Update report: LEO-II version 1.5

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Abstract. Recent improvements of the LEO-II theorem prover are presented. These improvements include a revised ATP interface, new translations into first-order logic, rule support for the axiom of choice, detection of defined equality, and more flexible strategy scheduling.

Keywords: Automated Theorem Proving, Classical Higher-Order Logic

1 Introduction

It has been five years since the last system description of Leo-II [6], and during the last months various improvements have been made to the system. In this article we outline the current system and describe the recent improvements.

2 System overview

LEO-II is written in OCaml and implements a RUE calculus [12] which relies on a ‘Boolean aware’ (or, more generally, ‘theory aware’ [3]) extensional preunification engine. LEO-II accepts problems encoded in the CNF (clausal first-order form) and FOF (first-order form) languages from the TPTP [15], but its principal input language is THF0, core typed higher-order form [16].

The logical organisation of the prover is illustrated in Figure 1 and this roughly corresponds to the modular organisation of the code. It is structured into four layers, as the figure shows:

Operating mode. The prover can be operated in two ways: (i) LEO-II can be used as a proof assistant when run in interactive mode. It provides a command interface through which the user can inspect and manipulate the prover’s state, making calls to the calculus’ rules as needed. This mode is very valuable for exploring logical problems and for debugging the prover’s automatic mode. (ii) The prover is usually run in automatic mode: this comprises a set of strategy schedules, and a main loop which drives applications of the calculus’ rules.

Prover interface. Both modes use a common infrastructure: they parse a problem and load it into the prover’s state, then further manipulate the state by executing commands. A command might involve carrying out an inference, inspecting the state, switching flags, calling external provers, etc. Each command makes calls to lower levels of the prover.
Logic. The main component in this level consists of the calculus: a collection of functions which accept and return clauses. This level also contains LEO-II’s main loop, and an interface to external ATPs (which also translates problems to other formats).

Basis. The lowest level of LEO-II defines the representation of terms and types, and associated operations (e.g. substitution, unification, matching, etc).

3 Improvements

The TPTP problem set is the canonical benchmark by which theorem provers are presently evaluated. We accompany the description of improvements in this section with TPTP problem names whose solution is affected by the feature. These problems consist of THF problems drawn from TPTP 5.4.0. We have used E version 1.6 as the backend ATP. Our tests were run on a 2GHz AMD Opteron with 4GB RAM, and given 60-second timeout. LEO-II was compiled with OCaml 3.11.2.

3.1 ATP interface

LEO-II cooperates with other provers in order to maximise its potential. We improved LEO-II’s translation to FOL in recognition of this. Version 1.5 includes a better translation into FOF, an experimental translation into TFF [14], and supports additional backend ATPs.

Translation into FOL. Alongside the old translations which were previously implemented in LEO-II, version 1.5 features a new translation module which was written from scratch. This module contains an intermediate language to which problems are first translated, before being transformed further and printed into a specific target syntax. HOL-to-FOL translations consist of a pipeline of functions which bring HOL formulas into this intermediate language, applying
analyses and transformations along the way. We are also experimenting with lighter encoding of type information. We have closely followed Claessen et al [7] to implement their monotonicity analysis by producing a SAT encoding, which we send to MiniSat using an interface adapted from Satallax [2].

LEO-II’s old and new FOF encodings can be used via the command-line arguments --translation fully-typed and --translation fof_full respectively. The gain of fof_full over fully-typed is due to improved handling of formulas—for instance, the new FOF translation implements full λ-lifting, which the old translation didn’t. The fof_full translation is now set as default.

Backend ATPs. LEO-II is mainly used in combination with E [11], and version LEO-II 1.5 features small improvements in how it interacts with E. In version 1.5 we improved LEO-II’s ATP interface and added support for various other backend ATPs, including remote provers on SystemOnTPTP [15].

3.2 Support for Axiom of Choice

The default semantics for THF0 is Henkin semantics with choice. Until version 1.5, LEO-II did not support reasoning with choice, unless naïve Skolemization was used—that is, first-order Skolemization without employing further restrictions (as investigated by Miller [8]). This enables limited reasoning with choice, and succeeds in some example cases, but it fails in many others [5, Section 3.2].

