On the boundedness of Hausdorff operators on real Hardy spaces $H^1$ over homogeneous spaces of groups with local doubling property

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Abstract. We give conditions for boundedness of Hausdorff operators on real Hardy spaces $H^1$ over homogeneous spaces of locally compact groups with local doubling property. The special case of the hyperbolic plane is considered.

Keywords. Hausdorff operator, locally compact group, homogeneous space, atomic Hardy space, hyperbolic plane.

MSC classes: 47G10, 43A85, 47A30, 51M10, 22E30

1 Introduction

Hausdorff operators on the finite interval were introduced by Hardy [1, Chapter XI]. Its natural $n$-dimensional generalization due to Lerner and Liflyand [2]. For more information on the history of the issue, see the survey articles [3], [4].

Below we shall denote by $\text{Aut}(G)$ the space of all topological automorphisms of a topological group $G$ endowed with its natural topology (see, e.g. [5, (26.3)]).

In [6], [7] the next definition was proposed

Definition. Let $(\Omega, \mu)$ be a measure space, $G$ a topological group, $A : \Omega \to \text{Aut}(G)$ a measurable map, and $\Phi$ a locally $\mu$-integrable function on $\Omega$. We define the Hausdorff operator with the kernel $\Phi$ over the group $G$ by the formula

$$(\mathcal{H}f)(x) = \int_{\Omega} \Phi(u)f(A(u)(x))d\mu(u).$$

By [6, Lemma 1] for a locally compact group $G$ this operator is bounded on $L^p(G)$ $(1 \leq p \leq \infty)$ provided $\Phi(u)(\text{mod}A(u))^{-1/p} \in L^1(\Omega, \mu)$ and

$$\|\mathcal{H}\|_{L^p \to L^p} \leq \int_{\Omega} |\Phi(u)|(\text{mod}A(u))^{-1/p}d\mu(u).$$

The case of a Hausdorff operator on $p$-Adic vector spaces was considered earlier in [8], the special case of a Hausdorff operator on the Heisenberg group in the sense of this definition was considered in [9], and [10].
In [7, Theorem] sufficient conditions were given for boundedness of a Hausdorff operator on atomic Hardy space $H^1(G)$.

In the case $G = \mathbb{R}$ there are many classical operators in analysis which are special cases of the Hausdorff operator in the sense of previous definition for suitable measure spaces $(\Omega, \mu)$ (see, e.g., [11], [12], [13], [14] and the bibliography therein). The next example shows that the Harish-Chandra transform leads to a Hausdorff operator on the group $SO(2, \mathbb{R})$.

**Example 1.** One of the form of the Harish-Chandra transform for the group $G = SL(2, \mathbb{R})$ looks as follows: $H^K f(x) = D(x)(Hf)(x)$ ($x \in SO(2, \mathbb{R})$) where $D(x) = 2i \sin \theta$ if the matrix $x$ represents the rotation of the plane by an angle $\theta$. Here $H$ stands for a Hausdorff operator

$$(Hf)(x) = \int_{A^+} \frac{\alpha(u) + \alpha(u^{-1})}{2} f(u^{-1}xu)d\mu(u),$$

where $A^+$ denotes the set of $2 \times 2$ matrices of the form $u = \text{diag}(a, a^{-1})$, $a \geq 1$, $\alpha(u) = a^2$, and $d\mu(u) = da$ (see, e.g., [15, Chapter VII, §5]).

In [6], [7] classical results on the boundedness of Hausdorff operators on the Hardy space $H^1$ over finite-dimensional real space were generalized to the case of a Hardy space over locally compact metrizable groups with the doubling property. In [16] Hausdorff operators on Lebesgue and real Hardy spaces over homogeneous spaces of locally compact groups with doubling property were considered. On the other hand, as was shown by T. Kawazoe, non compact semisimple Lie groups should not satisfy the doubling property but often enjoy the less restrictive so called local doubling property (see condition (LDP) below) [17, Lemma 2.6]. [18, Lemma 3.2]. The aim of this work, is to give conditions for boundedness of Hausdorff operators on real Hardy spaces $H^1$ over homogeneous spaces of locally compact groups with local doubling property (in particular, over Riemannian symmetric spaces $G/K$, where $G$ is a connected semisimple Lie group with finite center and $K$ its maximal compact subgroup).

