The Glimm space of the minimal tensor product of $C^*$-algebras

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Abstract

We show that for $C^*$-algebras $A$ and $B$, there is a natural open bijection from $\text{Glimm}(A) \times \text{Glimm}(B)$ to $\text{Glimm}(A \otimes_\alpha B)$ (where $A \otimes_\alpha B$ denotes the minimal $C^*$-tensor product), and identify a large class of $C^*$-algebras $A$ for which the map is continuous for arbitrary $B$. As a consequence we determine the structure space of the centre of the multiplier algebra $ZM(A \otimes_\alpha B)$ in terms of $\text{Glimm}(A)$ and $\text{Glimm}(B)$, and give necessary and sufficient conditions for the inclusion $ZM(A) \otimes ZM(B) \subseteq ZM(A \otimes_\alpha B)$ to be surjective. Further we show that when the Glimm spaces are considered as sets of ideals, the map $(G,H) \mapsto G \otimes_\alpha B + A \otimes_\alpha H$ implements the above bijection, extending a result of Kaniuth from [19] by eliminating the assumption of property (F).

The focus of our work is the relationship between a $C^*$-algebra $A$ and its collection of primitive ideals $\text{Prim}(A)$, a topological space in the hull kernel topology, the associated complete regularisation space $\text{Glimm}(A)$ of $\text{Prim}(A)$, and in particular how $\text{Glimm}(A \otimes_\alpha B)$ relates to $\text{Glimm}(A)$ and $\text{Glimm}(B)$. Here $A \otimes_\alpha B$ is the usual (minimal) tensor product of two $C^*$-algebras $A$ and $B$ and our work is motivated by a desire to extend earlier work such as [19], [9], [22].

The problem of representing a $C^*$-algebra as the section algebra of a bundle over a suitable base space may be viewed as that of finding a non-commutative analogue of the Gelfand-Naimark theorem. Significant contributions to the theory were made by Fell [11], Tomiyama [27], Dauns and Hofmann [9], Lee [24] and others. The topological space $\text{Prim}(A)$ arises in this context as a natural choice for the base space. However, a major obstacle to this approach is that $\text{Prim}(A)$ is in general not sufficiently well-behaved as a topological space from the point of view of bundle theory.

We define the topological space $\text{Glimm}(A)$ as the complete regularisation of $\text{Prim}(A)$ and denote by $\tau_{cr}$ the (completely regular) topology on this space induced by the continuous functions on $\text{Prim}(A)$. As a set, this space is the quotient of $\text{Prim}(A)$ modulo the equivalence relation $\approx$ of inseparability by continuous functions on $\text{Prim}(A)$. By the Glimm ideals of $A$ we mean the ideals given by taking the intersection of the primitive ideals in each $\approx$-equivalence class, see §1.

In [9], Dauns and Hofmann showed that any $C^*$-algebra $A$ may be represented as the section algebra of an upper semicontinuous $C^*$-bundle over $\text{Glimm}(A)$ if this space is locally compact, or over its Stone-Čech compactification otherwise. Under this representation the fibre algebras are given by the Glimm quotients of $A$. Thus in the case of the minimal tensor product of $C^*$-algebras $A$ and $B$, a natural question that arises is to determine $\text{Glimm}(A \otimes_\alpha B)$ in terms of $\text{Glimm}(A)$ and $\text{Glimm}(B)$, both topologically and as a collection of ideals of $A \otimes_\alpha B$. A related problem (over more general base spaces) was studied by Kirchberg and Wassermann in [20],

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and later by Archbold in [3], by considering the fibrewise tensor product of the corresponding bundles of $A$ and $B$.

We will denote by $\text{Id}'(A)$ the set of all proper norm-closed two sided ideals of $A$. By $\text{Fac}(A)$ we mean the space of kernels of factor representations of $A$, which is a topological space in the null-kernel topology.

There is a natural embedding of $\text{Id}'(A) \times \text{Id}'(B)$ into $\text{Id}'(A \otimes \alpha B)$ sending $(I, J) \mapsto \ker(q_I \otimes q_J)$, where $q_I$ and $q_J$ are the quotient maps. The restrictions of this map to the spaces of primitive and factorial ideals are known to be homeomorphisms onto dense subspaces of $\text{Prim}(A \otimes \alpha B)$ and $\text{Fac}(A \otimes \alpha B)$ respectively, see [11], [22]. Recently A.J. Lazar has shown in [22] that any continuous map $f : \text{Prim}(A) \times \text{Prim}(B) \to Y$, where $Y$ is a $T_1$ space has a continuous 'extension' to $\text{Prim}(A \otimes \alpha B)$, where we identify $\text{Prim}(A) \times \text{Prim}(B)$ with its image under the above embedding.

We begin in §1 with considerations of the complete regularisation of a product $X \times Y$ of topological spaces, gathering together and extending results on this theme from the literature. Central to this is the theory of $w$-compact spaces, introduced by Ishii in [17]. We establish (Proposition 1.9) conditions on a $C^*$-algebra $A$ that ensure that the complete regularisation of $\text{Prim}(A) \times \text{Prim}(B)$ is homeomorphic to the product space $\text{Glimm}(A) \times \text{Glimm}(B)$ for any $C^*$-algebra $B$. In the presence of a countable approximate unit for $A$, a necessary and sufficient condition for this to occur is that $\text{Glimm}(A)$ be locally compact. In the general case, sufficient conditions include compactness of $\text{Prim}(A)$ (e.g. if $A$ is unital), or that the complete regularisation map of $\text{Prim}(A)$ is open (e.g. if $A$ is quasi-standard, see [4]). Local compactness of $\text{Glimm}(A)$ is always a necessary condition.

Using the extension result of Lazar together with the universal property of the complete regularisation of a topological space described in §1, we show (Theorem 2.2) that as a topological space $\text{Glimm}(A \otimes \alpha B)$ is the same as (or can be identified in a natural way with) the complete regularisation of $\text{Prim}(A) \times \text{Prim}(B)$. We investigate conditions under which the latter coincides with the cartesian product space $\text{Glimm}(A) \times \text{Glimm}(B)$, while showing that the underlying sets always agree. In Corollary 2.3 we give some rather general conditions on $A$ or on $B$ for this coincidence but show an example in §6 where it fails.

A well-known Corollary of the Dauns-Hofmann Theorem is the existence of a $\alpha$-isomorphism identifying the centre $ZM(A)$ of the multiplier algebra of $A$ with the $C^*$-algebra of bounded continuous functions on $\text{Glimm}(A)$ (equivalently, on $\text{Prim}(A)$). We show in Theorem 3.6 that for any $C^*$-algebras $A$ and $B$, $ZM(A \otimes \alpha B)$ can be identified with the bounded continuous functions on (the complete regularisation of) $\text{Prim}(A) \times \text{Prim}(B)$, and give necessary and sufficient conditions (Theorem 3.6) for $ZM(A) \otimes ZM(B) = ZM(A \otimes \alpha B)$.

In §4, we determine the set of $\text{Glimm}$ ideals of $A \otimes \alpha B$ in terms of the $\text{Glimm}$ ideals of $A$ and $B$. In order to do this, we use an alternative construction of $\text{Glimm}(A)$ based on the complete regularisation of $\text{Fac}(A)$ (rather than $\text{Prim}(A)$) first considered by Kaniuth in [19]. The reason for this is the fact that there exists a continuous surjection $\text{Fac}(A \otimes \alpha B) \to \text{Fac}(A) \times \text{Fac}(B)$, while it is not known if the restriction of this map to $\text{Prim}(A \otimes \alpha B)$ has range $\text{Prim}(A) \times \text{Prim}(B)$.

We show in Theorem 4.8 that the map $\text{Glimm}(A) \times \text{Glimm}(B) \to \text{Glimm}(A \otimes \alpha B)$ sending $(G, H) \mapsto G \otimes \alpha B + A \otimes \alpha H$ determines the homeomorphism of these spaces when considered as sets of ideals. This extends Kaniuth’s result [19 Theorem 2.3], which was proved under the assumption that $A \otimes \alpha B$ satisfies Tomiyama’s property (F), defined below.

In §5 we consider the problem of determining conditions for which the canonical upper semicontinuous bundle representation of $A \otimes \alpha B$ over $\text{Glimm}(A \otimes \alpha B)$ of [9] is in fact continuous (i.e. $A \otimes \alpha B$ defines a maximal full algebra of operator fields in the sense of Fell [13]). The main result of this section (Theorem 5.2) shows that, under the assumption that $\ker(q_G \otimes q_H) = G \otimes \alpha B + A \otimes \alpha H$ for all pairs of $\text{Glimm}$ ideals $(G, H)$ of $A$ and $B$, this representation of
$A \otimes_{\alpha} B$ is continuous precisely when the corresponding bundle representations of $A$ and $B$ (over $\text{Glimm}(A)$ and $\text{Glimm}(B)$ respectively) are continuous. We also show that, under a different assumption that does not require that these ideals be equal, continuity of $A$ and $B$ is a necessary condition for continuity of $A \otimes_{\alpha} B$ (Proposition 5.4).

Let $\alpha$ and $\beta$ be states of $A$ and $B$ respectively. Then the product state $\alpha \otimes \beta$ of $A \otimes_{\alpha} B$ is defined via $(\alpha \otimes \beta)(a \otimes b) = \alpha(a)\beta(b)$ on elementary tensors $a \otimes b$ and extended to $A \otimes_{\alpha} B$ by linearity and continuity. If $I, J \in \text{Id}'(A \otimes_{\alpha} B)$ with $I \not\subset J$, then we say that a state $\gamma$ of $A \otimes_{\alpha} B$ separates $I$ and $J$ if $\gamma(J) = \{0\}$ and there exists $c \in I \cap J$ with $\gamma(c) = 1$. The minimal tensor product $A \otimes_{\alpha} B$ is said to satisfy Tomiyama’s property (F) if given any pair $I, J \in \text{Id}'(A \otimes_{\alpha} B)$ with $I \not\subset J$, there is a product state of $A \otimes_{\alpha} B$ separating $I$ and $J$. There are many equivalent characterisations of property (F), see [22] Proposition 5.1 for example.

For any $C^*$-algebra $A$ we denote by $M(A)$ its multiplier algebra and by $Z(A)$ its centre. We say that $A$ is $\sigma$-unital if it admits a countable approximate identity. For $I \in \text{Id}'(A)$ we denote by $\text{hull}(I) = \{P \in \text{Prim}(A) : P \supseteq I\}$ and by $\text{hull}_f(I) = \{M \in \text{Fac}(A) : M \supseteq I\}$. For a collection $S \subseteq \text{Id}'(A)$ we let $k(S) = \cap\{I : I \in S\}$, the kernel of $S$.

For any topological space $X$ we will denote by $C(X)$ (resp. $C^b(X)$) the $\ast$-algebra of continuous (resp. bounded continuous) complex valued functions on $X$. If $\alpha : X \to Y$ is a continuous map between topological spaces we will denote by $\alpha^* : C(Y) \to C(X)$ the unique $\ast$-homomorphism given by $\alpha^*(f) = f \circ \alpha$ for $f \in C(Y)$.

## 1 The complete regularisation of a product of topological spaces

Our terminology is that a topological space $X$ is said to be completely regular (or a Tychonoff space) if it is a Hausdorff space, and given any closed subset $F \subseteq X$ and a point $x \in X \setminus F$, there is a continuous function $f : X \to \mathbb{R}$ with $f(x) = 1$ and $f(F) = \{0\}$.

For $f \in C(X)$, denote by $\text{coz}(f) = \{x \in X : f(x) \neq 0\}$, the cozero set of $f$. Replacing $f$ with $\min(|f|, 1)$, we may assume that any cozero set in $X$ is the cozero set of some continuous function $f : X \to [0, 1]$.

Recall that for any topological space $X$ there is a canonically associated completely regular space $\rho X$, called the complete regularisation of $X$, with the property that $C(X) \equiv C(\rho X)$ and $C^b(X) \equiv C^b(\rho X)$. Furthermore the assignment of $\rho X$ to $X$ defines a covariant functor, the Tychonoff functor, from the category of topological spaces and continuous maps to the category of completely regular spaces.

Define an equivalence relation on $X$ as follows: for $x_1, x_2 \in X$ write $x_1 \approx x_2$ if $f(x_1) = f(x_2)$ for all $f \in C^b(X)$. Let $\rho X = X/\approx$ and let $\rho X : X \to \rho X$ be the quotient map. Each $f \in C^b(X)$ defines a function $f^\rho$ on $\rho X$ by setting $f^\rho([x]) = f(x)$, where $[x]$ denotes the $\approx$-equivalence class of $x$. Denote by $\tau_{cr}$ the weak topology on $\rho X$ induced by the functions $\{f^\rho : f \in C^b(X)\}$.

It is shown in [12] Theorem 3.9 that the space $\rho X$ constructed in this way is completely regular, and that the collection $\{\text{coz}(f^\rho) : f \in C^b(\rho X)\}$ forms a basis for the topology $\tau_{cr}$. Moreover, the map $g \mapsto g \circ \rho X$ is an isomorphism of $C^b(\rho X)$ onto $C^b(X)$.

An alternative construction of $\rho X$ is as follows: let $I$ denote the closed unit interval and $I^X$ the set of all continuous maps $f : X \to I$. Let $P(X) = \prod\{I_f : f \in I^X\}$, where $I_f = I$ for all $f \in I^X$, and let $\tau : X \to P(X)$ be defined via $\tau(x) = (f(x))_{f \in I^X}$.

Defining $P^\rho X, \rho X$ and $\tau^\rho : \rho X \to P(\rho X)$ analogously, it is clear that $P^\rho X = \{f^\rho : f \in I^X\}$ and hence $P(\rho X) = P(X)$. The definition of $f^\rho$, where $f \in I^X$, ensures that $\tau^\rho \circ \rho X = \tau$, hence $\tau(X)$ is equal to $\tau^\rho(\rho X)$. On the other hand, it is well-known that $\tau^\rho$ is a homeomorphism onto its image [29] Lemma 1.5, so that $\tau(X)$ is homeomorphic to $\rho X$.

The topological space $(\rho X, \tau_{cr})$, or more precisely the triple $(\rho X, \tau_{cr}, \rho X)$ constructed in this way is the complete regularisation of $X$.  

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Now let $X$ and $Y$ be topological spaces and $\phi : X \to Y$ a continuous map. Then setting $\phi^\#(\rho_X(x)) = (\rho_Y \circ \phi(x))$ gives a map $\phi^\# : \rho_X \to \rho_Y$ such that the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\phi} & Y \\
\rho_X \downarrow & & \downarrow \rho_Y \\
\rho_X & \xrightarrow{\phi^\#} & \rho_Y
\end{array}
$$

commutes. To see that $\phi^\#$ is continuous, let $U = \text{coz}(f)$ be a cozero set in $\rho_Y$. Then $(\rho_Y \circ \phi)^{-1}(U)$ is precisely the cozero set in $X$ of the continuous function $f \circ \rho_Y \circ \phi$. It follows that $(\phi^\#)^{-1}(U) = \text{coz}(f \circ \rho_Y \circ \phi)^\#$ is open in $\rho_X$ by the definition of $\tau_{cr}$.

Thus the assignment of $\rho_X$ to $X$ defines a covariant functor from the category of topological spaces to the subcategory of completely regular spaces, called the Tychonoff functor. The term Tychonoff functor was first used by Morita in [25, p. 32].

**Lemma 1.1.** Let $X$ and $Y$ be topological spaces. Then there is an open bijection

$$
\rho(X) \times \rho(Y) \to \rho(X \times Y)
$$

sending $(\rho_X(x), \rho_Y(y)) \mapsto \rho_{X \times Y}(x, y)$.

**Proof.** We first show that $(X/ \approx) \times (Y/ \approx) = (X \times Y)/ \approx$ as sets; specifically that $(x_1, y_1) \approx (x_2, y_2)$ if and only if $x_1 \approx x_2$ and $y_1 \approx y_2$. Indeed, if $(x_1, y_1) \approx (x_2, y_2)$ and $f \in C^b(X \times Y)$ then $f \circ \pi_X \in C^b(X \times Y)$, hence $f(x_1) = f \circ \pi_X(x_1, y_1) = f \circ \pi_X(x_2, y_2) = f(x_2)$, and $x_1 \approx x_2$.

On the other hand if both $x_1 \approx x_2$ and $y_1 \approx y_2$, take $g \in C^b(X \times Y)$; then $g(x_1, y_1) = g(x_2, y_1) = g(x_2, y_2)$. It follows that the mapping $(\rho_X(x), \rho_Y(y)) \mapsto \rho_{X \times Y}(x, y)$ is a well-defined bijection.

