The first cohomology, derivations and the reductivity of a (meromorphic open-string) vertex algebra

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Abstract

We give a criterion for the complete reducibility of modules satisfying a composability condition for a meromorphic open-string vertex algebra $V$ using the first cohomology of the algebra. For a $V$-bimodule $M$, let $\hat{H}^1_\infty(V, M)$ be the first cohomology of $V$ with the coefficients in $M$, which is canonically isomorphic to the quotient space of the space of derivations from $V$ to $W$ by the subspace of inner derivations. Let $\hat{Z}^1_\infty(V, M)$ be the subspace of $\hat{H}^1_\infty(V, M)$ canonically isomorphic to the space of derivations obtained from the zero mode of the right vertex operators of weight 1 elements such that the difference between the skew-symmetric opposite action of the left action and the right action on these elements are Laurent polynomials in the variable. If $\hat{H}^1_\infty(V, M) = \hat{Z}^1_\infty(V, M)$ for every $\mathbb{Z}$-graded $V$-bimodule $M$, then every left $V$-module satisfying a composability condition is completely reducible. In particular, since a lower-bounded $\mathbb{Z}$-graded vertex algebra $V$ is a special meromorphic open-string vertex algebra and left $V$-modules are in fact what has been called generalized $V$-modules with lower-bounded weights (or lower-bounded generalized $V$-modules), this result provides a cohomological criterion for the complete reducibility of lower-bounded generalized modules for such a vertex algebra. We conjecture that the converse of the main theorem above is also true. We also prove that when a grading-restricted vertex algebra $V$ contains a subalgebra satisfying some familiar conditions, the composability condition for grading-restricted generalized $V$-modules always holds and we need $\hat{H}^1_\infty(V, M) = \hat{Z}^1_\infty(V, M)$ only for every $\mathbb{Z}$-graded $\mathbb{Z}$-graded $V$-bimodule $M$ generated by a grading-restricted subspace in our complete reducibility theorem.

1 Introduction

In the representation theory of various algebras, one of the main tools is the cohomological method. The powerful tool of homological algebra often provides a unified treatment of many results in representation theory. Such a unified treatment not only gives solutions to open problems, but also provides a conceptual understanding of the results.

In the representation theory of vertex (operator) algebras, though the cohomology for a grading-restricted vertex algebra has been introduced by the first author in [H7], the
cohomological method in the representation theory of vertex (operator) algebras still needs to be fully developed. In this paper, we study the relation between the first cohomology and the complete reducibility of suitable modules for a grading-restricted vertex algebra or more generally for a meromorphic open-string vertex algebra introduced by the first author in \[H6\].

In \[H7\] and \[H8\], the first author introduced the cohomology of a grading-restricted vertex algebra and proved that the first cohomology and second cohomology indeed have the properties that they should have. The cohomology introduced in \[H7\] can be viewed as an analogue of the Harrison cohomology of a commutative associative algebra. In particular, the first author also introduced in \[H7\] a cohomology that should be viewed as an analogue of the Harrison cohomology of a commutative associative algebra. The generalization of this cohomology to a meromorphic open-string vertex algebra satisfies the composability condition using the first cohomology of this analogue of the Hochschild cohomology. In particular, we obtain a criterion for such a module (in fact such a generalized module in the terminology of \[HLZ1\]) for a lower-bounded or grading-restricted vertex algebra to be completely reducible.

We describe the main results of this paper more precisely here: For a meromorphic open-string vertex algebra \(V\) and a \(V\)-bimodule \(M\) (note that we do not require that the algebra \(V\) and \(V\)-modules be grading restricted, though they are lower bounded), let \(\hat{H}^1_\infty(V, M)\) be the first cohomology of \(V\) with the coefficients in \(M\) introduced in \[H7\], \[Q2\] and \[Q3\]. The first cohomology \(\hat{H}^1_\infty(V, M)\) is in fact canonically isomorphic to the quotient space of the space of derivations from \(V\) to \(M\) by the subspace of inner derivations. Let \(\hat{Z}^1_\infty(V, M)\) be the subspace of \(\hat{H}^1_\infty(V, M)\) canonically isomorphic to the space of what we call zero-mode derivations, that is, derivations obtained from the zero mode of the right vertex operators of weight 1 elements of \(M\) such that the difference between the skew-symmetric opposite action of the left action and the right action on these elements are Laurent polynomials in the variable. For a left \(V\)-module \(W\), a left \(V\)-submodule \(W_2\) of \(W\) and a graded subspace \(W_1\) of \(W\) such that as a graded vector space, \(W = W_1 \oplus W_2\), let \(\pi_{W_1}\) and \(\pi_{W_2}\) be the projections from \(W\) to \(W_1\) and \(W_2\), respectively. For a left \(V\)-module \(W\) and a left \(V\)-submodule \(W_2\), we say that the pair \((W, W_2)\) satisfies the composability condition if there exists a graded subspace \(W_1\) of \(W\) such that \(W = W_1 \oplus W_2\) and such that for \(k, l \in \mathbb{N}, w'_2 \in W_2, w_1 \in W_1, v_1, \ldots, v_{k+l}, v \in V\), the series

\[
\langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k)\pi_{W_2} Y_{W}(v, z)\pi_{W_1} Y_{W}(v_{k+1}, z_{k+1}) \cdots \pi_{W_1} Y_{W}(v_{k+l}, z_{k+l})w_1 \rangle
\]

is absolutely convergent the region \(|z_1| > \cdots > |z_k| > |z| > \cdots > |z_{k+l}| > 0\) to a suitable rational function. We say that a left \(V\)-module \(W\) satisfies the composability condition if for every proper nonzero left \(V\)-submodule \(W_2\) of \(W\), the pair \((W, W_2)\) satisfies the composability condition. We prove in this paper that if \(\hat{H}^1_\infty(V, M) = \hat{Z}^1_\infty(V, M)\) for every \(\mathbb{Z}\)-graded \(V\)-bimodule \(M\), then every left \(V\)-module satisfying the composability condition is completely
reducible. Since the first cohomology of $V$ with coefficients in $M$ is the quotient of the space of derivations from $V$ to $M$ by the space of inner derivations, the condition $\hat{H}_\infty^1(V, M) = \hat{Z}_\infty^1(V, M)$ in our main theorem above can also be formulated as the condition that every derivation from $V$ to $M$ is the sum of an inner derivation and a zero-mode derivation.

In particular, since a lower-bounded $\mathbb{Z}$-graded vertex algebra $V$ is a special meromorphic open-string vertex algebra and left $V$-modules are in fact what has been called generalized $V$-modules with lower-bounded weights (or lower-bounded generalized $V$-modules), this result provides a cohomological criterion for the complete reducibility of lower-bounded generalized modules for such a vertex algebra $V$. We also prove that when the grading-restricted vertex algebra $V$ contains a subalgebra $V_0$ such that products of intertwining operators are convergent absolutely in the usual region and the sums can be analytically extended, the associativity of intertwining operators holds and grading-restricted $V_0$-modules (grading-restricted generalized $V_0$-modules in the terminology of [HLZ1]) are completely reducible, the composability condition holds for every $V$-modules. We also prove that in this case, we need $\hat{H}_\infty^1(V, M) = \hat{Z}_\infty^1(V, M)$ only for every $\mathbb{Z}$-graded $V$-bimodule $M$ generated by a grading-restricted subspace in our complete reducibility theorem.

The composability condition on left $V$-modules needed in our results is in fact a condition on convergence and analytic extension. In the representation theory of vertex operator algebras, the convergence and analytic extension of suitable series always play a crucial role. The main difficult parts of the proofs of a number of major results are in fact on suitable convergence and analytic extension. For example, in the proof of the associativity of intertwining operators, the main difficult part of the proof in the paper [H2] is to use the $C_1$-cofiniteness of the modules to prove the convergence and extension property of the products and iterates of intertwining operators. Another important example is the modular invariance of intertwining operators proved in [H3]. The most difficult part of the proof of this modular invariance is the proof of the convergence and analytic extension of the $q$-traces of products of at least two intertwining operators. In fact, the method of proving such convergence and analytic extension by reducing it using formal series recurrent relations to the convergence of $q$-traces of one vertex operator does not work for general intertwining operators. One needs to derive modular invariant differential equations satisfied by these $q$-traces directly from the $C_2$-cofiniteness of the modules. Note that the associativity of intertwining operators and the modular invariance of intertwining operators are the two main conjectures stated in the important paper of Moore and Seiberg [MS] that led to the Verlinde formula and the modular tensor category structures (see [H4] and [H5] for the mathematical proof and mathematical construction, respectively, based on the associativity and modular invariance of intertwining operators proved in [H2] and [H3], respectively).

In fact, in Section 6, we prove that every grading-restricted left $V$-module satisfies the composability condition using the theory of intertwining operators for a subalgebra of a grading-restricted vertex algebra $V$ satisfying suitable conditions. This reveals a deep connection between the composability condition and the theory of intertwining operators. It is also proved in Section 6 that when $V$ is a grading-restricted vertex algebra containing a subalgebra satisfying suitable conditions, the relevant $\mathbb{Z}$-graded $V$-bimodule constructed from
a grading-restricted $V$-module and a $V$-submodule appearing in the proofs of our results is in fact generated by a grading-restricted subspace and thus we need consider only $\mathbb{Z}$-graded $V$-bimodules generated by such subspaces. This proof uses the $Q(z)$-tensor product first introduced in \cite{HL2} and \cite{HL3} and studied further in the more general nonsemisimple case in \cite{HLZ}. This use also reveals a deep connection between the cohomology theory and the tensor category theory for module categories for a suitable vertex (operator) algebra. We expect that the study of the composability condition introduced in this paper and the $V$-bimodules constructed in Section 3 from two left $V$-modules will lead to deep results in the cohomological method to the representation theory of (meromorphic open-string) vertex (operator) algebras.

The present paper is organized as follows: In Section 2, we recall the notions of meromorphic open-string vertex algebra, left module, right module and bimodule for such an algebra. We also recall briefly the cohomology theory for such an algebra. We prove that the first cohomology $\hat{H}^1_\infty(V,W)$ of such an algebra $V$ with coefficients in a bimodule $W$ is isomorphic to the quotient space of the space of derivations from $V$ to $W$ by the subspace of inner derivations. We introduce what we call zero-mode derivations and the subspace $\hat{Z}^1_\infty(V,W)$ of $\hat{H}^1_\infty(V,W)$. We construct $\mathbb{Z}$-graded $V$-bimodules from two left $V$-modules in Section 3. In Section 4, we construct a 1-cocycle from a left $V$-module and a left $V$-submodule. This cocycle is in fact the obstruction for the left $V$-module to be decomposed as the direct sum of the left $V$-submodule and another left $V$-submodule. We prove our main theorem on the complete reducibility in Section 5. In Section 6, we prove that the composability condition holds for suitable modules for a grading-restricted vertex algebra containing a subalgebra satisfying suitable conditions. We also prove in this section that in this case, we need $\hat{H}^1_\infty(V,M) = \hat{Z}^1_\infty(V,M)$ only for every $V$-bimodule $M$ generated by a grading-restricted subspace in the complete reducibility theorem.

\section{Meromorphic open-string vertex algebras, modules and cohomology}

We first recall the notion of meromorphic open-string vertex algebra.

\textbf{Definition 2.1} A \textit{meromorphic open-string vertex algebra} is a $\mathbb{Z}$-graded vector space $V = \bigoplus_{n \in \mathbb{Z}} V(n)$ (graded by weights) equipped with a \textit{vertex operator map}

$$Y_V : V \otimes V \to V[[x, x^{-1}]]$$

$$u \otimes v \mapsto Y_V(u, x)v$$

and a \textit{vacuum} $1 \in V$, satisfying the following axioms:

1. Axioms for the grading: (a) \textit{Lower bound condition}: When $n$ is sufficiently negative, $V(n) = 0$. (b) \textit{d-commutator formula}: Let $d_V : V \to V$ be defined by $d_V v = nv$ for $v \in V(n)$. Then

$$[d_V, Y_V(v, x)] = \frac{d}{dx}Y_V(v, x) + Y_V(d_V v, x)$$

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for $v \in V$.

2. Axioms for the vacuum: (a) Identity property. Let $1_V$ be the identity operator on $V$. Then $Y_V(1, x) = 1_V$. (b) Creation property. For $u \in V$, $Y_V(u, x)1 \in V[[x]]$ and $\lim_{x \to 0} Y_V(u, x)1 = u$.

3. $D$-derivative property and $D$-commutator formula: Let $D_V : V \to V$ be the operator given by

$$D_Vv = \lim_{x \to 0} \frac{d}{dx} Y_V(v, x)$$

for $v \in V$. Then for $v \in V$,

$$\frac{d}{dx} Y_V(u, x) = Y_V(D_Vu, x) = [D_V, Y_V(u, x)].$$

4. Rationality. Let $V' = \bigsqcup_{n \in \mathbb{Z}} V_{(n)}^*$ be the graded dual of $V$. For $u_1, \cdots, u_n, v \in V, v' \in V'$, the series

$$\langle v', Y_V(u_1, z_1) \cdots Y_V(u_n, z_n)v \rangle$$

converges absolutely when $|z_1| > \cdots > |z_n| > 0$ to a rational function in $z_1, \cdots, z_n$, with the only possible poles at $z_i = 0, i = 1, \cdots, n$ and $z_i = z_j, 1 \leq i \neq j \leq n$. For $u_1, u_2, v \in V$ and $v' \in V'$, the series

$$\langle v', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v \rangle$$

converges absolutely when $|z_2| > |z_1 - z_2| > 0$ to a rational function with the only possible poles at $z_1 = 0, z_2 = 0$ and $z_1 = z_2$.

5. Associativity. For $u_1, u_2, v \in V$ and $v' \in V'$, we have

$$\langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)v \rangle = \langle v', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v \rangle$$

when $|z_1| > |z_2| > |z_1 - z_2| > 0$.

A meromorphic open-string vertex algebra is said to be grading restricted if $\dim V_{(n)} < \infty$ for $n \in \mathbb{Z}$.

The meromorphic open-string vertex algebra just defined is denoted by $(V, Y_V, 1)$ or simply by $V$.

**Remark 2.2** Let $V$ be a meromorphic open-string vertex algebra. We say that $V$ satisfies the skew-symmetry if $Y_V(u, x)v = e^{xD_V}Y_V(v, -x)u$ for $u, v \in V$. A meromorphic open-string vertex algebra satisfies the skew-symmetry is a lower-bounded vertex algebra, that is, a vertex algebra with a lower-bounded $\mathbb{Z}$-grading. A grading-restricted meromorphic open-string vertex algebra satisfying the skew-symmetry is a grading-restricting vertex algebra, that is, a vertex algebra with a $\mathbb{Z}$-grading satisfying the two grading restriction conditions.
The following notion of left $V$-module was introduced in [H6]:

**Definition 2.3** A left module for $V$ or a left $V$-module is a $C$-graded vector space $W = \bigoplus_{n \in C} W[n]$ (graded by weights), equipped with a vertex operator map

$$Y_W : V \otimes W \rightarrow W[[x, x^{-1}]]$$

$$u \otimes w \mapsto Y_W(u, x)w,$$

and an operator $D_W$ of weight 1, satisfying the following axioms:

1. Axioms for the grading: (a) *Lower bound condition*: When the real part $\Re(n)$ of $n$ is sufficiently negative, $W[n] = 0$. (b) *$d$-commutator formula*: Let $d_W : W \rightarrow W$ be defined by $d_W w = nw$ for $w \in W[n]$. Then for $u \in V$,

$$[d_W, Y_W(u, x)] = Y_W(d_W u, x) + x \frac{d}{dx} Y_W(u, x).$$

2. The *identity property*: $Y_W(1, x) = 1_W$.

3. The *$D$-derivative property* and the *$D$-commutator formula*: For $u \in V$,

$$\frac{d}{dx} Y_W(u, x) = Y_W(D_V u, x) = [D_W, Y_W(u, x)].$$

4. *Rationality*: For $u_1, \ldots, u_n, w \in W$ and $w' \in W'$, the series

$$\langle w', Y_W(u_1, z_1) \cdots Y_W(u_n, z_n)w \rangle$$

converges absolutely when $|z_1| > \cdots > |z_n| > 0$ to a rational function in $z_1, \ldots, z_n$ with the only possible poles at $z_i = 0$ for $i = 1, \ldots, n$ and $z_i = z_j$ for $i \neq j$. For $u_1, u_2, w \in W$ and $w' \in W'$, the series

$$\langle w', Y_W(Y_V(u_1, z_1 - z_2)u_2, z_2)w \rangle$$

converges absolutely when $|z_2| > |z_1 - z_2| > 0$ to a rational function with the only possible poles at $z_1 = 0$, $z_2 = 0$ and $z_1 = z_2$.

5. *Associativity*: For $u_1, u_2, w \in W$, $w' \in W'$,

$$\langle w', Y_W(u_1, z_1)Y_W(u_2, z_2)w \rangle = \langle w', Y_W(Y_V(u_1, z_1 - z_2)u_2, z_2)w \rangle$$

when $|z_1| > |z_2| > |z_1 - z_2| > 0$.

A left $V$-module is said to be *grading restricted* if $\dim W[n] < \infty$ for $n \in \mathbb{C}$. 

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We denote the left $V$-module just defined by $(W, Y_W, D_W)$ or simply $W$.

The following notions of right $V$-module and $V$-bimodule were introduced by the first author and were explicitly written down in $[Q1]$ and $[Q3]$ by the second author:

**Definition 2.4** A *right module for $V$* or a *right $V$-module* is a $\mathbb{C}$-graded vector space $W = \bigsqcup_{n \in \mathbb{C}} W[n]$ (graded by weights), equipped with a *vertex operator map*

\[
Y_W : W \otimes V \rightarrow W[[x, x^{-1}]]
\]

\[w \otimes u \mapsto Y_W(w, x)u,
\]

and an operator $D_W$ of weight 1, satisfying the following axioms:

1. **Axioms for the grading:** (a) *Lower bound condition:* When $\Re(n)$ is sufficiently negative, $W[n] = 0$. (b) *d-commutator formula:* Let $d_W : W \rightarrow W$ be defined by $d_Ww = nw$ for $w \in W[n]$. Then for $w \in W$,

\[
d_WY_W(w, x) - Y_W(w, x)d_W = Y_W(d_Ww, x) + \frac{d}{dx}Y_W(w, x).
\]

2. **The creation property:** For $w \in W$, $Y_W(w, x)1 \in W[[x]]$ and $\lim_{x \to 0} Y_W(w, x)1 = w$.

3. **The $D$-derivative property and the $D$-commutator formula:** For $u \in V$,

\[
\frac{d}{dx}Y_W(w, x) = Y_W(D_Ww, x) = D_WY_W(w, x) - Y_W(w, x)D_V.
\]

4. **Rationality:** For $u_1, \ldots, u_n, w \in W$ and $w' \in W'$, the series

\[
\langle w', Y_W(w, z_1)Y_V(u_1, z_2) \cdots Y_V(u_{n-1}, z_n)u_n \rangle
\]

converges absolutely when $|z_1| > \cdots > |z_n| > 0$ to a rational function in $z_1, \ldots, z_n$ with the only possible poles at $z_i = 0$ for $i = 1, \ldots, n$ and $z_i = z_j$ for $i \neq j$. For $u_1, u_2, w \in W$ and $w' \in W'$, the series

\[
\langle w', Y_W(Y_W(w, z_1 - z_2)u_1, z_2)u_2 \rangle
\]

converges absolutely when $|z_2| > |z_1 - z_2| > 0$ to a rational function with the only possible poles at $z_1 = 0$, $z_2 = 0$ and $z_1 = z_2$.

5. **Associativity:** For $u_1, u_2, w \in W$, $w' \in W'$,

\[
\langle w', Y_W(w, z_1)Y_V(u_1, z_2)u_2 \rangle = \langle w', Y_W(Y_W(w, z_1 - z_2)u_1, z_2)u_2 \rangle
\]

when $|z_1| > |z_2| > |z_1 - z_2| > 0$.

A right $V$-module is said to be *grading restricted* if $\dim W[n] < \infty$ for $n \in \mathbb{C}$.
We also denote the generalized right $V$-module just defined by $(W, Y, D)$ or simply $W$.

We now give the definition of $V$-bimodule. Roughly speaking, a $V$-bimodule is a $\mathbb{C}$-graded vector space equipped with a left $V$-module structure and a right-module structure such that these two structures are compatible. More precisely, we have:

**Definition 2.5** A $V$-bimodule is a $\mathbb{C}$-graded vector space $W = \bigsqcup_{n \in \mathbb{C}} W_n$ (graded by weights) equipped with a left vertex operator map

$$Y^L_W : V \otimes W \to W[[x, x^{-1}]]$$

$$u \otimes w \mapsto Y^L_W(u, x)w,$$

a right vertex operator map

$$Y^R_W : W \otimes V \to W[[x, x^{-1}]]$$

$$w \otimes u \mapsto Y^R_W(w, x)u,$$

and $D_W$ on $W$ satisfying the following conditions.

1. $(W, Y^L_W, D_W)$ is a left $V$-module.
2. $(W, Y^R_W, D_W)$ is a right $V$-module.
3. **Compatibility:**

   (a) **Rationality for left and right vertex operator maps:** For $u_1, \ldots, u_{m+n} \in V$, $w \in W$ and $w' \in W'$, the series

   $$\langle w', Y^L_W(u_1, z_1) \cdots Y^L_W(u_k, z_m)Y^R_W(w, z_{m+1})Y_V(v_{m+1}, z_{m+2}) \cdots Y_V(v_{m+n-1}, z_{m+n})v_{m+n} \rangle$$

   is absolutely convergent in the region $|z_1| > \cdots |z_{m+n}| > 0$ to a rational function in $z_1, \ldots, z_{m+n}$ with the only possible poles $z_i = 0$ for $i = 1, \ldots, m+n$ and $z_i = z_j$ for $i, j = 1, \ldots, m+n, i \neq j$. For $u, v \in V$, $w \in W$ and $w' \in W'$, the series

   $$\langle w', Y^R_W(Y^L_W(u, z_1 - z_2)w, z_2) \rangle$$

   converges absolutely when $|z_2| > |z_1 - z_2| > 0$ to a rational function with the only possible poles at $z_1 = 0$, $z_2 = 0$ and $z_1 = z_2$.

   (b) **Associativity for left and right vertex operator maps:** For $u, v \in V$, $w \in W$ and $w' \in W'$,

   $$\langle w', Y^L_W(u, z_1)Y^R_W(w, z_2)v \rangle = \langle w', Y^R_W(Y^L_W(u, z_1 - z_2)w, z_2)v \rangle$$

   when $|z_1| > |z_2| > |z_1 - z_2| > 0$. 

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The $V$-bimodule just defined is denoted by $(W, Y^L_W, Y^R_W, D_W)$ or simply by $W$. There is also a notion of $V$-bimodule with different $D_W$ for the left and right $V$-module structure. Since we do not need such $V$-bimodules in this paper, we shall not discuss this more general notion.

We shall use the notations and terminology in [H7] and [H9]. For example, we shall denote the rational function that converges to by

$$R(⟨w', Y^L_W(u, z_1)Y^R_W(w, z_2)v⟩).$$

In particular, we have, for example,

$$R(⟨w', Y^L_W(u, z_1)Y^R_W(w, z_2)v⟩) = R(⟨w', Y^R_W(Y^L_W(u, z_1 - z_2)w, z_2)v⟩).$$

We shall also use, for example, $Y^L_W(u, z_1)Y^R_W(w, z_2)v$ to denote the element of $\mathcal{W} \subset (W')^*$ given by $w' \mapsto ⟨w', Y^L_W(u, z_1)Y^R_W(w, z_2)v⟩$ for $z_1$ and $z_2$ satisfying $|z_1| > |z_2| > 0$ (see Remark 2.7). In particular, we shall say, for example, that $Y^L_W(u, z_1)Y^R_W(w, z_2)v$ is equal to $Y^R_W(Y^L_W(u, z_1 - z_2)w, z_2)v$ in the region $|z_1| > |z_2| > |z_1 - z_2| > 0$. In addition, we shall use, for example, $E(Y^L_W(u, z_1)Y^R_W(w, z_2)v)$ to denote the the element of $\mathcal{W}$ given by $w' \mapsto R(⟨w', Y^L_W(u, z_1)Y^R_W(w, z_2)v⟩)$ for $z_1, z_2 \in \mathbb{C}^*$ satisfying $z_1 \neq z_2$. In particular, we have, for example,

$$E(Y^L_W(u, z_1)Y^R_W(w, z_2)v) = E(Y^R_W(Y^L_W(u, z_1 - z_2)w, z_2)v).$$

The following pole-order conditions on a meromorphic open-string vertex algebra $V$ and various $V$-modules are introduced in [Q2] and [Q3]:

**Definition 2.6** A meromorphic open-string vertex algebra $V$ is said to satisfy the pole-order condition if for $v_1, \ldots, v_n, u \in V$, there exist $r_i \in \mathbb{N}$ depending only on the pair $(v_i, u)$ for $i = 1, \ldots, n$, $p_{ij} \in \mathbb{N}$ depending only on the pair $(v_i, v_j)$ for $i, j = 1, \ldots, n$, $i \neq j$ and $g(z_1, \ldots, z_n) \in V[[z_1, \ldots, z_n]]$ such that for $u' \in V'$,

$$\prod_{i=1}^n z_i^{r_i} \prod_{1 \leq i < j \leq n} (z_i - z_j)^{p_{ij}} R(⟨u', Y_V(v_1, z_1) \cdots Y_V(v_n, z_n)u⟩)$$

is a polynomial and is equal to $⟨u', g(z_1, \ldots, z_n)⟩$. A left $V$-module $W$ is said to satisfy the pole-order condition if for $v_1, \ldots, v_n \in V$ and $w \in W$, there exist $r_i \in \mathbb{N}$ depending only on the pair $(v_i, w)$ for $i = 1, \ldots, n$, $p_{ij} \in \mathbb{N}$ depending only on the pair $(v_i, v_j)$ for $i, j = 1, \ldots, n$, $i \neq j$ and $g(z_1, \ldots, z_n) \in W[[z_1, \ldots, z_n]]$ such that for $w' \in W'$,

$$\prod_{i=1}^n z_i^{r_i} \prod_{1 \leq i < j \leq n} (z_i - z_j)^{p_{ij}} R(⟨w', Y_W(v_1, z_1) \cdots Y_W(v_n, z_n)w⟩)$$

is a polynomial and is equal to $⟨w', g(z_1, \ldots, z_n)⟩$. A right $V$-module $W$ is said to satisfy the pole-order condition if for $v_1, \ldots, v_n \in V$ and $w \in W$, there exist $r_i \in \mathbb{N}$ depending only on the pair $(w, v_n)$, $r_i \in \mathbb{N}$ depending only on the pair $(v_{i-1}, v_n)$ for $i = 2, \ldots, n$, $p_{ij} \in \mathbb{N}$
depending only on the pair \((w, v_{j-1})\) for \(j = 2, \ldots, n\), \(p_{ij} \in \mathbb{N}\) depending only on the pair \((v_{i-1}, v_{j-1})\) for \(i, j = 2, \ldots, n\), \(i \neq j\) and \(g(z_1, \ldots, z_n) \in W[[z_1, \ldots, z_n]]\) such that for \(w' \in W'\),

\[
\prod_{i=1}^{n} z_i^{p_{i1}} \prod_{1 \leq i < j \leq n} (z_i - z_j)^{p_{ij}} \prod_{l=1}^{k+l} (z_i - z_l)^{s_i}
\]

\[
R((w', Y_W(w, z_1)Y_V(v_1, z_2) \cdots Y_V(v_{n-1}, z_n)v_n))
\]

is a polynomial and is equal to \(\langle w', g(z_1, \ldots, z_n) \rangle\). A \(V\)-bimodule \(W\) is said to satisfy the pole-order condition if for \(v_1, \ldots, v_{k+l} \in V\) and \(w \in W\), there exist \(r_i \in \mathbb{N}\) depending only on the pair \((v_i, w)\) for \(i = 1, \ldots, k + l\), \(m \in \mathbb{N}\) depending on the pair \((w, v_n)\), \(p_{ij} \in \mathbb{N}\) depending only on the pair \((v_i, v_j)\) for \(i, j = 1, \ldots, k + l\), \(i \neq j\), \(s_i \in \mathbb{N}\) depending only on the pair \((v_i, v)\) for \(i = 1, \ldots, k + l\) and \(g(z_1, \ldots, z_{k+l}, z) \in W[[z_1, \ldots, z_{k+l}]]\) such that for \(w' \in W'\),

\[
z^m \prod_{i=1}^{k+l} z_i^{r_i} \prod_{1 \leq i < j \leq k+l} (z_i - z_j)^{p_{ij}} \prod_{i=1}^{k+l} (z_i - z)^{s_i}
\]

\[
R((w', Y^R_W(v_1, z_1) \cdots Y^L_W(v_{k}, z_k)Y^R_W(v_{k+1}, z)Y_V(v_{k+1}, z_{k+1}) \cdots Y_V(v_{k+l}, z_{k+l})w))
\]

is a polynomial and is equal to \(\langle w', g(z_1, \ldots, z_{k+l}, z) \rangle\).

