A FOURIER TRANSFORM FOR HIGGS BUNDLES

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ABSTRACT. We define a Fourier-Mukai transform for Higgs bundles on smooth curves (over \( \mathbb{C} \) or another algebraically closed field) and study its properties. The transform of a stable degree-0 Higgs bundle is an algebraic vector bundle on the cotangent bundle of the Jacobian of the curve. We show that the transform admits a natural extension to an algebraic vector bundle over projective compactification of the base. The main result is that the original Higgs bundle can be reconstructed from this extension.

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1. INTRODUCTION

Higgs bundles on a compact Riemann surface \( X \) are pairs \((E, \theta)\) consisting of a holomorphic vector bundle \( E \) and a holomorphic one-form \( \theta \) with values in \( \text{End}(E) \) on \( X \). They originated essentially simultaneously in Nigel Hitchin’s study \([13]\) of dimensionally reduced self-duality equations of Yang-Mills gauge theory and, over general Kähler manifolds, in Carlos Simpson’s work \([26]\) on Hodge theory.

To explain Hitchin’s point of view, we consider solutions of the self-dual Yang-Mills equations on \( \mathbb{R}^4 \) that are invariant under translations in one or more directions in \( \mathbb{R}^4 \). Invariance in one direction reduces the SDYM equations to the Bogomolnyi equations describing magnetic monopoles in \( \mathbb{R}^3 \), while invariance in three directions leads to Nahm’s equation on \( \mathbb{R} \). Invariance in two directions leads to the conformally invariant Hitchin’s equations (or Higgs bundle equations), which the conformal invariance allows to be considered on compact Riemann surfaces. A

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solution to Hitchin’s equations has an interpretation as a triple \((E, \theta, h)\) with \(E\) a holomorphic vector bundle, \(\theta\) a holomorphic section of \(\text{End}(E) \otimes \omega_X\) and \(h\) a Hermitian metric on \(E\), satisfying \(F + [\theta, \theta^*] = 0\), where \(F\) is the curvature of the connection determined by the metric. It was shown in Hitchin \[13\] for rank-2 bundles on Riemann surfaces, and in Simpson \[26\] in general, that a pair \((E, \theta)\) admits a unique such metric precisely when \(E\) has vanishing Chern classes and the pair \((E, \theta)\) is stable in a sense which generalises the usual stability for vector bundles. For an excellent overview of Simpson’s viewpoint of non-Abelian cohomology, see Simpson \[28\].

An important class of transforms in Yang-Mills theory, including the ADHM construction and the Fourier transform for instantons (Donaldson-Kronheimer \[6\]) and the Nahm transform for monopoles, is based on using the kernel of the Dirac operator coupled to the (twisted) connection. A recent work of Marcos Jardim \[16, 17\] uses a version of the Nahm transform to establish a link between singular Higgs bundles on a 2-torus and doubly-periodic instantons. Our goal is to generalise this work to Riemann surfaces of genus \(g \geq 2\). In this paper we shall consider the purely holomorphic aspect of the transform; we plan to return to the properly gauge-theoretic questions in a future paper.

The holomorphic side of the Nahm transform is captured by the (generalised) Fourier-Mukai transform, which originated in the work of Shigeru Mukai \[22\] on Abelian varieties. Let \(X\) be a complex torus and \(\hat{X}\) its dual, and let \(D(X)\) and \(D(\hat{X})\) denote the derived categories of the categories of coherent sheaves on \(X\) and \(\hat{X}\) respectively. Using the Poincaré sheaf \(\mathcal{P}\) on \(X \times \hat{X}\), Mukai defined a functor \(M: D(X) \to D(\hat{X})\) by

\[ M(\cdot) = R\text{pr}_{\hat{X}}^*(\text{pr}_X^*(\cdot) \otimes \mathcal{P}), \]

and showed that it is a category equivalence. This construction can be generalised to any varieties \(X\) and \(Y\) together with a sheaf \(\mathcal{P}\) on \(X \times Y\). The properties of these generalisations have been studied by A. Bondal and D. Orlov \[1, 2\], A. Maciocia \[19\], T. Bridgeland \[4, 5\], and others.

We interpret the endomorphism-valued one-form \(\theta\) as a bundle map, making a Higgs bundle \((E, \theta)\) into a sheaf complex \(\mathcal{E} \to \mathcal{E} \otimes \omega_X\), where \(\omega_X\) is the canonical sheaf of \(X\). Hence a Higgs bundle gives us an object of the derived category \(D(X)\), and we can use the general machinery of Fourier-Mukai transforms. The key observation is that it is necessary to consider relative transforms of families of Higgs bundles twisted by adding a scalar term to the Higgs field \(\theta\). Our transform produces sheaves on \(J(X) \times H^0(X, \omega_X)\), where \(J(X)\) is the Jacobian of \(X\). This base can be identified with the cotangent bundle of the Jacobian.

While the motivation for the present work comes from differential and complex analytic geometry, we are actually working within the framework of algebraic geometry, noting that on an algebraic curve the Higgs bundle data is purely algebraic. Translation between these categories is provided by Serre’s GAGA \[25\]. Our approach has the advantage that all constructions are automatically algebraic (or holomorphic), while the fact that we are dealing with rather high-dimensional base spaces would make some of the analytic techniques of Jardim hard to use.

The first part of this paper develops the machinery of generalised Fourier-Mukai transforms. While some of the material presented in section 2 cannot be found in the literature, it is mostly well known. The main new contributions are the
definition of a relative Fourier transform for curves and the reduction of it to the original Mukai transform.

The transform for Higgs bundles is developed in section 3. The first interesting application is that our Fourier transform takes stable Higgs bundles of degree zero to vector bundles on $J(X) \times H^0(X, \omega_X)$. Our approach has the advantage of giving directly an algebraic (holomorphic) extension of this bundle over the projective completion $J(X) \times \mathbb{P}(H^0(X, \omega_X) \oplus \mathbb{C})$ of the base space, without a need to separately compactify a bundle on $J(X) \times H^0(X, \omega_X)$. Denote this extension of the transform of a Higgs bundle $E = (E, \theta)$ by $TF(E)$. The main theorem of this paper is the following:

**Inversion Theorem (3.2.1).** — Let $E$ and $F$ be two Higgs bundles on a curve $X$ of genus $g \geq 2$. If $TF(E) \cong TF(F)$, then $E \cong F$ as Higgs bundles.

We in fact prove this theorem by exhibiting a procedure for recovering a Higgs bundle from its transform. Furthermore, it follows easily from the theorem that the transform functor is in fact fully faithful.

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**Notation and conventions.** Unless otherwise specified, all rings and algebras are commutative and unital. We fix an algebraically closed field $k$. All schemes are assumed to be of finite type over $k$. All morphisms are $k$-morphisms and all products are products over Spec($k$) unless stated otherwise. A curve always means a smooth irreducible complete (i.e., projective) curve over $k$. If $\mathcal{F}$ is an $\mathcal{O}_X$-module, $\mathcal{F}^\vee$ denotes its dual $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. The canonical sheaf of a curve $X$ is denoted by $\omega_X$.

$D(X)$ denotes the derived category of the (Abelian) category of $\mathcal{O}_X$-modules. $D^{-}(X)$ (resp. $D^{+}(X)$, resp. $D^b(X)$) is the full subcategory of objects cohomologically bounded above (resp. bounded below, resp. bounded). $D_{qcoh}(X)$ and $D_{coh}(X)$ are the full subcategories of objects with quasi-coherent and coherent cohomology objects, respectively. These superscripts and subscripts can be combined in the obvious way. The category of $\mathcal{O}_X$-modules is denoted by $\text{Mod}(X)$, and $\text{Qcoh}_X$ is the thick subcategory of quasi-coherent sheaves.

A commutative square

$$
\begin{array}{ccc}
Z & \xrightarrow{v} & X \\
\downarrow{g} & & \downarrow{f} \\
Y & \xrightarrow{u} & S \\
\end{array}
$$

is called Cartesian if the mapping $(v, g)_S: Z \to X \times_S Y$ is an isomorphism. We denote canonical isomorphisms often by "\isom".

**2. Fourier-Mukai Transforms**

We develop here the general machinery of Fourier-Mukai transforms that will be necessary for our application to Higgs bundles.
We will be using the theory of derived categories; the main reference to derived categories in algebraic geometry remains Hartshorne’s seminar [10] on Grothendieck’s duality theory. Further references include Gelfand-Manin [7], Kashiwara-Shapira [18] and Weibel [30]. For a nice informal introduction, see Illusie [15] or the introduction of Verdier’s thesis [29].

