Production of a pion in association with a high-\(Q^2\) dilepton pair in \(\bar{p}p\) annihilation at GSI-FAIR

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We evaluate the cross section for \(\bar{p}p \rightarrow \ell^+\ell^-\pi^0\) in the forward direction and for large lepton pair invariant mass. In this kinematical region, the leading-twist amplitude factorises into a short-distance matrix element, long-distance dominated antiproton Distribution Amplitudes and proton to pion Transition Distribution Amplitudes (TDA). Using a modelling inspired from the chiral limit for these TDAs, we obtain a first estimate of this cross section, thus demonstrating that this process can be measured at GSI-FAIR.

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Transition Distribution Amplitudes (TDAs) \(^1\) are universal non-perturbative objects describing the transitions between two different particles (e.g. \(p \rightarrow \pi, \pi \rightarrow \gamma, \pi \rightarrow \rho\)). They appear in the study of backward electro-production of a pion \(^2\), of \(\gamma^*\gamma \rightarrow p\pi\) and \(\gamma^*\gamma \rightarrow \pi\pi\) reactions \(^3\) as well as in hard exclusive production of a \(\gamma^*\pi\) pair in \(\bar{p}p\) annihilation:

\[
\bar{p}(p_\bar{p})p(p_\pi) \rightarrow \gamma^*(q)\pi(p_\pi) \rightarrow \ell^+(p_{\ell^+})\ell^-(p_{\ell^-})\pi(p_\pi) \quad (1)
\]

at small \(t = (p_{\pi} - p_p)^2\) (or at small \(u = (p_{\pi} - p_\bar{p})^2\)), which is the purpose of the present work. The TDAs are an extension of the concept of Generalised Parton Distributions (GPDs), as already advocated in \(^4\). The proton to meson TDAs are defined from the Fourier transform of a matrix element of a three-quark-light-cone operator between a proton and a meson state. They obey QCD evolution equations which follow from the renormalisation-group equation of the three-quark operator. Their \(Q^2\) dependence is thus completely under control.

Whereas in the pion to photon case, models used for GPDs \(^5\) \(^6\) \(^7\) \(^8\) could be applied to TDAs since they are defined from matrix elements of the same quark-antiquark operators, the situation is clearly different for the nucleon to meson TDAs. Before estimates based on models such as the meson-cloud model \(^9\) become available, it is important to use as much model-independent information as possible. In \(^2\), we derived constraints from the chiral limit on the TDAs \(p \rightarrow \pi\) and made a first evaluation of the cross section for the backward electro-production of a pion in the large-\(\xi\) (or small \(E_\pi\)) region. Related processes were also recently studied in \(^10\) similarly to what was proposed in \(^11\). In this work, we apply the same setting to evaluate the cross sections for \(\bar{p}p \rightarrow \ell^+\ell^-\pi^0\) in the kinematical region accessible by GSI-FAIR \(^12\) in the forward limit and at moderate energy of the meson.

In the scaling regime where \(Q^2 = q^2\) is of the order of \(W^2 = (p_\bar{p} + p_p)^2\), the amplitude for the process \((1)\) at small \(t\) or CM angle of the pion \(\theta_\pi\) close to \(0\) – involves the \(p \rightarrow \pi\) TDAs \(A^{p\pi}(x_1, \xi, \Delta^2)\), \(T^{p\pi}(x_1, \xi, \Delta^2)\), where \(x_i (i = 1, 2, 3)\) denote the light-cone-momentum fractions carried by participant quarks and \(\xi\) is the skewedness parameter such that \(\Delta^2 = x_1 + x_2 + x_3\). The amplitude is a convolution of the antiproton DAs, a perturbatively-calculable-hard-scattering amplitude and the \(p \rightarrow \pi\) TDAs.

We express the particle momenta through a Sudakov decomposition:

\[
\alpha = \frac{2}{1 + \xi},  \quad p = \frac{(1 + \xi) + \Delta_T}{\Delta_T},  \quad p_p = (1 + \xi) + \frac{M^2}{1 + \xi} n,  \quad p_\pi = (1 - \xi) + \frac{n_\pi^2 - \Delta_T^2}{1 - \xi} n + \Delta_T
\]

FIG. 1: The factorisation of the annihilation process \(\bar{p}p \rightarrow \gamma^*\pi\) into antiproton-distribution amplitudes (DA), the hard-subprocess amplitude \((M_h)\) and proton \(\rightarrow\) pion transition distribution amplitudes (TDA).

