The Randomly Driven Ising Ferromagnet

Part II: One and two dimensions

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We consider the behavior of an Ising ferromagnet obeying the
Glauber dynamics under the influence of a fast switching, random ex-
ternal field. In Part I, we introduced a general formalism for describing
such systems and presented the mean-field theory. In this article we
derive results for the one dimensional case, which can be only partially
solved. Monte Carlo simulations performed on a square lat-
tice indicate
that the main features of the mean field theory survive the presence of
strong fluctuations.

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This paper considers the randomly driven Ising model (RDIM), introduced and
discussed from a general point of view in Part I in one and two dimensions. The
main interest in studying such highly nonlinear nonequilibrium statistical physical
systems lies in their possible applications for storage and other information pro-
cessing tasks. Many biological systems, especially networks of real neurons, share
common features with the RDIM in that they form strongly coupled systems driven
by external stimuli acting in unison over a macroscopic number of elements in char-
acteristic times shorter than the typical system relaxation time. Therefore, the
Gaussian or Poissonian external “noise” is transformed radically within the coupled
system, leading to a “correlated noise” with a lot of “strange” properties. The mean
field theory was developed in Part I for a random binary switching external field.
Applications to cortical neurons will be discussed elsewhere.

The order parameter of the RDIM is a nonequilibrium stationary magnetization
distribution, which undergoes a symmetry breaking bifurcation at some critical field
strength. The analytic structure of this distribution also changes in character as
a function of the temperature and field parameters. Hence, transitions between a
singular function with fractal support to a singular function with euclidean support
and further, to an absolutely continuous distribution can be observed. They can be
seen in finite size effects and in the variance (the fluctuations) of the free energy.

Similar transitions can be seen also in one dimension. However, the critical lower
dimensionality of the RDIM remains one. Although the static critical exponent is
a function of the field/temperature ratio, the dynamic exponent $\Delta = 2$ remains
unchanged. Some simple arguments will be given, supporting these results.

The situation is much more difficult in two-dimensions, where only a Monte Carlo
simulation approach was possible. The main difficulty is that computing the sta-
tionary properties of the system requires a large population of different trajectories.
We found that it is more convenient to use systems of moderate size than to rely
on only few dynamic trajectories. Even so, the simulations require a vast amount
of computer resources. We solved this problem by using the Siemens-Nixdorf neu-
rocomputer SYNAPSE-1N110 for quite a different task than what it was conceived
for. This machine is basically a matrix-computer, allowing us to run many dif-
ferent systems in parallel. Even so, we were able only to determine roughly the
phase diagram itself: we have no means at the moment for calculating the critical
exponents.

The paper is organized as following: in Section I we consider the one dimensional
case. Although it cannot be fully solved, many interesting exact results can be derived. Section II deals with the Monte Carlo simulations we performed on finite square lattices. Many features of the mean-field dynamics are shown to survive the strong fluctuations characteristic to two dimensional systems. However, in contrast to the mean-field approach, the two dimensional model displays also an interesting spatial structure related to droplet dynamics. Comparisons between mean-field theory and two-dimensional results are systematically presented, including some preliminary results for hysteresis. Finally, we discuss our results in Section III.

I. 1DIM IN ONE DIMENSION

A. The Master Equation

In his pioneering work, Glauber defined a stochastic dynamics where only single spin flips are allowed and hence neither the magnetization (the order parameter) nor the energy is preserved (model A universality class, see ). Although introduced mainly for mathematical convenience, this dynamics is believed to describe appropriately many Ising-like systems.

The energy of an Ising chain with periodic boundary conditions is given by

$$ E = -J \sum_{i=1}^{N} s_{i} s_{i+1} + \mu_{B} B(t) \sum_{i=1}^{N} s_{i} $$  \hspace{1cm} (1)

Following the notation of Part I, we denote by the vector $\vec{\mu} = (s_{1}, s_{2}, \ldots, s_{i}, \ldots, s_{N})$ a given configuration of spins and by $\vec{\mu}_{i} = (s_{1}, s_{2}, \ldots, -s_{i}, \ldots, s_{N})$ the same configuration but with the spin $s_{i} \rightarrow -s_{i}$ flipped. The external field is sampled from $\rho(B) = \frac{1}{2}\delta(B - B_{0}) + \frac{1}{2}\delta(B + B_{0})$ at time intervals of length $\tau_{B}$,

$$ B(t) = \mu_{B} B \rho(B) \sum_{n=0}^{\infty} \Theta(t - n\tau_{B})\Theta((n+1)\tau_{B} - t) $$  \hspace{1cm} (2)

