MAXIMUM LINEARIZATIONS OF LOWER SETS IN $\mathbb{N}^m$ WITH APPLICATION TO MONOMIAL IDEALS

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Abstract. We compute the type (maximum linearization) of the well partial order of bounded lower sets in $\mathbb{N}^m$, ordered under inclusion, and find it is $\omega^{\omega m - 1}$; we give two proofs of this statement. Moreover we compute the type of the set of all lower sets in $\mathbb{N}^m$, a topic studied by Aschenbrenner and Pong in [3], and find that it is equal to
\[ \sum_{k=1}^{m} \omega^{m-k} (\binom{m}{k-1}) + 1. \]
As a consequence we deduce corresponding lower bounds on effectively given sequences of lower sets and effectively given sequences of monomial ideals in $F[X,Y]$ where $F$ is a field.

1. Introduction

In this paper we compute the type of several well partial orders. The type of a well partial order $X$, denoted $o(X)$, is the largest order type of a well-order extending the order on $X$; this was proven to exist by De Jongh and Parikh [5]. The type $o(X)$ can also be characterized inductively as the smallest ordinal greater than $o(Y)$ for any proper lower set $Y$ of $X$. The theory has been rediscovered several times; the term “type” comes from Kriz and Thomas [7].

In this paper we are interested in well partial orders whose elements are lower sets in the partial order $\mathbb{N}^m$. Here, a lower set is a set that is downwardly closed, that is, a set $S$ such that if $t \leq s$ and $s \in S$ then $t \in S$; they are also known as initial segments. We define:

Definition 1.1. If $X$ is a partial order, we define $I(X)$ to be the poset of lower sets in $X$ ordered under inclusion, and define $D(X)$ to consist of those elements of $I(X)$ that can be obtained as the downward closure of finitely many elements. (Such lower sets are also said to be finitely generated initial segments; a set is said to generate its downward closure as an initial segment.)

Then we are interested in $D(\mathbb{N}^m)$ and $I(\mathbb{N}^m)$. (Note that in the case of $\mathbb{N}^m$, we could equivalently define $D(\mathbb{N}^m)$ to be the set of bounded lower sets, or the set of finite lower sets.) We prove the following three theorems:

Theorem 1.2. For $m \geq 1$,
\[ o(D(\mathbb{N}^m)) = \omega^{\omega m - 1}. \]

Theorem 1.3. For $m \geq 1$,
\[ o(D(\mathbb{N}^m \times k)) = \omega^{\omega m - 1} k. \]
Theorem 1.4.
\[ o(I(\mathbb{N}^m)) = \omega\sum_{k=1}^{m-1} \omega^{m-k} + 1. \]

The last of these questions, that of determining \( o(I(\mathbb{N}^m)) \), was asked earlier by Aschenbrenner and Pong [3], who provided upper and lower bounds. Theorem 1.4 provides an exact answer to this question.

Theorem 1.2, the case of \( D(\mathbb{N}^m) \), is the “core” case, handled by means of the inductive characterization of \( o(X) \) above. Meanwhile, the cases of \( D(\mathbb{N}^m \times k) \) and \( I(\mathbb{N}^m) \) are handled combinatorially, by using Theorem 1.2 in combination with De Jongh and Parikh’s theorems that \( o(X \sqcup Y) = o(X) \square o(Y) \) and \( o(X \times Y) = o(X) \otimes o(Y) \), where \( \square \) and \( \otimes \) are the natural (or Hessenberg) sum and product of ordinals.

More specifically, the case of \( D(\mathbb{N}^m \times k) \) is handled by putting together \( k \) copies of \( D(\mathbb{N}^m) \), while the case of \( I(\mathbb{N}^m) \) is handled by putting together \( D(\mathbb{N}^m) \) together with \( D(\mathbb{N}^C) \), where \( C \) ranges over nonempty subsets of \( \{1, \ldots, m\} \).

In Section 4 we will show that the lengths of effectively given sequences of lower sets and effectively given sequences of monomial ideals in \( F[X, Y] \) are bounded from below by Hardy functions whose levels are determined by the type of the underlying well partial ordering.

In a future paper [2], we will extend these results to lower sets in products of larger ordinals as well.

2. Bounded lower sets in \( \mathbb{N}^m \)

In this section we show how to compute \( o(D(\mathbb{N}^m)) \), proving Theorem 1.2.

First, we recall some basic properties of the type:

Proposition 2.1 (De Jong and Parikh, [5]). Let \( X \) and \( Y \) be well partial orders. If \( X \) embeds in \( Y \), then \( o(X) \leq o(Y) \). Similarly, if there is a weakly increasing surjection from \( Y \) onto \( X \), then \( o(X) \leq o(Y) \). In particular, if \( \leq \) and \( \leq' \) are two well partial orderings on the set \( X \), and \( \leq \leq' \) extends \( \leq \), then \( o(X, \leq') \leq o(X, \leq) \).

