Research Article

Free Vibration Investigations of Rotating FG Beams Resting on Elastic Foundation with Initial Geometrical Imperfection in Thermal Environments

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In practical operations of mechanical structures, it is not difficult to meet some large components such as helicopter rotors, gas turbine blades of marine engines, and rotating railway bridges, where these elements can be seen as beam models rotating around one fixed axis. Therefore, mechanical explorations of these structures with and without the effect of temperature will guide the design, manufacture, and use of them in practice. This is the first paper that uses the shear deformation theory-type hyperbolic sine functions and the finite element method to analyze the free vibration response of rotating FGM beams with initial geometrical imperfections resting on elastic foundations considering the effect of temperature. The material properties are assumed to be varied in the thickness direction of the beam based on the power law function and temperature changes. The proposed theory and mathematical model are verified by comparing the results with other exact solutions. The numerical investigations have taken into account some geometrical and material parameters to evaluate the effects on the vibration behavior of the structure such as the rotational speed, temperature, as well as initial geometrical imperfections. The drawn comments have numerous scientific and practical implications for rotating beam structures.

1. Introduction

Functionally graded materials (FGM) are made of two or more different materials, where the common type is the combination of ceramic and metal with the changing law of material properties in one direction. Due to the ceramic component in the composition, the structures made of FGM materials have good heat resistance, and they are usually designed for use in high-temperature environments. Therefore, the research on the mechanical behavior of structures made of these materials, including FGM beams, has interested scientists worldwide [1–5]. Besides, in engineering practice, some parts of turbines, aircraft propellers, and so on can be seen as beam structures with rotational movements. The following publications related to these structures can be counted. Gunjal and Dixit [6] studied the minimum of the vibrations of rotating beams through the shape optimization of these beams, in which the finite element method was used to model the rotating beam and sequential quadratic programming was used for the optimization process. Pradhan and Murmu [7] used Eringen’s nonlocal elasticity theory and a single nonlocal beam model to investigate the mechanical behavior of a rotating nanobeam, where the differential quadrature method was adopted in this work. Li et al. [8] examined the free vibration response of a rotating functionally graded material (FGM) beam using a dynamic model; the results took the effects of the bending and stretching phenomena into account. Amir et al. [9] explored the lead-lag vibration problem of rotating microbeams based on Euler–Bernoulli and Timoshenko beam theories and the finite element method. Jung-Woo and Jung-Youn [10] studied the effects of cracks on the natural frequencies of a rotating Bernoulli–Euler beam by employing a new numerical approach, where these effects could be computed simply using the transfer matrix method. Das [11] employed the Ritz method to calculate the in-plane
and out-of-plane mechanical behavior of the rotating FGM beam, which was based on the Timoshenko beam theory and the Coriolis acceleration. Xu et al. [12] investigated the dynamic model and vibration suppression of a rotating cantilever beam under magnetic excitations based on the Hamilton principle and Galerkin method. The free vibration response analysis of the rotating rod based on Eringen's nonlocal elasticity was carried out by Alireza and Cai [13]. Liang et al. [14] studied the vibration control of a rotating piezoelectric FGM beam in thermal environments by using the high-order coupling modeling theory. Dejin et al. [15] introduced a combination of the Timoshenko beam and re-modified couple stress theories to capture the free vibration response of rotating microbeams, in which the beams were made from multilayered composite components with geometric imperfections. Euler–Bernoulli’s beam, Eringen’s nonlocal elasticity theories, and Galerkin method were employed by Atanasov and Stojanovic [16] to investigate the free vibration of rotating nanobeams, in which a wide range of parameter studies was carried out, especially the speed of rotation around one fixed axis.

In addition, for structures made of pore-defect materials, structures resting on elastic foundations and structures taking into account the effect of temperature have been also investigated by scientists worldwide. Jalaei and Civalek [17] used Navier’s solution and Bolotin’s approach to study the dynamic instability of viscoelastic porous functionally graded nanobeams embedded in visco-Pasternak medium. Demir and Civalek [18] investigated the bending of nano/microbeams under the concentrated and distributed loads using Euler–Bernoulli beam theory via the enhanced Eringen differential model. Akgoz and Civalek [19] introduced the exact solution for the static bending and buckling problems of simply supported microbeams embedded in an elastic medium. The exact solution was also presented by Hanten et al. [20] to analyze the free vibration analysis of beams made of fibre-metal laminated beams. The research on the mechanical behavior of plate and beam structures resting on different elastic foundations has been studied by scientists, typically in works [21–30]. In works [27, 29, 31–35], the authors explored the mechanical response of beams and plates taking into account the effect of temperature with numerous different theories.

Based on the review above, it can be seen that there are not any publications on free vibration responses of rotating FGM beams resting on elastic foundations in thermal environments, in which geometrical imperfections are taken into account. Hence, this paper is about to explore the free vibration behavior of these structures by using the combination of the finite element method and the new type of hyperbolic sine functions of shear deformation theory. The proposed theory is simple and easy to establish the stress and strain relations, which does not need to add any shear correction factors, while the mechanical phenomena are still desired exactly.

The rest of this paper is structured as follows. Finite element formulations of the rotating FGM beam in a thermal environment are presented in Section 2, where geometrical imperfections and temperature effects are considered. Section 3 conducts verification examples to evaluate the accuracy of the proposed theory and mechanical models. Numerical data and discussions are located in Section 4. Section 5 concludes some remarkable points of this work.

2. Finite Element Model of the Rotating FGM Beam in a Thermal Environment

Consider a functionally graded material (FGM) beam rotating around one fixed axis with the speed ω. One side of the beam is at distance r from the fixed axis A. The beam is resting on a two-parameter elastic foundation k_w and k_v, as shown in Figure 1. The beam has dimensions such as the length L, width b, and thickness h, and an initial geometric imperfection in the z-direction is w_{im}(x).

