THE LOOP HOMOLOGY ALGEBRA OF SPHERES AND PROJECTIVE SPACES

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Abstract. In [3] Chas and Sullivan defined an intersection product on the homology \( H^*(LM) \) of the space of smooth loops in a closed, oriented manifold \( M \). In this paper we will use the homotopy theoretic realization of this product described by the first two authors in [2] to construct a second quadrant spectral sequence of algebras converging to the loop homology multiplicatively, when \( M \) is simply connected. The \( E_2 \) term of this spectral sequence is \( H^*(M; H^*(\Omega M)) \) where the product is given by the cup product on the cohomology of the manifold \( H^*(M) \) with coefficients in the Pontryagin ring structure on the homology of its based loop space \( H_*(\Omega M) \). We then use this spectral sequence to compute the ring structures of \( H_*(LS^n) \) and \( H_*(L\mathbb{C}P^n) \).

Introduction

The loop homology of a closed orientable manifold \( M^d \) of degree \( d \) is the ordinary homology of the free loop space \( LM = \text{Map}(S^1, M^d) \), with degree shifted by \(-d\), i.e.

\[
\mathbb{H}_*(LM; \mathbb{Z}) = H_{*+d}(LM; \mathbb{Z}).
\]

In [3], Chas and Sullivan defined a type of intersection product on the chains of \( LM \), yielding an algebra structure on \( H_*(LM) \).

Roughly, the loop product is defined as follows. Let \( \alpha : \Delta^p \to LM \) and \( \beta : \Delta^q \to LM \) be singular simplices in \( LM \). The evaluation at \( 1 \in S^1 \subset \mathbb{C} \) defines a map \( \text{ev} : LM \to M \). Assume that \( \text{ev} \circ \alpha \) and \( \text{ev} \circ \beta \) define a map

\[
\Delta^p \times \Delta^q \to LM \times LM \to M \times M
\]

that is transverse to the diagonal. At each point \((s, t) \in \Delta^p \times \Delta^q\) where \( \text{ev} \circ \alpha \) intersects \( \text{ev} \circ \beta \), one can define a single loop by first traversing the loop \( \alpha(s) \) and then traversing the loop \( \beta(t) \). This then defines a chain \( \alpha \circ \beta \in C_{p+q-d}(LM) \). In [3] Chas and Sullivan showed that this procedure defines a chain map

\[
C_p(LM) \otimes C_q(LM) \to C_{p+q-d}(LM)
\]

which induces an associative, commutative algebra structure on the loop homology, \( \mathbb{H}_*(LM) \). Chas and Sullivan also described other structures this pairing induces, such as a Lie algebra structure on the equivariant homology of the loop space. In [3], the first two authors used the Pontryagin -

Date: May 30, 2018.
The first author was partially supported by a grant from the NSF.
Thom construction to show that the loop product is realized on the homotopy level on Thom spaces (spectra) of bundles over the loop space. In particular let $TM$ denote the tangent bundle of $M$, and $-TM$ denotes its inverse as a virtual bundle in $K$-theory. Let $M^{-TM}$ denote the Thom spectrum of this bundle, and $LM^{-TM}$ the Thom spectrum of $ev^*(-TM)$. Then in \[2\] it was shown that $LM^{-TM}$ is a homotopy commutative ring spectrum with unit, whose product realizes the Chas-Sullivan product in homology, after applying the Thom isomorphism, $H_q(LM^{-TM}) \cong H_{q+d}(LM) \cong Hq(LM)$.

The goal of this paper is to describe a spectral sequence of algebras converging to the loop homology algebra of a manifold, and to use it to compute the loop homology algebra of spheres and projective spaces. More specifically we shall prove the following theorems.

**Theorem 1.** Let $M$ be a closed, oriented, simply connected manifold. There is a second quadrant spectral sequence of algebras \(\{E^r_{p,q}, d^r : p \leq 0, q \geq 0\}\) such that

1. \(E^r_{*,*}\) is an algebra and the differential \(d^r : E^r_{*,*} \to E^r_{*,*-r+r-1}\) is a derivation for each \(r \geq 1\).
2. The spectral sequence converges to the loop homology \(H_*(LM)\) as algebras. That is, \(E^\infty_{*,*}\) is the associated graded algebra to a natural filtration of the algebra \(H_*(LM)\).
3. For \(m, n \geq 0\),
   \[
   E^2_{-m,n} \cong H^m(M; H_n(\Omega M)).
   \]
   Here \(\Omega M\) is the space of base point preserving loops in \(M\). Furthermore the isomorphism \(E^2_{*,*} \cong H^*(M; H_*(\Omega M))\) is an isomorphism of algebras, where the algebra structure on \(H^*(M; H_*(\Omega M))\) is given by the cup product on the cohomology of \(M\) with coefficients in the Pontrjagin ring \(H_*(\Omega M)\).
4. The spectral sequence is natural with respect to smooth maps between manifolds.

We then use this spectral sequence to do the following calculations. Let \(\Lambda[x_1, \ldots, x_n]\) denote the exterior algebra (over the integers) generated by \(x_1, \ldots, x_n\), and let \(\mathbb{Z}[a_1, \ldots, a_m]\) denote the polynomial algebra generated by \(a_1, \ldots, a_m\).

