Robustness versus sensitivity in non-Hermitian topological lattices probed by pseudospectra

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Introduction.—Non-Hermitian Hamiltonians in classical and quantum physics [1] describe the dynamics of open systems under the influence of dissipation and/or amplification. One of their intriguing characteristics is the existence of unique non-Hermitian degeneracies [2] the so-called exceptional points (EPs) [3], where two or more eigenvalues and eigenvectors coalesce for a particular value of the system’s parameter [4, 5], forming thus a higher order exceptional point (HEP). Motivated by the recent introduction of the concept of parity-time (PT)-symmetry [6–8] in optics [9–13], where the spatial mixing of gain and loss is physically accessible, the new area of non-Hermitian photonics [14–26] has emerged [27–31]. A plethora of experimental realizations of photonic devices that operate around the HEPs is evident, and is mainly based on the enhanced response of the system around such degeneracies. Among the most impressive experiments are these related to ultra sensitive sensors [25] and non-Hermitian gyroscopes [26].

On the other hand, topological photonics [32–35] relies on the key property of topological protection of the zero eigenstate. Especially in two dimensions, such an effect leads to transport of optical waves along the edge of a photonic topological insulator, even under the presence of strong external perturbations. Most of studies are so far devoted to Floquet systems with broken time reversal symmetry that induce effective pseudomagnetic fields [36].

Recently however, a new frontier of non-Hermitian topological photonics [39–43] has emerged, based on the synergy between the two aforementioned areas. This new direction has led to an explosion of theoretical and experimental results that exploit the existence of chiral and non-Hermitian symmetries on the same lattice. Among the recent experiments that define this field is the demonstration of PT-symmetry breaking in a non-Hermitian Su-Schrieffer-Heeger (NHSSH) lattice [44, 45], the topological insulator lasers [40, 41] and the non-Hermitian Haldane lattice [43]. Nonlinearity also plays an important role and provides a new degree of freedom, as a relevant recent experiment demonstrated [46]. In fact, it allows us to locally control not only the real part of the index modulation but the imaginary part as well. In such systems the inclusion of gain and loss elements make the topological lattice non-Hermitian, and thus extends the physics of topological insulators to the complex domain, with no analog whatsoever in condensed matter physics. Since most of previous concepts of topological physics have been built on the assumption of conservation laws, with underlying Hermitian operators, it means that one has to derive everything from first principles by incorporating the intricate properties of non-Hermitian algebra. Indeed, Zak phases, Chern numbers, bulk-edge correspondence and all relevant topological quantities must be redefined through the prism of non-Hermitian physics, leaving thus many open questions for further investigation [47, 48].

In this context of non-Hermitian topological photonics, we examine the interplay between robustness (due to topology) and sensitivity (due to non-Hermiticity) in the prototypical system of an NHSSH lattice around the underlying HEPs [46]. In order to systematically examine the lattice sensitivity, we provide a mathematical framework ideal for non-Hermitian systems, that is the complex and structured pseudospectra theory [49–52]. In particular, we consider three different lattices, namely an infinite, a finite and a hetero structure of two NHSSH lattices (see Fig. 1). The intricate relation between the lattice’s symmetry and the symmetries of the global perturbations is revealed based on pseudospectra [49, 50]. We find that for chiral structured perturbations, the topological zero state is indeed robust below the underlying EPN. Furthermore, we find that exactly at EP the chiral symmetry leads to a suppressed sensitivity that corresponds to an EP(N-1). Counterintuitively, the zero states turn out to be most sensitive for unstructured complex perturbations,
revealing the order of the pertinent exceptional point. At least we consider the lattice’s sensitivity due to single-site local perturbations of the interface channel, something that has been experimentally demonstrated using optical photorefractive nonlinearity [46].

