Retardation Effect and Dark State in a Waveguide QED Setup With Rectangle Cross Section

Yang Xue and Zhihai Wang*

This paper investigates the dynamics of two two-level atoms, which simultaneously couple to a quasi-1D waveguide with rectangular cross section. The waveguide modes serve as environment, which induces the interaction and collective dissipation between the two distant atoms. When both of the two atoms are located in the middle of the waveguide, a retardation effect is observed, which can be broken by moving one of the atoms away from the center of the waveguide. To preserve the complete dissipation of the system via dark state mechanism, a scheme where the connection of the atoms is perpendicular to the axis of the waveguide is proposed.

1. Introduction

The waveguide quantum electrodynamics (QED), which studies the light–matter interaction in confined structure, has attracted a lot of attention due to its interesting theoretical and experimental applications.\[1,2\] In the waveguide QED community, how to realize the mutual control between the photon and (artificial) atom is a central task in constructing the quantum network. On one hand, the propagation of the flying photon can be controlled by the frequency of the atom, which is widely used to realize coherent quantum device such as photon transistor\[3–5\] and router.\[6–13\] On the other hand, the waveguide can serve as a data bus, which induces the interaction between different atoms\[14–22\] and is utilized to realize remote quantum entanglement.

Due to the possible slow velocity of light in the waveguide, the time needed for the photon propagating from one atom to the other can be comparable to the lifetime of the atom. Therefore, the retardation effect, which will induce some non-Markovian dynamics, is becoming a hot topic recently. Such retardation effect will occur in multiple atoms system\[23–27\] or only one atom in front of a mirror,\[28–37\] and even a giant atom system which interacts with the waveguide via more than one connecting point.\[38–40\]

In these setups, the two small atoms or one giant atom can serve as atomic mirror, to trap the photon inside the middle regime.\[39,41\] As a result, the atomic population usually exhibits an oscillation behavior beyond the Markovian process.

In most of the previous studies about the retardation effect in the waveguide system, the waveguide is usually theoretically considered as 1D. Therefore, the atom is resonantly coupled with only one flying photon mode in the waveguide. However, in the realistic physical system, the waveguide can never be 1D. Therefore, it is productive to investigate the effect in the waveguide with finite cross section.

To tackle this issue, we here discuss the dynamics of two two-level atoms which couple to a waveguide with rectangular cross section.\[42–44\] The finite cross section area of the waveguide generates two effects. On the one hand, it supports more than one TM modes, which supply multiple emission channels for the atom. On the other hand, whether the atom is centered or off-centered in the waveguide will lead to dramatically different dynamical behaviors. For example, as both of the atoms are centered in the waveguide, the dynamical behavior is similar to that in the 1D waveguide, and we observe the non-Markovian retardation effect. Meanwhile, we recover the Markovian process by deviating one of the atom to make it off-centered. We also find a dark state when the connection of the two atoms is properly perpendicular to the axis of the waveguide, in which both of the atoms will retain some excitation even after the evolution time tends to be infinity.

The rest of the paper is organized as follows. In Section 2, we illustrate our model and give the general amplitudes equations. In Section 3, we discuss the non-Markovian dynamics when the two atoms are both centered in the waveguide. In Section 4, we consider the situation that one of the atom is off-centered. In Section 5, we reveal a dark state mechanism when the connection of the atoms is perpendicular to the axis of the waveguide. In Section 7, we arrive at the conclusion.

2. Model and Amplitude Equations

As schematically shown in Figure 1a–d, we consider a system composed of two two-level atoms, which couples to a common waveguide with \(a \times b\) rectangle cross section and being infinite in \(z\) direction. The two atoms are located at \(\vec{r}_1 = (x_1, y_1, z_1)\) and \(\vec{r}_2 = (x_2, y_2, z_2)\), respectively. The Hamiltonian of the whole system including the atoms and the waveguide is written as
the atoms. The energy spectrum of the waveguide between the two atoms is the three TM modes are shown in Figure 1e. For simplicity, we use a needed to be considered. The dispersion relation for the lowest TE modes in the waveguide, that is, only the TM modes are along the z direction. In this paper, we consider that the dipole moment of the atoms are along the z direction, therefore, they are decoupled with the TE modes in the waveguide, that is, only the TM modes are needed to be considered. The dispersion relation for the lowest three TM modes are shown in Figure 1e. For simplicity, we use a single notation j to denote the TM modes. We denote j = 1 for m = 1, n = 1, j = 2 for m = 2, n = 1 and j = 3 for m = 3, n = 1. Then, the atom-waveguide coupling strength and the dispersion relation of the waveguide are

\begin{align}
\gamma_j &= \Omega_j \mu_1 \sin(\frac{\pi m}{a}) \sin(\frac{\pi n}{b}) \sqrt{\frac{\hbar^2}{2\epsilon_0}} \\
\omega_j &= \sqrt{\Omega_j^2 + c^2 k^2}
\end{align}

respectively. Here, \(\Omega_{\text{max}} = c \sqrt{(m \pi / a)^2 + (n \pi / b)^2}\) is the cutoff frequency for a traveling wave for the TM_{mn} mode. \(A = ab\) is the area of the rectangular cross section and \(\mu_1 = \mu_2 = \mu\) is the magnitude of the transition dipole moment of the atom, which is assumed to be real. \(c\) is the light velocity and \(\epsilon_0\) is the permittivity of the vacuum.

