Logarithmic Corrections to the entropy of Hairy black hole in (2+1) dimension gravity

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Abstract. The entropy recives the logarithmic corrections in the black hole due to the quantum fluctuation close to the horizon. We study these logarithmic corrections of entropy for hairy BTZ black hole. By applying the WKB approximation calculation of the states is being done which is used to determine free energy of the scalar field. It is seen that the entropy and the free energy are diverging logarithmically in both cut off parameters and these divergences can be engrossed in the renormalized equation of the gravitational constant.

1. Introduction

Black holes are known as the thermodynamics object after the introduction of the black hole entropy which was introduced by the Bekenstein [1] on the basis of black hole thermodynamics. These laws are analogous to the laws of thermodynamics [3]. The entropy is identified by the area of event horizon and temperature is analogous to the surface gravity \( \kappa \). The behaviour of microscopic states are responsible for the thermal properties of normal physical systems which are characterized by the no hair theorem. The issue of which microstates contribute to the entropy is a challenging problem in black hole physics.

Classically the black hole does not radiates but when quantum mechanical effect taken into account the black hole radiates [4, 5]. The entropy for black holes receives logarithmic corrections arise in all thermodynamic systems when small stable fluctuations around equilibrium are taken into account [6, 7]. These logarithmic corrections can result from a microscopic theory of quantum gravity. This ensures that the logarithmic terms are indeed much smaller compared to the leading term. The divergences which appears due to the horizon in the density of states can be engrossed in the renormalization equation of the gravitational coupling constant. According to the suggestion of t Hooft one can get rid of these divergences if one impose a brick wall cut off of the horizon[8, 9, 10, 11].

It is considered that to obtain statistical black hole entropy the brick wall estimation method is suitable by counting the possible modes of quantum field very near to the event horizon, as a result the entropy of black holes can be intimates in terms of area law. In the present work, we use this method to study the entropy of black holes for hairy black hole [13] in (2+1)dimensional gravity. The expression of the entropy proportional to the area of the event horizon and logarithmic corrections to the entropy. The another methods to study the quantum fields in the presence of black hole are Euclidean path integral [14, 15, 16], entanglement entropy [17, 18, 19, 20, 21, 22], D-brane statistical method [23, 24, 25] and holographic entanglement entropy method [26, 27].

The sequence of the work is as follows: We demonstrate a mechanism of black hole entropy by using the brick wall model for the black hole is presented in the section II. We apply this method to the hairy BTZ black hole which is presented in Sec. III and results and conclusion are given in Sec. IV.

1. Hairy black hole in (2+1) dimension gravity

The charged AdS BTZ black hole solution in (2+1) gravity coupled to non-minimal scalar field can be written as

\[
\text{ds}^2 = -f(r)dt^2 + f(r)dr^2 + r^2d\phi^2, \quad \text{with} \quad f(r) = -M \left( 1 + \frac{2B}{r} \right) + \frac{r^2}{l^2}
\]
where $M$ is mass of the black hole, $B$ is related to the free parameter of scalar field and $\sqrt{l}$ is the AdS length which is related with the cosmological constant $l = -\lambda$. The field equation for the massless scalar field in the background of static spherically symmetric black hole can be written as [4, 5]

$$
\frac{1}{\sqrt{-g}} \partial_{\mu} \left( \sqrt{-g} g^{\mu \nu} \partial_{\nu} \Phi \right) = 0
$$

(2)

where $g^{\mu \nu}$ is the metric components of the black hole solution. The mode decomposition of the scalar field $\Phi$ in term of spherical harmonics is written as [4, 5]

$$
\Phi(t, r, \theta, \phi) = Y_{\ell m}(\theta, \phi) f_{\ell}(r)e^{-iEt}.
$$

(3)

