Isomorphism between automorphism groups of finitely generated groups

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Abstract. Let \( G \) be a finitely generated group and let \( C^* \) denote the group of all central automorphisms of \( G \) fixing the center of \( G \) elementwise. Azhdari and Malayeri [J. Algebra Appl., 6(2011), 1283-1290] gave necessary and sufficient conditions on \( G \) such that \( C^* \cong \text{Inn}(G) \). We prove a technical lemma and, as a consequence, obtain a short and easy proof of this result of Azhdari and Malayeri. Subsequently, we also obtain short proofs of some other existing and some new related results.

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1 Introduction. Let \( G \) be a finitely generated group and let \( \text{Inn}(G) \) denote the inner automorphism group of \( G \). For normal subgroups \( X \) and \( Y \) of \( G \), let \( \text{Aut}^X(G) \) and \( \text{Aut}_Y(G) \) denote the subgroups of \( \text{Aut}(G) \) centralizing \( G/X \) and \( Y \) respectively. We denote the intersection \( \text{Aut}^X(G) \cap \text{Aut}_Y(G) \) by \( \text{Aut}^{X,Y}(G) \). Let \( C^* \), in particular, denote the group \( \text{Aut}^{Z(G),Z(G)}(G) \), where \( Z(G) \) is the center of \( G \). For a finite group \( G \), let \( G_p \) and \( \pi(G) \) respectively denote the Sylow \( p \)-subgroup and the set of prime divisors of \( G \). For a finite \( p \)-group \( G \), Attar [2, Main Theorem] proved that \( C^* = \text{Inn}(G) \) if and only if either \( G \) is abelian or \( G \) is nilpotent of class 2 and \( Z(G) \) is cyclic. Azhdari and Malayeri [4, Theorem 0.1] (see also [5, Theorem 2.3] for correct version) generalized this result of Attar and proved that if \( G \) is a finitely generated nilpotent group of class 2, then \( C^* \cong \text{Inn}(G) \) if and only if \( Z(G) \) is infinite cyclic or \( Z(G) \cong C_m \times H \times Z^r \), where \( C_m \cong \prod_{p \in \pi(G/Z(G))} Z(G)_p \), \( H \cong \prod_{p \not\in \pi(G/Z(G))} Z(G)_p \), \( r \geq 0 \) is the torsion-free rank of \( Z(G) \) and \( G/Z(G) \) is of finite exponent dividing \( m \). We prove a technical lemma, Lemma 2.1, and as a consequence give a short and easy proof of this main theorem of Azhdari and Malayeri. We also obtain short and alternate proofs of Corollary 2.1 of [5], and Proposition 1.11 and Theorem 2.2(i) of [8]. Some other related results for finitely generated and finite \( p \)-groups are also obtained.

By \( C_p \) we denote a cyclic group of order \( p \) and by \( X^n \) we denote the direct product of \( n \)-copies of a group \( X \). By \( \text{Hom}(G,A) \) we denote the group of all homomorphisms of \( G \) into an abelian group \( A \). The rank of \( G \) is the smallest cardinality of a generating set of \( G \). The torsion rank and torsion-free rank of \( G \) are respectively denoted as \( d(G) \) and \( \rho(G) \). By \( \exp(G) \) we denote the exponent of torsion part of \( G \). All other unexplained

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notations, if any, are standard. The following well known results will be used very frequently without further referring.

**Lemma 1.1.** Let \( U, V \) and \( W \) be abelian groups. Then

(i) if \( U \) is torsion-free of rank \( m \), then \( \text{Hom}(U, V) \simeq V^m \), and

(ii) if \( U \) is torsion and \( V \) is torsion-free, then \( \text{Hom}(U, V) = 1 \).