**Remark 2.7** In the definition above, \(p_{ij}\) are required to be dependent only on the pair \((v_i, v_j)\). This condition is automatically satisfied by lower-bounded vertex algebras and lower-bounded modules for such vertex algebras because of the commutativity satisfied by them. This condition allows us to formulate equivalent definitions of meromorphic open-string vertex algebras and modules using formal variables. On the other hand, the condition that there exists \(g(z_1, \ldots, z_n)\) guarantees that the product of vertex operators acting on an element of the algebra or module is in its algebraic completion, which is smaller than the full dual space of the graded dual of the algebra or module when its homogeneous subspaces are not finite dimensional. For example, from this existence, it can be shown easily that the element of \((V')^*\) given by \(v' \mapsto \langle v', Y_V(v_1, z_1) \cdots Y_V(v_n, z_n)u \rangle\) for \(v' \in V'\) is in fact in \(\overline{\mathcal{V}}\). Similarly for modules. See [Q2] and [Q3] for detailed discussions.

In this paper, we consider only those meromorphic open-string vertex algebras, left modules, right modules and bimodules satisfying these pole-order conditions. From now on, when we mention such algebras and modules, we shall omit the phrase “satisfying the pole-order condition.”

**Remark 2.8** If \(V\) is a lower-bounded vertex algebra, then a left \(V\)-module (or a right-\(V\)-module) \(W\) has a \(V\)-bimodule structure whose right vertex operator map (or left vertex operator map) is defined by \(Y^R_W(v, x)v = e^{Dw}Y^L_W(v, -x)v\) for \(v \in V\) and \(w \in W\) (or \(Y^L_W(v, x)w = e^{Dw}Y^R_W(w, -x)v\) for \(v \in V\) and \(w \in W\). The proof was in fact given in [FHL].

We now briefly review the cohomology \(H^*_\infty(V, W)\) for a meromorphic open-string vertex algebra \(V\) and a \(V\)-bimodule \(W\). We refer the reader to [H7], [Q2] and [Q3] for details.
Let $V$ be a meromorphic open-string vertex algebra and $W$ a $V$-bimodule. Note that since we do not assume that $W$ is grading restricted, $\tilde{W} = \prod_{n \in \mathbb{C}} W_n[n]$ might not be isomorphic to $(W')^*$. For $n > 0$, a map $f$ from the configuration space $F_n \mathbb{C} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n | z_i \neq z_j, i, j = 1, \ldots, n, i \neq j\}$ to $\tilde{W} = \prod_{n \in \mathbb{C}} W_n[n]$ is called a $\tilde{W}$-valued rational functions in $z_1, \ldots, z_n$ with the only possible poles $z_i = z_j$ for $i, j = 1, \ldots, n, i \neq j$ if for $w' \in W'$, $\langle w', f(z_1, \ldots, z_n) \rangle$ is a rational function of such a form. For example, for $v_1, \ldots, v_n \in V$, $E(Y_{\Gamma}(v_1, z_1) \cdots Y_{\Gamma}(v_n, z_n) 1)$ is a $\tilde{V}$-valued rational functions in $z_1, \ldots, z_n$ with the only possible poles $z_i = z_j$ for $i, j = 1, \ldots, n, i \neq j$. Let $\tilde{W}_{z_1, \ldots, z_n}$ be the space of such $\tilde{W}$-valued rational functions.

For $n > 0$, let $\tilde{C}_n(V, W)$ be the subspace of $\text{Hom}(V^\otimes n, \tilde{W}_{z_1, \ldots, z_n})$ satisfying the $D$-derivative property, the $d$-conjugation property and being composable with arbitrary numbers of vertex operators. In this paper we are mainly concerned with the first cohomology. So we shall give the definitions of the $D$-derivative property, the $d$-conjugation property and being composable with an arbitrary number of vertex operators only in the case of $n = 1$. In the general case, see [H7], [Q2] and [Q3] for the precise meaning of these properties. An element $\Psi$ of $\text{Hom}(V, \tilde{W}_{z_1})$ is said to satisfy the $D$-derivative property if for $v_1 \in V$ and $z_1 \in \mathbb{C}$,

$$\frac{d}{dz_1}(\Psi(v_1))(z_1) = (\Psi(D_V v_1))(z_1) = D_W(\Psi(v_1))(z_1)$$

and is said to satisfy the $d$-conjugation property if for $a \in \mathbb{C}^\times$, $v_1 \in V$ and $z_1 \in \mathbb{C}$,

$$a^{d_{W}}(\Psi(v_1))(z_1) = (\Psi(a^{d_{V}} v_1))(az_1).$$

Such an element is said to be composable with an arbitrary number of vertex operators if for $k, l, m \in \mathbb{N}$, $v_1, \ldots, v_{k+l+m} \in V$, $w' \in W'$,

$$\langle w', Y_{\Gamma}(v_1, z_1) \cdots Y_{\Gamma}(v_k, z_k) Y_{\Gamma}(v_{k+1}, z_{k+1} - \xi) \cdots Y_{\Gamma}(v_{k+l}, z_{k+l} - \xi) 1, (\xi - \zeta), \zeta) \rangle$$

is absolutely convergent in the region given by $|z_1| > \cdots > |z_k|$, $|z_{k+1} - \xi| > \cdots > |z_{k+l} - \xi|$, $|z_{k+l+1}| > \cdots > |z_{k+l+m}|$, $|z_i| > |z_{k+j} - \xi| + |\xi - \zeta| + |\xi| + |\zeta|$ and $|\xi| > |z_{k+j} - \xi| + |z_{k+l+p}| + |\xi - \zeta|$ for $i = 1, \ldots, k$, $j = 1, \ldots, l$ and $p = 1, \ldots, m$ to a rational function of the form $\langle w', f(z_1, \ldots, z_{k+l+m}) \rangle$, where $f$ is a $\tilde{V}$-valued rational function in $z_1, \ldots, z_{k+l+m}$, with the only possible poles $z_i = z_j$ for $i, j = 1, \ldots, k + l + m, i \neq j$. Moreover, we also require that there exist $p_{ij} \in \mathbb{N}$ depending only on $v_i$ and $v_j$ for $i, j = 1, \ldots, k + l + m, i \neq j$, and $g(z_1, \ldots, z_{k+l+m}) \in W[[z_1, \ldots, z_{k+l+m}]]$ such that for $w' \in W'$,

$$\prod_{1 \leq i < j \leq k+l+m} (z_i - z_j)^{p_{ij}} \langle w', f(z_1, \ldots, z_{k+l+m}) \rangle$$

is a polynomial and is equal to $\langle w', g(z_1, \ldots, z_{k+l+m}) \rangle$. Note that the last part of the condition of being composable with an arbitrary number of vertex operators implies that the order of the pole $z_i = z_j$ is bounded by a number depending only on $v_i$ and $v_j$. Using the $d$-conjugation property and this last part of the condition of being composable with an arbitrary
number of vertex operators, we can also show that the expansions of \( f(z_1, \ldots, z_{k+i+m}) \) in various regions are in fact in \( \mathcal{W} \) instead of just \((W')^*\). Note that when \( W \) is not grading restricted, \( \mathcal{W} \neq (W')^* \).

The formulation of the property above that an element of \( \text{Hom}(V, \mathcal{W}_{z_i}) \) is composable with an arbitrary number of vertex operators uses directly the left and right vertex operators. In \( \textbf{Q2} \) and \( \textbf{Q3} \), the second author uses the skew-symmetry opposite vertex operators instead of the right vertex operators. It is easy to see from the definitions that the formulation here and the formulation in \( \textbf{Q2} \) and \( \textbf{Q3} \) are the same.

For \( n = 0 \), let \( \hat{\mathcal{C}}_0^0(V, W) \) be the subspace of \( W \) (which is in fact isomorphic to \( \text{Hom}(\mathcal{C}, W) \)) consisting of elements, say \( w \), such that \( a^dw = w \) (that is, \( w \in W_{[0]} \) and can be interpreted as a version of the \( \mathcal{d} \)-conjugation property), \( D_w w = 0 \) (can be interpreted as a version of the \( \mathcal{D} \)-derivative property). Note that because \( D_w w = 0 \), the \( \mathcal{D} \)-derivative property of \( Y_w^R \) implies that for \( v \in V \), \( Y_w^R(v, -z)v \) is independent of \( z \) and hence \( e^{zD_w}Y_w^R(w, -z)v \) is a power series in \( z \). Also, using the \( \mathcal{D} \)-derivative property, the \( \mathcal{D} \)-commutator formula and \( D_w w = 0 \), we see that for \( v \in V \), the derivative of \( e^{zD_w}Y_w^R(v, z)w \) with respect to \( z \) is in fact 0. Thus for \( v \in V \), \( Y_w^R(v, z)w = e^{zD_w}(e^{-zD_w}Y_w^L(v, z)w) \) is a power series in \( z \). An element \( w \) of \( \hat{\mathcal{C}}_0^0(V, W) \) has almost all the properties that the vacuum \( 1 \in V \) has and can be called a vacuum-like element.

For \( n > 0 \) and \( \Psi \in \hat{\mathcal{C}}_n^0(V, W) \), we define \( \delta_n \Psi \in \hat{\mathcal{C}}_{n+1}^1(V, W) \) by

\[
\langle w', ((\delta_n^\mathcal{d})(\Psi))(v_1 \otimes \cdots \otimes v_{n+1}))(z_1, \ldots, z_{n+1}) \rangle \\
= R(\langle w', Y_w^L(v_1, z_1)(\Psi(v_2 \otimes \cdots \otimes v_{n+1}))(z_2, \ldots, z_{n+1}) \rangle) \\
+ \sum_{i=1}^{n} R(\langle w', ((\Psi(v_1 \otimes \cdots \otimes v_{i-1} \otimes Y_V(v_i, z_i-z_{i+1})v_{i+1} \\
\otimes \cdots \otimes v_{n+1}))(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{n+1}) \rangle) \\
+ (-1)^{n+1} R((w', e^{z_{n+1}D_w}Y_w^R(\Psi(v_1 \otimes \cdots \otimes v_n))(z_1, \ldots, z_n, -z_{n+1})v_{n+1})) \tag{2.1}
\]

For \( w \in \hat{\mathcal{C}}_d^0(V, W) \), we define \( \delta^0 w \in \hat{\mathcal{C}}_1^1(V, W) \) by

\[
((\delta^0(w))(v))(z) = Y_w^L(v, z)w - e^{zD_w}Y_w^R(w, -z)v
\]

for \( v \in V \).

It was proved in \( \textbf{H7} \), \( \textbf{Q2} \) and \( \textbf{Q3} \) that \( \delta_n^\mathcal{d} \circ \delta_n^{\mathcal{d}-1} = 0 \) for \( n \in \mathbb{Z}_+ \). Thus we have the \( n \)-th cohomology \( H_n^\mathcal{d}(V, W) = \ker \delta_n^\mathcal{d}/(\delta_n^{\mathcal{d}-1}(\hat{\mathcal{C}}_n^{\mathcal{d}-1}(V, W))) \) for \( n \in \mathbb{Z}_+ \).

We now discuss the relation between \( H^\mathcal{d}_\infty(V, W) \) and derivations from \( V \) to \( W \).

**Definition 2.9** Let \( W \) be a \( V \)-bimodule. A derivation from \( V \) to \( W \) is a grading-preserving linear map \( f : V \to W \) satisfying

\[
f(Y_V(u, z)v) = Y_w^R(f(u), z)v + Y_w^L(u, z)f(v)
\]

for \( u, v \in V \).
Let \( w \in \hat{C}_\infty^0(V, W) \). By definition, for \( v \in V \), \( Y_L^W(v, z)w \) and \( e^{zDw}Y_R^W(w, -z)v \) are power series in \( z \). In particular, \( \lim_{z \to 0}(Y_L^W(v, z)w - e^{zDw}Y_R^W(w, -z)v) \) exists. Let \( f_w : V \to W \) be defined by these limits. That is,

\[
f_w(v) = \lim_{z \to 0}(Y_L^W(v, z)w - e^{zDw}Y_R^W(w, -z)v)
\]

for \( v \in V \). By the \( d \)-conjugation property for \( Y_L^W \) and \( Y_R^W \) and the fact that the weight of \( w \) is 0,

\[
adw f_w(v) = a dw \lim_{z \to 0}(Y_L^W(v, z)w - e^{zDw}Y_R^W(w, -z)v) = \lim_{z \to 0}(Y_L^W(a dv, az) a dw w - e^{azDw}Y_R^W(a dw w, -az) a dv v) = \lim_{z' \to 0}Y_L^W(a dv, z') w - e^{z'Dw}Y_R^W(w, -z') a dv v = f_w(a dv v)
\]

for \( a \in \mathbb{C}^x \). So \( f_w \) preserves weights.

From the \( D \)-derivative property of \( Y_L^W \) and \( Y_R^W \), \( D_w w = 0 \) and Taylor’s theorem, we obtain

\[
e^{z'Dw}(Y_L^W(v, z)w - e^{zDw}Y_R^W(w, -z)v) = e^{z'Dw}Y_L^W(v, z)w - e^{(z+z')Dw}Y_R^W(w, -z)v = Y_L^W(e^{z'Dw}v, z) w - e^{(z+z')Dw}Y_R^W(e^{-z'Dw}w, -z)v = Y_L^W(v, z + z')w - e^{(z+z')Dw}Y_R^W(w, -(z + z'))v
\]

for \( z' \in \mathbb{C} \). Taking the limit \( z \to 0 \) on both sides and then replacing \( z' \) by \( z \), we obtain

\[
Y_L^W(v, z)w - e^{zDw}Y_R^W(w, -z)v = e^{zDw}f_w(v).
\]

Let \( \Psi_w(v, z) = Y_L^W(v, z)w - e^{zDw}Y_R^W(w, -z)v \) for \( v \in V \). Then \( \Psi_w(v, z) = e^{zDw}f_w(v) \) and

\[
f_w(v) = \lim_{z \to 0} \Psi_w(v, z).
\]

**Proposition 2.10** For \( w \in \hat{C}_\infty^0(V, W) \), \( f_w \) is a derivation from \( V \) to \( W \).

**Proof.** We already know that \( f_w \) preserves weights.

For \( u, v \in V \) and \( w' \in W' \),

\[
R(\langle w', \Psi_w(Y_V(u, z_1 - z_2)v, z_2) \rangle) = R(\langle w', Y_L^W(Y_V(u, z_1 - z_2)v, z_2) \rangle) - R(\langle w', e^{z_2Dw}Y_R^W(w, -z_2)Y_V(u, z_1 - z_2)v \rangle) = R(\langle w', Y_L^W(u, z_1)Y_L^W(v, z_2)w \rangle) - R(\langle w', e^{z_2Dw}Y_R^W(Y_R^W(w, -z_1)u, z_1 - z_2)v \rangle) = R(\langle w', Y_L^W(u, z_1)Y_L^W(v, z_2)w \rangle) - R(\langle w', Y_L^W(u, z_1)e^{z_2Dw}Y_R^W(w, -z_2)v \rangle) + R(\langle w', Y_L^W(u, z_1)e^{z_2Dw}Y_R^W(w, -z_2)v \rangle) - R(\langle w', e^{z_2Dw}Y_R^W(Y_R^W(w, -z_1)u, z_1 - z_2)v \rangle)
\]
be a region containing a disk with 0 deleted in the variable $\zeta$

The convergence of the Laurent series of $f$.

Proof.

Then the multisum of (2.2) is equal to the Laurent series of $f(\zeta_1, \ldots, \zeta_n)$ in Condition 3. In particular, the multisum of (2.2) is convergent absolutely to $f(\zeta_1, \ldots, \zeta_n)$ in the region of convergence of the Laurent series of $f(\zeta_1, \ldots, \zeta_n)$ in Condition 3.

Proof. Taking $i = n$ in Condition 2, we know that (2.2) has only finitely many negative powers of $\zeta_n$. Since (2.2) is convergent absolutely in a region to $f(\zeta_1, \ldots, \zeta_n)$, there must be a region containing a disk with 0 deleted in the variable $\zeta_n$ such that (2.2) is convergent.
In particular, the first cohomology \( \hat{H}^1 \) is an isomorphism from the space of closed 1-cochains to the space of derivations from \( \hat{V} \). If \( \Psi \) is closed, then \( \hat{H}^1 \) is a derivation from \( \hat{V} \) to \( \hat{W} \). Also the conditions in Lemma 2.11 are what we can see easily in our proofs of the results in this paper. Thus the lemma is proved.

Remark 2.12 Lemma 2.11 can also be derived from Lemma 4.5 or Lemma 4.7 in [Q1]. The rational function in Lemma 2.11 is more special but is exactly what we need in this paper. Also the conditions in Lemma 2.11 are what we can see easily in our proofs of the results in this paper.

Theorem 2.13 Let \( V \) be a meromorphic open-string vertex algebra and \( W \) a \( V \)-bimodule. For \( \Psi \in \hat{C}_\infty^1(V,W) \), let \( f_\Psi : V \to W \) be defined by \( f_\Psi(v) = (\Psi(v))(0) \) for \( v \in V \). Then if \( \Psi \) is closed, \( f_\Psi \) is a derivation from \( V \) to \( W \) and the map given by \( \Psi \mapsto f_\Psi \) is a linear isomorphism from the space of closed 1-cochains to the space of derivations from \( V \) to \( W \). In particular, the first cohomology \( \hat{H}^1_\infty(V,W) \) of \( V \) with coefficients in \( W \) is isomorphic to the quotient of the space of derivations from \( V \) to \( W \) by the space of inner derivations.

Proof. Given an element \( \Psi \in \hat{C}_\infty^1(V,W) \), by definition,
\[
\langle w', ((\delta^1 \Psi)(v_1 \otimes v_2))(z_1, z_2) \rangle = R(\langle w', Y^L_W(v_1, z_1)(\Psi(v_2))(z_2) \rangle) - R(\langle w', (\Psi(Y^L(v_1, z_1 - z_2)v_2))(z_2) \rangle)
+ R\langle w', e^{2Dw}Y^R_W((\Psi(v_1))(z_1), -z_2)v_2) \rangle.
\]
Then the \( d \)-conjugation property satisfied by \( \Psi \) implies that \( f_\Psi \) preserves weights. The \( D \)-derivative property satisfied by \( \Psi \) gives \( (\Psi(v))(z) = e^{2Df_\Psi(v)} \) for \( v \in V \). If \( \Psi \) is closed, then we obtain
\[
R(\langle w', (\Psi(Y^L(v_1, z_1 - z_2)v_2))(z_2) \rangle)
= R(\langle w', Y^L_W(v_1, z_1)(\Psi(v_2))(z_2) \rangle) + R\langle (w', e^{2zDw}Y^R_W((\Psi(v_1))(z_1), -z_2)v_2) \rangle
= R(\langle w', Y^L_W(v_1, z_1)(\Psi(v_2))(z_2) \rangle) + R\langle (w', e^{2zDw}Y^R_W(f_\Psi(v_1), z_1 - z_2)v_2) \rangle.
\]
Letting \( z_2 = 0 \) on both sides of (2.3), we obtain
\[
\langle w', f_\Psi(Y^L(v_1, z_1)v_2) \rangle = \langle w', Y^L_W(v_1, z_1)f_\Psi(v_2) \rangle + \langle w', Y^R_W(f_\Psi(v_1), z_1)v_2 \rangle.
\]
Since (2.4) holds for all \( w' \in W' \), we have proved that \( f_\Psi \) is a derivation from \( V \) to \( W \). We obtain a linear map defined by \( \Psi \mapsto f_\Psi \) from the space of closed 1-cochains to the space of derivations from \( V \) to \( W \).
It is clear that given any derivation \( f \) from \( V \) to \( W \), for \( v \in V \), \( e^{z D_W} f(v) \) is a \( W \)-valued rational function in \( z \). To see that the linear map defined by \( \Psi : \text{Hom}(V, W) \to f_\psi \) is invertible, we first prove that \( \Psi f \in \text{Hom}(V, W) \) defined by \( (\Psi f)(v) = e^{z D_W} f(v) \) for \( v \in V \) is a closed 1-cochain. We have \( \frac{d}{dz} e^{z D_W} f(v) = D_W e^{z D_W} f(v) \). On the other hand, the same proof as the proof of Lemma 2.1 in [H8] shows that \( f(1) = 0 \). Thus we obtain

\[
\begin{align*}
f(D_W v) &= \lim_{z \to 0} \frac{d}{dz} f(e^{z D_W} v) \\
&= \lim_{z \to 0} \frac{d}{dz} f(Y_V(v, z)1) \\
&= \lim_{z \to 0} \frac{d}{dz} Y_{W}^R(f(v), z)1 + \lim_{z \to 0} \frac{d}{dz} Y_{W}^L(v, z)f(1) \\
&= \lim_{z \to 0} Y_{W}^R(D_W f(v), z)1 \\
&= D_W f(v)
\end{align*}
\]

for \( v \in V \). So \( \Psi f \) satisfies the \( D \)-derivative property. Since \( D_W \) is an operator of weight 1 and \( f \) preserves weights, we have \( a^d W e^{z D_W} f(v) = e^{az W} D_W a^{-d W} a^d W f(v) = e^{az} f(a^d W v) \) for \( a \in \mathbb{C}^* \), proving the \( d \)-conjugation property for \( \Psi f \).

For \( k, l, m \in \mathbb{N}, v_1, \ldots, v_{k+l+m} \in V, w' \in W' \),

\[
\begin{align*}
\langle w', Y_{W}^L(v_1, z_1) \cdots Y_{W}^L(v_k, z_k) Y_{W}^R(e^{(\xi - \zeta) D_W} f(Y_V(v_{k+1}, z_{k+1} - \xi) \cdots Y_V(v_{k+l}, z_{k+l} - \xi))1, \zeta) \cdots Y_Y(v_{k+l+m}, z_{k+l+m+1})1 \rangle \\
= \sum_{i=1}^{l} \langle w', Y_{W}^L(v_1, z_1) \cdots Y_{W}^L(v_k, z_k) Y_{W}^R(e^{(\xi - \zeta) D_W} Y_{W}^L(v_{k+1}, z_{k+1} - \xi) \cdots Y_{W}^L(v_{k+i-1}, z_{k+i-1} - \xi)) \cdots Y_{W}^R(f(v_{k+i}), z_{k+i} - \xi) Y_{W}^L(v_{k+i+1}, z_{k+i+1} - \xi) \cdots Y_V(v_{k+l}, z_{k+l} - \xi)1, \zeta) \cdots Y_Y(v_{k+l+m}, z_{k+l+m+1})1 \rangle
\end{align*}
\]

where in the right-hand side of (2.5), the negative powers of \( (z_{k+j} - \xi) + (\xi - \zeta) = z_{k+j} - \zeta \) for \( j = 1, \ldots, l \) are expanded as power series in \( \xi - \zeta \). Using (2.5) and the properties of the \( V \)-bimodule \( W \), we now prove that \( \Psi f \) can be composed with an arbitrary number of vertex operators.

