THE CONJUGATE GRADIENT ALGORITHM ON WELL-CONDITIONED WISHART MATRICES IS ALMOST DETERMINISTIC

PERCY DEIFT AND THOMAS TROGDON

Abstract. We prove that the number of iterations required to solve a random positive definite linear system with the conjugate gradient algorithm is almost deterministic for large matrices. We treat the case of Wishart matrices $W = XX^*$ where $X$ is $n \times m$ and $n/m \sim d$ for $0 < d < 1$. Precisely, we prove that for most choices of error tolerance, as the matrix increases in size, the probability that the iteration count deviates from an explicit deterministic value tends to zero. In addition, for a fixed iteration, we show that the norm of the error vector and the norm of the residual converge exponentially fast in probability, converge in mean and converge almost surely.

1. Introduction

The conjugate gradient algorithm (CGA) [HS52] is arguably the most effective iterative method from numerical linear algebra. In exact arithmetic, the algorithm requires at most $n$ iterations to solve a $n \times n$ positive-definite linear system and it often requires many less iterations to compute a good approximate solution. It is exceedingly simple to implement and there are well-known error bounds available. And, despite the fact that the CGA is sensitive to round-off errors these error bounds still effectively hold for floating point arithmetic [Gre89]. While we present the algorithm in full below (see Algorithm 1), the variational characterization of the method is summarized as follows: Consider the linear system $Wx = b$, $W > 0$. Given an initial guess $x_0$, find the unique vector $x_k$ that satisfies

$$\| x - x_k \|_W = \min_{y \in X_k} \| y - x \|_W,$$

$$X_k = x_0 + \text{span}\{r_0, Wr_0, \ldots, W^{k-1}r_0\}, \quad \| y \|_W^2 = y^*Wy, \quad r_0 = b - WX_0.$$

At each step $k$ of the iteration one can easily construct $x_k$ and the algorithm itself computes $r_k = b - WX_k$, $k = 0, 1, 2, \ldots, n$. One has to then determine a computable stopping criterion, and typically, the algorithm is halted when $\| r_k \|_2 < \epsilon$, $\| y \|_2^2 = y^*y$ for a chosen error tolerance $\epsilon$.

Here we focus on two main measures of the error, $e_k(W, b) = e_k := x - x_k$:

$$\| e_k \|_W \quad \text{and} \quad \| r_k \|_2 = \| e_k \|_{W^2}, \quad r_k(W, b) = r_k = b - Ax_k,$$
when $x_0 = 0$. The associated halting times are

$$
t^{(1)}(W, b) = \min\{k : \|e_k\|_W < \epsilon\},
$$

$$
t^{(2)}(W, b) = \min\{k : \|r_k\|_2 < \epsilon\}.
$$

We emphasize the importance of analyzing both quantities because $r_k$ is what is observed throughout the iteration and, of course, $e_k$ is the true error.

While our results (Theorems 3.1, 3.2, 3.3) also hold for complex Gaussian matrices, we state some consequences for real matrices for simplicity. Assume

$$
W = XX^*/m,
$$

where $X$ is an $n \times m$ matrix whose entries are iid standard normal random variables. This is the real Wishart distribution. Suppose further that $m = \lfloor n/d \rfloor$ for $0 < d \leq 1$. Then if $b$ is a random unit vector, independent of $W$ as $n \to \infty$

$$
\|r_k(W, b)\|_2 \text{ almost surely} \to d^{k/2}.
$$

If $d < 1$ then as $n \to \infty$

$$
\|e_k(W, b)\|_W \text{ almost surely} \to \frac{d^{k/2}}{\sqrt{1-d}}.
$$

Furthermore, for almost every choice of $\epsilon$, with $\epsilon$ fixed, we prove that

$$
\lim_{n \to \infty} \mathbb{P}(t^{(1)}(W, b) = \left\lfloor \frac{2\log \epsilon + \log(1-d)}{\log d} \right\rfloor) = 1, \quad \epsilon < (1-d)^{-1},
$$

$$
\lim_{n \to \infty} \mathbb{P}(t^{(2)}(W, b) = \left\lfloor \frac{2\log \epsilon}{\log d} \right\rfloor) = 1, \quad \epsilon < 1.
$$

Therefore, the halting time becomes effectively deterministic. We also present estimates that demonstrate the probability that the errors deviate from their means decays exponentially with respect $n$. In the case $d = 1$, a consequence of our results is that for any fixed $k > 0$ and $\epsilon < 1$,

$$
\lim_{n \to \infty} \mathbb{P}(t^{(2)}(W, b) > k) = 1.
$$

Remark 1.1. It is important to point out that $W$ in (1.2) is not necessarily a near-identity matrix. Indeed, the eigenvalues typically lie in the interval

$$
[(1 - \sqrt{d})^2, (1 + \sqrt{d})^2],
$$

and have an asymptotic density given by the famed Marchenko–Pastur law, see Definition 2.1.

Our proofs make critical use of the invariance of the Wishart distribution and the relation between Householder bidiagonalization and the Lanczos iteration. This allows one to use classical estimate on chi-distributed random variables in a crucial way. The specific tools and results we incorporate from random matrix theory include global eigenvalue estimates [RV10], the convergence of the empirical spectral measure [BMP07] and the central limit theorem for linear statistics [LP09].

The remainder of the paper is setup as follows. In Section 1.1 we compare our analysis with facts already known about the conjugate gradient algorithm. We also demonstrate our results with numerical examples. In Section 2 we introduce our random matrix ensembles, the basic definitions from random matrix theory and we state some consequences for real matrices for simplicity. Assume

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Furthermore, for almost every choice of $\epsilon$, with $\epsilon$ fixed, we prove that

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The remainder of the paper is setup as follows. In Section 1.1 we compare our analysis with facts already known about the conjugate gradient algorithm. We also demonstrate our results with numerical examples. In Section 2 we introduce our random matrix ensembles, the basic definitions from random matrix theory and
review the Householder bidiagonalization procedure applied to these ensembles. We also review the connections between the conjugate gradient algorithm, the Lanczos iteration and the Householder bidiagonalization procedure. In Section 3 we present our main theorems. In Section 4 we introduce the results from probability and random matrix theory that are required to prove our theorems. In Section 5 we give the proofs of the theorems.

1.1. Comparison and demonstration. We now give a demonstration and discussion of the results. In what follows $⟨·⟩$ denotes the sample average of a random variable using $20,000$ samples. We will refer to the matrix $W = XX^*/m$ where $X$ is an $n \times m$ matrix, having iid entries, $X_{11} = \pm 1$ with equal probability, as the Bernoulli ensemble.

1.1.1. A numerical demonstration. To demonstrate our main results, in Figure 2 we plot the following quantities as a function of $k$ for different values of $n$

$$\langle \|e_k(W, b)\|_W \rangle \quad \text{and} \quad \frac{d^{k/2}}{\sqrt{1 - d}}$$

with error bars that indicate where 99.9% of the samples lie. In Figure 1 we plot the same statistics for $\langle \|r_k(W, b)\|_2 \rangle$ compared with $d^{k/2}$.

Both Figure 2 and Figure 1 demonstrate the concentration of the errors about their means. We demonstrate the limiting behavior of the halting times $t^{(j)}$ in Figure 3.

In all of these figures we have included computations for distributions of random matrices that are beyond the class for which our results apply. Nonetheless, it is clear that the behavior persists. This universality will be investigated in future work.

1.1.2. Relation to previous work. The classical error estimate for the CGA is [HS52, Gre89]

$$\|e_k(W, b)\|_W \leq 2 \left[ \left( \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^k + \left( \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^{-k} \right]^{-1} \|e_0(W, b)\|_W,$$

where $\kappa = \lambda_1/\lambda_n$ is the condition number of $W$. Here $\lambda_1 \geq \cdots \geq \lambda_n > 0$ are the eigenvalues of $W$. It is a classical result in random matrix theory [BY93] that the condition number of (1.2) converges almost surely to $(1 + \sqrt{d})^2/(1 - \sqrt{d})^2$. Roughly, one then obtains

$$\|e_k(W, b)\|_W \lesssim 2 \left[ d^{k/2} + d^{-k/2} \right]^{-1} \|e_0(W, b)\|_W,$$

which is often just simplified to

$$\|e_k(W, b)\|_W \lesssim 2d^{k/2} \|e_0(W, b)\|_W.$$

This overestimates the actual error by just a factor of 2.

In [MT16], the authors used (1.3) and tail bounds on the condition number to estimate the halting times (1.1) in the case $d = 1 + o(1)$. A key observation was that the actual number of iterations appears to be of the same asymptotic order as the estimate obtained using (1.1). This is something that will indeed be true if the
Figure 1. A demonstration that $\|r_k\|_2$ concentrates strongly around is mean, which is nearly equal to $d^{k/2}$ (solid). This plot is for $d = 0.2$. The dots give the sample mean over 20,000 samples and the error bars give the symmetric interval where 99.9% of the samples lie. It is clear that this interval shrinks rapidly as $n$ increases.

error estimate used decays exponentially and turns out to be an overestimate by a constant factor.

Remark 1.2. Of particular interest is this case where $d$ depends on $n$ and $d \to 1$ as $n \to \infty$. For example, $d = 1 - 1/n^{1/2}$ was seen in [DMT16, DMOT14] to produce universal fluctuations for the halting times. Similarly, one would want to treat the case $\epsilon = \epsilon(n) \to 0$ as $n \to \infty$.

Remark 1.3. Our calculations in this work apply only to matrices with Gaussian entries. An important question, one of universality, is if our results hold if this assumption is relaxed. Indeed, one expects this to be true by the computations in Figures 2 and 5 and the wealth of theoretical results from random matrix theory [PY14, BKYY16, BMP07].