Also, if \( X \) and \( Y \) are any two well partial orders, one has
\[ o(X \sqcup Y) = o(X) \square o(Y) \]
and
\[ o(X \times Y) = o(X) \otimes o(Y), \]
where \( \square \) and \( \otimes \) are the natural (or Hessenberg) sum and product of ordinals. As such, if \( X \) is a well partial order and \( S, T \subseteq X \), then \( o(S \cup T) \leq o(S) \square o(T) \).

In order to prove Theorem 1.2 we apply a similar lemma (which had been communicated by Schnoebelen and Schmitz to the authors and which recently was made available on arXiv [11]).

Lemma 2.2 ([11]). Let \( X \) be a well partial order. Then
\[ o(D(X)) \leq 2^{o(X)}. \]

Note that Abriola et al. actually stated their lemma not for \( D(X) \), but rather for the more commonly-studied \( (\psi_{\text{fin}}(X), \leq_m) \), where \( \psi_{\text{fin}}(X) \) denotes the set of finite subsets of \( X \) and where we define \( S \leq_m T \) if for every \( s \in S \), there is some \( t \in T \) with \( s \leq t \). Of course, \( (\psi_{\text{fin}}(X), \leq_m) \) is not actually isomorphic to \( D(X) \), as the former lacks antisymmetry, being only a quasi-order rather than a partial order;
but after quotienting out by equivalences the resulting partial order is isomorphic to $D(X)$. So in essence these are the same.

For convenience of the reader we include a proof of Lemma 2.2.

Proof. We use standard arguments from [9] (following the lines of [10]). First note that if $o(X) = 0$ (i.e. $X$ is empty), the statement is trivial.

Now suppose that $o(X)$ is a limit ordinal. Then $2^o(X)$ is a power of $\omega$, i.e., additively closed. Given $a \in X$, let $X^a := \{x \in X : x \not\geq a\}$, a proper lower subset of $X$; so $o(X^a) < o(X)$. Let $I$ be an element of $D(X)$ and let

$$S := \{ J \in D(X) : J \not\supseteq I \};$$

we need to show that $o(S) < 2^{o(X)}$. Take a finite set $A$ such that $I$ is the downward closure of $A$. Then for each $J \in S$ we have $J \not\supseteq A$, so $J \in D(X^a)$ for some $a \in A$. This shows that $S \subseteq \bigcup_{a \in A} D(X^a)$ and therefore that $o(S) \leq \bigoplus_{a \in A} o(D(X^a))$.

By the inductive hypothesis, for each $a \in X$ we have $o(D(X^a)) \leq 2^{o(X^a)} < 2^{o(X)}$.

As $2^{o(X)}$ is additively closed, we obtain $o(S) < 2^{o(X)}$ as desired.

Finally suppose that $o(X)$ is a successor; say $o(X) = \eta + 1$. Then there exists an $x \in X$ such that $x$ is maximal in $X$ and $o(X \setminus \{x\}) = \eta$; this $x$ exists by Theorem 3.2 of De Jongh and Parikh [5]. So if $I \in D(X)$, then either $I \in D(X \setminus \{x\})$ or $x \in I$. Moreover, we have an increasing surjection

$$J \mapsto J \cup \{ x \} : D(X \setminus \{ x \}) \to \{ I \in D(X) : x \in I \}.$$

So, applying the inductive hypothesis, $o(D(X)) \leq 2^\eta \oplus 2^\eta$. Since $2^\eta$ necessarily contains only a single distinct power of $\omega$ in its Cantor normal form, one has $2^\eta \oplus 2^\eta = 2^{\eta+1}$, which in turn is equal to $2^{o(X)}$; thus $o(D(X)) \leq 2^{o(X)}$. This completes the proof. □

Corollary 2.3. $o(D(\mathbb{N}^m)) \leq \omega^{m-1}$.

Proof. One has $o(\mathbb{N}^m) = \omega^m$, so

$$o(D(\mathbb{N}^m)) \leq 2^{\omega^m} = (2^\omega)^{m-1} = \omega^{m-1}.\tag*{□}$$

Now we prove the lower bound:

Proposition 2.4. $o(D(\mathbb{N}^m)) \geq \omega^{m-1}$.

Proof. For a sequence $a = (a_1, \ldots, a_m)$ of length $m - 1$ we define $ord(a) = \omega^{m-2} \cdot a_1 + \cdots + \omega^{0} \cdot a_m$. For a finite non-empty downward closed subset $F$ in $\mathbb{N}^m$ assume that $F$ is the downward closure of $s(F) = \{(a_1, b_1), \ldots, (a_l, b_l)\}$ where $a_i$ is in $\mathbb{N}^{m-1}$ and $b_i$ is in $\mathbb{N}$ and each $(a_i, b_i)$ is maximal with respect to the pointwise ordering. Let $ord(s(F))$ be the natural sum over $1 \leq i \leq l$ of the terms $\omega^{ord(a_i)} \cdot b_i$. Let $ord(F) := 1 + ord(s(F))$. If $F$ is empty then $ord(F) := 0$. (Note the the zero vector describes the second minimal element.) We prove by induction on the cardinality of $s(G)$ that $F \subseteq G$ implies $ord(F) \leq ord(G)$; the proposition then follows from this.

