A UNIVERSAL FAMILY OF DEFORMATIONS FOR THE UNIFORMISING HIGGS BUNDLE

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ABSTRACT. Fix a simple complex Lie group $G$ and a principal $\mathfrak{sl}(2, \mathbb{C})$ subalgebra of $\text{Lie}(G)$. Then the moduli space of semi-stable, topologically trivial $G$-Higgs bundles on a hyperbolic, spin Riemann surface acquires a marked point. This is the unique $\mathbb{C}^*$-fixed point on the Hitchin section. We describe a universal analytic family of deformations which provides holomorphic Darboux coordinates in a neighbourhood of the section. This is a special case of a more general deformation-theoretic construction in the spirit of Kuranishi theory. As a toy example of the latter we consider the tautological family of centralisers over the Kostant slice.

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1. Introduction

1.1. Motivating example. The affine line of companion matrices

$$\Sigma = \left\{ \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}, \alpha \in \mathbb{C} \right\} \subset \mathfrak{sl}(2, \mathbb{C})$$

provides a section for $-\det : \mathfrak{sl}(2, \mathbb{C}) \to \mathbb{C} \simeq \mathfrak{sl}(2, \mathbb{C}) \sslash SL(2, \mathbb{C}) \simeq \mathfrak{t}/(\mathbb{Z}/2)$, where $\mathfrak{t}$ is the Cartan subalgebra of $\mathfrak{sl}(2, \mathbb{C})$. Nigel Hitchin observed ([Hit87a]) that $\Sigma$ can be promoted to a $(3g_X - 3)$-dimensional family of Higgs fields on the vector bundle $K_X^{1/2} \oplus K_X^{-1/2}$, where $X$ is a Riemann surface of genus $g_X \geq 2$. The Higgs fields in this family are given by the above formula but with $\alpha \in H^0(X, K_X^{\pm 1/2})$. This observation has numerous far-reaching consequences and generalisations. On the
other hand, the tautological family of centralisers over $\Sigma$ is isomorphic to $T_\nu(\mathbb{Z}/2)$ and can be trivialised by
\[
\mathbb{C}^2 \simeq \left\{ \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \alpha \xi \\ \xi & 0 \end{pmatrix} \right\} \subset \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C}).
\]
The trace gives a complex symplectic form on $\mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$, and the above trivialisation provides Darboux coordinates for the bundle of centralisers. Among other things, in this note we show how to construct a $(6g_X - 6)$-dimensional family of Higgs bundles by twisting appropriately the above formula.

1.2. **Background.** Let $G$ be a simple complex Lie group, and $X$ a smooth, compact Riemann surface of genus at least two. A $G$-Higgs bundle on $X$ is a pair $(\mathbf{P}, \theta)$, where $\mathbf{P}$ is a holomorphic principal $G$-bundle, and $\theta \in H^0(X, \text{ad}\mathbf{P} \otimes K_X)$. The moduli space $M_{\text{Dol}}(G)$ of topologically trivial, semi-stable $G$-Higgs bundles on $X$ was constructed by Hitchin ([Hit87a], [Hit87b]) and Simpson ([Sim92], [Sim94]). It admits a proper map, $\chi$, called the Hitchin map, to a vector space, $\mathcal{B}_g$, the Hitchin base. The Hitchin map admits a section which is a “global analogue” of the well-known “Kostant slice” from Lie theory. The latter generalises the notion of “companion matrices” and is a section of the adjoint quotient morphism $\text{Lie } G = g \to g/\mathcal{B}_g$. The Kostant section is not canonical, but depends on a choice of Lie-algebraic data (a principal $\mathfrak{sl}(2, \mathbb{C})$-subalgebra). Similarly, Hitchin’s section depends on such a choice, as well as on a choice of a theta-characteristic $\zeta = K_X^{1/2}$. We assume that these choices are fixed once and for all, and hence shall talk about the Hitchin section. The interested reader can find more details in the original paper [Hit92], as well as in [DP06] or [Ngô10]. It should be noted that $M_{\text{Dol}}(G)$ is a holomorphic symplectic variety, $\chi$ is a complex Lagrangian fibration, and the section is Lagrangian.

With the choice of a principal $\mathfrak{sl}(2, \mathbb{C})$ and a theta-characteristic the moduli space acquires a marked point as follows. There is a natural $\mathbb{C}^\times$-action on $M_{\text{Dol}}(G)$, given by $\lambda \cdot ([\mathbf{P}, \theta]) = ([\mathbf{P}, \lambda \theta])$ ([Sim92], [Hit87a]). The marked point is the unique $\mathbb{C}^\times$-fixed point lying on the (image of the) Hitchin section. In the special case $G = \text{SL}(2, \mathbb{C})$ it appeared in Hitchin’s original paper [Hit87a] (Example 1.5 on p.8) as the first nontrivial example of a stable Higgs bundle. We call it “the uniformising Higgs bundle” (the terminology goes back to [Sim88]) since the Hermitian-Yang-Mills metric on this bundle is obtained from the uniformising metric of the curve $X$. In physics (the logarithm of) this metric is known as a Toda field. This is also an example of a “system of Hodge bundles” in the terminology of [Sim92]. It corresponds, by the non-abelian Hodge theorem ([Sim92]) to a variation of Hodge structures, and in fact, a very special one, a $G$-oper. The uniformising Higgs bundle carries a (regular) nilpotent Higgs field, i.e., belongs to the “global nilpotent cone” $\chi^{-1}(0)$, the 0-fibre of the Hitchin map. More about systems of Hodge bundles and uniformisation can be found in [Hit87a], [Sim88], [Hit92], [Sim92], [Sim10].

All these special properties of the uniformising Higgs bundle impose restrictions on its deformation theory. Their rôle is discussed in Section 2.

1.3. **Results and contents of the paper.** The main result in this paper is contained in Section 6, where we describe a universal analytic family of deformations of the uniformising Higgs bundle, with base the germ $(\mathcal{B}_g \times \mathcal{B}_g^\vee, 0)$. Here $\mathcal{B}_g^\vee$ is the dual vector space to the Hitchin base. Our family has the property that the holomorphic symplectic form on $M_{\text{Dol}}(G)^{\text{reg}}$ induces the canonical symplectic form on
$B_g \times B_\hat{g}$, so we obtain holomorphic Darboux coordinates in an analytic neighbourhood of $[\mathcal{P}, \theta]$. As a by-product, we obtain a formula for the flow of the Hitchin section under linear Hamiltonian functions on $B_g$. This generalises an unpublished observation of C. Teleman for the case of structure group $GL(n, \mathbb{C})$ ([Tel07]).

Our approach uses several analytical pieces of data. First, we work with (analytic) differential graded Lie algebras (dgla), so holomorphic bundles are described in terms of their Dolbeault operators. And second, we use a small amount of Hodge theory. On the other hand, our final formulae are polynomial and are of Lie-algebraic origin, so a purely algebraic description of the flow may also be feasible.

Section 5 is devoted to a toy-version of the main example: we give there a trivialisation of the tautological family of centralisers over the Kostant slice.

In Section 2 we make some general remarks about deformation theory via dgla’s. We also describe the special features of the controlling dgla and sketch a general strategy that one can follow in order to understand such deformation problems.

The results from Sections 5 and 6 are a consequence of the special form of the dgla’s controlling the corresponding deformation problems. In Section 3 we give sufficient conditions on the controlling dgla under which similar (weaker) results hold.

The remaining sections are supplementary. In Section 4 we recall results from Lie theory and set up notation, and in Appendix 7 we review for our reader’s convenience the basics of Kuranishi theory. In Section 8 we give a glossary of notation.

Our main results are as follows.

Let $MC_{L, \bullet}$ (respectively $Def_{L, \bullet}$) denote the Maurer-Cartan (respectively, deformation) functor of a dgla $L^\bullet$, and let $pr$ be the natural projection $MC_{L, \bullet} \to Def_{L, \bullet}$. Suppose $L^1 = L' \oplus L''$ satisfies assumptions (1), (2), (3) from Section 3. The two inclusions (resp. projections) are denoted by $' , ' ''$ (resp. $\pi', \pi''$). Let $\mathcal{H}^1 = \mathcal{H}' \oplus \mathcal{H}'' \subset L^1$ be “harmonic representatives” of $H^1(L^\bullet)$ (see 7) and let $H : L^1 \to \mathcal{H}^1$ be the corresponding projection, $H = H' + H''$. Let $F_{L, \bullet} : Art_{\mathbb{C}} \to Sets$ be the formal Kuranishi map. We define a functor $S_L = MC_{L, \bullet} \cap \ker [(1 - H')\pi'] : Art_{\mathbb{C}} \to Sets$.

**Theorem A (3.6.3.10).** Let $L^\bullet$ be a dgla with $L^3 = 0$, $H^2(L^\bullet) = 0$ and $L^1 = L' \oplus L''$, satisfying (1), (2), (3) from Section 3. Let $\Pi''$ be a splitting of $d_1'$ and $\pi : L^2 \to \text{Im } d_1'$ a projection. Assume that the formal series $\Gamma \in L^1 \otimes \lim_{m \to \infty} \text{Sym}^m(\mathcal{H}^{1\vee})/m^k$ defined by $\Gamma(h, v) := (h, (1 + P\pi a_0)^{-1}(v))$ satisfies $[\Gamma, \Gamma] \in \text{Im } d_1' \otimes \lim_{m \to \infty} \text{Sym}^m(\mathcal{H}^{1\vee})/m^k$. Then:

- The natural transformation
  \[
  \Phi : S_L \to \mathcal{H}^1 = H' \oplus H''
  \]
  \[
  \Phi_A(h, v) = (h, (1 + P\pi a_0)v) \in H^1 \otimes m_A
  \]
  is an isomorphism in $FArt_{\mathbb{C}}$ and $\Phi^{-1} = \Gamma$. The composition
  \[
  pr \circ F^{-1} \circ \Phi : S_L \to Def_L
  \]
  is étale. If moreover $L^\bullet$ is normed and $\text{Im } d_1' \subset L^2$ is closed, then $S_L$ is prerepresented by the germ $(S, 0)$, where
  \[
  S = MC(L) \cap \ker [(1 - H')\pi'] \subset H' \oplus L'',
  \]
  and $\Phi : (S, 0) \simeq (H^1, 0)$. 


\textbullet{} Suppose that $L'$ and $L''$ are in (weak) duality by a pairing $\langle \cdot, \cdot \rangle$ and let $\omega_{\text{can}}$ be the canonical symplectic form on $L^1$. Then $\Gamma^* \omega_{\text{can}} = \omega_{\text{can}}$, provided $\text{Im } P \pi \text{ad}_h \subset H'_{\perp}$ for all $h \in H'$. In the normed case, $\Phi : S \rightarrow H^1$ gives holomorphic Darboux coordinates on $(S, 0)$.

Theorem B (5.1). Let $\mathfrak{g}$ be a simple complex Lie algebra, $\{y, h, x\}$ a principal $\mathfrak{sl}(2, \mathbb{C})$ subalgebra, and $\pi$ the projection onto $\text{Im } \text{ad}_y$, $\Sigma = y + \mathfrak{z}(x)$ the Kostant slice and $I$ the tautological family of centralisers. Then $\Phi : S \equiv I|_{\Sigma} \rightarrow \mathfrak{z}(x) \times \mathfrak{z}(y)$

is an isomorphism. Moreover,

$\Phi(h, u) = (h, (1 + P \pi \text{ad}_h)u)$

and

$\Gamma(h, v) := \Phi^{-1}(h, v) = \left( h, \left( \sum_{k=0}^{\hat{k}} (-1)^k (P \circ \text{ad}_h)^k \right)(v) \right),$

where $\hat{k}$ is the Coxeter number. Finally, $\Gamma^* \omega_{\text{can}} = \omega_{\text{can}}$, where $\omega_{\text{can}}$ denotes the canonical symplectic form on $\mathfrak{g} \times \mathfrak{g}$, as well as its restrictions to $I$ and $\mathfrak{z}(x) \times \mathfrak{z}(y)$.

Theorem C (6.5). Let $(P, \theta)$ denote the uniformising Higgs bundle. The notation and assumptions are from Section 6, in particular, we denote by $P$ the splitting of $\text{ad}_\theta$ induced by a principal $\mathfrak{sl}(2, \mathbb{C})$-subalgebra. Consider the holomorphic family of Higgs bundles

$\Gamma : H' \times H'' \rightarrow A^{1,0}(\text{ad}P) \oplus A^{0,1}(\text{ad}P),$

$\Gamma(h, v) = (h, \Phi^{-1}_h(v)) = \left( h, \sum_{k=0}^{\hat{k}} (-1)^k ((s^{-1} P \otimes \mathbb{C} 1) \circ \text{ad}_h)^k(v) \right),$

where $(h, v) \in H' \times H'' \simeq H^1(L^*) \simeq \mathcal{B}_0 \times \mathcal{B}_0^\vee$ and

$\Phi_h = 1 + s^{-1}(P \otimes \mathbb{C} 1) \pi \text{ad}_h \in \text{End}(A^{0,1}(\text{ad}P)).$

The family $\Gamma$ is a miniversal deformation of the uniformising Higgs bundle $(P, \theta)$. An explicit description of $H' \times H'' \subset A^1(\text{ad}P)$ is given in Theorem 6.5.

There exists an open neighbourhood $U \subset \mathcal{B}_0 \times \mathcal{B}_0^\vee$ containing $0$, for which $\Gamma|_U$ is a universal deformation. Moreover, $\Gamma^* \omega_{\text{can}} = \omega_{\text{can}}$.

At this point it may seem utterly unclear why is it possible to describe such a family of deformations. In short, the reason is the very special nature of our marked point, and, respectively, of the controlling dgla. In Section 2, after reviewing the basics of deformations via dglas, we describe why our results are in fact natural.

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2. Deformations via dgla’s

2.1. Basics. We start with some remarks on deformation theory via differential graded Lie algebras. This is by now very classical, and there are many great references. The ones which seem both pedagogical and closest to our purposes are [Fuk03], [GM88], [Man99], [Man04]. We state only the bare minimum of results and definitions, without motivate them in any way. All vector spaces and tensor products are over $\mathbb{C}$.

A differential graded Lie algebra (dgla) is a triple $(L^\bullet, d, [\ , \ ]).$ Here $L^\bullet = \bigoplus_{k \in \mathbb{N}} L^k[-k]$ is a graded vector space, endowed with a bracket $[\ , \ ] : L^i \times L^j \to L^{i+j}$. The bracket is graded skew-symmetric and satisfies a graded Jacobi identity. Finally, $d : L^\bullet \to L^{\bullet+1}$ is a differential ($d^2 = 0$), which is a graded derivation of the bracket. The set of Maurer-Cartan elements in a dgla is the zero set of the quadric $Q : L^1 \to L^2$, $Q(u) = du + \frac{1}{2}[u, u]$. We write $MC(L) := Q^{-1}(0)$. To a dgla $L^\bullet$ one associates a Maurer-Cartan functor $MC_{L^\bullet} : Art_\mathbb{C} \to Sets$, defined as

$$MC_{L^\bullet}(A) = MC(L^\bullet \otimes A) = \left\{ u \in L^1 \otimes m_A : du + \frac{1}{2}[u, u] = 0 \right\}.$$ 

Given $\gamma \in L^1$, one can check ([GM88], Section 1.3) that $d_\gamma := d + ad\gamma \in Der_1L$ satisfies $(d + ad\gamma)^2 = adQ(\gamma)$, and hence, if $\gamma \in MC(L) = Q^{-1}(0)$, $d_\gamma$ is differential, giving a new dgla structure on $L^\bullet$. There is a Lie algebra homomorphism $L^0 \to \mathfrak{aff}(L^1)$ (the affine vector fields on $L^1$), given by $\lambda \mapsto (\gamma \mapsto -d_\gamma(\lambda))$, and this affine vector field preserves the set of Maurer-Cartan elements. We define an action of $\exp(L^0 \otimes m_A)$ on $L^1 \otimes m_A$ by

$$\exp(\lambda) : u \mapsto \exp(ad\lambda)(u) + \frac{I - \exp(ad\lambda)}{ad\lambda}(d\lambda)$$

and define the deformation functor $\text{Def}_{L^\bullet} : Art_\mathbb{C} \to Sets$ by

$$\text{Def}_{L^\bullet}(A) = MC_{L^\bullet}(A)/\exp(L^0 \otimes m_A).$$

Then $MC_{L^\bullet}(A)$ can be considered as (the set of objects of) a groupoid, whose morphisms are determined by the gauge action; this is often referred to as the Deligne-Goldman-Millson groupoid. Details about it can be found in any of the references, e.g., Section 2.2. of [GM88]. Deformation problems are described by deformation functors $Art_\mathbb{C} \to Sets$, assigning to $A \in Art_\mathbb{C}$ the set of isomorphism classes of deformations over Spec$A$. We say that a problem is governed (controlled) by a dgla, if its deformation functor is isomorphic to $\text{Def}_{L^\bullet}$ for some dgla $L^\bullet$.

