Starlikeness of the generalized Bessel function

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ABSTRACT
For a fixed $a \in \{1, 2, 3, \ldots\}$, the radius of starlikeness of positive order is obtained for each of the normalized analytic functions

$f_{a,\nu}(z) := \left(2^{a\nu-a+1} a - \frac{a(a\nu+a+1)}{2} \Gamma(a\nu + 1) aB_{2a-1,a\nu-a+1,1}(a^{s/2} z) \right) \frac{1}{a^{a-a+1}},$

$g_{a,\nu}(z) := 2^{a\nu-a+1} a - \frac{a(a\nu+a+1)}{2} \Gamma(a\nu + 1) a^{a-a\nu} aB_{2a-1,a\nu-a+1,1}(a^{a/2} z),$

$h_{a,\nu}(z) := 2^{a\nu-a+1} a - \frac{a(a\nu+a+1)}{2} \Gamma(a\nu + 1) a^{1/2(1+a-a\nu)} aB_{2a-1,a\nu-a+1,1}(a^{a/2} \sqrt{z})$

in the unit disk, where $aB_{b,p,c}$ is the generalized Bessel function

$aB_{b,p,c}(z) := \sum_{k=0}^{\infty} \frac{(-c)^k}{k! \Gamma(ak + p + \frac{s+1}{2})} \left(\frac{z}{2}\right)^{2k+p}.$

The best range on $\nu$ is also obtained for a fixed $a$ to ensure the functions $f_{a,\nu}$ and $g_{a,\nu}$ are starlike of positive order in the unit disk. When $a = 1$, the results obtained reduced to earlier known results.

KEYWORDS
Bessel function; generalized Bessel function; starlike function; radius of starlikeness

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1. Introduction

There is a vast literature describing the importance and applications of the Bessel function of the first kind of order $p$ given by

$J_p(x) := \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k + p + 1)} \left(\frac{x}{2}\right)^{2k+p},$

where $\Gamma$ is the familiar gamma function. Various generalizations of the Bessel function have also been studied. Perhaps a more complete generalization is that given by Baricz
In this case, the generalized Bessel function takes the form

\[ a_{b,p,c}(x) := \sum_{k=0}^{\infty} \frac{(-c)^k}{k! \Gamma(ak + p + \frac{b+1}{2})} \left( \frac{x}{2} \right)^{2k+p} \]  

for \( a \in \mathbb{N} = \{1, 2, 3, \ldots\} \), and \( b, p, c, x \in \mathbb{R} \). It is evident that the function \( a_{b,p,c} \) converges absolutely at each \( x \in \mathbb{R} \). This generalized Bessel function was further investigated in [1,2] for \( z \in \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \). It was shown in [2] that the generalized Bessel function \( a_{b,p,c} \) is a solution of an \((a+1)\)-order differential equation

\[
(D - p) \prod_{j=1}^{a} \left(D + \frac{2p+b+1-2j}{a} - p\right)y(x) + \frac{cx^2}{a^{a+1-a}}y(x) = 0,
\]

where the operator \( D \) is given by \( D := x(d/dx) \). For \( a = 1 \), the differential equation reduces to

\[
x^2y''(x) + bxy'(x) + (cx^2 - p^2 + (1 - b)p)y(x) = 0.
\]

Thus it yields the classical Bessel differential equation for \( b = c = 1 \). Interesting functional inequalities for \( a_{b,p,-\alpha^2} \) were obtained, particularly for the case \( a = 2 \).

In [3], Baricz et al. investigated geometric properties involving the Bessel function of the first kind in \( \mathbb{D} \) for the following three functions:

\[
\begin{align*}
 f_{\nu}(z) &= (2^\nu \Gamma(\nu + 1)J_{\nu}(z))^{\frac{1}{\nu}}, \\
 g_{\nu}(z) &= 2^\nu \Gamma(\nu + 1)z^{1-\nu}J_{\nu}(z), \\
 h_{\nu}(z) &= 2^\nu \Gamma(\nu + 1)z^{1-\frac{\nu}{2}}J_{\nu}(\sqrt{z}).
\end{align*}
\]

Each function is suitably normalized to ensure that it belongs to the class \( \mathcal{A} \) consisting of analytic functions \( f \) in \( \mathbb{D} \) satisfying \( f(0) = f'(0) - 1 = 0 \). Here the principal branch is assumed, which is positive for \( z \) positive.

An important geometric feature of a complex-valued function is starlikeness. For \( 0 \leq \beta < 1 \), the class of starlike functions of order \( \beta \), denoted by \( \mathcal{S}^*(\beta) \), are functions \( f \in \mathcal{A} \) satisfying

\[
\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \beta \quad \text{for all} \quad z \in \mathbb{D}.
\]

In the case \( \beta = 0 \), these functions are simply said to be starlike (with respect to the origin). Geometrically \( f \in \mathcal{S}^* := \mathcal{S}^*(0) \) if the linear segment \( tw, 0 \leq t \leq 1 \), lies completely in \( f(\mathbb{D}) \) whenever \( w \in f(\mathbb{D}) \). A starlike function is necessarily univalent in \( \mathbb{D} \).

The three functions given by [2] do not possess the property of starlikeness in the whole disk \( \mathbb{D} \). Thus it is of interest to find the largest subdisk in \( \mathbb{D} \) that gets mapped by these functions onto starlike domains. In general, the radius of starlikeness of order \( \beta \) for a given class \( \mathcal{G} \) of \( \mathcal{A} \), denoted by \( r^*_\mathcal{G}(\beta) \), is the largest number \( r_0 \in (0, 1) \) such that
\( r^{-1} f(rz) \in \mathcal{S}^*(\beta) \) for \( 0 < r \leq r_0 \) and for all \( f \in \mathcal{G} \). Analytically,

\[
r^*_\beta(\mathcal{G}) := \sup \left\{ r > 0 : \Re \left( \frac{zf'(z)}{f(z)} \right) > \beta, \quad z \in \mathbb{D}_r, f \in \mathcal{G} \right\},
\]

where \( \mathbb{D}_r = \{ z : |z| < r \} \).

