Graph Homomorphisms, Circular Colouring, and Fractional Covering by $H$-cuts

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Abstract

A graph homomorphism is a vertex map which carries edges from a source graph to edges in a target graph. The instances of the Weighted Maximum $H$-Colourable Subgraph problem (MAX $H$-COL) are edge-weighted graphs $G$ and the objective is to find a subgraph of $G$ that has maximal total edge weight, under the condition that the subgraph has a homomorphism to $H$; note that for $H = K_k$ this problem is equivalent to MAX $k$-CUT. Färnqvist et al. have introduced a parameter on the space of graphs that allows close study of the approximability properties of MAX $H$-COL. Specifically, it can be used to extend previously known (in)approximability results to larger classes of graphs. Here, we investigate the properties of this parameter on circular complete graphs $K_{p/q}$ where $2 \leq p/q \leq 3$. The results are extended to $K_4$-minor-free graphs and graphs with bounded maximum average degree. We also consider connections with Šámal’s work on fractional covering by cuts: we address, and decide, two conjectures concerning cubical chromatic numbers.

Keywords: graph $H$-colouring, circular colouring, fractional colouring, combinatorial optimisation

1 Introduction

Denote by $\mathcal{G}$ the set of all simple, undirected and finite graphs. A graph homomorphism from $G \in \mathcal{G}$ to $H \in \mathcal{G}$ is a vertex map which carries the edges in $G$ to edges in $H$. The existence of such a map will be denoted by $G \rightarrow H$. For a graph $G \in \mathcal{G}$, let $W(G)$ be the set of weight functions $w : E(G) \rightarrow \mathbb{Q}^+$ assigning weights to edges of $G$. Now, Weighted Maximum $H$-Colourable Subgraph (MAX $H$-COL) is the maximisation problem with

Instance: An edge-weighted graph $(G, w)$, where $G \in \mathcal{G}$ and $w \in W(G)$.

Solution: A subgraph $G'$ of $G$ such that $G' \rightarrow H$.

Measure: The weight of $G'$ with respect to $w$.

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Given an edge-weighted graph \((G, w)\), denote by \(mc_H(G, w)\) the measure of the optimal solution to the problem \(\text{MAX } H\text{-COL}\). Denote by \(mc_k(G, w)\) the (weighted) size of a largest \(k\)-cut in \((G, w)\). This notation is justified by the fact that \(mc_k(G, w) = mc_{K_k}(G, w)\). In this sense, \(\text{MAX } H\text{-COL}\) generalises \(\text{MAX } k\text{-CUT}\) which is a well-known and well-studied problem that is computationally hard when \(k > 1\). Since \(\text{MAX } H\text{-COL}\) is a hard problem to solve exactly, efforts have been made to find suitable approximation algorithms. Färnqvist et al. [2] introduce a method that can be used to extend previously known (in)approximability bounds on \(\text{MAX } H\text{-COL}\) to new and larger classes of graphs. For example, they present concrete approximation ratios for certain graphs (such as the odd cycles) and near-optimal asymptotic results for large graph classes. The fundament of this promising technique is the ability to compute (or closely approximate) a function \(s: \mathcal{G} \times \mathcal{G} \rightarrow \mathbb{R}\) defined as follows:

\[
s(M, N) = \inf_{G \in \mathcal{G}} \omega \in \mathcal{W}(G) \frac{mc_M(G, \omega)}{mc_N(G, \omega)}.
\]

(1)

It is not surprising that estimating \(s(M, N)\) is, in many cases, non-trivial. One way is to solve a certain linear program that we present in Section 2: the program can be tedious to write down since it is based on the structure of \(N\)’s automorphism group, and can be prohibitively large. Another way is to use the following lemma:

**Lemma 1.1** ([2]). Let \(M \rightarrow H \rightarrow N\). Then, \(s(M, H) \geq s(M, N)\) and \(s(H, N) \geq s(M, N)\).

It is apparent that in order to use this result effectively, we need a large selection of graphs \(M, N\) that are known to be close to each other with respect to \(s\). For the moment, the set of such examples is quite meagre. Hence, we set out to investigate how the function \(s\) behaves on certain classes of graphs. In Section 3, we will take a careful look at 3-colourable circular complete graphs and, amongst other things, find that \(s\) is constant between a large number of these graphs. Moreover, we will extend bounds on \(s\) to other classes of graphs using known results about homomorphisms to circular complete graphs; examples include \(K_4\)-minor-free graphs and graphs with bounded maximum average degree.

Yet another way of estimating the function \(s\) is to relate it to other graph parameters. In this vein, Section 4 is dedicated to generalising the work of Šámal [8, 9] on fractional covering by cuts to obtain a new family of ‘chromatic numbers’. This reveals that \(s(M, N)\) and the new chromatic numbers \(\chi_M(N)\) are closely related quantities, which provides us with an alternative way of computing \(s\). We also use our knowledge about the behaviour of \(s\) to disprove a conjecture by Šámal concerning the cubical chromatic number and, finally, we decide in the positive another conjecture by Šámal concerning the same parameter. We conclude the paper, in Section 5, by discussing open problems and directions for future research. To improve readability some proofs are deferred to the appendices.

**2 A Linear Program for \(s\)**

Färnqvist et al. [2] have identified an alternative expression for \(s(M, N)\) which depends on the automorphism group of \(N\). Let \(M\) and \(N \in \mathcal{G}\) be graphs and let \(A = \text{Aut}^*(N)\) be the (edge) automorphism group of \(N\), i.e., \(\pi \in A\) acts on \(E(N)\) by permuting the edges. Let \(\mathcal{W}(N)\) be the set of weight functions \(\omega \in \mathcal{W}(N)\) which satisfy \(\sum_{e \in E(N)} \omega(e) = 1\) and for which \(\omega(e) = \omega(\pi \cdot e)\) for all \(e \in E(N)\) and \(\pi \in \text{Aut}^*(N)\). That is, the weight functions in \(\mathcal{W}(N)\) are constant over the edges belonging to each orbit of \(\text{Aut}^*(N)\).

**Lemma 2.1** ([2]). Let \(M, N \in \mathcal{G}\). Then, \(s(M, N) = \inf_{\omega \in \mathcal{W}(N)} mc_M(N, w)\). In particular, when \(N\) is edge-transitive, \(s(M, N) = mc_M(N, 1/|E(N)|)\).
Lemma 2.1 shows that in order to determine \( s(M, N) \), it is sufficient to minimise \( mc_M(N, \omega) \) over \( \mathcal{W}(N) \), and it follows that \( s(M, N) \) can be computed by solving a linear program. For \( i \in \{1, \ldots, r\} \), let \( A_i \) be the orbits of \( \text{Aut}^*(N) \) and, for \( f : V(N) \rightarrow V(M) \), define

\[
f_i = |\{uv \in A_i \mid f(u)f(v) \in E(M)\}|.
\]

That is, \( f_i \) is the number of edges in \( A_i \) which are mapped to an edge in \( M \) by \( f \). The measure of a solution \( f \) when \( \omega \in \mathcal{W}(N) \) is equal to \( \sum_{i=1}^{r} \omega_i \cdot f_i \) where \( \omega_i \) is the weight of an edge in \( A_i \). Given an \( \omega \), the measure of a solution \( f \) depends only on the vector \( (f_1, \ldots, f_r) \in \mathbb{N}^r \). We call this vector the signature of \( f \). When there is no risk of confusion, we will let \( f \) denote the signature as well. Since we have seen that the measure of a solution only depends on its signature the solution space is taken to be the set of possible signatures

\[
F = \{ f \in \mathbb{N}^r \mid f \text{ is a signature of a solution to } (N, \omega) \text{ of } \text{MAX-M-COL} \}.
\]

The variables of the linear program are \( \omega_1, \ldots, \omega_r \) and \( s \), where \( \omega_i \) represents the weight of each element in the orbit \( A_i \) and \( s \) is an upper bound on the signatures measure.

\[
\begin{align*}
\min \ s \\
\sum_i f_i \cdot \omega_i \leq s & \quad \text{for each } (f_1, \ldots, f_r) \in F \\
\sum_i |A_i| \cdot \omega_i = 1 & \quad \text{and } \omega_i, s \geq 0
\end{align*}
\]

Gives a solution \( \omega, s \) to this program, \( \omega(e) = \omega_i \) when \( e \in A_i \) is a weight function which minimises \( mc_M(G, \omega) \). The value of this solution is \( s = s(M, N) \).

3 Solutions to \((\text{LP})\) for Circular Complete Graphs

A circular complete graph \( K_{p/q} \) is a graph with vertex set \( \{v_0, v_1, \ldots, v_{n-1}\} \) and edge set \( E(K_{p/q}) = \{v_i v_j \mid q \leq |i - j| \leq p - q\} \). This can be seen as placing the vertices on a circle and connecting two vertices by an edge if they are at a distance at least \( q \) from each other. A fundamental property of these graphs is that \( K_{p/q} \rightarrow K_{p'/q'} \) if \( p/q \leq p'/q' \). Due to this fact, when we write \( K_{p/q} \), we will assume that \( p \) and \( q \) are relatively prime. We will denote the orbits of the action of \( \text{Aut}^*(K_{p/q}) \) by \( A_c = \{v_i v_j \in E(K_{p/q}) \mid j - i \equiv q + c - 1 \pmod{p}\} \), for \( c = 1, \ldots, \lceil p/2q + 1 \rceil \). We finally note that a homomorphism from a graph \( G \) to \( K_{p/q} \) is called a (circular) \((p/q)\)-colouring of \( G \). More information on this topic can be gained from the book by Hell and Nešetřil [3] and from the survey by Zhu [10].

