CRITICAL GROUPS OF COVERING, VOLTAGE AND SIGNED GRAPHS

VICTOR REINER AND DENNIS TSENG

Abstract. Graph coverings are known to induce surjections of their critical groups. Here we describe the kernels of these morphisms in terms of data parametrizing the covering. Regular coverings are parametrized by voltage graphs, and the above kernel can be identified with a naturally defined voltage graph critical group. For double covers, the voltage graph is a signed graph, and the theory takes a particularly pleasant form, leading also to a theory of double covers of signed graphs.

1. Introduction

This paper studies graph coverings and critical groups for undirected multigraphs $G = (V, E)$; here $E$ is a multiset of edges, with self-loops allowed. An example graph covering $\tilde{G} \to G$ is shown here, where the map sends an edge or vertex of $\tilde{G}$ to the corresponding edge or vertex of $G$ by ignoring the $+/−$ subscript:

The critical group $K(G)$ is a subtle isomorphism invariant of $G$ in the form of a finite abelian group, whose cardinality is the number of maximal forests in $G$. To present $K(G)$, one can introduce the (signed) node-edge incidence matrix $\partial := \partial_G$ for $G$ having rows indexed by $V$, columns indexed by $E$, as we now

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explain. One defines $\partial$ by first fixing an arbitrary orientation of the edge set $E$. Then one lets the column of $\partial$ indexed by an edge $e$ in $E$ that has been oriented from vertex $u$ to $v$ be the difference vector $+u - v$, regarding each $v$ in $V$ as a standard basis vector for $\mathbb{R}^V$. One can regard $\partial$ as a map $\mathbb{Z}^E \to \mathbb{Z}^V$, and define $K(G)$ via either of these equivalent presentations (see Proposition 2.2 below)

\begin{align}
K(G) &:= \text{im}\partial/\text{im}\partial^t \\
&\cong \mathbb{Z}^E / (\text{im}\partial^t + \ker \partial)
\end{align}

where $\partial^t$ is the map $\mathbb{Z}^V \to \mathbb{Z}^E$ corresponding to the transpose matrix of $\partial$. The presentation (1.1) allows one to compute the structure of $K(G)$ from the nonzero entries $d_1, d_2, \ldots, d_i$ in the Smith normal form of the graph Laplacian matrix $L(G) := \partial\partial^t$ appearing above:

$$K(G) \cong \bigoplus_{i=1}^t \mathbb{Z}_{d_i}$$

where $\mathbb{Z}_{d} := \mathbb{Z}/d\mathbb{Z}$ denotes the cyclic group of order $d$.

**Example.** The graphs in the above covering $\tilde{G} \to G$ have node-edge incidence matrices

$$\begin{align}
\partial_G &= \begin{pmatrix}
a & b & c & d & e & f & g & h & i \\
1 & 2
\end{pmatrix}
\begin{pmatrix}
+1 & +1 & +1 & +1 & +1 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{align}$$

From which one obtains the Laplacian matrices

$$L(G) = \partial_G \partial_G^t = \frac{1}{2} \begin{pmatrix}
1 & 2 \\
6 & -6
\end{pmatrix}$$

and

$$L(\tilde{G}) = \partial_{\tilde{G}} \partial_{\tilde{G}}^t = \begin{pmatrix}
1_+ & 1_- & 2_+ & 2_- \\
1_+ & 1_- & 2_+ & 2_- \\
1_+ & 1_- & 2_+ & 2_- \\
1_+ & 1_- & 2_+ & 2_-
\end{pmatrix}$$

whose Smith normal forms

$$\begin{pmatrix}
3 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 \\
0 & 0 & 36 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

allow one to read off their critical groups:

$$K(G) \cong \mathbb{Z}_6 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3, \quad K(\tilde{G}) \cong \mathbb{Z}_3^2 \oplus \mathbb{Z}_{36} \cong \mathbb{Z}_4 \oplus \mathbb{Z}_4^2 \oplus \mathbb{Z}_0.$$
We focus here on the interaction of critical groups with graph coverings. The above work of Berman \cite{Berman} and Treumann \cite{Treumann} already showed that covering maps of graphs $\tilde{G} \to G$ induce surjections $K(\tilde{G}) \to K(G)$ of their critical groups. Our goal is to study this surjection, describing its kernel, and use this to gain information about $K(\tilde{G})$ from knowledge of $K(G)$.

Section 3 reviews graph coverings, and proves the easy result that for an $m$-sheeted graph covering $\tilde{G} \to G$, the induced surjection $K(\tilde{G}) \to K(G)$ splits at all $p$-primary components for primes $p$ that do not divide $m$.

Section 4 deals with graph coverings which are regular, in the sense that $G$ is the quotient $\tilde{G}/H$ for a finite group $H$ acting freely on $\tilde{G}$. Here one can take advantage of the Gross-Tucker \cite{GT} encoding of a regular covering $\tilde{G} \to G$ via an $H$-valued voltage assignment $\beta: E \to H$ that simply assigns an arbitrary voltage $\beta(e)$ in $H$ to each edge $e$ of $G$; one often calls such extra structure on $G$ an $H$-voltage graph $G_\beta$. For such voltage graphs $G_\beta$, we will introduce matrices with coefficients in the group algebra $\mathbb{Z}H$ that allows us to define (in Sections 4, 5) a notion of voltage graph critical group $K(G_\beta)$, a finite abelian group that naturally extends the notion of critical group $K(G)$ for graphs. More importantly, our first main result shows that this voltage graph critical group $K(G_\beta)$ fills the role of presenting the kernel of the surjection $K(\tilde{G}) \to K(G)$.

**Theorem 1.1.** Any $H$-voltage assignment $G_\beta$ with regular covering $\tilde{G} \to G$ has a short exact sequence

$$0 \to K(G_\beta) \to K(\tilde{G}) \to K(G) \to 0$$

which splits when restricted to $p$-primary components for primes $p$ not dividing $|H|$.

In particular, the numbers of maximal forests of $G, \tilde{G}$ are related by a factor of $|K(G_\beta)|$:

$$|K(\tilde{G})| = |K(G_\beta)| \cdot |K(G)|.$$  

As an important special case, double (2-sheeted) coverings $\tilde{G} \to G$ are always regular, with $G = \tilde{G}/H$ for the two element group $H = \mathbb{Z}_2 = \{+, -\}$. One can then interpret the $H$-valued voltage assignment on the edges of $G$ as a signed graph $G_{\pm}$ in the sense of Zaslavsky \cite{Zaslavsky}. The double cover $\tilde{G} \to G$ parametrized by a signed graph $G_{\pm}$ is particularly simple: there are two vertices $v_+, v_-$ lying above each vertex $v$ of $G$, and each edge $e = \{u, v\}$ in $G$ gives rise to two edges in $\tilde{G}$, namely

- $e_+ = \{u_+, v_+\}, e_- = \{u_-, v_-\}$ if $e$ is labelled $+$ in $G_{\pm}$, and
- $e_+ = \{u_+, v_\}$, $e_- = \{u_-, v_\}$ if $e$ is labelled $-$ in $G_{\pm}$.

Zaslavsky \cite{Zaslavsky} associated to a signed graph $G_{\pm}$ an node-edge incidence matrix $\partial = \partial_{G_{\pm}}$ in $\mathbb{Z}^{|V| \times |E|}$ generalizing the definition for graphs. In his $\partial$, the column indexed by an edge $e$ in $E$ having positive sign $+$ (resp. negative sign $-$) that has been oriented from vertex $u$ to $v$ will be the vector $+u - v$ (resp. $+u + v$), where again one regards each $v$ in $V$ as a standard basis vector in $\mathbb{R}^V$. Regarding $\partial$ as a map $\mathbb{Z}^E \to \mathbb{Z}^V$, as before, one can define $K(G_{\pm})$ via the equivalent presentations

$$K(G_{\pm}) = \text{im} \partial / \text{im} \partial^t \cong \mathbb{Z}^E / (\text{im} \partial^t + \ker \partial)$$

where $\partial^t$ is the transpose matrix considered as a map $\mathbb{Z}^V \to \mathbb{Z}^E$. The signed graph Laplacian matrix $L(G_{\pm}) := \partial \partial^t$ appearing here already figured into Zaslavsky’s signed version of the matrix tree theorem \cite{Zaslavsky} Thm. 8.A.4, allowing us to interpret the cardinality $|K(G_{\pm})|$ as a weighted count of objects that one can think of as maximal forests in $G_{\pm}$; see Section 5.3 below. Theorem 1.1 then specializes as follows.

**Theorem 1.2.** For each signed graph $G_{\pm}$, parametrizing a graph double covering $\tilde{G} \to G$, one has a short exact sequence of critical groups

$$0 \to K(G_{\pm}) \to K(\tilde{G}) \to K(G) \to 0,$$

splitting on restriction to $p$-primary components for odd primes $p$. In particular, $|K(\tilde{G})| = |K(G_{\pm})| \cdot |K(G)|$.

**Example.** Our earlier double covering $\tilde{G} \to G$ is parametrized by this signed graph $G_{\pm}$:

\[\text{Disallowing half-loops for the moment, although they will be incorporated eventually in Section 9}\]
having node-edge incidence matrix $\partial$ and Laplacian matrix

\[
\begin{pmatrix}
a & b & c & d & e & f & g & h & i \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2
\end{pmatrix}
\]

and

\[
L(G) = \partial^t =
\begin{pmatrix}
1 & 18 & 0 \\
2 & 0 & 6
\end{pmatrix}
\]

Consequently

\[
\mathbb{Z}^2 / \text{im} \partial \cong \mathbb{Z}_6 \oplus \mathbb{Z}_{18} \cong \mathbb{Z}_2^2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_9.
\]

One can check that $\text{im} \partial$ here is the sublattice $\mathbb{Z}_2 \oplus \mathbb{Z}_{18}$ of index 2 inside $\mathbb{Z}^2$ where the sum of coordinates is even. Therefore $K(G_{\pm})$ is an index 2 subgroup of $\mathbb{Z}^2 / \text{im} \partial$. Thus the answer for $\mathbb{Z}^2 / \text{im} \partial$ given above forces

\[
K(G_{\pm}) \cong \text{im} \partial / \text{im} \partial^t \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_9.
\]

and the short exact sequence from Theorem 1.2 takes this form:

\[
0 \rightarrow K(G_{\pm}) \rightarrow K(\tilde{G}) \rightarrow K(G) \rightarrow 0
\]

\[
\begin{array}{cc}
\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_9 & \mathbb{Z}_4 \oplus \mathbb{Z}_9 \\
\mathbb{Z}_2 \oplus \mathbb{Z}_3 & \mathbb{Z}_2 \oplus \mathbb{Z}_3
\end{array}
\]

Note that its $p$-primary component splits at the odd prime $p = 3$

\[
0 \rightarrow \mathbb{Z}_3 \oplus \mathbb{Z}_9 \rightarrow \mathbb{Z}_3^2 \oplus \mathbb{Z}_9 \rightarrow \mathbb{Z}_3 \rightarrow 0
\]

but does not split at the prime $p = 2$

\[
0 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_4 \rightarrow \mathbb{Z}_2 \rightarrow 0.
\]

Having developed this theory in the earlier sections, Section 8 describes a class of nontrivial examples of regular graph coverings where the theory is particularly easy to apply, because the relevant voltage graph critical group $K(G_{\beta})$ has the peculiar property that its presentation involves a diagonal Laplacian matrix!

Sections 9, 10, 11 return to the special case of signed graphs, but generalize Theorem 1.2 in a different direction than Theorem 1.1 The idea is to allow half-loops as in Zaslavsky’s original paper [20], and also to introduce a notion of double-covering of signed graphs in which all three players involved (the base, the cover, the voltage assignment) are signed graphs. This allows us to prove a more flexible double covering result, Theorem 11.6, which we apply to two more families of examples in Section 12.

2. Review of critical groups

2.1. Presentations of the critical group. Given a multigraph $G = (V, E)$, as mentioned in the Introduction, we will let $\mathbb{Z}^E, \mathbb{Z}^V$ have $\mathbb{Z}$-bases indexed by $E, V$, and then fix an orientation $e = (u, v)$ for each edge $e$ in $E$, so as to define the $\mathbb{Z}$-linear node-edge map and incidence matrix via

\[
\begin{array}{c}
\mathbb{Z}^E \\
\partial
\end{array}
\begin{array}{c}
\rightarrow \\
\mathbb{Z}^V
\end{array}
\begin{array}{c}
e \\
\mapsto +u - v
\end{array}
\]

Then $\partial = \partial_G = \partial_\Sigma$ is represented by a matrix in $\mathbb{Z}^V \times \mathbb{E}$ with respect to these bases. One will also sometimes want to think of the associated $\mathbb{R}$-linear map $\mathbb{R}^E \xrightarrow{\partial_\Sigma} \mathbb{R}^V$.

Choosing inner products $\langle \cdot, \cdot \rangle_E, \langle \cdot, \cdot \rangle_V$ on $\mathbb{R}^E, \mathbb{R}^V$ that make the above bases each orthonormal, the transpose matrix $\partial^t$ represents the adjoint map $\mathbb{Z}^V \xrightarrow{\partial^t} \mathbb{Z}^E$ or $\mathbb{R}^V \xrightarrow{\partial^t} \mathbb{R}^E$ defined by

\[
\langle \partial x, y \rangle_V = \langle x, \partial^t y \rangle_E
\]
for all $x$ in $\mathbb{Z}^E$ and $y$ in $\mathbb{Z}^V$. It is easily seen that

$$\mathbb{R}^E = \text{im} \partial^t \oplus \ker \partial \tag{2.1}$$

is an orthogonal direct sum decomposition. Here $\ker \partial \subset \mathbb{R}^E$ and $\ker \partial \subset \mathbb{Z}^E$ are called the cycle space and cycle lattice of $G$, while $\text{im} \partial^t \subset \mathbb{R}^E$ and $\text{im} \partial^t \subset \mathbb{Z}^E$ are called the bond or cut space and bond or cut lattice of $G$; see [2, 8, 11]. The critical group $K(G)$ can be thought of as measuring the failure of equality in (2.1) when working with the lattices instead of the $\mathbb{R}$-linear spaces that they span.

**Definition 2.1.** Define the critical group

$$K(G) := \mathbb{Z}^E / (\text{im} \partial^t + \ker \partial).$$

The agreement between the two presentations of $K(G)$ given in (1.1) and (1.2), as well as the two presentations of $K(G_2)$ in (1.3), is explained by the following.

**Proposition 2.2.** Given abelian group homomorphisms $A \xrightarrow{f} B$, the map $f$ induces an isomorphism

$$A/ (\text{im} f + \ker f) \rightarrow \text{im} f / \text{im} f g.$$  

In particular, applying this to $\mathbb{Z}^E \xrightarrow{\partial} \mathbb{Z}^V$, the map $\partial$ induces an isomorphism

$$K(G) = \mathbb{Z}^E / (\text{im} \partial^t + \ker \partial) \rightarrow \text{im} \partial / \text{im} \partial \partial^t.$$  

**Proof.** The composite of two surjections $A \xrightarrow{f} \text{im} f \xrightarrow{g} \text{im} f / \text{im} f g$ annihilates both $\ker f$ and $\text{im} f$, inducing a surjection $A/ (\text{im} f + \ker f) \rightarrow \text{im} f / \text{im} f g$. To see that this surjection is also injective, note that for $a$ in $A$ to represent an element in the kernel of this surjection means that $f(a)$ lies in $\text{im} f g$, so that $f(a) = f(g(b))$ for some $b$ in $B$. This means $a = g(b)$ lies in $\ker(f)$, and hence the expression $a = g(b) + (a - g(b))$ shows that $a$ represents the zero coset of $A/ (\text{im} f + \ker f)$. \qed

2.2. **Functoriality and Pontryagin duality.** Given multigraphs $G_i = (V_i, E_i)$, for $i = 1, 2$, and a $\mathbb{Z}$-linear map $\mathbb{Z}^{E_1} \xrightarrow{f} \mathbb{Z}^{E_2}$ satisfying

$$f(\text{im} \partial_{G_1}^t) \subset \text{im} \partial_{G_2}^t$$

$$f(\ker \partial_{G_1}) \subset \ker \partial_{G_2},$$

the presentation of $K(G)$ in Definition 2.1 shows that $f$ induces a homomorphism $K(G_1) \xrightarrow{\hat{f}} K(G_2)$. Such homomorphisms will be our fundamental tools.

An important feature in this setting is the fact ([7, Proposition 2.3], [16, Proposition 5]) that the assumptions (2.2) are closed under taking adjoints/transposes: the adjoint map $\mathbb{Z}^{E_2} \xrightarrow{f^t} \mathbb{Z}^{E_1}$ will also satisfy

$$f^t(\text{im} \partial_{G_2}^t) \subset \text{im} \partial_{G_1}^t$$

$$f^t(\ker \partial_{G_2}) \subset \ker \partial_{G_1}.$$  

2.3. **Pontryagin duality.** Given the homomorphism $K(G_1) \xrightarrow{\hat{f}} K(G_2)$ induced by a map $\mathbb{Z}^{E_1} \xrightarrow{f} \mathbb{Z}^{E_2}$, we will often wish to apply the Pontryagin duality isomorphism

$$K(G) \cong \hat{K}(G) := \text{Hom}(K(G), \mathbb{Q}/\mathbb{Z})$$

to both of the finite abelian groups $K(G_i)$, and instead consider the dual morphism $\hat{K}(G_2) \xrightarrow{\hat{f}} \hat{K}(G_1)$. In the case of critical groups, the isomorphism (2.3) is very natural.