In order to show that the above map is open we use the fact that in a completely regular space, the cozero sets of continuous functions form a base for the topology [12, 3.4]. Consider a basic open set $\text{coz}(f^\#) \times \text{coz}(g^\#)$ in $\rho_X \times \rho_Y$, where $f \in C^b(X), g \in C^b(Y)$. Then $h(x, y) = (f^\# \circ \rho_X)(x)(g^\# \circ \rho_Y)(y)$ defines an element of $C^b(X \times Y)$, and hence gives $h^\# \in C^b(\rho(X \times Y))$ with $h^\# \circ \rho_{X \times Y} = h$. Then $\text{coz}(h^\#) = \text{coz}(f^\#) \times \text{coz}(g^\#)$, so that $\text{coz}(f^\#) \times \text{coz}(g^\#)$ is an open subset of $\rho(X \times Y)$.

In view of Lemma [14] for any topological spaces $X$ and $Y$ we identify $\rho(X) \times \rho(Y)$ and $\rho(X \times Y)$ as sets. This canonical map is not a homeomorphism in general, however (see Example [6.1]). Thus in what follows, we will denote by $\rho_X \times \rho_Y$ this product space with the (possibly weaker) product topology $\tau_p$, and by $\rho(X \times Y)$ the space with the topology $\tau_{cr}$ induced by the functions in $C^b(X \times Y)$.

The following result was obtained originally in [16]:

**Theorem 1.2.** (Hoshina and Morita) [18, Theorem 2.4]

Let $X$ and $Y$ be topological spaces. The following are equivalent:

(i) $\rho_X \times \rho_Y = \rho(X \times Y)$,

(ii) For any cozero set $G$ of $X \times Y$ and any point $(x, y) \in G$ there are cozero sets $U$ and $V$ of $X$ and $Y$ respectively with $(x, y) \in U \times V \subseteq G$.

**Definition 1.3.** Let $(X, T)$ be a topological space. For a subspace $Y \subseteq X$ denote by $\tau_Y$ the topology on $Y$ generated by $\{\text{coz}(f) : f \in C(Y)\}$.
For a topological space \((X, \mathcal{T})\) and a subset \(U \subseteq X\) we denote by \(\mathcal{T} \upharpoonright U\) the subspace topology on \(U\) inherited from \(\mathcal{T}\). For two topologies \(\mathcal{T}_1, \mathcal{T}_2\) on \(X\) we write \(\mathcal{T}_1 \leq \mathcal{T}_2\) to say that \(\mathcal{T}_1\) is weaker than \(\mathcal{T}_2\).

For any subset \(U \subseteq X\), we denote by \(\overline{U}\) the \(\mathcal{T}\)-closure of \(U\) and by \(\text{cl}_{\mathcal{T}_X}(U)\) the \(\mathcal{T}_X\)-closure of \(U\). The following lemma establishes some basic properties of the \(\tau\)-topologies on subspaces of a topological space \((X, \mathcal{T})\):

**Lemma 1.4.** Let \((X, \mathcal{T})\) be a topological space and let \(U \subseteq X\). Then:

(i) \(\tau_U \leq \mathcal{T} \upharpoonright U\),

(ii) \(U\) is \(\tau_X\) open if and only if \(U = \rho_X^{-1}(W)\) for some open subset \(W \subseteq \rho X\),

(iii) If \(U\) is \(\tau_X\)-open then \(U\) is saturated with respect to the relation \(\approx\) on \(X\) (hence \(X \setminus U\) is saturated also),

(iv) \(\text{cl}_{\tau_X}(U) = \rho_X^{-1}\left(\rho_X(U)\right)\), where \(\rho_X(U)\) is the \(\tau_{cr}\)-closure of \(\rho_X(U)\) in \(\rho X\),

(v) If \(U \subseteq V \subseteq X\) then \(\tau_V \upharpoonright U \leq \tau_U\).

**Proof.** (i) Every basic \(\tau_U\)-open set is of the form \(\{x \in U : f(x) \neq 0\}\) with \(f : U \to [0,1]\) continuous, hence is open in \(\mathcal{T} \upharpoonright U\).

(ii) Note that for every continuous \(f : X \to [0,1]\), \(\text{coz}(f) = \rho_X^{-1}(\text{coz}(f'))\) by the construction of \(f'\). Hence \(U \subseteq X\) is \(\tau_X\)-open if and only if there exist continuous functions \(f_i : X \to [0,1]\) for all \(i\) in some index set \(I\) such that

\[
U = \bigcup_{i \in I} \text{coz}(f_i) = \bigcup_{i \in I} \rho_X^{-1}(\text{coz}(f_i')) = \rho_X^{-1}\left(\bigcup_{i \in I} \text{coz}(f_i')\right).
\]

Since the \(\tau_{cr}\)-open subsets of \(\rho X\) are unions of cozero sets, the conclusion follows.

(iii) Suppose \(U\) is \(\tau_X\)-open and \(x \in U\). Then there is a cozero set neighbourhood \(\text{coz}(f)\) of \(x\) contained in \(U\), where \(f : X \to [0,1]\) is continuous. Thus for any \(y \in X \setminus U\), \(f(y) = 0\) and \(f(x) \neq 0\). Hence \(x \neq y\) for any such \(y\), and so \([x] \subseteq U\).

(iv) By (ii) \(\rho_X^{-1}\left(\rho_X(U)\right)\) is \(\tau_X\) closed. Now suppose \(F \subseteq X\) is \(\tau_X\) closed and \(U \subseteq F\). Then \(\rho_X(F)\) is closed in \(\rho X\) by (ii), and contains \(\rho_X(U)\), hence contains \(\rho_X(U)\). By (iii), this gives \(F = \rho_X^{-1}(\rho_X(F)) \supseteq \rho_X^{-1}\left(\rho_X(U)\right)\), as required.

(v) If \(U \subseteq V \subseteq X\), then every \(f \in C(V)\) has \(f \upharpoonright U \in C(U)\). Hence \(\text{coz}(f) \cap U = \text{coz}(f \upharpoonright U)\) is a cozero set of \(U\), so that the subspace topology \(\tau_V \upharpoonright U\) is weaker than \(\tau_U\). \(\square\)

**Definition 1.5.** A topological space \((X, \mathcal{T})\) is said to be \(w\)-compact if given any \(\mathcal{T}\)-open covering \(\{U_\alpha\}_{\alpha \in A}\) of \(X\), then there exist \(\alpha_1, \ldots, \alpha_n \in A\) such that \(X = \text{cl}_{\tau_X}(U_{\alpha_1} \cup \ldots \cup U_{\alpha_n})\).

It is shown in [18, Proposition 3.3] that \(X\) is \(w\)-compact if and only if any family \(\{Q_\alpha\}\) of \(\tau_X\)-open subsets of \(X\) with the finite intersection property has \(\bigcap Q_\alpha \neq \emptyset\).

The class of \(w\)-compact spaces was introduced by Ishii in [17] to characterise the topological spaces \(X\) for which \(\rho(X \times Y) = \rho X \times \rho Y\), for any topological space \(Y\):

**Theorem 1.6.** [18, Theorem 4.1] For a topological space \(X\) the following are equivalent:

(i) \(\rho(X \times Y) = \rho X \times \rho Y\) for any space \(Y\),

(ii) For each \(x \in X\) there is a cozero set neighbourhood \(U\) of \(x\) such that \(\overline{U}\) is \(w\)-compact.
We will show in Proposition 1.9 that condition (ii) of Theorem 1.6 is satisfied by Prim(A) for a large class of C*-algebras A. The following Lemma gives a sufficient condition for a point \( x \) in a general topological space \( X \) to have a cozero set neighbourhood with w-compact closure.

**Lemma 1.7.** Let \((X, \mathcal{T})\) be a topological space and suppose that \( \rho_X(x) \in \rho X \) has a compact neighbourhood \( K \) such that there is a compact \( C \subseteq X \) with \( \rho_X(C) = K \). Then \( x \) has a cozero set neighbourhood \( U \) in \( X \) with \( U \) w-compact.

**Proof.** Choose \( f \in C(\rho X) \) with \( f(\rho_X(x)) = 1 \) and \( f(\rho X \setminus \text{int} K) = \{0\} \). Let \( U = \rho_X^{-1}(\text{coz}(f)) = \text{coz}(f \circ \rho_X) \), a cozero set neighbourhood of \( x \) in \( X \). We claim that \( U \cap C \) is \( \tau_U \)-dense in \( U \).

Let \( V \) be a cozero set of \( U \), then \( V \) is also a cozero set of \( X \) by \cite{IS} Lemma 3.9. Choose \( g \in C(\rho X) \) such that \( V = \text{coz}(g \circ \rho_X) \). Note that for any \( v \in \rho_X(V) = \text{coz}(g) \) there is \( y \in C \) such that \( \rho_X(y) = v \), hence \( g \circ \rho_X(y) = g(v) \neq 0 \). Thus for every finite subcollection \( U \cap C \) is open by \cite{IS} Lemma 1.4 (iv). In particular, \( V \cap U \) is nonempty, and moreover is \( \tau_U \)-dense.

If \( Q \subseteq \overline{U} \) is a nonempty \( \tau_U \)-open subset of \( U \), then it is relatively open (in the subspace topology \( \mathcal{T} \mid \overline{U} \)) by Lemma 1.4 (i). In particular, \( Q \cap U \) is nonempty, and moreover is \( \tau_U \)-dense.

Take a collection \( \{Q_\alpha\} \) of \( \tau_U \)-open subsets of \( U \) with the finite intersection property. Then for every finite subcollection \( \{Q_{\alpha_j}\}_{j=1}^n \), the intersection \( \bigcap_{j=1}^n (Q_{\alpha_j} \cap U) = (\bigcap_{j=1}^n Q_{\alpha_j}) \cap U \) is nonempty. It follows that \( \{Q_\alpha \cap U\} \) is a collection of \( \tau_U \)-open subsets of \( U \) with the finite intersection property. Since \( U \cap C \) is \( \tau_U \)-dense,

\[
\bigcap_{j=1}^n (Q_{\alpha_j} \cap U \cap C) = \left( \bigcap_{j=1}^n Q_{\alpha_j} \cap U \right) \cap C
\]

is nonempty for every such subcollection. Thus \( \{Q_\alpha \cap U \cap C\} \) is a collection of subsets of \( C \) with the finite intersection property. Since \( C \) is compact, \( \bigcap (Q_\alpha \cap U \cap C) \cap C \neq \emptyset \). As

\[
\overline{(Q_\alpha \cap U \cap C)} \cap C \subseteq Q_\alpha
\]

for each \( \alpha \), this implies that \( \bigcap \overline{Q_\alpha} \neq \emptyset \). Hence \( U \) is w-compact. \qed

**Lemma 1.8.** Let \( X \) be a topological space and suppose that every \( x \in X \) has a cozero set neighbourhood with w-compact closure. Then \( \rho X \) is locally compact.

**Proof.** If \( A \subseteq X \) is w-compact, then \( \rho X(A) \) is w-compact by \cite{IS} Proposition 3.10. But then since \( \rho X(A) \) is completely regular it is homeomorphic to its complete regularisation \( \rho(\rho X(A)) \), hence is compact by \cite{IS} Proposition 3.4.

For each point \( x \in X \), let \( U_x \) be a cozero set neighbourhood of \( x \) with \( \overline{U_x} \) w-compact. Then \( \rho X(U_x) \) is compact, and is a neighbourhood of \( \rho X(x) \) since \( \rho X(U_x) \) is open by Lemma 1.4 (iv). \qed

Now let \( A \) be a C*-algebra and Prim(\( A \)) the space of primitive ideals of \( A \) with the hull-kernel topology. We define Glimm(\( A \)) as the complete regularisation \( \rho \text{Prim}(A) \) of Prim(\( A \)), and denote by \( \rho_A : \text{Prim}(A) \to \text{Glimm}(A) \) the complete regularisation map.

Let \( p \in \text{Glimm}(A) \) and choose \( P \in \text{Prim}(A) \) with \( \rho_A(P) = p \). We associate to the point \( p \) the norm closed two sided ideal \( G_p \) of \( A \) given by

\[
G_p = \bigcap \{Q \in \text{Prim}(A) : Q \approx P\} = \bigcap \{Q \in \text{Prim}(A) : \rho_A(Q) = p\} = k([P])
\]

Note that since \( [P] \) is closed in \( \text{Prim}(A) \) and \( G_p = k([P]) \), each equivalence class in \( \text{Prim}(A) / \approx \) is of the form

\[
[P] = \text{hull}(k([P])) = \text{hull}(G_p),
\]

6
by the definition of the hull-kernel topology.

The collection \{G_p : p \in \text{Glimm}(A)\} are known as the \textit{Glimm ideals} of \(A\). Since the assignment \(p \mapsto G_p\) is injective, we will regard elements of \text{Glimm}(A) as either points of a topological space or as ideals of \(A\), depending on the context.

**Proposition 1.9.** Let \(A\) be a \(C^*\)-algebra such that one of the following conditions hold:

(i) \(\text{Prim}(A)\) is compact,

(ii) the complete regularisation map \(\rho_A : \text{Prim}(A) \to \text{Glimm}(A)\) is open, or

(iii) \(A\) is \(\sigma\)-unital and \(\text{Glimm}(A)\) is locally compact.

Then every \(P \in \text{Prim}(A)\) has a cozero set neighbourhood with \(w\)-compact closure. Hence for any \(C^*\)-algebra \(B\), the complete regularisation \(\rho(\text{Prim}(A) \times \text{Prim}(B))\) of \(\text{Prim}(A) \times \text{Prim}(B)\) is homeomorphic to the product space \(\text{Glimm}(A) \times \text{Glimm}(B)\).

Conversely, if \(\text{Glimm}(A)\) is not locally compact then there is \(P \in \text{Prim}(A)\) that does not have a cozero set neighbourhood with \(w\)-compact closure.

**Proof.** Note that if (i) holds then \(\text{Glimm}(A)\), being the continuous image of the compact space \(\text{Prim}(A)\), is compact. The proposition is then immediate by Lemma 1.7 with \(C = \text{Prim}(A)\) and \(K = \text{Glimm}(A)\).

In cases (ii) and (iii), take \(P \in \text{Prim}(A)\) with \(\rho_A(P) = x\) and let \(K'\) be a compact neighbourhood of \(x\) in \(\text{Glimm}(A)\). By [23, Theorem 2.1 and Proposition 2.5], \(K'\) is contained in a compact subset of \(\text{Glimm}(A)\) of the form

\[
K := \{G \in \text{Glimm}(A) : \|a + G\| \geq \alpha\} = \rho_A(\{P \in \text{Prim}(A) : \|a + P\| \geq \alpha\}),
\]

for some \(a \in A\) and \(\alpha > 0\), and the set \(\{P \in \text{Prim}(A) : \|a + P\| \geq \alpha\}\) is compact by [10, Proposition 3.3.7]. Then \(K\) is a compact neighbourhood of \(x\), and the conclusion thus follows from Lemma 1.7.

It then follows from Theorem 1.9 that if any of the conditions (i) to (iii) hold, \(\rho(\text{Prim}(A) \times Y) = \rho(\text{Prim}(A)) \times \rho(Y)\) for any space \(Y\). In particular, if \(B\) is a \(C^*\)-algebra then we have

\[
\rho(\text{Prim}(A) \times \text{Prim}(B)) = \rho(\text{Prim}(A)) \times \rho(\text{Prim}(B)) = \text{Glimm}(A) \times \text{Glimm}(B).
\]

On the other hand if \(\text{Glimm}(A)\) is not locally compact, then by Lemma 1.8 there is \(P \in \text{Prim}(A)\) for which no cozero set neighbourhood of \(P\) has \(w\)-compact closure.

\[\square\]

**Remark 1.10.** Suppose that \(A\) is a \(C^*\)-algebra such that \(\text{Prim}(A)\) does not satisfy condition (ii) of Theorem 1.9. Then there is a topological space \(Y\) for which \(\rho(\text{Prim}(A) \times Y) \neq \text{Glimm}(A) \times \rho(Y)\). It is not immediately evident whether this space \(Y\) can be chosen as \(\text{Prim}(B)\) for some \(C^*\)-algebra \(B\). Thus the partial converse in Proposition 1.9 does not preclude the possibility that \(\rho(\text{Prim}(A) \times \text{Prim}(B)) = \text{Glimm}(A) \times \text{Glimm}(B)\) for all \(C^*\)-algebras \(A\) and \(B\).

We will show in Example 1.11 however that \(\rho(X \times Y) \neq \rho(X) \times \rho(Y)\) is indeed possible when \(X\) and \(Y\) are primitive ideal spaces of \(C^*\)-algebras. Specifically, we construct a \(C^*\)-algebra \(A\) for which \(\rho(\text{Prim}(A) \times \text{Prim}(A)) \neq \text{Glimm}(A) \times \text{Glimm}(A)\).

**Remark 1.11.** Another natural topology on the complete regularisation \(\rho X\) of a space \(X\) is the quotient topology \(\tau_q\) induced by the complete regularisation map \(\rho_X\); that is, the strongest topology on \(\rho X\) for which \(\rho_X\) is continuous. Since \(\rho_X\) is continuous as a map into \((\rho X, \tau_{cr})\), it always holds that \(\tau_{cr} \leq \tau_q\). However, there is an example due to D.W.B. Somerset of a
space $X$ for which $\tau_{cr} \neq \tau_q$ on $\rho X$, and a C*-algebra $A$ with Prim($A$) homeomorphic to $X$ [21, Appendix].

