Introduction

The cyclotomic Hecke algebras of type A are a much studied class of algebras that include, as special cases, the group algebras of the symmetric groups and the Iwahori-Hecke algebras of types A and B. They have a rich representation theory that can be approached using algebraic combinatorics, standard tools of representation theory, or via the theory of Lie and algebraic groups, which brings deep methods from geometry into play.

In 2008 Khovanov and Lauda [74,75] and Rouquier [121] introduced the quiver Hecke algebras, or KLR algebras. These are a remarkable family \( \mathcal{R}_n(\Gamma) \) of \( \mathbb{Z} \)-graded algebras defined by generators and relations depending on a quiver \( \Gamma \). The motivation for defining and studying these algebras came, at least in part, from questions in geometry and 2-representation theory. The algebras \( \mathcal{R}_n(\Gamma) \) categorify the negative
part of the associated quantum group $U_q(\mathfrak{g})$ \[122, 132\]. That is, there are natural isomorphisms

$$U_q^- (\mathfrak{g}) \cong \bigoplus_{n \geq 0} \text{Proj}(\mathcal{R}_n (\Gamma)),$$

where $[\text{Proj}(\mathcal{R}_n (\Gamma))]$ is the Grothendieck group of the category of finitely generated graded projective $\mathcal{R}_n (\Gamma)$-modules. For each dominant weight $\Lambda$ the quiver Hecke algebra $\mathcal{R}_n (\Gamma)$ has a cyclotomic quotient $\mathcal{R}_n (\Gamma) / (\Lambda)$ that categorifies the highest weight module $L(\Lambda)$ \[67, 122, 134\]. These results can be thought of as far reaching generalizations of Ariki’s Categorification Theorem in type $A$ \[3\].

Spectacularly, Brundan and Kleshchev \[21, 121\] proved that each cyclotomic Hecke algebra of type $A$ is isomorphic to a cyclotomic quotient of a quiver Hecke algebra of type $A$. Thus, the KLR algebras give a new window for understanding the cyclotomic Hecke algebras of type $A$. This chapter is an attempt to open this window and show how the “classical” ungraded representation theory and the emerging graded representation theory of the cyclotomic quiver Hecke algebras interact.

With the advent of the KLR algebras the cyclotomic Hecke algebras can now be studied from many different perspectives including:

a) As ungraded cyclotomic Hecke algebras.

b) As graded cyclotomic quiver Hecke algebras or KLR algebras.

c) Geometrically as the ext-algebras of Lusztig sheaves \[98, 123, 132\].

d) Through the lens of 2-representation theory using Rouquier’s theory of 2-Kac Moody algebras \[67, 121, 134\].

Here we focus on (a) and (b) taking an unashamedly combinatorial approach, although we will see shadows of geometry and 2-representation theory.

For every quiver there is a corresponding family of KLR algebras, however, the quiver Hecke algebras attached to the quivers of type $A$ are special because these are the only quiver Hecke algebras that existed in the literature prior to \[74, 121\] — all of the other quiver Hecke algebras are “new” algebras. In type $A$, when we are working over a field, the quiver Hecke algebras are isomorphic to affine Hecke algebras of type $A$ \[122\] and the cyclotomic quiver Hecke algebras are isomorphic to the cyclotomic Hecke algebras of type $A$ \[21\]. The cyclotomic Hecke algebras of type $A$ have a uniform description but, historically, they have been studied either as Ariki-Koike algebras ($v \neq 1$), or as degenerate Ariki-Koike algebras ($v = 1$). The existence of gradings on Hecke algebras, at least in the “abelian defect case”, was predicted by Rouquier \[120\], Remark 3.11 and Turner \[130\].

The cyclotomic quiver Hecke algebras of type $A$ are better understood than other types because we already know a lot about the isomorphic, but ungraded, cyclotomic Hecke algebras \[107\]. For example, by piggy-backing on the ungraded theory, homogeneous bases have been constructed for the cyclotomic quiver Hecke algebras of type $A$ \[54\] but such bases are not yet known in other types. Many of the major results for general quiver Hecke algebras were first proved in type $A$ and then generalized to other types. In fact, the type $A$ algebras, through Ariki’s theorem and Chuang and Rouquier’s seminal work on $\mathfrak{sl}_2$-categorifications \[28\], has motivated many of these developments.

The first section starts by giving a uniform description of the degenerate and non-degenerate cyclotomic Hecke algebras, recalling some structural results from the ungraded representation theory of these algebras. Everything mentioned in this section is applied later in the graded setting.

The second section introduces the cyclotomic KLR algebras as abstract algebras given by generators and relations. We use the relations in a series of extended examples to give the reader a feel for these algebras. In particular, using just the relations we show that the semisimple cyclotomic quiver Hecke algebras of type $A$ are direct sums of matrix rings. From this we deduce Brundan and Kleshchev’s Graded Isomorphism Theorem in the semisimple case.

The third section starts with Brundan and Kleshchev’s Graded Isomorphism Theorem \[21\]. We develop the representation theory of the cyclotomic quiver Hecke algebras as graded cellular algebras, focusing on the graded Specht modules. The highlight of this section is a self-contained proof of Brundan and Kleshchev’s Graded Categorification Theorem \[22\], starting from the graded branching rules for the graded Specht modules and then using Ariki’s Categorification Theorem \[3\] to make the link with canonical bases. We also give a new treatment of graded adjustment matrices using a cellular algebra approach.

In the final section we sketch one way of proving Brundan and Kleshchev’s Graded Isomorphism Theorem using the classical theory of seminormal forms. As an application we describe how to construct a new graded cellular basis for the cyclotomic quiver Hecke algebras that appears to have remarkable properties. We end with a conjecture for the $q$-characters of the graded simple modules.

Although the experts will find some new results, most of the novelty is in our approach and the arguments that we use. We include many examples and a comprehensive survey of the literature. For a different perspective we recommend Kleshchev’s survey article \[80\] on the applications of quiver Hecke algebras to symmetric groups.
Acknowledgements. This chapter grew out of a series of lectures at the IMS at the National University of Singapore. I thank the organizers for inviting me to give these lectures and to write this chapter. The direction taken in these notes, and the conjecture formulated in §4.4, is partly motivated by the author’s joint work with Jun Hu and I thank him for his implicit contributions. I thank Susumu Ariki, Anton Evseev, Matthew Fayers, Jun Hu, Kai Meng Tan and the referee for their comments on earlier versions of this manuscript and Jon Brundan for some helpful discussions about crystal bases. This chapter was written while visiting Universität Stuttgart and Charles University, Prague.

1. Cyclotomic Hecke algebras of type A

This section surveys the representation theory of the cyclotomic Hecke algebras of type $A$ and, at the same time, introduces the results and the combinatorics that we need later.

1.1. Cyclotomic Hecke algebras and Ariki-Koike algebras. Hecke algebras of the complex reflections groups $G_{l,n} = \mathbb{Z}/(\mathbb{Z}^l \oplus \mathbb{S}_n)$ of type $G(\ell, 1, n)$ were introduced by Ariki-Koike [10], motivated by the Iwahori-Hecke algebras of Coxeter groups [58]. Soon afterwards, Broué and Malle [16] defined Hecke algebras for arbitrary complex reflection groups. The following refinement of the definition of these algebras unifies the treatment of the degenerate and non-degenerate cyclotomic Hecke algebras of type $G(\ell, 1, n)$.

Let $\mathcal{Z}$ be a commutative domain with one.

1.1.1. Definition (Hu-Mathas [57, Definition 2.2]). Fix integers $n \geq 0$ and $\ell \geq 1$. The cyclotomic Hecke algebra of type $A$, with Hecke parameter $v \in \mathbb{Z}^\times$ and cyclotomic parameters $Q_1, \ldots, Q_\ell \in \mathcal{Z}$, is the unitary associative $\mathcal{Z}$-algebra $\mathcal{H}_n = \mathcal{H}_n(\mathbb{Z}, v, Q_1, \ldots, Q_\ell)$ with generators $L_1, \ldots, L_n, T_1, \ldots, T_{n-1}$ and relations

\[
\prod_{i=1}^\ell (L_1 - Q_i) = 0, \quad (T_r + v^{-1})(T_r - v) = 0, \quad L_{r+1} = T_s T_r + T_r, \\
L_r L_t = L_t L_r, \quad T_r T_s = T_s T_r \text{ if } |r - s| > 1, \\
T_r T_{n+1} T_s = T_{n+1} T_r T_{n+1}, \quad T_r L_t = L_t T_r \text{ if } t \neq r, r+1,
\]

where $1 \leq r < n$, $1 \leq s < n-1$ and $1 \leq t \leq n$.

By definition, $\mathcal{H}_n$ is generated by $L_1, T_1, \ldots, T_{n-1}$ but we prefer including $L_2, \ldots, L_n$ in the generating set.

Let $\mathcal{S}_n$ be the symmetric group on $n$ letters. For $1 \leq r < n$ let $s_r = (r, r+1)$ be the corresponding simple transposition. Then $\{s_1, \ldots, s_{n-1}\}$ is the standard set of Coxeter generators for $\mathcal{S}_n$. A reduced expression for $w \in \mathcal{S}_n$ is a word $w = s_{r_1} \cdots s_{r_k}$ with $k$ minimal and $1 \leq r_j < n$ for $1 \leq j \leq k$. If $w = s_{r_1} \cdots s_{r_k}$ is reduced then set $T_w = T_{r_1} \cdots T_{r_k}$. Then $T_w$ is independent of the choice of reduced expression by Matsumoto’s Monoid Lemma [110] since the braid relations hold in $\mathcal{H}_n$; see, for example, [104, Theorem 1.8]. Arguing as in [10, Theorem 3.3], it follows that $\mathcal{H}_n$ is free as a $\mathcal{Z}$-module with basis

\[
\{ L_1^{a_1} \cdots L_n^{a_n} T_w \mid 0 \leq a_1, \ldots, a_n < \ell \text{ and } w \in \mathcal{S}_n \}. \]

Consequently, $\mathcal{H}_n$ is free as a $\mathcal{Z}$-module of rank $\ell^n n!$, which is the order of the complex reflection group $G_{\ell,n} = \mathbb{Z}/(\mathbb{Z}^\ell \oplus \mathcal{S}_n)$ of type $G(\ell, 1, n)$.

Definition 1.1.1 is different from Ariki and Koike’s [10] definition of the cyclotomic Hecke algebras of type $G(\ell, 1, n)$ because we have changed the commutation relation for $T_r$ and $L_r$. Ariki and Koike [10] defined their algebra to be the unital associative algebra generated by $T_0, T_1, \ldots, T_{n-1}$ subject to the relations

\[
\prod_{i=1}^\ell (T_0 - Q_i) = 0, \quad (T_r + v^{-1})(T_r - v) = 0, \\
T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0, \quad T_s T_{s+1} T_s = T_{s+1} T_s T_{s+1}, \\
T_r T_s = T_s T_r \text{ if } |r - s| > 1
\]

We have renormalised the quadratic relation for the $T_r$, for $1 \leq r < n$, so that $q = v^2$ in the notation of [10]. Ariki and Koike then defined $L_1' = T_0$ and set $L_{r+1}' = T_1 L_r' T_1$ for $1 \leq r < n$. In fact, if $v = v^{-1}$ is invertible in $\mathcal{Z}$ then $\mathcal{H}_n$ is (isomorphic to) the Ariki-Koike algebra with parameters $Q_1' = 1 + (v - v^{-1})Q_1$ for $1 \leq l \leq \ell$. To see this set $L_1' = 1 + (v - v^{-1})L_1$ in $\mathcal{H}_n$, for $1 \leq r < n$. Then $L_r' L_r T_1 = (v - v^{-1})T_1 L_r' T_1 + T_1' = L_{r+1}'$, which implies our claim. Therefore, over a field, $\mathcal{H}_n$ is an Ariki-Koike algebra whenever $v^2 \neq 1$. On the other hand, if $v^2 = 1$ then $\mathcal{H}_n$ is a degenerate cyclotomic Hecke algebra [13,79].

We note that the Ariki-Koike algebras with $v^2 = 1$ include as a special the group algebras $\mathbb{Z}G_{\ell,n}$ of the complex reflection groups $G_{\ell,n}$, for $n \geq 0$. One consequence of the last paragraph is that $\mathbb{Z}G_{\ell,n}$ is not a specialization of $\mathcal{H}_n$. This said, if $F$ is a field such that $\mathcal{H}_n$ and $FG_{\ell,n}$ are both split semisimple then $\mathcal{H}_n \cong FG_{\ell,n}$. On the other hand, the algebras $\mathcal{H}_n$ always fit into the spetses framework of Broué, Malle and Michel [17].

The algebras $\mathcal{H}_n$ with $v^2 = 1$ are the degenerate cyclotomic Hecke algebras of type $G(\ell, 1, n)$ whereas if $v^2 \neq 1$ then $\mathcal{H}_n$ is an Ariki-Koike algebra in the sense of [10]. Our definition of $\mathcal{H}_n$ is more natural in...
the sense that many features of the algebras $\mathcal{H}_n$ have a uniform description in both the degenerate and non-degenerate cases:

- The centre of $\mathcal{H}_n$ is the set of symmetric polynomials in $L_1, \ldots, L_n$ (Brundan [19] in the degenerate case when $v^2 = 1$ and announced when $v^2 \neq 1$ by Graham and Francis building on [42]).
- The blocks of $\mathcal{H}_n$ are indexed by the same combinatorial data (Lyle and Mathas [96] when $v^2 \neq 1$ and Brundan [19] when $v^2 = 1$).
- The irreducible $\mathcal{H}_n$-modules are indexed by the crystal graph of the integral highest weight module $L(\Lambda)$ for $U_q(\mathfrak{sl}_n)$ (Ariki [3] when $v^2 \neq 1$ and Brundan and Kleshchev [23] when $v^2 = 1$).
- The algebras $\mathcal{H}_n$ categorify $L(\Lambda)$. Moreover, in characteristic zero the projective indecomposable $\mathcal{H}_n$-modules correspond to the canonical basis of $L(\Lambda)$. (Ariki [3] when $v^2 \neq 1$ and Brundan and Kleshchev [23] when $v^2 = 1$).
- The algebra $\mathcal{H}_n$ is isomorphic to a cyclotomic quiver Hecke algebras of type $A$ (Brundan and Kleshchev [21]).

In contrast, the Ariki-Koike algebras with $v^2 = 1$ do not share any of these properties: their center can be larger than the set of symmetric polynomials in $L_1, \ldots, L_n$ (Ariki [3]); if $\ell > 1$ then they have only one block (Lyle and Mathas [96]); their irreducible modules are indexed by a different set (Mathas [103]); they do not categorify $L(\Lambda)$ and no non-trivial grading on these algebras is known. In this sense, the definition of the Ariki-Koike algebras from [10] gives the wrong algebras when $v^2 = 1$. Definition 1.1.1 corrects for this.

Historically, many results for the cyclotomic Hecke algebras $\mathcal{H}_n$ were proved separately in the degenerate ($v^2 = 1$) and non-degenerate cases ($v^2 \neq 1$). Using Definition 1.1.1 it should now be possible to give uniform proofs of all of these results. In fact, in the cases that we have checked uniforms arguments can now be given for the degenerate and non-degenerate cases.

1.2. Quivers of type $A$ and integral parameters. Rather than work with arbitrary cyclotomic parameters $Q_1, \ldots, Q_\ell$, as in Definition 1.1.1, we now specialize to the integral case using the Morita equivalence results of Dipper and the author [32] (when $v^2 \neq 1$) and Brundan and Kleshchev [20] (when $v^2 = 1$). First, however, we need to introduce quivers and quantum integers.

Fix $e \in \{1, 2, 3, 4, \ldots\} \cup \{\infty\}$ and let $\Gamma_e$ be the quiver with vertex set $I_e = \mathbb{Z}/e\mathbb{Z}$ and edges $i \rightarrow i + 1$, for $i \in I_e$, where we adopt the convention that $e \mathbb{Z} = \{0\}$ when $e = \infty$. If $i, j \in I_e$ and $i$ and $j$ are not connected by an edge in $\Gamma_e$, then we write $i \leftrightarrow j$. When $e$ is fixed we write $\Gamma = \Gamma_e$ and $I = I_e$. Hence, we are considering either the linear quiver $\mathbb{Z}$ ($e = \infty$) or a cyclic quiver ($e < \infty$):

In the literature the case $e = \infty$ is often written as $e = 0$, however, we prefer $e = \infty$ because then $e = |I_e|$. There are also several results that hold when $e > n$ — using the “$e = 0$ convention” this condition must be written as $e > n$ or $e = 0$. We write $e \geq 2$ to mean $e \in \{2, 3, 4, 5, \ldots\} \cup \{\infty\}$.

To the quiver $\Gamma_e$ we attach the symmetric Cartan matrix $(c_{ij})_{i,j \in I_e}$, where

$$c_{ij} = \begin{cases} 2, & \text{if } i = j, \\ -1, & \text{if } i \rightarrow j \text{ or } i \leftarrow j, \\ -2, & \text{if } i \leftrightarrow j, \\ 0, & \text{otherwise}. \end{cases}$$

Following [66, Chapter 1], let $\hat{\mathfrak{sl}}_e$ be the Kac-Moody algebra of $\Gamma_e$ [66] with simple roots $\{\alpha_i | i \in I\}$, fundamental weights $\{\Lambda_i | i \in I\}$, positive weight lattice $P^+ = \bigoplus_{i \in I} \mathbb{N}\Lambda_i$ and positive root lattice $Q^+ = \bigoplus_{i \in I} \mathbb{N}\alpha_i$. Let $(\cdot, \cdot)$ be the usual invariant form associated with this data, normalised so that $(\alpha_i, \alpha_j) = c_{ij}$ and $(\Lambda_i, \alpha_j) = \delta_{ij}$, for $i, j \in I$.

Fix a sequence $\kappa = (\kappa_1, \ldots, \kappa_e) \in \mathbb{Z}^e$, the multicharge, and define $\Lambda = \Lambda(\kappa) = \Lambda_{\kappa_1} + \cdots + \Lambda_{\kappa_e}$, where $\mathbb{N} = a + e\mathbb{Z} \subseteq I$ for $a \in \mathbb{Z}$. Then $\Lambda \in P^+$ is dominant weight of level $\ell$. The integral cyclotomic Hecke algebras defined below depend only on $\Lambda$, however, the bases and our combinatorics often depend on the choice of multicharge $\kappa$.

Recall that $\mathcal{Z}$ is an integral domain. For $t \in \mathcal{Z}$ and $k \in \mathbb{Z}$ define the $t$-quantum integer $[k]_t$ by

$$[k]_t = \begin{cases} t + t^3 + \cdots + t^{2k-1}, & \text{if } k \geq 0, \\ -(t^{-1} + t^{-3} + \cdots + t^{-2k+1}), & \text{if } k < 0. \end{cases}$$

When $t$ is understood we simply write $[k] = [k]_t$. Unpacking the definition, if $t^2 \neq 1$ then $[k] = (t^{2k} - 1)/(t - t^{-1})$ whereas $[k] = \pm k$ if $t = \pm 1$.

The quantum characteristic of $v$ is the smallest element of $e \in \{2, 3, 4, 5, \ldots\} \cup \{\infty\}$ such that $[e]_v = 0$, where we set $e = \infty$ if $[k]_v \neq 0$ for all $k > 0$. ANDREW MATHAS
1.2.1. Definition. Suppose that \( \Lambda = \Lambda(\kappa) \in P^+ \), for \( \kappa \in \mathbb{Z}^\ell \), and that \( v \in \mathbb{Z} \) has quantum characteristic \( e \). The integral cyclotomic Hecke algebra of type \( A \) of weight \( \Lambda \) is the cyclotomic Hecke algebra \( \mathcal{H}_n^{\Lambda} = \mathcal{H}_n(\mathbb{Z}, v, Q_1, \ldots, Q_r) \) with Hecke parameter \( v \) and cyclotomic parameters \( Q_r = [\kappa_r]_e \), for \( 1 \leq r \leq \ell \).

When \( v^2 \neq 1 \) the parameter \( Q_r \) corresponds to the Ariki-Koike parameters \( Q'_r = v^{2\kappa_r} \), for \( 1 \leq r \leq \ell \), where we use the notation of §1.1.

As observed in [57, §2.2], translating the Morita equivalence theorems of [32, Theorem 1.1] and [20, Theorem 5.19] into the current setting explains the significance of the integral cyclotomic Hecke algebras.

1.2.2. Theorem (Dipper-Mathas [32], Brundan-Kleshchev [20]). Every cyclotomic quiver Hecke algebra \( \mathcal{H}_n^{\Lambda} \) is Morita equivalent to a direct sum of tensor products of integral cyclotomic Hecke algebras.

Brundan and Kleshchev treated the degenerate case when \( v^2 = 1 \) using very different arguments than those in [32]. With the benefit of Definition 1.1.1 the argument of [32] now applies uniformly to both the degenerate and non-degenerate cases. The Morita equivalences in [20, 32] are described explicitly, with the equivalence being determined by orbits of the cyclotomic parameters. See [20, 32] for more details.

In view of Theorem 1.2.2, it is enough to consider the integral cyclotomic Hecke algebras \( \mathcal{H}_n^{\Lambda} \) where \( v \in \mathbb{Z}^\times \) has quantum characteristic \( e \) and \( \Lambda \in P^+ \). This said, for most of Section 1 we consider the general case of a not necessarily integral cyclotomic Hecke algebra because we will need this generality in §4.2.

1.3. Cellular algebras. For convenience we recall Graham and Lehrer’s cellular algebra framework [48]. This will allow us to define Specht modules for \( \mathcal{H}_n^{\Lambda} \) as cell modules. Significantly, the cellular algebra machinery endows the Specht modules with an associative bilinear form. Here is the definition.

1.3.1. Definition (Graham and Lehrer [48]). Suppose that \( A \) is a \( \mathbb{Z} \)-algebra that is \( \mathbb{Z} \)-free and of finite rank as a \( \mathbb{Z} \)-module. A cell datum for \( A \) is an ordered triple \( (P, T, C) \), where \( (P, \triangleright) \) is the weight poset, \( T(\lambda) \) is a finite set for \( \lambda \in P \), and

\[
C : \prod_{\lambda \in P} T(\lambda) \times T(\lambda) \rightarrow A; (s, t) \mapsto c_{st},
\]

is an injective map of sets such that:

(\( GC_1 \)) \{ \( c_{\lambda} \mid s, t \in T(\lambda) \text{ for } \lambda \in P \} \) is a \( \mathbb{Z} \)-basis of \( A \).

(\( GC_2 \)) If \( s, t \in T(\lambda) \), for some \( \lambda \in P \), and \( a \in A \) then there exist scalars \( r_{\lambda}(a) \), which do not depend on \( s \), such that

\[
c_{st}a = \sum_{\nu \in T(\lambda)} r_{\lambda}(a)c_{\nu}(\text{mod } A^{P_{\lambda}}),
\]

where \( A^{P_{\lambda}} = \{ c_{\mu} \mid \mu \triangleright \lambda \text{ and } a, b \in T(\mu) \} \).

(\( GC_3 \)) The \( \mathbb{Z} \)-linear map \( * : A \rightarrow A \) determined by \( c_{st} = c_{ts} \), for all \( \lambda \in P \) and all \( s, t \in T(\lambda) \), is an anti-isomorphism of \( A \).

A cellular algebra is an algebra that has a cell datum. If \( A \) is a cellular algebra with cell datum \( (P, T, C) \) then the basis \( \{ c_{\lambda} \mid \lambda \in P \text{ and } s, t \in T(\lambda) \} \) is a cellular basis of \( A \) with cellular algebra anti-isomorphism \( * \).

König and Xi [86] have given an equivalent definition of cellular algebras that does not depend upon a choice of basis. Goodman and Gruber [45] have shown that (GC3) can be relaxed to the requirement that \( (c_{st})^\dagger = c_{ts} \) (mod \( A^{P_{\lambda}} \)) for some anti-isomorphism \( * \) of \( A \).

The prototypical example of a cellular algebra is a matrix algebra with its basis of matrix units, which we call a Wedderburn basis. As any split semisimple algebra is isomorphic to a direct sum of matrix algebras it follows that every split semisimple algebra is cellular. The cellular algebra framework is, however, most useful in studying non-semisimple algebras that are not isomorphic to a direct sum of matrix rings. In general, a cellular basis can be thought of as an approximation, or weakening, of a basis of matrix units. (This idea is made more explicit in [108].)

The cellular basis axioms determines a filtration of the cellular algebra, via the ideals \( A^{P_{\lambda}} \). As we will see, this leads to a quick construction of its irreducible representations.

For \( \lambda \in P \), let \( A^{P_{\lambda}} = \{ c_{ab} \mid a, b \in T(\mu) \} \). Then it follows from Definition 1.3.1 that \( A^{P_{\lambda}} \) is a two-sided ideal of \( A \).

Fix \( \lambda \in P \). The cell module \( \underline{\lambda} \) is the (right) \( A \)-module with basis \( \{ c_{\lambda} \mid t \in T(\lambda) \} \) and where \( a \in A \) acts on \( \underline{\lambda} \) by:

\[
c_{\lambda}a = \sum_{\nu \in T(\lambda)} r_{\lambda}(a)c_{\nu}, \quad \text{for } t \in T(\lambda),
\]

where the scalars \( r_{\lambda}(a) \in \mathbb{Z} \) are those appearing in (GC2). It follows immediately from Definition 1.3.1 that \( \underline{\lambda} \) is an \( A \)-module. Indeed, if \( s \in T(\lambda) \) then \( \underline{\lambda} \) is isomorphic to the submodule \( (c_{\lambda} + A^{P_{\lambda}})A \) of \( A/A^{\lambda} \) via
the map \( e_t \mapsto e_t + A^\lambda \), for \( t \in T(\lambda) \). The cell module \( C^\lambda \) comes with a symmetric bilinear form \( \langle \ , \ \rangle_\lambda \) that is uniquely determined by

\[
\langle c_t, c_v \rangle_\lambda \cdot c_{ab} \equiv c_{at} c_{vb} \pmod{A^{\lambda^+}},
\]

for \( a, b, t, v \in T(\lambda) \). By (GC2) of Definition 1.3.1, the inner product \( \langle c_t, c_v \rangle_\lambda \) depends only on \( t \) and \( v \), and not on the choices of \( a \) and \( b \). In addition, \( \langle xa, y \rangle_\lambda = \langle x, ya^* \rangle_\lambda \), for all \( x, y \in C^\lambda \) and \( a \in A \). Therefore,

\[
\text{rad } C^\lambda = \{ x \in C^\lambda \mid \langle x, y \rangle_\lambda = 0 \text{ for all } y \in C^\lambda \}
\]
is an \( A \)-submodule of \( C^\lambda \). Set \( D^\lambda = C^\lambda / \text{rad } C^\lambda \). Then \( D^\lambda \) is an \( A \)-module.

The following theorem summarizes some of the main properties of a cellular algebra. The proof is surprisingly easy given the strength of the result. In applications the main difficulty is in showing that a given algebra is cellular.

If \( M \) is an \( A \)-module and \( D \) is an irreducible \( A \)-module, let \([M : D]\) be the decomposition multiplicity of \( D \) in \( M \).

1.3.4. Theorem (Graham and Lehrer [48]), Suppose that \( Z = F \) is a field. Then:

a) If \( \mu \in \mathcal{P} \) then \( D^\mu \) is either zero or absolutely irreducible.

b) Let \( \mathcal{K} = \{ \mu \in \mathcal{P} \mid D^\mu \neq 0 \} \). Then \( \{ D^\mu \mid \mu \in \mathcal{K} \} \) is a complete set of pairwise non-isomorphic irreducible \( A \)-modules.

c) If \( \lambda \in \mathcal{P} \) and \( \mu \in \mathcal{K} \) then \( \langle C^\lambda : D^\mu \rangle \neq 0 \) only if \( \lambda \succeq \mu \). Moreover, \( \langle C^\lambda : D^\mu \rangle = 1 \).

If \( \mu \in \mathcal{K} \) let \( D^\mu \) be the projective cover of \( D^\mu \). It follows from Definition 1.3.1 that \( D^\mu \) has a filtration in which the quotients are cell modules such that \( C^\lambda \) appears with multiplicity \( \langle C^\lambda : D^\mu \rangle \). Consequently, an analogue of Brauer-Humphreys reciprocity holds for \( \lambda \). In particular, the Cartan matrix of \( A \) is symmetric.

1.4. Multipartitions and tableaux. A partition \( \lambda \) of \( m \) is a weakly decreasing sequence \( \lambda = (\lambda_1, \lambda_2, \ldots) \) of non-negative integers such that \( |\lambda| = \lambda_1 + \lambda_2 + \cdots = m \). An \((\ell-)\)multipartition of \( n \) is an \( \ell \)-tuple \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \) of partitions such that \( |\lambda^{(1)}| + \cdots + |\lambda^{(\ell)}| = n \). We identify the multipartition \( \lambda \) with its diagram, which is the set of nodes \( [\lambda] = \{(l, r, c) \mid 1 \leq c \leq \lambda^{(l)}_r \text{ for } 1 \leq l \leq \ell \} \). In this way, we think of \( \lambda \) as an ordered \( \ell \)-tuple of arrays of boxes in the plane and we talk of the components of \( \lambda \). Similarly, by the rows and columns of \( \lambda \) we will mean the rows and columns in each component. For example, if \( \lambda = (3, 1^22, 1^33, 2) \) then

\[
\lambda = [\lambda] = \begin{pmatrix}
\begin{array}{ccc}
\begin{array}{l}
\end{array} & \begin{array}{l}
\end{array} & \begin{array}{l}
\end{array}
\end{array}
\begin{array}{ccc}
\begin{array}{l}
\end{array} & \begin{array}{l}
\end{array} & \begin{array}{l}
\end{array}
\end{array}
\end{pmatrix}
\end{pmatrix}.
\]

A node \( A \) is an addable node of \( \lambda \) if \( A \not\in \lambda \) and \( \lambda \cup \{ A \} \) is the (diagram of) a multipartition of \( n + 1 \). Similarly, a node \( B \) is a removable node of \( \lambda \) if \( B \in \lambda \) and \( \lambda \setminus \{ B \} \) is a multipartition of \( n - 1 \). If \( A \) is an addable node of \( \lambda \) let \( \lambda + A \) be the multipartition \( \lambda \cup \{ A \} \) and, similarly, if \( B \) is a removable node let \( \lambda - B = \lambda \setminus \{ B \} \). Order the nodes lexicographically by \( \leq \).

The set of multipartitions of \( n \) becomes a poset under dominance where \( \lambda \) dominates \( \mu \), written as \( \lambda \succeq \mu \), if

\[
\sum_{k=1}^{l-1} |\lambda^{(k)}| + \sum_{j=1}^{i} \lambda^{(i)}_j \geq \sum_{k=1}^{l-1} |\mu^{(k)}| + \sum_{j=1}^{i} \mu^{(i)}_j,
\]

for \( 1 \leq l \leq \ell \) and \( i \geq 1 \). If \( \lambda \succeq \mu \) and \( \lambda \neq \mu \) then write \( \lambda \triangleright \mu \). Let \( \mathcal{P}^\lambda_n = \mathcal{P}^\lambda \cap \mathcal{P}_n \) be the set of multipartitions of \( n \). We consider \( \mathcal{P}^\lambda_n \) as a poset ordered by dominance.

Fix \( \lambda \in \mathcal{P}^\lambda_n \). A \( \lambda \)-tableau is a bijective map \( t : [\lambda] \rightarrow \{1, 2, \ldots, n\} \), which we identify with a labelling of (the diagram of) \( \lambda \) by \( \{1, 2, \ldots, n\} \). For example,

\[
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
9 & 10 & 11 \\
6 & 8 & 12 & 13
\end{pmatrix}
\]

are both \( \lambda \)-tableaux when \( \lambda = (3, 1^22, 1^33, 2) \).

A \( \lambda \)-tableau is standard if its entries increase along rows and down columns in each component. For example, the two tableaux above are standard. Let \( \text{Std}(\lambda) \) be the set of standard \( \lambda \)-tableaux. If \( \mathcal{P} \) is any set of multipartitions let \( \text{Std}(\mathcal{P}) = \bigcup_{\lambda \in \mathcal{P}} \text{Std}(\lambda) \). Similarly set \( \text{Std}^2(\mathcal{P}) = \{ (s, t) \mid s, t \in \text{Std}(\lambda) \text{ for } \lambda \in \mathcal{P} \} \).

If \( t \) is a \( \lambda \)-tableau set \( \text{Shape}(t) = \lambda \) and let \( t_{\lambda m} \) be the subtableau of \( t \) that contains the numbers \( \{1, 2, \ldots, m\} \). If \( t \) is a standard \( \lambda \)-tableau then \( \text{Shape}(t_{\lambda m}) \) is a multipartition for all \( m \geq 0 \). We extend the dominance ordering to \( \text{Std}(\mathcal{P}_n) \), the set of all standard tableaux, by defining \( s \triangleright t \) if \( \text{Shape}(s_{\lambda m}) \triangleright \text{Shape}(t_{\lambda m}) \), for \( 1 \leq m \leq n \). As before, write \( s \triangleright t \) if \( s \triangleright t \) and \( s \not\triangleright t \). Finally, define the strong dominance...
ordering on $\text{Std}^2(\mathcal{P}_n^\lambda)$ by $(s, t) \triangleright (u, v)$ if $s \supseteq u$ and $t \supseteq v$. Similarly, $(s, t) \triangleright (u, v)$ if $(s, t) \triangleright (u, v)$ and $(s, t) \neq (u, v)$.

It is easy to see that there are unique standard $\lambda$-tableaux $t^\lambda$ and $\mathbf{t}_\lambda$ such that $t^\lambda \triangleright t \triangleright \mathbf{t}_\lambda$, for all $t \in \text{Std}(\lambda)$. The tableau $t^\lambda$ has the numbers $1, 2, \ldots, n$ entered in order from left to right along the rows of $t^{\ell(\lambda)}$, and then $t^{\ell(\lambda)+1}, \ldots , t^{\ell(\lambda)}$. Similarly, $\mathbf{t}_\lambda$ is the tableau with the numbers $1, \ldots, n$ entered in order down the columns of $t^{\ell(\lambda)}, \ldots , t^{\ell(\lambda)+2}, t^{\ell(\lambda)}$. If $\lambda = (3, 1^2, 2, 1^3, 2, 1^3)$ then the two $\lambda$-tableaux displayed above are $t^\lambda$ and $\mathbf{t}_\lambda$, respectively.

Given a standard $\lambda$-tableau $t$ define permutations $d(t), d'(t) \in \mathfrak{S}_n$ by $t^\lambda d(t) = t = t^\lambda d'(t)$. Then $d(t)(d'(t))^{-1} = d(t)x_t \in \ell(d(t)) + \ell(d'(t))$ for all $t \in \text{Std}(\lambda)$. Let $\beta$ be the Bruhat order on $\mathfrak{S}_n$ with the convention that $1 \leq w$ for all $w \in \mathfrak{S}_n$. Independently, Ehresmann and James [59] showed that if $s, t \in \text{Std}(\lambda)$ then $s \triangleright t$ if and only if $d(s) \leq d(t)$ and if and only if $d'(s) \leq d'(t)$. A proof can be found, for example, in [104, Theorem 3.8].

Finally, we will need to know how to conjugate multipartitions and tableaux. The conjugate of a partition $\lambda$ is the partition $\lambda' = (\lambda_1', \lambda_2', \ldots)$ where $\lambda_i' = \# \{ s \geq 1 \mid \lambda_s \geq r \}$. That is, we swap the rows and columns of $\lambda$. The conjugate of a multipartition $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ is the multipartition $\lambda' = (\lambda^{(1)}', \ldots, \lambda^{(r)}')$. Similarly, the conjugate of a $\lambda$-tableau $t = (t^{(1)}, \ldots, t^{(r)})$ is the $\lambda'$-tableau $t' = (t^{(1)}', \ldots, t^{(r)'})$ where $t^{(r)'}$ is the tableau obtained by swapping the rows and columns of $t^{(k)}$, for $1 \leq k \leq r$. Then $\lambda \triangleright \mu$ if and only if $\mu' \triangleright \lambda'$.

1.5. The Murphy basis of $\mathcal{H}_n^A$. Graham and Lehrer [48] showed that the cyclotomic Hecke algebras (when $v^2 \neq 1$) are cellular algebras. In this section we recall another cellular basis for these algebras that was constructed in [31] when $v^2 = 1$ and in [13] when $v^2 = 1$. When $\ell = 1$ these results are due to Murphy [113].

First observe that Definition 1.1.1 implies that there is a unique anti-isomorphism $\ast$ on $\mathcal{H}_n^A$ that fixes each of the generators $T_1, \ldots, T_{n-1}, L_1, \ldots, L_n$ of $\mathcal{H}_n^A$. It is easy to see that $T_w = T_{w^{-1}}$, for $w \in \mathfrak{S}_n$.

Fix a multipartition $\lambda \in \mathcal{P}_n^A$. Following [31, Definition 3.14] and [13, §6], if $s, t \in \text{Std}(\lambda)$ define $m_{st} = T_{d(s)}^{-1}m_{\lambda}T_{d(t)}$, where $m_{\lambda} = u_{\lambda}x_{\lambda}$,

$$u_{\lambda} = \prod_{1 \leq r < t} |\lambda^{(r)}| \prod_{i=1}^{\ell(\lambda)} \frac{1}{Q_i} (L_{i} - [\kappa_{i+1}]) \quad \text{and} \quad x_{\lambda} = \sum_{w \in \mathfrak{S}_\lambda} v^{\ell(w)} T_w,$$

where $Q_i = 1 + (v^{-1} - v)$ as in §1.1. The renormalization of $u_{\lambda}$ by $1/Q_i$ is not strictly necessary. When $Q_{i+1} = 0$ this factor can be omitted from the definition of $u_{\lambda}$, at the expense of some formalisms in some of the integral case, which is what we care most about, this problem does not arise because $Q_i = v^{\kappa_i} \neq 0$ since $Q_1 = [\kappa_1]$, for $1 \leq i \leq \ell$.

Using the relations in $\mathcal{H}_n^A$ it is not hard to show that $u_{\lambda}$ and $x_{\lambda}$ commute. Consequently, $m_{st} = m_{ts}$, for all $(s, t) \in \text{Std}^2(\mathcal{P}_n^\lambda)$.

1.5.1. Theorem ([31, Theorem 3.26] and [13, Theorem 6.3]). The cyclotomic Hecke algebra $\mathcal{H}_n^A$ is free as a $\mathbb{Z}$-module with cardinal basis \{ $m_{st} \mid s, t \in \text{Std}(\lambda)$ for $\lambda \in \mathcal{P}_n^A$ \} with respect to the poset $(\mathcal{P}_n^\lambda, \triangleright)$.

Consequently, $\mathcal{H}_n^A$ is a cellular algebra so all of the theory in §1.3 applies. In particular, for each $\lambda \in \mathcal{P}_n^A$ there exists a Specht module $S_\lambda$ with basis \{ $m_{ts} \mid t \in \text{Std}(\lambda)$ \}. Concretely, we could take $m_t = m_{ta} + \mathcal{H}_n^Am_{\lambda}$ for $t \in \text{Std}(\lambda)$.

Let $D_\lambda^A = S_\lambda^A / \text{rad} S_\lambda^A$ be the quotient of $S_\lambda^A$ by the radical of its bilinear form. Set $K_n^A = \{ \mu \in \mathcal{P}_n^A \mid D_\mu^A \neq 0 \}$. By Theorem 1.3.4 we obtain:

1.5.2. Corollary ([13, 31, 48]). Suppose that $F$ is a field. Then \{ $D_\mu^A \mid \mu \in K_n^A$ \} is a complete set of pairwise non-isomorphic irreducible $\mathcal{H}_n^A$-modules.

The set of multipartitions $K_n^A$ has been determined by Ariki [4]; see also [8, 22]. We describe and recover his classification of the irreducible $\mathcal{H}_n^A$-modules in Corollary 3.5.28 below. When $\ell \geq 3$ the only known descriptions of $K_n^A$ are recursive. See [11, 29] for $\ell \leq 2$ and [103] when $\ell = 2$.

1.6. Semisimple cyclotomic Hecke algebras of type $A$. We now explicitly describe the semisimple representation theory of $\mathcal{H}_n^A$ using the seminormal coefficient systems introduced in [57]. As we are ultimately interested in the cyclotomic quiver Hecke algebras, which are intrinsically non-semisimple algebras, it is a little surprising that we are interested in these results. We will see, however, that the semisimple representation theory of $\mathcal{H}_n^A$ and the KLR grading are closely intertwined.

The Gelfand-Zetlin subalgebra of $\mathcal{H}_n^A$ is the subalgebra $\mathcal{L}_n^A = \mathcal{L}_n(A) = (L_1, L_2, \ldots, L_n)$. We believe that understanding this subalgebra is crucial to understanding the representation theory of $\mathcal{H}_n^A$. To explain how $\mathcal{L}_n^A$ acts on $\mathcal{H}_n^A$ we define two content functions for $t \in \text{Std}(\mathcal{P}_n^\lambda)$ by

$$(1.6.1) \quad c^2_t(t) = v^{2(c-b)}Q_t + [c-b]v \in \mathbb{Z} \quad \text{and} \quad c^2_t(t) = \kappa_t + c - b \in \mathbb{Z},$$
where \( t(l, b, c) = r \) and \( 1 \leq r \leq n \). In the special case of the integral parameters, where \( Q_l = [k]\), for \( 1 \leq l \leq \ell \), the reader can check that \( c^2_r(t) = [c^2_r(t)]_\ell \), for \( 1 \leq r \leq n \).

The next result is well-known and extremely useful.

1.6.2. Lemma (James-Mathas [62, Proposition 3.7]). Suppose that \( 1 \leq r \leq n \) and that \( s, t \in \text{Std}(\lambda) \), for \( \lambda \in \mathcal{P}^\Lambda_n \). Then

\[
m_{st}L_r \equiv c^2_r(t)m_{st} + \sum_{v \in C} a_v m_{sv} \quad (\text{mod } \mathcal{H}_n^{\Lambda, \lambda}),
\]

for some \( a_v \in \mathbb{Z} \).

