THE CHEV ALLEY-WEIL FORMULA ON NODAL CURVES

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Abstract. In this paper, we study the eigensubspace of the space of the holomorphic differentials of nodal curves over the algebraically closed field under the action of finite automorphism groups. We compute the Chevalley-Weil formula with some additional conditions of the quotient curve and give some examples.

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1. Introduction

Let $X$ be a connected projective smooth curve over an algebraically closed field $k$ and $G \subseteq \text{Aut}(X)$ be a finite subgroup. Then $G$ acts in a natural way on the space of the holomorphic differentials on $X$, thus we obtain a linear representation $G \to \text{GL}\left(H^0(X, \omega_X)\right)$. A basic problem is to determine how many times a given irreducible representation of $G$ occurs in $H^0(X, \omega_X)$.

This problem was first considered by Hurwitz [6] for $G$ cyclic over $k = \mathbb{C}$. Then in the 30s of the 20th century, Chevalley and Weil [3] solved this problem for general $G$ when $\pi: X \to X/G$ is unramified. Soon after, Weil [15] solved the case for general $\pi$. This result was named as the Chevalley-Weil formula and it remains valid for any algebraically closed field $k$ with $\text{char}(k) = p \nmid \#G$ [7].

When $\text{char}(k) = p > 0$ and $p \mid \#G$, the structure of $H^0(X, \Omega_X)$ becomes more complicated. Except the tame ramification case ([7], [13]), or weakly ramified case([8]), people focus on some special groups ([14] for the case of cyclic groups, [10] for abelian groups, [4] for $p$-groups or [2] for groups with a cyclic Sylow subgroup).

In the 1980s, Kani studied the projectivity of the logarithmic differentials space $H^0(X, \Omega_X(D))$ as $k[G]$–module in the tamely ramified case[7]. But most of his work was covered by Nakajima’s work([12],[13]). The latter improved Mumford’s method[11] II.5 Lemma 1] to study the
$H^i (X, \mathcal{G})$ of the coherent $G$-sheaf $\mathcal{G}$ in the tamely ramified Galois covering for any dimensional projective varieties. Nevertheless, Kani’s work gives us several valuable tools.

For smooth curves, the Chevalley-Weil formula was well understood by now. In this paper, we will follow Kani’s methods, and generalize the Chevalley-Weil formula to the nodal curves for one-dimensional $G$-representations with $\text{char}(k) = 0$ or prime to $\#G$.

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2. Preliminary

2.1. Notations. In this paper, we consider a finite group $G$ acting faithfully on a nodal curve $X$ over an algebraically closed field $k$. Let $\#G = n$ and $\text{char}(k) = p \nmid n$ or $\text{char}(k) = 0$, which implies that $k[G]$ is semi-simple. A curve means an equidimensional reduced projective scheme of finite type of dimension 1 over $k$.

Let $X$ be a nodal curve, $\omega_X$ the canonical (dualizing) sheaf of $X$. Let $\hat{X} \to X$ be the normalization of $X$, then it induces an immersion $H^0(X, \omega_X) \hookrightarrow H^0(\hat{X}, \Omega_{\hat{X}}(\hat{S}_X))$, where $\hat{S}_X$ is the preimage of singularities(nodes) of $X$. For a node $P \in \alpha(\hat{S}_X)$, we say $\{P_1, P_2\} = \alpha^{-1}(P)$ a pair of $P$.

An element $\varphi_0 \in H^0(X, \Omega_{\hat{X}}(\hat{S}_X))$ belongs to $H^0(X, \omega_X)$ if and only if $\text{Res}_{P_1} \varphi_0 + \text{Res}_{P_2} \varphi_0 = 0$ for any pair $\{P_1, P_2\}$. Such an element is called a holomorphic differential of $X$. It is known that $H^0(X, \omega_X)$ is a $k$-vector space of dimension $p_a$, the arithmetic genus of $X$.

Both the rational function field $K(X)$ and $H^0(X, \omega_X)$ are naturally (right)$k[G]$-modules, and every 1-dimensional representation is its character. Our goal is to compute the multiplicity of any 1-dimensional representation $\chi$, that is the dimension of $H^0(X, \omega_X)_\chi$ over $k$. Note that all the irreducible representations will be 1-dimensional when $G$ is abelian.

Let $X$ be smooth for the rest of this section. Now we recall some properties for smooth curves.

Consider the branched cover $\pi : X \to X/G = Y$ and let $e_P$ be the ramification index at $P \in X$, then we have the ramification divisor

$$ R_\pi = \sum_{P \in X} (e_P - 1)P. $$

For a divisor $D = \sum a_i P_i \in \text{Div}(X)$, define $\pi_* D \in \text{Div}(Y)$ by

$$ \pi_* D = \sum a_i \pi(P_i). $$

If $D = \sum a_i Q_i \in \text{Div}(Y)$ is a divisor and $r \in \mathbb{R}$, then define $[rD] \in \text{Div}(Y)$ by

$$ [rD] = \sum [ra_i] Q_i, $$

where $[ra_i]$ denotes the greatest integer $\leq ra_i$. And define $\pi^* D \in \text{Div}(Y)$ by

$$ \pi^* D = \sum_i a_i \left( \sum_{P \in \pi^{-1}(Q_i)} e_P P \right). $$

**Proposition 2.1 (Kani [7])**. Let $G$ be a finite group acting on a smooth curve $X$ with $R_\pi$ the ramification divisor of $\pi : X \to X/G = Y$. Consider a $G$-invariant divisor $D \in \text{Div}(X)$, then
for the trivial character $\chi = 1_G$, we have

\begin{align}
(1) \quad & H^0(X, \mathcal{O}_X(D))^G = \pi^* H^0(Y, \mathcal{O}_Y \left| n^{-1}\pi_* D \right|), \\
(2) \quad & H^0(X, \Omega_X(D))^G = \pi^* H^0(Y, \Omega_Y \left| n^{-1}\pi_*(D + R_\pi) \right|). 
\end{align}