We first prove that every term in the right-hand side of (2.5) is convergent absolutely in the region given by \( |z_1| > \cdots > |z_k|, |z_{k+1} - \xi| > \cdots > |z_{k+l} - \xi|, |z_{k+l+1} > \cdots > |z_{k+l+m}|, |z_a | > |z_{k+j} - \xi| + |\xi - \zeta| + |\xi| \) and \( |\zeta| > |z_{k+j} - \xi| + |z_{k+l+p}| + |\xi - \zeta| \) for \( a = 1, \ldots, k, j = 1, \ldots, l \).
and \( p = 1, \ldots, m \). We use induction on \( l \) to prove that the \( i \)-th term in the right-hand side of (2.5) is convergent absolutely. When \( l = 1 \), the only term in the right-hand side of (2.5) is

\[
\langle w', Y^L_W(v_1, z_1) \cdots Y^L_W(v_k, z_k) Y^R_W(Y^R_W(f(v_{k+1}), (z_{k+1} - \xi) + (\xi - \zeta)) \mathbf{1}, \zeta) \cdot Y_V(v_{k+2}, z_{k+2}) \cdots Y_V(v_{k+1+m}, z_{k+1+m}) \mathbf{1} \rangle.
\]

By the associativity of \( Y^R_W \), we know that \( Y^R_W(Y^R_W(f(v_{k+1}), (z_{k+1} - \xi) + (\xi - \zeta)) \mathbf{1}, \zeta) v \) for \( v \in V \) is convergent absolutely in the region \( |\zeta| > |z_{k+1} - \xi| + |\xi - \zeta| \) and is equal to \( Y^R_W(f(v_{k+1}), (z_{k+1} - \xi) + (\xi - \zeta)) \mathbf{1}, \zeta \) in (2.6). By (2.7) and one of the sums in the series (2.6) is convergent absolutely in the region \( |\zeta| > |z_{k+1} - \xi| + |\xi - \zeta| \) to the series (2.7). By Lemma (2.11), we see that this expansion must be equal to (2.6). In particular, we see that (2.6) must be convergent absolutely to this rational function in the region \( |z_i| > \cdots > |z_{k+1+m}| \) to a rational function in \( z_1, \ldots, z_{k+1+m} \) with the only possible poles at \( z_a = z_b \) for \( a \neq b \). Expand this rational function as a Laurent series in the variables \( z_1, \ldots, z_k, z_{k+1} - \xi, z_{k+2}, \ldots, z_{k+1+m}, \zeta \) and \( \xi - \zeta \) in the region \( |z_1| > \cdots > |z_k| > 0, |z_{k+2}| > |z_{k+1+m}|, |z_i| > |z_{k+1} - \xi| + |\zeta - \zeta| + |\xi| \) for \( i = 1, \ldots, k, |\zeta| > |z_{k+1} - \xi| + |z_{k+1+m} - \xi| + |\xi - \zeta| \) for \( p = 1, \ldots, m \). This expansion has the same form as the series (2.6) and one of the sums in the series (2.6) is convergent absolutely in the region \( |\zeta| > |z_{k+1} - \xi| + |\xi - \zeta| \). In the case \( l = 1 \), the right-hand side of (2.5) is convergent absolutely in the region given by \( |z_1| > \cdots > |z_k|, |z_{k+2}| > \cdots > |z_{k+1+m}|, |z_a| > |z_{k+1} - \xi| + |\xi - \zeta| + |\xi| \) and \( |\zeta| > |z_{k+1} - \xi| + |z_{k+1+m} - \xi| + |\xi - \zeta| \) for \( a = 1, \ldots, k \) and \( p = 1, \ldots, m \).

Now we assume that for \( l = q - 1 \), every term in the right-hand side of (2.5) is convergent absolutely in the region given by \( |z_1| > \cdots > |z_k|, |z_{k+1} - \xi| > \cdots > |z_{k+l} - \xi|, |z_{k+l+1}| > \cdots > |z_{k+l+m}|, |z_a| > |z_{k+j} - \xi| + |\xi - \zeta| + |\xi| \) and \( |\zeta| > |z_{k+j} - \xi| + |z_{k+l+p} - \xi| + |\xi - \zeta| \) for \( a = 1, \ldots, k, j = 1, \ldots, l \) and \( p = 1, \ldots, m \). In the case \( l = q \) and \( i = 1 \), \( Y^R_W(Y^R_W(f(v_{k+1}), (z_{k+1} - \xi) + (\xi - \zeta)) \mathbf{1}, \zeta) v \) for \( v, w \in V \) is convergent absolutely in the region \( |\zeta| > |z_{k+1} - \xi| + |\xi - \zeta| \) and is equal to \( Y^R_W(f(v_{k+1}), z_{k+1}) Y_V(u, \zeta) v \). In the case \( l = q \) and \( i > 1 \), \( Y^R_W(Y^R_W(v_{k+1}, (z_{k+1} - \xi) + (\xi - \zeta)) w, \zeta) v \) for \( v, w \in V \) is convergent absolutely in the region \( |\zeta| > |z_{k+1} - \xi| + |\xi - \zeta| \) and is equal to \( Y^R_W(v_{k+1}, z_{k+1}) Y^R_W(w, \zeta) v \). So in the region \( |\zeta| > |z_{k+1} - \xi| + |\xi - \zeta| \), the sum involving \( Y^R_W(Y^R_W(f(v_{k+1}), (z_{k+1} - \xi) + (\xi - \zeta)) w, \zeta) v \) in the case \( i = 1 \) or the sum involving \( Y^R_W(Y^R_W(v_{k+1}, (z_{k+1} - \xi) + (\xi - \zeta)) w, \zeta) v \) in the case \( i > 1 \) in the \( i \)-th term in (2.6) is convergent absolutely to the series

\[
\langle w', Y^L_W(v_1, z_1) \cdots Y^L_W(v_k, z_k) Y^R_W(f(v_{k+1}), z_{k+1}) \cdot Y_V(Y(v_{k+2}, (z_{k+2} - \xi) + (\xi - \zeta)) \cdots Y_V(v_{k+l}, (z_{k+l} - \xi) + (\xi - \zeta)) \mathbf{1}, \zeta) \cdot Y_V(v_{k+l+1}, z_{k+l+1}) \cdots Y_V(v_{k+1+m}, z_{k+1+m}) \mathbf{1} \rangle
\]
respectively.

We first discuss the case \( i = 1 \). By the induction assumption with \( W = V \), for \( v' \in V' \),

\[
\langle w', Y_W(v_{k+2}, z_{k+2} - \xi) \rangle \cdots Y_W(v_k, z_k) Y_W(f(v_{k+1}), z_{k+1}) Y_W(v_{k+1}, (z_{k+2} - \xi) + (\xi - \zeta)) Y_W(v_{k+2}, (z_{k+3} - \xi) + (\xi - \zeta)) \cdots Y_W(v_{k+l}, (z_{k+l+1} - \xi) + (\xi - \zeta)) Y_W(v_{k+l+1}, (z_{k+l+2} - \xi) + (\xi - \zeta)) \cdots Y_W(v_{k+l+m}, (z_{k+l+m} - \xi) + (\xi - \zeta)) Y_W(v_{k+1+m}, (z_{k+1+m} - \xi) + (\xi - \zeta)) \rangle,
\]

is convergent absolutely in the region \(|z_{k+1} - \xi| > \cdots > |z_{k+l+1} - \xi| > \cdots > |z_{k+l+m} - \xi|\) and \(|\zeta| > |z_{k+j} - \xi| + |z_{k+l+p} - \xi| + |\xi - \zeta|\) for \( j = 1, \ldots, l \) and \( p = 1, \ldots, m \) to the rational function

\[
R(\langle v', Y_V(v_{k+2}, z_{k+2}) \cdots Y_V(v_{k+1+m}, z_{k+1+m}) \rangle),
\]

or equivalently,

\[
Y_V(Y_V(v_{k+2}, z_{k+2} - \xi) + (\xi - \zeta)) \cdots Y_V(v_{k+l}, (z_{k+l} - \xi) + (\xi - \zeta)) Y_V(v_{k+l+1}, (z_{k+l+1} - \xi) + (\xi - \zeta)) Y_V(v_{k+l+m}, (z_{k+l+m} - \xi) + (\xi - \zeta)) Y_V(v_{k+1+m}, (z_{k+1+m} - \xi) + (\xi - \zeta)) \rangle
\]

is convergent absolutely in this region to

\[
E(Y_V(v_{k+2}, z_{k+2}) \cdots Y_V(v_{k+l+m}, z_{k+l+m})).
\]

But we know that in the region \(|z_1| > \cdots > |z_{k+l+m}|\),

\[
\langle w', Y_W(v_{k+2}, z_{k+2}) \cdots Y_W(v_k, z_k) Y_W(f(v_{k+1}), z_{k+1}) E(Y_V(v_{k+2}, z_{k+2}) \cdots Y_V(v_{k+l+m}, z_{k+l+m})) \rangle
\]

is convergent absolutely to a rational function in \( z_1, \ldots, z_{k+l+m} \) with the only poles at \( z_a = z_b \) for \( a \neq b \). Thus in the region \(|z_1| > \cdots > |z_{k+l+m}|\), \(|z_{k+2} - \xi| > \cdots > |z_{k+l} - \xi|\), \(|z_a| > |z_{k+j} - \xi| + |\xi - \zeta| + |\zeta|\) and \(|\zeta| > |z_{k+j} - \xi| + |z_{k+l+p} - \xi| + |\xi - \zeta|\) for \( a = 1, \ldots, k + 1, j = 2, \ldots, l \) and \( p = 1, \ldots, m \),

\[
\langle w', Y_W(v_1, z_1) \cdots Y_W(v_k, z_k) Y_W(f(v_{k+1}), z_{k+1}) \rangle Y_V(Y_V(v_{k+2}, z_{k+2} - \xi) + (\xi - \zeta)) \cdots Y_V(v_{k+l}, (z_{k+l} - \xi) + (\xi - \zeta)) Y_V(v_{k+l+1}, (z_{k+l+1} - \xi) + (\xi - \zeta)) Y_V(v_{k+l+m}, (z_{k+l+m} - \xi) + (\xi - \zeta)) Y_V(v_{k+1+m}, (z_{k+1+m} - \xi) + (\xi - \zeta)) \rangle
\]

is convergent absolutely to this rational function. Expand this rational function as a Laurent series in the variables \( z_1, \ldots, z_k, z_{k+1} - \xi, \ldots, z_{k+l} - \xi, z_{k+l+1}, \ldots, z_{k+l+m}, \zeta \) and \( \xi - \zeta \) in
the region $|z_1| > \cdots > |z_k|, |z_{k+1} - \xi| > \cdots > |z_{k+l} - \xi|, |z_{k+l+1}| > \cdots > |z_{k+l+m}|$, $|z_a| > |z_{k+j} - \xi| + |\xi - \zeta| + |\xi|$ and $| \zeta | > |z_{k+j} - \xi| + |z_{k+l+p}| + |\xi - \zeta|$ for $a = 1, \ldots, k$, $j = 1, \ldots, l$ and $p = 1, \ldots, m$. This expansion has the same form as the first term in the series (2.5) and one of the sums in the series (2.5) is convergent absolutely in the region $| \zeta | > |z_{k+1} - \xi| + |\xi - \zeta|$ to the series (2.5) which is in turn convergent absolutely to this rational function. By Lemma 2.11 we see that this expansion must be equal to the first term in the series (2.5). In particular, we see that the first term in the series (2.5) must be convergent absolutely to this rational function in $z_1, \ldots, z_k, z_{k+1} - \xi, \ldots, z_{k+l} - \xi, z_{k+l+1}, \ldots, z_{k+l+m}, \zeta$ and $\xi - \zeta$ in the region $|z_1| > \cdots > |z_k|, |z_{k+1} - \xi| > \cdots > |z_{k+l} - \xi|, |z_{k+l+1}| > \cdots > |z_{k+l+m}|$, $|z_a| > |z_{k+j} - \xi| + |\xi - \zeta| + |\zeta|$ and $| \zeta | > |z_{k+j} - \xi| + |z_{k+l+p}| + |\xi - \zeta|$ for $a = 1, \ldots, k$, $j = 1, \ldots, l$ and $p = 1, \ldots, m$. This proves that in the case $l = q$, the first term in the right-hand side of (2.5) is convergent absolutely to the rational function above in this region.

We now discuss the case $i > 1$ which is more straightforward than the case $i = 1$. Using the induction assumption, (2.3) is convergent absolutely in the region $|z_1| > \cdots > |z_{k+1}|, |z_{k+2} - \xi| > \cdots > |z_{k+l} - \xi|, |z_{k+l+1}| > \cdots > |z_{k+l+m}|, |z_a| > |z_{k+j} - \xi| + |\xi - \zeta| + |\zeta|$ and $| \zeta | > |z_{k+j} - \xi| + |z_{k+l+p}| + |\xi - \zeta|$ for $a = 1, \ldots, k$, $j = 2, \ldots, l$ and $p = 1, \ldots, m$ to a rational function in $z_1, \ldots, z_{k+l+m}$ with the only possible poles at $z_a = z_b$ for $a \neq b$. Expanding this rational function as a series in $z_1, \ldots, z_k, z_{k+1} - \xi, \ldots, z_{k+l} - \xi, z_{k+l+1}, \ldots, z_{k+l+m}, \zeta$ and $\xi - \zeta$ in the region $|z_1| > \cdots > |z_k|, |z_{k+1} - \xi| > \cdots > |z_{k+l} - \xi|, |z_{k+l+1}| > \cdots > |z_{k+l+m}|, |z_a| > |z_{k+j} - \xi| + |\xi - \zeta| + |\zeta|$ and $| \zeta | > |z_{k+j} - \xi| + |z_{k+l+p}| + |\xi - \zeta|$ for $a = 1, \ldots, k$, $j = 1, \ldots, l$ and $p = 1, \ldots, m$. The remaining steps are the same as in the $i = 1$ case except that we replace the first term in (2.5) and (2.8) by the $i$-th term and (2.5), respectively. This proves that in the case $l = q$, the $i$-th term in the right-hand side of (2.5) is convergent absolutely to the rational function above in the region above.

Since the poles of the rational function that (2.5) converges to are the poles of

$$
\sum_{i=1}^{l} R(\langle w', Y_W^L(v_1, z_1) \cdots Y_W^L(v_{k+i-1}, z_{k+i-1}) \rangle, $$

$$\cdot Y_W^R(f(v_{k+i}), z_{k+i}) Y_V(v_{k+i+1}, z_{k+i+l}) \cdots Y_V(v_{k+l+m}, z_{k+l+m} 1)), \quad (2.10)
$$

we see from the proof of the convergence above that the existence of $p_{ij}$ for $i, j = 1, \ldots, k+l+m$ follows from the pole-order condition satisfied by $W$. Thus $\Psi_f$ can be composed with an arbitrary number of vertex operators.

For $v_1, v_2 \in V$ and $w' \in W'$, by the definition of $\delta^1_\infty$ and the fact that $f$ is a derivation,

$$\langle w', ((\delta^1_\infty(\Psi_f))(v_1 \otimes v_2))(z_1, z_2) \rangle$$

$$= R(\langle w', Y_W^L(v_1, z_1)(e^{z_2 D_w} f(v_2)) \rangle)$$

$$- R(\langle w', (e^{z_2 D_w} f(Y_V(v_1, z_1 - z_2)v_2)) \rangle)$$

$$+ R(\langle w', e^{z_2 D_w} Y_W^R(e^{z_1 D_w} f(v_1), -z_2)v_2) \rangle)$$

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Proof. Since the weight of $v$ for Proposition 2.14 in our main theorem.

On the other hand, by (2.11) and the compatibility of the left and right actions, there exists $V$ a derivation from $V$ invertible with the inverse given by $f$.

So $Ψ$ such that

the quotient space of the space of derivations from $V$ is isomorphic to the space of derivations from $V$ to $W$.

By this fact, (2.12) and (2.13), Cauchy’s integral theorem and the convergence region of

We now discuss another type of derivations from $V$ to $W$ which play an important role in our main theorem.

**Proposition 2.14** Let $w$ be an element of $W_{[1]}$ such that

$$e^{x Dw} Y^L_w(v, -x) - Y^R_w(w, x)v \in W[x, x^{-1}]$$

for $v \in V$. Then $g_w : V \rightarrow W$ defined by $g_w(v) = \text{Res}_x Y^R_w(w, x)v = (Y^R_w)_0(w)v$ for $v \in V$ is a derivation from $V$ to $W$.

**Proof.** Since the weight of $w$ is 1, the weight of the map $g_w = (Y^R_w)_0(w) : V \rightarrow W$ is $wt w - 0 - 1 = 0$. So $g_w$ preserves weights.

For $u, v \in V$ and $w' \in W'$, by the definition of bimodule, we have the associativity

$$R(⟨w', Y^R_{W}(w, z_1)Y_{V}(u, z_2)v⟩) = R(⟨w', Y^R_{W}(Y^R_{W}(w, z_1 - z_2)u, z_2)v⟩). \tag{2.12}$$

On the other hand, by (2.11) and the compatibility of the left and right actions, there exists a rational function $h(z_1, z_2)$ in $z_1$ and $z_2$ with the only possible poles at $z_2 = 0$ and $z_1 = z_2$ such that

$$R(⟨w', Y^R_{W}(Y^R_{W}(w, z_1 - z_2)u, z_2)v⟩)$$

$$= R(⟨w', Y^R_{W}(e^{(z_1-z_2)Dw}Y^L_{W}(u, -(z_1 - z_2))w, z_2)v⟩) + h(z_1, z_2)$$

$$= R(⟨w', Y^R_{W}(Y^L_{W}(u, z_2 - z_1)w, z_1)v⟩) + h(z_1, z_2)$$

$$= R(⟨w', Y^L_{W}(u, z_2)Y^R_{W}(w, z_1)v⟩) + h(z_1, z_2). \tag{2.13}$$

Integrate the same rational function in (2.12) and (2.13) with respect to $z_1$ along a circle $C_\infty$ centered at 0 and containing $z_2$ in its interior. Since $h(z_1, z_2)$ is analytic in the disk $|z_1| < |z_2|$ as a function of $z_1$, its integral along a circle of radius less than $|z_2|$ is 0.

By this fact, (2.12) and (2.13), Cauchy’s integral theorem and the convergence region of

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\[ \langle w', Y^R_W(w, z_1)Y^L_W(u, z_2)v \rangle, \langle w', Y^R_W(Y^R_W(w, z_1 - z_2)u, z_2)v \rangle \] and \[ \langle w', Y^L_W(u, z_2)Y^R_W(w, z_1)v \rangle, \]
we obtain
\[
\oint_{C_\infty} \langle w', Y^R_W(w, z_1)Y^L_W(u, z_2)v \rangle \, dz_1
\]
\[
= \oint_{C_{z_2}} \langle w', Y^R_W(Y^R_W(w, z_1 - z_2)u, z_2)v \rangle \, dz_1 + \oint_{C_0} (\langle w', Y^L_W(u, z_2)Y^R_W(w, z_1)v \rangle + h(z_1, z_2)) \, dz_1
\]
\[
= \oint_{C_{z_2}} \langle w', Y^R_W(Y^R_W(w, z_1 - z_2)u, z_2)v \rangle \, dz_1 + \oint_{C_0} \langle w', Y^L_W(u, z_2)Y^R_W(w, z_1)v \rangle \, dz_1, \tag{2.14}
\]
where \( C_{z_2} \) and \( C_0 \) are circles centered at \( z_2 \) and 0, respectively, with radii less than \(|z_2|\). By the definition of \( g_w, \tag{2.14} \) gives
\[
g_w(Y^L_W(u, z_2)v) = Y^R_W(g_w(u), z_2)v + Y^L_W(u, z_2)g_w(v),
\]
proving that \( g_w \) is a derivation from \( V \) to \( W \).

We shall call the derivation in Proposition 2.14 a zero-mode derivation since it is in fact the zero-mode of the right vertex operator of the element \( w \). We shall denote the subspace of \( \hat{H}^1_\infty(V, W) \) consisting of the cosets containing zero-mode derivations by \( \hat{Z}^1_\infty(V, W) \).

3 A \( V \)-bimodule constructed from two left \( V \)-modules

In this section, we construct a \( V \)-bimodule \( H^N \) for \( N \in \mathbb{Z} \) from two left \( V \)-modules. The bimodule \( H^N \) is analogous to the bimodule \( \text{Hom}(M_1, M_2) \) for an associative algebra \( A \) constructed from left \( A \)-modules \( M_1 \) and \( M_2 \). In the category of modules for a vertex operator algebra, the first analogue of this bimodule was in fact given in the construction of the \( Q(z) \)-tensor product by Lepowsky and the first author in [HL1] (although the term \( Q(z) \)-tensor product was introduced later in [HL2]). This is the reason why a \( Q(z_0) \)-tensor product appears in Theorem 6.5 below. But even when \( V \) is a vertex operator algebra, the analogue of the bimodule \( \text{Hom}(M_1, M_2) \) needed in this paper is different. What we want is an analogue when \( V \) is viewed as a meromorphic open-string vertex algebra. This is the main reason why the construction in this section is difficult.

In the rest of this paper, we fix a meromorphic open-string vertex algebra \( V \). Let \( W_1 \) and \( W_2 \) be two left \( V \)-modules. Recall that by our convention, \( V, W_1 \) and \( W_2 \), in particular, satisfy the pole-order conditions. Let \((\widehat{W_2})_z\) be the space of \( W_2 \)-valued rational functions with the only possible pole at \( z = 0 \). Recall that \((\widehat{W_2})_z\) is the space of \( W_2 \)-valued holomorphic functions. Thus \((\widehat{W_2})_z \supset (\widehat{W_2})_z\).

Let \( H \) be the subspace of \( \text{Hom}(W_1, (\widehat{W_2})_z) \) spanned by elements, denoted by \( \phi \), satisfying the following conditions:

1. The \( d \)-conjugation property: There exists \( n \in \mathbb{Z} \) (called the weight of \( \phi \) and denoted by \( \text{wt} \ \phi \)) such that for \( a \in \mathbb{C}^\times \) and \( w_1 \in W_1 \),
\[
a^{d_{W_2}}(\phi(w_1))(z) = a^n(\phi(a^{d_{W_1}}w_1))(az).
\]
2. The *composability*; For $k, l \in \mathbb{N}$ and $v_1, \ldots, v_{k+l} \in V$, $w_1 \in W_1$ and $w'_2 \in W'_2$, the series

$$(w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k)(\phi(Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l}))w_1))(z)$$

is absolutely convergent in the region $|z_1| > \cdots > |z_k| > |z| > |z_{k+1}| > \cdots |z_{k+l}| > 0$ to a rational function

$$R((w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k)(\phi(Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l}))w_1))(z))$$

(3.1)

in $z_1, \ldots, z_{k+l}$ and $z$ with the only possible poles $z_i = 0$ for $i = 1, \ldots, k+l$, $z = 0$, $z_i = z_j$ for $i, j = 1, \ldots, k+l$, $i \neq j$ and $z_i = z$ for $i = 1, \ldots, k+l$. Moreover, there exist $r_i \in \mathbb{N}$ depending only on the pair $(v_i, w_1)$ for $i = 1, \ldots, k+l$, $m \in \mathbb{N}$ depending only on the pair $(\phi, w_1)$, $p_{ij} \in \mathbb{N}$ depending only on the pair $(v_i, v_j)$ for $i, j = 1, \ldots, k+l$, $i \neq j$, $s_i \in \mathbb{N}$ depending only on the pair $(v_i, \phi)$ for $i = 1, \ldots, k+l$ and $g(z_1, \ldots, z_{k+l}, z) \in W_2[[z_1, \ldots, z_{k+l}]]$ such that for $w'_2 \in W'_2$,

$$z^m \prod_{i=1}^{k+l} z_i^{r_i} \prod_{1 \leq i < j \leq k+l} (z_i - z_j)^{p_{ij}} \prod_{i=1}^{k+l} (z_i - z)^{s_i} \cdot R((w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k)(\phi(Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l}))w_1))(z))$$

is a polynomial and is equal to $(w'_2, g(z_1, \ldots, z_{k+l}, z))$.

Let $H_n$ be the subspace of $H$ consisting of elements of weight $n$. Then $H = \bigsqcup_{n \in \mathbb{Z}} H_n$.

Next we define the left and right vertex operator maps:

$$Y^L_H : V \otimes H \rightarrow H[[x, x^{-1}]]$$

$$v \otimes \phi \mapsto Y^L_H(v, x)\phi,$$

$$Y^R_H : H \otimes V \rightarrow H[[x, x^{-1}]]$$

$$\phi \otimes v \mapsto Y^R_H(\phi, x)v.$$

Heuristically we would like to define them using the formulas

$$\langle w'_2, (Y^L_H(v, z_1)\phi)(w_1))(z_2) \rangle = \langle w'_2, Y_{W_2}(v, z_1 + z_2)(\phi(w_1))(z_2) \rangle,$$

(3.2)

$$\langle w'_2, ((Y^R_H(\phi, z_1)v)(w_1))(z_2) \rangle = \langle w'_2, \phi(Y_{W_1}(v, z_2)w_1)(z_1 + z_2) \rangle$$

(3.3)

for $v \in V$, $\phi \in H$, $w_1 \in W_1$ and $w'_2 \in W'_2$. But we need to make these heuristic definitions precise.

We first give the precise definition of $Y^L_H$. Let $\phi \in H$. Since $\phi$ satisfies the composability, for $v \in V$, $w_1 \in W_1$ and $w'_2 \in W'_2$,

$$\langle w'_2, Y_{W_2}(v_1, z_1 + z_2)(\phi(w_1))(z_2) \rangle$$

is absolutely convergent in the region given by $|z_1 + z_2| > |z_2| > 0$ to a rational function

$$R((w'_2, Y_{W_2}(v, z_1 + z_2)(\phi(w_1))(z_2)))$$

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in \(z_1\) and \(z_2\) with the only possible poles at \(z_1 = 0, z_2 = 0, z_1 + z_2 = 0\). Expanding this rational function in the region \(|z_2| > |z_1| > 0\), we obtain a lower truncated Laurent series

\[
\sum_{p \in \mathbb{Z}} a^L_p (w'_2 \otimes v \otimes \phi \otimes w_1; z_2) z_1^{-p-1}
\]

in \(z_1\). The coefficients \(a^L_p (w'_2 \otimes v \otimes \phi \otimes w_1; z_2)\) for \(p \in \mathbb{Z}\) of this Laurent series are in fact Laurent polynomials in \(z_2\). On the other hand, by the composability, there exist \(r, s, m \in \mathbb{N}\) and \(g(z_1 + z_2, z_2) \in W_2[[z_1 + z_2, z_2]]\) such that

\[
z_2^m (z_1 + z_2)^r z_1^s R(\langle w'_2, Y_{W_2}(v, z_1 + z_2)(\phi(w_1))(z_2) \rangle) = \langle w'_2, g(z_1 + z_2, z_2) \rangle.
\]  
(3.4)

For fixed \(z_2 \neq 0\), the coefficients of the expansion in \(z_1\) of the left-hand side of (3.4) in the region \(|z_2| > |z_1| > 0\) for \(w'_2 \in W'_2\) define elements of \((W'_2)^*\). But since \(g(z_1 + z_2, z_2) \in W_2[[z_1 + z_2, z_2]]\), we see that these elements must be in the subspace \(\overline{W}_2\). Multiplying these elements by \(z_2^{-m}(z_1 + z_2)^{-r}z_1^{-s}\) in the region \(|z_2| > |z_1| > 0\), we see that the results are still in \(\overline{W}_2\), that is, for fixed \(z_2 \neq 0\), the maps given by \(w'_2 \mapsto a^L_p (w'_2 \otimes v \otimes \phi \otimes w_1; z_2)\) for \(p \in \mathbb{Z}\) are in fact elements of \(\overline{W}_2\). When \(z_2\) changes, we obtain elements of \((\overline{W}_2)_{z_2}\). Then for \(v \in V\) and \(\phi \in H\), we have elements \(\eta^L_{p,v,\phi} \in \text{Hom}(W_1, (\overline{W}_2)_{z_2})\) for \(p \in \mathbb{Z}\) such that

\[
\langle w'_2, (\eta^L_{p,v,\phi}(w_1))(z) \rangle = a^L_p (w'_2 \otimes v \otimes \phi \otimes w_1; z)
\]

for \(w_1 \in W_1\) and \(w'_2 \in W'_2\).