Since the number of the quanta is conserved in our system, the wave function in the single excitation subspace can be assumed as:

\[
|\psi(t)\rangle = e^{-i\omega_0 t} \left[ B_1(t) |\sigma_1^+\rangle |G, 0\rangle + B_2(t) |\sigma_2^+\rangle |G, 0\rangle \right] + \sum_j \int_{-\infty}^{\infty} dk e^{-i\omega_j t} B_{jk}(t) a_j^\dagger |G, 0\rangle
\]

where \(|G, 0\rangle\) represents the state that both of the atoms are in their ground states while the waveguide is in its vacuum state. \(B_1(t)\) and \(B_2(t)\) represent the excitation amplitudes for the first and second atom while \(B_{jk}\) is that for the kth mode in the waveguide for TM_{mn}. Based on the Schrödinger equation, these amplitudes satisfy

\[
\dot{B}_1(t) = -\sum_j \int_{-\infty}^{\infty} dk \frac{\gamma_j B_j(t)}{\omega_j} e^{-i\omega_0 t} e^{-i\omega_j t} \sqrt{\omega_j}
\]

\[
\dot{B}_2(t) = -\sum_j \int_{-\infty}^{\infty} dk \frac{\gamma_j B_j(t)}{\omega_j} e^{-i\omega_0 t} e^{-i\omega_j t} \sqrt{\omega_j}
\]

\[
\dot{B}_{jk}(t) = \frac{(B_1(t) g_{jk} + B_2(t) g_{jk} e^{ikz_2}) e^{ikz_1} e^{i(\omega_0 - \omega_j) t}}{\sqrt{\omega_j}}
\]

where \(z_0 = z_2 - z_1\). In the initial vacuum waveguide condition \(B_{jk}(0) = 0\), the excited amplitudes of the waveguide can be obtained formally as

\[
B_{jk}(t) = \int_0^t d\tau \frac{e^{ik\tau_2}}{\sqrt{\omega_{jk}}} [g_{jk} B_1(t) + g_{jk} B_2(t) e^{ik\tau_2}] e^{i(\omega_0 - \omega_j) \tau}
\]

It is well known that the electromagnetic modes whose frequencies are nearly resonant with the atoms make significant contributions to the atomic dissipation. Therefore, we will perform the Weisskopf–Wigner approximation by replacing \(\omega_j\) by \(\omega_\mu\). Substituting \(B_{jk}(t)\) into Equations (6) and (7), the retardation differential equations for the atomic amplitudes are obtained as

\[
(\partial_t + \sum_j \frac{\gamma_j^2 \pi}{\omega_j v_j}) B_1(t) = -\sum_j \frac{\gamma_j g_{jk}^2}{\omega_j v_j} B_2(t - \frac{\tau_2}{v_j}) e^{ik\tau_1} \Theta(t - \frac{\tau_1}{v_j})
\]

\[
(\partial_t + \sum_j \frac{\gamma_j^2 \pi}{\omega_j v_j}) B_2(t) = -\sum_j \frac{\gamma_j g_{jk}^2}{\omega_j v_j} B_1(t - \frac{\tau_2}{v_j}) e^{ik\tau_1} \Theta(t - \frac{\tau_1}{v_j})
\]
In the above equations, \( k_a = \sqrt{\omega_a^2 - \Omega_j^2} / c \) is the wave vector of the waveguide mode which is resonant with the atoms and is not close to the cutoff frequencies.\(^{[46]} \) \( d = |z_0| \) is the distance between the two atoms in the \( z \) direction. The corresponding group velocity \( v_j \) is

\[
v_j = \left. \frac{d\omega_j}{dk} \right|_{k = k_a} = \frac{c}{\omega_a} \sqrt{\omega_a^2 - \Omega_j^2} \tag{12}\]

We emphasize that Heaviside unit step function \( \Theta(x) \), which is defined as \( \Theta(x) = 1 \) for \( x > 0 \) and \( \Theta(x) = 0 \) for \( x \leq 0 \), represents the non-Markovian retardation effects, since \( d/v_j \) corresponds to the time needed for the photon propagating for the location of one atom to the other. The same retardation equation is also used to describe the dynamics of the giant atom system\(^{[39]} \) or one atom to the other. The same retardation equation is also used to describe the dynamics of the giant atom system\(^{[39]} \) or the time needed for the photon travelling for the location of one atom to the other. The same retardation equation is also used to describe the dynamics of the giant atom system\(^{[39]} \) or small atom when one end of the waveguide is blocked off by a mirror.\(^{[45]} \) In what follows, we set \( a = (\Omega_1 + \Omega_2) / 2 \), which is demonstrated in Figure 1e, so that the atoms are large detuned from TM\(_{31}\) mode, but is resonant with TM\(_{11}\) and TM\(_{21}\) with certain wave vector.

