Lie superalgebras of differential operators

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Abstract

We describe explicitly Lie superalgebra isomorphisms between the Lie superalgebras of first-order superdifferential operators on supermanifolds, showing in particular that any such isomorphism induces a diffeomorphism of the supermanifolds. We also prove that the group of automorphisms of such a Lie superalgebra is a semi-direct product of the subgroup induced by the supermanifold diffeomorphisms and another subgroup which consists of automorphisms determined by even superdivergences. These superdivergences are proven to exist on any supermanifold and their local form is explicitly described as well.

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1 Introduction

In [GKP09] the authors have described all Lie superalgebra isomorphisms \( \mathcal{X}(\mathcal{M}_1) \to \mathcal{X}(\mathcal{M}_2) \) of the Lie superalgebras of supervector fields on smooth supermanifolds \( \mathcal{M}_i \), \( i = 1, 2 \). It was shown that every such an isomorphism is induced by a diffeomorphism of supermanifolds \( \mathcal{M}_1 \to \mathcal{M}_2 \), unless the dimension of the manifolds is 1|1 or 0|2. In the latter low-dimensional cases there exist additional isomorphisms which have been also listed in [GKP09].

In the present paper the authors address a similar question, this time about isomorphisms of other natural and important Lie superalgebras growing on supermanifolds – the Lie superalgebras \( \mathcal{D}^1(\mathcal{M}) \) of first-order superdifferential operators. The structure of the Lie superalgebra automorphism group is much more complicated in this case, being a semi-direct product of the subgroup induced by the supermanifold diffeomorphisms and another subgroup which consists of automorphisms determined by even superdivergences.

The paper is organized as follows. In section 2 we review some basic facts about (super)differential operators on a smooth supermanifold and prove an important
In section 3 we prove that, if \( \dim \mathcal{M} \) differs from \( 0|1 \), the algebra \( \mathcal{A}(\mathcal{M}) \subset \mathcal{D}^1(\mathcal{M}) \) of smooth (super)functions on \( \mathcal{M} \) can be characterized as the unique maximal super Lie ideal in \( \mathcal{D}^1(\mathcal{M}) \) consisting of ad-nilpotent elements. In particular, any automorphism of the Lie superalgebra \( \mathcal{D}^1(\mathcal{M}) \) preserves \( \mathcal{A}(\mathcal{M}) \), thus induces an automorphism of the Lie superalgebra of supervector fields \( \mathcal{X}(\mathcal{M}) \cong \mathcal{D}^1(\mathcal{M})/\mathcal{A}(\mathcal{M}) \).

In section 4 we prove that, according to the canonical splitting \( \mathcal{D}^1(\mathcal{M}) = \mathcal{X}(\mathcal{M}) \oplus \mathcal{A}(\mathcal{M}) \), any automorphism of the Lie superalgebra \( \mathcal{D}^1(\mathcal{M}) \) is a product of an automorphism induced by a superdiffeomorphism of \( \mathcal{M} \) and an automorphism of the form \( (X + f) \mapsto (X + f + c(X)) \), determined by a 1-cocycle \( c: \mathcal{X}(\mathcal{M}) \to \mathcal{A}(\mathcal{M}) \) on the Lie superalgebra of vector fields \( \mathcal{X}(\mathcal{M}) \), with values in the algebra of functions on \( \mathcal{M} \). We also explain why only those automorphisms of \( \mathcal{X}(\mathcal{M}) \) which are induced by supermanifold diffeomorphisms can be extended from \( \mathcal{X}(\mathcal{M}) \) to \( \mathcal{D}^1(\mathcal{M}) \).

In section 5 we compute the 1st cohomology of the Lie superalgebra \( \mathcal{X}(\mathcal{M}) \) of supervector fields with values in \( \mathcal{A} \). We show that every 1-cocycle is a combination of a closed super 1-form and a fixed divergence.

Finally, in section 6, we review necessary facts about measures (Berezinian volumes) and divergences on supermanifolds and prove the existence of a divergence for each supermanifold.

## 2 Sheafs of superdifferential operators

Let us recall that a smooth supermanifold \( \mathcal{M} \) of dimension \( p|q \) is a (local) ringed superspace \( (\mathcal{M}, \mathcal{A}) \) over a topological space \( \mathcal{M} \) that is locally isomorphic to \( (\mathbb{R}^p, C^\infty(p|q)) \), where, for any open subset \( \mathcal{U} \subset \mathbb{R}^p, C^\infty(p|q)(\mathcal{U}) := C^\infty(\mathcal{U})[\xi^1, \ldots, \xi^q] \) – the \( \xi^\alpha \) being formal anticommuting generators. More precisely, we assume that \( \mathcal{A} \) is a sheaf of associative supercommutative \( \mathbb{R} \)-algebras with unit. The superalgebra \( \mathcal{A}(\mathcal{M}) = \Gamma(\mathcal{M}, \mathcal{A}) \) of global sections of \( \mathcal{A} \) is the algebra \( C^\infty(\mathcal{M}) \) of functions of the supermanifold \( \mathcal{M} \). It is well-known that, due to the local model condition, the locality condition for the stalks is automatically satisfied. Further, the considered data induce a smooth manifold structure of dimension \( p \) on \( \mathcal{M} \) and provide an embedding of the classical manifold \( \mathcal{M} \) into the supermanifold \( \mathcal{M} \).

An important result of smooth supergeometry \([\text{Ga}77]\) asserts that there exists a vector bundle \( V \) of rank \( q \) over \( \mathcal{M} \) such that \( \mathcal{M} \) is diffeomorphic as a supermanifold to \( PV \), that is, to the total space of \( V \) with the reversed parity of fibres. This implies that the algebra of smooth functions on \( \mathcal{M} \) is isomorphic (as a commutative superalgebra) to the algebra of functions on \( PV \), which is canonically identified with \( \Gamma(\Lambda^*V^*) \). This isomorphism is not canonical but it gives us an identification

\[ \mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \cong \Gamma(\Lambda^*V^*) \],

with

\[ \mathcal{A}_0 = \bigoplus_{i \geq 0} \mathcal{A}^{2i}, \quad \mathcal{A}_1 = \bigoplus_{i \geq 0} \mathcal{A}^{2i+1}, \quad \text{where} \quad \mathcal{A}^k \cong \Gamma(\Lambda^kV^*) \].

(2.2)
In particular, this gives a (non-canonical) embedding of the algebra $C^\infty(M)$ into $C^\infty(M)$ and a super-version of the partition of unity.

**Remark 1.** For simplicity, we assume in this text that $M$ is connected.

For any open subset $U \subset M$, we denote by $(\text{Der} \mathcal{A})(U)$ the $\mathcal{A}(U)$-module $\text{Der}(\mathcal{A}(U))$ of derivations of the superalgebra $\mathcal{A}(U)$. If $X \in (\text{Der} \mathcal{A})(U)$, there is, in view of the localization principle, for any open subset $V \subset U$, a unique derivation $X|_V \in (\text{Der} \mathcal{A})(V)$ such that $(Xf)|_V = X|_V f|_V$, for all $f \in \mathcal{A}(U)$. The assignment $U \to (\text{Der} \mathcal{A})(U)$ is actually a locally free sheaf of $\mathcal{A}$-modules, called the derivation sheaf $\text{Der} \mathcal{A}$ of the structure sheaf $\mathcal{A}$, or, also, the tangent sheaf $TM$ of the supermanifold $M$. The module $(TM)(M)$ of global sections of the supervector bundle $TM$ is the $C^\infty(M)$-module $\mathcal{X}(M)$ of vector fields of $M$ – which carries an obvious Lie superalgebra structure.

In the following we denote by $\text{End}(\mathcal{A}(U))$ the $\mathcal{A}(U)$-module of even and odd $\mathbb{R}$-linear maps from $\mathcal{A}(U)$ to itself. We can identify $\mathcal{A}(U)$ with a subalgebra of $\text{End}(\mathcal{A}(U))$ using the left-regular representation $f \mapsto m_f$, $m_f(g) = fg$. The $\mathcal{A}(U)$-module of $k$-th order differential operators $\mathcal{D}^k(U)$, $k \in \mathbb{N}$, is then defined inductively by

\[
\mathcal{D}^k(U) := \{ D \in \text{End}(\mathcal{A}(U)) : [D, \mathcal{A}(U)] \subset \mathcal{D}^{k-1}(U) \},
\]

where $[-,-]$ is the supercommutator and where $\mathcal{D}^{-1}(U) = \{0\}$.

