ASYMPTOTIC PROFILE OF SOLUTIONS TO A CERTAIN CHEMOTAXIS SYSTEM

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Abstract. We consider a Cauchy problem for a two-dimensional model of chemotaxis and we show that large time behavior of solution is given by a multiple of the heat kernel.

1. Introduction. Chemotaxis, i.e. biological process of directed movement (e.g. of cells) towards a chemically more favorable environment appears to be in a great interest of mathematical study in recent years (see i.e. survey [3] and references therein). The most classical model describing this phenomenon is the system of parabolic equations derived by Keller and Segel ([4]). Usually, the chemical substrate is produced by cells. In this paper, we consider the opposite case i.e. when the chemical substrate is consumed by them. A typical example of such a substrate is an oxygen (see [9]). More precisely, we consider the following system of partial differential equations, describing a population of bacteria

\[
\begin{align*}
    n_t &= \Delta n - \nabla \cdot (n \nabla c) \quad \text{with} \quad x \in \mathbb{R}^2, \quad t > 0, \\
    c_t &= \Delta c - nc,
\end{align*}
\]

supplemented with nonnegative initial conditions

\[
    n(x, 0) = n_0(x), \quad c(x, 0) = c_0(x).
\]

Here, the function \( n = n(x, t) \) represents the density of cells whereas the function \( c = c(x, t) \) denotes the concentration of oxygen.

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This system of equations was extensively studied in different contexts. First of all, it can be treated as a fluid-free version of the following, so called chemotaxis-Navier-Stokes model, initially proposed by Tuval et al. [9]

\begin{align}
  n_t + u \cdot \nabla n &= \Delta n - \nabla \cdot (n \nabla c), \\
  c_t + u \cdot \nabla c &= \Delta c - nc, \\
  u_t + u \cdot \nabla u &= \Delta u - \nabla P - n \nabla \varphi, \\
  \nabla \cdot u &= 0.
\end{align} 

(4)

In this model the additional equation describing the velocity field (denoted by \( u \)) of the fluid determined by the incompressible Navier-Stokes model with some given gravitational force \( \nabla \varphi \) and scalar function \( P \) (representing the pressure of the fluid, recovered from \( n \) and \( u \) via Calderón-Zygmund operators) is added in order to cover the situation in which both bacteria and oxygen are also transported with fluid. It is easy to see that the system (1)–(2) can be obtained by putting \( u(x, 0) = \nabla \varphi(x) = 0 \) in (4) and hence results to the full chemotaxis-Navier-Stokes model applies also to our problem (1)–(2).

The models above were mostly studied in bounded domain with the standard Neuman boundary conditions. The existence and boundedness results for (1)–(2) for \( n \geq 2 \) can be found in [7]. Considering the system (4) and its generalisation we refer the reader to [6, 10, 11] and references therein for results devoted to the problem of local and global-in-time existence and also uniqueness of solutions together with some asymptotic behavior properties.

The issue in the whole plane is considered in a much smaller number of publications. The global existence of the solution to the problem (4) was proved by Zhang and Zheng in [12]. The global existence of classical bounded solutions for the same system was proved in [6]. Under different initial conditions the authors obtained solutions \((n, c, u)\) in \( C^{2+\theta, 1+\frac{\theta}{2}}(\mathbb{R}^2 \times (0, \infty)) \) for some \( \theta \in (0, 1) \).

The model (4) enriched with the additional equation describing the evolution of the concentration of chemical attractant was examined by Kozono, Miura and Sugiyama in the paper [5]. Except the existence of the mild solutions they proved the estimates for decay rate for \( n(t) \) and \( \nabla c(t) \). For \( n \geq 2 \) and sufficiently small initial data it occurs

\begin{align}
  \|n(t) - e^{t\Delta}n_0\|_{L^p(\mathbb{R}^n)} &\leq O(t^{-\frac{n}{2}(\frac{2}{n}-\frac{1}{p})}) \text{ for all } t > 0, \\
  \|\nabla c(t) - \nabla e^{t\Delta}c_0\|_{L^p(\mathbb{R}^n)} &\leq O(t^{-\frac{n}{2}(\frac{4}{n}-\frac{1}{p})}) \text{ for all } t > 0.
\end{align} 

(5)

(6)

For some generalisation of the system (4) the existence of classical solution and temporal decay was proved under some smallness assumptions on \( L^\infty \)-norm of \( c_0 \) ([2]).

