Polynomial time algorithms for bi-criteria, multi-objective and ratio problems in clustering and imaging
Part I: Normalized cut and ratio regions

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Abstract
Partitioning and grouping of similar objects plays a fundamental role in image segmentation and in clustering problems. In such problems a typical goal is to group together similar objects, or pixels in the case of image processing. At the same time another goal is to have each group distinctly dissimilar from the rest and possibly to have the group size fairly large. These goals are often combined as a ratio optimization problem. One example of such problem is the normalized cut problem, another is the ratio regions problem. We devise here the first polynomial time algorithms solving these problems optimally. The algorithms are efficient and combinatorial. This contrasts with the heuristic approaches used in the image segmentation literature that formulate those problems as nonlinear optimization problems, which are then relaxed and solved with spectral techniques in real numbers. These approaches not only fail to deliver an optimal solution, but they are also computationally expensive. The algorithms presented here use as a subroutine a minimum \( s,t \)-cut procedure on a related graph which is of polynomial size. The output consists of the optimal solution to the respective ratio problem, as well as a sequence of nested solution with respect to any relative weighting of the objectives of the numerator and denominator.

An extension of the results here to bi-criteria and multi-criteria objective functions is presented in part II.

1 Introduction

The leading challenge in the field of imaging is vision grouping, or segmentation. Grouping is to recognize and delineate, automatically, the salient objects in an image. Image segmentation is equivalent to partitioning the set of pixels forming the image, or to clustering its pixels. High quality clustering is often defined by multiple attributes. As an optimization problem this requires attaining more than one objective. The motivation for studying the normalized cut problem is an example of setting an optimization criterion in order to attain two goals. One goal is to have the selected group’s pixels to be as dissimilar to the remainder of the image as possible, and the second is to maximize the similarity of the pixels within the group. These two objectives are presented as a minimization of the ratio of the first to the second.

A great deal of the literature is concerned only with the bipartitioning of the image. That is, the separation of one object segment from the rest of the image - the background. Even this
modest goal presents a number of computational difficulties. While presenting an image as a graph and the similarity between pairs of objects as a weight of an edge, a simple minimum 2-cut problem will achieve a partition that minimizes similarity between the two parts. An adverse phenomenon associated with the minimum 2-cut optimal solution, that was noted however by Shi and Malik, [19], and others, is that often the selected part tends to be very small. To compensate and correct for the phenomenon of small segments Shi and Malik introduced the notion of *normalized cut*.

Graph theoretical framework is suitable for representing image segmentation and grouping problems. The image segmentation problem is presented on an undirected graph \( G = (V, E) \), where \( V \) is the set of pixels and \( E \) are the pairs of which similarity information is available. Typically one considers a planar image with pixels arranged along a grid. The 4-neighbors set up is commonly used with each pixel having 4 neighbors two along the vertical axis and two along the horizontal axis. This set up forms a planar grid graph. The 8-neighbors arrangement is also used, but the planar structure is no longer preserved, and complexity of the various heuristic algorithms is increasing, sometimes significantly. Images can of course be also 3-dimensional, and in general clustering problems there is no grid structure. The algorithms presented here do not assume any specific property of the graph \( G \) - they work for general graphs.

The edges in the graph representing the image carry *similarity* weights. There is a great deal of literature on how to generate similarity weights, and we do not discuss this issue here. We only use the fact that similarity is inversely increasing with the difference in attributes between the pixels. In terms of the graph, each edge \([i, j]\) is assigned a similarity weight \( w_{ij} \) that is increasing as the two pixels \( i \) and \( j \) are perceived to be more similar. Low values of \( w_{ij} \) are interpreted as dissimilarity. However, in some contexts one might want to generate *dissimilarity* weights independently. In that case each edge has two weights, \( w_{ij} \) for similarity, and \( \hat{w}_{ij} \) for dissimilarity.

Two applications of efficient algorithms for ratio problems are presented: One for the problem of "*normalized cut*", which is to minimize the ratio of the similarity between the set of objects and its complement and the similarity within the set of objects. The second problem is that of "*ratio-regions*" which is to minimize the ratio of the similarity between the set of objects and its complement and the number (or weight) of the objects within the set. The algorithms not only provide an optimal solution to the ratio problem, but also deliver a sequence of solutions for all possible relative weighting of the two objectives. These solutions are often more informative than the optimal solution to the ratio problem alone.