As explained in Part I, the Master Equation has the form

$$ \dot{P}({s_{i}}; t) = -\dot{L}_{B(t)} P({s_{i}}; t) $$

$$ = \sum_{i}^{N} w(\vec{\mu}_{i}|\vec{\mu}_{i}) P(\vec{\mu}_{i}; t) - P(\vec{\mu}; t) \sum_{i}^{N} w(\vec{\mu}_{i}|\vec{\mu}) $$  \hspace{1cm} (3)

where $w(s_{i}) = w(\vec{\mu}_{i}|\vec{\mu}_{i})$ is the transition rate from a configuration $\{s_{i}\}$ into the state where only the $i$-th spin is flipped, $\{-s_{i}\}$.

Strictly speaking, the system we are going to consider here will never reach the equilibrium Boltzmann distribution $e^{-\beta E}$, where $\beta = 1/k_{B}T$. Nevertheless, we require the detailed balance condition to be fulfilled for a constant value of the external field.

The transition probability $w(s_{i})$ is determined by the constraint of detailed balance only up to a positive arbitrary function of the neighboring spins, $f(s_{i-1}, s_{i+1})$. Since in one dimension the phase transition is at $T_{c} = 0$, the choice of the transition probability influences the analytic form of the critical singularities. In what follows we will use the form

$$ w(s_{i}) = \frac{1}{2\alpha}[1 - s_{i}\tanh(K \sum_{j \in \langle i,j \rangle} s_{j} + H)] $$  \hspace{1cm} (4)
where $K = \beta J$, $H = \beta \mu B$, $\alpha$ sets the time constant, and $(i,j)$ denotes next neighbor pairs. Glauber introduced this form for $H = 0$ but used a slightly different one for $H \neq 0$. In one dimension one has then

$$w(s_i) = \frac{1}{2\alpha} [1 - s_i \tanh(K(s_{i-1} + s_{i+1}) + H)]$$  \hspace{1cm} (5)$$

For $H = 0$ the Liouville operator $\hat{L}_B$ can be mapped onto a free-fermion spin-chain Hamiltonian. This explains why the equations for the averaged spin products, $\langle \prod_{j \in \alpha} s_j \rangle$, decouple in subspaces which can be diagonalized by appropriate Fourier transformations. As shown below, this property is inherited by the first moments, $[\pi_{\alpha}(t)]$ of the stationary distribution $P_{\alpha}$.

**B. Magnetization and correlation functions**

Consider, for example, the time evolution of the local magnetization

$$m_i(t) = \langle s_i \rangle_t = \sum_{\{s_i\}} s_i P(\{s_i\}; t)$$  \hspace{1cm} (6)$$

which can be obtained from (3) as

$$\dot{m}_i = -\frac{2}{\alpha} (s_i w(s_i)) = \frac{1}{\alpha} [-m_i(t) + \langle \tanh(K(s_{i-1} + s_{i+1}) + H) \rangle_t]$$  \hspace{1cm} (7)$$

In order to make the relationship to chaotic maps more evident, we use now the ‘coarse grained’ form of (3). Formally, this procedure corresponds to a forward Euler discretization of (7), setting the time step equal to $\tau_B$, and measuring time in units of $\alpha = \tau_B$:

$$m_i(t+1) = \langle \tanh(K(s_{i-1} + s_{i+1}) + H) \rangle_t$$

$$= a + \gamma \frac{m_{i-1}(t) + m_{i+1}(t)}{2} + b \langle s_{i-1} s_{i+1} \rangle_t$$  \hspace{1cm} (8)$$

where $\gamma = \tanh 2K$, $h = \tanh H$, and

$$a = \frac{h}{2} \left(\frac{1 - \gamma^2}{1 - h^2 \gamma^2} + 1\right)$$  \hspace{1cm} (9)$$

$$\tilde{\gamma} = \frac{1 - h^2}{1 - h^2 \gamma^2}$$  \hspace{1cm} (10)$$

$$b = \frac{h}{2} \left(\frac{1 - \gamma^2}{1 - h^2 \gamma^2} - 1\right)$$  \hspace{1cm} (11)$$

