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Tomaszewski’s problem on randomly signed sums, revisited

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Abstract
Let $v_1, v_2, \ldots, v_n$ be real numbers whose squares add up to 1. Consider the $2^n$ signed sums of the form $S = \sum \pm v_i$. Boppana and Holzman (2017) proved that at least $\frac{13}{32} = 0.40625$ of these sums satisfy $|S| \leq 1$. Here we improve their bound to 0.427685.

Mathematics Subject Classifications: 60E15, 60G50, 60C05, 05A20

1 Introduction

Let $v_1, v_2, \ldots, v_n$ be real numbers such that the sum of their squares is at most 1. Consider the $2^n$ signed sums of the form $S = \sum \pm v_i$. In 1986, B. Tomaszewski (see Guy [4]) asked the following question: is it always true that at least $\frac{1}{2}$ of these sums satisfy $|S| \leq 1$?

Boppana and Holzman [2] proved that at least $\frac{13}{32} = 0.40625$ of the sums satisfy $|S| \leq 1$. Actually, they proved a slightly better bound of 0.406259. See their paper for a discussion of earlier work on Tomaszewski’s problem.

In this note, we will improve the lower bound to 0.427685. We will sharpen the Boppana-Holzman argument by using a Gaussian bound due to Bentkus and Dzindza-lieta [1].
After we wrote this note, two further improvements appeared. Dvořák, van Hintum, and Tiba [3] strengthened the lower bound to 0.46. Keller and Klein [5] completely solved Tomaszewski’s problem by proving a lower bound of $\frac{4}{3}$.

We will use the language of probability. Let $\Pr[A]$ be the probability of an event $A$. A random sign is a random variable whose probability distribution is the uniform distribution on the set $\{-1, +1\}$. With this language, we can state our main result.

**Main Theorem.** Let $v_1, v_2, \ldots, v_n$ be real numbers such that $\sum_{i=1}^{n} v_i^2 \leq 1$. Let $a_1, a_2, \ldots, a_n$ be independent random signs. Let $S = \sum_{i=1}^{n} a_i v_i$. Then $\Pr[|S| \leq 1] > 0.427685$.

## 2 Proof of the improved bound

In this section, we will prove the bound of 0.427685. We will follow the approach of Boppana and Holzman [2], replacing their fourth-moment method with a Gaussian bound.

Let $Q$ be the upper tail function of the standard normal (Gaussian) distribution:

$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-t^2/2} dt.$

Note that $Q$ is a decreasing, positive function.

Bentkus and Dzindzalieta [1] proved the following Gaussian bound on randomly-signed sums. See their paper for a discussion of earlier work on such bounds.

**Theorem 1** (Bentkus and Dzindzalieta). Let $x$ be a real number. Let $v_1, v_2, \ldots, v_n$ be real numbers such that $\sum_{i=1}^{n} v_i^2 \leq 1$. Let $a_1, a_2, \ldots, a_n$ be independent random signs. Let $S = \sum_{i=1}^{n} a_i v_i$. Then

$$\Pr[S \geq x] \leq \frac{Q(x)}{4Q(\sqrt{2})}.$$

Given a positive number $c$, define $F(c)$ by

$$F(c) := \frac{1}{2} - \frac{Q(1/\sqrt{c})}{4Q(\sqrt{2})}.$$

Note that $F$ is a decreasing function bounded above by $\frac{1}{2}$. A calculation shows that $F(\frac{1}{4}) > 0.427685$.

We will need the following lemma, which quantitatively improves Lemma 3 of Boppana and Holzman [2]. Roughly speaking, this lemma is used to show that if a partial sum is a little less than 1 in absolute value, then the final sum has a decent chance of remaining less than 1 in absolute value.

**Lemma 2.** Let $c$ be a positive number. Let $x$ be a real number such that $|x| \leq 1$. Let $v_1, v_2, \ldots, v_n$ be real numbers such that

$$\sum_{i=1}^{n} v_i^2 \leq c(1 + |x|)^2.$$
Let $a_1, a_2, \ldots, a_n$ be independent random signs. Let $Y = \sum_{i=1}^{n} a_i v_i$. Then
\[
\Pr[|x + Y| \leq 1] \geq F(c).
\]

