ON COMPACTNESS OF ONE CLASS OF SOLUTIONS OF THE DIRICHLET PROBLEM

OLEKSANDR DOVHOPIATYI, EVGENY SEVOST'YANO

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Abstract

We consider the Dirichlet problem for the Beltrami equation in some simply connected domain. We consider the class of all homeomorphic solutions of such a problem with a normalization condition and set-theoretic constraints on their complex characteristics. We have proved the compactness of this class in terms of prime ends for an arbitrary continuous function in the Dirichlet condition.

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1 Introduction

Relatively recently, different authors have obtained theorems on the compactness of the classes of solutions of the Beltrami equations with different types of restrictions on their complex characteristics, see, for example, [Dyb1], [LGR] and [L1]–[L2]. In particular, the results of [L1]–[L2] were used in the variational method, which is reflected in full volume in the article [LGR]. The problem of the compactness of the classes of solutions of the Dirichlet problem for this equation is quite close in relation to the question of the compactness of solutions to the Beltrami equation. The authors of this manuscript obtained some results on this topic for the case when the area in which the Dirichlet problem is studied is Jordanian, see, for example, [DS] and [SD]. This article contains results on the compactness of the classes of solutions to the Dirichlet problem for the case when the given domain is simply connected (not necessarily Jordan), and the complex characteristics of the solutions satisfy set-theoretic constraints.

In what follows, a mapping $f : D \rightarrow \mathbb{C}$ is assumed to be sense-preserving, moreover, we assume that $f$ has partial derivatives almost everywhere. Put $f_x = (f_x + if_y) / 2$ and $f_z = (f_x - if_y) / 2$. The complex dilatation of $f$ at $z \in D$ is defined as follows: $\mu(z) =$
\( \mu_f(z) = \frac{f_z}{f_{\overline{z}}} \) for \( f_z \neq 0 \) and \( \mu(z) = 0 \) otherwise. The \textit{maximal dilatation} of \( f \) at \( z \) is the following function:

\[
K_\mu(z) = K_{\mu_f}(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|}.
\]

(1.1)

Given a Lebesgue measurable function \( \mu : D \to \mathbb{D} \), \( \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \), we define the \textit{maximal dilatation} of \( f \) at \( z \) the function \( K_\mu(z) \) in (1.1). Note that the Jacobian of \( f \) at \( z \in D \) is calculated by the formula

\[
J(z, f) = |f_z|^2 - |f_{\overline{z}}|^2.
\]

It is easy to see that \( K_{\mu_f}(z) = \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|} \) whenever partial derivatives of \( f \) exist at \( z \in D \) and, in addition, \( J(z, f) \neq 0 \).

We will call the \textit{Beltrami equation} the differential equation of the form

\[
f_{\overline{z}} = \mu(z) \cdot f_z,
\]

(1.2)

where \( \mu = \mu(z) \) is a given function with \( |\mu(z)| < 1 \) a.a. The \textit{regular solution} of (1.2) in the domain \( D \subset \mathbb{C} \) is a homeomorphism \( f : D \to \mathbb{C} \) of the class \( W^{1,1}_{\text{loc}}(D) \) such that \( J(z, f) \neq 0 \) for almost all \( z \in D \).

In the extended Euclidean space \( \mathbb{R}^n = \mathbb{R}^n \cup \{\infty\} \), we use the so-called \textit{chordal metric} \( h \) defined by the equalities

\[
h(x, y) = \frac{|x - y|}{\sqrt{1 + |x|^2} \sqrt{1 + |y|^2}}, \quad x \neq \infty \neq y, \quad h(x, \infty) = \frac{1}{\sqrt{1 + |x|^2}},
\]

(1.3)

see e.g. [Va, Definition 12.1]. For a given set \( E \subset \mathbb{R}^n \), we set

\[
h(E) := \sup_{x,y \in E} h(x, y).
\]

(1.4)

The quantity \( h(E) \) in (1.4) is called the \textit{chordal diameter} of the set \( E \). As usual, the family \( \mathfrak{F} \) of mappings \( f : D \to \mathbb{C} \) is called \textit{normal}, if from each sequence \( f_n \in \mathfrak{F}, n = 1, 2, \ldots \), one can choose a subsequence \( f_{n_k}, k = 1, 2, \ldots \), converging to some mapping \( f : D \to \mathbb{C} \) locally uniformly with respect to the metric \( h \). If, in addition, \( f \in \mathfrak{F} \), the family \( \mathfrak{F} \) is called \textit{compact}.

A set \( A \subset \mathbb{D} \) is called \textit{invariantly convex} if the set \( g(A) \) is convex for any fractional-linear automorphism \( g \) of the unit disk.

