GLOBAL WELL-POSEDNESS OF THE MHD EQUATIONS VIA THE COMPARISON PRINCIPLE

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ABSTRACT. In this paper, we prove the global well-posedness of the incompressible MHD equations near a homogeneous equilibrium in the domain $\mathbb{R}^k \times \mathbb{T}^{d-k}, d \geq 2, k \geq 1$ by using the comparison principle and constructing the comparison function.

1. Introduction

In this paper, we consider the incompressible magneto-hydrodynamics (MHD) equations in $[0, T) \times \Omega, \Omega \subseteq \mathbb{R}^d, d \geq 2$:

\[
\begin{aligned}
\partial_t v - \nu \Delta v + v \cdot \nabla v + \nabla p &= b \cdot \nabla b, \\
\partial_t b - \mu \Delta b + v \cdot \nabla b &= b \cdot \nabla v, \\
\text{div } v &= \text{div } b = 0,
\end{aligned}
\]

where $v$ denotes the velocity field and $b$ denotes the magnetic field, and $\nu \geq 0$ is the viscosity coefficient, $\mu \geq 0$ is the resistivity coefficient. If $\nu = \mu = 0$, (1.1) is the so called ideal MHD equations; If $\nu > 0$ and $\mu = 0$, (1.1) is reduced to the Navier-Stokes equations.

It is well-known that the 2-D MHD equations with full viscosities (i.e., $\nu > 0$ and $\mu > 0$) have global smooth solution. In the absence of resistivity (i.e., $\mu = 0$), the global existence of weak solution and strong solution of the MHD equations is still an open question. Recently, Cao and Wu [5] proved the global regularity of the 2-D MHD equations with partial dissipation and magnetic diffusion. Motivated by numerical observation [3]: the energy of the MHD equations is dissipated at a rate independent of the ohmic resistivity, there are a lot of works [1, 7, 8, 9, 11] devoted to the global well-posedness of the MHD equations without resistivity in a homogeneous magnetic field $B_0$.

In high temperature plasmas, both $\nu$ and $\mu$ are usually very small. Thus, it is very interesting to investigate the long-time dynamics of the MHD equations in such case. In this paper, we consider the case of $\mu = \nu$. In terms of the Elsässer variables

\[ Z_+ = v + b, \quad Z_- = v - b, \]

the MHD equations (1.1) can be written as

\[
\begin{aligned}
\partial_t Z_+ + Z_- \cdot \nabla Z_+ &= \mu \Delta Z_+ - \nabla p, \\
\partial_t Z_- + Z_+ \cdot \nabla Z_- &= \mu \Delta Z_- - \nabla p, \\
\text{div } Z_+ &= \text{div } Z_- = 0.
\end{aligned}
\]

We introduce the fluctuations

\[ z_+ = Z_+ - B_0, \quad z_- = Z_- + B_0. \]
Then the system (1.2) can be reformulated as

\[
\begin{align*}
\partial_t z_+ + Z_- \cdot \nabla z_+ &= \mu \Delta z_+ - \nabla p, \\
\partial_t z_- + Z_+ \cdot \nabla z_- &= \mu \Delta z_- - \nabla p, \\
\text{div} z_+ = \text{div} z_- &= 0.
\end{align*}
\]

(1.3)

Bardos-Sulem-Sulem [2] proved the global well-posedness of (1.3) for \( \Omega = \mathbb{R}^d \) and \( \mu = 0 \) when the initial data is small in a weighed Hölder space. Recently, He-Xu-Yu [6] proved the global well-posedness of (1.3) for any \( \mu \geq 0 \) and \( \Omega = \mathbb{R}^3 \) by using some ideas from nonlinear stability of Minkowski space-time in general relativity. Cai and Lei [4] proved similar result for \( \Omega = \mathbb{R}^d, d = 2, 3 \) by using Alinhac’s ghost weight method. Wei and Zhang [10] dealt with more physical case, which allows \( \nu \neq \mu \) and \( \Omega \) to be a strip. For all these results, the key mechanism leading to the global well-posedness is that the nonlinear terms \( z_- \cdot \nabla z_+ \) and \( z_+ \cdot \nabla z_- \) are essentially neglected after a long time, because \( z_\pm \) are transported along the opposite direction.

The goal of this paper is twofold: (1) include the domain \( \mathbb{R}^k \times \mathbb{T}^{d-k}, k \geq 1 \). Previous results required \( k \geq 2 \) at least; (2) develop an elementary and much simpler method via the comparison principle.

Without loss of generality, we take the background magnetic field \( B_0 = (1, 0, \cdots, 0) \). We will identify a function in \( \mathbb{R}^k \times \mathbb{T}^{d-k} \) with a function in \( \mathbb{R}^d \) periodic in \( d-k \) directions \( e_1, \cdots, e_{d-k} \), where \( e_1, \cdots, e_{d-k}, B_0 \) are orthogonal.

We introduce

\[
\rho_\pm(t, X)^2 = \sum_{k=0}^{N} \int_{\mathbb{R}^d} |\nabla^k z_\pm|^2(t, Y)\theta(|X-Y|)dY,
\]

where \( \theta(r) \) is a smooth cut-off function so that

\[
\theta(r) = \begin{cases} 
1 & \text{for } |r| \leq 1, \\
0 & \text{for } |r| \geq 2, \\
|\theta'(r)|^2 & \leq C \theta(r).
\end{cases}
\]

Let

\[
J_\pm(0) = \int_{\mathbb{R}^{d-1}} \sup_{y \in \mathbb{R}^{d-1}} \rho_\pm(0, x, y)dx.
\]

Our result is stated as follows.