**Proposition 2.3.** ([7, Prop. 2.5], [16, Prop. 9]) For multigraphs $G$, the Pontryagin duality isomorphism in (2.3) can be chosen\(^3\) so that any map $\mathbb{Z}^{E_1} \xrightarrow{f} \mathbb{Z}^{E_2}$ satisfying the assumptions (2.2) will make the following diagram commute:

\(^3\)Although not needed, the description of the isomorphism in (2.3) is as follows. Letting $\pi_{\text{im} \partial_{G}} : \mathbb{R}^E \rightarrow \text{im} \partial_{G}^t$ be orthogonal projection onto the bond space, send an element of $K(G) = \mathbb{Z}^E / (\text{im} \partial^t + \ker \partial)$ represented by $x$ in $\mathbb{Z}^E$ to the homomorphism $K(G) \rightarrow \mathbb{Q}/\mathbb{Z}$ which maps an element of $K(G)$ represented by $y$ to the additive coset $(\pi_{\text{im} \partial_{G}}(x), y)_{\mathbb{E}} + \mathbb{Z}$ in $\mathbb{Q}/\mathbb{Z}$.  

The level of $\pi$ is, for inducing orientations of edges in $\tilde{G}$, $\pi$ induces a chain map, that is, one has a commutative square

$$
\begin{array}{ccc}
K(G_2) & \xrightarrow{f'} & K(G_1) \\
\downarrow & & \downarrow \\
\tilde{K}(G_2) & \xrightarrow{\tilde{f}} & \tilde{K}(G_1)
\end{array}
$$

3. Graph coverings, surjections, and splittings

Here we recall the notion of a graph covering as in Gross and Tucker [12 §2], and then prove a refinement of results of Treumann [16] and of Baker and Norine [5] for coverings.

**Definition 3.1.** Given two multigraphs $\tilde{G} = (\tilde{V}, \tilde{E})$ and $G = (V, E)$ a graph map is a continuous map $\tilde{G} \to G$ of their underlying topological spaces that maps the interior of each edge of $\tilde{G}$ homeomorphically onto the interior of some edge of $G$.

In particular, a graph map $\pi$ induces a set map $\tilde{E} \to E$; considering what happens via continuity at the endpoints of each edge, it also induces a set map $\tilde{V} \to V$. Note that when one has a graph map $\tilde{G} \to G$, any orientation of the edges of $G$ pulls back to a compatible orientation of the edges of $\tilde{G}$ in such a way that $f$ preserves orientation. Henceforth we will always assume that $\tilde{G}, G$ are oriented compatibly in this fashion when writing down node-edge incidence matrices.

**Definition 3.2.** Say that a graph map $\tilde{G} \to G$ is a graph covering if every vertex of $\tilde{G}$ has a neighborhood on which the restriction of $\pi$ is a homeomorphism.

It is not hard to see that within a fixed connected component of the base graph $G$, every vertex $v$ and edge $e$ will have the same cardinality $m$ for the inverse image sets $\pi^{-1}(v), \pi^{-1}(e)$.

**Definition 3.3.** Say $\tilde{G} \to G$ is an $m$-sheeted cover if $|\pi^{-1}(v)| = |\pi^{-1}(e)| = m$ for every component of $G$.

We come now to the main observation of this section.

**Proposition 3.4.** (cf. Baker-Norine [5] §4, Berman [8] Thm. 5.7, Treumann [16] Prop. 19) An $m$-sheeted covering $\tilde{G} \to G$ of finite multigraphs gives rise to a surjection of critical groups $K(\tilde{G}) \to K(G)$.

Furthermore, the backward map $K(G) \to K(\tilde{G})$ satisfies $\pi^t = m \cdot 1_{K(G)}$, and hence splits off the primary component $K(G)$ as a direct summand for each prime $p$ that does not divide $m$.

**Proof.** We first need to check that $\pi$ satisfies the two conditions (2.2).

For the first condition, note that for any graph map $\tilde{G} \to G$, the associated set maps $\tilde{E} \to E$ and $\tilde{V} \to V$ induce a chain map, that is, one has a commutative square

$$
\begin{array}{ccc}
\mathbb{Z}^E & \xrightarrow{\partial_E} & \mathbb{Z}^\tilde{V} \\
\downarrow & & \downarrow \\
\mathbb{Z}^E & \xrightarrow{\partial_{\tilde{G}}} & \mathbb{Z}^V.
\end{array}
$$

Consequently, the left vertical map $\pi$ in this square sends sends (oriented) cycles of $\tilde{G}$ to cycles of $G$, that is, $\pi(\ker \partial_{\tilde{G}}) \subset \ker \partial_G$.

For the second condition, note that when $\pi$ is not just a graph map but a graph covering, our conventions for inducing orientations of edges in $\tilde{E}$ from orientations in $E$ lead to a similar commutative square

$$
\begin{array}{ccc}
\mathbb{Z}^{\tilde{E}} & \xleftarrow{\partial_{\tilde{G}}} & \mathbb{Z}^\tilde{V} \\
\downarrow & & \downarrow \\
\mathbb{Z}^E & \xleftarrow{\partial_G} & \mathbb{Z}^V.
\end{array}
$$

Consequently the left vertical map $\pi$ similarly satisfies $\pi(\text{im} \partial^{\tilde{G}}) \subset \text{im} \partial_G$, as desired.

Thus $\pi$ induces a map $K(\tilde{G}) \to K(G)$. It is surjective because $\mathbb{Z}^E \to \mathbb{Z}^{\tilde{E}}$ is already surjective. The assertion $\pi^t = m \cdot 1_{K(G)}$ for the induced maps on critical groups follows because the same holds on the level of $\mathbb{Z}^E$: one has $\pi^t = m \cdot 1_{\mathbb{Z}^E}$ because every edge $e$ of $G$ has exactly $m$ preimages in $\pi^{-1}(e) \subset \tilde{E}$. □
Remark 3.5. Both Berman and Treumann considered a situation somewhat more general than a covering that leads to a surjection of critical groups. Berman [8] p.9 defined what it means for a graph \( \tilde{G} \) to be divisible by \( G \), leading to a graph map \( \tilde{G} \to G \) which Treumann [16] Definition 16] called a Berman bundle. Most of our results can be made to work, with extra technicallity, at the level of Berman bundles; see the second author’s REU report [17]. We have not yet found sufficiently interesting applications requiring this extra level of generality, and so we suppress this discussion here.

Similarly, Baker and Norine consider harmonic maps which are more general than coverings [5] §2, Example 3.4], showing that the assertions of Proposition 3.4 hold (with modified statements) in that setting; see their Lemmas 4.1 and Lemma 4.12, and their Theorem 4.13.

4. Regular coverings and voltage graphs

We recall here the notion of regular graph coverings from Gross and Tucker [12 §1].

Definition 4.1. For a multigraph \( G \), a graph map \( G \xrightarrow{h} G \) is called a graph endomorphism. If it has an inverse \( G \xleftarrow{h^{-1}} G \) which is also a graph endomorphism, then \( h \) is called a graph automorphism.

Say that a group \( H \) acts on the right on \( G \) if every \( h \) in \( H \) corresponds to a graph automorphism of \( G \), in such a way that \( h_1(h_2(x)) = (h_2h_1)(x) \) for all \( h_1, h_2 \) in \( H \) and all edges \( e \) of \( G \).

Say that a graph covering \( \tilde{G} \xrightarrow{\pi} G \) is regular (or normal or Galois) if there exists a group \( H \) acting on the right on \( \tilde{G} \) with the property that \( H \) acts simply transitively on all fibers \( \pi^{-1}(v) \) and \( \pi^{-1}(e) \) for every vertex \( v \) and edge \( e \) of \( G \). In this situation, \( H \) is called the transformation group of the regular covering \( \tilde{G} \xrightarrow{\pi} G \).

Remark 4.2. An alternative way to phrase a regular covering is to say that there is a group \( H \) acting via cellular automorphisms on the cell complex \( G \), with the action being free on the associated topological space. Then \( G \xrightarrow{\pi} G/H := G \) is the quotient mapping; see Gross and Tucker [12 Thm. 4].

We next review the encoding from [12 §4] of a regular graph covering \( \tilde{G} \xrightarrow{\pi} G \) with transformation group \( H \) and base graph \( G = (V, E) \), via an \( H \)-voltage assignment or \( H \)-voltage graph \( G_\beta \), which is nothing more than a set map \( \beta : E \to H \).

From a regular covering to a voltage assignment.

Given a regular graph covering \( \tilde{G} \xrightarrow{\pi} G \), arbitrarily choose for each vertex \( v \) in \( V \) one vertex \( v_1 \) in \( \pi^{-1}(v) \) to be labelled by the identity element \( 1 \) of \( H \). Since \( H \) acts simply transitively on \( \pi^{-1}(v) \), the remaining elements in the fiber can be labelled uniquely as \( v_h := h(v_1) \). Since \( H \) acts on the right, this forces that \[
(4.1) \quad h_1(v_{h_2}) = h_1(h_2(v_1)) = (h_2h_1)(v_1) = v_{h_2h_1}.
\]

To get the voltage assignment \( \beta(e) \) for an edge \( e \) in \( E \), first assume that the orientation of \( e = (u, v) \) has been pulled back to all of the edges in the fiber \( \pi^{-1}(e) \). There will be a unique such edge \( e_1 \) having source \( u_1 \); if this edge has target \( v_h \), then decree that \( \beta(e) = h \). Since \( H \) acts by automorphisms, one can use this to label the remaining edges in the same fiber: for any \( h' \) in \( H \) the edge \( e_{h'} := h'(e_1) \) must have source \( h'(u_1) = u_{h'} \) and target \( h'(v_h) = v_{h'h} \). In other words, the edges of \( \tilde{E} \) in \( \pi^{-1}(e) \) are all of the form \( e_{h'} = (u_{h'}, v_{\beta(e)h'}) \) as \( h' \) ranges through \( H \), and the \( H \)-action on them follows this rule: \[
(4.2) \quad h_1(e_{h_2}) = e_{h_2h_1}.
\]

From a voltage assignment to a regular covering.

Given a multigraph \( G = (V, E) \), with an arbitrary orientation on \( E \), and an arbitrary \( H \)-voltage assignment \( G_\beta \) as a map \( \beta : E \to H \), one creates \( \tilde{G} = (\tilde{V}, \tilde{E}) \) as follows:

\[
\tilde{V} := \{v_h\}_{v \in V, h \in H}
\]

\[
\tilde{E} := \{e_{h} = (u_{h}, v_{\beta(e)h})\}_{e = (u, v) \in E, h \in H}
\]

The regular graph covering \( \tilde{G} \xrightarrow{\pi} G \) simply forgets the subscripts: \( e_h \mapsto e \) and \( v_h \mapsto v \).

Example 4.3. Let \( \tilde{G} \) be the graph of the octahedron, which carries a free action of the cyclic group \( H = \{1, h, h^2\} \cong \mathbb{Z}_3 \), in which \( h \) rotates 120° around an axis passing through the centers of two opposite triangular faces. One finds that the associated regular covering \( \tilde{G} \xrightarrow{\pi} G \) is as shown below, described by
a voltage graph $G_\beta$ on an underlying multigraph $G = (V, E)$ with two vertices $V = \{u, v\}$ and four edges $E = \{a, b, c, d\}$. Here edges $a, b$ are both directed from $u$ to $v$ while $c, d$ are loops on vertices $u, v$, respectively, with voltage assignments $\beta(a) = 1, \beta(b) = \beta(c) = \beta(d) = h$.

Example 4.4. Call a graph covering $\hat{G} \xrightarrow{\pi} G$ a double cover if it is 2-sheeted. We claim that graph double covers are always regular, with transformation group $H = \mathbb{Z}_2 = \{+,-\}$: picking an arbitrary labelling of the two vertices in each fiber $\pi^{-1}(v) = \{v_+, v_-\}$ and $\pi^{-1}(e) = \{e_+, e_-\}$, one finds that the involution $h$ which simultaneously swaps all $v_+ \leftrightarrow v_-$ and $e_+ \leftrightarrow e_-$ is a graph automorphism generating the transformation group $H = \{1, h\} \cong \mathbb{Z}_2$ that satisfies the Definition 4.1 for a regular covering. In this setting, the voltage assignment $G_\beta$ as a function $E \rightarrow H = \mathbb{Z}_2 = \{+,-\}$ can be thought as a signed graph $G_\pm$ as in the Introduction.

We can now use this $H$-voltage assignment encoding of regular coverings to reformulate the critical group $K(\hat{G})$ using the group algebra of $H$. This reformulation will be useful in the proof of Theorem 1.1 below.

Definition 4.5. Recall that the group algebra $\mathbb{Z}H$ is the free $\mathbb{Z}$-module on $\mathbb{Z}$-basis elements $\{T_h\}_{h \in H}$ with multiplication defined $\mathbb{Z}$-linearly via $T_h T_h' := T_{hh'}$.

For any $H$-voltage assignment $G_\beta$ and associated regular covering $\hat{G} \xrightarrow{\pi} G$, the action of $H$ on the right of $\hat{G} = (\hat{V}, \hat{E})$ endows $\mathbb{Z}\hat{E}$ and $\mathbb{Z}\hat{V}$ with the structures of right-$\mathbb{Z}H$-modules:

$e_h T_h' := e_{hh'}$
$v_h T_h' := v_{hh'}$

We will also work with free right-$\mathbb{Z}H$-modules $(\mathbb{Z}H)^E$ and $(\mathbb{Z}H)^V$ having $\mathbb{Z}H$-basis elements indexed by $e$ in $E$ and $v$ in $V$. This means, for example, that $(\mathbb{Z}H)^E$ is a free $\mathbb{Z}$-module with $\mathbb{Z}$-basis elements $\{eT_h\}_{e \in E, h \in H}$, and its right-$\mathbb{Z}H$-module structure can be defined $\mathbb{Z}$-linearly by

$(eT_h)T_h' := eT_{hh'}$
Proposition 4.6. For any $H$-voltage assignment $G_{\beta}$ and associated regular covering $\tilde{G} \xrightarrow{\pi} G$, the following $\mathbb{Z}$-module maps give isomorphisms of right-$\mathbb{Z}H$-modules:

$$
\begin{align*}
\mathbb{Z}^E & \xrightarrow{\partial G_{\beta}} (\mathbb{Z}H)^E \\
E_h & \mapsto eT_h \\
\mathbb{Z}^V & \xrightarrow{\partial G_{\beta}} (\mathbb{Z}H)^V \\
V_h & \mapsto vT_h.
\end{align*}
$$

Proof. This follows from the fact that we have labelled the elements within the fibers $\pi^{-1}(v) = \{v_h\}_{h \in H}$ and $\pi^{-1}(e) = \{e_h\}_{h \in H}$ in such a way that the right-$H$-actions satisfy the rules (4.1) and (4.2). \hfill \Box

Here is the point of working with right-actions and right-$\mathbb{Z}H$-modules:

- one can regard elements of $(\mathbb{Z}H)^E$ and $(\mathbb{Z}H)^V$ as column vectors having entries in $\mathbb{Z}H$, and then
- specify right-$\mathbb{Z}H$-module maps between these free right-$\mathbb{Z}H$-modules via multiplication on the left by matrices with entries in $\mathbb{Z}H$.

For example, define $\partial G_{\beta}$ to be the matrix in $(\mathbb{Z}H)^{V \times E}$ representing the right-$\mathbb{Z}H$-module map

$$
(\mathbb{Z}H)^E \xrightarrow{\partial G_{\beta}} (\mathbb{Z}H)^V \\
e \mapsto +u - vT_{\beta(e)}T_h = +uT_h - vT_{\beta(e)h}
$$

for each edge $e$ of $G$ which is oriented $e = (u, v)$. We will also need a map in the other direction

$$
(\mathbb{Z}H)^V \xrightarrow{\partial G^*_\beta} (\mathbb{Z}H)^E
$$

which is represented by the matrix $\partial G^*_\beta$ in $(\mathbb{Z}H)^{E \times V}$ obtained from $\partial G_{\beta}$ by first transposing the matrix, and then applying to each $\mathbb{Z}H$-entry the anti-automorphism $\mathbb{Z}H \xrightarrow{\pi} \mathbb{Z}$ sending $T_h \mapsto T_{h^{-1}}$.

