A GENERALIZATION OF TÓTH IDENTITY IN THE RING OF ALGEBRAIC INTEGERS INVOLVING A DIRICHLET CHARACTER

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ABSTRACT. The \( k \)-dimensional generalized Euler function \( \varphi_k(n) \) is defined to be the number of ordered \( k \)-tuples \((a_1, a_2, \ldots, a_k) \in \mathbb{N}^k \) with \( 1 \leq a_1, a_2, \ldots, a_k \leq n \) such that both the product \( a_1 a_2 \cdots a_k \) and the sum \( a_1 + a_2 + \cdots + a_k \) are co-prime to \( n \). Tóth proved that the identity

\[
\sum_{\substack{a_1, a_2, \ldots, a_k = 1 \\ \gcd(a_1 a_2 \cdots a_k, n) = 1}} \gcd(a_1 + a_2 + \cdots + a_k - 1, n) = \varphi_k(n) \sigma_0(n), \quad \text{where } \sigma_s(n) = \sum_{d|n} d^s \text{ holds.}
\]

This identity can also be viewed as a generalization of Menon’s identity. In this article, we generalize this identity to the ring of algebraic integers involving arithmetical functions and Dirichlet characters.

1. Introduction

For a positive integer \( n \), the classical Menon’s identity \([9]\) states that

\[
\sum_{\substack{a=1 \\ \gcd(a, n) = 1}}^n \gcd(a - 1, n) = \varphi(n) \sigma_0(n),
\]

where \( \varphi(n) \) is the Euler’s totient function and \( \sigma_0(n) = \sum_{d|n} 1 \).

Sury \([13]\) generalized \([1]\) and proved that, for every integers \( n \geq 1 \) and \( s \geq 0 \),

\[
\sum_{\substack{1 \leq a, b_1, b_2, \ldots, b_s \leq n \\ \gcd(a, n) = 1}} \gcd(a - 1, b_1, \ldots, b_s, n) = \varphi(n) \sigma_s(n), \quad \text{where } \sigma_s(n) = \sum_{d|n} d^s.
\]

Li, Hu and Kim \([6]\) generalized \((2)\) and proved that

\[
\sum_{\substack{1 \leq a, b_1, b_2, \ldots, b_s \leq n \\ \gcd(a, n) = 1}} \gcd(a - 1, b_1, \ldots, b_s, n) \chi(a) = \varphi(n) \sigma_s \left( \frac{n}{d} \right),
\]

where \( \chi \) is a Dirichlet character modulo \( n \) with conductor \( d \).

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Miguel extended equations (1) and (2) in [10, 11] from $\mathbb{Z}$ to any residually finite Dedekind domain $\mathcal{D}$. Menon’s identity has been generalized in many direction by several authors (cf. [2, 5, 8, 15, 16, 17]).

For every positive integer $k$, Tóth [14] introduced the $k$-dimensional generalized Euler’s totient function $\varphi_k$ as,

$$\varphi_k(n) = \sum_{\substack{a_1, a_2, \ldots, a_k = 1 \\ \gcd(a_1, \ldots, a_k, n) = 1 \\ \gcd(a_1 + \cdots + a_k, n) = 1}} 1.$$

For every positive integer $k$, he proved that the function $\varphi_k$ is multiplicative and

$$\varphi_k(n) = \varphi(n)^k \prod_{p|n} \left( 1 - \frac{1}{p-1} + \frac{1}{(p-1)^2} + \cdots + \frac{(-1)^{k-1}}{(p-1)^{k-1}} \right).$$

He also proved that the following Menon-type identity

$$(4) \sum_{\substack{a_1, a_2, \ldots, a_k = 1 \\ \gcd(a_1, \ldots, a_k, n) = 1 \\ \gcd(a_1 + \cdots + a_k, n) = 1}} \gcd(a_1 + \cdots + a_k - 1, n) = \varphi_k(n) \sigma_0(n)$$

holds.

Here we note that the function $\phi_2(n)$ was introduced by Arai and Gakuen [1], and Carlitz [3] proved the corresponding formula for $\phi_2(n)$. Also for $k = 2$, the identity (4),

$$(5) \sum_{\substack{a_1, a_2 = 1 \\ \gcd(a_1, a_2, n) = 1 \\ \gcd(a_1 + a_2, n) = 1}} \gcd(a_1 + a_2 - 1, n) = \varphi_2(n) \sigma_0(n)$$

was deduced by Sita Ramaiah [12].

Recently, Ji and Wang [7] and Chattopadhyay and Sarkar [4] generalized Sita Ramaiah’s identity in the ring of algebraic integers involving a Dirichlet character. Before we state the results of [4] and [7], we introduce the fundamental functions defined on the integral ideals of $\mathcal{O}_K$. In what follows, for an element $\alpha \in \mathcal{O}_K$, the principal ideal $\alpha \mathcal{O}_K$ is denoted by $\langle \alpha \rangle$ and $\gcd(\alpha, a)$ will denote $\gcd(\langle \alpha \rangle, a)$.