Non-Hermitian SSH lattices. Our starting point is the prototypical SSH model, in the context of coupled mode theory. More specifically, the lattices that we consider are schematically depicted in Fig. 1. For the infinite lattice, the Hamiltonian in k-space reads, $H^{inf}(k) = (c_2 + c_1 \cos k) \cdot \sigma_x + c_1 \sin k \cdot \sigma_y + i \gamma \cdot \sigma_z$, where, $\sigma_x$, $\sigma_y$, $\sigma_z$ are the Pauli matrices and $k$ is the Bloch wavevector in the first Brillouin zone. The coupling constants are denoted by $c_1$ and $c_2$ for intra- and inter-cell coupling, respectively. The global gain-loss amplitude of each waveguide channel is described by the parameter $\gamma$, and thus making the whole system non-Hermitian. For the finite lattice the Hamiltonian matrix elements are: $H^{fin}_{nm} = \delta_{n+1,m} \cdot c_1 \cdot \delta_{n,m+1} \cdot e^{i\gamma} + \delta_{nm} \cdot c_1 \cdot i\gamma$, where $\gamma_n = (-1)^n \gamma$, and $n, m = 1, ..., N$. Regarding the last case of an interface SSH lattice, we consider two SSH chains that are connected with an extra channel at the interface that has a tunable gain/loss amplitude $\gamma_0$. In such a case the Hamiltonian matrix elements are given by the last expression with the only difference that $n, m = 1, ..., N/2, N/2 + 2, ..., N + 1$ with even $N$ and $H^{inf}_{N/2+1,N/2+1} = i\gamma_0$.

In all the above three cases, the associated matrices are non-Hermitian and symmetric meaning $H^T = H$, unlike the Hatano-Nelson problems [53]. Thus their eigenvalue spectrum ($\{\lambda_n\}$) is in general complex (Supplementary information). Depending on the value of the global gain-loss amplitude $\gamma$, we find that the first two lattices exhibit an EP2 and the interface lattice an EP3.

Complex unstructured Pseudospectra. In the vast majority of previous studies, the sensitivity of a non-Hermitian system under external perturbations was analyzed using the semi-analytical techniques based on perturbation theory, for systems of usually only a small number of waveguides. For lattices of our type such an approach is rather problematic and is not easily applicable. Therefore we introduce an alternative and general computational framework based on pseudospectra [49]. The so-called geometrical spectrum or pseudospectrum is a systematic mathematical way to study the sensitivity of a matrix/operator on external perturbations, without relying on perturbation theory. It has been extensively used in the context of fluid mechanics [50, 52] non-normal networks [51], and transient growth physics [53, 54]. However, it is largely unknown in optics despite being ideal for studying non-Hermitian systems [50, 62]. For Hermitian matrices, the spectrum and the pseudospectrum are almost identical, whereas for non-Hermitian could be significantly different. The measure of how different the two spectra might be, depends on the degree of the non-orthogonality of the corresponding eigenmodes. Thus the pseudospectrum of a non-normal matrix provides us with complete information, beyond the conventional spectrum.

