INDEX THEORY FOR TRAVELING WAVES IN REACTION DIFFUSION SYSTEMS WITH SKEW GRADIENT STRUCTURE

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Abstract. A unified geometric approach for the stability analysis of traveling pulse solutions for reaction diffusion equations with skew-gradient structure has been established in a previous paper [9], but essentially no results have been found in the case of traveling front solutions. In this work, we will bridge this gap. For such cases, a Maslov index of the traveling wave is well-defined, and we will show how it can be used to provide the spectral information of the waves. As an application, we use the same index providing the exact number of unstable eigenvalues of the traveling front solutions of FitzHugh-Nagumo equations.

Keywords: heteroclinic orbits; stability of traveling wave; Maslov index; spectral flow; FitzHugh–Nagumo equations

1. Introduction and main results

The setting of this paper is the systems of reaction-diffusion equations of the form

$$w_t = w_{xx} + Q \nabla F(w), \tag{1.1}$$

where \(x, t \in \mathbb{R}\) are space and time, respectively, \(w \in \mathbb{R}^n\) and \(\nabla F\) are the gradients of the function \(F : \mathbb{R}^n \to \mathbb{R}\), and \(Q \in \text{Mat}(n, \mathbb{R})\) has the form \(Q = \begin{bmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{bmatrix}\), where \(\text{Mat}(n, \mathbb{R})\) is the set of all \(n \times n\) matrices. Such a system is of the activator-inhibitor type and will be referred to as a skew-gradient system [18]. A traveling wave solution \(w^*\) of one variable \(\xi = x - ct\) is a solution to (1.1) with asymptotic behavior

$$w_\pm \text{ as } \xi \to \pm \infty.$$ 

Here, \(w_\pm\) are the constant equilibria of (1.1); that is, \(\nabla F(w_\pm) = 0\). If \(w_+ = w_-\), then \(w^*\) is called a pulse (and a front otherwise). In this paper, we are focused primarily on fronts. We will always consider the case of \(c > 0\): otherwise, the direction of motion can be reversed.

Traveling waves being solutions of (1.1) is an important subject in dynamical systems, and many aspects of traveling waves have been studied in extensive works[11, 6]. One of the key issues of concern is whether a steady state is stable with respect to disturbances under initial conditions since this directly determines whether it can be observed in nature. Motivated by [9], in this paper, we establish a unified geometric approach for the stability analysis of traveling front solutions for (1.1).

Written in a moving frame, a traveling front solution \(w^*\) of (1.1) can be regarded as a heteroclinic solution \(w^*\) of the following equation:

\[
\begin{aligned}
\begin{cases}
w_{\xi\xi} + cw_\xi + Q \nabla F(w) &= 0, \\
\lim_{\xi \to \pm \infty} w(\xi) &= w_\pm.
\end{cases}
\end{aligned}
\tag{1.2}
\]

The stability analysis is directly related to the spectral information of the operator \(L := \frac{d^2}{d\xi^2} + c \frac{d}{d\xi} + QB(\xi)\) by linearizing (1.2) along \(w^*\), where \(B(\xi) = \nabla^2 F(w^*)\); note that the matrices \(B_\pm :=
\]
\[
\lim_{\xi \to \pm \infty} B(\xi) \text{ are well-defined. Moreover, the previous description guarantees that there exists } C >, \text{ such that }
\]
\[
\langle QB(\xi)v, v \rangle \leq C|v|^2 \text{ for all } (\xi, v) \in \mathbb{R} \times \mathbb{R}^n. \tag{1.3}
\]
For \( z \in \mathbb{C} \), denote by \( \mathbb{R}_L \) and \( \mathbb{R}_I \) the real and imaginary parts of \( z \), respectively, \( \mathbb{R}^+ := (0, +\infty), \mathbb{R}^- := (-\infty, 0), \mathbb{C}^+ := \{ z \in \mathbb{C} | \mathbb{R}_L > 0 \} \) and \( \mathbb{C}^- := \{ z \in \mathbb{C} | \mathbb{R}_L < 0 \} \). The pair \((\mathbb{R}^n, \langle , \rangle)\) denotes the \( n \)-dimensional Euclidean space. Moreover, \# denotes the closure of the set \#.

Since a wave solution of (1.1) possesses a translation invariance property, it is said to be nondegenerate if zero is a simple eigenvalue of \( L \).

**Definition 1.1.** A nondegenerate wave solution of (1.1) is spectrally stable if all the nonzero eigenvalues of \( L \) are in \( \mathbb{C}^- \).

In this paper, we study the following special case.

(H1). Suppose that \( \sigma(QB_{\pm}) \subset \mathbb{C}^- \).

Thus, \( w_+ \) and \( w_- \) are both stable equilibria of (1.1).

From now on, we focus on studying the eigenvalue problem
\[
L \phi = \lambda \phi. \tag{1.4}
\]
Denoted by \( \sigma_p(L) \) is the set of isolated eigenvalues with finite multiplicity, and \( \sigma_{ess}(L) = \sigma(L) \setminus \sigma_p(L) \) is the essential spectrum of \( L \). It is known (Cf. Lemma 2.3) that \( \sigma_{ess}(L) \subset \mathbb{C}^- \). As was commonly performed in [9], we set \( y = \begin{bmatrix} \phi \\ \dot{\phi} \end{bmatrix} \) to convert (1.4) to
\[
\dot{y} = A_\lambda(\xi)y, \tag{1.5}
\]
where \( A_\lambda(\xi) = \begin{bmatrix} -c & \lambda I - QB(\xi) \\ I & 0 \end{bmatrix} \), and there are well-defined matrices \( A_\lambda(\pm \infty) = \lim_{\xi \to \pm \infty} A_\lambda(\xi) \).

Throughout this paper, the dots denote differentiation with respect to \( \xi \).

For any \( M \in \text{Mat}(\mathbb{R}, \mathbb{R}^n) \), denote by \( V^+(M) \) (\( V^-(M) \)) the positive (negative) spectral space corresponding to the eigenvalues of \( M \) having positive (negative) real parts. We provide the next two conditions:

(H2). \( \langle QB_{\pm}v, v \rangle < 0 \), for all \( v \in V^-(Q) \setminus \{0\} \),

(H2'). \( \langle QB_{\pm}v, v \rangle < 0 \), for all \( v \in \mathbb{R}^n \setminus \{0\} \).

Letting \( B \in \text{Mat}(n, \mathbb{R}) \), in the following remark, we will explain the relationship between (H1) and (H2').

**Remark 1.2.** (1) We claim that (H2') implies that (H1). Let \( \lambda \) be the eigenvalue of \( QB \) with eigenvector \( v \). If \( \lambda \in \mathbb{R} \), then we have that \( |\langle v, v \rangle| = |\langle QBv, v \rangle| < 0 \), which implies that \( \lambda < 0 \). If \( \lambda \in \mathbb{C} \setminus \mathbb{R} \), then \( \bar{\lambda} \) is also the eigenvalue of \( QB \) with eigenvector \( \bar{v} \), where \( \lambda \) and \( \bar{v} \) denote the complex conjugate of \( \lambda \) and \( v \), respectively. Now, let \( \lambda = a + bi \) and \( v = u + iw \): then, we have that
\[
\lambda(|u|^2 + |w|^2) = \lambda \langle v, v \rangle = \langle QBv, v \rangle = \langle QBu, u \rangle - i \langle QBu, w \rangle + i \langle QBw, u \rangle + \langle QBw, w \rangle.
\]
Similarly, we have that
\[
\bar{\lambda}(|u|^2 + |w|^2) = \langle QBu, u \rangle + i \langle QBu, w \rangle - i \langle QBw, u \rangle + \langle QBw, w \rangle.
\]
We thus determine that \( a = \frac{1}{2}(\lambda + \bar{\lambda}) = \frac{\langle QBu, u \rangle + \langle QBw, w \rangle}{|u|^2 + |w|^2} < 0 \). This implies that \( \lambda \) has a negative real part.