In the case of the primitive ideal space of a C*-algebra $A$, there are many conditions known to ensure that $\tau_{cr} = \tau_q$ on Glimm($A$). A.J. Lazar has shown in [21, Theorem 2.6] that if $X$ is locally compact and $\sigma$-compact (that is, $X$ is a countable union of compact subsets) then $\tau_{cr} = \tau_q$ on $\rho X$. In particular this holds for the space Prim($A$) whenever $A$ is a $\sigma$-unital C*-algebra, or when Prim($A$) is compact. The two topologies also coincide if $\rho_A$ is either $\tau_{cr}$ or $\tau_q$-open [4, p. 351].

One particular consequence of these results is that if $A$ is a C*-algebra satisfying one of the conditions (i) to (iii) of Proposition 1.9, then necessarily $\tau_{cr} = \tau_q$ on Glimm($A$).

The topology $\tau_{cr}$ is in some sense the more natural topology on $\rho X$, since it is by definition the unique topology for which $\rho^*_{X} : C^b(\rho X) \to C^b(X)$ is a $*$-isomorphism. This allows us to apply the results of Ishii from [18] on the complete regularisation of a product space. Moreover, in the case of the primitive ideal space of a C*-algebra $A$, we will apply the Dauns-Hofmann identification (see [3] of $ZM(A)$ with $C^b$(Glimm($A$), $\tau_{cr}$) to determine $ZM(A \otimes_\alpha B)$ in terms of continuous functions on the Glimm spaces of $A$ and $B$.

2 The Glimm space of the minimal tensor product of C*-algebras

In this section we show that, as a topological space Glimm($A \otimes_\alpha B$) can be naturally identified with Glimm($A$) $\times$ Glimm($B$), when the latter space is considered as the complete regularisation of Prim($A$) $\times$ Prim($B$). We first discuss the canonical embedding of Prim($A$) $\times$ Prim($B$) in Prim($A \otimes_\alpha B$).

Let $\pi : A \to A'$ and $\sigma : B \to B'$ be $*$-homomorphisms of C*-algebras. Then there is a unique $*$-homomorphism $\pi \otimes \sigma : A \otimes_\alpha B \to A' \otimes_\alpha B'$, such that $(\pi \otimes \sigma)(a \otimes b) = \pi(a) \otimes \sigma(b)$ for all elementary tensors $a \otimes b \in A \otimes B$. In particular let $(I,J) \in \text{Id}'(A) \times \text{Id}'(B)$ and denote by $q_I : A \to A/I, q_J : B \to B/J$ the quotient homomorphisms. Then we have a $*$-homomorphism $q_I \otimes q_J : A \otimes_\alpha B \to (A/I) \otimes_\alpha (B/J)$.

We now define two natural maps $\Phi, \Delta : \text{Id}'(A) \times \text{Id}'(B) \to \text{Id}'(A \otimes_\alpha B)$ via

$$\Phi(I,J) = \ker(q_I \otimes q_J) \quad (1)$$

$$\Delta(I,J) = I \otimes_\alpha B + A \otimes_\alpha J.$$  \hspace{1cm} (2)

The following Proposition lists some known properties of the map $\Phi$.

**Proposition 2.1.** Let $A$ and $B$ be C*-algebras and $A \otimes_\alpha B$ their minimal C*-tensor product. Then the map $\Phi$ defined by (1) has the following properties:

(i) If $I, K \in \text{Id}'(A)$ and $J, L \in \text{Id}'(B)$ are such that $I \supseteq K$ and $J \supseteq L$ then $\Phi(I,J) \supseteq \Phi(K,L)$ [22, Lemma 2.2],

(ii) The restriction of $\Phi$ to Prim($A$) $\times$ Prim($B$) is a homeomorphism onto its image which is dense in Prim($A \otimes_\alpha B$) [31, lemme 16],

(iii) The restriction of $\Phi$ to Fac($A$) $\times$ Fac($B$) is a homeomorphism onto its image which is dense in Fac($A \otimes_\alpha B$) [22, Corollary 2.7],

(iv) For $I, J \in \text{Id}'(A) \times \text{Id}'(B)$, $\Phi(\text{hull}(I) \times \text{hull}(J))$ is dense in $\text{hull}(\Phi(I,J))$ [22, Corollary 2.3].
(v) For $I,J \in \text{Id}'(A) \times \text{Id}'(B)$, we have

$$
\Phi(I,J) = \bigcap \{ \Phi(P,Q) : (P,Q) \in \text{hull}(I) \times \text{hull}(J) \}
$$

[22, Remark 2.4]

Theorem 2.2 below identifies the complete regularisation of $\text{Prim}(A) \times \text{Prim}(B)$ with that of $\text{Prim}(A \otimes_\alpha B)$. As discussed in the remarks proceeding Lemma [11], we need to take into account the appropriate topology on the former space. Thus we will refer to $\rho(\text{Prim}(A) \times \text{Prim}(B))$ as $\text{Glimm}(A) \times \text{Glimm}(B)$, and $\rho(\text{Prim}(A)) \times \rho(\text{Prim}(B))$ as $\text{Glimm}(A) \times \text{Glimm}(B), \tau_p$.

**Theorem 2.2.** Let $A$ and $B$ be $C^*$-algebras, $A \otimes_\alpha B$ their minimal $C^*$-tensor product and denote by $\rho_A, \rho_B$ and $\rho_\alpha$ the complete regularisation maps of $\text{Prim}(A), \text{Prim}(B)$ and $\text{Prim}(A \otimes_\alpha B)$ respectively. Then there is a homeomorphism $\psi : \text{Glimm}(A \otimes_\alpha B) \to (\text{Glimm}(A) \times \text{Glimm}(B), \tau_\text{cr})$ given by

$$
(\psi \circ \rho_\alpha)(\Phi(P,Q)) = (\rho_A(P), \rho_B(Q)).
$$

It follows that $\psi$ defines a continuous bijection $\text{Glimm}(A \otimes_\alpha B) \to (\text{Glimm}(A) \times \text{Glimm}(B), \tau_p)$.

**Proof.** The map $\rho_A \times \rho_B : \text{Prim}(A) \times \text{Prim}(B) \to (\text{Glimm}(A) \times \text{Glimm}(B), \tau_\text{cr})$ is the complete regularisation map of $\text{Prim}(A) \times \text{Prim}(B)$ by Lemma [11]. For the remainder of the proof we will consider $\text{Glimm}(A) \times \text{Glimm}(B)$ with this topology (from which the second assertion will follow since $\tau_p$ is weaker).

By [22, Theorem 3.2], the map $(\rho_A \times \rho_B) \circ \Phi^{-1} : \Phi(\text{Prim}(A) \times \text{Prim}(B)) \to \text{Glimm}(A) \times \text{Glimm}(B)$ extends uniquely to a continuous map $(\rho_A \times \rho_B) : \text{Prim}(A \otimes_\alpha B) \to \text{Glimm}(A) \times \text{Glimm}(B)$.

Since $\text{Glimm}(A) \times \text{Glimm}(B)$ is completely regular, $(\rho_A \times \rho_B)$ induces a continuous (surjective) map $\psi : \text{Glimm}(A \otimes_\alpha B) \to \text{Glimm}(A) \times \text{Glimm}(B)$ with the property that $\psi \circ \rho_\alpha = (\rho_A \times \rho_B)$ [29, Corollary 1.8].

$$
\xymatrix{ 
\text{Prim}(A) \times \text{Prim}(B) \ar[r]^\Phi \ar[d]_{\rho_A \times \rho_B} & \text{Prim}(A \otimes_\alpha B) \ar[d]_{\rho_\alpha} \\
\text{Glimm}(A) \times \text{Glimm}(B) \ar[r]_\psi & \text{Glimm}(A \otimes_\alpha B)
}
$$

To show that $\psi$ is in fact a homeomorphism, it suffices to show that the $\ast$-homomorphism $\psi^* : C^b(\text{Glimm}(A) \times \text{Glimm}(B)) \to C^b(\text{Glimm}(A \otimes_\alpha B))$, $\psi^*(f) = f \circ \psi$ is surjective [12, Theorem 10.3 (b)].

To this end, let $f \in C^b(\text{Glimm}(A \otimes_\alpha B))$, so that $f \circ \rho_\alpha \in C^b(\text{Prim}(A) \times \text{Prim}(B))$ and hence $f \circ \rho_\alpha \circ \Phi \in C^b(\text{Prim}(A) \times \text{Prim}(B))$. Denote by $g \in C^b(\text{Glimm}(A) \times \text{Glimm}(B))$ the unique function such that $g \circ (\rho_A \times \rho_B) = f \circ \rho_\alpha \circ \Phi$. Then $f \circ \rho_\alpha$ and $g \circ (\rho_A \times \rho_B)$ are both continuous extensions of $g \circ (\rho_A \times \rho_B) \circ (\Phi^{-1})_\text{Prim}(A \otimes_\alpha B)$, hence must agree by [22, Theorem 3.2].

Take $m \in \text{Glimm}(A \otimes_\alpha B)$ and $M \in \text{Prim}(A \otimes_\alpha B)$ such that $\rho_\alpha(M) = m$. Then

$$
\psi^*(g)(m) = (g \circ \psi)(m) = (g \circ (\rho_\alpha))(M) = (g \circ (\rho_A \times \rho_B))(M) = (f \circ \rho_\alpha)(M) = f(m)
$$

It follows that $\psi^*(g) = f$, hence $\psi^*$ is surjective. \qed
Note that Theorem 2.2 shows that \( \rho_s \circ \Phi \) is surjective. In particular, given any \( M \in \text{Prim}(A \otimes \alpha B) \) there exist \( (P, Q) \in \text{Prim}(A) \times \text{Prim}(B) \) such that \( M \approx \Phi(P, Q) \).

**Corollary 2.3.** Let \( A \) and \( B \) be \( C^* \)-algebras such that either \( A \) or \( B \) satisfies one of the conditions (i)-(iii) of Proposition 1.9. Then \( \tau_{cr} = \tau_p \) on \( \text{Glimm}(A) \times \text{Glimm}(B) \), and hence \( \text{Glimm}(A \otimes \alpha B) \) is homeomorphic to \( (\text{Glimm}(A) \times \text{Glimm}(B), \tau_p) \) via the map \( \psi \) of Theorem 2.2.

**Proof.** Immediate from Proposition 1.9 and Theorem 2.2. \( \square \)

### 3 The central multipliers of \( A \otimes \alpha B \)

In this section we apply Theorem 2.2 to determine the centre of the multiplier algebra of \( A \otimes \alpha B \) in terms of the topological space \( (\text{Glimm}(A) \times \text{Glimm}(B), \tau_{cr}) \). We show in Theorem 3.3 that \( ZM(A \otimes \alpha B) \) is \( * \)-isomorphic to the \( C^* \)-algebra of continuous functions on the Stone-Cech compactification of \( (\text{Glimm}(A) \times \text{Glimm}(B), \tau_{cr}) \). Further in Theorem 3.6 we give necessary and sufficient conditions on this space for which \( ZM(A) \otimes ZM(B) \approx ZM(A \otimes \alpha B) \).

The embedding of \( (M(A) \otimes \alpha M(B)) \subset M(A \otimes \alpha B) \) is discussed in [1], we include a proof in Lemma 3.1 below for completeness. It is shown in [15] Corollary 1 that for \( C^* \)-algebras \( C \) and \( D \) we have \( Z(C \otimes \alpha D) = Z(C) \otimes Z(D) \) (where \( Z(C) \otimes Z(D) \) is the unique \( C^* \)-completion of the algebraic tensor product \( Z(C) \otimes Z(D) \) by nuclearity). In particular it follows that for any \( C^* \)-algebras \( A \) and \( B \) we may identify \( Z(M(A) \otimes \alpha M(B)) = ZM(A) \otimes ZM(B) \). Thus in this section we are concerned with relating the centre of the larger algebra \( M(A \otimes \alpha B) \) with that of \( M(A) \otimes \alpha M(B) \).

Suppose that \( C \) is a \( C^* \)-algebra and \( z \in M(C) \) such that \( zc = cz \) for all \( c \in C \), and take \( m \in M(C) \). Then

\[
(zm)c = z(mc) = (mc)z = m(zc) = m(z)c = (zm)c,
\]

and similarly \( c(zm) = c(mz) \) for all \( c \in C \). Thus \( zm = mz \) and so we may identify \( \text{ZM}(C) = \{ z \in M(C) : zc = cz \text{ for all } c \in C \} \).

Recall that an ideal \( I \) of a \( C^* \)-algebra \( C \) is said to be essential in \( C \) if given any nonzero ideal \( J \) of \( C \), \( J \cap I \neq \{0\} \). Equivalently for any \( c \in C \), \( cI = Ic = \{0\} \) implies \( c = 0 \).

**Lemma 3.1.** There is a canonical embedding \( \Theta : M(A) \otimes \alpha M(B) \rightarrow M(A \otimes \alpha B) \) such that \( \Theta(x \otimes y)(a \otimes b) = xa \otimes yb \) and \( (a \otimes b)(\Theta(x \otimes y)) = ax \otimes by \)

for all \( a \in A, b \in B, x \in M(A), y \in M(B) \). Moreover, \( \Theta(ZM(A) \otimes ZM(B)) \subseteq ZM(A \otimes \alpha B) \).

**Proof.** Clearly \( M(A) \otimes \alpha M(B) \) contains \( A \otimes \alpha B \) as a two-sided ideal. Suppose \( J \) is a nonzero ideal of \( M(A) \otimes \alpha M(B) \). Then by [2] Proposition 4.5, \( J \) contains a nonzero elementary tensor \( x \otimes y \) where \( x \in M(A), y \in M(B) \). Since \( A \) is essential in \( M(A) \), there is \( a \in A \) with either \( ax \neq 0 \) or \( xa \neq 0 \). Suppose w.l.o.g. that \( xa \neq 0 \), so that \( \|[(xa)^*xa]\| = \|xa\|^2 \neq 0 \). Setting \( a' = (xa)^* \), we then have an element \( a' \in A \) with \( a'xa \neq 0 \). Similarly there are \( b, b' \in B \) with \( b'yb \neq 0 \). It follows that

\[
a'xa \otimes b'yb = (a' \otimes b')(x \otimes y)(a \otimes b)
\]

is a nonzero element of \( J \cap (A \otimes \alpha B) \). Hence \( A \otimes \alpha B \) is essential in \( M(A) \otimes \alpha M(B) \).

By [3] Proposition 3.7 (i) and (ii), there is a unique \( * \)-homomorphism \( \Theta : M(A) \otimes \alpha M(B) \rightarrow M(A \otimes \alpha B) \) extending the canonical inclusion of \( A \otimes \alpha B \) into \( M(A \otimes \alpha B) \), which is injective.
since $A \otimes_{\alpha} B$ is essential in $M(A) \otimes_{\alpha} M(B)$. For elementary tensors $x \otimes y \in M(A) \otimes_{\alpha} M(B)$ and $a \otimes b \in A \otimes_{\alpha} B$ we have

$$(\Theta(x \otimes y))(a \otimes b) = \Theta(x \otimes y)\Theta(a \otimes b) = \Theta(xa \otimes yb) = xa \otimes yb,$$

(since $\Theta$ is the identity on $A \otimes_{\alpha} B$), and similarly $(a \otimes b)(\Theta(x \otimes y)) = ax \otimes by$.

For elementary tensors $z_1 \otimes z_2 \in ZM(A) \otimes ZM(B)$ and $a \otimes b \in A \otimes_{\alpha} B$ we have

$$\Theta(z_1 \otimes z_2)(a \otimes b) = z_1a \otimes z_2b = az_1 \otimes bz_2 = (a \otimes b)\Theta(z_1 \otimes z_2),$$

from which it follows that for any $z \in ZM(A) \otimes ZM(B)$ and $c \in A \otimes_{\alpha} B$, $\Theta(z)c = c\Theta(z)$. Hence by \[3\] we see that $\Theta(ZM(A) \otimes ZM(B)) \subseteq ZM(A \otimes_{\alpha} B)$. \hfill \Box

We remark that it was shown in \[1\] Theorem 3.8 that if $A$ is $\sigma$-unital and non-unital, and $B$ is infinite dimensional, then $\Theta$ is not surjective. In what follows, we will suppress mention of $\Theta$ and simply consider $M(A) \otimes M(B) \subseteq M(A \otimes_{\alpha} B)$.