Although the multi-segmentation problem is a partition to multiple sets, the ratio problems discussed here do not directly addressed as such, due to computational issues. Instead these bipartitions have been used recursively to generate any desired number of segments. It is shown in part II that all the problems presented here have optimal solutions that are bipartition, and how to characterize ratio problems in general with this property. For multiple segments, the Markov Random Fields is one model that has been popular in that context. It has been studied by numerous authors, e.g. [2], and has been established for the first time to be polynomial time solvable for convex objectives in [15].
2 Notation

Let the weights of the edges in the graph be $w_{ij}$ for $[i, j] \in E$. If the edges have two sets of weights, these will be denoted by $w_{ij}^1$ and $w_{ij}^2$.

A bipartition of the graph is called a cut, $(S, \bar{S}) = \{(i, j) | i \in S, j \in \bar{S}\}$, where $\bar{S} = V \setminus S$. We define the capacity of a cut $(S, \bar{S})$, and the capacity for any pair of sets $(A, B)$ to be $C(A, \bar{A}) = \sum_{i \in A, j \in B} w_{ij}$. We define the capacity of a set $A \subset V$ to be $C(A) = C(A, A) = \sum_{i,j \in A} w_{ij}$. For inputs with two sets of edge weights we let $C_1(A, B) = \sum_{i \in A, j \in B} w_{ij}^1$ and $C_2(A, B) = \sum_{i \in A, j \in B} w_{ij}^2$.

Given a partition of a graph into $k$ disjoint components, $\{V_1, \ldots, V_k\}$ the $k$-cut value is $C(V_1, \ldots, V_k) = \frac{1}{2} \sum_{i=1}^k C(V_i, \bar{V}_i)$. The problem of partitioning a graph to $k$ nonempty components that minimize the $k$-cut value is polynomial time solvable for fixed $k$, [12].

For graphs with weighted nodes, we let the weight of node $j$ be $v_j$. The weight of a set of nodes $A$ is denoted by $V(A) = \sum_{j \in A} v_j$.

3 Several ratio problems

We list here four types of ratio problems. This include, in addition to the normalized cut problem and the ratio regions problem, also the densest subgraph problem and the “ratio cut” problem. We solve here only the first two. The third problem has been known to be polynomial time solvable, and the last problem is NP-hard.

3.1 The normalized cut problem

Shi and Malik noted in their work on segmentation that cut procedures tend to create segments that may be very small in size. To address this issue they proposed several versions of objective functions that provide larger segments in an optimal solution. Among the proposed objective they formulated the normalized cut as the optimization problem

$$\min_{S \subset V} C(S, \bar{S}) \cdot \left(\frac{1}{|S|} + \frac{1}{|\bar{S}|}\right).$$

This problem is equivalent to finding the expander ratio of the graph discussed in the next subsection.

This objective function drives the segment $S$ and its complement to be approximately of equal size. Indeed, like the balanced cut problem the problem was shown to be NP-hard, [19], by reduction from set partitioning. A variant of the problem also defined by Shi and Malik is

$$\min_{S \subset V} C(S, \bar{S}) \cdot \left(\frac{1}{C(S, V)} + \frac{1}{C(\bar{S}, V)}\right).$$

Another variant yet of the problem is the quantity $h_G = \min_{S \subset V} \frac{C(S, V \setminus S)}{\min[C(S, S), C(V \setminus S)]}$, also known as the Cheeger constant, [5, 6]. More frequently, for minor variants of the problem, the denominator is $\min(|S|, |V \setminus S|)$, or $|S|$ is replaced by a quantity representing the volume of $S$. This Cheeger constant is approximated by the second largest eigenvalue of a certain related adjacency matrix of the graph. This eigenvalue $\lambda_1$ is related to the Cheeger constant by the inequalities: $2h_G \geq \lambda_1 \geq h_G^2/2$. Computing the value of the Cheeger constant is NP-hard -
it is the same as finding the expander ratio of a graph and again it drives to a roughly equal or balanced partition of the graph. The dominant techniques in vision grouping are spectral in nature. That is, they compute the eigenvalues and the eigenvectors and then some type of rounding process, see e.g. [21] [20].

Instead of the sum problem, there are other related optimization problems used for image segmentation. Sharon et al. [20] define the normalized cut as

$$\min_{S \subseteq V} \frac{C(S, \bar{S})}{C(S, S)}$$

Sharon et al. [20] state that:

A salient segment in the image is one for which the similarity across its border is small, whereas the similarity within the segment is large (for a mathematical description, see Methods). We can thus seek a segment that minimizes the ratio of these two expressions. Despite its conceptual usefulness, minimizing this normalized cut measure is computationally prohibitive, with cost that increases exponentially with image size.

