Abstract. Darboux transformation is developed to systematically find variable separation solutions for the Nizhnik-Novikov-Veselov equation. Starting from a seed solution with some arbitrary functions, the once Darboux transformation yields the variable separable solutions which can be obtained from the truncated Painlevé analysis and the twice Darboux transformation leads to some new variable separable solutions which are the generalization of the known results obtained by using a guess ansatz to solve the generalized trilinear equation.

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To find some exact solutions of integrable systems has focused many mathematicians and physicists’ attention since the soliton theory came into being at the 1960’s. There are many important methods to obtain the special solutions of a given soliton equation. Some of the most important methods are the inverse scattering transformation (IST) approach[1], the bilinear form[2], symmetry reduction[3], Bäcklund transformation and Darboux transformation[5] etc. In comparison with the linear case, it is known that IST is an extension of the Fourier transformation in the nonlinear case. In addition to the Fourier transformation, there is another powerful tool called the variable separation method[6] in the linear case. Recently, some kinds of variable separation approaches had been developed to find new exact solutions of nonlinear models, say, the classical method, the differential Stäckel matrix approach[7], the geometrical method[8], the ansatz-based method[4,8], functional variable separation approach[10], the derivative dependent functional variable separation approach[11], the formal variable separation approach (nonlinearization of the Lax pairs or symmetry constraints)[12] and the informal variable separation methods[13][14].
Especially, for various (2+1)-dimensional nonlinear physics models, a quite universal formula

\[ U \equiv \frac{2(a_1 a_2 - a_0 a_3) q_y p_x}{(a_0 + a_1 p + a_2 q + a_3 p q)^2}, \]  

(1)

where \( a_i, (i = 0, 1, 2, 3) \) are arbitrary constants and \( p = p(x, t) \) and \( q = q(y, t) \) are arbitrary constants of the indicated variables, is found by using the informal variable separation approach[13]–[15]. Starting from the universal formula (1), abundant localized excitations like the dromions, lumps, ring solitons, breathers, instantons, solitoffs, fractal and chaotic patterns are found. Now a very important question is can we find the universal formula from other well known powerful methods like the IST approach, dressing method, Darboux transformation (DT) etc?

DT is one of the most powerful methods to construct a broad class of considerable physical interest and important nonlinear evolution equations such as the well-known Korteweg-de Vries (KdV) equation, the Kadomtsev-Petviashvili (KP) equation, the Davey-Stewartson (DS) equation, the sine-Gordon (SG) equation [4] and so on. In this letter, we use the Darboux transformation to study the variable separable solutions for the (2+1) dimensional Nizhnik-Novikov-Veselov system.[16]

\[ u_t = u_{xxx} + u_{yyy} + 3(vu)_x + 3(uw)_y, \]  

(2)

\[ u_x = v_y, \quad u_y = w_x. \]  

(3)

The (2+1)-dimensional NNV equation is an only known isotropic Lax integrable extension of the well-known (1+1)-dimensional KdV equation. Many authors have studied the solutions of the NNV equation. For example, Boiti et al[17] solved the NNV equation via the IST; Tagami and Hu and Li obtained the soliton-like solutions of the NNV equation by means of Bäcklund transformation[18]; Hu also gave out the nonlinear superposition formula of the NNV equation[19]; Some special types of multi-dromion
solutions were found by Radha and Lakshmanan\cite{20}; The generalized localized excitations expressed by (1) were given in \cite{13} and \cite{15} and the special binary Darboux transformation was given in \cite{4}.

It is known that the NNV equation system (2) and (3) can be represented as a compatibility condition of the linear system

\[
\Phi_{xy} + u\Phi = 0, \tag{4}
\]

\[
\Phi_t = \Phi_{xxx} + \Phi_{yyy} + 3v\Phi_x + 3w\Phi_y. \tag{5}
\]

In general, in order to construct the solutions of a given equation by means of Darboux transformation, one may use a fixed solution for a special spectrum parameter of the Lax pair. Then one can get a new solution from an old one with Darboux transformation. Without spectral parameter in the Lax pair of the NNV equation, we have to construct a binary Darboux transformation for the NNV equation. At the same time, with the variable separated approach, two arbitrary functions can be entered into the solution.

