Miura transformations for Toda–type integrable systems, with applications to the problem of integrable discretizations

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Abstract. We study lattice Miura transformations for the Toda and Volterra lattices, relativistic Toda and Volterra lattices, and their modifications. In particular, we give three successive modifications for the Toda lattice, two for the Volterra lattice and for the relativistic Toda lattice, and one for the relativistic Volterra lattice. We discuss Poisson properties of the Miura transformations, their permutability properties, and their role as localizing changes of variables in the theory of integrable discretizations.
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## Relativistic Toda lattice

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### 12.1 Equations of motion

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### 12.4 Application: localizing change of variables for dMRTL(\(+\))\((\alpha; \epsilon)\)

## Relativistic Volterra lattice

### 13.1 Equations of motion

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1 Introduction

1.1 The original Miura transformation

This paper is devoted to lattice Miura transformations. Let us start with recalling the basic facts about the original Miura transformation, as applied to the Korteweg–de Vries (KdV) equation.

The KdV equation,
\[ u_t = 6uu_x - u_{xxx} , \] (1.1)
is a bi–Hamiltonian system, i.e. admits two Hamiltonian formulations with respect to two compatible local Poisson brackets:
\[ \{ F, G \}_1 = \int \frac{\delta F}{\delta u} J_1 \frac{\delta G}{\delta u} dx , \quad J_1 = \partial , \] (1.2)
and
\[ \{ F, G \}_2 = \int \frac{\delta F}{\delta u} J_2 \frac{\delta G}{\delta u} dx , \quad J_2 = -\partial^3 + 2u\partial + 2\partial u . \] (1.3)

In other words, KdV may be represented in the Hamiltonian form in two different ways:
\[ u_t = J_1(3u^2 - u_{xx}) = J_1 \frac{\delta}{\delta u} \left( \frac{1}{2} u_x^2 + u^3 \right) , \] (1.4)
\[ = J_2(u) = J_2 \frac{\delta}{\delta u} \left( \frac{1}{2} u_x^2 \right) . \] (1.5)

The Miura transformation with parameter (also known as the Gardner transformation) reads:
\[ u = w + \epsilon w_x + \epsilon^2 w^2 . \] (1.6)

Let us list its most important properties. The Miura transformation is actually a differential substitution and therefore, generally speaking, noninvertible. Therefore nothing guarantees \textit{á priori} that there exists a differential equation for \( w \) which is pushed to (1.1) by this transformation. Nevertheless, this is the case, and the corresponding equation is the \textit{modified Korteweg–de Vries equation} MKdV(\( \epsilon \)):
\[ w_t = 6w(1 + \epsilon^2 w)w_x - w_{xxx} . \] (1.7)

Nothing guarantees \textit{á priori} also that there exist local Poisson brackets in the \( w \)–space pushed by (1.6) to either of the brackets (1.2), (1.3). Indeed, this is not the case. However, this is the case for a certain linear combination of these brackets. Namely,
\[ \{ \cdot, \cdot \}_1 + \epsilon^2 \{ \cdot, \cdot \}_2 \]
is the push–forward under (1.6) of the bracket
\[ \{ F, G \} = \int \frac{\delta F}{\delta w} J_1 \frac{\delta G}{\delta w} dx \] (1.8)
with the same \( J_1 = \partial \) as before. Correspondingly, the equation \( \text{MKdV}(\epsilon) \) is Hamiltonian with respect to the latter bracket, i.e. it may be presented as

\[
\frac{dw}{dt} = J_1 \left( 3w^2 + 2\epsilon^2 w^3 - w_{xx} \right) = J_1 \frac{\delta}{\delta w} \left( \frac{1}{2} w^2 + \frac{1}{2} \epsilon^2 w^4 \right). 
\]  

(1.9)

One can invert (1.6) formally, arriving at the formal power series

\[
w = u - \epsilon u_x + \epsilon^2 (u_{xx} - u^2) + \ldots.
\] 

(1.10)

Since \( w \) is trivially a density of a conservation law for \( \text{MKdV}(\epsilon) \), we get an infinite series of conservation laws densities for KdV, as coefficients of the above power series. The coefficients by odd powers of \( \epsilon \) deliver trivial conservation laws (as \( u_x \) by the first power of \( \epsilon \)), but the coefficients by even powers give nontrivial ones (as \( u, u^2 - u_{xx} \sim u^2, \ldots \)).

Finally, it should be mentioned that the origin of the Miura transformation (1.6) lies in the factorization of Lax operators. Indeed, it is well known that the Schrödinger operator

\[
\partial^2 - u
\]

is a Lax operator for KdV. Now the formula (1.6) is equivalent to the following factorization:

\[
1 - 4\epsilon^2 (\partial^2 - u) = (1 + 2\epsilon \partial + 2\epsilon^2 w)(1 - 2\epsilon \partial + 2\epsilon^2 w).
\] 

(1.11)

1.2 Lattice Miura transformations

The main aim of this paper is to collect a huge bulk of variable transformations for lattice systems, which can be considered as analogs of the Miura transformation, and to point out a remarkable similarity of their properties with the above mentioned ones.

We consider in this paper integrable lattice systems with local interactions:

\[
\dot{x}_k = f_{k \mod m}(x_{k-s}, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_{k+s})
\] 

(1.12)

with a fixed \( s \in \mathbb{N} \). They may be considered either on an infinite lattice \( (k \in \mathbb{Z}) \), or on a periodic one \( (k \in \mathbb{Z}/N\mathbb{Z}) \). The number \( m \in \mathbb{N} \) is called the number of fields. It is important that the number \( s \), measuring the locality of interaction, is one and the same for all \( k \) (and hence is independent on the total number \( N \) of particles, or lattice sites, in the periodic case).

As two simple examples of such systems, consider the Toda lattice:

\[
\dot{b}_k = a_k - a_{k-1}, \quad \dot{a}_k = a_k(b_{k+1} - b_k),
\] 

(1.13)

and the relativistic Toda lattice:

\[
\dot{b}_k = (1 + \alpha b_k)(a_k - a_{k-1}), \quad \dot{a}_k = a_k(b_{k+1} + \alpha a_{k+1} - b_k - \alpha a_{k-1}).
\] 

(1.14)

Both these systems are two–field, i.e. they have the form (1.12) with \( m = 2 \) (to see this, set \( b_k = x_{2k-1}, a_k = x_{2k} \). For the Toda lattice we have \( s = 1 \), while for the relativistic Toda lattice \( s = 2 \). These two will be among our basic examples considered in detail in the main text. Other examples include the Volterra lattice and the relativistic Volterra lattice, and also modifications of these systems.
The notion of modification is intimately related to the notion of Miura transformations. We do not give here a general definition of a Miura transformation, but instead briefly summarize the properties of the Miura transformations studied in this paper. It turns out that for all our systems (1.12) the corresponding Miura transformations appear in pairs. One of these maps always has the form

\[ M_1(\epsilon) : \quad x_k = y_k + \epsilon \Phi_{k \text{mod} m}(y_{k-s}, \ldots, y_{k-1}, y_k; \epsilon), \quad (1.15) \]

while the second one always has the form

\[ M_2(\epsilon) : \quad x_k = y_k + \epsilon \Psi_{k \text{mod} m}(y_k, y_{k+1}, \ldots, y_{k+s}; \epsilon). \quad (1.16) \]

The dependence on the (small) parameter \( \epsilon \) is analytic in some neighborhood of zero. Moreover, there holds the following property:

\[
\Psi_{k \text{mod} m}(x_k, x_{k+1}, \ldots, x_{k+s}; 0) - \Phi_{k \text{mod} m}(x_k, x_{k+1}, \ldots, x_{k+s}; 0) =
\]

\[
= f_{k \text{mod} m}(x_{k-s}, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_{k+s}). \quad (1.17)
\]

For instance, the Miura transformations for the Toda lattice (1.13) read:

\[
M_1(\epsilon) : \begin{cases} 
  b_k = p_k + \epsilon q_{k-1}, \\
  a_k = q_k(1 + \epsilon p_k), 
\end{cases} \quad M_2(\epsilon) : \begin{cases} 
  b_k = p_k + \epsilon q_k, \\
  a_k = q_k(1 + \epsilon p_{k+1}). 
\end{cases} \quad (1.18)
\]

So, the Miura maps are always given by local formulas. It is easy to understand that in the case of infinite lattices such maps are noninvertible. Of course, in the periodic case the inverse maps always exist, but are given in general by nonlocal formulas. The (small) parameter \( \epsilon \) is called a modification parameter. The remarkable properties of the Miura maps may be summarized as follows.

First, nothing guarantees a priori the existence of a local system of the type (1.12) in the variables \( y_k \) which is pushed to the original system (1.12) by a map of the type (1.13) or (1.16). As we shall be mostly working with the periodic case, we express the above statement also in this way: nothing guarantees a priori that the pull–back of the differential equations of motion (1.12) under the change of variables (1.13) or (1.16) will be given again by local formulas. However, this is the case for the Miura maps, and can be accepted as their (somewhat informal) definition. Of course, the very existence of such Miura maps is rather mystifying. The above mentioned local systems in \( y_k \) form one–parameter families of integrable deformations, or modifications, of lattice systems. So, the modified system (1.12) has the form

\[ \dot{y}_k = F_{k \text{mod} m}(y_{k-s}, \ldots, y_{k-1}, y_k, y_{k+1}, \ldots, y_{k+s}; \epsilon), \quad (1.19) \]

where

\[ F_{k \text{mod} m}(x_{k-s}, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_{k+s}; 0) = f_{k \text{mod} m}(x_{k-s}, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_{k+s}). \quad (1.20) \]

For instance, the equations of the modified Toda lattice read:

\[ \dot{p}_k = (1 + \epsilon p_k)(q_k - q_{k-1}), \quad \dot{q}_k = q_k(p_{k+1} - p_k). \quad (1.21) \]
Further, integrable lattice systems (1.12) often admit a Hamiltonian formulation, the corresponding invariant Poisson bracket \( \{ \cdot , \cdot \} \) being given by local formulas. That means that
\[
\{ x_j , x_k \} = 0
\]
for \( |j-k| \) large enough. Nothing guarantees a priori that the pull–backs of these brackets under the Miura maps (1.13), (1.16) are also given by local formulas. Indeed, as a rule these pull–backs are non–local. However, in the multi–Hamiltonian case, when (1.12) is Hamiltonian with respect to several compatible local Poisson brackets, it turns out that pull–backs of certain linear combinations of these brackets are local again. These pull–backs serve then as invariant local Poisson brackets in the Hamiltonian formulation of the modified systems (1.19). A general observation is: the number of invariant local Poisson brackets for a modified system is, as a rule, by one less than for the original one. For instance, the Toda lattice (1.13) is a tri–Hamiltonian system, and the modified Toda lattice (1.21) is a bi–Hamiltonian system.

The integrals of motion for the modified systems are always easily obtainable as compositions of the Miura maps with the integrals of motion for the original systems. Interestingly, just one integral of the modified system often allows to find the whole series of integrals for the original one. To this end one has to invert the Miura maps formally, as power series in \( \epsilon \). For example, the formal inversion of \( M_1(\epsilon) \) for the Toda lattice gives:

\[
\begin{align*}
p_k &= b_k - \epsilon a_{k-1} + \epsilon^2 b_{k-1} a_{k-1} - \epsilon^3 (b_{k-1}^2 a_{k-1} + a_{k-1} a_{k-2}) + \ldots \\
q_k &= a_k - \epsilon b_k a_k + \epsilon^2 (b_k^2 a_k + a_k a_{k-1}) - \ldots
\end{align*}
\]

Now it is easy to see that \( \log(1 + \epsilon p_k) \) is a density of a conservation law for the modified Toda lattice. The expansion of this density in powers of \( \epsilon \) reads:
\[
\log(1 + \epsilon p_k) = \epsilon b_k - \epsilon^2 \left( \frac{1}{2} b_k^2 + a_{k-1} \right) + \epsilon^3 \left( \frac{1}{3} b_k^3 + b_k a_{k-1} + b_{k-1} a_{k-1} \right) - \ldots. \tag{1.22}
\]

The coefficient by \( \epsilon^n \) here is nothing but the density of the \( n \)th conservation law for the Toda lattice. This is, probably, the most direct way to demonstrate the existence of an infinite series of integrals for this system.

Finally, we point out that the Miura maps turn out to provide localizing changes of variables for integrable discretizations. We refer the reader to the review [S] for a general construction of integrable discretizations for lattice systems, based on Lax representations and their \( r \)–matrix interpretation. We point out here that this construction leads, as a rule, to discretizations governed by nonlocal equations. For instance, equations of motion of the discrete time Toda lattice include continued fractions:

\[
\begin{align*}
\tilde{b}_k &= b_k + h \left( \frac{a_k}{\beta_k} - \frac{a_{k-1}}{\beta_{k-1}} \right), & \tilde{a}_k &= a_k \frac{\beta_{k+1}}{\beta_k},
\end{align*}
\]

where
\[
\beta_k = 1 + h b_k - \frac{h^2 a_{k-1}}{1 + h b_{k-1} - \frac{h^2 a_{k-2}}{1 + h b_{k-2} - \ldots}}.
\]

One way to repair this drawback is based on the notion of localizing changes of variables and will be one of our main themes here. It turns out that the change of variables \( \mathcal{M}_1(h) \),
\[
x_k = \mathbf{x}_k + h \Phi_{k \text{mod} \, m}(\mathbf{x}_{k-s}, \ldots, \mathbf{x}_{k-1}, \mathbf{x}_k; h), \tag{1.24}
\]

...
conjugates our discretizations with the maps described by the *local* formulas:

\[ \tilde{x}_k + h \Phi_{k \mod m}(\tilde{x}_{k-s}, \ldots, \tilde{x}_{k-1}, \tilde{x}_k; h) = x_k + h \Psi_{k \mod m}(x_k, x_{k+1}, \ldots, x_{k+s}; h) . \tag{1.25} \]

So, the local discretizations belong to the modified hierarchies, with the value of the modification parameter equal to the time step of the discretization. For instance, for the Toda lattice case the Miura map

\[
M_1(h) : \begin{cases} 
  b_k = b_k + h a_{k-1} , \\
  a_k = a_k (1 + h b_k) ,
\end{cases}
\]

conjugates the discrete time Toda lattice (1.23) with the following map belonging to the modified Toda lattice hierarchy:

\[
\tilde{b}_k + h \tilde{a}_{k-1} = b_k + h a_k , \quad \tilde{a}_k (1 + h \tilde{b}_k) = a_k (1 + h b_{k+1}) . \tag{1.26}
\]

Such implicit local discretizations are much more satisfying from the esthetical point of view and are much better suited for the purposes of numerical simulation. (If, for instance, one uses the Newton’s iterative method to solve (1.25) for \( \tilde{x} \), then one has to solve only linear systems whose matrices are triangular and have a band structure, i.e. only \( s \) nonzero diagonals).

In the main body of this paper we provide Miura transformations and formulate their properties for a wide set of Toda–like systems. After the results are found and formulated, it is rather straightforward to prove them, therefore all proofs are omitted. For the notions and notations concerning the Lax representations, the reader should consult [S] (including them here would unnecessarily increase the length of the paper, which is already much longer than the author would like it). Many of the results about the nonrelativistic Toda–like already appeared in the literature, while the most of the results concerning the relativistic Toda–like systems seem to be original. The paper ends with some bibliographical remarks.
Part I
Nonrelativistic systems

2 Toda lattice

2.1 Equations of motion

The system known as the *Toda lattice*, abbreviated as TL, lives on the space $\mathcal{T} = \mathbb{R}^{2N}(a, b)$ and is described by the following equations of motion:

$$
\dot{b}_k = a_k - a_{k-1}, \quad \dot{a}_k = a_k(b_{k+1} - b_k). \tag{2.1}
$$

2.2 Tri–Hamiltonian structure

We adopt once and forever the following conventions: the Poisson brackets are defined by writing down *all nonvanishing* brackets between the coordinate functions; the indices in the corresponding formulas are taken (mod $N$).

**Proposition 2.1**

a) The relations

$$
\{b_k, a_k\}_1 = -a_k, \quad \{a_k, b_{k+1}\}_1 = -a_k \tag{2.2}
$$

define a Poisson bracket on $\mathcal{T}$. The system TL is a Hamiltonian system on $(\mathcal{T}, \{\cdot, \cdot\}_1)$ with the Hamilton function

$$
H_2(a, b) = \frac{1}{2} \sum_{k=1}^{N} b_k^2 + \sum_{k=1}^{N} a_k. \tag{2.3}
$$

b) The relations

$$
\{b_k, a_k\}_2 = -b_k a_k, \quad \{a_k, b_{k+1}\}_2 = -a_k b_{k+1}, \\
\{b_k, b_{k+1}\}_2 = -a_k, \quad \{a_k, a_{k+1}\}_2 = -a_k a_{k+1} \tag{2.4}
$$

define a Poisson bracket on $\mathcal{T}$ compatible with the linear bracket $\{\cdot, \cdot\}_1$. The system TL is a Hamiltonian system on $(\mathcal{T}, \{\cdot, \cdot\}_2)$ with the Hamilton function

$$
H_1(a, b) = \sum_{k=1}^{N} b_k. \tag{2.5}
$$

c) The relations

$$
\{b_k, a_k\}_3 = -a_k(b_k^2 + a_k), \quad \{a_k, b_{k+1}\}_3 = -a_k(b_{k+1}^2 + a_k), \\
\{b_k, b_{k+1}\}_3 = -a_k(b_k + b_{k+1}), \quad \{a_k, a_{k+1}\}_3 = -2a_k a_{k+1} b_{k+1}, \\
\{b_k, a_{k+1}\}_3 = -a_k a_{k+1}, \quad \{a_k, b_{k+2}\}_3 = -a_k a_{k+1} \tag{2.6}
$$

define a Poisson bracket on $\mathcal{T}$ compatible with $\{\cdot, \cdot\}_1$ and $\{\cdot, \cdot\}_2$. The system TL is a Hamiltonian system on $(\mathcal{T}, \{\cdot, \cdot\}_3)$ with the Hamilton function

$$
H_0(a, b) = \frac{1}{2} \sum_{k=1}^{N} \log(a_k). \tag{2.7}
$$
2.3 Lax representation
Lax matrix $T : T \mapsto g$:

$$T(a, b, \lambda) = \lambda^{-1} \sum_{k=1}^{N} a_k E_{k,k+1} + \sum_{k=1}^{N} b_k E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} .$$

(2.7)

Lax representation for TL:

$$\dot{T} = [T, B] = [A, T] ,$$

(2.8)

where

$$B = \pi_+(T) = \sum_{k=1}^{N} b_k E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} ,$$

(2.9)

$$A = \pi_-(T) = \lambda^{-1} \sum_{k=1}^{N} a_k E_{k,k+1} .$$

(2.10)

2.4 Discretization
Lax representation for the map dTL:

$$\tilde{T} = B^{-1}TB = ATA^{-1}$$

(2.11)

with

$$B = \Pi_+(I + hT) = \sum_{k=1}^{N} \beta_k E_{kk} + h \lambda \sum_{k=1}^{N} E_{k+1,k} ,$$

(2.12)

$$A = \Pi_-(I + hT) = I + h\lambda^{-1} \sum_{k=1}^{N} \alpha_k E_{k,k+1} .$$

(2.13)

3 Modified Toda lattice

3.1 Equations of motion
The phase space of the modified Toda lattice $\text{MTL}(\epsilon)$ will be denoted by $\mathcal{MT} = \mathbb{R}^{2N}(q, r)$, the parameter $\epsilon$ will be called the modification parameter. Equations of motion of $\text{MTL}(\epsilon)$ read:

$$\dot{p}_k = (1 + \epsilon p_k)(q_k - q_{k-1}) , \quad \dot{q}_k = q_k(p_{k+1} - p_k) .$$

(3.1)

3.2 Bi–Hamiltonian structure

Proposition 3.1 a) The relations

$$\{p_k, q_k\}_{12} = -q_k(1 + \epsilon p_k) , \quad \{q_k, p_{k+1}\}_{12} = -q_k(1 + \epsilon p_{k+1})$$

(3.2)

define a Poisson bracket on $\mathcal{MT}$. The system $\text{MTL}(\epsilon)$ is Hamiltonian with respect to this bracket, with the Hamilton function

$$G_1(q, p) = \epsilon^{-1} \sum_{k=1}^{N} p_k + \sum_{k=1}^{N} q_k .$$

(3.3)
b) The relations
\[
\{p_k, q_k\}_{23} = -q_k(p_k + \epsilon q_k)(1 + \epsilon p_k),
\]
\[
\{q_k, p_{k+1}\}_{23} = -q_k(p_{k+1} + \epsilon q_k)(1 + \epsilon p_{k+1}),
\]
\[
\{p_k, p_{k+1}\}_{23} = -q_k(1 + \epsilon p_k)(1 + \epsilon p_{k+1}),
\]
\[
\{q_k, q_{k+1}\}_{23} = -q_k q_{k+1}(1 + \epsilon p_{k+1})
\]
(3.4)
define a Poisson bracket on $\mathcal{MT}$ compatible with (3.2). The system $\mathcal{MTL}(\epsilon)$ is Hamiltonian with respect to this bracket, with the Hamilton function
\[
G_0(q, p) = \epsilon^{-1} \sum_{k=1}^{N} \log(1 + \epsilon p_k).
\]
Notice that the function $G_1(q, p)$ is singular in $\epsilon$; it may be done regular, and, moreover, an $O(\epsilon)$–perturbation of $H_2(q, p)$ from (2.3), by subtracting
\[
\epsilon^{-2} \sum_{k=1}^{N} \log(1 + \epsilon p_k),
\]
which is a Casimir function of the bracket $\{\cdot, \cdot\}_{12}$. 

