CHIP-FIRING GROUPS OF ITERATED CONES

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ABSTRACT. Let \( \Gamma \) be a finite graph and let \( \Gamma_n \) be the “\( n \)th cone over \( \Gamma \)” (i.e., the join of \( \Gamma \) and the complete graph \( K_n \)). We study the asymptotic structure of the chip-firing group \( \text{Pic}^0(\Gamma_n) \).

1. Introduction

The chip-firing groups \( \text{Pic}^0(\Gamma) \subset \text{Pic}(\Gamma) \) of a finite graph \( \Gamma \) are classical objects of combinatorial study. Baker [Bak08] developed the connection between line bundles on a semistable arithmetic curve \( \mathcal{X} \) and \( \text{Pic}^0(\Gamma) \), where \( \Gamma \) is the dual graph of the special fiber of \( \mathcal{X} \), and with various coauthors [BN07, BN09] discovered that the cornerstone theorems satisfied by algebraic curves (e.g., Riemann–Roch and Clifford’s theorem) admit non-trivial analogous theorems for graphs.

The technology transfer flows both ways; chip-firing (and variants and tools from tropical geometry) have emerged as a central tool in recent results across several subfields of algebraic/arithmetic geometry and number theory, including the maximal rank conjecture for quadrics [JP16], the Gieseker–Petri theorem [JP14], the Brill–Noether theorem [CDPR12], and the uniform boundedness conjecture [KRZB16]; see [BJ15] for an extensive survey.

An interest in the computational properties of \( \text{Pic}^0(\Gamma) \) has recently emerged. Several authors, including [BMM+12, JNR03, Lor08], have worked to compute \( \text{Pic}^0(\Gamma) \) (or, failing that, \( |\text{Pic}^0(\Gamma)| \), which is equal to the number of spanning trees of \( \Gamma \) [BS13, Theorem 6.2]) for various families of graphs; we refer the reader to [AV12, pg. 1155] for nearly a complete list of authors contributing to this area.

Our question of interest is the behavior of the chip-firing group of the \( n \)th cone \( \Gamma_n \) over \( \Gamma \), where \( \Gamma_n \) is defined as the join of \( \Gamma \) with the complete graph \( K_n \). Recall, the join of two graphs \( \Gamma_1 \) and \( \Gamma_2 \) is a graph obtained from \( \Gamma_1 \) and \( \Gamma_2 \) by joining each vertex of \( \Gamma_1 \) to all vertices of \( \Gamma_2 \). In [AV12], the authors interpret the chip-firing group of the \( n \)th cone of the Cartesian product of graphs as a function of the chip-firing group of the cone of their factors. As a consequence, they completely describe the chip-firing group of the \( n \)th cone over the \( d \)-dimensional hypercube.

Our main theorem concerns the the chip-firing group of the \( n \)th cone over a fixed graph.

**Theorem A.** Let \( \Gamma \) be a graph on \( k \geq 1 \) vertices. Let \( n \geq 1 \) be an integer, and let \( \Gamma_n \) be the \( n \)th cone over \( \Gamma \) defined above. Then there is a short exact sequence of abelian groups

\[
0 \to \left( \mathbb{Z} / (n+k)\mathbb{Z} \right)^{n-1} \to \text{Pic}^0(\Gamma_n) \to H_n \to 0
\]

where the order of \( H_n \) is \( |P_\Gamma(-n)| \) and \( P_\Gamma(x) \) is the characteristic polynomial of the rational Laplacian operator.

In particular, this immediately gives an exact formula for the number of spanning trees of \( \Gamma_n \).

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Corollary B. Let \( \Gamma \) be a graph on \( k \geq 1 \) vertices. Let \( n \geq 1 \) be an integer, and let \( \Gamma_n \) be the \( n \)th cone over \( \Gamma \) defined above. There is a subgroup of \( \text{Pic}^0(\Gamma_n) \) isomorphic to \( (\mathbb{Z}/(n+k)\mathbb{Z})^{n-1} \), and
\[
|\text{Pic}^0(\Gamma_n)| = (n+k)^{n-1}|P_\Gamma(-n)|
\]
where \( P_\Gamma(x) \) is the characteristic polynomial of the rational Laplacian operator.