The momenta of the subprocess \(\bar{p}p \rightarrow \gamma^*\pi\) are defined as shown in Fig. 1. The \(z\)-axis is chosen along the colliding proton and antiproton and the \(x\)-\(z\) plane is identified with the collision or hadronic plane. We define the light-cone vectors \(p\) and \(\bar{p}\) such that \(2\xi = x_1 + x_2 + x_3\). The amplitude is a convolution of the antiproton DAs, a perturbatively-calculable-hard-scattering amplitude and the \(p \rightarrow \pi\) TDAs.
\[ \Delta = -2\xi p + \left[ \frac{m^2 - \Delta_T^2}{1 - \xi} - \frac{M^2}{1 - \xi} \right] n + \Delta_T \]
\[ q \simeq 2\xi p + \frac{M^2}{W^2}(1 + \xi) + \left[ \frac{W^2}{1 + \xi} - \frac{m^2 - \Delta_T^2}{1 - \xi} \right] n - \Delta_T \]
\[ \Delta_T^2 = 1 - \xi \left( 2 - 2\xi \frac{M^2}{1 + \xi} - \frac{m^2}{1 - \xi} \right), \tag{3} \]

where \( \alpha = W^2 - 2M^2 + W\sqrt{W^2 - 4M^2} \simeq 2W^2 \); the approximate expression for \( q \) is obtained with \( M \ll W \).

For \( \Delta_T = 0 \), \( M \ll W \) and \( m_\pi = 0 \), one gets
\[ p_\pi = (1 + \xi)p, \quad p_\pi = \frac{W^2}{1 + \xi} n, \quad p_\pi = (1 - \xi)p, \]
\[ t = \frac{2\xi M^2}{1 + \xi}, \quad \xi = \frac{Q^2}{2W^2 - Q^2}. \tag{4} \]

In the fixed-target mode, the maximal reachable value for \( W^2 = 2M^2 + 2ME_p \) at GSI will be \( \simeq 30 \text{ GeV}^2 \) (for \( E_p = 15 \text{ GeV} \)). The highest invariant mass of the photon could be \( Q^2_{\text{max}} \simeq 30 \text{ GeV}^2 \). We refer to Ref. 13 for a complete discussion of the kinematically allowed domain.

Let us now turn to the kinematics of \( \bar{p} p \to \ell^+ \ell^- \pi^0 \). In general, we have for the unpolarised differential cross section:
\[ \frac{d\sigma}{dt dQ^2 d\cos \theta_\ell} = \frac{4}{3} \pi^2 \pi^2 \frac{\bar{M}^2}{16\pi W^2} \left( \frac{W^2 - 4M^2}{2\pi} \right)^2 \tag{10} \]

Alternatively, by integrating on \( \varphi_\pi^* \) and on \( \varphi_\ell \),
\[ \frac{d\sigma}{dt dQ^2 d\cos \theta_\ell} = \int d\varphi_\ell \bar{M}_\ell \bar{M}_p \left( \frac{W^2 - 4M^2}{2\pi} \right)^2 \tag{11} \]

to be compared with the cross section for \( \bar{p} p \to \gamma^* \pi^0 \).

At \( \Delta_T = 0 \), the leading-twist TDAs for the \( p \to \pi^0 \) transition, \( V_1^P(x_1, \xi, \Delta^2) \), \( A_1^P(x_1, \xi, \Delta^2) \) and \( T_1^P(x_1, \xi, \Delta^2) \) are defined as (see Appendix for details):
\[ \mathcal{F} \left( \left\{ \pi^0(p_\pi) e^{ijk} a_i(z_1 n) a_j(z_2 n) a_k(z_3 n) | P(p_\pi, s_\pi) \right\} \right) = \]
\[ i \frac{\kappa_n}{4} \left[ V_1^{\pi^0}(p \gamma C)_{\alpha\beta}(u^+(p_\pi, s_\pi)) \gamma \right. \]
\[ + A_1^{\pi^0}(p \gamma C)_{\alpha\beta}(u^+(p_\pi, s_\pi)) \gamma \]
\[ \gamma \left. + T_1^{\pi^0}(p \gamma C)_{\alpha\beta}(u^+(p_\pi, s_\pi)) \right], \tag{12} \]

