Abstract

We consider the sharp interface limit of a coupled Stokes/Allen-Cahn system, when a parameter $\varepsilon > 0$ that is proportional to the thickness of the diffuse interface tends to zero, in a two dimensional bounded domain. For sufficiently small times we prove convergence of the solutions of the Stokes/Allen-Cahn system to solutions of a sharp interface model, where the interface evolution is given by the mean curvature equation with an additional convection term coupled to a two-phase Stokes system with an additional contribution to the stress tensor, which describes the capillary stress. To this end we construct a suitable approximation of the solution of the Stokes/Allen-Cahn system, using three levels of the terms in the formally matched asymptotic calculations, and estimate the difference with the aid of a suitable refinement of a spectral estimate of the linearized Allen-Cahn operator. Moreover, a careful treatment of the coupling terms is needed.

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1 Introduction, Main Result, and Overview

Two-phase flows of macroscopically immiscible fluids play an important role in real world applications e.g. in chemistry and engineering sciences. They lead to many interesting fundamental questions concerning modeling, numerical simulations and their mathematical analysis. There are two basic model classes: The so-called sharp and diffuse interface models. In sharp interface models the interface separating two fluids is described as a lower dimensional surface. In numerical simulations and the mathematical analysis the interface is usually either parametrized explicitly or described with the aid of the level set or characteristic function of one fluid domain. This leads to fundamental problems, when the interface develops singularities e.g. due to droplet collision or pinch-off. In diffuse interface models (also called phase field models) a partial mixing of the two fluids on a small length scale proportional to a parameter $\varepsilon > 0$ is taken into account. To this end an order parameter, which will be denoted by $c_\varepsilon$, is introduced, which is close to one of two distinct values (e.g. $\pm 1$) in the bulk phases of the fluids and which varies smoothly between these two values in an interfacial region, which has – at least heuristically/in sufficiently smooth situations – a thickness proportional to $\varepsilon$. A fundamental diffuse interface model for a two-phase flow of two macroscopically immiscible, viscous Newtonian fluids with same densities is given by the so-called model $H$, which yields the Navier-Stokes/Cahn-Hilliard system

$$\begin{align*}
\rho \partial_t \mathbf{v}_\varepsilon + \rho \mathbf{v}_\varepsilon \cdot \nabla \mathbf{v}_\varepsilon &- \text{div}(2\nu(c_\varepsilon)D\mathbf{v}_\varepsilon) + \nabla p_\varepsilon = -\varepsilon \text{div}(\nabla c_\varepsilon \otimes \nabla c_\varepsilon) \quad \text{in } \Omega \times (0, T_1), \\
\text{div} \mathbf{v}_\varepsilon & = 0 \quad \text{in } \Omega \times (0, T_1), \\
\partial_t c_\varepsilon + \mathbf{v}_\varepsilon \cdot \nabla c_\varepsilon & = m_\varepsilon \Delta c_\varepsilon \quad \text{in } \Omega \times (0, T_1), \\
\mu_\varepsilon & = -\varepsilon \Delta c_\varepsilon + \frac{1}{\varepsilon} f'(c_\varepsilon) \quad \text{in } \Omega \times (0, T_1)
\end{align*}$$

(1.1)-(1.4)

together with suitable boundary and initial conditions for a suitable double well potential $f$, e.g. $f(s) = \frac{1}{2}(s^2 - 1)^2$. Here $\mathbf{v}_\varepsilon, p_\varepsilon$ are the velocity and the pressure of the fluid mixture, $\mu_\varepsilon$ is a chemical potential and $c_\varepsilon$ is related to the concentration difference of the fluids. Moreover, $D\mathbf{v}_\varepsilon = \frac{1}{2}(\nabla \mathbf{v}_\varepsilon + \nabla \mathbf{v}_\varepsilon^T)$, $\rho > 0$ is a constant, $\nu : \mathbb{R} \to (0, \infty)$ a suitable function describing the viscosity, $\nabla = \nabla_x$, $\text{div} = \text{div}_x$, $\Delta = \Delta_x$ are always taken with respect to $x \in \Omega$.

Nowadays there are many results on existence of smooth solutions for sharp interface models of two-phase flows of viscous incompressible fluids for short times and sufficiently smooth initial data and for large times and initial data close to a stable equilibrium, cf. e.g. \[\{15, 22, 23, 24\}\]. Moreover, there are some results on existence of generalized/weak solutions, cf. \[\{25, 34, 1, 2, 6, 27\}\]. But in most cases a satisfactory theory on weak solutions is unknown. On the other hand diffuse interface models for two-phase flows, such as \[\{11, 14\}\] and generalizations of it, were studied intensively during the last decade. There are many results on existence and uniqueness of weak and strong solutions as well as long time behavior, which are comparable to the known results for the classical incompressible Navier-Stokes system in two and three space dimensions and phase field models without fluid mechanics, cf. e.g. \[\{11, 12, 3, 22, 23\}\]. We also refer to \[\{25\}\] Section on “Weak Solutions and Diffuse Interface Models for Incompressible Two-phase Flows”.

Although there are many analytic results on sharp and diffuse interface models for two-phase flows in fluid mechanics, there are only few rigorous results on convergence of solutions of diffuse interface models to sharp interface models as $\varepsilon \to 0$. Most results so far are based on the method of formally matched asymptotic expansions, where the validity of certain power series expansions close to the interface is assumed, cf. \[\{10, 32, 4\}\] for the model \[\{11, 14\}\]. First results on convergence for large times to so-called varifold solutions of the sharp interface models were obtained in \[\{6, Appendix\}\] and \[\{5\}\] for the model $H$ and a generalization of it for fluids of different densities. A disadvantage of these results is that the notion of varifold solutions, which
are comparable to measure-valued solutions, is rather weak and no convergence rates can be shown. We note that results on non-convergence for certain scalings of a mobility coefficient were obtained in [5] and [7] or [38] Chapter 5. A sharp interface limit for a Navier-Stokes/phase field system to a Navier-Stokes/Stefan system was proved by Starovoitov and Hoffmann [26] on the level of weak solutions. But the technique relies rather specifically on structural properties of the systems.

A similar sharp interface limit also arises in the theory of liquid crystals [21]. In the low temperature regime, the Landau-De Gennes theory predicts the co-existence of an isotropic phase and a nematic phase. Rescaling the corresponding hydrodynamic system (also referred to as Beris-Edward system) near the isotropic-nematic interface will lead to a limit system that is nematic and is governed by the Ericksen-Leslie system on one side and purely isotropic on the other side. However, a rigorous justification remains open. We believe that insights from the following analysis might also be helpful to solve this problem and other more complicated sharp interface limits in fluid mechanics.

It is the purpose of this contribution to establish a first rigorous convergence result with convergence rates in strong norms for the sharp interface limit $\varepsilon \to 0$ in the case of a two-phase flow in fluid mechanics, which is comparable to results known for single phase field models like the Allen-Cahn, the Cahn-Hilliard, or the phase field model equation, cf. De Mottoni and Schatzman [17], Alikakos et al. [9], Caginalp and Chen [13], respectively.

More precisely, on a bounded domain $\Omega \subseteq \mathbb{R}^2$ we consider the asymptotic limit $\varepsilon \to 0$ of the following system:

\begin{align}
-\Delta \varepsilon + \nabla p_\varepsilon &= -\varepsilon \text{div}(\nabla c_\varepsilon \otimes \nabla c_\varepsilon) \quad \text{in } \Omega \times (0, T_1), \quad (1.5) \\
\text{div } \varepsilon &= 0 \quad \text{in } \Omega \times (0, T_1), \quad (1.6) \\
0 &= \varepsilon \Delta c_\varepsilon - \frac{1}{\varepsilon^2} f'(c_\varepsilon) \quad \text{in } \Omega \times (0, T_1), \quad (1.7) \\
\varepsilon |_{\partial \Omega} &= 0 \quad \text{on } \partial \Omega \times (0, T_1), \quad (1.8) \\
c_\varepsilon|_{t=0} &= c_{0,\varepsilon} \quad \text{in } \Omega \quad (1.9)
\end{align}

for a suitable double well potential $f$ and for suitable “well-prepared” initial data $c_{0,\varepsilon}$ specified below.

Let us note that every sufficiently smooth solution of (1.5)-(1.9) satisfies the energy identity

\begin{align*}
E_\varepsilon(c_\varepsilon(t)) &= \int_0^t \int_\Omega \left( |\nabla \varepsilon|^2 + \frac{1}{\varepsilon} |\mu_\varepsilon|^2 \right) dx \, dt \quad (1.10)
\end{align*}

for all $t \in (0, T_1)$, where $\mu_\varepsilon = -\varepsilon \Delta c_\varepsilon + \frac{1}{\varepsilon^2} f'(c_\varepsilon)$ and

\begin{align*}
E_\varepsilon(c_\varepsilon(t)) &= \int_\Omega \left[ \frac{|\nabla c_\varepsilon(x,t)|^2}{2} + \frac{f(c_\varepsilon(x,t))}{\varepsilon} \right] dx,
\end{align*}

which provides some limited control as $\varepsilon \to 0$.

The sharp interface limit of (1.5)-(1.9) is the system

\begin{align}
-\Delta \nu + \nabla p &= 0 \quad \text{in } \Omega^\pm(t), t \in [0, T_0], \quad (1.11) \\
\text{div } \nu &= 0 \quad \text{in } \Omega^\pm(t), t \in [0, T_0], \quad (1.12) \\
[2D\nu - p] n_{\nu_t} &= -\sigma H_{\Gamma_t} n_{\nu_t} \quad \text{on } \Gamma_t, t \in [0, T_0], \quad (1.13) \\
|\nu| &= 0 \quad \text{on } \Gamma_t, t \in [0, T_0], \quad (1.14) \\
\nu|_{\partial \Omega} &= 0 \quad \text{on } \partial \Omega \times (0, T_0), \quad (1.15) \\
V_{\Gamma_t} &= n_{\nu_t} \cdot \nu|_{\nu_t} = H_{\Gamma_t} \quad \text{on } \Gamma_t, t \in [0, T_0]. \quad (1.16)
\end{align}

Here $\Omega$ is the disjoint union of $\Omega^+(t), \Omega^-(t)$, and $\Gamma_t$ for every $t \in [0, T_0]$. $\Omega^\pm(t)$ are smooth domains, $\Gamma_t = \partial \Omega^\pm(t)$, $n_{\nu_t}$ is the interior normal of $\Gamma_t$ with respect to $\Omega^+(t)$. Moreover,

\begin{align*}
[u](p, t) &= \lim_{h \to 0^+} [u(p + n_{\nu_t}(p) h) - u(p - n_{\nu_t}(p) h)].
\end{align*}
is the jump of a function $u: \Omega \times [0, T_0] \to \mathbb{R}^2$ at $\Gamma_t$ in direction of $\mathbf{n}_{\Gamma_t}$, $H_{\Gamma_t}$, and $V_{\Gamma_t}$ are the curvature and the normal velocity of $\Gamma_t$, both with respect to $\mathbf{n}_{\Gamma_t}$. Furthermore, $D\mathbf{v} = \frac{1}{\rho} (\nabla \mathbf{v} + (\nabla \mathbf{v})^T)$ and $\sigma = \int_{\mathbb{R}} \theta_0(\rho)^2 \, d\rho$, where $\theta_0$ is the so-called optimal profile that is the unique solution of

$$\begin{align*}
-\theta_0''(\rho) + f'(\theta_0(\rho)) &= 0 \quad \text{for all } \rho \in \mathbb{R}, \quad (1.17) \\
\lim_{\rho \to \pm \infty} \theta_0(\rho) &= \pm 1, \quad \theta_0(0) = 0. \quad (1.18)
\end{align*}$$

If the material derivative $\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}$ is added to the right-hand side of (1.5) (i.e., the Navier-Stokes equations are considered), the system (1.5)-(1.9) was already suggested by Liu and Shen in [31] as an alternative approximation of a classical sharp interface model for a two-phase flow of viscous, incompressible, Newtonian fluids, which has advantages for numerical simulations since the Allen-Cahn equation is of second order and not of fourth order as the Cahn-Hilliard equation. On the other hand, for solutions of (1.5)-(1.9) the total mass $\int_{\Omega} \rho_c(x, t) \, dx$ is in general not preserved in time, in contrast to solutions of (1.1)-(1.4), which is a disadvantage if the model is used to approximate a two-phase flow without phase transitions. However, (1.5)-(1.9) can be considered as a simplified model for a two-phase flow with phase transitions. Such models can yield systems of Navier-Stokes/Allen-Cahn type, e.g., Blesgen [70]. Finally, let us mention that in [43] a rigorous result on the sharp interface limit of (1.5)-(1.9) was announced, which was not published so far to the best of the author’s knowledge.

The limit system (1.11)-(1.16) was also studied by Liu, Sato, and Tonegawa in [30] if the Stokes equation on the right-hand side is replaced by a modified Navier-Stokes equation for a shear thickening non-Newtonian fluid of power-law type. They constructed weak solutions for this system using a Galerkin approximation by a corresponding Navier-Stokes/Allen-Cahn system. In the proof they pass to the limit in the Galerkin approximation and pass to the limit $\varepsilon \to 0$ simultaneously. Although the authors do not perform a separate sharp interface limit in the latter non-Newtonian Navier-Stokes/Allen-Cahn system, the result is close to it. The analysis depends heavily on the fact that a shear thickening fluid is used. Moreover, uniqueness of the limit is unknown and no convergence rates are given. Finally, we note that in [43] Takasao and Tonegawa study existence and regularity of global in time solutions of the mean curvature flow with a given convection term. To this end an approximation with a convective Allen-Cahn equation is used as well.

Throughout the paper we assume that $(\mathbf{v}, p, \Gamma)$ is a smooth solution of the limit equation (1.11)-(1.16) for some $T_0 > 0$, where $(\Gamma_t)_{t \in [0, T_0]}$ is a family of smoothly evolving compact, non-self-intersecting, closed curves in $\Omega$. More precisely, we assume that

$$\Gamma := \bigcup_{t \in [0, T_0]} \Gamma_t \times \{t\}$$

is a smooth two-dimensional submanifold of $\Omega \times \mathbb{R}$ (with boundary), and $\mathbf{v}|_{\Omega^\pm} \in C^\infty((\Omega^\pm)^2)$, $p|_{\Omega^\pm} \in C^\infty((\Omega^\pm)^2)$, where

$\Omega^\pm = \bigcup_{t \in [0, T_0]} \Omega^\pm(t) \times \{t\}$.

In particular, we assume that $\Gamma_t \subseteq \Omega$ for every $t \in [0, T_0]$, which excludes contact angle problems. Moreover, for $T_1 \geq T_0$ let $(\mathbf{v}_c, p_c, c_t)$ be the (classical) solution of (1.5)-(1.9) with smooth initial values $c_0, c_{\varepsilon}: \Omega \to \mathbb{R}$, which will be specified in the main result below. Existence of classical solutions can be shown by standard methods. We could work with less regular initial data, but for simplicity of the presentation we assume smoothness.

We note that existence of local strong solutions of (1.11)-(1.16) can e.g. be obtained by adapting the strategy in [9], where a coupled Navier-Stokes/Mullins-Sekerka system was treated. This was carried out by Moser in [33] in the case where the Stokes system (1.11)-(1.14) is replaced by the instationary Navier-Stokes system. By standard arguments from the regularity theory of parabolic equations and the Stokes system, one can prove that the solution is indeed smooth for smooth initial values.

As in [9] and [10] we assume in the following that $f: \mathbb{R} \to \mathbb{R}$ is smooth and satisfies the assumptions

$$f'(\pm 1) = 0, \quad f''(\pm 1) > 0, \quad f(s) = f(-s) > 0 \quad \text{for all } s \in (-1, 1).$$
We note that \( f' \) in the present paper corresponds to \( f \) in [9] and to \(-f\) in [16]. Then there is a unique solution \( \theta_0: \mathbb{R} \to \Omega \) of \( \text{(1.14)-(1.15)} \), which is monotone and will play a central role in the following. Moreover, for every \( m \in \mathbb{N}_0 \), there is some \( C_m > 0 \) such that
\[
|\psi^m(\theta_0(p) \neq 1)| \leq C_m e^{-\alpha |p|} \quad \text{for all } p \in \mathbb{R} \text{ with } p \geq 0,
\]
where \( \alpha = \min(\sqrt{f'}(-1), \sqrt{f'}(1)) \). In the case \( 0 < \alpha < \min(\sqrt{f'}(-1), \sqrt{f'}(1)) \) a detailed proof can be found in [38, Lemma 2.6.1]. One can choose e.g. \( f(s) = \frac{1}{2}(1 - s^2)^2 \). Then \( \theta_0(s) = \tanh(s) \) for all \( s \in \mathbb{R} \) and \( \alpha = 1 \), cf. e.g. [13]. For simplicity we will assume that \( f \) is even. This implies that \( \theta_0 \) is odd and \( \theta_0 \) is even.

For the statement of our main result, we need tubular neighborhoods of \( \Gamma_t \). For \( \delta > 0 \) and \( t \in [0, T_0] \) we defined
\[
\Gamma_t(\delta) := \{ y \in \Omega : \text{dist}(y, \Gamma_t) < \delta \}, \quad \Gamma(\delta) = \bigcup_{t \in [0, T_0]} \Gamma_t(\delta) \times \{ t \}.
\]
Moreover, we define the signed distance function
\[
d_{\Gamma}(x, t) := \text{sdist}(\Gamma, x) = \begin{cases}
\text{dist}(\Omega^+(t), x) & \text{if } x \notin \Omega^+(t) \\
-\text{dist}(\Omega^-(t), x) & \text{if } x \in \Omega^-(t)
\end{cases}
\]
for all \( x \in \Omega, t \in [0, T_0] \). Since \( \Gamma \) is smooth and compact, there is some \( \delta > 0 \) sufficiently small, such that \( d_{\Gamma}: \Gamma(3\delta) \to \mathbb{R} \) is smooth, cf. Section 2.1 below.

Our main result is:

**THEOREM 1.1** Let \( N = 2, (v, \Gamma) \) be a smooth solution of \( \text{(1.11)-(1.13)} \) for some \( T_0 \in (0, \infty) \) and let
\[
c_{\epsilon, 0}(x) = \zeta(d_{\Gamma}(x))\theta_0 \left( \frac{d_{\Gamma}(x)}{\epsilon} \right) + (1 - \zeta(d_{\Gamma}(x))) \left( \chi_{\Omega^+(0)}(x) - \chi_{\Omega^-(0)}(x) \right) \quad \text{for all } x \in \Omega,
\]
where \( d_{\Gamma} = d_{\Gamma}|_{t=0} \) is the signed distance function to \( \Gamma_0 \) and \( \zeta \in C^\infty(\mathbb{R}) \) such that
\[
\zeta(s) = 1, \text{ if } |s| \leq \delta; \quad \zeta(s) = 0, \text{ if } |s| > 2\delta; \quad 0 \leq -s' \zeta(s) \leq 4 \text{ if } \delta \leq |s| \leq 2\delta.
\]
Moreover, let \( c_{\epsilon, 0}: \Omega \to \mathbb{R}, 0 < \epsilon \leq 1 \), be smooth such that
\[
\|c_{\epsilon, 0} - c_{\epsilon, 0}\|_{L^2(\Omega)} \leq C\epsilon^{N+\frac{1}{2}} \quad \text{for all } \epsilon \in (0, 1]
\]
and some \( C > 0, \sup_{0 < \epsilon \leq 1} \|c_{\epsilon, 0}\|_\infty(\Omega) < \infty \) and \( v_\epsilon, c_\epsilon \) be the corresponding solutions of \( \text{(1.5)-(1.7)} \). Then there are some \( \epsilon_0 \in (0, 1], R > 0, T \in (0, T_0] \), and \( c_A: \Omega \times [0, T_0] \to \mathbb{R}, v_A: \Omega \times [0, T_0] \to \mathbb{R}^2 \) (depending on \( \epsilon \)) such that
\[
\sup_{0 \leq t \leq T} \|c_\epsilon(t) - c_A(t)\|_{L^2(\Omega)} + \|\nabla (c_\epsilon - c_A)\|_{L^2(\Omega \times (0, T) \cap \Gamma(\delta))} \leq \epsilon R^{N+\frac{1}{2}} \tag{1.21a}
\]
\[
\|\nabla^+(c_\epsilon - c_A)\|_{L^2(\Omega \times (0, T) \cap \Gamma(2\delta))} + \epsilon \|\partial_n (c_\epsilon - c_A)\|_{L^2(\Omega \times (0, T) \cap \Gamma(2\delta))} \leq \epsilon R^{N+\frac{1}{2}} \tag{1.21b}
\]
and for any \( q \in [1, 2) \)
\[
\|v_\epsilon - v_A\|_{L^2((0, T) \cap \Gamma(\delta))} \leq C(q, R)\epsilon^2 \tag{1.22}
\]
hold true for all \( \epsilon \in (0, \epsilon_0] \) and some \( C(q, R) > 0 \). Moreover,
\[
\lim_{\epsilon \to 0} c_A = \pm 1 \quad \text{uniformly on compact subsets of } \Omega^\pm.
\]
and
\[
v_A = v + O(\epsilon) \quad \text{in } L^\infty(\Omega \times (0, T)) \text{ as } \epsilon \to 0.
\]
More precise information on \( c_A \) can be found in Section 3. In particular, the result implies
\[
c_\epsilon \to \pm 1 \quad \text{in } L^{2}_{\text{loc}}(\Omega^\pm).
\]
Here \( N = 2 \) is the basic convergence order (w.r.t. the \( L^\infty(\Omega) \)-norm). Although this order is fixed, we will write \( N + \frac{1}{2}, N - \frac{1}{2} \) etc. instead of \( \frac{1}{2}, \frac{1}{2} \) etc. since in this way the relations between the different orders become more transparent.
Remark 1.2 It is easy to show that the solutions \((v_\varepsilon, c_\varepsilon)\) of (1.5)-(1.9) satisfy
\[
|c_\varepsilon(x, t)| \leq \max\left(\sup_{0 < \varepsilon \leq 1} \|c_\varepsilon\|_{L^\infty(0, 1)}\right) \quad \text{for all } x \in \Omega, t \in [0, T_0]. \tag{1.23}
\]
To this end one can e.g. argue by contradiction and apply the same arguments as in the proof of the weak maximum principle.

For the proof of the main result, we will follow the same basic strategy, which was already successfully used in [17] for the Allen-Cahn equation, in [9] for the Cahn-Hilliard equation, and in [10] for the mass-preserving Allen-Cahn equation. In many details we will follow the constructions in [10]. Following this strategy the proof consists of two parts. In the first part a suitable approximate solution, which will be denoted by \((v_A, c_A)\) in the following, for (1.5)-(1.9) up to an error term of a certain order in \(\varepsilon\) is constructed. To this end finitely many terms of an expansion in \(\varepsilon > 0\), using the method of formally matched asymptotics, are used. In the second step the error of the approximate \((v_A, c_A)\) and the exact solutions \((v_\varepsilon, c_\varepsilon)\) are estimated with the aid of a suitable estimate for the linearized Allen-Cahn operator \(\mathcal{L}_\varepsilon\), defined by
\[
\mathcal{L}_\varepsilon u = -\Delta u + \frac{1}{\varepsilon^2} f''(c_A) u, \quad \text{for all } u \in H^2(\Omega). \tag{1.24}
\]
However, in order to adapt this strategy to the present system, several new difficulties, which are mainly related to the coupling of the Allen-Cahn and the Stokes system, have to be overcome. More precisely, in order to estimate the difference \(u := c_\varepsilon - c_A\), a suitable estimate of the convection term \(v_\varepsilon \cdot \nabla c_\varepsilon\) is required. To this end it will be essential how this term is approximated in the equation of \(c_A\). More precisely, we will construct \(c_A\) such that we have the following result:

**THEOREM 1.3** Let the assumption of Theorem 1.1 be satisfied and \(R \geq 1\). Then for every \(\varepsilon \in (0, 1)\) there are
\[
v_A, w_1, w_2 : \Omega \times [0, T_0] \to \mathbb{R}^2, \quad c_A : \Omega \times [0, T_0] \to \mathbb{R}, \quad r_A : \Omega \times [0, T_0] \to \mathbb{R}
\]
(depending on \(\varepsilon \in (0, 1]\)) such that \(v_\varepsilon = v_A + \varepsilon^2 w_1 + \varepsilon^2 w_2\) and
\[
\partial_t c_A + (v_A + \varepsilon^2 w_2) \cdot \nabla c_A + \varepsilon^2 w_1 |\cdot\nabla c_A = \Delta c_A - \frac{f''(c_A)}{\varepsilon^2} + r_A \quad \text{in } \Omega \times [0, T_0] \tag{1.25}
\]
as well as \(c_A|_{\partial \Omega} = -1, v_A|_{\partial \Omega} = 0\). Moreover, there are some \(\varepsilon_0 > 0, T_1 > 0\) and \(M_\varepsilon : (0, 1) \times (0, T_0) \to (0, \infty)\), which is increasing with respect to both variables, such that \(M_\varepsilon(\varepsilon, T) \rightarrow \infty\) if \(T \rightarrow 0\) and if
\[
\sup_{0 \leq t \leq T_1} \|c_\varepsilon(t) - c_A(t)\|_{L^2(\Omega)} + \|\nabla(c_\varepsilon - c_A)\|_{L^2(\Omega \times [0, T_1] \cap \Gamma(\varepsilon))} \leq R_\varepsilon^{N+\frac{1}{2}}, \tag{1.26a}
\]
\[
\|\nabla(c_\varepsilon - c_A)\|_{L^2(\Omega \times [0, T_1] \cap \Gamma(\varepsilon))} \leq R_\varepsilon^{N+\frac{1}{2}} \tag{1.26b}
\]
hold true for some \(T_\varepsilon \in (0, T_0), \varepsilon_0 \in (0, 1]\), and all \(\varepsilon \in (0, \varepsilon_0]\), then
\[
\int_0^T \int_{\Gamma_\varepsilon} r_A(x, t)(c_\varepsilon(x, t) - c_A(x, t)) dx \, dt \leq M_\varepsilon(\varepsilon, T) \varepsilon^{2(N+\frac{1}{2})}, \tag{1.27}
\]
for all \(T \in (0, \min(T_\varepsilon, T_1))\), \(\varepsilon \in (0, \varepsilon_0]\).

Here \(w_1\) will be the leading part of the error \(\mathbf{w} = \mathbf{w}_A + \mathbf{w}_1\) and \(w_1|_{\partial \Omega} = w_1(P_T(x), t)\) for \(x \in \Gamma_\varepsilon(2\varepsilon)\), where \(P_T\) denotes the orthogonal projection onto \(\Gamma_\varepsilon\), cf. Section 2.1 below. The tangential gradient \(\nabla_{\varepsilon} \) and \(\partial_{\varepsilon}\) will be defined precisely in the same subsection. In (1.26) \(\varepsilon^2 w_2 \cdot \nabla c_A\) can also be omitted since it is of the same order as \(r_A\). But the presence of the term \(\varepsilon^2 w_1 \cdot \nabla c_A\) is essential for the error estimates, cf. Section 5 and Lemma 5.1 below for the details.

We will prove Theorem 1.1 with the aid of the latter theorem by considering the equation for \(u = c_\varepsilon - c_A\) and using suitable estimates for \(\mathcal{L}_\varepsilon\), which refine the results of [13] in tangential
directions, as well as careful estimates of all remainder terms. In this proof a continuation argument is used to show that \((1.26)\) is valid for \(T_s \geq T_1 > 0\) if \(T_1\) is sufficiently small.

**Remark:** Let us comment on the orders of \(\varepsilon\) on the right-hand side of \((1.26)\). Because of \((1.19)\) and a simple change of variables, we have

\[
\|\theta_0(\tau) - (\chi_{[0,\infty)} - \chi_{(-\infty,0)})\|_{L^2(\mathbb{R})} = M \varepsilon^{\frac{1}{2}}.
\]

Hence the power \(\varepsilon^{\frac{1}{2}}\) appears naturally, when estimating differences in \(L^2\) in normal direction. Moreover, applying the one-dimensional interpolation inequality

\[
\|u\|_{L^\infty(-2\delta,2\delta)} \leq C_\delta \|u\|^\frac{1}{2}_{L^2(-2\delta,2\delta)} \|u\|^\frac{1}{2}_{H^1(-2\delta,2\delta)} \text{ for all } u \in H^1(-2\delta,2\delta),
\]

to \((1.26)\) yield a control of \(u\) of the order \(\varepsilon^{N}\) in the \(L^\infty\)-norm in normal direction (and \(L^2\)-norms in the other directions).

As mentioned above the form of the convection terms in \((1.25)\) will be essential for the remainder estimates. In order to obtain this we will construct \(c_A\) such that

\[
c_A(x,t) = \theta_0 \left( \frac{d\tau(x,t)}{\varepsilon} - h_s(S(x,t),t) \right) + O(\varepsilon^2), \quad \text{where } h_s = h_1 + \varepsilon h_2,\epsilon,
\]

in \(\Gamma(\delta)\) for some suitable \(h_1, h_2, \varepsilon; \mathbb{T}^1 \times [0, T_0] \to \mathbb{R}\), cf. Section 4 for the details. Here \(S\) is defined in Section 2.4 below. For every \(t \in [0, T_0]\), \(S(\cdot, t)|_{\Gamma_t} : \Gamma_t \to \mathbb{T}^1\) is the pull-back of a suitable parametrization of \(\Gamma_t\). In order to include the term \(w_1|_{\Gamma_t} \cdot \nabla c_A\) in \((1.26)\), we will use a suitable choice of \(h_2, \varepsilon\), which will depend on \(w_1\), cf. \((1.9)\) below. Here \(w_1\) depends on \(\varepsilon\) and we have only certain norms of \(w_1\) under control as \(\varepsilon \to 0\). This is in sharp contrast to the term \(h_1\), which is defined by an equation independent of \(\varepsilon\), cf. \((1.5)\) below. The latter equation basically depends only on the (smooth) limit solution \((\Gamma, v, p)\) as it is the case in the expansions in \((1.3)\). The careful treatment of \(h_2, \varepsilon\) (as well as a similar higher order term \(c_3, \varepsilon\) in the expansion of \(c_A\)) is one of the essential novelties in this contribution compared to \((1.3)\) and \((1.5)\) and a key to the proof of our main result. Furthermore, treatment of the error \(v - v_A\) using careful estimates for (very) weak solutions of the Stokes system and the errors in the capillary term on the right-hand side of \((1.3)\) are another essential ingredient. Finally let us note that another novelty of the present contribution in comparison to \((1.3)\) is that, in the expansion of \(c_A\) we will only use three terms of orders \(O(1)\), \(O(\varepsilon^2)\) and \(O(\varepsilon^3)\), respectively, which reduces the number of levels in the asymptotic expansion significantly.

The structure of this article is as follows. In Section 2 we will discuss several preliminary results concerning suitable coordinates close to \(\Gamma\), evolution equations for \(h_1, h_2, \varepsilon\), results on ODEs needed in the asymptotic expansions, and some kind of spectral estimate for \(L_\varepsilon\) uniformly in \(\varepsilon \in (0, \varepsilon_0]\). Then \(v_A\) is constructed in Section 3 assuming that the leading part of \(c_A, 0\) of \(c_A\) (which depends on \(h_1, h_2, \varepsilon\)) is known. Moreover, \(v_A \cdot \nabla c_{A, 0}\) is expanded and using the knowledge of the latter expansion, \(c_A\) is constructed in Section 4 and Theorem \((1.3)\) is proved. Finally, the main result is proven in Section 5. Some lengthy but straightforward calculations related to the matched asymptotic expansions are given in details in the Appendix.

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2 Preliminaries

2.1 Coordinates
We will parametrize \((\Gamma_t)_{t \in [0,T_0]}\) with the aid of a family of smooth diffeomorphisms \(X_0 : \mathbb{T}^1 \times [0,T_0] \rightarrow \Omega\) such that \(\partial_s X_0(s,t) \neq 0\) for all \(s \in \mathbb{T}^1, t \in [0,T_0]\). Moreover, let
\[
\tau(s,t) = \frac{\partial_s X_0(s,t)}{\partial_s X_0(s,t)} \quad \text{and} \quad n(s,t) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tau(s,t) \quad \text{for all} \ (s,t) \in \mathbb{T}^1 \times [0,T_0]
\]
be the normalized tangent and normal vectors on \(\Gamma_t\) at \(X_0(s,t)\).

We choose the orientation of \(\Gamma_t\) (induced by \(X_0(\cdot,t)\)) such that \(n(s,t)\) is the exterior normal with respect to \(\Omega^-(t)\). Moreover, we denote
\[
n_{\Gamma_t}(x) := n(s,t) \quad \text{for all} \ x = X_0(s,t) \in \Gamma_t.
\]
Furthermore, \(V_{\Gamma_t}\) and \(H_{\Gamma_t}\) should be the normal velocity and (mean) curvature of \(\Gamma_t\) (with respect to \(n_{\Gamma_t}\)) and we define
\[
V(s,t) = V_{\Gamma_t}(X_0(s,t)), \quad H(s,t) = H_{\Gamma_t}(X_0(s,t)) \quad \text{for all} \ s \in \mathbb{T}^1, t \in [0,T_0].
\]
Hence \(H_{\Gamma_t} \leq 0\) if \(\Omega^-(t)\) is convex. Moreover, by definition,
\[
V_{\Gamma_t}(X_0(s,t)) = V(s,t) = \partial_s X_0(s,t) \cdot n(s,t) \quad \text{for all} \ (s,t) \in \mathbb{T}^1 \times [0,T_0].
\]

In the following we will need a tubular neighborhood of \(\Gamma_t\): For \(\delta > 0\) sufficiently small, the orthogonal projection \(P_{\Gamma_t}(x)\) of all
\[
x \in \Gamma_t(3\delta) = \{y \in \Omega : \text{dist}(y, \Gamma_t) < 3\delta\}
\]
is well-defined and smooth. Moreover, we choose \(\delta\) so small that \(\text{dist}(\partial \Omega, \Gamma_t) > 3\delta\) for every \(t \in [0,T_0]\). Every \(x \in \Gamma_t(3\delta)\) has a unique representation
\[
x = P_{\Gamma_t}(x) + r n_{\Gamma_t}(P_{\Gamma_t}(x))
\]
where \(r = \text{sdist}(\Gamma_t, x)\). Here
\[
dr(x,t) := \text{sdist}(\Gamma_t, x) = \begin{cases} \text{dist}(\Omega^-(t), x) & \text{if} \ x \notin \Omega^-(t), \\ -\text{dist}(\Omega^+(t), x) & \text{if} \ x \in \Omega^-(t). \end{cases}
\]
For the following we define for \(\delta' \in (0,3\delta]\)
\[
\Gamma(\delta') = \bigcup_{t \in [0,T_0]} \Gamma_t(\delta') \times \{t\}.
\]
Throughout this contribution we will often use
\[
\int_{\Gamma_t(\delta')} f(x) \, dx = \int_{-\delta'}^{\delta'} \int_{\Gamma_t} f(p + r n_{\Gamma_t}(p)) J(r,p,t) \, ds(p) \, dr
\]
for any \(\delta' \in (0,3\delta]\), where \(J : (-3\delta, 3\delta] \times \mathbb{T}^1 \rightarrow (0,\infty)\) is a smooth function depending on \(\Gamma\).

We introduce new coordinates in \(\Gamma(3\delta)\) which we denote by
\[
X : (-3\delta, 3\delta] \times \mathbb{T}^1 \times [0,T_0] \rightarrow \Gamma(3\delta) \quad \text{by} \ X(r,s,t) := X_0(s,t) + r n(s,t),
\]
where
\[
r = \text{sdist}(\Gamma_t, x), \quad s = X_0^{-1}(P_{\Gamma_t}(x), t) =: S(x,t).
\]
Differentiating the identity
\[
dr(X_0(s,t) + r n(s,t), t) = r,
\]
one obtains
\[
\nabla d_r(x,t) = n_{\Gamma_t}(P_{\Gamma_t}(x)), \quad \partial_r d_r(x,t) = -V_{\Gamma_t}(P_{\Gamma_t}(x)), \quad \Delta d_r(q,t) = -H_{\Gamma_t}(q)
\]
(2.4)
for all \((x, t) \in \Gamma(3\delta)\), \((q, t) \in \Gamma\), resp., cf. Chen et al. Section 4.1]. Moreover, we define
\[
\partial_r u(x, t) := \tau(S(x, t), t)u(x, t), \quad \nabla_r u(x, t) := \partial_r u(x, t)\tau(S(x, t), t)
\] (2.5)
for all \((x, t) \in \Gamma(3\delta)\).

In the following we associate a function \(\phi(x, t)\) to \(\tilde{\phi}(r, s, t)\) such that
\[
\phi(x, t) = \tilde{\phi}(d_r(x, t), S(x, t), t) \quad \text{or} \quad \phi(X_0(s, t) + r\mathbf{n}(s, t), t) = \tilde{\phi}(r, s, t).
\]

Then it follows from chain rule together with (2.3) that
\[
\partial_r \phi(x, t) = -V_1(x)(\mathbf{P}_1(x))\partial_r \tilde{\phi}(r, s, t) + \partial_r^3 \tilde{\phi}(r, s, t)
\]
\[
\nabla \phi(x, t) = \mathbf{n}_r(x)(\mathbf{P}_1(x))\partial_r \tilde{\phi}(r, s, t) + \nabla_r \tilde{\phi}(r, s, t)
\] (2.6)
\[
\partial_t \phi(x, t) = \partial_t \tilde{\phi}(r, s, t) + \Delta \mathbf{v}_1(x)\partial_r \tilde{\phi}(r, s, t) + \Delta^3 \tilde{\phi}(r, s, t),
\]

where \(r, s\) are as in (2.3) and we used the notation
\[
\partial_r^3 \tilde{\phi}(r, s, t) = \partial_r \partial_s \tilde{\phi}(r, s, t) + \partial_s \partial_r \tilde{\phi}(r, s, t)
\]
\[
\nabla_r \tilde{\phi}(r, s, t) = (\nabla S)(x, t)\partial_r \tilde{\phi}(r, s, t)
\] (2.7)
\[
\Delta^3 \tilde{\phi}(r, s, t) = (|\nabla S(x, t)|^2)^2 \partial_r \tilde{\phi}(r, s, t) + (\Delta S)(x, t)\partial_r \tilde{\phi}(r, s, t),
\]

cf. Section 4.1 for more details. In (2.7), \(x = \mathbf{n}(s, t)r + X_0(s, t)\).
It can be seen from (2.7) that, if \(g: \mathbb{T}^1 \times [0, T_0] \to \mathbb{R}\) depends only on \((s, t)\), then \(\nabla_r g\) will be a function of \((r, s, t)\) instead of merely depending on the surface coordinate \(s\) and \(t\):
\[
\nabla_r g(r, s, t) = (\nabla S)(x, t)\partial_r g(s, t), \quad \text{where} \quad x = X(r, s, t).
\] (2.8)
We note that, since \(\Gamma\) is given and smooth, we have \(|\nabla_r g| \leq C|\partial_s g|\). Moreover, differentiating
\[
s = S(X_0(x, t) + r\mathbf{n}(s, t), t) \quad \text{for all} \quad s \in \mathbb{T}^1, r \in (-3\delta, 3\delta)
\]
with respect to \(r\) we obtain \(\nabla S(x, t) \cdot \mathbf{n}(s, t) = 0\) for all \((x, t) \in \Gamma(3\delta)\), \(s = S(x, t)\). Motivated by this, we define for every \(h: \mathbb{T}^1 \times [0, T_0] \to \mathbb{R}\)
\[
(\nabla_r h)(s, t) := (\nabla_r^2 h)(0, s, t),
\]
\[
(\Delta_r h)(s, t) := (\Delta_r^2 h)(0, s, t),
\]
\[
(D_r h)(s, t) := (D_r^2 h)(0, s, t),
\] (2.9)
which are the restrictions to \(r = 0\) of the operators above, and we define the differences
\[
(L^r h)(r, s, t) := (\nabla_r^2 h)(r, s, t) - (\nabla_r h)(s, t),
\]
\[
(L^s h)(r, s, t) := (\Delta_r^2 h)(r, s, t) - (\Delta_r h)(s, t),
\]
\[
(L^t h)(r, s, t) := (D_r^2 h)(r, s, t) - (D_r h)(s, t).
\] (2.10)
So the coefficients of the latter operators vanish for \(r = 0\), which corresponds to \(x \in \Gamma_1\).

Throughout this contribution we will frequently use that, if \(\alpha: \Gamma(3\delta) \to \mathbb{R}\) is smooth in normal direction and vanishes on \(\Gamma\), then \(\tilde{\alpha}: \Gamma(3\delta) \to \mathbb{R}\) with
\[
\tilde{\alpha}(x, t) = \begin{cases} 
\frac{\alpha(x, t)}{d_r(x, t)} & \text{if} \ (x, t) \in \Gamma(3\delta) \setminus \Gamma, \\
\partial_n \alpha \cdot (x, t) & \text{if} \ (x, t) \in \Gamma
\end{cases}
\]
is smooth in normal direction as well. Moreover, regularity in tangential directions is preserved. In particular \(\tilde{\alpha}\) is smooth if \(\alpha\) is smooth. These statements can be easily proved with the aid of a Taylor expansion with respect to \(dr\).