**Notations.** Throughout the paper \( \mathbb{R}^+ = (0, \infty) \) and \( C \) stands for a constant which may vary from line to line. The function \( G \) denotes the heat kernel \( G(x, t) = (4\pi t)^{-\frac{n}{2}}e^{-\frac{|x|^2}{4t}} \) and the symbol \( e^{t\Delta} \) denotes the heat semigroup given as a convolution with the heat kernel \( G(x, t) \).

2. **Results and comments.** Since the considered model is simplification of that one investigated in [12], we recall that existence result and concentrate on large time asymptotics of the solutions, what is the main subject of the paper.
Indeed taking $\nabla \varphi = 0$ and putting $u_0(x) \equiv 0$ we simplify the system (4) to the considered one (1)-(3) (by [12, Remark 1.3] the initial condition $u_0(x) \equiv 0$ implies that $u(x,t) = 0$ for all pairs $(x,t)$).

Restricting the space for the initial conditions let us define (following [12]) the space

$$X_0 = \{ n_0 \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2), n_0 > 0, \quad \nabla \sqrt{n_0} \in L^2(\mathbb{R}^2), c_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2), c_0 > 0 \}.$$ 

Thus, we rewrite [12, Theorem 1.1] as follows

**Theorem 2.1** (Existence of global-in-time solution). Assume that $(n_0, c_0) \in X_0$. Then the initial value problem (1)–(3) has the unique, nonnegative, global-in-time solutions $(n, c)$ such that

$$n \in L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^2)) \cap L^\infty_{\text{loc}}(\mathbb{R}^+; L^2(\mathbb{R}^2)) \cap L^2_{\text{loc}}(\mathbb{R}^+; H^1(\mathbb{R}^2)),$$

$$c \in L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)) \cap L^\infty_{\text{loc}}(\mathbb{R}^+; H^1(\mathbb{R}^2)) \cap L^2_{\text{loc}}(\mathbb{R}^+; H^2(\mathbb{R}^2)).$$

Note that with more assumptions on the initial conditions (see [6]) we get the regular, classical solutions to the problem.

The goal of this work is to study large time behaviour of such obtained solution.

In the proof of the main result, we use the mild solutions, i.e. solutions which fulfill the following Duhamel formula

$$n(t) = e^{t \Delta} n_0 + \int_0^t \nabla e^{(t-s) \Delta} n(s) \nabla c(s) \, ds, \quad (7)$$

$$c(t) = e^{t \Delta} c_0 - \int_0^t e^{(t-s) \Delta} p(s) c(s) \, ds. \quad (8)$$

Thus, let us formulate the main result of this work.

**Theorem 2.2** (Large-time asymptotics). Assume that $(n_0, c_0) \in X_0$ and denote by $(n(t), c(t))$ the corresponding solution of problem (1)–(3) provided by Theorem 2.1. Then, for large times, the solutions $n = n(x,t)$ and $c = c(x,t)$ behave as a multiple of a heat kernel i.e. for every $p \in [1,\infty]$ and every $q \in [1,\infty]$ there exists constant $C_p > 0$ such that

$$\|n(t) - e^{t \Delta} n_0\|_p \leq C_p t^{-\frac{3}{2} + \frac{1}{p}} \quad \text{for all } t, \quad (9)$$

$$\|c(t) - M_c G(t)\|_q = o(t^{-1 + \frac{1}{q}}) \quad \text{as } t \to \infty, \quad (10)$$

where the number

$$M_c \equiv \int_{\mathbb{R}^2} c_0 \, dx - \int_0^\infty \int_{\mathbb{R}^2} nc \, dx$$

is nonnegative and finite.

With the additional condition on the second moment of the initial condition $n_0$, i.e. assuming $n_0 \in L^1(\mathbb{R}^2; (1+|x|^2) \, dx)$ we get

$$\|n(t) - M_n G(t)\|_p \leq C_p t^{-\frac{3}{2} + \frac{1}{p}} \quad \text{for all } t, \quad (11)$$

with

$$M_n \equiv \int_{\mathbb{R}^2} n_0 \, dx \geq 0.$$
Let us notice that the decay estimates received in the paper [5] were obtained under smallness assumption on initial data while the decay rate for \( n(t) \) and \( c(t) \) (compare the estimates (5), (6)) are worse than these proved in this paper.

The reminder of this paper is devoted to the proof of Theorem 2.2. In Section 3 we provide decay estimates of solutions using energy methods. In Section 4 we use an integral formulation of the initial value problem (1)–(3) to prove the main result.