2.1. A base change result.

(2.1.1) Consider the following diagram of schemes (here not necessarily of finite type over a field):

\[
\begin{array}{ccc}
X_1 \times_S X_2 & \xrightarrow{p_1} & X_1 \\
\downarrow f & & \downarrow p_1 \\
Y_1 \times_S Y_2 & \xrightarrow{q_1} & Y_1 \\
\end{array}
\]

with \( f = f_1 \times_S f_2 \). Recall the external tensor product over \( S \) of an \( \mathcal{O}_{X_1} \)-module \( \mathcal{F}_1 \) and an \( \mathcal{O}_{X_2} \)-module \( \mathcal{F}_2 \):

\[
\mathcal{F}_1 \boxtimes_S \mathcal{F}_2 = (p_1^* \mathcal{F}_1) \otimes_{\mathcal{O}_{X_1} \times_S X_2} (p_2^* \mathcal{F}_2).
\]

We get the corresponding left-derived bifunctor

\[
(\bullet) \boxtimes_S (\bullet) : D^- (X_1) \times D^- (X_2) \rightarrow D^- (X_1 \times_S X_2).
\]

The following theorem should be part of folklore; we include a proof of it for the lack of a suitable reference. It is essentially the derived-category version of a part of Grothendieck’s theory of "global hypertor functors" (EGA III [1], §6).

**Theorem (2.1.2) (Künneth formula).** — For \( i = 1, 2 \) let \( \mathcal{F}_i \) be an object of \( D^-_{qcoh}(X_i) \). Assume that the schemes are Noetherian and of finite dimension, and that the \( f_i \) are separated. Then

\[
(Rf_{i*} \mathcal{F}_i)^L \boxtimes_S (Rf_{2*} \mathcal{F}_2) = Rf_{1*} \left( \mathcal{F}_1 \boxtimes_S \mathcal{F}_2 \right)
\]

if either \( \mathcal{F}_1 \) or \( \mathcal{F}_2 \) is quasi-isomorphic to a complex of \( S \)-flat sheaves. This is true in particular if either \( X_1 \) or \( X_2 \) is flat over \( S \).

**Proof.** The Noetherian and dimensional hypotheses guarantee that the derived direct images are defined for complexes not bounded below. There are natural "adjunction" maps \( 1 \rightarrow Rf_{1*}Lf_{1*} \) giving

\[
(Rf_{1,*} \mathcal{F}_1)^L \boxtimes_S (Rf_{2*} \mathcal{F}_2) \rightarrow Rf_{*}Lf_{*}^* \left( (Rf_{1*} \mathcal{F}_1)^L \boxtimes_S (Rf_{2*} \mathcal{F}_2) \right).
\]
Notice that
\[
\mathbf{L}f^\ast \left( \mathbf{R}f_{1\ast} \mathcal{F}_1 \boxtimes_S (\mathbf{R}f_{2\ast} \mathcal{F}_2) \right) = (\mathbf{L}f^\ast \mathbf{L}q_1^\ast \mathbf{R}f_{1\ast} \mathcal{F}_1) \boxtimes (\mathbf{L}f^\ast \mathbf{L}q_2^\ast \mathbf{R}f_{2\ast} \mathcal{F}_2)
\]
\[
= (\mathbf{L}p_1^\ast \mathbf{L}f_{1\ast} \mathbf{R}f_{1\ast} \mathcal{F}_1) \boxtimes (\mathbf{L}p_2^\ast \mathbf{L}f_{2\ast} \mathbf{R}f_{2\ast} \mathcal{F}_2).
\]

Now the adjunctions \( \mathbf{L}f_{1\ast} \mathbf{R}f_{1\ast} \rightarrow 1 \) give a natural map
\[
(\mathbf{L}p_1^\ast \mathbf{L}f_{1\ast} \mathbf{R}f_{1\ast} \mathcal{F}_1) \boxtimes (\mathbf{L}p_2^\ast \mathbf{L}f_{2\ast} \mathbf{R}f_{2\ast} \mathcal{F}_2) \rightarrow (\mathbf{L}p_1^\ast \mathcal{F}_1) \boxtimes (\mathbf{L}p_2^\ast \mathcal{F}_2)
\]
\[
= \mathcal{F}_1 \boxtimes_S \mathcal{F}_2.
\]

Composing gives us a natural transformation
\[
(\mathbf{R}f_{1\ast} \mathcal{F}_1) \boxtimes_S (\mathbf{R}f_{2\ast} \mathcal{F}_2) \rightarrow \mathbf{R}f_s \left( \mathcal{F}_1 \boxtimes_S \mathcal{F}_2 \right).
\]

Whether this is an isomorphism is a local question; hence we may assume that
\( S = \text{Spec}(A) \) and \( Y_i = \text{Spec}(B_i) \) for \( i = 1, 2 \).

Suppose \( \mathcal{F}_1 \) is quasi-isomorphic to a complex of \( S \)-flat sheaves; replace \( \mathcal{F}_1 \) with this flat resolution. Then \( \mathcal{F}_1 \boxtimes_S \mathcal{F}_2 = \mathcal{F}_1 \boxtimes_S \mathcal{F}_2 \).

For \( i = 1, 2 \) let \( U_i = (U, F_i) \) be an affine open cover of \( X_i \). Let \( U \) denote the open affine cover \( (U, F_i) \) of \( X \). Notice that in all these covers arbitrary intersections of the covering sets are affine. Let \( \mathcal{C}^\ast(U_i, \mathcal{F}_i) \) denote the simple complex associated to the Cech double complex of \( \mathcal{F}_i \) with respect to \( U_i \). Similarly, let \( \mathcal{C}^\ast(U, \mathcal{F}_1 \boxtimes_S \mathcal{F}_2) \) be the Cech complex associated to \( U \).

Now \( \mathbf{R}\Gamma(X, \mathcal{F}_i) \) is quasi-isomorphic to \( \mathcal{C}^\ast(U_i, \mathcal{F}_i) \), and hence \( \mathbf{R}f_{1\ast} \mathcal{F}_i \) is quasi-isomorphic to \( \mathcal{C}^\ast(U_i, \mathcal{F}_i) \). But the sheaves of these complexes are \( S \)-flat by construction, whence
\[
(\mathbf{R}f_{1\ast} \mathcal{F}_1) \boxtimes_S (\mathbf{R}f_{2\ast} \mathcal{F}_2) = (\mathcal{C}^\ast(U_1, \mathcal{F}_1) \boxtimes (\mathcal{C}^\ast(U_2, \mathcal{F}_2)))^\wedge.
\]

Similarly
\[
\mathbf{R}f_s \left( \mathcal{F}_1 \boxtimes_S \mathcal{F}_2 \right) = (\mathcal{C}^\ast(U, \mathcal{F}_1 \boxtimes_S \mathcal{F}_2))^\wedge.
\]

Hence we are reduced to showing that the complex \( \mathcal{C}^\ast(U_1, \mathcal{F}_1) \boxtimes (\mathcal{C}^\ast(U_2, \mathcal{F}_2)) \) is quasi-isomorphic to \( \mathcal{C}^\ast(U, \mathcal{F}_1 \boxtimes_S \mathcal{F}_2) \). But this is showed in the proof of (6.7.6) of EGA III [5].

**Remark (2.1.3).** — If one wants to avoid the Noetherian hypothesis in the theorem, one could work with objects \( \mathcal{F}_i \) of \( \mathbf{D}^-(\Omega \mathcal{Coh}(X)) \) and require the \( f_i \) to be quasi-compact. This is essentially the viewpoint of EGA III.

**Corollary (2.1.4).** — Let \( f : X \rightarrow S \) and \( g : T \rightarrow S \) be morphisms of finite-dimensional Noetherian schemes. Let \( f' : X \times_S T \rightarrow T \) and \( g' : X \times_S T \rightarrow X \) be the projections, and let \( \mathcal{F} \) belong to \( \mathbf{D}^-_{\mathcal{Qcoh}}(X) \).

1. If \( \mathcal{F} \) is quasi-isomorphic to a complex of \( S \)-flat sheaves (in particular, if \( f \) is flat), then
\[
\mathbf{L}g^\ast \mathbf{R}f_s \mathcal{F} = \mathbf{R}f'_s \mathbf{L}g'^\ast \mathcal{F}.
\]
(2) If $g$ is flat, then
\[ g^* Rf_* \mathcal{F} = Rf_1^* g^* g^* Rf_* \mathcal{F}. \]

Proof. Apply the Künneth formula with $X_1 = X$, $Y_1 = S$, $f_1 = f$, $X_2 = Y_2 = T$, $f_2 = 1_T$, $\mathcal{F}_1 = \mathcal{F}$ and $\mathcal{F}_2 = \mathcal{O}_T$. \hfill \square

2.2. Integral transforms.

Definition (2.2.1). — Let $S$ be a separated $k$-scheme and let $X$ and $Y$ be flat $S$-schemes. If $P$ is an object of $D^b_{coh}(X \times_SY)$, the relative integral transform defined by $P$ is the functor $\Phi_P^{\mathcal{X}}: D^+(X) \to D^+(Y)$ given by
\[ \Phi_P^{\mathcal{X}}(\bullet) = R\text{pr}_2^* \left( \text{pr}_1^* (\bullet) \otimes \mathcal{P} \right), \]
where $\text{pr}_1$ and $\text{pr}_2$ are the canonical projections of $X \times_SY$. When $S = \text{Spec}(k)$ we call the transform the absolute integral transform and denote it by $\Phi_P^{\mathcal{X}}$.