In this section we start out by investigating \( s(K_r, K_t) \) for rational numbers \( 2 \leq r < t \leq 3 \). In Section 3.1, we fix \( r = 2 \) and choose \( t \) so that \( \text{Aut}^*(K_t) \) has few orbits. We find some interesting properties of these numbers which lead us to look at the case \( r = 2 + 1/k \) in Section 3.2 Our approach is based on relaxing the linear program \((\text{LP})\) that was presented in Section 2 combined with arguments that our chosen relaxations in fact find the optimum in the original program.

3.1 Maps to \( K_2 \)

We consider \( s(K_2, K_t) \) for \( t = 2 + n/k \) with \( k > n \geq 1 \), where \( n \) and \( k \) are integers. The number of orbits of \( \text{Aut}^*(K_t) \) then equals \( \lceil (n + 1)/2 \rceil \). We choose to begin our study of \( s(K_2, K_t) \) using small values of \( n \). When \( n = 1 \), \( K_{2+1/k} \) is isomorphic to the cycle \( C_{2k+1} \). The value of \( s(K_2, C_{2k+1}) = 2k/(2k + 1) \), for \( k \geq 1 \) was obtained in [2]. Combined with the following result, where we set \( t = 2 + 2/(2k - 1) = \frac{4k}{2k - 1} \), this has an immediate and perhaps surprising consequence.
Proposition 3.1. Let \( k \geq 1 \) be an integer, then \( s(K_2, K_{\frac{4k}{2k+1}}) = \frac{2k}{2k+1} \).

Proof. Let \( V(K_{\frac{4k}{2k+1}}) = \{ v_0, v_1, \ldots, v_{4k-1} \} \) and \( V(K_2) = \{ w_0, w_1 \} \). We will present two maps \( f, h : V(K_{\frac{4k}{2k+1}}) \rightarrow V(K_2) \). \( f \) sends a vertex \( v_i \) to \( w_0 \) if \( 0 \leq i < 2k \) and to \( w_1 \) if \( 2k \leq i < 4k \). It is not hard to see that \( f = (4k - 2, 2k) \). The map \( h \) sends \( v_i \) to \( w_0 \) if \( i \) is even and to \( w_1 \) if \( i \) is odd. Then, \( h \) maps all of \( A_1 \) to \( K_2 \) but none of the edges in \( A_2 \), so \( h = (4k, 0) \). It remains to argue that these two solutions suffice to determine \( s \). But we see that any map \( g \) with \( g_2 > 0 \) must cut at least two edges in the even cycle \( A_1 \), leading to \( g_1 < 4k - 2 \), thus \( g \leq f \), componentwise. The proposition now follows by solving the relaxation of \( (LP) \) using only the two inequalities obtained from \( f \) and \( h \).

Corollary 3.2. Let \( k \geq 1 \) and \( 2 \leq r < \frac{2k+1}{k} \leq t \leq \frac{4k}{2k-1} \). Then, \( s(K_r, K_t) = \frac{2k}{2k+1} \).

Proof. Note that we have the chain of homomorphisms \( K_2 \rightarrow K_r \rightarrow K_{2k+1} \rightarrow K_t \rightarrow K_{\frac{4k}{2k+1}} \). By Lemma [1] we get \( s(K_r, K_{\frac{4k}{2k+1}}) \geq s(K_2, K_{\frac{4k}{2k+1}}) = \frac{2k}{2k+1} \). But since \( K_{\frac{4k}{2k+1}} \not\rightarrow K_r \), and \( K_{\frac{4k}{2k+1}} \) is edge-transitive with \( 2k+1 \) edges, \( s(K_r, K_{\frac{4k}{2k+1}}) \leq \frac{2k}{2k+1} \), and therefore \( s(K_r, K_{\frac{4k}{2k+1}}) = \frac{2k}{2k+1} \). Again by Lemma [1] we have \( s(K_2, K_{\frac{4k}{2k+1}}) = s(K_r, K_{\frac{4k}{2k+1}}) \geq s(K_r, K_t) \geq s(K_2, K_{\frac{4k}{2k+1}}) = \frac{2k}{2k+1} \).

We find that there are intervals \( I_k = \{ t \in \mathbb{Q} \mid 2 + 1/k \leq t \leq 2 + 2/(2k - 1) \} \) where \( s(t) = s(K_r, K_t) \) is constant. In Figure 1 the intervals are shown for the first few values of \( k \). The intervals \( I_k \) form an infinite sequence with endpoints tending to \( 2 \). Similar intervals appear throughout the space of circular complete graphs. More specifically, Färnqvist et al. [2] have shown that \( s(K_n, K_{2m-1}) = s(K_n, K_{2m}) \) for arbitrary integers \( n, m \geq 2 \). Furthermore, it can be proved that \( s(K_2, K_n) = s(K_{8/3}, K_{n}) \) for \( n \geq 3 \). Two applications of Lemma [1] now shows that \( s(K_r, K_t) \) is constant on the regions \( [2, 8/3) \times J_m \), where \( J_m = \{ t \in \mathbb{Q} \mid 2m - 1 \leq t \leq 2m \} \).

As we proceed with determining \( s(K_2, K_t) \) we can now, thanks to Corollary 3.2, disregard those \( t \) which fall inside these constant intervals. For \( t = 2 + 3/k \), we see that if \( k \equiv 0 \) (mod 3), then \( r \) is an odd cycle, and if \( k \equiv 2 \) (mod 3), then \( t \) is in \( I_{k+1} \). Therefore, we assume that \( t \) is of the form \( 2 + 3/(3k + 1) = \frac{6k+5}{3k+1} \) for an integer \( k \geq 1 \).

Proposition 3.3. Let \( k \geq 1 \) be an integer. Then, \( s(K_2, K_{\frac{6k+5}{3k+1}}) = \frac{6k^2 + 8k + 3}{6k^2 + 11k + 5} = 1 - \frac{3k+2}{(k+1)(6k+3)} \).

For \( t = 2 + 4/k \), we find that we only need to consider the case when \( k \equiv 1 \) (mod 4). We then have graphs \( K_t \) with \( t = 2 + 4/(4k + 1) = \frac{8k+4}{4k+1} \) for integers \( k \geq 1 \).

Proposition 3.4. Let \( k \geq 1 \) be an integer. Then, \( s(K_2, K_{\frac{8k+4}{4k+1}}) = \frac{8k^2 + 2k + 2}{8k^2 + 6k + 3} = 1 - \frac{4k+1}{(k+1)(8k+6)} \).

The expressions for \( s \) in Proposition 3.3 and 3.4 have some interesting similarities, but for \( n \geq 5 \) it becomes harder to pick out a suitable set of solutions which guarantee that the relaxation has the same optimum as \( (LP) \) itself. Using computer calculations, we have however determined the first two values \((k = 1, 2)\) for the case \( t = 2 + 5/(5k+1) \) and the first value \((k = 1)\) for the case \( t = 2 + 6/(6k+1) \).
\[
s(K_2, K_{17/6}) = 322/425 \quad s(K_2, K_{27/11}) = 5/6 \quad s(K_2, K_{20/7}) = 67/89 \quad (4)
\]
3.2 Maps to Odd Cycles

It was seen in Corollary 3.2 that \( s(K_r, K_t) \) is constant on the region \((r, t) \in [2, 2 + 1/k] \times I_k \). In this section, we will study what happens when \( t \) remains in \( I_k \), but \( r \) is set to \( 2 + 1/k \). A first observation is that the absolute jump of the function \( s(K_r, K_t) \) when \( r \) goes from being less than \( 2 + 1/k \) to \( r = 2 + 1/k \) must be largest for \( t = 2 + 2/(2k - 1) \). Let \( V(K_{2+2/(2k-1)}) = \{v_0, \ldots, v_{4k-1}\} \) and \( V(K_{2+1/k}) = \{w_0, \ldots, w_{2k}\} \). The map \( f(v_i) = w_i \) with the indices of \( w \) taken modulo \( 2k + 1 \) has the signature \( f = (4k - 1, 2k) \). Since the subgraph induced by the orbit \( A_1 \) is isomorphic to \( C_{4k} \), any map to an odd cycle must exclude at least one edge from \( \sigma \). It follows that \( f \) alone determines \( s \), and we can solve \( \text{LP} \) to obtain \( s(K_{2+1/k}, K_{2+2/(2k-1)}) = (4k - 1)/4k \). Thus, for \( r < 2 + 1/k \), we have

\[
s(K_{2+1/k}, K_{2+2/(2k-1)}) - s(K_r, K_{2+2/(2k-1)}) = (2k - 1)/4k(2k + 1)
\]  
(5)

Smaller \( t \) will be expressed as \( t = 2 + 1/(k - x) \), where \( 0 < x < 1/2 \). We will write \( x = m/n \) for positive integers \( m \) and \( n \) which implies the form \( t = 2 + n/(kn - m) \), with \( m < n/2 \). For \( m = 1 \), it turns out to be sufficient to keep two inequalities from \( \text{LP} \) to get an optimal value of \( s \). From this we get the following result:

**Proposition 3.5.** Let \( k, n \geq 2 \) be integers. Then, \( s(C_{2k+1}, K_{2(2(k-1)+n)}) = 4(2(k-1)+n)(4k-1)/(2(k-1)+n)(4k-1+4k-2) \).