2. The bidiagonalization of Wishart matrices and invariance

Definition 2.1. For $0 < d \leq 1$ set $m = \lfloor n/d \rfloor$. Let $X$ be an $n \times m$ matrix of iid standard normal random variables ($\beta = 1$) or $X = X_1 + iX_2$ where $X_1$ and $X_2$ are independent copies of an $n \times m$ matrix of iid standard normal random variables ($\beta = 2$). Then

$$W_{n, \beta, d} := \frac{1}{\beta m} XX^*$$
Figure 2. A demonstration that $\|e_k\|_W$ concentrates strongly around its mean, which is nearly equal to $d^{k/2}/\sqrt{1-d}$ (solid). This plot is for $d = 0.2$. The dots give the sample mean over 20,000 samples and the error bars give the symmetric interval where 99.9% of the samples lie. It is clear that this interval shrinks rapidly as $n$ increases.

Figure 3. A demonstration that $t_{e}^{(2)}$ becomes almost deterministic as $n$ increases for $d = 0.2$ and $\beta = 1$ using 20,000 samples. Each panel gives the empirical halting time distribution for the indicated value of $n$. The histogram is extremely concentrated.

The CGA is almost deterministic

$\beta = 1, n = 200$

$\beta = 1, n = 800$

BE, $n = 200$

BE, $n = 800$

$k$

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Figure 4. A demonstration of how the Marchenko–Pastur law relates to the spectrum of a Wishart matrix for \( n = 400 \). A histogram for the eigenvalues of one sampled matrix closely approximates the Marchenko–Pastur density.

where \( \lambda_1(n, \beta, d) \geq \lambda_2(n, \beta, d) \geq \cdots \geq \lambda_n(n, \beta, d) \) are the eigenvalues of \( W_{n, \beta, d} \).

Define the averaged EMS \( E_{\mu_{n, \beta}} \) (or density of states) by

\[
(2.2) \quad \int f(\lambda)E_{\mu_{n, \beta,d}}(d\lambda) := E\left(\int f(\lambda)\mu_{n, \beta,d}(d\lambda)\right)
\]

for every \( f \in C_c(\mathbb{R}^+) \).

**Definition 2.2.** Marchenko–Pastur law \( \mu_{MP,d} \) on \( \mathbb{R} \) is given by the density

\[
(2.3) \quad \rho_{MP,d}(x) = \frac{1}{2\pi d} \sqrt{(d_+ - x)(x - d_-)} \frac{1}{x} \mathbbm{1}_{(d_-, d_+)}(x), \quad d_\pm = (1 \pm \sqrt{d})^2.
\]

The relation of the Marchenko–Pastur law to the eigenvalues of a Wishart matrix is given in the following section. But we demonstrate this relationship in Figure 4.

The Wishart matrix the distribution is invariant under orthogonal (\( \beta = 1 \)) or unitary (\( \beta = 2 \)) conjugation. Using \( \beta = 2 \), This means that if \( U \) is a random unitary matrix that is independent of \( W_{n, \beta, d} \) then

\[
UW_{n, \beta, d}U^* \overset{\text{dist.}}{=} W_{n, \beta, d}.
\]

For \( W_{n, \beta, d} = \frac{1}{\beta n}XX^* \) the Householder bidiagonalization procedure [TB97] operates on \( X \) on the left and the right with Householder reflections \( R_1, R_2, \ldots, R_n, \tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_n \) so that

\[
(2.3) \quad [H_{n, \beta, d} \ 0] = R_n R_{n-1} \cdots R_1 X \tilde{R}_1 \tilde{R}_2 \cdots \tilde{R}_n
\]

\[
(2.4) \quad = \begin{bmatrix}
\zeta_{11} & 0 & \cdots & 0 \\
\zeta_{21} & \zeta_{22} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\zeta_{n,n-1} & \zeta_{nn} & 0 & \cdots & 0
\end{bmatrix}
\]

\( C_c(\mathbb{R}^+) \) denote continuous functions with compact support in \([0, \infty)\).
where all entries are non-negative. Because of invariance, \( \{ \zeta_{ij} \} \) are independent \( \chi \)-distributed random variables, see [DE02] and the references therein. Specifically,

\[
H_{n,\beta,d} \overset{\text{dist.}}{=} \begin{bmatrix}
\chi_{\beta m} & \chi_{\beta (n-1)} & \chi_{\beta (n-2)} & \cdots & \chi_{\beta (m-1)} & \chi_{\beta (m-2)} & \cdots & \chi_{\beta (m-n+1)} \\
\chi_{\beta (n-1)} & \chi_{\beta (n-2)} & \cdots & \chi_{\beta (m-2)} & \cdots & \chi_{\beta (m-n+1)} \\
\chi_{\beta (n-2)} & \cdots & \chi_{\beta (m-2)} & \chi_{\beta (m-n+1)} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\chi_{\beta} & \chi_{\beta} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\chi_{\beta} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\chi_{\beta} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}
\]

where all entries are independent. Define the infinite matrix \( T_d \) by the entry-wise limit

\[
(T_d)_{ij} := \lim_{n \to \infty} \frac{1}{\beta m} \mathbb{E} \left[ (H_{n,\beta,d} H^*_{n,\beta,d})_{ij} \right], \quad 1 \leq i, j.
\]

Therefore

\[
T_d = \mathbb{H}_d \mathbb{H}_d^*,
\]

\[
\mathbb{H}_d = \begin{bmatrix}
1 & \frac{1}{\sqrt{d}} & 1 & \cdots & \cdots & \cdots & \cdots \\
\frac{1}{\sqrt{d}} & 1 & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}
\]

Lastly, define \( T_{k,d} \) to be the upper-left \( k \times k \) submatrix of \( T_d \).

2.1. Householder bidiagonalization, the Lanczos iteration and the CG algorithm. The conjugate gradient algorithm (CGA) for the iterative solution of

\[
Wx = b, \quad W > 0
\]

is given by

\[
\text{Algorithm 1: Conjugate Gradient Algorithm}
\]

\begin{itemize}
  \item (1) \( x_0 \) is the initial guess.
  \item (2) Set \( r_0 = b - Wx_0, \quad p_0 = r_0. \)
  \item (3) For \( k = 1, 2, \ldots, n \)
    \begin{itemize}
      \item (a) Compute \( a_{k-1} = \frac{r_{k-1}^* r_{k-1}}{r_{k-1}^* W p_{k-1}}. \)
      \item (b) Set \( x_k = x_{k-1} + a_{k-1} p_{k-1} \).
      \item (c) Set \( r_k = r_{k-1} - a_{k-1} W p_{k-1} \).
      \item (d) Compute \( b_{k-1} = -\frac{r_k^* r_k}{r_{k-1}^* r_{k-1}}. \)
      \item (e) Set \( p_k = r_k - b_{k-1} p_{k-1} \).
    \end{itemize}
\end{itemize}

The error at step \( k \) is given by \( e_k = x - x_k \). Define the norm \( \| y \|^2_W = e_k^* W e_k \). A variational characterization of the CGA is that

\[
\| e_k \|_W = \| p_k^* (W) e_0 \|_W = \min_{p_k \in \mathbb{P}^{(0)}_k} \| p_k (W) e_0 \|_W,
\]

where \( \mathbb{P}^{(0)}_k = \{ p : p \text{ is a polynomial of degree } k, \ p(0) = 1 \} \). The unique minimizer \( p_k^* \) in \( \mathbb{P}^{(0)}_k \) can be described the the Lanczos algorithm. The Lanczos algorithm is a
tridiagonalization algorithm given by

\begin{algorithm}[H]
\textbf{Algorithm 2: Lanczos Iteration}

1. $y_1$ is the initial vector. Suppose $\|y_1\|_2^2 = y_1^*y_1 = 1$

2. Set $\beta_0 = 0$

3. For $k = 1, 2, \ldots, n$
   a. Compute $\alpha_k = (Wy_k - \beta_{k-1}y_{k-1})^*y_k$.
   b. Set $v_k = Wy_k - \alpha_ky_k - \beta_{k-1}y_{k-1}$.
   c. Compute $\beta_k = \|v_k\|_2$ and if $\beta_k \neq 0$, set $y_{k+1} = v_k/\beta_k$.

The Lanczos algorithm produces a tridiagonal matrix $T$

\[ T = T(W, y_1) = \begin{bmatrix}
\alpha_1 & \beta_1 \\
\beta_1 & \alpha_2 & \ddots \\
& \ddots & \ddots & \beta_{n-1} \\
& & \beta_{n-1} & \alpha_n
\end{bmatrix} \]

and $T = QWQ^*$. We use $T_k = T_k(W, y_1), k = 1, 2, \ldots, n$ to denote the upper-left $k \times k$ subblock of $T$. Then, it is well-known that

\[ p_k^*(\lambda) = \frac{\det \left( T_k \left( W, \frac{r_0}{\|r_0\|} \right) - \lambda I \right)}{\det T_k \left( W, \frac{r_0}{\|r_0\|} \right)} \]

Then, write $W = U\Lambda U^*$, $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and

\[ \|e_k\|_{W^*}^2 = \|W^{t/2}p_k^*(W)e_0\|^2 = \|\Lambda^{t/2}p_k^*(\Lambda)U^*e_0\|^2 = \|\Lambda^{t/2}p_k^*(\Lambda)\Lambda^{-1}U^*b_0\|^2 \]

\[ = \sum_{j=1}^n \lambda_j^{t/2}p_k^*(\lambda_j)^2\omega_j, \quad \omega = |U^*b_0|^2. \]

Now, consider the special case $x_0 = 0$ and $b = b_0 := [1, 0, \ldots, 0]^T$, so that $r_0 = b_0$. We further analyze the relation $Q^*TQ = W$ with $y_1 = b_0$. The Lanczos algorithm gives the matrix representation of $W$ in the orthonormal basis found by applying the Gram–Schmidt process to the sequence $y_1, Wy_1, \ldots, W^{n-1}y_1$.