The beam structure is made of ceramic (denoted by c) and metal (denoted by m), in which volume proportions (V_c and V_m) are varied according to the thickness direction based on the following power law function [1, 36–39]:

\[
V_c = \left( \frac{z}{h} + \frac{1}{2} \right)^n, \quad V_m = 1 - V_c, \quad \text{with } n \geq 0, \tag{1}
\]

where z is the thickness coordinate variable with \(-h/2 \leq z \leq h/2\) and n is the volume fraction index and its value depends on the mechanical behavior of the beam. The beam is assumed to be placed in a thermal environment; as a result, the material properties are varied by the temperature as follows [40]:

\[
E(z) = E_m + (E_c - E_m) \left( \frac{1}{2} + \frac{z}{h} \right)^n, \\
\rho(z) = \rho_m + (\rho_c - \rho_m) \left( \frac{1}{2} + \frac{z}{h} \right)^n, \\
\nu(z) = \nu_m + (\nu_c - \nu_m) \left( \frac{1}{2} + \frac{z}{h} \right)^n, \\
\alpha(z) = \alpha_m + (\alpha_c - \alpha_m) \left( \frac{1}{2} + \frac{z}{h} \right)^n. \tag{2}
\]

The beam is assumed to be placed in a thermal environment; as a result, the material properties are varied by the temperature as follows [40]:

\[
P = P_0 \left( P_{-1} T^{-1} + 1 + P_1 T + P_2 T^2 + P_3 T^3 \right), \tag{3}
\]

where T = T_0 + ΔT, ΔT is the temperature increment, T_0 = 300K is the room temperature (this temperature also does not affect the stress of the material), and P_0, P_{-1}, P_1, P_2, and P_3 are constants depending on different materials.
To represent properly the mechanical behaviors of FGM beams, this work uses a new shear deformation theory-type hyperbolic sine functions; thus, the displacements \( u \) and \( w \) in the \( x \)- and \( z \)-directions at any point with the coordinate \((x, z)\)

are expressed as follows \cite{41, 42}:

\[
\begin{align*}
\begin{cases}
  u(x, z) &= u_0(x, y) - z \frac{du_b}{dx} - f(z) \frac{d^2w_b}{dx^2} \\
  w(x, z) &= w_b + \omega_z + w_{im}(x),
\end{cases}
\end{align*}
\] (4)

in which \( f(z) = z - h \sinh(z/h) + z \cosh(1/2) \).

The longitudinal and shear strains of the beam are defined as follows:

\[
\begin{align*}
  \varepsilon_{xx} &= \frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} - z \frac{\partial^2 w_b}{\partial x^2} - f(z) \frac{\partial^3 w_b}{\partial x^3} + \frac{\partial w_{im}}{\partial x} \frac{\partial (w_b + \omega_z)}{\partial x} - \varepsilon_T, \\
  \varepsilon_{bx} &= \frac{\partial u_0}{\partial x} + z \frac{\partial^2 w_b}{\partial x^2} + f(z) \left( -\frac{\partial^2 w_b}{\partial x^2} \right) + \frac{\partial w_{im}}{\partial x} \frac{\partial (w_b + \omega_z)}{\partial x} - \varepsilon_T, \\
  \gamma_{xz} &= \frac{\partial f(z) \partial w_z}{\partial z} = \frac{\partial f(z)}{\partial z} \gamma_{xz},
\end{align*}
\]

(5a)

in which the strain components have the following expressions:

\[
\begin{align*}
  \varepsilon_{0x} &= \frac{\partial u_0}{\partial x}, \\
  \varepsilon_{bx} &= \left( -\frac{\partial^2 w_b}{\partial x^2} \right), \\
  \varepsilon_{xx} &= \left( -\frac{\partial^2 w_z}{\partial x^2} \right), \\
  \varepsilon_{im} &= \frac{\partial w_{im}}{\partial x} \frac{\partial (w_b + \omega_z)}{\partial x}, \\
  \gamma_{sxz} &= \frac{\partial w_z}{\partial x}.
\end{align*}
\]

(5b)

The normal stresses relating to the mechanical strain \( \sigma_{xx} \), the thermal strain \( \sigma_T \), and the shear stress \( \tau_{xz} \) have the following expressions:

\[
\begin{align*}
  \sigma_{xx} &= E(z)\varepsilon_{xx}; \\
  \sigma_T &= E(z)\varepsilon_T, \\
  \tau_{xz} &= \frac{E(z)}{2(1 + \nu(z))} \gamma_{xz},
\end{align*}
\]

(6)

where the strain relating to the temperature is \( \varepsilon_T = \alpha(z)(T - T_0) \).

The energy of the FGM beam is calculated as

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**Figure 1:** A rotating FGM beam with the initial geometric imperfection.
The energy of the elastic foundation is defined as
\[ U^F = \frac{1}{2} \int_V \left( \epsilon_{xx}^T \epsilon_{xx} + \tau_{xx}^T \gamma_{xx} \right) dV = \frac{1}{2} \int_V \left( \epsilon_{xx}^T \epsilon_{xx} + \partial f(z) \epsilon_{xx} f(z) + f(z) \epsilon_{xx} \epsilon_{im} + \epsilon_{im} \epsilon_{xx} \right) dV, \]

where each node contains four degrees of freedom: \( u, w, \varphi_1, \varphi_2 \).

The kinetic energy of the beam is expressed as follows:
\[ U^E = \frac{1}{2} \int_V \left( \sigma_{xx} \epsilon_{xx} + \tau_{xx}^T \gamma_{xx} \right) dV = \frac{1}{2} \int_V \left( \epsilon_{xx}^T \epsilon_{xx} + \partial f(z) \epsilon_{xx} f(z) + f(z) \epsilon_{xx} \epsilon_{im} + \epsilon_{im} \epsilon_{xx} \right) dV, \]

To determine the equilibrium equation of the FGM beam, this work uses Hamilton’s principle:
\[ \delta \int_{t_1}^{t_2} \left[ D - U^E - U^F - U^{Ro} + U^{Th} \right] dt = 0. \]

In this paper, a two-node beam element is employed, where each node contains four degrees of freedom:
\[ q_e = \sum_{i=1}^{2} \left[ \begin{array}{c} u_{oi} \\ w_{oi} \\ \varphi_1 \\ \varphi_2 \end{array} \right], \]