3.3 Miura relations

**Theorem 3.2** Define the Miura maps $M_{1,2}(\epsilon) : \mathcal{MT}(q, p) \mapsto \mathcal{T}(a, b)$ by:
\[
M_1(\epsilon) : \begin{cases} 
    b_k = p_k + \epsilon q_{k-1}, \\
    a_k = q_k(1 + \epsilon p_k), 
\end{cases}
\]
\[
M_2(\epsilon) : \begin{cases} 
    b_k = p_k + \epsilon q_k, \\
    a_k = q_k(1 + \epsilon p_{k+1}). 
\end{cases}
\]
(3.6)
Both maps $M_{1,2}(\epsilon)$ are Poisson, if $\mathcal{MT}(q, p)$ is equipped with the bracket $\{\cdot, \cdot\}_{12}$, and $\mathcal{T}(a, b)$ is equipped with $\{\cdot, \cdot\}_1 + \epsilon\{\cdot, \cdot\}_2$, and also if $\mathcal{MT}(q, p)$ is equipped with the bracket $\{\cdot, \cdot\}_{23}$, and $\mathcal{T}(a, b)$ is equipped with $\{\cdot, \cdot\}_2 + \epsilon\{\cdot, \cdot\}_3$. The pull–back of the flow $\mathcal{T}_L$ under either of the Miura maps $M_{1,2}(\epsilon)$ coincides with $\mathcal{MTL}(\epsilon)$. 

3.4 Lax representation

Lax matrix $(P, Q) : \mathcal{MT} \mapsto \mathfrak{g} \otimes \mathfrak{g}$:
\[
P(q, p, \lambda) = \sum_{k=1}^{N} (1 + \epsilon p_k)E_{kk} + \epsilon \lambda \sum_{k=1}^{N} E_{k+1,k},
\]
\[
Q(q, p, \lambda) = I + \epsilon \lambda^{-1} \sum_{k=1}^{N} q_k E_{k,k+1}.
\]
(3.7) (3.8)
Notice that the formulas for the Miura map $M_1(\epsilon)$ are equivalent to the factorization
\[
I + \epsilon T(a, b, \lambda) = P(q, p, \lambda)Q(q, p, \lambda),
\]
(3.9)
while the formulas for the Miura map $M_2(\epsilon)$ are equivalent to the factorization
\[
I + \epsilon T(a, b, \lambda) = Q(q, p, \lambda)P(q, p, \lambda).
\]
(3.10)
Lax representation for $\text{MTL}(\epsilon)$:

$$
\begin{align*}
\dot{P} &= PB_2 - B_1 P = A_1 P - P A_2 , \\
\dot{Q} &= QB_1 - B_2 Q = A_2 Q - Q A_1 ,
\end{align*}
$$

(3.11)

where

$$
B_1 = \pi_+ \left( (PQ - I) / \epsilon \right) = \sum_{k=1}^{N} (p_k + \epsilon q_{k-1}) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} ,
$$

(3.12)

$$
B_2 = \pi_+ \left( (QP - I) / \epsilon \right) = \sum_{k=1}^{N} (p_k + \epsilon q_k) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} ,
$$

(3.13)

$$
A_1 = \pi_- \left( (PQ - I) / \epsilon \right) = \lambda^{-1} \sum_{k=1}^{N} q_k (1 + \epsilon p_k) E_{k,k+1} ,
$$

(3.14)

$$
A_2 = \pi_- \left( (QP - I) / \epsilon \right) = \lambda^{-1} \sum_{k=1}^{N} q_k (1 + \epsilon p_{k+1}) E_{k,k+1} .
$$

(3.15)

The formulas (3.12), (3.13) have to be compared with (2.9) and the definition of the Miura maps $M_{1,21}(\epsilon)$, while the formulas (3.14), (3.15) have to be compared with (2.10).

3.5 Discretization

Lax representation for the map $d\text{MTL}(\epsilon)$:

$$
\bar{P} = B_1^{-1} P B_2 = A_1 P A_2^{-1} , \quad \bar{Q} = B_2^{-1} Q B_1 = A_2 Q A_1^{-1}
$$

(3.16)

with

$$
B_1 = \Pi_+ \left( I + \frac{h}{\epsilon} (PQ - I) \right) = \sum_{k=1}^{N} \beta_k^{(1)} E_{kk} + h \lambda \sum_{k=1}^{N} E_{k+1,k} ,
$$

(3.17)

$$
B_2 = \Pi_+ \left( I + \frac{h}{\epsilon} (QP - I) \right) = \sum_{k=1}^{N} \beta_k^{(2)} E_{kk} + h \lambda \sum_{k=1}^{N} E_{k+1,k} ,
$$

(3.18)

$$
A_1 = \Pi_- \left( I + \frac{h}{\epsilon} (PQ - I) \right) = I + h \lambda^{-1} \sum_{k=1}^{N} \alpha_k^{(1)} E_{k,k+1} ,
$$

(3.19)

$$
A_2 = \Pi_- \left( I + \frac{h}{\epsilon} (QP - I) \right) = I + h \lambda^{-1} \sum_{k=1}^{N} \alpha_k^{(2)} E_{k,k+1} .
$$

(3.20)

3.6 Application: localizing change of variables for $d\text{TL}$

Consider again the discretization $d\text{TL}$ of the Toda lattice. Compare the formula

$$
I + h T = B A ,
$$
following from (2.12), (2.13), with (3.9). This comparison shows immediately that the Miura map $M_1(h)$ plays the role of the localizing change of variables for $dTL$. Indeed, define the change of variables $\mathcal{T}(a, b) \mapsto \mathcal{T}(a, b)$ by the formulas:

$$M_1(h) : \begin{cases} b_k = b_k + h a_{k-1} , \\ a_k = a_k (1 + h b_k) . \end{cases}$$ (3.21)

Then we find:

$$B = P(a, b, \lambda) = \sum_{k=1}^{N} (1 + h b_k) E_{kk} + h \lambda \sum_{k=1}^{N} E_{k+1,k} ,$$

$$A = Q(a, b, \lambda) = I + h \lambda^{-1} \sum_{k=1}^{N} a_k E_{k,k+1} ,$$

so that we immediately get local expressions for the entries of the factors $B, A$:

$$\beta_k = 1 + h b_k , \quad \alpha_k = a_k .$$ (3.22)

**Theorem 3.3** The change of variables (3.21) conjugates the map $dTL$ with the map on $\mathcal{T}(a, b)$ described by the following local equations of motion:

$$\tilde{b}_k + h \tilde{a}_{k-1} = b_k + h a_k , \quad \tilde{a}_k (1 + h \tilde{b}_k) = a_k (1 + h b_{k+1}) .$$ (3.23)

So, the local form of $dTL$ (3.23) actually belongs to the hierarchy $MTL(h)$.

**Corollary.** The local form of $dTL$ (3.23) is Poisson with respect to the following brackets on $\mathcal{T}(a, b)$:

$$\{ b_k, a_k \} = -a_k (1 + h b_k) , \quad \{ a_k, b_{k+1} \} = -a_k (1 + h b_{k+1}) ,$$ (3.24)

which is the pull–back of the bracket

$$\{ \cdot, \cdot \}_1 + h \{ \cdot, \cdot \}_2$$ (3.25)

on $\mathcal{T}(a, b)$ under the change of variables (3.21), and

$$\{ b_k, a_k \} = -a_k (b_k + h a_k) (1 + h b_k) ,$$

$$\{ a_k, b_{k+1} \} = -a_k (b_{k+1} + h a_k) (1 + h b_{k+1}) ,$$

$$\{ b_k, b_{k+1} \} = -a_k (1 + h b_k) (1 + h b_{k+1}) ,$$

$$\{ a_k, a_{k+1} \} = -a_k a_{k+1} (1 + h b_{k+1}) ,$$ (3.26)

which is the pull–back of the bracket

$$\{ \cdot, \cdot \}_2 + h \{ \cdot, \cdot \}_3$$ (3.27)

on $\mathcal{T}(a, b)$ under the change of variables (3.21).
4 Double modified Toda lattice

4.1 Equations of motion

The phase space of the double modified Toda lattice $M^2TL(\epsilon, \delta)$ will be denoted by $M^2T = \mathbb{R}^{2N}(r, s)$. The parameters $\epsilon, \delta$ will be called the modification parameters. The equations of motion read:

\[
\dot{r}_k = (1 + \epsilon r_k)(1 + \delta r_k)(s_k - s_{k-1}), \quad \dot{s}_k = s_k(1 + \epsilon \delta s_k)(r_{k+1} - r_k).
\]

Obviously, the parameters $\epsilon, \delta$ enter into the system $M^2TL(\epsilon, \delta)$ symmetrically.

4.2 Hamiltonian structure

Proposition 4.1 The relations

\[
\{r_k, s_k\}_{123} = -s_k(1 + \epsilon \delta s_k)(1 + \epsilon r_k)(1 + \delta r_k),
\]

\[
\{s_k, r_{k+1}\}_{123} = -s_k(1 + \epsilon \delta s_k)(1 + \epsilon r_{k+1})(1 + \delta r_{k+1})
\]

define a Poisson bracket on $M^2T$. The system $M^2TL(\epsilon, \delta)$ is Hamiltonian with respect to this bracket, with the Hamilton function

\[
F_0(r, s) = (\epsilon \delta)^{-1} \sum_{k=1}^{N} \log(1 + \delta r_k) + (\epsilon \delta)^{-1} \sum_{k=1}^{N} \log(1 + \epsilon \delta s_k).
\]

The Hamilton function $F_0(r, s)$ is given in the form in which the $\delta \to 0$ limit is transparent. By adding

\[
(2\epsilon \delta)^{-1} \sum_{k=1}^{N} \log \frac{1 + \epsilon r_k}{1 + \delta r_k},
\]

which is a Casimir function for the bracket $\{\cdot, \cot\}_{123}$, we find an equivalent Hamilton function symmetric in $\epsilon, \delta$.

4.3 Miura relations

Theorem 4.2 Define the Miura maps $M_{1,2}(\epsilon; \delta) : M^2T(r, s) \mapsto \mathcal{M}T(q, p)$ by:

\[
M_1(\epsilon; \delta) : \begin{cases} p_k = r_k + \delta s_{k-1}(1 + \epsilon r_k), \\ q_k = s_k(1 + \delta r_k), \end{cases} \quad M_2(\epsilon; \delta) : \begin{cases} p_k = r_k + \delta s_{k}(1 + \epsilon r_k), \\ q_k = s_k(1 + \delta r_{k+1}). \end{cases}
\]

Both maps $M_{1,2}(\epsilon; \delta)$ are Poisson, if $M^2T(y, z)$ is equipped with the bracket $\{\cdot, \cdot\}_{123}$, and $\mathcal{M}T(q, r)$ is equipped with $\{\cdot, \cdot\}_{12} + \delta\{\cdot, \cdot\}_{23}$. The pull–back of the flow $M^2TL(\epsilon)$ under each one of the Miura maps $M_{1,2}(\epsilon; \delta)$ coincides with $M^2TL(\epsilon, \delta)$.

Note that introducing the Miura maps $M_{1,2}(\epsilon; \delta)$ we actually break the symmetry of $M^2TL(\epsilon, \delta)$ in the modification parameters. This symmetry may be explained with the help of the following statement concerning the permutability of the Miura maps.
Theorem 4.3 The following diagram is commutative for all \( i = 1, 2, j = 1, 2 \):

\[
\begin{array}{ccc}
\mathcal{M}^2 \mathcal{T} & \xrightarrow{M_j(\delta; \epsilon)} & \mathcal{M} \mathcal{T} \\
\downarrow M_i(\epsilon; \delta) & & \downarrow M_i(\delta) \\
\mathcal{M} \mathcal{T} & \xrightarrow{M_j(\epsilon)} & \mathcal{T}
\end{array}
\]

In particular, the compositions \( M_j(\epsilon) \circ M_j(\epsilon; \delta) \) for \( j = 1, 2 \) have to be symmetric in \( \epsilon, \delta \). Indeed, we have:

\[
M_1(\epsilon) \circ M_1(\epsilon; \delta) : \begin{cases}
b_k = r_k + (\epsilon + \delta)s_{k-1} + \epsilon \delta s_{k-1}(r_{k-1} + r_k), \\
a_k = s_k(1 + \epsilon r_k)(1 + \delta r_k)(1 + \epsilon \delta s_{k-1})
\end{cases}
\quad (4.5)
\]

\[
M_2(\epsilon) \circ M_2(\epsilon; \delta) : \begin{cases}
b_k = r_k + (\epsilon + \delta)s_k + \epsilon \delta s_k(r_k + r_{k+1}) , \\
a_k = s_k(1 + \epsilon r_{k+1})(1 + \delta r_{k+1})(1 + \epsilon \delta s_{k+1}) \, .
\end{cases}
\quad (4.6)
\]

This explains also the symmetry of the Poisson bracket \( \{\cdot, \cdot\}_{123} \) (4.2), since the above compositions are Poisson maps between \( \mathcal{M}^2 \mathcal{T}(r, s) \) carrying this bracket, and \( \mathcal{T}(a, b) \) carrying the bracket

\[\{\cdot, \cdot\}_1 + (\epsilon + \delta)\{\cdot, \cdot\}_2 + \epsilon \delta \{\cdot, \cdot\}_3 .\]

4.4 Lax representation and discretization

Unfortunately, we do not know any interpretation of the Miura maps (4.4) in terms of factorization of Lax matrices. The only way we can define the discretization \( d\mathcal{M}^2 \mathcal{T}(\epsilon, \delta) \) is to pull–back the map \( d\mathcal{M} \mathcal{T}(\epsilon) \) under either of the Miura transformations \( M_{1,2}(\epsilon; \delta) \), or to pull–back the map \( d\mathcal{M} \mathcal{T}(\delta) \) under either of the Miura transformations \( M_{1,2}(\delta; \epsilon) \).

4.5 Application: localizing change of variables for \( d\mathcal{M} \mathcal{T}(\epsilon) \)

It turns out that the Miura map \( M_1(\epsilon; h) \) plays the role of the localizing change of variables for \( d\mathcal{M} \mathcal{T}(\epsilon) \). To demonstrate this, consider the following change of variables \( \mathcal{M} \mathcal{T}(q, p) \mapsto \mathcal{M} \mathcal{T}(q, p) \):

\[
M_1(\epsilon; h) : \begin{cases}
p_k = p_k + hq_{k-1}(1 + \epsilon p_k) \\
q_k = q_k(1 + h p_k)
\end{cases}
\quad (4.7)
Then the following expressions may be found for the entries of the factors \( B_j, A_j \) \((j = 1, 2)\):

\[
\beta_k^{(1)} = (1 + h\mathbf{p}_k)(1 + h\epsilon\mathbf{q}_{k-1}), \quad \alpha_k^{(1)} = \mathbf{q}_k(1 + \epsilon\mathbf{p}_k), \quad (4.8)
\]

\[
\beta_k^{(2)} = (1 + h\mathbf{p}_k)(1 + h\epsilon\mathbf{q}_k), \quad \alpha_k^{(2)} = \mathbf{q}_k(1 + \epsilon\mathbf{p}_{k+1}). \quad (4.9)
\]

**Theorem 4.4** The change of variables (4.7) conjugates the map \( \text{dMTL}(\epsilon) \) with the map on \( \mathcal{MT}(q, p) \) described by the following local equations of motion:

\[
\mathbf{p}_k + h\mathbf{q}_{k-1}(1 + \epsilon\mathbf{p}_k) = \mathbf{p}_k + h\mathbf{q}_k(1 + \epsilon\mathbf{p}_k), \quad \mathbf{q}_k(1 + h\mathbf{p}_k) = \mathbf{q}_k(1 + h\mathbf{p}_{k+1}). \quad (4.10)
\]

So, the local form of \( \text{dMTL}(\epsilon) \) (4.10) actually belongs to the hierarchy \( \text{M}^2\text{TL}(\epsilon; h) \).

**Corollary.** The local form of \( \text{dMTL}(\epsilon) \) (4.10) is a Poisson map with respect to the following bracket on \( \mathcal{MT}(q, r) \):

\[
\{\mathbf{p}_k, \mathbf{q}_k\} = -\mathbf{q}_k(1 + h\epsilon\mathbf{q}_k)(1 + \epsilon\mathbf{p}_k)(1 + h\mathbf{p}_k),
\]

\[
\{\mathbf{q}_k, \mathbf{p}_{k+1}\} = -\mathbf{q}_k(1 + h\epsilon\mathbf{q}_k)(1 + \epsilon\mathbf{p}_{k+1})(1 + h\mathbf{p}_{k+1}),
\]

which is the pull–back of the bracket

\[
\{\cdot, \cdot\}_1 + h\{\cdot, \cdot\}_2\}
\]

on \( \mathcal{MT}(q, p) \) under the change of variables (4.7).

There is an interesting question on the relation between the local forms of \( \text{dTL} \) and \( \text{dMTL}(\epsilon) \). In other words, how do the Miura maps \( M_{1,2}(\epsilon) \) look when seen through the localizing changes of variables? The answer is given by Theorem 4.3, which in the present case may be reformulated as follows:

**Theorem 4.5** a) The following diagram is commutative for \( j = 1, 2 \):

\[
\begin{array}{cccc}
\mathcal{MT}(q, p) & \xrightarrow{M_j(h; \epsilon)} & \mathcal{T}(a, b)
\end{array}
\]

\[
\mathcal{MT}(q, p) \xrightarrow{M_j(\epsilon)} \mathcal{T}(a, b)
\]

\[
\mathcal{MT}(q, p) \xrightarrow{M_1(\epsilon; h)} \mathcal{T}(a, b)
\]

\[
\mathcal{MT}(q, p) \xrightarrow{M_1(h)} \mathcal{T}(a, b)
\]
where the maps $M_{1,2}(h; \epsilon)$ are given by
\begin{align*}
M_1(h; \epsilon) : & \quad \begin{cases} b_k = p_k + \epsilon q_k(1 + h p_k), \\ a_k = q_k(1 + \epsilon p_k), \end{cases} \\
M_2(h; \epsilon) : & \quad \begin{cases} b_k = p_k + \epsilon q_k(1 + h p_k), \\ a_k = q_k(1 + \epsilon p_{k+1}). \end{cases}
\end{align*}

b) Either of the Miura maps $M_{1,2}(h; \epsilon)$ conjugates the local form of $dMTL(\epsilon)$ (4.10) with the local form of $dTL$ (3.23).

5 Triple modified Toda lattice

5.1 Equations of motion

The phase space of the triple modified Toda lattice $M^3TL(\epsilon, \delta, \gamma)$ will be denoted by $M^3T = \mathbb{R}^{2N}(a, b)$. Three parameters $\epsilon, \delta, \gamma$, all with equal rights, will be called the modification parameters. Equations of motion of this system read:
\begin{align*}
\dot{b}_k & = (1 + \epsilon b_k)(1 + \delta b_k)(1 + \gamma b_k)
\frac{a_k}{1 - \epsilon \delta \gamma b_k a_k} - \frac{a_{k-1}}{1 - \epsilon \delta \gamma b_k a_{k-1}}, \\
\dot{a}_k & = a_k(1 + \epsilon \delta a_k)(1 + \epsilon \gamma a_k)(1 + \delta a_k)
\frac{b_{k+1}}{1 - \epsilon \delta \gamma b_{k+1} a_k} - \frac{b_k}{1 - \epsilon \delta \gamma b_k a_k}.
\end{align*}

(5.1)

5.2 Miura relations

Actually, the available information about $M^3TL(\epsilon, \delta, \gamma)$ is not very rich. Even a local Hamiltonian structure is not known (and presumably does not exist). All we know is the Miura relation to $M^2TL(\epsilon, \delta)$.