## 2 The main result

We shall assume in this section that $G$ is a locally compact metrizable group with left invariant metric $\rho$ and left Haar measure $\nu$, $K$ its compact subgroup with normalized Haar measure $\beta$.

In [16] Hausdorff operators on the homogeneous space $G/K$ were introduced in the following way. Recall that the quotient space $G/K$ consists of

\[2\]The author thanks Professor R. Daher for these references.
Left cosets $\dot{x} := xK = \pi_K(x)$ $(x \in G)$ where $\pi_K : G \to G/K$ stands for a natural projection. We assume that the measure $\nu$ is normalized in such a way that generalized Weil’s formula

$$\int_G g(x)dx = \int_{G/K} \left( \int_K g(xk)d\beta(k) \right) d\lambda(\dot{x}) \quad (1)$$

holds for all $g \in L^1(G)$, where $\lambda$ denotes some left $G$-invariant measure on $G/K$ (see [19, Chapter VII, §2, no. 5, Theorem 2] and especially remark c) after this theorem). Here and below we write $dx$ instead of $d\nu(x)$ and $dk$ instead of $d\beta(k)$. We shall write also $d\dot{x}$ instead of $d\lambda(\dot{x})$.

The function $g : G \to \mathbb{C}$ is called right $K$-invariant if $g(xk) = g(x)$ for all $x \in G$, $k \in K$. For such a function we put $\dot{g}(\dot{x}) := g(x)$. This definition is correct and, since $\int_K dk = 1$, for $g \in L^1(G)$ formula (1) implies that

$$\int_G g(x)dx = \int_{G/K} \dot{g}(\dot{x})d\dot{x}. \quad (2)$$

The map $g \mapsto \dot{g}$ is a bijection between the set of all right $K$-invariant functions on $G$ (all right $K$-invariant functions from $L^p(G) = L^p(G/K, \lambda)$).

Let an automorphism $A \in \text{Aut}(G)$ maps $K$ onto itself. Since $A(\dot{x}) := A(xK) = \{A(x)A(k) : k \in K\} = A(x)K = \pi_K(A(x))$,

we get a homeomorphism $\dot{A} : G/K \to G/K$, $\dot{A}(\dot{x}) := \pi_K(A(x))$. Then for every right $K$-invariant function $g$ on $G$ we have $\dot{g}(\dot{A}(\dot{x})) = g(A(x))$.

We put

$$\text{Aut}_K(G) := \{\dot{A} : A \in \text{Aut}(G), A(K) = K\}.$$

**Definition 1.** [16] Let $(\Omega, \mu)$ be a measure space, $(\dot{A}(u))_{u \in \Omega} \subset \text{Aut}_K(G)$ a family of homeomorphisms of $G/K$, and $\Phi \in L^1_{\text{loc}}(\Omega, \mu)$. For a function $f$ on $G/K$ we define a Hausdorff operator on $G/K$ as follows

$$(\mathcal{H}_{\Phi, \dot{A}}f)(\dot{x}) := \int_{\Omega} \Phi(u)f(\dot{A}(u)(\dot{x}))d\mu(u).$$

It was shown in [16, Theorem 1] that under the conditions of Definition 1 for $p \in [1, \infty]$ the following inequality holds

$$\|\mathcal{H}_{\Phi, \dot{A}}\|_{L^p(G/K) \to L^p(G/K)} \leq \int_{\Omega} |\Phi(u)||\text{mod}A(u)|^{-1/p}d\mu(u).$$

$^3$G-left invariance of $\lambda$ means that $\lambda(xE) = \lambda(E)$ for every Borel subset $E$ of $G/K$ and for every $x \in G$. This measure is unique up to a constant multiplier.
Our goal is to give conditions for boundedness of operators $H_{\Phi,A}$ on atomic Hardy spaces $H^1$ over $G/K$.

Following [20] we shall assume that the group $G$ possesses the following properties:

(LDP) (local doubling property): for every $b \in \mathbb{R}_+$ there exists a constant $D_b$ such that for every ball $B$ in $G$ with radius $r_B < b$ the following inequality holds

$$\nu(2B) \leq D_b \nu(B);$$

where $2B$ denotes the ball in $G$ with the same center and radius $2r_B$.