## 3. Two Atoms Centered in the Waveguide

As shown in Figure 1a, we now consider that the two atoms are both located in the middle of the waveguide, that is, \( \tau_1 = (a/2, b/2, z_1) \) and \( \tau_2 = (a/2, b/2, z_2) \). A direct observation shows that \( g_{ij} = 0 \) for \( i = 1, 2 \). Therefore, both of the atoms are only coupled to the TM\(_{11}\) mode and the coupling strengths are obtained as

\[
g_{11} = g_{12} = \frac{\Omega_1 \mu}{\sqrt{\lambda \pi \epsilon_0}} \tag{13}\]

As a result, the amplitude equation in Equations (10) and (11) becomes

\[
(\partial_t + \gamma_{11})B_1(t) = -\gamma_{11} B_2(t - \tau_1) e^{i\omega_{12}t} \Theta(t - \tau_1) \tag{14a}\]

\[
(\partial_t + \gamma_{11})B_2(t) = -\gamma_{11} B_1(t - \tau_1) e^{i\omega_{12}t} \Theta(t - \tau_1) \tag{14b}\]

where \( \gamma_{11} = g_{11}^2 \pi / v_1 \) is the effective decay rate of the atoms (equal for each atom), \( g_{11} = g_{12} = \sqrt{\omega_{12}} \) is the renormalized coupling strength under the Weisskopf–Wigner approximation.\(^{[46]} \) \( \tau_1 = d / v_1 \) is the delay time for the photon travelling from the location of one atom to the other in the waveguide with group velocity \( v_1 \).

In the viewpoint of the quantum open system, the electromagnetic field in the waveguide serves as an environment, which induces the dissipation and indirect interaction between the atoms. Under the Born–Markovian approximation, the dynamics of the two atoms are governed by the master equation (ME)\(^{[47]} \)

\[
\dot{\rho} = -i[H_1, \rho] + \sum_{i,j=1}^2 \gamma_{ij} \left[ 2(\sigma_j^+ \rho \sigma_i^+ - \sigma_i^+ \sigma_j^- \rho - \rho \sigma_i^+ \sigma_j^-) \right] \tag{15}\]

The dynamics of the system, which is characterized by the atomic population \( P_i = \langle \sigma_i^+ \sigma_i^- \rangle = |B_i|^2 \) for \( i = 1, 2 \) is shown in Figure 2 for different atomic distance. Here, the system is
initially prepared in the product state \(|\psi(0)\rangle = \sigma_1^+|G, 0\rangle\), in which the first atom is in the excited state, the second atom is in the ground state, while the waveguide is in the vacuum state.

In Figure 2a, we consider the situation with \(d = 12/k_{10}\), in which the ME yields a monotonous decay for \(P_1\) and an increase–decrease transition for \(P_2\). However, the results based on Equation (14) reveal the non-Markovian nature of the system which is induced by the retardation effect during the photon propagation in the waveguide. For example, at the moment \(t = \tau_1\), the emitted photon by the first atom arrives at the location of the second atom and excites it, so \(P_2\) acquires a non-zero value. Then, it also emits photon, which in turn arrives at the first atom during another time interval \(\tau_1\), and the decreasing population \(P_1\) revivals along with the reabsorption of the travelling photon. Repeating such photon emitting, propagating, and absorbing process, both of the population \(P_1\) and \(P_2\) oscillate with the period \(\tau_1\). Moreover, due to the waveguide induced dissipation, both of the two populations will finally achieve zero after a sufficient long time. The similar behavior can also be found for a larger atomic distance \(d = 24/k_{10}\) as shown in Figure 2b. Comparing with the former situation, we find that the population will undergo a longer time to stay at the zero value (see the black solid curve nearby \(t/\tau_1 = 2\) and the red dashed curve nearby \(t/\tau_1 = 3\) ) due to the longer retardation time.

4. Effects of TM\(_{21}\) Mode

Now, we consider that the first atom is located in the middle of the atom, while the second atom is off-centered as shown in Figure 1b, that is, \(\vec{r}_1 = (a/2, b/2, z_i)\) and \(\vec{r}_2 = (a/2 + \Delta x, b/2, z_j)\) \((0 < \Delta x < a/2)\). An immediate result is the change in atom-waveguide coupling strength, which yields

\[
g_{11} = \frac{\Omega \mu}{\sqrt{\alpha \epsilon_0}}, \quad g_{12} = g_{11} \cos\left(\frac{\Delta x \pi}{a}\right), \quad g_{12}' = \frac{g_{12}}{\sqrt{\alpha \epsilon_0}}
\]  

(18)

More interesting, it is non-trivial that the second atom couples to the TM\(_{31}\) mode simultaneously besides TM\(_{11}\) mode and the coupling strength reads

\[
g_{22} = \frac{\Omega \mu}{\sqrt{\alpha \epsilon_0}} \sin\left(\frac{2 \Delta x \pi}{a}\right)
\]  

(19)