Finally, we denote
\[
(X_0^* u)(s, t) := u(X_0(s, t), t) \quad \text{for all} \quad s \in \mathbb{T}^1, t \in [0, T_0],
\]
\[
(X_0^* v)(p, t) := v(X_0^{-1}(p, t), t) \quad \text{for all} \quad (p, t) \in \Gamma
\]
if \(u: \Gamma \to \mathbb{R}^N\) and \(v: \mathbb{T}^1 \times [0, T_0] \to \mathbb{R}^N\) for some \(N \in \mathbb{N}\).
2.2 Tangential Differential Operators

In view of (2.9), we can express the tangential derivative in the following way
\[ \nabla_{\tau} = (I - n(S(\cdot), \cdot) \otimes n(S(\cdot), \cdot)) \nabla. \] (2.11)

Then we have
\[
[\partial_n, \nabla_{\tau}]g := \partial_n((I - n \otimes n)\nabla g) - (I - n \otimes n)\nabla(\partial_n g) \\
= (I - n \otimes n)\partial_n g - (I - n \otimes n)\nabla(n \cdot \nabla g) \\
= 2 \left((I - n \otimes n)\nabla n_j + \partial_{s_j} g = \tau(\partial_s n \cdot \nabla g). \right. \] (2.12)

This shows that the commutator \([\partial_n, \nabla_{\tau}]\) is a tangential differential operator. For the sake of integrating by parts for functions defined near the interface, we need the formula
\[
\text{div}(I - n(S(\cdot), \cdot) \otimes n(S(\cdot), \cdot)) = -\text{div}(n(S(\cdot), \cdot) \nabla n(S(\cdot), \cdot) - n(S(\cdot)) \cdot \nabla n(S(\cdot), \cdot) = \kappa n(S(\cdot), \cdot). \]

This together with Gauß’ Theorem implies the following lemma:

**Lemma 2.1** Let \( t \in [0, T_0] \). For any \( u \in H^2_0(\Gamma_t(2\delta)) \), \( v \in H^1_0(\Gamma_t(2\delta))^2 \) we have
\[
\int_{\Gamma_t(2\delta)} u \text{div}_{\tau} v dx = -\int_{\Gamma_t(2\delta)} \nabla_{\tau} u \cdot v dx - \int_{\Gamma_t(2\delta)} \kappa n \cdot v dx \] (2.13)

where \( \nabla_{\tau} \) is defined by (2.11) and \( \kappa = -\text{div} n(S(\cdot), \cdot) \).

2.3 The Stretched Variable

In the sequel, for given functions \( h_1, h_2 : T^1 \times [0, T_0] \to \mathbb{R} \), we shall define the stretched variable \( \rho \) by
\[
\rho(x, t) = \frac{dr(x, t)}{\epsilon} - h_1(S(x, t), t) - \epsilon h_2(S(x, t), t). \] (2.14)

Moreover, we denote \( h_\epsilon := h_1 + \epsilon h_2 \). As in [16, Section 4.2] we consider the Taylor expansion of \( \Delta d_T \) in the normal direction and obtain
\[
\Delta d_T(x, t) = -H_T(s) - \epsilon(\rho + h_\epsilon(s, t))\kappa_1(s, t) \\
+ \epsilon^2 \kappa_2(s, t)(\rho + h_\epsilon(s, t))^2 + \epsilon^3 \kappa_3(s, s, t), \] \] (2.15)
where \( s \) is understood via (2.3) and
\[
\kappa_1(s, t) = -\nabla d_T(X_0(s, t)) \cdot \nabla \Delta d_T(X_0(s, t)) = H(s, t)^2, \]
\( \kappa_2(s, t) \) is smooth and \( \kappa_{3, s} \) is a smooth function satisfying
\[
|\kappa_{3, s}(\rho, s, t)| \leq C|\rho + h_\epsilon(s, t)|^3 \text{ for all } \rho \in \mathbb{R}, s \in T^1, t \in [0, T_0], \epsilon \in (0, 1). \] (2.16)

The following lemma is due to the chain rule and [26], cf. [16, Section 4.2]:

**Lemma 2.2** Let \( \dot{w} : \mathbb{R} \times T^1 \times [0, T_0] \to \mathbb{R} \) be sufficiently smooth and let
\[
w(x, t) = \dot{w} \left( \frac{dx(x, t)}{\epsilon} - h_\epsilon(S(x, t), t), S(x, t), t \right) \text{ for all } (x, t) \in \Gamma(2\delta). \]

Then for each \( \epsilon > 0 \)
\[
\partial_{\tau} w(x, t) = -\left( V_T\left(\frac{dr(x, t)}{\epsilon}\right) + \partial_T h_\epsilon(r, s, t) \right) \partial_\rho \dot{w}(\rho, s, t) + \partial_T^2 \dot{w}(\rho, s, t) \\
\nabla w(x, t) = \left( \nabla_T\left(\frac{dr(x, t)}{\epsilon}\right) - \nabla_T h_\epsilon(r, s, t) \right) \partial_\rho \dot{w}(\rho, s, t) + \nabla_T^2 \dot{w}(\rho, s, t) \\
\Delta w(x, t) = \epsilon^2 + \|\nabla_T h_\epsilon(r, s, t)\|^2 \partial_\rho^2 \dot{w}(\rho, s, t) \\
+ \left( \epsilon^{-1} \Delta d_T(x, t) - \Delta_T h_\epsilon(r, s, t) \right) \partial_\rho \dot{w}(\rho, s, t) \\
- 2\nabla_T h_\epsilon(r, s, t) \cdot \nabla_T \partial_\rho \dot{w}(\rho, s, t) + \Delta_T \dot{w}(\rho, s, t), \] (2.17)
2.4 A Result from ODE-Theory

The following result on solvability of the linearized equation for the optimal profile \( \theta_0 \) will be essential for the asymptotic expansion. The proof can be found in [16, Lemma 3].

Proposition 2.3 Assume that \( g: \mathbb{R} \times \mathbb{T}^1 \times [0, T_0] \to \mathbb{R} \) and \( g^\pm: \mathbb{T}^1 \times [0, T_0] \to \mathbb{R} \) are smooth and for some \( i \in \mathbb{N}_0 \), \( a > 0 \) satisfying

\[
\sup_{(s,t) \in \mathbb{T}^1 \times [0,T_0]} \left| \partial^k_p \partial^l_s \partial^m_t [g(\rho, s, t) - g^\pm(s, t)] \right| \leq C_{k,l,m}(1 + |\rho|)^{a} e^{-a|\rho|} \quad \text{for all } \rho \geq 0.
\]

for all \( k, l, m \in \mathbb{N}_0 \) and some \( C_{k,l,m} > 0 \). Then for given \( (s, t) \in \mathbb{T}^1 \times [0, T_0] \) the ODE

\[
-\partial^2_p u + f''(\theta_0) u = g(\cdot, s, t) \quad \text{in } \mathbb{R}, \quad u(0, s, t) = 0
\]

has a unique bounded solution \( u(\cdot, s, t) \) if and only if

\[
\int_{\mathbb{R}} g(p, s, t) \theta'_0(p) dp = 0.
\]

If the solution exists for all \( (s, t) \in \mathbb{T}^1 \times [0, T_0] \), then for all \( (k, l, m) \in \mathbb{N}^3 \) there is some \( C_{k,l,m} \) such that

\[
\sup_{(s,t) \in \mathbb{T}^1 \times [0,T_0]} \left| \partial^k_p \partial^l_s \partial^m_t \left( u(\rho, s, t) - g^\pm(s, t) \right) \right| \leq C_{k,l,m}(1 + |\rho|)^{a} e^{-a|\rho|} \quad \text{for all } \rho \geq 0.
\]

2.5 Remainder Terms

In the sequel we shall use for fixed \( t \in [0, T_0] \) and \( 1 \leq p < \infty \)

\[
L^p(\Gamma_1(2\delta)) := \left\{ f: \Gamma_1(2\delta) \to \mathbb{R} \text{ measurable } : \| f \|_{L^p(\Gamma_1(2\delta))} < \infty \right\}, \quad \text{where}
\]

\[
\| f \|_{L^p(\Gamma_1(2\delta))} := \left( \int_{\Gamma_1(2\delta)} \text{ess sup}_{|r| \leq 2\delta} |f(X_0(s, t) + r n(s, t))|^p ds \right)^{\frac{1}{p}}.
\]

Moreover, the standard \( L^p \)-Sobolev space of order \( m \in \mathbb{N}_0 \) on an open set \( U \subseteq \mathbb{R}^N \) will be denoted by \( W^m_p(U) \) and \( L^p(U) \) is the usual Lebesgue space with respect to the Lebesgue measure. Furthermore, \( H^s(U) \) denotes the \( L^2 \)-Sobolev space of order \( s \in \mathbb{R} \) and \( H^s_0(U) \) is the closure of \( C_0^\infty(U) \) in \( H^s(U) \). The \( X \)-valued variants are denoted by \( W^m_p(U; X) \), \( L^p(U; X) \), and \( H^s(U; X) \), respectively. We note that

\[
H^1(\Gamma_1(2\delta)) \hookrightarrow L^{1,\infty}(\Gamma_1(2\delta)),
\]

which follows from the interpolation inequality

\[
\| f \|_{L^\infty(-2\delta,2\delta)} \leq C \| f \|_{H^1(-2\delta,2\delta)}^{\frac{1}{2}} \| f \|_{H^1(-2\delta,2\delta)}^{\frac{1}{2}} \quad \text{for } f \in H^1(-2\delta,2\delta).
\]

The following lemma provides a first useful general estimate of typical remainder terms.

Lemma 2.4 Let \( g: \mathbb{T}^1 \times [0, T_0] \to \mathbb{R} \) be continuous. There is some \( C > 0 \), independent of \( g \), such that

\[
\| \eta \left( \frac{d_R(t)}{s} \right) - g(S(\cdot, t), t) \psi \|_{L^1(\Gamma_1(2\delta))} \leq C e^{\frac{s}{2}} \| \eta \|_{L^2(\partial \delta)} \| u \|_{L^2(\Gamma_1(2\delta))} \| \psi \|_{L^2,\infty(\Gamma_1(2\delta))}
\]

holds for any \( u \in L^2(\Gamma_1(2\delta)), \psi \in L^{2,\infty}(\Gamma_1(2\delta)) \), continuous \( \eta \in L^2(\mathbb{R}) \), and \( \varepsilon \in (0, 1) \).
Proof: For fixed $t \in [0,T_0]$ we perform the change of variables $(x,t) \mapsto (s,r,t)$ and get

$$
\int_{T(2\delta)} \left| \eta \left( \frac{d\ell(x,t)}{\varepsilon} - g(S(x,t),t) \right) u(x) \psi(x) \right| \, dx
= \int_{T} \int_{-2\delta}^{2\delta} \left| \eta \left( \frac{\xi}{\varepsilon} - g(s,t) \right) u(X(r,s,t)) \psi(X(r,s,t)) \right| J_l(r,s) \, dr \, ds,
$$

where $J_l(r,s)$ denotes the square root of the Gram determinant. Since $|J_l(r,s)| \leq C$,

$$
\left\| \eta \left( \frac{d\ell(x,t)}{\varepsilon} - g(S(\cdot,t),t) \right) u \psi \right\|_{L^1(T(2\delta))} 
\leq C \int_{T} \| \eta \left( \frac{\xi}{\varepsilon} - g(s,t) \right) \|_{L^2(\mathbb{R})} \sup_{|r| \leq 2\delta} |\psi(X(r,s,t))| \sqrt{\int_{|r| \leq 2\delta} |u(X(r,s,t))|^2 J_l(r,s) \, dr \, ds} 
\leq C \varepsilon \| \eta \|_{L^2(\mathbb{R})} \sqrt{\sup_{|r| \leq 2\delta} |\psi(X(r,s,t))|^2 \int_{T} \int_{|r| \leq 2\delta} |u(X(r,s,t))|^2 J_l(r,s) \, dr \, ds},
$$

which implies the statement of the lemma. $
$

For a systematic treatment of the remainder terms, we introduce:

**Definition 2.5** For any $k \in \mathbb{R}$ and $\alpha > 0$, $\mathcal{R}_{k,\alpha}$ denotes the vector space of all families of continuous functions $\hat{r}_\varepsilon : \mathbb{R} \times \Gamma(2\delta) \to \mathbb{R}, \varepsilon \in (0,1)$, which are continuously differentiable with respect to $\eta r$, for all $t \in [0,T_0]$ such that

$$
|\partial_{\eta r} \hat{r}_\varepsilon (\rho, x, t) | \leq C e^{-\alpha |\rho|} \varepsilon^{-k} \quad \text{for all } \rho \in \mathbb{R}, (x,t) \in \Gamma(2\delta), j = 0,1, \varepsilon \in (0,1) \quad (2.21)
$$

for some $C > 0$ independent of $\rho \in \mathbb{R}, (x,t) \in \Gamma(2\delta), \varepsilon \in (0,1)$. Moreover, $\mathcal{R}_{k,\alpha}$ is equipped with the norm

$$
\| (\hat{r}_\varepsilon)_{\varepsilon \in (0,1)} \|_{\mathcal{R}_{k,\alpha}} = \sup_{\varepsilon \in (0,1), (x,t) \in \Gamma(2\delta), \rho \in \mathbb{R}, j = 0,1} |\partial_{\eta r} \hat{r}_\varepsilon (\rho, x, t) | e^{\alpha |\rho|} \varepsilon^{-k}.
$$

Finally, $\mathcal{R}_{k,\alpha}^n$ is the subspace of all $(\hat{r}_\varepsilon)_{\varepsilon \in (0,1)} \in \mathcal{R}_{k,\alpha}$ such that

$$
\hat{r}_\varepsilon (\rho, x, t) = 0 \quad \text{for all } \rho \in \mathbb{R}, x \in \Gamma, t \in [0,T_0]. \quad (2.22)
$$

**Lemma 2.6** Let $k \in \mathbb{R}, \alpha > 0$, $1 \leq p < \infty$, $h_\delta : T^1 \times [0,T_0] \to \mathbb{R}$ such that

$$
M := \sup_{0 < \varepsilon < \varepsilon_0, (x,t) \in T^1 \times [0,T_0]} |h_\varepsilon(s,t)| < \infty
$$

for some $T_0 \in (0,T_0], \varepsilon_0 \in (0,1), (\hat{r}_\varepsilon)_{\varepsilon \in (0,1)} \in \mathcal{R}_{k,\alpha}$ and

$$
r_\varepsilon(x,t) := \hat{r}_\varepsilon \left( \frac{d\ell(x,t)}{\varepsilon} - h_\varepsilon \left( S(x,t),t \right),x,t \right) \quad \text{for all } (x,t) \in \Gamma(2\delta).
$$

Then there is a constant $C > 0$, independent of $M,T_0,0 < \varepsilon \leq \varepsilon_0$, and $\varepsilon_0 \in (0,1)$, such that

$$
\sup_{(x,t) \in \Gamma(2\delta)} \| \hat{r}_\varepsilon (\xi, x, t) \|_{L^p(\mathbb{R})} \leq C \varepsilon^{k+\frac{p}{2}} \quad \text{for all } 0 < \varepsilon < \varepsilon_0. \quad (2.23)
$$

Moreover, if even $(\hat{r}_\varepsilon)_{\varepsilon \in (0,1)} \in \mathcal{R}_{k,\alpha}^n$, then there is a constant $C > 0$, independent of $M,T_0,0 < \varepsilon \leq \varepsilon_0$, and $\varepsilon_0 \in (0,1)$, such that

$$
\sup_{0 \leq t \leq T_0, \varepsilon \in (0,1)} \| r_\varepsilon(X(\cdot,s,t),t) \|_{L^p(-2\delta,2\delta)} \leq C(M+1) \varepsilon^{k+\frac{p}{2}+1} \quad \text{for all } \varepsilon \in (0,\varepsilon_0). \quad (2.24)
$$
2.5 Remainder Terms

**Proof:** First let \((\tilde{r}_s)_{s \in (0,1)} \in \mathcal{R}_{k,\alpha}\) and \(1 \leq p < \infty\). Then

\[
\| \sup_{(x,t) \in \Gamma(2\delta)} |r_s(x,t)| \|_{L^p(\mathbb{R})} \leq C \varepsilon^k \varepsilon \int_{-\infty}^{\infty} e^{-\alpha |r/\varepsilon - h_s(t)|} dr = C \varepsilon^{k+1} \int_{-\infty}^{\infty} e^{-\alpha |z|} dz = C \varepsilon^{k+1}
\]

uniformly in \(\varepsilon \in (0,1)\). This implies (2.23). If \(p = \infty\), then (2.23) follows immediately from (2.21).

Now, if even \((\tilde{r}_s)_{s \in (0,1)} \in \mathcal{R}_{k,\alpha}^0\), then (2.22) implies

\[
r_s(X(r,s,t), t) = \int_0^r \partial_\alpha r_s \tilde{r}_s \left(\frac{r}{\varepsilon} - h_s(s,t), X(z,s,t), t \right) dz \quad \text{for all } r \in (-2\delta, 2\delta).
\]

Hence

\[
|r_s(X(r,s,t), t)| \leq C \varepsilon^k |r| e^{-\alpha |r/\varepsilon - h_s(s,t)|} \\
\leq C \varepsilon^{k+1} \varepsilon ^i \varepsilon e^{-\alpha |r/\varepsilon - h_s(s,t)|} + C M \varepsilon^{k+1} e^{-\alpha |r/\varepsilon - h_s(s,t)|},
\]

Using this estimate together with \(\int_{-\infty}^{\infty} |z|^p e^{-\alpha |z|^p} dz < \infty\) for all \(j \in \mathbb{N}_0\), one proves (2.24) in a similar way as before. 

**Corollary 2.7** Let \(r_s, h_s, M \) and \(T_\varepsilon, 0 < \varepsilon \leq \varepsilon_0 \) be as in Lemma 2.4. \((\tilde{r}_s)_{0 < s < 1} \in \mathcal{R}_{k,\alpha}\) for some \(\alpha > 0\), \(k \in \mathbb{R}\) and let \(j = 1\) if even \((\tilde{r}_s)_{0 < s < 1} \in \mathcal{R}_{k,\alpha}^0\) and \(j = 0\) else. Then there is some \(C > 0\), independent of \(T_\varepsilon, 0 < \varepsilon \leq \varepsilon_0, \varepsilon_0 \in (0,1)\) such that

\[
\|a(P_{T_\varepsilon}(\cdot)) r_s \varphi \|_{L^1(\Gamma_s(2\delta))} \leq \left( 1 + M \right)^j \varepsilon^{1+k+j} \| \varphi \|_{H^1(\Omega)} \| a \|_{L^2(\Gamma_1)}
\]

and

\[
\|a(P_{T_\varepsilon}(\cdot)) r_s \varphi \|_{L^\infty(\Gamma_s(2\delta))} \leq \left( 1 + M \right)^j \varepsilon^{1+k+j} \| a \|_{L^2(\Gamma_1)}
\]

uniformly for all \(\varphi \in H^1(\Omega), a \in L^2(\Gamma_1), t \in [0, T_\varepsilon], \) and \(\varepsilon \in (0, \varepsilon_0]\).

**Proof:** Using a change of variables \((x, t) \rightarrow (r, s, t)\) and then Lemma 2.6 we obtain

\[
\|a(P_{T_\varepsilon}(\cdot)) r_s \varphi \|_{L^1(\Gamma_s(2\delta))} = \int_{-2\delta}^{2\delta} \int_{T_\varepsilon} |a(X_0(s,t))| |r_s(\frac{r}{\varepsilon} - h_s(s,t), X(r,s,t), t)| |\varphi(X(r,s,t))| J_2(r,s) ds dr \\
\leq C \int_{T_\varepsilon} |a(X_0(s,t))| \sup_{(x,t) \in \Gamma(2\delta)} \left| r_s \left(\frac{r}{\varepsilon} - h_s(s,t), X(z,s,t), t \right) \right| \sup_{r \in (-2\delta, 2\delta)} |\varphi(X(r,s,t))| ds \\
\leq \left( 1 + M \right)^j \varepsilon^{1+k+j} \int_{T_\varepsilon} \sup_{|r| \leq 2\delta} |\varphi(X(r,s,t))|^2 ds \left\| a \right\|_{L^2(\Gamma_1)} \\
\leq \left( 1 + M \right)^j \varepsilon^{1+k+j} \left\| \varphi \right\|_{H^1(\Omega)} \left\| a \right\|_{L^2(\Gamma_1)}
\]

for all \(a \in L^2(\Gamma_1), \varepsilon \in (0,1], t \in [0, T_\varepsilon], \) and \(\varphi \in H^1(\Omega), \) which proves the first estimate. Similarly, we obtain

\[
\|a(P_{T_\varepsilon}(\cdot)) r_s \varphi \|_{L^\infty(\Gamma_s(2\delta))} = \left( \int_{-2\delta}^{2\delta} \int_{T_\varepsilon} |a(X_0(s,t))|^2 |r_s(X(r,s,t), t)|^2 J_2(r,s) ds dr \right)^{\frac{1}{2}} \\
\leq C \left( \int_{T_\varepsilon} \sup_{0 \leq s \leq T_\varepsilon, x \in \Gamma_1} |r_s(X(\cdot, s,t), t)|^2 ds \\
\leq C \left( 1 + M \right)^j \varepsilon^{1+k+j} \left\| a \right\|_{L^2(\Gamma_1)}
\]

for all \(a \in L^2(\Gamma_1), \varepsilon \in (0,1), \) and \(t \in [0, T_\varepsilon]. \)
Remark 2.8 In the following we will apply the results of this subsection to remainder terms \((\hat{r}_\varepsilon)_{\varepsilon \in (0, \varepsilon_0)}\) for some \(\varepsilon_0 \in (0, 1]\). This case can easily be reduced to the former case by considering \((\hat{r}_\varepsilon)_{\varepsilon \in (0, 1)}\) with
\[
\hat{r}_\varepsilon = \begin{cases} 
\hat{r}_\varepsilon & \text{if } \varepsilon \in (0, \varepsilon_0), \\
0 & \text{else}.
\end{cases}
\]
In this way we can apply all definitions and statements above with \(\varepsilon \in (0, \varepsilon_0)\) instead of \(\varepsilon \in (0, 1]\).

2.6 Parabolic Equations on Evolving Hypersurfaces

Throughout this subsection \(0 < T < \infty\) is arbitrary, but fixed. In later applications we will choose \(T = T_0\). We shall denote the function space
\[
X_T := L^2(0, T; H^{5/2}(\mathbb{T}^1)) \cap H^1(0, T; H^{1/2}(\mathbb{T}^1)),
\tag{2.25}
\]
equipped with the norm
\[
\|u\|_{X_T} = \|u\|_{L^2(0, T; H^{5/2}(\mathbb{T}^1))} + \|u\|_{H^1(0, T; H^{1/2}(\mathbb{T}^1))} + \|u\|_{t=0}\|H^{3/2}(\mathbb{T}^1)}.
\]
We note that
\[
X_T \hookrightarrow BUC([0, T]; H^{3/2}(\mathbb{T}^1)) \cap L^1(0, T; H^2(\mathbb{T}^1))
\tag{2.26}
and the operator norm of the embedding is uniformly bounded in \(T\). Likewise, we define
\[
Y_T := L^2(\mathbb{R} \times \mathbb{T}^1 \times (0, T)).
\tag{2.27}
\]

**Theorem 2.9** Let \(w: \mathbb{T}^1 \times [0, T] \to \mathbb{R}^2\) and \(a: \mathbb{T}^1 \times [0, T] \to \mathbb{R}\) be smooth. For every \(g \in L^2(0, T; H^\frac{3}{2}(\mathbb{T}^1))\) and \(h_0 \in H^\frac{3}{2}(\mathbb{T}^1)\) there is a unique solution \(h \in X_T\) of
\[
D_t h + w \cdot \nabla h - \Delta h + ah = g \quad \text{on } \mathbb{T}^1 \times [0, T],
\tag{2.28}
\]
\[
h|_{t=0} = h_0 \quad \text{on } \mathbb{T}^1.
\tag{2.29}
\]

**Proof:** According to (2.9), equation (2.28) is equivalent to
\[
\partial_t h - L h = g \quad \text{on } \mathbb{T}^1 \times [0, T]
\]
for some uniformly elliptic operator \(L\) on \(\mathbb{T}^1\) with smooth coefficients depending on \((s, t) \in \mathbb{T}^1 \times [0, T]\). More precisely for every fixed \(t \in [0, T]\) the operator is \(\Lambda(\theta)\)-elliptic in the sense of [20] Definition 3.3 for every \(0 < \theta < \pi\).

Now the result follows from known results for parabolic equations. E.g. one can argue as follows: Let \(\mathcal{L}(t_0)\), with \(t_0 \in [0, T]\), denote the same operator, where \(t\) in the coefficients is replaced by a fixed \(t_0 \in [0, T]\). Moreover, let
\[
\mathcal{A}(t_0): D(\mathcal{A}(t_0)) := H^\frac{3}{2}(\mathbb{T}^1) \subseteq H^\frac{3}{2}(\mathbb{T}^1) \to H^\frac{3}{2}(\mathbb{T}^1)
\]
be its realization on \(H^\frac{3}{2}(\mathbb{T}^1)\). Then there is some \(\lambda_0 \geq 0\) such that \(\lambda_0 + \mathcal{A}(t_0)\) possesses a bounded \(H^\infty\)-calculus due to [20] Theorem 4.10. This implies that for every \(\mathcal{A}(t_0)\) has maximal \(L^p\)-regularity on every finite time interval and for every \(1 < p < \infty\) due to [19] Theorem 3.2. Now the theorem follows from [11] Theorem 2.7.

In order to couple (2.28) to the two-phase Stokes system, we need

**Lemma 2.10** For every \(t \in [0, T]\), \(\mathbf{v} \in H^\frac{3}{2}(\Gamma_t)\) \(^2\) and \(f \in L^2(\Omega)^2\) there is a unique solution \((\mathbf{v}, p) \in H^\frac{3}{2}(\Omega)^2 \times L^2(\Omega)^2\) with \(\mathbf{v}|_{\partial \Omega^+(t)} \in H^\frac{3}{2}(\Omega^+(t))^2\), \(p|_{\partial \Omega^+(t)} \in H^1(\Omega^+(t))^2\), with \(\int_\Omega p \, dx = 0\) of
\[
-\Delta \mathbf{v} + \nabla p = \mathbf{f} \quad \text{in } \Omega^+(t),
\tag{2.30}
\]
\[
\text{div } \mathbf{v} = 0 \quad \text{in } \Omega^+(t),
\tag{2.31}
\]
\[
[2\mathbf{D} \mathbf{v} - p] m_{\Gamma_t} = \mathbf{a} \quad \text{on } \Gamma_t.
\tag{2.32}
\]
Moreover, there is a constant $C > 0$ independent of $t \in [0,T]$ such that
\[
\|v\|_{H^1(\Omega)} + \sum_{\pm} \left( \|v\|_{H^2(\Omega; \pm)} + \|p\|_{H^1(\Omega; \pm)} \right) \leq C \left( \|f\|_{L^2(\Omega)} + \|a\|_{H^{1/2}(\Gamma_1)} \right).
\] (2.33)

**Proof:** First of all, [39, Theorem 1.1] implies the existence of a unique solution $(v, p)$ and for some $C > 0$, which might depend on $t \in [0,T]$. Hence it only remains to prove that $C > 0$ can be chosen independently of $t \in [0,T]$. Standard perturbation arguments imply that for every $t_0 \in [0,T]$ there is some $\varepsilon > 0$ such that (2.33) holds true for some $C > 0$ and any $t \in [0,T] \cap (t_0 - \varepsilon, t_0 + \varepsilon)$. (Alternatively, one can verify that the constants and cut-off functions in the proof of [39] Theorem 1.1 can be chosen independently of $t \in [0,T] \cap (t_0 - \varepsilon, t_0 + \varepsilon)$ if $\varepsilon > 0$ is sufficiently small.) Since $[0,T]$ is compact, there is some $C > 0$ such that (2.33) holds true for any $t \in [0,T]$. \qed

**Corollary 2.11** Let $w: \mathbb{T}^3 \times [0,T] \to \mathbb{R}^2$, $a: \mathbb{T}^3 \times [0,T] \to \mathbb{R}$ and $b: \Gamma \to \mathbb{R}^2$ be smooth. For every $g \in L^2(0,T; H^2(\mathbb{T}^3))$ and $h_0 \in H^2(\mathbb{T}^3)$ there is a unique solution $h \in X_T$ of
\[
D_t h + w \cdot \nabla h - \Delta h + ah = X_0^\ast(v_n) + g \quad \text{on} \ \mathbb{T}^3 \times [0,T],
\]
$h|_{t=0} = h_0$ \quad \text{on} \ \mathbb{T}^3$, \quad \text{(2.34)}
where for every $t \in [0,T]$, $v = v(\cdot, t) \in H^1(\Omega)^2$ with $v|_{\Omega; \pm} \in H^2(\Omega; \pm)(t)$ and $p = p(\cdot, t) \in H^1(\Omega \setminus \Gamma_1) \cap L^2(\Omega)$ with $\int_{\Omega} p \, dx = 0$ are determined by
\[
-\Delta v + \nabla p = 0 \quad \text{in} \ \Omega^\pm(t), t \in [0,T], \quad \text{(2.36)}
\]
\[
\text{div } v = 0 \quad \text{in} \ \Omega^\pm(t), t \in [0,T], \quad \text{(2.37)}
\]
\[
[2Dv - p] n_{\Omega; \pm} = bX_0^\ast(h) - aX_0^\ast(\Delta h) n_{\Gamma_1} \quad \text{on} \ \Gamma_1, t \in [0,T]. \quad \text{(2.38)}
\]
Here $\sigma = \int_{\Omega} \theta_0(\rho) \, d\rho$ as before and $v_n = n \cdot v$. Moreover, if $g$ and $h_0$ are smooth, then $h$ is smooth and $v|_{\Omega; \pm}, p|_{\Omega; \pm}$ are smooth in $\Omega^\pm$. \quad \text{(2.39)}

**Proof:** For given $h \in H^\frac{7}{2}(\mathbb{T}^3)$ and fixed $t \in [0,T]$, let $v \in H^2(\Omega \setminus \Gamma_1)^2 \cap H^1(\Omega)^2$, $p \in H^1(\Omega \setminus \Gamma_1) \cap L^2(\Omega)$ with $\int_{\Omega} p \, dx = 0$ be determined as solution of (2.36) - (2.38). Then testing (2.36) by $v$, integration by parts, and using (2.38) yields
\[
\|v\|_{H^1(\Omega)} \leq C \|(h, \Delta h)\|_{H^{-\frac{1}{2}}(\mathbb{T}^3)} \leq C' \|h\|_{H^\frac{7}{2}(\mathbb{T}^3)}
\]
for some $C, C'$ independent of $h$ and $t \in [0,T]$.

Now we define $B(t): H^\frac{7}{2}(\mathbb{T}^3) \to H^\frac{7}{2}(\mathbb{T}^3)$ by $B(t)h := X_0^\ast(v_n)$. Then the previous estimate implies
\[
\|B(t)h\|_{H^\frac{7}{2}(\mathbb{T}^3)} \leq C \|h\|_{H^\frac{7}{2}(\mathbb{T}^3)}
\]
for some $C$ independent of $h$ and $t \in [0,T]$. Hence we can extend $B(t)$ to $B(t): H^\frac{7}{2}(\mathbb{T}^3) \to H^\frac{7}{2}(\mathbb{T}^3)$ by continuity. Moreover, $H^\frac{7}{2}(\mathbb{T}^3)$ is relatively closer to $H^\frac{7}{2}(\mathbb{T}^3)$ compared with $H^\frac{7}{2}(\mathbb{T}^3)$ in the sense of [10] because of the compactness of the embedding $H^\frac{7}{2}(\mathbb{T}^3) \hookrightarrow H^\frac{1}{2}(\mathbb{T}^3)$, cf. [10] Example 2.9. (d)). Hence the existence of a unique solution $h \in X_T$ as in the statement of the corollary follows from [10] Theorem 2.11 and the previous theorem.

Finally, smoothness of $h$ and $v|_{\Omega; \pm}$ in $\Omega^\pm$ can be shown by standard localization techniques. To see this one transforms $\Omega^\pm$ and $\Gamma$ locally (in space and time) to the situation of a flat interface and applies e.g. difference quotients in tangential directions or the time direction using the a priori estimates in $L^2$-Sobolev spaces obtained before. This yields one order of higher regularity of $h$, $v|_{\Omega; \pm}$ and $p|_{\Omega; \pm}$ in tangential and time directions. Then one order higher regularity in normal directions follows from (2.39) and (2.40). Repeating this arguments yields the claimed smoothness. \quad \text{(2.41)}
Theorem 2.12 Let $g: \mathbb{R} \times T^1 \times (0, T) \to \mathbb{R}$ be a smooth function with $g(\cdot, s, t) \in L^2(\mathbb{R})$ and
\[
\int_{\mathbb{R}} g(\rho, s, t) \theta'_0(\rho) d\rho = 0 \quad \text{for all } (s, t) \in T^1 \times [0, T]. \tag{2.39}
\]
Then there is a unique smooth solution $c = c(\rho, s, t)$ solving
\[
\varepsilon^2(D_t c - \Delta_{T^1} c) - \partial^2_{\rho} c + f'(\theta_0) c = g \quad \text{on } \mathbb{R} \times T^1 \times [0, T],
\]
\[
c|_{t=0} = 0 \quad \text{on } \mathbb{R} \times T^1. \tag{2.40}
\]
Moreover,
\[
\int_{\mathbb{R}} c(\rho, s, t) \theta'_0(\rho) d\rho = 0 \quad \text{for all } (s, t) \in T^1 \times [0, T] \tag{2.41}
\]
and there is a constant $C_1$ independent of $g, c, T' \in (0, T]$ and $\varepsilon \in (0, 1]$ such that
\[
\varepsilon \|c\|_{L^\infty(0,T',L^2(T^1 \times \mathbb{R}))} + \|\nabla (c, \partial_{\rho} c, \varepsilon \partial_{\rho} c)\|_{L^2(0,T',L^2(T^1 \times \mathbb{R}))} \leq C \|g\|_{L^2(0,T',L^2(T^1 \times \mathbb{R}))},
\]
\[
\varepsilon \|
abla^2 (c, \partial_{\rho} c, \varepsilon \partial_{\rho} c)\|_{L^2(0,T',L^2(T^1 \times \mathbb{R}))} \leq C \|g\|_{L^2(0,T',L^2(T^1 \times \mathbb{R}))}.
\]
Furthermore, if additionally $\|((1 + |\rho|)^k g)\|_{L^2(0,T',L^2(T^1 \times \mathbb{R}))}$ is finite for some $k \in \mathbb{N}$, then there is some $C_p$ independent of $g, c, T' \in (0, T]$ and $\varepsilon \in (0, 1]$ such that
\[
\varepsilon \|\rho^k c\|_{L^\infty(0,T',L^2(T^1 \times \mathbb{R}))} + \|\nabla (\rho^k c, \partial_{\rho} (\rho^k c), \varepsilon \partial_{\rho} (\rho^k c))\|_{L^2(0,T',L^2(T^1 \times \mathbb{R}))} \leq C_p \|((1 + |\rho|)^k g)\|_{L^2(0,T',L^2(T^1 \times \mathbb{R}))}.
\]
Finally, for any $1 < p \leq \infty$ there is a constant $C_p$ independent of $g, c, T' \in (0, T]$ and $\varepsilon \in (0, 1]$ such that
\[
\varepsilon \frac{1}{p} \sup_{(s, t) \in T^1 \times [0, T']} \|c(\cdot, s, t)\|_{H^1(\mathbb{R})} \leq C_p \sup_{s \in T^1} \|\nabla (\cdot, s, t)\|_{L^2(\mathbb{R})} \|g(\cdot, s, t)\|_{L^2(\mathbb{R})}. \tag{2.44}
\]

Proof: The proof of existence is similar to the proof of Theorem 2.9 If we denote
\[
\mathcal{L} \phi := -\partial^2_{\rho} \phi + f'(\theta_0(\rho)) \phi \quad \text{for all } \phi = \phi(\rho) \in H^2(\mathbb{R}),
\]
then equation (2.40) is equivalent to
\[
\varepsilon^2(D_t - \mathcal{A}) c + \mathcal{L} c = g \quad \text{on } T^1 \times [0, T]. \tag{2.45}
\]

for some uniformly elliptic operator $\mathcal{A}$ on $T^1$ with smooth coefficients depending on $(s, t) \in T^1 \times [0, T]$. Then $\mathcal{L}$ is a non-negative self-adjoint operator on $L^2(\mathbb{R})$ with ker $\mathcal{L} = \text{span} \{\theta'_0\}$, which is invertible on the orthogonal complement of $\text{span} \{\theta'_0\}$, cf. e.g. [17, (1.8)]. By [20, Theorem 3.11 and Theorem 4.10] for every $\theta \in (0, \pi)$ there are some $\lambda_0, \lambda_1 \geq 0$ such that $\lambda_0 + \mathcal{A}(t_0)$ and $\lambda_1 + \mathcal{L}$ possess a bounded $H^\infty$-calculus on $L^2(T^1)$, $L^2(\mathbb{R})$, respectively, and a sector $C \setminus \Lambda(\theta)$, where $\Lambda(\theta) = \{ z \in C : \arg(z) \in [0, 2\pi - \theta] \}$. The same is true if $L^2(T^1)$ and $L^2(\mathbb{R})$ are replaced $L^2(T^1 \times \mathbb{R})$ since $\mathcal{A}(t_0)$ does not act on $\rho \in \mathbb{R}$ and $\mathcal{L}$ does not act on $s \in T^1$. Hence [25, Theorem 6.3] implies that $\lambda_0 + \lambda_1 + \mathcal{A}(t_0) + \mathcal{L}$ is $\mathcal{R}$-sectorial on $\Lambda(\theta)$ for every $t_0 \in [0, T]$. By choosing $\theta < \frac{\pi}{4}$, we obtain that $\mathcal{A}(t_0) + \mathcal{L}$ has maximal $L^2$-regularity on every finite time interval and for every $1 < p < \infty$ due to [19, Theorem 3.2]. Now the existence of a unique solution
\[
c \in H^1(0, T; L^2(T^1 \times \mathbb{R})) \cap L^2(0, T; H^2(T^1 \times \mathbb{R}))
\]
follows again from [10, Theorem 2.7]. Furthermore, standard results on parabolic equations imply smoothness.
In order to prove (2.41), we multiply the equation (2.45) with \( \theta'_0(\rho) \), use (2.39), and the fact that \( L \) is self-adjoint. This yields
\[
\varepsilon^2(\partial_t - A) \int_\mathbb{R} c(\rho, s, t) \theta'_0(\rho) d\rho = 0 \quad \text{on} \quad \mathbb{T}^1 \times [0, T],
\]
\[
\int_\mathbb{R} c(\rho, s, t) \theta'_0(\rho) d\rho \bigg|_{t=0} = 0 \quad \text{on} \quad \mathbb{T}^1.
\]
Hence, (2.41) follows from the unique solvability of the latter system. Consequently, we also obtain
\[
\int_\mathbb{R} c(\rho, s, t)(Lc)(\rho, s, t) d\rho \geq C\|c(\cdot, s, t)\|_{H^1_1(\mathbb{R})}^2
\]
(2.46)
since \( L \) is positive on the orthogonal complement of \( \text{span}\{\theta'_0\} \). Now the proof of (2.42) follows in a straightforward manner by testing the equation with \( c \), differentiating with respect to \( \rho, s \) and testing with \( \partial_\rho c, \partial_s c \), respectively, and integration by parts. The details are omitted here. Moreover, in order to prove (2.43) one uses that
\[
\varepsilon^2(D_1 - \Delta_T)(\rho^k c) - \partial^2_t (\rho^k c) + f'(\theta_0)\rho^k c = -2k\rho^{k-1}\partial_\rho c - k(k-1)\rho^{k-2}c + \rho^k g.
\]
Now testing with \( \rho^k c \) and using the interpolation inequalities
\[
\|\rho^{-1}\partial_\rho c\|_{L^2} \leq \|\rho^k \partial_\rho c\|_{L^2}^{\frac{1}{k}} \|\partial_\rho c\|_{L^2}^{\frac{k-1}{k}}, \quad \|\rho^{-2} c\|_{L^2} \leq \|\rho^k c\|_{L^2}^{\frac{1}{k}} \|c\|_{L^2}^{\frac{k-2}{k}},
\]
Young’s inequality, and (2.46) one derives (2.43) in the same way as before.