So assume $\emptyset \neq F \subseteq G$ and assume that $s(F) = \{(a_1, b_1), \ldots, (a_k, b_k)\}$ and $s(G) = \{(c_1, d_1), \ldots, (c_l, d_l)\}$. Let $S_1 := \{(a, b) \in s(F) : \neg(a, b) \leq (c_1, d_1)\}$ and $S_2 := \{(a, b) \in s(F) : (a, b) \leq (c_1, d_1)\}$. Then $S_1 \subseteq s(G) \setminus \{(c_1, d_1)\}$ and by induction hypothesis we may assume that $ord(s(S_1)) \leq ord(s(G) \setminus \{(c_1, d_1)\})$ if $S_1$ is not empty. It thus suffices to show $ord(S_2) \leq ord(s(G) \setminus \{(c_1, d_1)\})$. If $S_2$ is a
singleton then the assertion follows easily. Problems might occur when $S_2$ is not a singleton because $\text{ord} (\{(c_1, d_1)\})$ is in general not additively closed. We may assume after renumbering that $S_2 = \{(a_1, b_1), \ldots, (a_n, b_n)\}$. Assume that there is an $(a_i, b_i) \in S_2$ such that $a_i = c_1$. (The case that $a_i \neq c_1$ for all $i$ is similar but easier.) Then $b_i = d_i$ is excluded because if $(a_j, b_j) \in S_2$ is another element then $(a_j, b_j) \leq (c_1, d_1) = (a_i, b_i)$ and $(a_j, b_j)$ would not be maximal. Therefore $b_i < d_i$.

Now pick any $(a_j, b_j) \in S_2$ different from $(a_i, b_i)$. Then $a_j = a_i$ is impossible since then either $(a_i, b_i)$ is not maximal if $b_i < b_j$ or $(a_j, b_j)$ is not maximal if $b_j < b_i$. Since $(a_i, b_i) \leq (c_1, d_1)$ we conclude $a_j \leq c_1 = a_i$. Since $a_j \neq c_1$ we conclude that $a_j$ is lexicographically smaller than $c_1$ so that $\text{ord}(a_j) < \text{ord}(c_1)$. This means that all such terms $(a_j, b_j)$ get assigned ordinals $\omega^{\text{ord}(a_j)}, b_j < \omega^{\text{ord}(c_1)}$.

Summing up all terms for elements in $S_2$ we get a strict upper bound provided by $\omega^{\text{ord}(c_1)} \cdot b_i + \omega^{\text{ord}(c_1)} \leq \omega^{\text{ord}(c_1)} \cdot d_1 = \text{ord}(\{(c_1, d_1)\})$. □

This proves Theorem 1.2.

Proof of Theorem 1.2. We have $o(D(\mathbb{N}^m)) \leq \omega^{\omega^{m-1}}$ from Corollary 2.3 and the reverse inequality from Proposition 2.4 so $o(D(\mathbb{N}^m)) = \omega^{\omega^{m-1}}$. □

2.1. Bounded lower sets in $\mathbb{N}^m \times k$. Before we move on to $J(\mathbb{N}^m)$, let’s briefly consider $D(\mathbb{N}^m \times k)$. We stated the type of this in Theorem 1.3. In this subsection we prove it. First some notation:

Notation 2.5. For $X$ a partially-ordered set and $x \in X$, we define the upward closure $U_X(x)$ to be $\{y \geq x : y \in X\}$; this is the smallest upward closed subset of $X$ containing $x$. We may also just write $U(x)$ when $X$ is clear.

Now the proof:

Proof of Theorem 1.3. To prove the upper bound, note that there’s an obvious embedding of $D(\mathbb{N}^m \times k)$ into $D(\mathbb{N}^m)^k$, by mapping

$$S \mapsto (S \cap (\mathbb{N}^m \times \{0\}), \ldots, S \cap (\mathbb{N}^m \times \{k-1\})), $$

so

$$o(D(\mathbb{N}^m \times k)) \leq o(D(\mathbb{N}^m))^k = \omega^{\omega^{m-1}k}. $$

This leaves the lower bound. For this, we induct on $k$. The case $k = 1$ has already been proven above, so that leaves the inductive step.

We will construct a total order extending $D(\mathbb{N}^m \times k)$ that has the required order type. First, choose a total order extending $D(\mathbb{N}^m \times \{k-1\})$ of order type $\omega^{\omega^{m-1}}$; this is possible by the above. We will sort the elements $S$ of $D(\mathbb{N}^m \times k)$ first by the value of $S \cap (\mathbb{N}^m \times \{k-1\})$ (according to this order), and then find some way to break the ties.

So consider some element $T \in D(\mathbb{N}^m \times \{k-1\})$ and consider the set $P_T$ of $S \in D(\mathbb{N}^m \times k)$ such that $S \cap (\mathbb{N}^m \times \{k-1\}) = T$. What is the maximum extending ordinal of this set? To answer this, observe that there is some element $x \in \mathbb{N}^m$ such that $(x, k-1) \notin T$. So in fact $U(x) \times \{k-1\}$ is disjoint from $T$; and $U(x)$ is isomorphic to $\mathbb{N}^m$. This gives us an inclusion of $D(\mathbb{N}^m \times (k-1))$ into $P_T$, so $o(P_T)$ is (by the induction hypothesis) at least $\omega^{\omega^{m-1}(k-1)}$.