3. Global-in-time estimates. We begin with the proof of the certain global-in-time estimates of solutions to problem (1)–(3) which will be necessary in our study of their large time asymptotics.

**Lemma 3.1.** Let \((n(t), c(t))\) be the solution obtained by Theorem 2.1 for given \(n_0, c_0 \in X_0\). Then for all \(t > 0\) we have

\[ n(t) > 0, \ c(t) > 0. \]  \(\text{(12)}\)

**Proof.** This is an immediate consequence of a standard reasoning for parabolic equation, see [12, Proposition 3.1] for more details. \(\square\)

**Lemma 3.2.** Let \(n(t)\) and \(c(t)\) be the solutions obtained by Theorem 2.1 for given \(n_0(x)\) and \(c_0(x)\). The solution \(n(t)\) as a function of \(t\) conserves the “mass”, i.e.

\[ M \equiv \|n(t)\|_1 = \int_{\mathbb{R}} n(x, t) \, dx = \int_{\mathbb{R}} n_0(x) \, dx = \|n_0\|_1 \quad \text{for all} \quad t \geq 0 \]  \(\text{(13)}\)

while the solution \(c(t)\) is bounded from above, i.e.

\[ \|c(t)\|_{\infty} \leq \|c_0\|_{\infty}. \]  \(\text{(14)}\)

**Proof.** The proof of the equality (13) is standard for chemotaxis models and obtained by integration over the whole plane of the first equation of the system, while the estimates (14) is based on the positivity of \(n(x, t)\), \(n_0(x)\) and \(c(x, t)\) applied to Duhamel formula for the solution \(c(t)\). \(\square\)

The next step is estimating the decay of the norms \(\|n(t)\|_p\) and \(\|c(t)\|_p\) for all \(p \in [1, \infty)\).

First, we recall classical estimates for solutions to the heat equation.

**Lemma 3.3.** For all \(1 \leq q \leq p \leq +\infty\), there exist constants \(C_1 = C_1(p, q)\), \(C_2 = C_2(p, q)\), such that

\[ \|e^{t\Delta}f\|_{L^p} \leq C_1t^{-\left(\frac{1}{q} - \frac{1}{p}\right)}\|f\|_{L^q} \quad \text{for all} \quad t > 0, \]  \(\text{(15)}\)

\[ \|\nabla e^{t\Delta}f\|_{L^p} \leq C_2t^{-\left(\frac{1}{q} - \frac{1}{p}\right) - \frac{1}{2}}\|f\|_{L^q} \quad \text{for all} \quad t > 0, \]  \(\text{(16)}\)

for each \(f \in L^q(\mathbb{R}^2)\).

We have

**Lemma 3.4 (\(L^p\)-decay of \(c\)).** Let the assumptions of Theorem 2.2 holds true. For each \(p \in [1, \infty]\) there exists a number \(C > 0\) independent of \(t\) such that

\[ \|c(t)\|_p \leq Ct^{-\left(1 - \frac{1}{p}\right)}\|c_0\|_1 \quad \text{for all} \quad t > 0. \]  \(\text{(17)}\)
Proof. Since $c(x,t)$ and $n(x,t)$ are positive function, using the comparison principle for the heat equation we obtain

$$0 \leq c(x,t) \leq e^{t\Delta}c_0 \quad \text{for} \quad x \in \mathbb{R}^2, t > 0.$$ 

The decay of the heat semigroup (15) implies the required statement and finishes the proof of this Lemma.

Lemma 3.5 ($L^p$-decay of $n$). Let the assumptions of Theorem 2.2 holds true. For each $p \in [1, \infty)$ there exists a number $C > 0$ independent of $t$ such that

$$\|n(t)\|_p \leq Ct^{-\frac{1}{p}} \quad \text{for all} \quad t > 0.$$  

Moreover for $p \in [1, 2]$ we have

$$\|n(t)\|_p \leq C(1 + t)^{-\frac{1}{p}} \quad \text{for all} \quad t \geq 0.$$  

To prove Lemma 3.5 we need boundedness of some norm of the solution $c(t)$.

Lemma 3.6. Assume that $c(t)$ is the solution of problem (1)–(3). Then, there exist constant $C > 0$ such that

$$\int_0^\infty \int_{\mathbb{R}^2} |\nabla c(x,t)|^4 \, dx \, dt \leq C. \quad (20)$$

Proof. This estimate is direct consequence of an energy inequality (see inequality (21) below) which has been often used in the study of problem (1)–(3), see eg. [8, Lemma 3.1] and [12, Proposition 4.1]. In the proof we will rely on (and recall) the reasoning from [12], (where authors considered the same system) which leads to similar estimate. Note that in [12] the authors obtained only the exponentially growth of the considered term while we need its boundedness.

