POSITIVE SOLUTIONS FOR A SECOND ORDER EXTENDED FISHER-KOLMOGOROV'S EQUATION

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Abstract: We consider the existence of positive solutions of extended Fisher-Kolmogorov second order differential equation. Using a variational method and an approach of Verzini, we obtain the positive bounded solutions of this ODE.

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1. Introduction

In the present paper we study the existence of positive solutions of a second-order ordinary differential equation (ODE)

$$u'' + cu' + f(t, u) = 0, \quad t \in (a, +\infty),$$

(1)
coupled with the boundary conditions

$$u(a) = u(+\infty) = 0,$$

(2)
where $c > 0$ is a constant, $a \in \mathbb{R}$. We suppose that $f(t, s) : \mathbb{R}^2 \to \mathbb{R}$ and

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\[ f_s(t, s) = \frac{\partial f}{\partial s}(t, s), \quad f_t(t, s) = \frac{\partial f}{\partial t}(t, s) \]
are continuous functions which satisfy the following conditions:

\[ c_1 |s|^{1+q} \leq |f(t, s)| \leq c_2 |s|^{1+q}, \quad sf(t, s) \geq 0, \quad \forall (t, s) \in \mathbb{R}^2, \quad (H1) \]

\[ f_s(t, s)s - Af(t, s) \geq 0, \quad \forall (t, s) \in \mathbb{R}^2, \quad (H2) \]

\[ f(t, s) - (2 + \alpha)F(t, s) \geq 0, \quad \forall (t, s) \in \mathbb{R}^2, \quad (H3) \]

\[ c_3 |s|^{q+2} \geq f_t(t, s) \geq 0, \quad \forall (t, s) \in \mathbb{R}^2, \quad (H4) \]

where \( A > 1, \quad \alpha > 0, \quad c_j > 0, \quad j = 1, 2, 3, \) and \( F(t, s) := s \int_0^s f(t, \tau) d\tau. \)

An example of a function \( f \), which satisfies these conditions is \( f(t, s) = Cs^q \) with \( A \in (1, q+1], \quad \alpha = q, \quad c_1 = c_2 = c_3 = C > 0. \)

We will look for positive solutions \( u \) of (1.1) such that \( u \in H^1_0(a, +\infty) \cap H_{c,a} \), where \( H^1_0(a, +\infty) \) is the usual Sobolev space and

\[ H_{c,a} := \left\{ u \in H^1_{loc}(a, +\infty) : \int_a^{+\infty} e^{ct} u'(t)^2 \, dt < +\infty, \quad u(+\infty) = 0 \right\} \quad (3) \]

with norm

\[ ||u||_{c,a} = \int_a^{+\infty} e^{ct} \left( u'(t)^2 + u(t)^2 \right) \, dt \]

and \( H^0_{c,a} := H_{c,a} \cap H^1_0(a, +\infty) = \{ u \in H_{c,a} : u(a) = 0 \}. \)

Equation (1) is obtained by the Fisher-Kolmogorov’s equation \( u_t = u_{xx} + f(t, u) \), looking for the traveling waves \( u(x, t) = U(x - ct) \) with speed \( c. \)

There is a vast studies on heteroclinic solutions of eq. (1). We refer the reader to Kolmogorov, Petrovsky and Piskunov [3], Aronson and Weinberger [2], Arias and al. [1], Nehari [7], Versini [9], Szulkin [8], Li and Wang [4] and references therein. Solutions of (1) with initial data with compact support are studied in [2]. Fast solutions of Eq. (1) are studied in the paper of Arias and al. [1] via variational methods. Heteroclinic solutions for non-autonomous second order differential equations are studied in [5, 10]. Verzini also studied equation of type (1), when \( c = 0, \) and she proves the existence of many oscillating solutions belonging to \( L^\infty(\mathbb{R}). \) She used the variational method and the approach of Nehari [7]. In the present paper we will prove the existence of positive solution of Eq. (1), belonging to \( H^1_0(a, +\infty), \) using the methods of [9].

Note that by (H1) it follows

\[ \frac{c_1}{q + 2} |s|^{q+2} \leq F(t, s) \leq \frac{c_2}{q + 2} |s|^{q+2}, \quad \forall (t, s) \in \mathbb{R}^2, \quad (4) \]

for some positive constants \( c_1 \) and \( c_2. \)
We introduce the energy functional associated with the problem (1), (2), further referred as (P):

\[
J_{[a,b]} (u) := \int_a^b e^{ct} \left( \frac{(u'(t))^2}{2} - F(t, u(t)) \right) dt,
\]

\[
J (u) = J_{[a, +\infty)} (u) := \int_a^{+\infty} e^{ct} \left( \frac{(u'(t))^2}{2} - F(t, u(t)) \right) dt,
\]

where \(a \in \mathbb{R}\) is a fixed real number and \(b \in (a, +\infty)\). Let

\[
\mu (u) = \mu_{[a, +\infty)} (u) := \sup_{\lambda > 0} J_{[a, +\infty)} (\lambda u) = \sup_{\lambda > 0} J (\lambda u). \tag{5}
\]

We will prove that for each nonzero function \(u\), \(\mu (u) = \sup_{\lambda > 0} J (\lambda u) \geq C > 0\).

There exists unique number \(\lambda = \lambda (u) > 0\), for which \(\sup_{\lambda > 0} J (\lambda u)\) is attained, i.e.

\[
\mu (u) = \sup_{\lambda > 0} J (\lambda u) = J (\lambda (u) u).
\]

We introduce the following set, analogous to the Nehari manifold:

\[
N (a, +\infty) := \{ u \in H^0_{c,a} \setminus (0) : \lambda (u) = 1 \}
\]

\[
= \{ u \in H^0_{c,a} \setminus (0) : \nabla J (u) . u = 0 \},
\]

as in Verzini [9]. Since we are looking for the solutions of (1), which are non-negative on \([a, +\infty)\), we introduce the set

\[
N^+ (a, +\infty) := \{ u \in N (a, +\infty) : u \geq 0 \}.
\]

Define the function

\[
\varphi^+ (a, +\infty) := \inf \left\{ \sup_{\lambda > 0} J (\lambda u) : u \in H^0_{c,a} \setminus (0), u \geq 0 \right\}.
\]

Our main result is:

**Theorem 1.** Let the conditions (H1)-(H4) hold.

Then \(\varphi^+ (a, +\infty)\) is attained by at least one function \(u_+ \in N^+ (a, +\infty), u_+ > 0\) on \((a, +\infty)\) and \(u_+ (t)\) is a solution of the problem (P) for \(t \in (a, +\infty)\).

