MODULAR TRANSFORMATIONS AND THE ELLIPTIC FUNCTIONS OF SHEN

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Abstract. We employ Weierstrassian modular transformations to compute fundamental periods for the elliptic functions $dn_2$ and $dn_3$ of Shen.

An Introduction

Ramanujan’s theories of elliptic functions to alternative bases were provided with specific elliptic functions by Li-Chien Shen: an elliptic function $dn_3$ [2004] in signature three and an elliptic function $dn_2$ [2014] in signature four. The definition of each of these functions involves inverting an incomplete hypergeometric integral on the real line; in each case, the resulting function is seen to satisfy a differential equation whose solutions are known to be elliptic.

When an elliptic function arises as a solution to a differential equation, its periods are often expressed as integrals. Archetypically, when a Weierstrass $\wp$ function appears as a solution to

$$(f')^2 = 4f^3 - g_2 f - g_3$$

with $g_2$ and $g_3$ real, its real fundamental half-period has the form

$$\int_{e_1}^{\infty} (4t^3 - g_2 t - g_3)^{-\frac{1}{2}} dt$$

where $e_1$ is the largest zero of the cubic $4t^3 - g_2 t - g_3$; its imaginary fundamental half-period has a similar integral expression.

Because $dn_3$ and $dn_2$ are recognized as solutions to differential equations, their fundamental half-periods may be expressed in the way just described; recasting them hypergeometrically often calls for tricky and seemingly ad hoc manipulations. Our purpose here is to show that the half-periods of $dn_3$ and $dn_2$ may be expressed in explicit hypergeometric terms quite simply and indeed naturally.

On the one hand, the hypergeometric origins of $dn_3$ and $dn_2$ give immediate explicit form to their real fundamental half-periods, on account of the standard integral identity

$$\int_0^{\frac{1}{2}\pi} F(a, b; \frac{1}{2}; \kappa^2 \sin^2 t) \, dt = \frac{1}{2} \pi F(a, b; 1; \kappa^2).$$

On the other hand, their imaginary fundamental half-periods may also be given explicit hypergeometric form without the need for further integration: we show that they may be derived from the real fundamental half-periods by the use of Weierstrassian modular transformations that are associated to trimidiation and dimidiation.

Two Modular Transformations

We prepare our analysis of the elliptic functions $dn_2$ and $dn_3$ by assembling certain facts regarding modular transformations as they pertain to Weierstrass $\wp$ functions. Thus, let $\wp$ be a Weierstrass $\wp$ function: specifically, let it be the Weierstrass function with invariants $g_2$
and \( g_3 \); as an alternative description, let it be the Weierstrass function having \((2\omega;2\omega')\) as a fundamental pair of periods. We may name \( p \) by its invariants or by its half-periods, writing

\[ p = \wp(\bullet; g_2, g_3) = \wp(\bullet; \omega, \omega'). \]

The Weierstrass function

\[ q = \wp(\bullet; \omega, \frac{1}{4}\omega') = \wp(\bullet; h_2, h_3) \]

obtained from \( p \) upon division of a period by the positive integer \( n \) is said to arise from \( p \) via a modular transformation. We are especially interested in the effect of such a modular transformation on the invariants of a Weierstrass function: that is, we wish to determine the invariants \( h_2 \) and \( h_3 \) of \( q \) in terms of the invariants \( g_2 \) and \( g_3 \) of \( p \); in fact, we shall only require this information in the cases \( n = 2 \) and \( n = 3 \). In each case we merely state the results, referring to [1973] and [1989] for proofs.

The effect of a quadratic transformation is as follows. Here,

\[ q = \wp(\bullet; \omega, \frac{1}{3}\omega'). \]

**Theorem 1.** If \( n = 2 \) and \( b = p(\omega') \) then

\[ h_2 = 60b^2 - 4g_2 \]

and

\[ h_3 = 56b^3 + 8g_3. \]

**Proof.** This proceeds from an inspection of the identity

\[ q(z) = p(z) + p(z + \omega') - p(\omega'). \]

Details of the derivation may be found in [1973] Section 65 and in [1989] Section 9.8. □

The effect of a cubic transformation is as follows. Here,

\[ q = \wp(\bullet; \omega, \frac{1}{3}\omega'). \]

**Theorem 2.** If \( n = 3 \) and \( b = p(\frac{2}{3}\omega') \) then

\[ h_2 = 120b^2 - 9g_2 \]

and

\[ h_3 = 280b^3 - 42bg_2 - 27g_3. \]

**Proof.** This proceeds from an inspection of the identity

\[ q(z) = p(z) + p(z + \frac{2}{3}\omega') + p(z - \frac{2}{3}\omega') - 2p(\frac{2}{3}\omega'). \]

Details of the derivation may be found in Section 68 of [1973]; see also Exercises 8 and 9 of [1989] Chapter 9. □

**Signature three**

We begin by briefly reviewing the origin of the elliptic function \( \mathrm{dn}_3 \). For further details, we refer the reader to [2004].