Let \( D \) be a domain in \( \mathbb{R}^n \), \( n \geq 2 \). Recall some definitions (see, for example, [KR1], [KR2], [IS] or [ISS]). Let \( \omega \) be an open set in \( \mathbb{R}^k, k = 1, \ldots, n-1 \). A continuous mapping \( \sigma : \omega \to \mathbb{R}^n \) is called a \textit{k-dimensional surface} in \( \mathbb{R}^n \). A \textit{surface} is an arbitrary \((n-1)\)-dimensional surface \( \sigma \) in \( \mathbb{R}^n \). A surface \( \sigma \) is called a \textit{Jordan surface}, if \( \sigma(x) \neq \sigma(y) \) for \( x \neq y \). In the following, we will use \( \sigma \) instead of \( \sigma(\omega) \subset \mathbb{R}^n \), \( \overline{\sigma} \) instead of \( \overline{\sigma(\omega)} \) and \( \partial \sigma \) instead of \( \partial(\sigma(\omega)) \setminus \sigma(\omega) \). A Jordan surface \( \sigma : \omega \to D \) is called a \textit{cut} of \( D \), if \( \sigma \) separates \( D \), that is \( D \setminus \sigma \) has more than one component, \( \partial \sigma \cap D = \emptyset \) and \( \partial \sigma \cap \partial D \neq \emptyset \).
A sequence of cuts $\sigma_1, \sigma_2, \ldots, \sigma_m, \ldots$ in $D$ is called a chain, if:

(i) the set $\sigma_{m+1}$ is contained in exactly one component $d_m$ of the set $D \setminus \sigma_m$, wherein $\sigma_{m-1} \subset D \setminus (\sigma_m \cup d_m)$; (ii) $\bigcap_{m=1}^{\infty} d_m = \emptyset$.

Two chains of cuts $\{\sigma_m\}$ and $\{\sigma_k\}$ are called equivalent, if for each $m = 1, 2, \ldots$ the domain $d_m$ contains all the domains $d'_k$, except for a finite number, and for each $k = 1, 2, \ldots$ the domain $d'_k$ also contains all domains $d_m$, except for a finite number.

The end of the domain $D$ is the class of equivalent chains of cuts in $D$. Let $K$ be the end of $D$ in $\mathbb{R}^n$, then the set $I(K) = \bigcap_{m=1}^{\infty} \overline{d_m}$ is called the impression of the end $K$. Throughout the paper, $\Gamma(E, F, D)$ denotes the family of all paths $\gamma: [a, b] \to \mathbb{R}^n$ such that $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in D$ for every $t \in [a, b]$. In what follows, $M$ denotes the modulus of a family of paths, and the element $dm(x)$ corresponds to the Lebesgue measure in $\mathbb{R}^n$, $n \geq 2$, see [Va]. Following [Na2], we say that the end $K$ is a prime end, if $K$ contains a chain of cuts $\{\sigma_m\}$ such that $\lim_{m \to \infty} M(\Gamma(C, \sigma_m, D)) = 0$ for some continuum $C$ in $D$. In the following, the following notation is used: the set of prime ends corresponding to the domain $D$, is denoted by $E_D$, and the completion of the domain $D$ by its prime ends is denoted $\overline{D}_P$.

Consider the following definition, which goes back to Nåkki [Na2], see also [KR1]–[KR3]. We say that the boundary of the domain $D$ in $\mathbb{R}^n$ is locally quasiconformal, if each point $x_0 \in \partial D$ has a neighborhood $U$ in $\mathbb{R}^n$, which can be mapped by a quasiconformal mapping $\varphi$ onto the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$ so that $\varphi(\partial D \cap U)$ is the intersection of $\mathbb{B}^n$ with the coordinate hyperplane.

For the sets $A, B \subset \mathbb{R}^n$ we set, as usual,

$$\text{diam } A = \sup_{x,y \in A} |x - y|, \quad \text{dist } (A, B) = \inf_{x \in A, y \in B} |x - y|.$$  

Sometimes we also write $d(A)$ instead of diam $A$ and $d(A, B)$ instead of dist $(A, B)$, if no misunderstanding is possible. The sequence of cuts $\sigma_m$, $m = 1, 2, \ldots$, is called regular, if $\overline{\sigma_m} \cap \overline{\sigma_{m+1}} = \emptyset$ for $m \in \mathbb{N}$ and, in addition, $d(\sigma_m) \to 0$ as $m \to \infty$. If the end $K$ contains at least one regular chain, then $K$ will be called regular. We say that a bounded domain $D$ in $\mathbb{R}^n$ is regular, if $D$ can be quasiconformally mapped to a domain with a locally quasiconformal boundary whose closure is a compact in $\mathbb{R}^n$, and, besides that, every prime end in $D$ is regular. Note that space $\overline{D}_P = D \cup E_D$ is metric, which can be demonstrated as follows. If $g : D_0 \to D$ is a quasiconformal mapping of a domain $D_0$ with a locally quasiconformal boundary onto some domain $D$, then for $x, y \in \overline{D}_P$ we put:

$$\rho(x, y) := |g^{-1}(x) - g^{-1}(y)|, \quad (1.5)$$

where the element $g^{-1}(x), x \in E_D$, is to be understood as some (single) boundary point of the domain $D_0$. The specified boundary point is unique and well-defined by [IS] Theorem 2.1, Remark 2.1], cf. [Na2] Theorem 4.1]. It is easy to verify that $\rho$ in (1.5) is a metric on $\overline{D}_P$. 


and that the topology on $\overline{D}_P$, defined by such a method, does not depend on the choice of the map $g$ with the indicated property.

