ASYMPTOTIC STABILITY OF SOLITARY WAVES FOR
THE 1D CUBIC-QUINTIC SCHRÖDINGER EQUATION
WITH NO INTERNAL MODE

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Abstract. For the Schrödinger equation with a cubic-quintic, focusing-
defocusing nonlinearity in one space dimension, we prove the asymptotic
stability of solitary waves for a large range of admissible frequencies. For
this model, the linearized problem around the solitary waves does not
have internal mode nor resonance.

1. Introduction

In this article, we consider the one-dimensional Schrödinger equation with
focusing cubic and defocusing quintic nonlinearities

\begin{equation}
  i\partial_t \psi + \partial_x^2 \psi + |\psi|^2 \psi - |\psi|^4 \psi = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}.
\end{equation}

The corresponding Cauchy problem is globally well-posed in the energy
space $H^1(\mathbb{R})$ (see e.g. [6]) and this model enjoys the conservation laws

\begin{align*}
  \mathcal{M}[\psi] &= \int |\psi|^2 \, dx \\
  \mathcal{P}[\psi] &= \Im \int \psi \partial_x \bar{\psi} \, dx \\
  \mathcal{E}[\psi] &= \int \left( \frac{1}{2} |\partial_x \psi|^2 - \frac{1}{4} |\psi|^4 + \frac{1}{6} |\psi|^6 \right) \, dx.
\end{align*}

We recall the Galilean transform, translation and phase invariances of (1):
if $\psi(t, x)$ is a solution then, for any $\beta, \sigma, \gamma \in \mathbb{R}$, the function

\begin{equation}
  \tilde{\psi}(t, x) = e^{i(\beta x - \beta^2 t + \gamma)} \psi(t, x - 2\beta t - \sigma)
\end{equation}

is also a solution.

It follows from well-known arguments (see e.g. [1]) that for any $\omega \in (0, \frac{3}{16})$, there exists a unique positive even solution $\phi_\omega \in H^1(\mathbb{R})$ of the equation

\begin{equation}
  \phi''_\omega + \phi^3_\omega - \phi^5_\omega = \omega \phi_\omega, \quad x \in \mathbb{R},
\end{equation}

whereas for $\omega \geq \frac{49}{3}$, there exists no solution of (3) in $H^1(\mathbb{R}) \setminus \{0\}$. Moreover, for $\omega \in (0, \frac{3}{16})$, the solution is actually explicit (see e.g. [5], [30], [38] and [44, Chapter 5])

\[\phi_\omega(x) = \sqrt{\frac{4\omega}{1 + a_\omega \cosh(2\sqrt{\omega}x)}} \quad \text{where} \quad a_\omega = \sqrt{1 - \frac{16}{3} \omega}.\]
For any \( \omega \in (0, \frac{3}{10}] \), the function \( \psi(t, x) = e^{i\omega t} \phi_\omega(x) \) is a standing wave solution of (1). The invariances (2) generate a family of traveling waves, of the form
\[
\psi(t, x) = e^{i(\beta x - \beta^2 t + \omega t + \gamma)} \phi_\omega(x - 2\beta t - \sigma)
\]
for \( \beta, \sigma, \gamma \in \mathbb{R} \). The stability of these solutions by perturbation of the initial data in the energy space \( H^1(\mathbb{R}) \) is a classical question. We recall the orbital stability result for standing waves from [36] and we refer to [7, 18, 20, 42] for previous related works.

**Proposition 1** ([36] Theorem 3). For any \( \omega_0 \in (0, \frac{3}{10}) \) and any \( \varepsilon > 0 \), there exists \( \delta > 0 \) with the following property: if \( \psi_0 \in H^1(\mathbb{R}) \), \( \|\psi_0 - \phi_{\omega_0}\|_{H^1(\mathbb{R})} < \delta \) and \( \psi \) is the solution of (1) with \( \psi(0) = \psi_0 \), then
\[
\sup_{\varepsilon \in \mathbb{R}} \inf_{(\gamma, \sigma) \in \mathbb{R}^2} \|\psi(t, \cdot) + e^{i\gamma} \phi_{\omega_0}\|_{H^1(\mathbb{R})} < \varepsilon.
\]

To complement this stability result, we prove the asymptotic stability of a large range of standing waves of (1) by perturbations in the energy space.

**Theorem 1.** For any \( \omega_0 \in (0, \frac{1}{8}] \), there exists \( \delta > 0 \) with the following property: if \( \psi_0 \in H^1(\mathbb{R}) \), \( \|\psi_0 - \phi_{\omega_0}\|_{H^1(\mathbb{R})} < \delta \) and \( \psi \) is the solution of (1) with \( \psi(0) = \psi_0 \), then there exist \( \beta_+ \in \mathbb{R} \) and \( \omega_+ \in (0, \frac{3}{10}) \) such that for any bounded interval \( I \) of \( \mathbb{R} \)
\[
\lim_{t \to +\infty} \inf_{(\gamma, \sigma) \in \mathbb{R}^2} \sup_{x \in I} |\psi(t, x + \sigma) - e^{i\gamma} e^{i\beta_+ x} \phi_{\omega_+}(x)| = 0.
\]

**Remark 1.** The orbital stability property [4] means that the solution stays for all time close to the family of solitary waves and more precisely close to the initial solitary wave \( \phi_{\omega_0} \), up to phase and translation. In particular, in [4] one can replace \( \phi_{\omega_0} \) by \( e^{i\beta x} \phi_\omega \) for \( \beta \) small and \( \omega \) close to \( \omega_0 \). In contrast, the asymptotic stability property [5] says that as \( t \to +\infty \), the solution converges locally in space to a final asymptotic soliton, characterized by \( \beta_+ \) and \( \omega_+ \), up to phase and translation. As a consequence, the values of \( \beta_+ \) and \( \omega_+ \) such that (5) holds are unique and depend in an intricate way of the initial data. By the time reversibility of equation (1), the same holds as \( t \to -\infty \), with possibly different parameters \( \beta_- \) and \( \omega_- \). From the orbital stability, it follows in the context of Theorem 1 that \( |\beta_\pm| \) and \( |\omega_\pm - \omega_0| \) are arbitrarily small for \( \|\psi_0 - \phi_{\omega_0}\|_{H^1(\mathbb{R})} \) small.

The asymptotic stability property [5] is stated in the sup norm on any compact interval for the sake of simplicity. The proof provides additional information on the asymptotic behavior of the solution: there exist \( C^1 \) time dependent functions \( (\beta, \sigma, \gamma, \omega) : [0, \infty) \to \mathbb{R}^4 \times (0, \frac{3}{10}) \) with \( \lim_{\infty} \beta = \beta_+ \), \( \lim_{\infty} \omega = \omega_+ \), such that the function \( u \) defined by
\[
u(t, x) = e^{-i\gamma(t)} e^{-i\beta(t)x} \psi(t, x + \sigma(t)) - \phi_{\omega(t)}(x)
\]
satisfies for some constant \( c_0 > 0 \),
\[
\int_0^\infty \int_\mathbb{R} e^{-c_0 |x|} \left( |\partial_x u(t, x)|^2 + |u(t, x)|^2 \right) \, dx \, dt < \infty.
\]
Therefore, there exists a sequence \( t_n \to +\infty \) such that for any compact interval \( I \), \( \lim_{n \to +\infty} \int_I |\partial_x u(t_n, x)|^2 \, dx = 0 \). The convergence as \( t \to +\infty \) is conjectured, but we do not pursue this issue here. Recall that in general the global norm \( \|u(t)\|_{H^1(\mathbb{R})} \) does not converge to zero since by the stability result and the time reversibility of the equation, this would imply that \( \psi \) is exactly a solitary wave. From the proof, one also shows that \( \lim_{n \to +\infty} \dot{\sigma} = 2\beta_+ \) and \( \lim_{n \to +\infty} \dot{\gamma} = \beta_+^2 + \omega_+ \).

For \( H^1(\mathbb{R}) \) perturbations, it is unlikely that more can be said in general about the decay rate of \( u(t) \) and the asymptotic behaviors of \( \sigma \) and \( \gamma \). We refer to [19, Theorem 1.3] for the lack of decay rate for the perturbation part and to [30, Theorem 2] for a discussion on this issue for the Korteweg-de Vries equation.

**Remark 2.** By the Galilean transform, translation and phase invariances, the family of traveling solitary waves enjoys the same stability properties. In particular, for \((\beta_0, \sigma_0, \gamma_0, \omega_0) \in \mathbb{R}^3 \times (0, \frac{1}{8}]\), the result of Theorem 1 holds for initial data sufficiently close to \( e^{i\gamma_0} e^{i\beta_0 x} \phi_{\omega_0}(x - \sigma_0) \) for any \( \beta_0, \sigma_0, \gamma_0 \in \mathbb{R} \).

**Remark 3.** It would simplify the proof to show Theorem 1 only for small values of \( \omega_0 \), instead of considering the explicit range \((0, \frac{1}{8}]\). The value \( \frac{1}{8} \) is chosen for its simplicity and it is not sharp in our approach. However, a proof for the full range \((0, \frac{3}{16}]\) would certainly require additional arguments because of the specific behavior in the limit \( \omega_0 \to \frac{3}{10} \).

For the integrable one-dimensional focusing cubic Schrödinger equation

\[
(7) \quad i\partial_t \psi + \partial_x^2 \psi + |\psi|^2 \psi = 0,
\]

the Inverse Scattering Transform theory was successfully applied in [15] to prove the asymptotic stability of solitons in \( L^2 \) weighted spaces. See also [2] and the references in these papers concerning the IST theory. Note that the asymptotic stability of solitary waves for initial perturbations in the energy space \( H^1(\mathbb{R}) \) as stated in Theorem 1 is not true in the integrable case. Indeed, counterexamples are provided by the family of multi-solitons constructed in [45, §5]. More explicitly, formula (2.15) page 333 of [37] with the choice of parameters \( b_1 = b_2 = 1, \xi_1 = \xi_2 = 0, \eta_1 = \frac{1}{2}, \eta_2 = \frac{1}{8} \ll 1 \) gives an explicit periodic solution of (7) with a two soliton structure, which is arbitrarily close in \( H^1(\mathbb{R}) \) to the soliton \( \sqrt{2} \text{sech}(x) \) when \( \eta_1 \) parameter related to the size of the small soliton, is small. In the limit \( \omega_0 \) small, Theorem 1 shows that asymptotic stability of solitary waves in the energy space holds for models perturbative of (7). This observation is related to results of asymptotic stability of kinks proved for wave-type models close to the sine-Gordon equation in [25, Theorems 8 and 9].

We refer to [16, 17] for a description of the long time behavior of solutions of the defocusing cubic Schrödinger equation and some of its perturbations. Several articles are concerned with the one-dimensional cubic Schrödinger equation with a potential, see the most recent ones [9, 10, 14, 19, 33, 34].
and their references. More generally, there is a vast literature about the asymptotic stability of waves for nonlinear Schrödinger equations, with or without potential, for any space dimension and various nonlinearities; see the surveys [13, 23, 40]. We refer to [3, 4, 8, 11, 26, 27] for results directly related to the one-dimensional case with no potential.

The presence of internal modes is known to greatly complicate the analysis of asymptotic stability by shifting the problem to the nonlinear level, where a condition, called the Fermi golden rule, then enters into play (see e.g. the review [13]). Roughly, internal modes (as defined in [11, 12, 38]) generate linear periodic solutions to the linearized evolution equation around the solitary wave. For the Schrödinger equation with power nonlinearity $|u|^{p-1}u$ with $p \neq 3$ close to 3, the existence of internal modes bifurcating from the resonance for $p = 3$ is proved in [11] (see [8] for the case of $p$ close to 5). For the Schrödinger equation $i\partial_t \psi + \partial^2_x \psi + |\psi|^2 \psi + |\psi|^4 \psi = 0$, with focusing cubic-quintic nonlinearity, the existence of internal modes is shown in [38]. It is also shown there that for equation (1), there does not exist internal mode. This observation and the proximity of model (1) to the integrable case were the original motivations to study it in the present paper. Concerning its relevance in Physics, the reader may consult for instance [5, 21, p.769], [36, 38, 41], [44, Chapter 5].

Our approach to prove Theorem 1 is directly inspired by [24, 25] (see also [22]) proving the asymptotic stability of solitons and kinks for one-dimensional wave-type models, in the absence of resonance and internal mode, by using virial estimates and a transformed problem. We also point out that algebraic facts on the linearized operator around a solitary wave, established in [8] (see Lemma 2 in the present paper) are decisive in this approach. We refer to §2.3 and §3.2 for more comments on the proof. For the one-dimensional cubic Schrödinger equation with a real potential, the article [14] uses a similar approach, suitably combined with the notion of refined profiles introduced in [12, 13], to deal with the presence of more than one discrete mode for the potential.

Finally, we point out that similar virial estimates were used extensively to prove the asymptotic stability of solitons of the subcritical generalized Korteweg-de Vries equations (see [28, 30] and references therein), but also to study the singularity formation for the mass critical generalized Korteweg-de Vries equation and nonlinear Schrödinger equation (see [29, 31, 52]). Indeed, a sharp description of the bubbling phenomenon for these equations requires sharp estimates on the error term out of reach of standard energy methods.

**Notation.** The letters $u, v, w$ and $z$ denote complex-valued functions with e.g., $u = u_1 + iu_2$, $u_1, u_2 \in \mathbb{R}$. The letters $g$ and $h$ denote real-valued functions. The Fourier transform of a function $w$ is denoted by $\hat{w}$. For $\alpha > 0$, set

$$X_\alpha = (1 - \alpha \partial^2_x)^{-1} \quad i.e. \quad \widehat{X_\alpha w}(\xi) = \frac{\hat{w}(\xi)}{1 + \alpha \xi^2} \quad \text{for} \ \xi \in \mathbb{R} \ \text{and} \ w \in L^2(\mathbb{R}).$$
Denote \( \langle u, v \rangle = \Re \int u \overline{v} \, dx \) and \( \| u \| = \sqrt{\langle u, u \rangle} \). Last, we denote
\[
f(u) = |u|^2 u - |u|^4 u, \quad F(u) = \frac{|u|^4}{4} - \frac{|u|^6}{6}.
\]
In this paper, \( C \) denotes various positive constants which do not depend on the parameters \( \omega_0, \varepsilon, \alpha, A \) and \( B \), except when such parameters (like \( B \) and \( \alpha \)) at the end of the proof of Proposition \( \text{[1]} \) are eventually fixed.

