Short Zero-Sum Sequences Over Abelian $p$-Groups of Large Exponent

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Abstract

Let $G$ be a finite abelian group with exponent $n$. Let $\eta(G)$ denote the smallest integer $\ell$ such that every sequence over $G$ of length at least $\ell$ has a zero-sum subsequence of length at most $n$. We determine the precise value of $\eta(G)$ when $G$ is a $p$-group whose Davenport constant is at most $2n - 1$. This confirms one of the equalities in a conjecture by Schmid and Zhuang from 2010.

1 Introduction

Throughout this paper, let $G$ be an additive finite abelian group. By the Fundamental Theorem of Finite Abelian Groups, we can uniquely write

$$G \cong C_{n_1} \oplus \cdots \oplus C_{n_r},$$

where $1 < n_1 | \cdots | n_r$. Here $r = r(G)$ is called the rank of $G$, and $n_r = \exp(G)$ is called the exponent of $G$. The exponent is, for a finite abelian group, the largest order attained by an element of $G$.

We consider sequences of elements of $G$. A sequence of length $k$ over $G$ is a tuple $S = (g_1, \ldots, g_k)$ of $k$ elements of $G$, where repetition is allowed and the order of elements is disregarded. Algebraically, we can consider sequences over $G$ as elements of the free abelian monoid over $G$, a viewpoint that facilitates many algebraic applications. Accordingly, we can write sequences in multiplicative notation; for example, $S = g_1 \cdots g_k$. We say that $S$ is a zero-sum sequence (ZSS) if $\sum_{i=1}^{k} g_i = 0$. A zero-sum sequence of length at most $\exp(G)$ is called a short zero-sum sequence.

One important class of problems on zero-sum sequences asks for extremal conditions on a sequence $S$ guaranteeing the containment of a zero-sum subsequence with specified properties, such as having a length satisfying various constraints. The exploration of this class of problems centers on a family of invariants associated with a group $G$, called the zero-sum invariants $s_L(G)$.

Definition 1. For $L \subseteq \mathbb{N}^+$, let $s_L(G)$ be the minimum length $\ell \in \mathbb{N}^+$ such that every sequence $S$ over $G$ of length at least $\ell$ contains a zero-sum subsequence of some length $t \in L$. Let $s_L(G) = \infty$ if such a length $\ell$ does not exist.
A few examples of zero-sum invariants are of particular interest.

- $s_{N+}(G) = D(G)$ is called the Davenport constant.
- $s_{\{\exp(G)\}}(G) = s(G)$ is called the Erdős-Ginzburg-Ziv (EGZ) constant.
- $s_{[1,\exp(G)]}(G) = \eta(G)$ is called the $\eta$-invariant.

The study of these invariants has applications ranging from non-unique factorizations [7, 8] to zero-sum Ramsey theory [11] and finite geometry [1].

The precise values of $s(G)$ and $\eta(G)$ are known only for some very special types of groups $G$. For example, for groups of rank at most 2, we have the following result.

**Theorem 1.1** (Geroldinger, Halter-Koch, [7, Theorem 5.8.3]). Let $G = C_{n_1} \oplus C_{n_2}$ with $1 \leq n_1|n_2$. Then

$$s(G) = 2n_1 + 2n_2 - 3 \text{ and } \eta(G) = 2n_1 + n_2 - 2.$$ 

In other cases, some bounds on $s(G)$ and $\eta(G)$ are known. Schmid and Zhuang [10] obtained the following result for the case of a $p$-group with $D(G) \leq 2\exp(G) - 1$. We will call such a group a $p$-group with large exponent.

**Theorem 1.2** (Schmid, Zhuang, [10, Theorem 1.2]). Let $p$ be an odd prime and let $G$ be a finite abelian $p$-group with $D(G) \leq 2\exp(G) - 1$. Then

$$2D(G) - 1 \leq \eta(G) + \exp(G) - 1 \leq s(G) \leq D(G) + 2\exp(G) - 2.$$ 

Note that equality holds everywhere when $D(G) = 2\exp(G) - 1$. Schmid and Zhuang conjectured that equality holds between the three leftmost expressions without this extra condition.

**Conjecture 1.3** (Schmid, Zhuang, [10, Conjecture 4.1]). Let $G$ be a finite abelian $p$-group with $D(G) \leq 2\exp(G) - 1$. Then

$$2D(G) - 1 = \eta(G) + \exp(G) - 1 = s(G).$$

Gao, Han, and Zhang made progress on this conjecture in [5], proving the first equality under additional conditions on the rank and Davenport constant of $G$. Their result is stated here.