So, the first vector is $b_0$, and so the first column of $Q$ is $b_0$. The main consequence of this is that that first components of the eigenvectors of $W$ are the same as those of $T$.

Finally, we make a simple observation that the Householder bidiagonalization procedure [TB97] applied to $X$ where $W = XX^*$ (or Householder tridiagonalization applied to $W$) leaves the eigenvalues of $W$ unchanged and also leaves the first components of the eigenvectors unchanged. So, provided that Lanczos completes ($\beta_k \neq 0$ for $k = 1, \ldots, n-1$), the Householder bidiagonalization must produce $T(W, b_0)$. This is indeed true because a Jacobi matrix is uniquely defined by eigenvalues and first-components of eigenvectors [DLT85].

\footnote{This is true modulo permutations and normalizations.}
3. Main results

The proof of our main theorem is given in Section 5. The convention used in this paper is that $\beta$ and $d$ are fixed constants. The symbols $C, c, C', c'$ with an assortment subscripts will be used to denote constants and their (possible) dependencies. We suppress any dependence of these constants on $\beta$ but include dependence on $d$, with a view to forthcoming work where we will allow $d$ to vary.

**Theorem 3.1.** Assume the conjugate gradient algorithm is applied to solve $W_{n,\beta, d}x = b$ where $\|b\|_2 = 1$ is a (possibly) random vector, independent of $W_{n,\beta, d}$ and $0 < d < 1$. Let $e_k = x - x_k$, $k = 0, 1, 2, \ldots$ be the associated error vectors.

1. For any fixed $\ell \in \mathbb{Z}$ and $n > 1$ there exists a constant $C_{\ell,d,k}$ such that
   \[
   \mathbb{E} \left[ \|e_k\|_{W^\ell}^2 - \int \lambda^{\ell-2} \det(T_{k,d} - \lambda I)^2 \mu_{MP,d}(d\lambda) \right] \leq C_{\ell,d,k} \frac{\log n}{\sqrt{n}}.
   \]

2. Furthermore
   \[
   \mathbb{P} \left( \|e_k\|_{W^\ell}^2 - \int \lambda^{\ell-2} \det(T_{k,d} - \lambda I)^2 \mu_{n,\beta, d}(d\lambda) > t \right) \leq C'_{\ell,d,k} \frac{n e^{-c_{\ell,d,k} n h(t)}}{t},
   \]
   for some constants $C'_{\ell,d,k}, C_{\ell,d,k} > 0$ and a non-decreasing function $h(t)$ that satisfies $h(t) > 0$ for $t > 0$.

3. Lastly if $d = 1$ and $\ell \geq 2$ (1) and (2) hold.

**Theorem 3.2.** In the setting of Theorem 3.1, for $0 < d < 1$, and $\ell \in \mathbb{Z}$ define

\[
\ell_{\ell,k,d} := \int \lambda^{\ell-2} \det(T_{k,d} - \lambda I)^2 \mu_{MP}(d\lambda).
\]

Then as $k \to \infty$, $\ell_{\ell,k,d} \to 0$. Furthermore,

\[
\ell_{\ell,k,d} = \frac{d^k}{1 - d}.
\]

For $d = 1$, (3.1) holds if $\ell \geq 2$ and for $0 < d \leq 1$

\[
e_{\ell,k,d}^2 = d^k,
\]

\[
e_{3,k,d}^2 = d^k \begin{cases} 1 + d & k \geq 1, \\ 1 & k = 0. \end{cases}
\]

**Corollary 3.2.1.** In the setting of Theorem 3.1, for $0 < d < 1$ and $\ell \in \mathbb{Z}$ and $n > 1$

\[
\|e_k\|_{W^\ell} \stackrel{a.s.}{\to} \ell_{\ell,k,d},
\]

\[
\mathbb{E} [\|e_k\|_{W^\ell} - \ell_{\ell,k,d}] \leq C_{\ell,d,k} \frac{\log n}{\sqrt{n}}.
\]

If $d = 1$ this holds for $\ell \geq 2$.

**Proof.** The first claim follows from the Borel–Cantelli Lemma. The second follows from the observation

\[
\|e_k\|_{W^\ell}^2 - \ell_{\ell,k,d}^2 = (\|e_k\|_{W^\ell} - \ell_{\ell,k,d}) (\|e_k\|_{W^\ell} + \ell_{\ell,k,d}),
\]

which gives

\[
\|e_k\|_{W^\ell} - \ell_{\ell,k,d} \leq \ell_{\ell,k,d}^{-1} \|e_k\|_{W^\ell}^2 - \ell_{\ell,k,d}^2.
\]
The last of our main results is almost just a corollary of the above theorems and it concerns halting times (i.e. runtimes or iteration counts, recall (1.1)):

\[ t^1_\epsilon = t^1_\epsilon(W_{n,\beta,d}, b) = \min \{ k : \| e_k \|_{W_{n,\beta,d}} < \epsilon \}, \]

\[ t^2_\epsilon = t^2_\epsilon(W_{n,\beta,d}, b) = \min \{ k : \| r_k \|_2 < \epsilon \}. \]

Since \( \| e_k \|_W \) converges almost surely to \( \epsilon_{1,k,d} \) and \( \| r_k \|_2 = \| e_k \|_{W^2} \) converges almost surely to \( \epsilon_{2,k,d} \) we produce the candidate limit halting times

\[ \tau^1_\epsilon(\beta, d) = \left\lceil \frac{2 \log \epsilon + \log(1 - d)}{\log d} \right\rceil, \quad \epsilon < (1 - d)^{-1}, \]

\[ \tau^2_\epsilon(\beta, d) = \left\lceil \frac{2 \log \epsilon}{\log d} \right\rceil, \quad \epsilon < 1 \]

**Theorem 3.3.** In the setting of Theorem 3.1, for \( 0 < d < 1 \), for \( \ell = 1, 2 \) suppose that \( \epsilon \neq \epsilon_{\ell,k,d} \) for \( k = 0, 1, 2, \ldots \), \( \epsilon < \epsilon_{\ell,0,d} \), then

\[ \lim_{n \to \infty} \mathbb{P} \left( t^\ell_\epsilon(W_{n,\beta,d}, b) = \tau^\ell_\epsilon(\beta, d) \right) = 1. \]

If \( \epsilon = \epsilon_{\ell,k,d} \), i.e. \( k = \tau^\ell_\epsilon(\beta, d) \), for \( k > 0 \) then

\[ \lim_{n \to \infty} \mathbb{P} \left( t^\ell_\epsilon(W_{n,\beta,d}, b) = \tau^\ell_\epsilon(\beta, d) \right) = 1. \]

**Proof.** First assume \( \epsilon \neq \epsilon_{\ell,k,d} \) for \( k = 0, 1, 2, \ldots \), and let \( \delta \geq \min_k |\epsilon^2 - \epsilon^2_{\ell,k,d}| > 0 \). Note that if \( \kappa = \tau^\ell_{n,d,\epsilon,\beta} \) then

\[ \epsilon_{\ell,n,1,d}^2 > \epsilon^2 > \epsilon_{\ell,n,d}^2. \]

Then

\[ \mathbb{P} \left( t^\ell_\epsilon \neq \tau^\ell_\epsilon(\beta, d) \right) \leq \mathbb{P} \left( t^\ell_\epsilon < \tau^\ell_\epsilon(\beta, d) \right) + \mathbb{P} \left( t^\ell_\epsilon > \tau^\ell_\epsilon(\beta, d) \right) \]

We estimate these two terms individually. First,

\[ \mathbb{P} \left( t^\ell_\epsilon < \tau^\ell_\epsilon(\beta, d) \right) = \mathbb{P} \left( \| e_{k-1} \|_{W^\ell} < \epsilon^2, k = \tau^\ell_\epsilon(\beta, d) \right) \]

\[ \leq \mathbb{P} \left( \| e_{k-1} \|_{W^\ell} < \epsilon_{\ell,k-1,d}^2 - \delta, k = \tau^\ell_\epsilon(\beta, d) \right) \]

For sufficiently large \( n \), by Lemma 4.6

\[ \left| \epsilon_{\ell,k-1,d}^2 - \int \lambda^{\ell-2} \det(T_{k,d} - \lambda I)^2 \mu_{n,\beta,d}(d\lambda) \right| \leq \frac{\delta}{2} \]

and for such a value of \( n \)

\[ \mathbb{P} \left( \| e_{k-1} \|_{W^\ell} < \epsilon_{\ell,k-1,d}^2 - \delta, k = \tau^\ell_\epsilon(\beta, d) \right) \]

\[ \leq \mathbb{P} \left( \| e_{k-1} \|_{W^\ell} < \int \lambda^{\ell-2} \det(T_{k,d} - \lambda I)^2 \mu_{n,\beta,d}(d\lambda) - \frac{\delta}{2}, k = \tau^\ell_\epsilon(\beta, d) \right) \to 0 \]

by Theorem 3.1(2). The estimate for \( \mathbb{P} \left( t^\ell_\epsilon > \tau^\ell_\epsilon(\beta, d) \right) \) is analogous. \( \square \)

**Remark 3.4.** We conjecture that if \( \epsilon = \epsilon_{\ell,k,d} \), i.e \( k = \tau^\ell_\epsilon(\beta, d) \), for \( k > 0 \) then

\[ \lim_{n \to \infty} \mathbb{P} \left( t^\ell_\epsilon(W_{n,\beta,d}, b) = \tau^\ell_\epsilon(\beta, d) \right) = \frac{1}{2} = \lim_{n \to \infty} \mathbb{P} \left( t^\ell_\epsilon(W_{n,\beta,d}, b) = \tau^\ell_\epsilon(\beta, d) + 1 \right). \]
THE CGA IS ALMOST DETERMINISTIC

The CGA is almost deterministic.