Theorem 5.1 Define the Miura maps $M_{1,2}(\epsilon, \delta; \gamma) : M^3T(a, b) \mapsto M^2T(r, s)$ by:
\begin{align*}
M_1(\epsilon, \delta; \gamma) : & \quad \begin{cases} r_k = \frac{b_k + \gamma a_{k-1} + \gamma(\epsilon + \delta)b_k a_{k-1}}{1 - \epsilon \delta \gamma b_k a_{k-1}}, \\ s_k = \frac{a_k(1 + \gamma b_k)}{1 - \epsilon \delta \gamma b_k a_k}, \end{cases} \\
M_2(\epsilon, \delta; \gamma) : & \quad \begin{cases} r_k = \frac{b_k + \gamma a_k + \gamma(\epsilon + \delta)b_k a_k}{1 - \epsilon \delta \gamma b_k a_k}, \\ s_k = \frac{a_k(1 + \gamma b_{k+1})}{1 - \epsilon \delta \gamma b_{k+1} a_k}. \end{cases}
\end{align*}

(5.2) (5.3)

Than the pull–back of the flow $M^2TL(\epsilon, \delta)$ under either of the Miura maps $M_{1,2}(\epsilon, \delta; \gamma)$ coincides with $M^3TL(\epsilon, \delta, \gamma)$.
For these Miura maps again a permutability statement holds:

**Theorem 5.2** The following diagram is commutative for all \( i = 1, 2, j = 1, 2 \):

\[
\begin{array}{ccc}
\mathcal{M}^3 \mathcal{T} & \xrightarrow{M_j(\epsilon, \gamma; \delta)} & \mathcal{M}^2 \mathcal{T} \\
\downarrow M_i(\epsilon, \delta; \gamma) & & \downarrow M_i(\epsilon; \gamma) \\
\mathcal{M}^2 \mathcal{T} & \xrightarrow{M_j(\epsilon; \delta)} & \mathcal{M} \mathcal{T}
\end{array}
\]

It is of some interest to consider the compositions of the Miura maps \( M_j(\epsilon) \circ M_j(\epsilon; \delta) \circ M_j(\epsilon, \delta; \gamma) : \mathcal{M}_3^3 \mathcal{T}(a, b) \to \mathcal{T}(a, b) \) bringing \( M^3 \mathcal{T} \epsilon, \delta, \gamma \) directly into \( \mathcal{T} \). These compositions depend symmetrically on all three modification parameters \( \epsilon, \delta, \gamma \). We give here only the formulas for the case \( j = 1 \):

\[
a_k = \frac{a_k (1 + \epsilon b_k) (1 + \delta b_k) (1 + \gamma b_k) (1 + \epsilon \delta a_{k-1}) (1 + \epsilon \gamma a_{k-1}) (1 + \delta \gamma a_{k-1})}{(1 - \epsilon \delta \gamma b_k a_k) (1 - \epsilon \delta \gamma b_k a_{k-1})^2 (1 - \epsilon \delta \gamma b_{k-1} a_{k-1})},
\]

\[
b_k = b_k + \frac{\epsilon \delta \gamma a_{k-1}}{1 - \epsilon \delta \gamma b_{k-1} a_{k-1}} \left( b_{k-1} + \frac{b^2_k + a_{k-1}}{1 - \epsilon \delta \gamma b_{k-1} a_{k-1}} + \frac{b^2_{k-1} + a_{k-2}}{1 - \epsilon \delta \gamma b_{k-1} a_{k-2}} \right) + \frac{(\epsilon + \delta + \gamma) a_{k-1}}{1 - \epsilon \delta \gamma b_{k-1} a_{k-1}} \left( \frac{1}{1 - \epsilon \delta \gamma b_{k-1} a_{k-1}} + \frac{1}{1 - \epsilon \delta \gamma b_{k-1} a_{k-2}} - 1 \right).
\]

### 5.3 Application: localizing change of variables for \( dM^2 \mathcal{T}(\epsilon, \delta) \)

It turns out that the Miura map \( M_1(\epsilon, \delta; h) \) plays the role of the localizing change of variables for the map \( dM^2 \mathcal{T}(\epsilon, \delta) \). Indeed, consider the following change of variables \( \mathcal{M}^2 \mathcal{T}(r, s) \to \mathcal{M}^2 \mathcal{T}(r, s) \):

\[
M_1(\epsilon, \delta; h) : \quad \left\{ \begin{array}{l}
r_k = \frac{r_k + h s_{k-1} + h (\epsilon + \delta) r_k s_{k-1}}{1 - h e \delta r_k s_{k-1}}, \\
s_k = \frac{s_k (1 + hr_k)}{1 - h e \delta r_k s_k}.
\end{array} \right.
\]

(5.4)

It is useful to notice that the first formula in (5.4) may be equivalently rewritten as

\[
1 + e r_k = \frac{(1 + \epsilon r_k) (1 + h e s_{k-1})}{1 - h e \delta r_k s_{k-1}} \iff 1 + \delta r_k = \frac{(1 + \delta r_k) (1 + h \delta s_{k-1})}{1 - h e \delta r_k s_{k-1}},
\]

(5.5)
or else as
\[
\frac{1 + \epsilon r_k}{1 + \delta r_k} = \frac{(1 + \epsilon r_k)(1 + h\epsilon s_{k-1})}{(1 + \delta r_k)(1 + h\delta s_{k-1})},
\] (5.6)
while the second formula in (5.4) has the following two equivalent forms:
\[
1 + \epsilon s_k = \frac{1 + \epsilon \delta s_k}{1 - h\epsilon \delta r_k s_k},
\] (5.7)
and
\[
\frac{s_k}{1 + \epsilon \delta s_k} = \frac{s_k}{1 + \epsilon \delta s_k} (1 + h r_k).
\] (5.8)

**Theorem 5.3** The change of variables (5.4) conjugates the map \( dM^2 TL(\epsilon, \delta) \) with the map on \( M^2 T(r, s) \) described by the following local equations of motion:
\[
\frac{1 + \tilde{\epsilon} r_k}{1 + \delta \tilde{r}_k} \cdot \frac{1 + h\epsilon \tilde{s}_{k-1}}{1 + h\delta \tilde{s}_{k-1}} = \frac{1 + \epsilon r_k}{1 + \delta r_k} \cdot \frac{1 + h\epsilon s_k}{1 + h\delta s_k},
\] (5.9)
\[
\frac{\tilde{s}_k}{1 + \epsilon \delta \tilde{s}_k} (1 + h \tilde{r}_k) = \frac{s_k}{1 + \epsilon \delta s_k} (1 + h r_{k+1}).
\] (5.10)

So, the local form (5.9), (5.10) belongs actually to the hierarchy \( M^3 TL(\epsilon, \delta, h) \). We discuss now the relation between the local forms of the maps \( dMTL(\epsilon) \) and \( dM^2 TL(\epsilon, \delta) \). We have to determine how do the Miura transformations \( M_{1,2}(\epsilon; \delta) \) look through the localizing changes of variables. The answer is given by Theorem 5.2, which now takes the following form.

**Theorem 5.4** a) The following diagram is commutative for \( j = 1, 2 \):

\[
\begin{array}{ccc}
M^2 T(r, s) & \xrightarrow{M_j(\epsilon, h; \delta)} & MT(q, p) \\
\downarrow M_1(\epsilon, \delta; h) & & \downarrow M_1(\epsilon; h) \\
M^2 T(r, s) & \xrightarrow{M_j(\epsilon; \delta)} & MT(q, p)
\end{array}
\]

where the maps \( M_{1,2}(\epsilon, h; \delta) : M^2 T(r, s) \mapsto MT(q, p) \) are given by
\[
M_{1}(\epsilon, h; \delta) : \begin{cases}
p_k = \frac{r_k + \delta s_{k-1} + \delta(\epsilon + h)r_k s_{k-1}}{1 - h\epsilon \delta r_k s_{k-1}}, \\
q_k = \frac{s_k (1 + \delta r_k)}{1 - h\epsilon \delta r_k s_k} 
\end{cases}
\] (5.11)
\[
M_2(\epsilon, h; \delta) : \begin{cases}
p_k = \frac{r_k + \delta s_k + \delta (\epsilon + h)r_k s_k}{1 - h \epsilon \delta r_k s_k}, \\
q_k = \frac{s_k (1 + \delta r_{k+1})}{1 - h \epsilon \delta r_{k+1} s_k}.
\end{cases}
\] (5.12)

b) Either of the Miura maps \( M_{1,2}(\epsilon, h; \delta) \) conjugates the local form of \( dM^2TL(\epsilon, \delta) \) \( (5.9), (5.10) \) with the local form of \( dMTL(\epsilon) \) \( (4.10) \).

6 Volterra lattice

6.1 Equations of motion

Actually, the Volterra lattice \( VL \) is a more symmetric (and parameter free) version of the modified Toda lattice. However, as we shall see later on, analogous statement does not hold anymore for relativistic generalizations. The phase space of \( VL \) will be denoted through \( V = \mathbb{R}^{2N} (u, v) \). The equations of motion read:

\[
\dot{u}_k = u_k (v_k - v_{k-1}) , \quad \dot{v}_k = v_k (u_{k+1} - u_k) .
\] (6.1)

6.2 Bi–Hamiltonian structure

Proposition 6.1 a) The relations

\[ \{ u_k, v_k \}_2 = -u_k v_k , \quad \{ v_k, u_{k+1} \}_2 = -v_k u_{k+1} \] (6.2)

define a Poisson bracket on \( V \). The flow \( VL \) is a Hamiltonian system on \( (V, \{ \cdot, \cdot \}_2) \) with the Hamilton function

\[ H_1(u, v) = \sum_{k=1}^{N} u_k + \sum_{k=1}^{N} v_k . \] (6.3)

b) The relations

\[
\{ u_k, v_k \}_3 = -u_k v_k (u_k + v_k) , \quad \{ v_k, u_{k+1} \}_3 = -v_k u_{k+1} (v_k + u_{k+1}) ,
\]

\[
\{ u_k, u_{k+1} \}_3 = -u_k v_k u_{k+1} , \quad \{ v_k, v_{k+1} \}_3 = -v_k u_{k+1} v_{k+1}
\] (6.4)

define a Poisson bracket on \( V \) compatible with \( \{ \cdot, \cdot \}_2 \). The flow \( VL \) is a Hamiltonian system on \( (V, \{ \cdot, \cdot \}_3) \) with the Hamilton function

\[
H_0(u, v) = \sum_{k=1}^{N} \log(u_k) , \quad \text{or} \quad H_0(u, v) = \sum_{k=1}^{N} \log(v_k) .
\] (6.5)

(The difference of these two functions is a Casimir of the bracket \( \{ \cdot, \cdot \}_3 \).)
6.3 Miura relations

Define the Miura maps \( M_{1,2} : V \mapsto T \):

\[
M_1 : \begin{cases}
  b_k = u_k + v_{k-1} , \\
  a_k = u_kv_k ,
\end{cases} \quad M_2 : \begin{cases}
  b_k = u_k + v_k , \\
  a_k = u_{k+1}v_k .
\end{cases}
\]  \hspace{1cm} (6.6)

**Theorem 6.2**  
\( a) \) Both maps \( M_{1,2} : V \mapsto T \) are Poisson, if \( V \) is equipped with the bracket (6.2), and \( T \) is equipped with the bracket (2.4), and also if \( V \) is equipped with the bracket (6.4), and \( T \) is equipped with the bracket (2.6).

\( b) \) The pull-back of the flow \( TL \) with respect to either of the Miura maps \( M_{1,2} \) coincides with \( VL \).

6.4 Lax representation

Lax matrix: \((U,V) : V \mapsto g \otimes g\):

\[
U(u,v,\lambda) = \sum_{k=1}^{N} u_k E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} ,
\]

\[
V(u,v,\lambda) = I + \lambda^{-1} \sum_{k=1}^{N} v_k E_{k,k+1} .
\]  \hspace{1cm} (6.7) \hspace{1cm} (6.8)

The origin of the Miura transformations \( M_{1,2} \) lies in the following factorizations of the Toda Lax matrix \( T(a,b,\lambda) \): the formulas for the map \( M_1 \) are equivalent to

\[
T(a,b,\lambda) = U(u,v,\lambda)V(u,v,\lambda) ,
\]  \hspace{1cm} (6.9)

while the formulas for the map \( M_2 \) are equivalent to

\[
T(a,b,\lambda) = V(u,v,\lambda)U(u,v,\lambda) .
\]  \hspace{1cm} (6.10)

Lax representation for \( VL \):

\[
\begin{cases}
  \dot{U} = UB_2 - B_1U = C_1U - UC_2 , \\
  \dot{V} = VB_1 - B_2V = C_2V - VC_1 ,
\end{cases}
\]  \hspace{1cm} (6.11)

with

\[
B_1 = \pi_{+}(UV) = \sum_{k=1}^{N} (u_k + v_{k-1})E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} ,
\]  \hspace{1cm} (6.12)

\[
B_2 = \pi_{+}(VU) = \sum_{k=1}^{N} (u_k + v_k)E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} ,
\]  \hspace{1cm} (6.13)

\[
C_1 = \pi_{-}(UV) = \lambda^{-1} \sum_{k=1}^{N} u_kv_k E_{k,k+1} ,
\]  \hspace{1cm} (6.14)

\[
C_2 = \pi_{-}(VU) = \lambda^{-1} \sum_{k=1}^{N} u_{k+1}v_k E_{k,k+1} .
\]  \hspace{1cm} (6.15)
6.5 Discretization

Lax representation for the map dVL:

\[
\tilde{U} = B_1^{-1} U B_2 = C_1 U C_2^{-1}, \quad \tilde{V} = B_2^{-1} U B_1 = C_2 V C_1^{-1}
\]  

(6.16)

with

\[
B_1 = \Pi_+(I + hUV) = \sum_{k=1}^{N} \beta_k^{(1)} E_{kk} + h\lambda \sum_{k=1}^{N} E_{k+1,k},
\]  

(6.17)

\[
B_2 = \Pi_+(I + hVU) = \sum_{k=1}^{N} \beta_k^{(2)} E_{kk} + h\lambda \sum_{k=1}^{N} E_{k+1,k},
\]  

(6.18)

\[
C_1 = \Pi_-(I + hUV) = I + h\lambda^{-1} \sum_{k=1}^{N} \gamma_k^{(1)} E_{k,k+1},
\]  

(6.19)

\[
C_2 = \Pi_-(I + hVU) = I + h\lambda^{-1} \sum_{k=1}^{N} \gamma_k^{(2)} E_{k,k+1}.
\]  

(6.20)

7 Modified Volterra lattice

7.1 Equations of motion

The phase space of the modified Volterra lattice MVL(\(\epsilon\)) will be denoted by \(\mathcal{M}(y, z)\), the modification parameter here is \(\epsilon\), and the equations of motion read:

\[
\dot{y}_k = y_k(1 + \epsilon y_k)(z_k - z_{k-1}), \quad \dot{z}_k = z_k(1 + \epsilon z_k)(y_{k+1} - y_k).
\]  

(7.1)

7.2 Hamiltonian structure

Proposition 7.1 The relations

\[
\{y_k, z_k\} = -y_k z_k(1 + \epsilon y_k)(1 + \epsilon z_k),
\]  

\[
\{z_k, y_{k+1}\} = -z_k y_{k+1}(1 + \epsilon z_k)(1 + \epsilon y_{k+1})
\]  

(7.2)

define a Poisson bracket on \(\mathcal{M}(y, z)\). The system MVL(\(\epsilon\)) (7.1) is Hamiltonian with respect to this bracket, with the Hamilton function

\[
G_0(y, z) = \epsilon^{-1} \sum_{k=1}^{N} \log(1 + \epsilon y_k) + \epsilon^{-1} \sum_{k=1}^{N} \log(1 + \epsilon z_k).
\]  

(7.3)

7.3 Miura relations

The modified Volterra lattice is Miura related to the Volterra lattice as well as to the modified Toda lattice.

Theorem 7.2 a) The flow MVL(\(\epsilon\)) (7.1) is the pull–back of the flow VL (5.1) under either of the Miura transformations \(M_{1,2}(\epsilon) : \mathcal{M}(y, z) \mapsto \mathcal{V}(u, v)\) defined by

\[
M_1(\epsilon) : \begin{cases} 
    u_k = y_k(1 + \epsilon z_{k-1}), \\
    v_k = z_k(1 + \epsilon y_k),
\end{cases} \quad M_2(\epsilon) : \begin{cases} 
    u_k = y_k(1 + \epsilon z_k), \\
    v_k = z_k(1 + \epsilon y_{k+1}).
\end{cases}
\]  

(7.4)
Both Miura transformations $M_{1,2}(\epsilon)$ are Poisson, if $\mathcal{M}(y, z)$ is equipped with the bracket \{\cdot, \cdot\}$_{23}$ and $\mathcal{V}(u, v)$ is equipped with the bracket \{\cdot, \cdot\}$_{2} + \epsilon\{\cdot, \cdot\}$_{3}.

b) The flow $MVL(\epsilon)$ (7.1) is the pull–back of the flow $MTL(\epsilon)$ (3.1) under either of the Miura transformations $M_{1,2}(\epsilon): \mathcal{M}(y, z) \mapsto \mathcal{M}(q, p)$ defined by

\[
M_{1}(\epsilon) : \begin{cases} 
  p_k = y_k + z_{k-1} + \epsilon y_k z_{k-1}, \\
  q_k = y_k z_k,
\end{cases} \quad M_{2}(\epsilon) : \begin{cases} 
  p_k = y_k + z_k + \epsilon y_k z_k, \\
  q_k = y_{k+1} z_k.
\end{cases}
\] (7.5)

Both Miura transformations $M_{1,2}(\epsilon)$ are Poisson, if $\mathcal{M}(y, z)$ and $\mathcal{M}(q, p)$ carry the corresponding brackets \{\cdot, \cdot\}$_{23}$.

c) The following diagram is commutative for all $i = 1, 2$, $j = 1, 2$:

\[
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{M_{j}(\epsilon)} & \mathcal{M} \\
\downarrow & & \downarrow \\
\mathcal{V} & \xrightarrow{M_{j}} & \mathcal{T}
\end{array}
\]

7.4 Discretization

The map $\mathcal{M} \mapsto \mathcal{M}$ denoted by $dMVL(\epsilon)$ is by definition conjugated with $dVL$ by means of $M_{1,2}(\epsilon)$, and conjugated with $dMTL(\epsilon)$ by means of $M_{1,2}(\epsilon)$.

7.5 Application: localizing change of variables for $dVL$

The Miura map $M_{1}(h)$ turns out to play the role of the localizing change of variables for the map $dVL$. Indeed, consider the following change of variables $\mathcal{V}(u, v) \mapsto \mathcal{V}(u, v)$:

\[
M_{1}(h) : \begin{cases} 
  u_k = u_k(1 + h v_{k-1}), \\
  v_k = v_k(1 + h u_k),
\end{cases}
\] (7.6)

Then the entries of the factors $B_{j}, C_{j}$ (see (3.17)–(3.20)) admit local expressions in the coordinates $(u, v)$:

\[
\beta_{k}^{(1)} = (1 + h u_k)(1 + h v_{k-1}), \quad \gamma_{k}^{(1)} = u_k v_k,
\] (7.7)

\[
\beta_{k}^{(2)} = (1 + h v_k)(1 + h u_k), \quad \gamma_{k}^{(2)} = u_{k+1} v_k.
\] (7.8)
Theorem 7.3 The change of variables (7.6) conjugates the map dVL with the map on 
\( \mathcal{V}(u, v) \) described by the following local equations of motion:
\[
\tilde{u}_k(1 + h\tilde{v}_k) = u_k(1 + hv_k), \quad \tilde{v}_k(1 + h\tilde{u}_k) = v_k(1 + hu_{k+1}).
\] (7.9)
So, the local form of dVL (7.9) belongs actually to the hierarchy dMV L(h).

Corollary. The local form of dVL (7.9) is a Poisson map with respect to the following bracket on \( \mathcal{V}(u, v) \):
\[
\{u_k, v_k\} = -u_k v_k(1 + hu_k)(1 + hv_k),
\]
\[
\{v_k, u_{k+1}\} = -v_k u_{k+1}(1 + hv_k)(1 + hu_{k+1}),
\] (7.10)
which is the pull–back of the bracket
\[
\{\cdot, \cdot\}_2 + h\{\cdot, \cdot\}_3
\]
(7.11)
on \( \mathcal{V}(u, v) \) under the change of variables (7.6).

Now we translate the Miura relations existing between dTL and dVL, as members of the TL and VL hierarchies, into the localizing variables. This is achieved by the following reformulation of the part c) of Theorem 7.2

Theorem 7.4 a) The following diagram is commutative for \( j = 1, 2 \):

\[
\begin{array}{ccc}
\mathcal{V}(u, v) & \xrightarrow{M_j(h)} & \mathcal{T}(a, b) \\
\downarrow M_1(h) & & \downarrow M_1(h) \\
\mathcal{V}(u, v) & \xrightarrow{M_j} & \mathcal{T}(a, b)
\end{array}
\]
where the maps \( M_{1,2}(h) : \mathcal{V}(u, v) \mapsto \mathcal{T}(a, b) \) are given by the formulas
\[
M_1(h) : \begin{cases}
b_k = u_k + v_{k-1} + hu_k v_{k-1}, \\
a_k = u_k v_k,
\end{cases}
\] (7.12)
and
\[
M_2(h) : \begin{cases}
b_k = u_k + v_k + hu_k v_k, \\
a_k = u_{k+1} v_k.
\end{cases}
\] (7.13)
b) Both maps \( M_{1,2}(h) \) are Poisson, if \( \mathcal{V}(u, v) \) is equipped with the bracket (7.10), and \( \mathcal{T}(a, b) \) is equipped with the bracket (3.26).
c) The local form of dVL (7.9) is conjugated with the local form of dTL (3.23) by either of the maps \( M_{1,2}(h) \).
8 Double modified Volterra lattice

8.1 Equations of motion

The phase space of the double modified Volterra lattice $M^2V = \mathbb{R}^{2N}(u, v)$, the modification parameters by $\epsilon, \delta$. The equations of motion read:

$$
\dot{u}_k = u_k(1 + \epsilon u_k)(1 + \delta u_k) \left( \frac{v_k}{1 - \epsilon \delta v_k u_k} - \frac{v_{k-1}}{1 - \epsilon \delta u_k v_{k-1}} \right),
$$

$$
\dot{v}_k = v_k(1 + \epsilon v_k)(1 + \delta v_k) \left( \frac{u_k}{1 - \epsilon \delta u_k v_k} - \frac{u_{k-1}}{1 - \epsilon \delta u_k v_{k-1}} \right).
$$

(8.1)

8.2 Miura relations

Theorem 8.1

(a) Define the Miura maps $M_{1,2}(\epsilon; \delta) : M^2V(u, v) \mapsto M\mathcal{V}(y, z)$ by:

$$
M_1(\epsilon; \delta) : \begin{cases}
y_k = \frac{u_k(1 + \delta v_{k-1})}{1 - \epsilon \delta u_k v_{k-1}}, \\
z_k = \frac{v_k(1 + \delta u_k)}{1 - \epsilon \delta v_k u_k},
\end{cases}
$$

(8.2)

$$
M_2(\epsilon; \delta) : \begin{cases}
y_k = \frac{u_k(1 + \delta v_k)}{1 - \epsilon \delta v_k u_k}, \\
z_k = \frac{v_k(1 + \delta u_{k+1})}{1 - \epsilon \delta u_{k+1} v_k}.
\end{cases}
$$

(8.3)

Then the pull-back of the flow $M^2V(\epsilon)$ under either of the Miura maps $M_{1,2}(\epsilon; \delta)$ coincides with $M^2V(\epsilon, \delta)$.

