FINITE BASIS PROBLEMS FOR STALACTIC, TAIGA, SYLVESTER AND BAXTER MONOIDS

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Abstract. Stalactic, taiga, sylvester and Baxter monoids arise from the combinatorics of tableaux by identifying words over a fixed ordered alphabet whenever they produce the same tableau via some insertion algorithm. In this paper, three sufficient conditions under which semigroups are finitely based are given. By applying these sufficient conditions, it is shown that all stalactic and taiga monoids of rank greater than or equal to 2 are finitely based and satisfy the same identities, that all sylvester monoids of rank greater than or equal to 2 are finitely based and satisfy the same identities and that all Baxter monoids of rank greater than or equal to 2 are finitely based and satisfy the same identities.

1. Introduction

Knuth introduced the tableaux algebra [14] in the 1970s and this algebra was later studied in detail by Lascoux and Schützenberger under the name plactic monoid [18]. Plactic monoid arise from the combinatorics of tableaux by identifying words over a fixed ordered alphabet whenever they produce the same tableau via Schensted’s insertion algorithm [25]. Plactic-like monoids which arise from the combinatorics of tableaux as the plactic monoid include the hypoplactic monoid [15, 22], the stalactic monoid [10, 24], the taiga monoid [24], the sylvester monoid [9] and the Baxter monoid [8]. These monoids have attracted much attention due to their interesting connection with combinatorics [16] and applications in symmetric functions [21], representation theory [7], Kostka-Foulkes polynomials [17, 18], Schubert polynomials [19, 20], and musical theory [12].

Each of these plactic-like monoids can be obtained by factoring the free monoid \(A^*\) over the infinite ordered alphabet \(A = \{1 < 2 < 3 < \cdots\}\) by a congruence that can be defined by a so-called insertion algorithm that computes a combinatorial object from a word. For example, for the stalactic monoid, the corresponding combinatorial objects are stalactic tableaus. We introduce the definitions of combinatorial objects and insertion algorithms used to construct the stalactic, taiga, sylvester and Baxter monoids.

A \textit{stalactic tableau} is a finite array of symbols of \(A\) in which columns are top-aligned, and two symbols appear in the same column if and only if they are equal. The associated insertion algorithm is as follows:

\begin{algorithm}
\textbf{Algorithm 1.} [10, § 3.7] Input: A stalactic tableau \(T\) and a symbol \(a \in A\). If \(a\) does not appear in \(T\), add \(a\) to the left of the top row of \(T\); if \(a\) does appear in \(T\), add \(a\) to the bottom of the column in which \(a\) appears. Output the new tableau.
\end{algorithm}
Let $w_1, \cdots, w_k \in \mathcal{A}$ and $w = w_1 \cdots w_k \in \mathcal{A}^*$. Then the combinatorial object $P_{\text{stal}}(w)$ of $w$ is obtained as follows: reading $w$ from right-to-left, one starts with an empty tableau and inserts each symbol in $w$ into a stalactic tableau according to Algorithm 1. For example, $P_{\text{stal}}(3613151265)$ is given as follows:

$$
\begin{array}{c}
3 \\
1 \\
6 \\
5 \\
2 \\
1 \\
\end{array}
$$

Notice that the order in which the symbols appear along the first row in $P_{\text{stal}}(w)$ is the same as the order of the rightmost instances of the symbols that appear in $w$.

A binary search tree with multiplicities is a labelled binary search tree in which each label appears at most once, where the label of each node is greater than or equal to the label of every node in its left subtree, and strictly less than every node in its right subtree, and where a non-negative integer called the multiplicity is assigned to each node label. The associated insertion algorithm is as follows:

**Algorithm 2.** \cite{24} Algorithm 3] Input: A binary search tree with multiplicities $T$ and a symbol $a \in \mathcal{A}$. If $T$ is empty, create a node, label it by $a$, and assign it multiplicity 1. If $T$ is non-empty, examine the label $x$ of the root node: if $a > x$, recursively insert $a$ into the right subtree of the root node; otherwise recursively insert $a$ into the left subtree of the root node. Increment by 1 the multiplicity of the node label $x$.

Let $w_1, \cdots, w_k \in \mathcal{A}$ and $w = w_1 \cdots w_k \in \mathcal{A}^*$. Then the combinatorial object $P_{\text{taig}}(w)$ of $w$ is obtained as follows: reading $w$ from right-to-left, one starts with an empty tree and inserts each symbol in $w$ into a binary search tree with multiplicities according to Algorithm 2. For example, $P_{\text{taig}}(3613151265)$ is given as follows:

$$
\begin{array}{c}
5 \\
2 \\
6 \\
5 \\
\end{array}
$$

A right strict binary search tree is a labelled rooted binary tree where the label of each node is greater than or equal to the label of every node in its left subtree, and strictly less than every node in its right subtree. The associated insertion algorithm is as follows:

**Algorithm 3.** \cite{9, § 3.3] Input: A right strict binary search tree $T$ and a symbol $a \in \mathcal{A}$. If $T$ is empty, create a node and label it $a$. If $T$ is non-empty, examine the label $x$ of the root node: if $a > x$, recursively insert $a$ into the right subtree of the root node; otherwise recursively insert $a$ into the left subtree of the root node. Output the resulting tree.