Figure 5. A demonstration that $g_k := \|r_k\|^2 - d^k$ is appears asymptotically normal as $n$ increases. The top row demonstrates this for case $\beta = 1$ and the bottom row demonstrates this for the case where $X_{11} = \pm 1$ with equal probability ($X$ still has iid entries), i.e. the Bernoulli Ensemble. Specifically, we plot histograms for $g_k/(\langle g_k^2 \rangle)^{1/2}$ against a standard Gaussian density (black).

Indeed Figure 5 indicates this is true because

$$
\|r_k\|^2 - d^k
$$

appears to be asymptotically normal with a variance that decays like $1/n$. We note that this is related to, but not a consequence of, the central limit theorem for linear spectral statistics (CLT for LSS). For the CLT for LSS the variance decays as $1/n^2$.

In the case at hand, the fluctuations that occur in the random weights (see $\omega_j$ in (5.1)) and the fluctuations that occur in the random polynomial $p^*_k$ contribute to the variance on the order of $1/n$. This conjecture will be resolved in a forthcoming publication.

4. Technical results from random matrix theory

Lemma 4.1. Let $\chi_k$ be a chi distributed random variable with $k$ degrees of freedom. Then for any fixed positive integers $p$ and $q$ there exists $C_{q,p} > 0$ such that

$$
E \left| \left( \frac{\chi_k}{\sqrt{k}} \right)^p - 1 \right|^{2q} \leq C_{q,p} k^{-q}.
$$

Furthermore, for $t \geq 0$,

$$
P \left( \frac{\chi_k}{\sqrt{k}} - 1 \geq t \right) \leq \frac{1 + O(k^{-1})}{\sqrt{2\pi}} \int_{\sqrt{k}t}^{\infty} e^{-x^2/2} dx.
$$

For $-1 + \epsilon \leq t \leq 0$

$$
P \left( \frac{\chi_k}{\sqrt{k}} - 1 \leq t \right) \leq \frac{1 + O(k^{-1})}{\epsilon \sqrt{2\pi}} \int_{\sqrt{k}t}^{\infty} e^{-x^2/2} dx.
$$
Proof. Because $\chi_k$ has a density given by
\[
\frac{x^{k-1}e^{-x^2/2}}{2^{k/2-1}\Gamma(k/2)}
\]
we are led to analyze
\[
\frac{1}{2^{k/2-1}\Gamma(k/2)} \int_0^\infty \left| \left( \frac{x}{\sqrt{k}} \right)^p - 1 \right|^2 q x^{k-1}e^{-x^2/2} d\lambda.
\]
The result follows by the change of variables $x = \sqrt{k}y$ and applying the method of steepest descent (Laplace’s method) for integrals along with Stirling’s approximation. For the second inequality one has to use
\[
\frac{(y + 1)^2}{2} - \log(y + 1) - 1/2 \geq \frac{y^2}{2}.
\]
The last follows from
\[
\frac{1}{2^{k/2-1}\Gamma(k/2)} \int_0^{\sqrt{k}(t+1)} x^{k-1}e^{-x^2/2} dx \leq \frac{1}{2^{k/2-1}\Gamma(k/2)} \int_0^{\sqrt{k}(t+1)} x^{k-1} dx \leq 1 + O(k^{-1}) e^{k/2} (t + 1)^k,
\]
and $1/2 + \log(t + 1) \leq 1/2 + t$. $\square$

A good reference for the next classical result is [Ver18, Section 2.8].

**Theorem 4.2** (Bernstein’s inequality for sub-exponential random variables). Let $(X_i)_{i \geq 1}$ be a sequence of independent mean zero random variables satisfying
\[
\|X_i\|_{\psi_1} := \inf\{t > 0 : \mathbb{E} \exp(|X|/t) \leq 2\}.
\]
Then for $t \geq 0$
\[
\mathbb{P} \left( \left| \sum_{j=1}^n a_i X_i \right| \geq t \right) \leq 2 \exp \left( -c \min \{ \frac{t^2}{K^2\|a\|_2^2}, \frac{t}{K\|a\|_\infty} \} \right),
\]
and $K = \max_{1 \leq i \neq n} \|X_i\|_{\psi_1}$.

This theorem then gives the estimate on $\chi_{n,\beta}^2 = \sum_{j=1}^n \chi_\beta^2$
\[
\mathbb{P} \left( \frac{\chi_{n,\beta}^2}{\beta n} - 1 \geq t \right) \leq 2 \exp \left( -c \min \{ \frac{t^2}{\beta^2 K^2}, \frac{t}{K} \} \right).
\]

We will use three elementary facts that are encapsulated in the following lemma.

**Lemma 4.3.** Let $Z_1, Z_2, Y$ be random variables and assume $\mathbb{P}(Y = 0) = 0$. The following inequalities hold
\[(1) \quad \mathbb{P} \left( \left| \frac{Z_1}{Y} \right| \geq t \right) \leq \mathbb{P} \left( \left| Z_1 \left[ \frac{1}{|Y|} - \frac{1}{\mu} \right]_+ \right| + \frac{|Z_1|}{\mu} \geq t \right) \quad \text{where } [\cdot]_+ \text{ denotes the positive part and } \mu > 0,
\[(2) \quad \mathbb{P}(|Z_1| + |Z_2| \geq t) \leq \mathbb{P}(|Z_1| \geq t/p) + \mathbb{P}(|Z_2| \geq t/q), \quad 1/p + 1/q = 1 \text{ and}
\[(3) \quad \mathbb{P}(|Z_1||Z_2| \geq t) \leq \mathbb{P}(|Z_1| \geq t^{1/2}) + \mathbb{P}(|Z_2| \geq t^{1/2}).\]
Lemma 4.4. Suppose \(-\infty < \lambda_{1,n} < \lambda_{2,n} < \cdots \lambda_{n,n} < \infty\). Let \((\chi_{\beta}^{(j)})_{j\geq 1}\) be independent chi-distributed random variables with \(\beta\) degrees of freedom. Define weights

\[
\omega_j = \frac{(\chi_{\beta}^{(j)})^2}{\sum_k (\chi_{\beta}^{(k)})^2}.
\]

Then the Kolmogorov–Smirnov distance

\[
d_{\text{KS}}(\mu, \nu) := \sup_{x \in \mathbb{R}} |\mu((-\infty, x]) - \nu((-\infty, x])|
\]

of

\[
\mu_n = \sum_j \delta_{\lambda_j,n} \omega_j, \quad \nu_n = \frac{1}{n} \sum_j \delta_{\lambda_j,n},
\]

satisfies

\[
\mathbb{E}[d_{\text{KS}}(\mu_n, \nu_n)] \leq C \frac{\log n}{\sqrt{n}},
\]

and the tail estimate

\[
P(d_{\text{KS}}(\mu_n, \nu_n) \geq t) \leq C_1 e^{-c_1n(\beta t^2 + 1) + C_2ne^{-c_2n(\beta t^2)}}
\]

for absolute constants \(C, C_1, C_2, c_1, c_2 > 0\).

Proof. First, it follows that

\[
d_{\text{KS}}(\mu_n, \nu_n) = \max_{1 \leq k \leq n} \left| \frac{1}{n} \sum_{j=1}^{k} \omega_j - \frac{1}{n} \right|
\]

\[
= \max_{1 \leq k \leq n} \frac{1}{\sum_k (\chi_{\beta}^{(k)})^2} \left| \frac{1}{n} \sum_{j=1}^{k} \left((\chi_{\beta}^{(j)})^2 - \frac{1}{n} \sum_k (\chi_{\beta}^{(k)})^2\right) \right|
\]

\[
= \max_{1 \leq k \leq n} \frac{1}{\sum_k (\chi_{\beta}^{(k)})^2} \left| \frac{n-k}{n} \sum_{j=1}^{k} \left((\chi_{\beta}^{(j)})^2 - \frac{k}{n} (\chi_{\beta}^{(j)})^2\right) \right|
\]