Similarly, for the correlation function

$$c_{i,j} = \langle s_i s_j \rangle$$  \hspace{1cm} (12)$$

one obtains

$$\dot{c}_{i,j} = -\frac{2}{\alpha} \langle s_i s_j (w(s_i) + w(s_j)) \rangle_t$$  \hspace{1cm} (13)$$

leading to the map

$$c_{i,j}(t+1) = \frac{1}{2} \langle s_j \tanh(K(s_{i-1} + s_{i+1}) + H) + s_i \tanh(K(s_{j-1} + s_{j+1}) + H) \rangle_t$$  \hspace{1cm} (14)$$
where the time unit is now set to $\alpha = 2\tau_B$. These recursions can be written again in terms of the variables $a$, $b$ and $\tilde{\gamma}$, Eqs. (8-11).

For $H = 0$ one has $\tilde{\gamma} = \gamma$ and, as shown by Glauber, the slowest relaxation time equals the inverse of the smallest eigenvalue of the magnetization subspace, Eq. (7).

This relaxation time diverges with the square of the static correlation length (the dynamic critical exponent is $z = 2$).

For the binary field distribution of (2) the map of the local magnetization (8) has two branches:

$$m_i(t+1) = \begin{cases} a + \frac{\tilde{\gamma}}{2}m_{i-1}(t) + m_{i+1}(t) + bc_{i-1,i+1}(t) & \text{with prob. } \frac{1}{2} \\ -a + \frac{\tilde{\gamma}}{2}m_{i-1}(t) + m_{i+1}(t) - bc_{i-1,i+1}(t) & \text{with prob. } \frac{1}{2} \end{cases}$$

where it is implicitly assumed that the time needed to switch the field is negligible, $\tau_{\text{switch}} << \tau_B$.

From Eq. (13) it is evident that the map for the local magnetization couples to a two-spin correlation, which in turn couples to higher order correlations, etc. Hence, the full dynamic map lives in a $2^N$ dimensional space, as stated in Part I.

Nevertheless, some partial results can be obtained for the stationary distribution. Define a ‘thermal’ and a ‘dynamical’ average, $\langle \ldots \rangle$ resp. $[\ldots]$. In the stationary state one has $[\langle A(t+1) \rangle] = [\langle A(t) \rangle]$ for any spin-function $A$. The average of the local magnetization obeys

$$[m_i] = [\tilde{\gamma}](c_{m-1} + c_{m+1})$$

where (recall that $\gamma = \tanh 2K$, $h_0 = \tanh H_0$)

$$[\tilde{\gamma}] = \frac{1 - h_0^2}{1 - h_0^2 \gamma^2}$$

For the translation invariant magnetization $m = \frac{1}{N} \sum_i m_i$ and the two-spin correlation function $c_m = \frac{1}{N^2} \sum_i c_{i,i+m}$ one obtains equations formally similar to the ones solved by Glauber. From (14), except for $[\tilde{\gamma}] = 1$, the magnetization vanishes. The stationary two-spin correlation function obeys

$$[c_m] = [\tilde{\gamma}](c_{m-1} + c_{m+1})$$

which leads to

$$[c_m] = \eta^{m|m|}$$

where $\eta = \frac{1 - \sqrt{1 - [\tilde{\gamma}]^2}}{[\tilde{\gamma}]}$.

In general, when expressing the Frobenius-Perron operator in the basis formed by all moments of spin-products $[\pi_q(t)]$, the subspace of the first moments ($q = 1$) is closed and can be diagonalized by a Fourier transform. The remaining part, however, is intractable. For example, the second moment of the magnetization reads

$$[m^2] = [a^2] + 2[a][b] + [\tilde{\gamma}]^2[m^2] + [b^2][c_2]$$

and couples both to the first and to the second moment of the translation invariant correlation functions, $[c_2]$ and $[c_2^2]$, respectively.
C. The $T_c = 0$ phase transition