Remark 1.1. In a previous version of this paper, the authors erroneously claimed that this exact sequence was split for odd values of \( n+k \), and conjectured it was split in general. We are very grateful to Gopal Goel for pointing out this error and providing a counter example. If \( \Gamma \) is the graph given in Figure 1, a computer calculation shows that \( \text{Pic}^0(\Gamma_3) \cong \mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/27\mathbb{Z} \oplus (\mathbb{Z}/16\mathbb{Z})^2 \oplus \mathbb{Z}/19\mathbb{Z} \). In particular, the map \( (\mathbb{Z}/9\mathbb{Z})^2 \to \text{Pic}^0(\Gamma_3) \) cannot be split.

A more elusive question is the precise structure of the groups \( \text{Pic}^0(\Gamma_n) \) and \( H_n \).

Question 1.2. For which \( \Gamma \) and \( n \), is the short exact sequence in Theorem A split?

If \( \Gamma \) is a tree, then we are able to determine an upper bound on the number generators for the subgroup \( H_n \) appearing above. Recall that a leaf of a graph is a vertex of degree 1.

Theorem C. Let \( \Gamma \) be a tree with \( l+1 \geq 2 \) leaves and let \( \Gamma_n \) and \( H_n \) be as in Theorem A. Then, \( H_n \) can be generated by \( l \) elements.

It is possible that \( H_n \) may be generated by fewer elements, as in Figure 2.

Finally, we can slightly generalize Theorem A we determine the order of chip-firing group for the join of \( l \) graphs.
Theorem D. Let $\Gamma_1, \ldots, \Gamma_l$ be non-empty graphs with $k_1, \ldots, k_l$ vertices, and let $\Gamma_J$ be the join. Then,

$$|\text{Pic}^0(\Gamma_J)| = k^l - 2 \prod_{i \leq l} |P_{\Gamma_i}(k_i - k)|,$$

where $k = \sum k_i$ is the number of vertices of $\Gamma_J$.

2. Notation

All graphs are assumed to be non-empty, finite, and connected. Given a graph $\Gamma$, we denote by $V(\Gamma)$ and $E(\Gamma)$, respectively, the vertex and edge set of $\Gamma$.

We denote by $\text{Div}(\Gamma)$ the free abelian group on $V(\Gamma)$, and refer to $D \in \text{Div}(\Gamma)$ as divisors on $\Gamma$. The degree map $\deg : \text{Div}(\Gamma) \to \mathbb{Z}$ given by $\sum a_v v \mapsto \sum a_v$ is a group homomorphism, and we denote the kernel of this map by $\text{Div}^0(\Gamma)$.

An ordering $V(\Gamma) = \{v_1, \ldots, v_n\}$ of the vertices of $\Gamma$ determines a basis for $\text{Div}(\Gamma)$. We define the Laplacian operator $\Delta(\Gamma) : \text{Div}(\Gamma) \to \text{Div}^0(\Gamma)$ on a basis via the formula

$$v \mapsto (\deg v) v - \sum_{wv \in E(\Gamma)} w.$$

Given an ordering of $V(\Gamma)$, we define the Laplacian matrix $L(\Gamma)$ to be the matrix of $\Delta(\Gamma)$ with respect to the associated basis; $L(\Gamma)$ is equal to $D(\Gamma) - A(\Gamma)$, where $D(\Gamma)$ and $A(\Gamma)$ are, respectively, the degree and adjacency matrices of $\Gamma$. The matrix $L(\Gamma)$ is symmetric (and in particular diagonalizable with real eigenvalues) and has rank $n - c$, where $c$ is the number of connected components of $\Gamma$. If $\Gamma$ is connected, the kernel of $L(\Gamma)$ is spanned by the vector $(1, \ldots, 1)$.