Therefore $o(D(\mathbb{N}^m \times k)) \geq \omega^{\omega^{m-1}(k-1)}\omega^{\omega^{m-1}} = \omega^{\omega^{m-1}k}$. This completes the proof. □
3. General lower sets in $\mathbb{N}^m$

In this section we compute $o(I(\mathbb{N}^m))$. As we will see, $I(\mathbb{N}^m) \setminus \{\mathbb{N}^m\}$ can be approximately decomposed as a product over nonempty $C \subseteq \{1, \ldots, m\}$ of $D(\mathbb{N}^C)$; however, the exact nature of this decomposition will be slightly different in the upper bound proof and in the lower bound proof.

3.1. The upper bound proof. In this section we prove a proposition that expresses one half of this decomposition.

We will need the following lemma, which is an easy consequence of some known facts:

**Lemma 3.1.** Let $P = \alpha_1 \times \ldots \times \alpha_m$ be a finite Cartesian product of well-orders. Then any lower set of $P$ is a finite union of rectangles $\beta_1 \times \ldots \times \beta_m$ for some $\beta_i \leq \alpha_i$.

**Proof.** In general, a lower set in a well partial order is a finite union of ideals, which is a downward-closed set $I$ with the additional property that if $x, y \in I$, there exists $z \geq x, y$ with $z \in I$; one may see e.g. [6] for a proof, where this is a combination of Lemma 2.6 and Proposition 2.10. Moreover, the ideals of $X \times Y$ are precisely the sets $I \times J$ where $I$ is an ideal of $X$ and $J$ is an ideal of $Y$; again one may see [6], where this appears as Proposition 4.8. Since obviously an ideal of $\alpha_i$ is an ordinal $\beta_i \leq \alpha_i$, the result follows. $\square$

We also define $I_0(X)$ to denote $I(X) \setminus \{X\}$, as a bit of notation we will occasionally use.

Now, we can prove:

**Proposition 3.2.** Consider $\mathbb{N}^m$, and for a subset $C$ of $[m] := \{1, \ldots, m\}$, let $\pi_C$ be the projection from $\mathbb{N}^{|m|}$ onto $\mathbb{N}^C$. Define a map

$$\varphi : \prod_{\emptyset \neq C \subseteq [m]} D(\mathbb{N}^C) \to I_0(\mathbb{N}^{|m|})$$

by

$$\varphi((S_C)_{C \in \wp([m]) \setminus \{\emptyset\}) = \bigcup_{\emptyset \neq C \subseteq [m]} \pi^{-1}_C(S_C).$$

Then $\varphi$ is monotonic, surjective, and well-defined (i.e., its image lies within $I(\mathbb{N}^{|m|})$, and does not contain $\mathbb{N}^{|m|}$ as an element).

**Proof.** That $\varphi$ is monotonic is obvious. To see that $\varphi$ is well-defined, note that any finitely generated lower set $T \subseteq \mathbb{N}^C$, is, so long as $C \neq \emptyset$, a proper lower set, and thus its inverse image $\pi^{-1}_C(T)$ is a proper lower subset of $\mathbb{N}^{|m|}$. Moreover, the union of any two proper lower subsets of $\mathbb{N}^{|m|}$ is again a proper lower subset, since if one excludes a point $x$ and the other excludes a point $y$, then their union will exclude any point that is at least both $x$ and $y$, such as their join.

This leaves surjectivity. So say we have some $T \in I_0(\mathbb{N}^{|m|})$, by Lemma [5.1], write it as a finite union of rectangles $T = \bigcup_{k=1}^r T_k$, where each $T_k$ can be written as $\alpha_{k,1} \times \ldots \times \alpha_{k,m}$ for $\alpha_{k,i} \leq \omega$.

Now, it’s easy to see that if we have a tuple $((U_C)_{C \in \wp([m]) \setminus \{\emptyset\})$ and another tuple $((V_C)_{C \in \wp([m]) \setminus \{\emptyset\})$, then

$$\varphi((U_C \cup V_C)_{C \in \wp([m]) \setminus \{\emptyset\}) = \varphi((U_C)_{C \in \wp([m]) \setminus \{\emptyset\}) \cup \varphi((V_C)_{C \in \wp([m]) \setminus \{\emptyset\}).$$
As such it suffices to prove that each rectangle $T_k$ lies in the image of $\varphi$; so for simplicity just assume $T = T_1$ and write $T = \alpha_1 \times \ldots \times \alpha_m$.

So define $C_0 = \{ i : \alpha_i < \omega \}$; since $T \neq N^{|m|}$, this means that $C_0 \neq \emptyset$. Then define $S_{C_0} = \pi_{C_0}(T)$, and $S_C = \emptyset$ for $C \neq C_0$. Then since $\pi_{C_0}^{-1}(S_{C_0}) = T$, we have $\varphi((S_C)_{C \in \wp(|m|) \setminus \{\emptyset\}}) = T$, as needed.