It is straightforward to see that the NNV system (2) and (3) possesses the following trivial solution

\[
u = 0, \ v = v_0(x, t), \ w = w_0(y, t), \tag{6}\]

where \(v_0(x, t)\) and \(w_0(y, t)\) are arbitrary functions of \(\{x, t\}\) and \(\{y, t\}\) respectively.

In order to find some new solutions via Darboux transformation and the seed solution (6), the key step is to find the fixed solution of the Lax pair (4) and (5) with the seed (6).

It is evident that Eq.(4) have a variable separation solution in the form

\[
\Phi_0 = p + q \tag{7}\]
where \( p = p(x,t), q = q(y,t) \) are two arbitrary functions when \( u = 0 \). Substituting Eq. (7) and Eq. (8) into Eq. (5) yields

\[
p_t + q_t = p_{xxx} + q_{yyy} + 3v_0p_x + 3w_0q_y.
\]

It is clear that Eq. (8) can be solve by the usual variable separable approach and the result reads

\[
p_t = p_{xxx} + 3v_0p_x + c(t),
\]

\[
q_t = q_{yyy} + 3w_0q_y - c(t)
\]

where \( c(t) \) is an arbitrary function of \( t \). For given functions \( v_0, w_0 \) and \( c(t) \), it is still difficult to solve (9) and (10). However, because of the arbitrariness of the functions \( v_0 \) and \( w_0 \), we can solve the problem in an alternative way. If we consider the functions \( p \) and \( q \) as arbitrary functions, then \( v_0 \) and \( w_0 \) can be solved out from (9) and (10)

\[
v_0 = (3p_x)^{-1}(p_t - p_{xxx} - c(t))
\]

\[
w_0 = (3q_y)^{-1}(q_t - q_{yyy} + c(t)).
\]

As usual, the binary Darboux transformation for the NNV equation can be constructed by introducing the closed 1-form

\[
\omega(\Phi, \Phi_0) = (\Phi\Phi_{0x} - \Phi_x\Phi_0)dx - (\Phi\Phi_{0y} - \Phi_0\Phi_y)dy + (\Phi(\Phi_0_{xxx} - \Phi_{0yyy})
\]

\[
+ \Phi_0(\Phi_{yyy} - \Phi_{xxx}) + 2(\Phi_{xx}\Phi_{0x} - \Phi_x\Phi_{0xx} + \Phi_{0yy}\Phi_y - \Phi_0\Phi_{yy})
\]

\[
+ 3v(\Phi_0\Phi_x - \Phi_x\Phi_0) + 3w(\Phi_0\Phi_y - \Phi_0\Phi_y))dt.
\]

Then the new wave function is

\[
\Phi[1] = \Phi_0^{-1} \int_4 \omega
\]
and the equation system Eq.(4) and (5) is covariant with respect to the transformation (14) and the transformed coefficients $u[1]$, $v[1]$ and $w[1]$ are

$$u[1] = 2(\ln \Phi_0)_{xy} = -\frac{2p_xq_y}{(p + q)^2},$$ (15)

$$v[1] = v_0 + 2(\ln \Phi_0)_{xx} = v_0 + \frac{2p_x}{p + q} - \frac{p_x^2}{(p + q)^2},$$ (16)

$$w[1] = w_0 + 2(\ln \Phi_0)_{yy} = w_0 + \frac{2q_y}{p + q} - \frac{q_y^2}{(p + q)^2},$$ (17)

where $v_0$ and $w_0$ determined by Eqs.(11) and (12). From (13), we see that the solution $u[1]$ obtained by the once Darboux transformation is only a special case of the universal expression (1) with $a_3 = 0$. The solution (15)–(17) can also be obtained by the truncated Painlevé expansion using the similar method as for the AKNS system[21].