**Theorem 1.1.** Let \( d \geq 2, 1 \leq k \leq d \) and \( z_\pm(0) \in H^N(\mathbb{R}^k \times \mathbb{T}^{d-k}) \) for some integer \( N > \frac{d}{2} + 1 \). There exists \( \epsilon_1 > 0 \) so that if \( J_\pm(0) \leq \epsilon_1 \), then there exists a global unique solution \( (z_+, z_-) \in C([0, +\infty), H^N(\mathbb{R}^k \times \mathbb{T}^{d-k})) \) to the MHD equations (1.3) satisfying

\[
||z_\pm(t)||_{H^N(\mathbb{R}^k \times \mathbb{T}^{d-k})} \leq C ||z_\pm(0)||_{H^N(\mathbb{R}^k \times \mathbb{T}^{d-k})} \text{ for any } t \in [0, +\infty).
\]

Let us give some remarks on our result.

1. We only require the initial data to decay at infinity in \( B_0 \) direction, which is the key for the global well-posedness in \( \mathbb{R} \times \mathbb{T}^{d-1} \).
2. It is easy to check that \( J_\pm(0) \leq C \epsilon \) for the initial data considered in [4] and [6]. Indeed, for the data in [4], we have \( \rho_\pm(0, X) \leq C \epsilon (1 + |X|)^{-\delta} \). For the data in [6], we have \( \rho_\pm(0, X) \leq C \epsilon (R^2 + |X|^2)^{-\frac{1}{4}} (\ln(R^2 + |X|^2))^{-2} \). Here \( \delta > 1, R \geq 100 \) and \( \epsilon > 0 \) is small.
3. At a first glance, there seems no difference between \( \mathbb{R}^d \) and \( \mathbb{R}^k \times \mathbb{T}^{d-k} \). From the proofs in [4] and [6], it seems that the case of \( d = 3 \) is easier than the case of \( d = 2 \). However, for a solution \( (z_+, z_-) \in C([0, +\infty), H^N(\mathbb{R}^2)) \), \( (z_+(t, x), 0, z_-(t, x), 0) \) is also a solution in \( C([0, +\infty), H^N(\mathbb{R}^2 \times \mathbb{T})) \). Thus, the case of \( \mathbb{R}^2 \times \mathbb{T} \) is not easier than the case of \( \mathbb{R}^2 \). In this sense, the case of \( \mathbb{R} \times \mathbb{T}^{d-1} \) may be harder.
Throughout this paper, we denote by $C$ a constant independent of $t, \mu$, which may be different from line to line.

2. Local energy inequality

In this section, we derive the following local energy inequalities

\begin{equation}
(2.1) \quad \partial_t \rho_{\pm} + B_0 \cdot \nabla \rho_{\pm} - \mu \Delta \rho_{\pm} \leq CF,
\end{equation}

where $F(t, X)$ is given by

\begin{equation}
(2.2) F(t, X) = \sum_{k+j \leq N+1, 0 \leq k, j \leq N} \|\nabla^k z_+ \| \|\nabla^j z_- \| (t) \| L^2(B(X, 2)) + \sum_{k=0}^N \|\nabla^{k+1} p(t)\| L^2(B(X, 2)).
\end{equation}

Here $B(X, r)$ denotes a ball in $\mathbb{R}^d$ with the center at $X$ and radius $r$.

We only prove (2.1) for $\rho_+$, the case of $\rho_-$ is similar. For any multi-index $a \in \mathbb{N}^d$, using the same notations as in [4], one can easily deduce from (1.3) that

\begin{equation}
\begin{cases}
\partial_t \nabla^a z_+ - \mu \Delta \nabla^a z_+ - B_0 \cdot \nabla \nabla^a z_+ + \sum_{b+c=a} C_a^b (\nabla^b z_- \cdot \nabla^c z_+) + \nabla^a p = 0, \\
\partial_t \nabla^a z_- - \mu \Delta \nabla^a z_- + B_0 \cdot \nabla \nabla^a z_- + \sum_{b+c=a} C_a^b (\nabla^b z_+ \cdot \nabla^c z_-) + \nabla^a p = 0, \\
\text{div} \nabla^a z_+ = \text{div} \nabla^a z_- = 0.
\end{cases}
\end{equation}

Taking inner product of the first equation with $\nabla^a z_+$, we obtain

\begin{equation}
\begin{align*}
&\partial_t \|\nabla^a z_+\|^2 - \mu \Delta \|\nabla^a z_+\|^2 + 2\mu \|\nabla \nabla^a z_+\|^2 - B_0 \cdot \nabla \|\nabla^a z_+\|^2 \\
&+ 2 \sum_{b+c=a} C_a^b (\nabla^b z_- \cdot \nabla^c z_+ \nabla^a z_+) + 2\nabla^a z_+ \cdot \nabla^a p = 0,
\end{align*}
\end{equation}

from which, we deduce that

\begin{equation}
(\partial_t \rho_{\pm}^2 - \mu \Delta \rho_{\pm}^2 - B_0 \cdot \nabla \rho_{\pm}^2)(t, X) + 2\mu \int_{|a| \leq N} \sum |\nabla \nabla^a z_+|^2(t, Y) \theta(|X - Y|) dY \\
+ 2 \int_{|a| \leq N} \sum \sum \sum C_a^b (\nabla^b z_- \cdot \nabla^c z_+ \nabla^a z_+) (t, Y) \theta(|X - Y|) dY \\
+ 2 \int_{|a| \leq N} \sum \nabla^a z_+ \cdot \nabla^a p(t, Y) \theta(|X - Y|) dY = 0.
\end{equation}