The paper is organized as follows. In Section 2 we give preliminaries on the function spaces, embedding inequalities and three lemmas for the corresponding functional \(J\). In Section 3 we give the proof of Theorem 1 and some comments.
2. Preliminaries

By [1] for each \( u \in H_{c,a} \) the following inequality holds:

\[
\int_{a}^{+\infty} e^{ct} u'(t)^2 \, dt \geq \frac{c}{2} e^{ct_0} u(t_0)^2 + \frac{c^2}{4} \int_{a}^{+\infty} e^{ct} u(t)^2 \, dt
\]

for any \( t_0 \in [a, +\infty) \). The inequality (6) shows that in the linear space \( H_{c,a} \) we can introduce the norm

\[
\|u\|_{H_{c,a}} = \left( \int_{a}^{+\infty} e^{ct} u'(t)^2 \, dt \right)^{\frac{1}{2}},
\]

corresponding to the scalar product \( \langle u, v \rangle_{H_{c,a}} = \int_{a}^{+\infty} e^{ct} u'(t) v'(t) \, dt \).

For each function \( u \in H_{c,a} \subset L^\infty[a, +\infty), \sup_{t \in [a, +\infty)} |u(t)| < +\infty \) and it is attained, i.e.

\[
\sup_{t \in [a, +\infty)} |u(t)| = \max_{t \in [a, +\infty)} |u(t)| = |u(t_1)|
\]

for some point \( t_1 \in [a, +\infty) \). Further by \( C \) we will denote various positive constants not depending on \( u \).

We have the following lemma.

**Lemma 1.** Let \( u \in H_{c,a} \). Then the inequality

\[
\int_{a}^{+\infty} e^{ct} u'(t)^2 \, dt \geq C(u(t_1)^2 + \int_{a}^{+\infty} e^{ct} u(t)^2 \, dt + \int_{a}^{+\infty} \frac{e^{ct} |u(t)|^{q+2}}{|u(t_1)|^q} \, dt)
\]

\[(7)\]

holds for a constant \( C > 0 \).

**Proof.** We have by (6) that

\[
\int_{a}^{+\infty} e^{ct} u'(t)^2 \, dt \geq \frac{c}{2} e^{ct_1} u(t_1)^2,
\]

\[
\int_{a}^{+\infty} e^{ct} u'(t)^2 \, dt \geq \frac{c^2}{4} \int_{a}^{+\infty} e^{ct} u(t)^2 \, dt.
\]

By Young’s inequality we get

\[
\int_{a}^{+\infty} e^{ct} u'(t)^2 \, dt \geq C(u(t_1)^2 + \int_{a}^{+\infty} e^{ct} u(t)^2 \, dt)
\]
\[
\begin{align*}
&\geq C \left( u(t_1)^2 + \int_a^{+\infty} e^{ct} |u(t)|^{q+2} \, dt \right) \\
&\geq C \left( u(t_1)^2 \right)^{\frac{q}{q+2}} \left( \int_a^{+\infty} e^{ct} |u(t)|^{q+2} \, dt \right)^{\frac{2}{q+2}} \\
&\geq C \left( \int_a^{+\infty} e^{ct} |u(t)|^{q+2} \, dt \right)^{\frac{2}{q+2}}, \quad C > 0
\end{align*}
\]

for each nonzero function \( u \in H_{c,a} \). These inequalities imply (7).

Next, we have the following

**Lemma 2.** Let the function \( f(t, s) \in C^1 (\mathbb{R}^2) \) satisfy the conditions (H1), (H4) and \( \mu(u) \) is defined by (5). Then, for every nonzero function \( u(t) \in H_{c,a} \),

\[
\mu_{[a, +\infty)}(u) \geq C > 0,
\]

where the constant \( C \) does not depend on \( u \).

**Proof.** By (4),

\[
\begin{align*}
J(\lambda u) &= \frac{1}{2} \lambda^2 \int_a^{+\infty} e^{ct} u'(t)^2 \, dt - \int_a^{+\infty} e^{ct} F(t, \lambda u(t)) \, dt \\
&\geq \frac{1}{2} \lambda^2 \int_a^{+\infty} e^{ct} u'(t)^2 \, dt - \left( \frac{c_2}{q + 2} \int_a^{+\infty} e^{ct} |u(t)|^{q+2} \, dt \right) \lambda^{q+2} \\
&= A_1 \lambda^2 - B \lambda^{q+2},
\end{align*}
\]

where \( A_1 = \frac{1}{2} \int_a^{+\infty} e^{ct} u'(t)^2 \, dt > 0 \), \( B = \frac{c_2}{q + 2} \int_a^{+\infty} e^{ct} |u(t)|^{q+2} \, dt > 0 \), since \( u(t) \in H_{c,a} \) is nonzero function. We have that for every \( \lambda \in [0, +\infty) \),

\[
\mu_{[a, +\infty)}(u) = \sup_{\lambda > 0} J(\lambda u)
\]

\[
\begin{align*}
&\geq c_4(q) \frac{2^q \cdot q}{(q + 2)^{1+\frac{2}{q}}} \left( \frac{1}{2} \int_a^{+\infty} e^{ct} u'(t)^2 \, dt \right)^{\frac{2}{q+2}} \\
&\geq c_5(q) > 0,
\end{align*}
\]

where the constants \( c_4(q) \) and \( c_5(q) \) depend only on \( q \). In the conclusion of the last inequality, we took into account (7). The lemma is proved.