By means of the iteration of the Darboux transformation, we can construct the second Darboux transformation from the first Darboux transformation. We also take two special solutions of the Lax pair (4) and (5) as the variable separable ones, $\Phi_1 = p_1(x, t) + q_1(y, t)$ and $\Phi_2 = p_2(x, t) + q_2(y, t)$, then the transformed wave function is

$$\Phi[2] = \int \omega(\Phi_1, \Phi_2)$$ (18)

which leads to the formula

$$u[2] = 2(\ln \int \omega(\Phi_1, \Phi_2))_{xy},$$ (19)

$$v[2] = v_0 + 2(\ln \int \omega(\Phi_1, \Phi_2))_{xx},$$ (20)

$$w[2] = w_0 + 2(\ln \int \omega(\Phi_1, \Phi_2))_{yy}$$ (21)

According to the selection for the arbitrary functions, the potential functions $v_0$ and $w_0$ are related to the functions $p_1$, $p_2$, $q_1$ and $q_2$ by

$$p_{1t} - p_{1xxx} - c_1(t) - v_0p_{1x} = 0,$$ (22)

$$q_{1t} - q_{1yyy} + c_1(t) - w_0q_{1y} = 0,$$ (23)

$$p_{2t} - p_{2xxx} - c_2(t) - v_0p_{2x} = 0,$$ (24)

$$q_{2t} - q_{2yyy} + c_2(t) - w_0q_{2y} = 0.$$ (25)
Now we can consider one of the functions \( p_1 = p_1(x, t) \) and \( p_2 = p_2(x, t) \) as an arbitrary function of \( \{x, t\} \) and one of the functions \( q_1 = q_1(y, t) \) and \( q_2 = q_2(y, t) \) as an arbitrary function of \( \{y, t\} \) while the remained functions of \( v_0, w_0, p_1, p_2, q_1 \) and \( q_2 \) should be determined by (22), (23), (24) and (25).

It is not easy to solve the equation system (22), (23), (24) and (25). However, if we choose

\[
q_1 = 0, \quad p_2 = 0, \quad (26)
\]

then the Eqs. (19)-(21) can be simplified into

\[
\begin{align*}
\frac{\partial u}{\partial t} &= 2 \ln(c + p_1 q_2)_{xy}, \\
\frac{\partial v}{\partial t} &= v_0 + 2 \ln(c + p_1 q_2)_{xx}, \\
\frac{\partial w}{\partial t} &= w_0 + 2 \ln(c + q_1 q_2)_{yy},
\end{align*}
\]

where \( c \) is an arbitrary integral constant, \( p_1 \) and \( q_2 \) are arbitrary functions of \( \{x, t\} \) and \( \{y, t\} \) while \( v_0 \) and \( w_0 \) are fixed by (22) and (25). It is easy to prove that the results (27)–(29) are equivalent to those obtained via the usual variable separation of the multi-linear equation[13]. Actually, by re-writing \( c, p_1 \) and \( q_2 \) as

\[
\begin{align*}
c &= -a_0 - \frac{a_1 a_2}{a_3}, \quad p_1 = a_2 + a_3 p, \quad q_2 = q + \frac{a_1}{a_3},
\end{align*}
\]

the negative value of the right hand side of (27) is transformed to the universal quantity expressed by (1).

Various types of coherent localized structures for the physical field \( u \) of the NNV system had been described in [13] and [15] thanks to the arbitrariness of the functions \( p \) and \( q \). We do not repeated these known localized excitations but write down only one new type of localized solutions for the field \( u \) expressed by (27) with (30) to complement the results of [13].

In addition to the continuous localized excitations in (1+1)-dimensional nonlinear systems, some types of significant weak solutions like the peakons[22, 23] and compactons[25] have been attract much attention of both mathematicians and physicists.
In [15] and [23], the possible (2+1)-dimensional localized peakons had been given. The (1+1)-dimensional compacton solutions which describes the typical (1+1)-dimensional soliton solutions with finite wavelength when the nonlinear dispersion effects were firstly given by Rosenau et al. [24] and may have many interesting properties and possible physical applications [25]–[26]. For instance, the compacton equations may be used to study the motion of ion-acoustic waves and a flow of a two layer liquid [27]. In [26], the Painlevé integrability of two sets of Korteweg-de Vries (KdV) type and modified KdV type compacton equations are proved. Because of the entrance of arbitrary functions in the (2+1)-dimensional nonlinear physics models, it is easy to find some types of multiple compacton solutions by selecting the arbitrary functions appropriately. For instance, if we fixed the functions $p$ and $q$ as