Thus we can conclude the upper bound:

**Theorem 3.3.**

\[ o(I(N^{|m|})) \leq \omega^{\sum_{k=1}^{m} \omega^{-k}\binom{m}{k-1}} + 1. \]

**Proof.** Applying Proposition 3.2 with $\alpha_i = \omega$ for all $i$, together with Theorem 1.2 yields that

\[ o(I_0(N^{|m|})) \leq \omega^{\sum_{k=1}^{m} \omega^{-k}\binom{m}{k-1}}; \]

since $I(N^{|m|}) = I_0(N^{|m|}) \cup \{N^{|m|}\}$, we conclude

\[ o(I_0(N^{|m|})) \leq \omega^{\sum_{k=1}^{m} \omega^{-k}\binom{m}{k-1}} + 1. \]

3.2. The lower bound proof. For the proof of the lower bound, we will need some additional definitions. Rather than deal with fully specified lower sets in $I(N^{|m|})$, we will also define “partial specifications” of such sets.

**Definition 3.4.** Given a function $f : S \to T$ and $A \subseteq S$, define the “intersection image” $\overline{T}(A)$ to be $T \setminus f(S \setminus A)$, or equivalently to be $\{p \in T : f^{-1}(p) \subseteq A\}$.

**Definition 3.5.** A partial specification $X$ on $N^{|m|}$ consists of a nonempty collection $\mathcal{C}$ of subsets of $[m]$ and, for each $C \in \mathcal{C}$, some $X_C \in I_0(N^{|C|})$, such that:

- if $D \subseteq [m]$ and $C \in \mathcal{C}$ with $|D| < |C|$, then $D \in \mathcal{C}$, and
- if $D \subseteq C \in \mathcal{C}$, then $X_D = \pi_D(X_C)$. (Here of course $\pi_D$ denotes the projection from $N^D$ to $N^C$.)

Given a partial specification $X$ on $N^{|m|}$ and a set $S \in I_0(N^{|m|})$, we will say that $S$ is compatible with $X$ if $\pi_C(S) = X_C$ for each $C \in \mathcal{C}$. We define $\mathcal{A}_X$ to be the set of all $S \in I_0(N^{|m|})$ compatible with $X$.

Observe that, for a partial specification $X$ with domain $\mathcal{C}$, if $X_C$ is known for all maximal elements $C \in \mathcal{C}$ (under inclusion), then $X_D$ is known for all $D \in \mathcal{C}$.

We will show here how to get a lower bound on $o(\mathcal{A}_X)$ for any partial specification $X$, based only on the domain of $X$. Then, to get a lower bound on $o(I_0(N^{|m|}))$, we need only take $X$ to be the unique partial specification on $N^{|m|}$ with domain $\{\emptyset\}$, since every proper lower set in $N^{|m|}$ is compatible with this specification. (Conversely, if the domain of $X$ is $\wp([m])$, then $X_{[m]}$ is the unique element of $I_0(N^{|m|})$ that is compatible with $X$.)

With both the components of the upper and lower bounds laid out, we can now prove the theorem.

Now the proof:

**Proposition 3.6.** Let $X$ be a partial specification $X$ on $N^{|m|}$ with domain $\mathcal{C}$. Then

\[ o(\mathcal{A}_X) \geq \bigotimes_{C \in \mathcal{C}} \omega^{\omega^{(|C|-1)}}. \]
In fact, by the arguments above, this lower bound will actually be an equality, but we only care about the lower bound. Note \( o(\mathcal{A}_X) \) increases as the domain of \( X \) gets smaller; the less-specified \( X \) is, the more sets are compatible with it.

**Proof.** We prove this by downward induction on the size of the domain. It’s trivially true for any partial specification \( X \) on \( \mathbb{N}^m \) with domain \( \varnothing([m]) \), since in this case one will have \( |\mathcal{A}_X| = 1 \) and the product will be 1 as well. So suppose \( \mathcal{C} \subseteq [m] \) is a valid domain for a partial specification and that the proposition holds for all partial specifications on \( \mathbb{N}^m \) with that domain. Pick some \( C \in \mathcal{C} \) of maximum cardinality; we want to show the statement holds for any partial specification with domain \( \mathcal{C} \setminus \{ C \} \).

So let \( X \) be a partial specification with domain \( \mathcal{C} \setminus \{ C \} \). We want to put a total order on \( \mathcal{A}_X \) in order to get a lower bound on \( o(\mathcal{A}_X) \). Given any \( S \in \mathcal{A}_X \), we can obtain a partial specification \( Y \) with domain \( \mathcal{C} \) by taking \( Y_D = \pi_D(S) \) for \( D \in \mathcal{C} \); observe then that \( S \in \mathcal{A}_Y \). Obviously, any such \( Y \) has \( Y_D = X_D \) for any \( D \neq C \); the only distinguishing feature of \( Y \) is the value of \( Y_C \).