Let $w_1, \cdots, w_k \in \mathcal{A}$ and $w = w_1 \cdots w_k \in \mathcal{A}^*$. Then the combinatorial object $P_{\text{sylv}}(w)$ of $w$ is obtained as follows: reading $w$ from right-to-left, one starts with an empty tree and inserts each symbol in $w$ into a right strict binary search tree according to Algorithm 3. For example, $P_{\text{sylv}}(3613151265)$ is given as follows:

$$
\begin{array}{c}
6 \\
2 \\
5 \\
6 \\
1 \\
3 \\
\end{array}
$$
A left strict binary search tree is a labelled rooted binary tree where the label of each node is strictly greater than the label of every node in its left subtree, and less than or equal to every node in its right subtree. The associated insertion algorithm is as follows:

**Algorithm 4.** Input: A left strict binary search tree $T$ and a symbol $a \in A$. If $T$ is empty, create a node and label it $a$. If $T$ is non-empty, examine the label $x$ of the root node: if $a < x$, recursively insert $a$ into the left subtree of the root node; otherwise recursively insert $a$ into the right subtree of the root node. Output the resulting tree.

Let $w_1, \ldots, w_k \in A$ and $w = w_1 \cdots w_k \in A^*$. Then the combinatorial object $P_{sylv}(w)$ of $w$ is obtained as follows: reading $w$ from left-to-right, one starts with an empty tree and inserts each symbol in $w$ into a left strict binary search tree according to Algorithm 4. For example, $P_{sylv}(3613151265)$ is given as follows:

```
      3
     / \   \\
    1   6
   / \  /   \\
  5  2  1   5
```

Let $w_1, \ldots, w_k \in A$ and $w = w_1 \cdots w_k \in A^*$. Then the combinatorial object $P_{baxt}(w)$ of $w$ is obtained by the Algorithms 3 and 4 that is, $P_{baxt}(w) = (P_{sylv}(w), P_{sylv}(w))$.

For each $M \in \{\text{stal}, \text{taig}, \text{sylv}, \text{sylv}^2, \text{baxt}\}$, define the relation $\equiv_{M_{n\infty}}$ by $u \equiv_{M_{n\infty}} v \iff P_{M_{n\infty}}(u) = P_{M_{n\infty}}(v)$ for any $u, v \in A^*$. In each case, the relation $\equiv_{M_{n\infty}}$ is a congruence on $A^*$. The stalastic monoid $\text{stal}_{\infty}$ [resp. taiga monoid $\text{taig}_{\infty}$, sylvester monoid $\text{sylv}_{\infty}$, 5-sylvester monoid $\text{sylv}^5_{\infty}$, Baxter monoid $\text{baxt}_{\infty}$] is the factor monoid $A^*/\equiv_{M_{\infty}}$. The rank-$n$ analogue $\text{stal}_n$ [resp. $\text{taig}_n$, $\text{sylv}_n$, $\text{sylv}^5_n$, $\text{baxt}_n$] is the factor monoid $A^n*/\equiv_{M_{n\infty}}$, where the relation $\equiv_{M_{n\infty}}$ is naturally restricted to $A^n_0 \times A^n_0$ and $A_n = \{1 < 2 < \cdots < n\}$ is set of the first $n$ natural numbers viewed as a finite ordered alphabet. It follows from the definition of $\equiv_{M_{n\infty}}$ for any $M \in \{\text{stal}, \text{taig}, \text{sylv}, \text{sylv}^5, \text{baxt}\}$ that each element $[u]_{M_{n\infty}}$ of the factor monoid $M_{n\infty}$ can be identified with the combinatorial object $P_{M_{n\infty}}(u)$. In each case, $M_1$ is a free monogenic monoid $\langle a \rangle = \{1, a, a^2, a^3, \ldots \}$ and thus commutative. Note that $M_1 \subset M_2 \subset \cdots \subset M_i \subset M_{i+1} \subset \cdots \subset M_{\infty}$.

The evaluation of a word $u \in A^*$, denoted by $ev(u)$, is the infinite tuple of non-negative integers, indexed by $A$, whose $a$-th element, denoted by $|u|_a$, is the number of times the symbol $a$ appears in $u$: thus this tuple describes the number of each symbol in $A$ that appears in $u$. It is immediate from the definition of the monoids above that if $u \equiv_{M_{n\infty}} v$, then $ev(u) = ev(v)$, and hence it makes sense to define the evaluation of an element $p$ of one of these monoids to be the evaluation of any word representing it. We write $ev(u) \leq ev(v)$ [resp. $ev(u) < ev(v)$] if $|u|_a \leq |v|_a$ [resp. $|u|_a < |v|_a$] for each non-negative integer $a$.

A basis for an algebra $A$ is a set of identities satisfied by $A$ that axiomatize all identities of $A$. An algebra $A$ is said to be finitely based if it has some finite basis. Otherwise, it is said to be non-finitely based. The finite basis problem, that is the problem of classifying algebras according to the finite basis property, is one of the most prominent research problems in universal algebra. Since the first example of
non-finitely based finite semigroup was discovered by Perkins \[23\] in the 1960s, the finite basis problem for semigroups has attracted much attention. Now there exist several powerful methods to attack the finite basis problem for finite semigroups (see Volkov \[26\] for detail).

In contrast with the finite case, the finite basis problem for infinite semigroups is less explored. On the one hand, infinite semigroups usually arise in mathematics as transformation semigroups of an infinite set, or semigroups of relations on an infinite domain, or matrix semigroups over an infinite ring. And all these semigroups are too big to satisfy any non-trivial identity. On the other hand, when an infinite semigroup does satisfy non-trivial identities, then deciding if there is a finite basis remains difficult. Indeed, many of methods designed for finite semigroups do not apply so that fresh techniques are required.

