Automorphic $L$-Functions and Functoriality

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Abstract

This is a report on the global aspects of the Langlands-Shahidi method which in conjunction with converse theorems of Cogdell and Piatetski-Shapiro has recently been instrumental in establishing a significant number of new and surprising cases of Langlands Functoriality Conjecture over number fields. They have led to striking new estimates towards Ramanujan and Selberg conjectures.

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1. Preliminaries

Let $F$ be a number field. For each place $v$ of $F$, let $F_v$ be its completion at $v$. Assume $v$ is a finite place and let $O_v$ denote the ring of integers of $F_v$. Denote by $P_v$ its maximal ideal and fix a uniformizing parameter $\varpi_v$ generating $P_v$. Let $[O_v : P_v] = q_v$ and fix and absolute value $| \cdot |_v$ for which $|\varpi_v|_v = q_v^{-1}$.

Let $G$ be a quasisplit connected reductive algebraic group over $F$. Fix an $F$-Borel subgroup $B = TU$, where $T$ is a maximal torus of $B$ and $U$ is its unipotent radical. Let $A_0 \subset T$ be the maximal split subtorus of $T$. Throughout this article, $P$ is a maximal parabolic subgroup of $G$, defined over $F$, with a Levi decomposition $P = MN$, where $M$ is a Levi subgroup of $P$ and $N$ is its unipotent radical. We will assume $P$ is standard in the sense that $N \subset U$. We fix $M$ by assuming $T \subset M$. We finally use $W$ to denote the Weyl group of $A_0$ in $G$.

Let $A_F$ denote the ring of adeles of $F$ and for every algebraic group $H$ over $F$, let $H = H(A_F)$. Considering $H$ as a group over each $F_v$, we then set $H_v = H(F_v)$.

Let $A$ denote the split component of $M$, i.e., the maximal split subtorus of the connected component of the center of $M$. For every group $H$ defined over $F$, let

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$X(H)_F$ be the group of $F$-rational characters of $H$. We set $a = \text{Hom}(X(M)_F, \mathbb{R})$. Then $a^* = X(M)_F \otimes \mathbb{R} = X(A)_F \otimes \mathbb{R}$ and $a_\mathbb{C}^* = a^* \otimes \mathbb{C}$ is the complex dual of $a$.

When $G$ is unramified over a place $v$, we let $K_v = G(O_v)$. Otherwise, we shall fix a special maximal compact subgroup $K_v \subset G_u$ for which $G_v = P_v K_v = B_v K_v$. Let $K = \oplus_v K_v$ Then $G = PK = BK$. Let $K_M = K \cap M$.

For each $v$, the embedding $X(M)_F \hookrightarrow X(M)_{F_v}$ induces a map

$$a_v = \text{Hom}(X(M)_{F_v}, \mathbb{R}) \to a.$$

There exists a homomorphism $H_M : M \to a$ defined by

$$\exp(\chi, H_M(m)) = \prod_v [\chi(m_v)]_v$$

for every $\chi \in X(M)_F$ and $m = (m_v)$. We extend $H_M$ to $H_P$ on $G$ by making it trivial on $N$ and $K$.

Let $\alpha$ denote the unique simple root of $A$ in $N$. It can be identified by a unique simple root of $A_0$ in $U$. If $\rho$ is half the sum of $F$-roots in $N$, we set $\tilde{\alpha} = \langle \rho, \alpha \rangle^{-1} \rho \in a^*$, where for each pair of non-restricted roots $\alpha$ and $\beta$ of $T$, $\langle \alpha, \beta \rangle = 2(\alpha, \beta)/\langle \beta, \beta \rangle$ is the Killing form.

Given a connected reductive algebraic group $H$ over $F$, let $^LH$ be its $L$-group. Considering $H$ as a group over $F_v$, we then denote by $^LH_v$ its $L$-group over $F_v$. Let $^LH^0 = ^LH^0_v$ be the corresponding connected component of 1. We then have a natural homomorphism from $^LH_v$ into $^LH$. We let $\eta_v : ^L M_v \to ^L M$ be this map for $M$ (cf. [4]).

Let $^L N$ be the $L$-group of $N$ defined naturally in [4]. Let $^L n$ be its (complex) Lie algebra, and let $r$ denote the adjoint action of $^L M$ on $^L n$. Decompose $r = \bigoplus_{i=1}^m r_i$ to its irreducible subrepresentations, indexed according to the values $\langle \tilde{\alpha}, \beta \rangle = i$ as $\beta$ ranges among the positive roots of $T$. More precisely, $X_{\tilde{\alpha}^*} \subset ^L n$ lies in the space of $r_i$ if and only if $\langle \tilde{\alpha}, \beta \rangle = i$. Here $X_{\tilde{\alpha}^*}$ is a root vector attached to the coroot $\tilde{\alpha}^*$, considered as a root of the $L$-group. The integer $m$ is equal to the nilpotence class of $^L n$. We let $r_{i, v} = r_i \cdot \eta_v$ for each $i$ (cf. [34,40,41]).