There is still a non-zero jump of \( s(K_r, K_t) \) when we move from \( K_r < 2 + 1/k \) to \( K_r = 2 + 1/k \), but it is obviously smaller than that of \( \text{5} \) and tends to 0 as \( n \) increases. For \( m = 2 \), we have \( 2(kn - m) + n \) and \( kn - m \) relatively prime only when \( n \) is odd. In this case, it turns out that we need to include an increasing number of inequalities to obtain a good relaxation. Furthermore, we are not able to ensure that the obtained value is the optimum of the original \( \text{LP} \). We will therefore have to settle for a lower bound for \( s \). Explicit calculations have shown that, for small values of \( k \) and \( n \), equality holds in Proposition 3.6. We conjecture this to be true in general.

**Proposition 3.6.** Let \( k \geq 2 \) be an integer and \( n \geq 3 \) be an odd integer. Then,

\[
s(C_{2k+1}, K_{2(2k-2)+n}) \geq \frac{(2(kn - 2) + n)(\xi_n(4k - 1) + (2k - 1))}{(2(kn - 2) + n)(\xi_n(4k - 1) + (2k - 1)) + (4k - 2)(1 - \xi_n)},
\]  
(6)

where \( \xi_n = \alpha_1^{(n-1)/2} + \alpha_2^{(n-1)/2} \) / 4, and \( \alpha_1, \alpha_2 \) are the reciprocals of the roots of \( 4k - 2 \).  

3.3 Extending the Results

We will now take a look at one possible way of extending the results in the previous sections. To do this, we need to find graphs or classes of graphs we can homomorphically sandwich between graphs with known \( s \) value. Clearly, \( K_2 \) has a homomorphism to all non-empty graphs, and that if a graph \( G \) has circular chromatic number \( \chi_c(G) \leq r \) it has a homomorphism to \( K_r \). These facts, together with Lemma 3.4 combine into the following easily proved lemma:

**Lemma 3.7.** Let \( G \) be a non-empty graph with \( \chi_c(G) \leq r \). Then, \( s(K_2, G) \geq s(K_2, K_r) \). If, additionally, \( G \) has odd girth no greater than \( 2k + 1 \), then \( s(C_{2k+1}, G) \geq s(C_{2k+1}, K_r) \).

We can now make use of known results about bounds on the circular chromatic number for certain classes of graphs. Much of the extensive study conducted in this direction was instigated by the restriction of a conjecture by Jaeger [4] to planar graphs, which is equivalent to the claim that every
planar graph of girth at least $4k$ has a circular chromatic number at most $2 + 1/k$, for $k \geq 2$. The case $k = 1$ is Grötzsch’s theorem; that every triangle-free planar graph is 3-colourable. Currently, the best proven bound for when the circular chromatic number of a planar graph is guaranteed to be at most $2 + 1/k$ is $\frac{20k-2}{3}$ and due to Borodin et al. [1]. This result was used by Färmqvist et al. to achieve the bound $s(K_2, G) \leq \frac{4k}{4k+1}$ for planar graphs $G$ of girth at least $(40k - 2)/3$. Here, we significantly improve this bound by considering $K_4$-minor-free graphs, for which Pan and Zhu [5] have shown how their circular chromatic number is upper-bounded by their odd girth.

**Proposition 3.8.** Let $G$ be a $K_4$-minor-free graph, and $k \geq 1$ an integer. If $G$ has an odd girth of at least $6k - 1$, then $s(K_2, G) \leq \frac{4k}{4k+1}$. If $G$ has an odd girth of at least $6k + 3$, then $s(K_2, G) \leq \frac{4k+2}{4k+3}$.

Of course, it is a big limitation to only consider $K_4$-minor-free graphs. Almost all work on the circular chromatic number for planar graphs have focused on finding limits when $\chi_c(G) \leq 2 + 1/k$, that is, when there exists a homomorphism to the odd cycle $C_{2k+1}$. However, Corollary [22] implies that for two graphs $G$ and $H$, if $\chi_c(G) = 2 + 1/k$ and $\chi_c(H) = 2 + 2/(2k - 1)$ then $s(K_2, G) = s(K_2, H)$, so for our purposes it would be interesting to have more results when $\chi_c(G) \leq 2 + 2/(2k - 1)$. For general graphs, we can use results from Raspaud and Roussel [7] relating the circular chromatic number of graphs to their maximum average degree. Specifically, they show that for a general graph $G$ of girth at least 12, 11, or 10, its circular chromatic number is bounded from above by 8/3, 11/4, and 14/5, respectively, which translates into corresponding upper bounds 4/5, 17/22, and 16/21 on $s(K_2, G)$ (using Propositions [3-4] and Lemma [57]).

## 4 Fractional Covering by $H$-cuts

In the following, we slightly generalise the work of Šámal [8, 9] on fractional covering by cuts to obtain a complete correspondence between $s(H, G)$ and a family of ‘chromatic numbers’ $\chi_H(G)$ which generalise Šámal’s cubical chromatic number $\chi_d(G)$. The latter corresponds to the case when $H = K_2$. First, we recall the notion of a fractional colouring of a (hyper-) graph. Let $G$ be a (hyper-) graph with vertex set $V(G)$ and edge set $E(G) \subseteq \mathcal{P}(V(G)) \setminus \{\emptyset\}$. A subset $I$ of $V(G)$ is called independent in $G$ if no edge $e \in E(G)$ is a subset of $I$. Let $\mathcal{I}$ denote the set of all independent sets of $G$ and for a vertex $v \in V(G)$, let $\mathcal{I}_v$ denote all independent sets which contain $v$. Then, the fractional chromatic number $\chi_f(G)$ of $G$ is given by the linear program:

$$\begin{align*}
\text{Minimise} & \quad \sum_{I \in \mathcal{I}} f(I) \\
\text{subject to} & \quad \sum_{I \in \mathcal{I}_v} f(I) \geq 1 \quad \text{for all } v \in V(G), \\
\text{where} & \quad f : \mathcal{I} \to \mathbb{R}^+.
\end{align*}$$

(7)

The definition of fractional covering by cuts mimics fractional colouring, but replaces vertices with edges and independent sets with certain cut sets of the edges. Let $G$ and $H$ be undirected simple graphs and $f$ be an arbitrary vertex map from $G$ to $H$. The map $f$ induces a partial map from $E(G)$ to $E(H)$ and we will call the preimage of this map an $H$-cut in $G$. When $H$ is a complete graph $K_k$, this is precisely the notion of a $k$-cut. Let $\mathcal{C}$ denote the set of $H$-cuts in $G$ and for an edge $e \in E(G)$, let $\mathcal{C}(e)$ denote all $H$-cuts which contain $e$. The following definition is the generalisation of cut $n/k$-covers [9] to arbitrary $H$-cuts:

**Definition 4.1.** An $H$-cut $n/k$-cover of $G$ is a collection $X_1, \ldots, X_N$ of $H$-cuts in $G$ such that every edge of $G$ is in at least $k$ of them. The graph parameter $\chi_H$ is defined as:

$$\chi_H(G) = \inf\left\{ \frac{n}{k} \mid \text{there exists an } H\text{-cut } n/k\text{-cover of } G \right\}$$

(8)
By reasoning analogous to that of Šámal [9] Lemma 5.1.3, $\chi_H$ is also given by the following linear program:

\[
\begin{align*}
\text{Minimise} & \quad \sum_{X \in C} f(X) \\
\text{subject to} & \quad \sum_{X \in C(e)} f(X) \geq 1 \quad \text{for all } e \in E(G), \\
\text{where} & \quad f : C \to \mathbb{R}^+.
\end{align*}
\]

(9)

For $H = K_2$, an alternative definition of $\chi_H(G) = \chi_q(G)$ was obtained in [9] by taking the infimum (actually minimum due to the formulation in (9)) over $n/k$ for $n$ and $k$ such that $G \to Q_{n/k}$. Here, $Q_{n/k}$ is the graph on vertex set $\{0, 1\}^n$ with an edge $uv$ if $d_H(u, v) \geq k$, where $d_H$ denotes the Hamming distance. We generalise this family as well to produce a scale for each $\chi_H$. Namely, let $H^n_k$ be the graph on vertex set $V(H)^n$ and an edge between $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ when $|\{i \mid (u_i, v_i) \in E(H)\}| \geq k$. A moments thought shows that we can express $\chi_H$ as:

$$\chi_H(G) = \inf \{\frac{n}{k} \mid G \to H^n_k\}.$$

(10)

Šámal also notes that $\chi_q(G)$ is given by the fractional chromatic number of a certain hypergraph associated to $G$. For the general case, let $G'$ be the hypergraph obtained from $G$ by taking $V(G') = E(G)$ and letting $E(G')$ be the set of minimal subgraphs $S \subseteq G$ such that $S \not\rightarrow H$. A short argument shows that indeed $\chi_f(G') = \chi_H(G)$.