Let \( u(t) \in H_{c,a} \) be an arbitrary fixed nonzero function and the conditions of Lemma 2 be fulfilled. By (4) we have
\[ J(\lambda u) = \frac{1}{2} \lambda^2 \int_{a}^{+\infty} e^{ct} u'(t)^2 \, dt - \int_{a}^{+\infty} e^{ct} F(t, \lambda u(t)) \, dt \]

\[ \leq \lambda^2 A_1 - \left( \frac{c_1}{q + 2} \right) \int_{a}^{+\infty} e^{ct} |u(t)|^{q+2} \, dt \lambda^{q+2} \]

\[ = A_1 \lambda^2 - B_1 \lambda^{q+2}. \]

Then \( J(\lambda u) < 0 \) for sufficiently large \( \lambda > 0 \). Since \( J(0) = 0 \) and \( J(\lambda u) \) is continuous function in \( \lambda \), then \( \mu(u) = \sup_{\lambda > 0} J(\lambda u) \) is attained.

Moreover \( \mu(u) = \mu(ku) \) for every constant \( k > 0 \).

Thus for the given nonzero function \( u(t) \in H_{c,a} \), there exists a positive number \( \lambda_0 > 0 \), such that

\[ \mu(\lambda_0 u) = J(\lambda_0 u) = \sup_{\lambda > 0} J(\lambda u). \]

If we denote the function \( \lambda_0 u \in H_{c,a} \setminus \{0\} \) again by \( u \), then the last equality can be written as

\[ \mu(u) = J(u). \quad (8) \]

We show that for any nonzero function \( v \in H_{c,a} \), i.e. function belonging to \( H_{c,a} \setminus \{0\} \), there exists a function \( u \in H_{c,a} \setminus \{0\} \) such that \( u = kv \), with suitable constant \( k > 0 \), such that (8) holds. As in [9, p.2017] it follows that

\[ \frac{\partial}{\partial \lambda} J(\lambda u) \big|_{\lambda=1} = \nabla J(u) u = \int_{a}^{+\infty} e^{ct} \left( u'(t)^2 - f(t, u(t)) u(t) \right) \, dt = 0, \quad (9) \]

which holds for critical points of \( J(\lambda u) \) as a function of \( \lambda \).

**Lemma 3.** Let the function \( f(t, s) \in C^1(\mathbb{R}^2) \) satisfy the conditions \( (H1), (H2) \) and \( (H4) \) and \( u \in H_{c,a} \) be nonzero function, for which (9) holds. Then

\[ J''(u)[u, u] < 0. \]

Moreover there exists unique number \( \lambda = \lambda(u) > 0 \) such that \( \mu(u) = J(\lambda(u)) \) and the function \( u \to \lambda(u) \) is of class \( C^1 \).

**Proof.** We suppose that for the nonzero function \( u \in H_{c,a} \), (9) holds, but \( J''(u)[u, u] \geq 0 \). Then

\[ \int_{a}^{+\infty} e^{ct} \left( u'(t)^2 - f_s(t, u(t)) u(t)^2 \right) \, dt \geq 0. \quad (10) \]
Subtracting (10) from (9), we obtain that
\[
\int_a^{+\infty} e^{ct} \left( f_s(t, u(t)) u(t)^2 - f(t, u(t)) u(t) \right) dt \leq 0.
\]
Taking into account (H1) and (H2), we get
\[
0 \geq \int_a^{+\infty} e^{ct} \left( f_s(t, u(t)) u(t)^2 - f(t, u(t)) u(t) \right) dt \\
\quad \geq \int_a^{+\infty} e^{ct} (A - 1) f(t, u(t)) u(t) dt \\
\quad \geq c_5 (A - 1) \int_a^{+\infty} e^{ct} |u(t)|^{2+q} dt,
\]
where the constant \( c_5 > 0 \) and \( A > 1 \). Thus we proved that \( \int_a^{+\infty} e^{ct} |u(t)|^{2+q} dt \leq 0 \). But it is impossible for nonzero function \( u(t) \). The obtained contradiction shows that the considered function \( u(t) \) satisfies the inequality \( J''(u) [u, u] < 0 \).

The rest of the proof of the lemma is as in [9, Proposition 3.1]. Exactly, the unique number \( \lambda = \lambda(u) > 0 \) such that \( \mu(u) = J(\lambda(u) u) \), satisfies the equation
\[
\Phi(\lambda(u), u) := \nabla J(\lambda(u) u) . u = 0.
\]
Also \( \frac{\partial}{\partial \lambda} \Phi(\lambda, u) = \frac{\partial}{\partial \lambda} (\nabla J(\lambda u) . u) = J''(\lambda u) [u, u] < 0 \) for \( \lambda = \lambda(u) \), where \( \Phi \) is of class \( C^1 \) and the function \( \lambda = \lambda(u) \) can be locally implicitly defined. From the implicit function theorem \( \lambda(u) \) is of class \( C^1 \). Lemma 3 is proved.

The considerations in the proof of Lemma 3 show that
\[
\varphi^+(a, +\infty) = \inf_{N^+[a, +\infty)} J(u).
\]

3. Proof of the main result

Let \( \{u_n\} \subset N^+[a, +\infty) \) be a minimizing sequence for \( \varphi^+(a, +\infty) \). Without loss of generality, we can suppose that
\[
\varphi^+(a, +\infty) + \varepsilon \geq J(u_n) \longrightarrow \varphi^+(a, +\infty), \quad n \to +\infty \quad (11)
\]
for sufficiently small number \( \varepsilon > 0 \).