where the displacement components at each point within the beam element are approximated through Lagrange and Hermite interpolation functions \( N_i \) and \( H_z \).
\[
\begin{align*}
\mathbf{u}_0 &= \sum_{i=1}^{2} \mathbf{N}_i \mathbf{u}_i = \mathbf{N}_u \mathbf{q}_e, \\
\mathbf{w}_b &= \sum_{i=1}^{2} \left( H_i \mathbf{w}_{bi} + H_{i+1} \left( \frac{\partial \mathbf{w}_b}{\partial x} \right)_i \right) = \mathbf{H}_b \mathbf{q}_e, \\
\mathbf{w}_s &= \sum_{i=1}^{2} \left( H_i \mathbf{w}_{si} + H_{i+1} \left( \frac{\partial \mathbf{w}_s}{\partial x} \right)_i \right) = \mathbf{H}_s \mathbf{q}_e, \\
\frac{\partial \mathbf{w}_b}{\partial x} &= \sum_{i=1}^{2} \left( \frac{\partial H_i}{\partial x} \mathbf{w}_{bi} + \frac{\partial H_{i+1}}{\partial x} \left( \frac{\partial \mathbf{w}_b}{\partial x} \right)_i \right) = \mathbf{H}_{bx} \mathbf{q}_e, \\
\frac{\partial \mathbf{w}_s}{\partial x} &= \sum_{i=1}^{2} \left( \frac{\partial H_i}{\partial x} \mathbf{w}_{si} + \frac{\partial H_{i+1}}{\partial x} \left( \frac{\partial \mathbf{w}_s}{\partial x} \right)_i \right) = \mathbf{H}_{xs} \mathbf{q}_e, \\
\frac{\partial^2 \mathbf{w}_b}{\partial x^2} &= \sum_{i=1}^{2} \left( \frac{\partial^2 H_i}{\partial x^2} \mathbf{w}_{bi} + \frac{\partial^2 H_{i+1}}{\partial x^2} \left( \frac{\partial \mathbf{w}_b}{\partial x} \right)_i \right) = \mathbf{H}_{bxx} \mathbf{q}_e, \\
\frac{\partial^2 \mathbf{w}_s}{\partial x^2} &= \sum_{i=1}^{2} \left( \frac{\partial^2 H_i}{\partial x^2} \mathbf{w}_{si} + \frac{\partial^2 H_{i+1}}{\partial x^2} \left( \frac{\partial \mathbf{w}_s}{\partial x} \right)_i \right) = \mathbf{H}_{ssx} \mathbf{q}_e.
\end{align*}
\]

or, in the matrix form as

\[
\mathbf{u} = \begin{bmatrix}
\mathbf{u}_0 \\
\mathbf{w}_b \\
\mathbf{w}_s \\
\frac{\partial \mathbf{w}_b}{\partial x} \\
\frac{\partial \mathbf{w}_s}{\partial x}
\end{bmatrix} = \begin{bmatrix}
\mathbf{N}_u \\
\mathbf{H}_b \\
\mathbf{H}_s \\
\mathbf{H}_{bx} \\
\mathbf{H}_{xs}
\end{bmatrix} \mathbf{q}_e = \mathbf{H} \mathbf{q}_e. \quad (15)
\]

And, strain components are calculated according to the nodal displacement as follows:

\[
\begin{align*}
\epsilon_{0x} &= \frac{\partial \mathbf{u}_0}{\partial x} = \frac{\partial \mathbf{N}_u}{\partial x} \mathbf{q}_e = \mathbf{B}_u \mathbf{q}_e, \\
\epsilon_{bx} &= \frac{\partial^2 \mathbf{w}_b}{\partial x^2} = -\mathbf{H}_{bxx} \mathbf{q}_e = \mathbf{B}_b \mathbf{q}_e, \\
\epsilon_{ss} &= \frac{\partial^2 \mathbf{w}_s}{\partial x^2} = -\mathbf{H}_{ssx} \mathbf{q}_e = \mathbf{B}_s \mathbf{q}_e, \\
\epsilon_{\text{imp}} &= \frac{\partial \left( \mathbf{w}_b + \mathbf{w}_s \right)}{\partial x} \frac{\partial \mathbf{w}_{\text{im}}}{\partial x} + \frac{\partial \mathbf{w}_{\text{im}}}{\partial x} \left( \mathbf{H}_b + \mathbf{H}_s \right) \mathbf{q}_e = \mathbf{B}_{\text{im}} \mathbf{q}_e, \\
\gamma_{xx} &= \frac{\partial \mathbf{w}_s}{\partial x} = \mathbf{H}_{xs} \mathbf{q}_e. \quad (17)
\end{align*}
\]

One can get the energy expression of the FGM beam element as follows:
mined as follows:

\[
U_c^E = \frac{1}{2} \int_V \left( B_u^T \bf{E} B_u + B_u^T z E B_b + B_u^T f(z) E B_s + B_u^T E B_m \right) dV q_e.
\]

In the equation for determining the specific vibration frequency as well as vibration mode, the mass matrix \( M \) should also be noted that, in equation (23), the global vector \( q \) is expressed in the short form as follows:

\[
q = \begin{bmatrix} q_e^T \\ q_f^T \\ q_s^T \\ q_l^T \end{bmatrix},
\]

where \( \sigma_T \) is given in equation (6).

The kinetic energy of the FGM beam element is determined as follows:

\[
D_e = \frac{1}{2} \int_V (u^T \rho(z) u) dV = \frac{1}{2} q_e^T \left( \int_V (H^T H) dV \right) q_e = \frac{1}{2} q_e^T (M_e) q_e.
\]

in which \( M_e = \int_V \rho(z) (H^T H) dV \) is the element mass matrix.

By substituting equations (18)–(20) into (13), the equation to get the fundamental frequency as well as vibration mode shape of the rotating FGM beam is obtained as follows:

\[
U_c^E = \frac{1}{2} \int_V \left( \left( \frac{\partial f(z)}{\partial z} \right)^2 \frac{E}{2(1+\nu)} H^T H \right) dV q_e.
\]

The energy generated by the thermal strain is calculated as follows:

\[
U_c^F = \frac{1}{2} q_e^T K_c^F q_e.
\]

The energy of the centrifugal inertia force and the elastic foundation is calculated as follows:

\[
U_c^R = \frac{1}{2} q_e^T \left( \int_V \left( F_A(x) \right) \left( H_{bx} + H_{sx} \right)^T (H_{bx} + H_{sx}) \right) dx q_e = \frac{1}{2} q_e^T K_c^R q_e.
\]

The energy generated by the thermal strain is calculated as follows:

\[
U_c^{Th} = \frac{1}{2} \int_V \left( \int_V (H_b + H_s)^T (H_b + H_s) \right) dV = \frac{1}{2} q_e^T K_c^{Th} q_e.
\]

Equation (23) proves that all components related to rotational speed, elastic foundation parameter, initial geometrical imperfection, and temperature are presented in the equation for determining the specific vibration response of the FGM beam, which is completely different from traditional beams, which make the computation more complicated in comparison with previous studies. It should also be noted that, in equation (23), the global stiffness matrix \( K = \sum_c (k_e^E + k_e^F + k_e^R - K_e^{Th}) \), the global mass matrix \( M = \sum_c M_e \), and the global vector \( q = \sum_c q_e \) depend on specific boundary conditions. Here is the deletion of rows and columns corresponding to zero degrees of freedom at the boundary nodes, which are described as follows:

(i) Simply supported (represented by S):

\[
\sum_c (k_e^E + k_e^F + k_e^R - K_e^{Th}) - \omega^2 \sum_c M_e \right) \sum_c q_e = 0.
\]
\[ u_0 = 0, \]
\[ w_b = 0, \]
\[ w_s = 0. \]

(ii) Clamped (represented by \( C \)):
\[ u_0 = 0, \]
\[ w_b = 0, \]
\[ w_s = 0, \]
\[ \frac{\partial w_b}{\partial x} = 0, \]
\[ \frac{\partial w_s}{\partial x} = 0. \]

(iii) Free (represented by \( F \)):
\[ u_0 \neq 0, \]
\[ w_b \neq 0, \]
\[ w_s \neq 0, \]
\[ \frac{\partial w_b}{\partial x} \neq 0, \]
\[ \frac{\partial w_s}{\partial x} \neq 0. \]

This work calculates for FGM beams with three boundary conditions.