(2) We remark that (H2') cannot be derived from (H1). We consider \( B \in \text{Mat}(2, \mathbb{R}) \) with the form \( B = \begin{bmatrix} 1 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 1 \end{bmatrix} \) and \( Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \). It is easy to determine that \( \sigma(QB) \subset \mathbb{C}^- \), but
\[
\langle QB \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle = 1 > 0.
\]
(3) Supposing that \( \sigma(QB) \subset \mathbb{C}^- \), we claim that there exists a nonsingular matrix \( T \in \text{Mat}(n, \mathbb{R}) \), such that
\[
\langle T^{-1}QBv, v \rangle < 0, \forall v \in \mathbb{R}^n \setminus \{0\}.
\]
We only consider a special case, where the general case is analogous. Letting $\mathcal{E} \in \text{Mat}(6, \mathbb{R})$, suppose that $\lambda_1 \in \mathbb{R}$ and $\lambda_2 = a + ib$ are generalized eigenvalues of $\mathcal{E}$ and have the same algebraic multiplicity 2: then, for $\epsilon < -\max\{\lambda_1, a\}$, there exists $T_\epsilon$, such that

$$T_\epsilon^{-1} \mathcal{E} T_\epsilon = \begin{bmatrix}
\lambda_1 & \epsilon & 0 & 0 & 0 & 0 \\
0 & \lambda_1 & 0 & 0 & 0 & 0 \\
0 & 0 & a & -b & \epsilon & 0 \\
0 & 0 & b & a & 0 & \epsilon \\
0 & 0 & 0 & 0 & a & -b \\
0 & 0 & 0 & 0 & b & a
\end{bmatrix}.$$ 

By a simple calculation, we have that

$$\langle T_\epsilon^{-1} \mathcal{E} T_\epsilon, v \rangle, v \rangle = \lambda_1 v_1^2 + \lambda_2 v_2^2 + \epsilon v_1 v_2 + av_3^2 + av_4^2 + av_5^2 + av_6^2 + \epsilon v_3 v_5 + \epsilon v_4 v_6$$

$$\leq 2 \sum_{i=1}^3 (\lambda_1 + \frac{1}{2}\epsilon)v_i^2 + \sum_{j=3}^6 (a + \frac{1}{2}\epsilon)v_j^2 < 0.$$ 

Following Definition 1.1, we focus our attention on $\lambda \in \mathbb{R}$: suppose that $\Phi_{\tau, \lambda}(\xi)$ is the matrix solution of (1.5), such that $\Phi_{\tau, \lambda}(\tau) = I$. We recall that the stable and unstable subspaces of (1.5) are

$$E^s_\lambda(\tau) := \left\{ v \in \mathbb{R}^2 \mid \lim_{\tau \to +\infty} \Phi_{\tau, \lambda}(\xi)v = 0 \right\}$$

and

$$E^u_\lambda(\tau) := \left\{ v \in \mathbb{R}^2 \mid \lim_{\tau \to -\infty} \Phi_{\tau, \lambda}(\xi)v = 0 \right\}.$$ 

Throughout this paper, we abbreviate $E^s_\lambda(\tau)$ and $E^u_\lambda(\tau)$ as $E^s(\tau)$ and $E^u(\tau)$, respectively.

To realize the symplectic structure, we introduce the matrix $J := \begin{bmatrix} 0 & -Q \\ Q & 0 \end{bmatrix}$. Since $Q^2 = I$ and $Q^T = Q$, it follows that $J^2 = -I$ and $J^T = -J$. Therefore, $J$ is a complex structure on $\mathbb{R}^2$, and $\omega(\cdot, \cdot) := (J \cdot, \cdot)$ defines a symplectic form on $\mathbb{R}^2$. $\Lambda(n)$ denotes the set of all Lagrangian subspaces of $(\mathbb{R}^2, \omega)$.

By invoking [13, Lemma 3.1], Remarks 2.4 and 2.6 imply that the subspaces $E^s_\lambda(\tau)$ and $E^u_\lambda(\tau)$ are both Lagrangian for $(\tau, \lambda) \in \mathbb{R} \times \mathbb{R}$. Moreover, those results can be obtained from the same discussion in [9, Theorem 2.3].

We denote by $\mathcal{P}([0, 1]; \mathbb{R}^2)$ the space of all ordered pairs of continuous maps of Lagrangian subspaces $L : [0, 1] \ni t \mapsto L(t) := (L_1(t), L_2(t)) \in \Lambda(n) \times \Lambda(n)$ equipped with the compact-open topology. Following the authors in [4], we are in the position to briefly recall the definition of the Maslov index for pairs of Lagrangian subspaces, which will be denoted throughout the paper by the symbol $\iota_{\text{CLM}}$. Loosely speaking, given the pair $L = (L_1, L_2) \in \mathcal{P}([0, 1]; \mathbb{R}^2)$, this index counts with signatures and multiplicities the number of instants $t \in [0, 1]$ that $L_1(t) \cap L_2(t) \neq \emptyset$.

**Definition 1.3.** The **CLM-index** is the unique integer valued function

$$\iota_{\text{CLM}} : \mathcal{P}([0, 1]; \mathbb{R}^2) \ni L \mapsto \iota_{\text{CLM}}(L; [0, 1]) \in \mathbb{Z}$$

satisfying Properties I-VI given in [4, Section 1].

For the sake of the reader, we list a couple of properties of the $\iota_{\text{CLM}}$-index that we shall use throughout the paper.

- **(Reversal)** Let $L := (L_1, L_2) \in \mathcal{P}([a, b]; \mathbb{R}^2)$. Denoting by $\hat{L} \in \mathcal{P}([-b, -a]; \mathbb{R}^2)$ the path traveled in the opposite direction, and by setting $\hat{L} := (L_1(-s), L_2(-s))$, we obtain

$$\iota_{\text{CLM}}(\hat{L}; [-b, -a]) = -\iota_{\text{CLM}}(L; [a, b]).$$

- **(Stratum homotopy relative to the ends)** Given a continuous map

$$L : [a, b] \ni s \mapsto L(s) \in \mathcal{P}([a, b]; \mathbb{R}^2)$$

where $L(s)(t) := (L_1(s, t), L_2(s, t))$ such that $\dim L_1(s, a) \cap L_2(s, a)$ and $\dim L_1(s, b) \cap L_2(s, b)$ are both constant, and then,

$$\iota_{\text{CLM}}(L; [a, b]) = \iota_{\text{CLM}}(L(1); [a, b]).$$

Moreover, one efficient way to study the Maslov index is via the crossing form introduced by [17] as follows.

Let $L(t) : [0, 1] \to \Lambda(n)$ be a smooth curve with $L(0) = L_0$. Let $W$ be a fixed Lagrangian complement of $L(t)$. For $v \in L_0$ and small $t$, define $w(t) \in W$ by $v + w(t) \in L(t)$. The form
Q(v) = \frac{d}{dt}\bigg|_{t=0} \omega(v, w(t)) is independent of the choice of \(W\). A crossing for \(L(t)\) is some \(t\) for which \(L(t)\) intersects \(V \in \text{Lag}(n)\) nontrivially. At each crossing, the crossing form is defined as

\[ \Gamma(L(t), V; t) = Q|_{L(t)\cap V}. \]

A crossing is called regular if the crossing form is nondegenerate. For a quadratic form \(Q\), we use the notation \(\text{sign}(Q)\) for its signature. We also write \(m^+(Q)\) and \(m^-(Q)\) for the positive and negative indices of inertia of \(Q\), so that

\[ \text{sign}(Q) = m^+(Q) - m^-(Q). \]

From [20], if the path \(L(t)\) has only regular crossing with respect to \(V\), then we have that

\[ \text{c}_{\text{CLM}}(V, L(t), t \in [a, b]) = m^+(\Gamma(L(a), V; a)) + \sum_{a < t < b} \text{sign}(L(t), V; t) - m^-(\Gamma(L(b), V; b)). (1.6) \]

We recall the definition of Maslov index for a traveling pulse wave \(w^*\) of (1.1) defined in [9]. The only requirement is that \(\tau_0\) is large enough so that

\[ E^u(\tau_0) \cap E^s(\tau) = \{0\} \quad \text{for all} \quad \tau \geq \tau_0. \] (1.7)

Moreover, since the transformation used in the conversion of (1.4) to (1.5) is different from that in [9], then the crossing form defined in the previous paper [9] differs from ours by a negative sign, and so, we provide [9, Definition 3.6] in the following form:

**Definition 1.4.** [9, Definition 3.6] Let \(w^*\) be a traveling pulse solution of (1.1), where the Maslov index of \(w^*\) is given by

\[ \text{Maslov}(w^*) := \sum_{\tau < \tau_0} \text{sign}(E^s(\tau), E^u(\tau_0); \tau) - m^-(\Gamma(E^s(\tau), E^u(\tau_0); \tau_0)). \]

By using the Maslov index, this paper [9] provides a unified geometric treatment for the stability analysis of the traveling pulses solution and has been successfully employed in [11, 10] to obtain some elegant stability results. Motivated by [9], it is natural to consider the case of the traveling front solutions. We remark that Definition 1.4 is dependent on the condition (1.7). It is easy to check that this condition may be inefficient for the traveling front solutions, and we sidestep this difficulty by following [13, Definition 1.2] and define the Maslov index for the traveling waves as following.

**Definition 1.5.** Let \(w^*\) be a traveling wave of (1.1), and define the Maslov index of \(w^*\) as

\[ \iota(w^*) = - \text{c}_{\text{CLM}}(E^s(\tau), E^u(-\tau); \tau \in \mathbb{R}^+). \]

**Remark 1.6.** If Definition 1.4 is well-defined, then it should be independent of the choice of \(\tau_0\). It was shown in [7] that this definition is independent of \(\tau_0\), as long as (1.7) is satisfied. However, for the traveling front solution, there may be an intersection point between \(E^s(\tau)\) and \(E^u(-\tau)\) at \(+\infty\). Thus, (1.7) may be inefficient. We sidestep this difficulty by following [13, Definition 1.2], and the following proposition shows that Definition 1.5 generalizes Definition 1.4 to the front case.

**Proposition 1.7.** Letting \(w^*\) be a traveling pulse solution of (1.1) and supposing that (H1) holds, then there exists \(\tau_0 > 0\), such that \(E^u(-\infty) \cap E^s(+\infty) = \{0\}\) for all \(\tau \geq \tau_0\) and

\[ \iota(w^*) = \text{Maslov}(w^*). \]

Lemmas 2.10 and 2.5 show that the set of nonnegative, real eigenvalues of \(L\) is bounded above, and then, the spectrum of \(L\) in \(\mathbb{R}^+\) consists of isolated eigenvalues of finite multiplicity (Cf. page 172 of [2]), and so, it follows that the quantities

- \(N_r(L) := \) the number of real, positive eigenvalues of \(L\) counting algebraic multiplicity
- \(\overline{N}_r(L) := \) the number of real, nonnegative eigenvalues of \(L\) counting algebraic multiplicity

are both well-defined.