For a 1-dimensional character $\chi$, let $f_\chi \in K(X)^*$ be such that $\sigma f_\chi = \chi(\sigma) f_\chi$ for all $\sigma \in G$ (whose existence is guaranteed by Hilbert’s theorem 90). Then

\begin{align}
(3) \quad & H^0(X, \mathcal{O}_X(D))_\chi = f_\chi \cdot \pi^* H^0(Y, \mathcal{O}_Y \left| n^{-1}\pi_*(D + (f_\chi)) \right|), \\
(4) \quad & H^0(X, \Omega_X(D))_\chi = f_\chi \cdot \pi^* H^0(Y, \Omega_Y \left| n^{-1}\pi_*(D + (f_\chi) + R_\pi) \right|).
\end{align}

Proof. (More details here than in [7].) Note that $D \geq \pi^* \left| n^{-1}\pi_* D \right|$ and hence $H^0(X, \mathcal{O}_X(D))^G \supseteq \pi^* H^0(Y, \mathcal{O}_Y \left| n^{-1}\pi_* D \right|)$. Conversely, if $f \in H^0(X, \mathcal{O}_X(D))^G$, then $f = \pi^* e$ with some $e \in K(Y)$. Hence $\pi_*((f) + D) = n(e) + \pi_* D \geq 0$, which implies $(e) + \left| n^{-1}\pi_* D \right| \geq 0$. This proves (I).

To prove (2), fix a meromorphic differential $0 \neq \varphi \in \Omega(Y)$, which always exists by Riemann-Roch. By $H^0(X, \Omega_X(D)) = H^0(X, \mathcal{O}_X(D) + (\pi^* \varphi)) \cdot \pi^* \varphi = \pi^* H^0(X, \mathcal{O}_X(D + \pi^*(\varphi) + R_\pi)) \cdot \pi^* \varphi$, we have

\begin{align}
H^0(X, \Omega_X(D))_\chi &= f_\chi \cdot \pi^* H^0(Y, \mathcal{O}_Y \left| n^{-1}\pi_*((D + \pi^*(\varphi) + R_\pi)) \right|) \cdot \pi^* \varphi \\
&= \pi^* [H^0(Y, \mathcal{O}_Y \left| n^{-1}\pi_* D + R_\pi \right|) + (\varphi) \cdot \varphi] \\
&= \pi^* H^0(Y, \mathcal{O}_Y \left| n^{-1}\pi_* D + R_\pi \right|).
\end{align}

Finally, (3) and (4) for general $\chi$ is followed by

\begin{align}
H^0(X, \mathcal{O}_X(D))_\chi &= f_\chi \cdot H^0(X, \mathcal{O}_X(D + (f_\chi)))^G, \\
H^0(X, \Omega_X(D))_\chi &= f_\chi \cdot H^0(X, \Omega_X(D + (f_\chi)))^G.
\end{align}

\[ \square \]

2.2. Ramification modules. [7, Kani] Let $Bl(Y)$ be the branch locus of $\pi : X \to Y$.

Fix a point $P \in X$, and let $G_P$ be the stabilizer subgroup of $G$ at $P$, which is a cyclic of order $e_P$. Then there is a unique character $\theta_P : G_P \to k^*$ such that for any $f \in K(X)^*$,

\[ \frac{\sigma f}{f} \equiv \theta_P(\sigma)^{v_P(f)} \pmod{m_P}, \quad \forall \sigma \in G_P, \]

where $v_P$ denotes the valuation at $P$ and $m_P$ the maximal ideal of the local ring $\mathcal{O}_P$.

Set

\[ R_{G,P} := \text{Ind}^G_{G_P} \left( \bigoplus_{d=0}^{e_P-1} d \cdot \theta_P^d \right). \]

**Definition 2.2.** For a point $Q \in Y$, define the ramification module of $Q$

\[ R_{G,Q} := \bigoplus_{P \in \pi^{-1}(Q)} R_{G,P}, \]

and the ramification module of $\pi$

\[ R_G := \bigoplus_{Q \in Y} R_{G,Q}. \]

Note that this is a finite sum because $R_{G,Q} = 0$ for $Q \notin Bl(Y)$. 

Consider an $f_\chi \in K(X)^*$ such that $\sigma f_\chi = \chi(\sigma)f_\chi$ for any $\sigma \in G$ in Proposition 2.1. Since $f_\chi^n \in \pi^*(\mathcal{O}(Y))$, write $f_\chi^n = \pi^*(nA + B)$ where $A, B \in \text{Div}(Y)$ and $|n^{-1}B| = 0$.

Note that $\text{Supp}(B) \subseteq \text{Bl}(Y)$, so we write $B = \sum_{Q \in \text{Bl}} b_Q Q$. By definition, we have

$$b_Q = n \left\langle \frac{v_Q(f_\chi^n)}{n} \right\rangle,$$

where $\left\langle r \right\rangle = r - \lfloor r \rfloor$ denotes the fractional part of $r$. The following lemma shows that this $B$ is independent of the choices of $f_\chi$.

