In this paper we study the cosmological dynamics of Randall- Sundrum braneworld type scenarios in which the five - dimensional Weyl tensor has a non - vanishing projection onto the three - brane where matter fields are confined. Using dynamical systems techniques, we study how the state space of Friedmann-Lemaître-Robertson-Walker (FLRW) and Bianchi type I scalar field models with an exponential potential is affected by the bulk Weyl tensor, focusing on the differences that appear with respect to standard general relativity and also Randall- Sundrum cosmological scenarios without the Weyl tensor contribution.

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I. INTRODUCTION

The notion that we live on a three - dimensional brane embedded in a higher - dimensional spacetime has attracted a considerable amount of interest over the last few years. These ideas have had a long history (for references see [1]) but its recent revival is due to recent work by Randall and Sundrum [2].

In such braneworld scenarios the ordinary matter is confined to the brane while gravity can propagate in the whole spacetime (known as the bulk). The effective four - dimensional gravity on the brane is modified by extra terms in the gravitational equations, one quadratic in the energy - momentum tensor and the other representing the electric part of the five - dimensional Weyl tensor. The dynamics of a braneworlds filled with a perfect fluid have been intensively investigated during last three years. In particular Campus and Sopuerta [3, 6] have recently studied the complete dynamics of Friedmann-Lemaître-Robertson-Walker (FLRW) and the Bianchi type I and V cosmological models with a barotropic equation of state using an approach first introduced by Goliath and Ellis [4] (see also [5] for a detailed discussion of the application of dynamical systems techniques to cosmology).

Their analysis led to the discovery of new critical points corresponding to the Binétruy - Deffayet - Langlois (BDL) models [4], representing the evolution at very high energies, where effects due to the extra dimension become dominant. These solutions appear to be a generic feature of the state space of more general cosmological models. They also showed that the state space contains new bifurcations, demonstrating how the dynamical character of some of the critical points changes relative to the general - relativistic case. Finally, they showed that for models satisfying all the ordinary energy conditions, causality requirements and for $\gamma > 1$, the anisotropy is negligible near the initial singularity, a result first demonstrated by Maartens et. al. [4].

Scalar field dynamics on the brane is considerably more interesting due to the fact that the equation of state parameter $\gamma$ is now dynamical. In a recent paper [10] we considered the dynamics of inflationary models with an exponential potential $V(\phi) = \exp(b\phi)$, an important class of inflationary models first considered by Burd and Barrow [11] and Haliwell [12] in the context of standard general relativity (GR). These models have a richer dynamical structure in the brane - world scenario and have the nice feature that inflation can take place with potentials ordinarily too steep to sustain inflation which result from the high - energy corrections to the Friedmann equation [3]. Indeed for $b < 0$ we found past attractors representing high energy steep inflationary models in which inflation ends naturally as the energy drops below the brane tension and the condition for inflation no longer holds.

In this paper we complete this analysis (hereafter referred to as Paper I), studying the effects of the five - dimensional Weyl tensor on FLRW and Bianchi type I cosmological models. In the case of FLRW models this study will be completely general whereas in the Bianchi type I case we will neglect (or constrain) the Weyl tensor components for which the theory does not provide evolution equations. The paper is organized as follows. In section II we briefly outline the geometric formulation of brane - world scenarios and give the main dynamical equations for FLRW and Bianchi type I cosmological models where the matter is described by a dynamical scalar field $\phi$ with an exponential potential $V(\phi) = \exp(b\phi)$. The dynamics of the FLRW and Bianchi type I cosmological models are presented in section III and IV respectively. Finally we end with a discussion of the main results of the analysis in section V. Through this paper we follow the following notation: upper - case Latin indices denote coordinates in the bulk spacetime $(A, B, \ldots = 0 - 4)$ whereas lower - case Latin indices denote coordinates on the brane $(a, b, \ldots = 0-3)$. We use physical units in which $c = 1$.

II. PRELIMINARIES

To begin with we introduce the geometrical framework of braneworld cosmological models and the main assump-
tions used to study FLRW and Bianchi type I cosmological models.

A. Basic equations of the brane-world

In Randall-Sundrum brane-world type scenarios matter fields are confined in a three-brane embedded in a five-dimensional spacetime (bulk). It is assumed that the metric of this spacetime, $g_{AB}^{(5)}$, obeys the Einstein equations with a negative cosmological constant $\Lambda^{(5)}$ 

\[ G_{AB}^{(5)} = -\Lambda g_{AB}^{(5)} + \kappa^{(5)} S_{AB} - \mathcal{E}_{ab} , \]  

where $\mathcal{E}_{ab}$ is the Einstein tensor, $\kappa^{(5)}$ denotes the five-dimensional gravitational coupling constant and $T_{AB}$ represents the energy-momentum tensor of the matter with the Dirac delta function reflecting the fact that matter is confined to the spacelike hypersurface $x^4 = \chi = 0$ (the brane) with induced metric $g_{AB}$ and tension $\lambda$.

Using the Gauss-Codacci equations, the Israel junction conditions and the $Z_2$ symmetry with respect to the brane the effective Einstein equations on the brane are

\[ G_{ab} = -\Lambda g_{ab} + \kappa^2 T_{ab} + \kappa^{(5)} S_{ab} - \mathcal{E}_{ab} , \] 

where $G_{ab}$ is the Einstein tensor of the induced metric $g_{ab}$. The four-dimensional gravitational constant $\kappa$ and the cosmological constant $\Lambda$ can be expressed in terms of the fundamental constants in the bulk ($\kappa^{(5)}, \Lambda^{(5)}$) and the brane tension $\lambda$ [20].

As mentioned in the introduction, there are two corrections to the general-relativistic equations. Firstly $S_{ab}$ represent corrections quadratic in the matter variables due to the form of the Gauss-Codacci equations:

\[ S_{ab} = \frac{1}{12} T T_{ab} - \frac{1}{4} T a^c T_{bc} + \frac{1}{8} g_{ab} [3 T cd T_{cd} - T^2] . \]  

Secondly $\mathcal{E}_{ab}$, corresponds to the “electric” part of the five-dimensional Weyl tensor $E_{ABCD}$ with respect to the normals, $n_A$ ($n^4 n_A = 1$), to the hypersurface $\chi = 0$, that is

\[ \mathcal{E}_{AB} = C_{ACBD} n^C n^D , \] 

representing the non-local effects from the free gravitational field in the bulk. The modified Einstein equations together with the conservation of energy-momentum equations $\nabla^a T_{ab} = 0$ lead to a constraint on $S_{ab}$ and $\mathcal{E}_{ab}$:

\[ \nabla^a (\mathcal{E}_{ab} - \kappa^{(5)} S_{ab}) = 0 . \]

We can decompose $\mathcal{E}_{ab}$ into its ineducable parts relative to any timelike observers with 4-velocity $u^a$ ($u^a u_a = -1$):

\[ \mathcal{E}_{ab} = - \left( \frac{\kappa^{(5)}}{\kappa} \right)^4 \left[ (u_a u_b + \frac{1}{3} h_{ab} + 2 u_a Q_b) + P_{ab} \right] , \] 

where

\[ Q_a u^a = 0 , \quad P_{(ab)} = P_{ab} , \quad P^a_{ab} = 0 , \quad P_{ab} u^b = 0 . \] 

Here $\mathcal{U}$ has the same form as the energy-momentum tensor of a radiation perfect fluid and for this reason is referred to as the “dark” energy density of the Weyl fluid. $Q_a$ is a spatial and $P_{ab}$ is a spatial, symmetric and trace-free tensor. $Q_a$ and $P_{ab}$ are analogous to the usual energy flux vector $q^a$ and anisotropic stress tensor $\sigma_{ab}$ in General Relativity. The constraint equation (15) leads to evolution equations for $\mathcal{U}$ and $Q_a$, but not for $P_{ab}$ (see [14]).

B. Scalar field dynamics on the brane

In this paper we consider both FLRW and Bianchi I cosmological models on the brane where the matter is described by a dynamical scalar field $\phi$ with an exponential potential $V(\phi) = \exp(b \phi)$. In this case the fluid 4-velocity can be written as

\[ u^a = -\frac{\nabla_a \phi}{\phi} , \quad u^a u_a = -1 , \]

which makes it automatically orthogonal to the hypersurfaces of homogeneity (surfaces of constant $\phi$). With this choice of 4-velocity, the energy-momentum tensor $T_{\mu\nu}$ for a scalar field $\phi$ takes the form of a perfect fluid (See page 17 in [16] for details):

\[ T_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu} , \]

with

\[ \rho = \frac{1}{2} \dot{\phi}^2 + V(\phi) , \]

and

\[ p = \frac{1}{2} \dot{\phi}^2 - V(\phi) . \]

where $\phi$ is the momentum density of the scalar field and $V(\phi)$ is its potential energy. If the scalar field is not minimally coupled this simple representation is no longer valid, but it is still possible to have an imperfect fluid form for the energy-momentum tensor [17].