We say that a sequence $x_m \in D$, $m = 1, 2, \ldots$, converges to a prime end of $P \in E_D$ as $m \to \infty$, write $x_m \to P$ as $m \to \infty$, if for any $k \in \mathbb{N}$ all elements $x_m$ belong to $d_k$ except for a finite number. Here $d_k$ denotes a sequence of nested domains corresponding to the definition of the prime end $P$. Note that for a homeomorphism of a domain $D$ onto $D'$, the end of the domain $D$ uniquely corresponds to some sequence of nested domains in the image under the mapping.

Consider the following Dirichlet problem:

$$
 f_\pi = \mu(z) \cdot f_z, \\
 \lim_{\zeta \to P} \text{Re } f(\zeta) = \varphi(P) \quad \forall \ P \in E_D,
$$

(1.6)

where $\varphi : E_D \to \mathbb{R}$ is a predefined continuous function. In what follows, we assume that $D$ is some simply connected domain in $\mathbb{C}$. The solution of the problem (1.6)–(1.7) is called regular, if one of two is fulfilled: or $f(z) = \text{const}$ in $D$, or $f$ is an open discrete $W^{1,1}_\text{loc}(D)$-mapping such that $J(z, f) \neq 0$ for almost any $z \in D$.

Let $D$ be a domain in $\mathbb{R}^n$, $n \geq 2$. We say that a function $\varphi : D \to \mathbb{R}$ has a finite mean oscillation at a point $x_0 \in D$, write $\varphi \in F\text{MO}(x_0)$, if

$$
 \limsup_{\varepsilon \to 0} \frac{1}{\Omega_n \varepsilon^n} \int_{B(x_0, \varepsilon)} |\varphi(x) - \overline{\varphi}_\varepsilon| \, dm(x) < \infty,
$$

(1.8)

where

$$
 \overline{\varphi}_\varepsilon = \frac{1}{\Omega_n \varepsilon^n} \int_{B(x_0, \varepsilon)} \varphi(x) \, dm(x).
$$

Observe that, as known, $\Omega_n \varepsilon^n = m(B(x_0, \varepsilon))$, and that under condition (1.8), a situation is possible when $\overline{\varphi}_\varepsilon \to \infty$ при $\varepsilon \to 0$. We also say that a function $\varphi : D \to \mathbb{R}$ has a finite mean oscillation in $D$, write $\varphi \in F\text{MO}(D)$, or simply $\varphi \in F\text{MO}$, if $\varphi$ has a finite mean oscillation at any point $x_0 \in D$.

Let $M(z) \subset \mathbb{D}$, $z \in \mathbb{C}$ be some system of sets (that is, for each $z_0 \in \mathbb{C}$ the symbol $M(z_0)$ denotes some set in $\mathbb{D}$). Denote by $\mathfrak{M}_M$ the set of all complex measurable functions $\mu : \mathbb{C} \to \mathbb{D}$, such that $\mu(z) \in M(z)$ for almost all $z \in \mathbb{C}$.

We fix a point $z_0 \in D$ and a function $\varphi$. Let $M(z) \subset \mathbb{D}$, $z \in D$, be some system of sets. Let $\mathfrak{F}_{\varphi,M,z_0}(D)$ be the class of all regular solutions $f : D \to \mathbb{C}$ of the Dirichlet problem (1.6)–(1.7), which satisfy the condition $\text{Im } f(z_0) = 0$ such that $\mu \in \mathfrak{M}_M$. We define a function $Q_M(z)$ by the relation

$$
 Q_M(z) = \frac{1 + q_M(z)}{1 - q_M(z)}, \quad q_M(z) = \sup_{\nu \in M(z)} |\nu|,
$$

(1.9)
and we consider that \( Q_M(z) \equiv 1 \) for \( z \in \mathbb{C} \setminus D \). The following statement generalizes \([\text{Dyb}_1]\) Theorem 2 for the case of arbitrary simply connected Jordan domains.

**Theorem 1.1.** Let \( D \) be a simply connected domain in \( \mathbb{C} \), and let \( \varphi \) be a continuous function in \([L,7]\). Let \( M(z), z \in D, \) be a family of convex compact sets, and let \( Q_M \) be integrable in \( D \) and satisfies at least one of the following conditions: either \( Q_M \in FMO(\overline{D}) \), or

\[
\int_0^{\delta_0} \frac{dt}{tq_{M,x_0}(t)} = \infty
\]

(1.10)

for any \( x_0 \in \overline{D} \) and some \( \delta_0 = \delta(x_0) > 0 \), where \( q_{M,x_0}(t) = \frac{1}{2\pi} \int_0^{2\pi} Q_M(x_0 + te^{i\theta}) \, d\theta \). Then the family \( \mathfrak{F}_{\varphi,M,x_0}(D) \) is compact in \( D \).

