Primordial power spectrum reconstruction

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Abstract. In order to reconstruct the initial conditions of the universe it is important to devise a method that can efficiently constrain the shape of the power spectrum of primordial matter density fluctuations in a model-independent way from data. In an earlier paper we proposed a method based on the wavelet expansion of the primordial power spectrum. The advantage of this method is that the orthogonality and multiresolution properties of wavelet basis functions enable information regarding the shape of \(P_{\text{in}}(k)\) to be encoded in a small number of non-zero coefficients. Any deviation from scale invariance can then be easily picked out. Here we apply this method to simulated data to demonstrate that it can accurately reconstruct an input \(P_{\text{in}}(k)\), and present a prescription for how this method should be used on future data.

Keywords: CMBR experiments, CMBR theory
1. Introduction

Cosmic microwave background (CMB) and large scale structure (LSS) data are powerful probes of the universe. They are sensitive not only to the values of cosmological parameters but also to the initial conditions of the universe that may have arisen due to inflation. While the general concept of inflation solves many problems in cosmology and provides a framework within which the data are very well fitted, not much is known about the details of inflation.

The power spectrum of density perturbations that results from inflation is usually assumed to be scale invariant, or sometimes a power law or a power law with a running tilt, and such parametrizations fit the data well so far. Simple models of inflation generically predict Gaussian adiabatic perturbations with an early scale-invariant spectrum. However there are other possibilities, for example if multiple scalar fields were present during inflation, or if the slow-rolledness of the inflaton field was temporarily interrupted due to a feature in the inflaton potential, etc. Such models can predict power spectra that are different from scale invariance/power law \[ [12, 30, 23, 1, 15, 7, 2, 16, 10, 6] \].

Since the details of inflation are unclear, we could use precision observations of the CMB and LSS to constrain not only the cosmological parameters but also the shape of the primordial power spectrum \( (P_{\text{in}}(k)) \) in a non-parametric way. The motivation is to test the assumptions that are usually made regarding the shape of \( P_{\text{in}}(k) \), and to check if there are any features in \( P_{\text{in}}(k) \) that can thus be unravelled. The aim is to try to see if, as the data continue to improve, something new can be learnt about interesting cosmologically relevant functions such as \( P_{\text{in}}(k) \) besides deriving stronger constraints on cosmological parameters based on the same set of assumptions. We presented an efficient method for reconstructing the shape of \( P_{\text{in}}(k) \) in \[ 20 \]. In this paper, we demonstrate that this method is able to accurately reconstruct an input \( P_{\text{in}}(k) \) by applying it to simulated data.

That \( P_{\text{in}}(k) \) be probed model independently was initially suggested by Wang et al [31] and [32], who used conventional top-hat binning and linear interpolation respectively in the parametrization of \( P_{\text{in}}(k) \). Follow-up analysis using pre-WMAP data and WMAP data was done by Mukherjee et al [18, 19]. In [20] we proposed the wavelet expansion method and used it on WMAP data. This method is superior to binning methods in that
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it has fewer correlated errors. The method also has various other advantages: due to the simultaneous and adaptive localization property of orthogonal wavelets in both scale and position spaces, the information content of most signals is sparsely represented in wavelet space with useful information being encoded only in a small number of coefficients that are large, while the others are small, or in the case of noisy data consistent with zero. One can then hope to recover the large coefficients more easily and thus constrain the function more efficiently. Function reconstruction and denoising are thus efficiently achieved with wavelets, as we demonstrate further in this paper.

Since complex projection effects and other degeneracies limit our ability to constrain a function in $k$ space using data in multipole space \[28\], the use of a different basis set, that is at the same time adaptive to the signal at hand, is certainly a useful complementary method to any other model-independent method. Since the release of the WMAP first year data \[4,26,21\], a number of analyses with emphasis on the primordial power spectrum have been undertaken \[5,3,24\]. Model-independent methods have been pursued in \[5,11,13,17,25,29,14\].

In this paper, we use simulated data to explicitly demonstrate the potential and reliability of the wavelet expansion method in the primordial power spectrum reconstruction. Section 2 describes our method. Section 3 contains results. We conclude in section 4.