Of course, $\mathcal{D}^0(U) = \mathcal{A}(U)$, and thus 0-order operators are local. This entails by induction that any superdifferential operator is local. Indeed, if $D \in \mathcal{D}^k(U)$, if the restriction $f|_V$ of $f \in \mathcal{A}(U)$ to an open $V \subset U$ vanishes, and if $v \in V$, let $\gamma \in \mathcal{A}_0(U)$ be a bump superfunction with support $\text{supp} \gamma \subset V$ (in the supercontext the support can be defined as usual as the complement in $U$ of the set of those points $u \in U$ for which the restriction of $\gamma$ to some neighborhood of $u$ vanishes) and restriction $\gamma|_W = 1$, for some neighborhood $W \subset V$ of $v$, see localization principle [Lei80] Corollary 3.1.8]. It then follows from the defining property of differential operators applied to $[D, \gamma]f$, the induction assumption, and the fact $\gamma f = 0$, that $(Df)|_W = 0$. We can now show that there exists, just as in the case of vector fields, for any $D \in \mathcal{D}^k(U)$ and any open $V \subset U$, a unique $D|_V \in \mathcal{D}^k(V)$ such that $(Df)|_V = D|_V f|_V$, for all $f \in \mathcal{A}(U)$. Indeed, if $f \in \mathcal{A}(V)$ and $v \in V$, it is possible to choose a function $F \in \mathcal{A}(U)$ (of the same parity as $f$) such that $\text{supp} F \subset \text{supp} f$ and $F|_V = f|_W$, for some neighborhood $W \subset V$ of $v$. Locality entails that $(DF)|_W \in \mathcal{A}(W)$ and $(DF')|_W' \in \mathcal{A}(W')$, defined for two points $v, v' \in V$, depend only on $f$ and coincide in the intersection $W \cap W'$. Thus these local functions define a unique global function $D|_V f \in \mathcal{A}(V)$ such that

\[
(D|_V f)|_W = (DF)|_W.
\]

Since, obviously, $D|_V \in \text{End}(\mathcal{A}(V))$ (note that $D|_V$ has the same parity as $D$), it suffices – to prove the above claim – to observe that, for any $f_1, \ldots, f_{k+1} \in \mathcal{A}(V)$, we have

\[
[[\ldots [[D|_V, f_1], f_2], \ldots, f_{k+1}]|_W = [[\ldots [[D, F_1], F_2], \ldots, F_{k+1}]|_W = 0,
\]

with self-explaining notations.

In view of the just detailed restrictions of differential operators, the assignment $U \to \mathcal{D}^k(U)$ is a presheaf and obviously also a sheaf – as $\mathcal{A}$ is a sheaf.
Proposition 1. For any \( k \in \mathbb{N} \), the presheaf \( \mathcal{D}^k \) of \( k \)-th order superdifferential operators over the base manifold \( M \) (the body) of a smooth supermanifold \( \mathcal{M} = (M, \mathcal{A}) \) of dimension \( p|q \) is a locally free sheaf of \( \mathcal{A} \)-modules, with local basis

\[
\partial_x^\alpha \partial_\xi^\beta := \partial_x^{\alpha_1} \ldots \partial_x^{\alpha_p} \partial_\xi^{\beta_1} \ldots \partial_\xi^{\beta_q},
\]

where \( (x, \xi) \) are local coordinates, \( \beta_a \in \{0,1\} \), and \( |\alpha| + |\beta| \leq k \).

Proof. The method used to prove local freeness of the sheaf of vector fields goes through in the case of differential operators. Let us give some details because of the increased technicality.

If \( M = U^{p|q} \) is a superdomain, if \( D \in \mathcal{D}^k(U) \) is of the type

\[
\sum_{i=0}^k D^i = \sum_{i=0}^k \sum_{|\alpha| + |\beta| = i} D_{\alpha\beta}^i(x, \xi) \partial_x^\alpha \partial_\xi^\beta \in \mathcal{D}^k(U),
\]  

(2.4)

and if \( m_{\alpha\beta} = (1/\alpha!) x^\alpha \xi^\beta \), where the odd coordinates are written in increasing order, then necessarily

\[
D_{\alpha\beta}^i = D^i m_{\alpha\beta} = Dm_{\alpha\beta} - \sum_{j=0}^{i-1} D^j m_{\alpha\beta},
\]  

(2.5)

and an induction on \( i \) immediately shows that the coefficients \( D_{\alpha\beta}^i \), if they exist, are unique.

Take now an arbitrary \( D \in \mathcal{D}^k(U) \) and set \( \Delta = D - \sum \in \mathcal{D}^k(U) \), where \( \sum \) denotes the RHS of (2.4) with the coefficients defined in (2.5). This operator \( \Delta \) vanishes by construction on the polynomials of degree \( \leq k \) in \( x, \xi \).

For any \( f_1, \ldots, f_{\ell-1}, h \in \mathcal{A}(U), \ell \geq k + 1 \), we have

\[
\Delta(f_1 \ldots f_{\ell-1} h) = \sum_{b=1}^{\ell-1} \sum \pm f_1 \ldots f_b \Delta(f_{b+1} \ldots f_{\ell-1} h) + F(h),
\]  

(2.6)

as immediately seen when developing \( F(h) := [\ldots[\Delta, f_1], f_2], \ldots, f_{\ell-1} h) \). If \( \ell > k + 1 \), the term \( F(h) \) vanishes, whereas in the case \( \ell = k + 1 \) it is given by \( F(h) = F(1)h \). Equation (2.6) shows that \( \Delta = 0 \) on any polynomial of degree \( k + 1 \), then, by induction, that \( \Delta = 0 \) on an arbitrary polynomial in \( x, \xi \). Further, this equation entails that \( \Delta \mathcal{T}_m^{k+c} \subset \mathcal{T}_m^{k+1}, m \in U, c \geq 1 \), where \( \mathcal{T}_m \) is the unique homogeneous maximal ideal of the stalk \( \mathcal{A}_m \). However, in view of Hadamard’s lemma, we can, for any \( f \in \mathcal{A}(U) \) and any \( m \in U \), find a polynomial \( P_{f,m} \) in \( x, \xi \) such that \( f - P_{f,m} \in \mathcal{T}_m^{k+q+1} \). It follows that \( \Delta f = \Delta(f - P_{f,m}) \in \mathcal{T}_m^{q+1}, \) for all \( m \in U \), so that \( \Delta f = 0 \).

Remark 2. It follows easily from the above proof that the order of a differential operator can be determined locally by looking at supercommutators with the multiplications by coordinate functions: \( D \in \mathcal{D}^k(U) \) if and only if, for a given system \( (u^1, u^2, \ldots, u^{p+q}) \) of local coordinates in \( U \),

\[
[[\ldots[[D(u, u^{i_1}], u^{i_2}], \ldots], u^{i_{k+1}}] = 0
\]

for any sequence \( i_1, \ldots, i_{k+1} \in \{1, \ldots, p+q\} \).
Remark 3. The idea of defining differential operators on an abstract (super)commutative (super)algebra by the formula (2.3) goes back to Grothendieck and Vinogradov [V72].