## 2 Preliminaries

In what follows, \( M(\Gamma) \) denotes the conformal modulus of a family of paths \( \Gamma \) (see \([\text{Va}_1]\) section 6]), and the element \( dm(x) \) corresponds to a Lebesgue measure in \( \mathbb{R}^n, n \geq 2 \), see \([\text{Va}_1]\). Everywhere below, unless otherwise stated, the boundary and the closure of a set are understood in the sense of an extended Euclidean space \( \overline{\mathbb{R}^n} \). Let \( x_0 \in \overline{D}, x_0 \not\in \infty \),

\[
S(x_0, r) = \{ x \in \mathbb{R}^n : |x - x_0| = r \}, S_i = S(x_0, r_i), \quad i = 1, 2,
\]

\[
A = A(x_0, r_1, r_2) = \{ x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2 \}.
\]

(2.1)

Let \( Q : \mathbb{R}^n \to \mathbb{R}^n \) be a Lebesgue measurable function satisfying the condition \( Q(x) \equiv 0 \) for \( x \in \mathbb{R}^n \setminus D \), and let \( p \geq 1 \). Given sets \( E \) and \( F \) and a given domain \( D \) in \( \mathbb{R}^n = \mathbb{R}^n \cup \{ \infty \} \), we denote by \( \Gamma(E, F, D) \) the family of all paths \( \gamma : [0, 1] \to \mathbb{R}^n \) joining \( E \) and \( F \) in \( D \), that is, \( \gamma(0) \in E, \gamma(1) \in F \) and \( \gamma(t) \in D \) for all \( t \in (0,1) \). According to \([\text{MRSY}]\) Chap. 7.6], a mapping \( f : D \to \mathbb{R}^n \) is called a ring \( Q \)-mapping at the point \( x_0 \in \overline{D} \setminus \{ \infty \} \), if the condition

\[
M(f(\Gamma(S_1, S_2, D))) \leq \int_{A \cap \overline{D}} Q(x) \cdot \eta^n(|x - x_0|) \, dm(x)
\]

(2.2)

holds for all \( 0 < r_1 < r_2 < d_0 := \sup_{x \in D} |x - x_0| \) and all Lebesgue measurable functions \( \eta : (r_1, r_2) \to [0, \infty] \) such that

\[
\int_{r_1}^{r_2} \eta(r) \, dr \geq 1.
\]

(2.3)

The mapping \( f : D \to \mathbb{R}^n \) is called a ring \( Q \)-mapping in \( \overline{D} \setminus \{ \infty \} \) if (2.2) holds for any \( x_0 \in \overline{D} \setminus \{ \infty \} \). This definition can also be applied to the point \( x_0 = \infty \) by inversion: \( \varphi(x) = \frac{x}{|x|^2}, \infty \mapsto 0 \). In what follows, \( h \) denotes the so-called chordal metric defined by (1.3).

The next important lemma follows by \([\text{RSS}]\) Theorems 4.1 and 4.2].
Lemma 2.1. Let $D$ be a domain in $\mathbb{R}^n$, $n \geq 2$, and let $Q : D \rightarrow [1, \infty]$ be a Lebesgue measurable function. In addition, let, $f_k, k = 1, 2, \ldots$ be a sequence of homeomorphisms of $D$ into $\mathbb{R}^n$, which satisfy conditions (2.2)–(2.3) at any point $x_0 \in D$ that converges to some mapping $f : D \rightarrow \mathbb{R}^n$ locally uniformly in $D$ with respect to the chordal metric $h$. Assume that the function $Q$ satisfies at least one of two following conditions: either $Q \in FMO(D)$, or

$$\int_0^{\delta_0} \frac{dt}{tq_{x_0}^{-1}(t)} = \infty, \quad (2.4)$$

for any $x_0 \in D$ and some $\delta_0 = \delta(x_0) > 0$, where $q_{x_0}(t) = \frac{1}{\omega_{n-1}t^{n-1}} \int_{s(x_0,t)} Q(x)d\mathcal{H}^{n-1}$, and $\mathcal{H}^{n-1}$ denotes $(n-1)$-dimensional Hausdorff measure. Then the mapping $f$ is either a homeomorphism $f : D \rightarrow \mathbb{R}^n$, or a constant $c \in \mathbb{R}^n$.

Let $I$ be a fixed set of indices and let $D_i, i \in I$, be some sequence of domains. Following [NP Sect. 2.4], we say that a family of domains $\{D_i\}_{i \in I}$ is equi-uniform if for any $r > 0$ there exists a number $\delta > 0$ such that the inequality

$$M(\Gamma(F^*, F, D_i)) \geq \delta \quad (2.5)$$

holds for any $i \in I$ and any continua $F, F^* \subset D$ such that $h(F) \geq r$ and $h(F^*) \geq r$. If $D$ is one domain satisfying condition (2.5), then it is called uniform.

Given numbers $\delta > 0$, a domain $D \subset \mathbb{R}^n$, $n \geq 2$, a point $a \in D$ and a Lebesgue measurable function $Q : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $Q(x) \equiv 0$ for $x \in \mathbb{R}^n \setminus D$, denote by $\mathcal{F}_{Q,a,\delta}(D)$ the family of all homeomorphisms $f : D \rightarrow \mathbb{R}^n$ satisfying (2.2)–(2.3) in $\overline{D}$ such that $h(f(a), \partial f(D)) \geq \delta$, $h(\mathbb{R}^n \setminus f(D)) \geq \delta$. The following statement holds (see [SevSkv Theorem 2]).