As expected, in the stationary state the odd moments of the magnetization vanish at $T > 0$. By expanding $\tilde{\gamma}$ and $\eta$ at low temperature one obtains after straightforward calculations that close to $T_c = 0$ the correlation length is in leading order

$$\xi = -\frac{1}{\ln \eta} \begin{cases} \frac{1}{2} e^{2K - H_0} & \text{if } 2K > H_0 \\ \frac{1}{2} H_0^{-1} & \text{if } 2K < H_0 \end{cases}$$

where $H_0 = \beta \mu B_0$ and $B_0$ is defined as in Eq. (2). Hence, in one dimension the RDIM has a critical line with continuously changing singularities at $T_c = 0$ for $\kappa = H_0/2K \in [0,1]$. The slowest relaxation time corresponds to the magnetization (order parameter) decay and can be computed as

$$\tau_{sys}^{-1} = 2(1 - [\tilde{\gamma}]) \begin{cases} \xi^{-2} & \text{if } \kappa < 1 \\ 2 & \text{if } \kappa > 1 \end{cases}$$

Consider first the case of a strong field, $\kappa > 1$. The field will align all spins in one iteration step, as evident from Eq. (22). Hence, the spins are almost always parallel to the driving field. Since $[\tilde{\gamma}]$ vanishes, there is no spontaneously broken symmetry and $[m] = 0$ due to the symmetry of the field distribution, $\rho(B) = \rho(-B)$.

For fields smaller than the critical field $\kappa_c = 1$ one obtains a true symmetry breaking ferromagnetic phase. Interesting enough, while the divergence of the correlation length decreases continuously according to Eq. (21), the critical dynamic exponent remains $z = 2$ up to and including $\kappa \leq 1$. A physical argument explaining this result is presented below.

D. Kink dynamics at $T = 0$

Consider now the transition probability $w(s_i)$ at $T = 0$, Eq. (2), which is a function of the three spins $s_i$, $s_{i-1}$, and $s_{i+1}$. Let us call an interface between two oppositely oriented spin domains a kink. If $\kappa = H_0/2K = 0$ and $T = 0$ each kink performs a random walk, moving with equal probability to the left or to the right. When two kinks become nearest neighbors, they annihilate because in the next time step the single spin left between them will flip with probability one.

Due to this annihilation process the number of kinks decreases steadily and in the end only very few are left. Two kinks situated at the typical distance $\xi$ (the correlation length) will meet via diffusive motion in the characteristic time $\tau \sim \xi^2$, which explains why the critical dynamic index is $z = 2$. The situation is similar if one is close to (but not at) $T_c = 0$.

How does this picture change if we switch on the random external field ($\kappa > 0$)? The domains parallel to the external field start growing - both ends of such a cluster will move outwards. Once the field changes sign, these domains shrink again and the clusters of oppositely aligned spins grow. During a longer period of time these effects compensate each other and the surviving kinks perform effectively a random walk. However, the annihilation rate of kink-pairs is highly increased. Assume, for example, that after $2T$ iteration steps the external field had the value $+H_0 (T + n)$-times. The probability for this to happen is given by the Bernoulli distribution, $B_{2n}^{2T} = \left(\begin{array}{c} 2T \\ n \end{array}\right) p^n (1-p)^{2T-n}$. During this time, a down-oriented domain whose original length was $L_0$ shrinks on average by $2n(2p - 1)$, where both kinks associated with the ends of the domain have moved inward during each of the $n$ time steps with probability.
\( p > 1/2 \) and outward with probability \( q = 1 - p \). Hence, if \( 2n(2p - 1) \sim L_0 \), the domain will be eliminated. \( p \) is a function of \( \kappa = \frac{H_0}{K} \), e.g. \( p(\kappa = 0.2) = 0.982 \). Due to the depletion of small clusters, the number of kinks decreases much faster than in the absence of the external field. This effect is illustrated by a numerical simulation in Fig. 1, showing the \( T = 0 \) dynamics of walls for \( \kappa = 0 \) and \( \kappa = 0.2 \), respectively.

Once only a few large clusters remain, however, their width becomes macroscopic. On this scale the kinks perform again a random walk and asymptotically one regains \( z = 2 \).

At finite temperatures, however, the presence of the field term facilitates the nucleation of new clusters, so that the correlation length \( \langle 21 \rangle \) (the mean cluster size) is less divergent when \( H_0 > 0 \).