**Proof.** By symmetry, we may assume that $x \geq 0$. Let $w_i = \frac{-v_i}{\sqrt{c(1+x)}}$. Then $\sum_{i=1}^{n} w_i^2 \leq 1$. Let $S = \sum_{i=1}^{n} a_i w_i$. Then $Y = -\sqrt{c} (1 + x) S$. Because $Y$ has a symmetric distribution, we have
\[
\Pr[Y > 1 - x] \leq \Pr[Y > 0] \leq \frac{1}{2}.
\]
By the Bentkus-Dzindzalieta inequality (Theorem 1), we have
\[
\Pr[Y < -(1 + x)] = \Pr[S > \frac{1}{\sqrt{c}}] \leq \frac{Q(1/\sqrt{c})}{4Q(\sqrt{2})}.
\]
Therefore
\[
\Pr[|x + Y| > 1] = \Pr[Y > 1 - x] + \Pr[Y < -(1 + x)] \leq \frac{1}{2} + \frac{Q(1/\sqrt{c})}{4Q(\sqrt{2})}.
\]
Taking the complement, we obtain
\[
\Pr[|x + Y| \leq 1] = 1 - \Pr[|x + Y| > 1] \geq \frac{1}{2} - \frac{Q(1/\sqrt{c})}{4Q(\sqrt{2})} = F(c).
\]

We will also need the following lemma, which says that $F$ satisfies a certain weighted-average inequality. This lemma is used to show that a weighted average of lower bounds from Lemma 2 is still a good lower bound.

**Lemma 3.** Let $K$ be an integer such that $K \geq 2$. Then
\[
\frac{1}{2^{K-1}} F \left( \frac{(K+1)^2 - K}{(2K+1)^2} \right) + \left(1 - \frac{1}{2^{K-1}}\right) F \left( \frac{(K+1)^2 - (K+2)}{(2K+1)^2} \right) \geq F\left(\frac{1}{4}\right).
\]

**Proof.** Let
\[
c_1 = \frac{(K+1)^2 - K}{(2K+1)^2} = \frac{1}{4} + \frac{3}{4} \frac{1}{(2K+1)^2};
\]
\[
c_2 = \frac{(K+1)^2 - (K+2)}{(2K+1)^2} = \frac{1}{4} - \frac{5}{4} \frac{1}{(2K+1)^2}.
\]
Since $c_1 \geq c_2$ and $F$ is a decreasing function, we see that for $K \geq 2$ we have
\[
\frac{1}{2^{K-1}} F(c_1) + \left(1 - \frac{1}{2^{K-1}}\right) F(c_2) \geq \frac{1}{2} F(c_1) + \frac{1}{2} F(c_2).
\]
Therefore it is sufficient to show that the following inequality holds for $0 \leq \xi \leq 1/25$:
\[
\frac{1}{2} F \left( \frac{1}{4} + \frac{3}{4} \xi \right) + \frac{1}{2} F \left( \frac{1}{4} - \frac{5}{4} \xi \right) \geq F\left(\frac{1}{4}\right).
\]
Once we show that $F(x)$ is a concave function in the region $0 < x \leq 1/4 + 3/100$, we conclude that the left hand side of the inequality is also concave in $\xi$ in the region $0 \leq \xi \leq 1/25$ and we need only check the inequality for $\xi = 0$ and for $\xi = 1/25$. We will show that $Q(1/\sqrt{x})$ is convex in $x$ in the region $0 < x \leq 1/3$. Recall that $Q$ satisfies the ordinary differential equation $Q''(x) = -xQ'(x)$ and that $Q'(x) < 0$ for all $x$. Thus, for $x > 0$

$$\frac{d^2}{dx^2} Q(x^{-1/2}) = Q''(x^{-1/2}) \left(-\frac{1}{2}x^{-3/2}\right)^2 + Q'(x^{-1/2}) \left(\frac{3}{4}x^{-5/2}\right) = -\frac{1}{4}Q'(x^{-1/2})x^{-7/2}(1 - 3x),$$

which is positive if $1 - 3x > 0$. It follows that $Q(x^{-1/2})$ is convex in the region $0 < x \leq 1/3$. Therefore $F(x)$ is concave in the region $0 < x \leq 1/3$. Inequality (1) holds trivially for $\xi = 0$, and one can check by calculation that it also holds for $\xi = 1/25$ (and even for $\xi = 1/9$).

Finally, we will use these two lemmas to prove our main theorem.

**Proof of Main Theorem.** We will follow the proof of Theorem 4 of Boppana and Holzman [2] nearly line for line. Their proof uses a different function $F$. Closely examining their proof, we see that they use four properties of $F$: it is bounded above by $\frac{1}{2}$, satisfies their Lemma 3 (our Lemma 2), is a nonincreasing function (on the set of positive numbers), and satisfies the weighted-average inequality of Lemma 3. Our function $F$ has those same four properties. Hence we reach the same conclusion: $\Pr[|S| \leq 1] \geq F(\frac{1}{4})$. A calculation shows that $F(\frac{1}{4}) > 0.427685$.

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