In the case of FLRW models, the effective Einstein Equations (13) lead to the conditions

\[ Q_a = P_{ab} = 0 , \]

and this, through the constraint (15), further implies

\[ D_a \mathcal{U} = 0 \iff \mathcal{U} = \mathcal{U}(t) , \]

where $D_a$ denotes the covariant derivative associated with the induced metric on the hypersurfaces of homogeneity ($h_{ab} \equiv g_{ab} + u_a u_b$). The situation is somewhat different when one considers Bianchi type I models. This time we obtain $Q_a = 0$ but we do not get any restriction.
which is an evolution equation for the Hubble parameter $H^2 \equiv \nabla_a u^a$, and the Friedmann equation constraint (a first integral of (13):

$$H^2 = \frac{1}{3} \kappa^2 \rho \left(1 + \frac{\rho}{2\lambda}\right) - \frac{1}{6} R + \frac{1}{3} \sigma^2 + \frac{1}{3} \Lambda + \frac{2\mu}{\lambda \kappa^2},$$

(14)

where $^3R$ is the scalar curvature of the hypersurfaces orthogonal to the fluid flow and $2\sigma^2 = \sigma^{ab} \sigma_{ab}$ is the shear scalar ($\sigma_{ab} = h^a_c h^b_d \nabla_c u_d - H h_{ab}$). We consider only the case of a positive four-dimensional cosmological constant, i.e. $\Lambda \geq 0$. For Bianchi type I models $^3R$ vanishes whereas for FLRW models it is given by $^3R = 6ka^{-2}(t)$. On the other hand, the shear vanishes for FLRW models and for Bianchi type I models the evolution of the shear scalar is

$$\langle \sigma^2 \rangle = -6H \sigma^2.$$

In both cases the evolution equation for $\mathcal{U}$ is given by

$$\dot{\mathcal{U}} = -4H \mathcal{U},$$

and substituting for $\rho$ and $p$ from (10) and (11) into the energy conservation equation

$$\dot{\rho} + \Theta(p + \rho) = 0,$$

(15)

leads to the 1+3 form of the Klein-Gordon equation

$$\ddot{\phi} + \Theta \dot{\phi} + V'(\phi) = 0,$$

(16)

an exact ordinary differential equation for $\phi$ once the potential has been specified. It is convenient to relate $p$ and $\rho$ by the index $\gamma$ defined by

$$p = (\gamma - 1)\rho \quad \Leftrightarrow \quad \gamma = \frac{p + \mu}{\rho} = \frac{\dot{\phi}^2}{\rho}.$$  

(17)

This index would be constant in the case of a simple one-component fluid, but in general will vary with time in the case of a scalar field:

$$\dot{\gamma} = \Theta \gamma (\gamma - 2) - 2\gamma \frac{V'}{\phi}. $$

(18)

Notice that this equation is well-defined even for $\dot{\phi} \to 0$, since $\dot{\phi}^2 = \frac{\dot{\phi}^2}{\rho}$.

In the next two sections we study in detail the dynamics of (a) FLRW models with $\mathcal{U} \neq 0$ and (b) the anisotropic Bianchi I models with $\mathcal{U} \neq 0$ thus extending our recent work which only considered FLRW models with vanishing non-local energy density $\mathcal{U}$. The key difference between this work and a similar study carried out by Campos and Sopuerta is that we have a dynamical equation of state parameter $\gamma$ which evolves according to equation (15) above. In the case of an exponential potential $V(\phi) = \exp(b\phi)$, where $b \leq 0$ we obtain

$$\dot{\gamma} = 3H \gamma (\gamma - 2) + \sqrt{3\gamma (\gamma - 2)} b\sqrt{\frac{\rho}{3\kappa^2}}.$$  

(19)

In order to obtain a compact state space we will re-write the above dynamical equations in terms of dimensionless coordinates that are appropriately expansion normalized.

### III. ANALYSIS OF FLRW MODELS WITH EXPONENTIAL POTENTIALS

In the first paper of this series we had to distinguish between the case $^3R \leq 0$ and $^3R \geq 0$ when introducing appropriate expansion normalized coordinates for the FLRW models. In this paper, we have to consider four different subcases when studying the FLRW models and each one will have to be normalized by different quantities in order to obtain a compact state space: (A) $\mathcal{U} \geq 0$ and $^3R \leq 0$; (B) $\mathcal{U} \geq 0$ and $^3R \geq 0$; (C) $\mathcal{U} \leq 0$ and $^3R \leq 0$; (D) $\mathcal{U} \leq 0$ and $^3R \geq 0$. The total state space is composed of these 4 sectors, which are disconnected since trajectories cannot leave one sector and enter another. For a detailed description of the state space, in particular the invariant sets and the unusual geometry of sector D we refer to [7].

#### A. $\mathcal{U} \geq 0$ and $^3R \leq 0$

As explained by several authors (see e.g. [1, 2]), a compact state space is obtained by using the dimensionless general relativistic variables

$$\Omega_\rho \equiv \frac{\kappa^2 \rho}{3H^2}, \quad \Omega_k \equiv -\frac{^3R}{6H^2}, \quad \Omega_\Lambda \equiv \frac{\Lambda}{3H^2}$$

(20)

together with the following new variables:

$$\Omega_\gamma \equiv \frac{\kappa^2 \rho^2}{6\lambda H^2}, \quad \Omega_\mathcal{U} \equiv \frac{2\mu}{\lambda \kappa^2 H^2}.$$  

(21)

The coordinates represent the fractional contributions of the ordinary energy density, the curvature, the brane tension, and the dark radiation energy to the total energy density, normalized with respect to the Hubble parameter $H(t)$. In addition, we use the variable $\gamma \in [0, 2]$ in order to describe the dynamics of the scalar field $\phi$. Using these variables, the Friedmann constraint (4) becomes:

$$\Omega_\rho + \Omega_k + \Omega_\Lambda + \Omega_\gamma + \Omega_\mathcal{U} = 1.$$  

(22)
Since all of the terms in the sum are non-negative, each of the variables $\Omega_\rho$, $\Omega_k$, $\Omega_\Lambda$, $\Omega_\psi$ takes values in the interval $[0, 1]$, which means that we obtain a compact state space.

In order to decouple the evolution equation for $H$ from the evolution equations for the density parameters $\Omega_i$, we introduce the dimensionless time derivative

$$
\tau' \equiv |H|^{-1} d/dt .
$$

(23)

Inserting the Friedmann constraint (22) in order to keep the dimensionality of the state space as low as possible, it follows that the dynamics of the open or flat models with positive $\mathcal{U}$ are described by the following equations:

$$
\gamma' = \epsilon \sqrt{3 \gamma (\gamma - 2)} [\sqrt{3 \gamma} + \epsilon b \sqrt{1 - \Omega_k - \Omega_\Lambda - \Omega_\psi - \Omega_d}],
$$

$$
\Omega_k' = \epsilon (3 \gamma - 2) (1 - \Omega_k) + 3 \gamma (\Omega_\Lambda - \Omega_\psi) + (4 - 3 \gamma) \Omega_d |\Omega_k|,
$$

$$
\Omega_\Lambda' = \epsilon [3 \gamma (1 + \Omega_\Lambda - \Omega_\psi - \Omega_d) + 2 \Omega_k + 4 \Omega_d |\Omega_k|],
$$

(24)

$$
\Omega_\psi' = \epsilon (3 \gamma (\Omega_\Lambda - \Omega_\psi - 1 - \Omega_d) + 2 \Omega_k + 4 \Omega_d |\Omega_k|),
$$

$$
\Omega_d' = \epsilon (3 \gamma - 4) (1 - \Omega_d) + (2 - 3 \gamma) \Omega_k + 3 \gamma (\Omega_\Lambda - \Omega_\psi) |\Omega_d| .
$$

The equilibrium points of this dynamical system, their coordinates in state space and their eigenvalues are given in TABLE I below. Most of the equilibrium points of the system (23) are the same as those obtained in Paper I: the flat FLRW universe $F^2$ with stiff matter and $a(t) = t^{1/3}$; the Milne universe $M^2$ with stiff matter and $a(t) = t$; the Milne universe with $\gamma = 0$ ($M^0$); the flat non-relativistic, non-relativistic model $m^2$ with $\gamma = 2$ and scale factor $a(t) = t^{1/6}$, which has been discussed in detail in [3]; the flat FLRW universe $F^2_{1/3}$ with $\gamma = b^2$ and $a(t) = t^{2/b^2}$; and a set of universe models $X_{1/3}^2(b)$ with $\gamma = \frac{b^2}{3}$ and curvature $\Omega_k = 1 - \frac{2}{b^2}$ depending on the value of $b$.

Furthermore, we find the de-Sitter model $dS^2$ with $\gamma = 2$ and scale factor $a(t) = \exp(\sqrt{3/2} \lambda)$, and a set of non-relativistic critical points with $\gamma = 0$ which extends both in $\Omega_\Lambda$ and $\Omega_\psi$ direction. This set is denoted by $m^0_0(\Omega_\Lambda, \Omega_\psi)$; it contains the flat general relativistic FLRW model and the general relativistic de-Sitter model with constant energy density $\rho$ respectively. This 2-parameter set of equilibrium points contains the line of non-relativistic, non-relativistic critical points $m^0_0(\Omega_\Lambda)$ obtained in our previous paper [10].

In addition to these equilibrium points with vanishing $\mathcal{U}$, we find the following points with $\mathcal{U} > 0$: the points $R^2_+ \text{ and } R^2_-$ which have the same metric $a(t) = t^{1/2}$ as the flat the FLRW model with $\gamma = \frac{b^2}{3}$ (radiation), but have $\gamma = 0$, $2$ respectively and $\rho = k = \lambda = 0$. The point $A_{1/3}^2(b)$ describes a flat expanding model with $\gamma = \frac{b^2}{3}$ and vanishing brane-tension ($\Omega_\Lambda = 0$), but which has in general non-vanishing energy density and non-local energy density contributions $\Omega_\psi$, $\Omega_d > 0$. For $b = 0$, this model has maximal positive $\mathcal{U}$ ($\Omega_d = 1$, $\Omega_\psi = 0$), whereas for $b^2 = 4$ the model coincides with the flat FLRW model with $\gamma = 4/3$ ($\Omega_\rho = 1$, $\Omega_d = 0$). The scale factor of this model is proportional to $t^{1/2}$ for all values of $b$.

Notice that $E_{1/3}^{b^2/3}$ only occurs for $0 \leq b^2 \leq 6$, $X_{1/3}^{b^2/3}(b)$ occurs in this sector of state space for $b^2 \geq 2$, and $A_{1/3}^{b^2/3}(b)$ occurs only for $b^2 \geq 4$ in this sector of state space. All three points move in state space as we vary the parameter value $b$, but independent of the value of $b$ only occur in the expanding sector $\epsilon = +1$.

Note that equilibrium points $M^2_0$, $m^0_0(\Omega_\Lambda, \Omega_\psi)$ and $R^0_0$ are non-hyperbolic; we analyzed their nature using the perturbative methods described in Paper I [10].