The radial function is written as

$$
f_{\ell, j} \sim e^{\pm \int_{0}^{r} k_{E, \ell}(r)dr} \quad \text{with} \quad k_{E, 1}^{2} = \left[ \frac{E^{2} - f(r) \left( \ell + \frac{1}{2} \right) + \mu^{2}}{f^{2}(r)} \right].
$$

(4)

where $k_{E, \ell}$ is the radial momentum. The number of density of state is given by

$$
g(E) = \sum_{\text{modes}} n_{E, \ell} \sim \int_{0}^{\ell_{\text{max}}} (2\ell + 1) d\ell \int_{r_{0} + \epsilon}^{L} k_{E, \ell}(r)dr
$$

(5)

where $n_{E, \ell}$ is the radial modes related to the momentum by.

Now we are able to calculate the free energy of the system, it can be calculated by using the free energy

$$
F = -\int_{0}^{\infty} g(E) \frac{1}{e^{\beta E} - 1} dE
$$

$$
= -\frac{1}{\pi} \int_{0}^{\ell_{\text{max}}} (2\ell + 1) d\ell \int_{0}^{\infty} g(E) \frac{1}{e^{\beta E} - 1} dE \int_{r_{0} + \epsilon}^{L} k_{E, \ell}(r)dr
$$

$$
= -\frac{2}{3\pi} \int_{0}^{\infty} \frac{1}{e^{\beta E} - 1} dE \int_{r_{0} + \epsilon}^{L} k_{E, \ell}(r)dr
$$

$$
= -\frac{2}{3\pi} \int_{0}^{\infty} \frac{dE}{e^{\beta E} - 1} I(E),
$$

where $I(E)$ is

$$
I(E) = \int_{r_{0} + \epsilon}^{L} k_{E, \ell}(r)dr
$$

$$
= \int_{r_{0} + \epsilon}^{L} \frac{r^{2}}{f(r)} \left[ E^{2} - \mu^{2} f(r) \right] dr
$$

(8)
To evaluate $I(E)$ we can expand $f(r)$ for a non-extremallimit about the event horizon by using the Taylor expansion

$$f(r) = f(r_e) + f'(r_e)(r - r_e) + f''(r_e)(r - r_e)^2 + O(r - r_e)^3$$

(9)

The first term is zero as $g''(r_e) = f(r_e) = 0$. Therefore, the integral $I(E)$ is calculated to be

$$I(E) = \int_{r_{e} + \epsilon}^{r_{e}} \frac{E^3 r^2_e}{f'(r_+)^2} \left( 1 + \frac{2}{r_+} - \frac{f''(r_+)}{f'(r_+)} - \frac{3 \mu^2}{2 E^2} f'(r_+) \right) \log \frac{L}{\epsilon} + O(\epsilon^2)$$

$$= \frac{E^3 r^2_e}{f'(r_+)^2} \left( \frac{2}{r_+} - \frac{f''(r_+)}{f'(r_+)} - \frac{3 \mu^2}{2 E^2} f'(r_+) \right) \log \frac{L}{\epsilon} + O(\epsilon^2)$$

(10)

Substitute the value of $I(E)$ from Eq. (10) to Eq. (7), which in turn results into the free energy of the form

$$F = -\frac{2}{3 \pi} \frac{r^2_e}{f'(r_+)^2} \left( \frac{3! \zeta(4)}{\beta^4} \left[ \frac{1}{\epsilon} - \frac{2}{r_+} - \frac{f''(r_+)}{f'(r_+)} \right] \log \frac{L}{\epsilon} \right) + \zeta(2) \frac{3 \mu^2}{2 \beta^2 E^2} f'(r_+) \log \frac{L}{\epsilon} + O(\epsilon^2)$$

$$= -\frac{2}{3 \pi} \frac{r^2_e}{f'(r_+)^2} \left( \frac{\pi^4}{15 \beta^4} \left[ \frac{1}{\epsilon} - \frac{2}{r_+} - \frac{f''(r_+)}{f'(r_+)} \right] \log \frac{L}{\epsilon} \right) + \frac{\pi^2 \mu^2}{4 \beta^2} f'(r_+) \log \frac{L}{\epsilon} + O(\epsilon^2)$$