Proof of Theorem 1:
Step 1. We have
\[
c_6 \int_a^{+\infty} e^{ct} u_n'(t)^2 \, dt \leq J(u_n) \leq c_7 \int_a^{+\infty} e^{ct} u_n'(t)^2 \, dt, \tag{12}
\]
\[
c_8 \int_a^{+\infty} e^{ct} F(t, u_n(t)) \, dt \leq J(u_n) \leq c_9 \int_a^{+\infty} e^{ct} F(t, u_n(t)) \, dt \tag{13}
\]
for some positive constants \(c_i, i = 6, 7, 8, 9\). Since \(\nabla J(u_n) \cdot u_n = 0\), as in (9),
\[
\int_a^{+\infty} e^{ct} \left( u_n'(t)^2 - f(t, u_n(t)) u_n(t) \right) \, dt = 0.
\]
We have \(J(u_n) = \int_a^{+\infty} e^{ct} \left( \frac{1}{2} u_n'(t)^2 - F(t, u_n(t)) \right) \, dt\). By (H3) and (4),
\[
2J(u_n) = \int_a^{+\infty} e^{ct} \left( f(t, u_n(t)) u_n(t) - 2F(t, u_n(t)) \right) \, dt 
\geq \alpha \int_a^{+\infty} e^{ct} F(t, u_n(t)) \, dt \geq \frac{\alpha c_1}{q + 2} \int_a^{+\infty} e^{ct} |u_n(t)|^{2+q} \, dt.
\]
This inequality shows that \(J(u_n) \geq 0\). Hence
\[
\frac{1}{2} \int_a^{+\infty} e^{ct} u_n'(t)^2 \, dt = \int_a^{+\infty} e^{ct} F(t, u_n(t)) \, dt + J(u_n) 
\leq \frac{c_2}{q + 2} \int_a^{+\infty} e^{ct} |u_n(t)|^{2+q} \, dt + J(u_n) 
\leq \left( 1 + \frac{2c_2}{\alpha c_1} \right) J(u_n)
\]
and
\[
J(u_n) \geq \frac{\alpha c_1}{2\alpha c_1 + 4c_2} \int_a^{+\infty} e^{ct} u_n'(t)^2 \, dt.
\]
By the definition of \(J(u_n)\) and (4), \(J(u_n) \leq \frac{1}{2} \int_a^{+\infty} e^{ct} u_n'(t)^2 \, dt\). Thus, we proved the first inequality (12) with \(c_6 = \frac{\alpha c_1}{2\alpha c_1 + 4c_2}\) and \(c_7 = \frac{1}{2}\). The inequality (13) holds with \(c_8 = \frac{\alpha}{2}\) and \(c_9 = \frac{c_2(q+2)}{2c_1}\) by (H1) and (4) since
\[
2J(u_n) \leq \int_a^{+\infty} e^{ct} u_n'(t)^2 \, dt = \int_a^{+\infty} e^{ct} f(t, u_n(t)) u_n(t) \, dt 
\leq \frac{c_2(q+2)}{c_1} \int_a^{+\infty} e^{ct} F(t, u_n(t)) \, dt.
\]
These assertions show that $$\int_a^{+\infty} e^{ct} u'_n(t)^2 \, dt$$ and $$\int_a^{+\infty} e^{ct} |u_n(t)|^{2+q} \, dt$$ are bounded by constant, which does not depend on $$n$$. From the inequality (6), the same is true for $$\int_a^{+\infty} e^{ct} u_n(t)^2 \, dt$$. Hence

$$\int_a^{+\infty} e^{ct} (u_n(t)^2 + u'_n(t)^2) \, dt$$

and

$$\int_a^{+\infty} e^{ct} |u_n(t)|^{2+q} \, dt$$

are bounded by a constant, which does not depend on $$n$$. Then, the sequence $$\{u_n\}$$ is bounded in $$H_{c,a}$$ equipped by the norm $$\|u\|_{H_{c,a}} = \left(\int_a^{+\infty} e^{ct} u'(t)^2 \, dt\right)^{\frac{1}{2}}$$.

**Step 2.** There exists a function $$u_0 \in H^0_{c,a} := H_{c,a} \cap H^1_0[a, +\infty)$$ and subsequence $$\{u_{n_k}\}$$ still denoted by $$\{u_n\}$$, such that $$u_n \to u_0$$ for $$n \to +\infty$$ in the weak $$H^0_{c,a}$$-topology; $$u_n \to u_0$$ for $$n \to +\infty$$ in the strong $$L^2$$-topology and on any bounded and closed subinterval of $$[a, +\infty)$$. Moreover $$u_0 \geq 0$$ is nonzero function. Let us remind that $$\{u_n\} \subset N^+(a, +\infty)$$ and then $$u_n \geq 0, n = 1, 2, \ldots$$. Hence $$u_0 \geq 0$$.

We will prove that $$u_0$$ is nonzero function. Suppose the contrary, i.e., that $$u_0 \equiv 0$$. This means that $$u_n \to 0$$ for $$n \to +\infty$$ in the weak $$H^0_{c,a}$$-topology and $$u_n \to 0$$ for $$n \to +\infty$$ in the strong $$L^2$$-topology on any bounded and closed subinterval of $$[a, +\infty)$$. Let $$b > a$$ be an arbitrary number. From [1, p.321],

$$|u_n(b)| \leq \left(\frac{e^{-cb}}{c} \int_b^{+\infty} e^{ct} u'_n(t)^2 \, dt\right)^{\frac{1}{2}} \leq c_{10} e^{-\frac{c}{2} b}, \quad \forall b > a$$  \tag{14}

and the constant $$c_{10} > 0$$ does not depend on $$n \in \mathbb{N}$$ and $$b$$. Let $$b \in (a, +\infty)$$ be a fixed (sufficiently large) number. Then for every $$\varepsilon > 0$$, there exists $$n_0 \in \mathbb{N}$$, depending on $$b$$ and $$\varepsilon$$, such that

$$\int_a^b e^{ct} u_n(t)^2 \, dt \leq e^{cb} \int_a^b u_n(t)^2 \, dt < \varepsilon, \quad \forall n \geq n_0.$$

From (14), replacing $$b$$ by $$t$$, $$|u_n(t)| \leq c_{10} e^{-\frac{c}{2} t}, \quad \forall t \in [a, +\infty)$$ and for some $$n_1 > n_0, n_1 \in \mathbb{N}$$, depending on $$b$$, we get by $$q > 0$$

$$\int_a^b e^{ct} |u_n(t)|^{2+q} \, dt \leq c_{10} \int_a^b e^{ct} |u_n(t)|^2 \, dt < \frac{\varepsilon}{2}, \quad \forall n \geq n_1.$$  \tag{15}