Let $\rho_{\pm}^{(\epsilon)}(\rho_{\pm}^2 + \epsilon)^{\frac{1}{2}}$ for $\epsilon > 0$. We have

\begin{align*}
2(\rho_{\pm}^{(\epsilon)} \nabla \rho_{\pm}^{(\epsilon)})(t, X) &= (\nabla \rho_{\pm}^2)(t, X) = 2 \int_{|a| \leq N} (\nabla \nabla^a z_+ \cdot \nabla^a z_+)^2(t, Y) \theta(|X - Y|) dY \\
&\leq 2 \left( \int_{|a| \leq N} \sum |\nabla \nabla^a z_+|^2(t, Y) \theta(|X - Y|) dY \right)^{\frac{1}{2}} \\
&= 2 \left( \int_{|a| \leq N} \sum |\nabla \nabla^a z_+|^2(t, Y) \theta(|X - Y|) dY \right)^{\frac{1}{2}} \rho_{\pm}(t, X),
\end{align*}
which implies that
\[ \int_{\mathbb{R}^d} \sum_{|a| \leq N} |\nabla \nabla^a z_+|^2(t, Y) \theta(|X - Y|) dY \geq |\nabla \rho_+^{(e)}(t, X)|^2. \]

If $|a| \leq N$, $b + c = a$, $c < a$, then $|b|, |c| + 1 \leq N$ and

\[
- \int_{\mathbb{R}^d} (\nabla^b z_+ \cdot \nabla \nabla^c z_+ \nabla^a z_+)(t, Y) \theta(|X - Y|) dY \\
\leq \left( \int_{B(X, 2)} |\nabla^{|b|} z_+|^2 \nabla^{|c|+1} z_+|^2(t, Y) dY \right) \frac{1}{2} \\
\leq F(t, X) \rho_+(t, X).
\]

If $|a| \leq N$, $b + c = a$, $c = a$, then $b = (0, \cdots, 0)$ and

\[
-2 \int_{\mathbb{R}^d} (\nabla^b z_+ \cdot \nabla \nabla^c z_+ \nabla^a z_+)(t, Y) \theta(|X - Y|) dY \\
= - \int_{\mathbb{R}^d} (z_+ \cdot \nabla |\nabla^a z_+|^2)(t, Y) \theta(|X - Y|) dY \\
= - \int_{\mathbb{R}^d} z_+ \cdot \nabla \theta(|X - Y|)|\nabla^a z_+|^2(t, Y) dY \\
\leq C \left( \int_{B(X, 2)} |z_+|^2 |\nabla^{[a]} z_+|^2(t, Y) dY \right) \frac{1}{2} \\
\leq CF(t, X) \rho_+(t, X).
\]

If $|a| \leq N$, then

\[
- \int_{\mathbb{R}^d} \nabla^a z_+ \cdot \nabla \nabla^a p(t, Y) \theta(|X - Y|) dY \\
\leq \left( \int_{B(X, 2)} |\nabla^{[a]} z_+|^2(t, Y) dY \right) \frac{1}{2} \\
\leq F(t, X) \rho_+(t, X).
\]

Summing up, we obtain

\[
(\partial_t \rho_+^2 - \mu \Delta \rho_+^2 - B_0 \cdot \nabla \rho_+^2)(t, X) + 2\mu |\nabla \rho_+^{(e)}(t, X)|^2 \leq CF(t, X) \rho_+(t, X),
\]

which gives

\[
(\partial_t \rho_+^{(e)} - \mu \Delta \rho_+^{(e)} - B_0 \cdot \nabla \rho_+^{(e)})(t, X) \leq CF(t, X).
\]

Now, the local energy inequality (2.1) follows by letting $\varepsilon \to 0$.

Let us conclude this section by the following estimate for $F(t, X)$:

(2.4)
\[
F(t, X) \leq C \int_{\mathbb{R}^d} \frac{\rho_+(t, Y) \rho_-(t, Y)}{1 + |X - Y|^{d+1}} dY.
\]

We need the following fact.

**Lemma 2.1.** It holds that

\[
\|f\|_{L^2(B(X, 2))} \leq C \int_{B(X, 3)} \|f\|_{L^2(B(Y, \frac{1}{4}))} dY.
\]
Proof. By Fubini theorem, we have
\[
\int_{B(X, \frac{1}{2})} \|f\|_{L^2(B(X, 2) \cap B(Y, 3))}^2 dY = \int_{B(X, 3)} \|f\|_{L^2(B(X, 2) \cap B(Y, \frac{1}{2}))}^2 dY \leq \int_{B(X, 3)} \|f\|_{L^2(B(X, 2))} \|f\|_{L^2(B(Y, \frac{1}{2}))} dY.
\]
As \(B(X, 2) \cap B(Y, 3) = B(X, 2)\) for \(Y \in B(X, \frac{1}{2})\), we infer that
\[
\frac{\omega_d}{2^d} \|f\|_{L^2(B(X, 2))}^2 \leq \|f\|_{L^2(B(X, 2))} \int_{B(X, 3)} \|f\|_{L^2(B(Y, \frac{1}{2}))} dY,
\]
where \(\omega_d\) is the volume of the unit ball in \(\mathbb{R}^d\), and this gives the result. \(\square\)