B. $\mathcal{U} \geq 0$ and $3\mathcal{R} \geq 0$

Here we will use the variables $\gamma, Q, \tilde{\Omega}_\rho, \tilde{\Omega}_\Lambda, \tilde{\Omega}_d$, where we define

$$
Q \equiv \frac{H}{D}, \text{ with } D^2 = H^2 + \frac{1}{6} \mathcal{R}
$$

(25)

and the variables with a tilde are the analogues of those in [20] and [21] but normalized with respect to $D$ instead of $H$. The Friedmann equation (14) then becomes

$$
\tilde{\Omega}_\rho + \tilde{\Omega}_\Lambda + \tilde{\Omega}_d = 1 ,
$$

(26)

from which it can be seen that the quantities $\tilde{\Omega}_i$ take values in $[0, 1]$, whereas $Q$ takes values in $[-1, 1]$. Hence these coordinates define a compact state space. Introducing the time derivative

$$
\tau' \equiv D^{-1} d/dt ,
$$

the evolution equation for $D$ decouples from the evolution equations for $\tilde{\Omega}_i$. Inserting the Friedmann constraint (24), we find that the dynamics of the closed and flat models are described by

$$
\gamma' = \sqrt{3 \gamma (\gamma - 2)} [\sqrt{3 \gamma} Q + b \sqrt{1 - \tilde{\Omega}_\Lambda - \tilde{\Omega}_d}] ,
$$

$$
Q' = [1 - \frac{3}{2} \gamma (1 + \tilde{\Omega}_\Lambda - \tilde{\Omega}_\Lambda - \tilde{\Omega}_d) - 2 \tilde{\Omega}_d |\tilde{\Omega}_d| (1 - Q^2) ,
$$

$$
\tilde{\Omega}_\Lambda' = [3 \gamma (1 + \tilde{\Omega}_\Lambda - \tilde{\Omega}_d) + 4 \tilde{\Omega}_d |\tilde{\Omega}_d| Q \tilde{\Omega}_\Lambda] ,
$$

$$
\tilde{\Omega}_d' = [(3 \gamma - 4)(1 - \tilde{\Omega}_d) + 3 \gamma (\tilde{\Omega}_\Lambda - \tilde{\Omega}_\Lambda)] Q \tilde{\Omega}_d .
$$

(27)

The equilibrium points of this system, their coordinates in state space and their eigenvalues are given in TABLE II. We recover the equilibrium points obtained in the previous subsection which corresponded to flat models. In particular, for $b^2 \geq 4$ we find the equilibrium point $A_{1/3}^{b^2/3}(b)$, which for $b^2 \neq 4$ describes a model with non-vanishing local energy density $\mathcal{U}$.

In addition, we find a 2-parameter set of static ($Q = 0$) models $E$. These models occur for any fixed values of coordinates $(\gamma^*, 0, \tilde{\Omega}_\Lambda^*, \tilde{\Omega}_\Lambda^*, \tilde{\Omega}_d^*)$ subject to the constraints

$$
1 - \frac{3}{2} \gamma^* (1 + \tilde{\Omega}_\Lambda^* - \tilde{\Omega}_\Lambda^* - \tilde{\Omega}_d^*) - 2 \tilde{\Omega}_d^* = 0
$$

(28)
TABLE I: This table gives the coordinates and eigenvalues of the critical points with $U \geq 0$ and $3R \leq 0$. We have defined $\psi = \sqrt{\frac{b_1^2}{\chi} - 3}$, $\zeta = \sqrt{\frac{b_2^2}{\chi} - 15}$.

| Model | Coordinates | Eigenvalues |
|-------|-------------|-------------|
| $F_2^b$ | $(2, 0, 0, 0, 0)$ | $(6e + \sqrt{6b}, 4e, 6e, -6e, 2e)$ |
| $M_1^b$ | $(0, 1, 0, 0, 0)$ | non-hyperbolic |
| $M_2^b$ | $(2, 1, 0, 0, 0)$ | $2e(3, -2, 1, -5, -1)$ |
| $dS_5^2$ | $(2, 0, 1, 0, 0)$ | $(6e, -2e, -6e, -12e, -4e)$ |
| $m_1^b(\Omega_\Lambda, \Omega_\Xi)$ for $b = 0$ | $(0, 0, \Omega_\Lambda, \Omega_\Xi, 0)$ | $-2e(3, 1, 0, 0, 2)$ |
| $m_2^b(\Omega_\Lambda, \Omega_\Xi)$ for $b \neq 0$ | $(0, 0, \Omega_\Lambda, \Omega_\Xi, 0)$ | $\left(\frac{b_2^2}{2} - 3, b_2^2 - 2, b_2^2 - b_2^2, b_2^2 - 4\right)$ |

\[a\] The eigenvalues of these points can only be evaluated for $\Omega_\Lambda + \Omega_\Xi \neq 1$. For $\Omega_\Lambda + \Omega_\Xi = 1$ a perturbative analysis has to be carried out.

\[b\] This actually reads $\lim_{\gamma \to 0} (-6e - \sqrt{3b} \sqrt{1 - \Omega_\Lambda - \Omega_\Xi}) = \infty$.

TABLE II: This table gives the coordinates and eigenvalues of the critical points with $U \geq 0$ and $3R \geq 0$. We have defined the real quantities $\chi = \sqrt{\frac{1 - 3b^2}{\xi^2}}, \xi = \sqrt{2b(\xi - \frac{1}{b})} \sqrt{3\gamma(1 - \Omega_\Lambda - \Omega_\Xi) + 2\Omega_\Xi - 1}$, $\alpha = \sqrt{9\gamma^2(\Omega_\Lambda + \Omega_\Xi - 1) + 6\gamma(1 - 2\Omega_\Xi) + 4(1 - \Omega_\Xi)}$. Notice that $\alpha$ is a real positive quantity within the allowed parameter range.

| Model | Coordinates | Eigenvalues |
|-------|-------------|-------------|
| $F_2^b$ | $(2, \epsilon, 0, 0, 0)$ | $(6e + \sqrt{6b}, 4e, 6e, -6e, 2e)$ |
| $dS_5^2$ | $(2, \epsilon, 1, 0, 0)$ | $(6e, -2e, -6e, -12e, -4e)$ |
| $m_1^b(\Omega_\Lambda, \Omega_\Xi)$ for $b = 0$ | $(0, \epsilon, \Omega_\Lambda, \Omega_\Xi, 0)$ | $-2e(3, 1, 0, 0, 2)$ |
| $m_2^b(\Omega_\Lambda, \Omega_\Xi)$ for $b \neq 0$ | $(0, \epsilon, \Omega_\Lambda, \Omega_\Xi, 0)$ | $\left(\frac{b_2^2}{2} - 3, b_2^2 - 2, b_2^2 - b_2^2, b_2^2 - 4\right)$ |
| $R_2^b$ | $(2, 0, 0, 0, 0)$ | $(6e, -2e, -6e, -12e, -4e)$ |
| $R_2^b$ | $(2, 0, 0, 0, 1)$ | $2e(3, 1, 2, -4, -1)$ |
| $E$ | $(\gamma^* \epsilon, \Omega_\Lambda^*, \Omega_\Xi^*, \Omega_\Xi^*)$ | $\left(\frac{b_2^2}{2} - 3, b_2^2 - 2, b_2^2 - b_2^2, b_2^2 - 4\right)$ |
| $F_2^{b^{3/2}}$ | $\left(\frac{b_2^2}{2} - 3, b_2^2 - 2, b_2^2 - b_2^2, b_2^2 - 4\right)$ |
| $X_2^{1/3}(b)$ | $(\frac{b_2^2}{2} - 3, b_2^2 - 2, b_2^2 - b_2^2, b_2^2 - 4\right)$ |
| $A_4^{1/3}(b)$ | $\left(\frac{b_2^2}{2} - 3, b_2^2 - 2, b_2^2 - b_2^2, b_2^2 - 4\right)$ |

\[a\] The eigenvalues of these points can only be evaluated for $\Omega_\Lambda + \Omega_\Xi \neq 1$. For $\Omega_\Lambda + \Omega_\Xi = 1$ a perturbative analysis has to be carried out.

\[b\] This actually reads $\lim_{\gamma \to 0} (-6e - \sqrt{3b} \sqrt{1 - \Omega_\Lambda - \Omega_\Xi}) = \infty$.

\[\alpha\] This actually reads $\lim_{\gamma \to 0} (-6e - \sqrt{3b} \sqrt{1 - \Omega_\Lambda - \Omega_\Xi}) = \infty$.

\[\beta\] The same line element as the Einstein universe ($H = 0, k = +1$), but in general also non-zero brane tension and positive nonlocal energy density $U$. The quantities $a, \rho, U$ are constants $a^*, \rho^*, U^*$. Requiring positive energy density $\rho^*$, we find from (14) as well as

$$\sqrt{\gamma^*(\gamma^* - 2)b}\sqrt{1 - \Omega_\Lambda^* - \Omega_\Xi^* - \Omega_\Xi^*} = 0. \quad (29)$$

Again, we have already inserted the Friedmann constraint (23), i.e. note that the Jacobian of the dynamical system (40) is in general not well-defined for $\gamma = 0$ (35). In that case a perturbative study confirms that the points $E |_{\gamma^* = 0}$ are indeed saddle points in state space.
that the constants \( a^*, U^* \) must satisfy
\[
U^* \leq \frac{\lambda \kappa^2}{2} \left( 1 - \frac{\Lambda}{a^*} \right).
\]

Note that the constraint (29) comes from the \( \gamma' \) - equation in (27), hence from the dynamics of the scalar field, and does not occur in the studies of a similar model with constant equation of state, i.e. without a dynamical equation of state parameter.

From (28) we can see that the static models \( E \) can occur for any values of \( \gamma \in [0, 2] \). Equation (29) is however in general only satisfied for all values of \( \gamma \) for the vacuum models (\( \Omega_\rho = 0 \) [22]).