Since the plactic monoid of infinite rank does not satisfy any non-trivial identity \[4, Proposition 3.1\], the plactic monoid of infinite rank is finitely based. The plactic monoid of rank 2 satisfies exactly the same identities as the bicyclic monoid \[13, Remark 4.6\] the monoid of all $2 \times 2$ upper triangular tropical matrices. Thus the plactic monoid of rank 2 is non-finitely based by the result of Chen et al. \[5, Corollary 5.6\]. The plactic monoid of rank 3 satisfies exactly the same identities as the monoid of all $3 \times 3$ upper triangular tropical matrices \[13, Corollary 4.5\]. Thus the plactic monoid of rank 3 is non-finitely based by the result of Han et al. \[11\]. The finite basis problems for the plactic monoids of rank greater than or equal to 4 are still open. Cain et al. proved that all hypoplactic monoids of rank greater than or equal to 2 are finitely based and satisfy the same identities \[3\]. For each $M \in \{stal, taig, sylv, sylv^\#, baxt\}$, $M_1$ is a free monogenic monoid and commutative, and so $M_1$ is finitely based by \[23, Theorem 9\]. However the finite basis problems for $M_n$ with $2 \leq n \leq \infty$ are still open.

In this paper, we investigate the finite basis problems for all stalactic, taiga, sylvester and Baxter monoids of rank greater than or equal to 2. It is shown that all stalactic and taiga monoids of rank greater than or equal to 2 are finitely based and satisfy the same identities, that all sylvester monoids of rank greater than or equal to 2 are finitely based and satisfy the same identities and that all Baxter monoids of rank greater than or equal to 2 are finitely based and satisfy the same identities.

This paper is organized as follows. Notation and background information of the paper are given in Section 2. In Section 3, three sufficient conditions under which semigroups are finitely based are given. By applying these sufficient conditions, we solve the finite basis problems for all stalactic, taiga, sylvester and Baxter monoids of rank greater than or equal to 2 in Section 4.

2. Preliminaries

Most of the notation and background material of this article are given in this section. Refer to the monograph of Burris and Sankappanavar \[1\] for more information.

Let $\mathcal{X}$ be a countably infinite alphabet. Elements of $\mathcal{X}$ are called letters and elements of the free monoid $\mathcal{X}^*$ are called words. Let $w \in \mathcal{X}^*, x, y, x_1, x_2, \ldots, x_m \in \mathcal{X}$. Then

- the content of $w$, denoted by $\text{con}(w)$, is the set of letters occurring in $w$;
- $\text{occ}(x, w)$ is the number of occurrences of the letter $x$ in $w$;
- $\delta_{\text{occ}}(x, w)$ [resp. $\bar{\delta}_{\text{occ}}(x, w)$] is the number of occurrences of $x$ before [resp. after] the first [resp. last] occurrence of $y$ in $w$;
- $w$ is said to be simple if $\text{occ}(x, w) = 1$ for any $x \in \text{con}(w)$;
• the initial part [resp. final part] of $w$, denoted by $ip(w)$ [resp. fp($w$)], is the simple word obtained from $w$ by retaining the first [resp. last] occurrence of each letter;
• mix($w$) is the word obtained from $w$ by retaining the first and the last occurrences of each letter;
• $w[x_1, x_2, \ldots, x_m]$ denote the word obtained from $w$ by retaining only the occurrences of the letters $x_1, x_2, \ldots, x_m$.

A semigroup identity is a formal expression $u \approx v$ where $u, v$ are words over the alphabet $X$. An identity $u \approx v$ is said to be non-trivial if $u \neq v$ and trivial otherwise. A semigroup $S$ satisfies an identity $u \approx v$ if the equality $\varphi(u) = \varphi(v)$ holds in $S$ for every possible substitution $\varphi : X \to S$. Denote by $id(S)$ the set of all non-trivial identities satisfied by $S$.

Clearly any monoid that satisfies an identity $s \approx t$ also satisfies the identity $s[x_1, x_2, \ldots, x_n] \approx t[x_1, x_2, \ldots, x_n]$ for any $x_1, x_2, \ldots, x_n \in X$, since assigning the unit element to a letter $x$ in an identity is effectively the same as removing all occurrences of $x$.

An identity system $\Sigma$ is a collection of non-trivial identities. An identity $u \approx v$ is said to be derived from $\Sigma$ or is a consequence of $\Sigma$ if there is a sequence of words $u = u_1, u_2, \ldots, u_{n-1}, u_n = v$ over the alphabet $X$ such that for every $i = 1, 2, \ldots, n-1$, $u_i = a_i \varphi_i(p_i)b_i, u_{i+1} = a_i \varphi_i(q_i)b_i$ with some words $a_i, b_i \in X^*$, some endomorphism $\varphi_i : X^+ \to X^+$ and some identity $p_i \approx q_i \in \Sigma$.

Given an identity system $\Sigma$, we denote by $id(\Sigma)$ the set of all consequences of $\Sigma$. An identity basis for a semigroup $S$ is any set $\Sigma \subseteq id(S)$ such that $id(\Sigma) = id(S)$, that is, every identity satisfied by $S$ can be derived from $\Sigma$. A semigroup $S$ is called finitely based if it possesses a finite identity basis, that is, all identities satisfied by $S$ can be derived from a finite subset of $id(S)$; otherwise $S$ is called non-finitely based.