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**2. Preliminaries**

2.1. **Solitary waves.** We gather basic properties of \( \phi_\omega \) and \( \Lambda_\omega = \omega \frac{\partial \phi_\omega}{\partial \omega} \).

**Lemma 1.** For any \( k \geq 0 \), there exists \( C_k > 0 \) such that for any \( \omega \in (0, \frac{1}{8}] \) and any \( x \in \mathbb{R} \),
\[
|\phi_\omega^{(k)}(x)| \leq C_k \omega \frac{1+k}{2} e^{-\sqrt{\omega} |x|}, \quad |\Lambda_\omega^{(k)}(x)| \leq C_k \omega \frac{1+k}{2} (1 + \sqrt{\omega} |x|) e^{-\sqrt{\omega} |x|}.
\]
Moreover, there exists \( c > 0 \) such that \( \langle \phi_\omega, \Lambda_\omega \rangle \geq c \sqrt{\omega} \).

**Proof.** The bounds follow from the explicit expressions of \( \phi_\omega \) and \( \Lambda_\omega \). The restriction \( \omega \in (0, \frac{1}{8}] \) implies \( a_\omega \geq 1/\sqrt{3} \) and so it allows us to obtain constants \( C_k \) independent of \( \omega \).

By change of variable, we compute \( \| \phi_\omega \|^2 = 2 \sqrt{\omega} \int (1 + a_\omega \cosh(y))^{-1} \, dy \).

The lower bound on \( \langle \phi_\omega, \Lambda_\omega \rangle = \frac{1}{2} \omega \frac{\partial}{\partial \omega} \| \phi_\omega \|^2 \) follows from the observation that the map \( \omega \in (0, \frac{3}{16}) \mapsto a_\omega \) is decreasing. The fact that \( \langle \phi_\omega, \Lambda_\omega \rangle > 0 \) is related to orbital stability (\([18, 20, 36, 43]\)). \( \square \)

2.2. **Spectral properties.** The linearization of \( \text{[1]} \) around \( \phi_\omega \) involves the operators (see e.g. \([8], [42]\))
\[
L_+ = -\partial_x^2 + \omega - 3\phi_\omega^2 + 5\phi_\omega^4 \quad \text{and} \quad L_- = -\partial_x^2 + \omega - \phi_\omega^2 + \phi_\omega^4.
\]
We recall a few properties of the operators \( L_+ \) and \( L_- \) and refer to \([12, 43]\) and \([8, \text{Lemma 2.2}] \) for details. The operator \( L_+ \) has exactly one negative eigenvalue. Moreover, the kernel of \( L_+ \) is generated by \( \phi_\omega' \). Differentiating the equation of \( \phi_\omega \) with respect to \( \omega \), we obtain \( L_+ \Lambda_\omega = -\omega \phi_\omega \). Last, \( L_- \geq 0 \) and its kernel is generated by \( \phi_\omega \).

2.3. **Conjugate identity.** We adapt to the present context an identity from \([8, \text{§3.4}] \). Let
\[
S = \phi_\omega \cdot \partial_x \cdot \frac{1}{\phi_\omega}, \quad S^* = -\frac{1}{\phi_\omega} \cdot \partial_x \cdot \phi_\omega,
\]
\[
M_+ = -\partial_x^2 + \omega - \frac{1}{3} \phi_\omega^4, \quad M_- = -\partial_x^2 + \omega + \phi_\omega^4.
\]
The above definitions of $S$ and $S^*$ mean that for a function $g$, $Sg = \phi_\omega(\frac{g}{\phi_\omega})'$ and $S^*g = -\frac{1}{\phi_\omega}(g\phi_\omega)'$.

**Lemma 2.** For any $\omega \in (0, \frac{3}{16})$, $S^2L_+L_- = M_+M_-S^2$.

The identity proved in Lemma 2 is different from the ones involved in [14, 21, 25] to define the transformed problem for the Schrödinger equation with a real-valued potential or for wave-type equations. See also Remark 4.

The interest of this identity lies on the properties of the operators $M_+$ and $M_-$. Indeed, the potential of $M_-$ is repulsive (in the sense that $x(\phi_\omega')' \leq 0$ on $\mathbb{R}$) while the potentials of $L_+$ and $L_-$ are not repulsive. Recall that the repulsive nature of a potential is decisive to apply a virial argument (see [39, Theorem XIII.60]). The potential of $M_+$ is not repulsive, but being in absolute value three times less than the one of $M_-$, it is possible to control it for a large range of values of $\omega$ (see the proof of Proposition 4 and in particular, the definition of the functional $\mathcal{K}$).

The intuition that $S^2$ can be factorized at the left of the operator $S^2L_+L_-$ comes from the relations

\begin{equation}
L_\omega = 0, \quad L_+(x\phi_\omega) = L_+(xL_\omega - 2\phi_\omega') = -2L_\omega' = 0, \quad S\phi_\omega = 0 \quad \text{and} \quad S^2(x\phi_\omega) = S\phi_\omega = 0.
\end{equation}

The identity of Lemma 2 is general and not specific to the nonlinearity in (1). For example, in the case of the integrable equation (7), one obtains $M_+ = M_- = -\partial^2_x + \omega$. This was a strong motivation to work in some sense close to the integrable case in the present paper. We refer to similar observations on the sine-Gordon equation in [25]. Last, we point out that such approach by factorization seems limited to one-dimensional problems.

**Proof of Lemma 2.** We recall from (3.25)-(3.26) of [8] (with a slight change of notation) the general formula

\[(\partial_\omega - R)(\partial^2_x - V_+)(\partial_\omega + R) = (\partial_\omega + R)(\partial^2_x - V_-)(\partial_\omega - R)\]

where

\[V_\pm = R^2 \pm 3R' + \frac{R''}{R}.\]

We set $R = \frac{\phi'_\omega}{\phi_\omega}$. Using the identities

\[\phi''_\omega = \omega\phi_\omega - \phi^3_\omega + \phi^5_\omega \quad \text{and} \quad (\phi'_\omega)^2 = \omega\phi^2_\omega - \frac{1}{2}\phi^4_\omega + \frac{1}{3}\phi^6_\omega\]

we compute

\[R^2 = \omega - \frac{1}{2}\phi^2_\omega + \frac{1}{3}\phi^4_\omega, \quad R' = -\frac{1}{2}\phi^2_\omega + \frac{2}{3}\phi^4_\omega, \quad \frac{R''}{R} = -\phi^2_\omega + \frac{8}{3}\phi^4_\omega,\]

and thus

\[V_+ = \omega - 3\phi^2_\omega + 5\phi^4_\omega, \quad V_- = \omega + \phi^4_\omega.\]

Observing that

\[\partial_\omega - R = S, \quad \partial_\omega + R = -S^*, \quad L_+ = -\partial^2_x + V_+, \quad M_- = -\partial^2_x + V_-,\]

we have

\[S^2L_+L_- = M_+M_-S^2.\]
the general formula implies that $SL_+ S^* = S^* M_+ S$. Following [8], we also check that $L_- = S^* S$ and $M_+ = SS^*$ using the identities in [9]. Thus, composing $SL_+ S^* = S^* M_- S$ by $S$ on the left and on the right yields the identity $S^2 L_+ L_- = M_+ M_- S^2$.

2.4. Invertibility of $L_+$ and $M_-$. We briefly discuss the invertibility of the operators $L_+$ and $M_-$. Let $G$ be the even solution of $L_+ G = 0$ such that $\phi''_G G - \phi'_G G' = 1$ on $\mathbb{R}$. We check that $|G^{(k)}(x)| \leq C_k \omega^{-\frac{k+2}{2}} e^{\sqrt{\omega}|x|}$, for constants $C_k > 0$. For any bounded continuous function $W$, define

$$I_+[W](x) = \begin{cases} -\phi'_G(x) \int_0^x GW - G(x) \int_x^\infty \phi'_G W & \text{for } x \geq 0 \\
\phi'_G(x) \int_0^x GW + G(x) \int_x^{-\infty} \phi'_G W & \text{for } x < 0 \end{cases}$$

Note that if $\langle W, \phi_G \rangle = 0$, then we have $-\int_x^{\infty} \phi'_G W = \int_{-\infty}^x \phi'_G W$ so that the two expressions coincide at $x = 0$ and provide a solution to $L_+ U = W$.

Last, denote by $H_1$ and $H_2$ two solutions of $M_- H_1 = M_- H_2 = 0$ satisfying

$$|H_1^{(k)}(x)| \leq C_k \omega^{-\frac{k+2}{4} + \frac{k}{2}} e^{-\sqrt{\omega} x}; \quad |H_2^{(k)}(x)| \leq C_k \omega^{-\frac{k+2}{4}} e^{\sqrt{\omega} x}$$

for $C_k > 0$ and $H_1 H_2' - H_2 H_1' = 1$ on $\mathbb{R}$. Two such independent solutions exist because $M_- > 0$ and so the equation $M_- U = 0$ does not have an $H^1$ solution. For any bounded continuous function $W$, define

$$J_-[W](x) = H_1(x) \int_{-\infty}^{x} H_2 + H_2(x) \int_{x}^{\infty} H_1 W.$$

This formula defines a solution to $M_- U = W$.

3. Asymptotic stability

3.1. Modulation. We fix $\omega_0 \in (0, \frac{3}{10})$ and an initial data $\psi_0 \in H^1(\mathbb{R})$ close to $\phi_{\omega_0}$. By Proposition [1] the global solution $\psi$ of [1] is close to the family of solitary waves for all time. It is standard to decompose $\psi$ as

$$\psi(t, y) = e^{i(\beta(t)(y-\sigma(t))+\gamma(t))} \left[ \phi_{\omega(t)}(y-\sigma(t)) + u(t, y - \sigma(t)) \right]$$

(see also the equivalent formulation [6] in the Introduction) where the time-dependent functions $\beta$, $\sigma$, $\gamma$ and $\omega$ are of class $C^1$ and uniquely fixed so that, for all $t \geq 0$, the following orthogonality relations hold

$$\langle u, \phi_\omega \rangle = \langle u, x \phi_\omega \rangle = \langle u, i \Lambda_\omega \rangle = \langle u, i \phi'_\omega \rangle = 0.$$

This specific choice of orthogonality relations is known to yield quadratic estimates for the time derivative of the modulation parameters, for all $t \geq 0$,

$$\frac{\|\dot{\beta}\|}{\sqrt{\omega}} + \frac{|\dot{\omega}|}{\omega} + \sqrt{\omega}|\dot{\sigma} - 2\beta| + |\dot{\gamma} - \omega - \beta^2| \leq C \sqrt{\omega} \| \text{sech}(\sqrt{\omega} x/2) u \|^2.$$

See [12] (i) of Proposition 2.4 and e.g. [35] proof of Lemma 12]. Moreover, for $\varepsilon > 0$ small, we can formulate the orbital stability result of Proposition [1] as follows, for all $t \geq 0$,

$$\|\partial_x u\| + \|u\| + |\beta| + |\omega - \omega_0| \leq \varepsilon.$$
Using the equation of $\psi$, we check that $u = u_1 + iu_2$ satisfies

\begin{align}
\begin{cases}
\partial_t u_1 &= L_- u_2 + \theta_2 + m_2 - q_2 \\
\partial_t u_2 &= -L_+ u_1 - \theta_1 - m_1 + q_1
\end{cases}
\end{align}

where

\begin{align*}
\theta_1 &= \dot{\beta} x \phi_\omega + (\dot{\gamma} - \omega - \beta^2)\phi_\omega \\
\theta_2 &= -\frac{\dot{\omega}}{\omega} A_\omega + (\dot{\sigma} - 2\beta)\phi_\omega' \\
m_1 &= \dot{\beta} xu_1 + (\dot{\gamma} - \omega - \beta^2)u_1 - (\dot{\sigma} - 2\beta)\partial_x u_2 \\
m_2 &= \dot{\beta} xu_2 + (\dot{\gamma} - \omega - \beta^2)u_2 + (\dot{\sigma} - 2\beta)\partial_x u_1
\end{align*}

and

\begin{align*}
q_1 &= \Re \left\{ f(\phi_\omega + u) - f(\phi_\omega) - f'(\phi_\omega)u_1 \right\} \\
q_2 &= \Im \left\{ f(\phi_\omega + u) - i\frac{f'(\phi_\omega)}{\phi_\omega}u_2 \right\}.
\end{align*}

3.2. Outline of the proof of Theorem 1. The proof of Theorem 1 relies on two localized virial estimates. By localized, we mean that the functionals make sense for $H^1(\mathbb{R})$ functions, by truncating the function $x$ involved in a virial computation at two different large spatial scales $A \gg B \gg 1$.

The first virial estimate is performed on the function $u$, solution of (14). Since the operators $L_+$ and $L_-$ are not repulsive, a virial computation is not sufficient to prove directly convergence to zero of $u$, but it allows to estimate it at any large spatial scale $A$ by a norm of $u$ with a weight $\rho$ related to the spatial decay of $\phi_\omega$. See Proposition 2.

The second virial estimate is performed on the transformed function $v$, whose definition originates from the identity in Lemma 2. See §3.4 for the definition of $v$ and Proposition 3. The equation (18) of $v$ involves the operators $M_+$ and $M_-$ which are better suited for a virial computation. This virial estimate in $v$ at the spatial scale $B$ contains error terms in $u$, in particular, nonlinear terms and modulation terms. In this step, we fix the constants $B$ and $\alpha$, independently of $\omega_0$, $\varepsilon$ and $A$.

Last, in Proposition 4 we estimate the function $u$ by the transformed function $v$ for suitable weighted norms, using the special orthogonality relations (11).

Gathering the estimates of Propositions 2, 3 and 4, adjusting the choice of $A \gg 1$ and taking $\varepsilon > 0$ small enough, we complete the proof of Theorem 1 in §3.7.