One of our contributions here is to show that the problem of minimizing this ratio is in fact solvable in polynomial time, and with a combinatorial algorithm.

The typical solution approach used when addressing optimization problems for image segmentation is to approximate the problem objective by a nonlinear (quadratic) expression for which the eigenvectors of an associated matrix form an optimal solution. Let binary variables $x_i$ for $i \in V$ be defined so that $x_i = 1$ if node $i$ in the selected side of the cut – the segment. The following nonlinear formulation is the relaxation that has been used by Sharon et al. and others, [20] [21] [19]:

$$\min \sum w_{ij}(x_i - x_j)^2 \sum w_{ij}x_i \cdot x_j = x^T L x$$

where $L$ is the Laplacian matrix of the graph and $W$ is a matrix appropriately defined. The use of spectral techniques involves real number computations with the associated numerical issues. Even an exact solution to the nonlinear problem is a vector of real numbers whereas the original problem is discrete and binary.

However, this normalized cut problem (without the “balanced” requirement) is polynomial time solvable. We show an algorithm solving the problem in the same complexity as a single minimum $s,t$-cut on a related graph on $O(n + m)$ nodes and $O(n + m)$ edges.

3.2 Ratio regions and expanders

Consider the objective function $\min_{|S| \leq \frac{n}{2}} \frac{C(S, \bar{S})}{|S|}$. This value of the optimal solution, for a graph $G$, is known as the expansion ratio of $G$. This problem is NP-hard as the limit on the size of $|S|$ makes it difficult, and drives the solution towards a balanced cut – a known NP-hard problem. The objective function can also be written as $\min_{S \subseteq V} \frac{C(S, \bar{S})}{\min(|S|, |\bar{S}|)}$.

A variant of this problem $\min_{S \subseteq V} \frac{C(S, \bar{S})}{|S|}$ has been considered under the name ratio regions by Cox et al. [7]. The ratio region problem is motivated by seeking a segment, or region, where
the boundary is of low cost and the segment itself has high weight. The problem studied by Cox et al. is restricted to planar graphs and thus to planar grid images with 4 neighbors only. Though the boundary of the region is defined as a path, the path corresponds to a cut in the dual graph. For graph nodes of weight \( d_i \) the problem is \( \min_{S \subset V} \frac{C(S,S)}{|S|} \sum_{i \in S} d_i \).

This problem is shown here to be polynomially solvable by a parametric cut procedure, in the complexity of a single minimum cut. The problem is in fact equivalent to a binary and linear version of the Markov Random Fields problem, called the maximum \( s \)-excess problem in [14]. It is interesting to note that the pseudoflow algorithm in [14] is set to solve the maximum \( s \)-excess problem directly. Our algorithm for the ratio regions problem applies for node weights that can be either positive or negative. This generalizes the application context of Cox et al. the node weights were all positive.

### 3.3 Densest subgraph

Sarkar and Boyer [18] defined the problem \( \min_{S \subset V} \frac{C(S,S)}{|S|} \). This objective is of interest for weights that reflect dissimilarity. In that case the goal is to minimize the dissimilarity within the selected segment while the size of the segment tends to be large. For similarity weights the objective would be to maximize this quantity. Both this problem, and its maximization version are solved in polynomial time. Also the nodes can carry arbitrary weights and the algorithm is still applicable with no change in the running time.

This problem has been known for a long time as the maximum density subgraph is the subgraph induced by the subset of nodes \( D \) maximizing,

\[
\max_{S \subset V} \frac{C(S,S)}{|S|}.
\]

This problem was shown to be solvable in polynomial time by Goldberg [10]. Gallo, Grigoriadis and Tarjan [9] showed how the problem would be solved as a parametric minimum \( s, t \)-cut in the complexity of a single \( s, t \)-cut.

A node weighted version of the problem is \( \max_{S \subset V} \frac{C(S,S)}{|S|} \). This problem is solved by a minor extension of the densest subgraph approach in the same run time.