Similarly, since \(\phi\) satisfies the composability, for \(v \in V\), \(w_1 \in W_1\) and \(w'_2 \in W'_2\),

\[
\langle w'_2, \phi(Y_{W_1}(v, z_2)w_1)(z_1 + z_2) \rangle
\]

is absolutely convergent in the region given by \(|z_1 + z_2| > |z_2| > 0\) to a rational function

\[
R(\langle w'_2, \phi(Y_{W_1}(v, z_2)w_1)(z_1 + z_2) \rangle)
\]

in \(z_1\) and \(z_2\) with the only possible poles at \(z_1 = 0, z_2 = 0, z_1 + z_2 = 0\). Expanding this rational function in the region \(|z_2| > |z_1| > 0\), we obtain a lower truncated Laurent series

\[
\sum_{p \in \mathbb{Z}} a^R_p (w'_2 \otimes v \otimes w_1; z_2) z_1^{-p-1}
\]

in \(z_1\). The coefficients \(a^R_p (w'_2 \otimes v \otimes w_1; z_2)\) for \(p \in \mathbb{Z}\) of this Laurent series are in fact Laurent polynomials in \(z_2\). For fixed \(z_2 \neq 0\), the same argument as above shows that the maps given by \(w'_2 \mapsto a^R_p (w'_2 \otimes v \otimes w_1; z_2)\) for \(p \in \mathbb{Z}\) are in fact elements of \(\overline{W}_2\). When \(z_2\) changes, we obtain elements of \((\overline{W}_2)_{z_2}\). The for \(v \in V\) and \(\phi \in H\), we have elements \(\eta^R_{p,v,\phi} \in \text{Hom}(W_1, (\overline{W}_2)_{z_2})\) for \(p \in \mathbb{Z}\) such that

\[
\langle w'_2, \eta^R_{p,v,\phi}(w_1) \rangle = a^R_p (w'_2 \otimes v \otimes w_1; z)
\]

for \(w_1 \in W_1\) and \(w'_2 \in W'_2\).
Proposition 3.1 The maps $\eta_{p,v,\phi}^L$ and $\eta_{p,v,\phi}^R$ are elements of $H$. When both $v$ and $\phi$ are homogeneous, $\eta_{p,v,\phi}^L$ and $\eta_{p,v,\phi}^R$ are also homogeneous of weight $\text{wt} \ v + \text{wt} \ \phi - p - 1$. In addition, if $\phi$ satisfies the D-derivative property

$$\frac{d}{dz}(\phi(w_1))(z) = D_{W_2}(\phi(w_1))(z) - (\phi(D_{W_2}w_1))(z),$$

or $w_1 \in W_1$, where $D_{W_2}$ is the natural extension of $D_{W_2}$ on $W_2$ to $\overline{W}_2$, $\eta_{p,v,\phi}^L$ and $\eta_{p,v,\phi}^R$ also satisfy the D-derivative property

$$\frac{d}{dz}(\eta_{p,v,\phi}^L(w_1))(z) = (D_{W_2}(\eta_{p,v,\phi}^L(w_1))(z) - (\eta_{p,v,\phi}^L(D_{W_2}w_1))(z))$$

and

$$\frac{d}{dz}(\eta_{p,v,\phi}^R(w_1))(z) = (D_{W_2}(\eta_{p,v,\phi}^R(w_1))(z) - (\eta_{p,v,\phi}^R(D_{W_2}w_1))(z))$$

for $w_1 \in W_1$.

Proof. We prove the result only for $\eta_{p,v,\phi}^L$. The proof for $\eta_{p,v,\phi}^R$ is similar and is omitted.

In the case that $v$ and $\phi$ are homogeneous, for $w_1 \in W_1$, $w_2' \in W_2'$ and $a \in \mathbb{C}^*$, in the region $|z_2| > |z_1| > 0$, we have

$$\sum_{p \in \mathbb{Z}} (w_2', a^{d_{w_2}(\eta_{p,v,\phi}^L(w_1))(z_2)}) z_1^{-p-1}$$

$$= \sum_{p \in \mathbb{Z}} (a^{d_{w_2}w_2'}(\eta_{p,v,\phi}^L(w_1))(z_2)) z_1^{-p-1}$$

$$= \sum_{p \in \mathbb{Z}} a_p^L (a^{d_{w_2}w_2'} \otimes v \otimes \phi \otimes w_1; z_2) z_1^{-p-1}$$

$$= R((a^{d_{w_2}w_2'}, Y_{W_2}(v, z_1 + z_2)(\phi(w_1))(z_2)))$$

$$= R((w_2', a^{d_{w_2}Y_{W_2}(v, z_1 + z_2)(\phi(w_1))(z_2)))$$

$$= R((w_2', Y_{W_2}(a^{d_{w_2}(v, az_1 + az_2)}a^{d_{w_2}(\phi(w_1))(z_2))))$$

$$= a^{\text{wt} \ v + \text{wt} \ \phi} R((w_2', Y_{W_2}(v, az_1 + az_2)(\phi(a^{d_{w_1}w_1}))(az_2)))$$

$$= a^{\text{wt} \ v + \text{wt} \ \phi - p - 1} \sum_{p \in \mathbb{Z}} a_p^L (w_2' \otimes v \otimes \phi \otimes a^{d_{w_1}w_1}; az_2) z_1^{-p-1}$$

$$= a^{\text{wt} \ v + \text{wt} \ \phi - p - 1} \sum_{p \in \mathbb{Z}} (w_2', (\eta_{p,v,\phi}(a^{d_{w_1}w_1}))(az_2)) z_1^{-p-1},$$

proving the d-conjugation property for $\eta_{p,v,\phi}^L$ with weight $\text{wt} \ v + \text{wt} \ \phi - p - 1$.

Now we prove the composability of $\eta_{p,v,\phi}^L$. Since $\phi \in H$, it satisfies the composability. Then

$$\langle w_2', Y_{W_2}(v, z_1) \cdots Y_{W_2}(v, z_k) \rangle Y_{W_2}(v, z_0 + z)(\phi(Y_{W_1}(v, z_{k+1}, z_{k+1}) \cdots Y_{W_1}(v, z_{k+i}, z_{k+i})w_1))(z)$$

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is absolutely convergent in the region $|z_1| > \cdots > |z_k| > |z_0 + z| > |z| > |z_{k+1}| > \cdots > |z_{k+l}| > 0$ to a rational function

$$R((\langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k)Y_{W_2}(v, z_0 + z)(\phi(Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l})w_1))(z)\rangle)$$

(3.5)

in $z_0, z_1, \ldots, z_{k+l}$ and $z$ with the only possible poles $z_i = 0$ for $i = 1, \ldots, k + l$, $z_0 + z = 0$, $z = 0$, $z_i = z_j$ for $i \neq j$, $z_i = z_0 + z_i$ and $z_i = z$ for $i = 1, \ldots, k + l$ and $z_0 = 0$.

For $q \in \mathbb{C}$, let $\{e_{W_2}^{W_2}\}_{\lambda \in \Lambda_q}$ be a basis of $(W_2)_{[q]}$ and $\{(e_{W_2}^{W_2})'\}_{\lambda \in \Lambda_q}$ be the subset of $W'_2$ defined by

$$\langle (e_{W_2}^{W_2})', e_{W_2}^{W_2} \rangle = \delta_{\lambda_1, \lambda_2}.$$ 

Putting $\{e_{W_2}^{W_2}\}_{\lambda \in \Lambda_q}$ for $q \in \mathbb{C}$ together, we see that $\{e_{W_2}^{W_2}\}_{q \in \mathbb{C}, \lambda \in \Lambda_q}$ is a basis of $W_2$. For $q \in \mathbb{C}$, let $\pi_q$ be the projection from $W_2$ to $(W_2)_{[q]}$ and use the same notation to denote its natural extension to $\tilde{W}_2$. Then for $\tilde{w}_2 \in \tilde{W}_2$, we have

$$\pi_q \tilde{w}_2 = \sum_{\lambda \in \Lambda_q} \langle (e_{W_2}^{W_2})', \tilde{w}_2 \rangle e_{W_2}^{W_2}.$$ 

For $k, l \in \mathbb{N}$ and $v_1, \ldots, v_{k+l} \in V$, we know that

$$\sum_{q \in \mathbb{C}} \langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k) \cdot \pi_q Y_{W_2}(v, z_0 + z)(\phi(Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l})w_1))(z) \rangle$$

$$= \sum_{q \in \mathbb{C}} \left( \sum_{\lambda \in \Lambda_q} \langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k) e_{W_2}^{W_2} \rangle \cdot \langle (e_{W_2}^{W_2})', Y_{W_2}(v, z_0 + z)(\phi(Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l})w_1))(z) \rangle \right),$$

is absolutely convergent in the region $|z_1| > \cdots > |z_k| > |z_0 + z| > |z| > |z_{k+1}| > \cdots > |z_{k+l}| > 0$ to the rational function (3.5) and

$$\langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k) e_{W_2}^{W_2} \rangle$$

and

$$\langle (e_{W_2}^{W_2})', Y_{W_2}(v, z_0 + z)(\phi(Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l})w_1))(z) \rangle$$

are absolutely convergent in the regions $|z_1| > \cdots > |z_k|$ and $|z_0 + z| > |z| > |z_{k+1}| > \cdots > |z_{k+l}| > 0$, respectively, to the rational functions

$$R(\langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k) e_{W_2}^{W_2} \rangle)$$

and

$$R(\langle (e_{W_2}^{W_2})', Y_{W_2}(v, z_0 + z)(\phi(Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l})w_1))(z) \rangle),$$

(3.6)
respectively. So

\[
\sum_{q \in \mathbb{C}} \left( \sum_{\lambda \in \Lambda_q} R(\langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k) e_{(q; \lambda)}^{W_2} \rangle) \cdot R((e_{(q; \lambda)}^{W_2})', Y_{W_2}(v, z_0 + z) (\phi(Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l}) w_1))(z)) \right)
\]

is absolutely convergent in the region \(|z_1| > \cdots > |z_k| > |z_0 + z| > |z| > |z_{k+1}| > \cdots > |z_{k+l}| > 0\) to (3.5). But the expansion of the rational function (3.5) in the region \(|z_1|, \ldots, |z_k| > |z_0 + z|, |z|, |z_{k+1}|, \ldots, |z_{k+l}| > 0\) is a series of rational functions of the same form as that of (3.7). By Lemma 2.11 we see that this expansion must be equal to (3.7). In particular, (3.7) is absolutely convergent in the region \(|z_1|, \ldots, |z_k| > |z_0 + z|, |z|, |z_{k+1}|, \ldots, |z_{k+l}| > 0\) to the rational function (3.5).

But each term in the right-hand side of (3.7) can be further expanded in the region \(|z| > |z_0| > 0\) and \(|z_i - z| > |z_0| > 0\) for \(i = k + 1, \ldots, k + l\) as a Laurent series in \(z_0\). In particular, in the region \(|z_1| > \cdots > |z_k| > |z| > |z_{k+1}| > \cdots > |z_{k+l}| > 0, |z| > |z_0| > 0\) and \(|z_i - z| > |z_0| > 0\) for \(i = k + 1, \ldots, k + l\), the series in the right-hand side of (3.7) is equal to

\[
\sum_{p \in \mathbb{Z}} \sum_{q \in \mathbb{C}} \left( \sum_{\lambda \in \Lambda_q} \langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k) e_{(q; \lambda)}^{W_2} \rangle \cdot a_p^L((e_{(q; \lambda)}^{W_2})' \otimes v \otimes \phi \otimes Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l}) w_1; z) \right) z_0^{-p-1}
\]

\[
= \sum_{p \in \mathbb{Z}} \sum_{q \in \mathbb{C}} \left( \sum_{\lambda \in \Lambda_q} \langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k) e_{(q; \lambda)}^{W_2} \rangle \cdot (e_{(q; \lambda)}^{W_2})' (\eta_{p;v,\phi}^L(Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l}) w_1))(z) \right) z_0^{-p-1}
\]

\[
= \sum_{p \in \mathbb{Z}} \langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k) \rangle \cdot (\eta_{p;v,\phi}^L(Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l}) w_1))(z) z_0^{-p-1}.
\]

Thus each term in the right-hand side of (3.8) is absolutely convergent in the region given by \(|z_1| > \cdots > |z_k| > |z| > |z_{k+1}| > \cdots > |z_{k+l}| > 0\) to a rational function in \(z_1, \ldots, z_{k+l}\) and \(z\) with the only possible poles \(z_i = 0\) for \(i = 1, \ldots, k + l\), \(z_0 = 0\), \(z_i = z_j\) for \(i, j = 1, \ldots, k + l\), \(i \neq j\) and \(z_i = z\) for \(i = 1, \ldots, k + l\), proving the first part of the composability of \(\eta_{p;v,\phi}^L\). Since \(\phi\) satisfies the second part of the composability and the right-hand side of (3.8) is absolutely convergent to (3.5), it is clear that \(\eta_{p;v,\phi}^L\) also satisfies the second part of the composability.

Finally when \(\phi\) satisfies the \(D\)-derivative property, for \(w_1 \in W_1\) and \(w'_2 \in W'_2\), in the
region $|z_2| > |z_1| > 0$, we have

\[
\sum_{p \in \mathbb{Z}} \left( w_2' \frac{d}{dz_2} (\eta^L_{p,v,\phi}(w_1))(z_2) \right) z_1^{-p-1} \\
= \frac{d}{dz_2} \sum_{p \in \mathbb{Z}} a_p^L(w_2' \otimes v \otimes \phi \otimes w_1; z_2) z_1^{-p-1} \\
= \frac{d}{dz_2} R(\langle w_2', Y_{W_2}(v, z_1 + z_2)(\phi(w_1))(z_2) \rangle)
\]

\[
= R \left( \left( w_2', \frac{d}{dz_2} Y_{W_2}(v, z_1 + z_2) (\phi(w_1))(z_2) \right) \right) \\
+ R \left( \left( w_2', Y_{W_2}(v, z_1 + z_2) (\phi(w_1))(z_2) \right) \frac{d}{dz_2} \right) \\
= R(\langle w_2, D_{W_2} Y_{W_2}(v, z_1 + z_2)(\phi(w_1))(z_2) \rangle) - R(\langle w_2', Y_{W_2}(v, z_1 + z_2) D_{W_2} (\phi(w_1))(z_2) \rangle) \\
+ R(\langle w_2', Y_{W_2}(v, z_1 + z_2)(\phi(w_1))(z_2) \rangle) - R(\langle w_2', Y_{W_2}(v, z_1 + z_2)(\phi(D_{W_1} w_1))(z_2) \rangle) \\
= \sum_{p \in \mathbb{Z}} a_p^L(D_{W_2} w_2' \otimes v \otimes \phi \otimes w_1; z_2) z_1^{-p-1} - \sum_{p \in \mathbb{Z}} a_p^L(w_2' \otimes v \otimes \phi \otimes D_{W_1} w_1; z_2) z_1^{-p-1} \\
= \sum_{p \in \mathbb{Z}} \langle D_{W_2} w_2', (\eta^L_{p,v,\phi}(w_1))(z_2) \rangle - \sum_{p \in \mathbb{Z}} \langle w_2', (\eta^L_{p,v,\phi}(D_{W_1} w_1))(z_2) \rangle z_1^{-p-1} \\
= \sum_{p \in \mathbb{Z}} \langle w_2', (D_{W_2} (\eta^L_{p,v,\phi}(w_1))(z_2) - (\eta^L_{p,v,\phi}(D_{W_1} w_1))(z_2)) z_1^{-p-1},
\]

proving the $D$-derivative property for $\eta^L_{p,v,\phi}$.

For $v \in V$ and $\phi \in H$, we define

\[
Y^L_H(v, x) \phi = \sum_{p \in \mathbb{Z}} \eta^L_{p,v,\phi} x^{-p-1}
\]

and

\[
Y^R_H(\phi, x) v = \sum_{p \in \mathbb{Z}} \eta^R_{p,\phi,v} x^{-p-1}.
\]

Using our notations for components of vertex operators, we have $(Y^L_H)_p(v) \phi = \eta^L_{p,v,\phi}$ and $(Y^R_H)_p(\phi) v = \eta^R_{p,\phi,v}$.

The proof of Proposition 3.1 in fact has also proved the first half of the following result (the second half can be proved similarly and its proof is omitted):

**Proposition 3.2** Let $k, l \in \mathbb{N}$, $v_1, \ldots, v_{k+l} \in V$, $w_1 \in W_1$ and $w'_2 \in W'_2$. Then the series

\[
\langle w_2', Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k)((Y^L_H(v, z_0) \phi)(Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l}) w_1))(z) \rangle \quad (3.9)
\]

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is absolutely convergent in the region $|z_1| > \cdots > |z_k| > |z| > |z_{k+1}| > \cdots > |z_{k+l}|$, $|z| > |z_0| > 0$, $|z_i| > |z + z_0| > 0$ for $i = 1, \ldots, k$, $|z_i - z| > |z_0| > 0$ for $i = k + 1, \ldots, k + l$ to the rational function

$$R(\langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k)Y_{W_2}(v, z_0 + z)(\phi(Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l})w_1))(z)\rangle). \quad (3.10)$$

Similarly, the series

$$\langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k)(Y^R_H(\phi, z_0)v)(Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l})w_1))(z)\rangle$$

is absolutely convergent in the region $|z_1| > \cdots > |z_k| > |z| > |z_{k+1}| > \cdots > |z_{k+l}|$, $|z| > |z_0| > 0$, $|z_i| > |z + z_0| > 0$ for $i = 1, \ldots, k$, $|z_i - z| > |z_0| > 0$ for $i = k + 1, \ldots, k + l$ to the rational function

$$R(\langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k)(\phi(Y_{W_1}(v, z)Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l})w_1))(z_0 + z)\rangle). \quad (3.11)$$

In particular, if $w'_2$ and $w_0$ hold in the region $|z_1 + z_2| > |z_2| > |z_1| > 0$ and

$$R(\langle w'_2, ((Y^L_H(v, z_1)\phi)(w_1))(z_2)\rangle) = R(\langle w'_2, Y_{W_2}(v_1, z_1 + z_2)(\phi(w_1))(z_2)\rangle), \quad (3.12)$$

and

$$R(\langle w'_2, (Y^R_H(\phi, z_1)v)(w_1))(z_2)\rangle) = R(\langle w'_2, \phi(Y_{W_1}(v, z_2)w_1)(z_1 + z_2)\rangle). \quad (3.13)$$

We also have the following result:

**Proposition 3.3** Let $w'_2 \in W'_2$, $w_1 \in W_1$, $v_1, \ldots, v_{k+l} \in V$ and $\phi \in H$. Then

$$\langle w'_2, ((Y^L_H(v_1, z_1) \cdots Y^L_H(v_k, z_k)Y^R_H(\phi, z)Y_V(v_{k+1}, z_{k+1}) \cdots Y_V(v_{k+l-1}, z_{k+l-1})v_{k+l})(w_1))(z_{k+l})\rangle$$

is absolutely convergent in the region $|z_{k+l}| > |z_1| > \cdots > |z_k| > |z| > |z_{k+1}| > \cdots > |z_{k+l-1}| > 0$ to the same rational function in $z_1, \ldots, z_{k+l}$ and $z$ as the rational function that the series

$$\langle w'_2, Y_{W_2}(v_1, z_1 + z_{k+l}) \cdots Y_{W_2}(v_k, z_k + z_{k+l}) \cdot \phi(Y_{W_1}(v_{k+1}, z_{k+1} + z_{k+l}) \cdots Y_{W_1}(v_{k+l-1}, z_{k+l-1} + z_{k+l})Y_{W_1}(v_{k+l}, z_{k+l})w_1)(z + z_{k+l}) \rangle. \quad (3.14)$$

is absolutely convergent in the region $|z_1 + z_{k+l}| > \cdots > |z_k + z_{k+l}| > |z + z_{k+l}| > |z_{k+1} + z_{k+l}| > \cdots > |z_{k+l-1} + z_{k+l}| > |z_{k+l}| > 0$. In particular, we have the equality

$$R(\langle w'_2, ((Y^L_H(v_1, z_1) \cdots Y^L_H(v_k, z_k) \cdot Y^R_H(\phi, z)Y_V(v_{k+1}, z_{k+1}) \cdots Y_V(v_{k+l-1}, z_{k+l-1})v_{k+l})(w_1))(z_{k+l})\rangle)$$

$$= R(\langle w'_2, Y_{W_2}(v_1, z_1 + z_{k+l}) \cdots Y_{W_2}(v_k, z_k + z_{k+l}) \cdot \phi(Y_{W_1}(v_{k+1}, z_{k+1} + z_{k+l}) \cdots Y_{W_1}(v_{k+l-1}, z_{k+l-1} + z_{k+l})Y_{W_1}(v_{k+l}, z_{k+l})w_1)(z + z_{k+l}) \rangle). \quad (3.15)$$

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of rational functions and the only possible poles of the rational function (in both sides of) \(3.15\) are \(z_i + z_{k+l} = 0\) for \(i = 1, \ldots, k+l-1, z + z_{k+l} = 0, z_i = z_j\) for \(i, j = 1, \ldots, k+l-1, i \neq j, z_i = 0\) for \(i = 1, \ldots, k+l, z_i = z\) for \(i = 1, \ldots, k+l-1\) and \(z = 0\). More explicitly, \(3.15\) is of the form

\[
g(z_1, \ldots, z_{k+l}, z)z^{-m}(z + z_{k+l})^{-p} \prod_{i=1}^{k+l-1} (z_i + z_{k+l})^{-p_i} \prod_{1 \leq i < j \leq k+l-1} (z_i - z_j)^{-p_{ij}} \prod_{i=1}^{k+l-1} z_i^{-r_i} \prod_{i=1}^{k+l-1} (z_i - z)^{-s_i} \tag{3.16}
\]

where \(g(z_1, \ldots, z_{k+l}, z)\) is a polynomial in \(z_1, \ldots, z_{k+l}\) and \(z\) and \(m, n, p_i, p_{ij}, r_i, s_i \in \mathbb{N}\) depend only on the pairs \((v_{k+l}, \phi), (\phi, w_1), (v_i, w_1), (v_i, v_j), (v_i, v_{k+l}), (v_i, \phi)\), respectively.

**Proof.** Since \(\phi\) satisfies the compositability, the series

\[
\langle w_2', Y_{W_2}(v_1, z_1 + z_{k+l}) \cdots Y_{W_2}(v_k, z_k + z_{k+l}) \rangle \cdot \phi(Y_{W_1}(v_{k+l}, z_{k+l+1} + z_{k+l}) \cdots Y_{W_1}(v_{k+l-1}, z_{k+l-1} + z_{k+l})Y_{W_1}(v_{k+l}, z_{k+l})w_1)(z + z_{k+l}) \rangle \tag{3.17}
\]

is absolutely convergent in the region \(|z_1 + z_{k+l}| > \cdots > |z_k + z_{k+l}| > |z + z_{k+l}| > |z_{k+l} + z_{k+l}| > \cdots |z_{k+l-1} + z_{k+l}| > |z_{k+l}| > 0\) to a rational function

\[
R(\langle w_2', Y_{W_2}(v_1, z_1 + z_{k+l}) \cdots Y_{W_2}(v_k, z_k + z_{k+l}) \rangle \cdot \phi(Y_{W_1}(v_{k+l}, z_{k+l+1} + z_{k+l}) \cdots Y_{W_1}(v_{k+l-1}, z_{k+l-1} + z_{k+l})Y_{W_1}(v_{k+l}, z_{k+l})w_1)(z + z_{k+l}) \rangle) \tag{3.18}
\]

in \(z_1, \ldots, z_{k+l}\) and \(z\) with the only possible poles \(z_i + z_{k+l} = 0\) for \(i = 1, \ldots, k+l-1, z + z_{k+l} = 0, z_i = z_j\) for \(i, j = 1, \ldots, k+l-1, i \neq j, z_i = 0\) for \(i = 1, \ldots, k+l, z_i = z\) for \(i = 1, \ldots, k+l-1\) and \(z = 0\). The rational function \(3.18\) is of the form \(3.16\) such that \(g(z_1, \ldots, z_{k+l}, z)\) is a polynomial in \(z_1, \ldots, z_{k+l}\) and \(z\) and \(m, n, p_i, p_{ij}, r_i, s_i \in \mathbb{N}\) depend only on the pairs \((v_{k+l}, \phi), (\phi, w_1), (v_i, w_1), (v_i, v_j), (v_i, v_{k+l}), (v_i, \phi)\), respectively.