Two semigroups $S_1$ and $S_2$ are called equationally equivalent if $id(S_1) = id(S_2)$.

For any semigroup $S$, let $S^1$ be the monoid obtained from $S$ by adjoining a unit element. Denote by $L_2, R_2, M$ the left-zero semigroup of order 2, the right-zero semigroup of order 2 and the free monogenic monoid, whose presentations are given as follows:

\[
L_2 = \langle a, b \mid a^2 = ab = a, b^2 = ba = b \rangle,
\]
\[
R_2 = \langle a, b \mid a^2 = ba = a, b^2 = ab = b \rangle,
\]
\[
M = \langle a \rangle = \{1, a, a^2, a^3, \ldots \}.
\]

The following results are well-known.

**Lemma 2.1.** Let $u \approx v$ be any non-trivial identity. Then

(i) $L_2^1$ satisfies $u \approx v$ if and only if $ip(u) = ip(v)$;
(ii) $R_2^1$ satisfies $u \approx v$ if and only if $fp(u) = fp(v)$;
(iii) $M$ satisfies $u \approx v$ if and only if $occ(x, u) = occ(x, v)$ for any $x \in X$.

3. **Three sufficient conditions for a semigroup to be finitely based**

In this section, we give three sufficient conditions under which a semigroup is finitely based. The next three theorems are the main results in this section.

**Theorem 3.1.** Suppose that a semigroup $S$ satisfies the identity $xyz \approx yz^2$ and for any identity $u \approx v$ satisfied by $S$,

(i) $occ(x, u) = occ(x, v)$ for any $x \in X$;
(ii) $fp(u) = fp(v)$.
Then the identity $yx ≈ y x^2$ is an identity basis for $S$, and so $S$ is finitely based.

**Proof.** It suffices to show that any identity $u ≈ v$ satisfied by $S$ can be derived from $yx ≈ y x^2$. For any $x ∈ \text{con}(u)$, if $\text{occ}(x, u) ≥ 2$, then the identity $yx ≈ y x^2$ can be used to gather any non-last $x$ in $u$ with the last $x$ in $u$, that is $u$ can be written into the form

$$u = x_1^{e_1} x_2^{e_2} \cdots x_m^{e_m},$$

where $fp(u) = x_1 x_2 \cdots x_m$. By the same argument, $v$ can be written into the form

$$v = y_1^{f_1} y_2^{f_2} \cdots y_n^{f_n},$$

where $fp(v) = y_1 y_2 \cdots y_n$. It follows from (ii) that $m = n, x_i = y_i$ for $i = 1, 2, \ldots, m$. And $e_i = f_i$ for $i = 1, 2, \ldots, m$ can be obtained from (i). Therefore $u = v$.

Consequently, every identity satisfied by $S$ is a consequence of the identity $yx ≈ y x^2$, and so the identity $yx ≈ y x^2$ is an identity basis for $S$.

**Theorem 3.2.** Suppose that a semigroup $S$ satisfies the identity $yx x y ≈ y x x y$ and for any identity $u ≈ v$ satisfied by $S$,

(i) $\text{occ}(x, u) = \text{occ}(x, v)$ for any $x ∈ X'$;

(ii) $\text{occ}_u(x, u) = \text{occ}_y(x, v)$ for any $x, y ∈ X'$;

(iii) $fp(u) = fp(v)$.

Then the identity $yx x y ≈ y x x y$ is an identity basis for $S$, and so $S$ is finitely based.

**Proof.** It suffices to show that any identity $u ≈ v$ satisfied by $S$ can be derived from $yx x y ≈ y x x y$. Since $S$ satisfies the identity $yx x y ≈ y x x y$, it follows that $u$ can be written into the form

$$u = x_1^{e_1} x_2^{e_2} \cdots x_m^{e_m},$$

where $fp(u) = x_1 \cdots x_m$, $e_i ≥ 1$ and $\text{con}(u_i) ⊆ \{x_i, \ldots, x_m\}$ for $1 ≤ i ≤ m$. For each $1 ≤ i ≤ m$, the letters in $u_i$ are not the last occurrences and so can be moved in any manner by using the identity $yx x y ≈ y x x y$. In particular, any occurrence of $x_i$ in $u_i$ can be moved to the right and combined with $x_i^{e_i}$ that immediately follows $u_i$. Therefore, we may further assume that $u_i = x_i^{h_{i+1}} \cdots x_i^{h_n}$ with $g_{i+1}, \ldots, g_m ≥ 0$ for $1 ≤ i ≤ m - 1$ and $u_m = \emptyset$. By the same argument, $v$ can be written into the form

$$v = y_1^{f_1} y_2^{f_2} \cdots y_n^{f_n},$$

where $fp(v) = y_1 y_2 \cdots y_n$, $f_i ≥ 1$ for $1 ≤ i ≤ n$, $v_i = y_i^{g_{i+1}} \cdots y_i^{g_n}$ with $g_{i+1}, \ldots, g_n ≥ 0$ for $1 ≤ i ≤ n - 1$ and $v_n = \emptyset$. It follows from (iii) that $m = n$ and $x_i = y_i$ for $1 ≤ i ≤ m$. It follows from (i) that $e_i = f_i$. For $1 ≤ i ≤ m$, since $x_i ∉ \text{con}(u_i, v_i)$, it follows that $e_i = \text{occ}_x(x_i, u)$ and $f_i = \text{occ}_x(x_i, v)$. Hence by (ii) $e_i = f_i$ for $1 ≤ i ≤ m$.