(AMP) (approximate midpoint property): there exist $R_0 \in [0,1)$ and $\beta \in (1/2,1)$ such that for every pair of points $x, y \in G$ with $\rho(x,y) > R_0$ there exists a ball $B$ containing $x$ and $y$ with radius $r_B < \beta \rho(x,y)$.

By [20, Remark 2.3] the property (LDP) implies that for each $r \geq 2$ and for each $b \geq 0$ there exists a constant $C > 0$ such that

$$\nu(B') \leq C \nu(B)$$

(LD)

for each pair of balls $B$ and $B'$, with $B \subset B'$, $r_B \leq b$, and $r_{B'} \leq \tau r_B$. In the following $D_{r,b}$ denotes the smallest constant for which (LD) holds.

**Remark 1.** As was noted in [20, p. 2] important examples of metric measure spaces which are locally doubling (but not doubling) are complete Riemannian manifolds with Riemannian distance $\rho$ and Riemannian density $\nu$, and with Ricci curvature bounded from below, a class which includes all Riemannian symmetric spaces of the noncompact type and Damek–Ricci spaces. As was mentioned in the introduction, a connected non compact semisimple Lie group with finite center also possesses the property (LDP). It is well known also that every complete metric space with path metric possesses the approximate midpoint property.

Under conditions (LDP) and (AMP) we prove the theorem on boundedness of Hausdorff operators on atomic Hardy spaces $H^1$ over homogeneous spaces of locally compact groups.

First recall that a function $a$ on $G$ is an $((1,\infty)-)$atom if

(i) the support of $a$ is contained in a ball $B(x,r)$;

(ii) $\|a\|_{\infty} \leq \frac{1}{\nu(B(x,r))}$;

(iii) $\int_G a(x)d\nu(x) = 0$.

In case $\nu(G) < 1$ we shall assume $\nu(G) = 1$. Then the constant function having value 1 is also considered to be an atom.

According to [20] an $H^1_b$ atom is an atom supported in a ball of radius at most $b$. Using $H^1_b$ atoms for each $b > 0$ the spaces $H^1_b = H^1_b(G) := H^1_{b,\infty}(G)$ on the group $G$ are defined in [20] in the same manner as in the case of spaces of homogeneous type considered in [21], the only difference being that it is
required that the balls involved have at most radius \( b \). Furthermore, due to \cite{20} Proposition 4.3 for \( b > R_0 / (1 - \beta) \) we have \( H_1^b = H_1^c \) for all \( c > b \). So, we put \( H_1^b(G) := H_1^b(G) \) for such \( b \). In the following the constant \( b > R_0 / (1 - \beta) \) will be fixed.

**Definition 2.** (cf. \cite{16}). We define the Hardy space \( H_1^b(G/K) \) as a space of such functions \( f = ˙g \) on \( G/K \) that \( g \) admits an atomic decomposition of the form

\[
g = \sum_{j=1}^{\infty} \alpha_j a_j, \tag{3}
\]

where \( a_j \) are right \( K \)-invariant \( H_1^b \) atoms, and \( \sum_{j=1}^{\infty} |\alpha_j| < \infty \). In this case,

\[
\|f\|_{H_1^b(G/K)} := \inf \sum_{j=1}^{\infty} |\alpha_j|,
\]

and infimum is taken over all decompositions above of \( g \).

Thus a function \( f = ˙g \) from \( H_1^b(G/K) \) admits an atomic decomposition \( f = \sum_{j=1}^{\infty} \alpha_j a_j \) such that \( \sum_{j=1}^{\infty} |\alpha_j| < \infty \), and \( \|f\|_{H_1^b(G/K)} = \|g\|_{H_1^b(G)} \).

**Proposition 1.** The space \( H_1^b(G/K) \) is Banach. If for some \( h \in H_1^b(G) \) the inequality \( \int_K h(k)dk \neq 0 \) holds the space \( H_1^b(G/K) \) is nontrivial.

The proof of this proposition is similar to the proof of Proposition 2 in \cite{16}.

We need the following two lemmas to prove our main result.

**Lemma 1.** \cite{7} Every automorphism \( A \in \mbox{Aut}(G) \) of a locally compact metrizable group \( (G, \rho) \) is Lipschitz. Moreover, one can choose the Lipschitz constant to be

\[
L_A = \kappa_\rho \mod A,
\]

where the constant \( \kappa_\rho \) depends on the metric \( \rho \) only.