**Theorem 2.** There exist isomorphisms of graded algebras,

1. \(H_*(LS^1) \cong \Lambda[a] \otimes \mathbb{Z}[t, t^{-1}]\)
   where \(a \in \mathbb{H}_{-1}(LS^1)\) and \(t, t^{-1} \in \mathbb{H}_0(LS^1)\).
2. For \(n > 1\),
   \[
   H_*(LS^n) = \begin{cases}
   \Lambda[a] \otimes \mathbb{Z}[u], & \text{for } n \text{ odd} \\
   (\Lambda[b] \otimes \mathbb{Z}[a, v])/(a^2, ab, 2av), & \text{for } n \text{ even},
   \end{cases}
   \]
   where \(a \in \mathbb{H}_{-n}(LS^n)\), \(b \in \mathbb{H}_{-1}(LS^n)\), \(u \in \mathbb{H}_{n-1}(LS^n)\), and \(v \in \mathbb{H}_{2n-2}(LS^n)\).
Theorem 3. There is an isomorphism of algebras,
\[ \mathbb{H}_*(L\mathbb{C}P^n) \cong (\Lambda[w] \otimes \mathbb{Z}[c, u])/(c^{n+1}, (n+1)c^n u, wc^n) \]
where \( w \in \mathbb{H}_{-1}(L\mathbb{C}P^n) \), \( c \in \mathbb{H}_{-2}(L\mathbb{C}P^n) \), and \( u \in \mathbb{H}_{2n}(L\mathbb{C}P^n) \).

The organization of this paper is as follows. In section 1 we will review the construction of the loop product that was used in [2] and describe it on the chain level. In section 2 we will construct the spectral sequence and prove theorem 1. In section 3 we use this spectral sequence to do the calculations presented in theorems 2 and 3.

1. The Loop Product

For the remainder of the paper, let \( M \) be a closed, connected, oriented manifold of dimension \( d \). The goal of this section is to give a description of the loop product on \( \mathbb{H}_*(LM) \) at the chain level. Of course this was done originally by Chas and Sullivan in [3], but our approach will be slightly different. It will be more amenable to the construction and the analysis of the loop homology spectral sequence to be done in the next section.

We begin by recalling a chain level description of the intersection product arising from Poincare duality on the homology of the manifold,
\[ \langle, \rangle : H_q(M) \otimes H_p(M) \to H_{p+q-d}(M). \]

For pairs of spaces \((X, A)\) with \( A \subset X \), denote by \( C_\ast(X, A) \) and \( C^\ast(X, A) \) the groups of singular chains and cochains.

Recall that the normal bundle of the diagonal embedding \( \Delta : M \to M \times M \) is naturally isomorphic to the tangent bundle \( TM \). Let \( M^{TM} \) denote the Thom space of this bundle. The Thom-Pontryagin map for the diagonal embedding is therefore a map
\[ \tau : M \times M \to M^{TM}, \]
which collapses everything outside a tubular neighbourhood of the diagonal to the base point of \( M^{TM} \).

Choose a Riemannian metric on \( TM \) and let \( DT \) and \( ST \) be the unit disk and sphere bundles of \( TM \). Since \( ST \) has a collar, \( ST \times I \to DT \), there is a chain equivalence
\[ \theta_t : C_\ast(DT/ST, *, \ast) \to C_\ast(DT \cup cST, cST) \to C_\ast(DT, ST) \]
where \( * \) is the base point of \( DT/ST \), \( cST \) is the cone on \( ST \), and \( DT \cup cST \) is the mapping cone of the inclusion \( ST \to DT \). Let \( t \in C^d(DT, ST) \) be a cochain that represents the Thom class \( [t] \in H^d(DT, ST) \). Then taking the cap product with \( t \) at the chain level followed by the projection from \( DT \) to \( M \),
\[ \sigma_t : C_\ast(DT, ST) \xrightarrow{\cap t} C_{\ast-d}(DT) \xrightarrow{\pi_*} C_{\ast-d}(M) \]
induces the Thom isomorphism
\[ \tilde{H}_s(M^TM) \xrightarrow{\theta_s} H_*(D TM, S TM) \xrightarrow{\sigma_s} H_{*+d}(M) \]
in homology.

Let \( \epsilon : C_*(D TM/S TM) \to C_*(D TM/S TM, *) \) be the projection map onto the quotient complex, and denote by \( u_\sharp \) the composition of the chain maps
\[ u_\sharp : C_*(M^TM) \xrightarrow{\epsilon_\sharp} C_*(D TM/S TM, *) \xrightarrow{\theta_\sharp} C_*(D TM, S TM) \xrightarrow{\sigma_\sharp} C_{*-d}(M). \]

Now let \( A \) be a graded ring. Recall that the intersection product structure on \( H_*(M; A) \) is defined so that the Poincaré duality isomorphism
\[ D : H^*(M; A) \to H_{d-*}(M; A) \]
\[ \alpha \to \alpha \cap [M] \]
is an isomorphism of graded algebras. Here \([M] \in H_d(M; A)\) is the fundamental class determined by the orientation of \( M \). The commutativity of the following diagram describes the well-known relation between the intersection product, the Thom-Pontryagin map and the Thom isomorphism:

\[ H_{p+q}(M \times M) \xrightarrow{\tau_p} H_{p+q}(M^TM) \xrightarrow{u_*} H_{p+q-d}(M) \]

where \(< \cdot, \cdot >\) is the intersection product on \( H_*(M) \), and \( \times \) denotes the cross product. We therefore have the following chain description of the intersection pairing:

**Proposition 4.** Let \( M \) be as above. Then the following composition of chain maps
\[ C_p(M) \otimes C_q(M) \xrightarrow{\times} C_{p+q}(M \times M) \xrightarrow{\tau_{p+q}} C_{p+q}(M^TM) \xrightarrow{u_*} C_{p+q-d}(M) \]
is a representative of the intersection product
\[ (-1)^{d(d-p)}< \cdot, \cdot > : H_p(M) \otimes H_q(M) \to H_{p+q-d}(M). \]

We next turn our attention to the loop space and the loop product. Let \( LM = C^\infty(S^1, M) \) be the free loop space of \( M \), where \( S^1 \) denotes the unit circle in the complex line, parametrized by \( \exp : [0, 1] \to S^1 \) with \( 1 \in S^1 \) chosen as the basepoint. We now recall some constructions in [2].