**Lemma 2.3.** Let $\chi : G \to \mathbb{K}^*$ be a 1-dimensional character. Then for any $Q \in \text{Bl}(Y)$, we have

$$n \left\langle \frac{v_Q(f_\chi^n)}{n} \right\rangle = \langle \chi, R_{G,Q} \rangle_G.$$

**Proof.** Let $P \in \pi^{-1}(Q)$. Then by Frobenius reciprocity, we have

$$\langle \chi, R_{G,P} \rangle_G = \left\langle \frac{v_P(f_\chi^n)}{n} \right\rangle = \left\langle \chi|_{G_P}, \bigoplus_{d=0}^{e_P-1} d \cdot \theta_P^d \right\rangle_{G_P}.$$

Note that $\theta_P^d$ runs through are all the irreducible representations of $G_P$, hence we have

$$\langle \chi, R_{G,P} \rangle_G = a \iff \chi|_{G_P} = \theta_P^a$$

with $0 \leq a < e_P$. Choose a generator $\sigma$ of $G_P$, then by definition of $f_\chi$, we have

$$\sigma f_\chi = \chi(\sigma)f_\chi = \theta_P(\sigma)^a f_\chi.$$

Furthermore, by the definition of $\theta_P$, we have

$$\theta_P(\sigma)^a = \frac{\sigma f_\chi}{f_\chi} \equiv \theta_P(\sigma)^{v_P(f_\chi)}(\text{mod } m_P),$$

which implies $a \equiv v_P(f_\chi)(\text{mod } e_P)$ since $\theta_P(\sigma)$ has order $e_P$ in $\mathbb{K}^*$. Finally,

$$\frac{\langle \chi, R_{G,P} \rangle_G}{e_P} = \left\langle \frac{v_P(f_\chi)}{e_P} \right\rangle = \left\langle \frac{v_Q(f_\chi^n)}{n} \right\rangle = \frac{b_Q}{n},$$

namely we have $\langle \chi, R_{G,Q} \rangle_G = b_Q$.

3. **Irreducible nodal curves**

Let $X$ be an irreducible nodal curve in this section.

3.1. **The $G$-invariant differentials.** Let $G$ be a finite group acting on an irreducible nodal curve $X$ and $Y = X/G$ the quotient curve. For the space $H^0(X, \omega_X)^G$ of $G$-invariant differentials, it is a classical fact that

**Proposition 3.1.** If $X$ is smooth, then

$$\dim_k H^0(X, \Omega_X)^G = g_Y.$$
Proof. With the notations of \([2.1]\), let \(e_Q := e_P\) for any \(P \in \pi^{-1}(Q)\). Note that
\[
\left| n^{-1}\pi_*R_{\pi}\right| = \sum_{Q \in Y} \left\lfloor \frac{e_Q - 1}{e_Q} \right\rfloor Q = 0.
\]
By Proposition \([2.1 \; (2)]\), we have
\[
H^0(X, \Omega_X)^G = \pi^*H^0(Y, \Omega_Y \left| n^{-1}\pi_*R_{\pi}\right|) = \pi^*H^0(Y, \Omega_Y).
\]
\(\square\)

Here comes a natural question that for the covering \(\pi : X \to X/G = Y\) of nodal curves, do we still have the equality
\[
\dim_k H^0(X, \omega_X)^G = p_a(Y)? \tag{8}
\]
Consider the normalizations \(\hat{X} \to X\) and \(\hat{Y} \to Y\), respectively. We have \(\hat{X}/G = \hat{Y}\), so there is a commutative diagram
\[
\begin{array}{ccc}
\hat{X} & \xrightarrow{\hat{\pi}} & \hat{Y} \\
\downarrow & & \downarrow \\
X & \xrightarrow{\pi} & Y.
\end{array}
\]
This induces the corresponding morphisms of differentials
\[
\begin{array}{ccc}
H^0(X, \omega_X) & \leftarrow & \pi^*H^0(Y, \omega_Y) \\
\downarrow & & \downarrow \\
H^0(\hat{X}, \Omega_{\hat{X}}(\hat{S}_X)) & \leftarrow & \hat{\pi}^*H^0(\hat{Y}, \Omega_{\hat{Y}}(\hat{S}_Y)),
\end{array} \tag{9}
\]
because \(X \to X/G\) takes smooth points to smooth points, so \(\hat{\pi}^{-1}(\hat{S}_Y) \subseteq \hat{S}_X\).

**Lemma 3.2.** The upper row
\[
\pi^*H^0(Y, \omega_Y) \subseteq H^0(X, \omega_X)
\]
of \((9)\) exists if and only if the ramification indexes \(e_{P_1} = e_{P_2}\) for all pairs \(\{P_1, P_2\} \subset \hat{S}_X\).

Proof. Given some \(\varphi_Y \in H^0(Y, \omega_Y)\), we have \(\operatorname{Res}_{\hat{\pi}(P_1)} \varphi_Y = \operatorname{Res}_{\hat{\pi}(P_2)} \varphi_Y\). Note that for any \(P \in \hat{X}\), we have \(\operatorname{Res}_{\hat{P}}(\hat{\pi}^* \varphi_Y) = e_{P} \cdot \operatorname{Res}_{\hat{\pi}(P_2)} \varphi_Y\). Hence for any pair \(\{P_1, P_2\} \subset \hat{S}_X\), we have
\[
\operatorname{Res}_{P_1}(\hat{\pi}^* \varphi_Y) = \operatorname{Res}_{P_2}(\hat{\pi}^* \varphi_Y) \tag{10}
\]
if and only if \(e_{P_1} = e_{P_2}\). \(\square\)

Note that the points of \(\hat{S}_X - \hat{\pi}^{-1}(\hat{S}_Y)\) are mapped to the smooth part of \(Y\).