Note that the calculations and estimates below should be done on the level of the regularized problem (as in [12]), which gives more regularity than claimed in the statement of Theorem 2.1. For simplicity we write it without introducing standard mollifier $\rho^\varepsilon$.

The following energy inequality holds true for every sufficiently regular solution of problem (1)–(3):

$$\frac{d}{dt} \left\{ \int_{\mathbb{R}^2} (n + 1) \log(n + 1) \, dx + 2 \int_{\mathbb{R}^2} |\nabla \sqrt{n}|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^2} c^2 \, dx \right\}$$

$$+ 3 \int_{\mathbb{R}^2} \Delta \sqrt{n + 1} \, dx + \frac{4}{3} \int_{\mathbb{R}^2} |\Delta \sqrt{c}|^2 \, dx + \frac{2}{5} \int_{\mathbb{R}^2} \frac{|\nabla \sqrt{c}|^4}{c} \, dx$$

$$+ 2 \int_{\mathbb{R}^2} n |\nabla \sqrt{c}|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^2} |\nabla c|^2 \, dx \leq 0. \quad (21)$$

To prove it, we follow the calculation in [12, Proposition 4.1]. First, rewriting the equation (1) as

$$\frac{d}{dt} (n + 1) = \Delta(n + 1) - \nabla \cdot ((n + 1)\nabla c) - \Delta c,$$

then multiplying by the function $1 + \log(n + 1)$ and integrating over $\mathbb{R}^2$ we arrive at

$$\frac{d}{dt} \int_{\mathbb{R}^2} (n + 1) \log(n + 1) \, dx + 4 \int_{\mathbb{R}^2} |\nabla \sqrt{n + 1}|^2 \, dx$$

$$= \int_{\mathbb{R}^2} \nabla n \nabla c \, dx + \int_{\mathbb{R}^2} \Delta c \log(n + 1) \, dx. \quad (22)$$

$$= \int_{\mathbb{R}^2} \nabla n \nabla c \, dx + \int_{\mathbb{R}^2} \Delta c \log(n + 1) \, dx. \quad (23)$$
Similarly, using the equality
\[ \Delta c = 2|\nabla \sqrt{c}|^2 + 2\sqrt{c}\Delta \sqrt{c}, \]
we rewrite the equation (2) as
\[
\frac{d}{dt}(\sqrt{c}) = (\sqrt{c})^{-1}|\nabla \sqrt{c}|^2 + \Delta \sqrt{c} - \frac{1}{2}\sqrt{c} n.
\]
Multiplying the above equation by \(-\Delta \sqrt{c}\) and integrating over \(\mathbb{R}^2\) we get
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |\nabla \sqrt{c}|^2 \, dx + \int_{\mathbb{R}^2} |\Delta \sqrt{c}|^2 \, dx
= -\int_{\mathbb{R}^2} (\sqrt{c})^{-1}|\nabla \sqrt{c}|^2 \Delta \sqrt{c} \, dx + \frac{1}{2} \int_{\mathbb{R}^2} n \sqrt{c} \Delta \sqrt{c} \, dx = I_1 + I_2.
\]
(24)

Next, by calculations analogous to those in [12, Proposition 4.1] we have
\[
I_1 \leq -\frac{1}{6} \int_{\mathbb{R}^2} \frac{|\nabla \sqrt{c}|^4}{c} \, dx + \frac{2}{3} \int_{\mathbb{R}^2} |\Delta \sqrt{c}|^2 \, dx,
\]
\[
I_2 \leq -\frac{1}{4} \int_{\mathbb{R}^2} \nabla n \nabla c \, dx - \frac{1}{2} \int_{\mathbb{R}^2} n |\nabla \sqrt{c}|^2 \, dx.
\]
(25)

Hence, plugging inequalities (25) into equation (24) we obtain
\[
2 \frac{d}{dt} \int_{\mathbb{R}^2} |\nabla \sqrt{c}|^2 \, dx + \frac{4}{3} \int_{\mathbb{R}^2} |\Delta \sqrt{c}|^2 \, dx + \frac{2}{3} \int_{\mathbb{R}^2} \frac{|\nabla \sqrt{c}|^4}{c} \, dx + 2 \int_{\mathbb{R}^2} n |\nabla \sqrt{c}|^2 \, dx
\leq -\int_{\mathbb{R}^2} \nabla n \nabla c \, dx.
\]
(26)