As a result, the retardation differential equations for the atomic amplitudes become

\[
\begin{align*}
(\partial_t + \gamma_{11})B_1(t) &= -\gamma_{12} B_2(t - \tau_1)e^{i\theta(t - \tau_1)} \\
(\partial_t + \gamma_{22} + \gamma_{21})B_2(t) &= -\gamma_{12} B_1(t - \tau_1)e^{i\theta(t - \tau_1)}
\end{align*}
\]  

(20a)

(20b)

where

\[
\begin{align*}
\gamma_{11} &= \frac{g_{11}' \pi}{v_1}, \quad \gamma_{12} = \frac{g_{11}' g_{12}' \pi}{v_1} \\
\gamma_{22} &= \frac{g_{11}' g_{12}' \pi}{v_1}, \quad \gamma_{21} = \frac{g_{11}' \pi}{v_2}
\end{align*}
\]  

(21)

(22)

with \(g_{11'} = g_{11}/\sqrt{\alpha \epsilon_0}\). It is obvious that \(\gamma_{11}, \gamma_{12}, \gamma_{22}\) come from the coupling to the TM\(_{11}\) mode while \(\gamma_{21}\) comes from the effect of the TM\(_{21}\) mode.

To demonstrate the effect of the TM\(_{21}\) mode to the second atom, we plot the atomic populations for \(\gamma_{21} = 0\) and \(\gamma_{21} \neq 0\) based on Equation (20) in Figure 3a, with the initial state being same with that in the last section. When \(\gamma_{21}\) is considered to be zero, that is, the TM\(_{21}\) mode is neglected, both \(P_1\) and \(P_2\) will experience an oscillation, which is similar to the situation when the two atoms are both centered in the waveguide, and the difference comes from the modification of the coupling strength between the second atom and the waveguide due to its deviation from the center. However, when the effect of the TM\(_{21}\) mode is taken into consideration \((\gamma_{21} \neq 0)\), \(P_1\) experiences an exponential
Equation (23). When the TM$_{21}$ mode is considered, the second retardation differential equations, Equation (20) and ME in

\[ \dot{\rho} = -i[H, \rho] + \sum_{ij=1}^{2} \frac{\Gamma_{ij}^{+}}{2} (2\sigma_{ji}^{-} \rho \sigma_{ij}^{+} - \sigma_{ij}^{+} \rho \sigma_{ji}^{-} - \sigma_{ji}^{-} \rho \sigma_{ij}^{+} - \sigma_{ij}^{+} \rho \sigma_{ji}^{-}) + \gamma_{212}(2\sigma_{x} \rho \sigma_{x}^{\dagger} - \sigma_{x}^{\dagger} \sigma_{x} \rho - \rho \sigma_{x}^{\dagger} \sigma_{x}) \]  

Here, the last term represents the dissipation of the second atom induced by the TM$_{21}$ mode in the waveguide. The Hamiltonian $H_2$ for the interaction between the two atoms reads

\[ H_2 = \sum_{i=1}^{2} \omega_{a} \sigma_{i}^{\dagger} \sigma_{i} + \sum_{i,j=1}^{2} \frac{U_{ij}^{+}}{2} (\sigma_{i}^{\dagger} \sigma_{j}^{\dagger} + \sigma_{i} \sigma_{j}) \]  

where

\[ U_{ij}^{+} = \gamma_{ij} \sin |z_{i} - z_{j}|, \Gamma_{ij}^{+} = \frac{2\pi g_{ij}}{c} \sqrt{\omega_{a}^{2} - \Omega_{i}^{2}} \]  

In Figure 4b, we show the agreement of the results between the retardation differential equations, Equation (20) and ME in Equation (23). When the TM$_{21}$ mode in considered, the second atom immediately decays after it is excited by the photon emitted by the first atom, so that we can barely observe the oscillation. Meanwhile, the photon emitted by the first atom can propagate via TM$_{11}$ mode, but it can not reflected by the second atom due to its dissipation via TM$_{21}$ mode; therefore, the ME works well and $P_{1}$ exhibits an exponential decay.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{Dynamical evolution of the atomic populations when the connection of the two atoms are perpendicular to the axis of the waveguide.}
\end{figure}

\[ \text{where the Hamiltonian} \quad H_3 = \sum_{i=1}^{2} \omega_{a} \sigma_{i}^{\dagger} \sigma_{i} \]  

implies that the two atoms do not coherently couple to each other. However, the nonzero value of

\[ \Gamma_{ij}^{+} = \frac{2\pi g_{ij}}{c} \sqrt{\omega_{a}^{2} - \Omega_{i}^{2}} \]  

indicates that they will undergo a collective dissipation due to their couplings to the waveguide. Figure 4a shows the comparison between Equation (26) and ME for dynamical evolution of the atomic populations. The absence of the retardation yields the agreement of the two results as shown in the figure. Furthermore, $P_{1}$ experiences an exponential decay and $P_{2}$ nearly keeps zero during the time evolution; therefore, the second atom is nearly frozen in the ground state.