Finally, we can work out the correspondence to $s(H, G)$. Consider the dual program of (9):

\[
\begin{align*}
\text{Maximise} & \quad \sum_{e \in E(G)} g(e) \\
\text{subject to} & \quad \sum_{e \in X} g(e) \leq 1 \quad \text{for all } H\text{-cuts } X \in C, \\
\text{where} & \quad g : E(G) \to \mathbb{R}^+.
\end{align*}
\]

(11)

Let $s = \sum_{e \in E(G)} g(e)$ and make the substitution $g' = g/s$ in (11). Comparing with (LP), we have

$$\chi_H(G) = 1/s(H, G).$$

(12)

We now move on to address two conjectures by Šámal [9] on the cubical chromatic number $\chi_q = \chi_{K_2}$. In Section 4.1 we discuss an upper bound on $s$ which relates to the first conjecture, Conjecture 5.5.3 [9]. This is the suspicion that $\chi_q(G)$ can be determined by measuring the maximum cut over all subgraphs of $G$. We show that this is false by providing a counterexample from Section 5.1. We then consider Conjecture 5.4.2 [9], concerning “measuring the scale”, i.e., determining $\chi_q$ for the graphs $Q_{n/k}$ themselves. We prove that this conjecture is true, and state it as Proposition 4.2 in Section 4.2.

### 4.1 An Upper Bound on $s$

In Section 3 we obtained lower bounds on $s$ by relaxing the linear program (LP). In most cases, the corresponding solution was proven feasible for the original (LP), and hence optimal. Now, we take a look at the only known source of upper bounds for $s$.

Let $G, H \in \mathcal{G}$, with $G \rightarrow H$ and take an arbitrary $S$ such that $G \rightarrow S \rightarrow H$. Then, applying Lemma 1.1 followed by Lemma 2.1 gives

$$s(G, H) \leq s(G, S) = \inf_{w \in W(S)} \text{mc}_G(S, w) \leq \text{mc}_G(S, 1/|E(S)|).$$

(13)

When $G = K_2$ it follows that

$$s(K_2, H) \leq \min_{S \subseteq G} b(S),$$

(14)
where \( b(S) \) denotes the bipartite density of \( S \). Šámal [9] conjectured that this inequality, expressed on the form \( \chi_q(S) \geq 1/(\min_{G \subseteq G} b(S)) \), can be replaced by an equality. We answer this in the negative, using \( K_{11/4} \) as our counterexample. Lemma 4.4 with \( k = 1 \) gives \( s(K_2, K_{11/4}) = 17/22 \). If \( s(K_2, K_{11/4}) = b(S) \) for some \( S \subseteq K_{11/4} \) it means that \( S \) must have at least 22 edges. Since \( K_{11/4} \) has exactly 22 edges, then \( S = K_{11/4} \). However, a cut in a cycle must contain an even number of edges. Since the edges of \( K_{11/4} \) can be partitioned into two cycles, we have that the maximum cut in \( K_{11/4} \) must be of even size, hence \( |E(K_{11/4})| \cdot b(K_{11/4}) \neq 17 \). This is a contradiction.

### 4.2 Confirmation of a Scale

As a part of his investigation of \( \chi_q \), Šámal [9] set out to determine the value of \( \chi_q(Q_{n/k}) \). We complete the proof of his Conjecture 5.4.2 [9] to obtain the following result.

**Proposition 4.2.** Let \( k, n \) be integers such that \( k \leq n < 2k \). Then, \( \chi_q(Q_{n/k}) = n/k \) if \( k \) is even and \((n + 1)/(k + 1)\) if \( k \) is odd.

Šámal provides the upper bound and an approach to the lower bound using the largest eigenvalue of the Laplacian of a subgraph of \( Q_{n/k} \). The computation of this eigenvalue boils down to an inequality (Conjecture 5.4.6 [9]) involving some binomial coefficients. We first introduce the necessary notation and then prove the remaining inequality in Lemma 4.4 whose second part, for odd \( k \), corresponds to one of the formulations of the conjecture. Proposition 4.2 then follows from Theorem 5.4.7 [9] conditioned on the result of this lemma.

Let \( k, n \) be positive integers such that \( k \leq n \), and let \( x \) be an integer such that \( 1 \leq x \leq n \). For \( k \leq n < 2k \), let \( S_e(n, k, x) \) denote the set of all \( k \)-subsets of \( \{1, \ldots, n\} \) that have an odd number of elements in common with the set \( \{n - x + 1, \ldots, n\} \). Define \( S_o(n, k, x) \) analogously as the \( k \)-subsets with an even number of common elements. Let \( N_o(n, k, x) = |S_o(n, k, x)| \) and \( N_e(n, k, x) = |S_e(n, k, x)| \). Then,

\[
N_o(n, k, x) = \sum_{\text{odd } t} \binom{x}{t} \binom{n-x}{k-t}, \quad N_e(n, k, x) = \sum_{\text{even } t} \binom{x}{t} \binom{n-x}{k-t}.
\]  

(15)

When \( x \) is odd, the function \( f : S_e(2k, k, x) \to S_e(2k, k, x) \) given by the complement \( f(\sigma) = \{1, \ldots, n\} \setminus \sigma \) is a bijection. Since \( N_o(n, k, x) + N_e(n, k, x) = \binom{n}{k} \), we have

\[
N_o(2k, k, x) = N_e(2k, k, x) = \frac{1}{2} \binom{2k}{k}.
\]  

(16)

**Lemma 4.3.** Let \( 1 \leq x < n = 2k - 1 \) with \( x \) odd. Then, \( N_e(n, k, x) = N_e(n, k, x + 1) \) and \( N_o(n, k, x) = N_o(n, k, x + 1) \).

**Proof.** First, partition \( S_e(n, k, x) \) into \( A_1 = \{\sigma \in S_e(n, k, x) \mid n-x \notin \sigma\} \) and \( A_2 = S_e(n, k, x) \setminus A_1 \). Similarly, partition \( S_e(n, k, x + 1) \) into \( B_1 = \{\sigma \in S_e(n, k, x + 1) \mid n-x \notin \sigma\} \) and \( B_2 = S_e(n, k, x + 1) \setminus B_1 \). Note that \( A_1 = B_1 \). We argue that \( |A_2| = |B_2| \). To prove this, define the function \( f : \mathcal{P}(\{1, \ldots, n\}) \to \mathcal{P}(\{1, \ldots, n\}) \) by \( f(\sigma) = (\sigma \cap \{1, \ldots, n-x-1\}) \cup \{s-1 \mid s \in \sigma, s > n-x\} \), i.e., \( f \) acts on \( \sigma \) by ignoring the element \( n-x \) and renumbering subsequent elements so that the image is a subset of \( \{1, \ldots, n-1\} \). Note that \( f(A_2) = S_e(2k-2, k-1, x) \) and \( f(B_2) = S_o(2k-2, k-1, x) \). Since \( x \) is odd, it follows from (16) that \( |f(A_2)| = |f(B_2)| \). The first part of the lemma now follows from the injectivity of the restrictions \( f|_{A_2} \) and \( f|_{B_2} \). The second equality is proved similarly.
Lemma 4.4. Choose \( k, n \) and \( x \) so that \( k \leq n < 2k \) and \( 1 \leq x \leq n \). For odd \( k \),
\[
N_e(n, k, x) \leq \binom{n-1}{k-1} \quad \text{and for even } k, \quad N_o(n, k, x) \leq \binom{n-1}{k-1}. \tag{17}
\]

Proof. We will proceed by induction over \( n \) and \( x \). The base cases are given by \( x = 1, x = n, \) and \( n = k \). For \( x = 1 \), \( N_o(n, k, x) = \binom{n-1}{k-1} \) and \( N_e(n, k, x) = \binom{n-1}{k-1} \), where the inequality holds for all \( n < 2k \). For \( x = n \) and odd \( k \), we have \( N_e(n, k, x) = 0 \), and for even \( k \), we have \( N_o(n, k, x) = 0 \). For \( n = k \), \( N_e(n, k, x) = 1 - N_o(n, k, x) = 1 \) if \( x \) is even and 0 otherwise. Let \( 1 < x < n \) and consider \( N_e(n, k, x) \) for odd \( k \) and \( k < n < 2k - 1 \). Partition the sets \( \sigma \in S_e(n, k, x) \) into those for which \( n \in \sigma \) on the one hand and those for which \( n \not\in \sigma \) on the other hand. These parts contain \( N_o(n-1, k-1, x-1) \) and \( N_e(n-1, k, x-1) \) sets, respectively. Since \( k - 1 \) is even, and since \( k - 1 \leq n - 1 < 2(k - 1) \) when \( k < n < 2k - 1 \), it follows from the induction hypothesis that \( N_e(n, k, x) = N_o(n-1, k-1, x-1) + N_e(n-1, k, x-1) \leq \binom{n-2}{k-2} + \binom{n-2}{k-1} = \binom{n-1}{k-1} \). The case for \( N_o(n, k, x) \) and even \( k \) is treated identically.