From (14), we have

$$\int_b^{+\infty} e^{ct} |u_n(t)|^{2+q} \, dt \leq \max_{t \in [b, +\infty)} |u_n(t)|^q \cdot \int_b^{+\infty} e^{ct} |u_n(t)|^2 \, dt \leq c_{11} e^{-\frac{c}{2} bq}$$  \tag{16}
and the constant \( c_{11} > 0 \) does not depend on \( b \in (a, +\infty) \) and \( n \in \mathbb{N} \). Now let \( \varepsilon > 0 \) be a fixed, sufficiently small number. We choose the number \( b \in (a, +\infty) \) so large, such that \( c_{11} e^{-\frac{bq}{2}} < \frac{\varepsilon}{2} \). Then by (16),

\[
\int_{b}^{+\infty} e^{ct} |u_n(t)|^{2+q} dt < \frac{\varepsilon}{2}, \quad \forall n \in \mathbb{N}.
\]

We choose the number \( n_1 \in \mathbb{N}, n_1 > n_0 \), such that \( (15) \) holds and hence

\[
\int_{a}^{+\infty} e^{ct} |u_n(t)|^{2+q} dt < \varepsilon, \quad \forall n \geq n_1.
\]

Thus we prove that \( \lim_{n \to +\infty} \int_{a}^{+\infty} e^{ct} |u_n(t)|^{2+q} dt = 0 \).

By Step 1, (4) and (13) it follows \( \lim_{n \to +\infty} J(u_n) = 0 \). This contradicts to Lemma 2, according which \( J(u_n) = \mu_{(a, +\infty)}(u_n) \geq C > 0 \). The contradiction shows that \( u_0 \) is a nonzero function.

**Step 3.** \( \int_{a}^{+\infty} e^{ct} u_n'(t)^2 dt \geq \int_{a}^{+\infty} e^{ct} u_0'(t)^2 dt + o(1) \).

We have

\[
\int_{a}^{+\infty} e^{ct} u_n'(t)^2 dt = \int_{a}^{+\infty} e^{ct} \left[ u_0'(t) + (u_n'(t) - u_0'(t)) \right]^2 dt \tag{17}
\]

\[
= \int_{a}^{+\infty} e^{ct} u_0'(t)^2 dt + 2 \int_{a}^{+\infty} e^{ct} (u_n'(t) - u_0'(t)) u_0'(t) dt \]

\[
+ \int_{a}^{+\infty} e^{ct} (u_n'(t) - u_0'(t))^2 dt.
\]

By \( u_n \rightarrow u_0 \) for \( n \to +\infty \) weakly in \( H^0_{c,a} \) we have

\[
\lim_{n \to +\infty} \int_{a}^{+\infty} e^{ct} (u_n'(t) - u_0'(t)) u_0'(t) dt = \lim_{n \to +\infty} \langle u_n'(t) - u_0'(t), u_0'(t) \rangle_{H_{c,a}} = 0.
\]

Then

\[
2 \int_{a}^{+\infty} e^{ct} (u_n'(t) - u_0'(t)) u_0'(t) dt = o(1),
\]

and by (17),

\[
\int_{a}^{+\infty} e^{ct} u_n'(t)^2 dt \geq \int_{a}^{+\infty} e^{ct} u_0'(t)^2 dt + o(1).
\]

**Step 4.** \( \lim_{n \to +\infty} \int_{a}^{+\infty} e^{ct} F(t, u_n(t)) dt = \int_{a}^{+\infty} e^{ct} F(t, u_0(t)) dt \).

As in Step 2, replacing \( u_n \) by \( u_n - u_0 \), we can prove that

\[
\lim_{n \to +\infty} \int_{a}^{+\infty} e^{ct} |u_n(t) - u_0(t)|^{2+q} dt = 0. \tag{18}
\]
By (H1) and \( u_n(t) \geq 0 \), \( u_0(t) \geq 0 \),

\[
|F(t, u_n(t)) - F(t, u_0(t))| = \left| \int_{u_0(t)}^{u_n(t)} f(t, \tau) \, d\tau \right| \leq c_2 \int_{u_0(t)}^{u_n(t)} \tau^{1+q} \, d\tau
\]

\[
= \frac{c_2}{q + 2} \left| u_n(t)^{2+q} - u_0(t)^{2+q} \right|.
\]

Hence

\[
\left| \int_a^{+\infty} e^{ct} F(t, u_n(t)) \, dt - \int_a^{+\infty} e^{ct} F(t, u_0(t)) \, dt \right|
\]

\[
\leq \frac{c_2}{q + 2} \int_a^{+\infty} e^{ct} \left| u_n(t)^{2+q} - u_0(t)^{2+q} \right| \, dt
\]

\[
\leq K_1 \int_a^{+\infty} e^{ct} \left| u_n(t) - u_0(t) \right| \left( u_n(t)^{1+q} + u_0(t)^{1+q} \right) \, dt.
\]

Using the Hölder inequality, it is easy to obtain that

\[
\left| \int_a^{+\infty} e^{ct} F(t, u_n(t)) \, dt - \int_a^{+\infty} e^{ct} F(t, u_0(t)) \, dt \right|
\]

\[
\leq K_1 \left( \int_a^{+\infty} e^{ct} (u_n(t)^{1+q} + u_0(t)^{1+q})^{\frac{2+q}{1+q}} \, dt \right)^{\frac{1+q}{2+q}}\]

\[
\left( \int_a^{+\infty} e^{ct} \left| u_n(t) - u_0(t) \right|^{2+q} \, dt \right)^{\frac{1}{2+q}}
\]

\[
\leq K_2 \left( \int_a^{+\infty} e^{ct} u_n(t)^{2+q} \, dt + \int_a^{+\infty} e^{ct} u_0(t)^{2+q} \, dt \right)^{\frac{1+q}{2+q}}\]

\[
\left( \int_a^{+\infty} e^{ct} \left| u_n(t) - u_0(t) \right|^{2+q} \, dt \right)^{\frac{1}{2+q}}
\]

\[
\leq K_3 \left( \int_a^{+\infty} e^{ct} \left| u_n(t) - u_0(t) \right|^{2+q} \, dt \right)^{\frac{1}{2+q}},
\]

where \( K_j, j = 1, 2, 3 \) are constants not depending on \( u \) and \( n \). Taking into account (18), we obtain the assertion of Step 4.