Lemma 2.2. Let $D$ be regular, and let $D'_f = f(D)$ be bounded domains with a locally quasiconformal boundary which are equi-uniform over all $f \in \mathcal{F}_{Q,a,\delta}(D)$. If $Q \in FMO(D)$, or the condition (2.4) holds, then any $f \in \mathcal{F}_{Q,a,\delta}(D)$ has a continuous extension $\overline{f} : \overline{D}_P \rightarrow \mathbb{R}^n$ and, in addition, the family $\mathcal{F}_{Q,a,\delta}(\overline{D})$ of all extended mappings $\overline{f} : \overline{D}_P \rightarrow \mathbb{R}^n$ is equicontinuous in $\overline{D}_P$.

We also need to formulate a similar statement for homeomorphisms inverse to (2.2). For this purpose, consider the following definition.

For domains $D \subset \mathbb{R}^n$ and $D' \subset \mathbb{R}^n$, $n \geq 2$, points $a \in D$, $b \in D'$ and a Lebesgue measurable function $Q : \mathbb{R}^n \rightarrow [0, \infty]$ such that $Q(x) \equiv 0$ for $x \notin D$, we denote by $\mathcal{G}_{a,b,Q}(D, D')$ the family of all homeomorphisms $h$ of $D'$ onto $D$ such that the mapping $f = h^{-1}$ satisfies the condition (2.2) in $\overline{D}$, while $f(a) = b$.

The boundary of the domain $D$ is called weakly flat at the point $x_0$, if for every number $P > 0$ and every neighborhood $U$ of this point there is a neighborhood $V$ of point $x_0$ such that $M(\Gamma(E, F, D)) > P$ for any continua $E$ and $F$, satisfying conditions $F \cap \partial U \neq \emptyset \neq F \cap \partial V$. 

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The boundary of domain \( D \) is called \textit{weakly flat} if it is such at each of its point. The following statement holds (see e.g. [SSD, Theorem 7.1]).

**Lemma 2.3.** Assume that \( D \) is a regular domain, and that \( D' \) has a weakly flat boundary, none of the components of which degenerates into a point. Then any \( h \in \mathcal{G}_{a,b,Q}(D,D') \) has a continuous extension \( \overline{h} : \overline{D'} \to \overline{D}_p \), while \( \overline{h}(D') = \overline{D}_p \) and, in addition, the family \( \mathcal{G}_{a,b,Q}(D,\overline{D}') \) of all extended mappings \( \overline{h} : \overline{D'} \to \overline{D}_p \) is equicontinuous in \( \overline{D}' \).

3 Proof of Theorem 1.1

In general, we will use the scheme of proving Theorem 1.2 in [SD].

\[ I. \text{ Let } f_m \in \mathcal{F}_{\varphi,D,z_0}(D), \ m = 1, 2, \ldots. \] By Stoilow’s factorization theorem (see, e.g., [St, 5(III).V]) a mapping \( f_m \) has a representation

\[ f_m = \varphi_m \circ g_m, \quad \tag{3.1} \]

where \( g_m \) is some homeomorphism, and \( \varphi_m \) is some analytic function. By Lemma 1 in [Sev], the mapping \( g_m \) belongs to the Sobolev class \( W^{1,1}_{\text{loc}}(D) \) and has a finite distortion. Moreover, by [A, (1).C, Ch. I]

\[ f_{mz} = \varphi_{mz}(g_m(z))g_{mz}, \quad \tag{3.2} \]

for almost all \( z \in D \). Therefore, by the relation (3.2), \( J(z, g_m) \neq 0 \) for almost all \( z \in D \), in addition, \( K_{\mu f_m}(z) = K_{\mu g_m}(z) \).

\[ II. \text{ We prove that } \partial g_m(D) \text{ contains at least two points. Suppose the contrary. Then either } g_m(D) = \mathbb{C}, \text{ or } g_m(D) = \mathbb{C} \setminus \{a\}, \text{ where } a \in \mathbb{C}. \] Consider first the case \( g_m(D) = \mathbb{C} \). By Picard’s theorem \( \varphi_m(g_m(D)) \) is the whole plane, except perhaps one point \( \omega_0 \in \mathbb{C} \). On the other hand, for every \( m = 1, 2, \ldots \) the function \( u_m(z) := \Re f_m(z) = \Re (\varphi_m(g_m(z))) \) is continuous on the compact set \( \overline{D} \) under the condition (1.1) by the continuity of \( \varphi \). Therefore, there exists \( C_m > 0 \) such that \( |\Re f_m(z)| \leq C_m \) for any \( z \in D \), but this contradicts the fact that \( \varphi_m(g_m(D)) \) contains all points of the complex plane except, perhaps, one. The situation \( g_m(D) = \mathbb{C} \setminus \{a\}, a \in \mathbb{C} \), is also impossible, since the domain \( g_m(D) \) must be simply connected in \( \mathbb{C} \) as a homeomorphic image of the simply connected domain \( D \).