A dgla is called normed ([GM90]), if it is endowed with a norm, with respect to which $d$ and $[,]$ are continuous. It is called an analytic dgla, if moreover it is endowed with continuous splitting $\delta$ , compatible with the other structures. See Appendix 7 or [GM90] for the definition of splitting and details about compatibility.

If $L^\bullet$ is normed, by a holomorphic family of deformations of $\text{Def}_{L^\bullet}(\mathbb{C})$ over a (pointed) complex manifold $(o, U)$ we mean a holomorphic map $\Gamma : U \to MC(L) \subset L^1$, $\Gamma(o) = 0$. Holomorphicity makes sense even if $L^1$ is infinite dimensional, since it means continuous differentiability together with $\mathbb{C}$-linearity of $d\Gamma$. If $U$ is an open subset of a vector space and $\Gamma$ is a polynomial map, then holomorphicity makes sense even if $L^1$ has no topology.

One defines analogously deformations over a germ of an analytic subspace of $\mathbb{C}^N$ or more general analytic spaces, see e.g. [Fuk03], section 8.2. The Kodaira-Spencer
map \( KS : T_M \to H^1(L^\bullet) \) is defined by \( KS(\xi) = [\xi(\Gamma)(o)] \), where \( \xi \) is thought of as a derivation.

### 2.2. Broader Context of the Paper

One of the main outcomes of this paper is the explicit description of a convenient universal family of deformations of a particular marked point in a particular moduli space. Why is this possible at all? The reason is the very special nature of our marked point, i.e., the very special form of the controlling differential graded Lie algebra \( L^\bullet \). This “speciality” manifests itself in three ways:

1. The dgla \( L^\bullet \) is the total complex of a double complex. The bigrading (by Hodge type) and its interaction with the bracket put a restriction on the set of Maurer-Cartan elements.
2. The HYM metric provides a space \( L^1 \supset H^1 \simeq H^1(L^\bullet) \) of harmonic representatives. It comes with a decomposition \( H^1 \simeq H^1 \oplus H'' \) into Lagrangian subspaces which consist of Maurer-Cartan elements. The natural maps to Def\(_L\) are étale onto their images, see 3.1.
3. The dgla \( L^\bullet \) has an extra (finite length) grading and one of the differentials of the double complex is a shift with respect to it.

Item (1) holds for the dgla controlling the deformations of any Higgs bundle. Items (2) and (3) are related to the fact that \( (P, \theta) \) is a \( \mathbb{C}^\times \)-fixed point and hence, by the non-abelian Hodge theorem, corresponds to a (polarised) \( \mathbb{C} \)-VHS (\cite{Sim92}). The latter carries two pieces of data: polarisation and Hodge filtration. Item (2) uses the particular form of the polarisation and the Hodge structure on \( L^\bullet \). The reason is the very special form of our marked point, i.e., the very special form of the double complex is a shift with respect to it.

We now argue that in such a situation there is a natural strategy for writing a (semi-universal) family of deformations.

For that we look at the above three items from a more general perspective, which is partially influenced by the discussion of monads in [DK90] (Sections 3.1.3 and 3.2.1).

Suppose that \( L^2 \) and \( L = L' \oplus L'' \) are two complex vector spaces and that \( Q : L \to L^2 \) is an origin-preserving, “off-diagonal” quadratic map. This means that \( Q = Q_1 + \frac{1}{2}Q_2 \), where \( Q_1 = Q_1'' \in \text{Hom}(L, L^2) \) and \( Q_2 \in \text{Hom}(L' \otimes L'', L^2) \). Consider the quadric \( M = Q^{-1}(0) \subset L \) and its “tangent bundle” \( T_M = \text{ker} dQ|_M \subset L \times L \), \( dQ_\lambda = Q_1 + \lambda Q_2 \).  

The quadric \( M \) contains lots of affine spaces. In particular, \( T_{M,0} = \text{ker} Q_1 \) contains two distinguished subspaces, \( T' = \text{ker} Q_1'' \subset M \) and \( T'' = \text{ker} Q_2'' \subset M \). The addition map \( L \times L \to L \) identifies \( p_2^* L''|_L \subset T_L \) with \( L \) and \( T_M \cap p_2^* L''|_{L \times \{0\}} \) with the “slice” \( E := M \cap (T' \times L'') \). The latter is a family of vector spaces parametrised by \( T' \). One may require (or look for conditions) that \( E \) be a vector bundle (possibly, after suitable completion), so that all fibres \( E_h, h \in T' \) will be isomorphic to \( E_0 = T'' \). Suppose now \( P : \text{Im} Q_1'' \to L'' \) is a splitting of the linear map \( Q_1'' \), and that \( \pi : L^2 \to \text{Im} Q_1'' \) is a projection onto its image. Then the family of linear maps \( \Phi_h = 1 + P \pi(h, \cdot) Q_2 \) gives an

\[\text{If } L \text{ is infinite-dimensional and is not equipped with topology we do not have a notion of vector bundle, hence the use of inverted commas.}\]
identification $\Phi : E \simeq T' \times T''$, or rather, $E|_{U} \simeq U \times T''$, for some set $U$ around $0 \in T'$, determined by the condition that $\Phi_{h}$ be invertible. If $L$ is equipped with a topology in which the inverse function theorem holds, then $U$ can be taken to be an (analytic) open set. If there is a (weak) duality pairing $L' \times L'' \to \mathbb{C}$, and $L$ is equipped with the corresponding canonical symplectic form $\omega_{\text{can}}$, then, under certain mild "orthogonality" conditions, $(\Phi^{-1})^{*}\omega_{\text{can}} = \omega_{\text{can}}|_{T' \times T''}$.

We shall apply this general strategy to the Maurer-Cartan quadratic $Q(x) = dx + \frac{1}{2}[x,x]$. It is here that Item (3) enters: the choice of a principal $\mathfrak{gl}(2, \mathbb{C})$-subalgebra gives a natural splitting $P$ of the differential $Q_{1}^{0} = \text{ad} \theta$, and the formal series for $\Phi^{-1}$ terminates due to the finite length of the filtration, i.e., the nilpotence of $\theta$. Due to Item (2), we have harmonic representatives of $H^{1}(L^{*})$ and can restrict $\Phi^{-1}$ to the subspace $\mathcal{H}^{1}$.

This construction is formally similar to the standard construction of the Kuranishi family, but the rôle of Green's operator $G$ is played by the much simpler splitting $P$.

3. Symplectic Kuranishi Map

In this section we abstract some basic properties of the dgla's which occur in our examples of interest and explore their deformation theory.

Consider a dgla, $L^{*}$, whose $L^{1}$-term admits a non-trivial decomposition into a direct sum $L^{1} = L' \oplus L''$, such that the two subspaces are:

1. Isotropic for the bracket: $[L', L'] = 0 = [L'', L'']$
2. Preserved under $\text{ad} L^{0}$: $[L^{0}, L'] \subseteq L'$, $[L^{0}, L''] \subseteq L''$.

Hence $(L^{*}, d)$ contains as a subcomplex (not sub-dgla!) the total complex of

$$
\begin{array}{c}
L'' \xrightarrow{d'_{1}} L^{2} \\
\downarrow d'_{0} \quad \downarrow d'_{0} \\
L^{0} \xrightarrow{d''_{1}} L'
\end{array}
$$

If $L^{*}$ is an analytic dgla, we assume that the two subspaces $L'$ and $L''$ are closed, and the (co)product is in the category of topological vector spaces. We denote by $d'_{k}$ the horizontal differentials, and by $d''_{k}$ the vertical ones. Notice that $\text{ker} d'_{1} \subseteq L''$ and $\text{ker} d''_{1} \subseteq L'$.

**Example 3.1.** Let $\mathfrak{g}$ be a complex Lie algebra and let $L^{*} = \bigoplus_{k} \mathfrak{g} \otimes L^{k}[-k]$, where $L^{0} = \mathfrak{g}$, $L^{1} = \mathfrak{g} \oplus \mathfrak{g}$, $L^{2} = \mathfrak{g}$. Fix $y \in \mathfrak{g}$ and endow $L^{*}$ with differentials $d_{0} = (ad_{y}, 0)^{T}$, $d_{1} = (0, ad_{y})$. There is a unique bracket on $L^{*}$ for which the above assumptions hold and which coincides with the Lie bracket on $L^{0}$.

**Example 3.2.** Let $X$ be a smooth compact curve, $E$ a holomorphic vector bundle on it, and $\theta \in H^{0}(X, \text{End} E \otimes K_{X})$ a Higgs field. Let $L^{*} = \bigoplus_{p+q=0} A^{p,q}(\text{End} E)$ with differential $\bar{\partial}_{E} + \text{ad} \theta$. Then we can take $L' = A^{1,0}(\text{End} E)$, $L'' = A^{0,1}(\text{End} E)$. The conditions on the bracket are satisfied for type reasons.

Finally, we impose the following crucial assumption

3. Suppose $\text{Im} d'_{1} \subseteq L^{2}$ is split. Suppose that $L^{*}$ admits a splitting $\delta$ (see Appendix 7) for which the direct sum decomposition of $L^{1}$ induces a non-trivial decomposition of $\mathcal{H}^{1}$ into $\mathcal{H}^{1} = \mathcal{H}' \oplus \mathcal{H}''$ and $\mathcal{H}'' = \text{ker} d'_{1}$. Fix one such $\delta$. Denote by $H = H' + H'' : L^{1} \to \mathcal{H}^{1}$ the harmonic projection.
Proposition 3.1. Suppose that $H^2(L^\bullet) = 0$ and (1), (2), (3) hold. Then there exist natural morphisms in $\text{FArt}_\mathbb{C}$

$$\mathcal{H}'[-1] \subset MC_{L^\bullet} \to \text{Def}_{L^\bullet},$$

$$\mathcal{H}''[-1] \subset MC_{L^\bullet} \to \text{Def}_{L^\bullet},$$

which are étale onto their images.

Proof:

By (1) we have $\ker d' \subset MC(L)$ and $\ker d'' \subset MC(L)$. But $\mathcal{H}' \subset \ker d'$ and $\mathcal{H}'' \subset \ker d''$, and the resulting inclusions $\mathcal{H}' \subset MC(L)$ and $\mathcal{H}'' \subset MC(L)$ induce the above-stated morphisms in $\text{FArt}_\mathbb{C}$. Since $H^2(L^\bullet) = 0$ (obstructions vanish), the Kuranishi map equals the identity on $MC_L \cap \mathcal{H}^1$, and the (formal) Kuranishi functor $K_L$ equals $H^1$. By [GM90], Section 3 or [Man99], Theorem 4.7 (see also the Appendix 7) we have an étale morphism

$$\mathcal{K}_L \xrightarrow{\mathcal{Y}_L} \text{Def}_L.$$ 

We shall now digress and make some elementary remarks on dgla’s with vanishing $L^3$ and $H^2(L^\bullet)$.

Theorem 3.1. Let $L^\bullet$ be a dgla with $L^3 = 0$ and $H^2(L^\bullet) = 0$. Let $\tilde{\delta} : L^2 \to L^1$ be a splitting of $d_1$, that is, $d_1 \tilde{\delta} = 1_{L^2}$. Fix a subspace $\mathcal{H}^1 \subset L^1$, isomorphic to $H^1(L^\bullet)$, and consider the formal power series $\Gamma \in L^1 \otimes \lim_{\leftarrow} \text{Sym}^* (\mathcal{H}^1^\vee)/m^k$, $\Gamma = \sum_{k=1}^\infty \Gamma_k$, where $\Gamma_k \in L^1 \otimes m^k/m^{k-1}$ is defined inductively by

$$(1) \quad \Gamma_1(x) = x, \quad \Gamma_k(x) = -\frac{1}{2} \delta \sum_{n=1}^{k-1} [\Gamma_n(x), \Gamma_{k-n}(x)].$$

Then $\Gamma$, thought of as a formal map $(\mathcal{H}^1, 0) \to L^1$, determines a formal miniversal family of deformations of $\text{Def}_{L^\bullet}(\mathbb{C})$ over $(\mathcal{H}^1, 0)$.

If $L^\bullet$ is a normed dgla and the above series converges in some neighbourhood, $U$, of $0 \in \mathcal{H}^1$, then the corresponding family $\Gamma : U \to L^1$ is a miniversal analytic family of deformations of $\text{Def}_{L^\bullet}(\mathbb{C})$.

Remark 3.2. In coordinates $\Gamma$ is described as follows. We fix a basis, $\{t_i\}, i = 1...d$, of $(\mathcal{H}^1)^\vee$. Then $\Gamma = \sum_{k=1}^\infty \Gamma_k \in L^1 \otimes \mathbb{C}[t_1, \ldots, t_d]$, and $\Gamma_k = \sum_{|J|=k} \Gamma_{J,k} t^J$, where $J$ is a multi-index.

Proof:

This is a statement about power series which can be related to some classical deformation-theoretic calculations (see, e.g. [KNS58] or [Kur62]). Since the proof is easy and instructive, we are going to give it here anyway.

On one hand, reading the Maurer-Cartan equation “up to order $k$” we see that any formal power series solution has to satisfy

$$d_1 \Gamma_k + \frac{1}{2} \sum_{n=1}^{k-1} [\Gamma_n, \Gamma_{k-n}] = 0.$$
On the other hand, applying $d_1$ to both sides of the proposed recursive formula for $\Gamma_k$ we get

$$d_1 \Gamma_k = -\frac{1}{2} d \delta \sum_{n=1}^{k-1} [\Gamma_n, \Gamma_{k-n}] = -\frac{1}{2} \sum_{n=1}^{k-1} [\Gamma_n, \Gamma_{k-n}].$$

By construction (1), the (formal or analytic) Kodaira-Spencer map of this family is the identity, and hence it is isomorphic to the Kuranishi family, which is a miniversal deformation. See e.g., [Fuk03] or Appendix 7 for other references and comments. We emphasise that this family need not be the Kuranishi family.

Remark 3.3. Kuranishi theory for dgla's (or $L_\infty$-algebras) is based on a choice of "splitting" (or passing to a minimal model), see Appendix 7 for the relevant definitions. This involves a degree $-1$ endomorphism of $L^\bullet$, $\delta$, which in particular satisfies $d \delta + \delta d = 1 - H$, where $H$ is a “harmonic projection”. There is a well-known power-series solution of the Maurer-Cartan equation (the inverse of the formal Kuranishi map, see the Appendix 7), known from the works of Kuranishi, Kodaira-Nirenberg-Spencer, Huebschmann-Stasheff and many others. It is given exactly by the above formula (1) but with $\delta$ instead of $\delta$ (i.e., by (11)). The latter formula involves only $\delta_2$ and none of the other $\delta_i$! To verify that the series (11) gives a formal solution, one proceeds essentially as in the above proof. The main difference is that now instead of $d_1 \delta = 1$ we have $d_1 \delta_2 = 1 - \delta_3 d_2 - H_2$. But $H_2 = 0$ since $H^2(L^\bullet) = 0$, and the term involving $\delta_3 d_2$ vanishes due to the fact that $d$ is a derivation of the bracket, combined with associativity (graded Jacobi identity). If $L^3 = 0$, this latter term is not present at all, so $d_1 \delta_2 = 1$. Since the series involves only $\delta_2$, we could start with any splitting (and ignore the remaining $\delta_i$) and will still get a formal solution.

