A closed-form analytical approach for stress concentrations at elliptical holes in moderately thick plates

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In this work, a modified plate model situated in-between the classical Kirchhoff-Love and first-order shear deformation plate theory (FSDT) is proposed. The modified plate model includes transverse shear deformation and the governing system of partial differential equations can be tackled employing a complex potential approach. This allows for using the powerful tool of conformal mapping enabling closed-form analytical solutions on complex domains. Based on the established complex potential methodology, the distribution of bending moments at an elliptical hole in an infinite plate under bending loading is derived. Solutions of the modified plate model are compared against numerical reference data employing FSDT.

1 Introduction

Along with a progressively increasing application of lightweight design concepts, efficient methods enabling a detailed analysis of stress concentrations are needed. In contrast to plane problems which have been comprehensively investigated in literature, cf. [1, 2], the study of stress raisers in out-of-plane situations still poses challenges. Neglecting transverse shear deformation, the Kirchhoff-Love plate theory becomes inadequate for notches with a characteristic length of the same order as the plate thickness [3, 4]. Applying FSDT yields a proper description of the stress state at notches. However, the governing system of partial differential equations prevents closed-form analytical solutions on complex domains. In the present work, a modified plate model accounting for transverse shear deformation is proposed which additionally allows for formulating three physically sound boundary conditions being crucial for correctly rendering the local stress field at notches.

2 Modified plate model

Consider a plate with applied bending moments at infinity that contains an elliptical hole and obeys the kinematics according to FSDT. The presence of a notch induces localised stress concentrations associated to large stress-gradients. Assuming that terms associated to higher-order derivatives play a dominant role in regions where such large gradients are present, the following system of partial differential equations can be derived employing the principle of minimum potential energy [3]:

\[
K \left( \psi_{x,xx} + \nu \psi_{y,xy} + \frac{1 - \nu}{2} \psi_{x,yy} + \psi_{y,yy} \right) = G h_s w_{,x},
\]

\[
K \left( \psi_{y,yy} + \nu \psi_{x,xy} + \frac{1 - \nu}{2} \psi_{x,xx} + \psi_{x,xx} \right) = G h_s w_{,y}, \quad \Delta w = 0.
\]

(1)

Here, \(w\) denotes the deflection of the mid-plane and \(\psi_x, \psi_y\) represent the inclination angles, respectively.

3 Complex potential method

A solution of the system (1) is derived expressing the deformation quantities \(w, \psi_x, \psi_y\) in terms of three holomorphic potentials \(\Phi_1(z), \Phi_2(z)\) and \(\Phi_3(z)\) with \(z = x + iy\). Requiring that Eq. (1) is satisfied for an arbitrary choice of the complex potentials finally yields the following general solution (with \(\kappa = (3 - \nu)/(1 + \nu)\)):

\[
\psi_x = \frac{1}{K(\nu - 1)} \text{Re} \left[ \Phi_1(z) - \kappa \Phi_2(z) + \frac{\nu - 1}{1 + \nu} \Phi_3(z) + \overline{\Phi_2'(z)} \right],
\]

\[
\psi_y = -\frac{1}{K(\nu - 1)} \text{Im} \left[ \Phi_1(z) + \kappa \Phi_2(z) - \frac{\nu - 1}{1 + \nu} \Phi_3(z) + \overline{\Phi_2'(z)} \right], \quad w = \frac{1}{G h_s} \text{Re} \left[ \Phi_2'(z) \right].
\]

(2)

The corresponding bending moments and transverse shear forces read:

\[
M_{xx} + M_{yy} = 4 \text{Re} \left[ \Phi_2'(z) \right] + 2 \text{Re} \left[ \Phi_3'(z) \right], \quad M_{xy} = M_{yx} = 2 \left( \Phi_1'(z) + \overline{\Phi_2'(z)} \right),
\]

\[
Q_x - i Q_y = \Phi_2'(z).
\]

(3)

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4 Results and discussion

In order to solve the boundary value problem of an elliptical hole in an infinite plate under bending loading, the complex potentials have to be chosen such that the boundary conditions at infinity ($M_{yy} = M_0$) and along the stress-free hole contour are satisfied. This is done employing the method of conformal mapping as shown in Fig. 1, which allows for formulating boundary conditions with respect to the ζ-plane [1]. Starting with an infinite Laurent-series expansion of the complex potentials, the proper incorporation of boundary conditions ultimately yields the closed-form solution:

\[
\begin{align*}
\Phi_1(\zeta) &= \frac{M_0 r}{2} \left( \zeta + \frac{1}{m\zeta} - \frac{1 + m^2}{m} \left( \frac{\zeta}{\zeta^2 - m} \right) \right), \\
\Phi_3(\zeta) &= 0, \quad r = \frac{a + b}{2}, \quad m = \frac{a - b}{a + b}.
\end{align*}
\]

The resulting distribution of the bending moment $M_{\phi\phi}$ along the elliptical hole is depicted in Fig. 2. According to the modified plate theory, a maximum bending moment $M_{\phi\phi}/M_0 = 7$ is observed at $\phi = \pm \pi/2$. For small values of the semi-axis to thickness ratio $a/h$, results are in a very good agreement with the numerical FSDT reference solution. For an increasing ratio $a/h$, however, the modified theory tends to overestimate the bending moments along the hole’s contour. In fact, the proposed model represents an asymptotic solution of FSDT for $a/h \to 0$. This is further illustrated in Fig. 3 for the case of a circular hole. For small $R/h$-ratios, the modified plate model correctly captures the maximum bending moment $M_{\phi\phi}^{max}$ whereas the classical Kirchhoff-Love theory becomes inadequate and underestimates the actual bending moments significantly.

5 Conclusion

It has been shown, that the proposed modified plate model allows for analytically calculating the bending moment distribution along an elliptical hole in an infinite plate under uniform bending loading. Due to the simplified plate kinematics, the model’s validity range is restricted to small values of the semi-axis to thickness ratio. In this region, a very good agreement with numerical data according to FSDT is obtained. Since a complex potential approach has been used, the derived solution can be extended to anisotropic elasticity and thus enables a very efficient treatment of laminated composite plates.

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