Therefore, the boundary of the domain \( g_m(D) \) contains at least two points. Then, according to Riemann’s mapping theorem, we may transform the domain \( g_m(D) \) onto the unit disk \( \mathbb{D} \) using the conformal mapping \( \psi_m \). Let \( z_0 \in D \) be a point from the condition of the theorem. By using an auxiliary conformal mapping

\[ \widetilde{\psi}_m(z) = \frac{z - (\psi_m \circ g_m)(z_0)}{1 - z(\psi_m \circ g_m)(z_0)} \]
of the unit disk onto itself we may consider that \((\psi_m \circ g_m)(z_0) = 0\). Now, by \([3.1]\) we obtain that
\[
f_m = \varphi_m \circ g_m = \varphi_m \circ \psi_m^{-1} \circ \psi_m \circ g_m = F_m \circ G_m, \quad m = 1, 2, \ldots ,
\]
where \(F_m := \varphi_m \circ \psi_m^{-1}, \ F_m : \mathbb{D} \to \mathbb{C}\), and \(G_m = \psi_m \circ g_m\). Obviously, a function \(F_m\) is analytic, and \(G_m\) is a regular Sobolev homeomorphism in \(D\). In particular, \(\text{Im} \ F_m(0) = 0\) for any \(m \in \mathbb{N}\).

**III.** Observe that
\[
\int_D K_{\mu G_m}(z) \, dm(z) \leq \int_D Q_M(z) \, dm(z) < \infty , \quad (3.3)
\]
because by the condition \(\mu(z) \in M(z)\) for almost any \(z \in D\), moreover, the inequality \(K_{\mu G_m}(z) \leq Q_M(z)\) holds for almost any \(z \in D\), where \(Q_M(z)\) does not depend on \(m = 1, 2, \ldots \) and is integrable.

**IV.** We prove that each map \(G_m, m = 1, 2, \ldots \), has a continuous extension to \(E_D\), in addition, the family of extended maps \(\overline{G}_m, m = 1, 2, \ldots \), is equicontinuous in \(\overline{D}_p\). Indeed, as proved in item **III**, \(K_{\mu G_m} \in L^1(D)\). By \([KPRS\text{, Theorem 3}]\) (see also \([LSS\text{, Theorem 3.1}]\)) each \(G_m, m = 1, 2, \ldots \), is a ring \(Q\)-homeomorphism in \(\overline{D}\) for \(Q = K_{\mu G_m}(z)\), where \(\mu\) is defined in \([1,6]\), and \(K_{\mu m}\) may be calculated by the formula \([1,1]\). Note that the unit disk \(\mathbb{D}\) is a uniform domain as a finitely connected flat domain at its boundary with a finite number of boundary components (see, for example, \([Na]\text{, Theorem 6.2 and Corollary 6.8}]\). Then the desirable the conclusion is a statement of Lemma \([2,2]\)

**V.** Observe that, the inverse homeomorphisms \(G_m^{-1}, m = 1, 2, \ldots \), have a continuous extension \(\overline{G}^{-1}_m\) to \(\partial \mathbb{D}\) in terms of prime ends in \(D\), and \(\{\overline{G}^{-1}_m\}_m=1\) is equicontinuous in \(\overline{D}\) as a family of mappings from \(\overline{D}\) to \(\overline{D}_p\). Indeed, by the item **IV** mappings \(G_m, m = 1, 2, \ldots \), are ring \(K_{\mu G_m}(z)\)-homeomorphisms in \(D\) such that \(G_m^{-1}(0) = z_0\) for any \(m = 1, 2, \ldots \). In this case, the possibility of a continuous extension of \(G_m^{-1}\) to \(\partial \mathbb{D}\), and the equicontinuity of \(\{\overline{G}^{-1}_m\}_m=1\) as mappings \(G_m^{-1} : \overline{D} \to \overline{D}_p\) follows by Lemma \([2,3]\)

**VI.** Since, as proved above the family \(\{G_m\}_m=1\) is equicontinuous in \(D\), according to Arzela-Ascoli criterion there exists an increasing subsequence of numbers \(m_k, k = 1, 2, \ldots \), such that \(G_{m_k}\) converges locally uniformly in \(D\) to some continuous mapping \(G : D \to \overline{C}\) as \(k \to \infty\) (see, e.g., \([Va\text{, Theorem 20.4}]\)). By Lemma \([2,1]\) either \(G\) is a homeomorphism with values in \(\mathbb{R}^n\), or a constant in \(\mathbb{R}^n\). Let us prove that the second case is impossible. Let us apply the approach used in proof of the second part of Theorem 21.9 in \([Va]\). Suppose the contrary: let \(G_{m_k}(x) \to c = \text{const}\) as \(k \to \infty\). Since \(G_{m_k}(z_0) = 0\) for all \(k = 1, 2, \ldots \), we have that \(c = 0\). By item **V**, the family of mappings \(G_m^{-1}, m = 1, 2, \ldots \), is equicontinuous in \(\mathbb{D}\). Then
\[
h(z, G_{m_k}^{-1}(0)) = h(G_{m_k}^{-1}(G_{m_k}(z)), G_{m_k}^{-1}(0)) \to 0
\]
as \( k \to \infty \), which is impossible because \( z \) is an arbitrary point of the domain \( D \). The obtained contradiction refutes the assumption made above. Thus, \( G : D \to \mathbb{C} \) is a homeomorphism.