Now, we introduce a symplectic invariant called triple index (Cf. Definition 4.1) which can be used to compute the Maslov index, and we denote by \(\iota(L_1, L_2; L_3)\) the triple index for any \(L_1, L_2, L_3 \in \text{Lag}(n)\).
Theorem 1.8. If (H1) and (H2) hold, then
\[ |\iota(w^*)| + \iota(E^u(-\infty), E^s(+\infty); L_R) | \leq N_0(L), \text{ where } L_R = \left\{ \begin{bmatrix} p \\ q \end{bmatrix} | p \in V^+(Q) \text{ and } q \in V^-(Q) \right\}. \]

Remark 1.9. Under the conditions (H1) and (H2), if (1.7) holds, then Definition 1.4 is well-defined for the traveling front solution. Using the notation of Maslov box in [9, Figure 4.1], by the same discussion in the proof of Proposition 1.7, it is easy to prove that the Maslov index contribution along the “bottom shelf” \( \alpha_4 \) is equal to \( \iota(E^u(-\infty), E^s(+\infty); L_R) \). Based on this discussion and Theorem 1.8, the distinction between the pulse and front is nontrivial, and we generalize [9, Theorem 4.1] to the front case.

Now, we present the central result of this paper.

Theorem 1.10. If (H2') holds, then we have that
\[ |\iota(w^*)| \leq N_0(L). \]

Remark 1.11. If (H2') holds, by Lemmas 3.4 and 3.5, we have that \( E^s(+\infty) \cap E^u(-\infty) = \{0\} \), and this implies that if (1.7) holds, then Definition 1.4 is well-defined. By invoking (3.4) and Lemma 3.6, (H2') provides a sufficient condition for the Maslov index contribution along the “bottom shelf” when \( \alpha_4 \) is equal to 0. This can be easily verified for \( \lambda = 0 \), and it remains true for \( \lambda > 0 \). This is important for practitioners who want to use the Maslov index to prove stability. As shown in [11], an employed strategy can be used to compute Maslov\((w^*)\) for the doubly-diffusive FitzHugh-Nagumo system. Based on those, Definition 1.4 is more practical than Definition 1.5 if one wants to compute the Maslov index for this case. Moreover, from the same discussion in the proof of Proposition 1.7, under (H2'), Definitions 1.4 and 1.5 are equivalent. The strategy employed in [11] may be valid for computing \( \iota(w^*) \) for the doubly diffusive FitzHugh-Nagumo system, such as to consider a traveling front solution established in [11, Theorem 2.2].

For the following FitzHugh-Nagumo equations
\[
\begin{align*}
\begin{cases}
u_t & = u_{xx} + \frac{1}{2}(f(u) - v), \\
v_t & = v_{xx} + u - \gamma v,
\end{cases}
\end{align*}
\]
(1.8)
where \( d, \gamma > 0 \) and \( f(u) = u(1 - u)(u - a) \) with \( 0 < a < \frac{1}{2} \). If \( \gamma \) is large enough, then the equation \( u = \gamma f(u) \) has three solutions \( 0 = u_1 < u_2 < u_3 \), and we remark that it is easy to check that \( f'(0) < 0 \) and \( f'(u_3) < 0 \). Now, we consider a traveling front solution \( w^* \) of the FitzHugh-Nagumo equation (1.8): its existence has been established in [6], and we remark that such waves are obtained as local minimizers of an energy functional, and based on the variational characterization, authors [5] utilize the spectral flow to define and calculate a stability index. As a supplement, we provide a geometric insight into the stability of the traveling front solution for the FitzHugh-Nagumo equation (1.8).

Making a comparison with (1.1), then, (1.8) can be expressed as
\[ w_t = w_{xx} + QD\nabla F(w), \]
where \( D = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \) and \( \nabla F(w) = \begin{bmatrix} f(u) - v \\ \gamma v - u \end{bmatrix} \). The eigenvalue problem
\[ \ddot{\phi} + c\dot{\phi} + QD\nabla^2 F(w^*)\phi = \lambda\phi \]
or its equivalent eigenvalue problem
\[ L\phi = \lambda\phi \]
is studied to determine the stability of \( w^* \), where \( L = \frac{d^2}{dx^2} + c\frac{d}{dx} + QB(\xi) \) and \( B(\xi) = D\ddot{\xi} \nabla^2 F(w^*)D\ddot{\xi} \).

Theorem 1.12. Letting \( w^* \) be a traveling front solution of the FitzHugh-Nagumo equation (1.8) satisfying the following asymptotic condition:
\[ \lim_{\xi \to -\infty} w^* = (u_3, \frac{u_3}{\gamma}), \quad \lim_{\xi \to +\infty} w^* = (0, 0) \]
and \( d > \gamma^{-2} \), then we have that
\[ N_+(L) = \iota(w^*). \]
2. Preliminary: an index formula

As is commonly done for the Sturm-Liouville operators \([7, 14]\), in this section, by applying \([13, \text{Theorem 1}]\), we derive an index formula (Cf. Proposition 2.12) for the Hamiltonian system (2.3), which plays a crucial role in obtaining the spectral information of \(L\).

The notation of \textit{spectral flow} was introduced by Atiyah, Patodi and Singer in their study of index theory on manifolds with a boundary [3]. For the reader's convenience, we provide a brief description of the basic properties of spectral flow. Suppose that \(E\) is a real separable Hilbert space, and denote by \(\mathcal{C}(E)\) the space of all closed self-adjoint and Fredholm operators equipped with the gap topology. Let \(A : [0, 1] \to \mathcal{C}(E)\) be a continuous curve. The \(\text{Sf}(A_t; t \in [0, 1])\) counts the algebraic multiplicities of the spectral flow of \(A_t\) across the line \(t = -\epsilon\) with some small positive number \(\epsilon\).

For each \(A_t\), there is an orthogonal splitting
\[
E = E_-(A_t) \oplus E_0(A_t) \oplus E_+(A_t),
\]
where \(E_0\) is the null space of \(A_t\), \(\langle Av, v \rangle \geq 0\) if \(v \in E_+\) and \(\langle Av, v \rangle \leq 0\) if \(v \in E_-\). There is an efficient way to compute the spectral flow through what are called crossing forms. Let \(P_t\) be the orthogonal projector from \(E\) to \(E_0(A_t)\). When \(E_0(A_{t_0}) \neq \{0\}\), we call \(t_0\) a crossing instant. In this case, we defined the crossing from \(Cr[A_{t_0}]\) as
\[
\begin{align*}
Cr[A_{t_0}] &= P_{t_0} \frac{\partial}{\partial t} P_{t_0} : E_0(A_{t_0}) \to E_0(A_{t_0}).
\end{align*}
\]
We call the crossing instant \(t_0\) \textit{regular} if the crossing form \(Cr[A_{t_0}]\) is nondegenerate. In this case we define the signature simply as
\[
\text{sgn} (Cr(A_{t_0})) := \dim E_+ (Cr(A_{t_0})) - \dim E_- (Cr(A_{t_0})).
\]
We assume that all of the crossings are regular. Then, the crossing instants are isolated (and, hence, those on a compact interval are finite in number), and the spectral flow is given by the following formula
\[
\text{Sf} (A_t; t \in [0, 1]) = \sum_{t_0 \in S_\ast} \text{sgn} (Cr(A_{t_0})) - \dim E_- (Cr(A_0)) + \dim E_+ (Cr(A_1))
\]
(2.1)
where \(S_\ast := S \cap (a, b)\), and \(S\) denotes the set of all crossings.

We list some basic properties of spectral flow:

\textbf{Proposition 2.1.} \(\text{(1)}\) If \(t_0 \in [0, 1]\), then
\[
\text{Sf} (A_t; t \in [0, 1]) = \text{Sf} (A_t; t \in [0, t_0]) + \text{Sf} (A_t; t \in [t_0, 1]).
\]
\(\text{(2)}\) If \(S_\ast := S \cap (0, 1)\) and \(S\) denotes the set of all crossings, then
\[
\text{Sf} (A_t; t \in [0, 1]) \leq \sum_{t \in S_\ast} \text{dim} E_0(A_t).
\]

Now, we return to the problem of stability analysis. If there is not a Hamiltonian structure for \(L\) by the presence of the \(\frac{d^2}{d\xi^2}\) term, we can circumvent this by considering instead the operator
\[
L := e^{-\frac{1}{2}C}Le^{\frac{1}{2}C} = \frac{d^2}{d\xi^2} - \frac{\epsilon^2}{4} I + QB,
\]
as is done in [12, 9]. We now proceed to the study of the following eigenvalue problem
\[
L\varphi = \lambda\varphi,
\]
(2.2)
Setting \(\psi = \varphi - \frac{1}{2}C\varphi\) and \(z = \begin{bmatrix} \psi \\ \varphi \end{bmatrix}\) converts Equation (2.2) to the following Hamiltonian system
\[
\begin{align*}
\dot{z} &= JH_\lambda (\xi)z,
\end{align*}
\]
(2.3)
where \(H_\lambda = -J A_\lambda - \frac{1}{2}CJ = \begin{bmatrix} Q & \frac{\epsilon}{2}CQ \\ \frac{\epsilon}{2}CQ & B - \lambda Q \end{bmatrix}\). We define the self-adjoint operators
\[
F_\lambda := -J \frac{d}{d\xi} - H_\lambda : \text{dom} F_\lambda \subset L^2(\mathbb{R}, \mathbb{R}^{2n}) \subset L^2(\mathbb{R}, \mathbb{R}^{2n}),
\]
where...
where $\text{dom} F_\lambda = W^{1,2}(\mathbb{R}, \mathbb{R}^{2n})$ and $\lambda \in [0, C]$. Here, $C$ is given in (1.3). The following Proposition 2.2 shows that the Hamiltonian system (2.3) has close contact with the system (1.5).