$$ p = \sum_{i=1}^{N} \begin{cases} 0 & x + c_i t \leq x_{0i} - \frac{\pi}{2k_i} \\ b_i \sin(k_i(x + c_i t - x_{0i})) + b_i & x_{0i} - \frac{\pi}{2k_i} < x + c_i t \leq x_{0i} + \frac{\pi}{2k_i} \\ 2b_i & x + c_i t > x_{0i} + \frac{\pi}{2k_i} \end{cases}, \quad (31) $$

and

$$ q = \sum_{j=1}^{M} \begin{cases} 0 & y \leq y_{0j} - \frac{\pi}{2l_j} \\ d_j \sin(l_j(y - y_{0j})) + d_j & y_{0j} - \frac{\pi}{2l_j} < y \leq y_{0j} + \frac{\pi}{2l_j} \\ 2d_j & y > y_{0j} + \frac{\pi}{2l_j} \end{cases}, \quad (32) $$

where $b_i$, $k_i$, $x_{0i}$, $d_j$, $l_j$ and $y_{0j}$ are all arbitrary constants, then the solution (27) with (30) becomes a multi-compacton solution.

From (31) and (32), one can see that the piecewise functions $p$ and $q$ of the compacton solutions are once differentiable

$$ p_x = \sum_{i=1}^{N} \begin{cases} 0 & x + c_i t \leq x_{0i} - \frac{\pi}{2k_i} \\ b_i k_i \cos(k_i(x + c_i t - x_{0i})) & x_{0i} - \frac{\pi}{2k_i} < x + c_i t \leq x_{0i} + \frac{\pi}{2k_i} \\ 0 & x + c_i t > x_{0i} + \frac{\pi}{2k_i} \end{cases}, \quad (33) $$

and

$$ q_y = \sum_{j=1}^{M} \begin{cases} 0 & y \leq y_{0j} - \frac{\pi}{2l_j} \\ d_j l_j \cos(l_j(y - y_{0j})) & y_{0j} - \frac{\pi}{2l_j} < y \leq y_{0j} + \frac{\pi}{2l_j} \\ 0 & y > y_{0j} + \frac{\pi}{2l_j} \end{cases}, \quad (34) $$

So the multi-compacton solutions expressed by (31) (i.e., (27) with (30)) are still continuous any where though there are some isolated non-differentiable lines.
In (1+1)-dimensions, a non-differentiable solution, \( u = u_0 \), like the the compacton (and/or peakon) is called a weak solution of a nonlinear (1+1)-dimensional PDE (partial differential equation)

\[
F(u, u_x, u_t, u_{xx}, \ldots) \equiv F(u) = 0
\]  

under the meaning that though the compacton (and/or peakon) solution is non-differentiable at some points \( x = x_i(t) \), the distribution, \( F(u_0) = f(\delta(x - x_i(t))) \) (where \( \delta(x - x_i(t)) \) is a Dirac \( \delta \) function and \( f \) is a function of the Dirac \( \delta \) function and its derivatives) is really a zero distribution that means \( \int_{-\infty}^{+\infty} f\psi(x, t) = 0 \) for arbitrary \( \psi \).

However, in (2+1)-dimensions, the (2+1)-dimensional compactons (and the peakons reported in [15]) are exact solutions of some (2+1)-dimensional nonlinear equations are guaranteed by the variable separation procedure. When we substituting the piecewise solutions into the nonlinear PDEs, the Dirac delta function(s) in \( x \) (\( y \)) directions will be vanished by the differential operator in other direction, \( \partial_y (\partial_x) \). In other words, we can take (have taken) \( \partial_y \psi(x, t) = 0 \) no matter the function \( \psi \) is a continuous function of \( \{x, t\} \) or a generalized distribution functions (with some Dirac delta functions) of \( \{x, t\} \).