**Theorem 1.4** (Gao, Han, Zhang, [5, Theorem 1.2]). Let $G$ be a finite abelian $p$-group with $\exp(G) = n$ and $D(G) \leq 2n - 1$. Write $G = C_n \oplus H$, and let $G' = C_{an} \oplus H$ for some $a \geq 1$. Then

$$2D(G') - 1 = \eta(G') + \exp(G') - 1,$$

provided that $p > 2r(H)$ and $\left\lceil \frac{2D(H)}{\exp(H)} \right\rceil$ is either even or at most 3.

Here taking $a = 1$ yields the relevant progress towards Conjecture 1.3.

The second equality in the conjecture is a special case of a conjecture by Gao.
**Conjecture 1.5** (Gao, [3, Conjecture 2.3]). For any finite abelian group $G$,

$$\eta(G) = s(G) - \exp(G) + 1.$$  

This conjecture holds for every $G$ for which $\eta(G)$ and $s(G)$ have been determined, and is also known to hold in a few other special cases (see Theorem 6.6 in [4]). In this paper, we prove the first equality in Conjecture 1.3 in general.

**Theorem 1.6.** Let $G$ be a finite abelian $p$-group with $D(G) \leq 2 \exp(G) - 1$. Then

$$2D(G) - 1 = \eta(G) + \exp(G) - 1.$$  

We also give an extension of Theorem 1.6 to groups of the form $G' = C_a \oplus G$, where $G$ is a $p$-group with $D(G) \leq 2 \exp(G) - 1$ and $a$ is not divisible by $p$.

In the process of proving Theorem 1.6 we introduce some additional zero-sum invariants and techniques for using them, whose study may prove interesting in their own right or when applied to proving the second equality in Conjecture 1.3. In the next section, we recall some useful previous results on zero-sum invariants in various settings. Afterwards, we give a simple proof of our main result and the extension. Finally, we extend the methods we use to make some progress on tackling the second half of Conjecture 1.3 and discuss some ideas for further work.

## 2 Background and Tools

We start by giving some bounds on the invariants $D(G)$, $\eta(G)$, and $s(G)$ for general $G$.

**Theorem 2.1.** For any finite abelian group $G$,

$$D(G) \leq \eta(G) \leq s(G) - \exp(G) + 1 \leq |G|.$$  

This result can be found in Lemma 5.7.2 and Theorem 5.7.4 of [7]. Conjecture 1.5 is just the claim that equality always holds in the second inequality.

Let $D^*(G) = \sum_{i=1}^r (n_i - 1) + 1$. A simple construction shows that $D^*(G) \leq D(G)$ for any $G$. We have already seen that equality holds for a group of rank at most two. Equality also holds for $p$-groups and in another case of interest.

**Theorem 2.2** (Kruyswijk, Olson, [9]). If $G$ is a $p$-group, then $D(G) = D^*(G)$.

**Theorem 2.3** (Geroldinger, [8, Corollary 4.2.13]). If $G' = C_a \oplus G$, where $G$ is a $p$-group with $D(G) \leq 2 \exp(G) - 1$ and $a \in \mathbb{N}^+$, then $D(G') = D^*(G')$.

We will need the following result on short zero-sum subsequences of zero-sum sequences.
Theorem 2.4 (Fan, Gao, Wang, Zhong, Zhuang, [2, Theorem 2]). If \( G = C_{mp^n} \oplus H \) for some \( m, n \geq 1 \), where \( H \) is a \( p \)-group with \( D(H) \leq p^n \), then any zero-sum sequence over \( G \) of length at least \( D(G) + 1 \) contains a short zero-sum subsequence.

We are particularly interested in the case \( m = 1 \).

Corollary 2.5. If \( G \) is a \( p \)-group and \( D(G) \leq 2 \exp(G) - 1 \), then any zero-sum sequence over \( G \) of length at least \( D(G) + 1 \) contains a short zero-sum subsequence.

Proof. A \( p \)-group \( G \) with \( D(G) \leq 2 \exp(G) - 1 \) can be written in the form \( G = C_{p^n} \oplus H \), where \( p^n = \exp(G) \) and, by Theorem 2.2, \( D(H) = D(G) - (p^n - 1) \leq p^n \). So, the conclusion of Theorem 2.4 holds for such \( p \)-groups \( G \).