The loop evaluation map
\[ ev : LM \to M \]
\[ \gamma \to \gamma(1), \]
is a Serre fibration. Let $LM \times_M LM$ denote the pull back of the product of this fibration with itself along the diagonal embedding $\Delta : M \to M \times M$, denoted by $LM \times_M LM$:

$$LM \times_M LM \xrightarrow{\Delta} LM \times LM \xrightarrow{\ev_{\infty}} M \to M \times M.$$ 

Notice that $LM \times_M LM$ is the space of pairs of loops having the same basepoint. The map $\ev_{\infty}$ in this fiber square is given by $\ev_{\infty}(\alpha, \beta) = \alpha(1) = \beta(1)$. The map $\tilde{\Delta}$ is an embedding of a codimension $d$ infinite dimensional submanifold of the infinite dimensional manifold $LM \times LM$.

As shown in [2] this pullback square allows for a Thom - Pontryagin map $\tilde{\tau} : LM \times LM \to (LM \times_M LM)^{TM}$, where $(LM \times_M LM)^{TM}$ denotes the Thom space of the pull back bundle $ev_{\infty}^*(TM)$, which is the normal bundle of the embedding $\tilde{\Delta}$. Notice that we have a commutative diagram of Thom - Pontryagin maps:

$$LM \times LM \xrightarrow{\tau} (LM \times_M LM)^{TM} \xrightarrow{\ev_{\infty}} M^{TM}.$$ 

In this diagram the map $\ev_{\infty}$ is the induced map on the Thom spaces.

Loop composition $\alpha \beta$ is defined for two loops $\alpha$ and $\beta$ having the same base point by first traversing the loop $\alpha$, then the loop $\beta$, i.e.

$$\alpha \beta(t) = \begin{cases} 
\alpha(2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\
\beta(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1.
\end{cases}$$

Denote this operation by

$$\gamma : LM \times_M LM \to LM$$

$$(\alpha, \beta) \to \alpha \beta.$$ 

Notice that $\gamma$ preserves the base points of loops in $LM \times_M LM$ and $LM$, thus the composition

$$LM \times_M LM \xrightarrow{\gamma} LM \xrightarrow{\ev} M$$ 

coincides with $\ev_{\infty}$. Therefore $\gamma$ induces a map of bundles, $\gamma : ev_{\infty}^*(TM) \to ev^*(TM)$, and therefore an induced map of Thom spaces,

$$\tilde{\gamma} : (LM \times_M LM)^{TM} \to LM^{TM}.$$ 

Putting the above maps together, we obtain the following commutative diagram

$$LM \times LM \xrightarrow{\tilde{\tau}} (LM \times_M LM)^{TM} \xrightarrow{\tilde{\gamma}} LM^{TM}$$ 

$$M \times M \xrightarrow{\tau} M^{TM} \xrightarrow{=} M^{TM}.$$
Let $D_L M$ and $S_L M$ be the pulls back of $D_T M$ and $S_T M$ via $ev$. They are the unit disk and sphere bundles of $ev^*(TM)$. Recall that the cochain $t$ in $C^d(D_T M, S_T M)$ represents the Thom class and therefore its pull back $\tilde{t} = ev^*(t)$ is a cochain in $C^d(D_L M, S_L M)$ representing the Thom class of $ev^*(TM)$. Similar to (1.2), capping with $\tilde{t}$ at the chain level followed by the projection of $D_L M$ to $LM$

$$\tilde{\sigma}_t : C_\ast(D_L M, S_L M) \xrightarrow{\gamma \tilde{t}} C_{\ast-d}(D_L M) \xrightarrow{\tilde{\pi}_\ast} C_{\ast-d}(LM)$$

induces the Thom isomorphism

$$\tilde{H}_\ast(LMT M) \xrightarrow{\tilde{\sigma}_t} H_\ast(D_L M, S_L M) \xrightarrow{\tilde{\sigma}_t} H_{\ast-d}(LM)$$

in homology. Similar to the argument used for (1.3), there is a chain map

$$\tilde{u}_\ast : C_\ast(LMT M) \xrightarrow{\tilde{\varepsilon}_t} C_\ast(D_L M/S_L M, \ast) \xrightarrow{\tilde{\theta}_t} C_\ast(D_L M, S_L M) \xrightarrow{\tilde{\sigma}_t} C_{\ast-d}(LM),$$

such that the following diagram commutes:

$$\begin{array}{ccc}
H_\ast(LMT M) & \xrightarrow{\tilde{u}_\ast} & H_{\ast-d}(LM) \\
\downarrow ev_* & & \downarrow ev_* \\
H_\ast(MTM) & \xrightarrow{\tilde{u}_\ast} & H_{\ast-d}(M).
\end{array}$$

In [2] the first two authors proved that the composition of $\tilde{\tau}_\ast$, $\tilde{\gamma}_\ast$, and $\tilde{u}_\ast$ realizes the Chas - Sullivan loop product. That is, the following diagram commutes:

$$\begin{array}{ccc}
H_p(LM) \otimes H_q(LM) & \xrightarrow{(-1)^d(d-p)(\circ)} & H_{p+q-d}(LM) \\
\downarrow & & \uparrow \tilde{u}_\ast \\
H_{p+q}(LM \times LM) & \xrightarrow{\tilde{\tau}_\ast} & H_{p+q}(LM \times M)TM \xrightarrow{\tilde{\tau}_\ast} H_{p+q}(LTM)
\end{array}$$

where $\circ : H_p(LM) \otimes H_q(LM) \to H_{p+q-d}(LM)$ is the Chas - Sullivan loop product. We therefore have the following proposition, which should be viewed as the analogue of proposition [4].