**Lemma 2.** \cite{6} Let \((X; m)\) be a measure space and \( \mathcal{F}(X) \) be some Banach space of \( m \)-measurable functions on \( X \). Assume that the convergence of a sequence strongly in \( \mathcal{F}(X) \) yields the convergence of some subsequence to the same function for \( m \)-almost all \( x \in X \). Let \((\Omega, q, \mu)\) be \( \sigma \)-compact quasi-metric space with quasi-metric \( q \) and positive Radon measure \( \mu \), and \( F(u, x) \) be a function such that \( F(u, \cdot) \in \mathcal{F}(X) \) for \( \mu \)-a.e. \( u \in \Omega \) and the map \( u \mapsto F(u, \cdot) : \Omega \to \mathcal{F}(X) \) is Bochner integrable with respect to \( \mu \). Then for \( m \)-a.e. \( x \in X \)

\[
\left( (B) \int_{\Omega} F(u, \cdot) d\mu(u) \right)(x) = \int_{\Omega} F(u, x) d\mu(u).
\]

\[4\text{Real Hardy spaces over compact connected (not necessary quasi-metric) Abelian groups were defined in } \cite{22}.\]
Now we are in position to state and prove the following.

**Theorem 1.** Let $\Omega$ be a compact quasi-metric space with positive Radon measure $\mu$. Let $G$ be a locally compact group with left Haar measure $\nu$ and $\rho$ a left invariant metric which is compatible with the topology of $G$ and conditions (LDP) and (AMP) hold. Let for some constant $C_1 > 0$ the family $(\hat{A}(u))_{u \in \Omega} \subset \text{Aut}_K(G)$ satisfies $\text{mod} A(u) \geq C_1$. Then for $\Phi \in L^1(\Omega, \mu)$ the Haussdorff operator $H_{\Phi, \hat{A}}$ is bounded on the real Hardy space $H^1(G/K)$ and for $\tau = \max(2, \kappa_\rho / C_1)$ and some constant $\gamma_{\tau, b} > 0$ (depending only on $b$ and $\tau$) the next estimate is valid

$$\|H_{\Phi, \hat{A}}\| \leq \gamma_{\tau, b} \|\Phi\|_{L^1(\Omega, \mu)}.$$  

Proof. If we set $X = G/K$ and $m = \lambda$ the pair $(X, m)$ satisfies the conditions of Lemma 1 with $H^1(G/K)$ in place of $\mathcal{F}(X)$. Indeed, let $f_n = g_n \in H^1(G/K)$, $f = g \in H^1(G/K)$, and $\|f_n - f\|_{H^1(G/K)} \to 0 (n \to \infty)$. Since

$$\|f_n - f\|_{L^1(G/K)} = \int_{G/K} |\pi_K(g_n - g)| d\lambda$$

$$= \int_G |g_n(x) - g(x)| dx \leq \|g_n - g\|_{H^1(G)} = \|f_n - f\|_{H^1(G/K)} \to 0$$

there is a subsequence of $f_n$ that converges to $f$ $\lambda$-a.e.

Then Definition 2 and lemma 1 imply for $f \in H^1(G/K)$ that

$$H_{\Phi, \hat{A}} f = \int_{\Omega} \Phi(u) f \circ \hat{A}(u) d\mu(u)$$

(the Bochner integral).

Therefore (below $f = g$, $g \in H^1(G) = H^1_b(G)$)

$$\|H_{\Phi, \hat{A}} f\|_{H^1(G/K)} \leq \int_{\Omega} |\Phi(u)| \|f \circ \hat{A}(u)\|_{H^1(G/K)} d\mu(u)$$

$$= \int_{\Omega} |\Phi(u)| \|g \circ A(u)\|_{H^1(G)} d\mu(u).$$  \hspace{1cm} (4)

If $g$ has representation (3) then

$$g \circ A(u) = \sum_{j=1}^{\infty} \alpha_j a_j \circ A(u).$$  \hspace{1cm} (5)

We claim that $a'_{j,u} := (1 / D_{\tau, b}) a_j \circ A(u)$ is an atom from $H^1_{\nu, \rho}(G)$. Indeed, if $a_j$ is supported in $B(x_j, r_j)$ ($r_j < b$) lemma 1 implies that $a_j \circ A(u)$ is supported in

$$A(u)^{-1}(B(x_j, r_j)) \subset B \left( A(u)^{-1}x_j, \frac{\kappa_\rho}{\text{mod} A(u)} r_j \right) \subset B(A(u)^{-1}x_j, \tau r_j).$$
And thus $a'_{j,u}$ is supported in $B(A(u)^{-1}x_j, \tau r_j)$ ($\tau r_j < \tau b$).