## 2 Main results

Let \( G \) be a finitely generated group and \( M \) be an abelian subgroup of \( G \) with \( \pi(M) = \{q_1, q_2, \ldots, q_c\} \). Let \( L \) and \( N \) be normal subgroups of \( G \) such that \( G' \leq N \leq L \) and \( \pi(G/L) = \{p_1, p_2, \ldots, p_d\} \). Let \( X, Y, Z \) be respective torsion parts and \( a, b, c \) be respective torsion-free ranks of \( G/L, G/N \) and \( M \). Let \( X_{p_i} \simeq \prod_{j=1}^{n_i} C_{p_i}^{\alpha_{ij}} \), \( Y_{p_i} \simeq \prod_{j=1}^{l_i} C_{p_i}^{\beta_{ij}} \) and \( Z_{q_j} \simeq \prod_{j=1}^{m_j} C_{q_j}^{\gamma_{ij}} \), where for each \( i \), \( \alpha_{ij} \geq \alpha_{i(j+1)} \), \( \beta_{ij} \geq \beta_{i(j+1)} \) and \( \gamma_{ij} \geq \gamma_{i(j+1)} \) are positive integers, respectively denote the Sylow subgroups of \( X, Y \) and \( Z \). Then

\[
G/L \simeq X \times Z^a \simeq \prod_{i=1}^{d} X_{p_i} \times \prod_{i=1}^{d} C_{p_i}^{\alpha_{ij}} \times Z^a,
\]

\[
G/N \simeq Y \times Z^b \simeq \prod_{i=1}^{d} Y_{p_i} \times \prod_{i=1}^{d} C_{p_i}^{\beta_{ij}} \times Z^b,
\]

and

\[
M \simeq Z \times Z^c \simeq \prod_{i=1}^{c} Z_{q_i} \times \prod_{i=1}^{c} C_{q_i}^{\gamma_{ij}} \times Z^c.
\]

Since \( G/L \) is a quotient group of \( G/N \), it follows that \( a \leq b, l_i \leq n_i \) and \( \alpha_{ij} \leq \beta_{ij} \) for all \( i \), \( 1 \leq i \leq d \) and for all \( j, 1 \leq j \leq l_i \). We begin with the following lemma.

**Lemma 2.1.** Let \( G, L, M \) and \( N \) be as above. Then \( \text{Hom}(G/N, M) \simeq G/L \) if and only if one of the following conditions hold:

(i) \( G \) is torsion-free, \( M \) is infinite cyclic and both \( G/L \) and \( G/N \) are torsion-free of same rank.

(ii) \( G \) is torsion, \( M \simeq \prod_{i=1}^{d} p_i^{\gamma_{i1}} \times \prod_{i=d+1}^{c} Z_{q_i} \), \( l_i = n_i \) and either \( \alpha_{ij} = \beta_{ij} \leq \gamma_{i1} \) for each \( j \) or \( \alpha_{ij} = \gamma_{i1} \) for \( 1 \leq j \leq r_i \) and \( \alpha_{ij} = \beta_{ij} \) for \( r_i + 1 \leq j \leq l_i \), where \( r_i \) is the largest positive integer between \( 1 \) and \( l_i \) such that \( \beta_{ir_i} > \gamma_{i1} \) for each fixed \( i, 1 \leq i \leq d \).

(iii) \( G \) is a mixed group, \( M \simeq \prod_{i=1}^{d} p_i^{\gamma_{i1}} \times \prod_{i=d+1}^{c} Z_{q_i} \times Z^c \), both \( G/L \) and \( G/N \) are finite, \( l_i = n_i \) and either \( \alpha_{ij} = \beta_{ij} \leq \gamma_{i1} \) for each \( j \) or \( \alpha_{ij} = \gamma_{i1} \) for \( 1 \leq j \leq r_i \) and \( \alpha_{ij} = \beta_{ij} \) for \( r_i + 1 \leq j \leq l_i \), where \( r_i \) is the largest positive integer between \( 1 \) and \( l_i \) such that \( \beta_{ir_i} > \gamma_{i1} \) for each fixed \( i, 1 \leq i \leq d \).