### 3.4 “Ratio cuts”

This problem was introduced by Wang and Siskind [22]. In the ratio cut problem the edges have two weights associated with each. Wang and Siskind studied the case where \( w_{ij}^1 \) are positive and \( w_{ij}^2 \) are equal to 1 for all \([i, j] \in E\). The goal is to minimize the ratio,

\[
\min_{S \subset V} \frac{C_1(S, \bar{S})}{C_2(S, S)}.
\]

This problem was shown in [22] to be at least as hard as the sparsest cut problem, and therefore NP-hard. On the other hand, for planar graphs, Wang and Siskind demonstrated that the problem is solvable in polynomial time.
3.5 Overview

The problems, that the methodology paradigm presented here solve in polynomial time, are summarized in the Table 1:

| Problem name          | Objective | Reference |
|-----------------------|-----------|-----------|
| Normalized cut        | min       | [19, 20]  |
| Normalized cut’       | min       | [19]      |
| “Density”             | min       | [18]      |
| Ratio regions         | min       | [7]       |
| Weighted ratio regions| min       | [7]       |

Table 1: Ratio optimization problems in image segmentation

We present here in detail only the algorithm for the normalized cut problem, which is the hardest one on the list. The construction for the ratio regions is described briefly. For the “density” problem with either a minimization or maximization objective function we studied the problem for the entire sequence of solutions in the context of dynamically evolving facility set, see [17]. Thus for the ”density” case we only point out an efficient algorithm.

4 The solution approach

4.1 Monotone integer programming formulation

The key is to formulate the problem as an integer linear programming problem, a 0-1 integer programming here, with monotone inequalities constraints. It was shown in [16] that any integer programming formulation on monotone constraints has a corresponding graph where the minimum cut solution corresponds to the optimal solution to the integer programming problem. Thus the formulation is solvable in polynomial time.

To convert the ratio objective to a linear objective we utilize the reduction of the ratio problem to a linearized optimization problem.

4.2 Linearizing ratio problems

A general approach for maximizing a fractional (or as it is sometimes called, geometric) objective function over a feasible region $\mathcal{F}$, $\min_{x \in \mathcal{F}} \frac{f(x)}{g(x)}$, is to reduce it to a sequence of calls to an oracle that provides the yes/no answer to the $\lambda$-question:

Is there a feasible subset $V' \subset V$ such that $\sum_{x \in \mathcal{F}} f(x) - \lambda \sum_{x \in \mathcal{F}} g(x) < 0$?

If the answer to the $\lambda$-question is yes then the optimal solution has value smaller than $\lambda$. Otherwise, the optimal value is greater than or equal to $\lambda$. A standard approach is then to utilize a binary search procedure that calls for the $\lambda$-question $O(\log(UF))$ times in order to solve the problem, where $U = \sum_{[i,j] \in E} w_{ij}$, and $F = \sum_{[i,j] \in E} w'_{ij}$ for the weights at the denominator $w'_{ij}$.
Therefore, if the linearized version of the problem, that is the \( \lambda \)-question, is solved in polynomial time, then so is the ratio problem. Note that the number of calls to the linear optimization is not strongly polynomial but rather, if binary search is employed, depends on the logarithm of the magnitude of the numbers in the input. In some cases however there is an efficient procedure that uses the solution for one parameter value to compute the value for another parameter value more efficiently. Such is the case for the densest subgraph problem which has an efficient parametric procedure ([9]).

It is important to note that not all ratio problems are solvable in polynomial time. One prominent example of such ratio problem is the ratio cut introduced by Wang and Siskind, [22]. That criterion applies in a graph with two sets of weights for each edge. Let \( w^1_{ij}, w^2_{ij} \) be the two weights assigned to edge \([i,j]\). Then a cut with respect to \( w^1_{ij} \) separating \( S \) from \( \bar{S} \) is \( C_1(S,\bar{S}) \) and with respect to \( w^2_{ij} \) is is denoted by \( C_2(S,\bar{S}) \). The ratio criterion defined by Wang and Siskind is to minimize \( \frac{C_1(S,\bar{S})}{C_2(S,\bar{S})} \). For the weight \( w^2_{ij} \) they use the value 1, so this objective is to find a cut minimizing the cut value divided by the number of edges in the cut. As in other cases, the rationale is to try and increase the number of edges in the cut, and hence the size of the cluster/segment. This particular criterion was shown in [22] to be NP-hard, and polynomial time solvable on planar graphs. For the ratio cut problem the linearized problem is NP-hard, and the ratio cut problem is NP-hard as well. The linearized problem is NP-hard by reduction from maximum cut, and the ratio problem by reduction from the sparsest cut problem, [22]. However the ratio cut problem has a polynomial time algorithm for planar graphs. For planar graphs the \( \lambda \)-question is solved by finding a maximum weight non-bipartite matching in a related graph. The procedure of [22] indeed makes repeated calls to solving non-bipartite matching problems, where for each value of \( \lambda \) another graph has to be constructed.