So, we are led to analyze the sums

\[
S_k = \sum_{j=1}^{k} \left(\frac{n-k}{n}\right) (\chi_{\beta}^{(j)})^2 - \sum_{j=k+1}^{n} \frac{k}{n} (\chi_{\beta}^{(j)})^2,
\]

\[
S = \sum_{j=1}^{n} (\chi_{\beta}^{(j)})^2.
\]

which has expected value zero. Bernstein’s inequality gives

\[
P(|S_k| \geq t) \leq 2 \exp\left(-c \min\left\{\frac{4t^2}{K^2n}, \frac{t}{K}\right\}\right) =: F(t).
\]

for absolute constants \(c, K > 0\). From the moment generating function for a chi-square distribution, we have

\[
P(S \leq t) \leq \min_{s > 0} e^{st}(1 + 2s)^{-\frac{n\beta}{2}}.
\]
This minimum occurs at \( s = \frac{n\beta}{2t} - \frac{1}{2} \), giving
\[
P(S \leq n\beta t) \leq e^{\frac{n\beta}{2t}(1-t)\frac{t^2}{2}} = (te^{1-t})^{\frac{n\beta}{t}}.
\]
Then we write for \( 0 \leq s \leq 1 \)
\[
P(S \leq n\beta (1-s)) \leq e^{\frac{n\beta}{2t}(1-s)\frac{t^2}{2}}.
\]
So that
\[
P\left(\frac{1}{S} - \frac{1}{n\beta} \geq \frac{1}{n\beta} \left[ \frac{1}{1-s} - 1 \right]\right) \leq e^{\frac{n\beta}{2t}(1-s)\frac{t^2}{2}}.
\]
Now set \( t = \frac{s}{1-s} - 1 = \frac{s}{t+1} \) to find
\[
P\left(\frac{1}{S} - \frac{1}{n\beta} \geq \frac{t}{n\beta}\right) \leq \left(e^{\frac{t}{t+1}} \left[1 - \frac{t}{t+1}\right]\right)^{\frac{n\beta}{2t}}.
\]
Then it is easy to see that for \( 0 \leq t \leq 1 \)
\[
e^{\frac{t}{t+1} \left[1 - \frac{t}{t+1}\right]} \leq e^{-t^2/2},
\]
giving the estimate
\[
P\left(\frac{n\beta}{S} - 1 \geq t\right) \leq e^{-n\beta t^2/4} =: G(n\beta t).
\]
We then write \( \tilde{S}_k = S_k/(n\beta) \) and \( \tilde{S} = S/(n\beta) \) so that
\[
\frac{|S_k|}{|S|} = \frac{|\tilde{S}_k|}{|\tilde{S}|}
\]
and then we apply each property of Lemma 4.3 in order, to obtain
\[
P\left(\frac{|S_k|}{|S|} \geq t\right) \leq G\left(n\beta \frac{t^{1/2}}{\sqrt{p}}\right) + F\left(n\beta \frac{t^{1/2}}{\sqrt{p}}\right) + G\left(n\beta \frac{t}{q}\right),
\]
for \( 1/p + 1/q = 1 \). The tail estimate (4.2) follows by a union bound. We examine \( F \) more closely, and get a crude bound
\[
F\left(n\beta \frac{t^{1/2}}{\sqrt{p}}\right) = 2\exp\left(-\frac{c}{Kp^{1/2}}n\beta \min\left\{\frac{4\beta t}{Kp^{1/2}}, t^{1/2}\right\}\right).
\]
While we do not specifically need the value, it follows that for a \( \chi \)-squared random variable \( \chi^2_\beta \) with \( \beta \) degrees of freedom \( \|\chi^2_\beta\|_{\psi_1} = \frac{2}{1-(1/2)^{1/2}} \) gives \( K = \|\chi^2_\beta - \beta\|_{\psi_1} < \infty \). Then, choosing \( p \) large enough so that \( r\beta/(Kp^{1/2}) < 1 \) we obtain
\[
F\left(n\beta \frac{t^{1/2}}{\sqrt{p}}\right) \leq 2\exp(-Cnt), \quad 0 \leq t \leq 1.
\]
Then, we can estimate moments
\[
E\left[\left(\frac{S_k}{S}\right)^m\right] = m \int_0^1 t^{m-1}P\left(\frac{|S_k|}{|S|} \geq t\right) dt,
\]
\[
\leq \frac{2m}{(3n/4)^m}\Gamma(m) + \frac{2m}{(Cn)^m}\Gamma(m) + \frac{2m}{(\beta n/4)^{m/2}}\Gamma(m/2).
\]
The $m$th root of this is bounded by $C'n^{-1/2}m$ for some $C' > 0$. Thus $\|n^{1/2} \frac{S_n}{S_0}\|_\psi \leq C''$ for some absolute constant $C''$. From the usual estimate
\[
\mathbb{E} \left[ \max_k |X_k| \right] \leq s \log \mathbb{E} \left[ \exp \max_k |X_k|/s \right] \leq s \log \sum_k \mathbb{E} \left[ \exp |X_k|/s \right],
\]
and choosing $s = \max_k \|X_k\|_{\psi_1}$ we obtain
\[
\mathbb{E} \left[ \max_k |X_k| \right] \leq \log 2n(\max_k \|X_k\|_{\psi_1}).
\]
Thus $\mathbb{E}[d_{KS}(\mu_n, \nu_n)] \leq C\log n$, for some new constant $C$. Hence $d_{KS}(\mu_n, \nu_n)$ converges to zero in $L^1$, in probability and almost surely.\[\square\]

**Theorem 4.5** (Global eigenvalue bounds, [RV10]). For the eigenvalues $\lambda_n \leq \cdots \leq \lambda_1$ of a $\beta$-Wishart distribution
\[
\mathbb{P} \left( 1 - \sqrt{\frac{n}{m}} - t \leq \lambda_n^{1/2} \leq 1 + \sqrt{\frac{n}{m}} + t \right) \geq 1 - 2e^{-cnt^2},
\]
for an absolute constant $c$.

This immediately implies that for any interval $(a, b)$ such that $[(1 - \sqrt{d})^2, (1 + \sqrt{d})^2] \subset (a, b)$ there exists a constant $\gamma = \gamma(a, b)$ such that
\[
(4.3) \quad \mathbb{P}(\lambda_n < a \text{ or } \lambda_1 > b) \leq 2e^{-\gamma t}.
\]
And it also implies the bound on the distribution function for $\lambda_n$. Define $d_n = \frac{n}{m} = d + O(1)$ so
\[
\mathbb{P}(\lambda_1 \geq t) \leq \begin{cases} 
1 & \text{if } t \leq (1 + \sqrt{d_n})^2 \leq (1 + \sqrt{d_n})^2, \\
2e^{-c n \min(s^2/32, s^+)} & \text{if } t > (1 + \sqrt{d_n})^2, s^+ \leq 0,
\end{cases}
\]
\[
\mathbb{P}(\lambda_n \leq t) \leq \begin{cases} 
1 & \text{if } t \geq (1 - \sqrt{d_n})^2 \leq (1 - \sqrt{d_n})^2, \\
2e^{-c n \min(s^2/32, s^-)} & \text{if } t < (1 - \sqrt{d_n})^2, s^- \geq 0,
\end{cases}
\]
where $s^\pm = t - (1 \pm \sqrt{d_n})^2$. And the important conclusion from this is that
\[
(4.4) \quad \mathbb{E}[\lambda_n^{k}] \leq C_k < \infty, \quad k = 0, 1, 2, \ldots,
\]
where $C_k$ is independent of $n$.

**Lemma 4.6.** For fixed $k \in \mathbb{Z}$
\[
m_{k,d,n} := \frac{1}{n} \mathbb{E} \left[ \text{tr} W_{n,\beta,d}^k \right] \rightarrow \int \lambda^k \rho_{MP,\beta}d(\lambda)d\lambda =: m_{k,d}
\]
\[
m_{k,d,n} - m_{k,d} = \begin{cases} 
O(n^{-1}) & k \geq 0, \\
O(n^{-1/2}) & k < 0,
\end{cases}
\]
\[
\frac{1}{n^2} \mathbb{E} \left( \text{tr} W_{n,\beta,d}^k - nm_{k,d,n} \right)^2 = \begin{cases} 
O(n^{-2}) & k \geq 0, \\
O(n^{-1}) & k < 0,
\end{cases}
\]
as $n \rightarrow \infty$.

\[\text{Because } d_{KS}(\mu, \nu) \text{ is always less than or equal to unity, almost sure convergence gives } L^1\text{ convergence, but we have obtained a rate.}\]
Proof. The case of $k \geq 0$ is classical and implies weak convergence of the ESM to the Marchenko–Pastur law, see [BS10, Section 3.1], for example. For $k < 0$ more work is required. Recall (2.2) and the useful fact that

$$\int f(\lambda)\mathbb{E}_{\mu_{n,\beta,d}}(d\lambda) = \frac{1}{n}\mathbb{E} \text{ tr } f(W_{n,\beta,d}).$$

So, we need to choose $f(\lambda) = \lambda^{-k}$ for $k \geq 1$. We introduce a continuous truncation of $f$:

$$g(\lambda) = \begin{cases} 
\left(\frac{2}{d}\right)^k & 0 \leq \lambda \leq \frac{d}{2}, \\
\lambda^{-k} & \text{otherwise}.
\end{cases}$$

Then, consider

$$\left| \int \lambda^{-k}(\mathbb{E}_{\mu_{n,\beta,d}}(d\lambda) - \mu_{\text{MP},d}(d\lambda)) \right| = \left| \int x^{-k}\mathbb{E}_{\mu_{n,\beta,d}}(d\lambda) - \int g(x)\mu_{\text{MP},d}(d\lambda) \right|$$

$$\leq \left| \int (\lambda^{-k} - g(\lambda))\mathbb{E}_{\mu_{n,\beta,d}}(d\lambda) \right| + \left| \int g(\lambda)(\mathbb{E}_{\mu_{n,\beta,d}}(d\lambda) - \mu_{\text{MP},d}(d\lambda)) \right|.$$

We estimate each of these terms separately. First, we use that

$$\left| \int (\lambda^{-k} - g(\lambda))\mathbb{E}_{\mu_{n,\beta,d}}(d\lambda) \right| \leq \int_0^{d_-/2} \lambda^{-k}\mathbb{E}_{\mu_{n,\beta,d}}(d\lambda)$$

$$\leq \mathbb{E} \left[ \lambda_{n}^{-k}(n,\beta,d) \mathbb{I}(\lambda_{n}(n,\beta,d) \leq d_-/2) \right].$$