The super $\mathbb{R}$-vector space $\text{End}(\mathcal{A}(U))$ carries natural associative and Lie superalgebra structures $\circ$ and $[-,-]$ (we often omit the symbol $\circ$). An induction on $k + \ell$ allows seeing that $\mathcal{D}^k(U) \circ \mathcal{D}^\ell(U) \subset \mathcal{D}^{k+\ell}(U)$ and $[\mathcal{D}^k(U), \mathcal{D}^\ell(U)] \subset \mathcal{D}^{k+\ell-1}(U)$, so that the supervector space $\mathcal{D}(U) := \bigcup_{k \in \mathbb{N}} \mathcal{D}^k(U)$ of all differential operators inherits associative and Lie superalgebra structures that have weight 0 and $-1$, respectively, with respect to the filtration degree. It is easily checked that $\mathcal{D} : U \to \mathcal{D}(U)$ (resp. $\mathcal{D}^1 : U \to \mathcal{D}^1(U)$) is a locally free sheaf of $\mathcal{A}$-modules and associative and Lie superalgebras (resp. of $\mathcal{A}$-modules and sub Lie superalgebras) over $M$. The algebra $\mathcal{D}(M)$ (resp. $\mathcal{D}^1(M)$) is the Lie superalgebra of differential operators (resp. first-order differential operators) of the supermanifold $M$. In the sequel we denote this algebra also by $\mathcal{D}(M)$ or even by $\mathcal{D}$ (resp. by $\mathcal{D}^1(M)$ or $\mathcal{D}^1$).

The usual splitting of the space of first-order differential operators holds true in the supersetting.

Proposition 2. Let $M = (M, \mathcal{A})$ be a smooth supermanifold. An endomorphism $D \in \text{End}(\mathcal{A})$ is a first-order differential operator if and only if, for any $f, g \in \mathcal{A}$, we have
\[
D(fg) = (Df)g + (-1)^{|f||g|} f(Dg) - (D1)f g,
\] (2.7)
so that supervector fields are those first-order differential operators $D$ that verify $D1 = 0$. Moreover, the supervector space $\mathcal{D}^1$ admits a canonical splitting
\[
\mathcal{D}^1 = \mathcal{A} \oplus \mathcal{X}
\] (2.8)
given by $D \mapsto D1 + (D - D1)$.

Proof. To prove the direct implication (resp. converse implication), it suffices to compute $[[D,f], g](1)$ (resp. $[D,f](g)$), where $[-,-]$ denotes the supercommutator of endomorphisms and where 1 is the unit of $\mathcal{A}$. If $X \in \mathcal{X}$, we have $[X,f] = Xf \in \mathcal{A}$, so that $\mathcal{X} \subset \mathcal{D}^1$. The second claim now follows from Equation (2.7). It entails that the sum $\mathcal{A} + \mathcal{X}$ is direct. Finally, any $D \in \mathcal{D}^1$ decomposes in the form $D = D1 + (D - D1)$, where $D1 \in \mathcal{A}$ and $D - D1 \in \mathcal{X}$.

For purely even manifolds it is a well-known fact that the derived ideal $\mathcal{X}(M)' = [\mathcal{X}(M), \mathcal{X}(M)]$ is the whole Lie algebra $\mathcal{X}(M)$ of vector fields of $M$ [Gra78]. More precisely, if $X \in \mathcal{X}(M)$, supp $X \subset U$, $U$ open in $M$, then $X = \sum_{i=1}^n [X_i, Y_i]$, where $n$ is independent of the considered $X$ and where $X_i, Y_i$ are vector fields of $M$ with support supp $X_i$, supp $Y_i \subset U$ [Pon04]. The next theorem extends these result to the supercontext and will be used as a technical tool in the sequel.

Theorem 1. Let $M$ be a smooth supermanifold. Then, every $X \in \mathcal{X}(M)$ can be written as a finite sum of supercommutators,
\[
X = \sum_i [X_i, Y_i],
\] (2.9)
where \( X_i \in \mathcal{X}(\mathcal{M}) \) and \( Y_i \in \mathcal{X}_0(\mathcal{M}) \). Moreover, if \( \text{supp} \, X \subset U \), \( U \) open in \( M \), then \( X_i, Y_i \) can be chosen so that \( \text{supp} Y_i \subset \text{supp} \, X \) and \( \text{supp} (\pi X_i) \subset U \) for any \( i \). In particular, the derived algebra \( \mathcal{X}'(\mathcal{M}) = [\mathcal{X}(\mathcal{M}), \mathcal{X}(\mathcal{M})] \) equals \( \mathcal{X}(\mathcal{M}) \).

**Proof.** Using the theorem stating that any smooth supermanifold \( \mathcal{M} = (M, \mathcal{A}) \) is diffeomorphic to the supermanifold \( \Pi V \) for some vector bundle \( V \) over \( M \), we can assume that \( \mathcal{M} = \Pi V \). The algebras of functions and of vector fields of supermanifolds of the type \( \Pi V \) carry a \( \mathbb{Z}_2 \)-compatible \( N \)-grading. We denote the parity-supergrading by subscripts and the \( N \)-grading by superscripts. The grading is recognized by the Euler vector field \( \varepsilon \), in the sense that \( [\varepsilon, X] = nX \) for \( X \in \mathcal{X}_n(\Pi V) \).

Let us recall that supervector fields of degree 0 which, in local coordinates \((x, \xi)\), read

\[
X = \sum_i X^i(x) \partial_{x^i} + \sum_{a,b} X^a_b(x) \xi^b \partial_{\xi^a},
\]

can be identified with the sections of the Atiyah algebroid of \( V \). This identification is a Lie algebra isomorphism, so that

\[
0 \to \ker \pi \to \mathcal{X}^0(\Pi V) \xrightarrow{\pi} \mathcal{X}(\mathcal{M}) \to 0
\]

is a split short exact sequence of Lie algebras, so that \( \mathcal{X}^0(\Pi V) = \ker \pi \oplus \mathcal{X}(\mathcal{M}) \).

Note that in coordinates

\[
\pi \left( \sum_i X^i(x) \partial_{x^i} + \sum_{a,b} X^a_b(x) \xi^b \partial_{\xi^a} \right) = \sum_i X^i(x) \partial_{x^i}
\]

and the Euler vector field reads \( \sum_a \xi^a \partial_{\xi^a} \).

We first prove the theorem for \( X \in \mathcal{X}(\mathcal{M}) \), \( \text{supp} \, X \subset U \). According to [Gr93, Theorems (3.2) and (4.1)],

\[
X \in [\mathcal{X}(\mathcal{M}), [\mathcal{X}(\mathcal{M}), X]],
\]

so that there are \( X_i, Y'_i \in \mathcal{X}(\mathcal{M}) \) such that

\[
X = \sum_i [X_i, [Y'_i, X]].
\]

Putting \( Y_i = [Y'_i, X] \), we can write \( X \) in the form (2.10) with \( \text{supp} \, Y_i = [Y'_i, X] \subset \text{supp} \, X \). Let us observe that this is also true for any \( X \in \mathcal{X}(\Pi V) \). Indeed, if \( X \in \mathcal{X}^k(\Pi V) \), \( k \neq 0 \), then

\[
X = \frac{1}{k} [\varepsilon, X],
\]

where \( \varepsilon \) is the Euler vector field. If \( X \in \mathcal{X}^0(\Pi V) \), \( X \in \ker \pi \), then \( X \) vanishes on \( C^\infty(M) \). It is well known that there are vector fields \( Z_i \in \mathcal{X}(\mathcal{M}) \) and smooth functions \( f_i \in C^\infty(M) \) such that \( \sum_i (Z_i)(f_i) = 1 \) [Gra78]. As \( X(f_i) = 0 \) by assumption, we can write

\[
X = \left( \sum_i Z_i(f_i) \right) X = \sum_i ([Z_i, f_i X] - [f_i Z_i, X])
\]
and it is clear that \( \text{supp } f_i X \subset \text{supp } X \). We can therefore assume the decomposition \((2.9)\) with \( \text{supp } Y_i \subset \text{supp } X \). Consider now a function \( \psi \in C^\infty(M) \) with the support in \( U \) that takes value 1 on \( \text{supp } X \). We have
\[
\sum_i [\psi X_i, Y_i] = \sum_i (\psi [X_i, Y_i] \pm (Y_i \psi)) X_i = \psi \sum_i [X_i, Y_i] = \psi X = X,
\]
since \( Y_i(\psi) = 0 \), as \( \psi = 1 \) on the support of \( Y_i \).