Note that in contrast to dynamical models, it is not possible to have \( \tilde{\Omega} = 0 \) and \( \tilde{\Omega}_\lambda \neq 0 \) when considering static models. This is due to the fact that for static models \( a = a^*, \rho = \rho^* \) are constants and therefore \( \tilde{\Omega}_\rho = 0 \) implies \( \rho^* = 0 \), which leads to \( \tilde{\Omega}_\lambda = 0 \). This has the interesting consequence that there exist no vacuum static solutions in the \( \Lambda = 0 \) - subset. Only when allowing for the additional degree of freedom \( \tilde{\Omega}_\lambda \) do we find that (28) can be satisfied for \( \tilde{\Omega}_\rho = 0 \).

We emphasize that the static model denoted by \( E^{1/3} \) in our previous paper [10] is in that sense unphysical, since that model found in the \( \Lambda = 0 \) case has \( Q = 0, \tilde{\Omega}_\rho = 0 \) and \( \tilde{\Omega}_\lambda = 1 \) which can only be satisfied in this extended scenario allowing for non-zero cosmological constant \( \Lambda \).

Hence the only static vacuum model occurring in this sector is the model with \( \tilde{\Omega}_\lambda = 0 \), which means that \( \tilde{\Omega}_\rho = 1/2, \tilde{\Omega}_\lambda = 1/2 \). This is a line of equilibrium points extending in \( \gamma \) - direction. The other static models are the ones with \( \gamma = 0 \) or \( \gamma = 2 \). The former are characterized by \( \tilde{\Omega}_\lambda = 1/2 \), the latter have \( \tilde{\Omega}_\rho = 2 + 3(\tilde{\Omega}_\lambda - \tilde{\Omega}_\rho) \).

Note that the only static models with \( \tilde{\Omega}_\lambda = 0 \) are the ones with \( \gamma = 0 \), \( \tilde{\Omega}_\rho = 1/2, \tilde{\Omega}_\lambda + \tilde{\Omega}_\rho = 1/2 \).

For constant potential (\( b = 0 \)) equation (29) is automatically fulfilled for all values of \( \gamma^*, \tilde{\Omega}_\rho, \tilde{\Omega}_\lambda, \tilde{\Omega}_\rho^* \). Thus for \( b = 0 \) \( E \) degenerates into a 3-dimensional surface containing static models for all values of \( \gamma \in [0, 2] \).

Notice that in the \( \tilde{\Omega}_\lambda = 0 \) - subset much stronger constraints must again be satisfied; it can be shown that here \( \gamma \in [0, \frac{2}{3}] \) is required even for \( b = 0 \).

Although these sets of equilibrium points form geometrically interesting objects in state space, they are not of interest for our stability analysis, since all these equilibrium points are unstable saddle points.

C. \( U \leq 0 \) and \( ^3R \leq 0 \)

In this case we obtain a compact state space by introducing the dynamical variables \( Z, \Omega_\rho, \Omega_\kappa, \Omega_\lambda, \bar{\Omega}_\lambda \), where
\[
Z \equiv \frac{H}{N}, \quad N^2 \equiv H^2 - \frac{2U}{\lambda \kappa^2},
\]
and the variables \( \bar{\Omega}_i \) are defined as in in (20) and (21) but normalized with respect to \( N \) instead of \( H \). Using these variables, the Friedmann constraint reads
\[
\tilde{\Omega}_\rho + \tilde{\Omega}_\kappa + \tilde{\Omega}_\lambda + \bar{\Omega}_\lambda = 1
\]
(32)

As before, all the terms in that sum are non-negative by definition, hence the variables \( \bar{\Omega}_\rho, \bar{\Omega}_\kappa, \bar{\Omega}_\lambda, \bar{\Omega}_\lambda \) take values in the interval \([0, 1]\). Since \( \gamma \in [0, 2] \) and \( Z \in [-1, 1] \), we find that the state space is again compact.

Furthermore, introducing the time derivative
\[
\dot{\gamma} = \frac{N^{-1}d}{dt},
\]
the evolution of \( N \) decouples from the rest of the variables. We obtain
\[
\gamma' = \sqrt{3\gamma(\gamma - 2)}\sqrt{3\gamma Z + b}\sqrt{1 - \tilde{\Omega}_\lambda - \tilde{\Omega}_\kappa - \bar{\Omega}_\lambda},
\]
\[
Z' = -\left[\frac{3}{2}(1 + \tilde{\Omega}_\lambda - \bar{\Omega}_\lambda) + (1 - \frac{3}{2})\tilde{\Omega}_\kappa - 2(1 - Z^2)\right],
\]
\[
\bar{\Omega}_\kappa = \left[2\gamma(1 - \tilde{\Omega}_\kappa - \tilde{\Omega}_\lambda + \bar{\Omega}_\lambda) + 3\gamma\tilde{\Omega}_\kappa - 2\tilde{\Omega}_\kappa\right]H\bar{\Omega}_\kappa,
\]
\[
\bar{\Omega}_\lambda = \left[3\gamma(1 - \tilde{\Omega}_\kappa - \tilde{\Omega}_\lambda + \bar{\Omega}_\lambda) + 2\tilde{\Omega}_\kappa\right]H\bar{\Omega}_\lambda.
\]
(33)

The equilibrium points of this system, their coordinates in state space and their eigenvalues are given in TABLE III. Again we recover the models with \( U = 0 \) that we have obtained in the previous two sections. Notice that the point \( A_{1/3}^b \) now only occurs for \( b^2 \in [0, 4] \). This is due to the fact that the point represents a model with nonlocal energy density depending on the value of the parameter \( b \). For \( b^2 < 4 \), the point corresponds to a model with nonlocal energy density \( U \), whereas for \( b^2 > 4 \) it represents a model with positive \( U \). For \( b^2 = 4 \), the point coincides with the expanding FLRW model \( P_{1/3}^{b^2/3} \) \( |b^2 = 4 \) with \( U = 0 \) and \( \gamma = \frac{1}{3} \). Thus the point \( A_{1/3}^b \) moving in state space is leaving the sector describing models with \( U \leq 0 \) and entering the sector of models with \( U \geq 0 \) when the parameter value \( b^2 = 4 \) \( (b = -2) \).

In addition to these points, we find the set \( S \) of static models \( (H = 0) \) whose coordinates \( (\gamma^*, 0, \tilde{\Omega}_\kappa, \tilde{\Omega}_\lambda, \bar{\Omega}_\lambda) \) satisfy the constraints
\[
2 - \frac{3}{2}\gamma^*(1 + \tilde{\Omega}_\lambda - \tilde{\Omega}_\kappa - \bar{\Omega}_\lambda - \bar{\Omega}_\kappa) = 0
\]
(35)
and
\[
\sqrt{\gamma^2(\gamma^* - 2)}b\sqrt{1 - \tilde{\Omega}_\kappa - \tilde{\Omega}_\lambda - \bar{\Omega}_\lambda} = 0,
\]
(36)
where we have already inserted the Friedmann constraint (23).

Requiring non-negative energy density \( \rho^* \), we also find from (14) that the constants \( a^*, U^* \) must satisfy
\[
-U^* \geq \frac{\lambda \kappa^2}{2} \left( \frac{\Lambda}{3} - \frac{k}{a^*} \right).
\]
(37)
TABLE III: This table gives the coordinates and eigenvalues of the critical points with $\mathcal{U} \leq 0$ and $\dot{3}R \leq 0$. We have defined $\psi = \sqrt{\frac{b}{b^2} - 3}$, $\xi = \sqrt{b^2 - 15b^2}$, $\eta = \sqrt{2b(\frac{b}{b^2} - 1)}\sqrt{3\gamma(1 - \Omega_\Lambda - \Omega_K) + \Omega_K - 2}$, $\Gamma = \sqrt{9\gamma^2(\Omega_\Lambda + \Omega_K - 1) + 6\gamma(2 - \Omega_K) + 4(1 - \Omega_K)}$. Notice that $\Gamma$ is a real positive quantity within the allowed range of variables.

| Model | Coordinates | Eigenvalues |
|-------|-------------|-------------|
| $F^2_0$ | $(2, e, 0, 0)$ | $(6e + \sqrt{6b}, 2e, 4e, 6e, -6e)$ |
| $M^0_0$ | $(0, e, 1, 0, 0)$ | non-hyperbolic |
| $M^2_0$ | $(2, e, 1, 0, 0)$ | $2e(3, -1, -2, 1, -5)$ |
| $\mathrm{d}S^0_2$ | $(0, e, 0, 1, 0)$ | $(6e, -4e, -2e, -6e, -12e)$ |
| $m^0_2(\Omega_\Lambda, \Omega_K)$ for $b = 0$ | $(0, e, 0, 0)$ | $-2e(3, 2, 0, 0)$ |
| $m^0_2(\Omega_\Lambda, \Omega_K)$ for $b \neq 0$ | $(-\infty^b, -4e, -2e, 0)$ | $2e(3, 4, 5, 6, 3)$ |
| $S$ | $(\gamma^*, 0, \Omega_\Lambda, \Omega_K)$ | $(\eta, \Gamma, 0, -\Gamma, 0)$ |
| $F^{b^2/3}$ | $(\frac{b^2}{3}, 1, 0, 0, 0)$ | $(\frac{b^2}{3} - 3, b^2 - 2, b^2 + 2, -b^2)$ |
| $X^{2/3}_0(b)$ | $\left(\frac{2}{3}, 1, 1 - \frac{2}{b^2}, 0, 0\right)$ | $(-1 - \psi, -2, -1 + \psi, 2, -2)$ |
| $A^{4/3}_0(b)$ | $(\frac{2}{3}, -\frac{b}{2}, 0, 0)$ | $(\frac{2}{3}(b - \xi), \frac{2}{3}(b + \xi), -b, -2b, 2b)$ |

$^a$The eigenvalues of these points can only be evaluated for $\Omega_\Lambda + \Omega_K \neq 1$. For $\Omega_\Lambda + \Omega_K = 1$ a perturbative analysis has to be carried out.

$^b$This actually reads $\lim_{\gamma \to 0}(6e - \sqrt{3b}(1 - \Omega_\Lambda - \Omega_K)) = \infty$.