where \( \sigma_{\mu\nu} = \frac{1}{2} [\gamma^{\mu}, \gamma^{\nu}] \), \( C \) is the charge conjugation matrix, \( f_\pi = 131 \text{ MeV} \) is the pion decay constant and \( f_N \sim 5.2 \times 10^{-3} \text{ GeV} \), \( u^+ \) is the upper component of the nucleon spinor: \( u(p_\pi, s_\pi) = (\gamma^\mu + i\gamma^5 \gamma^\mu)u(p_\pi, s_\pi) = u^-(p_\pi, s_\pi) + u^+(p_\pi, s_\pi) \) with \( u^+(p_\pi, s_\pi) \sim \sqrt{p^+} \) and \( u^-(p_\pi, s_\pi) \sim \sqrt{1/p^+} \).

For the three TDAs \( V_1^{\pi^0} \), \( A_1^{\pi^0} \) and \( T_1^{\pi^0} \), contributing in the limit \( \Delta_T \to 0 \), we use the following expressions for \( \Delta_T = 0 \) and large \( \xi \) (see Appendix):
\[ \left\{ V_1^{\pi^0}, A_1^{\pi^0}, T_1^{\pi^0} \right\}(x_1, x_2, x_3, \xi, \Delta^2) = \]
\[ \frac{1}{4\xi} \left\{ V_1^P, A_1^P, T_1^P \right\}(x_1, 2\xi \cdot x_2, 2\xi \cdot x_3), \tag{13} \]

where \( V_1^P \), \( A_1^P \) and \( T_1^P \) are the proton DAIs \[ 14 \].

At the leading order in \( \alpha_s \) and at \( \Delta_T = 0 \), the amplitude \( M_\lambda^{\pi^0 \pi^0} \) for \( \bar{p} (p_\pi, s_\pi) p(p_\pi, s_\pi) \rightarrow \gamma^* (\gamma, \lambda) \pi^0 (p_\pi) \) reads
\[ M_\lambda^{\pi^0 \pi^0} = -\frac{i}{(4\pi\alpha_s)^2} \frac{\sqrt{4\pi\alpha_s} m^3}{54f_\pi Q^4} S_\lambda^{\pi^0 \pi^0} \mathcal{I} \tag{14} \]

with \( S_\lambda^{\pi^0 \pi^0} = \bar{v}^+(p_\pi, s_\pi) v^+(\lambda) \gamma^5 u^+(p_\pi, s_\pi) \), and
\[ \mathcal{I} = \int \left[ dx \right] \int \left[ dy \right] \left( \frac{1}{x_a} \sum_{\alpha=1}^{14} R_\alpha + \frac{2}{x_a} \sum_{\alpha=8}^{14} R_\alpha \right), \tag{15} \]

where \( [dx] = dx_1 dx_2 dx_3 \delta(x_1 - x_2 - x_3) \) and \( [dy] = dx_2 dx_3 dy_1 \delta(1 - x_1 - y_1) \); the coefficients \( R_\alpha \) (\( \alpha = 1, ..., 14 \)) exactly correspond to \( T_{\alpha} \) in \[ 14 \] after the replacement \( -ic \rightarrow ic \) due to the presence of the \( \gamma^* \) in the final instead of initial state. Even though the TDA formalism can be applied at any value of \( \xi \) (or \( E_z \)), we have for now at our disposal estimates for the \( p \rightarrow \pi \) TDAs only.
at large $\xi$. In the following, we shall therefore limit ourselves to the computation of the cross section for reaction (1) in this region. At large $\xi$, the ERBL regime ($x_t > 0$) covers most of the integration domain. Therefore it is legitimate to approximate the cross section only from the ERBL contribution, i.e. when the integration range of the momentum fractions is restricted to $[0, 2\xi]$.