Our next goal is to explain how the supermanifold structure of \( \mathcal{M} \) is encoded in the Lie superalgebra structure of \( \mathcal{D}^1(M) \) and, further, to describe all the automorphisms of this superalgebra. More precisely, we will show that every Lie superalgebra isomorphism \( \mathcal{D}^1(M_1) \rightarrow \mathcal{D}^1(M_2) \), where the dimension of manifolds is different from 0, respects the filtration \( \mathcal{D}^0(M_i) \subset \mathcal{D}^1(M_i), \ i = 1, 2 \) and covers a supermanifold diffeomorphism \( \mathcal{M}_1 \rightarrow \mathcal{M}_2 \). We prove that the group of automorphisms of the Lie superalgebra \( \mathcal{D}^1(M) \) is a semi-direct product of the subgroup induced by supermanifold diffeomorphisms and another subgroup which consists of automorphisms determined by a 1-cocycle on the Lie superalgebra of vector fields \( \mathcal{X}(M) \) with values in the algebra of functions on \( \mathcal{M} \).

**Remark 4.** In contrast with even manifolds, the Lie superalgebra structure on the Lie superalgebra of all differential operators \( \mathcal{D}(M) \) for a general supermanifold \( \mathcal{M} \) does not recognize \( \mathcal{D}(M) \) as the Lie superalgebra of superdifferential operators on \( \mathcal{M} \): there are Lie superalgebra isomorphisms \( \mathcal{D}(M_1) \rightarrow \mathcal{D}(M_1) \) not respecting the canonical filtration of differential operators \( \mathcal{D}^k(M) \subset \mathcal{D}^{k+1}(M), k \geq 0 \). This is because, in the pure odd situation, any linear operator on superfunctions is a differential operator. For instance, let \( \mathcal{M} \) be \( \Pi V \) for a finite-dimensional vector space \( V \). Then \( \mathcal{D}(M) = \text{End}(\Lambda^*(V)) \) where \( \Lambda^*V \) is viewed as a supervector space with the canonical \( \mathbb{Z}_2 \)-grading. Indeed, let \( \xi^i \) be linear coordinates on \( V \) counted as odd variables. It is clear that \( \mathcal{D}(M) \) is generated as an associative superalgebra by \( \xi^i \) and \( \partial \xi^i \) subject to the Clifford relations
\[
\xi^i \partial \xi^j + \partial \xi^i \xi^j = \delta^i_j.
\]
Thus \( \mathcal{D}(M) \) is isomorphic as an associative superalgebra (correspondingly, as a Lie superalgebra) to the Clifford algebra (correspondingly, the underlying Lie superalgebra) of \( V \oplus V^* \) supplied with the canonical pairing. The latter coincides up to an isomorphism with the superalgebra of all linear endomorphisms of the spinor module \( \Lambda^*V \). On the other hand, \( V \) can be replaced with any maximal isotropic subspace of \( V \oplus V^* \). For instance, one can interchange \( V \) and \( V^* \); this will produce a Lie superalgebra automorphism of \( \mathcal{D}(M) \) which apparently breaks the filtration. Therefore the automorphism have not so nice geometrical description that is available for the Lie superalgebra of first-order differential operators.

### 3 Algebraic characterization of functions

This section provides a Lie algebraic characterization of superfunctions inside first-order superdifferential operators. In what follows, \( \mathcal{A}, \mathcal{X}, \mathcal{D}^1 \) denote the algebras of
superfunctions, supervector fields, first-order superdifferential operators of a smooth supermanifold \( \mathcal{M} \) of dimension \( p|q \).

**Remark 5.** In this paper we can assume that the odd dimension \( q \) of \( \mathcal{M} \) is at least \( 1 \) – otherwise we investigate the already studied purely even situation [GP04].

**Theorem 2.** If \( \dim \mathcal{M} \) differs from \( 0|1 \), the algebra \( \mathcal{A} \subset \mathcal{D}^1 \) is the unique maximal super Lie ideal of \( \mathcal{D}^1 \) consisting of ad-nilpotent elements. In particular, any automorphism of the Lie superalgebra \( \mathcal{D}^1 \) preserves \( \mathcal{A} \), thus induces an automorphism of the Lie superalgebra \( \mathcal{X} = \mathcal{D}^1/\mathcal{A} \).

**Proof.** Apparently, \( \mathcal{A} \subset \mathcal{D}^1 \) is an ideal made up by ad-nilpotent elements. It thus suffices to prove that any another ideal with the same property is contained in \( \mathcal{A} \).

First we identify \( \mathcal{D}^1/\mathcal{A} \) as a Lie superalgebra with \( \mathcal{X} \) via the canonical isomorphism \( \mathcal{D}^1/\mathcal{A} \ni [D] \to D - D1 \in \mathcal{X} \). Further, we choose any diffeomorphism of supermanifolds between \( \mathcal{M} \) and \( IV \), where \( V \) is a certain vector bundle \( V \to M \) of rank \( q \). This choice provides a non canonical isomorphism between the superalgebras of functions, \( \mathcal{A} \) and \( \Gamma(\wedge V^*) \), which implements a \( \mathbb{Z}_2 \)-compatible \( \mathbb{N} \)-grading on \( \mathcal{A} \). The adjoint action of the corresponding derivation or Euler vector field \( \epsilon \in \text{Der}_0 \mathcal{A} = \mathcal{X}_0 \) supplies \( \mathcal{X} \) with a \( \mathbb{Z}_2 \)-compatible \( \mathbb{Z} \)-grading \( \mathcal{X}^k \) = \( \{ X \in \mathcal{X} : [\epsilon, X] = kX \} \), \( k \in \mathbb{Z} \), \( k \geq -1 \) [GKP09].

Suppose now that \( \mathcal{J} \) is a super Lie ideal of \( \mathcal{D}^1 \) made up by ad-nilpotent elements and denote by \( p : \mathcal{D}^1 \to \mathcal{D}^1/\mathcal{A} \simeq \mathcal{X} \) the canonical surjective Lie superalgebra morphism. Then \( \mathcal{I} := p(\mathcal{J}) \) is a super Lie ideal of \( \mathcal{X} \) whose elements are ad-nilpotent as well and that is moreover \( \mathbb{Z} \)-graded with respect to \( \epsilon \). To explain the last claim it suffices to prove that, if an element \( X \in \mathcal{I} \subset \mathcal{X} \) decomposes as \( X = \sum_{j=1}^s X_{kj} \), \( X_{kj} \in \mathcal{X}^{k_i} \), \( i > j \Rightarrow k_i > k_j \), then \( X_{kj} \in \mathcal{I} \) for all \( j \). As \( Y^m := \text{ad}_m^\epsilon X \in \mathcal{I} \) for all \( m \), we obtain

\[
\sum_{j=1}^s k_j^m X_{kj} = Y^m \in \mathcal{I}, \quad m = 0, \ldots, s - 1.
\]

This linear system can be inverted, since the corresponding Vandermonde matrix is nondegenerate, so that all \( X_{kj} \) are actually in \( \mathcal{I} \). Furthermore, it follows from the proof of [GKP09 Proposition 2] that \( \mathcal{I} \cap \mathcal{X}^0 = \{ 0 \} \). Indeed, if \( X \in \mathcal{I} \cap \mathcal{X}^0 \), then \( X \in \mathcal{I}_0 \), which is a Lie ideal of \( \mathcal{X}_0 \) that is made up by ad-nilpotent elements. The mentioned proof then implies that \( X \in \mathcal{X}^0 \cap \oplus_{i>0} \mathcal{X}^{2i} \), so that \( X = 0 \).