The most basic definition of the $\epsilon$-pseudospectrum of a non-Hermitian matrix $H$, with $\sigma(H)$-spectrum, is the union of all spectra of the matrices $H + E$, where $E$ is a full complex random matrix (with respect to its matrix elements), with $||E|| < \epsilon$. More specifically, $\sigma_{\epsilon}(H)$ is the set of $z \in \mathbb{C}$ such that $z \in \sigma(H + E)$ for some $E \in \mathbb{C}^{N \times N}$ with $||E|| < \epsilon$. In particular, $E = \epsilon \frac{E}{||E||}$, where, $\epsilon$ defines the perturbation strength and the matrix $E$ is the perturbation matrix before the normalization. Our results for the three NHSSH lattices, are shown in Fig. 1. The red dots represent the spectrum of the non-Hermitian system and the yellow dots the corresponding pseudospectra on the complex plane. The global gain-loss amplitude is close and below the EP2 for the infinite and finite lattices, as well as, below the EP3 for the interface lattice. As we can see the three spectra are entirely real (unbroken PT-symmetry regime) and thus lying on the real axis (the gap is open but small). However, the corresponding pseudospectra are extended on the complex plane, and their size is related to the sensitivity of the lattice. In particular, the eigenstates close to the edges of the gap are most sensitive and these away from that gap are robust. The non-orthogonality of the eigenmodes close to the gap is significantly higher than the rest of the eigenstates. From the complex eigenvalue bifurcation curves vs $\gamma$ (see SI), we can identify the modes that coalesce and form the EP2 for the infinite and finite lattices and the EP3 for the interface lattice. The geometrical size of the main lobe of the complex pseudospectrum as a function of the perturbation strength $\epsilon$, has a square-root and cubic-root dependence, that are characteristics of the EP2 and EP3, respectively. Another conclusion we can draw, is that as we can see, in terms of the bulk-edge correspondence, the sensitivity is similar for both finite and infinite lattices. This is not the case for asymmetrically coupled lattices (like the Hatano-Nelson model [53]), because of the non-Hermitian skin effect [63].

Structured Pseudospectra. Since the applied perturbations are complex and applied everywhere, even in the zero entries of the $H$-matrix, we would like to examine more realistic and experimentally relevant perturbations. Such perturbations are called structured perturbations and they define the structured pseudospectrum [49], which is ideal for studying the sensitivity of our NHSSH lattice. We define the structured pseudospectrum $\sigma_{\epsilon, str}^2(H)$ of the Hamiltonian $H$, as $\sigma_{\epsilon, str}^2(H) \equiv \bigcup_{j=1..s \text{-structured}} ||E_j|| < \epsilon \sigma(H + E_j)$, where $s$ is the number of different realizations of the structured perturbations. The main difference from the unstructured pseudospectrum is that the matrix $E$ is not full but has a particular structure that stems from the physics of the problem. For example if the external perturbations are applied only on the index modulation, then $E$ is diagonal. If they are applied to the coupling coefficients, then the ±1-diagonals are non-zero. In particular, we consider perturbations on the coupling coefficients i.e., the elements of the matrix are $E_{nm} = \delta_{n+1,m} \cdot \epsilon_n + \delta_{n,m+1} \cdot \epsilon_m$, diagonal perturbations with
FIG. 1. Complex pseudospectra of the three NHSSH lattices: infinite (left), finite (center, $N = 80$) for $\gamma = 2$, and interface (right, $N = 81$) for $\gamma = 2.0159$ and $\gamma_0 = 0$. In all cases the coupling constants are $c_1 = 3$ and $c_2 = 1$. On the top row the red, green and gray colors illustrate the gainy, lossy, and neutral waveguide, respectively. On the middle row, the spectra (red dots) and corresponding 0.0001-pseudospectra (100 realizations-yellow dots), for the three lattices are presented on the complex plane. Insets show a magnified view of the white-dashed area (for 1000 realizations). We have to note here that in the interface lattice, the central red dot at $[0, 0]$ consists of three eigenvalues. We do not show the edge states of the finite SSH, which are located at $[0, \pm 2i]$ and have perturbations of the order of half the $\varepsilon$. On the bottom row, the order of the exceptional point that each lattice exhibits is shown.

FIG. 2. Structured pseudospectra $\sigma_{0.01}^\text{str}(H)$, $s = 60$ of infinite (black dots) and finite (green dots) NHSSH lattices. The spectrum (red dots) in all cases is also plotted for comparison in the complex plane. (a,b,c) correspond to off-diagonal (on coupling constants), on-diagonal and combined perturbations, respectively. For each type of perturbations one of the edge states of the finite lattice is shown in right column with $s = 1000$ realizations.

