The Fidelity of Recovery is Multiplicative

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Fawzi and Renner (arXiv:1410.0664) recently established a lower bound on the conditional quantum mutual information of tripartite quantum states $\rho_{ABC}$ in terms of the fidelity of recovery, i.e. the maximal fidelity of the state $\rho_{ABC}$ with a state reconstructed from its marginal $\rho_{BC}$ by acting only on the $C$ system. In this brief note we show that the fidelity of recovery is multiplicative by utilizing semi-definite programming duality. This allows us to simplify an operational proof by Brandao et al. (arXiv:1411.4921) of the above-mentioned lower bound that is based on quantum state redistribution. In particular, in contrast to the previous approaches, our proof does not rely on de Finetti reductions.

I. INTRODUCTION

The conditional quantum mutual information (CQMI) of a tripartite quantum state $\rho_{ABC}$ is defined as

$$I(A:B|C)_\rho := H(AC)_\rho + H(BC)_\rho - H(ABC)_\rho - H(C)_\rho,$$ \hspace{1cm} (1)

where $H(X)_\rho := -\text{tr}[\rho_X \log \rho_X]$ denotes the von Neumann entropy. The CQMI is a measure for the correlations between $A$ and $B$ from the perspective of $C$ and has an operational interpretation as the optimal quantum communication cost in quantum state redistribution [31]. Apart from that the CQMI has found manifold applications in information theory [6, 11], physics [15, 21], as well as computer science [8–10, 25].

A celebrated result known as strong subadditivity of entropy states that the CQMI is always non-negative [18],

$$I(A:B|C)_\rho \geq 0.$$ \hspace{1cm} (2)

Following a line of works (see [5, 15, 17, 32] and references therein), Fawzi and Renner have shown in a recent breakthrough result that the lower bound (2) can be improved to [13],

$$I(A:B|C)_\rho \geq -\log F(A:B|C)_\rho,$$ \hspace{1cm} (3)

where we have the fidelity of recovery,

$$F(A:B|C)_\rho := \sup_{\Gamma_{C\rightarrow AC}} F(\rho_{ABC}, (\mathcal{I}_B \otimes \Gamma_{C\rightarrow AC}) \rho_{BC}).$$ \hspace{1cm} (4)

Here, $\mathcal{I}_B$ denotes the identity channel on $B$, the supremum is taken over all quantum channels (completely positive and trace-preserving (cptp) maps) from $C$ to $AC$, and we use $F(\cdot, \cdot)$ to denote Uhlmann’s fidelity [26]: $F(\rho, \sigma) := (\text{tr}[\sqrt{\sqrt{\rho} \sigma \sqrt{\rho}}])^2$. In this note we investigate the fidelity of recovery as defined in (4) and give a simplified proof of the lower bound in (3).

In the following we denote the set of quantum states on a Hilbert space $A$ by $\mathcal{S}(A)$ and consequently use $\mathcal{S}(ABC)$ to denote states on a tripartite quantum system $ABC$. We use subscripts to indicate on which Hilbert spaces an operator acts. The remainder of the manuscript is structured as follows. In Section II we give a semi-definite programming (SDP) formulation of the fidelity of recovery and find the dual minimization problem. Based on this, we establish our main technical result in Section III and show that this quantity is multiplicative for product states. Namely, for any two states $\rho_{ABC} \in \mathcal{S}(ABC)$ and $\tau_{A'B'C'} \in \mathcal{S}(A'B'C')$, we establish the identity

$$F(AA'; BB'\,|\,CC')_{\otimes\tau} = F(A; B|C)_\rho \cdot F(A'; B'|C')_{\tau}.$$ \hspace{1cm} (5)

This in particular implies that there exists an optimal recovery map that has product structure as well. Using this finding we provide a simple operational proof of (3) in Section IV. This proof utilizes a connection between the fidelity of recovery and one-shot quantum state redistribution, using the ideas of Brandao et al. [7].