The relationship between the central multipliers of a C*-algebra and its Glimm space was established by Dauns and Hofmann as a corollary to their work on sectional representation for C*-algebras:

**Corollary 3.2.** \[8, III Corollary 8.16\] For any C*-algebra $A$, there is a homeomorphism of $\text{Prim}(ZM(A))$ onto $\beta\text{Glimm}(A)$, and hence a *-isomorphism $\mu_A : C(\beta\text{Glimm}(A)) \rightarrow ZM(A)$. Moreover, $\mu_A$ satisfies

$$\mu_A(f)a - f(p)a \in G_p , \text{ for all } f \in C(\beta\text{Glimm}(A)), p \in \text{Glimm}(A), a \in A,$$

where $G_p = \bigcap \{P \in \text{Prim}(A) : \rho_A(P) = p\}$ is the Glimm ideal of $A$ corresponding to $p$.

Applying this identification to $A \otimes_{\alpha} B$ together with the homeomorphism $\psi$ of Theorem 2.2 allows us to determine $ZM(A \otimes_{\alpha} B)$ in terms of Glimm($A$) and Glimm($B$):

**Theorem 3.3.** Let $A$ and $B$ be C*-algebras and denote by $\psi$ the homeomorphism of Theorem 2.2. For each point $p \in \text{Glimm}(A \otimes_{\alpha} B)$ let $G_p$ denote the Glimm ideal of $A \otimes_{\alpha} B$ corresponding to $p$. Then there is a canonical *-isomorphism $\Theta_\alpha : C(\beta(\text{Glimm}(A) \times \text{Glimm}(B), \tau_{cr})) \rightarrow ZM(A \otimes_{\alpha} B)$ with the property that

$$\Theta_\alpha(f)c - (f \circ \psi)(p)c \in G_p , \text{ for all } f \in C(\beta(\text{Glimm}(A) \times \text{Glimm}(B), \tau_{cr})), p \in \text{Glimm}(A \otimes_{\alpha} B) \text{ and } c \in A \otimes_{\alpha} B.$$

**Proof.** Since $\psi$ is a homeomorphism the induced map $\psi^*$ is a *-isomorphism. Denote by $\mu_\alpha$ the *-isomorphism of Corollary 3.2 applied to $A \otimes_{\alpha} B$, and by $\Theta_\alpha$ the composition of the *-isomorphisms

$$C(\beta(\text{Glimm}(A) \times \text{Glimm}(B), \tau_{cr})) \xrightarrow{\psi^*} C(\beta(\text{Glimm}(A \otimes_{\alpha} B)) \xrightarrow{\mu_\alpha} ZM(A \otimes_{\alpha} B).$$

Then $\Theta_\alpha$ clearly has the required properties since $\mu_\alpha$ does. \hfill \Box

On the other hand, applying the identification of Corollary 3.2 to $A$ and $B$ separately gives *-isomorphisms

$$C(\beta\text{Glimm}(A) \times \beta\text{Glimm}(B))) \xrightarrow{\nu} C(\beta\text{Glimm}(A)) \otimes C(\beta\text{Glimm}(B)) \xrightarrow{\mu_{A \otimes_{\alpha} B}} ZM(A) \otimes ZM(B),$$
where \( \nu \) is the canonical identification satisfying \( \nu^{-1}(f \otimes g)(x, y) = f(x)g(y) \) for all elementary tensors \( f \otimes g \) and \( (x, y) \in \beta \mathrm{Glimm}(A) \times \beta \mathrm{Glimm}(B) \).

Let \( X \) and \( Y \) be completely regular spaces. Then since the product \( \beta X \times \beta Y \) is compact, the universal property of the Stone-Cech compactification [12, Theorem 6.5 (I)] ensures that the inclusion \( \iota : X \times Y \rightarrow \beta X \times \beta Y \) has a continuous extension to \( \iota^\beta : \beta(X \times Y) \rightarrow \beta X \times \beta Y \). Moreover, since \( \iota \) has dense range, compactness of \( \beta(X \times Y) \) implies that \( \iota^\beta \) is necessarily surjective.

Considering \( \mathrm{Glimm}(A) \subseteq \beta \mathrm{Glimm}(A) \) and \( \mathrm{Glimm}(B) \subseteq \beta \mathrm{Glimm}(B) \), Lemma [13] gives a continuous map
\[
\phi : (\mathrm{Glimm}(A) \times \mathrm{Glimm}(B), \tau_{cr}) \rightarrow (\mathrm{Glimm}(A) \times \mathrm{Glimm}(B), \tau_p) \subseteq \beta \mathrm{Glimm}(A) \times \beta \mathrm{Glimm}(B),
\]
and \( \phi \) is a homeomorphism onto its range if and only if \( \tau_p = \tau_{cr} \) on \( \mathrm{Glimm}(A) \times \mathrm{Glimm}(B) \). Again by the universal property of the Stone-Cech compactification, \( \phi \) extends to a continuous surjection
\[
\phi^\beta : \beta(\mathrm{Glimm}(A) \times \mathrm{Glimm}(B), \tau_{cr}) \rightarrow \beta \mathrm{Glimm}(A) \times \beta \mathrm{Glimm}(B).
\]
Dual to this map is an injective *-homomorphism
\[
(\phi^\beta)^* : C(\beta \mathrm{Glimm}(A) \times \beta \mathrm{Glimm}(B)) \rightarrow C(\beta(\mathrm{Glimm}(A) \times \mathrm{Glimm}(B), \tau_{cr})),
\]
sending \( f \mapsto f \circ \phi^\beta \).

The situation is summarised in the following diagram:

\[
\begin{array}{ccc}
C(\beta \mathrm{Glimm}(A) \times \beta \mathrm{Glimm}(B)) & \overset{(\phi^\beta)^*}{\longrightarrow} & C(\beta(\mathrm{Glimm}(A) \times \mathrm{Glimm}(B), \tau_{cr})) \\
\nu \downarrow & & \psi^* \\
C(\beta \mathrm{Glimm}(A)) \otimes C(\beta \mathrm{Glimm}(B)) & \overset{\mu_A \otimes \mu_B}{\longrightarrow} & C(\beta \mathrm{Glimm}(A \otimes_\alpha B)) \\
\end{array}
\]

\[
\begin{array}{ccc}
ZM(A) \otimes ZM(B) & \overset{\mu_\alpha}{\longrightarrow} & ZM(A \otimes_\alpha B) \\
\end{array}
\]

**Corollary 3.4.** For any \( C^* \)-algebras \( A \) and \( B \), \( ZM(A) \otimes ZM(B) = ZM(A \otimes_\alpha B) \) if and only if the canonical map
\[
\phi^\beta : \beta(\mathrm{Glimm}(A) \times \mathrm{Glimm}(B), \tau_{cr}) \rightarrow \beta \mathrm{Glimm}(A) \times \beta \mathrm{Glimm}(B)
\]
is injective. Moreover, when this occurs we have \( \tau_p = \tau_{cr} \) on \( \mathrm{Glimm}(A) \times \mathrm{Glimm}(B) \).

**Proof.** We first show that the preceding diagram commutes; that is, for any \( h \in C(\beta \mathrm{Glimm}(A) \times \beta \mathrm{Glimm}(B)) \) the multipliers \( z_1 \) and \( z_2 \) of \( A \otimes_\alpha B \) given by
\[
z_1 = (\mu_\alpha \circ \psi^* \circ (\phi^\beta)^*)(h), \quad z_2 = ((\mu_A \otimes \mu_B) \circ \nu)(h)
\]
are equal. By linearity and continuity it suffices to check equality for functions of the form \( h = \nu^{-1}(f \otimes g) \), where \( f \in C(\beta \mathrm{Glimm}(A)) \), \( g \in C(\beta \mathrm{Glimm}(B)) \).

Consider an elementary tensor \( a \otimes b \in A \otimes_\alpha B \), and a pair \((P, Q) \in \mathrm{Prim}(A) \times \mathrm{Prim}(B)\). We will show that \((z_1 - z_2)(a \otimes b) + \Phi(P, Q) = 0\). Set \((p, q) = (\rho_A \times \rho_B)(P, Q)\), and note that by Theorem 2.2 \((\psi \circ \rho_\alpha \circ \Phi)(P, Q) = (p, q)\). In particular it follows that
\[
(\psi^* \circ (\phi^\beta)^*)(h)(\rho_\alpha(\Phi(P, Q))) = (\phi^\beta)^*(h)(\psi \circ \rho_\alpha \circ \Phi(P, Q)) = h(p, q) = f(p)g(q).
\]
Firstly by applying the $*$-isomorphism of Corollary 3.2 to the element $(\psi^* \circ (\phi^\beta)^*)(h)$ of $C(\beta(\text{Glimm}(A \otimes_\alpha B)))$ see that

$$z_1(a \otimes b) + \Phi(P,Q) = (\mu_a \circ \psi^* \circ (\phi^\beta)^*)(h)(a \otimes b) + \Phi(P,Q)$$

$$= f(p)g(q)(a \otimes b) + \Phi(P,Q)$$

$$= f(p)a \otimes g(q)b + \Phi(P,Q).$$

On the other hand, applying Corollary 3.2 to $f \in C(\beta\text{Glimm}(A))$ gives $\mu_A(f)a - f(p)a \in P$, so that

$$(\mu_A(f)a - f(p)a) \otimes g(q)b = \mu_A(f)a \otimes g(q)b - f(p)a \otimes g(q)b \in \ker(q_P \otimes q_Q) = \Phi(P,Q),$$

from which it follows that $\mu_A(f)a \otimes g(q)b + \Phi(P,Q) = f(p)a \otimes g(q)b + \Phi(P,Q)$. A similar argument applied to $B$ gives $\mu_A(f)a \otimes g(q)b + \Phi(P,Q) = \mu_A(f)a \otimes \mu_B(g)b + \Phi(P,Q)$, and we conclude that

$$z_1(a \otimes b) + \Phi(P,Q) = \mu_A(f)a \otimes \mu_B(g)b + \Phi(P,Q)$$

$$= (\mu_A \otimes \mu_B)(f \otimes g)(a \otimes b) + \Phi(P,Q)$$

$$= (\mu_A \otimes \mu_B \circ \nu)(h)(a \otimes b) + \Phi(P,Q)$$

$$= z_2(a \otimes b) + \Phi(P,Q).$$

In particular $(z_1 - z_2)(a \otimes b) \in \Phi(P,Q)$ for all $a \in A, b \in B$ and $(P,Q) \in \text{Prim}(A) \times \text{Prim}(B)$. Since $\bigcap \{\Phi(P,Q) : (P,Q) \in \text{Prim}(A) \times \text{Prim}(B)\} = \{0\}$, it follows that $(z_1 - z_2)(a \otimes b) = 0$ for all $a \in A, b \in B$. Thus $(z_1 - z_2)(A \otimes_\alpha B) = \{0\}$, that is, $z_1 = z_2$.

Since the vertical arrows of the diagram all describe $*$-isomorphisms, the inclusion $ZM(A) \otimes ZM(B) \subseteq ZM(A \otimes_\alpha B)$ will be surjective if and only if $(\phi^\beta)^*$ is. By [12 Theorem 10.3], $(\phi^\beta)^*$ is surjective if and only if $\phi^\beta$ is a homeomorphism.

But then $\phi^\beta$, being a continuous surjection from a compact Hausdorff space to a Hausdorff space, is thus a homeomorphism if and only if it is injective.

The final assertion follows from the fact that $\phi^\beta$ is the identity on $\text{Glimm}(A) \times \text{Glimm}(B)$. \[\Box\]

Let $X$ and $Y$ be completely regular spaces. The question of establishing conditions on $X,Y$ and $X \times Y$ for which canonical surjection $\iota^\beta : \beta(X \times Y) \to \beta X \times \beta Y$ is injective (and hence a homeomorphism) has been studied by several authors. If either $X$ or $Y$ is finite, then this is trivially true. The most well-known characterisation in the infinite case is due to Glicksberg [13].

**Definition 3.5.** Let $X$ be a completely regular space. We say that $X$ is pseudocompact if every $f \in C(X)$ is bounded.

Glicksberg’s Theorem [13 Theorem 1] states that, for infinite completely regular spaces $X$ and $Y$ the canonical map $\beta(X \times Y) \to \beta X \times \beta Y$ is a homeomorphism if and only if $X \times Y$ is pseudocompact.

**Theorem 3.6.** For any $C^*$-algebras $A$ and $B$, $ZM(A) \otimes ZM(B) = ZM(A \otimes_\alpha B)$ if and only if one of the following conditions hold:

(i) $\text{Glimm}(A)$ or $\text{Glimm}(B)$ is finite, or

(ii) $\tau_p = \tau_{cr}$ on $\text{Glimm}(A) \times \text{Glimm}(B)$ and $\text{Glimm}(A) \times \text{Glimm}(B)$ is pseudocompact.
Proof. If (i) holds, w.l.o.g. Glimm(B) is finite, hence discrete and compact. In particular $\rho_B$ is an open map, so by Proposition 1.3(ii), $\tau p = \tau_{cr}$ on $\text{Glimm}(A) \times \text{Glimm}(B)$. Then by [29 Proposition 8.2] the map $\phi^3$ is a homeomorphism and hence $ZM(A) \otimes ZM(B) = ZM(A \otimes_A B)$ by Corollary 3.4.

If Glimm(A) and Glimm(B) are infinite then by [13 Theorem 1], $\beta ((\text{Glimm}(A) \times \text{Glimm}(B), \tau_p))$ is canonically homeomorphic to $\beta \text{Glimm}(A) \times \beta \text{Glimm}(B)$ if and only if $(\text{Glimm}(A) \times \text{Glimm}(B), \tau_p)$ is pseudocompact. Hence in the infinite case, Corollary 5.4 gives $ZM(A) \otimes ZM(B) = ZM(A \otimes_A B)$ if and only if (ii) holds. \hfill \Box

Clearly if $M(A) \otimes_A M(B) = M(A \otimes_A B)$ then $ZM(A) \otimes ZM(B) = Z(M(A) \otimes_A M(B)) = ZM(A \otimes_A B)$. We will show in Example 6.2 that the converse is not true; we construct C*-algebras A and B such that $ZM(A) \otimes ZM(B) = Z(M(A \otimes_A B))$, but $M(A) \otimes_A M(B) \subsetneq M(A \otimes_A B)$.

Remark 3.7. It is easily seen that the continuous image of a pseudocompact space is pseudocompact. In particular if X and Y are completely regular spaces such that $X \times Y$ is pseudocompact, then since the projection maps $\pi_X$ and $\pi_Y$ are continuous we have necessarily that both X and Y are pseudocompact.

In the other direction, it is not always true that a product of pseudocompact spaces is pseudocompact; see [12 Example 9.15] for a counterexample. However, for a product of pseudocompact spaces X and Y, one of which is also locally compact, then $X \times Y$ is pseudocompact by [29 Proposition 8.21].

Remark 3.8. In the particular case that A and B are unital, then Prim(A) and Prim(B) are compact so that $\tau_{cr} = \tau_p$ on $\text{Glimm}(A) \times \text{Glimm}(B)$ by Proposition 1.3(i). Moreover, $\text{Glimm}(A) \times \text{Glimm}(B)$ is compact, hence pseudocompact. Thus Theorem 3.6 implies the Haydon-Wassermann result [15 Corollary 1] in the unital case.

4 Glimm ideals

We now turn to the question of determining the Glimm ideals of $A \otimes_A B$ in terms of those of $A$ and $B$. More precisely Theorem 1.8 shows that, when the Glimm spaces considered as sets of ideals of $A, B$ and $A \otimes_A B$, then the map $\Delta$ of Equation 2 satisfies $\Delta = \psi^{-1}$.

We define a new map $\Psi : \text{Id}'(A \otimes_A B) \rightarrow \text{Id}'(A) \times \text{Id}'(B)$, which is a left inverse of the map $\Phi$ of Equation 1. For $M \in \text{Id}'(A \otimes_A B)$ we define closed two-sided ideals $M^A$ and $M^B$ of A and B respectively via

$$M^A = \{ a \in A : a \otimes B \subseteq M \}, M^B = \{ b \in B : A \otimes b \subseteq M \}. \quad (4)$$

The assignment $\Psi(M) = (M^A, M^B)$ gives a map $\Psi : \text{Id}'(A \otimes_A B) \rightarrow \text{Id}'(A) \times \text{Id}'(B)$.

Proposition 4.1. Let A and B be C*-algebras and $A \otimes_A B$ their minimal C*-tensor product. Then the map $\Psi : \text{Id}'(A \otimes_A B) \rightarrow \text{Id}'(A) \times \text{Id}'(B)$ satisfies the following properties:

(i) $\Psi \circ \Phi$ is the identity on $\text{Id}'(A) \times \text{Id}'(B)$,

(ii) $\Psi(\text{Fac}(A \otimes_A B)) = \text{Fac}(A) \times \text{Fac}(B)$,

(iii) The restriction of $\Psi$ to $\text{Fac}(A \otimes_A B)$ is continuous in the hull-kernel topologies,

(iv) For any $M \in \text{Fac}(A \otimes_A B)$, the inclusion $M \subseteq \Phi \circ \Psi(M)$ holds.
Proof. (i) and (iii) are shown in the proof of [22, Theorem 2.6]. To prove (iii), [14, Proposition 1] shows that $\Psi(\text{Fac}(A \otimes \alpha B) \subseteq \text{Fac}(A) \times \text{Fac}(B)$. Surjectivity then follows from Proposition 2.1(iii) and part (ii).