Proposition 4.7. The isomorphisms in Proposition 4.6 make the following diagrams of right-$\mathbb{Z}H$-module morphisms commute:

$$
\begin{array}{ccc}
\mathbb{Z}^E & \xrightarrow{\partial G_{\beta}} & (\mathbb{Z}H)^E \\
\downarrow & & \downarrow \\
\mathbb{Z}^V & \xrightarrow{\partial G_{\beta}} & (\mathbb{Z}H)^V \\
\end{array}
\begin{array}{ccc}
(\mathbb{Z}H)^E & \xrightarrow{\partial G_{\beta}} & (\mathbb{Z}H)^V \\
\downarrow & & \downarrow \\
(\mathbb{Z}H)^E & \xrightarrow{\partial G_{\beta}} & (\mathbb{Z}H)^E
\end{array}
$$

Proof. To see the commutativity of the left diagram, note that the basis element of $\mathbb{Z}^E$ corresponding to a directed edge $e_h = (u_h, v_{\beta(e)h})$ in $E$, lying above $\pi(e_h) = e = (u, v)$ in $E$, will map under $\partial G_{\beta}$ to the vector $+u_h - v_{\beta(e)h}$ in $\mathbb{Z}^V$. Since the horizontal isomorphisms send $e_h \mapsto eT_h$ and $+u_h - v_{\beta(e)h} \mapsto +uT_h - vT_{\beta(e)h}$, commutativity follows from the last line of (4.3).

The commutativity of the right diagram then follows from a general fact: the horizontal isomorphisms carry the inner products on $\mathbb{Z}^V, \mathbb{Z}^E$ to inner products on $(\mathbb{Z}H)^V, (\mathbb{Z}H)^E$ that make $\{vT_h\}_{v \in V, h \in H}$ and $\{eT_h\}_{v \in E, h \in H}$ orthonormal bases. This implies that a right-$\mathbb{Z}H$-module map $(\mathbb{Z}H)^E \xrightarrow{M} (\mathbb{Z}H)^V$ represented by a matrix $M = (m_{e,v})$ in $(\mathbb{Z}H)^{V \times E}$, has its adjoint map represented by the matrix $M^*$ in the above notation. To check this, write $m_{e,v} = \sum_{h \in H} \mu_{v,e,h}T_h$ for some $\mu_{v,e,h}$ in $\mathbb{Z}$, so the $(e, v)$-entry of $M^* = (m^*_{e,v})$ is $\sum_{h \in H} \mu_{v,e,h^{-1}}T_h$, and then

$$
\begin{align*}
M(eT_{h_1}) &= \sum_{v \in V} \sum_{h \in H} \mu_{v,e,h}vT_{h_1}h, \\
M^*(vT_{h_2}) &= \sum_{e \in E} \sum_{h \in H} \mu_{v,e,h^{-1}}eT_{h_2}.
\end{align*}
$$

Therefore

$$
\langle M(eT_{h_1}), vT_{h_2} \rangle_{(\mathbb{Z}H)^V} = \mu_{v,e,h_1^{-1}} = \langle eT_{h_1}, M^*(vT_{h_2}) \rangle_{(\mathbb{Z}H)^V}. \quad \Box
$$

The following corollary is immediate.
Corollary 4.8. For any $H$-voltage assignment $G_\beta$ and associated regular covering $\tilde{G} \xrightarrow{\pi} G$, one has
\[
K(\tilde{G}) \cong (\mathbb{Z}H)^E / \left( \text{im} \partial_{G_\beta}^* + \ker \partial_{G_\beta} \right)
\]
where here
- $\text{im} \partial_{G_\beta}^*$ and $\ker \partial_{G_\beta}$ are $\mathbb{Z}$-sublattices of $(\mathbb{Z}H)^E$,
- $\text{im} \partial_{G_\beta}$ and $\text{im} \partial_{G_\beta}^* \partial_{G_\beta}^*$ are $\mathbb{Z}$-sublattices of $(\mathbb{Z}H)^V$.

5. The short exact sequence for a regular covering

For an $H$-voltage assignment $G_\beta$ and associated regular covering $\tilde{G} \xrightarrow{\pi} G$, we can now identify the kernel of the surjection $K(\tilde{G}) \xrightarrow{\pi} K(G)$ in Proposition 3.4.

5.1. The reduced group algebra.

Definition 5.1. Inside the group algebra $\mathbb{Z}H$, consider the (central) element $c := \sum_{h \in H} T_h$, and the 2-sided ideal $I = \mathbb{Z}c$ consisting of the $\mathbb{Z}$-multiples of $c$. In other words, $I$ is the $\mathbb{Z}$-submodule of $\mathbb{Z}H$ where all $\mathbb{Z}$-basis elements $T_h$ have the same coefficient. Define the reduced group algebra $\overline{\mathbb{Z}H}$ to be the quotient ring $\overline{\mathbb{Z}H} := \mathbb{Z}H/I = \mathbb{Z}H/\mathbb{Z}c$.

Note that, just as $\mathbb{Z}H$ is a free $\mathbb{Z}$-module of rank $m := |H|$, the ring $\overline{\mathbb{Z}H}$ is a free $\mathbb{Z}$-module of rank $m - 1$. As $c$ is invariant under $T_h \mapsto T_{h^{-1}}$, the ring $\overline{\mathbb{Z}H}$ inherits an anti-automorphism $\overline{\mathbb{Z}H} \xrightarrow{^*} \overline{\mathbb{Z}H}$ sending $T_h \mapsto T_{h^{-1}}$.

In general we will use $\overline{(\_)}$ for the quotient operation $\mathbb{Z}H \to \overline{\mathbb{Z}H}$ which reduces right-$\mathbb{Z}H$-modules and morphisms modulo $I$. For example, one has right-$\overline{\mathbb{Z}H}$-module maps
\[
\overline{\mathbb{Z}H}^E \xrightarrow{\overline{\partial}_{G_\beta}} \overline{\mathbb{Z}H}^V \\
\overline{\mathbb{Z}H}^V \xrightarrow{\overline{\partial}_{G_\beta}} \overline{\mathbb{Z}H}^E
\]
used in the following definition.

Definition 5.2. For $H$-voltage assignment $G_\beta$ with regular covering $\tilde{G} \xrightarrow{\pi} G$, define the critical group of $G_\beta$
\[
K(G_\beta) := \overline{\mathbb{Z}H}^E / \left( \text{im} \overline{\partial}_{G_\beta}^* + \ker \overline{\partial}_{G_\beta} \right)
\]
where $\text{im} \overline{\partial}_{G_\beta}^*$, $\text{im} \overline{\partial}_{G_\beta}^* \overline{\partial}_{G_\beta}^*$ are considered as $\overline{\mathbb{Z}H}$-submodules of $\overline{\mathbb{Z}H}^V$. We also name the matrix in $\overline{\mathbb{Z}H}^{V \times V}$ appearing in the definition of $K(G_\beta)$ the voltage graph Laplacian, so one can rewrite this as
\[
L(G_\beta) := \overline{\partial}_{G_\beta} \overline{\partial}_{G_\beta}^*
\]
(5.1)

We can now prove our first main result, which was stated in the Introduction, and which we recall here.

Theorem 1.1. Any $H$-voltage assignment $G_\beta$ with regular covering $\tilde{G} \xrightarrow{\pi} G$ has a short exact sequence
\[
0 \to K(G_\beta) \to K(\tilde{G}) \to K(G) \to 0
\]
which splits when restricted to $p$-primary components for primes $p$ not dividing $|H|$. In particular, $|K(\tilde{G})| = |K(G_\beta)| \cdot |K(G)|$.

Proof. It suffices to show that the surjection $K(\tilde{G}) \xrightarrow{\pi} K(G)$ from Proposition 3.4 has kernel isomorphic to $K(G_\beta)$. Instead we will show the equivalent statement that $K(G_\beta)$ is isomorphic to the cokernel of the Pontryagin dual injection $K(G) \xrightarrow{\pi^*} K(\tilde{G})$ (using Proposition 2.20). This is equivalent since $\text{coker}(\pi^*)$ is Pontryagin dual to $\ker \pi$, and hence they are (abstractly) isomorphic abelian groups.
Recall that

\[ K(G) = \mathbb{Z}^E / (\text{im} \partial_G + \ker \partial_G), \]

\[ K(\tilde{G}) \cong (\mathbb{Z}H)^E / (\text{im} \partial_{\tilde{G}} + \ker \partial_{\tilde{G}}) \]

from Definition 2.1 and Corollary 4.8. Consequently

\[ \text{coker}(\pi^t) \cong (\mathbb{Z}H)^E / \left( \text{im} \partial_{G_\beta} + \ker \partial_{G_\beta} + \pi^t(\mathbb{Z}^E) \right) \]

Recall an edge e of G has fiber \( \pi^{-1}(e) = \{ e_h \}_{h \in H} \), hence its basis element of \( \mathbb{Z}^E \) maps under \( \pi^t \) to the sum \( \sum_{h \in H} e_h \) in \( \mathbb{Z}^E \). This sum corresponds under the isomorphism \( \mathbb{Z}^E \to (\mathbb{Z}H)^E \) of Proposition 1.7 to

\[ \sum_{h \in H} eT_h = e \left( \sum_{h \in H} T_h \right) = e \cdot c. \]

Hence \( \pi^t(\mathbb{Z}^E) = (\mathbb{Z}G)^E = I^E \) inside \( (\mathbb{Z}H)^E \), so that

\[ (\mathbb{Z}H)^E / \pi^t(\mathbb{Z}^E) \cong (\mathbb{Z}H)^E / I^E \cong (\mathbb{Z}H/I)^E = \mathbb{Z}H^E \]

and using Noether’s third isomorphism theorem, one concludes that

\[ \text{coker}(\pi^t) \cong \mathbb{Z}H^E / \left( \text{im} \partial_{G_\beta} + \ker \partial_{G_\beta} \right) \cong K(G_\beta). \]

\[ \square \]

**Remark 5.3.** Although not needed later, for primes \( p \) not dividing \( m = |H| \), one can be more precise about the summand splitting off the \( p \)-primary component (or \( Sylow \ p\)-subgroup) \( Syl_pK(\tilde{G}) \) of the critical group \( K(\tilde{G}) \), isomorphic to \( Syl_pK(G) \). Since the group \( H \) acts on the graph \( \tilde{G} \) via graph automorphisms, it also acts on \( \mathbb{Z}^E \), preserving \( \text{im} \partial \) and \( \ker \partial \), and inducing a (right-)action on the abelian group \( K(\tilde{G}) \). Thus one can consider the subgroup of \( H \)-invariants within \( K(\tilde{G}) \):

\[ K(\tilde{G})^H := \{ x \in K(\tilde{G}) : h(x) = x \text{ for all } h \in H \}. \]

**Proposition 5.4.** In the setting of Theorem 1.4 for primes \( p \) that do not divide \( m = |H| \), the map \( \pi^t \) sends \( Syl_pK(G) \) isomorphically onto \( Syl_pK(\tilde{G})^H \).

**Proof.** Note that the orbit-sum map \( \mathbb{Z}^E \xrightarrow{\Omega} \mathbb{Z}^E \) sending \( e \mapsto \sum_{h \in H} h(e) \) has the same image, namely the \( H \)-invariants \( (\mathbb{Z}^E)^H \), as does the map \( \mathbb{Z}^E \xrightarrow{\pi^t} \mathbb{Z}^E \). Since \( \Omega \) is a sum of group automorphisms, it induces a map \( K(\tilde{G}) \xrightarrow{\Omega} K(\tilde{G}) \), which again has the same image as \( K(G) \xrightarrow{\pi^t} K(\tilde{G}) \). This image lies in \( K(\tilde{G})^H \). Note that the map \( \Omega \) when restricted from \( K(\tilde{G}) \) to \( K(\tilde{G})^H \) will act as multiplication by \( m \). Hence for primes \( p \) that do not divide \( m \), it induces an isomorphism \( Syl_pK(\tilde{G})^H \xrightarrow{\Omega} Syl_pK(\tilde{G}) \). Consequently one has

\[ Syl_pK(\tilde{G})^H = Syl_p(\text{im} \Omega) = Syl_p(\text{im} \pi^t). \]

\[ \square \]

On the other hand, the map \( \pi^t \) generally fails to induce an isomorphism between \( p \)-primary components of \( K(G) \) and \( K(\tilde{G})^H \) for primes \( p \) dividing \( m = |H| \). This occurs already for \( H = \mathbb{Z}/2\mathbb{Z} \) in the double covering \( \tilde{G} \to G \) of an \( n \)-cycle \( G \) by a \( 2n \)-cycle \( \tilde{G} \), where one can check that \( K(\tilde{G})^H = K(\tilde{G}) = \mathbb{Z}_{2n} \), while \( K(G) = \mathbb{Z}_m \). More generally, if \( H = \mathbb{Z}/m\mathbb{Z} \) in the \( m \)-covering \( \tilde{G} \to G \) of an \( n \)-cycle by an \( mn \)-cycle \( \tilde{G} \), then \( K(\tilde{G})^H = K(\tilde{G}) = \mathbb{Z}_{mn} \), while \( K(G) = \mathbb{Z}_n \).

6. **Voltage groups of prime order**

When the voltage group \( H \) is abelian, the group algebra \( \mathbb{Z}H \) is a commutative ring, as is the quotient ring \( \mathbb{Z}H \), and the distinctions between right and left modules over these rings disappear, simplifying some of the considerations of Sections 4 and 5.

Things simplify even further if the group \( H \) has prime order \( p \), as \( H \) is cyclic, say with generator \( h \):

\[ H = \{ 1, h, h^2, \ldots, h^{p-1} \} \cong \mathbb{Z}_p. \]
Letting $\zeta$ denote a primitive $p^{th}$ root of unity in $\mathbb{C}$, one has a well-defined surjective ring map induced by
\[
\mathbb{Z}H \cong \mathbb{Z}[T_h]/(T_h^p - 1) \rightarrow \mathbb{Z}[\zeta]
\]
Since $\zeta$ has minimal polynomial $1 + x + x^2 + \cdots + x^{p-1}$ over $\mathbb{Q}$, the kernel of the above map is exactly $I = \mathbb{Z}(1 + T_h + T_{h^2} + \cdots + T_{h^{p-1}}) = \mathbb{Z}c$, and hence it induces an isomorphism
\[
\mathbb{Z}H \cong \mathbb{Z}[\zeta].
\]
Consequently one can regard the matrices $\overline{G}_\beta$ and $\overline{\partial}_G$ as elements of $\mathbb{Z}[\zeta]^{V \times E}$ and $\mathbb{Z}[\zeta]^{E \times V}$, and one can present the critical group for the voltage graph $G_\beta$ as
\[
K(G_\beta) := \mathbb{Z}[\zeta]/(\text{im}\overline{\partial}_G + \ker \overline{G}_\beta)
\]
\[
\cong \text{im}\overline{\partial}_G/\text{im}\overline{G}_\beta\overline{\partial}_G
\]
where $\text{im}\overline{\partial}_G$ and $\text{im}\overline{G}_\beta\overline{\partial}_G$ are $\mathbb{Z}\zeta$-submodules of $\mathbb{Z}[\zeta]^V$. Note that under the isomorphism $\mathbb{Z}H \cong \mathbb{Z}[\zeta]$, the (anti-)automorphism $\overline{T}_h \mapsto \overline{T}_{h^{-1}}$ of $\mathbb{Z}H$ corresponds to complex conjugation $z \mapsto \bar{z}$. Hence the matrix operation $M \mapsto M^*$ is now the usual conjugate-transpose operation $M^* = M^T$.

**Example 6.1.** Consider the regular cover $\hat{G} \rightarrow G$ of Example 4.3, where $\hat{G}$ is the graph of the octahedron, and the transformation group $H = \{1, h, h^2\}$ has prime order $p = 3$. The map $T_h \mapsto \zeta = e^{2\pi i/3}$ identifies $\mathbb{Z}H \cong \mathbb{Z}[\zeta]$, and under this identification one has
\[
\overline{\partial}_G = u\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \overline{G}_\beta \overline{\partial}_G = u\begin{pmatrix} u & v \\ -1 - \zeta & 0 \end{pmatrix}
\]
To understand $\text{im}\overline{\partial}_G$, $\text{im}\overline{G}_\beta\overline{\partial}_G$, one can use row and column operations invertible over the *principal ideal domain* $\mathbb{Z}[\zeta]$ to bring these two matrices to their unique Smith normal forms over $\mathbb{Z}[\zeta]$, namely
\[
\text{im}\overline{\partial}_G/\text{im}\overline{G}_\beta\overline{\partial}_G \cong \mathbb{Z}[\zeta]/24\mathbb{Z}[\zeta] \cong \mathbb{Z}_2 \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3
\]
This shows that
\[
\mathbb{Z}[\zeta]^V/\text{im}\overline{\partial}_G \cong \mathbb{Z}[\zeta]/24\mathbb{Z}[\zeta] \cong \mathbb{Z}_2 \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3
\]
and therefore one must have
\[
\text{im}\overline{\partial}_G = \text{im}\overline{\partial}_G \overline{\partial}_G \cong \mathbb{Z}_3.
\]
An easy calculation shows that $K(G) = \mathbb{Z}_2$ (e.g. observe that $|K(G)| = 2$ since $G$ has only two spanning trees). Hence the exact sequence from Theorem 4.1 must look as follows:
\[
0 \rightarrow K(G) \rightarrow K(\hat{G}) \rightarrow K(G_\beta) \rightarrow 0
\]
\[
\mathbb{Z}_2 \mathbb{Z}_3 \oplus \mathbb{Z}_3
\]
Since the theorem tells us that this sequence splits at the $p$-primary components for $p \neq 3$, one concludes from this that the octahedron graph $\hat{G}$ has
\[
K(\hat{G}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_{24}
\]
in agreement with the known answer (see e.g. [7] §9.4.2).