**Proposition 5.** The composition of the four chain maps

$$\begin{array}{c}
\times : C_\ast(LM) \otimes C_\ast(LM) \to C_\ast(LM \times LM), \\
\tilde{\tau}_t : C_\ast(LM \times LM) \to C_\ast(LM \times M LM)^TM, \\
\tilde{\gamma}_t : C_\ast(LM \times M LM)^TM \to C_\ast(LTM), \\
\tilde{u}_t : C_\ast(LTM) \to C_{\ast-d}(LM).
\end{array}$$

gives a chain representative of the Chas - Sullivan loop product up to sign

$$H_p(LM) \otimes H_q(LM) \to H_{p+q-d}(LM)$$

$$\alpha \otimes \beta \to \alpha \circ \beta.$$
2. The Loop Algebra Spectral Sequence

In this section we will describe the loop algebra spectral sequence and prove theorem 1. To do this we describe a filtration of simplicial sets arising from the fibration $\Omega M \hookrightarrow LM \to M$, which will induce the Serre spectral sequence for this fibration. We then analyze how the loop product behaves with respect to this spectral sequence, using the chain level description of the product given in the last section. We then apply Poincare duality to regrade the spectral sequence (and in particular change a first quadrant spectral sequence into a second quadrant one), and prove theorem 1.

Given a topological space $X$, let $S.X$ denote its singular simplicial set. The $p$ - simplices are given by singular simplices $S_p(X) = \{\sigma : \Delta^p \to X\}$, where $\Delta^p$ is the standard $p$ - simplex, $\Delta^p = \{(x_0, \ldots, x_p) \in \mathbb{R}^{p+1} : x_i \geq 0 \text{ and } \sum x_i = 1\}$. $S.X$ has the usual face and degeneracy operations, and it is well known that its geometric realization $|S.X|$ has the weak homotopy type of $X$. See [4] for details.

Number the vertices of $\Delta^p \{0, 1, \ldots, p\}$. Then for a non-decreasing sequence of integers $0 \leq i_0 \leq \ldots \leq i_r \leq p$, define a map of simplices $(i_0, i_1, \ldots, i_r) : \Delta^r \to \Delta^p,$ by requiring that $(i_0, i_1, \ldots, i_r)$ be a linear map that sends the vertex $k$ of $\Delta^r$ to the vertex $i_k$ of $\Delta^p$. Given a singular $p$ - simplex $\sigma : \Delta^p \to X$, the composition of $\sigma$ with $(i_0, i_1, \ldots, i_r)$ defines an $r$ - simplex

$$\sigma(i_0, i_1, \ldots, i_r) : \Delta^r \to \Delta^p \to X.$$  

Now let $F \hookrightarrow E \xrightarrow{\pi} B$ be a fibration. There is a filtration of the simplicial set $S.E$ defined as follows.

**Definition 1.** Let $F_p(S.E) \subset S.E$ be the subsimplicial set whose $r$ simplices are given by

$$F_p(S_r(E)) = \{T : \Delta^r \to E : \pi \circ T = \sigma(i_0, \ldots, i_r), \text{ for some } \sigma \in S_q(B), \ q \leq p, \ \text{and some sequence } 0 \leq i_0 \leq \ldots \leq i_r \leq q\}.$$  

Given a simplicial set $Y$, let $C_*(Y)$ be the associated simplicial chain complex, whose $q$ - chains $C_q(Y)$ are the free abelian group on the $q$ - simplices $Y_q$, and whose boundary homomorphisms are given by the alternating sum of the face maps. Again, see [4] for details. In particular for a space $X$, we have $C_*(S.X)$ is the singular chain complex which we previously denoted simply by $C_*(X)$. The following is is verified in [4].

**Proposition 6.** Let $F \hookrightarrow E \to B$ be a fibration as above. Consider the filtration of chain complexes,

$$\{0\} \hookrightarrow \cdots \hookrightarrow F_{p-1}(C_*(E)) \hookrightarrow F_p(C_*(E)) \hookrightarrow \cdots \hookrightarrow C_*(E)$$  

defined by $F_p(C_*(E)) = C_*(F_p(S.E))$ where $F_p(S.E)$ is the $p$th filtration of the singular simplicial set defined above. Then this filtration induces the Serre spectral sequence converging to $H_*(E)$. 

We will study this spectral sequence in the examples of the fibrations
\[ ev: LM \to M, \]
\[ ev_\infty : LM \times_M LM \to M, \]
\[ ev : DLM \to DTM, \] and
\[ ev : SLM \to STM \]
described in the last section. In particular notice that by taking the filtration of pairs
\[ F_p(C_\ast(DLM, SLM)) = F_p(C_\ast(DLM))/F_p(C_\ast(SDM)) \]
we get a spectral sequence (the relative Serre spectral sequence) converging to
\[ \tilde{H}_\ast(DLM, SLM) = \tilde{H}_\ast(\tilde{LM}, \tilde{STM}) \]
and whose \(E_2\) term is
\[ E_2^{p,q} = H^{p}_\ast(DTM, STM; H^q_\ast(\Omega M)) = H^{p}_\ast(DLM, STM; H^q_\ast(\Omega M)). \]

The following is an immediate observation based on the chain descriptions in the last section.