In [4], Baricz et. al. obtained the radius of starlikeness of order \( \beta \) for each of the three functions \( f_\nu, g_\nu, \) and \( h_\nu \) given by [2]. This extends the earlier work of Brown in [7] who obtained the radius of starlikeness (of order 0) for functions \( f_\nu \) and \( g_\nu \).

For \( a \in \mathbb{N} \), we consider the following extension of the three functions in [2] involving the generalized Bessel function:

\[
f_{a,\nu}(z) := \left(2^{a\nu-1}a^{-\frac{a\nu-1}{2}}\Gamma(a\nu+1)\right) \frac{1}{a^{a\nu+1}} \Gamma(2a\nu+1),
\]

\[
g_{a,\nu}(z) := 2^{a\nu-1}a^{-\frac{a\nu-1}{2}}\Gamma(a\nu+1)z^{a-\nu}B_{2a-1,a\nu-a+1,1}(a^{a/2}z),
\]

\[
h_{a,\nu}(z) := 2^{a\nu-1}a^{-\frac{a\nu-1}{2}}\Gamma(a\nu+1)z^{\frac{1}{2}(1-a-\nu)}B_{2a-1,a\nu-a+1,1}(a^{a/2}\sqrt{z}).
\]

Here the function \( f_{a,\nu} \) is taken to be the principal branch (see the following section). Evidently for \( a = 1 \), these functions are those given by [2] treated by Baricz et. al in [4]. Denote by \( r^*_\beta(f) \) to be the radius of starlikeness of order \( \beta \) for a given function \( f \).

In this paper, we find \( r^*_\beta(f_a) \) when \( f_a \) is either one of the three functions in [3]. The best range on \( \nu \) is also obtained for a fixed \( a \) to ensure the functions \( f_{a,\nu} \) and \( g_{a,\nu} \) are starlike of order \( \beta \) in the unit disk. When \( a = 1 \), the results obtained reduced to earlier known results.

For \( a = 1 \), the generalized Bessel function (1) is simply written as \( B_{b,p,c} := B_{b,p,c} \). Thus

\[
B_{b,p,c}(z) := \sum_{k=0}^{\infty} \frac{(-c)^k}{k!(k+p+b+1/2)} \left( \frac{z}{2} \right)^{2k+p}.
\]

**Proposition 1.1.** [2, Proposition 2.2] Let \( a \in \mathbb{N} \), and \( b, p, c, \in \mathbb{R} \). Then

\[
aB_{b,p,c}(z) = (2\pi)^{\frac{a-1}{2}} \frac{a}{a^p} \left( \frac{z}{2} \right)^{\frac{b}{2}} \prod_{j=1}^{a} \left( \frac{z}{2a^{a/2}} \right)^{-\frac{p+b}{a}} B_{b+1-a, p+j-1, a, c, \frac{z}{a^{a/2}}},
\]

where \( B_{b,p,c} \) is given by (1).

In [2], the generalized Bessel function was also shown to satisfy the following relations:

\[
z \frac{d}{dz} aB_{b,p,c}(z) = pB_{b,p,c}(z) - c \left( \frac{z}{2} \right)^{b-1} aB_{b+p,c}(z),
\]

and

\[
z \frac{d}{dz} aB_{b,p,c}(z) = \left( \frac{2p+b-1}{a} - p \right) aB_{b,p,c}(z),
\]

which together lead to the following result.
Proposition 1.2. [2, Proposition 2.3] Let \( a \in \mathbb{N}, b, p, c \in \mathbb{R} \) and \( z \in \mathbb{D} \). Then

\[
\frac{z}{a} aB_{b,p-1,c}(z) + c \left( \frac{z}{2} \right)^{1-a} z aB_{b,p+a,c}(z) = \left( \frac{2p+b-1}{a} \right) aB_{b,p,c}(z).
\]

2. Radius of starlikeness of generalized Bessel functions

The following preliminary result sheds insights into the zeros of the three functions given by (3).

Theorem 2.1. Let \( \nu > (a-1)/a, a \in \mathbb{N} \). Then all zeros of \( aB_{2a-1,av-a+1,1}(a^{a/2}z) \) are real. Further the origin is the only zero of \( aB_{2a-1,av-a+1,1}(a^{a/2}z) \) in the unit disk \( \mathbb{D} \).

Proof. Proposition 1.1 shows that

\[
aB_{2a-1,av-a+1,1}(a^{a/2}z) = (2\pi) \left( \frac{a-1}{a} \right) (av-a+1) \left( \frac{z}{2} \right)^{av-a+1} \times \prod_{j=1}^{\infty} \left( \frac{z}{2} \right)^{-(\nu-1)2^{-j}} B_{1,\nu-1+j/a,1}(z).
\]

Since

\[
B_{1,\nu-1+j/a,1}(z) = J_{\nu-1+j/a}(z),
\]

it readily follows that

\[
aB_{2a-1,av-a+1,1}(a^{a/2}z) = (2\pi) \left( \frac{a-1}{a} \right) \left( a^{2\nu-2av-a^2+a-1} \right) \left( \frac{z}{2} \right)^{-\frac{1}{2}(a-1)} \times J_{\nu-1+1/2}(z) J_{\nu-1+2/3}(z) \ldots J_{\nu}(z).
\]

Now, \( \nu - 1 + (j/a) \geq \nu - 1 + (1/a) > 0, j = 1, \ldots, a \). Further for \( p > -1 \), it is known [12, p. 483] that the zeros of \( J_p \) are all real. If \( j_{p,k} \) denotes the \( k \)-th positive zero of \( J_p \), it is also known [12, p. 508] that when \( p \) is positive, the positive zeros of \( J_p \) increases as \( p \) increases. Thus we infer that the zeros of \( aB_{2a-1,av-a+1,1}(a^{a/2}z) \) are all real. Since

\[
j_{\nu,1} > j_{\nu-1,1} > \ldots > j_{\nu-1+(1/a),1} > j_{0,1} \approx 2.40483,
\]

the only zero in \( \mathbb{D} \) occurs at the origin.