(4.7)

and show that this tends to zero exponentially. To estimate this expectation, we use estimates on the marginal density $\rho(\lambda)$ for $\lambda_n$. Specifically, (4.3) implies that for any $p > 0$

$$\int_{\frac{d_-}{2}}^{d_-/2} \lambda^{-k}\rho(\lambda)d\lambda \leq C_{k,d} e^{-nc_{k,d}},$$

(4.8)

for some constants $C_{k,p}, c_{k,p}$ that do not depend on $n$. And so, for this term we are left estimating

$$\int_{0}^{\frac{d_-}{2}} \lambda^{-k}\rho(\lambda)d\lambda.$$

From [ES05], it follows that the density $R(\lambda)$ for $\beta m\lambda_1$ is bounded by

$$R(\lambda) \leq n \frac{\Gamma \left(1 + \frac{\beta}{2}\right) \Gamma \left(\frac{\beta}{2}(m + 1)\right)}{\Gamma \left(1 + \frac{\beta}{2}n\right) \Gamma (p + 1)} 2^{-p} \frac{\lambda^{p-1} e^{-\lambda/2}}{\Gamma(p)}.$$

From [BS10], it follows that the density $R(\lambda)$ for $\beta m\lambda_1$ is bounded by
We then have that $n/d \leq m < n/d+1$ and expression we use that $\Gamma(z+a) \sim \Gamma(z) z^a$ for fixed $a$ as $z \to \infty$:

$$n \frac{\Gamma\left(\frac{z}{2} \left(\frac{n}{2} + 2\right)\right)}{\Gamma\left(1 + \frac{z}{2} n\right) \Gamma(p+1)} \sim n \frac{\Gamma\left(\frac{z}{2} m\right) \left(\frac{z}{2} m\right)^{\beta/2}}{\Gamma\left(\frac{z}{2} n\right) \Gamma\left(\frac{z}{2} (m-n)\right) \left(\frac{z}{2} (m-n)\right)^{\beta}}$$

$$\sim 2^{\beta} \left(\frac{\beta}{\pi}\right)^{\beta} \left(\frac{\beta}{\pi}\right)^{(m-n)} \left(\frac{\beta}{\pi}\right)^{(m-n)} \left(\frac{\beta}{\pi}\right)^{(m-n)} =: D_{n,\beta}.$$ 

Then, we estimate

$$D_{n,\beta} \leq 2^{\beta} \frac{1}{\left(\frac{\beta}{\pi}(1-d)\right)^{\beta}} \frac{\Gamma\left(\frac{z}{2} n\right)}{\Gamma\left(\frac{z}{2} (1-d) n\right) \Gamma\left(\frac{z}{2} n\right)}$$

Then the assuming $y = \alpha x + O(1)$ and applying Stirling’s formula gives

$$\frac{\Gamma(x+\alpha x)}{\Gamma(x) \Gamma(\alpha x)} \sim \frac{1}{\sqrt{2\pi}} \left(\frac{\Gamma(1+\alpha)x}{\Gamma(\alpha)\Gamma(x)\alpha x}\right)^{x-1/2}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{(1+\alpha)^{\alpha x-1/2}(1+\alpha)x^x x^{\alpha x-1/2}}{x^{1/2} x^{\alpha x-1/2} x^{\alpha x-1/2}}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{(1+\alpha)^{\alpha x-1/2}}{x^{1/2} x^{\alpha x-1/2}}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{(1+\alpha)^{\alpha x-1/2}}{x^{1/2} x^{\alpha x-1/2}} = O\left((\frac{1+\alpha}{\alpha})^{(1+\alpha)x}\right).$$

To estimate $D_{n,\beta}$ we choose $x = n, y = (1/d - 1)n$ so that

$$D_{n,\beta} = O\left((1-d)^{-n/d}\right).$$

We then bound for some $C_d > 0$

$$\rho(\lambda) = m\beta R(m\beta \lambda) \leq C_d e^{n/d \log(1-d)^{-1}} \frac{2^{-p}}{\Gamma(p)}(m\beta)^p \lambda^{p-1} e^{-m\beta \lambda/2}.$$ 

Then we may estimate

$$E \left[\lambda_n^{k}(n, \beta, d) \mathbb{1}\{\lambda_n(n, \beta, d) \leq \frac{\alpha}{\pi}\}\right] \leq C_d e^{n/d \log(1-d)^{-1}} (m\beta)^k \int_0^1 \lambda^{p-k-1} e^{-\lambda} \frac{d\lambda}{\Gamma(p)}$$

$$\leq C_d e^{n/d \log(1-d)^{-1}} (m\beta)^k \frac{1}{\Gamma(p)} \leq C_{k,d} e^{-c_{k,d} n},$$

for some $C_{k,d}, c_{k,d} > 0$. Indeed, this converges to zero super exponentially. From (4.8) we obtain

$$E \left[\lambda_n^{k}(n, \beta, d) \mathbb{1}\{\frac{\alpha}{\pi(n, \beta, d)} \leq \frac{\alpha}{\pi}\}\right] \leq C_{k,d} e^{-c_{k,d} n},$$
and we may assume these constants are the same. Therefore \( I_1 \leq C_{k,d} e^{-ck,dn} \). To estimate \( I_2 \) we write

\[
I_2 \leq \left( \int_0^\infty |g'(\lambda)| d\lambda \right) d_{KS}(E_{\mu_{n,\beta,d}, \mu_{MP,d}})
\]

Then the Kolmogorov–Smirnov distance is given by [BST10, Theorem 8.10]

\[
d_{KS}(E_{\mu_{n,\beta,d}, \mu_{MP,d}}) = O(n^{-1/2}), \quad n \to \infty.
\]

And therefore \( I_2 = O(n^{-1/2}) \). Finally, to the variance estimate (4.6) for \( k < 0 \) can be established from [LP09, (4.16) and Remark 4.1]

\[
\text{Var} \left( \int g(\lambda) \mu_{n,\beta,d}(d\lambda) \right) \leq C \|g'\|_\infty n^{-2}.
\]

Then (4.6) follows from (4.5). \( \square \)

The final results from random matrix theory come from [GZ00, Corollary 1.8]

**Theorem 4.7.** For any Lipschitz function \( f : \mathbb{R}^+ \to \mathbb{R} \)

\[
P \left( \left| \int f(\lambda) \mu_{n,\beta,d}(d\lambda) - \int f(\lambda) E_{\mu_{n,\beta,d}}(d\lambda) \right| \geq t \right) \leq C_{f,d} e^{-nc_{f,d}},
\]

for some constants \( C_{f,d}, c_{f,d} \).

**Corollary 4.7.1.** Let \( f \) be a continuous function on \((0, \infty)\), Lipschitz in a neighborhood of \([(1 - \sqrt{d})^2, (1 + \sqrt{d})^2]\), with at most polynomial growth at \( 0 \) and at \( \infty \), then

\[
P \left( \left| \int f(\lambda) \mu_{n,\beta,d}(d\lambda) - \int f(\lambda) E_{\mu_{n,\beta,d}}(d\lambda) \right| \geq t \right) \leq C_{f,d} e^{-ntc_{f,d}},
\]

for some constants \( C_{f,d}, c_{f,d} \).

**Proof.** Let \([(1 - \sqrt{d})^2, (1 + \sqrt{d})^2] \subset (a,b), a > 0 \) such that \( f \) is Lipschitz on \([a,b]\). Then define

\[
\tilde{f}(x) = \begin{cases} f(x) & a \leq x \leq b \\ f(a) & x < a \\ f(b) & x > b. \end{cases}
\]

And estimate

\[
\left| \int f(x) \mu_{n,\beta,d}(d\lambda) - E_{\mu_{n,\beta,d}}(d\lambda) \right| \leq \left| \int (f(x) - \tilde{f}(x)) \mu_{n,\beta,d}(d\lambda) - E_{\mu_{n,\beta,d}}(d\lambda) \right| + \left| \int \tilde{f}(x) \mu_{n,\beta,d}(d\lambda) - E_{\mu_{n,\beta,d}}(d\lambda) \right|.
\]

By Theorem 4.7

\[
P \left( \left| \int \tilde{f}(x) \mu_{n,\beta,d}(d\lambda) - E_{\mu_{n,\beta,d}}(d\lambda) \right| \geq t \right) \leq C_{f,d} e^{-ntc_{f,d}},
\]

Using Theorem 4.5

\[
P \left( \int (f(x) - \tilde{f}(x)) \mu_{n,\beta,d}(d\lambda) \neq 0 \right) \leq C_0 e^{-c_0 n}
\]

(4.11)
By assumption, there exists \( p, q > 0 \) such that
\[
f(x) = \begin{cases} 
  C_1 x^{-p} & 0 \leq x \leq 1, \\
  C_2 x^q & x \geq 1.
\end{cases}
\]

Following the proof of Lemma 4.6, we find that
\[
\int_0^\alpha |f(x) - \tilde{f}(x)| \mu_{n, \beta, d}(d\lambda) \leq C_3 e^{-c_3 n}.
\]

Similarly, using Theorem 4.5 (using the same constants, for convenience)
\[
\int_\beta^\infty |f(x) - \tilde{f}(x)| \mu_{n, \beta, d}(d\lambda) \leq C_3 e^{-c_3 n}.
\]

The corollary follows by applying Lemma 4.3(2) twice.