Next, note that $\|a_j\| \leq 1/\nu(B(x_j, r_j))$. Since $\nu$ and $\rho$ are left invariant, condition (LD) yields that

$$\nu(B(A(u)^{-1}x_j, \tau r_j)) = \nu(B(x_j, \tau r_j)) \leq D_{\tau,b} \nu(B(x_j, r_j)).$$

It follows that

$$\|a'_{j,u}\|_{\infty} = \frac{1}{D_{\tau,b}} \|a_j\|_{\infty} \leq \frac{1}{D_{\tau,b}(B(x_j, r_j))} \leq \frac{1}{\nu(B(A(u)^{-1}x_j, \tau r_j))}$$

and $a'_{j,u}$ satisfies (i) and (ii) from the definition of an atom. The property (iii) follows from the equality

$$\int_G a'_{j,u} d\nu = (1/D_{\tau,b}) \text{mod} A(u)^{-1} \int_G a_j d\nu.$$

We conclude that the function

$$g \circ A(u) = \sum_{j=1}^{\infty} (\alpha_j D_{\tau,b}) a'_{j,u}$$

belongs to $H^1_{\tau b}(G)$ and

$$\|g \circ A(u)\|_{H^1_{\tau b}} \leq D_{\tau,b} \sum_{j=1}^{\infty} |\alpha_j|.$$

So $\|g \circ A(u)\|_{H^1_{\tau b}} \leq D_{\tau,b}\|g\|_{H^1_b}$.

On the other hand, by [20, Proposition 4.3] for $b > R_0/(1 - \beta)$ we get $H^1_{\tau b} = H^1_b$ and for some constant $C_{\tau,b} > 0$ depending only on $b$ and $\tau$

$$\|g \circ A(u)\|_{H^1_b} \leq \|g \circ A(u)\|_{H^1_{\tau b}} \leq \|g \circ A(u)\|_{H^1_{\tau b}} \leq C_{\tau,b} \|g \circ A(u)\|_{H^1_{\tau b}}.$$

Then

$$\|g \circ A(u)\|_{H^1_b} \leq C_{\tau,b} \|g \circ A(u)\|_{H^1_{\tau b}} \leq C_{\tau,b}D_{\tau,b}\|g\|_{H^1_{\tau b}}.$$

and thus

$$\|\mathcal{H}_{\Phi,A}f\|_{H^1(G/K)} = \|\mathcal{H}_{\Phi,A}\hat{g}\|_{H^1(G/K)} \leq \int_{\Omega} |\Phi(u)||\hat{g} \circ \hat{A}(u)||_{H^1(G/K)}d\mu(u)$$

$$= \int_{\Omega} |\Phi(u)||g \circ A(u)||_{H^1(G)}d\mu(u) \leq C_{\tau,b}D_{\tau,b}\|\Phi\|_{L^1(\Omega, \mu)}\|g\|_{H^1(G)}$$

$$= C_{\tau,b}D_{\tau,b}\|\Phi\|_{L^1(\Omega, \mu)}\|f\|_{H^1(G/K)}.$$
and the proof is complete.

In [16] Cesaro operator over a homogeneous space $G/K$ was defined in the following way:

$$(C_{\hat{A},\mu} f)(\hat{x}) := \int_{\{\mod A(u) \geq 1\}} \frac{f(\hat{A}(u)(\hat{x}))}{\mod A(u)} d\mu(u).$$

**Corollary 1.** Under the conditions of theorem 1 we have that

$$\|C_{\hat{A},\mu}\|_{H^1 \rightarrow H^1} \leq \gamma_{\tau,b} \int_{\{\mod A(u) \geq 1\}} \frac{d\mu(u)}{\mod A(u)}.$$

Indeed, this follows from theorem 1, since for the Cesaro operator

$$\Phi(u) = \frac{\chi_{\{\mod A(u) \geq 1\}}(u)}{\mod A(u)}$$

($\chi_E$ denotes the indicator of the subset $E \subset \Omega$).