**Lemma 3.3.** For the left column of \((9)\), we have
\[
H^0(X, \omega_X)^G \hookrightarrow H^0(\hat{X}, \Omega_{\hat{X}}(\pi^{-1}(\hat{S}_Y)))^G.
\]

Proof. Let \(\varphi \in H^0(X, \omega_X)^G\), and a pair \(\{P_1, P_2\} \subset \hat{S}_X - \hat{\pi}^{-1}(\hat{S}_Y)\). Then there exists some \(\sigma \in G\) such that \(\sigma(P_1) = P_2\), which implies that \(\operatorname{Res}_{\hat{P}_1} \varphi = \operatorname{Res}_{\hat{P}_2} \sigma \varphi = \operatorname{Res}_{\sigma(P_1)} \varphi = \operatorname{Res}_{\hat{P}_2} \varphi\). As \(\operatorname{Res}_{\hat{P}_1} \varphi = -\operatorname{Res}_{\hat{P}_2} \varphi\), we get \(\operatorname{Res}_{\hat{P}_1} \varphi = \operatorname{Res}_{\hat{P}_2} \varphi = 0\), and \(\varphi\) is holomorphic at \(\{P_1, P_2\}\). \(\square\)

If for all pairs \(\{P_1, P_2\} \subset \hat{S}_X\), we have \(e_{P_1} = e_{P_2}\), then we can give a positive answer to \((8)\).
Proposition 3.4. With the notations above, we have
\[ \pi^* H^0(\hat{Y}, \Omega_{\hat{Y}}(\hat{S}_Y)) = H^0(\hat{X}, \Omega_{\hat{X}}(\pi^{-1}(\hat{S}_Y)))^G. \]

Moreover, if the ramification indexes for any pair \( \{P_1, P_2\} \subseteq \hat{S}_X \) are equal, namely \( e_{P_1} = e_{P_2} \), then we have the canonical isomorphism
\[ H^0(X, \omega_X)^G \overset{\pi^*}{\longrightarrow} H^0(Y, \omega_Y) \]
and the rows are both isomorphisms. In particular, we have \( \dim H^0(X, \omega_X)^G = p_a(Y) \).

Proof. Now we treat \( \hat{S}_Y \) as a positive divisor here. By Proposition 3.4, we have
\[ H^0(\hat{X}, \Omega_{\hat{X}}(\pi^{-1}(\hat{S}_Y)))^G = \pi^* H^0(\hat{Y}, \Omega_{\hat{Y}}(\left[n^{-1} \pi_*(\pi^{-1}(\hat{S}_Y) + R_\pi)\right])). \]

Consider the coefficient of prime divisors in \( \left[n^{-1} \pi_*(\pi^{-1}(\hat{S}_Y) + R_\pi)\right] = \sum a_Q Q \).

1. If \( Q \in Bl(Y) - \hat{S}_Y \), then \( a_Q = \left[\frac{e_Q - 1}{e_Q}\right] = 0 \);
2. If \( Q \in \hat{S}_Y - Bl(Y) \), then \( a_Q = 1 \);
3. If \( Q \in \hat{S}_Y \cap Bl(Y) \), then \( a_Q = \left[\frac{1}{e_Q} + \frac{e_Q - 1}{e_Q}\right] = 1 \).

So we have \( \left[n^{-1} \pi_*(\pi^{-1}(\hat{S}_Y) + R_\pi)\right] = \hat{S}_Y \), hence the isomorphism on the lower row. Note that both \( H^0(X, \omega_X)^G \) and \( H^0(Y, \omega_Y) \) are the subspaces satisfying the residue relations, then we have the isomorphism of the upper row. \( \square \)

3.2. Chevalley-Weil formula for irreducible nodal curves. Let \( \chi \) be a 1-dimensional character of \( G \). With the notations in (10), we consider the embedding
\[ H^0(X, \omega_X)_\chi \rightarrow H^0(\hat{X}, \Omega_{\hat{X}}(\hat{S}_X))_\chi. \]

Proposition 3.5. Let \( Y \) be smooth, set
\[ \hat{S}_X^\chi = \left\{ \hat{P} \in \hat{S}_X \mid \exists \tau \in G_{\alpha(P)} \text{ s.t. } \tau(\hat{P}) \neq \hat{P} \text{ and } \chi(\tau) = -1 \right\}. \]
Then the image of \( H^0(X, \omega_X)_\chi \) in \( H^0(\hat{X}, \Omega_{\hat{X}}(\hat{S}_X))_\chi \) is equal to \( H^0(\hat{X}, \Omega_{\hat{X}}(\hat{S}_X^\chi))_\chi \). So we have an isomorphism
\[ H^0(X, \omega_X)_\chi \overset{\sim}{\rightarrow} H^0(\hat{X}, \Omega_{\hat{X}}(\hat{S}_X^\chi))_\chi. \]
We call \( \hat{S}_X^\chi \) the singular \( \chi \)-set of \( X \).

Proof. Assume \( \varphi \in H^0(\hat{X}, \Omega_{\hat{X}}(\hat{S}_X^\chi))_\chi \), and let \( \alpha^{-1}(P) = \{P_1, P_2\} \subseteq \hat{S}_X^\chi \). So there is a \( T \in G_P \) with \( T(P_1) = P_2 \) and \( \chi(T) = -1 \) by the definition of \( \hat{S}_X^\chi \). Hence \( - \text{Res}_{P_1} \varphi = \text{Res}_{P_2} T \varphi = \text{Res}_{\alpha^{-1}(P)} \varphi = \text{Res}_{P_2} \varphi \).

Conversely, given some \( \varphi_0 \in H^0(X, \omega_X)_\chi \) with poles on a pair \( \alpha^{-1}(P) = \{P_1, P_2\} \subseteq \hat{S}_X \) and some \( T \in G_P \) with \( T(P_1) = P_2 \), which exists by the smooth of \( Y \), we have \( \chi(T) \text{Res}_{P_1}(\varphi_0) = \text{Res}_{P_2}(T \varphi_0) = \text{Res}_{P_2}(\varphi_0) = - \text{Res}_{P_1}(\varphi_0) \).