Proof. Let \( (l, b, c) = t^{-1}(r) \). Using our notation, [62, Proposition 3.7] says that \( m_{st}L_r' = Q_l^r t^{2(c-b)} m_{st} \) plus linear combination of more dominant terms, where \( Q_l^r = 1 + (v - v^{-1})Q_l \). As \( L_r = 1 + (v - v^{-1})L_r' \), this easily implies the result when \( v^2 \neq 1 \). The case when \( v^2 = 1 \) now follows by specialization — or, see [13, Lemma 6.6].

In the integral case, \( m_{st}L_r \equiv [c^2_r(t)]_\lambda m_{st} + \sum_{v \in C} a_v m_{sv} \quad (\text{mod } \mathcal{H}_n^{\Lambda, \lambda}) \). This agrees with [57, Lemma 2.9].

The Hecke algebra \( \mathcal{H}_n \) is content separated if whenever \( s, t \in \text{Std}(\mathcal{P}^\Lambda_n) \) are standard tableaux, not necessarily of the same shape, then \( s = t \) if and only if \( c^2_r(s) = c^2_r(t) \), for \( 1 \leq r \leq n \). The following is an immediate corollary of Lemma 1.6.2 using the theory of JM-elements developed in [108, Theorem 3.7].

1.6.3. Corollary (57, Proposition 3.4). Suppose that \( Z = F \) is a field and that \( \mathcal{H}_n \) is content separated. Then, as an \((L_n, L_n)\)-bimodule,

\[
\mathcal{H}_n = \bigoplus_{(s, t) \in \text{Std}(\mathcal{P}^\Lambda_n)} H_{st},
\]

where \( H_{st} = \{ h \in \mathcal{H}_n \mid L_r h = c^2_r(s)h \text{ and } hL_r = c^2_r(t)h, \text{ for } 1 \leq r \leq n \} \).

For the rest of §1.6 we assume that \( \mathcal{H}_n \) is content separated. Corollary 1.6.3 motivates the following definition.

1.6.4. Definition (Hu-Mathas [57, Definition 3.7]). Suppose that \( Z = K \) is a field. A \(*\)-seminormal basis of \( \mathcal{H}_n \) is a basis of the form

\[
\{ f_{st} \mid 0 \neq f_{st} \in H_{st} \text{ and } f^*_n = f_{ts}, \text{ for } (s, t) \in \text{Std}(\mathcal{P}^\Lambda_n) \}.
\]

There is a vast literature on seminormal bases. This story started with Young’s seminormal forms for the symmetric groups [137] and has now been extended to Hecke algebras and many other diagram algebras including the Brauer, BMW and partition algebras; see, for example, [105, 115, 118].

Suppose that \( \{ f_{st} \} \) is a \(*\)-seminormal basis and that \( (s, t), (u, v) \in \text{Std}(\mathcal{P}^\Lambda_n) \). Let \( \mathcal{C}_n = \{ c^2_r(s) \mid s \in \text{Std}(\mathcal{P}^\Lambda_n) \} \) for \( 1 \leq r \leq n \) be the set of all possible contents for the tableaux in \( \text{Std}(\mathcal{P}^\Lambda_n) \). Following Murphy [108, 112], for a standard tableau \( s \in \text{Std}(\mathcal{P}^\Lambda_n) \) define

\[
F_s = \prod_{r=1}^{n} \prod_{c \neq c^2_r(s)} \frac{L_r - c}{c^2_r(s) - c}.
\]

By Definition 1.6.4, if \( (s, t), (u, v) \in \text{Std}(\mathcal{P}^\Lambda_n) \) then \( \delta_{su} \delta_{tv} f_{st} = F_u f_{st} F_v \). In particular, \( F_s \) is a non-zero element of \( \mathcal{H}_n \). It follows that \( F_s \) is a scalar multiple of \( f_{ss}^\Lambda \), which implies that \( \{ F_s \mid s \in \text{Std}(\mathcal{P}^\Lambda_n) \} \) is a complete set of pairwise orthogonal idempotents in \( \mathcal{H}_n \). (In fact, in [108] these properties are used to establish Corollary 1.6.3.) Consequently, there exists a non-zero scalar \( \gamma_s \in F \) such that \( F_s = \frac{1}{\gamma_s} f_{ss} \). If \( (s, t), (u, v) \in \text{Std}(\mathcal{P}^\Lambda_n) \) then

\[
f_{st} f_{uv} = f_{st} F_u F_t f_{uv} = \delta_{uu} \gamma_t f_{sv},
\]

(1.6.5)

The next definition allows us to classify all seminormal bases and to describe how \( \mathcal{H}_n^{\Lambda, \lambda} \) acts on them.

1.6.6. Definition (Hu-Mathas [57, §3]). A \(*\)-seminormal coefficient system is a collection of scalars

\[
\alpha = \{ \alpha_r(t) \mid t \in \text{Std}(\mathcal{P}^\Lambda_n) \text{ and } 1 \leq r \leq n \}
\]

such that \( \alpha_r(t) = 0 \) if \( v = t(r, r + 1) \) is not standard, if \( v \in \text{Std}(\mathcal{P}^\Lambda_n) \) then

\[
\alpha_r(v) \alpha_r(t) = \frac{(1 - v^{-1} c^2_r(t) + v c^2_r(v)) (1 + v c^2_r(t) - v^{-1} c^2_r(v))}{(c^2_r(t) - c^2_r(v))(c^2_r(v) - c^2_r(t))},
\]

and \( \alpha_r(t) \alpha_{r+1}(ts_r) = \alpha_{r+1}(t) \alpha_r(ts_{r+1}) \alpha_{r+1}(ts_{r+1}s_r) \) and if \( |r - r'| > 1 \) then \( \alpha_r(t) \alpha_{r'}(ts_r) = \alpha_r(t) \alpha_{r'}(ts_{r'}) \), for \( 1 \leq r, r' < n \).
As the reader might guess, the conditions on the scalars $\alpha_r(t)$ in Definition 1.6.6 correspond to the quadratic relations $(T_r - v)(T_r + v^{-1}) = 0$ and the braid relations for $T_1, \ldots, T_n$. The simplest example of a seminormal coefficient system is

$$\alpha_v(t) = \frac{1 - v^{-1}c^{s}_{r+1}(t) + vc^{s}_{r}(t)}{(c^{s}_{r+1}(t) - c^{s}_{r}(t))},$$

whenever $1 \leq r < n$ and $t, t(r, r + 1) \in \text{Std}(P^\Delta_n)$. Another seminormal coefficient system is given in (1.7.1) below.

Seminormal coefficient systems arise because they describe the action of $H_n$ on a seminormal basis. More precisely, we have the following:

1.6.7. **Theorem** (Hu-Mathas [57]). Suppose that $Z = K$ is a field and that $H_n$ is content separated and that $\{f_{\lambda t} \mid (s, t) \in \text{Std}^2(P^\Delta_n)\}$ is a seminormal basis of $\mathcal{H}_n$. Then $\{f_{\lambda t}\}$ is a cellular basis of $\mathcal{H}_n$ and there exists a unique seminormal coefficient system $\alpha$ such that

$$f_{\lambda t}T_r = \alpha_v(t)f_{sv} + \frac{1 + (v - 1)c^{s}_{r+1}(t)}{(c^{s}_{r+1}(t) - c^{s}_{r}(t))}f_{\lambda t},$$

where $v = t(r, r + 1)$. Moreover, if $s \in \text{Std}(\lambda)$ then $F_s = \frac{1}{n}f_{\lambda t}$ is a primitive idempotent and $\sum_{\lambda} \cong F_{\lambda}H_n$ is irreducible for all $\lambda \in P^\Delta_n$.

**Sketch of proof.** By definition, $\{f_{\lambda t}\}$ is a basis of $\mathcal{H}_n$ such that $f_{\lambda t}^* = f_{\lambda t}$ for all $(s, t) \in \text{Std}^2(P^\Delta_n)$. Therefore, it follows from (1.6.5) that $\{f_{\lambda t}\}$ is a cellular basis of $\mathcal{H}_n$ with cellular automorphism $\ast$.

It is an amusing application of the relations in Definition 1.1.1 to show that there exists a seminormal coefficient system that describes the action of $T_r$ on the seminormal basis. See [57, Lemma 3.13] for details. The uniqueness of $\alpha$ is clear.

We have already observed in (1.6.5) that $F_s = \frac{1}{n}f_{\lambda t}$, for $s \in \text{Std}(\lambda)$, so it remains to show that $F_s$ is primitive and that $S^\Delta \cong F_sH_n$. By what we have already shown, $F_sH_n$ is contained in the span of $\{f_{\lambda t} \mid t \in \text{Std}(\lambda)\}$. On the other hand, if $f = \sum_{s} f_{s}f_{\lambda t} \in F_sH_n$ and $r_v \neq 0$ then $r_vf_{sv} = fF_v \in F_sH_n$. It follows that $F_sH_n = \sum_{s} Kf_{sv}$, as a vector space. Consequently, $F_sH_n$ is irreducible and $F_s$ is a primitive idempotent in $H_n$. Finally, $S^\Delta \cong F_sH_n$ by Lemma 1.6.2 since $H_n$ is content separated. $lacksquare$

1.6.8. **Corollary** ([57, Corollary 3.17]). Suppose that $\alpha$ is a seminormal coefficient system and that $s \triangleright t = s(r, r + 1)$, for tableaux $s, t \in \text{Std}(P^\Delta_n)$ and where $1 \leq r < n$. Then $\alpha_r(s)\gamma_t = \alpha_r(t)\gamma_s$.

Consequently, if the seminormal coefficient system $\alpha$ is known then fixing $\gamma_t$, for some $t \in \text{Std}(\lambda)$, determines $\gamma_t$ for all $s \in \text{Std}(\lambda)$.

1.6.9. **Corollary** (Classifying seminormal bases [57, Theorem 3.14]). There is a one-to-one correspondence between the $s$-seminormal bases of $\mathcal{H}_n$ and the pairs $(\alpha, \gamma)$ where $\alpha = \{\alpha_r(s) \mid 1 \leq r < n \text{ and } s \in \text{Std}(P^\Delta_n)\}$ is a seminormal coefficient system and $\gamma = \{\gamma_\lambda \mid \lambda \in P^\Delta_n\}$.

Finally, the seminormal basis machinery in this section can be used to classify the semisimple cyclotomic Hecke algebras $H_n$, thus re-proving Ariki’s semisimplicity criterion [2], when $v^2 \neq 1$, and [13, Theorem 6.11], when $v^2 = 1$.

1.6.10. **Theorem** (Ariki [2] and [13, Theorem 6.11]). Suppose that $F$ is a field. The following are equivalent:

a) $\mathcal{H}_n = \mathcal{H}_n(F, v, Q_1, \ldots, Q_l)$ is semisimple.

b) $\mathcal{H}_n$ is content separated.

c) $[1]_v[2]_v \cdots [n]_v \prod_{1 \leq r < s \leq n \text{ and } c^s_r} (v^{2d}Q_r + [d]_v - Q_s) \neq 0$.

We want to rephrase the semisimplicity criterion of Theorem 1.6.10 for the integral cyclotomic Hecke algebras $\mathcal{H}_n$, for $\Lambda \in P^+$. For each $i \in I$ define the $i$-string of length $n$ to be $\alpha_t = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{i+n-1}$. Then $\alpha_t \in Q^+$.

1.6.11. **Corollary.** Suppose that $\Lambda \in P^+$ and that $Z = F$ is a field. Then $\mathcal{H}_n^\Lambda$ is semisimple if and only if $e > n$ and $(\Lambda, \alpha_t) \leq 1$, for all $i \in I$.

**Proof.** As $Q_r = [\kappa_r]_v$, for $1 \leq r \leq \ell$, the statement of Theorem 1.6.10(c) simplifies because $v^{2c}Q_r + [d]_v - Q_s = v^{-2c}[d + \kappa_r - \kappa_s]_v$. Therefore, $\mathcal{H}_n^\Lambda$ is semisimple if and only if

$$[1]_v[2]_v \cdots [n]_v \prod_{1 \leq r < s \leq n \text{ and } c^s_r} [d + \kappa_r - \kappa_s]_v \neq 0.$$
1.7. Gram determinants and the Jantzen sum formula. For future use, we now recall the closed formula for the Gram determinants of the Specht modules $S^\lambda$ and the connection between these formulas and Jantzen filtrations. Throughout this section we assume that $\mathcal{H}_n$ is content separated over the field $K = \mathbb{Z}$.

For $\lambda \in \mathcal{P}_n$ let $G^\lambda = \{(m_s, m_t)\}_{s \in \text{Std}(\lambda)}$ be the Gram matrix of the Specht module $S^\lambda$, where we fix an arbitrary ordering of the rows and columns of $G^\lambda$.

For $(s, t) \in \text{Std}^2(\mathcal{P}_n)$ set $f_{st} = F_s m_{st} F_t$. By Lemma 1.6.2 and (1.6.5),

$$f_{st} = m_{st} + \sum_{(u, v) \in (s, t)} r_{uv} m_{uv},$$

for some $r_{uv} \in K$. By construction, $\{f_{st}\}$ is a seminormal basis of $\mathcal{H}_n$. By [57, Proposition 3.18] this basis corresponds to the seminormal coefficient system given by

$$\alpha_r(t) = \begin{cases} 1, & \text{if } t \succ (r, r + 1), \\ \frac{(1 - v r c_r(t) + v c_r(v)) (1 + v c_r(t) - r c_r(v))}{c_r(t) - c_r(v)}, & \text{otherwise}, \end{cases}$$

for $t \in \text{Std}(\mathcal{P}_n)$ and $1 \leq r < n$ such that $ts_r$ is standard. The $\gamma$-coefficients $\{\gamma_t\}$ for this basis are explicitly known by [62, Corollary 3.29]. Moreover,

$$\det G^\lambda = \prod_{t \in \text{Std}(\lambda)} \gamma_t$$

By explicitly computing the scalars $\gamma_t$, and using an intricate inductive argument based on the semisimple branching rules for the Specht modules, James and the author proved the following:

1.7.3. Theorem (James-Mathas [62, Corollary 3.38]). Suppose that $\mathcal{H}_n$ is content separated. Then there exist explicitly known scalars $g_{\lambda\mu}$ and signs $\varepsilon_{\lambda\mu} = \pm 1$ such that

$$\det G^\lambda = \prod_{\mu \in \mathcal{P}_n} \varepsilon_{\lambda\mu} g_{\lambda\mu} \dim S^\mu.$$

The scalars $g_{\lambda\mu}$ are described combinatorially as the quotient of at most two hook lengths. The sign $\varepsilon_{\lambda\mu}$ is the parity of the sum of the leg lengths of these hooks.

Theorem 1.7.3 gives a very pretty closed formula for the Gram determinant $G^\lambda$, generalizing a classical result of James and Murphy [64]. One problem with this formula is that $G^\lambda$ is a polynomial in $v, v^{-1}, Q_1, \ldots, Q_l$ whereas Theorem 1.7.3 computes this determinant as a rational function in $v, Q_1, \ldots, Q_l$.

On the other hand, as we now recall, Theorem 1.7.3 has an impressive module theoretic application in the Jantzen sum formula.

Fix a modular system $(K, Z, F)$, where $Z$ is a discrete valuation ring with maximal ideal $p$ and such that $Z$ contains $v, v^{-1}, Q_1, \ldots, Q_l$. Let $K$ be the field of fractions of $Z$ and let $F = Z/p$ be the residue field of $Z$. Let $\mathcal{H}_n^Z$, $\mathcal{H}_n^K \cong \mathcal{H}_n^Z \otimes_Z K$ and $\mathcal{H}_n^F = \mathcal{H}_n^Z \otimes_Z F$ be the corresponding Hecke algebras. Therefore, $\mathcal{H}_n^F$ has Hecke parameter $v + p$ and cyclotomic parameters $Q_i + p$, for $1 \leq i \leq l$.

Let $\lambda \in \mathcal{P}_n$ and let $S^\lambda_Z$ and $S^\lambda_K \cong S^\lambda_Z \otimes_Z K$ be the corresponding Specht modules for $\mathcal{H}_n^Z$ and $\mathcal{H}_n^K$, respectively. Define a filtration of the Specht module $S^\lambda_K$ by $J_k(S^\lambda_K) = \{ x \in S^\lambda_K \mid (x, y)_\lambda \in p^k \text{ for all } y \in S^\lambda_Z \}$, for $k \geq 0$. The Jantzen filtration of $S^\lambda_K$ is the filtration

$$S^\lambda_K = J_0(S^\lambda_K) \supseteq J_1(S^\lambda_K) \supseteq \cdots \supseteq J_k(S^\lambda_K) = 0,$$

where $J_k(S^\lambda_K) = (J_k(S^\lambda_Z) + p S^\lambda_Z^k) / p S^\lambda_Z^k$ for $k \geq 0$. (As $S^\lambda_Z$ is finite dimensional, $J_z(S^\lambda_Z) = 0$ for $z \gg 0$.)

Let $\text{Rep}(\mathcal{H}_n)$ be the category of finitely generated $\mathcal{H}_n$-modules and let $[\text{Rep}(\mathcal{H}_n)]$ be its Grothendieck group. Let $[M]$ be the image of the $\mathcal{H}_n$-module $M$ in $[\text{Rep}(\mathcal{H}_n)]$. Let $v_\mathfrak{p}$ be the $p$-adic valuation map on $Z^\times$.

1.7.4. Theorem (James-Mathas [62, Theorem 4.6]). Let $(K, Z, F)$ be a modular system and suppose that $\lambda \in \mathcal{P}_n$. Then, in $[\text{Rep}(\mathcal{H}_n^F)]$,

$$\sum_{k \geq 0} [J_k(S^\lambda_K)] = \sum_{\lambda \subseteq \mu} \varepsilon_{\lambda\mu} v_\mathfrak{p}(g_{\lambda\mu}) [S^\mu].$$

Intuitively, the proof of Theorem 1.7.4 amounts to taking the $p$-adic valuation of the formula in Theorem 1.7.3. In fact, this is exactly how Theorem 1.7.4 is proved except that you need the corresponding formulas for the Gram determinants of the weight spaces of the Weyl modules of the cyclotomic Schur algebras of [31]. This is enough because the dimensions of the weight spaces of a module uniquely determine its image in the Grothendieck group of the Schur algebra. The proof given in [62] is stated only for the non-degenerate case $v^2 \neq 1$, however, the arguments apply equally well for the degenerate case when $v^2 = 1$. 

The main point that we want to emphasize in this section is that the rational formula for $\det Q^\Lambda_n$ in Theorem 1.7.3 corresponds to writing the left-hand side of the Jantzen sum formula sum as a $\mathbb{Z}$-linear combination of Specht modules. Therefore, when the right-hand side of the sum formula is written as a linear combination of simple modules some of the terms must cancel. We give a cancellation free sum formula in §4.1.

Theorem 1.7.4 is a useful inductive tool because it gives an upper bound on the decomposition numbers $d_{\lambda\mu}$. Let $j_{\lambda\mu} = \varepsilon_{\mu\nu} |_{\nu = \lambda}$, for $\lambda, \mu \in \mathcal{P}_n^\Lambda$ and set $d_{\lambda\mu}^F = [S^\Lambda_n : D^\mu]$. Using Theorem 1.7.4 to compute the multiplicity of $D^\mu$ in $\bigoplus_{k>0} J_k(S^\Lambda_n)$ yields the following.

1.7.5. Corollary. Suppose that $\lambda, \mu \in \mathcal{P}_n^\Lambda$. Then $0 \leq d_{\lambda\mu}^F \leq \sum_{\nu \in \mathcal{P}_n^\Lambda, \lambda = \nu} j_{\lambda\mu} d_{\nu\mu}$.

As a second application, Theorem 1.7.4 classifies the irreducible Specht modules $S^\Lambda_n$, for $\lambda \in \mathcal{K}_n^\Lambda$.

1.7.6. Corollary (James-Mathas [62, Theorem 4.7]). Suppose that $F$ is a field and $\lambda \in \mathcal{K}_n^\Lambda$. Then the Specht module $S^\Lambda_n$ is irreducible if and only if $j_{\lambda\lambda} = 0$ for all $\mu > \lambda$.

1.8. The blocks of $\mathcal{H}_n^F$. The most important application of the Jantzen Sum Formula (Theorem 1.7.4) is to the classification of the blocks of $\mathcal{H}_n^F$. The algebra $\mathcal{H}_n^F$, and in fact any finite-dimensional algebra over a field, can be written as a direct sum of indecomposable two-sided ideals: $\mathcal{H}_n^F = B_1 \oplus \cdots \oplus B_d$. The subalgebras $B_1, \ldots, B_d$, which are the blocks of $\mathcal{H}_n^F$, are uniquely determined up to permutation. Any $\mathcal{H}_n^F$-module $M$ splits into a direct sum of block components $M = M_1 \oplus \cdots \oplus M_d$, where we allow some of the summands to be zero. The module $M$ belongs to the block $B_i$ if $M = M_i$. It is a standard fact that two simple modules $D^\lambda$ and $D^\mu$ belong to the same block if and only if they are in the same linkage class.

That is, there exists a sequence of indecomposable modules $M_1, \ldots, M_d$ and a sequence of multipartitions $\nu_0 = \lambda, \nu_1, \ldots, \nu_z = \mu$ such that $[M_i : D^{\nu_r}] \neq 0$ and $[M_i : D^{\nu_{r+1}}] \neq 0$, for $0 \leq r < z$. In fact, we can assume that the $M_i$ are Specht modules, even though the Specht modules are not necessarily indecomposable.

We want an explicit combinatorial description of the blocks of $\mathcal{H}_n^F$. Define two equivalence relations $\sim_C$ and $\sim_J$ on $\mathcal{P}_n^\Lambda$ as follows. First, $\lambda \sim_C \mu$ if there is an equality of multisets $\{ e_{\lambda_i}(r) \mid 1 \leq r \leq n \} = \{ e_{\mu_i}(r) \mid 1 \leq r \leq n \}$. The second relation, Jantzen equivalence, is more involved: $\lambda \sim_J \mu$ if there exists a sequence $\nu_0 = \lambda, \nu_1, \ldots, \nu_z = \mu$ of multipartitions in $\mathcal{P}_n^\Lambda$ such that $j_{\nu, \nu_{i+1}} \neq 0$ or $j_{\nu_{i+1}, \nu_r} \neq 0$, for $0 \leq r < z$.

1.8.1. Theorem (Lyle-Mathas [96], Brundan [19]). Suppose that $F$ is a field and that $\lambda, \mu \in \mathcal{P}_n^\Lambda$. Then the following are equivalent:

a) $D^\lambda$ and $D^\mu$ are in the same $\mathcal{H}_n^F$-block.
b) $S^\lambda_n$ and $S^\mu_n$ are in the same $\mathcal{H}_n^F$-block.
c) $\lambda \sim_J \mu$.
d) $\lambda \sim_C \mu$.

Parts (a) and (b) are equivalent by the general theory of cellular algebras [48] whereas the equivalence of parts (b) and (c) is a general property of Jantzen filtrations from [96]. (In fact, part (c) is general property of the standard modules of a quasi-hereditary algebra.) In practice, part (d) is the most useful because it is easy to compute.

The hard part in proving Theorem 1.8.1 is in showing that parts (c) and (d) are equivalent. The argument is purely combinatorial with work of Fayers [36,37] playing an important role.

In the integral case, when $\mathcal{H}_n^F = \mathcal{H}_n^\Lambda$ for some $\Lambda \in \mathcal{P}^+$, there is a nice reformulation of Theorem 1.8.1. The residue sequence of a standard tableau $t$ is $F = (i^1_1, \ldots, i^1_n) \in \mathcal{P}^+$ where $i^1_i = e_{\nu_i}(t) + e \mathbb{Z}$. If $t \in \text{Std}(\lambda)$, for $\lambda \in \mathcal{P}_n^\Lambda$, define $\beta^\lambda = \sum_{r=1}^n \alpha^r_{e_{\nu_i}(t)} = \sum_{r=1}^n \alpha^r_{e_{\nu_i}(t)} \in Q^+$. By definition, $\beta^\lambda \in Q^+$ depends only on $\lambda$, and not on the choice of $t$. Moreover, $\lambda \sim_C \mu$ if and only if $\beta^\lambda = \beta^\mu$. Hence, we have the following:

1.8.2. Corollary. Suppose that $\Lambda \in \mathcal{P}^+$ and $\lambda, \mu \in \mathcal{P}_n^\Lambda$. Then $S^\lambda$ and $S^\mu$ are in the same $\mathcal{H}_n^\Lambda$-block if and only if $\beta^\lambda = \beta^\mu$.

2. Cyclotomic quiver Hecke algebras of type $A$

This section introduces the quiver Hecke algebras, and their cyclotomic quotients. We use the relations to reveal some of the properties of these algebras. The main aim of this section is to give the reader an appreciation of, and some familiarity with, the KLR relations without appealing to any general theory.
2.1. Graded algebras. In this section we quickly review the theory of graded (cellular) algebras. For more details the reader is referred to [15, 54, 114]. Throughout, $\mathcal{Z}$ is a commutative integral domain. Unless otherwise stated, all modules and algebras will be free and of finite rank as $\mathcal{Z}$-modules.

In this chapter a graded module will always mean a $\mathcal{Z}$-graded module. That is, a $\mathcal{Z}$-module $M$ that has a decomposition $M = \bigoplus_{d \in \mathcal{Z}} M_d$ as a $\mathcal{Z}$-module. A positively graded module is a graded module $M = \bigoplus_d M_d$ such that $M_d = 0$ if $d < 0$.

A graded algebra is a unital associative $\mathcal{Z}$-algebra $A = \bigoplus_{d \in \mathcal{Z}} A_d$ that is a $\mathcal{Z}$-graded $\mathcal{Z}$-module such that $A_d A_c \subseteq A_{d+c}$, for all $d, e \in \mathcal{Z}$. It follows that $1 \in A_0$ and that $A_0$ is a $\mathcal{Z}$-graded subalgebra of $A$. A graded (right) $\mathcal{A}$-module is a graded $\mathcal{Z}$-module $M$ such that $M$ is an $\mathcal{A}$-module and $M_d A_e \subseteq M_{d+e}$, for all $d, e \in \mathcal{Z}$. Here $M_0$ is the (ungraded) module, and $\mathcal{A}$ is the (ungraded) algebra, obtained by forgetting the $\mathcal{Z}$-gradings on $M$ and $A$ respectively. Graded submodules, graded left $\mathcal{A}$-modules and so on are all defined in the obvious way.

Suppose that $M$ is a graded $\mathcal{A}$-module. If $m \in M_d$, for $d \in \mathcal{Z}$, then $m$ is homogeneous of degree $d$ and we set $\deg m = d$. Every element $m \in M$ can be written uniquely as a linear combination $m = \sum d m_d$ of its homogeneous components, where $\deg m_d = d$ and $m_d \in M$.

A homomorphism of graded $\mathcal{A}$-modules $M$ and $N$ is an $\mathcal{A}$-module homomorphism $f : M \rightarrow N$ such that $\deg f(m) = \deg m$, whenever $m \in M$ is homogeneous. That is, $f$ is a degree preserving $\mathcal{A}$-module homomorphism.

Let $\text{Rep}(\mathcal{A})$ be the category of finitely generated graded $\mathcal{A}$-modules together with degree preserving homomorphisms. Similarly, $\text{Proj}(\mathcal{A})$ is the category of finitely generated projective $\mathcal{A}$-modules with degree preserving maps. A graded functor between such categories is any functor that commutes with the grading shift functor that sends $M$ to $M(1)$.

If $M$ is a graded $\mathcal{Z}$-module and $s \in \mathcal{Z}$ let $M(s)$ be the graded $\mathcal{Z}$-module obtained by shifting the grading on $M$ up by $s$; that is, $M(s)_d = M_{d-s}$, for $d \in \mathcal{Z}$. If $M \neq 0$ then $M \cong M(s)$ as $\mathcal{A}$-modules if and only if $s = 0$. In contrast, $M(d) \cong M(s)$ as $\mathcal{A}$-modules, for all $s \in \mathcal{Z}$.

Let $\mathcal{K}(M, N)$ be the space of (degree preserving) $\mathcal{A}$-module homomorphisms and set

\[ \mathcal{K}(M, N) = \bigoplus_{d \in \mathcal{Z}} \mathcal{K}(M_{-d}, N) \cong \bigoplus_{d \in \mathcal{Z}} \mathcal{K}(M(d), N). \]

The reader may check that $\mathcal{K}(M, N) \cong \mathcal{K}(M, N)$ as $\mathcal{Z}$-modules.

Suppose that $q$ is an indeterminate and that $\mathcal{M}$ is a graded module. The graded dimension of $M$ is the Laurent polynomial

\[ \dim_q M = \sum_{d \in \mathcal{Z}} (\dim M_d)q^d \in \mathbb{N}[q, q^{-1}]. \]

If $M$ is a graded $\mathcal{A}$-module, and $D$ is an irreducible graded $\mathcal{A}$-module, then the graded decomposition number is the Laurent polynomial

\[ [M : D]_q = \sum_{s \in \mathcal{Z}} [M : D(s)] q^s \in \mathbb{N}[q, q^{-1}]. \]

By definition, the (ungraded) decomposition multiplicity $[M : D]$ is given by evaluating $[M : D]_q$ at $q = 1$.

Suppose that $A$ is a graded algebra and that $m$ is an (ungraded) $\mathcal{A}$-module. A graded lift of $m$ is any graded $\mathcal{A}$-module $M$ such that $M(s) \cong m$ as $\mathcal{A}$-modules. If $M$ is a graded lift of $m$ then so is $M(s)$, for any $s \in \mathcal{Z}$, so graded lifts are not unique. By Fitting's Lemma, if $m$ is indecomposable then its graded lift, if it exists, is unique up to grading shift [15, Lemma 2.5.3].

Following [54], the theory of cellular algebras from §1.3 extends to the graded setting in a natural way.

2.1.1. Definition ([54, §2]). Suppose that $A$ is a $\mathcal{Z}$-graded $\mathcal{Z}$-algebra that is free of finite rank over $\mathcal{Z}$. A graded cell datum for $A$ is a cell datum $(\mathcal{P}, T, C)$ together with a degree function

\[ \deg : \prod_{\lambda \in \mathcal{P}} T(\lambda) \rightarrow \mathcal{Z} \]

such that

\begin{itemize}
  \item[(GC)] the element $c_{\gamma \lambda}$ is homogeneous of degree $\deg c_{\gamma \lambda} = \deg(s) + \deg(t)$, for all $\lambda \in \mathcal{P}$ and $s, t \in T(\lambda)$.\end{itemize}

Then, $A$ is a graded cellular algebra with graded cellular basis $\{c_{\gamma \lambda}\}$.

We use $\star$ for the homogeneous cellular algebra involution of $A$ that is determined by $c_{\gamma \lambda} \star = c_{\gamma \lambda}$, for $s, t \in T(\lambda)$.

2.1.2. Example (Toy example) The most basic example of a graded algebra is the truncated polynomial ring $A = F[x]/(x^{n+1})$, for some integer $n > 0$, where $\deg x = 2$. As an ungraded algebra, $A$ has exactly one simple module, namely the field $F$ with $x$ acting as multiplication by zero. This algebra is a graded cellular algebra with $\mathcal{P} = \{0, 1, \ldots, n\}$, with its natural order, and $T(d) = \{d\}$ and $c_{\gamma \lambda} = x^d$. The irreducible graded $\mathcal{A}$-modules are $F(d)$, for $d \in \mathcal{Z}$, and $\dim_q A = 1 + q^2 + \cdots + q^{2n}$. \(\diamondsuit\)
2.1.3. Example Let $A = \text{Mat}_n(\mathbb{Z})$ be the $\mathbb{Z}$-algebra of $n \times n$-matrices. The basis of matrix units $\{e_{st} \mid 1 \leq s, t \leq n\}$ is a cellular basis for $A$, where $\mathcal{P} = \{\emptyset\}$ and $T(\emptyset) = \{1, 2, \ldots, n\}$. We want to put a non-trivial grading on $A$. Let $\{d_1, \ldots, d_n\} \subset \mathbb{Z}$ be a set of integers such that $d_s + d_{n-s+1} = 0$, for $1 \leq s \leq n$. Set $e_{st} = e_{s(t-n+t+1)}$ and define a degree function $\deg: T(\emptyset) \to \mathbb{Z}$ by $\deg s = d_s$. Then $\{e_{st} \mid 1 \leq s, t \leq n\}$ is a graded cellular basis of $A$ and $\dim_q A = \sum_{s=1}^n q^{d_s}$. Consequently, semisimple algebras can have non-trivial gradings.

Exactly as in §1.3, for each $\lambda \in \mathcal{P}$ we obtain a graded cell module $C_\lambda$ with homogeneous basis $\{e_t \mid t \in T(\lambda)\}$ and $\deg e_t = \deg t$. Generalizing (1.3.2), the graded cell module $C_\lambda$ comes equipped with a homogeneous symmetric bilinear form $\langle \cdot , \cdot \rangle$, of degree zero. Therefore, if $x$ and $y$ are homogeneous elements of $C_\lambda$ then $\langle x, y \rangle_\lambda \neq 0$ only if $\deg x + \deg y = 0$. Moreover, $\langle xa, y \rangle_\lambda = \langle x, ya^* \rangle_\lambda$, for all $x, y \in C_\lambda$ and all $a \in A$.

Consequently, if $M$ is an $A$-module then its (graded) dual is the $A$-module
\[(2.1.4) \quad M^\oplus = \mathcal{H}_{\text{hom}}(M, \mathbb{Z}),\]
with $A$-action $(f \cdot a)(m) = f(ma^*)$, for $f \in M^\oplus$, $a \in A$ and $m \in M$.

2.1.5. Theorem ([54, Theorem 2.10]). Suppose that $Z$ is a field and that $A$ is a graded cellular algebra. Then:

a) If $\lambda \in \mathcal{P}$ then $D_\lambda$ is either 0 or an absolutely irreducible graded $A$-module. If $D_\lambda \neq 0$ then $(D_\lambda)^\oplus \cong D_\lambda$.

b) Let $K = \{\mu \in \mathcal{P} \mid D_\mu \neq 0\}$. Then $\{D_\lambda(s) \mid \lambda \in K$ and $s \in \mathbb{Z}\}$ is a complete set of pairwise non-isomorphic irreducible (graded) $A$-modules.

c) If $\lambda \in \mathcal{P}$ and $\mu \in K$ then $[C_\lambda : D_\mu]_q \neq 0$ only if $\lambda \geq \mu$. Moreover, $[C^\mu : D^\mu]_q = 1$.

Forgetting the grading, the basis $\{e_t\}$ is still a cellular basis of $A$. Comparing Theorem 1.3.4 and Theorem 2.1.5 it follows that every (ungraded) irreducible $A$-module has a graded lift that is unique up to shift. Conversely, if $D$ is an irreducible graded $A$-module then $D$ is an irreducible $A$-module. (This holds more generally whenever a grading is put on a finite dimensional algebra; see [114, Theorem 4.4.4].) It is an instructive exercise to prove that if $A$ is a finite dimensional graded $A$-module then every simple $A$-module has a graded lift and, up to shift, every graded simple $A$-module is of this form.

By [47, Theorems 3.2 and 3.3] every projective indecomposable $\mathcal{H}_{\text{mod}}^A$-module has a graded lift. More generally, as shown in [114, §4], if $M$ is a finitely generated graded $A$-module then the Jacobson radical of $M$ has a graded lift.

The matrix $D_A(q) = ([C_\lambda : D_\mu]_q)_{\lambda, \mu \in \mathcal{P}}$ is the graded decomposition matrix of $A$. For each $\mu \in K$ let $P^\mu$ be the projective cover of $D^\mu$ in $\text{Rep}(A)$. The matrix $C_A(q) = ([P^\lambda : D_\mu]_q)_{\lambda, \mu \in K}$ is the graded Cartan matrix of $A$.

An $A$-module $M$ has a cell filtration if there exists a filtration $M = M_0 \supset M_1 \supset \cdots \supset M_z \supset 0$ such that each subquotient $M_r/M_{r+1}$ is isomorphic, up to shift, to some graded cell module. Fixing isomorphisms $M_r/M_{r+1} \cong C^{\lambda_r}(d_r)$, for some $\lambda_r \in \mathcal{P}$ and $d_r \in \mathbb{Z}$, define $(M : C^{\lambda_r}d_r) = \sum q^{m_r}d_r^q$, where $m_\beta = \# \{1 \leq \tau \leq z \mid \lambda_\tau = \lambda$ and $d_\tau = d \}$. In general, the multiplicities $(M : C^{\lambda_r}d_r)$ depend upon the choice of filtration and the labelling of the isomorphisms $M_r/M_{r+1} \cong C^{\lambda_r}(d_r)$ because the cell modules are not guaranteed to be pairwise non-isomorphic, even up to shift.

2.1.6. Corollary ([54, Theorem 2.17]). Suppose that $Z = F$ is a field. If $\mu \in K$ then $P^\mu$ has a cell filtration such that $(P^\mu : C^{\lambda_r})_q = [C^{\lambda_r} : D^\mu]_q$, for all $\lambda \in \mathcal{P}$. Consequently, $C_A(q) = D_A(q)^{tr}D_A(q)$ is a symmetric matrix.

2.2. Cycloptomic quiver Hecke algebras. We are now ready to define cyclotomic quiver Hecke algebras. We start by defining the affine versions of these algebras and then pass to the cyclotomic quotients. Throughout this section we will make extensive use of the Lie theoretic data that is attached to the quiver $\Gamma_e$ in §1.2.

If $\beta \in \mathbb{Q}^+$ let $I_\beta = \{i \in I^n \mid 1 = \alpha_i + \cdots + \beta \alpha_i \}.$

2.2.1. Definition (Khovanov and Lauda [74, 75] and Rouquier [121]). Suppose that $n \geq 0$, $e \geq 1$, and $\beta \in \mathbb{Q}^+$. The quiver Hecke algebra, or Khovanov-Lauda–Rouquier algebra, $\mathcal{H}_\beta(Z)$ of type $\Gamma_e$ is the untangled associative $Z$-algebra with generators

\[\{\psi_1, \ldots, \psi_{n-1}\} \cup \{y_1, \ldots, y_n\} \cup \{e(i) \mid i \in I^\beta\}\]

and relations

\[e(i)e(j) = \delta_{ij} e(i),\]
\[e(i)e(i) = e(i)e_r,\]
\[\psi_r e(i) = e(s_r^{-1}i)\psi_r,\]
\[y_r y_s = y_s y_r,\]
\[ \psi_r \psi_s = \psi_s \psi_r, \quad \text{if } |r-s| > 1, \]
\[ \psi_r y_s = y_s \psi_r, \quad \text{if } s \neq r, r+1, \]

(2.2.2)
\[ \psi_r y_{r+1} e(i) = (y_r \psi_r + \delta_{i,r+1}) e(i), \]
\[ y_{r+1} \psi_r e(i) = (y_r \psi_r + \delta_{i,r+1}) e(i), \]

(2.2.3)
\[ \psi_r^2 e(i) = \begin{cases} 
(y_{r+1} - y_r)(y_r - y_{r+1}) e(i), & \text{if } i_r \rightleftharpoons i_{r+1}, \\
(y_r - y_{r+1}) e(i), & \text{if } i_r \rightarrow i_{r+1}, \\
(y_{r+1} - y_r) e(i), & \text{if } i_r \leftarrow i_{r+1}, \\
e(i), & \text{if } i_r = i_{r+1}, \\
0, & \text{otherwise},
\end{cases} \]

and \((\psi_r \psi_{r+1} \psi_r - \psi_{r+1} \psi_r \psi_r + 1)e(i)\) is equal to
\[ \begin{cases} 
(y_r + y_{r+2} - 2y_{r+1}) e(i), & \text{if } i_{r+2} = i_r \rightleftharpoons i_{r+1}, \\
-e(i), & \text{if } i_{r+2} = i_r \rightarrow i_{r+1}, \\
e(i), & \text{if } i_{r+2} = i_r \leftarrow i_{r+1}, \\
0, & \text{otherwise},
\end{cases} \]

(2.2.4)

for \(1, j \in I^\beta\) and all admissible \(r\) and \(s\).

Part of the point of these definitions is that \(\mathcal{R}_\beta\) is a \(\mathbb{Z}\)-graded algebra with degree function determined by
\[ \deg e(i) = 0, \quad \deg y_r = 2 \quad \text{and} \quad \deg \psi_s e(i) = -c_{i_r, i_{r+1}}, \]
for \(1 \leq r \leq n, 1 \leq s < n\) and \(i \in I^n\).

Suppose that \(n \geq 0\). Then \(I^n = \bigcup_{\beta} I^\beta\) is the decomposition of \(I^n\) into a disjoint union of \(S_n\)-orbits. Define
\[ \mathcal{R}_n = \bigoplus_{\beta \in Q^+} \mathcal{R}_\beta. \]

Set \(\mathcal{R}_\beta = \{ e_\beta \mathcal{R}_n e_\beta \mid e_\beta \in Q^+ \} \), \(\mathcal{R}_\beta\) is a two-sided ideal of \(\mathcal{R}_n\), and \(\mathcal{R}_n\) is the decomposition of \(\mathcal{R}_n\) into blocks. That is, \(\mathcal{R}_\beta\) is indecomposable for all \(\beta \in Q^+\).

Khovanov and Lauda \[74, 75\] and Rouquier \[121\] define quiver Hecke algebras for quivers of arbitrary type. In the short time since their inception a lot has been discovered about these algebras. The first important result is that these algebras categorify the negative part of the corresponding quantum group \[22, 74, 122, 132\].