**Proposition 2.2.** Letting $\lambda \in \mathbb{C}^+$ and $z \in H^2(\mathbb{R}, \mathbb{C}^{2n})$, then $z \in \ker \left( \frac{d}{d\xi} - A_\lambda \right)$ if and only if $e^{\frac{i}{2}z} \xi \in \ker \left( -J \frac{d}{d\xi} - H_\lambda \right)$.

It is well-known [16, Lemma 3.1.10] that the essential spectrum is given by

$$\sigma_{\text{ess}}(L) = \{ \lambda \in \mathbb{C} | A_\lambda(+\infty) \text{ or } A_\lambda(-\infty) \text{ has a pure imaginary eigenvalue} \}.$$ 

Following the same method as that in paper [11, Section 2], we here give the proof of the following Lemma for convenience of the reader.

**Lemma 2.3.** Under the condition (H1), we have that

1. $\sigma_{\text{ess}}(L) \subset \mathbb{C}^-$,
2. if $\lambda \in \mathbb{C}^+$, then $A_\lambda(\pm \infty)$ are hyperbolic.

**Proof.** A simple calculation shows the eigenvalues of $A_\lambda(+\infty)$ and $A_\lambda(-\infty)$, such that

$$\mu(\lambda) = \frac{1}{2} \{ -c \pm \sqrt{c^2 + 4(\lambda - \alpha)} \} \quad \text{and} \quad \nu(\lambda) = \frac{1}{2} \{ -c \pm \sqrt{c^2 + 4(\lambda - \beta)} \}$$

respectively, where $\alpha$ is the eigenvalue of $QB_+$, and $\beta$ is the eigenvalue of $QB_-$. To prove that (1) holds, we only need to show that $A_\lambda(+\infty)$ and $A_\lambda(-\infty)$ have no purely imaginary eigenvalues if $\lambda \in \mathbb{C}^+$, which is equivalent to showing that $\Re \sqrt{c^2 + 4(\lambda - \alpha)} \neq -c$ and $\Re \sqrt{c^2 + 4(\lambda - \beta)} \neq -c$.

From the formula $\Re \sqrt{a + i b} = (\frac{1}{2} \sqrt{a^2 + b^2} + \frac{1}{2} a^2)^{\frac{1}{2}}$, we have that

$$\Re \sqrt{c^2 + 4(\lambda - \alpha)} = \left( \frac{1}{2} \sqrt{(c^2 + 4 \Re (\lambda - \alpha))^2 + (\Im (\lambda - \alpha))^2} \right)^{\frac{1}{2}} + c^2 + 4 \Re (\lambda - \alpha)^{\frac{1}{2}}.$$  \hspace{1cm} (2.4)

we note that $\Re (\lambda - \alpha) > 0$, so Equation (2.4) implies that

$$\Re \sqrt{c^2 + 4(\lambda - \alpha)} > c.$$

This calculation actually proves that $A_\lambda(+\infty)$ has exactly $n$ eigenvalues of the positive real part and $n$ eigenvalues of the negative real part for $\Re \lambda \geq 0$. Additionally, $A_\lambda(-\infty)$. We label the eigenvalues of $A_\lambda(\pm \infty)$ in order of increasing real part and observe that

$$\Re \mu_1(\lambda) \leq \cdots \leq \Re \mu_n(\lambda) \leq -c < 0 < \Re \mu_{n+1}(\lambda) \leq \cdots \leq \Re \nu_n(\lambda),$$

$$\Re \nu_1(\lambda) \leq \cdots \leq \Re \nu_n(\lambda) < -c < 0 < \Re \nu_{n+1}(\lambda) \leq \cdots \leq \Re \nu_n(\lambda).$$

From these, we complete the proof. \hspace{1cm} $\square$

**Remark 2.4.** We note that $JH_\lambda(\pm \infty) = A_\lambda(\pm \infty) + \frac{1}{2} c \xi$: from Lemma 2.3, if (H1) holds, then $JH_\lambda(\pm \infty)$ are both hyperbolic for all $\lambda \in \mathbb{C}^+$, and this, together with [14, Theorem 1], tells us that for each $\lambda \in [0, C]$, the operator $F_\lambda$ is a self-adjoint Fredholm operator, and in particular, it is possible to associate it to path

$$\lambda \rightarrow F_\lambda \hspace{1cm} (2.5)$$

the topological invariant: spectral flow.

**The proof of Proposition 2.2.** If $z \in \ker \left( \frac{d}{d\xi} - A_\lambda \right)$. By a simple calculation,

$$\frac{d}{d\xi} \left( e^{\frac{i}{2}z} \xi \right) = \frac{1}{2} c e^{\frac{i}{2}z} \xi + e^{\frac{i}{2}z} \xi \xi = J \left( -\frac{1}{2} c J - J A_\lambda(\xi) \right) e^{\frac{i}{2}z} \xi = J H_\lambda(\xi) e^{\frac{i}{2}z} \xi.$$

From Lemma 2.3, $z$ must decay as fast as $e^{\nu_n(\lambda) \xi}$ as $\xi \rightarrow +\infty$, and we note that if $\nu_n(\lambda) < -c$, then $e^{\frac{i}{2}z} \xi$ and $e^{\frac{i}{2}z} \xi$ both exponentially decay to 0 as $\xi \rightarrow +\infty$, and hence, $e^{\frac{i}{2}z} \xi \in L^2(\mathbb{R}, \mathbb{C}^{2n})$ and $e^{\frac{i}{2}z} \xi \in L^2(\mathbb{R}, \mathbb{C}^{2n})$. Then, $e^{\frac{i}{2}z} \xi \in H^2(\mathbb{R}, \mathbb{C}^{2n})$: this together with Equation (2.6) implies that

$$e^{\frac{i}{2}z} \xi \in \ker \left( -J \frac{d}{d\xi} - H_\lambda \right).$$

Conversely, if $e^{\frac{i}{2}z} \xi \in \ker \left( -J \frac{d}{d\xi} - H_\lambda \right)$, then by a simple calculation,

$$\dot{z} = \frac{1}{2} c z + \frac{1}{2} \frac{d}{d\xi} \left( e^{\frac{i}{2}z} \xi \right) = e^{-\frac{i}{2}z} \xi \left( \frac{1}{2} e^{\frac{i}{2}z} \xi + e^{\frac{i}{2}z} \xi \right) - \frac{1}{2} c \xi = e^{-\frac{i}{2}z} \xi \left( \frac{d}{d\xi} \left( e^{\frac{i}{2}z} \xi \right) \right) = -\frac{1}{2} c \xi = A_\lambda(z).$$
We note that when $J\mathcal{H}_\lambda(\pm \infty) = A_\lambda(\pm \infty) + \frac{1}{4} c$, for each $\mu \in \sigma(J\mathcal{H}_\lambda(\pm \infty)) \cap \mathbb{C}^+$, invoking Lemma 2.5, it is easy to check that $\Re \mu > \frac{1}{4} c$, then, $e^{\pm \xi t} z$ must decay at least as fast as $e^{\frac{1}{4} \xi t}$ as $\xi \to -\infty$. We thus determine that $z$ and $\dot{z}$ both exponentially decay as $\xi \to \pm \infty$, and then, $z, \dot{z} \in L^2(\mathbb{R}, \mathbb{C}^{2n})$, and this together with Equation (2.7), implies that $z \in \ker \left( d \frac{d}{d\xi} - A_\lambda \right)$. 

□

As a direct consequence, the following result holds.

**Corollary 2.5.** Letting $\lambda \in \overline{\mathbb{C}^+}$ and $\phi \in H^2(\mathbb{R}, \mathbb{C}^n)$, then $\phi \in \ker(L - \lambda I)$ if and only if $e^{\pm \xi t} \phi \in \ker(\mathcal{L} - \lambda I)$.

**Remark 2.6.** Under the condition (H1), if $y$ satisfies (1.5) with $\lim_{\xi \to +\infty} y = 0$, then a similar discussion in the proof of Proposition 2.2 guarantees that $e^{\pm \xi t} y$ satisfies (2.3) with $\lim_{\xi \to +\infty} e^{\pm \xi t} y = 0$.