II. FIDELITY OF RECOVERY AS AN SDP

Mostly to simplify notation, we slightly generalize the concept of the fidelity of recovery as follows. Consider any two states $\rho_{AB}$ and $\sigma_{AC}$ (that may or may not have the same marginal on $A$) and define the fidelity of recovery as

$$F_{C\rightarrow B}(\rho_{AB}, \sigma_{AC}) := \sup_{\Gamma_{C\rightarrow B}} F(\rho_{AB}, (\mathcal{I}_C \otimes \Gamma_{C\rightarrow B}) \sigma_{AC}).$$ \hspace{1cm} (6)

We will drop the subscript $C \rightarrow B$ in the following when it is evident from the context on which systems the maps act. The fidelity of recovery of (4) is then simply given as

$$F_{C\rightarrow AC}(\rho_{ABC}, \rho_{BC}) = F(A; B|C)_\rho.$$ \hspace{1cm} (7)

We note that if $\rho_{AB} = \Phi_{AB}$ is a (normalized) maximally entangled state (MES) with $d_B = d_A$ then we get by

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1 The fidelity of recovery was independently defined and explored by Seshadreesan and Wilde [22].
standard SDP duality the conditional min-entropy [16, Theorem 2],
\[
F(\Phi_{AB}\|\sigma_{AC}) = \frac{1}{d_A} 2^{-H_{\text{min}}(A(C))} \\
= \frac{1}{d_A} \inf_{\omega_D \in \mathcal{S}(C)} \left\| \omega_C^{-1/2} \sigma_{AC} \omega_C^{-1/2} \right\|
\]
where \( \| \cdot \| \) denotes the operator norm. In general we do not know much about the optimal \( \omega_C \) in this expression except that for tensor-product states it is of tensor-product form as well. That is, the conditional min-entropy is additive [16]. Furthermore if \( \rho_{AB} = \psi_{AB} \) is a pure state with \( d_B = d_A \) then we get by standard SDP duality [16, Remark 1],
\[
F(\psi_{AB}\|\sigma_{AC}) = \inf_{\omega_D \in \mathcal{S}(C)} \left\| \psi_A^{1/2} \otimes \omega_C^{-1/2} \sigma_{AC} \psi_A^{1/2} \otimes \omega_C^{-1/2} \right\|
\]
This quantity was also studied by Barnum and Knill in the context of quantum error correction [2]. It turns out that the fidelity of recovery can be formulated as a SDP in general.

**Lemma 1.** Let \( \rho_{AB} \in \mathcal{S}(AB) \) and \( \sigma_{AC} \in \mathcal{S}(AC) \) and let \( \sigma_{ACD} \) be a purification of \( \sigma_{AC} \). Then, \( F_{C\to B}(\rho_{AB}\|\sigma_{AC}) \) as in (6) is the solution to the following minimization problem:
\[
\begin{align*}
\text{minimize} & : \quad \text{tr} \left[ \rho_{AB} R_{AB}^{-1} \right] \cdot \text{tr} \left[ \sigma_{AD} Q_{AD} \right] \\
\text{subject to} & : \quad R_{AB} > 0, \quad Q_{AD} > 0, \quad 1_B \otimes Q_{AD} \geq R_{AB} \otimes 1_D .
\end{align*}
\]

The proof uses SDP duality following the footsteps of Watrous’ lecture notes [27]. In particular Watrous discusses the dual SDP for the fidelity [28] in the form of Alberti [1],
\[
F(\rho,\sigma) = \inf_{R > 0} \text{tr} \left[ \rho R^{-1} \right] \cdot \text{tr} \left[ \sigma R \right].
\]
Our resulting dual program (11) can be thought of as the Alberti form of the fidelity of recovery (since it simplifies to (12) for trivial \( B \) and \( C \) systems).\(^2\)