$^c\xi^2 \in [2, \infty[$

$^d\xi^2 \in [0, 4]$  

It is important to note that these static models in general have negative curvature ($k = -1$).

We can see from (35) and (32) that there are no static vacuum models. The only static models for $b \neq 0$ are the ones with $\gamma^* = 2$ and $\Omega_\Lambda = \frac{1}{2} + \frac{b}{2}(\Omega_\Lambda - \Omega_K)$. These models occur both in the $\bar{\Omega}$ subset and outside of that region. Again this is a 2-parameter surface.

For $b = 0$, this 2-dimensional set degenerates into a 3-dimensional surface; in this special case we find that static models occur for all values of $\gamma^* \geq 2/3$.

Notice that the constraints on these models, in contrast to the constraints on the models $E$ found in the previous section, do not change when we restrict ourselves to the $\Lambda = 0$-subset: for $b = 0$ there are static models for all $\gamma \geq 2/3$ even for $\bar{\Omega}_\Lambda = 0$.

Dynamically these models are again not very interesting, since they all represent saddle points in state space.

D. $\mathcal{U} \leq 0$ and $\dot{3}R \geq 0$

As explained in [3], we need to take into account that the Friedmann equation (34) now has two non-positive terms. It turns out that we can obtain a compact state space introducing the dimensionless dynamical variables $W, \Omega_\rho, \bar{\Omega}_\Lambda, \bar{\Omega}_\Lambda, \bar{\Omega}_\Lambda$, where

$$W = \frac{H}{P}, \quad P^2 = H^2 + \frac{1}{6} \dot{3}R - \frac{2\mathcal{U}}{\Lambda\Phi^2},$$

and the variables with a hat are defined as those in (21) and (22) but normalized with respect to $P$ instead of $H$.

The Friedmann constraint now reduces to

$$\dot{\Omega}_\rho + \dot{\bar{\Omega}}_\Lambda + \dot{\bar{\Omega}}_\Lambda = 1.$$  \hspace{1cm} (38)

It is important to note that $\Omega_\mathcal{U}$ does not appear in the Friedmann equation (38). Indeed, it can be seen from its definition that $\Omega_\mathcal{U}$ is negative and belongs to the interval $[-1, 0]$. As before, we can see from (38) that $\Omega_\rho, \bar{\Omega}_\Lambda, \bar{\Omega}_\Lambda \in [0, 1]$. Together with $W \in [-1, 1]$ and $\gamma \in [0, 2]$, we find that these variables define a compact state space.

Using the time derivative

$$\dot{t} = \frac{1}{P} dt,$$  \hspace{1cm} (39)

we obtain

$$\dot{\gamma} = \sqrt{3\gamma(\gamma - 2)[\sqrt{3\gamma}W + b\sqrt{1 + \bar{\Omega}_\Lambda - \bar{\Omega}_\Lambda}]},$$

$$W' = [1 - \frac{3}{2}\gamma(1 + \bar{\Omega}_\Lambda - \bar{\Omega}_\Lambda)](1 - \dot{W}^2) - \bar{\Omega}_\mathcal{U},$$

$$\dot{\bar{\Omega}}_\Lambda = 3\gamma(1 + \bar{\Omega}_\Lambda - \bar{\Omega}_\Lambda)W\bar{\Omega}_\Lambda,$$

$$\dot{\bar{\Omega}}_\Lambda = 3\gamma(\bar{\Omega}_\Lambda - \bar{\Omega}_\Lambda - 1)W\bar{\Omega}_\Lambda,$$

$$\dot{\bar{\Omega}}_\mathcal{U} = 3[\gamma(1 + \bar{\Omega}_\Lambda - \bar{\Omega}_\Lambda) - 4W\bar{\Omega}_\Lambda].$$  \hspace{1cm} (40)

The equilibrium points of this system, their coordinates in state space and their eigenvalues are given in TABLE IV. Again, we recover the equilibrium points that represent flat models obtained in the previous subsection. In addition, we find another set of equilibrium points representing static models denoted by $E$.

These models correspond to the Einstein-universe-like models found in subsection B, since they also have the same line element as the Einstein universe ($H = 0, k = +1$) and have non-zero brane tension and non-local energy density $\mathcal{U}$, except now $\mathcal{U} \leq 0$.

Their coordinates $(\gamma^*, 0, \Omega_\Lambda^{\ast}, \bar{\Omega}_\Lambda^{\ast}, \bar{\Omega}_\Lambda^{\ast})$ satisfy the constraints

$$1 - \frac{3}{2}\gamma^*(1 + \bar{\Omega}_\Lambda^{\ast} - \bar{\Omega}_\Lambda^{\ast}) - \bar{\Omega}_\mathcal{U} = 0.$$  \hspace{1cm} (41)
This stands in strong contrast to the results obtained in the presence of a scalar field with exponential potential. \[ \beta = \sqrt{9\gamma^2 \Delta + 4} \] Notice that \( \beta \) is a real positive quantity within the allowed range of variables.

### TABLE IV

This table gives the coordinates and eigenvalues of the critical points with \( U \leq 0 \) and \( \delta R \geq 0 \). We have defined \( \chi = \frac{2 - 3b}{12} \) and \( \xi = \sqrt{64 - 15b^2} \). \[ \delta = \sqrt{6b/1 - \Omega \Omega} - \Omega \delta, \quad \beta = \sqrt{9\gamma^2 \Delta_\Lambda + 4} \]

| Model | Coordinates | Eigenvalues |
|-------|-------------|-------------|
| \( F^2 \) | \((2, \epsilon, 0, 0, 0)\) | \((6e + \sqrt{6b/1 - \Omega \Omega} - \Omega \delta, 6e, -6e, 2\epsilon)\) |
| \( dS^2 \) | \((2, 1, 0, 0)\) | \((-2\epsilon(3, 1, 0, 0, 2)\) |
| \( m^2_0(\Omega, \Omega) \) for \( b = 0 \) \( a \) | \((0, \epsilon, \Omega, \Omega, 0)\) | \((-2\epsilon(3, 1, 0, 0, 2)\) |
| \( m^2_0(\Omega, \Omega) \) for \( b \neq 0 \) \( a \) | \((0, \epsilon, \Omega, \Omega, 0)\) | \((-2\epsilon(3, 1, 0, 0, 2)\) |
| \( m^2_0(\Omega, \Omega) \) for \( b = 0 \) \( a \) | \((2, \epsilon, 0, 1, 0)\) | \((-2\epsilon(3, 1, 0, 0, 2)\) |
| \( E \) | \((\gamma^*, 0, \Omega^*, \Omega^*, \Omega^*)\) | \((\delta, \beta, 0 = \beta, 0)\) |
| \( F^{2/3} \) | \((\frac{k^2}{2}, 1, 0, 0, 0)\) | \((-2\epsilon(3, 1, 0, 0, 2)\) |
| \( X^{2/3}(b) \) | \((\frac{k}{2} - \frac{k}{\sqrt{2}}, 0, 0, 0)\) | \((-2\epsilon(3, 1, 0, 0, 2)\) |
| \( A^{2/3}(b) \) | \((\frac{k}{2} - \frac{k}{\sqrt{2}}, 0, 0, 0)\) | \((-2\epsilon(3, 1, 0, 0, 2)\) |

\[ a \) The eigenvalues of these points can only be evaluated for \( \Omega + \Omega \neq 1 \). For \( \Omega + \Omega = 1 \) a perturbative analysis has to be carried out.

\[ b \) This actually reads \( \lim_{\gamma \to 0}(-6\epsilon - \sqrt{3b/1 - \Omega \Omega} = \infty. \)

\[ c \) \( k^2 \in [0, 2] \)

\[ d \) \( h^2 \in [0, 4] \)

\[ e \) \( k^2 \in [0, 4] \)

and

\[ \sqrt{\gamma^*(\gamma^* - 2)b\sqrt{1 - \Omega^* \Omega^*}} = 0. \]

where we have again inserted the Friedmann constraint [3].

We can see from (11) that there exist no vacuum solutions and no solutions for \( \gamma^* = 0 \). Hence for a non-flat potential the only static models are the ones with \( \gamma^* = 2 \) and \( \Omega^* \Omega^* = -2 - 3(\Omega^* + \Omega^*) \). In particular there are no static models for \( \Omega^* = 0 \).

For \( b = 0 \), we find static models for all \( \gamma \geq 1/3 \); the ones in the \( \Omega^* = 0 \) - subset occur for \( \gamma \in [1/3, 4/3] \).

### E. Qualitative Analysis

We use the results obtained in the previous subsections in order to determine the dynamical character of the equilibrium points found above. We summarize the results in TABLE V below. For all values of \( b \) the past attractors of the FLRW models are the BDL models \( m^2_0 \), with \( \gamma = 2 \) (\( V < \delta^2 \)) and the set of non-general relativistic models \( m^2_0(\Omega, \Omega) \) with \( \gamma = 0 \) (\( \delta^2 \ll V \)). For a non-flat potential (\( b \neq 0 \)) the contracting BDL model \( m^2_0 \) with \( \gamma = 2 \) is the unique future attractor. In particular the expanding de Sitter model is not a future attractor in the presence of a scalar field with exponential potential. This stands in strong contrast to the results obtained in [11] for the scalar field - free case where the expanding de Sitter model \( dS^2_+ \) is a future attractor for all values of \( \gamma \).

In the special case of a flat potential (\( b = 0 \)) we find that the models \( m^2_0(\Omega, \Omega) \) including the de Sitter model with \( \gamma = 0 \) form another set of future attractors. Note that in the \( \Lambda = 0 \)-subset the general-relativistic models \( F^{2/3} \) and \( X^{2/3}(b) \) are also future attractors. To be precise, the flat FLRW model \( F^{2/3} \) is an attractor for \( b^2 < 2 \) and the model \( X^{2/3}(b) \) for \( b^2 > 2 \).