2.2.6. Remark. We have defined only a special case of the quiver Hecke algebras defined in \[74, 121\]. In addition to allowing arbitrary quivers, Khovanov and Lauda allow a more general choice of signs. Rouquier’s definition, which is the most general, defines the quiver Hecke algebras in terms of a matrix \(Q = (Q_{ij})_{i,j \in I}\) with entries in a polynomial ring \(\mathbb{Z}[u,v]\) with the properties that \(Q_{ii} = 0, \ Q_{ij}\) is not a zero divisor in \(\mathbb{Z}[u,v]\) for \(i \neq j\) and \(Q_{ij}(u,v) = Q_{ji}(v,u)\), for \(i, j \in I\). For an arbitrary quiver \(\Gamma\), Rouquier \[121\], Definition 3.2.1 defines \(\mathcal{A}_\beta(\Gamma)\) to be the algebra generated by \(\psi, y, e(i)\) subject to the relations above except that the quadratic and braid relations are replaced with \(\psi^2 e(i) = Q_{rr+1}(y_r y_{r+1}) e(i)\) and \((\psi_r \psi_{r+1} \psi_r - \psi_{r+1} \psi_r \psi_r + 1)e(i)\) is equal to
\[ \begin{cases} 
Q_{rr+1}(y_r y_{r+1})^2 - Q_{rr+1} y_{r+2}^2 - y_r, & \text{if } i_{r+2} = i_r, \\
0, & \text{otherwise},
\end{cases} \]

The assumptions on \(Q\) ensure that the last expression is a polynomial in the generators. In general, \(y_r e(i)\) is homogeneous of degree \((\alpha_i, \alpha_i)\), for \(1 \leq r \leq n\) and \(i \in I^n\). Under some mild assumptions, the isomorphism type of \(\mathcal{R}_\beta\) is independent of the choice of \(Q\) by \[121, Proposition 3.12\]. We leave it to the reader to find a suitable matrix \(Q\) for Definition 2.2.1.

For the rest of these notes for \(w \in S_n\) we arbitrarily fix a reduced expression \(w = s_{r_1} \ldots s_{r_k}\), with \(1 \leq r_j < n\). Using this fixed reduced expression for \(w\) define \(\psi_w = \psi_{r_1} \ldots \psi_{r_k}\).

2.2.7. Example As the \(\psi\)-generators of \(\mathcal{R}_n\) do not satisfy the braid relations the element \(\psi_w\) will, in general, depend upon the choice of reduced expression for \(w \in S_n\). For example, by \(\psi_{r+1}^2 = \psi_{r+1} - \psi_{r+2} \), \(n = 3\) and \(w = s_1 s_2 s_1 = s_2 s_1 s_2\) then \(\psi_1 \psi_2 \psi_1 e(0,2,0) = \psi_2 \psi_1 \psi_2 e(0,2,0) + e(0,2,0)\), by \(\psi_{r+1}^2 = \psi_{r+1} - \psi_{r+2}\). Therefore, these two reduced expressions determine different elements of \(\mathcal{R}_n\).

Khovanov and Lauda \[74, Theorem 2.5\] and Rouquier \[121, Theorem 3.7\] proved the following.

2.2.8. Theorem (Khovanov-Lauda \[74\] and Rouquier \[121\]). Suppose that \(\beta \in Q^+\). Then \(\mathcal{R}_\beta(\mathbb{Z})\) is free as an \(\mathbb{Z}\)-algebra with homogeneous basis \(\{ \psi_w y_{a_1}^{\alpha_1} \ldots y_{a_n}^{\alpha_n} e(i) \mid w \in S_n, a_1, \ldots, a_n \in \mathbb{N} \text{ and } i \in I^\beta \} \).
Li [92, Theorem 4.3.10] has constructed a graded cellular basis of $R_n$. In the special case when $e = \infty$, that Kleshchev, Loubert and Miemietz [85] give a graded affine cellular basis of $R_n$, in the sense of König and Xi [87].

In these notes we are not directly concerned with the quiver Hecke algebras $R_n$. Rather, we are more interested in cyclotomic quotients of these algebras.

2.2.9 Definition (Brundan-Kleshchev [21]). Suppose that $\Lambda \in P^+$. The cyclotomic quiver Hecke algebra of type $\Gamma$ and weight $\Lambda$ is the quotient algebra $R_\Lambda = R_n / \langle y_1(\Lambda, \alpha_i) e(i) \mid 1 \in I^n \rangle$.

We abuse notation and identify the KLR generators of $R_\Lambda$ with their images in $R_\Lambda$. That is, we consider the algebra $R_\Lambda$ to be generated by $\psi_1, \ldots, \psi_{n-1}, y_1, \ldots, y_n$ and $e(i)$, for $i \in I^n$, subject to the relations in Definition 2.2.1 and Definition 2.2.9. From this point onwards, $\Lambda \in P^+$.

When $\Lambda$ is a weight of level 2, the algebras $R_\Lambda$ first appeared in the work of Brundan and Stroppel [26] in their series of papers on the Khovanov diagram algebras. In full generality, the cyclotomic quotients of $R_n$ were introduced by Khovanov-Lauda [74] and Rouquier [121]. Brundan and Kleshchev were the first to systematically study the cyclotomic quiver Hecke algebras $R_\Lambda$, for any $\Lambda \in P^+$.

Although we will not need this here we note that, rather than working algebraically, it is often easier to work diagrammatically by identifying the elements of $R_\Lambda$ with certain planar diagrams. In these diagrams, the end-points of the strings are labeled by $\{1, 2, \ldots, n, 1', 2', \ldots, n'\}$ and the strings themselves are coloured by $I^n$. For example, following [74], the KLR generators can be identified with the diagrams:

$$e(i) = \begin{array}{c}
\begin{array}{c}
\bigotimes
\end{array}
\end{array} \quad \psi_r e(i) = \begin{array}{c}
\begin{array}{c}
\oplus
\end{array}
\end{array} \quad y_s e(i) = \begin{array}{c}
\begin{array}{c}
\ominus
\end{array}
\end{array} \ .$$

Multiplication of diagrams is given by concatenation, read from top to bottom, subject to the relations above that are also interpreted diagrammatically. As an exercise, we leave it to the reader to identify the two relations in Definition 2.2.1 that correspond to the following ‘local’ relations on strings inside braid diagrams:

$$(e + \delta_{ij}) x = x$$

and

$$x = x \pm \delta_{ij}.$$ (For the second relation, $e \neq 2$.)

For more rigorous definitions of such diagrams, and non-trivial examples of their application, we refer the reader to the papers [53, 81, 92, 97] for examples of these diagrams in action.

2.2.10 Example (Rank one algebras) Suppose that $n = 1$ and $\Lambda \in P^+$. Then $R_\Lambda = \langle y_1, e(i) \mid y_1 e(i) = e(i) y_1 \rangle$, for $i \in I$, with $\deg y_1 = 2$ and $\deg e(i) = 0$, for $i \in I$. Therefore, there is an isomorphism of graded algebras

$$R_1^\Lambda \cong \bigoplus_{(\alpha, \beta) > 0} Z[y_1]/y_1^{(\Lambda, \alpha)} Z[y],$$

where $y = y_1$ is in degree 2. Armed with this description of $R_\Lambda$ it is now straightforward to show that $R_\Lambda \cong R_\Lambda$ when $Z$ is a field and $n = 1$.

2.3. Nilpotence and small representations. In this section and the next we use the KLR relations to prove some results about the cyclotomic quiver Hecke algebras $R_\Lambda$ for particular $\Lambda$ and $n$.

By Theorem 2.2.8 the algebra $R_n$ is infinite dimensional, so it is not obvious from the relations that the cyclotomic Hecke algebra $R_\Lambda$ is finite dimensional — or even that $R_\Lambda$ is non-zero. The following result shows that $y_i$ is nilpotent, for $1 \leq r \leq n$, which implies that $R_\Lambda$ is finite dimensional.

2.3.1 Lemma (Brundan and Kleshchev [21, Lemma 2.1]). Suppose that $1 \leq r \leq n$ and $i \in I^n$. Then $y_r^N e(i) = 0$ for $N \gg 0$.

Proof. We argue by induction on $r$. If $r = 1$ then $y_1^{(\Lambda, \alpha_i)} e(i) = 0$ by Definition 2.2.9, proving the base step of the induction. Now consider $y_{r+1} e(i)$. By induction, we may assume that there exists $N \gg 0$ such that $y_{r+1}^N e(j) = 0$, for all $j \in I^n$. There are three cases to consider.

Case 1. $i_{r+1} \neq i_r$.

By (2.2.3) and (2.2.2),

$$y_{r+1}^N e(i) = y_{r+1}^N \psi_r^2 e(i) = \psi_r y_r^N \psi_r e(i) = \psi_r y_r^N e(s_r \cdot 1) \psi_r = 0,$$
where the last equality follows by induction.

**Case 2.** \( i_{r+1} = i_r \pm 1 \).

Suppose first that \( e \neq 2 \). This is a variation on the previous case, with a twist. By (2.2.3) and (2.2.2), again

\[
y_{r+1}^{2N} e(i) = \sum_{j=0}^{N-1} y_{r+1} e(i) + y_{r+1}^N (y_{r+1} - y_r) e(i)
\]

\[
= y_r y_{r+1}^{2N-1} e(i) + y_r y_{r+1}^{2N-1} \psi_r e(i)
\]

\[
y_r y_{r+1}^{2N-1} e(i) + \psi_r y_r y_{r+1}^{2N-1} e(s_r \cdot i) \psi_r
\]

\[
y_r y_{r+1}^{2N-1} e(i) \equiv y_r y_{r+1}^{2N-1} e(i) = 0.
\]

The case when \( e = 2 \) is similar. First, observe that

\[
y_{r+1}^{3N} e(i) = (2y_{r+1} - y_r^2 - \psi_r^2) e(i) \text{ by (2.2.3).}
\]

Therefore, arguing as before,

\[
y_{r+1}^{3N} e(i) = y_r (2y_{r+1} - y_r) y_{r+1}^{3N-2} e(i) = \cdots = y_r (2y_{r+1} - y_r)^N y_{r+1}^{N} e(i) = 0.
\]

**Case 3.** \( i_{r+1} = i_r \).

Let \( \phi_r = \psi_r (y_r - y_{r+1}) \). Then \( \phi_r \psi_r e(i) = -2 \psi_r e(i) \) by (2.2.2), so that \((1 + \phi_r)^2 e(i) = e(i)\). Moreover,

\[
(1 + \phi_r) y_r (1 + \phi_r) e(i) = (y_r + \phi_r y_r + y_r \phi_r + \phi_r y_r \phi_r) e(i) = y_{r+1} e(i),
\]

where the last equality uses (2.2.2). Now we are done because

\[
y_r^{N+1} e(i) = ((1 + \phi_r) y_r (1 + \phi_r))^N e(i) = (1 + \phi_r)^N y_r (1 + \phi_r) e(i) = 0,
\]

since \( \phi_r \) commutes with \( e(i) \) and \( y_r e(i) = 0 \) by induction. \( \Box \)

We have marginally improved on Brundan and Kleshchev’s original proof of Lemma 2.3.1 because the argument above gives an upper bound for the nilpotency index of \( y_r \). In general, this bound is far from sharp. For a better estimate of the nilpotency index of \( y_r \) see [57, Corollary 4.6] (and [53] when \( e = \infty \)). See [67, Lemma 4.4] for another argument that applies to cyclotomic quiver Hecke algebras of arbitrary type.

Combining Theorem 2.2.8 and Lemma 2.3.1 we have:

**2.3.2. Corollary** (Brundan and Kleshchev [21, Corollary 2.2]). Suppose \( Z \) is an integral domain. Then \( \mathcal{H}_n^\Lambda \) is finite dimensional.

As our next exercise we classify the one dimensional representations of \( \mathcal{H}_n^\Lambda \) when \( Z = F \) is a field. For \( i \in I \) let \( i_n^i = (i, i+1, \ldots, i+n-1) \) and \( i_n = (i, i-1, \ldots, i-n+1) \). Then \( i_n \) is in \( I^n \). If \( (\Lambda, \alpha_i) \neq 0 \) then \( e(i_n) = 0 \) by Definition 2.2.9. However, if \( \{ \Lambda, \alpha_i \} \neq 0 \) then using the relations it is easy to see that \( \mathcal{H}_n^\Lambda \) has unique one dimensional representations \( D_{i,n}^\Lambda = F d_{i,n}^\Lambda \) and \( D_{i,n} = F d_{i,n} \) such that

\[
d_{i,n} e(i) = \delta_{i,1} d_{i,n}^\Lambda \quad \text{and} \quad d_{i,n} y_r = 0 = d_{i,n} \psi_s,
\]

for \( i \in I^n, 1 \leq r \leq n \) and \( 1 \leq s \leq n \) and such that \( \text{deg} d_{i,n}^\Lambda = 0 \). In particular, this shows that \( e(i_n^\Lambda) \neq 0 \) and hence that \( \mathcal{H}_n^\Lambda \neq 0 \). If \( e \neq 2 \) then \( D_{i,n}^\Lambda | i \in I \) and \( \{ \Lambda, \alpha_i \} \neq 0 \) are pairwise non-isomorphic irreducible representations of \( \mathcal{H}_n^\Lambda \). If \( e = 2 \) then \( i_n^\Lambda = i_n \) so that \( D_{i,n}^\Lambda = D_{i,n} \).

**2.3.3. Proposition.** Suppose that \( Z = F \) is a field and that \( D \) is a one dimensional graded \( \mathcal{H}^\Lambda_n \)-module. Then \( D \cong D_{i,n}^\Lambda(k) \), for some \( k \in Z \) and \( i \in I \) such that \( (\Lambda, \alpha_i) \neq 0 \).

**Proof.** Let \( d \) be a non-zero element of \( D \) so that \( D = F d \). Then \( d = \sum_{i \in I} d e(i) \) so that \( d e(i) \neq 0 \) for some \( i \in I \). Moreover, \( d e(j) = 0 \) if and only if \( j = i \) since otherwise \( d e(i) \) and \( d e(j) \) are linearly independent elements of \( D \), contradicting assumption that \( D \) is one dimensional. Now, \( \text{deg} y_r = 2 + \text{deg} d \), so \( \text{deg} y_r = 0 \), for \( 1 \leq r \leq n \), since \( D \) is one dimensional. Similarly, \( \text{deg} \psi_r = \text{deg} e(i) \psi_r = 0 \) if \( i_r = i_{r+1} \) or \( i_r = i_{r+1} \) since in these cases \( \text{deg} e(i) \psi_r \neq 0 \).

It remains to show that \( i_n^\Lambda \neq 0 \) and that \( (\Lambda, \alpha_i) \neq 0 \). First, since \( 0 \neq d = d e(i) \) we have that \( e(i) \neq 0 \) so that \( (\Lambda, \alpha_i) \neq 0 \) by Definition 2.2.9. To complete the proof we show that if \( d \neq i_n^\Lambda \) then \( d = 0 \), which is a contradiction. First, suppose that \( i_r = i_{r+1} \) for some \( r \), with \( 1 \leq r < n \). Then \( d = d e(i) = d(\psi_r y_{r+1} - y_r \psi_r) e(i) = 0 \) by (2.2.2), which is not possible, so \( i_r \neq i_{r+1} \). Next, suppose that \( i_r = i_{r+1} = i_r \neq i_{r+1} \). Then \( d = d e(i) = d(\psi_r^2) e(i) = d \psi_r (s_r \cdot i) \psi_r = 0 \) because \( D \) is one dimensional and \( d e(j) = 0 \) if \( j \neq i \). This is another contradiction, so we must have \( i_{r+1} = i_r \pm 1 \) for \( 1 \leq r \leq n \). Therefore, if \( i \neq i_n^\Lambda \) then \( e \neq 2 \), \( n > 2 \) and \( i_r = i_{r+2} = i_{r+1} \pm 1 \) for some \( r \). Applying the braid relation (2.2.4),

\[
d = d e(i) = \pm d \cdot (\psi_r \psi_{r+1} \psi_r \psi_{r+1}) e(i) = 0,
\]

a contradiction. Hence, \( D \cong D_{i,n}^\Lambda(\text{deg} d) \), completing the proof. \( \Box \)
2.4. Semisimple KLR algebras. Now that we understand the one dimensional representations of $\mathcal{R}_n^\Lambda$, we consider the semisimple representation theory of the cyclotomic quiver Hecke algebras. These results do not appear in the literature, but there are few surprises here because everything we do can be easily deduced from results that are known. The main idea is to show by example how to use the quiver Hecke algebra relations.

In this section we fix $e > n$ and $\Lambda \in P^+$ such that $(\Lambda, \alpha, \lambda) \leq 1$, for all $i \in I$, and we study the algebras $\mathcal{R}_n^\Lambda$. Notice that these conditions ensure that $\mathcal{H}_n^\Lambda$ is semisimple by Corollary 1.6.11.

Recall from §1.8 that $\Gamma = (\Gamma_1, \ldots, \Gamma_n)$ is the residue sequence of $t \in \text{Std}(\mathcal{P}_n^\Lambda)$, where $\Gamma_i = (\gamma_i)(t) + e\mathbb{Z}$. We caution the reader that if $t$ is a standard tableau then the contents $\gamma_i(t) \in \mathbb{Z}$ and the residues $\gamma_i \in I$ are in general different.

If $i \in I$ then a node $A = (l, r, c)$ is an $i$-node if $e = k_l + c - r + e\mathbb{Z}$. Therefore, extending the definitions of §1.4, we can now talk of admissible and removable $i$-nodes.

2.4.1. Lemma. Suppose that $e > n$ and $(\Lambda, \alpha, \lambda) \leq 1$, for all $i \in I$. Let $s, t \in \text{Std}(\mathcal{P}_n^\Lambda)$. Then $s = t$ if and only if $P = P$.

Proof. Observe that if $i \in I$ and $\mu \in \text{Std}^\Lambda_n$, where $0 \leq m < n$, then $\mu$ has at most one admissible $i$-node since $(\Lambda, \alpha, \lambda) \leq 1$. Hence, it follows easily by induction on $n$ that $\mu = t$ if and only if $i^* = i^!$.

Lemma 2.4.1 also follows from Theorem 1.6.10 and Corollary 1.6.11.

Let $I_n^\Lambda = \{ t \mid t \in \text{Std}(\mathcal{P}_n^\Lambda) \}$ be the set of residue sequences of all of the standard tableaux in $\text{Std}(\mathcal{P}_n^\Lambda)$. By the proof of Lemma 2.4.1, if $i = t \in I_n^\Lambda$ and $i_{r+1} = i_r \pm 1$ then $r$ and $r + 1$ must be in either the same row or in the same column of $t$. Hence, we have the following useful fact.

2.4.2. Corollary. Suppose that $e > n$ and that $(\Lambda, \alpha, \lambda) \leq 1$, for all $i \in I$, and that $t \in I_n^\Lambda$ such that $i_{r+1} = i_r \pm 1$. Then $s_r \cdot t \notin I_n^\Lambda$.

When $\Lambda = \Lambda_0$ the next result is due to Brundan and Kleshchev [21, §5.5]. More generally, Kleshchev and Ram [83, Theorem 3.4] prove similar results for quiver Hecke algebras of simply laced type.

2.4.3. Proposition (Seminormal representations of $\mathcal{R}_n^\Lambda$). Suppose that $Z = F$ is a field, $e > n$ and that $\Lambda \in P^+$ with $(\Lambda, \alpha, \lambda) \leq 1$, for all $i \in I$. Then for each $\lambda \in \mathcal{P}_n^\Lambda$ there is a unique irreducible graded $\mathcal{R}_n^\Lambda$-module $S^\lambda$ with homogeneous basis $\{ \psi_1 \mid t \in \text{Std}(\lambda) \}$ such that $\text{deg} \psi_1 = 0$, for all $t \in \text{Std}(\lambda)$, and where the $\mathcal{R}_n^\Lambda$-action is given by

$$\psi_t(e(i)) = \delta_{i,p} \psi_t, \quad \psi_t y_r = 0 \quad \text{and} \quad \psi_t y_r = \psi_t y_{(r,r+1)},$$

where we set $\psi_t y_{(r,r+1)} = 0$ if $t(r, r + 1)$ is not standard.

Proof. By Lemma 2.4.1, if $s, t \in \text{Std}(\lambda)$ then $s = t$ if and only if $P = P$. Moreover, $i_r + 1 = i^* \pm 1$ if and only if $r$ and $r + 1$ are in the same row or in the same column of $t$. Similarly, $\psi_\lambda = \psi_t(\lambda)$ almost all of the relations in Definition 2.2.1 are trivially satisfied. In fact, all that we need to check is that $\psi_1, \ldots, \psi_{n-1}$ satisfy the braid relations of the symmetric group $\mathfrak{S}_n$ with $\psi_\lambda$ acting as zero when $i_r + 1 = i^* \pm 1$, which follows automatically by Corollary 2.2.2. By the same reasoning if $t(r, r + 1)$ is standard then $e(i)^t \psi_t = 0$. Hence, we can set $\psi_t = 0$, for all $t \in \text{Std}(\lambda)$. This proves that $S^\lambda$ is a graded $\mathcal{R}_n^\Lambda$-module.

It remains to show that $S^\lambda$ is irreducible. If $s, t \in \text{Std}(\lambda)$ then $s = t^\lambda d(s) = td(t)^{-1} d(s)$, so $\psi_s = \psi_t \psi_{d(s)}$. Suppose that $x = \sum_{r} r \psi_t$ is a non-zero element of $S^\lambda$. If $r \neq 0$ then $\psi_t = \frac{1}{r} e(\hat{t})$, so it follows that $\psi_t \in x \mathcal{R}_n^\Lambda$, so $S^\lambda$ is irreducible as claimed.

Hence, $\psi_t \neq 0$ in $\mathcal{R}_n^\Lambda$, for all $i \in I_n^\Lambda$. This was not clear until now.

We want to show that Proposition 2.4.3 describes all of the graded irreducible representations of $\mathcal{R}_n^\Lambda$, up to degree shift. To do this we need a better understanding of the set $I_n^\Lambda$. Okounkov and Vershik [117, Theorem 6.7] explicitly described the set of all content sequences $(\gamma_i(t), \ldots, \gamma_n(t))$ when $t = 1$. This combinatorial result easily extends to higher levels and so suggests a description of $I_n^\Lambda$.

If $i \in I_n$ and $1 \leq m \leq n$ set $i_{lm} = (i_1, \ldots, i_m)$. Then $i_{lm} \in I^m$ and $I_n^\Lambda = \{ i_{lm} \mid l \in I_n^\Lambda \}$.

2.4.4. Lemma (cf. Ogievetsky-d’Andecy [116, Proposition 5]). Suppose that $e > n$ and $(\Lambda, \alpha, \lambda) \leq 1$, for all $i \in I$. Let $i \in I_n^\Lambda$. Then $i \in I_n^\Lambda$ if and only if it satisfies the following three conditions:

(a) $(\Lambda, \alpha, \lambda) \neq 0$.

(b) If $1 \leq r \leq n$ and $(\Lambda, \alpha, \lambda) = 0$ then $\{i_r, i_r - 1, i_r + 1\} \cap \{i_1, \ldots, i_{r-1}\} \neq \emptyset$.

(c) If $1 \leq s \leq r \leq n$ and $i_r = i_s$ then $\{i_r, i_r - 1\} \subseteq \{i_{s+1}, \ldots, i_{r-1}\}$.

Proof. Suppose that $t \in \text{Std}(\mathcal{P}_n^\Lambda)$ and let $\lambda = (t)$. We prove by induction on $r$ that $i_{lr} \in I_n^\Lambda$. By definition, $i_1 = k_l + e\mathbb{Z}$ for some $t$ with $1 \leq t \leq l$, so (a) holds. By induction we may assume that the subsequence $(i_1, \ldots, i_{r-1})$ satisfies properties (a)–(c). If $(\Lambda, \alpha_i) = 0$ then $r$ is not in the first row or in the first column.
of any component of \( t \), so \( t \) has an entry in the row directly above \( r \) or in the column immediately to the left of \( r \) — or both! Hence, there exists an integer \( s \) with \( 1 \leq s < r \) such that \( i_s = i^r_s \pm 1 \). Hence, (b) holds.

Finally, suppose that \( i_s = i_r \) as in (c). As the residues of the nodes in different components of \( t \) are disjoint it follows that \( s \) and \( r \) are in same component of \( t \) and on the same diagonal. In particular, \( r \) is not in the first row or in the first column of its component in \( t \). As \( t \) is standard, the entries in \( t \) that are immediately above or to the left of \( r \) are both larger than \( s \) and smaller than \( r \). Hence, (c) holds.

Conversely, suppose that \( i \in I^n \) satisfies properties (a)–(c). We show by induction on \( m \) that \( i_{1m} \in I^*_\Lambda \), for \( 1 \leq m \leq n \). If \( m = 1 \) then \( i_{11} \in I^*_\Lambda \) by property (a). Now suppose that \( 1 < m < n \) and that \( i_{1m} \in I^*_\Lambda \). By induction \( i_{1m} = i^r \), for some \( s \in \text{Std}(P^{\Lambda}_{m}) \). Let \( v = \text{Shape}(s) \). If \( i \in I \) then \( (\Lambda, \alpha_i, m) \leq 1 \), so the multipartition \( v \) can have at most one addable \( m \)-node. On the other hand, reversing the argument of the last paragraph, using properties (b) and (c) with \( r = m + 1 \), shows that \( v \) has at least one addable \( i_{m+1} \)-node. Let \( A \) be the unique addable \( i_{m+1} \)-node of \( v \). Then \( i_{1(m+1)} = i^t \) where \( t \in \text{Std}(P^{\Lambda}_{m+1}) \) is the unique standard tableau such that \( t_{1m} = s \) and \( t(A) = m + 1 \). Hence, \( i \in I^{m+1}_{\Lambda} \) as required.

By Proposition 2.4.3, if \( i \in I^m_\Lambda \) then \( e(i) \neq 0 \). We use Lemma 2.4.4 to show that \( e(i) = 0 \) if \( i \notin I^m_\Lambda \). First, a result that holds for all \( \Lambda \in P^+ \).

2.4.5. Lemma. Suppose that \( \Lambda \in P^+ \) and that \( e(i) \neq 0 \), for \( i \in I^n \). Then \( (\Lambda, \alpha_i, r) \neq 0 \). Moreover, \( \{i_1, \ldots, i_r \} = \emptyset \) whenever \( (\Lambda, \alpha_i) \neq 0 \). and \( 1 < r \leq n \).}

**Proof.** By Definition 2.2.9, \( e(i) = 0 \) whenever \( (\Lambda, \alpha_i) = 0 \). To prove the second claim suppose that \( (\Lambda, \alpha_i) = 0 \) and \( i_1 \neq i_r \). Applying (2.3) \( r \)-times,

\[
eq \cdots \psi_{r-1} \cdots \psi_{1} \psi_{i_r} \psi_{i_1} \cdots \psi_{i_{r-1}} \cdots \psi_{r} \psi_{1} \cdots \psi_{r-1} \psi_{r} = 0,
\]

where the last equality follows because \( (\Lambda, \alpha_r) = 0 \).

2.4.6. Proposition. Suppose that \( 1 \leq m \leq n \) and that \( (\Lambda, \alpha_i, m) \leq 1 \), for all \( i \in I \). Then \( e(i) = 0 \) whenever \( i \in I^m_\Lambda \). Moreover, if \( i \in I^n \) then \( e(i) = 0 \) only if \( i_{1m} \in I^*_\Lambda \).

**Proof.** We argue by induction on \( m \) to show that \( e(i) = 0 \) whenever there exists an integer \( 1 \leq s < r \) such that \( i_s = i_r \) and \( \{i_{s-1}, i_s, i_{s+1} \} \subseteq \{i_{r-1}, i_r, i_{r+1} \} \). We may assume that \( s \) is maximal such that \( i_s = i_r \) and \( 1 \leq s < r \). There are three cases to consider.

**Case 1.** \( r = s + 1 \).

By (2.2.2), \( e(i) = (y_{s+1}^{\psi} - \psi y_s^{\psi}) e(i) = y_{s+1}^{\psi} e(i) \), since \( y_s = 0 \) by induction. Using this identity twice, reveals that \( e(i) = y_{s+1}^{\psi} e(i) = y_{s+1}^{r} e(i) \psi_s = y_{r+1}^{s} e(i) \psi_s = y_{s+1}^{r} e(i) \psi_s = 0 \), where the last equality comes from (2.2.3). Therefore, \( e(i) = 0 \) as we wanted to show.

**Case 2.** \( s < r - 1 \) and \( \{i_1, \ldots, i_r \} \subseteq \{i_{s+1}, \ldots, i_{r-1} \} \).

By the maximality of \( s \), \( i_s \notin \{i_{s+1}, \ldots, i_{r-1} \} \). Therefore, as argued in the proof of Lemma 2.4.5, there exists a permutation \( w \in \mathcal{S}_r \) such that \( e(i) = \psi_w e(i, i_1, i_2, \ldots, i_s, i_s, i_{s+1}, \ldots, i_{r-1}, i_r, i_{r+1}, \ldots, i_n) \psi_w \). Hence, \( e(i) = 0 \) by Case 1.

**Case 3.** \( s < r - 1 \) and \( \{i_1, \ldots, i_r \} \subseteq \{i_{s+1}, \ldots, i_{r-1} \} \).

Let \( t \) be an index such that \( i_t = j = i_r \pm 1 \) and \( s < t < r \). Note that if there exists an integer \( t' \) such that \( i_{t'} = i_r \) and \( s < t < t' < r \) then we may assume that \( i_s \notin \{i_{s+1}, \ldots, i_{r-1} \} \) by Lemma 2.4.4(c) and induction. Therefore, since \( s \) was chosen to be maximal, \( t \) is the unique integer such that \( i_t = j \) and \( s < t < r \). Hence, arguing as in Case 2, there exists a permutation \( w \in \mathcal{S}_r \) such that

\[
eq \psi_w e(i, i_1, i_2, \ldots, i_n) \psi_w = \psi_w e(i) \psi_w \psi_w = \psi_w e(j, i) \psi_w = 0,
\]

where the last equality follows by Case 1.

Combining Cases 1-3, if \( e(i) \neq 0 \) then \( \{i_1, \ldots, i_r \} \subseteq \{i_{s+1}, \ldots, i_{r-1} \} \) whenever there exists an integer \( s \) such that \( i_s = i_r \) and \( 1 \leq s < r \). Hence, as remarked above, induction, Lemma 2.4.5 and Lemma 2.4.4 show that \( e(i) \neq 0 \) only if \( i_{1r} \notin I^*_\Lambda \).
To complete the proof of the inductive step (and of the proposition), it remains to show that $y_r = 0$. Using what we have just proved, it is enough to show that $y_r e(i) = 0$ whenever $i_{r-1} \in I_{\Lambda}$. If $i_{r-1} = i_r \pm 1$ then, by induction and (2.2.3),

$$y_r e(i) = (y_r - y_{r-1}) e(i) = \pm \psi_{r-1}(e(i)) = \pm \psi_{r-1}(e(s_{r-1} \cdot i)) = 0,$$

where the last equality follows because $(s_{r-1} \cdot i)_{\Lambda} \notin I_{\Lambda}$ by Corollary 2.4.2. If $i_{r-1} \neq i_r \pm 1$ then $i_{r-1} \neq i_r$ by Lemma 2.4.4 since $i_r \in I_{\Lambda}$. Therefore, $y_r e(i) = y_r \psi_{r-1}(e(i)) = \psi_{r-1} y_{r-1} \psi_{r-1}(e(i)) = 0$ since $y_{r-1} = 0$ by induction. This completes the proof.

Before giving our main application of Proposition 2.4.6 we interpret it means for the cyclotomic quiver Hecke algebras of the symmetric groups.

2.4.7. Example (Symmetric groups) Suppose that $\Lambda = \Lambda_0, n \geq 0$ and set $f = \min\{e, n\}$. Then $(\Lambda, \alpha_{i-1}, \Lambda) \leq 1$ for all $i \in I$. Therefore, Proposition 2.4.6 shows that $y_r = 0$ for $1 \leq r < f$ and that $e(i) \neq 0$ only if $i_{f-1} \in I_{\Lambda}^{-1}$. In addition, we also have $\psi_1 = 0$ because if $i \in I^n$ then $\psi_1 e(i) = e(s_1 \cdot i) \psi_1 = 0$ because if $i_{f-1} \in I_{\Lambda}^{-1}$ then $(s_1 \cdot i)_{f-1} \notin I_{\Lambda}^{-1}$.

Translating the proof of Proposition 2.4.6 back to Lemma 2.4.1, the reason why $\psi_1 = 0$ is that if $i = 1^{1}$ is the residue sequence of some standard tableau $t \in \text{Std}(P_{\Lambda}^1)$ then $i_1 = 0$ and $i_2 \neq 0$, so that $s_1 \cdot i \notin I_{\Lambda}^1$ is not a residue sequence and, consequently, $\psi_1 e(i) = e(s_1 \cdot i) \psi_1 = 0$. By the same reasoning, $\psi_1 \neq 0$ if $\Lambda$ has level $\ell > 1$.

We now completely describe the structure of the KLR algebras $\mathcal{R}_{\Lambda}^n$ when $e > n$ and $\Lambda \in P^+$ such that $(\Lambda, \alpha_{i,n}) \leq 1$, for all $i \in I$. For $(s, t) \in \text{Std}^2(P_{\Lambda}^n)$ define $e_{st} = \psi_{(s,t)-1}(e(P)) \psi_{(s,t)}$, where $P = 1^{1}$. 2.4.8. Theorem. Suppose that $e > n$ and $\Lambda \in P^+$ with $(\Lambda, \alpha_{i,n}) \leq 1$, for all $i \in I$. Then $\mathcal{R}_{\Lambda}^n$ is a graded cellular algebra with graded cellular basis $\{e_{st} \mid (s, t) \in \text{Std}^2(P_{\Lambda}^n)\}$ with $\text{deg} e_{st} = 0$ for all $(s, t) \in \text{Std}^2(P_{\Lambda}^n)$.

Proof. By Proposition 2.4.6, $y_r = 0$ for $1 \leq r < n$ and $e(i) = 0$ if $i \notin I_{\Lambda}^1$. In particular, this implies that $\psi_1, \ldots, \psi_{n-1}$ satisfy the braid relations for the symmetric group $\mathcal{S}_n$ because, by Lemma 2.4.4, if $i \in I_{\Lambda}^1$ then $(i, i \pm 1, i)$ is not a subsequence of $i$, for any $i \in I$. Therefore, $\mathcal{R}_{\Lambda}^n$ is spanned by the elements $\psi_1 e(i) \psi_w$, where $v, w \in \mathcal{S}_n$ and $i \in I_{\Lambda}^1$. Moreover, if $j \in I^n$ then $e(j) \psi_1 e(i) \psi_w = 0$ unless $j = v \cdot i \in I_{\Lambda}^1$. Therefore, $\mathcal{R}_{\Lambda}^n$ is spanned by the elements $\{e_{st} \mid (s, t) \in \text{Std}^2(P_{\Lambda}^n)\}$ as required by the statement of the theorem. Hence, $\mathcal{R}_{\Lambda}^n$ has rank at most $\sum_{\Lambda \in P_{\Lambda}^n} |\text{Std}(\Lambda)|^2 = \ell^n n!$, where this combinatorial identity comes from Theorem 1.6.7.

Let $\text{K}$ be the algebraic closure of the field of fractions of $\mathcal{Z}$. Then $\mathcal{R}_{\Lambda}^n(K) \cong \mathcal{R}_{\Lambda}^n(\mathcal{Z}) \otimes \text{K}$. By the last paragraph, the dimension of $\mathcal{R}_{\Lambda}^n$ is at most $\ell^n n!$. Let $\text{rad} \mathcal{R}_{\Lambda}^n(K)$ be the Jacobson radical of $\mathcal{R}_{\Lambda}^n(K)$. For each multipartition $\Lambda \in P_{\Lambda}^n$, Proposition 2.4.3 constructs an irreducible graded Specht module $S_{\Lambda}^K$. By Lemma 2.4.1, if $\lambda, \mu \in P_{\Lambda}^n$ and $d \in \mathcal{Z}$ then $S_{\lambda}^K \cong S_{\mu}^K(d)$ if and only if $\lambda = \mu$ and $d = 0$. By the Wedderburn theorem,

$$\ell^n n! \geq \dim \mathcal{R}_{\Lambda}^n(K)/\text{rad} \mathcal{R}_{\Lambda}^n(K) \geq \sum_{\Lambda \in P_{\Lambda}^n} (\dim S_{\lambda}^K)^2 = \sum_{\Lambda \in P_{\Lambda}^n} |\text{Std}(\Lambda)|^2 = \ell^n n!.$$

Hence, we have equality throughout, so $\{e_{st} \mid (s, t) \in \text{Std}^2(P_{\Lambda}^n)\}$ is a basis of $\mathcal{R}_{\Lambda}^n(K)$. As the elements $\{e_{st}\}$ span $\mathcal{R}_{\Lambda}^n(\mathcal{Z})$, and their images in $\mathcal{R}_{\Lambda}^n(K)$ are linearly independent, so $\{e_{st}\}$ is a basis of $\mathcal{R}_{\Lambda}^n(\mathcal{Z})$.

It remains to prove that $\{e_{st}\} = \psi_1 e(i) \psi_w\}$ span $\mathcal{R}_{\Lambda}^n(\mathcal{Z})$, and their images in $\mathcal{R}_{\Lambda}^n(K)$ are linearly independent, so $\{e_{st}\}$ is a basis of $\mathcal{R}_{\Lambda}^n(\mathcal{Z})$.

Consequently, $\mathcal{R}_{\Lambda}^n$ is a direct sum of matrix rings, for any integral domain $\mathcal{Z}$, and $\{e_{st}\}$ is a basis of $\mathcal{R}_{\Lambda}^n$. Finally, we need to show that $e_{st}$ is homogeneous of degree zero. This will follow if we show that $\text{deg} \psi_1 e(i) = 0, f \leq r < n$ and $\mathcal{I}_{\Lambda}^n$. In fact, this is already clear because if $i \in I_{\Lambda}^n$ then $i_r \neq i_{r+1}$, by Lemma 2.4.4, and if $i_{r+1} = i_r \pm 1$ then $\psi_1 e(i) = 0$ by Corollary 2.4.2 and Proposition 2.4.6.

By definition, $e_{st} e_{uv} = \delta_{ut} e_{sv}$. Let $\text{Mat}_{d}(\mathcal{Z})$ be the ring of $d \times d$ matrices over $\mathcal{Z}$. Hence, the proof of Theorem 2.4.8 also yields the following.

2.4.9. Corollary. Suppose that $\mathcal{Z}$ is an integral domain $e > n$ and that $\Lambda \in P^+$ with $(\Lambda, \alpha_{i,n}) \leq 1$, for all $i \in I$. Then $\mathcal{R}_{\Lambda}^n(\mathcal{Z}) \cong \bigoplus_{\Lambda \in P_{\Lambda}^n} \text{Mat}_{s_{\lambda}}(\mathcal{Z}),$

where $s_{\lambda} = \# \text{Std}(\Lambda)$ for $\lambda \in P_{\Lambda}^n$.

Another consequence of Theorem 2.4.8 is that the KLR relations simplify in the semisimple case — giving a non-standard presentation for a direct sum of matrix rings.
2.4.10. Corollary. Suppose that \( Z \) is an integral domain, \( e > n \) and that \( \Lambda \in P^+ \) with \( (\Lambda, \alpha_i, n) \leq 1 \), for all \( i \in I \). Then \( \mathcal{A}_n^\Lambda \) is the unital associative \( Z \)-graded algebra generated by \( \psi_1, \ldots, \psi_{n-1} \) and \( e(i) \), for \( i \in I^n \), subject to the relations
\[
e(i)^{e(\Lambda, \alpha_i)} = 0 \quad \sum_{i \in I^r} e(i) = 1, \quad e(i) e(j) = \delta_{ij} e(i),
\]
\[
\psi_r e(i) = e(s_r \cdot i) \psi_r, \quad e(i) = 0 \text{ if } i_r = i_{r+1}, \quad e(i) e(i) = e(i)
\]
\[
\psi_r \psi_s = \psi_s \psi_r, \quad \text{if } |r-s| > 1,
\]
\[
\psi_r \psi_{r+1} \psi_r e(i) = \begin{cases} (\psi_r \psi_{r+1} \psi_{r+1} - 1) e(i), & \text{if } i_{r+2} = i_r \rightarrow i_{r+1}, \\ (\psi_r \psi_{r+1} \psi_{r+1} + 1) e(i), & \text{if } i_{r+2} = i_r \leftarrow i_{r+1}, \\ \psi_{r+1} \psi_r \psi_{r+1} e(i), & \text{otherwise}, \end{cases}
\]
for all \( i, j \in I^n \) and admissible \( r \) and \( s \). Moreover, \( \mathcal{A}_n^\Lambda \) is concentrated in degree zero.

The reader is encouraged to check the details here. Note that these relations, together with the argument of Proposition 2.4.6, imply that \( e(i) \neq 0 \) if only if \( i \in I_n^\Lambda \). In particular, the combinatorics of tableau content sequences is partially encoded in the failure of the braid relations for the \( \psi_r \).

As a final application, we prove Brundan and Kleshchev’s graded isomorphism theorem in this special case.

2.4.11. Corollary. Suppose that \( Z = K \) is a field, \( e > n \), and that \( \Lambda \in P^+ \) with \( (\Lambda, \alpha_i, n) \leq 1 \), for all \( i \in I \). Then \( \mathcal{A}_n^\Lambda \cong \mathcal{H}_n^\Lambda \).

Proof. By Corollary 2.4.10 and Theorem 1.6.7, there is a well-defined homomorphism \( \Theta : \mathcal{A}_n^\Lambda \rightarrow \mathcal{H}_n^\Lambda \) determined by \( e(\mathfrak{P}) \mapsto F_s \) and
\[
\psi_r e(\mathfrak{P}) \mapsto \begin{cases} \frac{1}{\alpha_r(s)} \left( T_r - \frac{c_{x+1}(s) - c_{x}(s)}{1 + (r-s) c_{x+1}(s)} \right) F_s, & \text{if } \alpha_r(s) \neq 0, \\ 0, & \text{otherwise}, \end{cases}
\]
for \( s \in \text{Std}(\mathcal{P}_n^\Lambda) \) and \( 1 \leq r < n \). Using Theorem 2.4.8, or Proposition 2.4.3, it follows that \( \Theta \) is an isomorphism.

We emphasize that it is essential to work over a field in Corollary 2.4.11 because Corollary 2.4.9 says that \( \mathcal{A}_n^\Lambda \) is always a direct sum of matrix rings whereas if \( n > 1 \) this is only true of \( \mathcal{H}_n^\Lambda \) when it is defined over a field.