If $\Delta$ denotes the set of simple roots of $A_0$ in $U$, we use $\theta \subset \Delta$ to denote the subset generating $M$. Then $\Delta = \theta \cup \{\alpha\}$. There exists a unique element $\tilde{w}_0 \in W$ such that $\tilde{w}_0(\theta) \subset \Delta$, while $\tilde{w}_0(\alpha) < 0$. We will always choose a representative $w_0$ for $\tilde{w}_0$ in $G(F)$ and use $w_0$ to denote each of its components.

2. Eisenstein series and $L$-functions

Let $\pi = \otimes_v \pi_v$ be a cusp form on $M$. Given a $K_M$-finite function $\varphi$ in the space of $\pi$, we extend $\varphi$ to a function $\tilde{\varphi}$ on $G$ as in Section 2 of [39] as well as in [17], and for $s \in \mathbb{C}$, set

$$\phi_s(g) = \tilde{\varphi}(g) \exp(s\alpha + \rho \rho, H_P(g)).$$

(2.1)
The corresponding Eisenstein series is then defined by

\[ E(s, \phi_s, g, P) = \sum_{\gamma \in \mathbf{P}(F) \backslash \mathbf{G}(F)} \phi_s(\gamma g) \]  
\[ (2.2) \]

(cf. [17,33,34,35]).

Let \( I(s, \pi) = \otimes_v I(s, \pi_v) \) be the representation parabolically induced from \( \pi \otimes \exp(\mathfrak{s}_0, \mathcal{H}_p()) \).

Let \( \mathbf{M}' \) be the Levi subgroup of \( \mathbf{G} \) generated by \( \bar{\omega}(\theta) \). There exists a parabolic subgroup \( \mathbf{P}' \supset \mathbf{B} \) which has \( \mathbf{M}' \) as a Levi factor. Let \( \mathbf{N}' \) be its unipotent radical. Given \( f \) in the space of \( I(s, \pi) \) and \( \text{Re}(s) \gg 0 \), define the global intertwining operator \( M(s, \pi) \) by

\[ M(s, \pi)f(g) = \int_{\mathbf{N}'} f(w_0^{-1}n'g)dn' \]  
\[ (2.3) \]

Observe that if \( f = \otimes_v f_v \), then for almost all \( v \), \( f_v \) is the unique \( K_v \)-fixed functions normalized by \( f_v(e_v) = 1 \). Finally, if at each \( v \) we define a local intertwining operator by

\[ A(s, \pi_v, w_0)g_v(g) = \int_{\mathbf{N}_v'} f_v(w_0^{-1}n'g)dn' \]  
\[ (2.4) \]

then

\[ M(s, \pi) = \otimes_v A(s, \pi_v, w_0) \]  
\[ (2.5) \]

It follows form the general theory of Eisenstein series that the poles of \( E(s, \bar{\varphi}, g, P) \), as \( \bar{\varphi} \) and \( g \) vary, are the same as those of \( M(s, \pi) \), and for \( \text{Re}(s) \geq 0 \), they are all simple and finite in number, with none on the line \( \text{Re}(s) = 0 \) (cf. [17,33,35]).

By construction each \( \phi_s \) belongs to the space of \( I(s, \pi) \). Consequently, one can consider \( M(s, \pi)\phi_s \) which is a member of \( I(-s, w_0(\pi)) \). The Eisenstein series \( E(s, \bar{\varphi}, g, P) \) then satisfies the functional equation

\[ E(s, \phi_s, g, P) = E(-s, M(s, \pi)\phi_s, g, P') \]  
\[ (2.6) \]

Suppose that \( \mathbf{G} \) splits over \( L \), where \( L \) is a finite Galois extension of \( F \). For every unramified \( v \), there exists a unique Frobenius conjugacy class in \( \text{Gal}(L_w/F_v) \), \( \tau_v \) which we denote by \( \tau_v \). Moreover, if \( v \) is such that \( \pi_v \) and \( \mathbf{G} \) are both unramified, then there exists and \( L \mathbf{M} \) semisimple conjugacy class in \( L \mathbf{M} \times \tau_v \) which determines \( \pi_v \) uniquely ([40]). We may identify, as we in fact do, this conjugacy class with an element \( A_v \in LT^0 \) which may be assumed to be fixed by \( \tau_v \) (cf. \S 6.3 and 6.5 of [4]). The local Langlands \( L \)-function defined by \( \pi_v \) and \( r_v, r_v = r \cdot \eta_v \), where \( r \) is a complex analytic representation of \( L \mathbf{M} \), is then defined to be (cf. [4,34,40]),

\[ L(s, \pi_v, r_v) = \det(I - r_v(A_v \times \tau_v)q_v^{-1})^{-1} \]  
\[ (2.7) \]