Notice that in General Relativity, the scenario that we are describing here (matter described by a dynamical scalar field with exponential potential) only admits a static universe in the special case of a constant potential (\( b = 0 \)). This model had the same line element as the Einstein universe (\( H = 0, k = +1 \)) and only occurred at \( \gamma = 2/3 \).

If we allow for non-zero brane tension but neglect the bulk effects (\( U = 0 \), see Paper I [11]), this condition is relaxed: we find static universe models characterized by \( H = 0, k = +1 \) for all values of \( \gamma \in [1/3, 2/3] \) for \( b = 0 \). For a non-flat potential (\( b \neq 0 \)) there are still no physical static equilibrium points [2].

In this paper taking into account the bulk effects as well as allowing for a non-zero cosmological constant \( \Lambda \), we find that there are not only static Einstein universe like models (\( k = +1 \)), but also static saddle points that are flat (\( k = 0 \)) or even negatively curved (\( k = -1 \)).

For \( b = 0 \) there are static models for all values of \( \gamma \) even in the \( \Lambda = 0 \)-subset, which shows that these are purely due to the bulk effects. Allowing for \( \Lambda \neq 0 \) we find that the constraints on the static models are further relaxed. We then find that Einstein static models occur for all values of \( \gamma \) when \( U > 0 \) and for all \( \gamma \geq 1/3 \) when \( U \leq 0 \). Flat and open static models occur for all \( \gamma \geq 2/3 \) but only if \( U \leq 0 \).

More interestingly we also find static models for \( b \neq 0 \). For \( \Lambda = 0 \) we find Einstein static models with \( \gamma = 0 \) in the \( U \geq 0 \)-sector and for \( \gamma = 2 \) these are open and flat models in the \( U \leq 0 \)-sector.
When allowing for a cosmological constant, we find that the Einstein static universe occurs for all values of $\gamma$ when $U \geq 0$ and for $\gamma = 2$ when $U \leq 0$. The open and flat models still only occur for $\gamma = 2$ and only in the $U \leq 0$-sector.

Altogether, the conditions for allowing for static models in General Relativity are changed dramatically when considering the terms corresponding to brane tension and non-local energy density, and are further relaxed when including a cosmological constant. Instead of finding only the static Einstein universe with $\gamma = \frac{2}{3}$, and this only in the special case of a flat potential, we now find static models for all values of $b$ and all $\gamma \in [0, 2]$.

### IV. BIANCHI I MODELS WITH AN EXPONENTIAL POTENTIALS

We now turn our attention to the dynamics of Bianchi type I models in the brane-world scenario with exponential potential. This class of models is characterized by a metric of the form

$$ds^2 = -dt^2 + \sum_{\alpha=1}^{3} A_\alpha^2(t)(dx^\alpha)^2.$$  \hspace{1cm} (43)

Again, we follow very closely the analysis done in [3], the only difference being that we have the additional equation (19) describing the dynamics of the scalar field.

As we have discussed above, the non-zero contributions from the five-dimensional Weyl tensor are $U$ and $\mathcal{P}_{ab}$ but since the second one has no evolution equation

| Model          | $b = 0$ | $0 < b^2 < 2$ | $b^2 = 2$ | $2 < b^2 < 4$ | $b^2 = 4$ | $4 < b^2 < 6$ | $b^2 = 6$ | $b^2 > 6$ |
|----------------|---------|---------------|-----------|---------------|-----------|---------------|-----------|-----------|
| $m_0^a (\Omega_\lambda)$ | sink   | saddle        | saddle    | saddle        | saddle    | saddle        | saddle    | saddle    |
| $m_0^b (\Omega_\lambda)$ | source | source        | source    | source        | source    | source        | source    | source    |
| $E$             | saddle  | saddle        | saddle    | saddle        | saddle    | saddle        | saddle    | saddle    |
| $S$             | saddle  | saddle        | saddle    | saddle        | saddle    | saddle        | saddle    | saddle    |
| $F^2_0$         | saddle  | saddle        | saddle    | saddle        | saddle    | saddle        | saddle    | saddle    |
| $m^2_a$         | source  | source        | source    | source        | source    | source        | source    | source    |
| $m^2_b$         | sink    | sink          | sink      | sink          | sink      | sink          | sink      | sink      |
| $R^0_0$         | saddle  | saddle        | saddle    | saddle        | saddle    | saddle        | saddle    | saddle    |
| $R^2_2$         | saddle  | saddle        | saddle    | saddle        | saddle    | saddle        | saddle    | saddle    |
| $F^{b+2/3}_a$   | sink    | sink          | saddle    | saddle        | saddle    | saddle        | saddle    | saddle    |
| $X^{2+3}_a (b)$ | saddle  | saddle        | saddle    | saddle        | saddle    | saddle        | saddle    | saddle    |
| $A^{4+3}_a (b)$ | saddle  | saddle        | saddle    | saddle        | saddle    | saddle        | saddle    | saddle    |

$a$ Notice that this model has negative non-local energy density $U$ for $b^2 < 4$, $U = 0$ for $b^2 = 4$ and $U > 0$ for $b^2 > 4$.

| Model          | $b = 0$ | $0 < b^2 < 2$ | $b^2 = 2$ | $2 < b^2 < 4$ | $b^2 = 4$ | $4 < b^2 < 6$ | $b^2 = 6$ | $b^2 > 6$ |
|----------------|---------|---------------|-----------|---------------|-----------|---------------|-----------|-----------|
| $m_0^a (\Omega_\Lambda, \Omega_\lambda)$ | sink   | saddle        | saddle    | saddle        | saddle    | saddle        | saddle    | saddle    |
| $m_0^b (\Omega_\Lambda, \Omega_\lambda)$ | source | source        | source    | source        | source    | source        | source    | source    |
| $E$             | saddle  | saddle        | saddle    | saddle        | saddle    | saddle        | saddle    | saddle    |
| $S$             | saddle  | saddle        | saddle    | saddle        | saddle    | saddle        | saddle    | saddle    |
| $F^2_0$         | saddle  | saddle        | saddle    | saddle        | saddle    | saddle        | saddle    | saddle    |
| $dS^2_a$        | saddle  | saddle        | saddle    | saddle        | saddle    | saddle        | saddle    | saddle    |
| $m^2_a$         | source  | source        | source    | source        | source    | source        | source    | source    |
| $m^2_b$         | sink    | sink          | sink      | sink          | sink      | sink          | sink      | sink      |
| $R^0_0$         | saddle  | saddle        | saddle    | saddle        | saddle    | saddle        | saddle    | saddle    |
| $R^2_2$         | saddle  | saddle        | saddle    | saddle        | saddle    | saddle        | saddle    | saddle    |
| $F^{b+2/3}_a$   | sink    | sink          | saddle    | saddle        | saddle    | saddle        | saddle    | saddle    |
| $X^{2+3}_a (b)$ | saddle  | saddle        | saddle    | saddle        | saddle    | saddle        | saddle    | saddle    |
| $A^{4+3}_a (b)$ | saddle  | saddle        | saddle    | saddle        | saddle    | saddle        | saddle    | saddle    |

$a$ Notice that this model has negative non-local energy density $U$ for $b^2 < 4$, $U = 0$ for $b^2 = 4$ and $U > 0$ for $b^2 > 4$.
we will assume for simplicity that $P_{ab}a^a b^b = 0$.

When introducing appropriate variables, we have to consider two different cases: (A) $\mathcal{U} \geq 0$; (B) $\mathcal{U} \leq 0$.

### A. $\mathcal{U} \geq 0$

This case is formally very similar to subsection III A. We obtain a compact state space using the dimensionless variables $\gamma, \Omega_\rho, \Omega_\Lambda, \Omega_\sigma, \Omega_\kappa, \Omega_\mu$, where

$$\Omega_\sigma \equiv \frac{\sigma^2}{3H^2}. \quad (44)$$

defines a normalized variable for the shear contribution, and the remaining variables are defined by (20) and (21).

The Friedmann constraint (44) reduces to

$$\Omega_\rho + \Omega_\Lambda + \Omega_\sigma + \Omega_\kappa + \Omega_\mu = 1. \quad (45)$$

Using the time derivative

$$' \equiv |H|^{-1} d/dt$$

the system of dynamical equations is given by

$$\gamma' = \sqrt{3\gamma(\gamma - 2)}[\sqrt{3\gamma} Z + b\sqrt{1 - \Omega_\kappa - \Omega_\Lambda - \Omega_\mu}],$$

$$\Omega'_\Lambda = \epsilon(3\gamma(1 + \Omega_\kappa - \Omega_\Lambda) + (4 - 3\gamma)\Omega_\sigma + (6 - 3\gamma)\Omega_\mu)|\Omega_\Lambda|,$$

$$\Omega'_\sigma = \epsilon(3\gamma - 6)(1 - \Omega_\kappa) + 3\gamma(\Omega_\kappa + \Omega_\Lambda - \Omega_\mu) + 3\gamma(\Omega_\kappa + \Omega_\Lambda - \Omega_\mu)|\Omega_\sigma|,$$

$$\Omega'_\mu = \epsilon(3\gamma - 4)(1 - \Omega_\mu) + (6 - 3\gamma)\Omega_\sigma + 3\gamma(\Omega_\kappa + \Omega_\Lambda - \Omega_\mu)|\Omega_\mu|$$

The equilibrium points of this system, their coordinates in state space and their eigenvalues are given in TABLE VI. The only new equilibrium points that we obtain in this subsection are the anisotropic models $K$. These have been identified in \[13\] as the general relativistic vacuum Kasner models with line element II where

$$A_\alpha = t^{2p_\alpha} \quad \text{and} \quad \sum_{\alpha=1}^{3} p_\alpha = \sum_{\alpha=1}^{3} p^2_\alpha = 1. \quad (47)$$