**Proof.** It is easy to see that if any of the three conditions hold, then \( \text{Hom}(G/N, M) \simeq G/L \). Conversely suppose that \( \text{Hom}(G/N, M) \simeq G/L \). Then

\[
\text{Hom}(Y \times Z^b, Z \times Z^c) \simeq X \times Z^a.
\]

We prove only (i) and (ii), because (iii) can be proved using similar arguments. First assume that \( G \) is torsion-free. Then \( N \) is also torsion-free and therefore by (1) \( \text{Hom}(Y \times
$\mathbb{Z}^b, \mathbb{Z}^c \simeq X \times \mathbb{Z}^a$. Thus $X = 1$ and since $a \leq b$, $c = 1$ and $a = b$. It follows that $M$ is infinite cyclic and both $G/N$ and $G/L$ are torsion-free of same rank. Next assume that $G$ is torsion. Then $\text{Hom}(Y, Z) \simeq X$ by (1). Since $\pi(X) = \pi(Y)$ and $d(X) \leq d(Y)$, therefore $q_i = p_i$ and $m_i = 1$ for all $i, 1 \leq i \leq d$. Thus $M \simeq \prod_{i=1}^{d} C_{p_i^{\gamma_i} \times \prod_{i=d+1}^{\infty} C_{q_i^{\gamma_i}}}$. Also, observe that

$$\text{Hom}(Y, Z) \simeq \text{Hom}(\prod_{i=1}^{d} C_{p_i^{\gamma_i}} \times \prod_{i=d+1}^{\infty} C_{q_i^{\gamma_i}})$$

and $X \simeq \prod_{i=1}^{d} \prod_{j=1}^{l_i} C_{p_i^{\alpha_{ij}}}$, therefore $\text{Hom}(\prod_{j=1}^{l_i} C_{p_i^{\alpha_{ij}}}, C_{p_i^{\gamma_i}}) \simeq \prod_{j=1}^{l_i} C_{p_i^{\beta_{ij}}}$ for each $i, 1 \leq i \leq d$, and hence $l_i = n_i$. It thus follows that for each fixed $i, 1 \leq i \leq d$,

$$\text{Hom}(\prod_{j=1}^{l_i} C_{p_i^{\alpha_{ij}}}, C_{p_i^{\gamma_i}}) \simeq \prod_{j=1}^{l_i} C_{p_i^{\gamma_i}}.$$  (2)

Now, if $\exp(Y_{p_i}) \leq \exp(Z_{p_i})$, then $\beta_{ij} \leq \gamma_{i1}$ for each $j$ and $\text{Hom}(\prod_{j=1}^{l_i} C_{p_i^{\beta_{ij}}}, C_{p_i^{\gamma_i}}) \simeq \prod_{j=1}^{l_i} C_{p_i^{\beta_{ij}}}$. It therefore follows from (2) that $\alpha_{ij} = \beta_{ij}$ for each $j$. And, if $\exp(Y_{p_i}) > \exp(Z_{p_i})$, then there exists largest positive integer $r_i$ between 1 and $l_i$ such that $\beta_{ir_i} > \gamma_{i1}$ and $\beta_{ij} \leq \gamma_{i1}$ for each $j, r_i + 1 \leq j \leq l_i$. Therefore $\text{Hom}(\prod_{j=1}^{l_i} C_{p_i^{\beta_{ij}}}, C_{p_i^{\gamma_i}}) \simeq \prod_{j=1}^{l_i} C_{p_i^{\gamma_i}} \times \prod_{j=r_i+1}^{l_i} C_{p_i^{\beta_{ij}}}$. It then follows by (2) that $\alpha_{ij} = \gamma_{i1}$ for $1 \leq j \leq r_i$ and $\alpha_{ij} = \beta_{ij}$ for $r_i + 1 \leq j \leq l_i$. \hfill \qed

**Remark 2.2.** Observe that if $N = L$ and $\text{exp}(G/N) \mid \text{exp}(M)$, then $\text{exp}(Y_{p_i}) \leq \exp(Z_{p_i})$ for all $i$ and hence $\text{Hom}(G/L, M) \simeq G/L$ if and only if either $M$ is infinite cyclic or $M \simeq \prod_{i=1}^{d} Z_{q_i^{\gamma_i}} \times \prod_{i=d+1}^{\infty} Z_{q_i^{\gamma_i}}$, where $c \geq 0$ is the torsion-free rank of $M$.

The next lemma is a little modification of arguments of Alperin [1] Lemma 3] and Fournelle [17 Section 2].

**Lemma 2.3.** Let $G$ be any group and $Y$ be a central subgroup of $G$ contained in a normal subgroup $X$ of $G$. Then the group of all automorphisms of $G$ that induce the identity on both $X$ and $G/Y$ is isomorphic to $\text{Hom}(G/X, Y)$.