Finally, we prove (2.44). We multiply the differential equation for \( c \) in (2.44) with \( Lc \), integrate with respect to \( \rho, s \), and obtain
\[
\varepsilon^2(\partial_t - A) \int_\mathbb{R} c(\rho, s, t)(Lc)(\rho, s, t) d\rho + \int_\mathbb{R} (Lc)^2(\rho, s, t) d\rho = \int_\mathbb{R} g(\rho, s, t)(Lc)(\rho, s, t) d\rho
\]
\[
\leq \|g(\cdot, s, t)\|_{L^2(\mathbb{R})} \|Lc(\cdot, s, t)\|_{L^2(\mathbb{R})}.
\]
Using the fact that
\[
\|(Lc)(\cdot, s, t)\|_{L^2(\mathbb{R})} \geq 2\kappa \|(L^2 c)(\cdot, s, t)\|_{L^2(\mathbb{R})} = 2\kappa \int_\mathbb{R} c(\cdot, s, t)(Lc)(\cdot, s, t) d\rho
\]
for some \( \kappa > 0 \) (since \( c(\cdot, s, t) \) is orthogonal to \( \theta'_0 \)) and Young’s inequality we deduce that
\[
\varepsilon^2(\partial_t - A + \frac{\kappa}{\varepsilon^2}) \int_\mathbb{R} c(\rho, s, t)(Lc)(\rho, s, t) d\rho \leq \sup_{s \in \mathbb{T}^1} \|g(\cdot, s, t)\|_{L^2(\mathbb{R})}^2 =: M(t).
\]
Since the function \( w(t) := \frac{1}{\varepsilon^2} \int_0^t e^{\frac{\kappa}{\varepsilon^2}(s-t)} M(s) ds \) satisfies \( \varepsilon^2(w' + \frac{\kappa}{\varepsilon^2}w) = M \), it solves the equation
\[
\varepsilon^2(\partial_t - A + \frac{\kappa}{\varepsilon^2})w = M \quad \text{on} \quad \mathbb{T}^1 \times [0, T_0],
\]
\[
w|_{t=0} = 0 \quad \text{on} \quad \mathbb{T}^1.
\]
Thus, by the comparison principle and Hölder’s inequality, we have
\[
\int_\mathbb{R} c(\rho, s, t)(Lc)(\rho, s, t) d\rho \leq w(t) \leq C_p \|M\|_{L^p(0, T')} \frac{\varepsilon^2}{\varepsilon^2 t_p} \quad \text{for all} \quad t \in [0, T'],
\]
which combined with (2.46) implies (2.44).
2.7 Spectral Estimate

In this subsection we assume that $\Omega \subseteq \mathbb{R}^d$ is a bounded domain and $\Gamma_t \subseteq \Omega$, $t \in [0, T_0]$, $T_0 > 0$, are given smoothly evolving closed and compact $C^\infty$-hypersurfaces, dividing $\Omega$ in disjoint domains $\Omega^+(t)$ and $\Omega^-(t)$ as before, and

\[
c_{A}(x) = c_{A,0}(x) + \varepsilon^2 c_{A,2+}(x), \quad \text{for all } x \in \Omega,
\]

\[
c_{A,0}(x) = \zeta \, d\tau \theta_t + (1 - \zeta \, d\tau) \left( \chi_{\Omega^+} - \chi_{\Omega^-} \right) \quad \text{for all } x \in \Omega,
\]

where $\zeta$ satisfies \((2.20)\). Moreover, we assume that dist$(\Gamma_t, \partial \Omega) > 2\delta$ for all $t \in [0, T_0]$. In this subsection, for given continuous functions $(\hat{h}_t)_{0 < c < 1} : \Gamma \to \mathbb{R}$ with $\hat{h}_t := \bigcup_{t \in [0, T_0]} \Gamma_t \times \{t\}$, we define the stretched variable $\rho$ by

\[
\rho = \frac{d\tau(x, t)}{\varepsilon} - \hat{h}_t(P_{\Gamma_t}(x), t).
\]

Finally, we assume that

\[
\sup_{\varepsilon \in (0, 1)} \left( \sup_{(p, t) \in \Gamma} |\hat{h}_t(p, t)| + \sup_{\varepsilon \in [0, T_0]} |c_{A,2+}(x, t)| \right) \leq M \quad (2.47)
\]

for some $M > 0$. We remark that later we will apply the results of this subsection to $\hat{h}_t(p, t) = \hat{h}_t(\chi_{\Omega_0}^+(p, t), t)$ for some $\hat{h}_t : [0, T_0] \to \mathbb{R}$. But for the following proof the present formulation is more convenient. Moreover, the result holds true in any dimension $d \geq 2$.

The following spectral estimate will be a key ingredient for the proof of convergence.

**THEOREM 2.13** Let $c_{A}$ be as above and \((2.24)\) be satisfied for some $M > 0$. Then there are some $C, \varepsilon_0 > 0$, independent of $\hat{h}_t, c_{A}$, such that for every $\psi \in H^1(\Omega)$, $t \in [0, T_0]$, and $\varepsilon \in (0, \varepsilon_0]$ we have

\[
\int_{\Omega} (|\nabla \psi(x)|^2 + \varepsilon^{-2} f''(c_{A}(x, t))\psi^2(x)) \, dx \geq -C \int_{\Omega} \psi^2 \, dx + \int_{\Omega_{\Gamma}(t)} |\nabla \psi|^2 \, dx + \int_{\Gamma_{\tau}(t)} |\nabla \tau \psi|^2 \, dx.
\]

**Proof:** In the sequel, we shall fix $t \in [0, T_0]$. For sufficiently small $\varepsilon_0 \in (0, 1]$ and $\varepsilon \in (0, \varepsilon_0]$, consider the set

\[
\Gamma_{\epsilon} := \{ x \in \Gamma_t(2\delta) \mid d\tau(x, t) \in I_{\epsilon}, p = P_{\Gamma_t}(x) \}
\]

where $I_{\epsilon}$ is defined via

\[
I_{\epsilon} := \left(-\frac{4\delta}{\varepsilon} + \varepsilon \hat{h}_t(p, t), \frac{4\delta}{\varepsilon} + \varepsilon \hat{h}_t(p, t)\right).
\]

As in \((15)\) we assume for notational simplicity that $\delta = 1$. Then it is evident that $x \in \Gamma_{\epsilon}$ is equivalent to $\rho \in I_{\epsilon} := \left(-\frac{4\delta}{\varepsilon} + \varepsilon \hat{h}_t(p, t), \frac{4\delta}{\varepsilon} + \varepsilon \hat{h}_t(p, t)\right)$. If we denote $\psi(r, p) = \psi(x), \varepsilon_{\epsilon}(r, p, t) = c_{A}(x, t)$, then

\[
\int_{\Omega} (|\nabla \psi(x)|^2 + \varepsilon^{-2} f''(c_{A}(x, t))\psi^2(x)) \, dx \\
\geq \int_{\Omega_{\Gamma}(t)} \left( |\nabla \psi(r, p)|^2 + \varepsilon^{-2} f''(c_{A}(r, p, t))\psi^2(r, p) \right) J_{\epsilon}(r, p) \, dr \, dp \\
+ \int_{\Omega \setminus \Gamma_{\epsilon}} |\nabla \psi|^2 \, dx + \int_{\Gamma_{\epsilon}} |\nabla \tau \psi|^2 \, dx,
\]

since $f''(c_{A}(x, t)) \geq 0$ for all $x \in \Omega \setminus \Gamma_t(\frac{1}{2}) \supseteq \Omega \setminus \Gamma_{\epsilon}, t \in [0, T_0]$ and $0 < \varepsilon \leq \varepsilon_0$ for sufficiently small $\varepsilon_0$. Moreover, since $|\nabla \psi(x)|^2 \geq |\nabla \tau \psi(x)|^2$ for all $x \in \Gamma_t(1) \setminus \Gamma_{\epsilon}$ and $\Gamma_t(1) \supseteq \Gamma_{\epsilon}$ for sufficiently small $\varepsilon$, we have

\[
\int_{\Omega \setminus \Gamma_{\epsilon}} |\nabla \psi|^2 \, dx + \int_{\Gamma_{\epsilon}} |\nabla \tau \psi|^2 \, dx \geq \int_{\Omega \setminus \Gamma_t(1)} |\nabla \psi|^2 \, dx + \int_{\Gamma_t(1)} |\nabla \tau \psi|^2 \, dx.
\]

Let us fix $(p, t) \in \Gamma$ and perform a change of variables in $r$:

\[
P_{\epsilon}^* : I_{\epsilon} \to I_{\epsilon}^*, \quad z \mapsto r = \varepsilon(z + \hat{h}_t(p, t)).
\]
This gives
\[
\int_{I_p} \left( |\partial_p \tilde{\psi}(r,p)|^2 + \varepsilon^{-2} f''(\tilde{\epsilon}_A(r,p,t))\tilde{\psi}^2(r,p) \right) J_t(r,p)dr
\]
\[
= \int_{I_t} \left( \varepsilon |\partial_p \tilde{\psi}(F_p^\varepsilon(z),p)|^2 + \varepsilon^{-1} f''(\tilde{\epsilon}_A(F_p^\varepsilon(z),p,t)) \tilde{\psi}^2(F_p^\varepsilon(z),p) \right) J_t(F_p^\varepsilon(z),p)dz
\]
\[
= \varepsilon^{-2} \int_{I_t} \left( |\partial_p \tilde{\psi}(z,p)|^2 + f''(\tilde{\epsilon}_A(F_p^\varepsilon(z),p,t)) \tilde{\psi}^2(z,p) \right) J_t(F_p^\varepsilon(z),p)dz,
\]
where \( \Psi(z,p) := \sqrt{\varepsilon} \tilde{\psi}(F_p^\varepsilon(z),p) \) and \( \tilde{J}_t(z,p) := J_t(F_p^\varepsilon(z),p) \). Furthermore, with \( \tilde{\Psi} := J_t^{1/2} \Psi \), we compute
\[
\int_{I_p} \left( |\partial_p \tilde{\psi}(r,p)|^2 + \varepsilon^{-2} f''(\tilde{\epsilon}_A(r,p,t))\tilde{\psi}^2(r,p) \right) J_t(r,p)dr
\]
\[
= \varepsilon^{-2} \int_{I_t} \left( |\partial_p \tilde{\psi}(z,p)|^2 + f''(\tilde{\epsilon}_A(z,p,t)) \tilde{\psi}^2(z,p) \right) J_t(z,p)dz
\]
\[
- \varepsilon^{-2} \int_{I_t} 2\partial_p \tilde{\psi}(z,p) J_t^{1/2}(z,p) \tilde{\psi}(z,p) \partial_p J_t^{1/2}(z,p) + \tilde{\psi}^2(z,p) \partial_p J_t^{1/2}(z,p)^2 dz
\]
\[
= \varepsilon^{-2} \int_{I_t} \left( |\partial_p \tilde{\psi}(z,p)|^2 + f''(\tilde{\epsilon}_A(z,p,t)) \tilde{\psi}^2(z,p) \right) J_t(z,p)dz
\]
\[
- \varepsilon^{-2} \int_{I_t} \frac{1}{2} \partial_p J_t^{1/2}(z,p) \partial_p \tilde{J}_t(z,p) + \frac{1}{4} \tilde{\psi}^2(z,p) \partial_p \tilde{J}_t^{1/2}(z,p)^2 dz
\]
\[
= \varepsilon^{-2} \int_{I_t} \left( |\partial_p \tilde{\psi}(z,p)|^2 + f''(\tilde{\epsilon}_A(z,p,t)) \tilde{\psi}^2(z,p) \right) J_t(z,p)dz
\]
\[
+ \varepsilon^{-2} \int_{I_t} \frac{1}{4} \tilde{\psi}^2(z,p) \partial_p \tilde{J}_t(z,p)/\tilde{J}_t^{1/2}(z,p) dz
\]
\[
- \varepsilon^{-3} \int_{I_t} \frac{1}{2} \tilde{\psi}^2(z,p) \partial_p \tilde{J}_t(z,p)/\tilde{J}_t^{1/2}(z,p) \bigg|_{z=-3/4}^{z=-3/4e},
\]
where we used integration by parts in the last step. We conclude that
\[
\int_{I_p} \left( |\partial_p \tilde{\psi}(r,p)|^2 + \varepsilon^{-2} f''(\tilde{\epsilon}_A(r,p,t))\tilde{\psi}^2(r,p) \right) J_t(r,p)dr
\]
\[
= \varepsilon^{-2} \int_{I_t} \left( |\partial_p \tilde{\psi}(z,p)|^2 + f''(\tilde{\epsilon}_A(z,p,t)) \tilde{\psi}^2(z,p) \right) J_t(z,p)dz
\]
\[
+ \int_{I_t} \tilde{q}(z,p,t) \tilde{\psi}^2(z,p)dz - \varepsilon^{-1} \frac{1}{2} \tilde{\psi}^2(z,p) (\partial_p J_t)(F_p^\varepsilon(z),p)/J_t(F_p^\varepsilon(z),p) \bigg|_{z=-3/4}^{z=-3/4e},
\]
where
\[
\tilde{q}(z,p,t) := \varepsilon^{-2} (f''(\tilde{\epsilon}_A(z,p,t)) + f''(\tilde{\epsilon}_A(z,p,t)))
\]
\[
+ \varepsilon^{-3} (2\partial_p \tilde{J}_t)(F_p^\varepsilon(z),p)/J_t(F_p^\varepsilon(z),p) - ((\partial_p J_t)(F_p^\varepsilon(z),p))/J_t^{1/2}(F_p^\varepsilon(z),p).\]

Since \( \Gamma \) is smooth and compact and due to (2.47), we have
\[
|\tilde{q}(z,p,t)| \leq \sup_{|\xi| \leq 2M+2} |f''(\xi)| \sup_{(r,t) \in \Omega} |\epsilon_{A,2}(x,t)| + C \leq C',
\]
for all \( z \in \mathbb{R}, (p,t) \in \Gamma \), where \( M \) is the uniform bound in (2.47). Now, we can proceed as in (2.15) Proof of Theorem 2.3 to show that
\[
\int_{I_t} \int_{I_p} \left( |\partial_p \tilde{\psi}(r,p)|^2 + \varepsilon^{-2} f''(\tilde{\epsilon}_A(r,p,t)) \tilde{\psi}^2(r,p) \right) J_t(r,p)drdp
\]
\[
\geq -\frac{1}{C} \int_{I_t} \int_{I_p} \tilde{\psi}^2(z,p) \tilde{J}_t(z,p)dzdp \geq -\frac{1}{C} \int_{I_t} \tilde{\psi}^2(r,p)drdp \geq -\frac{1}{C} \int_{\Omega} \psi^2(x)dx
\]
for some \( C > 0 \) independent of \( \psi, t \in [0,T_0], \) and \( \varepsilon \in (0,c_0]. \) Altogether we obtain the claimed estimate. \[ \square \]
3 Approximation of the Stokes System

3.1 The Leading Part of the Velocity

The aim of this section is to construct an approximation of the following system

\[
\begin{align*}
-\Delta \hat{v}_A + \nabla \hat{p}_A &= -\varepsilon \text{div}(\nabla c_{A,0} \otimes \nabla c_{A,0}), \\
\text{div} \hat{v}_A &= 0, \\
\hat{v}_A |_{\partial \Omega} &= 0
\end{align*}
\]  

(3.1)

with the aid of the method of formally matched asymptotics, where

\[
c_{A,0}(x, t) = \zeta \circ d_t c_{0}^{\text{in}} + (1 - \zeta \circ d_t) \left( c_{+}^{\text{out}} \chi_+ + c_{-}^{\text{out}} \chi_- \right) \text{ with } c_{0}^{\text{in}}(\rho) := \theta_0(\rho)
\]  

(3.2)

and \( c_{+}^{\text{out}}, c_{-}^{\text{out}} = \pm 1, \chi_\pm = \chi_{\Omega \varepsilon(t)}(x) \),

\[
\rho(x, t) := \frac{d \nu(x, t)}{\varepsilon} - h_1(S(x, t), t), \\
h_\varepsilon(s, t) := h_1(s, t) + \varepsilon h_2(s, t),
\]  

(3.3)

and \( \zeta \) is the cutoff function defined by (1.20). Let \( X_T \) be the function space defined by (2.28). Then we assume that \( h_1, h_2 \) satisfy

\[
h_1 \in C^\infty(T^1 \times [0, T_0]), \quad \sup_{0 < \varepsilon \leq \varepsilon_0} \| h_2, \varepsilon \|_{X_T} \leq M,
\]  

(3.4)

for some \( \varepsilon_0 \in (0, 1), M \geq 1, T_\varepsilon \in (0, T_0) \) for all \( \varepsilon \in (0, \varepsilon_0] \). Note that \( h_1 \) is independent of \( \varepsilon \) but \( h_2 = h_2, \varepsilon \) does.

In the following we construct approximate solutions \((v_A, p_A)\) to \((\hat{v}_A, \hat{p}_A)\) as

\[
\begin{align*}
v_A(x, t) &:= \zeta \circ d_t v_A^{\text{in}}(\rho_A, x, t) + (1 - \zeta \circ d_t) \left( v_A^{\pm}(x, t) \chi_{\pm} + v_A^0(x, t) \right), \\
p_A(x, t) &:= \zeta \circ d_t p_A^{\text{in}}(\rho_A, x, t) + (1 - \zeta \circ d_t) \left( p_A^{\pm}(x, t) \chi_{\pm} + p_A^0(x, t) \right),
\end{align*}
\]  

(3.5)

where \( \rho_A \) is understood via (3.3) and \( v_A^{\pm}, p_A^{\pm}, v_A^0, p_A^0 \) are defined by

\[
\begin{align*}
v_A^{\pm}(x, t) &:= v_A^{\pm}(x, t) + \varepsilon v_A^{\pm}(x, t), \\
p_A^{\pm}(x, t) &:= p_A^{\pm}(x, t) + \varepsilon p_A^{\pm}(x, t),
\end{align*}
\]  

(3.6)

and

\[
v_A^0(\rho, x, t) := v_0(\rho, x, t) + \varepsilon v_1(\rho, x, t) + \varepsilon^2 v_2(\rho, x, t), \\
p_A^0(\rho, x, t) := \varepsilon^{-1} p_{-1}(\rho, x, t) + p_0(\rho, x, t) + \varepsilon p_1(\rho, x, t).
\]  

(3.7)

After determining \( v_A^{\pm}, \Omega^{\pm}(t) \to \mathbb{R}^2 \) and \( p_A^{\pm}, \Omega^{\pm}(t) \to \mathbb{R} \) in the sequel, these functions will be extended to smooth functions (denoted again by \( v_A^{\pm} \) and \( p_A^{\pm} \)) on \( \Omega \times [0, T_0] \) such that \( \text{div} v_A^{\pm} = 0 \) in \( \Omega \times [0, T_0] \) and \( v_A^0 |_{\partial \Omega} = 0 \). These extensions can e.g. be obtained by using the extension operator in [24] Chapter VI, §3] and correcting the divergence of the extension to obtain \( \text{div} v_A^{\pm} = 0 \) with the aid of the Bogovskii operator, cf. e.g. [21] Chapter III, Theorem 3.2].

Here \( v_j^{\pm}, p_j^{\pm}, v_j, p_j \) are defined as follows: \( v_0 := v_{0|\Omega^{\pm}(t)}; p_0 := p_{0|\Omega^{\pm}(t)} \), where \((v, p)\) is the smooth solution of (1.11)-(1.15). Hence \((v_0^{\pm}, p_0^{\pm})\) solve

\[
\begin{align*}
-\Delta v_0^{\pm} + \nabla p_0^{\pm} &= 0 & \text{in } \Omega^{\pm}(t), t \in (0, T_0), \\
\text{div} v_0^{\pm} &= 0 & \text{in } \Omega^{\pm}(t), t \in (0, T_0), \\
[2Dv_0^{\pm} - p_0^{\pm} \mathbf{I}] n_{\Gamma_t} &= -\sigma H n_{\Gamma_t}, & \text{on } \Gamma_t, t \in (0, T_0), \\
v_0^{\pm} |_{\partial \Omega} &= 0 & \text{on } \partial \Omega \times (0, T_0),
\end{align*}
\]  

(3.8a)-(3.8e)
where we employed the notation

\[ [f^±] = f^+|_{r_1} - f^-|_{r_1} \quad \text{on } \Gamma_t. \] (3.9)

In order to determine \( v_j, p_j \) satisfying suitable matching conditions for all \((x, t) \in \Gamma(3\delta)\), we use the ansatz

\[
\begin{align*}
v_j(x, t) &= \tilde{v}_j(x, t) + \eta(p) \frac{d}{dt} v_j(x, t), \quad j = 0, 1, 2, \\
p_j(x, t) &= \tilde{p}_j(x, t) + \eta(p) \frac{d}{dt} \tilde{p}_j(x, t), \quad j = -1, 0, 1,
\end{align*}
\] (3.10)

where \( \eta(p) := -1 + \frac{2}{\tau} \int_0^\infty \theta_0^2(s)^2 ds \) for all \( p \in \mathbb{R} \). Then \( \eta : \mathbb{R} \to \mathbb{R} \) is a smooth, non-decreasing, and odd function such that

\[ |\eta(p) + 1| \leq C e^{-\alpha \rho} \quad \text{if } \rho \geq 0 \] (3.11)

for some \( \alpha > 0 \). Now we define for all \((p, x, t) \in \mathbb{R} \times \Gamma(3\delta)\)

\[
\begin{align*}
\tilde{p}_{-1}(x, t) &= p_{-1}(x, t) = -\left(\theta_0'(0)\right)^2, \quad \tilde{p}_{-1} \equiv 0, \quad p_{-1}^\pm \equiv 0, \\
\tilde{v}_0(x, t) &= \frac{1}{\tilde{v}}(v_0^+(x, t) + v_0^-(x, t)), \quad \tilde{v}_0(x, t) = \frac{1}{\tilde{v}}(v_0^+(x, t) - v_0^-(x, t)), \\
\tilde{p}_0(x, t) &= \frac{1}{\tilde{v}}(p_0^+(x, t) + p_0^-(x, t)) - \frac{2}{\tilde{v}} \Delta_t(x, t) \eta(p), \\
\tilde{p}_0(x, t) &= \frac{1}{\tilde{v}}(p_0^+(x, t) - p_0^-(x, t)) + \sigma \Delta_t(x, t) \eta(p),
\end{align*}
\] (3.12a-d)

where \((v_1^\pm, p_1^\pm)\) is the solution of the linear two-phase Stokes system

\[
\begin{align*}
-\Delta v_1^+ + \nabla p_1^+ &= 0 \quad \text{in } \Omega^+(t), t \in (0, T_0), \\
\text{div } v_1^+ &= 0 \quad \text{in } \Omega^+(t), t \in (0, T_0), \\
[2Dv_1^+ - p_1^+ \mathbf{n}]_{r_1} &= 2X_0^{-1}(h_1)(\mathbf{n}_r, \tilde{p}_0 - 2\partial_n \tilde{v}_0) \\
-\sigma X_0^{-1}(\Delta_t h_1) \mathbf{n}_{r_1} &= 0 \quad \text{on } \Gamma_t, t \in (0, T_0), \\
[v_1^+] &= 0 \quad \text{on } \Gamma_t, t \in (0, T_0), \\
\nu_1^+ &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\] (3.13a-e)

Here and in the following, for a function \( f \) that vanishes on \( \Gamma_t \), \( \frac{1}{\partial_t} f(x, t) \) is understood as

\[
\lim_{\partial_t \to 0} \frac{1}{\partial_t} f(x, t) = \partial_n f(x, t), \quad \forall (x, t) \in \Gamma.
\]

Moreover, we define

\[
\tilde{v}_1(x, t) = \frac{1}{\tilde{v}}(v_1^+(x, t) + v_1^-(x, t)), \quad \tilde{v}_1(x, t) = \frac{1}{\tilde{v}}(v_1^+(x, t) - v_1^-(x, t)).
\] (3.14)

Furthermore \((v_2, p_1)\) and \(v_2^\pm\) are determined by

\[
\begin{align*}
v_2^\pm(x, t) &= 0 \quad \text{on } \Omega \times [0, T_0], \\
v_2^\pm(x, t) &= -\partial_n \tilde{v}_0, n(x, t) \int_{\mathbb{R}} (z + h_1(S(x, t), t))^2 \eta'(z)dz, \\
v_2(x, t) &= \frac{\partial_n (\tilde{v}_0, n(x, t))}{\tilde{v}} \int_{\mathbb{R}} (z + h_1(S(x, t), t))^2 \eta'(z)dz + \nabla_2^\pm(x, t) - \nabla_2^\pm(x, t),
\end{align*}
\] (3.15a-d)

for all \((x, t) \in \Gamma(3\delta), \rho \in \mathbb{R} \), and

\[
\begin{align*}
\tilde{v}_{2, \rho}(x, t) &= -\partial_n \tilde{v}_0, \rho^\pm(y_1, x, t) \int_{-\infty}^0 \int_{-\infty}^y ((z^2 - h_1^2)\eta''(z) + 4z\eta'(z)) dz dy, \\
v_{2, \rho}(x, t) &= 0, \quad \tilde{v}_{2, \rho}(x, t) = \tilde{v}_{2, \rho}(x, t), \quad \tilde{v}_{2, \rho}(x, t) = \lim_{\rho \to \pm \infty} \tilde{v}_{2, \rho}(x, t),
\end{align*}
\] (3.16a-b)
Finally following facts: if $p$ respect to $(3.8)$

Lemma 3.2

We first show that these definitions do not leads to circular reasoning: First of all

Proof:

In the above formula, $h_2$ should be interpreted as a function of $(x, t)$ via $h_2 = h_2(S(x), t)$. The motivation for defining $a_1$ can be seen at (3.50) in the appendix.

Remark 3.1 From the construction above one observes that $v_0$ and $p_0$ only depend on the solutions $(v_0^+, p_0^+)$ of the limit sharp interface problem (3.3) or equivalently (3.11)-(3.16). Moreover, $v_1$ and $v_2$ depend only on the choice of $h_1$ and $p_1$ depends on $h_1, h_2, \varepsilon$.

Lemma 3.2 Under the assumption (3.3), the functions $(v_0^+ \cdot, p_0^+)$, $(v_1^+, p_1^+)$, $(v_{2, \varepsilon}^+, p_{2, \varepsilon}^+)$ and

\[ \{ \tilde{p}_{-1}, \tilde{p}_0, \tilde{p}_1, \tilde{v}_0, \tilde{v}_1, \tilde{v}_2, \} \]

can be defined through (3.8), (3.12), (3.13), and (3.16) and they are all smooth. The functions $\tilde{p}_1, \tilde{p}_1$ are continuous with respect to $(\rho, x, t) \in \Gamma \times (3\varepsilon)$ and continuously differentiable with respect to $(\rho, x, t) \in \Gamma \times (3\varepsilon)$ for almost every $t \in (0, T)$. 

Proof: We first show that these definitions do not lead to circular reasoning: First of all $\Gamma$ and $(v_0^+, p_0^+)$ are given. Hence all the functions in (3.12) as well as $v_{2, \varepsilon}$ are well defined. Moreover, $(v_0^+, p_0^+)$ are determined by solving (3.13) for given $h_1$, which can be done because of Lemma 2.10 and Corollary 2.11. Hence $v_1$ and thus $v_1, v_2, \varepsilon$ are well defined as well. With all the previous information, the first three formulae in (3.10) make sense and thus also (3.13). Finally $\tilde{p}_1, \tilde{p}_1$ and $p_1$ can be defined via (3.16).

In order to show that these functions are continuously differentiable/smooth, we need the following facts: if $f(x) \in C^4(\Gamma \times T)$ and vanishes on $\Gamma$, then we can redefine $\frac{\partial f}{\partial n}$ on $\Gamma$ by

\[ \lim_{t \to 0} \frac{f(x) - f(\Gamma_t(x))}{t} \in C^{4-1}(\Gamma \times T). \]

One can verify that all the functions defined in (3.12) are smooth, using (3.13). $\frac{\partial v_0}{\partial n}$ is smooth because $v_0 |_{\Gamma}$ vanishes on the interface due to the divergence-free condition of $v_0^+$ and $|v_0^+| := v_0^+ |_{\Gamma}, -v_0^+ |_{\Gamma} = 0$.

Now we consider $v_2$ and $\tilde{v}_2$. We substitute the formula for $\tilde{v}_2$ in (3.16) into that of $v_2, \varepsilon$ and denote that, for all $x \in \Omega \setminus \Gamma$, $t \in [0, T]$

\[ v_2(x, t) = \frac{1}{2\varepsilon} \left[ \frac{\partial v_0}{\partial n}(x, t) - \delta h_v_0(p_1(x, t)) \right] \int_\Gamma (\varepsilon + h_1)^2 \eta(z) dz + \varepsilon \int_\Gamma (\varepsilon + h_1)^2 \eta(z) dz + \varepsilon \int_\Gamma (\varepsilon + h_1)^2 \eta(z) dz \]

and the expression in the rectangle bracket vanishes on $\Gamma$, which implies that $v_2, \varepsilon$ as well as $v_2, \varepsilon$ are smooth. On the other hand, it can be verified that the double integral defining (3.16a) is well-defined as the integral $\int_{-\infty}^{r_f} f(r) dr$ vanishes when $g \leq -1$ because of (3.11). So $v_2, \varepsilon$ is smooth in the tangent direction and is constant on the normal direction.
3.1 The Leading Part of the Velocity

The conclusion for \( \hat{\rho}_1 \) follows from (3.2), together with the following formula

\[
\hat{\rho}_1(x,t) = \frac{1}{\omega_{\mathbb{R}^2}} \left( p_1^+ (x,t) - p_1^- (x,t) - \int_{\mathbb{R}} a_1(\rho, x, t) d\rho \right) - p_1^+ (P_T (x), t) + p_1^- (P_T (x), t) + \int_{\mathbb{R}} a_1(\rho, P_T (x), t) d\rho \right),
\]

where \( a_1 \) is smooth in normal direction and continuous in all directions since it depends on smooth quantities and the push-forward of \( h_{2,\varepsilon} \in X_{T_\varepsilon} \mapsto BU\mathcal{C}([0,T_\varepsilon]; H^2(\mathbb{T}^1)) \cap L^4(0,T_\varepsilon; C^1(\mathbb{T}^1)), H^2(\mathbb{T}^1) \hookrightarrow C^0(\mathbb{T}^1) \).

Now we construct the outer expansion in (3.6). The following statement can be readily verified:

**Lemma 3.3** There exists some \( \alpha > 0 \) and \( C > 0 \) such that for all \( \rho \in \mathbb{R} \) and \( m, \ell = 0, 1, 2, n = 0, 1 \),

\[
\sup_{(x,t) \in \Gamma(3\delta)} |D_\rho^{\alpha} \partial_\rho \partial_\rho (v_1 (\pm \rho, x, t) - v_1^+ (x,t))| \leq Ce^{-\alpha |\rho|} \quad \text{for all } 0 \leq i \leq 2,
\]

\[
\sup_{(x,t) \in \Gamma(3\delta)} |D_\rho^{\alpha} (p_1 (\pm \rho, x, t) - p_1^+ (x,t))| \leq Ce^{-\alpha |\rho|} \quad \text{for all } -1 \leq i \leq 1.
\]

The following lemma is crucial in the inner expansion and its proof is given in Section A.2 below.

**Lemma 3.4** If we define \( (v_1^{in}, p_1^{in}) \) through (3.7), then

\[
-\Delta v_{1, A}^{in} + \nabla p_{1, A}^{in} = -\varepsilon \text{div} (\nabla v_1^{in} \otimes \nabla v_0^{in}) + r_{\varepsilon} \left( d_x - h_\varepsilon, x, t \right) + \bar{r}_{\varepsilon} \left( d_x - h_\varepsilon, x, t \right)
+ \sum_{i' \leq 2 \delta \leq i, j, j' \leq 1} \varepsilon^2 R_{j, j'}^{\alpha} (x,t)(\partial_\rho)^{\alpha} (\partial_\rho^i_h)^{\alpha} (\partial_\rho^{j'}_h)^{\alpha} + \sum_{0 \leq i', i, j, j' \leq 1} \varepsilon^2 \tilde{R}_{\varepsilon, j'}^{\alpha} (x,t)(\partial_\rho)^{\alpha} (\partial_\rho^i_h)^{\alpha} (\partial_\rho^{j'}_h)^{\alpha} (\partial_\rho^{i'}_h)^{\alpha}
\]

\[
\text{div } v_{1, A}^{in} = \sum_{0 \leq i, j \leq 1} g_{\varepsilon}^{i,j} \left( d_x - h_\varepsilon, x, t \right) (\partial_\rho^{i'}_h)^{\alpha} + \varepsilon^2 \bar{g}_\varepsilon (x,t)
\]

in \( \Gamma(3\delta) \) where \( h_\varepsilon \) is defined through (3.4) and

\[
(r_{\varepsilon})_{0 < \varepsilon < 1} \in \mathcal{R}_{0,\alpha}, \quad (\bar{r}_{\varepsilon})_{0 < \varepsilon < 1} \in \mathcal{R}_{1,\alpha}, \quad (g_{\varepsilon}^{i,j})_{0 < \varepsilon < 1} \in \mathcal{R}_{0,\alpha}^0
\]

for some \( \alpha > 0 \) and \( R_{\varepsilon}^{\alpha} \), \( \bar{R}_{\varepsilon}^{\alpha} \), \( \tilde{R}_{\varepsilon}^{\alpha} \), \( g_{\varepsilon}^{\alpha} \) are uniformly bounded with respect to \( \varepsilon \in (0,1) \), \( (x,t) \in \Gamma(3\delta) \).

The main result of this section is:

**Theorem 3.5** Under the assumptions (3.4), there is some \( C(M) > 0 \), independent of \( (T_\varepsilon)_{0 < \varepsilon \leq \varepsilon_0} \) and \( \varepsilon_0 \in (0,1] \), such that

\[
\| v_{1, A} - \bar{v}_{1, A} \|_{L^2(0,T; H^1(\mathbb{T}))} \leq C(M)(T^{\frac{1}{4}} + \varepsilon^{\frac{1}{4}})^2
\]

for all \( 0 < T \leq T_\varepsilon \) and \( \varepsilon \in (0,\varepsilon_0] \).

**Proof:** We will use that

\[
\left| \frac{d_x(x,t)}{\varepsilon} - h_\varepsilon (S(x,t), t) \right| \geq \frac{\delta}{2\varepsilon} \quad \text{for all } (x,t) \in \Gamma(3\delta) \setminus \Gamma(\delta), t \leq T_\varepsilon
\]

(23.3)
if $0 < \varepsilon \leq \min(\varepsilon_1, \varepsilon_0)$ for sufficiently small $\varepsilon_1 \in (0, 1)$ in dependence of $M$. Moreover, we will choose $\varepsilon_1$ such that $\varepsilon_1 M \leq 1$. First of all, of because of (3.19) and (2.20), we have for $1 \leq i' \leq 2, 0 \leq j' \leq 1,$

$$
\| \partial_x^i (\omega \cdot \partial_x^j f) \partial_x^j (v_{\chi}^a - v_{\chi}^a + v_{\chi}^a) \|_{L^\infty((0, T), L^2(\Omega^s))} \leq C e^{-\frac{\alpha}{\varepsilon}},
$$

$$
\| \partial_x^i (\omega \cdot f) (p_{\chi}^a + p_{\chi}^a + p_{\chi}^a) \|_{L^\infty((0, T), L^2(\Omega^s))} \leq C e^{-\frac{\alpha}{\varepsilon}}.
$$

(3.24)

So it follows from (3.25) that

$$
\Delta v_A \equiv (\varepsilon \cdot \partial_x f) \Delta v_A + (1 - \varepsilon \cdot \partial_x f) (\Delta v_A^a + \Delta v_A^a - v_A^a - v_A^a) \\
+ 2(\varepsilon \cdot \partial_x f) \cdot \nabla \left( v_A^a - v_A^a + v_A^a - v_A^a \right) + \Delta(\varepsilon \cdot \partial_x f) \left( v_A^a - v_A^a + v_A^a - v_A^a \right) \\
\nabla p_A = (\varepsilon \cdot \partial_x f) \nabla p_A + (1 - \varepsilon \cdot \partial_x f) \left( \nabla p_A^a + \nabla p_A^a \right) + \nabla(\varepsilon \cdot \partial_x f) \left( p_A^a - p_A^a + p_A^a - p_A^a \right).
$$

On the other hand, it follows from (3.26) and the first equation of (3.18) and (3.19) that

$$
-\Delta v_A^a + \nabla p_A^a + \Delta v_A^a + \nabla p_A^a - O(\varepsilon^2) \quad \text{in} \quad L^\infty(\Omega \times (0, T)).
$$

The above two formulas together with (3.1) imply

$$
-\Delta(v_A - \tilde{v}_A) + \nabla(p_A - \tilde{p}_A) \\
= \varepsilon \cdot \partial_x f \left( -\Delta v_A^a + \nabla p_A^a + \varepsilon \cdot \nabla(c_{\chi}^0) \right) + O(\varepsilon^2) \quad \text{in} \quad L^\infty(0, T; L^2(\Omega)).
$$

(3.25)

To expand the concentration term, we need the following formula which can be easily derived from (2.2) together with \nabla c_{\chi}^0 = 0:

$$
\nabla c_{\chi}^0 = \nabla(\varepsilon \cdot \partial_x f) \left( c_{\chi}^0 - c_{\chi}^0 - c_{\chi}^0 \right) + \varepsilon \cdot \partial_x f \nabla c_{\chi}^0.
$$

(3.26)

Applying the above formula to the second term on the right hand side of (3.26) and extract the leading terms, we get

$$
-\Delta(v_A - \tilde{v}_A) + \nabla(p_A - \tilde{p}_A) \\
= \varepsilon \cdot \partial_x f \left( -\Delta v_A^a + \nabla p_A^a + \varepsilon \cdot \nabla(c_{\chi}^0) \right) + O(\varepsilon^2) \quad \text{in} \quad L^\infty(0, T; L^2(\Omega)).
$$

(3.27)

Next we multiply (3.26) by an arbitrary $\varphi \in C^\infty_{0, \sigma}(\Omega)$ and we employ Lemma 3.4 to estimate the first part on the right hand side of (3.27),

$$
\int_{\Gamma_1} \left( \varepsilon \cdot \partial_x f \left( -\Delta v_A^a + \nabla p_A^a + \varepsilon \cdot \nabla(c_{\chi}^0) \right) \right) \cdot \varphi \, dx
$$

$$
\leq \int_{\Gamma_1} \left| \frac{\partial \varepsilon}{\partial x} - h_a(x, t) \right| \cdot \varphi \, dx + \int_{\Gamma_1} \left| \frac{\partial \varepsilon}{\partial x} - h_a(x, t) \right| \cdot \varphi \, dx
$$

$$
+ \int_{\Gamma_1} \left| \varepsilon^2 R_{a}^{ij}(x, t)(\partial_x h_a)^i(\partial_x h_a)^j \varphi \, dx
$$

$$
+ \int_{\Gamma_1} \left| \sum_{0 \leq l, k, j, i' \leq 1} \varepsilon^2 R_{a}^{ij}(x, t)(\partial_x h_a)^i(\partial_x h_a)^j \varphi \, dx
$$

$$
+ C e^2 \| \varphi \|_{L^2(\Omega)} \quad \text{for all} \quad t \in (0, T).
$$

Moreover, we have the embeddings $H^{3/2}(\mathbb{T}^1) \hookrightarrow W_{\text{loc}}^1(\mathbb{T}^1)$ for every $1 \leq p < \infty$ and the estimate

$$
\left( \int_{\Gamma_1} \frac{\partial \varepsilon}{\partial x} + \left( X_0(s, t) + r n(s, t) \right) \right) \quad \text{for} \quad t \in (0, T).
$$

$$
\leq C \| \varphi \|_{H^2(\Gamma_1(25))}
$$
3.2 The Leading Error in the Velocity

for some $C > 0$ independent of $\varphi$. This estimate is proved in the same way as (2.20) with the role of $r \in (-2\delta, 2\delta)$ and $s \in T^3$ interchanged. These together with (2.28), (3.4), and the first inequality in Corollary 3.2 yields

$$\left| \int \zeta \, dt \left( -\Delta \nabla v^m_{-\varphi} + \nabla p^m_{-\varphi} + \epsilon \text{div}(\nabla c^m_{-\varphi} \otimes \nabla c^m_{-\varphi}) \right) \cdot \varphi \, dx \right| \leq C(M) \epsilon^2 \left(1 + \|h_2(t)\|_{H^2(T^3)} \|h_2(t)\|_{H^1(T^3)} + \|h_2(t)\|_{H^3(T^3)} \right) \|\varphi\|_{H^1(\Omega)}$$

(3.28)

Combining (3.27) and (3.28) yields

$$\left| \int \nabla (v_A - \tilde{v}_A) : \nabla \varphi \, dx \right| = \left| \int \nabla (\Delta (v_A - \tilde{v}_A) + \nabla (p_A - \tilde{p}_A)) \cdot \varphi \, dx \right| \leq C(M) \epsilon^2 \left(1 + \|h_2(t)\|_{H^2(T^3)} \right) \|\varphi\|_{H^1(\Omega)}$$

(3.29)

where $h_2 \in X_{T^3} \to L^4(0, T; H^2(T^3))$ is bounded. On the other hand, using (3.6) together with $\text{div} v_j^m = 0$ for $j = 0, 1$, we obtain

$$\text{div} v_A = \text{div} \left( \zeta \, dt \nabla v^m_{+\varphi} + (1 - \zeta \, dt) \left( \nabla v^m_{+\varphi} + \nabla \tilde{v}_{-\varphi} \right) \right)$$

$$= \nabla \left( \zeta \, dt \right) \cdot \left( \nabla v^m_{+\varphi} - \nabla \tilde{v}_{+\varphi} \right) + (\zeta \, dt) \text{div} \nabla v^m_{+\varphi} + O(\epsilon)$$

in $L^\infty(\Omega \times [0, T])$. So we can apply (3.21) to the above estimate and then the second inequality in Corollary 3.2 as well as (3.21):

$$\|\text{div} (v_A - \tilde{v}_A)\|_{L^2(0, T; H^1)} \leq C(M) \epsilon^2 \left( T^{1/2} + \epsilon^{1/2} \right) \left( \|h_2\|_{L^2(0, T; H^{5/2}(\Omega))} \right) + 1.$$  (3.30)

Combining (3.29) and (3.30) together with standard results on the weak solutions of the Stokes equation with general divergence leads to

$$\|v_A - v_A\|_{L^2(0, T; H^1)} \leq C(M) \epsilon^2 \left( T^{1/2} + \epsilon^{1/2} \right).$$

This implies the desired result due to (3.4).  

3.2 The Leading Error in the Velocity

Within this subsection we denote $u_1 = c_{-\varphi} - c_{+\varphi}$, where $c_{+\varphi}, c_{-\varphi}$ is defined by (3.2) for given $h_1, h_2, \epsilon = h_2$ satisfying (3.3) for some $\epsilon_0 \in (0, 1)$ and $M \geq 1$. Moreover, we will often write $n$ instead of $n_r$ for simplicity.