### B. $\mathcal{U} \leq 0$

In analogy to subsection III C, we consider the variables $\gamma, Z, \Omega_\rho, \Omega_\Lambda, \Omega_\sigma, \Omega_\kappa$, where $\Omega_\sigma$ is defined as $\Omega_\sigma$ in equation (44), but normalized with respect to $N$ defined in (24), and the remaining variables are as in subsection III C. With these variables the Friedmann constraint (44) reads

$$\Omega_\rho + \Omega_\Lambda + \Omega_\sigma + \Omega_\kappa = 1. \quad (48)$$

With respect to the time derivative $' \equiv N^{-1} d/dt$, the dynamical system becomes

$$\gamma' = \sqrt{3\gamma(\gamma - 2)}[\sqrt{3\gamma} Z + b\sqrt{1 - \Omega_\kappa - \Omega_\Lambda - \Omega_\mu}],$$

$$Z' = -\left[\frac{(3\gamma - 6) + (6 - 3\gamma)\Omega_\kappa + 3\gamma(\Omega_\kappa + \Omega_\Lambda - \Omega_\mu)}{2\gamma}\right](1 - Z^2),$$

$$\Omega'_\Lambda = [(6 - 3\gamma)\Omega_\sigma + 3\gamma(1 + \Omega_\kappa + \Omega_\Lambda - \Omega_\mu)]Z\Omega_\Lambda,$$

$$\Omega'_\sigma = [(3\gamma - 6)(1 - \Omega_\kappa) + 3\gamma(1 + \Omega_\kappa + \Omega_\Lambda - \Omega_\mu)]Z\Omega_\sigma,$$

$$\Omega'_\mu = [(6 - 3\gamma)\Omega_\sigma + 3\gamma(\Omega_\kappa - \Omega_\Lambda - 1)]Z\Omega_\mu. \quad (49)$$

The equilibrium points of this system, their coordinates in state space and their eigenvalues are given in TABLE VII. In addition to the anisotropic Kasner models we also obtain the set of saddle points $nE$ whose coordinates $(\gamma^*, 0, \Omega^*_\Lambda, \Omega^*_\sigma, \Omega^*_\kappa)$ satisfy

$$2 - \frac{3}{2}\gamma^*(1 + \Omega^*_\kappa - \Omega^*_\Lambda - \Omega^*_\mu) - 3\Omega^*_\kappa = 0 \quad (50)$$

and

$$\sqrt{\gamma^*}(\gamma^* - 2)b\sqrt{1 - \Omega^*_\kappa - \Omega^*_\Lambda - \Omega^*_\mu}. \quad (51)$$

Furthermore the constants $\mathcal{U}^*, \sigma^{*2}$ must satisfy the condition

$$-\mathcal{U}^* \geq \frac{\lambda k \kappa}{6}(\sigma^{*2} + \Lambda) \quad (52)$$

in order for the energy density $\rho^*$ to be non-negative.

We can see from (50) and (51) that there are vacuum solutions with $\Omega^*_\kappa = 2/3$ for all values of $\gamma^*$. Furthermore we find non-vacuum models for $\gamma^* = 0$ and $\gamma^* = 2$.

The set of static models $nE$ forms a 2-parameter set of equilibrium points which for $b = 0$ degenerates into a 3-dimensional surface containing static equilibrium points for all values of $\gamma^*$.

Note that the $\Omega_\Lambda = 0$-subset only contains static models for $\gamma^* \in [0, \frac{3}{2}]$ if $b \neq 0$ and for $\gamma^* \in [0, \frac{1}{2}]$ if $b = 0$.

For each fixed value of $\gamma^*$ the models $nE$ are the same as the ones found in \[13\] with scale factors

$$A_\alpha(t) = e^{q_\alpha t},$$

where $q_\alpha$ are constants which satisfy

$$\sum_{\alpha=1}^{3} q_\alpha = 0 \quad \text{and} \quad \sum_{\alpha=1}^{3} q^2_\alpha = 2\sigma^{*2}.$$

We repeat that although $H = 0$, the scale factors $A_\alpha(t)$ are in general not static. It is only the overall scale factor $\prod_{\alpha=1}^{3} A_\alpha(t) = 1$ which remains constant. The matter can still expand or contract along the principal shear axis, hence forming a pancake singularity if one $q_\alpha$ is negative and a cigar type singularity if two of them are negative.

Note that the Jacobian of the dynamical system (49) is in general not well defined for $\gamma^* = 0 \quad (24)$. In that case a perturbative study confirms that the points $nE \mid \gamma^* = 0$ are also saddle points in state space.
TABLE VIII: This table gives the coordinates and eigenvalues of the critical points with $\mathcal{U} \geq 0$. We have defined $\zeta = \sqrt{\frac{2\xi}{\gamma}} - 15$.

| Model       | Coordinates | Eigenvalues    |
|-------------|-------------|----------------|
| $K^2_\gamma$ | (0, 0, 1, 0) | non-hyperbolic |
| $K^2_\gamma(\Omega_*)$ | (2, 0, $\Omega_*$, 0) | (6$e + \sqrt{6b}/1 - \Omega_*$, 6$e$, 0, $-6e, 2e$) |
| $dS^2_\gamma$ | (2, 1, 0, 0) | (6$e$, $-6e$, $-6e$, $-12e$, $-4e$) |
| $m^0(\Omega_+, \Omega_+)$ for $b = 0$ | (0, 0, 0, 0) | $-2(3, 0, 3, 0, 2)$ |
| $m^0(\Omega_+, \Omega_+)$ for $b \neq 0$ | (0, 0, 0, 0, 0) | $(\infty, 0, 0, -6e, -4e)$ |
| $m^2$ | (2, 0, 0, 1, 0) | $2e(3, 6, 3, 4)$ |
| $R^0$ | (0, 0, 0, 0, 1) | non-hyperbolic |
| $R^2$ | (2, 0, 0, 0, 1) | $2e(3, 2, -1, -4, -1)$ |
| $F^{b/3}_c$ | $\left(\frac{b^2}{3}, 0, 0, 0, 0\right)$ | $(b^2 - 3, b^2, b^2 - 6, -b^2, b^2 - 4)$ |
| $A^{4/3}_n(b)$ | $\left(\frac{4}{3}, 0, 0, 0, 1 - \frac{4}{3}\right)$ | $(-\frac{1}{2}(1 + \zeta), 4, -2, -4, -\frac{1}{2}(1 - \zeta))$ |

The eigenvalues of these points can only be evaluated for $\Omega_+ + \Omega_+ \neq 1$. For $\Omega_+ + \Omega_+ = 1$ a perturbative analysis has to be carried out.

This actually reads $\lim_{\gamma \to 0}(-6e - \sqrt{3b}\sqrt{\frac{1 - \Omega_- - \Omega_+}{\gamma}}) = \infty$.

TABLE VIII: This table gives the coordinates and eigenvalues of the critical points with $\mathcal{U} \leq 0$. We have defined $\xi = \sqrt{64 - 15b^2, \phi = \sqrt{2}b(\frac{\xi}{3} - \frac{1}{3})}\sqrt{2\gamma(1 - \Omega_+ - \Omega_*) + 3\Omega_+ - 2}$. $\varphi = \sqrt{\gamma^2(\Omega_+ + \Omega_*) - 1} - 18\gamma(\Omega_\sigma + 12\gamma + 4)$. Notice that $\varphi$ is a real positive quantity within the allowed range of variables.

| Model       | Coordinates | Eigenvalues    |
|-------------|-------------|----------------|
| $K^2_\gamma$ | (0, $e$, 0, 1, 0) | non-hyperbolic |
| $K^2_\gamma(\Omega_*)$ | (2, $e$, 0, $\Omega_*$, 0) | (6$e + \sqrt{6b}/1 - \Omega_*$, 2$e$, 6$e$, 0, $-6e$) |
| $dS^2_\gamma$ | (2, $e$, 1, 0, 0) | (6$e$, $-6e$, $-6e$, $-12e$) |
| $m^0(\Omega_+, \Omega_+)$ for $b = 0$ | (0, 0, 0, 0, 0) | $-2(3, 2, 0, 3, 0)$ |
| $m^0(\Omega_+, \Omega_+)$ for $b \neq 0$ | (0, 0, 0, 0, 0, 0) | $(\infty, -4e, 0, -6e, 0)$ |
| $m^2$ | (2, $e$, 0, 0, 1) | $2(e(3, 4, 6, 3, 3))$ |
| $nE$ | $\left(\gamma, 0, \Omega_+, \Omega_*, \Omega_\sigma\right)$ | $(\varphi, 0, -\varphi, 0)$ |
| $F^{b/3}_c$ | $\left(\frac{b^2}{3}, 1, 0, 0, 0\right)$ | $(b^2 - 3, b^2, b^2 - 6, -b^2)$ |
| $A^{4/3}_n(b)$ | $\left(\frac{4}{3}, -\frac{4}{3}, 0, 0, 0\right)$ | $(\frac{2}{3}(b - \xi), \frac{2}{3}(b + \xi), -2b, b, 2b)$ |

The eigenvalues of these points can only be evaluated for $\Omega_+ + \Omega_+ \neq 1$. For $\Omega_+ + \Omega_+ = 1$ a perturbative analysis has to be carried out.

This actually reads $\lim_{\gamma \to 0}(-6e - \sqrt{3b}\sqrt{\frac{1 - \Omega_- - \Omega_+}{\gamma}}) = \infty$.

C. Qualitative Analysis

We summarize the dynamical character of the equilibrium points obtained in this section in TABLE VIII below. The main question we want to address here is whether the initial singularity is isotropic or not when restricting our analysis to Bianchi I models. We obtain the following result.

For $b = 0$ we find that the only equilibrium points of the dynamical system $\{16, 19\}$ that are sources and therefore present stable initial configurations are the expanding non-general relativistic BDL model with $\gamma = 2$ $(m^2_\gamma)$ and the line of the flat and collapsing models with $\gamma = 0$ $(m^0(\Omega_\lambda, \Omega(\lambda)))$ including the flat FLRW model and the maximally non-general relativistic BDL model.

All these models are isotropic.

If $b \leq 0$, the dynamical system possesses in addition to the sources above one further source denoted by $K^0_n$. This equilibrium point represents the anisotropic Kasner model with $\gamma = 0$.