These results suggest that \( \mathcal{A}_n^\Lambda \) should be considered as the “idempotent completion” of the algebra \( \mathcal{H}_n^\Lambda \) obtained by adjoining idempotents \( e(\lambda) \), for \( i \in I^n \). We will see how to make sense of the idempotents \( e(\lambda) \in \mathcal{H}_n^\Lambda \) for any \( i \in I^n \) in Theorem 3.1.1 and Lemma 4.2.2 below.

2.5. The nil-Hecke algebra. Still working just with the relations we now consider the shadow of the nil-Hecke algebra in the cyclotomic KLR setting. For the affine KLR algebras the nil-Hecke algebras case has been well-studied [74, 121].

For this section fix \( i \in I \) and set \( \beta \equiv n\alpha \) and \( \Lambda \equiv n\alpha \). Following (2.2.5), set \( \mathcal{A}_\beta^\Lambda = e(1) \mathcal{A}_\lambda^\Lambda e(1) \), where \( i = 1^\beta = (\ell^\beta) \). Then \( \mathcal{A}_\beta^\Lambda \) is a direct summand of \( \mathcal{A}_\lambda^\Lambda \) and, moreover, it is a non-unital subalgebra with identity element \( e(1) \). As \( e(1) \) is the unique non-zero KLR idempotent in \( \mathcal{A}_\lambda^\Lambda \), \( \psi_r = \psi_r e(1) \) and \( y_s = ye(1) \). Therefore, \( \mathcal{A}_\beta^\Lambda \) is the unital associative graded algebra generated by \( \psi_r \) and \( y_s \), for \( 1 \leq r < n \) and \( 1 \leq s \leq n \), with relations
\[
y_1^0 = 0, \quad \psi_2^2 = 0, \quad y_r y_s = y_s y_r, \quad \psi_r y_{r+1} = y_r \psi_r + 1, \quad y_{r+1} \psi_r = \psi_r y_r + 1,
\]
\[
\psi_r \psi_s = \psi_s \psi_r, \quad \text{if } |r-s| > 1, \quad \psi_r y_s = y_s \psi_r, \quad \text{if } s \neq r, r+1, \quad \psi_r \psi_{r+1} = \psi_{r+1} \psi_r \psi_{r+1}.
\]
The grading on \( \mathcal{A}_\beta^\Lambda \) is determined by \( \text{deg } \psi_r = -2 \) and \( \text{deg } y_s = 2 \). Some readers will recognize this presentation as defining as a cyclotomic quotient of the nil-Hecke algebra of type \( A \) [88]. Note that the argument from Case 3 of Lemma 3.1.1 shows that \( y_r^0 = 0 \) for \( 1 \leq r \leq \ell \).

Let \( \lambda = [(11) \ldots (1)] \in \mathcal{P}_\beta^\Lambda \). Then the map \( t \mapsto d(t) \) defines a bijection between the set of standard \( \lambda \)-tableaux and the symmetric group \( S_n \). For convenience, we identify the standard \( \lambda \)-tableaux with the set of (non-standard) tableaux of partition shape \( n \) by concatenating their components. In other words, if \( d = d(t) \) then \( t = [t_1, t_2, \ldots, t_\ell] \), where \( d = d_1, \ldots, d_\ell \) is the permutation written in one-line notation.

If \( v, s \in \text{Std}(\lambda) \) then write \( s \triangleright v \) if \( s \triangleright v \) and \( d(v) = d(s) + 1 \). To make this more explicit write \( t \prec_m m \) if \( t \) is in an earlier component of \( v \) than \( m \) — that is, \( t \) is to the left of \( m \) in \( v \). The reader can check that \( s \triangleright v \) if and only if there exist integers \( 1 \leq m < t \leq n \) such that \( s = v(m, t), \ m \ll t \) and if \( m < l < t \) then either \( l \ll m \) or \( m \ll l \).
2.5.1. Example Suppose that \( n = 6 \). Let \( \nu = [4\ 6\ 5\ 3\ 1\ 2] \) and take \( t = 3 \). Then
\[
\left\{ \begin{array}{c} 3\ 6\ 5\ 4\ 1\ 2 \quad 4\ 6\ 3\ 5\ 1\ 2 \quad 4\ 6\ 5\ 2\ 1\ 3 \quad 4\ 6\ 5\ 1\ 3\ 2 \end{array} \right\}
\]
is the set of \( \lambda \)-tableaux \( \{ s \mid s = \nu \langle s, r \rangle v \text{ for } 1 \leq r \leq n \} \).

We can now state the main result of the section.

2.5.2. Proposition Suppose that \( \beta = n\alpha_i \) and \( \Lambda = n\Lambda_i \), for \( i \in I \). Then there is a unique graded \( R_{\beta}^\Lambda \)-module \( S^\Lambda \) with homogeneous basis \( \{ \psi_s \mid s \in \text{Std}(\Lambda) \} \) such that \( \deg \psi_s = \binom{n}{2} - 2\ell(d(s)) \) and
\[
\psi_s \psi_r = \begin{cases} \psi_{s(r,r+1)}, & \text{if } s \triangleright (s(r,r+1)) \in \text{Std}(\Lambda), \\ 0, & \text{otherwise}, \end{cases}
\]
\[
\psi_s \psi_{yt} = \sum_{1 \leq k < \ell \leq n} \psi_s - \sum_{t < k \leq n} \psi_s,
\]
for \( s, v \in \text{Std}(\Lambda) \), \( 1 \leq r < n \) and \( 1 \leq t \leq n \). Moreover, if \( Z \) is a field then \( S^\Lambda \) is irreducible.

Proof. The uniqueness is clear. To show that \( S^\Lambda \) is an \( R_{\beta}^\Lambda \)-module we check that the action respects the relations of \( R^\Lambda_{\beta} \). By definition, if \( v \in \text{Std}(\Lambda) \) then \( \psi_v = \psi(v) \psi(d(v)) \) and \( \psi_v \psi_v^2 = 0 \) since \( \psi_v \psi_v = 0 \) if \( v \) is \( n \times (r,r+1) \triangleright v \). In particular, this implies that the action of \( \psi_1, \ldots, \psi_{n-1} \) on \( S^\Lambda \) respects the braid relations of \( S_n \) and that \( \psi_v \) has the specified degree. Further, note that if \( u \triangleright v \) then \( \ell(d(v)) = \ell(d(u)) + 1 \) so that \( \deg \psi_u = \deg \psi_v + 2 \).

By the last paragraph, the action of \( R_{\beta}^\Lambda \) is compatible with the grading on \( S^\Lambda \), but we still need to check the relations involving \( y_1, \ldots, y_n \). First consider \( \psi_v \psi_{yt} = \psi_v y_{yt} \), for \( 1 \leq r, t \leq n \) and \( v \in \text{Std}(\Lambda) \). If \( r = t \) there is nothing to prove so suppose \( r \neq t \). By definition,
\[
\psi_v y_{yt} = \sum_{u \triangleright v} \sum_{s \triangleright u} \epsilon_1(v,u) \epsilon_1(u,s) \psi_s,
\]
for appropriate choices of the signs \( \epsilon_1(v,u) \) and \( \epsilon_1(u,s) \). Suppose that \( \psi_s \) appears with non-zero coefficient in this sum. Then we can write \( u = v(m, t) \) and \( s = \psi(d(v)) \), for some \( m \) such that \( s \triangleright u \triangleright v \). Suppose first that \( l \neq m \). Then the permutations \( (m, t) \) and \( (l, r) \) commute and, as their lengths add, we have \( s \triangleright v(l, r) \triangleright v \). Therefore, \( \psi_s \) appears with the same weight \( \psi_v y_{yt} \) and \( \psi_s y_{yt} \). If \( l = m \) then \( s \triangleright u \triangleright v \) only if \( m \) is in between \( r \) and \( t \) in \( v \). That is, either \( r \prec_s m \prec_s t \) or \( t \prec_s m \prec_s r \). However, this implies that either \( s \not\triangleright u \) or \( u \not\triangleright v \), so that \( \psi_s \) does not appear in either \( \psi_s y_{yt} \) or \( \psi_s y_{yt} \). Hence, the actions of \( y_r \) and \( y_t \) on \( S^\Lambda \) commute.

Similar, but easier, calculations with tableaux show that the action defined on \( S^\Lambda \) respects the three relations \( \psi_v \psi_{yt} = y_r \psi_v + 1 \), \( y_r \psi_{yt} = y_t \psi_v + 1 \) and \( \psi_v y_{yt} = y_t \psi_v \) when \( r \neq t \). To complete the verification of the relations in \( R_{\beta}^\Lambda \) it remains to show that \( \psi_v y_{yt} = 0 \) for all \( v \in \text{Std}(\Lambda) \). This is clear, however, because \( \psi_v y_{yt} \) is equal to a linear combination of terms \( \psi_s \) where \( 1 \) appears in an earlier component of \( s \) than it does in \( v \).

Finally, it remains to prove that \( S^\Lambda \) is irreducible over a field. First we need some more notation. Let \( t_k = \binom{n}{k} - 2k \) and set \( w_\Lambda = d(t_k) \). Then \( w_\Lambda \) is the unique element of maximal length in \( S_n \). Recall from \S 1.4, that \( d(t) \) is the unique permutation such that \( t = x_\Lambda d(t) \) and, moreover, \( d(t) d(t)^{-1} = w_\Lambda \) with the lengths adding. Therefore, if \( \ell(d(s)) \leq \ell(d(t)) \) then \( \psi_s \psi_{yt} = \delta_{s,t} \).

We are now ready to show that \( S^\Lambda \) is irreducible. Suppose that \( x = \sum_{s \neq t} r_s \psi_s \) is a non-zero element of \( S^\Lambda \). Let \( t \) be any tableau such that \( r_t \neq 0 \) and \( \ell(d(t)) \) is minimal. Then, by the last paragraph, \( x \psi_{yt} = r_t \psi_{yt} \). We have already observed that \( y_t \) acts by moving \( 1 \) to an earlier component. Therefore, \( \psi_{t_1} y_{t_1} = (1)^{n-1} \psi_{t_1} \), where \( t_1 = \binom{n}{1} - \frac{n(n-1)}{2} \psi_{t_1} \), \( t_2 = \binom{n}{2} - \frac{n(n-1)(n-2)}{6} \psi_{t_2} \), \( t_3 = \binom{n}{3} - \frac{n(n-1)(n-2)(n-3)}{24} \psi_{t_3} \). Continuing in this way shows that \( \psi_{y_1} (y_2 - n-2) \psi_{n-1} = (1)_{\frac{n(n-1)}{2}} \psi_{x} \). Hence, \( x S^\Lambda = S^\Lambda \), so that \( S^\Lambda \) is irreducible as claimed.

The proof of Proposition 2.5.2 shows that \( y_{n-2} \cdots y_2 \psi_1 \psi_n \) is a non-zero element of \( R_{\beta}^\Lambda \). Using the relations, and a bit of ingenuity, it is possible to show that \( \psi_w y_1 \cdots y_{n-1} \in \text{Std}(\Lambda) \) and \( 0 \leq a_w \leq n - r \), for \( 1 \leq r \leq n \) is a basis of \( R_{\beta}^\Lambda \). Alternatively, it follows from [22, Theorem 4.20] that \( \dim R_{\beta}^\Lambda = (n!)^2 \). Hence, we obtain the following.

2.5.3. Corollary Suppose that \( \beta = n\alpha_i \) and \( \Lambda = n\Lambda_i \), for \( i \in I \). Let \( \lambda = [1] \cdots [1] \) and for \( s, t \in \text{Std}(\Lambda) \) define \( \psi_{st} = \psi_{st(d)} \psi_{s(t)} \), where \( \lambda^\Lambda = \lambda^\Lambda \) and \( y^\Lambda = y_{n-1} \cdots y_2 \psi_1 \). Then \( \{ \psi_{st} \mid s, t \in \text{Std}(\Lambda) \} \) is a graded cellular basis of \( R_{\beta}^\Lambda \).

The basis of the Specht module \( S^\Lambda \) in Proposition 2.5.2 is well-known because it is really a disguised version of the basis of Schubert polynomials of the covariant algebra of the symmetric group \( S_n \) [90, 101]. The
covariant algebra \( \mathcal{C}_n \) is the quotient of the polynomial ring \( \mathbb{Z}[x] = \mathbb{Z}[x_1, \ldots, x_n] \) by the ideal generated by the symmetric polynomials in \( x_1, \ldots, x_n \) of positive degree. Then \( \mathcal{C}_n \) is free of rank \( n! \). As we have quotiented out by a homogeneous ideal, \( \mathcal{C}_n \) inherits a grading from \( \mathbb{Z}[x] \), where we set \( \deg x_r = 2 \) for \( 1 \leq r \leq n \). Identify \( x_r \) with its image in \( \mathcal{C}_n \), for \( 1 \leq r \leq n \). There is a well-defined action of \( \mathcal{H}_3^\mathbb{A} \) on \( \mathcal{C}_n \) where \( y_r \) acts as multiplication by \( x_r \), and \( \psi_r \) acts from the right as a divided difference operator:

\[
    f(x)\psi_r = \partial_r f(x) = \frac{f(x) - f(s_r \cdot x)}{x_r - x_{r+1}},
\]

where \( x = (x_1, \ldots, x_n) \) and \( s_r \cdot x = (x_1, \ldots, x_{r+1}, x_r, \ldots, x_n) \) for \( 1 \leq r < n \). Here we are secretly thinking of \( \mathcal{H}_3^\mathbb{A} \) as being a quotient of the nil-Hecke algebra, where this action is well-known.

For \( d \in \mathfrak{S}_n \) define \( \sigma_d = (x_1^{r-1}x_2^{r-2} \cdots x_{n-1})\psi_{\text{w}0d} \). Then \( \{ \sigma_d \mid d \in \mathfrak{S}_n \} \) is the basis of the Schubert polynomials of \( \mathcal{C}_n \). The Specht module is isomorphic to \( \mathcal{C}_n \) as an \( \mathcal{H}_3^\mathbb{A} \)-module, where an isomorphism is given by \( \psi_t \mapsto \sigma_d(t) \). To see this it is enough to know that the Schubert polynomials satisfy the identity

\[
    \partial_r \sigma_d = \begin{cases} 
    \sigma_{s_r d}, & \text{if } \ell(s_r d) = \ell(d) - 1, \\
    0, & \text{otherwise.}
    \end{cases}
\]

By the last paragraph of the proof of Proposition 2.5.2, if \( t \in \mathfrak{S}_n \) then

\[
    \psi_t = \psi_{\text{w}t} \psi_{d(t)} = \psi_{\text{w}t} y_1^{t_1-1} y_2^{t_2-1} \cdots y_n^{t_n-1} \psi_{d(t)}.
\]

Therefore, our claim follows by identifying \( \psi_t \) with the polynomial \( 1 \in \mathcal{C}_n \).

Finally, we remark that the formula for the action of \( y_1, \ldots, y_n \) in Proposition 2.5.2 is a well-known corollary of Monk’s rule; see, for example, [101, Exercise 2.7.3].

3. Isomorphisms, Specht modules and categorification

In the last section we proved that the algebras \( \mathcal{H}_{\mathbb{A}}^\Lambda \) and \( \mathcal{H}_{\mathbb{A}}^\Lambda \) are isomorphic when \( e > n \) and \( (\Lambda, \alpha_{1+n}) \leq 1 \), for all \( i \in I \). In this section we state Brundan and Kleshchev’s Graded Isomorphism Theorem, which says that \( \mathcal{H}_{\mathbb{A}}^\Lambda \cong \mathcal{H}_{\mathbb{A}}^\Lambda \), and we start to investigate the consequences of this result for both algebras.

3.1. The Graded Isomorphism Theorem. One of the most fundamental results for the cyclotomic Hecke algebras \( \mathcal{H}_{\mathbb{A}}^\Lambda \) is Brundan and Kleshchev’s spectacular isomorphism theorem.

3.1.1. Theorem (Graded Isomorphism Theorem [21,121]). Suppose that \( Z = F \) is a field, \( v \in F \) has quantum characteristic \( e \) and that \( \Lambda \in P^+ \). Then there is an isomorphism of algebras \( \mathcal{H}_{\mathbb{A}}^\Lambda \cong \mathcal{H}_{\mathbb{A}}^\Lambda \).

Suppose that \( F \) is a field of characteristic \( p > 0 \) and that \( e = pf \), where \( f > 1 \). Then \( F \) cannot contain an element \( v \) of quantum characteristic \( e \), so Theorem 3.1.1 says nothing about the quiver Hecke algebra \( \mathcal{H}_{\mathbb{A}}^\Lambda(F) \).

As a first consequence of Theorem 3.1.1, by identifying \( \mathcal{H}_{\mathbb{A}}^\Lambda \) and \( \mathcal{H}_{\mathbb{A}}^\Lambda \) we can consider \( \mathcal{H}_{\mathbb{A}}^\Lambda \) as a graded algebra.

3.1.2. Corollary. Suppose that \( \Lambda \in P^+ \) and \( Z = F \) is a field. Then there is a unique grading on \( \mathcal{H}_{\mathbb{A}}^\Lambda \) such that \( \deg e(i) = 0 \), \( \deg y_r = 2 \) and \( \deg \psi_r e(i) = -c_{1+r, i+1} \), for \( 1 \leq r \leq n, 1 \leq s < n \) and \( i \in I^n \).

Brundan and Kleshchev prove Theorem 3.1.1 by constructing family of isomorphisms \( \mathcal{H}_{\mathbb{A}}^\Lambda \to \mathcal{H}_{\mathbb{A}}^\Lambda \), together with their inverses, and then painstakingly checking that these isomorphisms respect the relations of both algebras. Their argument starts with the well-known fact that \( \mathcal{H}_{\mathbb{A}}^\Lambda \) decomposes into a direct sum of simultaneous generalized eigenspaces for the Jucys-Murphy elements \( L_1, \ldots, L_n \). These eigenspaces are indexed by \( I^n \), so for each \( i \in I^n \) there is an element \( e(i) \in \mathcal{H}_{\mathbb{A}}^\Lambda \), possibly zero, such that \( e(i)e(j) = \delta_{ij} e(i) \). We describe these idempotents explicitly in Lemma 4.2.2 below.

Translating through Definition 1.1.1, Brundan and Kleshchev’s isomorphism is given by \( e(i) \mapsto e(i) \) and

\[
    y_r \mapsto \sum_{i \in I^n} v^{r-i} (L_r - [i]_r \cdot e(i)), \quad \text{and} \quad \psi_i \mapsto \sum_{i \in I^n} \left( T_s + P_r(i) \frac{1}{Q_s} \right) e(i),
\]

for \( 1 \leq r \leq n, 1 \leq s < n \) and \( i \in I^n \) and where \( P_r(i) \) and \( Q_r(i) \) are certain rational functions in \( y_r \) and \( y_{r+1} \) that are well-defined because \( (L_r - [i]_r \cdot e(i)) \) is nilpotent in \( \mathcal{H}_{\mathbb{A}}^\Lambda \), for \( 1 \leq t \leq n \); see [21, §3.3 and §4.3]. We are abusing notation by identifying the KLR generators with their images in \( \mathcal{H}_{\mathbb{A}}^\Lambda \). The inverse isomorphism is given by \( e(i) \mapsto e(i) \),

\[
    L_r \mapsto \sum_{i \in I^n} (v^{r-i} y_r + [i]_r \cdot e(i)) \quad \text{and} \quad T_s \mapsto \sum_{i \in I^n} (\psi_s Q_s(i) - P_s(i)) e(i),
\]

for \( 1 \leq r \leq n, 1 \leq s < n \) and \( i \in I^n \).

Rouquier [121, Corollary 3.20] has given a more direct proof of Theorem 3.1.1 by first showing that the (non-cyclotomic) quiver Hecke algebra \( \mathcal{H}_n \) is isomorphic to the (extended) affine Hecke algebra of type \( A \). Following [57], we sketch another approach to Theorem 3.1.1 in §4.2 below.
Theorem 3.1.1 was a surprise (at least to the author!).

3.1.3. Corollary. Suppose that \( F = \mathbb{C} \) is a field and that \( v, v' \in F \) are two elements of quantum characteristic \( e \). Then \( \mathcal{H}_n^\Lambda(F,v) \cong \mathcal{H}_n^\Lambda(F,v') \).

Proof. By Theorem 3.1.1, \( \mathcal{H}_n^\Lambda(F,v) \cong \mathcal{H}_n^\Lambda(F,v') \).

Consequently, up to isomorphism, the algebra \( \mathcal{H}_n^\Lambda \) depends only on \( e, \Lambda \) and the field \( F \). Therefore, because \( \mathcal{H}_n^\Lambda \) is cellular, the decomposition matrices of \( \mathcal{H}_n^\Lambda \) depend only on \( e, \Lambda \) and \( p \), where \( p \) is the characteristic of \( F \). In the special case of the symmetric group, when \( \Lambda = \Lambda_0 \), this weaker statement for the decomposition matrices was conjectured in [104, Conjecture 6.38].

When \( F = \mathbb{C} \) it is easy to prove Corollary 3.1.3 because there is a Galois automorphism of \( \mathbb{Q}(v) \), as an extension of \( \mathbb{Q} \), which interchanges \( v \) and \( v' \). It is not difficult to see that this automorphism induces an isomorphism \( \mathcal{H}_n^\Lambda(F,v) \cong \mathcal{H}_n^\Lambda(F,v') \). This argument fails for fields of positive characteristic because such fields have fewer automorphisms.

3.2. Graded Specht modules. As we noted in §2.1, if we impose a grading on an algebra \( A \) then it is not true that every (ungraded) \( A \)-module has a graded lift, so there is no reason to expect that graded lifts of Specht modules \( S^\Lambda \) always exist. Of course, graded Specht modules do exist and this section describes one way to define them.

Recall from §1.5 that the ungraded Specht module \( S^\Lambda \), for \( \Lambda \in \mathcal{P}_n^\Lambda \), has basis \( \{ m_\tau \mid \tau \in \text{Std}(\Lambda) \} \). By construction, \( S^\Lambda = m_\Lambda \mathcal{H}_n^\Lambda \). Brundan, Kleshchev and Wang [25] proved that \( S^\Lambda \) has a graded lift essentially by declaring that \( m_\Lambda \) should be homogeneous and then showing that this induces a grading on the Specht module \( S^\Lambda = m_\Lambda \mathcal{H}_n^\Lambda \).

Partly inspired by [25], Jun Hu and the author [54] showed that \( \mathcal{H}_n^\Lambda \) is a graded cellular algebra. The graded cell modules constructed from this cellular basis coincide exactly with those of [25]. Perhaps most significantly, the construction of the graded Specht modules using cellular algebra techniques endows the graded Specht modules with a homogeneous bilinear form of degree zero.

Following Brundan, Kleshchev and Wang [25, §3.5] we now define the degree of a standard tableau. Suppose that \( \mu \in \mathcal{P}_n^\Lambda \). For \( i \in I \) let \( \text{Add}_i(\mu) \) be the set of addable \( i \)-nodes of \( \mu \) and let \( \text{Rem}_i(\mu) \) be its set of removable \( i \)-nodes.

3.2.1. Definition. If \( A \) is an addable or removable \( i \)-node of \( \mu \) define:

\[
\begin{align*}
  d^A(\mu) &= \# \left\{ B \in \text{Add}_i(\mu) \mid B > A \right\} - \# \left\{ B \in \text{Rem}_i(\mu) \mid B > A \right\}, \\
  d_A(\mu) &= \# \left\{ B \in \text{Add}_i(\mu) \mid A < B \right\} - \# \left\{ B \in \text{Rem}_i(\mu) \mid A < B \right\}, \\
  d_{\mu i} &= \# \text{Add}_i(\mu) - \# \text{Rem}_i(\mu).
\end{align*}
\]

If \( t \) is a standard \( \mu \)-tableau then its codegree and degree are defined inductively by setting codeg\(_e\) \( \tau = 0 = \text{deg}_e \tau \) if \( n = 0 \) and if \( n > 0 \) then

\[
\text{codeg}_e \tau = \text{codeg}_e \tau_{(n-1)} + d^A(\mu) \quad \text{and} \quad \text{deg}_e \tau = \text{deg}_e \tau_{(n-1)} + d_A(\mu),
\]

where \( A = t^{-1}(n) \). If \( e \) is fixed we write \( \text{codeg}_e \tau = \text{codeg}_e \tau \) and \( \text{deg} \tau = \text{deg}_e \tau \).

Implicitly, all of these definitions depend on the choice of multicharge \( \kappa \). The definition of the degree and codegree of a standard tableau is due to Brundan, Kleshchev and Wang [25], however, the underlying combinatorics dates back to Misra and Miwa [111] and their work on the crystal graph and Fock space representations of \( U_q(\widehat{s\ell}_n) \).

Recall that we have fixed an arbitrary reduced expression for each permutation \( w \in S_n \). In §1.4 for each standard tableau \( t \in \text{Std}(\Lambda) \) we have defined permutations \( d'(t), d(t) \in S_n \) by \( t d'(t) = t^\Lambda d(t) \).

3.2.2. Definition ([54, Definitions 4.9 and 5.1]). Suppose that \( \mu \in \mathcal{P}_n^\Lambda \). Define non-negative integers \( d^\mu_1, \ldots, d^\mu_n \) and \( d_{\mu 1}, \ldots, d_{\mu n} \) recursively by requiring that \( d^\mu_1 + \cdots + d^\mu_n = \text{codeg}(t^\mu_{\lambda k}) \) and \( d_{\mu 1} + \cdots + d_{\mu n} = \text{deg}(t^\mu_{\lambda k}) \), for \( 1 \leq k \leq n \). Now set \( i_0 = 1^\mu, \ i_1 = 1^\mu, \ y_0 = y_1 = \cdots = y_{n-1} = y_n = y_1^\mu = \cdots = y_n^\mu \). For \( (s, t) \in \text{Std}^2(\mu) \) define

\[
\psi^\Delta_{st} = \psi^D(s)(\iota(\mu)) y_0^\mu \psi^D(t) \quad \text{and} \quad \psi_{st} = \psi^D(s)(\iota(\mu)) y^\mu \psi^D(t),
\]

where \( * \) is the unique (homogeneous) anti-isomorphism of \( \mathcal{H}_n^\Lambda \) that fixes the KLR generators.

3.2.3. Example Suppose that \( e = 3 \), \( \Lambda = \Lambda_0 + \Lambda_2 \) and \( \mu = (7, 6, 3, 2/4, 3, 1) \), with multicharge \( \kappa = (0, 2) \). Then

\[
\begin{array}{cccccccc}
 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
 19 & 20 & 21 & 22 & 23 & 24 & 25 \\
 14 & 15 & 16 & 17 & 18 \\
 26 & 27 & 28 & 29 & 30 & 31 & 32
\end{array}
\]
The reader may check that \( e(\mu^i) = e(01201202012011200120121200) \). We have shaded the nodes in \( t^\mu \) when they have column index divisible by \( e \) and when they have residue \( 2 = \text{res}_\mu(19) \). This should convince the reader that \( y^\mu = y^\mu_1y^\mu_2y^\mu_3y^\mu_4y^\mu_5y^\mu_6 \). With analogous shadings,

\[
\begin{array}{cccccccc}
9 & 13 & 17 & 20 & 22 & 24 & 26 \\
10 & 14 & 18 & 21 & 23 & 25 & 22 & 25 & 28 \\
12 & 16 & \end{array}
\]

Hence, reading right to left, \( y_\mu = y^\mu_1y^\mu_2y^\mu_3y^\mu_4y^\mu_5y^\mu_6 \). Note that \( \text{res}_\mu(9) = 0 \). 

\[\begin{array}{c}
3.2.4. \textbf{Theorem} (Hu-Mathas [54, Theorem 5.8]). Suppose that \( Z = F \) is a field. Then \( \{ \psi_\mu \mid (s,t) \in \text{Std}^2(P_n^\Lambda) \} \)

is a graded cellular basis of \( R_\beta^\Lambda \) with \( \psi_\mu^s = \psi_\mu \) and \( \deg \psi_\mu = \deg s + \deg t \), for \( (s,t) \in \text{Std}^2(P_n^\Lambda) \).

\[\begin{array}{c}
3.2.5. \textbf{Example} \text{ Let } \beta = \alpha_n, \Lambda = n\Lambda_1, \text{ for some } i \in I, \text{ so that } R_\beta^\Lambda \text{ is the nil-Hecke algebra } R^\Lambda_\beta \text{ of } §2.5. \text{ Let } \lambda = (11\ldots 1). \text{ Then the definitions give } y_\lambda = y_\lambda^{n-1} \ldots y_\lambda^{2}y_\lambda^{1} - 1. \text{ Hence, the basis } \{ \psi_\mu \} \text{ of } R^\Lambda_\beta \text{ coincides with that of Corollary 2.5.3.}
\end{array}\]

\[\begin{array}{c}
3.2.6. \textbf{Example} \text{ As in Example 2.2.7, in general, the basis element } \psi_\mu \text{ depends on the choices of reduced expressions that we have fixed for the permutations } d(s) \text{ and } d(t). \text{ For example, let } \Lambda = 2\Lambda_0 + \Lambda_1, \kappa = (0,1,0), \mu = (11\ldots 1), \text{ and consider the standard } \mu \text{-tableaux } t^\mu = (1\ 2\ 3) \text{ and } t_\mu = (3\ 2\ 1) \text{. Then } d(t^\mu) = 1 \text{ and } d(t_\mu) = (1,3) = s_1s_2s_1 = s_2s_1s_2 \text{ has two different reduced expressions. Let } \psi_{t^\mu} = \psi_1^\mu \psi_2^\mu e(\mu^i) \psi_1^\mu \psi_2^\mu \psi_1^\mu \psi_2^\mu. \text{ Then the calculation in Example 2.2.7 implies that } \psi_\mu = \psi_{t^\mu} + \psi_{t^\mu} + \psi_{t^\mu} + \psi_{t^\mu} + \psi_{t^\mu}.
\end{array}\]

This is probably the simplest example where different reduced expressions lead to different \( \psi \)-basis elements, but examples occur for almost all \( R^\Lambda_\beta \). This said, in view of Proposition 2.4.3, if \( \psi_\mu \) is independent of the choice of reduced expressions for \( d(s) \) and \( d(t) \) whenever \( e > n \) and \( (\Lambda, \alpha_i, n) \leq 1, \text{ for all } i \in I \). The \( \psi \)-basis can be independent of the choice of reduced expressions even when \( R^\Lambda_\beta \) is not semisimple. For example, this is always the case when \( e > n \) and \( \ell = 2 \) by [55, Appendix], yet these algebras are typically not semisimple.

Using the theory of graded cellular algebras from [§2.1, Theorem 3.2.4] allows us to construct a family \( \{ S_\lambda^\Lambda \mid \lambda \in P_\alpha^\Lambda \} \) of graded Specht modules for \( \mathcal{H}^\Lambda_\alpha \). By [54, Corollary 5.10] the graded Specht modules attached to the \( \psi \)-basis coincide with those constructed by Brundan, Kleshchev and Wang [25]. When \( e > n \) and \( (\Lambda, \alpha_i, n) \leq 1, \text{ for all } i \in I \), it is not hard to show that these Specht modules coincide with those we constructed in Proposition 2.4.3. Similarly, for the nil-Hecke algebra considered in §2.5, the graded Specht module \( S_\lambda^\Lambda \), with \( \Lambda = (11\ldots 1) \), is isomorphic to the graded module constructed in Proposition 2.5.2. Moreover, on forgetting the grading \( S_\lambda^\Lambda \) coincides exactly with the ungraded Specht module \( S_\lambda^\Lambda \) constructed in §1.5, for \( \Lambda \in P_\alpha^\Lambda \).

If \( \lambda \in P_\alpha^\Lambda \) the graded Specht module \( S_\lambda^\Lambda \) has basis \( \{ \psi_\mu \mid (s,t) \in \text{Std}(\lambda) \} \), with \( \deg \psi_\mu = \deg t \). The reader should be careful not to confuse \( \psi_\mu \in S_\lambda^\Lambda \) with \( \psi_{(s,t)} \in R^\Lambda_\beta \). By Theorem 3.2.4 we recover [22, Theorem 4.20]:

\[
\dim_{q} \mathcal{H}^\Lambda_\alpha = \sum_{(s,t) \in \text{Std}(\lambda)} q^{\deg s + \deg t} = \sum_{\lambda \in P_\alpha^\Lambda} \left( \dim_{q} S_\lambda^\Lambda \right)^2.
\]

In essence, Theorem 3.2.4 is proved in much the same way that Brundan, Kleshchev and Wang [25] constructed a grading on the Specht modules: we proved that the transition matrix between the \( \psi \)-basis and the Murphy basis of Theorem 1.5.1 is triangular. In order to do this we needed the correct definition of the elements \( y^\mu \), which we discovered by first looking at the one dimensional two-sided ideals of \( \mathcal{H}^\Lambda_\alpha \) (which are necessarily homogeneous). We then used Brundan and Kleshchev’s Graded Isomorphism Theorem 3.1.1, together with the seminormal forms (Theorem 1.6.7), to show that \( e(\mu^i) y^\mu \neq 0 \). This established that the basis of Theorem 3.2.4 is a graded cellular basis. Finally, the combinatorial results of [25] are used to determine the degree of \( \psi \)-basis elements.

Following the recipe in §2.1, for \( \mu \in P_\alpha^\Lambda \) define \( D^\mu = S^\mu / \text{rad } S^\mu \), where \( \text{rad } S^\mu \) is the radical of the homogeneous bilinear form on \( S^\mu \). This yields the classification of the graded irreducible \( \mathcal{H}^\Lambda_\alpha \)-modules. The main point of the next result is that the labelling of the graded irreducible \( \mathcal{H}^\Lambda_\alpha \)-modules agrees with Corollary 1.5.2.

\[\begin{array}{c}
3.2.7. \textbf{Corollary} \text{ [22, Theorem 5.13], [54, Corollary 5.11]. Suppose that } \Lambda \in P^+ \text{ and that } Z = F \text{ is a field. Then } \{ D^\mu_{\psi(d)} \mid \mu \in K^\Lambda, \text{ and } d \in Z \} \text{ is a complete set of pairwise non-isomorphic graded } \mathcal{H}^\Lambda_\alpha \text{-modules. Moreover, } D^\mu_{\psi(d)} \cong D^\mu_{\psi(d)} \text{ and } D^\mu_{\psi(d)} \text{ is absolutely irreducible, for all } \mu \in K^\Lambda. \text{ The graded decomposition numbers are the Laurent polynomials}
\end{array}\]

\[
d^\mu_{\psi(d)}(q) = [S^\mu : D^\mu_{\psi(d)}] = \sum_{d \in Z} [S^\mu : D^\mu_{\psi(d)}] q^d,
\]
for $\lambda \in P_n^A$ and $\mu \in K_n^A$. Write $S^\lambda = S_n^\lambda$, $D^\mu = D_n^\mu$ and $d_{\lambda \mu}(q) = d_{\lambda \mu}^F(q)$ when $F$ is understood. By definition, $d_{\lambda \mu}(q) \in \mathbb{N}[q, q^{-1}]$ is a Laurent polynomial with non-negative coefficients. Let $d_i = (d_{\lambda \mu}(q))_{\lambda \in P_n^A, \mu \in K_n^A}$ be the graded decomposition matrix of $h_n^A$.

The KLR algebra $h_n^A$ is always $\mathbb{Z}$-free, however, it is not clear whether the same is true for the cyclotomic KLR algebra $h_n^A$. To prove this you cannot use the Graded Isomorphism Theorem 3.1.1 because this result holds only over a field. Using extremely sophisticated diagram calculus, Li [92] proved the following.

3.2.9. Theorem (Li [92]). Suppose that $\Lambda \in P^+$. Then the quiver Hecke algebra $h_n^A(\Lambda)$ is free as a $\mathbb{Z}$-module of rank $\ell^n!$. Moreover, $h_n^A(\Lambda)$ is a graded cellular algebra with graded cellular basis $\{ \psi_{st} | (s, t) \in \text{Std}(P_n^A) \}$.

Therefore, $h_n^A$ is free over any commutative ring and any field is a splitting field for $h_n^A$. Moreover, the graded Specht modules, together with their homogeneous bilinear forms, are defined over $\mathbb{Z}$. The integrality of the graded Specht modules can also be proved using Theorem 3.6.2 below.

The next result lists some important properties of the $\psi$-basis.

3.2.10. Proposition. Suppose that $(s, t) \in \text{Std}(P_n^A)$ and that $\mathbb{Z}$ is an integral domain. Then:

a) [54, Lemma 5.2] If $i, j \in I^n$ then $\psi_i = d_i \psi_j e(i) \psi_e(j)$.

b) [55, Lemma 3.17] Suppose that $\psi_{st}$ and $\psi_{uv}$ are defined using different reduced expressions for the permutations $d(s), d(t) \in S_n$. Then there exist $a_{uv} \in \mathbb{Z}$ such that

$$\tilde{\psi}_{st} = \psi_{st} + \sum_{(u,v) \triangleright (s,t)} a_{uv} \psi_{uv},$$

where $a_{uv} \neq 0$ only if $I^u = I^v$, $I^v = I^t$ and $deg u + deg v = deg s + deg t$.

c) [56, Corollary 3.11] If $1 \leq r \leq n$ then there exists $b_{uv} \in \mathbb{Z}$ such that

$$\tilde{\psi}_{st} y_r = \sum_{(u,v) \triangleright (s,t)} b_{uv} \psi_{uv},$$

where $b_{uv} \neq 0$ only if $I^u = I^v$, $I^v = I^t$ and $deg u + deg v = deg s + deg t + 2$.

Part (a) follows quickly using the relations in Definition 2.2.1 and the definition of the $\psi$-basis. In contrast, parts (b) and (c) are proved by using Theorem 3.1.1 to reduce to the seminormal basis. With part (c), it is fairly easy to show that $b_{uv} \neq 0$ only if $u \triangleright v$. The difficult part is showing that $b_{uv} \neq 0$ only if $v \triangleright u$. Again, this is done using seminormal bases.

Finally, we note that Theorem 3.2.9 implies that $e(i) \neq 0$ in $h_n^A$ if and only if $I^i = \{ i \}$, generalizing Proposition 2.4.6. In fact, if $F$ is a field and $h_n^A(F) \cong h_n^A(\Lambda)$ then it is shown in [54, Lemma 4.1] that the non-zero KLR idempotents are a complete set of primitive (central) idempotents in the Gelfand-Zetlin algebra $L_n(F)$ and that $L_n(F) = \langle y_1, \ldots, y_n, e(i) | i \in I^n \rangle$. It follows that $L_n(F)$ is a positively graded commutative algebra with one dimensional irreducible modules indexed by $I^n$, up to shift. It would be interesting to find a (homogeneous) basis of $L_n(F)$. The author would also like to know whether $h_n^A$ is projective as a graded $L_n$-module.

3.3. Blocks and dual Specht modules. This section shows that the blocks of $h_n^A$ are graded symmetric algebra and it sketches the proof of an analogous statement that relates the graded Specht modules and their graded duals.

Theorem 1.8.1 describes the block decomposition of $h_n^A$ so, by Theorem 3.1.1, it gives the block decomposition of $h_n^A$. As in (2.2.5), set

$$\mathcal{B}_n^A = \mathcal{B}_n^A e_\beta,$$

where $e_\beta = \sum_{i \in I^\beta} e(i)$.

It follows from Definition 2.2.1 that $e_\beta$ is central in $h_n^A$, so $\mathcal{B}_n^A = e_\beta h_n^A e_\beta$ is a two-sided ideal of $h_n^A$. Let $Q_n^+ = Q_n^+(\Lambda) = \{ \beta \in Q^+ | e_\beta \neq 0 \}$ in $h_n^A$. Similarly, let $P_n^A = \{ \lambda \in P_n^A | I^\lambda \in I^J \} = \{ \lambda \in P_n^A | \beta_n = \beta \}$.

Combining Theorem 3.2.9, Theorem 3.1.1 and Corollary 1.8.2 we obtain the following.

3.3.1. Theorem. Suppose that $\Lambda \in P^+$. Then $h_n^A = \bigoplus_{\beta \in Q_n^+} \mathcal{B}_n^A$ is the decomposition of $h_n^A$ into indecomposable two-sided ideals. Moreover, $\mathcal{B}_n^A$ is a graded cellular algebra with cellular basis $\{ \psi_{st} | (s, t) \in \text{Std}(P_n^A) \}$ and weight poset $P_n^A$.

By virtue of Theorem 3.2.9, the block decomposition of $h_n^A$ holds over $\mathbb{Z}$, even though we cannot think about the blocks as linkage classes of simple modules in this case. Compare with Theorem 2.4.8 in the semisimple case.

Suppose that $A$ is a graded $\mathbb{Z}$-algebra. Then $A$ is a graded symmetric algebra if there exists a homogeneous non-degenerate trace form $\tau : A \to \mathbb{Z}$, where $\mathbb{Z}$ is in degree zero. That is, $\tau(ab) = \tau(ba)$ and if
There is a non-degenerate pairing we defined two sets of elements $\psi_{st}$ and $\psi'_{st}$ in $\mathcal{H}^A_n$. Just as there are two versions of the Murphy basis, $(m_n)$ and $(m'_n)$, that are built from the trivial and sign representations of $\mathcal{H}^A_n$ [106], respectively, there are two versions of the $\psi$-basis. By [54, Theorem 6.17], $\{ \psi'_{st} \mid (s, t) \in \text{Std}^2(P^A_n) \}$ is a second graded cellular basis of $\mathcal{H}^A_n$ with weight poset $(P^A_n, \leq)$ and with deg $\psi'_{st} = \text{deg} s + \text{codeg} t$. We warn the reader that we are following the conventions of [55], rather than the notation of [54]. See [55, Lemma 3.15 and Remark 3.12] for the translation.

The bases $\{ \psi_{st} \}$ and $\{ \psi'_{st} \}$ of $\mathcal{H}^A_n$ are dual in the sense that if $(s, t), (u, v) \in \text{Std}^2(P^A_n)$ then, by [56, Theorem 6.17],

$$
(3.3.3) \quad \psi_{st} \psi'_{ts} \neq 0 \quad \text{and} \quad \psi_{st} \psi'_{uv} \neq 0 \quad \text{only if} \quad i^t = i^u \quad \text{and} \quad u \geq t.
$$

Let $\tau$ be the usual non-degenerate trace form on $\mathcal{H}^A_n$ [20, 100]. We can write $\tau = \sum_d \tau_d$, where $\tau_d$ is homogeneous of degree $d \in \mathbb{Z}$. Let $\tau_d = \tau_{-d} \neq \text{def}$. By [56, Theorem 6.17], if $(s, t) \in \text{Std}^2(P^A_n)$ then $\tau_d(\psi_{at} \psi'_{st} a) \neq 0$, so (3.3.3) implies the following.