Fix \( \kappa \in (0, 1) \) as modulus and \( \lambda = \sqrt{1 - \kappa^2} \) as complementary modulus. Write \( \phi : \mathbb{R} \to \mathbb{R} \) for the inverse to the strictly increasing surjective function

\[ \mathbb{R} \to \mathbb{R} : T \mapsto \int_0^T \frac{1}{2} F\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; \kappa^2 \sin^2 t \right) \, dt \]

and write

\[ K = \int_0^{\frac{\pi}{2}} F\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; \kappa^2 \sin^2 t \right) \, dt = \frac{1}{2}\pi F\left(\frac{1}{3}, \frac{2}{3}; 1; \kappa^2 \right). \]

Elementary calculations show that \( \phi \) satisfies

\[ \phi(u + 2K) = \phi(u) + \pi \]

whence its derivative \( \phi' : \mathbb{R} \to \mathbb{R} \) has (least positive) period \( 2K \). We shall write \( \delta = \phi' \) for this derivative, writing \( \delta_\kappa \) when we wish to draw attention to the modulus \( \kappa \).
The function $\delta$ satisfies the initial condition $\delta(0) = 1$ because the function inverse to $\phi$ plainly has derivative 1 at the origin; with rather more work, it may be shown that $\delta$ satisfies the differential equation

$$9(\delta')^2 = 4(1 - \delta)(\delta^3 + 3\delta^2 - 4\lambda^2).$$

As the right-hand side of this complex differential equation is a quartic with simple zeros, its solutions are elliptic functions; the specific solution with $\delta(0) = 1$ is singled out as follows.

**Theorem 3.** The function $\delta_\kappa = \phi'$ satisfies

$$(1 - \delta_\kappa)(\frac{1}{3} + p_\kappa) = \frac{4}{9}\kappa^2$$

where $p_\kappa = \wp(\kappa; g_2, g_3)$ is the Weierstrass function with invariants

$$g_2 = \frac{4}{27}(9 - 8\kappa^2) = \frac{4}{27}(8\lambda^2 + 1)$$

and

$$g_3 = \frac{8}{729}(27 - 36\kappa^2 + 8\kappa^4) = \frac{8}{729}(8\lambda^4 + 20\lambda^2 - 1).$$

**Proof.** See [2004]; the proof is effected by reference to Section 20.6 of the classic [1927].

Thus, $\delta$ is the restriction to $\mathbb{R}$ of an elliptic function; this elliptic extension of $\delta$ is the function $d_3$ of Shen.

The elliptic function $d_3 = \delta_\kappa$ and the Weierstrass function $p_\kappa$ are plainly coperiodic. We shall denote by $(2\omega_\kappa, 2\omega'_\kappa)$ their fundamental pair of periods for which $\omega_\kappa$ and $-i\omega'_\kappa$ are strictly positive; we may then also write $p_\kappa = \wp(\kappa; \omega_\kappa, \omega'_\kappa)$. We have already identified the real half-period $\omega_\kappa$: it is precisely $K$ as displayed above; that is,

$$\omega_\kappa = \frac{1}{4}\pi F\left(\frac{1}{3}, \frac{2}{3}; 1; \kappa^2\right).$$

We now proceed to evaluate the imaginary half-period $\omega'_\kappa$. A customary method for performing such an evaluation involves the calculation of an integral. We propose to depart from this custom: instead, we shall make use of a modular transformation of the sort that is appropriate to Weierstrass functions.

Explicitly, alongside the Weierstrass function $p_\kappa = \wp(\kappa; \omega_\kappa, \omega'_\kappa)$ we introduce the Weierstrass function

$$q_\kappa = \wp(\kappa; \omega_\kappa, \frac{1}{3}\omega'_\kappa)$$

that results upon division of its imaginary period by three.