Hence \( \chi(T) = -1 \) and \( \{P_1, P_2\} \subseteq \hat{S}_X^\chi \). \( \square \)
Remark 3.6. If $\chi = 1$ is the trivial representation, then $\hat{S}_X^\chi = \emptyset$, which is consistent with Lemma 3.4 in the case $\hat{S}_Y = \emptyset$.

Remark 3.7. Suppose $\varphi' \in H^0(\hat{X}, \Omega_{\hat{X}}(S'))$ has a pole at $P_1 \in S'$, namely $\text{Res}_{P_1} \varphi' \neq 0$. If $\varphi' \in H^0(X, \omega_X)$, then for the pair $\{P_1, P_2\}$, it requires

a) $v_{P_2}(\varphi') = v_{P_1}(\varphi') = -1$;  
b) $\text{Res}_{P_2} \varphi' = -\text{Res}_{P_1} \varphi'$.

For a), in general, we can’t determine $v_{P_2}(\varphi')$ from the value of $v_{P_1}(\varphi')$. But if $Y$ is smooth, then $v_{P_2}(\varphi') = v_{P_1}(\varphi')$ for $\forall P \in \hat{\pi}^{-1}(P_1)$.

For b), under the hypothesis of smoothness of $Y$, we use the criterion from the singular $\chi$-set to delete these points that can not be poles.

Assume $Y = X/G$ is smooth for the rest of this section.

Now we compute the dimension of $H^0(\hat{X}, \Omega_X(\hat{S}_X^\chi))$. Let $f_\chi$ be a rational function on $\hat{X}$ such that $\sigma f_\chi = \chi(\sigma)f_\chi, \forall \sigma \in G$ and set $D_\chi = \left[n^{-1}\hat{\pi}_*\left(\hat{S}_X^\chi + (f_\chi) + R_\hat{\pi}\right)\right]$. By Proposition 2.1 (4), we have

$$H^0(\hat{X}, \Omega_X(\hat{S}_X^\chi))_\chi = f_\chi \cdot \hat{\pi}^*H^0(Y, \Omega_Y(D_\chi)).$$

By Riemann-Roch Theorem, we have

$$\dim_k H^0(Y, \Omega_Y(D_\chi)) = \dim_k H^0(Y, \mathcal{O}_Y(-D_\chi)) + \deg D_\chi + g_Y - 1.$$ (13)

**Lemma 3.8.** The space $H^0(Y, \mathcal{O}_Y(-D_\chi))$ vanishes except when $\chi = 1_G$, and in this case, we have $\dim_k H^0(Y, -D_\chi_G) = 1$.

**Proof.** We will show that $\deg D_\chi > 0$ if $\hat{S}_X^\chi \neq \emptyset$ and $D_\chi$ is principal if and only if $\chi = 1$. Assume

$$\hat{\pi}_*(f_\chi) = \sum_{Q \in Y} \frac{n}{e_Q} b_Q \cdot Q$$

where $b_Q = v_P(f_\chi), \forall P \in \hat{\pi}^{-1}(Q)$. Note that

$$\left[n^{-1}\hat{\pi}_*((f_\chi) + R_\hat{\pi})\right] = \sum_Q \left[\frac{b_Q + e_Q - 1}{e_Q}\right] Q \geq \sum_Q \frac{b_Q}{e_Q} Q,$$

hence we have

$$\deg \left[n^{-1}\hat{\pi}_*\left(\hat{S}_X^\chi + (f_\chi) + R_\hat{\pi}\right)\right] \geq \deg \left[n^{-1}\hat{\pi}_*((f_\chi) + R_\hat{\pi})\right] \geq n^{-1}\deg(f_\chi) = 0.$$ (14)

Wirte $\hat{\pi}_*(\hat{S}_X^\chi) = \sum_{Q \in Y} \frac{n}{e_Q} c_Q \cdot Q$. If $\hat{S}_X^\chi \neq \emptyset$, then there is some $c_{Q'} \geq 1$, hence

$$\deg D_\chi = \sum_{Q \neq Q'} \left[\frac{c_Q + b_Q + e_Q - 1}{e_Q}\right] + \left[\frac{c_{Q'} + b_{Q'} + e_{Q'} - 1}{e_{Q'}}\right] \geq \sum_{Q \neq Q'} \left[\frac{c_Q + b_Q + e_Q - 1}{e_Q}\right] + \left[\frac{b_{Q'} + e_{Q'}}{e_{Q'}}\right] > \sum_Q \frac{b_Q}{e_Q} = 0.$$ (15)

Hence $\dim_k H^0(Y, \mathcal{O}_Y(-D_\chi)) = 0$ provided $\hat{S}_X^\chi \neq \emptyset$. 


Now we suppose $\hat{S}_X^k = \emptyset$, and $\deg \left[ n^{-1}\hat{\pi}_*\left((f_\chi) + R_\pi\right) \right] = 0$, then we have
\[
\left\lfloor \frac{b_Q + e_Q - 1}{e_Q} \right\rfloor = b_Q/e_Q
\]
which implies $b_Q = \lambda_Q e_Q$ for some integer $\lambda_Q$ and $D_\chi = n^{-1}\hat{\pi}_*(f_\chi)$. If $D_\chi$ is principal, namely $\dim_k H^0(Y, \mathcal{O}_Y(-D_\chi)) = 1$, then we have $D_\chi = (h)$ for some rational function $h \in K(Y)$. Hence $(f_\chi) = (\hat{\pi}^* h)$, which implies $f_\chi \in K(X)^G$, namely $\chi = 1$. Conversely, if $\chi = 1$, then $f_\chi = \hat{\pi}^* h$ for some rational function $h \in K(Y)$ and $D_\chi = (h)$, hence $\dim_k H^0(Y, \mathcal{O}_Y(-D_\chi)) = 1$. 