5. Dark State

In the above sections, we have considered the situation with $d = |z_{1} - z_{2}| \neq 0$, in which the dynamics of the system is demonstrated by the retardation differential equations. Another interesting situation is that the connection between the two atoms is perpendicular to the central axis of the waveguide. We first consider the case illustrated in Figure 1c, where both of the atoms are located in the position $z = z_{0}$ but the second atom is deviated from the first one in the $x$ direction, that is, $r_{1} = (a/2, b/2, z_{0})$ and $r_{2} = (a/2 + \Delta x, b/2, z_{0})$ ($0 < \Delta x < a/2$). As a result, there is no retardation effect and the amplitudes satisfy the differential equations

\[ (\partial_{t} + y_{12}) B_{1}(t) = -\gamma_{12} B_{2}(t) \]  

\[ (\partial_{t} + y_{22} + y_{212}) B_{2}(t) = -\gamma_{12} B_{1}(t) \]  

where the parameters $y_{11}, y_{22}, y_{212}$, and $y_{12}$ are same with those given in Equations (21) and (22). Correspondingly, the Markovian master equation becomes

\[ \dot{\rho} = -i[H, \rho] + \sum_{ij=1}^{2} \frac{\Gamma_{ij}^{+}}{2} (2\sigma_{ji}^{-} \rho \sigma_{ij}^{+} - \sigma_{ij}^{+} \rho \sigma_{ji}^{-} - \sigma_{ji}^{-} \rho \sigma_{ij}^{+} - \sigma_{ij}^{+} \rho \sigma_{ji}^{-}) + \gamma_{212}(2\sigma_{x} \rho \sigma_{x}^{\dagger} - \sigma_{x}^{\dagger} \sigma_{x} \rho - \rho \sigma_{x}^{\dagger} \sigma_{x}) \]  

and

\[ \text{ann. physik} \quad \text{www.ann-phys.org} \]  

\[ \text{www.advancedsciencenews.com} \]  

\[ \text{© 2023 Wiley-VCH GmbH} \]
The above dynamical process can be broken if the second atom is deviated from the first one in the y direction as shown in Figure 1d, instead of the x direction, that is, \( \vec{r}_2 = (a/2, b/2, z_0) \) and \( \vec{r}_2 = (a/2, b/2 + \Delta y, z_0) \) \((0 < \Delta y < b/2)\). In this case, the amplitude equations and the ME are same as Equations (26) and (27), respectively; the only difference is \( \gamma_{212} = 0 \) since the second atom is decoupled from the TM\(_{21}\) mode. As shown in Figure 4b, the ME describes the dynamics of the system perfectly and the atomic populations will achieve nonzero fixed values as the evolution time \( t \) tends to be infinite. It means that the system finally reaches a dark state which protects the atoms from decaying to the ground state. The underlying physics can be extracted from the effective interaction Hamiltonian, which is simplified as

\[
H_i = i \int_{-\infty}^{\infty} dk \frac{1}{\sqrt{\omega_{1k}}} [g_{11} \sigma_1^+ + g_{12} \sigma_2^+] a_{1k}^e^{\pm ikz} - \text{H.c.} \tag{30}
\]

It means that the two atoms simultaneously couple to the same mode in the waveguide, and the phase difference between the coupling strengths is wave vector independence. Therefore, the two dissipation channels for each atom interfere with each other destructively, leading to a dark state, which is immune to the environment provided by the waveguide.

We denote the wave function of the dark state as \( |D\rangle \); it satisfies \( H_i |D\rangle = 0 \) and can be expressed as

\[
|D\rangle = \frac{1}{\sqrt{g_{12}^2 + g_{11}^2}} [g_{12} \sigma_1^+ - g_{11} \sigma_2^+] |G, 0\rangle \tag{31}
\]

Due to the effective coupling induced by the waveguide, there will exist mode hybridization. Actually, the dark state is a hybrid mode in the single excitation subspace. The other hybrid state is orthogonal to the dark state, and will dissipate off due to the coupling between the atoms and the waveguide environment.

Therefore, when the time tends to be infinity, we will have

\[
P_1(t = \infty) = \frac{g_{12}^2}{g_{11}^2} = \cos^2 \left( \frac{\Delta y \pi}{b} \right) \tag{32}
\]

which coincides with the results given in Figure 4b.

### 6. Discussion about the Dark State

In the previous sections, we have discussed the dynamical evolution of the atomic populations in a two two-level atom system which couples to a waveguide with rectangle cross section. For the four configurations as shown in Figure 1a–d, we list the main results in Table 1, and we only find a dark state in the scenario with \( \Delta x = \Delta z = 0, \Delta y = b/4 \), which is given in Figure 1d. Now, we compare four cases in detail to discuss the conditions for the appearance of the dark state in view of the interaction Hamiltonian \( H_i \).