In the following $\bar{w}_1, \bar{w}_2 : \Omega \times [0, T_0] \to \mathbb{R}^2, q_1, q_2 : \Omega \times [0, T_0] \to \mathbb{R}$ are the solutions of

$$-\Delta \bar{w}_1 + \nabla q_1 = -\epsilon \text{div}((\nabla c_{+\varphi} - g) \otimes \nabla u_1) - \epsilon \text{div}(\nabla u_1 \otimes (\nabla c_{+\varphi} - g)) \quad \text{in } \Omega, \quad (3.31)$$

$$\text{div} \bar{w}_1 = 0 \quad \text{in } \Omega, \quad (3.32)$$

$$-\Delta \bar{w}_2 + \nabla q_2 = -\epsilon \text{div}(g \otimes \nabla u_1) - \epsilon \text{div}(\nabla u_1 \otimes g) - \epsilon \text{div}(\nabla u_1 \otimes \nabla u_1) \quad \text{in } \Omega, \quad (3.33)$$

$$\text{div} \bar{w}_2 = 0 \quad \text{in } \Omega, \quad (3.34)$$

$$\bar{w}_1|_{\partial \Omega} = \bar{w}_2|_{\partial \Omega} = 0 \quad \text{on } \partial \Omega \quad (3.35)$$

for every $t \in [0, T_0]$, where

$$g = -\zeta \, dt \, t_0(\frac{d}{dt} - h_x) \epsilon \nabla_x h_{2, \epsilon}.$$

Then $v_A = v_A + \bar{w}_1 + \bar{w}_2$ because of the unique solvability of the Stokes systems. According to (3.20),

$$\nabla c_{+\varphi} - g = \zeta \, dt \, t_0(\frac{d}{dt} - h_x)(\nabla_x h_{1}) + \nabla d \zeta \, dt \, t_0 \left( t_0(\frac{d}{dt} - h_x) - \chi_+ + \chi_-. \right). \quad (3.36)$$
It will turn out that $\tilde{w}_1$ is the leading part in the error estimates for $v - \tilde{v}_A$ and it plays an important role in the construction of $c_A$. Before proceeding the estimates, we give a few words about the Stokes equations above. These systems will be understood in the weak sense. E.g., $v = \tilde{w}_1 \in H^1_0(\Omega)^2 \cap L^2_0(\Omega)$ solves

$$
\int_{\Omega} \nabla v : \nabla \varphi dx = (F, \varphi) \quad \text{for all } \varphi \in H^1_0(\Omega)^2 \cap L^2_0(\Omega),
$$

(3.37)

where $F \in (H^1_0(\Omega)^2 \cap L^2_0(\Omega))'$ is defined by

$$
(F, \varphi) := \varepsilon \int_{\Omega} \left( (\nabla c_{A,0} - g) \otimes \nabla u_1 + \nabla u_1 \otimes (\nabla c_{A,0} - g) \right) : \nabla \varphi dx
$$

for all $\varphi \in H^1(\Omega)^2 \cap L^2_0(\Omega)$. Here $L^2_0(\Omega)$ is the closure of divergence free $C^\infty(\Omega)$-vector fields in $L^2(\Omega)$. Existence of weak solutions for every $\varepsilon > 0$ is a consequence of the Lax-Milgram Theorem. Moreover, if $v \in H^1_0(\Omega)^2 \cap L^2_0(\Omega)$ solves (3.37) for some $F \in (H^1_0(\Omega)^2 \cap L^2_0(\Omega))'$ and $1 < p \leq \infty$, then there is some $C_p > 0$ independent of $v$ and $F$ such that

$$
\|v\|_{L^p(\Omega)} \leq C \sup_{0 \neq \varphi \in W^{1,p}_0(\Omega) \cap L^2(\Omega)} \frac{|(F, \varphi)|}{\|\varphi\|_{W^{1,p}_0(\Omega)}}
$$

(3.38)

where $\frac{1}{p} + \frac{1}{q} = 1$. The latter estimate follows by duality from the well-known fact that the Stokes operator $A_p : W^{1,p}_0(\Omega)^2 \cap W^{1,0}_0(\Omega)^2 \cap L^2(\Omega) \to L^q(\Omega)$ is bijective.

**Proposition 3.6** Let $N = 2$, $1 < q < 2$ and $1 \leq r \leq 2$. We assume that there are some $R \geq 1$, $\varepsilon_0 \in (0, 1)$, $T_\varepsilon \in (0, T_0)$ such that

$$
\|c_{\varepsilon} - c_{A,0}\|_{L^1((0,T_\varepsilon);L^2(\Omega))} + \|\nabla (c_{\varepsilon} - c_{A,0})\|_{L^2((0,T_\varepsilon);\Gamma'(\Omega))} \leq 2R \varepsilon^{N+\frac{1}{2}},
$$

(3.39a)

$$
\|c_{\varepsilon} - c_{A,0}\|_{L^1((0,T_\varepsilon);\Gamma'((\varepsilon\delta)))} + \|\nabla (c_{\varepsilon} - c_{A,0})\|_{L^2((0,T_\varepsilon);\Gamma'((\varepsilon\delta)))} \leq 2R \varepsilon^{N+\frac{1}{2}}
$$

(3.39b)

for all $\varepsilon \in (0, \varepsilon_0)$ and (3.41) holds true for some $M \geq 1$. Then there are some $C(R) > 0$, $\varepsilon_1 \in (0, 1)$, independent of $T_\varepsilon$, $\varepsilon_0$ such that

$$
\|\tilde{w}_1\|_{L^2((0,T_\varepsilon);H^1(\Omega))} \leq C(R) \varepsilon^N, \quad (3.40a)
$$

$$
\|\tilde{w}_1\|_{L^r((0,T_\varepsilon);L^q(\Omega))} \leq C(R) \left( T_\varepsilon^{\frac{1}{r}} + \varepsilon^{\frac{1}{q}} \right) \varepsilon^N, \quad (3.40b)
$$

$$
\|\tilde{w}_2\|_{L^r((0,T_\varepsilon);L^q(\Omega))} \leq C(R) \varepsilon^{\frac{1}{q}}, \quad (3.40c)
$$

for all $0 < T \leq T_\varepsilon$, $\varepsilon \in (0, \min(\varepsilon_0, \varepsilon_1)]$. Moreover, $C(R)$ is independent of $M$.

**Proof:** Within this proof we will often write $h_\varepsilon$ instead of $h_\varepsilon(S(x,t), t)$ for brevity. We also recall that $u_1 = c_{\varepsilon} - c_{A,0}$. Because of (3.24), we have

$$
\left| \theta_\varepsilon \left( \frac{d_\varepsilon(x,t)}{\varepsilon} - h_\varepsilon(S(x,t), t) \right) - \chi_+(x,t) + \chi_-(x,t) \right| + \left| \delta_\varepsilon \left( \frac{d_\varepsilon(x,t)}{\varepsilon} - h_\varepsilon(S(x,t), t) \right) \right| \leq C e^{-\frac{\varepsilon^2}{M}}
$$

for all $(x,t) \in \Gamma(3\delta) \setminus \Gamma(\delta)$ and $0 < \varepsilon \leq \min(\varepsilon_0, \varepsilon_1)$ because of (1.19). Provided $\varepsilon_1$ is chosen sufficiently small (in dependence on $M$). In particular, we choose $\varepsilon_1 \leq \frac{1}{3M}$ such that $\varepsilon \|h_{2\varepsilon}\|_{L^r(\varepsilon\delta)} \leq 1$ independent of the choice of $M$. Here $C$ can be chosen independent of $M$. According to the definition of $\zeta$ at (1.23), all terms involving derivatives of $\zeta \circ d_T$ will provide terms of order $O(e^{-\frac{\varepsilon^2}{M}})$ in the following and will be negligible.

**Proof of (3.40a):** Using identity (3.35) and $\nabla u_1 = \nabla \tau u_1 + n \partial_\nu u_1$ in $\Gamma(3\delta)$, we have for all
3.2 The Leading Error in the Velocity

\[ \psi \in H^1(\Omega)^2 \cap L^2(\Omega) \text{ with } \|\psi\|_{H^1(\Omega)} \leq 1 \text{ that} \]

\[ \int_\Omega \varepsilon((\nabla c_{A_0} - g) \otimes \nabla u_1 : \nabla \psi) dx = \int_{\Gamma_1(2\delta)} \zeta \circ dT \varepsilon((\nabla c_{A_0} - g) \otimes \nabla u_1 : \nabla \psi) dx + O(\varepsilon^{\frac{3}{2}}) \]

\[ = \int_{\Gamma_1(2\delta)} \zeta \circ dT \theta_0 \left( \frac{\nabla \psi}{\varepsilon} - h_\varepsilon(S(x,t),t) \right) (n - \varepsilon \nabla h_1(S(x,t),t)) \otimes \nabla u_1 : \nabla \psi dx + O(\varepsilon^{\frac{3}{2}}) \]

\[ = \int_{\Gamma_1(2\delta)} \zeta \circ dT \theta_0 \left( \frac{\nabla \psi}{\varepsilon} - h_\varepsilon \right) n \otimes \nabla \psi u_1 : \nabla \psi dx + \int_{\Gamma_1(2\delta)} \zeta \circ dT \theta_0 \left( \frac{\nabla \psi}{\varepsilon} - h_\varepsilon \right) n \otimes n \partial_\alpha u_1 : \nabla \psi dx \]

\[ - \int_{\Gamma_1(2\delta)} \zeta \circ dT \theta_0 \left( \frac{\nabla \psi}{\varepsilon} - h_\varepsilon \right) (\varepsilon \nabla h_1) \otimes \nabla u_1 : \nabla \psi dx + O(\varepsilon^{\frac{3}{2}}) \]

\[ \equiv I + II + III + O(\varepsilon^{\frac{3}{2}}) \text{ in } L^2(0,T_\varepsilon). \quad (3.41) \]

Therefore the following estimates are due to Cauchy-Schwarz inequality:

\[ |I| \leq C \|\nabla u_1(t)\|_{L^2(\Gamma_1(2\delta))} \|\psi\|_{H^1(\Omega)}, \quad |III| \leq C \|\nabla u_1(t)\|_{L^2(\Gamma_1(2\delta))} \|\psi\|_{H^1(\Omega)}. \]

To estimate II, we employ the formula \( n \otimes n : \nabla \psi = \partial_\alpha \psi n = -\nabla \psi \), a consequence of the divergence-free condition \( \nabla \psi = 0 \). This together with (2.13) implies

\[ II = -\int_{\Gamma_1(2\delta)} \zeta \circ dT \theta_0 \left( \frac{\nabla \psi}{\varepsilon} - h_\varepsilon \right) \partial_\alpha u_1 \div \psi dx \]

\[ = \int_{\Gamma_1(2\delta)} \zeta \circ dT \theta_0 \left( \frac{\nabla \psi}{\varepsilon} - h_\varepsilon \right) (\nabla \psi \partial_\alpha u_1 + \kappa n \partial_\alpha u_1) \cdot \psi dx \]

\[ - \int_{\Gamma_1(2\delta)} \zeta \circ dT \theta_0' \left( \frac{\nabla \psi}{\varepsilon} - h_\varepsilon \right) \nabla h_\varepsilon \partial_\alpha u_1 \cdot \psi dx \]

\[ = \int_{\Gamma_1(2\delta)} \zeta \circ dT \theta_0 \left( \frac{\nabla \psi}{\varepsilon} - h_\varepsilon \right) \partial_\alpha \nabla \psi u_1 \cdot \psi dx + \int_{\Gamma_1(2\delta)} \zeta \circ dT \theta_0 \left( \frac{\nabla \psi}{\varepsilon} - h_\varepsilon \right) [\partial_\alpha, \nabla] u_1 \cdot \psi dx \]

\[ + \int_{\Gamma_1(2\delta)} \zeta \circ dT \left( -\theta_0' \left( \frac{\nabla \psi}{\varepsilon} - h_\varepsilon \right) \nabla h_\varepsilon \right) \partial_\alpha u_1 \cdot \psi dx + O(\varepsilon^{\frac{3}{2}}) \]

\[ = -\int_{\Gamma_1(2\delta)} \zeta \circ dT \left( \theta_0' \left( \frac{\nabla \psi}{\varepsilon} - h_\varepsilon \right) \nabla h_\varepsilon \right) \partial_\alpha u_1 \cdot \psi dx - \int_{\Gamma_1(2\delta)} \zeta \circ dT \theta_0 \left( \frac{\nabla \psi}{\varepsilon} - h_\varepsilon \right) \nabla \psi u_1 \cdot \partial_\alpha \psi dx \]

\[ + \int_{\Gamma_1(2\delta)} \zeta \circ dT \theta_0 \left( \frac{\nabla \psi}{\varepsilon} - h_\varepsilon \right) [\partial_\alpha, \nabla] u_1 \cdot \psi dx \]

\[ + \int_{\Gamma_1(2\delta)} \zeta \circ dT \left( -\theta_0' \left( \frac{\nabla \psi}{\varepsilon} - h_\varepsilon \right) \nabla h_\varepsilon \right) \partial_\alpha u_1 \cdot \psi dx + O(\varepsilon^{\frac{3}{2}}) \]

\[ \equiv II_1 + II_2 + II_3 + II_4 + O(\varepsilon^{\frac{3}{2}}) \text{ in } L^2(0,T_\varepsilon). \]

In the above calculation, \([\cdot, \cdot]\) denotes the commutator of two differential operators. So it follows from assumption (3.34) and Lemma (2.14) that

\[ |II_1| \leq \frac{1}{\varepsilon} \|\theta_0'\|_{L^2(\Omega)} \|\nabla \psi u_1\|_{L^2(\Gamma_1(2\delta))} \|\psi\|_{L^{\infty}(\Gamma_1(2\delta))} \leq C \frac{\varepsilon}{\varepsilon^2} \|\nabla \psi u_1\|_{L^2(\Gamma_1(2\delta))} \|\psi\|_{H^1(\Omega)}, \]

\[ |II_2| \leq C \|\nabla \psi u_1\|_{L^2(\Gamma_1(2\delta))} \|\psi\|_{H^1(\Omega)}, \]

\[ |II_3| \leq C \|\theta_0'\|_{L^2(\Omega)} \|\nabla \psi u_1\|_{L^2(\Gamma_1(2\delta))} \|\psi\|_{H^1(\Omega)} \leq C' \sqrt{\varepsilon} \|\nabla \psi u_1\|_{L^2(\Gamma_1(2\delta))} \|\psi\|_{H^1(\Omega)}, \]

\[ |II_4| \leq C \sqrt{\varepsilon} \|\theta_0'\|_{L^2(\Omega)} \|\nabla \psi u_1\|_{L^2(\Gamma_1(2\delta))} \|\psi\|_{L^{\infty}(\Gamma_1(2\delta))} \leq C \sqrt{\varepsilon} \|\nabla \psi u_1\|_{L^2(\Gamma_1(2\delta))} \|\psi\|_{H^1(\Omega)}, \]

where we have used (2.12), (2.24) and the following imbedding theorem

\[ H^1(\Gamma_1(2\delta)) \hookrightarrow L^{4,\infty}(\Gamma_1(2\delta)), \quad H^2(\Omega^1) \hookrightarrow W^1_2(\Omega^1). \]
Similarly, we obtain
\[\int_{\Omega} \varepsilon \nabla u_1 \otimes (\nabla c_{A,0} - g) : \nabla \psi dx + \int_{\Gamma_1(28)} \varepsilon \nabla u_1 \otimes n : \nabla \psi dx + \int_{\Gamma_1(28)} \varepsilon \nabla u_1 \cdot n \nabla \psi dx + \int_{\Gamma_1(28)} \varepsilon \nabla u_1 \otimes \nabla \psi dx = 0\]
where each term can be estimated as before. Hence we obtain
\[\left\| \tilde{w}_1(t) \right\|_{H^1(\Omega)} \leq C \left( \varepsilon^{\frac{1}{2}} \| \nabla u_1 \|_{L^2(\Gamma_1(28))} + \sqrt{\varepsilon} \| \nabla u_1 \|_{L^2(\Gamma_1(28))} + \varepsilon^{-\frac{3}{2}} \right) \quad \text{in} \quad L^2(0, T_e),\]
and this together with (3.40a) leads to (3.40b).

**Proof of (3.40a):** We still need to estimate the same terms as in (3.41). But it is sufficient to prove Theorem 4.5 in Section 4.1 for the inner expansions defined by (4.1) in the sequel, which will be glued to the outer expansion in $L^2(\Omega)$. Then
\[II = \int_{\Gamma_1(28)} \varepsilon \nabla u_1 \cdot \nabla \psi dx + \int_{\Gamma_1(28)} \varepsilon \nabla u_1 \cdot \nabla \psi dx + \int_{\Gamma_1(28)} \varepsilon \nabla u_1 \cdot \nabla \psi dx + O(\varepsilon^{-\frac{3}{2}}) \quad \text{in} \quad L^2(0, T_e)\]
and therefore Lemma 2.4 implies
\[|II| \leq C \left\| u_1(t) \right\|_{L^2(\Omega)} \left( \| \psi \|_{L^2(\Omega)} + \frac{1}{\sqrt{\varepsilon}} \| \theta_0' \|_{L^2(\Omega)} \right) \| \nabla u_1 \|_{L^2(\Omega)} + O(\varepsilon^{-\frac{3}{2}}) \]
\[\leq C \varepsilon^{-\frac{1}{2}} \left\| u_1(t) \right\|_{L^2(\Omega)} \| \psi \|_{H^2(\Omega)} + O(\varepsilon^{-\frac{3}{2}}) \quad \text{in} \quad L^2(0, T_e).\]
Since we have \( \| u_1 \|_{L^2(0,T,L^2)} \leq C(R)T^\frac{1}{2}\varepsilon^{N+\frac{1}{2}} \) for all \( 0 < T \leq T_e \) due to (3.40b), we obtain (3.40a).

**Proof of (3.40b):** Because of \( W^2_0(\Omega) \hookrightarrow H^0(\Omega) \), we have the imbedding \( L^2(\Omega) \hookrightarrow (W^2_0(\Omega))' \) and it follows from (3.38) that
\[\left\| \tilde{w}_2(t, \cdot) \right\|_{L^2(0,T,L^2)} \leq C \left( \| \nabla u_1 \otimes \nabla u_1 \|_{L^2(0,T,L^1)} + \| g \otimes \nabla u_1 \|_{L^2(0,T,L^1)} \right).\]
On the other hand
\[\varepsilon^\frac{1}{2} \left\| \nabla u_1 \right\|_{L^2(0,T,L^2)} \leq \varepsilon^\frac{1}{2} \left\| \nabla u_1 \right\|_{L^2(0,T,L^2)} + \varepsilon^\frac{1}{2} \left\| c_{A,0} \right\|_{L^\infty(0,T,L^2)} \leq C\]
uniformly in \( T \in (0, T_e), \varepsilon \in (0, \min(\varepsilon_0, \varepsilon_1)] \) because of the energy estimate (3.10) for (1.4)- (1.5) and the explicit form of \( c_{A,0} \) at (3.2). Altogether we conclude
\[\left\| \tilde{w}_2 \right\|_{L^2(0,T,L^2)} \leq C \left( \| \nabla u_1 \otimes \nabla u_1 \|_{L^2(0,T,L^1)} + \| g \otimes \nabla u_1 \|_{L^2(0,T,L^1)} \right) \]
\[\leq C \varepsilon^\frac{1}{2} \left( \| \nabla u_1 \|_{L^2(0,T,L^2)} \| \nabla u_1 \|_{L^2(0,T,L^1)} + \varepsilon \| \partial_t h_2 \|_{L^2(0,T,L^2)} \| \theta_0' \|_{L^2(\Omega)} \| \nabla u_1 \|_{L^2(0,T,L^2)} \right) \]
\[\leq C \left( \varepsilon^{1+\frac{3}{2}-1+\frac{1}{2}} + M\varepsilon^{N+\frac{3}{2}} \right) \leq C' \varepsilon^\frac{1}{2} \quad \text{for all} \quad 0 < t \leq T_e, \varepsilon \in (0, \varepsilon_0].\]
Therefore we have completed the proof. \[\blacksquare\]

4 Approximate Solution for the Allen-Cahn Part

Let us start with an overview of this central section. The goal of this section is to construct the full expansion of (4.17) through Ansatz using inner and outer expansions, cf. (4.25) below, and to prove Theorem 4.3. The crucial step is to prove Theorem 4.4 in Section 4.1 for the inner expansions defined by (4.1) in the sequel, which will be glued to the outer expansion in
4.1 Inner Expansion

In this section we will construct the inner expansion of the approximate solution \( c_{4} \) with the aid of the following ansatz:

\[
    e^{n}(x,t) = \hat{e}^{n}(\rho, s, t) = \theta_{0}(\rho) + \varepsilon^{2} \hat{c}_{2}(\rho, S(x,t), t) + \varepsilon^{3} \hat{c}_{3}(\rho, S(x,t), t)
\]

where \( s = S(x,t) \) and \( \rho \) is related to \((x,t)\) by

\[
    \rho = \frac{d_{r}(x,t)}{\varepsilon} - h_{1}(S(x,t), t) - \varepsilon h_{2}(S(x,t), t).
\]

We note that in the functions which will be defined in the following, \( \rho \) is often an independent variable. But in the final definition of the approximate solutions \( \rho \) is related to \((x,t)\) by (4.2).

In Lemma 4.7, we deal with estimates away from the interface, which is less subtle. For a first reading, the proofs of these technical lemmas can be skipped.

Furthermore, we define

\[
    \hat{v}_{b} := 1 - \frac{s}{n} \in (0, 1) \quad \text{and extended smoothly to } \Omega \setminus [0, T_{0}]
\]

as before and \( \tilde{v}_{b} \) is defined as the bounded solution of the following ordinary differential equation:

\[
    - \partial_{s} \hat{c}_{2}(\rho, s, t) + f'(\theta_{0}(\rho))\hat{c}_{2}(\rho, s, t) = \left| \nabla \tau_{h_{1}}(s,t) \right| \hat{\theta}_{0}(\rho) - \rho \hat{\theta}_{0}(\rho)(\kappa_{1}(s, t) - (\text{div}_{\tau_{\rho}})(X_{0}(s,t), t))
\]

together with the initial data \( h_{1}|_{s=0} = 0 \), cf. Corollary 2.11. Note that here we use the notation (4.1) and \((\nu, \gamma)\) is the solution of (4.11) - (4.15). \( \kappa_{1} \) is a curvature term defined in (4.13). Using \( h_{1} \), we shall define \( \hat{c}_{2} = \hat{c}_{2}(\rho, s, t) \) as the bounded solution of the following ordinary differential equation:

\[
    \frac{\partial}{\partial s} \hat{c}_{2}(\rho, s, t) + f'(\theta_{0}(\rho))\hat{c}_{2}(\rho, s, t) = \left| \nabla \tau_{h_{1}}(s,t) \right| \hat{\theta}_{0}(\rho) - \rho \hat{\theta}_{0}(\rho)(\kappa_{1}(s, t) - (\text{div}_{\tau_{\rho}})(X_{0}(s,t), t))
\]

Furthermore, we define

\[
    D_{1}(r, \rho, s, t) = -2\nabla \tau_{h_{1}}(s,t) \cdot L \nabla h_{1}(r, s, t) \theta_{0}(\rho) - |L \nabla h_{1}(r, s, t)|^{2} \theta_{0}(\rho)
\]

where \( r \in (-3\delta, 3\delta), \rho \in \mathbb{R}, \) and \( \rho \in \mathbb{R} \). Moreover,

\[
    \mathcal{B}(s,t) := \frac{1}{\int_{0}^{1} \theta_{0}(\rho) d\rho} \int_{\mathbb{R}^{2}} \left[ (b(\rho, s, t) - \kappa_{2}(s, t) \rho^{2}) \theta_{0}(\rho) + (\rho + h_{1}(s, t)) \mathcal{D}(\rho, s, t) \right] \theta_{0}(\rho) d\rho.
\]
Now we define $h_2 = h_{2,s} : T^1 \times [0, T_0] \to \mathbb{R}$ as solution of the following parabolic equation
\[
D_t h_2 - \Delta h_2 - \kappa_1 h_2 - X_0^*(\mathbf{v}) \cdot \nabla h_2 + X_0^*(\text{div}_{\nabla} \mathbf{v}) h_2 = \mathfrak{B} - \kappa_2 h_1^2 + X_0^*(\mathbf{n} \cdot \mathbf{w}_1)
\]
(4.9) on $T^1 \times [0, T_0]$ together with initial condition $h_2|_{t=0} = 0$, where $\mathbf{w}_1 = \tilde{\mathbf{v}}_A - \mathbf{v}_A + \tilde{\mathbf{w}}_1$ with $\tilde{\mathbf{w}}_1 \in L^\infty(0, T_0; H_0^1(\Omega)^2 \cap L_2^2(\Omega))$
(4.10) is determined by (2.31)-(2.32) and $\mathbf{v}_A, \mathbf{v}_A$ are as in the beginning of the previous section. Finally, we shall define $c_3 = c_3(\rho, s, t)$ as the solution of
\[
\varepsilon^2 (D_t \tilde{c}_3 - \Delta \tilde{c}_3 - \partial^2_{ss} \tilde{c}_3 + f'(\theta_0) \tilde{c}_3) = 2\nabla h_1 \cdot \nabla h_2 \theta_0^\prime - (b - \mathfrak{B} - \kappa_2(\rho^2 + 2\rho h_1)) \theta_0^\prime - (\rho + h_1)\mathcal{D} \quad \text{on } \mathbb{R} \times T^1 \times [0, T_0]
\]
(4.11) together with initial condition $\tilde{c}_3|_{t=0} = 0$.

The next lemma provides important estimates to treat remainder terms in the following.

**Lemma 4.1** The functions $h_1, \tilde{c}_2, h_2, \tilde{c}_3$ are well defined through the above formulæ. For $\tilde{c}_2$ we have
\[
\partial^k_s \partial^j\partial^\ell \tilde{c}_2(\rho, S(x, t), t) \in \mathcal{R}_{0,\alpha}, \quad \forall i, j, k \geq 0.
\]
(4.12) Moreover, for $\tilde{c}_3$ we have for every $k \in \mathbb{N}_0$, $\theta > 0$, $\varepsilon \in (0, 1)$ and $T_\varepsilon \in (0, T_0)$ that
\[
\varepsilon \||\rho^k \tilde{c}_3, \partial_s \tilde{c}_3||L^\infty(0, T_\varepsilon; L^2(T^1 \times \mathbb{R})) + 
\varepsilon^\theta \sup_{(s, t) \in T^1 \times (0, T_\varepsilon)} \||\tilde{c}_3(\rho, s, t)||H^1(\mathbb{R}) + 
\varepsilon^\theta \||\tilde{c}_3||L^\infty((0, T_\varepsilon) \times T^1 \times \mathbb{R})
\]
(4.13)
\[
+ \||\rho^k \tilde{c}_3, \partial^k \partial^j \tilde{c}_3, \partial^k \partial^j \partial^\ell \tilde{c}_3, \partial \partial^k \tilde{c}_3, \partial^k \partial^j \partial^\ell \tilde{c}_3, \varepsilon \partial^2 \tilde{c}_3, \varepsilon^s \partial^k \partial^j \tilde{c}_3||L^2((0, T_\varepsilon) \times T^1 \times \mathbb{R}) \leq C_{k, \theta} (1 + \|\varepsilon\|_X_\varepsilon),
\]
where $C_{k, \theta}$ is independent of $\varepsilon, T_\varepsilon$ and $\varepsilon_k$.

**Proof:** First of all, we will show that all terms are well-defined. Because of Corollary 2.11 $h_1$ can be uniquely determined by solving the coupled system involving (2.13) and (2.20) together with $h_1|_{t=0} = 0$. Moreover, we can obtain $\tilde{c}_2 = \tilde{c}_2(\rho, s, t)$ by solving (4.13) for every $s \in T^1$.

\begin{equation}
\int_\mathbb{R} ((\nabla h_1(s, t))^2 \theta_0^\prime(\rho) - \theta_0^\prime(\rho))(\kappa_1(s, t) - (\text{div}_s \mathbf{v})(X_0(s, t))) \theta_0^\prime(\rho) \, d\rho = 0,
\end{equation}

following from the fact that $\theta_0^\prime$ is an even function and $\theta_0^\prime(\rho) \to |\rho| \to \infty 0$. This leads to a smooth function $\mathfrak{B}$ defined by (2.14). Moreover, according to (2.19) and Definition (2.5) we can use Proposition 2.10 to show (4.12), where $g^s = 0$. The existence of a unique solution $h_2 \in X_{T_\varepsilon}$ of (4.11) is shown in the next lemma. Note that, as $\tilde{w}_1$ depends on $h_2$, the equation for $h_2$ is non-linear.

Finally, we need to establish the estimate for $\tilde{c}_3$ satisfying (4.11). In order to apply Theorem 2.12 we need to estimate
\[
\tilde{g}(\rho, s, t) := 2\nabla h_1 \cdot \nabla h_2 \theta_0^\prime(\rho) - (b - \mathfrak{B} - \kappa_2(\rho^2 + 2\rho h_1)) \theta_0^\prime(\rho) - (\rho + h_1)\mathcal{D}(\rho, s, t).
\]

We note that the compatibility condition (4.11) is fulfilled due to the definition of $\mathfrak{B}$. It follows from (4.13), (4.17), (4.18) and the decay estimates (1.18) that
\[
\varepsilon \||\tilde{c}_3, \partial_s \tilde{c}_3||L^\infty(0, T_{0}; L^2(T^1 \times \mathbb{R})) + \||\tilde{c}_3, \partial^2_s \tilde{c}_3, \partial^2_s \tilde{c}_3, \partial^2_s \partial^2_s \tilde{c}_3, \varepsilon \partial^2 \tilde{c}_3ahi||L^2(\mathbb{R} \times T^1 \times (0, T_{0})) \leq C(1 + \|\varepsilon\|_X_\varepsilon).
\]

Moreover, using the exponential decay of $\theta_0^\prime(\rho), \theta_0^\prime(\rho)$ and $\tilde{g}(\rho, s, t)$ as $\rho \to \infty$ and (2.33), it is easy to observe that for any $k \in \mathbb{N}$ there is a constant $C_k$ independent of $\varepsilon, h_s$ such that
\[
\varepsilon \||\rho^k \tilde{c}_3||L^\infty(0, T_{0}; L^2(T^1 \times \mathbb{R})) + \||\rho^k \tilde{c}_3, \rho^k \partial_s \tilde{c}_3||L^2(\mathbb{R} \times T^1 \times (0, T_{0})) \leq C(1 + \|\varepsilon\|_X_\varepsilon).
\]
The second and third estimate in (4.13) follows from (4.14) and the boundness of
\[ h_{2,\varepsilon} \in X_T, \quad BUC([0,T_\varepsilon]; H^\frac{1}{2}(\mathbb{T}^1)) \cap L^2(0,T_\varepsilon; H^-\frac{1}{2}(\mathbb{T}^1)) \rightarrow L^2(0,T_\varepsilon; C^1(\mathbb{T}^1)) \]
for any \( 1 \leq p < \infty \).

In the last proof we used the following lemma which is concerned with the solvability of (4.10).

**Lemma 4.2** Let \( \varepsilon \in (0,1) \). Then there is a unique solution \( h_{2,\varepsilon} \in X_{T_0} \) of (4.10), where \( w_1 \) is determined by \( \tilde{w}_1 \) being determined by (3.31)-(3.32). Moreover, there are some \( \varepsilon_1 \in (0,1), T_1 \leq T_0 \) and \( M = M(R) > 0 \), independent of \( \varepsilon \), such that the solution \( h_\varepsilon \) satisfies (4.14) (for some \( T_\varepsilon \in (0,T_0] \)) if (3.39) is valid and \( \varepsilon \leq \varepsilon_1, T_\varepsilon \leq T_1 \).

**Proof:** First of all, we note that \( \tilde{w}_1 \) and therefore \( w_1 \) depends on \( h_{2,\varepsilon} \) since the definition of \( c_{A,0} \) in (3.34) contains the term
\[ t_0 \left( \frac{dX(t)}{\varepsilon} - h_1(s,t) - \varepsilon h_{2,\varepsilon}(s,t) \right). \]
Using Theorem (4.9) one can reduce (4.10) to a fixed point equation
\[ h_{2,\varepsilon} = S_T(h_{2,\varepsilon}) \quad \text{in} \quad X_T. \]
To solve it, one can first apply the contraction mapping principle to obtain a solution \( h_{2,\varepsilon} \in X_T \) for some \( T = T_\varepsilon \in (0,T_0] \). To this end one uses that for every \( h_{2,\varepsilon} \in X_T \)
\[
\sup_{0 \leq t \leq T} \left| \varepsilon \int_\Omega (\nabla (c_{e} - c_{A,0}) \cdot (\nabla c_{A,0} - \varphi)) : D\varphi(x)dx \right| \\
\leq C(\varepsilon) \left( \sup_{0 \leq t \leq T} \| h_{2,\varepsilon}(\cdot,t) \|_{H^1(\mathbb{T}^1)} + 1 \right) \| \varphi \|_{H^1(\Omega)}
\]
for all \( \varphi \in H^1(\Omega)^2 \) because of (3.20), (4.9), and since \( c_{e} \) is a known smooth function. Therefore the \( L^2(0,T;H^{1/2}(\mathbb{T}^1)) \)-norm of \( X_0(n \cdot w_1) \) on the right-hand side of (4.10) can be estimated by
\[
C' \varepsilon T_\frac{1}{2} \left( \sup_{0 \leq t \leq T} \| h_{2,\varepsilon}(\cdot,t) \|_{H^1(\mathbb{T}^1)} + 1 \right). \tag{4.14}
\]
Similarly one shows that the right-hand side is Lipschitz-continuous with respect to \( h_{2,\varepsilon} \) with Lipschitz constant \( C(\varepsilon) T_\frac{1}{2} \). Hence choosing \( T = T_\varepsilon \in (0,T_0] \) sufficiently small one obtains a contraction. Moreover, because of the linear growth of the bound in (4.11), there is an a priori bound of \( \sup_{0 \leq t \leq T} \| h_{2,\varepsilon}(\cdot,t) \|_{H^1(\mathbb{T}^1)} \), which depends only on \( \varepsilon \) and \( T_\varepsilon \). Therefore the solution can be extended on \( [0,T_\varepsilon] \) to a unique solution \( h_{2,\varepsilon} \in X_{T_\varepsilon} \). The details are left to the reader.

Now we assume that (3.39) is valid for some \( T_\varepsilon \in (0,T_0] \). In order to show the validity of (3.4), we use that
\[
\| h_{2,\varepsilon} \|_{X_{T_\varepsilon}} \leq C(1 + \| X_0(n \cdot w_1) \|_{L^2(0,T_{1+\varepsilon}H^{1/2}(\mathbb{T}^1))})
\]
because of Theorem (4.9). To estimate the right-hand side of the above inequality, we use (4.10), Theorem (4.3) and Proposition (3.6) and obtain
\[
C(1 + \| X_0(n \cdot w_1) \|_{L^2(0,T_{1+\varepsilon}H^{1/2}(\mathbb{T}^1))}) \\
\leq C'(1 + \varepsilon^{-2}) \| v_A - \tilde{v}_A \|_{L^2(0,T_{1+\varepsilon}H^{1/2}(\mathbb{T}^1))} + \varepsilon^{-2} \| \tilde{w}_1 \|_{L^2(0,T_{1+\varepsilon}H^{1/2}(\mathbb{T}^1))} \leq C(R) + C(M)(T_\frac{1}{4} + \varepsilon^{1/2})
\]
for some \( C', C(R) \) independent of \( \varepsilon, T_\varepsilon, M, h_{2,\varepsilon} \) due to (3.39i) as long as \( \varepsilon \leq \min(\varepsilon_0, \varepsilon_1) \), where \( \varepsilon_1 \) is as in Proposition (3.6) and depends on \( M \). Note that \( C(R) \) is independent of \( M \). Now we choose \( M = C(R) + 1 \) and \( T_\varepsilon \in (0,T_0], \varepsilon_1 \in (0,1] \) such that \( C(M)(T_\frac{1}{4} + \varepsilon^{1/2}) \leq 1 \) and Proposition (3.6) can be applied. This determines \( \varepsilon_1 \) and \( T_\varepsilon \) and finishes the proof.

The next lemma is concerned with the estimate of the inner expansion defined by (4.11).
Lemma 4.3 Assume that (4.3) holds true for some $M > 0$, $T_0 \in [0, T_0]$, $\in (0, \varepsilon_0]$, and $\varepsilon_0 \in (0, 1]$. Then there is some $C(M) > 0$ independent of $T_0$, $\varepsilon \in (0, \varepsilon_0]$, and $\varepsilon_0 \in (0, 1]$ such that

$$
\varepsilon^3 \|c_{3}^{(\cdot, t)}\|_{L^\infty(0,T_0;L^2(\Gamma_t(23)))} \leq C(M)\varepsilon^{N + \frac{1}{2}},
$$

$$
\varepsilon^3 \|\partial_\nu c_{3}^{(\cdot, t)}\|_{L^\infty(0,T_0;L^2(\Gamma_t(23)))} \leq C(M)\varepsilon^{N + \frac{1}{2}},
$$

$$
\varepsilon^3 \|\nabla c_{3}^{(\cdot, t)}\|_{L^\infty(0,T_0;L^2(\Gamma_t(23)))} \leq C(M, \theta)\varepsilon^{N + \frac{1}{2} - \theta},
$$

$$
\varepsilon^3 \|\nabla c_{3}^{(\cdot, t)}\|_{L^2(0,T_0;L^2(\Gamma_t(23)))} \leq C(M)\varepsilon^{N + \frac{1}{2}},
$$

for any $\theta \in (0, 1)$.

Proof: The estimates of $c_{3}^{(\cdot, t)}$ follow from (4.12), (4.14), and Lemma 2.6 in a straightforward manner. Using Lemma 4.1 and the same change of variables as in the proof of Lemma 2.6, we conclude

$$
\varepsilon^3 \|c_{3}^{(\cdot, t)}\|_{L^\infty(0,T_0;L^2(\Gamma_t(23)))} \leq C(M)\varepsilon^{N + \frac{1}{2}} \sup_{t \in [0,T_0]} \|\tilde{c}_3(\cdot, t)\|_{L^\infty(T^1 \times \mathbb{R})} \leq C(M, \theta)\varepsilon^{N + \frac{1}{2} - \theta},
$$

as well as

$$
\varepsilon^3 \|\nabla c_{3}^{(\cdot, t)}\|_{L^\infty(0,T_0;L^2(\Gamma_t(23)))} \leq C(M)\varepsilon^{N + \frac{1}{2}} \left( \frac{1}{\varepsilon} \sup_{t \in [0,T_0], s \in T_1} \|\partial_\rho \tilde{c}_3(\cdot, s, t)\|_{L^2(\mathbb{R})} + \|\partial_s \tilde{c}_3\|_{L^\infty(0,T_0;L^2(T^1 \times \mathbb{R}))} \right) \leq C(M, \theta)\varepsilon^{N + \frac{1}{2} - \theta}.
$$

Here we first applied (4.14) to $\tilde{c}_3$ and then employed (4.13).

Finally, the last inequality follows from the boundedness of $h_2 \in X_T \hookrightarrow BUC([0, T_0]; W^1_T(T^1))$, (2.17) and

$$
\varepsilon^3 \|\nabla c_{3}^{(\cdot, t)}\|_{L^2(0,T_0;L^2(\Gamma_t(23)))} \leq C(M)\varepsilon^{N + \frac{1}{2}} \left( \|\tilde{c}_3(\cdot, \cdot, t)\|_{L^2(0,T_0)} + \|\partial_\rho \tilde{c}_3\|_{L^2(0,T_0;L^2(T^1 \times \mathbb{R}))} \right) \leq C(M)\varepsilon^{N + \frac{1}{2}}.
$$

The following result gives an important expansion formula for the convection term. Recall that, $L^N$ is defined by (2.10).

Lemma 4.4 Let $\mathbf{v}_0$ be defined via (3.7), (3.12), (5.10) and assume that (3.4) and (5.3) hold true. Then

$$
\mathbf{v}_0^\nu(\rho, x, t) \cdot \nabla c_{3}^{(\cdot, t)}(x, t) = (\frac{1}{\varepsilon}\mathbf{v}_0 + \mathbf{v}_1^\nu)(\rho, x, t) \cdot \theta_0(\rho)
$$

$$
+ \left( \mathbf{v}_1 \cdot \nabla h_{1} + \mathbf{v}_1 | \cdot \nabla h_{1} - (\text{div}_\Gamma \mathbf{v}) | \cdot h_{1} + h_{1} \right)(x, t) \theta_0(\rho) + \varepsilon b(\rho, s, t) \theta_0(\rho)
$$

$$
+ \varepsilon (\mathbf{v} \cdot \nabla h_{1} + \mathbf{v} | \cdot \nabla h_{1} - (\text{div}_\Gamma \mathbf{v}) | \cdot h_{1} + h_{1}) \theta_0(\rho) + \varepsilon \mathbf{v}_1 \cdot \nabla c_{3}(x, t) \partial_\rho \tilde{c}_3(\rho, s, t) + R_s(x, t)
$$

in $\Gamma(3\delta)$, where $s = S(x, t)$, $\rho$ as in (1.2), and $b = b(\rho, s, t)$ is defined in (5.3). Moreover,

$$
\|R_s\|_{L^2(0,T_0;L^2(\Gamma_t(23)))} \leq C(R, M)\varepsilon^{N + 1/2} \quad \text{for all } \varepsilon \in (0, \varepsilon_0],
$$

where $C(R, M)$ is independent of $T_0$, $\varepsilon$, and $\varepsilon_0$.