The differential cross section for unpolarised protons and antiprotons is calculated as usual using Eq. (11). From the averaged-squared amplitudes,

$$|\mathcal{M}_{\Delta T}|^2 = \frac{1}{4} \sum_{s_p,s_p} \mathcal{M}_{s_p,s_p}^\lambda (\mathcal{M}_{s_p,s_p}^\lambda)^*.$$  \hspace{0.5cm} (16)

$|\mathcal{M}_{00}|^2$ vanishes at the leading-twist accuracy, as in the nucleon-form-factor case. The same is true for $|\mathcal{M}_{+1}|^2$ and $|\mathcal{M}_{-1}|^2$, etc., since the $x$ and $y$ directions are not distinguishable when $\Delta T$ is vanishing. We then define

$$|\mathcal{T}|^2 = |\mathcal{M}_{+1}|^2 + |\mathcal{M}_{-1}|^2.$$

To compute $\mathcal{I}$, we need to choose models for the DAs and the deduced TDAs. For the sake of coherence with experimental data, we shall use reasonable parametrisations of CZ [15] and KS [16], which are both based on an analysis of QCD sum rules. For CZ, they are

$$V^p(x_i) = \varphi_\alpha [11.35(x_i^2 + x_i^2) + 8.82x_i^3 - 1.68x_i - 2.94],$$

$$A^p(x_i) = \varphi_\alpha [6.72(x_i^2 - x_i^2)],$$

$$T^p(x_i) = \varphi_\alpha [13.44(x_i^2 + x_i^2) + 4.62x_i^3 + 0.84x_i - 3.78],$$

and for KS (which we used in Fig. 2),

$$V^p(x_i) = \varphi_\alpha [17.64(x_i^2 + x_i^2) + 22.68x_i^3 - 6.72x_i - 5.04],$$

$$A^p(x_i) = \varphi_\alpha [2.52(x_i^2 - x_i^2) + 1.68(x_i - x_i)],$$

$$T^p(x_i) = \varphi_\alpha [21.42(x_i^2 + x_i^2) + 15.12x_i^3 + 0.84x_i - 7.56],$$

and we evaluate our model TDAs from Eq. (13). This gives $\mathcal{I} \simeq 1.28 \cdot 10^5$ for CZ and $\mathcal{I} \simeq 2.15 \cdot 10^5$ for KS; this yields an induced uncertainty of order 3 for our estimates of the cross section. In the following, we use $\alpha_\alpha = 0.3$ as suggested in [13].

Altogether, we have the following analytic results for the dominant ERBL contribution:

$$|\mathcal{M}_T|^2 = \frac{(4\pi\alpha_\alpha)^4(4\pi\alpha_{em})f_\pi^2}{54^2f_\pi^2} \frac{2(1 + \xi)|\mathcal{I}|^2}{\xi Q^6}.$$  \hspace{0.5cm} (19)

From this, we straightforwardly obtain $d\sigma/dt$, whose $W^2$ evolution is displayed on Fig. 2(a) at $\Delta T = 0$ for the two extreme values of meson longitudinal momentum where one may trust the soft pion limit, corresponding to $p_{\pi}^2 = 0$ or $|p_{\pi}^2| = M/3$ in the laboratory frame ($\xi = 1$ or $1/2$).

For the process (1), the averaged-squared amplitude is:

$$|\mathcal{M}_{pp-\ell+\ell-\pi^0}|^2 = \frac{1}{4} \sum_{s_p,s_p,\lambda,\lambda'} \mathcal{M}_{s_p,s_p}^\lambda \mathcal{M}_{s_p,s_p}^{\lambda'} \frac{L^{\lambda\lambda'} 1}{Q^2} (\mathcal{M}_{s_p,s_p}^{\lambda'})^*,$$

with $L^{\lambda\lambda'} = e^2 \text{Tr}(\gamma_\mu \gamma_\ell(p_{\pi} - \ell)(\gamma_\mu \gamma_\ell)^*)(\lambda,\lambda')$. Integrating on the lepton azimuthal angle $\varphi_\ell$, we have

$$\int d\varphi_\ell |\mathcal{M}_{pp-\ell+\ell-\pi^0}|^2 = |\mathcal{M}_T|^2 \frac{2\pi e^2(1 + \cos^2 \theta_\ell)}{Q^2}.$$  \hspace{0.5cm} (20)

from which we get, via Eq. (10) and integrating over $\theta_\ell$ the differential cross section displayed in Fig. 2(b).