As for (iv), it is shown in [7, Lemma 2.13(iv)] that for any prime $M$ of $A \otimes \alpha B$ we have $M \subseteq \Phi \circ \Psi(M)$. But then [6, Proposition II.6.1.11] shows that every factorial ideal of a $C^*$-algebra is prime, from which (iv) follows.

We remark that Proposition 4.1(ii) shows that $\Psi$ maps $\text{Prim}(A \otimes \alpha B)$ to $\text{Fac}(A) \times \text{Fac}(B)$. It is not known in general whether $\Psi$ maps $\text{Prim}(A \otimes \alpha B)$ onto $\text{Prim}(A) \times \text{Prim}(B)$. For this reason, we will need to use an alternative construction of the space of Glimm ideals of a $C^*$-algebra, which was first considered by Kaniuth in [19].

It is shown in [19, Section 2] how for any $C^*$-algebra $A$, Glimm($A$) can be constructed as $\rho(\text{Fac}(A))$. For $I, J \in \text{Fac}(A)$ we write $I \approx_f J$ if $f(I) = f(J)$ for all $f \in C^b(\text{Fac}(A))$, and denote by $[I]_f$ the equivalence class of $I$ in $\text{Fac}(A)$.

**Proposition 4.2.** Let $A$ be a $C^*$-algebra. Then the relation $\approx_f$ on $\text{Fac}(A)$ has the following properties:

(i) For $I \in \text{Fac}(A)$ and $P \in \text{hull}(I)$ we have $[I]_f \cap \text{Prim}(A) = [P]$ and $k([I]_f) = k([P])$,

(ii) $\text{Fac}(A)/\approx_f$ is homeomorphic to $\text{Prim}(A)/\approx$ via the map $[I]_f \mapsto [P]$, where $P \in \text{hull}(I)$, when both spaces are considered with the quotient topology,

(iii) Each Glimm ideal of $A$ is of the form $G_I = k([I]_f)$ for some $I \in \text{Fac}(A)$.

(iv) The equivalence classes of $\approx_f$ satisfy $[I]_f = \text{hull}_f(G_I)$.

**Proof.** Parts (i) and (ii) are shown in [19, Lemma 2.2]. (iii) is immediate from (i).

To prove (iv) take $I \in \text{Fac}(A)$. It follows from the definition of $\approx_f$ that the equivalence class $[I]_f$ is a closed subset of $\text{Fac}(A)$. By the definition of the hull-kernel topology and by part (iii) we then have $[I]_f = \text{hull}_f(k([I]_f)) = \text{hull}_f(G_I)$.

As a consequence of Proposition 4.2(ii), we shall consider the set of equivalence classes $\text{Fac}(A)/\approx_f$ as Glimm($A$), and denote by $\rho'_A : \text{Fac}(A) \rightarrow \text{Glimm}(A)$ the corresponding quotient map. Moreover, we may unambiguously speak of the quotient topology $\tau_q$ on Glimm($A$) as the strongest topology on this space for for which either $\rho_A$ or $\rho'_A$ is continuous.

For $C^*$-algebras $A$ and $B$ and two pairs of ideals $(P, Q), (R, S) \in \text{Fac}(A) \times \text{Fac}(B)$, we will write $(P, Q) \approx_f (R, S)$ when $g(P, Q) = g(R, S)$ for all $g \in C^b(\text{Fac}(A) \times \text{Fac}(B))$. By Lemma 4.1 this is equivalent to saying $P \approx_f R$ and $Q \approx_f S$.

Lemmas 4.3, 4.4 and Proposition 4.5 below relate equivalence classes of the relation $\approx_f$ in $\text{Fac}(A) \times \text{Fac}(B)$ with those in $\text{Fac}(A \otimes \alpha B)$, via the maps $\Phi$ and $\Psi$.

**Lemma 4.3.** Let $(I, J) \in \text{Fac}(A) \times \text{Fac}(B)$, and denote by $G_{I,J} = k([I]_f)$ and $G_I = k([I]_f)$ the corresponding Glimm ideals of $A$ and $B$ respectively. Then $G_{\Phi(I,J)} := k([\Phi(I,J)]_f)$, the Glimm ideal of $A \otimes \alpha B$ corresponding to $[\Phi(I,J)]_f$, satisfies $G_{\Phi(I,J)} \subseteq \Phi(G_I, G_J)$. 

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Proof. The fact that $\Phi(I,J) \in \text{Fac}(A \otimes \alpha B)$ follows from Proposition 2.1(iii). Taking $(P,Q) \in \text{hull}(I) \times \text{hull}(J)$ we have $\Phi(P,Q) \in \text{hull}(\Phi(I,J))$ by Proposition 2.1(i). In this case Proposition 4.2(i) gives
\[ k([I]_f) = k([P])_f = k([Q])_f = k((\Phi(I,J))_f) = k((\Phi(P,Q)))_f, \]
and so we may replace $(I,J)$ with $(P,Q)$.

Note that if $(R,S) \in \text{Prim}(A) \times \text{Prim}(B)$ such that $(R,S) \approx (P,Q)$, and $f \in C^b(\text{Prim}(A \otimes \alpha B)$ then $f \circ \Phi \in C^b(\text{Prim}(A) \times \text{Prim}(B))$, hence $f(\Phi(R,S)) = f(\Phi(P,Q))$, so that $\Phi(R,S) \approx \Phi(P,Q)$. It then follows from Proposition 2.1(v) that
\[ \Phi(G_I,G_J) = \Phi(k([P]),k([Q])) = \bigcap \{ \Phi(R,S) : (R,S) \in \text{hull}(k(P)) \times \text{hull}(k(Q)) \} \]
\[ = \bigcap \{ \Phi(R,S) : (R,S) \approx (P,Q) \} \]
\[ \supseteq \bigcap \{ M \in \text{Prim}(A \otimes \alpha B) : M \approx \Phi(P,Q) \} \]
\[ = k((\Phi(P,Q)))_f = G_{\Phi(I,J)} \]

\[ \square \]

Lemma 4.4. For any $M \in \text{Fac}(A \otimes \alpha B)$, $\Phi \circ \Psi(M) \in \text{Fac}(A \otimes \alpha B)$ and $M \approx_f \Phi \circ \Psi(M)$.

Proof. The fact that $\Phi \circ \Psi(M) \in \text{Fac}(A \otimes \alpha B)$ follows from Propositions 2.1(iii) and 4.1(ii). By Proposition 4.1(iv), we have $M \subseteq \Phi \circ \Psi(M)$. Hence $\Phi \circ \Psi(M) \in \text{hull}(M) = \{\overline{M}\}$, so that $M \approx_f \Phi \circ \Psi(M)$.

\[ \square \]

Note that the proof of Proposition 4.5 below requires that we base the definition of Glimm ideals on the complete regularisation of the space of factorial ideals (since $\Psi$ maps factorial ideals to factorial ideals).

Proposition 4.5. Let $(I,J) \in \text{Fac}(A) \times \text{Fac}(B)$, $M \in \text{Fac}(A \otimes \alpha B)$ and denote by $(M^A,M^B) = \Psi(M)$. Then $M \approx_f \Phi(I,J)$ if and only if $(M^A,M^B) \approx_f (I,J)$. Hence with $G_I,G_J$ and $G_{\Phi(I,J)}$ as defined in Lemma 4.3 we have
\[ M \in \text{hull}_f(G_{\Phi(I,J)}) \text{ if and only if } (M^A,M^B) \in \text{hull}_f(G_I) \times \text{hull}_f(G_J) \]

Proof. Suppose $M \approx_f \Phi(I,J)$, and take $g \in C^b(\text{Fac}(A) \times \text{Fac}(B))$. Using Proposition 4.1(ii) and (iii), we have $g \circ \Psi \in C^b(\text{Fac}(A \otimes \alpha B))$. Hence
\[ g(M^A,M^B) = (g \circ \Psi)(M) = (g \circ \Psi)(\Phi(I,J)) = g(I,J), \]

since $\Psi \circ \Phi$ if the identity on $\text{Fac}(A) \times \text{Fac}(B)$ by Proposition 4.1(i). It follows that $(M^A,M^B) \approx_f (I,J)$.

Since $G_{\Phi(I,J)} = k((\Phi(I,J))_f)$, Proposition 4.2(iv) shows that $[\Phi(I,J)]_f = \text{hull}_f(G_{\Phi(I,J)})$. Similarly $[I]_f = \text{hull}_f(G_I)$ and $[J]_f = \text{hull}_f(G_J)$.

To prove the converse, suppose that $(M^A,M^B) \approx_f (I,J)$. Then by Lemma 4.3, $M^A \approx_f I$ and $M^B \approx_f J$, so that $M^A \supseteq G_I$ and $M^B \supseteq G_J$. Together with Lemma 4.3 this gives the inclusion
\[ \Phi \circ \Psi(M) = \Phi(M^A,M^B) \supseteq \Phi(G_I,G_J) \supseteq G_{\Phi(I,J)}, \]

and since by Proposition 2.1(iii) $\Phi \circ \Psi(M) \in \text{Fac}(A \otimes \alpha B)$, it follows from Proposition 4.2(iv) that $\Phi \circ \Psi(M) \approx_f \Phi(I,J)$. Then by Lemma 4.3 we have $M \approx_f \Phi(I,J)$.

The final assertion of the statement follows from Proposition 4.2(iv).

\[ \square \]
In what follows we make use of the map $\Delta : \text{Id}'(A) \times \text{Id}'(B) \to \text{Id}'(A \otimes_{\alpha} B)$ defined via \cite{2}. For $(I, J) \in \text{prime}(A) \times \text{prime}(B)$, $(\Psi \circ \Delta)(I, J) = (I, J)$ \cite[Lemma 2.13(i)]{2}. We will extend this to general $(I, J) \in \text{Id}'(A) \times \text{Id}'(B)$ in Lemma \ref{lem:extension} below. On the other hand, if $K \in \text{Id}'(A \otimes_{\alpha} B)$ then

$$(\Delta \circ \Psi)(K) = K^A \otimes_{\alpha} B + A \otimes_{\alpha} K^B \subseteq K$$

by the definition of $K^A$ and $K^B$ in \cite{3}.

**Lemma 4.6.** Let $I$ and $J$ be proper ideals of $A$ and $B$ respectively. Then

(i) $\text{hull}_f \Delta(I, J) = \Psi^{-1}(\text{hull}_f(I) \times \text{hull}_f(J))$.

(ii) $\Psi \circ \Delta(I, J) = (I, J)$

**Proof.** To show (i), take $F \in \text{hull}_f \Delta(I, J)$, then $\Psi(F) = (F^A, F^B) \in \text{Fac}(A) \times \text{Fac}(B)$ and $F^A \supseteq I, F^B \supseteq J$. Hence $\Psi(F) \in \text{hull}_f(I) \times \text{hull}_f(J)$.

On the other hand, suppose $F \in \text{Fac}(A \otimes_{\alpha} B)$ and $\Psi(F) \in \text{hull}_f(I) \times \text{hull}_f(J)$. Then using \cite{3}, $\Delta(I, J) \subseteq \Delta(F^A, F^B) \subseteq F$, as required.

To prove (ii), denote by $K = \Delta(I, J)$ and $(K^A, K^B) = \Psi(K)$. Then if $a \in I$, $a \otimes B \subseteq I \otimes_{\alpha} B \subseteq K$ and hence $a \in K^A$ and hence $I \subseteq K^A$. On the other hand, suppose $a \in K^A$ and $b \in B \setminus J$, so that $a \otimes b \in K$. Choose a bounded linear functional $\lambda$ on $B$ vanishing on $J$ such that $\lambda(b) = 1$. Let $L_\lambda : A \otimes_{\alpha} B \to A$ be the corresponding left slice map defined via $L_\lambda(a' \otimes b') = \lambda(b')a'$ on elementary tensors and extended to $A \otimes_{\alpha} B$ by linearity and continuity \cite[Theorem 1]{28}. Then $L_\lambda(A \circ J) = \{0\}$ and $L_\lambda(I \circ B) \subseteq I$, so that $L_\lambda(K) \subseteq I$. In particular $L_\lambda(a \circ b) = a \in I$, hence $K^A \subseteq I$ and so $K^A = I$. A similar argument shows that $K^B = J$, which completes the proof. \hfill $\square$

**Corollary 4.7.** Let $(I, J) \in \text{Fac}(A) \times \text{Fac}(B)$. Then with $G_I, G_J$ and $G_{\Phi(I, J)}$ defined as in Lemma \ref{lem:extension} we have

$$G_{\Phi(I, J)} = G_I \otimes_{\alpha} B + A \otimes_{\alpha} G_J$$

**Proof.** Take $M \in \text{Fac}(A \otimes_{\alpha} B)$. By Proposition \ref{prop:extension} M $\supseteq G_{\Phi(I, J)}$ if and only if $M \in \Psi^{-1}(\text{hull}_f(G_I) \times \text{hull}_f(G_J))$. Hence by Lemma \ref{lem:extension}(i), $M \supseteq G_{\Phi(I, J)}$ if and only if $M \supseteq \Delta(G_I, G_J)$, so $G_{\Phi(I, J)} = \Delta(G_I, G_J)$. \hfill $\square$

As a consequence of Corollary \ref{cor:extension} we are now in a position to prove a similar result to \cite[Theorem 2.3]{19};

**Theorem 4.8.** Let $A$ and $B$ be $C^*$-algebras and denote by $\psi : Glimm(A \otimes_{\alpha} B) \to (Glimm(A) \times Glimm(B), \tau_{cr})$ the homeomorphism of Theorem \ref{thm:homeo}. Then identifying the Glimm spaces with the corresponding sets of ideals we have

(i) $\Delta = \psi^{-1}$ on $Glimm(A) \times Glimm(B)$, hence $\Delta$ is a homeomorphism of $(Glimm(A) \times Glimm(B), \tau_{cr})$ onto $Glimm(A \otimes_{\alpha} B)$,

(ii) $\psi$ is given by the restriction of $\Psi$ to $Glimm(A \otimes_{\alpha} B)$.

**Proof.** Following the notation of Theorem \ref{thm:homeo}, Proposition \ref{prop:extension}(i) and Corollary \ref{cor:extension} show that the diagram

\[
\begin{array}{ccc}
\text{Prim}(A) \times \text{Prim}(B) & \xrightarrow{\Phi} & \text{Prim}(A \otimes_{\alpha} B) \\
\rho_A \times \rho_B \downarrow & & \downarrow \rho_{\alpha} \\
\text{Glimm}(A) \times \text{Glimm}(B) & \xrightarrow{\Delta} & \text{Glimm}(A \otimes_{\alpha} B)
\end{array}
\]

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commutes, i.e., that \( \Delta \circ (\rho_A \times \rho_B) = \rho_A \circ \Phi \). Therefore if we can show that \( \psi^{-1} \circ (\rho_A \times \rho_B) = \rho_A \circ \Phi \) also, then it will follow necessarily that \( \Delta = \psi^{-1} \) (since \( \rho_A \times \rho_B \) is surjective).

From Theorem 2.2 \( (\rho_A \times \rho_B) \circ \Phi = \rho_A \times \rho_B \) and \( \psi \circ \rho_A = (\rho_A \times \rho_B) \), so we have

\[
\psi^{-1} \circ (\rho_A \times \rho_B) = \psi^{-1} \circ (\rho_A \times \rho_B) \circ \Phi = \psi^{-1} \circ (\psi \circ \rho_A) \circ \Phi = \rho_A \circ \Phi,
\]

which proves (i).

Assertion (ii) is then immediate from (i) and Lemma 4.6 (ii). □

Suppose that \( A \) and \( B \) are \( C^* \)-algebras such that \( A \otimes_{\alpha} B \) satisfies Tomiyama’s property (F). Under this assumption, Kaniuth’s result [19, Theorem 2.3] shows that the map \( \Delta : (\text{Glimm}(A), \tau_q) \times (\text{Glimm}(B), \tau_q) \to (\text{Glimm}(A \otimes_{\alpha} B), \tau_q) \) is an open bijection, where \( \tau_q \) denotes the quotient topology on the Glimm space as discussed in Remark [14.11].

In order to extend this to arbitrary minimal tensor products, we first need some Lemmas:

**Lemma 4.9.** Suppose that the complete regularisation maps \( \rho_A \) and \( \rho_B \) are open with respect to either \( \tau_q \) or \( \tau_{cr} \) on \( \text{Glimm}(A) \) and \( \text{Glimm}(B) \) respectively. Then

(i) \( \tau_q = \tau_{cr} \) on each of \( \text{Glimm}(A) \) and \( \text{Glimm}(B) \),

(ii) \( \tilde{\tau}_q = \tau_{cr} = \tau_p \) on \( \text{Glimm}(A) \times \text{Glimm}(B) \), where \( \tilde{\tau}_q \) is the topology induced by the product map \( \rho_A \times \rho_B \),

(iii) \( \tau_q = \tau_{cr} \) on \( \text{Glimm}(A \otimes_{\alpha} B) \).