$E_{nm} = \delta_{n,m} \cdot \epsilon_n$, and combination of the previous two with $E_{nm} = \delta_{n,m} \cdot \epsilon_m + \delta_{n+1,m} \cdot \epsilon_n + \delta_{n,m+1} \cdot \epsilon_m$. For the case of the interface lattice we perturb the coupling constants as follows, $E_{nm}^\text{int} = \delta_{n+1,m+1} \cdot \epsilon_{(n \text{ mod } 2) + 1} + \delta_{n,m+1} \cdot \epsilon_{(m \text{ mod } 2) + 1}$. The complex numbers $\epsilon$ are drawn from the standard normal distribution.

Structured pseudospectra of the infinite and finite lattices—Let us now consider, the infinite and finite NHSSH. For these lattices we have calculated three different types of structured pseudospectra $\sigma_{0.01}^\text{str}(H)$, that are shown in Fig. 2. In particular, Figs. 2(a,b,c) correspond to coupling, diagonal and combined perturbations, respectively. In terms of the EP’s sensitivity, both approaches give us similar results, something that is expected [49] for symmetric matrices. The difference from the complex pseudospectra (Fig. 1) is that the structured perturbations on the couplings reveal the topological robustness of the edge states, as is evident from Fig. 2(a).

Structured pseudospectra of the interface lattice—Now we study the effect of different structured perturbations on the spectrum of the interface lattice. Let us start with chiral complex perturbations that physically correspond to changes on the coupling coefficients $c_1, c_2$. Since they respect the chiral symmetry of the lattice, we expect the zero-mode to be robust to such external perturbations. In order to systematically examine such a problem, we calculate the corresponding structured pseudospectrum. This time the perturbation matrix is again of size $N \times N$ but with random complex elements that respect chiral symmetry. Our results are shown in Figs. 3(a,b). In particular, we plot on the complex plane the eigenvalue spectrum of the lattice (red dots) and the corresponding structured pseudospectrum (black dots). As we can see, it is indeed true that the zero-mode is topologically protected, since the size of the corresponding pseudoeigenvalue cloud (black dots) is zero. The rest of the supermodes of the

Structured pseudospectra of the infinite and finite lattices—Let us now consider, the infinite and finite NHSSH.
The blue line is $\approx \frac{1}{3}$, and the associated pseudospectra have the perturbations are real or complex doesn’t apparently survive, and non-Hermitian sensitivity determines the behavior of our lattice close to the EP3. The fact that the topological robustness of the zero mode does not survive, and non-Hermiticity has an immediate impact on lattice’s sensitivity rather than its topological structure. More specifically, we studied the effect of symmetries of the applied perturbations on the overall sensitivity of various lattices. In particular, the complex-unstructured and diagonal-structured ($PT$ or not) pseudospectra describe the enhanced sensitivity (algebraic root dependence) around the EP’s. On the other hand, the topological robustness of the zero mode is revealed only by the structured pseudospectra. In other words, below the HEP’s (non-zero gap), the chiral structured perturbations uncover the topological protection of the zero mode. Exactly at the HEP’s (zero gap) the situation is more complex. Thus depending on the modal content of the HEP (meaning if the zero mode participates on the EP or not) the topological state is less sensitive on the infinite and finite NHSSH lattices, but it is the most sensitive case of the interface NHSSH. Furthermore, the chiral symmetry of the applied perturbation matrix $E$, leads to reduction of the apparent sensitivity by one order (form EP3 to EP2, in the interface lattice), when the lattice is exactly at the EP3. Finally, we have examined

![Diagram](image-url)

**FIG. 3.** Structured pseudospectra $\sigma_{10^{-6}}(H)$ of the interface NHSSH lattice. (a) We include only chiral complex perturbations on the $\pm 1$ diagonals for $s = 1000$ realizations, and global gain/loss amplitude $\gamma = 2.0155$ (below the EP3). As we can see the zero mode indeed remains robust. (b) Magnified view of the selected area (green dashed line) which corresponds to the five modes closer to the origin of the complex plane. The size of the gap is $\approx 0.1$. (c) Complex diagonal perturbations for $s = 1000$ realizations and $\gamma = 2.0159293$. (d) Pseudospectral radius (red line) of (c) as a function of $\epsilon \in [10^{-6}, 10^{-4}]$. The blue line is $\approx \epsilon^{1/3}$ and shown for comparison.