Theorem 2.1 shows that the function

\[
f_{a,\nu}(z) = z \left( 1 + \sum_{k=1}^{\infty} \frac{(-1)^k \Gamma(a\nu + 1)a^{ak}}{k!2^{2k}\Gamma(ak + a\nu + 1)} z^{2k} \right)^{\frac{1}{av-a+1}}
\]

have only one zero inside \( \mathbb{D} \) whenever \( \nu - 1 + (1/a) > 0 \). Thus in this instance, we may take the principal branch for \( f_{a,\nu} \in \mathcal{A} \). It is also readily verified that the functions \( g_{a,\nu} \)

\[
\left( \frac{a-1}{a} \right) (av-a+1) \left( \frac{z}{2} \right)^{av-a+1} \times \prod_{j=1}^{\infty} \left( \frac{z}{2} \right)^{-(\nu-1)2^{-j}} B_{1,\nu-1+j/a,1}(z).
\]
and
\[ h_{a,\nu}(z) = z - \frac{\Gamma(\nu + 1)}{1! 2^2 \Gamma(a + \nu + 1)} a^a z^2 + \frac{\Gamma(\nu + 1)}{2! 2^4 \Gamma(2a + \nu + 1)} a^{2a} z^3 + \ldots \]
\[ + (-1)^k \frac{\Gamma(\nu + 1)}{k! 2^{2k} \Gamma(ak + \nu + 1)} a^{ak} z^{k+1} + \ldots \]
are both analytic and belong to the normalized class \( \mathcal{A} \).

The following is another preliminary result required in the sequel.

**Lemma 2.2.** Let \( a \in \mathbb{N} \) and \( \nu > -1/a \). Then
\[ \frac{z a^{B_{2a-1, a\nu-a+1, 1}(a^{a/2} z)}}{a^{B_{2a-1, a\nu-a+1, 1}(a^{a/2} z)}} = \frac{z J_{\nu-1}(z)}{J_\nu(z)} - (2 - a)\nu + 1 - a. \]

**Proof.** Since
\[ B_{0,0,1}(z) = \left( \frac{z}{2} \right) \frac{b-1}{2} \frac{z^b}{x} J_{b-1,0}(z), \]
it follows from Proposition 1.1 that
\[ a^{B_{2a-1, a\nu-a+1, 1}(z)} = (2\pi)^{a-1} a^{2a+1} \left( \frac{z}{2} \right)^{a\nu-a+1} a^{a-2} \prod_{j=1}^{\infty} \left( 1 - \frac{1}{2} \right)^{a\nu-a+j} \left( \frac{2}{z} \right) \frac{a^{a\nu-a+j}}{J_{a\nu-a+j}(z)}, \]
and
\[ a^{B_{2a-1, a\nu-a+1, 1}(z)} = (2\pi)^{a-1} a^{2a+1} \left( \frac{z}{2} \right)^{a\nu-a} a^{a-2} \prod_{j=1}^{\infty} \left( 1 - \frac{1}{2} \right)^{a\nu-a+j-1} \left( \frac{2}{z} \right) \frac{a^{a\nu-a+j-1}}{J_{a\nu-a+j-1}(z)}. \]

Expanding the above products, a routine calculation shows that
\[ \frac{a^{B_{2a-1, a\nu-a+1, 1}(z)}}{a^{B_{2a-1, a\nu-a+1, 1}(z)}} = a^{1 - \frac{a}{2} \frac{J_{\nu-1}(z)}{J_\nu(z)}}. \]

With \( b = 2a - 1 \) and \( p = a\nu - a + 1 \), the recurrence relation (5) gives
\[ \frac{z}{d} \frac{d}{dz} a^{B_{2a-1, a\nu-a+1, 1}(z)} = \frac{z}{a^{B_{2a-1, a\nu-a+1, 1}(z)}} - (\nu(2 - a) + a - 1) a^{B_{2a-1, a\nu-a+1, 1}(z)}. \]

Replacing \( z \) by \( a^{a/2} z \) leads to
\[ \frac{z a^{B_{2a-1, a\nu-a+1, 1}(a^{a/2} z)}}{a^{B_{2a-1, a\nu-a+1, 1}(a^{a/2} z)}} = \frac{a^{a/2} z}{a} \frac{a^{B_{2a-1, a\nu-a+1, 1}(a^{a/2} z)}}{a^{B_{2a-1, a\nu-a+1, 1}(a^{a/2} z)}} - (2 - a)\nu + 1 - a \]
\[ = \frac{z J_{\nu-1}(z)}{J_\nu(z)} - (2 - a)\nu + 1 - a, \]
which proves the assertion. \( \square \)
A result on the modified Bessel function of order $p$ given by

$$I_p(z) = \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(k + p + 1)} \left( \frac{z}{2} \right)^{2k+p}$$

is the final preliminary result required in the sequel.

**Proposition 2.3.** Let $\alpha, \nu \in \mathbb{R}$ satisfy $-1 < \nu < -\alpha$. Then the equation $r I'_\nu(r) + \alpha I_\nu(r) = 0$ has a unique root in $(0, \infty)$.

**Proof.** Consider the function

$$q(r) := \frac{r I'_\nu(r)}{I_\nu(r)} + \alpha.$$

It is known from [2, Theorem 3.1(c)] that $r I'_\nu(r) / I_\nu(r)$ is increasing on $(0, \infty)$. Further, the asymptotic properties show that $r I'_\nu(r) / I_\nu(r) \to \nu$ as $r \to 0$, and $r I'_\nu(r) / I_\nu(r) \to \infty$ as $r \to \infty$. This implies that $q(r) \to \nu + \alpha < 0$ as $r \to 0$, and $q(r) \to \infty$ for $r \to \infty$. Thus $q$ has exactly one zero.

We also recall additional facts on the zeros of the Dini functions.