(i) One side is clamped, and the other side is free: C-F
(ii) Fully simply supported beam: S-S
(iii) Fully clamped beam: C-C

3. Verification Study

This section considers 4 examples to evaluate the accuracy of the proposed theory and mechanical model, where the numerical results of this work are compared with those of other trustful papers.

Example 1. Firstly, this example compares the results of frequencies of the FGM (Al/Al₂O₃) beam resting on a two-parameter elastic foundation with simply supported boundaries at two sides of the beam. Geometrical and material parameters are the length \( L \), thickness \( h \), \( L/h = 100 \), width \( b \), \( E_m = 70 \) GPa, \( \rho_m = 2702 \) kg/m³, \( E_c = 380 \) GPa, and \( \rho_c = 3960 \) kg/m³. Two non-dimensional frequencies of the elastic foundation are defined as follows:

\[
K_w = \frac{K_1}{E_m I},
\]
\[
K_s = \frac{K_2}{E_m I},
\]

with
\[
I = \frac{bh^3}{12},
\]

The nondimensional frequency of the beam is defined by the following formula:
\[
\tilde{\omega} = \omega_1 \frac{L^2}{h} \sqrt{\frac{\rho_m}{E_m}}.
\]

Table 1 presents the results of this work and the analytical method [44], where this work carries out some different mesh sizes. It can be observed that, with the 10-element mesh size, the accuracy is acceptable; therefore, this work will use this mesh for all following sections.

Example 2. Next, this problem carries the comparison of frequencies of the fully simply supported FGM beam with initial geometrical imperfection. The beam has the following parameter: \( L = 288.7 \) h, \( h = 0.02 \) m, \( b = 0.04 \) m, \( E = 971 \) GPa, and mass density \( \rho = 2300 \) kg/m³. The initial imperfection of the beam is \( w_{im}(x) = f_0 \sin(\pi x) \), where \( f_0 \) is the amplitude of imperfection. The first nondimensional frequency and the initial imperfection coefficient \( \varsigma_0 \) are defined as follows:

\[
\omega^* = \omega_1 \left[ \frac{12pL^4}{Eh^2} \right].
\]

\[
\varsigma_0 = \frac{f_0 L}{L}.
\]

The first nondimensional frequencies of the S-S FGM beam with initial geometrical imperfection obtained from this work and the pseudo-arclength continuation technique [45] in the case of increasing gradually the value of \( \varsigma_0 \) are plotted in Figure 2. It can be seen that they meet a very good agreement.

Example 3. Now, nondimensional frequencies of the rotating beam with the speed \( \chi \) are compared. Consider an cantilever beam with the length \( L \), thickness \( h = b = L/100 \), \( r = 0 \), \( E = 70 \) GPa, and mass density \( \rho = 2700 \) kg/m³. The nondimensional frequency is calculated as follows:

\[
\tilde{\omega}_i = \omega_1 L^2 \sqrt{\frac{12\rho}{Eh^2}}.
\]
Table 2 presents the first three nondimensional frequencies of the beam with different rotational speed ratios \( \vartheta \) \( L/2\chi \rho/Er^2 \) obtained from this work, exact solution \([46]\), new dynamic modeling method (DMM) \([47]\), and isogeometric analysis \([15]\).

Table 1: The comparative frequencies (\( \tilde{\omega} \)) of the FGM resting on the two-parameter elastic foundation, \( K_w = 10 \), \( K_s = 1 \), and S-S.

| Mode | Exact [44] | This work |
|------|------------|-----------|
|      | a/h = 100  | 6 elements| 8 elements| 10 elements| 12 elements| 14 elements|
| 0    | 1 5.483    | 5.483     | 5.483     | 5.483       | 5.483       | 5.483 |
| 2    | 2 21.933   | 21.931    | 21.924    | 21.922      | 21.921      | 21.921 |
| 3    | 3 49.350   | 49.389    | 49.318    | 49.296      | 49.288      | 49.284 |
| 1    | 1 4.221    | 4.220     | 4.220     | 4.220       | 4.220       | 4.220 |
| 2    | 2 16.884   | 16.884    | 16.879    | 16.877      | 16.876      | 16.876 |
| 3    | 3 37.989   | 38.030    | 37.975    | 37.958      | 37.952      | 37.949 |
| 1    | 1 3.852    | 3.851     | 3.851     | 3.851       | 3.851       | 3.851 |
| 2    | 2 15.407   | 15.408    | 15.403    | 15.402      | 15.401      | 15.401 |
| 3    | 3 34.666   | 34.707    | 34.657    | 34.641      | 34.635      | 34.633 |

Without elastic foundation \( K_w = 0 \) and \( K_s = 0 \)

| Mode | Exact [44] | This work |
|------|------------|-----------|
| 0    | 1 6.015    | 6.014     | 6.014     | 6.014       | 6.014       | 6.014 |
| 2    | 2 22.445   | 22.442    | 22.436    | 22.434      | 22.433      | 22.433 |
| 3    | 3 49.858   | 49.891    | 49.824    | 49.803      | 49.796      | 49.792 |
| 1    | 1 5.008    | 5.007     | 5.007     | 5.007       | 5.007       | 5.007 |
| 2    | 2 17.666   | 17.664    | 17.660    | 17.659      | 17.658      | 17.658 |
| 3    | 3 38.777   | 38.801    | 38.753    | 38.738      | 38.732      | 38.729 |
| 1    | 1 4.752    | 4.752     | 4.752     | 4.752       | 4.752       | 4.752 |
| 2    | 2 16.316   | 16.315    | 16.311    | 16.310      | 16.310      | 16.309 |
| 3    | 3 35.577   | 35.606    | 35.564    | 35.550      | 35.545      | 35.543 |