E. The magnetization distribution

Consider again the map (15). In the translational invariant sector one has

\[
m(t + 1) = \begin{cases} 
  a + \tilde{\gamma}m(t) + bc_2(t) & \text{with prob. } \frac{1}{2} \\
  -a + \tilde{\gamma}m(t) - bc_2(t) & \text{with prob. } \frac{1}{2}
\end{cases} \tag{23}
\]

As already discussed, the magnetization couples to the correlation function \( c_2(t) \), etc. A simple approximation to decouple the magnetization sector is using for \( c_2(t) \) the stationary value \( c_2 = \eta^2 \). The resulting map corresponds to a Bernoulli-shift and is shown graphically in Fig. 2. If the gap between the two branches is positive \( \Delta > 0 \)

\[
\Delta = \frac{2(a + bc_2)(1 - 2\tilde{\gamma})}{1 - \tilde{\gamma}} > 0 \tag{24}
\]

the corresponding stationary magnetization distribution is a Cantor set.

Since \( a, b, (1 - \tilde{\gamma}) \) are positive and \( c_2 > 0 \) for ferromagnetic interactions, the demarcation line between a fractal and a nonfractal magnetization distribution is given by

\[
\tilde{\gamma} = \frac{1}{2}, \tag{25}
\]

independently of the actual value \( c_2(t) \) might have. The time dependence of \( c_2(t) \) induces nonlinearities in the map. Therefore, although the distribution remains fractal for \( \tilde{\gamma} < 1/2 \), in general it is not a homogeneous Cantor set.

II. MONTE CARLO SIMULATIONS IN TWO DIMENSIONS

We simulated the RDIM on a two dimensional square lattice on the neurocomputer SYNAPSE-1/N110. In the following sections we first describe a Monte Carlo Algorithm (MCA) designed to make use of the computational power of SYNAPSE-1 and then present our numerical results. They include a phase diagram in the H-K-plane, magnetization distributions in the para- and ferromagnetic regime, and a series of snapshots documenting the behavior of the system.
A. The algorithm

SYNAPSE-1 is a workstation-driven coprocessor consisting of a systolic array of eight MA16 Neural Signal Processors. It was kindly put at our disposal by the ZFE of Siemens AG. Its hardware was designed to tackle typical problems encountered when simulating neural networks, namely calculations involving very large matrices. In order to efficiently make use of the C++-library interface supplied with SYNAPSE-1 for the RDIM, we devised a MCA that simulates multiples of eight lattices in parallel.

Consider a square lattice of spins \( s_{ij}^k \in \{-1, 1\} \) of linear dimension \( L \), where \( k \) numbers the system and \( i, j = 1 \ldots L \) denotes the lattice position. Each of eight lattices is represented as a column vector \( s_\nu \) by renumbering indices \( \nu = iL + j \).

The eight systems can thus be treated as one \( 8 \times L^2 \)-matrix. By setting \( \nu \mapsto \nu + L^2 \) for \( \nu \leq 0 \) and \( \nu \mapsto \nu - L^2 \) for \( \nu > L^2 \) we enforce helical boundary conditions. The neighbors of spin \( \nu \) are \( \mu = \nu \pm 1, \mu = \nu \pm L \).

In order to avoid metastable states induced by a simultaneous update of neighboring spins we split the lattices into black and white sites in a checkerboard fashion, leading to two matrices encoding the eight systems. Note that when a lattice is divided up in this fashion, if \( L \) is odd, the sites in the first and last row have neighbors of their own color. If \( L \) is even, the same is true for the first and last column of the lattice. For technical reasons we chose \( L \) to be odd.

To further simplify the updating scheme, a copy of the first and last \( L \) components of each lattice are included at the end respectively beginning of each column vector. A Monte Carlo step now consists of a parallel update of all black sites followed by an update of all white sites (or vice versa).

From the well known Glauber dynamic rule Eq. (4), a spin is flipped under the following condition. Given a random number \( z \in [0, 1] \) drawn from a uniform distribution, spin \( s_\nu \) is updated according to

\[
\begin{align*}
  s_\nu &\mapsto \\
  -s_\nu &\quad \text{if } z < \frac{1}{2}(1 - \tanh(s_\nu(K\Sigma_\mu s_\mu + H))) \\
  s_\nu &\quad \text{if } z \geq \frac{1}{2}(1 - \tanh(s_\nu(K\Sigma_\mu s_\mu + H)))
\end{align*}
\]  

(26)