**Lemma 2.4.** [12, p. 482] If $\nu > -1$ and $\alpha, \gamma \in \mathbb{R}$, then the Dini function $z \mapsto \alpha J_\nu(z) + \gamma z J'_\nu(z)$ has all its zeros real whenever $((\alpha/\gamma) + \nu) \geq 0$. In the case $((\alpha/\gamma) + \nu) < 0$, it also has two purely imaginary zeros.

**Lemma 2.5.** [9, Theorem 6.1] Let $\alpha \in \mathbb{R}$, $\nu > -1$ and $\nu + \alpha > 0$. Further let $x_{\nu,1}$ be the smallest positive root of $\alpha J_\nu(z) + z J'_\nu(z) = 0$. Then

$$x_{\nu,1}^2 < j_{\nu,1}^2.$$ 

**Lemma 2.6.** [8, p. 78] Let $-1 < \nu < -\alpha$, and $\pm i \zeta$ be the single pair of conjugate purely imaginary zeros of the Dini function $z \mapsto \alpha J_\nu(z) + z J'_\nu(z)$. Then

$$\zeta^2 < -\frac{\alpha + \nu}{2 + \alpha + \nu j_{\nu,1}^2}.$$ 

We are now ready to present the radius of starlikeness for each function given in

**Theorem 2.7.** Let $0 \leq \beta < 1$, and $a \in \mathbb{N}$. If $\nu > (a - 1)/a$, then $r^a_\beta(f_{a,\nu}) = j_{\nu,\beta,1}^a$, where $j_{\nu,\beta,1}^a$ is the smallest positive root of the equation

$$ra^{a/2} J'_\nu(r) - \left( (\nu - 1)(1 - a)a^{a/2} + \beta (av - a + 1) \right) J_\nu(r) = 0. \quad (6)$$

If $\nu \in (-1/a, (a-1)/a)$ and

$$\frac{(av - a + 1) (a^{a/2} - \beta)}{2a^{a/2} + (av - a + 1) (a^{a/2} - \beta)} > -1,$$ \quad (7)
then \( r_\beta^*(f_{a,\nu}) = i_{\nu,\beta}^{a,f} \), where \( i_{\nu,\beta}^{a,f} \) is the unique positive root of the equation

\[
ra^{a/2}I_\nu(r) - \left((\nu - 1)(1-a)a^{a/2} + \beta(\nu - a + 1)\right) I_\nu(r) = 0.
\] (8)

**Proof.** Differentiating logarithmically, Lemma 2.2 shows that

\[
\frac{zf'_{a,\nu}(z)}{f_{a,\nu}(z)} = \frac{a^{a/2}}{av-a+1} \frac{z B_{a-1,av-a+1,1}(a^{a/2}z)}{a B_{a-1,av-a+1,1}(a^{a/2}z)}
\]

\[
= \frac{a^{a/2}}{av-a+1} \left( \frac{zJ_{\nu-1}(z)}{J_\nu(z)} - \nu(2-a) + 1 - a \right). \tag{9}
\]

Since \( J_{1,\nu,1}(z) = J_\nu(z) \), the relation (5) leads to the well-known recurrence relation

\[
zJ'_\nu(z) = zJ_{\nu-1}(z) - \nu J_\nu(z),
\]

and whence (9) reduces to

\[
\frac{zf'_{a,\nu}(z)}{f_{a,\nu}(z)} = \frac{a^{a/2}}{av-a+1} \left( \frac{zJ'_\nu(z)}{J_\nu(z)} - (\nu - 1)(1-a) \right). \tag{10}
\]

With \( j_{\nu,n} \) as the n-th positive zero of the Bessel function \( J_\nu \), the Bessel function \( J_\nu \) admits the Weierstrassian decomposition [12, p.498]

\[
J_\nu(z) = \frac{z^\nu}{2^\nu \Gamma(\nu+1)} \prod_{n=1}^\infty \left(1 - \frac{z^2}{j_{\nu,n}^2} \right).
\]

Thus

\[
z \frac{J'_\nu(z)}{J_\nu(z)} = \nu - \sum_{n=1}^\infty \frac{2z^2}{j_{\nu,n}^2 - z^2},
\]

which reduces (10) to

\[
\frac{zf'_{a,\nu}(z)}{f_{a,\nu}(z)} = a^{a/2} - \frac{a^{a/2}}{av-a+1} \sum_{n=1}^\infty \frac{2z^2}{j_{\nu,n}^2 - z^2}.
\] (11)

For \( \nu > (a-1)/a \) and \( |z| < j_{\nu,n} \), evidently

\[
\text{Re} \frac{zf'_{a,\nu}(z)}{f_{a,\nu}(z)} = a^{a/2} - \frac{a^{a/2}}{av-a+1} \text{Re} \sum_{n=1}^\infty \frac{2z^2}{j_{\nu,n}^2 - z^2}
\]

\[
\ge a^{a/2} - \frac{a^{a/2}}{av-a+1} \sum_{n=1}^\infty \frac{2|z|^2}{j_{\nu,n}^2 - |z|^2} = \frac{|z|f'_{a,\nu}(|z|)}{f_{a,\nu}(|z|)}.
\]
Equality holds for $|z| = r$, and by the minimum principle for harmonic functions,

$$\text{Re} \frac{zf'_\nu(z)}{f_\nu(z)} \geq \beta \iff |z| \leq j_{\nu,1}^{a,f},$$

where $j_{\nu,1}^{a,f}$ is the smallest positive root of equation (6). Since

$$\nu - \left( (\nu - 1)(1 - a) + \frac{\beta(a\nu - a + 1)}{a\nu^2} \right) = (a\nu - a + 1) \left( 1 - \frac{\beta}{a \nu} \right) > 0$$

for all $\nu > (a - 1)/a$, we infer from Lemma 2.4 and Lemma 2.5 that $j_{\nu,1}^{a,f} < j_{\nu,n} < j_{\nu,1}^{a,f}$. Consider next the case $-1/a < \nu < (a - 1)/a$. It is known from [4, p. 2023] that for $z \in \mathbb{C}$ and $\alpha \in \mathbb{R}$ with $\alpha \geq |z|$, then

$$\text{Re} \left( \frac{z}{\alpha - z} \right) \geq - \frac{|z|}{\alpha + |z|},$$

which in turn implies that

$$\text{Re} \left( \frac{z^2}{j_{\nu,n}^2 - z^2} \right) \geq - \frac{|z|^2}{j_{\nu,n}^2 + |z|^2}$$

whenever $|z| < j_{\nu,1} < j_{\nu,n}$.