Observe that $C^* \simeq \text{Hom}(G/Z(G), Z(G))$ by Lemma 2.3. If $G$ is nilpotent of class 2, then $\exp(G') = \exp(G/Z(G))$. Now taking $L = M = N = Z(G)$ in Lemma 2.1, we get the following main result of Azhdari and Malayeri [14] Theorem 0.1] (see [15] Theorem 2.3) for correct version).

**Corollary 2.4.** Let $G$ be a finitely generated nilpotent group of class 2. Then $C^* \simeq \text{Im}(G)$ if and only if either $Z(G)$ is infinite cyclic or $Z(G) \simeq C_{\prod_{i=1}^{d} Z_{q_i^{\gamma_i}}} \times \prod_{i=d+1}^{\infty} Z_{q_i^{\gamma_i}}$, where $c$ is the torsion-free rank of $Z(G)$.  

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Corollary 2.5 ([5 Corollary 2.1]). Let $G$ be a finitely generated non-abelian group and let $M$ and $N$ be normal subgroups of $G$ such that $M \leq Z(G) \leq N$ and $G/Z(G)$ is finite. Then $\text{Aut}^M_G(N) = \text{Inn}(G)$ if and only if $G$ is a nilpotent group of class 2, $N = Z(G)$, $G' \leq M$ and $M \simeq C_{\prod_{i=1}^c p_i^{\gamma_i}} \times \prod_{i=d+1}^e \mathbb{Z}_{q_i} \times \mathbb{Z}'$, where $c \geq 0$ is the torsion-free rank of $M$.

Proof. First suppose that $\text{Aut}^M_G(N) = \text{Inn}(G)$. Observe that $\text{Aut}^M_G(N) \simeq \text{Hom}(G/N, M)$ by Lemma 2.3. It follows that $\text{Inn}(G)$ is abelian and therefore nilpotence class of $G$ is 2. For any $[a, b] \in G'$, $[a, b] = a^{-1} b(a) \in M$ and thus $G' \leq M$. Also, for any $n \in N$, $L_k(n) = n$ for all $x \in G$ and therefore $N = Z(G)$. Now since $\exp(G/Z(G)) = \exp(G')$ divides $\exp(M)$, the result follows from Lemma 2.1 by taking $L = Z(G)$. The converse follows easily.

In 1911, Burnside [6] Note B. p. 463 gave the notion of pointwise inner automorphism of a group $G$. An automorphism $\alpha$ of $G$ is called pointwise inner automorphism of $G$ if $x$ and $\alpha(x)$ are conjugate for each $x \in G$. Let $H$ be a characteristic subgroup of $G$. As defined in [3], an automorphism $\alpha$ of $G$ is called $H$-pointwise inner if for each element $x \in G$, there exists $h \in H$ such that $\alpha(x) = x^h = x[x, h]$. For convenience, we denote $\gamma_k(G)$-pointwise inner automorphism of $G$ by $\text{Aut}_{k - \text{pwi}}(G)$. As another application of Lemma 2.1, we get the following two results of Azhdari [3]. The second one generalizes Theorem 2.2(i) of [3].

Corollary 2.6 ([3 Prop. 1.11]). Let $G$ be a finitely generated nilpotent group of class $k + 1 \geq 2$. Then $\text{Hom}(G/\zeta_k(G), \gamma_{k+1}(G)) \simeq G/\zeta_k(G)$ if and only if $\gamma_{k+1}(G)$ is cyclic. In particular, if $\gamma_{k+1}(G) = [x, \gamma_k(G)]$ for all $x \in G \setminus C_G(\gamma_k(G))$ is cyclic, then $\text{Aut}_{k - \text{pwi}}(G)$ is isomorphic to a quotient group of $\text{Inn}(G)$.

Proof. It follows from [9] Cor. 2.6, Cor. 3.16, Cor. 3.17] that $\exp(G/\zeta_k(G)) = \exp(\gamma_{k+1}(G))$ and $G/\zeta_k(G)$ is finite if and only if $\gamma_{k+1}(G)$ is finite. The result now follows from Lemma 2.1 (see Remark 2.2) by taking $L = N = \zeta_k(G)$ and $M = \gamma_{k+1}(G)$. In particular, if $\gamma_{k+1}(G) = [x, \gamma_k(G)]$ for all $x \in G \setminus C_G(\gamma_k(G))$ is cyclic, then using the arguments as in [10] Prop. 3.1, we can prove that $\text{Aut}_{k - \text{pwi}}(G) \simeq \text{Hom}(G/\zeta_k(G), \gamma_{k+1}(G))$.