In order to extend LEO-II to support the axiom of choice (AC), instances of AC could be automatically added to the input problem. An example is the following instance of AC for type \((ι \rightarrow o) \rightarrow ι\):

\[
\exists E (ι \rightarrow o) \rightarrow ∀P (ι \rightarrow o) \Rightarrow P (E P) \quad (1)
\]

However, such kinds of impredicative axioms should generally be avoided in automated proof search since they allow for simulation of the cut rule in any Henkin-complete THF prover [4].

Our approach involves adding two new rules to LEO-II: detectChoiceFn and choice. The first rule detects and removes instances of AC, such as (1) above, and keeps a register of choice functions CFs. CFs always contains at least one choice function symbol for each choice type. The second rule gives the semantics to choice functions. Taken together, these rules allow AC-valid reasoning without the risk of cut-simulation.

In more detail, rule detectChoiceFn removes choice-axiom clauses from the search space and registers the corresponding choice function symbols \(f\) in CFs.

\[
\frac{[PX]^\# \lor [P(f(\alpha \rightarrow o) \rightarrow \alpha) P]^\#}{\text{detectChoiceFn}}
\]

Rule choice investigates whether a term \(\epsilon(\alpha \rightarrow o) \rightarrow ι B_{\alpha \rightarrow o}\) (where \(\epsilon \in \text{CFs}\) is a registered choice function or a free variable) is contained as a subterm of a literal \([A]^\#\) in a clause \(C\). In this case it adds the instantiation of AC at type
(α \rightarrow o) \rightarrow α$, and with term $B$, to the search space. Side-conditions guard against unsound reasoning, such as the ‘uncapturing’ of free variables in $B$:

$$C := C' \lor \left[ A[E(\alpha \rightarrow o) \rightarrow αB] \right]^p \quad \text{freeVars}(C) \subseteq \text{freeVars}(B), Y \text{ fresh}$$

Rules $\text{detectChoiceFn}$ and $\text{choice}$ are obviously sound: $\text{detectChoiceFn}$ simply removes clauses from the search space, and for any choice function $f$, the rule $\text{choice}$ only introduces new instances of the corresponding choice axiom.

There is a correspondence with the handling of choice in Satallax. Satallax too considers only selective instantiations of AC in order to avoid cut-simulation. For instance, when $[1]$ is assumed, the terms $T$ which Satallax considers to be eligible instantiations for variable $P$ are those occurring in formulas of the following forms in a tableau branch (and where $ε$ is a choice function): $(ε T) S_1 \ldots S_n$ or $\neg((ε T) S_1 \ldots S_n)$, or the disequations $(ε T)S_1 \ldots S_n \neq S$ or $S \neq (ε T)S_1 \ldots S_n$.

It is easy to see that our rule $\text{choice}$, which is less restrictive, subsumes these cases. We also experimented with Satallax’s approach in $\text{Leo-II}$ but this led to worse results. Our choice rule is more closely related to that of Mints $[9]$. Use of the choice rules can be disabled using the $-\text{nuc}$ command-line switch.

### 3.3 Detection of defined equality

*Primitive equality* in HOL refers to the use of the interpreted constant ‘$=$’. Equality can also be defined in HOL—for example, as $\lambda X_α\lambda Y_α\forall P_α→o. P X \Rightarrow P Y$ or $\lambda X_α\lambda Y_α\forall Q_α→o. \forall Z_α(Q_α Z_α) \Rightarrow Q X Y$. The former is known as Leibniz equality and the latter we call Andrews equality (cf. $[1]$, Exercise X5303). Both Leibniz and Andrews equality support cut-simulation due to their impredicative nature $[4]$, and should thus be avoided in proof automation. In fact, using primitive, rather than defined, equality may save many primitive substitution steps in proofs. Such steps involve instantiations of set variables, and this generally involves blind guessing. Examples of the benefit of using primitive, rather than defined, equality have been given in the literature $[5, \text{Sections 5.1 and 5.2}]$. In order to address this issue we added the following two rules to $\text{Leo-II}$’s calculus; they instantiate the set variable $P$ with primitive equality:

$$\frac{C \lor [P A]^Γ \lor [P B]^Γ}{C(\lambda X. A = X/P) \lor [A = B]^Γ} \text{LeibEQ} \quad \frac{C \lor [P A A]^Γ}{C(\lambda X\lambda Y. X = Y/P)} \text{AndrEQ}$$