**Proof.** (i) is shown in the discussion in [4, p. 351].

(ii) As a consequence of (i), we may consider \( \tau_p \) as the product of the quotient topologies. Since \( \rho_A \times \rho_B \) is necessarily \( \tau_p \) continuous, \( \tau_p \) is always weaker than \( \tilde{\tau}_q \).

Consider a \( \tilde{\tau}_q \) open subset \( U \) of \( \text{Glimm}(A) \times \text{Glimm}(B) \) and let \( (x, y) \in U \). Choose \( (P, Q) \in \text{Prim}(A) \times \text{Prim}(B) \) with \( (\rho_A \times \rho_B)(P, Q) = (x, y) \). Then since \( (\rho_A \times \rho_B)^{-1}(U) \) is an open subset of \( \text{Prim}(A) \times \text{Prim}(B) \), we can find open neighbourhoods \( W \) of \( P \) and \( S \) of \( Q \) such that \( W 	imes S \subseteq (\rho_A \times \rho_B)^{-1}(U) \). Then we have

\[
(\rho_A \times \rho_B)(P, Q) = (x, y) \in \rho_A(W) \times \rho_B(S) \subseteq U,
\]

and \( \rho_A(W) \times \rho_B(S) \) is \( \tau_p \)-open since \( \rho_A \) and \( \rho_B \) are both \( \tau_{cr} \)-open. In particular \( (x, y) \) is a \( \tau_p \)-interior point of \( U \), and hence \( U \) is \( \tau_p \)-open.

The fact that \( \tau_p = \tau_{cr} \) follows from condition (ii) of Proposition 1.9.

As for (iii), it is always true that \( \tau_q \) is stronger than \( \tau_{cr} \), thus we need to prove that any \( \tau_q \) open subset \( U \) of \( \text{Glimm}(A \otimes_{\alpha} B) \) is \( \tau_{cr} \)-open. By part (ii) and Corollary 2.3, \( \psi \) is a homeomorphism of \( (\text{Glimm}(A \otimes_{\alpha} B), \tau_{cr}) \) onto \( (\text{Glimm}(A) \times \text{Glimm}(B), \tilde{\tau}_q) \). So given a \( \tau_{cr} \)-open subset \( U \) of \( \text{Glimm}(A \otimes_{\alpha} B) \), it will suffice to prove that \( \psi(U) \) is \( \tilde{\tau}_q \)-open, that is, that \( (\rho_A \times \rho_B)^{-1}(\psi(U)) \) is open.

Denote by \( W = (\rho_A \times \rho_B)^{-1}(\psi(U)) \). We will show that \( W = \Phi^{-1}(\rho_A^{-1}(U)) \). Since \( U \) is \( \tau_{cr} \)-open and \( \Phi \) is continuous on \( \text{Prim}(A) \times \text{Prim}(B) \) by Proposition 2.1 (ii), this will imply that \( W \) is open. For any \( (P, Q) \in \text{Prim}(A) \times \text{Prim}(B) \), Theorem 2.2 gives

\[
\psi \circ \rho_A \circ \Phi(P, Q) = (\rho_A \times \rho_B)(P, Q),
\]

so that \( (P, Q) \in W \) if and only if \( \rho_A \circ \Phi(P, Q) \in \psi^{-1}(\psi(U)) = U \). It follows that \( W = \Phi^{-1}(\rho_A^{-1}(U)) \), so that \( \psi(U) \) is \( \tilde{\tau}_q \)-open and hence \( U \) is \( \tau_{cr} \)-open. □
Lemma 4.10. Let \( \rho_A', \rho_B' \) and \( \rho_A'' \) denote the complete regularisation maps of \( \text{Fac}(A) \), \( \text{Fac}(B) \) and \( \text{Fac}(A \otimes \alpha B) \) respectively, and let \( \psi : \text{Glimm}(A \otimes \alpha B) \to \text{Glimm}(A) \times \text{Glimm}(B) \) be the map of Theorem 2.2. Then it holds that

\[
\psi \circ \rho_A' \circ \Phi = \rho_A'' \times \rho_B'
\]
on \( \text{Fac}(A) \times \text{Fac}(B) \).

Proof. Let \( (I, J) \in \text{Fac}(A) \times \text{Fac}(B) \) and denote by \( (p, q) = (\rho_A' \times \rho_B')(I, J) \). Take \( (P, Q) \in \text{hull}(I) \times \text{hull}(J) \), so that \( (\rho_A \times \rho_B)(P, Q) = (p, q) \) by Proposition 4.1(i). Then by Proposition 4.2(ii), the preimages \( \Phi((P, Q)) \in \text{hull}(\Phi(I, J)) \), so that \( (\rho_A \circ \Phi)(P, Q) = (\rho_A' \circ \Phi)(I, J) \) by Proposition 4.2(i) applied to \( A \otimes \alpha B \).

Finally, Theorem 2.2 gives

\[
(\psi \circ \rho_A \circ \Phi)(P, Q) = (\rho_A \times \rho_B)(P, Q),
\]
so that

\[
(\psi \circ \rho_A' \circ \Phi)(I, J) = (\rho_A' \times \rho_B')(I, J),
\]
as required. \( \square \)

We are now in a position to extend [19, Theorem 2.3], which required the assumption that \( A \otimes \alpha B \) satisfies property (F).

Theorem 4.11. The map \( \psi \) of Theorem 2.2 defines a continuous bijection of \( (\text{Glimm}(A \otimes \alpha B), \tau_q) \) onto the product space \( (\text{Glimm}(A), \tau_q) \times (\text{Glimm}(B), \tau_q) \), where \( \tau_q \) denotes the quotient topology induced by the complete regularisation map. It follows that its inverse \( \Delta \) is an open bijection. Moreover, \( \Delta \) is a homeomorphism whenever the complete regularisation maps \( \rho_A \) and \( \rho_B \) are open with respect to the quotient topologies on \( \text{Glimm}(A) \) and \( \text{Glimm}(B) \).

Proof. Let \( U \times V \) be a basic open subset of the product space \( (\text{Glimm}(A), \tau_q) \times (\text{Glimm}(B), \tau_q) \). Then by Proposition 4.2(ii), the preimages \( W := (\rho_A')^{-1}(U) \) and \( S := (\rho_B')^{-1}(V) \) are open subsets of \( \text{Fac}(A) \) and \( \text{Fac}(B) \) respectively. We claim that \( \psi^{-1}(U \times V) \) is a \( \tau_q \)-open subset of \( \text{Glimm}(A \otimes \alpha B) \), that is, that \( (\rho_A'')^{-1}(\psi^{-1}(U \times V)) \) is an open subset of \( \text{Fac}(A \otimes \alpha B) \). Since the map \( \Psi \) is continuous by Proposition 4.1(iii), it will suffice to show that

\[
(\rho_A'')^{-1}(\psi^{-1}(U \times V)) = \Psi^{-1}(W \times S).
\]

Let \( M \in \Psi^{-1}(W \times S) \). By Lemma 4.4 \( M \approx_f \Phi \circ \Psi(M) \), so that \( \rho_A'(M) = \rho_A''(\Phi \circ \Psi(M)) \). Then Lemma 4.10 gives

\[
\psi \circ \rho_A'(M) = (\psi \circ \rho_A' \circ \Phi)(\Psi(M)) = (\rho_A' \times \rho_B'')(\Psi(M)) \in U \times V.
\]

Hence \( \rho_A''(M) \in \psi^{-1}(U \times V) \), and we have \( \Psi^{-1}(W \times S) \subseteq (\rho_A'')^{-1}(\psi^{-1}(U \times V)) \).

To show the reverse inclusion, let \( M \in (\rho_A'')^{-1}(\psi^{-1}(U \times V)) \), and denote by \( (p, q) = (\psi \circ \rho_A''(M) \in U \times V \). Choose \( (I, J) \in W \times S \) with \( (\rho_A' \times \rho_B')(I, J) = (p, q) \). Then invoking Lemma 4.10 again we have

\[
(\psi \circ \rho_A''(\Phi(I, J)) = (\rho_A' \times \rho_B')(I, J) = (p, q).
\]

Since \( \psi \) is injective and \( (\psi \circ \rho_A''(M) = (p, q) \), it follows that \( \rho_A''(M) = \rho_A''(\Phi(I, J)) \) and hence \( M \approx_f \Phi(I, J) \). By Proposition 4.35 this implies that \( \Psi(M) \approx_f (I, J) \), so that \( (\rho_A' \times \rho_B'')(\Psi(M)) =
$(p,q)$. In particular $\Psi(M) \in (\rho_A^f \times \rho_B^f)^{-1}(U \times V) = W \times S$, so that $M \in \Psi^{-1}(W \times S)$ and hence $(\rho_A^f)^{-1}(\psi^{-1}(U \times V)) \subseteq \Psi^{-1}(W \times S)$, as required.

If in addition the complete regularisation maps $\rho_A$ and $\rho_B$ are open, then by Lemma 1.9 we have $\tau_q = \tau_{cr}$ on each of Glimm$(A)$, Glimm$(B)$ and Glimm$(A \otimes_\alpha B)$. Applying Corollary 2.3 and Theorem 4.8 it follows that $\Delta$ is a homeomorphism of $(\text{Glimm}(A), \tau_q) \times (\text{Glimm}(B), \tau_q)$ onto Glimm$(A \otimes_\alpha B)$.

\[\Box\]

5 Sectional representation

We first describe the Dauns-Hofmann Theorem on sectional representation for $\mathbb{C}^*$-algebras. This originally appeared as [9, Corollary 8.13]; however we adopt a version that follows from [26, Theorem 3.3] for convenience. For any $\mathbb{C}^*$-algebra $A$ there is a canonically associated upper semicontinuous $\mathbb{C}^*$-bundle (in the sense of [26, Section 1]) $\mathcal{A}$ over Glimm$(A)$, if this space is locally compact, or $\beta\text{Glimm}(A)$ otherwise. Under this representation,

(i) the fibre algebras are $\ast$-isomorphic to the Glimm quotients $A/G_p$ of $A$, where we define $G_p = A$ for $p \in \beta\text{Glimm}(A) \setminus \text{Glimm}(A)$ if necessary, and

(ii) the map sending $a \mapsto \hat{a}$, where

$$\hat{a}(p) = a + G_p \quad (p \in \beta\text{Glimm}(A))$$

is a $\ast$-isomorphism of $A$ onto the $\mathbb{C}^*$-algebra of sections of the bundle $\mathcal{A}$ vanishing at infinity on Glimm$(A)$.

As a consequence of Theorem 4.8 we can see that for any $\mathbb{C}^*$-algebras $A$ and $B$ the canonical bundle associated with $A \otimes_\alpha B$ has base space homeomorphic to $(\text{Glimm}(A) \times \text{Glimm}(B), \tau_{cr})$ (or its Stone-Cech compactification), and fibre algebras $\ast$-isomorphic to

$$\frac{A \otimes_\alpha B}{\Delta(G_p,G_q)}$$

for $(p,q) \in \text{Glimm}(A) \times \text{Glimm}(B)$, and zero otherwise.

Lee’s Theorem [24, Theorem 4] asserts that the bundle $\mathcal{A}$ is in fact continuous (i.e. $A$ is $\ast$-isomorphic to a maximal full algebra of operator fields in the sense of Fell [11]) if and only if the complete regularisation map $\rho_A$ is open (see also [4, Theorem 2.1] and [26, Theorem 3.3]). Thus in the case of a minimal tensor product of $\mathbb{C}^*$-algebras, it is natural to ask whether $\rho_\alpha$ being open is equivalent to $\rho_A$ and $\rho_B$ being open. It follows from [19, Lemma 2.2 and Theorem 2.3] that this is indeed the case when $A \otimes_\alpha B$ satisfies property (F). In this section we consider this question under more general hypotheses.

It is well known that $A \otimes_\alpha B$ satisfies property (F) if and only if $\Phi(I,J) = \Delta(I,J)$ for all $(I,J) \in \text{Id}^f(A) \times \text{Id}^f(B)$; see [28, Theorem 5 (2)] for example. The assumption that $\Phi(G,H) = \Delta(G,H)$ for all Glimm ideals of $A$ and $B$ is weaker in general. For example, if $H$ is an infinite dimensional Hilbert space then $B(H) \otimes_\alpha B(H)$ does not satisfy property (F) [30, Corollary 7]. However, Glimm$(B(H))$ is a one point space consisting of the zero ideal, and clearly $\Phi({\{0\}},{\{0\}}) = \Delta({\{0\}},{\{0\}})$. It appears to be unknown whether there exist $\mathbb{C}^*$-algebras $A$ and $B$ and Glimm ideals $(G,H) \in \text{Glimm}(A) \times \text{Glimm}(B)$ such that $\Delta(G,H) \subsetneq \Phi(G,H)$.

The condition that $\Delta = \Phi$ on Glimm$(A)$ is equivalent to requiring that the fibre algebras of the canonical bundle associated with $A \otimes_\alpha B$ are given by the minimal tensor products of the corresponding fibres of the bundles of $A$ and $B$, that is

$$\{(A/G_p) \otimes_\alpha (B/G_q) : (p,q) \in \text{Glimm}(A) \times \text{Glimm}(B)\}, \quad (6)$$
(with topology inherited from \((\text{Glimm}(A) \times \text{Glimm}(B), \tau_{cr})\) by Theorem 4.8). Indeed one may always consider an element \(c \in A \otimes_{\alpha} B\) as a cross-section of the fibre space given by \([\mathcal{D}]\) via \((p,q) \mapsto c + \Phi(G_p,G_q)\) for \((p,q) \in \text{Glimm}(A) \times \text{Glimm}(B)\) [3, p. 136-137]. In the case that the bundles of \(A\) and \(B\) are continuous, it was shown in [3, Corollary 3.1] that this representation of \(A \otimes_{\alpha} B\) defines a continuous \(C^*\)-bundle over \((\text{Glimm}(A) \times \text{Glimm}(B), \tau_p)\) in the obvious way if and only if \(\Delta = \Phi\) on \(\text{Glimm}(A) \times \text{Glimm}(B)\).

Theorem 5.2 below asserts that under the assumption \(\Delta = \Phi\) on \(\text{Glimm}(A) \times \text{Glimm}(B)\), the Dauns-Hofmann representation of \(A \otimes_{\alpha} B\) defines a continuous \(C^*\)-bundle over \(\text{Glimm}(A \otimes_{\alpha} B)\) if and only if \(A\) and \(B\) define continuous \(C^*\)-bundles over \(\text{Glimm}(A)\) and \(\text{Glimm}(B)\) respectively.

**Lemma 5.1.** Let \(A\) and \(B\) be \(C^*\)-algebras such that \(\Delta(G,H) = \Phi(G,H)\) for all \((G,H) \in \text{Glimm}(A) \times \text{Glimm}(B)\), and let \(\mathcal{U} \subseteq \text{Prim}(A \otimes_{\alpha} B)\) be open. Then

\[
\rho_{\alpha}(\mathcal{U}) = \rho_{\alpha}(\mathcal{U} \cap \Phi(\text{Prim}(A) \times \text{Prim}(B))).
\]

**Proof.** Let \(m \in \rho_{\alpha}(\mathcal{U})\) and choose \(M \in \mathcal{U}\) such that \(\rho_{\alpha}(M) = m\). Denote by \(G_m = k([M])\) the corresponding Glimm ideal of \(A \otimes_{\alpha} B\). Then there exist \((p,q) \in \text{Glimm}(A) \times \text{Glimm}(B)\) and corresponding Glimm ideals \(G_p\) and \(G_q\) of \(A\) and \(B\) respectively with \(G_m = \Phi(G_p,G_q) = \Delta(G_p,G_q)\) by Theorem 4.8.

Now \(M \in \text{hull}(G_m) \cap \mathcal{U}\), which is a nonempty relatively open subset of \(\text{hull}(G_m)\). By Proposition 2.1(iv), \(\Phi(\text{hull}(G_p) \times \text{hull}(G_q))\) is dense in \(\text{hull}(G_m) = \rho_{\alpha}^{-1}\{\{m\}\}\). Hence there exists \((P,Q) \in \text{hull}(G_p) \times \text{hull}(G_q)\) such that \(\Phi(P,Q) \in \text{hull}(G_m) \cap \mathcal{U}\). In particular \(\Phi(P,Q) \in \mathcal{U} \cap \Phi(\text{Prim}(A) \times \text{Prim}(B))\), and \(\rho_{\alpha} \circ \Phi(P,Q) = m\). It follows that \(\rho_{\alpha}(\mathcal{U}) \subseteq \rho_{\alpha}(\mathcal{U} \cap \Phi(\text{Prim}(A) \times \text{Prim}(B)))\), and the reverse inclusion is trivial.