Proposition (2.2.2). — Let $i: X \times_SY \to X \times_k Y$ be the morphism $(\text{pr}_1, \text{pr}_2)_k$. Then $Ri_* = i_*$ and
\[ \Phi_P^{\mathcal{X}}(\bullet) = \Phi_{i_*}^{\mathcal{P}}(\bullet). \]

Proof. We have the commutative diagram
\[
\begin{array}{ccc}
X \times_SY & \xrightarrow{i} & X \\
\text{pr}_1 \downarrow & & \text{pr}_2 \downarrow \\
X & \xrightarrow{i} & Y \\
p \downarrow & & q \downarrow \\
X \times_k Y & \xrightarrow{\Delta_{S/k}} & S \times_k S.
\end{array}
\]

Notice that because both $\text{pr}_1$ and $p$ are flat morphisms, we have
\[ \text{pr}_1^* = \mathcal{L} \text{pr}_1^* = \mathcal{L}(i^* \circ p^*) = \mathcal{L}i^* \circ \mathcal{L}p^* = \mathcal{L}i^* \circ p^*. \]

Using this and the projection formula, we have
\[ \Phi_P^{\mathcal{X}}(\bullet) = R\text{pr}_2^* (\text{pr}_1^* (\bullet) \otimes \mathcal{P}) \]
\[ = R\text{q}_s R\text{i}_* (\text{Li}^* (p^* (\bullet)) \otimes \mathcal{P}) \]
\[ = R\text{q}_s (p^* (\bullet) \otimes R\text{i}_* \mathcal{P}). \]

But $i$ fits in a Cartesian square
\[
\begin{array}{ccc}
X \times_SY & \xrightarrow{i} & X \times_k Y \\
\downarrow & & \downarrow \\
S & \xrightarrow{\Delta_{S/k}} & S \times_k S.
\end{array}
\]

As $S/k$ is separated, $\Delta_{S/k}$ is a closed immersion, and consequently so is $i$. In particular, $i_*$ is an exact functor and therefore equal to $R\text{i}_*$. Hence
\[ \Phi_P^{\mathcal{X}}(\bullet) = R\text{q}_s (p^* (\bullet) \otimes i_* \mathcal{P}) = \Phi_{i_*}^{\mathcal{P}}(\bullet) \]
as claimed. \hfill \square
Remark (2.2.3). — We cannot avoid using the derived tensor product in the above result, even if $\mathcal{P}$ is a locally free sheaf, because $i_x \mathcal{P}$ is not flat in general. However, as $i$ is proper, $i_x \mathcal{P}$ belongs always to $\mathcal{D}_c(X \times Y)$.

(2.2.4) For flat $S$-schemes $X$ and $Y$ and for $x \in X$, let $Y_x$ denote the fibre $\text{pr}_1^{-1}(x)$, where $\text{pr}_1 : X \times_S Y \to X$ is the canonical projection. We have then a commutative diagram

$$
\begin{array}{ccc}
Y_x & \xrightarrow{j} & X \times_S Y \\
\downarrow & & \downarrow \text{pr}_1 \\
\kappa(x) & \xrightarrow{} & X \\
\end{array}
$$

in which all squares are Cartesian. Let $i$ denote the composition of the top arrows. For an object $\mathcal{F}$ of $\mathcal{D}_{coh}(X \times_S Y)$ (resp. $\mathcal{D}_{coh}(Y)$), we denote by $\mathcal{F}_x$ the “restriction” $Lj^* \mathcal{F}$ (resp. $Li^* \mathcal{F}$) to $Y_x$. For complexes of locally free sheaves these are just ordinary restrictions to $Y_x$. If $\mathcal{P}$ is a locally free sheaf on $X \times S$, then for each $x \in X$

$$
\Phi^\mathcal{P}_{X \to Y/S}(k(x)) = i_x \mathcal{P}_x,
$$

where $k(x)$ is the skyscraper sheaf $k$ at $x$. Indeed, consider the commutative diagram above: the claim follows from flat base change around the left-hand square and the projection formula applied to $j$. Notice that $i_x$ is exact.

Example (2.2.5). — Let $X$ be an Abelian variety, $\hat{X}$ its dual, and let $S$ be a separated scheme. Let $\mathcal{P}$ be the Poincaré sheaf on $X \times \hat{X}$, normalised so that both $\mathcal{P}|_{X \times \{0\}}$ and $\mathcal{P}_{\{0\} \times \hat{X}}$ are the trivial line bundles. Denote by $\mathcal{P}_S$ the pull-back of this Poincaré sheaf to $X \times \hat{X} \times S = (X \times S) \times_S (\hat{X} \times S)$. The relative Mukai transform functor $M_S : \mathcal{D}_{coh}(X \times S) \to \mathcal{D}_{coh}(\hat{X} \times S)$ is the relative integral transform functor $\Phi^\mathcal{P}_{(X \times S) \to (\hat{X} \times S)/S}$. If $S = \text{Spec}(k)$, we denote the transform by $M$.

The following theorem of Mukai plays a crucial role in the proof of our invertibility result (2.2.1).

Theorem (2.2.6). — If $S$ is a smooth projective variety, then the relative Mukai transform $M_S$ is an equivalence of categories from $\mathcal{D}_{coh}(X \times S)$ to $\mathcal{D}_{coh}(\hat{X} \times S)$.

Proof. See Mukai [23]. The proof is a generalisation of Mukai’s original proof of this result for the absolute transform $M$ in [22].

Proposition (2.2.7). — Let $X$ and $Y$ be flat $S$-schemes and $\mathcal{P}$ an object of $\mathcal{D}_{coh}(X \times S)$. Let $u : T \to S$ be a morphism of schemes. Let $i_X : X(T) \to X$, $i_Y : Y(T) \to Y$, and $j : (X \times_S Y)(T) = X(T) \times_T Y(T) \to X \times_Y Y$ be the canonical projections. Then

$$
Li^*_Y \circ \Phi^\mathcal{P}_{X/Y/S} = \Phi^\mathcal{P}_{X(T)/Y(T)/T} \circ Li^*_X.
$$

Moreover, if $u$ is a flat morphism, then all derived pull-backs above can be replaced with normal pull-backs.
Proof. Consider the commutative diagram

\[
\begin{array}{cccccc}
X_{(T)} \times_Y Y_{(T)} & \overset{p}{\longrightarrow} & X_{(T)} \\
\downarrow q & & \downarrow \pi_X \\
X \times_S Y & \overset{pr_1}{\longrightarrow} & X \\
\downarrow pr_2 & & \downarrow \pi_Y \\
Y & \overset{u}{\longrightarrow} & S \\
\uparrow i_Y & & \uparrow \pi_S \\
Y_{(T)} & \overset{u}{\longrightarrow} & T
\end{array}
\]

It is immediate that all squares are Cartesian. If \(u\) is flat, then so are \(i_X, i_Y\) and \(j\); this proves the claim about replacing derived pull-backs. Since in any case \(X/S\) is flat, \(pr_2\) is also flat. So by (2.1.4) we can do a base change around the leftmost square. We get

\[
\begin{align*}
Li_Y^* \Phi_{X \to Y/S}^p(\mathcal{E}) &= Li_Y^* R pr_{2*} (pr_1^* \mathcal{L} \otimes \mathcal{P}) \\
&= Rq_* Li_X^* (pr_1^* \mathcal{L} \otimes \mathcal{P}) \\
&= Rq_* (Lj^* (pr_1^* \mathcal{L}) \otimes Lj^* \mathcal{P}) \\
&= Rq_* (p^* Li_X^* \mathcal{L} \otimes Lj^* \mathcal{P}) = \Phi_{X_{(T)} \to Y_{(T)}/T}^{Lj^* \mathcal{P}}(Li_X^* \mathcal{L}).
\end{align*}
\]

□

Proposition (2.2.8). — Let \(X\) and \(Y\) be flat \(S\)-schemes and \(\mathcal{P}\) an object of \(D_{coh}^b(X \times_S Y)\). Then

\[
R \Gamma(Y, \Phi_{X \to Y/S}^p(\mathcal{E})) = R \Gamma(X, \mathcal{L} \otimes Rpr_{1*} \mathcal{P}).
\]