Then, $\{ e^{\pm \xi t} y(\tau)|y \text{ solve (1.5) and } y \to 0 \text{ as } \tau \to +\infty \}$ of the Hamiltonian system (2.3), so we can say that $E^s_\lambda(\tau)$ is also the stable space of the Hamiltonian system (2.3). By the same reasoning, $E^s_\lambda(\tau)$ is also the unstable space of the Hamiltonian system (2.3). Moreover, letting $E^s_\lambda(\pm \infty) := \{ v \in \mathbb{R}^{2n} \mid \lim_{\xi \to +\infty} \exp (\xi A_\lambda(\pm \infty)) v = 0 \}$ and $E^u_\lambda(\pm \infty) := \{ v \in \mathbb{R}^{2n} \mid \lim_{\xi \to -\infty} \exp (\xi A_\lambda(\pm \infty)) v = 0 \}$, a similar discussion in Proposition 2.2 shows that the following facts

$E^s_\lambda(\pm \infty) = \{ v \in \mathbb{R}^{2n} \mid \lim_{\xi \to +\infty} \exp (\xi J\mathcal{H}_\lambda(\pm \infty)) v = 0 \}$ and $E^u_\lambda(\pm \infty) = \{ v \in \mathbb{R}^{2n} \mid \lim_{\xi \to -\infty} \exp (\xi J\mathcal{H}_\lambda(\pm \infty)) v = 0 \}$

hold.

Under the condition (H1), we determine that

$E^s_\lambda(\pm \infty) = \lim_{\tau \to +\infty} E^s_\lambda(\tau)$ and $E^u_\lambda(\pm \infty) = \lim_{\tau \to -\infty} E^u_\lambda(\tau)$,

where the convergence is meant in the gap (norm) topology of the Lagrangian Grassmannian (Cf. [1] for further details).

Given $\tau \geq 0$, let $B_1(\xi) = B(\tau + \xi)$ with $\xi \in \mathbb{R}^+$ and $B_2(\xi) = B(\xi - \tau)$ with $\xi \in \mathbb{R}^-$. The aim of the next part is to provide some sufficient condition for the coefficient of (2.5) to obtain the nondegeneracy.

**Lemma 2.7.** Let

$\mathbb{L}^+_\lambda M := \frac{d^2}{d\xi^2} - \left\{ \frac{c^2}{4} + \lambda \right\} I + QB_1 : W^{2,2}(\mathbb{R}^+, \mathbb{R}^n) \to L^2(\mathbb{R}^+, \mathbb{R}^n)$

and

$\mathbb{L}^-_\lambda M := \frac{d^2}{d\xi^2} - \left\{ \frac{c^2}{4} + \lambda \right\} I + QB_2 : W^{2,2}(\mathbb{R}^-, \mathbb{R}^n) \to L^2(\mathbb{R}^-, \mathbb{R}^n)$.

With $C$ given in (1.3), assuming that (H1) holds, and $\lambda \geq C$, then the system

$\begin{cases} 
\mathbb{L}^+_\lambda M \dot{\varphi}_1 = 0 = \mathbb{L}^-_\lambda M \dot{\varphi}_2, \\
\varphi_1(0) = \varphi_2(0), \quad \varphi_1(0) = \varphi_2(0)
\end{cases}$

has only the zero solution.

**Proof.** Assuming that the system has a solution $(\varphi_1, \varphi_2)$, then we have that

$\langle \mathbb{L}^+_\lambda M \varphi_1, \varphi_1 \rangle_{L^2} + \langle \mathbb{L}^-_\lambda M \varphi_2, \varphi_2 \rangle_{L^2} = 0$

Integrating by part, we obtain

$\langle \mathbb{L}^+_\lambda M \varphi_1, \varphi_1 \rangle_{L^2} = -\| \varphi_1 \|^2_{L^2} - \left\{ \frac{c^2}{4} + \lambda \right\} \| \varphi_1 \|^2_{L^2} + \int_0^{+\infty} (QB_1 \varphi_1, \varphi_1) d\xi - \langle \varphi_1(0), \varphi_1(0) \rangle$

and

$\langle \mathbb{L}^-_\lambda M \varphi_2, \varphi_2 \rangle_{L^2} = -\| \varphi_2 \|^2_{L^2} - \left\{ \frac{c^2}{4} + \lambda \right\} \| \varphi_2 \|^2_{L^2} + \int_{-\infty}^0 (QB_2 \varphi_2, \varphi_2) d\xi + \langle \varphi_2(0), \varphi_2(0) \rangle$
Let $I_{\lambda,1} := -\|\varphi_1\|_{L^2}^2 - \left\{ \frac{\sigma^2}{4} + \lambda \right\} \|\varphi_1\|_{L^2}^2 + \int_0^{\infty} (QB_1(\varphi_1, \varphi_1) d\xi$ and $I_{\lambda,2} := -\|\varphi_2\|_{L^2}^2 - \left\{ \frac{\sigma^2}{4} + \lambda \right\} \|\varphi_2\|_{L^2}^2 + \int_{-\infty}^{0} (QB_2(\varphi_2, \varphi_2) d\xi$. It is easy to see that $I_i \leq -\|\varphi_i\|_{L^2}^2 - \left\{ \frac{\sigma^2}{4} + \lambda - C \right\} \|\varphi_i\|_{L^2}^2$. Then, by using the second condition in the above boundary value problem, we obtain

$$0 = I_1 + I_2 \leq \sum_{i=1,2} \left( -\|\varphi_i\|_{L^2}^2 - \left\{ \frac{\sigma^2}{4} + \lambda - C \right\} \|\varphi_i\|_{L^2}^2 \right).$$

If $\lambda \geq C$, then we infer that

$$I_1 + I_2 = 0$$

if and only if $\varphi_i = \varphi_i = 0$ for $i = 1, 2$. This concludes the proof.

Let us now consider the associated first order differential operators $F^{+}_{\lambda, M}$ and $F^{-}_{\lambda, M}$ of $L^{1.3}_{\lambda, M}$ and $L^{-1.3}_{\lambda, M}$. A similar result holds.

**Lemma 2.8.** With $C$ given in (1.3), assuming that (H1) holds, and $\lambda \geq C$, then the system

$$\begin{cases}
F^{+}_{\lambda, M}z_1 = F^{-}_{\lambda, M}z_2 = 0 \\
z_1(0) = z_2(0)
\end{cases}$$

has only the zero solution.

**Lemma 2.9.** With $C$ given in (1.3), assuming that (H1) holds, we have that

$$E^s_{\lambda}(\tau) \cap E^u_{\lambda}(-\tau) = \{0\} \quad \text{for all } (\lambda, \tau) \in [C, \infty) \times \mathbb{R}^+.$$

**Proof.** Let $B_1(x) = B(x + \tau)$ with $\xi \in \mathbb{R}^+$ and $B_2(\xi) = B(x - \tau)$ with $\xi \in \mathbb{R}^-$. Then, the stable subspace of the equation $F^{+}_{\lambda, M} = 0$ at 0 is $E^s_{\lambda}$, the unstable subspace of the equation $F^{-}_{\lambda, M} = 0$ at 0 is $E^u_{\lambda}$, and there exists a linear bijection from the set of solutions of the system

$$\begin{cases}
F^{+}_{\lambda, M}z_1 = F^{-}_{\lambda, M}z_2 = 0 \\
z_1(0) = z_2(0)
\end{cases}$$

with the subspace $E^s_{\lambda}(\tau) \cap E^u_{\lambda}(-\tau)$. By invoking once again Lemma 2.8 and Lemma 2.7, we conclude that the initial value problem only admits the trivial solution for every $\lambda \geq C$. This concludes the proof.

By setting $B_1(\xi) = B(\xi)$ for every $\xi \geq 0$ and $B_2(\xi) = B(\xi)$ for every $\xi \leq 0$, the following result holds.

**Lemma 2.10.** With $C$ given in (1.3), assuming that (H1) holds, if $\lambda \geq C$, then $\ker (L - \lambda) = \{0\}$ and $\ker F_\lambda = \{0\}$.

From Lemma 2.10, we determine that the following result holds.

**Corollary 2.11.** With $C$ given in (1.3), assuming that (H1) holds, and if $\lambda \geq C$, then we have that $\ker S_\lambda = \{0\}$, where $S_\lambda := -Q \frac{d^2}{dx^2} + \frac{\sigma^2}{4} Q - B + \lambda Q$.

It is well-known that for each $\lambda \in [0, C]$, the operator $S_\lambda$ is closed and self-adjoint with dense domain in $L^2(\mathbb{R}, \mathbb{R}^n)$. As a byproduct of condition (H1) and [14, Theorem 1], $S_\lambda$ is also a Fredholm operator.

Finally, from [13, Theorem 1], we obtain that

$$Sf(F_\lambda, \lambda \in [0, C]) = \lambda^{CLM}(E^s_\lambda(\tau), E^u_\lambda(-\tau); \tau \in \mathbb{R}^+) - \lambda^{CLM}(E^s_\lambda(\tau), E^u_\lambda(-\tau); \tau \in \mathbb{R}^-)$$

$$- \lambda^{CLM}(E^s_\lambda(+\infty), E^u_\lambda(-\infty); \lambda \in [0, C]).$$

As a direct consequence of Lemma 2.9 and Equation (2.8), we obtain the following result.