Let now \( X \) be a nonvanishing element of \( \mathcal{I} \) and let \( X_k \in \mathcal{X}^k \cap \mathcal{I} \) be one of its nonvanishing \( \mathbb{Z} \)-homogeneous terms. It is always possible to find \( Y_1, \ldots, Y_r \in \mathcal{X} \) such that

\[
[Y_1, \ldots, [Y_r, X_k], \ldots] \in \mathcal{X}^0 - \{ 0 \}.
\] (3.1)

Indeed, it is easily seen that \( X_k \) must be 0, if the preceding multibracket vanishes, for \( X_k \in \mathcal{X}^k \), \( k \geq 1 \), (resp. for \( X_k \in \mathcal{X}^{-1} \)) and for all \( Y_i \in \mathcal{X}^{-1} \) (resp. \( Y_i \in \mathcal{X}^1 \)) (a problem arises only if \( \dim \mathcal{M} = 0|1 \) and the degree of \( X \) is \(-1 \), as then \( \mathcal{X}^1 = \{ 0 \} \)). Since the multibracket (3.1) is again in \( \mathcal{I} \), we get a contradiction with the fact that \( \mathcal{I} \cap \mathcal{X}^0 = \{ 0 \} \). Therefore \( p(\mathcal{J}) = \mathcal{I} = \{ 0 \} \), so that eventually \( \mathcal{J} \subset \mathcal{A} \).
Remark 6. The assumption that the dimension of $\mathcal{M}$ differs from $0|1$ is crucial. Indeed, for a manifold of dimension $0|1$ there are automorphisms of the Lie algebra $\mathcal{D}^1$ not preserving $\mathcal{A}$. For, let $\xi$ be an odd coordinate on $\mathcal{M}$. Then, $\mathcal{D}^1$ is spanned by superfunctions $1, \xi$ and supervector fields $\partial_{\xi}, \xi\partial_{\xi}$. It is easy to see that the linear map on $\mathcal{D}^1$ for which $\xi\partial_{\xi} \mapsto -\xi\partial_{\xi}$, $\partial_{\xi} \leftrightarrow \xi$, $1 \mapsto 1$ is an automorphism.

4 Reduction of the automorphism-problem

In this section we fix a supermanifold $\mathcal{M}$ of dimension different from $0|1$ and we reduce the quest for the automorphisms of the Lie algebra of first-order superdifferential operators to the computation of the even 1-cocycles of the Lie algebra of supervector fields valued in the associative algebra of superfunctions.

Clearly, any diffeomorphism $\varphi$ of the supermanifold $\mathcal{M}$ induces an automorphism of the Lie superalgebra of first-order differential operators of $\mathcal{M}$. Indeed, any $\varphi \in \text{Diff}(\mathcal{M})$ defines in particular an associative superalgebra automorphism $\varphi^* : \mathcal{A} \rightarrow \mathcal{A}$ and the induced Lie superalgebra automorphism is given by

$$\varphi^* : \mathcal{D}^1 \ni D \mapsto \varphi^{*^{-1}} \circ D \circ \varphi^* \in \mathcal{D}^1.$$  

In the following we use the canonical splitting $\mathcal{D}^1 = \mathcal{A} \oplus \mathcal{X}$ and we denote the projection $\mathcal{D}^1 \ni D \mapsto D - D1 \in \mathcal{X}$ by $\rho$. Using the fact that, for $D \in \mathcal{D}^1$ and $f \in \mathcal{A}$, we have $[D, f] = Df - (D1)f$, we immediately check that $\rho$ is a representation by derivations of the Lie superalgebra $\mathcal{D}^1$ on the supervector space $\mathcal{A}$.

Proposition 1. Any Lie superalgebra automorphism $\phi : \mathcal{D}^1 \rightarrow \mathcal{D}^1$ splits into a product $\varphi_\ast \circ \phi_c$, where $\varphi_\ast$ is the automorphism induced by a diffeomorphism $\varphi : \mathcal{M} \rightarrow \mathcal{M}$, and where the automorphism $\phi_c$ has the form $\phi_c = \text{id} + c$, with $c$ being an even 1-cocycle of $\mathcal{D}^1$ represented upon $\mathcal{A}$ by $\rho$.

Proof. It follows from Theorem 2 that any automorphism $\phi : \mathcal{D}^1 \rightarrow \mathcal{D}^1$ preserves $\mathcal{A}$. Thus $\phi$ induces a Lie superalgebra automorphism

$$\tilde{\phi} : \mathcal{X} \ni X \mapsto \phi(X) - \phi(1)X \in \mathcal{X}. \quad (4.1)$$

All automorphisms of $\mathcal{X}$ are classified in [GKP09], see Theorem 2 and Proposition 5. It follows from the proof of Proposition 5 that they are implemented by a diffeomorphism of $\mathcal{M}$, unless $\dim \mathcal{M} = 1|1$ or $0|2$. In the latter case there exist additional automorphisms. Let us assume that $\dim \mathcal{M} = 1|1$ or $0|2$. Then the Euler vector field $\epsilon \in \mathcal{X}_0$ is well-defined up to a sign, such that any automorphism of $\mathcal{X}$, which is not coming from a diffeomorphism of $\mathcal{M}$, interchanges $\epsilon$ with $-\epsilon$. Now we can make a canonical choice of the Euler vector field $\epsilon$ such that $\epsilon$ acts on $\mathcal{A}_1$ as the identity. Apparently, any automorphism of $\mathcal{X}$, which is induced by an automorphism of $\mathcal{D}^1$, must preserve the canonically chosen $\epsilon$. Thus, in both exceptional (low-dimensional) cases an automorphism of $\mathcal{X}$ which exchanges $\epsilon$ and $-\epsilon$ does not
admit an extension to first-order differential operators. Therefore, each \( \tilde{\phi} \) is always induced by a diffeomorphism \( \varphi: M \to M \).

Let us take a diffeomorphism \( \varphi \) of \( M \), such that
\[
\tilde{\phi} : \mathcal{X} \ni X \mapsto \varphi^{-1} \circ X \circ \varphi^* \in \mathcal{X}.
\]
The automorphism \( \tilde{\psi} = \tilde{\varphi}^{-1} \circ \tilde{\phi} \) of \( \mathcal{X} \), induced by the automorphism \( \psi := \varphi^{-1} \circ \phi \) of \( D^1 \), is the identity map. Hence, for \( X \in \mathcal{X} \), we have \( \psi(X) = X + \psi(X)1 \) and, for \( D = f + X \in D^1 \), we thus get \( \psi(D) = f + X + \psi(X)1 + \psi(f) - f = D + c(D) \), where \( c : D^1 \to A \) is an even linear map. The injectivity and surjectivity of \( \psi \) are equivalent to the corresponding property of \( (\text{id} + c)|_A \). To prove that the surjectivity of this restriction implies that of \( \psi \), it suffices, if \( \Delta = g + Y \in D^1 \), to set \( g - c(Y) = f + c(f) \), for some \( f \in A \), and to observe that \( \psi(f + Y) = \Delta \). Eventually, the Lie algebra homomorphism property of \( \psi = \text{id} + c \) is clearly equivalent with the 1-cocycle condition of \((D^1, \rho)\) for \( c \).

**Theorem 3.** The automorphism \( \phi_c \) can be uniquely written in the form
\[
\phi_c(f + X) = (\kappa f + \gamma(X)) + X,
\]
where \( \kappa \) is a non-zero constant and \( \gamma : \mathcal{X} \to A \) is an even 1-cocycle of the Lie superalgebra \( \mathcal{X} \) canonically represented on \( A \).

**Proof.** Note first that, in view of the preceding proof, any even 1-cocycle \( c \) of the Lie superalgebra \( D^1 \) represented by \( \rho \) on \( A \), such that \((\text{id} + c)|_A \) is bijective, defines an automorphism \( \phi_c = \text{id} + c \). The 1-cocycle condition obviously splits into the intertwining condition
\[
c(Xf) = X(c(f)), \quad \forall X \in \mathcal{X}, f \in A,
\]
and the 1-cocycle condition of the Lie superalgebra \( \mathcal{X} \) canonically represented upon \( A \). We prove below in Lemma 1 that the intertwining condition means that \( c|_A = \lambda \cdot \text{id}, \lambda \neq -1 \), where the exclusion of the value \(-1\) is due to the condition that \((\text{id} + c)|_A \) be a bijection. Thus,
\[
\phi_c(f + X) = f + X + c(f) + c(X) = ((1 + \lambda)f + c(X)) + X
\]
and it is easy to see that \( \gamma : \mathcal{X} \to A, \gamma = c|_\mathcal{X} \), is a 1-cocycle with coefficients in the canonical representation. \( \square \)

**Lemma 1.** Any even linear map \( c : A \to A \) that verifies the intertwining condition
\[
c(Xf) = X(c(f))
\]
for all \( X \in \mathcal{X} \) and \( f \in A \) is of the form \( c(f) = \lambda f, \lambda \in \mathbb{R} \).