Let \( S \) be a finite set of places of \( F \), including all the archimedean ones, such that for every \( v \notin S \), \( \pi_v \) and \( \mathbf{G} \) are both unramified. Set

\[ L_S(s, \pi, r) = \prod_{v \notin S} L(s, \pi_v, r_v) \]  
\[ (2.8) \]
The main result of [34, also see 40] is that
\[ M(s, \pi)f = \otimes_{v \in S} A(s, \pi_v, w_0)f_v \otimes \otimes_{v \in S} \bar{f}_v \]
\[ \times \prod_{i=1}^{m} L_S(is, \pi, \bar{r}_i)/L_S(1 + is, \pi, \bar{r}_i), \]  
(2.9)

where \( f = \otimes_v f_v \) is such that for each \( v \notin S \), \( f_v \) is the unique \( K_v \)-fixed function in \( I(s, \pi_v) \) normalized by \( f_v(e_v) = 1 \) and for each \( i \), \( \bar{r}_i \) denotes the contragredient of \( r_i, i = 1, \ldots, m \), the irreducible components of the adjoint action of \( {}^k M \) or \( {}^k N \). Here \( \bar{f}_v \) is the \( K_v \)-fixed function in the space of \( I(-s, w_0(\pi_v)) \), normalized the same way. Moreover \( f_v \) and \( \bar{f}_v \) are identified as elements in spherical principal series.

3. Generic representations and the non-constant term

Suppose \( F \) is a field, either local or global, and \( G \) is as before, with a Borel subgroup \( B = TU \) over \( F \). Fix an \( F \)-splitting \( \{X_{\alpha}^i\} \), i.e., a collection of root vectors as \( \alpha' \) ranges over simple roots of \( T \) in \( U \) which is invariant under the action of \( \Gamma_F = \text{Gal}(F/F) \). This then determines a map \( \phi \) form \( U \) to \( \Pi G_{\alpha}, \varphi(u) = (x_{\alpha'})_{\alpha'} \), where \( x_{\alpha'} \) is the \( \alpha' \)-coordinate of \( u \) with respect to \( \{X_{\alpha}^i\} \). Let \( \{\kappa_{\alpha'}\} \) be a collection of elements in \( F^\times \) such that \( \sigma(\kappa_{\alpha'}) = \kappa_{\sigma\alpha'} \) for every \( \sigma \in \Gamma_F \). Set \( f(u) = \sum_{\alpha'} \kappa_{\alpha'} x_{\alpha'} \).

Observe that \( f \) is \( F \)-rational. If \( F \) is global, we extend \( f \) to a map on \( U(\mathbb{A}_F) \). Let \( \varphi \) be a non-trivial character of \( F \) (\( F \setminus \mathbb{A}_F \) if \( F \) is global). A character \( \chi \) of \( U(F)(U(F) \setminus U(\mathbb{A}_F) \setminus U(\mathbb{A}_F)) \) if \( F \) is global) is called non-degenerate or generic if \( \chi(u) = \varphi(f(u), u \in U(F)(u \in U(F) \setminus U(\mathbb{A}_F) \) if \( F \) is global).

We now continue to assume \( F \) is a number field. Let \( \chi = \otimes_v \varphi_v \) be a generic character of \( U(F) \setminus U \).

Let \( U^0 = U \cap M \) and let \( \chi \) also denote the restriction of \( \chi \) to \( U^0 \). Choose a function \( \varphi \) in the space of \( \pi = \otimes_v \pi_v \), a cuspidal representation of \( M \), and \( U^0(F) \setminus U^0 \) being compact, set
\[ W_\varphi(m) = \int_{U^0(F) \setminus U^0} \varphi(um)\chi(u)du. \]  
(3.1)

We shall say \( \pi \) is (globally) \( \chi \)-generic if \( W_\varphi \neq 0 \) for some \( \varphi \). The representation \( \pi \) is (globally) generic if it is \( \chi \)-generic with respect to some generic \( \chi \). Then each \( \pi_v \) will be \( \chi_v \)-generic in the sense that there exists a non-zero Whittaker functional \( \lambda_v \), i.e., a continuous (in the semi-norm topology if \( v = \infty \)) functional satisfying \( \langle \pi_v(u)x, \lambda_v \rangle = \chi_v(u)(x, \lambda_v), x \in \mathcal{H}(\pi_v), u \in U^0_v \). Choosing \( \varphi \) appropriately, i.e., if \( \varphi = \otimes_v \varphi_v, \varphi_v \in \mathcal{H}(\pi_v) \), then \( W_\varphi(m) = \prod_v (\pi_v(m_v)\varphi_v, \lambda_v) \), for \( m = (m_v) \).