3.3. First virial estimate. The following definitions are taken from [24]. We fix a smooth even function $\chi : \mathbb{R} \to \mathbb{R}$ satisfying

$\chi = 1$ on $[0, 1]$, $\chi = 0$ on $[2, +\infty)$, $\chi' \leq 0$ on $[0, +\infty)$.
For $K > 0$, define
\[ \chi_K(x) = \chi \left( \frac{\omega_0 \sqrt{\omega_0}}{K} x \right), \quad \eta_K(x) = \text{sech} \left( \frac{2\omega_0 \sqrt{\omega_0}}{K} x \right), \]
\[ \zeta_K(x) = \exp \left( -\frac{\omega_0 \sqrt{\omega_0}}{K} |x| (1 - \chi(\sqrt{\omega_0}x)) \right), \quad \Phi_K(x) = \int_0^x \zeta_K^2(y) \, dy. \]

Let $1 \ll B \ll A$ be large constants to be defined later. Define \[ \Psi_{A,B} = \chi_A^2 \Phi_B. \]

Technically, in the definitions of $\chi_K$, $\eta_K$ and $\zeta_K$, the multiplicative factor $\omega_0 \sqrt{\omega_0}/K$ will allow us to choose a parameter $B$ large enough independent of $\omega_0$ in the proof of Proposition 3.

Last, we introduce a weight function, related to the solitary wave $\phi_\omega$
\[ \rho(x) = \text{sech} \left( \frac{\sqrt{\omega_0} x}{10} \right). \]

Since $|\omega - \omega_0| \leq \varepsilon$ by (13), for $\varepsilon < \frac{\omega_0}{2}$, we have $\frac{\omega_0}{2} \leq \omega < \frac{3}{2} \omega_0$, which allows by Lemma 1 to control $\phi_\omega$, $\Lambda_\omega$ and their derivatives in terms of powers of $\rho$.

**Proposition 2.** There exists $C > 0$ such that for $\varepsilon > 0$ small enough, for any $T \geq 0$,
\[ \int_0^T \left( \| \eta_A \partial_x u \|^2 + \omega_0^3 \| \eta_A u \|^2 \right) \, dt \leq C \varepsilon + C \omega_0 \int_0^T \| \rho^2 u \|^2 \, dt. \]

**Proof.** Define
\[ \mathcal{I} = \int u_1 \left( 2\Phi_A \partial_x u_2 + \Phi'_A u_2 \right) \quad \text{and} \quad w = \zeta_A u. \]

We claim, for all $t \geq 0$,
\[ \dot{\mathcal{I}} \geq \int |\partial_x w|^2 - C \omega_0 \| \rho^2 u \|^2. \] (15)

From the equation (14) of $(u_1, u_2)$ and $\int (2\Phi_A \partial_x u_k + \Phi'_A u_k) u_k = 0$,
\[ \dot{\mathcal{I}} = -\sum_{k=1,2} \int \left( 2\Phi_A \partial_x u_k + \Phi'_A u_k \right) \partial_x^2 u_k \]
\[ - \Re \left\{ \int \left( 2\Phi_A \partial_x \tilde{u} + \Phi'_A \tilde{u} \right) (f(\phi_\omega + u) - f(\phi_\omega)) \right\} \]
\[ + \sum_{k=1,2} \int \left( 2\Phi_A \partial_x u_k + \Phi'_A u_k \right) (\theta_k + m_k). \]

Integrating by parts (see e.g. [24] Lemma 1), for $k = 1, 2$, we compute
\[ - \int (2\Phi_A \partial_x u_k + \Phi'_A u_k) \partial_x^2 u_k = 2 \int (\partial_x w_k)^2 + \int (\ln \zeta_A)'' w_k^2 \]
where
\[ (\ln \zeta_A)'' = \frac{\omega_0^2}{A} \left( \sqrt{\omega_0} |x| \chi''(\sqrt{\omega_0}x) + 2 \chi'(\sqrt{\omega_0}x) \text{sgn}(x) \right) \]
Using the following inequality from [24, Claim 1]

\[ |(\ln \zeta_A)''| \leq \frac{C\omega_0^2}{A} 1_{[1,2]}(\sqrt{\omega_0}|x|) \leq \frac{C}{A}\omega_0^2 \rho^4(x) \]

Thus,

\[-\int (2\Phi_{A}\partial_x u_k + \Phi'_{A}u_k) \partial_x^2 u_k \geq 2 \int (\partial_x w_k)^2 - \frac{C}{A}\omega_0^2 \rho^2 w_k \|

For the next term in the expression of \( \tilde{\mathcal{L}} \), we note that

\[ \partial_x (F(\phi_\omega + u) - F(\phi_\omega) - f(\phi_\omega)u) = \mathcal{R} \{ (\partial_x \tilde{u})(f(\phi_\omega + u) - f(\phi_\omega)) \} \]

and so by integration by parts, we decompose

\[-\mathcal{R} \left\{ \int (2\Phi_{A}\partial_x \tilde{u} + \Phi'_{A}\tilde{u}) (f(\phi_\omega + u) - f(\phi_\omega)) \right\} = 2 \int \Phi'_{A} \mathcal{R} \{ F(\phi_\omega + u) - F(\phi_\omega) - f(\phi_\omega)u \}

+ 2 \int \Phi_{A} \mathcal{R} \{ \phi_\omega'(f(\phi_\omega + u) - f(\phi_\omega) - f'(\phi_\omega)u) \}

- \int \Phi'_{A} \mathcal{R} \{ \tilde{u} (f(\phi_\omega + u) - f(\phi_\omega)) \} = I_1 + I_2 + I_3.

To estimate \( I_1, I_2 \) and \( I_3 \), we use the following observations

\[ 0 < \Phi'_{A}(x) = \zeta^2 \leq 1, \quad |\Phi_{A}(x)| \leq |x| \quad \text{on } \mathbb{R}, \]

and so (the estimate below is not optimal in terms of power of \( \rho \))

\[ |\Phi_{A}(x)\phi_\omega| \leq \sqrt{\omega}|x| \sech(\sqrt{\omega}x) \leq C\rho^4(x). \]

Moreover, by Lemma 1

\[ \|u\|_{L^\infty} \leq C\|u\|_{H^1(\mathbb{R})} \leq C\varepsilon \leq C\omega_0. \]

Thus,

\[ I_1 \leq C \int \zeta^2_{A}(\phi_\omega^2|u|^2 + |u|^4) \leq C\omega_0 \int \rho^4|u|^2 + C \int \zeta^2_{A}|u|^4, \]

\[ I_2 \leq C\sqrt{\omega_0} \int |\Phi_{A}|\phi_\omega (\phi_\omega^2|u|^2 + |u|^3) \leq C\omega_0 \int \rho^4|u|^2, \]

\[ I_3 \leq C \int \zeta^2_{A}(\phi_\omega^2|u|^2 + |u|^4) \leq C\omega_0 \int \rho^4|u|^2 + C \int \zeta^2_{A}|u|^4. \]

Using the following inequality from [24] Claim 1

\[ \int \zeta^2_{A}|u|^4 \leq \frac{CA^2}{\omega_0^3} \|u\|^2_{L^\infty} \int |\partial_x w|^2 \leq \frac{CA^2\varepsilon^2}{\omega_0^3} \int |\partial_x w|^2 \]

we obtain

\[ |I_1| + |I_2| + |I_3| \leq C\omega_0 \int \rho^4|u|^2 + \frac{CA^2\varepsilon^2}{\omega_0^3} \int |\partial_x w|^2. \]
Next, by integration by parts and then using (12), for \( k = 1, 2 \),
\[
\left| \int (2\Phi_A \partial_x u_k + \Phi'_A u_k) \theta_k \right| = \left| \int u_k (2\Phi_A \partial_x \theta_k + \Phi'_A \theta_k) \right|
\leq \|u\|_{L^\infty} \int (|x| |\partial_x \theta_k| + |\theta_k|) \leq C \varepsilon \| \rho^2 u \|^2 \leq C \omega_0 \| \rho^2 u \|^2.
\]

For the last term in the expression of \( \hat{I} \), we integrate by parts
\[- \int (2\Phi_A \partial_x u_1 + \Phi'_A u_1) m_1 = \beta \int \Phi_A u_1^2 + (\hat{\sigma} - 2\beta) \int (2\Phi_A \partial_x u_1 + \Phi'_A u_1) \partial_x w.
\]
Combining this identity with its analogue for \( \int (2\Phi_A \partial_x u_2 + \Phi'_A u_2) m_2 \), we obtain
\[- \int (2\Phi_A \partial_x u_1 + \Phi'_A u_1) m_1 - \int (2\Phi_A \partial_x u_2 + \Phi'_A u_2) m_2 = \hat{\beta} \int \Phi_A |u|^2 + (\hat{\sigma} - 2\beta) \int \Phi'_A (u_2 \partial_x u_1 - u_1 \partial_x u_2) \cdot
\]
Thus, using \( \| \Phi_A \|_{L^\infty} + \| \phi \Phi'_A \|_{L^\infty} \leq CA \omega_0^{-\frac{3}{2}}, \ |\Phi'_A| \leq 1, \ |u|_{H^1(\mathbb{R})} \leq C \varepsilon \) and (12),
\[
\left| \sum_{k=1,2} \int (2\Phi_A \partial_x u_k + \Phi'_A u_k) m_k \right| \leq \frac{CA}{\omega_0^3} \varepsilon \| \rho^2 u \|^2.
\]
Gathering these estimates, we have proved
\[
\hat{I} \geq 2 \left( 1 - \frac{CA^2 \varepsilon^2}{\omega_0^3} \right) \int |\partial_x w|^2 - C \left( \omega_0 + \frac{A \varepsilon^2}{\omega_0^3} \right) \| \rho^2 u \|^2.
\]
Taking \( \varepsilon \) small such that
\[
\frac{CA^2 \varepsilon^2}{\omega_0^3} \leq \frac{1}{2},
\]
the estimate (15) is proved.

Now, for any \( T \geq 0 \), using the above estimates for \( \Phi_A \) and (13), we estimate
\[
|\hat{I}(T)| \leq \left( \| \Phi_A \|_{L^\infty} + \| \phi \Phi'_A \|_{L^\infty} \right) \| u \|_{H^1(\mathbb{R})} \leq \frac{CA}{\omega_0} \varepsilon^2 \leq C \varepsilon.
\]
Therefore, integrating on \([0, T] \), we obtain
\[
\int_0^T \int |\partial_x w|^2 \leq C \varepsilon + C \omega_0 \int_0^T \| \rho^2 u \|^2.
\]
Now, we use the elementary inequality (see e.g. [24, Lemma 4])
\[
\int \eta_A |w|^2 \leq \frac{CA^2}{\omega_0^3} \int |\partial_x w|^2 + \frac{CA}{\omega_0^3} \int \rho^2 |w|^2,
\]
which implies
\[
\frac{\omega^3}{A^2} \int_0^T \int \eta_A^2 |u|^2 \leq C \varepsilon + \frac{C \omega^2}{A} \int_0^T \|\rho^2 u\|^2.
\]
Last, recalling \( w = \zeta_A u \), by integration by parts,
\[
\int \zeta_A^2 |\partial_x w|^2 = \int \zeta_A^4 |\partial_x u|^2 - \int \zeta_A^3 \zeta''_A |u|^2 - 2 \int \zeta''_A (\zeta_A)^2 |u|^2
\]
and so using \( \frac{1}{C} \eta_A \leq \zeta_A \leq C \eta_A \) and \(|\zeta''_A| + |\zeta'_A|^2 \leq C \omega^2 A^{-2} \zeta_A\),
\[
\int \eta_A^2 |\partial_x u|^2 \leq C \int |\partial_x w|^2 + \frac{C \omega^3}{A^2} \int \eta_A^2 |u|^2,
\]
which is sufficient to complete the proof. \( \square \)

3.4. Transformed problem. For \( \alpha > 0 \) small to be fixed, we introduce the function \( v = v_1 + iv_2 \) defined by
\[
v_1 = X^2 \alpha M - S^2 u_2 \quad \text{and} \quad v_2 = -X^2 \alpha S^2 L_+ u_1.
\]

By direct computations, using
\[
S^2 = \partial_x^2 - 2 \frac{\phi^4}{\phi} \partial_x + \omega - \frac{1}{3} \phi^4
\]
and the identities (21), we have
\[
M_- S^2 = -\partial_x^3 + 2 \partial_x^2 \cdot \frac{\phi'}{\phi} \cdot \partial_x + \frac{4}{3} \partial_x \cdot \phi^4 \cdot \partial_x + \left( -2 \omega \frac{\phi'}{\phi} - \frac{14}{3} \phi^3 \phi' \right) \cdot \partial_x
\]
\[
+ \omega^2 + 6 \omega \phi^4 - \frac{10}{3} \phi^6 + \frac{7}{3} \phi^8
\]
and
\[
S^2 L_+ = -\partial_x^3 + 2 \partial_x^2 \cdot \frac{\phi'}{\phi} \cdot \partial_x + \partial_x \cdot \left( -\phi^2 + \frac{8}{3} \phi^4 \right) \cdot \partial_x
\]
\[
+ \left( -2 \omega \frac{\phi'}{\phi} - 2 \phi \phi' + 14 \phi^3 \phi' \right) \cdot \partial_x
\]
\[
+ \omega^2 + 3 \omega \phi^2 + 3 \phi^4 + \frac{134}{3} \omega \phi^4 - 38 \phi^6 + 25 \phi^8.
\]
For future use, we introduce
\[
Q_- = 2 \partial_x^3 \cdot \left( \frac{\Lambda \phi - \phi' \Lambda}{\phi^2} \right) \cdot \partial_x + \frac{16}{3} \partial_x \cdot (\Lambda \phi^3) \cdot \partial_x
\]
\[
+ \left[ -2 \omega \frac{\phi'}{\phi} - 2 \omega \left( \frac{\Lambda \phi - \phi' \Lambda}{\phi^2} \right) - \frac{14}{3} \left( 3 \phi^2 \Lambda \phi' + \phi^3 \phi' \right) \right] \cdot \partial_x
\]
\[
+ 2 \omega^2 + 6 \omega \phi^4 + 24 \omega \Lambda \phi^3 - \frac{60}{3} \Lambda \phi^5 + \frac{56}{3} \Lambda \phi^7
\]
and

\[ Q_+ = 2\partial_x^2 \left( \frac{\Lambda_\omega^\prime \phi_\omega - \phi_\omega^\prime \Lambda_\omega}{\phi_\omega^2} \right) \cdot \partial_x + \partial_x \left( -2\Lambda_\omega \phi_\omega + \frac{32}{3} \Lambda_\omega \phi_\omega^3 \right) \cdot \partial_x \]
\[ + \left[ -2\Lambda_\omega^\prime \phi_\omega^2 - 2\omega \left( \frac{\Lambda_\omega^\prime \phi_\omega - \phi_\omega^\prime \Lambda_\omega}{\phi_\omega^2} \right) \right] \cdot \partial_x \]
\[ + \left( -2\Lambda_\omega^\prime \phi_\omega^2 - 2\phi_\omega^\prime \Lambda_\omega^\prime + 42\Lambda_\omega \phi_\omega^2 \phi_\omega^\prime + 14\phi_\omega^3 \Lambda_\omega^\prime \right) \cdot \partial_x \]
\[ + 2\omega^2 - 3\omega \phi_\omega^2 - 6\Lambda_\omega \phi_\omega + 12\Lambda_\omega \phi_\omega^3 + \frac{134}{3} \omega \phi_\omega^4 + \frac{536}{3} \omega \Lambda_\omega \phi_\omega^3 \]
\[ - 228\Lambda_\omega \phi_\omega^5 + 200\Lambda_\omega \phi_\omega^7 \]

Note that the operators \( Q_- \) and \( Q_+ \) are obtained from \( M_- S^2 \) and \( S^2 L_+ \) by differentiation with respect to \( \omega \). Their exact expressions are not so important, only their specific structures (similar to the ones of \( \omega \) by differentiation with respect to \( \omega \)) are used in the proof of Lemma 6.