**Proposition 2.12.** Under the previous notations and assuming that (H1) holds, the following equation holds:

$$-Sf(F_\lambda, \lambda \in [0, C]) = \lambda^{CLM}(E^s_\lambda(\tau), E^u_\lambda(-\tau); \tau \in \mathbb{R}^+) + \lambda^{CLM}(E^s_\lambda(+\infty), E^u_\lambda(-\infty); \lambda \in [0, C]).$$
3. The proof of the main result

The goal of this section is to provide a detailed proof of Theorem 1.8, Theorem 1.10 and Theorem 1.12. Before showing the proof, we start by analyzing the distribution of the eigenvalues of $L$.

As in [8], the same discussion can be used here to obtain the distribution of eigenvalues of $L$, which serves a crucial role of counting the number of nonnegative eigenvalues of $L$ via spectral flow. Let $Q^+$ and $Q^-$ be the orthogonal projections from $E$ to $E_+(Q)$ and $E_-(Q)$, respectively. Define $\mathcal{F}_1 = Q^+SQ^+$, $\mathcal{F}_2 = Q^-SQ^-$ and $\mathcal{F}_3 = Q^+SQ^-$. In other words, $S$ can be decomposed as

$$
\begin{bmatrix}
\mathcal{F}_1 & \mathcal{F}_2 \\
\mathcal{F}_3 & \mathcal{F}_2
\end{bmatrix},
$$

where $\mathcal{F}_3^* = \mathcal{F}_3^T$ and $\mathcal{F}_3$ denotes the complex conjugate of $\mathcal{F}_3$. For a linear self-adjoint operator $A$ defined on a Hilbert space $E$, denoted by $A > 0$, if $\langle Av, v \rangle > 0$ for all $v \in E \setminus \{0\}$, two linear operators $A$ and $A^*$ are denoted by $A > A$ if $A - A^* > 0$.

**Lemma 3.1.** [8] Supposing that $\mathcal{F}_1 > 0$ and $I > \mathcal{F}_3^2\mathcal{F}_1^{-2}\mathcal{F}_3$, then $\sigma(L) \cap C^+ \subset \mathbb{R}$. The same assertion holds if $-\mathcal{F}_2 > 0$ and $I > \mathcal{F}_3^2(-\mathcal{F}_2)^{-2}\mathcal{F}_3$.

**Proposition 3.2.** Under the condition (H1), we have that

1. if $-\mathcal{F}_2 > 0$ and $I > \mathcal{F}_3^2(-\mathcal{F}_2)^{-2}\mathcal{F}_3$, then $\text{Sf}(S_\lambda; \lambda \in [0, C]) = N_+(L)$,
2. if $\mathcal{F}_1 > 0$ and $I > \mathcal{F}_3^2(-\mathcal{F}_2)^{-2}\mathcal{F}_3$, then $\text{Sf}(S_\lambda; \lambda \in [0, C]) = -N_+(L)$.

**Proof.** We only prove (1), while the other is analogous. From Remark 2.6, we know that $N_+(L) = N_+(\mathcal{L})$. Next, we prove that $\text{Sf}(S_\lambda; \lambda \in [0, C]) = N_+(\mathcal{L})$. Suppose that along the spectral flow, there is a crossing at $S_\lambda$ for some $\lambda \in [0, C]$ and $\phi \in \ker S_\lambda$: that is,

$$-Q\dot{\phi} + \left(\frac{1}{4}c^2Q - B + \lambda Q\right)\phi = 0. \quad (3.1)$$

Letting $\phi_+ = Q^+\phi$ and $\phi_- = Q^-\phi$, we can rewrite Equation (3.1) as

$$\mathcal{F}_1\phi_+ + \mathcal{F}_3\phi_- = -\lambda\phi_+ \quad (3.2)$$

$$\mathcal{F}_2\phi_+ + \mathcal{F}_3\phi_- = \lambda\phi_- \quad (3.3)$$

Solving Equation (3.3), we obtain that $\phi_- = (\lambda - \mathcal{F}_2)^{-1}\mathcal{F}_3\phi_+$, and this, together with Equation (3.2), obtains

$$
\frac{d}{d\lambda}(S_\lambda, \phi) = (Q\phi, \phi) = \langle \phi_+, \phi_+ \rangle - \langle \phi_-, \phi_- \rangle = \langle \phi_+, \phi_+ \rangle - \langle \mathcal{F}_3^2(\lambda - \mathcal{F}_2)^{-2}\mathcal{F}_3\phi_+, \phi_+ \rangle
$$

We note that $I > \mathcal{F}_3^2(-\mathcal{F}_2)^{-2}\mathcal{F}_3 > \mathcal{F}_3^2(\lambda - \mathcal{F}_2)^{-2}\mathcal{F}_3$ for all $\lambda \geq 0$. This indicates that the sign of the crossing form has to be positive whenever a crossing occurs at $\lambda \in [0, C]$. In view of Equation (2.1), we conclude from Lemma that

$$\text{Sf}(S_\lambda; \lambda \in [0, C]) = \sum_{\lambda \in [0, C]} \dim \ker S_\lambda.$$

For (2), a slightly modified argument shows that the sign of the crossing operator must be negative if a crossing occurs at $\lambda \in [0, C]$, and then

$$\text{Sf}(S_\lambda; \lambda \in [0, C]) = -\sum_{\lambda \in [0, C]} \dim \ker S_\lambda.$$

This completes the proof. \qed

The aim of the next part is to prove some transversal properties about some invariant subspaces that are useful in our proof.

**Lemma 3.3.** Under the conditions (H1) and (H2), we have that

$$E_\lambda^+(+\infty) \cap L_R \text{ and } E_\lambda^-(\infty) \cap L_R,$$
Proof. We provide the proof of $E^\lambda_\pm(\infty) \cap L_R$ in completely similar fashion. Let $\begin{bmatrix} p \\ q \end{bmatrix} \in E^\lambda_\pm(\infty) \cap L_R$, and noting that $E^\lambda_\pm(\infty)$ is invariant under $JH^\lambda(\infty)$, then $JH^\lambda(\infty) \begin{bmatrix} p \\ q \end{bmatrix} \in E^\lambda_\pm(\infty)$. From (H2), a direct computation yields that

$$0 = \omega \left( JH^\lambda(\infty) \begin{bmatrix} p \\ q \end{bmatrix} \right) = -\left\langle \begin{bmatrix} Q \ & \frac{1}{2} \varphi Q \\ \frac{1}{2} \varphi Q \ & B_+ - \lambda Q^{-1} \end{bmatrix}, \begin{bmatrix} p \\ q \end{bmatrix} \right\rangle = -\left( \langle p, p \rangle - \lambda \langle q, q \rangle + \langle QB_+q, q \rangle \right) \leq 0,$$

we determine that $E^\lambda_\pm(\infty) \cap L_R$. This completes the proof. \hfill \Box

Now, following Lemma 4.4 and Equation (4.1), we have that

$$\iota^{\text{CLM}}(E^\lambda_\pm(\infty), E^\nu_\pm(-\infty); \lambda \in [0, C]) = \iota(E^\nu_\pm(\infty), E^\nu_\pm(\infty); L_R) - \iota(E^\nu_\pm(\infty), E^\nu_\pm(\infty); L_R) \quad (3.4)$$

$$= m^+\left( \Omega(E^\nu_\pm(\infty), E^\nu_\pm(\infty); L_R) \right) - m^+\left( \Omega(E^\nu_\pm(\infty), E^\nu_\pm(\infty); L_R) \right).$$

We recall that a Lagrangian frame for a Lagrangian subspace $L$ is an injective linear map $T : \mathbb{R}^n \to \mathbb{R}^{2n}$ whose image is $L$ (Cf. page 828 of [17]). Such a frame has the form

$$T = \begin{bmatrix} X \\ Y \end{bmatrix},$$

where $X$ and $Y$ are both $n \times n$ matrices and $X^T Q Y = Y^T Q X$.

We introduce the following notations $T_\lambda^+ = \begin{bmatrix} X_{\lambda^+} & Y_{\lambda^+} \\
0 & I_2 \\
I_1 & 0 \\
Y^T_{\lambda^+} & Z_{\lambda^+} \end{bmatrix}$, $T_\lambda^- = \begin{bmatrix} X_{\lambda^-} & Y_{\lambda^-} \\
0 & I_2 \\
I_1 & 0 \\
Y^T_{\lambda^-} & Z_{\lambda^-} \end{bmatrix}$, $M_{\lambda^+} = \begin{bmatrix} X_{\lambda^+} & Y^T_{\lambda^+} \\
Y_{\lambda^+} & Z_{\lambda^+} \end{bmatrix}$ and $M_{\lambda^-} = \begin{bmatrix} X_{\lambda^-} & Y^T_{\lambda^-} \\
Y_{\lambda^-} & Z_{\lambda^-} \end{bmatrix}$, where $X_{\lambda^\pm}$, $Y_{\lambda^\pm}$, $Z_{\lambda^\pm}$ are $r \times r$ matrices, $Y_{\lambda^\pm}$ are $(n-r) \times r$ matrices, $Z_{\lambda^\pm}$ are $(n-r) \times (n-r)$ matrices, $I_1$ is the $r \times r$ identity matrix, and $I_2$ is the $(n-r) \times (n-r)$ identity matrix. Here, $r = \dim V^+(Q)$. For each $\lambda \in \mathbb{R}^+$, from Lemma 3.3, we can use notations $T_{\lambda^+}$ and $T_{\lambda^-}$ for the Lagrangian frames of $E^\lambda_\pm(\infty)$ and $E^\nu_\pm(\infty)$, respectively.