The expression (11) yields

$$\text{Re} \frac{zf'_a(z)}{f_a(z)} \geq a^{a/2} + \frac{a^{a/2}}{a\nu - a + 1} \sum_{n=1}^{\infty} \frac{2|z|^2}{j_{\nu,n}^2 + |z|^2} = \frac{i|z|f'_a(i|z|)}{f_a(i|z|)}.$$

Equality holds for $|z| = i|z| = ir$. Hence

$$\text{Re} \frac{zf'_a(z)}{f_a(z)} \geq \beta$$

if $|z| \leq i_{\nu,1}^{a,f}$, where $i_{\nu,1}^{a,f}$ is a root of $i|z|f'_a(i|z|) = \beta f_a(i|z|)$, that is, $i_{\nu,1}^{a,f}$ is a root of

$$\frac{a^{a/2}}{a\nu - a + 1} \left( \frac{i|z|f'_a(i|z|)}{f_a(i|z|)} - (\nu - 1)(1 - a) \right) = \beta.$$

Since $I_{\nu}(z) = i^{-\nu}J_{\nu}(iz)$, the above equation is equivalent to (8). It also follows from Proposition 2.3 that the root $i_{\nu,1}^{a,f}$ is unique. Finally, that $i_{\nu,1}^{a,f} < j_{\nu,n}$ is a consequence of Lemma 2.6 and assumption (7). Indeed,

$$\left( i_{\nu,1}^{a,f} \right)^2 < - \frac{(a\nu - a + 1)(a^{a/2} - \beta)}{2a^{a/2} + (a\nu - a + 1)(a^{a/2} - \beta)} j_{\nu,n}^2 < j_{\nu,1}^2 < j_{\nu,n}^2,$$

which completes the proof.

Interestingly, Theorem 2.7 reduces to earlier known result for $a = 1$. 

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Corollary 2.8. Let $0 \leq \beta < 1$. If $\nu > 0$, then $r^*_\beta(f_{1,\nu})$ is the smallest positive root $j^{1,1}_{\nu,1}$ of the equation

$$r J'_\nu(r) - \beta \nu J_\nu(r) = 0.$$ 

In the case $\nu \in (-1,0)$, then $r^*_\beta(f_{1,\nu})$ is the unique positive root $j^{1,1}_{\nu,1}$ of the equation

$$r I'_\nu(r) - \beta \nu I_\nu(r) = 0.$$ 

The next two results find the radius of starlikeness of order $\beta$ for the functions $g_{a,\nu}$ and $h_{a,\nu}$ given in (3).

**Theorem 2.9.** Let $\beta \in [0,1)$, $a \in \mathbb{N}$, and $\nu > -1/a$. If $a(\nu - 1)(a^{\alpha/2} - 1) + a^{\alpha/2} - \beta \geq 0$, then $r^*_\beta(g_{a,\nu}) = \overline{J}^a_{\nu,1}$, where $\overline{J}^a_{\nu,1}$ is the smallest positive root of the equation

$$ra^{a/2} J'_\nu(r) - (\nu - 1)(1 - a)a^{a/2} - a(1 - \nu) + \beta) J_\nu(r) = 0. \quad (12)$$

**Proof.** It follows from (3) that

$$\frac{zg'_{a,\nu}(z)}{g_{a,\nu}(z)} = a(1 - \nu) + a^{a/2} \left( \frac{zJ_{\nu-1}(z)}{J_\nu(z)} - \nu(2 - a) + 1 - a \right).$$

As in the proof of Theorem 2.7 (see (9) and (10)), it is readily shown that

$$\frac{zg'_{a,\nu}(z)}{g_{a,\nu}(z)} = a(1 - \nu) + a^{a/2} \left( \frac{zJ'_\nu(z)}{J_\nu(z)} - (\nu - 1)(1 - a) \right)$$

$$= a(1 - \nu) + a^{a/2} \left[ a\nu + 1 - a - \sum_{n=1}^{\infty} \frac{2z^2}{J^2_{\nu,n} - z^2} \right]. \quad (13)$$

This implies that

$$\text{Re} \frac{zg'_{a,\nu}(z)}{g_{a,\nu}(z)} \geq a(1 - \nu) + a^{a/2} \left[ a\nu + 1 - a - \sum_{n=1}^{\infty} \frac{2|z|^2}{J^2_{\nu,n} - |z|^2} \right] = \frac{|z|g'_{a,\nu}(|z|)}{g_{a,\nu}(|z|)}$$

provided $|z| < j_{\nu,n}$. Equality holds at $|z| = r$. The minimum principle for harmonic functions leads to

$$\text{Re} \frac{zg'_{a,\nu}(z)}{g_{a,\nu}(z)} \geq \beta \iff |z| \leq r^*_\beta(g_{a,\nu}).$$

The exact value of $r^*_\beta(g_{a,\nu})$ is obtained from the equation $rg'_{a,\nu}(r) = \beta g_{a,\nu}(r)$. From (13), this is equivalent to determining the root of (12).