We recall some technical estimates from [25].

**Lemma 3.** There exists \( C > 0 \) such that for any \( \alpha > 0 \) small and \( h \in L^2(\mathbb{R}) \)

\[ |X_\alpha h| \leq |h|, \quad |\partial_x X_\alpha^{1/2} h| \leq \alpha^{-1/2} |h|, \]
\[ |ho X_\alpha h| \leq C |X_\alpha [\rho h]|, \quad |\rho^{-1} X_\alpha [\rho h]| \leq C |X_\alpha h|, \]
\[ |\eta_A X_\alpha h| \leq C |X_\alpha [\eta_A h]|, \quad |\eta_A^{-1} X_\alpha [\eta_A h]| \leq C |X_\alpha h|, \]
\[ |\rho^{-1} X_\alpha \partial_x^2 [\rho h]| \leq C \alpha^{-1} |h|, \quad |\rho^{-1} X_\alpha \partial_x [\rho h]| \leq C \alpha^{-1} |h|, \]
\[ |\eta_A X_\alpha \partial_x^2 h| \leq C \alpha^{-1} |\eta_A h|, \quad |\eta_A X_\alpha \partial_x h| \leq C \alpha^{-1} |\eta_A h|. \]

**Proof.** These estimates follow directly from [25, Lemma 4.7] (see also [24, Lemma 5]) except the last two lines. We prove the estimates \( |\eta_A X_\alpha \partial_x^2 h| \) and \( |\eta_A X_\alpha \partial_x h| \). First,

\[ |\eta_A X_\alpha \partial_x^2 h| \leq |X_\alpha [\eta_A \partial_x^2 h]|. \]

Using \( \eta_A \partial_x^2 h = \partial_x^2 (\eta_A h) - 2\partial_x (\eta_A^\prime h) + \eta_A^\prime h \) and \( |\eta_A^\prime| + |\eta_A^\prime| \leq C \eta \), we have

\[ |X_\alpha [\eta_A \partial_x^2 h]| \leq \alpha^{-1} |\eta_A h| + 2\alpha^{-1} |\eta_A h| + |\eta_A h| \leq C \alpha^{-1} |\eta_A h|. \]

Similarly,

\[ |X_\alpha [\eta_A \partial_x h]| \leq |X_\alpha [\partial_x (\eta_A h)]| + |X_\alpha [\partial_x h]| \leq C \alpha^{-1} |\eta_A h|. \]

The estimates on \( |\rho^{-1} X_\alpha \partial_x^2 [\rho h]| \), \( |\rho^{-1} X_\alpha \partial_x [\rho h]| \) are proved similarly. \( \square \)

**Lemma 4.** There exists \( C > 0 \) such that for any \( \alpha > 0 \) small and \( g \in H^1(\mathbb{R}) \)

\[ |\eta_A X_\alpha^2 M_- S^2 g| + |\eta_A X_\alpha^2 S^2 L_+ g| \leq C (\alpha^{-2} |\eta_A \partial_x g| + \omega_\alpha^2 |\eta_A g|), \]
\[ |\eta_A \partial_x X_\alpha^2 M_- S^2 g| + |\eta_A \partial_x X_\alpha^2 S^2 L_+ g| \leq C (\alpha^{-2} |\eta_A \partial_x g| + \omega_\alpha^2 |\eta_A g|). \]
Proof. We prove the two estimates for $X^2 \alpha \, M \, S^2 \, g$. The proof for $X^2 \, S^2 \, L_+ \, g$ is identical. By Lemma 3

$$\| \eta_A X^2_\alpha \partial^1_x g \| \leq C \alpha^{-\frac{3}{2}} \| \eta_A \partial_x g \|, \quad \| \eta_A X^2_\alpha \partial^2_x g \| \leq C \alpha^{-2} \| \eta_A \partial_x g \|.$$ 

By Lemma 3 and $|\frac{\partial^j}{\partial \omega^j}| \leq C \sqrt{\omega^0} \leq C$ (see (9)), we have

$$\Big\| \eta_A X^2_\alpha \partial^1_x \frac{\phi'}{\phi} \cdot \partial_x g \Big\| \leq C \alpha^{-1} \| \eta_A \partial_x g \|,$$

$$\Big\| \eta_A X^2_\alpha \partial^2_x \frac{\phi'}{\phi} \cdot \partial_x g \Big\| \leq C \alpha^{-\frac{3}{2}} \| \eta_A \partial_x g \|.$$ 

Similarly,

$$\| \eta_A X^2_\alpha \partial_x \cdot \phi^4 \cdot \partial_x g \| \leq C \alpha^{-\frac{4}{3}} \| \eta_A \partial_x g \|,$$

$$\| \eta_A X^2_\alpha \partial_x \cdot \phi^1 \cdot \partial_x g \| \leq C \alpha^{-1} \| \eta_A \partial_x g \|,$$

and

$$\Big\| \eta_A X^2_\alpha \Big( 2 \omega \frac{\phi'}{\phi} + \frac{14}{3} \phi^3 \phi' \Big) \cdot \partial_x g \Big\| \leq C \| \eta_A \partial_x g \|,$$

$$\Big\| \eta_A X^2_\alpha \partial_x \Big( 2 \omega \frac{\phi'}{\phi} + \frac{14}{3} \phi^3 \phi' \Big) \cdot \partial_x g \Big\| \leq C \alpha^{-\frac{1}{2}} \| \eta_A \partial_x g \|.$$ 

Moreover,

$$\Big\| \eta_A X^2_\alpha \Big( \omega^2 + 6 \omega \phi^4 - \frac{10}{3} \phi^6 + \frac{7}{3} \phi^8 \Big) g \Big\| \leq C \omega^0 \| \eta_A g \|.$$ 

Last, we observe that

$$\partial_x \left[ \left( \omega^2 + 6 \omega \phi^4 - \frac{10}{3} \phi^6 + \frac{7}{3} \phi^8 \right) g \right] = \left( \omega^2 + 6 \omega \phi^4 - \frac{10}{3} \phi^6 + \frac{7}{3} \phi^8 \right) \partial_x g + \left( 24 \omega \phi^3 - 20 \phi^5 + \frac{56}{3} \phi^7 \right) \phi' \phi' \cdot g.$$ 

As before,

$$\Big\| \eta_A X^2_\alpha \Big( \omega^2 + 6 \omega \phi^4 - \frac{10}{3} \phi^6 + \frac{7}{3} \phi^8 \Big) \partial_x g \Big\| \leq C \| \eta_A \partial_x g \|$$

and since $|\phi'| \leq C \rho^2$,

$$\Big\| \eta_A X^2_\alpha \Big( 24 \omega \phi^3 - 20 \phi^5 + \frac{56}{3} \phi^7 \Big) \phi' \phi' \cdot g \Big\| \leq C \omega^2 \| \rho^2 \|.$$ 

This completes the proof of Lemma 4. 

Applying Lemma 4 to $u_2$ and $u_1$, we obtain the following result.

Lemma 5. There exists $C > 0$ such that for any $\alpha > 0$ small,

$$\| \eta_A v \| \leq C \left( \alpha^{-\frac{3}{2}} \| \eta_A \partial_x u \| + \omega^0 \| \eta_A u \| \right),$$

$$\| \eta_A \partial_x v \| \leq C \left( \alpha^{-2} \| \eta_A \partial_x u \| + \omega^0 \| \rho^2 u \| \right).$$
Lemma 6. There exists $C > 0$ such that for any $\alpha > 0$ small and $g \in H^1(\mathbb{R})$

$$\|\eta A X_\alpha^2 Q - g\| + \|\eta A X_\alpha^2 Q + g\| \leq C (\alpha^{-1} \omega_0^2 \|\eta A \partial_x g\| + \omega_0^2 \|\eta A g\|),$$

$$\|\eta A \partial_x X_\alpha^2 Q - g\| + \|\eta A \partial_x X_\alpha^2 Q + g\| \leq C (\alpha^{-3} \omega_0^2 \|\eta A \partial_x g\| + \omega_0^2 \|\rho^2 g\|).$$

Proof. The proof is similar to the one of Lemma 4. \qed

3.5. Second virial estimate. From the equation (14) of $u$ and the identity of Lemma 2, $v$ satisfies

\begin{align}
\partial_t v_1 &= M_- v_2 + Y_\alpha v_2 + X_\alpha^2 n_2 - X_\alpha^2 r_2 \\
\partial_t v_2 &= -M_+ v_1 + \frac{1}{2} Y_\alpha v_1 - X_\alpha^2 n_1 + X_\alpha^2 r_1
\end{align}

where we have used $S^2 \theta_1 = S^2 L_+ \theta_2 = 0$ (see (2.2) and (5)) and the notation

$$n_1 = S^2 L_+ m_2 + \frac{\omega}{\omega} Q_+ u_1, \quad n_2 = -M_- S^2 m_1 + \frac{\omega}{\omega} Q_- u_2,$$

$$r_1 = S^2 L_+ q_2, \quad r_2 = -M_- S^2 q_1,$$

and

$$Y_\alpha = X_\alpha^2 \cdot \phi_\alpha^4 \cdot X_\alpha^{-2} - \phi_\alpha^4.$$

Remark 4. The identity used for the transformed problem is different from the one for the wave-type equations in [24, 25] and for the Schrödinger equation with a real potential in [14]. However, the underlying idea is similar: the system for $(v_1, v_2)$ has the same structure as the original system in $(u_1, u_2)$, but with more favorable operators $M_+, M_-.$

The next proposition provides the key estimate on the function $v$. Thanks to the special structure of the operators $M_+$ and $M_-$, a virial estimate on $v$ controls a weighted $L^2$ norm of $v$ by higher order terms in $u$.

Proposition 3. Assume that $\omega_0 \in (0, \frac{1}{8}].$ There exists $C > 0$ such that, for $B > 0$ large enough, $\alpha > 0$ and $\varepsilon > 0$ small enough, for any $T > 0,$

$$\omega_0^2 \int_0^T \|\rho v\|^2 \, dt \leq C \varepsilon + \frac{C}{A} \int_0^T \left( \|\eta A \partial_x u\|^2 + \frac{\omega_0^2}{A} \|\eta A u\|^2 + \omega_0 \|\rho^2 u\|^2 \right) \, dt.$$

Proof. First, we introduce

$$\mathcal{J} = \int v_1 \left( 2 \Psi_{A,B} \partial_x v_2 + \Psi'_{A,B} v_2 \right) \quad \text{and} \quad z = \chi_A \zeta_B v.$$

By the equation (13) of $v$ and direct computations (see [24] Proof of Proposition 2, §4.3), we compute

$$\dot{\mathcal{J}} = \int \left( 2 (\partial_x z_1)^2 - P_B z_1^2 \right) + \int \left( 2 (\partial_x z_2)^2 + 3 P_B z_2^2 \right) + \sum_{j=1}^5 J_j,$$

where

$$P_B = -\frac{1}{3} \frac{\Phi_B}{\zeta_B^2} (\phi_{\omega_0})'.$$
and

\[ J_1 = \sum_{k=1}^{2} \int (\ln \zeta_B)'' z_k^2, \]

\[ J_2 = -\sum_{k=1}^{2} \int \left( \frac{1}{2} (\chi_A')' (\zeta_B')' + (3(\chi_A')^2 + \chi''_A) \zeta_B^2 + \frac{1}{2}(\chi_A''') \Phi_B \right) v_k^2 \]

\[ + \sum_{k=1}^{2} \int (\chi_A') \Phi_B (\partial_k v_k)^2, \]

\[ J_3 = -\frac{1}{3} \int (2\Psi_{A,B} \partial_1 v_1 + \Psi_{A,B}' v_1) Y_a v_1 + \int (2\Psi_{A,B} \partial_2 v_2 + \Psi_{A,B}' v_2) Y_a v_2, \]

\[ J_4 = \sum_{k=1}^{2} \int (2\Psi_{A,B} \partial_k v_k + \Psi_{A,B}' v_k) (X^2_n k - X^2_r k) \]

\[ J_5 = -\frac{1}{3} \int \phi^4_{\omega_0} - \phi^4_{\omega} (z_1^2 - 3z_2^2). \]

Second, we set

\[ K = -\int z_1 z_2 R_B, \]

where the function \( R_B \) is the unique (time-independent) bounded solution of

\[ -\frac{1}{2} R'' + \omega_0 R_B = \frac{3}{2} P_B \quad \text{on } \mathbb{R}. \]

Using the equation (10) of \( v \), integrating by parts and using the equation of \( R_B \), we obtain

\[ \dot{K} = \frac{3}{2} \int (z_1^2 - z_2^2) P_B + \sum_{j=1}^{5} K_j, \]

where

\[ K_1 = \sum_{k=1}^{2} (-1)^k \int v_k^2 \left( (\chi_A \zeta_B)' \chi_A \zeta_B R_B + (\chi_A \zeta_B)''^2 R_B \right) \]

\[ K_2 = \int \left( (\partial_1 z_1)^2 - (\partial_2 z_2)^2 \right) R_B - \int \left( \frac{1}{3} z_1^2 + z_2^2 \right) \phi^4 v_B \]

\[ K_3 = -\int \left( \frac{1}{3} (Y_a v_1 v_1 + (Y_a v_2) v_2) \chi_A^2 R_B, \right) \]

\[ K_4 = \sum_{k=1}^{2} (-1)^{k-1} \int (X^2_n k - X^2_r k) v_k \chi_A^2 R_B, \]

\[ K_5 = (\omega - \omega_0) \int (z_1^2 - z_2^2) R_B. \]
Moreover,
\[ \omega R \text{ into the above expression of } z \text{ provided that the error terms } J_j \text{ and } K_j \text{ are controlled.} \]

Comparing the expressions of \( \dot{J} \) and \( \dot{J} + \dot{K} \), we observe that the functional \( \mathcal{K} \) was used to exchange \( \frac{1}{2} \int P_B z^2 \) by \( \frac{3}{2} \int P_B z^2 \) at the cost of additional error terms.