We now return to our discussion of dgla's with a decomposition and state a version of the above theorem based on splitting $d_1'$ only.

Let $L^\bullet$ be a dgla with $H^2(L^\bullet) = 0$ and $L^3 = 0$, satisfying the assumptions (1), (2) and (3). Let $\pi : L^2 \rightarrow \operatorname{Im} d_1'$ be a projector and $P_{\delta''}$ a splitting of $d_1'$, so the linear map $\tilde{\delta} = \begin{pmatrix} 0 & \pi \end{pmatrix} : L^2 \rightarrow L^1$ satisfies $d_1 \tilde{\delta} = \pi$.

Theorem 3.4. The formal power series $\Gamma \in L^1 \otimes \lim_{\leftarrow} \operatorname{Sym}^\bullet(\mathcal{H}^{1\vee})/m^k$ given by

$$\Gamma(h, v) = \left( \begin{array}{c} h \\ (1 + P \pi ad_1) h(v) \end{array} \right) + \sum_{k=1}^{\infty} (-1)^{k-1} \left( \begin{array}{c} 0 \\ (P \pi ad_1)^{k-1}(v) \end{array} \right),$$

$(h, v) \in \mathcal{H}' \oplus \mathcal{H}''$, is a formal deformation of $\operatorname{Def}_{L^\bullet}(\mathbb{C})$ over $(\mathcal{H}^1, 0)$ if and only if $[\Gamma, \Gamma] \in \operatorname{Im} d_1' \otimes \lim_{\leftarrow} \operatorname{Sym}^\bullet(\mathcal{H}^{1\vee})/m^k$.

If moreover $L^\bullet$ is a normed dgla and $\operatorname{Im} d_1' \subset L^2$ is closed, then there exists a neighbourhood of the origin, $\mathcal{U} \subset \mathcal{H}^1$, such that the family $\Gamma : \mathcal{U} \rightarrow L^1$ is a miniversal analytic family of deformations of $\operatorname{Def}_{L^\bullet}(\mathbb{C})$.

Proof: The formal statement is proved exactly as in the previous theorem. Indeed, due to the isotropy of the bracket and the choice of $\delta$, the formula (1) reduces to the formula (2). But since here $d \delta = \pi$, we have that $\Gamma$ satisfies $d \Gamma = -\pi [\Gamma, \Gamma]$, which
coincides with the Maurer-Cartan equation if and only if the right hand side is $[\Gamma, \Gamma]$, i.e., $(\pi - 1)[\Gamma, \Gamma] = 0$.

For the analytic statement notice that the power series is essentially the geometric series, and since $\ad, \pi$ and $P$ are continuous, the series will converge for $h$ sufficiently small, that is, for $x = (h, v) \in U = B_1 \times \mathcal{H}'$, where $B_1 \ni 0$ is a ball of sufficiently small radius $\epsilon$. The Kodaira-Spencer map of the family is the identity, so it is miniversal by Theorem 1.3.3., [Fuk03].

**Corollary 3.5.** Let $L^\bullet$ be as in the statement of the theorem, and assume that $\pi, P, \ad$ extend to continuous linear maps on some completion $\widehat{L}^\bullet$. Let $\delta$ be a compatibly chosen splitting of $\widehat{L}^\bullet$. If $P\pi\ad_h$ is (locally) nilpotent for all $h \in \mathcal{H}'$, then $\Gamma : \mathcal{H} \rightarrow L^1 \subset \widehat{L}^1$ is a miniversal analytic family of deformations of $\Def_L(C)$.

**Proof:**
If $P\pi\ad_h$ is locally nilpotent for all $h$, then $(1 + P\pi\ad_h)^{-1}(\mathcal{H}') \subset L \subset \widehat{L}$. □

**Proposition 3.6.** Let $L^\bullet$ be a dgla satisfying assumptions (1), (2), (3). Let $P\pi''$ be a splitting of $d_1''$ and $\pi : L^2 \rightarrow \text{Im} \ d_1''$ a projector. Assume that $L^1 = 0$, $H^2(L^\bullet) = 0$, and $[\Gamma, \Gamma] \in \text{Im} \ d_1'' \otimes \lim \text{Sym}^n(\mathcal{H}^{1,1})/\mathfrak{m}^n$.

Let $\mathcal{S}_L \in \FArt_C$ be the functor $\mathcal{S}_L = MC_L \cap \ker [(1 - \mathcal{H}')\pi']$. Then

$$\Phi : \mathcal{S}_L \rightarrow \mathcal{H} = \mathcal{H}' \oplus \mathcal{H}''$$

$$\Phi_A(h, v) = (h, (1 + P\pi\ad_h)v) \in \mathcal{H} \otimes \mathfrak{m}_A$$

is an isomorphism in $\FArt_C$ and $\Phi^{-1} = \Gamma$. The composition

$$\pr \circ F^{-1} \circ \Phi : \mathcal{S}_L \rightarrow \Def_L$$

is étale. If additionally $L^\bullet$ is a normed and $\text{Im} \ d_1'' \subset L^2$ is closed, then $\mathcal{S}_L$ is prrepresented by the germ $(\mathcal{S}, 0)$, where

$$\mathcal{S} = MC(L) \cap \ker [(1 - \mathcal{H}')\pi'] \subset \mathcal{H}' \oplus L''$$

and $\Phi : (\mathcal{S}, 0) \simeq (\mathcal{H}', 0)$. □

**Proof:**
We have that $(h, v) \in MC_L(A) \cap \ker [(1 - \mathcal{H}')\pi'](A) \iff h \in \mathcal{H}' \otimes \mathfrak{m}_A$ and $v \in \ker(d' + \ad_h)$.

Now, $1 + P\pi\ad_h$ maps $\ker(d' + \ad_h)$ to $\ker d'$, since on the former $\ad_h$ equals $-d'$, and $1 + P\pi\ad_h$ equals $(1 - Pd')$, the projector onto $\ker d'$. Since $\mathfrak{m}_A$ is nilpotent, $1 + P\pi\ad_h$ is invertible for all $h$, and hence it maps injectively $\ker(d' + \ad_h)$ to $\ker d'$.

The condition $(\pi - 1)[\Gamma, \Gamma] = 0$ means, by Theorem 3.4 that if $(h, v) \in \mathcal{H} \otimes \mathfrak{m}_A = (\mathcal{H}' \oplus \ker d') \otimes \mathfrak{m}_A$, then $(h, (1 + P\pi\ad_h)^{-1}v) \in MC_L(A)$, i.e., belongs to $\mathcal{S}_L(A)$. Hence $\Phi_A$ is an isomorphism.

The composition $\pr \circ F^{-1} \circ \Phi$ is étale since $\Phi$ is an isomorphism and $\pr \circ F^{-1}$ is étale by [GM90], Section 3 or [Man99], Theorem 4.7. □

For the next two corollaries, assume that $\widehat{L}^\bullet$ is a normed dgla, which is the completion of a dgla $L^\bullet$ with respect to some norm. Also, assume that $\delta$ is a (compatibly chosen) splitting and that $P, \pi$ and $\ad$ extend to continuous operators on $\widehat{L}^\bullet$.

**Corollary 3.7.** Let both $L^\bullet$ and $\widehat{L}^\bullet$ satisfy the assumptions of the theorem. Then, if $P\pi\ad_h$ is (locally) nilpotent for all $h \in \mathcal{H}'$, the slice $\mathcal{S}$ satisfies $\mathcal{S} = \Phi^{-1}(\mathcal{H}^1) \subset L^1 \subset \widehat{L}^1$. □
Corollary 3.8. Let $L^\bullet$ be a normed dgla satisfying the assumptions of the Theorem, except possibly the condition $[\Gamma,\Gamma] \in \text{Im} \ d_1 \otimes \lim \text{Sym}^n(\mathcal{H}^1)/m^k$. Then $\Phi : (S,0) \simeq (\mathcal{H}',0) \times \mathcal{H}''$ is a holomorphic vector bundle isomorphism over $(\mathcal{H}',0)$ if and only if $[\Gamma,\Gamma] \in \text{Im} \ d_1 \otimes \lim \text{Sym}^n(\mathcal{H}^1)/m^k$. In particular, if dim $L^\bullet < \infty$, dim ker($d' + ad_h$) is constant in a (connected) neighbourhood of $0 \in U \subset \mathcal{H}'$ if and only if $ad_h(1 + P\pi ad_h)^{-1}(v) \in \text{Im} d_1'$ for all $h \in U$ and all $v \in \mathcal{H}''$.

Proof: By the proof of Theorem 3.4 $(1 + P\pi ad_h)$ is invertible for $h \in B_\epsilon$, some $\epsilon > 0$. By the inverse function theorem, its inverse is analytic in some (possibly smaller) open, which we shall still denote by $B_\epsilon$. By the proof of Theorem 3.6 $M(C(L) \cap \ker (1 - \mathcal{H}'')^\pi)$ is a family of kernels, which we have trivialised by a holomorphic family of projectors. By [ZKKP75], §1, Theorem 1.5, Theorem 2.7 and §3, the image of a holomorphic family of projectors is a Banach vector bundle. Thus $\mathcal{S} \cap B_\epsilon + L''$ is a Banach vector bundle precisely when $\Phi$ is an isomorphism. Let us underline here that dim $H^\bullet(L^\bullet) < \infty$, so both the base and the fibre of this vector bundle are finite dimensional vector spaces! Of course, if also dim $L^\bullet < \infty$, then $\mathcal{S} \cap B_\epsilon + L''$ is a vector bundle if and only if $\text{rk}(1 + P\pi ad_h) = \text{const}$ on $B_\epsilon$. \qed

Remark 3.9. Since $\mathcal{H}'$ is finite dimensional, probably some clarification is needed regarding the appearance of Banach vector bundles. Our setup is the following. We have a holomorphic family of linear maps between two (possibly) infinite dimensional vector spaces, $L''$ and $L^2$, a priori without topology: this is $\mathcal{H}' \to \text{Hom}(L'', L^2)$, $h \mapsto (d_1' + ad_h)$. We are interested in the collection of kernels, $\mathcal{S}$. We gave conditions for the kernels to be of finite, constant dimension ($\text{dim} \mathcal{H}'''$) and gave an explicit formal trivialisation, $\Phi$, of $\mathcal{S}$. If we want to put a topology on $\mathcal{S}$, make it into an honest vector bundle and have that $\Phi$ is a vector bundle trivialisation, then we have to pass to a completion of $L^\bullet$. In the intended applications $\Phi_h$ is in fact a polynomial in $h$ due to nilpotence, and $S \subset L^1 \subset \hat{L}^1$.

We are ultimately interested in situations where the “local moduli space” corresponding to Def $L^\bullet$ is symplectic, and the symplectic form is induced by a (constant) symplectic form on $L^1$ for which the two subspaces $L'$ and $L''$ are isotropic. The motivating example is the case when $L^1$ is a (weak) cotangent bundle.

Lemma 3.1. Let $L^\bullet$ be as in Theorem 3.6. Let $\omega$ be a skew-symmetric bilinear form on $L^1$ for which the subspaces $L'$ and $L''$ are isotropic. Then $(\Phi^{-1})^* \omega$ vanishes on the subbundles $\mathcal{H}' \times \mathcal{H}' \subset T_{\mathcal{H}'}$ and $\mathcal{H}' \times \mathcal{H}'' \subset T_{\mathcal{H}''}$. Moreover, $(\Phi^{-1})^* \omega$ on $\mathcal{H}'$ is invariant under translations along $\mathcal{H}''$.

Proof: Let $\Phi^{-1}$ denote the holomorphic map $\mathcal{H}' \to \text{End}(L^2)$, $h \mapsto \Phi^{-1}_h = (1 + P\pi ad_h)^{-1}$. Then $(d\Phi^{-1})_{(h,v)}(\xi', \xi'') = (\xi', 0) + (0, (d\Phi^{-1})_h(\xi'')(v)) + (0, (\Phi^{-1}_h(\xi''))$, so $d\Phi^{-1}$ preserves ker $d_1'$ and ker $d''_1$ and the first statement follows. But the second of the three terms vanishes identically due to assumption (1): $(d\Phi^{-1})_h(\xi'')(v) = -P\pi[\xi'', v] = 0$, and so $(\Phi^{-1})^* \omega_{(h,v)}$ is independent of $v \in \mathcal{H}''$. \qed

Proposition 3.10. Let $L^\bullet$ (resp. $\hat{L}^\bullet$) be a dgla, satisfying the assumptions of Theorem 3.6. Suppose that $L'$ and $L''$ are placed in (weak) duality by a pairing $(\cdot, \cdot)$ and let $\omega_{\text{can}}$ be the canonical symplectic form on $L^1 = L' \oplus L''$. If Im $P\pi ad_h \subset \mathcal{H}'$ for all $h \in \mathcal{H}'$, then $(\Phi^{-1})^* \omega_{\text{can}} = \omega_{\text{can}}$. In the normed case, $\Phi : S \to \mathcal{H}'$ gives holomorphic Darboux coordinates on $(S,0)$. 


Proof:
The canonical symplectic form on \( L^1 \) is \( \omega_{can}((\xi', \xi''), (\eta', \eta'')) = \langle \xi', \eta'' \rangle - \langle \xi'', \eta' \rangle \). Using the formula for \( d\Phi^{-1} \) from the previous Lemma and the isotropy of \( \mathcal{H}' \) and \( \mathcal{H}'' \) we get

\[
(\Phi^{-1})^* \omega_{can}((\xi', \xi''), (\eta', \eta'')) = \langle \xi', \Phi^{-1}(\eta'') \rangle - \langle \Phi^{-1}(\xi''), \eta' \rangle.
\]

Substituting \( (1 + P\text{ad}_h)^{-1} = 1 - P\text{ad}_h(1 + P\text{ad}_h)^{-1} \) into the previous formula and using the orthogonality assumption we get

\[
(\Phi^{-1})^* \omega_{can}((\xi', \xi''), (\eta', \eta'')) = \langle \xi', \eta'' \rangle - \langle \xi'', \eta' \rangle.
\]

\[\square\]

Remark 3.11. With the above assumptions, \( S \) is a Lagrangian foliation, with space of leaves (the germ of) \( \mathcal{H}' \). Such a foliation carries a torsion-free flat connection along the leaves. Since \( T_{\mathcal{H}'}^S \simeq S \) (as symplectic manifolds), the affine structures on the leaves is induced by the vector space structure on the fibres, and we have described it in terms of the controlling dgla.

4. Lie-algebraic preliminaries

We review here some relevant facts from Lie theory mostly to set up notation. Details can be found in [CG97] or [Kos63]. Let \( G \) be a simple complex Lie group, \( \mathfrak{g} = \text{Lie}(G) \) and \( \text{rank}(\mathfrak{g}) = l \). An element of \( \mathfrak{g} \) is regular if its centraliser is of the smallest possible dimension, \( l \). An element \( \varphi \in \mathfrak{g} \) is semisimple (respectively, nilpotent) if \( \text{ad}_{\varphi} \in \text{End}(\mathfrak{g}) \) is semisimple (respectively, nilpotent). If \( \mathfrak{g} = \mathfrak{sl}(l+1) \), the regular elements are trace-free matrices with a single Jordan block per eigenvalue. A regular nilpotent \( \varphi \) is one which is conjugate to a single Jordan block with zeros on the diagonal. We denote by \( \mathfrak{g}^{reg} \), \( \mathfrak{g}^{ss} \) and \( \mathfrak{g}^{reg, ss} \) the sets of regular, semisimple and regular semisimple elements of \( \mathfrak{g} \). One has \( \mathfrak{g}^{reg, ss} \subset \mathfrak{g}^{reg} \subset \mathfrak{g} \) and \( \mathfrak{g}\backslash \mathfrak{g}^{reg, ss} \subset \mathfrak{g} \) is a divisor while \( \mathfrak{g}\backslash \mathfrak{g}^{reg} \subset \mathfrak{g} \) is of codimension 3.