In the configuration illustrated by Figure 1a with \( \vec{r}_1 = (a/2, b/2, z_0) \) and \( \vec{r}_2 = (a/2, b/2, z_0) \), the interaction Hamiltonian is written as

\[
H_i = i \int_{-\infty}^{\infty} dk \frac{1}{\sqrt{\omega_{1k}}} [(\sigma_1^+ e^{ikz_1} + \sigma_2^+ e^{ikz_2}) a_{1k}^e - \text{H.c.}] \tag{33}
\]

Table 1. Comparison of four atomic configuration.

| Atomic configuration | Retardation effect | Dark state |
|----------------------|-------------------|------------|
| \( \vec{r}_1 = (a/2, b/2, z_1), \vec{r}_2 = (a/2, b/2, z_2) \) | Yes | No |
| \( \vec{r}_1 = (a/2, b/2, z_1), \vec{r}_2 = (a/2, b/2 + \Delta x, b/2, z_2) \) | No | No |
| \( \vec{r}_1 = (a/2, b/2, z_1), \vec{r}_2 = (a/2 + \Delta x, b/2, z_2) \) | No | No |
| \( \vec{r}_1 = (a/2, b/2, z_1), \vec{r}_2 = (a/2 + \Delta x, b/2, z_0) \) | No | Yes |

It shows that the phase difference in the atom–waveguide coupling of the two atoms \( e^{\pm ikz_1} \) is wave vector \( k \) dependent and the dark state is absent. In the configuration illustrated by Figure 1b with \( \vec{r}_1 = (a/2, b/2, z_1) \) and \( \vec{r}_2 = (a/2 + \Delta x, b/2, z_2) \), the interaction Hamiltonian is written as

\[
H_i = i \int_{-\infty}^{\infty} dk \frac{1}{\sqrt{\omega_{1k}}} [(g_{11} \sigma_1^+ e^{ikz_1} + g_{12} \sigma_2^+ e^{ikz_2}) a_{1k}^e - \text{H.c.}] + i \int_{-\infty}^{\infty} dk \frac{1}{\sqrt{\omega_{2k}}} [g_{12} \sigma_2^+ e^{ikz_2} a_{2k}^e - \text{H.c.}] \tag{34}
\]

On the one hand, similar to the previous case, the phase difference in the atom-TM\(_{11}\) mode coupling of the two atoms \( e^{\pm ikz_1} \) is wave vector \( k \) dependent. On the other hand, the second atom is also coupled to the TM\(_{31}\) mode, to make the second atom decay to its ground state after a long time. This two elements make the disappearance of the dark state.

In the configuration illustrated by Figure 1c with \( \vec{r}_1 = (a/2, b/2, z_1) \) and \( \vec{r}_2 = (a/2 + \Delta x, b/2, z_0) \), the interaction Hamiltonian is written as

\[
H_i = i \int_{-\infty}^{\infty} dk \frac{1}{\sqrt{\omega_{1k}}} [(g_{11} \sigma_1^+ + g_{12} \sigma_2^+) e^{ikz_1} a_{1k}^e - \text{H.c.}] + i \int_{-\infty}^{\infty} dk \frac{1}{\sqrt{\omega_{2k}}} [g_{12} \sigma_2^+ e^{ikz_2} a_{2k}^e - \text{H.c.}] \tag{35}
\]

Although the phase difference of the coupling to the TM\(_{11}\) mode is \( k \) independent, the coupling of the second atom to the TM\(_{21}\) mode makes the dark state disappear. In the configuration illustrated by Figure 1d with \( \vec{r}_1 = (a/2, b/2, z_1) \) and \( \vec{r}_2 = (a/2, b/2 + \Delta y, z_0) \), the interaction Hamiltonian is written in Equation (30) and the two atoms couple to the same mode in the waveguide with no coupling phase difference. As a result, we will obtain the final steady state as given by Equation (32) in Section 5.

The above dark state can also be achieved for the case in which there exists a \( k \) independent atom-waveguide coupling phase difference \( \phi \). In this case, the atom-waveguide interaction Hamiltonian can be obtained as

\[
H_i = i \int_{-\infty}^{\infty} dk \frac{1}{\sqrt{\omega_{1k}}} [(g_{11} \sigma_1^+ + g_{12} e^{i\phi} \sigma_2^+) a_{1k}^e e^{ikz_1} - \text{H.c.}] \tag{36}
\]
and the corresponding dark state is obtained as

\[
|D\rangle = \frac{1}{\sqrt{\delta_{11}^2 + \delta_{12}^2}} (\delta_{11}^2 \sigma_1^+ - \delta_{12}^2 \sigma_2^+) |G, 0\rangle
\]  

Further, this dark state is also robust to certain type of external environment. For example, if the two qubits yield a common decoherence environment except for the waveguide and the phase difference between the atom-environment is independent of the wave vector \( k \), there will still exist the ead state.

7. Conclusion

In this paper, we investigate the time evolution of two two-level atom system, which couples to a quasi 1D waveguide with rectangular cross section. We find that the dynamics of the system can be controlled by adjusting the relative location of the two atoms in an on-demand manner, and the main results are listed in Table 1. Similar to the modeled waveguide system, where the effect of cross section is neglected, the dynamics exhibits an obvious non-Markovian retardation character when the atoms are both located on the central axis of the waveguide. This mode not only induces the dissipation but also plays as a data bus to indirectly couple the two atoms. As one of the atom is off-centered, an additional mode in the waveguide acts as pure dissipation environment, which erodes the retardation effect and, therefore, the Markovian master equation will capture the main physics in an analytical way. More interestingly, when the connection of the atoms are perpendicular to the axis of the waveguide, we find a dark state mechanism which prevents the complete decay of the system.