Proof: In the proof, all identities should be interpreted in terms of the variable $x$ instead of the surface coordinate $s \in T^1$. For example, according to our definition, the function $\nabla h_{3} = (\nabla h_{3})\circ s(\cdot, t) \in \mathbb{R}^3$, is a function defined on the chart of the interface. However, in this proof, it is understood as $\nabla h_{3} = (\nabla h_{3})(S(x, t), t)$. Similarly, $\mathbf{v}_1 | \cdot \nabla = \mathbf{v}_1 | \nabla$ is the restriction of $\mathbf{v}$ on the interface $\Gamma$. However, it should be understood via $\mathbf{v}_1 | \cdot = \mathbf{P}_r(1)(x, t)$, as a function of $x$. In the sequel the dependence on variables $(x, t)$ is emphasized occasionally for clarity.
4.1 Inner Expansion

In the following $R_\varepsilon$ will denote any term such that $\|R_\varepsilon\|_{L^2(\Gamma(2\delta))} \leq \varepsilon^{N+1/2}$ for all $\varepsilon \in (0, \varepsilon_0]$ and for some $C > 0$, which depends only on the quantity $M$ in (4.14), $R$ in (3.39), $\v, \v_1$, and $\Gamma$. We use that

$$v_0^\varepsilon(p, x, t) \cdot \nabla c_0^\varepsilon(x, t) =$$

$$v_0^\varepsilon \left( \frac{d_r}{\varepsilon} - h_3(S(x, t), t) \cdot \frac{(\varepsilon h_1(x))}{\varepsilon} - (\nabla^T h_0)(d_r, S(x, t), t) \right) \theta_0^\varepsilon \left( \frac{d_r}{\varepsilon} - h_3(S(x, t), t) \right),$$

where $d_r = d_r(x, t)$. Furthermore, we have $\partial_\varepsilon v_0^\varepsilon |_{\Gamma} = - \nabla \cdot v_0^\varepsilon |_{\Gamma}$ and therefore $\partial_\varepsilon v_0^\varepsilon |_{\Gamma} = \nabla \cdot v_0^\varepsilon |_{\Gamma}$. Thus a Taylor expansion of $v_0^\varepsilon$ in terms of distance function $r = d_r(x, t)$ gives

$$v_0^\varepsilon(x, t) = v_0 |_{\Gamma}(x, t) - \nabla \cdot v_0 |_{\Gamma}(x, t) d_r(x, t) + \frac{\partial_\varepsilon^2 v_0^\varepsilon}{2} |_{\Gamma}(x, t) d_r(x, t)^2 + \cdots,$$

where the remainder $r^\varepsilon$ satisfies

$$\sup_{(x, t) \in \Gamma(2\delta)} |r^\varepsilon(x, t)| \leq C.$$  \hfill (4.16)

On the other hand, it follows from (4.11) and (4.12) that,

$$v_0(p, x, t) = \frac{1}{\varepsilon}(v_0^\varepsilon(x, t) + v_0(x, t)) + \frac{\eta(p)}{2}(v_0^\varepsilon(x, t) - v_0(x, t)).$$  \hfill (4.17)

Combining the above two identities leads to

$$\frac{1}{\varepsilon} v_0(p, x, t) \cdot n(s, t)$$

$$= \frac{1}{\varepsilon} v_0 |_{\Gamma}(x, t) - \nabla \cdot v_0 |_{\Gamma}(x, t) (\rho + h_3(s, t) + \varepsilon h_2, e(s, t))$$

$$+ \frac{\varepsilon}{2} (\partial_\varepsilon^2 v_0^\varepsilon |_{\Gamma} + \partial_\varepsilon^2 v_0 |_{\Gamma})(x, t) (\rho + h_3(s, t) + \varepsilon h_2, e(s, t))^2$$

$$+ \frac{\varepsilon}{2} (\partial_\varepsilon^2 v_0^\varepsilon |_{\Gamma} - \partial_\varepsilon^2 v_0 |_{\Gamma})(x, t) \eta(p) (\rho + h_3(s, t) + \varepsilon h_2, e(s, t))^2$$

$$+ \frac{\varepsilon^2}{2} \left( r^\varepsilon(x, t) + r^\varepsilon(x, t) \right) \eta(p) (\rho + h_3(s, t) + \varepsilon h_2, e(s, t))^3$$

where $s = S(x, t)$ and $\rho$ is defined via (4.2). Lemma 2.6 implies that the terms in the last line of the above identity give a contribution to $R_\varepsilon$ after multiplication with $\theta_0^\varepsilon (d_r(x, t) - h_3(s, t))$. Moreover, one can replace $(\rho + h_3(s, t) + \varepsilon h_2, e(s, t))$ by $(\rho + h_3(s, t))$ in all terms except the second since the remainders give another contribution to $R_\varepsilon$. As a result

$$\frac{1}{\varepsilon} v_0(p, x, t) \cdot n(s, t) \theta_0^\varepsilon (\frac{d_r(x, t)}{\varepsilon} - h_3(s, t))$$

$$= \frac{1}{\varepsilon} v_0 |_{\Gamma}(x, t) - \nabla \cdot v_0 |_{\Gamma}(x, t) (\rho + h_3)$$

$$+ \frac{\varepsilon}{2} (\partial_\varepsilon^2 v_0^\varepsilon |_{\Gamma} + \partial_\varepsilon^2 v_0 |_{\Gamma})(x, t) \eta(p) (\rho + h_3)^2 \theta_0^\varepsilon (\rho)$$

$$+ \frac{\varepsilon}{2} (\partial_\varepsilon^2 v_0^\varepsilon |_{\Gamma} - \partial_\varepsilon^2 v_0 |_{\Gamma})(x, t) \eta(p) (\rho + h_3)^2 \theta_0^\varepsilon (\rho) + R_\varepsilon$$

Moreover, a Taylor expansion of $v_0^\varepsilon$ near the interface is given by

$$v_0^\varepsilon(x, t) = v_0 |_{\Gamma}(x, t) + \partial_\varepsilon v_0^\varepsilon |_{\Gamma}(x, t) d_r(x, t) + \cdots,$$

where $\sup_{(x, t) \in \Gamma(2\delta)} |r^\varepsilon(x, t)| \leq C$. This together with (4.17) leads to

$$v_0(p, x, t) \cdot (\nabla^T h_0)(d_r(x, t), S(x, t), t)$$

$$= v_0 |_{\Gamma} \cdot \nabla h_3 + v_0 |_{\Gamma} \cdot (L^0 h_1)(d_r, s, t) + \varepsilon v_0 |_{\Gamma} \cdot \nabla h_2, e + \varepsilon v_0 |_{\Gamma} \cdot (L^0 h_2, e)(d_r, s, t)$$

$$+ \frac{\varepsilon}{2} \left[ (\partial_\varepsilon v_0^\varepsilon |_{\Gamma} + \partial_\varepsilon v_0^\varepsilon |_{\Gamma}) - (\partial_\varepsilon v_0^\varepsilon |_{\Gamma} - \partial_\varepsilon v_0^\varepsilon |_{\Gamma}) \eta(p) \cdot \nabla h_3(s, t)(\rho + h_3 + \varepsilon h_2, e)$$

$$+ \frac{\varepsilon}{2} \left[ (\partial_\varepsilon v_0^\varepsilon |_{\Gamma} + \partial_\varepsilon v_0^\varepsilon |_{\Gamma}) + (\partial_\varepsilon v_0^\varepsilon |_{\Gamma} - \partial_\varepsilon v_0^\varepsilon |_{\Gamma}) \eta(p) \cdot L^0 h_3(d_r, s, t)(\rho + h_3 + \varepsilon h_2, e)$$

$$+ \frac{\varepsilon^2}{2} \left[ (\partial_\varepsilon v_0^\varepsilon |_{\Gamma} + \partial_\varepsilon v_0^\varepsilon |_{\Gamma}) + (\partial_\varepsilon v_0^\varepsilon |_{\Gamma} - \partial_\varepsilon v_0^\varepsilon |_{\Gamma}) \eta(p) \right] \cdot (\nabla^T h_3)(d_r, s, t)(\rho + h_3 + \varepsilon h_2, e)^2.$$

$$+ \frac{\varepsilon}{2} \left( \frac{r^\varepsilon(x, t) + r^\varepsilon(x, t)}{2} - \frac{r^\varepsilon(x, t) - r^\varepsilon(x, t)}{2} \eta(p) \right) \cdot (\nabla^T h_3)(d_r, s, t)(\rho + h_3 + \varepsilon h_2, e)^2.$$
Using Lemma 2.6 and $L^\infty v_1|\Gamma \equiv 0$, it is easy to observe that the last three terms give rise to terms $R_\varepsilon$ after multiplication with $\theta_0'(\frac{d(r(x,t))}{\varepsilon} - h_\varepsilon(s,t))$. Hence we obtain

$$v_0(v,\rho, x, t) \cdot \nabla v_0(x, t) = \frac{1}{\varepsilon} v_n|\Gamma \theta_0'(\rho) + \left( v|\Gamma \cdot \nabla h_1 + v\cdot L^\infty h_1(d,\rho, t) - \nabla \cdot v|\Gamma v(h_1 + h_\varepsilon) \right) \theta_0'(\rho)$$

$$+ \varepsilon (v|\Gamma \cdot \nabla h_{2,x} \cdot - \nabla v|\Gamma h_{2,x} \cdot) \theta_0'(\rho)$$

$$+ \frac{\varepsilon}{2} \left( \partial^2_{v_n} v_0|\Gamma + \partial^2_{v_{0,n}} v_0|\Gamma \right)(\rho + h_\varepsilon) \theta_0'(\rho) + \frac{\varepsilon}{2} \left( \partial^2_{v_n} v_0|\Gamma + \partial^2_{v_{0,n}} v_0|\Gamma - \partial^2_{v_n} v_0|\Gamma \eta(\rho) \cdot \nabla h_1(\rho + h_\varepsilon) \theta_0'(\rho) + R_\varepsilon.\right.\)$$

Similarly we derive

$$\varepsilon v_1(v,\rho, x, t) \cdot \nabla v_0(x, t) = v_1^\perp(x, t) \theta_0'(\rho) + \varepsilon \left( v_1^\perp \cdot \nabla h_1 + v_1^\perp \cdot L^\infty h_1(d,\rho, t) - \nabla v_1^\perp \cdot |h_{1}| \theta_0'(\rho)\right)$$

$$+ \varepsilon^2 \left( \nabla h_{2,x} \cdot - \nabla v_1^\perp \cdot |h_{2,x} \cdot \theta_0'(\rho)\right)$$

$$+ \varepsilon^2 \left( \frac{r_{1,n}^+ + r_{1,n}^-}{2} - \frac{r_{1,n}^+ - r_{1,n}^-}{2} \eta(\rho) \right)(\rho + h_\varepsilon) \theta_0'(\rho)$$

$$+ \varepsilon^2 \left( \frac{r_{1,n}^+ + r_{1,n}^-}{2} - \frac{r_{1,n}^+ - r_{1,n}^-}{2} \eta(\rho) \right) \cdot \nabla h_{2,x}(d,\rho, t)(\rho + h_\varepsilon) \theta_0'(\rho)$$

$$= v_1^\perp \theta_0'(\rho) + \varepsilon \left( v_1^\perp \cdot \nabla h_1 - \nabla v_1^\perp \cdot |h_{1}| \theta_0'(\rho) + R_\varepsilon.\right)$$

as well as

$$\varepsilon^2 v_2(v,\rho, x, t) \cdot \nabla v_0(x, t) = v_2^\perp(x, t) \theta_0'(\rho) - \varepsilon^2 v_2(v,\rho, x, t) \cdot \nabla h_1(\rho + h_\varepsilon) \theta_0'(\rho)$$

$$+ \varepsilon^2 \left( \left( \frac{v_2(x, t) - v_0(x, t)}{d} \right) \cdot \eta(\rho) \nabla v_2(x, t) \right) \theta_0'(\rho)$$

$$= \varepsilon v_2(v,\rho, x, t) \theta_0'(\rho) + R_\varepsilon.$$
4.1 Inner Expansion

**Proof:** In the Appendix [A3] it is shown by careful, but lengthy calculations (see equation (4.17)) that

\[ \mathcal{E}(x, t) = R_e(x, t) + \sum_{k=1}^5 \varepsilon \Re_k \left( \frac{d^k}{dt^k} - h_s(S(x, t), t), S(x, t), t \right) + \Re_1 \left( \frac{d^k}{dt^k} - h_s(S(x, t), t), S(x, t), t \right) + \Re_2 \left( \frac{d^k}{dt^k} - h_s(S(x, t), t), S(x, t), t \right) \]

where \( R_e(x, t) \) is defined as in Lemma [4.4] and \( b = b(\rho, s, t) \) is defined in (4.6). Moreover,

\[ \Re_1 := -\frac{\varepsilon^2}{2} f''(\theta_0 + \xi(\rho, s, t)(\varepsilon^2 \bar{\sigma}^2 + \varepsilon^3 \bar{\sigma}^3)) \bar{\sigma}_0, \]

\[ \Re_2 := \partial_t^2 \bar{\sigma}_0 - \Gamma^1 \bar{\sigma}_0 + 2\nabla^{T} h_1 \cdot \nabla \bar{\sigma}_0 - \nabla^{0} h_1 \cdot \bar{\sigma}_0 \bar{\sigma}_0 - \nabla^{0} h_1 \cdot \left( \nabla^{2} h_1 + \nabla^{3} h_1 \right) \]

\[ = a_1(\rho, x, t) \partial_t^2 \bar{\sigma}_0 + a_2(\rho, x, t) \partial_t^2 \bar{\sigma}_0, \]

where \( \xi \) is some function with \( |\xi(\rho, s, t)| \leq 1 \), \( \bar{\sigma}_0 \) is as in (A.63) below, and \( \kappa_1, \kappa_2, \kappa_3, \varepsilon \) are defined by (2.15). We recall that \( \kappa_1(s, t) \) and \( \kappa_2(s, t) \) are smooth and \( \varepsilon \)-independent functions while the estimate of \( \kappa_3, \varepsilon \) is given by (4.16):

\[ |\kappa_3(\rho, s, t)| \leq C|\rho + h_s(s, t)|^3 \quad \text{for all } \rho \in \mathbb{R}, s \in T^1, t \in [0, T_0]. \]

In order to prove (4.19) we estimate \( \Re \) and \( \Re_k, k = 1, \ldots, 5 \) individually.

**To \( \Re \):** By the Definition (2.6) and (2.10), we have

\[ \varepsilon(L^2 h_2 - L^1 h_2)(d_t(x, t), S(x, t), t) \theta_0(\rho) \]

\[ = a_1(\rho, x, t) \partial_t^2 \bar{\sigma}_0 + a_2(\rho, x, t) \partial_t^2 \bar{\sigma}_0, \]

where \( a_1, a_2 \in \Re_0^0, \alpha \). So we can apply Corollary (2.7) to deduce

\[ \int_0^T \| \varepsilon(L^2 h_2 - L^1 h_2) \theta_0(\rho) \| L^2(\Gamma_1(2)) \, dt \leq C(M) T^{\frac{1}{4}} T^{\frac{1}{2}} \varepsilon N + \frac{1}{2} \]

and

\[ \int_0^T \| 2\varepsilon \theta_0(\rho) (d_t(x, t)) \| L^2(\Gamma_1(2)) \, dt \leq C(M) T^{\frac{1}{4}} T^{\frac{1}{2}} \varepsilon N + \frac{1}{2} \]

for all \( T \in (0, T_0], \varepsilon \in (0, \varepsilon_0] \). All the rest terms in \( \Re \) are multiplied by \( \varepsilon^2 \), depend only on \( h_2(s, t) \) (and not on its derivatives), have exponential decay as \( |\rho| \to \infty \) uniformly in \( (s, t), \varepsilon \), and can be estimated with the help of Lemma (2.4) because of \( X_T \to C^0(\Omega \times [0, T]) \) and (A.67). So all these estimates together imply

\[ \int_0^T \| \Re(\frac{d^k}{dt^k} - h_s(S(\cdot, t), S(\cdot, t)) \| L^2(\Gamma_1(2))) \, dt \leq C(M, \varepsilon, T) \varepsilon N + \frac{1}{2}, \]

where \( C(M, \varepsilon, T) \to (\varepsilon, T) \to 0. \)
To $\mathcal{R}_1$: First of all, because of (4.13), $\varepsilon\|\tilde{c}_3\|_{L^\infty((0,T_\varepsilon)\times\mathbb{T}^1 \times \mathbb{S})}$ is bounded. Therefore there is some $C > 0$ such that

$$\|\mathcal{R}_1(\rho, s, t)\| \leq C\varepsilon^2(\varepsilon\tilde{c}_3(\rho, s, t))^2$$

for all $\rho \in \mathbb{R}, s \in \mathbb{T}^1, t \in [0, T_\varepsilon]$.

Hence, using a change of variable, (4.12), and (4.13) again, we obtain

$$\int_0^T \|\mathcal{R}_1(\frac{\partial}{\partial t} - h_s(\cdot, t), \cdot)\|_{L^2(\Gamma_i(2\varepsilon))} dt \leq C\varepsilon^N + \frac{1}{4}\left( T + \varepsilon\|\tilde{c}_3\|_{L^1(0,T;L^2(\mathbb{R}^2 \times \mathbb{T}^1))} \right) \leq C(M)(T + \varepsilon)\varepsilon^N + \frac{1}{4}$$

for all $T \in (0, T_\varepsilon), \varepsilon \in (0, \varepsilon_0]$.

To $\mathcal{R}_2$: Using (4.12) and the smoothness of $h_1$, we can show that all the terms in $\mathcal{R}_2$ that are related to $\partial \tilde{c}_3$ belong to $\mathcal{R}_{0,a}$, i.e.,

$$\mathcal{R}_{0,a} \ni \mathcal{R}_2(\rho, s, t) + [\nabla \varphi h^2]\bar{\theta}_0^2(\rho) + \nabla \varphi h_1\bar{\theta}_0 \cdot \mathbf{w}_1|_T + (V + \Delta \varphi \partial \tilde{c}_3(\rho, s, t)).$$

It remains to estimate the last three terms on the right hand side. A change of variable together with (4.13) implies that

$$\int_0^T \left\|\nabla \varphi h_2\bar{\theta}_0^2(\rho) - \nabla \varphi h(S(\cdot, t), t)\right\|_{L^2(\Gamma_i(2\varepsilon))} dt \leq CT^\frac{1}{2}\sqrt{\varepsilon}\|\partial \tilde{c}_3\|_{L^2(\mathbb{R}^2 \times (0,T^\varepsilon))} \leq C'T^\frac{1}{2}\sqrt{\varepsilon}$$

for all $T \in (0, T_\varepsilon), \varepsilon \in (0, \varepsilon_0]$. The estimate for $[\nabla \varphi h_2]^2\bar{\theta}_0^2(\rho)$ follows from (3.3) and (2.8) together with Corollary 2.7.

$$\int_0^T \left\|\partial \tilde{c}_3(\rho, s, t)\right\|_{L^2(\mathbb{R}^2 \times (0,T^\varepsilon))} dt \leq C\varepsilon\int_0^T \|\partial \tilde{c}_3\|_{L^2(\mathbb{R}^2 \times (0,T^\varepsilon))} dt \leq C\varepsilon\|h_2\|_{L^1(\mathbb{T}^1)}$$

for all $T \in (0, T_\varepsilon), \varepsilon \in (0, \varepsilon_0]$. Similarly, it follows from the smoothness of $h_1$, Corollary 2.7 and trace estimate that

$$\int_0^T \left\|\nabla \varphi h_1\bar{\theta}_0 \cdot \mathbf{w}_1|_T\right\|_{L^2(\mathbb{R}^2(2\varepsilon))} dt \leq C\int_0^T \|\bar{\theta}_0 \cdot \mathbf{w}_1|_T\|_{L^2(\mathbb{R}^2(2\varepsilon))} dt \leq C\varepsilon\int_0^T \|\mathbf{w}_1|_T\|_{L^2(\mathbb{R}^2(2\varepsilon))} dt \leq C\varepsilon\|\mathbf{w}_1\|_{L^2(0,T;H^1(\mathbb{T}^1(0)))}$$

Since we assume (4.4) and (4.39), Proposition 3.10 and Theorem 3.4 are applicable. These together with (4.10) imply the estimate of $\mathbf{w}_1$

$$\|\mathbf{w}_1\|_{L^2(0,T;H^1(0))} \leq \varepsilon^{-2} \left(C(R)\varepsilon^N + C(M)(T^\frac{1}{4} + \varepsilon^\frac{1}{4})\varepsilon^N\right) = C(R) + C(M)(T^\frac{1}{4} + \varepsilon^\frac{1}{4})$$

for all $T \in (0, T_\varepsilon), \varepsilon \in (0, \varepsilon_0]$. The above two estimates together imply the estimate for $\mathcal{R}_2$.

To $\mathcal{R}_3$: We rearrange the terms of $\mathcal{R}_3$ as

$$\mathcal{R}_3 = 2\nabla \varphi h_2 \partial \tilde{c}_3 + \Delta \varphi h_2 \partial \tilde{c}_3 - 2\nabla \varphi h_1 \cdot \nabla \varphi h_2 \partial \tilde{c}_3 - \partial \tilde{c}_3 \partial \tilde{c}_3 + \nabla \varphi h_1 \cdot \mathbf{w}_1|_T \bar{\theta}_0 + 2\nabla \varphi h_1 \cdot \nabla \varphi \partial \tilde{c}_3 - \Delta \varphi h_1 \partial \tilde{c}_3 + |\nabla \varphi h_1|^2 \partial \tilde{c}_3 - \partial \tilde{c}_3 + (L^1 - L^2)\tilde{c}_3.$$

(4.23)
The common feature of terms in $\mathcal{R}_4^0$ is that derivatives of $h_2$ are only multiplied with derivatives of $\hat{c}_2$ but not that of $\hat{c}_3$. So they can be estimated by

$$\int_0^T |\varepsilon^3 \mathcal{R}_4^0| \, dt \lesssim C_\varepsilon |T^\frac{3}{2} \|h_2\|_{X_T} \lesssim C(M) \varepsilon^{N + \frac{3}{2}},$$

for all $T \in (0, T_\varepsilon]$, $\varepsilon \in (0, \varepsilon_0)$. Moreover, $\mathcal{R}_4^0$ consists of terms that includes derivatives of $\hat{c}_3$. Note that every terms here is multiplied by derivatives of smooth and $\varepsilon$-independent functions and according to (2.10), $L^1 - L^\infty$ is a second order operator with coefficients vanishing on $\Gamma$. They can be estimated using (4.13):

$$\int_0^T \varepsilon^3 |\mathcal{R}_4^0| \, dt \lesssim C(M) \varepsilon^{N + \frac{3}{2}} \|\hat{c}_3\|_{L^2(\Omega \times T), \varepsilon} \|\partial_{\rho} \hat{c}_3, \partial_{\rho} \hat{c}_3, \partial_{\rho} \hat{c}_3, \partial_{\rho} \hat{c}_3\|_{L^2(\Omega \times T)} \lesssim C(M) \varepsilon^{N + \frac{3}{2}} \|\mathcal{R}_4^0\|$$

for all $T \in (0, T_\varepsilon]$, $\varepsilon \in (0, \varepsilon_0)$. Finally, the estimate of the last summand in (4.23) follows from Corollary 2.8 and (4.24) successively:

$$\int_0^T \varepsilon^3 \|
abla f(x, \cdot, \cdot)\|_{L^2(\Omega \times T)} \, dt \lesssim C \|f\|_{H^{1/2}(\Omega)} \|g\|_{H^{1/2}(\Omega)} \quad \text{for all } f, g \in H^1(\Omega).$$

To $\mathcal{R}_4$: For the first term in $\mathcal{R}_4$, we use that

$$\sup_{r \in (-2, 2)} \|\nabla h_2(r, \cdot, \cdot)\|_{L^2(\Omega \times T)} \lesssim C \|\partial_x \hat{c}_3(h_2(t, \cdot, \cdot))\|_{L^2(\Omega \times T)}$$

for almost every $t \in [0, T]$ and $X_T \hookrightarrow L^1(0, T; C^1(\Omega))$). Therefore it follows from (1.13) that

$$\varepsilon^4 \int_0^T \|
abla h_2(\partial_x \hat{c}_3, h_2, h_2, h_2)\|_{L^2(\Omega \times T)} \, dt \lesssim C \|\partial_x \hat{c}_3(h_2(t, \cdot, \cdot))\|_{L^2(\Omega \times T)} \|\hat{c}_3\|_{L^2(\Omega \times T)} \lesssim C(M) \varepsilon^{N + \frac{5}{2}}$$

for all $T \in (0, T_\varepsilon]$, $\varepsilon \in (0, \varepsilon_0)$. For the second term in $\mathcal{R}_4$, we apply (1.21), Hölder's inequality, and Sobolev interpolation inequality:

$$\sup_{r \in (-2, 2)} \|\nabla h_2(\partial_x \hat{c}_3, h_2, h_2)\|_{L^2(\Omega \times T)} \lesssim C \|\partial_x h_2(t, \cdot, \cdot)\|_{H^{1/2}(\Omega)} \|\partial_x \hat{c}_3(h_2(t, \cdot, \cdot))\|_{H^{1/2}(\Omega)}$$

for almost every $t \in [0, T_\varepsilon]$. This together with a change of variable and (4.13) leads to

$$\varepsilon^4 \|\mathcal{R}_4\|_{L^1(0, T; L^2(\Omega \times T))} \lesssim C(M) \varepsilon^{N + \frac{5}{2}}$$

for all $T \in (0, T_\varepsilon]$, $\varepsilon \in (0, \varepsilon_0)$. The remaining terms can be treated in a similar manner and finally we get

$$\varepsilon^4 \|\mathcal{R}_4\|_{L^1(0, T; L^2(\Omega \times T))} = o(\varepsilon^{N + \frac{5}{2}}) \quad \text{as } \varepsilon \to 0.$$

To $\mathcal{R}_5$: It follows from (3.23) and Sobolev imbedding that $\|h_2\|_{H^1(\Omega \times T)}$ is uniformly bounded in $\varepsilon$. So we can show similarly as before that

$$\varepsilon^5 \|\mathcal{R}_5\|_{L^1(0, T; L^2(\Omega \times T))} \lesssim C(M) \varepsilon^{5 + \frac{3}{2}}$$

for all $T \in (0, T_\varepsilon]$, $\varepsilon \in (0, \varepsilon_0)$. 

4.2 The Full Expansion

In this subsection, we shall use the shorter notation \( \chi_{\pm} := \chi_{\Omega^z(\Omega)} \). After determination of the inner expansion, we define the approximate solutions \( c_{A,\varepsilon} \) in \( \Omega \times [0, T_0] \) as

\[
c_{A}(x, t) = \zeta \circ d \Gamma t \left[ (1 - \zeta \circ d \Gamma) \left( c_{\pm}^{\text{out}} \chi_+ + c_{-}^{\text{out}} \chi_- \right) \right] = c_{\pm}^{\text{out}} \chi_+ + c_{-}^{\text{out}} \chi_- + \left( c_{\pm}^{\text{in}}(x, t) - c_{-}^{\text{out}} \chi_+ - c_{-}^{\text{out}} \chi_- \right) \zeta \circ d \Gamma \tag{4.25}\]

where \( c_{\pm}^{\text{out}} = \pm 1 \) and \( \varepsilon \) is as in \( \text{(4.2)} \). Moreover, \( \rho \) is as in \( \text{(4.20)} \) and we define \( c_{A,\varepsilon} \) by

\[
c_{A,\varepsilon}(x, t) = \zeta \circ d \Gamma t \varepsilon \chi_{\pm}^{\text{in}}(x, t), \quad j = 2, 3 \tag{4.26}\]

with \( \varepsilon \) as in \( \text{(4.1)} \).

Remark 4.6 We note that we have chosen \( c_{\pm}^{\text{out}} = \pm 1 \) as approximation of \( c_{\pm} \) in \( \{ \Omega \times [0, T_0] \} \setminus \Gamma(\delta) \) (the region of the outer expansion). One can also derive this (formally) by expanding \( c_{\pm} \) in this outer region in the form \( \sum_{k=0}^{N} \delta^k \chi_{\pm}^{\text{out}}(x, t) \). Since the calculations are simple and standard, we omit them.

Lemma 4.7 Assume that \( \text{(4.4)} \) holds true for some \( M > 0 \) and let \( k \in \mathbb{N} \). Then there are \( C(M), C_{\varepsilon}(M) > 0 \), independent of \( T_0, \varepsilon \), and \( \varepsilon_0 \) such that for all \( 0 < \varepsilon \leq \varepsilon_0 \) and \( \delta \in (0, 1) \)

\[
\sup_{0 \leq t \leq T_0} \left\| \varepsilon \chi_{\pm}^{\text{in}}(t) - c_{\pm}^{\text{in}} \chi_+ - c_{-}^{\text{in}} \chi_- \right\|_{L^2(\Gamma(t) \setminus \Gamma_{\varepsilon})} \leq C(M)\varepsilon^{3N},
\]

\[
\sup_{0 \leq t \leq T_0} \varepsilon \left\| \chi_{\pm}^{\text{in}}(t) \right\|_{L^2(\Gamma(t) \setminus \Gamma_{\varepsilon})} + \sup_{0 \leq t \leq T_0} \left\| \nabla \chi_{\pm}^{\text{in}}(t) \right\|_{L^2(\Gamma(t) \setminus \Gamma_{\varepsilon})} \leq C(M)\varepsilon^{3N}.
\]

Proof: First of all \( h_{a,\varepsilon} \) is uniformly bounded since \( X_{T_{a,\varepsilon}} \to L^\infty([0, T_{a,\varepsilon}] \times \Omega) \) with operator norm bounded independently of \( T_{a,\varepsilon} \in (0, T_0] \). On the other hand, if \( 2\delta > |d \Gamma(t, x)| > \delta \), then

\[
|\rho(t, x)| = \left| d \Gamma t - h_{1}(s, t) - \varepsilon h_{2}(s, t) \right| \geq \frac{\delta}{2\varepsilon},
\]

for all \( \varepsilon \in (0, \varepsilon_0) \) if \( \varepsilon_0 \in (0, 1) \) is chosen sufficiently small. Moreover, we have, because of \( \text{(4.19)}, \text{(4.12)}, \text{and (4.13)} \), that

\[
|\theta_0(\rho) - c_{\text{out}}^{\text{in}}(\rho)| + |\theta_0(\rho)| \leq C e^{-\alpha |\rho|} \quad \text{if } \rho \geq 0,
\]

\[
|\varepsilon \theta_0' c_\theta(\rho, s, t)| + |\theta_0' c_\theta(\rho, s, t) + |\theta_0' c_\theta(\rho, s, t)| \leq C e^{-\alpha |\rho|},
\]

\[
(4.31a)
\]

\[
(4.31b)
\]

\[
(4.31c)
\]

for all \( (\rho, s, t) \in \mathbb{R} \times \Omega \times [0, T_0] \) and some \( C, C(M, \theta), \alpha > 0 \), where \( \theta > 0 \) is arbitrary.

For the following we denote \( \Sigma = \Gamma(2\delta) \setminus \Gamma_\delta \). Because of \( \text{(4.27), (4.30a), and (4.30b)} \), one easily obtains

\[
\sup_{0 \leq t \leq T_{a,\varepsilon}} \left[ \left\| \chi_{\pm}^{\text{in}}(t) - c_{\pm}^{\text{out}} \chi_+ - c_{-}^{\text{out}} \chi_- \right\|_{L^2(\Sigma_{\varepsilon})} + \varepsilon^2 \left\| \varepsilon \chi_{\pm}^{\text{in}}(t) \right\|_{L^2(\Sigma_{\varepsilon})} \right] \leq C e^{-\frac{\delta}{\varepsilon}} \leq C'e^{3N},
\]

\[
\sup_{0 \leq t \leq T_{a,\varepsilon}} \left[ \left\| \nabla \chi_{\pm}^{\text{in}}(t) \right\|_{L^2(\Sigma_{\varepsilon})} + \varepsilon^2 \left\| \nabla \chi_{\pm}^{\text{in}}(t) \right\|_{L^2(\Sigma_{\varepsilon})} \right] \leq C e^{-\frac{\delta}{\varepsilon}} \leq C'e^{3N}
\]

for all \( \varepsilon \in (0, \varepsilon_0) \) and some \( C, C', \alpha > 0 \). This shows \( \text{(4.27) - (4.28)} \).

To prove the first inequality in \( \text{(4.29)} \), we employ \( \text{(4.30)} \) and a change of variable to deduce

\[
\sup_{0 \leq t \leq T_{a,\varepsilon}} \varepsilon^3 \left\| \chi_{\pm}^{\text{in}}(t) \right\|_{L^2(\Sigma_{\varepsilon})} \leq C e^{-\frac{\delta}{\varepsilon}} \sup_{0 \leq t \leq T_{a,\varepsilon}} \left\| \rho^b \chi_{\pm}^{\text{in}}(t) \right\|_{L^2(\Sigma_{\varepsilon})} \leq C e^{-\frac{\delta}{\varepsilon}} \sup_{0 \leq t \leq T_{a,\varepsilon}} \left\| \rho^b \chi_{\pm}^{\text{in}}(t) \right\|_{L^2(\Omega^z)}.
\]
4.2 The Full Expansion

So the inequality follows by applying (4.13). The proof of the second inequality in (4.20) is done in the same way using additionally

\[ \| p^{k} \partial_{p} e_{3} \|_{L^{4}(\Omega)} \leq C \left( \sup_{t \in [0,T_{c}]} \| \partial_{p} e_{3}(\cdot, s, t) \|_{L^{2}(\Omega)} \right)^{1/4} \| p^{k} \partial_{p} e_{3} \|_{L^{2}(\Omega)} \]

to estimate the leading term related to \( \partial_{p} e_{3} \nabla^{T} h_{2,\varepsilon} \).

\[ \text{Lemma 4.8 Let (4.20) hold true. Then there are some } \varepsilon_{1}, T_{1} > 0 \text{ independent of } \varepsilon, T_{c}, e_{c}, e_{A} \text{ such that} \]

\[ \| c_{e}(t) - c_{A,0}(t) \|_{L^{2}(\Omega)} \leq \| c_{e}(t) - c_{A}(t) \|_{L^{2}(\Omega)} + \| c_{A}(t) - c_{A,0}(t) \|_{L^{2}(\Omega)} \]

and because of the assumptions (4.20), we only need to estimate the last term suitably. We also know from (4.24) and (4.2) that

\[ c_{A} - c_{A,0} = (\varepsilon^{3} e_{n}(x, t) - \theta_{0}(\rho)) \zeta \circ \partial_{T} + (\varepsilon^{3} e_{n}(x, t) + \varepsilon^{3} c_{n}(x, t)) \zeta \circ \partial_{T}. \]

Now let \( M = M(R), \varepsilon_{1}, \text{ and } T_{1} \) be as in Lemma 4.4. Because of \( h_{2,\varepsilon} \|_{t=0} = 0 \), there is some \( T'_{c} \in (0, T_{c}] \) such that \( h_{2,\varepsilon} \) satisfies (3.3) with \( T'_{c} \) instead of \( T_{c} \). Hence we can apply Lemma 4.3 and Lemma 4.2 to conclude

\[ \| c_{A}(t) - c_{A,0}(t) \|_{L^{2}(\Omega)} \leq \| c_{A}(t) - c_{A}(t) \|_{L^{2}(\Omega)} + \| c_{A}(t) - c_{A,0}(t) \|_{L^{2}(\Omega)} \]

\[ \| \nabla c_{A}(\cdot, t) \|_{L^{2}(\Omega)} \leq C(\varepsilon, T',) \varepsilon^{N+\frac{1}{4}}, \]

where \( C(\varepsilon, T',) \rightarrow (\varepsilon, T') \rightarrow 0 \). Choosing \( \varepsilon_{1}, T_{1} > 0 \) possibly even smaller we can achieve that \( C(\varepsilon, T) \leq \frac{1}{2} \) provided \( \varepsilon \leq \varepsilon_{1}, T \leq T_{1} \). This shows (4.32) for some \( T'_{c} \in (0, T_{c}] \) instead of \( T_{c} \). In order to show the estimate for \( T_{c} \), let

\[ T_{c} := \sup \{ T'_{c} \in (0, T_{c}) : (4.32) \text{ holds true for } T'_{c} \text{ instead of } T_{c} \}. \]

The previous step implies \( T_{c} > 0 \). Now assume that \( T_{c} < T_{c} \). Then (4.32) holds true for \( T_{c} \) instead of \( T_{c} \). Hence there is some \( T'_{c} \in (T_{c}, T_{c}) \) such that (4.32) holds true with \( T'_{c} \) instead of \( T_{c} \). Then Lemma 4.2 implies that \( h_{2,\varepsilon} \) satisfies (4.2) with \( T'_{c} \) instead of \( T_{c} \) and the first part of the proof shows (4.32) for \( T'_{c} \) instead of \( T_{c} \) provided \( T_{c} \leq T_{1} \) and \( \varepsilon \leq \varepsilon_{1} \). This is a contradiction to the definition of \( T_{c} \). Hence \( T_{c} = T_{c} \) and (4.32) holds true provided \( T_{c} \leq T_{1} \) and \( \varepsilon \leq \varepsilon_{1} \).

\[ \text{Corollary 4.9 Let (4.20) hold true for some } T_{\varepsilon} \leq T_{1} \text{ and } \varepsilon_{0} \leq \varepsilon_{1}, \text{ where } T_{1}, \varepsilon_{1} \text{ are as in Lemma 4.3. Then} \]

\[ \left| \int_{0}^{T} \int_{\Omega} \zeta \circ \partial_{T} \hat{w}_{2} \cdot \nabla c_{e}(\cdot, \cdot) dx dt \right| \leq C(\varepsilon, T, \varepsilon) \varepsilon^{N+1} \]

for every \( T \in (0, T_{c}] \) and \( \varepsilon \in (0, \varepsilon_{0}] \), where \( C(\varepsilon, T, \varepsilon) \) is independent of \( T_{c} \) and \( \varepsilon_{0} \) and \( C(\varepsilon, T, \varepsilon) \rightarrow (\varepsilon, T) \rightarrow 0 \).
Proof: We shall denote $u = c_e - c_A$ as before. We use
\[
\bar{w}_2 \cdot \nabla c^{3n} = \bar{w}_2 \cdot \nabla (\theta_0 + \varepsilon^2 c_2 + \varepsilon^3 c_3)
\] (4.33)
and estimate each term separately.

**Step 1:** We have
\[
\bar{w}_2 \cdot \nabla c_0^{3n} = \bar{w}_2 \cdot \left( \frac{\varepsilon}{\varepsilon} \nabla \bar{h}_1(s, t) - \varepsilon^2 \nabla \bar{h}_2(s, t) \right) \frac{\partial \theta_0}{\partial \varepsilon} - \bar{h}_e(s, t).
\]
Moreover, we use that
\[
h_{e,s} \in X_T \rightarrow BUC([0, T_e]; H^\frac{2}{3}(T^1)) \rightarrow BUC([0, T_e]; W^{3,1}(T^1)),
\]
is bounded with respect to $\varepsilon \in (0, \varepsilon_0)$ for any $s \in (1, \infty)$. Proposition 3.6 and Hölder’s inequality imply that
\[
\left\| \frac{\partial \theta_0}{\partial \varepsilon} \left( \frac{d}{d \varepsilon} - \bar{h}_s(s, t) \right) (\bar{u} - \varepsilon \nabla \bar{h}_u, h_u) \right\|_{L^r(0, T; L^s)} \leq \frac{C}{\varepsilon} \left\| \bar{w}_2 \right\|_{L^r(0, T; L^s)} \leq C(R)\varepsilon^{\frac{1}{s} - \frac{1}{r} - 1}, \quad (4.34)
\]
where $1 < r < 2$, $1 < q < q' < 2$, $1 < s < \infty$ and $\frac{1}{r} = \frac{1}{s} + \frac{1}{q}$. In the following let $r', q' \in (1, \infty)$ be such that $\frac{1}{r} + \frac{1}{r'} = \frac{1}{s} + \frac{1}{q'} = 1$. Moreover, the Gagliardo-Nirenberg interpolation inequality and (1.26) imply
\[
\left\| u \right\|_{L^r(0, T; L^q')} \leq C \left\| u \right\|_{L^\infty(0, T; L^s)}^{\frac{1}{r'} - \frac{1}{s} - \frac{1}{r} + 1} \left\| u \right\|_{L^2(0, T; H^1)}^{\frac{1}{r'} + \frac{1}{s} - \frac{1}{r} - 1} = C(R)\varepsilon^{\frac{2}{r'} - \frac{2}{r} - \frac{1}{q'}} \quad (4.35)
\]
if we choose $q' \in (2, \infty)$ (and therefore $q \in (1, 2)$) such that $\frac{1}{r'} = \frac{1}{2} - \frac{1}{q'}$. Hence
\[
\left| \int_0^T \int_\Omega \xi \partial_t \bar{w}_2 \cdot \left( \frac{d}{d \varepsilon} - \bar{h}_s(s, t) \right) (\bar{u} - \varepsilon \nabla \bar{h}_u, h_u) u \, dx \, dt \right| \leq C(R)\varepsilon^{\frac{1}{r'} + \frac{1}{s} - \frac{1}{r} - 1} = C(R)\varepsilon^{\frac{2}{r'} - \frac{2}{r} - \frac{1}{q'}}
\]
for all $T \in (0, T_e)$, $\varepsilon \in (0, \varepsilon_0)$. If we choose now $r \in (1, \frac{1}{q'})$, we have $\frac{1}{r} - \frac{1}{2} > 5$ and therefore
\[
\left| \int_0^T \int_\Omega \xi \partial_t \bar{w}_2 \cdot \left( \frac{d}{d \varepsilon} - \bar{h}_s(s, t) \right) (\bar{u} - \varepsilon \nabla \bar{h}_u, h_u) u \, dx \, dt \right| \leq C(R, T, \varepsilon)\varepsilon^{2N + 1},
\]
where $C(R, T, \varepsilon) \rightarrow (T, \varepsilon) \rightarrow 0$.