It is also important to note that linearizing is not always the right approach to use for a ratio problem. For example, the problem of finding a partition of a graph to \( k \) components minimizing the \( k \)-cut between components for \( k \geq 2 \) divided by the number of components \( k \), always has an optimal solution with \( k = 2 \) which is attained by a minimum 2-cut algorithm. On the other hand, the linearized problem is much harder to solve (though it can be solved in polynomial time.) Additional details are provided in part II of this paper.

5 The normalized cut formulation

We first provide a formulation for the problem, \( \min_{S \subset V} \frac{C_1(S,\bar{S})}{C_2(S,\bar{S})} \). We use different similarity weights for the numerator \( w^1_{ij} \) and denominator \( w_{ij} \).

We begin with an integer programming formulation of the problem. Let

\[
x_i = \begin{cases} 
1 & \text{if } i \in S \\
0 & \text{if } i \in \bar{S}.
\end{cases}
\]

We define additional binary variables: \( z_{ij} = 1 \) if exactly one of \( i \) or \( j \) is in \( S \); \( y_{ij} = 1 \) if both \( i \) or \( j \) are in \( S \).

\[
z_{ij} = \begin{cases} 
1 & \text{if } i \in S, j \in \bar{S}, \text{ or } i \in \bar{S}, j \in S \\
0 & \text{if } i, j \in S \text{ or } i, j \in \bar{S}.
\end{cases}
\]
\[ y_{ij} = \begin{cases} 1 & \text{if } i, j \in S \text{ or } i, j \in \bar{S} \\ 0 & \text{otherwise.} \end{cases} \]

With these variables the following is a valid formulation (NC) of the normalized cut problem:

\[ \text{(NC)} \quad \min \frac{\sum w_{ij} z_{ij}}{\sum w'_{ij} y_{ij}} \]
subject to
\[ x_i - x_j \leq z_{ij} \text{ for all } [i, j] \in E \]
\[ x_j - x_i \leq z_{ji} \text{ for all } [i, j] \in E \]
\[ y_{ij} \leq x_i \text{ for all } [i, j] \in E \]
\[ y_{ij} \leq x_j \]
\[ 1 \leq \sum_{[i,j] \in E} y_{ij} \leq |E| - 1 \]
\[ x_j \text{ binary } j \in V \]
\[ z_{ij}, y_{ij} \text{ binary } j \in V. \]

To verify the validity of the formulation notice that the objective function drives the values of \( z_{ij} \) to be as small as possible, and the values of \( y_{ij} \) to be as large as possible. With the constraints, \( z_{ij} \) cannot be 0 unless both endpoints \( i \) and \( j \) are in the same set. On the other hand \( y_{ij} \) cannot be equal to 1 unless both endpoints \( i \) and \( j \) are in \( S \). The sum constraint ensures that at least one edge is in the segment \( S \) and at least one edge is in the complement - the background. Otherwise the ratio is not be defined in the first case and the optimal solution is to choose \( S = V \) in the second. We remove the sum constraint from the formulation and replace it by setting for some edge in the object to serve as “seed” and some edge in the background to serve as “seed”. Adding seeds is an approach often used in segmentation, see e.g. [3] where the user keeps adding “seeds” until the bipartition is satisfactory. Here the choice is made once, and can be replaced by enumerating the possible pairs of edges that serve as object and background edges. Since for both the object and it complement the cut value is the same, the solution will always be in terms of the larger segment in the bipartition that is likely to contain higher total similarity weights. The edge that we set to be in the source is therefore usually the one in the background and the one in the sink would be in the object. We thus replace the sum constraint by setting \( y_{i'j'}, y_{i'j'} = 1 \) for some pair of edges in the background and the object respectively.

Once the sum constraint has been removed, the problem formulation is easily recognized as a monotone integer programming with up to three variables per inequality according to the definition provided in Hochbaum’s [16]. Any such problem was shown there to be solvable as a minimum cut problem on a certain associated graph. Because the objective function is a ratio, we first “linearize” the problem.

### 5.1 Linearizing the objective function

The \( \lambda \)-question for the normalized cut problem is:

Is there a feasible subset \( V' \subset V \) such that \( \sum_{[i,j] \in E} w_{ij} z_{ij} - \lambda \sum_{[i,j] \in E} w'_{ij} y_{ij} < 0? \)