(b) Define the Miura maps $M_{1,2}(\epsilon, \delta) : M^2V(u, v) \mapsto M^2\mathcal{T}(r, s)$ by:

$$
M_1(\epsilon, \delta) : \begin{cases}
r_k = \frac{u_k + v_{k-1} + (\epsilon + \delta)u_k v_{k-1}}{1 - \epsilon \delta u_k v_{k-1}}, \\
s_k = \frac{u_k v_k}{1 - \epsilon \delta u_k v_k},
\end{cases}
$$

(8.4)

$$
M_2(\epsilon, \delta) : \begin{cases}
r_k = \frac{u_k + v_k + (\epsilon + \delta)u_k v_k}{1 - \epsilon \delta u_k v_k}, \\
s_k = \frac{u_k + v_{k+1}}{1 - \epsilon \delta u_{k+1} v_k}.
\end{cases}
$$

(8.5)

Then the pull-back of the flow $M^2\mathcal{T}(\epsilon, \delta)$ under either of the Miura maps $M_{1,2}(\epsilon, \delta)$ coincides with $M^2V(\epsilon, \delta)$.

(c) The following diagram is commutative for all $i = 1, 2, j = 1, 2$: 
d) The following diagram is commutative for all $i = 1, 2$, $j = 1, 2$:

\[
\begin{array}{c}
M^2 \mathcal{V} \\
\downarrow M_i(\epsilon; \delta) \\
\downarrow \\
M \mathcal{V}
\end{array}
\quad 
\begin{array}{c}
\quad M_j(\delta; \epsilon) \\
\quad M_j(\delta) \\
\quad M_j(\epsilon) \\
\quad \mathcal{V}
\end{array}
\]

8.3 Application: localizing change of variables for $dMVL(\epsilon)$

It turns out that the Miura map $M_1(\epsilon; h)$ plays the role of the localizing change of variables for the map $dMVL(\epsilon)$. Indeed, consider the following change of variables $M \mathcal{V}(y, z) \mapsto M \mathcal{V}(\tilde{y}, \tilde{z})$:

\[
M_1(\epsilon; h) : \quad y_k = \frac{y_k(1 + hz_{k-1})}{1 - h\epsilon y_k z_{k-1}}, \quad z_k = \frac{z_k(1 + hy_k)}{1 - h\epsilon z_k y_k}.
\]  

(8.6)

Theorem 8.2 The change of variables (8.6) conjugates the map $dMVL(\epsilon)$ with the map on $M \mathcal{V}(y, z)$ described by the following local equations of motion:

\[
\frac{\tilde{y}_k}{1 + \epsilon \tilde{y}_k} (1 + h\tilde{z}_{k-1}) = \frac{y_k}{1 + \epsilon y_k} (1 + h z_k),
\]

(8.7)

\[
\frac{\tilde{z}_k}{1 + \epsilon \tilde{z}_k} (1 + h\tilde{y}_k) = \frac{z_k}{1 + \epsilon z_k} (1 + h y_{k+1}).
\]
We see that the local form of the map $dMVL(\epsilon)$ (8.7) belongs actually to the hierarchy $M^2VL(\epsilon, h)$.

Finally, we give the translations of the Miura relations of $dMVL(\epsilon)$ to both $dVL$ and $dMTL(\epsilon)$ into the language of localizing variables. This is achieved by reformulating the parts c) and d) of Theorem 8.1.

**Theorem 8.3**
a) The following diagram is commutative for $j = 1, 2$:

```
\[
\begin{array}{ccc}
\mathcal{M}V(y, z) & \xrightarrow{M_{1,2}(h; \epsilon)} & V(u, v) \\
\downarrow M_1(\epsilon; h) & & \downarrow M_1(h) \\
\mathcal{M}V(y, z) & \xrightarrow{M_{1,2}(\epsilon)} & V(u, v)
\end{array}
\]
```

Here the maps $M_{1,2}(h; \epsilon) : \mathcal{M}V(y, z) \mapsto V(u, v)$ are given by the formulas

\[
M_1(h; \epsilon) : \begin{cases}
  u_k = \frac{y_k(1 + \epsilon z_{k-1})}{1 - h\epsilon y_k z_{k-1}}, \\
  v_k = \frac{z_k(1 + \epsilon y_k)}{1 - h\epsilon z_k y_k},
\end{cases}
\]

and

\[
M_2(h; \epsilon) : \begin{cases}
  u_k = \frac{y_k(1 + \epsilon z_k)}{1 - h\epsilon z_k y_k}, \\
  v_k = \frac{z_k(1 + \epsilon y_{k+1})}{1 - h\epsilon y_{k+1} z_k},
\end{cases}
\]

Either of the maps $M_{1,2}(h; \epsilon)$ conjugates the local form of $dMVL(\epsilon)$ (8.7) with the local form of $dVL$ (7.9).

b) The following diagram is commutative for $j = 1, 2$:
Here the maps $M_{1,2}(\epsilon, h) : \mathcal{MV}(y, z) \mapsto \mathcal{MT}(q, p)$ are given by the formulas

\begin{align*}
M_1(\epsilon, h) : & \quad \begin{cases}
p_k = \frac{y_k + z_{k-1} + (\epsilon + h)y_k z_{k-1}}{1 - h\epsilon y_k z_{k-1}}, \\
q_k = \frac{y_k z_k}{1 - h\epsilon y_k z_k},
\end{cases} \\
M_2(\epsilon, h) : & \quad \begin{cases}
p_k = \frac{y_k + z_k + (\epsilon + h)y_k z_k}{1 - h\epsilon y_k z_k}, \\
q_k = \frac{y_{k+1} z_k}{1 - h\epsilon y_{k+1} z_k},
\end{cases}
\end{align*}

Either of the maps $M_{1,2}(\epsilon, h)$ conjugates the local form of $d\text{MVL}(\epsilon)$ (8.7) with the local form of $d\text{MTL}(\epsilon)$ (4.10).
Part II
Relativistic systems

9 Relativistic Toda lattice

9.1 Equations of motion

We now turn to a tower of modifications connected with the relativistic Toda lattice hierarchy, or RTL(\(\alpha\)). Here \(\alpha\) is a (small) parameter playing the role of the inverse speed of light. A typical phenomenon encountered for these relativistic systems is the splitting of some notions and results. For example, in the RTL(\(\alpha\)) hierarchy there exist actually two simple flows approximating the usual TL as \(\alpha \to 0\). The phase space of this hierarchy will be denoted by \(\mathcal{RT} = \mathbb{R}^{2N}(a,b)\).

The flow RTL\(_{+}(\alpha)\) is a polynomial perturbation of TL:

\[
\dot{b}_k = (1 + ab_k)(a_k - a_{k-1}) , \quad \dot{a}_k = a_k(b_{k+1} - b_k + \alpha a_{k+1} - \alpha a_{k-1}) . \tag{9.1}
\]

The flow RTL\(_{-}(\alpha)\) is a rational perturbation of TL:

\[
\dot{b}_k = \frac{a_k}{1 + ab_{k+1}} - \frac{a_{k-1}}{1 + ab_{k-1}} , \quad \dot{a}_k = a_k \left( \frac{b_{k+1}}{1 + ab_{k+1}} - \frac{b_k}{1 + ab_k} \right) . \tag{9.2}
\]

The first one of these flows resembles the modified Toda lattice MTL(\(\alpha\)), however its properties are somewhat different. In particular, it is a tri–Hamiltonian system, while MTL(\(\alpha\)) admits only two local hamiltonian formulations.

9.2 Tri–Hamiltonian structure

**Proposition 9.1** a) The relations

\[
\{b_k, a_k\}_{1\alpha} = -a_k , \quad \{a_k, b_{k+1}\}_{1\alpha} = -a_k , \quad \{b_k, b_{k+1}\}_{1\alpha} = \alpha a_k \tag{9.3}
\]

define a Poisson bracket on \(\mathcal{RT}\). The flows RTL\(_{\pm}(\alpha)\) are Hamiltonian systems on \((\mathcal{RT}, \{\cdot, \cdot\}_{1\alpha})\) with the Hamilton functions

\[
H^{(+)}_{2\alpha}(a,b) = \sum_{k=1}^{N} \left( \frac{1}{2} b_k^2 + a_k \right) + \alpha \sum_{k=1}^{N} (b_k + b_{k+1})a_k + \alpha^2 \sum_{k=1}^{N} \left( \frac{1}{2} a_k^2 + a_k a_{k+1} \right) , \tag{9.4}
\]

\[
H^{(-)}_{2\alpha}(a,b) = -\alpha^{-2} \sum_{k=1}^{N} \log(1 + ab_k) , \tag{9.5}
\]

respectively. The functions \(H^{(+)}_{2\alpha}\) and \(H^{(-)}_{2\alpha}\) are in involution in the bracket \(\{\cdot, \cdot\}_{1\alpha}\).

b) The relations

\[
\{b_k, a_k\}_{2\alpha} = -b_k a_k , \quad \{a_k, b_{k+1}\}_{2\alpha} = -a_k b_{k+1} , \quad \{b_k, b_{k+1}\}_{2\alpha} = -a_k , \quad \{a_k, a_{k+1}\}_{2\alpha} = -a_k a_{k+1} \tag{9.6}
\]
define a Poisson bracket on $\mathcal{RT}(a, b)$, compatible with $\{\cdot, \cdot\}_{1\alpha}$. The flows $\text{RTL}_\pm(\alpha)$ are Hamiltonian on $\big(\mathcal{RT}, \{\cdot, \cdot\}_{2\alpha}\big)$ with the Hamilton functions

$$H_{1\alpha}^{(+)}(a, b) = \sum_{k=1}^{N} b_k + \alpha \sum_{k=1}^{N} a_k ,$$  \hspace{1cm} \text{(9.7)}

$$H_{1\alpha}^{(-)}(a, b) = \alpha^{-1} \sum_{k=1}^{N} \log(1 + \alpha b_k) ,$$  \hspace{1cm} \text{(9.8)}

respectively. The functions $H_{1\alpha}^{(+)}$ and $H_{1\alpha}^{(-)}$ are in involution in the bracket $\{\cdot, \cdot\}_{2\alpha}$.

c) The relations

$$\{b_k, a_k\}_{3\alpha} = -a_k (b_k^2 + a_k) - \alpha b_k a_k^2 ,$$

$$\{a_k, b_{k+1}\}_{3\alpha} = -a_k (b_{k+1}^2 + a_k) - \alpha a_k b_{k+1} ,$$

$$\{b_k, b_{k+1}\}_{3\alpha} = -a_k (b_k + b_{k+1}) - \alpha b_k a_k b_{k+1} ,$$

$$\{a_k, a_{k+1}\}_{3\alpha} = -2 a_k b_{k+1} a_{k+1} - \alpha a_k a_{k+1} (a_k + a_{k+1}) ,$$

$$\{b_k, a_{k+1}\}_{3\alpha} = -a_k a_{k+1} - \alpha b_k a_k a_{k+1} ,$$

$$\{a_k, b_{k+2}\}_{3\alpha} = -a_k a_{k+1} - \alpha a_k a_{k+1} b_{k+2} ,$$

$$\{a_k, a_{k+2}\}_{3\alpha} = -\alpha a_k a_{k+1} a_{k+2}$$

define a Poisson bracket on $\mathcal{RT}(a, b)$ compatible with $\{\cdot, \cdot\}_{1\alpha}$ and $\{\cdot, \cdot\}_{2\alpha}$. The flows $\text{RTL}_\pm(\alpha)$ are Hamiltonian on $\big(\mathcal{RT}, \{\cdot, \cdot\}_{3\alpha}\big)$ with the Hamilton functions

$$H_{0\alpha}^{(+)}(a, b) = \frac{1}{2} \sum_{k=1}^{N} \log(a_k) ,$$  \hspace{1cm} \text{(9.10)}

$$H_{0\alpha}^{(-)}(a, b) = \frac{1}{2} \sum_{k=1}^{N} \log(a_k) - \sum_{k=1}^{N} \log(1 + \alpha b_k) ,$$  \hspace{1cm} \text{(9.11)}

respectively. The functions $H_{0\alpha}^{(+)}$ and $H_{0\alpha}^{(-)}$ are in involution in the bracket $\{\cdot, \cdot\}_{3\alpha}$.

Notice that the function $H_{2\alpha}^{(-)}(a, b)$, singular in $\alpha$, becomes regular, and, moreover, an $O(\alpha)$–perturbation of $H_{2}(a, b)$ from (2.3), upon adding

$$\alpha^{-1} \sum_{k=1}^{N} b_k + \sum_{k=1}^{N} a_k ,$$

which is a Casimir function of the bracket $\{\cdot, \cdot\}_{1\alpha}$.

Notice also that the bracket $\{\cdot, \cdot\}_{2\alpha}$ is actually independent on $\alpha$ in coordinates $(a, b)$ and literally coincides with the invariant quadratic bracket (2.4) of the TL hierarchy.
9.3 Lax representation

Lax matrix \( (L, W^{-1}) : \mathcal{RT} \to g \otimes g \):

\[
L(a, b, \lambda) = \sum_{k=1}^{N} (1 + \alpha b_k) E_{kk} + \alpha \lambda \sum_{k=1}^{N} E_{k+1,k} ,
\]

\[
W(a, b, \lambda) = I - \alpha \lambda^{-1} \sum_{k=1}^{N} a_k E_{k,k+1} .
\]

Lax representation of RTL\(_+\)(\(\alpha\)):

\[
\dot{L} = LA_2 - A_1 L , \quad \dot{W} = WA_2 - A_1 W ,
\]

with

\[
A_1 = \pi_+ \left( (LW^{-1} - I) / \alpha \right) = \sum_{k=1}^{N} (b_k + \alpha a_{k-1}) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} ,
\]

\[
A_2 = \pi_+ \left( (W^{-1} L - I) / \alpha \right) = \sum_{k=1}^{N} (b_k + \alpha a_k) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} .
\]

Lax representation of RTL\(_-\)(\(\alpha\)):

\[
\dot{L} = C_1 L - LC_2 , \quad \dot{W} = C_1 W - WC_2 ,
\]

with

\[
C_1 = \pi_- \left( (I - WL^{-1}) / \alpha \right) = \lambda^{-1} \sum_{k=1}^{N} \frac{a_k}{1 + \alpha b_{k+1}} E_{k,k+1} ,
\]

\[
C_2 = \pi_- \left( (I - L^{-1} W) / \alpha \right) = \lambda^{-1} \sum_{k=1}^{N} \frac{a_k}{1 + \alpha b_k} E_{k,k+1} .
\]

9.4 Discretization

Lax representation of the map \( \text{dRTL}_+(\alpha) \):

\[
\tilde{L} = A_1^{-1} L A_2 , \quad \tilde{W} = A_1^{-1} W A_2 ,
\]

where

\[
A_1 = \Pi_+ \left( I + \frac{h}{\alpha} (LW^{-1} - I) \right) = \sum_{k=1}^{N} a_k E_{kk} + h \lambda \sum_{k=1}^{N} E_{k+1,k} ,
\]

\[
A_2 = \Pi_+ \left( I + \frac{h}{\alpha} (W^{-1} L - I) \right) = \sum_{k=1}^{N} b_k E_{kk} + h \lambda \sum_{k=1}^{N} E_{k+1,k} .
\]

Lax representation of the map \( \text{dRTL}_-(\alpha) \):

\[
\tilde{L} = C_1 L C_2^{-1} , \quad \tilde{W} = C_1 W C_2^{-1} ,
\]

where

\[
C_1 = \Pi_- \left( I + \frac{h}{\alpha} (I - WL^{-1}) \right) = I + h \lambda^{-1} \sum_{k=1}^{N} c_k E_{k,k+1} ,
\]

\[
C_2 = \Pi_- \left( I + \frac{h}{\alpha} (I - L^{-1} W) \right) = I + h \lambda^{-1} \sum_{k=1}^{N} d_k E_{k,k+1} .
\]
10 Modified relativistic Toda lattice I

As a next example of the “relativistic splitting”, let us mention the existence of two different ways to modify the RTL(\(\alpha\)) hierarchy. In other words, there exist two different modified relativistic Toda hierarchies. In this section we consider one of them.

10.1 Equations of motion

The phase space of the MRTL(\((\alpha; \epsilon)\)) hierarchy will be denoted by \(\mathcal{MRT} = \mathbb{R}^{2N}(q,p)\). As usual, in this relativistic hierarchy there exist two simplest flows.

Equations of motion of MRTL\(_{+}^{(\alpha; \epsilon)}\):

\[
\dot{p}_k = (1 + \alpha p_k)(1 + \epsilon p_k)(q_k - q_{k-1}) ,
\]

\[
\dot{q}_k = q_k(1 + \epsilon \alpha q_k)(p_{k+1} + \alpha q_{k+1} + \epsilon \alpha p_{k+1} q_{k+1} - p_k - \alpha q_{k-1} - \epsilon \alpha p_k q_{k-1}) .
\]

Equations of motion of MRTL\(_{-}^{(\alpha; \epsilon)}\):

\[
\dot{p}_k = (1 + \epsilon p_k)\left(\frac{q_k}{(1 + \alpha p_{k+1})(1 + \epsilon \alpha q_k)} - \frac{q_{k-1}}{(1 + \alpha p_{k-1})(1 + \epsilon \alpha q_{k-1})}\right) ,
\]

\[
\dot{q}_k = q_k\left(\frac{p_{k+1}}{1 + \alpha p_{k+1}} - \frac{p_k}{1 + \alpha p_k}\right) .
\]

Obviously, the equations of MRTL\(_{+}^{(\alpha; \epsilon)}\) are similar to (and slightly more complicated than) the equations of motion of the double modification M\(^2\)TL(\(\alpha, \epsilon\)) of the Toda lattice. However, these systems have somewhat different properties. Most obvious are the non–symmetric roles played in MRTL\(_{+}^{(\alpha; \epsilon)}\) by the parameters \(\alpha\) and \(\epsilon\). More serious differences will become apparent later on, in particular, the system MRTL\(_{+}^{(\alpha; \epsilon)}\) possesses two local invariant Poisson brackets, while for M\(^2\)TL(\(\alpha, \epsilon\)) only one local invariant Poisson bracket is known.

10.2 Bi–Hamiltonian structure

Proposition 10.1 a) The relations

\[
\{p_k, q_k\}_{12\alpha} = -q_k(1 + \epsilon p_k) ,
\]

\[
\{q_k, p_{k+1}\}_{12\alpha} = -q_k(1 + \epsilon p_{k+1}) ,
\]

\[
\{p_k, p_{k+1}\}_{12\alpha} = \alpha \frac{q_k}{1 + \epsilon \alpha q_k}(1 + \epsilon p_k)(1 + \epsilon p_{k+1})
\]

define a Poisson bracket on \(\mathcal{MRT}(q,p)\). The systems MRTL\(_{\pm}^{(\alpha; \epsilon)}\) are Hamiltonian with respect to the bracket (10.3), with the Hamilton functions

\[
G_{1\alpha}^{(\pm)}(q, p) = \epsilon^{-1} \sum_{k=1}^{N} p_k + (1 + \epsilon^{-1} \alpha) \sum_{k=1}^{N} q_k + \alpha \sum_{k=1}^{N} (p_k + p_{k+1}) q_k + \alpha^2 \sum_{k=1}^{N} q_k q_{k+1}(1 + \epsilon p_{k+1}) .
\]
respectively. The functions \(G_{1\alpha}^{(+)}\) and \(G_{1\alpha}^{(-)}\) are in involution in the bracket \(\{\cdot, \cdot\}_{12\alpha}\).

b) The relations

\[
\{p_k, q_k\}_{23\alpha} = -q_k(p_k + \epsilon q_k + \epsilon \alpha p_k q_k)(1 + \epsilon p_k),
\]

\[
\{q_k, p_{k+1}\}_{23\alpha} = -q_k(p_{k+1} + \epsilon q_k + \epsilon \alpha p_{k+1} q_k)(1 + \epsilon p_{k+1}),
\]

\[
\{p_k, p_{k+1}\}_{23\alpha} = -q_k(1 + \epsilon p_k)(1 + \epsilon p_{k+1}),
\]

\[
\{q_k, q_{k+1}\}_{23\alpha} = -q_k q_{k+1}(1 + \epsilon p_{k+1})(1 + \epsilon \alpha q_{k+1})
\]

define a Poisson bracket on \(\mathcal{MRT}\) compatible with \((10.3)\). The systems \(\text{MRTL}_{\pm}^{(+)}(\alpha; \epsilon)\) are Hamiltonian with respect to this bracket with the Hamilton function

\[
G_{0\alpha}^{(+)}(q, p) = \epsilon^{-1} \sum_{k=1}^{N} \log(1 + \epsilon p_k) + \epsilon^{-1} \sum_{k=1}^{N} \log(1 + \epsilon \alpha q_k),
\]

\[
G_{0\alpha}^{(-)}(q, p) = (\epsilon - \alpha)^{-1} \sum_{k=1}^{N} \log \left( \frac{1 + \epsilon p_k}{1 + \epsilon \alpha p_k} \right),
\]

respectively. The functions \(G_{0\alpha}^{(+)}\) and \(G_{0\alpha}^{(-)}\) are in involution in the bracket \(\{\cdot, \cdot\}_{23\alpha}\).