Fig.1 is the evolution plot of a three compacton solution (27) with (30), (31), (32) and

\[
N = 3, \ M = 1, \ a_0 = 20, \ a_1 = a_2 = 25a_3 = 1, \ b_1 = b_3 = -c_2 = c_3 = -2, \nonumber \\
-b_2 = -c_1 = d_1 = k_1 = k_2 = k_3 = l_1 = 1, \ x_{01} = x_{02} = x_{03} = y_{01} = 0. \tag{36}
\]

In [15], we have pointed out that (i) the interaction between two travelling ring shape soliton solutions is completely elastic and (ii) the interaction between two travelling peakons is not completely elastic, two peakons may completely exchange their shapes. From Fig. 1, we see that the interaction between two compactons exhibits a new phenomenon. The interaction is nonelastic but two compactons do not completely exchange their shapes.
Figure 1. Evolution plot for the quantity $G \equiv -1000u[2]$ related to the three compacton solution (27) with (30), (31), (32) and (36) at times (a) $t = -6$, (b) $t = -2$, (c) $t = 0$, (d) $t = 3$, (e) $t = 6$. 
Figure 2. Evolution plot for the quantity \( u_2 \equiv u[2] \) expressed by (39) at times \( t = 0 \).

Because of the arbitrariness of the functions \( p \) and \( q \) may have also quite rich structures. For instance, (2+1)-dimensional compactons may be only compacted at one direction. For convenience later we call this type of compactons the partial compactons. Fig. 2 is a plot of a special partial compacton solution (27) with (30), (31), (33),

\[
N = 1, \; a_0 = 20, \; a_1 = a_2 = 25a_3 = -c_1 = k_1 = 1, \; x_{01} = 0.
\]

and

\[
q = 10 \tanh(y - t),
\]

at \( t = 0 \). For the partial compacton shown by Fig. 2, the quantity \( G \) possesses the following quite simple form (\( u_2 \equiv u[2] \))

\[
u_2 = \begin{cases} 
0 & \text{if } |x - t| \geq \frac{1}{2} \pi, \\
\left(100 \cos(x - t) \sech^2(y - t) \right) & \text{if } |x - t| < \frac{1}{2} \pi.
\end{cases}
\]

(39)

Similar to the first kind of compacton (full compacted) solution as shown in Fig. 1, the detailed study of the interaction between partial compactons is also non-completely elastic and they will partially exchange their shapes.

If the arbitrary functions \( q_1 \) and \( p_2 \) are not taken as in (26) but are solved out from (23) and (24), then we can obtain some further new types of exact solutions which can not be obtained by the usual variable separation approach. Actually, when the arbitrary functions \( p_1 \) and \( q_2 \) are fixed by (22) and (25) at the same time, then the
corresponding solution(s) for the functions \( q_1 \) and \( p_2 \) can be found by solving the linear equations (24) and (23).

In principle, the more kinds of exact solutions can be obtained from further Darboux transformations starting from the seed solution \{\[(18), (19), (20), (21)\}\}. However, we do not discuss these types of solutions further because of their complexity.

In summary, the variable separation solutions can be obtained not only by the truncated Painlevé expansion and the generalized multi-linear equations, but also by other well known approaches especially by the Darboux transformation. In this short paper, the Darboux transformation is successfully used to find the variable separable solutions of the (2+1)-dimensional NNV equation. The recursive Darboux transformation may yield further new types of variable separable solutions while the new variable separable function should satisfy a further constrained condition, say, (24) with \( v_0 \) being given by (22).

By selecting the arbitrary functions appropriately, one may obtain abundant localized excitations like the dromions, lumps, ring solitons, breathers, instantons, solitoffs, peakons, fractal and chaotic patterns. In addition to these types of localized excitations, a further type of the localized excitations, compactons, is given in this paper. The (2+1)-dimensional compactons discussed here possess the different type of interaction properties as that of the ring solitons and peakons. The interactions among compactons are not completely elastic and do not exchange their shapes completely.

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