With elastic foundation \( K_w = 10 \) and \( K_s = 1 \)

| Mode | Exact [44] | This work |
|------|------------|-----------|
| 0    | 1 7.015    | 7.014     | 7.014     | 7.014       | 7.014       | 7.014 |
| 2    | 2 22.445   | 22.442    | 22.436    | 22.434      | 22.433      | 22.433 |
| 3    | 3 50.858   | 50.891    | 50.824    | 50.803      | 50.796      | 50.792 |
| 1    | 1 5.008    | 5.007     | 5.007     | 5.007       | 5.007       | 5.007 |
| 2    | 2 18.666   | 18.664    | 18.660    | 18.659      | 18.658      | 18.658 |
| 3    | 3 40.777   | 40.801    | 40.753    | 40.738      | 40.732      | 40.729 |
| 1    | 1 4.752    | 4.752     | 4.752     | 4.752       | 4.752       | 4.752 |
| 2    | 2 17.316   | 17.315    | 17.311    | 17.310      | 17.310      | 17.309 |
| 3    | 3 38.577   | 38.606    | 38.564    | 38.550      | 38.545      | 38.543 |

4. Numerical Results

This section is about to present the numerical explorations on free vibration analysis of rotating FGM beams in thermal environments, where the initial geometrical imperfection is considered. The beam is resting on a two-parameter elastic foundation. Temperature-dependent material properties are listed in Table 3, and Young’s moduli of ceramic and metal changing by temperature are plotted in Figure 3. The imperfection of the beam is \( \omega_0(x) = \Gamma_0 \sin (\pi x) \), in which \( \Gamma_0 \) is the amplitude of the imperfection and the imperfection ratio is \( \gamma_0 = \Gamma_0/L \).

The nondimensional frequency of the FGM beam and other parameters are normalized as follows:

\[ T a b l e \ 1: \ T h e \ c o m p a r a t i v e \ f r e q u e n c i e s \ (\tilde{\omega}) \ of \ t h e \ F G M \ r e s t i n g \ on \ t h e \ t w o - p a r a m e t e r \ e l a s t i c \ f o u n d a t i o n , K_w = 10 , K_s = 1 , \ a n d \ S - S . \]
ω = ω₀L² \sqrt{\frac{12ρ₀}{E₀h^3}}

\begin{align*}
K_w &= k_w L^4 \sqrt{\frac{12}{D₀}} \\
K_s &= k_s L^2 \sqrt{\frac{12}{D₀}} \\
η &= L^2 \sqrt{\frac{12P₀}{E₀h^3}} \\
D₀ &= \frac{E₀h^3}{12}, \text{ with } ρ₀ = 3800 \text{kg/m}^3, E₀ = 3.2024.10^{11} \text{ N/m}^2.
\end{align*}

(31)

4.1. Free Vibration Explorations

4.1.1. Effect of Rotational Speed. Consider a geometrically imperfect beam with dimensions L/h = 20, imperfect parameter η₀ = 0.001, K_w = 10, K_s = 1, distance ratio r/L = 0, and ΔT = 20 K. Changing the value of the rotational speed χ of the beam so that η varies in a range of 0 to 10, and the volume fraction index n gets the values from 0 to 10. Table 2 presents the dependences of the first three frequencies of C-F and S-S beamson the rotational speed and volume fraction index.

Table 2: The first three nondimensional frequencies (ω) of the C-F beam with different rotational speed ratios ϑ.

| Mode | Method | ϑ = 0 | ϑ = 3 | ϑ = 6 | ϑ = 12 |
|------|--------|-------|-------|-------|--------|
| ω₁  | This work | 3.5161 | 4.6790 | 7.0734 | 12.6098 |
|      | Exact [46] | 3.5160 | 4.7973 | 7.3604 | 13.1702 |
|      | DMM [47] | 3.5160 | 4.7973 | 7.3604 | —      |
|      | Isogeometric [15] | 3.5118 | 4.7733 | 7.3276 | —      |
| ω₂  | This work | 22.0247 | 23.1197 | 26.1110 | 36.4500 |
|      | Exact [46] | 22.0345 | 23.3203 | 26.8091 | 37.6031 |
|      | DMM [47] | — | — | — | — |
|      | Isogeometric [15] | — | — | — | — |
| ω₃  | This work | 61.6254 | 62.7602 | 66.0242 | 77.4512 |
|      | Exact [46] | 61.6972 | 62.9850 | 66.6840 | 79.6145 |
|      | DMM [47] | — | — | — | — |
|      | Isogeometric [15] | — | — | — | — |
| ω₄  | This work | 120.647 | 121.843 | 125.348 | 138.295 |
|      | Exact [46] | 120.902 | 122.236 | 126.140 | 140.534 |
|      | DMM [47] | — | — | — | — |
|      | Isogeometric [15] | — | — | — | — |

Table 3: Temperature-dependent material properties [40].

| Materials | P₀ (Pa) | P₁ (300K) | P₂ (300K) | P₃ (300K) |
|-----------|--------|-----------|-----------|-----------|
| Ceramic aluminum oxide (Al₂O₃) | 349.55×10⁹ | 3.853×10⁻¹ | 4.027×10⁻¹ | 1.673×10⁻¹ |
| Metal stainless steel SUS304 | 201.04×10⁹ | 3.079×10⁻¹ | 6.534×10⁻¹ | 0.392×10⁻¹ |

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when the beam rotates at a small speed, the difference of the first natural frequency with different values of distance \( r \) is not much; the greater the speed, the more this difference is.

(iii) Both rotational speed and distance \( r \) affect the vibration mode shapes of the FGM beam, especially to the first frequency. From Figure 6, the readers can find that, as the distance \( r \) increases, the first vibration mode shape of the beam tends to deviate to the right due to the influence of the greater inertial force. This shows that the distance \( r \) and the rotational speed of the beam have a significant effect on the frequency as well as the vibration mode shape of the structure.

4.1.2. Influence of the Temperature. In this section, the influence of the temperature on the free vibration response of the beam is evaluated. Consider a geometrically imperfect beam with geometrical parameters \( L/h = 20 \), imperfect parameter \( \zeta_0 = 0.001 \), \( K_w^* = 10 \), \( K_r^* = 1 \), distance ratio \( r/L = 0 \), and rotational speed \( \eta = 5 \) changing the temperature so that \( \Delta T = 0 - 100 \) K, and the volume fraction index \( n = 0 - 10 \). The dependence of the first three dimensional frequencies of the cantilever FGM beam on the temperature and volume fraction index \( n \) is shown in Figure 10. The first three dimensional frequencies of C-F and S-S FGM beams on the temperature and rotational speed are presented in Figure 11. Some discussions of these numerical results are drawn as follows:

(i) When increasing the temperature, frequencies of FGM beams decrease. This can be explained that Young’s moduli of ceramic and metal decrease as the temperature increases; therefore, the structure becomes softer.