We define the chip-firing group $\text{Pic}^0(\Gamma)$ of $\Gamma$ to be $\text{Div}^0(\Gamma) / \text{im} \Delta(\Gamma)$; this is also frequently called the Jacobian of $\Gamma$ and denoted by $\text{Jac}(\Gamma)$ (and also often called the critical group, or the sandpile group). As mentioned above, it has order equal to the number of spanning trees of $\Gamma$, and by Kirchhoff’s matrix-tree theorem, this is equal to the product of the non-zero eigenvalues of $L(\Gamma)$ divided by the number of vertices [Sta13, Corollary 9.10(a)]. (Some authors take the dual point of view and define $\mathcal{M}(\Gamma) := \text{Hom}(V(\Gamma), \mathbb{Z})$ and the Laplacian as an operator $\mathcal{M}(\Gamma) \to \text{Div}(\Gamma)$; in our arguments, we immediately identify $\mathcal{M}(\Gamma)$ and $\text{Div}(\Gamma)$ anyway, so we skip directly to our convention.)

Remark 2.1. The name chip-firing group is closely related to chip-firing games or dollar games [BLS91]. Given a divisor $D \in \text{Div}(\Gamma)$, one can think of an integer $a_v$ as the number of dollars assigned to each vertex $v$. A chip-firing move consists of picking a vertex and having it either borrow one dollar from each neighbor or giving one dollar to each of its neighbors. For $D_1, D_2 \in \text{Div}(\Gamma)$, $D_1 - D_2 \in \text{im} \Delta(\Gamma)$ if and only if starting from the configuration $D_1$, one can reach the configuration $D_2$ through a sequence of chip-firing moves (cf. [BS13, Section 4]). This result illustrates the relationship between chip-firing moves and the chip-firing group.

Let $\text{Div}^0_Q(\Gamma) := \text{Div}^0(\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}$. The Laplacian operator $\Delta(\Gamma)$ induces a linear endomorphism of $\text{Div}^0_Q(\Gamma)$ that we denote by $\Delta(\Gamma)_Q$. We let $P_T(x)$ be the characteristic polynomial
of $\Delta(\Gamma)_Q$; $P_\Gamma(x)$ has integer coefficients and degree $n - 1$. We adopt the convention that $P_\Gamma(x) = 1$ if $\Gamma$ has only one vertex.

**Definition 2.2.** Given a graph $\Gamma$, we say that a subset $S = \{w_1, \ldots, w_n\} \subset V(\Gamma)$ has the **conformity property** if the induced subgraph on $S$ is either completely disconnected or complete, and for every vertex $x$ outside $S$, $w_ix$ is an edge if and only if $w_jx$ is an edge, for all $i, j$.

### 2.3. Sketch of proof.

As stated above, $\Pic^0(\Gamma_n)$ has order equal to the determinant of $\Delta(\Gamma_n)$. To prove Theorem [A] we not only compute this determinant, but we also determine all of the eigenvalues of $\Delta(\Gamma_n)$. More precisely, we first isolate eigenvectors $v_k^{n+k} - v_k^{n+k}$ of $\Delta(\Gamma_n)$ that come from $K_n$, and then use the conformity property (cf. Lemmas 3.1 and 3.2) to prove that these eigenvectors generate a subgroup of $\Pic^0(\Gamma_n)$ isomorphic to $(\mathbb{Z}/(n + k)\mathbb{Z})^{n-1}$. To conclude, we show that the remaining eigenvectors of $\Delta(\Gamma_n)$ come from eigenvectors of $\Delta(\Gamma)$.

### 3. Proofs

We begin with a pair of computational lemmas that facilitate our proofs.

**Lemma 3.1.** Assume $\Gamma$ is connected with at least 3 vertices. Suppose $v_1, v_2$ are a pair of vertices of degree $d$ with the conformity property. Let $e_{12} = v_1 - v_2$ as an element of $\Pic^0(\Gamma)$. Then if $v_1v_2$ is an edge, $e_{12}$ has order $d + 1$, and otherwise $e_{12}$ has order $d$.