**Definition 1.1.** Let $K$ be an algebraic number field with ring of integers $\mathcal{O}_K$ and $I(\mathcal{O}_K)$ the set of all nonzero integral ideals of $\mathcal{O}_K$. Then for any two arithmetical functions $f$ and $g$ on $I(\mathcal{O}_K)$ and for any $n \in I(\mathcal{O}_K)$, the Dirichlet convolution “$*$” is defined by

$$f * g(n) = \sum_{\mathfrak{d} | n} f(\mathfrak{d}) g(n/\mathfrak{d}).$$

**Definition 1.2.** Let $K$ be an algebraic number field with ring of integers $\mathcal{O}_K$ and $n \in I(\mathcal{O}_K)$. Then we define the following functions on the set $I(\mathcal{O}_K)$.
The Möbius \( \mu \) function is defined as

\[
\mu(n) := \begin{cases} 
1 & \text{if } n = O_K, \\
(-1)^t & \text{if } n = \prod_{i=1}^t p_i; \text{ where each } p_i \subseteq O_K \text{ is a prime ideal and } p_i \neq p_j \text{ for } i \neq j, \\
0 & \text{otherwise.}
\end{cases}
\]

The Euler totient function is defined as

\[
\varphi(n) := \left| (O_K/\mathfrak{n})^* \right| = N(\mathfrak{n}) \prod_{p | n} \left( 1 - \frac{1}{N(p)} \right),
\]

where for any ideal \( \mathfrak{a} \subseteq O_K \), \( N(\mathfrak{a}) \) stands for the absolute norm \( |O_K/\mathfrak{a}| \).

For an integer \( s \geq 0 \), the function \( \sigma_s \) is defined by

\[
\sigma_s(n) := \sum_{d | n} N(\mathfrak{d})^s.
\]

The function \( \varphi_2(n) \) is defined as

\[
\varphi_2(n) := \sum_{a,b \in (O_K/\mathfrak{n})^*} 1.
\]

It is also known that (cf. [7]), \( \varphi_2(n) = \phi^2(n) \sum_{d | n} \frac{\mu(d)}{\varphi(d)} = \phi^2(n) \prod_{p | n} \left( 1 - \frac{1}{N(p)} \right). \]

**Definition 1.3.** Let \( K \) be an algebraic number field with ring of integers \( O_K \) and \( n \in I(O_K) \). Let \( \chi \) be any Dirichlet character modulo \( n \).

(i) An integral divisor \( \mathfrak{d} \) of \( n \) is said to be an induced modulus for \( \chi \) if \( \chi(a) = \chi(b) \) whenever \( a, b \in (O_K/\mathfrak{n})^* \) and \( a \equiv b \pmod{\mathfrak{d}} \). The unique induced modulus \( \mathfrak{d}_0 \) such that for any induced modulus \( \mathfrak{d} \), we have \( \mathfrak{d}_0 \mid \mathfrak{d} \), is said to be the conductor of \( \chi \).

(ii) A character \( \chi \) modulo \( n \) is said to be primitive modulo \( n \) if it has no induced modulus \( \mathfrak{d} \nmid n \). Let \( \chi \) be any character modulo \( n \) with conductor \( \mathfrak{d} \). Then there is a unique primitive character \( \psi \) modulo \( \mathfrak{d} \) such that \( \chi(a) = \psi(a) \) for all \( a \in (O_K/\mathfrak{n})^* \).

A special case of the result in [4] is the following generalization of Sita Ramaiah’s identity:

\[
\sum_{a_1,a_2,a_1+a_2 \in (O_K/\mathfrak{n})^*, \ b_1,b_2,...,b_s \in O_K/\mathfrak{n}} N(\gcd(a_1 + a_2 - 1, b_1, b_2, \ldots, b_s, n)) \chi(a_1) = \mu(\mathfrak{d}) \varphi \left( \frac{n^3}{\mathfrak{d}} \right) \varphi_2 \left( \frac{n}{n_0} \right) \sigma_s \left( \frac{n}{\mathfrak{d}} \right),
\]

where \( \chi \) is a Dirichlet character modulo \( n \) with conductor \( \mathfrak{d} \), \( n_0 \mid n \) is such that \( n_0 \) has the same prime ideal factors as that of \( \mathfrak{d} \) and \( \gcd \left( n_0, \frac{n}{n_0} \right) = 1. \)
In this article, we generalize Tóth’s identity \( \square \) to the ring of algebraic integers \( \mathcal{O}_K \) concerning arithmetical functions on \( I(\mathcal{O}_K) \) and Dirichlet characters. But before that, we need to define the \( k \)-dimensional generalized Euler function \( \varphi_k(n) \) on \( I(\mathcal{O}_K) \).

**Definition 1.4.** Let \( K \) be an algebraic number field with ring of integers \( \mathcal{O}_K \) and \( n \in I(\mathcal{O}_K) \). For any positive integer \( k \), we define the \( k \)-dimensional generalized Euler function \( \varphi_k \) as,

\[
\varphi_k(n) := \sum_{a_1, a_2, \ldots, a_k \in (\mathcal{O}_K / n)^*} 1.
\]

Note that \( \varphi_1(n) = \varphi(n) \).

The main results of this article are as follows.