### Theorem (Hu-Mathas [54, Corollary 6.18])

Let $\beta \in Q^+_n$. Then $\mathcal{H}^A_n$ a graded symmetric algebra with homogeneous trace form $\tau_\beta$ of degree $-2 \text{def} \beta$.

It would be better to have an intrinsic definition of $\tau_\beta$ for $\mathcal{H}^A_n(\mathbb{Z})$. Webster [134, Remark 2.27] has given a diagrammatic description of a trace form on an arbitrary cyclotomic KLR algebra. It is unclear to the author how these two forms on $\mathcal{H}^A_n$ are related.

The $\psi'$-basis is a graded cellular basis of $\mathcal{H}^A_n$ so it defines a collection of graded cell modules. For $\lambda \in P^A_n$, the dual graded Specht module $S_\lambda$ is the corresponding graded cell module determined by the $\psi'$-basis. The dual Specht module $S_\lambda$ has basis $\{ \psi'_t \mid t \in \text{Std}(\lambda) \}$, with deg $\psi'_t = \text{codeg} t$, and

$$
\dim \lambda S_{\lambda} = \sum_{t \in \text{Std}(\lambda)} q^{\text{codeg} t}.
$$

We can identify $S_{\lambda}(\text{codeg} t)_{\lambda}$ with $\psi'_{s\lambda t_{\lambda}} + \mathcal{H}_{\mathcal{H}^A_n}^{\text{deg} A, \lambda}$, where $\mathcal{H}_{\mathcal{H}^A_n}^{\text{deg} A, \lambda}$ is the two-sided ideal of $\mathcal{H}^A_n$ spanned by $\psi'_{st}$ where $(s, t) \in \text{Std}(\mu)$ for some multipartition $\mu$ such that $\lambda \supset \mu$. Similarly, we can identify $S_\lambda(\text{deg} A)_{\lambda}$ with $\psi'_{s\lambda t_{\lambda}} + \mathcal{H}_{\mathcal{H}^A_n}^{\text{deg} A, \lambda}$. By (3.3.3) there is a non-degenerate pairing

$$
\{ \text{, } \} : S_\lambda(\text{deg} A) \times S_\lambda(\text{codeg} t) \rightarrow \mathbb{Z}
$$
given by $\{ a + \mathcal{H}_{\mathcal{H}^A_n}^{\text{deg} A, \lambda}, b + \mathcal{H}_{\mathcal{H}^A_n}^{\text{codeg} A, \lambda} \} = \tau_\beta(ab)$. Hence, Lemma 3.3.2 implies:

### Corollary (Hu-Mathas [54, Proposition 6.19])

Suppose that $\lambda \in P^A_n$. Then $S_\lambda \cong S_\lambda(\text{codeg} A)$ and $S_{\lambda} = (S_\lambda(\text{deg} A))^{\text{def} (\text{deg} A)}$.

This result holds for the Specht modules defined over $\mathbb{Z}$ by Theorem 3.2.9 or by [81, Theorem 7.25].

There is an interesting byproduct of the proof of Corollary 3.3.5. In the ungraded setting the Specht module $\Sigma^A_\lambda$ is isomorphic to the submodule of $\mathcal{H}^A_n$ generated by an element $m_{\lambda} T_{u_{\lambda} m_{\lambda}^\prime}$; see [33, Definition 2.1 and Theorem 2.9]. By [54, Corollary 6.21], $m_{\lambda} T_{u_{\lambda} m_{\lambda}^\prime}$ is homogeneous. In fact, $\psi'_{s\lambda t_{\lambda}} \psi'_{s\lambda t_{\lambda}} = m_{\lambda} T_{u_{\lambda} m_{\lambda}^\prime}$ and $\psi'_{s\lambda t_{\lambda}} \psi'_{s\lambda t_{\lambda}} \mathcal{H}_{\mathcal{H}^A_n}^{\text{deg} A, \lambda} \cong S_\lambda(\text{def} A + \text{codeg} t)$.

### 3.4. Induction and restriction

The cyclotomic Hecke algebra $\mathcal{H}^A_n$ is naturally a subalgebra of $\mathcal{H}^A_{n+1}$, and $\mathcal{H}^A_{n+1}$ is free as an $\mathcal{H}^A_n$-module by (1.1.2). This gives rise to the usual induction and restriction functors. These functors can be decomposed into the “classical” $i$-induction and $i$-restriction functors, for $i \in I$, by projecting onto the blocks of these two algebras. As we will see, these functors are implicitly built into the graded setting.
Recall that $I = \mathbb{Z}/e\mathbb{Z}$ and $\Lambda \in P^+$. For each $i \in I$ define

$$e_{n,i} = \sum_{j \in I^n} e(j \lor i) \in \mathcal{A}_{n+1}^\Lambda.$$ 

The relations for $\mathcal{A}_{n+1}^\Lambda$ in Definition 2.2.1 imply that $e_{n,i}$ is an idempotent and that $\sum_{i \in I} e_{n,i} = \sum_{i \in I_{n+1}} e(i)$ is the identity element of $\mathcal{A}_{n+1}^\Lambda$.

Let $\text{Rep}(\mathcal{A}_n^\Lambda)$ and $\text{Rep}(\mathcal{A}_{n+1}^\Lambda)$ for $\beta \in Q^+$, be the category of finite dimension (graded) $\mathcal{A}_n^\Lambda$-modules, respectively, $\mathcal{A}_{n+1}^\Lambda$-modules. Similarly, let $\text{Proj}(\mathcal{A}_n^\Lambda)$ and $\text{Proj}(\mathcal{A}_{n+1}^\Lambda)$ be the categories of finitely generated projective modules for these algebras.

### 3.4.1. Lemma.

Suppose that $i \in I$ and that $Z$ is an integral domain. Then there is a (non-unital) embedding of graded algebras $\mathcal{A}_n^\Lambda \hookrightarrow \mathcal{A}_{n+1}^\Lambda$ given by

$$e(j) \mapsto e(j \lor i), \quad y_r \mapsto e_{n,i}y_r \quad \text{and} \quad \psi_s \mapsto e_{n,i}\psi_s,$$

for $j \in I^n$, $1 \leq r \leq n$ and $1 \leq s < n$. This map induces an exact functor

$$i\text{-Ind} : \text{Rep}(\mathcal{A}_n^\Lambda) \rightarrow \text{Rep}(\mathcal{A}_{n+1}^\Lambda); M \mapsto M \otimes_{\mathcal{A}_n^\Lambda} e_{n,i}\mathcal{A}_{n+1}^\Lambda.$$

Moreover, $\text{Ind} = \bigoplus_{i \in I} i\text{-Ind}$ is the graded induction functor from $\text{Rep}(\mathcal{A}_n^\Lambda)$ to $\text{Rep}(\mathcal{A}_{n+1}^\Lambda)$.

**Proof.** The images of the homogeneous generators of $\mathcal{A}_n^\Lambda$ under this embedding commute with $e_{n,i}$, which implies that this map defines a non-unital degree preserving homomorphism from $\mathcal{A}_n^\Lambda$ to $\mathcal{A}_{n+1}^\Lambda$. This map is an embedding by Theorem 3.2.9. The remaining claims follow because, by definition, $e_{n,i}$ is an idempotent and $\sum_{i \in I} e_{n,i}$ is the identity element of $\mathcal{A}_{n+1}^\Lambda$. $\square$

The $i$-induction functor $i\text{-Ind}$ functor is obviously a left adjoint to the $i$-restriction functor $i\text{-Res}$, which sends an $\mathcal{A}_{n+1}^\Lambda$-module $M$ to

$$i\text{-Res} M = M e_{n,i} \cong \text{Hom}_{\mathcal{A}_n^\Lambda}(e_{n,i}\mathcal{A}_{n+1}^\Lambda, M).$$

A much harder fact is that these functors are two-sided adjoints.

### 3.4.2. Theorem (Kashiwara [71, Theorem 3.5]).

Suppose $i \in I$. Then $(E_i, F_i)$ is a biadjoint pair.

Kashiwara proves this theorem by constructing explicit homogeneous adjunctions. He does this for any cyclotomic quiver Hecke algebras defined by a symmetric Cartan matrix. As we do not need this result we feel justified in stating it now, even though its proof builds upon Kang and Kashiwara’s proof that the cyclotomic quiver Hecke algebras of arbitrary type categorify the integrable highest weight modules of the corresponding quantum group [67]: compare with Proposition 3.5.12 and Corollary 3.5.27 below. This biadjointness property is also a consequence of Rouquier’s Kac–Moody categorification axioms [121, Theorem 5.16]. Theorem 3.4.2 was conjectured by Khovanov–Lauda [74].

Recall from (2.1.4) that $\otimes$ defines a graded duality on $\text{Rep}(\mathcal{A}_n^\Lambda)$. Similarly, define $\#$ to be the graded functor given by

$$M^\# = \text{Hom}_{\mathcal{A}_n^\Lambda}(M, \mathcal{A}_n^\Lambda), \quad \text{for } M \in \text{Rep}(\mathcal{A}_n^\Lambda),$$

where the action of $\mathcal{A}_n^\Lambda$ on $M^\#$ is given by

$$(f \cdot h)(m) = h^\bullet f(m), \quad \text{for } f \in M^\#, \, h \in \mathcal{A}_n^\Lambda \text{ and } m \in M.$$ 

We consider $\otimes$ and $\#$ as endofunctors of $\text{Rep}(\mathcal{A}_n^\Lambda) = \bigoplus_{\beta} \text{Rep}(\mathcal{A}_\beta^\Lambda)$ and $\text{Proj}(\mathcal{A}_n^\Lambda) = \bigoplus_{\beta} \text{Proj}(\mathcal{A}_\beta^\Lambda)$. As noted in [22, Remark 4.7], Theorem 3.3.4 implies that these two functors agree up to shift.

### 3.4.4. Lemma.

As endofunctors of $\text{Rep}(\mathcal{A}_\beta^\Lambda)$, there is an isomorphism of functors $\# \cong (2 \text{ def } \beta) \circ \otimes$.

**Proof.** By Theorem 3.3.4, $\mathcal{A}_\beta^\Lambda \cong (\mathcal{A}_\beta^\Lambda)^{\otimes}(2 \text{ def } \beta)$. If $M \in \text{Rep}(\mathcal{A}_\beta^\Lambda)$ then

$$M^\# = \text{Hom}_{\mathcal{A}_\beta^\Lambda}(M, \mathcal{A}_\beta^\Lambda) \cong \text{Hom}_{\mathcal{A}_\beta^\Lambda}(M, (\mathcal{A}_\beta^\Lambda)^{\otimes}(2 \text{ def } \beta)) \cong \text{Hom}_{\mathcal{A}_\beta^\Lambda}(M, \text{Hom}_{\mathcal{A}_\beta^\Lambda}(\mathcal{A}_\beta^\Lambda, Z))(2 \text{ def } \beta) \cong \text{Hom}_{\mathcal{A}_\beta^\Lambda}(M \otimes_{\mathcal{A}_\beta^\Lambda} \mathcal{A}_\beta^\Lambda, Z)(2 \text{ def } \beta) \cong M^\#(2 \text{ def } \beta),$$

where the third isomorphism is the standard adjointness of tensor and hom. As all of these isomorphisms are functorial, the lemma follows. $\square$

By well-known arguments, $(M^\#)^\# \cong M$ for all $M \in \text{Rep}(\mathcal{A}_n^\Lambda)$. Hence, $(M^\#)^\# \cong M$ by Lemma 3.4.4. Therefore, $\otimes$ and $\#$ define self-dual equivalences on $\text{Rep}(\mathcal{A}_n^\Lambda)$ and $\text{Proj}(\mathcal{A}_n^\Lambda)$.
Lemma 3.3.2

That the functors

Definition 3.2.1

Lemma 3.5.13

Lemma 3.4.4

Part (a), which was conjectured by Brundan, Kleshchev and Wang [28 ANDREW MATHAS]

Recall the integers

Recall that if

Observe that parts (a) and (c), and parts (b) and (d), are equivalent by Corollary 3.3.5 (and Lemma 3.3.2).

Part (b) is proved using the fact that the action of \( \mathcal{H}_n^A \) on the \( \psi \)-basis is compatible with restriction. Part (a), which was conjectured by Brundan, Kleshchev and Wang [25, Remark 4.12], is proved by extending elegant ideas of Ryom-Hansen [125] to the graded setting using [54].

3.4.5. Proposition. Suppose that \( \beta \in Q^+ \) and \( i \in I \). Then there are functorial isomorphisms

\[
\circ \circ i\text{-Res} \cong i\text{-Res} \circ \circ : \text{Rep}(R_{n+1}^A) \longrightarrow \text{Rep}(R_n^A),
\]

\[
\circ \circ i\text{-Ind} \cong i\text{-Ind} \circ \circ : \text{Proj}(R_n^A) \longrightarrow \text{Proj}(R_{n+1}^A).
\]

Proof. The isomorphism \( \circ \circ i\text{-Res} \cong i\text{-Res} \circ \circ \) is immediate from the definitions. For the second isomorphism, recall that if \( P \in \text{Proj}(R_n^A) \) then \( \text{Hom}_{R_n^A}(P, M) \cong \text{Hom}_{R_n^A}(M, R_n^A) \otimes_{R_n^A} M \), for any \( R_n^A \)-module \( M \). Now,

\[
(e_{n,i}R_{n+1}^A) \cong \text{Hom}_{R_n^A}(e_{n,i}R_{n+1}^A, R_n^A) \cong e_{n,i}R_n^A,
\]

the last isomorphism following because \( e_{n,i}^* = e_{n,i} \). Therefore,

\[
i\text{-Ind}(P) \circ \circ = \text{Hom}_{R_n^A}(P, R_n^A) \cong \text{Hom}_{R_n^A}(P, e_{n,i}R_{n+1}^A) \cong \text{Hom}_{R_n^A}(P, e_{n,i}R_{n+1}^A) \cong \text{Hom}_{R_n^A}(P \otimes R_n^A, e_{n,i}R_{n+1}^A) \cong (i\text{-Ind} P) \circ \circ,
\]

where the second last isomorphism is the usual tensor-hom adjointness. \( \square \)

It follows from Proposition 3.4.5 and Lemma 3.4.4 that the functors \( \circ \circ \) and \( i\text{-Ind} \), and \( \circ \circ \) and \( i\text{-Res} \), commute only up to shift. This difference in degree shift is what makes Lemma 3.5.13 work below.

We next describe the effect of the \( i \)-induction and \( i \)-restriction functors on the graded Specht modules, for \( i \in I \). This result generalizes the well-known (ungraded) branching rules for the symmetric group [59, Example 17.16] and the cyclotomic Hecke algebras [12, 109, 125].

Recall the integers \( d_A^A(\lambda) \) and \( d_A^A(\lambda) \) from Definition 3.2.1.

3.4.6. Theorem. Suppose that \( Z \) is an integral domain and \( \lambda \in P_n^A \).

a) [56, Main theorem] Let \( A_1 < A_2 \cdots < A_z \) be the addable \( i \)-nodes of \( \lambda \). Then \( i\text{-Ind} S^A \lambda \) has a graded Specht filtration

\[
0 = \mu_0 \subset \mu_1 \subset \cdots \subset \mu_z = i\text{-Ind} S^A \lambda,
\]

such that \( I_j/I_{j-1} = S^{A+J_j}(d_{J_j}(\lambda)) \).

b) [25, Theorem 4.11] Let \( B_1 > B_2 > \cdots > B_y \) be the removale \( i \)-nodes of \( \lambda \). Then \( i\text{-Res} S^A \lambda \) has a graded Specht filtration

\[
0 = \mu_0 \subset \mu_1 \subset \cdots \subset \mu_y = i\text{-Res} S^A \lambda,
\]

such that \( R_j/R_{j-1} = S^{A-B_j}(d_{B_j}(\lambda)) \), for \( 1 \leq j \leq y \).

c) [56, Corollary 4.7] Let \( A_1 > A_2 > \cdots > A_z \) be the addable \( i \)-nodes of \( \lambda \). Then \( i\text{-Ind} S^A \lambda \) has a graded Specht filtration

\[
0 = \mu_0 \subset \mu_1 \subset \cdots \subset \mu_z = i\text{-Ind} S^A \lambda,
\]

such that \( I_j/I_{j-1} = S^{A+J_j}(d_{A_j}(\lambda)) \), for \( 1 \leq j \leq z \).

d) Let \( B_1 < B_2 < \cdots < B_y \) be the removale \( i \)-nodes of \( \lambda \). Then \( i\text{-Res} S^A \lambda \) has a graded Specht filtration

\[
0 = \mu_0 \subset \mu_1 \subset \cdots \subset \mu_y = i\text{-Res} S^A \lambda,
\]

such that \( R_j/R_{j-1} = S^{A-B_j}(d_{B_j}(\lambda)) \), for \( 1 \leq j \leq y \).

Observe that parts (a) and (c), and parts (b) and (d), are equivalent by Corollary 3.3.5 (and Lemma 3.3.2).

3.5. Grading Ariki’s Categorification Theorem. The aim of this section is to prove the Ariki-Brundan-Kleshchev Categorification Theorem [3] that connects the canonical bases of \( U_q(\mathfrak{gl}_l) \)-modules with the simple and projective indecomposable \( \mathfrak{sl}_l \)-modules in characteristic zero. We give a new proof of Brundan and Kleshchev’s theorem [22] that the cyclotomic KLR algebras of type \( A \) categorify the integrable highest weight modules of \( U_q(\mathfrak{sl}_l) \). Our argument runs parallel to Brundan and Kleshchev’s with the key difference being that we use the representation theory of \( \mathcal{H}_n^A \) and in particular the graded branching rules, to construct a bar involution on the Fock space. In this way we are able to show that the canonical basis is categorified by the basis of simple \( \mathcal{H}_n^A \)-modules if and only if the graded decomposition numbers are polynomials. As a consequence, Ariki’s categorification theorem [3] lifts to the graded setting.

Throughout this section we assume that the Hecke algebra \( \mathcal{H}_n^A \) is defined over a field \( \mathbb{F} \). In the end we will assume that \( \mathbb{F} \) is a field of characteristic zero, however, almost all of the results in this section hold over an arbitrary field. We delay introducing the quantum group \( U_q(\mathfrak{sl}_l) \) until we actually need it because we want to emphasize the role that the quantum group is playing in the representation theory of \( \mathcal{H}_n^A \).
For the time being fix an integer \( n \geq 0 \). Very soon we will vary \( n \). Let \( \mathcal{A} = \mathbb{Z}[q, q^{-1}] \) be the ring of Laurent polynomials in \( q \) over \( \mathbb{Z} \).

Let \( \text{Rep}(\mathcal{H}_n^\Lambda) \) be the category of finitely generated graded \( \mathcal{H}_n^\Lambda \)-modules and let \( \text{Proj}(\mathcal{H}_n^\Lambda) \) be the category of finitely generated projective graded \( \mathcal{H}_n^\Lambda \)-modules. Let \( \text{Rep}(\mathcal{H}_n^\Lambda) \) and \( \text{Proj}(\mathcal{H}_n^\Lambda) \) be the Grothendieck groups of these categories. If \( M \) is a finitely generated \( \mathcal{H}_n^\Lambda \)-module let \( [M] \) be its image in \( \text{Rep}(\mathcal{H}_n^\Lambda) \).

Abusing notation slightly, if \( M \) is projective we also let \( [M] \) be its image in \( \text{Proj}(\mathcal{H}_n^\Lambda) \). Consider \( \text{Rep}(\mathcal{H}_n^\Lambda) \) and \( \text{Proj}(\mathcal{H}_n^\Lambda) \) as \( \mathcal{A} \)-modules by letting \( q \) act as the grading shift functor: \([M(d)] = q^d[M] \), for \( d \in \mathbb{Z} \).

### 3.5.1. Definition

Suppose that \( \mu \in K_n^\Lambda \). Let \( Y^\mu \) be the projective cover of \( D^\mu \) in \( \text{Rep}(\mathcal{H}_n^\Lambda) \).

Importantly, the module \( Y^\mu \) is graded. Since \( Y^\mu \) is indecomposable, the grading on \( Y^\mu \) is uniquely determined by the surjection \( Y^\mu \twoheadrightarrow D^\mu \), for \( \mu \in K_n^\Lambda \). We use the notation \( Y^\mu \) because these modules are special cases of the graded lifts of the Young modules constructed in [105]; see [55, §5.1] and [99, §2.6]. (The symbol \( P^\mu \) is usually reserved for the projective indecomposable modules of the cyclotomic Schur algebras [20, 31, 55, 128].)

By definition, the Grothendieck groups \( \text{Rep}(\mathcal{H}_n^\Lambda) \) and \( \text{Proj}(\mathcal{H}_n^\Lambda) \) are free \( \mathcal{A} \)-modules with distinguished bases:

\[
\text{Rep}(\mathcal{H}_n^\Lambda) = \bigoplus_{\mu \in K_n^\Lambda} \mathcal{A}[D^\mu] \quad \text{and} \quad \text{Proj}(\mathcal{H}_n^\Lambda) = \bigoplus_{\mu \in K_n^\Lambda} \mathcal{A}[Y^\mu],
\]

respectively. Recall from (3.2.8) that \( d_q = (d_{\lambda\mu}(q)) \) is the graded decomposition matrix of \( \mathcal{H}_n^\Lambda \). If \( \lambda \in P_n^\Lambda \) and \( \mu \in K_n^\Lambda \) then in \( \text{Rep}(\mathcal{H}_n^\Lambda) \),

\[
[S^\lambda] = \sum_{\tau \in K_n^\Lambda} d_{\lambda\tau}(q)[D^\tau] \quad \text{and} \quad [Y^\mu] = \sum_{\sigma \in P_n^\Lambda} d_{\sigma\mu}(q)[S^\sigma],
\]

where the second formula comes from Corollary 2.16. By Theorem 2.15(c), the submatrix \( d^K_q = (d_{\lambda\mu}(q))_{\lambda, \mu \in K_n^\Lambda} \) of the graded decomposition matrix \( d_q \) is invertible over \( \mathcal{A} \) with inverse

\[
e^K_q = (d^K_q)^{-1} = (e_{\lambda\mu}(-q))_{\lambda, \mu \in K_n^\Lambda}.
\]

(The reason why we consider \( e_{\lambda\mu}(-q) \) as a Laurent polynomial in \(-q\) is explained after Corollary 3.5.27 below.) Hence, if \( \lambda \in K_n^\Lambda \) then

\[
[D^\lambda] = \sum_{\mu \in K_n^\Lambda} e_{\lambda\mu}(-q)[S^\mu].
\]

Consequently, \( \{ [S^\mu] \mid \mu \in K_n^\Lambda \} \) is a second \( \mathcal{A} \)-basis of \( \text{Rep}(\mathcal{H}_n^\Lambda) \).

The set of projective indecomposable \( \mathcal{H}_n^\Lambda \)-modules \( \{ [Y^\mu] \} \) is the only natural basis of the split Grothendieck group \( \text{Proj}(\mathcal{H}_n^\Lambda) \). Somewhat artificially, but motivated by the formulas above, for \( \mu \in K^\Lambda \) define

\[
X_\mu = \sum_{\lambda \in K^\Lambda} e_{\lambda\mu}(-q)[Y^\lambda] \in \text{Proj}(\mathcal{H}_n^\Lambda).
\]

Then \( \{ X_\mu \mid \mu \in K^\Lambda \} \) is an \( \mathcal{A} \)-basis of \( \text{Proj}(\mathcal{H}_n^\Lambda) \). We will use the bases \( \{ [S^\mu] \mid \mu \in K^\Lambda \} \) and \( \{ X_\mu \mid K^\Lambda \} \) of \( \text{Rep}(\mathcal{H}_n^\Lambda) \) and \( \text{Proj}(\mathcal{H}_n^\Lambda) \), respectively, to construct new distinguished bases of the Grothendieck groups.

The bar involution on \( \mathcal{A} = \mathbb{Z}[q, q^{-1}] \) is the unique \( \mathbb{Z} \)-linear map such that \( q^d = q^{-d} \), for \( d \in \mathbb{Z} \). In particular, \( \dim_q M^\oplus = \dim_q M \), for any \( \mathcal{H}_n^\Lambda \)-module \( M \). A semilinear map of \( \mathcal{A} \)-modules is a \( \mathbb{Z} \)-linear map \( \theta : M \rightarrow N \) such that \( \theta(f(q)m) = f(q)\theta(m) \), for all \( f(q) \in \mathcal{A} \) and \( m \in M \).

A sesquilinear map \( f : M \times N \rightarrow \mathcal{A} \), where \( M \) and \( N \) are \( \mathcal{A} \)-modules, is a function that is semifinite in the first variable and linear in the second. Let \( \langle \ , \ : \rangle : [\text{Proj}(\mathcal{H}_n^\Lambda)] \times [\text{Rep}(\mathcal{H}_n^\Lambda)] \rightarrow \mathcal{A} \) be the sesquilinear pairing

\[
\langle [P], [M] \rangle := \dim_q \text{Hom}_{\mathcal{H}_n^\Lambda}(P, M),
\]

for \( P \in \text{Proj}(\mathcal{H}_n^\Lambda) \) and \( M \in \text{Rep}(\mathcal{H}_n^\Lambda) \). This pairing is naturally sesquilinear because \( \text{Hom}_{\mathcal{H}_n^\Lambda}(P(k), M) \cong \text{Hom}_{\mathcal{H}_n^\Lambda}(P, M(-k)) \), for any \( k \in \mathbb{Z} \).

The functors \( \oplus \) and \( \# \), of (2.1.4) and (3.4.3), induce semilinear automorphisms of the Grothendieck groups \( \text{Rep}(\mathcal{H}_n^\Lambda) \) and \( \text{Proj}(\mathcal{H}_n^\Lambda) \):

\[
[P]# = [P\#], \quad \text{and} \quad [M]^\oplus = [M^\oplus]
\]

for \( M \in \text{Rep}(\mathcal{H}_n^\Lambda) \) and \( P \in \text{Proj}(\mathcal{H}_n^\Lambda) \). The next result is fundamental.

### 3.5.3. Lemma

Suppose that \( [P] \in [\text{Proj}^\Lambda_\mathcal{A}] \) and \( [M] \in [\text{Rep}^\Lambda_\mathcal{A}] \). Then

\[
\langle [P], [M]^\oplus \rangle = \langle [P\#], [M] \rangle.
\]
Lemma 3.5.3

Proof. Applying the definitions, and tensor-hom adjointness,
\[
\langle [P], [M] \rangle = \dim_q \mathcal{H}_{\mathcal{H}_n}^\Lambda (P, M^\otimes) = \dim_q \mathcal{H}_{\mathcal{H}_n}^\Lambda \left( P, \mathcal{H}_{\mathcal{H}_n}^\Lambda (M, \mathbb{F}) \right) \\
= \dim_q \mathcal{H}_{\mathcal{H}_n}^\Lambda (P \otimes_{\mathcal{H}_n} M, \mathbb{F}) = \dim_q (P \otimes_{\mathcal{H}_n} M)^\otimes \\
= \dim_q P \otimes_{\mathcal{H}_n} M = \dim_q \mathcal{H}_{\mathcal{H}_n}^\Lambda (P^\#, \mathcal{H}_{\mathcal{H}_n}^\Lambda (M^\#, \mathbb{F})) \\
= \dim_q \mathcal{H}_{\mathcal{H}_n}^\Lambda (P^#, M) = \langle [P^#], [M] \rangle.
\]
For the second last line, note that \( \mathcal{H}_{\mathcal{H}_n}^\Lambda (Q, M) \cong \mathcal{H}_{\mathcal{H}_n}^\Lambda (Q, \mathcal{H}_{\mathcal{H}_n}^\Lambda ) \otimes_{\mathcal{H}_n} M \) whenever \( Q \) is projective. \( \square \)

3.5.4 Lemma. Suppose that \( \lambda, \mu \in K_n^\Lambda \). Then
\[
\langle [Y^\Lambda], [D^\mu] \rangle = \delta_{\lambda \mu} = \langle [X^\lambda], [S^\mu]^\otimes \rangle.
\]
Proof. The first equality is immediate from the definition of the sesquilinear form \( \langle , \rangle \) because \( Y^\Lambda \) is the projective cover of \( D^\lambda \), for \( \lambda \in K_n^\Lambda \). For the second equality, using the fact that \( \otimes \) is semilinear,
\[
\langle X^\lambda, [S^\mu]^\otimes \rangle = \sum_{\sigma \in K_n^\Lambda} e_{\sigma \lambda}(-q) \langle [Y^\sigma], [S^\mu]^\otimes \rangle \\
= \sum_{\sigma, \tau \in K_n^\Lambda} e_{\sigma \lambda}(-q) d_{\sigma \tau}(q) \langle [Y^\sigma], [D^\tau] \rangle \\
= \sum_{\sigma \in K_n^\Lambda} d_{\sigma \lambda}(q) e_{\sigma \lambda}(-q) = \delta_{\lambda \mu},
\]
where the last equality follows because \( e_{\sigma \lambda}^\Lambda_q = (d_{\sigma \lambda}^\Lambda_q)^{-1} \). \( \square \)

3.5.5 Lemma. Suppose that \( \mu \in K_n^\Lambda \). Then \( [Y^\mu]^\# = [Y^\mu], [D^\mu]^\# = [D^\mu] \),
\[
(X^\mu)^\# = X^\mu + \sum_{\sigma \in K_n^\Lambda, \sigma > \mu} a_{\sigma \mu}(q) X^\sigma \quad \text{and} \quad [S^\mu]^\# = [S^\mu] + \sum_{\tau \in K_n^\Lambda, \tau > \mu} a^{\mu \tau}(q) [S^\tau],
\]
for some Laurent polynomials \( a_{\sigma \mu}(q), a^{\mu \tau}(q) \in A \).

Proof. That \( [D^\mu]^\# = [D^\mu] \) is immediate by Corollary 3.2.7, whereas \( [Y^\mu]^\# = Y^\mu \) because \( Y^\mu \) is a direct summand of \( \mathcal{H}^\Lambda \) — alternately, use Lemma 3.5.4 and Lemma 3.5.3. If \( \mu \in K_n^\Lambda \) then, by Theorem 2.1.5,
\[
[S^\mu]^\# = \left( \sum_{\nu \in K_n^\Lambda, \mu \geq \nu} d_{\nu \mu}(q) [D^\nu] \right)^\# = \sum_{\nu \in K_n^\Lambda, \mu \geq \nu} d_{\nu \mu}(q) [D^\nu] \\
= [S^\mu] + \sum_{\tau \in K_n^\Lambda, \mu > \tau} \left( \sum_{\nu \in K_n^\Lambda, \mu > \nu > \tau} d_{\nu \mu}(q) e_{\nu \tau}(-q) \right) [S^\tau]
\]
as claimed. Note that \( d_{\mu \mu}(q) = 1 = e_{\mu \mu}(-q) \).

Finally, we can compute \( (X^\mu)^\# \) by writing \( X^\mu = \sum_{\mu} e_{\mu \mu}(-q) [Y^\mu] \) and then using essentially the same argument to show that \( (X^\mu)^\# \) can be written in the required form. Alternatively, use Lemma 3.5.4 and Lemma 3.5.3.

The triangularity of the action of \( \otimes \) and \# on \([\text{Rep}^\Lambda_n]\) and \([\text{Proj}^\Lambda_n]\), respectively, has the following easy but important consequence.

3.5.6 Proposition. Suppose that \( \mathbb{F} \) is a field. Then there exist unique bases \( \{ B^\mu \mid \mu \in K_n^\Lambda \} \) and \( \{ B^\mu \mid \mu \in K_n^\Lambda \} \) of \([\text{Proj}^\Lambda_n]\) and \([\text{Rep}^\Lambda_n]\), respectively, such that \( (B^\mu)^\# = B^\mu, \ (B^\mu)^\otimes = B^\mu \)
\[
B^\mu = X^\mu + \sum_{\sigma \in K_n^\Lambda, \sigma > \mu} b_{\sigma \mu}(q) X^\sigma \quad \text{and} \quad B^\mu = [S^\mu] + \sum_{\tau \in K_n^\Lambda, \mu > \tau} b^{\mu \tau}(q) [S^\tau]
\]
for polynomials \( b^{\mu \sigma}(q), b_{\sigma \mu}(q) \in \delta_{\sigma \mu} + q\mathbb{Z}[q] \).

Proof. The existence and uniqueness of these two bases follows immediately from Lemma 3.5.5 by Lusztig’s Lemma [95, Lemma 24.2.1]. We give a variation of Lusztig’s argument for the basis \( \{ B^\mu \} \).

Fix a multipartition \( \mu \in K_n^\Lambda \), for some \( n \geq 0 \), and suppose that \( B^\mu \) and \( \tilde{B}^\mu \) are two elements of \([\text{Rep}^\Lambda_n]\) with the required properties. By assumption the element \( B^\mu - \tilde{B}^\mu \) is \( \otimes \)-invariant and we can write
\[
B^\mu - \tilde{B}^\mu = \sum_{\mu > \tau} i^{\mu \tau}(q) [S^\tau],
\]
for some polynomials $\tilde{b}^\mu\tau(q) \in \mathbb{Z}[q]$. As the left-hand side is $\otimes$-invariant and \( \overline{\tilde{b}^\mu\tau(q)} \in q^{-1}\mathbb{Z}[q^{-1}] \), so Lemma 3.5.5 forces $B^\mu = \tilde{B}^\mu$.

To prove existence, we argue by induction on dominance. If $\mu$ is minimal in $K^\Lambda_n$ then we can take $B^\mu = [S^\mu] = [D^\mu]$ by Lemma 3.5.5. If $\mu \in K^\Lambda_n$ is not minimal with respect to dominance then set $\tilde{B}^\mu = [D^\mu]$. Then
\[
(B^\mu)^\oplus = \tilde{B}^\mu \quad \text{and} \quad \tilde{B}^\mu = [S^\mu] + \sum_{\tau \in K^\Lambda_n \setminus \mu} \tilde{b}^\mu\tau(q)[S^\tau],
\]
for some Laurent polynomials $\tilde{b}^\mu\tau(q) \in \mathbb{Z}[q, q^{-1}]$. If $\tilde{b}^\mu\tau(q) \in \mathbb{Z}[q]$, for all $\mu \triangleright \tau$, then $B^\mu = \tilde{B}^\mu$ has all of the required properties. Otherwise, pick any multipartition $\mu \triangleright \nu$ that is maximal with respect to dominance such that $\tilde{b}^\mu\nu(q) \notin \mathbb{Z}[q]$. By induction, there exists an element $B^\nu$ with all of the required properties. Replace $\tilde{B}^\mu$ with the element $\tilde{B}^\mu - p^\nu\mu(q)(B^\nu)$, where $p^\nu\mu(q)$ is the unique Laurent polynomial such that $\overline{p^\nu\mu(q)} = \bar{p}^\nu\mu(q)$ and $\tilde{b}^\mu\nu(q) - p^\nu\mu(q) \in \mathbb{Z}[q]$. Then $(\tilde{B}^\mu)^\oplus = B^\mu$ and the coefficient of $[S^\nu]$ in $\tilde{B}^\mu$ belongs to $\mathbb{Q}[q]$. Continuing in this way, after finitely many steps we will construct an element $B^\mu$ with the required properties.

3.5.7. Corollary. Suppose that $\lambda, \mu \in K^\Lambda_n$. Then
\[
\langle B_\mu, B^\lambda \rangle = \sum_{\sigma \in K^\Lambda_n, \Lambda^\sigma \otimes \mu} b^\sigma(q) b_{\sigma\mu}(q) = \delta_{\lambda\mu}.
\]

Proof. If $\sigma, \tau \in K^\Lambda_n$ then $\langle X_{\sigma}, [S^\tau]\rangle = \delta_{\sigma\tau}$ by Lemma 3.5.4. Therefore, since the form $\langle \cdot , \cdot \rangle$ is sesquilinear and $B^\lambda\oplus = B^\lambda$,
\[
\langle B_\mu, B^\lambda \rangle = \langle B_\mu, B^\lambda \rangle = \sum_{\sigma \in K^\Lambda_n, \Lambda^\sigma \otimes \mu} b_{\sigma\mu}(q) \langle X_{\sigma}, [S^\tau]\rangle = \sum_{\lambda \leq \sigma} \langle b^\sigma(q) b_{\sigma\mu}(q) \rangle.
\]
In particular, $(B_\mu, B^\lambda) \in \delta_{\lambda\mu} + q^{-1}\mathbb{Z}[q^{-1}]$. On the other hand,
\[
\langle B_\mu, B^\lambda \rangle = \langle B_\mu^\oplus, B^\lambda \rangle = \langle B_\mu, B^\lambda \rangle = \langle B_\mu, B^\lambda \rangle
\]
by Lemma 3.5.3. Therefore, $(B_\mu, B^\lambda) = \delta_{\lambda\mu}$ as this is the only bar invariant polynomial in $\delta_{\lambda\mu} + q^{-1}\mathbb{Z}[q^{-1}]$.

Applying Lemma 3.4.4 to Proposition 3.5.6 we obtain.

3.5.8. Corollary. Suppose that $\mu \in K^\Lambda_n$. Then $q^{-\text{def}\mu} B_\mu \oplus = q^{-\text{def}\mu} B_\mu$ and $(q^{\text{def}\mu} B_\mu)^\# = q^{\text{def}\mu} B_\mu$

In order to link the bases $\{B_\mu\}$ and $\{B^\mu\}$ with the representation theory of $\mathcal{H}_n^\Lambda$ we need to introduce the quantum group $U_q(\mathfrak{sl}_n)$.

The quantum group $U_q(\mathfrak{sl}_n)$ associated with the quiver $\Gamma_c$ is the $\mathbb{Q}(q)$-algebra generated by $\{E_i, F_i, K_i^\pm \mid i \in I\}$, subject to the relations:
\[
K_i K_j = K_j K_i, \quad K_i K_i^{-1} = 1, \quad [E_i, F_j] = \delta_{ij} K_i - K_i^{-1} = q^{c_{ij}} E_j, \quad K_i F_j K_i^{-1} = q^{-c_{ij}} F_j,
\]
\[
\sum_{0 \leq c < 1 - c_{ij}} (-1)^c \left[ \frac{1 - c_{ij}}{c} \right] q^{1-c_{ij}} F_j E_j = 0, \quad \sum_{0 \leq c < 1 - c_{ij}} (-1)^c \left[ \frac{1 - c_{ij}}{c} \right] q^{1-c_{ij}} F_j E_j = 0
\]
where $[c]_q = [[d]]/[[c]] [[d - c]]$ and $[m]_q = \prod_{k=1}^m (q^{k} - q^{-k})/(q-q^{-1})$, for integers $c < d, m \in \mathbb{N}$. Then $U_q(\mathfrak{sl}_n)$ is a Hopf algebra with coproduct determined by $\Delta(K_i) = K_i \otimes K_i$, $\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i$ and $\Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i$, for $i \in I$. A self-contained account of much of what we need can be found in Ariki’s book [5]. See also [95, §3.1] and [22].

The combinatorial Fock space $\mathcal{F}_Q^A$ is the free $A$-module with basis the set of symbols $\{| \lambda \rangle \mid \lambda \in \mathcal{P}^\Lambda_n \}$, where $\mathcal{P}^\Lambda_n = \bigcup_{n \geq 0} \mathcal{P}_n^\Lambda$. For future use, let $K^\Lambda_n = \bigcup_{n \geq 0} K^\Lambda_n$. Set $\mathcal{F}_Q^A = \mathcal{F}_Q^A \otimes_A \mathbb{Q}(q)$. Then, $\mathcal{F}_Q^A$ is an infinite dimensional $\mathbb{Q}(q)$-vector space. We consider $\{| \lambda \rangle \mid \lambda \in \mathcal{P}^\Lambda_n \}$ as a basis of $\mathcal{F}_Q^A$ by identifying $| \lambda \rangle$ and $| \lambda \rangle \otimes 1_{\mathbb{Q}(q)}$.

Recall the integers $d^A(\lambda)$, $d_B(\lambda)$ and $d_i(\lambda)$ from Definition 3.2.1.
3.5.9. Theorem (Hayashi [52,111]). Suppose that $\Lambda \in P^+$. Then $\mathcal{F}_\Lambda^{\Lambda}(q)$ is an integrable $U_q(\hat{\mathfrak{sl}}_e)$-module with $U_q(\mathfrak{sl}_e)$-action determined by

$$E_i|\lambda\rangle = \sum_{B \in \text{Rem}_i(\lambda)} q^{d_{\beta}(\lambda)|\lambda - B|} \text{ and } F_i|\lambda\rangle = \sum_{A \in \text{Add}_i(\lambda)} q^{-d_{\alpha}(\lambda)|\lambda + A|},$$

and $K_i|\lambda\rangle = q^{d_{\beta}(\lambda)|\lambda\rangle}$, for all $i \in I$ and $\lambda \in \mathcal{P}_n^{\Lambda}$.

3.5.10. Remark. A slightly different action of $U_q(\hat{\mathfrak{sl}}_e)$ on the Fock space is used in many places in the literature, such as [5,89,104]. As is already evident, and will be made precise in Proposition 3.5.16 below, the $U_q(\mathfrak{sl}_e)$-action on the Fock space is closely related to induction and restriction for the graded Specht modules. The $U_q(\mathfrak{sl}_e)$-action on the Fock space used in [5,89,104] corresponds to the action of the induction and restriction functors on the dual graded Specht modules. Equivalently, this difference in the $U_q(\hat{\mathfrak{sl}}_e)$-action arises because, ultimately, we will work with an action of $U_q(\mathfrak{sl}_e)$ on the Grothendieck groups of the finitely generated $\mathcal{P}_n^{\Lambda}$-modules, whereas these other sources consider the corresponding adjoint action on the projective Grothendieck groups.