Now let us prove (2.4). By Sobolev embedding, we have
\[
\|\nabla^k z_\pm(t)\|_{L^{p_k}(B(X, 1))} \leq C\|z_\pm(t)\|_{H^N(B(X, 1))} \leq C \rho_+(t, X),
\]
where \(\frac{1}{p_k} = \frac{k-1}{2(N-1)}\) for \(1 \leq k \leq N\) and \(\frac{1}{p_k} = 0\) for \(k = 0\). Thus, for \(k+j \leq N+1, 0 \leq k, j \leq N\), we have \(\frac{1}{p_k} + \frac{1}{p_j} \leq \frac{1}{2}\) and
\[
\|\nabla^k z_+ \nabla^j z_- (t)\|_{L^2(B(X, 1))} \leq C\|\nabla^k z_+(t)\|_{L^{p_k}(B(X, 1))} \|\nabla^j z_-(t)\|_{L^{p_j}(B(X, 1))} \leq C \rho_+(t, X) \rho_-(t, X).
\]
Due to \(\text{div} z_\pm = 0\), we infer from the first equation of (1.3) that
\[
-\Delta p = \partial_i (\partial^1_i f^j z^i_\pm) + \partial_i z^i_\pm \partial_j z^j_- = \partial_i \partial_j (z^i_+ z^j_-).
\]
Then by the interior elliptic estimates, we get
\[
\|\nabla p(t)\|_{H^N(B(X, \frac{1}{2}))} \leq C\|\nabla p(t)\|_{L^\infty(B(X, 1))} + C\|\Delta p(t)\|_{H^{N-1}(B(X, 1))} \leq C\|\nabla p(t)\|_{L^\infty(B(X, 1))} + C \sum_{k+j \leq N-1, k, j \geq 0} \|\nabla^{k+1} z_+ \nabla^{j+1} z_- (t)\|_{L^2(B(X, 1))} \leq C\|\nabla p(t)\|_{L^\infty(B(X, 1))} + C \rho_+(t, X) \rho_-(t, X),
\]
from which and Lemma 2.1, we infer that
\[
(2.5) \quad F(t, X) \leq C\|\nabla p(t)\|_{L^\infty(B(X, A))} + C \int_{B(X, 3)} \rho_+(t, Y) \rho_-(t, Y) dY.
\]
It remains to estimate \(\|\nabla p(t)\|_{L^\infty(B(X, A))}\). For this, we use the following representation formula of the pressure \(p(t, X)\):
\[
-\nabla p(t, X) = \int_{\mathbb{R}^d} \nabla N(X - Y) \theta(|X - Y|) |\Delta p(t, Y)| dY + \int_{\mathbb{R}^d} \partial_i \partial_j \left(\nabla N(X - Y)(1 - \theta(|X - Y|))\right)(z^i_+ z^j_-)(t, Y) dY,
\]
where \(N(X)\) is the Newton potential. Thanks to \(|\nabla^k N(X)| \leq C|X|^{2-d-k}\) and Sobolev embedding, we obtain
\[
|\nabla p(t, X)| \leq C \int_{B(X, 2)} \frac{dY}{|X - Y|^{d-1}} |\Delta p(t)|_{L^\infty(B(X, 2))} + C \int_{\mathbb{R}^d} |z^i_+ z^j_- (t, Y)| dY \leq \frac{1}{1 + |X - Y|^{d+1}}.
\]
\[ \leq C\|\triangle p(t)\|_{H^{n-1}(B(X,2))} + C \int_{\mathbb{R}^d} \frac{\rho_+(t,Y)\rho_-(t,Y)\rho_-(t,Y)dY}{1 + |X - Y|^{d+1}}. \]

Notice that \( \|\triangle p(t)\|_{H^{n-1}(B(Y,\frac{1}{2}))} \leq C\rho_+(t,Y)\rho_-(t,Y) \), which along with Lemma 2.1 gives
\[
\|\triangle p(t)\|_{H^{n-1}(B(X,2))} \leq C \int_{B(X,3)} \rho_+(t,Y)\rho_-(t,Y)dY.
\]

This shows that
\[
|\nabla p(t,X)| \leq C \int_{\mathbb{R}^d} \frac{\rho_+(t,Y)\rho_-(t,Y)dY}{1 + |X - Y|^{d+1}}.
\]

Thanks to \( \frac{1}{1 + |X' - Y'|^{d+1}} \leq C \frac{1}{1 + |X - Y|^{d+1}} \) for \( X' \in B(X,4) \), we have
\[
\|\nabla p(t)\|_{L^\infty(B(X,4))} \leq C \int_{\mathbb{R}^d} \frac{\rho_+(t,Y)\rho_-(t,Y)dY}{1 + |X - Y|^{d+1}}.
\]

Inserting this into (2.5), we arrive at (2.4).

3. Comparison principle

It follows from (2.1) and (2.3) that
\[
\partial_t \rho_+ + B_0 \cdot \nabla \rho_+ - \mu \triangle \rho_+ \leq C_1(\rho_+ \rho_-) * N_1,
\]
where \( N_1(X) = (1 + |X|^{d+1})^{-1} \).

To control \( \rho_\pm \), we establish the following comparison principle.

**Lemma 3.1.** Let \( 0 \leq \rho_1 \in L^\infty \cap C^0([0,T) \times \mathbb{R}^d) \) satisfy
\[
(3.2) \quad \partial_t \rho_1 + B_0 \cdot \nabla \rho_1 - \mu \triangle \rho_1 \geq C_1(\rho_+ \rho_-) * N_1.
\]

If \( \rho_\pm(0) \leq \rho_1(0) \) in \( \mathbb{R}^d \), then \( \rho_\pm \leq \rho_1 \) in \([0,T) \times \mathbb{R}^d\).