Given \( f_v \in V(s, \pi_v) \), the space of \( I(s, \pi_v) \), define
\[ \lambda_{\lambda_v}(s, \pi_v)(f_v) = \int_{N_v^0} \langle f_v(w_0^{-1}n'), \lambda_v \rangle du', \]  
(3.2)
a canonical Whittaker functional for $I(s, \pi_v)$. Changing the splitting we now assume $\kappa_{\alpha'} = 1$. It now follows from Rodier’s theorem that there exists a complex function (of $s$), $C_{\chi_v}(s, \pi_v)$, depending on $\pi_v, \chi_v$ and $w_0$ such that (cf. [41,42,43])

$$\lambda_{\chi_v}(s, \pi_v) = C_{\chi_v}(s, \pi_v) \lambda_{\chi_v}(-s, w_0(\pi_v)) \cdot A(s, \pi_v, w_0). \quad (3.3)$$

This is what we call the Local Coefficient attached to $s, \pi_v, \chi_v$ and $w_0$. The choice of $w_0$ is now specified by our fixed splitting as in [43].

Finally, if

$$E_\chi(s, \phi_s, g, P) = \int_{U(F) \backslash U} E(s, \phi_s, ug, P) \chi(u) du \quad (3.4)$$

is the $\chi$-nonconstant term of the Eisenstein series, then ([7,41,42])

$$E_\chi(s, \phi_s, e, P) = \prod_{v \in S} W_v(e) \prod_{i=1}^m L_G(1 + is, \pi, r_i)^{-1}, \quad (3.5)$$

where now $S$ is assumed to have the property that if $v \not\in S$, then $\chi_v$ is also unramified.

Applying Definition (3.4) to both sides of (2.6), using (3.5) now implies the crude functional equation ([40,41])

$$\prod_{i=1}^m L_G(is, \pi, r_i) = \prod_{v \in S} C_{\chi_v}(s, \pi_v) \prod_{i=1}^m L_G(1 - is, \pi, \overline{r_i}). \quad (3.6)$$

4. The main induction, functional equations and multiplicativity

To prove the functional equation for each $r_i$ with precise root numbers and $L$-function, we use (cf. [42]):

**Proposition 4.1.** Given $1 < i \leq m$, there exists a quasisplit group $G_i$ over $F$, a maximal $F$-parabolic subgroup $P_i = M_i N_i$, both unramified for every $v \not\in S$, and a cuspidal automorphic form $\pi'$ of $M_i = M_i(\mathbb{A}_F)$, unramified for every $v \not\in S$, such that if the adjoint action $r'$ of $L M_i$ on $L n_i$ decomposes as $r' = \bigoplus_{j=1}^{m'} r'_j$, then

$$L_S(s, \pi, r_i) = L_S(s, \pi', r'_i).$$

Moreover $m' < m$.

**Remark 4.2.** As was observed by Arthur [1], each $M_i$ can be taken equal to $M$ and $\pi' = \pi$. In fact each $G_i$ can be taken to be an endoscopic group for $G$, sharing $M$ as a Levi subgroup. We shall record this as

**Proposition 4.3.** Given $i$, $1 < i \leq m$, there exist a quasisplit connected reductive $F$-group with $M$ as a Levi subgroup and $m' < m$ for which $r'_i = r_i$. 