Corollary 2.7 (cf. [3 Theorem 2.2(i)]). Let $G$ be a finitely generated nilpotent group of class $k + 1 \geq 2$. Then $\text{Hom}(G/\zeta_k(G), \gamma_{k+1}(G)) \simeq \text{Inn}(G)$ if and only if $G$ is nilpotent of class 2 and $G'$ is cyclic. In particular, if $\gamma_{k+1}(G) = [x, \gamma_k(G)]$ for all $x \in G \setminus C_G(\gamma_k(G))$, then $\text{Aut}_{k - \text{pwi}}(G) \simeq \text{Inn}(G)$ if and only if $G$ is nilpotent of class 2 and $G'$ is cyclic.

Proof. Observe that if $\text{Hom}(G/\zeta_k(G), \gamma_{k+1}(G)) \simeq \text{Inn}(G)$, then $G/Z(G)$ is abelian, and therefore nilpotence class of $G$ is 2. It follows that $\zeta_k(G) = Z(G)$ and $\gamma_{k+1}(G) = G'$. The result now follows from above corollary by taking $k = 1$.

For $g \in G$ and $\alpha \in \text{Aut}(G)$, the element $[g, \alpha] = g^{-1} \alpha(g)$ is called the automcommutator of $g$ and $\alpha$. Inductively, define

$$[g, \alpha_1, \alpha_2, \ldots, \alpha_n] = [[g, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}], \alpha_n],$$

where $\alpha_i \in \text{Aut}(G)$. The absolute center $L(G)$ of $G$ is defined as

$$L(G) = \{g \in G \mid [g, \alpha] = 1, \text{ for all } \alpha \in \text{Aut}(G)\}.$$

Let $L_1(G) = L(G)$, and for $n \geq 2$, define $L_n(G)$ inductively as

$$L_n(G) = \{g \in G \mid [g, \alpha_1, \alpha_2, \ldots, \alpha_n] = 1 \text{ for all } \alpha_1, \alpha_2, \ldots, \alpha_n \in \text{Aut}(G)\}.$$
The automocommutator subgroup $G^*$ of $G$ is defined as

$$G^* = \langle g^{-1}\alpha(g) \mid g \in G, \alpha \in \text{Aut}(G) \rangle.$$ 

It is easy to see that $L_n(G) \leq Z_n(G)$ for all $n \geq 1$ and $G' \leq G^*$. An automorphism $\alpha$ of $G$ is called an automcentral automorphism if $g^{-1}\alpha(g) \in L(G)$ for all $g \in G$. The group of all automcentral automorphisms of $G$ is denoted by $\text{Var}(G)$. A group $G$ is called an autonilpotent group of class at most $n$ if $L_n(G) = G$ for some natural number $n$. Observe that if $G$ is an autonilpotent group of class 2, then $G^* \leq L(G)$. Nasrabad and Farimani proved that if $G$ is a finite autonilpotent group of class 2, then $\text{Var}(G) = \text{Inn}(G)$ if and only if $L(G) = Z(G)$ and $Z(G)$ is cyclic. Observe that $\text{Var}(G) \simeq \text{Hom}(G/L(G), L(G))$ by Lemma 2.3. As a final consequence of Lemma 2.1, we get the following result which generalizes the main result of Nasrabad and Farimani. The proof follows from Lemma 2.1 by taking $M = N = L(G)$ and $L = Z(G)$.