**Theorem 1.1.** Let \( K \) be an algebraic number field with ring of integers \( \mathcal{O}_K \) and \( n \) be a non-zero ideal in \( \mathcal{O}_K \). Then for every positive integer \( k \), we have

\[
\varphi_k(n) = \varphi(n)^k \prod_{p \mid n} \left( 1 - \frac{1}{N(p) - 1} + \frac{1}{(N(p) - 1)^2} + \cdots + \frac{(-1)^{k-1}}{(N(p) - 1)^{k-1}} \right).
\]

**Theorem 1.2.** Let \( K \) be an algebraic number field with ring of integers \( \mathcal{O}_K \) and \( n \) be a non-zero ideal in \( \mathcal{O}_K \). Let \( \chi \) be a Dirichlet character modulo \( n \) with conductor \( \mathfrak{d} \). Then for a fixed element \( r \in \mathcal{O}_K \) with \( (r, n) = 1 \) and for any arithmetical function \( f \) on \( I(\mathcal{O}_K) \), we have

\[
\sum_{\substack{a_1, a_2, \ldots, a_k \in (\mathcal{O}_K / n)^* \cr a_1 + a_2 + \cdots + a_k \in (\mathcal{O}_K / n)^* \cr b_1, b_2, \ldots, b_s \in \mathcal{O}_K / n}} f(\gcd(a_1 + a_2 + \cdots + a_k - r, b_1, b_2, \ldots, b_s, n)) \chi(a_1)
\]

\[
= \mu(\mathfrak{d})^{k-1} \psi(r) \varphi \left( \frac{n_0^k}{\mathfrak{d}^{k-1}} \right) \varphi_k \left( \frac{n}{n_0} \right) \sum_{\mathfrak{d} \mid e|n} \frac{\mu * f(e)}{\varphi(e)},
\]

where \( \psi \) is the primitive character modulo \( \mathfrak{d} \) that induces \( \chi \), \( n_0 \mid n \) is such that \( n_0 \) has the same prime ideal factors as that of \( \mathfrak{d} \) and \( \gcd \left( n_0, \frac{n}{n_0} \right) = 1 \).

Let \( f \) be the norm function \( f(n) = N(n) \) in Theorem 1.2. Then we get the following

**Corollary 1.1.** Let \( K \) be an algebraic number field with ring of integers \( \mathcal{O}_K \) and \( n \) be a non-zero ideal in \( \mathcal{O}_K \). Let \( \chi \) be a Dirichlet character modulo \( n \) with conductor \( \mathfrak{d} \). Then for a fixed element \( r \in \mathcal{O}_K \) with \( (r, n) = 1 \), we have

\[
\sum_{\substack{a_1, a_2, \ldots, a_k \in (\mathcal{O}_K / n)^* \cr a_1 + a_2 + \cdots + a_k \in (\mathcal{O}_K / n)^* \cr b_1, b_2, \ldots, b_s \in \mathcal{O}_K / n}} \sigma_s \left( \frac{n}{n_0} \right) \varphi_k \left( \frac{n}{n_0} \right) \mu(\mathfrak{d})^{k-1} \psi(r) \varphi \left( \frac{n_0^k}{\mathfrak{d}^{k-1}} \right) \varphi_k \left( \frac{n}{n_0} \right) 
\]
where \( \psi \) is the primitive character modulo \( d \) that induces \( \chi \), \( n_0 \mid n \) is such that \( n_0 \) has the same prime ideal factors as that of \( d \) and \( \gcd\left(\frac{n_0}{n_0}, \frac{n}{n_0}\right) = 1 \).

In the special case of \( r = 1 \), Corollary 1.1 reduces to the following generalization of the equation (6).

**Corollary 1.2.** Let \( K \) be an algebraic number field with ring of integers \( \mathcal{O}_K \) and let \( n \) be a non-zero ideal in \( \mathcal{O}_K \). Let \( \chi \) be a Dirichlet character modulo \( n \) with conductor \( d \). Then

\[
\sum_{a_1, a_2, \ldots, a_k \in (\mathcal{O}_K/n)^*} a_1 + a_2 + \cdots + a_k \in (\mathcal{O}_K/\mathfrak{n})^* \sum_{b_1, b_2, \ldots, b_s \in \mathcal{O}_K/n} \chi(a_1)
\]

\[
= \mu(d)^{k-1} \varphi\left(\frac{n_0}{d^{k-1}}\right) \varphi_k\left(\frac{n}{n_0}\right) \sigma_s\left(\frac{n}{\mathfrak{n}}\right),
\]

where \( n_0 \mid n \) is such that \( n_0 \) has the same prime ideal factors as that of \( d \) and \( \gcd\left(\frac{n_0}{n_0}, \frac{n}{n_0}\right) = 1 \).

**Remark 1.1.** Note that, for \( K = \mathbb{Q} \) and \( \chi = 1 \), the trivial character, then Corollary 1.2 reduces to

\[
\sum_{a_1, a_2, \ldots, a_k \in (\mathcal{O}_K/n)^*} \gcd(a_1 + \cdots + a_k - 1, b_1, b_2, \ldots, b_s, n) = \varphi_k(n) \sigma_s(n),
\]

which is a generalization of Tóth’s identity (4).

2. Preliminaries

In this section, we fix an algebraic number field \( K \) with ring of integers \( \mathcal{O}_K \) and a non-zero ideal \( n \) in \( \mathcal{O}_K \). We need the following lemmas that have been proved in [17], and we record it here.