Notice that the function \(G_{1\alpha}^{(+)}(q, p)\) becomes regular in \(\epsilon\) upon subtracting

\[
\epsilon^{-2} \sum_{k=1}^{N} \log(1 + \epsilon p_k) + \epsilon^{-2} \sum_{k=1}^{N} \log(1 + \epsilon \alpha q_k),
\]

which is a Casimir of the bracket \(\{\cdot, \cdot\}_{12\alpha}\).

### 10.3 Miura relations

The relation between \(\text{MRTL}_{\pm}^{(+)}(\alpha; \epsilon)\) and \(\text{RTL}(\alpha)\) hierarchies is established in the following statement.

**Proposition 10.2** Define the Miura maps \(M_{1,2}^{(+)}(\alpha; \epsilon) : \mathcal{MRT}(q, p) \mapsto \mathcal{R}(a, b)\) by:

\[
M_{1}^{(+)}(\alpha; \epsilon) : \quad \begin{cases} 
  b_k = p_k + \epsilon q_k + \epsilon \alpha p_k q_k, \\
  a_k = q_k(1 + \epsilon p_k)(1 + \epsilon \alpha q_k),
\end{cases}
\]

\[
M_{2}^{(+)}(\alpha; \epsilon) : \quad \begin{cases} 
  b_k = p_k + \epsilon q_k + \epsilon \alpha p_k q_k, \\
  a_k = q_k(1 + \epsilon p_{k+1})(1 + \epsilon \alpha q_{k+1}).
\end{cases}
\]
Both maps $M_{1,2}^{(+)}(\alpha; \epsilon)$ are Poisson, if $\mathcal{MRT}(q, p)$ is equipped with the bracket $\{\cdot, \cdot\}_{12\alpha}$, and $\mathcal{RT}(a, b)$ is equipped with $\{\cdot, \cdot\}_{1\alpha} + \epsilon\{\cdot, \cdot\}_{2\alpha}$, and also if $\mathcal{MRT}(q, p)$ is equipped with the bracket $\{\cdot, \cdot\}_{23\alpha}$, and $\mathcal{RT}(a, b)$ is equipped with $\{\cdot, \cdot\}_{2\alpha} + \epsilon\{\cdot, \cdot\}_{3\alpha}$. The pull–back of the flows $\text{RTL}_\pm(\alpha)$ under either of the Miura maps $M_{1,2}^{(+)}(\alpha; \epsilon)$ coincides with $\text{MRTL}_\pm^{(+)}(\alpha; \epsilon)$.

In the rest of this section we shall sometimes denote by $a_k^{(1)}(q, r), b_k^{(1)}(q, r)$ the functions given by the formulas (10.9), and by $a_k^{(2)}(q, r), b_k^{(2)}(q, r)$ the functions given by the formulas (10.10).

### 10.4 Lax representations

The next interesting issue of the “relativistic splitting” is the following: the MRTL hierarchy possesses two different Lax representations.

The first Lax matrix $(P_1, W_1^{-1}, Q_1) : \mathcal{MRT} \mapsto g \otimes g \otimes g$:

\[
P_1(q, p, \lambda) = \sum_{k=1}^{N} (1 + \epsilon p_k)(1 + \epsilon \alpha q_{k-1})E_{kk} + \epsilon \lambda \sum_{k=1}^{N} E_{k+1,k}, \tag{10.11}
\]

\[
Q_1(q, p, \lambda) = I + (\epsilon - \alpha)\lambda^{-1} \sum_{k=1}^{N} q_k E_{k,k+1}, \tag{10.12}
\]

\[
W_1(q, p, \lambda) = I - \alpha \lambda^{-1} \sum_{k=1}^{N} q_k (1 + \epsilon p_k)(1 + \epsilon \alpha q_{k-1})E_{k,k+1}. \tag{10.13}
\]

Notice that the formulas (10.9) for the Miura map $M_{1}^{(+)}(\alpha; \epsilon)$ are equivalent to the factorization

\[
(1 - \frac{\epsilon}{\alpha})W(a^{(1)}, b^{(1)}, \lambda) + \frac{\epsilon}{\alpha} L(a^{(1)}, b^{(1)}, \lambda) = P_1(q, p, \lambda)Q_1(q, p, \lambda), \tag{10.14}
\]

along with the formula $W(a^{(1)}, b^{(1)}, \lambda) = W_1(q, p, \lambda)$.

The second Lax matrix $(P_2, Q_2, W_2^{-1}) : \mathcal{MRT} \mapsto g \otimes g \otimes g$:

\[
P_2(q, p, \lambda) = \sum_{k=1}^{N} (1 + \epsilon p_k)(1 + \epsilon \alpha q_k)E_{kk} + \epsilon \lambda \sum_{k=1}^{N} E_{k+1,k}, \tag{10.15}
\]

\[
Q_2(q, p, \lambda) = I + (\epsilon - \alpha)\lambda^{-1} \sum_{k=1}^{N} q_k E_{k,k+1}, \tag{10.16}
\]

\[
W_2(q, p, \lambda) = I - \alpha \lambda^{-1} \sum_{k=1}^{N} q_k (1 + \epsilon p_{k+1})(1 + \epsilon \alpha q_{k+1})E_{k,k+1}. \tag{10.17}
\]

The formulas (10.10) for the Miura map $M_{2}^{(+)}(\alpha; \epsilon)$ are equivalent to the factorization

\[
(1 - \frac{\epsilon}{\alpha})W(a^{(2)}, b^{(2)}, \lambda) + \frac{\epsilon}{\alpha} L(a^{(2)}, b^{(2)}, \lambda) = Q_2(q, p, \lambda)P_2(q, p, \lambda), \tag{10.18}
\]

along with the formula $W(a^{(2)}, b^{(2)}, \lambda) = W_2(q, p, \lambda)$.

The relation between these two Lax representation is established via the following formulas:

\[
Q_1 = Q_2, \quad W_1^{-1}P_1 = P_2W_2^{-1}. \tag{10.19}
\]
The first Lax representation of $MRTL^+_+(\alpha; \epsilon)$:

$$
\dot{P}_1 = P_1A_3 - A_1P_1 , \quad (10.20)
$$

$$
\dot{W}_1 = W_1A_2 - A_1W_1 , \quad (10.21)
$$

$$
\dot{Q}_1 = Q_1A_2 - A_3Q_1 , \quad (10.22)
$$

where

$$
A_1 = \pi_+((P_1Q_1W_1^{-1} - I)/\epsilon) = \sum_{k=1}^{N} (b_k^{(1)} + \alpha a_{k-1}^{(1)}) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} ,
$$

$$
A_2 = \pi_+((W_1^{-1}P_1Q_1 - I)/\epsilon) = \sum_{k=1}^{N} (b_k^{(1)} + \alpha a_k^{(1)}) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} ,
$$

$$
A_3 = \pi_+((Q_1W_1^{-1}P_1 - I)/\epsilon) = \sum_{k=1}^{N} (b_k^{(2)} + \alpha a_{k-1}^{(2)}) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} .
$$

The second Lax representation of $MRTL^+_+(\alpha; \epsilon)$:

$$
\dot{P}_2 = P_2B_3 - B_1P_2 , \quad (10.23)
$$

$$
\dot{Q}_2 = Q_2B_1 - B_2Q_2 , \quad (10.24)
$$

$$
\dot{W}_2 = W_2B_3 - B_2W_2 , \quad (10.25)
$$

where

$$
B_1 = \pi_+((P_2W_2^{-1}Q_2 - I)/\epsilon) = \sum_{k=1}^{N} (b_k^{(1)} + \alpha a_k^{(1)}) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} ,
$$

$$
B_2 = \pi_+((Q_2P_2W_2^{-1} - I)/\epsilon) = \sum_{k=1}^{N} (b_k^{(2)} + \alpha a_{k-1}^{(2)}) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} ,
$$

$$
B_3 = \pi_+((W_2^{-1}Q_2P_2 - I)/\epsilon) = \sum_{k=1}^{N} (b_k^{(2)} + \alpha a_k^{(2)}) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} .
$$

The relations (10.19) assure the identities

$$
A_2 = B_1 , \quad A_3 = B_2 .
$$

Let us mention the following expressions for the entries of the matrices $A_j, B_j$ above:

$$
b_k^{(1)} + \alpha a_{k-1}^{(1)} = p_k + (\epsilon + \alpha)q_k - 1 + \epsilon \alpha (p_{k-1}q_{k-1} + p_kq_k - 1 + \alpha q_{k-1}q_{k-2} + \epsilon \alpha q_{k-1}p_{k-1}q_{k-2}) ,
$$

$$
b_k^{(1)} + \alpha a_k^{(1)} = p_k + \alpha q_k - 1 + \epsilon \alpha (p_kq_k + p_{k+1}q_{k+1} + \alpha q_{k-1}q_{k-2} + \epsilon \alpha q_{k-1}p_{k-1}q_{k-2}) ,
$$

$$
b_k^{(2)} + \alpha a_{k-1}^{(2)} = p_k + \alpha q_{k-1} - 1 + \epsilon \alpha (p_kq_k + p_{k-1}q_{k-1} + \alpha q_{k-1}q_{k-1} + \epsilon \alpha q_{k-1}p_{k-1}q_{k-2}) ,
$$

$$
b_k^{(2)} + \alpha a_k^{(2)} = p_k + (\epsilon + \alpha)q_k - 1 + \epsilon \alpha (p_kq_k + p_{k+1}q_{k+1} + \alpha q_{k-1}q_{k} + \epsilon \alpha q_{k-1}p_{k+1}q_{k} + \epsilon \alpha q_{k+1}q_{k} + \epsilon \alpha q_{k+1}p_{k+1}q_{k}) .
$$
The first Lax representation of $\text{MRTL}^{(+)}(\alpha; \epsilon)$:

$$
\dot{P}_1 = C_1 P_1 - P_1 C_3 , \\
\dot{W}_1 = C_1 W_1 - W_1 C_2 , \\
\dot{Q}_1 = C_3 Q_1 - Q_1 C_2 ,
$$

with

$$
C_1 = \lambda^{-1} \sum_{k=1}^{N} \frac{a_k^{(1)}}{1 + \alpha b_{k+1}^{(1)}} E_{k,k+1} = \lambda^{-1} \sum_{k=1}^{N} q_k (1 + \epsilon p_k) (1 + \epsilon \alpha q_{k-1}) (1 + \alpha p_k) E_{k,k+1} , \\
C_2 = \lambda^{-1} \sum_{k=1}^{N} \frac{a_k^{(1)}}{1 + \alpha b_{k+1}^{(1)}} E_{k,k+1} = \lambda^{-1} \sum_{k=1}^{N} q_k (1 + \epsilon p_k) (1 + \epsilon \alpha q_k) E_{k,k+1} , \\
C_3 = \lambda^{-1} \sum_{k=1}^{N} \frac{a_k^{(2)}}{1 + \alpha b_{k+1}^{(2)}} E_{k,k+1} = \lambda^{-1} \sum_{k=1}^{N} q_k (1 + \epsilon p_{k+1}) (1 + \epsilon \alpha q_k) E_{k,k+1} .
$$

The second Lax representation of $\text{MRTL}^{(+)}(\alpha; \epsilon)$:

$$
\dot{P}_2 = D_1 P_2 - P_2 D_3 , \\
\dot{Q}_2 = D_2 Q_2 - Q_2 D_1 , \\
\dot{W}_2 = D_2 W_2 - W_2 D_3 ,
$$

with

$$
D_1 = \lambda^{-1} \sum_{k=1}^{N} \frac{a_k^{(1)}}{1 + \alpha b_{k+1}^{(1)}} E_{k,k+1} = \lambda^{-1} \sum_{k=1}^{N} \frac{q_k (1 + \epsilon p_k)}{1 + \alpha p_k} E_{k,k+1} , \\
D_2 = \lambda^{-1} \sum_{k=1}^{N} \frac{a_k^{(2)}}{1 + \alpha b_{k+1}^{(2)}} E_{k,k+1} = \lambda^{-1} \sum_{k=1}^{N} \frac{q_k (1 + \epsilon p_{k+1})}{1 + \alpha p_{k+1}} E_{k,k+1} , \\
D_3 = \lambda^{-1} \sum_{k=1}^{N} \frac{a_k^{(2)}}{1 + \alpha b_{k+1}^{(2)}} E_{k,k+1} = \lambda^{-1} \sum_{k=1}^{N} \frac{q_k (1 + \epsilon p_{k+1})(1 + \epsilon \alpha q_{k+1})}{(1 + \alpha p_k)(1 + \epsilon \alpha q_k)} E_{k,k+1} .
$$

### 10.5 Discretization

We discuss here only the discretization of the flow $\text{MRTL}^{(+)}(\alpha; \epsilon)$.

The first Lax representation of $\text{dMRTL}^{(+)}(\alpha; \epsilon)$:

$$
\tilde{P}_1 = A_1^{-1} P_1 A_3 , \quad \tilde{W}_1 = A_1^{-1} W_1 A_2 , \quad \tilde{Q}_1 = A_3^{-1} Q_1 A_2
$$

with

$$
A_1 = \Pi_+ (I + \frac{h}{\epsilon} (P_1 Q_1 W_1^{-1} - I)) = \sum_{k=1}^{N} a_k E_{kk} + h \lambda \sum_{k=1}^{N} E_{k+1,k} ,
$$
\[ A_2 = \Pi_p \left( I + \frac{h}{\epsilon} (W^{-1} P_1 Q_1 - I) \right) = \sum_{k=1}^{N} b_k E_{kk} + h \lambda \sum_{k=1}^{N} E_{k+1,k} , \quad (10.34) \]

\[ A_3 = \Pi_p \left( I + \frac{h}{\epsilon} (Q_1 W^{-1} P_1 - I) \right) = \sum_{k=1}^{N} c_k E_{kk} + h \lambda \sum_{k=1}^{N} E_{k+1,k} . \quad (10.35) \]

The second Lax representation of \( dMRTL^{(+)}(\alpha; \epsilon) \):

\[ \bar{P}_2 = B_1^{-1} P_2 B_3 , \quad \bar{Q}_2 = B_2^{-1} Q_2 B_1 , \quad \bar{W}_2 = B_2^{-1} W_2 B_3 \quad (10.36) \]

with

\[ B_1 = \Pi_p \left( I + \frac{h}{\epsilon} (P_2 W^{-1} Q_2 - I) \right) = \sum_{k=1}^{N} b_k E_{kk} + h \lambda \sum_{k=1}^{N} E_{k+1,k} , \quad (10.37) \]

\[ B_2 = \Pi_p \left( I + \frac{h}{\epsilon} (Q_2 P_2 W^{-1} - I) \right) = \sum_{k=1}^{N} c_k E_{kk} + h \lambda \sum_{k=1}^{N} E_{k+1,k} , \quad (10.38) \]

\[ B_3 = \Pi_p \left( I + \frac{h}{\epsilon} (W^{-1} Q_2 P_2 - I) \right) = \sum_{k=1}^{N} f_k E_{kk} + h \lambda \sum_{k=1}^{N} E_{k+1,k} . \quad (10.39) \]

### 10.6 Application: localizing change of variables for \( dRTL^{(+)}(\alpha) \)

The role of the localizing change of variables for the map \( dRTL^{(+)}(\alpha) \) is played by the Miura transformation \( M_1^{(+)}(\alpha; h) \). Indeed, we have:

\[ \Pi_p \left( I + \frac{h}{\alpha} \left( LW^{-1} - I \right) \right) = \Pi_p \left( \left( 1 - \frac{h}{\alpha} \right) W + \frac{h}{\alpha} L \right) . \]

Compare this with (10.14). We see that if we define the change of variables \( \mathcal{R}T(a, b) \mapsto \mathcal{R}T(a, b) \) by the formulas

\[ M_1^{(+)}(\alpha; h) : \left\{ \begin{array}{l}
  b_k = b_k + h a_{k-1} + h \alpha b_k a_{k-1} , \\
  a_k = a_k (1 + h b_k) (1 + h \alpha a_{k-1}) ,
\end{array} \right\} \quad (10.40) \]

then

\[ \Pi_p \left( I + \frac{h}{\alpha} \left( LW^{-1} - I \right) \right) = P_1(a, b, \lambda) = \sum_{k=1}^{N} (1 + h b_k) (1 + h \alpha a_{k-1}) E_{kk} + h \lambda \sum_{k=1}^{N} E_{k+1,k} . \]

So, we get the following local expressions for the entries of the factors (9.21), (9.22):

\[ a_k = (1 + h b_k) (1 + h \alpha a_{k-1}) , \quad b_k = (1 + h b_k) (1 + h \alpha a_k) . \]

**Theorem 10.3** The change of variables (10.40) conjugates \( dRTL^{(+)}(\alpha) \) with the map on \( \mathcal{R}T(a, b) \) described by the following local equations of motion:

\[ \bar{b}_k + h \bar{a}_{k-1} (1 + \alpha \bar{b}_k) = b_k + h a_k (1 + \alpha b_k) , \quad (10.41) \]

\[ \bar{a}_k (1 + h \bar{b}_k) (1 + h \alpha \bar{a}_{k-1}) = a_k (1 + h b_{k+1}) (1 + h \alpha a_{k+1}) . \]
So, the local form of dRTL$_+ (\alpha)$, i.e. the map (10.41), belongs actually to the hierarchy MRTL$^{(+)} (\alpha; \hbar)$.

**Corollary.** The local form of dRTL$_+ (\alpha)$ (10.41) is a Poisson map with respect to the following Poisson bracket on $\mathcal{RT} (a, b)$:

$$\{ b_k, a_k \} = -a_k (1 + h b_k),$$
$$\{ a_k, b_{k+1} \} = -a_k (1 + h b_{k+1}),$$
$$\{ b_k, b_{k+1} \} = \frac{a_k}{1 + \hbar \alpha} (1 + h b_k) (1 + h b_{k+1}),$$

which is the pull–back of the bracket

$$\{ \cdot, \cdot \}_{1 \alpha} + h \{ \cdot, \cdot \}_{2 \alpha}$$

on $\mathcal{RT} (a, b)$ under the change of variables (10.40), and also with respect to the following Poisson bracket on $\mathcal{RT} (a, b)$, compatible with (10.42):

$$\{ b_k, a_k \} = -a_k (b_k + h a_k + h \alpha b_a k)(1 + h b_k),$$
$$\{ a_k, b_{k+1} \} = -a_k (b_{k+1} + h a_k + h \alpha b_{k+1} a_k)(1 + h b_{k+1}),$$
$$\{ b_k, b_{k+1} \} = -\frac{a_k}{1 + h \alpha a_k} (1 + h b_k) (1 + h b_{k+1}),$$
$$\{ a_k, a_{k+1} \} = -a_k a_{k+1} (1 + h \alpha a_k) (1 + h b_{k+1} a_k) (1 + h a_{k+1}),$$

which is the pull–back of the bracket

$$\{ \cdot, \cdot \}_{2 \alpha} + h \{ \cdot, \cdot \}_{3 \alpha}$$

on $\mathcal{RT} (a, b)$ under the change of variables (10.40).

### 11 Modified relativistic Toda lattice II

We introduce now a modified relativistic Toda hierarchy MRTL$^{(-)} (\alpha; \epsilon)$ different from the one considered in the previous section. The presentation will be parallel to that of the previous section, and there will be several similar objects denoted by the same letters. This should not lead to a confusion.

#### 11.1 Equations of motion

In particular, we shall use the notation $\mathcal{MT} = \mathbb{R}^{2N} (q, p)$ for the phase space of the MRTL$^{(-)} (\alpha; \epsilon)$ hierarchy.