**Proof.** It follows from (3.1) and (3.2) that
\[
\partial_t(\rho_\pm - \rho_1) + B_0 \cdot \nabla(\rho_\pm - \rho_1) - \mu \triangle(\rho_\pm - \rho_1) \leq C_1(\rho_+ \rho_- - \rho_1 \rho_1) * N_1.
\]

Let \( f^+ = \max(f,0) \). By maximum principle, we deduce that for \( t \in [0,T) \),
\[
\|\rho_\pm - \rho_1\|_{L^\infty(\mathbb{R}^d)} \leq \int_0^t C_1 \left( \|(\rho_+ \rho_- - \rho_1 \rho_1)(s)\| * N_1 \right)_{L^\infty(\mathbb{R}^d)} ds
\]
\[
\leq \int_0^t C_1 \left( \|(\rho_+ \rho_- - \rho_1 \rho_1)(s)\| * N_1 \right)_{L^\infty(\mathbb{R}^d)} ds
\]
\[
\leq C \int_0^t \|(\rho_+ \rho_- - \rho_1 \rho_1)(s)\|_{L^\infty(\mathbb{R}^d)} ds
\]
\[
\leq CM \int_0^t \left( \|(\rho_+ - \rho_1)(s)\|_{L^\infty(\mathbb{R}^d)} + \|(\rho_- - \rho_1)(s)\|_{L^\infty(\mathbb{R}^d)} \right) ds,
\]
where \( M = \|\rho_+ + \rho_1\|_{L^\infty([0,T) \times \mathbb{R}^d)} < +\infty \). This implies that \( \|\rho_\pm - \rho_1\|_{L^\infty(\mathbb{R}^d)} = 0 \), hence \( \rho_\pm(t) \leq \rho_1(t) \) for \( t \in [0,T) \). 

To construct comparison functions \( \rho_1 \), we need the following observation.
Lemma 3.2. There exists an absolute constant $C_0 > 1$ so that
\[ C_0^{-1} N_1 \leq N_1 \ast N_1 \leq C_0 N_1, \quad \rho_{\pm}(0) \leq C_0 \rho_{\pm}(0) \ast N_1. \]
Here $\|N_1\|_{L^1(\mathbb{R}^d)} \leq C_0$.

Proof. As $N_1(X) \leq C N_1(Y)$ for $|X - Y| \leq 1$, we have
\[ N_1 \ast N_1(X) \geq \int_{B(X, 1)} N_1(Y) N_1(X - Y) dY \geq C^{-1} N_1(X). \]
Using the fact that $\min(N_1(Y), N_1(X - Y)) \leq C_N(X)$,
we infer that
\[ N_1 \ast N_1(X) = \int_{\mathbb{R}^d} N_1(Y) N_1(X - Y) dY \]
\[ = \int_{\mathbb{R}^d} \min(N_1(Y), N_1(X - Y)) \max(N_1(Y), N_1(X - Y)) dY \]
\[ \leq \int_{\mathbb{R}^d} C N_1(X)(N_1(Y) + N_1(X - Y)) dY = 2CN_1(X)\|N_1\|_{L^1(\mathbb{R}^d)}, \]
which gives the first inequality.
Using $\|z_0(0)\|_{H^N(B(Y, 1))} \leq C \rho_{\pm}(0, Y)$ and Lemma 2.1, we deduce that
\[ \rho_{\pm}(0, X) \leq \|z_0(0)\|_{H^N(B(X, 2))} \leq C \int_{B(X, 3)} \rho_{\pm}(0, Y) dY \leq C \rho_{\pm}(0) \ast N_1(X), \]
which gives the second inequality.

Now let us construct the comparison function.

Lemma 3.3. Let $\rho_{\pm}^0 \in L^1 \cap C(\mathbb{R})$, $\rho_{\pm}^{00} \in L^2 \cap L^\infty \cap C(\mathbb{R}^k \times \mathbb{T}^{d-k})$. Assume that
\[ 0 \leq \rho_{\pm}^{00}(x, y) \leq \rho_{\pm}^0(x) \quad \text{for } x \in \mathbb{R}, \ y \in \mathbb{R}^{d-1}, \]
\[ C_0^{-1} \rho_{\pm}^{00} \leq \rho_{\pm}^{00} \ast N_1 \leq C_0 \rho_{\pm}^{00}. \]
Then there exists $\epsilon_0 > 0$ depending only on $C_0, C_1$ such that if $\|\rho_{\pm}^0\|_{L^1(\mathbb{R})} < \epsilon_0$, then there exists
\[ 0 \leq \rho_{\pm}^1 \in L^\infty \cap C([0, +\infty) \times \mathbb{R}^d) \]
satisfying (3.2) and $\rho_{\pm}^0 \leq \rho_{\pm}^1(0)$. Moreover,
\[ \|\rho_{\pm}^1(t)\|_{L^2(\mathbb{R}^k \times \mathbb{T}^{d-k})} \leq C \|\rho_{\pm}^{00}\|_{L^2(\mathbb{R}^k \times \mathbb{T}^{d-k})}. \]