If $a(\nu - 1)(a^{\alpha/2} - 1) + a^{\alpha/2} - \beta \geq 0$, then Lemma 2.4 shows that all roots of (12) are real. In this case, $r^*_\beta(g_{a,\nu})$ is its smallest positive root $\overline{J}^a_{\nu,1}$. Finally, Lemma 2.4 shows that $j_{\nu,1} < j_{\nu,1}$, and whence $|z| < r^*_\beta(g_{a,\nu}) < j_{\nu,1}$. $\blacksquare$
Theorem 2.10. Let $\beta \in [0, 1)$, $a \in \mathbb{N}$, and $\nu > -1/a$. If $(a^{\alpha/2} - 1)(1 - a + a\nu) + 2(1 - \beta) > 0$, then $r^{\ast}_{\beta}(h_{a, \nu}) = j_{\nu, \beta, 1}^{a, h}$, where $j_{\nu, \beta, 1}$ is the smallest positive root of the equation
\[
a^{\alpha/2} r_j'(r) + \left((a^{\alpha/2} - 1)(1 - a + a\nu) - a^{\alpha/2}\nu + 2(1 - \beta)\right) J_\nu(r) = 0. \tag{14}
\]

Proof. It follows from (3) that
\[
\beta 
\]
and thus Lemma 2.2 yields
\[
\begin{aligned}
zh_{a, \nu}(z) & = 1 + a - a\nu + a^{\alpha/2} \frac{B_{2a-1, a\nu-a+1, 1}^{(a^{\alpha/2} \sqrt{z})}}{2\sqrt{z}} \\
& = 1 + a - a\nu + a^{\alpha/2} \frac{(\sqrt{z} J_\nu'((\sqrt{z})) - (\nu - 1)(1 - a))}{2} \\
& = 1 - \frac{(a\nu + 1 - a)(1 - a^{\alpha/2})}{2} - a^{\alpha/2} \sum_{n=1}^{\infty} \frac{z}{J_{\nu, n}^2}.
\end{aligned}
\]

Proceeding similarly as in the proof of Theorem 2.7, it is readily shown that
\[
\text{Re} \frac{zh_{a, \nu}(z)}{h_{a, \nu}(z)} \geq 1 - \frac{(a\nu + 1 - a)(1 - a^{\alpha/2})}{2} - a^{\alpha/2} \sum_{n=1}^{\infty} \frac{|z|}{J_{\nu, n}^2} = |z|h_{a, \nu}(|z|) = \beta
\]
if and only if $|z| \leq r^{\ast}(h_{\nu, \beta}) < j_{\nu, \beta}$. Here $r^{\ast}(h_{\nu, \beta})$ is the smallest root of the equation
\[
r h_j'(r)/h_{a, \nu}(r) = \beta,
\]
that is, a root of
\[
\frac{1 + a - a\nu}{2} + a^{\alpha/2} \frac{(\sqrt{z} J_\nu'((\sqrt{z})) - (\nu - 1)(1 - a))}{2} = \beta,
\]
or equivalently, of the equation
\[
a^{\alpha/2} r_j'(r) + \left((a^{\alpha/2} - 1)(1 - a + a\nu) - a^{\alpha/2}\nu + 2(1 - \beta)\right) J_\nu(r) = 0.
\]

Thus by Lemma 2.3 $r^{\ast}(h_{\nu, \beta})$ is the smallest positive root $j_{\nu, \beta, 1}^{a, h}$ of (14) when $(a^{\alpha/2} - 1)(1 - a + a\nu) + 2(1 - \beta) > 0$. \hfill $\square$

Remark 1. In the case $a = 1$, the condition $a(\nu - 1)(a^{\alpha/2} - 1) + a^{\alpha/2} - 1 = 1 - \beta > 0$ and $(a^{\alpha/2} - 1)(1 - a + a\nu) + 2(1 - \beta) = 2(1 - \beta) > 0$ both hold trivially for all $\beta \in [0, 1)$. Both theorems therefore coincide with the earlier results in [4].

Further, it is of interest to determine the radius of starlikeness $r^{\ast}_{\beta}(g_{a, \nu})$ in Theorem 2.9 in the event that $a(\nu - 1)(a^{\alpha/2} - 1) + a^{\alpha/2} - 1 < 0$, as well as that of $r^{\ast}_{\beta}(h_{a, \nu})$ in Theorem 2.10 when $(a^{\alpha/2} - 1)(1 - a + a\nu) + 2(1 - \beta) < 0$. 

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3. Starlikeness of the generalized Bessel function

In this final section, the best range on $\nu$ is obtained for a fixed $a \in \mathbb{N}$ to ensure the functions $f_{a,\nu}$ and $g_{a,\nu}$ given by (3) are starlike of order $\beta$ in $\mathbb{D}$.

**Theorem 3.1.** For a fixed $a \in \mathbb{N}$, the function $f_{a,\nu}$ given by (3) is starlike of order $\beta \in [0,1]$ in $\mathbb{D}$ if and only if $\nu \geq \nu_f(a,\beta)$, where $\nu_f(a,\beta)$ is the unique root of

$$(a\nu - a + 1)(a^{\alpha/2} - \beta)J_{\nu}(1) = a^{\alpha/2}J_{\nu+1}(1)$$

in $((a - 1)/a, \infty)$.

**Proof.** For $\nu > (a - 1)/a$ and $|z| = r \in [0,1)$, it follows from (11) that

$$\Re\left(\frac{zf'_{a,\nu}(z)}{f_{a,\nu}(z)}\right) = \frac{r^2 f'_{a,\nu}(r)}{f_{a,\nu}(r)} = a^{\alpha/2} - \frac{a^{\alpha/2}}{a\nu - a + 1} \sum_{n=1}^{\infty} \frac{2r^2}{\nu,n} \geq 0.$$

The above inequality holds since $r < 1$ and it is known (12, p. 508), (11, p. 236) that the function $\nu \mapsto j_{\nu,n}$ is increasing on $(0, \infty)$ for each fixed $n \in \mathbb{N}$, and whence $j_{\nu,1} \geq j_{(a-1)/a,1} \geq j_{0,1} \approx 2.40483\ldots$

A computation yields

$$\frac{d}{dr} \left( \frac{r f'_{a,\nu}(r)}{f_{a,\nu}(r)} \right) = -\frac{2a^{\alpha/2}}{a\nu - a + 1} \sum_{n=1}^{\infty} 2rJ_{\nu,n}^2 \geq 0.$$