The notion of a regularity makes sense for reductive Lie algebras as well. In particular, if \( \varphi \in \mathfrak{g}(n, \mathbb{C})^{reg} \), its centraliser \( \mathfrak{z}(\varphi) \) is spanned by \( \{\varphi, \varphi^2, \ldots, \varphi^n\} \). We do not have such a convenient description of the centraliser for other Lie algebras.

By the Jacobson-Morozov lemma any nilpotent \( x \in \mathfrak{g} \) can be embedded in an \( \mathfrak{sl}(2, \mathbb{C}) \)-subalgebra of \( \mathfrak{g} \). A principal \( \mathfrak{sl}(2, \mathbb{C}) \) subalgebra is one which is spanned by two regular nilpotent elements, \( x \) and \( y \), and a semisimple \( h \in \mathfrak{g} \). The inclusion \( \mathfrak{sl}(2, \mathbb{C}) \hookrightarrow \mathfrak{g} \) exponentiates to a homomorphism \( g : SL(2, \mathbb{C}) \to G \), a “principal homomorphism”. The maximal compact \( SU(2) \subset SL(2, \mathbb{C}) \) maps to a compact form of \( G \).

Under the adjoint action of \( \mathfrak{sl}(2, \mathbb{C}) \) \( \mathfrak{g} \) decomposes into \( l \) odd-dimensional irreducible representations:

\[
\mathfrak{g} = \bigoplus_{i=1}^{l} W_{m_i}, \quad W_{m_i} = \text{Sym}^{2m_i}(\mathbb{C}^2),
\]

where \( \mathbb{C}^2 \) is the standard representation of \( \mathfrak{sl}(2, \mathbb{C}) \). The spaces \( W_{m_i} \) are \((2m_i + 1)\)-dimensional, so the restriction \( SL(2, \mathbb{C}) \to \text{Aut}(W_{m_i}) \) of the adjoint representation to each \( W_{m_i} \) factors through \( PGL(2, \mathbb{C}) \). The restriction to the maximal compact makes \( W_{m_i} \) into a representation of \( PSU(2) = SO(3) \). On each \( W_{m_i} \), the eigenvalues of \( \text{ad}_h \) are even integers \( 2m \), where \(-m_i \leq m \leq m_i \). The highest weight vectors span the centraliser \( \mathfrak{z}(x) \). We shall label the eigenspaces by half of the corresponding
eigenvalue and shall let $g_m$ stand for the eigenspace of $ad_h$ with eigenvalue $2m$. The decomposition

$$ g = \bigoplus_{m=-i}^i g_m, $$

is called the principal grading of $g$. The filtration $W_+ g$, $W_p g = \oplus_{2m < p} g_m$ is the canonical (Deligne) filtration of the nilpotent endomorphism $ad_g$. Intersecting 3 and 4 we get a bigrading $g = \oplus g_{k,i} = g_k \cap W_{m,i}$. Then $\mathfrak{z}(x) = \oplus_i g_{m,i}$ and $\mathfrak{z}(y) = \oplus_i g_{-m,i}$.

The numbers $m_i$ are the exponents of $g$ (or $G$). For a simple Lie algebra they are all distinct except if $g = D_{2n}$, when the largest exponent has multiplicity two. We order the exponents, so that $m_i \leq m_j$ for $i < j$ and for the most part we shall write $W_i$ instead of $W_{m,i}$. In particular, as $G$ is simple, $m_1 = 1$ and $W_1 = \mathfrak{sl}(2, \mathbb{C})$ is the principal subalgebra.

The motivating example is the $l$-th symmetric power embedding $\mathfrak{sl}(2, \mathbb{C}) \hookrightarrow \mathfrak{sl}(l + 1, \mathbb{C})$. Notice that it maps the standard generators $\{y_0 = E_{21}, h_0 = E_{11} - E_{22}, x_0 = E_{12}\}$ of $\mathfrak{sl}(2, \mathbb{C})$ to the $(l + 1) \times (l + 1)$ matrices $y, h, x$, where $y = \sum_{p=1}^l E_{p+1,p}$, $h = \sum_{p=1}^{l+1} (l - 2p + 2)E_{p,p}$ and $x = \sum_{p=1}^l p(l - p + 1)E_{p,p+1}$. In particular, $x \neq y^T$!

Let $\mathbb{C}[g]^G \subset \mathbb{C}[g]$ be the ring of $G$-invariants for the adjoint action. The GIT quotient is $g / G := \text{Spec} \mathbb{C}[g]^G$, and its points correspond to closures of $G$-orbits. The closure of each $G$-orbit contains a unique open (regular) and a unique closed (semisimple) orbit. By a theorem of Chevalley, $\mathbb{C}[g]^G$ is isomorphic to a polynomial ring, i.e., $g / G$ is non-canonically isomorphic to a vector space. We can fix one such isomorphism by choosing a basis for the $G$-invariant polynomials on $g$, say $\{p_1, \ldots, p_l\}$, $\deg(p_i) = m_i + 1$. We assume that our choice of invariant polynomials is compatible with the decompositions $\mathfrak{z}(x) = \oplus_i \mathfrak{z}(x) \cap W_i$ induced by the principal subalgebra. This means that there exists a basis for $\mathfrak{z}(x)$ consisting of highest weight vectors $v_i \in W_i \cap g_m$, such that $p_i(y + a_1 v_1 + \ldots + a_l v_l) = a_i$. This gives an identification $\mathbb{C}[g]^G \simeq \mathbb{C}[p_1, \ldots, p_l]$ and the Chevalley projection $g \to g / G$ can be interpreted as a map $g \to \mathbb{C}^l$. For $g = \mathfrak{sl}(n, \mathbb{C})$ this map sends a matrix to the (non-leading) coefficients of its characteristic polynomial.

Let $t \supset \mathfrak{h}$ be a Cartan subalgebra and $W$ the corresponding Weyl group. Chevalley proved that $t \to g$ induces an isomorphism $t / W \simeq g / G$. In [Kos63] it is shown that the adjoint quotient $g \to t / W$ becomes an isomorphism when restricted to the Kostant slice $\Sigma = y + \mathfrak{z}(x) \subset g^{reg}$. Thus $\Sigma$ provides a splitting $t / W \to g$ of the Chevalley projection. We shall also write $s$ for the affine-linear map $s : \mathfrak{z}(x) \to \Sigma$, $s(a) = a + y$.

We shall use one particular principal $\mathfrak{sl}(2, \mathbb{C})$-subalgebra $\{y, \mathfrak{h}, x\}$ which is the standard one in the literature on opers ([Fre07]) and which we describe now.

Fix Chevalley generators $\{f_i, h_i, e_i\}$, $t = \text{span}\{h_i\}$, $i = 1 \ldots l$, and assume $\kappa(e_i, f_i) > 0$. Fix positive roots $\Delta^+$. Let $\rho = \sum_i \rho_i^+ h_i$ be the dual Weyl vector, i.e., half the sum of the positive roots. We take $y = \sum_i f_i$, a regular nilpotent element, and $\mathfrak{h} = 2\rho^\vee \in t$. The unique $x$ for which $\text{span}\{x, 2\rho^\vee, y\} \simeq \mathfrak{sl}(2, \mathbb{C})$ is $x = \sum_i 2\rho_i^+ e_i$.

The choice of Chevalley generators determines a split and a compact real form of $g$ ([Bou82], IX.16 §3). The former is the real subalgebra generated by $\{e_i, f_i, h_i\}$. The latter is the $+1$ eigenspace of the anti-linear extension, $\eta$, of $e_j \mapsto -f_j$, $f_j \mapsto$
On the other hand, the hamiltonian reduction of $\mathfrak{sl}_m$ principal $\mathfrak{sl}_m$ the actual coefficient, depending on $z$ symplectic form on $I_{\mathfrak{ad}} W$ probably well-known that the different irreducible representations $W_i$ are orthogonal with respect to this inner product, but for lack of reference we have proved it in [Dal08].

Notice that by construction the principal $\mathfrak{sl}(2, \mathbb{C})$ (and all the representations $W_i$) are all real with respect to $\eta$ and in particular $y^* = x, h^* = h$.

5. Universal Centralisers

Consider now the tautological family of centralisers of regular elements $I = \{(u, v) : [v, u] = 0, v \in \mathfrak{g}^{reg}, u \in \mathfrak{g}\} \subset \mathfrak{g}^{reg} \times \mathfrak{g}$.

The projection $\text{pr}_1 : I \to \mathfrak{g}^{reg}$ makes this locally closed subvariety into a rank $l$ vector bundle, a subbundle of the trivial bundle $T_{\mathfrak{g}^{reg}}$. The group $G$ acts on $I$ diagonally by the adjoint action, and the quotient is the universal centraliser. It is a hamiltonian reduction of $T_{\mathfrak{g}}^\vee \simeq \mathbb{R}_0$, and $I / G \simeq T_{I / W}$ is a symplectic isomorphism. On the other hand, $I / G \simeq I|_{y^2 + z(x)} = \mathfrak{s}^* I$, and we shall see that the choice of a principal $\mathfrak{sl}(2, \mathbb{C})$ provides a natural trivialisation $\mathfrak{s}^* I \simeq \mathfrak{z}(x) \times \mathfrak{z}(y)$, with the property that the symplectic form on $I|_{y^2 + z(x)} \subset T_{\mathfrak{g}}$ pulls back to the standard symplectic form on $\mathfrak{z}(x) \times \mathfrak{z}(y)$.

The subspace $\mathfrak{z}(x) \simeq \text{coker}(\text{ad}_y)$ provides a splitting, $P \in \text{Hom}(\text{Im} \text{ad}_y, \mathfrak{g})$, of $\text{ad}_y$. To compute $P$ in examples one can use that each $W_m$ is an irreducible $\mathfrak{sl}(2, \mathbb{C})$-representation, so a suitable multiple of $\text{ad}_x$ inverts $\text{ad}_y$ on $\text{Im} (\text{ad}_y)$. For the actual coefficient, depending on $m_i$ and $k$, see [FH91], Lecture 11. The bigrading of $\mathfrak{g}$ provides natural projections $\pi : \mathfrak{g} \to \text{Im} \text{ad}_y$ and $p^r_p : \mathfrak{g}_r \to \mathfrak{g}_{r,p}$. Note that $\pi, p^r_p \in \text{End}_0(\mathfrak{g})$, while $P \in \text{End}_1(\mathfrak{g})$. Consequently, for all $h \in \mathfrak{z}(x)$, $P \pi \text{ad}_h \in \text{End}_2(\mathfrak{g})$ and is hence nilpotent. Note in passing that in this setup we also have a natural splitting of $\text{ad}_x$, say $Q \in \text{Hom}(\text{Im} (\text{ad}_x), \mathfrak{g}), \text{ad}_x \circ Q = 1$.

We now formulate a technical Lemma.

**Lemma 5.1.** Let $\mathfrak{g}$ be a simple complex Lie algebra, $\{y, h, x\}$ a principal $\mathfrak{sl}(2, \mathbb{C})$ subalgebra, and $P$ the canonical splitting of $\text{ad}_y$ determined by it. Let $0 \neq h \in \mathfrak{z}(x)$.

Then, $\forall k \geq 0$, $\text{ad}_h (P \text{ad}_h)^k (\mathfrak{z}(y)) \subset \text{Im} \text{ad}_y$.

Equivalently, $\forall k \geq 0$, $(P \pi \text{ad}_h)^k (\mathfrak{z}(y)) = (P \text{ad}_h)^k (\mathfrak{z}(y))$.

**Proof:**

For notational simplicity assume that $\mathfrak{g} \neq D_{2n}$. This is the only simple Lie algebra with a repeated exponent (the largest exponent appears twice), and in that case the proof is exactly as the one that follows below, but one has to choose the two...
$W_i$’s corresponding to the maximal exponent in a way that they be orthogonal with respect to the inner product induced by the Killing form.

We work by induction on $k$, and use an observation from Clebsch-Gordan theory of $\text{SL}(2, \mathbb{C})$ ([Hit92], p.458) regarding commutators of elements from different $W_i$. Namely,

$$pr^{m+n}_p ([g_{m,i}, g_{n,j}]) = 0$$

unless $m_i + m_j + m_p = 1 \mod 2$

For the case $k = 1$ we have to show that $\text{ad}_h : \mathfrak{z}(y) \to \text{Im}(\text{ad}_y)$. Let $v \in \mathfrak{z}(y)$. Since $3 = \oplus_3 \mathfrak{z}(y) \cap W_i$, and similarly for $\mathfrak{z}(x)$, we may assume $v \in g_{m_j, 1} \subset \mathfrak{z}(y)$ and $h \in V_{m_p} = g_{m, i} \subset \mathfrak{z}(x)$. Then $[h, v] = [e_{m_j}, e_{-m_j}] \in g_{m_j, m_j}$, where $e_{m_j}$ (resp. $e_{-m_j}$) is a highest (resp. lowest) weight vector in $W_i$ (resp. $W_j$). We claim that this commutator can never be in some $V_{m_p} = g_{m, p, p}$, that is $pr^{m_{j,m_j}}_{m_p}([e_{m_j}, e_{-m_j}]) = 0$. Indeed, if there were such a term, there would be an exponent $m_p$, such that $m_i - m_j = m_p$ and $m_i + m_j + m_p = 1 \mod 2$, which would mean that $2m_i = 1 \mod 2$. So the base case is proved and $P o \pi o \text{ad}_h(v) = P o \text{ad}_h(v) \in g_{m_i, m_j, 1}$.

For the inductive step, let $(P o \pi o \text{ad}_h)^k(v) = (P o \text{ad}_h)^k(v), k \geq 1$. Then we can write it as a linear combination of elements in the $(km_i - m_j + k)$-th graded piece of $g$. Such an element has a nonzero projection in some $W_{m_p}$ if

$$m_i + m_j + m_p = 2l + 1, m_i + m_j + m_p = 2l + 1, \ldots, m_i + m_{p-1} + m_p = 2l + 1,$$

where $l \in \mathbb{Z}$.

Adding these up gives

$$(5) \quad km_i + m_j + m_p = \sum_{r=1}^{k-1} 2m_r + m_p = \sum_{r=1}^k 2l_r + 1$$

If $\text{ad}_h (P o \text{ad}_h)^k(v)$ has a nonzero projection in some $V_{m_i}$, then it must be the case that

$$(k + 1)m_i - m_j + k = m_i, m_i + m_p + m_i = 1 \mod 2,$$

and adding these we get

$$(6) \quad (k + 2)m_i + m_p - m_j + k = 1 \mod 2.$$
where \( k \) is the Coxeter number. Finally, \( \Gamma^*\omega_{\text{can}} = \omega_{\text{can}} \), where \( \omega_{\text{can}} \) denotes the canonical symplectic form on \( \mathfrak{g} \times \mathfrak{g} \), as well as its restrictions to \( I \) and \( \mathfrak{g}(x) \times \mathfrak{g}(y) \).