7. Voltage groups of order 2: double covers and signed graphs

The situation is particularly simple when $p = 2$, that is, for *double coverings*. As mentioned in Example 4.3, graph double covers are always regular, with transformation group $H = \mathbb{Z}_2 = \{1, h\}$ identified with the two voltages $\{+, -\}$. Thus a voltage graph $G_\beta$ as a function $E \rightarrow H = \mathbb{Z}_2 = \{+, -\}$ is the same as a signed graph $G_\beta$ as defined in the Introduction.

Note that here $\zeta = -1$ and the isomorphism $\mathbb{Z}H \cong \mathbb{Z}[\zeta] = \mathbb{Z}$ sends $h \mapsto \zeta = -1$. Also the anti-automorphism $T_h \mapsto T_{h^{-1}}$ of $\mathbb{Z}H$ and of $\mathbb{Z}H$ has become trivial, so that $\partial^*_G = \partial_G$.


Proposition 7.1. For a double cover corresponding to a signed graph $G_\pm$, the matrix $\partial_{G_\pm}$ from the Introduction is the same as $\partial_{G_\beta}$ as in Definition 7.1. In particular, the critical group $K(G_\beta)$ is exactly $K(G_\pm)$ as defined in [13] in the Introduction.

\textbf{Proof.} In the Introduction, $\partial_{G_\pm}$ mapped a positive (resp. negative) edge $e$ directed as $(u, v)$ to $+u - v$ (resp. $+u + v$), which agrees with the action of $\partial_{G_\beta}$ as $e \mapsto +u - vT_\beta(e)$, since $T(\beta(e)) \mapsto +1, -1$ depending upon whether $e$ is a positive, negative edge.

\hfill $\Box$

Consequently, Theorem 1.2 immediately implies the following result from the Introduction.

\textbf{Theorem 1.2.} For each signed graph $G_\pm$, parametrizing a graph double covering $\tilde{G} \to G$, one has a short exact sequence of critical groups

$$0 \to K(G_\pm) \to K(\tilde{G}) \to K(G) \to 0$$

splitting on restriction to $p$-primary components for odd primes $p$. In particular, $|K(\tilde{G})| = |K(G_\pm)| \cdot |K(G)|$.

Theorem 1.2 will be generalized in a different direction in Theorem 11.6 below, after we generalize (in Section 10) the notion of double coverings of unsigned graphs to double coverings of signed graphs.

7.1. Example: Bipartite double covers and crowns.

\textbf{Definition 7.2.} Given an unsigned multigraph $G = (V, E)$, its \textit{bipartite double cover} (see, e.g. Waller [19]) is the double cover $\tilde{G} \to G$ associated to the signed graph which Zaslavsky [20, §7.D] calls the \textit{all-negative assignment} $G_\beta = G_\pm = -G$, in which every edge $e$ in $E$ has $\beta(e) = -$. The bipartite double cover of $G$ is sometimes also called the tensor product or categorical product $G \times K_2$, where $K_2$ is the unsigned graph consisting of a single edge between two vertices.

When $G$ is highly symmetric, the same is true of the all negative signed graph $-G$, sometimes leading to an easy computation of both $K(G), K(-G)$, where Theorem 1.2 is easy to apply.

\textbf{Example 7.3.} The $n$-\textit{crown graph} $\text{Crown}_n$ is the unsigned graph obtained from the complete bipartite graph $K_{n,n}$ on bipartitioned vertex set $V = \{v_1^{(1)}, \ldots, v_1^{(n)}\} \cup \{v_2^{(1)}, \ldots, v_2^{(n)}\}$ by removing the perfect matching of edges $M = \{\{v_i^{(1)}, v_i^{(2)}\} : i = 1, 2, \ldots, n\}$. More generally, define $\text{Crown}_n^{(k)}$ to be the multigraph obtained from $\text{Crown}_n$ by adding back in $k$ copies of each edge from the perfect matching $M$ that was removed. Equivalently, $\text{Crown}_n^{(k)}$ is the multigraph obtained from $K_{n,n}$ by adding $k - 1$ copies of the perfect matching $M$. In particular, taking $k = 1$, the graph $\text{Crown}_n^{(1)}$ recovers $K_{n,n}$ itself.

Let $K_n^{(m)}$ be the multigraph obtained from the complete graph $K_n$ on vertex set $\{v_1^{(1)}, \ldots, v_n^{(n)}\}$ by adding $m$ multiple copies of a self-loop to every vertex $v_i^{(1)}$.

The following proposition is then straightforward.

\textbf{Proposition 7.4.} When $k$ is even, $\text{Crown}_n^{(k)}$ provides the bipartite double covering of $K_n^{(\frac{k}{2})}$, via the map

$$\tilde{G} := \text{Crown}_n^{(k)} \xrightarrow{\pi} K_n^{(\frac{k}{2})} : G$$

$$v_i^{(j)} \mapsto v_i^{(j)}$$

$$\{v_i^{(j)}, v_i^{(j')}\} \mapsto \{v_i^{(j)}, v_i^{(j')}\}$$

that also sends the extra $k$ copies of the matching edge $\{v_+^{(i)}, v_-^{(i)}\}$ to the $\frac{k}{2}$ copies of the loop edge on $v_i^{(j)}$.

\textbf{Example 7.5.} For $n = 4, k = 2$, here is a depiction of the bipartite double covering $\text{Crown}_4^{(2)} \to K_4^{(1)}$: 
Corollary 7.6. For $k$ even and $n$ odd,

$$K(\text{Crown}_n^{(k)}) \cong \mathbb{Z}^{n-2} \oplus \mathbb{Z}^{n-2}_{n-2+2k} \oplus \mathbb{Z}^{n-2}_{(n-1+k)(n-2+2k)}$$

Proof. For $G = \text{Crown}_n^{(k)}$, both the unsigned graph Laplacian $L(G) = \partial_G \partial_G^t$ and the all-negative signed graph Laplacian $L(-G) = \partial_{-G} \partial_{-G}^t$ are $n \times n$ matrices of the form $M_n(b, a) = bI_n - aJ_n$ where $I, J$ are the identity and all ones matrices, respectively.

Specifically,

$$L(G) = M_n(n, 1)$$
$$L(-G) = M_n(n - 2 + 2k, -1).$$

Hence one can begin the calculation of $K(G)$ and $K(-G)$ with an easy general computation (see [10, Prop 4.2(v)]) showing $M_n(b, a)$ has Smith normal form entries

$$\begin{pmatrix}
    \text{gcd}(a, b), & b, b, \ldots, b \\
    n - 2 \text{ times} & \frac{b(b - na)}{\text{gcd}(a, b)}
\end{pmatrix}.$$

For $L(G)$ this gives Smith normal form entries $(1, n, \ldots, n, 0)$ and $K(G) = \text{im} \partial_G / \text{im} L(G) \cong \mathbb{Z}_n^{n-2}$, as is well-known. For $L(-G)$ it gives Smith entries

$$(1, n - 2 + 2k, \ldots, n - 2 + 2k, 2(n - 1 + k)(n - 2 + 2k))$$

and hence

$$(7.1) \quad \mathbb{Z}_n / \text{im} L(-G) \cong \mathbb{Z}_{n-2+2k}^{n-2} \oplus \mathbb{Z}_{2(n-1+k)(n-2+2k)}^{n-2}.$$

One can also easily check (see Proposition 9.7 below) that $\text{im} \partial_{-G}$ is the index two sublattice $\mathbb{Z}_n^{n \equiv \equiv 0 \mod 2}$ of $\mathbb{Z}^n$ where the sum of the entries is even. Hence $K(-G) = \text{im} \partial_{-G} / \text{im} L(-G)$ must be a subgroup of index two within the group $\mathbb{Z}_n / \text{im} L(-G)$ described in (7.1) above. If one assumes that $n$ is odd, which we will do for the remainder of this calculation, so that $n - 2 + 2k$ is also odd, then the only summand in (7.1) having a subgroup of index 2 is the last summand $\mathbb{Z}_{2(n-1+k)(n-2+2k)}$. Hence this forces

$$K(-G) \cong \mathbb{Z}_{n-2+2k}^{n-2} \oplus \mathbb{Z}_{(n-1+k)(n-2+2k)}^{(n-2+2k)}$$

for $n$ odd. Thus the short exact sequence from Theorem 1.2 takes the form

---

4 We will be able to remove this assumption that $k$ is even in Section 12.1 below, after allowing for negative half-loops in signed graphs and double covers.

5 Actually, with a bit more matrix manipulation, one can draw this same conclusion for all $n$; see Tseng [17, §8.1].
whose underlying multigraph is $mG$ assignment $\beta K$ (7.3) For a positive integer $n$, we define $8.1$. The construction $\beta$. Proven correct for all $n$, we apply it to three families of examples.

Once the theorem also tells us this sequence splits at $p$-primary components for all odd primes $p$, and since $K(G) = \mathbb{Z}^{n-2}_n$ only has odd primary components for $n$ odd, the sequence must split at all primes. Therefore

$$K(Crown_n^{(k)}) = K(\tilde{G}) \cong K(G) \oplus K(-G) \cong \mathbb{Z}^{n-2}_n \oplus \mathbb{Z}^{n-2}_{n-2+2k} \oplus \mathbb{Z}^{n-2}_{n-2+2k}.$$}

We remark that, for $k = 0$, this answer for $n$ odd agrees with a result of Machacek \[14, Theorem 14\]

$$K(Crown_n) \cong \mathbb{Z}^{n-2}_n \oplus \mathbb{Z}^{n-3}_n \oplus \mathbb{Z}^{n-1}_n,$$

proven correct for all $n$ (not just $n$ odd) via Smith normal forms. See also Remark \[12.2\] below.

8. Application: when the voltage graph Laplacian is diagonal.

The voltage graph Laplacian $L(G_\beta)$ defined in \[5.3\] has a peculiar feature that happens only when the voltage group $H$ is nontrivial: nonempty voltage graphs $G_\beta$ can have a diagonal $L(G_\beta)$. We describe such a situation, giving a result that uses this diagonal structure, then apply it to three families of examples.

8.1. The construction.

**Definition 8.1.** For a positive integer $m \geq 2$ and a multigraph $G = (V,E)$, let $mG = (V,mE)$ denote the multigraph on the same vertex set $V$ in which each edge $e$ in $E$ has been replicated into $m$ copies.

Given a group $H$ of order $|H| = m$, and a multigraph $G = (V,E)$, let $HG$ denote the $H$-voltage graph whose underlying multigraph is $mG$, so that its edges can be labelled $\{e(h)\}_{r \in E, h \in H}$, and with voltage assignment $\beta(e(h)) = h$.

**Proposition 8.2.** Consider a group $H$ of order $m \geq 2$, and a connected multigraph $G = (V,E)$ with degree sequence $(d_1, \ldots, d_{|V|})$ of $G$, in which loops count 2 toward the degree of a vertex. Then the voltage graph Laplacian $L(HG)$ in $\mathbb{Z}^{V \times V}_H$ is the diagonal matrix whose entries are $(md_1, \ldots, md_{|V|})$.

Furthermore, after uniquely expressing $\bigoplus_{i=1}^{V} \mathbb{Z}_{s_i}$, for positive integers $s_1, \ldots, s_{|V|}$ with $s_i$ dividing $s_{i+1}$, one has

$$K(HG) \cong \mathbb{Z}_{s_1} \oplus \mathbb{Z}^{m-2}_{s_1} \oplus \bigoplus_{i=2}^{V} \mathbb{Z}^{m-1}_{s_i}.$$ In particular, whenever $m$ is relatively prime to all the degrees $d_i$, one can rewrite this as

$$K(HG) \cong \mathbb{Z}^{m-1}_{m|V|-1} \oplus \bigoplus_{i=1}^{V} \mathbb{Z}^{m-1}_{d_i}.$$ Proof. For the description of the entries of $L(HG)$, first note that $L(HG)$ is diagonal since a pair of vertices $u, v$ with $u \neq v$ having $d$ edges between them will have $(u,v)$ entry in $L(HG)$ given by

$$d \sum_{h \in H} (-1) T_h = -d \sum_{h \in H} T_h = -d \cdot c = 0 \quad \text{in} \quad \mathbb{Z}^{V \times V}_H := \mathbb{Z}^{V \times V} / \mathbb{Z}^{V \times V} c.$$
Thus we only need to compute the diagonal \((v, v)\) entry corresponding to each vertex \(v\) in \(V\). If \(v\) has \(\ell\) loops attached and is incident to \(d\) nonloop edges, then this \((v, v)\) entry in \(L(HG) := \partial G_\beta \partial G_\beta\) is given by the sum
\[
d \sum_{h \in H} (-T_h)(-T_{h^{-1}}) + \ell \sum_{h \in H} (1 - T_h)(1 - T_{h^{-1}})
\]
\[
= d \sum_{h \in H} 1 + \ell \sum_{h \in H} (2 - (T_h + T_{h^{-1}}))
\]
\[
= dm + 2\ell m
\]
\[
= md_v.
\]

For the assertions about \(K(HG)\), we use its presentation from (5.2) as \(K(HG) = \text{im} \partial_{HG}/\text{im}L(HG)\), and start by describing \(\text{im} \partial_{HG}\) more explicitly. Note that \(\partial_{HG}\) fits into this commutative square, where the vertical maps are both quotient maps:
\[
\begin{array}{ccc}
(\mathbb{Z}H)^E & \xrightarrow{\partial_{HG}} & (\mathbb{Z}H)^V \\
\kappa_H \downarrow & & \kappa_V \\
(\mathbb{Z}H)^E & \xrightarrow{\partial_{HG}} & (\mathbb{Z}H)^V.
\end{array}
\]

Therefore \(\text{im} \partial_{HG} = \kappa_V(\text{im} \partial_{HG})\), and it helps to first analyze \(\text{im} \partial_{HG}\). If \(\tilde{m}G = (\tilde{V}, \tilde{E})\) denotes the total space in the covering \(mG \to mG\), then one easily checks (or see Proposition 8.3 below) that connectivity of \(G\) implies connectivity of \(\tilde{m}G\). Hence \(\text{im} \partial_{HG}\) is the sublattice \(\mathbb{Z}^{L_0}\) of \(\mathbb{Z}^V\) where the coordinates sum to zero. Under the isomorphism of \(\mathbb{Z}^V\) with \((\mathbb{Z}H)^V\) in Proposition 4.6, this sublattice \(\text{im} \partial_{HG}\) corresponds to the sublattice of \((\mathbb{Z}H)^V\) consisting of those elements \(x = (x_1, \ldots, x_{|V|})\) whose sum of coordinates \(x_1 + \cdots + x_{|V|} = \sum_{h \in H} a_h T_h\), when considered as an element of \(\mathbb{Z}H\), satisfies \(\sum_{h \in H} a_h = 0\). Then \(\text{im} \partial_{HG}\) is the image of this sublattice \(\text{im} \partial_{HG}\) of \((\mathbb{Z}H)^V\) under the quotient map \(\kappa_V\) that mods out by multiples of \(c := \sum_{h \in H} T_h\). Since \(c\) has its sum of coordinates equal to \(|H| = m\), one concludes that \(\text{im} \partial_{HG}\) is the sublattice \(\Lambda\) of \((\mathbb{Z}H)^V\) consisting of the elements \(x = (x_1, \ldots, x_{|V|})\) whose sum of coordinates \(x_1 + \cdots + x_{|V|} = \sum_{h \in H} a_h T_h\), when considered as an element of \(\mathbb{Z}H\), satisfies \(\sum_{h \in H} a_h \equiv 0 \mod m\).

We next compute that
\[
K(HG) = \text{im} \partial_{HG}/\text{im}L(HG)
\]
\[
(8.2)
\]
\[
= \Lambda / \bigoplus_{i=1}^{|V|} md_i \mathbb{Z}^H/n \biggl/ \bigoplus_{i=1}^{|V|} md_i \mathbb{Z}^{m-1}
\]
in which \(\mathbb{Z}^{n \mod m}\) denotes the sublattice of \(\mathbb{Z}^n\) where the sum of coordinates is 0 modulo \(m\). Then the last expression in (8.2) is isomorphic to the right-side of (8.1) via Lemma 8.3 below.