By adding equation (22) to inequality (26) we get
\[
\frac{d}{dt} \left\{ \int_{\mathbb{R}^2} (n + 1) \log(n + 1) \, dx + 2 \int_{\mathbb{R}^2} |\nabla \sqrt{c}|^2 \, dx \right\} + 4 \int_{\mathbb{R}^2} |\nabla \sqrt{n + 1}|^2 \, dx
+ \frac{4}{3} \int_{\mathbb{R}^2} |\Delta \sqrt{c}|^2 \, dx + \frac{2}{3} \int_{\mathbb{R}^2} \frac{|\nabla \sqrt{c}|^4}{c} \, dx + 2 \int_{\mathbb{R}^2} n |\nabla \sqrt{c}|^2 \, dx \leq \int_{\mathbb{R}^2} \Delta c \log(n + 1) \, dx.
\]
(27)

To estimate the integral of right-hand side of the inequality (27) we integrate by parts and use the Cauchy inequality in the following way
\[
\int_{\mathbb{R}^2} \Delta c \log(n + 1) \, dx = -\int_{\mathbb{R}^2} \nabla c \frac{\nabla n}{n + 1} \, dx \leq \frac{1}{2} \int_{\mathbb{R}^2} \frac{|\nabla c|^2}{n + 1} \, dx + \frac{1}{2} \int_{\mathbb{R}^2} \left( \frac{|\nabla n|}{\sqrt{n + 1}} \right)^2 \, dx
\leq \frac{1}{2} \int_{\mathbb{R}^2} |\nabla c|^2 \, dx + \int_{\mathbb{R}^2} |\nabla \sqrt{n + 1}|^2 \, dx,
\]
(28)

since \(n\) is positive. Now, we multiple equation (2) by \(c\) and integrate over \(\mathbb{R}^2\) to obtain
\[
\frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}^2} c^2 \, dx + \int_{\mathbb{R}^2} |\nabla c|^2 \, dx = -\int_{\mathbb{R}^2} nc^2 \, dx \leq 0.
\]
(29)

Adding inequality (29) to (27) and using estimate (28) we obtain (21).
Lemma 3.7. Assume that $f \in C^1((0, \infty))$ is nonnegative and satisfies
\begin{equation}
\frac{d}{dt} f + C_1 t^{p-2} f^2 \leq C_2 g f, \tag{31}
\end{equation}
with constants $C_1, C_2 > 0, p \geq 2$ and a function $g(t) \geq 0$ such that
\begin{equation}
\int_0^\infty g(s) \, ds \leq C_0 \quad \text{for a constant } \quad C_0 > 0. \tag{32}
\end{equation}
Then there exist $C = C(p, C_0, C_1, C_2) > 0$ such that
\begin{equation}
f(t) \leq C t^{-(p-1)} \quad \text{for all } \quad t > 0. \tag{33}
\end{equation}

Proof. For $G(t) = C_2 \int_0^t g(s) \, ds$, we rewrite inequality (31) in the form
\begin{equation}
\frac{d}{dt} \left( f e^{-G(t)} \right) + C_1 t^{p-2} e^{-G(t)} f^2 \leq 0. \tag{34}
\end{equation}
Now, let $h(t) = f(t) e^{-G(t)}$. Then, we have
\begin{equation}
\frac{d}{dt} h(t) + C_1 t^{p-2} e^{G(t)} h^2(t) \leq 0. \tag{35}
\end{equation}
Since $h$ is nonnegative, solving differential inequality (35) we obtain
\begin{equation}
h(t) \leq \left( C_1 \int_0^t s^{p-2} e^{G(s)} \, ds \right)^{-1}. \tag{36}
\end{equation}
Applying the estimate $1 \leq e^{G(t)} \leq C_2 e^{C_0}$ we finally get
\begin{equation}
f(t) \leq \frac{e^{G(t)}}{C_1 \int_0^t s^{p-2} e^{G(s)} \, ds} \leq C_1 e^{C_0-C_2 t^{-(p-1)}}. \tag{37}
\end{equation}