To estimate the error terms, we need several technical estimates.

**Lemma 7.** For any \( \omega_0 \in (0, \frac{1}{8}] \),
\[
\| P_B \|_{L^\infty} \leq \frac{1}{5} \omega_0, \quad \| R_B \|_{L^\infty} \leq \frac{7}{18}.
\]

Moreover,
\[
0 \leq P_B(x) \leq C \omega_0 \phi_{\omega_0}^2, \quad 0 \leq R_B(x) \leq C \phi_{\omega_0}^2 \quad |R'_B(x)| \leq C \sqrt{\omega_0} \phi_{\omega_0}^2 \quad \text{on } \mathbb{R}.
\]

**Proof.** Using \( |\Phi_B| \leq |x| \) and \( \zeta_B(x) \geq e^{-\frac{\omega_0}{2} \sqrt{2} |x|} \geq e^{-\omega_0 e^{2 \sqrt{2} |x|}} \), we have
\[
0 \leq P_B(x) \leq \frac{1}{3} |x| e^{\frac{\omega_0}{2} |x|} (\phi_{\omega_0}^4)' - \phi_{\omega_0}^4 e^{-\omega_0 |x|} \sinh(2 \sqrt{\omega_0} x).
\]

Note that
\[
\phi_{\omega_0}^4 = \frac{16 \omega_0^2}{(1 + a_{\omega_0} \cosh(2 \sqrt{\omega_0} x))^2}, \quad (\phi_{\omega_0}^4)’ = -a_{\omega_0} \phi_{\omega_0}^6 e^{-\frac{1}{\sqrt{3}} \omega_0} \sinh(2 \sqrt{\omega_0} x).
\]

Moreover, \( \omega_0 \leq \frac{1}{8} \) implies \( a_{\omega_0} \geq \frac{1}{3\sqrt{3}} \). By the inequality (21) proved in Appendix A, we obtain \( 0 \leq P_B(x) \leq \frac{32}{9} \omega_0 \phi_{\omega_0}^2 \). For any \( \omega_0 \in (0, \frac{1}{8}] \), we have \( \frac{32}{9} \omega_0 a_{\omega_0} \leq \frac{4}{3\sqrt{3}} \leq \frac{7}{27} \) and thus \( 0 \leq P_B(x) \leq \frac{7}{27} \omega_0 \). From (20) and using first \( B \geq 1, \frac{1}{\sqrt{3}} \leq a_{\omega} \leq 1 \) and then \( |\sinh y| \leq \cosh y \) and \( |y| e^{\frac{|y|}{3}} \leq C \cosh(y) \), we have
\[
P_B(y) \leq \frac{16}{3} \omega_0 \phi_{\omega_0}^2 \frac{\sqrt{\omega_0} |x| e^{\frac{\omega_0}{3} |x|} \sinh(2 \sqrt{\omega_0} x)}{(1 + \frac{1}{\sqrt{3}} \cosh(2 \sqrt{\omega_0} x))^2} \leq C \omega_0 \phi_{\omega_0}^2.
\]

Next, we recall the explicit expression of \( R_B \)
\[
R_B = \frac{3}{2} \int_{-\infty}^{x} e^{\sqrt{2} \omega_0 (y-x)} P_B(y) \, dy + \int_{x}^{\infty} e^{\sqrt{2} \omega_0 (x-y)} P_B(y) \, dy
\]
and so \( \| R_B \|_{L^\infty} \leq \frac{4}{3} \omega_0^{-1} \| P_B \|_{L^\infty} \leq \frac{7}{18} \). Inserting \( 0 \leq P_B(x) \leq C \omega_0 e^{-2 \sqrt{\omega_0} |x|} \) into the above expression of \( R_B \), we also find \( 0 \leq R_B(x) \leq C \omega_0 e^{-2 \sqrt{\omega_0} |x|} \) and \( |R'_B(x)| \leq C \omega_0 e^{-2 \sqrt{\omega_0} |x|} \).

\[\square\]
Lemma 8. For any $\omega_0 \in (0, \frac{1}{8}]$, for any function $h \in H^1(\mathbb{R})$, 
\[
\int \phi^4_{\omega_0} h^2 \leq \frac{19}{6} \int P_B h^2 + 3 \int (h')^2.
\]

Proof. By the definition of $P_B$ and integration by parts, we have 
\[
3 \int P_B h^2 = - \int \frac{\Phi_B}{\zeta_B^2} (\phi^4_{\omega_0})' h^2 = \int \left( \frac{\Phi_B}{\zeta_B^2} \phi^4_{\omega_0} h^2 + 2 \int \frac{\Phi_B}{\zeta_B^2} \phi^4_{\omega_0} hh' \right).
\]
Using 
\[
\int \left( \frac{\Phi_B}{\zeta_B^2} \phi^4_{\omega_0} \right)' h^2 = \int \phi^4_{\omega_0} h^2 - 2 \int \frac{\Phi_B}{\zeta_B^2} \phi^4_{\omega_0} h^2 \geq \int \phi^4_{\omega_0} h^2
\]
and 
\[
2 \left| \int \frac{\Phi_B}{\zeta_B^2} \phi^4_{\omega_0} h h' \right| \leq \frac{1}{3} \int \frac{\Phi_B}{\zeta_B^2} \phi^8_{\omega_0} h^2 + 3 \int (h')^2,
\]
we obtain 
\[
\int \phi^4_{\omega_0} h^2 \leq \left( 3 + \frac{1}{3} \left\| \frac{\Phi_B \phi^8_{\omega_0}}{\zeta_B^4 P_B} \right\|_{L^\infty} \right) \int P_B h^2 + 3 \int (h')^2.
\]
We claim the following estimate 
\[
\left\| \frac{\Phi_B \phi^8_{\omega_0}}{\zeta_B^4 P_B} \right\|_{L^\infty} \leq 4 \omega_0 \leq \frac{1}{2}
\]
which is sufficient to complete the proof. To prove the above estimate, we recall that $a_{\omega_0} \geq \frac{1}{\sqrt{3}}$, $|\Phi_B| \leq |x|$, $\zeta_B \geq e^{-\frac{\sqrt{x^2}}{3h}}$, so that 
\[
\left| \frac{\Phi_B^2 \phi^8_{\omega_0}}{\zeta_B^4 P_B} \right| = \frac{3|\Phi_B| \phi^8_{\omega_0}}{\zeta_B^4(P_B)^4} = \frac{3\sqrt{\omega_0} \Phi_B \phi^2_{\omega_0}}{\zeta_B^2 a_{\omega_0} \sqrt{\omega_0} x \sinh(2 \sqrt{\omega_0} x)}
\]
\[
\leq 6 \omega_0 \left\| \frac{ye^{-\frac{\sqrt{y}}{3}}}{\sinh(y)(1 + \frac{1}{\sqrt{3}} \cosh(y))} \right\|_{L^\infty} \leq 4 \omega_0
\]
using in the last step the inequality (25) proved in Appendix A. \qed

Lemma 9. There exists $c > 0$ such that, for any $\omega_0 \in (0, \frac{1}{8}]$, for any $x \in \mathbb{R}$, 
\[
P_B(x) \geq c \omega_0^2 1_{[1,2]}(\sqrt{\omega_0} |x|).
\]
Moreover, there exists $C > 0$ such that for any $h \in H^1(\mathbb{R})$, 
\[
\omega_0^2 \int \rho h^2 \leq C \omega_0 \int (h')^2 + C \int P_B h^2.
\]

Proof. The function $\zeta : [0, \infty) \to [0, 1]$ being non increasing, for $x \geq 0$, $\Phi_B(x) = \int_0^x \zeta_B \geq |x| \zeta_B^2(x)$. By parity $|\Phi_B(x)| \geq |x| \eta_B^2(x)$ on $\mathbb{R}$. Thus, 
\[
P_B = - \frac{1}{3} \frac{\Phi_B}{\zeta_B^2} (\phi^4_{\omega_0})' \geq c \omega_0^2 \frac{\sqrt{\omega_0} |x| \sinh(2 \sqrt{\omega_0} x)}{1 + \cosh(2 \sqrt{\omega_0} x)} \geq c \omega_0^2 1_{[1,2]}(\sqrt{\omega_0} |x|),
\]
where $c$ denotes positive constants. The inequality (21) then follows from standard arguments, see e.g. [24, Lemma 4]. \qed
Last, we prove an estimate on \( v \) in terms of \( z \), plus an error term in \( u \).

**Lemma 10.** There exists \( C > 0 \) such that
\[
\int \rho^2 (|\partial_x v|^2 + \omega_0^2 |v|^2) \leq C \int (|\partial_x z|^2 + P_B |z|^2) + C \left( \frac{\alpha - 4}{A} \|\eta_A \partial_x u\|^2 + \frac{\omega_0^2}{A^2} \|\eta_A u\|^2 \right).
\]

**Proof.** We claim
\[
\int_{\omega_0^2 |x| \leq A} \rho^2 (|\partial_x v|^2 + \omega_0^2 |v|^2) \leq C \int (|\partial_x z|^2 + P_B |z|^2).
\]

By definition, for \( \omega_0^2 |x| \leq A \), one has \( z = \zeta_B v \) and so (for \( B \) large)
\[
\int_{\omega_0^2 |x| \leq A} \rho^2 |v|^2 \leq C \int_{\omega_0^2 |x| \leq A} \rho |\zeta_B^2 v|^2 \leq C \int_{\omega_0^2 |x| \leq A} \rho |z|^2.
\]

For \( \omega_0^2 |x| \leq A \), using \( \partial_x z = \zeta_B \partial_x v + \zeta_B \partial_x \) and \( \|\zeta_B\| \leq C \omega_0^{-2} B^{-1} \zeta_B \), we also have
\[
\rho^2 |\partial_x v|^2 \leq C \rho |\zeta_B^2 v|^2 \leq C \rho |\partial_x z|^2 + C \omega_0^2 B^{-2} \rho \zeta_B^2 |v|^2 \leq C \rho |\partial_x z|^2 + C \omega_0^2 B^{-2} \rho |z|^2
\]
and so
\[
\int_{\omega_0^2 |x| \leq A} \rho^2 |\partial_x v|^2 \leq C \int |\partial_x z|^2 + C \frac{\omega_0^2}{B^2} \int \rho |z|^2.
\]

We complete the proof of the claim using (21).

Now, using Lemma 5,
\[
\int_{\omega_0^2 |x| \geq A} \rho^2 (|\partial_x v|^2 + |v|^2) \leq C e^{-\frac{A}{10 \omega_0}} \left( \|\eta_A \partial_x v\|^2 + \|\eta_A v\|^2 \right)
\leq C \frac{\omega_0^2}{A^2} \left( \alpha - 4 \|\eta_A \partial_x u\|^2 + \|\eta_A u\|^2 \right),
\]
which implies the desired estimate. \( \square \)

**Estimate of \( J_1 \).** By (16) and then Lemma 9 we have
\[
|\ln \zeta_B''| \leq C \frac{\omega_0^2}{B} |1, z| (|\sqrt{\omega_0} |x| |) \leq C \frac{B}{B} P_B.
\]

Thus, for \( B \) large enough (independent of \( \alpha, \omega_0, A, \) and \( \epsilon \)), we have
\[
|J_1| \leq \frac{1}{100} \int P_B |z|^2.
\]

**Estimate of \( K_1 \).** Using \( |\chi_A'| \leq C \omega_0^{-3} A^{-1} \leq C \omega_0^{-3} B^{-1} \), \( |\zeta_B'| \leq C \omega_0^{-3} B^{-1} \zeta_B \), and Lemma 7 we estimate
\[
\left| (\chi_A \zeta_B)' \chi_A \zeta_B R_B + (\chi_A \zeta_B')^2 R_B \right| \leq \frac{\omega_0^3}{B^2} \rho^2.
\]
Thus,
\[ |K_1| \leq \frac{C\omega_0^3}{B} \int \rho^2 |v|^2, \]
and using Lemma 10 taking \( B \) large enough (independent of \( \alpha, \omega_0, A \) and \( \varepsilon \)),
\[ |K_1| \leq \frac{1}{100} \int \left( |\partial_x z|^2 + P_B |z|^2 \right) + \frac{C}{A} \left( \alpha^{-4} \|\eta_A \partial_x u\|^2 + \frac{\omega_0^3}{A^2} \|\eta_A u\|^2 \right). \]

From now on, \( B \) is fixed so that the above estimates on \( J_1 \) and \( K_1 \) hold.