Discussion and Conclusions.— As a general conclusion, non-Hermiticity has an immediate impact on lattice’s sensitivity rather than its topological structure. More specifically, we studied the effect of symmetries of the applied perturbations on the overall sensitivity of various lattices. In particular, the complex-unstructured and diagonal-structured ($PT$ or not) pseudospectra describe the enhanced sensitivity (algebraic root dependence) around the EP’s. On the other hand, the topological robustness of the zero mode is revealed only by the structured pseudospectra. In other words, below the HEP’s (non-zero gap), the chiral structured perturbations uncover the topological protection of the zero mode. Exactly at the HEP’s (zero gap) the situation is more complex. Thus depending on the modal content of the HEP (meaning if the zero mode participates on the EP or not) the topological state is less sensitive on the infinite and finite NHSSH lattices, but it is the most sensitive case of the interface NHSSH. Furthermore, the chiral symmetry of the applied perturbation matrix $E$, leads to reduction of the apparent sensitivity by one order (form EP3 to EP2, in the interface lattice), when the lattice is exactly at the EP3. Finally, we have examined

![](image-url)

**Table:**

| Parameter     | Value   |
|---------------|---------|
| $\gamma$      | 2.0155  |
| $s$           | 1000    |
| $\epsilon$    | $10^{-6}$ to $10^{-4}$ |
FIG. 4. Topological robustness and non-Hermitian sensitivity for chiral perturbations, by locally varying the interface channel’s potential strength by nonlinearity. Magenta dots correspond to a lattice with $\gamma = 0.001$, gap size $\approx 4$ and $|\gamma_0| = 1$. Yellow dots are for $\gamma = 1$ with gap size $3.5$ and $|\gamma_0| = 1$. Finally, blue dots are for $\gamma = 2$ with gap size $\approx 0.5$ and $|\gamma_0| = 0.2$. For each of these defect-mode eigenvalues we add perturbations on the coupling coefficients to study their sensitivity. Green, black and red dots denote the corresponding eigenvalue fluctuation for 200 realizations of added perturbation of strength 20%, 20% and 0.85% respectively. Inset depicts a magnified view for the non-Hermitian case with $\gamma = 2$.

the nonlinearly controlled pseudospectrum of the zero mode, where sensitivity and memory of the topological robustness co-exist. Our results highlight for the first time the fundamental question of the interplay between ultra sensitivity and topological protection in the unique framework of pseudospectra theory and may provide insight for the study of other lattices of non-Hermitian topological physics.