Proof. We simply use the composition property of derived functors and the projection formula:

\[
\begin{align*}
R \Gamma(Y, \Phi_{X \to Y/S}^p(\mathcal{E})) &= R \Gamma(Y, R pr_{2*} (pr_1^* \mathcal{L} \otimes \mathcal{P})) \quad \text{(by definition)} \\
&= R \Gamma(X \times_S Y, pr_1^* \mathcal{L} \otimes \mathcal{P}) \quad \text{(composition)} \\
&= R \Gamma(X, Rpr_{1*} (pr_1^* \mathcal{L} \otimes \mathcal{P})) \quad \text{(composition)} \\
&= R \Gamma(X, \mathcal{L} \otimes Rpr_{1*} \mathcal{P}) \quad \text{(by projection formula)}.
\end{align*}
\]

□

2.3. WIT complexes.

Notation (2.3.1). — Let \(X\) and \(Y\) be proper flat \(S\) schemes. We fix a locally free sheaf \(\mathcal{P}\) on \(X \times_S Y\), and denote by \(F_S\) the relative integral transform functor \(\Phi_{X \to Y/S}^p : D_{coh}^b(X) \to D_{coh}^b(Y)\). We leave it to the reader to generalise the results of this subsection to a more general setting.
**Definition (2.3.2).** — We say that an object $\mathcal{E}$ of $\mathrm{D}_{\mathrm{coh}}^b(X)$ is a WIT$_{\mathcal{P}}(n)$-complex$^1$ if $H^p(F_S(\mathcal{E})) = 0$ for all $p \neq n$. If $\mathcal{P}$ is clear from the context, we shall omit the explicit reference to it. An object of $\mathrm{D}_{\mathrm{coh}}^b(X)$ is a WIT-complex if it is a WIT$(n)$-complex for some $n$.

If $\mathcal{E}$ is a WIT$(n)$-complex on $X$, the (coherent) sheaf $H^n(F_S(\mathcal{E}))$ on $Y$ is called the integral transform of $\mathcal{E}$, and is denoted by $\hat{\mathcal{E}}$.

**Definition (2.3.3).** — We say that an object $\mathcal{E}$ of $\mathrm{D}_{\mathrm{coh}}^b(X)$ is an IT$_{\mathcal{P}}(n)$-complex$^2$ if for each (closed) point $y \in Y$ and each $p \neq n$ we have

$$H^p(X_y, \mathcal{E}_y \otimes \mathcal{P}_y) = 0,$$

where we are using the notation of (2.2.4) for $\mathcal{E}_y$, $\mathcal{P}_y$ and $X_y$.

**Lemma (2.3.4).** — Let $f : X \to Y$ be a proper morphism of (Noetherian) schemes and let $\mathcal{E}$ be an object of $\mathrm{D}_{\mathrm{coh}}^b(X)$ which has a $Y$-flat resolution. Let $y \in Y$. Then:

1. if the natural map $\varphi^p(y) : R^p f_*(\mathcal{E}) \otimes \kappa(y) \to H^p(X_y, \mathcal{E}_y)$ is surjective, then it is an isomorphism.
2. If $\varphi^p(y)$ is an isomorphism, then $\varphi^{p-1}$ is also an isomorphism if and only if $R^p f_*(\mathcal{E})$ is free in a open neighbourhood of $y$.

**Proof.** This follows from EGA III [9] §7. However, that part of EGA can be somewhat hard to read; one could also follow the simpler proof of Hartshorne [11] Theorem III.12.11, making the fairly minor and obvious adjustments for hypercohomology. \hfill $\square$

**Proposition (2.3.5).** — Let $\mathcal{E}$ be an IT$(n)$ complex. Then $\mathcal{E}$ is a WIT$(n)$-complex, and $\hat{\mathcal{E}}$ is locally free on $Y$.

**Proof.** Our schemes are Jacobson, and so it suffices to restrict our attention to closed points. Since $\text{pr}_2$ is flat, $\text{pr}_1^* \mathcal{E}$ is quasi-isomorphic to a complex of sheaves flat over $Y$. Moreover, $X$ is proper over $S$, and so $\text{pr}_2$ is a proper morphism. We are then in position to use (2.3.4). Let $y \in Y$ be a closed point. Now

$$(\text{pr}_1^* \mathcal{E} \otimes \mathcal{P})_y \cong \mathcal{E}_y \otimes \mathcal{P}_y$$
on $(X \times_S Y)_y = X_y$. Hence by hypothesis the natural map

$$\varphi^p(y) : R^p \text{pr}_2^*(\text{pr}_1^* \mathcal{E} \otimes \mathcal{P}) \otimes \kappa(y) \to H^p(X_y, (\text{pr}_1^* \mathcal{E} \otimes \mathcal{P})_y)$$

is trivially surjective, and by the base change theorem in fact isomorphic, for all $p \neq n$. As the hyper direct images of a complex of coherent sheaves are coherent for a proper map, we have

$$R^p \text{pr}_2^*(\text{pr}_1^* \mathcal{E} \otimes \mathcal{P}) = 0$$

for $p \neq n$ by Nakayama’s lemma. This proves the first part of the proposition.

Now in particular $R^{n+1} \text{pr}_2^*(\text{pr}_1^* \mathcal{E} \otimes \mathcal{P}) = 0$. Thus, by the second part of the base change theorem, $\varphi^n(y)$ is an isomorphism. But as $\varphi^{n-1}(y)$ is also surjective and thus isomorphic, $R^n \text{pr}_2^*(\text{pr}_1^* \mathcal{E} \otimes \mathcal{P})$ is free in a neighbourhood of $y$, again by the second part of (2.3.4). \hfill $\square$

---

$^1$Following Mukai, "WIT" stands for "weak index theorem".

$^2$"IT" stands for "Index theorem".
Proposition (2.3.6). — Let \( X, Y \) and \( S \) be as in (2.3.1), and let \( u: T \to S \) be a morphism of schemes. Suppose that \( \mathcal{E} \) is an IT\((n)\)-complex on \( X \). Then, in the notation of (2.2.7), \( \text{Li}_X \mathcal{E} \) is a WIT\((n)\)-complex with respect to the pull-back \( j^* \mathcal{P} \) of \( \mathcal{P} \) to \( (X \times_S Y)_{(T)} \). Furthermore, if \( \text{Li}_X \mathcal{E} \) denotes the corresponding Fourier transform, then

\[
i_Y^* \left( \mathcal{E} \right) = \text{Li}_X \mathcal{E}.
\]

Proof. By the assumptions and (2.3.5), \( \Phi_{X \to Y/S}(\mathcal{E}) \) is a locally free sheaf shifted \( n \) places to the right. Hence (2.2.7) gives

\[
i_Y^* \left( \Phi_{X \to Y/S}(\mathcal{E}) \right) = \Phi_{X \to Y/S}^*(\text{Li}_X \mathcal{E}).
\]

But this shows that \( \Phi_{X \to Y/S}^*(\text{Li}_X \mathcal{E}) \) is also a locally free sheaf shifted \( n \) places to the right. Both statements of the proposition are now immediate. \( \square \)

2.4. Fourier transform for curves. To fix terminology and notation, we first recall some basic facts about Jacobians of curves; for details, see Milne [20, 21].

Notation (2.4.1). — Let \( X \) be a smooth projective curve of genus \( g \). We denote by \( J(X) \) a Jacobian of \( X \), i.e., a scheme representing the functor \( T \mapsto \text{Pic}^0(\mathcal{O}_X/T) \). Let \( \mathcal{M} \) be the corresponding universal sheaf on \( X \times J(X) \). Recall that \( J(X) \) is an Abelian variety of dimension \( g \); let \( J(X) \) denote its dual Abelian variety, and let \( \mathcal{P} \) be the Poincaré sheaf on \( J(X) \times J(X) \), normalised as in (2.2.5).

(2.4.2) Choosing a base point \( P \in X \) gives the Abel-Jacobi map \( i_P: X \to J(X) \), taking the base point to 0. Notice that \( i_P \) is a closed immersion. Furthermore, this choice gives \( J(X) \) a principal polarisation and hence an isomorphism \( \varphi_P: J(X) \to J(X) \), which we use henceforth to identify \( J(X) \) with its dual. Under this identification, the pull-back \( (i_P \times 1_{J(X)})^* \mathcal{P} \) is just the universal sheaf \( \mathcal{M} \) on \( X \times J(X) \).