Let \(\{e_{(q; \gamma)}^W\}_{q \in \mathbb{C}, \gamma \in \Gamma_q}\) be a homogeneous basis of \(W_1\) and \(\{(e_{(q; \gamma)}^W)'\}_{q \in \mathbb{C}, \gamma \in \Gamma_q}\) the subset of \(W_1^*\) given by

\[
\langle (e_{(q; \gamma)}^W)' \rangle, (e_{(q; \gamma)}^W) \rangle = \delta_{\gamma_1 \gamma_2}.
\]

By the associativity of vertex operators, for \(q \in \mathbb{C}, \gamma \in \Gamma_q\)

\[
\langle (e_{(q; \gamma)}^W)' \rangle Y_{W_1}(v_{k+l+1}, z_{k+l} + z_{k+l}) \cdots Y_{W_1}(v_{k+l-1}, z_{k+l-1} + z_{k+l})Y_{W_1}(v_{k+l}, z_{k+l})w_1 \rangle = \langle (e_{(q; \gamma)}^W)' \rangle Y_{W_1}(Y_{V}(v_{k+l+1}, z_{k+l}) \cdots Y_{V}(v_{k+l-1}, z_{k+l-1})v_{k+l}, z_{k+l})w_1 \rangle
\]

in the region \(|z_{k+l+1} + z_{k+l}| > \cdots > |z_{k+l-1} + z_{k+l}| > |z_{k+l}| > 0, |z_{k+l}| > |z_{k+l}| > \cdots > |z_{k+l-1}| > 0\). Then by the convergence of \(3.17\), we see that

\[
\langle w_2', Y_{W_2}(v_1, z_1 + z_{k+l}) \cdots Y_{W_2}(v_k, z_k + z_{k+l}) \rangle \cdot \phi(Y_{W_1}(Y_{V}(v_{k+l+1}, z_{k+l}) \cdots Y_{V}(v_{k+l-1}, z_{k+l-1})v_{k+l}, z_{k+l})w_1)(z + z_{k+l}) \rangle \tag{3.19}
\]

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is convergent absolutely to the same rational function (3.18) or (3.16) in the region $|z_1 + z_{k+l}| > \cdots > |z_k + z_{k+l}| > |z + z_{k+l}| > |z_{k+2} + z_{k+l}| > \cdots \cdots > |z_{k+l-1} + z_{k+l}| > 0$, $|z_{k+l}| > |z_{k+l}| > \cdots > |z_{k+l-1}| > 0$.

Expand (3.16) as a Laurent series in $z_i + z_{k+l}$ for $i = 1, \ldots, k, z_{k+j}$ for $j = 1, \ldots, l$, $z + z_{k+l}$ in the region $|z + z_{k+l}| > |z_{k+l}| > |z_{k+1}| > \cdots > |z_{k+l-1}| > 0$, $|z_1 + z_{k+l}| > \cdots > |z_k + z_{k+l}| > |z_{k+1} + z_{k+l}|, \ldots, |z_{k+l-1} + z_{k+l}| > 0$, $|z + z_{k+l}| > |z_{k+1} + z_{k+l}|, \ldots, |z_{k+l-1} + z_{k+l}| > 0$. Since (3.19) is a Laurent series of the same form, by Lemma 2.11 it must be convergent absolutely to (3.16) in the same region.

Let $\{e_{\lambda(q;\mu)}\}_{q \in \mathbb{Z}, \mu \in M_q}$ be a homogeneous basis of $V$ and $\{(e_{\lambda(q;\mu)}')\}_{q \in \mathbb{Z}, \mu \in M_q}$ a subset of $V'$ given by

$$
\langle (e_{\lambda(q;\mu)})', (e_{\lambda(q;\mu)}) \rangle = \delta_{\mu_1 \mu_2}.
$$

Let $\{e_{\lambda(q;\lambda)}\}_{\lambda \in \Lambda_q}$ be a basis of $(W_2)_q$ and $\{(e_{\lambda(q;\lambda)}')\}_{\lambda \in \Lambda_q}$ the subset of $W'_2$ as above. Then the series (3.19) can be written as

$$
\sum \sum \sum \left( \sum \sum \left( \sum \sum \langle w'_2, Y_{W_2}(v_1, z_1 + z_{k+l}) \cdots Y_{W_2}(v_k, z_k + z_{k+l}) e_{\lambda(q;\lambda)} \rangle \cdot \langle (e_{\lambda(q;\lambda)})', \phi(Y_{W_1}(e_{(q;\lambda)})(z + z_{k+l})) \rangle \right) \right) \left( \sum \sum \left( \sum \sum \langle w'_2, Y_{W_2}(v_1, z_1 + z_{k+l}) \cdots Y_{W_2}(v_k, z_k + z_{k+l}) e_{\lambda(q;\lambda)} \rangle \right) \right)
$$

Because of (3.3), we can replace

$$
\langle (e_{\lambda(q;\lambda)}'), \phi(Y_{W_1}(e_{(q;\lambda)})(z + z_{k+l})) \rangle
$$

in (3.20) by

$$
\langle (e_{\lambda(q;\lambda)}'), ((Y_{W_2} R(\phi, z)e_{(q;\lambda)})(w_1))(z + z_{k+l}) \rangle
$$

when $|z + z_{k+l}| > |z_{k+l}| > |z| > 0$. Thus

$$
\sum \sum \sum \left( \sum \sum \langle w'_2, Y_{W_2}(v_1, z_1 + z_{k+l}) \cdots Y_{W_2}(v_k, z_k + z_{k+l}) e_{\lambda(q;\lambda)} \rangle \right)
$$

is absolutely convergent to (3.16) in the region $|z + z_{k+l}| > |z_{k+l}| > |z| > |z_{k+1}| > \cdots > |z_{k+l-1}| > 0$, $|z_1 + z_{k+l}| > \cdots > |z_k + z_{k+l}| > |z_{k+1} + z_{k+l}|, \ldots, |z_{k+l-1} + z_{k+l}|, \ldots, |z_{k+l} + z_{k+l}| > 0$. But the form of the Laurent series (3.21) is
the same as the form of the expansion of (3.16) as a Laurent series in $z_i + z_{k+l}$ for $i = 1, \ldots, k$, $z_{k+j}$ for $j = 1, \ldots, l$, $z$ in the larger region $|z_{k+l}| > |z| > |z_{k+1}| > \cdots > |z_{k+l-1}| > 0$, $|z_1 + z_{k+l}| > \cdots > |z_k + z_{k+l}| > |z_{k+1} + z_{k+l}|, \ldots, |z_{k+l-1} + z_{k+l}|, |z + z_{k+l}| > 0$, $|z + z_{k+l}| > |z_{k+1} + z_{k+l}|, \ldots, |z_{k+l-1} + z_{k+l}| > 0$. Thus by Lemma 2.11 (3.21) is in fact absolutely convergent to (3.16) in this larger region.

The series in the right-hand side of (3.21) can be written as

$$
\sum_{q_1 \in \mathbb{Z}} \sum_{q_2 \in \mathbb{Z}} \left( \sum_{\mu \in M_{q_1}} \sum_{\gamma \in \Gamma_{q_2}} \langle w_2^\mu, Y W_2(v_1, z_1 + z_{k+l}) \cdots Y W_2(v_{k-1}, z_{k-1} + z_{k+l}) e^{W_2}_{(q_2; \lambda)} \rangle \cdot \langle (e^{W_2}_{(q_2; \lambda)})', Y W_2(v_k, z_k + z_{k+l}) ((Y_R^P e^{V}_{(q_1; \mu)}(v_1))(z_{k+l})) \rangle \cdot \langle (e^{V}_{(q_1; \mu)})', Y V(v_{k+1}, z_{k+1}) \cdots Y V(v_{k+l-1}, z_{k+l-1}) v_{k+l} \rangle \right)
$$

$$= \sum_{q_1 \in \mathbb{Z}} \sum_{q_2 \in \mathbb{Z}} \sum_{p \in \mathbb{Z}} \left( \sum_{\mu \in M_{q_1}} \sum_{\gamma \in \Gamma_{q_2}} \langle w_2^\mu, Y W_2(v_1, z_1 + z_{k+l}) \cdots Y W_2(v_{k-1}, z_{k-1} + z_{k+l}) e^{W_2}_{(q_2; \lambda)} \rangle \cdot \langle (e^{W_2}_{(q_2; \lambda)})', Y W_2(v_k, z_k + z_{k+l}) (\eta^{R}_{p; \phi, (q_1; \mu)}(v_1))(z_{k+l}) \rangle \cdot \langle (e^{V}_{(q_1; \mu)})', Y V(v_{k+1}, z_{k+1}) \cdots Y V(v_{k+l-1}, z_{k+l-1}) v_{k+l} \rangle \right) z^{-p-1} \quad (3.22)
$$

Because of (3.2), we can replace

$$\langle (e^{W_2}_{(q_2; \lambda)})', Y W_2(v_k, z_k + z_{k+l}) (\eta^{R}_{p; \phi, (q_1; \mu)}(v_1))(z_{k+l}) \rangle$$

in (3.22) by

$$\langle (e^{W_2}_{(q_2; \lambda)})', (Y^L_R(v_k, z_k) \eta^{R}_{p; \phi, (q_1; \mu)}(v_1))(z_{k+l}) \rangle$$
when $|z_k + z_{k+l}| > |z_{k+l}| > |z_k| > 0$. Thus

$$
\sum_{q_1 \in \mathbb{Z}} \sum_{q_2 \in \mathbb{Z}} \sum_{p \in \mathbb{Z}} \left( \sum_{\mu \in M_{q_1}} \sum_{\gamma \in \Gamma_{q_2}} \langle w'_2, Y_{W_2}(v_1, z_1 + z_{k+l}) \cdots Y_{W_2}(v_{k-1}, z_{k-1} + z_{k+l}) e^{W_2}_{(q_2; \lambda)} \rangle \cdot \langle (e^{W_2}_{(q_2; \lambda)})', (Y^L_{H}(v_k, z_k) \eta_{p, \psi, r}^{R_{(q_1; \mu)}})(w_1) \rangle (z_{k+l}) \rangle \cdot \langle (e^{V}_{(q_1; \mu)})', Y_{V}(v_{k+1}, z_{k+1}) \cdots Y_{V}(v_{k+l-1}, z_{k+l-1}) v_{k+l} \rangle \right) z^{-p-1}
$$

$$
= \sum_{q_1 \in \mathbb{Z}} \sum_{q_2 \in \mathbb{Z}} \left( \sum_{\mu \in M_{q_1}} \sum_{\gamma \in \Gamma_{q_2}} \langle w'_2, Y_{W_2}(v_1, z_1 + z_{k+l}) \cdots Y_{W_2}(v_{k-1}, z_{k-1} + z_{k+l}) e^{W_2}_{(q_2; \lambda)} \rangle \cdot \langle (e^{W_2}_{(q_2; \lambda)})', (Y^L_{H}(v_k, z_k) Y^R_{H}(\phi, z) e^{V}_{(q_1; \mu)})(w_1) \rangle (z_{k+l}) \rangle \cdot \langle (e^{V}_{(q_1; \mu)})', Y_{V}(v_{k+1}, z_{k+1}) \cdots Y_{V}(v_{k+l-1}, z_{k+l-1}) v_{k+l} \rangle \right)
$$

$$
= \langle w'_2, Y_{W_2}(v_1, z_1 + z_{k+l}) \cdots Y_{W_2}(v_{k-1}, z_{k-1} + z_{k+l}) e^{W_2}_{(q_2; \lambda)} \rangle \cdot \langle (Y^L_{H}(v_k, z_k) Y^R_{H}(\phi, z) Y_{V}(v_{k+1}, z_{k+1}) \cdots Y_{V}(v_{k+l-1}, z_{k+l-1}) v_{k+l}) \rangle (w_1)(z_{k+l}) \rangle
$$

(3.23)

is absolutely convergent to (3.16) in the larger region $|z_k + z_{k+l}| > |z_{k+l}| > |z_k| > |z| > |z_{k+1}| > \cdots > |z_{k+l-1}| > 0$, $|z_k + z_{k+l}| > |z_{k-1} + z_{k+l+1}| > |z_k + z_{k+l+1}| > \cdots > |z_{k+l-1} + z_{k+l}| > 0$. But the form of the Laurent series (3.23) is the same as the form of the expansion of (3.16) as a Laurent series in $z_i + z_{k+l}$ for $i = 1, \ldots, k - 1, z_k + j$ for $j = 0, \ldots, l$, $z$ in the region $|z_{k+l}| > |z_k| > |z| > |z_{k+1}| > \cdots > |z_{k+l-1}| > 0$, $|z_k + z_{k+l}| > \cdots > |z_{k-1} + z_{k+l}| > |z_{k+1} + z_{k+l}| > \cdots > |z_{k+l-1} + z_{k+l}| > 0$, $|z + z_{k+l}| > |z_{k+1} + z_{k+l}| > \cdots > |z_{k+l-1} + z_{k+l}| > 0$. Thus by Lemma 2.11 (3.23) is absolutely convergent to (3.16) in this larger region.

Repeating this last step, we see that the series

$$
\langle w'_2, (Y^L_{H}(v_1, z_1) \cdots Y^L_{H}(v_k, z_k) Y^R_{H}(\phi, z) Y_{V}(v_{k+1}, z_{k+1}) \cdots Y_{V}(v_{k+l-1}, z_{k+l-1}) v_{k+l}) \rangle (w_1)(z_{k+l}) \rangle
$$

is absolutely convergent to (3.16) in the region $|z_{k+l}| > |z_1| > |z| > |z_{k+1}| > \cdots > |z_{k+l-1}| > 0$.

Since $H$ is $\mathbb{Z}$-graded, we have an operator $d_H$ on $H$ defined by $d_H \phi = n \phi$ for $\phi \in H^n$. We define $D_H$ on $H$ by

$$
((D_H \phi)(w_1))(z) = \frac{\partial}{\partial z} (\phi(w_1))(z)
$$

for $\phi \in H$, $w_1 \in W_1$.

Though $H$ is $\mathbb{Z}$-graded, the grading is in general not lower bounded and hence it cannot be a $V$-bimodule. We now consider subspaces of $H$ which indeed have $V$-bimodule structures.

For $N \in \mathbb{Z}$, let $H^N$ be the subspace of $H$ spanned by homogeneous elements, say $\phi$, satisfying the following condition:
4. The $N$-weight-degree condition: For $k, l \in \mathbb{N}$, $v_1, \ldots, v_{k+l} \in V$, $w_1 \in W_1$ and $w'_2 \in W'_2$, expand the rational function

$$R((w'_2, Y_{W_2}(v_1, z_1)) \cdots Y_{W_2}(v_k, z_k)(\phi(Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l}, z_{k+l})w_1))(z))) \quad (3.24)$$

in the region $|z_{k+l}| > |z_i - z_{k+l}| > \cdots > |z_k - z_{k+l}| > |z - z_{k+l}| > |z_{k+1} - z_{k+l}| > \cdots > |z_{k+l-1} - z_{k+l}| = 0$ as a Laurent series in $z_i - z_{k+l}$ for $i = 1, \ldots, k + l - 1$ and $z - z_{k+l}$ with Laurent polynomials in $z_{k+l}$ as coefficients. Then the total degree of each monomial in $z_i - z_{k+l}$ for $i = 1, \ldots, k + l - 1$ and $z - z_{k+l}$ (that is, the sum of the powers of $z_i - z_{k+l}$ for $i = 1, \ldots, k + l - 1$ and $z - z_{k+l}$) in the expansion is larger than or equal to $N - \sum_{i=1}^{k+l} \text{wt } v_i - \text{wt } \phi$.

**Theorem 3.4** The $\mathbb{Z}$-graded space $H^N$, equipped with the actions of the restrictions of $Y^L_H$, $Y^R_H$, $d_H$ and $D_H$ to $H^N$, is a $V$-bimodule.

**Proof.** We need to prove that $H^N$ is closed under the actions of $Y^L_H$, $Y^R_H$ to $H^N$, $d_H$ and $D_H$ and all axioms for $V$-modules hold. We prove them one by one as follows:

1. $H^N$ is closed under the actions of $Y^L_H$, $Y^R_H$, $H^N$, $d_H$ and $D_H$: For homogeneous $v \in V$, $\phi \in H^N$, we prove $Y^L_H(v, x)\phi \in H^N[[x, x^{-1}]]$. We need only prove that $(Y^L_H)p(v)\phi$ satisfies the $N$-weight-degree condition. For $k, l \in \mathbb{N}$, homogeneous $v_1, \ldots, v_{k+l} \in V$, $w_1 \in W_1$ and $w'_2 \in W'_2$, by Proposition 3.2, the series (3.9) is absolutely convergent in the region $|z_1| > \cdots |z_k| > |z| > |z_{k+l}| > \cdots > |z_0| > 0$, $|z_i| > |z + z_0| > 0$ for $i = 1, \ldots, k$, $|z_i - z| > |z_0| > 0$ for $i = k + 1, \ldots, k + l$ to the rational function (3.10). The rational function (3.10) can be written as

$$g(z_1, \ldots, z_{k+l}, z_0, z):
\begin{equation}
  z^{-m} \prod_{i=1}^{k+l} z_i^{-r_i}(z_0 + z)^{-n} \prod_{1 \leq i < j \leq k+l} (z_i - z_j)^{-p_{ij}} \prod_{i=1}^{k+l} (z_i - z - z_0)^{-q_i} \prod_{i=1}^{k+l} (z_i - z - z_0)^{-s_i} z_0^{-t},
\end{equation}
\quad (3.25)$$

where $g(z_1, \ldots, z_{k+l}, z_0, z)$ is a polynomial in $z_i$ (for $i = 1, \ldots, k + l$), $z_0$ and $m, r_i, n, p_{ij}, q_i, s_i, t \in \mathbb{N}$. Since $g(z_1, \ldots, z_{k+l}, z_0, z)$ is a polynomial, we can write (3.25) as a linear combination of rational functions of the form

$$z^{-m} \prod_{i=1}^{k+l} z_i^{-r_i}(z_0 + z)^{-n} \prod_{1 \leq i < j \leq k+l} (z_i - z_j)^{-p_{ij}} \prod_{i=1}^{k+l} (z_i - z - z_0)^{-q_i} \prod_{i=1}^{k+l} (z_i - z - z_0)^{-s_i} z_0^{-t},
\quad (3.26)$$
where \( m, r_i, t \in \mathbb{Z} \). We can write (3.26) as

\[
\begin{aligned}
&z_{k+l}^{-m} \left( 1 + \frac{z - z_{k+l}}{z_{k+l}} \right)^{-m} \left( \prod_{i=1}^{k+l-1} z_{k+l}^{-r_i} \left( 1 + \frac{z_i - z_{k+l}}{z_{k+l}} \right) \right)^{-r_{k+l}} \cdot \\
&\cdot z_{k+l}^{-n} \left( 1 + \frac{z_0 + z - z_{k+l}}{z_{k+l}} \right)^{-n} \prod_{1 \leq i < j \leq k+l-1} \left( z_i - z_{k+l} \right)^{-p_{ij}} \left( 1 - \frac{z_j - z_{k+l}}{z_i - z_{k+l}} \right)^{-p_{ij}} \cdot \\
&\cdot \prod_{i=1}^{k+l-1} \left( z_i - z_{k+l} \right)^{-s_i} \left( 1 - \frac{z - z_{k+l}}{z_i - z_{k+l}} \right)^{-s_i} \prod_{i=k+1}^{k+l-1} \left( -\left( z - z_{k+l} \right) \right)^{-q_i} \left( 1 - \frac{z_i - z_{k+l}}{z + z_0 - z_{k+l}} \right)^{-q_i} \cdot \\
&\cdot \left( -\left( z - z_{k+l} \right) \right)^{-s_{k+l}} \left( z_0 + z - z_{k+l} \right)^{-t} \left( 1 - \frac{z - z_{k+l}}{z_0 + z - z_{k+l}} \right)^{-t}.
\end{aligned}
\]

When (3.27) is expanded in the region \(|z_{k+l}| > |z_1 - z_{k+l}| > \cdots > |z_{k-1} - z_{k+l}| > |z_{k+l}| > |z_{k+1} - z_{k+l}| > \cdots > |z_{k+l-1} - z_{k+l}| > 0\) as a Laurent series in \( z_i - z_{k+l} \) for \( i = 1, \ldots, k+l-1 \), \( z = z_{k+l} \) and \( z_0 = z - z_{k+l} \) with Laurent polynomials in \( z_{k+l} \) as coefficients, the lowest total degree is

\[
- \sum_{1 \leq i < j \leq k+l} p_{ij} - \sum_{i=1}^{k+l} q_i - \sum_{i=1}^{k+l} s_i - t.
\]

Since \( \phi \) satisfies the \( N \)-weight-degree condition, we have

\[
- \sum_{1 \leq i < j \leq k+l} p_{ij} - \sum_{i=1}^{k+l} q_i - \sum_{i=1}^{k+l} s_i - t \geq N - \sum_{i=1}^{k+l} \text{wt } v_i - \text{wt } v - \text{wt } \phi.
\]
We can also write (3.26) as
\[
\sum_{i=1}^{k} \prod_{j<i} (z_i - z_{k+l})^{-p_{ij}} \left( 1 - \frac{z_j - z_{k+l}}{z_i - z_{k+l}} \right)^{-q_{ij}} \prod_{i=1}^{k+l-1} \left( 1 - \frac{z_0}{z_i - z_{k+l}} \right)^{-q_i} \prod_{i=k+1}^{k+l-1} \left( 1 - \frac{z_i - z_{k+l}}{z_{i-1} - z_{k+l}} \right)^{-s_i} \frac{z^{s_{k+l}}}{z_0^t} \tag{3.29}
\]

Now expand (3.29) first as a Laurent series in \( z_0 \) in the region \( |z| > |z_0| > 0, |z_i - z| > |z_0| > 0 \) and then expand the coefficient of the \(-p - 1\) power of \( z_0 \) in the region \( |z_{k+l}| > |z_1 - z_{k+l}| > \cdots > |z_{k-1} - z_{k+l}| > |z - z_{k+l}| > |z_{k+l-1} - z_{k+l}| > \cdots > |z_{k+l-1} - z_{k+l}| > 0 \) as a Laurent series in \( z_i - z_{k+l} \) for \( i = 1, \ldots, k+l-1 \) and \( z - z_{k+l} \) with Laurent polynomials in \( z_{k+l} \) as coefficients. The \(-p - 1\) powers of \( z_0 \) in the expansion of (3.29) comes from the expansion of
\[
\sum_{i=1}^{k+l-1} \prod_{j<i} (z_i - z_{k+l})^{-p_{ij}} \left( 1 - \frac{z_j - z_{k+l}}{z_i - z_{k+l}} \right)^{-q_{ij}} \prod_{i=1}^{k+l-1} \left( 1 - \frac{z_0}{z_i - z_{k+l}} \right)^{-q_i} \prod_{i=k+1}^{k+l-1} \left( 1 - \frac{z_i - z_{k+l}}{z_{i-1} - z_{k+l}} \right)^{-s_i} \frac{z^{s_{k+l}}}{z_0^t} \tag{3.30}
\]

The expansion of (3.30) is an infinite linear combination of the monomial of \( z_0 \) of the form
\[
\sum_{i=1}^{k+l-1} \prod_{j<i} \left( \frac{z_i}{z_{k+l}} \left( 1 + \frac{z - z_{k+l}}{z_{k+l}} \right) \right)^{a_{ij}} \prod_{i=1}^{k+l-1} \left( \frac{z_0}{z_i - z_{k+l}} \right)^{b_i} \left( \frac{z_{k+l}}{z - z_{k+l}} \right)^{c_i} \frac{z_0^t}{z_0^t} \tag{3.31}
\]

for \( a, b_i, c \in \mathbb{N} \). In the case that the power of \( z_0 \) is \(-p - 1\), we have
\[
a + \sum_{i=1}^{k+l-1} b_i = c - t = -p - 1. \tag{3.32}
\]
In this case, the total degree of \((3.31)\) in \(z_i - z_{k+l}\) for \(i = 1, \ldots, k + l\) and \(z - z_{k+l}\) is \(-\sum_{i=1}^{k+l-1} b_i - c\). From \((3.29)\), \((3.31)\) and \((3.32)\), and this fact, we see that the total degrees of each monomial in the expansion of the coefficient of the \(-p - 1\) power of \(z_0\) is larger than or equal to

\[
-\sum_{1 \leq i < j \leq k+l} p_{ij} - \sum_{i=1}^{k+l} q_i - \sum_{i=1}^{k+l-1} b_i - c - \sum_{i=1}^{k+l} s_i.
\] (3.33)

Using \((3.32)\), we see that \((3.33)\) is equal to

\[
-\sum_{1 \leq i < j \leq k+l} p_{ij} - \sum_{i=1}^{k+l} q_i + a - t + p + 1 - \sum_{i=1}^{k+l} s_i.
\] (3.34)

Using \(a \in \mathbb{N}\) and \((3.28)\), we see that \((3.34)\) is larger than or equal to

\[
-\sum_{1 \leq i < j \leq k+l} p_{ij} - \sum_{i=1}^{k+l} q_i - t + p + 1 - \sum_{i=1}^{k+l} s_i
\geq N - \sum_{i=1}^{k+l} \text{wt } v_i - \text{wt } v - \text{wt } \phi + p + 1
= N - \sum_{i=1}^{k+l} \text{wt } v_i - \text{wt } (Y_{H}^L)_{p}(v)\phi,
\]

where in the last step we have used the formula

\[
\text{wt } (Y_{H}^L)_{p}(v)\phi = \text{wt } v + \text{wt } \phi - p - 1,
\]

which in turn follows from the \(d\)-conjugation property of \(Y_{H}^L\) that we shall prove below. Since the proof the \(d\)-conjugation property of \(Y_{H}^L\) does not need the result that \(H^N\) is closed under the actions of \(Y_{H}^L\) and its proof, we can indeed use this formula here. This proves that \((Y_{H}^L)_{p}(v)\phi\) satisfies the \(N\)-weight-degree condition.

We omit the proof that \(Y_{H}^R(\phi, x)v \in H^N[[x, x^{-1}]]\) for \(v \in V\) and \(\phi \in H^N\). It is similar to the proof above for \(Y_{H}^L\).

Since for homogeneous \(\phi \in H^N\), \(d_{H}\phi = (\text{wt } \phi)\phi\), \(d_{H}\phi\) also satisfies the \(N\)-weight-degree condition. So \(d_{H}\) maps \(H^N\) to \(H^N\).