\(\square\)

**Theorem 5.2.** Let \(A\) and \(B\) be \(C^*\)-algebras such that \(\Delta(G,H) = \Phi(G,H)\) for all \((G,H) \in \text{Glimm}(A) \times \text{Glimm}(B)\). Then the following are equivalent:

(i) \(\rho_{\alpha}\) is an open map with respect to \(\tau_{cr}\) on \(\text{Glimm}(A \otimes_{\alpha} B)\),

(ii) \(\rho_{\alpha}\) is an open map with respect to \(\tau_q\) on \(\text{Glimm}(A \otimes_{\alpha} B)\),

(iii) \(\rho_A\) and \(\rho_B\) are open maps with respect to \(\tau_{cr}\) on \(\text{Glimm}(A)\) and \(\text{Glimm}(B)\) respectively,

(iv) \(\rho_A\) and \(\rho_B\) are open maps with respect to \(\tau_q\) on \(\text{Glimm}(A)\) and \(\text{Glimm}(B)\) respectively.

**Proof.** Note that by Lemma 4.9(ii), (i) is equivalent to (ii), and (iii) is equivalent to (iv). We will show that (i) implies (iv) and that (iii) implies (i).

Suppose that (i) holds. We first claim that \(\rho_A \times \rho_B\) is open as a map into \((\text{Glimm}(A) \times \text{Glimm}(B), \tau_{cr})\). Take an open subset \(\mathcal{U} \subseteq \text{Prim}(A) \times \text{Prim}(B)\). Then since the restriction of \(\Phi\) to \(\text{Prim}(A) \times \text{Prim}(B)\) is a homeomorphism onto its image by Proposition 2.1(ii), there is an open subset \(\mathcal{U}' \subseteq \text{Prim}(A \otimes_{\alpha} B)\) with \(\mathcal{U}' \cap \Phi(\text{Prim}(A) \times \text{Prim}(B)) = \Phi(\mathcal{U})\). By Lemma 5.1, \(\rho_{\alpha}(\mathcal{U}') = \rho_{\alpha}(\Phi(\mathcal{U}))\).

Then by Theorem 2.2

\[
(\rho_A \times \rho_B)(\mathcal{U}) = (\psi \circ \rho_{\alpha} \circ \Phi)(\mathcal{U}) = (\psi \circ \rho_{\alpha})(\mathcal{U}'),
\]

which is \(\tau_{cr}\)-open since \(\rho_{\alpha}\) is an open map and \(\psi\) is a homeomorphism.

As in the proof of [4, p.351], \(\tau_{cr}\) must agree with the quotient topology \(\tilde{\tau}_q\) on \(\text{Glimm}(A) \times \text{Glimm}(B)\) induced by the map \(\rho_A \times \rho_B\). In particular \(\rho_A \times \rho_B\) is a \(\tilde{\tau}_q\)-open map. To see that \(\rho_{\alpha}\) is open, let \(\mathcal{W} \subseteq \text{Prim}(A)\) be open. Then

\[
(\rho_A \times \rho_B)(\mathcal{W} \times \text{Prim}(B)) = \rho_A(\mathcal{W}) \times \text{Glimm}(B)
\]
is $\tilde{\tau}_q$-open, hence $(\rho_A \times \rho_B)^{-1}(\rho_A(W) \times \Glimm(B))$ is open. By Lemma 1.1 we have

$$(\rho_A \times \rho_B)^{-1}(\rho_A(W) \times \Glimm(B)) = \rho_A^{-1}(\rho_A(W)) \times \Prim(B),$$

so that in particular, $\rho_A^{-1}(\rho_A(W))$ is open. It follows that $\rho_A$ is a $\tau_q$-open map. A similar argument shows that $\rho_B$ is also $\tau_q$-open, hence (i) implies (iv).

Assume that (iii) holds and take an open subset $U \subseteq \Prim(A \otimes_A B)$. Then $U \cap \Phi(\Prim(A) \times \Prim(B))$ is a relatively open subset of $\Phi(\Prim(A) \times \Prim(B))$. Again by Proposition 2.1(ii), there is an open subset $V \subseteq \Prim(A) \times \Prim(B)$ such that $\Phi(V) = U \cap \Phi(\Prim(A) \times \Prim(B))$. By Theorem 2.2 we then have

$$(\psi \circ \rho_\alpha)(U \cap \Phi(\Prim(A) \times \Prim(B))) = (\psi \circ \rho_\alpha \circ \Phi)(V) = (\rho_A \times \rho_B)(V),$$

which is $\tau_{cr}$-open since $\rho_A \times \rho_B$ is a $\tau_{cr}$-open map by Lemma 5.3(ii). Together with Lemma 5.3 this shows that

$$\rho_\alpha(U) = \rho_\alpha(U \cap \Phi(\Prim(A) \times \Prim(B))) = \psi^{-1}((\rho_A \times \rho_B)(V)),$$

which is open since $\psi$ is a homeomorphism. Hence $\rho_\alpha$ is an open map.

Following a suggestion of R.J. Archbold, below we give a similar result to [3, Proposition 4.1]. Under the assumption that $A$ and $B$ each have at least one Glimm quotient containing a nonzero projection, we show in Proposition 5.4 that the implication (i) of Theorem 5.2 does not require $\Delta = \Phi$ on $\Glimm(A \otimes_A B)$. We establish as a corollary that under the same assumptions on $A$ and $B$, if $A \otimes_A B$ is quasi-standard then $A$ and $B$ must be quasi-standard.

**Lemma 5.3.** Let $X$ and $Y$ be topological spaces. Then for any $y_0 \in Y$, the map sending $\rho_X(x) \mapsto \rho_{X \times Y}(x, y_0)$ is a homeomorphic embedding of $\rho_X$ into $\rho(X \times Y)$, with respect to the corresponding $\tau_{cr}$-topologies on each space.

**Proof.** By Lemma 1.1 we may identify $\rho(X \times Y)$ with $(\rho X \times \rho Y, \tau_{cr})$ under the canonical mapping $\rho_{X \times Y}(x, y) \mapsto (\rho_X(x), \rho_Y(y))$. Clearly the map sending $\rho_X(x) \mapsto (\rho_X(x), \rho_Y(y_0))$ is a homeomorphic embedding of $\rho X$ into $\rho X \times \rho Y$ with the product topology $\tau_p$. Thus we must show that the restrictions of the $\tau_p$ and $\tau_{cr}$ topologies to the subspace $\rho X \times \{\rho_Y(y_0)\}$ are equal. Since $\tau_p \leq \tau_{cr}$ it will suffice to show that for any $\tau_{cr}$-open subset $O$ of $\rho X \times \rho Y$ and $x_0 \in X$ such that $(\rho_X(x_0), \rho_Y(y_0)) \in O$, there is a cozero set neighbourhood $U$ of $\rho_X(x_0)$ in $\rho X$ such that $U \times \{\rho_Y(y_0)\} \subseteq O$.

Since $O$ is $\tau_{cr}$-open there is $g \in C^b(X \times Y)$ such that $\text{coz}(g^p)$ is a neighbourhood of $(\rho_X(x_0), \rho_Y(y_0))$ contained in $O$. Define $f \in C^b(X)$ via $f(x) = g(x, y_0)$, then $f^p \in C^b(\rho X)$ and $f^p(\rho_X(x)) = g^p(\rho_X(x), \rho_Y(y_0))$ for all $x \in X$. In particular, $f^p(\rho_X(x)) = 0$ if and only if $g^p(\rho_X(x), \rho_Y(y_0)) = 0$, so that

$$\text{coz}(f^p) \times \{\rho_Y(y_0)\} = \text{coz}(g^p) \cap (\rho X \times \{\rho_Y(y_0)\}),$$

as required. 

**Proposition 5.4.** Suppose that $A$ and $B$ are $C^*$-algebras such that the complete regularisation map $\rho_\alpha : \Prim(A \otimes_A B) \to \Glimm(A \otimes_A B)$ is open. If there exists a point $q_0 \in \Glimm(B)$ (resp. $p_0 \in \Glimm(A)$) such that the quotient $C^*$-algebra $B/G_{q_0}$ (resp. $A/G_{p_0}$) contains a nonzero projection, then $\rho_A$ (resp. $\rho_B$) is open.
Proof. Let \( e \in B \) such that \( e + G_{q_0} \) is a nonzero projection in \( B/G_{q_0} \). Then the map \( \Theta_p : A \to (A \otimes_\alpha B)/\Delta(G_p, G_{q_0}) \) defined by

\[
\Theta_p(a) = a \otimes e + \Delta(G_p, G_{q_0})
\]

is a \(*\)-homomorphism for each \( p \in \text{Glimm}(A) \). We first claim that \( \ker \Theta_p = G_p \).

Indeed, it is clear that if \( a \in G_p \) then \( a \otimes e \in \Delta(G_p, G_{q_0}) \), so that \( G_p \subseteq \ker \Theta_p \). Now choose a state \( \lambda \) of \( B \) vanishing on \( G_{q_0} \) such that \( \lambda(e) = 1 \), and consider the associated left slice map \( L_\lambda : A \otimes_\alpha B \to A \) defined on elementary tensors via \( L_\lambda(a \otimes b) = \lambda(b)a \), and extended to \( A \otimes_\alpha B \) by linearity and continuity. Then since \( L_\lambda(A \otimes G_{q_0}) = \{0\} \) and \( L_\lambda(G_p \otimes B) \subseteq G_p \), we have \( L_\lambda(\Delta(G_p, G_{q_0})) \subseteq G_p \). In particular, if \( a \in \ker \Theta_p \) then \( a \otimes e \in \Delta(G_p, G_{q_0}) \), so that

\[
L_\lambda(a \otimes e) = \lambda(e)a = a \in G_p,
\]

hence \( \ker \Theta_p = G_p \). It follows that for any \( a \in A \) and \( p \in \text{Glimm}(A) \), \( \|a + G_p\| = \|\Theta_p(a)\| \).

By [4, Theorem 2.1 (i) \( \Rightarrow \) (ii)] the function on \( \text{Glimm}(A \otimes_\alpha B) \) sending \( x \mapsto \|a \otimes e + G_p\| \) is continuous. Since by Theorem [5.3] \( \Delta : (\text{Glimm}(A) \times \text{Glimm}(B), \tau_{cr}) \to \text{Glimm}(A \otimes_\alpha B) \) is a homeomorphism, the map sending \((p, q) \mapsto \|a \otimes e + \Delta(G_p, G_{q_0})\| \) is \( \tau_{cr}\)-continuous on \( \text{Glimm}(A) \times \text{Glimm}(B) \).

Finally, by Lemma [5.3] the map \( p \mapsto (p, q_0) \) is a homeomorphic embedding of \( \text{Glimm}(A) \) into \( (\text{Glimm}(A) \times \text{Glimm}(B), \tau_{cr}) \). It follows that for each \( a \in A \) the function \( p \mapsto \|a + G_p\| \) agrees with the composition of continuous maps given by

\[
\text{Glimm}(A) \hookrightarrow (\text{Glimm}(A) \times \text{Glimm}(B), \tau_{cr}) \xrightarrow{\Delta} \text{Glimm}(A \otimes_\alpha B) \xrightarrow{\Theta_p} \mathbb{R},
\]

hence is continuous. By [4, Theorem 2.1 (ii) \( \Rightarrow \) (i)], this implies that \( \rho_A \) is open. \( \square \)

An ideal \( I \) of a \( C^*\)-algebra \( A \) is said to be \emph{primal} if whenever \( n \geq 2 \) and \( J_1, \ldots, J_n \) are ideals of \( A \) with zero product then \( J_i \subseteq I \) for at least one \( i \) between 1 and \( n \). We say that \( A \) is \emph{quasi-standard} if the complete regularisation map \( \rho_A \) is open, and every Glimm ideal of \( A \) is primal. There are many equivalent definitions of quasi-standard \( C^*\)-algebras, see [4, Theorems 3.3 and 3.4] for example.

\begin{corollary}
Suppose that \( A \) and \( B \) are \( C^*\)-algebras such that \( A \otimes_\alpha B \) is quasi-standard. If there exists a point \( q_0 \in \text{Glimm}(B) \) (resp. \( p_0 \in \text{Glimm}(A) \)) such that the quotient \( C^*\)-algebra \( B/G_{q_0} \) (resp. \( A/G_{p_0} \)) contains a nonzero projection, then \( A \) (resp. \( B \)) is quasi-standard.
\end{corollary}

\begin{proof}
Since \( A \otimes_\alpha B \) is quasi-standard, the Glimm ideals of \( A \) and \( B \) are primal by [3, Lemma 4.1]. The fact that \( \rho_A \) and \( \rho_B \) are open under the respective hypotheses then follows from Proposition [5.4]. \( \square \)

We remark that if \( Z(B) \neq \{0\} \) then there is \( q_0 \in \text{Glimm}(B) \) such that \( Z(B) \not\subseteq G_{q_0} \) (otherwise \( Z(B) \subseteq \bigcap \{G_q : q \in \text{Glimm}(B)\} = \{0\} \) ). It then follows from [4, Proposition 2.2(ii)] that \( B/G_{q_0} \) is in fact unital, and in particular contains a nonzero projection.

This condition is not necessary for the assumptions of Proposition [5.4] and Corollary [5.5] to hold, as can be seen by taking \( B = K(H) \) for a separable infinite dimensional Hilbert space \( H \). Then \( \text{Glimm}(B) \) consists of the zero ideal, so that \( B \) is a Glimm quotient of itself. We have \( Z(B) = \{0\} \), while \( B \) contains all of the finite rank projections on \( H \).
6 Examples

Our first example shows that the topologies $\tau_p$ and $\tau_{cr}$ can indeed differ for the complete regularisation of a product of primitive ideal spaces when condition (ii) of Theorem 1.6 fails. We show that the primitive ideal space of the (separable) $C^*$-algebra $A$ of [8, III, Example 9.2] admits a point $P_0$ for which no cozero set neighbourhood of $P_0$ in Prim($A$) has w-compact closure. Further we exhibit a cozero set neighbourhood $U$ of $(P_0, P_0)$ in Prim($A$) $\times$ Prim($A$) which does not contain a product $V \times W$ of cozero sets $V, W \subseteq$ Prim($A$). Thus $\rho_A \times \rho_V (U)$ is a $\tau_{cr}$-open subset of Glimm($A$) $\times$ Glimm($B$) which is not $\tau_p$-open. In particular we deduce that Glimm($A \otimes_{\alpha} A$) is not homeomorphic to (Glimm($A$) $\times$ Glimm($A$), $\tau_p$).