**Corollary 2.8.** Let $G$ be a finitely generated non-abelian group such that $G' \leq L(G)$ and $\pi(G/L(G)) = \pi(G/Z(G))$. Then $\text{Var}(G) \simeq \text{Inn}(G)$ if and only if one of the following holds

(i) $G$ is torsion-free, $L(G)$ is infinite cyclic and $\rho(G/L(G)) = \rho(G/Z(G))$;

(ii) $G$ is torsion, $L(G) \simeq C_{\prod_{i=1}^d p_i^{\gamma_i}} \times \prod_{i=d+1}^e Z_{q_i}$, and either $L(G) = Z(G)$ or $l_i = n_i$, $\alpha_{ij} = \gamma_{i1}$ for $1 \leq j \leq r_i$, and $\alpha_{ij} = \beta_{ij}$ for $r_i + 1 \leq j \leq l_i$, where $r_i$ is the largest positive integer between $1$ and $l_i$ such that $\beta_{ir_i} > \gamma_{i1}$ for each fixed $i, 1 \leq i \leq d$.

(iii) $G$ is a mixed group, both $G/L(G)$ and $G/Z(G)$ are finite, $L(G) \simeq C_{\prod_{i=1}^d p_i^{\gamma_i}} \times \prod_{i=d+1}^e Z_{q_i} \times Z^r$ and either $L(G) = Z(G)$ or $l_i = n_i$, $\alpha_{ij} = \gamma_{i1}$ for $1 \leq j \leq r_i$, and $\alpha_{ij} = \beta_{ij}$ for $r_i + 1 \leq j \leq l_i$, where $r_i$ is the largest positive integer between $1$ and $l_i$ such that $\beta_{ir_i} > \gamma_{i1}$ for each fixed $i, 1 \leq i \leq d$.

Let $G$ be a finite $p$-group such that $G' \leq L(G)$. Let $G/Z(G) \simeq \prod_{i=1}^k C_{p^{\alpha_i}}$, $G/L(G) \simeq \prod_{i=1}^l C_{p^{\beta_i}}$, and $L(G) \simeq \prod_{i=1}^n C_{p^{\gamma_i}}$, where $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_r$, $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_s$ and $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_t$ are positive integers. Since $G/Z(G)$ is a quotient group of $G/L(G)$, $r \leq s$ and $\alpha_i \leq \beta_i$ for $1 \leq i \leq r$.

**Corollary 2.9.** Let $G$ be a finite non-abelian $p$-group. Then $\text{Var}(G) = \text{Inn}(G)$ if and only if $G' \leq L(G)$, $L(G)$ is cyclic and either $L(G) = Z(G)$ or $d(G/L(G)) = d(G/Z(G))$, $\alpha_i = \gamma_1$ for $1 \leq i \leq k$ and $\alpha_i = \beta_i$ for $k + 1 \leq i \leq r$, where $k$ is the largest positive integer such that $\beta_k > \gamma_1$.

**Proof.** Observe that if $\text{Var}(G) = \text{Inn}(G)$, then for any $[a, b] \in G'$, $[a, b] = a^{-1}b(a) \in L(G)$ and thus $G' \leq L(G)$. The result now follows from Cor. 2.8. \qed

**Corollary 2.10 (Theorem 3.2).** Let $G$ be a non-abelian autonilpotent finite $p$-group of class 2. Then $\text{Var}(G) = \text{Inn}(G)$ if and only if $L(G) = Z(G)$ and $L(G)$ is cyclic.

**Proof.** Suppose that $\text{Var}(G) = \text{Inn}(G)$. Observe that if $g^{-1}\alpha(g) \in G^*$, then $\alpha(g) = gl$ for some $l \in L(G)$ and hence $(g^{-1}\alpha(g))^m = g^{-m}\alpha(g)^m$ for all $m \geq 1$. Let $\exp(G/L(G)) = d$ and $\exp(G^*) = k$. Then $1 = (g^{-1}\alpha(g))^k = g^{-k}\alpha(g)^k$ implies that $g^k \in L(G)$ and hence $d \leq k$. Conversely, if $g^k \in L(G)$, then $g^k \in L(G)$ and thus $1 = g^{-d}\alpha(g)^d = (g^{-1}\alpha(g))^d$. It follows that $k \leq d$ and hence $\exp(G/L(G)) = \exp(G^*)$. Since $G^* \leq L(G)$, $\exp(G/L(G)) \mid \exp(L(G))$. Therefore $\text{Var}(G) \simeq \text{Hom}(G/L(G), L(G)) \simeq G/L(G)$, because $L(G)$ is cyclic by Corollary 2.9, and hence $L(G) = Z(G)$. \qed
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