Using this induction and local-global arguments (cf. Proposition 5.1 of [42]), it was proved in [42] that

**Theorem 4.4.** (Theorems 3.5 and 7.7 of [42]) a) For each $i$, $1 \leq i \leq m$, and each $v$, there exist a local $L$-function $L(s, \pi_v, r_{i,v})$, which is the inverse of a polynomial in $q_v^{-s}$ whose constant term is 1, if $v < \infty$, and is the Artin $L$-function attached to $r_i \cdot \varphi'_v$, where $\varphi'_v : W_{F_v} \to L^{M_v}$ is the homomorphism of the Deligne-Weil group into $L^{M_v}$ parametrizing $\pi_v$, if either $v = \infty$ or $\pi_v$ has an Iwahori-fixed vector; and a root number $\varepsilon(s, \pi_v, r_{i,v}, \varphi_v)$ satisfying the same provisions, such that

$$L(s, \pi, r_i) = \prod_v L(s, \pi_v, r_{i,v})$$

(4.1)

and

$$\varepsilon(s, \pi, r_i) = \prod_v \varepsilon(s, \pi_v, r_{i,v}, \psi_v),$$

(4.2)

then

$$L(s, \pi, r_i) = \varepsilon(s, \pi, r_i)L(1 - s, \pi, r_i).$$

(4.3)

b) Let

$$\gamma(s, \pi_v, r_{i,v}, \psi_v) = \varepsilon(s, \pi_v, r_{i,v}, \psi_v)L(1 - s, \pi_v, r_{i,v})/L(s, \pi_v, r_{i,v}).$$

(4.4)

Then each $\gamma(s, \pi_v, r_{i,v}, \psi_v)$ is multiplicative in the sense of equation (2.13) in Theorem 3.5 of [42]. (See below.) If $\pi_v$ is tempered, then $\gamma(s, \pi_v, r_{i,v}, \psi_v)$ determines the corresponding root number and $L$-function uniquely and in fact that is how they are defined. Suppose $\pi_v$ is non-tempered, then each $L(s, \pi_v, r_{i,v})$ is determined by means of the analytic continuation of its quasi-tempered Langlands parameter and multiplicativity of corresponding $\gamma$-functions. More precisely, if $\pi_v$ is the quasitempered Langlands parameter that gives $\pi_v$ as a subrepresentation, then

$$L(s, \pi_v, r_{i,v}) = \prod_{j \in S_i} L(s, \overline{\pi}_j(\sigma_v), r'_{i(j),v}),$$

(4.5)

where the notation is as in part 3) of Theorem 3.5 of [42], provided that every $L$-function on the right hand side is holomorphic for $\Re(s) > 0$, whenever $\pi_v$ is (unitary) tempered (Conjecture 7.1 of [42], proved in many cases [3.6.42]). The set $S_i, \overline{\pi}_j$ and $r'_{i(j)}$ are defined as follows in which we drop the index $v$. Assume $\pi \subset \text{Ind}_{M_\theta(N_\theta \cap M)}M_{\sigma} \otimes 1$, where $M_{\theta}(N_\theta \cap M)$ is a parabolic subgroup of $M$ defined by a subset $\theta \subset \Delta$, the set of simple roots of $A_0$. Let $\theta' = \overline{w}_0(\theta) \subset \Delta$ and fix a reduced decomposition $\overline{w}_0 = \overline{w}_{n-1} \cdots \overline{w}_1$ of $\overline{w}_0$ (Lemma 2.1.1 of [41]). For each $j$, there exists a unique root $\alpha_j \in \Delta$ such that $\overline{w}_j(\alpha_j) < 0$. For each $j$, $2 \leq j \leq n - 1$, let $\overline{\pi}_j = \overline{w}_{j-1} \cdots \overline{w}_1$. Set $\overline{w}_1 = 1$. Let $\Omega_j = \theta_j \cup \{\alpha_j\}$, where $\theta_1 = \theta$, $\theta_n = \theta'$, and $\theta_{j+1} = \overline{w}_j(\theta_j), 1 \leq j \leq n - 1$. Then $M_{\Omega_j}$ contains $M_{\theta_j}(N_{\theta_j} \cap M_{\Omega_j})$ as a maximal parabolic subgroup and $\overline{w}_j(\sigma)$ is a representation of $M_{\theta_j}$. The $L$-group $L^{M_{\theta_j}}$ acts on the space of $r_i$, but no longer necessarily irreducibly. Given an irreducible constituent of this action, there exists a unique $j$, $1 \leq j \leq n - 1$, which under $\overline{w}_j$ is equivalent to an irreducible constituent of the action of $L^{M_{\theta_j}}$ on the Lie algebra of the $L$-group of $N_{\theta_j} \cap M_{\Omega_j}$. Let $i(j)$ be the index of this subspace and denote by
$r'_i(j)$ the action of $L^iM_{r'_i}$ on it. Finally, let $S_i$ denote the set of all such $j$'s for a given $i$. (See Theorem 3.5 and Section 7 of [42]. Also see the discussion just before Proposition 5.2 of [2.8].)