Note that not every \( T \in I_0(\mathbb{N}^C) \) is a possible value of \( Y_C \), since if \( T = Y_C \) we have the restriction that for \( D \subseteq C \) we have \( \pi_D(T) = X_D \). But given such a \( T \) we can define \( Y(T) \) to be \( Y \) obtained by setting \( Y_C = T \). So we will put a total order on \( \mathcal{A}_X \) by first putting a total order on the set of such \( T \) (call this set \( \mathcal{T} \)), and sorting elements \( S \) of \( \mathcal{A}_X \) by the value of \( \pi_C(S) \); and then, for each such \( T \), putting a total order on \( \mathcal{A}_Y(T) \). So we will get a lower bound on \( o(\mathcal{A}_X) \) of the form \( \sum_{T \in \mathcal{T}} o(\mathcal{A}_Y(T)) \) (using the total order on \( \mathcal{T} \) that we have picked).

In fact, by the inductive hypothesis, for any \( T \in \mathcal{T} \), we know that

\[
o(\mathcal{A}_Y(T)) \geq \bigotimes_{D \notin \mathcal{C}} \omega^{\omega^{\omega^{\omega^{\cdots} - 1} - 1} - 1}.
\]

Thus, we immediately get that

\[
o(\mathcal{A}_X) \geq \bigotimes_{D \notin \mathcal{C}} \omega^{\omega^{\omega^{\omega^{\cdots} - 1} - 1} - 1} o(\mathcal{T}).
\]

It then remains to show that \( o(\mathcal{T}) \geq \omega^{\omega^{\omega^{\omega^{\cdots} - 1} - 1}} \). Once we know this, we will have

\[
o(\mathcal{A}_X) \geq \bigotimes_{D \notin \mathcal{C}} \omega^{\omega^{\omega^{\omega^{\cdots} - 1} - 1}},
\]

because, by assumption, \( |C| \leq |D| \) for any \( D \notin \mathcal{C} \), and so the ordinary product here coincides with the natural product.

So let \( A \subseteq \mathbb{N}^C \) be defined by \( A = \bigcup_{D \subseteq C} \pi_D^{-1}(X_D) \). Then for any \( V \in D(\mathbb{N}^C) \), \( A \cup V \in \mathcal{T} \). Pick some \( b \in (\mathbb{N}^C) \setminus A \), and consider \( U(b) := U_{\mathbb{N}^C}(b) \). Given \( V \in D(U(b)) \) let \( L(V) \) be the downward closure of \( V \) in \( \mathbb{N}^C \). Observe that the map from \( D(U(b)) \) to \( \mathcal{T} \) given by \( V \mapsto A \cup L(V) \) is injective and indeed an embedding. Also observe that \( U(b) \) is isomorphic to \( \mathbb{N}^C \). So by Theorem \( \ref{thm:embedding} \) \( o(D(U(b))) = \omega^{\omega^{\omega^{\omega^{\cdots} - 1} - 1}} \), and so \( o(\mathcal{T}) \geq \omega^{\omega^{\omega^{\omega^{\cdots} - 1} - 1}} \), as needed. This completes the proof.

We can now prove the lower bound:

**Theorem 3.7.**

\[
o(I(\mathbb{N}^m)) \geq \omega^{\sum_{k=1}^m \omega^{m-k} \binom{m}{k}} + 1.
\]
Proof. Let $X$ be the unique partial specification on $\mathbb{N}^m$ with domain $\{\emptyset\}$; then $I_0(\mathbb{N}^m) = \mathcal{A}_X$. By Proposition 3.6 then,

$$o(I_0(\mathbb{N}^m)) = o(\mathcal{A}_X) \geq \omega^2 \cdot (m - k).$$

Therefore

$$o(I(\mathbb{N}^m)) \geq \omega^2 \cdot (m - k) + 1,$$

proving the theorem.

3.3. Putting together the proof. Finally, we can put the above together to yield the proof.

Proof of Theorem 1.4. The upper bound is Theorem 3.3 and the lower bound is Theorem 3.7.

4. Application to monomial ideals

We now discuss applications to computational complexity and provide complementary results to Corollary 3.27 in [3]. In the sequel we work with ordinals below $\omega^\omega$. For these ordinals we look at the multiply recursive functions defined by the Hardy functions $H_\omega : \mathbb{N} \to \mathbb{N}$ which are defined recursively as follows. Let $H_0(x) := x$, $H_{\alpha+1}(x) := H_\alpha(x + 1)$ and for a limit $\lambda$ let $H_\lambda(x) := H_\lambda(x + 1)$ where $\lambda[x]$ denotes the $x$-th member of the canonical fundamental sequence for $\lambda$. These fundamental sequences are defined by recursion as follows. If $\lambda = \omega^\lambda$ with $\lambda'$ a limit then $\lambda[x] = \omega^{\lambda'[x]}$. If $\lambda = \omega^{\beta+1}$ then $\lambda[x] = \omega^{\beta+1} \cdot x$. If $\lambda = \omega^{\beta} + \lambda'$ with $\lambda' < \lambda$ a limit then $\lambda[x] = \omega^{\beta} + \lambda'[x]$.

For technical reasons we also define $(\alpha + 1)[x] := \alpha$.

By standard results (see, for example, Lemma 4 in [4]) it is known that $H_{\omega^\omega}$ is a variant of the non-primitive recursive Ackermann function, and $H_{\omega^{\omega+2}}$ is roughly the result of iterating the Ackermann function twice.