**Proposition 7.** The chain maps
\[ \times : C_\ast(LM) \otimes C_\ast(LM) \to C_\ast(LM \times LM) \]
\[ \tilde{\tau} : C_\ast(LM \times LM) \to C_\ast(LM \times_M LM)^{TM}, \] and
\[ \tilde{\gamma} : C_\ast(LM \times_M LM)^{TM} \to C_\ast(LM^{TM}) \]
described in the last section all preserve the above filtrations:
\[ \times : F_p(C_\ast(LM)) \otimes F_q(C_\ast(LM)) \to F_{p+q}(C_\ast(LM \times LM)), \]
\[ \tilde{\tau} : F_m(C_\ast(LM \times LM)) \to F_m(C_\ast(ev_\infty^{\ast}(DTM), ev_\infty^{\ast}(STM))), \]
\[ \tilde{\gamma} : F_m(C_\ast(ev_\infty^{\ast}(STM))) \to F_m(C_\ast(DLM, SLM)), \]
and therefore induce maps of the associated Serre spectral sequences.

The following is a bit more delicate.

**Theorem 8.** The chain map
\[ \tilde{u}_\ast : C_\ast(LM^{TM}) \to C_\ast^{-d}(LM) \]
induces a map of filtered chain complexes that lowers the filtration by \(d\),
\[ \tilde{u}_\ast : F_p(C_\ast(DLM, SLM)) \to F_{p-d}(C_\ast(LM)) \]
and therefore induces a map of the associated Serre spectral sequences that shifts grading,
\[ \tilde{u}_\ast : E^{\ast}_{p,q}(DLM, SLM) \to E^{\ast}_{p-d,q}(LM). \]
In particular on the $E_2$-level $\tilde{u}$ is the Thom isomorphism,
\[ H_p(M^TM; H_q(\Omega M)) \to H_{p-d}(M; H_q(\Omega M)). \]

Proof. By the definition of the chain map $\tilde{u}_\#$, to prove this theorem it suffices to show that taking the cap product with the Thom class $\tilde{t} \in \text{C}^d(D_LM, S_LM)$ induces a map of filtered chain complexes that lowers the filtration by $d$,
\[ F_p(\text{C}_*(D_LM, S_LM)) \xrightarrow{\cap \tilde{t}} F_{p-d}(\text{C}_*(D_LM)). \]

To verify this, recall that the cap product has the following chain level description. Consider the operations on singular $n$-simplices,
\[ \lfloor p \rfloor \text{ and } p \lfloor \cdot \rfloor : S_n(X) \to S_p(X) \]
defined by
\[ \lfloor p \rfloor \sigma = (d_{p+1})^{n-p}(\sigma), \quad p \lfloor \sigma \rfloor = (d_0)^{n-p}(\sigma). \]
That is, $\lfloor p \rfloor \sigma$ is the restriction of $\sigma$ to the “front” $p$-face, and $p \lfloor \sigma \rfloor$ is the restriction of $\sigma$ to the “back” $p$-face.

Now we can choose our cochain $\tilde{t} \in \text{C}^*(D_LM, S_LM)$ to represent the Thom class so that if we view it as an element in $\text{Hom}(\text{C}_d(D_LM); \mathbb{Z})$ it satisfies

(2.1) \[ C_d(S_LM) \subset \text{Ker}(\tilde{t}), \]
(2.2) \[ \text{Im}(s_j : S_{d-1}(D_LM) \to S_d(D_LM) \subset C_d(D_LM)) \subset \text{Ker}(\tilde{t}), \]
That is, $\tilde{t}$ vanishes on chains on $S_LM$ and degenerate chains on $D_LM$. Let $\sigma \in \text{S}_p(D_LM)$ for some $p \geq d$, then the cap product can be described as

(2.3) \[ \sigma \cap \tilde{t} = (-1)^{d(p-d)} \tilde{t}(d[p] \lfloor \sigma \rfloor \lfloor p \rfloor \sigma), \]
This gives a well defined chain map $\cap \tilde{t} : \text{C}_p(D_LM, S_LM) \to \text{C}_{p-d}(D_LM)$ that represents capping with the Thom class in cohomology.

The map $\cap \tilde{t}$ on $\text{C}_*(D_LM, S_LM)$ is completely determined by its composition with the projection $C_*(D_LM) \to C_*(D_LM, S_LM)$. So to prove the lemma it now suffices to prove that $\cap \tilde{t}$ maps $F_p(C_*(D_LM))$ to $F_{p-d}(C_*(D_LM))$.

Let $T \in F_p(S_r(D_LM))$ be a singular $r$-simplex in filtration $p$. Then, by definition,
\[ \text{ev}(T) = \sigma(i_0, i_1, \cdots, i_r) \]
for some $q$-simplex $\sigma \in S_q(D_TM)$, and some sequence $i_0 \leq \cdots \leq i_r \leq q \leq p$. By the above formula for the cap product, we then have
\[ \text{ev}(T \cap \tilde{t}) = \text{ev}(T) \cap t = \pm t(\sigma(i_{r-d}, \cdots, i_r))\sigma(i_0, \cdots, i_{r-d}), \]
where, as above, \( t \in C^d(DTM, STM) \) represents the Thom class of the tangent bundle \( T M \to M \).