5. Proofs of the main theorems

Proof of Theorem 3.1. We first use invariance. It follows that the errors \( \|e_k\|_{W_n, \beta, d} \) realized in the CGA are invariant under unitary transformations, i.e. for \( \tilde{W} = UWU^* \)
\[
\|e_k(W, b)\|_{W_n, \beta, d} = \|e_k(UWU^*, Ub)\|_{UWU^*}.
\]

for any unitary matrix \( U \). This follows because if \( p_k(\lambda) \) a polynomial of degree \( k \) (assuming \( x_0 = 0 \)) then
\[
\|p_k(W)e_0(W, b)\|^2_{W_n, \beta, d} = \|W^{\ell/2} p_k(W)x\|_2 = \|U^*(UWU^*)^{\ell/2} p_k(UWU^*)UX\|_2.
\]

And so, the minimum over \( p_k \in \mathbb{P}_k^{(0)} \) must be same in both cases. So, by invariance of \( W_n, \beta, d \) it suffices to solve
\[
W_n, \beta, d x = b_0 = [1, 0, \ldots, 0]^T.
\]

We then recall the formula with \( T_k = T_k(W_n, \beta, d, b_0) \)
\[
\|e_k\|^2_{W_n, \beta, d} = \sum_{j=1}^n \lambda_j^{-2} p_k^2(\lambda_j) \omega_j = \int \lambda_j^{-2} \frac{\det(T_k - \lambda J)^2}{\det T_k^2} \nu_{n, \beta, d}(d\lambda).
\]

where
\[
\nu_{n, \beta, d} = \sum_{j=1}^n \delta_{\lambda_j} \omega_j.
\]

Here the distribution of \( \omega \) is parameterized by
\[
\omega = \frac{\nu}{\|\nu\|_1},
\]

where \( \nu \) is a vector of iid \( \chi_\beta \)-squared random variables. The variable \( \nu \) is the square of the first components of the eigenvectors of \( W_n, \beta, d \). It is well-known that the eigenvalues and eigenvectors of \( W_n, \beta, d \) are independent. But, \( T_k \) is dependent on both the eigenvalues and eigenvectors.

Lemma 5.1. For \( n > 0 \),
\[
\mathbb{E} \left[ \left| \int \lambda_j^{-2} \frac{\det(T_k - \lambda J)^2}{\det T_k^2} \nu_{n, \beta, d}(d\lambda) - \int \lambda_j^{-2} \nu_{n, \beta, d}(d\lambda) \right| \right] \leq C \sqrt{n}.
\]
Proof. We begin with a simple observation

\[ \lambda^{j-2} \det(T_k - \lambda I)^2 = \sum_{j=0}^{2k} t_j \lambda^j. \]  

(5.2)

By Lemma 4.1 it follows that for \( q > 0 \), \( E[t^q_j] \leq C_{j,q} \), where the bound is independent of \( n \). Similarly \( E[\lambda^q_i] \leq C_q \) regardless of if \( q \) is positive or negative, see (4.4) and (4.7). Therefore, it suffices to show that

\[ \frac{1}{\det T_k^2} \to 1. \]

This is clear because

\[ \frac{1}{\det T_k^2} - 1 = \sum_{j=1}^{k} \left( \frac{\beta^2 m^2}{\chi^4_{(m-j+1)} \chi^4_{(m-j)}} - 1 \right) \left( \prod_{i=j}^{k} \frac{\beta^2 m^2}{\chi^4_{(m-j)}} \right) \]

and the first term tends to zero in any \( L^p \) norm by Lemma 4.1 at a rate \( n^{-1/2} \), and the second term is bounded uniformly in any \( L^p \) norm. \( \square \)

Next, we argue that while the measure is still random, we can replace the integrand with a deterministic one.

Lemma 5.2. For \( n > 0 \),

\[ E \left[ \left| \int \lambda^{j-2} \det(T_k - \lambda I)^2 \nu_{n,\beta,d}(d\lambda) - \int \lambda^{j-2} \det(T_k - \lambda I)^2 \mu_{n,\beta,d}(d\lambda) \right| \right] \leq C \sqrt{n}. \]

Proof. Write \( \det(T - \lambda I)^2 = \sum_{j=0}^{2k} t_j \lambda^j \). Using the notation of (5.2), it suffices to show that \( |t_j - t_j| \to 0 \) in \( L^2 \) at a rate \( n^{-1/2} \). Consider the product

\[ P = \prod_{j=1}^{k} \chi^{p_j}_{(n-d_j)} \chi^{q_j}_{(m-s_j)} \]

where \( p_j, q_j \in \{0, 1, 2, 3, 4\} \) and \( 0 \leq d_j, s_j \leq k \). Then for \( n > 0 \)

\[ \frac{1}{(\beta m)^{2p+2}} E \left[ \left| P - \prod_{j=1}^{k} (\beta m)^{p_j/2} (\beta m)^{q_j/2} \right|^2 \right]^{1/2} \leq C \sqrt{n}. \]

(5.3)

where \( p = \sum_j p_j \) and \( q = \sum_j q_j \). This follows from Lemma 4.1. Then, one notes that the \( L^2 \) norm of \( t_j - t_j \) can be bounded by a sum of terms of the form (5.3). This establishes this lemma. \( \square \)

Lemma 5.3. For \( n > 1 \),

\[ E \left[ \left| \int \lambda^{j-2} \det(T_k - \lambda I)^2 \nu_{n,\beta,d}(d\lambda) - \int \lambda^{j-2} \det(T_k - \lambda I)^2 \mu_{n,\beta,d}(d\lambda) \right| \right] \leq C \frac{\log n}{\sqrt{n}}. \]
Proof. Write \( f(\lambda) = \lambda^{\ell-2} \det(T_k - \lambda I)^2 \) and integrate by parts

\[
I(f) := \int f(\lambda)(\nu_{n,\beta,d}(d\lambda) - \mu_{n,\beta,d}(d\lambda)) = \int_{\lambda_n}^{\lambda_1} f(\lambda)(\nu_{n,\beta,d}(d\lambda) - \mu_{n,\beta,d}(d\lambda))
\]

\[
= \int_{\lambda_n}^{\lambda_1} f'(\lambda)F_{\beta,d}(x)d\lambda
\]

where \( F_{\beta,d}(x) = \mu_{n,\beta,d}((-\infty, x]) - \nu_{n,\beta,d}((-\infty, x]) \). Therefore

\[
|I(f)| \leq \left( \int_{\lambda_n}^{\lambda_1} |f'(\lambda)|d\lambda \right) d_{KS}(\mu_{n,\beta,d}, \nu_{n,\beta,d}).
\]

Therefore

\[
\mathbb{E}[|I(f)|] = \mathbb{E}\left[ \int_{\lambda_n}^{\lambda_1} |f'(\lambda)|d\lambda \right] \mathbb{E}[d_{KS}(\mu_{n,\beta,d}, \nu_{n,\beta,d})],
\]

by the independence of eigenvalues and eigenvectors \( d_{KS}(\mu_{n,\beta,d}, \nu_{n,\beta,d}) \) is independent of the eigenvectors. Then, we just note that there exists power \( p, q \geq 0 \) such that

\[
|f'(\lambda)| \leq C_k(\lambda^{-p} + \lambda^q),
\]

and therefore

\[
\mathbb{E}\left[ \int_{\lambda_n}^{\lambda_1} |f'(\lambda)|d\lambda \right]
\]

is bounded uniformly in \( n \) by (4.4) and (4.7). The lemma follows from Lemma 4.4.

These three lemmas combined with Lemma 4.6 establishes the Theorem 3.1(1). For the second part, we again establish a series of lemmas.

**Lemma 5.4.** For \( n \geq 0 \)

\[
\mathbb{P}\left( \left| \lambda^{\ell-2} \frac{\det(T_k - \lambda I)^2}{\det T_k^2} \nu_{n,\beta,d}(d\lambda) - \lambda^{\ell-2} \frac{\det(T_k - \lambda I)^2}{\det T_k^2} \nu_{n,\beta,d}(d\lambda) \right| \geq t \right) \leq C e^{-g(t)n}.
\]

for a non-decreasing function \( g(t) \) that satisfies \( g(t) > 0 \) for \( t > 0 \).

**Proof.** For \( t \geq 0 \), let \( \Lambda_d(C) \) be the event on which \( C^{-1} \leq \lambda_n \leq \lambda_1 \leq C \) for \( C > (1 + \sqrt{d})^2 \) and \( 1/C < (1 - \sqrt{d})^{-2} \). Then

\[
\mathbb{P}(\Lambda_d(C)) \geq 1 - 2e^{-ng_d(C)}
\]

where \( g_d(C) > 0 \). This follows from (4.3). Now, we make two elementary observations about

\[
\lambda^{\ell-2} \frac{\det(T_k - \lambda I)^2}{\det T_k^2} = \sum_{j=0}^{2k} \tau_j \lambda^{l+j-2}.
\]

Recall (2.3) and it follows that \( \tau_j = \tau_j(H_n,\beta,d/\sqrt{m}) \) is a Lipschitz function of the entries \((h_{ij})_{i,j} \neq 0 \) of \( H_n,\beta,d/\sqrt{m} \) in any closed \( \epsilon \)-neighborhood \( 0 < \epsilon < 1 \) of \( \mathbb{H}_d \) in
the max norm on lower-triangular matrices. Let $L_{\epsilon,j}$ be the Lipschitz constant. The second observation is to let $Z_{d}(t)$ be the event where

$$\max \left| \mathbb{H}_{d} - \frac{H_{n,\beta,d}}{\sqrt{\beta m}} \right| \leq |t|.$$  

By Lemma 4.1 for $0 < \epsilon \leq t \leq 1$, $P(Z(t)) \geq 1 - C_{k,d} \epsilon^{-n_{k,d}}$ for some constants $C_{k,d}, c_{k,d} > 0$. Therefore

$$1 - C_{k,d} \epsilon^{-n_{k,d}} - 2 \epsilon^{-ng_{d}(C)} \leq P(Z_{d}(t), \Lambda_{d}(C)) \leq P \left( \sum_{j=0}^{k} \tau_{j}(H_{n,\beta,d}/\sqrt{\beta m}) - \tau_{j}(\mathbb{H}_{d}) \right) \leq |t| C_{k,d} \sum_{j=0}^{2k} L_{\epsilon,j}$$