**Step 2:** Now we estimate $\varepsilon^2 \int_0^T \int_\Omega \xi \partial_t \bar{w}_2 \cdot \nabla c_2^{3n} u \, dx \, dt$. Because of (4.12), we obtain in a similar way as for (4.31) that
\[
\left\| \xi \partial_t \bar{w}_2 \cdot \nabla c_2^{3n} \right\|_{L^r(0, T; L^q)} \leq \left\| \xi \partial_t \bar{w}_2 \cdot \left( \frac{d}{d \varepsilon} - \nabla \bar{h}_u, h_u \right) \partial_\varepsilon \bar{c}_2 + \nabla \bar{c}_2 \right\|_{L^r(0, T; L^q)} \leq C(R)\varepsilon^{\frac{1}{r'} - 1}.
\]
This together with (4.35) implies
\[
\varepsilon^2 \int_0^T \int_\Omega \xi \partial_t \bar{w}_2 \cdot \nabla c_2^{3n} (c_e - c_A) \, dx \, dt \leq C(R, T, \varepsilon)\varepsilon^{2N + 1}.
\]

**Step 3:** We treat
\[
\varepsilon^3 \int_0^T \int_\Omega \xi \partial_t \bar{w}_2 \cdot \nabla c_2^{3n} (c_e - c_A) \, dx \, dt.
\]
In view of (1.23) and (4.31), we know that $|c_e - c_A|$ is uniformly bounded in $(x, t)$ and $\varepsilon$. On the other hand, it follows from (1.13) and the Gagliardo-Nirenberg interpolation inequality inequality that
\[
\left\| \nabla \bar{c}_2^{3n} \right\|_{L^1(0, T; L^8(\Gamma_{(2)}))} \leq \left\| \varepsilon \partial_\varepsilon \bar{c}_2^{3n} \right\|_{L^8(0, T; L^8(\Gamma_{(2)}))}^{\frac{1}{8}} \left\| \nabla \bar{c}_2^{3n} \right\|_{L^1(0, T; L^8(\Gamma_{(2)}))}^{\frac{7}{8}} \left( \varepsilon^2 \partial_\varepsilon \bar{c}_2^{3n}, \varepsilon \partial_\varepsilon \bar{c}_2^{3n} \right) \leq C\varepsilon^{1+\frac{1}{r} - \frac{1}{2} + \theta} ||(\varepsilon \partial_\varepsilon \bar{c}_2^{3n}, \varepsilon \partial_\varepsilon \bar{c}_2^{3n})||_{L^4(0, T; L^4(\Gamma_{(2)}))}
\]
is bounded, where we choose $\theta = \frac{1}{2}$. Hence
\[
\left| \varepsilon^3 \int_0^T \int_\Omega \xi \partial_t \bar{w}_2 \cdot \nabla c_2^{3n} (c_e - c_A) \, dx \, dt \right| \leq C\varepsilon^{1+\frac{1}{r} - \frac{1}{2} + \theta} ||(\varepsilon \partial_\varepsilon \bar{c}_2^{3n}, \varepsilon \partial_\varepsilon \bar{c}_2^{3n})||_{L^4(0, T; L^4(\Gamma_{(2)}))}.
\]
Combining this with (4.30a), \( r = \frac{q}{2} \), leads to
\[
|\varepsilon^3 \int_0^T \int_{\Omega} \zeta \partial_t \bar{w}_2 \cdot \nabla c_0^{in} (c_e - c_A) \, dx \, dt| \leq C(R)\varepsilon^{d-1+\frac{1}{2}} \leq C(R, T, \varepsilon)\varepsilon^{2N+1}
\]
with \( C(R, T, \varepsilon) \to_\varepsilon 0 \).

**Proof of Theorem 1.3** Again we denote \( u = c_e - c_A \). We first note that, since (1.26) is assumed, it follows from Lemma 1.8 that both (3.30) and (4.32) and thus (4.33) are valid if \( \varepsilon \leq \min\{\varepsilon_0, \varepsilon_1\} \) and \( T_e \leq \min\{T_0, T_e\} \). So the assumptions for applying Proposition 5.6 and Theorem 4.13 are fulfilled. These together with (4.11) imply the estimate of \( w_1 \):
\[
\|w_1\|_{L^2(0, T; L^2(\Omega))} \leq \varepsilon^{-2} (C(R)\varepsilon^N + C(M)(T_T^2 + \varepsilon^7)\varepsilon^N) = C(R) + C(M)(T_T^2 + \varepsilon^7) \quad (4.36)
\]
for all \( T \in (0, T_e), \varepsilon \in (0, \min\{\varepsilon_0, \varepsilon_1\}] \). To proceed, we shall calculate
\[
\partial_t c_A + (v_A + \varepsilon^2 w_2) \cdot \nabla c_A + \varepsilon^2 w_1 |_{\Gamma} \cdot \nabla c_{A, 0} - \Delta c_A + \frac{f'(c_A)}{\varepsilon^2}.
\]

Using (4.25) and Lemma 4.7, we have the following asymptotics in \( L^2(0, T; L^2(\Omega)) \):
\[
\Delta c_A = \zeta \partial_t \Delta c^{in} + 2\nabla(\zeta \partial_t) \cdot \nabla c^{in} + \Delta(\zeta \partial_t) (c^{in} - c_{e+}^{out} c_{\cdot} - c_{e-}^{out} c_{\cdot})
\]
\[
\partial_t c_A = \partial_t \partial_t c^{in} + O(\varepsilon^{N+\frac{3}{2}}),
\]
\[
\partial_t c_A = \partial_t \partial_t c^{in} + O(\varepsilon^{N+\frac{3}{2}}) + \varepsilon^2 w_1 |_{\Gamma} \cdot \nabla c_{A, 0} - \Delta c_A + \frac{f'(c_A)}{\varepsilon^2}.
\]

It follows from (4.31a), (4.31b), and (4.31c) that
\[
|c^{in} - c_{e+}^{out} c_{\cdot} - c_{e-}^{out} c_{\cdot}| \leq C(M)\varepsilon^{N+\frac{3}{2}} \text{ in } (\Gamma(3\delta) \setminus (\Gamma(\delta))) \cap (\Omega \times (0, T_e)). \quad (4.37)
\]

This together with (4.10) and (4.30) implies
\[
(v_A + \varepsilon^2 w_2) \cdot \nabla c_A = \zeta \partial_t (v_A + \varepsilon^2 w_2) \cdot \nabla c^{in} + (v_A + \varepsilon^2 w_2) \cdot \nabla (\zeta \partial_t) (c^{in} - c_{e+}^{out} c_{\cdot} - c_{e-}^{out} c_{\cdot})
\]
\[
= \zeta \partial_t (v_A + \varepsilon^2 w_2) \cdot \nabla c^{in} + (v_A + \varepsilon^2 w_2) \cdot \nabla (\zeta \partial_t) (c^{in} - c_{e+}^{out} c_{\cdot} - c_{e-}^{out} c_{\cdot}) + O(\varepsilon^{N+\frac{3}{2}} + \varepsilon^2 w_1 |_{\Gamma} \cdot \nabla c_{A, 0})
\]
\[
\text{in } L^2(0, T; L^2(\Omega)).
\]

The trace estimate, (4.31a), and (4.30a) imply
\[
\varepsilon^2 w_1 |_{\Gamma} \cdot \nabla c_{A, 0} = \zeta \partial_t \varepsilon^2 w_1 |_{\Gamma} \cdot \nabla c^{in} + O(\varepsilon^{N+\frac{3}{2}} + \varepsilon^2 w_1 |_{\Gamma} \cdot \nabla (\zeta \partial_t) (c^{in} - c_{e+}^{out} c_{\cdot} - c_{e-}^{out} c_{\cdot}))
\]
\[
= \zeta \partial_t \varepsilon^2 w_1 |_{\Gamma} \cdot \nabla c^{in} + O(\varepsilon^{N+\frac{3}{2}} + \varepsilon^2 w_1 |_{\Gamma} \cdot \nabla c_{A, 0}) \text{ in } L^2(0, T; L^2(\Omega)).
\]

To treat the bulk term we use (4.25) and (4.37) and obtain
\[
\frac{1}{\varepsilon^3} f'(c_A) = \begin{cases} \frac{1}{\varepsilon^3} f'(c^{in}), & \text{in } \Gamma(\delta), \\ O(\varepsilon^{N+1}), & \text{in } \Gamma(2\delta) \setminus \Gamma(\delta), \\ 0, & \text{in } \Omega \times (0, T_e) \setminus \Gamma(2\delta), \end{cases}
\]
with respect to the \( L^\infty(0, T; L^2) \)-norm. Actually, we only need to verify the case when \( \delta < |\partial_t| < 25 \). With a Taylor expansion, (4.27), and (4.29) we obtain
\[
f'(c_A) = f'(c_{e+}^{out} c_{\cdot} + c_{e-}^{out} c_{\cdot}) + f''(\xi(x, t)) \left[ (c^{in} - c_{e+}^{out} c_{\cdot} - c_{e-}^{out} c_{\cdot}) \zeta \partial_t \right] = 0 + O(\varepsilon^{N+3})
\]
in \( L^\infty(0, T; L^2(\Gamma(2\delta) \setminus \Gamma(\delta))) \), where \( \xi(x, t) \) is uniformly bounded. In particular, we obtain
\[
\frac{1}{\varepsilon^3} f'(c_A) = \zeta \partial_t \frac{1}{\varepsilon^3} f'(c^{in}) + O(\varepsilon^{N+1}) \text{ in } L^\infty(0, T; L^2(\Omega)).
\]

By collecting the above asymptotics, we arrive at
\[
\partial_t c_A + (v_A + \varepsilon^2 w_2) \cdot \nabla c_A + \varepsilon^2 w_1 |_{\Gamma} \cdot \nabla c_{A, 0} - \Delta c_A + \frac{f'(c_A)}{\varepsilon^3}
\]
\[
= \zeta \partial_t \left( \partial_t c^{in} + v_A \cdot \nabla c^{in} + \varepsilon^2 w_1 |_{\Gamma} \cdot \nabla c^{in} + \partial_t \frac{f'(c^{in})}{\varepsilon^3} \right)
\]
\[
+ \zeta \partial_t (\varepsilon^2 w_2 + v_A \cdot \nabla c^{in} + s_A), \quad (4.38)
\]
where \( s_A = O(\varepsilon^{N+\frac{1}{2}}) \) in \( L^2(0, T; L^2(\Omega)) \). This together with Theorem 4.13 leads to
\[
\partial_t c_A + (v_A + \varepsilon^3 w_2) \cdot \nabla c_A + \varepsilon^3 w_1 \cdot \nabla v_A - \Delta c_A + \frac{1}{\varepsilon} f'(c_A) = \zeta \circ dt \cdot c + \zeta \circ dt (\varepsilon^3 w_2 + v_A - v_A^n) \cdot \nabla v_A^n + \varepsilon^3 w_1 \cdot \nabla (c_A - c_{A,0}) + s_A.
\]
Due to (4.35), we have
\[
v_A - v_A^n = (\zeta \circ dt - 1)(v_A^n - v_A^n c_A + v_A^n - v_A^n - \zeta). = \sum_{i=0}^2 (\zeta \circ dt - 1)\varepsilon^i (v_i - v_i^c c_i + v_i^c - v_i^c c_i).
\]
This together with (4.39) and Lemma 5.3 implies that, the term \( \zeta \circ dt (v_A - v_A^n) \cdot \nabla c_A^n \) in (4.35) can be absorbed into \( s_A \). Moreover, because of (1.26), Theorem 4.5 and Corollary 4.9, we obtain
\[
\int_0^T \left| \int_\Omega \zeta \circ dt \left( c + \varepsilon^3 w_2 \cdot \nabla c_A^n \right)(c - c_A)(x, t) \, dt \right| \leq C(R, T, \varepsilon)\varepsilon^{2N+1}
\]
for all \( \varepsilon \in (0, \varepsilon_0] \) and \( T \in (0, \min(T_1, T_2)) \), where \( C(R, T, \varepsilon) \rightarrow (T, \varepsilon) \rightarrow 0 \).

To show (1.27) it remains to estimate \( \varepsilon^3 w_1 \cdot \nabla (c_A - c_{A,0}) \) since
\[
\int_0^T \int_\Omega |u_A(x, t)u(x, t)\, dt \leq C_{\epsilon_N+\frac{1}{2}} \|u\|_{L^2(0, T; L^2)} \leq C(R)T^{\frac{1}{2}} \varepsilon^{2N+1}.
\]

With the aid of Sobolev embeddings we obtain
\[
\int_0^T \int_\Omega |u_{\varepsilon^3 w_1}| |\nabla (c_A - c_{A,0})| \, dt \leq C T^{\frac{1}{2}} \|u\|_{L^2(0, T; H^1)} \|\nabla (c_A - c_{A,0})\|_{L^\infty(0, T; L^2)} \|u\|_{L^2(0, T; L^2)} \leq C T^{\frac{1}{2}} \|u\|_{L^2(0, T; H^1)} \|\nabla (c_A - c_{A,0})\|_{L^\infty(0, T; L^2)} \|u\|_{L^2(0, T; H^1)} \frac{1}{2}
\]
due to \( \|u\|_{L^4(\Omega)} \leq C \|u\|_{L^2(\Omega)} \|u\|_{L^1(\Omega)}^{\frac{1}{2}} \). Because of (4.1) and (4.25), we deduce
\[
\nabla (c_A - c_{A,0}) = \nabla (\zeta \circ dt)(c_{c_{A,0}} - c_{c_{A,0}}) + \zeta \circ dt \nabla \nabla (c_{c_{A,0}} - c_{c_{A,0}}) = \nabla (\zeta \circ dt)(\varepsilon^3 c_{c_{A,0}} + \varepsilon^3 c_{c_{A,0}}) + \zeta \circ dt \nabla (\varepsilon^3 c_{c_{A,0}} + \varepsilon^3 c_{c_{A,0}}).
\]

Hence Lemma 5.3 yields
\[
\|\nabla (c_A - c_{A,0})\|_{L^\infty(0, T; L^2)} \leq C(R)\varepsilon^{N-\frac{1}{2}}.
\]

This together with (1.26) and (4.39) implies
\[
\int_0^T \int_\Omega |u_{\varepsilon^3 w_1}| |\nabla (c_A - c_{A,0})| \, dt \leq C(R)\varepsilon^{3N-1/2} = C(R)\varepsilon^{2N+3/2}.
\]

Therefore the proof of Theorem 1.3 is finished.

5 Estimates of Approximate and Exact Solutions

5.1 Estimates of the Error in the Velocity and the Remainder in the Linearization of \( f' \)

We first recall that \( c_{A,0} \) is defined in (3.2), \( u_1 = c_{c_{A,0}} \) and \( v_1 = \hat{v}_A + \hat{w}_1 + \hat{w}_2 \), where \( \hat{v}_A \) is the solution of (3.1) and \( \hat{w}_1, \hat{w}_2 \) are as in Section 3.2, i.e., solve (3.34)-(3.35). Moreover, we define
\[
\hat{w}_1 := \frac{\hat{v}_A - v_A + \hat{w}_1}{\varepsilon^2} \quad \text{and} \quad \hat{w}_2 = \frac{\hat{w}_2}{\varepsilon^2}.
\]
5.1 Estimates of the Error in the Velocity and the Remainder in the Linearization of $f'$

Then $v = v_A + \varepsilon^2 (w_1 + w_2)$. We note that, under the assumption \((1.26)\), it follows from Lemma 4.8 that both (3.4) and (4.32) and thus (3.39) are valid if $\varepsilon \leq \min \{ \varepsilon_0, \varepsilon_1 \} \text{ and } T_2 \leq \min \{ T_0, T_1 \}$. So the assumptions for applying Proposition 3.3 and Theorem 4.5 are fulfilled and we have the following estimate on $w_1$:

$$
\|w_1\|_{L^2(0,T; H^1(\Omega))} \leq C(R), \quad \|w_1\|_{L^2(0,T; L^2(\Omega))} \leq C(R) \left( T^{\frac{1}{2}} + \varepsilon^{\frac{3}{2}} \right) \tag{5.1}
$$

for all $T \in (0, \min(T_1, T_2))$, $\varepsilon \in (0, \min(\varepsilon_0, \varepsilon_1))$ provided \((1.26)\) holds true.

**Lemma 5.1** Let $\varepsilon_1, T_1$ be as in Lemma 4.8 Then there is some $C(R)$ such that under the assumptions \((1.26)\), we have for all $T \in (0, \min(T_1, T_2))$, $\varepsilon \in (0, \min(\varepsilon_0, \varepsilon_1))$

$$
\varepsilon^4 \int_0^T \int_{\Gamma_{\varepsilon}} \left( \frac{1}{2} (w_{1,n} - w_{1,n}) |r| \right) \theta_0'(p)(c_e - c_A) dx \text{dt} \leq C(R) \left( T^{\frac{1}{2}} + \varepsilon^{\frac{3}{2}} \right) \varepsilon^{2N+1},
$$

$$
\varepsilon^2 \int_0^T \int_{\Gamma_{\varepsilon}} \theta_0'(p)(w_{1,r} - w_{1,r}) |r| \cdot \nabla x h_e(S(x, t), t)(c_e - c_A) dx \text{dt} \leq C(R) \varepsilon^{2N+1}. \tag{5.4}
$$

**Proof:** For the sake of simplifying the presentation, let us denote $u = c_e - c_A$ and $w = \varepsilon^2 w_1$.

**Proof of the first inequality:** It follows from \((4.19)\) and $\text{div } w = - \text{div } v_A$ that

$$
\int_{\Gamma_{\varepsilon}} \frac{1}{2} (w_{1,n} - w_{1,n}) |r| \theta_0'(p) u(x, t) dx \text{dt} = \int_{-2\delta}^{2\delta} \int_{\Gamma_{\varepsilon}} \frac{1}{2} (w_{1,n} - w_{1,n}) \theta_0'(p)(\delta - h_e) u(r, p, t) J(r, p, t) d\sigma(p) dr
$$

$$
= \int_{-2\delta}^{2\delta} \int_{\Gamma_{\varepsilon}} \frac{1}{2} \theta_0'(p)(\delta - h_e) u(r, p, t) J(r, p, t) d\sigma(p) dr \tag{5.5}
$$

Using Lemma 5.4 one can show in a straightforward manner that

$$
| \text{div } v_A(r', p, t)| \leq C(M)|\varepsilon| r' | + \varepsilon^2 \left( |h_{2,\delta}(t)|_{C^1(\Gamma_{\varepsilon})} + 1 \right) \quad \text{for all } r' \in (-2\delta, \delta), p \in \Gamma(t)
$$

and $t \in (0, \min(T_1, T_2))$, $\varepsilon \in (0, \min(\varepsilon_0, \varepsilon_1))$. Hence

$$
\int_{-2\delta}^{2\delta} \int_{\Gamma_{\varepsilon}} \frac{1}{2} \text{div } v_A(r', p, t) dr' \theta_0'(\delta - h_e) u(r, p, t) J(r, p, t) d\sigma(p) dr
$$

$$
\leq C(M) \int_{-2\delta}^{2\delta} \int_{\Gamma_{\varepsilon}} (\varepsilon^2 + |r'|) |\theta_0'(\delta - h_e)| |u(r, p, t)| J(r, p, t) d\sigma(p) dr \left( |h_{2,\delta}(t)|_{C^1(\Gamma_{\varepsilon})} + 1 \right)
$$

$$
\leq C(M) \varepsilon^{N+\frac{1}{2}} \|u(t)\|_{L^2(\Omega)} \left( |h_{2,\delta}(t)|_{C^1(\Gamma_{\varepsilon})} + 1 \right)
$$
Moreover, we note that
\[
\left| \frac{1}{\varepsilon} \int_0^T \mathbf{w}(r', p, t) \, dr' \right| \leq C \| \mathbf{w}(\cdot, p, t) \|_{L^\infty(\cdot, -\varepsilon, \varepsilon)} \frac{1}{\varepsilon}. \tag{5.2}
\]
Hence we arrive at
\[
\begin{align*}
& \left| \int_{\Gamma(t)} \frac{1}{\varepsilon} (\mathbf{w}_n - \mathbf{w}_n|_\Gamma) \delta_0(p) u(x, t) \, dx \right| \\
& \leq C(M) \varepsilon^{N+\frac{1}{2}} \| u(t) \|_{L^2(\Gamma)} \left( \| h_{2,\varepsilon}(t) \|_{C^1(\Gamma_1)} + 1 \right) \\
& + C \int_{-\varepsilon}^{\varepsilon} \left( \| \mathbf{w}(\cdot, p, t) \|_{L^\infty(\cdot, -\varepsilon, \varepsilon)} \left( 1 + \| (\partial_t h_\varepsilon)(X_0^{-1}(p, t), t) \|\right) \\
& \quad + \frac{1}{\varepsilon} \left( \| \theta_0^\varepsilon \| - h_\varepsilon \| + |\theta_0^\varepsilon| - h_\varepsilon \| \right) \right) \| u, \nabla_r u \| (r, p, t) \| d\sigma(p) \, dr + O(e^{-\frac{\varepsilon^2}{\varepsilon'}}) \\
& \leq C(M) \varepsilon^{N+\frac{1}{2}} \| u(t) \|_{L^2(\Gamma)} \left( \| h_{2,\varepsilon}(t) \|_{C^1(\Gamma_1)} + 1 \right) \\
& + C \left( \| \mathbf{w}(\cdot, P_{\varepsilon}(\cdot), t) \|_{L^\infty(\cdot, -\varepsilon, \varepsilon)} \left( 1 + \| (\partial_t h_\varepsilon)(S(\cdot, t), t) \| \right) \| \frac{\partial}{\partial r} \left( \| \theta_0^\varepsilon \| - h_\varepsilon \| + |\theta_0^\varepsilon| - h_\varepsilon \| \right) \right) \| \mathbf{w}(\cdot, P_{\varepsilon}(\cdot), t) \|_{L^2(\Gamma_1)} \\
& + \left( \| \mathbf{w}(\cdot, P_{\varepsilon}(\cdot), t) \|_{L^\infty(\cdot, -\varepsilon, \varepsilon)} \| u, \nabla_r u \| (r, p, t) \| d\sigma(p) \, dr + O(e^{-\frac{\varepsilon^2}{\varepsilon'}}) \right) \text{ in } L^2(0, T).
\end{align*}
\]
Since \( h_\varepsilon = h_1 + \varepsilon h_{2,\varepsilon} \) is uniformly bounded and
\[
\| f \|_{L^\infty(\cdot, -\varepsilon, \varepsilon)} \leq C \| f \|_{L^2(\cdot, -\varepsilon, \varepsilon)} \| f \|_{H^1(\cdot, -\varepsilon, \varepsilon)} \cdot
\]
applying the second inequality of Corollary \([2, 7]\) yields
\[
\begin{align*}
& \left| \int_{\Gamma(t)} \frac{1}{\varepsilon} (\mathbf{w}_n - \mathbf{w}_n|_\Gamma) \theta_0^\varepsilon(p) u(x, t) \, dx \right| \\
& \leq C \varepsilon^\frac{1}{2} \| \mathbf{w}(\cdot, P_{\varepsilon}(\cdot), t) \|_{L^\infty(\cdot, -\varepsilon, \varepsilon)} \left( 1 + \| (\partial_t h_\varepsilon)(S(\cdot, t), t) \| \right) \left( \| \theta_0^\varepsilon \| - h_\varepsilon \| + |\theta_0^\varepsilon| - h_\varepsilon \| \right) \left( \| \mathbf{w}(\cdot, P_{\varepsilon}(\cdot), t) \|_{L^2(\Gamma_1)} \right) \\
& \leq C \varepsilon^\frac{1}{2} \left( \| \mathbf{w}(\cdot, P_{\varepsilon}(\cdot), t) \|_{L^2(\Gamma_1)} + \varepsilon \| \mathbf{w}(\cdot, P_{\varepsilon}(\cdot), t) \|_{H^2(\Gamma_1)} \| h_{2,\varepsilon} \|_{L^2(\Gamma_1)} \right) .
\end{align*}
\]
Combining the above estimates with \([1, 20], \ (3.4), \ (5.1)\), we conclude
\[
\int_0^T \int_{\Gamma(t)} \left( \frac{1}{\varepsilon} (\mathbf{w}_n - \mathbf{w}_n|_\Gamma) \theta_0^\varepsilon(p) u(x, t) \, dx \right) \, dt \leq C(M, R) \left( T^\varepsilon + \varepsilon^\frac{1}{2} \right) \varepsilon^{2N+1}. \tag{5.3}
\]
Proof of the second inequality:
\[
\begin{align*}
& \int_{\Gamma(t)} \theta_0^\varepsilon(p) \mathbf{w}_n(r, p, t) \cdot \nabla_r h_\varepsilon(S(x, t), t) u(x, t) \, dx \\
& = \int_{-\varepsilon}^{\varepsilon} \int_{\Gamma(t)} \theta_0^\varepsilon(p) \mathbf{w}_n(r, p, t) \cdot \nabla_r h_\varepsilon(X_0^{-1}(p, t), t) u(r, p, t) J(r, p, t) \, d\sigma(p) \, dr \\
& = \int_{-\varepsilon}^{\varepsilon} \int_{\Gamma(t)} \theta_0^\varepsilon(p) \, \int_{\Gamma(t)} \partial_r \mathbf{w}_n(r', p, t) \, dr' \cdot \nabla_r h_\varepsilon(X_0^{-1}(p, t), t) u(r, p, t) J(r, p, t) \, d\sigma(p) \, dr.
\end{align*}
\]
This along with \( h_\varepsilon \in X_{T_\varepsilon} \hookrightarrow L^2(0, T_\varepsilon; C^1(\Gamma_1)) \) leads to
\[
\begin{align*}
& \left| \int_{\Gamma(t)} \mathbf{w}_n(r, p, t) \cdot \nabla_r h_\varepsilon(S(x, t), t) u(x, t) \, dx \right| \\
& \leq C \int_{-\varepsilon}^{\varepsilon} \int_{\Gamma(t)} \sqrt{\tau} \left( \| \theta_0^\varepsilon \| - h_\varepsilon \| + |\theta_0^\varepsilon| - h_\varepsilon \| \right) \| \mathbf{w}(\cdot, p, t) \|_{H^1(\cdot, -\varepsilon, \varepsilon)} \| \partial_r h_\varepsilon(X_0^{-1}(p, t), t) \| \, d\sigma(p) \, dr \\
& \leq C \varepsilon^\frac{1}{2} \sqrt{\tau} \left( \| \theta_0^\varepsilon \| - h_\varepsilon \| + |\theta_0^\varepsilon| - h_\varepsilon \| \right) \| \mathbf{w}(\cdot, P_{\varepsilon}(\cdot), t) \|_{H^1(\cdot, -\varepsilon, \varepsilon)} \| u(\cdot, t) \|_{L^2(\Gamma(\cdot, -\varepsilon, \varepsilon))} \| h_\varepsilon(t) \|_{C^1(\Gamma_1)} \\
& \leq C \varepsilon^\frac{1}{2} \sqrt{\tau} \left( \| \theta_0^\varepsilon \| - h_\varepsilon \| + |\theta_0^\varepsilon| - h_\varepsilon \| \right) \| \mathbf{w}(\cdot, P_{\varepsilon}(\cdot), t) \|_{L^2(\Gamma(\cdot, -\varepsilon, \varepsilon))} \| h_\varepsilon(t) \|_{C^1(\Gamma_1)} .
\end{align*}
\]
5.1 Estimates of the Error in the Velocity and the Remainder in the Linearization of $f'$

In this last step, we employed Corollary 2.27 to gain the factor $\varepsilon^{2/3}$. Then the second inequality follows by integrating the above estimate, (5.14), and (5.20).

Using the above lemma, we can prove the following important estimate:

**Lemma 5.2** Let $\varepsilon_1, T_1$ be as in Lemma 4.8 Then there is some $C(R)$ such that under the assumptions (4.20), we have

$$
\varepsilon^2 \int_0^T \left| \int_\Omega (w_1 - w_1 |r|) \cdot \nabla c_A (c_e - c_A) \text{d}x \right| \text{d}t \leq C(R) (T^{1/8} + \varepsilon^{1/4}) \varepsilon^{2N+1}
$$

provided $0 < \varepsilon \leq \min(\varepsilon_0, \varepsilon_1)$ and $0 < T \leq \min(T_0, T_1)$.

**Proof:** We shall denote $R := \varepsilon^2 (w_1 - w_1 |r|) \cdot \nabla c_A$. As before, it follows from Proposition 3.6 and Lemma 4.8 that, under the assumption (1.26) the estimate (3.40a) holds true. We shall prove the statement by distinguishing the cases $|d| \leq \delta$, $\delta < |d| < 2\delta$ and $|d| \geq 2\delta$. According to the definition of $c_A$ in (4.25)

$$
\int_0^T \int_{\Gamma(\gamma_2(2\delta)) \setminus \Gamma_1(\delta)} |R| \text{d}x \text{d}t = 0,
$$

where $u = c_e - c_A$. For the integral over the domain $\Gamma_1(2\delta) \setminus \Gamma_1(\delta)$, we first note that $\|w_1\|_{L^2(0, T, H^1)}$ is uniformly bounded due to (5.11). Hence $\|w_1 - w_1 |r|\|_{L^2(0, T, L^2)}$ is uniformly bounded because $H^2(\Gamma_1) \hookrightarrow L^4(\Gamma_1)$ and $H^1(\Omega) \hookrightarrow L^4(\Omega)$. Hence (1.26), (4.30) and Lemma 4.7 imply

$$
\int_0^T \int_{\Gamma_1(2\delta) \setminus \Gamma_1(\delta)} |R| \text{d}x \text{d}t \leq C\varepsilon^2 \|\nabla c_A\|_{L^\infty(0, T, L^2(\Gamma_1(2\delta) \setminus \Gamma_1(\delta)))} \|u\|_{L^2(0, T, H^1(\Omega \setminus \Gamma_1(\delta)))} \leq C\varepsilon^{2N+2}
$$

for any $T \in (0, T_1)$. So it remains to estimate the integral in $\Gamma_1(\delta)$. For any $x \in \Gamma_1(\delta)$, it follows from (3.20) that $c_{A,0} = c_{a_0}^0$ and thus we can decompose $R$ into

$$
R = -\varepsilon^2 (w_1 - w_1 |r|) \cdot \nabla (c_A - c_{A,0}) - \varepsilon^2 (w_1 - w_1 |r|) \cdot \nabla c_{A,0} = -\varepsilon^2 (w_1 - w_1 |r|) \cdot \nabla (c_A - c_{A,0}) = R_1 \quad \text{and} \quad -\varepsilon^2 (w_1 - w_1 |r|) \cdot \nabla c_{A,0} = R_2 + \varepsilon^2 (w_1 - w_1 |r|) \cdot \nabla c_{A,0}.
$$

The terms $R_2$ and $R_3$ are treated in Lemma 4.7. For $R_1$, we proceed in a similar way as in the proof of Theorem 4.3 using Sobolev embeddings,

$$
\int_0^T \int_{\Gamma_1(\delta)} |R_{1u}| \text{d}x \text{d}t \leq C\varepsilon^2 \|w_1\|_{L^2(0, T; H^1)} \|\nabla (c_A - c_{A,0})\|_{L^\infty(0, T; L^2)} \|u\|_{L^2(0, T; L^4)}
$$

$$
\leq C\varepsilon^2 \|w_1\|_{L^2(0, T; H^1)} \|\nabla (c_A - c_{A,0})\|_{L^\infty(0, T; L^2)} \|u\|_{L^2(0, T; H^1)}^2.
$$

In view of (4.20), (4.31), and (4.39)

$$
\int_0^T \int_{\Gamma_1(\delta)} |R_{1u}| \text{d}x \text{d}t \leq C\varepsilon^{2N+3/2} = C\varepsilon^{2N+1/2}.
$$

**Lemma 5.3** Under the assumptions (4.20), for every $0 \leq T \leq T_1$ and $\varepsilon \in (0, \varepsilon_0]$ we have

$$
\frac{1}{\varepsilon^2} \left| \int_0^T \int_\Omega \left[ f'(c_e) - f'(c_A) - f''(c_A)(c_e - c_A) \right] (c_e - c_A) \text{d}x \text{d}t \right| \leq C(R) (T^{1/8} + \varepsilon^{1/4}) \varepsilon^{2N+1}
$$

provided $\varepsilon_0 \in (0, 1)$ is sufficiently small.

The proof is based on:
5 ESTIMATES OF APPROXIMATE AND EXACT SOLUTIONS

Lemma 5.4 There is some $C > 0$ such that for every $u \in H^1(\Omega)$ and $t \in [0, T_0], \delta' \in (0, 2\delta]$: 

$$\|u\|_{L^3(\Gamma_t(\delta'))}^3 \leq C \left(\|u\|_{L^2(\Gamma_t(\delta'))} + \|\partial_n u\|_{L^2(\Gamma_t(\delta'))}\right)^{1/2} \left(\|u\|_{L^2(\Gamma_t(\delta'))} + \|\nabla u\|_{L^2(\Gamma_t(\delta'))}\right) \|u\|_{L^3(\Gamma_t(\delta'))}^2.$$ 

Proof: We use that

$$\|u\|_{L^3(\Gamma_t(\delta'))}^2 \leq C \int_0^{T_0} \int_{\Gamma_t} |u(p, r, t)|^3 \, dp \, dr = C \left(\left(\int_{\Omega} \|u(r, \cdot, t)\|_{H^1(\Gamma_t)}\right)^2 L^2(\delta') \right) \|u\|_{L^3(\Gamma_t(\delta'))},$$

where $\|u\|_{L^3(\Gamma_t)} \leq C\|u\|_{H^1(\Gamma_t)}\|u\|_{L^2(\Gamma_t)}$ since $\Gamma_t$ is one-dimensional. Hence Hölder’s inequality together with Minkowski inequality imply

$$\|u\|_{L^3(\Gamma_t(\delta'))}^2 \leq C \left(\|u(r, \cdot, t)\|_{H^1(\Gamma_t)}\right)^2 L^2(\delta') \|u\|_{L^3(\Gamma_t(\delta'))},$$

since $\|f\|_{L^2(\delta', \delta')} \leq C \|f\|_{L^2(\delta', \delta')}^2 \|f\|_{L^2(\delta', \delta')}^2$ for all $f \in H^1(-\delta', \delta').$}

Proof of Lemma 5.3 As before, we shall denote $u = c_e - c_A$ for simplicity. The estimate in $\Gamma_t(\delta')$ follows from Lemma 5.4 and (1.26)

$$\int_0^T \|u\|^3_{L^3(\Gamma_t(\delta'))} \, dt \leq C \left(\left(\int_0^T \|u(t)\|^3_{L^2(\Gamma_t(\delta'))}\right)^{1/2} \left(\int_0^T \|u(t)\|^2_{L^2(\Gamma_t(\delta'))}\right)^{1/2}\right) \left(\int_0^T \|u(t)\|^2_{L^2(\Gamma_t(\delta'))}\right)^{1/2},$$

and in $\Omega \setminus \Gamma_t(\delta)$ we can use the Gagliardo-Nirenberg inequality in two dimensions and (1.26)

$$\int_0^T \|u\|^3_{L^3(\Omega \setminus \Gamma_t(\delta))} \, dt \leq C \int_0^T \|\nabla u\|_{L^2(\Omega \setminus \Gamma_t(\delta))} \|u\|^2_{L^2(\Omega \setminus \Gamma_t(\delta))} \, dt \leq C R^{-N+1/2}.$$

The above two estimates together with the following formula imply the desired result:

$$\|f'(a) - f'(b) - f''(a)(a-b)(a-b)| = \frac{1}{2} |f''((1-\theta)b + \theta a) - (a-b)| \leq C |a-b|^3$$

for all $a, b \in [-L, L]$, where $\theta = \theta(a, b) \in [0, 1]$. Here we used that

$$\sup_{x \in \Omega, t \in [0, T_0]} |c_e(x, t)| \leq L := \max(2, \sup_{x \in [0, 1]} \|c_e(x, t)\|_{L^\infty(\Omega)})$$

cf. Remarks 1.2 and

$$\sup_{x \in \Omega, t \in [0, T_0]} |c_A(x, t)| \leq 2$$

for all $\varepsilon \in (0, \varepsilon_0]$ if $\varepsilon_0$ is sufficiently small. The later assertion follows from

$$\lim_{\varepsilon \to 0} \sup_{x \in \Omega, t \leq 0} |c_A(x, t)| = 1,$$

due to the construction of $c_A.$
5.2 Proof of Theorem 1.1

We first note that \( h_1, h_2, \) and \( \hat{c}_3 \) are determined by (4.3), (4.9), and (4.11), resp., with initial data \((h_1, h_2, \hat{c}_3)\) \(t=0\equiv 0\). This together with the construction of \( c_A \), more precisely (4.3) and (4.26) implies that

\[
    c_A(x, 0) = \zeta(d_{r_0}(x)) \left( \theta_0\left( \frac{d_{r_0}(x)}{\varepsilon} \right) + \varepsilon^2 \bar{c}_2 \left( \frac{d_{r_0}(x)}{\varepsilon}, S(x, 0), 0 \right) \right) + (1 - \zeta(d_{r_1}(x))) \left( \chi_{\Omega^+}(x) - \chi_{\Omega^-}(x) \right).
\]

Since the equation (4.3) that determines \( \hat{c}_2 \) has a smooth solution, we have

\[
    c_A(x, 0) = c_{A, 0}(x) + O(\varepsilon^{2+\frac{1}{2}}) \quad \text{in } L^2(\Omega),
\]

where we gained a factor \( \sqrt{\varepsilon} \) through the same change of variable as before. This together with the assumption that \( \|c_{x, 0} - c_{A, 0}\|_{L^2(\Omega)} \leq C\varepsilon^{N+\frac{1}{2}} \) implies the existence of some \( R \geq 1 \) such that

\[
    \sup_{\varepsilon \in (0, 1)} \|c_{0, \varepsilon} - c_A\|_{L^2(\Omega)} \leq \frac{R^2}{4} \varepsilon^{2N+1} + 1 \quad \text{for sufficiently small } \varepsilon, (5.4)
\]

In the following let \( c_A \) be as in Theorem 1.3 with \( R \) as determined before. We consider the validity of

\[
    \sup_{0 \leq t \leq \tau} \|c_{x, t}(t)\|_{L^2(\Omega)} + \|\nabla(c_x - c_A)\|_{L^2(\Omega \times (0, \tau))} \leq \frac{R^2}{2} \varepsilon^{2N+1}, \quad (5.5a)
\]

\[
    \|\nabla(c_x - c_A)\|_{L^2(\Omega \times (0, \tau))} + \varepsilon^2 \|\partial_t(c_x - c_A)\|_{L^2(\Omega \times (0, \tau))} \leq \frac{R^2}{2} \varepsilon^{2N+1} \quad (5.5b)
\]

for some \( \tau = \tau(\varepsilon) \in (0, T_0) \) and all \( \varepsilon \in (0, \varepsilon_0) \) for sufficiently small \( \varepsilon_0 \in (0, 1) \), which imply (4.21) for \( T = \tau \). Since the statement of Theorem 1.1 is for sufficiently short time and small \( \varepsilon \), we can assume that \( \tau \leq T_1 \) and \( \varepsilon_0 \leq \varepsilon_1 \), where \( T_1, \varepsilon_1 \) are as in Lemma 4.3. Hence (4.3) and (4.22) hold true as well. In the following we will prove that (5.5a) remain valid as long as \( \tau < T \) and \( \varepsilon \in (0, \varepsilon_0) \) for some \( T \in (0, T_0] \) independent of \( \varepsilon \) and some sufficiently small \( \varepsilon_0 \in (0, 1) \).