Equations of motion of MRTL$^{(-)} (\alpha; \epsilon)$:

$$\dot{p}_k = (1 + (\epsilon + \alpha) p_k) (q_k - q_{k-1}),$$
$$\dot{q}_k = q_k \left( p_{k+1} + \alpha q_{k+1} + \frac{\epsilon \alpha p_{k+1} q_{k+1}}{1 + \alpha p_{k+1}} - p_k - \alpha q_{k-1} - \frac{\epsilon \alpha p_k q_{k-1}}{1 + \alpha p_k} \right).$$

(11.1)
Equations of motion of $\text{MRTL}_\pm^-(\alpha; \epsilon)$:

\begin{align*}
\dot{p}_k &= (1 + (\epsilon + \alpha)p_k) \left\{\frac{q_k}{(1 + \alpha p_k)(1 + \alpha p_{k+1}) + \epsilon\alpha q_k} \right\} - \left\{\frac{q_{k-1}}{(1 + \alpha p_{k-1})(1 + \alpha p_k + \epsilon\alpha q_{k-1})}\right\}, \\
\dot{q}_k &= \frac{q_k(p_{k+1} - p_k)}{(1 + \alpha p_k)(1 + \alpha p_{k+1}) + \epsilon\alpha q_k}.
\end{align*}

(11.2)

### 11.2 Bi–Hamiltonian structure

**Proposition 11.1**  

a) The relations

\begin{align*}
\{p_k, q_k\}_{12\alpha} &= -q_k \left(1 + (\epsilon + \alpha)p_k\right), \\
\{q_k, p_{k+1}\}_{12\alpha} &= -q_k \left(1 + (\epsilon + \alpha)p_{k+1}\right), \\
\{q_k, q_{k+1}\}_{12\alpha} &= -\alpha q_k q_{k+1} \left(1 + \frac{\epsilon p_{k+1}}{1 + \alpha p_{k+1}}\right).
\end{align*}

(11.3)

define a Poisson bracket on $\mathcal{MRT}(q, p)$. The systems $\text{MRTL}_\pm^-(\alpha; \epsilon)$ are Hamiltonian with respect to the bracket (11.3), with the Hamilton functions

\begin{align*}
G_{1\alpha}^+(q, p) &= (\epsilon + \alpha)^{-1} \sum_{k=1}^{N} p_k + \sum_{k=1}^{N} q_k, \\
G_{1\alpha}^-(q, p) &= (\epsilon\alpha)^{-1} \sum_{k=1}^{N} \log(1 + \alpha p_k) + (\epsilon\alpha)^{-1} \sum_{k=1}^{N} \log \left(1 + \frac{\epsilon\alpha q_k}{(1 + \alpha p_k)(1 + \alpha p_{k+1})}\right),
\end{align*}

(11.4)  

(11.5)

respectively. The functions $G_{1\alpha}^+$ and $G_{1\alpha}^-$ are in involution in the bracket $\{\cdot, \cdot\}_{12\alpha}$.  

b) The relations

\begin{align*}
\{p_k, q_k\}_{23\alpha} &= -q_k \left(p_k + (\epsilon + \alpha)q_k\right) \left(1 + (\epsilon + \alpha)p_k\right), \\
\{q_{k+1}, p_{k+1}\}_{23\alpha} &= -q_k \left(p_{k+1} + (\epsilon + \alpha)q_k\right) \left(1 + (\epsilon + \alpha)p_{k+1}\right), \\
\{q_k, q_{k+1}\}_{23\alpha} &= -\alpha q_k q_{k+1} \left(1 + 2\alpha p_{k+1} + \alpha(\epsilon + \alpha)(q_k + q_{k+1})\right) \left(1 + \frac{\epsilon p_{k+1}}{1 + \alpha p_{k+1}}\right), \\
\{p_k, q_{k+1}\}_{23\alpha} &= -\alpha q_k q_{k+1} \left(1 + (\epsilon + \alpha)p_k\right) \left(1 + \frac{\epsilon p_{k+1}}{1 + \alpha p_{k+1}}\right), \\
\{q_k, q_{k+2}\}_{23\alpha} &= -\alpha q_k q_{k+1} \left(1 + (\epsilon + \alpha)p_{k+2}\right) \left(1 + \frac{\epsilon p_{k+1}}{1 + \alpha p_{k+1}}\right), \\
\{q_k, q_{k+2}\}_{23\alpha} &= -\alpha^2 q_k q_{k+1} q_{k+2} \left(1 + \frac{\epsilon p_{k+1}}{1 + \alpha p_{k+1}}\right) \left(1 + \frac{\epsilon p_{k+2}}{1 + \alpha p_{k+2}}\right).
\end{align*}

(11.6)

define a Poisson bracket on $\mathcal{MRT}$ compatible with (11.3). The systems $\text{MRTL}_\pm^-(\alpha; \epsilon)$ are Hamiltonian with respect to this bracket with the Hamilton function

\begin{align*}
G_{1\alpha}^+(q, p) &= (\epsilon + \alpha)^{-1} \sum_{k=1}^{N} \log \left(1 + (\epsilon + \alpha)p_k\right),
\end{align*}

(11.7)
\[ G_{1\alpha}^{-}(q, p) = \epsilon^{-1} \sum_{k=1}^{N} \log \left( 1 + \frac{\epsilon p_k}{1 + \alpha p_k} \right) - \epsilon^{-1} \sum_{k=1}^{N} \log \left( 1 + \frac{\epsilon \alpha q_k}{(1 + \alpha p_k)(1 + \alpha p_{k+1})} \right), \] (11.8)

respectively. The functions \( G_{0\alpha}^{(+)} \) and \( G_{0\alpha}^{-} \) are in involution in the bracket \( \{ \cdot, \cdot \}_{23\alpha} \).

Notice that the function \( G_{1\alpha}^{-}(q, p) \) becomes regular in \( \epsilon \) upon subtracting

\[ (\epsilon \alpha)^{-1} \sum_{k=1}^{N} \log \left( 1 + (\epsilon + \alpha)p_k \right), \]

which is a Casimir function of the bracket \( \{ \cdot, \cdot \}_{12\alpha} \).

### 11.3 Miura relations

The relation between MRTL\(^{(-)}\)(\(\alpha; \epsilon\)) and RTL(\(\alpha\)) hierarchies is established in the following statement.

**Proposition 11.2** Define the Miura maps \( M_{1,2}^{(-)}(\alpha; \epsilon) : \) MRTL\((q, p) \leftrightarrow \) RTL\((a, b) \) by:

\[
M_{1}^{(-)}(\alpha; \epsilon) : \begin{cases} 
    b_k = p_k + \frac{\epsilon q_{k-1}}{1 + \alpha p_{k-1}}, \\
    a_k = q_k \left(1 + \frac{\epsilon p_k}{1 + \alpha p_k}\right),
\end{cases} \tag{11.9}
\]

\[
M_{2}^{(-)}(\alpha; \epsilon) : \begin{cases} 
    b_k = p_k + \frac{\epsilon q_{k}}{1 + \alpha p_{k+1}}, \\
    a_k = q_k \left(1 + \frac{\epsilon p_{k+1}}{1 + \alpha p_{k+1}}\right),
\end{cases} \tag{11.10}
\]

Both maps \( M_{1,2}^{(-)}(\alpha; \epsilon) \) are Poisson, if \( \text{MRTL}(q, p) \) is equipped with the bracket \( \{ \cdot, \cdot \}_{12\alpha} \), and \( \text{RTL}(a, b) \) is equipped with \( \{ \cdot, \cdot \}_{1a} + (\epsilon + \alpha)\{ \cdot, \cdot \}_{2a} \), and also if \( \text{MRTL}(q, p) \) is equipped with the bracket \( \{ \cdot, \cdot \}_{23\alpha} \), and \( \text{RTL}(a, b) \) is equipped with \( \{ \cdot, \cdot \}_{2a} + (\epsilon + \alpha)\{ \cdot, \cdot \}_{3a} \). The pull–back of the flows RTL\(_{\pm}(\alpha) \) under either of the Miura maps \( M_{1,2}^{(-)}(\alpha; \epsilon) \) coincides with MRTL\(_{\pm}^{(-)}(\alpha; \epsilon) \).

In the rest of this section we shall sometimes denote by \( a_k^{(1)} = a_k^{(1)}(q, r), b_k^{(1)} = b_k^{(1)}(q, r) \) the functions given by the formulas (11.9), and by \( a_k^{(2)} = a_k^{(2)}(q, r), b_k^{(2)} = b_k^{(2)}(q, r) \) the functions given by the formulas (11.10).

### 11.4 Lax representations

The MRTL\(^{(-)}\) hierarchy, like the MRTL\(^{(+)}\) one, possesses two different Lax representations.
The first Lax matrix \((P_1, W_{1}^{-1}, Q_1) : \mathcal{MRT} \mapsto g \otimes g \otimes g\):

\[
P_1(q, p, \lambda) = \sum_{k=1}^{N} \left(1 + (\epsilon + \alpha)p_k\right)E_{kk} + (\epsilon + \alpha)\lambda \sum_{k=1}^{N} E_{k+1,k}, \quad (11.11)
\]

\[
Q_1(q, p, \lambda) = I + \epsilon\lambda^{-1} \sum_{k=1}^{N} \frac{q_k}{1 + \alpha p_k} E_{k,k+1}, \quad (11.12)
\]

\[
W_1(q, p, \lambda) = I - \alpha\lambda^{-1} \sum_{k=1}^{N} q_k \left(1 + \frac{\epsilon p_k}{1 + \alpha p_k}\right) E_{k,k+1}. \quad (11.13)
\]

The formulas (11.9) for the Miura map \(M_1^{(-)}(\alpha; \epsilon)\) are equivalent to the factorization

\[
\left(1 + \frac{\epsilon}{\alpha}\right)L(a^{(1)}, b^{(1)}, \lambda) - \frac{\epsilon}{\alpha} W(a^{(1)}, b^{(1)}, \lambda) = P_1(q, p, \lambda)Q_1(q, p, \lambda), \quad (11.14)
\]

along with the formula \(W(a^{(1)}, b^{(1)}, \lambda) = W_1(q, p, \lambda)\).

The second Lax matrix \((P_2, Q_2, W_2^{-1}) : \mathcal{MRT} \mapsto g \otimes g \otimes g\):

\[
P_2(q, p, \lambda) = \sum_{k=1}^{N} \left(1 + (\epsilon + \alpha)p_k\right)E_{kk} + (\epsilon + \alpha)\lambda \sum_{k=1}^{N} E_{k+1,k}, \quad (11.15)
\]

\[
Q_2(q, p, \lambda) = I + \epsilon\lambda^{-1} \sum_{k=1}^{N} \frac{q_k}{1 + \alpha p_k+1} E_{k,k+1}, \quad (11.16)
\]

\[
W_2(q, p, \lambda) = I - \alpha\lambda^{-1} \sum_{k=1}^{N} q_k \left(1 + \frac{\epsilon p_k+1}{1 + \alpha p_k+1}\right) E_{k,k+1}. \quad (11.17)
\]

The formulas (11.10) for the Miura map \(M_2^{(-)}(\alpha; \epsilon)\) are equivalent to the factorization

\[
\left(1 + \frac{\epsilon}{\alpha}\right)L(a^{(2)}, b^{(2)}, \lambda) - \frac{\epsilon}{\alpha} W(a^{(2)}, b^{(2)}, \lambda) = Q_2(q, p, \lambda)P_2(q, p, \lambda), \quad (11.18)
\]

along with the formula \(W(a^{(2)}, b^{(2)}, \lambda) = W_2(q, p, \lambda)\).

The relation between these two Lax representation is established via the following formulas:

\[
P_1 = P_2, \quad Q_1W_1^{-1} = W_2^{-1}Q_2. \quad (11.19)
\]

The first Lax representation of \(\mathcal{MRTL}^{(-)}_+(\alpha; \epsilon)\):

\[
\dot{P}_1 = P_1A_3 - A_1P_1, \quad (11.20)
\]

\[
\dot{W}_1 = W_1A_2 - A_1W_1, \quad (11.21)
\]

\[
\dot{Q}_1 = Q_1A_2 - A_3Q_1, \quad (11.22)
\]

where

\[
A_j = \pi_+((\epsilon + \alpha)^{-1}(T_j - I)), \quad j = 1, 2, 3,
\]
with

\[ T_1 = P_1 Q_1 W_1^{-1} , \quad T_2 = W_1^{-1} P_1 Q_1 , \quad T_3 = Q_1 W_1^{-1} P_1 . \]

So, we have the following expressions:

\[
A_1 = \sum_{k=1}^{N} \left( b_k^{(1)} + \alpha a_{k-1}^{(1)} \right) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} \\
= \sum_{k=1}^{N} \left( p_k + (\epsilon + \alpha) q_{k-1} \right) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} ,
\]

\[
A_2 = \sum_{k=1}^{N} \left( b_k^{(1)} + \alpha a_{k-1}^{(1)} \right) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} \\
= \sum_{k=1}^{N} \left( p_k + \alpha q_k + \frac{\epsilon q_{k-1}}{1 + \alpha p_{k-1}} + \frac{\epsilon \alpha q_k p_k}{1 + \alpha p_k} \right) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} ,
\]

\[
A_3 = \sum_{k=1}^{N} \left( b_k^{(2)} + \alpha a_{k-1}^{(2)} \right) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} \\
= \sum_{k=1}^{N} \left( p_k + (\epsilon + \alpha) q_k \right) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} .
\]

The second Lax representation of \(MRTL_+ (\alpha; \epsilon)\):

\[
\dot{P}_2 = P_2 B_3 - B_1 P_2 , \quad (11.23)
\]

\[
\dot{Q}_2 = Q_2 B_1 - B_2 Q_2 , \quad (11.24)
\]

\[
\dot{W}_2 = W_2 B_3 - B_2 W_2 , \quad (11.25)
\]

where

\[
B_j = \pi_+ \left( (\epsilon + \alpha)^{-1} (T_j - I) \right) , \quad j = 1, 2, 3 ,
\]

with

\[
T_1 = P_2 W_2^{-1} Q_2 , \quad T_2 = Q_2 P_2 W_2^{-1} , \quad T_3 = W_2^{-1} Q_2 P_2 .
\]

So, we have now the following expressions:

\[
B_1 = \sum_{k=1}^{N} \left( b_k^{(1)} + \alpha a_{k-1}^{(1)} \right) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} \\
= \sum_{k=1}^{N} \left( p_k + (\epsilon + \alpha) q_{k-1} \right) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} ,
\]

\[
B_2 = \sum_{k=1}^{N} \left( b_k^{(2)} + \alpha a_{k-1}^{(2)} \right) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} \\
= \sum_{k=1}^{N} \left( p_k + \alpha q_{k-1} + \frac{\epsilon q_k}{1 + \alpha p_{k-1}} + \frac{\epsilon \alpha q_k p_k}{1 + \alpha p_k} \right) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} ,
\]

\[
B_3 = \sum_{k=1}^{N} \left( b_k^{(2)} + \alpha a_{k-1}^{(2)} \right) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} \\
= \sum_{k=1}^{N} \left( p_k + (\epsilon + \alpha) q_k \right) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k} .
\]
The relations (11.19) assure the identities

\[ A_1 = B_1, \quad A_3 = B_3. \]

Turning to the flow \( MRTL_+^{(+)} (\alpha; \epsilon) \), we have the first Lax representation:

\[ \dot{P}_1 = C_1 P_1 - P_1 C_3, \quad (11.26) \]
\[ \dot{W}_1 = C_1 W_1 - W_1 C_2, \quad (11.27) \]
\[ \dot{Q}_1 = C_3 Q_1 - Q_1 C_2, \quad (11.28) \]

with

\[
C_1 = \lambda^1 \sum_{k=1}^{N} \frac{a_k^{(1)}}{1 + \alpha b_k^{(1)}} E_{k,k+1}
\]
\[
= \lambda^1 \sum_{k=1}^{N} q_k \left(1 + (\epsilon + \alpha) p_k\right) E_{k,k+1},
\]
\[
C_2 = \lambda^1 \sum_{k=1}^{N} \frac{a_k^{(2)}}{1 + \alpha b_k^{(2)}} E_{k,k+1}
\]
\[
= \lambda^1 \sum_{k=1}^{N} \frac{q_k \left(1 + (\epsilon + \alpha) p_k\right) \left(1 + \alpha p_{k-1}\right)}{(1 + \alpha p_k)(1 + \alpha p_{k-1}) + \epsilon \alpha q_k} E_{k,k+1},
\]
\[
C_3 = \lambda^1 \sum_{k=1}^{N} \frac{a_k^{(2)}}{1 + \alpha b_k^{(2)}} E_{k,k+1}
\]
\[
= \lambda^1 \sum_{k=1}^{N} \frac{q_k \left(1 + (\epsilon + \alpha) p_{k+1}\right)}{(1 + \alpha p_{k+1})(1 + \alpha p_k) + \epsilon \alpha q_k} E_{k,k+1}.
\]

The second Lax representation of \( MRTL_+^{(+)} (\alpha; \epsilon) \):

\[ \dot{P}_2 = D_1 P_2 - P_2 D_3, \quad (11.29) \]
\[ \dot{Q}_2 = D_2 Q_2 - Q_2 D_1, \quad (11.30) \]
\[ \dot{W}_2 = D_2 W_2 - W_2 D_3, \quad (11.31) \]

with

\[
D_1 = \lambda^1 \sum_{k=1}^{N} \frac{a_k^{(1)}}{1 + \alpha b_k^{(1)}} E_{k,k+1}
\]
\[
= \lambda^1 \sum_{k=1}^{N} \frac{q_k \left(1 + (\epsilon + \alpha) p_k\right)}{(1 + \alpha p_{k+1})(1 + \alpha p_k) + \epsilon \alpha q_k} E_{k,k+1},
\]
\[
D_2 = \lambda^1 \sum_{k=1}^{N} \frac{a_k^{(2)}}{1 + \alpha b_k^{(2)}} E_{k,k+1}
\]
\[ D_3 = \lambda^{-1} \sum_{k=1}^{N} \frac{a_k^{(2)}}{1 + \alpha b_k^{(2)}} E_{k,k+1} \]

\[ = \lambda^{-1} \sum_{k=1}^{N} \frac{q_k (1 + (\epsilon + \alpha)p_{k+1})}{(1 + \alpha p_{k+2})(1 + \alpha p_{k+1}) + \epsilon \alpha q_{k+1}} (1 + \alpha p_{k+1}) E_{k,k+1} \]

11.5 Discretization

We discuss here only the discretization of the flow \( \text{MRTL}^{(+)}_+(\alpha; \epsilon) \).

The first Lax representation of \( \text{dMRTL}^{(-)}_+ (\alpha; \epsilon) \):

\[ \tilde{P}_1 = A_1^{-1} P_1 A_3, \quad \tilde{W}_1 = A_1^{-1} W_1 A_2, \quad \tilde{Q}_1 = A_3^{-1} Q_1 A_2 \quad (11.32) \]

with

\[ A_1 = \Pi_+ \left( I + \frac{h}{\epsilon + \alpha} (P_1 Q_1 W_1^{-1} - I) \right) = \sum_{k=1}^{N} a_k E_{kk} + h \lambda \sum_{k=1}^{N} E_{k+1,k}, \]

\[ A_2 = \Pi_+ \left( I + \frac{h}{\epsilon + \alpha} (W_1^{-1} P_1 Q_1 - I) \right) = \sum_{k=1}^{N} b_k E_{kk} + h \lambda \sum_{k=1}^{N} E_{k+1,k}, \]

\[ A_3 = \Pi_+ \left( I + \frac{h}{\epsilon + \alpha} (Q_1 W_1^{-1} P_1 - I) \right) = \sum_{k=1}^{N} \epsilon_k E_{kk} + h \lambda \sum_{k=1}^{N} E_{k+1,k}. \]

The second Lax representation of \( \text{dMRTL}^{(+)}_+ (\alpha; \epsilon) \):

\[ \tilde{P}_2 = B_1^{-1} P_2 B_3, \quad \tilde{Q}_2 = B_2^{-1} Q_2 B_1, \quad \tilde{W}_2 = B_2^{-1} W_2 B_3 \quad (11.33) \]

with

\[ B_1 = \Pi_+ \left( I + \frac{h}{\epsilon + \alpha} (P_2 W_2^{-1} Q_2 - I) \right) = \sum_{k=1}^{N} b_k E_{kk} + h \lambda \sum_{k=1}^{N} E_{k+1,k}, \]

\[ B_2 = \Pi_+ \left( I + \frac{h}{\epsilon + \alpha} (Q_2 P_2 W_2^{-1} - I) \right) = \sum_{k=1}^{N} \epsilon_k E_{kk} + h \lambda \sum_{k=1}^{N} E_{k+1,k}, \]

\[ B_3 = \Pi_+ \left( I + \frac{h}{\epsilon + \alpha} (W_2^{-1} Q_2 P_2 - I) \right) = \sum_{k=1}^{N} j_k E_{kk} + h \lambda \sum_{k=1}^{N} E_{k+1,k}. \]

11.6 Application: localizing change of variables for \( \text{dRTL}(\alpha) \)

The Miura transformation \( \text{M}^{(-)}_1 (\alpha; h) \) plays the role of the localizing change of variables for \( \text{dRTL} (\alpha) \). The reason for this lies in the following identity:

\[ \Pi_- \left( I + \frac{h}{\alpha} (I - L^{-1} W) \right) = \Pi_- \left( (1 + \frac{h}{\alpha}) L - \frac{h}{\alpha} W \right). \]
This has to be compared with (11.14). This comparison implies that, if we define the change of variables $\mathcal{R}\mathcal{T}(a, b) \mapsto \mathcal{R}\mathcal{T}(a, b)$ by the formulas

$$M_1(-)(\alpha; h) : \begin{cases} b_k = b_k + \frac{h a_{k-1}}{1 + \alpha b_{k-1}}, \\
a_k = a_k \left(1 + \frac{h b_k}{1 + \alpha b_k}\right), \end{cases} \quad (11.34)$$

then

$$\Pi_-(I + \frac{h}{\alpha} (I - L^{-1}W)) = Q_1(a, b, \lambda) = I + h\lambda^{-1} \sum_{k=1}^{N} \frac{a_k}{1 + \alpha b_k} E_{k,k+1}.$$ 

So, we get for the entries of the factors (9.24), (9.25) the following local expressions:

$$c_k = \frac{a_k}{1 + \alpha b_{k+1}}, \quad \delta_k = \frac{a_k}{1 + \alpha b_k}.$$

**Theorem 11.3** The change of variables (11.34) conjugates $d\mathcal{R}\mathcal{T}_-(\alpha)$ with the map on $\mathcal{R}\mathcal{T}(a, b)$ described by the following local equations of motion:

$$\bar{b}_k + \frac{h \bar{a}_{k-1}}{1 + \alpha \bar{b}_{k-1}} = b_k + \frac{h a_k}{1 + \alpha b_{k+1}},$$

$$\bar{a}_k \left(1 + \frac{h b_k}{1 + \alpha b_k}\right) = a_k \left(1 + \frac{h b_{k+1}}{1 + \alpha b_{k+1}}\right). \quad (11.35)$$

So, the local form of $d\mathcal{R}\mathcal{T}_-(\alpha)$ (11.35) belongs to the hierarchy $\mathcal{M}\mathcal{R}\mathcal{T}(-)(\alpha; h)$.