**Proof.** Define

$$
\mu := \begin{cases} 
(d+1)e_{12} & \text{if } v_1v_2 \text{ is an edge, and} \\
\frac{d}{e_{12}} & \text{otherwise.} 
\end{cases}
$$

By the conformity condition, $\Delta(\Gamma)(v_1 - v_2) = \mu$. It remains to show that no smaller multiple of $e_{12}$ is trivial in $\Pic^0(\Gamma)$. Suppose $m$ is an integer such that $me_{12}$ is trivial in $\Pic^0(\Gamma)$. Then there exists $D = \sum a_i v_i \in \Div(\Gamma)$ such that $\Delta(\Gamma)(D) = me_{12}$. Since

$$
m\Delta(\Gamma)(v_1 - v_2) = \mu\Delta(\Gamma)(D)
$$

in $\Pic^0(\Gamma)$, and since the kernel of $\Delta(\Gamma)_Q$ is spanned by $\sum v_i$, there exists a rational number $r \in \mathbb{Q}$ such that

$$
\mu \sum a_i v_i - m(v_1 - v_2) = r \sum v_i.
$$

In particular $\mu a_1 = r + m, \mu a_2 = r - m,$ and $\mu a_i = r$ for $i > 2$. But then, since $\mu, m,$ and each $a_i$ are integers, $r$ is also an integer. Subtracting $r \sum v_i$ from both sides then gives

$$
\mu(a'_1 v_1 - a'_2 v_2) = m(v_1 - v_2)
$$

where $a'_i = a_i - r/\mu \in \mathbb{Z}$. But then $\mu a'_i = m$, and we conclude that $\mu$ divides $m$. \hfill \Box

Next, we generalize Lemma 3.1 to proper subgraphs with the conformity property.

**Lemma 3.2.** Let $j \geq 1$ and let $S^1 = \{v_1^1, \ldots, v_{n_1}^1\}, \ldots, S^j = \{v_1^j, \ldots, v_{n_j}^j\}$ be $j$ mutually disjoint vertex sets, each with the conformity property. Assume $\Gamma$ is connected and the sets $S^1, \ldots, S^j$ do not completely cover $\Gamma$. Then if the elements $e_{1k}^i = v_i^i - v_k^i$, where $i$ ranges from 1 to $j$, and $k$ ranges from 2 to $m_i$, satisfy a relation $\sum_{i=1}^{j} \sum_{k=2}^{m_i} \alpha_{ij} e_{1k}^i = 0$ in $\Pic^0(\Gamma)$, then each $\alpha_{ij} e_{1k}^i = 0$ in $\Pic^0(\Gamma)$.

The order of $e_{1k}^i$ is specified by the previous lemma (and only depends on $i$).
Proof. Let $\mu^i_k$ be the order of the element $e^i_{1k}$ in $Pic^0(\Gamma)$. If $n^i_k$ are integers such that

$$\sum_{i=1}^j \sum_{k=2}^{m_i} n^i_k e^i_{1k} = 0 \text{ in } Pic^0(\Gamma),$$

then there exists $D \in Div \Gamma$ such that

$$\Delta(\Gamma)(D) = \sum_{i=1}^j \sum_{k=2}^{m_i} n^i_k e^i_{1k}.$$

But for each $i$ and $k$, Lemma 3.1 asserts that

$$\Delta(\Gamma)(v^i_1 - v^i_k) = \mu^i_k e^i_{1k},$$

and thus

$$\sum_{i=1}^j \sum_{k=2}^{m_i} n^i_k e^i_{1k} = \Delta(\Gamma) \left( \sum_{i=1}^j \sum_{k=2}^{m_i} \frac{n^i_k}{\mu^i_k} (v^i_1 - v^i_k) \right).$$