Hence

$$\Re\left(\frac{zf'_{a,\nu}(z)}{f_{a,\nu}(z)}\right) \geq a^{\alpha/2} - \frac{a^{\alpha/2}}{a\nu - a + 1} \sum_{n=1}^{\infty} \frac{2}{J_{\nu,n}^2} = f'_{a,\nu}(1) = f_{a,\nu}(1).$$

The monotonicity property of $\nu \mapsto j_{\nu,n}$ leads to

$$\frac{f'_{a,\mu}(1)}{f_{a,\mu}(1)} = a^{\alpha/2} - \frac{a^{\alpha/2}}{a\mu - a + 1} \sum_{n=1}^{\infty} \frac{2}{J_{\mu,n}^2} \geq a^{\alpha/2} - \frac{a^{\alpha/2}}{a\nu - a + 1} \sum_{n=1}^{\infty} \frac{2}{J_{\nu,n}^2} = f'_{a,\nu}(1) = f_{a,\nu}(1),$$

$\mu \geq \nu > -1$. Since $\nu \mapsto f'_{a,\nu}(1)/f_{a,\nu}(1)$ is increasing in $((a-1)/a, \infty)$, and from consideration of the asymptotic behavior of $f'_{a,\nu}(1)/f_{a,\nu}(1)$, evidently $f'_{a,\nu}(1)/f_{a,\nu}(1) \geq \beta$ if and only if $\nu \geq \nu_f(a,\beta)$, where $\nu_f(a,\beta)$ is the unique root of the equation $f'_{a,\nu}(1) = \beta f_{a,\nu}(1)$. From (3), the latter equation is equivalent to

$$a^{\alpha/2}J_{\nu-1}(1) = \left(a^{\alpha/2}(\nu(2-a) + a - 1) + \beta(a\nu - a + 1)\right)J_{\nu}(1).$$

The recurrence relation in Proposition 1.2 now shows that $\nu_f(a,\beta)$ is a unique root of $(a\nu - a + 1)(a^{\alpha/2} - \beta)J_{\nu}(1) = a^{\alpha/2}J_{\nu+1}(1)$. Since all inequalities are sharp, it follows that the value $\nu_f(a,\beta)$ is best.

**Remark 2.** With regards to Theorem 3.1, we tabulate the best value $\nu$ for a fixed $\beta$ and $a$ for which $f_{a,\nu}$ is starlike of order $\beta$. These values are given in Table 1.
Using (6), we tabulate the radius of starlikeness for $f_{a, \nu}$ in Theorem 2.7 for a fixed $\nu = 0.7$, $a = 1, 2, 3$, and respectively $\beta = 0$ and $\beta = 1/2$. These are given in Table 2. Here the value of $j_{\nu,1}$ at $\nu = 0.7$ is $j_{0.7,1} = 3.42189$. With reference to Table 1, we expect the radius of starlikeness to be less than 1 whenever $\nu = 0.7$ is less than the given values of $\nu$ in Table 1.

| $\beta = 0$ | $\beta = 1/2$ | $\beta = 0.95$ |
|-------------|---------------|-----------------|
| $a = 1$     | $r_0^*(f_{1,0.7}) = 1.44678$ | $r_{1/2}^*(f_{1,0.7}) = 1.05621$ | $r_{0.95}^*(f_{1,0.7}) = 0.343848$ |
| $a = 2$     | $r_0^*(f_{2,0.7}) = 1.12397$ | $r_{1/2}^*(f_{2,0.7}) = 0.982365$ | $r_{0.95}^*(f_{2,0.7}) = 0.828745$ |
| $a = 3$     | $r_0^*(f_{3,0.7}) = 0.577726$ | $r_{1/2}^*(f_{3,0.7}) = 0.549716$ | $r_{0.95}^*(f_{3,0.7}) = 0.523133$ |

Table 2. Radius of starlikeness for $f_{a, \nu}$ when $\nu = 0.7$

Letting

$$F(r) := \frac{rJ'_\nu(r)}{J_\nu(r)},$$

then (6) takes the form $F(r) = -\alpha$, where

$$\alpha := \alpha(a, \beta, \nu) = -(\nu - 1)(1 - a) + \frac{\beta(\nu a - a + 1)}{a^{\alpha/2}}.$$

For $\nu > 0$, it is known that $F(r)$ is strictly decreasing on $(0, \infty)$ except at the zeros of $J_\nu(r)$. Differentiating with respect to $\beta$, it is clear that $\alpha$ is decreasing with respect to $\beta$ so long as $\nu a - a + 1 > 0$, and thus $r_0^*$ is decreasing. Further, for a fixed $\nu < 1$ and $\beta = 0$, then $\alpha$ is monotonically decreasing with respect to $a$, that is, $r_0^*$ is decreasing as a function of $a$. However, for $\beta$ near 1, then $\alpha$ is no longer monotonic. For instance, choosing $\nu = 0.7$ and $\beta = 0.95$, Table 2 illustrates the fact that $r_{0.95}^*$ is not monotonic with respect to the parameter $a$.

**Theorem 3.2.** Let $a \in \mathbb{N}$, $\nu > -1/a$, and $j_{\nu,1}$ be the first positive zero of $J_\nu$. Then the function $g_{a, \nu}$ given by (3) is starlike of order $\beta \in [0, 1)$ in $D$ if and only if $\nu \geq \nu_g(a, \beta)$, where $\nu_g(a, \beta)$ is the unique root in $(\max\{\tilde{\nu}, -1/a\}, \infty)$ of

$$(\alpha a - 1)(a^{\alpha/2} - 1) + a^{\alpha/2} - \beta)J_\nu(1) = a^{\alpha/2}J_{\nu+1}(1),$$

and $\tilde{\nu} \simeq -0.7745\ldots$ is the unique root of $j_{\nu,1} = 1$. 