We now consider the following operator:

$$F_\lambda^+ := -J \frac{d}{d \xi} - H^\lambda(\infty)$$

and the associated second order operator $L^\lambda_{\lambda^+}$ and let $\varphi$ be a solution of $L^\lambda_{\lambda^+} \varphi = 0$, where $L^\lambda_{\lambda^+}$ denotes the operator $L^\lambda_{\lambda^+}$ defined on the maximal domain $W^{2,2}(\mathbb{R}^\tau, \mathbb{R}^n)$. Then, the map $\varphi \mapsto (\varphi^T(0) - \frac{1}{2} c \varphi^T(0), \varphi^T(0))^T$ provides a linear bijection from $\ker L^\lambda_{\lambda^+}$ to $E^\lambda_\pm(\infty) = V^-(JH^\lambda(\infty))$.

We note that for each $z \in E^\lambda_\pm(\infty)$, there exists $u = \begin{bmatrix} p \\ q \end{bmatrix} \in V^+(Q) \oplus V^-(Q) \cong \mathbb{R}^n$, such that $z = T_{\lambda^+}(u)$.

Let $\varphi(\xi) \in \ker L^\lambda_{\lambda^+}$ with $(\varphi^T(0) - \frac{1}{2} c \varphi^T(0), \varphi^T(0))^T = T_{\lambda^+}^+ u \in E^\lambda_\pm(\infty)$, where $u = \begin{bmatrix} p \\ q \end{bmatrix} \in V^+(Q) \oplus V^-(Q) \cong \mathbb{R}^n$. A simple calculation shows that

$$0 = \left( L^\lambda_{\lambda^+} \varphi(\xi), \varphi(\xi) \right)_{L^2} = \langle \dot{\varphi}, \varphi \rangle_{L^2} - c(\varphi, \varphi)_{L^2} + \frac{2}{4} \| \varphi \|^2_{L^2} + \lambda \| \varphi \|^2_{L^2} - \langle QB_+ \varphi, \varphi \rangle_{L^2} + \langle \dot{\varphi}(0) - \frac{c}{2} \varphi(0), \varphi(0) \rangle_{L^2}$$

$$= \left\| \varphi - \frac{c}{2} \varphi \right\|^2_{L^2} - \int_{0}^{\infty} \langle QB_+ \varphi, \varphi \rangle d \xi + \lambda \| \varphi \|^2_{L^2} + \left( M_{\lambda^+} \begin{bmatrix} p \\ q \end{bmatrix}, \begin{bmatrix} p \\ q \end{bmatrix} \right) \geq \left\| \varphi - \frac{c}{2} \varphi \right\|^2_{L^2} + (\lambda - C) \| \varphi \|^2_{L^2} + (M_{\lambda^+} u, u),$$

this equation, together with (H1), (H2) and (H2'), shows the following Lemma:
Lemma 3.4. With $C$ given in (1.3), the following results hold:
(1) if (H1) and (H2) hold, we have that $M_{\lambda,+}(s)$ is negative definite for all $\lambda \geq C$.
(2) if (H2') holds, we have that $M_{\lambda,+}(s)$ is negative definite for all $\lambda \geq 0$.

Similarly, we have that

Lemma 3.5. With $C$ given in (1.3), the following results hold:
(1) if (H1) and (H2) hold, we have that $M_{\lambda,-}$ is positive definite for all $\lambda \geq C$.
(2) if (H2') holds, we have that $M_{\lambda,-}$ is positive definite for all $\lambda \geq 0$.

Letting $z \in E_{\lambda}^s(-\infty)$, then there is $u = \begin{pmatrix} p \\ q \end{pmatrix} \in V^+(Q) \oplus V^-(Q) \cong \mathbb{R}^n$, such that $z = T_{\lambda,+}u$, and $z$ can be rewritten as $z = T_{\lambda,+}u + (T_{\lambda,-} - T_{\lambda,+})u \in E_{\lambda}^s + L_R$. From a simple calculation, we have that

$$
\Omega(E_{\lambda}^s(-\infty), E_{\lambda}^s(+\infty); L_R) (u, u)
= \omega\left[ \begin{bmatrix} X_{\lambda,+}p + Y_{\lambda,+}^T q \\ p \\ Y_{\lambda,+}p + Z_{\lambda,+}q \end{bmatrix}, \begin{bmatrix} (X_{\lambda,-} - X_{\lambda,+}) p + (Y_{\lambda,-}^T - Y_{\lambda,+}^T) q \\ 0 \\ (Y_{\lambda,-} - Y_{\lambda,+}) p + (Z_{\lambda,-} - Z_{\lambda,+}) q \end{bmatrix} \right]
= \langle (X_{\lambda,+} - X_{\lambda,-})p, p \rangle + 2 \langle (Y_{\lambda,+}^T - Y_{\lambda,-}^T) p, q \rangle + \langle (Z_{\lambda,+} - Z_{\lambda,-})q, q \rangle
= \langle (M_{\lambda,+} - M_{\lambda,-})u, u \rangle.
$$

From Lemma 3.4, Lemma 3.5, Equation (3.5) and Equation (4.1), the following result holds.

Lemma 3.6. With $C$ given in (1.3), the following results hold:
(1) if (H1) and (H2) hold, then we have that
$$
\iota(E_{\lambda}^s(-\infty), E_{\lambda}^s(+\infty), L_R) = 0 \text{ for } \lambda \geq C.
$$
(2) if (H2') holds, then we have that
$$
\iota(E_{\lambda}^s(-\infty), E_{\lambda}^s(+\infty), L_R) = 0 \text{ for } \lambda \geq 0.
$$

Before finishing the preparation of our proof of our main results, we recall the definition of positive curve.

Definition 3.7. [15] Let $A : [0, 1] \to \mathcal{F}^{sa}(E)$ be a continuous curve. The curve $A$ is named a positive curve if $\{ t \mid \ker A_t \neq 0 \}$ is finite and
$$
\text{Sf}(A_t; t \in [0, 1]) = \sum_{0 < t < 1} \dim \ker A_t.
$$

The Proof of Proposition 1.7. For some $a \in \mathbb{R}$, we construct the following homotopy Lagrangian path

$$
(E^s(\tau + sa), E^u(-\tau + sa)), (\tau, s) \in \mathbb{R}^+ \times [0, 1].
$$

We point out that $\dim (E^s(sa) \cap E^u(sa))$ is constant for all $s \in [0, 1]$ and $E^s(+\infty) \cap E^u(-\infty)$.

By the stratum homotopy invariance property of the Maslov index, we have that

$$
\iota^{\text{CLM}}(E^s(\tau), E^u(-\tau); \tau \in \mathbb{R}^+) = \iota^{\text{CLM}}(E^s(\tau + a), E^u(-\tau + a); \tau \in \mathbb{R}^+)
= \iota^{\text{CLM}}(E^s(\tau + 2a), E^u(-\tau); \tau \in [-a, +\infty)).
$$

We know that $E^s(+\infty) \cap E^u(-\infty)$ and $E^s(\tau) \to E^s(+\infty)$ as $\tau \to +\infty$ under the gap topology of the Lagrangian Grassmannian, so we can choose $\tau_0$, such that $E^s(\tau) \cap E^u(-\infty)$ for all $\tau \geq \tau_0$, and the path $E^s(\tau) : (-\infty, \tau_0) \to \text{Lag}(n)$ has only regular crossing with respect to $E^s(\tau_0)$. Letting $a = \tau_0$, we construct the following homotopy Lagrangian path:

$$
(E^s(\tau_0 + s(\tau_0 + \tau)), E^u(-\tau)), (\tau, s) \in [-\tau_0, +\infty) \times [0, 1].
$$
By the stratum homotopy invariance, reversal property of Maslov index and Equation (1.6), we determine that
\[
\iota^{clm}(E^s(\tau + 2\tau_0), E^u(-\tau); \tau \in [-\tau_0, +\infty)) = \iota^{clm}(E^s(\tau_0), E^u(-\tau); \tau \in [-\tau_0, +\infty))
\]
\[
= -\iota^{clm}(E^s(\tau_0), E^u(\tau); \tau \in (-\infty, \tau_0]) = -\text{Maslov}(w^*),
\]
and this, together with Equation (3.6), completes the proof.

\[\square\]