One possible approach is to utilize a binary search procedure that calls for the \( \lambda \)-question \( O(\log(UF)) \) times in order to solve the problem, where \( U = \sum_{[i,j] \in E} w_{ij} \), and \( F = \sum_{[i,j] \in E} w'_{ij} \) for the weights at the denominator \( w'_{ij} \).
With the construction of the graph we observe that one can use instead a parametric approach which is significantly more efficient. We note that the $\lambda$-question is the following monotone optimization problem,

\[
(\lambda\text{-NC}) \quad \min \sum_{[i,j] \in E} w_{ij} z_{ij} - \sum_{[i,j] \in E} \lambda w'_{ij} y_{ij}
\]

subject to

\begin{align*}
  x_i - x_j &\leq z_{ij} \quad \text{for all } [i,j] \in E \\
  x_j - x_i &\leq z_{ji} \quad \text{for all } [i,j] \in E \\
  y_{ij} &\leq x_i \quad \text{for all } [i,j] \in E \\
  y_{ij} &\leq x_j \\
  y_{i^*j^*} &= 1 \text{ and } y_{i'j'} = 0 \\
  x_j &\text{ binary } j \in V \\
  z_{ij}, y_{ij} &\text{ binary } j \in V.
\end{align*}

If the optimal value of this problem is negative then the answer is yes, otherwise the answer is no. This problem is an integer optimization problem on a totally unimodular constraint matrix. That means that we can solve the linear programming relaxation of this problem and get a basic optimal solution that is integer. Instead we will use a much more efficient algorithm described in [16] which relies on the monotone property of the constraints.

5.2 Solving the $\lambda$ question with a minimum cut procedure

We construct a directed graph $G' = (V', A')$ with a set of nodes $V'$ that has a node for each variable $x_i$ and a node for each variable $y_{ij}$. The nodes $y_{ij}$ carry a negative weight of $-\lambda w_{ij}$. The arc from $x_i$ to $x_j$ has capacity $w'_{ij}$ and so does the arc from $x_j$ to $x_i$ as in our problem $w_{ij} = w_{ji}$. The two arcs from each edge-node $y_{ij}$ to the endpoint nodes $x_i$ and $x_j$ have infinite capacity. Figure 1 shows the basic gadget in the graph $G'$ for each edge $[i,j] \in E$.

![Figure 1: The basic gadget in the graph representation.](image)

We claim that any finite cut in this graph, that has $y_{i^*j^*}$ on one side of the bipartition and $y_{i'j'}$ on the other, corresponds to a feasible solution to the problem $\lambda$-NC. Let the cut $(S,T)$, where $T = V' \setminus S$, be of finite capacity $C(S,T)$. We set the value of the variable $x_i$ or $y_{ij}$ to be equal to 1 if the corresponding node is in $S$, and 0 otherwise. Because the cut is finite, then $y_{ij} = 1$ implies that $x_i = 1$ and $x_j = 1$. 

9
Next we claim that for any finite cut the sum of the weights of the $y_{ij}$ nodes in the source set and the capacity of the cut is equal to the objective value of problem $\lambda$-NC. Notice that if $x_i = 1$ and $x_j = 0$ then the arc from the node $x_i$ to node $x_j$ is in the cut and therefore the value of $z_{ij}$ is equal to 1.

We next create a source node $s$ and connect all $y_{ij}$ nodes to the source with arcs of capacity $\lambda w'_{ij}$. The node $y_{i*,j*}$ is then shrunk with a source node $s$ and therefore also its endpoints nodes. The node $y_{i'j'}$ and its endpoints nodes are shrunk with the sink $t$. We denote this graph illustrated in Figure 2 $G'_{st}$.

Figure 2: The graph $G'_{st}$ with edge $[1, 2]$ as source seed and edge $[4, 5]$ as sink seed.

**Theorem 5.1** A minimum $s, t$-cut in the graph $G'_{st}$, $(S, T)$, corresponds to an optimal solution to $\lambda$-NC by setting all the variables whose nodes fall in $S$ to 1 and zero otherwise.

**Proof:** Note that whenever a node $y_{ij}$ is in the sink set $T$ the arc connecting it to the source is included in the cut. Let the set of $x$ variable nodes be denoted by $V_x$ and the set of $y$ variable nodes, excluding $y_{i*,j*}$, be denoted by $V_y$. Let $(S, T)$ be any finite cut in $G'_{st}$ with $s \in S$ and $t \in T$ and capacity $C(S, T)$.

\[
C(S, T) = \sum_{y_{ij} \in T \cap V_y} \lambda w'_{ij} + \sum_{i \in V_x \cap S, j \in V_x \cap T} w_{ij} = \sum_{v \in V_y} \lambda w'_v - \sum_{y_{ij} \in S \cap V_y} \lambda w'_{ij} + \sum_{x_i \in V_x \cap S, x_j \in V_x \cap T} w_{ij} = \lambda W' + \left[ \sum_{i \in V_x \cap S, j \in V_x \cap T} w_{ij} - \sum_{y_{ij} \in S \cap V_y} \lambda w'_{ij} \right].
\]