**Estimate of \( J_2 \).** First we recall some bounds on the functions involved in the definition of \( J_2 \). We have
\[ |\chi_A'| \leq \frac{C\omega_0^3}{A}, \quad |\chi_A''| \leq \frac{C\omega_0^3}{A^2}, \quad |\chi_A'''| \leq \frac{C\omega_0^3}{A^4}, \]
and
\[ \chi_A'(x) = \chi_A''(x) = \chi_A'''(x) = 0 \quad \text{if } \omega_0^3 |x| < A \quad \text{or if } \omega_0^3 |x| > 2A. \]

Moreover,
\[ |\zeta_B(x)| \leq Ce^{-\frac{4}{B}}, \quad |\zeta_B'(x)| \leq \frac{C\omega_0^3}{B} e^{-\frac{4}{B}} \quad \text{for } \omega_0^3 |x| > A. \]

Thus,
\[ |\chi_A''(\zeta_B^2)| \leq \frac{C\omega_0^3}{AB} e^{-\frac{4}{B}} \frac{\omega_0^3}{A} \zeta_B^2, \quad |\chi_A''(\zeta_B^2)| \frac{C\omega_0^3}{AB} e^{-\frac{4}{B}} \frac{\omega_0^3}{A} \zeta_B^2 \leq \frac{C\omega_0^3}{A^2} e^{-\frac{4}{B}} \frac{\omega_0^3}{A} \zeta_B^2. \]

Using also \( |\Phi_B| \leq CB\omega_0^{-\frac{3}{2}} \), we obtain
\[ |\chi_A''(\zeta_B^2)\Phi_B| \leq \frac{CB}{A} \omega_0^2 \zeta_B^2, \quad |\chi_A''(\zeta_B^2)\Phi_B| \leq \frac{CB}{A^2} \omega_0^3 \zeta_B^2. \]

Therefore (recall that \( B \) has been fixed)
\[ |J_2| \leq \frac{C}{A} \left( \|\eta_A \partial_x v\|^2 + \frac{\omega_0^3}{A^2} \|\eta_A v\|^2 \right). \]

Using Lemma 5 it follows that
\[ |J_2| \leq \frac{C}{A} \left( \alpha^{-4} \|\eta_A \partial_x u\|^2 + \frac{\omega_0^3}{A^2} \|\eta_A u\|^2 + \omega_0 \|\rho^2 u\|^2 \right). \]

**Estimate of \( K_2 \).** Using Lemmas 11 and 12, we have
\[
|K_2| \leq \|R_B\| L = \left( \int |\partial_x z|^2 + \frac{1}{3} \int z^2 \phi_\omega^4 + \int z^2 \phi_\omega^4 + \int |z|^2 |\phi_\omega^4 - \phi_\omega^4| \right) \
\leq (1 + C \varepsilon) \int \left( \frac{7}{9} (\partial_x z_1)^2 + \frac{133}{324} P_B z_1^2 + \frac{14}{9} (\partial_x z_2)^2 + \frac{133}{108} P_B z_2^2 \right) \
\leq \frac{9}{10} \int \left( 2(\partial_x z_1)^2 + \frac{1}{2} P_B z_1^2 + 2(\partial_x z_2)^2 + \frac{3}{2} P_B z_2^2 \right) 
\]
for \( \varepsilon \) small enough.
Estimate of $J_3$ and $K_3$. By the Cauchy-Schwarz inequality, and the bounds
\[ |\Psi_{A,B}| \leq CB\omega_0^{-\frac{3}{2}} \leq C\omega_0^{-\frac{3}{2}}, \quad |\Psi'_{A,B}| \leq C, \]
we have for $k = 1, 2$
\[ \left| \int (2\Psi_{A,B}\partial_x v_k + \Psi'_{A,B}v_k)Y_\alpha v_k \right| \leq C\left( \omega_0^{-\frac{3}{2}} \|\rho \partial_x v_k\| + \|\rho v_k\| \right) \|\rho^{-1}Y_\alpha v_k\|. \]
We rewrite $Y_\alpha$ as
\[ Y_\alpha = 2\alpha X_\alpha^2 [2\partial_x \cdot (\phi_\alpha^4)' - (\phi_\alpha^4)'] + \alpha^2 X_\alpha^2 [-4\partial_x^2 \cdot (\phi_\alpha^4)' + 6\partial_x^2 \cdot (\phi_\alpha^4)'' - 4\partial_x \cdot (\phi_\alpha^4)''' - 2(\phi_\alpha^4)^{(4)}]. \]
Using Lemma 4 and Lemma 6, we obtain
\[ \|\rho^{-1}Y_\alpha v_k\| \leq C\alpha^\frac{3}{2}\omega_0^\frac{3}{2}\|\rho v_k\|. \]
Thus,
\[ |J_3| \leq C\alpha^\frac{3}{2}\omega_0 \left( \|\rho \partial_x v\| + \omega_0^\frac{3}{2}\|\rho v\| \right) \|\rho v\|. \]
Therefore, using Lemma 10
\[ |J_3| \leq C\alpha^\frac{1}{2} \int \left( \|\partial_x z\|^2 + P_B|z|^2 \right) + \frac{C}{A} \left( \alpha^{-4}\|\eta_A\partial_x u\|^2 + \frac{\omega_0^3}{A^2}\|\eta_A u\|^2 \right). \]
The estimate of $K_3$ is similar and easier.
We fix $\alpha > 0$ (independent of $\omega_0$, $A$ and $\varepsilon$) so that
\[ |J_3| + |K_3| \leq \frac{1}{100} \int \left( \|\partial_x z\|^2 + P_B|z|^2 \right) + \frac{C}{A} \left( \|\eta_A\partial_x u\|^2 + \frac{\omega_0^3}{A^2}\|\eta_A u\|^2 \right). \]
Estimate of $J_4$ and $K_4$. Using Lemma 4 and Lemma 5 we have for $k = 1, 2$,
\[ \|\eta_A X_\alpha^2 m_k\| \leq C \left( \|\eta_A\partial_x m_k\| + \|\eta_A m_k\| \right) + C\omega_0^{-\frac{3}{2}} \left( \|\eta_A\partial_x u\| + \|\eta_A u\| \right). \]
By the expression of $m_k$, $|x\eta_A| \leq CA$ and (12), (13), we obtain
\[ \|\eta_A\partial_x m_k\| + \|\eta_A m_k\| \leq CA\varepsilon\|u\|^2 \leq CA\varepsilon\|\eta_A u\|. \]
Thus, by the Cauchy-Schwarz inequality, Lemma 5 and (12),
\[ \left| \int (2\Psi_{A,B}\partial_x v_k + \Psi'_{A,B}v_k) (X_\alpha^2 m_k) \right| \leq C\|\eta_A X_\alpha^2 m_k\| \left( \|\eta_A\partial_x v\| + \|\eta_A v\| \right) \leq CA\varepsilon^2 \left( \|\eta_A\partial_x u\| + \|\eta_A u\| \right)^2. \]
Using Lemmas 4 we have
\[ \sum_{k=1,2} \|\eta_A X_\alpha^2 q_k\| \leq C \sum_{k=1,2} \left( \|\eta_A\partial_x q_k\| + \|\eta_A q_k\| \right) \]
Moreover, by (13),
\[ |q_1| + |q_2| \leq C|u|^2 \leq C\varepsilon|u| \]
and so
\[ \|\eta_A\partial_x q_k\| + \|\eta_A q_k\| \leq C\varepsilon \left( \|\eta_A\partial_x u\| + \|\eta_A u\| \right) \]
Thus, as before,
\[
\left| \int (2\Psi_{A,B} \partial_x v_k + \Psi'_{A,B} v_k) (X^2 r_k) \right| \leq C \varepsilon \left( \| \eta_A \partial_x u \| + \| \eta_A u \| \right)^2.
\]
Thus, for \( \varepsilon \) small (depending on \( A \) and \( \omega_0 \)),
\[
|J_4| \leq \frac{C}{A} \left( \| \eta_A \partial_x u \|^2 + \frac{\omega_0^3}{A^2} \| \eta_A u \|^2 \right).
\]
The estimate of \( K_4 \) is similar to the one of \( J_4 \).

**Estimate of \( J_5 \) and \( K_5 \).** By \( |\Phi_B| \leq |x| \), Lemma 7 and (13), we have
\[
|J_5| + |K_5| \leq C \varepsilon \omega_0 \| \rho^2 z \|.
\]
For \( \varepsilon \) small, these terms are controlled using (21),
\[
|J_5| + |K_5| \leq \frac{1}{100} \int (|\partial_x z|^2 + |P_B|^2).
\]

**Conclusion of the proof of Proposition 3.** Combining the identity (19) with the above estimates on the error terms, we have
\[
\dot{J} + \dot{K} \geq \frac{1}{100} \int (|\partial_x z|^2 + |P_B|^2)
\]
\[
- \frac{C}{A} \left( \| \eta_A \partial_x u \|^2 + \frac{\omega_0^3}{A^2} \| \eta_A u \|^2 + \omega_0 \| \rho^2 u \|^2 \right).
\]
Moreover, we see that for any \( T \geq 0 \), using (17),
\[
|\mathcal{J}(T)| + |\mathcal{K}(T)| \leq C (\| \Phi_B \|_{L^\infty} + \| \Phi'_{B} \|_{L^\infty} + \| R_B \|_{L^\infty}) \left( |\partial_x v|^2 + |v|^2 \right)
\]
\[
\leq C B \omega_0^{-\frac{3}{2}} \varepsilon^2 \leq C \varepsilon.
\]
Therefore, integrating (22) on \([0, T]\), we obtain
\[
\int_0^T \int (|\partial_x z|^2 + |P_B|^2) \leq C \varepsilon
\]
\[
+ \frac{C}{A} \int_0^T \left( \| \eta_A \partial_x u \|^2 + \frac{\omega_0^3}{A^2} \| \eta_A u \|^2 + \omega_0 \| \rho^2 u \|^2 \right).
\]
Using Lemma 10, it follows that
\[
\omega_0^2 \int_0^T \| \rho v \|^2 \leq C \varepsilon + \frac{C}{A} \int_0^T \left( \| \eta_A \partial_x u \|^2 + \frac{\omega_0^3}{A^2} \| \eta_A u \|^2 + \omega_0 \| \rho^2 u \|^2 \right),
\]
which completes the proof of the proposition.

**3.6. Coercivity property.** Lemma 5 gives estimates of \( v \) in terms of \( u \). The next proposition, inspired by [24, Lemma 6], shows conversely that \( u \) is controlled by \( v \) in the \( L^2 \) norm with weight \( \rho^2 \) and some loss. This result uses the orthogonality conditions (11) on \( u \).

**Proposition 4.** There exists \( C > 0 \) such that, for all \( t \geq 0 \),
\[
\omega_0^2 \| \rho^2 u \| \leq C \| \rho v \|.
\]
The proof of Proposition \[\text{Proposition}\] follows from the next two lemmas.

**Lemma 11.** There exists $C > 0$ such that for any $\omega \in (0, \frac{1}{8}]$ and $g \in L^2(\mathbb{R})$, if

$$\langle g, \phi_\omega \rangle = \langle g, x\phi_\omega \rangle = 0$$

then

$$\|\rho^2 g\| \leq C\omega_0^{-2}\|\rho(X_0^2 S^2 L_g)\|.$$  

**Proof.** Let $g$ be as in the statement of the lemma and let $h = X_0^2 S^2 L_g$. We have

$$\partial_x^2 \left( \frac{L_g}{\phi_\omega} \right) = \alpha^2 \frac{h''''}{\phi_\omega} - 2\alpha \frac{h''}{\phi_\omega} + \frac{h''}{\phi_\omega}.$$  

To simplify notation, we denote by $f_j$ functions of class $C^\infty$ whose expression may change from line to line, and satisfying

$$|f_j(x)| \leq C\omega^{-\frac{1}{2}}e^{|x|}$$

on $\mathbb{R}$. We also denote $\partial_x^{-1} = \int_0^x$ and $\partial_x^{-2} = \partial_x^{-1} \cdot \partial_x^{-1}$. We check

$$h'' = \left( \frac{h}{\phi_\omega} \right)'' + (f_3 h)' + f_2 h,$$

$$h''' = \left( \frac{h}{\phi_\omega} \right)''' + (f_3 h)'' + (f_2 h)' + f_1 h.$$  

Thus,

$$\partial_x^2 \left( \frac{L_g}{\phi_\omega} \right) = \alpha^2 \left( \frac{h}{\phi_\omega} \right)'' + \alpha^2 (f_3 h)'' + \alpha (f_2 h)' + f_1 h.$$  

By integration and multiplication by $\phi_\omega$, we obtain

$$L_g = a(\omega^2 x)\phi_\omega + b\phi_\omega + \phi_\omega \sum_{k=-2}^2 \alpha^{\frac{k+1}{2}} |\partial_x^k (f_{k+2}h)|$$

where $a$ and $b$ are integration constants. We claim that

$$|a| + |b| \leq C\omega^{-\frac{1}{2}}\|\rho h\|.$$  

To prove this, we note that $\langle L_g, \Lambda_\omega \rangle = \langle L_g, \phi_\omega \rangle = 0$, by $L_0 \Lambda_\omega = -\omega \phi_\omega$, $L_0 \phi_\omega = 0$ and $\langle g, \phi_\omega \rangle = 0$. Moreover, $\langle \Lambda_\omega, x\phi_\omega \rangle = 0$ by parity. Taking the scalar product of the above expression of $L_g$ by $\Lambda_\omega$, we have

$$|b| \leq \frac{C}{|\langle \phi_\omega, \Lambda_\omega \rangle|} \sum_{k=-2}^2 \alpha^{\frac{k+1}{2}} |\langle \phi_\omega \Lambda_\omega, \partial_x^k (f_{k+2}h) \rangle|.$$  

We recall from Lemma \[\text{Lemma}\] that $|\langle \phi_\omega, \Lambda_\omega \rangle| \geq C\omega^{\frac{1}{2}}$. For $k = 0$, we have by the Cauchy-Schwarz inequality, $|\langle \phi_\omega \Lambda_\omega, f_2 h \rangle| \leq C\omega^{\frac{1}{2}}\|\rho h\|$. For $k = 1, 2$, integrating by parts,

$$|\langle \phi_\omega \Lambda_\omega, \partial_x^k (f_{k+2}h) \rangle| = |\langle \partial_x^k (\phi_\omega \Lambda_\omega), f_{k+2}h \rangle| \leq C\omega^{\frac{1}{2} + \frac{k}{2}}\|\rho h\|. $$
For $k = -1$, we have
\[
\left| \phi_\omega \Lambda_\omega \int_0^x f_1 h \right| \leq C \omega^{\frac{1}{2}} \rho \int_0^x \rho^2 |h| \leq C \omega^{\frac{3}{4}} \rho \| \rho h \|,
\]
and so \(|\langle \phi_\omega \Lambda_\omega, \partial_x^{-1} (f_1 h) \rangle| \leq C \omega^{-\frac{1}{4}} \| \rho h \|\). Last, for $k = -2$, we have
\[
\left| \phi_\omega \Lambda_\omega \int_0^x \int_0^y f_0 h \right| \leq C \omega^{\frac{1}{2}} \rho \int_0^x \rho \int_0^y \rho^2 |h| \leq C \omega^{-\frac{1}{4}} \rho \| \rho h \|,
\]
and so \(|\langle \phi_\omega \Lambda_\omega, \partial_x^{-2} (f_0 h) \rangle| \leq C \omega^{-\frac{3}{4}} \| \rho h \|\). Thus, \(|b| \leq C \omega^{-\frac{3}{4}} \| \rho h \|\). The proof of the estimate for $a$ is similar.