**Proof of Lemma 1.** Without loss of generality, we assume that the marginals \( \rho_B \) and \( \sigma_C \) have full support. First note that using the purification \( \sigma_{ACD} \), we can write
\[
\Gamma_{C\to B}(\sigma_{AC}) = \text{tr}_D \left[ \Gamma_{C\to B}(\sigma_{ACD}) \right] = \text{tr}_D \left[ \Gamma_{C\to B}(\sqrt{\sigma_{AD}} \Psi_{AD:C} \sqrt{\sigma_{AD}}) \right],
\]
where we denote by \( \Psi_{AD:C} \) the (unnormalized) maximally entangled state between \( AD : C \) in the Schmidt decomposition of \( \sigma_{ACD} \). Then, define the Choi-Jamiliokowski state (unnormalized) of the map \( \Gamma \) as
\[
\tau_{ABD} = \Gamma_{C\to B}(\Psi_{AD:C}), \quad \text{tr}_B[\tau_{ABD}] = \Pi^\sigma_{AD},
\]
where \( \Pi^\sigma_{AD} \) is the projector onto the support of \( \sigma_{AD} \). Hence, we can write
\[
\Gamma_{C\to B}(\sigma_{AC}) = \text{tr}_D \left[ \sqrt{\tau_{ABD}} \sqrt{\tau_{ABD}} \right],
\]
and thus we can express the optimization problem for \( \Gamma_{C\to B} \) in terms of the Choi-Jamiliokowski state in (15). On the other hand, every state \( \tau_{ABD} \) satisfying (15) corresponds to a cptp map \( \Gamma_{C\to B} \). Hence, we can optimize over all Choi-Jamiliokowski states of the form (15) instead of cptp maps from \( B \) to \( C \).\(^3\) This leads to the following expression for the fidelity of recovery:
\[
F(\rho_{AB}\|\sigma_{AC}) = \sup \left\{ F(\rho_{AB}, \text{tr}_D \left[ \sqrt{\tau_{ABD}} \sqrt{\tau_{ABD}} \right]) : \tau_{ABD} \succeq 0, \quad \text{tr}_B[\tau_{ABD}] = \Pi^\sigma_{AD} \right\}.
\]

The primal problem above is obtained by considering a SDP for the root fidelity \( \sqrt{F(\rho,\sigma)} = \| \sqrt{\rho} \sqrt{\sigma} \|_1 \) as in [28, Sec. 2.1]. The square root of the fidelity of recovery in (17) is then written as
\[
\begin{align*}
\text{maximize} & : \quad \frac{1}{2} \text{tr} \left[ Z_{AB} + Z^\dagger_{AB} \right] \\
\text{subject to} & : \quad \tau_{ABD} \succeq 0, \quad Z_{AB} \in \mathcal{L}(AB), \quad \text{tr}_B[\tau_{ABD}] = \Pi^\sigma_{AD}, \\
& \quad \left( \frac{\rho_{AB}}{\text{tr}_D \left[ \sqrt{\tau_{ABD}} \sqrt{\tau_{ABD}} \right]} \right) \succeq 0,
\end{align*}
\]
where \( \mathcal{L} \) denotes the set of linear operators. In the next step we bring this program into standard form. We want to write the primal problem as a maximization over \( X \geq 0 \) of the functional \( \text{tr}[XA] \) subject to \( \Phi(X) = B \). Hence, we set
\[
X = \begin{pmatrix} X_{11} & Z^\dagger_{AB} & 0 \\ Z_{AB} & X_{22} & \cdot \\ \cdot & \cdot & \tau_{ABD} \end{pmatrix}, \quad A = \frac{1}{2} \begin{pmatrix} 0 & 1_{AB} & 0 \\ 1_{AB} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]
\[
B = \begin{pmatrix} \rho_{AB} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Pi^\sigma_{AD} \end{pmatrix},
\]
as well as
\[
\Phi(X) = \begin{pmatrix} X_{11} & 0 & 0 \\ 0 & X_{22} & \tau_D \left[ \sqrt{\tau_{ABD}} \sqrt{\tau_{ABD}} \right] \cdot \tau_B[\tau_{ABD}] \end{pmatrix}.
\]

\(^2\) Interestingly this Alberti form (11) does not directly simplify to (8) and (10) for the special case of MES and pure states.