Although the cross sections are evaluated at $\Delta T = 0$, we do not anticipate any dramatic $\Delta T$-dependence of the TDAs below a few hundred MeV, so that our estimates are likely to be valid in a not-too-narrow $\Delta T$ region. To evaluate a magnitude of the integrated cross section we take as an example the kinematical region with $W^2 = 10$ GeV$^2$ in Fig. 2(b) accompanied by the $Q^2$ window 7 GeV$^2 < Q^2 < 8$ GeV$^2$, which corresponds to $\xi \approx 1/2$ or to the pion momentum of the order 310 MeV. Integrating over this $Q^2$-bin and in a $t$-bin corresponding to $\Delta T < 500$ MeV leads then to a cross section around $100$ femtobarns. Such a cross section is sizable and seems to be accessible to experimental setups such as PANDA with the designed value of the FAIR luminosity.

The calculations done till now and which involve the proton $\to \pi^0$ TDA are valid for the small $t$ region. Let us however stress that - due to the charge symmetry - an identical result will be obtained in the small $u$ region but with the $\tilde{p} \to \pi^0$ TDA. In the laboratory frame at GSI-FAIR, this second region is quite different from the previous one since the $\pi^0$ meson is boosted in the forward direction. A precise detection of the particles...
of the final state, either in the proton or the antiproton “fragmentation” kinematics, will depend on the detector performances in the respective regions.

In conclusion, we have demonstrated that the study of proton-antiproton exclusive annihilation into a lepton pair and a pion is feasible at large values of the lepton pair invariant mass in the forthcoming PANDA or PAX experiments at GSI-FAIR. We believe that such a study will bring unique information about the inner structure of the proton, and particularly about the pion content of a proton, provided that the predictions (scaling behaviour, angular dependence of the lepton pair) of the factorised framework used here are shown to be valid. Expressing the cross section of this process as the convolution of a hard-scattering amplitude for quarks with a photon with DAs and proton to pion TDAs will allow to evaluate these new hadronic matrix elements which contain much information about the confinement dynamics. It has to be emphasised that the same hadronic matrix elements appear also in a similar description of backward electroproduction of a pion [2].

Note that other channels are also of much interest, such as \( p \bar{p} \to l^+l^- \eta \) or \( p \bar{p} \to l^+l^- \rho^0 \). The theoretical treatment of the \( \eta \) case is identical to the one for the \( \pi^0 \) case, but for the isosinglet nature of \( \eta \). In the \( \rho^0 \) case, one should distinguish between the longitudinally polarised meson where the TDA has the same structure as for the \( \pi^0 \) and the transversally polarised case which leads to more TDAs. Needless to say, we are strongly lacking of model estimates for these \( p \to \eta \) and \( p \to \rho \) TDAs but their experimental determination (or at least the measurement of their ratios to the \( p \to \pi \) TDAs) opens a fascinating window on the properties of the sea quarks in the proton wave function.

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**Appendix.** We now derive the general limit of the three contributing TDAs at \( \Delta_T = 0 \) in the soft-pion limit, when \( \xi \) gets close to 1. In that limit, the soft-meson theorem [17] derived from current algebra applies [11], which allows us to express these 3 TDAs in terms of the 3 Distribution Amplitudes (DAs) of the corresponding baryon. Conventionally [15], the three proton DAs are defined through the decomposition of the following matrix element of the 3-quark operator in terms of three invariant functions of the scalar product of the light-like separation \( z,n \equiv \tilde{z}, \tilde{n} \) with the proton momentum \( p_p \), \( V^p(\tilde{z}, p_p) \), \( A^p(\tilde{z}, p_p) \) and \( T^p(\tilde{z}, p_p) \).