(ii) When the value of the rotational speed is small, the effect of the centrifugal force presents slightly; thus, the slope of the dependence curve of frequency on the temperature is higher. Besides, the higher the speed of the beam is, the more clearly is the frequency dependence on the temperature shown is.

4.1.3. Influence of the Imperfect Ratio. Next, the influence of the imperfect ratio is investigated. Consider a geometrically imperfect beam with \( L/h = 20 \), \( K_w^* = 10 \), \( K_r^* = 1 \), distance ratio \( r/L = 2 \), rotational speed \( \eta = 10 \), temperature \( \Delta T = 20 \) K, the initial imperfect parameter \( \zeta_0 \) changes in a range of 0 to 0.002, and the volume fraction index \( n = 0 - 10 \). Figure 12 plotted the first three vibration mode shapes of FGM beams on the initial imperfect parameter and volume fraction index. Figure 13 presents the dependence of the first three frequencies on the initial imperfect parameter, temperature, and rotational speed. One can see as follows:

(i) When increasing the initial imperfect parameter \( \zeta_0 \) of the beam, frequencies increase; however, this change is presented very slightly

(ii) For each value of the initial imperfect parameter \( \zeta_0 \), the change of the frequency of the FGM beam is presented most when the volume fraction \( n \) gets the values in a range of 0 to 2

4.1.4. Influence of Elastic Foundation Parameters. This last investigation is about to evaluate the effect of elastic foundation parameters on vibration responses of FGM beams. To see more of this influence, the two following parameters are changed as follows: \( K_w^* \) varies in a range of 0 to 20 and \( K_r^* \) gets the values from 0 to 2. The first
Figure 4: The dependence of the first three nondimensional frequencies of the C-F geometrically imperfect beam (C-F) on the rotational speed and \(n\), \(\omega^*_0 = 0.001\), \(K^*_w = 10\), and \(K^*_s = 1\). (a) The first frequency. (b) The second frequency. (c) The third frequency.

Figure 5: Continued.
nondimensional frequencies of FGM beams are listed in Tables 5 and 6, where these parameters are taken into calculations: $L/h = 20$, distance ratio $r/L = 2$, rotational speed $\eta = 10$, temperature $\Delta T = 20K$, and initial imperfect parameter $\varsigma_0 = 0.001$. It can be observed that when increasing the elastic foundation parameters, the stiffness of the structure increases; as a result, nondimensional frequencies of FGM beams increase correspondingly. Similarly, one can see that when the volume fraction index $n$ changes in a range of 0 to 2, the change of nondimensional frequencies is presented more sharply. This change is faded gradually when $n$ gets the values from 2 to 10.

Figure 5: The dependence of the first three nondimensional frequencies of the C-F geometrically imperfect beam (S-S) on the rotational speed and $n$, $\varsigma_0 = 0.001$, $K^*_w = 10$, and $K^*_s = 1$. (a) The first frequency. (b) The second frequency. (c) The third frequency.

Figure 6: The dependence of the first nondimensional frequency of the geometrically imperfect FGM beam on the distance ratio $r/L$. 
Figure 7: The dependence of the first vibration mode shape of the geometrically imperfect FGM beam on the distance ratio $r/L$.

Figure 8: The first vibration mode shape of the FGM beam depends on the rotational speed, $n = 0.5$ and $r = 0$. 

Figure 9: The dependence of the first four vibration mode shapes of the FGM beam on the rotational speed, \( n = 0.5 \) and \( r = 0 \). (a) C-F, \( \eta = 0 \), and \( r = 0 \). (b) C-F, \( \eta = 5 \), and \( r = 0 \). (c) S-S, \( \eta = 0 \), and \( r = 0 \). (d) S-S, \( \eta = 5 \), and \( r = 0 \).
Figure 10: The dependence of the first three frequencies of the cantilever FGM beam on the temperature, $r/L = 0$ and $\eta = 5$. (a) $\omega_1^*$. (b) $\omega_2^*$. (c) $\omega_3^*$.

Figure 11: The dependence of the first frequency of FGM beams on the temperature and rotational speed. (a) C-F. (b) S-S.
Figure 12: The first three nondimensional frequencies of the cantilever FGM beam on the initial imperfect parameter $\varsigma_0$. (a) $\omega^*_1$. (b) $\omega^*_2$. (c) $\omega^*_3$.

Figure 13: The first nondimensional frequency of the cantilever FGM beam on the initial imperfect parameter $\varsigma_0$ and the temperature.
5. Conclusions

This paper uses a combination of the simple shear deformation theory (type hyperbolic sine functions) and the finite element method to establish the finite element formulations of the free vibration problem of rotating FGM beams resting on two-parameters' elastic foundations, in which the effects of the initial geometrical imperfection and temperature are taken into calculations. The numerical results of this work are truly novel explorations, which can be used in computational design and in practical engineering.

(i) When the speed of rotation of the FGM beam increases, the inertial force increases; therefore, the frequencies of the structure increase. The rotational speed has a significant influence on the vibration responses of the beam. In addition, the distance \(r\) from the top of the beam to the axis of rotation also has a significant influence on the natural frequency of the beam; the greater the distance, the greater the frequency of the beam. At the same time, the shape of the first natural frequency tends to move to the right as this distance increases.

(ii) As the temperature increases, the mechanical properties of the material change and the elastic modulus decreases; as a result, the frequencies of the FGM beam also decrease.

(iii) The nondimensional frequencies of FGM beams increase when the initial imperfect parameter increases slightly. Also, as the elastic foundation parameters increase, nondimensional frequencies of FGM beams enhance. The change of nondimensional frequencies is presented most when the volumefraction index \(n\) gets the values in a range of 0 to 2.

From the numerical results of this study, the design and use of a rotating beam structure performing in a temperature environment need to pay attention to many parameters, in which special attention should be paid to the rotation speed of the beam and the distance \(r\) from the top of the beam to the fixed axis.

### Data Availability

Data used to support the findings of this study are included within the article.

### Conflicts of Interest

All authors declare that there are no conflicts of interest regarding the publication of this paper.