Now we define

\[
    T_0 := \sup \{ \tau \in [0, T_0] \mid \text{ (5.5a) holds true.} \}.
\]

Because of (4.3), since \( c_x, c_A \) are smooth, and since \( \|c_{x, t}(t)\|_{L^2(\Omega)} \) is continuous in \( t \in [0, T_0] \), we have \( T_0 \). It remains to show that \( T_0 \) has a positive lower bound that is independent of \( \varepsilon \). To this end, we apply Theorem 1.2 and obtain

\[
    \partial_t c_A + (v_A + \varepsilon^2 w_2) \cdot \nabla c_A + \varepsilon^2 w_1 \cdot \nabla c_A - \Delta c_A + \varepsilon^{-2} f'(c_A) = r_A \quad (5.6)
\]

where \( v_A = v_A + \varepsilon^2 (w_1 + w_2) \) and

\[
    \int_0^T \int_\Omega r_A(x, t)(c_A(x, t) - c_A(x, t)) \, dx \, dt \leq M_R(\varepsilon, T) \varepsilon^{2N+1} \quad (5.7)
\]

for all \( T \in (0, T_0] \) and \( \varepsilon \in (0, \varepsilon_0] \), where \( M_R(\varepsilon, T) \rightarrow (\varepsilon, T) \rightarrow 0 \). For the following let \( u = c_x - c_A \) and \( L_x = -\Delta + \frac{1}{\varepsilon^2} f''(c_A) \). Then we have

\[
    v_A \cdot \nabla c_x - (v_A + \varepsilon^2 w_2) \cdot \nabla c_A = v_A \cdot \nabla u + \varepsilon^2 w_1 \cdot \nabla c_A.
\]

Subtracting (5.6) from (1.7) and substituting the latter formula leads to

\[
    \partial_t u + v_A \cdot \nabla u + L_x u = -r_x(c_x, c_A) - r_A + \mathcal{R} \quad (5.8)
\]

where \( L_x \) is as in (4.24) and

\[
    r_x(c_x, c_A) = \frac{1}{\varepsilon^2} \left( f'(c_x) - f'(c_A) - f''(c_A)(c_x - c_A) \right),
\]

\[
    \mathcal{R} = -\varepsilon^2 w_1 \cdot \nabla c_A + \varepsilon^2 w_1 |_{\Gamma} \cdot \nabla c_A.
\]
Taking the inner product of (5.8) with $u$ in $L^2(\Omega)$ yields

$$\frac{1}{2} \|u(t)\|_{L^2(\Omega)}^2 + \int_0^t \int_{\Omega} \left| \nabla u \right|^2 dx ds \leq \int_0^t \int_{\Omega} |r_e(c, A)u| dx ds + \int_0^t \int_{\Omega} |r_c(c, A)u| dx ds + \int_0^t \int_{\Omega} |\mathcal{R} u| dx ds + \frac{1}{2} \|u(0)\|_{L^2(\Omega)}^2 \tag{5.9}$$

for all $t \in [0, T]$. Now using Theorem 2.3, (5.6), and (5.7) we obtain

$$\sup_{0 \leq s \leq T} \|u(s, \cdot)\|_{L^2(\Omega)}^2 + \|\nabla_r u\|_{L^2(\Omega \times (0, t) \cap \Gamma(\delta))}^2 + \|\nabla u\|_{L^2(\Omega \times (0, t) \cap \Gamma(\delta))}^2 \leq C \int_0^T \int_{\Omega} |r_e(c, A)u| dx ds + \int_0^T \int_{\Omega} |r_c(c, A)u| dx ds \leq \frac{R^2}{4} + M_R(\varepsilon, T) \varepsilon^{2N+1} \tag{5.10}$$

for all $t \in [0, T]$. Here we used that $\|\nabla_r u\|_{L^2(\Omega \cap \Gamma(\delta))} + \|\nabla u\|_{L^2(\Omega \cap \Gamma(\delta))}$ is equivalent to $\|\nabla_r u\|_{L^2(\Omega \cap \Gamma(\delta))} + \|\nabla u\|_{L^2(\Omega \cap \Gamma(\delta))}$. So we can apply Gronwall’s inequality on the interval $[0, T]$ with $T < T_1$ and Young’s inequality to obtain

$$\sup_{0 \leq s \leq T} \|u(t)\|_{L^2(\Omega)}^2 \leq C \left( \int_0^T \int_{\Omega} |r_e(c, A)u| dx ds + \int_0^T \int_{\Omega} |r_c(c, A)u| dx ds \right) \leq \frac{R^2}{4} + M_R(\varepsilon, T) \varepsilon^{2N+1} \tag{5.11}$$

By the definition of $T_\varepsilon$, the first two terms on the right-hand side can be estimated using Lemma 5.2 and Lemma 5.3

$$e^{CT} \left( \int_0^T \int_{\Omega} |r_e(c, A)u| dx ds + \int_0^T \int_{\Omega} |r_c(c, A)u| dx ds \right) \leq e^{CT} \left( C(R)(T^{1/8} + \varepsilon^{1/4}) \varepsilon^{2N+1} + \left( \frac{R^2}{4} + M_R(\varepsilon, T) \varepsilon^{2N+1} \right) \right) \leq \frac{R^2}{3} \varepsilon^{2N+1} \tag{5.12}$$

for all $\varepsilon \in (0, \varepsilon_0]$ provided $T \leq \min(T_1, T_\varepsilon)$ and $T_1 > 0, \varepsilon_0 \in (0, 1]$ are sufficiently small. This together with (5.11) implies

$$\sup_{0 \leq s \leq T} \|u(t)\|_{L^2(\Omega)}^2 \leq \frac{R^2}{3} \varepsilon^{2N+1} \leq \frac{R^2}{2} \varepsilon^{2N+1}$$

for all $T \in (0, \min(T_1, T_\varepsilon))$ and $\varepsilon \in (0, \varepsilon_0]$ if $T_1 \in (0, T_0]$ and $\varepsilon_0$ are sufficiently small. To complete the proof of (5.5) with strict inequality, it remains to estimate $\partial_t u(x, t)$. To this end we use (5.9), $\inf_{s \in \mathbb{R}} f''(s) > -\infty$, (5.12) and the previous estimates:

$$\varepsilon^2 \|\partial_t u(c, A)\|_{L^2(\Omega \times (0, t) \cap \Gamma(\delta))} \leq \varepsilon^2 \|\nabla u\|_{L^2(\Omega \times (0, t) \cap \Gamma(\delta))} \leq \frac{R^2}{3} \varepsilon^{2N+1} + CT \sup_{0 \leq t \leq T} \|u(t)\|_{L^2(\Omega)}^2 \leq \frac{R^2}{3} \varepsilon^{2N+1} + CT \frac{R^2}{2} \varepsilon^{2N+1} < \frac{R^2}{6} \varepsilon^{2N+1}$$

for all $0 < T \leq \min(T_1, T_\varepsilon)$ and $0 < \varepsilon \leq \varepsilon_0$ provided $T_1 \in (0, T_0]$ and $\varepsilon_0 \in (0, 1]$ are sufficiently small.

Altogether we obtain (5.5) with $\tau = \min(T_1, T_\varepsilon)$ and strict inequality for all $\varepsilon \in (0, \varepsilon_0]$, provided $T_1 \in (0, T_0]$ and $\varepsilon_0 \in (0, 1]$ are sufficiently small. Because of the definition of $T_\varepsilon$ and since the norms in (5.5) depend continuously on $\tau$, this implies $T_\varepsilon > T_1$ for all $\varepsilon \in (0, \varepsilon_0]$. Hence the estimate (1.14) of Theorem 1.3 is proved.

Now (1.22) follows directly from Proposition 4.6 and Theorem 5.5 since $v_e - v_A = v_A - v_A + w_1 + w_2$. Finally, $\lim_{\varepsilon \to 0} \sup_{x \in \Omega \times [0, T]} |c(x, t)| = 1$ follows easily from the construction of $c_A$ as well as (4.13). The last statement on convergence of $v_A$ follows easily from the constructions. This finishes the proof of Theorem 1.3.
## A Appendix

### A.1 Formally Matched Asymptotics for Stokes System

The main task of this and the next subsection is to prove the Lemma 3.4. One approach is to plug (3.12) and (3.19) into (3.7) and to verify. However, in order to show the reader how the obtained formula, we shall first assume some ansatz and see what kind of equations these terms in the expansion should satisfy. To this end we will construct an approximate solution with the inner expansion

\[
\hat{v}_j^\pm (\rho, x, t) = v_0(\rho, x, t) + \varepsilon v_1(\rho, x, t) + \varepsilon^2 v_2(\rho, x, t),
\]

(A.1)

\[
p_j^\pm (\rho, x, t) = \varepsilon^{-1} p_{-1}(\rho, x, t) + p_0(\rho, x, t) + \varepsilon p_1(\rho, x, t),
\]

where we assume that \((v_j, p_j)\) are given by

\[
v_j(\rho, x, t) = v_j(\rho, x, t) + d_t(x, t) \hat{v}_j(x, t) \eta(\rho), \quad 0 \leq j \leq 2,
\]

\[
p_j(\rho, x, t) = p_j(\rho, x, t) + d_t(x, t) \hat{p}_j(x, t) \eta(\rho), \quad -1 \leq j \leq 1.
\]

(A.2)

where \(\eta(\rho)\) satisfies (3.11). Moreover, we assume that (3.19) holds, which implies the matching conditions

\[
\lim_{\rho \to \pm \infty} v_j(\rho, x, t) = v_j^\pm (x, t), \quad \lim_{\rho \to \pm \infty} \partial_\rho v_j(\rho, x, t) = \lim_{\rho \to \pm \infty} \partial_\rho v_j(\rho, x, t) = 0, \quad 0 \leq j \leq 2 \quad (A.3)
\]

\[
\lim_{\rho \to \pm \infty} p_j(\rho, x, t) = p_j^\pm (x, t), \quad \lim_{\rho \to \pm \infty} \partial_\rho p_j(\rho, x, t) = \lim_{\rho \to \pm \infty} \partial_\rho p_j(\rho, x, t) = 0, \quad -1 \leq j \leq 1. \quad (A.4)
\]

(A.5)

After determining \(v_j^\pm\), they are extended smoothly from \(\Omega^\pm(t)\) to \(\Omega\) such that \(\text{div} v_j^\pm = 0\) is preserved. The existence of such an extension can be seen as in the discussion before (3.8).

The general routine is: By matching terms with the same powers of \(\varepsilon\) of the divergence equation, we get \(\hat{v}_k\), and take the normal component in the Stokes equation to get \(\hat{p}_{k-1}\) and finally use this to solve the Stokes equation to get \(v_k\).

### A.1.1 Divergence equation for \(v_0\):

In the following we will use:

**Lemma A.1** Let \(k = 0, 1\). If \(\hat{v}_k\) is independent of \(\rho\), then the following formulas hold

\[
\hat{v}_k = \frac{1}{2} (v_k^+ + v_k^-) \text{ on } \Gamma(3\delta), \quad \hat{v}_k = \frac{1}{2\varepsilon} (v_k^+ - v_k^-) \text{ on } \Gamma(3\delta) \setminus \Gamma.
\]

Moreover, if \(v_k^\pm \in C^\infty(\overline{\Omega})\) and \(v_k^+ = v_k^-\text{ on } \Gamma\), then we can define

\[
\hat{v}_k := \frac{1}{2} \partial_n(v_k^+ - v_k^-) \text{ on } \Gamma_1
\]

and \(\hat{v}_k \in C^\infty(\overline{\Omega})\). The same statements also hold for the normal and tangential components of \(v_k\).

**Proof:** To prove the first part, since \(\hat{v}_k(\rho, x, t)\) is independent of \(\rho\), we take \(\rho \to \pm \infty\) in the first equation of (A.2) and use (A.3) to conclude

\[
\hat{v}_k(\rho, x, t) \pm d_t(x, t) \hat{v}_k(x, t) = v_k^\pm (x, t) \quad \text{for all } (x, t) \in \Gamma(3\delta).
\]

Solving the linear equations leads to the first statement. The proof of the last statement follows from a Taylor expansion with respect to \(d_t\) as described in the end of Subsection 2.1.  

[49]
Moreover, we have for \( j = 0, 1, 2 \)
\[
\text{div} \left( \frac{\delta}{\varepsilon} \mathbf{v}_j \left( \frac{\delta}{\varepsilon} h_x \right), x, t \right) = \frac{1}{\varepsilon} \partial_\rho \mathbf{v}_{j,0}(\rho, x, t) - \nabla^T h_x (d_T, s, t) \cdot \partial_\rho \mathbf{v}_{j,\tau}(\rho, x, t) \\
+ \frac{d\rho}{\varepsilon} \eta'(\rho) \mathbf{v}_{j,\eta}(x, t) - d_T \eta'(\rho) \nabla^T h_x (d_T, s, t) \cdot \mathbf{v}_{j,\tau}(x, t) + \text{div}_x (\mathbf{v}_j(\rho, x, t) - \mathbf{v}_j(x, t) d_T \mathbf{\eta}(\rho))
\]

Therefore it follows that
\[
\text{div} \left( \frac{\delta}{\varepsilon} \mathbf{v}^+_0 \left( \frac{\delta}{\varepsilon} - h_x \right), x, t \right) = \frac{1}{\varepsilon} \partial_\rho \mathbf{v}^+_0(\rho, x, t) \\
- \nabla^T h_x \cdot \partial_\rho \mathbf{v}^+_0(\rho, x, t) + (\rho + h_x) \eta'(\rho) \mathbf{v}^+_0(\rho, x, t) \\
- d_T \eta'(\rho) \nabla^T h_x \cdot \mathbf{v}^+_0(\rho, x, t) + \text{div}_x (\mathbf{v}_0(\rho, x, t) + \mathbf{v}_0(x, t) d_T \mathbf{\eta}(\rho)) + \partial_\rho \mathbf{v}^+_1(\rho, x, t) \\
- \varepsilon \nabla^T h_x \cdot \partial_\rho \mathbf{v}^+_1(\rho, x, t) + \varepsilon (\rho + h_x) \eta'(\rho) \mathbf{v}^+_1(\rho, x, t) - \varepsilon d_T \eta'(\rho) \nabla^T h_x \cdot \mathbf{v}^+_1(\rho, x, t) \\
+ \varepsilon \text{div}_x (\mathbf{v}_1(\rho, x, t) + \mathbf{v}_1(x, t) d_T \mathbf{\eta}(\rho)) + \varepsilon \partial_\rho \mathbf{v}^+_2(\rho, x, t) \\
- \varepsilon^2 \nabla^T h_x \cdot \partial_\rho \mathbf{v}^+_2(\rho, x, t) + \varepsilon^2 (\rho + h_x) \eta'(\rho) \mathbf{v}^+_2(\rho, x, t) \\
- \varepsilon^2 d_T \eta'(\rho) \nabla^T h_x \cdot \mathbf{v}^+_2(\rho, x, t) + \varepsilon^2 \text{div}_x (\mathbf{v}_2(\rho, x, t) + \mathbf{v}_2(x, t) d_T \mathbf{\eta}(\rho)).
\]  

Hence equating terms with the same power of \( \varepsilon \) in the expansion of \( \text{div} \mathbf{v}^+_0 \), we obtain at order \( O(\varepsilon^1) \) that
\[
\partial_\rho \mathbf{v}^+_0(\rho, x, t) = 0 \quad \text{for all } \rho \in \mathbb{R}, (x, t) \in \Gamma(3\delta) \tag{A.7}
\]
and this together with Lemma A.1 implies
\[
\mathbf{v}^+_0 = \frac{1}{\varepsilon} (\mathbf{v}^{+_0}_{0,n}(x, t) + \mathbf{v}^{+_0}_{0,n}(x, t)) \quad \text{in } \Gamma(3\delta), \quad \mathbf{v}^+_0 = \frac{1}{\varepsilon} (\mathbf{v}^{+_0}_{0,n}(x, t) - \mathbf{v}^{+_0}_{0,n}(x, t)) \quad \text{in } \Gamma(3\delta) \setminus \Gamma, \tag{A.8}
\]
where \( \mathbf{v}^{+_0}_{0,n} \) are extended smoothly from \( \Omega^s(t) \) to \( \Gamma_\delta \) such that \( \text{div} \mathbf{v}^{+_0}_{0,n} = 0 \) is preserved. So we can define:
\[
\mathbf{v}^+_0 = \frac{1}{\varepsilon} \partial_\rho (\mathbf{v}^{+_0}_{0,n} - \mathbf{v}^{+_0}_{0,n}) \quad \text{on } \Gamma.
\]

For the expansion of the Stokes system, we obtain:

Lemma A.2 If we relate \( \mathbf{h}_1 - \varepsilon \mathbf{h}_2 \) in (A.1), then expanding
\[
- \Delta \mathbf{v}^+_0 + \nabla \mathbf{p}^0_0 \quad \text{and} \quad - \varepsilon \text{div}(\nabla c^0_0 \otimes \nabla c^0_0) \tag{A.9}
\]
and equating the terms with the same power of \( \varepsilon \) yields:
\[
O(\varepsilon^{1/2}) : - \partial^2_\rho \mathbf{v}_0 + \partial_\rho \mathbf{p}_{-1} \mathbf{n} = - \partial_\rho (\mathbf{v}_0(\rho))^2 \mathbf{n} \tag{A.10}
\]
and, if \( \partial_\rho \mathbf{v}_0 = \mathbf{v}_x \mathbf{p}_{-1} = 0 \) and \( \mathbf{p}_{-1} = 0 \),
\[
O(\varepsilon) : - \partial^2_\rho \mathbf{v}_1 - \nabla^T h_x \partial_\rho \mathbf{p}_{-1} + \mathbf{n} \partial_\rho \mathbf{p}_0 - (\rho + h_1) \mathbf{v}_0 \eta''(\rho) - 2\eta'(\rho) \partial_\rho (d_T \mathbf{v}_0) \\
= \partial_\rho (\mathbf{v}_0(\rho))^2 \nabla^T h_x - (\mathbf{v}_0(\rho))^2 \mathbf{n} d_T. \tag{A.11}
\]
A.1 Formally Matched Asymptotics for Stokes System

If additionally \( \partial_{\rho} \hat{\nu}_1 = 0 \) and \((\partial_{\rho} \hat{p}_1, \partial_{\rho} \hat{v}_2, \partial_{\rho}^2 \hat{v}_2) \in R_{0,0}, \) then

\[
O(1) := - \partial_{\rho}^2 \hat{v}_2 - \eta''(\rho)(h_2 \hat{v}_0 + (p + h_1) \hat{v}_1) + \eta'(\rho) \hat{p}_0 \mathbf{n}(\rho + h_1) + n \partial_{\rho} \hat{p}_1 - \partial_{\rho} \hat{v}_0 \nabla_f h_1 \\
- \partial_{\rho} \hat{v}_1 \nabla_f h_2 + \eta(\rho) \nabla_x(d_\rho \hat{v}_0) - \Delta_x(d_\rho \hat{v}_0) - \Delta_x \hat{v}_0 + \nabla_x \hat{p}_0 + 2\eta(\rho) \nabla_f h_1 : \nabla_x(d_\rho \hat{v}_0) \\
- 2\eta(\rho) \partial_{\rho} \hat{v}_2 - (\eta'(\rho) \hat{h}_1 \nabla_f h_0 + \eta(\rho) \hat{v}_0 \nabla_f h_0) \Delta_x d_\rho \\
= \partial_{\rho} \hat{\theta}_0(\partial_{\rho}^2 \hat{v}_2 - \eta''(\rho) \hat{v}_0)^2 \nabla_f h_1 + \eta(\rho) \hat{v}_0)^2 \left( \Delta_x d_\rho \nabla_f h_1 + \nabla_f h_1 \right) - \eta(\rho) \hat{v}_0)^2 \left( \Delta_x d_\rho \nabla_f h_1 + \nabla_f h_1 \right),
\]

(A.12)

\[
O(\varepsilon) := - \eta''(\rho)(\rho \hat{v}_2 + h_2 \hat{v}_1 + \hat{v}_2) + \eta(\rho) \mathbf{n}(\mathbf{h}_2 \hat{p}_0 + h_1 \hat{p}_1 + p \hat{p}_1) - \eta(\rho) \nabla_f h_1 \nabla_f h_2 + \eta'(\rho)(\rho + h_1) \hat{v}_0 + \eta(\rho) \nabla_x(d_\rho \hat{v}_1) \\
+ 2\eta(\rho) \nabla_f h_2 \cdot \nabla_x(d_\rho \hat{v}_0) + 2\eta'(\rho) \nabla_f h_1 \cdot \nabla_x(d_\rho \hat{v}_1) - 2\eta(\rho) \partial_{\rho} \hat{v}_2 + n \partial_{\rho} \hat{v}_2 \\
+ \nabla_x \hat{p}_1 - \eta(\rho)(\rho \hat{v}_1 + h_2 \hat{v}_0) \Delta_x d_\rho - \partial_{\rho} \hat{v}_1 - \Delta_x d_\rho \eta(\rho)(\rho \hat{v}_1 + h_2 \hat{v}_0) - \Delta_x \hat{v}_1 \\
= \left( \Delta_x d_\rho \nabla_f h_1 - 2(\nabla_f h_1 \cdot \nabla_f h_2) \mathbf{n} \right) \partial_{\rho} \hat{\theta}(\rho)^2 \\
+ \left( \Delta_f h_0 \nabla_f h_1 - \nabla_f h_1 \Delta_f h_1 + \Delta_f d_\rho \nabla_f h_2 - \nabla_f \hat{v}_1 \nabla_f h_1 \right) \nabla_f \hat{v}_1
\]

(A.13)

\[
O(\varepsilon^2) := \sum_{0 \leq i' \leq 2} \sum_{0 \leq i,j \leq 1} \hat{R}_0^{i,j}(x,t)(\partial_{\rho}^2 \hat{h}_2)^i(\partial_{\rho}^i \hat{h}_2)^j \\
+ \sum_{0 \leq i,j,k \leq 1} \hat{R}_0^{i,j,k}(x,t)(\partial_{\rho}^2 \hat{h}_2)^i(\partial_{\rho}^i \hat{h}_2)^j(\partial_{\rho}^k \hat{h}_2)^k
\]

(A.14)

where \( \hat{R}_0^{i,j}, \hat{R}_0^{i,j,k} \) are uniformly bounded with respect to \( \varepsilon \in (0,1], (x,t) \in \Gamma(3\delta) \).

**Proof:** Since \( \hat{c}_0^{\varepsilon} := \theta_0(\rho), \) it follows from (A.17) that:

\[
-\varepsilon \frac{\partial}{\partial \rho} \left( \nabla \theta_0(\rho \hat{v}_2 - \hat{h}_2) \otimes \nabla \theta_0(\rho \hat{v}_2 - \hat{h}_2) \right) = -\varepsilon \Delta \theta_0(\rho \hat{v}_2 - \hat{h}_2) \nabla \theta_0(\rho \hat{v}_2 - \hat{h}_2) - 2\varepsilon \nabla \theta_0(\rho \hat{v}_2 - \hat{h}_2)^2 \\
= - \left( \frac{\partial}{\partial \rho} \theta_0(\rho \hat{v}_2 - \hat{h}_2) \right)^2 + \theta_0(\rho \hat{v}_2 - \hat{h}_2)^2 \Delta \theta_0(\rho \hat{v}_2 - \hat{h}_2) \left( \mathbf{n} - \varepsilon \nabla h_1 \right) \theta_0(\rho) \\
- \frac{\varepsilon}{2} \left( \nabla \theta_0(\rho \hat{v}_2 - \hat{h}_2) \right)^2 \theta_0(\rho \hat{v}_2 - \hat{h}_2)^2 \\
= - \left( \frac{\partial}{\partial \rho} \theta_0(\rho \hat{v}_2 - \hat{h}_2) \right)^2 \mathbf{n} + \theta_0(\rho \hat{v}_2 - \hat{h}_2)^2 \nabla \theta_0(\rho \hat{v}_2 - \hat{h}_2)^2 \mathbf{n} \\
- \theta_0(\rho \hat{v}_2 - \hat{h}_2)^2 \Delta \theta_0(\rho \hat{v}_2 - \hat{h}_2)^2 \mathbf{n} \\
+ \varepsilon \theta_0(\rho \hat{v}_2 - \hat{h}_2)^2 \left( \Delta \theta_0(\rho \hat{v}_2 - \hat{h}_2)^2 \mathbf{n} \right) \theta_0(\rho \hat{v}_2 - \hat{h}_2)^2 \\
+ \varepsilon \theta_0(\rho \hat{v}_2 - \hat{h}_2)^2 \left( \Delta \theta_0(\rho \hat{v}_2 - \hat{h}_2)^2 \mathbf{n} \right) \theta_0(\rho \hat{v}_2 - \hat{h}_2)^2 \\
+ \varepsilon^2 \sum_{i,j,k \leq 1} \hat{R}_0^{i,j,k}(x,t) \left( \partial_{\rho}^2 \hat{h}_2 \right)^i \left( \partial_{\rho}^i \hat{h}_2 \right)^j \left( \partial_{\rho}^k \hat{h}_2 \right)^k
\]

where \( \hat{R}_0^{i,j}, \hat{R}_0^{i,j,k} \) are uniformly bounded with respect to \( \varepsilon \in (0,1], (x,t) \in \Gamma(3\delta) \). On the other hand,

\[
\Delta \hat{v}_j = \left( \frac{\partial}{\partial \rho} + \nabla h_1 \hat{v}_j \right) \mathbf{n} \partial_{\rho} \hat{v}_j + \left( \Delta \hat{v}_j \right) \partial_{\rho} \hat{v}_j \\
= 2(\nabla h_1 \cdot \nabla h_1) + \Delta \hat{v}_j + \Delta \hat{v}_j, \quad 0 \leq j \leq 2
\]

\[
\nabla \hat{p}_j = \left( \frac{\partial}{\partial \rho} + \nabla h_1 \hat{v}_j \right) \partial_{\rho} \hat{p}_j + \nabla \hat{p}_j, \quad -1 \leq j \leq 1
\]

Furthermore we have

\[
\Delta(d_\rho \hat{v}_j(x,t) \eta(\rho)) = d_\rho \hat{v}_j \left( \frac{\partial}{\partial \rho} + \nabla h_1 \hat{v}_j \right) \eta(\rho) + \left( \Delta \hat{v}_j \right) \eta(\rho) \\
+ \eta(\rho) \Delta(d_\rho \hat{v}_j) + 2\eta'(\rho) \left( \frac{\partial}{\partial \rho} + \nabla h_1 \hat{v}_j \right) \nabla(d_\rho \hat{v}_j),
\]

\[
\nabla(d_\rho \hat{p}_j(x,t) \eta(\rho)) = \eta(\rho) \nabla_x(d_\rho \hat{p}_j) + d_\rho \hat{p}_j \left( \frac{\partial}{\partial \rho} + \nabla h_1 \hat{v}_j \right) \eta(\rho).
\]
If we substitute the above two sets of formulas into (A.10), we get the desired expansion. The last statement (A.14) follows from Lemma A.1 and the exponential decay of $\theta'_0, \theta''_0, \eta', \eta''$, $\partial_\rho \hat{\rho}_1, \partial_\rho \hat{\rho}_2$ and $\partial_\rho^2 \hat{v}_2$.

**A.1.2 Stokes equation for $v_0$ and $p_{-1}$:**

Taking the normal component of (A.10) and using (A.6) gives $\partial_\rho \tilde{p}_{-1} = -\partial_\rho (\theta'_0(\rho))^2$ and therefore

$$\tilde{p}_{-1}(\rho, x, t) = -\theta'_0(\rho)^2 + \tilde{p}_{-1}(x, t)$$

for some $\tilde{p}_{-1} : \Gamma(3\delta) \to \mathbb{R}$. The matching condition (A.4) implies that

$$\tilde{p}_{-1}(\rho, x, t) = p_{-1}(\rho, x, t) = -\theta'_0(\rho)^2, \quad \tilde{p}_{-1} = 0$$

for all $(x, t) \in \Gamma(3\delta)$. (A.15)

Now we go back to (A.10) and deduce that $-\partial_\rho^2 \hat{v}_0 = 0$ and this implies

$$\dot{v}_0(\rho, x, t) = \hat{v}_0(x, t) \quad \text{for all } (x, t) \in \Gamma(3\delta)$$

(A.16)

since the only bounded solution of $-\partial_\rho^2 \hat{v}_0 = 0$ is independent of $\rho$. This together with Lemma A.1 yields

$$\dot{v}_0 = \frac{1}{2}(v_0^+ + v_0^-) \text{ on } \Gamma(3\delta), \quad \dot{v}_0 = \frac{1}{2\eta}(v_0^+ - v_0^-) \text{ on } \Gamma(3\delta) \setminus \Gamma$$

(A.17)

and

$$v_0(\rho, x, t) = \frac{1}{2}(v_0^+(x, t) + v_0^-(x, t)) + \frac{\delta \rho}{\eta^2}(v_0^+(x, t) - v_0^-(x, t)), \quad \forall (x, t) \in \Gamma(3\delta) \setminus \Gamma, \rho \in \mathbb{R}.$$  

On the interface $\Gamma$ the matching condition yields

$$\lim_{\rho \to \pm \infty} \dot{v}_0(\rho, x, t) = \lim_{\rho \to \pm \infty} v_0(\rho, x, t) = v_0^\pm(x, t) \quad \text{for all } (x, t) \in \Gamma.$$  

Hence we have

$$[v_0^+](x, t) := v_0^+(x, t) - v_0^-(x, t) = 0, \quad \text{for all } (x, t) \in \Gamma$$

(A.18)

and it follows from Lemma A.1 that

$$\dot{v}_0 = \frac{1}{\eta} \partial_\rho (v_0^+ - v_0^-) \text{ on } \Gamma.$$  

(A.19)

**A.1.3 Divergence equation for $v_1$:**

Using (A.10) and (A.6), we obtain at order $O(1)$ of the expansion for $\text{div} \hat{v}_1^\rho$ that:

$$(\rho + h_1)\eta'(\rho)\hat{v}_{0,n}(x, t) - d \eta'(\rho) \nabla^\tau h_1 \cdot \hat{v}_{0,n}(x, t)$$

$$+ \text{div}_x (\hat{v}_0(\rho, x, t) + \hat{v}_0(x, t) d \eta(\rho)) + \partial_\rho \hat{v}_{1,n}(\rho, x, t) = 0$$

(A.20)

By restricting (A.20) on $\Gamma$ we can show that $\partial_\rho \hat{v}_{1,n}(\rho, x, t) = 0$, which implies

$$\hat{v}_{1,n}(\rho, x, t) = \hat{v}_{1,n}(x, t) \text{ on } \Gamma,$$

(A.21)

by showing that

$$(\rho + h_1)\eta'(\rho)\hat{v}_{0,n}(x, t) + \text{div}_x (\hat{v}_0(x, t) + \hat{v}_0(x, t) d \eta(\rho)) = 0 \text{ on } \Gamma.$$
Actually, it follows from (A.18), (A.19) and \(\text{div} \mathbf{v}_\pm^n = 0\) that
\[
2\mathbf{v}_0,n = \partial_n (\mathbf{v}_0^n - \mathbf{v}_0^n) = \text{div}_\nu [\mathbf{v}_0^n] = 0 \quad \text{on } \Gamma
\] (A.22)
and from (A.17) that
\[
\text{div}_\nu \mathbf{v}_0(x,t) = \frac{1}{2}(\text{div}_\nu \mathbf{v}_0^n + \text{div}_\nu \mathbf{v}_0^n) = 0 \quad \text{on } \Gamma.
\]
Hence it follows from (A.21) and (A.23) that \(\mathbf{v}_1,n(x,t) = \mathbf{v}_1^n(x,t)\) and therefore
\[
[\mathbf{v}_1^n] := \mathbf{v}_{1,n} - \mathbf{v}_{1,n} = 0 \quad \text{on } \Gamma.
\]
A similar result holds on \(\Gamma(3\delta) \setminus \Gamma\) following from (A.17), (A.5) and the extension process afterwards:
\[
\text{div}_\nu (\mathbf{v}_0 + \delta_t \mathbf{v}_0 \eta(\rho)) = \frac{1}{2} \text{div}_\nu (\mathbf{v}_0^n + \mathbf{v}_0^n) + \frac{1}{2} \eta(\rho) \text{div}_\nu (\mathbf{v}_0^n - \mathbf{v}_0^n) = 0.
\]
Since the first two terms on the left hand side of (A.20) vanish on \(\Gamma\), we can compensate them in (A.6) by terms with higher powers of \(\varepsilon\) using \(1 = \varepsilon \frac{\partial}{\partial \varepsilon}\). Therefore we simply solve
\[
\partial_\nu \mathbf{v}_1,n(\rho, x, t) = 0 \quad \text{on } \Gamma(3\delta) \setminus \Gamma.
\] (A.23)
Thus we obtain from Lemma A.1 that
\[
\mathbf{v}_1,n = \frac{1}{2} (\mathbf{v}_1^n + \mathbf{v}_1^n), \quad \mathbf{v}_1,n = \frac{1}{2\varepsilon} (\mathbf{v}_1^n - \mathbf{v}_1^n) \quad \text{on } \Gamma(3\delta) \setminus \Gamma
\] (A.24)
and the \(O(\varepsilon)\) order equation of (A.6) becomes:
\[
\frac{\mu + h_1}{\delta_t} \eta'(\rho) \mathbf{v}_0,n - (\rho + h_1) \eta'(\rho) \nabla^T h_1 \cdot \mathbf{v}_{0,\tau} - \nabla^T h_1 \cdot \partial_n \mathbf{v}_{1,\tau} + h_2 \eta'(\rho) \mathbf{v}_0,n + (\rho + h_1) \eta'(\rho) \mathbf{v}_1,n - \delta_t \eta'(\rho) \nabla^T h_2 \cdot \mathbf{v}_{0,\tau} + \text{div}_\nu (\mathbf{v}_1 + \delta_t \delta_\nu \eta(\rho)) + \partial_n \mathbf{v}_2,n = 0.
\] (A.25)

### A.1.4 Stokes equation for \(\mathbf{v}_1\) and \(p_0\):

We substitute (A.15) and (A.17) into (A.11). This leads to
\[
O(1) : -\partial^2_\nu \mathbf{v}_1 + \mathbf{n} \partial_\nu \mathbf{p}_0 - \mathbf{v}_0(\rho + h_1) \eta''(\rho) - 2\delta_\nu (\delta_t \mathbf{v}_0) \eta'(\rho) = - (\theta'_0(\rho))^2 \mathbf{n} \Delta d_\tau \text{ in } \Gamma(3\delta). \quad (A.26)
\]
Restricting (A.20) on \(\Gamma\), integrating with respect to \(\rho \in \mathbb{R}\), and using (A.3), (A.4), (A.15) and the last formula in (2.4) imply
\[
\mathbf{n}[\mathbf{p}_0] - 2 \mathbf{v}_0 = \sigma H \mathbf{n} \quad \text{on } \Gamma.
\]
To proceed, we need the following lemma

**Lemma A.3** Let \(j = 0, 1\). Under the condition that \([\mathbf{v}_j^n] := \mathbf{v}_j^n - \mathbf{v}_j^n = 0 \quad \text{on } \Gamma\), it holds that
\[
[\mathbf{p}_j^n] = [\mathbf{p}_0], \quad 2 \mathbf{v}_j = [\partial_n \mathbf{v}_j^n] = 2 \left( D \mathbf{v}_j^n - D \mathbf{v}_j^n \right) \quad \text{on } \Gamma.
\]

**Proof:** We shall only prove \([\partial_n \mathbf{v}_j^n] = 2 \left( D \mathbf{v}_j^n - D \mathbf{v}_j^n \right)\) since the others are consequences of the matching condition (A.4) and Lemma A.1 (see also (A.33)). On \(\Gamma\) we have
\[
2(D \mathbf{v}_j^n - D \mathbf{v}_j^n) \mathbf{n} = \partial_n (\mathbf{v}_j^n - \mathbf{v}_j^n) - (\nabla \mathbf{n}) \cdot (\mathbf{v}_j^n - \mathbf{v}_j^n) + \nabla (\mathbf{v}_j^n - \mathbf{v}_j^n)
= \partial_n (\mathbf{v}_j^n - \mathbf{v}_j^n) + \mathbf{n} \partial_n (\mathbf{v}_j^n - \mathbf{v}_j^n) + \mathbf{\tau} \partial_\nu (\mathbf{v}_j^n - \mathbf{v}_j^n)
\]
The last two components vanish due to \([\mathbf{v}_j^n] = 0\) and \(\text{div} \mathbf{v}_j^n = 0\).

These altogether imply
\[
2(D \mathbf{v}_0^n - D \mathbf{v}_0^n) \mathbf{n} - (\mathbf{p}_0^n - \mathbf{p}_0^n) \mathbf{n} = - \sigma H \mathbf{n} \quad \text{on } \Gamma.
\] (A.27)
Due to (A.29), the last two terms on the left hand side of (A.26) vanish on $\Gamma$ and we omit them and get
\begin{equation}
\begin{aligned}
&-\Delta \mathbf{v}_0^+ + \nabla p_0^+ = 0 \quad \text{in } \Omega^\pm(t), \ t \in (0, T_0), \\
&\operatorname{div} \mathbf{v}_0^+ = 0 \quad \text{in } \Omega^\pm(t), \ t \in (0, T_0), \\
&[2D\mathbf{v}_0^+ - p_0^+ \mathbf{n}] = -\sigma H \mathbf{n} \quad \text{on } \Gamma, \ t \in (0, T_0), \\
&[\mathbf{v}_0^+] = 0 \quad \text{on } \Gamma, \ t \in (0, T_0), \\
&\mathbf{v}_0|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega \times (0, T_0).
\end{aligned}
\end{equation}

Now we can take the inner product of (A.29) with $\mathbf{n}$ and use the above equations together with (A.22), (A.21), and (A.17) and this implies
\begin{equation}
\partial_\rho \tilde{p}_0 = (\theta_0'(\rho))^2 H \quad \text{on } \Gamma. \tag{A.28}
\end{equation}

We integrate the above formula, use the matching condition, and obtain
\begin{equation}
\tilde{p}_0(\rho, x, t) = H \int_{-\infty}^\rho (\theta_0'(z))^2 dz + p_0^-(x, t) \quad \text{on } \Gamma.
\end{equation}

Taking $\rho \to +\infty$ in the above formula and combining it with (A.19) and (A.27) leads to
\begin{equation}
2\tilde{\mathbf{v}}_0 = \partial_\mathbf{n} \mathbf{v}_0^+ - \partial_\mathbf{n} \mathbf{v}_0^- = 0, \ p_0^+ - p_0^- = \sigma H \quad \text{on } \Gamma. \tag{A.29}
\end{equation}

Using this, (A.16) and (A.28), we deduce from (A.26) that
\begin{equation}
\partial^2_\rho \tilde{\mathbf{v}}_1(\rho, x, t) = 0 \quad \text{on } \Gamma.
\end{equation}

Hence we can use (A.3) to deduce $\partial_\rho \tilde{\mathbf{v}}_1 = 0$ and
\begin{equation}
[\mathbf{v}_1] = \mathbf{v}_1^+ - \mathbf{v}_1^- = 0 \quad \text{on } \Gamma. \tag{A.30}
\end{equation}

Due to (A.29), the last two terms on the left hand side of (A.26) vanish on $\Gamma$ and we omit them and get
\begin{equation}
O(\epsilon^2) : -\partial^2_\rho \tilde{\mathbf{v}}_1 + \mathbf{n} \partial_\rho \tilde{p}_0 = -((\theta_0'(\rho))^2 n) \Delta d_\Gamma \quad \text{in } \Gamma(T_0)
\end{equation}
but we have to compensate the omitted terms into (A.12) and (A.13). Meanwhile, we use (A.15) and (A.17) to simplify (A.12) into:
\begin{equation}
O(1) : -\partial^2_\rho \tilde{\mathbf{v}}_2 - \eta''(\rho)(h_2 \mathbf{v}_0 + (\rho + h_1) \tilde{\mathbf{v}}_1) + \eta'(\rho) \tilde{p}_0 \mathbf{n}(\rho + h_1) \\
+ \eta'(\rho) (\nabla_x (\partial \tilde{\mathbf{v}}_0) - \Delta x \mathbf{v}_0) - \Delta_x \mathbf{v}_0 + \nabla_h \tilde{p}_0 + 2\eta'(\rho) \nabla^\Gamma \mathbf{h}_1 \cdot (\nabla_x (\partial \mathbf{v}_0) \\
- 2\eta'(\rho) \partial_\mathbf{n}(\partial \tilde{\mathbf{v}}_1) - \eta''(\rho) (h_1 \mathbf{v}_0 + \tilde{p}_0 \hat{\mathbf{n}}) \Delta_x d_\Gamma \\
- \frac{\rho}{2\eta'} \partial_\rho ((\rho + h_1)^2 \eta'(\rho) - 2(\rho + h_1) \partial_\rho \tilde{\mathbf{v}}_0 \eta'(\rho))
\end{equation}
new terms due to compensation
\begin{equation}
= -\partial_\rho (\theta_0'(\rho))^2 [\nabla^\Gamma \mathbf{h}_1]^2 \mathbf{n} + (\theta_0'(\rho))^2 \left( \Delta d_\Gamma \nabla^\Gamma \mathbf{h}_1 + \Delta^\Gamma \mathbf{h}_1 \mathbf{n} \right),
\end{equation}

while the following terms must be added to (A.13):
\begin{equation}
- \frac{\rho}{2\eta'} h_2 (\rho + h_1) \eta''(\rho) - 2\frac{\rho}{\eta'} h_2 \eta'(\rho) - 2h_2 \partial_\mathbf{n} \tilde{\mathbf{v}}_0 \eta'(\rho). \tag{A.33}
\end{equation}

More precisely, the last two terms on the left-hand side of (A.26) can be treated as follows:
\begin{equation}
- \mathbf{v}_0(\rho + h_1) \eta''(\rho) - 2\partial_\mathbf{n}(\partial \mathbf{v}_0) \eta'(\rho) = -\frac{\rho}{\eta'} \partial_\rho \left( \mathbf{v}_0(\rho + h_1) \eta''(\rho) + 2\partial_\mathbf{n}(\partial \mathbf{v}_0) \eta'(\rho) \right) \\
eq \varepsilon \left( \frac{\rho}{\eta'} \partial_\rho ((\rho + h_1)^2 \eta'(\rho) - 2(\rho + h_1) \partial_\mathbf{n} \tilde{\mathbf{v}}_0 \eta'(\rho)) \right) \\
+ \varepsilon^2 \left( \frac{\rho}{\eta'} h_2 (\rho + h_1) \eta''(\rho) - 2\frac{\rho}{\eta'} h_2 \eta'(\rho) - 2h_2 \partial_\mathbf{n} \tilde{\mathbf{v}}_0 \eta'(\rho) \right).
\end{equation}

Now we take the normal component of (A.31) and employ (A.23). This leads to
\begin{equation}
\partial_\rho \tilde{p}_0 + (\theta_0'(\rho))^2 \Delta d_\Gamma = 0 \tag{A.34}
\end{equation}
and this together with matching conditions (A.1) implies
\[ \tilde{p}_0 = \frac{1}{\partial_t} (p_0^+ - p_0^- + \sigma \Delta d t) \cdot \tilde{p}_0 = \frac{1}{2} (p_0^+ + p_0^-) - \frac{\sigma}{2} \Delta d t \eta (\rho) \text{ in } \Gamma (3 \delta) \setminus \Gamma \] (A.35)
where \( \sigma = \int_\Gamma (\theta (s)) d s \). Here we have used \( \frac{\sigma}{2} \partial_t \eta (\rho) = \theta (\rho) \eta (\rho) \) by the definition of \( \eta \). Thus
\[ p_0 = \frac{1}{2} (p_0^+ + p_0^-) + \frac{\Delta d t}{2} (p_0^+ - p_0^-) \] (A.36)
So (A.31) reduces to \( \partial_t^2 \tilde{v}_1 = 0 \) in \( \Gamma (3 \delta) \). Since the only bounded solution of this equation is independent of \( \rho \), we can use Lemma A.1 to deduce
\[ \hat{v}_1 = \frac{1}{2 \sigma (\rho + h_1)^2} (v_1^+ - v_1^-), \quad \tilde{v}_1 = \frac{1}{2} (v_1^+ + v_1^-), \quad v_1 = \frac{1}{2} (v_1^+ + v_1^-) + \frac{\rho (\rho)}{2} (v_1^+ - v_1^-) \] (A.37)
This combined with (A.30) implies
\[ 2 \tilde{v}_1 = \partial_\eta (v_1^+ - v_1^-) \quad \text{on } \Gamma. \] (A.38)

### A.1.5 Divergence equation for \( v_2 \):

Now consider the order \( O (\varepsilon) \) equation for the divergence-free equation (A.25). Restricting (A.25) on \( \Gamma \) and using (A.17), (A.37), (A.22), (A.30) we conclude that
\[ \lim_{d t \to 0} \frac{\partial_\eta (\rho + h_1)^2 \eta (\rho) + (\rho + h_1) \eta (\rho) \tilde{v}_1, n + \partial_\rho \tilde{v}_2, n = 0 \quad \text{on } \Gamma. \] (A.39)
Note that the second term in the above equation also vanishes due to (A.38), (A.30) and the divergence-free condition of \( v_1^+ \):
\[ \tilde{v}_1, n = \frac{1}{2} \partial_\eta (v_1^+, n - v_1^-, n) = - \frac{1}{2} \text{div}_\tau (v_1^+ - v_1^-) = 0 \quad \text{on } \Gamma. \] (A.40)
Using this in (A.39) and employing (A.22), we obtain
\[ \partial_\rho \tilde{v}_2, n (\rho, x, t) + (\rho + h_1)^2 \eta (\rho) \partial_\rho \tilde{v}_0, n = 0 \quad \text{on } \Gamma. \] (A.41)
In order to determine \( v_2^+ \), instead of solving a certain sharp interface system as we did for \( (v_1^+, p_1^+) \), we choose \( v_2^+ \equiv 0 \) and integrate (A.11):
\[ (v_2^+ - v_2^-) + \int_\Gamma (\rho + h_1)^2 \eta (\rho) d \rho \partial_\rho \tilde{v}_0, n = 0 \quad \text{on } \Gamma. \]
Altogether, we define
\[ v_2 \equiv 0 \text{ in } \Omega \times [0, T_0], \quad v_2 = - \partial_\eta \tilde{v}_0, n \int_\Gamma (\rho + h_1)^2 \eta (\rho) d \rho \quad \text{in } \Gamma (3 \delta) \] (A.42)
and extend \( v_2^+ \) smoothly to \( \Omega \times [0, T_0] \). Note that, in contrast to \( v_1^+ \), we do not assume \( v_2^+ \) to be divergence-free.