For the last assertion of the proposition, when \(m\) happens to be relatively prime to all the vertex degrees \(d_i\), it is also relatively prime to all of the \(s_i\), and hence one has
\[
K(HG) \cong \mathbb{Z}^{s_1} \oplus \mathbb{Z}^{m-2} \oplus \bigoplus_{i=2}^{|V|} \mathbb{Z}^{m-1}_{s_i}
\]
\[
\cong \mathbb{Z}^{(m-1)|V|-1} \oplus \bigoplus_{i=1}^{|V|} \mathbb{Z}^{m-1}_{d_i}.
\]

The following numerical lemma was used in the preceding proof.
Theorem 8.3. Given positive integers $d_1, \ldots, d_n$, if one uniquely expresses $\bigoplus_{i=1}^{n} \mathbb{Z}_{d_i} \cong \bigoplus_{i=1}^{n} \mathbb{Z}_{s_i}$, for positive integers $s_1, \ldots, s_{|V|}$ with $s_i$ dividing $s_{i+1}$, then

$$
\mathbb{Z}_{\equiv 0 \mod m} \bigoplus_{i=1}^{n} \mathbb{Z}_{md_i} \cong \bigoplus_{i=1}^{n} \mathbb{Z}_{s_i} \bigoplus \mathbb{Z}_{ms_i}.
$$

Proof. If $\mathbb{Z}^n$ has standard basis $e_1, \ldots, e_n$, then the sublattice $\mathbb{Z}_{\equiv 0 \mod m}^n$ has a $\mathbb{Z}$-basis given by $\delta_1 = me_1$ and $\delta_i = e_i - e_{i-1}$ for $i = 2, 3, \ldots, n$. With respect to this basis, one can express $mc_i = \delta_1 + m(\delta_2 + \delta_3 + \cdots + \delta_i)$ for $i = 1, 2, \ldots, n$. Hence $\mathbb{Z}_{\equiv 0 \mod m}^n / \bigoplus_{i=1}^{n} \mathbb{Z}_{d_i} \mathbb{Z}$ is isomorphic to the cokernel of this matrix $A$ in $\mathbb{Z}^{n \times n}$:

$$
A = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & m & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & 0 \\
0 & 0 & \cdots & 0 & m
\end{bmatrix} \cong \begin{bmatrix}
d_1 & d_2 & d_3 & \cdots & d_n \\
0 & m & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & 0 \\
0 & 0 & \cdots & 0 & m
\end{bmatrix}.
$$

One can easily check that for $k = 1, 2, \ldots, m$, the gcd of the set of all $k \times k$ minor subdeterminants of $A$ is $m^{k-1}$ times the gcd of all products $d_1 \cdots d_k$ with $1 \leq i_1 < \cdots < i_k \leq n$, and hence equals $m^{k-1}s_1s_2\cdots s_k$. This implies the Smith normal form entries for $A$ are $s_1, ms_2, ms_3, \ldots, ms_n$, proving the lemma.

Having determined the critical group $K(HG)$ for these special voltage graphs $HG$, we now wish to consider their associated regular covering of the underlying graph $mG$.

Proposition 8.4. For $m \geq 2$, the total space $\tilde{mG} = (\tilde{V}, \tilde{E})$ in the graph covering $\tilde{G} \to mG$ associated to $HG$ has the following description as an undirected graph, which depends only on $m = |H|$, and not on the structure of $H$ as a group.

The vertex set $V$ contains $m$ vertices $\{v_h\}_{h \in H}$ for each vertex $v$ in $V$.

The edge set $E$ contains for each nonloop edge $e = (u,v)$ of $E$, a copy of the complete bipartite graph $K_{m,m}$ on the bipartitioned vertex set $\{u_h\}_{h \in H} \cup \{v_h\}_{h \in H}$. For each loop edge on a vertex $v$ in $V$, the edge set $E$ also contains a loop on each vertex in $\{v_h\}_{h \in H}$, as well as two copies of the complete graph $K_m$ on vertex set $\{v_h\}_{h \in H}$.

Proof. An edge $(u_{h_1}, v_{h_1})$ within a copy of the bipartite graph $K_{m,m}$ corresponding to an edge $e$ represents in the regular cover the edge labelled $e^{(h)}_{h_1}$, lying above the copy $e^{(h)}$ of $e$ in $mG$ that has been assigned voltage $\beta(e^{(h)}) = h$.

For each loop $e$ at a vertex $v$ in $V$, the loop on $v_h$ represents the regular cover edge labelled $e^{(1)}_h$ lying above the copy $e^{(1)}$ of $e$ in $mG$ that has been assigned voltage $\beta(e^{(1)}) = 1$. The two copies of the complete graphs $K_m$ on $\{v_h\}_{h \in H}$ come from the regular cover edges labelled $e^{(h)}_{h'}$ as the ordered pairs $(h, h')$ run through $H \times (H \setminus \{1\})$: for $h_1 \neq h_2$ in $H$ one will obtain the undirected edge $\{v_{h_1}, v_{h_2}\}$ twice, once from taking $h = h_1$ and $h' = h_2h_1^{-1}$, and once from taking $h = h_1$ and $h' = h_2h_1^{-1}$.

Corollary 8.5. Fix $m \geq 2$ a positive integer, and let $G = (V, E)$ be a connected multigraph with no loops, with critical group $K(G) \cong \bigoplus_{i=1}^{V-1} \mathbb{Z}_{|v_i|}$. Assume $G$ has vertex degrees $d_1, \ldots, d_{|V|}$, all relatively prime to $m$.

Then the short exact sequence of Theorem 7.3 becomes

$$
0 \to K(mG) \to K(\tilde{mG}) \to K(HG) \to 0.
$$

(8.3)

$$
\bigoplus_{i=1}^{V-1} \mathbb{Z}_{m|v_i|} \oplus \mathbb{Z}_{m-1}^{(m-1)|V|} \oplus \bigoplus_{i=1}^{V} \mathbb{Z}_{m^{|v_i|}}.
$$

If one further assumes that $m$ is relatively prime to the determinant of the adjacency matrix of $G$, then

$$
K(\tilde{mG}) \cong \mathbb{Z}_{m^{(|V|-1)}} \oplus \bigoplus_{i=1}^{V} \mathbb{Z}_{m|v_i|} \oplus \bigoplus_{i=1}^{V} \mathbb{Z}_{m^{(|v_i|)-1}}.
$$

(8.4)
Proof. The fact that \( K(G) \cong \bigoplus_{i=1}^{|V|} \mathbb{Z}_{k_i} \), if and only if \( K(mG) \cong \bigoplus_{i=1}^{|V|-1} \mathbb{Z}_{m k_i} \), is a well-known consequence of the presentation (1.1) of \( K(G) \), as one has this relation between Laplacian matrices: \( L(mG) = mL(G) \).

The description of \( K(HG) \) comes from Proposition 8.1. This explains the exact sequence (8.3).

For the last assertion, we first consider primes \( p \) that divide \( m \), noting that the \( p \)-primary part of the sequence (8.3) looks like

\[
0 \to \text{Syl}_p \left( \bigoplus_{i=1}^{|V|-1} \mathbb{Z}_{m k_i} \right) \to \text{Syl}_p \tilde{K}(mG) \to \text{Syl}_p \left( \mathbb{Z}_{m(|V|-1)} \right) \to 0
\]

where recall that here \( \text{Syl}_p(A) \) denotes the \( p \)-subgroup or \( p \)-primary component of a finite abelian group \( A \). Since \( m \geq 2 \), one can pick distinct elements \( h_1 \neq h_2 \) in \( H \), and check that the \( |V| \times |V| \) submatrix of the Laplacian \( L(mG) \) with rows indexed by \( \{v_{h_1}\}_{v \in V} \) and columns indexed by \( \{v_{h_2}\}_{v \in V} \) is the negative of the adjacency matrix of \( G \). Therefore under the additional assumption that \( m \) is relatively prime to the determinant of this adjacency matrix, for every prime \( p \) dividing \( m \), the \( p \)-primary component \( \text{Syl}_p \tilde{K}(mG) \) appearing in the middle of the sequence (8.3) has its number of generators bounded by

\[
(|\tilde{V}| - 1) - |V| = (m|V| - 1) - |V| = (m - 1)|V| - 1
\]

matching the exponent on \( \mathbb{Z}_m \) in the right term in the sequence. This then forces

\[
\text{Syl}_p \tilde{K}(mG) = Syl_p \left( \bigoplus_{i=1}^{|V|-1} \mathbb{Z}_{m^2 k_i} \oplus \mathbb{Z}_{m(|V|-2)} \right)
\]

By Proposition 8.4 the sequence (8.3) splits at \( p \)-primary components for all the other primes \( p \) that do not divide \( m \). Collating the various \( p \)-primary components then gives the description (8.4) for \( K(mG) \). \( \square \)

We next apply Corollary 8.5 in three families of examples.

8.2. Example: when \( G \) is a path.

Proposition 8.6. Let \( G = (V, E) \) be a path, with \( |V| \) even, and let \( m \geq 3 \) be an odd integer. Then the regular covering \( \tilde{mG} \to mG \) has

\[
K(\tilde{mG}) = \mathbb{Z}_{m^2}^{(|V|-2)} \oplus \mathbb{Z}_{m}^{(|V|-1)} \oplus \mathbb{Z}_{2}^{(m-1)(|V|-2)}.
\]

Here is a picture of \( \tilde{mG} \to mG \) in the case where \( m = 3 \) and \( |V| = 4 \).

\[
\begin{array}{c}
a_3 \quad b_3 \\
b_2 \quad c_2 \\
a_1 \quad b_1 \quad c_1 \\
d_3 \\
d_2 \\
d_1
\end{array}
\]

Proof. Check Corollary 8.5 applies. The vertex degrees are \( (d_1, \ldots, d_{|V|}) = (1, 2, 2, \ldots, 2, 2, 1) \), all relatively prime to the odd number \( m \). One can calculate by induction on \( |V| \) that a path has the determinant of its adjacency matrix either 0 for \( |V| \) odd, or \pm 1 for \( |V| \) even, and hence relatively prime to \( m \) when \( |V| \) is even. Having checked that the corollary applies, one needs to know that \( K(G) \) is the trivial group (since \( G \) is a tree), with invariant factors \( (k_1, \ldots, k_{|V|-1}) = (1, 1, \ldots, 1) \). \( \square \)
8.3. Example: when $G$ is a cycle.

**Proposition 8.7.** Let $G = (V, E)$ be a cycle with $|V| \neq 0 \mod 4$, and let $m \geq 3$ be an odd integer. Then the regular covering $\tilde{mG} \to mG$ has

$$K(\tilde{mG}) = \mathbb{Z}_{m}^{(m-2)|V|} \oplus \mathbb{Z}_{m^2} |V| \oplus \mathbb{Z}_{m^2} |V| \oplus \mathbb{Z}_{2}^{(m-1)|V|}.$$ 

Note that in the previous figure, gluing together vertices $a, d$ and gluing $a_i, d_i$ to each other for $i = 1, 2, 3$, gives the picture for $m = 3 = |V|$.

**Proof.** Again check Corollary 8.3 applies. The vertex degrees $(d_1, \ldots, d_{|V|}) = (2, 2, \ldots, 2)$, relatively prime to the odd number $m$. One can calculate by induction on $|V|$ (or see [1, Prop. 2.4]) that a cycle has the determinant of its adjacency matrix either

$$\begin{cases} 
0 & \text{if } |V| \equiv 0 \mod 4, \\
-4 & \text{if } |V| \equiv 2 \mod 4, \\
\pm 2 & \text{if } |V| \equiv 1, 3 \mod 4,
\end{cases}$$

relatively prime to the odd number $m$ if $|V| \neq 0 \mod 4$. Having checked that the corollary applies, one needs the well-known fact that $K(G) = \mathbb{Z}_{|V|}$, with invariant factors $(k_1, \ldots, k_{|V|-1}) = (1, 1, \ldots, 1, |V|)$. \hfill \Box

8.4. Example: when $G$ is a complete graph.

**Proposition 8.8.** Let $G = (V, E)$ be a complete graph (without loops) having $|V| - 1$ relatively prime to the positive number $m \geq 2$. Then the regular covering $\tilde{mG} \to mG$ has

$$K(\tilde{mG}) = \mathbb{Z}_{m}^{(m-2)|V|} \oplus \mathbb{Z}_{m^2} |V| \oplus \mathbb{Z}_{m^2} |V| \oplus \mathbb{Z}_{2}^{(m-1)|V|}.$$ 

**Proof.** Again check Corollary 8.3 applies. The vertex degrees $(d_1, \ldots, d_{|V|})$ are all $|V| - 1$, relatively prime to $m$. The adjacency matrix for $G$ is $J - I$ of the $|V| \times |V|$ all ones matrix $J$ and the identity matrix $I$, with eigenvalues $(|V|, 0, 0, \ldots, 0) - (1, 1, 1, \ldots, 1) = (|V| - 1, -1, -1, \ldots, -1)$, and hence determinant $\pm(|V| - 1)$, relatively prime to $m$. Having checked the corollary applies, one needs the well-known fact that $K(G) = \mathbb{Z}_{|V|}^{|V|-2}$, with invariant factors $(k_1, \ldots, k_{|V|-1}) = (1, |V|, |V|, \ldots, |V|)$. \hfill \Box

We remark that that in this example, the graph $\tilde{mG}$ is the complete $|V|$-partite graph $K_{m, m, \ldots, m}$, whose critical group was computed for all $|V|$ and $m$ in [10, Cor. 5]. One can check that the answer given in Proposition 8.8 agrees with this computation when $|V| - 1$ and $m$ are relatively prime.

9. Signed graphs in general: allowing half-loops

This section reviews the more general notion of signed graphs $G_{\pm}$, as in Zaslavsky [20], in which one allows positive and negative half-loops, with the goal of generalizing our definition of the critical group $K(G_{\pm})$ to this case. This gives us the flexibility to consider in the next section a more general notion of double covering, both for unsigned and signed graphs, in which a half-loop can be doubly covered by a single edge.

For example, in Section 10.2, we will use this to re-interpret a calculation of H. Bai on the critical group of the $n$-dimensional cube graph $Q_n$: the obvious projection $Q_n \to Q_{n-1}$ can be regarded as such a double cover, in which each edge of $Q_n$ parallel to the direction of projection doubly covers a half-loop added to its image vertex in $Q_{n-1}$; see Figure 10.1.

9.1. Definition of a general signed graph critical group.

**Definition 9.1.** An unsigned multigraph with half-loops $G = (V, E)$ is a multigraph in which some of the self-loops have been designated as half-loops. A signed graph $G_{\pm}$ consists of an underlying multigraph with half-loops $G = (V, E)$ together with an assignment $\beta : E \to \{+1, -1\}$ (=: $\{+, -\}$), designating edges positive or negative.

For these more general signed graphs, we will need two closely related versions of an node-edge-incidence matrix, $\partial = \partial_{G_{\pm}}$ and $\delta = \delta_{G_{\pm}}$, both lying in $\mathbb{Z}^{V \times E}$, that is, both regarded as $\mathbb{Z}$-linear maps $\mathbb{Z}^{E} \to \mathbb{Z}^{V}$. As before, one first chooses an arbitrary orientation of the edges $E$ to write them down.
Definition 9.2. The map $\partial$ treats loops and half-loops the same, sending an edge $e$ directed from $u$ to $v$ to $+u - \beta(e)v$, even if $u = v$. This means that $\partial$ sends positive loops and positive half-loops to 0, and sends both a negative loop and negative half-loop on vertex $v$ to $+2v$.

The map $\delta$ is almost the same, except that it treats negative loops and negative half-loops unequally. Just as with $\partial$, the map $\delta$ sends an edge $e$ directed from $u$ to $v$ to $+u - \beta(e)v$ when $u \neq v$. Also just as with $\partial$, the map $\delta$ send both positive loops and positive half-loops to 0, and $\delta$ sends a negative loop on vertex $v$ to $+2v$. However, $\delta$ sends a negative half-loop on vertex $v$ to $+v$.

Remark 9.3. The map $\delta$ is the signed graph incidence matrix used by Zaslavsky in [20 §8A]. Note that $\delta = \partial$ if and only if $G_{\pm}$ contains no negative half-loops.

Definition 9.4. For a signed graph $G_{\pm}$, define its critical group
\begin{equation}
K(G_{\pm}) := \text{im}\partial/\text{im}\delta^t
\end{equation}
and its signed graph Laplacian matrix
\begin{equation}
L(G_{\pm}) := \partial\delta^t
\end{equation}
where we will call the matrix $L(G_{\pm})$ := $\partial\delta^t$ appearing above a signed graph Laplacian.

9.2. Issues of well-definition. Note that this definition of $K(G_{\pm})$ generalizes our earlier definition for the more restrictive signed graphs in the Introduction, where half-loops were disallowed. The next proposition answers some other obvious questions which are not as familiar or transparent as for unsigned graphs.

Proposition 9.5. The signed graph Laplacian matrix $L(G_{\pm}) = \partial\delta^t$ has entries
\begin{equation}
L(G_{\pm})_{u,v} = \begin{cases}
\#\{\text{negative edges with endpoints } u, v\} & \text{if } u \neq v, \\
-\#\{\text{positive edges with endpoints } u, v\} & \\
\#\{\text{non-loop edges (positive or negative) incident to } v\} & \\
+4\#\{\text{negative (full) loops at } v\} + 2\#\{\text{negative half-loops at } v\} & \text{if } u = v.
\end{cases}
\end{equation}

In particular,
\begin{itemize}
\item $L(G_{\pm})$ is symmetric, and
\item both the matrix $L(G_{\pm})$ and the isomorphism type of the abelian group $K(G_{\pm})$ do not depend upon the choice of orientation of the edges $E$ used to write down $\partial$ and $\delta$.
\end{itemize}

Proof. The matrix entry calculation for $L(G_{\pm})$ is straightforward, and does not depend on the orientations. Note also that the sublattice $\text{im}\partial$ inside $\mathbb{Z}^V$ does not depend upon the orientations, as a typical column of $\text{im}\partial$ for an oriented edge $e = (u, v)$ is $\delta(e) = +u - \beta(e)v = \pm(+v - \beta(e)u)$. Thus $K(G_{\pm}) = \text{im}\partial/\text{im}\delta^t$ does not change when one reorients edges.