Recall the notation $I_+$ from 2.4. We have $g = I_+ [L_+ g] + c \phi_\omega$ where $c$ is a constant. In particular, we obtain
\[
g = a I_+ [(\omega^{\frac{3}{4}} x) \phi_\omega] + b I_+ [\phi_\omega] + \sum_{k=-2}^2 \alpha^{\frac{1+k}{2}} I_+ [\phi_\omega \partial^k_x (f_{k+2} h)] + c \phi_\omega.
\]

We estimate each term above for $x \geq 0$. We check easily that
\[
|I_+ [(\omega^{\frac{1}{4}} x) \phi_\omega]| + |I_+ [\phi_\omega]| \leq C \omega^{-\frac{1}{4}}
\]
and so $|a I_+ [(\omega^{\frac{1}{4}} x) \phi_\omega]| + |b I_+ [\phi_\omega]| \leq C \omega^{-\frac{3}{4}} \| \rho h \|$. Next, for $k = 0$, we have
\[
I_+ [\phi_\omega f_2 h] = -\phi_\omega' \int_0^x G \phi_\omega f_2 h - G \int_0^\infty \phi_\omega' \phi_\omega f_2 h.
\]
Thus,
\[
|I_+ [\phi_\omega f_2 h]| \leq C \omega^{-\frac{1}{4}} e^{-\sqrt{\omega} x} \int_0^x e^\sqrt{\omega} |h| + C \omega^{-\frac{1}{4}} e^\sqrt{\omega} x \int_x^\infty e^{-\sqrt{\omega} y} |h| \\
\leq C \omega^{-\frac{1}{4}} \rho^{-\frac{1}{2}} \int_0^\infty \rho^\frac{1}{2} |h| \leq C \omega^{-\frac{1}{4}} \rho^{-\frac{1}{2}} \| \rho h \|.
\]

For $k = 1$, by integration by parts,
\[
I_+ [\phi_\omega \partial_x (f_3 h)] = -\phi_\omega' \int_0^x G \phi_\omega \partial_x (f_3 h) - G \int_0^\infty \phi_\omega' \phi_\omega \partial_x (f_3 h) \\
= \phi_\omega' \int_0^x (G \phi_\omega)' f_3 h + G \int_0^\infty (\phi_\omega' \phi_\omega)' f_3 h + c_3 \phi_\omega'
\]
where $c_3 = G(0) \phi_\omega (0) f_3 (0) h(0)$. Proceeding as before, we obtain
\[
|I_+ [\phi_\omega \partial_x (f_3 h)] - c_3 \phi_\omega' | \leq C \rho^{-\frac{1}{4}} \omega^{-\frac{1}{4}} \| \rho h \|.
\]

For $k = 2$, by integration by parts and using $\phi_\omega'' G - \phi_\omega' G' = 1$, we compute
\[
I_+ [\phi_\omega \partial_x^2 (f_4 h)] = -\phi_\omega' \int_0^x G \phi_\omega \partial_x^2 (f_4 h) - G \int_0^\infty \phi_\omega' \phi_\omega \partial_x^2 (f_4 h) \\
= -\phi_\omega f_4 h - \phi_\omega' \int_0^x (G \phi_\omega)' f_4 h - G \int_0^\infty (\phi_\omega' \phi_\omega)'' f_4 h + c_4 \phi_\omega'
\]
where \( c_4 = -G(0)\phi_\omega(0)(f_4 h)'(0) + (G\phi_\omega)'(0)f_4(0)h(0) \). Proceeding as before, we obtain for \( x \geq 0 \),
\[
|I_+[\phi_\omega \partial_x^2 (f_1 h)] - c_4 \phi_\omega'| \leq C|h| + \rho^{-\frac{3}{2}}\omega^{\frac{1}{2}}\|\rho h\|.
\]
For \( k = -1 \), we have
\[
I_+[\phi_\omega \partial_x^{-2} (f_1 h)] = -\phi_\omega' \int_0^x G\phi_\omega \int_0^y (f_1 h) \, dy - G \int_x^\infty \phi_\omega \phi_\omega \int_0^y (f_1 h) \, dy.
\]
Thus,
\[
|I_+[\phi_\omega \partial_x^{-2} (f_1 h)]| \leq C\omega^{-\frac{1}{2}} \rho^{-\frac{3}{2}} \int_0^\infty \rho^\frac{3}{4} \int_0^y \rho^\frac{1}{4} |h| \, dy \leq C\omega^{-\frac{1}{2}} \rho^{-\frac{3}{2}} \|\rho h\|.
\]
Proceeding similarly,
\[
|I_+[\phi_\omega \partial_x^{-2} (f_0 h)]| \leq C\omega^{-\frac{1}{2}} \rho^{-\frac{3}{2}} \|\rho h\|.
\]
We have just proved that for \( x \geq 0 \)
\[
|g - \tilde{c} \phi_\omega'| \leq C|h| + \omega^{-\frac{3}{2}} \rho^{-\frac{3}{2}} \|\rho h\|
\]
where \( \tilde{c} = c + c_3 + c_4 \). This estimate also holds for \( x \leq 0 \) with the same constant \( \tilde{c} \). Taking the scalar product by \( x \phi_\omega \) and using \( \langle g, x\phi_\omega \rangle = 0 \), we obtain
\[
|\tilde{c}| \leq C\omega^{-2} \|\phi_\omega\|^{-2} \|\rho h\| \leq C\omega^{-\frac{3}{2}} \|\rho h\|.
\]
Therefore,
\[
|g| \leq C|g - \tilde{c} \phi_\omega'| + C|\tilde{c}| \|\phi_\omega'\| \leq C|h| + C\omega^{-\frac{3}{2}} \rho^{-\frac{3}{2}} \|\rho h\|.
\]
Multiplying by \( \rho^2 \) and taking the \( L^2 \) norm, we obtain the result. \( \square \)

**Lemma 12.** There exists \( C > 0 \) such that for any \( \omega \in (0, \frac{1}{8}] \) and \( g \in L^2(\mathbb{R}) \), if
\[
\langle g, A_\omega \rangle = \langle g, \phi_\omega' \rangle = 0
\]
then
\[
\|\rho^2 g\| \leq C\omega_0^{-2} \|\rho(X^2_\alpha M_- S^2 g)\|.
\]

**Proof.** Let \( g \) be as in the statement of the lemma and let \( h = X^2_\alpha M_- S^2 g \). We have
\[
M_- S^2 g = h - 2\alpha h'' + \alpha^2 h'''
\]
and so using the notation \( J_- \) from (2.4)
\[
\partial_x^2 \left( \frac{g}{\phi_\omega} \right) = \frac{1}{\phi_\omega} \left( J_- [h] - 2\alpha J_- [h''] + \alpha^2 J_- [h'''] \right).
\]
By integration, we obtain
\[
g = a_1(\omega^\frac{1}{2} x)\phi_\omega + b_1 \phi_\omega + \phi_\omega \int_0^x \int_0^y \frac{1}{\phi_\omega} J_- [h]
\]
\[
- 2\alpha \phi_\omega \int_0^x \int_0^y \frac{1}{\phi_\omega} J_- [h''] + \alpha^2 \phi_\omega \int_0^x \int_0^y \frac{1}{\phi_\omega} J_- [h'''].
\]
First, we note that
\[ |J_-[h]| \leq C\omega^{-\frac{3}{4}}\rho^{-\frac{3}{4}} \int \rho^{\frac{3}{4}} |h| \leq C\omega^{-\frac{3}{4}}\rho^{-\frac{3}{4}} \|\rho h\|. \]

For \( x \geq 0 \),
\[ \phi_\omega \int_0^x \int_0^y \frac{1}{\phi_\omega} |J_-[h]| \leq C\omega^{-\frac{3}{4}}\rho^{-\frac{3}{4}} \int_0^x \int_0^y \rho^{-\frac{3}{4}} \leq C\omega^{-\frac{3}{4}}\rho^{-\frac{3}{4}} \|\rho h\|. \]

For the term with \( h'' \), we use integration by parts and \( H_1 H_2' - H_1'H_2 = 1 \),
\[ J_-[h''] = -h + H_1 \int_{-\infty}^x H_2'' h + H_2 \int_x^\infty H_1'' h \]

Thus,
\[ |J_-[h'']| \leq C|h| + C\omega^{-\frac{3}{4}}\rho^{-\frac{3}{4}} \|\rho h\| \]

and for \( x \geq 0 \),
\[ \phi_\omega \int_0^x \int_0^y \frac{1}{\phi_\omega} |J_-[h'']| \leq C\omega^{-\frac{3}{4}}\rho^{-\frac{3}{4}} \|\rho h\|. \]

We continue with the term involving \( h''' \), integrating by parts and using the relation \( H_1'' H_2 - H_1'H_2'' = 0 \),
\[ J_-[h'''] = h'' + (H_1'H_2'' - H_1''H_2')h + H_1 \int_{-\infty}^x H_2''' h + H_2 \int_x^\infty H_1''' h. \]

For the last three terms in the right hand side, we proceed as before. For the first term, we further compute by integration by parts
\[ \phi_\omega \int_0^x \int_0^y \frac{1}{\phi_\omega} h''' = a_2(\omega \frac{1}{2} x) \phi_\omega + b_2 \phi_\omega + h + 2\phi_\omega \int_0^x \frac{h \phi_\omega'}{\phi_\omega^3} \]

where we have used \( \phi_\omega'' \phi_\omega - 2(\phi_\omega')^2 = -\omega \phi_\omega^2 + \frac{1}{3} \phi_\omega^6 \) (from (9)) and we proceed as before. We obtain, for \( a = a_1 + a_2 \) and \( b = b_1 + b_2 \), for \( x \geq 0 \)
\[ |g - a(\omega \frac{1}{2} x) \phi_\omega - b\phi_\omega| \leq C|h| + C\omega^{-\frac{3}{4}}\rho^{-\frac{3}{4}} \|\rho h\|. \]