**Definition 3.9.** Let $S \subseteq \hat{X}$ be a finite subset stable by $G$, and define
\[
m_\chi(S) = \#\hat{\pi}(S) + \sum_{Q \notin \hat{\pi}(S)} \left\lfloor \frac{e_Q - 1}{e_Q} \right\rfloor + \sum_{Q \notin \hat{\pi}(S)} \frac{b_Q}{n} - \frac{b_Q}{n}.
\]

**Lemma 3.10.** We have $\deg D_\chi = m_\chi(\hat{S}_X^k)$, which is independent of the choice of $f_\chi$.

**Proof.** Denote $s_\chi = \#\hat{\pi}(\hat{S}_X^k)$. Write $n^{-1}\hat{\pi}_*(\hat{S}_X^k) = U^X + V^X$, where $\text{Supp}(V^X) = Bl(Y) \cap \hat{\pi}(\hat{S}_X^k)$. Suppose $(f_\chi^n) = \hat{\pi}^*(nA + B)$ and $[n^{-1}B] = 0$ as in lemma 2.3. Write $B = \sum_{Q \in Bl(Y)} b_Q Q$. By
\[
\left\lfloor n^{-1}\hat{\pi}^*(\hat{S}_X^k + (f_\chi) + R_\pi) \right\rfloor = U^X + A + [V^X + n^{-1}B + n^{-1}\hat{\pi}^*R_\pi],
\]
and $\deg B = -\deg nA$, we have
\[
\deg D_\chi = \#U^X + \sum_{Q \notin \hat{\pi}(\hat{S}_X^k)} \left\lfloor 1 + \frac{b_Q}{n} \right\rfloor + \sum_{Q \notin \hat{\pi}(\hat{S}_X^k)} \left\lfloor \frac{e_Q - 1}{e_Q} + \frac{b_Q}{n} \right\rfloor - \sum_{Q} \frac{b_Q}{n}
\]
\[
= s_\chi + \sum_{Q \notin \hat{\pi}(\hat{S}_X^k)} \left\lfloor \frac{e_Q - 1}{e_Q} + \frac{b_Q}{n} \right\rfloor - \sum_{Q} \frac{b_Q}{n}.
\]

By Lemma 2.3 [5], we have $b_Q = (\chi, R_{G,Q})_G$, hence
\[
\deg D_\chi = s_\chi + \sum_{Q \notin \hat{\pi}(\hat{S}_X^k)} \left\lfloor \frac{e_Q - 1}{e_Q} + \frac{1}{n} (\chi, R_{G,Q})_G \right\rfloor - \frac{1}{n} (\chi, R_{G})_G = m_\chi(\hat{S}_X^k).
\]

Now we summary above discussion.

For a quotient map $\pi : X \to X/G = Y$ from an irreducible nodal curve to a smooth curve, we have the induced covering of curves $\hat{\pi} : \hat{X} \to Y$ with $\hat{S}_X \subseteq \hat{X}$ the preimage of singular locus. Let $Bl(Y)$ be the branch locus, $R_{G}$ the ramification module of $\hat{\pi}$, and $R_{G,Q}$ the ramification module of $Q \in Y$.

**Theorem 3.11** (The Chevalley-Weil formula on irreducible nodal curves). Let $f : X \to X/G$ be the quotient map of irreducible nodal curves by a finite group $G$ of order $n$. Assume $Y = X/G$ is smooth and $\text{char}(k) \nmid n$, then the multiplicity of a given irreducible character $\chi$ is given by
\[
\dim_k H^0(X, \omega_X)^\chi = g_Y - 1 + m_\chi(\hat{S}_X^k) + (\chi, 1_G).
\]
where $\hat{S}_X^k$ is the singular $\chi$-set of $X$ defined in Proposition 3.5 and
\[
m_\chi(\hat{S}_X^k) = s_\chi + \sum_{Q \notin \hat{\pi}(\hat{S}_X^k)} \left\lfloor \frac{e_Q - 1}{e_Q} + \frac{1}{n} (\chi, R_{G,Q})_G \right\rfloor - \frac{1}{n} (\chi, R_{G})_G
\]
is defined in Definition 3.9.
In particular, when $\chi = 1_G$, we have $\hat{S}_X^\chi = \emptyset$ and $\dim_k H^0(X, \omega_X)^G = g_Y$, which is a special case of Proposition 3.4.

**Example 3.12** (Hyperelliptic stable curves). A hyperelliptic stable curve $C$ is a stable curve with a hyperelliptic involution $J : C \to C$, which is an order 2 automorphism satisfying $C/\langle J \rangle = \mathbb{P}^1$.

Suppose $C$ is an irreducible hyperelliptic stable curve with $N(\geq 1)$ nodes. Let $\hat{C}$ be the normalization of $C$ with genus $g$, then $\hat{\pi} : \hat{C} \to \mathbb{P}^1$ has $2g + 2$ fixed (ramification) points, which are all of ramification index 2, hence $\# \text{Bl}(\mathbb{P}^1) = 2g + 2$.

There are no branch points in $\hat{\pi}(\hat{S}_C)$, and the Galois group $G = \langle J \rangle \cong \mathbb{Z}_2$ has two irreducible representations $1_G$ and $\chi^-$ where $\chi^-(J) = -1$.