Example 6.1. Let $H$ be a separable infinite dimensional Hilbert space, $\{e_n : n = 1, 2, \ldots\}$ a fixed orthonormal basis for $H$, and let $K(H)$ denote the compact operators on $H$. The subset $D(H)$ of all compact operators that are diagonalisable with respect to the basis $\{e_n\}$ is a $C^*$-subalgebra of $K(H)$. Let

$$A = \{ F \in C([-1, 1], K(H)) : F(t) \in D(H) \text{ for all } t \geq 0 \}.$$ 

With pointwise operations and norm $\|F\| = \sup \{|F(t)| : t \in [-1, 1]\}$, $A$ is a $C^*$-algebra. Each element $F$ of $A$ is given by an infinite matrix of continuous functions $F_{ij} : [-1, 1] \to \mathbb{C}$, such that

$$F(t)e_j = \sum_i F_{ij}(t)e_i : i = 1, 2, \ldots.$$ 

The set Prim($A$) consists of the ideals

$$P(t) = \{ F \in A : F(t) = 0 \} \text{ for } t < 0$$

$$P(t, n) = \{ F \in A : F_{n, n}(t) = 0 \} \text{ for } t \geq 0, n \in \mathbb{N}$$

The topology on Prim($A$) is as follows:

- For $t < 0$, a neighbourhood basis for $P(t)$ is given by the sets
  $$N(t, \varepsilon) := \{ P(q) : |q - t| < \varepsilon, q < 0 \}$$

- For $t = 0$ and $n \in \mathbb{N}$, $P(0, n)$ has a neighbourhood basis of sets
  $$M(0, n, \varepsilon) := \{ P(q) : -\varepsilon < q < 0 \} \cup \{ P(q, n) : 0 < q < \varepsilon \}$$

- For $t > 0$ and $n \in \mathbb{N}$, a neighbourhood basis for $P(t, n)$ is given by the sets
  $$M(t, n, \varepsilon) := \{ P(q, n) : q > 0, |q - t| < \varepsilon \}$$

For each $n$ let $I_n = [0, 1] \times \{ n \}$, and for $t \in [0, 1]$ let $t^{(n)} = (t, n) \in I_n$. Then

$$\text{Prim}(A) \equiv [-1, 0) \cup \left( \bigcup_{n=1}^{\infty} I_n \right),$$

where each point $0^{(n)}$ has a neighbourhood basis of intervals $(-\varepsilon, 0) \cup [0^{(n)}, \varepsilon^{(n)})$. It is easy to see that for a pair of points $0^{(n)}$ and $0^{(m)}$, with $n \neq m$, any open neighbourhood of $0^{(n)}$ will intersect every open neighbourhood of $0^{(m)}$ in the subset $[-1, 0)$.
The complete regularisation map \( \rho : \text{Prim}(A) \to \text{Glimm}(A) \) fixes the sets \([-1,0) \) and \((0^{(n)}, 1^{(n)}) \) for all \( n \). The set \( \{0^{(n)} : n \in \mathbb{N} \} \) is mapped to a single point \( 0^{(0)} \). Moreover, \( 0^{(0)} \) does not have a compact neighbourhood in \( \text{Glimm}(A) \). For a proof of these facts, see [9, Examples 3.4 and 9.2]. Hence

\[
\text{Glimm}(A) = [-1, 0] \cup \bigcup_{n=1}^{\infty} (0^{(n)}, 1^{(n)}),
\]

where a neighbourhood basis of \( 0^{(n)} \) is given by the sets \( (-\delta_0, 0) \cup \bigcup_{n=1}^{\infty} (0^{(n)}, \delta_n^{(n)}) \) where \( \delta_j > 0 \) for all \( j \geq 0 \). \( \text{Glimm}(A) \) is homeomorphic to the subset of \( \mathbb{C} \) given by

\[
[-1,0] \cup \{re^{i\theta} : 0 < r \leq 1, \theta = \frac{2n+1}{2n}\pi \text{ for } n \geq 1 \}.
\]

We will show that no \( 0^{(n)} \) has a cozero set neighbourhood in \( \text{Prim}(A) \) with w-compact closure. Indeed any such neighbourhood is of the form \( \rho^{-1}(V) \) where \( V \) is a cozero set neighbourhood of \( 0 \) in \( \text{Glimm}(A) \), and is hence a neighbourhood of \( 0^{(j)} \) for all \( j \in \mathbb{N} \). Hence for every \( j \in \mathbb{N} \cup \{0\} \) there is \( \varepsilon_j > 0 \) such that

\[
U := (-\varepsilon_0, 0) \cup \bigcup_{n=1}^{\infty} (0^{(n)}, \varepsilon_n^{(n)}) \subseteq \rho^{-1}(V).
\]

Note that

\[
\overline{U} = [-\varepsilon_0, 0) \cup \bigcup_{n=1}^{\infty} (0^{(n)}, \varepsilon_n^{(n)}),
\]

\[
\rho(U) = (-\varepsilon_0, 0) \cup \bigcup_{n=1}^{\infty} (0^{(n)}, \varepsilon_n^{(n)}),
\]

and that \( \rho(U) \) is clearly a cozero set of \( \text{Glimm}(A) \), hence \( U \) is a cozero set of \( \text{Prim}(A) \).

We claim that it suffices to consider a cozero set neighbourhood of \( 0^{(n)} \) of the form \( U \). Indeed, if \( \rho^{-1}(V) \) were w-compact, then \( \overline{U} \), being the closure of a cozero set of a w-compact space, would also be w-compact by [18, Proposition 3.8]. Therefore if \( \overline{U} \) is not w-compact, then no cozero set neighbourhood \( \rho^{-1}(V) \) of \( 0^{(n)} \) can be w-compact.

For each \( n \) let \( U_n = [-\varepsilon_0, 0) \cup (0^{(n)}, \varepsilon_n^{(n)}) \). Then the collection \( \{U_n\} \) is an open cover of \( \overline{U} \).

If we take any finite subcollection, w.l.o.g \( U_1, \ldots, U_n, \) then \( \text{cl}_{\tau_{\text{gr}}} (U_1 \cup \ldots \cup U_n) \) consists of all points \( x \in \overline{U} \) such that any cozero set neighbourhood of \( x \) intersects \( U_1 \cup \ldots \cup U_n \). If \( m > n \) and \( y^{(m)} \in (0^{(m)}, \varepsilon_m^{(m)}) \) then we can choose \( g \in C^b((0^{(m)}, \varepsilon_m^{(m)}]) \) with \( g(y^{(m)}) = 1 \), vanishing off a compact neighbourhood of \( y^{(m)} \) in \( (0, \varepsilon_m) \). Extending \( g \) to be zero elsewhere on \( \overline{U} \) gives a cozero set neighbourhood of \( y^{(m)} \) disjoint from \( U_1 \cup \ldots \cup U_n \). It follows that

\[
\text{cl}_{\tau_{\text{gr}}} (U_1 \cup \ldots \cup U_n) = U_1 \cup \ldots \cup U_n \cup \{0^{(m)} : m > n\},
\]

a proper subset of \( \overline{U} \). So \( 0^{(n)} \) does not have a cozero set neighbourhood with w-compact closure for any \( n \).

We now show that the product topology \( \tau_p \) on \( \text{Glimm}(A) \times \text{Glimm}(A) \) is strictly weaker than \( \tau_{\text{gr}} \). Note first that

\[
\text{Prim}(A) \times \text{Prim}(A) = [-1,0) \times [-1,0) \cup \left( \bigcup_{n=1}^{\infty} I_n \right) \cup \left( \bigcup_{m,n=1}^{\infty} I_m \times [-1,0) \right) \cup \left( \bigcup_{m,n=1}^{\infty} I_m \times I_n \right).
\]

The neighbourhood basis of the following types of points will be of interest:
• \((0^{(m)}, 0^{(n)})\): sets of the form

\[
\left((-\delta, 0) \cup [0^{(m)}, \delta^{(m)})\right) \times \left(( -\varepsilon, 0) \cup [0^{(n)}, \varepsilon^{(n)})\right)
\]

where \(\delta, \varepsilon > 0\).

• \((x, y) \in \{0^{(m)}\} \times \{0^{(n)}, 1^{(n)}\}\): neighbourhoods of the form

\[
\left((-\delta, 0) \cup [0^{(n)}, \delta)\right) \times \left((y - \varepsilon)^{(n)}, (y + \varepsilon)^{(n)}\right)
\]

where \(0 < \delta < 1, 0 < \varepsilon < |y|\).

• \((x, y) \in \{0^{(m)}, 1^{(m)}\} \times \{0^{(n)}\}\): neighbourhoods of the form

\[
\left((x - \delta)^{(m)}, (x + \delta)^{(m)}\right) \times \left((-\varepsilon, 0) \cup [0^{(n)}, \varepsilon^{(n)})\right)
\]

For each \(m, n \in \mathbb{N}\) define \(f_{m,n} : I_m \times I_n \to [0,1]\) via

\[
f_{m,n}(x, y) = \max(1 - mnx, 1 - mny, 0).
\]

Then \(f_{m,n}(x, y) > 0\) when \(x < \frac{1}{mn}\) or \(y < \frac{1}{mn}\), i.e. \(\text{coz}(f_{m,n}) = [0^{(m)}, (\frac{1}{mn})^{(m)}] \times [0^{(n)}, (\frac{1}{mn})^{(n)}]\).

Now define \(f : \text{Prim}(A) \times \text{Prim}(A) \to [0,1]\) via

\[
f(x, y) = \begin{cases} f_{m,n}(x, y) & \text{if } (x, y) \in I_m \times I_n \\ 1 & \text{otherwise} \end{cases}
\]

Then \(f\) is continuous since \(f_{m,n}(0^{(m)}, 0^{(n)}) = f_{m,n}(0^{(m)}, y^{(n)}) = f_{m,n}(x^{(m)}, 0^{(n)}) = 1\), and by the neighbourhood bases of these points constructed above. Moreover, the cozero set of \(f\) is

\[
[-1,0) \times [-1,0) \cup \left([-1,0) \times \bigcup_{n=1}^\infty I_n\right) \cup \left(\bigcup_{m=1}^\infty I_m \times [-1,0)\right) \cup \left(\bigcup_{m,n=1}^\infty [0^{(m)}, (\frac{1}{mn})^{(m)}] \times [0^{(n)}, (\frac{1}{mn})^{(n)}]\right)
\]

We show that \(\text{coz}(f)\) is not a union of cozero set rectangles. Indeed, for \(i, j \in \mathbb{N}\), suppose \(U \times V\) were a cozero set neighbourhood of \((0^{(i)}, 0^{(j)})\) contained in \(\text{coz}(f)\). As before we may assume w.l.o.g. that

\[
U = (-\delta_0, 0) \cup \bigcup_{m=1}^\infty [0^{(m)}, \delta_m^{(m)}]
\]

\[
V = (-\varepsilon_0, 0) \cup \bigcup_{n=1}^\infty [0^{(n)}, \varepsilon_n^{(n)}],
\]

where \(\delta_j, \varepsilon_j > 0\) for all \(j \geq 0\). If \(U \times V \subseteq \text{coz}(f)\) then in particular it must be true that

\[
[0^{(m)}, \delta_m^{(m)}] \times [0^{(n)}, \varepsilon_n^{(n)}] \subseteq [0^{(m)}, (\frac{1}{mn})^{(m)}] \times [0^{(n)}, (\frac{1}{mn})^{(n)}]
\]

for all \(m, n \geq 1\). In other words, \(\delta_m \leq \frac{1}{mn}\) for all \(n \geq 1\) and \(\varepsilon_n \leq \frac{1}{mn}\) for all \(m \geq 1\). But then \(\delta_m = \varepsilon_n = 0\) for all \(m, n \geq 1\). \(\square\)
In what follows we denote by $\omega_0$ the first infinite ordinal and by $\omega_1$ the first uncountable ordinal. For $i = 0, 1$ we let $[0, \omega_i)$ be the space of all ordinals $\gamma < \omega_i$ and $[0, \omega_i] = [0, \omega_i + 1)$. These spaces will be considered with the order topology, with basic open sets given by

$$(\alpha, \beta) := \{ \gamma \in [0, \omega_i] : \alpha < \gamma < \beta \},$$

where $\alpha, \beta \in [0, \omega_i]$ for $i = 0, 1$. A useful property of the space $[0, \omega_1]$ is that $\beta([0, \omega_1]) = [0, \omega_1]$. \[12\] 5.13.

It follows from \[12\] 5.11(c) and 5.12(c) that the space $[0, \omega_1)$ is a non-compact pseudocompact space. On the other hand $[0, \omega_0]$ is homeomorphic to $\mathbb{N}$, which being infinite and discrete cannot be pseudocompact.

Our second example is a nontrivial application of Theorem 3.6. First we describe the C$^*$-algebra $A$ of \[21\] Appendix, which has the property that Glimm($A$) is pseudocompact but non-compact. We then construct a (non-unital) $\sigma$-unital C$^*$-algebra $B$ with Glimm($B$) compact, such that $M(A) \otimes_A M(B) \neq M(A \otimes B)$, while $ZM(A) \otimes ZM(B) = ZM(A \otimes B)$.

**Example 6.2.** Let $Y = [0, \omega_1) \times [0, \omega_0) \setminus \{(\omega_1, \omega_0)\}$ and denote by $S = \{\omega_1\} \times [0, \omega_0)$ and $T = [0, \omega_1) \times \{\omega_0\}$. Define a new space $X = Y \cup \{y\}$, where $y \not\in Y$ with topology such that $Y$ is embedded homeomorphically into $X$, and $\{y\}$ is a singleton set whose closure is $S \cup \{y\}$.

Let $C = C_0(Y)$, $D = C_0(S)$ and let $\pi_1 : C \to D$ be the restriction map. Let $H$ be an infinite dimensional separable Hilbert space and $\{p_n\}$ a sequence of infinite dimensional mutually orthogonal projections on $H$. Define an injective $*$-homomorphism $\lambda : D \to B(H)$ via $\lambda(f) = \sum_{n=1}^{\infty} f(\omega_1, n)p_n$, and note that $\lambda(D) \cap K(H) = \{0\}$.

Set $E = \lambda(D) + K(H)$ and let $\pi_2 : E \to D$ be the quotient map. Let $A = \{(c, e) \in C \oplus E : \pi_1(c) = \pi_2(e)\}$. Then Prim($A$) is homeomorphic to $X$.

The complete regularisation map $\rho_A$ maps $Y \setminus S$ to itself, and $S \cup \{y\}$ to a single point which we will denote by $z$. Thus Glimm($A$) = $(Y \setminus S) \cup \{z\}$, where a neighbourhood basis of $z$ is given by the collection of sets of the form

$$\left( \bigcup_{n=1}^{\omega_0} (\alpha_n, \omega_1) \right) \cup \{z\},$$

for some ordinals $0 < \alpha_n < \omega_1$ for all $1 \leq n \leq \omega_0$.

Note that $Y \setminus S = [0, \omega_1) \times [0, \omega_0)$, being the product of a pseudocompact space and a compact space, is necessarily pseudocompact. It follows that Glimm($A$) is pseudocompact.

Consider the C$^*$-algebra $B$ of sequences $(T_n) \in B(H)$ such that $T_n \to T_\infty \in K(H)$, with pointwise operations and supremum norm. Then Prim($B$) consists of the ideals

$$(\alpha_n, \omega_1) = \{(T_n) : T_{\alpha_n} = 0\}, K_{\infty} = \{(T_n) : T_{\infty} \in K(H)\}$$

for $n_0 \in \mathbb{N}$, and $K_\infty = \{(T_n) : T_\infty = 0\}$. The $\approx$-equivalence classes in Prim($B$) then consist of pairs $(P_{n_0}, K_{n_0})$ for $n_0 \in \mathbb{N}$, and $(P_\infty)$. As in the proof of \[21\] Proposition 3.6, the complete regularisation map $\rho_B : \text{Prim}(B) \to \text{Glimm}(B)$ is open and Glimm($B$) is homeomorphic to $\mathbb{N} \cup \infty$, with $G_q = P_q$ for all $1 \leq q \leq \infty$.

We claim that $B$ is a $\sigma$-unital C$^*$-algebra. Fix an orthonormal basis $\{e_m : m \in \mathbb{N}\}$ of $H$. For each $n \in \mathbb{N}$ let $1_n$ denote the projection onto the $n$-dimensional subspace of $H$ spanned by $e_1, \ldots, e_n$. Then $\{1_n : n \in \mathbb{N}\}$ is an increasing approximate identity for $K(H)$.

Now define sequences $E^{(n)} = (E^{(n)}_m)$ in $B$ via

$$E^{(n)}_m = \begin{cases} 1 & \text{if } m \leq n \\ 1_n & \text{if } m > n \end{cases}$$

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for each \( n \in \mathbb{N} \). We will show that the sequence \( \{E^{(n)}\}_n \) is an approximate identity for \( B \). Take \( T = (T_m) \in B \) and let \( \varepsilon > 0 \) be given. Note that for each \( n \),

\[
\|T - E^{(n)}T\| = \sup_{m \geq 1} \|T_m - E^{(n)}_mT_m\| = \sup_{m > n} \|T_m - 1_nT_m\|
\]

by definition of \( E^{(n)} \). Then

- there exists \( m_0 \geq 1 \) such that \( \|T_m - T_\infty\| < \frac{\varepsilon}{4} \) whenever \( m \geq m_0 \), and
- there exists \( n_1 \geq 1 \) such that \( \|T_\infty - 1_nT_\infty\| < \frac{\varepsilon}{4} \) whenever \( n \geq n_1 \) (since \( T_\infty \in K(H) \)).

Set \( n_0 = \max(m_0, n_1) \). Then if \( n \geq n_0 \) and \( m > n \) we have

\[
\|T_m - 1_nT_m\| \leq \|T_m - T_\infty\| + \|T_\infty - 1_nT_m\| + \|1_nT_\infty - 1_nT_m\|
\]

\[
\leq \|T_m - T_\infty\| + \|T_\infty - 1_nT_\infty\| + \|1_nT_\infty - T_m\|
\]

\[
< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{3\varepsilon}{4}.
\]

In particular, for all \( n \geq n_0 \) we have

\[
\|T - E^{(n)}T\| = \sup_{m > n} \|T_m - 1_nT_m\| \leq \frac{3\varepsilon}{4} < \varepsilon,
\]

so that \( \{E^{(n)}\}_n \) is a countable approximate identity for \( B \).

Now take the tensor product \( A \otimes_\alpha B \). Since \( B \) is \( \sigma \)-unital, the inclusion \( M(A) \otimes_\alpha M(B) \subseteq M(A \otimes_\alpha B) \) is strict by [1, Theorem 3.8]. Since \( \rho_B \) is open, Proposition 1.9 (ii) shows that \( \tau_{cr} = \tau_p \) on \( \text{Glimm}(A) \times \text{Glimm}(B) \). Hence by Theorem 2.2 \( \text{Glimm}(A \otimes_\alpha B) \) is homeomorphic to \( \text{Glimm}(A) \times \text{Glimm}(B) \) with the product topology. Moreover, as a product of a pseudocompact space and a compact space, \( \text{Glimm}(A) \times \text{Glimm}(B) \) is pseudocompact [29, Proposition 8.21]. It then follows from Theorem 3.6 that \( \text{ZM}(A) \otimes \text{ZM}(B) = \text{ZM}(A \otimes_\alpha B) \).

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