Clearly, $u_m = v_m = \emptyset$. To show that $u_i = v_i$ for $i = 1, 2, \ldots, m - 1$, it suffices to show that $\text{occ}(x_j, u_i) = \text{occ}(x_j, v_i)$ for $i + 1 ≤ j ≤ m$. Since

\[
\text{occ}(x_j, u_i) = \text{occ}(x_j, u) - \text{occ}_x(x_j, u)
= \text{occ}(x_j, v) - \text{occ}_x(x_j, v)
= \text{occ}(x_j, v_1)
\]

by (i) and (ii), it follows that $u_1 = v_1$. Since

\[
\text{occ}(x_j, u_1) = \text{occ}_x(x_j, u) - \text{occ}_x(x_j, u)
= \text{occ}(x_j, v_1)
\]

by (i) and (ii), it follows that $u_1 = v_1$. Since

\[
\text{occ}(x_j, u_1) = \text{occ}_x(x_j, u) - \text{occ}_x(x_j, u)
= \text{occ}(x_j, v_1)
\]
by (ii), it follows that $u_i = v_i$ for $1 < i \leq m - 1$. Therefore $u = v$. Consequently, every identity satisfied by $S$ is a consequence of the identity $xysxt \approx yxsxy$, and so the identity $xysxy \approx yxsxy$ is an identity basis for $S$.

\[\square\]

**Theorem 3.3.** Suppose that a semigroup $S$ satisfies the identities

\begin{align*}
(3.1a) & \quad xysxt \approx yxsxy, \\
(3.1b) & \quad xysxy \approx yxsxy,
\end{align*}

and for any identity $u \approx v$ satisfied by $S$,

(i) $\text{occ}(x, u) = \text{occ}(x, v)$ for any $x \in X'$;

(ii) $\delta \text{occ}_y(x, u) = \delta \text{occ}_y(x, v)$ for any $x, y \in X'$;

(iii) $\text{ip}(u) = \text{ip}(v) = \text{fp}(u) = \text{fp}(v)$.

Then the identities (3.1) constitute an identity basis for $S$, and so $S$ is finitely based.

**Proof.** It suffices to show that any identity $u \approx v$ satisfied by $S$ can be derived from (3.1). Suppose that $\text{con}(u) = \{x_1, x_2, \ldots, x_r\}$ and $\text{mix}(u) = a_1a_2\cdots a_{m+1}$ with $a_i \in \text{con}(u)$ for $i = 1, 2, \ldots, m + 1$. Clearly, $u$ can be written into the form

$u = a_1u_1a_2u_2\cdots a_mu_ma_{m+1}$

where $u_0, u_1, \ldots, u_m \in X^*$. Since each occurrence of letters in $u_i$ is neither its first occurrence nor its last occurrence in $u$, the letters in $u_i$ can be permuted within $u_i$ by the identities (3.1) in any manner. Therefore, we may assume that $u_i = x_1^{i_1}x_2^{i_2}\cdots x_r^{i_r}$ with some non-negative integers $i_1, i_2, \ldots, i_r$. Suppose that $\text{con}(v) = \{y_1, y_2, \ldots, y_s\}$ and $\text{mix}(v) = \{b_1, b_2, \ldots, b_{n+1}\}$ with $b_i \in \text{con}(v)$ for $i = 1, 2, \ldots, n + 1$. By the same argument, $v$ can be written into the form

$v = b_1v_1b_2v_2\cdots b_nv_{n+1}$

where $v_i = y_1^{i_1}y_2^{i_2}\cdots y_s^{i_s}$ with some non-negative integers $i_1, i_2, \ldots, i_s$. In the following, we will show that if $u \approx v$ satisfies the conditions (i)–(iii), then $u = v$.

It follows from (i) that either $\text{occ}(x, u) = \text{occ}(x, v) = 1$ or $\text{occ}(x, u) = \text{occ}(x, v) \geq 2$. Hence $m = n$.

Next we show that $\text{mix}(u) = \text{mix}(v)$. Clearly, $a_1 = b_1$ by (iii). Proceeding by induction, suppose that $a_1 \cdots a_{k-1} = b_1 \cdots b_{k-1}$. Assume that $a_k = x$ and $b_k = y$. If both $a_k$ and $b_k$ are the first occurrences of $x$ in $u$ and $y$ in $v$ respectively, then it follows from (i) $\text{ip}(u) = \text{ip}(v)$ that $a_k = b_k$; if both $a_k$ and $b_k$ are the last occurrences of $x$ in $u$ and $y$ in $v$ respectively, then it follows from (ii) $\text{fp}(u) = \text{fp}(v)$ that $a_k = b_k$.

Otherwise, by symmetry, we may assume that $a_k$ is the first occurrence of $x$ in $u$ and $b_k$ is the last occurrence of $y$ in $v$. Then by the above arguments the result $a_k = b_k$ still holds when either $\text{occ}(x, u) = 1$ or $\text{occ}(y, v) = 1$. Therefore, we may assume that $\text{occ}(x, u), \text{occ}(y, v) \geq 2$.