**Step 5.** There exists a nonzero function \( u_+ \in N^+(a, +\infty) \), which \( u_+ \geq 0 \) and \( J_{(a, +\infty)}(u_+) = \varphi^+(a, +\infty) = \inf_{N^+(a, +\infty)} J_{(a, +\infty)}(u) > 0 \).
From (4), Step 3 and Step 4 it follows that \( J(u_n) \geq J(u_0) + o(1) \) (i.e., the functional \( J \) is weakly lower semi continuous). Hence the inequality \( J_{[a, +\infty)}(\lambda u_n) \leq J_{[a, +\infty)}(\lambda u_0) + o(1), \forall \lambda > 0 \) holds.

Define

\[
  u_+ := \lambda(u_0)u_0.
\]

We obtain, that \( u_+ \) satisfies the conditions of Step 5.

We will prove that \( u_+ > 0 \) on \((a, +\infty)\). For this purpose, we adapt the proof of Theorem 3.1 of [9, pp. 2019-2020] to our case.

**Claim 1.** \( u_+ \in C^1(a, +\infty) \). We assume that there exists \( \tau \in (a, +\infty) \), for which \( u_+(\tau) = 0 \) and \( u_+'(t) \) is not continuous for \( t = \tau \). Let \( u_+'(\tau - 0) < 0 \) and the constants \( \rho > 0 \) and \( \varepsilon > 0 \) are sufficiently small numbers. For \( \lambda \in [1 - \varepsilon, 1 + \varepsilon] \) we consider the class of problems

\[
\inf \left\{ \begin{array}{l} J_{[\tau - \rho, \tau + \rho]}(v) : v \in H^1(\tau - \rho, \tau + \rho), \quad v(\tau - \rho) = \lambda u_+(\tau - \rho) \\ \|v\|_\infty \leq 1 \end{array} \right\}
\]

where \( \tau - \rho, \tau + \rho \in (a, +\infty) \). We have

\[
\frac{d^2}{d\lambda^2}J_{[\tau - \rho, \tau + \rho]}(u + \lambda \varphi)_{\lambda=0} = \int_{\tau - \rho}^{\tau + \rho} e^{ct} \left( \varphi'(t)^2 - f_s'(t, u(t)) \varphi(t)^2 \right) dt
\]

\[
\geq \int_{\tau - \rho}^{\tau + \rho} e^{ct} \varphi'(t)^2 dt - c_{13} \int_{\tau - \rho}^{\tau + \rho} e^{ct} \varphi(t)^2 dt
\]

\[
\geq \left( \frac{c_{14}}{\rho^2} - c_3 \right) \int_{\tau - \rho}^{\tau + \rho} e^{ct} \varphi(t)^2 dt > 0,
\]

where the constants \( c_{13} := \sup \{ f_s'(t, u) : a \leq t < +\infty, -1 \leq u \leq 1 \} \), \( c_{14} > 0 \), and the function \( \varphi(t) \) vanishes on at least one point \( t = t_0 \in [\tau - \rho, \tau + \rho] \). As in [9, p. 2026, Lemma 5.1], we can conclude, that \( J \) is strictly convex, and thus the minimum of (19) is uniquely achieved by a function \( v_\lambda \). We will show that \( v_\lambda \) satisfies Eq. (1.) and \( \lim_{\rho \to 0} \|v_\lambda\|_{H^1(\tau - \rho, \tau + \rho)} = 0 \). For this purpose, we need to prove that \( |v_\lambda(t)| < 1 \) and \( v_\lambda(t) \geq 0 \) for every \( t \in [\tau - \rho, \tau + \rho] \).

First, we prove that \( \|v_\lambda\|_\infty = \max_{t \in [\tau - \rho, \tau + \rho]} |v_\lambda(t)| < 1 \). Suppose the contrary, \( \|v_\lambda\|_\infty = 1 \). By inclusion \( H^1(\tau - \rho, \tau + \rho) \subset C[\tau - \rho, \tau + \rho] \), \( u_+ \in C[\tau - \rho, \tau + \rho] \) and \( v_\lambda \in C[\tau - \rho, \tau + \rho] \). By \( u_+(\tau) = 0 \), \( v_\lambda(\tau \pm \rho) = \lambda u_+(\tau \pm \rho) = o(1) \) for \( \rho \to 0^+ \). If \( v_\lambda(\tau_1) = \pm 1 \) for some \( \tau_1 \in (\tau - \rho, \tau + \rho) \), then
\begin{align*}
|v_\lambda (\tau_1) - v_\lambda (\tau - \rho)| &= 1 - o(1) = \left| \int_{\tau - \rho}^{\tau_1} v_\lambda' (t) \, dt \right| \\
&\leq \int_{\tau - \rho}^{\tau + \rho} e^{-\frac{ct}{2} e^{\frac{ct}{2}}} |v_\lambda' (t)| \, dt \\
&\leq \left( \int_{\tau - \rho}^{\tau + \rho} e^{-ct} \, dt \right)^{\frac{1}{2}} \left( \int_{\tau - \rho}^{\tau + \rho} e^{ct} v_\lambda' (t)^2 \, dt \right)^{\frac{1}{2}} \\
&= \left( \frac{e^{-c(\tau - \rho)} - e^{-c(\tau + \rho)}}{c} \right)^{\frac{1}{2}} \left( \int_{\tau - \rho}^{\tau + \rho} e^{ct} v_\lambda' (t)^2 \, dt \right)^{\frac{1}{2}} \\
&\leq c_{15} \sqrt{\rho} \left( \int_{\tau - \rho}^{\tau + \rho} e^{ct} v_\lambda' (t)^2 \, dt \right)^{\frac{1}{2}},
\end{align*}

where the constant \( c_{15} > 0 \) is close to \( (2e^{-ct})^{\frac{1}{2}} \) for small \( \rho \), i.e. \( c_{15} \) depends only on \( \tau \). Since \( 1 - o(1) \geq \sqrt{2} \) for small \( \rho \), then

\[ \int_{\tau - \rho}^{\tau + \rho} e^{ct} v_\lambda' (t)^2 \, dt \geq \frac{1}{2c_{15}^2 \rho}. \]