Soundness of $\text{LeibEQ}$ and $\text{AndrEQ}$ is obvious, since both rules simply realise specific instances of primitive substitution. For improved configurability, either rule can be individually disabled from the command-line by using the switches $-\text{nrleq}$ and $-\text{nraeq}$ respectively. If $\text{LeibEQ}$ is used in combination with the new FOF translations (see Section 5.1) several TPTP problems whose previous SZS $[13]$ status was ‘Unknown’ can now be solved by $\text{Leo-II}$. Examples include SYO246*5.p, SYO244*5.p, NUM817*5.p, NUM816*5.p, and NUM814*5.p.
There are also many problems that can now be solved with primitive substitution (blind guessing) disabled when LeibEQ and AndrEQ are available. Overall, these two new rules lead to significantly better coverage using the lighter primitive-substitution search modes -ps 0 or -ps 1.

3.4 Strategy scheduling

Strategy schedules were added to LEO-II in version 1.2 and the catalogue of schedules has slowly increased in the versions that followed. In version 1.5 we recoded the strategy-scheduling feature to facilitate the encoding of new strategies, to improve code reuse with other parts of LEO-II, and to have greater flexibility when encoding strategies.

We are also interested in computing strategies on-the-fly based on problem characteristics, and version 1.5 carries out some small initial checks (e.g. size of the problem, and whether it contains instances of AC), and schedules strategies based on that limited analysis. Optimising this further remains as future work.

3.5 Other improvements

Numerous other additions were made to LEO-II. Previously, LEO-II was entirely focused on refutation: that is, until version 1.5, in terms of the SZS classification, LEO-II would judge a problem to be a Theorem (if a refutation exists), Unsatisfiable (if the problem’s axioms themselves can be refuted), or diverge (by extending the preunification depth and reattempting a refutation). It can now classify Satisfiable problems and detect CounterSatisfiable problems, thus improving both LEO-II’s precision and termination behaviour. The added support for choice was very relevant for achieving this.

LEO-II’s unification algorithm has been redone, and can be set (from the command-line) to disregard Boolean and functional extensionality. This has strengthened LEO-II’s behaviour in non-extensional problems, since disabling the extensional behaviour shrinks the search space.

Numerous other improvements and fixes have been made: these range from system features (such as the parser, status reporting, avoiding redundant computations, etc) to deeper areas in the calculus and main loop (including factorisation, subsumption, and clause selection).

4 Future work

We have started experimenting with using term orderings to influence literal selection. We also plan to revise LEO-II’s internals to make full use of the potential benefit they offer. For instance, the shared term graph is currently underutilised.

More work is needed to compute better schedules, paired with better problem analyses. Such analyses can determine the scheduling of specific strategies, which can be better tuned to the problem.
The ATP interface can be improved further to call multiple backend ATPs in parallel. Experiments comparing 30-second invocations of LEO-II on all THF problems, supported by provers E (version 1.6), SPASS (version 3.5) [17] and Vampire (version 2.6) [10] showed us that there were 37, 5 and 20 theorems that were proved exclusively by LEO-II(E), LEO-II(SPASS) and LEO-II(Vampire), respectively. And there were 31, 95 and 98 theorems that LEO-II(E), LEO-II(SPASS) and LEO-II(Vampire) missed, but which one of the others could prove.

Supporting various ATP backends increases the scope for peephole optimisation; we have not yet investigated this. The translation module can be optimised further, and extended to target more formats. Table 1 shows how the new HOL-to-FOL translation (fof\_full) and its lighter variant (fof\_experiment) are superior to LEO-II’s preexisting encoding (fully\_typed). In future work we plan to improve fof\_experiment further and make it the default translation.

## 5 Conclusion

Version 1.5 of LEO-II includes various improvements which affect its performance and completeness. To obtain a broader picture, we compared the results of using LEO-II version 1.5 with earlier versions, and the results are shown in Table 2.