Suppose that $a_k \neq b_k$. Then $x \not\in \text{con}(a_1u_1\cdots a_{k-1}u_{k-1})$ by $a_k$ being the first occurrence of $x$ in $u$ and $y \not\in \text{con}(v_kb_{k+1}\cdots v_nb_{n+1})$ by $b_k$ being the last occurrence of $y$ in $v$. Hence it follows from (i) $\text{occ}(y, v) \geq 2$ that $\text{occ}(y, b_1\cdots b_{k-1}) = 1$, so that $\text{occ}(y, a_1\cdots a_{k-1}) = \text{occ}(y, a_{k+1}\cdots a_{n+1}) = 1$. Clearly, $x \not\in \text{con}(a_1\cdots a_{k-1}) = \text{con}(b_1\cdots b_{k-1})$. Hence $\text{occ}(y, v) = \delta \text{occ}_x(y, v)$. Now by (i) and (ii),

$\text{occ}(y, u) = \text{occ}(y, v) = \delta \text{occ}_x(y, v) = \delta \text{occ}_x(y, u).$

But $\text{occ}(y, u) = \delta \text{occ}_x(y, u)$ is impossible since $\text{occ}(y, a_{k+1}\cdots a_{n+1}) = 1$, hence $a_k = b_k$. Therefore $a_i = b_i$ for all $i = 1, \ldots, n + 1$ by induction, and so $\text{mix}(u) = \text{mix}(v)$.

Finally, we show that $u_k = v_k$ for each $k = 1, \ldots, n$. By the forms of $u$ and $v$, it suffices to show that $\text{occ}(z, u_k) = \text{occ}(z, v_k)$ for any $z \in \text{con}(u_k, v_k)$ and $k = 1, 2, \ldots, n$. Let $\text{occ}(z, u_k) = s$ and $\text{occ}(z, v_k) = t$. There are two cases.
Thus $s \geq t$ in both Case 1. and Case 2.

Hence $s \geq t$ in both Case 1. and Case 2.

2.2. $a_k$ is the first occurrence of $x$ and $a_{k+1}$ is the last occurrence of $y$ in both $u$ and $v$. Clearly, $z \neq y$. If $z = x$, then by (i),

$$1 + s = \delta \overleftarrow{cc}_y(x, u) = \delta \overleftarrow{cc}_y(x, v) = 1 + t.$$

Thus $s = t$ follows from (ii).

2.3. $a_k$ is the last occurrence of $x$ and $a_{k+1}$ is the first occurrence of $y$ in both $u$ and $v$. Clearly, $z \neq x, y$. Then by (i),

$$\delta \overleftarrow{cc}_x(z, u) + \delta \overleftarrow{cc}_x(z, v) - s = \delta \overleftarrow{cc}_x(z, u) = \delta \overleftarrow{cc}_x(z, v) + \delta \overleftarrow{cc}_y(z, v) - t.$$

Thus $s = t$ follows from (ii).

Hence $u_k = v_k$ for $k = 1, 2, \ldots, n$. Therefore $u = v$. Consequently, every identity satisfied by $S$ is a consequence of the identities in $\{3.1\}$, and so the identities\footnote{\cite{3.1}} constitute an identity basis for $S$. \hfill \Box

4. Finite basis problems for stalactic, taiga, sylvester and Baxter monoids

In this section, by applying the sufficient conditions given in Section 3, we solve the finite basis problems for all stalactic, taiga, sylvester and Baxter monoids of rank greater than or equal to 2.

4.1. Finite basis problems for stalactic and taiga monoids.

Lemma 4.1. \footnote{\cite{3.1} Propositions 15 and 16] } Both the stalactic monoid $\text{stal}_\infty$ and the taiga monoid $\text{taig}_\infty$ satisfy the identity $xyx \approx yx^2$.

Theorem 4.2. The identity $xyx \approx yx^2$ is a finite identity basis for the monoids $\text{stal}_n$ and $\text{taig}_n$, whenever $2 \leq n \leq \infty$. Therefore all stalactic and taiga monoids of rank greater than or equal to 2 are equationally equivalent.
Proof. Clearly, we only need to show that each of the monoids $\text{stal}_n$ and $\text{taig}_n$ for any $2 \leq n \leq \infty$ can be defined by the identity $xyx \approx yx^2$. First we show that each of the monoids $\text{taig}_n$ for any $2 \leq n \leq \infty$ can be defined by the identity $xyx \approx yx^2$. Note that

$$\text{taig}_1 \subset \text{taig}_2 \subset \cdots \subset \text{taig}_n \subset \cdots \subset \text{taig}_\infty.$$ 

By Theorem 3.1 it suffices to show that $\text{taig}_\infty$ satisfies the identity $xyx \approx yx^2$ and $\text{taig}_2$ satisfies the conditions (i) and (ii) in Theorem 3.1. Clearly, $\text{taig}_\infty$ satisfies the identity $xyx \approx yx^2$ by Lemma 4.1.

Let $u \approx v$ be any identity satisfied by the monoid $\text{taig}_2$. Since $\text{taig}_1$ is a free monogenic monoid, it follows from Lemma 2.1 (iii) that $\text{occ}(x, u) = \text{occ}(x, v)$ for any $x \in X$, and so the condition (i) holds in $\text{taig}_2$. Suppose that $\text{fp}(u) \neq \text{fp}(v)$. Then there exist some $x, y$ such that $\text{taig}_2$ satisfies $axy^s = u[x, y] \approx v[x, y] = bxy^t$ for some $s, t \geq 1$ and $a, b \in \{x, y\}^*$. Let $\varphi$ be a substitution such that $x \mapsto 2, y \mapsto 1$. Then $\varphi(u[x, y])$ ends with 2 and $\varphi(v[x, y])$ ends with 1. Since the rightmost symbol in a word $w$ determines the root node of $P_{\text{taig}_2}(w)$, it follows that $\varphi(u[x, y]) \neq \varphi(v[x, y])$. This implies that $\text{taig}_2$ does not satisfy $u[x, y] \approx v[x, y]$, a contradiction. Hence $\text{fp}(u) = \text{fp}(v)$, and so the condition (ii) holds.