Since the kernel of $\Delta(\Gamma)$ is spanned by $\sum_{v \in V(\Gamma)} v$, there thus exists a rational number $r$ such that

$$D - \sum_{i=1}^j \sum_{k=2}^{m_i} \frac{n^i_k}{\mu^i_k} (v^i_1 - v^i_k) = r \sum_{v \in V(\Gamma)} v.$$

Since the vertex sets $S^i$ do not completely cover $\Gamma$, there is at least one vertex $v$ not included among the $v^i_k$; comparing the coefficient of $v$, we conclude that $r$ is an integer (since the coefficient of $v$ in $D$ is an integer). Thus each fraction $\frac{n^i_k}{\mu^i_k}$ must also be an integer, so each $n^i_k$ is divisible by $\mu^i_k$.

We now prove our main theorems.

Proof of Theorem A. When $k = 1$, $\Gamma_n = K_{n+1}$, and it is well known that $Pic^0(K_{n+1})$ is isomorphic to $(\mathbb{Z}/(n-1)\mathbb{Z})^{n+1}$, in which case we directly observe that the theorem is true. Thus we may assume that $k \geq 2$.

Consider the matrix $B_n = L(\Gamma_n) - L(K_{n+k})$. Every entry is 0 except for the upper $k$ by $k$ submatrix, which we denote by $B_0$. Now, $B_0 = L(\Gamma) - L(K_k)$.

Now, note that $B_n$ acts on $Div^0_Q(\Gamma_n)$. Let $u \in Div^0_Q(\Gamma_n)$ be an eigenvector of $B_n$ with eigenvalue $\mu$. As $K_{n+k}$ is a complete graph, the matrix $L(K_{n+k})$ acts as multiplication by $n+k$ on all elements of $Div^0_Q(\Gamma_n)$. Thus $u$ is an eigenvector of the operator $\Delta(K_{n+k})$ with eigenvalue $n+k$, so it is an eigenvalue of $\Delta(\Gamma_n)$ with eigenvalue $n+k+\mu$.

Choose $k-1$ eigenvectors $u_1, \ldots, u_{k-1} \in Div^0_Q(\Gamma)$ of $B_0$, with corresponding eigenvalues $\mu_i$. Then, by appending $n$ zeros to each vector, we get eigenvectors of $\Delta(\Gamma_n)$ with eigenvalues $n+k+\mu_i$. On the other hand, $u_1, \ldots, u_{k-1}$ are eigenvectors of $\Delta(\Gamma)$ with eigenvalues $k+\mu_i$.

For $i > k+1$, the $n-1$ vectors $u_i = v_i - v_{k+1}$ are eigenvectors of $\Delta(\Gamma_n)$ with eigenvalue $n+k$. Finally, the vector

$$n \sum_{i=1}^k v_i - k \sum_{i=k+1}^{n+k} v_i$$

is an eigenvector, also with eigenvalue $n+k$. We have given a basis for $Div^0_Q(\Gamma_n)$ in eigenvectors of $\Delta(\Gamma_n)$.
By the matrix-tree theorem, the order of \( \text{Pic}^0(\Gamma_n) \) is the product of these eigenvalues, divided by \( n + k \), which is \((n + k)^{n-1}\prod(n + k + \mu_i)\).

Now, the elements \( v_{k+1} - v_{k+i} \) for \( i > 0 \) generate a subgroup isomorphic to \((\mathbb{Z}/(n + k)\mathbb{Z})^{n-1}\) by Lemma 3.2. The quotient has order \( \prod(n + k + \mu_i) \). But the \( k + \mu_i \) are the eigenvalues of \( \Delta(\Gamma) \) acting on \( \text{Div}^0_Q(\Gamma) \), so this expression is equal to \(|P_T(-n)|\).

\[\Box\]