The lemma follows. □

**Lemma 5.5.** For $n \geq 0$

$$P \left( \left| \int \chi^{\ell-2} \det(T_{k} - \lambda I)^{2} \nu_{n,\beta,d}(d\lambda) - \int \chi^{\ell-2} \det(T_{k} - \lambda I)^{2} \mu_{n,\beta,d}(d\lambda) \right| \geq t \right) \leq C e^{-c_{g}(t)n}.$$  

**Proof.** Recalling the notation $\Lambda_{d}(C)$ of the proof of the previous lemma, we then define the event

$$K_{d}(t) = \{d_{KS}(\mu_{n,\beta,d}, \nu_{n,\beta,d}) \geq t\}.$$  

Using the notation of (4.2)

$$1 - C_{1} \epsilon^{-c_{1} n \beta t^{2}} - C_{2} \epsilon^{-c_{2} n \beta t} \leq P(K_{d}(t)).$$

Then

$$1 - C_{1} \epsilon^{-c_{1} n \beta t^{2}} - C_{2} \epsilon^{-c_{2} n \beta t} - 2 \epsilon^{-ng_{d}(C)} \leq P(\Lambda_{d}(C), K_{d}(t))$$

and then we find for a constant $C_{k} > 0$

$$P(\Lambda_{d}(C), K_{d}(t)) \leq P \left( \sup_{\lambda \in [C^{-1}, C]} \left| \frac{d}{d\lambda} \chi^{\ell-2} \det(T_{k} - \lambda I) \right| d_{KS}(\mu_{n,\beta,d}, \nu_{n,\beta,d}) \leq C_{k} t \right).$$

Therefore

$$1 - C_{1} \epsilon^{-c_{1} n \beta t^{2}} - C_{2} \epsilon^{-c_{2} n \beta t} - 2 \epsilon^{-ng_{d}(C)} \leq P \left( \int \chi^{\ell-2} \det(T_{k} - \lambda I)^{2} \nu_{n,\beta,d}(d\lambda) - \int \chi^{\ell-2} \det(T_{k} - \lambda I)^{2} \mu_{n,\beta,d}(d\lambda) \right) \leq C_{k} t$$

and this establishes the lemma. □

Applying Corollary 4.7.1 establish along with these two lemmas establishes Theorem 3.1(2). The last claim follows from [LP09, Theorem 4.2], using (4.10) and the estimate (4.11).

**Proof of Theorem 3.3.** To evaluate

$$\epsilon_{k,d}^{2} := \int \chi^{\ell-2} \det(T_{k,d} - \lambda I)^{2} \mu_{MP,d}(d\lambda)$$

The max norm gives the maximum entry, in modulus.
we make a simple change of variable \(\lambda = d_{x-d-1} - x + d_{x+d} = 2x\sqrt{d} + 1 + d\) so that

\[
(5.5) \quad \epsilon_{\ell,k,d} := \frac{2}{\pi} \int_{-1}^{1} (2x\sqrt{d} + 1 + d)^{\ell-3} \text{det}(T_{k,d} - (2x\sqrt{d} + 1 + d)I) ^2 \sqrt{1-x^2} \, dx.
\]

Then examine

\[
\frac{1}{\sqrt{d}} \left( T_{k,d} - (2x\sqrt{d} + 1 + d)I \right) = \begin{bmatrix} -\sqrt{d} - 2x & 1 \\ 1 & -2x & \ddots \\ & \ddots & \ddots & 1 \\ & & 1 & -2x \end{bmatrix} := D_{k,d}(-x).
\]

Next, we define \(\det D_{0,d}(x) = 1\) and compute

\[
\det D_{1,d}(x) = 2x - \sqrt{d}, \\
\det D_{k+1,d}(x) + \det D_{k-1,d}(x) = 2x \det D_{k,d}(x), \quad k \geq 1.
\]

We need some elementary properties of the Chebyshev polynomials of \(T_n\) and \(U_n\) of the first and second kind, respectively [MH03]:

\[
U_n(\cos \theta) = \frac{\sin((n + 1)\theta)}{\sin \theta}, \quad T_n(\cos \theta) = \cos n\theta,
\]

\[
\frac{2}{\pi} \int_{-1}^{1} U_j(x)U_k(x) \sqrt{1-x^2} \, dx = \delta_{jk},
\]

\[
U_j(x)U_k(x) = \sum_{\ell=0}^{\min(j,k)} U_{j-k+2\ell}(x),
\]

\[
\frac{1}{x+a} = \frac{2}{\sqrt{a^2-1}} \sum_{j=0}^{\infty} (a - \sqrt{a^2-1})^j T_j(-x),
\]

where the \(\prime\) denotes that the \(j = 0\) term is halved. From the last equality it follows by differentiation that

\[
\frac{1}{(x+a)^2} = \frac{2}{\sqrt{a^2-1}} \sum_{j=1}^{\infty} (a - \sqrt{a^2-1})^j jU_{j-1}(-x).
\]

Recalling (5.5) with \(\ell = 1\) we have

\[
\frac{1}{(2x\sqrt{d} + d + 1)^2} = \frac{1}{\sqrt{d}(1-d)} \sum_{j=0}^{\infty} d^{j/2+1/2}(j + 1)U_j(-x).
\]

The formula for \(D_{k,d}\) is simplified using

\[
\det \left( T_{k,d} - (2x\sqrt{d} + 1 + d)I \right) = d^{k/2} \left[ U_k(-x) - \sqrt{d}U_{k-1}(-x) \right].
\]

Therefore

\[
\frac{2}{\pi} \int_{-1}^{1} \frac{1}{(2x\sqrt{d} + d + 1)^2} \left( U_k(-x) - \sqrt{d}U_{k-1}(-x) \right)^2 \sqrt{1-x^2} \, dx
\]

\[
= \frac{1}{\sqrt{d}(1-d)} \left( (2n+1)\sqrt{d^{2n+1} + \sum_{k=0}^{n-1} [(1+d)(2k+1) - 2d(2k+2)] \sqrt{d^{2k+1}} \right)
\]
Continuing,
\[
\sum_{k=0}^{n-1} [(1 + d)(2k + 1) - 2d(2k + 2)] \sqrt{d}^{2k+1} = \sqrt{d} \sum_{k=0}^{n-1} [(1 - d)(2k + 1) - 2d] \, d^k
\]
\[
= \sqrt{d}(1 - 3d) \sum_{k=0}^{n-1} d^k + 2(1 - d)d^{3/2} \sum_{k=1}^{n-1} kd^{k-1}
\]
\[
= \sqrt{d}(1 - 3d) \frac{1 - d^n}{1 - d} + 2(1 - d)d^{3/2} \frac{-nd^{n-1}(1 - d) + 1 - d^n}{(1 - d)^2}
\]
\[
= \sqrt{d}(1 - 3d) \frac{1 - d^n}{1 - d} + 2d^{3/2} \frac{nd^n - nd^{n-1} + 1 - d^n}{1 - d}
\]
\[
= \sqrt{d}(1 - d^n) + 2d^{3/2} \frac{nd^{n-1}(d - 1)}{1 - d}
\]
\[
= \sqrt{d} - (2n + 1)d^{n+1/2}
\]
and this gives
\[
\epsilon_{1,k,d}^2 = \frac{d^k}{1 - d}.
\]
For \(\epsilon_{2,k,d}\), we use \(T_k(x) = \frac{1}{2} U_k(x) - \frac{1}{2} U_{k-2}(x)\) for \(k \geq 1\) and \(U_0(x) = T_0(x)\) to find
\[
\frac{1}{2x\sqrt{d} + 1 + d} = \frac{1}{1 - d} \sum_{j=0}^{\infty} d^{j/2} [U_j(-x) - U_{j-2}(-x)]
\]
\[
= \sum_{j=0}^{\infty} d^{j/2} U_j(-x).
\]
Then
\[
\frac{2}{\pi} \int_{-1}^{1} \frac{1}{2x\sqrt{d} + 1 + d} (U_k(-x) - \sqrt{d} U_{k-1}(-x))^2 \sqrt{1 - x^2} \, dx
\]
\[
= d^k + (1 + d) \sum_{j=0}^{k-1} d^j - 2d \sum_{j=0}^{k-1} d^j
\]
\[
= d^k + (1 - d) \sum_{j=1}^{k-1} d^j
\]
\[
= d^k + 1 - d^k
\]
\[
= 1.
\]
And this gives
\[
\epsilon_{2,k,d}^2 = d^k.
\]
For \(\epsilon_{3,k,d}\) we find
\[
\frac{2}{\pi} \int_{-1}^{1} (U_k(-x) - \sqrt{d} U_{k-1}(-x))^2 \sqrt{1 - x^2} \, dx = \begin{cases} 1 + d & k \geq 1, \\ 1 & k = 0. \end{cases}
\]
and this gives
\[ e_{3,k,d}^2 = d^k \begin{cases} 1 + d & k \geq 1, \\ 1 & k = 0. \end{cases} \]

Lastly, one can use the bound \(|U_k(x)| \leq k|\) to see that \(e_{l,k,d} \to 0\) as \(k \to \infty\) provided \(0 < d < 1\).

\[ \square \]

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NEW YORK UNIVERSITY
E-mail address: deift@cims.nyu.edu

UNIVERSITY OF CALIFORNIA, IRVINE
E-mail address: ttrogdon@math.uci.edu