**Theorem 4.** The Weierstrass functions $p$ and $q$ are related by

$$q_\kappa(z) = -3p_\lambda(\sqrt{3}iz).$$

**Proof.** Note the passage to the complementary modulus. First, apply Theorem 2 and take into account the fact that

$$b = p_\kappa\left(\frac{1}{3}\omega'_\kappa\right) = -\frac{4}{3};$$

this nontrivial fact is proved in Section 5 of [2004]. From the $\kappa$-dependent formulae for $g_2$ and $g_3$ in Theorem 3 it follows by substitution that the invariants $h_2$ and $h_3$ of $q_\kappa$ are given by

$$h_2 = 120b^2 - 9g_2 = \frac{4}{3}(1 + 8\kappa^2)$$

and

$$h_3 = 280b^3 - 42bg_2 - 27g_3 = \frac{8}{27}(1 - 20\kappa^2 - 8\kappa^4).$$

It is now convenient to write $g_2(f)$ and $g_3(f)$ for the quadrinvariant and cubinvariant of any Weierstrass function $f$. With this understanding, we have just established that

$$g_2(q_\kappa) = 9g_2(p_\lambda) = (\sqrt{3}i)^4g_2(p_\lambda)$$

and

$$g_3(q_\kappa) = -27g_3(p_\lambda) = (\sqrt{3}i)^6g_3(p_\lambda).$$
by reference to Theorem 3 for the complementary modulus. The homogeneity relation for Weierstrass functions carries us to the announced conclusion

\[ q_\kappa(z) = (\sqrt{3}i)^2 p_\lambda(\sqrt{3}i z). \]

This relationship between Weierstrass functions entails a relationship between their half-periods. Explicitly, \( q_\kappa \) has fundamental half-periods \( \omega_\kappa \) and \( \frac{1}{3} \omega'_\kappa \) while \( p_\lambda \) has fundamental half-periods \( \omega_\lambda \) and \( \omega'_\lambda \). Accordingly, we deduce the relationship

\[ \omega'_\kappa = \sqrt{3}i \omega_\lambda. \]

**Theorem 5.** The fundamental half-periods of \( \text{dn}_3 = \delta_\kappa \) and \( p_\kappa \) are given by

\[ \omega_\kappa = \frac{1}{2} \pi F\left(\frac{1}{3}, \frac{2}{3}; 1; \kappa^2\right) \]

and

\[ \omega'_\kappa = i\sqrt{3} \pi F\left(\frac{1}{3}, \frac{2}{3}; 1; 1 - \kappa^2\right). \]

**Proof.** The real half-period has already been identified; the imaginary half-period follows at once from the relationship displayed immediately prior to the present Theorem, on account of the fact that \( \lambda^2 = 1 - \kappa^2 \). \( \square \)

Thus the shape of the period lattice is given by the period ratio

\[ \frac{\omega'_\kappa}{\omega_\kappa} = i\sqrt{3} F\left(\frac{1}{3}, \frac{2}{3}; 1; 1 - \kappa^2\right) \]

**Signature four**

We begin by briefly reviewing the origin of the elliptic function \( \text{dn}_2 \). For further details, we refer the reader to [2014].

Fix \( \kappa \in (0, 1) \) as modulus and \( \lambda = \sqrt{1 - \kappa^2} \) as complementary modulus. Write \( \phi : \mathbb{R} \to \mathbb{R} \) for the inverse to the strictly increasing surjective function

\[ \mathbb{R} \to \mathbb{R} : T \mapsto \int_0^T F\left(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}; \kappa^2 \sin^2 t\right) \; dt \]

and write

\[ K = \int_0^{\pi/2} F\left(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}; \kappa^2 \sin^2 t\right) \; dt = \frac{1}{2} \pi F\left(\frac{1}{4}, \frac{3}{4}; 1; \kappa^2\right). \]

Elementary calculations show that \( \phi \) satisfies

\[ \phi(u + 2K) = \phi(u) + \pi \]

whence if

\[ \psi = \arcsin(\kappa \sin \phi) \]

then the function \( d = \cos \psi \) has (least positive) period \( 2K \). When we wish to place the modulus \( \kappa \) in evidence, it shall appear as a subscript.

The function \( d \) satisfies the initial condition \( d(0) = 1 \) quite plainly; less plainly, it also satisfies the differential equation

\[ (d')^2 = 2(1 - d)(d^2 - \lambda^2). \]

The solution to this initial value problem extends to an elliptic function that may be expressed in terms of its coperiodic Weierstrass \( p \) function, as follows.
Theorem 6. The function \( d_\kappa = \cos \psi \) satisfies
\[
(1 - d_\kappa)(\frac{1}{3} + p_\kappa) = \frac{1}{2} \kappa^2
\]
where \( p_\kappa = \wp(\bullet; g_2, g_3) \) is the Weierstrass function with invariants
\[
g_2 = \frac{4}{3} - \kappa^2 = \lambda^2 + \frac{1}{3}
\]
and
\[
g_3 = \frac{8}{27} - \frac{1}{3} \kappa^2 = \frac{1}{3} \lambda^2 - \frac{1}{27}.
\]

Proof. See [2014]; again, the proof refers to Section 20.6 of [1927].