**Proof:**

Since \( P^{\text{rad}}_h \) is nilpotent for all \( h \in \mathfrak{g}(x) \), then \( 1 + P^{\text{rad}}_h \) is invertible. Then \( \Phi \) is an isomorphism by Theorem 3.4 applied to Example 3.1: the series (2) is convergent for all \( h \in \mathcal{U} = \mathfrak{g}(x) \) and is actually a polynomial of degree at most \( k \). But by Lemma 5.1 this polynomial equals the one from the statement of the Theorem. The condition \((\pi - 1)[\Gamma, \Gamma] = 0\) clearly holds, since all elements from the Kostant slice are regular. Finally, the statement about the symplectic form holds by Proposition 3.10, applied to the dgla under consideration. Indeed, the Killing form is non-zero only on \( \mathfrak{g}_{m,i} \times \mathfrak{g}_{-m,i} \) and \( \mathfrak{g}_{-m,i} \times \mathfrak{g}_{m,i} \), while \( P^{\text{rad}}_h \in \text{End}_2(\mathfrak{g}) \), so the orthogonality condition from 3.10 is satisfied.

**Example 5.1.** Let \( \mathfrak{g} = A_1 = \mathfrak{sl}(2, \mathbb{C}) \) with the standard generators \( y = E_{21}, 2\rho^i = E_{11} - E_{22}, x = E_{12} \). Then \( \tilde{P}(y) = \rho^i, P(h) = -x \) and \( \Gamma : \mathbb{C}^2 \simeq \mathfrak{g}(x) \times \mathfrak{g}(y) \rightarrow \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C}) \) is given by

\[
\Gamma(h, v) = (y + \alpha x, \xi y + \alpha \xi x) = \begin{pmatrix}
0 & \alpha \\
1 & 0
\end{pmatrix}, \begin{pmatrix}
0 & \alpha \xi \\
\xi & 0
\end{pmatrix}, \alpha, \xi \in \mathbb{C}.
\]

6. The Uniformising Higgs Bundle

6.1. The Uniformising Higgs bundle. Let us fix a theta-characteristic \( K^{1/2}_X \). This is a line bundle \( \zeta \in \text{Pic}_X^{1/2} \), together with an isomorphism \( \zeta^{\otimes 2} \simeq K_X \). It is well-known that such a \( \zeta \) always exists: \( \zeta \) is a spin-structure and \( X \) is spin, since \( w_2(X) = 0 \). There are \( 2^{2g} \) choices of \( \zeta \): the different theta-characteristics form a torsor over the points of order 2 in \( \text{Pic}_X^0 \). Consider the \( SL(2, \mathbb{C}) \)-Higgs pair \((\zeta \otimes \zeta^{-1}, \theta_0), \theta_0 = \begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix}\), where 1 is considered as a global section of \( \zeta^{-2} \otimes \mathcal{O}_X \). Consider then \( \text{Isom}(\zeta \otimes \zeta^{-1}, \mathcal{O}_X^{1/2}) \), the \( SL(2, \mathbb{C}) \)-frame bundle of \( \zeta \otimes \zeta^{-1} \), and set \( P = \text{Isom}(\zeta \otimes \zeta^{-1}, \mathcal{O}_X^{1/2}) \times_G G \). Assuming all the Lie-algebraic data from Section 4 fixed, we equip \( P \) with the Higgs field \( \theta = d\theta(\theta_0) \), which can be identified with the matrix \( y = d\theta(y_0) = \sum_i f_i \in \mathfrak{g} \). We shall discuss this identification in more detail in the next subsection.

Specifying a complex structure on \( X \) is equivalent to specifying a conformal class of Riemannian metrics. A metric \( g \) within that class induces an hermitian metric on all tensor powers \( K^m_X \), and more generally, on \( \zeta^{\otimes m} = K^{m/2}_X \), \( m \in \mathbb{Z} \), so we get a reduction of the structure group of \( P \) to \( U(1) = \text{diag}(U(1)) \subset G \). If \( \nabla \) is the corresponding Chern connection, \( F(\nabla) \) its curvature, and \( F_1 \) the curvature of the Levi-Civita connection, then Hitchin’s equation

\[
F(\nabla) + [\theta, \theta^*] = 0
\]

reduces to (rk \( \mathfrak{g} \) copies of) the equation \( F_1 - 4i\omega_X = 0 \). In other words, the \( U(1) \)-reduction gives the harmonic metric for \((P, \theta)\) if and only if the Gauss curvature \( K_g = -4 \). This is shown for \( G = SL(2, \mathbb{C}) \) in [Hit87a]. The extension to other groups is trivial and will be clear from the discussion that follows. It can also be deduced from the functoriality (with respect to \( G \)) of non-abelian Hodge theory. From the works of Poincaré and Koebe it is known that there is a unique such metric in a given conformal class: it descends from the standard hyperbolic metric on the upper half-plane after identifying the latter (biholomorphically) with the
universal cover, $\tilde{X}$, of $X$. In this sense the harmonic (Hermite-Yang-Mills) metric on $(P, \theta)$ “is” the uniformising metric, and we call $(P, \theta)$ the uniformising Higgs bundle, following Simpson ([Sim88]).

The choice of a Killing form and a compact real form determine an hermitian product on $\mathfrak{g}$ (see Section 4). The harmonic reduction of $P$ gives rise to an hermitian inner product on $\text{ad} P$, which is the harmonic metric for the Higgs (vector) bundle $(\text{ad} P, \text{ad} \theta)$. We also get $L^2$-inner products on $A^p(\text{ad} P \otimes K_X)$ for various $p$.

The infinitesimal deformations of the uniformising Higgs bundle (as well as those of any Higgs bundle, [BR94]) are computed by the Dolbeault complex

\[ \text{ad} P \xrightarrow{\theta} \text{ad} P \otimes \mathcal{O}_X K_X. \]

Taking its Dolbeault resolution and passing to global sections we obtain the double complex

\[ A^{0,1}(\text{ad} P) \xrightarrow{-\text{ad} \theta} A^{1,1}(\text{ad} P), \]

\[ A^{0,0}(\text{ad} P) \xrightarrow{-\text{ad} \theta} A^{1,0}(\text{ad} P) \]

whose total complex is

\[ 0 \to A^0(\text{ad} P) \xrightarrow{d_0} A^{1,0}(\text{ad} P) \oplus A^{0,1}(\text{ad} P) \xrightarrow{d_1} A^{1,1}(\text{ad} P) \to 0 \]

with differentials $d_0 = \left( \begin{smallmatrix} \text{ad} \theta \\ \mathcal{J}_P \end{smallmatrix} \right)$, $d_1 = \left( \begin{smallmatrix} \mathcal{J}_P, & -\text{ad} \theta \end{smallmatrix} \right)$.

The dgla controlling the deformations of $([P, \theta]) \in M_{Dol}(G)$ is the deformation complex (7), i.e., \( L^* = A^*(\text{ad} P) \), with $d = \mathcal{J}_P + \text{ad} \theta$ and the standard bracket. Notice that one can think of $\theta$ either as a twisted section of $\text{ad} P$, or as 1-form with values in $\text{ad} P$, and alternating between the two viewpoints may cause sign changes. The complex (7) is a slightly generalised version of Example 3.2, and it is immediate to see that conditions (1) and (2) from Section 3 are satisfied. The Maurer-Cartan equation is

\[ \mathcal{J}_P h + [\theta + h, v] = 0, \]

$(h, v) \in A^{1,0}(\text{ad} P) \oplus A^{0,1}(\text{ad} P)$. One sees immediately that if $(h, v) \in \text{MC}(L^*)$ and $h$ is holomorphic for $\mathcal{J}_P$, then $v \in \mathfrak{g}(\theta + h)$. This suggests that we can use the results and setup from Sections 3 and 5. For that, we have to identify harmonic representatives of $H^1(L^*)$ and see if condition (3) from Section 3 is satisfied. First we discuss the structure of $\text{ad} P$ in more detail.

6.2. Filtrations, gradings and adjoints. The homomorphisms between filtered (graded) objects in an abelian category carry a filtration (grading), and hence the principal gradings on $\mathfrak{g}$ and $\text{ad} P$ induce gradings on their respective endomorphisms. In particular, we have $\text{ad} \in \text{Hom}(\mathfrak{g}, \text{End}(\mathfrak{g}))$, i.e., $\text{ad} \in \text{Hom}(\mathfrak{g}_m, \text{End}_m(\mathfrak{g}))$ for all $m$. For the adjoint bundle and its endomorphism bundle we have

\[ \text{ad} P = \bigoplus_{m=-\delta} \text{ad}_m P = \bigoplus_{m=-\delta} \mathfrak{g}_m \otimes \mathbb{C} K_X \otimes^m, \]
\[ \text{End}(\text{adP}) = \bigoplus_{m=-\delta}^{\delta} \text{End}_m(\text{adP}) = \bigoplus_{m=-\delta}^{\delta} \text{End}_m(g) \otimes \mathcal{O}_X \otimes \mathcal{O}^m_X, \]

and \textit{mut.mut.} for \( A^{p,q}(\text{adP}) \) and \( A^{p,q}(\text{End}(\text{adP})) \).

Tensoring with powers of \( K_X \) we obtain plenty of trivial bundles: for all \( m \in \mathbb{Z} \),

\[ \text{ad}_m \mathcal{P} \otimes \mathcal{O}_X K_{-m}^m = g_m \otimes \mathcal{O}_X, \]

and for every \( m \in \mathbb{Z} \), \( K^m \otimes \mathcal{O}_X K_{-m}^m \) has a canonical section \( 1_m \), namely, the image of \( 1 \in \mathcal{H}^0(X, \mathcal{O}_X) \) under \( \mathcal{O}_X \cong K^m \otimes \mathcal{O}_X K_{-m}^m \) and we have a commutative diagramme.

\[ \begin{array}{ccc}
\mathfrak{g}_m \otimes \mathfrak{m} \rightarrow & \Gamma(X, \text{ad}_m \mathcal{P} \otimes K_{-m}^m) \\
\text{ad} & & \text{ad} \\
\text{End}_m \mathfrak{g} \otimes \mathfrak{m} \rightarrow & \Gamma(X, \text{End}_m(\text{adP}) \otimes K_{-m}^m) \rightarrow & \text{Hom}_{C^\infty}(A^\bullet(\text{adP}), A^\bullet(\text{adP} \otimes K_{-m}^m)).
\end{array} \]

In particular, for every \( m \in \mathbb{Z} \) there are inclusions

\[ \mathfrak{g}_m \hookrightarrow \text{End}_m \mathfrak{g} \hookrightarrow \Gamma(\text{End}_m(\text{adP}) \otimes K_{-m}^m) \hookrightarrow \text{Hom}_{C^\infty}(A^\bullet(\text{adP}), A^\bullet(\text{adP} \otimes K_{-m}^m)), \]

\[ \mathfrak{g}_m \ni \lambda \mapsto \iota(\text{ad}_\lambda \mathfrak{m} 1_m) \in \text{Hom}_{C^\infty}(A^\bullet(\text{adP}), A^\bullet(\text{adP} \otimes K_{-m}^m)). \]

For readability, we may occasionally suppress \( \iota \) or the subscript \( m \) in \( 1_m \).

We get similar inclusions if we fix a Kähler metric \( h \in A^{0,1}(K_X) \) with Kähler form \( \omega_X \in A^{1,1}_X \) and

\[ \mathfrak{g}_m \otimes \omega_m \rightarrow A_{<1,X}^{1,1}(\text{ad}_m \mathcal{P} \otimes K_{-m}^m) \]

and

\[ \mathfrak{g}_m \otimes h_m \rightarrow A_{<1,X}^{0,1}(\text{ad}_m \mathcal{P} \otimes K_{-m+1}^m). \]

The natural isomorphism \( A^0 = A^{1,0}(K_X^{-1}) \) gives rise to a “shift isomorphism” \( s : A^{p,q}(\text{adP} \otimes \mathcal{O}_X K_{-m}^m) \cong A^{p+1,q}(\text{adP} \otimes \mathcal{O}_X K_{-m}^m) \). Again, we are going to suppress \( s \) occasionally, but one should keep in mind that for \( S \in \text{End}_{-1,\mathfrak{g}} \), \( \overline{\partial}(s \partial S) + (s \partial S) \overline{\partial} = 0 \), in particular, \( \overline{\partial} \) anti-commutes with \( \text{ad}_\lambda \). This is a consequence of \([\text{Voi07}], \text{Remark 5.11}\) : the \( \overline{\partial} \) operators on \( A^{p,q} \) and \( A^{0,0}(K_{\mathbb{R}}^m) \) differ by \((-1)^p\).

We now make some comments on adjoints and Hodge stars in order to clarify conventions.

We are going to denote the hermitian metric on \( T_X \) by \( h \). In a local chart \((U, z)\) it is given by \( h = h dz \otimes d\bar{z} \), and the Kähler form is \( \omega_X = \frac{1}{2} h dz \wedge d\bar{z} \). The Riemannian metric \( g \) on \( T_{X,R} \) can be extended sesqui-linearly to an hermitian pairing \( \langle \cdot, \cdot \rangle \) on \( T_{X,C} \), which can then be restricted to \( T_{X,R}^{1,0} \). Similarly for \( T_{X,C}^{1,0} \) and its exterior powers. The pairing on \( T_{X,C}^{1,0} \) equals half of the direct sum of hermitian metrics on \( A^{1,0} \oplus A^{0,1} \); see for example, \([\text{Voi07}], \text{Lemma 5.6 or [Huy05], \text{Lemma 1.2.17.}}\)

The Riemannian metric \( g \) induces a dual metric, \( g^\vee \) on \( T_X^{\vee} \), and, consequently, hermitian metrics \( \tilde{h} \) (on \( K_X \)) and \( (g^\vee)^\vee \) (on \( T_{X,C}^{\vee} \)). One can check easily that \( (g^\vee)^\vee = (g\overline{\omega})^\vee \). However, \( h = 4h^\vee \), where \( h^\vee = h^{-1} \partial_z \otimes \partial_{\overline{z}} \) is the dual metric to \( h \).

We are going to use the convention that the Hodge star is \textit{anti-linear}, \( * : A^{p,q} \rightarrow A^{1-p,1-q} \), satisfying \( \beta \wedge * \alpha = g(\beta, \alpha) \overline{\omega}_X \). On 1-forms * coincides with conjugation.
up to \( \pm i \): we have \( \alpha = \alpha^* \) for \( \alpha \in A^{1,0} \). An hermitian bundle, \( E \), comes with an anti-linear isomorphism \( \#: E \cong E^\vee \), \( e \mapsto (\cdot, e) \), where \( (\cdot, \cdot) \) is the hermitian metric. Notice that for \( \alpha \in \Gamma_U(T_X^\vee L) \), \( \alpha = -\frac{\alpha^\vee}{\omega_X} \). We extend \( * \) and define \( *: A^{p,q}(E) \to A^{1-p,-q}(E^\vee) \) by \( *(\alpha \otimes e) = *\alpha \otimes e^\vee \).

Let \( L \) and \( M \) be hermitian line bundles on \( X \), \( U \subset X \) a trivialising analytic open set, and \( \lambda \in H^0(U, L) \simeq \text{Hom}_U(M, M \otimes L) \) a nowhere vanishing section. Then it is immediate to check that

\[
\lambda^* = ||\lambda||^2 \lambda^\vee = \# \lambda.
\]

Here \( \lambda^* \in A^0(\text{Hom}(L \otimes M, M)) \) is the hermitian adjoint of \( \lambda \) and \( \lambda^\vee = \lambda^{-1} \in \text{Hom}_U(L \otimes M, M) \) is the unique section pairing to 1 with \( \lambda \). In particular, for a (nonvanishing) section \( \lambda \in H^0(U, K_X) \) we have \( \# \lambda = \frac{1}{2} \lambda^* \).

Let \( * \) be a real structure on a vector bundle \( E \) (compatible with the hermitian structure). We extend \( * \) to \( A^{p,q}(E) \) by complex conjugation: \( (\alpha \otimes v)^* = \overline{\alpha} \otimes v^\vee \).