\(^3\) We could as well replace \( \Pi^\sigma_{AD} \) with \( 1_{AD} \) above. This allows a larger class of maps but since we only consider their action on \( \sigma_{AD} \) this does not increase our solution space.
The variables with the placeholder ‘·’ are of no interest to us. The dual SDP is a minimization over self-adjoint $Y$ of the functional $\text{tr}[YB]$ subject to $\Phi(Y) \geq A$. The dual variables and adjoint map can be determined to be

$$Y = \begin{pmatrix} L_{AB} & \cdot & \cdot \\
\cdot & R_{AB} & \cdot \\
\cdot & \cdot & Q_{AD} \end{pmatrix}$$

(22)

with

$$\Phi(Y) = \begin{pmatrix} L_{AB} & 0 & 0 \\
0 & R_{AB} & 0 \\
0 & \cdot & -\sqrt{\sigma_{AD}}(R_{AB} \otimes 1_D)\sqrt{\sigma_{AD} + 1_B \otimes Q_{AD}} \end{pmatrix}.$$  

(23)

This leads to the following dual problem:

$$\begin{array}{l}
\text{minimize} : \quad \text{tr}[\rho_{AB}L_{AB}] + \text{tr}[\Pi_{AD}Q_{AD}] \\
\text{subject to :} \quad L_{AB}, R_{AB} \in \mathcal{H}(AB), \quad Q_{AD} \in \mathcal{H}(AD), \\
\quad 1_B \otimes Q_{AD} \geq \sqrt{\sigma_{AD}}(R_{AB} \otimes 1_D)\sqrt{\sigma_{AD}}, \\
\quad \begin{pmatrix} L_{AB} & 0 \\
0 & R_{AB} \\
0 & \cdot \end{pmatrix} \geq \frac{1}{2} \begin{pmatrix} 1_{AB} & 0 \\
0 & 1_{AB} \\
0 & \cdot \end{pmatrix},
\end{array}$$

(24)

where $\mathcal{H}$ denotes the set of self-adjoint operators. The Slater condition (cf. [27]) for strong duality is satisfied. The program (24) can be simplified further by the substitutions $L_{AB} \rightarrow \frac{1}{2}L_{AB}, \quad R_{AB} \rightarrow \frac{1}{2}R_{AB}$ and $Q_{AD} \rightarrow \frac{1}{2}\sqrt{\sigma_{AD}}Q_{AD}\sqrt{\sigma_{AD}}$, leaving us with

$$\begin{array}{l}
\text{minimize} : \quad \frac{1}{2}\text{tr}[\rho_{AB}L_{AB}] + \frac{1}{2}\text{tr}[\sigma_{AD}Q_{AD}] \\
\text{subject to :} \quad L_{AB}, R_{AB} \in \mathcal{H}(AB), \quad Q_{AD} \in \mathcal{H}(AD), \\
\quad 1_B \otimes Q_{AD} \geq R_{AB} \otimes 1_D, \\
\quad \begin{pmatrix} L_{AB} & 0 \end{pmatrix} \geq \frac{1}{2} \begin{pmatrix} 1_{AB} & \cdot \end{pmatrix}.
\end{array}$$

(25)

Now we note that the above matrix inequality holds if and only if $L_{AB}, R_{AB} \geq 0$ and $R_{AB} \geq L_{AB}^{-1}$. Without loss of generality we can choose $R_{AB} = L_{AB}^{-1}$, and our problem simplifies to

$$\begin{array}{l}
\text{minimize} : \quad \frac{1}{2}\text{tr}[\rho_{AB}R_{AB}^{-1}] + \frac{1}{2}\text{tr}[\sigma_{AD}Q_{AD}] \\
\text{subject to :} \quad R_{AB}, Q_{AD} \geq 0, \\
\quad 1_B \otimes Q_{AD} \geq R_{AB} \otimes 1_D.
\end{array}$$

(26)

Finally, we follow the argument leading Watrouss to Alberti’s expression for the fidelity [27, Lecture 8]. We first remark that

$$\frac{1}{2}\text{tr}[\rho_{AB}R_{AB}^{-1}] + \frac{1}{2}\text{tr}[\sigma_{AD}Q_{AD}] \geq \sqrt{\text{tr}[\rho_{AB}R_{AB}^{-1}] \cdot \text{tr}[\sigma_{AD}Q_{AD}]}$$