**Lemma 2.1.** [17] Let \( a \) be a nonzero ideal in \( \mathcal{O}_K \) such that \( a \mid n \). Then for any \( u \in \mathcal{O}_K \), we have

\[
\sum_{a \equiv u \pmod{a}} 1 = \begin{cases} \frac{\varphi(n)}{\varphi(a)} & \text{if } (u, a) = 1 \\ 0 & \text{otherwise} \end{cases}
\]

**Lemma 2.2.** [17] Let \( a \) and \( b \) be two nonzero ideals in \( \mathcal{O}_K \) such that \( a \mid n \) and \( b \mid n \). Then for any \( u, v \in \mathcal{O}_K \), we have

\[
\sum_{a \equiv u \pmod{a}} 1 = \begin{cases} \frac{\varphi(n)}{\varphi((u, b))} & \text{if } (u, a) = (v, b) = 1 \text{ and } u \equiv v \pmod{(a, b)} \\ 0 & \text{otherwise} \end{cases}
\]
Lemma 2.3. \[17\] Let \( \psi \) a primitive character modulo \( n \) and \( r \in \mathcal{O}_K \) with \( (r, n) = 1 \). Then for any \( d \mid n \),

\[
\sum_{\substack{a \in (\mathcal{O}_K/n)^* \\
 a \equiv r \pmod{d}}} \psi(a) \neq 0 \quad \text{if and only if} \quad d = n.
\]

In particular, if \( d = n \), then

\[
\sum_{\substack{a \in (\mathcal{O}_K/n)^* \\
 a \equiv r \pmod{d}}} \psi(a) = \psi(r).
\]

3. Proof of Theorem \[1.1\]

Let \( K \) be an algebraic number field with ring of integers \( \mathcal{O}_K \) and \( m, n \) two nonzero ideals in \( \mathcal{O}_K \). For every integer \( k \geq 1 \), we define the following more general function than \( \varphi_k(n) \):

\[
\varphi_k(n, m) := \sum_{\substack{a_1, a_2, \ldots, a_k \in (\mathcal{O}_K/n)^* \\
a_1 + a_2 + \cdots + a_k \in (\mathcal{O}_K/m)^*}} 1.
\]

If \( m = n \), then clearly \( \varphi_k(n, n) = \varphi_k(n) \). To prove Theorem \[1.1\], we need the following recursion lemma.

Lemma 3.1. Let \( k \geq 2 \) and \( m \mid n \). Then

\[
\varphi_k(n, m) = \varphi(n) \sum_{d \mid m} \frac{\mu(d)}{\varphi(d)} \varphi_{k-1}(n, d).
\]

Proof. We have

\[
\varphi_k(n, m) = \sum_{\substack{a_1, a_2, \ldots, a_k \in (\mathcal{O}_K/n)^* \\
a_1 + a_2 + \cdots + a_k \in (\mathcal{O}_K/m)^*}} 1 = \sum_{\substack{a_1, a_2, \ldots, a_k \in (\mathcal{O}_K/n)^* \\
a_1 + a_2 + \cdots + a_k \equiv 0 \pmod{d}}} \sum_{d \mid m} \mu(d)
\]

\[
= \sum_{d \mid m} \mu(d) \sum_{\substack{a_1, a_2, \ldots, a_k \in (\mathcal{O}_K/n)^* \\
a_1 + a_2 + \cdots + a_k \equiv 0 \pmod{d}}} 1
\]

\[
= \sum_{d \mid m} \mu(d) \sum_{\substack{a_1, a_2, \ldots, a_k-1 \in (\mathcal{O}_K/n)^* \\
a_k \equiv -a_1 - a_2 - \cdots - a_k-1 \pmod{d}}} 1
\]

\[
= \sum_{d \mid m} \mu(d) \sum_{\substack{a_1, a_2, \ldots, a_k \in (\mathcal{O}_K/n)^*}} 1
\]
By using Lemma 2.1 we get,

\[ \varphi_k(n, m) = \sum_{\mathfrak{d} | m} \frac{\mu(\mathfrak{d})}{\varphi(\mathfrak{d})} \sum_{a_1, a_2, \ldots, a_k \in (\mathcal{O}_K/n)^* \atop a_1 + a_2 + \cdots + a_k \in (\mathcal{O}_K/\mathfrak{d})^*} \frac{\varphi(n)}{\varphi(\mathfrak{d})} = \varphi(n) \sum_{\mathfrak{d} | m} \frac{\mu(\mathfrak{d})}{\varphi(\mathfrak{d})} \sum_{a_1, a_2, \ldots, a_k \in (\mathcal{O}_K/n)^* \atop a_1 + a_2 + \cdots + a_k \in (\mathcal{O}_K/\mathfrak{d})^*} 1 = \varphi(n) \sum_{\mathfrak{d} | m} \frac{\mu(\mathfrak{d})}{\varphi(\mathfrak{d})} \varphi_{k-1}(n, \mathfrak{d}). \]