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I. REFERENCES

[1] N. Moiseyev, Non-Hermitian Quantum Mechanics (Cambridge, New York, 2011).
[2] M. V. Berry, Czechoslovak J. Phys. 54, 1039 (2004).
[3] W. D. Heiss, J. Phys. A: Math. Gen. 37, 2455 (2004).
[4] J. Wiersig, S.-W. Kim, and M. Hentschel, Phys. Rev. A 78, 053809 (2008).
[5] S.-B. Lee et al., Phys. Rev. Lett. 103, 134101 (2009).
[6] C. M. Bender and S. Boettcher, Phys. Rev. Lett. 80, 5243 (1998).
[7] C. M. Bender, S. Boettcher, and P. N. Meisinger, J. Math. Phys. 40, 2201 (1999).
[8] C. M. Bender, D. C. Brody, and H. F. Jones, Phys. Rev. Lett. 89, 270401 (2002).
[9] R. El-Ganainy, K. G. Makris, D. N. Christodoulides, and Z. H. Musslimani, Opt. Lett. 32, 2632 (2007).
[10] K. G. Makris, R. El-Ganainy, D. N. Christodoulides, and Z. H. Musslimani, Phys. Rev. Lett. 100, 103904 (2008).
[11] Z. H. Musslimani, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, Phys. Rev. Lett. 100, 030402 (2008).
[12] A. Guo, G. J. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. A. Siviloglou, and D. N. Christodoulides, Phys. Rev. Lett. 103, 093902 (2009).
[13] C. E. Rüter, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, M. Segev, and D. Kip, Nat. Phys. 6, 192 (2010).
[14] Y. D. Chong, L. Ge, and A. D. Stone, Phys. Rev. Lett. 106, 093902 (2011).
[15] L. Ge, Y. D. Chong, and A. D. Stone, Phys. Rev. A 85, 023802 (2012).
[16] P. Ambichl, K. G. Makris, L. Ge, Y. D. Chong, A. D. Stone, and S. Rotter, Phys. Rev. X 3, 041030 (2013).
[17] A. Regensburger, C. Bersch, M.-A. Miri, G. Onishchukov, D. N. Christodoulides, and U. Peschel, Nature 488, 167 (2012).
[18] L. Feng, Y.-L. Xu, W. S. Fegadolli, M.-H. Lu, J. E. B. Oliveira, V. R. Almeida, Y.-F. Chen, and A. Scherer, Nat. Mater. 12, 108 (2013).
[19] B. Peng, S. K. Özdemir, F. Lei, F. Monifi, M. Gianfreda, G. L. Long, S. Fan, F. Nori, C. M. Bender, and L. Yang, Nat. Phys. 10, 394 (2014).
[20] L. Feng, Z. Jing Wong, R.-M. Ma, Y. Wang, and X. Zhang, Science 346, 972 (2014).
[21] H. Hodaei, M.-A. Miri, M. Heinrich, D. N. Christodoulides, and M. Khajavikhan, Science 346, 975 (2014).
[22] B. Peng, S. K. Özdemir, S. Rotter, H. Yilmaz, M. Liertzer, F. Monifi, C. M. Bender, F. Nori, and L. Yang Science 346, 328 (2014).
[23] S. Assawaworrarit, X. Yu, and S. Fan, Nature 546, 387 (2017).
[24] J. Zhang, B. Peng, S.K. Özdemir, K. Pichler, D.O. Krimer, G. Zhao, F. Nori, Y. Liu, S. Rotter, and L. Yang, Nat. Phot. 12, 479 (2018).
[25] H Hodaei, AU Hassan, S Wittek, H Garcia-Gracia, R El-Ganainy, D. N. Christodoulides, and M. Khajavikhan, Nature 548, 187 (2017).
[26] M.P. Hokmabadi, A. Schumer, D.N. Christodoulides, and M. Khajavikhan Nature 576, 70 (2019).
[27] T. Kottos, Nat. Phys. 6, 166 (2010).
[28] D.F. Pile, and D. N. Christodoulides, Nat. Phot. 11, 742 (2017).
[29] R. El-Ganainy, K.G. Makris, M. Khajavikhan, Z.H. Musslimani, S. Rotter, and D. N. Christodoulides, Nat. Phys. 14, 11 (2018).
[30] S.K. Özdemir, S. Rotter, F. Nori, and L. Yang, Nat. mater. 18, 783 (2019).