(2.4.3) Let \( S \) be a separated \( k \)-scheme, \( X_S = X \times S \), and let \( J(X)_S = J(X) \times S \) be the relative Jacobian of the trivial family \( X_S \). We have a Cartesian square

\[
\begin{array}{c}
\text{X} \times 
\overset{\text{pr}_2}{\longrightarrow} 
\text{J}(X)_S \\
\downarrow \text{pr}_1 \\
\text{X}_S \\
\downarrow \\
\text{S}.
\end{array}
\]

Let \( \mathcal{M}_S \) be the pull-back of \( \mathcal{M} \) to \( X \times J(X) \times S \). The relative integral transform functor \( \Phi_{\text{X}_S \to J(X)_S/S}^\#: \mathcal{D}_{\text{coh}}^b(\text{X}_S) \to \mathcal{D}_{\text{coh}}^b(\text{J}(X)_S) \) is given by

\[
\Phi_{\text{X}_S \to J(X)_S/S}^\#(\bullet) = \text{Rpr}_2_*(\text{pr}_1^*(\bullet) \otimes \mathcal{M}_S),
\]

where we can use the ordinary tensor product since \( \mathcal{M}_S \) is locally free.

Definition (2.4.4). — The relative integral transform \( \Phi_{\text{X}_S \to J(X)_S/S}^\# \) is called the relative Fourier functor on \( X \times S \) and is denoted by \( \Phi_S \). If \( \mathcal{E} \) is WIT with respect to \( \Phi_S \), the integral transform \( \mathcal{E} \) is called the Fourier transform of \( \mathcal{E} \).
Proposition (2.4.5). — Let $M_S : D^b_{coh}(J(X) \times S) \to D^b_{coh}(J(X) \times S)$ denote the relative Mukai transform. Then

$$F_S = M_S \circ (i_p \times 1_S)_*.$$  

Proof. Consider the diagram

$$\begin{array}{cccc}
X_S \times_S J(X)_S & \xrightarrow{j} & J(X)_S \times J(X)_S & \xrightarrow{p_2} & J(X)_S \\
\downarrow{pr_1} & & \downarrow{p_1} & & \\
X_S & \xrightarrow{i_p \times 1_S} & J(X)_S & \longrightarrow & S,
\end{array}$$

where the right-hand square is the fibre-product diagram and $j = (i_p \times 1_S) \times 1_{J(X)_S}$.

It is clear that the left-hand square is also commutative, and that the composition of the two top arrows is just the canonical projection $pr_2$. But this means that the big rectangle is Cartesian, and hence so is the left-hand square too.

By definition,

$$M_S(\bullet) = R_{P_{2*}} (p_2^* (\bullet) \otimes \mathcal{P}_S),$$

where $\mathcal{P}_S$ is the pull-back of the Poincaré sheaf onto $J(X)_S \times_S J(X)_S$. Clearly $\mathcal{M}_S = j^* \mathcal{P}_S$. Now by the projection formula

$$R j_*(\bullet \otimes \mathcal{M}_S) = R j_*(\bullet) \otimes \mathcal{P}_S.$$

Because $p_1$ is flat as a base extension of a flat morphism, we can do a base change (2.1.4) around the left-hand square to get

$$p_1^* \circ R(i_p \times 1_S)_* = R j_* \circ pr_1^*.$$  

But $i_p \times 1_S$ is a closed immersion and thus $R(i_p \times 1_S)_* = (i_p \times 1_S)_*$. Putting these observations together, we get

$$M_S ((i_p \times 1_S)_*(\bullet)) = R p_{2*} (p_2^* ((i_p \times 1_S)_*(\bullet)) \otimes \mathcal{P}_S)$$

$$= R p_{2*} (R j_* (pr_1^* (\bullet)) \otimes \mathcal{P}_S)$$

$$= R p_{2*} (R j_* (pr_1^* (\bullet) \otimes \mathcal{M}_S))$$

$$= R pr_{2*} (pr_1^* (\bullet) \otimes \mathcal{M}_S) = F_S(\bullet).$$

\[ \square \]

Proposition (2.4.6). — Let $X$ be a curve of genus $g$ and choose a base point $P \in X$ as in (2.4.2); we suppose made the identifications given loc. cit. Let $S$ be a $k$-scheme, and denote by $j$ the embedding $S \cong (X \times S)_P \to X \times S$ of the fibre over $P$. Let $\mathcal{E}^\bullet$ be a bounded complex of locally free sheaves on $X \times S$. Then

$$H^p(J(X) \times S, F_S(\mathcal{E}^\bullet)) = \bigoplus_{i=1}^g H^{p-i}(S_p, j^* \mathcal{E}^\bullet)^{\otimes (-i)}.$$

Proof. By (2.2.8) we have natural isomorphisms

$$H^p(J(X) \times S, F_S(\mathcal{E}^\bullet)) = H^p(X \times S, \mathcal{E}^\bullet \otimes R pr_{1*} \mathcal{M}_S)$$

for all $p$.

Lemma (2.4.6.1). — With the notation of the proposition, $R pr_{1*} \mathcal{M}_S$ is the zero-differential complex $\mathcal{E}^\bullet$ where $\mathcal{E}^i$ is the direct sum of $\binom{g-1}{i-1}$ copies of $j_* \mathcal{O}_S$ for $1 \leq i \leq g$, zero otherwise.
Consider the Cartesian square
\[
\begin{array}{c}
X \times S \times J(X) \xrightarrow{\rho'} X \times J(X) \\
\downarrow p_1 \quad \quad \downarrow q \\
X \times S \xrightarrow{p} X.
\end{array}
\]
By flat base change around the square we get
\[
R_{pr_1}^* \mathcal{M}(S) = R_{pr_1}^* p'^* \mathcal{M} = p^* R_{q*} \mathcal{M}.
\]
In order to compute \( R_{q*} \mathcal{M} \) on \( X \), we consider the Cartesian square
\[
\begin{array}{c}
X \times J(X) \xrightarrow{i \times 1} J(X) \times J(X) \\
\downarrow q \quad \quad \downarrow \pi_i \\
X \xrightarrow{i_P} \pi_1.
\end{array}
\]
Now by the general base-change (2.1.4) we have
\[
R_{q*} \mathcal{M} = R_{q*} (i_P \times 1)^* \mathcal{P} = L_{i_P}^* R_{\pi_1} \mathcal{P}.
\]
But \( R_{\pi_1} \mathcal{P} = k(0)[-g] \), the skyscraper sheaf at 0 shifted \( g \) places to the right (see the proof of the theorem of §13 in Mumford [24]). Notice that \( i_P \) is a regular embedding; using Koszul resolutions it follows that \( L_{i_P}^* i_P^* \mathcal{O}_X = \bigwedge^* \mathcal{N}_{J(X)/X} \), the zero-differential exterior-algebra complex of the conormal sheaf of \( X \) in \( J(X) \), concentrated in degrees \(-g+1\) to 0. Similarly \( L_P^* k(0) = \bigwedge^* \mathcal{N}_{J(X)/X}(P) \), the exterior algebra of the fibre at \( P \), whence the lemma follows immediately taking into account the shift by \(-g\).

Using the projection formula we have
\[
H^p(X \times S, \mathcal{E}^\bullet \otimes j_* \mathcal{O}_{S_P}) = H^p(S_P, \mathcal{E}^\bullet).
\]
The proposition now follows from the lemma because hypercohomology commutes with direct sums. \( \square \)

3. Transforms of Higgs bundles

We shall now apply the Fourier-transform machinery developed in the previous section to stable Higgs bundles on curves.

3.1. Definitions and basic properties.

Definition (3.1.1). — A Higgs bundle on a smooth projective curve is a pair \( E = (\mathcal{E}, \theta) \), where \( \mathcal{E} \) is a locally free sheaf on \( X \), and \( \theta \) is a morphism \( \mathcal{E} \rightarrow \mathcal{E} \otimes \omega_X \). The morphism \( \theta \) is often called the Higgs field. The Higgs bundle \( \mathcal{O}_X \xrightarrow{0} \omega_X \) is called trivial.

The rank and degree (i.e., the first Chern class) of a Higgs bundle \( (\mathcal{E}, \theta) \) mean the rank and degree of the underlying sheaf \( \mathcal{E} \). If \( E = (\mathcal{E} \xrightarrow{\theta} \mathcal{E} \otimes \omega_X) \) and \( F = \)
Higgs bundles, by a morphism $E \to F$ we understand a morphism of sheaves $\varphi : E \to F$ making the square

\[
\begin{array}{c}
E \xrightarrow{\theta} E \otimes \omega_X \\
\downarrow \varphi \downarrow \varphi \otimes 1 \\
F \xrightarrow{\eta} F \otimes \omega_X
\end{array}
\]

commutative.