Proof of Theorem 1.1. Let \( k \geq 1 \) and \( m \) and \( n \) be two nonzero ideals of \( \mathcal{O}_K \) such that \( m | n \). Then we show the following more general result

\[ \varphi_k(n, m) = \varphi(n)^k \prod_{p | m} \left( 1 - \frac{1}{N(p) - 1} + \frac{1}{(N(p) - 1)^2} + \cdots + \frac{(-1)^{k-1}}{(N(p) - 1)^{k-1}} \right), \]

by induction on \( k \). If \( k = 1 \), then \( \varphi_1(n, m) = \varphi(n) \), by the definition. Let \( k \geq 2 \), and assume that the result is true for \( k - 1 \). Using Lemma 3.1 we have

\[ \varphi_k(n, m) = \varphi(n) \sum_{\mathfrak{d} | m} \frac{\mu(\mathfrak{d})}{\varphi(\mathfrak{d})} \varphi_{k-1}(n, \mathfrak{d}) \]

\[ = \varphi(n) \sum_{\mathfrak{d} | m} \frac{\mu(\mathfrak{d})}{\varphi(\mathfrak{d})} (\varphi(n))^{k-1} \prod_{p | \mathfrak{d}} \left( 1 - \frac{1}{N(p) - 1} + \frac{1}{(N(p) - 1)^2} + \cdots + \frac{(-1)^{k-2}}{(N(p) - 1)^{k-2}} \right) \]

\[ = \varphi(n)^k \prod_{p | m} \left( 1 - \frac{1}{N(p) - 1} + \frac{1}{(N(p) - 1)^2} + \cdots + \frac{(-1)^{k-2}}{(N(p) - 1)^{k-2}} \right), \]

which proves the result.

Now if we take \( m = n \), then we have the proof of Theorem 1.1. \( \square \)

4. Proof of Theorem 1.2

Let \( K \) be an algebraic number field with ring of integers \( \mathcal{O}_K \) and \( n \) a nonzero ideal in \( \mathcal{O}_K \). We are given that \( \chi \) is a Dirichlet character modulo \( n \) with conductor \( \mathfrak{d} \) and \( \psi \) is the primitive character modulo \( \mathfrak{d} \) that induces \( \chi \). Let \( a_1 \in (\mathcal{O}_K/n)^* \) and \( u \in (\mathcal{O}_K/\mathfrak{d})^* \) be such that \( a_1 \equiv u \) (mod \( \mathfrak{d} \)). Since \( \psi \) is the primitive character modulo \( \mathfrak{d} \) that induces \( \chi \), we have \( \chi(a_1) = \psi(a_1) = \psi(u) \). \( \square \)
ψ(u). Let \( M_{\chi,k}(n) \) denote the sum in left hand side of the equation (7). Then

\[
M_{\chi,k}(n) = \sum_{a_1,a_2,\ldots,a_k \in (\mathcal{O}_K/n)^*} f(\gcd(a_1 + a_2 + \cdots + a_k - r, b_1, b_2, \ldots, b_s, n)) \chi(a_1)
\]

By using the convolution identity \( f = (\mu * f) * 1 \), we get

\[
M_{\chi,k}(n) = \sum_{u \in (\mathcal{O}_K/\mathfrak{d})^*} \sum_{\chi \in (\mathcal{O}_K/\mathfrak{d})^*} \sum_{a_1,a_2,\ldots,a_k \in (\mathcal{O}_K/n)^*} f(\chi(a_1 + a_2 + \cdots + a_k - r, b_1, b_2, \ldots, b_s, n)) \mu(g)
\]

\[
= \sum_{\epsilon | n} (\mu * f)(\epsilon) \sum_{\chi \in (\mathcal{O}_K/\mathfrak{d})^*} \sum_{u \in (\mathcal{O}_K/\mathfrak{d})^*} \sum_{a_1,a_2,\ldots,a_k \in (\mathcal{O}_K/n)^*} f(\chi(a_1 + a_2 + \cdots + a_k - r, b_1, b_2, \ldots, b_s, n)) \mu(g)
\]

\[
= \sum_{\epsilon | n} (\mu * f)(\epsilon) \sum_{\chi \in (\mathcal{O}_K/\mathfrak{d})^*} \sum_{u \in (\mathcal{O}_K/\mathfrak{d})^*} \sum_{a_1,a_2,\ldots,a_k \in (\mathcal{O}_K/n)^*} f(\chi(a_1 + a_2 + \cdots + a_k - r, b_1, b_2, \ldots, b_s, n)) \mu(g)
\]

\[
= \sum_{\epsilon | n} (\mu * f)(\epsilon) \sum_{\chi \in (\mathcal{O}_K/\mathfrak{d})^*} \sum_{u \in (\mathcal{O}_K/\mathfrak{d})^*} \sum_{a_1,a_2,\ldots,a_k \in (\mathcal{O}_K/n)^*} f(\chi(a_1 + a_2 + \cdots + a_k - r, b_1, b_2, \ldots, b_s, n)) \mu(g)
\]

(8) \[
= \sum_{\epsilon | n} (\mu * f)(\epsilon) \sum_{\chi \in (\mathcal{O}_K/\mathfrak{d})^*} \sum_{u \in (\mathcal{O}_K/\mathfrak{d})^*} \sum_{a_1,a_2,\ldots,a_k \in (\mathcal{O}_K/n)^*} f(\chi(a_1 + a_2 + \cdots + a_k - r, b_1, b_2, \ldots, b_s, n)) \mu(g)
\]

where \( N_k(n, \epsilon, g, \mathfrak{d}, u) = \sum_{a_1,a_2,\ldots,a_k \in (\mathcal{O}_K/n)^*} \mu(g) \sum_{u \in (\mathcal{O}_K/\mathfrak{d})^*} u N_k(n, \epsilon, g, \mathfrak{d}, u) \)