(27)

by the arithmetic–geometric mean inequality, with equality when the two terms are equal. However, it is easy to see that for any feasible pair $(R_{AB}, Q_{AD})$, there exists a constant $\lambda \in \mathbb{R}$ such that two trace terms evaluated for $(\lambda R_{AB}, \lambda Q_{AD})$ are equal (and clearly $(\lambda R_{AB}, \lambda Q_{AD})$ is also feasible). Hence, restricting our optimization to such rescaled pairs of operators, and going from the root fidelity to the fidelity again we find that

$$F(\rho_{AB}||\sigma_{AC}) = \inf_{1_B \otimes Q_{AD} \geq R_{AB} \otimes 1_D} \text{tr}[\rho_{AB}R_{AB}^{-1}] \cdot \text{tr}[\sigma_{AD}Q_{AD}].$$

(28)

This concludes the proof.

$$\square$$

### III. FIDELITY OF RECOVERY IS MULTIPLICATIVE

As a direct consequence of this formulation of the problem we see that the fidelity of recovery is multiplicative.

**Proposition 2.** For any $\rho_{AB} \in S(AB), \quad \tau_{A'B'} \in S(A'B'), \quad \sigma_{AC} \in S(AC)$ and $\omega_{AC'} \in S(A'C')$, we have

$$F(\rho_{AB} \otimes \tau_{A'B'} || \sigma_{AC} \otimes \omega_{AC'}) = F(\rho_{AB} || \sigma_{AC}) \cdot F(\tau_{A'B'} || \omega_{AC'}).$$

(29)

**Proof.** From the definition in (6) it is evident that if we restrict to recovery maps that have a product structure, we immediately find

$$F(\rho_{AB} \otimes \tau_{A'B'} || \sigma_{AC} \otimes \omega_{AC'}) \geq F(\rho_{AB} || \sigma_{AC}) \cdot F(\tau_{A'B'} || \omega_{AC'}).$$

(30)

To establish the equality, we take a closer look at (28). Here we simply note the following. For every two pairs of feasible operators $(R_{AB}, Q_{AD})$ and $(R_{A'B'}, Q_{A'D'})$ for $F(\rho_{AB} || \sigma_{AC})$ and $F(\tau_{A'B'} || \omega_{AC'})$, respectively, we have

$$1_B \otimes Q_{AD} \geq R_{AB} \otimes 1_D \quad \land \quad 1_B \otimes Q_{A'D'} \geq R_{A'B'} \otimes 1_D \implies 1_{BB'} \otimes Q_{AD} \otimes Q_{A'D'} \geq R_{AB} \otimes R_{A'B'} \otimes 1_{DD'}.$$

(31)

Hence, $(R_{AB} \otimes R_{A'B'}, Q_{AD} \otimes Q_{A'D'})$ is a feasible pair for $F(\rho_{AB} \otimes \tau_{A'B'} || \sigma_{AC} \otimes \omega_{AC'})$, and, thus,

$$F(\rho_{AB} \otimes \tau_{A'B'} || \sigma_{AC} \otimes \omega_{AC'}) \leq \text{tr}[\rho_{AB}R_{AB}^{-1}] \cdot \text{tr}[\tau_{A'B'}R_{A'B'}^{-1}] \cdot \text{tr}[\sigma_{AD}Q_{AD}] \cdot \text{tr}[\omega_{A'D'}Q_{A'D'}].$$

(32)

Since this holds for all feasible operators, we conclude that

$$F(\rho_{AB} \otimes \tau_{A'B'} || \sigma_{AC} \otimes \omega_{AC'}) \leq F(\rho_{AB} || \sigma_{AC}) \cdot F(\tau_{A'B'} || \omega_{AC'}).$$

(33)

$$\square$$

### IV. THE FAWZI-RENNER BOUND WITHOUT DE FINETTI REDUCTIONS

Brandao et al. [7] show that the Fawzi-Renner lower bound (3) can be deduced from the operational interpretation of the conditional quantum mutual information.
as the optimal quantum communication cost in quantum state redistribution [31]. We simplify their proof and in particular get rid of the continuity and representation theoretic arguments (de Finetti reductions). Instead we leverage on the multiplicativity of the fidelity of recovery (Proposition 2).