(11)

where, $\zeta(n)$ is the Riemann zeta function of order $n$. Substitute the value of $f(r)$ from Eq. (1), we get the free energy of hairy BTZ black hole

$$F = -\frac{2}{3 \pi} \frac{r^2_e}{f'(r_+)^2} \left( \frac{\pi^4}{15 \beta^4} \left[ \frac{1}{\epsilon} - \frac{2}{r_+} - \frac{f''(r_+)}{f'(r_+)} \right] \log \frac{L}{\epsilon} \right) + \frac{3 \pi^2 \mu^2}{4 \beta^2} f'(r_+) \log \frac{L}{\epsilon} + O(\epsilon^2)$$

(12)

This is free energy of the hairy BTZ black hole and $\beta_+ = 2 \pi / f'(r_e)$ is the inverse of temperature. The entropy of the hairy BTZ black hole can be derived from the free energy

$$S = \beta^2 \frac{\partial F}{\partial \beta} |_{\beta = \beta_+}$$

$$= \frac{2}{3 \pi} \frac{r^2_e}{f'(r_+)^2} \left( \frac{4 \pi^4}{15 \beta^4} \left[ \frac{1}{\epsilon} - \frac{2}{r_+} - \frac{f''(r_+)}{f'(r_+)} \right] \log \frac{L}{\epsilon} \right) + \frac{3 \pi^2 \mu^2}{4 \beta_+} f'(r_+) \log \frac{L}{\epsilon} + O(\epsilon^2)$$

(13)

Therefore, the entropy of static spherically symmetric black hole close to the the horizon is calculated to be

$$S = \frac{2}{3 \pi} \frac{r^2_e}{f'(r_+)^2} \left( \frac{4 \pi^4}{240} \left[ \frac{1}{\epsilon} - \frac{2}{r_+} - \frac{f''(r_+)}{f'(r_+)} \right] \log \frac{L}{\epsilon} \right) + \frac{2 \pi^2 \mu^2}{16 \pi} f'^2 \log \frac{L}{\epsilon} + O(\epsilon^2)$$

$$= \frac{r^2_e}{360 \epsilon} \left( \frac{4 \pi^4}{240} \left[ \frac{1}{\epsilon} - \frac{2}{r_+} - \frac{f''(r_+)}{f'(r_+)} \right] \log \frac{L}{\epsilon} \right) + \frac{2 \pi^2 \mu^2}{16 \pi} f'^2 \log \frac{L}{\epsilon} + O(\epsilon^2)$$

(14)

(15)
So in terms of the proper distance the entropy of the hairy BTZ black hole is obtained by substituting the value of $f(r)$ from Eq. (1) in Eq. (15), we get

$$S_{BW} \approx \frac{r_+^2}{90 h_c^2} - \left[ \frac{r_+^2}{90} \left( \frac{2 M_{BM}}{r_+^2} - \frac{2}{L^2} \right) - \frac{r_+^2 \mu^2}{6} \right] \log \left( \frac{r_+ + \mu}{h_c^2} \right)$$

(17)

$S_{BW}$ denotes the entropy of the hairy BTZ black hole. The leading order term in Eq. (17) is the standard result. If the infrared cutoff approaches the event horizon, i.e., if $L \to r_+$, then the entropy.

2. Conclusions

In the present work we have estimated the thermodynamics by taking a scalar field in Hairy BTZ black hole background in two parts. Firstly, we study the conventional radial brick wall cut-off parameter which governs the solution near the event horizon and secondly, we studied the cut-off parameter $h_c$ in the angular coordinate. Entropy has been calculated by using the WKB approximation and is found to be the logarithmically divergent in both of these cut-off parameters and these divergences term can be engrossed in the renormalized equation of the gravitational constant.

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