(1 - (b + c_I)/(\lambda_I - \varepsilon)) > \lambda_A - \delta - c_A - b\).

Suppose that the parameters have been chosen so that \(WA\) and \(WI\) lie in \(D_1\) and are stable, hence DFE is a source point. In order to construct a Jordan curve, we choose suitable ellipse-like neighbourhoods \(U_1, U_2\) and \(U_3\) around DFE, \(WA\) and \(WI\), respectively, such that the vector field is tangent or inward on \(\Gamma = \partial((D_1 \cup U_1 \cup U_3) \setminus U_1)\) (see Figure 6). Thus \(I(\Gamma) = 1\), hence there is a saddle point in the interior of \(D_1\).

Finding examples of bistability for these special cases of the system helps us to find some examples of bistability for the general model. More specifically, by applying a suitable perturbation, the saddle endemic equilibrium remains in \(D_1\) and the Morse index of DFE does not change. Now, the Poincaré Index argument shows that there are two stable endemic equilibria in \(\text{int}\,(D_1)\) and the proof of the following proposition is complete.

**Proposition 5.7** If \(b + c_A + \delta < \lambda_A\), \(b + c_I + \varepsilon < \lambda_I\), \((\lambda_I - \delta)(1 - (b + c_A)/(\lambda_A - \delta)) > \lambda_I - \varepsilon - c_I - b\) and \((\lambda_A - \varepsilon)(1 - (b + c_I)/(\lambda_I - \varepsilon)) > \lambda_A - \delta - c_A - b\), then the system is bistable for sufficiently small values of \(b_{AA}, b_{AI}, b_{IA}, b_{II}, p, q, \alpha\) and \(1 - \beta\).

**Remark 5.8** The above proposition asserts that when \(\alpha = 0, \beta = 1\), DFE is a source point and \(WA\) and \(WI\) are sinks, a small perturbation of the system leads to bistability. It can be shown that if \(WA\) or \(WI\) are saddle points, then the perturbation of the system leads to monostability.
6. Numerical examples

When DFE is the unique stable equilibrium of the system, by starting from any initial condition the solution to the system converges to the stable DFE. Hence spreading the disease does not need any excess action or method to be controlled and finally the disease dies out. Moreover, if there are a stable DFE and a stable endemic equilibria, then by using the hysteresis behaviour of bistable systems, we can control spreading the disease [17]. Hysteresis behaviour could also been seen in the case of bistable endemic equilibria. In the following numerical examples, we have used Proposition 5 to find values of parameters for which the system has two stable endemic equilibria. We fix $b_{AA} = b_{AI} = b_{IA} = b_{II} = \alpha = 0, b_{SS} = b_{AS} = b_{IS} = 0.01, \beta = 0.1, \epsilon = 0.05, c_A = 0.05, p = 0.002, q = 0.001, \delta = 0.1, \lambda_A = 0.5, \lambda_I = 1$ and let $c_I$ be the control parameter. We start with $c_I = 0.2$ for which the system is bistable (Figure 7(a)).

If we increase $c_I$, then the system undergoes a saddle-node bifurcation at $c_I = 0.211$ and then $E_2$ disappears and all the solutions tend to $E_1$ (see Figure 7(b) for $c_I = 0.22$). This makes our system exhibit hysteresis behaviour as it is shown in Figure 8(a).

In this figure, we start from $c_I = 0.2$ and increase it up to $c_I = 0.22$ and then decrease it to its initial value. Indeed, We assume that $c_I$ is a function of variable $t$, for instance $c_I = (-0.01)(|t - 40| - |t - 30| + |t - 150| - |t - 160|) + 2$. If we choose $(a, i) = (0.05, 0.92)$ as the initial condition of the system, then the solution first tends to $E_2$ with $i = 0.73$ and then to $E_1$ with $i = 0.05$ by increasing $c_I$ up to $c_I = 0.22$. When we decrease $c_I$ to its initial value, the system becomes bistable, but the current state of the system is in the basin of attraction of $E_1$ instead of $E_2$, hence the solution remains around $E_1$. Similarly if we decrease $c_I$, then the

![Figure 7](image_url)

Figure 7. (a) For $c_I = 0.2$ the system is bistable with two stable endemic equilibria, $E_1$ and $E_2$. (b) $E_1$ is globally stable for $c_I = 0.22$. (c) $E_2$ is globally stable for $c_I = 0.18$. Circles show stable points.
Figure 8. The hysteresis behaviour is observed by starting from a point in the basin of attraction of (a) $E_2$, (b) $E_1$, when $c_I$ changes and the other parameters are fixed as in Figure 7.

system undergoes another saddle-node bifurcation at $c_I = 0.1899$ and $E_1$ annihilates and then $E_2$ would be globally asymptotically stable (see Figure 7(c) for $c_I = 0.18$). It is easy to see that this system exhibits hysteresis behaviour in this case by starting from a point in the basin of attraction of $E_1$ (Figure 8(b)). In this figure we assume that $c_I$ is a function of variable $t$, for instance $c_I = (0.01)(|t - 50| - |t - 40| + |t - 50| - |t - 160|) + 2$.