Finally, we will use the following tools to prove the extension of our main result.

Lemma 2.6 (Edel, Elsholtz, Geroldinger, Kubertin, Rackham, [1, Lemma 3.2]). Let \( H \) be a finite abelian group and \( G' = C_n \oplus H \), where \( \exp(H) \mid n \). Then
\[
\eta(G') \geq 2(D(H) - 1) + n.
\]

Proposition 2.7 (Geroldinger, Halter-Koch, [7, Proposition 5.7.11]). If \( H \) is a subgroup of \( G \) such that \( \exp(G) = \exp(H) \exp(G/H) \), then
\[
\eta(G) \leq (\eta(H) - 1) \exp(G/H) + \eta(G/H).
\]

Proposition 2.7 is an example of an application of the inductive method [7], an approach used to obtain bounds on the zero-sum invariants of a group \( G \) in terms of the zero-sum invariants of its subgroups.

3 Proof of the Main Result

3.1 The Invariants \( \zeta_i \)

As the first step in the proof of Theorem 1.6 we define the invariants \( \zeta_i(G) \) and compute them for a \( p \)-group \( G \).

Definition 2. Let \( G \) be a finite abelian group and let \( n = \exp(G) \). For \( 1 \leq i \leq n \), let
\[
\zeta_i(G) = s_{[i-n,0]+n\mathbb{N}^+}(G).
\]

That is, \( \zeta_i(G) \) is the shortest length \( \ell \) for which any sequence over \( G \) of length at least \( \ell \) contains a nonempty zero-sum subsequence \( T \) such that \( T \) is congruent mod \( \exp(G) \) to some residue between \( i \) and \( \exp(G) \), inclusive.

Note that \( \zeta_1(G) = D(G) \) by definition, and \( \zeta_\exp(G) = s_{\exp(G)\mathbb{N}^+}(G) \). When \( G \) is a \( p \)-group, the precise value of \( s_{\exp(G)\mathbb{N}^+}(G) \) is known.
Lemma 3.1 (Gao, Geroldinger, [4 Theorem 6.7]). If $G$ is a $p$-group with $n = \exp(G)$, then
\[ S_{n+1}(G) = D(G) + n - 1. \]

This allows us to compute $\zeta_i(G)$ for all $i \in [1, n]$ for a $p$-group $G$.

Lemma 3.2. Let $G$ be a $p$-group and let $n = \exp(G)$. For $1 \leq i \leq n$, we have
\[ \zeta_i(G) = D(G) + i - 1. \]

**Proof.** Let $1 \leq i \leq n - 1$. For any $t > 1$ and sequence $S$, if the sequence $0S$ obtained by prepending 0 to $S$ contains a zero-sum subsequence of length $t$, then $S$ contains a zero-sum sequence of length $t - 1$ or $t$. So $\zeta_i(G) \leq \zeta_i(G) + 1$. Thus, by Lemma 3.1,
\[ D(G) = \zeta_1(G) < \zeta_2(G) < \cdots < \zeta_n(G) = D(G) + n - 1, \]

implying that $\zeta_i(G) = D(G) + i - 1$ for each $i$, as desired. \qed

Now we can proceed with the proof of Theorem 1.6 itself.

3.2 Main Proof

**Proof of Theorem 1.6.** Let $G$ be a $p$-group with $D(G) \leq 2\exp(G) - 1$. Let $n = \exp(G)$. By Theorem 1.2, $\eta(G) \geq 2D(G) - n$, so it suffices to show that $\eta(G) \leq 2D(G) - n$.

Let $S$ be a sequence of length $2D(G) - n$ over $G$. We wish to show that $S$ contains a short zero-sum subsequence. Note that $1 \leq D(G) - n + 1 \leq n$, so by Lemma 3.2 $|S| = \zeta_{D(G) - n + 1}(G)$. Thus, $S$ contains a nonempty zero-sum subsequence $T$ whose length $t$ is congruent mod $n$ to some residue between $D(G) - n + 1$ and $n$, inclusive. Then either $t \leq n$ or $t \geq D(G) + 1$.

If $t \leq n$, then $T \subseteq S$ is a short zero-sum subsequence of $S$, as desired.

Otherwise, $t \geq D(G) + 1$, so by Corollary 2.5 $T$ contains a short zero-sum subsequence, meaning $S$ contains a short zero-sum subsequence. So $S$ contains a short zero-sum subsequence in any case.