### Table 1

| SZS Status | fully-typed | fof\_full | fof\_experiment |
|------------|-------------|-----------|-----------------|
| Thm        | 64.8        | 64.9      | 65.3            |
| All        | 60.9        | 61        | 61.3            |

Table 1. Comparing FOL encodings in LEO-II 1.5 (30s timeout). Table shows the percentage of matches between LEO-II’s SZS output and the ‘Status’ field of problems.

### Table 2

| Timeout (s) | v1.2 | v1.4.3 | v1.5 |
|-------------|------|--------|------|
|             | Thm  | All    | Thm  | All  |
| 30          | 58.4 | 51.1   | 62.1 | 54.4 |
| 60          | 58.7 | 51.3   | 65   | 56.9 |

Table 2. Percentage match between different versions of LEO-II and the Status field of TPTP problems. LEO-II version 1.2 was the winner of the CASC competition in 2010, and version 1.4.3 was the last public release. Version 1.5 was run with the fof\_experiment encoding.

This also means that ‘Unknown’ problems which LEO-II now classifies as ‘Theorem’ count against us, but this experiment was only intended to offer a rough idea of progress.
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References

1. P.B. Andrews. An Introduction to Mathematical Logic and Type Theory: To Truth Through Proof. Applied Logic Series. Springer, 2002.
2. J. Backes and C.E. Brown. Analytic tableaux for higher-order logic with choice. Journal of Automated Reasoning, 47(4):451–479, 2011.
3. C. Benzmüller. Comparing approaches to resolution based higher-order theorem proving. Synthese, 133(1-2):203–235, 2002.
4. C. Benzmüller, C.E. Brown, and M. Kohlhase. Cut-simulation and impredicativity. Logical Methods in Computer Science, 5(1:6):1–21, 2009.
5. C. Benzmüller and C.E. Brown. A Structured Set of Higher-Order Problems. Proc. of TPHOLs 2005, number 3603 in LNCS, pp. 66–81. Springer, 2005.
6. C. Benzmüller, F. Theiss, L. Paulson, and A. Fietzke. LEO-II - a cooperative automatic theorem prover for higher-order logic. Proc. of IJCAR 2008, vol. 5195 of LNCS, pp. 162–170. Springer, 2008.
7. K. Claessen, A. Lillieström, and N. Smallbone. Sort it out with monotonicity: translating between many-sorted and unsorted first-order logic. Proc. of CADE 2011, vol. 6803 of LNCS, pp. 207–221. Springer, 2011.
8. D.A. Miller. Proofs in Higher-Order Logic. PhD thesis, Carnegie Mellon U., 1983.
9. G. Mints. Cut-elimination for simple type theory with an axiom of choice. Journal of Symbolic Logic, 64(2):479-485, 1999.
10. A. Riazanow and A. Voronkov. The design and implementation of VAMPIRE. AI Commun. 15(2):91–110, 2002.
11. S. Schulz. E – A Brainiac Theorem Prover. AI Commun., 15(2/3):111–126, 2002.
12. N. Sultana and C. Benzmüller. Understanding LEO-II’s proofs. The 9th International Workshop on the Implementation of Logics (IWIL-2012, affiliated with LPAR-2012), Merida, Venezuela, 2012.
13. G. Sutcliffe. The SLS ontologies for automated reasoning software. Proc. of the LPAR Workshops: Knowledge Exchange: Automated Provers and Proof Assistants, and The 7th International Workshop on the Implementation of Logics, vol. 418, pp. 38–49. CEUR Workshop Proc., 2008.
14. G. Sutcliffe, S. Schulz, K. Claessen, and P. Baumgartner. The TPTP typed first-order form with arithmetic. Proc. of LPAR 2012, vol. 7180 of LNCS, pp. 406–419. Springer, 2012.
15. G. Sutcliffe. The TPTP problem library and associated infrastructure. Journal of Automated Reasoning, 43(4):337–362, 2009.
16. G. Sutcliffe and C. Benzmüller. Automated Reasoning in Higher-Order Logic using the TPTP THF Infrastructure. Journal of Formalized Reasoning, 3(1):1, 2010.
17. C. Weidenbach, D. Dimova, A. Fietzke, R. Kumar, M. Suda and P. Wischnewski. SPASS Version 3.5. Proc. of CADE 2009, vol. 5663 of LNCS, pp. 140–145, Springer.