For homogeneous \(\phi \in H^N\), \(k, l \in \mathbb{N}\), homogeneous \(v_1, \ldots, v_{k+l} \in V\), \(w_1 \in W_1\) and \(w_2' \in W_2'\), by definition, we have

\[
R((w_2', Y_{W_2}(v_1, z_1)) \cdots Y_{W_2}(v_k, z_k))(D_{H}\phi)(Y_{W_1}(v_{k+1}, z_{k+1})) \cdots Y_{W_1}(v_{k+l}, z_{k+l+1})(w_1)(z))
= \frac{\partial}{\partial z} R((w_2', Y_{W_2}(v_1, z_1)) \cdots Y_{W_2}(v_k, z_k)(\phi(Y_{W_1}(v_{k+1}, z_{k+1})) \cdots Y_{W_1}(v_{k+l}, z_{k+l+1})(w_1))(z)).
\] (3.35)

Then the expansion of \((3.35)\) in the region \(|z_{k+1}| > |z_i - z_{k+l}| > \cdots > |z_k - z_{k+l}| > |z - z_{k+l}| > |z_k| > \cdots > |z_{k+1} - z_{k+l}| > 0\) as a Laurent series in \(z_i - z_{k+l}\) for \(i = 1, \ldots, k + l - 1\)
and \( z - z_{k+l} \) with Laurent polynomials in \( z_{k+l} \) as coefficients is equal to the derivative with respect to \( z \) of the same expansion of (3.24). In particular, the total degree of each monomial in \( z_{i} - z_{k+l} \) for \( i = 1, \ldots, k + l - 1 \) and \( z - z_{k+l} \) in the expansion of (3.35) is equal to the total degree of a monomial in the expansion of (3.24) minus 1. Thus the total degree of each monomial in \( z_{i} - z_{k+l} \) for \( i = 1, \ldots, k + l - 1 \) and \( z - z_{k+l} \) in the expansion of (3.35) is larger than or equal to \( N - \sum_{i=1}^{k+l} \text{wt } v_{i} - \text{wt } \phi - 1 \). But from the definition of \( D_{H} \) and \( \text{wt } \phi \), we have

\[
a^{d_{w_{2}}}((D_{H}\phi)(w_{1}))(z) = a^{d_{w_{2}}} \frac{\partial}{\partial z}(\phi(a^{d_{w_{1}}}w_{1}))(az) \\
= a^{\text{wt } \phi} \frac{\partial}{\partial z}(\phi(a^{d_{w_{1}}}w_{1}))(az) \\
= a^{\text{wt } \phi+1} \frac{\partial}{\partial z'}(\phi(a^{d_{w_{1}}}w_{1}))(z') \bigg|_{z'=az} \\
= a^{\text{wt } \phi+1}((D_{H}\phi)(a^{d_{w_{1}}}w_{1}))(az),
\]

proving \( \text{wt } \phi + 1 = \text{wt } D_{H}\phi \). Therefore the total degree of each monomial in \( z_{i} - z_{k+l} \) for \( i = 1, \ldots, k + l - 1 \) and \( z - z_{k+l} \) in the expansion of (3.35) is larger than or equal to \( N - \sum_{i=1}^{k+l} \text{wt } v_{i} - \text{wt } (D_{H}\phi) \). So \( D_{H}\phi \) satisfies the \( N \)-weight-degree condition and is in \( H^{N} \).

2. Axioms for the grading: Take \( v_{1} = \cdots = v_{k+l} = 1 \) in (3.24). Then (3.24) becomes

\[
\langle w'_{2}, (\phi(w_{1}))(z) \rangle.
\]

Expand this as a power series in \( z - z_{k+l} \) in the region \( |z_{k+l}| > |z - z_{k+l}| \). Then the lowest degree term in this expansion is the constant term \( \langle w'_{2}, (\phi(w_{1}))(z_{k}) \rangle \). Thus the lowest of the degrees of the monomials in \( z_{i} - z_{k+l} \) for \( i = 1, \ldots, k + l - 1 \) and \( z - z_{k+l} \) is 0. So we obtain \( 0 \geq N - \sum_{i=1}^{k+l} v_{i} + \text{wt } \phi = N - \text{wt } \phi \) or \( \text{wt } \phi \geq N \). This proves the lower bound condition of \( H^{N} \).

We now prove the \( d \)-conjugation property of \( Y_{H}^{L} \) which is equivalent to the \( d \)-commutator formula for \( Y_{H}^{L} \). Let \( v \in V \) and \( \phi \in H^{N} \). Then for \( a \in \mathbb{R}, w'_{2} \in W'_{2} \) and \( w_{1} \in W_{1} \), we have

\[
\langle w'_{2}, ((a^{d_{H}}Y_{H}^{L}(v, z_{1})\phi)(w_{1}))(z_{2}) \rangle \\
= \langle w'_{2}, a^{d_{w_{2}}}((Y_{H}^{L}(v, z_{1})\phi)(-d_{w_{1}}w_{1}))(a^{-1}z_{2}) \rangle \\
= \langle w'_{2}, a^{d_{w_{2}}}Y_{W_{2}}(v, z_{1} + a^{-1}z_{2})(\phi(-d_{w_{1}}w_{1}))(a^{-1}z_{2}) \rangle \\
= \langle w'_{2}, Y_{W_{2}}(a^{d_{v}}v, az_{1} + z_{2})a^{d_{w_{2}}}(\phi(-d_{w_{1}}w_{1}))(a^{-1}z_{2}) \rangle \\
= \langle w'_{2}, ((a^{H}\phi)w_{1})(z_{2}) \rangle \\
= \langle w'_{2}, ((Y_{H}(a^{d_{v}}v, az_{1})a^{d_{H}}\phi)(w_{1}))(z_{2}) \rangle.
\]

Thus we have

\[
a^{d_{H}}Y_{H}^{L}(v, z_{1})\phi = Y_{H}(a^{d_{v}}v, az_{1})a^{d_{H}}\phi,
\]

proving the \( d \)-conjugation property of \( Y_{H}^{L} \). Similarly we can prove the \( d \)-conjugation property of \( Y_{H}^{R} \) which is equivalent to the \( d \)-commutator formula for \( Y_{H}^{R} \). We omit the proof.

3. The identity property of \( Y_{H}^{L} \) and the creation property of \( Y_{H}^{R} \). These properties follows directly from (3.2) and (3.3).
4. The rationality and the pole-order condition: Let \( w'_2 \in W'_2, w_1 \in W_1, v_1, \ldots, v_{k+l} \in V \) and \( \phi \in H^N \) be homogeneous. Since (3.15) is of the form (3.16),

\[
z^m \prod_{1 \leq i < j \leq k+l-1} (z_i - z_j)^{p_{ij}} \prod_{i=1}^{k+l-1} z_i^{r_i} \prod_{i=1}^{k+l-1} (z_i - z)^{s_i}.
\]

\[
\cdot \left( w'_2, (Y^L_H(v_1, z_1) \cdots Y^L_H(v_k, z_k)Y^R_H(\phi, z)Y_V(v_{k+1}, z_{k+1}) \cdots Y_V(v_{k+l-1}, z_{k+l-1})v_k l) (w_1) \right)(z_{k+l})
\]

(3.36)

must be convergent absolutely in the region \(|z_{k+l}| > |z_1| > \cdots > |z_k| > |z| > |z_{k+1}| > \cdots > |z_{k+l-1}| > 0\) to

\[
g(z_1, \ldots, z_{k+l}, z)(z + z_{k+l})^{-n} \prod_{i=1}^{k+l-1} (z_i + z_{k+l})^{-p_i} z_{k+l}^{r_i}.
\]

(3.37)

So (3.36) is the expansion of the rational function (3.37) obtained by expanding the negative powers of \( z + z_{k+l} \) and \( z_i + z_{k+l} \) for \( i = 1, \ldots, k+l-1 \) as power series of \( z \) and \( z_i \), respectively. Thus (3.36) must be a power series in \( z \) and \( z_i \) for \( i = 1, \ldots, k+l-1 \) and must be absolutely convergent to (3.37) in the larger region \(|z_{k+l}| > |z|, |z_1|, \ldots, |z_{k+l-1}| > 0\) than \(|z_{k+l}| > |z_1| > \cdots > |z_k| > |z| > |z_{k+1}| > \cdots > |z_{k+l-1}| > 0\).

Since (3.36) is a power series in \( z \) and \( z_i \) for \( i = 1, \ldots, k+l-1 \) for all \( w'_2 \in W'_2 \) and \( w_1 \in W_1 \) and all \( z_{k+l} \) satisfying \(|z_{k+l}| > |z|, |z_1|, \ldots, |z_{k+l-1}| > 0\) and \( m, p_{ij} \) (for \( 1 \leq i < j \leq k+l-1 \), \( r_i \) (for \( i = 1, \ldots, k+l-1 \), \( s_i \) (for \( i = 1, \ldots, k+l-1 \)) are independent of \( w'_2 \in W'_2 \) and \( w_1 \in W_1 \),

\[
z^m \prod_{1 \leq i < j \leq k+l-1} (z_i - z_j)^{p_{ij}} \prod_{i=1}^{k+l-1} z_i^{r_i} \prod_{i=1}^{k+l-1} (z_i - z)^{s_i}.
\]

\[
\cdot Y^L_H(v_1, z_1) \cdots Y^L_H(v_k, z_k)Y^R_H(\phi, z)Y_V(v_{k+1}, z_{k+1}) \cdots Y_V(v_{k+l-1}, z_{k+l-1})v_k l
\]

(3.38)

is also a power series in \( z \) and \( z_i \) for \( i = 1, \ldots, k+l-1 \) with coefficients in \( H^N \). Let \( \phi' \in (H^N)' \). Since the grading of \( (H^N)' \) is bounded from below and the \( d \)-commutator formula for \( Y^L_H \) holds,

\[
\langle \phi', Y^L_H(v_1, z_1) \cdots Y^L_H(v_k, z_k)Y^R_H(\phi, z)Y_V(v_{k+1}, z_{k+1}) \cdots Y_V(v_{k+l-1}, z_{k+l-1})v_k l \rangle
\]

(3.39)

has only finitely many positive power terms in \( z_1 \). Thus

\[
z^m \prod_{1 \leq i < j \leq k+l-1} (z_i - z_j)^{p_{ij}} \prod_{i=1}^{k+l-1} z_i^{r_i} \prod_{i=1}^{k+l-1} (z_i - z)^{s_i}.
\]

\[
\cdot \langle \phi', Y^L_H(v_1, z_1) \cdots Y^L_H(v_k, z_k)Y^R_H(\phi, z)Y_V(v_{k+1}, z_{k+1}) \cdots Y_V(v_{k+l-1}, z_{k+l-1})v_k l \rangle
\]

(3.40)

also has only finitely many positive power terms in \( z_1 \). So (3.40) is a polynomial in \( z_1 \).

Take the coefficient of a fixed nonnegative power of \( z_1 \) in (3.40). Since

\[
z^m \prod_{1 \leq i < j \leq k+l-1} (z_i - z_j)^{p_{ij}} \prod_{i=1}^{k+l-1} z_i^{r_i} \prod_{i=1}^{k+l-1} (z_i - z)^{s_i}
\]

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is a polynomial in \( z_1 \), this coefficient involves only finitely many terms in the Laurent series 
\( Y_H^L(v_1, z_1) \). Then this coefficient must be of the form

\[
\left\langle \phi', \left( \sum_{n=n_1}^{n_2} f_n(z_2, \ldots, z_{k+l-1}, z)(Y_H^L)_n(v_1) \right) Y_H^L(v_2, z_2) \cdots Y_H^L(v_k, z_k) \right\rangle \cdot Y_H^R(\phi, z) Y_V(v_{k+1}, z_{k+1}) \cdots Y_V(v_{k+l-1}, z_{k+l-1}) v_{k+l},
\]  

(3.41)

where \( n_1 \leq n_2 \) are integers and \( f_n(z_2, \ldots, z_{k+l-1}, z) \) for \( n = n_1, \ldots, n_2 \) are polynomials in 
\( z_2, \ldots, z_{k+l-1}, z \). Since the grading of \((H^N)\)' is bounded from below, the d-commutator formula for 
\( Y_H^L \) holds and \( f_n(z_2, \ldots, z_{k+l-1}, z) \) for \( n = n_1, \ldots, n_2 \) are polynomials in 
\( z_2, \ldots, z_{k+l-1}, z \), (3.41) has only finitely many positive power terms in \( z_2 \). Since (3.41) as a coefficient of a fixed 
nonnegative power of \( z_1 \) in the power series (3.40) must be a power series in \( z_2, \ldots, z_{k+l-1}, z \), it must be a polynomial in \( z_2 \).

Since the coefficient of every fixed nonnegative power of \( z_1 \) in (3.40) is a polynomial in \( z_2 \) and since (3.40) is a polynomial in \( z_1 \), (3.40) is also a polynomial in \( z_2 \).

Repeating these steps, we see that (3.40) is in fact a polynomial \( h(z_1, \ldots, z_{k+l-1}, z) \) in \( z \) and \( z_i \) for \( i = 1, \ldots, k + l - 1 \). Note that the form of the series (3.39) is the same as 
the form of the Laurent series expansion in the region \(|z_1| > \cdots > |z_k| > |z| > z_{k+1}| > \cdots > |z_{k+l-1}| > 0 \) of a rational function in \( z \) and \( z_i \) for \( i = 1, \ldots, k + l - 1 \) with the only possible poles \( z, z_i = 0 \) (for \( i = 1, \ldots, k + l - 1 \), \( z_i = z_j \) (for \( 1 \leq i < j \leq k + l - 1 \)) and 
\( z = z_i \) (for \( i = 1, \ldots, k + l - 1 \)). Thus (3.39) must be absolutely convergent in the region

\[
|z_1| > \cdots > |z_k| > |z| > z_{k+1}| > \cdots > |z_{k+l-1}| > 0 \text{ to } h(z_1, \ldots, z_{k+l-1}, z),
\]

\[
z^m \prod_{1 \leq i < j \leq k+l-1} (z_i - z_j)^{p_{ij}} \prod_{i=1}^{k+l-1} z_i^{r_i} \prod_{i=1}^{k+l-1} (z_i - z)^{s_i}.
\]

Thus the rationality is proved.

The pole-order condition follows immediately from the facts that (3.38) is an element of 
\( H^N[[z_1, \ldots, z_{k+l-1}, z]] \) and that \( m, p_{ij}, r_i, s_i \in \mathbb{N} \) depend only on the pairs \( (v_{k+l}, \phi), (v_i, v_j), 
(v_i, v_{k+l}), (v_i, \phi) \), respectively.

5. The associativity: We prove only the associativity of \( Y_H^L \). The associativity of \( Y_H^R \) and 
the associativity of \( Y_H^L \) and \( Y_H^R \) can be proved similarly and are omitted. Take \( k = 2, l = 1, 
\varepsilon_3 = 1 \) and \( z = 0 \) in (3.17). Then we have proved above that for \( w_1 \in W_2, v_1, v_2 \in V, 
w_1 \in W_1, \phi \in H^N \)

\[
\langle w'_1, Y_{W_2}(v_1, z_1 + z_3)Y_{W_2}(v_2, z_2 + z_3)(\phi(w_1))(z_3) \rangle
\]

is absolutely convergent in the region \(|z_1 + z_2| > |z_2 + z_3| > |z_3| > 0 \) to a rational function

\[
R(\langle w'_1, Y_{W_2}(v_1, z_1 + z_3)Y_{W_2}(v_2, z_2 + z_3)(\phi(w_1))(z_3) \rangle)
\]

(3.42)
in \( z_1, z_2, z_3 \) with the only possible poles \( z_1 + z_3 = 0, z_2 + z_3 = 0, z_1, z_2, z_3 = 0 \) and \( z_1 = z_2 \). Moreover, we have proved above that
\[
\langle w'_2, ((Y^L_H(v_1, z_1)Y^L_H(v_2, z_2)\phi)(w_1))(z_3) \rangle
\]
is also absolutely convergent in the region \(|z_3| > |z_1| > |z_2| > 0\) to (3.42). In particular, we have
\[
R(\langle w'_2, Y_{W_2}(v_1, z_1 + z_3)Y_{W_2}(v_2, z_2 + z_3)(\phi(w_1))(z_3) \rangle) = R(\langle w'_2, ((Y^L_H(v_1, z_1)Y^L_H(v_2, z_2)\phi)(w_1))(z_3) \rangle).
\] (3.43)

On the other hand, by the associativity of \( Y_{W_2} \), we have
\[
R(\langle w'_2, Y_{W_2}(v_1, z_1 + z_3)Y_{W_2}(v_2, z_2 + z_3)(\phi(w_1))(z_3) \rangle) = R(\langle w'_2, Y_{W_2}(Y_V(v_1, z_1 - z_2)v_2, z_2 + z_3)(\phi(w_1))(z_3) \rangle).
\] (3.44)

Using the homogeneous basis \( \{e^V_{(q,\mu)}\}_{q \in \mathbb{Z}, \mu \in M^q} \) of \( V' \) as above and then using (3.11), the right-hand side of (3.44) can be expanded as the series
\[
\sum_{q \in \mathbb{Z}} \left( \sum_{\mu \in M^q} R(\langle w'_2, Y_{W_2}(e^V_{(q,\mu)}, z_2 + z_3)(\phi(w_1))(z_3) \rangle)(e^V_{(q,\mu)}', Y_V(v_1, z_1 - z_2)v_2) \right)
\]
\[
= \sum_{q \in \mathbb{Z}} \left( \sum_{\mu \in M^q} R(\langle w'_2, ((Y^L_H(e^V_{(q,\mu)}, z_2)\phi)(w_1))(z_3) \rangle)(e^V_{(q,\mu)}', Y_V(v_1, z_1 - z_2)v_2) \right).
\] (3.45)

in the region \( |z_2 + z_3| > |z_1 - z_2| > 0 \). But the right-hand side of (3.45) is absolutely convergent in the region \( |z_2| > |z_1 - z_2| > 0 \) to the rational function
\[
R(\langle w'_2, ((Y^L_H(Y_V(v_1, z_1 - z_2)v_2, z_2)\phi)(w_1))(z_3) \rangle).
\] (3.46)

Using the calculation from (3.43)–(3.46), we obtain
\[
R(\langle w'_2, ((Y^L_H(v_1, z_1)Y^L_H(v_2, z_2)\phi)(w_1))(z_3) \rangle) = R(\langle w'_2, ((Y^L_H(Y_V(v_1, z_1 - z_2)v_2, z_2)\phi)(w_1))(z_3) \rangle).
\] (3.47)

Taking \( k = 2, l = 1, v_3 = 1 \) and \( z = 0 \) in (3.16), we see that for \( \tilde{p}_{12} \geq p_{12}, \tilde{r}_1 \geq r_1 \) and \( \tilde{r}_2 \geq r_2 \)
\[
(\tilde{z}_1 - \tilde{z}_2)^{\tilde{p}_{12}}\tilde{z}_1^{\tilde{r}_1}\tilde{z}_2^{\tilde{r}_2}\langle w'_2, ((Y^L_H(v_1, z_1)Y^L_H(v_2, z_2)\phi)(w_1))(z_3) \rangle
\] (3.48)
is absolutely convergent in the region \( |z_3| > |z_1|, |z_2| > 0 \) to the rational function
\[
\tilde{g}(z_1, z_2, z_3)\frac{z_3^{n_0 + r_3}(z_1 + z_3)^{p_1}(z_2 + z_3)^{p_2}}{z_3^{n_0 + r_3}(z_1 + z_3)^{p_1}(z_2 + z_3)^{p_2}},
\] (3.49)

where \( \tilde{g}(z_1, z_2, z_3) \) is a polynomial in \( z_1, z_2, z_3 \). From (3.47), we also see that
\[
(\tilde{z}_1 - \tilde{z}_2)^{\tilde{p}_{12}}\tilde{z}_1^{\tilde{r}_1}\tilde{z}_2^{\tilde{r}_2}\langle w'_2, ((Y^L_H(Y_V(v_1, z_1 - z_2)v_2, z_2)\phi)(w_1))(z_3) \rangle
\] (3.50)
must be absolutely convergent in the region $|z_3| > |z_2| > |z_1 - z_2| > 0$ to (3.49). Since there is no negative power term in (3.49), (3.50) is in fact absolutely convergent in the larger region $|z_3| > |z_1 = z_2 + (z_1 - z_2)|, |z_2| > 0$ than $|z_3| > |z_2| > |z_1 - z_2| > 0$.

We have proved that (3.48) is the expansion of (3.49) by expanding the negative powers of $z_1 + z_3$ and $z_2 + z_3$ as power series of $z_1$ and $z_2$, respectively. We have also proved that (3.50) is the expansion of (3.49) by expanding the negative powers of $z_1 + z_3$ and $z_2 + z_3$ as power series of $z_1$ and $z_2$, respectively, and then expanding positive powers of $z_1 = z_2 + (z_1 - z_2)$ using the binomial expansion as polynomials in $z_2$ and $z_1 - z_2$. Thus as a power series in $z_2$ and $z_1 - z_2$, (3.50) can be obtained by expanding the positive powers of $z_1 = z_2 + (z_1 - z_2)$ in the series (3.48) using the binomial expansion as polynomials in $z_2$ and $z_1 - z_2$. By the composability, $p_{12}, r_1$ and $r_2$ are independent of $w_2', w_1$ and $z_3$. Hence

$$(z_1 - z_2)^{p_{12}z_1^{r_1}z_2^{r_2}}Y_H^L(v_1, z_1)Y_H^L(v_2, z_2) \phi$$

(3.51)
is a power series in $z_1$ and $z_2$ and

$$(z_1 - z_2)^{p_{12}z_1^{r_1}z_2^{r_2}}Y_H^L(Y_V(v_1, z_1 - z_2)v_2, z_2) \phi$$

(3.52)
is a power series in $z_2$ and $z_1 - z_2$. Moreover, (3.52) can also be obtained from (3.51) by expanding positive powers of $z_1 = z_2 + (z_1 - z_2)$ using the binomial expansion as polynomials in $z_2$ and $z_1 - z_2$. Let $\phi' \in (H^N)'$. Then the same is also true for

$$(z_1 - z_2)^{p_{12}z_1^{r_1}z_2^{r_2}}\langle \phi', Y_H^L(v_1, z_1)Y_H^L(v_2, z_2) \phi \rangle$$

(3.53)
and

$$(z_1 - z_2)^{p_{12}z_1^{r_1}z_2^{r_2}}\langle \phi', Y_H^L(Y_V(v_1, z_1 - z_2)v_2, z_2) \phi \rangle.$$ 

(3.54)
But as a special case of the rationality proved above, (3.53) is a polynomial $h(z_1, z_2)$ in $z_1$ and $z_2$. So (3.54) must be a polynomial in $z_2$ and $z_1 - z_2$ obtained from (3.53) by expanding the positive powers of $z_1 = z_2 + (z_1 - z_2)$ using the binomial expansion as polynomials in $z_2$ and $z_1 - z_2$. Thus

$$R(\langle \phi', Y_H^L(Y_V(v_1, z_1 - z_2)v_2, z_2) \phi \rangle)$$

$$= \frac{h(z_1, z_2)}{(z_1 - z_2)^{p_{12}z_1^{r_1}z_2^{r_2}}}$$

$$= R(\langle \phi', Y_H^L(v_1, z_1)Y_H^L(v_2, z_2) \phi \rangle).$$

This is the associativity of $Y_H^L$.

6. The $D$-derivative property and the $D$-commutator formula: We prove only the $D$-derivative property and $D$-commutator formula for $Y_H^R$. The proof of these properties of
$Y_{H}^R$ are similar and are omitted. Calculating the derivatives and using (3.12) and the definition of $D_{H}$, we obtain

$$
R \left( \left\langle w'_2, \left( \left( \frac{d}{dz_1} Y_{H}^R(\phi, z_1)v \right)(w_1) \right)(z_2) \right\rangle \right)
= \frac{\partial}{\partial z_1} R(\langle w'_2, (Y_{H}^R(\phi, z_1)v)(w_1)\rangle(z_2))
= \frac{\partial}{\partial z_1} R(\langle w'_2, \phi(Y_{W_1}(v, z_2)w_1)(z_1 + z_2)\rangle)
= \left( \frac{\partial}{\partial z_0} R(\langle w'_2, \phi(Y_{W_1}(v, z_2)w_1)(z_0)\rangle) \right)_{z_0 = z_1 + z_2}
= R(\langle w'_2, ((D_{H}\phi)(Y_{W_1}(v, z_2)w_1))(z_1 + z_2)\rangle)
= R(\langle w'_2, ((Y_{H}^R(D_{H}\phi, z_1)v)(w_1))(z_2)\rangle)
$$

(3.55)

for $w'_2 \in W'_2$ and $w_1 \in W_1$. Thus we obtain the $D$-derivative property

$$
\frac{d}{dz_1} Y_{H}^R(\phi, z_1)v = Y_{H}^R(D_{H}\phi, z_1)v.
$$

Using the equality that the left-hand side of (3.55) is equal to the fourth line in (3.55), we obtain

$$
R \left( \left\langle w'_2, \left( \left( \frac{d}{dz_1} Y_{H}^R(\phi, z_1)v \right)(w_1) \right)(z_2) \right\rangle \right)
= \left( \frac{\partial}{\partial z_0} R(\langle w'_2, \phi(Y_{W_1}(v, z_2)w_1)(z_0)\rangle) \right)_{z_0 = z_1 + z_2}
= \frac{\partial}{\partial z_2} R(\langle w'_2, \phi(Y_{W_1}(v, z_2)w_1)(z_1 + z_2)\rangle) - R \left( \left\langle w'_2, \phi \left( \frac{\partial}{\partial z_2} Y_{W_1}(v, z_2)w_1 \right)(z_1 + z_2) \right\rangle \right)
= \frac{\partial}{\partial z_2} R(\langle w'_2, ((Y_{H}^R(\phi, z_1)v)(w_1))(z_2)\rangle) - R(\langle w'_2, ((Y_{H}^R(D_{H}\phi, z_1)v)(w_1))(z_2)\rangle)
= R(\langle w'_2, ((D_{H}Y_{H}^R(\phi, z_1)v)(w_1))(z_2)\rangle) - R(\langle w'_2, ((Y_{H}^R(\phi, z_1)D_{V}v)(w_1))(z_2)\rangle)
$$

for $w'_2 \in W'_2$ and $w_1 \in W_1$. Thus we obtain the $D$-commutator formula for $Y_{H}^R$:

$$
\frac{d}{dz_1} Y_{H}^R(\phi, z_1)v = D_{H}Y_{H}^R(\phi, z_1)v - Y_{H}^R(\phi, z_1)D_{V}v.
$$

All the axioms have been verified and thus the theorem is proved.

For $S \subset H^N$, let $H^{(N,S)}$ be the $V$-subbimodule of $H^N$ generated by $S$. 