Hayashi [52] considered only the special case when $\Lambda = \Lambda_0$, however, this implies the general case using the coproduct $U_q(\hat{\mathfrak{sl}}_e)$ because

$$\mathcal{F}_\Lambda^{\Lambda}(q) \cong \mathcal{F}_\Lambda^{\Lambda}(q) \otimes \cdots \otimes \mathcal{F}_\Lambda^{\Lambda}(q)$$

as $U_q(\hat{\mathfrak{sl}}_e)$-modules. The crystal and canonical bases of $\mathcal{F}_\Lambda^{\Lambda}(q)$, which were first studied in [65,111,131], play an important role in what follows. A self-contained proof of Theorem 3.5.9, stated with similar language, can be found in Ariki’s book [6, Theorem 10.10].

An element $x \in \mathcal{F}_\Lambda^{\Lambda}(q)$ has weight $\text{wt}(x) = \Gamma$ if $K_i x = q^{(\Gamma, \alpha_i)} x$, for $i \in I$. In particular, if $0_\ell = (0|0\ldots|0) \in \mathcal{P}_\Lambda$ is the empty multipartition of level $\ell$ then $K_i|0_\ell\rangle = q^{(\Lambda, \alpha_i)}|0_\ell\rangle$, for $i \in I$, so that $|0_\ell\rangle$ has weight $\Lambda$. More generally, if $\beta \in Q^+$ then writing $\lambda = \mu + A$ it follows by induction that

$$d_i(\beta) = (\Lambda - \beta, \alpha_i).$$

Therefore, $\text{wt}(|\lambda\rangle) = \Lambda - \beta$ by Theorem 3.5.9. Set $d_i(\beta) = (\Lambda - \beta, \alpha_i)$.

For each dominant weight $\Lambda \in P^+$ let $L(\Lambda) = U_q(\mathfrak{sl}_e)v_\Lambda$ be the irreducible integrable highest weight module of highest weight $\Lambda$, where $v_\Lambda$ is a highest weight vector of weight $\Lambda$. By Theorem 3.5.9, $|0_\ell\rangle$ is a highest vector of weight $\Lambda$ in $\mathcal{F}_\Lambda^{\Lambda}(q)$. In fact, it follows from Theorem 3.5.9 that

$$L(\Lambda) \cong U_q(\hat{\mathfrak{sl}}_e)|0_\ell\rangle.$$ 

For example, see [5, Theorem 10.10]. Henceforth, we set $v_\Lambda = |0_\ell\rangle$.

To compare the Grothendieck groups $[\text{Rep}(\mathcal{H}_n^{\Lambda})]$ and $[\text{Proj}(\mathcal{H}_n^{\Lambda})]$ with the Fock space we need to consider all $n \geq 0$ simultaneously. Define

$$[\text{Rep}_n^{\Lambda}] = \bigoplus_{n \geq 0} [\text{Rep}(\mathcal{H}_n^{\Lambda})] \text{ and } [\text{Proj}_n^{\Lambda}] = \bigoplus_{n \geq 0} [\text{Proj}(\mathcal{H}_n^{\Lambda})].$$

Set $[\text{Rep}_Q^{\Lambda}(q)] = [\text{Rep}_Q^{\Lambda}] \otimes_A \mathbb{Q}$ and $[\text{Proj}_Q^{\Lambda}(q)] = [\text{Proj}_Q^{\Lambda}] \otimes_A \mathbb{Q}(q)$.

3.5.12. Proposition. Suppose that $\Lambda \in P^+$. Then the $i$-induction and $i$-restriction functors of $[\text{Rep}_Q^{\Lambda}(q)]$ induce a $U_q(\hat{\mathfrak{sl}}_e)$-module structure on $[\text{Proj}_Q^{\Lambda}(q)]$ and $[\text{Rep}_Q^{\Lambda}(q)]$ such that, as $U_q(\hat{\mathfrak{sl}}_e)$-modules,

$$[\text{Proj}_Q^{\Lambda}(q)] \cong L(\Lambda) \cong [\text{Rep}_Q^{\Lambda}(q)].$$

Proof. Recall that $d_\mu$ is the graded decomposition matrix of $\mathcal{H}_n^{\Lambda}$ and $d_\mu^T$ is its transpose. Abusing notation slightly by simultaneously using these matrices for all $n \geq 0$, define linear maps

$$[\text{Proj}_Q^{\Lambda}(q)] \xrightarrow{d_\mu^T} \mathcal{F}_Q^{\Lambda}(q) \xleftarrow{c_\mu} [\text{Rep}_Q^{\Lambda}(q)],$$

where $d_\mu^T(|\gamma\mu\rangle) = \sum_{\lambda} d_{\lambda\mu}(q)|\lambda\rangle$, $d_\mu(|\lambda\rangle) = \sum_{\mu} d_{\lambda\mu}(q)|\gamma\mu\rangle$ and where $c_\mu = d_\mu \circ d_\mu^T$ is the Cartan map. As vector space homomorphisms, $d_\mu^T$ is injective and $d_\mu$ is surjective. As defined these maps are only vector space homomorphisms, however, we claim that both maps can be made into $U_q(\hat{\mathfrak{sl}}_e)$-module homomorphisms.
The $i$-induction and $i$-restriction functors are exact, for $i \in I$, because they are exact when we forget the grading [49, Corollary 8.9]. Therefore, they send projective modules to projectives and they induce endomorphisms of the Grothendieck groups $[\text{Rep}_{\mathbb{Z}[(q)]}]$ and $[\text{Proj}_{\mathbb{Z}[(q)]}]$. By Theorem 3.4.6,

$$[i\text{-Res} S^\lambda] = \sum_{B \in \text{Rem}_i(\lambda)} q^{d_B(\lambda)}[S^{\lambda-B}],$$

$$[i\text{-Ind} S^\lambda(1-d_i(\lambda))] = \sum_{A \in \text{Add}_i(\lambda)} q^{d_A(\lambda)+1-d_i(\lambda)}[S^{\lambda+A}]$$

$$= \sum_{A \in \text{Add}_i(\lambda)} q^{-d_i(\lambda)}[S^{\lambda+A}],$$

where the last equality uses Lemma 3.3.2(a). Identifying $E_i$ with $i$-Res, and $F_i$ with $q$ $i$-Ind $K_i^{-1}$, the linear maps $d_q$ and $d_q^T$ become well-defined $U_q(\mathfrak{sl}_n)$-module homomorphisms by Theorem 3.5.9. As $U_q(\mathfrak{sl}_n)$-modules, $[\text{Rep}_{\mathbb{Z}((q))}]$ and $[\text{Proj}_{\mathbb{Z}((q))}]$ are both cyclic because they are both generated by $[Y^0] = [S^0] = [D^0]$. By definition, $d_q^n([Y_n^0]) = v_A$ and $d_q^T(v_A) = [S_A^0]$, so the proposition follows because $L(\Lambda) \cong U_q(\mathfrak{sl}_n)v_A$ is irreducible.

Let $U_A(\hat{\mathfrak{sl}_n})$ be Lusztig’s $A$-form of $U_q(\hat{\mathfrak{sl}_n})$, which is the $A$-subalgebra of $U_q(\hat{\mathfrak{sl}_n})$ generated by the quantised divided powers $E^{(k)} = F^{(k)}/[k]$, and $F^{(k)} = F^{(k)}/[k]$, for $i \in I$ and $k \geq 0$. Theorem 3.5.9 implies that $U_A(\mathfrak{sl}_n)$ acts on the $A$-submodule $\mathcal{F}_A^\Lambda$ of $\mathcal{F}_{\mathbb{Z}((q))}^\Lambda$, compare with [89, Lemma 6.2] and with [104, Lemma 6.16]. Therefore, by Proposition 3.5.12, $[\text{Rep}_{\mathbb{Z}((q))}]$ and $[\text{Proj}_{\mathbb{Z}((q))}]$ are $U_A(\mathfrak{sl}_n)$-modules. Moreover, if we set $L_A(\Lambda) = U_A(\hat{\mathfrak{sl}_n})v_A$ then there are $U_A(\mathfrak{sl}_n)$-module homomorphisms $[\text{Proj}_{\mathbb{Z}((q))}] \hookrightarrow L_A(\Lambda) \to [\text{Rep}_{\mathbb{Z}((q))}]$. In particular, $L_A(\Lambda) \cong [\text{Proj}_{\mathbb{Z}((q))}]$ as $U_A(\mathfrak{sl}_n)$-modules by Proposition 3.5.12.

3.5.13. **Lemma.** Suppose that $i \in I$. The involution $\circ$ commutes with the actions of $E_i$ and $F_i$ on $[\text{Rep}_{\mathbb{Z}((q))}]$ and on $[\text{Proj}_{\mathbb{Z}((q))}]$.

**Proof.** By Proposition 3.4.5, there are isomorphisms $i\text{-Res} \circ \cong \circ \circ i\text{-Res}$ and $i\text{-Ind} \circ \cong \# \circ i\text{-Ind}$. In particular, the actions of $E_i = i\text{-Res}$ and $\circ$ commute. Fix $\beta \in Q^+$. Recall from after (3.5.11) that $d_i(\beta) = (\Lambda-\beta,\alpha_i)$. Identifying $F_i$ with the functor $q \circ i\text{-Ind} \circ K_i^{-1} = q^{-d_i(\beta)}i\text{-Ind}$ on $\text{Rep}(\mathcal{H}_\beta^\Lambda)$, there are isomorphisms

$$F_i \circ \cong q \circ i\text{-Ind} K_i^{-1} \circ q^{-2\text{det} \beta} \# \quad \text{by Lemma 3.4.4},$$

$$\cong q^{-d_i(\beta)-2\text{det} \beta} i\text{-Ind} \circ \# \quad \text{by Proposition 3.4.5},$$

$$\cong q^{-2\text{det} \beta+\alpha_i} \# \circ q^{d_i(\beta)-1} \circ i\text{-Ind} \quad \text{by Lemma 3.3.2(a)},$$

$$\cong \circ \circ q^{-1} i\text{-Ind} K_i \cong \circ \circ F_i \quad \text{by Lemma 3.4.4}.$$

Hence, $E_i$ and $F_i$ commute with $\circ$ on $[\text{Rep}_{\mathbb{Z}((q))}]$ and $[\text{Proj}_{\mathbb{Z}((q))}]$ as claimed.

In contrast, $E_i$ and $F_i$ on $[\text{Rep}_{\mathbb{Z}((q))}]$ and $[\text{Proj}_{\mathbb{Z}((q))}]$ do not commute with $\#$. We want to relate the Cartan pairing $(\ , \ )$ on $[\text{Proj}_{\mathbb{Z}((q))}] \times [\text{Rep}_{\mathbb{Z}((q))}]$ with the representation theory of $L_A(\Lambda)$. Define a non-degenerate symmetric bilinear form $(\ , \ )$ on the Fock space $\mathcal{F}_A^\Lambda$ by

$$(\lambda|\mu) = \delta_{\lambda\mu}q^{\text{def} \lambda}, \quad \lambda, \mu \in P^\Lambda.$$  

By Theorem 3.5.9, $(K_i x, y) = q^{w_i(y)}(x, y) = (x, K_i y)$, for weight vectors $x, y \in \mathcal{F}_A^\Lambda$ and $i \in I$. By restriction, we also consider $(\ , \ )$ as a bilinear form on $L_A(\Lambda)$.

3.5.14. **Lemma.** The bilinear form $(\ , \ )$ on $L_A(\Lambda)$ is characterised by the three properties: $(v_A, v_A) = 1$, $(E_i x, y) = (x, F_i y)$ and $(F_i x, y) = (x, E_i y)$, for all $i \in I$ and $x, y \in L_A(\Lambda)$.

**Proof.** By definition, $(v_A, v_A) = 1$. If $i \in I$ then in order to check that $E_i$ and $F_i$ are biadjoint with respect to $(\ , \ )$ it is enough to consider the cases when $x = |\lambda\rangle$ and $y = |\mu\rangle$, for $\lambda, \mu \in P^\Lambda$. By Theorem 3.5.9, $(F_i |\lambda\rangle, |\mu\rangle) = 0 = (|\lambda\rangle, E_i |\mu\rangle)$ unless $\mu = \lambda + A$ for some $A \in \text{Add}_i(\Lambda)$. On the other hand, if $A \in \text{Add}_i(\Lambda)$ and $\mu = \lambda + A$ then, using Lemma 3.3.2(a) for the second equality,

$$(F_i |\lambda\rangle, |\mu\rangle) = q^{\text{def} \mu-d \lambda}(\lambda) = q^{\text{def} \lambda+d_i(\lambda)-1-d \lambda}(\lambda)$$

$$= q^{\text{def} \lambda+ d \lambda}(\mu) = (|\lambda\rangle, E_i |\mu\rangle).$$

A similar calculation shows that $(E_i |\lambda\rangle, |\mu\rangle) = (|\lambda\rangle, F_i |\mu\rangle)$, for all $\lambda, \mu \in P^\Lambda$. As $v_A$ is the highest weight vector in the irreducible module $L_A(\Lambda)$, these three properties uniquely determine the bilinear form $(\ , \ )$ on $L_A(\Lambda)$ by induction on weight.
By restriction, the next result categorifies the pairing $(, \ )$ on $L_A(\Lambda)$.

3.5.16. Proposition. Let $x \in [\text{Proj}_A] \Lambda$ and $y \in \mathcal{F}_A^\Lambda$ with $\text{wt}(\lambda) = \beta$. Then

$$(x, d_\beta(y)) = q^{-\text{def} \beta} (A_f^\tau(x^\#), y).$$

Proof. As $(, \ )$ is bilinear and $(, \ )$ is sesquilinear it is enough to verify this identity when $x = X_\mu$ and $y = |\lambda\rangle$, for $\mu \in K^\Lambda$ and $\lambda \in P_\beta$. Then

$$\langle x^\#, d_\beta(y) \rangle = \langle X_\mu, [S^\lambda] \rangle = \sum_{\sigma \in K^\Lambda_\beta} d_{\lambda\sigma}(q)(X_\mu, [D^\sigma])$$

$$= \sum_{\tau \in K^\Lambda_\beta} \sum_{\sigma \in K^\Lambda_\beta} d_{\lambda\sigma}(q) e_{\sigma\tau}(-q) \langle X_\mu, [S^\tau] \rangle$$

$$= \sum_{\sigma \in K^\Lambda_\beta} d_{\lambda\sigma}(q) e_{\sigma\mu}(-q),$$

where the last equality uses Lemma 3.5.4. For the right hand side,

$$(d_f^\tau(x^\#), y) = (d_f^\tau(X_\mu^\#), |\lambda\rangle) = \sum_{\sigma \in K^\Lambda_\beta} e_{\sigma\mu}(-q) (d_f^\tau([Y^\sigma]), |\lambda\rangle)$$

$$= \sum_{\nu \in P_\beta} \sum_{\sigma \in K^\Lambda_\beta} d_{\nu\sigma}(q) e_{\sigma\mu}(-q) (\tau \lambda \sigma, |\lambda\rangle)$$

$$= q^{\text{def} \beta} \langle x, d_\beta(y) \rangle,$$

by (3.5.14) and calculation above. The proof is complete. \qed

Now we can prove the results that we are really interested in.

3.5.17. Corollary. Let $P \in \text{Proj}(\mathcal{A}_n^\Lambda)$, $y \in \text{Rep}(\mathcal{A}_n^\Lambda)$, and $i \in I$. Then

$$\langle (i\text{-Ind} x, y) = \langle x, i\text{-Res} y \rangle \text{ and } (i\text{-Res} x, y) = \langle x, i\text{-Ind} y \rangle \rangle$$

Proof. By Theorem 3.4.2, $(i\text{-Res}, i\text{-Ind})$ is a biadjoint pair so the corollary follows directly from the definition of the Cartan pairing in (3.5.2). As it is non-trivial to show that $i\text{-Res}$ is left adjoint to $i\text{-Ind}$ we prove this at the level of Grothendieck groups. Write $\hat{x} = d_f^\tau(x)$ and $y = d_f^\tau(y)$ where $\hat{x}, \hat{y} \in L_A(\Lambda)$ and $\text{wt}(\hat{y}) = \Lambda - \beta$. Then $\langle i\text{-Res} x, y \rangle = 0$ unless $\text{wt}(x) = \Lambda - (\beta + \alpha_i)$. To improve readability, identify $x$ and $\hat{x} = d_f^\tau(x)$ below. Then,

$$\langle i\text{-Res} x, y \rangle = q^{-\text{def} \beta} (E_i x^\#, \hat{y})$$

by Proposition 3.5.16,

$$= q^{\text{def} \beta} (E_i x^\#, \hat{y}),$$

by 3.4.4 and 3.5.13,

$$= q^{\text{def} \beta} (\hat{x}, F_i \hat{y})$$

by Lemma 3.5.15,

$$= q^{\text{def} \beta - \text{def} (\beta + \alpha_i)} (x^\#, F_i \hat{y})$$

by Lemma 3.4.4,

$$= q^{\text{def} \beta - \text{def} (\beta + \alpha_i)} (x^\#, F_i y)$$

by Proposition 3.5.16,

$$= \langle x, i\text{-Ind} y \rangle,$$

where the last equality uses Lemma 3.3.2 and the identification of $F_i$ and $q\text{-Ind} \circ K_i^{-1}$ on $[\text{Rep}_A]$, via Proposition 3.5.12. \qed

Let $\tau$ be the unique semilinear anti-isomorphism of $U_\Lambda(\mathfrak{sl}_L)$ such that $\tau(K_i) = K_i^{-1}$, $\tau(E_i) = q F_i K_i^{-1}$ and $\tau(F_i) = q^{-1} K_i E_i$, for all $i \in I$. Then the biadjointness of induction and restriction with respect to the Cartan pairing translates into the following more Lie theoretic statement.

3.5.18. Corollary. Suppose that $x \in [\text{Proj}_A] \Lambda$ and $y \in [\text{Rep}_A] \Lambda$. Then

$$\langle u x, y \rangle = \langle x, \tau(u) y \rangle,$$

for all $u \in U_A(\mathfrak{sl}_L)$. We extend the bar involution of $A$ extends to a semilinear involution of $U_A(\mathfrak{sl}_L)$ determined by $\overline{K}_i = K_i^{-1}$, $\overline{F}_i = E_i$ and $\overline{E}_i = F_i$, for all $i \in I$. Similarly, define a bar involution on $L_A(\Lambda)$ by

$$\overline{v}_\Lambda = v_{\Lambda} \text{ and } \overline{u}_\Lambda = (u_{\Lambda})^\tau,$$

for $u \in U_A(\mathfrak{sl}_L)$ and $x \in L_A(\Lambda)$.

As noted in [22, §3.1], it follows from the relations that $\tau \circ = \circ \circ \tau^{-1}$.

As in [22, §3.3], the Shapovalov form on $L(\Lambda)$ is the sesquilinear map

$$\langle x, y \rangle = q^{\text{def} \beta} (\overline{x}, y),$$
for \(x, y \in L(\Lambda)\) with \(\text{wt}(y) = \Lambda - \beta\), for \(\beta \in Q^+\). As our notation suggests, the Shapovalov form is categorified by the Cartan pairing.

3.5.19. **Corollary.** Suppose that \(x \in [\text{Proj}_{\Lambda}^A]\) and \(y \in L_A(\Lambda)\). Then

\[
\langle d_y^\Lambda(x), y \rangle = \langle x, d_y^\Lambda(y) \rangle.
\]

**Proof.** By Lemma 3.5.15, the pairing \(\langle . , . \rangle\) on \(L_A(\Lambda)\) is unique symmetric bilinear map on \(L_A(\Lambda)\) that is biadjoint with respect to \(E_i\) and \(F_i\) and such that \(\langle r_{\Lambda}, v_{\Lambda} \rangle = 1\). This implies that the Shapovalov form is the unique sesquilinear form on \(L_A(\Lambda)\) such that \(\langle v_{\Lambda}, v_{\Lambda} \rangle = 1\) and \(\langle u, y \rangle = \langle x, \tau(u)y \rangle\), for \(x, y \in L_A(\Lambda)\) and \(u \in U_A(\mathfrak{sl}_e)\). Hence, the result follows from Corollary 3.5.18.

The module \(L_A(\Lambda) = U_A(\mathfrak{sl}_e)\) is the standard \(A\)-form of the irreducible \(U_q(\mathfrak{sl}_e)\)-module \(L(\Lambda)\). The costandard \(A\)-form of \(L(\Lambda)\) is the dual lattice

\[
L_A(\Lambda)^* = \{ y \in L(\Lambda) \mid (x, y) \in A \text{ for all } x \in L_A(\Lambda) \} = \{ y \in L(\Lambda) \mid (x, y) \in A \text{ for all } x \in L_A(\Lambda) \}
\]

We can now identify both \([\text{Proj}_{\Lambda}^A]\) and \([\text{Rep}_{\Lambda}^A]\) as \(U_A(\mathfrak{sl}_e)\)-modules.

3.5.20. **Corollary.** Suppose that \(\Lambda \in Q^+\). Then, as \(U_A(\mathfrak{sl}_e)\)-modules,

\[
[\text{Proj}_{\Lambda}^A] \cong L_A(\Lambda) \quad \text{and} \quad [\text{Rep}_{\Lambda}^A] \cong L_A(\Lambda)^*.
\]

**Proof.** The first isomorphism we noted already after Proposition 3.5.12. The second isomorphism follows from Corollary 3.5.19 and Lemma 3.5.4.

By Lemma 3.5.13, the action of \(F_i\) on \([\text{Rep}_{\Lambda}^A]\) and \([\text{Proj}_{\Lambda}^A]\), for \(i \in I\), commutes with \(\oplus\). In the language of [22, §3.1], \(\oplus\) is a compatible bar-involution. As is easily proved by induction on weight, every integrable \(U_A(\mathfrak{sl}_e)\)-module has a unique bar-compatible involution, so

\[
d_y(\mathcal{F}) = d_y(y)\quad \text{for all } y \in \mathcal{F}_{\mathcal{Q}(q)}.
\]

It follows that \(\{ B^\mu \mid \mu \in K^\Lambda \}\) is Kashiwara’s upper global basis at \(q = 0\) [69], or Lusztig’s dual canonical basis [94, §14.4], of \(L(\Lambda)\). By Corollary 3.5.8, \(q^{-\text{def}}B_{\mu} \) is bar invariant and, thinking of \(\langle . , . \rangle\) as a pairing from \([\text{Proj}_{\Lambda}^A] \times [\text{Rep}_{\Lambda}^A]\) to \(A\), we have

\[
(q^{-\text{def}}B_{\mu}, B^\lambda) = (B^\mu, B^\lambda) = (B_{\mu}, B^\lambda) = \delta_{\lambda\mu},
\]

by Proposition 3.5.16 and Corollary 3.5.19. Hence, \(\{ q^{-\text{def}}B_{\mu} \mid \mu \in K^\Lambda \}\) is the canonical basis, or the upper global basis, of \(L(\Lambda)\).

We could have proved the equivalence of parts (a)–(d) of the next result before we introduced the quantum group \(U_q(\mathfrak{sl}_e)\). For (e), however, we need Kashiwara’s theory of crystal bases [69, 70] and the work of Misra and Miwa [111] relate crystal bases of the Fock space and crystal bases of \(L(\Lambda)\).

3.5.22. **Proposition.** Suppose that \(\mathbb{F}\) is an arbitrary field and that \(n \geq 0\). Then the following are equivalent:

a) For all \(\mu \in K^\Lambda\), \(B^\mu = [D^\mu]\).

b) For all \(\lambda, \mu \in K^\Lambda\), \(e_{\mu \tau}(-q) = e_{\lambda \mu} + qN[\mu]\).

c) For all \(\mu \in K^\Lambda\), \(B^\mu = [Y^\mu]\).

d) For all \(\lambda, \mu \in K^\Lambda\), \(d_{\mu \lambda}(q) = e_{\lambda \mu} + qN[\mu]\).

e) For all \(\lambda \in P^\Lambda\) and \(\mu \in K^\Lambda\), \(d_{\lambda \mu}(q) = e_{\lambda \mu} + qN[\mu]\).

**Proof.** In the Grothendieck groups, \([D^\mu] = [S^\mu] + \sum_{\rho \in \tau} e_{\rho \tau}(-q)[S^\rho]\) and \([Y^\mu] = X^\mu + \sum_{\rho \in \tau} d_{\rho \mu}(q)X^\rho\), where in the sums \(\rho, \tau \in K^\Lambda\). Moreover, by Lemma 3.5.5, \([Y^\mu]^\# = [Y^\mu]\) and \([D^\mu]^\# = [D^\mu]\), for all \(\mu \in K^\Lambda\).

By definition, \(d_{\lambda \mu}(q) \in \mathbb{N}[q, q^{-1}]\) and \(e_{\lambda \mu}(-q) \in \mathbb{Z}[q, q^{-1}]\). Hence, parts (a) and (b), and parts (c) and (d), are equivalent by Proposition 3.5.6. Moreover, \(e^\mu\frac{Y^\mu}{X^\mu} = (d^\mu)^{-1}\), \(d_{\mu \lambda}(1) = 1 = e_{\mu \lambda}(-q)^\tau\) and the Laurent polynomials \(d_{\lambda \mu}(q)\) and \(e_{\mu \lambda}(-q)\) are non-zero only if \(\lambda \geq \mu\) by Theorem 1.3.4, so parts (b) and (d) are also equivalent. Certainly, (e) implies (d) so to complete the proof it is enough to show that (a) implies (e).

Suppose that (a) holds so that \(B_{\mu} = D^\mu\), for all \(\mu \in K^\Lambda\). To prove that (e) holds we need the machinery of crystal bases [69, 70] in the special case of the Fock space \(\mathcal{F}_{\mathcal{Q}(q)}\). We will refer the reader to the literature for the definitions and results that we need.

Following [70, §2] define rings \(\mathbb{A} = \mathbb{Q}[q, q^{-1}] = \mathbb{Q} \otimes \mathbb{Z} A\), \(A_0 = \mathbb{A}(q)\) and \(A_\infty = \mathbb{A}(q^{-1})\), so that \(A_0\) and \(A_\infty\) are the rational functions in \(\mathbb{Q}(q)\) that are regular at 0 and \(\infty\), respectively. Set

\[
L_0(\Lambda) = \bigoplus_{\mu \in K^\Lambda} A_0 B^\mu = \bigoplus_{\mu \in K^\Lambda} A_0[S^\mu]
\]
and $B_0(\Lambda) = \{ [S^\mu] + qL_0(\Lambda) \mid \mu \in \mathcal{K}^\Lambda \}$. As $B_\mu$ is the upper crystal basis, the pair $(L_0(\Lambda), B_0(\Lambda))$ is an upper crystal base at $q = 0$ for $L(\Lambda)$ as defined by Kashiwara [69, §2]. Similarly, in the Fock space define 
\[
\mathcal{F}_0^\Lambda = \bigoplus_{\lambda \in \mathcal{P}^\Lambda} \mathbf{k}_0[\lambda] \quad \text{and} \quad C_0^\Lambda = \{ [\lambda] + qq_0^\Lambda \mid \lambda \in \mathcal{P}^\Lambda \} .
\]

Misra and Miwa [111] showed that $(\mathcal{F}_0^\Lambda, C_0^\Lambda)$ is an upper crystal basis for $\mathcal{F}_Q^{\Lambda(q)}$. (As we discuss below they explicitly described the crystal graph of $\mathcal{F}_Q^{\Lambda(q)}$.) By [69, Theorem 7] the Fock space $\mathcal{F}_Q^{\Lambda(q)}$ has a unique basis $\{ C^\Lambda \mid \lambda \in \mathcal{P}^\Lambda \}$, Kashiwara’s upper global basis, such that
\[
(3.5.23) \quad C^\Lambda = C_0^\Lambda \quad \text{and} \quad C^\Lambda \equiv [\lambda] \quad (\text{mod } qq_0^\Lambda), \quad \text{for } \lambda \in \mathcal{P}^\Lambda .
\]

Let $\mathcal{F}_*^\Lambda = (\mathcal{F}_0^\Lambda, \mathcal{F}_0^\Lambda, \mathcal{F}_0^\Lambda)$ and $L_*(\Lambda) = (L_0(\Lambda), L_0(\Lambda), L_0(\Lambda)^\oplus)$, where $\mathcal{F}_0^\Lambda = \mathcal{A} \otimes \mathbf{k}_0, \mathcal{F}_0^\Lambda$ and $L_0(\Lambda) = \mathcal{A} \otimes \mathbf{k}_0 L_0(\Lambda)$. Then $\mathcal{F}_*^\Lambda$ and $L_*(\Lambda)$ are balanced triples in the sense of [70, §2]. The $\Lambda$-weight space of $\mathcal{F}_Q^{\Lambda(q)}$ is $Q(q)v_\Lambda$ so, up to a scalar, the decomposition map $d_q$ is the unique $U_q(\mathfrak{sl}_n)$-module homomorphism $d_q : \mathcal{F}_Q^{\Lambda(q)} \rightarrow [\text{Rep}_Q^{\Lambda}(q)]$. By [69, Proposition 5.2.1], the image of $\mathcal{F}_0^\Lambda$ under $d_q$ is a balanced triple contained in $L(\Lambda)$. In fact, we have $d_q(\mathcal{F}_0^\Lambda) = L_*(\Lambda)$ by [69, Proposition 5.2.2] because $d_q$ sends $v_\Lambda = \{ 0 \}$ to $[S^\Lambda]$. Consequently, if $\lambda \in \mathcal{P}^\Lambda$ then $d_q([\lambda]) \in L_0(\Lambda)$. That is,
\[
[S^\Lambda] = d_q([\lambda]) \in \bigoplus_{\mu \in \mathcal{K}^\Lambda} \mathbf{k}_0[S^\mu] = \bigoplus_{\mu \in \mathcal{K}^\Lambda} \mathbf{k}_0[D^\mu].
\]

As $[S^\Lambda] = \sum_\mu d_\mu(q)[D^\mu]$ it follows that $d_\mu(q) \in \mathbb{N}[q,q^{-1}] \cap \mathbf{k}_0 = \mathbb{N}[q]$. Moreover, because of $(3.5.21)$, $d_q$ sends canonical basis elements in $\mathcal{F}_0^\Lambda$ to canonical basis elements in $L_0(\Lambda)$, or to zero. It follows that
\[
d_q(C^\Lambda) = \begin{cases} B^\Lambda = [D^\Lambda] & \text{if } \lambda \in \mathcal{K}^\Lambda \\ 0 & \text{otherwise.} \end{cases}
\]

By $(3.5.23)$, $|\lambda| - C^\Lambda \in qq_0^\Lambda$ for all $\lambda \in \mathcal{P}^\Lambda$. Consequently, if $\lambda \notin \mathcal{K}^\Lambda$ then
\[
[S^\Lambda] = d_q([\lambda]) = d_q([\lambda] - C^\Lambda) = d_q(qq_0^\Lambda) = qL_0(\Lambda).
\]

Hence, $d_\mu(q) \in \delta_\mu + q\mathbb{N}[q]$, for all $\lambda \in \mathcal{P}^\Lambda$ and all $\mu \in \mathcal{K}^\Lambda$. Thus, (e) holds and the proposition is proved.

\[\Box\]

3.5.24. Remark. The difference between the upper and lower crystal bases, or the dual canonical and canonical bases, can be interpreted as changing between the bases of Specht modules and dual Specht modules. The global bases and their crystal lattices are:

- upper: $q = 0$, $B^\mu \equiv [S^\mu] \pmod{\sum_{\lambda \in \mathcal{K}^\Lambda} \mathbf{k}_0[S^\lambda]}$
- lower: $q = \infty$, $c_q(q^{-d^\mu}B^\mu) \equiv [S^\mu] \pmod{\sum_{\lambda \in \mathcal{K}^\Lambda} \mathbf{k}_0[S^\lambda]}$

where $m$ is an involution on $\mathcal{K}^\Lambda$ that generalises the well-known Mullineux map for the symmetric groups. See Theorem 3.6.6 below.

3.5.25. Remark. As mentioned in Remark 3.5.10, a different action on the Fock space is commonly used in the literature. With respect to the Cartan pairing, as in Corollary 3.5.18, this action is the adjoint of the action in Theorem 3.5.9. As a consequence, the papers that use a different $U_q(\mathfrak{sl}_n)$-action also use a different coproduct for $U_q(\mathfrak{sl}_n)$, as they have to if they want Kashiwara’s tensor product rule to connect the crystal bases at different levels for a fixed $\Lambda \in P^+$. In the dual set up, $\#$ categorifies the bar involution on $L(\Lambda)$, $\{ B^\mu \mid \mu \in \mathcal{K}^\Lambda \}$ is the canonical basis, or lower global crystal basis at $q = 0$ for $L(\Lambda)$ and $\{ B^\mu \mid \mu \in \mathcal{K}^\Lambda \}$ is its dual canonical basis.

It is natural to ask when the equivalent conditions of Proposition 3.5.22 are satisfied. In general, this is a difficult open problem. The next result shows that these properties hold whenever $F$ is a field of characteristic zero.

We can now state Ariki’s celebrated Categorification Theorem. By specializing $q = 1$ the quantum group $U_\Lambda(\mathfrak{sl}_n) \otimes \mathbb{Q}$ becomes the Kac-Moody algebra $U(\mathfrak{sl}_n)$. Let $L_1(\Lambda)$ be the irreducible integrable highest weight $U(\mathfrak{sl}_n)$-module of high weight $\Lambda$. The canonical bases of $L_1(\Lambda)$ are obtained by specializing $q = 1$ in the canonical bases of $L_0(\Lambda)$. Forgetting the grading in the results above, $\text{Rep}_q^\Lambda \cong L_1(\Lambda) \cong \text{Proj}_q^\Lambda$, where $\text{Rep}_q^\Lambda = \bigoplus_n \text{Rep}(\mathcal{M}_n^\Lambda) \otimes \mathbb{Z} \otimes \mathbb{Q}$ and $\text{Proj}_q^\Lambda = \bigoplus_n \text{Proj}(\mathcal{M}_n^\Lambda) \otimes \mathbb{Z} \otimes \mathbb{Q}$.

3.5.26. Theorem (Ariki’s Categorification Theorem [3, 23]). Suppose that $F$ is a field of characteristic zero. Then the canonical basis of $L_1(\Lambda)$ coincides with the basis of (ungraded) projective indecomposable $\mathcal{M}_n^\Lambda$-modules $\{ [S^\mu] \mid \mu \in \mathcal{K}^\Lambda \}$ of $\text{Proj}_q^\Lambda$.
This theorem was proved by Ariki [3, Theorem 4.4] when $v^2 \neq 1$ and by Brundan and Kleshchev when $v^2 = 1$ [23, Theorem 3.10]. For a detailed proof of this important result when $v^2 \neq 1$ see [5, Theorem 12.5]. For an overview and historical account of Ariki’s theorem see [44].

Combining Theorem 3.5.26 with Proposition 3.5.22 we obtain the main result of this section.

3.5.27. Corollary (Brundan and Kleshchev [22, Theorem 5.14]). Suppose that $F$ is a field of characteristic zero. Then the canonical basis of $L_A(\Lambda)$ coincides with the basis $\{ q^{-\text{det}\mu}[Y^\mu] \mid \mu \in K^A \}$ of $\text{Proj}^A_{Q[q]}$. In particular, $d_{\lambda\mu}(q) \in \delta_{\lambda\mu} + q\mathbb{N}[q]$, for all $\lambda \in P^A$ and $\mu \in K^A$.

When $\Lambda$ is a weight of level 2 and $e = \infty$ this was first proved by Brundan and Stroppel [26, Theorem 9.2]. For extensions of this result to cyclotomic quiver Hecke algebras of arbitrary type see [67, 91, 122, 134]. Corollary 3.5.27 implies that the graded decomposition numbers $d_{\lambda\mu}(q) = [S^\lambda : D^\mu]_q = b_{\lambda\mu}(q)$ are parabolic Kazhdan-Lusztig polynomials. Explicit formulas are given in [55, Appendix A] and [99, Lemma 2.46].

For the canonical basis $\{ B_\mu \}$ it is immediate that the Laurent polynomials $b_{\lambda\mu}(q) \in \mathbb{Z}[q]$ are polynomials, for $\lambda \in P^A$ and $\mu \in K^A$, however, it is a deep fact that their coefficients are non-negative integers. In contrast, it is immediate that $d_{\lambda\mu}(q) \in \mathbb{N}[q, q^{-1}]$ but it is a deep fact that the graded decomposition numbers are polynomials rather than Laurent polynomials. Thus, the difficult result changes from positivity of coefficients in the ungraded setting, to positivity of exponents in the graded setting. In fact, it is also true when $F = \mathbb{C}$.

Brundan and Kleshchev’s proof of Corollary 3.5.27 is quite different to the one given here. They have to work quite hard to define triangular bar involutions on $L_A(\Lambda)$. They have to do this by exploiting the representation theory of $\mathcal{H}_n^A$. One benefit of Brundan and Kleshchev’s approach is that they have an explicit description of the bar involution on $F^A_{\lambda}$. In contrast, we have no hope of working with our bar involution unless we already know the graded decomposition matrices. On the other hand, the approach here works for an arbitrary multicharge $\kappa$.

To complete the proof of Corollary 3.5.27, Brundan and Kleshchev lift Grojnowski’s approach [49] to the representation theory of $\mathcal{H}_n^A$ to the graded setting. As a result they obtain graded analogues of Kleshchev’s modular branching rules [18, 77, 78]. Under categorification, these branching rules correspond to the action of the crystal operators on the crystal graph of $L(\Lambda)$; see [22, Theorem 4.12]. By invoking Ariki’s theorem they deduce an analogue of Corollary 3.5.27, although with a possibly different labelling of the irreducible modules. Finally, they prove that the labelling of the irreducible $\mathcal{H}_n^A$-modules coming from the branching rules agrees with the labelling of Corollary 1.5.2; compare with [6, 9].

We have not yet given an explicit description of the labelling of the irreducible $\mathcal{H}_n^A$-modules because we defined $K_n^A = \{ \mu \in P_n^A \mid D^\mu \neq 0 \}$. Extending Definition 3.2.1, if $\mu \in P_n^A$ and given nodes $A < C$ define $d_{\lambda\mu}^A(\mu) = \# \{ B \in \text{Add}_A(\mu) : A < B < C \} - \# \{ B \in \text{Rem}_A(\mu) : A < B < C \}$.

Following Misra and Miwa [111] (but using Kleshchev’s terminology [76]), a removable $i$-node $A$ is normal if $d_{\lambda\mu}^A(\mu) \leq 0$ and $d_{\lambda\mu}^A(\mu) < 0$ whenever $C \in \text{Rem}_A(\mu)$ and $A < C$. A normal $i$-node $A$ is good if $A \leq B$ whenever $B$ is a normal $i$-node. Write $\lambda \xrightarrow{\text{good}} \mu$ if $\mu = \lambda + A$ for some good node $A$. Misra and Miwa [111, Theorem 3.2] show that the crystal graph of $L_A(\Lambda)$, considered as a submodule $F^A_{\lambda}$, is the graph with vertex set $\mathcal{L}^A_{\lambda} = \{ \mu \in P^A \mid \mu = v_\lambda \text{ or } \lambda \xrightarrow{\text{good}} \mu \text{ for some } \lambda \in \mathcal{L}^A_0 \}$, and with labelled edges $\lambda \xrightarrow{i} \mu$ whenever $\mu$ is obtained from $\lambda$ by adding a good $i$-node, for some $i \in I$. See [5, Theorem 11.11] for a self-contained proof of this result, couched in similar language.

3.5.28. Corollary (Ariki [4, 8, 22]). Suppose that $F$ is an arbitrary field and that $\mu \in P_n^A$. Then $K_n^A = \mathcal{L}^A_0$. That is, if $\mu \in P_n^A$ then $D^\mu \neq 0$ if and only if $\mu \in \mathcal{L}^A_0$.

Proof. If $F$ is a field of characteristic zero then $\{ \mu + qL_A(\Lambda) \mid \mu \in K^A \}$ is a basis of $L_A(\Lambda)/qL_A(\Lambda)$ by Proposition 3.5.22 and Corollary 3.5.27. This basis of $L_A(\Lambda)/qL_A(\Lambda)$ is exactly the crystal basis of $L(\Lambda)$ by Corollary 3.5.27, so $K_n^A = \mathcal{L}^A_0$ in characteristic zero. If $F$ is a field of positive characteristic then a straightforward modular reduction argument shows that $D_{\lambda\mu}^A \neq 0$ only if $D_{\lambda'\mu}^A \neq 0$, for $\mu \in P_n^A$ (for example, see §3.7 below). So, $K_n^A \subseteq \mathcal{L}^A_0$. By Proposition 3.5.12, the number of irreducible $\mathcal{H}_n^A$-modules depends only on $\kappa$, and not on the field $F$, so $K_n^A = \mathcal{L}^A_0$ as required.

The idea in Proposition 3.5.12 that over any field the natural bases $\{ [D^\mu] \}$ and $\{ [Y^\mu] \}$ of $[\text{Rep}^A_{\lambda}]$ and $[\text{Proj}^A_{\lambda}]$, respectively, are distinguished bases of $L(\Lambda)$ goes back to at least Lascoux, Leclerc and Thibon [89]. This was generalized to higher levels by Ariki [3] and it played an important role in the classification of the irreducible $\mathcal{H}_n^A$-modules [4, 12] and in Grojnowski and Vazirani’s work [49, 51, 79, 133]. The role of the crystal graphs in the representation theory of $\mathcal{H}_n^A$ is explored further in [68, 91].
3.6. Homogeneous Garnir relations. By Theorem 3.2.9, $\mathcal{D}_n^\Lambda$ is a graded cellular algebra and, as a consequence, that there exist graded lifts of the Specht modules for arbitrary $\Lambda \in P^+$. However, at this point we cannot really compute inside the graded Specht modules because we do not know how to write basis elements indexed by non-standard tableaux in terms of standard ones. This section shows how to do this. First, some combinatorics.