Proof. Step 1. Construction of the data
For $\mu = 0$, we set $g_{\pm}^{00} = \rho_{\pm}^{00}$, $g_{\pm}^0 = \rho_{\pm}^0$. For $\mu > 0$, we set
\[ g_{\pm}^{00}(X) = \frac{1}{2\mu} \int_0^{+\infty} e^{-\frac{y}{2\mu}} \rho_{\pm}^{00}(X \mp B_0 y) dy, \quad g_{\pm}^0(x) = \frac{1}{2\mu} \int_0^{+\infty} e^{-\frac{y}{2\mu}} \rho_{\pm}^0(x \mp y) dy. \]
It is easy to check that
\[ g_{\pm}^0 \pm 2\mu \partial_x g_{\pm}^0 = \rho_{\pm}^0, \quad g_{\pm}^{00} \pm 2\mu B_0 \cdot \nabla g_{\pm}^{00} = \rho_{\pm}^{00}, \]
\[ \|g_{\pm}^0\|_{L^1(\mathbb{R})} = \|\rho_{\pm}^0\|_{L^1(\mathbb{R})}, \quad \|g_{\pm}^{00}\|_{L^\infty(\mathbb{R}^d)} \leq \|\rho_{\pm}^{00}\|_{L^\infty(\mathbb{R}^d)}, \]
and $0 \leq g_{\pm}^{00}(x, y) \leq g_{\pm}^0(x)$ for $x \in \mathbb{R}, \ y \in \mathbb{R}^{d-1}$.
Let
\[ h_{\pm}^0(x) = \frac{1}{2\epsilon_0} \int_0^{+\infty} (\rho_{\pm}^0 + g_{\pm}^0)(x \mp y) dy. \]
Then we have
\[ 0 \leq h_\pm^0 < 1, \quad \pm \partial_x h_\pm^0 = (\rho_\pm^0 + g_\pm^0)/(2\epsilon_0). \]

**Step 2.** Construction of comparison function

Let \((\rho_{\pm}^{01}, g_{\pm}^{01})(t, X)\) be the solution to
\[ \partial_t f + B_0 \cdot \nabla f - \mu \Delta f = 0, \quad f(0, X) = (\rho_{\pm}^{00}, g_{\pm}^{00})(X), \]
and \((\rho_{\pm}^{11}, g_{\pm}^{11}, h_{\pm}^1)(t, x)\) be the solution to
\[ \partial_t f + \partial_x f - \mu \partial_x^2 f = 0, \quad f(0, x) = (\rho_{\pm}^0, g_{\pm}^0, h_{\pm}^0)(x). \]

Thanks to the construction of the data, we find that
\[ 0 \leq h_{\pm}^1 < 1, \quad \pm \partial_x h_{\pm}^1 = (\rho_{\pm}^{11} + g_{\pm}^1)/(2\epsilon_0), \]
\[ g_{\pm}^{01} \pm 2\mu B_0 \cdot \nabla g_{\pm}^{01} = \rho_{\pm}^{01}, \]
\[ 0 \leq \rho_{\pm}^{01}(t, x, y) \leq \rho_{\pm}^{11}(t, x), \quad 0 \leq g_{\pm}^{01}(t, x, y) \leq g_{\pm}^1(t, x). \]

Now we take \(\rho_{\pm}^{10}(t) = C_0\rho_{\pm}^{10}(t) \ast N_1, \) where
\[ \rho_{\pm}^{10}(t, X) = \rho_{\pm}^{01}(t, X) + g_{\pm}^{01}(t, X)h_{\pm}^1(t, x). \]

**Step 3.** Verification of the conditions

By our construction, it is easy to check that
\[ \partial_t \rho_{\pm}^{10} + B_0 \cdot \nabla \rho_{\pm}^{10} - \mu \Delta \rho_{\pm}^{10} = (\partial_t \rho_{\pm}^{00} + B_0 \cdot \nabla \rho_{\pm}^{00} - \mu \Delta \rho_{\pm}^{00}) \]
\[ + (\partial_t g_{\pm}^{01} + B_0 \cdot \nabla g_{\pm}^{01} - \mu \Delta g_{\pm}^{01})h_{\pm}^{1} + g_{\pm}^{01}(\partial_t h_{\pm}^{1} + \partial_x h_{\pm}^{1} - \mu \partial_x^2 h_{\pm}^{1}) \]
\[ - 2\mu B_0 \cdot \nabla g_{\pm}^{01} \partial_x h_{\pm}^{1} \]
\[ = 0 + 2g_{\pm}^{01} \partial_x h_{\pm}^{1} - 2\mu B_0 \cdot \nabla g_{\pm}^{01} \partial_x h_{\pm}^{1} \]
\[ = (g_{\pm}^{01} + \rho_{\pm}^{01})(\rho_{\pm}^{11} + g_{\pm}^1)/(2\epsilon_0) \geq (g_{\pm}^{01} + \rho_{\pm}^{01})(\rho_{\pm}^{01} + g_{\pm}^{01})/(2\epsilon_0), \]
which implies that
\[ (\partial_t \rho_{\pm}^{01} + B_0 \cdot \nabla \rho_{\pm}^{01} - \mu \Delta \rho_{\pm}^{01} \geq C_0/(2\epsilon_0)((g_{\pm}^{01} + \rho_{\pm}^{01})(\rho_{\pm}^{01} + g_{\pm}^{01})) \ast N_1. \]

As \(0 \leq h_\pm^1 \leq 1,\) we have
\[ \rho_{\pm}^{10}(t) \leq C_0\rho_{\pm}^{10}(t) \ast N_1 + C_0g_{\pm}^{01}(t) \ast N_1. \]