(3.1.2) Let $E = (E \xrightarrow{\theta} E \otimes \omega_X)$ be a Higgs bundle on $X$. Then we can consider it as a complex of sheaves concentrated in degrees 0 and 1, and hence as an object in $\text{D}^b_{\text{coh}}(X)$. When we write $E \otimes F$ or $\text{H}^p(X, E)$ etc., we consider the Higgs bundle as a sheaf complex this way. Notice that the image of $E$ in $\text{D}^b_{\text{coh}}(X)$ does not uniquely determine the isomorphism class of the Higgs bundle $(E \xrightarrow{\theta} E \otimes \omega_X)$. In fact, multiplying $\theta$ by a non-zero constant gives a quasi-isomorphic complex; however, the resulting Higgs bundle is not in general isomorphic.

**Definition (3.1.3).** — A Higgs bundle $(E \xrightarrow{\theta} E \otimes \omega_X)$ is called stable if for any locally free subsheaf $F$ of $E$ satisfying $\theta(F) \subset F \otimes \omega_X$, we have

\[
\frac{\deg F}{\text{rk} F} < \frac{\deg E}{\text{rk} E}.
\]

**Theorem (3.1.4).** — Let $E = (E \xrightarrow{\theta} E \otimes \omega_X)$ be a non-trivial stable Higgs bundle on $X$ with $\deg(E) = 0$. Then

\[
\text{H}^p(X, E) = 0
\]

for $p \neq 1$.

**Proof.** Hausel [12] Corollary (5.1.4.). Notice that $\text{H}^p(X, E) = 0$ automatically for $p > 2$ because $\dim(X) = 1$ and the length of the complex $E$ is 2.

**Proposition (3.1.5).** — If a Higgs bundle $E$ is stable, then so is $E \otimes \mathcal{L}$, where $\mathcal{L}$ is an element of $\text{Pic}^0(X)$.

**Proof.** Let $\mathcal{F} \subset E \otimes \mathcal{L}$ be a subbundle stable under $\theta \otimes 1_{\mathcal{L}}$. Then $\mathcal{F} \otimes \mathcal{L}^{-1}$ is a subbundle of $E$ stable under $\theta$. But tensoring with $\mathcal{L}$ affects neither the ranks nor the degrees of $E$ and $\mathcal{F}$, and hence the lemma follows from the stability of the Higgs bundle $E$.

(3.1.6) Let $E = (E \xrightarrow{\theta} E \otimes \omega_X)$ be a Higgs bundle and $\alpha \in H^0(X, \omega_X)$ a global 1-form. Then $1_{E} \otimes \alpha$ is canonically identified with a morphism $E \to E \otimes \omega_X$. We denote the Higgs bundle $(E \xrightarrow{\theta + 1_{E} \otimes \alpha} E \otimes \omega_X)$ by $E(\alpha)$.

**Lemma (3.1.7).** — Let $E$ be a stable Higgs bundle. Then $E(\alpha)$ is also stable for any $\alpha \in H^0(X, \omega_X)$.

**Proof.** Let $\mathcal{F} \subset E$ be a subbundle stable under $\theta = \theta + 1 \otimes \alpha$. Let $t \in \Gamma(U, \mathcal{F})$. Then $\theta(t) = \theta(t) + t \otimes \alpha \in \Gamma(U, \mathcal{F} \otimes \omega_X)$. But $t \otimes \alpha \in \Gamma(U, \mathcal{F} \otimes \omega_X)$ too, and hence $\theta(t) \in \Gamma(U, \mathcal{F} \otimes \omega_X)$. Thus $\mathcal{F}$ is stable under $\theta$, and the lemma follows from the stability of $E$. □
We shall now introduce an important construction of algebraic families of Higgs bundles. For details about projective bundles see for example EGA II [8] §4.

(3.1.8) Let \( E = (\mathcal{E} \stackrel{b}{\to} \mathcal{E} \otimes \omega_X) \) be a Higgs bundle on a curve \( X \) of genus \( g \), and let \( \pi: X \to \text{Spec}(k) \) be the structural morphism. Then the \( k \)-rational points of the vector bundle (or affine space) \( V((\pi_* \omega_X)^\vee) \) are canonically identified with the elements of \( H^0(X, \omega_X) \); we use the notation \( H^0(X, \omega_X) \) also for this scheme if no confusion seems likely. Let \( \mathcal{D} = \pi^*((\pi_* \omega_X)^\vee) = (\pi^* \pi_* \omega_X)^\vee \); we have the canonical adjunction morphism

\[
\varphi: \mathcal{D}^\vee = \pi^* \pi_* \omega_X \to \omega_X.
\]

Let \( \tilde{\varphi}: \mathcal{D}^\vee \to \omega_X \otimes \text{End}(\mathcal{E}) \) be the morphism

\[
t \mapsto \varphi(t) \otimes 1_{\mathcal{E}}.
\]

On the other hand, let \( \psi: \Theta_X \to \omega_X \otimes \text{End}(\mathcal{E}) \) be the map that takes \( 1 \) to \( \theta \). Putting these together we get a morphism

\[
\gamma = \tilde{\varphi} + \psi: \mathcal{D}^\vee \oplus \Theta_X \to \omega_X \otimes \text{End}(\mathcal{E}).
\]

Because \( \mathcal{D} \oplus \Theta_X = \pi^*((\pi_* \omega_X)^\vee \oplus k) \), we have a canonical isomorphism

\[
P_X(\mathcal{D} \oplus \Theta_X) = X \times P_k((\pi_* \omega_X)^\vee \oplus k) = X \times P(H^0(X, \omega_X) \oplus k) \cong X \times P^e_k.
\]

Let \( p: P = P_X(\mathcal{D} \oplus \Theta_X) \to X \) be the projection. There is the canonical surjection \( p^*(\mathcal{D} \oplus \Theta_X) \to \Theta_P(1) \), and so by dualising a canonical \( \Theta_P(-1) \to p^*(\mathcal{D}^\vee \oplus \Theta_X) \).

Composing this morphism with \( p^* \gamma \) we get a morphism

\[
\Theta_P(-1) \to p^*(\omega_X \otimes \text{End}(\mathcal{E})),
\]

or in other words a global section of \( p^*(\omega_X \otimes \text{End}(\mathcal{E})) \otimes \Theta_P(1) \). We interpret this section as a morphism

\[
\Theta: p^* \mathcal{E} \to p^* \mathcal{E} \otimes p^* \omega_X \otimes \Theta_P(1),
\]

and denote this complex of sheaves (in degrees 0 and 1) on \( P \) by \( \mathcal{E}^*(E) \).

In more pedestrian terms, let \( (\alpha_i) \) be a basis of \( H^0(X, \omega_X) \), and let \( (\alpha_i^*) \) be the dual basis of \( H^0(X, \omega_X)^\vee \). Let \( t: k \to k \) be the canonical coordinate on \( k \); then \( (t, \alpha_1^*, \ldots, \alpha_g^*) \) forms a basis of the global sections of \( \Theta_P(1) \), and \( H^0(X, \omega_X) \) corresponds to the open affine subscheme of \( P^e \) with \( t \neq 0 \). Now

\[
\Theta = \theta \otimes t + \sum_{i=1}^g \alpha_i \otimes 1 \otimes \alpha_i^*.
\]

Remark (3.1.9). — Notice that for \( \alpha \in H^0(X, \omega_X) \) the restriction of \( \mathcal{E}^*(E) \) to \( X \times \{ \alpha \} \) is just \( E(\alpha) \) of (3.1.6).

Proposition (3.1.10). — Let \( E \) be a stable Higgs bundle of degree 0 and rank \( \geq 2 \) on a curve \( X \) of genus \( g \geq 2 \). Then the complex \( \mathcal{E}^*(E) \) on \( X \times P^e \) is WIT(1) with respect to the relative Fourier functor \( F_{P^e}: D^b_{\text{coh}}(X \times P^e) \to D^b_{\text{coh}}(J(X) \times P^e) \).

Moreover, the Fourier transform \( (\mathcal{E}^*(E)) \) is a locally free sheaf on \( J(X) \times P^e \).

Proof. By (3.3.5) we are reduced to showing that \( \mathcal{E}^*(E) \) is IT(1) with respect to \( \mathcal{H}_{(P^e)} \). We consider two cases. Let \( U \) denote the open subset \( H^0(X, \omega_X) \) in \( P^e \).
A) Let $(\xi, \alpha) \in J(X) \times U$. Then (using the notation of (2.2.4))

$$(\mathcal{H}^\bullet(E))_{(\xi, \alpha)} \cong E(\alpha),$$

and we need to show that

$$H^p(X, E(\alpha) \otimes \mathcal{M}_\xi) = 0$$

for $p \neq 1$. But this follows from (3.1.5), (3.1.7) and (3.1.4). Notice that for a rank-1 Higgs bundle $E$ one of the bundles $E(\alpha)$ would be trivial, and the vanishing theorem (3.1.4) would fail.