Fix a multipartition $\lambda$ and a node $A = (l, r, c) \in \lambda$. A (row) Garnir node of $\lambda$ is any node $A = (l, r, c)$ such that $(l, r + 1, c) \in \lambda$. The $(e, A)$-Garnir belt is the set of nodes

$$B_A = \{ (l, r, k) \in \lambda \mid k \geq c \text{ and } e[\frac{k-e+c}{e}] \leq \lambda_r(l) - c + 1 \} \cup \{ (l, r + 1, k) \in \lambda \mid k \leq c \text{ and } c \geq e[\frac{k-e+c}{e}] \}.$$ 

Let $b_A = \# B_A/e$ and write $a_A = c_A + c_A$ where $eA$ is the number of nodes in $B_A$ in row $(l, r)$. Let $\mathcal{D}_A$ be the set of minimal length right coset representatives of $\mathfrak{S}_{\mathfrak{d} A} \times \mathfrak{S}_{c A}$ in $\mathfrak{S}_{b A}$; see, for example, [104, Proposition 3.3]. When $e = \infty$ these definitions should be interpreted as $B_A = \emptyset$, $b_A = 0 = a_A = c_A$ and $\mathcal{D}_A = 1$.

Suppose $A$ is a Garnir node of $\lambda$. The rows of $\lambda$ are indexed by pairs $(l, r)$, corresponding to row $r$ in $\lambda^{(l)}$ where $1 \leq l \leq l$ and $r \geq 1$. Order the row indices lexicographically. Let $t_A$ be the $\lambda$-tableau that agrees with $t^A$ for all numbers $k < t^A(A) = t^A(l, r, c)$ and $k > t^A(l, r + 1, c)$ and where the remaining entries in rows $(l, r)$ and $(l, r + 1)$ are filled in increasing order from left to right first along the nodes in row $(l, r + 1)$ that are in the first $c$ columns but not in $B_A$, then along the nodes in row $(l, r)$ of $B_A$ followed by the nodes in row $(l, r + 1)$ of $B_A$, and then along the remaining nodes in row $(l, r)$.

3.6.1. Example As Garnir belts are contained in consecutive rows of the same component, the general case can be understood by looking at a two-rowed partition (of level one), so we consider the case $e = 3$, $\lambda = (14, 6)$ and $A = (1, 1, 4)$. Then

$$t_A = \begin{array}{ccccccccccccc}
1 & 2 & 3 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 17 & 18 \\
4 & 14 & 15 & 16 & 19 & 20
\end{array}$$

The lines in $t_A$ show how the $(3, A)$-Garnir belt decomposes into a disjoint union of “$c$-bricks”. In general, $b_A$ is equal to the number of $c$-bricks in the Garnir belt and $a_A$ is the number of $c$-bricks in its first row. In this case, $b_A = 4$ and $a_A = 3$. Therefore, $\mathcal{D}_A = \{1, 3, 4, 5, 6, 7, 8, 9\}$. \(
\)

Let $k_A = t_A(A)$ be the number occupying $A$ in $t_A$ and define

$$w^A_r = \prod_{a=k_A+e(r-1)}^{k_A+e(r-1)} (a, a+e),$$

for $1 \leq r < b_A$. The subgroup $\langle w^A_r \mid 1 \leq r < b_A \rangle$ of $\mathfrak{S}_n$ is isomorphic to $\mathfrak{S}_{b_A}$ via the map $w^A_r \mapsto s_r$, for $1 \leq r < b_A$. Set $t^A = t^A$ and $\tau^A = e(1^A)(\psi_{w^A_r} + 1)$, for $1 \leq r < b_A$. If $d \in \mathcal{D}_A$ choose a reduced expression $d = s_r, \ldots, s_{r_d}$ for $d$ and define

$$\tau^A_d = \tau^A_{r_1} \ldots \tau^A_{r_d} \in e(1^A)\mathcal{D}_n^\Lambda.$$ 

The elements $\tau^A_d$ of $\mathcal{D}_n^\Lambda$ seem to be very special and deserving of further study. They are homogeneous elements in $\mathcal{D}_n^\Lambda$ of degree zero that are independent of all choices of reduced expressions. Moreover, by [81, Theorem 4.13], the elements $\{ \tau^A \mid 1 \leq r < b_A \}$ satisfy the braid relations when they act on $S_2^\Lambda$ and they generate a copy of $\mathfrak{S}_{b_A}$ inside $\text{End}_\mathbb{Z}(S_2^\Lambda))$!

3.6.2. Theorem (Kleshchev, Mathas and Ram [81, Theorem 6.23]). Suppose that $\lambda \in \mathcal{D}_n^\Lambda$ and that $Z$ is an integral domain. The graded Specht module $S_2^\Lambda$ of $\mathcal{D}_n^\Lambda(\mathcal{Z})$ is isomorphic to the graded $\mathcal{D}_n^\Lambda$-module generated by a homogeneous element $v_\lambda$ of degree $\text{deg} t^A X$ subject to the relations:

- $v_\lambda e(1) = e(1) v_\lambda.$
- $v_\lambda s_r = 0$, for $1 \leq s \leq n$.
- $v_\lambda \psi_r = 0$ whenever $s_r \in \mathfrak{S}_\lambda$, for $1 \leq r \leq n$.
- $\sum_{d \in \mathcal{D}_A} v_\lambda \psi_A \tau^A_d = 0$, for all Garnir nodes $A \in \lambda$.

Relations (a)–(c) already appear in [25] and, in terms of the cellular basis machinery, they are a consequence of Proposition 3.2.10.

The relations in part (d) are the homogeneous Garnir relations. These relations are a homogeneous form of the well-known Garnir relations of the symmetric group [59, Theorem 7.2]. There is an analogous description of the dual Specht modules $S_\lambda$ in terms of column Garnir relations [81, §7]. Using Dyck tilings, Fayers [41] has shown how to write the homogeneous Garnir relations in terms of $\psi$-basis of the Specht module.

The most difficult part of the proof of Theorem 3.6.2 is showing that the $\tau^A$ satisfy the braid relations. This is proved using the Khovanov-Lauda diagram calculus that was briefly mentioned in §2.2. Like Theorem 3.2.9 this result holds over an arbitrary ring. To prove that the graded module defined by Theorem 3.6.2 has the
correct rank the construction of the graded Specht module \( S^\lambda \) over a field in Theorem 3.2.4, from [25,54], is used.

One of the main points of Theorem 3.6.2 is that it makes it possible to calculate in the graded Specht module over any ring. Prior to Theorem 3.6.2 the only way to compute inside the graded Specht modules was, in effect, to use the isomorphism \( R_n^\lambda \cong R_n^\beta \) of Theorem 3.1.1 to work in the ungraded setting then use the inverse isomorphism \( R_n^\beta \cong R_n^\lambda \) to get back to the graded setting. This made it difficult to keep track of, and to exploit, the grading on \( S^\lambda \) — and it was only possible to work with Specht modules defined over a field.

Theorem 3.6.2 also gives the relations for \( S^\lambda \) as an \( R_e \)-module. From this perspective Theorem 3.6.2 can be used to give another construction of the graded Specht modules. For \( \alpha, \beta \in Q^+ \) let \( R_{\alpha, \beta} = R_\alpha \otimes R_\beta \). Definition 2.2.1 implies that there is a non-unital embedding \( R_{\alpha, \beta} \hookrightarrow R_{\alpha+\beta} \) that maps \( e(\mathbf{i}) \otimes e(\mathbf{j}) \) to \( e(\mathbf{i} \cup \mathbf{j}) \), where \( \mathbf{i} \cup \mathbf{j} \) is the concatenation of \( \mathbf{i} \) and \( \mathbf{j} \). Under this embedding the identity element of \( R_{\alpha, \beta} \) maps to

\[
e_{\alpha, \beta} = \sum_{\mathbf{i} \in I^\alpha, \mathbf{j} \in I^\beta} e(\mathbf{i} \cup \mathbf{j}).
\]

Definition 2.2.1 implies that \( R_{\alpha+\beta} \) is free as an \( R_{\alpha, \beta} \)-module, so the functor

\[
\text{Ind}_{\alpha, \beta}^{\alpha+\beta} (M \otimes N) = (M \otimes N)e_{\alpha, \beta} \otimes_{R_{\alpha, \beta}} R_{\alpha+\beta}
\]

is a left adjoint to the natural restriction map. Iterating this construction, given \( \beta_1, \ldots, \beta_{\ell} \in Q^+ \) and \( R_{\beta_k} \) modules \( M_k \), for \( 1 \leq k \leq \ell \), define

\[
M_1 \circ \cdots \circ M_\ell = \text{Ind}_{\beta_1, \ldots, \beta_{\ell}}^{\beta_{\ell}} (M_1 \otimes \cdots \otimes M_\ell).
\]

The definition of the graded Specht modules by generators and relations in Theorem 3.6.2 makes the following result almost obvious. This description of the Specht modules is part of the folklore of these algebras with several authors [23,133] using it as the definition of Specht modules.

3.6.3. Corollary (Kleshchev, Mathas and Ram [81, Theorem 8.2]). Suppose that \( \lambda^{(k)} \in P_{1,\beta_k} \), for \( \beta_k \in Q^+ \) and \( 1 \leq k \leq \ell \), so that \( \lambda \in P_\beta \), where \( \beta = \beta_1 + \cdots + \beta_{\ell} \). Then there is an isomorphism of graded \( R_n^\lambda \)-modules (and graded \( R_e \)-modules),

\[
S^\lambda \langle \deg t^{\lambda_1} + \cdots + \deg t^{\lambda_{\ell}} \rangle \cong (S^\lambda_1 \circ \cdots \circ S^\lambda_{\ell}) \langle \deg t^\lambda \rangle,
\]

where on the right hand side \( S^\lambda(k) \) is considered as an \( R_{\beta_k} \)-module, for \( 1 \leq k \leq \ell \).

A second application of Theorem 3.6.2 is a generalization of James’ famous result [59, Theorem 8.15] for symmetric groups that describes what happens to the Specht modules when they are tensored with the sign representation. First some notation.

Following [81, §3.3], for \( i \in I^\beta \) let \(-i = (-i_1, \ldots, -i_n) \in I^n\). Recalling the multicharge \( \kappa \) from §1.2, set \( \kappa' = (-\kappa_1, \ldots, -\kappa_1) \) and let \( \Lambda' = \Lambda(\kappa') \in P^+ \). Similarly, if \( \beta = \sum a_i \alpha_i \in Q^+ \) let \( \beta' = \sum a_i \alpha_{-i} \). Inspecting Definition 2.2.9, there is a unique isomorphism of graded algebras

\[
\text{sgn} : R_{\beta} \rightarrow R_{\beta'} ; \quad e(1) \mapsto e(-1), \quad y_r \mapsto -y_r, \quad \psi_s \mapsto -\psi_s,
\]

for all admissible \( r \) and \( s \) and \( i \in I^\beta \). The involution \( \text{sgn} \) induces an equivalence of categories \( \text{Rep}(R_{\beta'}) \rightarrow \text{Rep}(R_{\beta}) \) that sends an \( R_{\beta'} \)-module \( M \) to the \( R_{\beta} \)-module \( M_{\text{sgn}} \), where the \( R_{\beta} \)-action is twisted by \( \text{sgn} \).

3.6.5. Corollary (Kleshchev, Mathas and Ram [81, Theorem 8.5]). Suppose that \( \mu \in P_\beta \), for \( \beta \in Q^+ \). Then

\[
S^\mu \cong (S_{\mu'})^\text{sgn} \quad \text{and} \quad S_\mu \cong (S_{\mu'})^\text{sgn}
\]

as \( R_{\beta} \)-modules.

In [81] this is proved by checking the relations in Theorem 3.6.2. As noted in [55, Proposition 3.26], this can be proved more transparently by noting that, up to sign, the involution \( \text{sgn} \) maps the \( \psi' \)-basis of \( R_{\beta}^\lambda \) to the \( \psi \)-basis of \( R_{\beta'}^\lambda \). Some care must be taken with the notation here. For example, if \( \mu \in P_\beta \) then \( \mu' \in P_{\beta'} \).

See [55, §3.7] for more details.

We give an application of these results to the graded decomposition numbers. First, by Corollary 3.5.28 if \( \mu \in K_\beta^\lambda \) there exists \( i \in I^n \) and a sequence of multipartitions \( \mu_0 = v_\lambda \mu_1, \ldots, \mu_n = \mu \) in \( K_\lambda \) such that \( \mu_{k+1} \) is obtained from \( \mu_k \) by adding a good \( i_k \)-node, for \( 0 \leq k < n \). It follows from the modular branching rules [22, Theorem 4.12], and properties of crystal graphs, that there exists a unique sequence of multipartitions \( \mu(\mu_0) = v_\lambda \mu_1, \ldots, \mu(\mu_n) = \mu(\mu) \) such that \( \mu(\mu_{k+1}) \) is obtained from \( \mu(\mu_k) \) by adding a good \(-i_k\)-node and \( \mu(\mu_{k+1}) \in K_{k+1}^\lambda \), for \( 1 \leq k \leq n \). The Mullineux conjugate of \( \mu \) is the
multipartition \( m(\mu) \). Thus, \( D^{m(\mu)} \) is a non-zero irreducible \( \mathcal{A}_\beta^{X'} \)-module. We emphasize that the \( \mathcal{A}_\beta^{X'} \)-module \( D^{m(\mu)} \) is defined using the \( \psi \)-basis of \( \mathcal{A}_\beta^{X'} \), and hence the crystal theory used in §3.5, with respect to the multicharge \( \kappa' \).

3.6.6. Theorem. Suppose that \( \mu \in K^X_\beta \), for \( \beta \in Q^+ \). Then

\[
(D^{m(\mu)})^{\mathrm{sgn}} \cong D^\mu
\]
as \( \mathcal{A}_\beta^X \)-modules.

Proof. As \( \mathrm{sgn} \) is an equivalence of categories, \( (D^{m(\mu)})^{\mathrm{sgn}} \cong D^\nu(d) \) for some \( \nu \in K^X_\beta \) and \( d \in \mathbb{Z} \) by Corollary 3.2.7. Since \( \mathrm{sgn} \) is homogeneous, by Theorem 2.1.5(a),

\[
dim_q(D^{m(\mu)})^{\mathrm{sgn}} = \dim_q D^{m(\mu)} = \dim_q D^{m(\mu)} = \dim_q(D^{m(\mu)})^{\mathrm{sgn}},
\]
so that \( d = 0 \) and \( (D^{m(\mu)})^{\mathrm{sgn}} \cong D^\nu \). To show that \( \nu = \mu \) it is now enough to work in the ungraded setting. Therefore, we can either use the modular branching rules of [6, 49], or their graded counterparts from [22, Theorem 4.12], together with what is by now a standard argument due to Kleshchev [77, Theorem 4.7], to show that \( \nu = \mu \). \( \square \)

The \( \mathrm{sgn} \) map induces an equivalence \( \text{Rep}(\mathcal{A}_\beta^{X'}) \to \text{Rep}(\mathcal{A}_\beta^X) \). As \( \mathrm{sgn} \) is an involution, we also write \( \mathrm{sgn} : \text{Rep}(\mathcal{A}_\beta^X) \to \text{Rep}(\mathcal{A}_\beta^X) \) for the inverse equivalence. The last two results can now be written as \( (S^X)^{\mathrm{sgn}} \cong S_X \) and \( (D^\mu)^{\mathrm{sgn}} \cong D^{m(\mu)} \) as \( \mathcal{A}_\beta^X \)-modules, for \( \lambda \in P^X_\beta \) and \( \mu \in K^X_\beta \).

3.6.7. Corollary. Suppose that \( F \) is a field and that \( \lambda \in P^X_\beta \) and \( \mu \in K^X_\beta \). Then \( d_{\mu \mu}(q) = 1, d_{m(\mu)\mu}(q) = q^{\text{def} \mu} \) and \( d_{\lambda \mu}(q) \neq 0 \) only if \( m(\mu) \triangleright \lambda \triangleright \mu \). Moreover, if \( F = \mathbb{C} \) then \( 0 < \text{deg} d_{\lambda \mu}(q) < \text{deg} \mu \) whenever \( m(\mu) \triangleright \lambda \triangleright \mu \).

Proof. Suppose that \( \lambda \in P^X_\beta \) and \( \mu \in K^X_\beta \). Then

\[
[S^X : D^\mu] = [S^X : (S^X)^{\mathrm{sgn}}] = [S^X : D^{m(\mu)}],
\]
by Corollary 3.6.5 and Theorem 3.6.6, respectively. Therefore, using Corollary 3.3.5 and Theorem 2.1.5(a),

\[
[S^X : D^\mu] = q^{\text{def} \mu}[(S^X)^{\otimes} : D^{m(\mu)}] = q^{\text{def} \mu}[S^X : D^{m(\mu)}].
\]
By Theorem 2.1.5(c), if \( \tau \in K^X_\beta \) and \( \sigma \in P^X_\beta \) then \( d_{\tau \tau}(1) = 1 \) and \( d_{\sigma \tau}(q) \neq 0 \) only if \( \sigma \triangleright \tau \). Therefore, \( d_{m(\mu)\mu}(q) = q^{\text{def} \mu} d_{m(\mu)\mu}(q) = q^{\text{def} \mu} \) and \( d_{\lambda \mu}(q) \neq 0 \) only if \( \lambda \triangleright \mu \). The argument so far is valid over any field. Now suppose that \( F = \mathbb{C} \). Then \( d_{\lambda \mu}(q) \in \delta_{\lambda \mu} + qN[q] \), by Corollary 3.5.27, so the remaining statement about the degrees of the graded decomposition numbers follows. \( \square \)

Corollary 3.6.7 is the easy half of a conjecture of Fayers [40], which he was interested in because it leads to a faster algorithm for computing the graded decomposition numbers of \( \mathcal{A}_\mu^X \). At the level of canonical bases the last two results correspond to the fact shifting by the defect transforms an upper crystal base into a lower crystal base [69, Lemma 2.4.1]. See also [22, Remark 3.19].

3.7. Graded adjustment matrices. All of the results in this section have their origin in the work of James [60] and Geck [43] on adjustment matrices. Brundan and Kleshchev have given two different approaches to graded decomposition matrices in [21, §6] and [22, §5.6]. In this section we give third cellular algebra approach. Even though our definitions and proofs are different, it is easy to see that everything in this section is equivalent to definitions or theorems of Brundan and Kleshchev — or to graded analogues of results of James and Geck.

Before we introduce the adjustment matrices, let \( A[I^n] \) be the free \( A \)-module generated by \( I^n \). The \( q \)-character of a finite dimensional \( \mathcal{A}_\mu \)-module \( M \) is

\[
\text{Ch}_q M = \sum_{i \in I^n} dim_q M_i : i \in A[I^n],
\]
where \( M_i = M e(i) \), for \( i \in I^n \). For example, \( \text{Ch}_q S^X = \sum_{\lambda \in \text{Std}(\lambda)} q^{\text{deg}(\lambda)} \cdot 1^\lambda \).

3.7.1. Theorem ([74, Theorem 3.17]). Suppose that \( Z \) is a field. Then the map

\[
\text{Ch}_q : [\text{Rep}(\mathcal{A}_\mu)] \to A[I^n] : [M] \mapsto \text{Ch}_q M
\]
is injective.
As every $\mathcal{R}_n^\Lambda$-module can be considered as an $\mathcal{R}_n$-module by inflation, it follows that the restriction of $\text{Ch}_q$ to $[\text{Rep}(\mathcal{R}_n^\Lambda)]$ is still injective. Extend the map $\oplus$ to $A[F^n]$ by defining $(\sum_i f_i(q) \cdot \mathbf{i})^\oplus = \sum_i f_i(q) \cdot \mathbf{i}$. Then $(\text{Ch}_q[M])^\oplus = \text{Ch}_q[M^\oplus]$, for all $M \in [\text{Rep}(\mathcal{R}_n^\Lambda)]$.

This section compares representations of cyclotomic KLR algebras over different fields. Write $S^\mu_n$ and $D^\mu_n$ to emphasize that these modules are $\mathcal{R}_n^\Lambda(\mathbb{Z})$-modules, for $\Lambda \in \mathcal{P}_n^\Lambda$ and $\mu \in \mathcal{K}_n^\Lambda$. If $Z = F$ is a field, and $K$ is an extension of $F$, then $D^\mu_K \cong D^\mu_F \otimes_K K$ since $D^\mu_F$ is absolutely irreducible by Theorem 2.1.5. Therefore $\text{Ch}_q D^\mu_K$ depends only on $\mu$ and the characteristic of $F$.

By Theorem 3.7.9, or by Theorem 3.6.2, the graded Specht module $S^\mu_n\otimes_Z$ is defined over $Z$ and $S^\mu_n\otimes_Z \cong S^\mu_D\otimes_Z Z$ for any commutative ring $Z$. The graded Specht module $S^\mu_n$ has basis $\{\psi_t \mid t \in \text{Std}(\mu)\}$ and it comes equipped with a $Z$-valued bilinear form $(\ , \ )$ that is determined by

\[ (\psi_t, \psi_s) = \psi_t \psi_s(1) = \psi_t \psi_s(0) e(I^{\Lambda}). \]

Following (1.3.3), define the radical of $S^\mu_n$ to be $\text{rad } S^\mu_n = \{ x \in S^\mu_n \mid (x, y) = 0 \text{ for all } y \in S^\mu_n \}$.

In fact, by (3.7.2), $\text{rad } S^\mu_n = \{ x \in S^\mu_n \mid xa = 0 \text{ for all } a \in (\mathcal{R}_n^\Lambda)^{\oplus \mu}\}.$

### 3.7.3. Definition. Suppose that $\mu \in \mathcal{P}_n^\Lambda$. Let $D^\mu_n = S^\mu_n / \text{rad } S^\mu_n$.

By definition, $\text{rad } S^\mu_n$ is a graded submodule of $S^\mu_n$, so $D^\mu_n$ is a graded $\mathcal{R}_n^\Lambda(\mathbb{Z})$-module. Hence, $D^\mu_n \otimes_Z Z$ is a graded $\mathcal{R}_n^\Lambda(\mathbb{Z})$-module for any ring $Z$.

The following result should be compared with [21, Theorem 6.5].

### 3.7.4. Theorem. Suppose that $\mu \in \mathcal{P}_n^\Lambda$. Then $D^\mu_n \otimes_Z Z$ is a $\mathbb{Z}$-lattice in $D^\mu_n$ and $D^\mu_n$ is a $\mathbb{Z}$-lattice in $D^\mu_n$. Consequently, $D^\mu_Z = D^\mu_n \otimes_Z \mathbb{Q}$ and $\text{Ch}_q D^\mu_Z = \text{Ch}_q D^\mu_n$.

**Proof.** Let $G^\mu_n = (\langle \psi_t, \psi_s \rangle)$ be the Gram matrix of $S^\mu_n$. As $Z$ is a principal ideal domain, by the Smith normal form there exists a pair of bases $\{a_r\}$ and $\{b_s\}$ of $S^\mu_n$ such that $(a_r , b_s)$ is a principal ideal of $\mathbb{Z}$ for some non-negative integers such that $d_1 | d_2 | \cdots | d_z$, where $d_i \neq 0$ only if $d_i = 0$ for all $s \geq r$. That is, $d_1, \ldots, d_z$ are the principal divisors of $G^\mu_n$. As the form is homogeneous, we may assume that the bases $\{a_r\}$ and $\{b_s\}$ are homogeneous with $a_r e(i) = \delta_{r,i}a_r$ and $b_s e(i) = \delta_{s,i}b_s$, for $1 \leq r, s \leq z$ and $i \in I^\Lambda$. Comparing with the definitions above, it follows that $\{a_r \mid d_r = 0\}$ is a basis of rad $S^\mu_n$ and that $\{a_r \mid d_r \neq 0\}$ is a basis of $D^\mu_n$. All of our claims now follow.

For an arbitrary field $F$, it is usually not the case that $D^\mu_F$ is isomorphic to $D^\mu_n \otimes_Z F$ as an $\mathcal{R}_n^\Lambda(F)$-module. Indeed, if $F$ is a field of characteristic $p > 0$ then the argument of Theorem 3.7.4 shows that

$$\text{dim}_F D^\mu_F = \{1 \leq r \leq z \mid d_r \neq 0 (\text{mod } p)\} \leq \text{rank}_\mathbb{Z} D^\mu_Z = \text{dim}_F D^\mu_n,$$

with equality if and only if all of the non-zero principal divisors of $G^\mu_n$ are coprime to $p$.

### 3.7.5. Definition (cf. Brundan and Kleshchev [22, §5.6]). Suppose that $F$ is a field. For $\lambda, \mu \in \mathcal{K}_n^\Lambda$ define Laurent polynomials $a^F_{\lambda, \mu}(q) \in \mathbb{N}[q, q^{-1}]$ by

$$a^F_{\lambda, \mu}(q) = \sum_{d \in \mathbb{Z}} [D^\mu_n \otimes_Z F : D^\mu_F(d)] q^d.$$

The matrix $a^F_E = (a^F_{\lambda, \mu}(q))$ is the graded adjustment matrix of $\mathcal{R}_n^\Lambda(F)$.

Recall that $d^F_{\lambda, \mu}(q)$ is a graded decomposition number of $\mathcal{R}_n^\Lambda$. If we want to emphasize the field $F$ then we write $d^F_{\lambda, \mu}(q) = [\mathcal{R}_n^\Lambda : D^\mu_F(d)]$. Note that $e$ is always fixed.

### 3.7.6. Theorem (cf. Brundan and Kleshchev [22, Theorem 5.17]). Suppose that $F$ is a field. Then:

a) If $\lambda, \mu \in \mathcal{K}_n^\Lambda$ then $a^F_{\lambda, \mu}(1) = 1$ and $a^F_{\lambda, \mu}(q) \neq 0$ only if $\lambda \sqsupseteq \mu$. Moreover, $a^F_{\lambda, \mu}(q) = a^F_{\mu, \lambda}(q)$.

b) We have, $a^E_E = d^E_E \circ a^F_E$. That is, if $\lambda \in \mathcal{P}_n^\Lambda$ and $\mu \in \mathcal{K}_n^\Lambda$ then

$$[\mathcal{R}_n^\Lambda : D^\mu_F(q)] = d^F_{\lambda, \mu}(q) = \sum_{\nu \in \mathcal{K}_n^\Lambda} d^F_{\nu, \mu}(q) a^F_{\lambda, \nu}(q).$$

**Proof.** By construction, every composition factor of $D^\mu_n \otimes F$ is a composition factor of $S^\mu_n$, so the first two properties of the Laurent polynomials $a^F_{\lambda, \mu}(q)$ follow from Theorem 2.1.5. By Theorem 3.7.4, the adjustment matrix induces a well-defined map of Grothendieck groups $a^F_E : [\text{Rep}(\mathcal{R}_n^\Lambda(Q))] \rightarrow [\text{Rep}(\mathcal{R}_n^\Lambda(F))]$ given by

$$a^F_E([D^\mu_n]) = [D^\mu_n \otimes F] = \sum_{\mu \in \mathcal{K}_n^\Lambda} a^F_{\lambda, \mu}(q)[D^\mu_F].$$
Taking $q$-characters, $\text{Ch}_q D^\lambda_F = \sum_{\mu} a^\mu_{\lambda\mu}(q) \text{Ch}_q D^\mu_F$. Applying $\oplus$ to both sides gives $\text{Ch}_q D^\lambda_Q = \sum_{\mu} a^\mu_{\lambda\mu}(q) \text{Ch}_q D^\mu_F$. Therefore, $a^\mu_{\lambda\mu}(q) = a^\mu_{\lambda\mu}(q)$ by Theorem 3.7.1, completing the proof of part (a). For (b), since $S^\lambda_F \cong S^\lambda_Q \otimes \mathbb{Z} F$, 

$$[S^\lambda_F] = a^\mu_{\lambda\mu}([S^\lambda_Q]) = a^\mu_{\lambda\mu} \left( \sum_{\nu \in \mathcal{K}^\mu} d^\nu_{\lambda\mu}(q) [D^\nu_Q] \right) = \sum_{\nu \in \mathcal{K}^\mu} \sum_{\mu \in \mathcal{K}^\mu} d^\nu_{\lambda\mu}(q) a^\mu_{\lambda\mu}(q) [D^\nu_F].$$

Comparing the coefficient of $[D^\nu_F]$ on both sides completes the proof. \hfill $\Box$

Corollary 3.5.27 determines the graded decomposition numbers of the cyclotomic Hecke algebras in characteristic zero. There are several different algorithms for computing the graded decomposition numbers in characteristic zero $[40, 46, 55, 82, 89, 131]$. To determine the graded decomposition numbers in positive characteristic it is enough to compute the adjustment matrices of Theorem 3.7.6. The simplest case will be when $a^\mu_{\lambda\mu}(q) = \delta_{\lambda\mu}$, for all $\lambda, \mu \in \mathcal{K}^\alpha_n$. Unfortunately, we currently have no idea when this happens. Two failed conjectures for when $a^\mu_{\lambda\mu}$ is the identity matrix are discussed in Example 3.7.10 and Example 3.7.11 below.

We now compute the integral Gram matrices $G^\lambda_2 = \langle \langle \psi, \psi \rangle \rangle$ and some adjustment matrix entries in several examples.

**3.7.7 Example** (Semisimple algebras) Suppose that $e > n$ and that $(\Lambda, \alpha_i, n_l) \leq 1$, for all $i \in I$. Let $\Lambda \in \mathcal{P}^\alpha_n$ and $s, t \in \text{Std}(\Lambda)$. Then $\langle \psi_s, \psi_t \rangle = \delta_{st}$ because $\mathbb{P} = \mathbb{I}^t$ if and only if $s = t$ by Lemma 2.4.1. Hence, $G^\lambda_2$ is the identity matrix for all $\Lambda \in \mathcal{P}^\alpha_n$. \hfill $\Box$

**3.7.8 Example** (Nil-Hecke algebras) Suppose that $\Lambda = n\Lambda_i$ and $\beta = n\alpha_i$, for some $i \in I$. Let $\Lambda = (11|\ldots|1) \in \mathcal{P}^\alpha_n$, as in §2.5, and suppose $s, t \in \text{Std}(\Lambda)$ then $\langle \psi_s, \psi_t \rangle = \psi_t \circ \psi^{s}(y_1^{n-1} y_2^{n-2} \ldots y_{n-1})$, by (3.7.2) and Example 3.2.5. By Proposition 2.5.2, $\psi_t \circ \psi^{s}(y_1^{n-1} y_2^{n-2} \ldots y_{n-1}) = 0$ if $u \neq t$ and $\psi_t \circ \psi^{s}(y_1^{n-1} y_2^{n-2} \ldots y_{n-1}) = (-1)^{n(n-2)/2} \psi_{t \circ s}$, by (3.7.2). Hence, $\langle \psi_s, \psi_t \rangle = \delta_{st}$, where $t' = t \circ d(t)$ is the tableau that is conjugate to $t$. Hence, $G^\alpha_2$ is $(-1)^{n(n-2)/2}$ times the anti-diagonal identity matrix. Consequently, $D^\alpha_2 = S^\alpha_2$ and $S^\alpha_2$ is irreducible for any field $F$. \hfill $\Box$

**3.7.9 Example** Suppose $e = 2$, $\Lambda = \Lambda_0$ and $\lambda = (2, 2, 1)$. Then $\text{Std}(\lambda)$ contains the five tableaux:

| $t_1$ | $t_2$ | $t_3$ | $t_4$ | $t_5$ |
|-------|-------|-------|-------|-------|
| $t$   | $t^\lambda$ | $t^2$ | $t^3$ | $t^4$ |
| $d(t)$ | 1 | 2 | 3 | 4 | 5 |
| $\deg t$ | 1 | 2 | 3 | 4 | 5 |

We want to compute the Gram matrix $G^\lambda_2 = \langle \langle \psi, \psi \rangle \rangle$. Now $\langle \psi_s, \psi_t \rangle \neq 0$ only if $t = s$, by Proposition 3.2.10(a), and only if $\deg s + \deg t = 0$, since the bilinear form is homogeneous of degree zero. Hence, the only possible non-zero inner products are

$$\langle \psi_t \circ \psi_s \rangle = \langle \psi_t, \psi_s \rangle,$$

together with $\langle \psi_t \circ \psi_s \rangle = \langle \psi_t, \psi_s \rangle$ and $\langle \psi_t \circ \psi_s \rangle$. If $a \in \{2, 4\}$ then

$$\langle \psi_t \circ \psi_s \rangle = \langle \psi_t, \psi_s \rangle = \langle \psi_t, \psi_s \rangle = \langle \psi_t, \psi_s \rangle = 0,$$

because $\psi_t \circ y_r = 0$ by (2.2.3), for $1 \leq r \leq 5$. To compute the remaining inner products we have to go back to the definition of the bilinear form (3.7.2). By Definition 3.2.2, $y^2 = y_2 y_4$ so

$$\langle \psi_t \circ \psi_s \rangle = \psi_t \circ \psi_s \circ \psi_2 y_4 y_2 = \psi_t \circ \psi_s \circ \psi_2 y_4 y_2 = \psi_t \circ \psi_s (y_3 y_2 + 1) = \psi_t \circ \psi_s,$$

by Proposition 3.2.10(c). Hence, $\langle \psi_t, \psi_s \rangle = 1 = \langle \psi_t \circ \psi_s \rangle$. Finally,

$$\langle \psi_t \circ \psi_s \rangle = \langle \psi_t \circ \psi_s \circ \psi_2 y_4 y_2 y_4 = \psi_t \circ \psi_s \circ \psi_2 y_4 y_2 y_4 = \psi_t \circ \psi_s \circ \psi_2 y_4 y_2 y_4,$$

where the second equality uses (2.2.3). Now $\psi_t \circ \psi_2 y_3 = \psi_t (y_2 \psi_1 + 1) = \psi_t$ and, similarly, $\psi_2 y_4 \psi_4 = -\psi_4$. Consequently $\psi_t \circ \psi_2 y_3^2 = 0$, for $a = 3, 4$, so it follows that $\psi_t \circ \psi_2 y_3 \psi_3^2 = -2\psi_2$ and hence that $\psi_t \circ \psi_3 = -2$. Therefore, the Gram matrix of $S^{(2, 2, 1)}$ is

$$G^\lambda_2 = \begin{pmatrix}
0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2
\end{pmatrix}.$$
Williamson’s counter-examples to the James and Lusztig conjectures suggest that there is no block theoretic criterion for the adjustment matrix of a block to be trivial, except asymptotically where the Lusztig conjecture is known to hold [1]. With hindsight, perhaps this is not so surprising because the condition given in Corollary 1.7.6 for a Specht module to be irreducible is rarely a block invariant. The failure of the James and Lusztig conjectures suggests that we should, instead, look for necessary and sufficient conditions for the \( \mathcal{B}_n(F) \)-modules \( D^\mu \otimes F \) to be irreducible, for \( \mu \in K^\Lambda_n \). Some steps towards such a criterion are made in Conjecture 4.4.1 below.
Brundan and Kleshchev [22, §5.6] remarked that $a^F_{\lambda\mu}(q) \in \mathbb{N}$ in all of the examples that they had computed. They asked whether this might always be the case. The next examples show that, in general, $a^F_{\lambda\mu}(q) \notin \mathbb{N}$.

**Example** (Evseev [35, Corollary 5]) Suppose that $e = 2$, $\Lambda = \mathcal{A}$ and let $\lambda = (3, 2^2, 1^2)$ and $\mu = (1^9)$. Take $F = \mathbb{F}_2$ to be a field of characteristic 2 and let $a^F_{\lambda\mu} = (a^F_{\lambda\mu}(q))$ be the corresponding adjustment matrix.

As part of a general argument Evseev shows that $a^F_{\lambda\mu}(q) \notin \mathbb{N}$. In fact, it is not hard to see directly that $a^F_{\lambda\mu}(q) = q + q^{-1}$. Comparing the decomposition matrix for $\mathbb{F}_2\mathcal{S}_9$ given by James [59] with the graded decomposition matrices when $e = 2$ given in [104], shows that $d^F_{\lambda\mu} = 0$, and that $a^F_{\lambda\mu}(1) = 2$. Now $D^\mu = D^\mu_{\mathbb{F}_2} e(i^\mu)$ is one dimensional, so any composition factor of $S^\lambda_{\mathbb{F}_2}$ that is isomorphic to $D^\mu_{\mathbb{F}_2} (d)$, for some $d \in \mathbb{Z}$, must be contained in $S^\lambda_{\mathbb{F}_2} e(i^\mu)$. There are exactly six standard $\lambda$-tableaux with residue sequence $i^\mu$, namely:

$$\begin{array}{cccccc}
\text{deg} t & 1 & 1 & 1 & 1 & 1 \\
\text{t} & 1 & 6 & 9 & 14 & 9 & 14 & 5 & 12 & 3 & 14 & 7 & 14 & 7 \\
2 & 7 & 2 & 5 & 2 & 7 & 4 & 7 & 5 & 5 & 5 & 5 & 5 & 5 \\
3 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
4 & 9 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
5 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
\end{array}$$

As $D^\mu$ is one dimensional, and concentrated in degree zero, it follows that $a_{\lambda\mu} = a^F_{\lambda\mu}(q) = q + q^{-1}$. We can see a shadow of the adjustment matrix entry in the Gram matrix of $S^\lambda_{\mathbb{F}_2} e(i^\mu)$, that is equal to

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}$$

The elementary divisors of this matrix are $2, 2, 0, 0, 0, 0$, with the $2$’s in degrees $\pm 1$. Therefore, the graded dimension of $D^\mu_{\mathbb{F}_2} e(i^\mu)$ decreases by $q + q^{-1}$ in characteristic 2.

**Example** (Motivated by the runner removable theorems of [27, 63] and Example 3.7.12, take $e = 3$, $F = \mathbb{F}_2$, $\lambda = (3, 2^4, 1^3)$ and $\mu = (1^14)$. (The partitions $\lambda$ and $\mu$ are obtained from the corresponding partitions in Example 3.7.12 by conjugating, adding an empty runner, and then conjugating again.) Again, we work over $\mathbb{F}_2$ and consider the corresponding adjustment matrices.

Calculating with Specht [102] we find that $d^\mu_{\lambda\mu} = 0$ and that $a^F_{\lambda\mu}(q) = q + q^{-1}$. Once again, it turns out that there are six $\lambda$-tableaux with 3-residue sequence $i^\mu$, with five of these having degree 1 and one having degree $-1$. (Moreover, the Gram matrix of $S^\lambda e(i^\mu)$ is the same as the Gram matrix given in Example 3.7.12.) Hence, as in Example 3.7.12, $a^F_{\lambda\mu}(q) = q + q^{-1} = d^F_{\lambda\mu}(q)$.

As the runner removable theorems compare blocks for different $e$ over the same field we cannot expect to find an example of a non-polynomial adjustment matrix entry in odd characteristic in this way. Nonetheless, it seems fairly certain that non-polynomial adjustment matrix entries exist for all $e$ and all $p > 0$.

Evseev [35, Corollary 5] gives three other examples of adjustment matrix entries that are equal to $q + q^{-1}$ when $e = p = 2$. All of them have similar analogues when $e = 3$ and $p = 2$. Finally, if we try adding further empty runners to the partitions $\lambda$ and $\mu$, so that $e \geq 4$, then the corresponding adjustment matrix entry is zero (all of these partitions have weight 4).

4. Seminormal bases and the KLR grading

In this final section we link the KLR grading on $\mathcal{H}_n^\Lambda$ with the semisimple representation theory of $\mathcal{H}_n^\Lambda$ using the seminormal bases. We start by showing that by combining information from all of the KLR gradings for different cyclic quivers leads to an integral formula for the Gram determinants of the ungraded Specht modules.

4.1. Gram determinants and graded dimensions. In Theorem 1.7.3 we gave a “rational” formula for the Gram determinant of the ungraded Specht modules $S^\lambda$, for $\lambda \in \mathcal{P}_n^\Lambda$. We now give an integral formula for these determinants and give both a combinatorial and a representation theoretic interpretation of this formula.

Suppose that the Hecke parameter $v$ from Definition 1.1.1 is an indeterminate over $\mathbb{Q}$ and consider an integer cyclotomic Hecke algebra $\mathcal{H}_n^\Lambda$ over the field $\mathbb{Z} = \mathbb{Q}(v)$ where $\Lambda \in \mathbb{P}^+$ such that $e > n$ and $(\Lambda, \alpha_{i,n}) \leq 1$, for all $i \in I$. Then $\mathcal{H}_n^\Lambda$ is semisimple by Corollary 1.6.11.

4.1.1. Definition. Suppose that $\lambda \in \mathcal{P}_n^\Lambda$. For $e \geq 2$ and $i \in I_e^\lambda$ define

$$\deg_{e,i}(\lambda) = \sum_{t \in \text{Std}_i(\lambda)} \deg_t(\lambda),$$

where $\text{Std}_i(\lambda) = \{ t \in \text{Std}(\lambda) \mid i^\lambda = i \}$. Set $\deg(\lambda) = \sum_{i \in I_e^\lambda} \deg_{e,i}(\lambda)$. For a prime integer $p > 0$ set $\text{Deg}_p(\lambda) = \sum_{k \geq 1} \deg_{p^k}(\lambda)$. 


By definition, \( \deg_e(\lambda), \deg_p(\lambda) \in \mathbb{Z} \). For \( e > 0 \) let \( \Phi_e(x) \in \mathbb{Z}[x] \) be the \( e \)th cyclotomic polynomial in the indeterminate \( x \).

4.1.2. **Theorem** (Hu-Mathas [57, Theorem C]). Suppose that \( \Lambda \in P^+ \), \( e > n \) and that \( (\Lambda, \alpha_i, n) \leq 1 \), for all \( i \in I \). Let \( \lambda \in P^+_n \). Then
\[
\det G^\lambda = \prod_{e > 1} \Phi_e(v^\lambda)^{\deg_e(\lambda)}.
\]

Consequently, if \( e = 1 \) then \( \det G^\lambda = \prod_{p \text{ prime}} p^{\deg_p(\lambda)} \).

Proving this result is not hard: it amounts to interpreting Definition 1.6.6 in light of the KLR degree functions on \( \text{Std}(\lambda) \). There is a power of \( v \) in the statement of this result in [57]. This is not needed here because we have renormalised the quadratic relations in the Hecke algebra given in Definition 1.1.1.

The Murphy basis is defined over \( \mathbb{Z}[v, v^{-1}] \). Therefore, \( \det G^\lambda \in \mathbb{Z}[v, v^{-1}] \) and Theorem 4.1.2 implies that \( \deg_e(\lambda) \geq 0 \) for all \( \lambda \in P^+_n \) and \( e \geq 2 \). In fact, [57, Theorem 3.24] gives an analogue of Theorem 4.1.2 for the determinant of the Gram matrix restricted to \( \bigoplus \mathbb{C} v(i) \), suitably interpreted, and the following is true:

4.1.3. **Corollary** ([57, Corollary 3.25]). Suppose that \( e \geq 2 \), \( \lambda \in P^+_n \) and \( i \in I^e_0 \). Then \( \deg_e(i, \lambda) \geq 0 \).