For $1_G$, we have

$$\dim_k H^0(C, \omega_C)^G = g(\mathbb{P}^1) = 0.$$  

For $\chi^-$, we have $\hat{S}_C^{\chi^-} = \hat{S}_C$, hence $s_{\chi^-} = N$.

And for any fixed point $P$, the induced character $\theta_P : G_P = \langle J \rangle \to k^*$, $J \mapsto -1$ is the generator of $\text{Hom}(\mathbb{Z}_2, k^*)$. So we have $\langle \chi^-, R_G, Q \rangle_G = 1$ for all $Q \in \text{Bl}(\mathbb{P}^1)$ and $\langle \chi^-, R_G \rangle_G = 2g + 2$, which gives

$$m_{\chi^-}(\hat{S}_C^{\chi^-}) = s_{\chi^-} + \sum_{Q \in \text{Bl}(\mathbb{P}^1)} \left[ \frac{e_Q - 1}{e_Q} + \frac{1}{2} \langle \chi^-, R_G, Q \rangle_G \right] - \frac{1}{2} \langle \chi^-, R_G \rangle_G$$

(20)

$$= N + 2g + 2 - (g + 1) = p_a(C) + 1.$$

Hence

$$\dim_k H^0(C, \omega_C)^{\chi^-} = g(\mathbb{P}^1) - 1 + m_{\chi^-}(\hat{S}_C^{\chi^-}) = p_a(C).$$

4. Nodal curves with several irreducible components

In this section, let $X$ be a connected nodal curve and $X = \bigcup_{i=1}^d X_i$ be the decomposition of irreducible components. Consider

$$\alpha : \coprod_{i=1}^d X_i \to X,$$

the partial normalization at the intersection locus, then we have the immersion

$$H^0(X, \omega_X) \hookrightarrow \oplus_{i=1}^d H^0(X_i, \omega_{X_i}(I_i)), \varphi \mapsto (\varphi|_{X_i}),$$

where $I_i$ is the set of intersection points of each $X_i$.

4.1. **Reduction.** Let $G$ be a finite subgroup of $\text{Aut}(X)$ and $\pi : X \to X/G = Y$ be the quotient map. Suppose $Y$ is irreducible, then $G$ acts transitively on $\{X_1, \cdots, X_d\}$ and all these components are isomorphic. Let $G_i$ be the stabling subgroup of $G$ at $X_i$ and note that the canonical map $X_i/G_i \to X/G = Y$ is an isomorphism.

**Proposition 4.1.** Given a 1-dimensional character $\chi$ of $G$, we have a commutative diagram:

$$\begin{array}{ccc}
H^0(X, \omega_X)^\chi & \hookrightarrow & \left[ \oplus_{i=1}^d H^0(X_i, \omega_{X_i}(I_i)) \right]^\chi \\
& \downarrow_{p_1} & \varphi \mapsto (\varphi|_{X_i})_{i=1}^d \\
H^0(X_1, \omega_{X_1}(I_1))^\chi & & \varphi|_{X_1}
\end{array}$$

(23)

where $\chi_1$ is the restriction of $\chi$ in $G_1$. Moreover, the projection $p_1$ is an isomorphism.
Proof. Fix some \((\varphi_1, \cdots, \varphi_d) \in \left[ \oplus_{i=1}^d H^0(X_i, \omega_{X_i}(I_i)) \right]_X\). Since \(G\) acts on \(\{X_1, \cdots, X_d\}\) transitively, then we always have some \(T_i : X_i \to X_1\) and \(T_i \varphi_1 = \chi(T_i) \varphi_i\), hence \((\varphi_1, \cdots, \varphi_d) = (\varphi_1, \chi(T_2)^{-1}T_2 \varphi_1, \cdots, \chi(T_d)^{-1}T_d \varphi_1 )\), namely \((\varphi_1, \cdots, \varphi_d)\) is uniquely determined by \(\varphi_1\). Note that \(T_1 \varphi_1 = \chi(T_1) \varphi_1\) for any \(T_1 \in G_1\), which implies \(\varphi_1 \in H^0(X_1, \omega_{X_1}(I_1))_X\), so the projection to the first component \(p_1\) is injective.

Conversely, we want to show \((\varphi_1, \chi(T_2)^{-1}T_2 \varphi_1, \cdots, \chi(T_d)^{-1}T_d \varphi_1 )\) is the image of \(\varphi_1 \in H^0(X_1, \omega_{X_1}(I_1))\). Note that for any two \(\sigma, \tau : X_1 \to X_1\), we have \(\chi(\sigma)^{-1} \sigma \varphi_1 = \chi(\tau)^{-1} \tau \varphi_1\). Hence for any \(T \in G\), we have
\[
T(\varphi_1, \chi(T_2)^{-1}T_2 \varphi_1, \cdots, \chi(T_d)^{-1}T_d \varphi_1 ) = \chi(T)(\varphi_1, \chi(T_2)^{-1}T_2 \varphi_1, \cdots, \chi(T_d)^{-1}T_d \varphi_1 ).
\]

\[\square\]

Remark 4.2. By the same arguments as in Proposition 3.3 and Proposition 4.1, we set
\[
I^X_1 = \{ P \in I_1 | \exists \tau \in G_P - G_i \text{ s.t. } \chi(\tau) = -1 \}
\]
to be those intersection points that could be the poles of \(\varphi|_{X_i}\) for \(\varphi \in H^0(X, \omega_X)_X\), and then we have the isomorphisms
\[
H^0(X, \omega_X)_X \cong [\oplus_{i=1}^d H^0(X_i, \omega_{X_i}(I^X_i))_X] \cong H^0(X_1, \omega_{X_1}(I^X_1))_X.
\]

So our research object has been reduced to the irreducible nodal curve acting by the subgroup \(G_1\) on \(X_1\).