Furthermore, if the endemic equilibrium is the unique stable equilibrium of the system, then all the solutions tend to this equilibrium and hence there are always some patients in the population. In the latter case, it is important to find an efficient method to control the disease. In the following, we give a numerical example and elaborate that how we can control the disease by varying a suitable parameter of the system.

Suppose that the parameters of the system (5)–(6) have been chosen such that the system is monostable and there is an endemic equilibrium as the unique stable equilibrium of the system and DFE is a saddle point. For instance, let $b_{AA} = b_{AI} = b_{II} = p = q = 0, b_{SS} = b_{AS} = b_{IS} = 0.0275, \alpha = \beta = 0.0009, \delta = 0.2735, \varepsilon = 0.1125, \lambda_I = 0.9196, \gamma_A = 0.0468, \gamma_I = 0.0093$ and $\lambda_A = 0.45$ (Figure 9).

We fix all the parameters and vary $\lambda_A$ as the control parameter. If we decrease $\lambda_A$, then a saddle-node bifurcation occurs by passing through the bifurcation value $\lambda_A = 0.42$ (Figure 10(a)), and the system becomes bistable with two stable endemic equilibria and a saddle endemic equilibrium (Figure 10(b)).

Notice that DFE is still a saddle point. Further decrease in $\lambda_A$ causes one of the stable endemic equilibria to get close to DFE and they finally collide at $\lambda_A = 0.3465$ and a transcritical bifurcation occurs (Figure 11(a)). By passing $\lambda_A$ through this bifurcation value, DFE becomes stable (Figure 11(b)).

If we decrease $\lambda_A$, then the system undergoes saddle-node bifurcation at $\lambda_A = 0.1$ (Figure 12(a)) and then the stable endemic equilibrium annihilates. The system is monostable and DFE is globally stable for $\lambda_A < 0.1$ (Figure 12(b)).
Figure 9. The endemic equilibrium is globally stable. Circle shows stable point.

Figure 10. (a) Saddle-node bifurcation at $\lambda_A = 0.42$. (b) The system is bistable when $\lambda_A = 0.36$.

Figure 11. (a) Transcritical bifurcation at $\lambda_A = 0.3465$. (b) DFE is stable together with an endemic equilibrium when $\lambda_A = 0.2$. 
Figure 12. (a) Saddle-node bifurcation at $\lambda_A = 0.1$. (b) DFE is globally stable when $\lambda_A = 0.08$.

For the above example we choose an initial condition $(x_0 = (0.1, 0.8))$ in $\text{int}(D_1)$ for $\lambda_A = 0.45$. Starting from $x_0$ the solution tends to the stable endemic equilibrium since this equilibrium is globally stable. If we decrease $\lambda_A$ to $\lambda_A = 0.08$, then starting from the current state of the system the solution to the system tends to DFE for sufficiently large amounts of time, for instance $t = 410$. Now we increase $\lambda_A$ to $\lambda_A = 0.34$. By passing $\lambda_A$ through the bifurcation value $\lambda_A = 0.1$, a saddle-node bifurcation occurs and then the system becomes bistable with a stable endemic equilibrium and a stable DFE. But the state of the system is now in the basin of attraction of stable DFE. Hence, the solution remains around DFE (Figure 13). In this figure we assume that $\lambda_A$ is

Figure 13. By changing $\lambda_A$, the hysteresis behavior is observed.
a function of the variable \( t \), 
\[
\lambda_A(t) = (0.0175)(|t - 220| - |t - 210| + |t - 350| - |t - 360|) + (0.0045)(|t - 360| - |t - 350|) + 0.385.
\]
Similar changes in the parameters are expected in most of short-time health propaganda programmes. During the programme, the transmission parameters decrease and after that it increases up to a level which is less than its initial value.

Therefore, by decreasing the parameter \( \lambda_A \) for a short time we can control spreading of the disease such that every solution tends to the DFE.

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