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Finally, Proposition 1.2 implies that equivalently, if and only if
\[ \nu = a(1 - \nu) + a^{a/2} \left( \frac{z J_\nu'(z)}{J_\nu(z)} - (\nu - 1)(1 - a) \right) = a(1 - \nu) + a^{a/2} \left( a\nu - a + 1 - \sum_{n=1}^{\infty} \frac{2z^2}{j_{2n,\nu}^2 - z^2} \right), \]
and
\[ \Re \left( \frac{z g_{a,\nu}'(z)}{g_{a,\nu}(z)} \right) > \frac{r g_{a,\nu}'(r)}{g_{a,\nu}(r)} = a(1 - \nu) + a^{a/2} \left( a\nu - a + 1 - \sum_{n=1}^{\infty} \frac{2r^2}{j_{2n,\nu}^2 - r^2} \right) \]
for \(|z| < j_{\nu,1}\). The function \( r \mapsto r g_{a,\nu}'(r)/g_{a,\nu}(r) \) is decreasing on \([0, 1]\). Since \( j_{\nu,1} > 1 \) for \( \nu > \max \{ \tilde{\nu}, -1/a \} \), it follows that \( j_{\nu,n} > 1 \) for each \( n \), and consequently
\[ \Re \left( \frac{z g_{a,\nu}'(z)}{g_{a,\nu}(z)} \right) > \frac{r g_{a,\nu}'(r)}{g_{a,\nu}(r)} > \frac{g_{a,\nu}(1)}{g_{a,\nu}(1)} = a(1 - \nu) + a^{a/2} \left( a\nu - a + 1 - \sum_{n=1}^{\infty} \frac{2}{j_{2n,\nu}^2 - 1} \right). \]
Further, for \( \mu \geq \nu > \max \{ \tilde{\nu}, -1/a \} \),
\[ \frac{g_{a,\nu}'(1)}{g_{a,\nu}(1)} = a \left( a^{a/2} - 1 \right) (\mu - 1) + a^{a/2} - \sum_{n=1}^{\infty} \frac{2a^{a/2}}{j_{2n,\nu}^2 - 1} \geq a \left( a^{a/2} - 1 \right) (\nu - 1) + a^{a/2} - \sum_{n=1}^{\infty} \frac{2a^{a/2}}{j_{2n,\nu}^2 - 1} = \frac{g_{a,\nu}'(1)}{g_{a,\nu}(1)}. \]
This implies that \( \nu \mapsto g_{a,\nu}'(1)/g_{a,\nu}(1) \) is increasing on \((\max \{ \tilde{\nu}, -1/a \}, \infty)\).

Thus
\[ \Re \left( \frac{z g_{a,\nu}'(z)}{g_{a,\nu}(z)} \right) > \frac{g_{a,\nu}'(1)}{g_{a,\nu}(1)} \geq \beta \]
if and only if \( \nu \geq \nu_g(a, \beta) \), where \( \nu_g(a, \beta) \) is the unique root of \( g_{a,\nu}'(1) = \beta g_{a,\nu}(1) \), or equivalently,
\[ a(1 - \nu) + a^{a/2} \left( \frac{J_\nu'(1)}{J_\nu(1)} - (\nu - 1)(1 - a) \right) = \beta. \]
Finally, Proposition 1.2 implies that \( \nu_g(a, \beta) \) is a unique root of
\[ (a(\nu - 1)(a^{a/2} - 1) + a^{a/2} - \beta) J_\nu(1) = a^{a/2} J_{\nu+1}(1). \]

\[ \Box \]

**Remark 3.** The best value \( \nu \) obtained from Theorem 3.2 for a fixed \( \beta \) and \( a \) for which \( g_{a,\nu} \) is starlike of order \( \beta \) is given in Table 3.2.

The radius of starlikeness for \( g_{a,\nu} \) drawn from Theorem 2.9 is tabulated in Table 4 for a fixed \( \nu = 0.7 \), \( a = 1, 2, 3 \), and respectively \( \beta = 0 \), \( \beta = 0.5 \) and \( \beta = 0.95 \). Here the
radius of starlikeness is expectedly less than 1 whenever \( \nu = 0.7 \) is less than the given values of \( \nu \) in Table 3. A similar situation occurs as for the function \( f_{a,\nu} \) with regard to the monotonicity of the radius of starlikeness with respect to either parameter \( \beta \) or \( a \).

Table 3. Values of \( \nu \) for \( g_{a,\nu} \) to be starlike

| \( a \) | \( \beta = 0 \) | \( \beta = 0.5 \) | \( \beta = 0.95 \) |
|---|---|---|---|
| 1 | \( \nu = -0.340092 \) | \( \nu = 0.122499 \) | \( \nu = 9.02272 \) |
| 2 | \( \nu = 0.39002 \) | \( \nu = 0.586273 \) | \( \nu = 0.772587 \) |
| 3 | \( \nu = 0.714616 \) | \( \nu = 0.751407 \) | \( \nu = 0.784626 \) |

Table 4. The radius of starlikeness for \( g_{a,\nu} \) when \( \nu = 0.7 \)

| \( a \) | \( \beta = 0 \) | \( \beta = 1/2 \) | \( \beta = 0.95 \) |
|---|---|---|---|
| 1 | \( r^*_0(g_{1,0.7}) = 1.68326 \) | \( r^*_1/2(g_{1,0.7}) = 1.24519 \) | \( r^*_0,0.95(g_{1,0.7}) = 0.410407 \) |
| 2 | \( r^*_0(g_{2,0.7}) = 1.44678 \) | \( r^*_1/2(g_{2,0.7}) = 1.1867 \) | \( r^*_0,0.95(g_{2,0.7}) = 0.856647 \) |
| 3 | \( r^*_0(g_{3,0.7}) = 0.939782 \) | \( r^*_1/2(g_{3,0.7}) = 0.763126 \) | \( r^*_0,0.95(g_{3,0.7}) = 0.549716 \) |

Remark 4. For \( a = 1 \), Theorem 3.1 and Theorem 3.2 respectively reduces to Theorem 1 and Theorem 2 in [5].

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