In view of (A.25), we can treat (A.29) in \( \Gamma (3 \delta) \) in the same manner as before by omitting the terms that vanish on the interface and just solve
\[ (\rho + h_1)^2 \eta (\rho) \frac{\partial v_2, n}{\partial t} + \partial_\rho \tilde{v}_2, n = 0 \quad \text{in } \Gamma (3 \delta). \] (A.43)
This together with (A.37), (A.17), (A.30) and the extension process after it will change the expansion of the divergence equation (A.8) into
\[ \text{div } \left( v_2^+ (\rho^+ - h_1, x, t) \right) = O (\varepsilon) : \quad - (\rho + h_1) \eta (\rho) \nabla h_1 \cdot \tilde{v}_0, \tau - d_t \eta (\rho) \nabla h_1 \cdot \tilde{v}_1, \tau + (\rho + h_1) \eta (\rho) \tilde{v}_1, n \]
\[ + h_2 \eta (\rho) \nabla h_2 \cdot \tilde{v}_0, \tau \]
\[ + O (\varepsilon^2) : \quad + (\rho + h_1) h_2 \eta (\rho) \frac{\partial v_0, n}{\partial t} - h_2 \eta (\rho) \nabla h_1 \cdot \tilde{v}_0, \tau + h_2 \eta (\rho) \tilde{v}_1, n - d_t \eta (\rho) \nabla h_2 \cdot \tilde{v}_1, \tau \]
\[ - \nabla \tau h_1 \cdot \partial_\rho \tilde{v}_2, \tau + (\rho + h_1) \eta (\rho) \tilde{v}_2, n (x, t) - d_t \eta (\rho) \nabla h_2 \cdot \tilde{v}_2, \tau (x, t) + \text{div}_\tau (v_2 (\rho, x, t) d \eta (\rho)) \]
\[ + O (\varepsilon) \] (A.44)
as remainder terms at the corresponding orders.

The above analysis together with (A.3) allow us to solve \( \tilde{v}_2, n \) and \( \tilde{v}_2, n \) uniquely:
Lemma A.4 The normal component of $\tilde{v}_2$ and $v_2$ are given by:

$$\tilde{v}_{2,n} = v_{2,n} + dr\dot{v}_{2,n} - \frac{v_{0,n}}{dt} \int_{-\infty}^{0} (z + h_1)\eta'(z)dz$$
on
$$\dot{v}_{2,n} = \frac{1}{2dt} \left[ \frac{v_{0,n}}{dt} \int_{-\infty}^{0} (z + h_1)\eta'(z)dz + v^+_2 n - v^-_2 n \right]$$
on
where $v^+_2 n$ is given by (A.42) and is extended smoothly to $\Omega \times [0, T_0]$.

A.1.6 Stokes equation for $v_2$ and $p_1$:

According to (A.17) and (A.35)

$$\Delta_x \tilde{v}_0 + \Delta_x (dr \tilde{v}_0) \eta(\rho) = \Delta_x v_0^+ \frac{1+\eta'(\rho)}{2} + \Delta_x v_0^- \frac{1-\eta'(\rho)}{2},$$
$$\nabla_x \tilde{p}_0 + \nabla_x (dr \tilde{p}_0) \eta(\rho) = \nabla_x p_0^+ \frac{1+\eta'(\rho)}{2} + \nabla_x p_0^- \frac{1-\eta'(\rho)}{2}.$$ 

Thus

$$- \Delta_x \tilde{v}_0 - \Delta_x (dr \tilde{v}_0) \eta(\rho) + \nabla_x \tilde{p}_0 + \nabla_x (dr \tilde{p}_0) \eta(\rho) = (- \Delta_x v_0^+ + \nabla_x p_0^+) \frac{1+\eta'(\rho)}{2} + (- \Delta_x v_0^- + \nabla_x p_0^-) \frac{1-\eta'(\rho)}{2}.$$ 

This together with (A.17), (A.34), (A.35), and (A.37) reduces (A.32) to

$$O(1) : - \partial^2_x \tilde{v}_2 - \eta''(\rho) (h_2 \tilde{v}_2 + (\rho + h_1) \tilde{v}_1) + \eta'(\rho) \tilde{p}_0 n (\rho + h_1) + n \delta x \tilde{p}_1$$
$$- (\Delta_x v_0^+ + \nabla_x p_0^+) \frac{1+\eta'(\rho)}{2} + (\Delta_x v_0^- + \nabla_x p_0^-) \frac{1-\eta'(\rho)}{2} + 2\eta'(\rho) \nabla^T h_1 \cdot \nabla_x (dr \tilde{v}_0)$$
$$- 2\eta'(\rho) \partial_x \tilde{v}_0 (dr x) - (h_1 \tilde{v}_0 + \tilde{v}_0 \rho) \eta'(\rho) \Delta_x d r$$
$$- \frac{\delta}{\delta x} \tilde{v}_0 ((\rho + h_1)^2 \eta' (\rho)) - 2(\rho + h_1) \partial_x \tilde{v}_0 \eta'(\rho)$$
$$= - \partial_x (\theta_0 (\rho))^2 \nabla^T h_1^2 n + (\theta_0 (\rho))^2 \Delta^T h_1 n.$$ 

Restricting (A.46) on $\Gamma$ and using (A.29) to eliminate the terms containing $\tilde{v}_0$ we derive

$$- \partial^2_x \tilde{v}_2 - \eta''(\rho) (\rho + h_1) \tilde{v}_1 + \eta'(\rho) (\rho + h_1) n \tilde{p}_0 + n \delta x \tilde{p}_1 - 2\eta'(\rho) \tilde{v}_1 - \partial_x \tilde{v}_0 \rho (\rho + h_1)^2 \eta'(\rho)$$
$$- 2(\rho + h_1) \partial_x \tilde{v}_0 \eta'(\rho) = - \partial_x (\theta_0 (\rho))^2 \nabla^T h_1^2 n + (\theta_0 (\rho))^2 \Delta^T h_1 n$$
on
at order $O(1)$. Integrating the above identity with respect to $\rho \in \mathbb{R}$ and then using (A.33) and (A.4) yields

$$- 2\tilde{v}_1 + n [\tilde{p}_1] + 2h_1 n \tilde{p}_0 - 4h_1 \partial_x \tilde{v}_0 = \sigma \Delta^T h_1 n$$
on
Now Lemma A.3 together with (A.38) and (A.48) implies

$$\left[ 2D \tilde{v}^+_2 - \tilde{p}_1 \Gamma \right] \cdot n = 2h_1 n \tilde{p}_0 - 4h_1 \partial_x \tilde{v}_0 - \sigma \Delta^T h_1 n$$

These together with (A.30) and the outer expansion (A.5) leads to the second order sharp interface limit (A.13).

Now we determine $\tilde{v}_{2,r}$ and $\tilde{p}_1$ such that the tangential part of (A.46) is fulfilled on $\Gamma$ and the normal part is fulfilled on $\Gamma'(3\delta)$, up to a high order term in $\varepsilon$. From the normal part of (A.46) we obtain

$$\partial_r \tilde{p}_1 (\rho, x, t) = \partial^2_r \tilde{v}_{2,n} + \eta''(\rho) (h_2 \tilde{v}_{0,n} + (\rho + h_1) \tilde{v}_{1,n}) - \eta'(\rho) \tilde{p}_0 (\rho + h_1)$$
$$- 2\eta'(\rho) \nabla^T h_1 \cdot \nabla_x (dr x \tilde{v}_{0,n}) + 2\eta'(\rho) \partial_x (dr x \tilde{v}_{1,n}) + (\rho + h_1) \tilde{v}_{0,n} \eta'(\rho) \Delta_x d r$$
$$+ \frac{v_{0,n}}{dt} \partial_x ((\rho + h_1)^2 \eta'(\rho)) + 2(\rho + h_1) \partial_x v_{0,n} \eta'(\rho)$$
$$- \partial_x (\theta_0 (\rho))^2 \nabla^T h_1^2 + (\theta_0 (\rho))^2 \Delta^T h_1 =: a_1 (\rho, x, t)$$
on
Note that we omitted the lower order term

$$\left( (- \Delta_x v^+_0 + \nabla_x p^+_0) \frac{1+\eta'(\rho)}{2} + (- \Delta_x v^-_0 + \nabla_x p^-_0) \frac{1-\eta'(\rho)}{2} \right) \cdot n.$$
that will be treated in (A.56) below. Furthermore, \( \lim_{\rho \to \pm \infty} \tilde{p}_1(\rho, x, t) = p_\pm^1(x, t) \) for \( x \in \Omega^\pm(t) \) because of (A.4). Hence we define for \((x, t) \in \Gamma(\delta^3)\)

\[
\tilde{p}_1(\rho, x, t) = \frac{1}{2} \left( p_1^+ (x, t) + p_1^- (x, t) - \int_{\mathbb{R}} a_1(z, x, t) dz \right) + \int_{-\infty}^{\rho} a_1(z, x, t) dz,
\]

(A.51)

where \( p_\pm^1 \) is determined by (3.13). Restricting (A.50) on the interface and integrating with respect to \( \rho \) leads to the following compatibility condition:

\[
p_1^+(P_{\Gamma_1}(x, t)) - p_1^-(P_{\Gamma_1}(x, t)) = \int_{\mathbb{R}} a_1(z, P_{\Gamma_1}(x, t)) dz. \tag{A.52}
\]

In the rest part of this subsection, we shall determine \((\tilde{v}_2, \tilde{v}_3)\), the tangential part of \((\tilde{v}_2, \tilde{v}_3)\). The tangential part of (A.48) implies \( v_{1, \rho} = -2h_1 \partial_n \tilde{v}_{0, \rho} \) on \( \Gamma \) and we can employ it to simplify the equation for \( v_{2, \rho} \) on \( \Gamma \)

\[
-\partial_n^2 \tilde{v}_{2, \rho} = \eta''(\rho)(\rho + h_1) v_{1, \rho} + 2\eta'(\rho) v_{1, \rho} + \partial_n \tilde{v}_{0, \rho} \partial_n \rho \left( (\rho + h_1)^2 \eta'(\rho) \right) + 2(\rho + h_1) \partial_n \tilde{v}_{0, \rho} \eta'(\rho) \\
= \left( (\rho^2 - h_1^2)^2 \eta''(\rho) + 4\rho \eta'(\rho) \right) \partial_n \tilde{v}_{0, \rho} \tag{A.53}
\]

To proceed, let us note that \( \xi(z) := \int_{-\infty}^z \rho \eta'(\rho) d\rho \) and \( \xi'(\rho) \) belongs to \( \mathcal{X}_{0, \alpha} \) for some \( \alpha > 0 \). This follows easily from the exponential decay of \( \xi'(\rho) = \rho \eta'(\rho) \) and \( \int_{-\infty}^\rho \rho \eta'(\rho) d\rho = 0 \) since \( \eta' \) is even. Hence we can integrate (A.53) to obtain \( \tilde{v}_{2, \rho} \):

\[
\partial_n \tilde{v}_{2, \rho} = -\int_{-\infty}^\rho \left( (z^2 - h_1^2) \eta''(z) + 4z \eta'(z) \right) dz \partial_n \tilde{v}_{0, \rho} \quad \text{on} \quad \Gamma.
\]

Therefore we define \( \tilde{v}_{2, \rho} \) on \( \Gamma(\delta^3) \) through

\[
\tilde{v}_{2, \rho}(\rho, x, t) = -\partial_n \tilde{v}_{0, \rho}(\rho, P_{\Gamma_1}(x, t)) \int_{-\infty}^\rho \int_{-\infty}^y \left( (z^2 - h_1^2) \eta''(z) + 4z \eta'(z) \right) dz \, dy \tag{A.54}
\]

in \( \Gamma(\delta^3) \) and

\[
\tilde{v}_{2, \rho}(x, t) \equiv 0, \quad \tilde{v}_{2, \rho}^\pm(x, t) := \lim_{\rho \to \pm \infty} \tilde{v}_{2, \rho}(\rho, x, t) \quad \text{for all} \quad (x, t) \in \Gamma(\delta^3), \rho \in \mathbb{R}.
\]

Moreover, we extend \( \tilde{v}_{2, \rho}^\pm \) smoothly to \( \Omega \times [0, T_0] \). It can be verified that the above definitions are compatible with (A.53) and (A.2).

### A.2 Proof of Lemma 3.4

The proof will heavily rely on Lemma 3.2.

**Proof of (3.21):** It follows from (A.7) that the order \( O(\varepsilon^{-1}) \) is eliminated from (A.9). Then it follows from (A.23) that the order \( O(1) \) is eliminated from (A.9) and change the \( O(\varepsilon) \) order terms of (A.0) to be as in (A.11). All \( O(\varepsilon) \)-terms in (A.11) vanish on \( \Gamma \) and decay exponentially in \( \rho \). Hence it follows from Lemma 3.2 that they will be included in the term \( g_e \left( \frac{x^2}{\varepsilon^2} - h_1, x, t \right) \) with \( (g_e)_{0 < \varepsilon < 1} \in \mathcal{X}_{0, \alpha} \). Moreover, the \( O(\varepsilon^2) \) order terms in (A.15) can also be included in a term \( \varepsilon \tilde{g} \), where \( \tilde{g} \) is uniformly bounded with respect to \( (x, t) \in \Gamma(\delta^3), \varepsilon \in (0, 1] \). So we proved (3.21).

**Proof of (3.20):** The construction (A.15) and (A.17) fulfill (A.10) and thus eliminate the \( O(\varepsilon^{-7}) \) order terms in the expansion of (A.0). The formula (A.36) and (A.37) balance the \( O(\varepsilon^{-1}) \) order terms in (A.11) and change the \( O(1) \) order terms in (A.12) to (A.46). It can be verified that, in (A.46), the terms that vanish on \( \Gamma \), except

\[
(-\Delta \tilde{v}_0^\pm + \nabla_x p_0^\pm) \frac{1 + \eta'(\rho)}{2} + (-\Delta \tilde{v}_0^\pm + \nabla_x p_0^\pm) \frac{1 - \eta'(\rho)}{2}, \tag{A.55}
\]
can be included in $\mathcal{R}_{0,\alpha}^0$ and the rest terms are given in (A.47). According to the construction, Lemma [A.4] together with formula (A.51), (A.54) determines $\tilde{v}_2, v_2, \tilde{p}_1, \tilde{p}_1$. Moreover, they belong to the class $\mathcal{R}_{0,\alpha}^0$ and will be included in $r_c$.

Although (A.55) vanishes on $\Gamma$ according to the outer expansion (A.3), it does not decay in $\rho$ exponentially. To treat it, we replace the factor $\frac{1}{2}|\pm j\rho|\varepsilon$ by some smooth $\zeta_\rho: \mathbb{R} \to \mathbb{R}$ such that $|\zeta_\rho(\rho)| \leq C e^{-\alpha|\rho|}$ and

$$
\int_{-\infty}^{\infty} \zeta_\rho(\rho) \, d\rho = 0, \quad \zeta_\rho(\rho) = \frac{1}{2}\frac{1}{\varepsilon} |\pm j\rho| \varepsilon \quad \text{for all } \rho \leq \pm N.
$$

Then for $j = 0, 1,$

$$
(-\Delta \nu_j^\pm + \nabla \cdot \nu_j^\pm \varepsilon^{1+\eta(\rho)})_{\rho = \frac{d_2}{\varepsilon} - h_0} = (-\Delta \nu_j^\pm + \nabla \nu_j^\pm) \zeta_\rho(\rho)_{\rho = \frac{d_2}{\varepsilon} - h_0}
$$

(A.56)

since $(-\Delta \nu_j^\pm + \nabla \nu_j^\pm)_{|z(t)| = 0}$ and $\frac{d_2}{\varepsilon} - h_0 \leq \pm N$ in $\Omega(t)$ if $N > 0$ is chosen large enough. Hence the term in (A.55) equals to

$$
\sum_{\pm} (-\Delta \nu_j^\pm + \nabla \nu_j^\pm) \zeta_\rho(\rho) \in \mathcal{R}_{0,\alpha}^0.
$$

Using Lemma [A.4] and the previous step, one can verify that the terms in (A.13) as well as those in (A.33) belong to $\mathcal{R}_{0,\alpha}^0$ and taking into account their level, we can write them in the general form $(\tilde{r}_c)_{0 \leq i < 1} \in \mathcal{R}_{1,\rho}$. Finally, the terms $R_c^{ij}$ and $R_c^{ijk}$ in (3.22) come from (A.11) by multiplication with $\varepsilon^2$.

### A.3 Expansion of the Allen-Cahn Part

We shall consider the inner expansion of $c$ as follows:

$$
c^m(x, t) = c^m(\rho, s, t) = \theta_0(\rho) + \varepsilon^2 c_2(\rho, S(x, t), t) + \varepsilon^3 c_3(\rho, S(x, t), t) = c_0^m(x, t) + \varepsilon^2 c_2^m(x, t) + \varepsilon^3 c_3^m(x, t).
$$

(A.57)

where $\rho$ is defined by

$$
\rho = \frac{d_1(x)}{\varepsilon} - h_1(S(x, t), t) - \varepsilon h_2, s(S(x, t), t).
$$

(A.58)

Here and in the following $x$ and $(\rho, s)$ will always be related by (A.58) and $s = S(x, t)$ if both variables appear. Moreover, we will for simplicity write $h_2$ instead of $h_{2,s}$ in the following.

It follows from

$$
S (X_0(s, t) + r n(s, t)) = s \quad \text{for all } s \in \mathbb{T}^1, t \in [0, T_0], r \in (-3\delta, 3\delta)
$$

that, by differentiating with respect to $r$, \[(\nabla S)(x, t) \cdot n(S(x, t), t) = 0 \quad \text{for all } (x, t) \in \Gamma(3\delta)\]

Therefore it follows from (2.8) that

$$
\left| \frac{n(s, t)}{\varepsilon} - (\nabla^T h_z)(r, s, t) \right|^2 = \frac{1}{\varepsilon^2} + |\nabla^T h_z (r, s, t)|^2
$$

for all $r \in (-3\delta, 3\delta), s = S(x, t) \in \mathbb{T}^1, t \in [0, T_0]$. So we can employ the formula (2.11) and notation (2.11, 2.22) to get

$$
\begin{align*}
\partial_t c_0^m(x, t) &= \theta_0(\rho) \left( \frac{\nu(x, t)}{\varepsilon} - \partial_1 h_z(r, s, t) \right) \\
\nabla c_0^m(x, t) &= \theta_0(\rho) \left( \frac{\nu(x, t)}{\varepsilon} - \nabla^T h_z(r, s, t) \right) \\
\Delta c_0^m(x, t) &= \theta_0(\rho) \left( \frac{\nu(x, t)}{\varepsilon^2} + |\nabla^T h_z(r, s, t)|^2 \right) + \theta_0(\rho) \left( \frac{\Delta \nu(x, t)}{\varepsilon^2} - \Delta^T h_z(r, s, t) \right).
\end{align*}
$$

(A.59)
A.3 Expansion of the Allen-Cahn Part

Here and in the following all functions as e.g. $\theta_0$, $\tilde{c}_2$, $h_1$, $\nabla^T \tilde{c}_2$, $\nabla^T h_1$ without arguments are evaluated at $\rho, (\rho, s, t)$, $(s, t)$, $(r, \rho, s, t)$, $(r, s, t)$, respectively. Moreover, Taylor expansion yields

$$\frac{1}{\varepsilon^2} f'(\varepsilon c_m(x, t)) = \frac{1}{\varepsilon^2} f'(\theta_0(\rho)) + f''(\theta_0(\rho))\tilde{c}_2(\rho, s, t) + \varepsilon f'''(\theta_0(\rho))\tilde{c}_3(\rho, s, t) + \mathcal{R}_1,$$

where

$$\mathcal{R}_1 = \frac{\varepsilon^2}{2} f'''(\theta_0(\rho) + \xi(\rho, s, t)(\varepsilon^2 \tilde{c}_2 + \varepsilon^3 \tilde{c}_3)(\rho, s, t)) (\tilde{c}_2(\rho, s, t) + \varepsilon \tilde{c}_3(\rho, s, t))^2$$

for some $\xi(\rho, s, t) \in [0, 1]$. This together with (A.59) and (4.17) applied to $\tilde{c}_2, \tilde{c}_3$ leads to

$$\begin{align*}
\partial_t c_m(x, t) - \nabla c_m(x, t) + \varepsilon^2 X_0^*(\mathbf{w}_1) \cdot \nabla c_0^m(x, t) + \frac{1}{\varepsilon^2} f'(\varepsilon c_m(x, t)) \\
= f'(\theta_0(\rho) - \theta_0''(\rho) \mathcal{D}_h(\rho, s, t)) - \nabla f'(\varepsilon c_m(x, t)) \\
= f'(\theta_0(\rho) - \theta_0''(\rho) \mathcal{D}_h(\rho, s, t)) - \nabla f'(\varepsilon c_m(x, t)) \\
= f'(\theta_0(\rho) - \theta_0''(\rho) \mathcal{D}_h(\rho, s, t)) - \nabla f'(\varepsilon c_m(x, t)) \\
+ \varepsilon (\nabla^T h_2 - h_2) \cdot \nabla R_1(\rho, s, t) + \mathcal{R}_1 + \sum_{k=2}^5 \varepsilon^k \mathcal{R}_k + \varepsilon^3 (\mathcal{D}_h - \Delta \varepsilon^3) \tilde{c}_3.
\end{align*}$$

(A.60)

where $\mathcal{R}_k, k = 2, \ldots, 5$, are defined as in the proof of Theorem 4.3.

First we eliminate the terms of order $O(\varepsilon^{-1})$ on the right hand side of (A.60). Because of (2.10) and (4.10), we have

$$V + \Delta \mathcal{D}_r(\mathbf{x}, t) - \mathbf{n} \cdot X_0^*(\mathbf{v}) = -\varepsilon(\rho + h) \kappa_1(s, t) + \varepsilon^2 \kappa_2(s, t)(\rho + h) + \varepsilon^3 \kappa_3(s, t).$$

(A.61)

Hence, using Lemma 4.3 we arrive at

$$\begin{align*}
\partial_t c_m(x, t) + \nabla c_m(x, t) \cdot \nabla c_m(x, t) + \varepsilon^2 X_0^*(\mathbf{w}_1) \cdot \nabla c_0^m(x, t) - \Delta c_m(x, t) + \frac{1}{\varepsilon^2} f'(\varepsilon c_m(x, t)) \\
= \theta_0'(\rho + h) \kappa_1 + \varepsilon h_2 \kappa_1 + \varepsilon \kappa_2(\rho + h) - \nabla^T h_1(r, s, t) - \partial_{h_1} h_1(r, s, t) \\
+ \theta_0'(X_0^*(\mathbf{v}^T) + X_0^*(\mathbf{v}) \cdot \nabla h_1 + X_0^*(\mathbf{v}) \cdot L^T h_1 - X_0^*(\mathbf{v} \cdot \nabla h_1)) \\
- \partial_{h_1} h_1 \cdot f''(\theta_0(\rho)) \tilde{c}_2 - |\nabla^T h_1|^2 \theta_0'' - 2 \mathbf{v} \cdot \nabla h_1 \cdot L^T h_1 \theta_0'' - |L^T h_1|^2 \theta_0'' \\
+ \varepsilon [\nabla^T h_2 - \partial_{h_1} h_2 + X_0^*(\mathbf{n} \cdot \mathbf{w}_1)] \theta_0'' - (V + \Delta h_1(r, s, t) \partial_{h_1} h_2 - \partial_{h_1} h_1) \\
+ \varepsilon X_0^*(\mathbf{n} \cdot \mathbf{w}_1) \partial_{\mathbf{w}_1} h_1 + R_1(\rho, s, t) + \mathcal{R}_1 + \sum_{k=2}^5 \varepsilon^k \mathcal{R}_k + \varepsilon^3 (\mathcal{D}_h - \Delta \varepsilon^3) \tilde{c}_3
\end{align*}$$

(A.62)

where $\mathbf{v}$ solves (1.11) and $R_1(\rho, s, t)$ is given in Lemma 4.3. Now, we want to eliminate all terms of order $O(1)$. To this end, we first list all the $O(1)$ terms in the right hand side of (A.62) that are multiplied by $\theta_0''$ and employ (2.10):

$$\begin{align*}
\mathfrak{A} := & (\rho + h) \kappa_1 + (\nabla^T h_1 - \partial_{h_1} h_1) + X_0^*(\mathbf{v}^T) + X_0^*(\mathbf{v}) \cdot \nabla h_1 + X_0^*(\mathbf{v}) \cdot L^T h_1 - X_0^*(\mathbf{v} \cdot \nabla h_1) \\
= & \kappa_1 + \nabla^T h_1 - \partial_{h_1} h_1 + X_0^*(\mathbf{v}^T) + X_0^*(\mathbf{v}) \cdot \nabla h_1 - X_0^*(\mathbf{v} \cdot \nabla h_1)
\end{align*}$$

vanishes according to (A.63) below

$$\begin{align*}
+ \rho \kappa_1 - X_0^*(\mathbf{v} \cdot \nabla h_1) + \Delta h_1 - L^T h_1 + X_0^*(\mathbf{v}) \cdot L^T h_1.
\end{align*}$$

Note that the terms in the second last line depend only on $(s, t)$. This motivates us to define $h_1$ as the solution of the following equation on $T^1 \times [0, T_0]$:

$$D_1 h_1 - X_0^*(\mathbf{v}) \cdot \nabla h_1 - \Delta h_1 - \kappa_1 h_1 + X_0^*(\mathbf{v} \cdot \nabla h_1) = X_0^*(\mathbf{v}^T)$$

(A.63)
together with the initial condition $h_1|_{t=0} = 0$. This changes (A.62) into

$$
\partial_t c^{in}(x, t) + v^{in}_\theta(x, t) \cdot \nabla c^{in}(x, t) + \varepsilon^2 X_0^\theta(w_1) \cdot \nabla c^{in}_0(x, t) - \Delta c^{in}(x, t) + \frac{1}{\varepsilon^2} f'(c^{in})
$$

$$
= \theta_0^\prime (h_2 \varepsilon - \kappa_2 (\rho + h_1)^2 + \rho (\kappa_1 - X_0^\theta (\nabla \cdot v)))
$$

$$
- \partial_{\rho}^2 \hat{c}_2 + f''(\theta_0) \hat{c}_2 - |\nabla \rho h_1|^2 \theta_0^\prime
$$

$$
2 \nabla \cdot h_1 \cdot L^\theta h_1 \theta_0^\prime - |L^\theta h_1|^2 \theta_0^\prime + \theta_0^\prime (L^\Delta h_1 - L^\Delta h_1 + X_0^\theta (v) \cdot \nabla h_1)
$$

$$
= \mathcal{D}_1(d_\rho, \rho, s, t)
$$

$$
+ \varepsilon [(\Delta^\Delta h_2(r, s, t) - \partial_\rho \hat{c}_2(h_2(r, s, t)) + X_0^\theta (\nabla \cdot w_1))(\theta_0^\prime - (V + \Delta d_\rho(x, t)) \partial_\rho \hat{c}_2 - \partial_\rho^2 \hat{c}_3]
$$

$$
+ \varepsilon [f''(\theta_0) \hat{c}_2 - 2 \nabla^2 \cdot h_1 (r, s, t) \cdot \nabla h_2 (r, s, t) \theta_0^\prime + (b + X_0^\theta (v) \cdot \nabla h_2 - X_0^\theta (\nabla \cdot v) h_2) \theta_0^\prime]
$$

$$
+ \varepsilon X_0^\theta (v) \partial_\rho \hat{c}_2 + R_e(x, t) + \mathcal{R}_1 + \sum_{k=2}^5 \varepsilon^k \mathcal{R}_k + \varepsilon^3 (D_1 - \Delta \rho \hat{c}_3) - \varepsilon^2 \kappa_3 \cdot \theta_0^\prime.
$$

(A.64)

By the definition in (2.10), $L^\Delta h_1$, $L^\theta h_1$ and $L^\theta h_1$ are smooth functions that vanish on $\Gamma$. Therefore, using a Taylor expansion with respect to $r = d_\rho$ in the coefficients of $L^\Delta$, $L^\theta$, and $L^\theta$ and using $d_\rho = \varepsilon (\rho + h_1) + \varepsilon^2 h_2$ we obtain

$$
\mathcal{D}_1(d_\rho, \rho, s, t) = d_\rho \mathcal{D}(\rho, s, t) + \varepsilon^2 \mathcal{D}_2(d_\rho, \rho, s, t) = \varepsilon (\rho + h_1) \mathcal{D}(\rho, s, t) + \varepsilon^2 \mathcal{D}_2(d_\rho, \rho, s, t),
$$

(A.65)

where $\mathcal{D}(\rho, s, t) = \partial_t \mathcal{D}_1(0, \rho, s, t) \in \mathcal{R}_0^\alpha$ is independent of $h_2$, but $\mathcal{D}_2(d_\rho, \rho, s, t)$ depends on $h_2$. Moreover, there is some $C(M) > 0$ such that

$$
|\mathcal{D}_2(d_\rho, \rho, s, t)| \leq C(M) e^{-\alpha \rho} \quad \text{for all } (\rho, s, t) \in \mathbb{R} \times \mathbb{T}^1 \times [0, T_0], \varepsilon \in (0, \varepsilon_0]
$$

(A.66)

provided (B.4) holds true. Hence $\mathcal{D}_1$ can be rewritten as a term of order $O(\varepsilon)$. In order to eliminate the remaining $O(1)$ terms, we only need to choose $\hat{c}_2$ such that

$$
- \partial_{\rho}^2 \hat{c}_2 + f''(\theta_0) \hat{c}_2 = |\nabla \rho h_1|^2 \theta_0^\prime - \theta_0^\prime (\rho (\kappa_1 - X_0^\theta (\nabla \cdot v)))
$$

for all $(\rho, s, t) \in \mathbb{R} \times \mathbb{T}^1 \times [0, T_0]$ and the solvability is guaranteed by Proposition 2.24. With this choice and (2.10), (A.64) reduces to

$$
\partial_t c^{in}(x, t) + v^{in}_\theta(x, t) \cdot \nabla c^{in}(x, t) + \varepsilon^2 X_0^\theta(w_1) \cdot \nabla c^{in}_0(x, t) - \Delta c^{in}(x, t) + \frac{1}{\varepsilon^2} f'(c^{in})
$$

$$
= \theta_0^\prime (h_2 \varepsilon - \kappa_2 (\rho + h_1)^2 + \rho (\kappa_1 - X_0^\theta (\nabla \cdot v)))
$$

$$
+ \varepsilon ((\Delta^\Delta h_2 - D_2 h_2 + X_0^\theta (\nabla \cdot w_1))(\theta_0^\prime - (V + \Delta d_\rho(x, t)) \partial_\rho \hat{c}_2 - \partial_\rho^2 \hat{c}_3]
$$

$$
- 2 \nabla \cdot h_1 \cdot \nabla h_2 \theta_0^\prime + \varepsilon |\nabla h_2 - X_0^\theta (\nabla \cdot v) h_2 + b| \theta_0^\prime + \varepsilon X_0^\theta (\nabla \cdot v) \partial_\rho \hat{c}_2
$$

$$
- \varepsilon^2 \kappa_2 (2 (\rho + h_1) h_2 + \varepsilon \hat{c}_2 \theta_0^\prime - 2 \varepsilon \nabla \cdot h_1 \cdot L^\theta h_2 + L^\Delta h_1 \cdot \nabla h_2 + \nabla \cdot h_1 \cdot L^\theta h_2) \theta_0^\prime
$$

$$
+ \varepsilon^2 \mathcal{D}_2 \varepsilon + \varepsilon (L^\Delta - L^\theta) h_2 \theta_0^\prime + R_e(x, t) + \mathcal{R}_1 + \sum_{k=2}^5 \varepsilon^k \mathcal{R}_k + \varepsilon^3 (D_1 - \Delta \rho \hat{c}_3) - \varepsilon^2 \kappa_3 \cdot \theta_0^\prime.
$$

(A.67)

Next we start to eliminate the terms of order $O(\varepsilon)$. To this end, we collect all the $O(\varepsilon)$ terms on the right hand side of (A.67) that are multiplied by $\theta_0^\prime$ (but not $\theta_0^\prime$) and do not vanish on $\Gamma$. These terms are included in:

$$
\mathcal{D}_2 := h_2 \varepsilon - \kappa_2 (\rho + h_1)^2 + \Delta \rho h_2 - D_2 h_2 + X_0^\theta (\nabla \cdot w_1) + X_0^\theta (v) \cdot \nabla h_2 - X_0^\theta (\nabla \cdot v) h_2 + b
$$

$$
= \mathfrak{B} + \kappa_1 h_2 - \kappa_2 h_2^2 + \Delta \rho h_2 - D_2 h_2 + X_0^\theta (\nabla \cdot w_1) + X_0^\theta (v) \cdot \nabla h_2 - X_0^\theta (\nabla \cdot v) h_2
$$

$$
- \mathfrak{B} - \kappa_2 (\rho^2 + 2ph_1) + b.
$$

(A.68)

Here all terms expect the last two ones depend only on $(s, t)$. Moreover, we added a term $\mathfrak{B} = \mathfrak{B}(s, t)$ in order to satisfy the compatibility condition for the equation for $\hat{c}_3$ in the sequel. To eliminate all terms except the last three in (A.68), we choose $h_2$ as the solution

$$
D_1 h_2 - \Delta \rho h_2 - \kappa_1 h_2 - X_0^\theta (v) \cdot \nabla h_2 + X_0^\theta (\nabla \cdot v) h_2 = \mathfrak{B} - \kappa_2 h_2^2 + X_0^\theta (\nabla \cdot v) h_2.
$$

(A.69)
on \( T^1 \times [0, T_0] \) together with the initial condition \( h_2|_{t=0} = 0 \). Existence of a solution is proved in Lemma \( \ref{lem:existence} \). Hence \( \ref{eq:8} \) reduces to \( \mathcal{D}_2 = -\mathfrak{B} - \kappa_\gamma (\rho^2 + 2ph_0) + b \) and \( \ref{eq:105} \) reduces to

\[
\partial_t c^{in}(x,t) + \nabla_x (\rho, x, t) \cdot \nabla c^{in}(x,t) + \varepsilon^2 X_0^0 (w_1) \cdot \nabla c_0^{in}(x,t) - \Delta c^{in}(x,t) + \frac{1}{\varepsilon} f'(e^{in})
\]

\[
= \varepsilon (\rho + h_1) \mathfrak{D} + \varepsilon \left( (b - \mathfrak{B} - \kappa_\gamma (\rho^2 + 2ph_0)) \phi_0 + \varepsilon^2 (\partial_\rho \phi_0) \right) + \varepsilon^2 \nabla \cdot \phi_0 - V + \Delta \phi_0 (x,t) - \n\cdot X_0^0 (v) \partial_\rho \phi_2
\]

\[
+ \varepsilon \kappa_\gamma (2(\rho + h_1)h_2 + \varepsilon h_2^2) \phi_0 + \varepsilon^2 \phi_0 + \varepsilon (L^\Delta - L^1) h_2 \phi_0
\]

\[
- 2 \varepsilon \left( \nabla \cdot \nabla h_2 + L^\Delta h_1 \cdot \nabla h_2 + L^\Delta h_1 \cdot L^\Delta h_2 \right) \phi_0 + R_v(x,t) + \mathfrak{R}_1 + \sum_{k=2}^5 \varepsilon^k \mathfrak{R}_k. \quad (A.70)
\]

In view of \( \ref{eq:83} \), we can further rewrite \( \ref{eq:70} \) as

\[
\partial_t c^{in}(x,t) + \nabla_x (\rho, x, t) \cdot \nabla c^{in}(x,t) + \varepsilon^2 X_0^0 (w_1) \cdot \nabla c_0^{in}(x,t) - \Delta c^{in}(x,t) + \frac{1}{\varepsilon} f'(e^{in})
\]

\[
= \varepsilon (\rho + h_1) \mathfrak{D} + \varepsilon \left( (b - \mathfrak{B} - \kappa_\gamma (\rho^2 + 2ph_0)) \phi_0 + \varepsilon^2 \nabla \cdot \phi_0 - V + \Delta \phi_0 (x,t) - \n\cdot X_0^0 (v) \partial_\rho \phi_2
\]

\[
+ \varepsilon \kappa_\gamma (2(\rho + h_1)h_2 + \varepsilon h_2^2) \phi_0 + \varepsilon^2 \phi_0 + \varepsilon (L^\Delta - L^1) h_2 \phi_0
\]

\[
- 2 \varepsilon \left( \nabla \cdot \nabla h_2 + L^\Delta h_1 \cdot \nabla h_2 + L^\Delta h_1 \cdot L^\Delta h_2 \right) \phi_0 + R_v(x,t) + \mathfrak{R}_1 + \sum_{k=2}^5 \varepsilon^k \mathfrak{R}_k. \quad (A.71)
\]

It remains to eliminate all the \( O(\varepsilon) \) terms by solving the following equation for \( \phi_3 \):

\[
\varepsilon^2 \nabla \cdot \phi_3 - \varepsilon^2 \nabla \cdot \phi_0 - \nabla \cdot \phi_0 = \varepsilon \kappa_\gamma (2(\rho + h_1)h_2 + \varepsilon h_2^2) \phi_0 + \varepsilon^2 \phi_0 + \varepsilon (L^\Delta - L^1) h_2 \phi_0
\]

\[
- 2 \varepsilon \left( \nabla \cdot \nabla h_2 + L^\Delta h_1 \cdot \nabla h_2 + L^\Delta h_1 \cdot L^\Delta h_2 \right) \phi_0
\]

\[
+ \varepsilon^2 (\rho + h_1) \phi_1 - \varepsilon \kappa_\gamma (2(\rho + h_1)) \partial_\rho \phi_2 + R_v(x,t) + \mathfrak{R}_1 + \sum_{k=2}^5 \varepsilon^k \mathfrak{R}_k. \quad (A.72)
\]

which is equivalent to \( \ref{eq:111} \). This is a system treated in Theorem \( \ref{thm:existence} \) and we shall choose \( \mathfrak{B} \) such that the compatibility condition of Theorem \( \ref{thm:existence} \) is valid, namely

\[
\mathfrak{B}(s,t) \int \phi_0^2 \, \, d\rho = \int \left[ \phi_0^2 - \kappa_\gamma (\rho^2 + 2ph_0) \right] \phi_0 \, \, d\rho \quad (A.73)
\]

since \( \int \rho \phi_0^2 \, \, d\rho = 0 \) and \( \int \phi_0 \, \, d\rho = 0 \). It is crucial that the right hand side of \( \ref{eq:83} \), especially \( \mathfrak{B} \) defined in \( \ref{eq:83} \), is uniquely determined by \( h_1, \phi_2 \) and the solution of \( \ref{eq:111} \). Hence \( \mathfrak{B} \) is uniquely determined by \( h_1, \phi_2 \) and the solution of \( \ref{eq:111} \). So we are able to solve \( \ref{eq:83} \) and then \( \ref{eq:82} \) without leading to a circular argument. Finally \( \ref{eq:70} \) becomes

\[
\partial_t c^{in}(x,t) + \nabla_x (\rho, x, t) \cdot \nabla c^{in}(x,t) + \varepsilon^2 X_0^0 (w_1) \cdot \nabla c_0^{in}(x,t) - \Delta c^{in}(x,t) + \frac{1}{\varepsilon} f'(e^{in})
\]

\[
= R_v(x,t) + \sum_{k=2}^5 \varepsilon^k \mathfrak{R}_k + \mathfrak{R}_1 - \varepsilon^2 \kappa_\gamma (\rho^2 + 2ph_0) + \varepsilon^2 \phi_0 + \varepsilon (L^\Delta - L^1) h_2 \phi_0
\]

\[
+ \varepsilon^2 (\rho + h_1) \phi_1 + \varepsilon \kappa_\gamma (2(\rho + h_1)) \partial_\rho \phi_2 + R_v(x,t) + \mathfrak{R}_1 \quad (A.74)
\]

which implies \( \ref{eq:20} \).
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