Since \( F(t, \nu_\lambda) \) is bounded for \( t \in [\tau - \rho, \tau + \rho] \), and \( \frac{1}{2c_{15}^2 \rho} - C \geq \frac{c_{16}}{\rho} \), for small \( \rho > 0 \), it implies

\[ J_{[\tau - \rho, \tau + \rho]} (v_\lambda) \geq \frac{c_{16}}{\rho}. \tag{21} \]

Remind that \( u_+ \in H^1 (\tau - \rho, \tau + \rho) \) and \( u_+ (\tau) = 0 \). As above

\[ \max_{[\tau - \rho, \tau + \rho]} |u_+ (t)| = \max_{[\tau - \rho, \tau + \rho]} |u_+ (t) - u_+ (\tau)| \]

\[ \leq c_{15} \sqrt{\rho} \left( \int_{\tau - \rho}^{\tau + \rho} e^{ct} u_+ ' (t)^2 \, dt \right)^{\frac{1}{2}} = o(1) \sqrt{\rho}, \]

because \( u_+ \in H^0_{c, a} \) and then \( \left( \int_{\tau - \rho}^{\tau + \rho} e^{ct} u_+ ' (t)^2 \, dt \right)^{\frac{1}{2}} = o(1) \) when \( \rho > 0 \) is sufficiently small. Thus

\[ |\lambda u_+ (\tau \pm \rho)| = o(\sqrt{\rho}), \quad \forall \lambda \in [1 - \varepsilon, 1 + \varepsilon]. \]

Now we consider the linear function

\[ w(t) = u_+ (\tau - \rho) - \frac{u_+ (\tau - \rho) - u_+ (\tau + \rho)}{2\rho} (t - \tau + \rho). \]
Evidently
\[ w(t - \rho) = u_+(\tau - \rho), \]
\[ w(t + \rho) = u_+(\tau + \rho) \] and \( w'(t) = \frac{u_+(\tau + \rho) - u_+(\tau - \rho)}{2\rho} \). As above we have
\[
|u_+(\tau - \rho) - u_+(\tau + \rho)| = \left| \int_{\tau - \rho}^{\tau + \rho} u'(s) \, ds \right|
\leq c_{15} \sqrt{\rho} \left( \int_{\tau - \rho}^{\tau + \rho} e^{c t} u'_+(t)^2 \, dt \right)^{\frac{1}{2}} = \sqrt{\rho} o(1),
\]
because \( u_+ \in H^0_{c,a} \). Then \( \left( \int_{\tau - \rho}^{\tau + \rho} e^{c t} u'_+(t)^2 \, dt \right)^{\frac{1}{2}} = o(1) \) when \( \rho > 0 \) is sufficiently small. Thus
\[
|w'(t)| \leq \frac{C}{\sqrt{\rho}} o(1),
\]
\[
0 \leq \int_{\tau - \rho}^{\tau + \rho} e^{c t} w'(t)^2 \, dt \leq \frac{C}{\rho} o(1) \int_{\tau - \rho}^{\tau + \rho} e^{c t} \, dt \leq o(1).
\]
Also,
\[
|w(t)| \leq \max(u_+(\tau - \rho), u_+(\tau + \rho))
\leq \max_{[\tau - \rho, \tau + \rho]} |u_+(t)| = o(1) \sqrt{\rho}, \quad \forall t \in [\tau - \rho, \tau + \rho].
\]
Hence \( \|w\|_{H^1[\tau - \rho, \tau + \rho]} = o(1) \) and
\[
J_{[\tau - \rho, \tau + \rho]}(w) = o(1), \quad J_{[\tau - \rho, \tau + \rho]}(\lambda w) = o(1) \quad \forall \lambda \in [1 - \varepsilon, 1 + \varepsilon]. \quad (22)
\]
From (21) and (22), \( J_{[\tau - \rho, \tau + \rho]}(\lambda w) << J_{[\tau - \rho, \tau + \rho]}(v_\lambda), \) \( \forall \lambda \in [1 - \varepsilon, 1 + \varepsilon] \) for sufficiently small \( \rho > 0 \), (where \( a << b \) means that \( \frac{a}{b} = o(1) \)). The last inequality contradicts to the fact, that infimum in (19) is attained by the function \( v_\lambda \). The contradiction is due to the assumption, that \( \|v_\lambda\|_\infty = 1 \). Therefore \( \|v_\lambda\|_\infty < 1 \) and thus \( v_\lambda \) is a solution of Eq.(1).