The definition of the integers \( \deg_e(i, \lambda) \) is purely combinatorial, so it should be possible to give a combinatorial proof of this result perhaps using Theorem 3.4.6. We think, however, that this is probably difficult.

Fix an integer \( e \geq 2 \) and a dominant weight \( \lambda \in P^+ \) and consider the Hecke algebra \( \mathcal{H}_n^A \) over a field \( F \). If \( \lambda \in P^+_n \) then, by definition,
\[
\text{Ch}_q S^\lambda = \sum_{\mu \in K^+_\lambda} d_{\lambda\mu}(q) \text{Ch}_q D^\mu \in A[I^n].
\]

Let \( \partial : A[I^n] \to \mathbb{Z}[I^n] \) be the linear map given by \( \partial(f(q) \cdot i) = f'(1)i \), where \( f'(1) \) is the derivative of \( f(q) \in A \) evaluated at \( q = 1 \). Then \( \partial \text{Ch}_q S^\lambda = \sum_{\lambda \in \Lambda} \deg_e(i, \lambda) \cdot i \). The KLR idempotents are orthogonal, so \( \dim_{\mathbb{C}} D^\mu = \dim_{\mathbb{C}} D^\mu \) since (\( D^\mu \) is independent of \( \mu \)). Therefore, \( \partial \text{Ch}_q D^\mu = 0 \). Hence, applying \( \partial \) to the formula for \( \text{Ch}_q S^\lambda \) shows that
\[
\sum_{\mu \in K^+_\lambda} \deg_e(i, \lambda) \cdot i = \partial \text{Ch}_q S^\lambda = \sum_{\lambda \in \Lambda} \sum_{\mu \in K^+_\lambda} d_{\lambda\mu}(1) \dim D^\mu_i \cdot i.
\]

Consequently, \( \deg_e(i, \lambda) = \sum_{\mu \in K^+_\lambda} d_{\lambda\mu}(1) \dim D^\mu_i \). So we have worked over an arbitrary field. If \( F = \mathbb{C} \) then \( d_{\lambda\mu}(q) \in \mathbb{N}[q] \), by Proposition 3.5.6, so that \( d_{\lambda\mu}(1) \geq 0 \). Therefore, \( \deg_e(i, \lambda) \geq 0 \) as claimed. (In fact, by Theorem 3.7.6, the right-hand side of (4.1.4) is independent of \( F \), as it must be.)

Theorem 1.7.4 shows that taking the \( p \)-adic valuation of the Gram determinant of \( S^\lambda \) leads to the Jantzen sum formula for \( S^\lambda \). Therefore, (4.1.4) suggests that
\[
\sum_{k > 0} |J_k(S^\lambda)| = \sum_{\mu \gg \lambda} d_{\lambda\mu}(1) \dim D^\mu_i,
\]
where we use the notation of Theorem 1.7.4. That is, Theorem 4.1.2 corresponds to writing the Jantzen sum formula as a non-negative linear combination of simple modules. In fact, we have not done enough to prove (4.1.5). (One way to do this would be to establish analogous statements for the Gram determinants of the Weyl modules of the cyclotomic Schur algebras [31].) Nonetheless, (4.1.5) is true, being proved by Ryom-Hansen [124, Theorem 1] in level one and by Yvonne [138, Theorem 2.11] in general.

A better interpretation of (4.1.4) is in terms of grading filtrations [15, §2.4]. Let \( \mathcal{H}_n^A = \text{Hom}_{\mathcal{A}_n^A}(Y, Y) \), where \( Y = \bigoplus_{\mu \in K^+_\lambda} Y^\mu \) is a progenitor for \( \mathcal{H}_n^A \). Then \( \mathcal{H}_n^A \) is a graded algebra for \( \mathcal{A}_n^A \) and the functor
\[
F_n : \text{Rep}(\mathcal{H}_n^A) \to \text{Rep}(\mathcal{A}_n^A) ; M \mapsto \text{Hom}_{\mathcal{A}_n^A}(Y, M), \quad \text{for } M \in \text{Rep}(\mathcal{A}_n^A),
\]
is a graded Morita equivalence; see, for example, [55, §2.3-2.4]. Recall that \( c_\lambda(q) = (c_{\lambda\mu}(q)) = d_{\lambda}^T \circ d_\lambda(q) \) is the Cartan matrix of \( \mathcal{H}_n^A \). By Corollary 2.1.6, \( c_{\lambda\mu}(q) = \dim_{\mathbb{C}} \text{Hom}_{\mathcal{A}_n^A}(Y^\lambda, Y^\mu) \) so that
\[
\dim_{\mathbb{C}} \mathcal{H}_n^A = \sum_{\lambda, \mu \in K^+_\lambda} c_{\lambda\mu}(q) \in \mathbb{N}[q, q^{-1}].
\]

For the rest of this section assume that \( F = \mathbb{C} \). Then \( c_{\lambda\mu}(q) \in \mathbb{N}[q] \) by Corollary 3.5.27. Therefore, \( \dim_{\mathbb{C}} \mathcal{H}_n^A \in \mathbb{N}[q] \) so that \( \mathcal{H}_n^A \) is a positively graded algebra. Let \( M = \bigoplus_{d \in \mathbb{N}} M_d \) be a graded \( \mathcal{H}_n^A \)-module. The grading filtration of \( M \) is the filtration \( M = G_d(M) \supseteq G_{d+1}(M) \supseteq \cdots \supseteq G_{s}(M) \supseteq 0 \), where
\[
G_d(M) = \bigoplus_{k \geq d} M_k.
\]
Then $G_r(M)$ is a graded $A^\Lambda_n$-module precisely because $A^\Lambda_n$ is positively graded. The grading filtration of an $A^\Lambda_n$-module $M$ is the filtration given by $G_r(M) = F_{r+1}^n(G_r(F_n(M)))$, for $r \in \mathbb{Z}$. By Corollary 3.6.7, $S^\Lambda = G_0(S^\Lambda)$ and $G_r(S^\Lambda) = 0$ for $r > \text{def } \Lambda$.

For $\lambda \in \mathcal{P}^\Lambda_n$ and $\mu \in \mathcal{K}^\Lambda_n$ write $d_{\lambda\mu}(q) = \sum_{r \geq 0} d_{\lambda\mu}(r) q^r$, for $d_{\lambda\mu}(r) \in \mathbb{N}$.

4.1.6. Lemma. Suppose that $F = \mathbb{C}$ and $\lambda \in \mathcal{P}^\Lambda_n$. If $0 \leq r \leq \text{def } \lambda$ then

$$G_r(S^\Lambda)/G_{r+1}(S^\Lambda) \cong \bigoplus_{\mu \in \mathcal{K}^\Lambda_n} (D^\mu(s))^\otimes d_{\lambda\mu}^{(r)}.$$  

Proof. This is an immediate consequence of the definition of the grading filtration and Corollary 3.5.27. \hfill \Box

Comparing this with (4.1.5) suggests that $J_r(S^\Lambda) = G_r(S^\Lambda)$, for $r \geq 0$. Of course, there is no reason to expect that $J_r(S^\Lambda)$ is a graded submodule of $S^\Lambda$. Nonetheless, establishing a conjecture of Rouquier [89, (16)], Shan has proved the following when $\Lambda$ is a weight of level 1.

4.1.7. Theorem ([Shan [126, Theorem 0.1]]). Suppose that $F$ is a field of characteristic zero, $\Lambda = \Lambda_0$, and that $\lambda \in \mathcal{P}^\Lambda_n$. Then $J_r(S^\Lambda) = G_r(S^\Lambda)$ is a graded submodule of $S^\Lambda$ and $[J_r(S^\Lambda)/J_{r+1}(S^\Lambda) : D^\mu(s)] = \delta_{rs} d_{\lambda\mu}^{(r)}$, for all $\mu \in \mathcal{K}^\Lambda_n$ and $r \geq 0$.

Shan actually proves that the Jantzen, radical and grading filtrations of graded Weyl modules coincide for the Dipper-James $\nu$-Schur algebras [30]. This implies the result above because the Schur functor maps Jantzen filtrations of Weyl modules to Jantzen filtrations of Specht modules. There is a catch, however, because Shan remarks that it is unclear how her geometrically defined grading relates to the grading on the $\nu$-Schur algebra given by Ariki [7] and hence to the KLR grading on $A^\Lambda_n$. As we now sketch, Theorem 4.1.7 can be deduced from Shan’s result using recent work.

Since Shan’s paper cyclotomic quiver Schur algebras have been introduced for arbitrary dominant weights [7, 55, 128], thus giving a grading on all of the cyclotomic Schur algebras introduced by Dipper, James and the author [31]. The key point, which is non-trivial, is that the module categories of the cyclotomic quiver Schur algebras are Koszul. When $e = \infty$ this is proved in [55] by reducing to parabolic category $\mathcal{O}$ for the general linear groups, which is known to be Koszul by [14, 15]. Using similar ideas, Maksimau [99] proves that Stroppel and Webster’s cyclotomic quiver Schur algebras are Koszul for arbitrary $e$ by using [123] to reduce to affine parabolic category $\mathcal{O}$.

As the module categories of the cyclotomic quiver Schur are Koszul, an elementary argument [15, Proposition 2.4.1] shows that the radical and grading filtrations of the graded Weyl modules of these algebras coincide. By definition, the analogue of Lemma 4.1.6 describes the graded composition factors of the grading (=radical) filtrations of the graded Weyl modules — compare with [55, Corollary 7.24] when $e = \infty$ and [99, Theorem 1.1] in general. The graded Schur functors of [55, 99] send graded Weyl modules to graded Specht modules, graded simple modules to graded simple $A^\Lambda_n$-modules (or zero), grading filtrations to grading filtrations and Jantzen filtrations to Jantzen filtrations. Combining these facts with Shan’s work [126] implies Theorem 4.1.7 when $\Lambda = \Lambda_0$. We note that the $\nu$-Schur algebras were first shown to be Koszul by Shan, Varagnolo and Vasserot [127]. It is also possible to match up Shan’s grading on the $\nu$-Schur algebras with the gradings of [7, 128] using the uniqueness of Koszul gradings [15, Proposition 2.5.1]. As these papers use different conventions, it is necessary to work with the graded Ringel dual.

The obstacle to extending Theorem 4.1.7 to arbitrary weights $\Lambda \in \mathbb{P}^+$ is in showing that the Jantzen and radical (=grading) filtrations of the graded Weyl modules of the cyclotomic quiver Schur algebras coincide. As the cyclotomic quiver Schur algebras are Koszul it is possible that this is straightforward. It seems to the author, however, that it is necessary to generalize Shan’s arguments [126] to realize the Jantzen filtration geometrically using the language of [123].

4.2. A deformation of the KLR grading. Following [57], especially the appendix, we now sketch how to use the seminormal basis to prove that $\mathcal{H}^\Lambda_n \cong \mathcal{H}^\Lambda_n$ over a field $F$ that has Hecke parameter $\nu \in F^\times$ of quantum characteristic $e \geq 2$. As in §1.2, $\Lambda \in \mathbb{P}^+$ is determined by a multicharge $\kappa \in \mathbb{Z}^\vee$. We set up a modular system for studying $\mathcal{H}^\Lambda_n = \mathcal{H}^\Lambda_n(F)$.

Let $\nu$ be an indeterminate over $F$ and let $\mathcal{O} = F[[\nu]]$ be the localization of $F[x]$ at the principal ideal generated by $x$. Let $K = F(x)$ be the field of fractions of $\mathcal{O}$. Let $\mathcal{H}^\mathcal{O}$ be the cyclotomic Hecke algebra with Hecke parameter $t = x + \nu$, a unit in $\mathcal{O}$, and cyclotomic parameters $Q_l = x^l + [\kappa l]_t$, for $1 \leq l \leq \ell$. Then
$\mathcal{H}_n^K = \mathcal{H}_n^\mathcal{O} \otimes \mathcal{O} K$ is a split semisimple algebra by Theorem 2.4.8. Moreover, by definition, $\mathcal{H}_n^\Lambda = \mathcal{H}_n^\Lambda(F) \cong \mathcal{H}_n^\mathcal{O} \otimes \mathcal{O} F$, where we consider $F$ as an $\mathcal{O}$-module by letting $x$ act as multiplication by $0$.

As the algebra $\mathcal{H}_n^K$ is semisimple, it has a seminormal basis $\{f_a\}$ in the sense of Definition 1.6.4. With our choice of parameters, the content functions from (1.6.1) become

$$c^Z_i(s) = t^{2(c-b)x^{l}} + [\kappa_l + c - b]t = t^{2(c-b)x^{l}} + [c^Z_i(s)]_l$$

if $s(l, b, c) = r$, for $1 \leq k \leq n$. Then, $L_rf_{st} = c^Z_i(s)f_{st}$, for $(s, t) \in \text{Std}(\mathcal{P}_n^\Lambda)$. By Corollary 1.6.9, the basis $\{f_{st}\}$ determines a seminormal coefficient system $\alpha = \{\alpha_r(t) \mid t \in \text{Std}(\mathcal{P}_n^\Lambda) \text{ and } 1 \leq r < n\}$ and a set of scalars $\{\kappa_l \mid t \in \text{Std}(\mathcal{P}_n^\Lambda)\}$.

For $i \in I^n$ let $\text{Std}(i) = \{s \in \text{Std}(\mathcal{P}_n^\Lambda) \mid p = i\}$ be the set of standard tableaux with residue sequence $i$. Define

\begin{equation}
(4.2.1)
J^O_i = \sum_{t \in \text{Std}(i)} F_t.
\end{equation}

By definition, $J^O_i \in \mathcal{H}_n^K$ but, in fact, $J^O_i \in \mathcal{H}_n^\mathcal{O}$. This idempotent lifting result dates back to Murphy [112] for the symmetric groups. For higher levels it was first proved in [108]. In [57] it is proved for a more general class of rings $\mathcal{O}$.

4.2.2. **Lemma** ([57, Lemma 4.4]). Suppose that $i \in I^n$. Then $J^O_i \in \mathcal{H}_n^\mathcal{O}$.

We will see that $J^O_i \otimes \mathcal{O} 1_F$ is the KLR idempotent $e(i)$, for $i \in I^n$. Notice that $1 = \sum_i J^O_i$ and, further, that $f^O J^O_i = \delta_i J^O_i$ for $i, j \in I^n$, by Theorem 1.6.7.

As detailed after Theorem 3.1.1, Brundan and Kleshchev construct their isomorphisms $\mathcal{H}_n^\Lambda \xrightarrow{\sim} \mathcal{H}_n^\Lambda$ using certain rational functions $P_i(\mathcal{I})$ and $Q_i(\mathcal{I})$ in $F[y_1, \ldots, y_n]$. The advantage of working with seminormal forms is that, at least intuitively, these rational functions “converge” and can be replaced with “nicer” polynomials. The main tool for doing this is the following result, generalizing Lemma 4.2.2.

Let $M_r = 1 - t^{-1}L_r + L_r + 1$, for $1 \leq r < n$. Then $M_r f_{st} = M_r^2(s) f_{st}$, where $M_r^2(s) = 1 - t^{-1} c^Z_i(s) + t c^Z_{i+1}(s)$. The constant term of $M_r^2(s)$ is equal to $v^2c^Z_i(1) - c^Z_i(s) + c^Z_{i+1}(s) 
eq 0$. Consequently, $M_r$ acts invertibly on $f_{st}$ whenever $s \in \text{Std}(i)$ and $1 - i_r + i_{r+1} \neq 0$ in $I = \mathbb{Z}/e\mathbb{Z}$. This observation is part of the proof of part (a) of the next result. Similarly, set $p^O_r(i) = c^Z_i(s) - c^Z_{i+1}(s)$. Then $p^O_r(i)$ is invertible in $\mathcal{O}$ if $i_r \neq i_{r+1}$.

4.2.3. **Corollary** ([Hu-Mathas [57, Corollary 4.6]). Suppose that $1 \leq r < n$ and $i \in I^n$.

a) If $i_r \neq i_{r+1} + 1$ then

$$\frac{1}{M_r} J^O_i = \sum_{s \in \text{Std}(i)} \frac{1}{M_r^2(s)} F_s \in \mathcal{H}_n^\mathcal{O}.\
$$

b) If $i_r \neq i_{r+1} + 1$ then

$$\frac{1}{L_r - L_{r+1}} J^O_i = \sum_{s \in \text{Std}(i)} \frac{1}{L_r^2(s)} F_s \in \mathcal{H}_n^\mathcal{O}.\
$$

The invertibility of $M_r J^O_i$, when $i_r \neq i_r + 1 + 1$, allows us to define analogues of the KLR generators of $\mathcal{H}_n^\Lambda$ in $\mathcal{H}_n^\mathcal{O}$. The invertibility of $(L_r - L_{r+1}) J^O_i$ is needed to show that these new elements generate $\mathcal{H}_n^\mathcal{O}$.

Define an embedding $I \hookrightarrow \mathbb{Z}$, $i \mapsto i$ by letting $i$ be the smallest non-negative integer such that $i = i + e\mathbb{Z}$, for $i \in I$.

4.2.4. **Definition.** Suppose that $1 \leq r < n$. Define elements $\psi^O_r = \sum_{i \in I^n} \psi^O_r J^O_i$ in $\mathcal{H}_n^\mathcal{O}$ by

$$\psi^O_r J^O_i = \begin{cases} (T_r + t^{-1}) \frac{1}{L_r} J^O_i, & \text{if } i_r = i_{r+1}, \\ (T_r L_r - L_r T_r) t^{-2i_r} J^O_i, & \text{if } i_r = i_{r+1} + 1, \\ (T_r L_r - L_r T_r) \frac{1}{L_r} J^O_i, & \text{otherwise}. \end{cases}$$

If $1 \leq r \leq n$ then define $\psi^O_r = \sum_{i \in I^n} t^{-2i_r} (L_r - [i_r]) J^O_i$.

We now describe an $\mathcal{O}$-deformation of cyclotomic KLR algebra $\mathcal{H}_n^\Lambda$. This is a special case of one of the main results of [57], which allows greater flexibility in the choice of the ring $\mathcal{O}$.

4.2.5. **Theorem** ([Hu-Mathas [57, Theorem A]). As an $\mathcal{O}$-algebra, the algebra $\mathcal{H}_n^\mathcal{O}$ is generated by the elements

$$\{ J^O_i \mid i \in I^n \} \cup \{ \psi^O_r \mid 1 \leq r < n \} \cup \{ y^O_r \mid 1 \leq r \leq n \}$$

subject only to the following relations:

$$\prod_{1 \leq \ell \leq n} (y^O_r - x^{l} - [\kappa_l - i_l]) J^O_i = 0,$$

$$\psi^O_r J^O_i = \delta_i J^O_i,$$

$$\sum_{i \in I^n} J^O_i = 1,$$

$$y^O_r J^O_i = J^O_i y^O_r,$$
Theorem 4.2.5 is slightly different to [57, Theorem A] because we are using a different choice of modular system \((K, O, F)\) and because Definition 1.1.1 renormalises the quadratic relations for the generators \(T_r\) of \(H^O\), for \(1 \leq r < n\).

The strategy behind the proof of Theorem 4.2.5 is quite simple: we compute the action of the elements defined in Definition 4.2.4 on the seminormal basis and use this to verify that they satisfy the relations in the theorem. To bound the rank of the algebra defined by the presentation in Theorem 4.2.5 we essentially count dimensions. By specializing \(x = 0\), we obtain Theorem 3.1.1 as a corollary of Theorem 4.2.5.

To give a flavour of the type of calculations that were used to verify that the elements in Definition 4.2.4 satisfy the relations in Theorem 4.2.5, for \(s \in \text{Std}(l)\) and \(1 \leq r < n\) define

\[
\beta_r(s) = \begin{cases} 
\frac{\alpha_r(s) \rho_r^2(s)}{M_r^Z(s)}, & \text{if } i_r = i_{r+1}, \\
\frac{\alpha_r(s) \rho_r^2(s)}{M_r^Z(s)}, & \text{if } i_r = i_{r+1} + 1, \\
\frac{\alpha_r(s) \rho_r^2(s)}{M_r^Z(s)}, & \text{otherwise},
\end{cases}
\]

(4.2.6)

Then Theorem 1.6.7 easily yields the following.

4.2.7. **Lemma.** Suppose that \(1 \leq r < n\) and that \((s, t) \in \text{Std}^2(D_n^\Lambda)\). Set \(i = i', j = i', u = s(r, r+1)\) and \(v = t(r, r+1)\). Then

\[
\psi_r^O f_{st} = \beta_r(s) f_{st} - \delta_{i_r, i_{r+1}} \frac{1}{\rho_r^2(s)} f_{st}.
\]

Moreover, if \(s(l, b, c) = r\) then

\[
y_r^{(d)} f_{st} = t^{-1} \left( t^{2(c-b-d-i_r)} x^l + \left[ c_r^2(s) + d - i_r\right] \right) f_{st},
\]

for \(1 \leq r \leq n\) and \(d \in \mathbb{Z}\).

Armed with Lemma 4.2.7, and Definition 1.6.6, it is an easy exercise to verify that all of the relations in Theorem 4.2.5 hold in \(H^O\). For the quadratic relations, Lemma 4.2.7 implies that \((\psi_r^O)^2 f_{st} = 0\) if \(s \in \text{Std}(l)\) and \(i_r = i_{r+1}\) whereas if \(i_r \neq i_{r+1}\) then \((\psi_r^O)^2 f_{st} = \beta_r(s) \beta_r(u) f_{st}\), where \(u = s(r, r+1)\). The quadratic relations in Theorem 4.2.5 now follow using (4.2.6) and Lemma 4.2.7. For example, suppose that \(i_r \rightarrow i_{r+1}\) and \(s \in \text{Std}(l)\). Pick nodes \((l, b, c)\) and \((l', b', c')\) such that \(s(l, b, c) = r\) and \(s(l', b', c') = r+1\). Then, using Lemma 4.2.7 and Definition 1.6.6,

\[
(\psi_r^O)^2 f_{st} = t^{-2i_{r+1}} \beta_r(s) \beta_r(u) f_{st} = t^{-2i_r} M_r^Z(u) f_{st}.
\]

On the other hand, by Lemma 4.2.7, \((y_r^{(1+\rho_r(i))} - y_{r+1}^O)\) acts on \(f_{st}\) as multiplication by the same scalar. It follows that

\[
(\psi_r^O)^2 f_{1}^O = (\psi_r^O)^2 \sum_{s \in \text{Std}(l)} \frac{1}{\gamma_s} f_{st} = (y_r^{(1+\rho_r(i))} - y_{r+1}^O) \sum_{s \in \text{Std}(l)} \frac{1}{\gamma_s} f_{st} = (y_r^{(1+\rho_r(i))} - y_{r+1}^O) f_{1}^O,
\]

when \(i_r \rightarrow i_{r+1}\). These calculations are perhaps not very pretty, but nor are they difficult. As indicated by Remark 2.2.6, the quadratic relations appear in, and simplify, the proof of the deformed braid relations.
4.3. A distinguished homogeneous basis. One of the advantages of Theorem 4.2.5 is that it allows us to
transplant questions about the KLR algebra \( H^\Lambda_n \) into the language of seminormal bases. Definition 1.6.6
defines \( * \)-seminormal bases, which provide a good framework for studying the semisimple cyclotomic Hecke
algebras. The algebra \( H^\Lambda_n \) comes with two cellular algebra automorphisms, \( \ast \) and \( \ast \), where \( \ast \) is the unique
anti-isomorphism fixing the homogeneous generators of Definition 2.2.9 and \( \ast \) is the unique anti-isomorphism
fixing the generators of Definition 1.1.1. In general, these automorphisms are different.

4.3.1. Definition (Hu-Mathas [57, §5]). A \( * \)-seminormal coefficient system is a collection of scalars
\( \beta = \{ \beta_i(t) \mid t \in \text{Std}(P_n^\Lambda) \text{ and } 1 \leq r \leq n \} \)
such that \( \beta_i(t) = 0 \) if \( i = t(r, r + 1) \) is not standard, if \( i \in \text{Std}(P_n^\Lambda) \) then \( \beta_i(v) \beta_i(t) \) is given by the product of
the particular \( \beta \)-coefficients in (4.2.6), and if \( \beta_i(t) = 1 \) then \( \beta_i(t) = \beta_i(t) \beta_i(t_r s_r + 1) = \beta_i(t) \beta_i(t_r s_r + 1) \beta_i(t_n s_n) \), and if \( |r - r'| > 1 \) then \( \beta_i(t) \beta_i(t_r s_r) = \beta_i(t) \beta_i(t_n s_n) \) for \( 1 \leq r, r' < n \).

Exactly as in Corollary 1.6.9, a \( * \)-seminormal coefficient system determines a \( * \)-seminormal basis \( \{ f_{st} \} \)
that, similar to Definition 1.6.4, consists of elements \( f_{st} \in H_{st} \) such that \( f_{st} = f_{ts} \) for \( (s, t) \in \text{Std}^2(P_n^\Lambda) \). The left (and right) the action of \( \psi^O \) on \( f_{st} \) is exactly as in Lemma 4.2.7 but where the coefficients come from an arbitrary \( * \)-seminormal coefficient system \( \beta \).

Definition 4.3.1 gives extra flexibility in choosing a \( * \)-seminormal basis. By [57, (5.8)] there exists a \( * \)-seminormal basis \( \{ f_{st} \} \) such that the \( \psi \)-basis of Theorem 3.2.4 lifts to a \( \psi^O \)-basis \( \{ \psi^O_{st} \} \) with the property that
\[
\psi^O_{st} = f_{st} + \sum_{(u, v) \triangleright (s, t)} r_{uv} f_{uv},
\]
for some \( r_{uv} \in K \). In this way we recover Theorem 3.2.4 and with quicker proof than the original arguments in [54]. Perhaps most significantly, by working with \( H^\Lambda_n \) we can improve upon the \( \psi \)-basis.

4.3.3. Theorem (Hu-Mathas [57, Theorem 6.2, Corollary 6.3]). Suppose that \( (s, t) \in \text{Std}^2(P_n^\Lambda) \). There exists
a unique element \( B^O_{st} \in H^\Lambda_n^O \) such that
\[
B^O_{st} = f_{st} + \sum_{(u, v) \in \text{Std}^2(P_n^\Lambda), (u, v) \triangleright (s, t)} p^O_{uv}(x^{-1}) f_{uv},
\]
where \( p^O_{uv}(x) \in xK[x] \). Moreover, \( \{ B^O_{st} \mid (s, t) \in \text{Std}^2(P_n^\Lambda) \} \) is a cellular basis of \( H^\Lambda_n^O \).

The existence and uniqueness of this basis essentially come down to Gaussian elimination, although for
technical reasons it is necessary to work over the \( xO \)-adic completion of \( O \). Proving that \( \{ B^O_{st} \} \) is a cellular
basis is more involved and, ultimately, this relies on the uniqueness properties of the \( B^O \)-basis elements.

As the \( B^O \)-basis is determined by a \( * \)-seminormal basis, the basis \( \{ B^O_{st} \} \) behaves well with respect to the
KLR grading on \( H^\Lambda_n^A \). The main justification for using this seminormal basis as a proxy for choosing a "nice"
basis for \( H^\Lambda_n^A \), apart from the fact that it works, is that Theorem 2.4.8 shows that the natural homogeneous
basis of the semisimple cyclotomic quiver Hecke algebras is a \( * \)-seminormal basis.

In characteristic zero the non-zero polynomials \( p^O_{uv}(x) \) satisfy
\[
0 < \deg p^O_{uv}(x) \leq \frac{1}{2} \left( \deg u - \deg s + \deg v - \deg t \right),
\]
whenever \( (u, v) \triangleright (s, t) \) by [57, Proposition 6.4]. Moreover, if \( s, t, u, v \) are all standard tableaux of the same
shape then \( p^O_{uv}(x) = p^O_{tv}(x)p^O_{sv}(x) \), where \( 0 < \deg p^O_{tv}(x) \leq \frac{1}{2} \left( \deg u - \deg s \right) \) and \( 0 < \deg p^O_{sv}(x) \leq \frac{1}{2} \left( \deg v - \deg t \right) \),
whenever \( u \triangleright v \) and \( v \triangleright t \), respectively.

As the basis \( \{ B^O_{st} \} \) is defined over \( O \) we can reduce modulo the ideal \( xO \) to obtain a basis \( \{ B^O_{st} \otimes xO, 1_K \} \)
of \( H^\Lambda_n^A \). This basis is hard to compute and we do not know if the elements of \( \{ B^O_{st} \otimes xO, 1_K \} \) are
homogeneous in general. Nonetheless, it is possible to construct a homogeneous basis \( \{ B_{st} \} \) of \( H^\Lambda_n^A \) from
\( \{ B^O_{st} \} \). If \( \Lambda \in P_n^\Lambda \) then define \( B_{st} \lambda \) to be the homogeneous component of \( B^O_{st} \otimes 1_K \) of degree \( 2t \deg \lambda \)

More generally, for \( s, t \in \text{Std}(\Lambda) \) we define \( B_{st} = D_t B_{st} \lambda \). \( D_s, D_t \in H^\Lambda_n^A \) are certain homogeneous
elements in \( H^\Lambda_n^A \). In characteristic zero, \( B_{st} \) is the homogeneous component of \( B^O_{st} \otimes 1_K \) of degree \( \deg \lambda + \deg t \),
and all other components are of larger degree. For any field, by (4.3.2) and Theorem 4.3.3,
\[
B_{st} = \psi_{st} + \sum_{(u, v) \triangleright (s, t)} a_{uv} \psi_{uv},
\]
for some \( a_{uv} \in K \) that are non-zero only if \( i^t = i^u, i^t = i^v \) and \( \deg u + \deg v = \deg s + \deg t \). Therefore, the
\( B \)-basis resolves the ambiguities of Proposition 3.2.10(b). More importantly, we have the following.
4.3.6. Theorem (Hu-Mathas [57, Theorem 6.9]). Suppose that K is a field. Then \{B_{s,t} \mid (s,t) \in \text{Std}^2(P_n^\Lambda)\} is a graded cellular basis of \mathcal{H}_n^\Lambda with weight poset \(P_n^\Lambda, \leq\), cellular algebra automorphism \(\psi\) and with \(\deg B_{s,t} = \deg s + \deg t\), for \((s,t) \in \text{Std}^2(P_n^\Lambda)\). Moreover, if \((s,t) \in \text{Std}^2(P_n^\Lambda)\) then \(B_{s,t} + \mathcal{H}_n^\Lambda\) depends only on \(s\) and \(t\) and not on the choice of reduced expressions for the permutations \(d(s), d(t) \in S_n\).

By construction, the basis \(\{B_{s,t}\}\) depends on the field \(F\). If \(F\) is a field of positive characteristic then \(B_{s,t}\) depends upon the choice of the elements \(D_s\) and \(D_t\), which are uniquely determined modulo the ideal \(\mathcal{H}_n^{>\Lambda}\).

4.4. A simple conjecture. The construction of the basis \(\{B_{s,t}\}\) of \(\mathcal{H}_n^\Lambda\) in Theorem 4.3.3, together with the degree constraints on the polynomials \(p_B^\Lambda(x)\) in (4.3.4), is reminiscent of the Kazhdan-Lusztig basis [73]. There is no known analogue of the Kazhdan-Lusztig bar involution in this setting. On the other hand, we do require that the basis elements \(B_{s,t}\) are homogeneous, which might be an appropriate substitute for being bar invariant in the graded setting. Partly motivated by this analogy with the Kazhdan-Lusztig basis, we now define analogues of cell representations for the \(B\)-basis.

The basis \(\{B_{s,t}\}\) of Theorem 4.3.6 is a graded cellular basis so it defines a new homogeneous basis \(\{B_t \mid t \in \text{Std}(\Lambda)\}\) of the graded Specht module \(S^\Lambda\). Let the pre-order \(\geq_B\) on \(S^\Lambda\) be the transitive closure of the relation \(\geq_B\) where \(t \geq_B v\) if there exists \(a \in \mathcal{H}_n^\Lambda\) such that \(B_{a,t} = \sum_{i} B_{t_i} B_{i,v}\) with \(r_v \neq 0\). (So \(\geq_B\) is reflexive and transitive but not anti-symmetric.) Let \(\sim_B\) be the equivalence relation on \(\text{Std}(\Lambda)\) determined by \(\geq_B\) so that \(t \sim_B v\) if and only if \(t \geq_B v \geq_B t\). For example, \(t^\Lambda \geq_B t \sim_B t_\kappa\), for all \(t \in \text{Std}(\Lambda)\).

Let \(S^\Lambda\) be the set of \(\sim_B\)-equivalence classes in \(\text{Std}(\Lambda)\). The set \(\text{Std}(\Lambda)\) is partially ordered by \(\geq_B\), where \(T \geq_B V\) if \(t \geq_B v\) for some \(t \in T\) and \(v \in V\). Write \(T \geq_B v\) if \(t \geq_B v\) for some \(t \in T\) and \(T \geq_B v\) if \(T \geq_B V\) and \(v \in V\). Define \(S^\Lambda_{\geq}\) to be the vector subspace of \(S^\Lambda\) with basis \(\{B_t \mid T \geq_B V\}\). Similarly, let \(S^\Lambda_{\geq}\) be the vector space with basis \(\{B_t \mid T \geq_B V\}\). The definition of \(\geq_B\) ensures that \(S^\Lambda_{\geq}\) and \(S^\Lambda_{\geq}\) are both graded \(\mathcal{H}_n^\Lambda\)-submodules of \(S^\Lambda\) and that \(S^\Lambda_{\geq}\subseteq S^\Lambda_{\geq}\). Therefore, \(S^\Lambda_{\geq} = \mathcal{H}_n^\Lambda_{\geq}\) is a graded \(\mathcal{H}_n^\Lambda\)-module. By choosing any total order on \(\text{Std}(\Lambda)\) that extends the partial order \(\geq_B\), it is easy to see that \(S^\Lambda\) has a filtration with subquotients being precisely the modules \(S^\Lambda_{\geq}\), for \(T \in \text{Std}(\Lambda)\).

For \(\lambda \in \mathcal{P}_n^\Lambda\) let \(T(\lambda) = \{ t \in \text{Std}(\lambda) \mid t \sim_B t^\Lambda\}\). In view of (3.7.2), if \((s,t) \in S^\Lambda\) and \((B_s,B_t) \neq 0\) then \(s \sim_B t^\Lambda \sim_B t\) so that \(t, s \in T(\lambda)\). Therefore, \(\dim D^\Lambda \leq |T(\lambda)|\). Of course, if \(\lambda \not\in \mathcal{K}_n^\Lambda\) then this bound is not sharp because \(D^\Lambda = 0\) whereas \(|T(\lambda)| \geq 1\).

4.4.1. Conjecture. Suppose that \(F\) is a field of characteristic zero and that \(\lambda \in \mathcal{P}_n^\Lambda\). Then \(S^\Lambda(\lambda)\) is an irreducible \(\mathcal{H}_n^\Lambda\)-module, for all \(T \in \text{Std}(\lambda)\).

As discussed in [57, §3.3], and is implicit in (4.1.4), by fixing a composition series for \(S^\Lambda\) and using a Gaussian elimination argument, it is possible to construct a basis \(\{G_i\}\) of \(S^\Lambda\) such that (1) each module in the composition series has a basis contained in \(\{G_i\}\), and (2), if \(t \in \text{Std}(\lambda)\) then \(C_t = \psi_t + \text{linear combination of \emph{higher terms}}\) with respect to some total order on \(\text{Std}(\lambda)\). This defines a partition of \(\text{Std}(\lambda) = T_1 \sqcup \cdots \sqcup T_{\ell}\) (disjoint union), where the tableaux in the set \(T_\ell\) are in bijection with a basis of the \(k\)th composition factor. Therefore, there exists an equivalence relation on \(\text{Std}(\lambda)\), together with an associated composition series, such that the analogue of Conjecture 4.4.1 holds for this equivalence relation. Our conjecture attempts to make this equivalence relation on \(\text{Std}(\lambda)\) explicit and canonical.

If \(T \subseteq \text{Std}(\lambda)\) define its character to be \(\chi_T = \sum_{i \in T} q^{d_{\text{deg}} T} i \in A[f^\Lambda]\). The point of this definition is that \(\chi_T\) is a purely combinatorial invariant of \(T\). As two examples, \(\chi_T^\lambda = \chi_T \text{Std}(\lambda)\) and \(\chi_T S^\Lambda\) is \(\chi_T\).

4.4.2. Proposition. Suppose that Conjecture 4.4.1 holds when \(F = \mathbb{C}\).

a) Suppose that \(\mu \in \mathcal{K}_n^\Lambda\). Then \(D^\mu = \mathcal{H}_c^\mu + \text{Ch}_h D^\mu\), and \(\text{Ch}_h D^\mu = \chi_T T^\mu\).

b) If \(\lambda \in \mathcal{P}_n^\Lambda\), \(T \subseteq \text{Std}(\lambda)\), then there is a unique pair \((\nu_T, d_T)\) in \(\mathcal{K}_n^\Lambda \times \mathbb{N}\) such that \(\chi_T = q^{d_T} \chi_T D^\nu_T = q^{d_T} \chi_T T^\nu_T\). Moreover,

\[
d_{\chi_T}(q) = \sum_{T \subseteq \text{Std}(\lambda)} q^{d_T}.
\]

Proof. By Corollary 3.2.7, \(D^\mu \neq 0\) since \(\mu \in \mathcal{K}_n^\Lambda\). The irreducible module \(D^\mu\) is generated by \(B_{\nu_T} + \text{rad} S^\mu = \psi_{\nu_T} + \text{rad} S^\mu\), so \(D^\mu \cong S^\mu\), since both modules are irreducible by Conjecture 4.4.1. Hence, \(a)\) follows.

For part \(b)\), \(S^\mu \cong D^\mu\) for some \(\nu \in \mathcal{K}_n^\Lambda\) and \(d \in \mathbb{Z}\), because \(S^\mu\) is irreducible by Conjecture 4.4.1. Therefore, \(\text{Ch}_h S^\mu = q^{d} \chi_T D_\nu^\mu\). The uniqueness of \((\nu_T, d_T) = (\nu, d) \in \mathcal{K}_n^\Lambda \times \mathbb{Z}\) now follows from Theorem 3.7.1. Moreover, \(d \geq 0\) by Corollary 3.5.27. As every composition factor of \(S^\mu\) is isomorphic to \(S^\nu\), for some \(T \subseteq \text{Std}(\lambda)\), the formula for \(d_{\chi_T}(q)\) is now immediate. \(\square\)

Proposition 4.4.2 shows that Conjecture 4.4.1 encodes closed formulas for the characters and graded dimensions of the irreducible \(\mathcal{H}_n^\Lambda\)-modules and for the graded decomposition numbers of \(\mathcal{H}_n^\Lambda\). For this result to be useful we need to first verify Conjecture 4.4.1 and then to explicitly determine the equivalence relation \(\sim_B\). Our last result is a step in this direction.
4.4.3. Lemma. Suppose that $s, t \in \text{Std}(\lambda)$ and that $t = s(r, r + 1)$ such that $i_{r+1}^s \neq i_r^s \pm 1$, where $1 \leq r < n$ and $\lambda \in \mathcal{T}_n$. Then $s \sim_B t$.

Proof. By assumption, either $s \triangleright t$ or $t \triangleright s$. Without loss of generality we assume that $s \triangleright t$. It follows from (4.3.5), and Theorem 3.6.2, that

$$B_s^t \psi_r = \psi_t + \sum_u a_u \psi_u = B_t + \sum_u b_u B_u,$$

where $a_u, b_u \in F$ are non-zero only if $\ell(d(u)) < \ell(d(s))$. Therefore, $s \triangleright_B t$. If $i_{r+1}^s \neq i_r^s$ then $\epsilon(i)^2 = \epsilon(i)$ by (2.2.3), so $s \sim_B t$. Now consider the more interesting case when $i_{r+1}^s = i_r^s$ or, equivalently, $i_{r+1}^t = i_r^t$. Using (2.2.2),

$$B_t^s \psi_r = \left( B_t^s - \sum_u b_u B_u \right) \psi_{r+1} = B_t^s (\psi_r \psi_{r+1} + 1) - \sum_u b_u B_u \psi_{r+1}.$$

In view of Proposition 3.2.10(c), $B_t$ appears on the right-hand side with coefficient 1. Hence, $t \triangleright_B s$ implying that $s \sim_B t$ as claimed. \hfill $\square$

Finally, we remark that it is easy to check that Conjecture 4.4.1 is true in the trivial cases considered in Example 3.7.7 and Example 3.7.8. With considerably more effort, using [26, Lemma 9.7] and results of [55, Appendix], it is possible to verify the conjecture when $\Lambda \in \mathcal{P}^+$ is a weight of level 2 and $e > n$. In all of these cases, the conjecture can be checked because $B_{st} = \psi_{st}$, for all $(s, t) \in \text{Std}^2(\mathcal{P}^n)$.

The $B$-basis, and hence Conjecture 4.4.1 and all of the results in this section (except that in positive characteristic we can only say that $dT \in \mathbb{Z}$ in Proposition 4.4.2, rather than $dT \in \mathbb{N}$), make sense over any field. We restrict our conjecture to fields of characteristic zero because it would be foolish to venture into the realms of positive characteristic without strong evidence. This said, whether or not our conjecture for the $B$-basis is true, we are convinced that, in all characteristics, there exists a “canonical” graded cellular basis $\{ C_{st} \}$ of $\mathcal{H}^n_{\Lambda}$ such that the analogous version of Conjecture 4.4.1 holds for the $\sim_C$ equivalence classes.

To put it another way, the results of [57, §3.3] show that the KLR-tableau combinatorics is rich enough to give closed combinatorial formulas for both the graded decomposition numbers and the graded dimensions of the irreducible representations of $\mathcal{H}^n_{\Lambda}$. We believe that over any field the graded Specht modules have a distinguished homogeneous basis that “canonically” determines these combinatorial formulas.

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