This proves that for a fixed constant $W' = \sum_{v \in V_y} w'_v$ the capacity of a cut is equal to a constant $W'\lambda$ plus the objective value corresponding to the feasible solution. Hence the partition $(S, T)$ minimizing the capacity of the cut minimizes also the objective function of $\lambda$-NC.
5.3 A parametric procedure for solving normalized cut

The source adjacent arcs in the graph $G'_{st}$ are monotone increasing with $\lambda$. As the value of $\lambda$ increases the source set of the respective minimum cuts are nested. This is called the nestedness lemma. Although the capacity of the cut is increasing with any increase in $\lambda$ the set of nodes in the source set can thus be incremented only $n' = |V'|$ times. We call the values of $\lambda$ where the source set expands by at least one node, the breakpoints of the parametric cut. Let the breakpoints be $\lambda_1 > \lambda_2 > \ldots > \lambda_\ell$, with corresponding minimal source sets, $S_1 \subset S_2 \subset \ldots \subset S_\ell$. As a result of the nestedness lemma $\ell \leq n'$ for a graph on $n'$ nodes since there can be no more than $n'$ different nested source sets. The capacity value of the minimum cut is increasing as a function of increasing values of $\lambda$ along a piecewise linear concave curve.

**Theorem 5.2** All breakpoints of the density graph can be found by solving a parametric minimum cut problem where the source adjacent capacities of arcs are linear functions of the parameter, $\lambda$.

Gallo, Grigoriadis and Tarjan showed in [9] how to find all the breakpoints and the corresponding minimum cuts in the same complexity as that required to solve a single minimum $s,t$-cut problem with the push-relabel algorithm of [11]. The pseudoflow algorithm for maximum flow and minimum cut (see Hochbaum [14]) also finds the parametric breakpoints in the complexity of a single minimum $s,t$-cut. Once all the breakpoints are generated, we search for the largest value of $\lambda$ among the breakpoints so that the optimal value of $\lambda$-NC is negative, or equivalently, the minimum $s,t$-cut value that is strictly less than $\lambda W'$. To summarize, let $T(n,m)$ be the running time required to solve the minimum cut problem on a graph with $n$ nodes and $m$ arcs. In the graph $G'_{st}$ the number of nodes is $n' = n + m$ where $m$ is the number of adjacencies or edges in the image graph. The number of arcs in $G'_{st}$, $m' = |A'|$, is $O(m)$. For general graph this running time is $O(m^2 \log m)$ with either the pseudoflow algorithm or the push-relabel algorithm. The degree of each node is constant for imaging applications so for that context $m' = O(n)$ and $n'$ is $O(n)$ and the running time is $O(n^2 \log n)$.

**Theorem 5.3** The normalized cut problem is solvable in the running time of a minimum $s,t$-cut problem, $T(n',m')$.

**Remark:** It may be desirable to solve ($\lambda$-NC) without specifying a source and a sink. The problem is then to partition the graph $G'_{st}$ to two nonempty components so that the cut separating them is minimum. This problem is the directed minimum 2-cut problem. It was shown by Hao and Orlin [13], that the directed minimum 2-cut problem is solved in the same complexity as a single minimum $s,t$ cut problem, with the push-relabel algorithm. (This was shown to hold also for the pseudoflow algorithm.) In order to solve the normalized cut problem, the algorithm produces a sequence of nested solutions for all possible values of the parameter $\lambda$. Each such solution represents a different weighting of the cut objective versus the similarity objective. As the value of the $\lambda$ grows the similarity objective is more prominent and the optimal solution $S_\lambda$ expands. Although the normalized cut ratio problem’s optimal solution comprises of a single connected component (see part II), the sequence of optimal solutions to the range of parameter values is not necessarily
formed of a single connected component. Such solutions could be more meaningful in medical images for instance, where lesions are the features sought, but they often appear as disjoint components in the image.

6 A sketch of the technique for Ratio Regions

The weighted ratio regions problem, once linearized, is an instance of the $s$-excess problem in [14].