Our proposal can be experimentally in Rydberg atom system. Taking the \(^{85}\)Rb atom as an example, the transition frequency between the Rydberg states \(|\nu S_{1/2}, m = 1/2\rangle \) and \(|\nu P_{1/2}, m = 1/2\rangle \) with the principle quantum number \( \nu \approx 80 \) is in the microwave regime of \( \omega_0 \approx 2\pi \times 7 \) GHz,\(^{[48]} \) and the atom-waveguide coupling strength can reach \( g_{ij} \approx 2\pi \times 500 \) MHz.\(^{[49]} \) With the recent technology, the cross rectangle section of the waveguide is achieved as by \( a = 2b = 2 \) cm, and the cut-off frequency is obtained as \( \Omega_{11} \approx 2\pi \times 6 \) GHz and the wave vector \( k_{10} \approx 80 \) m\(^{-1} \). The direct atom–atom coupling strength obeys the \( d^{-6} \) in \(^{85}\)Rb atom, \( U = -C_6/d^6 \) where \( C_6/2\pi = -4210 \) GHz \( \mu \)m\(^6 \) and \( d \) is atomic distance.\(^{[50]} \) In our consideration, the atomic distance is in the order of mm, and the atom–atom coupling strength is in the order of \( 10^{-6} \) Hz, which is much smaller than the atom–waveguide coupling and is neglected in our treatment in this paper. In the setup as shown in Figure 1a, the retarded effect is obvious when \( d = 12/k_{10} = 15 \) mm, and the retardation time is \( \tau_{r} \approx 0.5 \mu s \), it is much shorter than the lifetime of the Rydberg state which is in the order of 100 \( \mu s \).\(^{[51]} \) Therefore, the retardation effect can be observed successfully.

At last, we remark that in the presence of an external driving or chiral waveguide,\(^{[52,53]} \) we can still explore how to realize quantum information processing with the assistance of dark state.

Acknowledgements

The authors thank W. Mu and L. Du for warm help. This work was supported by the National Key R&D Program of China (No. 2021YFE0193500), and the National Natural Science Foundation of China (No. 11875011).

Conflict of Interest

The authors declare no conflict of interest.

Data Availability Statement

Research data are not shared.

Keywords

dark states, retardation effects, waveguide QED

Received: September 23, 2022
Revised: January 15, 2023
Published online: March 10, 2023