Now, we are ready to prove decay estimates of solutions to problem (1)–(3) in $L^p$-norms. First, we deal with the densities of bacteria $n(x, t)$.
Proof of Lemma 3.5. Decay for $p = 2$. We multiply equation (1) by $n$ and integrate over $\mathbb{R}^2$. After integration by parts and using the Cauchy inequality, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} n^2 \, dx + \int_{\mathbb{R}^2} |\nabla n|^2 \, dx = \int_{\mathbb{R}^2} n \nabla c \nabla n \, dx \leq \frac{1}{4} \int_{\mathbb{R}^2} |\nabla n|^2 \, dx + \int_{\mathbb{R}^2} |\nabla c|^2 n^2 \, dx.$$ 

Next, using the Hölder and Sobolev inequalities we get

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} n^2 \, dx + \frac{3}{4} \int_{\mathbb{R}^2} |\nabla n|^2 \, dx \leq \|\nabla c\|_4^2 \|n\|_4^2 \leq C \|\nabla c\|_4^2 \|\nabla n\|_2 \|n\|_2.$$ 

Again using the Cauchy inequality we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^2} n^2 \, dx + \int_{\mathbb{R}^2} |\nabla n|^2 \, dx \leq C \|\nabla c\|_4^4 \int_{\mathbb{R}^2} n^2 \, dx. \quad (34)$$ 

Now, by the following Nash inequality

$$\|n\|_2^2 \leq C \|\nabla n\|_2^2 \|n\|_1^2 \quad (35)$$ 

applied in (34) we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^2} n^2 \, dx + \frac{C}{M^2} \left( \int_{\mathbb{R}^2} n^2 \, dx \right)^2 \leq C \|\nabla c\|_4^4 \left( \int_{\mathbb{R}^2} n^2 \, dx \right),$$

where $M = \|n(t)\|_1$ is independent of $t$.

Note, that the regularity assumed in Lemma 3.7 can be obtain on the level of approximation and next passing to the limit.

Finally, substituting $f(t) = \int_{\mathbb{R}^2} n^2(x, t) \, dx$ and $g(t) = \|\nabla c(t)\|_4^4$ (note that the function $g \in L^1(\mathbb{R}^7)$ by Lemma 3.6) for $p = 2$ due Lemma 3.7 we prove that

$$\|n(\cdot, t)\|_2 \leq Ct^{-1/2} \quad \text{for all} \ t > 0. \quad (36)$$

Notice that, since $n_0 \in L^1 \cap L^2$, Theorem 2.1 together with the estimates above implies that $\|n(t)\|_p \leq C(1 + t)^{-\left(1 - \frac{1}{p}\right)}$ for all $1 \leq p \leq 2$, which gives us the second statement of Lemma 3.5. This implies also that there is no blow-up at $t = 0$ in (18).

Decay for $p = 2^k$ for each $k \in \mathbb{N}$. We proceed by induction assuming that decay estimate (18) holds true for $p = 2^{k-1}$. Let $p = 2^k$. Similarly as above, we multiply equation (1) by $n^{p-1}$ and integrate over $\mathbb{R}^2$. After integration by parts, using the Cauchy inequality, we obtain

$$\frac{1}{p} \frac{d}{dt} \int_{\mathbb{R}^2} n^p \, dx + (p - 1) \int_{\mathbb{R}^2} n^{p-2} |\nabla n|^2 \, dx = \int_{\mathbb{R}^2} n^{p-1} \nabla c \nabla n \, dx \leq \frac{p - 1}{2} \int_{\mathbb{R}^2} n^{p-2} |\nabla n|^2 \, dx + C \int_{\mathbb{R}^2} |\nabla c|^2 n^p \, dx.$$ 

Next, by the Hölder and Sobolev inequalities, we get

$$\frac{1}{p} \frac{d}{dt} \int_{\mathbb{R}^2} n^p \, dx + \frac{p - 1}{2} \int_{\mathbb{R}^2} n^{p-2} |\nabla n|^2 \, dx \leq C \|\nabla c\|_4^2 \|n^{p/2}\|_4^2 \leq C \|\nabla c\|_4^2 \|n^{p/2}\|_2 \|n^{p/2}\|_2.$$
Again using the Cauchy inequality we can find such constant $C > 0$ that
\[
\frac{d}{dt} \int_{\mathbb{R}^2} n^p \, dx + \int_{\mathbb{R}^2} |\nabla n^{p/2}|^2 \, dx \leq C \|n\|_{L^p}^2 \int_{\mathbb{R}^2} n^p \, dx.
\]
Now, the Nash inequality (35) with $n$ replaced by $n^{p/2}$ takes the form
\[
\|n\|_{L^p}^{2p} \leq C \|\nabla n^{p/2}\|_2^2 \|n\|_{L^{p/2}}^p.
\]
Hence, we obtain the following differential inequality
\[
\frac{d}{dt} \int_{\mathbb{R}^2} n^p \, dx + C \|n\|_{L^p}^{2p} \|n\|_{L^{p/2}}^{2p} \leq C \|\nabla c\|^2 \left( \int_{\mathbb{R}^2} n^p \, dx \right).
\]
By the recurrence hypothesis, we have $\|n\|_{L^{p/2}}^p \leq C t^{(-1 + \frac{2}{p})p} = C t^{2-p}$. Thus, as in the previous step, we use Lemma 3.7 with $f(t) = \int_{\mathbb{R}^2} n^p(x, t) \, dx$ and $g(t) = \|\nabla c(t)\|^4_4$ to complete the proof of the decay estimate (18) for $p = 2^k$ with $k \in \mathbb{N}$. To deal with other $p \in (2, \infty)$ it suffices to use the Hölder inequality.