We now evaluate the sum \( N_k(n, \epsilon, g, \mathfrak{d}, u) \), where \( \epsilon | n, g | n \) and \( u \in (\mathcal{O}_K/\mathfrak{d})^* \) be fixed. Since \( (r,n) = 1 \), if \( (\epsilon, g) > 1 \), then \( N_k(n, \epsilon, g, \mathfrak{d}, u) = 0 \), the empty sum. Hence we assume that \( (\epsilon, g) = 1 \).
If $k = 1$, using Lemma 2.2 we get
\begin{equation}
 N_1(n, e, g, d, u) = \sum_{\substack{a_1 \in (O_K/n)^* \
a_1 \equiv u \pmod{d} \
a_1 \equiv r \pmod{c} \
a_1 \equiv 0 \pmod{g}}} 1 = \begin{cases} \varphi(n) & \text{if } g = 1, (u, d) = (r, e) = 1 \text{ and } u \equiv r \pmod{(d, e)} \\ 0 & \text{otherwise.} \end{cases}
\end{equation}

**Lemma 4.1.** *(Recursion formula for $N_k(n, e, g, d, u)$)* Let $e$ and $g$ be two ideals in $O_K$ such that $e \mid n$, $g \mid n$ with $(e, g) = 1$. Then for every integer $k \geq 2$ and an element $u \in (O_K/d)^*$, we have
\begin{equation}
 N_k(n, e, g, d, u) = \frac{\varphi(n)}{\varphi(e)\varphi(g)} \sum_{j \mid e} \mu(j) \sum_{t \mid g} \mu(t) N_{k-1}(n, j, t, d, u).
\end{equation}

**Proof.** We have
\begin{align*}
 N_k(n, e, g, d, u) &= \sum_{\substack{a_1, a_2, \ldots, a_k \in (O_K/n)^* \
a_1 \equiv u \pmod{d} \
a_1 + a_2 + \ldots + a_k \equiv r \pmod{c} \
a_1 + a_2 + \ldots + a_k \equiv 0 \pmod{g}}} 1 \\
 &= \sum_{\substack{a_1, a_2, \ldots, a_k \in (O_K/n)^* \\
a_1 \equiv u \pmod{d} \\
a_1 + a_2 + \ldots + a_{k-1} \equiv r \pmod{c} \\
(a_1 + a_2 + \ldots + a_k - r, e) = 1}} \frac{\varphi(n)}{\varphi(e)\varphi(g)} \sum_{j \mid (a_1 + a_2 + \ldots + a_{k-1} - r, e)} \mu(j) \sum_{t \mid (a_1 + a_2 + \ldots + a_{k-1}, g)} \mu(t) \\
 &= \frac{\varphi(n)}{\varphi(e)\varphi(g)} \sum_{j \mid e} \mu(j) \sum_{t \mid g} \mu(t) N_{k-1}(n, j, t, d, u).
\end{align*}

Since $(e, g) = 1$, using Lemma 2.2 we get
\begin{align*}
 N_k(n, e, g, d, u) &= \sum_{\substack{a_1, a_2, \ldots, a_k \in (O_K/n)^* \\
a_1 \equiv u \pmod{d} \\
(a_1 + a_2 + \ldots + a_{k-1} - r, e) = 1}} \frac{\varphi(n)}{\varphi(e)\varphi(g)} \sum_{j \mid (a_1 + a_2 + \ldots + a_{k-1} - r, e)} \mu(j) \sum_{t \mid (a_1 + a_2 + \ldots + a_{k-1}, g)} \mu(t) \\
 &= \frac{\varphi(n)}{\varphi(e)\varphi(g)} \sum_{j \mid e} \mu(j) \sum_{t \mid g} \mu(t) N_{k-1}(n, j, t, d, u).
\end{align*}

\qed
Lemma 4.2. Let \( e \) and \( g \) be two nonzero ideals in \( O_K \) such that \( e \mid n \), \( g \mid n \) with \( (e, g) = 1 \). Then for every integer \( k \geq 2 \),

\[
\sum_{u \in (O_K/\mathfrak{d})^*} \psi(u)N_k(n, e, g, \mathfrak{d}, u) = \frac{\varphi(n)^k\psi(r)\mu(\mathfrak{d})^{k-1}}{\varphi(e)\varphi(g)} \prod_{p \mid \mathfrak{d}} \frac{1}{(N(p) - 1)^{k-1}} \prod_{p \mid \mathfrak{d}} \left(1 - \frac{1}{N(p) - 1} + \cdots + \frac{(-1)^{k-1}}{(N(p) - 1)^{k-1}}\right)
\]

if \( \mathfrak{d} \mid e \). Otherwise, the sum is 0.

Proof. We prove this lemma by induction on \( k \). If \( k = 2 \), then by the recursion Lemma 4.1 and the equation (9), we get

\[
\sum_{u \in (O_K/\mathfrak{d})^*} \psi(u)N_2(n, e, g, \mathfrak{d}, u) = \sum_{u \in (O_K/\mathfrak{d})^*} \psi(u) \frac{\varphi(n)}{\varphi(e)\varphi(g)} \sum_{j \mid e} \mu(j) \sum_{t \mid \mathfrak{g}} \mu(t)N_1(n, j, t, \mathfrak{d}, u)
\]

By Lemma 2.3, we have

\[
\sum_{u \in (O_K/\mathfrak{d})^*} \psi(u) \neq 0 \quad \text{if and only if } \quad (j, \mathfrak{d}) = \mathfrak{d}, \quad \text{that is, if and only if } \quad \mathfrak{d} \mid j.
\]