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| \(K^*\) | \(K^*_s\) | 0    | 0.2  | 1    | 2    | 5    | 10   |
|-------|--------|------|------|------|------|------|------|
| 0     | 1      | 17.0410 | 14.7543 | 13.9080 | 13.1900 | 12.1903 |
| 2     | 17.2951 | 14.9814 | 14.1244 | 13.3970 | 13.0941 |
| 10    | 17.2852 | 14.9677 | 14.1099 | 13.3821 | 13.0784 |
| 2     | 17.5358 | 15.1916 | 14.3233 | 13.5861 | 13.2793 |
| 20    | 17.5259 | 15.1781 | 14.3089 | 13.5714 | 13.2639 |
|       | 17.6517 | 15.2907 | 14.4162 | 13.6741 | 13.3650 |
|       | 17.7732 | 15.3990 | 14.5194 | 13.7726 | 13.4620 |

### Table 5: The first nondimensional frequency of the cantilever FGM beam depends on \(K^*_w\) and \(K^*_s\).

| \(K^*\) | \(K^*_s\) | 0    | 0.2  | 1    | 2    | 5    | 10   |
|-------|--------|------|------|------|------|------|------|
| 0     | 1      | 27.7426 | 23.7399 | 22.2960 | 21.0839 | 20.5570 |
| 2     | 28.1035 | 24.0662 | 22.6082 | 21.3838 | 20.8350 |
| 10    | 27.8930 | 23.8730 | 22.4223 | 21.2044 | 20.6753 |
| 2     | 28.2520 | 24.1975 | 22.7328 | 21.5026 | 20.9697 |
| 20    | 28.0427 | 24.0053 | 22.5479 | 21.3243 | 20.7930 |
|       | 28.2220 | 24.1674 | 22.7031 | 21.4733 | 20.9401 |
|       | 28.3998 | 24.3280 | 22.8567 | 21.6208 | 21.0857 |

### Table 6: The first nondimensional frequency of the S-S FGM beam depends on \(K^*_w\) and \(K^*_s\).
References

[1] J. N. Reddy, “Free vibration analysis of functionally graded ceramic-metal plates,” *Analysis and Design of Plated Structures*, vol. 1, pp. 293–321, 2007.

[2] Z. Ning, K. Tahir, G. Haomin, S. Shaoshuai, Z. Wei, and Z. Weip, “Functionally graded materials: an overview of stability, buckling, and free vibration analysis,” *Advances in Materials Science and Engineering*, vol. 2019, Article ID 1354150, 18 pages, 2019.

[3] D. V. Thom, D. H. Duc, P. V. Minh, and N. S. Tung, “Finite element modelling for free vibration response of cracked stiffened FGM plates,” *Vietnam Journal of Science and Technology*, vol. 58, no. 1, pp. 119–129, 2020.

[4] V. T. Do, D. K. Nguyen, N. D. Duc, D. H. Doan, and Q. T. Bui, “Analysis of bi-directional functionally graded plates by FEM and a new third-order shear deformation plate theory,” *Thin-Walled Structures*, vol. 119, pp. 687–699, 2017.

[5] H.-N. Nguyen, T. C. Tan, D. T. Luat, V.-D. Phan, D. V. Thom, and P. V. Minh, “Research on the buckling behavior of functionally graded plates with stiffeners based on the third-order shear deformation theory,” *Materials*, vol. 12, no. 8, p. 1262, 2019.

[6] S. K. Gunjal and U. S. Dixit, “Vibration analysis of shape-optimized rotating cantilever beams,” *Engineering Optimization*, vol. 39, no. 1, pp. 105–123, 2007.

[7] S. C. Pradhan and T. Murmu, “Application of nonlocal elasticity and DQM in the flapwise bending vibration of a rotating nanocantilever,” *Physica E: Low-Dimensional Systems and Nanostructures*, vol. 42, no. 7, pp. 1944–1949, 2010.

[8] L. Li, D. G. Zhang, and W. D. Zhu, “Free vibration analysis of a rotating hub–functionally graded material beam system with the dynamic stiffening effect,” *Journal of Sound and Vibration*, vol. 333, no. 5, pp. 1526–1541, 2014.

[9] M. D. S. Amir, B. J. Mehdi, and M. Dehrouyeh, “On size-dependent lead-lag vibration of rotating microcantilevers,” *International Journal of Engineering Science*, vol. 101, pp. 56–63, 2016.

[10] L. Jung-Woo and L. Jung-Youn, “In-plane bending vibration analysis of a rotating beam with multiple edge cracks by using the transfer matrix method,” *Meccanica*, vol. 52, pp. 1143–1157, 2017.

[11] D. Das, “Free vibration and buckling analyses of geometrically non-linear and shear-deformable FGM beam fixed to the inside of a rotating rim,” *Composite Structures*, vol. 179, pp. 628–645, 2017.

[12] X. Xu, Q. Han, F. Chu, and R. G. Parker, “Vibration suppression of a rotating cantilever beam under magnetic excitations by applying the magnetostrictive material,” *Journal of Intelligent Material Systems and Structures*, vol. 30, no. 4, pp. 576–592, 2019.

[13] B. Alireza and X. Y. Cai, “Vibration analysis of rotating rods based on the nonlocal elasticity theory and coupled displacement field,” *Microsystem Technologies*, vol. 25, pp. 1077–1085, 2019.

[14] L. Li, W.-H. Liao, D. Zhang, and Y. Zhang, “Vibration control and analysis of a rotating flexible FGM beam with a lumped mass in temperature field,” *Composite Structures*, vol. 208, pp. 244–260, 2019.

[15] C. Dejin, F. Kai, and Z. Shiijie, “Flapwise vibration analysis of rotating composite laminated timoshenko microbeams with geometric imperfection based on a re-modified couple stress theory and isogeometric analysis,” *European Journal of Mechanics-A/Solids*, vol. 76, pp. 25–35, 2019.

[16] M. S. Atanasov and V. Stojanović, “Nonlocal forced vibrations of rotating cantilever nano-beams,” *European Journal of Mechanics-A/Solids*, vol. 79, Article ID 103850, 2020.

[17] M. H. Jalaei and Ö. Civelek, “On dynamic instability of magnetically embedded viscoelastic porous FG nanobeam,” *International Journal of Engineering Science*, vol. 143, pp. 14–32, 2019.

[18] Ç. Demir and Ö. Civelek, “On the analysis of microbeams,” *International Journal of Engineering Science*, vol. 201, pp. 14–33, 2017.

[19] B. Akgöz and Ö. Civelek, “A novel microstructure-dependent shear deformable beam model,” *International Journal of Mechanical Sciences*, vol. 99, pp. 10–20, 2015.

[20] L. Hanten, G. Giunta, S. Belouettar, and V. Salnikov, “Free vibration analysis of fibre-metal laminated beams via hierarchical one-dimensional models,” *Mathematical Problems in Engineering*, vol. 2018, Article ID 2724781, 12 pages, 2018.

[21] G. Moustafa, S. Hayat, B. Fouad et al., “Influences of porosity distributions and boundary conditions on mechanical bending response of functionally graded plates resting on Pasternak foundation,” *Steel and Composite Structures*, vol. 38, no. 1, pp. 1–15, 2021.

