Onset of instability due to variable viscosity and dissipation in a plane porous channel

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Abstract. The effect of viscous dissipation is considered on modelling the fully-developed heat transfer in a parallel plane channel filled with a saturated porous medium. The basic Darcy’s flow in a regime of forced convection is analysed insofar as the variability of fluid viscosity with temperature is taken into account. The thermal boundary conditions at the impermeable channel walls are described by assuming external convection with a constant heat transfer coefficient, viz. by imposing Robin conditions for the temperature as parametrised through the Biot number. The emergence of a singular behaviour in the basic velocity and temperature profiles is found when the Péclet number and the variable viscosity parameter are large enough as to imply a failure of the linear fluidity model. A linear stability analysis of the basic parallel flow is carried out to detect the reaction of the system to small-amplitude external perturbations. Different odd or even normal modes of the longitudinal type are studied. It is shown that no instability arises until the parametric condition for the emergence of the singularity is approached. An argument to predict the behaviour of normal modes of oblique type is eventually presented.

1. Introduction
Viscous dissipation is an important effect in heat transfer processes involving the flow of Newtonian or non-Newtonian fluids with a large viscosity and relatively small thermal conductivity. Such fluids display, in most cases, large viscosity variations with temperature. Examples of fluid flows with variable viscosity and dissipation are frequent also in fluid saturated porous media, where viscous dissipation is not only an effect arising close to boundary walls, as for Navier-Stokes flows, but it arises also in the bulk region of the seepage flow. The reason is that viscous dissipation in Navier-Stokes flow is set off by the square strain, while for Darcy’s flows its intensity is directly proportional to the local square velocity [1, 2].

It is well established that viscous dissipation and variable viscosity may cause a thermal instability in the flow [2–10]. There are basically two physical causes for the instability, viz. the variability of viscosity with temperature and the effect of thermal buoyancy [2, 10]. While the former cause was recognised several years ago by Joseph [3], the role played by the thermal buoyancy has been identified only recently [9]. A wide work has been carried out in recent years on this topic, as surveyed by Barletta [2], dealing either with Navier-Stokes flows and with seepage flows in saturated porous media. On the interplay between the effect of variable viscosity and that of variable density (thermal buoyancy) in triggering the instability, an analysis has been recently carried out by Barletta and Nield [10]. In this paper, the instability of the fully developed parallel flow in a horizontal porous channel was proved to arise when the Péclet number exceeds a critical value, that depends both on the Gebhart number and on the slope parameter of the viscosity law. The configuration analysed by Barletta and Nield [10] is such
that the upper channel wall is isothermal and impermeable, while the lower wall is impermeable and adiabatic.

Our aim is to develop further the analysis of Barletta and Nield [10], in order to study a configuration where the boundary conditions are symmetric instead of asymmetric, and to focus our attention on the role played by the variable viscosity. To this end, we will assume conditions of forced convection, so that the thermal buoyancy will be neglected. The symmetric channel boundaries will be assumed impermeable and subject to Robin temperature conditions, so that a generic process of heat transfer to the external environment is envisaged, reducing to the purely isothermal case when the Biot number tends to infinity.

2. Governing equations

The analysis developed in this paper is relative to a plane porous channel with half-width $L$ (see Fig. 1). The channel walls, $y^* = \pm L$, are considered impermeable and cooled up by an external fluid environment at temperature $T_0$, with a constant heat transfer coefficient $h$. The porous medium is homogeneous and isotropic; it is saturated by a fluid under local thermal equilibrium conditions. Darcy’s law is satisfied and the fluid density is taken to be constant, so that the buoyancy force can be neglected.

Coordinates $x^*$ and $z^*$ define the streamwise and the spanwise directions, respectively. The velocity field $u^*$ has components $(u^*, v^*, w^*)$ along the $(x^*, y^*, z^*)$ axes. Time is denoted by $t^*$, temperature field by $T^*$, and pressure field by $p^*$. Asterisks denote dimensional fields, coordinates and time. Dimensionless quantities are defined through the scaling

$$
\frac{1}{L} (x^*, y^*, z^*) = (x, y, z), \quad \frac{\alpha}{\sigma L^2} t^* = t, \quad \frac{K}{\mu_0 \alpha} (p^* - p_0) = p,
$$

$$
\frac{L}{\alpha} u^* = \frac{L}{\alpha} (u^*, v^*, w^*) = (u, v, w) = u, \quad \rho_0 c K \frac{T^* - T_0}{\mu_0 \alpha} = T, \quad \frac{\mu_0}{\mu(T^*)} = \eta(T),
$$