Finally, let \( n = 2k - 1 \). If \( x \) is odd, then Lemma 4.3 is applicable, so we can assume that \( x \) is even. Now, as before \( N_e(2k-1, k, x) = N_o(2k-2, k-1, x-1) + N_e(2k-2, k, x-1) \leq \binom{2k-2}{k-2} + \binom{2k-3}{k-1} = \binom{2k-1}{k-1} \), where the first term is evaluated using (16). The same inequality can be shown for \( N_o(2k-1, k, x) \) and even \( k \), which completes the proof. \( \square \)

5 Conclusions and Open Problems

We have seen that for all integers \( k \geq 2 \), \( s(K_2, K_t) \) is constant on \( I_k \). It follows that our sandwich approach using Lemma 4.1 with \( M = K_2 \) and \( N = K_r \) can not distinguish between the class of graphs with circular chromatic number \( 2 + 1/k \) and the (larger) class with circular chromatic number \( 2 + 2/(2k - 1) \). As previously noted, Jaeger’s conjecture and subsequent research has provided partial information on the members of the former class. We remark that Jaeger’s conjecture implies a weaker statement in our setting. Namely, if \( G \) is a planar graph with girth greater than \( 4k \), then \( G \to C_k \) implies \( s(K_2, G) \geq s(K_2, C_k) = 2k/(2k + 1) \). Deciding this to be true would certainly provide support for the original conjecture, and would be an interesting result in its own right. Our starting observation shows that the slightly weaker condition \( G \to K_2 + 2/(2k-1) \) implies the same result.

When it comes to completely understanding how \( s \) behaves on circular complete graphs, even restricted to those between \( K_2 \) and \( K_3 \), there is still work to be done. For edge-transitive graphs \( K_t \), in our case the cycles and the complete graphs, it is not surprising that the expression \( s(K_r, K_t) \) assumes a finite number of values seen as a function of \( r \). Indeed, Lemma 4.1 says that \( s(K_r, K_t) = mc_{K_r}(K_t, 1/|E(K_t)|) \) which leaves at most \( |E(K_t)| \) values for \( s \). This produces a number of constant intervals which are partly responsible for the constant regions of Corollary 3.2 and the discussion following it. More surprising are the constant intervals that arise from \( s(K_r, K_{2r+2}/(2k-1)) \). They give some hope that the behaviour of \( s \) is possible to characterise more generally. One direction could be to identify additional constant regions, perhaps showing that they completely tile the entire space?

In Section 3 we generalised the notion of covering by cuts due to Šámal. By doing this, we have found a different interpretation of the \( s \)-numbers as an entire family of ‘chromatic numbers’. It is our belief that these alternate viewpoints can benefit from each other. The refuted conjecture in Section 3 is an immediate example of this. On the other hand, it would be interesting to determine when the generalised upper bound in (13) is tight. For \( H = K_2 \), the proof of Proposition 4.2 is precisely such a result for the graphs \( Q_{n/k} \), which is evident from studying the proof of Theorem 3.1. Following this, a natural step would be to calculate \( \chi_H(H^n) \) for more general graphs \( H \), starting with \( H = K_3. \)
It is fairly obvious that \( \text{Max H-COL} \) is a special case of the \textit{maximum constraint satisfaction} (Max CSP) problem; in this problem, one is given a finite collection of constraints on overlapping sets of variables, and the goal is to assign values from a given domain to the variables so as to maximise the number of satisfied constraints. By letting \( \Gamma \) be a finite set of relations, we can parameterise Max CSP with \( \Gamma \) (Max CSP(\( \Gamma \))) so that the only allowed constraints are those constructed from the relations in \( \Gamma \). By viewing a graph \( H \) as a binary relation, the problems Max CSP(\{\( H \}\)) and Max H-COL are virtually identical. Raghavendra \[6\] has presented an algorithm for Max CSP(\( \Gamma \)) based on semi-definite programming. Under the so-called unique games conjecture, this algorithm optimally approximates Max CSP(\( \Gamma \)) in polynomial-time, i.e. no other polynomial-time algorithm can approximate the problem substantially better. However, it is notoriously difficult to find out exactly how well the algorithm approximates Max CSP(\( \Gamma \)) for a given \( \Gamma \). It seems plausible that the function \( s \) can be extended into a function \( s' \) from pairs of sets of relations to \( \mathbb{Q}^+ \), and that \( s' \) can be used for studying the approximability of Max CSP by extending the approach in Färnqvist et al. \[2\]. This would constitute a novel method for studying the approximability of Max CSP — a method that, hopefully, may cast some new light on the performance of Raghavendra’s algorithm.

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APPENDIX

Let $0 < q \leq p$ be positive integers. We often assign names to the vertices, so that $V(K_{p/q}) = \{v_0, v_1, \ldots, v_{p-1}\}$. Then, we have $E(K_{p/q}) = \{v_i v_j \mid q \leq |i - j| \leq p - q\}$. Note that $K_{p/q}$ does not have any edges unless $p \geq 2q$, since the circular distance between two vertices is at most $p/2$.

For a fixed $p$, let $\delta(v_i, v_j) = j - i \pmod{p}$. $\delta(v_i, v_j)$ is then the directed circular distance (in positive direction) between $v_i$ and $v_j$. Furthermore let $\delta(v_i, v_j) = \min\{\delta(v_i, v_j), \delta(v_j, v_i)\}$. This is then the undirected circular distance. We do index arithmetics for circular complete graphs modulo $p$, e.g. $v_{-1} = v_{p-1}$. Even though $K_{2k+1/k}$ is isomorphic to $C_{2k+1}$, we distinguish them by letting $v_i v_j$ be an edge in $C_{2k+1}$ if $\delta_{2k+1}(v_i, v_j) = 1$, while $v_i v_j$ is an edge in $K_{2k+1/k}$ if $\delta_{2k+1}(v_i, v_j) = k$.

Let $M$ and $N$ be graphs and let $F$ be a set of signatures to $(N, \omega)$ of $\text{MAX } M$-$\text{COL}$. If $F' \subseteq F$ is a subset for which the relaxation of $\text{(LP)}$ has the same optimal solution as the original program, we will call $F'$ a complete set of signatures with respect to $(N, \omega)$ of $\text{MAX } M$-$\text{COL}$.

A Proofs of Results from Section 3.1

**Proposition 3.3**

**Proof.** Let $V(K_{6k+4}) = \{v_0, v_1, \ldots, v_{6k+4}\}$ and $V(K_2) = \{w_0, w_1\}$. Let $f$ be the solution with $f(v_i) = w_0$ if $0 \leq i < 3k + 3$ and $f(v_i) = w_1$ if $3k + 3 \leq i < 6k + 5$. From $A_1$ only the edges $v_0 v_{3k+1}$, $v_1 v_{3k+2}$ and $v_{3k+3} v_{6k+4}$ are mapped to a single vertex in $K_2$, so $f_1 = 6k + 2$. From $A_2$ only the edge $v_0 v_{3k+2}$ is mapped to a single vertex in $K_2$, so $f_2 = 6k + 4$. Thus, $f$ has the signature $f = (6k + 2, 6k + 4)$.

Note that since $6k + 5$ and $3k + 2$ are relatively prime, the edges of $A_1$, as well as $A_2$, form cycles of length $6k + 5$. Therefore, any solution which maps more than $6k + 2$ edges from $A_1$ to $K_2$ must map exactly $6k + 4$ to $K_2$. Let $g$ be such a solution. We will show that $g_2 = 2k + 2$. We may assume that $v_{3k+1} v_0$ is the edge in $A_1$ which is not mapped to $K_2$ by $g$. Note that, if $i \neq 3k + 1, 6k + 2$, then $v_i v_{i+3} v_{i+4}$ and $v_{i+3} v_{i+4} v_{i+1}$ are both mapped to $K_2$ by $g$ which implies that $g(v_i) = g(v_{i+3})$. Now, let $v_i v_{i+3}(k+1)$ be an edge in $A_2$ and let $S = \{l, l + 3, \ldots, l + 3k\}$. Then, this edge is mapped to $K_2$ by $g$, i.e., $g(v_i) \neq g(v_{i+3}(k+1))$ if and only if $\{3k + 1, 6k + 2\} \cap S \neq \emptyset$. Since $v_{3k+1}$ and $v_{6k+2}$ are adjacent in $A_1$, they can not both be in $S$. Therefore, there are $2 \cdot |S| = 2(k + 1)$ edges that are mapped to $K_2$ by $g$, so $g = (6k + 4, 2k + 2)$. We conclude that solving $\text{(LP)}$ with the inequalities obtained from $f$ and $g$ yields the correct value of $s$. \hfill $\Box$

**Proposition 3.4**

**Proof.** Let $V(K_{8k+5}) = \{v_0, v_1, \ldots, v_{8k+5}\}$ and $V(K_2) = \{w_0, w_1\}$. Define $f$ by $h(v_i) = w_0$ if $0 \leq i < 4k + 3$ and $f(v_i) = w_1$ if $4k + 3 \leq i < 8k + 6$. Here, the edges $v_0 v_{4k+1}$, $v_1 v_{4k+2}$, $v_{4k+3} v_{8k+4}$ and $v_{4k+4} v_{8k+5}$ in $A_1$ are mapped to a single vertex in $K_2$ by $f$. From $A_2$, $f$ maps edges $v_0 v_{4k+2}$ and $v_{4k+3} v_{8k+5}$ to a single vertex in $K_2$. Finally, $f$ maps all edges in $A_3$ to the edge in $K_2$. The signature of this solution is $f = (8k + 2, 8k + 4, 4k + 3)$.