It is fairly obvious for unsigned graphs that the critical group $K(G)$ is an isomorphism invariant of the graph $G = (V, E)$, since permuting or relabelling vertices corresponds to permuting the coordinates of the ambient space $\mathbb{R}^V \supset \mathbb{Z}^V \supset \text{im}\partial \supset \text{im}L(G)$, without altering $K(G)$ up to isomorphism. Of course, the same holds for permutation of the vertices in signed graphs $G_{\pm}$. However, there is a stronger notion of signed graph isomorphism that allows not only permuting or relabelling vertices, but in addition, at any vertex $v$ in $V$ one can perform the switch at $v$ (see [20 §3]) on $G_{\pm}$, which has the effect of exchanging $\beta(e)$ via $+ \leftrightarrow -$ for every non-loop, non-half-loop edge $e$ incident to $v$. Algebraically, this corresponds to a sign change in the $v$-coordinate of the ambient space $\mathbb{R}^V$, and again does not alter $K(G)$ up to isomorphism. We will take advantage of such signed graph isomorphisms in the next section.

9.3. Balanced cycles and the image of $\partial$. The following simple notion is an important signed graph isomorphism invariant (see [20 §2]), dictating the nature of $\text{im}\partial$ inside $\mathbb{Z}^V$.

Definition 9.6. For a signed graph $G_{\pm}$ with underlying multigraph $G = (V, E)$, consider a subset $C \subset E$ forming a cycle in $G$, with $C$ possibly a singleton (full) loop or half-loop. Call $C$ a balanced (resp. unbalanced) cycle of $G_{\pm}$ if the number of negative edges $e$ in $C$ (that is, those with $\beta(e) = -$) is even (resp. odd).

For unsigned graphs, the description of the sublattice $\text{im}\partial$ inside $\mathbb{Z}^V$ is fairly straightforward: when $G = (V, E)$ has connected components with vertex sets $V_1, V_2, \ldots, V_t$, one has compatible direct sum decompositions $\mathbb{Z}^V = \bigoplus Z_{V_i}$ and $\text{im}\partial = \bigoplus_{i=1}^t Z^V_{V_i \setminus 0}$ where $Z^V_{V_i \setminus 0} := \{ x \in \mathbb{Z}^V : \sum_{v \in V} x_v = 0 \}$. 
For a signed graph $G_\pm$ one again has the same reduction to each connected component of its underlying multigraph, which one can therefore assume is connected.

**Proposition 9.7.** A signed graph $G_\pm$ with connected underlying multigraph has two cases for $\text{im} \partial$:

(i) $\text{im} \partial = \mathbb{Z}_0^V$ $(\mod 2) := \{ x \in \mathbb{Z}^V : \sum_{v \in V} x_v = 0 \mod 2 \}$ if $G_\pm$ contains at least one unbalanced cycle; call $G_\pm$ **unbalanced** in this case.

(ii) $\text{im} \partial = \mathbb{Z}_0^V$ if $G_\pm$ has no unbalanced cycles; call $G_\pm$ **balanced** in this case.

Moreover, $G_\pm$ being balanced is equivalent to it being signed-graph isomorphic to an unsigned multigraph with half-loops, that is, a signed graph with no negative edges.

**Proof.** Straightforward, or see Zaslavsky [20, Prop. 2.1, Thm. 5.1].

### 9.4. The cardinality of the critical group.

The definition of the signed graph critical group $K(G_\pm)$ does not make it clear that it is a **finite** group. We pause here to show this, and to give a signed generalization of the formula for the cardinality of unsigned graph critical groups in terms of maximal forests. The methodology is straightforward, proven analogously to Zaslavsky’s Matrix-Tree Theorem for signed graphs [20, Thm. 8A.4], via the Binet-Cauchy Theorem[4].

To start, one needs to know the analogue of maximal forests in unsigned graphs.

**Proposition 9.8.** Given a signed graph $G_\pm$ having underlying multigraph $G = (V, E)$, consider a subset $B \subset E$. Then $B$ indexes a subset of columns of $\partial$ or $\delta$ forming a basis for the full column space if and only if

- its intersection with each balanced connected component of $G$ forms a spanning tree, and
- its intersection with each unbalanced connected component of $G$ is a collection of unicyclic connected components, (that is, each connected component contains a unique cycle) and the unique cycle of each component is unbalanced.

**Proof.** Straightforward, or see Zaslavsky [20 Thm. 5.1(g)].

With this in hand, we will be able to describe the cardinality of the critical group $K(G_\pm)$ as a sum over such bases $B \subset E$. Given such a base $B$ as described in Proposition 9.8, define the quantity $d(B)$ to be a product over the connected components $B_1, \ldots, B_t$ induced by the edges of $B$, where

- a component $B_i$ forming a spanning tree for a balanced component of $G$ contributes a factor of 1,
- a component $B_i$ which is unicyclic will either contribute a factor of 2 (resp. 4) if its unique cycle is a singleton negative half-loop (resp. is an unbalanced cycle that contains no negative half-loop).

**Proposition 9.9.** A signed graph $G_\pm$ having $c$ unbalanced connected components will have

$$|K(G_\pm)| = 2^{-c} \sum_B d(B),$$

where $B$ runs over the bases in Proposition 9.8. In particular, $K(G)$ is finite.

**Proof.** Note that by the definition of $d(B)$, the quantity $F(G_\pm)$ on the right side of the proposition which we wish to show equals $|K(G_\pm)|$ has the multiplicative property that $F(G_\pm) = \prod_{i=1}^t F(G_{\pm}^i)$ if the underlying multigraph $G_\pm = (V, E)$ has connected components $G_{\pm}^1, \ldots, G_{\pm}^t$. The compatible direct sum decompositions $Z^V = \bigoplus_{i=1}^t Z_{\pm}^{V_i}$ and $K(G_\pm) \cong \bigoplus_{i=1}^t K(G_{\pm}^i)$ imply that $|K(G_\pm)|$ has this same multiplicative property, so it suffices to prove the proposition when $G_\pm$ is connected.

When $G_\pm$ is connected and balanced, one can assume after applying a signed graph isomorphism, that $G_\pm$ is an unsigned graph. Then $K(G_\pm)$ is the usual critical group, which is finite, and has cardinality equal to the number of spanning trees, which agrees with $F(G_\pm)$.

When $G_\pm$ is connected and unbalanced, we calculate $\det L(G_\pm)$ explicitly and show that it is positive: this will in particular show that $K(G_\pm) := \text{im} \partial / \text{im} L(G_\pm)$ is finite, since it implies $\text{im} L(G_\pm)$ has full rank inside $Z^V$ and hence also inside $\text{im} \partial$. One starts by using the Binet-Cauchy Theorem to express $\det L(G_\pm)$

---

6Our answer differs somewhat from Zaslavsky’s because he dealt with a Laplacian matrix of the form $\partial \delta^t$ and computed the cardinality of $Z^V / \text{im} \partial \delta^t$, whereas we deal with our Laplacian $L(G) = \delta \partial^t$ and compute the cardinality of $K(G) = \text{im} \partial / \text{im} \partial^t$. 
as a sum over bases $B$:
\[
\det L(G\pm) = \det \partial = \sum_B \det \partial|_{\text{cols } B} \cdot \det \delta|_{\text{rows } B} = \sum_B \det \partial|_{\text{cols } B} \cdot \det \delta|_{\text{cols } B}
\]
\[
= \sum_B \prod_{\text{connected components } B_i \text{ of } B} \det \partial|_{\text{cols } B_i} \cdot \det \delta|_{\text{cols } B_i} = \sum_B d(B)
\]

The last equality used the following calculation, which one can reduce to the case where $B_i$ is an unbalanced cycle, via an induction that plucks off leaf vertices (vertices with only one incident edge):

\[
\det \partial|_{\text{cols } B_i} = \pm 2,
\]

so that one has

\[
\det \delta|_{\text{cols } B_i} = \begin{cases} 
\pm 1 & \text{if its unbalanced cycle is a negative half-loop,} \\
\pm 2 & \text{otherwise.}
\end{cases}
\]

A crucial point to be emphasized here is that signs on the $+2$ and $+4$ are always positive in (9.5) because the plus/minus signs on the $\pm 1, \pm 2$ always agree in (9.3) and (9.4): these signs will be determined by the choices of orientations of the edges in $B_i$. This shows

\[
|Z^V/\text{im}L(G\pm)| = \det L(G\pm) = \sum_B d(B) > 0
\]

for connected unbalanced signed graphs $G\pm$. But $F(G\pm) = \frac{1}{2} \sum_B d(B)$ in this case, in agreement with

\[
|K(G)| = |\text{im}\partial/\text{im}L(G\pm)| = \frac{1}{2} |Z^V/\text{im}L(G\pm)|
\]

where the $\frac{1}{2}$ arises here since $\text{im}\partial$ is the index two sublattice $Z^V_{\pm 0 \mod 2}$ inside $Z^V$. \hfill \Box

10. Doubly covering a signed graph

Our goal in this section is to define a notion of a signed graph double coverings, leading to a more flexible generalization of Theorem 1.2.

10.1. The double cover construction for signed graphs.

**Definition 10.1.** Given two signed graphs $G^{(i)}_\pm$ for $i = 1, 2$ with same underlying multigraph $G = (V, E)$, and edge orientation on $E$ chosen arbitrarily, define a signed graph

\[\tilde{G}_\pm := \text{Double}(G^{(1)}_\pm, G^{(2)}_\pm),\]

which we will think of as a double cover of the base $G^{(1)}_\pm$ parametrized by the voltage-assignment signed graph $G^{(2)}_\pm$. It has vertex set $\tilde{V} := \{v_+, v_-\}_{v \in V}$ and edge set $\tilde{E}$ defined and oriented as follows. For each edge $e = (u, v)$ in $G$ which is not a half-loop, (so possibly $u = v$ if $e$ is a full loop), create two full (directed) edges $e_+, e_-$ of $\tilde{G}_\pm$ having the same sign as $e$ in $G^{(1)}_\pm$, with these endpoints:

\[
e_+ = (u_+, u_-), e_- = (u_-, u_-) \quad \text{if } G^{(1)}_\pm, G^{(2)}_\pm \text{ agree on the sign of } e,
\]
\[
e_+ = (u_+, v_-), e_- = (u_-, v_-) \quad \text{if } G^{(1)}_\pm, G^{(2)}_\pm \text{ disagree on the sign of } e.
\]

For each half-loop edge $e$ at vertex $v$ in $G$ create either one or two edges of $\tilde{G}_\pm$ having the same sign as $e$ in $G^{(1)}_\pm$, with these endpoints:

\[
\left\{\begin{array}{ll}
\text{half-loops } e_+ = (v_+, v_-), e_- = (v_-, v_-) & \text{if } G^{(1)}_\pm, G^{(2)}_\pm \text{ agree on the sign of } e, \\
\text{edge } \tilde{e} = (v_+, v_-) & \text{if } G^{(1)}_\pm, G^{(2)}_\pm \text{ disagree on the sign of } e.
\end{array}\right.
\]

Figure 10.1 depicts an example, first showing $\tilde{G} = \text{Double}(G^{(1)}_\pm, G^{(2)}_\pm) \rightarrow G^{(1)}_\pm$, and then below it $G^{(2)}_\pm$. 
Figure 1. An example of a signed graph double covering

\[ \tilde{G}_\pm := \text{Double}(G^{(1)}_\pm, G^{(2)}_\pm) \xrightarrow{\pi^{(1)}} G^{(1)} \]

and the signed graph \( G^{(2)} \), in which all of the loop edges shown are intended to be half-loops. Note that the half-loop edges \( e \) in the underlying graph \( G \) where \( G^{(1)}, G^{(2)} \) disagree on their \( +/− \) voltage assignment are “doubly covered” under the projection by the edges of \( \tilde{G} \) that point in the third coordinate direction.
10.2. Properties of signed graph double coverings. As anticipated in the phrasing of Definition 10.1, we will speak of a double-covering map

\[ \tilde{G}_\pm \xrightarrow{\pi(1)} G^{(1)}_\pm \]

\[ v_+, v_- \mapsto v \]

\[ e_+, e_- \mapsto e \]

as also “doubly covering” each half-loop \( e = (v, v) \) with opposite signs in \( G^{(1)}_\pm, G^{(2)}_\pm \) via \( \tilde{e} = (v_+, v_-) \mapsto e \).

We first note that it generalizes the unsigned graph double coverings defined earlier. The proof of the following proposition is a straightforward exercise in the definitions.

**Proposition 10.2.** Let \( \tilde{G}_\pm = \text{Double}(G^{(1)}_\pm, G^{(2)}_\pm) \), be a signed graph double covering, in which \( G^{(1)}_\pm = G \) is an unsigned multigraph with no half-loops, meaning that \( G^{(1)} \) has \( \beta(e) = + \) and has no half-loops.

Then \( \tilde{G}_\pm = \tilde{G} \) is also an unsigned multigraph with no half-loops, and \( \tilde{G}_\pm \xrightarrow{\pi(1)} G^{(1)}_\pm \) is the same as the (regular) double covering \( \tilde{G} \xrightarrow{\pi} G \) corresponding to the voltage graph \( G^{(2)}_\pm \).

The asymmetry of the roles of \( G^{(1)}_\pm \) and \( G^{(2)}_\pm \) in constructing \( \text{Double}(G^{(1)}_\pm, G^{(2)}_\pm) \) turns out to be illusory.

**Proposition 10.3.** The two signed graphs \( \text{Double}(G^{(2)}_\pm, G^{(1)}_\pm) \) and \( \text{Double}(G^{(1)}_\pm, G^{(2)}_\pm) \) are actually signed graph isomorphic: one is obtained from the other by switching at every vertex in the subset \( \{v_\pm \}_{v \in V} \) of their common vertex set \( \tilde{V} = \{v_+, v_- \}_{v \in V} \).

**Proof.** Let \( \tilde{G}_\pm := \text{Double}(G^{(1)}_\pm, G^{(2)}_\pm) \), and let \( \tilde{G}_\pm' \) be the result of performing the signed isomorphisms described in the proposition. Then the edges whose voltage +/− signs will have changed from \( \tilde{G}_\pm \) to \( \tilde{G}_\pm' \) are the edges that cross the vertex cut from \( \{v_\pm \}_{v \in V} \) to \( \{v_+ \}_{v \in V} \). These are exactly the edges of \( G \) whose voltage signs in \( G^{(1)}_\pm, G^{(2)}_\pm \) disagreed, so that in \( \tilde{G}_\pm' \) they carry voltages that agree with \( G^{(2)}_\pm \). The remaining edges in \( \tilde{G}_\pm \) already agreed in voltage with \( G^{(2)}_\pm \), so all edges of \( \tilde{G}_\pm' \) agree with \( G^{(1)}_\pm \). In addition, those edges of \( \tilde{G}_\pm' \) which cross the vertex cut will still be the ones where the voltages on \( G^{(1)}_\pm, G^{(2)}_\pm \) disagree. Thus \( \tilde{G}_\pm' \) matches the description of \( \text{Double}(G^{(2)}_\pm, G^{(1)}_\pm) \).

This hidden symmetry between the “base graph” \( G^{(1)}_\pm \) and “voltage assignment” \( G^{(2)}_\pm \) becomes apparent only after generalizing graph double covers to signed graphs, and was one motivation for introducing such covers.

11. The short complex for a double covering of signed graphs

Our goal here is a second generalization of Theorem 1.2 that applies to signed graph double covers \( \text{Double}(G^{(1)}_\pm, G^{(2)}_\pm) \). When working with these signed graph critical groups \( K(G_\pm) \), we could in principle use the edge-presentation (1.2) as \( K(G_\pm) = \mathbb{Z}E/(\text{id}^\delta + \text{ker } \partial) \). However, we have found it more convenient in the proofs of this section to work with the vertex-presentation (9.1):

\[ K(G_\pm) = \text{id}/\text{id}^\delta = \text{id}/L(G_\pm). \]

Thus we define various maps on the level of the vertex groups \( \mathbb{Z}V, \mathbb{Z}\tilde{V} \), inducing morphisms of critical groups.

**Definition 11.1.** Given \( \tilde{G}_\pm = \text{Double}(G^{(1)}_\pm, G^{(2)}_\pm) \), as before, consider free \( \mathbb{Z} \)-modules \( \mathbb{Z}V, \mathbb{Z}\tilde{V} \) and \( \mathbb{Z}E, \mathbb{Z}\tilde{E} \), having \( \mathbb{Z} \)-basis elements indexed by vertices or edges in sets \( V, \tilde{V} \) and \( E, \tilde{E} \).

On the level of vertices, define \( \mathbb{Z} \)-linear maps

\[
\begin{align*}
\mathbb{Z}\tilde{V} & \xrightarrow{\pi(1)} \mathbb{Z}V \\
v_+ & \mapsto +v \\
v_- & \mapsto +v
\end{align*}
\]

\[
\begin{align*}
\mathbb{Z}\tilde{V} & \xrightarrow{\pi(2)} \mathbb{Z}V \\
v_+ & \mapsto +v \\
v_- & \mapsto -v
\end{align*}
\]
Also define an involution
\[
\begin{align*}
\mathbb{Z}^\tilde{V} & \xrightarrow{\tilde{\pi}} \mathbb{Z}^\tilde{V} \\
v_+ & \mapsto v_- \\
v_- & \mapsto v_+
\end{align*}
\]
and these two sublattices of \(\mathbb{Z}^\tilde{V}\)
\[
\begin{align*}
\mathbb{Z}^\tilde{V}_{\text{sym}} & := \{ x \in \mathbb{Z}^\tilde{V} : \iota(x) = x \}, \\
\mathbb{Z}^\tilde{V}_{\text{skew}} & := \{ x \in \mathbb{Z}^\tilde{V} : \iota(x) = -x \}.
\end{align*}
\]
We collect in the next proposition the various necessary technical properties of these maps \(\pi_{(i)}\) and \(\iota\).