**Remark 4.5.** If $G = GL_{t+n}, M = GL_t \times GL_n$ and $\pi = \otimes_v \pi_v$ and $\pi' = \otimes_v \pi'_v$ are cuspidal representations of $GL_t(\mathbb{A}_F)$ and $GL_n(\mathbb{A}_F)$, then $m = 1$ and $L(s, \pi \otimes \pi', r_i)$ is precisely the Rankin-Selberg product $L$-function $L(s, \pi \times \pi')$ attached to $(\pi, \pi')$ (cf. [21,43,44]). In this case each of the local $L$-functions and root numbers are precisely those of Artin through parametrization which is now available for $GL_N(F_v)$ for any $N$ due to the work Harris-Taylor [18] and Henniart [19]. As we explain later, this will also be the case for many of our local factors as a result of our new cases of functoriality which we shall soon explain. This is quite remarkable, since our factors are defined using harmonic analysis, as opposed to the very arithmetic nature of the definition given for Artin factors. This is a perfect example of how deep Langlands' conjectures are.

**Remark 4.6.** The multiplicativity of local factors, in the sense of Theorem 3.4, are absolutely crucial in establishing our new cases of functoriality throughout our proofs [12,23,28]. In fact, not only do we need them to prove our strong transfers, they are also absolutely necessary in establishing our weak ones.

## 5. Twists by highly ramified characters, holomorphy and boundedness

While the functional equations developed from our method are in perfect shape and completely general, nothing that general can be said about the holomorphy and possible poles of these $L$-functions. On the other hand, there has been some remarkable new progress on the question of holomorphy of these $L$-function, mainly due to Kim [24,25,31]. They rely on reducing the existence of the poles to that of existence of certain unitary automorphic forms, which in turn points to the existence of certain local unitary representations. One then disposes of these representations, and therefore the pole, by checking the corresponding unitary dual of the local group. In view of the functional equation, this needs to be checked only for $\text{Re}(s) \geq 1/2$. In fact, to carry this out, one needs to verify that:

$$\text{Certain local normalized (as in [41]) intertwining operators are holomorphic and non-zero for } \text{Re}(s) \geq 1/2,$$

in each case [24,25,31]. The main issue is that one cannot always get such a contradiction and rule out the pole. In fact, there are many unitary duals whose complementary series extend all the way to $\text{Re}(s) = 1$.

On the other hand, if one considers a highly ramified twist $\pi_\eta$ (see Theorem 5.1 below) of $\pi$, then it can be shown quite generally that every $L(s, \pi_\eta, r_i)$ is entire (cf. [45] for its local analogue). In fact, if $\eta$ is highly ramified, then $w_0(\pi_\eta) \not\equiv \pi_\eta$, whose negation is a necessary condition for $M(s, \pi_\eta)$ to have a pole, a basic fact from Langlands spectral theory of Eisenstein series (Lemma 7.5 of [33]). This was used by Kim [24], and in view of the present powerful converse theorems [8,9], that
is all one needs to prove our cases of functoriality [12,23,28,30]. To formalize this, we borrow the following proposition (Proposition 2.1) from [28], in order to state the result. It is a consequence of our general induction (Propositions 4.1 and 4.3) and [24].

**Theorem 5.1.** Assume (5.1) is valid. Then there exists a rational character \( \xi \in X(M_F) \) with the following property: Let \( S \) be a non-empty finite set of finite places of \( F \). For every globally generic cuspidal representation \( \pi \) of \( M = M(A_F) \), there exist non-negative integers \( f_v, v \in S \), depending only on the local central characters of \( \pi_v \) for all \( v \in S \), such that for every grössencharacter \( \eta = \otimes_v \eta_v \) of \( F \) for which conductor of \( \eta_v, v \in S \), is larger than or equal to \( f_v \), every \( L \)-function \( L(s, \pi_v, r_i), i = 1, \ldots, m \), is entire, where \( \pi_\eta = \pi \otimes (\eta \cdot \xi) \). The rational character \( \xi \) can be simply taken to be \( \xi(m) = \text{det}(\text{Ad}(m)|n), m \in M \), where \( n \) is the Lie algebra of \( N \).

The last ingredient in applying converse theorems is that of boundedness of each \( L(s, \pi_v, r_i) \) in every vertical strip of finite width, away from its poles, which are finite in number, again using the functional equation and under Assumption (5.1). This was proved in full generality by Gelbart-Shahidi [15], using the theory of Eisenstein series via [33] and [36]. The main theorem of [15] (Theorem 4.1) is in full generality, allowing poles for \( L \)-functions. Here we will state the version which applies to our \( \pi_\eta \).