Next we show that each of the monoids $\text{stal}_n$ for any $2 \leq n \leq \infty$ can be defined by the identity $xyx \approx yx^2$. Note that

$$\text{stal}_1 \subset \text{stal}_2 \subset \cdots \subset \text{stal}_n \subset \cdots \subset \text{stal}_\infty.$$ 

By Theorem 3.1 it suffices to show that $\text{stal}_\infty$ satisfies the identity $xyx \approx yx^2$ and $\text{stal}_2$ satisfies the conditions (i) and (ii) in Theorem 3.1. Clearly, $\text{stal}_\infty$ satisfies the identity $xyx \approx yx^2$ by Lemma 4.1. It is routine to show that $\text{stal}_2$ is isomorphic to $\text{taig}_2$. Hence $\text{stal}_2$ satisfies the conditions (i) and (ii) in Theorem 3.1 by the above arguments.

Consequently, each of monoids $\text{stal}_n, \text{taig}_n$ for any $2 \leq n \leq \infty$ can be defined by the identity $xyx \approx yx^2$, and so all of them are equationally equivalent. □

4.2. Finite basis problem for sylvester monoid.

Lemma 4.3. Let $u \approx v$ be any identity satisfied by the monoid $\text{sylv}_2$. Then the monoid $\text{sylv}_2$ satisfies the conditions (i)–(iii) in Theorem 3.2.

Proof. Let $u \approx v$ be any identity satisfied by $\text{sylv}_2$. Since $\text{sylv}_1$ is a free monogenic monoid, it follows from Lemma 2.1 (iii) that $\text{occ}(x, u) = \text{occ}(x, v)$ for any $x \in X$, and so the condition (i) holds in $\text{sylv}_2$. Suppose that $\text{occ}_\varphi(x, u) \neq \text{occ}_\varphi(x, v)$ for some $x, y \in X$. Then $\text{sylv}_2$ satisfies $axy^s = u[x, y] \approx v[x, y] = bxy^t$ for some $s, t \geq 1$, $s \neq t$ and $a, b \in \{x, y\}^*$. Without loss of generality, we may assume that $s < t$. Let $\phi$ be a substitution such that $x \mapsto 2, y \mapsto 1$. Using the Algorithm 3 one sees that

$$P_{\text{sylv}_2}(\phi(u[x, y])) =$$

$$P_{\text{sylv}_2}(\phi(v[x, y])) =$$

Consequently, each of monoids $\text{stal}_n, \text{taig}_n$ for any $2 \leq n \leq \infty$ can be defined by the identity $xyx \approx yx^2$, and so all of them are equationally equivalent. □
Then \( \varphi(u[x,y]) \neq \varphi(v[x,y]) \). This implies that \( \text{sylv}_2 \) does not satisfy \( u[x,y] \approx v[x,y] \), a contradiction. Hence \( \text{occ}_\varphi(x,u) = \text{occ}_\varphi(x,v) \) for any \( x,y \in X \), and so the condition (ii) holds.

Suppose that \( \text{fp}(u) \neq \text{fp}(v) \). Then there exists letters \( x,y \) such that \( \text{sylv}_2 \) satisfies \( ayx^s = u[x,y] \approx v[x,y] = bxy^t \) for some \( s,t \geq 1 \) and \( a,b \in \{x,y\}^* \). Let \( \varphi \) be a substitution such that \( x \mapsto 2, y \mapsto 1 \). Then \( \varphi(u[x,y]) \) ends with 2 and \( \varphi(v[x,y]) \) ends with 1. Since the rightmost symbol in a word \( w \) determines the root node of \( P_{\text{sylv}_2}(w) \), it follows that \( \varphi(u[x,y]) \neq \varphi(v[x,y]) \). This implies that \( \text{sylv}_2 \) does not satisfy \( u[x,y] \approx v[x,y] \), a contradiction. Hence \( \text{fp}(u) = \text{fp}(v) \), and so the condition (iii) holds.

**Lemma 4.4.** Let \( p,q,r \in \text{sylv}_\infty \) such that \( \text{ev}(p) = \text{ev}(q) \leq \text{ev}(r) \). Then \( pr = qr \).

**Proof.** In [2, Lemma 19], it is shown that if \( p,q,r \in \text{sylv}_\infty \) such that \( \text{ev}(p) = \text{ev}(q) = \text{ev}(r) \), then \( pr = qr \). In fact, by the proof of [2, Lemma 19], it is easy to see that the result still holds when \( \text{ev}(p) = \text{ev}(q) < \text{ev}(r) \). This is because every symbol \( d \) that from \( p \) or \( q \) is inserted into a particular previously empty subtree of \( P_{\text{sylv}_\infty}(r) \), dependent only on the value of the symbol \( d \) (and not on its position in \( p \) or \( q \)), and that unequal symbols are inserted into different subtrees. Since \( \text{ev}(p) = \text{ev}(q) \), the same number of symbols \( d \) are inserted for each such symbol \( d \). Hence if \( p,q,r \in \text{sylv}_\infty \) such that \( \text{ev}(p) = \text{ev}(q) \leq \text{ev}(r) \), then \( pr = qr \) still holds.