**Proposition 11.2.** Given \(\tilde{G}_\pm = \text{Double}(G^{(1)}_\pm, G^{(2)}_\pm)\), one has the following properties of \(\pi_{(i)}\) for \(i = 1, 2\).

(i) \(\pi_{(i)}(\text{im} \partial G_\pm) = \text{im} \partial G^{(i)}_\pm\).

(ii) \(\pi_{(i)}(\text{im} \partial G^{(i)}_\pm) \subset \text{im} \partial G_\pm\).

(iii) \(\pi_{(i)}(\text{im} \lambda(G_\pm)) = \text{im} \lambda(G^{(i)}_\pm)\).

(iv) This diagram commutes
\[
\begin{align*}
\mathbb{Z}^\tilde{V} & \xrightarrow{L(\tilde{G}_\pm)} \mathbb{Z}^\tilde{V} \\
\pi_{(i)} & \uparrow \pi_{(i)} \\
\mathbb{Z}^V & \xrightarrow{L(G^{(i)}_\pm)} \mathbb{Z}^V
\end{align*}
\]

(v) These two sequences are short exact:
\[
\begin{align*}
0 & \longrightarrow \mathbb{Z}^V \xrightarrow{\pi_{(1)}} \mathbb{Z}^\tilde{V} \xrightarrow{\pi_{(2)}} \mathbb{Z}^V \longrightarrow 0 \\
0 & \longrightarrow \mathbb{Z}^V \xrightarrow{\pi_{(2)}} \mathbb{Z}^\tilde{V} \xrightarrow{\pi_{(1)}} \mathbb{Z}^V \longrightarrow 0
\end{align*}
\]
with
\[
\text{im} \pi_{(1)} = \ker \pi_{(2)} = \mathbb{Z}^\tilde{V}_{\text{sym}} \\
\text{im} \pi_{(2)} = \ker \pi_{(1)} = \mathbb{Z}^\tilde{V}_{\text{skew}}
\]

(vi) As operators on \(\mathbb{Z}^\tilde{V}\), the map \(\iota\) commutes with \(L(\tilde{G}_\pm)\).

**Proof.** By Proposition 10.3 it suffices to check the assertions for \(\pi_{(1)}\); the assertions for \(\pi_{(2)}\) will then follow by applying sign switches at all vertices \(\{v_-\}_{v \in V}\) of \(\tilde{G}_\pm\).

In proving assertions (i),(ii),(iii),(iv), it is convenient to introduce two maps \(\mathbb{Z}^\tilde{E} \overset{\pi_{(1)}}{\longrightarrow} \mathbb{Z}^E\) defined as follows:

\[
\begin{align*}
\mathbb{Z}^\tilde{E} & \xrightarrow{\pi_{(1)}} \mathbb{Z}^E \\
\xi_+, \xi_- & \mapsto e \quad \text{if } e \text{ has two preimages } \xi_+, \xi_i \text{ in } \tilde{E} \\
\tilde{e} & \mapsto e \quad \text{if } e \text{ is a half-loop with preimage } \tilde{e}, \text{ and voltages } \beta(e) = -1, +1 \text{ in } G^{(1)}_\pm, G^{(2)}_\pm, \text{ resp.} \\
\breve{e} & \mapsto 0 \quad \text{if } e \text{ is a half-loop with preimage } \breve{e}, \text{ and voltages } \beta(e) = +1, -1 \text{ in } G^{(1)}_\pm, G^{(2)}_\pm, \text{ resp.}
\end{align*}
\]

\[
\begin{align*}
\mathbb{Z}^E & \xrightarrow{\rho_{(1)}} \mathbb{Z}^\tilde{E} \\
e & \mapsto e_+ + e_- \quad \text{if } e \text{ has two preimages } e_+, e_i \text{ in } \tilde{E} \\
e & \mapsto 2\tilde{e} \quad \text{if } e \text{ is a half-loop with preimage } \tilde{e}, \text{ with voltages } \beta(e) = -1, +1 \text{ in } G^{(1)}_\pm, G^{(2)}_\pm, \text{ resp.} \\
e & \mapsto 0 \quad \text{if } e \text{ is a half-loop with preimage } \breve{e}, \text{ with voltages } \beta(e) = +1, -1 \text{ in } G^{(1)}_\pm, G^{(2)}_\pm, \text{ resp.}
\end{align*}
\]

Note that \(\mathbb{Z}^E \overset{\rho_{(1)}}{\longrightarrow} \mathbb{Z}^\tilde{E}\) is close, but not quite equal, to the transpose \(\pi_{(1)}^t\) of the map \(\mathbb{Z}^\tilde{E} \overset{\pi_{(1)}}{\longrightarrow} \mathbb{Z}^E\). These maps \(\pi_{(1)}, \rho_{(1)}\) between edge lattices correspond to the maps \(\pi_{(1)}, \pi_{(1)}^t\) already defined between vertex lattices, in
the sense that one has these easily-checked commutative diagrams:

\[
\begin{array}{ccc}
\mathbb{Z} \overrightarrow{\partial} & \mathbb{Z} \overrightarrow{E} & \mathbb{Z} \overrightarrow{V} \\
\pi_{(1)} & \pi_{(1)} & \pi_{(1)} \\
\mathbb{Z} \overrightarrow{V} & \mathbb{Z} \overrightarrow{E} & \mathbb{Z} \overrightarrow{V}
\end{array}
\]

 Assertion (iv). This follows immediately from the commutative square \((11.1)\) and these elements lie in the kernel of \(Z \overrightarrow{\partial} \) with the properties that \(Z \overrightarrow{E} \) is surjective. Although it is not necessarily surjective, the only basis elements in \(Z \overrightarrow{E} \) not in the image of \(\pi_{(1)} \) correspond to half loops \(e \) which are positive in \(G^{(1)} \) and are covered by a single edge in \(\tilde{G}^{(1)} \), and these elements lie in the kernel of \(\partial_{G^{(1)}} \). Therefore \(\partial_{G^{(1)}} \) is surjective, and the equality follows.

Assertion (vi). This follows because \(\iota \) is a signed graph automorphism of \(\tilde{G}^{(1)} \) that involves no sign switches, only permutations of the coordinates. □

Corollary 11.3. Given \(\tilde{G}^{(1)} = \text{Double}(G^{(1)}, G^{(2)}) \), the maps \(Z \overrightarrow{V} \xrightarrow{\pi_{(i)}} Z \overrightarrow{V} \) for \(i = 1, 2 \) induce morphisms

\[
K(\tilde{G}^{(1)}) \xrightarrow{\pi_{(i)}} K(G^{(i)})
\]

with the properties that

(a) \(\pi_{(i)} \) is surjective, and
Case 1. The signed graph \( \tilde{G}_\pm \) is connected and unbalanced.

Case 2. The signed graph \( \tilde{G}_\pm \) is connected and balanced.

Case 3. The signed graph \( \tilde{G}_\pm \) has two connected components, exchanged by \( \iota \).

In Case 3, we claim one can perform a sequence of switches at various vertices \( v \) of \( G^{(2)}_\pm \), with the effect of exchanging the labels \( v_+ \leftrightarrow v_- \) in \( \tilde{G}_\pm = \text{Double}(G^{(1)}_\pm, G^{(2)}_\pm) \), until the two connected components of \( \tilde{G}_\pm \) have vertex sets \( \{v_+\}_{v \in V} \) and \( \{v_-\}_{v \in V} \). In other words, one can take \( G^{(2)}_\pm = G^{(1)}_\pm \) without loss of generality, so \( \tilde{G}_\pm \) is the disjoint union \( G^{(1)}_\pm \sqcup G^{(1)}_\pm \). We tacitly make this assumption whenever in Case 3.

Although not obvious, we also claim that in Case 2, one or the other of \( G^{(1)}_\pm \) or \( G^{(2)}_\pm \) (but not both) must be balanced, that is, signed isomorphic to an unsigned graph; this is proven in Proposition 11.3. For this we tacitly make the assumption in Case 3.

Proposition 11.4. If \( G \) is connected, the underlying multigraph \( \tilde{G} \) of \( \tilde{G}_\pm \) has at most two connected components, and when there are two components, they are exchanged by the involutive automorphism \( \iota \).

Proof. Fix a base vertex \( v \) of \( G \), with two lifts \( v_+ \) or \( v_- \). Since every other vertex \( u \) of \( G \) has a path to \( v \) in \( \tilde{G} \), every vertex \( u_+, u_- \) in \( \tilde{G} \) either has a lifted path to \( v_+ \) or \( v_- \) or to both, and hence lies in the component of one (or both) of \( v_+, v_- \). Note also that \( \iota \) must send the component of \( v_+ \) to one of \( v_- \).

This leaves three cases for \( \tilde{G}_\pm \) if \( G \) is connected:

Case 1. The signed graph \( \tilde{G}_\pm \) is connected and unbalanced.

Case 2. The signed graph \( \tilde{G}_\pm \) is connected and balanced.

Case 3. The signed graph \( \tilde{G}_\pm \) has two connected components, exchanged by \( \iota \).

Proposition 11.5. Given \( \tilde{G}_\pm = \text{Double}(G^{(1)}_\pm, G^{(2)}_\pm) \), create a third signed graph \( G^{(1,2)}_\pm \) with same underlying graph \( G \) as \( G^{(i)}_\pm \) for \( i = 1, 2 \), having voltage assignment \( \beta_{(1,2)}(e) = \beta_{(1)}(e)\beta_{(2)}(e) \) for each \( e \in E \).

Assuming \( G^{(1)}_\pm, G^{(2)}_\pm \) are both unbalanced, then either

- \( \tilde{G}_\pm \) has two components, if \( G^{(1,2)}_\pm \) is balanced (so we are in a subcase of Case 3), or
- \( \tilde{G}_\pm \) is unbalanced, if \( G^{(1,2)}_\pm \) is unbalanced (so we are in Case 1).

In particular, \( G^{(1)}_\pm, G^{(2)}_\pm \) both being unbalanced excludes being in Case 2.

Proof. Note that the edges \( e \) of \( G \) having \( \beta_{(1,2)}(e) \) negative are exactly the ones whose lifts in \( \tilde{G}_\pm \) go across the vertex cut from \( \{v_+\}_{v \in V} \) to \( \{v_-\}_{v \in V} \). Thus whenever \( G^{(1,2)}_\pm \) is balanced, any vertex \( v \) of \( G \) will have its two preimages \( v_+, v_- \) in \( \pi^{-1}(v) \) lying in different components of \( \tilde{G}_\pm \); the edges in a path from from \( v_+ \) to \( v_- \) in \( \tilde{G}_\pm \) would project to an unbalanced cycle for \( G^{(1,2)}_\pm \). Hence \( \tilde{G}_\pm \) has two components if \( G^{(1,2)}_\pm \) is balanced.

If \( G^{(1)}_\pm, G^{(2)}_\pm, G^{(1,2)}_\pm \) are all unbalanced, then we wish to show that there is a cycle \( C \) which is unbalanced for \( \tilde{G}_\pm \). This is the same as showing \( C \) is unbalanced for both \( G^{(1)}_\pm \) and \( G^{(2)}_\pm \) (and hence balanced for \( G^{(1,2)}_\pm \)). Parity considerations show that these are the only possible patterns of balance for cycles \( C \) in \( G \).
Note that, given two cycles $C_1, C_2$ in $G$, since $G$ is connected, one can create a third cycle $C_3$ going around $C_1$, following a path $P$ to $C_2$, then around $C_2$, and back along the reverse of $P$. This $C_3$ will have balance pattern the “mod 2 sum” of that for $C_1$ and $C_2$, reading \{ “balanced” , “unbalanced” \} as \{0,1\} in $\mathbb{Z}_2$.

Now one can complete the argument that there exists a cycle $C$ in $G$ unbalanced for $\tilde{G}_\pm$, that is, a cycle $C$ matching the fourth row of the table. We know $G^{(1)}_{\pm}$ contains some unbalanced cycle $C_1$ and $G^{(2)}_{\pm}$ contains some unbalanced cycle $C_2$. One must either have that one of the two cycles $C_1, C_2$ matches the fourth row of the table, in which case we are done, or $C_1, C_2$ can be combined to create a $C_3$ matching the fourth row of the table, and again we are done.

We can now prove the last main result, generalizing Theorem 1.2 to signed graph double covers.

**Theorem 11.6.** Given $\tilde{G}_\pm = \text{Double}(G^{(1)}_{\pm}, G^{(2)}_{\pm})$ with underlying multigraph connected, the maps $\pi_{(1)}, \pi_{(2)}$ from Corollary 11.3 fit in a short complex

\[
(11.3) \quad 0 \to K(G^{(1)}_{\pm}) \xrightarrow{\pi_{(1)}} K(\tilde{G}_\pm) \xrightarrow{\pi_{(2)}} K(G^{(2)}_{\pm}) \to 0
\]

which

- in Case 3, is split exact,
- in Case 2, is short exact, and
- in Case 1, is exact at the two ends, but has homology at the middle term equal to $\mathbb{Z}_2$.

In particular, in every case, for all odd primes $p$ one has the splitting

\[
(11.4) \quad \text{Syl}_p K(\tilde{G}_\pm) = \text{Syl}_p K(G^{(1)}_{\pm}) \oplus \text{Syl}_p K(G^{(2)}_{\pm}).
\]

**Proof.** Note that the asserted splitting (11.4) will follow from the splitting in Corollary 11.3 once the assertions about the short complex are verified.

We first deal with the easy Case 3, where our preparatory reductions allow one to assume that $G^{(2)}_{\pm} = G^{(1)}_{\pm}$ and $\tilde{G}_\pm = G^{(1)}_{\pm} \cup G^{(1)}_{\pm}$. Then setting $K := K(G^{(1)}_{\pm})$, the sequence (11.3) becomes

\[
0 \to K \xrightarrow{\pi_{(1)}} K \oplus K \xrightarrow{\pi_{(2)}} K \to 0
\]

\[
\begin{array}{c|c|c|c}
\text{in } & \text{in } & \text{in } & \text{in } \\
G^{(1)}_{\pm} & G^{(2)}_{\pm} & G^{(1,2)}_{\pm} & \tilde{G}_\pm \\
\hline
\text{balanced} & \text{balanced} & \text{balanced} & \text{balanced} \\
\text{balanced} & \text{unbalanced} & \text{unbalanced} & \text{not a cycle} \\
\text{unbalanced} & \text{balanced} & \text{unbalanced} & \text{not a cycle} \\
\text{unbalanced} & \text{unbalanced} & \text{balanced} & \text{unbalanced} \\
\end{array}
\]

which is easily seen to be split exact.

In Cases 1,2, the arguments will resemble each other, and proceed according to the following plan:

**Step 1.** Show $\pi_{(1)}$ maps $K(G^{(1)}_{\pm})$ isomorphically onto $\mathbb{Z}_{\text{sym}}^V/0/L(\tilde{G}_\pm)(\mathbb{Z}_{\text{sym}}^V)$ in Case 2, and isomorphically onto an index 2 subgroup of $\mathbb{Z}_{\text{sym}}^V/L(\tilde{G}_\pm)(\mathbb{Z}_{\text{sym}}^V)$ in Case 1.

**Step 2.** Show that

\[
\ker \left( K(\tilde{G}_\pm) \xrightarrow{\pi_{(2)}} K(G^{(2)}_{\pm}) \right) = \begin{cases} 
\mathbb{Z}_{\text{sym}}^V/0/L(\tilde{G}_\pm)(\mathbb{Z}_{\text{sym}}^V) & \text{in Case 2,} \\
\mathbb{Z}_{\text{sym}}^V/L(\tilde{G}_\pm)(\mathbb{Z}_{\text{sym}}^V) & \text{in Case 1.}
\end{cases}
\]

Note that these would imply the assertions of Case 1 and Case 2 from the theorem.
Step 1. In Case 2, starting with the commuting square of Proposition 11.2(iv),

\[
\begin{array}{c}
\mathbb{Z}^V & \xrightarrow{L(\tilde{G}_\pm)} & \mathbb{Z}^V \\
\pi_{t(1)} & & \pi_{t(1)} \\
\downarrow & & \downarrow \\
\mathbb{Z}^V & \xrightarrow{L(G_{(1)}^{(1)})} & \mathbb{Z}^V
\end{array}
\]

note that its bottom horizontal map restricts to a map \(\mathbb{Z}^V \xrightarrow{L(G_{(1)}^{(1)})} \text{im} \partial_{G_{(1)}^{(1)}}\), since \(\text{im} L(G_{(1)}) \subset \text{im} \partial_{G_{(1)}^{(1)}}\). Note also that its left vertical map restricts to an isomorphism \(\mathbb{Z}^V \xrightarrow{\pi_{t(1)}} \mathbb{Z}_\text{sym}\) according to Proposition 11.2(iv). Since we are in Case 2, Proposition 9.7 implies \(\text{im} \partial_{G_{(1)}^{(1)}} = \mathbb{Z}_\text{sym}^{(0)}\), and one can easily check that this isomorphism \(\mathbb{Z}^V \xrightarrow{\pi_{t(1)}} \mathbb{Z}_\text{sym}\) restricts to an isomorphism sending \(\text{im} \partial_{G_{(1)}^{(1)}} = \mathbb{Z}_\text{sym}^{(0)}\) isomorphically onto \(\mathbb{Z}_\text{sym}^{(0)}\). Thus one deduces that the commuting square of Proposition 11.2(iv) restricts to the following square in which both vertical maps are isomorphisms induced by \(\pi_{t(1)}\):

\[
\begin{array}{c}
\mathbb{Z}_\text{sym}^V & \xrightarrow{L(\tilde{G}_\pm)} & \mathbb{Z}_\text{sym}^V,=0 \\
\pi_{t(1)} & & \pi_{t(1)} \\
\downarrow & & \downarrow \\
\mathbb{Z}^V & \xrightarrow{L(G_{(1)}^{(1)})} & \text{im} \partial_{G_{(1)}^{(1)}}
\end{array}
\]

The five-lemma shows \(\pi_{t(1)}\) induces an isomorphism from the cokernel \(K(G_{(1)}^{(1)})\) of the bottom horizontal row here to the cokernel \(\mathbb{Z}_\text{sym}^V/L(\tilde{G}_\pm)(\mathbb{Z}_\text{sym}^V)\) of the top horizontal row here, as desired in Step 1 for Case 2.