[22] R. Abdelkrim, C. Abdelbaki, A. B. Abdelmoumen et al., “Physical stability response of a SLSG resting on viscoelastic medium using nonlocal integral first-order theory,” *Steel and Composite Structures*, vol. 37, no. 6, pp. 695–709, 2020.

[23] B. Noureddine, Z. Mohamed, A. B. Abdelmoumen et al., “Deflections, stresses and free vibration studies of FG-CNT reinforced sandwich plates resting on Pasternak elastic foundation,” *Computers and Concrete*, vol. 26, no. 3, pp. 213–226, 2020.

[24] R. Mohamed, H. B. Kouider, K. Abdelhakim et al., “A new innovative 3-unknowns HSDT for buckling and free vibration of exponentially graded sandwich plates resting on elastic foundations under various boundary conditions,” *Geomechanics and Engineering*, vol. 22, no. 2, pp. 119–132, 2020.

[25] B. Fouad, A. B. Abdelmoumen, T. Abdeldjabbar et al., “Stability and dynamic analyses of SW-CNT reinforced concrete beam resting on elastic-foundation,” *Computers and Concrete*, vol. 25, no. 6, pp. 485–495, 2020.

[26] C. C. Sara, K. Abdelhakim, A. B. Abdelmoumen et al., “A novel four-unknown integral model for buckling response of FG sandwich plates resting on elastic foundations under various boundary conditions using Galerkin’s approach,” *Geomechanics and Engineering*, vol. 21, no. 5, pp. 471–487, 2020.

[27] R. Salah, A. B. Abdelmoumen, B. Abdelhakim et al., “Effects of hygro-thermo-mechanical conditions on the buckling of FG sandwich plates resting on elastic foundations,” *Computers and Concrete*, vol. 25, no. 4, pp. 311–325, 2020.

[28] K. Miloud, K. Abdelhakim, A. B. Abdelmoumen et al., “A study on the structural behaviour of functionally graded porous plates on elastic foundation using a new quasi-3D model: bending and free vibration analysis,” *Computers and Concrete*, vol. 25, no. 1, pp. 37–57, 2020.

[29] T. Abdelouahed, S. U. Al-Dulaijan, M. A. Al-Osta et al., “A four variable trigonometric integral plate theory for hygro-thermo-mechanical bending analysis of AFG ceramic-metal plates resting on a two-parameter elastic foundation,” *Steel and Composite Structures*, vol. 34, no. 4, pp. 511–524, 2020.

[30] Y. A. Farouk, M. Mustapha, A. B. Abdelmoumen et al., “Influences of porosity on dynamic response of FG plates resting on Winkler/Pasternak/Kerr foundation using quasi
3D HSDT,” *Computers and Concrete*, vol. 24, no. 4, pp. 347–367, 2019.

[31] B. Hichem, M. S. Mahmoud, C. Abdelbaki et al., "Influence of porosity on thermal buckling behavior of functionally graded beams,” *Smart Structures and Systems*, vol. 27, no. 4, pp. 719–728, 2021.

[32] A. Ehsan, K. Mohammad, Z. Soleimani-Javid, A. Saeed, and T. Abdelouahed, "Porosity-dependent vibration analysis of FG microplates embedded by polymeric nanocomposite patches considering hygrothermal effect via an innovative plate theory,” *Engineering With Computers*, 2021.

[33] M. S. Al-Furjan, H. Mostafa, R. Alireza et al., “Chaotic simulation of the multi-phase reinforced thermo-elastic disk using GDQM,” *Engineering With Computers*, 2020.

[34] M. Hakima, A. B. Abdelmoumen, H. Houari et al., "Investigation on hygro-thermal vibration of P-FG and symmetric S-FG nanobeams using integral Timoshenko beam theory,” *Advances in Nano Research*, vol. 8, no. 4, pp. 293–305, 2020.

[35] B. Abderrafik, B. Belhadj, B. Mohamed et al., "A simple nth-order shear deformation theory for thermomechanical behavior analysis of different configurations of FG sandwich plates,” *Smart Structures and Systems*, vol. 25, no. 2, pp. 197–218, 2020.

[36] T. Yu, T. Q. Bui, S. Yin et al., “On the thermal buckling analysis of functionally graded plates with internal defects using extended isogeometric analysis,” *Composite Structures*, vol. 136, pp. 684–695, 2016.

[37] P. T. Dat, D. V. Thom, and D. T. Luat, “Free vibration of functionally graded sandwich plates with stiffeners based on the third-order shear deformation theory,” *Vietnam Journal of Mechanics*, vol. 38, no. 2, pp. 103–122, 2016.

[38] N. D. Duc, T. D. Trinh, T. Van Do, and D. H. Doan, “On the buckling behavior of multi-cracked FGM plates,” in *Proceedings of the International Conference on Advances in Computational Mechanics 2017*, pp. 29–45, Phu Quoc Island, Vietnam, August 2018.

[39] V. T. Do, D. H. Doan, N. D. Duc, and T. Q. Bui, “Phase-field thermal buckling analysis for cracked functionally graded composite plates considering neutral surface,” *Composite Structures*, vol. 182, pp. 542–548, 2017.

[40] Q. B. Tinh, V. D. Thom, H. T. T. Lan et al., “On the high temperature mechanical behaviors analysis of heated functionally graded plates using FEM and a new third-order shear deformation plate theory,” *Composites Part B: Engineering*, vol. 92, pp. 218–241, 2016.

[41] K. P. Soldatos, "A transverse shear deformation theory for homogeneous monoclinic plates,” *Acta Mechanica*, vol. 94, no. 3-4, pp. 195–220, 1992.

[42] T. Huu-Tai and C. Dong-Ho, "Finite element formulation of various four unknown shear deformation theories for functionally graded plates,” *Finite Elements in Analysis and Design*, vol. 75, pp. 50–61, 2013.

[43] S. P. Timoshenko and J. M. Gere, *Theory of Elastic Stability*, Dover Publications, Mineola, NY, USA, 1989.

[44] M. Avcar and K. M. M. Waleed, "Free vibration of functionally graded beams resting on Winkler-Pasternak foundation,” *Arabian Journal of Geosciences*, vol. 11, no. 232, 2018.

[45] M. H. Ghayesh and M. Amabili, "Coupled longitudinal-transverse behaviour of a geometrically imperfect microbeam,” *Composites Part B: Engineering*, vol. 60, pp. 371–377, 2014.

[46] A. D. Wright, C. E. Smith, R. W. Thresher, and J. L. C. Wang, "Vibration modes of centrifugally stiffened beams,” *Journal of Applied Mechanics*, vol. 49, no. 1, pp. 197–202, 1982.