**Theorem 4.5.** The Sylvester monoid \( \text{sylv}_\infty \) satisfies the identity \( xysxty \approx yxsxty \).

**Proof.** Let \( \varphi : X \rightarrow \text{sylv}_\infty \) be any substitution. Then it is obvious that \( \text{ev}(\varphi(xy)) = \text{ev}(\varphi(yx)) \leq \text{ev}(\varphi(sxty)) \). Hence it follows from Lemma 4.4 that \( \varphi(xysxty) = \varphi(yxsxty) \). Therefore the Sylvester monoid \( \text{sylv}_\infty \) satisfies the identity \( xysxty \approx yxsxty \).

**Theorem 4.6.** The identity \( xysxty \approx yxsxty \) is a finite identity basis for the monoids \( \text{sylv}_n \) whenever \( 2 \leq n \leq \infty \). Therefore all Sylvester monoids of rank greater than or equal to 2 are equationally equivalent.

**Proof.** Clearly, we only need to show that each of the monoids \( \text{sylv}_n \) for any \( 2 \leq n \leq \infty \) can be defined by the identity \( xysxty \approx yxsxty \). Note that 

\[ \text{sylv}_1 \subset \text{sylv}_2 \subset \cdots \subset \text{sylv}_n \subset \cdots \subset \text{sylv}_\infty . \]

By Theorem 4.5, it suffices to show that \( \text{sylv}_\infty \) satisfies the identity \( xysxty \approx yxsxty \) and \( \text{sylv}_2 \) satisfies the conditions (i)–(iii) in Theorem 3.2. Therefore, the results hold directly from Theorem 4.5 and Lemma 4.4.

Symmetrically, we have

**Theorem 4.7.** The identity \( ytxsyx \approx ytxsxy \) is a finite identity basis for the monoids \( \text{sylv}_n^{\prime} \) whenever \( 2 \leq n \leq \infty \). Therefore all Sylvester monoids of rank greater than or equal to 2 are equationally equivalent.

### 4.3. Finite basis problem for Baxter monoid.

**Lemma 4.8.** Let \( p,q,r \in \text{baxt}_\infty \) such that \( \text{ev}(p) = \text{ev}(q) \leq \text{ev}(r) \), \( \text{ev}(s) \). Then \( \text{spr} = \text{sqr} \).

**Proof.** Since \( \text{ev}(p) = \text{ev}(q) \leq \text{ev}(r) \), it follows from Lemma 1.4 that \( P_{\text{sylv}_\infty}(pr) = P_{\text{sylv}_\infty}(qr) \). Thus \( P_{\text{sylv}_\infty}(spr) = P_{\text{sylv}_\infty}(sqr) \). Since \( \text{ev}(p) = \text{ev}(q) \leq \text{ev}(s) \), it follows from the dual of Lemma 1.4 that \( P_{\text{sylv}_\infty}(sp) = P_{\text{sylv}_\infty}(sq) \). Thus \( P_{\text{sylv}_\infty}(spr) = P_{\text{sylv}_\infty}(sqr) \). Therefore \( P_{\text{baxt}_\infty}(spr) = P_{\text{baxt}_\infty}(sqr) \), so that \( spr = sqr \).

**Theorem 4.9.** The Baxter monoid \( \text{baxt}_\infty \) satisfies the identities (3.1).
Proof. Let \( \varphi : \mathcal{X} \to \text{baxt}_{\infty} \) be any substitution. It is obvious that \( \text{ev}(\varphi(xy)) = \text{ev}(\varphi(xy)) \leq \text{ev}(\varphi(yxtyhxky)) \). By Lemma 4.8, we have \( \varphi(yxtyhxky) = \varphi(yxtyhxky) \). Therefore the Baxter monoid \( \text{baxt}_{\infty} \) satisfies the identity (3.1a).

A similar argument can show that the Baxter monoid \( \text{baxt}_{\infty} \) satisfies the identity (3.1b). \( \square \)

**Theorem 4.10.** The identities (3.1) constitute a finite identity basis for the monoids \( \text{baxt}_n \) whenever \( 2 \leq n \leq \infty \). Therefore all Baxter monoids of rank greater than or equal to 2 are equationally equivalent.

Proof. Clearly, we only need to show that each of the monoids \( \text{baxt}_n \) for any \( 2 \leq n \leq \infty \) can be defined by the identities (3.1). Note that

\[ \text{baxt}_1 \subset \text{baxt}_2 \subset \cdots \subset \text{baxt}_n \subset \cdots \subset \text{baxt}_{\infty}. \]

By Theorem 3.3, it suffices to show that \( \text{baxt}_{\infty} \) satisfies the identities (3.1) and \( \text{baxt}_2 \) satisfies the conditions (i)–(iii) in Theorem 3.3.

Clearly, the Baxter monoid \( \text{baxt}_{\infty} \) satisfies the identities (3.1) by Theorem 4.9. Since both \( \text{syv}_2 \) and \( \text{syv}_1^2 \) are homomorphic images of \( \text{baxt}_2 \) by the definition of Baxter monoid, it follows from Lemma 4.8 and its dual that \( \text{baxt}_2 \) satisfies the conditions (i)–(iii) in Theorem 3.3. Consequently, each of monoids \( \text{baxt}_n \) for any \( 2 \leq n \leq \infty \) can be defined by the identities (3.1), and so all of them are equationally equivalent. \( \square \)

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