**Theorem 3.** For any $\rho_{BCD} \in S(BCD)$, we have

$$I(C : D|B)_\rho \geq -\log F(C; D|B)_\rho.$$  \hfill (34)

**Proof.** We use the same setup as in [7]. Let $\rho_{ABCD}$ be a purification of $\rho_{BCD}$ and consider a state redistribution protocol for sending $C$ from Alice (with $AC$) to Bob (with $B$) relative to the environment (the $D$ system). The protocol uses some entanglement $\Phi_{T_A T_B}$ between Alice ($T_A$) and Bob ($T_B$) and some quantum communication $Q$ from Alice to Bob. We denote the encoding map by $E_{ACT_A \rightarrow AQ}$, the decoding map by $D_{BT_B Q \rightarrow BC}$, and define

$$\sigma_{AQ T_B BD} := (E_{ACT_A \rightarrow AQ} \otimes D_{BT_B Q \rightarrow BC}) (\rho_{ABCD} \otimes \Phi_{T_A T_B}).$$ \hfill (35)

Brandao et al. observe that the existence of a protocol with small error is sufficient to give some bounds on the performance of the decoder even if the quantum communication is omitted. This is a consequence of the standard operator inequality $d_X \cdot I_X \diamond \omega_Y \geq \omega_{XY}$. In our case,

$$\sigma_{QT_B BD} \leq d_Q \cdot 1_Q \otimes \sigma_{TB BD}$$ \hfill (36)

$$= d_Q^2 \cdot 1_Q \otimes \frac{1_{TB}}{d_{TB}} \otimes \rho_{BD}.$$ \hfill (37)

which again implies

$$d_Q^2 \cdot D_{BT_B Q \rightarrow BC} \left( \frac{1}{d_{Q}} \otimes \frac{1_{TB}}{d_{TB}} \otimes \rho_{BD} \right)$$

$$\geq D_{BT_B Q \rightarrow BC} (\sigma_{QT_B BD}).$$ \hfill (38)

Due to the operator monotonicity of the square root function we then get

$$d_Q^2 \cdot F \left( \rho_{BCD}, D_{BT_B Q \rightarrow BC} \left( \frac{1}{d_{Q}} \otimes \frac{1_{TB}}{d_{TB}} \otimes \rho_{BD} \right) \right)$$

$$\geq F \left( \rho_{BCD}, D_{BT_B Q \rightarrow BC} (\sigma_{QT_B BD}) \right).$$ \hfill (39)

By the achievability of one-shot quantum state redistribution [3, Theorem 2] (see also [12] for an alternative bound), we find that for any $\varepsilon > 0$ small enough,

$$F \left( \rho_{BCD}, D_{BT_B Q \rightarrow BC} (\sigma_{QT_B BD}) \right) \geq 1 - 144 \varepsilon^2$$ \hfill (40)

for a quantum communication $Q$ of size

$$\log d_Q \leq \frac{1}{2} \left( H_{\text{max}}^\varepsilon (C|B)_\rho - H_{\text{min}}^\varepsilon (C|BD)_\rho \right) + \log(4/\varepsilon^4).$$ \hfill (41)

Here, $H_{\text{min}}^\varepsilon$ and $H_{\text{max}}^\varepsilon$ denote the smooth conditional min- and max-entropy, respectively.\(^4\) Hence, we can use (39) to estimate

$$H_{\text{max}}^\varepsilon (C|B)_\rho - H_{\text{min}}^\varepsilon (C|BD)_\rho$$

$$\geq - \log F \left( \rho_{BCD}, D_{BT_B Q \rightarrow BC} \left( \frac{1}{d_{Q}} \otimes \frac{1_{TB}}{d_{TB}} \otimes \rho_{BD} \right) \right)$$

$$- \log(16/\varepsilon^8) + \log(1 - 144 \varepsilon^2),$$ \hfill (42)

$$\geq - \log \sup_{\Gamma_{B \rightarrow BC}} F \left( \rho_{BCD}, \Gamma_{B \rightarrow BC} (\rho_{BD}) \right)$$

$$- \log(16/\varepsilon^8) + \log(1 - 144 \varepsilon^2),$$ \hfill (43)

and find that

$$H_{\text{max}}^\varepsilon (C|B)_\rho - H_{\text{min}}^\varepsilon (C|BD)_\rho$$

$$\geq - \log F(C; D|B)_\rho - O(\log(1/\varepsilon)).$$ \hfill (44)