The latter functions satisfy \( V^p(\tilde{z}, p_p = 0) = T^p(\tilde{z}, p_p = 0) = 1 \) and \( A^p(\tilde{z}, p_p = 0) = 0 \), which provides the interpretation of \( f_N \) as the value of the proton wave function at the origin. To go to momentum space one writes a Fourier transform [16] which enables to define functions of momentum fractions \( x_i \) (\( F = V^p, A^p, T^p \))

\[
\begin{align*}
\tilde{F}(x_i) & = \int_{-\infty}^{\infty} \frac{dz}{(2\pi)^3} e^{i \Sigma_k z_k \tilde{z}_k} \tilde{F}(\tilde{z}, p_p).
\end{align*}
\] 

Inspired by [11], which considered the related case of the distribution amplitude of the proton-meson system, we use the soft pion theorems [17] to write:

\[
\langle \pi^0(p_\pi) | O | P(p_1, s_1) \rangle = -i \int \frac{d\pi}{\pi} \langle 0 | Q_5^a O | P(p_1, s_1) \rangle,
\]

where we neglected the nucleon pole term, which does not contribute at threshold.

For the transition \( p \to \pi^0 \), \( Q_5^0 = Q_5^3 \) and \( O = u_\alpha u_\beta d_\gamma \). Since the commutator of the chiral charge \( Q_5 \) with the quark field \( \psi \) (\( \tau^a \) being the Pauli matrix)

\[
\langle Q_5^0, \psi \rangle = -\frac{\tau^a}{2} \gamma^5 \psi,
\]

the first term in the rhs of Eq. (23) gives three terms from \( \langle \gamma^\alpha u_\alpha u_\beta d_\gamma, u_\alpha (\gamma^\beta u_\beta) d_\gamma \rangle \) and \( u_\alpha u_\beta d_\gamma \). The corresponding multiplication by \( \gamma^5 \) (or \( \gamma^5 T \) when it acts on the index \( \beta \) on the vector and axial-vector structures of the DA (Eq. (21)) gives two terms which cancel and the third one, which remains, is the same as the one for the TDA up to the modification that on the DA decomposition, \( p_p \) is the proton momentum, whereas for the TDA one, \( p \) is the light-cone projection of \( P \), i.e. half the proton momentum if one neglects the pion one. This introduces a factor \( 2 \xi \) in the relations between the 2 DAs \( A^p \) and \( V^p \) and the 2 TDAs \( V^{p_\pi^0} \) and \( A^{p_\pi^0} \).

To what concerns the tensorial structure multiplying \( T^p \), the three terms are identical at leading-twist accuracy and correspond to the structure multiplying \( T^{p_\pi^0} \), this gives a factor 3. We eventually have the soft limit for our three TDAs at \( \Delta_T = 0 \):

\[
(V^{p_\pi^0}, A^{p_\pi^0} T^{p_\pi^0})(\tilde{z}, p_p) = \xi (V^p, A^p, T^p)(\tilde{z}, p_p).
\]

We will derive now this relation in the momentum representation of DAs and TDAs. To do so, we start from translational invariance which implies

\[
\langle 0 | u_\alpha (\tilde{z} + a) u_\beta (\tilde{z} + a) d_\gamma (\tilde{z} + a) | p_p \rangle = e^{-ia.p_p} \langle 0 | u_\alpha (\tilde{z}) u_\beta (\tilde{z}) d_\gamma (\tilde{z}) | p_p \rangle,
\]

and thus \( F((\tilde{z} + a), p_p) = e^{-ia.p_p} F(\tilde{z}, p_p) \). In momentum space, we correspondingly get

\[
\tilde{F}(x_i) = \int_{-\infty}^{\infty} \frac{d(\tilde{z} + a)}{(2\pi)^3} e^{i \Sigma_k (\tilde{z}_k + a) \cdot p_p} \tilde{F}((\tilde{z} + a), p_p) = e^{i (\Sigma_k \tilde{z}_k - a) \cdot p_p} \tilde{F}(x_i).
\]
This condition is conveniently expressed by the following redefinition: \( \hat{F}(x_i) = \delta(\sum_k x_k - 1)F(x_i) \). The inverse Fourier transform is then written as (\(|dx| = dx_1 dx_2 dx_3\)):

\[
F(\tilde{z}_i, p_p) = \int_0^1 [dx] e^{-i \Sigma_k x_k \tilde{z}_k p_p} \delta(\sum_k x_k - 1)F(x_i). (28)
\]