In particular, if \((U, z)\) is a local chart on \( X \) and \( \theta = \theta_z dz \in A^{1,0}(U, \text{ad} P) \), we have \( \theta^* = \theta_z^* dz = i \theta_z^* dz \). This agrees with the conventions in [Hit87a]; in [Sim92] the same quantity is denoted by \( \theta^* \).

As a special case, let us consider \( 1 \in H^0(O_X) \subset A^{1,0}(\text{Hom}(M, M \otimes K_X^{-1})) \), where \( M \) is an arbitrary hermitian line bundle. It is immediate to check that \( 1^* = h = \text{ad} z \otimes d\bar{z} \in A^{0,1}(\text{Hom}(M \otimes K_X^{-1}, M)) \). Here \( 1^* \) means, naturally, \( (s(1))^* = (1)^* \).

More generally, given \( \lambda \in g_{\text{ad}} \), \( \lambda \otimes 1 = s(\lambda \otimes 1_{-1}) \in A^{1,0}(X, \text{ad} P) \) and we have \((\lambda \otimes 1)^* = \lambda^* \otimes h \in A^{0,1}(\text{ad} P) \).

Similarly \( (\text{ad}_\lambda \otimes 1)^* = (\text{ad}_\lambda \otimes h) \lambda^* \in A^{0,1}(\text{End}_X \text{ad} P) \).

Finally, we can consider \( \text{ad}_\lambda \otimes 1 \) as an operator acting on \( A^*(\text{ad} P) \), in which case its adjoint then is \( \frac{1}{2} \text{ad}_\lambda \otimes 1 \). More pedantically, \( (\text{ad}_\lambda \otimes 1_{-1})^* = \frac{1}{2} i \text{ad}_\lambda \otimes 1 \).

6.3 Harmonic Representatives of cohomology. Now we return to the Dolbeault complex (7). We have

\[
(9) \quad 0 \rightarrow \bigoplus_m A^0(\text{ad} m P) \xrightarrow{d_0} \bigoplus_m A^1(\text{ad} m P) \oplus A^0(\text{ad} m P) \xrightarrow{d_1} \bigoplus_m A^1(\text{ad} m P) \rightarrow 0.
\]

with differentials \( d_0 = \left( \begin{array}{c} \text{ad}_y \\ \partial \end{array} \right) \) and \( d_1 = \left( \begin{array}{c} \partial \text{ad}_y \\ 1 \end{array} \right) \). For legibility, we have suppressed the obvious part of the nomenclature: \( \text{ad}_y \otimes 1 \) stands for \( i \text{ad}_\theta = i s(\text{ad}_\lambda \otimes 1_{-1}) \) and the Dolbeault operator \( \partial \) is \( \bigoplus m \partial \mathcal{K}_m \), the direct sum of the Dolbeault operators on \( \text{ad} m P = \mathfrak{g}_m \otimes K_X^{\otimes m} \).

**Theorem 6.1.** Let \( g = \text{Lie}(G) \) be a simple complex Lie algebra, equipped with \( \{ x_\alpha, x_{\alpha'}, y \} \) as in Section 4, so \( \mathfrak{z} \simeq \bigoplus \mathfrak{g}_{m,i} \) and \( \mathfrak{z}(y) = \bigoplus \mathfrak{g}_{-m,i} \), where \( \mathfrak{g}_{m,i} = \mathfrak{g}_{\pm m, i} \cap W_i \). Let \((P, \theta)\) be the uniformising \( G \)-Higgs bundle, equipped with the Hermite-Yang-Mills metric and let \( L^* \) be the Dolbeault complex (9). Then

\[ H^1(L^*) \simeq H^1(L^*) \subset A^{1,0}(\text{ad} P) \oplus A^{0,1}(\text{ad} P), \]

where

\[
H^1(L^*) = \bigoplus \mathfrak{g}_{m,i} \otimes \mathcal{H}^1(K_{X^m}) \bigoplus \bigoplus \mathfrak{g}_{-m,i} \otimes \mathcal{H}^{0,1}(K_{X^{-m}}).
\]

Hence

\[
H^1(L^*) \simeq H^1(\mathfrak{z}(x)_P) \oplus H^0(\mathfrak{z}(y)_P) \simeq \mathcal{E}_g \oplus \mathcal{E}_y^\vee,
\]

where \( \mathcal{E}_g = H^0(X, t \otimes K_X/W) \) denotes the Hitchin base.
Remark 6.2. The vector bundles $\mathcal{Z}(x)_P$ and $\mathcal{Z}(y)_P$ are (the obvious) twists of the centralisers $\mathcal{Z}(x)$ and $\mathcal{Z}(y)$ by $P$: $\mathcal{Z}(x)_P = \bigoplus_{\mathfrak{g}_{m,i}} \mathfrak{g} K_{m}^{-m}$ and $\mathcal{Z}(y)_P = \bigoplus_{\mathfrak{g}_{-m,i}} \mathfrak{g} K_{m}^{-m}$. Since $K_{m}^0$ are hermitian, we can talk about harmonic representatives of their cohomology, hence the notation. Explicitly, $\mathcal{H}_{\mathcal{Z}(x)}(\mathcal{Z}(x)_P) = \bigoplus_{\mathfrak{g}_{m,i}} \mathfrak{g} \mathcal{H}_{\mathcal{Z}(x)}(X, K_{m}^{-m})$ and $\mathcal{H}_{\mathcal{Z}(y)}(\mathcal{Z}(y)_P) = \bigoplus_{\mathfrak{g}_{-m,i}} \mathfrak{g} \mathcal{H}_{\mathcal{Z}(y)}(X, K_{-m}^{-m})$. Recall that $\dim \mathfrak{g}_{m,i} = 1$. We are assuming a fixed basis for the $G$-invariant polynomials on $\mathfrak{g}$, which fixes, by duality, bases for the spaces $\mathfrak{g}_{m,i}$, as discussed in Section 4. These give rise to bases of $\mathfrak{g}_{\pm m,i}$: either by taking hermitian conjugates or by applying $\text{ad}_{y}^{m,i}$ (the two choices differ by a combinatorial coefficient). The identification $\mathcal{B}_{\mathfrak{g}} \simeq \mathcal{H}_{\mathcal{Z}(x)}(\mathcal{Z}(x)_P) \oplus \mathcal{H}_{\mathcal{Z}(y)}(\mathcal{Z}(y)_P)$ depends on the choice of invariants polynomials. The identification $\mathcal{H}_{\mathcal{Z}(x)}(\mathcal{Z}(x)_P) \oplus \mathcal{H}_{\mathcal{Z}(y)}(\mathcal{Z}(y)_P)$ depends on the choice of basis for $\mathfrak{g}_{m,i}$ and uses the hermitian metric.

Proof: For the purposes of the proof, let us denote the summands in the decomposition

$$\mathfrak{g} = \mathcal{Z}(x) \oplus (\ker(\text{ad}_{x}) \cap \ker(\text{ad}_{y})) \oplus \mathcal{Z}(y)$$

by subscripts $x$, $o$ and $y$, so $\mathfrak{g} = \mathfrak{g}_{x} \oplus \mathfrak{g}_{o} \oplus \mathfrak{g}_{y}$. Use combinations of subscripts to denote projections on pairs of summands. If $\sigma = (\sigma', \sigma'')^T \in \ker d_1 \subset A^1(\text{ad}P)$, then

$$\sigma = d_0(P \otimes 1(\sigma'_{xy})) + (\sigma'_x, 0)^T + (0, \sigma''_y)^T, \quad \overline{\sigma''_x} = 0.$$ 

The first summand is a coboundary and the second term is never a coboundary, as $\mathcal{Z}(x) \cong \ker(\text{ad}_{y})$. The last summand, however, can contain a $\overline{\sigma''_x}$-exact term. By the Hodge decomposition on $A^p(X, K_{m}^0)$, we can write $\sigma''_{y} \in A^0(\mathcal{Z}(y)_P)$ as

$$\sigma''_{y} = \overline{\sigma''_x} \mathcal{G}_{\sigma''_y} + H(\sigma''_y),$$

where $\mathcal{G}$ is Green’s operator. Thus altogether

$$\sigma = (\sigma', \sigma'')^T = d_0 \left( P \otimes_{\mathbb{C}} 1(\sigma'_{xy}) + \overline{\sigma''_x} \mathcal{G}_{\sigma''_y} \right) + (\sigma'_x, H\sigma''_y)^T, \quad \overline{\sigma''_x} = 0,$$

and we obtain

$$\ker d_1 = \text{Im} d_0 \bigoplus \bigoplus_{\mathfrak{g}_{m,i}} \mathfrak{g} \mathcal{H}_{\mathcal{Z}(x)}(X, K_{m}^{-m-1}) \bigoplus \bigoplus_{\mathfrak{g}_{-m,i}} \mathfrak{g} \mathcal{H}_{\mathcal{Z}(y)}(X, K_{-m}^{-m+1}).$$

The second and third direct summands are hence isomorphic to $H^1(L^*)$, and are identified (via shifts) with $\mathcal{B}_{\mathfrak{g}} \oplus \mathfrak{g}_{y}$. In the next proposition we show that these are actually the harmonic representatives for $H^1(L^*)$. Explicitly, the isomorphism $H^1(L^*) \simeq \mathcal{H}_{\mathcal{Z}(x)}(\mathcal{Z}(x)_P) \oplus \mathcal{H}_{\mathcal{Z}(y)}(\mathcal{Z}(y)_P)$ is given by

$$[\sigma] = [(\sigma', \sigma'')] \mapsto (\sigma'_x, H\sigma''_y).$$

\[ \square \]

Remark 6.3. Using the explicit knowledge of the differentials of $L^*$, one can check easily that $H^2(L^*) = 0 = H^3(L^*)$. Moreover, $\text{Aut}(P, \theta) = Z(G)$, i.e., the pair has no “extra” automorphisms (it is regularly stable). This can be deduced for instance from Proposition 3.1.5 (ii), [BD91] and the non-abelian Hodge theorem ([Sim92]). Hence $[(P, \theta)]$ corresponds to a smooth point of $M_{\text{Dol}}(G)$. Of course, this is already contained in [Hit92] for the case when $G$ is of adjoint type.

Proposition 6.1. The vector space $\mathcal{H}^1(L^*) = \mathcal{H}_{\mathcal{Z}(x)}(\mathcal{Z}(x)_P) \oplus \mathcal{H}_{\mathcal{Z}(y)}(\mathcal{Z}(y)_P)$ is the space of harmonic representatives of $H^1(L^*)$. That is,

$$\mathcal{H}^1(L^*) = \ker d_1 \cap \ker d_0' \simeq H^1(L^*).$$
Proof:
We have
\[ d_0^* = (\text{ad}_g, \overline{\partial}) = (2 s^{-1} \text{ad}_x \otimes \mathbb{C} 1, \overline{\partial}) : A^{1,0}(\text{adP}) \oplus A^{0,1}(\text{adP}) \rightarrow A^0(\text{adP}), \]
and \( \sigma \in \ker d_0^* \) implies
\[ \sigma = (\sigma', \sigma'')^T = (\sigma'_x, 0)^T + (0, \sigma''_y)^T + \left( -\frac{1}{2} (Q \otimes \mathbb{C} 1) (\overline{\partial} \sigma''_x), \sigma''_y \right)^T, \overline{\partial} \sigma''_y = 0. \]
Applying the Hodge decompositions \( A^{0,1}(K_X^{-m_i+1}) = \text{Im} \overline{\partial} \oplus \mathcal{H}^{0,1}(K^{-m_i+1}) \) and \( A^{1,0}(K_X^{m_i+1}) = \mathcal{H}^{1,0} \oplus \text{Im} \overline{\partial} \) to \( \sigma'_y \) and \( \sigma''_y \), respectively, we get
\[ \ker d_0^* = \text{Im} \overline{\partial} \bigoplus \oplus_i \mathfrak{g}_{m_i, i} \otimes \mathbb{C} \mathcal{H}^{1,0}(K_X^{m_i}) \bigoplus \oplus_i \mathfrak{g}_{-m_i, i} \otimes \mathbb{C} \mathcal{H}^{0,1}(K_X^{-m_i}), \]
and the result follows. \( \square \)

Remark 6.4. For the proofs of Theorem 6.1 and Proposition 6.1 it is not essential that the principal \( \mathfrak{sl}(2, \mathbb{C}) \) is related in a specific way to some fixed Chevalley generators, but it is essential that principal subalgebra is real, the compact anti-involution maps \( g \) to \( -g \), and the different \( W_i \)'s are mutually orthogonal.

Proposition 6.2. The induced complex symplectic form on
\[ \mathcal{H}^{1}(L^*) \subset A^{1,0}(\text{adP}) \oplus A^{0,1}(\text{adP}) \]
is the canonical symplectic form on \( \mathcal{H}^{1,0}(\mathfrak{z}(x)_P) \oplus \mathcal{H}^{0,1}(\mathfrak{z}(y)_P) \) and agrees, up to Lie-theoretic normalisation factors, with the canonical symplectic form on \( \mathcal{B}_g \oplus \mathcal{B}'_g \).

Proof:
This is essentially clear from the construction. The Killing form \( \kappa \) places \( \mathfrak{z}(x) \) and \( \mathfrak{z}(y) \) in duality. Next, the complex symplectic form on \( H^1(L^*) \) is induced by the (weak) duality pairing
\[ (A^{1,0}(\text{adP}) \oplus A^{0,1}(\text{adP}))^2 \rightarrow A^{1,1} \rightarrow \mathbb{C}, \]
\[ ((u, \alpha), (v, \beta)) \mapsto \int_X \kappa(u \wedge \beta) - \kappa(v \wedge \alpha). \]
Evaluating it on pairs of harmonic representatives of \( H^0(K^{m_i+1}) \) and \( H^1(T_{X_{m_i}}) \), say, \( (u_i, \alpha_i), (v_i, \beta_i) \), we get an expression of the form \( \sum \kappa(e_{m_i}, e_{-m_i})(\beta_i(u_i) - \alpha_i(v_i)) \), where \( e_{m_i} \) and \( e_{-m_i} \) are bases of the 1-dimensional vector spaces \( \mathfrak{g}_{\pm m_i, i} \). If they are dual bases, then the pairing will coincide with the canonical symplectic form on \( \mathcal{B}_g \oplus \mathcal{B}'_g \), otherwise there will be extra coefficients \( \kappa(e_{m_i}, e_{-m_i}) \). \( \square \)

6.4. The Symplectic Kuranishi slice. Now we apply the results from the previous sections to the deformation theory of the uniformising Higgs bundle.

Theorem 6.5. Keep the notation and assumptions from the previous sections. Consider the holomorphic family of Higgs bundles
\[ \Gamma : \mathcal{B}_g \times \mathcal{B}'_g \rightarrow A^{1,0}(\text{adP}) \oplus A^{0,1}(\text{adP}), \]
\[ \Gamma(h, v) = (h, \Phi_h^{-1}(v)) = \left( h, \sum_{k=0}^s (-1)^k \left( (s^{-1}P \otimes \mathbb{C} 1) \circ \text{ad}_h \right)^k(v) \right), \]
where
\[ (h, v) \in \oplus_i \mathfrak{g}_{m_i, i} \otimes \mathbb{C} \mathcal{H}^{1,0}(K_X^{m_i}) \bigoplus \oplus_i \mathfrak{g}_{-m_i, i} \otimes \mathbb{C} \mathcal{H}^{0,1}(K_X^{-m_i}) \simeq \mathcal{B}_g \times \mathcal{B}'_g \]
and \[\Phi_h = 1 + s^{-1}(P \otimes \mathbb{C} 1)ad_h \in \text{End}(A^{0,1}(ad\mathbb{P})).\]

The family \(\Gamma\) is a miniversal deformation of the uniformising Higgs bundle \((\mathbb{P}, \theta)\). There exists an open neighbourhood \(\mathcal{U} \subset \mathcal{B}_\mathbb{P} \times \mathcal{B}_\mathbb{P}'\) containing 0, for which \(\Gamma|\mathcal{U}\) is a universal deformation. Moreover, \(\Gamma^*\omega_{\text{can}} = \omega_{\text{can}}\).