VII. According to VII, the family of mappings \( \{ \overline{G_{m_k}} \} \) is equicontinuous in \( \overline{D} \). By the Arzela-Ascoli criterion (see, e.g., [Va, Theorem 20.4]) we may consider that \( \overline{G_{m_k}} \) converges to some mapping \( \tilde{F} : \overline{D} \to \overline{D} \) as \( k \to \infty \) uniformly in \( \overline{D} \). Let us to prove that \( \tilde{F} = \overline{G}^{-1} \). For this purpose, we show that \( G(D) = \mathbb{D} \). Fix \( y \in \mathbb{D} \). Since \( G_{m_k}(D) = \mathbb{D} \) for every \( k = 1, 2, \ldots \), we obtain that \( G_{m_k}(x_k) = y \) for some \( x_k \in D \). Since \( D \) is regular, the metric space \( (\overline{D}_P, \rho) \) is compact. Thus, we may assume that \( \rho(x_k, x_0) \to 0 \) as \( k \to \infty \), where \( x_0 \in \overline{D}_P \). By the triangle inequality and the equicontinuity of \( \{ \overline{G_m} \} \) on \( \overline{D}_P \), see IV, we obtain that

\[
|\overline{G}(x_0) - y| = |\overline{G}(x_0) - \overline{G}_{m_k}(x_k)| \leq |\overline{G}(x_0) - \overline{G}_{m_k}(x_0)| + |\overline{G}_{m_k}(x_0) - \overline{G}_{m_k}(x_k)| \to 0 \quad \text{as} \quad k \to \infty.
\]

Thus, \( \overline{G}(x_0) = y \). Observe that \( x_0 \in D \), because \( G \) is a homeomorphism. Since \( y \in \mathbb{D} \) is arbitrary, the equality \( G(D) = \mathbb{D} \) is proved. In this case, \( G_{m_k}^{-1} \) is locally uniformly in \( \mathbb{D} \) as \( k \to \infty \) (see, e.g., [RSS, Lemma 3.1]). Thus, \( \tilde{F}(y) = G^{-1}(y) \) for every \( y \in \mathbb{D} \).

Finally, since \( \tilde{F}(y) = G^{-1}(y) \) for any \( y \in \mathbb{D} \) and, in addition, \( \tilde{F} \) has a continuous extension on \( \partial \mathbb{D} \), due to the uniqueness of the limit at the boundary points we obtain that \( \tilde{F}(y) = \overline{G}^{-1}(y) \) for \( y \in \overline{D} \). Therefore, we have proved that \( \overline{G}^{-1} \) is compact with respect to the metrics \( \rho \) in \( \overline{D}_P \).

VIII. By VII, for \( y = e^{i\theta} \in \partial \mathbb{D} \)

\[
\text{Re } F_{m_k}(e^{i\theta}) = \varphi\left( \overline{G_{m_k}^{-1}}(e^{i\theta}) \right) \to \varphi\left( \overline{G}^{-1}(e^{i\theta}) \right) \quad (3.4)
\]

as \( k \to \infty \) uniformly on \( \theta \in [0, 2\pi) \). Since by the construction \( \text{Im } F_{m_k}(0) = 0 \) for any \( k = 1, 2, \ldots \), by the Schwartz formula (see, e.g., [GK, section 8.III.3]) the analytic function \( F_{m_k} \) is uniquely restored by its real part, namely,

\[
F_{m_k}(y) = \frac{1}{2\pi i} \int_{S(0,1)} \varphi\left( \overline{G_{m_k}^{-1}}(t) \right) \frac{t + y}{t - y} \cdot \frac{dt}{t}. \quad (3.5)
\]

Set

\[
F(y) := \frac{1}{2\pi i} \int_{S(0,1)} \varphi\left( \overline{G}^{-1}(t) \right) \frac{t + y}{t - y} \cdot \frac{dt}{t}. \quad (3.6)
\]

Let \( K \subset \mathbb{D} \) be an arbitrary compact set, and let \( y \in K \). By (3.5) and (3.6) we obtain that

\[
|F_{m_k}(y) - F(y)| \leq \frac{1}{2\pi} \int_{S(0,1)} \left| \varphi(\overline{G_{m_k}^{-1}}(t)) - \varphi(\overline{G}^{-1}(t)) \right| \left| \frac{t + y}{t - y} \right| |dt|. \quad (3.7)
\]