Usually, spins are either treated sequentially or chosen randomly. SYNAPSE-1 permits the parallel generation of (pseudo-) random numbers in a single Elementary Operation (ELOP) which can be piped through a function lookup table at no extra computational cost. For this reason we transform the flip condition Eq. (26) into

\[
\begin{align*}
  s_\nu &\mapsto \\
  -s_\nu &\quad \text{if } \Sigma_\mu s_\nu s_\mu < \frac{1}{2K} \log\left(\frac{\frac{1}{2} - z'}{\frac{1}{2} + z'}\right) - s_\nu \frac{H}{K} \\
  s_\nu &\quad \text{if } \Sigma_\mu s_\nu s_\mu \geq \frac{1}{2K} \log\left(\frac{\frac{1}{2} - z'}{\frac{1}{2} + z'}\right) - s_\nu \frac{H}{K}
\end{align*}
\]  

(27)

where \( z' \) is drawn from a uniform distribution in \([\frac{1}{2}, \frac{1}{2}]\). The RHS of the flip condition is evaluated in two ELOPs: One to generate a matrix of random numbers piped through the function \( z' \mapsto \frac{1}{2K} \log\left(\frac{\frac{1}{2} - z'}{\frac{1}{2} + z'}\right) \) and one weighted matrix subtraction. The LHS also takes two ELOPs to calculate from the “black” and “white” matrices. Two further ELOPs are required to construct and evaluate a flip indicator matrix. Some more operations are necessary to fix boundary conditions and to evaluate the mean lattice magnetizations. This procedure is applied sequentially first to the matrix holding the “black” spins and then to the “white” one to accomplish a complete Monte Carlo step.

Initially, the spins in a lattice are set to \(+1\) with probability \( p \) and \(-1\) with probability \( 1 - p \), where different values of \( p \) can be used for each lattice in one.
simulation run. The results for systems of linear dimension \( L = 415 \) are initialized with \( p = 0, 0.2, 0.4, 0.5, 0.6, 0.8, \) and \( 1 \), in addition, there is one lattice in which the top half of all spins is set to +1 and the bottom half to −1. The external driving field is the same for each system. Simulations at smaller \( L \) consist of 64 systems initialized with \( p = \frac{1}{2} \), but each with its own driving field trajectory.

Temperature \( K \) is measured in units of the critical temperature, \( K \rightarrow K_c \), where 
\[
K_c = \frac{J}{k_B T_c} = \frac{1}{2} \ln(\sqrt{2} + 1) \approx 0.44069
\]
of the standard two dimensional Ising model, i.e. \( K = 1 \) corresponds to the critical temperature for \( H_0 = 0 \).

B. Dynamics and phase diagram

In order to understand the dynamics of the two-dimensional RDIM in more detail, it is useful to consider first what happens to a cluster of parallel spins at \( T = 0 \), in analogy to kink dynamics for the one dimensional case. Transition rates at \( T = 0 \) are either 0, 1, or \( \frac{1}{2} \). Recalling that \( K \) is measured in units of \( K_c \), define \( \kappa' := \frac{H_0}{2K} = \kappa K_c \). Consider now a square cluster of \( 2N \times 2N \) parallel spins under the influence of an anti-parallel external field. First, if \( 0 < \kappa' < 1 \), the spins at the corners of the cluster flip with probability \( p = 1 \), all other remain anti-parallel to the field (as shown in Fig. 3a). Thus the cluster disappears if the external field remains constant for \( 2N - 1 \) consecutive steps. Secondly, if \( 1 < \kappa' < 2 \), such a cluster will be destroyed in \( N \) steps, due to the fact that all but inner spins will flip with \( p = 1 \). Thirdly, at \( \kappa' > 2 \) we arrive at a driven paramagnetic phase. Regardless of their position, all spins will flip into the direction of the driving field with \( p = 1 \). For the case of \( \kappa' = 0 \), \( \kappa' = 1 \), or \( \kappa' = 2 \), corner, edge, and inner spins flip with \( p = \frac{1}{2} \).

This implies that, e.g. for a strong driving field with \( \kappa' = 2K_c \), nucleation flips (see Fig. 3c) may take place inside the cluster, creating magnetic swiss cheese.

The transition probabilities of the processes shown in Fig. 3 increase with increasing field strength. At small fields the a)-type flip is prevalent, resulting in a radial growth (shrinkage) of clusters. Although energetically more expensive, the b)-type flip has a large entropy contribution and results in long-wavelength growth of flat domain walls. The nucleation process shown in flip c) has the smallest probability.