**Proof.** Like in the proof of Theorem 1, we can assume that \( M = \text{IV} \), where \( V \) is a vector bundle over \( M \), and consider vector fields \( Z_i \in \mathcal{X}(\text{IV}) \) and functions \( f_i \in C^\infty(M) \), such that \( \sum_i Z_i(f_i) = 1 \). Since
\[
f = f \cdot 1 = \sum_i (fZ_i)(f_i),
\]
we see that \( c(f) = f \sum_i Z_i(c(f_i)) \). Let us put \( \lambda = \sum_i Z_i(c(f_i)) \in A \). As \( \lambda = c(1) \) and for each \( X \in \mathcal{X} \) we have \( X(\lambda) = X(c(1)) = c(X(1)) = 0 \), the superfunction \( \lambda \) is actually a constant. \( \square \)
Remark 7. The additional automorphisms of the Lie superalgebra $\mathcal{X}(\mathcal{M})$ in the exceptional cases can be described as follows. In the first case, when $\dim \mathcal{M} = 1|1$, the supermanifold $\mathcal{M}$ is isomorphic to $\Pi L$ for a real line bundle $L \to M$ and $A_1 \simeq \Gamma(L^*)$. Taking into account that the structure group $L$ can be reduced to $O(1) = \mathbb{Z}_2$, one can always choose a trivialization $\sigma_0$ of $L^\otimes 2$ and a flat connection $\nabla$ on $L$ (and thus a flat connection on $L^\otimes 2$, denoted by the same letter) such that $\nabla(\sigma_0) = 0$. On the other hand, $M$ is either $\mathbb{R}$ or $S^1$, so $M$ is always orientable.

Let $\mu$ be a volume form on $M$, then $\sigma_0 \otimes \mu$ determines a bundle isomorphism $L^* \otimes TM \to L$, which gives rise to a bundle automorphism $\chi$ of $L \oplus L^* \otimes TM$, interchanging $L$ and $L^* \otimes TM$ such that $\chi^2 = \text{Id}$. Taking into account that $\chi_1$ is isomorphic to $\Gamma(L \oplus L^* \otimes TM)$ as a vector space, we obtain an invertible linear map $\chi_1 \to \chi_1$. The choice of a flat connection $\nabla$ determines a Lie algebra isomorphism $\chi_0 \simeq D^1(M)$, where $D^1(M)$ is the Lie algebra of first-order differential operators on $M$. The commutator relations in $\mathcal{X}$ are given by the following formulas:

$$
[v + f, s] = \nabla_v(s) - fs
$$

$$
[v + f, \theta \otimes v'] = \nabla_v(\theta) \otimes v' + \theta \otimes [v, v'] + f \theta \otimes v',
$$

$$
[s, \theta \otimes v'] = \langle \theta, s \rangle v' + \langle \theta, \nabla_v s \rangle,
$$

where $v, v' \in \Gamma(TM)$, $s \in \Gamma(L)$, $\theta \in \Gamma(L^*)$, and $f \in C^\infty(M)$. Apparently, $\chi_1$ is a faithful $\chi_0$-module. The linear invertible map $\chi$ determines a Lie algebra automorphism of $\chi_0$ of the form $v + f \mapsto v + div_\mu(v) - f$, where $div_\mu(v) = L_\nu(\mu)^{-1}$. Using appropriate local supercoordinates $(t, \xi)$, such that $\mu = dt$, $\nabla(\xi) = 0$, and $\sigma_0 = \xi^\otimes(-2)$, we can write the whole transformation in the following form (here $h$ and $f$ are arbitrary local smooth functions):

$$
h(t)\partial_t \mapsto h(t)\xi\partial_t,
$$

$$
h(t)\xi\partial_t \mapsto h(t)\partial_t,
$$

$$
h(t)\partial_t + f(t)\xi\partial_t \mapsto h(t)\partial_t + (\partial_t h(t) - f(t))\xi\partial_t.
$$

It is easy to verify that such a transformation is a Lie superalgebra automorphism. In fact, the group of supermanifold diffeomorphisms of $\mathcal{M}$ acts freely and transitively on the set of automorphisms which interchanges $\epsilon$ and $-\epsilon$, so a particular choice of $\chi$ is nothing but the choice of "an origin".

In the second case, when $\dim \mathcal{M} = 0|2$, the supermanifold $\mathcal{M}$ is isomorphic to $\Pi V$ for a 2-dimensional vector space $V$. The even part of $\mathcal{X}$ is naturally isomorphic to $\mathfrak{gl}(V)$, while the odd part is isomorphic to $V \oplus \Lambda^2 V^* \otimes V$. Let us fix a constant volume form $c \in \Lambda^2 V^*$. The Lie algebra $\mathfrak{gl}(V)$ is a direct sum of $\mathfrak{sl}(V)$ and the center of $\mathfrak{gl}(V)$ spanned by $\text{Id}$. The subalgebra $\mathfrak{sl}(V)$ preserves $c$, which makes $V$ and $\Lambda^2 V^* \otimes V$ into isomorphic $\mathfrak{sl}(V)$-modules. Let us combine the isomorphism $V \overset{c}{\rightarrow} \Lambda^2 V^* \otimes V$ with the Lie algebra isomorphism of $\mathfrak{gl}(V)$ which preserves all elements of $\mathfrak{sl}(V)$ and interchanges $\text{Id}$ and $-\text{Id}$ (such an automorphism is a composition of a conjugation and the opposite to a transposition). One can easily check that the obtained linear map $\mathcal{X} \to \mathcal{X}$ is a Lie superalgebra automorphism.
5 Cohomology of supervector fields represented on functions

Let $\gamma : X(M) \to A(M)$ be an even 1-cocycle, so that, for any $X, Y \in X(M)$,
\[
\gamma([X, Y]) = X(\gamma(Y)) - (-1)^{|X||Y|}Y(\gamma(X)). \tag{5.1}
\]

**Proposition 3.** Every even 1-cocycle of the Lie superalgebra $X(M)$ represented upon $A(M)$ is a local operator.

**Proof.** Let $X \in X(M)$ such that $X|_U = 0$, $U \subset M$ open, and let $x_0 \in U$. According to Theorem II, the vector field $X$ reads $X = \sum_i [X_i, Y_i]$, for some $X_i \in X(M)$ and some $Y_i \in X_0(M)$ which are 0 in a neighbourhood of $x_0$ as well. Hence, in view of the cocycle condition,
\[
\gamma(X) = \sum_i \left( X_i(\gamma(Y_i)) - (-1)^{|X_i||Y_i|}Y_i(\gamma(X_i)) \right),
\]
$\gamma(X) = 0$ in a neighbourhood of $x_0$. \hfill $\square$

The next result gives the local form of the even 1-cocycles of $X(M)$ with coefficients in $A(M)$. We will denote by $\Omega^1(U)$, $U \subset M$ open, the super $A(U)$-module of superdifferential 1-forms over $U$ and $\Omega^1_0(U)$ will refer to its even part.

**Theorem 4.** Let $(U, u = (u^1, \ldots, u^{p+q}))$ be any coordinate chart of $M$. The restriction to $U$ of any even 1-cocycle $\gamma : X(M) \to A(M)$ is of the form
\[
\gamma|_U \left( \sum_k \partial_k \cdot g^k \right) = \sum_k \left( a \partial_k g^k + \omega_k g^k \right), \tag{5.2}
\]
where $\partial_k = \partial_{u^k}$, $g^k \in A(U)$, $a \in \mathbb{R}$, and $\omega = \sum_k du^k \omega_k$ is a closed even 1-form.

**Remark 8.** As in this text we use the Deligne sign convention for the wedge product, the super de Rham operator $d$ has parity 0.

**Proof.** Due to locality, the restriction $\gamma|_U$ is an even 1-cocycle of $X(U)$ valued in $A(U)$. In the following we often omit the restriction to $U$ and write simply $\gamma$, $X$, $A$, etc. We will of course show that $\gamma$ is a differential operator.