Thus, $\eta(G) = 2D(G) - n$, which rearranges to the desired equality. \qed

4 Extension to $C_a \oplus G$

As with Theorem 1.4, our main result can be extended to groups of the form $G' = C_a \oplus G$, where $G$ is a $p$-group with $D(G) \leq 2\exp(G) - 1$ and $a$ is not divisible by $p$. The proof is essentially the same as the derivation of Theorem 1.4 from the $a = 1$ case, given in [5].

**Theorem 4.1.** Let $G$ be a finite abelian $p$-group with $D(G) \leq 2\exp(G) - 1$. Then for any positive integer $a$ not divisible by $p$, $G' = C_a \oplus G$ satisfies
\[ 2D(G') - 1 = \eta(G') + \exp(G') - 1. \]
Proof. Let \( n = \exp(G) \). By Theorem 2.3 we have \( \Delta(G') = \Delta^*(G') = \Delta(G) + (a - 1)n \), and by Lemma 2.6 we have \( \eta(G') \geq 2\Delta(G) - n + an = 2\Delta(G') - an \).

By Proposition 2.7 we have

\[
\eta(G') \leq (\eta(C_a) - 1)\exp(G) + \eta(G) = (a - 1)n + \eta(G).
\]

Finally, by Theorem 1.6 we have

\[
\eta(G) = 2\Delta(G) - n,
\]

so

\[
\eta(G') \leq 2\Delta(G') - \exp(G'),
\]

which rearranges to the desired equality. \( \square \)

Note that Theorem 4.1 is an extension of Theorem 4.2.1 in [6]. Remark 4.3 in [6] pointed out that a result like this would follow given Theorem 1.6.

5 Towards Evaluating \( s(G) \)

It would be interesting to see if a similar method suffices for verifying the second half of Conjecture 1.3, showing that \( s(G) = 2\Delta(G) - 1 \) for a \( p \)-group with \( \Delta(G) \leq 2\exp(G) - 1 \). Here we describe an approach that seems promising and use it to make some progress.

Let \( n = \exp(G) \). For \( 1 \leq i \leq n \), let \( \eta_i(G) = s_{[i,n]}(G) \). Since a sequence \( S \) contains a zero-sum subsequence of length \( i \) or \( i + 1 \) if \( 0S \) contains a zero-sum subsequence of length \( i + 1 \), it is easy to see that

\[
\eta(G) = \eta_1(G) < \cdots < \eta_n(G) = s(G).
\]

So, if Conjecture 1.3 holds in general, \( \eta_i(G) = \eta_{i-1}(G) + 1 = \eta(G) + i - 1 \) should always hold for \( 2 \leq i \leq n \).

The following partial result is known.

**Lemma 5.1** (Gao, [3, Lemma 2.6]). For any finite abelian group \( G \), \( \eta_i(G) = \eta(G) + i - 1 \) for \( 2 \leq i \leq \lceil \frac{n}{2} \rceil + 1 \).

Unfortunately, the approach used to prove this lemma does not directly generalize to the case where \( i > \lceil \frac{n}{2} \rceil + 1 \). However, we can apply the same idea to get an improved result when specializing to a \( p \)-group of large exponent, via a result generalizing Corollary 2.3 on short zero-sum subsequences of zero-sum sequences.

**Theorem 5.2.** Let \( G \) be a \( p \)-group with \( \Delta(G) \leq 2\exp(G) - 1 \). For \( 1 \leq i \leq 2\exp(G) - \Delta(G) \), any zero-sum sequence \( S \) of length exactly \( \Delta(G) + i \) over \( G \) contains a short zero-sum subsequence of length at least \( i \).

**Proof.** We use strong induction on \( i \). The base case \( i = 1 \) is Corollary 2.3. Assume the result is true for \( 1 \leq i < j - 1 \), and take \( i = j \leq 2\exp(G) - \Delta(G) \).