When we are in Case 1, Proposition 9.7 implies \(\text{im} \partial_{G_{(1)}^{(1)}} = \mathbb{Z}_\text{sym}^{(0)} \mod 2\) is an index 2 sublattice of \(\mathbb{Z}^V\), and hence \(K(G_{(1)}^{(1)}) = \text{im} \partial_{G_{(1)}^{(1)}} / \text{im} L(G_{(1)}^{(1)})\) is an index 2 subgroup of \(\mathbb{Z}^V / \text{im} L(G_{(1)}^{(1)})\). Therefore our stated goal for Step 1 in Case 1 would be achieved if one could show that \(\pi_{t(1)}\) maps \(\mathbb{Z}^V / \text{im} L(G_{(1)}^{(1)})\) isomorphically onto \(\mathbb{Z}_\text{sym}^V / L(\tilde{G}_\pm)(\mathbb{Z}_\text{sym}^V)\). This is argued similarly to Case 2: restricting the commuting square of Proposition 11.2(iv) gives this square with vertical isomorphisms

\[
\begin{array}{c}
\mathbb{Z}_\text{sym}^V & \xrightarrow{L(\tilde{G}_\pm)} & \mathbb{Z}_\text{sym}^V \\
\pi_{t(1)} & & \pi_{t(1)} \\
\downarrow & & \downarrow \\
\mathbb{Z}^V & \xrightarrow{L(G_{(1)}^{(1)})} & \mathbb{Z}^V
\end{array}
\]

and then the five-lemma shows \(\pi_{t(1)}\) induces an isomorphism from the cokernel \(\mathbb{Z}^V / \text{im} L(G_{(1)}^{(1)})\) of the bottom horizontal row to the cokernel \(\mathbb{Z}_\text{sym}^V / L(\tilde{G}_\pm)(\mathbb{Z}_\text{sym}^V)\), of the top horizontal row, as desired.

Step 2. In both Cases 1,2, start reformulating \(\ker \left( K(\tilde{G}_\pm) \xrightarrow{\pi_{(2)}} K(G_{(2)}^{(2)}) \right)\) via a diagram of short complexes

\[
\begin{pmatrix}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
\text{im} L(\tilde{G}_\pm) \cap \mathbb{Z}_\text{sym}^V & \rightarrow & \text{im} L(\tilde{G}_\pm) & \xrightarrow{\pi_{(2)}} & \text{im} L(G_{(2)}^{(2)}) & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \rightarrow & \text{im} \partial_{G_{(1)}^{(1)}} \cap \mathbb{Z}_\text{sym}^V & \rightarrow & \text{im} \partial_{G_{(1)}^{(1)}} & \xrightarrow{\pi_{(2)}} & \text{im} \partial_{G_{(2)}^{(2)}} & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \rightarrow & \text{im} \partial_{G_{(1)}^{(1)}} \cap \mathbb{Z}_\text{sym}^V / \text{im} L(\tilde{G}_\pm) \cap \mathbb{Z}_\text{sym}^V & \rightarrow & K(\tilde{G}_\pm) & \xrightarrow{\pi_{(2)}} & K(G_{(2)}^{(2)}) & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
\end{pmatrix}
\]
in which the vertical sequences are all exact by definition. We argue here why its horizontal rows 1, 2, 3 are also exact. The horizontal maps in rows 1 and 2 come from an exact sequence derived from Proposition [11.2 (v)]

\begin{equation}
0 \rightarrow Z_{\text{sym}}^V \rightarrow Z^V \xrightarrow{\pi(2)} Z^V \rightarrow 0
\end{equation}

which one intersects with the first two terms in this tower of inclusions: \( \text{im} L(\tilde{G}_\pm) \subset \text{im} \partial_{G_\pm} \subset Z^V \). Thus rows 1 and 2 are exact at their left and middle positions due to the exactness of [11.6]. They are exact at their right positions due to Proposition [11.2 (i)] and (iii). Hence rows 1 and 2 are exact, and then by the nine-lemma, Row 3 is also exact.

Exactness of Row 3 lets one reformulate

\begin{equation}
\ker \left( K(\tilde{G}_\pm) \xrightarrow{\pi(2)} K(G^{(2)}_\pm) \right) = \text{im} \partial_{G_\pm} \cap Z_{\text{sym}}^V / \text{im} L(\tilde{G}_\pm) \cap Z_{\text{sym}}^V .
\end{equation}

Next we further simplify the numerator and denominator on the right side of (11.7). First we claim that one can reformulate the numerator on the right side of (11.7) as

\[ \text{im} \partial_{G_\pm} \cap Z_{\text{sym}}^V = \begin{cases} Z_{\text{sym}}^V & \text{in Case 1}, \\ Z_{\text{sym},=0} & \text{in Case 2}. \end{cases} \]

due to Proposition [9.3]. In Case 1, so that \( \tilde{G}_\pm \) is unbalanced, this proposition asserts that \( \text{im} \partial_{G_\pm} = Z_{\equiv 0 \mod 2}^V \), which contains \( Z_{\text{sym}}^V \) as a sublattice. In Case 2, so that our preparatory reductions have us assume \( G^{(1)}_\pm \) and \( \tilde{G}_\pm \) are unsigned, this proposition asserts that \( \text{im} \partial_{G_\pm} = Z_{=0}^V \).

We next argue that one can reformulate the denominator on the right side of (11.7) as

\[ \text{im} L(\tilde{G}_\pm) \cap Z_{\text{sym}}^V = L(\tilde{G}_\pm) \left( Z_{\text{sym}}^V \right) . \]

To see this, note that its elements are those of the form \( L(\tilde{G}_\pm)(x) \) lying in \( Z_{\text{sym}}^V \), meaning that

\[ 0 = \iota L(\tilde{G}_\pm)(x) - L(\tilde{G}_\pm)(x) = L(\tilde{G}_\pm)(\iota(x) - x) \]

via Proposition [11.2 (vi)]. In Case 1, so that \( \tilde{G}_\pm \) is unbalanced, equation [9.6] in the proof Proposition [9.9] showed that \( L(\tilde{G}_\pm) \) is invertible, so this is equivalent to \( 0 = \iota(x) - x \), that is, \( x \) lies in \( Z_{\text{sym}}^V \), as claimed. In Case 2, because \( \tilde{G}_\pm \) is unsigned, one has \( \delta_{G_\pm}^t = \delta_{G_\pm}^t \). Thus since \( \iota(x) - x \) lies in the kernel of

\[ L(\tilde{G}_\pm) = \partial_{G_\pm} \delta_{G_\pm} = \partial_{G_\pm} \delta_{G_\pm}^t, \]

it must also lie in the kernel of \( \partial_{G_\pm}^t \). As \( \tilde{G} \) is connected, this means \( \iota(x) - x \) has all its coordinates equal, which then forces \( \iota(x) - x = 0 \), that is, again \( x \) lies in \( Z_{\text{sym}}^V \). Hence in this case one has

\[ \text{im} L(\tilde{G}_\pm) \cap Z_{\text{sym}}^V = L(\tilde{G}_\pm)(Z_{\text{sym}}^V). \]

Thus we have reformulated the kernel (11.7) as

\begin{equation}
\ker \left( K(\tilde{G}_\pm) \xrightarrow{\pi(2)} K(G^{(2)}_\pm) \right) = \begin{cases} Z_{\text{sym}}^V / L(\tilde{G}_\pm) \left( Z_{\text{sym}}^V \right) & \text{in Case 1}, \\ Z_{\text{sym},=0} / L(\tilde{G}_\pm) \left( Z_{\text{sym}}^V \right) & \text{in Case 2} \end{cases}
\end{equation}

which was exactly our goal in Step 2. □

12. TWO APPLICATIONS OF SIGNED GRAPH DOUBLE COVERS

We conclude with two applications of Theorem [11.6]
12.1. **Application: Crowns revisited.** Recall from Example [7.3] that the unsigned multigraph $\text{Crown}_n^{(k)}$ is obtained from a complete bipartite graph $K_{m,n}$ by removing a perfect matching $M$ of edges, and then replacing $M$ with $k$ copies of this same matching, so that $\text{Crown}_n^{(k)}$ is $K_{m,n}$ together with $k-1$ added extra copies of each edge in the matching $M$. Also recall that Corollary [7.6] proved the following formula

$$K(\text{Crown}_n^{(k)}) \cong \mathbb{Z}_m^{-2} \oplus \mathbb{Z}_n^{-2} \oplus \mathbb{Z}_n^{-2} \oplus \mathbb{Z}_{(n-1+k)(n-2+2k)}^-,$$

under assumptions that

- $n$ is odd, and
- $k$ is even.

**Corollary 12.1.** Assuming $n$ is odd, this formula for $K(\text{Crown}_n^{(k)})$ is correct, regardless of the parity of $k$.

**Proof.** Now that we can allow half-loops in our graphs, regardless of the parity of $k$, one can define the unsigned multigraph $\hat{K}_n^{(k)}$ to be obtained from a complete graph $K_n$ by adding $k$ copies of a (positive) half-loop to each vertex $v$. Consider this as a signed graph $G_{\pm}^{(1)} := \hat{K}_n^{(k)}$ and introduce its negative $G_{\pm}^{(2)} := -\hat{K}_n^{(k)}$ as the signed graph obtained from a complete graph having all negative edges by adding $k$ copies of a negative half-loop to each vertex $v$.

One can then check that $\text{Crown}_n^{(k)}$ is exactly the associated signed graph double covering $\text{Double}(G_{\pm}^{(1)}, G_{\pm}^{(2)})$. Thus Case 2 of Theorem [11.6] recovers a short exact sequence generalizing (12.2)

$$0 \to K(\hat{K}_n^{(k)}) \to K(\text{Crown}_n^{(k)}) \to K(-\hat{K}_n^{(k)}) \to 0. $$

(12.1)

The remainder of the proof of Corollary [7.6] showing that

$$K(\hat{K}_n^{(k)}) = \mathbb{Z}_m^{-2}$$

$$K(-\hat{K}_n^{(k)}) = \mathbb{Z}_n^{-2} \oplus \mathbb{Z}_{(n-1+k)(n-2+2k)}^-$$

for $n$ odd, and that the sequence splits for $n$ odd, still applies unchanged.

**Remark 12.2.** When $n$ is even, things are trickier. However, with a bit more work the second author was able to use these methods to derive the following formula for the case when $n$ is even and $\gcd(k-1,n) = 1$:

$$K(\text{Crown}_n^{(k)}) = \mathbb{Z}_m^{-2+2k} \oplus \mathbb{Z}_n^{-2} \oplus \mathbb{Z}_{(n-1+k)(n-2+2k)}^- \oplus \mathbb{Z}_{n(n-2+2k)}^-.$$ 

See Tseng [17] Proposition 8.4]. In particular, when $k = 0$ and $k = 2$, this result applies to all even $n$, and the $k = 0$ case recovers the rest of the answer (7.3) computed by Machacek [14] Theorem 14, that was discussed for $n$ odd already in Example [7.3] above.

12.2. **Application: Reinterpreting Bai’s calculation for the $n$-cube.**

**Definition 12.3.** Let $Q_n$ denote the unsigned graph of the $n$-dimensional cube, that is, its vertices are all binary vectors in $\{0,1\}^n$, and two such vertices lie on an edge if they differ in exactly one coordinate.

H. Bai calculated the structure of the $p$-primary component $\text{Syl}_p Q_n$ for all odd primes $p$, using an induction on $n$, that proceeded via consideration of cokernels for a larger family of matrices. We use Theorem [11.6] to reinterpret his calculation here geometrically, identifying these matrices as Laplacians for a larger family of signed graphs, involved in a family of double covers.

**Definition 12.4.** For nonnegative integers $m,n$, consider the the signed graph $Q_n^{(m)}$ whose underlying unsigned graph is the $n$-cube $Q_n$ with $m$ added half-loops at each vertex, and with voltage assignment $\beta$ in which all (nonloop) cube edges $e$ of $Q_n$ have $\beta(e) = +$, and all the half-loops $e$ have $\beta(e) = -$.

**Proposition 12.5.** For $n \geq 1$, and for each odd prime $p$ one has

$$\text{Syl}_p K(Q_n^{(m)}) \cong \text{Syl}_p K(Q_{n-1}^{(m)}) \oplus \text{Syl}_p K(Q_{n-1}^{(m+1)})$$

**Proof.** Consider the signed graph double covering $\tilde{G}_{\pm} = \text{Double}(G_{\pm}^{(1)}, G_{\pm}^{(2)})$ in which $G_{\pm}^{(1)}$ is obtained from $Q_n^{(m)}$ by adding one positive half-loop at each vertex, and where $G_{\pm}^{(2)} = Q_n^{(m+1)}$. Then $\tilde{G}_{\pm} = Q_n^{(m)}$, since each positive half-loop on the vertices of $G_{\pm}^{(1)}$ will be double covered by a positive edge of $\tilde{G}$...
out” into the $n^{th}$ coordinate direction. The example with $n = 3, m = 2$ is pictured in Figure 10.1 with $G_± = Q_{3}^{(2)} \rightarrow G_±^{(1)}$, and $G_±^{(2)} = Q_{2}^{(3)}$.

As positive half-loops give rise to zero columns of $\partial$ and $\delta$, they have no effect on $K(G_±)$, and hence $K(G_±^{(1)}) = K(Q_{n-1}^{(m)})$. Therefore the splitting of (11.4) implies the assertion of the proposition. \hfill $\square$

Corollary 12.6. (cf. Bai [3] §2) For $n \geq 0$, and for each odd prime $p$ one has

$$Syl_p K(Q_{n}^{(m)}) = Syl_p \left( \bigoplus_{k=0}^{n} \mathbb{Z}_{k+m}^{(n)} \right),$$

and in particular, when $m = 0$, the $n$-cube $Q_n$ has

$$Syl_p K(Q_{n}) = Syl_p \left( \bigoplus_{k=0}^{n} \mathbb{Z}_{k}^{(n)} \right).$$

Proof. Induct on $n$. If $n = 0$, the signed graph $Q_0^{(m)}$ has one vertex with $m$ negative half-loops, so that

$$\partial = [2 \hspace{1em} 2 \hspace{1em} \cdots \hspace{1em} 2]$$

$$\delta = [1 \hspace{1em} 1 \hspace{1em} \cdots \hspace{1em} 1]$$

$$K(Q_0^{(m)}) = \text{im}\partial/\text{im}\partial\delta = 2\mathbb{Z}/2m\mathbb{Z} \cong \mathbb{Z}_m^{(n)} = \bigoplus_{k=0}^{n} \mathbb{Z}_{k+m}^{(n)} \text{ for } n = 0.$$  

In the inductive step, where $n > 1$, applying Proposition 12.4 gives

$$Syl_p K(Q_{n}^{(m)}) = Syl_p K(Q_{n-1}^{(m)}) \oplus Syl_p K(Q_{n-1}^{(m+1)})$$

$$= Syl_p \left( \bigoplus_{k=0}^{n-1} \mathbb{Z}_{k+m}^{(n-1)} \right) \oplus Syl_p \left( \bigoplus_{k=0}^{n-1} \mathbb{Z}_{k+m+1}^{(n-1)} \right)$$

$$= Syl_p \left( \mathbb{Z}_{m}^{(n)} \oplus \bigoplus_{k=1}^{n-1} \mathbb{Z}_{k+m}^{(n-1)} \right)$$

$$= Syl_p \left( \bigoplus_{k=0}^{n} \mathbb{Z}_{k+m}^{(n)} \right)$$

where the second equality used the inductive hypothesis. \hfill $\square$

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School of Mathematics, Univ. of Minnesota, Minneapolis, MN 55455

Dept. of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139

E-mail address: reiner@math.umn.edu
E-mail address: dennisctseng@gmail.com