Since \(\rho_{\pm}^{01}(t) \ast N_1\) satisfy \(\partial_t f + B_0 \cdot \nabla f - \mu \Delta f = 0\) and \(\rho_{\pm}^{01}(0) \ast N_1 = \rho_{\pm}^{00} \ast N_1 \leq C_0\rho_{\pm}^{00} \ast N_1 = C_0\rho_{\pm}^{00}(0),\) we conclude that \(\rho_{\pm}^{10}(t) \ast N_1 \leq C_0\rho_{\pm}^{01}(t).\) Thanks to \(g_{\pm}^{00} \ast N_1 \leq C_0g_{\pm}^{00},\) we similarly have \(g_{\pm}^{01}(t) \ast N_1 \leq C_0g_{\pm}^{01}(t).\) Thus,
\[ 0 \leq \rho_{\pm}^{10}(t) \leq C_0^2 \rho_{\pm}^{01}(t) + g_{\pm}^{01}(t), \]
which along with \(\text{(3.3)}\) gives
\[ \partial_t \rho_{\pm}^{10} + B_0 \cdot \nabla \rho_{\pm}^{10} - \mu \Delta \rho_{\pm}^{10} \geq (1 + \rho_{\pm}^{10})(t) \ast N_1/(2\epsilon_0 C_0^3). \]

Taking \(\epsilon_0 = 1/(2C_0^3 C_1),\) we find that \(0 \leq \rho_{\pm}^{10}(t) \in L^\infty \cap C([0, +\infty) \times \mathbb{R}^d)\) satisfies \(\text{(3.2)}\). As \(\rho_{\pm}^{10}(0) \geq \rho_{\pm}^{01}(0) = \rho_{\pm}^{00},\) we have \(\rho_{\pm}^{10}(0) = C_0\rho_{\pm}^{01}(0) \ast N_1 \geq C_0\rho_{\pm}^{00} \ast N_1 \geq \rho_{\pm}^{00}.\)

By standard energy estimate, we can deduce that
\[ \|\rho_{\pm}^{01}(t)\|_{L^2(\mathbb{R}^d \times T^{d-k})} \leq \|\rho_{\pm}^{01}(0)\|_{L^2(\mathbb{R}^d \times T^{d-k})} = \|\rho_{\pm}^{00}\|_{L^2(\mathbb{R}^d \times T^{d-k})}, \]
\[ \|g_{\pm}^{01}(t)\|_{L^2(\mathbb{R}^d \times T^{d-k})} \leq \|g_{\pm}^{00}\|_{L^2(\mathbb{R}^d \times T^{d-k})}. \]
For $\mu = 0$, $\|g_{\pm}^{00}\|_{L^2(\mathbb{R}^k \times \mathbb{T}^{d-k})} = \|\rho_{\pm}^{00}\|_{L^2(\mathbb{R}^k \times \mathbb{T}^{d-k})}$, and for $\mu > 0$,

$$
\|g_{\pm}^{00}\|_{L^2(\mathbb{R}^k \times \mathbb{T}^{d-k})} \leq \frac{1}{2\mu} \int_0^{+\infty} e^{-\frac{\nu}{2\mu}} \|\rho_{\pm}^{00}(\cdot \pm B_0y)\|_{L^2(\mathbb{R}^k \times \mathbb{T}^{d-k})} dy = \frac{1}{2\mu} \int_0^{+\infty} e^{-\frac{\nu}{2\mu}} \|\rho_{\pm}^{00}\|_{L^2(\mathbb{R}^k \times \mathbb{T}^{d-k})} dy = \|\rho_{\pm}^{00}\|_{L^2(\mathbb{R}^k \times \mathbb{T}^{d-k})}.
$$

Thus, we obtain

$$
\|\rho_{\pm}(t)\|_{L^2(\mathbb{R}^k \times \mathbb{T}^{d-k})} \leq C_0^2 (\|\rho_{\pm}^{01}(t)\|_{L^2(\mathbb{R}^k \times \mathbb{T}^{d-k})} + \|g_{\pm}^{01}(t)\|_{L^2(\mathbb{R}^k \times \mathbb{T}^{d-k})}) \\
\leq C_0^2 (\|\rho_{\pm}^{00}\|_{L^2(\mathbb{R}^k \times \mathbb{T}^{d-k})} + \|g_{\pm}^{00}\|_{L^2(\mathbb{R}^k \times \mathbb{T}^{d-k})}) \\
\leq 2C_0^2 \|\rho_{\pm}^{00}\|_{L^2(\mathbb{R}^k \times \mathbb{T}^{d-k})}.
$$

This completes the proof. \qed

4. Proof of Theorem 1.1

By the local well-posedness result, there exists a unique solution $z_{\pm} \in C([0, T^*), H^N(\mathbb{R}^k \times \mathbb{T}^{d-k})$ to the MHD equations (1.3), where $T^*$ is the maximal existence time of the solution.

Let $\rho_{\pm}^{00} = C_0 \rho_{\pm}(0) * N_1$. Then Lemma 3.2 ensures that

$$
\rho_{\pm}(0) \leq \rho_{\pm}^{00}, \quad C_0^{-1} \rho_{\pm}^{00} \leq \rho_{\pm}^{00} * N_1 \leq C_0 \rho_{\pm}^{00}.
$$

Let

$$
\rho_{\pm}^0(x) = C_0 \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \rho_{\pm}^+(x-x')N_1(x', y')dy'dx',
$$

where $\rho_{\pm}^+(x) = \sup_{y \in \mathbb{R}^{d-1}} \rho_{\pm}(0, x, y)$, thus $J_\pm(0) = \|\rho_{\pm}^+\|_{L^1(\mathbb{R})}$. Thanks to

$$
\rho_{\pm}^{00}(x, y) = C_0 \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \rho_{\pm}(0, x-x', y-y')N_1(x', y')dy'dx',
$$

we find that $\rho_{\pm}^{00}(x, y) \leq \rho_{\pm}^0(x)$ and

$$
\|\rho_{\pm}^0\|_{L^1(\mathbb{R})} = C_0 \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \rho_{\pm}^+(x-x')N_1(x', y')dy'dx' = C_0 \|\rho_{\pm}^+\|_{L^1(\mathbb{R})} \int_{\mathbb{R}^{d-1}} N_1(x', y')dy'dx' = C_0 J_\pm(0) \|N_1\|_{L^1(\mathbb{R}^{d})} \leq C_0^2 \epsilon_1 < \epsilon_0,
$$

if $J_\pm(0) \leq \epsilon_1 = \epsilon_0/2C_0^2$.