When looking at the cocycle condition for $X = u^i \partial_j$ and $Y = g \partial_k$, where $g \in A$, we are naturally led to introduce the map
\[
\gamma_k : A \ni g \mapsto (-1)^{|u^k||g|} \gamma(g \partial_k) = \gamma(\partial_k \cdot g) \in A.
\]
The cocycle equation then means that the map
\[
g \mapsto \gamma_k(u^i \partial_j g) - \delta^i_k \gamma_j g - (-1)^{|u^k|(|u^i| + |u^j|)} u^i \partial_j (\gamma_k g) \tag{5.3}
\]
is a differential operator of order 0 and parity $|u^i| + |u^j| + |u^k|$. Similarly, taking $X = \partial_j$ and $Y = g \partial_k$, we get that the map
\[
g \mapsto \gamma_k(\partial_j g) - (-1)^{|u^k||u^j|} \partial_j (\gamma_k g) \tag{5.4}
\]
is a differential operator of order 0 and parity \(|u^i| + |u^k|\). Thus, subtracting from operator (5.3) the operator (5.4) multiplied from the left by \((-1)^{|u^k||u^i|}u^j\), we obtain that

\[ T_{k,i,j} = [\gamma_k, u^i] \circ \partial_j - \delta_k^i \gamma_j \tag{5.5} \]

is a differential operator of order 0 and parity \(|u^i| + |u^j| + |u^k|\).

If \(i \neq k\), then \(T_{k,i,j}\) reduces to \([\gamma_k, u^i] \circ \partial_j\). The latter is of order 0 and vanishes on constants, so \([\gamma_k, u^i] \circ \partial_j = 0\), for all \(i \neq k\) and all \(j\). This in turn implies that

\[ [\gamma_k, u^i] = 0 \quad \text{for} \quad i \neq k. \tag{5.6} \]

Indeed, if there exists an even coordinate \(u^j\), we can integrate classical functions with respect to \(u^j\) and the result is obvious. In the pure odd case, we have in particular \([\gamma_k, u^i] \circ \partial_i = 0\), \(i \neq k\), so that

\[ 0 = [\gamma_k, u^i](\partial_i(u^j g)) = [\gamma_k, u^i](g) + (-1)^{|u^j|}[\gamma_k, u^i](u^i \partial_i g), \tag{5.7} \]

for all \(g \in \mathcal{A}\). But, as easily checked, for odd \(u^i\), the supercommutator \([\gamma_k, u^i]\) supercommutes with the multiplication by \(u^i\), so that

\[ [\gamma_k, u^i] \circ (u^i \partial_i) = (-1)^{|u^j|(|u^k|+|u^i|)}u^i[\gamma_k, u^i] \circ \partial_i = 0, \]

and, according to Equation (5.7), \([\gamma_k, u^i] = 0\), which completes the proof of Equation (5.6).

For \(i = k\), the differential operator \(T_{i,i,j}\) reads

\[ T_{i,i,j} = [\gamma_i, u^i] \circ \partial_j - \gamma_j \]

and, since it is of order 0, we have \([T_{i,i,j}, u^j] = 0\), i.e.,

\[ [\gamma_i, u^i] + (-1)^{|u^j|}[\gamma_i, u^i], u^j] \circ \partial_j - [\gamma_j, u^j] = 0. \tag{5.8} \]

As for \(i \neq j\), the Jacobi identity and Equation (5.6) entail

\[ [[\gamma_i, u^i], u^j] = (-1)^{|u^j||u^i|}[[\gamma_i, u^i], u^j] = 0, \]

we get

\[ [\gamma_i, u^i] = [\gamma_j, u^j]. \tag{5.9} \]

Choosing \(i = j\) in Equation (5.8), we obtain \([[\gamma_i, u^i], u^j] \circ \partial_i = 0\), which implies

\[ [[\gamma_i, u^i], u^j] = 0, \tag{5.10} \]

if \(u^i\) is even. However, it suffices to develop the LHS of Equation (5.10) to conclude that the claim holds true for odd \(u^i\) as well. Therefore, when taking into account Equation (5.6), we finally get

\[ [[[\gamma_i, u^j], u^k] = 0, \tag{5.11} \]
for all $i, j, k$. The latter equation means that $\gamma_k$ are first-order differential operators (see Remark 2).

According to Equation (5.6), $\gamma_k$ commute with multiplication by $u^i$, $i \neq k$, so they are of the form $\gamma_k = a_k \partial_k + \omega_k$, $a_k, \omega_k \in A$. Since, for any first-order operator and any function, we have $[D, f] = Df - D1 \cdot f$, Equation (5.9) shows that $a_i = a_j =: a$. Finally,

$$\gamma \left( \sum_k \partial_k \cdot g^k \right) = \gamma \left( \sum_k (-1)^{|u_k|g^k} \partial_k \right) = \sum_k \left( a \partial_k g^k + \omega_k g^k \right).$$  (5.12)

Note now that $|a| = 0$ and $|\omega_k| = |u^k|$, since $\gamma_k$ has parity $|u^k|$. Starting from

$$[f \partial_i, g \partial_j] = f \partial_i g \partial_j - (-1)^{|u_i||f|+|u_j||g|} g \partial_j f \partial_i,$$

we straightforwardly see that the corresponding cocycle condition provides, after simplification, an identically vanishing bidifferential operator in $f$ and $g$. When writing that its coefficients vanish, we get

$$\partial_i a = 0, \quad (5.13)$$
$$\partial_j \omega_j - (-1)^{|u_i||u_j|} \partial_j \omega_i = 0, \quad (5.14)$$

for all $i, j$. We thus conclude that the differential operator $\gamma$ defined by (5.12) is a 1-cocycle if and only if $a \in \mathbb{R}$ and the even superdifferential 1-form $\omega = \sum_k du^k \omega_k$ is closed.

It is easily checked that $\gamma = \gamma|_U$ defined by Equation (5.12) has the following property with respect to the right module structure of $\mathcal{X} = \mathcal{X}(U)$ over $\mathcal{A} = \mathcal{A}(U)$:

$$\gamma(X \cdot f) = \gamma(X) \cdot f + a X f. \quad (5.15)$$

**Proposition 4.** Any even 1-cocycle $\gamma : \mathcal{X}(\mathcal{M}) \to \mathcal{A}(\mathcal{M})$ verifies Equation (5.15), for all $X \in \mathcal{X}(\mathcal{M})$, all $f \in \mathcal{A}(\mathcal{M})$, and some $a \in \mathbb{R}$.

**Proof.** Indeed, the restriction $\gamma|_U$ to any chart domain $U$ is of the form (5.12) and thus satisfies the local equation (5.15) for some $a_U \in \mathbb{R}$. It follows that, for any global $X$ and $f$, we have

$$a_U(Xf)|_{U \cap V} = (\gamma(X \cdot f) - \gamma(X) \cdot f)|_{U \cap V} = a_V(Xf)|_{U \cap V},$$

where $V$ denotes a chart domain that intersects $U$. Since the base manifold $\mathcal{M}$ of $\mathcal{M}$ is connected, all $a_U$ coincide, which proves the claim.

Some authors, see e.g. [KSM02], define a divergence operator in $\mathcal{M}$ as an operator $\gamma : \mathcal{X}(\mathcal{M}) \to \mathcal{A}(\mathcal{M})$ that satisfies only (5.15) with $a = 1$. In this paper, we assume also the cocycle condition, which can be understood as a vanishing curvature condition for $\gamma$.

**Definition 1.** A divergence in a supermanifold $\mathcal{M}$ (or a superdivergence) is a 1-cocycle of $\mathcal{X}(\mathcal{M})$ valued in $\mathcal{A}(\mathcal{M})$ that satisfies Equation (5.15) with $a = 1$. Any even 1-cocycle will be called a generalized divergence.
Remark 9. We prove in the last section that in any smooth supermanifold there exists a divergence operator $\gamma_0$.

We are now able to describe the even part of the first cohomology group of the Lie superalgebra $\mathcal{X}(\mathcal{M})$ valued in $\mathcal{A}(\mathcal{M})$.