**Theorem 5.2.** Under Assumption (5.1), let \( \xi \) and \( \eta \) be as in Theorem 5.1. Assume \( \eta \) is ramified enough so that each \( L(s, \pi_\eta, r_i) \) is entire. Then, given a finite real interval \( I \), each \( L(s, \pi_\eta, r_i) \) remains bounded for all \( s \) with \( \text{Re}(s) \in I \).

The main difficulty in proving Theorem 5.2 is having to deal with reciprocals of each \( L(s, \pi_v, r_i) \), \( 2 \leq i \leq m \), near and on the line \( \text{Re}(s)=1 \), the edge of the critical strip, whenever \( m \geq 2 \), which is unfortunately the case for each of our cases of functoriality. We handle this by appealing to equations (3.5) and estimating the non-constant term (3.4) by means of [33,36].

### 6. New cases of functoriality

Langlands functoriality predicts that every homomorphism between \( L \)-groups of two reductive groups over a number field, leads to a canonical correspondence between automorphic representations of the two groups. The following instances of functoriality are quite striking and are consequences of applying recent ingenious converse theorems of Cogdell and Piatetski-Shapiro [8,9] to certain classes of \( L \)-functions whose necessary properties are obtained mainly from our method. (See [20] for an insightful survey.) We refer to [11] for more discussion of these results and the transfer from \( GL_2(A_F) \times GL_2(A_F) \) to \( GL_4(A_F) \), using Rankin-Selberg method by Ramakrishnan [37]. (See [23] for a proof using our method.)

**6.a.** Let \( \pi_1 = \otimes_v \pi_{1v} \) and \( \pi_2 = \otimes_v \pi_{2v} \) be cuspidal representations of \( GL_2(A_F) \) and \( GL_3(A_F) \), respectively. For each \( v \), let \( \rho_{iv} \) be the homomorphism of Deligne-Weil group into \( GL_{i+1}(\mathbb{C}) \), parametrizing \( \pi_{iv} \), \( i = 1,2 \). Let \( \pi_{1v} \otimes \pi_{2v} \) be the irreducible admissible representation of \( GL_6(F_v) \) attached to \( \rho_{1v} \otimes \rho_{2v} \) via [18,19]. Set \( \pi_1 \boxtimes \pi_2 = \otimes_v(\pi_{1v} \boxtimes \pi_{2v}) \), an irreducible admissible representation of \( GL_6(A_F) \).
Next, let $\pi = \pi_1$, $\pi_v = \pi_{1v}$ and $\rho_v = \rho_{1v}$. Let $\text{Sym}^3(\pi_v)$ be the irreducible admissible representation of $GL_4(F_v)$ attached to $\text{Sym}^3(\rho_v)$ and set $\text{Sym}^3(\pi) = \otimes_v \text{Sym}^3(\pi_v)$, an irreducible admissible representation of $GL_4(\mathbb{A}_F)$. We have:

**Theorem 6.1** [28,30]. a) The representations $\pi_1 \boxtimes \pi_2$ and $\text{Sym}^3(\pi)$ are automorphic.

b) $\text{Sym}^3(\pi)$ is cuspidal, unless $\pi$ is either of dihedral or of tetrahedral type.

In view of [9], one needs to show that $L(s,(\pi_1 \boxtimes \pi_2) \times (\sigma \otimes \eta))$ is nice in the sense that it satisfies the contentions of Theorems 4.4.a, 5.1 and 5.2 for a highly ramified Grössencharacter $\eta$, where $\sigma$ is a cuspidal representation of $GL_n(\mathbb{A}_F)$, $n = 1,2,3,4$, which is unramified in every place $v$ where either $\pi_{1v}$ or $\pi_{2v}$ is ramified. In particular for each $v$, one of $\pi_{1v}, \pi_{2v}$, or $\sigma_v$ is in the principal series. It then follows from multiplicativity (cf. Theorem 4.4) and the main results of [43,44], that these $L$-functions are equal to certain $L$-functions defined from our method. More precisely, we can take $(G,M)$ to be: a) $G = SL_5$, $M_D = SL_2 \times SL_3$; b) $G = \text{Spin}(10)$, $M_D = SL_3 \times SL_2 \times SL_2$; c) $G = E_6^{sc}$, $M_D = SL_3 \times SL_2 \times SL_3$; d) $G = E_7^{sc}$, $M_D = SL_3 \times SL_2 \times SL_4$, according as $n = 1,2,3,4$, respectively.

This leads to a proof that $\pi_1 \boxtimes \pi_2$ is weakly automorphic. The strong transfer requires a lot more work, involving base change, both normal [2] and non-normal [22], and finally a local result [5]. Automorphy of $\text{Sym}^3(\pi)$ is a consequence of applying the first part to $(\pi,\text{Ad}(\pi))$, where $\text{Ad}(\pi)$ is the adjoint of $\pi$, established by Gelbart-Jacquet [14]. It does not require the use of [5].