This estimate is also true for \( x \leq 0 \) with the same constants \( a \) and \( b \). Using the orthogonality relations \( (g, \Lambda_\omega) = (g, \phi_\omega') = 0 \) to estimate \( a \) and \( b \), we complete the proof as the one of Lemma 11. \( \square \)
3.7. End of the proof of Theorem 1 Using first Proposition 4 then Proposition 3 and last Proposition 2 we obtain, for all $T > 0$,
\[
\omega_0 \int_0^T \|\rho^2 u\|^2 \, dt \leq C \omega_0^2 \int_0^T \|\rho v\|^2 \, dt
\]
\[
\leq C \varepsilon + \frac{C}{A} \int_0^T \left( \|\eta_A \partial_x u\|^2 + \omega_0^2 \|\eta_A u\|^2 + \omega_0 \|\rho^2 u\|^2 \right) \, dt
\]
\[
\leq C \varepsilon + \frac{C \omega_0}{A} \int_0^T \|\rho^2 u\|^2 \, dt.
\]
Thus, for $A$ large enough, independent of $\varepsilon$, but dependent on $\omega_0$,
\[
\omega_0 \int_0^T \|\rho^2 u\|^2 \, dt \leq C \varepsilon \omega_0^{-5}.
\]
Now, $A$ is fixed to such value. Using again Proposition 2 and passing to the limit $T \to \infty$, we obtain
\[
\int_0^\infty \left( \|\eta_A \partial_x u\|^2 + \omega_0^2 \|\eta_A u\|^2 + \omega_0 \|\rho^2 u\|^2 \right) \, dt \leq C \varepsilon \omega_0^{-5}.
\]
From the equation (14) of $u$, we compute
\[
\frac{d}{dt} \int |u|^2 \rho^4 = \int (u_1 (\partial_x u_2) - (\partial_x u_1) u_2) (\rho^4)' + \int (2 \phi_0^2 - 4 \phi_0^4) u_1 u_2 \rho^4
\]
\[+ \int (\theta_2 u_1 + m_1 u_1 - q_2 u_1 - \theta_1 u_2 - m_1 u_2 + q_1 u_2) \rho^4.
\]
Thus, using $|\rho'| \leq C \rho$, $\|u\|_{L^\infty} \leq C$ and (12), we obtain
\[
\frac{d}{dt} \int |u|^2 \rho^4 \leq C \int \rho^4 \left( |\partial_x u|^2 + |u|^2 \right).
\]
Since $\int_0^\infty \|\rho^2 u\|^2 \, dt < \infty$, there exists a sequence $t_n \to +\infty$ such that
\[
\lim_{n \to +\infty} \|\rho^2 u(t_n)\| = 0.
\]
Let $t \geq 0$ and $n$ be such that $t_n > t$. Integrating (23) on $(t, t_n)$, we obtain
\[
\|\rho^2 u(t)\|^2 \leq \|\rho^2 u(t_n)\|^2 + C \int_0^{t_n} \left( \|\rho^2 \partial_x u\|^2 + \|\rho^2 u\|^2 \right) \, dt'.
\]
Passing to the limit $n \to +\infty$,
\[
\|\rho^2 u(t)\|^2 \leq C \int_t^{\infty} \left( \|\rho^2 \partial_x u\|^2 + \|\rho^2 u\|^2 \right) \, dt'.
\]
Since $\int_0^\infty \left( \|\rho^2 \partial_x u\|^2 + \|\rho^2 u\|^2 \right) \, dt \leq \int_0^\infty \left( \|\eta_A \partial_x u\|^2 + \|\eta_A u\|^2 \right) \, dt < \infty$, we have
\[
\lim_{t \to +\infty} \int_t^{\infty} \left( \|\rho^2 \partial_x u\|^2 + \|\rho^2 u\|^2 \right) \, dt' = 0
\]
and thus
\[
\lim_{t \to +\infty} \|\rho^2 u(t)\| = 0.
\]
Lemma 13. For any \( x, y \in \mathbb{R} \), write
\[
\rho^2(x)|u(t, x)|^2 = \rho^2(y)|u(t, y)|^2 + \int_y^x \left[ 2\Re \{ \bar{u}(t) \bar{u}_x u(t) \} \rho^2 + |u(t)|^2 (\rho^2)’ \right] dt
\]
so that by the Cauchy-Schwarz inequality,
\[
\rho^2(x)|u(t, x)|^2 \leq \rho^2(y)|u(t, y)|^2 + C\|u(t)\|_{H^1(\mathbb{R})} \| \rho^2 u(t) \|
\]
Integrating for \( y \in [0, 1] \) and then using (13), we obtain
\[
\rho^2(x)|u(t, x)|^2 \leq C\|u(t)\|_{H^1(\mathbb{R})} \| \rho^2 u(t) \| \leq C\varepsilon \| \rho^2 u(t) \|.
\]
Thus,
\[
\lim_{t \to +\infty} \sup_{x \in \mathbb{R}} \{ \rho(x)|u(t, x)| \} = 0.
\]
By (12), we have \( |\beta| + |\omega| \leq C\| \rho^2 u \|^2 \). From \( \int_0^\infty \| \rho^2 u \|^2 dt < \infty \), it follows that both \( \beta(t) \) and \( \omega(t) \) have finite limits as \( t \to +\infty \), denoted respectively by \( \beta_+ \) and \( \omega_+ \). By (13), we infer that \( |\beta_+| + |\omega_+ - \omega_0| \leq C\varepsilon \). Last, by (10) and the triangle inequality, we have
\[
|\psi(t, x + \sigma(t)) - e^{i\gamma(t)} e^{i\beta_+ x} \phi_{\omega_+}(x)| \leq |e^{i\beta(t)x} \phi_{\omega(t)}(x) - e^{i\beta_+ x} \phi_{\omega_+}(x)| + |u(t, x)|.
\]
The elementary observation
\[
\lim_{t \to +\infty} \sup_{x \in \mathbb{R}} \left| e^{i\beta(t)x} \phi_{\omega(t)}(x) - e^{i\beta_+ x} \phi_{\omega_+}(x) \right| = 0
\]
then completes the proof of Theorem 1.

Appendix A.

Lemma 13. There exists \( B_0 \geq 1 \) such that for any \( B \geq B_0 \), the following is true on \( \mathbb{R} \),
\[
0 \leq y \sinh(y) e^{\frac{1}{3t} |y|} \leq \frac{1}{3} \left( 1 + \frac{1}{\sqrt{3}} \cosh(y) \right)^3,
\]
(24)
\[
|y| e^{\frac{1}{3t} |y|} \leq \frac{2}{3} |\sinh(y)| \left( 1 + \frac{1}{\sqrt{3}} \cosh(y) \right).
\]
(25)

Proof. First, we check the following inequality
\[
y \sinh(y) < \frac{1}{3} \left( 1 + \frac{1}{\sqrt{3}} \cosh(y) \right)^3.
\]

For \( |y| \leq \frac{5}{4} \), we have \( y \sinh(y) \leq \frac{5}{4} \cosh(y) \) and the inequality in this case is a consequence of the fact that for any \( a \geq 0 \), \( 15a < 4(1 + a/\sqrt{3})^3 \).

For \( |y| \geq \frac{5}{4} \), it is easy to check that \( |y| \leq \frac{5}{4} |\sinh(y)| \) and so \( y \sinh(y) \leq \frac{5}{4} \sinh^2(y) = \frac{5}{4} (\cosh^2(y) - 1) \). The inequality is then a consequence of the fact that for \( a \geq 0 \), \( 12(a^2 - 1) < 5(1 + a/\sqrt{3})^3 \).

Second, the inequality
\[
1.05 |y| \leq \frac{2}{3} |\sinh(y)| \left( 1 + \frac{1}{\sqrt{3}} \cosh(y) \right)
\]

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For \( |y| \geq \frac{5}{4} \), it is easy to check that \( |y| \leq \frac{5}{4} |\sinh(y)| \) and so \( y \sinh(y) \leq \frac{5}{4} \sinh^2(y) = \frac{5}{4} (\cosh^2(y) - 1) \). The inequality is then a consequence of the fact that for \( a \geq 0 \), \( 12(a^2 - 1) < 5(1 + a/\sqrt{3})^3 \).

Second, the inequality
\[
1.05 |y| \leq \frac{2}{3} |\sinh(y)| \left( 1 + \frac{1}{\sqrt{3}} \cosh(y) \right)
\]
is checked easily since $\cosh(y) \geq 1$, $|\sinh(y)| \geq |y|$ and $\frac{2}{3}(1 + \frac{1}{\sqrt{3}}) > 1.05$. The existence of $B_0$ such that (24) and (25) hold for $B = B_0$ then follows from standard arguments. Last, (24) and (25) are also true for any $B \geq B_0$. □

REFERENCES

[1] H. Berestycki and P.-L. Lions, Nonlinear scalar field equations. I. Existence of a ground state. *Arch. Rational Mech. Anal.* **82**, (1983) 313–345.

[2] M. Borghese, R. Jenkins, K. T.-R. McLaughlin, Long-time asymptotic behavior of the focusing nonlinear Schrödinger equation, *Ann. I. H. Poincaré* **35** (2018) 887–920.

[3] V. Buslaev and G. Perelman, Scattering for the nonlinear Schrödinger equations: states close to a soliton. *St. Petersburg Math. J.* **4** (1993), no. 6, 1111–1142.

[4] V. Buslaev and G. Perelman, On the stability of solitary waves for nonlinear Schrödinger equations. *Nonlinear evolution equations*, 75–98, Amer. Math. Soc. Transl. Ser. 2, 164, AMS, Providence, RI, 1995.

[5] R. Carles and C. Sparber, Orbital stability vs. scattering in the cubic-quintic Schrödinger equation, *Reviews in Mathematical Physics* **33** No. 3 (2021) 2150004.

[6] T. Cazenave, *Semilinear Schrödinger equations*. Courant Lecture Notes in Mathematics **10**, Providence, RI: American Mathematical Society (AMS); New York, NY: Courant Institute of Mathematical Sciences (2003).

[7] T. Cazenave and P.L. Lions, Orbital stability of standing waves for some nonlinear Schrödinger equations, *Comm. Math. Phys.* **85** (1982), 549–561.

[8] S.-M. Chang, S. Gustafson, K. Nakanishi and T.-P. Tsai, Spectra of linearized operators for NLS solitary waves, *SIAM J. Math. Anal.* **39** (2007/08), 1070–1111.

[9] G. Chen and F. Pusateri, The 1d nonlinear Schrödinger equation with a weighted $L^1$ potential. To appear in *Analysis & PDE*.

[10] G. Chen, Long-time dynamics of small solutions to 1d cubic nonlinear Schrödinger equations with a trapping potential. Preprint [arXiv:2106.10106](https://arxiv.org/abs/2106.10106).

[11] M. Coles and S. Gustafson, A degenerate edge bifurcation in the 1D linearized nonlinear Schrödinger equation, *Discrete Contin. Dyn. Syst.* **36** (2016), 2991–3009.

[12] S. Cuccagna and M. Maeda, Coordinates at small energy and refined profiles for the Nonlinear Schrödinger Equation, *Ann. PDE* **7** (2021), no. 2, Paper No. 16, 34pp.

[13] S. Cuccagna and M. Maeda, A survey on asymptotic stability of ground states of nonlinear Schrödinger equations II, *Discrete Contin. Dyn. Syst., Series S* **14** (2021) 1693–1716.

[14] S. Cuccagna and M. Maeda, On selection of standing wave at small energy in the 1D Cubic Schrödinger Equation with a trapping potential. Preprint [arXiv:2109.08108](https://arxiv.org/abs/2109.08108).

[15] S. Cuccagna and D.E. Pelinovsky, The asymptotic stability of solitons in the cubic NLS equation on the line, *Applicable Analysis* **93** (2014), 791–822.

[16] P. Deift and X. Zhou, A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation, *Ann. Math.* (2) **137** (1993), 295–368.

[17] P. Deift and X. Zhou, Perturbation theory for infinite-dimensional integrable systems on the line. A case study, *Acta Math.* **188** (2002), 163–262.

[18] M. Grillakis, J. Shatah and W.A. Strauss, Stability Theory of solitary waves in the presence of symmetry, I, *J. Funct. Anal.* **74** (1987), 160–197.

[19] S. Gustafson, K. Nakanishi, and T.-P. Tsai, Asymptotic stability and completeness in the energy space for nonlinear Schrödinger equations with small solitary waves, *Int. Math. Res. Not.* (2004), no. 66, 3559–3584.

[20] I.D. Iliev and K.P. Kirchev, Stability and instability of solitary waves for one-dimensional singular Schrödinger equations, *Differential and Integral Equations* **6** (1993), 685–703.
[21] Y.S. Kivshar and B.A. Malomed, Dynamics of solitons in nearly integrable systems, Rev. Mod. Phys. **61**, 763 (1989).

[22] M. Kowalczyk, Y. Martel and C. Muñoz, Kink dynamics in the $\phi^4$ model: asymptotic stability for odd perturbations in the energy space. J. Amer. Math. Soc. **30** (2017), 769–798.

[23] M. Kowalczyk, Y. Martel and C. Muñoz, On asymptotic stability of nonlinear waves. Séminaire Laurent Schwartz – EDP et applications (2016-2017), Exp. No. 18, 27pp.

[24] M. Kowalczyk, Y. Martel and C. Muñoz, Soliton dynamics for the 1D NLKG equation with symmetry and in the absence of internal modes, J. Eur. Math. Soc. (2021).

[25] M. Kowalczyk, Y. Martel, C. Muñoz and H. Van Den Bosch, A sufficient condition for asymptotic stability of kinks in general (1+1)-scalar field models, Ann. PDE **7** (2021), No. 1, Paper No. 10, 98 pp.

[26] J. Krieger and W. Schlag, Stable manifolds for all monic supercritical focusing nonlinear Schrödinger equations in one dimension, J. Amer. Math. Soc. **19** (2006), 815–920.

[27] J. Krieger, K. Nakanishi and W. Schlag, Global dynamics above the ground state energy for the one-dimensional NLKG equation, Math. Z. **272** (2012), 297–316.

[28] Y. Martel, Linear problems related to asymptotic stability of solitons of the generalized KdV equations, SIAM J. Math. Anal. **38** (2006), 759-781.

[29] Y. Martel and F. Merle, Liouville theorem for the critical generalized Korteweg-de Vries equation, J. Math. Pures Appl., **79** (2000), 339–425.

[30] Y. Martel and F. Merle, Asymptotic stability of solitons of the subcritical gKdV equations revisited, Nonlinearity **18** (2005), 55-80.

[31] Y. Martel, F. Merle and P. Raphaël, Blow-up for the critical generalized Korteweg-de Vries equation I: dynamics near the soliton, Acta Math., **212** (2014), 59-140.

[32] F. Merle and P. Raphaël, The blow-up dynamic and upper bound on the blow-up rate for critical nonlinear Schrödinger equation, Ann. of Math. (2), **161** (2005), 157–222.

[33] T. Mizumachi, Asymptotic stability of small solitary waves to 1D nonlinear Schrödinger equations with potential, J. Math. Kyoto Univ. **48** (2008), 471–497.

[34] I.P. Naumkin, Sharp asymptotic behavior of solution for cubic nonlinear Schrödinger equations with a potential, J. Math. Phys. **57** (2016), 05501.

[35] T.V. Nguyen, Existence of multi-solitary waves with logarithmic relative distances for the NLS equation, C. R. Acad. Sci. Paris, Ser. I **357** (2019) 13–58.

[36] M. Ohta, Stability and instability of standing waves for one-dimensional nonlinear Schrödinger equations with double power nonlinearity, Kodai Math. J., **18** (1995), 68–74.

[37] E. Olmedilla, Multiple pole solutions of the nonlinear Schrödinger equation, Phys. D **25** (1987), 330–346.

[38] D. E. Pelinovsky, Y. S. Kivshar and V. V. Afanasjev, Internal modes of envelope solitons, Phys. D **116** (1998), 121–142.

[39] M. Reed and B. Simon, Analysis of Operators IV. Methods of Modern Mathematical Physics. Academic Press, 1978.

[40] W. Schlag, Dispersive estimates for Schrödinger operators: A survey. Mathematical aspects of nonlinear dispersive equations, 255–285, Ann. of Math. Stud., **163**, Princeton Univ. Press, Princeton, NJ, 2007.

[41] J. Soneson and A. Peleg, Effect of quintic nonlinearity on soliton collisions in optical fibers, Physica D **195** (2004) 123–140.

[42] M.I. Weinstein, Modulational stability of ground states of nonlinear Schrödinger equations, SIAM J. Math. Anal. **16** (1985), no. 3, 472–491.

[43] M.I. Weinstein, Lyapunov stability of ground states of nonlinear dispersive evolution equations, Comm. Pure Appl. Math. **29** (1986), 51–68.

[44] J. Yang, Nonlinear Waves in Integrable and Nonintegrable systems. Mathematical Modeling and Computation **16**, Philadelphia, PA: SIAM, 2010.
[45] T. Zakharov and A.B. Shabat. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media, *Sov. Phys. JETP* 34 (1972), 62–69.

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