By applying this bound to $\rho_{BCD}^\otimes n$, multiplying the resulting inequality by $1/n$, and letting $n \rightarrow \infty$ we find by the asymptotic equipartition property for the smooth conditional min- and max-entropy [23], and the multiplicativity of the fidelity of recovery (Proposition 2) that

$$I(C : D|B)_\rho = H(C|B)_\rho - H(C|BD)_\rho$$

$$\geq - \log F(C; D|B)_\rho.$$ \hfill (45)

By the achievability of one-shot quantum state redistribution [3, Theorem 2] (see also [12] for an alternative bound), we find that for any $\varepsilon > 0$ small enough,

$$F \left( \rho_{BCD}, D_{BT_B Q \rightarrow BC} (\sigma_{QT_B BD}) \right) \geq 1 - 144 \varepsilon^2$$ \hfill (40)

for a quantum communication $Q$ of size

$$\log d_Q \leq \frac{1}{2} \left( H_{\text{max}}^\varepsilon (C|B)_\rho - H_{\text{min}}^\varepsilon (C|BD)_\rho \right) + \log(4/\varepsilon^4).$$ \hfill (41)

\(^4\) See [24] for the precise definitions and their properties, which will however not be needed here.

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V. CONCLUSION

In this note we have shown that the fidelity of recovery is multiplicative (Proposition 2). From this we deduced a proof of the Fawzi-Renner lower bound on the conditional quantum mutual information without making use of de Finetti reductions (Theorem 3). We note that Brandao et al. [7] also show a potentially stronger lower bound in terms of a regularized relative entropy distance,

$$I(A : B|C)$$

$$\geq \lim_{n \rightarrow \infty} \frac{1}{n} \min_{\Gamma_{C^n \rightarrow A^n C^n}} D(\rho_{ABCD}^\otimes n || \Gamma_{C^n \rightarrow A^n C^n} (\rho_{BC}^\otimes n)).$$ \hfill (47)

They then use de Finetti reductions to get rid of the regularization and arrive at a bound in terms of a measured relative entropy distance. On the other hand, our Proposition 2 rephrased in terms of the (sandwiched) quantum Rényi divergence of order $\frac{1}{2}$ [19, 30] reads

$$\min_{\Gamma_{C^n \rightarrow A^n C^n}} \frac{1}{2} \left( D_{\frac{1}{2}}(\rho_{ABCD}^\otimes n || \Gamma_{C^n \rightarrow A^n C^n} (\rho_{BC}^\otimes n)) \right) = n \cdot \min_{\Gamma_{C \rightarrow AC}} \frac{1}{2} \left( D_{\frac{1}{2}}(\rho_{ABCD} || \Gamma_{C \rightarrow AC} (\rho_{BC})) \right).$$ \hfill (48)
And hence the Fawzi-Renner bound then follows from the
monotonicity of the quantum Rényi divergence in $\alpha$. Moreover, if such an additivity property would hold for any $\alpha \in (\frac{1}{2}, 1]$ we could find stronger bounds. We note that Li and Winter [17] have asked about a bound in terms of the relative entropy distance ($\alpha = 1$). However, our techniques do not extend to these cases since the corresponding optimization problems cannot be stated as SDPs. It also appears difficult to adapt the techniques of [14] to this setting.

Another interesting question is if we can estimate the performance of the optimal map $\Lambda_{C\rightarrow AC}$ in (4) with the Petz recovery map [20],

$$\Gamma^\text{Petz}_{C\rightarrow AC}(|\cdot|) := \rho_{AC}^{1/2} \rho_C^{-1/2} |\cdot| \rho_C^{-1/2} \rho_{AC}^{1/2}, \quad (49)$$

in the sense that the Petz map should perform nearly as good as the optimal map. This in analogy to what Barnum and Knill have shown for the special case of pure states, i.e. for the quantity (10). Finally, the results of Fawzi and Renner were recently generalized to the multiparty setting [29] as well as to a new lower bound on the monotonicity of the quantum relative entropy under ctp maps [4]. It might be insightful to study SPD techniques in these more general settings as well.

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