Observe that we have in fact proved that the homomorphisms $GL_2(\mathbb{C}) \otimes GL_3(\mathbb{C}) \subset GL_6(\mathbb{C})$ and $\text{Sym}^3: GL_2(\mathbb{C}) \rightarrow GL_4(\mathbb{C})$ are functorial. Neither are endoscopic.

6.b. Let $\Pi = \otimes_v \Pi_v$ be a cuspidal representation of $GL_4(\mathbb{A}_F)$ and let $\Lambda^2 : GL_4(\mathbb{C}) \rightarrow GL_6(\mathbb{C})$ be the exterior square map. Also with $\pi$ as in 6.a, let $\text{Sym}^4(\pi) = \otimes_v \text{Sym}^4(\pi_v)$, where $\text{Sym}^4(\pi_v)$ is attached to $\text{Sym}^4(\rho_v)$. Then

**Theorem 6.2** (cf. [23]). a) The map $\Lambda^2$ is weakly functorial, in the sense that there exists an automorphic form on $GL_6(\mathbb{A}_F)$ whose local components are equal to $\Lambda^2(\Pi_v)$ for all $v$, except if $v|2$ or $v|3$. Here $\Lambda^2(\Pi_v)$ is defined by the local Langlands conjecture [18,19].

b) $\text{Sym}^4(\pi)$ is an automorphic representation of $GL_6(\mathbb{A}_F)$.

We point out that b) is obtained by applying a) to $\text{Sym}^3(\pi)$. a) is proved by applying our method to Spin groups (Case $D_n - 1$ of [40], $n = k + 4$, $k = 0,1,2,3$).

**Proposition 6.3** (cf. [29]). $\text{Sym}^4(\pi)$ is cuspidal unless $\pi$ is either of dihedral, tetrahedral or octahedral type.

Let $\eta = \otimes_v \pi_v$ be a cuspidal form on $GL_2(\mathbb{A}_F)$. For each unramified $v$, let $\alpha_v$ and $\beta_v$ be the Hecke eigenvalues of $\pi_v$. Then as corollary to Proposition 6.3 we have the following striking improvements towards Ramanujan and Selberg conjectures.

**Corollary 6.4.** a) (cf. [29]) Assume $F$ is an arbitrary number field. Then $q_v^{-1/9} < |\alpha_v|$ and $|\beta_v| < q_v^{1/9}$. b) (cf. [27]). Assume $F = \mathbb{Q}$. Then $p^{-7/64} \leq |\alpha_p|$ and $|\beta_p| \leq p^{7/64}$. Similar estimates are valid for the Selberg conjecture. More precisely, the smallest positive eigenvalue $\lambda_1(\Gamma)$ of the Laplace operator on $L^2(\Gamma \mathbb{H})$ for every congruence subgroup $\Gamma$ satisfies $\lambda_1(\Gamma) \geq 975/4096 \approx 0.2380 \cdots$

6.c. Let $i : Sp_{2n}(\mathbb{C}) \hookrightarrow GL_{2n}(\mathbb{C})$ be the natural embedding. Let $\pi = \otimes_v \pi_v$
be a generic cuspidal representation of $SO_{2n+1}(\mathbb{A}_F)$. For each unramified $v$, let $\{A_v\} \subset Sp_{2n}(\mathbb{C})$ be the Hecke-Frobenius conjugacy class parametrizing $\pi_v$. Let $i(\pi_v)$ be the unramified representation of $GL_{2n}(F_v)$ attached to $\{i(A_v)\}$. Then the main theorem of [12] proves:

**Theorem 6.5 [12].** The embedding $i$ is weakly functorial, i.e., there exist an automorphic representation of $GL_{2n}(\mathbb{A}_F)$ whose components are equal to $i(\pi_v)$ for almost all $v$.

This is proved by applying our method to maximal parabolics of appropriate odd special orthogonal groups (Case $B_n$ of [40]). The strong transfer is now also established by Ginzburg-Rallis-Soudry [16] as well as Kim [26] by building upon Theorem 6.5.

**Final Comments.** Many other cases are in progress. Among them are a proof of the existence of the Asai transfer [32] using our method, which was originally proved by Ramakrishnan [38], using the Rankin-Selberg method. This is the first case where one needs to use quasisplit groups. Since the issue of stability of root numbers [10] (cf. [11]) seems to be close to being settled by means of our method [46], many others transfers should now be available. A similar approach for nongeneric representations was initiated in [13].

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