The function \( d_{n_2} \) of Shen is the elliptic extension of \( d \) guaranteed by this Theorem.

We write \( (2\omega_\kappa, 2\omega'_\kappa) \) for the fundamental pair of periods for \( d_{n_2} = d_\kappa \) and \( p_\kappa \) such that \( \omega_\kappa \) and \(-i\omega'_\kappa\) are strictly positive. The real half-period \( \omega_\kappa \) has already been identified in hypergeometric terms; the imaginary half-period \( \omega'_\kappa \) will now be similarly identified by means of an appropriate modular transformation.

Thus, as a companion to \( p_\kappa = \wp(\bullet; \omega_\kappa, \omega'_\kappa) \) we introduce the Weierstrass function
\[
q_\kappa = \wp(\bullet; \omega_\kappa, \frac{1}{2}\omega'_\kappa)
\]
that results upon halving its imaginary period.

Theorem 7. The Weierstrass functions \( p \) and \( q \) are related by
\[
q_\kappa(z) = -2p_\lambda(\sqrt{2i}z).
\]

Proof. Again, note the involvement of the complementary modulus. The proof follows the line of argument for Theorem 4. It is shown in Section 4 of [2014] that
\[
p_\kappa(\omega'_\kappa) = -\frac{1}{3}.
\]

Accordingly, by application of Theorem 1 along with reference to the \( \kappa \)-dependent formulae for \( g_2 \) and \( g_3 \) in Theorem 6 the invariants \( h_2 \) and \( h_3 \) of \( q_\kappa \) are found to be
\[
h_2 = 4(\frac{1}{3} + \kappa^2)
\]
(which is \( 4 = (\sqrt{2i})^4 \) times the quadrinvariant of \( p_\lambda \)) and
\[
h_3 = 8(\frac{1}{27} - \frac{1}{3} \kappa^2)
\]
(which is \( -8 = (\sqrt{2i})^6 \) times the cubinvariant of \( p_\lambda \)). Finally, the Weierstrassian homogeneity relation serves to conclude the proof.

As was the case in signature three, a relation between real and imaginary half-periods for complementary moduli follows here: thus
\[
\omega'_\kappa = \sqrt{2i} \omega_\lambda.
\]

Theorem 8. The fundamental half-periods of \( d_{n_2} = d_\kappa \) and \( p_\kappa \) are given by
\[
\omega_\kappa = \frac{1}{2\pi} F(\frac{1}{4}, \frac{3}{4}; 1; \kappa^2)
\]
and
\[
\omega'_\kappa = i\sqrt{2} \pi F(\frac{1}{4}, \frac{3}{4}; 1; 1 - \kappa^2).
\]

Proof. As for signature three, the relationship between \( \omega'_\kappa \) and \( \omega_\lambda \) allows us to derive the imaginary half-period at once from the previously identified real half-period.

The shape of the period lattice is thus given by the period ratio
\[
\frac{\omega'_\kappa}{\omega_\kappa} = i\sqrt{2} \frac{F(\frac{1}{4}, \frac{3}{4}; 1; 1 - \kappa^2)}{F(\frac{1}{4}, \frac{3}{4}; 1; \kappa^2)}.
\]
REFERENCES

[1927] E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis*, Fourth Edition, Cambridge University Press.

[1973] P. Du Val, *Elliptic Functions and Elliptic Curves*, L.M.S. Lecture Note Series 9, Cambridge University Press.

[1989] D.F. Lawden, *Elliptic Functions and Applications*, Applied Mathematical Sciences 80, Springer-Verlag.

[2004] Li-Chien Shen, *On the theory of elliptic functions based on* $\genfrac{[}{]}{0pt}{}{2}{1}\genfrac{.}{.}{0pt}{}{\frac{1}{3}, \frac{2}{3}, \frac{1}{2}}{z}$, Transactions of the American Mathematical Society 357 2043-2058.

[2014] Li-Chien Shen, *On a theory of elliptic functions based on the incomplete integral of the hypergeometric function* $\genfrac{[}{]}{0pt}{}{2}{1}\genfrac{.}{.}{0pt}{}{\frac{1}{4}, \frac{3}{4}}{\frac{1}{2}; z}$, Ramanujan Journal 34 209-225.

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