This means that if $b = 0$, the initial singularity must be isotropic, since the anisotropic models are not stable at early times. If on the other hand the potential $V(\phi)$ is not flat, i.e. $\varphi \neq 0$, we find that the initial singularity could be anisotropic, since the anisotropic Kasner model is stable at early times.
TABLE IX: Dynamical character of the critical points in the Bianchi I case for $\Lambda = 0$.

| Model       | $b = 0$ | $0 < b^2 < 4$ | $b^2 = 4$ | $4 < b^2 < 6$ | $b^2 = 6$ | $b^2 > 6$
|-------------|---------|--------------|-----------|--------------|-----------|---------
| $K_0^6$     | saddle  | source       | source    | source       | source    | source  |
| $K_0^6$     | saddle  | saddle       | saddle    | saddle       | saddle    | saddle  |
| $K_0^6(\Omega_0)$ | saddle | saddle       | saddle    | saddle       | saddle    | saddle  |
| $m_{\gamma}^1(\Omega_0)$ | sink    | sink         | sink      | sink         | sink      | sink    |
| $m_{\gamma}^2(\Omega_0)$ | source  | source       | source    | source       | source    | source  |
| $m_{\gamma}^2$ | sink    | sink         | sink      | sink         | sink      | sink    |
| $R_0^6$     | saddle  | saddle       | saddle    | saddle       | saddle    | saddle  |
| $R_0^6$     | saddle  | saddle       | saddle    | saddle       | saddle    | saddle  |
| $m_{\gamma}^2$ | sink    | sink         | sink      | sink         | sink      | sink    |
| $F^{b^2/3}$ | sink    | sink         | sink      | sink         | sink      | sink    |
| $A^{b^2/3}(b)^a$ | saddle  | saddle       | saddle    | saddle       | saddle    | saddle  |

*Notice that this point is an attractor for all general relativistic closed models.

$^a$Notice that this model has negative non-local energy density $\mathcal{U}$ for $b^2 < 4$, $\mathcal{U} = 0$ for $b^2 = 4$ and $\mathcal{U} > 0$ for $b^2 > 4$

| Model       | $b = 0$ | $0 < b^2 < 4$ | $b^2 = 4$ | $4 < b^2 < 6$ | $b^2 = 6$ | $b^2 > 6$
|-------------|---------|--------------|-----------|--------------|-----------|---------
| $K_0^6$     | saddle  | source       | source    | source       | source    | source  |
| $K_0^6$     | saddle  | saddle       | saddle    | saddle       | saddle    | saddle  |
| $K_0^6(\Omega_0)$ | saddle | saddle       | saddle    | saddle       | saddle    | saddle  |
| $dS^2$      | saddle  | saddle       | saddle    | saddle       | saddle    | saddle  |
| $m_{\gamma}^1(\Omega_0, \Omega_0)$ | sink    | sink         | sink      | sink         | sink      | sink    |
| $m_{\gamma}^2(\Omega_0, \Omega_0)$ | source  | source       | source    | source       | source    | source  |
| $m_{\gamma}^3$ | sink    | sink         | sink      | sink         | sink      | sink    |
| $R_0^6$     | saddle  | saddle       | saddle    | saddle       | saddle    | saddle  |
| $R_0^6$     | saddle  | saddle       | saddle    | saddle       | saddle    | saddle  |
| $m_{\gamma}^2$ | sink    | sink         | sink      | sink         | sink      | sink    |
| $F^{b^2/3}$ | sink    | sink         | sink      | sink         | sink      | sink    |
| $A^{b^2/3}(b)^a$ | saddle  | saddle       | saddle    | saddle       | saddle    | saddle  |

*Notice that this model has negative non-local energy density $\mathcal{U}$ for $b^2 < 4$, $\mathcal{U} = 0$ for $b^2 = 4$ and $\mathcal{U} > 0$ for $b^2 > 4$

V. DISCUSSION AND CONCLUSION

We end this paper with a comparison of our results with work previously done in this area. In particular, we focus on two issues: Firstly we discuss the occurrence and stability of equilibrium points in the dynamical systems analysis of FLRW and Bianchi I models with and without an exponential potential and compare our work to that done by Campos and Sopuerta [1]. Secondly we discuss the issue of isotropization in the past; here we mainly comment on the results recently obtained by Coley et al in [13, 19].

Addressing the first issue let us state the differences between this work and [1, 3] where the scalar field free analog of the scenario was studied. Formally, the new features of our work is contained in an additional evolution equation for $\gamma$ [14] thus enlarging the dynamical system analyzed with respect to the one studied in [3]. This means in particular that we should expect to obtain no more than the equilibrium points already obtained in [3]. The stability of these equilibrium points on the other hand is expected to be altered since the additional dynamical equation yields an additional eigenvalue in the dynamical systems analysis which can potentially destabilize the equilibrium points. In fact we found that we in general only recover the equilibrium points that Campos et al found for any linear barotropic equation of state for the special values of $\gamma = 0, 2$ which correspond to $\phi << V$ and $V << \phi$ respectively.

It is worthwhile mentioning that the “new” equilibrium points $F^{b^2/3}, X^{2/3}(b)$ and $A^{4/3}(b)$ which are moving in state space as the steepness of the potential (characterized by the value of $b$) is increased are in fact not new. The first point simply corresponds to the flat FLRW
model with equation of state parameter $\gamma$ depending on the value of $b$. More interestingly, the last two correspond to the bifurcations occurring for $\gamma = \frac{2}{3}, \frac{4}{3}$ in [1]: for each value of $b$ the model $X^{2/3}_+(b)$ corresponds to a point of the line of general-relativistic, non-static equilibrium points which occurs in [1] for $\gamma = \frac{2}{3}$ (see figure 4 in that paper), and similarly the model $A^{1/3}_+(b)$ corresponds for each value of $b$ to one of the points on the line of equilibrium points joining the models denoted by $F_+$ and $R_+$ for $\gamma = \frac{4}{3}$ in [1] (see figure 2 in that paper). There the stability of the equilibrium points forming the bifurcations was not discussed in much detail. We find that for sufficiently steep potential ($b^2 > 4$) the model $A^{1/3}_+(b)$ is a stable sink in the $\Lambda = 0$-subset of the state space of the Bianchi I models and the $\Lambda = k = 0$-subset of the state space of the FLRW models. This means that in that case this point which has scale factor $a(t) = t^{1/2}$ and non-vanishing positive non-local energy density $U$ is a future attractor with the equation of state of pure radiation. It is the only non-collapsing future attractor, but it is unstable for $\Lambda, k \neq 0$.

The second important issue we want to address here is the issue of isotropization. In [12] it was claimed that the initial singularity in the brane-world scenario is necessarily isotropic. It was shown that for $\gamma \geq 1$ the non-general relativistic BDL-model $m_+$ [20] with scale factor $a(t) = t^{1/\gamma}$ is a source in the state space of all Bianchi IX models, and it was also claimed that this is the only source with physically relevant values of $\gamma(\gamma \geq 1)$. In this paper we have considered the more general situation of $\gamma \in [0, 2]$. We have found all equilibrium points of the state space and identified the past attractors without imposing any constraints on $\gamma$. We confirm that the BDL model with $\gamma = 2$ is a generic past attractor in both the FLRW case and the Bianchi I case. We want to point out however, that in the analysis of the Bianchi I models the anisotropic Kasner model with $\gamma = 0$ ($K^0_0$) is another past attractor unless the potential is flat ($b = 0$). This model has the equation of state $p = -\rho$ and corresponds to a slow rolling scalar field ($\phi << V$). In particular no energy conditions are violated [23]. In summary, we find that if we adopt the assumption on the equation of state in the early universe $\gamma \geq 1$, the initial singularity must be isotropic. The benefit of our approach is that we first find all configurations that are stable in the past in a transparent and complete analysis. We can then exclude certain ones on physical grounds.

We conclude with the following remark on the future attractors in this scenario. In [10] we explained that the general relativistic models $F^{1/3}_+$ and $X^{2/3}_+(b)$ correspond to the equilibrium points obtained in [11] and [12] in the general relativistic analysis, and found that the stability of these models is not altered in the brane-world extension with vanishing non-local energy density $U$. In this paper we find that these models are also stable against perturbations in $U$. It is the perturbations in $\Lambda$ that grow and hence destabilize these models. This confirms the intuitive idea that the cosmological constant dominates at late times since we showed that allowing for non-zero $\Lambda$ destabilizes all expanding future attractors with $\Omega_\Lambda = 0$. It does not destabilize the contracting non-general relativistic future attractor $m_2$, since at high energies the $\Lambda$ term is negligible with respect to the $\rho^2$ term corresponding to $\Omega_\Lambda$. Hence the re-collapsing BDL model with stiff matter ($\gamma = 2$) is the unique future attractor for the models discussed in this paper.

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[19] A. Coley Class. Quant. Grav. 19 L45-L56 (2002).
[20] In order to recover conventional gravity on the brane $\lambda$ must be assumed to be positive.
[21] For $\gamma = b = 0$ the Jacobian is well-defined and the non-zero eigenvalues of $nE$ are given by $\pm \alpha |_{\rho = 0}$.
[22] If the potential is flat ($b = 0$) equation (23) is satisfied for any values of $\gamma$, $\Omega_\rho$. We will not discuss this case in much detail since it is physically not very interesting.
[23] As explained above, the static model $F^{1/3}_+$ with $\Omega_\rho = 0$ and $\Omega_\Lambda = 1$ must be excluded.
[24] For $\gamma = b = 0$ the Jacobian is well defined and the non-zero eigenvalues of $nE$ are given by $\pm \phi |_{\rho = 0}$.
[25] This model is referred to as the BRW solution $F_0$ in [13].
We also find that the line of the isotropic flat and collapsing models with $\gamma = 0 (m^0_\Omega, \Omega_\Lambda)$ including the flat FLRW model and the maximally non-general relativistic BDL model is a sink for all values of $b$ in both the FLRW case and the Bianchi I case, hence all of the models contained in that line are stable in the past and possible as initial configurations if we allow for $\gamma < 1$. 