Let $g$ be defined by

\[ g(v_0) = g(v_4) = \cdots = g(v_{8k+4}) = g(v_2) = \cdots = g(v_{4k-2}) = g(v_{8k+3}) = g(v_1) = \cdots = g(v_{4k-3}) = w_0 \]
and
\[ g(v_{4k+1}) = g(v_{4k+5}) = \cdots = g(v_{8k+5}) = g(v_3) = \cdots = g(v_{8k-1}) = g(v_{4k+2}) = g(v_{4k+6}) = \cdots = g(v_{8k+2}) = w_1. \]

From \( A_1 \) only the edges \( v_{4k+1}v_{8k+2} \) and \( v_{8k+3}v_{4k-2} \) are mapped to a single vertex in \( K_2 \). From \( A_2 \) we partition the edges which are mapped to the edge in \( K_2 \) by \( g \) into four sets, with \( k + 1 \) edges in each set. These are

- \( \{v_0v_{4k+2}, v_4v_{4k+6}, \ldots, v_{4k}v_{8k+2}\} \),
- \( \{v_{4k+2}v_{8k+4}, v_{4k+6}v_{2}, \ldots, v_{8k+2}v_{4k-2}\} \),
- \( \{v_{4k+1}v_{8k+3}, v_{4k+5}v_{1}, \ldots, v_{8k+1}v_{4k-3}\} \),
- \( \{v_{8k+3}v_{4k-1}, v_{1}v_{4k+3}, \ldots, v_{4k-3}v_{8k-1}\} \).

Finally, for \( A_3 \), \( g \) maps the \( k \) edges \( v_0v_{4k+3}, v_4v_{4k+7}, \ldots, v_{4k-4}v_{8k-1} \) as well as the \( k + 1 \) edges \( v_{4k+1}v_{8k+4}, v_{4k+5}v_{2}, \ldots, v_{8k+1}v_{4k-2} \) to the edge in \( K_2 \). In summary, \( g = (8k + 4, 4k + 4, 2k + 1) \).

The relaxation of (LP) corresponding to the two solutions \( f \) and \( g \) has the following solution:

\[ s = \frac{8k^2 + 6k + 2}{8k^2 + 10k + 3}, \quad \omega_1 = \frac{k}{8k^2 + 10k + 3}, \quad \omega_2 = \frac{1}{2(8k^2 + 10k + 3)}, \quad \omega_3 = 0. \]

We will now show that \( s, \omega_1, \omega_2 \) and \( \omega_3 \) is feasible in the original program. We will show that for all solutions \( h \), we must have \( h_2 \leq 8k + 4 \). We will also show that if \( h \) is such that \( h_1 = 8k + 4 \), then \( h_2 \leq 4k + 4 \). Finally, we will show that if \( h_1 = 8k + 6 \), then \( h_2 \) must be \( 0 \). In the final case, we note that \( \omega_1 \cdot h_1 + \omega_2 \cdot h_2 < s \).

The edges of \( A_2 \) connects vertices at a distance of \( 4k + 2 \). Since we have a common factor 2 in \( 4k + 2 \) and \( 8k + 6 \), the edges of \( A_2 \) consists of two odd cycles, each of length \( 4k + 3 \). Since a cut of a cycle must include an even number of edges, we can then at most have a solution that maps \( 8k + 4 \) edges to \( K_2 \).

For the second case, note that \( v_{i+4k+2} = v_{i+(2k+2)(4k+1)} \). This means that the shortest path between \( v_i \) and \( v_{i+4k+2} \) in \( A_1 \) is of length \( 2k + 2 \). The edge \( v_iv_{i+4k+2} \) is mapped to \( K_2 \) if and only if at least one edge in each of the paths from \( v_i \) to \( v_{i+4k+2} \) in \( A_1 \) is not mapped to \( K_2 \), since they are both of even length. If a solution \( h \) has \( h_1 = 8k + 4 \), only two edges from \( A_1 \) are not mapped to \( K_2 \). Therefore no more than \( 4k + 4 \) paths of length \( 2k + 2 \) can include at least one of these two edges, hence \( h_2 \leq 4k + 4 \).

Finally, if a solution \( h \) includes an edge from \( A_2 \) it means that \( h(v_i) \neq h(v_{i+4k+2}) \) for some \( i \). But since both paths from \( v_i \) to \( v_{i+4k+2} \) in \( A_1 \) are of even length, not all edges from \( A_1 \) can be mapped to \( K_2 \). So if \( h_2 > 0 \), then \( h_1 < 8k + 6 \).

\[ \square \]

**B Proof of Proposition 3.5**

The proof of Proposition 3.5 follows from Lemma B.1 and B.3 introduced and proved in this section.
**Proposition 3.5**

*Proof.* Let \( p = 2(kn - 1) + n \). From Lemma B.1 we get a solution \( f \), with

\[
f = (\alpha \cdot |A_1|, |A_2|, \ldots, |A_{\frac{2m+1}{2}}|),
\]
where \( \alpha = 1 - 1/p \). From Lemma B.3 we get another solution \( f' \), with

\[
f' = (|A_1|, \beta \cdot |A_2|, \ldots, \beta \cdot |A_{\frac{2m+1}{2}}|),
\]
where \( \beta = 1 - 2(2k - 1)/p \). The last constraint in (LP) can be written as

\[
\sum_{i \neq 1} \omega_k \cdot |A_i| = 1 - \omega_1 \cdot |A_1|.
\]

We now insert (20) into the inequalities obtained from \( f \) and \( f' \) to get the following relaxation of (LP):

\[
\begin{align*}
\omega_1 \cdot |A_1| \cdot (\alpha - 1) + 1 & \leq s \\
\omega_1 \cdot |A_1| \cdot (1 - \beta) + \beta & \leq s.
\end{align*}
\]

The solution to this is \( \frac{1-\alpha \beta}{1-\alpha - \beta} \), which yields the \( s \)-value in the proposition.

To show that this is optimal for the original program, let us consider the restriction of (LP) in which we force \( \omega_i = 0 \) for \( i = 3, \ldots, \lceil \frac{n+1}{2} \rceil \). Due to the second part of Lemma B.3 it suffices to keep the two inequalities from \( f \) and \( f' \) in the program. The equality constraint can now be written as

\[
\omega_2 \cdot |A_2| = 1 - \omega_1 \cdot |A_1|.
\]

By inserting (22) into the two remaining inequalities we again obtain (21). Thus, the solution to the relaxation gives the right value for \( s \).

*Lemma B.1.* Let \( k, n, m \) be integers with \( k, n \geq 2 \) and \( 1 \leq m \leq \min\{n/2, 2k+1\} \). Then, there exists a solution \( f \) to \((K_{kn-m+2m,n-m}, \omega)\) of MAX \( C_{2k+1} \)-COL with signature \((|A_1| - m, |A_2|, \ldots, |A_{\frac{2m+1}{2}}|)\).

*Proof.* Let \( V(K_{kn-m+2m,n-m}) = \{v_0, \ldots, v_{2kn-n-2m-1}\} \) and \( V(C_{2k+1}) = \{w_0, \ldots, w_{2k}\} \). The construction of \( f \) will depend on whether \( m \leq k \) or \( m > k \). When \( m \leq k \) we define \( f \) as follows.

\[
\begin{align*}
&f^{-1}(w_0) = \{v_0, v_1, \ldots, v_{n-1}\}, \\
&f^{-1}(w_2) = \{v_n, \ldots, v_{2n-1}\}, \\
&\quad \vdots \\
&f^{-1}(w_{2k-2m}) = \{v_{(k-m)n}, \ldots, v_{(k-m+1)n-1}\}, \\
&f^{-1}(w_{2k-2m+2}) = \{v_{(k-m+1)n}, \ldots, v_{(k-m+2)n-2}\}, \\
&\quad \vdots \\
&f^{-1}(w_{2k}) = \{v_{kn-m+1}, \ldots, v_{(k+1)n-m-1}\}, \\
&f^{-1}(w_1) = \{v_{(k+1)n-m}, \ldots, v_{(k+2)n-m-1}\}, \\
&\quad \vdots \\
&f^{-1}(w_{2k-2m-1}) = \{v_{(2k-2m-1)n-m}, \ldots, v_{(2k-2m)n-m-1}\}, \\
&f^{-1}(w_{2k-2m+1}) = \{v_{(2k-2m)n-m}, \ldots, v_{(2k-2m+1)n-m-2}\}, \\
&\quad \vdots \\
&f^{-1}(w_{2k-1}) = \{v_{2kn-2m+1}, \ldots, v_{(2k+1)n-2m-1}\}.
\end{align*}
\]
Note, in particular, that
\[ |f^{-1}(w_j)| = \begin{cases} n & \text{for } 0 \leq j \leq 2(k - m), \text{ and} \\ n - 1 & \text{for } 2(k - m) < j \leq 2k - 1. \end{cases} \]