Now, we prove the similar result for the gradient of the function $c = c(x, t)$.

**Lemma 3.8 (L^p-decay of $\nabla c$).** For each $p \in [1, \infty)$ there exists a number $C > 0$ independent of $t$ such that
\[
\|\nabla c(t)\|_p \leq Ct^{-\frac{3}{2} + \frac{1}{p}} \log(1 + t).
\]
Additionally, for $p = 2$ we have
\[
\|\nabla c(t)\|_2 \leq C (1 + t)^{-1} \log(1 + t).
\]
**Proof.** Using equation (8) and estimate (16) we obtain
\[
\|\nabla c(t)\|_p \leq \|\nabla c(t)\|_p + \int_0^t \|\nabla e^{(t-s)A} n(s)c(s)\|_{L^p} \, ds \leq C t^{-\frac{3}{2} + \frac{1}{p}} \|c_0\|_1 + I_1,
\]
where the integral $I_1$ is defined as follows
\[
I_1 = \int_0^{t/2} \|\nabla e^{(t-s)A} n(s)c(s)\|_{L^p} \, ds + \int_{t/2}^t \|\nabla e^{(t-s)A} n(s)c(s)\|_{L^p} \, ds = I_{1,1} + I_{1,2}.
\]
For the integral $I_{1,1}$ estimate (16) together with the Hölder inequality lead to
\[
I_{1,1} \leq C \int_0^{t/2} (t-s)^{-1 + \frac{3}{p} - \frac{1}{2}} \|n(s)c(s)\|_1 \, ds \leq C \int_0^{t/2} (t-s)^{-\frac{3}{2} + \frac{1}{p}} \|n(s)\|_2 \|c(s)\|_2 \, ds.
\]
Now, using Lemma 3.5 and Lemma 3.4 we get
\[
I_{1,1} \leq C \int_0^{t/2} (t-s)^{-\frac{3}{2} + \frac{1}{p}} (1 + s)^{-1} \, ds \leq Ct^{-\frac{3}{2} + \frac{1}{p}} \log(1 + t),
\]
for all $t > 0$. Similarly, for the term $I_{1,2}$ we have
\[
I_{1,2} \leq C \int_{t/2}^t (t-s)^{-\frac{3}{2} + \frac{1}{p}} \|n(s)c(s)\|_3 \, ds \leq C \int_{t/2}^t (t-s)^{-\frac{3}{2} + \frac{1}{p}} \|n(s)\|_4 \|c(s)\|_{L^2} \, ds
\]
\[
\leq C \int_{t/2}^t (t-s)^{-\frac{3}{2} + \frac{1}{p}} s^{-2 + \frac{1}{2}} \, ds \leq Ct^{-\frac{3}{2} + \frac{1}{p}},
\]
which gives the estimates (38).
To obtain the estimate (39) note that, by regularity of solutions provided by Theorem 2.1, Lemma 3.8 implies that \( \|\nabla c\|_2 \leq C(1 + t)^{-1} \log(1 + t) \), namely there is no blow-up at \( t = 0 \) in (38).