where $\alpha$ is the average thermal diffusivity of the saturated porous medium, $p_0$ is the reference pressure, $\rho_0$ is the reference fluid density evaluated at temperature $T_0$, $\mu_0$ is the reference dynamic viscosity evaluated at temperature $T_0$, $c$ is the specific heat of the fluid, $K$ is the permeability, and $\sigma$ is the heat capacity ratio of the saturated porous medium. The latter quantity is defined as the ratio between the average volumetric heat capacity of the saturated medium and the volumetric heat capacity of the fluid. In Eq. (1), the change of the fluid viscosity with temperature is accounted for by expressing the temperature-dependent dynamic viscosity $\mu(T^*)$ through a dimensionless function, $\eta(T)$, of the dimensionless temperature $T$. Function $\eta(T)$ is called fluidity [3]. By definition, fluidity is such that $\eta(0) = 1$, and $\eta(T) > 0$.

If we take into account the effects of viscous dissipation and variable viscosity, the local mass,
momentum and energy balance equations can be written in a dimensionless form as

\[ \nabla \cdot \mathbf{u} = 0, \]  
\[ \mathbf{u} = -\eta(T) \nabla p, \]  
\[ \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \nabla^2 T + \eta(T)^{-1} \mathbf{u} \cdot \nabla. \]

Equations (2) lead to a pressure-temperature formulation,

\[ \nabla \left[ \eta(T) \nabla p \right] = 0, \]  
\[ \frac{\partial T}{\partial t} - \eta(T) \nabla p \cdot \nabla T = \nabla^2 T + \eta(T) \nabla p \cdot \nabla p. \]

The boundaries \( y = \pm 1 \) are impermeable with a convective condition to the external fluid environment, as sketched in Fig. 1, so that we may write

\[ y = \pm 1 : \quad \frac{\partial p}{\partial y} = 0, \quad \frac{\partial T}{\partial y} \pm BT = 0, \]  

where

\[ B = \frac{hL}{\lambda} \]

is the Biot number, and \( \lambda \) is the average thermal conductivity of the saturated medium. We mention that \( B \to \infty \) describes the limiting case of a perfectly isothermal wall, while \( B \to 0 \) yields a perfectly insulated wall.

3. Basic flow
A stationary solution of Eqs. (3) and (4) exists such that the pressure gradient is a parallel vector field directed along the \( x \)-direction, and such that the temperature depends only on \( y \). This basic solution is given by

\[ \nabla p_b = (P, 0, 0), \]
\[ \frac{d^2 T_b}{dy^2} = -P^2 \eta(T_b), \quad \left( \frac{dT_b}{dy} \pm BT_b \right)_{y=\pm 1} = 0, \]

where \( P \) is a dimensionless constant expressing the Péclet number associated with the imposed pressure gradient along the \( x \)-direction, and the subscript "\( b \)" denotes the basic solution.

The basic solution, Eqs. (6), is fully determined if and only if the fluidity function \( \eta(T) \) is prescribed. Assuming that viscous dissipation does not cause very large temperature changes over the channel cross-section, a linear fluidity model is a reasonable assumption [3],

\[ \eta(T) = 1 + a^2 T, \]

where \( a \) is a positive constant parameter. The positive slope of the fluidity function, Eq. (7), implies that the viscosity of the fluid is a decreasing function of temperature, which is the behaviour experimentally observed for liquids. We mention that the linear fluidity model, Eq. (7), can be considered as a first-order approximation of the exponential, or Nahme, model [11].

On account of Eq. (7), Eqs. (6) yield

\[ \nabla p_b = (P, 0, 0), \]
\[ T_b(y) = \frac{B \cos(aP) - aP \sin(aP) - B \cos(aPy)}{a^2[aP \sin(aP) - B \cos(aP)]}, \]
\[ \eta_b(y) = \frac{B \cos(aPy)}{B \cos(aP) - aP \sin(aP)}. \]
where $\eta_b = \eta(T_b)$.