Let us define a complexity measure for downward closed sets in $\mathbb{N}^k$. For finite $\alpha$ put $M\alpha = \alpha$ and for $\alpha = \omega$ put $M\alpha = 0$. This measure is extended to cartesian products of initial intervals as follows: put $M(\alpha_1 \times \cdots \times \alpha_k) := \max\{M\alpha_i : i \leq k\}$. Almost the same measure can be applied when dealing with monomial ideals which can be indentified with upward closed sets in $\mathbb{N}^k$ (see the next theorem).

If a downward closed set $D$ is a finite union of $k$-times cartesian products of initial intervals $J_i$ then we put $MD := \max\{M(J_i)\}$. Then for any natural number $d$ there will only be finitely many downward closed sets of cardinality not exceeding $d$. In the sequel we stick for simplicity to cartesian products with two factors and so we let $k := 2$. We believe that the case for $k > 2$ can be carried out analogously.

Proposition 4.1. For a given $K \in \mathbb{N}$ there exists a sequence $(D_i)_{i=1}^L$ of downward closed sets contained in $\mathbb{N}^2$ such that $L \geq H_{\omega^{\omega+2}}(K) - K$, such that $MD_i \leq (K + i)^2$ for $1 \leq i \leq L$, and such that $D_i$ is not contained in $D_j$ for $1 \leq i < j \leq L$.

Proof. Let $\alpha_0 := \omega^{\omega+2}$ and let $\alpha_{i+1} := \alpha_i[K + i]$. Then $\alpha_i > 0$ yields $\alpha_i > \alpha_{i+1}$ and moreover we find

$$H_{\alpha_0}(K) = H_{\alpha_0[K+1]}(K+1) = \ldots = H_{\alpha_0[K+\ldots[K+L-1]]}(K+L) = K + L$$

where $L$ is minimal with $\alpha_L = 0$. 

For $\alpha = \omega^{\omega+1} \cdot p + \omega^r \cdot q + \omega^{a_1} \cdot b_1 + \ldots + \omega^{a_r} \cdot b_r$ in normal form where $p, q, r \geq 0$ and $a_1 > \ldots > a_r$ and $b_1, \ldots, b_r > 0$ let

$$N \alpha := p + q + b_1 + \cdots + b_r + \max \{a_i\}.$$ 

Then an induction on $i$ yields $N \alpha_i \leq (K + i)^2$.

For $\alpha = \omega^{\omega+1} \cdot p + \omega^r \cdot q + \omega^{a_1} \cdot b_1 + \ldots + \omega^{a_r} \cdot b_r$ in normal form define a downward closed set $D(\alpha)$ as follows:

$$D(\alpha) := p \times \mathbb{N} \cup \mathbb{N} \times q \cup \{(p + a_1 + 1, q + b_1), \ldots, (p + a_r + 1, q + b_1 + \ldots + b_r)\} \subseteq \mathbb{N}.$$ 

Then $M(D(\alpha)) \leq N \alpha$.

Assume that $\alpha' = \omega^{\omega+1} \cdot p' + \omega^r \cdot q' + \omega^{a_1'} \cdot b_1' + \ldots + \omega^{a_r'} \cdot b_r'$ is in normal form and assume $\alpha' < \alpha$. We show that $D(\alpha)$ is not a subset of $D(\alpha')$. The proof can be established by a simple case distinction.

Case 1. $p' < p$. Then $p \times \mathbb{N}$ is not contained in $D(\alpha')$.

Case 2. $p' = p$ and $q' < q$. Then $\mathbb{N} \times q$ is not contained in $D(\alpha')$.

Case 3. $p = p'$, $q = q'$ and there exists a $j_0$ such that $a_{j_0} < a_{j_0}$ or $(a_{j_0} = a'_{j_0}$ and $b'_{j_0} < b_{j_0})$ and for all $l < j_0$ we have $a_l = a'_l$ and $b_l = b'_l$.

Then $\{(p + a_{j_0} + 1, q + b_1 + \ldots + b_{j_0})\} \subseteq \mathbb{N}$ is not contained in $D(\alpha')$. This can be checked by verifying that $(p + a_{j_0} + 1, q + b_1 + \ldots + b_{j_0})$ is in no interval showing up in the representation of $D(\alpha')$. The first two intervals are left out since $p < p + a_{j_0} + 1$ and $q < q + b_1 + \ldots + b_{j_0}$. The intervals with index $i > j_0$ do not contain $p + a_{j_0} + 1$ in their left coordinates and the intervals with index $i < j$ do not contain $q + b_1 + \ldots + b_{j_0}$ in their right coordinates. A similar argument applies for $i = j_0$.

The result follows by putting $D_i := D(\alpha_i)$ for $i \geq 1$. \qed

Let us now consider polynomial rings in two variables over a field $F$. We believe that the cases $k > 2$ can be carried out analogously.