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4 A 1-cocycle constructed from a left $V$-module and a left $V$-submodule

In this section, given a left $V$-module $W$ and a left $V$-submodule $W_2$ of $W$ and assuming a composability condition, we construct a 1-cocycle in $\hat{C}_1^\infty(V, H^{(N,F(V))})$ where $N$ is a lower bound of the weights of the elements of $V$, $F(V)$ is the image of a suitable linear map $F$ from $V$ to $H^N$, $H^N$ is the $V$-bimodule constructed in the preceding section and $H^{(N,F(V))}$ is the $V$-subbimodule of $H^N$ generated by $F(V)$. In fact $F(V)$ is independent of $N$. Thus we shall denote $H^{(N,F(V))}$ simply by $H^F(V)$.

Let $W$ be a left $V$-module and $W_2$ a $V$-submodule of $W$. Let $W_1$ be a graded subspace of $W$ such that as a graded vector space, we have

$$W = W_1 \oplus W_2.$$ 

Then we can also embed $W'_1$ and $W'_2$ into $W'$ and we have

$$W' = W'_1 \oplus W'_2.$$ 

Let $\pi_{W_1} : W \to W_1$ and $\pi_{W_2} : W \to W_2$ be the projections given by this graded space decomposition of $W$. For simplicity, we shall use the same notations $\pi_{W_1}$ and $\pi_{W_2}$ to denote their natural extensions to operators on $\overline{W}_1$ and $\overline{W}_2$, respectively. By definition, we have $\pi_{W_1} + \pi_{W_2} = 1_W$, $\pi_{W_1} \circ \pi_{W_1} = \pi_{W_1}$, $\pi_{W_2} \circ \pi_{W_2} = \pi_{W_2}$, $\pi_{W_1} \circ \pi_{W_2} = \pi_{W_2} \circ \pi_{W_1} = 0$.

Since $W_2$ is a submodule of $W$, we have $\pi_{W_2} \circ Y_W \circ (1_V \otimes \pi_{W_2}) = Y_{W_2}$, $D_{W_2} = \pi_{W_2} \circ D_W \circ \pi_{W_2}$ and $d_{W_2} = \pi_{W_2} \circ d_W \circ \pi_{W_2}$. We also have $\pi_{W_1} \circ Y_W \circ (1_V \otimes \pi_{W_2}) = 0$.

Let $Y_{W_1} = \pi_{W_1} \circ Y_W \circ (1_V \otimes \pi_{W_1})$, $D_{W_1} = \pi_{W_1} \circ D_W \circ \pi_{W_1}$ and $d_{W_1} = \pi_{W_1} \circ d_W \circ \pi_{W_1}$ which is equal to the operator giving the grading on $W_1$. As we have done above, we use the same notations $D_{W_1}$ and $d_{W_1}$ to denote their natural extensions to $\overline{W}_1$. We also use the same convention for notations for extensions of operators on $W$ and $W_2$.

**Proposition 4.1** The graded vector space $W_1$ equipped with the vertex operator map $Y_{W_1}$ and the operator $D_{W_1}$ is a left $V$-module.

**Proof.** The axioms for the grading and the identity property are obvious.

For $v \in V$,

$$\frac{d}{dz} Y_{W_1}(v, z) = \pi_{W_1} \left( \frac{d}{dz} Y_W(v, z) \right) \pi_{W_1}$$

$$= \pi_{W_1} Y_W(D_V v, z) \pi_{W_1}$$

$$= Y_{W_1}(D_V v, z). \quad (4.1)$$
Also using (4.1), we have
\[
\frac{d}{dz} Y_{W_1}(v, z) = \pi_{W_1} Y_{W}(D_{W} v, z) \pi_{W_1}
\]
\[
= \pi_{W_1} D_{W} Y_{W}(v, z) \pi_{W_1} - \pi_{W_1} Y_{W}(v, z) D_{W} \pi_{W_1}
\]
\[
= \pi_{W_1} D_{W} \pi_{W_1} Y_{W}(v, z) \pi_{W_1} + \pi_{W_1} D_{W} \pi_{W_2} Y_{W}(v, z) \pi_{W_1}
\]
\[
- \pi_{W_1} Y_{W}(v, z) \pi_{W_1} D_{W} \pi_{W_1} - \pi_{W_1} Y_{W}(v, z) \pi_{W_2} D_{W} \pi_{W_1}
\]
\[
= D_{W} Y_{W_1}(v, z) + \pi_{W_1} D_{W} \pi_{W_2} Y_{W}(v, z) \pi_{W_1}
\]
\[
- Y_{W_1}(v, z) D_{W_1} - \pi_{W_1} Y_{W}(v, z) \pi_{W_2} D_{W} \pi_{W_1}. \tag{4.2}
\]

Since $W_2$ is a $V$-submodule of $W$, $\pi_{W_1} D_{W} \pi_{W_2} = \pi_{W_1} D_{W_2} \pi_{W_2} = 0$. Again since $W_2$ is a $V$-submodule of $W$, $\pi_{W_1} Y_{W}(v, z) \pi_{W_2} = \pi_{W_1} Y_{W}(v, z) \pi_{W_2} = 0$. So the right-hand side of (4.2) is equal to
\[
D_{W_1} Y_{W_1}(v, z) - Y_{W_1}(v, z) D_{W_1}.
\]
Thus both the $D$-derivative property and the $D$-commutator formula hold.

For $v, v_1, \ldots, v_k \in V$, $w'_1 \in W'_1$ and $w_1 \in W_1$, using the properties of $\pi_{W_1}$, $\pi_{W_2}$, $Y_{W_1}$, $Y_{W_2}$ given above, we have
\[
\langle w'_1, Y_{W_1}(v_1, z_1) \cdots Y_{W_1}(v_k, z_k) w_1 \rangle = \langle w'_1, Y_{W}(v_1, z_1) \cdots Y_{W}(v_k, z_k) w_1 \rangle. \tag{4.3}
\]
Since the right-hand side of (4.3) is absolutely convergent in the region $|z_1| > \cdots > |z_k| > 0$ to a rational function in $z_1, \ldots, z_k$ with the only possible poles $z_i = 0$ for $i = 1, \ldots, k$ and $z_i = z_j$ for $1 \leq i < j \leq k$, so is the left-hand side. This proves the rationality.

For $v, v_1, v_2 \in V$, $w'_1 \in W'_1$ and $w_1 \in W_1$, using the properties of $\pi_{W_1}$, $\pi_{W_2}$, $Y_{W_1}$, $Y_{W_2}$ again and the associativity for $Y_{W}$, we obtain
\[
\langle w'_1, Y_{W_1}(v_1, z_1) Y_{W}(v_2, z_2) w_1 \rangle = \langle w'_1, Y_{W}(v_1, z_1) Y_{W}(v_2, z_2) w_1 \rangle
\]
\[
= \langle w'_1, Y_{W}(Y_{W}(v_1, z_1 - z_2) v_2, z_2) w_1 \rangle
\]
\[
= \langle w'_1, Y_{W_1}(Y_{W}(v_1, z_1 - z_2) v_2, z_2) w_1 \rangle, \tag{4.4}
\]
proving the associativity for $Y_{W_1}$.

**Remark 4.2** Note that although $W_1$ is a graded subspace of $W$, $(W_1, Y_{W_1})$ is not a submodule of $W$ since the vertex operator $Y_{W_1}$ is not the restriction of the vertex operator $Y_{W}$ to $V \otimes W_1$.

We now have two left $V$-modules $W_1$ and $W_2$. From Theorem 3.4 for $N \in \mathbb{Z}$, we have a $V$-bimodule $H^N \subset \text{Hom}(W_1, (W_2)_{\pi_i})$.

We need the following assumption (called *composability condition*) on the map $\pi_{W_2} \circ Y_{W} \circ (1_V \otimes \pi_{W_1})$:
Assumption 4.3 (Composability condition) For \(v, v_1, \ldots, v_{k+l} \in V\), \(w'_2 \in W'_2\) and \(w_1 \in W_1\),

\[
\langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k) \pi_{W_2} Y_W(v, z) \pi_{W_1} Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l-1}, z_{k+l-1}) w_1 \rangle
\]

is absolutely convergent in the region \(|z_1| > \cdots |z_k| > |z| > |z_{k+1}| > \cdots > |z_{k+l}| > 0\) to a rational function in \(z_1, \ldots, z_{k+l}, z\) with the only possible poles \(z_i = 0\) for \(i = 1, \ldots, k + l\), \(z = 0\), \(z_i = z_j\) for \(1 \leq i < j \leq k + l\) and \(z_i = z\) for \(i = 1, \ldots, k + l\). Moreover, the orders of the poles \(z_i = z\) for \(i = 1, \ldots, k + l\) and \(z_i = z_j\) for \(i, j = 1, \ldots, k + l\), \(i \neq j\) are bounded above by nonnegative integers depending only on the pairs \((v_i, v)\) and \((v_i, v_j)\), respectively, and there exists \(N \in \mathbb{Z}\) such that when \([F, Y]\) is expanded as a Laurent series in the region \(|z_1| > |z_i - z_{k+l}| > \cdots > |z_k - z_{k+l}| > |z - z_{k+l}| > |z_{k+1}| > \cdots > |z_{k+l-1} - z_{k+l}| > 0\) as a Laurent series in \(z_i - z_{k+l}\) for \(i = 1, \ldots, k + l - 1\) and \(z - z_{k+l}\) with Laurent polynomials in \(z_{k+l}\) as coefficients, the total degree of each monomial in \(z_i - z_{k+l}\) for \(i = 1, \ldots, k + l - 1\) and \(z - z_{k+l}\) in the expansion is larger than or equal to \(N - \sum_{i=1}^{k+l} \omega(v_i)\).

We now assume that \(\pi_{W_2} \circ Y_W \circ (1_V \otimes \pi_{W_1})\) satisfies the composability condition. For \(v \in V\), let \(F(v) \in \text{Hom}(W_1, (\overline{W_2})_z)\) be given by

\[
((F(v))(w_1))(z) = \pi_{W_2} Y_W(v, z) \pi_{W_1} w_1
\]

for \(w_1 \in W_1\). Thus we obtain a linear map \(F : V \to \text{Hom}(W_1, (\overline{W_2})_z)\).

Proposition 4.4 For \(N \in \mathbb{Z}\) such that \(\omega(v) \geq N\) for any homogeneous \(v \in V\), the image of \(F\) is in fact in \(H^N\) and is thus a map from \(V\) to \(H^N\). Moreover, \(F\) preserves the gradings.

Proof. Let \(v \in V\) be homogeneous. For \(a \in \mathbb{C}^\times\) and \(w_1 \in W_1\),

\[
a^{dw_2}((F(v))(w_1))(z) = a^{dw_2} \pi_{W_2} Y_W(v, z) \pi_{W_1} w_1
\]

\[
= \pi_{W_2} Y_W(a^{dv} v, az) a^{dw_1} w_1
\]

\[
= a^{\omega(v)} (F(v))(a^{dw_1} w_1)(az),
\]

proving the \(d\)-conjugation property of \(F(v)\) and \(\omega(F(v)) = \omega(v)\).

For \(k, l \in \mathbb{N}\) and \(v_1, \ldots, v_{k+l} \in V\), \(w_1 \in W_1\) and \(w'_2 \in W'_2\), by the definition of \(F(v)\), we have

\[
\langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k)(F(v))(Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l-1}, z_{k+l-1}) w_1) \rangle(z)
\]

\[
= \langle w'_2, Y_{W_2}(v_1, z_1) \cdots Y_{W_2}(v_k, z_k) \pi_{W_2} Y_W(v, z) \pi_{W_1} Y_{W_1}(v_{k+1}, z_{k+1}) \cdots Y_{W_1}(v_{k+l-1}, z_{k+l-1}) w_1 \rangle
\]

Then by Assumption 4.3, the composability for \(F(v)\) holds.

Also by Assumption 4.3, the sum of the orders of the possible poles \(z_i = z\) for \(i = 1, \ldots, k + l\) and \(z_i = z_j\) for \(i, j = 1, \ldots, k + l\), \(i \neq j\) of the rational function that the right-hand side of (4.6) converges to is less than or equal to \(\sum_{i=1}^{k+l} \omega(v_i) + \omega(v) - N\). By (4.6) and the fact \(\omega(F(v)) = \omega(v)\), we see that the \(N\)-weight-degree condition for \(F(v)\) holds.
Proposition 4.4 says in particular that \( F(V) \) is independent of such lower bound \( N \) of the weights of \( V \). Thus we shall denote \( H^{(N,F(V))} \) simply by \( H^F(V) \). Now we construct a 1-cochain \( \Psi \in \hat{C}_\infty^1(V, H^F(V)) \). Since

\[
\hat{C}_\infty^1(V, H^F(V)) \subset \text{Hom}(V, (\widehat{H^F(V)})_z),
\]

to avoid confusion with the variable in \( H^F(V) \subset \text{Hom}(W_1, (\widehat{W_2})_z) \), we use different notations to denote these variables. For example, we might use \( z_1 \) and \( z_2 \) to denote the variables in \( \text{Hom}(W_1, (\widehat{W_2})_{z_1}) \) and \( \text{Hom}(V, (\widehat{H^F(V)})_{z_2}) \), respectively. For \( v \in V, w_1 \in W_1 \) and \( w'_2 \in W'_2 \),

\[
\langle w'_2, ((e^{z_2D_H}F(v))(w_1))(z_1) \rangle = \langle w'_2, ((F(v))(w_1))(z_1 + z_2) \rangle = \langle w'_2, \pi_{W_2}Y_W(v, z_1 + z_2)w_1 \rangle \]

in the region \(|z_1| > |z_2|\). Let \( E(e^{z_2D_H}F(v)) \in (\widehat{H^F(V)})_{z_2} \) be defined by

\[
\langle w'_2, ((E(e^{z_2D_H}F(v)))(w_1))(z_1) \rangle = \langle w'_2, \pi_{W_2}Y_W(v, z_1 + z_2)w_1 \rangle
\]

\( v \in V, w_1 \in W_1 \) and \( w'_2 \in W'_2 \) in the region \( z_1 + z_2 \neq 0 \).

We define

\[
(\Psi(v))(z_2) = E(e^{z_2D_H}F(v)).
\]

More explicitly, for \( v \in V, w_1 \in W_1 \) and \( w'_2 \in W'_2 \),

\[
\langle w'_2, (((\Psi(v))(z_2))(w_1))(z_1) \rangle = \langle w'_2, \pi_{W_2}Y_W(v, z_1 + z_2)w_1 \rangle. \tag{4.7}
\]

in the region \( z_1 + z_2 \neq 0 \). In the region \(|z_1| > |z_2|\), the series \( e^{z_2D_H}F(v) \) is convergent absolutely to \( \Psi(v) \). We shall also use \( e^{z_2D_H}F(v) \) to denote \( (\Psi(v))(z_2) \) in the region \(|z_1| > |z_2|\).

By definition, \( \Psi(v) \in (\widehat{H^F(V)})_{z_2} \) and thus \( \Psi \in \text{Hom}(V, (\widehat{H^F(V)})_{z_2}) \).

**Theorem 4.5** \( \Psi \in \ker \delta_\infty^1 \subset \hat{C}_\infty^1(V, H^F(V)) \).

**Proof.** From Theorem 2.13 we need only show that \( \Psi(\cdot)(0) = F \) is a derivation from \( V \) to \( H^F(V) \). Using (3.2) and (3.3) in the region \(|x + z_1| > |z_1| > |z| \neq 0 \), we have

\[
\langle w'_2, ((F(Y_V(u,z)v))(w_1))(z_1) \rangle
\]

\[
= \langle w'_2, \pi_{W_2}Y_W(Y_V(u, z)v, z_1)\pi_{W_1}w_1 \rangle
\]

\[
= \langle w'_2, \pi_{W_2}Y_W(u, z + z_1)Y_V(v, z_1)\pi_{W_1}w_1 \rangle
\]

\[
= \langle w'_2, \pi_{W_2}Y_W(u, z + z_1)\pi_{W_1}Y_W(v, z_1)\pi_{W_1}w_1 \rangle
\]

\[
+ \langle w'_2, \pi_{W_2}Y_W(u, z + z_1)\pi_{W_1}Y_W(v, z_1)\pi_{W_1}w_1 \rangle
\]

\[
= \langle w'_2, ((F(u))(Y_{W_1}(v, z_1)w_1))(z + z_1) \rangle
\]

\[
+ \langle w'_2, Y_{W_2}(u, z + z_1)((F(v))(w_1))(z_1) \rangle
\]

\[
= \langle w'_2, ((Y^R_H(F(u), z)v)(w_1))(z_1) \rangle
\]

\[
+ \langle w'_2, ((Y^L_H(F(u), z)v)(w_1))(z_1) \rangle \tag{4.8}
\]
for \( u, v \in V, w_1 \in W_1 \text{ and } w_2 \in W_2' \). By Proposition \ref{p.2} the left-hand side and the two terms in the right-hand side of (4.8) are absolutely convergent in the same region \(|z_1| > |z| > 0\). Hence the left-hand side and the right-hand side of (4.8) are equal in the larger region \(|z_1| > |z| > 0\). Thus we obtain

\[ F(Y_V(u, z)v) = Y_{H}^{R}(F(u), z)v + Y_{H}^{L}(u, z)F(v) \]

for \( u, v \in V \), proving that \( F \) is indeed a derivation.

\section{The main theorem}

In this section, we formulate and prove our main result on complete reducibility of modules for a meromorphic open-strong vertex algebra \( V \).

Let \( W \) be a left \( V \)-module. Assume that \( W \) is not irreducible. Then there exists a proper nonzero left \( V \)-submodule \( W_2 \) of \( W \). We say that the pair \((W, W_2)\) satisfies the composability condition if there exists a graded subspace \( W_1 \) of \( W \) such that \( W = W_1 \oplus W_2 \) as a graded vector space such that \( \pi_{W_2} \circ Y_W \circ (1_V \otimes \pi_{W_1}) \) satisfies Assumption \ref{a.3}. If for every proper nonzero left \( V \)-submodule \( W_2 \) of \( W \), the pair \((W, W_2)\) satisfies the composability condition, we say that \( W \) satisfies the composability condition.

\begin{proposition}
Let \( W \) be a completely reducible left \( V \)-module. Then \( W \) satisfies the composability condition.
\end{proposition}

\begin{proof}
Let \( W_2 \) be a left \( V \)-submodule of \( W \). Since \( W \) is completely reducible, there is a left \( V \)-submodule \( W_1 \) of \( W \) such that \( W \) as a left \( V \)-module is the direct sum of the left \( V \)-modules \( W_1 \) and \( W_2 \). Then \( \pi_{W_1} \) and \( \pi_{W_2} \) are module maps. Thus \( \pi_{W_2} \circ Y_W \circ (1_V \otimes \pi_{W_1}) \) satisfies Assumption \ref{a.3}.
\end{proof}

Now let \( W \) be a left \( V \)-module which is not irreducible and \( W_2 \) a proper nonzero left \( V \)-submodule \( W_2 \) of \( W \). Assume that the pair \((W, W_2)\) satisfies the composability condition. Then there exists a graded subspace \( W_1 \) of \( W \) such that as a graded vector space, \( W \) is the direct sum of \( W_1 \) and \( W_2 \) and \( \pi_{W_2} \circ Y_W \circ (1_V \otimes \pi_{W_1}) \) satisfies Assumption \ref{a.3}. By Theorem \ref{t.4}, Proposition \ref{p.4} and Theorem \ref{t.5} there exist a left \( V \)-module structure on \( W_1 \), a \( V \)-bimodule \( H^N \subset \text{Hom}(W_1, (W_2)_{z_1}) \) for a lower bound \( N \) of \( V \), a grading preserving linear map \( F : V \to H^N \) and \( \Psi \in \ker \delta_{\infty}^{1} \subset \hat{C}_{\infty}^{1}(V, H^{F(V)}) \), where \( H^{F(V)} \) is the \( V \)-subbimodule of \( H^N \) generated by \( F(V) \).

We see from Proposition \ref{p.14} that besides inner derivations, there are also zero-mode derivations obtained from suitable elements of a bimodule. For example, when \( V \) is a vertex algebra and \( W = V \), then every weight 1 element of \( V \) gives a derivation from \( V \) to \( V \). In particular, the first cohomology \( \hat{H}_{\infty}^{1}(V, V) \) is not 0, even if all \( V \)-modules are completely reducible. Moreover, we have the following result:
Proposition 5.2 Let $V$ be a grading-restricted vertex algebra generated by its homogeneous subspace $V(1)$ of weight 1 elements. Assume that the finite-dimensional Lie algebra $V(1)$ with the Lie bracket given by $(u, v) \mapsto (Y_V)_0(v)u$ for $u, v \in V(1)$ is semisimple. Then $\hat{H}_\infty^1(V, V) = \hat{Z}_\infty^1(V, V)$ and $\hat{Z}_\infty^1(V, V)$ is linearly isomorphic to $V(1)$. In particular, $\hat{H}_\infty^0(V, V)$ is linearly isomorphic to $V(1)$.

Proof. Since $V$ is a vertex algebra, an inner derivation from $V$ to $V$ is 0. To prove this result, we need only prove that every derivation from $V$ to $V$ is a zero-mode derivation and every zero-mode derivation can be identified linearly with an element of $V(1)$.

Let $f$ be a derivation from $V$ to $V$. For $u, v \in V(1)$, from $f(Y_V(u, x)v) = Y_V(f(u), x)v + Y_V(u, x)f(v)$, we obtain $f((Y_V)_0(u)v) = (Y_V)_0(f(u))v + (Y_V)_0(u)f(v)$. So the restriction of $f$ to $V(1)$ is a derivation of the Lie algebra $V(1)$. Since this Lie algebra is semisimple, $f$ must be an inner derivation of this Lie algebra. Thus there exists a unique $w \in V(1)$ such that $f(v) = (Y_V)_0(w)v$ for $v \in V(1)$. Since $w \in V(1)$, $g_w : V \to V$ defined by $g_w(v) = (Y_V)_0(w)v$ for $v \in V$ is a zero-mode derivation from $V$ to $V$. Since $V$ is generated by $V(1)$ and both $f$ and $g_w$ are derivations from $V$ to $V$, $f(v) = g_w(v)$ for $v \in V(1)$ implies $f = g_w$. 

From this result, we see that even for an affine Lie algebra vertex operator algebra $V$ associated to a finite-dimensional simple Lie algebra $g$ such that all weak $V$-modules are completely reducible, $\hat{H}_\infty^1(V, V) = \hat{Z}_\infty^1(V, V)$ which is in turn isomorphic to $g \neq 0$. However, we have the following result:

Theorem 5.3 Let $W$, $W_1$, $W_2$ and $H^{F(V)}$ be as above. If $\hat{H}_\infty^1(V, H^{F(V)}) = \hat{Z}_\infty^1(V, H^{F(V)})$ (see the end of Section 2), then there exists another left $V$-submodule $\tilde{W}_1$ of $W$ such that $W$ is the direct sum of the left $V$-submodules $\tilde{W}_1$ and $W_2$.

Proof. Since $\hat{H}_\infty^1(V, H^{F(V)}) = \hat{Z}_\infty^1(V, H^{F(V)})$, $\Psi$ constructed in the preceding section must be the sum of a coboundary and a 0-cochain obtained from a zero-mode derivation. That is, there exist $\Phi_1 \in \hat{C}_\infty^0(V, H)$ and $\Phi_2 \in H^{F(V)}_{[1]}$ satisfying

$$e^{zD_H}Y^L_H(v, -z)\Phi_2 - Y^R_H(\Phi_2, z)v \in H^{F(V)}[z, z^{-1}].$$

(5.1)

for $v \in V$ such that

$$(\Psi(v))(z_2) = (\delta^0_\infty \Phi_1)(v)(z_2) + e^{zD_H}\text{Res}_zY^R_H(\Phi_2, z)v.$$ (5.2)

Note that the 0-cochain $\Phi_1$ is an element of $H^{F(V)}_{[0]}$ such that $D_H\Phi_1 = 0$ and, in particular, $((\delta^0_\infty \Phi_1)(v))(z)$ is an $H^{F(V)}$-valued holomorphic function on $C$. By the definition of $((\delta^0_\infty \Phi_1)(v))(z_2)$, we obtain from (5.2)

$$(\Psi(v))(z_2) = Y^L_H(v, z_2)\Phi_1 - e^{zD_H}Y^R_H(\Phi_1, -z_2)v + e^{zD_H}\text{Res}_zY^R_H(\Phi_2, z)v.$$ (5.3)
Applying both sides of (5.3) to \( w_1 \in W_1 \), evaluating at \( z_1 \), pairing with \( w' \in W' \), using (4.7) and properties of \( D_H \) and then taking the limit \( z_2 \to 0 \), we obtain in the region \( |z_1| > |z_2| > 0 \),

\[
<w', \pi_{W_2} Y_W(v, z_1) w_1> = <w', (( \lim_{z_2 \to 0} Y_H^L(v, z_2) \Phi_1)(w_1))(z_1) > - <w', (( \lim_{z_2 \to 0} Y_H^R(\Phi_1, -z_2)v)(w_1))(z_1) > + \text{Res}_z <w', ((Y_H^R(\Phi_2, z)v)(w_1))(z_1) >. \tag{5.4}
\]

Note that since \( D_H \Phi_1 = 0 \), \((\Phi_1(w_1))(z_1) \) is in fact independent of \( z_1 \). In particular, \((\Phi_1(w_1))(0) \) exists and \((\Phi_1(w_1))(z) = (\Phi_1(w_1))(0) \) for \( z \in \mathbb{C} \). We now define a linear map \( \eta : W_1 \to W \) by

\[
\eta(w_1) = w_1 - (\Phi_1(w_1))(0) + \text{Res}_z (\Phi_2(w_1))(z) \tag{5.5}
\]

for homogeneous \( w_1 \in W_1 \). Note that the weight of \((\Phi_1(w_1))(0) \) is equal to the weight of \( w_1 \). It is clear that the weight of \((\text{Res}_z(\Phi_2(w_1))(z) \) is also equal to the weight of \( w_1 \). Thus the map \( \eta \) preserves the gradings. Let \( \tilde{W}_1 = \eta(W_1) \). We show that \( \tilde{W}_1 \) is in fact a left \( V \)-submodule of \( W \).