Proof of Theorem C. Let \( v_1, \ldots, v_k \) be the vertices of \( \Gamma \). Let \( \Gamma_n \) be the join of \( \Gamma \) with the complete graph \( K_n \) on the vertices \( w_1, \ldots, w_n \). The group \( H_n \) is the quotient of \( \text{Pic}^0(\Gamma_n) \) by relations generated by \( w_i - w_j \); in the following, for an element \( w \in \text{Pic}^0(\Gamma_n) \) we write \( \bar{w} \) for the image of \( w \) in \( H_n \). Thus \( H_n \) is generated by the elements \( \bar{v}_i - \bar{w}_1 \). Suppose \( v_1, \ldots, v_l \) are leaves of \( \Gamma \), and let \( H' \) be the subgroup of \( H_n \) generated by \( \bar{v}_j - \bar{w}_1 \), for \( 1 \leq j \leq l \). We will show that \( H' = H_n \).

Let \( S \) be the set of vertices \( v_i \) of \( \Gamma \) such that for every vertex \( v' \in \Gamma_n \) connected to \( v_i \), \( \bar{v}_i - \bar{v}' \in H' \). By construction, the vertices \( v_1, \ldots, v_l \) are in \( S \). Now suppose \( v \) is a vertex of \( \Gamma \) with the following property: at least one neighbor of \( v \) is in \( S \) and at most one neighbor of \( v \) is not in \( S \). Let \( w \) be a neighbor of \( v \) in \( \Gamma \) which is in \( S \). Then \( \bar{w} - \bar{w}_1 \) and \( \bar{w} - \bar{v} \) are in \( H' \), so \( \bar{v} - \bar{w}_1 \) is as well. Thus \( \bar{v} - \bar{w}_1 \) is as well, for any \( w_i \in K_n \), so we have accounted for every edge of \( v \) except possibly one. But firing \( v \) shows the image of this edge is also in \( H' \).

Since \( \Gamma \) was a tree, and \( S \) contained all but one of the leaves, we must have that \( S \) is the whole of \( \Gamma \).

\[\Box\]

\[\text{Figure 3. Theorem C in action: group relations for the cone } \Gamma_1 \text{ over the line graph } \Gamma \text{ on 5 vertices. We orient the edges to give a presentation for } \text{Pic}^0(\Gamma_1), \text{ such that each oriented edge represents the function which is 1 at the tip, } -1 \text{ at the tail, and 0 at all other vertices. The signed sum of the edges around every loop is trivial in } \text{Pic}^0(\Gamma_1), \text{ and the chip firing relations impose that the signed sum of the edges incident to each vertex is trivial as well. Firing at the cone vertex shows that } 55e \text{ is trivial in the group, and } \text{Pic}^0(\Gamma_1) \cong \mathbb{Z}/55\mathbb{Z}.\]

Proof of Theorem 2. Let \( B_i \) be the matrix \( \Delta(\Gamma_i) - \Delta(K_{k_i}) \). Then \( B_i \) acts on \( \text{Div}^0_Q(\Gamma_i) \), which admits a basis of eigenvectors \( u^i_1, \ldots, u^i_{k_i-1} \), with eigenvalues \( \mu^i_j \). These eigenvectors are also eigenvectors of \( \Delta(\Gamma_i) \), with eigenvalues \( \mu^i_j + k_i \).
Now let $B_J$ be the matrix $L(\Gamma_J) - L(K_k)$. Then $B_J$ is block diagonal, where the $i$th block is a copy of $B_i$. The eigenvalues of $B_J$ acting on $\text{Div}^0_Q(\Gamma_J)$ thus include the eigenvalues of each $B_i$. The remaining eigenvalues are all 0, since the sum of the rows of any $B_i$ is 0.

Thus the eigenvalues of $\Delta(\Gamma_J)$ acting on $\text{Div}^0_Q(\Gamma_J)$ are $\mu_i^j + k$, along with $l - 1$ copies of $k$. By the matrix-tree theorem, the order of $\text{Pic}^0(\Gamma_J)$ is $k^l - 2 \prod_j \prod_i (\mu_i^j + k)$, and note that $\prod_i (\mu_i^j + k) = |P_{\Gamma_i}(k_i - k)|$. □

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