**Remark 6.6.** Clearly, we also have a formal version of \(\Gamma\), i.e., a functor of Artin rings \(\Gamma: \mathcal{B}_\mathbb{P} \times \mathcal{B}_\mathbb{P}' \rightarrow \text{Def}_\mathbb{P}\), given by the same formula as above. As everywhere above, \(\mathcal{B}_\mathbb{P}\) should be understood in terms of harmonic representatives.

**Proof:**

From subsection 6.3 we know that the dgla \(L^\bullet\) satisfies conditions (1),(2) and (3) from Section 3, and that \(H^2(L^\bullet) = 0 = L^3\). In the notation of Section 3, \(d''_1 = ad\theta\), and we have a splitting, \(s^{-1}P \otimes \mathbb{C} 1\). By Lemma 5.1, the geometric series for \(\Phi_h^{-1}\) reduces to the given formula, i.e., \(\pi\) drops out of the expressions. The condition \((\pi - 1)[\Gamma, \Gamma] = 0\) holds (essentially) for the same reasons as in Section 5: since \(h\) (resp. \(v\)) is a highest (resp. lowest) weight vector, none of the sections from [\(\Gamma, \Gamma\)] will be contained in \(A^{1,1}(\mathfrak{g}(y)\mathbb{P})\). The Kodaira-Spencer map of this family is the identity, so, by [Fuk03], Theorem 1.3.3, this is a miniversal family. Since \((\mathbb{P}, \theta)\) is regularly stable, we obtain a universal family by restricting the domain of \(\Gamma\).

Finally, the statement that \(\omega_{\text{can}}\) pulls back to \(\omega_{\text{can}}\) follows from Theorem 3.10. The conditions in that theorem are satisfied (essentially) for the same reason as before: the pairing \(A^{1,0}(ad\mathbb{P}) \times A^{0,1}(ad\mathbb{P}) \rightarrow \mathbb{C}\) is obtained by combining cup product \(A^{1,0}(K_{m_1}) \times A^{0,1}(K_{m_2}) \rightarrow A^{1,1} \rightarrow \mathbb{C}\) with the Killing form \(\kappa: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}\). But the Killing form is non-zero only on \(\mathfrak{g}_{m_1} \times \mathfrak{g}_{m_2}\) and \(\mathfrak{g}_{m_1} \times \mathfrak{g}_{m_2}\), and since \(s^{-1}(P \otimes 1)\piad_h\) has degree 2 (with respect to the principal grading), the orthogonality condition from 3.10 is satisfied. \(\square\)

**Remark 6.7.** The description of the family from Theorem 6.5 is constructive, and one can write \(\Gamma\) explicitly once the Lie-algebraic data are fixed. After all, \(\Gamma\) is simply a “twisted” version of the analogous formula from Section 5 (see the example there.)

We now rephrase the result and draw some easy corollaries.

**Corollary 6.1.** Denote, as in Section 3, \(S = MC(L) \cap \ker[(1 - H')\pi']\). Then \(\Phi = \Gamma^{-1}: S \rightarrow \mathcal{B}_\mathbb{P} \times \mathcal{B}_\mathbb{P}'\) provides Darboux coordinates on \(S\).

Notice, once again, that if we want to consider \(S\) with its (somewhat useless) structure of a (germ of a) subvariety of an infinite-dimensional vector space, we should first complete \(L^\bullet\) with respect to a suitable (Hölder or Sobolev) norm. However, \(\Gamma(\mathcal{B}_\mathbb{P} \times \mathcal{B}_\mathbb{P}') \subset A^1(ad\mathbb{P})\) since \(s^{-1}(P \otimes \mathbb{C} 1)\piad_h\) is nilpotent. See also Corollary 3.5.

In [Hit92], N.Hitchin constructed a section, \(\varsigma: \mathcal{B}_\mathbb{P} \rightarrow M_{Dad}(G)\) as a “global version” of Kostant’s section \(t/W \rightarrow \mathfrak{g}\). We have identified \(\mathcal{B}_\mathbb{P} \simeq \bigoplus_{i} \mathcal{H}^{1,0}(X, K_X^{m_i})\), and have embedded the latter into \(A^{1,0}(ad\mathbb{P})\) via the basis vectors \(e_{m_i}\), spanning \(\mathfrak{g}(x)\). The section then is the holomorphic family of Higgs bundles, whose underlying bundle is \(\mathbb{P}\), and which carries the Higgs field \(\theta + \sum_{i} e_{m_i} \alpha_i, \alpha_i \in \mathcal{H}^{1,0}(X, K_X^{m_i})\). In terms of deformation functors, the section is given by \(\varsigma: \mathcal{B}_\mathbb{P} \rightarrow A^1(ad\mathbb{P}) \rightarrow \text{Def}_\mathbb{P}, \varsigma(h) = (h, 0)\).

**Remark 6.8.** What we give here is a somewhat non-canonical description of the section. To construct the section, one only needs a choice of theta-characteristic, \(\zeta\),
Corollary 6.2. The restriction of $\Gamma$ to $\mathcal{B}_g \times \{0\}$ is the Hitchin section. If we regard elements of $\mathcal{B}_g^\vee \simeq \oplus H^{0,1}(K_X^{-m_i})$ as linear Hamiltonian functions on the base, we get that $\Gamma(h, v) = \exp_{X_v}(s(h))$, where $X_v$ is the Hamiltonian vector field, corresponding to $v$.

Proof: We have $\Gamma(h, 0) = (h, 0) = s(h)$ by construction. The rest is immediate from the Theorem. □

Because of the above, one may refer to $\Gamma$ as a “holomorphic exponential map”.

Remark 6.9. We can also look at the image of $\{0\} \times \mathcal{B}_g^\vee$ under $\Gamma$. This is a family of Higgs bundles for which the Higgs field is constant (as a smooth twisted endomorphism), while the holomorphic structure varies in $\mathcal{B}_g^\vee$. For instance, if $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, the underlying vector bundles are extensions of $\zeta$ by $\zeta^{-1}$. They all come with a canonical inclusion $\mathbb{C} \hookrightarrow H^0(\text{End} E \otimes K_X)$, and the Higgs field is the image of $1 \in \mathbb{C}$.

Recall from Section 5 that $I \parallel G \simeq |y + (x) = s^* I$, where $I$ is the tautological family of centralisers and coincides with the set of Maurer-Cartan elements for a certain dgla. There is a “global version” of this statement. More precisely, recall that a “regular Higgs field” is one which is an everywhere regular section of $\text{ad} P \otimes K$, i.e., pointwise it takes values in $\mathfrak{g}^{\text{reg}}$.

Corollary 6.3. The image of the exponential map consists of all regular Higgs bundles in the connected component of the uniformising Higgs bundle.

Proof: Since the values of $\Gamma$ are regular by construction, the nontrivial statement is the opposite inclusion. We claim that every Higgs pair with a regular Higgs field is isomorphic to one in the image of $\Gamma$. Suppose $(Q, \varphi)$ is such a Higgs bundle. Then there exists a $C^\infty$-isomorphism $\text{ad} Q \otimes_{\mathcal{O}_X} K_X \simeq C^\infty \text{ad} P \otimes_{\mathcal{O}_X} K_X$, and $\overline{\mathcal{Q}} = \overline{\mathcal{P}} + \zeta$. Since $\varphi$ is regular, it can be conjugated to $s(\chi(\varphi))$. This replaces $\overline{\mathcal{Q}}$ by a gauge-equivalent Dolbeault operator, $\overline{\mathcal{P}} + v$, and the Maurer-Cartan equation states that $v \in \mathfrak{g}(s(\chi(\varphi)))$, i.e., $(s(\chi(\varphi)), v) \in \mathcal{S} \simeq H^1(L^*)$. Notice that we are not making any claim regarding stability of our bundles. □

Remark 6.10. In unpublished notes ([Tel07]) C. Teleman proved the same result for $\text{GL}(n, \mathbb{C})$.

7. Appendix: Kuranishi Theory

In this subsection we recall some relevant facts from formal and analytic Kuranishi theory. Our main references will be [GM90], [GM88] and [Man99], and, to a lesser extent, [Fuk03] and [Kon94].

Suppose $L^*$ is a dgla equipped with a splitting $\delta$. This is a linear map $\delta \in \text{Hom}^{-1}(L^*, L^*)$ which satisfies $\delta^2 = 0$, $d = d\delta$ and $\delta = \delta d\delta$. Notice that while $d$ is a derivation of the bracket, $\delta$ a priori need not be compatible with the bracket in any way. In fact, if $L^*$ admits a splitting which is a derivation, then it is formal ([Kos03], Theorem 4.2.1.). For comparison, any choice of splitting gives an isomorphism between $L^*$ and $H^*(L)$ as $L_\infty$-algebras: see e.g., [Kon94] or [Fuk03].
Specifying a splitting is equivalent to specifying a “Hodge decomposition” \( L^* = B^* \oplus \mathcal{H}^* \oplus C^* \), where \( B^* = \text{Im} (d) \), \( C^* = \text{Im} (\delta) \), \( \mathcal{H}^* = \text{ker} d \cap \text{ker} \delta \cong H^*(L^*) \) and \( \mathcal{H}^* \oplus C^* = \text{ker} \delta \). The map \( \delta \) gives a (co)chain homotopy between \( H = \text{pr}_H \) and the identity, i.e., \( d\delta + \delta d = 1 - H \). This can also be written as \( \text{pr}_B + \text{pr}_H + \text{pr}_C = 1 \) since \( d\delta = \text{pr}_B \) and \( \delta d = \text{pr}_C \). The choice of splitting gives a decomposition of \( (L^*, d) \) into a sum of 1-term complexes \( \mathcal{H}^i[-i] \) and 2-term contractible complexes \( C^i \longrightarrow B^{i+1} \).

For a given \( \psi \in B^* \), the equation \( d\varphi = \psi \) has unique solution, \( \varphi = \delta\psi \) in \( C[-1]^* \). However, if \( \mathcal{H}^* \neq (0) \), this equation will have infinitely many solutions in \( \ker \delta (-1) = (\mathcal{H}^* \oplus C^*)[-1] \), since \( \varphi = H(\varphi) + \delta\psi \), and the harmonic part \( H(\varphi) \) can be arbitrary.

One approaches the Maurer-Cartan equation \( d\varphi = -\frac{1}{2}[\varphi, \varphi] \) in a similar way: we look for all \( \varphi \in L^1 \) such that \( \varphi = H(\varphi) - \frac{1}{2} \delta [\varphi, \varphi] \). They constitute the zero locus of \( M : L^1 \rightarrow B^1 \oplus \mathcal{C}^1 \)

\[
M(\varphi) = (1 - H)(\varphi + \frac{1}{2} \delta [\varphi, \varphi]) = (1 - H)(F(\varphi)),
\]

where \( F(\varphi) = \varphi + \frac{1}{2} \delta [\varphi, \varphi] \) is the Kuranishi map \( F : L^1 \rightarrow L^1 \). So

\[
\ker \delta = \mathcal{H}^1 \oplus \mathcal{C}^1 \supset \left\{ \varphi : \varphi = H(\varphi) - \frac{1}{2} \delta [\varphi, \varphi] \right\} = M^{-1}(0) = F^{-1}(\mathcal{H}^1).
\]

A priori \( \varphi \in F^{-1}(\mathcal{H}^1) \) is not a Maurer-Cartan element. There is, however, an obvious necessary condition that Maurer-Cartan elements have to satisfy: \( d\varphi = -\frac{1}{2}[\varphi, \varphi] \Rightarrow H(\varphi) = 0 \). Hence we define \( k : L^1 \rightarrow \mathcal{H}^2 \) by \( k(\varphi) = H(\varphi) \) and look at the set \( F^{-1}(\mathcal{H}^1) \cap k^{-1}(0) \) and at its image under \( F \), \( K_L := F(F^{-1}(\mathcal{H}^1) \cap k^{-1}(0)) \subset \mathcal{H}^1 \).

Loosely speaking, if we work formally (or analytically), then (the germ of) \( F^{-1}(\mathcal{H}^1) \cap k^{-1}(0) \) consists of all Maurer-Cartan elements in (some neighbourhood of zero in) \( \mathcal{H}^1 \oplus \mathcal{C}^1 \) and provides a semi-universal family of deformations of \( \text{Def}_L(\mathbb{C}) \).

We first make some remarks about the Kuranishi map. Since \( \delta(F(x)) = \delta(x) \), we have that \( F(\ker \delta) \subset \ker \delta \), and, by (10), \( F^{-1}(\mathcal{H}^1) \subset \ker \delta \). Next, the “slice” \( Y_L := Q^{-1}_L(0) \cap \ker \delta \), consisting of Maurer-Cartan elements in \( \ker \delta \), gets mapped to \( \mathcal{H}^1 \) by \( F \). Indeed, \( y \in Y_L \Rightarrow dF(y) = \frac{1}{2} H[y, y] \in B^2 \cap H^2 = (0) \), so \( F(Y_L) \subset \mathcal{H}^1 \).

Moreover, \( Y_L = F_Y^{-1}(\mathcal{H}^1) \subset M^{-1}(0) \cap k^{-1}(0) \), so \( F(Y_L) \subset K_L \). Notice that \( F \), considered as a quadratic map between vector spaces (or subsets thereof) need not be invertible!

The next diagram illustrates the different inclusions:

\[
Y_L = \text{MC}(L) \cap \ker \delta \longrightarrow F^{-1}(\mathcal{H}^1) \cap k^{-1}(0) \longrightarrow F^{-1}(\mathcal{H}^1)^c \longrightarrow \ker \delta \longrightarrow L^1.
\]

In order to say more, we need a topology.

First, we turn to the formal setup and define a functor \( Y_L = \text{MC}(L) \cap \ker \delta \in \text{FArt}_{\mathbb{C}} \),

\[
Y_L(A) = Y_L \otimes m_A = \left\{ \eta \in \ker \delta \otimes m_A : d\eta + \frac{1}{2} [\eta, \eta] = 0 \right\}.
\]
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For every element in $\text{MC}_L(A)$ there is a unique gauge transformation, taking it to $Y_L(A)$, see e.g. [ES09], Lemma 2.6. This is a variant of the so-called Uhlenbeck slice (Coulomb gauge). Slice theorems have been widely used in gauge theory since the late 1970’s, most notably by Atiyah–Hitchin–Singer, Taubes and Uhlenbeck and before that by Parker and Mitter–Viallet.

The Kuranishi map gives rise to a functor $F \in \text{Fun}(\mathcal{L}, \mathcal{L})$, given by the same formula as before, mut. mut. One shows, using artinian induction, that $F$ is an isomorphism, see e.g., [GM90], Lemma 3.1 or [Man99], Lemma 4.2.

The Kuranishi functor $\mathbb{K}_L \in \text{FArt}_\mathbb{C}$ is defined as the kernel of $k \circ F^{-1} \in \text{Fun}(\mathcal{L}, \mathcal{L})$:

$$\mathbb{K}_L(A) = K_{L \otimes m_A} = \{x : H([F_A^{-1}(x), F_A^{-1}(x)]) = 0\} \subset \mathcal{H}^1 \otimes m_A.$$ 

Applying the earlier considerations to $L \otimes m_A$, we see that $F \in \text{Fun}(Y_L, \mathbb{K}_L)$, and it is in fact an isomorphism ([GM90], Section 3 or [Man99], Proposition 4.6). Then

$$\mathbb{K}_L \xrightarrow{g^{-1}} Y_L \xrightarrow{r} \text{Def}_L$$

is shown to be étale: see [GM90], Section 3 or [Man99], Theorem 4.7 for further details. The functor $Y_L$ is called a formal miniversal deformation or formal Kuranishi space. The isomorphism class of $Y_L$ is independent of the choice of $\delta$ and quasi-isomorphic dgla’s have isomorphic $Y_L$’s ([GM90]) It is clear that $Y_L$ is pro-representable, i.e., $Y_L = h_{R^1}$, for a complete local algebra $R$. An explicit description of $R$ can be obtained by fixing a basis of $\mathcal{H}^1$, say $\{\eta_1, \ldots, \eta_d\}$, with dual basis $\{t_1, \ldots, t_d\}$, and then taking $R = \mathbb{C}\{t_1, \ldots, t_d\}/I$. The ideal $I$ is generated by the components of $H[\sum_i t_i \eta_i, \sum_j t_j \eta_j] = 0$ with respect to some basis of $\mathcal{H}^2$.