By (2.2) \( t \) vanishes on degenerate simplices. Therefore this expression can only be nonzero if 
\[ i_{r-d} < \ldots < i_r \leq q, \]
and therefore \( r - d \leq q - d \leq p - d \). Hence \( T \cap \tilde{t} \in F_{p-d}(C_*(DLM)) \) as claimed. \( \square \)

Notice that if we compose the chain maps in proposition 7 and theorem 8, we have a map of filtered chain complexes
\[
\mu : F_p(C_*(LM)) \otimes F_q(C_*(LM)) \to F_{p+q-d}(C_*(LM))
\]
which, by proposition 3 induces the loop product in homology. Therefore \( \mu \) induces a map of spectral sequences
\[
(2.4) \quad \mu : E_{p,s}^r(LM) \otimes E_{q,t}^r(LM) \to E_{p+q-d,s+t}^r(LM).
\]
For simply connected \( M \), on the \( E_2 \)-level, \( \mu \) defines a map
\[
\mu : H_p(M; H_*(\Omega M)) \otimes H_q(M; H_*(\Omega M)) \to H_{p+q-d}(M; H_{s+t}(\Omega M))
\]
which we claim is given up to sign, by the intersection product with coefficients on the Pontryagin ring \( H_*(\Omega M) \). More explicitly,
\[
\mu((a \otimes g) \otimes (b \otimes h)) = \pm (a \cdot b) \otimes (gh)
\]
where \( a \in H_p(M) \), \( g \in H_*(\Omega M) \), \( b \in H_q(M) \), and \( h \in H_*(\Omega M) \), and where \( a \cdot b \) is the intersection product and \( gh \) is the Pontryagin product.

To see this, notice that the composition of chain maps used to define \( \mu \) is given by a composition of chain maps of fibrations (and pairs of fibrations). On the base space level this is given by the composition of maps described in proposition 3 realizing the intersection product. On the fiber level, the fact that this chain map induces the Pontrjagin product comes from the fact that map \( \gamma : LM \times_M LM \to LM \) as defined in (1.4) is a map of fibrations
\[
\begin{array}{ccc}
\Omega M \times \Omega M & \xrightarrow{\rho} & \Omega M \\
\downarrow & & \downarrow \\
LM \times_M LM & \xrightarrow{\gamma} & LM \\
\downarrow^{ev_\infty} & & \downarrow^{ev} \\
M & \xrightarrow{=} & M
\end{array}
\]
where \( \rho \) is the Pontryagin product.

Thus \( \mu \) defines a multiplicative structure on the Serre spectral sequence for the fibration \( \Omega M \to LM \to M \) which converges to the loop homology algebra structure on \( H_*(LM) \), and on the \( E_2 \)-level is given (up to sign) by the intersection product on \( M \) with coefficients in the Pontryagin ring \( H_*(\Omega M) \). However the grading is shifted in a way that is confusing for calculational purposes. To remedy this, define a second quadrant spectral \( \{ E_{s,t}^r(H_*(LM)) ; d_r : E_{s,t}^r \to E_{s-r,t+r-1}^r \} \) with \( s \leq 0 \),
where $d$ is the dimension of $M$, and the right hand side is the Serre spectral sequence for the fibration $\Omega M \to LM \to M$ we have been considering. Notice that $E_{r,s,t}^r(\mathbb{H}_s(LM))$ can only be nonzero for $-d \leq s \leq 0$. Moreover with the new indexing the spectral sequence converges to the loop homology in a grading preserving way, $E_{r,s,t}^r(\mathbb{H}_s(LM)) \Rightarrow H_{s+t}(LM)$. We also see that with this new indexing, the loop multiplication in the spectral sequence (2.4) preserves the bigrading, 

$$
\mu : E_{r,s,t}^r(\mathbb{H}_s(LM)) \otimes E_{p,q}^r(\mathbb{H}_s(LM)) \to E_{s+p,t+q}^r(\mathbb{H}_s(LM)).
$$

Finally notice that the $E^2$ - term is given by 

$$
E_{s,t}^2(\mathbb{H}_s(LM)) = H_{s+d}(M; H_t(\Omega M))
$$

for $-d \leq s \leq 0$. By applying Poincare duality we have 

$$
E_{s,t}^2(\mathbb{H}_s(LM)) = H^{-s}(M; H_t(\Omega M)).
$$

Since under Poincare duality the intersection pairing in homology coincides with the cup product in cohomology, on the $E^2$ - level the multiplication $\mu$ is given (up to sign) by cup product with coefficients in the Pontryagin ring $H_s(\Omega M)$.

$$
\mu : H^{-s}(M; H_t(\Omega M)) \otimes H^{-p}(M; H_q(\Omega M)) \to H^{-(s+p)}(M; H_{t+q}(\Omega M))
$$

for $-d \leq s, p \leq 0$ and $t, q \geq 0$. This completes the proof of theorem 1.

3. The Loop Product on $S^n$ and $\mathbb{C}P^n$

Our goal in this section is to use the loop homology spectral sequence constructed in the last section to perform the calculations described in theorems 2 and 3.

We first prove theorem 2 by calculating the ring structure of $H_*(LS^n)$. If $n = 1$, the fibration $\Omega S^1 \to LS^1 \to S^1$ is trivial. Since the components of the based loop space $\Omega S^1$ are all contractible, and $\pi_0(\Omega S^1) \cong \mathbb{Z}$, we have a homotopy equivalence 

$$
LS^1 \cong S^1 \times \Omega S^1 \cong S^1 \times \mathbb{Z}.
$$

Similarly, there is a homotopy equivalence $LS^1 \times S^1 \cong S^1 \times \mathbb{Z} \times \mathbb{Z}$. With respect to these equivalences, it is clear that the map $\gamma : LS^1 \times S^1 \to LS^1$ is given by 