**Theorem 5.** Let $\mathcal{M}$ be a supermanifold with a fixed divergence $\gamma_0$. Any even 1-cocycle $\gamma : \mathcal{X}(\mathcal{M}) \to \mathcal{A}(\mathcal{M})$, i.e. any generalized divergence, can be uniquely written as

$$\gamma = a \gamma_0 + i \omega,$$

where $a \in \mathbb{R}$ and $\omega$ is a closed even 1-form on $\mathcal{M}$. The cocycle $\gamma$ is a coboundary if and only if $a = 0$ and $\omega = df$, $f \in \mathcal{A}_0(\mathcal{M})$. In other words,

$$H_0^1(\mathcal{X}(\mathcal{M}), \mathcal{A}(\mathcal{M})) = \mathbb{R} \gamma_0 \oplus H_{\text{DR},0}^1(\mathcal{M}),$$

where $H_{\text{DR},0}^1(\mathcal{M})$ is the even part of the first super de Rham cohomology group of $\mathcal{M}$.

**Proof.** If $\gamma$ is a generalized divergence, it verifies Equation (5.15) for some $a \in \mathbb{R}$. Therefore, the difference $\gamma - a \gamma_0$ is a right $\mathcal{A}(\mathcal{M})$-module morphism from $\mathcal{X}(\mathcal{M})$ to $\mathcal{A}(\mathcal{M})$, so an even super 1-form $\omega \in \Omega_0^1(\mathcal{M})$. The cocycle equation for $\omega = \gamma - a \gamma_0$ and Cartan’s formula for the super de Rham operator $d$ show that $d \omega = 0$. Furthermore, $a \gamma_0 + \omega = \gamma = df$, $f \in \mathcal{A}_0(\mathcal{M})$, if and only if $a = 0$ and $\omega = df$. □

It follows from Proposition 11 and Theorems 3 and 5 that for supermanifolds we get an analog of Theorem 8 from [GP01].

**Theorem 6.** Let $\mathcal{M}$ be a smooth supermanifold of dimension different from 0|1 and let $\gamma_0$ be a fixed divergence in $\mathcal{M}$. Then, an even linear map $\phi : \mathcal{D}^1(\mathcal{M}) \to \mathcal{D}^1(\mathcal{M})$ is an automorphism of the Lie superalgebra $\mathcal{D}^1(\mathcal{M}) = \mathcal{A}(\mathcal{M}) \oplus \mathcal{X}(\mathcal{M})$ of first-order superdifferential operators of $\mathcal{M}$ if and only if it can be written in the form

$$\phi(f + X) = \varphi_* (X) + (\varphi^{-1})^* (\kappa f + a \gamma_0 (X) + i \omega (X)),$$

where $\varphi$ is a diffeomorphism of $\mathcal{M}$, $a, \kappa \in \mathbb{R}$, $\kappa \neq 0$, and $\omega$ is a closed even 1-form. All the objects $\varphi, a, \kappa$, and $\omega$ are uniquely determined by $\phi$.

## 6 Existence of superdivergences

Our aim in this section is to prove the existence of a divergence on each supermanifold. This will be done similarly to the proof of existence of a divergence on any standard (even) manifold with the use of a nowhere-vanishing 1-density understood as a class of volume forms up to a sign. The sheaf of top forms $\Omega^{\text{top}}$ of a classical differential manifold has to be replaced, in the case of a supermanifold $\mathcal{M} = (\mathcal{M}, \mathcal{A})$, by the Berezinian sheaf $\text{Ber} = \text{Ber}(\mathcal{M})$ whose nowhere-vanishing sections are Berezinian volumes. A homogeneous Berezinian volume $s$ defines a divergence $\gamma_s$ of a homogeneous vector field $X$ by the formula (see [KSM02])

$$\mathcal{L}_X s = (-1)^{|X||s|} s \cdot \gamma_s (X).$$
Lie superalgebras of differential operators

where the contraction of a section of $\Lambda$ over $M$ both sides by reads

$$\text{Vol}(\text{super(divergence)} \gamma V)$$

In other words, we have an embedding $S$ of the space of sections of $\text{Vol}(V)$ into the space of Berezinian densities. In local coordinates this embedding reads

$$dx^1 \wedge \ldots \wedge dx^p \otimes e_q \wedge \ldots \otimes e_1 \mapsto dx^1 \wedge \ldots \wedge dx^p \otimes \partial_{q} \circ \ldots \circ \partial_{e_1},$$

where $e_1, \ldots, e_q$ is a basis of local sections of $V$ and $\partial_{e_i}$ is the derivation of $\mathcal{A}$ being the contraction of a section of $\Lambda V$ with $e_i$. A linear change of coordinates in the vector bundle $V$, say $(x, \xi) \mapsto (y(x), \eta)$ with $\eta_j = \sum_i a^j_i(x) \xi_i$, results in multiplying both sides by

$$\det \left( \frac{\partial y^a}{\partial x^b}(x) \right) \det \left( a^j_i(x) \right)^{-1},$$

so the embedding is well defined globally. Since the line bundles are classified by $H^1(M, \mathbb{Z}_2)$, we can find an open covering $(U_a)$ on $M$ and local nowhere-vanishing sections $v_\sigma$ of $\text{Vol}(V)|_{U_a}$ such that $v_\sigma = \pm v_{\sigma'}$ on $U_a \cap U_{\sigma'}$. Hence, $S(v_\sigma) = \pm S(v_{\sigma'})$ over $U_a \cap U_{\sigma'}$. But the divergence $\gamma_\sigma$ does not depend on the sign of $\sigma$, so the collection $\sigma$ of local nowhere-vanishing Berezinian volumes $S(v_\sigma)$ gives rise to a super(divergence) $\gamma_\sigma$ on $\mathcal{M}$. Note that the collection of local nowhere-vanishing Berezinian volumes $S(v_\sigma)$ can be viewed as a nowhere-vanishing section of the 1-density sheaf $\mathcal{D}_1 = \text{Ber} \otimes \text{or}(M)$ of $\mathcal{M}$, defined as the Berezinian sheaf twisted by the orientation sheaf of the body. In the literature, the sections of $\mathcal{D}_1$ are sometimes referred to as nonoriented Berezinian sections. If $\sigma' = f \sigma$, with an even nowhere-vanishing function $f$, is another such section, then $\gamma_{\sigma'}(X) = \gamma_\sigma(X) + X(\ln f)$. Thus we get the following.

**Theorem 7.** Let $\mathcal{M} = (M, \mathcal{A})$ be a smooth supermanifold. Then, there is a non-degenerate global section $\sigma$ of $\mathcal{D}_1$ with even coefficients. Moreover, for any such $\sigma$,

$$\gamma_\sigma : \mathcal{X} \ni X \mapsto (\mathcal{L}_X \sigma)\sigma^{-1} \in \mathcal{A}$$

is a divergence of $\mathcal{M}$. Any other such section of $\mathcal{D}_1$ implements a divergence in the same cohomology class.

**References**

[Ga77] K. Gawędzki. Supersymmetries-mathematics of supergeometry. *Ann. Inst. Henri Poincare* XXVII (1977), 335-366.
Lie superalgebras of differential operators

[Gr93] J. Grabowski. Ideals of the Lie algebras of vector fields revisited. *Suppl. Rend. Circ. Mat. Palermo, Ser. II* 32 (1993), 89-95.

[GKP09] J. Grabowski, A. Kotov, N. Poncin. The Lie Superalgebra of a Supermanifold. *Journal of Lie Theory.* 20 (2010), no.4, 739-749.

[GP04] J. Grabowski, N. Poncin. Automorphisms of quantum and classical Poisson algebras. *Compositio Math.* 140 (2004), 511-527.

[Gra78] J. Grabowski. Isomorphisms and ideals of the Lie algebras of vector fields. *Invent. Math.* 50 (1978), 13-33.

[KSM02] Y. Kosmann-Schwarzbach, J. Monterde. Divergence operators and odd Poisson brackets. *Ann. Inst. Fourier (Grenoble)* 52 (2002), no. 2, 419-456.

[Lei80] D. A. Leites. Introduction to the theory of supermanifolds. *Russian Math. Surveys* 35 (1980), no. 1, 1-64.

[Pon04] N. Poncin. Equivariant operators between some modules of the Lie algebra of vector fields. *Comm. Algebra* 32 (2004), no. 7, 2559–2572.

[V72] A. M. Vinogradov. The logic algebra for the theory of linear differential operators. *Soviet. Mat. Dokl.* 13 (1972), 1058-1062.

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