3 Hausdorff operators over the hyperbolic plane

In this section we give an answer to the question, posed in [16].

Let $\mathbb{H}^2$ be the open upper half plane of the complex plane with the hyperbolic metric (the Poincaré model of the Lobachevsky plane). The group $G := SL(2) = SL(2, \mathbb{R})$ acts isometrically and transitively on $\mathbb{H}^2$ by the rule

$$\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \cdot z = \frac{az + b}{cz + d}.$$ 

Since the stabilizer in $SL(2)$ of $i$ is the (maximal compact) subgroup $K := SO(2) = SO(2, \mathbb{R})$ of $SL(2)$, one can identify $\mathbb{H}^2$ with the homogeneous space $SL(2)/SO(2)$ via the map $z = x \cdot i \mapsto \hat{x} := \pi_K(x) = xSO(2)$ ($x \in SL(2)$) (this is a diffeomorphism of $\mathbb{H}^2$ onto $G/K$; see, e.g., [15] Chapter III, §1]). It is easy to verify that for $z \in \mathbb{H}^2$

$$z = x(z) \cdot i$$

where

$$x(z) = \frac{1}{\sqrt{Imz}} \begin{pmatrix} Imz & Rez \\ 0 & 1 \end{pmatrix}.$$ 

We shall identify $z = x(z) \cdot i \in \mathbb{H}^2$ with $\pi_K(x(z))$.

It is known that $x \mapsto g^{-1} x g$ with $g \in GL(2, \mathbb{R})$ is the general form of automorphisms of the group $SL(2, \mathbb{R})$. Next, since $K = SO(2)$ is the connected
component of the unit in \( O(2) \), \( K \) is a normal subgroup of \( O(2) \). In other words every automorphism \( A(u)(x) := u^{-1}xu \ (u \in O(2)) \) of \( SL(2) \) maps \( K \) onto itself. By the previous identification, \( A(u)(\hat{x}(z)) = \pi_K(A(u)(x(z))) = (u^{-1}x(z)u) \cdot i \).

Hence for our \( G, K, \) and \( \Omega = O(2) \) the Definition 1 takes the form (we put \( x = x(z) \) in this definition and identify \( \hat{x}(z) \) with \( z )

\[
\left( \mathcal{H}_\Phi f \right)(z) := \int_{O(2)} \Phi(u)f((u^{-1}x(z)u) \cdot i)d\mu(u) \tag{6}
\]

where \( \mu \) stands for a (regular Borel) measure on \( O(2) \) and \( f \) is a function on \( H_2 \).

Since \( \text{mod} A(u) = \Delta_G(u) \) \[19\] Chapter VII, §1, n. 4] and \( SL(2) \) is unimodular \[19\] Chapter VII, §3, n. 3], we have \( \text{mod} A(u) = 1 \) for all \( u \). Let \( \Phi \in L^1(O(2), \mu) \). Then theorem 1 from \[16\] yields, that the operator (6) is bounded in \( L^p(H_2) \) for \( p \in [1, \infty] \) and \( \|\mathcal{H}_\Phi\|_{L^p \rightarrow L^p} \leq \|\Phi\|_{L^1} \).

Remark 1 implies that the group \( SL(2) \) endowed with the path (Riemannian) metric satisfies the conditions (LDP) and (AMP). Since \( \text{mod} A(u) = 1 \), theorem 1 can be applied to the group \( SL(2) \) with \( C_1 = 1 \). So if \( \Phi \in L^1(O(2), \mu) \) the operator (6) is bounded on \( H^1(H^2) \) and \( \|\mathcal{H}_\Phi\|_{H^1 \rightarrow H^1} \leq \gamma \|\Phi\|_{L^1(\mu)} \) for some absolute constant \( \gamma = \gamma_{\tau,b} \) (\( \tau = \max(2, \kappa_\rho) \), \( b > 0 \) is sufficiently large).