B) Let $(\xi, z) \in J(X) \times (\mathbb{P}^g - U)$. We consider the second hypercohomology spectral sequence:

$$H^p(X, H^q((\mathcal{H}^\bullet(E))_{(\xi, z)} \otimes \mathcal{M}_\xi)) \Rightarrow H^{p+q}(X, (\mathcal{H}^\bullet(E))_{(\xi, z)} \otimes \mathcal{M}_\xi).$$

But

$$(\mathcal{H}^\bullet(E))_{(\xi, z)} \cong (E \xrightarrow{1 \otimes \alpha} E \otimes \omega_X)$$

for a 1-form $\alpha \neq 0$, determined up to multiplication by a non-zero scalar. Now $1 \otimes \alpha$ is clearly an injective map of sheaves; let $\mathcal{S}$ be its cokernel. Thus the $E_2$-terms of the spectral sequence are

$$
\begin{array}{ccc}
E_2^{pq} & = & H^p(X, \mathcal{S} \otimes \mathcal{M}_\xi) \\
 & & H^1(X, \mathcal{S} \otimes \mathcal{M}_\xi) \\
 & & 0 \\
 & & 0
\end{array}
$$

But $\mathcal{S}$ is a direct sum of skyscraper sheaves supported on the divisor of zeroes of the one-form, and since skyscraper sheaves are flasque, we have $H^1(X, \mathcal{S} \otimes \mathcal{M}_\xi) = 0$. Hence $H^0(X, (\mathcal{H}^\bullet(E))_{(\xi, z)} \otimes \mathcal{M}_\xi) = H^2(X, (\mathcal{H}^\bullet(E))_{(\xi, z)} \otimes \mathcal{M}_\xi) = 0$. □

Definition (3.1.11). — Let $E$ be a stable Higgs bundle of degree 0 and rank $r \geq 2$ on a curve $X$ of genus $g \geq 2$. Then the locally free sheaf $(\mathcal{H}^\bullet(E))$ on $J(X) \times \mathbb{P}^g$ is called (by abuse of language) the total Fourier transform of $E$ and is denoted by $\text{TF}(E)$.

Proposition (3.1.12). — Let $E$ and $X$ be as in (3.1.10), and let $\alpha \in H^0(X, \omega_X)$. Then

$$\text{TF}(E)_\alpha \cong \widehat{E(\alpha)},$$

where the left-hand side denotes the absolute Fourier transform.

Proof. By the proof of (3.1.10), $\mathcal{H}^\bullet(E)$ is $IT(1)$. Now the proposition follows from Remark (3.1.9) and Proposition (2.3.6) applied to the immersion $\{\alpha\} \rightarrow H^0(X, \omega_X) \rightarrow \mathbb{P}^g$. □

Proposition (3.1.13). — Let $E = (E \xrightarrow{\theta} E \otimes \omega_X)$ be a non-trivial stable Higgs bundle of degree 0 on a curve $X$ of genus $g \geq 2$. Then the rank of the total Fourier transform $\text{TF}(E)$ is $(2g - 2)\text{rk}(E)$. 
Proof. It follows from (3.1.10) and (3.1.12) that \( \text{rk}(\mathbf{T}(E)) = \dim \mathcal{H}^1(X, E) \). Consider the first hypercohomology spectral sequence

\[
I_{E^p}^q = H^p(H^q(X, E)) \Rightarrow H^{p+q}(X, E).
\]

The \( E_1 \)-terms of the sequence are:

\[
\begin{array}{ccc}
I_{E^p}^q & = & q \\
H^1(X, \mathcal{E}) & \xrightarrow{H^1(\theta)} & H^1(X, \mathcal{E} \otimes \omega_X) \\
H^0(X, \mathcal{E}) & \xrightarrow{H^0(\theta)} & H^0(X, \mathcal{E} \otimes \omega_X) \\
\end{array}
\]

The sequence clearly degenerates at \( E_2 \), i.e., \( I_{E^p}^q = I_{E^p}^{p+1} \), and hence

\[
I_{E^p}^{0,0} = H^0(X, E) \quad \text{and} \quad I_{E^p}^{1,1} = H^2(X, E).
\]

But these hypercohomologies vanish by (3.1.4), and thus \( H^0(X, \theta) \) is injective and \( H^1(X, \theta) \) is surjective. On the other hand,

\[
\mathcal{H}^1(X, E) \cong I_{E^\infty}^{0,1} \oplus I_{E^\infty}^{1,0} = \ker H^1(X, \theta) \oplus \operatorname{coker} H^0(X, \theta),
\]

and hence

\[
\dim \mathcal{H}^1(X, E) = \dim H^1(X, \mathcal{E}) - \dim H^1(X, \mathcal{E} \otimes \omega_X)
= \dim H^0(X, \mathcal{E} \otimes \omega_X) - \dim H^0(X, \mathcal{E})
= \chi(\mathcal{E} \otimes \omega_X) - \chi(\mathcal{E}).
\]

But as \( \deg(\mathcal{E}) = 0 \), the Riemann-Roch theorem gives

\[
\chi(\mathcal{E}) = (1 - g) \operatorname{rk}(\mathcal{E}) \quad \text{and} \quad \chi(\mathcal{E} \otimes \omega_X) = (g - 1) \operatorname{rk}(\mathcal{E}),
\]

whence the result follows immediately. \( \square \)

Proposition (3.1.14). — Let \( E \) be a stable Higgs bundle of rank \( r \geq 2 \) and degree 0 on a curve \( X \) of genus \( g \geq 2 \). Then

\[
\dim_k H^p(J(X) \times \mathbb{P}^e, T\mathbf{F}(E)) = \operatorname{rg} \left( \begin{array}{c} g - 1 \\ p - 1 \end{array} \right),
\]

when \( 1 \leq p \leq g \), and zero otherwise.

Proof. Let \( P \in X \) be a base point giving an embedding \( i_P : X \to J(X) \), and denote by \( j \) the embedding \( \mathbb{P}^e \to X \times \mathbb{P}^e \) of the fibre \( \operatorname{pr}_X^{-1}(P) \). Then by (2.4.6)

\[
H^p(J(X) \times \mathbb{P}^e, T\mathbf{F}(E)) = H^{p+1}(J(X) \times \mathbb{P}^e, F_p(\mathcal{H}^\bullet(\mathcal{E})))
\]

(3.1.14.1)

\[
\mathcal{H}^p(J(X) \times \mathbb{P}^e, T\mathbf{F}(E)) = \bigoplus_{i=1}^g H^{p+1-i}(\mathbb{P}^e, j^* \mathcal{H}^\bullet(\mathcal{E})) \oplus (\mathcal{E}^\bullet).\]

We apply the first hypercohomology spectral sequence

\[
I_{E^p}^{q} = H^p(H^q(\mathbb{P}^e, j^* \mathcal{H}^\bullet(\mathcal{E}))) \Rightarrow H^{p+q}(\mathbb{P}^e, j^* \mathcal{H}^\bullet(\mathcal{E})).
\]
The $E_1$-terms are given by

\[
\begin{array}{ccc}
\mathcal{E}_{1}^{pq} & = & q \\
H^1(P^q, \mathcal{O}_{P^q}^r) & \longrightarrow & H^1(P^q, \mathcal{O}_{P^q}(1)^r) \\
H^0(P^q, \mathcal{O}_{P^q}^r) & \longrightarrow & H^0(P^q, \mathcal{O}_{P^q}(1)^r).
\end{array}
\]

The standard results on the cohomology of a projective space (Hartshorne [11], III.5.1) show that the $E_1^{0,1} = E_1^{1,0} = 0$. Furthermore, it is clear from the definition (3.1.8) of $\mathcal{H}^\ast(E)$ that $d = H^0(P^q, j^\ast\Theta)$ is an injection. Thus we see that

\[
\dim H^p(P^q, j^\ast\mathcal{H}^\ast(E)) = \begin{cases} 
rg & \text{if } p=1, \\
0 & \text{otherwise}.
\end{cases}
\]

Thus in the direct sum of (3.1.14.1) we have non-zero cohomology only when $i = p$, and the result follows immediately.

**Proposition (3.1.15).** — Let $E = (\mathcal{E} \xrightarrow{\theta} \mathcal{E} \otimes \omega_X)$ be a stable non-trivial Higgs bundle on a smooth projective curve $X$ of genus $g \geq 2$, with $r = \text{rk}(\mathcal{E}) \geq 2$ and $\text{deg}(\mathcal{E}) = 0$. Then

\[
\text{ch}(\mathcal{T}F(E)) = \text{rk}(E) \left( g - 1 + (g - 1)\text{pr}_p^r \text{ch}(\mathcal{O}_{P^q}(1)) + t.(1 - \text{pr}_p^r \text{ch}(\mathcal{O}_{P^q}(1))) \right),
\]

where $t$ is the class of the $\Theta$-divisor on $J(X)$.