For a given Biot number, $B$, Eqs. (8) imply that the basic solution may be singular, and hence unphysical, when $aP$ is a root of the equation

$$aP \sin(aP) - B \cos(aP) = 0.$$  

(9)

In the special case $B \to \infty$ (isothermal duct wall), the smallest singular value of $aP$ is $aP = \pi/2$. On gradually reducing the Biot number, the smallest singular value of $aP$ decreases approaching 0 when $B \to 0$. In fact, when the Biot number vanishes the numerator and denominator at the right hand side of Eq. (8b) are simplified and $T_b$ achieves a uniform distribution, with value $-1/a^2$. This uniform distribution is allowed only as a consequence of the variable viscosity model. When $a = 0$ (uniform viscosity), no stationary flow solution with a vanishing axial temperature gradient is possible. However, the uniform temperature distribution for the adiabatic case, $T_b = -1/a^2$, is a definitely unphysical limiting solution as it represents an identically vanishing fluidity $\eta_b = 0$ and, hence, an infinite viscosity.

Plots of the fluidity $\eta_b(y)$ for different values of $aP$, below the lowest singularity, and with given Biot numbers are reported in Fig. 2. We mention that, on account of Eqs. (2b) and (6a), these plots are also illustrative of the qualitative trend of the basic velocity profile, given by $u_b(y) = -P \eta_b(y)$.
4. Stability analysis

When the basic state is perturbed,

\[ p = p_b + \varepsilon \hat{p}, \quad T = T_b + \varepsilon \hat{T}, \tag{10} \]

with a small amplitude parameter \(|\varepsilon| \ll 1\), the system of governing equations (3) and (4) can be rewritten as

\[
\begin{align*}
\eta_b(y)\nabla^2 \hat{p} + \eta_b'(y) \frac{\partial \hat{p}}{\partial y} + a^2 P \frac{\partial \hat{T}}{\partial x} &= 0, \tag{11a} \\
\frac{\partial \hat{T}}{\partial t} - \eta_b(y)P \frac{\partial \hat{T}}{\partial x} - \frac{1}{a^2} \eta_b(y) \eta_b'(y) \frac{\partial \hat{p}}{\partial y} &= \nabla^2 \hat{T} + a^2 P^2 \hat{T} + 2P \eta_b(y) \frac{\partial \hat{p}}{\partial x}, \tag{11b} \\
y = \pm 1 : \quad \frac{\partial \hat{p}}{\partial y} = 0, \quad \frac{\partial \hat{T}}{\partial y} \pm BT = 0, \tag{11c}
\end{align*}
\]

4.1. Normal modes

Eqs. (11) will be solved by expressing \( \hat{p} \) and \( \hat{T} \) as linear combinations of normal modes, given by

\[ \hat{p} = f(y) e^{st} e^{i(k_x x + k_z z)}, \quad \hat{T} = g(y) e^{st} e^{i(k_x x + k_z z)}, \tag{12} \]

where \((k_x, k_z)\) is a pair of real parameters expressing the components of the wave vector with wavenumber \( k = (k_x^2 + k_z^2)^{1/2} \). Moreover, \( s = s_r - i \omega \) is a complex parameter, with \( s_r \) and \(-\omega\) denoting the real and imaginary parts of \( s \), respectively. The condition \( s_r > 0 \) yields instability, \( s_r = 0 \) neutral stability, and \( s_r < 0 \) denotes linear stability. The angular frequency \( \omega \) is given by the negative imaginary part of \( s \).

On account of Eq. (12), Eqs. (11) yield

\[
\begin{align*}
\eta_b(y) \left( f'' - k^2 f \right) + \eta_b'(y) f' + ik_x a^2 P g &= 0, \tag{13a} \\
g'' - [k^2 - a^2 P^2 + s - ik_x \eta_b(y)P] g + 2ik_x P \eta_b(y) f + \frac{1}{a^2} \eta_b(y) \eta_b'(y) f' &= 0, \tag{13b} \\
y = \pm 1 : \quad f' = 0, \quad g' \pm B g &= 0. \tag{13c}
\end{align*}
\]

5. Longitudinal modes

A special class of normal modes is that of perturbations expressed through Eq. (12) with \( k_x = 0 \) and \( k_z = k \). These modes are \( x \)-independent, or streamwise invariant. With \( k_x = 0 \), a dramatic simplification of Eqs. (13) occurs, namely

\[
\begin{align*}
\eta_b(y) \left( f'' - k^2 f \right) + \eta_b'(y) f' &= 0, \tag{14a} \\
g'' - (k^2 - a^2 P^2 + s) g + \frac{1}{a^2} \eta_b(y) \eta_b'(y) f' &= 0, \tag{14b} \\
y = \pm 1 : \quad f' = 0, \quad g' \pm B g &= 0. \tag{14c}
\end{align*}
\]