**Corollary.** The local form of $d\mathcal{R}\mathcal{T}_-(\alpha)$ (11.35) is a Poisson map with respect to the following Poisson bracket on $\mathcal{R}\mathcal{T}(a, b)$:

$$\begin{align*}
\{b_k, a_k\} &= -a_k \left(1 + (h + \alpha) b_k\right), \\
\{a_k, b_{k+1}\} &= -a_k \left(1 + (h + \alpha) b_{k+1}\right), \\
\{a_k, a_{k+1}\} &= -a_k a_{k+1} \left(1 + \frac{h b_{k+1}}{1 + \alpha b_{k+1}}\right),
\end{align*} \quad (11.36)$$

which is the pull–back of the bracket

$$\{\cdot, \cdot\}_{1\alpha} + (h + \alpha) \{\cdot, \cdot\}_{2\alpha} \quad (11.37)$$

on $\mathcal{R}\mathcal{T}(a, b)$ under the change of variables (11.34), and also with respect to the following Poisson bracket on $\mathcal{R}\mathcal{T}(a, b)$ compatible with (11.36):

$$\begin{align*}
\{b_k, a_k\} &= -a_k (b_k + (h + \alpha) a_k) \left(1 + (h + \alpha) b_k\right), \\
\{a_k, b_{k+1}\} &= -a_k (b_{k+1} + (h + \alpha) a_k) \left(1 + (h + \alpha) b_{k+1}\right), \\
\{b_k, b_{k+1}\} &= -a_k \left(1 + (h + \alpha) b_k\right) \left(1 + (h + \alpha) b_{k+1}\right),
\end{align*}$$

which is the pull–back of the bracket

$$\{\cdot, \cdot\}_{1\alpha} + (h + \alpha) \{\cdot, \cdot\}_{2\alpha} \quad (11.37)$$

on $\mathcal{R}\mathcal{T}(a, b)$ under the change of variables (11.34), and also with respect to the following Poisson bracket on $\mathcal{R}\mathcal{T}(a, b)$ compatible with (11.36):
\{a_k, a_{k+1}\} = -a_k a_{k+1} \left( 1 + 2 \alpha b_{k+1} + \alpha (h + \alpha) (a_k + a_{k+1}) \right) \left( 1 + \frac{h b_{k+1}}{1 + \alpha b_{k+1}} \right),

\{b_k, a_{k+1}\} = -\alpha a_k a_{k+1} \left( 1 + (h + \alpha) b_k \right) \left( 1 + \frac{h b_{k+1}}{1 + \alpha b_{k+1}} \right),

\{a_k, b_{k+2}\} = -\alpha a_k a_{k+1} \left( 1 + (h + \alpha) b_{k+2} \right) \left( 1 + \frac{h b_{k+1}}{1 + \alpha b_{k+1}} \right),

\{a_k, a_{k+2}\} = -\alpha^2 a_k a_{k+1} a_{k+2} \left( 1 + \frac{h b_{k+1}}{1 + \alpha b_{k+1}} \right) \left( 1 + \frac{h b_{k+2}}{1 + \alpha b_{k+2}} \right),

which is the pull-back of the bracket

\{\cdot, \cdot\}_{2\alpha} + (h + \alpha) \{\cdot, \cdot\}_3\alpha

on \mathcal{RT}(a, b) under the change of variables (11.34).

12 Double modified relativistic Toda lattice

12.1 Equations of motion

We shall not elaborate all branches of the tree of successive modifications of the relativistic Toda lattice, and restrict ourselves with one possible modification of one of the modified relativistic Todas, namely of MRTL\(^{(+)\alpha; \epsilon}\)). The phase space of the corresponding hierarchy \(M^2\mathcal{RTL}^{(+)\alpha; \epsilon, \delta}\) will be denoted by \(\mathcal{M}^2\mathcal{RT} = \mathbb{R}^{2N}(r, s)\). The equations of motion of the simplest representative of this hierarchy read:

\dot{r}_k = \frac{(1 + \alpha r_k)(1 + \epsilon r_k)(1 + \delta r_k)(s_k - s_{k-1})}{1 - \epsilon \delta \alpha D(s_k, r_k, s_{k-1})},

(12.1)

\dot{s}_k = s_k (1 + \epsilon \alpha s_k) (1 + \delta \alpha s_k) (1 + \epsilon \delta s_k) \left( \frac{r_{k+1} + \alpha s_{k+1} + \alpha (\epsilon + \delta) r_{k+1} s_{k+1}}{1 - \epsilon \delta \alpha D(s_{k+1}, r_{k+1}, s_k)} - \frac{r_k + \alpha s_{k-1} + \alpha (\epsilon + \delta) r_k s_{k-1}}{1 - \epsilon \delta \alpha D(s_k, r_k, s_{k-1})} \right),

(12.2)

where

\[ D(s_k, r_k, s_{k-1}) = r_k s_{k-1} + s_k r_k + \alpha s_k s_{k-1} + \alpha (\epsilon + \delta) s_k r_k s_{k-1}. \]

(12.3)

12.2 Hamiltonian structure

Proposition 12.1 The relations

\[ \{r_k, s_k\}_{123\alpha} = -s_k (1 + \epsilon \delta s_k) (1 + \epsilon r_k) (1 + \delta r_k), \]

\[ \{s_k, r_{k+1}\}_{123\alpha} = -s_k (1 + \epsilon \delta s_k) (1 + \epsilon r_{k+1}) (1 + \delta r_{k+1}), \]

\[ \{r_k, r_{k+1}\}_{123\alpha} = \alpha \frac{s_k (1 + \epsilon \delta s_k)}{(1 + \epsilon \alpha s_k) (1 + \delta \alpha s_k)} (1 + \epsilon r_k) (1 + \delta r_k) (1 + \epsilon r_{k+1}) (1 + \delta r_{k+1}). \]

(12.4)

define a Poisson bracket on \(\mathcal{M}^2\mathcal{RT}\). The flow \(\text{M}^2\text{RTL}^{(+)\alpha; \epsilon, \delta}\) is Hamiltonian with respect to this bracket with the Hamilton function

\[ F_{\alpha \epsilon}(r, s) = (\epsilon \delta)^{-1} \sum_{k=1}^{N} \log(1 + \delta r_k) + (\epsilon \delta)^{-1} \sum_{k=1}^{N} \log(1 + \epsilon \delta s_k) + (\epsilon \delta)^{-1} \sum_{k=1}^{N} \log(1 + \delta \alpha s_k). \]
\[-(\epsilon\delta)^{-1}\sum_{k=1}^{N} \log \left(1 - \epsilon\delta\alpha D(s_k, r_k, s_{k-1})\right).\] (12.5)

### 12.3 Miura relations

**Theorem 12.2** Define the Miura maps \(M_{1,2}^{(+)}(\alpha; \epsilon; \delta) : \mathcal{M}^2{\mathcal{R}\mathcal{T}} \hookrightarrow \mathcal{M}\mathcal{R}\mathcal{T}\) by the formulas:

\[
M_{1}^{(+)}(\alpha; \epsilon; \delta) : \begin{cases} 
 p_k = \frac{r_k + \delta s_{k-1} + \delta(\epsilon + \alpha)r_k s_{k-1}}{1 - \epsilon\delta\alpha r_k s_{k-1}}, \\
 q_k = \frac{s_k(1 + \delta r_k)(1 + \delta\alpha s_{k-1})}{1 - \epsilon\delta\alpha D(s_k, r_k, s_{k-1})},
\end{cases}
\] (12.6)

\[
M_{2}^{(+)}(\alpha; \epsilon; \delta) : \begin{cases} 
 p_k = \frac{r_k + \delta s_k + \delta(\epsilon + \alpha)r_k s_k}{1 - \epsilon\delta\alpha r_k s_k}, \\
 q_k = \frac{s_k(1 + \delta r_{k+1})(1 + \delta\alpha s_{k+1})}{1 - \epsilon\delta\alpha D(s_{k+1}, r_{k+1}, s_k)},
\end{cases}
\] (12.7)

Both maps \(M_{1,2}^{(+)}(\alpha; \epsilon; \delta)\) are Poisson, if \(\mathcal{M}^2{\mathcal{R}\mathcal{T}}\) is equipped with the bracket \(\{\cdot, \cdot\}_{123}\), and \(\mathcal{M}\mathcal{R}\mathcal{T}\) is equipped with \(\{\cdot, \cdot\}_{12a} + \delta\{\cdot, \cdot\}_{23}\). The pull-back of the flow \(\text{MRTL}^{(+)}(\alpha; \epsilon)\) with respect to either of the Miura maps \(M_{1,2}^{(+)}(\alpha; \epsilon; \delta)\) coincides with \(\text{M}^2\text{RTL}^{(+)}(\alpha; \epsilon, \delta)\).

The following statement holds concerning the permutability of the Miura maps.

**Theorem 12.3** The following diagram is commutative for all \(i = 1, 2, j = 1, 2\):

\[\begin{array}{ccc}
\mathcal{M}^2\mathcal{R}\mathcal{T} & \xrightarrow{M_j^{(+)}(\alpha; \delta; \epsilon)} & \mathcal{M}\mathcal{R}\mathcal{T} \\
\downarrow M_i^{(+)}(\alpha; \epsilon; \delta) & & \downarrow M_i^{(+)}(\alpha; \delta) \\
\mathcal{M}\mathcal{R}\mathcal{T} & \xrightarrow{M_j^{(+)}(\alpha; \epsilon)} & \mathcal{R}\mathcal{T}
\end{array}\]

### 12.4 Application: localizing change of variables for \(d\text{MRTL}^{(+)}_+(\alpha; \epsilon)\)

The map \(M_{1}^{(+)}(\alpha; \epsilon; h)\) turns out to play the role of the localizing change of variables for \(d\text{MRTL}^{(+)}_+(\alpha; \epsilon)\). Indeed, define the change of variables \(\mathcal{M}\mathcal{R}\mathcal{T}(q, p) \mapsto \mathcal{M}\mathcal{R}\mathcal{T}(q, p)\) as

\[
M_{1}^{(+)}(\alpha; \epsilon; h) : \begin{cases} 
 p_k = \frac{p_k + hq_k - 1 + h(\epsilon + \alpha)p_k q_{k-1}}{1 - h\epsilon\alpha p_k q_{k-1}}, \\
 q_k = \frac{q_k(1 + hp_k)(1 + h\alpha q_{k-1})}{1 - h\epsilon\alpha D(q_k, p_k, q_{k-1})},
\end{cases}
\] (12.8)
where, as before,

\[ D(q_k, p_k, q_{k-1}) = q_k p_k + p_k q_{k-1} + \alpha q_k q_{k-1} + (h + \epsilon)\alpha q_k p_k q_{k-1}. \]  \tag{12.9}

Let us notice that the expression for \( p_k \) is symmetric in \( \alpha, \epsilon \), and may be represented in two equivalent alternative forms:

\[ 1 + \epsilon p_k = \frac{(1 + \epsilon p_k)(1 + \epsilon q_{k-1})}{1 - \epsilon \alpha p_k q_{k-1}} \quad \Leftrightarrow \quad 1 + \alpha p_k = \frac{(1 + \alpha p_k)(1 + \alpha q_{k-1})}{1 - \epsilon \alpha p_k q_{k-1}}, \]  \tag{12.10}

or else as

\[ \frac{1 + \epsilon p_k}{1 + \alpha p_k} = \frac{(1 + \epsilon p_k)(1 + \epsilon q_{k-1})}{(1 + \alpha p_k)(1 + \epsilon q_{k-1})}. \]  \tag{12.11}

The expression for \( q_k \) in (12.8) does not enjoy the \( \alpha \leftrightarrow \epsilon \) symmetry anymore; it may be equivalently rewritten as

\[ 1 + \epsilon \alpha q_k = \frac{(1 + \epsilon \alpha q_k)(1 - \epsilon \alpha p_k q_{k-1})}{1 - \epsilon \alpha D(q_k, p_k, q_{k-1})}, \]  \tag{12.12}

or else as

\[ \frac{1 + \epsilon \alpha q_k}{1 + \epsilon \alpha q_k} = \frac{q_k(1 + \epsilon \alpha q_k)(1 + \epsilon q_{k-1})}{(1 + \epsilon \alpha q_k)(1 - \epsilon \alpha p_k q_{k-1})}. \]  \tag{12.13}

It can be proved that under this change of variables the following local expressions for the factors (10.33)-(10.35) and (10.37)-(10.39) hold:

\[ a_k = \frac{(1 + \epsilon p_k)(1 + \epsilon q_{k-1})(1 + \epsilon q_{k-2})}{(1 - \epsilon \alpha p_k q_{k-1}) (1 - \epsilon \alpha D(q_{k-1}, p_{k-1}, q_{k-2}))}, \]  \tag{12.14}

\[ b_k = \frac{(1 + \epsilon p_k)(1 + \epsilon q_{k-1})}{1 - \epsilon \alpha D(q_k, p_k, q_{k-1})}, \]  \tag{12.15}

\[ c_k = \frac{(1 + \epsilon p_k)(1 + \epsilon q_{k-2})}{1 - \epsilon \alpha D(q_{k-1}, p_{k-1})}. \]  \tag{12.16}

\[ d_k = \frac{(1 + \epsilon p_k)(1 + \epsilon q_{k-1})(1 + \epsilon q_{k-2})}{(1 - \epsilon \alpha p_k q_{k-1})(1 - \epsilon \alpha D(q_{k-1}, p_{k-1}, q_{k-2}))}. \]  \tag{12.17}

**Theorem 12.4** The change of variables (12.8) conjugates dMRTL\(_+^t\)(\(\alpha; \epsilon\)) with the map on MRTL\((q, p)\) described by the following local equations of motion:

\[ \frac{(1 + \epsilon p_k)(1 + \epsilon q_{k-1})}{1 - \epsilon \alpha p_k q_{k-1}} = \frac{(1 + \epsilon p_k)(1 + \epsilon q_k)}{1 - \epsilon \alpha q_k p_k}, \]  \tag{12.18}

\[ \tilde{q}_k(1 + \epsilon \tilde{p}_k)(1 + \epsilon q_{k-1}) = \frac{q_k(1 + \epsilon p_{k+1})(1 + \epsilon q_{k+1}^t)}{1 - \epsilon \alpha D(q_{k+1}, p_{k+1}, q_k)}. \]  \tag{12.19}

The equivalent form of the above equations of motion:

\[ \frac{1 + \epsilon \tilde{p}_k}{1 + \alpha \tilde{p}_k} \cdot \frac{1 + \epsilon \tilde{q}_{k-1}}{1 + \alpha \tilde{q}_{k-1}} = \frac{1 + \epsilon p_k}{1 + \alpha p_k} \cdot \frac{1 + \epsilon q_k}{1 + \alpha q_k}, \]  \tag{12.20}

\[ \frac{1 + \epsilon \tilde{q}_k}{1 + \epsilon \tilde{q}_k} \cdot \frac{1 + \epsilon \tilde{p}_{k+1}}{1 + \epsilon \tilde{p}_{k+1}} = \frac{1 + \epsilon q_k}{1 + \epsilon q_k} \cdot \frac{1 + \epsilon p_{k+1}}{1 + \epsilon p_{k+1}}. \]  \tag{12.21}
We see that the local form of $dMRTL_+^{(+)}(\alpha; \epsilon)$ given in the previous theorem belongs actually to the hierarchy $M^2RTL^{(+)}(\alpha; \epsilon, h)$.

**Corollary.** The local form of $dMRTL_+^{(+)}(\alpha; \epsilon)$ is a Poisson map with respect to the following Poisson bracket on $\mathcal{MRT}(q, r)$:

\[
\{ r_k, q_k \} = -q_k (1 + h\epsilon q_k)(1 + h r_k)(1 + \epsilon r_k),
\]

\[
\{ q_k, r_{k+1} \} = -q_k (1 + h\epsilon q_k)(1 + h r_{k+1})(1 + \epsilon r_{k+1}),
\]

\[
\{ r_k, r_{k+1} \} = \alpha \frac{q_k(1 + h\epsilon q_k)}{(1 + \epsilon \alpha q_k)(1 + h \alpha q_k)} (1 + h r_k)(1 + \epsilon r_k)(1 + h r_{k+1})(1 + \epsilon r_{k+1}),
\]

which is the pull–back of the bracket

\[
\{ \cdot, \cdot \}_{12\alpha} + h \{ \cdot, \cdot \}_{23\alpha}
\]

on $\mathcal{MRT}(q, r)$ under the change of variables (12.8).

Finally, let us consider the relation between the local forms of the maps $dRTL_+^{(+)}(\alpha)$ and $dMRTL_+^{(+)}(\alpha; \epsilon)$. In other words, how do the Miura maps $M_{1,2}^{(+)}(\alpha; \epsilon)$ look when seen through the localizing changes of variables? The answer is given by Theorem 12.3, which can be reformulated as follows:

**Theorem 12.5** The following diagram is commutative for $j = 1, 2$:

\[
\begin{array}{ccc}
\mathcal{MRT}(q, p) & \xrightarrow{M_j^{(+)}(\alpha; h; \epsilon)} & \mathcal{RT}(a, b) \\
M_1^{(+)}(\alpha; \epsilon; h) & \bigg\downarrow & M_1^{(+)}(\alpha; h) \\
\mathcal{MRT}(q, p) & \xrightarrow{M_j^{(+)}(\alpha; \epsilon)} & \mathcal{RT}(a, \epsilon) \\
\end{array}
\]

where the maps $M_1^{(+)}(\alpha; h; \epsilon)$ are given by

\[
M_1^{(+)}(\alpha; h; \epsilon) : \begin{cases} 
  b_k = \frac{p_k + \epsilon q_k + \epsilon(h + \alpha)p_k q_k}{1 - h\epsilon \alpha p_k q_k}, \\
  a_k = \frac{q_k(1 + \epsilon p_k)(1 + \epsilon \alpha q_k)}{1 - h\epsilon \alpha D(q_k, p_k, q_k)},
\end{cases}
\]

(12.24)

\[
M_2^{(+)}(\alpha; h; \epsilon) : \begin{cases} 
  b_k = \frac{p_k + \epsilon q_{k+1} + \epsilon(h + \alpha)q_{k+1}p_k}{1 - h\epsilon \alpha q_{k+1}p_k}, \\
  a_k = \frac{q_k(1 + \epsilon p_{k+1})(1 + \epsilon \alpha q_{k+1})}{1 - h\epsilon \alpha D(q_{k+1}, p_{k+1}, q_k)},
\end{cases}
\]

(12.25)
13 Relativistic Volterra lattice

13.1 Equations of motion

The systems related to the flows RTL±(\(\alpha\)) in the same way as VL is related to TL, are naturally called relativistic Volterra lattices RVL±(\(\alpha\)). Their phase space will be denoted by \(\mathcal{R}_V = \mathbb{R}^{2N}(u,v)\). At this point the next instance of the “relativistic splitting” appears: there are two systems playing the role of RVL_+(\(\alpha\)), namely

\[
\begin{align*}
\dot{u}_k &= u_k(v_k - v_{k-1} + \alpha u_kv_k - \alpha u_{k-1}v_{k-1}) , \\
\dot{v}_k &= v_k(u_{k+1} - u_k + \alpha u_{k+1}v_k - \alpha u_kv_k) ,
\end{align*}
\]

(13.1)

and

\[
\begin{align*}
\dot{u}_k &= u_k(v_k - v_{k-1} + \alpha u_{k+1}v_k - \alpha u_kv_k) , \\
\dot{v}_k &= v_k(u_{k+1} - u_k + \alpha u_{k+1}v_k - \alpha u_kv_k) .
\end{align*}
\]

(13.2)

Similarly, there are two systems playing the role of RVL_-(\(\alpha\)), namely

\[
\begin{align*}
\dot{u}_k &= u_k(v_k - v_{k-1} + \alpha u_kv_k - \alpha u_{k-1}v_{k-1}) , \\
\dot{v}_k &= v_k(u_{k+1} - u_k + \alpha u_{k+1}v_k - \alpha u_kv_k) ,
\end{align*}
\]

(13.3)

and

\[
\begin{align*}
\dot{u}_k &= u_k(v_k - v_{k-1} + \alpha u_{k+1}v_k - \alpha u_kv_k) , \\
\dot{v}_k &= v_k(u_{k+1} - u_k + \alpha u_{k+1}v_k - \alpha u_kv_k) .
\end{align*}
\]

(13.4)

Obviously, (13.1) and (13.2) are related by means of a simple shift

\[
\bar{u}_k = v_k , \quad \bar{v}_k = u_{k+1} ,
\]

(13.5)

and the same holds for (13.3) and (13.4). Therefore we restrict ourselves to considering only one system of each pair. We reserve the notation RVL_+(\(\alpha\)) for the flow (13.1), and RVL_-(\(\alpha\)) for the flow (13.3).