Let \( n = \exp(G) \) and consider a zero-sum sequence \( S \) of length \( \Delta(G) + j \) over \( G \). Let \( S' \) be the subsequence of \( S \) consisting of all but the last element, so \( S' \)}
Conjecture 5.4. Let $G$ be a finite abelian $p$-group with $\exp(G) = n$ and $D(G) \leq 2n - 1$. Let $\ell \in [1, D(G) + 1 - n]$, and let $S$ be a sequence over $G$ of length at least $D(G) + (n - 2) + \ell$. Then one of the following statements holds:

1. $S$ contains a zero-sum sequence $T$ whose length $|T|$ is congruent mod $n$ to some residue between $j$ and $n$, inclusive. Since $D(G) + j \leq 2n$, either $j \leq |T| \leq n$ or $n + j \leq |T| \leq 2n - 1$. In the former case, $T$ is a short zero-sum subsequence of $S$ of length at least $j$, as desired. In the latter case, the complement $T'$ of $T$ in $S$ is a nonempty zero-sum subsequence of $S$ of length $|T'| \leq (D(G) + j) - (n + j) = D(G) - n \leq n - j$.

2. If $|T'| \geq j$, we again have a short zero-sum subsequence of $S$ of length at least $j$, as desired. Otherwise, $|T| = D(G) + j - |T'| > D(G)$. By the inductive hypothesis, $T$ contains a short zero-sum subsequence $U$ of length at least $j - |T'|$. If $|U| \geq j$, we are again done, so assume $j - |T'| \leq |U| \leq j - 1$. Then

$$j \leq |T'| + |U| \leq n - 1,$$

so $T'U$ is a short zero-sum subsequence of $S$ of length at least $j$, as desired. So the result holds for $i = j$ as well, and thus by induction holds for all $i \leq 2 \exp(G) - D(G)$.

This gives the following extension of Theorem 1.6 as an immediate corollary.

**Corollary 5.3.** Let $G$ be a finite abelian $p$-group with $D(G) \leq 2 \exp(G) - 1$. Then for $1 \leq i \leq \max\left(2 \exp(G) - D(G), \left\lfloor \frac{\exp(G)}{2} \right\rfloor + 1\right)$, we have

$$\eta_i(G) = 2D(G) - \exp(G) + (i - 1).$$

**Proof.** Let $n = \exp(G)$, and let $R$ be a sequence of length $D(G) + n - 1$ over $G$. Since $D(G) + n - 1 < 3n$, by Lemma 3.1 $R$ contains a zero-sum subsequence $S$ of length $n$ or $2n$. If $|S| = 2n$, then Theorem 5.2 gives a short zero-sum subsequence $T$ of length at least $2n - D(G)$. So, in either case, $R$ contains a short zero-sum subsequence of length at least $2n - D(G)$, meaning $\eta_{2n-D(G)}(G) \leq D(G) + n - 1$. Since $\eta(G) = 2D(G) - n$, the bounds $\eta_{i-1}(G) < \eta_i(G)$ yield the desired result for all $i \leq 2 \exp(G) - D(G)$.

The result for $i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$ is a direct consequence of Theorem 1.6 and Lemma 5.1. □

The argument used in Theorem 3.1 of [10] to prove that $s(G) \leq D(G) + 2n - 2$ for a $p$-group $G$ of large exponent relies on the fact that a sequence $S$ over $G$ of length at least $D(G) + (n - 2) + \ell$ contains either a zero-sum subsequence of length $n$, or a zero-sum subsequence of length $2n$ that itself contains a short zero-sum subsequence of length at least $\ell$. Our Theorem 5.2 provides a strengthening of this result for $\ell = 1$, giving a short zero-sum subsequence of length at least $(2n - 1) - D(G) + \ell = 2n - D(G)$ in the zero-sum subsequence of length $2n$. It is conceivable that this strengthening could generalize to larger values of $\ell$, as the following conjecture voices.

Conjecture 5.4. Let $G$ be a finite abelian $p$-group with $\exp(G) = n$ and $D(G) \leq 2n - 1$. Let $\ell \in [1, D(G) + 1 - n]$, and let $S$ be a sequence over $G$ of length at least $D(G) + (n - 2) + \ell$. Then one of the following statements holds:
1. $S$ has a zero-sum subsequence $B$ with $|B| = n$.

2. $S$ has a zero-sum subsequence $B$ with $|B| = 2n$ such that $B$ has a zero-sum subsequence $B'$ with $|B'| \in [(2n - 1) - D(G) + \ell, n - 1]$.

This conjecture, if true, would yield the desired bound of $s(G) \leq D(G) + (n - 2) + (D(G) - n + 1) = 2D(G) - 1$. Perhaps in this or some other way, the argument used in [10], extended by the arguments we use, could lead to the full resolution of Conjecture 1.3.

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