The normalisation conditions then reads

\[
\int_0^1 [dx] \delta(\sum_k x_k - 1)(V^p, A^p, T^p)(x_i) = (1, 0, 1). \quad (29)
\]

Note that the delta function insuring translational invariance is exactly the one expected from the interpretation that \( x_i \) be the momentum fraction carried by the quark \( i \) off the proton of momentum \( p_p \). This shows that the natural conjugate variable to the momentum fractions \( x_i \)'s are the \( \tilde{z}_i, p_p \)'s, i.e. the spatial separation dotted by the proton momentum. Indeed, \( p_p \) enters in the exponential of the rhs of Eq. (27), via the conjugate variable to \( x_i, \tilde{z}_i, p_p \), and as the initial-state momentum of the matrix element. Another choice than \( \tilde{z}_i, p_p \) would not have provided the correct support for the \( x_i \)'s.

The case of TDAs is similar except for the choice of the natural conjugate variable. We start with translational invariance:

\[
\langle B(p_B)|e^{i k_{(\tilde{z} + a)} u^i_{(\tilde{z} + a)}(\tilde{z}_2 + a) d^k_{(\tilde{z} + a)}|A(p_A)\rangle = e^{-i a(\xi + a)}B(p_B)|e^{i k_{(\tilde{z} + a)} u^i_{(\tilde{z} + a)}(\tilde{z}_2 + a) d^k_{(\tilde{z} + a)}|A(p_A)\rangle.
\]

Now let us define a Fourier transform without specifying the momentum \( p_0 \) to which we dot the spatial separation:

\[
F_{A \rightarrow B}(x_i) = \int_{-\infty}^{\infty} [d(\tilde{z} + a)] e^{i \Sigma_k x_k (\tilde{z}_k + a). p_0} \times (30)
\]

\[
F_{A \rightarrow B}(\tilde{z}_i + a). p_0 = e^{i \Sigma_k x_k (\tilde{z}_k + a). p_0} \times F_{A \rightarrow B}(x_i).
\]

The condition derived from translational invariance would then be satisfied by \( \delta(\sum_k x_k a p_0 - a(\xi + a)) \). In order to get the correct support, i.e. \( \delta(2\xi - \sum_k x_k) \), for a translation \( a \) along \( n \), we have to choose \( p_0 \) such that \( 2\xi = \frac{n((p_A - p_B))}{n.p_0} \), which is satisfied by \( p_0 = (p_a + p_B)/2 = P \) or a light-cone vector \( p \). Then we choose \( p \) and have

\[
\delta(2\xi - \sum_k x_k)F_{A \rightarrow B}(x_i) \equiv F(\tilde{z}_i(n.p)) \equiv (31)
\]

\[
(n.p)^3 \int_{-\infty}^{\infty} [dz] e^{i \Sigma_k x_k (\tilde{z}_i + a. n.p)} F(z_i(n.p)).
\]

Let us now use the Fourier transform Eq. (22) on both side of Eq. (25). Defining \( \alpha \) such that \( p_p . n = \alpha P . n \), we get for instance for the \( V \'s:

\[
\delta(\sum_k x_k - 1)V^p(x_i) = \int_{-\infty}^{\infty} [dz]\frac{[d\tilde{z}]}{(2\pi)^3} e^{i \Sigma_k x_k (\tilde{z}_R \alpha) + \tilde{z}_R \alpha} V_{1}^{p\alpha}(\tilde{z}_R). (32)
\]

We conclude that translational invariance imposes naturally, through the delta function, that \( \alpha = 2\xi \); the change of variable \( x_i = \alpha x_i \) then yields

\[
\{V_{1}^{p\alpha}, A_{1}^{p\alpha}, T_{1}^{p\alpha}\}(x_1, x_2, x_3, \xi, \Delta^2) = \frac{1}{4\xi}\{V^p, A^p, T^p\}(\frac{x_1}{2\xi}, \frac{x_2}{2\xi}, \frac{x_3}{2\xi}). (33)
\]