Applying \( Y_W(v, z_1) \) to the right-hand side of (5.3) and pairing the result with \( w' \in W' \), using (5.4) and then using (3.11) and (3.12), we obtain

\[
<w', Y_W(v, z_1)(w_1 - (\Phi_1(w_1))(0) + \text{Res}_z (\Phi_2(w_1))(z))>
= <w', (\pi_{W_1} Y_W(v, z_1) w_1 + \pi_{W_2} Y_W(v, z_1) w_1 - Y_W(v, z_1)(\Phi_1(w_1))(0) + \text{Res}_z Y_W(v, z_1)(\Phi_2(w_1))(z))>
= <w', (Y_{W_1}(v, z_1) w_1 + (( \lim_{z_2 \to 0} Y_H^L(v, z_2) \Phi_1)(w_1))(z_1) - (( \lim_{z_2 \to 0} Y_H^R(\Phi_1, -z_2)v)(w_1))(z_1) + \text{Res}_z ((Y_H^R(\Phi_2, z)v)(w_1))(z_1)>
= <w', (Y_{W_1}(v, z_1) w_1 - (\Phi_1(Y_{W_1}(v, z_1) w_1))(0) + \text{Res}_z ((Y_H^R(\Phi_2, z)v)(w_1))(z_1) + \text{Res}_z Y_W(v, z_1)(\Phi_2(w_1))(z)). \tag{5.6}
\]

We now calculate the last two terms in (5.6). From (3.12), we have

\[
R(\langle w', ((Y_H^R(\Phi_2, z_2)v)(w_1))(z_1) \rangle) = R(\langle w', (\Phi_2(Y_{W_1}(v, z_1) w_1))(z_1 + z_2) \rangle). \tag{5.7}
\]

Using (5.6) and (3.11), we see that there exists a rational function \( g(z_1, z_2) \) with the only possible pole \( z_1 = 0 \) and \( z_2 = 0 \) such that

\[
R(\langle w', ((Y_H^R(\Phi_2, z_2)v)(w_1))(z_1) \rangle)
= R(\langle w', ((e^{z_2 D_H}Y_H^L(v, -z_2) \Phi_2)(w_1))(z_1) \rangle) + g(z_1, z_2)
= R(\langle w', (Y_H^L(v, -z_2) \Phi_2)(w_1))(z_1 + z_2) \rangle) + g(z_1, z_2)
= R(\langle w', Y_W(v, z_1)(\Phi_2(w_1))(z_1 + z_2) \rangle) + g(z_1, z_2). \tag{5.8}
\]
Let $z_2 = z - z_1$ in (5.7) and (5.8). Then we obtain

$$R(\langle w', ((Y^R_H(\Phi_2, z - z_1)v)(w_1))(z_1) \rangle)$$
$$= R(\langle w', (\Phi_2(Y_{W_1}(v, z_1)w_1))(z) \rangle)$$
$$= R(\langle w', Y_{W_2}(v, z_1)(\Phi_2(w_1))(z) \rangle) + g(z_1, z - z_1).$$

From (5.10) and (5.6), we obtain

$$\langle w', Y_{W1}(v, z_1)(w_1 - (\Phi_1(w_1))(0) + \text{Res}_z(\Phi_2(w_1))(z) \rangle$$
$$= \langle w', (Y_{W_1}(v, z_1)w_1 - (\Phi_1(Y_{W_1}(v, z_1)w_1))(0) + \text{Res}_z(\Phi_2(Y_{W_1}(v, z_1)w_1))(z) \rangle,$$

or equivalently,

$$Y_{W_1}(v, z_1)\eta(w_1) = \eta(Y_{W_1}(v, z_1)w_1).$$

The formula (5.11) means in particular that the space $\widetilde{W_1}$ is invariant under the action of $Y_{W_1}$. Thus $\widetilde{W_1}$ is a submodule of $W$. Moreover, the sum of $\widetilde{W_1}$ and $W_2$ is clearly $W$ and the intersection of $\widetilde{W_1}$ and $W_2$ is clearly 0. So $W$ is equal to the direct sum of $\widetilde{W_1}$ and $W_2$, proving the theorem.

We shall need the following result:
Proposition 5.4 Let $W$ be a left $V$-module satisfying the composability condition. Then every left $V$-submodule of $W$ also satisfies the composability condition.

Proof. Let $W_0$ be a left $V$-submodule of $W$. Then any proper nonzero left $V$-submodule $W_2$ of $W_0$ is also a proper nonzero left $V$-submodule of $W$. Then there is a graded subspace $W_3$ of $W$ such that $W = W_3 \oplus W_2$ as a graded vector space and $\pi_{W_2}^W \circ Y_W \circ (1_V \otimes \pi_{W_3}^W)$ satisfies Assumption 4.3 with $W_1$ in Assumption 4.3 replaced by $W_3$, where $\pi_{W_2}^W$ and $\pi_{W_3}^W$ are projections from $W$ to $W_2$ and $W_3$, respectively. Let $W_1 = W_3 \cap W_0$. Then $W_0 = W_1 \oplus W_2$ as a graded vector space. Let $\pi_{W_1}^W$ and $\pi_{W_2}^W$ be the projections from $W_0$ to $W_1$ and $W_2$, respectively. Then $\pi_{W_1}^{W_0} = \pi_{W_1}^W |_{W_0}$ and $\pi_{W_2}^{W_0} = \pi_{W_2}^W |_{W_0}$. So we have

$$\pi_{W_2}^W \circ Y_{W_0} \circ (1_V \otimes \pi_{W_1}^W) = \pi_{W_2}^W \circ Y_{W_0} \circ (1_V \otimes \pi_{W_1}^W) |_{V \otimes W_0}.$$ 

Since $\pi_{W_2}^W \circ Y_{W_0} \circ (1_V \otimes \pi_{W_1}^W)$ satisfies Assumption 4.3, $\pi_{W_2}^W \circ Y_{W_0} \circ (1_V \otimes \pi_{W_1}^W) |_{V \otimes W_0}$ satisfies Assumption 4.3 with $W$ in Assumption 4.3 replaced by $W_0$. Thus $\pi_{W_2}^{W_0} \circ Y_{W_0} \circ (1_V \otimes \pi_{W_1}^{W_0})$ satisfies Assumption 4.3 with $W$ replaced by $W_0$.

From Theorem 5.3 and Proposition 5.4 we obtain immediately the following main result of this paper:

Theorem 5.5 Let $V$ be a meromorphic open-string vertex algebra. If $\hat{H}_1(V, M) = \hat{Z}_\infty(V, M)$ for every $\mathbb{Z}$-graded $V$-bimodule $M$, then every left $V$-module satisfying the composability condition is completely reducible. Assume in addition that the following condition also holds: For every left $V$-module $W$ satisfying the composability condition and every nonzero proper left $V$-submodule $W_2$ of $W$, there exists a graded subspace $W_3$ of $W$ such that $W = W_3 \oplus W_2$ as a graded vector space, $\pi_{W_3}^W \circ Y_W \circ (1_V \otimes \pi_{W_2}^W)$ satisfies Assumption 4.3 and the submodule $H^{F(V)}$ of $H^N$ is grading restricted for the grading preserving linear map $F : V \to H^N$ given by Proposition 4.4. Then the conclusion still holds if $H_1(V, M) = \hat{Z}_\infty(V, M)$ only for every grading-restricted $\mathbb{Z}$-graded $V$-bimodule $M$.

Proof. Let $W$ be a left $V$-module satisfying the composability condition and let $W_0$ be the direct sum of all irreducible left $V$-submodule of $W$. We want to show $W_0 = W$. If $W_0 \neq W$, there exists $w \in W$ but $w \notin W_0$. By Zorn’s lemma, we can find a left $V$-submodule $W_2$ of $W$ such that $W_2$ is maximal among left $V$-submodules containing $W_0$ but not $w$. Clearly $W_2 \neq W$. Since $W$ satisfies the composability condition, there is a graded subspace $W_3$ of $W$ such that $W = W_1 \oplus W_2$ as a vector space and $\pi_{W_2}^W \circ Y_W \circ (1_V \otimes \pi_{W_1}^W)$ satisfies Assumption 4.3. By Theorem 5.3 there is a nonzero left $V$-submodule $W_1$ of $W$ such that $W = W_1 \oplus W_2$ as a left $V$-module. Since $W_0 \subset W_2$, $W_1$ cannot be irreducible. Let $W_4$ be a nonzero proper left $V$-submodule of $W_1$. By Proposition 4.4, $W_1$ also satisfies the composability condition. Therefore there is a graded subspace $W_3$ of $W$ such that $W_1 = W_3 \oplus W_4$ as a vector space and $\pi_{W_4}^W \circ Y_{W_1} \circ (1_V \otimes \pi_{W_3}^W)$ satisfies Assumption 4.3. By Theorem 5.3, there is a nonzero proper left $V$-submodule $W_3$ of $W_1$ such that $W_1 = W_3 \oplus W_4$ as a left $V$-module. Thus we have $W = W_3 \oplus W_4 \oplus W_2$.
Since $W_2$ is maximal among left $V$-submodules containing $W_0$ but not $w$ and $\widetilde{W}_3 \oplus W_2$ contains $W_2$, $\widetilde{W}_3 \oplus W_2$ must contain $w$. Similarly, $W_4 \oplus W_2$ must contain $w$. Thus $w \in (\widetilde{W}_3 \oplus W_2) \cap (W_4 \oplus W_2) = W_2$. Contradiction. So $W_0 = W$.

In the case that the additional condition also holds, since $H^F(V)$ is grading restricted, $\hat{H}^1_{\infty}(V, M) = \hat{Z}^1_{\infty}(V, M)$ for every grading-restricted $\mathbb{Z}$-graded $V$-bimodule $M$ implies in particular $\hat{H}^1_{\infty}(V, H^F(V)) = \hat{Z}^1_{\infty}(V, H^F(V))$. Then the conclusion of Theorem 5.3 still holds. Thus $W$ is completely reducible.

\begin{remark}
By Theorems 2.13 and 5.5, we can replace the condition $\hat{H}^1_{\infty}(V, M) = \hat{Z}^1_{\infty}(V, M)$ by the statement that every derivation from $V$ to $M$ is the sum of an inner derivation and a zero-mode derivation.
\end{remark}

We also have the following conjecture:

\begin{conjecture}
Let $V$ be a meromorphic open-string vertex algebra. If every left $V$-module is completely reducible, then $\hat{H}^1_{\infty}(V, M) = \hat{Z}^1_{\infty}(V, M)$ for every $V$-bimodule $M$.
\end{conjecture}

Note that because of Proposition 5.1, we do not need to require that left $V$-modules satisfy the composability condition in this conjecture.

\section{Application to the reductivity of vertex algebras}

In this section, we apply our results in the preceding section to lower-bounded and grading-restricted vertex algebras to obtain results on the complete reducibility of $V$-modules. See Remark 2.2 for our definitions of lower-bounded and grading-restricted vertex algebra.

We prove that when $V$ is a grading-restricted vertex algebra containing a vertex subalgebra $V_0$ satisfying certain natural conditions (see below), then every grading-restricted left $V$-module satisfies the composability condition and thus in this case, the results in the preceding sections holds for all grading-restricted left $V$-modules. We also prove in this section that in this case, $F(V)$ (see the preceding section) is grading restricted and thus in this case, the condition that $\hat{H}^1_{\infty}(V, M) = \hat{Z}^1_{\infty}(V, M)$ for every $\mathbb{Z}$-graded $V$-bimodule $M$ in Theorem 5.3 can be weakened to require that $\hat{H}^1_{\infty}(V, M) = \hat{Z}^1_{\infty}(V, M)$ only for every $\mathbb{Z}$-graded $V$-bimodule $M$ generated by a grading-restricted subspace.

Before we give our applications, we need to clarify the terminology on modules. Note that in Section 2, left modules, right modules and bimodules are graded. When $V$ is a vertex operator algebra (in particular, there are Virasoro operators acting on $V$), a lower-bounded generalized $V$-module $W$ (that is, a weak $V$ module $W$ with a grading given by generalized eigenspaces of $L(0)$ with a lower bound on the weights of the elements of $W$) is a left $V$-module defined in Section 2 when $V$ is viewed as a meromorphic open-string vertex algebra. A grading-restricted generalized $V$-module is a grading-restricted left $V$-module defined in Section 2. But to avoid confusion, we shall use the terminology of this paper to call these generalized $V$-modules left $V$-modules or grading-restricted left $V$-modules, though for a
vertex algebra, the notions of left module and right module are equivalent. The reader should not confuse left $V$-modules in this section with $V$-modules in the literature on vertex (operator) algebras. We shall also use the same terminology for modules for subalgebras of $V$. We also note that a $V$-bimodule when $V$ is viewed as a meromorphic open-string vertex algebra is not the same as a left $V$-module. See the discussion below.

First, Theorem 5.5 in the case that $V$ is lower-bounded vertex algebra becomes:

**Theorem 6.1** Let $V$ be a lower-bounded vertex algebra (in particular, a grading-restricted vertex algebra or a vertex operator algebra). If $\hat{H}_\infty^1(V, M) = \hat{Z}_\infty^1(V, M)$ (or every derivation from $V$ to $M$ is the sum of an inner derivation and a zero-mode derivation) for every $\mathbb{Z}$-graded $V$-bimodule $M$, then every left $V$-module satisfying the composability condition is completely reducible.

**Remark 6.2** Note that when $V$ is a grading-restricted vertex algebra (in particular, a vertex operator algebra) and $M$ is a left $V$-module, we have the chain complexes $C_n^\infty(V, M)$ and the cohomologies $H_n^\infty(V, M)$ introduced in [17]. In the case $n = 1$, if we view $M$ as a $V$-bimodule with the right $V$-module structure obtained from the left $V$-module structure using the skew-symmetry formula, then $C_1^\infty(V, M) = \hat{C}_1^\infty(V, M)$ and $H_1^\infty(V, M) = \hat{H}_1^\infty(V, M)$. But in Theorem 6.1, the assumption is that $\hat{H}_\infty^1(V, M) = \hat{Z}_\infty^1(V, M)$ for all $V$-bimodules $M$, including those $V$-bimodules for which the right actions are not obtained from the left actions. For a $V$-bimodule $M$ when $V$ is viewed as a meromorphic open-string vertex algebra, $M$ is in fact a $\mathbb{C}$-graded space $M$ equipped with two commuting $V$-module structures $Y_M^1$ and $Y_M^2$ (commuting in the sense of the commutativity) when $V$ is viewed as a grading-restricted vertex algebra such that the operator $D_M^1$ and $D_M^2$ for the two $V$-module structures are equal (denoted simply by $D_M$). Then an inner derivation from $V$ to $M$ is a linear map $f_w$ from $V$ to $W$ defined by

$$f_w(v) = \lim_{z \to 0} (Y_M^1(v, z)w - Y_M^2(v, z)w) = (Y_M^1)_{-1}(v)w - (Y_M^2)_{-1}(v)w$$

for $v \in V$, where $w \in W$ is of weight 0 and satisfies $D_Mw = 0$. On the other hand, a zero-mode derivation is a linear map $g_w$ from $V$ to $W$ defined by

$$g_w(v) = \text{Res}_z e^{zD_M} Y_M^2(v, -z)w = \sum_{n \in \mathbb{N}} \frac{(-1)^{n-1}}{n!} D_M^n(Y_M^2)_{n}(v)w$$

for $v \in V$, where $w \in W$ is of weight 1 and satisfies $Y_M^1(v, z)w - Y_M^2(v, z)w \in M[z, z^{-1}]$. The condition $\hat{H}_\infty^1(V, M) = \hat{Z}_\infty^1(V, M)$ is now equivalent to the statement that every derivation $f$ from $V$ to $M$ is of the sum of derivations of the forms $f_w$ and $g_w$ as above (certainly with different $w$’s).

The remark above provides a method to determine whether $\hat{H}_\infty^1(V, M) = \hat{Z}_\infty^1(V, M)$ when $V$ is a grading-restricted vertex algebra (in particular, a vertex operator algebra). But this method is practical only if we know all bimodules, not just those modules for which the
Assume the following conditions holds for intertwining operators among grading-restricted left $V$-modules: 

1. For any $n \in \mathbb{Z}_+$, products of $n$ intertwining operators among grading-restricted left $V_0$ modules evaluated at $z_1, \ldots, z_n$ are absolutely convergent in the region $|z_1| > \cdots > |z_n| > 0$ and can be analytically extended to (possibly multivalued) analytic functions.
in $z_1, \ldots, z_n$ with the only possible singularities (branch points or poles) $z_i = 0$ for $i = 1, \ldots, n$ and $z_i = z_j$ for $i, j = 1, \ldots, n$, $i \neq j$.

2. The associativity of intertwining operators among grading-restricted left $V_0$ modules holds.

Let $W$ be a grading-restricted left $V$-module and $W_2$ a left $V$-submodule of $W$. Then for any left $V_0$-submodule $W_1$ of $W$ such that $W = W_1 \oplus W_2$ as a left $V_0$-module, Assumption 4.3 holds for the map $\pi_{W_2} \circ Y_W \circ (1_V \otimes \pi_{W_1})$.

Proof. Since $Y_{W_1}$ and $Y_{W_2}$ are vertex operator maps for left $V$-modules, they are also intertwining operators among left $V_0$-modules. By Proposition 6.3, $\pi_{W_2} \circ Y_W \circ (1_V \otimes \pi_{W_1})$ is also an intertwining operator among left $V_0$-modules.

Then by Condition 1, (4.5) for $v, v_1, \ldots, v_{k+l} \in V$, $w'_2 \in W'_2$ and $w_1 \in W_1$ are absolutely convergent in the region $|z_1| > \cdots > |z_k| > |z| > |z_{k+1}| > \cdots > |z_{k+l}| > 0$ and can be analytically extended to a (possibly multivalued) analytic function with the only possible singularities (branch points or poles) $z_i = 0$ for $i = 1, \ldots, k + l$ and $z_i = z_j$ for $i, j = 1, \ldots, k + l$, $i \neq j$. The proof of Theorem 2.2 in [HLZ2] in fact shows that the commutativity of intertwining operators among grading-restricted left $V_0$-modules also holds. Since the intertwining operators $Y_{W_1}$, $Y_{W_2}$ and $\pi_{W_2} \circ Y_W \circ (1_V \otimes \pi_{W_1})$ involve only integral powers of the variables, using the associativity and commutativity of intertwining operators among grading-restricted left $V_0$-modules, we see that the singularities $z_i = 0$ for $i = 1, \ldots, k + l$ and $z_i = z_j$ for $i, j = 1, \ldots, k + l$, $i \neq j$ must be poles. Thus (4.5) is absolutely convergent in the region $|z_1| > \cdots > |z_k| > |z| > |z_{k+1}| > \cdots > |z_{k+l}| > 0$ to a rational function with the only possible poles $z_i = 0$ for $i = 1, \ldots, k + l$ and $z_i = z_j$ for $i, j = 1, \ldots, k + l$, $i \neq j$. So the first part of Assumption 4.3 for $\pi_{W_2} \circ Y_W \circ (1_V \otimes \pi_{W_1})$ holds. The second part of Assumption 4.3 holds for $\pi_{W_2} \circ Y_W \circ (1_V \otimes \pi_{W_1})$ also because of the associativity and commutativity of intertwining operators among grading-restricted left $V_0$-modules. 

We also have the following result:

Theorem 6.5 Let $W$ be a grading-restricted left $V$-module, $W_2$ a left $V$-submodule of $W$ and $W_1$ a left $V_0$-submodule of $W$ such that $W = W_1 \oplus W_2$ as a left $V_0$-module. Assume that for some $z_0 \in \mathbb{C}^\times$, the $Q(z_0)$-tensor product $W_2 \otimes_{Q(z_0)} W_1$ of left $V_0$-modules $W_2'$ and $W_1$ is still a grading-restricted generalized left $V_0$-module. Then for the map $F : V \to H^N$ defined in Proposition 4.4, the graded vector space $F(V)$ is grading restricted and, in particular, the $V$-bimodule $H^N(V)$ is generated by the grading-restricted submodule $F(V)$.

Proof. We consider the category $\mathcal{C}$ of grading-restricted left $V_0$-modules. By Proposition 5.69 in [HLZ2], since $W_2' \otimes_{Q(z_0)} W_1$ is still in $\mathcal{C}$, $W_2 \otimes_{Q(z_0)} W_1$ is also in $\mathcal{C}$. In particular, $W_2 \otimes_{Q(z_0)} W_1$ is grading restricted.

For $p \in \mathbb{Z}$, by Remark 5.62 and Proposition 5.63 in [HLZ2],

$$(F_{\pi_{W_2} \circ Y_W \circ (1_V \otimes \pi_{W_1})})^p(V) \subset W_2' \otimes_{Q(z_0)} W_1.$$
Since $W_{2}S_{Q_{0}(w_{0})}W_{1}$ is grading restricted, $\left(I_{Q_{0}(w_{0})}\right)_{W_{2}Y_{V\circ(1_{V}\otimes\pi_{W_{1}})}p}^{{\prime}}(V)$ is also grading restricted. On the other hand, for $v \in V$, $w_{2}' \in W_{2}'$ and $w_{1} \in W_{1}$, 
\[
((I_{Q_{0}(w_{0})}\pi_{W_{2}Y_{V\circ(1_{V}\otimes\pi_{W_{1}})}p})'(v))(w_{2}' \otimes w_{1}) \\
= \langle w_{2}', \pi_{W_{2}}Y_{V}(v, z_{0})\pi_{W_{1}}w_{1} \rangle \\
= \langle w_{2}', ((F(v))(w_{1}))(z_{0}) \rangle
\]
By the $D$-derivative property for $Y_{V}$, $((F(v))(w_{1}))(z) = ((F(\epsilon(z-z_{0})D_{V}v))(w_{1}))(z_{0})$, that is, $F(v)$ is determined completely by its evaluation at $z_{0}$. So we obtain a linear map from $(I_{Q_{0}(w_{0})}\pi_{W_{2}Y_{V\circ(1_{V}\otimes\pi_{W_{1}})}p})'(V)$ to $H^{F(V)}$ given by 
\[
(((I_{Q_{0}(w_{0})}\pi_{W_{2}Y_{V\circ(1_{V}\otimes\pi_{W_{1}})}p})'(v)) \mapsto F(v).
\]
It is clear from the definition that this map is a linear isomorphism and preserves weights. Since $(I_{Q_{0}(w_{0})}\pi_{W_{2}Y_{V\circ(1_{V}\otimes\pi_{W_{1}})}p})'(V)$ is grading restricted, the space $F(V)$ is also grading restricted. 

**Remark 6.6** Since $W_{1}$ is a $V_{0}$-submodule of $W$, it is invariant under the action of $D_{W}$. In particular, $D_{W}W_{1}$ is equal to the restriction of $D_{W}$ on $W_{1}$. From this fact, the definition of $F(v)$ for $v \in V$ and the $D$-derivative property of the vertex operator $Y_{V}(v, z)$, we see that $F(v)$ satisfies the the $D$-derivative property 
\[
\frac{d}{dz}((F(v))(w_{1}))(z) = D_{W}((F(v))(w_{1}))(z) - ((F(v))(D_{W}w_{1}))(z).
\]
Thus elements of $F(V)$ satisfy the $D$-derivative property.

From Theorems 6.4, 6.5 and 6.6 we obtain immediately our main theorem in this section:

**Theorem 6.7** Let $V$ be a grading-restricted vertex algebra and $V_{0}$ a vertex subalgebra of $V$. Assuming that the two conditions in Theorem 6.4 hold. Also assume that every grading-restricted left $V_{0}$-module is completely reducible and the $Q(z)$-tensor product of any two grading-restricted left $V_{0}$-modules is still a grading-restricted left $V_{0}$-module. Then $H^{1}_{\infty}(V, M) = \hat{Z}_{\infty}'(V, M)$ for every $\mathbb{Z}$-graded $V$-bimodule $M$ generated by a grading-restricted subspace implies the complete reducibility of every grading-restricted left $V$-module.

The two conditions in Theorem 6.4 and the condition in Theorem 6.5 indeed hold when $V_{0}$ and $V_{0'}$-modules satisfy some algebraic conditions. We discuss these conditions in the following remark:

**Remark 6.8** A left $V_{0}$-module $W$ is $C_{1}$-cofinite if $\dim W/C_{1}(W) < \infty$ where $C_{1}(W)$ is the subspace of $W$ spanned by elements of the form $(Y_{V})_{-1}(v)w$ for $v \in V_{0}$ and $w \in W$. Note that in [H2], a generalized module for a vertex (operator) algebra means a weak module.
graded by the eigenvalues of $L(0)$ but the grading does not have to be lower bounded and the homogeneous subspaces do not have to be finite dimensional. Here we call such a generalized module an $L(0)$-semisimple left weak module. By Theorems 3.5 and 3.7 in [H2], the assumptions in Theorems 6.4 and 6.5 hold when the following three conditions on (generalized) $V_0$-modules hold:

1. Every $L(0)$-semisimple left weak $V_0$-module is a direct sum of irreducible grading-restricted left $V_0$-modules.

2. There are only finitely many inequivalent irreducible grading-restricted left $V_0$-modules and they are all $\mathbb{R}$-graded.

3. Every irreducible grading-restricted left $V_0$-module satisfies the $C_1$-cofiniteness condition.

The vertex algebra $V_0$ is $C_2$-cofinite if $\dim V_0/C_2(V_0) < \infty$ where $C_2(V_0)$ is the subspace of $V_0$ spanned by elements of the form $(Y_{V_0})_{-2}(u)v$ for $u, v \in V_0$. By Theorem 3.9 in [H2], the three conditions above hold when the following three conditions holds:

1. For $n < 0$, $(V_0)_{(n)} = 0$ and $(V_0)_{(0)} = \mathbb{C}1$.

2. Every grading-restricted left $V_0$-module is completely reducible.

3. $V_0$ is $C_2$-cofinite.

By Proposition 12.5 in [HL4], Corollary 9.30 in [HLZ3], Theorem 11.8 in [HLZ3] and Theorem 3.1 in [H10], the assumptions in Theorems 6.4 and 6.5 also hold when the following three conditions hold:

1. Every grading-restricted left $V_0$-module is completely reducible.

2. For any left $V_0$-modules $W_1$ and $W_2$ and any $z \in \mathbb{C}^\times$, if the weak $V$-module $W_\lambda$ generated by a generalized eigenvector $\lambda \in (W_1 \otimes W_2)^*$ for $L_{P(z)}(0)$ satisfying the $P(z)$-compatibility condition is lower bounded, then $W_\lambda$ is a grading-restricted left $V_0$-module.

3. Every irreducible grading-restricted left $V_0$-module satisfies the $C_1$-cofiniteness condition.

Thus the conclusions of Theorems 6.4 and 6.5 hold when one of these three sets of three conditions holds.

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