Furthermore, if \( \mathfrak{d} \mid j \), then

\[
\sum_{u \in (O_K/\mathfrak{d})^* \atop u \equiv r \pmod{(j, \mathfrak{d})}} \psi(u) = \psi(r).
\]

Hence

\[
\sum_{u \in (O_K/\mathfrak{d})^*} \psi(u)N_2(n, e, g, \mathfrak{d}, u) = \frac{\varphi(n)^2\psi(r)}{\varphi(e)\varphi(g)} \sum_{j \mid e \atop \mathfrak{d} \mid j} \mu(j) \varphi([j, \mathfrak{d}]) = \frac{\varphi(n)^2\psi(r)}{\varphi(e)\varphi(g)} \sum_{j \mid e \atop \mathfrak{d} \mid j} \mu(j) \varphi([j, \mathfrak{d}])
\]

Therefore, it is clear that, if \( \mathfrak{d} \nmid e \) or if \( \mathfrak{d} \) is not square-free, then the sum is empty. If \( \mathfrak{d} \mid e \), then

\[
\sum_{u \in (O_K/\mathfrak{d})^*} \psi(u)N_2(n, e, g, \mathfrak{d}, u) = \frac{\varphi(n)^2\psi(r)\mu(\mathfrak{d})}{\varphi(e)\varphi(g)} \prod_{p \mid \mathfrak{d}} \frac{1}{N(p) - 1} \prod_{p \mid \mathfrak{d}} \left(1 - \frac{1}{N(p) - 1}\right).
\]
Hence the formula is true for $k = 2$. Assume that the formula is true for $k - 1$. Then by the recursion Lemma \[\text{(4.1)}\] we get

\begin{align*}
\sum_{u \in (O_K / \mathfrak{d})^*} \psi(u) N_k(n, c, g, \mathfrak{d}, u) & = \sum_{u \in (O_K / \mathfrak{d})^*} \psi(u) \frac{\varphi(n)}{\varphi(c) \varphi(g)} \sum_{j \mid c} \mu(j) \sum_{t \mid g} \mu(t) N_{k-1}(n, t, \mathfrak{d}, u) \\
& = \frac{\varphi(n)}{\varphi(c) \varphi(g)} \sum_{j \mid c} \mu(j) \sum_{t \mid g} \mu(t) \sum_{u \in (O_K / \mathfrak{d})^*} \psi(u) N_{k-1}(n, t, \mathfrak{d}, u) \\
& = \frac{\varphi(n)}{\varphi(c) \varphi(g)} \sum_{j \mid c} \mu(j) \sum_{t \mid g} \mu(t) \varphi(n)^{k-1} \psi(r) \mu(d) (N(p) - 1)^{k-2} \prod_{p \mid \mathfrak{d}} \frac{1}{(N(p) - 1)^{k-2}} \\
& \times \prod_{p \mid j} \left(1 - \frac{1}{N(p) - 1} + \ldots + \frac{(-1)^{k-2}}{(N(p) - 1)^{k-2}}\right) \prod_{p \mid t} \left(1 - \frac{1}{N(p) - 1} + \ldots + \frac{(-1)^{k-3}}{(N(p) - 1)^{k-3}}\right) \\
& = \frac{\varphi(n)^k \psi(r) \mu(d)^{k-2}}{\varphi(c) \varphi(g)} \prod_{p \mid \mathfrak{d}} \frac{1}{(N(p) - 1)^{k-2}} \sum_{j \mid c} \frac{\mu(j)}{\varphi(j)} \prod_{p \mid \mathfrak{d}} \left(1 - \frac{1}{N(p) - 1} + \ldots + \frac{(-1)^{k-2}}{(N(p) - 1)^{k-2}}\right) \\
& \times \prod_{p \mid c} \left(1 - \frac{1}{N(p) - 1} + \ldots + \frac{(-1)^{k-2}}{(N(p) - 1)^{k-2}}\right) \\
& \times \prod_{p \mid \mathfrak{d}} \left(1 - \frac{1}{N(p) - 1} + \ldots + \frac{(-1)^{k-3}}{(N(p) - 1)^{k-3}}\right) \\
& = \frac{\varphi(n)^k \psi(r) \mu(d)^{k-1}}{\varphi(c) \varphi(g)} \prod_{p \mid \mathfrak{d}} \frac{1}{(N(p) - 1)^{k-1}} \prod_{p \mid \mathfrak{d}} \left(1 - \frac{1}{N(p) - 1} + \ldots + \frac{(-1)^{k-1}}{(N(p) - 1)^{k-1}}\right) \\
& \times \prod_{p \mid \mathfrak{d}} \left(1 - \frac{1}{N(p) - 1} + \ldots + \frac{(-1)^{k-2}}{(N(p) - 1)^{k-2}}\right).
\end{align*}