What happens when switching instantly the field from the equilibrium state at \( -H_0 \) into the unfavorable direction \( H_0 \)? The system relaxes from the now metastable state to the new equilibrium value. Obviously, the lifetime of the metastable state depends on the strength of the applied field. This scenario has been discussed in detail in the ferromagnetic phase using droplet theory and Monte Carlo simulations (see, for example 7). Here, four distinct field intervals, shown schematically in Fig. 4, were, identified in which the lifetimes markedly differ due to different decay mechanisms. A numerical result for \( K = 1.25 \) is shown in Fig. 5 where we approximated the metastable lifetimes by measuring the average first passage times (FPT) from \( m = -1 \) to \( m = 0.7 \) in Monte Carlo steps. Figs. 6 and 7 show examples of the time development of the magnetization from the metastable state to th new equilibrium for the mean field and 2D model.

We calculated numerically a phase diagram in the \( K - \frac{H}{K} \) plane for the 2D RDIM, Fig 8. Similar to the mean field and one dimensional model, there is a paramagnetic, a ferromagnetic (and a driven paramagnetic) phase. Note that for \( K \rightarrow \infty \), the phase boundary should remain below \( \kappa' = 2 \). We are currently in no position to assess this. Also, the behavior around \( K = 1 \) doesn’t seem to correspond to the first order dynamic freezing transition seen in the mean field theory, rather a second order transition is likely.
C. The paramagnetic phase

If the external field is above its critical value, \( H_0 > H_c \), the stationary phase of the RDIM is paramagnetic. In this phase the system relaxes relatively fast to the equilibrium state, except close to \( H_c \), where critical slowing down sets in due to the type-I intermittency effects discussed in the previous section.

This behavior of the average magnetization is shown in Fig. 7 and is probably enhanced by local correlations not taken into account in the mean field approximation. If the random field is switched on, close but above \( H_c \) the critical slowing down is dramatically enhanced. Further away from the phase transition point the dynamics is - similarly to the one-dimensional case - determined by the nucleation and radial growth (shrinking) of droplet-like clusters.

As expected from the mean-field results, the RDIM can display a fractal magnetization distribution at higher fields. This is shown in Fig. 9. Thermal fluctuations and finite size effects wash out the fine structure of the multifractal magnetization distribution predicted by the mean-field calculations. However, the presence of sharp peaks in the distribution (and their scaling behavior) demonstrates that some of the main features of the magnetization distribution survive the thermal fluctuations.

These peaks are related to long-lived droplets whose radius is large enough to allow them to stay alive even when a long series of unfavorable external field draws makes them shrink. It is, however, the competition between the two thermodynamically stable states which leads to a chaotic dynamics and strange attractors.

D. The ferromagnetic phase

The situation is even more complex below \( H_c \). The schematic dependence of the average time spent in a thermodynamically unstable state vs. the inverse of the field strength is shown after in Fig. 7. The external-field sampling time \( \tau_B \) should be chosen in either the strong or multi-droplet regime (certainly not in the coexistence regime). By varying \( \tau_B \) it seems possible to explore these different dynamic mechanisms in more detail. Fig. 10 shows a simulation in the ferromagnetic phase where one can observe both a multi-droplet (series 1 and 3) and a domain-wall type (series 2) dynamics. Again, the spontaneous magnetization distribution shows well separated peaks, which can be seen in Fig. 11.

For the sake of completeness, we show also a Monte Carlo simulation of the hysteresis measurement described in Fig. 8 of Part I. Only the evolution of one system initialized with spins up/down with equal probability is displayed. Again, one can see that the thermal fluctuations are smoothing only the fine scale structure of the mean-field predictions – but the main features remain intact.

The results presented here leave open many questions regarding the 2D RDIM – an in-depth study by Monte Carlo simulation lies currently beyond our means. The theory and evolution of droplets and domains in the RDIM as well as hysteretic effects remain interesting research topics.