When \( m > k \), we define \( f \) as follows:
\[
\begin{align*}
  f^{-1}(w_0) &= \{v_0, v_1, \ldots, v_{n-2}\}, \\
  f^{-1}(w_2) &= \{v_{n-1}, \ldots, v_{2n-3}\}, \\
  f^{-1}(w_{4k-2m}) &= \{v_{(2k-m)(n-1)}, \ldots, v_{(2k-m+1)(n-1)-1}\}, \\
  f^{-1}(w_{4k-2m+2}) &= \{v_{(2k-m+1)(n-1)}, \ldots, v_{(2k-m+2)(n-1)-2}\}, \\
  \vdots \\
  f^{-1}(w_{2k}) &= \{v_{k(n-1)-m+k+1}, \ldots, v_{(k+1)(n-1)-m+k-1}\}, \\
  f^{-1}(w_{1}) &= \{v_{(k+1)(n-1)-m+k}, \ldots, v_{(k+2)(n-1)-m+k-1}\}, \\
  \vdots \\
  f^{-1}(w_{4k-2m+1}) &= \{v_{(4k-2m)(n-1)-m+k}, \ldots, v_{(4k-2m+1)(n-1)-m+k-1}\}, \\
  f^{-1}(w_{4k-2m+3}) &= \{v_{(4k-2m+1)(n-1)-m+k}, \ldots, v_{(4k-2m+2)(n-1)-m+k-2}\}, \\
  \vdots \\
  f^{-1}(w_{2k-1}) &= \{v_{2k(n-1)-2m+2k}, \ldots, v_{(2k+1)(n-1)-2m+2k}\}.
\]

In this case,
\[ |f^{-1}(w_j)| = \begin{cases} n - 1 & \text{for } 0 \leq j < 2(2k - m + 1), \text{ and} \\ n - 2 & \text{for } 2(2k - m + 1) \leq j \leq 2k - 1. \end{cases} \]

Now, consider a vertex \( v_i \) with \( f(v_i) = w_j \). Take one edge \( v_iv_l \in A_j \cup \cdots \cup A_{\frac{n+1}{2}} \). Then,
\[ kn - m + 1 \leq \delta(v_i, v_l) \leq 2(kn - m) + n - (kn - m + 1) = kn - m + n - 1. \quad (23) \]

Let \( a = \min \{ h \mid f(v_h) = w_{j-1} \} \). That is, \( v_a \) is the vertex with lowest index which is mapped to \( w_{j-1} \). Furthermore let \( b = \max \{ h \mid f(v_h) = w_{j+1} \} \). We then have
\[ f(\{v_a, v_{a+1}, \ldots, v_{b-1}, v_b \}) = \{w_{j-1}, w_{j+1} \}. \]

We now want to show that \( l \in \{a, \ldots, b\} \). It will then follow that \( f(v_i) f(v_l) \in E(C_{2k+1}) \), i.e., all edges outside of \( A_l \) are mapped to an edge in \( C_{2k+1} \). To do this, we will show that \( \delta(v_i, v_a) \leq \delta(v_i, v_l) \).

First, we bound \( \delta(v_i, v_a) \) from above by taking a walk along the vertices between \( v_i \) and \( v_a \). We need to pass at most \( |f^{-1}(w_j)| - 1 \) vertices to enter the set \( f^{-1}(w_{j+2}) \). We then continue until \( f^{-1}(w_{2k-1}) \) or \( f^{-1}(w_{2k}) \) depending on the parity of \( j \). Our walk continues from \( f^{-1}(w_0) \) or \( f^{-1}(w_1) \) up until we come to the last vertex in \( f^{-1}(w_{j-3}) \). Finally we take one last step into \( f^{-1}(w_{j-1}) \) and reach \( v_a \). We have then passed
\[ \delta(v_i, v_a) \leq |f^{-1}(w_j)| - 1 + |f^{-1}(w_{j+2})| + |f^{-1}(w_{j+4})| + \ldots + |f^{-1}(w_{j-3})| + 1 \]
vertices. There are \( k \) sets among \( f^{-1}(w_j), \ldots, f^{-1}(w_{j-3}) \). When \( m \leq k \) each set has either \( n \) or \( n - 1 \) vertices. However, at most \( \left\lfloor \frac{2(k-m)+1}{2} \right\rfloor = k - m + 1 \) of them can contain \( n \) vertices. Thus,
\[ \delta(v_i, v_a) \leq k(n-1) + k - m + 1 = kn - m + 1. \]
In the case of \( m > k \), each set has either \( n - 1 \) or \( n - 2 \) vertices but at most \( \frac{2(2k - m + 1)}{2} = 2k - m + 1 \) of them can contain \( n - 1 \) vertices. Thus,
\[
\delta(v_i, v_a) \leq k(n - 2) + 2k - m + 1 = kn - m + 1.
\]
When bounding \( \delta(v_i, v_b) \) from below, we take a similar walk, but now we want to determine the fewest possible vertices we will pass. Therefore, we assume that we immediately move into the set \( f^{-1}(w_{j+2}) \) and will go all the way to the last vertex in \( f^{-1}(w_{j+1}) \). We have then passed a total of
\[
\delta(v_i, v_b) \geq |f^{-1}(w_{j+2})| + |f^{-1}(w_{j+4})| + \ldots + |f^{-1}(w_{j+1})|
\]
vertices. There are \( k + 1 \) sets among \( f^{-1}(w_{j+2}), \ldots, f^{-1}(w_{j+1}) \). When \( m \leq k \) at least \( \frac{2(k - m)}{2} = k - m \) of the sets has \( n \) vertices. Thus,
\[
\delta(v_i, v_b) \geq (k + 1)(n - 1) + k - m = kn - m + n - 1.
\]
In the case of \( m > k \), at least \( \frac{2(k - m + 1)}{2} = 2k - m + 1 \) has \( n - 1 \) vertices. Thus,
\[
\delta(v_i, v_b) \geq (k + 1)(n - 2) + 2k - m + 1 = kn - m + n - 1.
\]
Combining the lower and upper bounds with (23), we find that
\[
\delta(v_i, v_a) \leq kn - m + 1 \leq \delta(v_i, v_b) \leq kn - m + n - 1 \leq \delta(v_i, v_b),
\]
hence \( f(v_i)f(v_l) \in E(C_{2k+1}) \). Since \( v_i v_l \) was an arbitrary edge in \( A_2 \cup \ldots \cup A_{\left\lfloor \frac{m}{2} \right\rfloor} \), this implies that \( f_j = |A_j| \) for \( j > 1 \).

It remains to determine \( f_1 \). Recall that \( A_1 = \{v_i v_{i+k-n-m} | 0 \leq i < 2(k-n) + n \} \). As before, we want to check if \( \delta(v_i, v_a) \leq \delta(v_i, v_l) = kn - m < kn - m + n - 1 \leq \delta(v_i, v_b) \) to determine if \( f(v_i)f(v_l) \in E(C_{2k+1}) \). This means that \( f(v_i)f(v_l) \) is a non-edge in \( C_{2k+1} \) if and only if \( \delta(v_i, v_a) = kn - m + 1 \). This, in turn, can only happen if the walk from \( v_i \) to \( v_a \) passes all \( |f^{-1}(w_j)| - 1 \) of the vertices from \( f^{-1}(w_j) \) (excluding \( v_i \)). Thus, \( v_i \) has to be the vertex with the lowest index in \( f^{-1}(w_j) \). In total there are \( 2k + 1 \) such vertices, one for each vertex in \( C_{2k+1} \). Furthermore, it must be the case that the walk fully passes the \( k - m + 1 \) sets \( f^{-1}(w_0), f^{-1}(w_2), \ldots, f^{-1}(w_{2k-2m}) \) with \( n \) vertices in the case when \( m \leq k \) and the \( 2k - m + 1 \) sets \( f^{-1}(w_0), f^{-1}(w_2), \ldots, f^{-1}(w_{2(2k-m)-1}) \) with \( n - 1 \) vertices when \( m > k \). When \( m \leq k \) this happens precisely when \( j \) is odd and \( 2(k-m)+3 \leq j \leq 2k-1 \), i.e. \( m \) times. When \( m > k \) it happens precisely when \( j \) is odd and \( 2(k-m)+1 \leq j \leq 2k-1 \) which is also \( m \) times. In all cases, there will be \( m \) edges in \( A_1 \) which are not mapped to edges in \( E(C_{2k+1}) \) so \( f_1 = |A_1| - m \) which concludes the proof.

Let \( p \) and \( q \) be relatively prime and let \( V(K_{p/q}) = \{v_0, \ldots, v_{p-1}\} \). Define a function \( \tau : [p] \to [p] \) by letting \( \tau(i) = j \) if \( 0 \leq j < p \) and \( jq \equiv i \) (mod \( p \)). Note that \( \tau \) is a bijection on \( [p] \). We will think of \( \tau \) as indicating the length of a path (in the positive direction) from \( v_0 \) to \( v_j \) in the cycle \( A_1 \). We will denote the length from \( v_k \) to \( v_l \) in \( A_1 \) by \( \delta_{\tau}(v_k, v_l) = \tau(l) - \tau(k) \) taken modulo \( p \). Note that \( \delta_{\tau}(v_i, v_{i+a}) = \delta_{\tau}(v_0, v_a) \) for all integers \( i \). Closed and half-open intervals are defined by \( [v_a, v_b]_{\tau} := \{v_l \mid \delta_{\tau}(v_a, v_l) \leq \delta_{\tau}(v_a, v_b)\} \) and \( (v_a, v_b]_{\tau} := \{v_l \mid 0 < \delta_{\tau}(v_a, v_l) \leq \delta_{\tau}(v_a, v_b)\} \), respectively.