4.2. Chevalley-Weil formula for connected nodal curves. With the notations above, note that \(I^X_1 = \emptyset\) when \(\chi = 1_G\). By Proposition 4.1, if the ramification indexes of any pair \(\{P_1, P_2\} \subseteq \hat{S}_{X_1}\) are equal for \(\pi_1 : X_1 \to X_1/G_1 = Y\), then we have
\[
H^0(X, \omega_X)^G \cong H^0(X_1, \omega_{X_1}(I^X_1))^G \cong H^0(Y, \omega_Y).
\]

In this case, we have \(\dim_k H^0(X, \omega_X)^G = p_a(Y)\).

Assume that \(Y\) is smooth for the rest of this section.

Let \(\hat{\pi}_1 : X_1 \to Y\) be the normalization of \(\pi_1\), \(Bl(Y)\) the branch locus, \(R_{G_1}\) the ramification module of \(\hat{\pi}_1\), and \(R_{G_1, Q}\) the ramification module of \(Q \in Y\).

Suppose \(\hat{S}^{X_1}_{X_1}\) is the singular \(\chi_1\)-set of \(X_1\) in Proposition 3.5, then we have the isomorphisms
\[
H^0(X, \omega_X)_X \cong H^0(X_1, \omega_{X_1}(I^X_1))_X \cong H^0(\hat{X}_1, \Omega_{\hat{X}_1}(\hat{S}^{X_1}_{X_1} \cup I^X_1))_X.
\]

Note that \(n_1 := \#G_1 = n/d\) and by Proposition 2.7 [4], again we have
\[
H^0(\hat{X}_1, \Omega_{\hat{X}_1}(\hat{S}^{X_1}_{X_1} \cup I^X_1))_X = f_{X_1} \cdot \hat{\pi}_1^* H^0(Y, \Omega_Y) \left[ \prod_{i=1}^{n_1-1} \pi_1^* \left( \hat{S}^{X_1}_{X_1} \cup I^X_1 + (f_{X_1}) + R_{\pi_1} \right) \right]
\]
where \(f_{X_1} \in K(X_1)^*\) is such that \(\sigma f_{X_1} = \chi_1(\sigma) f_{X_1}, \forall \sigma \in G_1\). The same argument in Lemma 3.10 shows that
\[
\deg \left[ \prod_{i=1}^{n_1-1} \pi_1^* \left( \hat{S}^{X_1}_{X_1} \cup I^X_1 + (f_{X_1}) + R_{\pi_1} \right) \right] = m_{X_1}(\hat{S}^{X_1}_{X_1} \cup I^X_1)
\]
where by Definition 3.2
\[
m_{X_1}(\hat{S}^{X_1}_{X_1} \cup I^X_1) = \# \hat{\pi}_1(\hat{S}^{X_1}_{X_1} \cup I^X_1) + \sum_{Q \notin \hat{\pi}_1(\hat{S}^{X_1}_{X_1} \cup I^X_1)} \left[ \frac{e_Q - 1}{e_Q} + \frac{d}{n} \langle \chi_1, R_{G_1, Q} \rangle_{G_1} \right] - \frac{d}{n} \langle \chi_1, R_{G_1} \rangle_{G_1}.
\]

By the same argument in Lemma 3.8 and Riemann-Roch Theorem, we have
Theorem 4.3 (The Chevalley-Weil formula on connected nodal curves). Let $X$ be a connected nodal curve of $d$ irreducible components and $G$ a finite group of order $n$ acting on $X$.

Assume the quotient curve $Y = X/G$ is smooth (hence irreducible), then we have a canonical map $X_1 \rightarrow Y = X_1/G_1 = Y$, where $X_1$ is an irreducible component and $G_1$ is the stablizer subgroup of $G$ at $X_1$.

With the notations above, the multiplicity of a 1-dimensional character $\chi$ is given by

$$\dim_k H^0(X, \omega_X)\chi = g_Y - 1 + m_{\chi_1}(\tilde{S}^{X_1} \cup I_1^X) + \delta_\chi,$$

where $\delta_\chi = 0$ or 1. And $\delta_\chi = 1$ if and only if $I_1^X = \emptyset$ and $\chi_1 = 1_{G_1}$. In particular, when $\chi = 1_G$, we have

$$\dim_k H^0(X, \omega_X)^G = g_Y.$$

Note that this theorem is exactly the direct generalization of Theorem 3.11 since if $d = 1$, then $I_1^X = \emptyset$.

Example 4.4. Let $C = C_1 \cup C_2$ be a hyperelliptic stable curve, where $C_1 \approx C_2 \approx \mathbb{P}^1$, and they intersect in $\#(C_1 \cap C_2) = m(> 2)$ points, then $p_a(C) = m - 1$. Consider the hyperelliptic involution $J$ permuting $C_1$ and $C_2$, we have $\pi : C \rightarrow C/J = \mathbb{P}^1$.

The covering map $\pi_1 : C_1 \rightarrow \mathbb{P}^1$ is an identity. For the representation $\chi^-$, we have $\chi_1^- = id$.

Hence $\tilde{S}^{C_1}_C = \emptyset$ and $m_{id}(\tilde{S}^{C_1}_C \cup I_1^{C_1}) = \#(I_1^{C_1}) = m$.

So by Theorem 4.3, we have

$$\dim_k H^0(C, \omega_C)\chi^- = g_{\mathbb{P}^1} - 1 + m = m - 1 = p_a(C),$$

and $\dim_k H^0(C, \omega_C)^G = p_a(\mathbb{P}^1) = 0$.

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