The degree of a monomial ideal is the maximum degree of the minimal generating set of monomials. We denote by $(m_1, \ldots, m_l)$ the monomial ideal generated by the monomials $m_i$. The degree of a monomial ideal with minimal representation $(m_1, \ldots, m_l)$ is equal to $\max\{\deg(m_i)\}$

**Theorem 4.2.** For a given $K \in \mathbb{N}$ there exists a sequence $(I_i)_{i=1}^L$ of monomial ideals contained in $F(X, Y)$ such that $L \geq H_{\omega+2}(K) - K$ and such that $\deg(I_i) \leq (K + i)^2$ for $1 \leq i \leq L$ and such for $i < j$ the ideal $I_i$ is not contained in the ideal $I_j$.

**Proof.** We associate with an ordinal smaller than $\omega^{\omega+2}$ a monomial ideal $I(\alpha)$ such that for the descending sequence of ordinals $\alpha_i$ from the last lemma we find $\deg(I(\alpha_i)) \leq (K + i)^2$ for $i \geq 1$ and that $i < j$ yields that $I(\alpha_j)$ is not contained in $I(\alpha_i)$.

Assume that $\alpha = \omega^{\omega+1} \cdot p + \omega^r \cdot q + \omega^{a_1} \cdot b_1 + \ldots + \omega^{a_r} \cdot b_r$ is in normal form. Let $c_j := b_1 + \cdots + b_r$. Let

$$I(\alpha) := (X^{a_1+p+1}Y, X^{a_2+p+1}Y^{c_1+q+1}, \ldots, X^{a_r+p+1}Y^{c_{r-1}+q+1}, X^{p}Y^{c_r+q+1}).$$ 

For $r = 0$ we put $I(\alpha) := (X^{p+1} \cdot X^q, X^p \cdot Y^{q+1})$.

Assume that $\alpha' = \omega^{\omega+1} \cdot p' + \omega^r \cdot q' + \omega^{a_1'} \cdot b_1' + \ldots + \omega^{a_r'} \cdot b_r'$ is in normal form and assume $\alpha' > \alpha$. Let $c'_j := b'_1 + \cdots + b'_r$. We show that $I(D(\alpha))$ is not a subset of $I(D(\alpha'))$.

The proof can be established by a simple case distinction.
Case 1. $p < p'$. Then $X^{p+1} \alpha_{p+1}$ is not an element of $I(\alpha')$ (even for $r = 0$) since all generators of $I(\alpha')$ contain a multiple of $X^{p'}$.

Case 2. $p = p'$ and $q < q'$. Then $X^{q+1} \alpha_{q+1}$ (or $X^{q+1} \alpha_{q+1}$ in the case $r = 0$) is not an element of $I(\alpha')$ since all generators of $I(\alpha')$ contain a multiple of $Y^{q'}$.

Case 3. $p = p'$ and there exists a $j_0$ such that $a_{j_0} < a'_{j_0}$ or $(a_{j_0} = a'_{j_0}$ and $b_{j_0} < b'_{j_0})$ and for all $l < j_0$ we have $a_l = a'_l$ and $b_l = b'_l$.

Case 3.1. $a_{j_0} < a'_{j_0}$. Then $X^{a_{j_0}+p+1}Y^{c_{j_0}+1+q+1} \not\in I(\alpha')$. Indeed, for $i < j_0$ we have $a_{j_0} < a_i$ and hence

$$X^{a_{j_0}+p+1}Y^{c_{j_0}+1+q+1} \not\in (X^{a'_i+p+1}Y^{c'_i+1+q+1}).$$

If $i = j_0$ then from $a_{j_0} < a'_j$ we conclude

$$X^{a_{j_0}+p+1}Y^{c_{j_0}+1+q+1} \not\in (X^{a'_j+p+1}Y^{c'_j+1+q+1}).$$

For $i > j_0$ we find $X^{a_{j_0}+p+1}Y^{c_{j_0}+1} \not\in (X^{a'_i+p+1}Y^{c'_i+1+q+1})$ because $c_{j_0+1} = c'_{j_0+1} < c'_i$.

Case 3.2. $(a_{j_0} = a'_j$ and $b_{j_0} < b'_{j_0})$. Then $X^{a_{j_0+1}+p+1}Y^{c_{j_0}+1+q+1} \not\in I(\alpha')$. Indeed, for $i < j_0$ we obtain $X^{a_{j_0+1}+p+1}Y^{c_{j_0}+1+q+1} \not\in (X^{a'_i+p+1}Y^{c'_i+1+q+1})$ since $a_{j_0+1} = a_i+1 \geq a_{j_0} > a_{j_0+1}$. For $i \geq j_0$ we conclude

$$X^{a_{j_0+1}+p+1}Y^{c_{j_0}+1+q+1} \not\in (X^{a'_i+p+1}Y^{c'_i+1+q+1}).$$

since $c_{j_0} < c'_{j_0} \leq c_i$. 

The lower bounds provided by previous lemmas are essentially sharp in the sense that $K \to L$ depends elementary recursively on $H_\alpha$ where $\alpha$ is the maximal order type under consideration. This can be shown by a reification analysis using the results on the upper bound for the maximal order type involved. For this one can exploit that the lengths of elementary descending sequences of ordinals can be bounded in terms of the Hardy functions as shown for example in [4].

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