2. Method

CMB data are sensitive to $P_{\text{in}}(k)$ as follows:

$$C_l(\{b_{j,l}\}) = (4\pi)^2 \int \frac{dk}{k} P_{\text{in}}(k) |\Delta T_l(k, \tau = \tau_0)|^2$$

where the cosmological model dependent transfer function $\Delta T_l(k, \tau = \tau_0)$ is an integral over conformal time $\tau$ of the sources which generate CMB temperature fluctuations, $\tau_0$ being the conformal time today.

As described in \[20\] we parametrize $P_{\text{in}}(k)$ by the coefficients of its wavelet expansion.

$$P_{\text{in}}(k_i) = \sum_{j=0}^{J-1} \sum_{l=0}^{2^j-1} b_{j,l} \psi_{j,l}(k_i),$$

where $\psi_{j,l}$ are the wavelet basis functions, constructed from the dilations and translations of a mother function $\psi(k)$. The resulting wavelet bases are discrete, compactly supported and orthogonal with respect to both the scale $j$ and the position $l$ indices, and hence their coefficients provide a complete and non-redundant (hence invertible) representation of the function $P_{\text{in}}(k)$. The scale index $j$ increases from 0 to $J-1$, and wavelet coefficients with increasing $j$ represent structure in $P_{\text{in}}(k)$ on increasingly smaller scales, with each scale a factor of two finer than the previous one. The index $l$, which runs from 0 to $2^j - 1$ for each $j$, denotes the position of the wavelet basis $\psi_{j,l}$ within the $j$th scale. (Note that the properties of the basis functions are completely different from the basis functions in binning methods.)

We take the sample points $k_i$ to be equally spaced in $\log(k)$, and use 16 wavelet coefficients to represent $P_{\text{in}}(k)$ in the range $0.0001 \lesssim k/(\text{Mpc}^{-1}) \lesssim 0.1$, with the spectrum...
for $k < 0.0001 \text{ Mpc}^{-1}$ and $k > 0.1 \text{ Mpc}^{-1}$ set to be scale invariant. We use the Daubechies-4 wavelet\(^3\) here [8, 22]. In general, different discrete orthogonal wavelets will give similar results.

We obtain constraints on the wavelet coefficients of $P_{\text{in}}(k)$, as well as the usual cosmological parameters, from simulated data, using a Markov chain Monte Carlo (MCMC) technique to trace the full posterior probability distribution of all the parameters. We use CAMB (appropriately modified) and CosmoMC for our computations. We simulate cosmic variance limited CMB temperature power spectrum data in a flat $\Lambda$CDM universe with $H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$, $\Omega_b h^2 = 0.022$, $\Omega_c h^2 = 0.12$ and $\tau = 0$, out to an $l_{\text{max}}$ of 2000, with some functional form for the primordial power spectrum. The forms we have considered are given in the next section, each normalized to $1.8 \times 10^{-9}$. The samples of coefficients can be mapped onto corresponding power spectra and the confidence limits on the power spectra can be obtained at each $k_i$, or equivalently constraints on the wavelet coefficients can be inverted to obtain constraints on the power spectrum with inter-correlations taken into account.

3. Results

We consider three different functional forms for the input $P_{\text{in}}(k)$, the scale-invariant, saw-tooth, and Gaussian dip models. The scale-invariant model is given by

$$P_{\text{in}}(k) = 1.$$  

(3)

The Gaussian dip model is given by

$$P_{\text{in}}(k) = 1 - \alpha e^{-[\log(k/0.05)]^2/\beta},$$  

(4)

where $\alpha = 0.7$ and $\beta = 0.1$. The saw-tooth model is given by

$$P_{\text{in}}(k) = a_1, \quad k < k_1$$
$$= a_N, \quad k > k_N$$
$$= \frac{k_i - k}{k_i - k_{i-1}} a_{i-1} + \frac{k - k_{i-1}}{k_i - k_{i-1}} a_i, \quad k_{i-1} < k < k_i,$$

(5)

where $a_1 = 1, a_2 = 0.65, a_3 = 1.35$ and $a_N = 1$, with the $k_i$ stretching between 0.005 and 0.1 equally spaced in $\log(k)$. Such spectra have also been used in [9, 11]. Note that these spectra have been selected independently of the discretization of the recovered power spectra. What the method efficiently fits for is the correspondingly discretized input spectra.