Since \( K \) is compact, there is \( 0 < R_0 = R_0(K) < \infty \) such that \( K \subset B(0, R_0) \). By the triangle inequality \( |t + y| \leq 1 + R_0 \) and \( |t - y| \geq |t| - |y| \geq 1 - R_0 \) for \( y \in K \) and any \( t \in S^1 \). Thus

\[
\left| \frac{t + y}{t - y} \right| \leq \frac{1 + R_0}{1 - R_0} := M = M(K).
\]
Put $\varepsilon > 0$. By (3.7) for a number $\varepsilon' := \frac{\varepsilon}{M}$ there is $N = N(\varepsilon, K) \in \mathbb{N}$ such that $|\varphi(G^{-1}_{mk}(t)) - \varphi(G^{-1}_{m_k}(t))| < \varepsilon'$ for any $k \geq N(\varepsilon)$ and $t \in \mathbb{S}^1$. Now, by (3.7)

$$|F_{mk}(y) - F(y)| < \varepsilon \quad \forall \ k \geq N. \quad (3.8)$$

It follows from (3.8) that the sequence $F_{mk}$ converges to $F$ as $k \to \infty$ in the unit disk locally uniformly. In particular, we obtain that $\text{Im} \ F(0) = 0$. Note that $F$ is analytic function in $\mathbb{D}$ (see remarks made at the end of item 8.III in [GK]), and

$$\text{Re} \ F(re^{i\psi}) = \frac{1}{2\pi} \int_0^{2\pi} \varphi\left(G^{-1}(e^{i\theta})\right) \frac{1 - r^2}{1 - 2r \cos(\theta - \psi) + r^2} \, d\theta$$

for $z = re^{i\psi}$. By [GK] Theorem 2.10.III.3

$$\lim_{\zeta \to z} \text{Re} \ F(\zeta) = \varphi(G^{-1}(z)) \quad \forall \ z \in \partial\mathbb{D}. \quad (3.9)$$

Observe that $F$ either is a constant or open and discrete (see, e.g., [St] Ch. V, I.6 and II.5]). Thus, $f_{mk} = F_{mk} \circ G_{mk}$ converges to $f = F \circ G$ locally uniformly as $k \to \infty$, where $f = F \circ G$ either is a constant or open and discrete. Moreover, by (3.9)

$$\lim_{\zeta \to P} \text{Re} \ f(\zeta) = \lim_{\zeta \to P} \text{Re} \ F(G(\zeta)) = \varphi(G^{-1}(G(P))) = \varphi(P). \quad (3.10)$$

IX. Since by VI $G$ is a homeomorphism, by [L2] Lemma 1 and Theorem 1] $G$ is a regular solution of the equation (1.6) for some function $\mu : \mathbb{C} \to \mathbb{D}$. Since the set of points of the function $F$, where its Jacobian is zero, consist only of isolated points (see [St] Ch. V, 5.II and 6.II]), $f$ is regular solution of the Dirichlet problem (1.6)–(1.7) whenever $F \neq \text{const}$. It remains to show that $\mu \in \mathfrak{M}_M$. If $f(z) = c = \text{const}$ in $D$, due to the condition (1.7) we obtain that $f_n(z) = c$ in $D$, and $\mu_n(z) = 0 \in M(z)$ for almost any $z \in D$. In this case, $\mu(z) = 0$ for almost any $z \in D$, in particular, $\mu \in \mathfrak{M}_M$.

Let $f(z) \neq \text{const}$. As proved above, $f$ is regular. Since $f_n(z)$ converge to $f(z)$ locally uniformly in $D$, in addition, the Jacobian of $f$ does not vanish almost everywhere, by [L2] Lemma 1] $\mu(z) \in \text{inv co} M_0(z)$ for almost any $z \in D$, where $\text{inv co} A$ denotes the invariant-convex hull of the set $A \subset \mathbb{C}$ (see, e.g., [Ryaz]), and $M_0(z)$ is a cluster set of $\mu_n(z)$, $n = 1, 2, \ldots$. Obviously, there is a set $D_0 \subset D$ such that $\mu_n(z) \in M(z)$ and $\mu(z) \in \text{inv co} M_0(z)$ for all $z \in D_0$ and any $n \in \mathbb{N}$, where $m(D \setminus D_0) = 0$. Fix $z_0 \in D_0$. Let $w_0 \in M_0(z_0)$. Then there is a subsequence of numbers $n_k, k = 1, 2, \ldots$, for which $\mu_{nk}(z_0)$ converge as $k \to \infty$ and $\lim_{k \to \infty} \mu_{nk}(z_0) = w_0$. Since, by the assumption, $\mu_{nk}(z_0) \in M(z_0)$ for any $k = 1, 2, \ldots$, in addition, $M(z_0)$ is closed, we obtain that $w_0 \in M(z_0)$. Thus,

$$M_0(z_0) \subset M(z_0). \quad (3.10)$$

It follows from (3.10) that

$$\text{inv co} M_0(z_0) \subset M(z_0), \quad (3.11)$$
because $M(z_0)$ is invariant-convex. Thus,

$$\mu(z_0) \in \text{inv co} \, M_0(z_0) \subset M(z_0)$$

for almost any $z_0 \in D$, that is desired conclusion. □

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Oleksandr Dovhopiatyi
1. Zhytomyr Ivan Franko State University,
40 Bol’shaya Berdichevskaya Str., 10 008 Zhytomyr, UKRAINE
alexdov1111111@gmail.com

Evgeny Sevost’yanov
1. Zhytomyr Ivan Franko State University,
40 Bol’shaya Berdichevskaya Str., 10 008 Zhytomyr, UKRAINE
2. Institute of Applied Mathematics and Mechanics
of NAS of Ukraine,
1 Dobrovol’skogo Str., 84 100 Slavyansk, UKRAINE
esevostyanov2009@gmail.com