**Proof.** This is an easy application of the Grothendieck-Riemann-Roch formula.

\[\square\]

**3.2. Invertibility.**

**Theorem (3.2.1).** — Let $E$ and $F$ be two Higgs bundles on a curve $X$ of genus $g \geq 2$. If $\mathcal{T}F(E) \cong \mathcal{T}F(F)$, then $E \cong F$ as Higgs bundles.

**Proof.** We show this by actually exhibiting a process of recovering a Higgs bundle $E$ from its total Fourier transform $\mathcal{T}F(E)$.

**Step 1.** Choose a base point $P \in X$ as in (2.4.2), and let $i_p : X \to J(X)$ be the corresponding embedding. Denote by $j$ the immersion $i_p \times 1_{J(X)}$. Then by (2.4.5)

$\mathcal{F}_p = \mathcal{M}_p \circ j_s$. By (2.2.6) $\mathcal{M}_p$ is a category equivalence; let $G$ be its inverse. Now by definition $\mathcal{T}F(E) = \mathcal{F}_p(\mathcal{H}^\ast(E))[1]$, and hence

$G(\mathcal{T}F(E))[-1] = j_s(\mathcal{H}^\ast(E))$.

**Lemma (3.2.1.1).** — The differential $\Theta$ of the complex $\mathcal{H}^\ast(E)$ is injective.

Let $U \subset X \times P^q$ be an open subset and $s \in \Gamma(U, \text{pr}^s \mathcal{E})$ a non-zero section. There is a point $z = (x, p) \in U$ for which $s(z) \neq 0$. Because $\mathcal{E}$ is locally free, it follows (using Nakayama’s lemma) that there is an open neighbourhood $V \subset U$ of $z$ such that $s(z') \neq 0$ for $z' \in V$. If $\Theta(z)(s(z)) = 0$, it follows from the definition of $\Theta$ that there is a point $y \in V$ with $\Theta(y)(s(y)) \neq 0$, and in particular $\Theta_U(s) \neq 0$. But this shows that $\Theta$ is injective as a morphism of presheaves and hence as a sheaf morphism too. Thus the lemma is proved.
By the lemma there is an exact sequence

\[
\begin{align*}
\text{(3.2.1.2)} & \quad 0 \to \text{pr}^* E \to \text{pr}^* (E \otimes \omega_X) \otimes \text{pr}^* \mathcal{O}_U(1) \to \mathcal{R} \to 0,
\end{align*}
\]

and consequently \( \mathcal{H}^* (E) \) is quasi-isomorphic to \( \mathcal{R}[-1] \). It follows from this that \( G(TF(E)) = j_* \mathcal{R} \) in \( D^b_{\text{coh}}(X \times \mathbb{P}^8) \). Since \( j_* \mathcal{R} \) is an honest sheaf, \( G(TF(E)) = j_* \mathcal{R} \) also in \( \text{Mod}(X \times \mathbb{P}^8) \). This means that we can recover the cokernel \( \mathcal{R} \) of \( \mathcal{H}^* (E) \) on \( X \times \mathbb{P}^8 \) as \( j^!(G(TF(E))) \).

**Step 2.** Tensor \((3.2.1.2)\) with \( \text{pr}^* \mathcal{O}_U \) and obtain the exact sequence

\[
\begin{align*}
\text{(3.2.1.3)} & \quad 0 \to \text{pr}^* E \otimes \text{pr}^* \mathcal{O}_U \to \text{pr}^* (E \otimes \omega_X) \to \mathcal{R} \otimes \text{pr}^* \mathcal{O}_U \to 0.
\end{align*}
\]

We shall use the long exact \( R\text{pr}_{X,*} \)-sequence associated to \((3.2.1.3)\). By the projection formula

\[
\begin{align*}
R\text{pr}_{X,*}(\text{pr}^*_X E \otimes \text{pr}^*_U \mathcal{O}_U) & = E \otimes R\text{pr}_{X,*}\text{pr}^*_U \mathcal{O}_U, \quad \text{and} \\
R\text{pr}_{X,*}(\text{pr}^*_X (E \otimes \omega_X)) & = E \otimes \omega_X \otimes \text{pr}^*_X \mathcal{O}_{X \times U}.
\end{align*}
\]

It follows then from the long exact sequence that \( \text{pr}^*_{X,*} (\mathcal{R} \otimes \text{pr}^*_U \mathcal{O}_U) \cong E \otimes \omega_X \), and that we may consequently recover the underlying sheaf \( E \) of \( \mathcal{R} \) by twisting by \( \mathcal{O}_U \), projecting down to \( X \), and twisting by \( \omega_X \).

**Step 3.** It remains to recover the Higgs field \( \theta \). This will be done after discarding much of the information contained in \( \mathcal{R} \). We choose a non-zero \( \alpha \in H^0(X, \omega_X) \), and we let \( U = \text{Spec}(A) \) be an open affine subscheme of \( X \) over which \( \alpha \) does not vanish; then \( \alpha \) gives a trivialisation of \( \omega_X \) on \( U \). Clearly it is enough to recover \( \theta \) over \( U \).

Let \( V \) be the subvector space of \( H^0(X, \omega_X) \) generated by \( \alpha \). We can consider \( V \) as a closed subscheme of the open subscheme \( H^0(X, \omega_X) \) of \( \mathbf{P}(H^0(X, \omega_X) \oplus k) \). Furthermore, we consider \( U \times V \) as a subscheme of \( U \times \mathbf{P}(H^0(X, \omega_X) \oplus k) \), and let \( \mathcal{S} \) be the restriction of \( \mathcal{R} \) to \( U \times V \); it is just the cokernel of \( \Theta \) restricted to \( U \times V \). Notice that \( U \times V \cong \text{Spec}(A[T]) \).

On \( U \) the underlying sheaf \( E \) of \( \mathcal{R} \) corresponds to an \( A \)-module \( M \) and \( \theta \) corresponds to an endomorphism \( u \) of \( M \). Furthermore, the pull-back of \( E \) to \( U \times V \) corresponds to \( M[T] = M \otimes_A A[T] \). By the definition of \( \Theta \) \((3.1.8)\), \( \Theta \mid_{U \times V} \) corresponds to the \( A[T] \)-linear map

\[
\Psi = 1_M \otimes T + u \otimes 1_{A[T]}.
\]

But \( \Psi \) fits into the exact sequence

\[
M[T] \xrightarrow{\Psi} M[T] \to M_u \to 0,
\]

where \( M_u \) is the \( A[T] \)-module with \( T \) acting on \( M \) as \( u \) (cf. Bourbaki \((3)\), Ch. III §8 no. 10). Hence \( \mathcal{S} = (M_u)^\sim \). But the structure of \( A[T] \)-module of \( M_u \) determines \( u \) and hence \( \theta \mid_U \).

\( \square \)
Remark (3.2.2). — Lemma 6.8 in Simpson [27] gives a description of Higgs bundles on $X$ as coherent sheaves on the total space of the cotangent bundle of $X$. The scheme $U \times V$ in Step 3 of the proof is the total space of the cotangent bundle of $U$, and the coherent sheaf $\mathcal{H}$ on $U \times V$ is the one that corresponds to $E|_U$ under Simpson’s correspondence.

Corollary (3.2.3). — The functor $\mathbf{TF}$ from the category of stable non-trivial Higgs bundles on $X$ with vanishing Chern classes to $\mathcal{M}(\mathcal{J}(X) \times \mathbb{P}^g)$ is fully faithful.

Proof. Let $E$ and $E'$ be Higgs bundles on $X$ and let $\mathcal{R}$ and $\mathcal{R}'$ be the cokernels of $\mathcal{H}^\bullet(E)$ and $\mathcal{H}^\bullet(E')$ respectively. Because the relative Mukai transform is an equivalence of categories, we have

$$\text{Hom}(\mathbf{TF}(E), \mathbf{TF}(E')) = \text{Hom}(\mathcal{R}, \mathcal{R}{'}).$$

Thus faithfulness is clear. On the other hand, let $\varphi: \mathcal{R} \rightarrow \mathcal{R}'$; using the notation of the proof of the theorem, the previous remark shows that $\varphi|_{U \times V}$ gives a morphism of Higgs bundles $E|_U \rightarrow E'|_U$. But as the genus of $X$ is at least 2, the canonical linear system $|\omega_X|$ has no base points. Hence we can cover $X$ by open sets like $U$; it is clear that the morphisms thus obtained glue to give a morphism $E \rightarrow E'$. □

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