The orthogonality of the wavelet basis functions in $k$ space is degraded due to the non-linear model-dependent mapping between $k$ and $l$ spaces and due to degeneracies with cosmological parameters. As a result, it is important to take into account the correlations between the constrained wavelet coefficients in deducing constraints on the primordial power spectrum. The correlations can be obtained with a sufficient number of MCMC samples. We obtained about $10^6$ samples for each case.

\(^3\) The number associated with the wavelet corresponds to the order of the wavelet; the larger the number the smoother the wavelet is and the better it is at representing smooth functions or higher order polynomials.
The $C_l$ spectra corresponding to these primordial power spectra are shown in figure 1. Corresponding $C_l$ observations are then simulated for the WMAP and Planck satellite missions. For the WMAP experiment three frequency channels are used with beam FWHM of 31.8, 21.0 and 13.8 arcmin, and pixel noise of 19.8, 30.0 and 45.6 $\mu$K, respectively. For the Planck experiment, three frequency channels with beam FWHM of 10.7, 8.0 and 5.5 arcmin, and pixel noise of 4.6, 5.4 and 11.7 $\mu$K, respectively, are used. For both experiments a sky fraction of 0.8 is assumed. The instrumental noise of these specifications together with cosmic variance uncertainties are used to simulate data (see for example [27]).

Figure 2 shows the input $P_{\text{in}}(k)$ (dashed curve), and mean power spectrum reconstructed from simulated $C_l$ data together with $2\sigma$ uncertainties (solid curves). The dot–dashed curves give the $5\sigma$ contours. This is shown for the scale-invariant, saw-tooth and Gaussian dip models of $P_{\text{in}}(k)$, for WMAP and Planck simulated observations of the CMB temperature anisotropy power spectrum. In each case the dashed curves shown are discretized versions of the corresponding continuous spectra. Figure 1 shows that the input $P_{\text{in}}(k)$ is accurately recovered by our method. The features are detected at high significance.

We have used CAMB with its original discretization in $k$ and $l$ spaces with interpolation in between; this gridding can be made smoother to improve the accuracy of reconstructions if needed.

When applied on future data, for example from the 4 year WMAP or Planck experiments, complemented by LSS power spectrum data, we recommend that our method be used as follows.

(a) Follow the procedure described in this paper to deduce constraints on $P_{\text{in}}(k)$.
(b) Look at constraints obtained from the larger (more significantly constrained) wavelet coefficients, i.e. those that deviate from zero at greater than say $1\sigma$, or $2\sigma$. If tentative
Figure 2. Each plot shows the input primordial power spectrum (dashed curve), and mean power spectrum reconstructed from simulated $C_l$ data together with $2\sigma$ uncertainties (solid curves) and the $5\sigma$ confidence contours (dot–dashed curves). These are shown for WMAP simulated observations in the (a) scale-invariant, (b) saw-tooth and (c) Gaussian dip cases, and similarly for Planck simulated observations in (d), (e) and (f).

Evidence of a feature is found in the thus-denoised $P_{in}(k)$ spectrum, i.e. if any wavelet coefficients other than the two that together indicate/reconstruct the amplitude of a scale-invariant function are found to be significantly non-zero, then there is indication of a feature or a deviation from scale-invariance.
(c) If step (b) finds the indication of a feature, one can use a parametric form for the kind of deviation from scale-invariance that is indicated, and repeat the likelihood analysis to better fit the feature and to obtain constraints on the relevant parameters. On the theoretical side, hopefully the feature will indicate something that is theoretically viable in certain scenarios so that it can be thus modelled. The relevant parametrization to use can also be derived in this way. If evidence calculations indicate that scale-invariance or power-law forms remain the best explanation for the data we will have achieved the task of checking model independently for this and ruling out or constraining more complex physics within inflationary models.

4. Conclusions

It is important to reconstruct the primordial power spectrum as a free function from cosmological data, the aim being to test the assumptions that are usually made about $P_{in}(k)$, and to constrain it model independently so as to be able to detect any features in it that may be signatures of new physics during inflation. Representing $P_{in}(k)$ by coefficients of its wavelet expansion is an effective and powerful way of approaching this problem as described above. We have demonstrated that input $P_{in}(k)$ is accurately recovered with this method. We have presented a prescription for how this method should be used on future data.

We have used CMB temperature data simulated according to WMAP and Planck specifications, with various forms for the primordial power spectra. Our results (see figure 2) indicate that future data hold great promise for probing the detailed shape of the primordial power spectrum.

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