Solution of Eqs. (14) can be sought analytically by recognising that Eq. (14a) does not contain \( h \), so that it can be solved independently of Eq. (14b) with the boundary conditions \( f'(\pm 1) = 0 \). This yields \( f(y) = constant \), where the constant is arbitrary, when \( k = 0 \). On the other hand, when \( k > 0 \), we get \( f(y) = 0 \). No other solutions of Eqs. (14a) and (14c) exist, as it is proved in Appendix A. Thus, Eq. (14b) is satisfied either with even modes

\[ g_n(y) = \cos(\gamma_n y), \quad n = 0, 1, 2, \ldots, \tag{15} \]

such that \( \gamma_n \) are the roots of

\[ B \cos(\gamma) - \gamma \sin(\gamma) = 0, \tag{16} \]
in increasing order, or with odd modes
\[ g_n(y) = \sin(\gamma_n y), \quad n = 0, 1, 2, \ldots, \] (17)
such that \( \gamma_n \) are the roots of
\[ B \sin(\gamma) + \gamma \cos(\gamma) = 0, \] (18)
in increasing order. A dispersion relation must hold expressed by
\[ \begin{aligned}
  a^2 P^2 &= k^2 + \gamma_n^2 + s, & \text{for even modes}, \\
  a^2 P^2 &= k^2 + \tilde{\gamma}_n^2 + s, & \text{for odd modes}.
\end{aligned} \] (19)
The imaginary part of \( s \) vanishes, \( i.e. \omega = 0 \), as it is proved in the following.

By taking into account that \( f' = 0 \), we multiply Eq. (14b) by \( \bar{g} \), where \( \bar{g} \) is the complex conjugate of \( g \), and integrate over \( y \in [-1, 1] \). Thus, by employing integration by parts for the first term, we obtain

\[ \bar{g}(1) g'(1) - \bar{g}(-1) h'(-1) - \int_{-1}^{1} |g'|^2 dy + \left( a^2 P^2 - s - k^2 \right) \int_{-1}^{1} |g|^2 dy = 0, \] (20)

while use of Eq. (14c) allows further simplification to
\[ B \left[ |g(1)|^2 + |g(-1)|^2 \right]^2 + \int_{-1}^{1} |g'|^2 dy - \left( a^2 P^2 - s - k^2 \right) \int_{-1}^{1} |g|^2 dy = 0. \] (21)

In order to satisfy Eq. (21), both the real and the imaginary parts of its right hand side must be zero. As for the imaginary part, one obtains
\[ \omega \int_{-1}^{1} |g|^2 dy = 0, \] (22)
which implies either \( \omega = 0 \) or \( g \) identically zero. Only, the former alternative is allowed, as the latter would lead to no perturbation of the basic flow. Thus, the dispersion relation (19) can be rewritten as
\[ \begin{aligned}
  a^2 P^2 &= k^2 + \gamma_n^2 + s_r, & \text{for even modes}, \\
  a^2 P^2 &= k^2 + \tilde{\gamma}_n^2 + s_r, & \text{for odd modes}.
\end{aligned} \] (23)
As a by-product of this argument, one can infer that the roots, $\gamma_n$, of Eq. (16), as well as the roots, $\tilde{\gamma}_n$, of Eq. (18) are real.

Instability arises when $s_r > 0$. On account of Eq. (23), this means $a^2P^2 > \gamma_n^2$ for even modes, or $a^2P^2 > \tilde{\gamma}_n^2$ for odd modes. The most restrictive among these inequalities being $a^2P^2 > \gamma_0^2$, which corresponds to the even mode with $n = 0$ and $k = 0$. The trend of $\gamma_n^2$ and $\tilde{\gamma}_n^2$ versus $B$ is plotted in Fig. 3 with $n = 0$, 1, 2. Thus, the neutral stability condition is defined by

$$a^2P^2 = \gamma_0^2.$$ 

Comparison between Eq. (9) and Eq. (16) allows one to reckon that the threshold for instability is at the lowest singularity of the basic solution, Eqs. (8). This means that the basic flow is linearly stable to longitudinal modes as long as the product $aP$ is smaller than its lowest value satisfying Eq. (9). Asymptotic values of $\gamma_0$ are attained in the limiting cases of $B \to 0$ and $B \to \infty$, where they coincide with $0$ and $\pi/2$, respectively.