It is well known that any matrix from \( O(2) \) looks like

\[
k(\theta) = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}, \quad \theta \in [0, 2\pi)
\]

(this matrix presents a rotation by \( \theta \) by the origin in the Euclidean plain), or like:

\[
v(\theta) = \begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix}, \quad \theta \in [0, 2\pi)
\]

(this matrix presents a reflection in the Euclidean plain across a line at an angle of \( \theta/2 \)).

Consider both of these cases.

1) If \( u = k(\theta) \) formula (6) takes the form

\[
\left( \mathcal{H}_{\Phi_1} f \right)(z) = \int_{0}^{2\pi} \Phi_1(\theta)f((k(\theta)^{-1}x(z)k(\theta)) \cdot i)d\mu_1(\theta)
\]

where \( \mu_1 \) stands for a regular Borel measure on \( [0, 2\pi) \).

But, since \( k(\theta) \cdot i = i \), we have

\[
(k(\theta)^{-1}x(z)k(\theta)) \cdot i = k(\theta)^{-1} \cdot (x(z) \cdot (k(\theta) \cdot i)) = k(-\theta) \cdot z = \frac{z \cos \theta + \sin \theta}{-z \sin \theta + \cos \theta}.
\]
Note that the Möbius transformation in the right-hand side induces a hyperbolic rotation of the half-plane $H^2$ by the angle $2\theta$ about $i$ (see, e.g., [23, Lemma 9.19]). So if $\theta$ runs over $[0, 2\pi)$, the point $\frac{z \cos \theta + \sin \theta}{-z \sin \theta + \cos \theta}$ runs twice over the hyperbolic circle centered at $i$, which passes through $z$ (this line is an Euclidean circle, too, see, e.g., [23, Corollary 5.3]).

So in this case the Hausdorff operator looks as follows:

$$
(\mathcal{H}_{\Phi_1} f)(z) = \int_{0}^{2\pi} \Phi_1(\theta) f(k(-\theta) \cdot z) d\mu_1(\theta)
$$

$$
= \int_{0}^{2\pi} \Phi_1(\theta) f\left(\frac{z \cos \theta + \sin \theta}{-z \sin \theta + \cos \theta}\right) d\mu_1(\theta). \tag{7}
$$

2) Let $u = v(\theta)$. Since $v(\theta) \cdot i = -i$ and $v(\theta)^{-1} = v(\theta)$, we have

$$(v(\theta)^{-1} x(z) v(\theta)) \cdot i = v(\theta) \cdot (x(z) \cdot (v(\theta) \cdot i)) = -v(\theta) \cdot z = \frac{z \cos \theta + \sin \theta}{-z \sin \theta + \cos \theta}$$

and we arrived at the same expression for a Hausdorff operator as in the previous case.

Thus, formula (7) gives us the general form of a Hausdorff operator on the hyperbolic plane. This operator is bounded on $L^p(H^2)$ and $H^1(H^2)$ if $\Phi_1 \in L^1([0, 2\pi), \mu_1)$ and by [16, Theorem 1] and theorem 1

$$
\|\mathcal{H}_{\Phi_1}\|_{L^p \to L^p} \leq \|\Phi_1\|_{L^1(\mu_1)}, \quad \|\mathcal{H}_{\Phi_1}\|_{H^1 \to H^1} \leq \gamma \|\Phi_1\|_{L^1(\mu_1)}.
$$

**Example 2.** Consider the Cesaro operator over the hyperbolic plane (see corollary 1). In our case we have $\text{mod} A(u) = 1$ for all $u$. So we put $\Phi_1(\theta) = 1$ in formula (7) and define the (generalized) Cesaro operator over the hyperbolic plane by the formula

$$
(C_{\mu_1} f)(z) := \int_{0}^{2\pi} f\left(\frac{z \cos \theta + \sin \theta}{-z \sin \theta + \cos \theta}\right) d\mu_1(\theta)
$$

(\text{perhaps the best choice here is } d\mu_1(\theta) = d\theta/2\pi). \text{ It follows from (8) that this operator is bounded on } L^p(H^2) \text{ and } H^1(H^2) \text{ if the measure } \mu_1 \text{ is finite and}

$$
\|C_{\mu_1}\|_{L^p \to L^p} = \|\mu_1\|, \quad \|C_{\mu_1}\|_{H^1 \to H^1} \leq \gamma \|\mu_1\|.
$$

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