This ends the proof of this Lemma. \( \Box \)

4. Large-time asymptotics. Now, we are in a position to prove the main result of this work.

Proof of Theorem 2.2. Step 1. (Large time behaviour of \( c(x,t) \)) At the beginning, let us notice that function \( nc \in L^1(\mathbb{R}^2 \times (0, \infty)) \). This is an immediate consequence of the fact that functions \( n(x,t) \) and \( c(x,t) \) are both finite and of the equality

\[
\int_{\mathbb{R}^2} c(x,t) \, dx = \int_{\mathbb{R}^2} c_0(x) \, dx - \int_0^t \int_{\mathbb{R}^2} n(x,t)c(x,t) \, dx, 
\]

which is obtained from equation (2) after integration it over \( \mathbb{R}^2 \times (0, t) \). In particular, the quantity

\[
M_c = \int_{\mathbb{R}^2} c_0(x) \, dx - \int_0^\infty \int_{\mathbb{R}^2} n(x,t)c(x,t) \, dx
\]

is finite and nonnegative.

Now, to prove formula (10) for \( p = 1 \) it is sufficient to use [1, Theorem 6.1], where authors consider a general linear nonhomogeneous heat equation. For \( p \in [1, \infty) \) we use the following interpolation inequality

\[
\|f\|_p \leq C \|f\|_n^{\frac{1}{p}} \|f\|_q^{1-\frac{1}{p}},
\]

where numbers \( p, q \) and \( a \) satisfies \( \frac{1}{p} = a + \frac{1-a}{q} \) in the following way

\[
\|c(t) - M_c \mathcal{G}(t)\|_p \leq C \|c(t) - M_c \mathcal{G}(t)\|_1^{1/p} \left( \|c(t)\|_\infty^{1-\frac{1}{p}} + \|M_c \mathcal{G}(t)\|_\infty^{1-\frac{1}{p}} \right).
\]

We have already explained that the first factor on the right-hand side of this inequality converges to zero as \( t \to \infty \). Using Lemma 3.4 and the following, elementary estimate of the heat kernel \( \|\mathcal{G}(t)\|_\infty \leq Ct^{-1} \), we finish the proof of this step.

Step 2. (Large time behaviour of \( n(x,t) \)) Now, we proceed to estimate (9). Using the Duhamel principle (8) and the estimate (16) we obtain

\[
\|n(t) - e^{\Delta t} n_0\|_p \leq \int_0^t \|\nabla e^{(t-s)}\Delta n(s)\nabla c(s)\|_p \, ds \\
= \int_0^{t/2} \|\nabla e^{(t-s)}\Delta n(s)\nabla c(s)\|_p \, ds + \int_{t/2}^t \|\nabla e^{(t-s)}\Delta n(s)\nabla c(s)\|_p \, ds \\
= I_1 + I_2.
\]

For the integral \( I_1 \), estimate (16) together with the Hölder inequality yields that

\[
I_1 \leq C \int_0^{t/2} (t-s)^{-\frac{3}{2} + \frac{1}{p} - \frac{1}{2}} \|n\nabla c\|_1 \, ds \leq C \int_0^{t/2} (t-s)^{-\frac{3}{2} + \frac{1}{p}} \|n\|_2 \|\nabla c\|_2 \, ds \\
\leq C \int_0^{t/2} (t-s)^{-\frac{3}{2} + \frac{1}{p}} (1 + s)^{-\frac{3}{2}} \log(1 + s) \, ds \leq Ct^{-\frac{3}{2} + \frac{1}{p}}
\]

since

\[
\int_0^{\infty} (1 + s)^{-\frac{3}{2}} \log(1 + s) \, ds \leq C.
\]
Similarly, for the term $I_2$ we have
\[
I_2 \leq C \int_{t/2}^t (t-s)^{-\frac{1}{2} + \frac{1}{p} - \frac{1}{2}} ||n\nabla c||_3 ds \leq C \int_{t/2}^t (t-s)^{-\frac{3}{2} + \frac{1}{p} + \frac{1}{4}} \log(1+s) ds
\]
\[
\leq C \int_{t/2}^t (t-s)^{-\frac{3}{2} + \frac{1}{p} + \frac{1}{4}} \log(1+s) ds \leq Ct^{-\frac{3}{2} + \frac{1}{p}} \log(1+t) \leq Ct^{-\frac{3}{2} + \frac{1}{p}}.
\]

The estimation of the term $I_1$ and $I_2$ finish the proof of the decay estimate (9).

To get the estimate (11) let us recall well known estimation of the difference in time decay of the heat semigroup and heat kernel. Assuming that $n_0 \in L^1(\mathbb{R}^2; (1 + |x|^2))$ and denoting by $M = ||n_0||_1$ we have
\[
||e^{t\Delta}n_0 - MG(t)||_p = o(t^{-\frac{3}{2} + \frac{1}{p}}).
\]
Triangle inequality applied to the difference $e^{t\Delta}n_0 - MG(t)$ gives the required statement.

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