Now we will prove that \( v_\lambda \geq 0 \) for \( t \in [\tau - \rho, \tau + \rho] \). Suppose the contrary. Then changing the sign, \( v_\lambda \) vanish at some point of \([\tau - \rho, \tau + \rho] \) and by \( v_\lambda(\tau \pm \rho) = \lambda u_+(\tau \pm \rho) \geq 0 \), it follows that \( v'_\lambda \) also vanish at some other point of \([\tau - \rho, \tau + \rho] \). Since \( v_\lambda \) satisfies (1), then
\[
(e^{c t} v'_\lambda) + e^{c t} f(t, v_\lambda) = 0 \implies v'_\lambda(t) = - \int_{t_1}^{t} e^{-c(t-s)} f(s, v_\lambda(s)) \, ds,
\]
where \( t_1 \in [\tau - \rho, \tau + \rho] \) is such that \( v'_{\lambda}(t_1) = 0 \). Then
\[
|v'_{\lambda}(t)| \leq \int_{t_1}^{t} e^{-c(t-s)} |f(s, v_{\lambda}(s))| \, ds
\]
\[
\leq c_{17} \int_{\tau - \rho}^{\tau + \rho} e^{-c(t-s)} \, ds = c_{17} \frac{e^{-c(\tau - \rho)} - e^{-c(\tau + \rho)}}{c} \leq c_{18} \rho,
\]
because \( f(s, v_{\lambda}(s)) \) is bounded, for \( s \in [\tau - \rho, \tau + \rho] \). Thus
\[
|v'_{\lambda}(t)| \leq c_{18} \rho \quad \forall t \in [\tau - \rho, \tau + \rho],
\tag{23}
\]
where the constant \( c_{18} > 0 \) depends only on \( \tau \). Recall that \( u_{+}(\tau) = 0 \) and \( u'_{+}(\tau - 0) < 0 \). Then for \( \rho > 0 \) sufficiently small, \( u_{+}(\tau - \rho) = u_{+}(\tau) - \rho u'_{+}(\tau - 0) = -\rho u'_{+}(\tau - 0) \). Hence
\[
v_{\lambda}(\tau - \rho) = \lambda u_{+}(\tau - \rho) = -\lambda \rho u'_{+}(\tau - 0),
\]
i.e. \( c_{19} \rho \leq v_{\lambda}(\tau - \rho) \leq c_{20} \rho \) for \( \rho > 0 \) sufficiently small. Then \( v_{\lambda}(t) = v_{\lambda}(\tau - \rho) + \int_{\tau - \rho}^{t} v'_{\lambda}(s) \, ds \) and from (23)
\[
\left| \int_{\tau - \rho}^{t} v'_{\lambda}(s) \, ds \right| \leq c_{18} \rho^2 << v_{\lambda}(\tau - \rho), \quad \forall t \in [\tau - \rho, \tau + \rho].
\]
Thus \( v_{\lambda}(t) \) cannot change the sign when \( t \in [\tau - \rho, \tau + \rho] \). This consideration is true also, when \( u'_{+}(\tau - 0) = -\infty \), because \( v_{\lambda}(\tau - \rho) \geq C \rho \) for sufficiently large constant \( C > 0 \) and \( v_{\lambda}(t) \) does not change the sign again. We prove, that \( v_{\lambda} \geq 0 \) for \( t \in [\tau - \rho, \tau + \rho] \).

Note that \( v_{\lambda}(t) \in C^1(\tau - \rho, \tau + \rho) \), because \( v_{\lambda}(t) \) satisfies Eq. (1) for \( t \in (\tau - \rho, \tau + \rho) \). Now we define the function
\[
\tilde{u}_{\lambda}(t) := \begin{cases} v_{\lambda}(t), & \quad t \in [\tau - \rho, \tau + \rho], \\ \lambda u_{+}(t), & \quad t \in [a, +\infty) \setminus [\tau - \rho, \tau + \rho]. \end{cases}
\]
We should note that \( \tilde{u}_{\lambda}(t) \geq 0 \quad \forall t \in [a, +\infty) \) and \( \tilde{u}_{\lambda}(t) \in H^0_{c,a} \), because \( v_{\lambda}(t) \) satisfies the boundary conditions in (19). We will prove, that \( \tilde{u}_{\lambda} \rightarrow \lambda u_{+} \) in \( H^0_{c,a} \), when \( \rho \rightarrow 0 \). Indeed, from (22), \( J_{[\tau - \rho, \tau + \rho]}(v_{\lambda}) \leq J_{[\tau - \rho, \tau + \rho]}(w) \leq o(1) \) and since
\[
\|v_{\lambda}\|_{\infty} < 1 \quad \implies \quad \int_{\tau - \rho}^{\tau + \rho} F(t, v_{\lambda}(t)) \, dt < \frac{2c_2}{q + 2} \rho,
\]
then \( \int_{\tau - \rho}^{\tau + \rho} e^{ct} v'_{\lambda}(t)^2 \, dt = o(1) \) when \( \rho \rightarrow 0 \). This proves that \( \tilde{u}_{\lambda} \rightarrow \lambda u_{+} \) in \( H^0_{c,a} \), when \( \rho \rightarrow 0 \). Further the proof that \( u_{+} \in C^1(a, +\infty) \) holds as in [9, p.2020]. Finally we should note that there exists finite right derivative \( u'_{+}(a + 0) \). Indeed,
if \( u_+ (t) \equiv 0 \) in a small right neighborhood of \( a \), then obviously \( u'_+ (a + 0) = 0 \). If \( u_+ (t) > 0 \) in a small right neighborhood of \( a \), then \( u_+ (t) \) satisfies (1) and it is easy to show that

\[
u'_+ (a + 0) : = \lim_{t \to a+0} u'_+ (t) = e^{-c(a-t_0)} \left[ u'_+ (t_0) + \int_a^{t_0} e^{c(s-t_0)} f (s, u_+ (s)) \, ds \right],\]

where \( t_0 > a \) is a point, close enough to \( a \). Thus we prove, that \( u_+ \in C^1 [a, +\infty) \).

Claim 2. \( u_+ (t) > 0, \forall t \in [a, +\infty) \).

We assume that \( u_+ (t_0) = 0 \) for some \( t_0 > a \), and \( u_+ (t) > 0 \) in a small right (left) neighbourhood of \( t_0 \). Since \( u_+ \geq 0 \), then \( t_0 \) is a point of local minimum for \( u_+ \) and since \( u_+ \in C^1 (a, +\infty) \), then \( u'_+ (t_0) = 0 \). Since \( f (t, 0) \equiv 0 \), then \( u \equiv 0 \) is a solution of (1) for the Cauchy conditions \( u (t_0) = u' (t_0) = 0 \) and from the uniqueness theorem of the local Cauchy problem, follows that \( u_+ (t) \equiv 0 \) in a small neighbourhood of \( t_0 \). Thus it implies \( u_+ (t) = 0, \forall t \in [a, +\infty) \), which contradicts to the definition of \( u_+ \) as a nonzero function. This contradiction proves the assertion of Claim 2. Theorem 1 is proved.

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