The Proof of Theorem 1.8. We first prove that \(\text{SF} (S_\lambda; \lambda \in [0, C]) = \text{SF} (F_\lambda; \lambda \in [0, C])\). We start by introducing the continuous map
\[
f: \mathcal{C}(\mathbb{R}^n) \to \mathcal{C}(\mathbb{R}^{2n})
\]
defined by \(f(S_\lambda) := F_\lambda\).
Let \(h(\lambda, s) = f(S_\lambda + sI)\) for \((\lambda, s) \in [0, C] \times [0, \varepsilon]\). Then, for every \(\lambda \in [0, C]\), \(h(\lambda, s)\) is a positive curve. Let \(\lambda_0 \in [0, C]\) be a crossing instant for the path \(\lambda \mapsto S_\lambda\), meaning that \(\ker S_{\lambda_0} \neq \{0\}\), and let us consider the positive path \(s \mapsto S_{\lambda_0} + sI\). Thus, there exists \(\delta > 0\), such that \(\ker (S_{\lambda_0} + \delta I) = \{0\}\), which is equivalent to \(\ker h(\lambda_0, \delta) = \{0\}\). Since \(S_{\lambda_0} + \delta I\) is a Fredholm operator, there exists \(\delta_1 > 0\), such that \(\ker (S_{\lambda_0} + \delta I) = \{0\}\) for every \(\lambda \in [\lambda_0 - \delta_1, \lambda_0 + \delta_1]\). By this argument, we determine that \(\ker h(\lambda, \delta) = \{0\}\) for every \(\lambda \in [\lambda_0 - \delta_1, \lambda_0 + \delta_1]\). Then, we determine that
\[
\begin{align*}
\text{SF} (S_\lambda + \delta I, \lambda \in [\lambda_0 - \delta_1, \lambda_0 + \delta_1]) &= 0, \\
\text{SF} (S_\lambda, \lambda \in [\lambda_0 - \delta_1, \lambda_0 + \delta_1]) &= 0.
\end{align*}
\]

By the homotopy invariance of the spectral flow, we infer that
\[
\text{SF} (S_\lambda, \lambda \in [\lambda_0 - \delta_1, \lambda_0 + \delta_1]) = \text{SF} (S_{\lambda_0} + sI, s \in [0, \delta]) - \text{SF} (S_{\lambda_0} + sI, s \in [0, \delta])
\]
and
\[
\text{SF} (h(\lambda, 0), \lambda \in [\lambda_0 - \delta_1, \lambda_0 + \delta_1]) = \text{SF} (h(\lambda_0 - \delta_1, s), s \in [0, \delta]) - \text{SF} (h(\lambda_0 - \delta_1, s), s \in [0, \delta])
\]
We observe that \(s \mapsto S_{\lambda_0} + sI\) and \(s \mapsto h(\lambda \pm \delta_1, s)\) are both positive curves. It follows that
\[
\text{SF} (S_{\lambda_0} + sI, s \in [0, \delta]) = \sum_{0 < s \leq \delta} \dim \ker (S_{\lambda_0} + sI) = \sum_{0 < s \leq \delta} \dim \ker h(\lambda \pm \delta_1, s)
\]
from Equations (3.7), (3.8) and (3.9), we have that
\[
\text{SF} (S_\lambda, \lambda \in [\lambda_0 - \delta_1, \lambda_0 + \delta_1]) = \text{SF} (h(\lambda, 0), \lambda \in [\lambda_0 - \delta_1, \lambda_0 + \delta_1])
\]
the crossing instants are isolated, and those on a compact interval are finite in number. From Equation (3.10) and the path additivity of spectral flow, we determine that \(\text{SF} (S_\lambda; \lambda \in [0, C]) = \text{SF} (F_\lambda; \lambda \in [0, C])\).

From Proposition 2.12 and Equation (3.4), we determine that
\[
-\text{SF} (S_\lambda; \lambda \in [0, C]) = \iota^{clm}(E^s(\tau), E^u(-\tau); \tau \in \mathbb{R}^+) - \iota (E^u(-\infty), E^s(+\infty); L_R).
\]
Since (H1) and (H2) hold, from Lemma 3.6, Equation (3.11) and Proposition 2.1, then
\[
|\iota(w^*) + \iota (E^u(-\infty), E^s(+\infty); L_R)| \leq \mathcal{N}_+(L),
\]
\[\square\]

The proof of Theorem 1.10. From Remark 1.2, \((H2')\) implies that (H1) holds, and then, from Theorem 1.8 and Lemma 3.6, we have that
\[
|\iota(w^*)| \leq \mathcal{N}_+(L).
\]
\[\square\]

The proof of Theorem 1.12. We note that \(S = - \frac{d^2}{dx^2} + \frac{\gamma}{\sqrt{d}} - \frac{\gamma}{\sqrt{H}}\), and then,
\[
\mathcal{J}_3 = \frac{1}{\sqrt{d}}, \quad \mathcal{J}_2 = \frac{d^2}{dx^2} - \frac{\gamma}{\sqrt{d}} - \frac{\gamma}{\sqrt{H}}.
\]
Since \(\gamma > 0\), it is easy to see that \(\mathcal{J}_2 > 0\). Moreover, if \(d > \gamma^{-2}\), then we have that \(I > \frac{1}{4} \left(- \frac{d^2}{dx^2} + \frac{\gamma}{\sqrt{d}} + \gamma\right)^2 = \mathcal{J}_3^2 (\mathcal{J}_2)^{-2} \mathcal{J}_3^2\), so Proposition 3.2 (1) holds, and
then, we have that \( Sf(Sx; \lambda \in [0, C]) = N_+ (L) \). Moreover, by a simple calculation and the facts \( f'(0) < 0 \) and \( f'(a_i) < 0 \), it is easy to check that the condition \((H^2)\) holds. Then, from Equation (3.11) and Lemma 3.6, we complete the proof. \( \square \)

4. The triple and Hörmander index

Recently, Zhu et al., in the interesting paper [19], deeply investigated the Hörmander index, particularly its relation with respect to the so-called triple index in a slightly generalized (in fact, isotropic) setting. Given three isotropic subspaces \( \alpha, \beta \) and \( \delta \) in \( (\mathbb{R}^{2n}, \omega) \), we define the quadratic form \( \Omega \) as follows:

\[
\Omega := \Omega(\alpha, \beta; \delta): \alpha \cap (\beta + \delta) \to \mathbb{R} \text{ given by } \Omega(x_1, x_2) = \omega(y_1, z_2),
\]

where for \( j = 1, 2 \), \( x_j = y_j + z_j \in \alpha \cap (\beta + \delta) \) and \( y_j, z_j \in \beta, z_j \in \delta \). By invoking [19, Lemma 3.3], in the particular case in which \( \alpha, \beta, \delta \) are Lagrangian subspaces, we obtain

\[
\ker \Omega(\alpha, \beta; \delta) = \alpha \cap \beta + \alpha \cap \delta.
\]

By [19, Lemma 3.13], we are in position to define the triple index in terms of the quadratic form \( \Omega \) defined above.

**Definition 4.1.** Let \( \alpha, \beta \) and \( \kappa \) be three Lagrangian subspaces of symplectic vector space \( (\mathbb{R}^{2n}, \omega) \). Then, the triple index of the triple \( (\alpha, \beta, \kappa) \) is defined by

\[
i(\alpha, \beta, \kappa) = m^+ (\Omega(\alpha, \beta; \kappa)) + \dim (\alpha \cap \kappa) - \dim (\alpha \cap \beta \cap \kappa),
\]

where \( m^+ \) is the Morse positive index of a quadratic form \( Q \).

Another closely related symplectic invariant is the so-called Hörmander index, which is particularly important for measuring the difference in the (relative) Maslov index computed with respect to two different Lagrangian subspaces (we refer the interested reader to the celebrated and beautiful paper [17] and the references therein).

Let \( V_0, V_1, L_0, L_1 \) be four Lagrangian subspaces and \( L \in \mathcal{C}^0([0, 1], \text{Lag}(n)) \) be such that \( L(0) = L_0 \) and \( L(1) = L_1 \).

**Definition 4.2.** Letting \( L, V \in \mathcal{C}^0([0, 1], \text{Lag}(n)) \) be such that \( L(0) = L_0, L(1) = L_1, V(0) = V_0 \) and \( V(1) = V_1 \), the Hörmander index is the integer defined by

\[
s(L_0, L_1; V_0, V_1) = e_{CLM}(V_1, L(t); t \in [0, 1]) - e_{CLM}(V_0, L(t); t \in [0, 1])
\]

\[
= e_{CLM}(V(t), L_1; t \in [0, 1]) - e_{CLM}(V(t), L_0; t \in [0, 1])
\]

**Remark 4.3.** As a direct consequence of the fixed endpoints homotopy invariance of the \( e_{CLM} \)-index, it is actually possible to prove that Definition 4.2 is well-posed, meaning that it is independent of the path \( L \) joining the two Lagrangian subspaces \( L_0, L_1 \). (Cf. [17] for further details).

Let us now be given four Lagrangian subspaces, namely \( \lambda_1, \lambda_2, \kappa_1, \kappa_2 \) of symplectic vector space \( (\mathbb{R}^{2n}, \omega) \). By [19, Theorem 1.1], the Hörmander index \( s(\lambda_1, \lambda_2; \kappa_1, \kappa_2) \) can be expressed in terms of the triple index as follows

\[
s(\lambda_1, \lambda_2; \kappa_1, \kappa_2) = i(\lambda_1, \lambda_2, \kappa_2) - i(\lambda_1, \lambda_2, \kappa_1) = i(\lambda_1, \kappa_1, \kappa_2) - i(\lambda_2, \kappa_1, \kappa_2).
\]

**Lemma 4.4.** [14] Let \( L_1(t) \) and \( L_2(t) \) be two paths in \( \text{Lag}(n) \) with \( t \in [0, 1] \), and assume that \( L_1(t) \) and \( L_2(t) \) are both transversal to the (fixed) Lagrangian subspace \( L \). We then obtain

\[
e_{CLM}(L_1(t), L_2(t); t \in [0, 1]) = i(L_2(1), L_1(1); L) - i(L_2(0), L_1(0); L).
\]

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