[1] D. Roy, C. M. Wilson, O. Firstenberg, Rev. Mod. Phys. 2017, 89, 021001.
[2] X. Gu, A. F. Kockum, A. Miranowicz, Y.-X. Liu, F. Nori, Phys. Rev. 2017, 718, 1.
[3] J. T. Shen, S. Fan, Phys. Rev. Lett. 2005, 95, 213001.
[4] D. E. Chang, A. S. Sørensen, E. A. Demler, M. D. Lukin, Nat. Phys. 2007, 3, 807.
[5] L. Zhou, Z. R. Gong, Y. X. Liu, C. P. Sun, F. Nori, Phys. Rev. Lett. 2008, 101, 100501.
[6] A. A. Abdumalikov Jr, O. Astafiev, A. M. Zagoskin, Y. A. Pashkin, Y. Nakamura, J. S. Tsai, Phys. Rev. Lett. 2010, 104, 193601.
[7] I.-C. Hoi, C. M. Wilson, G. Johansson, T. Palomaki, B. Peropadre, P. Delsing, Phys. Rev. Lett. 2011, 107, 073601.
[8] L. Zhou, L. P. Yang, Y. Li, C. P. Sun, Phys. Rev. Lett. 2013, 111, 103604.
[9] Z. H. Wang, L. Zhou, Y. Li, C. P. Sun, Phys. Rev. A 2014, 89, 053813.
[10] J. Lu, L. Zhou, L.-M. Kuang, F. Nori, Phys. Rev. A 2014, 89, 013805.
[11] I. Shomroni, S. Rosenblum, Y. Losovsky, O. Behcher, G. Guendelmann, B. Dayan, Science 2014, 345, 903.
[12] C.-H. Yan, Y. Li, H. Yuan, L. F. Wei, Phys. Rev. A 2018, 97, 023821.
[13] Y.-L. Ren, S.-L. Ma, J.-K. Xie, X.-K. Li, M.-T. Cao, F.-L. Li, Phys. Rev. A 2022, 105, 013711.
[14] K. Stannigel, P. Rabl, P. Zoller, New J. Phys. 2012, 14, 063014.
[15] A. González-Tudela, D. Porras, Phys. Rev. Lett. 2013, 110, 080502.
[16] C. Gonzales-Ballestero, F. J. Garcia-Vidal, E. Moreno, New J. Phys. 2013, 15, 073015.
[17] H. Pichler, T. Ramos, A. J. Daley, P. Zoller, Phys. Rev. A 2015, 91, 042116.
[18] G. Calajó, F. Ciccarello, D. Chang, P. Rabl, Phys. Rev. A 2016, 93, 033833.
[19] A. González-Tudela, J. I. Cirac, Phys. Rev. A 2017, 96, 043811.
[20] F. Galve, R. Zambrini, Phys. Rev. A 2018, 97, 033846.
[21] H. Z. Shen, S. Xu, H. T. Cui, X. X. Yi, Phys. Rev. A 2019, 99, 032101.
[22] E. Kim, X. Zhang, V. S. Ferreira, J. Banker, J. K. Iverson, A. Sipahigil, M. Bello, A. G.-Tudela, M. Mirhosseini, O. Painter, Phys. Rev. X 2021, 11, 011055.
[23] F. T. Arecchi, E. Courtens, Phys. Rev. A 1970, 2, 1730.
[24] P. W. Milonni, P. L. Knight, Phys. Rev. A 1974, 10, 1096.
[25] Q. Gulfoam, Z. Ficek, J. Evers, Phys. Rev. A 2012, 86, 022325.
[26] H. Pichler, P. Zoller, Phys. Rev. Lett. 2016, 116, 093601.
[27] K. Sinha, P. Meystre, E. A. Goldschmidt, F. K. Fatemi, S. L. Rolston, P. Solano, Phys. Rev. Lett. 2020, 124, 043603.
[28] J. Eschner, C. Raab, F. Schmidt-Kaler, R. Blatt, Nature 2001, 413, 495.
[29] U. Dorner, P. Zoller, Phys. Rev. A 2002, 66, 023816.
[30] P. Bushev, A. Wilson, J. Eschner, C. Raab, F. Schmidt-Kaler, C. Becher, R. Blattet, Phys. Rev. Lett. 2004, 92, 223602.
[31] F. Dubin, M. Mukherjee, C. Russo, J. Eschner, R. Blatt, Phys. Rev. Lett. 2007, 98, 183003.
[32] A. Glaetzle, K. Hammerer, A. Daley, R. Blatt, P. Zoller, Optics Communications 2010, 283, 758.
[33] T. Tufarelli, F. Ciccarello, M. S. Kim, Phys. Rev. A 2013, 87, 013820.
[34] T. Tufarelli, M. S. Kim, F. Ciccarello, Phys. Rev. A 2014, 90, 0212113.
[35] I.-C. Hoi, A. F. Kockum, L. Tornberg, A. Pourkabirian, G. Johansson, P. Delsing, C. M. Wilson, Nat. Phys. 2015, 11, 1045.
[36] Y.-L. L. Fang, H. U. Baranger, Phys. Rev. A 2015, 91, 053845.
[37] P. Y. Wen, O. V. Ivakhnenko, M. A. Nakonechny, B. Suri, J.-J. Lin, W.-J. Lin, J. C. Chen, S. N. Shevchenko, F. Nori, I.-C. Hoi, Phys. Rev. B 2020, 102, 075448.
[38] L. Guo, A. L. Grimsmo, A. F. Kockum, M. Pletyukhov, G. Johansson, Phys. Rev. A 2017, 95, 053821.
[39] L. Guo, A. F. Kockum, F. Marquardt, G. Johansson, Phys. Rev. Research 2020, 2, 043014.
[40] L. Du, M.-R. Cai, J.-H. Wu, Z. Wang, Y. Li, Phys. Rev. A 2021, 103, 053701.
[41] L. Zhou, H. Dong, Y. X. Liu, C. P. Sun, F. Nori, Phys. Rev. A 2008, 78, 063827.
[42] J.-F. Huang, T. Shi, C. P. Sun, F. Nori, Phys. Rev. A 2013, 88, 013836.
[43] J. Li, L. Hu, J. Lu, L. Zhou, Chin. Phys. B 2021, 30, 090307.
[44] L. Hu, G. Lu, J. Lu, L. Zhou, Quantum Inf. Process 2020, 19, 81.
[45] H.-X. Song, X.-Q. Sun, J. Lu, L. Zhou, Commun. Theor. Phys. 2018, 69, 59.
[46] M. O. Scully, M. S. Zubairy, Quantum Optics, 1st ed., Cambridge University Press, Cambridge 1997, p. 206.
[47] H. Breuer, F. Petruccione, The Theory of Open Quantum Systems, Oxford University Press, Oxford 2002, p. 130.
[48] A. Sanz-Mora, S. Wüster, J.-M. Rost, Phys. Rev. A 2017, 96, 013855.
[49] J. D. Thompson, T. G. Tiecke, N. P. de Leon, J. Feist, A. V. Akimov, M. Gullans, A. S. Zibrov, V. Vuletić, M. D. Lukin, Science 2013, 340, 1202.
[50] N. Šišalić, J. Pritchard, C. Adams, K. Weatherill, Comput. Phys. Commun. 2017, 220, 319.
[51] D. Barredo, V. Lienhard, P. Scholl, S. de Léséleuc, T. Boulier, A. Browaeys, T. Lahaye, Phys. Rev. Lett. 2020, 124, 023201.
[52] A. Ask, G. Johansson, Phys. Rev. Lett. 2022, 128, 083603.
[53] T. Li, A. Miranowicz, X. D. Hu, K. Y. Xia, F. Nori, Phys. Rev. A 2018, 97, 062318.