If $L$ is formal, then $\mathbb{K}_L$ is the quadratic cone in $\mathcal{H}^1$ determined by cup product. If $H^2(L) = 0$, $\mathbb{K}_L = \mathcal{H}^1$ and $R = \mathbb{C}\{t_1, \ldots, t_d\}$. If $H^0(L) = 0$, then $\text{Def}_L$ is pro-representable.

If $L^\bullet$ itself carries a topology we can go beyond the formal level and exhibit a germ of a complex space whose local ring completed at the origin prorepresents $\mathbb{K}_L$. Suppose $L^\bullet$ is an analytic dgla in the sense of [GM90]. This means that $L^\bullet$ is a normed dgla (i.e., for all $i \in \mathbb{N}$ there is a norm $\|\cdot\|$ on $L^i$ with respect to which $d$ and $|\cdot|$ are continuous) and the completion $\hat{L}^\bullet$ is equipped with a continuous splitting, $\delta$. In the case of “usual” Hodge theory, the norms are the Sobolev norms and $\delta = \overline{\partial} \mathbf{G}$, where $\mathbf{G}$ is Green’s operator. In the case of deformations of a complex manifold or deformations of a holomorphic vector bundle the norms are Hölder norms. The splitting $\delta$ has to be compatible with the inclusion $L^\bullet \subset \hat{L}^\bullet$, which means two things. First, we assume that $\mathcal{H} = \ker d \cap \ker \delta \subset L \subset \hat{L}$. And second, we assume that the three projections preserve $L^\bullet \subset \hat{L}^\bullet$ and $\text{pr}_B L = d(L[-1])$. Such a $\delta$ induces a splitting of $L^\bullet$ as well.

Then, by the implicit function theorem for Banach spaces $F : \hat{L}^1 \rightarrow \hat{L}^1$ is an analytic isomorphism between open balls around the origin: we have $dF_\xi = 1 + \delta ad$, and $dF_0 = 1$ (see [GM90], Lemma 2.2). We can introduce now the analytic versions of all of the above functors. First set

$$\mathcal{Y} = Y_L = \left\{\eta : \delta \eta = 0, d\eta + \frac{1}{2}[\eta, \eta] = 0\right\} \subset \mathcal{H}^1 \oplus \mathcal{C}^1.$$
Notice that $\mathcal{Y}$ is an algebraic subset of the (possibly) infinite-dimensional vector space $\ker \delta = \mathcal{H}^1 \oplus \mathcal{C}^1$. Next, let

$$\mathcal{K} = \mathcal{K}_L = \{ x \in \mathcal{H}^1 : \mathbf{H}(\{ F^{-1}(x), F^{-1}(x) \}) = 0 \} \subset \mathcal{H}^1.$$ 

The previous discussion, applied to $\mathcal{L}$, gives $F(\mathcal{Y}) \subset \mathcal{K}$.

The generalised Kuranishi’s theorem ([GM90], Theorem 2.3) states that the Kuranishi map induces an analytic isomorphism of germs $F : (\mathcal{Y}, 0) \simeq (\mathcal{K}, 0)$, and hence the functors $\mathcal{K}_L$ and $\mathcal{Y}_L$ are prorepresented by $\mathcal{O}_{(K,0)}$. To prove it one must show that for some open ball $B_0 \subset \mathcal{L}$ we have $F^{-1}(\mathcal{K} \cap B_0) = B_0' \cap F^{-1}(\mathcal{H}^1) \cap \mathcal{K}^{-1}(0) \subset \mathcal{Y}$. Indeed, if $F(\xi) \in \mathcal{H}^1$ and $H[\xi, \xi] = 0$, we get immediately that $\delta F(\xi) = \delta(\xi)0$. Then

$$dF(\xi) = d\xi + \frac{1}{2} \delta[\xi, \xi] = 0 = \left(d\xi + \frac{1}{2} \delta[\xi, \xi]\right) - \frac{1}{2} \delta d[\xi, \xi].$$ 

We have to show that the last summand is zero. The fact that $d$ is a derivation, combined with the Jacobi identity shows that $\delta d[\xi, \xi]$ satisfies

$$(1 + \delta \text{ad}_\xi) \delta d[\xi, \xi] = 0.$$

But if $\xi$ is small, $dF_\xi = 1 + \delta \text{ad}_\xi$ is invertible, so $\delta d[\xi, \xi] = 0$. Finally, we turn to the question of the miniversal family. For simplicity, we discuss only the unobstructed case, i.e. the case when $\mathcal{K} = \mathcal{H}^1$. If we fix a basis of $\mathcal{H}^1$, as above, then $\mathcal{O}_{(K,0)} \simeq \mathbb{C}[t_1, \ldots, t_d]$. The inverse of the Kuranishi map gives a formal family of deformations of $\text{Def}_L(\mathbb{C})$ over $(\mathcal{H}^1, 0)$,

$$\Gamma \in L^1 \otimes \mathbb{C}[t_1, \ldots, t_d],$$

$$\Gamma = \sum_{k=1}^{d} \Gamma_k, \quad \Gamma_k := \sum_{|J|=k} t^J \Gamma_J,$$

$J$ is a multi-index, and $\Gamma_k$ are determined inductively: for $x = \sum_i t_i \eta_i \in \mathcal{H}^1$,

$$\Gamma_1(x) = x, \quad \Gamma_2(x) = -\frac{1}{2} \delta[\Gamma_1, \Gamma_1], \quad \ldots, \quad \Gamma_k = -\frac{1}{2} \delta \sum_{n=1}^{k-1} [\Gamma_n, \Gamma_{k-n}].$$

This series has been known for a long time (in various contexts and different levels of generality), see e.g., [Kur62], [Kod86], [HS02], [Fuk03], and can be thought of as a reincarnation of Picard’s method of solving ODE’s by iterations.

In the case of a normed dgla, one shows first that the series converges (in $\mathcal{L}$) in a sufficiently small poly-disk around the origin. Next, using elliptic estimates, one proves that the family can be modified so that the convergence takes place in $L$. The prototypical example is the Kodaira-Spencer dgla, and the convergence was proved in [KNSS58]. In [Ita02] the author proves the convergence of this series in the case of the Barannikov-Kontsevich construction, which contains our setup as a special case.
8. Glossary of Notation

Art\(_C\): the category of local Artin \(C\)-algebras with residue field \(C\)

\(A\): an Arting ring

\(A^p, A^{p,q}\): sheaves of smooth forms of type \(p\) (resp. \((p,q)\))

\(A^p = H^0(X, A^p), A^{p,q} = H^0(X, A^{p,q})\): global sections

\(\text{ad}_u = [u, \cdot] = \text{ad}u\)

\(\text{ad}\mathbf{P} = \mathbf{P} \times \text{ad} \mathbf{g}\): the adjoint bundle of \(\mathbf{P}\)

\(B_\theta = H^0(X, t \otimes K_X/W) \simeq \bigoplus_i H^0(X, K^{m_i+1}_X)\) the Hitchin base

\(B^i \subset L^i\) boundaries; used only in Appendix 7

\(B_\varepsilon\): Ball of radius \(\varepsilon\), Section 3

\(C^i \subset L^i\): a complement to \(\ker d_i\), used only in Appendix 7

\(d\) or \(d_i\): differentials of a complex (always increasing the degree)

\(d\): differential of a map

\(\text{Def}_{L^\bullet}\): the deformation functor of a dgla \(L^\bullet\)

\(\delta\): splitting of a dgla, Appendix 7

\(\Delta^+\): positive simple roots, Section 4

\(e_i\): “upper nilpotent” Chevalley generators \(\{e_i, h_i, f_i\}\)

\(e_{m_i}\): basis vectors for the 1-dimensional subspaces \(g_{m_i}\)

\(\text{End} (\text{resp. } \text{End})\): Endomorphisms (resp. sheaf endomorphisms);

\(\text{End}_{m}\): \(m\)-th graded piece of \(\text{End}\)

\(F\text{Art}_C\): functors \(F : \text{Art}_C \to \text{Sets}\) for which \(F(C) = \{\ast\}\)

\(f_i\): “lower nilpotent” Chevalley generators \(\{e_i, h_i, f_i\}\)

\(\Phi\): trivialisation of the symplectic Kuranishi slice

\(G\): simple complex Lie group

\(G\): Green’s operator

\(g = \text{Lie} G\): simple Lie algebra

\(g_o = \text{Im}(\text{ad}_x) \cap \text{Im}(\text{ad}_y)\)

\(g_{m_i}\): \(m\)-th graded piece of \(g\) with respect to the principal grading

\(g_{k,i} = g_k \cap W_{m_i}\)

\(g(x)\): the centraliser of \(x \in g\)

\(g\): a Riemannian metric on the curve \(X\)

\(g_X\): the anti-linear extension of \(g\) to \(T_X\)

\(g_X\): the genus of the curve \(X\)

\(\Gamma = \Phi^{-1}\): the (formal) inverse of the trivialisation of the symplectic Kuranishi slice

\(\Gamma\): the global section functor

\(H\): harmonic projection, \(H', H''\) the two components of \(H\)

\(H\): harmonic representatives of cohomology

\(\mathfrak{h}\): one of the elements of a principal \(\mathfrak{sl}(2, \mathbb{C})\)-subalgebra \(\{y, \mathfrak{h}, x\}\)

\(\check{h}\): Coxeter number (largest exponent) of \(G\)

\(h\): Hermitian metric on \(T_X\)

\(h\): the matrix of the Hermitian metric \(h\), a positive real-valued function
\((h, v) \in L' \oplus L'' = L^1\) a typical element
\(h_i\): semisimple elements among the Chevalley generators \(\{e_i, h_i, f_i\}\)

I bundle of centralisers
\(i: \Gamma(\text{End}_m(\text{ad}P) \otimes K_X^{-m}) \rightarrow \text{Hom}_{C^\infty}(A^\bullet(\text{ad}P), A^\bullet(\text{ad}P \otimes K_X^{-m}))\)

\(i', i''\): the canonical inclusions of \(L'\) and \(L''\) into \(L = L' \oplus L''\), Section 3

\(k: t/W \rightarrow \mathfrak{g}\): Kostant section, Section 5

\(K\): formal Kuranishi functor, Appendix 7, Section 3

\(K\): analytic Kuranishi functor

\(K_X\): canonical bundle of \(X\)

\(\kappa\): Killing form

\(l = \text{rk}(\mathfrak{g})\): the rank of \(\mathfrak{g}\)

\(m_i\): the exponents of \(\mathfrak{g}\)

\(M = Q^{-1}(0)\): the vanishing set of the quadric \(Q\), Section 1.

\(M_{\text{Did}}(G)\): the Dolbeault moduli space

\(\mathfrak{m}_A\): the maximal ideal of \(A\)

\(\text{MC}(L) = Q^{-1}(0)\): Maurer-Cartan elements of a dgla \(L\), Section 2

\(\text{MC}_L\), the Maurer-Cartan functors of \(L\), Section 2; \(\text{MC}_L(A) = \text{MC}(L \otimes A)\)

\(o\): marked point

\(\mathcal{O}_X\): the structure sheaf of \(X\)

\(\{p_1, \ldots, p_l\}\): basis of homogeneous invariant polynomials on \(\mathfrak{g}\)

\(P\): splitting of \(\text{ad}_x\) determined by the choice of principal \(\mathfrak{sl}(2, \mathbb{C})\)

\(P\) the uniformising Higgs bundle, \(P = F \times_{\text{ad}(\mathfrak{g})} G\); also a principal bundle (in general)

\(\text{pr}_i\): the canonical projection \(\text{MC}_L \rightarrow \text{Def}_L\)

\(\text{pr}^{k\flat}_i\): the projection \(\mathfrak{g}_k \rightarrow \mathfrak{g}_{k,n}\), associated with a choice of \(\mathfrak{sl}(2, \mathbb{C})\)-subalgebra, Section 4

\(\pi\): a projection \(L^2 \rightarrow \text{Im} \, d'\) in a dgla with a splitting as in Section 3.

\(\pi', \pi''\): projections to the two factors \(L = L' \oplus L''\) in a dgla with a decomposition as in Section 3.

\(\text{pr}_i\): projection onto the \(i\)-th factor in a Cartesian product

\(Q\): splitting of \(\text{ad}_x\) determined by the choice of principal \(\mathfrak{sl}(2, \mathbb{C})\)

\(Q\): a quadric; also the Maurer-Cartan quadric \(Q(u) = du + \frac{1}{2}[u, u]\)

\(g: SL(2, \mathbb{C}) \rightarrow G\): principal embedding

\(\rho\): Weyl vector, \(\rho^\vee\): dual Weyl vector (half the sum of positive coroots)

\(S\): formal symplectic Kuranishi slice, Section 3

\(S\): analytic symplectic Kuranishi slice

\(s\): shift, Section 6

\(s: j(y) \hookrightarrow g\): affine-linear map, a variant of Kostant’s section

\(s: S_B \rightarrow M_{\text{Did}}(G)\): Hitchin’s section
A UNIVERSAL FAMILY OF DEFORMATIONS FOR THE UNIFORMISING HIGGS BUNDLE

\[ \Sigma \subset g: \text{Kostant's slice, Section 4} \]

\[ t \subset g: \text{Cartan subalgebra} \]
\[ \mathcal{L}: \text{the functor } \mathcal{L}(A) = V \otimes m_A, \text{ } V \text{ a vector space} \]

\[ \omega_{\text{can}}: \text{the canonical symplectic form on } V \times W, \text{ where } V \text{ and } W \text{ are two spaces in (weak) duality} \]
\[ \omega \text{ or } \omega_X: \text{Kähler form on } X \]
\[ W: \text{Weyl group} \]
\[ W_i \subset g: \text{irreducible representations for the principal } \mathfrak{sl}(2, \mathbb{C}) \text{-action on } g \]
\[ \mathcal{W}_*: \text{Deligne filtration on } g, \text{ Section 4} \]

\[ x \in g: \text{regular nilpotent, part of a principal } \mathfrak{sl}(2, \mathbb{C})\text{-subalgebra } \{x, h, y\} \]
\[ X: \text{smooth projective curve (over } \mathbb{C}) \text{ of genus at least two} \]

\[ y \in g: \text{regular nilpotent, part of a principal } \mathfrak{sl}(2, \mathbb{C})\text{-subalgebra } \{x, h, y\}, \text{ } y = \sum_i f_i \]
\[ \mathcal{Y}: \text{formal Kuranishi slice Appendix 7, Section 3} \]
\[ \mathcal{Y}: \text{analytic Kuranishi slice, Appendix 7, Section 3} \]

\[ z: \text{centraliser} \]
\[ z(x) = \mathfrak{g}_x = \bigoplus \mathfrak{g}_{m_i,i} \]
\[ z(y) = \mathfrak{g}_y = \bigoplus \mathfrak{g}_{-m_i,i} \]
\[ \zeta: \text{theta-characteristic} \]

\[ 1_m: \text{the canonical section of } \mathcal{O}_X \cong K_X^m \otimes K_X^{-m} \]

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