### 6. Three-dimensional modes

The analysis of instability to longitudinal modes revealed that the onset of instability is for a finite product $aP$. The use of a linear fluidity law is legitimate as long as the slope parameter $a$ is small enough, namely $a \ll 1$. Thus, the regime of instability is to be considered as one where simultaneously $a \ll 1$ and $P \gg 1$, so that $aP \sim O(1)$. This finding implies a drastic simplification of Eqs. (13), where Eq. (13a) is well approximated by Eq. (14a). The proof given in Appendix A allows us to conclude that $f = 0$, as for the oblique modes $k_x \neq 0$ and, hence, $k > 0$. Thus, Eq. (13b) can be approximated as

$$g'' - [k^2 - a^2P^2 + s - ik_x \eta_b(y)P] g = 0,$$

with the term $ik_x \eta_b(y)P$ being the dominant contribution. As a consequence, Eq. (25) implies that $g = 0$ as well. The conclusion is that, in a physically sensible regime for instability where $a \ll 1$ and $P \sim O(a^{-1})$, no instability to oblique normal modes is possible.

### 7. Conclusions

The stability analysis of forced convection with viscous dissipation in a plane parallel porous channel has been carried out. Fully developed conditions have been assumed for the basic flow. Viscous dissipation acts as source of the temperature gradient within the flow region, while variable viscosity yields a thermal coupling between velocity and temperature in the momentum balance, thus expressing the physical cause of a possible flow instability. The stability analysis has been carried out under the assumptions that Darcy’s law holds, that the law expressing the change of fluidity vs. temperature is linear, and that the boundary walls are impermeable while undergoing heat transfer to an external environment (Robin temperature conditions).

The linearised perturbation equations have been solved analytically for normal modes of the longitudinal type. Oblique normal modes have been shown to be ineffective, as for the onset of instability, in a physically sensible regime where the slope parameter of viscosity vs. temperature is very small, and the Péclet number is very large.

The instability of the basic flow to longitudinal normal modes arises for the same conditions leading to a singular behaviour of the basic flow solution. Hence, the main physical fact emerging from our stability analysis is that the basic forced convection flow with viscous dissipation remains stable in the whole parametric domain where the viscosity model adopted to describe it is physically conceivable. We mention that a similar conclusion was drawn by Joseph [3] on studying the dissipation instability of a basic Couette or Poiseuille flow in a clear fluid. As suggested by Joseph’s analysis, what is expected when the linear fluidity model is replaced by more realistic models (Nahme or Arrhenius) is that the basic solution implies dual flows (see also Magyari and Barletta [12] for a second-degree polynomial fluidity model). Joseph [3] proved that the principal branch of the dual flows is stable, while the second branch is not. We expect that a similar behaviour is to be found also for the flow in a porous channel examined in this paper.
We mention that our stability analysis refers just to the thermally developed regime of forced convection. The onset of instability in the thermal entrance region could imply more restrictive conditions for stability, depending on the inlet conditions assumed for the flow at the entrance cross-section.

References
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Appendix A.
A convenient way to express Eq. (14a) with the boundary condition given by Eq. (14c) is
\[
\begin{align*}
[\eta_b(y) f']' - k^2 \eta_b(y) f &= 0, \\
y = \pm 1 &: f' &= 0.
\end{align*}
\] (A.1)

We multiply Eq. (A.1a) by \(\tilde{f}\), where \(\tilde{f}\) is the complex conjugate of \(f\), and integrate over \(y \in [-1, 1]\). Thus, we obtain
\[
\int_{-1}^{1} \tilde{f} \left[ \eta_b(y) f' \right] dy - k^2 \int_{-1}^{1} \eta_b(y) |f|^2 dy = 0.
\] (A.2)

Integration by parts for the first term of Eq. (A.2), yields
\[
\eta_b(1) \tilde{f}(1) f'(1) - \eta_b(-1) \tilde{f}(-1) f'(-1) - \int_{-1}^{1} \eta_b(y) |f'|^2 dy - k^2 \int_{-1}^{1} \eta_b(y) |f|^2 dy = 0.
\] (A.3)

By employing the boundary condition, Eq. (A.1b), we further simplify to
\[
\int_{-1}^{1} \eta_b(y) |f'|^2 dy + k^2 \int_{-1}^{1} \eta_b(y) |f|^2 dy = 0.
\] (A.4)

By recalling that \(\eta_b(r) > 0\), we conclude that Eq. (A.4) can be satisfied only with \(f(r) = constant\), when \(k = 0\), or with \(f(r) = 0\), when \(k > 0\).