$$
S^1 \times \mathbb{Z} \times \mathbb{Z} \to S^1 \times \mathbb{Z}
$$

$$
(x, m, n) \mapsto (x, m + n).
$$
It is also clear that with respect to these equivalences, the Thom - Pontryagin map \( \tilde{\tau} : LS^1 \times LS^1 \to (LS^1 \times S^1 \times LS^1)^{TS^1} \) is given by
\[
\tau \times 1 : S^1 \times S^1 \times \mathbb{Z} \times \mathbb{Z} \to (S^1)^{TS^1} \wedge (\mathbb{Z} \times \mathbb{Z})_+
\]
where \( \tau : S^1 \times S^1 \to (S^1)^{TS^1} \) is the Thom - Pontryagin construction for the diagonal map \( \Delta : S^1 \to S^1 \times S^1 \). Thus by (1.7) the loop homology algebra structure on \( H_\ast(\text{LS}^1) \) is, with respect to the equivalence \( \text{LS}^1 \simeq S^1 \times \mathbb{Z} \), given by the tensor product of the intersection ring structure on \( H_\ast(S^1) \) with the group algebra structure, \( H_0(\mathbb{Z}) \cong \mathbb{Z}[t, t^{-1}] \). Using Poincare duality, we then have an algebra isomorphism
\[
(3.2) \quad \mathbb{H}_\ast(\text{LS}^1) \cong H^\ast(S^1) \otimes H_0(\mathbb{Z}) \cong \Lambda[a] \otimes \mathbb{Z}[t, t^{-1}]
\]
where \( a \in \mathbb{H}_1(\text{LS}^1) \) corresponds to the generator in \( H_0(S^1) \).

We now proceed with a calculation of \( H_\ast(\text{LS}^n) \) for \( n > 1 \). Consider the loop homology spectral sequence in this case. For dimension reasons, the only nontrivial differentials occur at the \( E^n \) level. Recall that there is an isomorphism of algebras, \( H_\ast(\Omega S^n) \cong \mathbb{Z}[x] \), where \( x \) has degree \( n - 1 \). It then follows that
\[
E^2_{p,q}(\mathbb{H}_\ast(\text{LS}^n)) \cong \cdots \cong E^n_{p,q} \cong H^p(S^n) \otimes H_q(\Omega S^n).
\]

The differentials \( d^n \) can be computed using the results in [1] and [7]. An exposition of this calculation (for the Serre spectral sequence of the fibration \( \Omega S^n \to LS^n \to S^n \)) is given in [5]. Inputting the change of grading used to define the loop homology spectral sequence, we have the following picture of the differentials:

The Spectral Sequences for \( LS^n \)
Denote by \( \iota, \sigma \) the generators of \( H^n(S^n) \) and \( H^0(S^n) \), respectively. Let \( 1_{\Omega} \) be the unit of \( H_*(\Omega S^n) \).

In both spectral sequences \( \sigma \otimes 1_{\Omega} \) is an infinite cycle and therefore represents a class in \( \mathbb{H}_0(LS^n) \). This class is the unit in the algebra \( \mathbb{H}_*(LS^n) \), which we denote by \( 1 \). Similarly \( 1 = \iota \otimes 1_{\Omega} \) represents a class in \( \mathbb{H}_-(LS^n) \), and for dimension reasons \( a^2 \) must vanish in \( \mathbb{H}_*(LS^n) \).

Let \( u = \sigma \otimes x \in E_{0,n-1}^n \). In the case when \( n \) is odd, the spectral sequence collapses. Since \( n > 1 \), for dimensional reasons there can be no extension problems, and so we have an isomorphism of algebras

\[
\mathbb{H}_*(LS^n) \cong E_{1,*}^2 \cong H^{-*}(S^n) \otimes H_*(\Omega S^n) \cong \Lambda[a] \otimes \mathbb{Z}[u].
\]

If \( n \) is even, the \( u^{2k} \) terms survive to \( E^\infty \), and

\[
d_n(u^{2k+1}) = 2au^{2k+2}
\]

for all nonnegative integers \( k \). Let \( v = u^2 = \sigma \otimes x^2 \). Being an infinite cycle it represents a class in \( \mathbb{H}_{2(n-1)}(LS^n) \). Then \( u^k \) represents a generator of a subgroup of \( \mathbb{H}_{2k(n-1)}(LS^n) \) represented by classes in \( E^\infty_{2k(n-1)} \). Notice that the ideal \( (2av) \) vanishes in \( E^\infty \). Thus \( \mathbb{Z}[a,v]/(a^2,2av) \) is a subalgebra in \( E^\infty_{1,*} \). Let \( b = \iota \otimes x \in E^\infty_{-n,n-1} \). Then \( E^\infty_{1,*} \) is generated by \( \mathbb{Z}[a,v]/(a^2,2av) \) and \( b \). Again for dimension reasons \( b^2 = ab = 0 \in E^\infty \), and so

\[
E^\infty_{1,*} \cong (\Lambda[b] \otimes \mathbb{Z}[a,v])/(a^2,ab,2av), \quad \text{if } n \text{ is even.}
\]

When \( n > 2 \), for dimensional reasons there can be no extension issues, so \( \mathbb{H}_*(LS^n) \cong (\Lambda[b] \otimes \mathbb{Z}[a,v])/(a^2,ab,2av) \) for \( n \) even and \( n > 2 \). For \( n = 2 \) we consider the potential extension problem. For filtration reasons, there are unique classes in \( \mathbb{H}_*(LS^2) \) represented by \( a \) and \( b \) in \( E^\infty_{1,*} \). The ambiguity in the choice of class represented by \( v \in E_{0,2}^\infty \) lies in \( E_{0,4}^\infty \cong \mathbb{Z} \) generated by \( av \). Since \( a^2 = 0 \) in \( \mathbb{H}_*(LS^2) \), any choice \( \tilde{v} \in \mathbb{H}_{2}(LS^2) \) will satisfy \( 2av = 0 \). Thus any choice of \( \tilde{v} \) together with \( a \) and \( b \) will generate the same algebra, namely \( \mathbb{H}_*(LS^2) \cong (\Lambda[b] \otimes \mathbb{Z}[a,\tilde{v}])/(a^2,ab,2av) \).