As before we formulate the problem first. Let

$$x_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \in \bar{S} \end{cases}$$

Let $z_{ij} = 1$ if exactly one of $i$ or $j$ is in $S$.

$$z_{ij} = \begin{cases} 1 & \text{if } i \in S, j \in \bar{S}, \text{ or } i \in \bar{S}, j \in S \\ 0 & \text{if } i, j \in S \text{ or } i, j \in \bar{S} \end{cases}$$

Let the similarity weight on each edge be $w_{ij}$ and the weight of node (pixel) $j$ be $v_j$. With these parameters the ratio regions problem formulation is,

$$(RR) \quad \text{min} \quad \sum_{ij} w_{ij} z_{ij} \sum_{j} v_j x_j$$
subject to
$$x_i - x_j \leq z_{ij} \text{ for all } [i, j] \in E$$
$$x_j - x_i \leq z_{ji} \text{ for all } [i, j] \in E$$
$$x_j \text{ binary } j \in V$$
$$z_{ij} \text{ binary } j \in V.$$

The corresponding $\lambda$-question is,

$$(\lambda\text{-RR}) \quad \text{min} \quad \sum_{[i,j] \in E} w_{ij} z_{ij} - \sum_{j \in V} \lambda v_j x_j$$
subject to
$$x_i - x_j \leq z_{ij} \text{ for all } [i, j] \in E$$
$$x_j - x_i \leq z_{ji} \text{ for all } [i, j] \in E$$
$$x_j \text{ binary } j \in V$$
$$z_{ij} \text{ binary } j \in V.$$

The graph constructed for that problem is of the same size as the original graph $G$. Each node representing a variable $x_j$ has an arc going to sink node with capacity $\lambda v_j$. One variable node, $x_s$ is selected arbitrarily as corresponding to a source “seed”.

The graph $G'_{st}$ has $O(n)$ nodes and $O(m)$ arcs. The algorithm solving the problem is then a simple parametric cut algorithm in that graph, with run time $T(n, m)$. Furthermore, the parametric $s, t$-cut algorithm delivers the sequence of optimal nested solutions for all values of $\lambda$, as well as the optimal solution to the ratio problem, in run time $T(n, m)$.  

12
7 Brief remarks about the other problems

The problem normalized cut’ is: \( \min_{S \subseteq V} \frac{C(S, \bar{S})}{C(S, V)} \). This problem is in fact identical to the normalized cut problem. To see that notice that \( C(S, V) = C(S, S) + C(S, \bar{S}) \). Substituting this we get:

\[
\frac{C(S, \bar{S})}{C(S, V)} = \frac{C(S, \bar{S})}{C(S, S) + C(S, \bar{S})} = \frac{1}{1 + \frac{C(S, S)}{C(S, \bar{S})}}.
\]

This quantity is minimized when \( \frac{C(S, S)}{C(S, \bar{S})} \) is maximized. Thus the same algorithm applies.

Concerning the maximum density problem, it is presented as a minimum \( s, t \)-cut problem on an unbalanced bipartite graph with nodes representing the edges of the graph \( G \) on one side of the bipartition and nodes representing the nodes of \( G \) on the other. (Details are available in [17].) That bipartite graph has \( m + n \) nodes, and \( m' = O(m) \) arcs. The complexity of a single minimum \( s, t \)-cut in such graph is therefore \( O(m^2 \log m) \). This however can be improved.

The number of iterations required by the push-relabel algorithm or the pseudoflow algorithm is bounded by a function of the length of the longest residual path in the graph – \( O(m'n') \) – where \( m' \) is the number of arcs in the bipartite graph and \( n' \) is the maximum residual path length. In the \( \lambda \)-network constructed for the \( \lambda \)-question, this length \( n' \) is at most \( 2n + 2 \) as each path alternates between the two sets in the partition.

This fact is used by Ahuja, Orlin, Stein and Tarjan, [4], who devised improved push-relabel algorithms for unbalanced bipartite graphs. Among those, the most efficient for parametric minimum cut is an adaptation of the parametric push-relabel algorithm of Gallo, Grigoriadis and Tarjan with run time \( O(m'n' \log \frac{n'^2}{m} + 2)) \). This run time translates to \( O(mn \log \frac{n^2}{m} + 2) \) for the parametric problem solving the minimum density problem on a general graph, improving on the \( O(m^2 \log m) \) complexity for a direct application of the parametric cut algorithm.
8 Experimental example

The normalized cut procedure was implemented using the pseudoflow algorithm, [14], which solves the minimum $s,t$-cut problem (and the maximum flow problem.) The pseudoflow algorithm is fast in theory and in practice, [4]. The code and its parametric version are available for download at [http://riot.ieor.berkeley.edu/riot/Applications/Pseudoflow/](http://riot.ieor.berkeley.edu/riot/Applications/Pseudoflow/).

The algorithm described here was applied to a synthetic image provided in Figure 4. The optimal solution differentiated precisely the feature from the background.

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