III. SUMMARY AND DISCUSSION

In Part I and in this article we have discussed in detail the properties of a simple Ising model subject to a fast switching external field. In many ways, the situation is just the opposite as in quenched random systems. While there the defects and
hence the (local) fields are frozen relative to the spin degrees of freedom, which are (in principle) free to relax, in the RDIM the external field is the fast variable compared to the interacting spin system. From a “technological” theoretical point of view the situation is, however, much easier. We expect that similar analytic results can be obtained for other random distributions as well. Strongly driven systems show rather peculiar properties, which one could use for increasing the storage properties of ferromagnetic materials. For example, it seems possible to use arithmetic coding in ‘preparing’ a semi-macroscopic ferromagnetic region to fall into a given distribution peak, as the ones shown in Fig. 3. An appropriately sensitive reading head could then discriminate between the different magnetization values. Such ‘devices’ could be tested first with the help of Monte Carlo simulations. However, the main application domain for randomly driven systems might well be in biology. Further work in that direction is under progress.

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FIG. 1. Diffusion of walls for $\kappa = 0$ (top) and $\kappa = 0.2$ (bottom), respectively. Note the different time scales.
**FIG. 2.** The truncated 1D magnetization map for $\Delta > 0$

**FIG. 3.** The basic spin-flip configurations, $H_0$ points upwards, up (down) spins are black (white), a spin turning up is drawn in grey. a) a ‘droplet’-flip, b) a ‘domain’-flip, c) nucleation-flip

**FIG. 4.** The average time spent in the metastable thermodynamic state as a function of inverse field-strength. The different domains are denoted according to their main relaxation mechanism.
FIG. 5. The average first-passage time from $m = -1$ to $m = 0.7$ at $K = 1.25$ as a function of inverse field-strength. $\langle t(m=0.7) \rangle$ is calculated from an ensemble of 64 systems with linear dimension $L = 143$.

FIG. 6. Mean field iteration $m$ vs. time $t$ in an unfavorable field for $\frac{H_0}{K} = 0.057$ and $K = 1.2$, which is slightly above the critical field.
FIG. 7. Monte Carlo simulation in an unfavorable field for $\frac{\mu_0}{K} = 0.08$, $K = 1.2$, $\tau_B = 1$, and linear dimension $L = 143$. The solid line shows the average magnetization in an ensemble of 64 systems, $m$, as a function of time $t$. The dashed lines mark a 1-σ range around the mean. All systems start with all spins down, i.e. $\forall \nu: s_\nu = -1$.

FIG. 8. Phase diagram of the 2D RDIM from Monte Carlo simulation of 64 systems of linear dimension $L = 63$. 
FIG. 9. Top: Magnetization distribution for the square lattice RDIM averaged from eight different initial conditions for $K = 0.4$, $\frac{H}{K} = 0.5$, $\tau_B = 1$, and linear dimension $L = 415$. Here, the simulation covers more than $2 \cdot 10^5$ Monte Carlo sweeps. Note the similarity to Fig. 1 of Part I. Bottom: $K = 1$, $\frac{H}{K} = 1$, $\tau_B = 1$, and $L = 415$. The simulation run covers close to $2 \cdot 10^5$ Monte Carlo sweeps. Compare this to Fig. 2 of Part I.
FIG. 10. Time series of the mean magnetization of the RDIM. Parameters are $K = 2$, $H/K = 1$, $\tau_B = 1$, and the linear dimension is 128. Note that here the boundary conditions are periodic. Snapshots of the system are shown in series 1, 2, and 3 on the next pages.
FIG. 11. Top: Magnetization distribution for the square lattice RDIM averaged from eight different initial conditions for $K = 2$, $\frac{H}{K} = 1$, $\tau_B = 1$, and linear dimension $L = 415$. We simulated more than $2 \cdot 10^5$ sweeps. Only the region from $m = 0.8$ to $m = 1.0$ is shown, the distribution is symmetric in $m$. It is similar to the mean field distribution around the critical driving field $H_c$ (c.f. Fig. 3 and 4 of Part I). Bottom: $K = 1.33$, $\frac{H}{K} = 0.75$, $\tau_B = 1$, and $L = 415$. We tracked $1.5 \cdot 10^5$ Monte Carlo sweeps. Here, the distribution resembles the mean field distribution close to the critical driving field $H_c$. 
FIG. 12. Hysteresis in the square lattice RDIM, showing the mean magnetization \( m(t) \) vs. the external driving field \( H(t) \), see Eq. (32) of Part I. Parameters are \( \frac{A}{K} = \frac{H_0}{K} = 1 \), \( K = 2 \), \( \Omega = \frac{2\pi}{1000} \), and linear dimension \( L = 415 \). The simulation ran for more than \( 10^5 \) sweeps. See Fig. 8 of Part I for comparison.