This proves the result. \qed
Proof of Theorem 1.2. We now continue the evaluation of $M_{\chi,k}(n)$. From the equation (8) and Lemma 4.2 we have

$$M_{\chi,k}(n) = \sum_{\epsilon|n} (\mu \ast f)(\epsilon) \sum_{b_1,b_2,...,b_s \in \mathcal{O}_K/n} \sum_{\epsilon|b_1,...,\epsilon|b_s} \sum_{g|n} \sum_{u \in (\mathcal{O}_K/\mathfrak{o})^*} \psi(u) N_k(n, \epsilon, g, \mathfrak{o}, u)$$

$$= \sum_{\epsilon|n} (\mu \ast f)(\epsilon) \sum_{b_1,b_2,...,b_s \in \mathcal{O}_K/n} \sum_{\epsilon|b_1,...,\epsilon|b_s} \sum_{g|n} \prod_{p|\mathfrak{d}} \left(1 - \frac{1}{N(p) - 1} + \cdots + \frac{(-1)^{k-1}}{(N(p) - 1)^{k-1}}\right) \prod_{p|\mathfrak{d}} \left(1 - \frac{1}{N(p) - 1} + \cdots + \frac{(-1)^{k-2}}{(N(p) - 1)^{k-2}}\right)$$

$$= \sum_{\epsilon|n} (\mu \ast f)(\epsilon) \sum_{b_1,b_2,...,b_s \in \mathcal{O}_K/n} \sum_{\epsilon|b_1,...,\epsilon|b_s} \sum_{g|n} \prod_{p|\mathfrak{d}} \left(1 - \frac{1}{N(p) - 1} + \cdots + \frac{(-1)^{k-1}}{(N(p) - 1)^{k-1}}\right) \prod_{p|\mathfrak{d}} \left(1 - \frac{1}{N(p) - 1} + \cdots + \frac{(-1)^{k-2}}{(N(p) - 1)^{k-2}}\right)$$

$$= \sum_{\epsilon|n} (\mu \ast f)(\epsilon) \sum_{b_1,b_2,...,b_s \in \mathcal{O}_K/n} \sum_{\epsilon|b_1,...,\epsilon|b_s} \sum_{g|n} \prod_{p|\mathfrak{d}} \left(1 - \frac{1}{N(p) - 1} + \cdots + \frac{(-1)^{k-1}}{(N(p) - 1)^{k-1}}\right) \prod_{p|\mathfrak{d}} \left(1 - \frac{1}{N(p) - 1} + \cdots + \frac{(-1)^{k-2}}{(N(p) - 1)^{k-2}}\right)$$

$$= \prod_{p|\mathfrak{d}} \left(1 - \frac{1}{N(p) - 1} + \cdots + \frac{(-1)^{k-1}}{(N(p) - 1)^{k-1}}\right) \sum_{b_1,b_2,...,b_s \in \mathcal{O}_K/n} \prod_{e|b_1,...,e|b_s} \sum_{g|n} \prod_{p|\mathfrak{d}} \left(1 - \frac{1}{N(p) - 1} + \cdots + \frac{(-1)^{k-1}}{(N(p) - 1)^{k-1}}\right)$$

$$= \prod_{p|\mathfrak{d}} \left(1 - \frac{1}{N(p) - 1} + \cdots + \frac{(-1)^{k-1}}{(N(p) - 1)^{k-1}}\right) \sum_{b_1,b_2,...,b_s \in \mathcal{O}_K/n} \prod_{e|b_1,...,e|b_s} \sum_{g|n} \prod_{p|\mathfrak{d}} \left(1 - \frac{1}{N(p) - 1} + \cdots + \frac{(-1)^{k-1}}{(N(p) - 1)^{k-1}}\right)$$

$$= \prod_{p|\mathfrak{d}} \left(1 - \frac{1}{N(p) - 1} + \cdots + \frac{(-1)^{k-1}}{(N(p) - 1)^{k-1}}\right) \sum_{b_1,b_2,...,b_s \in \mathcal{O}_K/n} \prod_{e|b_1,...,e|b_s} \sum_{g|n} \prod_{p|\mathfrak{d}} \left(1 - \frac{1}{N(p) - 1} + \cdots + \frac{(-1)^{k-1}}{(N(p) - 1)^{k-1}}\right)$$

Since

$$\varphi_k(n) = \varphi(n) \prod_{p|n} \left(1 - \frac{1}{N(p) - 1} + \frac{1}{(N(p) - 1)^2} + \cdots + \frac{(-1)^{k-1}}{(N(p) - 1)^{k-1}}\right),$$

we have


\[
M_{\chi,k}(n) = \varphi_k(n)\psi(r)\mu(\mathfrak{d})^{k-1}\prod_{p|\mathfrak{d}} \frac{1}{(N(p) - 1)^{k-1}} \sum_{e|\mathfrak{d}/e} \frac{(\mu * f)(e)}{\varphi(e)} \sum_{\mathfrak{b}_1, \mathfrak{b}_2, \ldots, \mathfrak{b}_s \in \mathcal{O}_K / n} \prod_{p|\mathfrak{b}} \left(1 - \frac{1}{N(p) - 1} + \cdots + (-1)^{k-1} \frac{(N(p) - 1)^{k-1}}{(N(p) - 1)^{k-1}} \right)^{-1}
\]

\[
= \mu(\mathfrak{d})^{k-1}\psi(r)\left(\frac{n_0^k}{\mathfrak{d}^{k-1}}\right)\varphi_k\left(\frac{n}{n_0}\right) \sum_{\mathfrak{d}|e|n} \frac{(\mu * f)(e)}{\varphi(e)}
\]

where \(n_0 | n\) is such that \(n_0\) has the same prime ideal factors as that of \(\mathfrak{d}\) and \(\gcd\left(n_0, \frac{n}{n_0}\right) = 1\). This completes the proof of Theorem 1.2.

\[\Box\]

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