13.2 Bi–Hamiltonian structure

Proposition 13.1 a) The formulas

\[
\{u_k, v_k\}_{2\alpha} = -u_kv_k , \quad \{v_k, u_{k+1}\}_{2\alpha} = -v_ku_{k+1} ,
\]

(13.6)

define a Poisson bracket on \(\mathcal{R}_V\). The systems RVL_±(\(\alpha\)) are Hamiltonian flows on \((\mathcal{R}_V, \{\cdot, \cdot\}_{2\alpha})\), with the Hamilton functions

\[
H^{(+)}_{1\alpha}(u,v) = \sum_{k=1}^{N}(u_k + v_k + \alpha u_kv_k)
\]

(13.7)
and
\[ H_{1\alpha}^{(-)}(u, v) = \alpha^{-1} \sum_{k=1}^{N} \log \left( 1 + \alpha(u_k + v_{k-1}) \right), \quad (13.8) \]
respectively. The functions \( H_{1\alpha}^{(+)} \) and \( H_{1\alpha}^{(-)} \) are in involution in the bracket \( \{\cdot, \cdot\}_{2\alpha} \).

b) The relations
\[ \{ u_k, v_k \}_3^{3\alpha} = -u_kv_k (u_k + v_k + \alpha u_kv_k), \]
\[ \{ v_k, u_{k+1} \}_3^{3\alpha} = -v_k u_{k+1} (v_k + u_{k+1}), \]
\[ \{ u_k, u_{k+1} \}_3^{3\alpha} = -u_kv_k u_{k+1} (1 + \alpha u_k), \quad (13.9) \]
\[ \{ v_k, v_{k+1} \}_3^{3\alpha} = -v_k u_{k+1} v_{k+1} (1 + \alpha v_{k+1}), \]
\[ \{ v_k, u_{k+2} \}_3^{3\alpha} = -\alpha v_k u_{k+1} v_{k+1} u_{k+2} \]
define a Poisson bracket on \( RV \) compatible with \( \{\cdot, \cdot\}_{2\alpha} \). The systems \( RVL_{\pm}^{\alpha} \) are Hamiltonian flows on \( (RV, \{\cdot, \cdot\}_{3\alpha}) \), with the Hamilton functions
\[ H_{0\alpha}^{(+)}(u, v) = \frac{1}{2} \sum_{k=1}^{N} \log (u_kv_k) \quad (13.10) \]
and
\[ H_{0\alpha}^{(-)}(u, v) = \frac{1}{2} \sum_{k=1}^{N} \log (u_kv_k) - \sum_{k=1}^{N} \log \left( 1 + \alpha(u_k + v_{k-1}) \right), \quad (13.11) \]
respectively. The functions \( H_{0\alpha}^{(+)} \) and \( H_{0\alpha}^{(-)} \) are in involution in the bracket \( \{\cdot, \cdot\}_{3\alpha} \).

13.3 Miura relations

The relativistic Volterra hierarchies are related to the relativistic Toda hierarchy by means of literally the same Miura transformations as the nonrelativistic one, i.e.
\[ M_1 : \begin{cases} b_k = u_k + v_{k-1}, \\ a_k = u_kv_k, \end{cases} \quad M_2 : \begin{cases} b_k = u_k + v_k, \\ a_k = u_{k+1}v_k. \end{cases} \quad (13.12) \]

It turns out that the flows (13.1), (13.3) are connected with \( M_1 \), while the flows (13.2), (13.4) are connected with \( M_2 \). We restrict ourselves with the \( M_1 \) case.

**Theorem 13.2** Both Miura maps \( M_{1,2} : RV(u, v) \mapsto RT(a, b) \) given in (13.12) are Poisson, provided both spaces \( RV \) and \( RT \) carry the corresponding brackets \( \{\cdot, \cdot\}_{2\alpha} \). The Miura map \( M_1 \) is also Poisson, provided both spaces \( RV \) and \( RT \) carry the corresponding brackets \( \{\cdot, \cdot\}_{3\alpha} \). The pull–back of the flows \( RTL_{\pm}^{\alpha} \) under the Miura map \( M_1 \) coincides with \( RVL_{\pm}^{\alpha} \).

The bracket on \( RV \) analogous to \( \{\cdot, \cdot\}_{3\alpha} \) and assuring the Poisson property of \( M_2 \) case can be obtained via the shift transformation (13.5).
13.4 Lax representation

Lax matrix \((U, W^{-1}, V) : \mathcal{RV} \mapsto g \otimes g \otimes g\):

\[U(u, v, \lambda) = \sum_{k=1}^{N} u_k E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k},\]
\[V(u, v, \lambda) = I + \lambda^{-1} \sum_{k=1}^{N} v_k E_{k,k+1},\]
\[W(u, v, \lambda) = I - \alpha \lambda^{-1} \sum_{k=1}^{N} u_k v_k E_{k,k+1}.\] (13.13)

Notice that the formulas

\[b_k = u_k + v_{k-1}, \quad a_k = u_{k-1} v_k\]

for the Miura map \(M_1\) are equivalent to the following factorization:

\[L(a, b, \lambda) = W(a, b, \lambda) - \alpha U(u, v, \lambda)V(u, v, \lambda),\]

while the formulas

\[b_k = u_k + v_k, \quad a_k = u_{k+1} v_k\]

for the Miura map \(M_2\) are equivalent to the following factorization:

\[L(a, b, \lambda) = W(a, b, \lambda) - \alpha V(u, v, \lambda)U(u, v, \lambda).\]

Lax representation of the flow \(RVL_+ (\alpha)\):

\[\dot{U} = U A_3 - A_1 U, \quad \dot{V} = V A_2 - A_3 V, \quad \dot{W} = W A_2 - A_1 W,\] (13.16)

with

\[A_1 = \pi_+ (UVW^{-1}) = \sum_{k=1}^{N} (u_k + v_{k-1} + \alpha u_{k-1} v_{k-1}) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k},\] (13.17)
\[A_2 = \pi_+ (W^{-1}UV) = \sum_{k=1}^{N} (u_k + v_{k-1} + \alpha u_k v_k) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k},\] (13.18)
\[A_3 = \pi_+ (VW^{-1}U) = \sum_{k=1}^{N} (u_k + v_k + \alpha u_k v_k) E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k}.\] (13.19)

Lax representation of the flow \(RVL_- (\alpha)\):

\[\dot{U} = C_1 U - U C_3, \quad \dot{V} = C_3 V - V C_2, \quad \dot{W} = C_1 W - W C_2,\] (13.20)

with

\[C_1 = \pi_- \left( f(UVW^{-1}) \right) = \lambda^{-1} \sum_{k=1}^{N} \frac{u_k v_k}{1 + \alpha (u_{k+1} + v_k)} E_{k,k+1},\] (13.21)
\[C_2 = \pi_- \left( f(W^{-1}UV) \right) = \lambda^{-1} \sum_{k=1}^{N} \frac{u_k v_k}{1 + \alpha (u_k + v_{k-1})} E_{k,k+1}.,\] (13.22)
\[C_3 = \pi_- \left( f(VW^{-1}U) \right) = \lambda^{-1} \sum_{k=1}^{N} \frac{u_{k+1} v_k}{1 + \alpha (u_{k+1} + v_k)} E_{k,k+1},\] (13.23)

where \(f(T) = (I - (I + \alpha T)^{-1})/\alpha\).
13.5 Discretization

The discrete Lax equations for the map \( d\text{RVL}_+ (\alpha) \):

\[
\tilde{U} = A_1^{-1}UA_3, \quad \tilde{V} = A_3^{-1}VA_2, \quad \tilde{W} = A_1^{-1}WA_2, \quad (13.24)
\]

where

\[
A_1 = \Pi_+ \left( I + hUVW^{-1} \right) = \sum_{k=1}^{N} \alpha_k E_{kk} + h\lambda \sum_{k=1}^{N} E_{k+1,k}, \quad (13.25)
\]

\[
A_2 = \Pi_+ \left( I + hW^{-1}UV \right) = \sum_{k=1}^{N} \beta_k E_{kk} + h\lambda \sum_{k=1}^{N} E_{k+1,k}, \quad (13.26)
\]

\[
A_3 = \Pi_+ \left( I + hVW^{-1}U \right) = \sum_{k=1}^{N} \gamma_k E_{kk} + h\lambda \sum_{k=1}^{N} E_{k+1,k}. \quad (13.27)
\]

The discrete Lax equations for the map \( d\text{RVL}_- (\alpha) \):

\[
\tilde{U} = C_1UC_3^{-1}, \quad \tilde{V} = C_3VC_2^{-1}, \quad \tilde{W} = C_1WC_2^{-1}, \quad (13.28)
\]

where

\[
C_1 = \Pi_- \left( I + hf(UVW^{-1}) \right) = I + h\lambda^{-1} \sum_{k=1}^{N} \epsilon_k E_{k,k+1}, \quad (13.29)
\]

\[
C_2 = \Pi_- \left( I + hf(W^{-1}UV) \right) = I + h\lambda^{-1} \sum_{k=1}^{N} \epsilon_k E_{k,k+1}, \quad (13.30)
\]

\[
C_3 = \Pi_- \left( I + hf(VW^{-1}U) \right) = I + h\lambda^{-1} \sum_{k=1}^{N} \epsilon_k E_{k,k+1}. \quad (13.31)
\]

14 Modified relativistic Volterra lattice

14.1 Equations of motion

The phase space of the \textit{modified relativistic Volterra hierarchy} \( \text{MRVL}(\alpha; \epsilon) \) will be denoted by \( \mathcal{MRV} = \mathbb{R}^{2N}(y, z) \). The modification parameter \( \epsilon \) plays a role somewhat different from the relativistic parameter \( \alpha \). The equations of motion of the flow \( \text{MRVL}_+ (\alpha; \epsilon) \) read:

\[
\dot{y}_k = y_k (1 + \epsilon y_k) \left( \frac{z_k + \alpha y_k z_k}{1 + \epsilon \alpha y_k z_k} - \frac{z_{k-1} + \alpha y_{k-1} z_{k-1}}{1 + \epsilon \alpha y_{k-1} z_{k-1}} \right), \quad (14.1)
\]

\[
\dot{z}_k = z_k (1 + \epsilon z_k) \left( \frac{y_{k+1} + \alpha y_{k+1} z_{k+1}}{1 + \epsilon \alpha y_{k+1} z_{k+1}} - \frac{y_k + \alpha y_k z_k}{1 + \epsilon \alpha y_k z_k} \right). \quad (14.2)
\]

The equations of motion of the flow \( \text{MRVL}_- (\alpha; \epsilon) \) read:

\[
\dot{y}_k = y_k (1 + \epsilon y_k) \left( \frac{z_k}{1 + \alpha (y_{k+1} + z_k) + \epsilon \alpha y_{k+1} z_k} - \frac{z_{k-1}}{1 + \alpha (y_k + z_{k-1}) + \epsilon \alpha y_k z_{k-1}} \right),
\]

\[
\dot{z}_k = z_k (1 + \epsilon z_k) \left( \frac{y_{k+1}}{1 + \alpha (y_{k+1} + z_k) + \epsilon \alpha y_{k+1} z_k} - \frac{y_k}{1 + \alpha (y_k + z_{k-1}) + \epsilon \alpha y_k z_{k-1}} \right).
\]
14.2 Hamiltonian structure

**Proposition 14.1** The relations
\[
\{y_k, z_k\}_{23\alpha} = -y_k z_k (1 + \epsilon y_k)(1 + \epsilon z_k),
\]

\[
\{z_k, y_{k+1}\}_{23\alpha} = -z_k y_{k+1} (1 + \epsilon z_k)(1 + \epsilon y_{k+1})
\]

(14.3)
define a Poisson bracket on MRV(y, z). The flows MRVL\(_{\pm}(\alpha; \epsilon)\) are Hamiltonian with respect to this bracket with the Hamilton functions
\[
G_{0+}(\alpha)(y, z) = \epsilon^{-1} \sum_{k=1}^{N} \log(1 + \epsilon y_k) + \epsilon^{-1} \sum_{k=1}^{N} \log(1 + \epsilon z_k) - \epsilon^{-1} \sum_{k=1}^{N} \log(1 - \epsilon \alpha y_k z_k)
\]

(14.4)

and
\[
G_{0-}(\alpha)(y, z) = (\epsilon - \alpha)^{-1} \sum_{k=1}^{N} \log(1 + \epsilon y_k) + (\epsilon - \alpha)^{-1} \sum_{k=1}^{N} \log(1 + \epsilon z_k) - (\epsilon - \alpha)^{-1} \sum_{k=1}^{N} \log \left(1 + \alpha (y_k + z_{k-1}) + \epsilon \alpha y_k z_{k-1}\right),
\]

(14.5)
respectively. The functions \(G_{0+}\) and \(G_{0-}\) are in involution in the bracket \(\{\cdot, \cdot\}_{23\alpha}\).

Notice that the bracket \(\{\cdot, \cdot\}_{23\alpha}\) actually does not depend on \(\alpha\).

14.3 Miura relations

**Theorem 14.2** a) The flows MRVL\(_{\pm}(\alpha; \epsilon)\) are the pull–backs of the flows MRTL\(_{\pm}^{(1)}(\alpha; \epsilon)\) under the Miura transformation \(M_1^{(\pm)}(\alpha; \epsilon) : MRV(y, z) \mapsto MRT(q, p)\) defined by
\[
M_1^{(+)}(\alpha; \epsilon) : \begin{cases} p_k = y_k + z_{k-1} + \epsilon y_k z_{k-1}, \\ q_k = \frac{y_k z_k}{1 - \epsilon \alpha y_k z_k}. \end{cases}
\]

(14.6)

This Miura transformation is Poisson, if both \(MRV(y, z)\) and \(MRT(q, r)\) carry the corresponding brackets \(\{\cdot, \cdot\}_{23\alpha}\).

b) The flows MRVL\(_{\pm}(\alpha; \epsilon)\) are the pull–backs of the flows RV\(_{\pm}(\alpha)\) under either of the Miura transformations \(M_1^{(+)}(\alpha; \epsilon) : MRV(y, z) \mapsto RV(u, v)\) defined by
\[
M_1^{(+)}(\alpha; \epsilon) : \begin{cases} u_k = \frac{y_k (1 + \epsilon z_{k-1})}{1 - \epsilon \alpha y_k z_{k-1}}, \\ v_k = \frac{z_k (1 + \epsilon y_k)}{1 - \epsilon \alpha y_k z_k}, \end{cases}
\]

(14.7)

\[
M_2^{(+)}(\alpha; \epsilon) : \begin{cases} u_k = \frac{y_k (1 + \epsilon z_k)}{1 - \epsilon \alpha y_k z_k}, \\ v_k = \frac{z_k (1 + \epsilon y_{k+1})}{1 - \epsilon \alpha y_{k+1} z_{k+1}}. \end{cases}
\]

(14.8)
Both Miura transformations \( M_{1,2}(\alpha; \epsilon) \) are Poisson, if \( MRV(y, z) \) is equipped with the bracket \( \{\cdot, \cdot\}_{23\alpha} \) and \( RV(u, v) \) is equipped with the bracket \( \{\cdot, \cdot\}_{2\alpha} + \epsilon\{\cdot, \cdot\}_{3\alpha} \).

c) the following diagram is commutative for \( j = 1, 2 \):

\[
\begin{array}{ccc}
MRV & \xrightarrow{M_1(\alpha; \epsilon)} & MRT \\
\downarrow M_j(\alpha; \epsilon) & & \downarrow M_j(\alpha; \epsilon) \\
RV & \Rightarrow & RT \\
\end{array}
\]

There is only one Miura transformation \( M_1(\alpha; \epsilon) \) relating the modified relativistic Volterra hierarchy \( MRV(\alpha; \epsilon) \) to the modified relativistic Toda hierarchy \( MRT(\alpha; \epsilon) \). Of course, this is no surprise: the situation with the unmodified systems was exactly the same: the hierarchies \( RVL(\alpha) \) and \( RTL(\alpha) \) are related by only one Miura transformation \( M_1 \). Actually, there exists a “twin” \( MRVL(\alpha; \epsilon) \) hierarchy related to \( MRTL(\alpha; \epsilon) \) via the “twin” Miura transformation \( M_2(\alpha; \epsilon) \).

14.4 Application: localizing change of variables for \( dRVL_+(\alpha) \)

The Miura map \( M_1(\alpha; h) \) plays the role of the localizing change of variables for the map \( dRVL_+(\alpha) \). Indeed, consider the change of variables \( RV(u, v) \mapsto RV(\tilde{u}, \tilde{v}) \):

\[
\begin{array}{c}
\tilde{u}_k = u_k + \frac{h v_{k-1}}{1 - h \alpha u_{k-1} v_{k-1}} + h u_k, \\
\tilde{v}_k = v_k + \frac{1 + h v_k}{1 - h \alpha u_k v_k},
\end{array}
\]

The the entries of the factors \( A_j, j = 1, 2, 3 \) (see (13.25)–(13.27)) admit local expressions in the coordinates \( (u, v) \):

\[
\begin{align}
\alpha_k &= \frac{(1 + h u_k)(1 + h v_{k-1})}{1 - h \alpha u_{k-1} v_{k-1}}, \\
\beta_k &= \frac{(1 + h u_k)(1 + h v_k)}{1 - h \alpha u_k v_k}, \\
\gamma_k &= \frac{(1 + h u_k)(1 + h v_{k+1})}{1 - h \alpha u_{k+1} v_{k+1}}.
\end{align}
\]

Theorem 14.3: The change of variables (14.9) conjugates \( dRVL_+(\alpha) \) with the map on \( RV(u, v) \) described by the following equations:

\[
\begin{align}
\tilde{u}_k &= u_k + \frac{h v_{k-1}}{1 - h \alpha u_{k-1} v_{k-1}}, \\
\tilde{v}_k &= v_k + \frac{1 + h u_{k+1}}{1 - h \alpha u_{k+1} v_{k+1}}.
\end{align}
\]
We see that the local form of $dRVL_+ (\alpha)$ lives in the hierarchy $MRVL_+(\alpha; h)$.

**Corollary.** The local form of $dRVL_+ (\alpha)$ is a Poisson map with respect to the following bracket on $RV(u, v)$:

\[
\{ u_k, v_k \} = -u_k v_k (1 + h u_k)(1 + h v_k),
\]

\[
\{ v_k, u_{k+1} \} = -v_k u_{k+1} (1 + h v_k)(1 + h u_{k+1}),
\]

which is the pull–back of the bracket

\[
\{ \cdot, \cdot \}_{2\alpha} + h \{ \cdot, \cdot \}_{3\alpha}
\]

on the space $RV(u, v)$ under the change of variables (14.9).

Finally, we give the translations of the Miura relations between the maps $dRVL_+ (\alpha)$ and $dRTL_+ (\alpha)$ into the language of localizing variables. This is achieved by reformulating the part c) of Theorem 14.2.

**Theorem 14.4** a) The following diagram is commutative:

\[
\begin{array}{ccc}
RV(u, v) & \xrightarrow{M_1^{(+)}(\alpha; h)} & RT(a, b) \\
\downarrow M_1^{(+)}(\alpha; h) & & \downarrow M_1^{(+)}(\alpha; h) \\
RV(u, v) & \xrightarrow{M_1} & RT(a, b)
\end{array}
\]

where the map $M_1^{(+)}(\alpha; h) : RV(u, v) \mapsto RT(a, b)$ is given by the formulas

\[
M_1^{(+)}(\alpha; h) : \begin{cases}
  a_k = \frac{u_k v_k}{1 - h \alpha u_k v_k}, \\
  b_k = u_k + v_{k-1} + h u_k v_{k-1}.
\end{cases}
\]

b) The map $M_1^{(+)}(\alpha; h)$ conjugates the local form of $dRVL_+ (\alpha)$ (14.13) with the local form of $dRTL_+ (\alpha)$ (10.41).

c) The map $M_1^{(+)}(\alpha; h)$ is Poisson, provided $RV(u, v)$ is equipped with the bracket (14.14), and $RT(a, b)$ is equipped with the bracket (10.44).

15 Bibliographical remarks.

a) Our account of the Miura maps is very close in spirit to that of Kupershmidt, see has works
In particular, our presentation of the original Miura (Gardner) transformation in the introduction follows [K1]. The Hamiltonian formalism for lattice systems is developed in [K3], in particular, the tri–Hamiltonian structure of the Toda lattice and the bi–Hamiltonian structure of the Volterra lattice was found there for the first time. The papers [K2], [K3] treat also the modified and the double modified Toda lattice, as well as the modified Volterra lattice. Accordingly, the corresponding Miura maps and their Poisson and permutability properties are given there. However, the role of Miura maps as localizing changes of variables for integrable discretizations is a novel aspect, not discussed in the above mentioned references.

b) The whole tower of modifications for the nonrelativistic Toda–like systems, including the triple modified Toda and the double modified Volterra lattices, was discovered by Yamilov in the course of classifying integrable lattice systems with nearest neighbors interactions:

[Y1] R. Yamilov. On the classification of discrete equations. In: Integrable systems, Ufa 1982, 95-114 (in Russian).

[Y1] R. Yamilov. Classification of discrete evolution equations. Uspekhi Matem. Nauk 38 (1983) 155–156 (in Russian).

[Y3] R. Yamilov. Construction scheme for discrete Miura transformations. J. Phys. A: Math. Gen. 27 (1994) 6839–6851.

The Hamiltonian properties of these systems and of Miura maps were not his main concern.

c) The local integrable discretizations of the Toda lattice (5.23), the Volterra lattice (7.9), and the modified Toda lattice (4.10) were found for the first time by Hirota, including the Miura relations between them:

[H1] R. Hirota, S. Tsujimoto, T. Imai. Difference scheme of soliton equations. In: ”Future directions of nonlinear dynamics in physical and biological systems”, Eds. P.L.Christiansen, J.C.Eilbeck, and R.D.Parmentier (Plenum, 1993), 7–15.

[H2] R. Hirota, S. Tsujimoto. Conserved quantities of discrete Lotka-Volterra equations. RIMS Kokyuroku 868 (1994) 31–38 (in Japanese); Conserved quantities of a class of nonlinear difference–difference equations. J. Phys. Soc. Japan 64 (1995) 3125–3127.

However, he did not identify these discretizations as the members of the modified hierarchies, and did not study their Hamiltonian properties.

d) The present paper is a companion for
Yu.B. Suris. Integrable discretizations for lattice systems: local equations of motion and their Hamiltonian properties. Rev. Math. Phys. (1999, to appear).

The reader should consult this paper for general concepts from the theory of $r$–matrix hierarchies, relevant for the problem of integrable discretizations, as well as for an extensive list of references. The concept of localizing changes of variables appeared there for the first time. The present paper contains more examples, in particular, almost all results of the “relativistic” part are new.