Now, Lemma 3.3 ensures that there exists $0 \leq \rho_{\pm}^1 \in L^\infty \cap C([0, +\infty) \times \mathbb{R}^d)$, which satisfies (3.2), and $\rho_{\pm}(0) \leq \rho_{\pm}^{00} \leq \rho_{\pm}^1(0)$ in $\mathbb{R}^d$, and

$$
\|\rho_{\pm}^1(t)\|_{L^2(\mathbb{R}^k \times \mathbb{T}^{d-k})} \leq C \|\rho_{\pm}^{00}\|_{L^2(\mathbb{R}^k \times \mathbb{T}^{d-k})} \leq C \|\rho_{\pm}(0)\|_{L^2(\mathbb{R}^k \times \mathbb{T}^{d-k})} \leq C \|z_{\pm}(0)\|_{H^N(\mathbb{R}^k \times \mathbb{T}^{d-k})}.
$$

Then we infer from Lemma 3.4 that $0 \leq \rho_{\pm} \leq \rho_{\pm}^1$ in $[0, T) \times \mathbb{R}^d$ for $0 < T < T^*$. Hence,

$$
\|z_{\pm}(t)\|_{H^N(\mathbb{R}^k \times \mathbb{T}^{d-k})} \leq C \|\rho_{\pm}(t)\|_{L^2(\mathbb{R}^k \times \mathbb{T}^{d-k})} \leq C \|\rho_{\pm}^1(t)\|_{L^2(\mathbb{R}^k \times \mathbb{T}^{d-k})} \leq C \|z_{\pm}(0)\|_{H^N(\mathbb{R}^k \times \mathbb{T}^{d-k})}
$$

for any $t \in [0, T)$, which implies $T^* = +\infty$. 

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5. Decay estimates

In this section, we provide some decay estimates of the solution in time when $\mu > 0$. Here we use the notations in section 3 and section 4.

Using the fact that $(\rho_{\pm 1}^{01}, g_{\pm 1}^{01})(t, X) = e^{\mu t \Delta} (\rho_{\pm 0}^{00}, g_{\pm 0}^{00})(X - B_0 t)$, we have
\[
g_{\pm 1}^{01}(t, X) = \frac{1}{2\mu} \int_{0}^{+\infty} e^{-\frac{\mu}{2} \rho_{\pm 1}^{01}(t, X + B_0 y) dy},
\]
which gives
\[
\|g_{\pm 1}^{01}(t)\|_{L^p(\mathbb{R}^k \times T^{d-k})} \leq \|\rho_{\pm 1}^{01}(t)\|_{L^p(\mathbb{R}^k \times T^{d-k})} = \|e^{\mu t \Delta} \rho_{\pm 1}^{00}\|_{L^p(\mathbb{R}^k \times T^{d-k})},
\]
Thus, we deduce that for $2 \leq p \leq \infty$,
\[
\|\rho_{\pm 1}^{01}(t)\|_{L^p(\mathbb{R}^k \times T^{d-k})} \leq C\|e^{\mu t \Delta} \rho_{\pm 1}^{00}\|_{L^p(\mathbb{R}^k \times T^{d-k})}.
\]
which along with the fact that $e^{\mu t \Delta} \rho_{\pm 0}^{00} = C_0 e^{\mu t \Delta} \rho_{\pm 0}(0) * N_1$, we infer that
\[
\|\rho_{\pm 1}^{01}(t)\|_{L^p(\mathbb{R}^k \times T^{d-k})} \leq C\|e^{\mu t \Delta} \rho_{\pm 0}(0)\|_{L^p(\mathbb{R}^k \times T^{d-k})},
\]
for $2 \leq p \leq \infty$. Thus, we obtain
\[
\|z_{\pm}(t)\|_{W^{1,\infty}(\mathbb{R}^k \times T^{d-k})} \leq C\|\rho_{\pm 1}^{01}(t)\|_{L^\infty(\mathbb{R}^k \times T^{d-k})} \leq C\|e^{\mu t \Delta} \rho_{\pm 0}(0)\|_{L^\infty(\mathbb{R}^k \times T^{d-k})}
\leq C(1 + \mu t)^{-\frac{k}{4}}\|z_{\pm}(0)\|_{H^\infty(\mathbb{R}^k \times T^{d-k})}.
\]
If $\rho_{\pm 0}(0) \in L^1(\mathbb{R}^d)$, we similarly have
\[
\|z_{\pm}(t)\|_{H^\infty(\mathbb{R}^k \times T^{d-k})} \leq C(1 + \mu t)^{-\frac{k}{4}},
\]
\[
\|z_{\pm}(t)\|_{W^{1,\infty}(\mathbb{R}^k \times T^{d-k})} \leq C(1 + \mu t)^{-\frac{k}{2}}.
\]

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