This completes the proof of theorem 3.

We now proceed with the proof of theorem 3 by calculating the ring structure of \( \mathbb{H}_*(\mathbb{C}P^n) \). Notice that if \( n = 1 \), \( \mathbb{H}_*(\mathbb{C}P^n) \cong \mathbb{H}_*(LS^2) \), and this case was already discussed above. So for what follows we assume \( n > 1 \). The \( E^2 \) -term of the loop homology spectral sequence is \( H^*(\mathbb{C}P^n; H_*(\Omega \mathbb{C}P^n)) \). The cohomology ring \( H^*(\mathbb{C}P^n) \cong \mathbb{Z}[c_1]/(c_1^{n+1}) \) is generated by the first Chern class \( c_1 \). We now recall the Pontryagin ring structure of \( H_*(\Omega \mathbb{C}P^n) \).

Consider the homotopy fibration

\[
\Omega S^{2n+1} \xrightarrow{\Omega \eta} \Omega \mathbb{C}P^n \to \Omega \mathbb{C}P^\infty \simeq S^1,
\]

where \( \eta : S^{2n+1} \to \mathbb{C}P^n \) is the Hopf map. Since this is a fibration of loop spaces that has a section (because \( \pi_1(\Omega \mathbb{C}P^n) \cong \pi_1(\Omega \mathbb{C}P^\infty) \)) we can conclude the following.

1. The fibration is homotopically trivial, \( \Omega \mathbb{C}P^n \simeq \Omega S^{2n+1} \times S^1 \).
2. The Serre spectral sequence of this fibration is a spectral sequence of algebras.
Therefore we have that in the Serre spectral sequence,

\[ E_\infty \cong H_*(S^1) \otimes H_*(\Omega S^{2n+1}) \]
\[ \cong \Lambda[t] \otimes \mathbb{Z}[x] \]

(3.3)

where \( t \in H_1(S^1) \) and \( x \in H_{2n}(\Omega S^{2n+1}) \) are generators. This isomorphism is one of algebras. Indeed it is clear that there are no extension issues in this spectral sequence so that

\[ H_*(\Omega \mathbb{C}P^n) \cong H_*(S^1) \otimes H_*(\Omega S^{2n+1}) \cong \Lambda[t] \otimes \mathbb{Z}[x]. \]

Now consider the loop homology spectral sequence for \( L\mathbb{C}P^n \). We therefore have

\[ E_2^{*,*}(\mathbb{H}_*(L\mathbb{C}P^n)) \cong H^*(\mathbb{C}P^n; H_*(\Omega \mathbb{C}P^n)) \]
\[ \cong \mathbb{Z}[c_1]/(c_1^{n+1}) \otimes \Lambda[t] \otimes \mathbb{Z}[x] \]

as algebras.

It was computed in [8] that

\[ H_k(L\mathbb{C}P^n) = \begin{cases} 
\mathbb{Z} & \text{if } k = 0, 1, \ldots, k \neq 2mn, m \geq 1, \\
\mathbb{Z} \oplus \mathbb{Z}_{n+1} & \text{if } k = 2mn, m \geq 1.
\end{cases} \]

This calculation implies the following pattern of differentials in the loop homology spectral sequence:

That is, the only nonzero differentials are \( d_{2n} \) and

\[ d_{2n}(y) = (n+1)c^nu, \]
where \( c = c_1 \otimes 1, \ u = \sigma \otimes x, \) and \( y = \sigma \otimes t. \) Here \( \sigma \) is a generator of \( H^0(\mathbb{CP}^n) \cong \mathbb{Z}. \) For dimension reasons \( d_{2n}(u) = 0, \) and since \( d_{2n} \) is a derivation,
\[
d_{2n}(yu^k) = d_{2n}(y)u^k = (n + 1)c_n u^{k + 1},
\]
and these are all the nonzero differentials.

The spectral sequence collapses beyond \( E^{2n} \) level, therefore \( c \) and \( u \) represent homology classes in \( \mathbb{H}_{-2}(L\mathbb{CP}^n) \) and \( \mathbb{H}_{2n}(L\mathbb{CP}^n) \) respectively. Notice also that the ideal \( (c^{n+1}, (n+1)c_n u) \) vanishes in \( E^\infty_{n,*} \). Therefore we have a subalgebra
\[
\mathbb{Z}[c, u]/(c^{n+1}, (n+1)c_n u) \subset E^\infty_{n,*}(\mathbb{H}_*(L\mathbb{CP}^n)).
\]
Let \( w = yc \in E^2_{n,*} \). \( w \) is an infinite cycle in this spectral sequence and represents a class in \( \mathbb{H}_{-1}(\mathbb{CP}^n). \)
Notice that \( w^2 = 0 \) in \( E^\infty_{2,2}. \) Similarly \( wc^n \) also vanishes in \( E^\infty_{n,*}. \) Thus the \( E^\infty \) term of the loop homology spectral sequence can be written as follows:
\[
E^\infty_{n,*}(\mathbb{H}_*(L\mathbb{CP}^n)) \cong (\Lambda[w] \otimes \mathbb{Z}[c, u])/((n+1)c_n u, wc^n).
\]
The extension issues are handled just as they were for \( \mathbb{H}_*(L\mathbb{S}^2) = \mathbb{H}_*(L\mathbb{CP}^1) \) above. We then conclude that
\[
\mathbb{H}_*(L\mathbb{CP}^n) \cong (\Lambda[w] \otimes \mathbb{Z}[c, u])/((n+1)c_n u, wc^n)
\]
as algebras. This completes the proof of theorem 3.

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