[31] M.A. Miri and A. Alu, Science 363, eaar7709 (2019).
[32] N. Malkova, I. Hromada, X. Wang, G. Bryant, Z. Chen, Opt. Lett. 34, 1633 (2009).
[33] Z. Wang, Y. Chong, J. D. Joannopoulos, M. Soljacic, Nature 461, 772 (2009).
[34] M.C. Rechtsman, J.M. Zeuner, Y. Plotnik, Y. Lumer, D. Podolsky, F. Dreisow, S. Nolte, M. Segev, and A. Szameit, Nature 496, 196 (2013).
[35] M.A. Bandres, M.C. Rechtsman, and M. Segev, Phys. Rev. X 6, 011016 (2016).
[36] A. Blanco-Redondo, B. Bell, D. Oren, B. J. Eggleton, M. Segev, Science 362, 856 (2020).
[37] S. Xia, D. Jukic, N. Wang, D. Smirnova, L. Smirnov, L. Tang, D. Song, A. Szameit, D. Leykam, J. Xu, Z. Chen, H. Buljan, Light Sci. Appl. 9, 147 (2020).
[38] M. Parto, S. Wittek, H. Hodaei, G. Harari, M.A. Bandres, J. Ren, M. Rechtsman, M. Segev, D. N. Christodoulides, and M. Khajavikhan Phys. Rev. Lett. 120, 113901 (2016).
[39] G. Harari, M. Bandres, Y. Lumer, M. Rechtsman, Y. Chong, M. Khajavikhan, D.N. Christodoulides and M. Segev, Science 359, 6381 (2018).
[40] M. Bandres, S. Wittek, G. Harari, M. Parto, J. Ren, M. Segev, D.N. Christodoulides and M. Khajavikhan, Science 359, 6381 (2018).
[41] H. Zhao, X. Qiao, T. Wu, B. Midya, S. Longhi, L. Feng, Science 365, 1163 (2019).
[42] Y.G.N. Liu, P.S. Jung, M. Parto, D.N. Christodoulides, and M. Khajavikhan Nat. Phys. 17, 704 (2021).
[43] W. P. Su, J. R. Schrieffer, A. J. Heeger, Phys. Rev. Lett. 42, 1698 (1979).
[44] S. Weimann, M. Kremer, Y. Plotnik, Y. Lumer, S. Nolte, K.G. Makris, M. Segev, M. Rechtsman, and A. Szameit Nat. Mat. 16, 433 (2017).
[45] S. Xia, D. Kaltsas, D. Song, I. Komis, J. Xu, A. Szameit, H. Buljan, K.G. Makris, and Z. Chen, Science 372 72 (2021).
[46] Z. Gong, Y. Ashida, K. Kawabata, K. Takasan, S. Higashikawa, and M. Ueda Phys. Rev. X 8, 031079 (2018).
[47] K. Kawabata, K. Shiozaki, M. Ueda, M. Sato, Phys. Rev. X 9, 041015 (2020).
[48] L. N. Trefethen and M. Embree, Spectra and Pseudospectra, (Princeton University Press, 2005).
[49] L.N. Trefethen, A.E. Trefethen, S.C. Reddy and T.A. Driscoll, Science 261, 578 (1993).
[50] L. N. Trefethen, SIAM Rev. 39 383 (1997).
[51] L. N. Trefethen, Acta Num. 8 247 (1999).
[52] N. Hatano, and D.R. Nelson, Phys. Rev. Lett. 77, 570 (1996).
[53] G. Baggio et al., Sci. Adv. 6: eaba2282 (2020).
[54] S. C. Reddy, P. J. Schmid and D. S. Henningson, SIAM J. Appl. Math 53, 15 (1993).
[55] S. C. Reddy, and D. S. Henningson, J. Fluid Mech. 252, 209 (1993).
[56] J. S. Baggett, T. A. Driscoll and L. N. Trefethen, Phys. Fluids 7, 833 (1995).
[57] P. J. Schmid, Phys. Plasmas. 7, 1788 (2000).
[58] K. G. Makris, L. Ge, and H. E. Türeci, Phys Rev. X 4, 041044 (2014).
[59] K. G. Makris, Phys Rev. E 104, 054218 (2021).
[60] S. Longhi, and P. Laporta, Phys. Rev. E 61 R 989 (2000).
[61] F. Papoff, G. D’Alessandro, and G.L. Oppo, Phys. Rev. lett. 100 123905 (2008).
[62] Y. Ashida, Z. Gong, and M. Ueda Adv. Phys. 69, 249 (2021).
[63] E.B. Davies, and M. Hager, J. Approx. Theory 156, 82 (2010).