Let \( V(C_{2k+1}) = \{w_0, \ldots, w_{2k}\} \). Given a subset \( S \subseteq \{v_0, \ldots, v_{p-1}\} \), we will now describe a general construction of a solution \( f = f_S \) to an instance \((K_{p/q}, \omega)\) of \( \text{MAX } C_{2k+1} \cdot \text{COL} \). The idea is to map the nodes \( v_{\tau(i)} \) in order of increasing \( i \) starting by \( f(v_{\tau(0)}) = f(v_0) = w_0 \). We then map \( v_{\tau(i)} \) to a node adjacent to \( f(v_{\tau(i-1)}) \), picking one of the two possibilities depending on whether \( i + 1 \in S \).
or not. To give the formal definition, it will be convenient to introduce the rotation $\rho$ on $C_{2k+1}$ defined as $\sigma(w_i) = w_{i+1}$. We then have,

$$f(v_{r(i)}) = \begin{cases} w_0 & \text{when } i = 0, \\ \rho^{-1}(f(v_{r(i-1)})) & \text{when } i > 0 \text{ and } v_i \in S, \\ \rho(f(v_{r(i-1)})) & \text{when } i > 0 \text{ and } v_i \notin S. \end{cases}$$

Note that the last vertex to be mapped is $v_{r^{-1}(p-1)} = v_{p-q}$. If the created solution has $f(v_{p-q}) = w_1$ or $w_{2k}$, then $f_1 = |A_1|$, otherwise $f_1 = |A_1| - 1$. In the latter case, it does not matter whether $v_0 \in S$ or not and we can assume that $v_0 \notin S$. However, to maintain consistency in the case of $f_1 = |A_1|$, we want to have $v_0 \in S$ if $f(v_{p-q}) = w_1$ and $v_0 \notin S$ otherwise. Therefore, $v_0 \in S$ if and only if $f(v_{p-q}) = w_1$.

**Example 1.** The solution $f : V(K_{22/9}) \to V(C_5)$ with $S = \{v_{14}, v_1, v_{11}, v_20, v_7, v_{17}, v_4, v_{13}\}$ looks as follows.

| $f^{-1}(w_0)$ | $f^{-1}(w_1)$ | $f^{-1}(w_2)$ | $f^{-1}(w_3)$ | $f^{-1}(w_4)$ |
|-------------|-------------|-------------|-------------|-------------|
| $v_0$       | $v_9$       | $v_{18}$    | $v_5$       |
| $v_1$       | $v_{14}$    | $v_{10}$    | $v_{19}$    | $v_6$       |
| $v_{15}$    | $v_2$       | $v_7$       | $v_{20}$    |
| $v_{11}$    | $v_{12}$    | $v_{21}$    | $v_8$       |
| $v_{13}$    | $v_4$       | $v_{17}$    |

Note that the $v_i$ are mapped in the order $v_0, v_9, v_{18}, v_5, \ldots, v_{13}$. $S$ is given in the order in which the vertices appear along the $A_1$. To start, we let $f(v_0) = w_0$. Neither of $v_9, v_{18}$ or $v_5$ appear in $S$, so these are mapped consecutively. Then, we get to $v_{14}$ which is in $S$. Since $f(v_5) = w_3$ we let $f(v_{14}) = w_2$. Finally, $f(v_{13}) = w_0$ so the signature of $f$ has $f_1 = |A_1| - 1 = 21$.

We will now give some basic properties of the solutions created using this construction for the case when $p = 2(kn - m) + n$ and $q = kn - m$. We will from now on assume that $f_1 = |A_1|$. This occurs when the construction has an equal number of applications of $\rho$ and $\rho^{-1}$ modulo $2k + 1$. That is, when $|S| \equiv p - |S| \pmod{2k+1}$. Solving for $|S|$ we get:

$$|S| \equiv 2k + 1 - m \pmod{2k+1}. \quad (24)$$

Assume that $f(v_i) = w_j, f(v_{i'}) = w_{j'}$. Then, the index $j'$ is determined by $\delta_{r}(v_i, v_{i'})$ and $S \cap (v_i, v_{i'})_r$ as follows:

$$j' \equiv j + \delta_{r}(v_i, v_{i'}) - 2 \cdot |S \cap (v_i, v_{i'})_r| \pmod{2k+1}. \quad (25)$$

Relation (25) implies the following useful lemma:

**Lemma B.2.** $f(v_i)f(v_{i'}) \in E(C_{2k+1})$ iff $|S \cap (v_i, v_{i'})_r| \equiv (k + 1)(\delta_{r}(v_i, v_{i'}) \pm 1) \pmod{2k+1}$. 

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Lemma B.3. Let $k, n \geq 2$ be integers. There exists a solution $f$ to $(K_{2kn+2}, \omega)$ of $\text{MAX } C_{2k+1}$-COL with $f_1 = |A_1|$, and

$$f = \begin{cases} 
\left( |A_1|, |A_2| - 2(2k - 1), \ldots, |A_{2^r+1}| - 2(2k - 1) \right) & \text{if } n \text{ is odd,} \\
\left( |A_1|, |A_2| - 2(2k - 1), \ldots, |A_{2^r+2}| - 2(2k - 1), |A_{2^{r+1}}| - (2k - 1) \right) & \text{if } n \text{ is even.}
\end{cases}$$

Furthermore, for any other solution $g$, if $g_1 = |A_1|$, then $g_2 \leq f_2$, componentwise.

Proof. Let $p = 2(kn - 1) + n, q = kn - 1$, and $V(K_{p/q}) = \{v_0, \ldots, v_{p-1}\}$. The desired solution $f$ is obtained from the construction $f = f(S)$ with $S = [v_{r-1(p-2k+1)}, v_0]$. As required by (24), we have $|S| = p - (p - 2k + 1) + 1 = 2k$ so that $f_1 = |A_1|$. It remains to determine $f_c$ for $c > 1$.

Let $v_iv'_v \in A_c, c > 1$ be an edge. In order to count the edges only once, we will assume that $i' = i + q + (c - 1) \pmod{p}$. To be able to use the condition in Lemma B.2 we need to determine $\delta_r(v_i, v'_v)$. But, $\tau(i') = \tau(i) + 1 + q^{-1}(c - 1) \pmod{p}$, where $q^{-1} := p - 2k - 1$, the inverse of $q$ modulo $p$. We then obtain $\delta_r(v_i, v'_v) = \tau(i') - \tau(i)$ by reducing $1 + (p - 2k - 1)(c - 1)$ modulo $p$.

$$\delta_r(v_i, v'_v) = \begin{cases} 
1 & \text{if } c = 1, \\
-1 + (2k + 1)(n - c + 1) & \text{otherwise.}
\end{cases}$$

Assuming $c \neq 1$, we have two cases in Lemma B.2. We conclude that $f(v_i)f(v'_v) \in E(C_{2k+1})$ if and only if either

$$|S \cap (v_i, v'_v)| = 0 \quad \text{or} \quad |S \cap (v_i, v'_v)| = (k + 1)(-2) = 2k \pmod{2k + 1}. \quad (26)$$

In both cases the condition is equivalent to $v_i, v'_v \notin S \setminus \{v_0\}$. Therefore, the edges $v_iv'_v$ which are not mapped to an edge in $C_{2k+1}$ by $f$ are the ones with an endpoint in $S \setminus \{v_0\}$. (There are no edges with both endpoints in this set.) When $n$ is even and $c = n/2 + 1$, this number equals $|S \setminus \{v_0\}| = 2k - 1$. In all other cases, there are $2(2k - 1)$ such edges. The first part of the lemma follows.

For the second part, we pick an arbitrary solution $g$ and show that we can find at least $2(2k - 1)$ edges in $A_2$ which cannot be mapped to $C_{2k+1}$, provided that $g_1 = |A_1|$. It is easy to see that, up to rotational symmetry, a $g$ with $g_1 = |A_1|$ must be constructible by $g = g(S)$ for some $S$. We already know that such an $S$ must satisfy $|S| = 2k \pmod{2k + 1}$. This implies $|S| \geq 2k$. From $p \equiv 2k - 1 \pmod{2k + 1}$, we also see that we must have $|V(K_{p/q}) \setminus S| \geq 2k$. As argued before, an edge from $A_c$ is mapped to $C_{2k+1}$ if and only if one of the congruences in (26) holds. Since $|S| \equiv 2k \pmod{2k + 1}$, we can equivalently write this as $f(v_i)f(v'_v) \in E(C_{2k+1})$ if and only if either

$$|S \cap (v_i, v'_v)| = 2k \quad \text{or} \quad |S \cap (v_i, v'_v)| = 0 \pmod{2k + 1}. \quad (27)$$

Hence, either the intersection of $S$ with $(v_i, v'_v)$ is empty or the latter is a subset of the former. As the two cases can be treated identically, we assume, without loss of generality, that the intersection is empty. Note that $|(v_i, v'_v)| = 2k$. We will now determine $2(2k - 1)$ edges which can not be mapped to edges in $C_{2k+1}$. Let $v_{j_1}$ be the first vertex in $S$ encountered following $A_1$ from $v_i$ in the positive direction. Similarly, let $v_{j_2}$ be the first vertex in $S$ encountered following $A_1$ from $v'_v$ in the negative direction. Then, $v_{j_1}, v_{j_1+q} \in (v_{j_1}+(a+1)q+1,v_{j_1}+aq)r$, for $a = 0, \ldots, 2k - 2$, but $v_{j_1} \in S$ and $v_{j_1-q} \notin S$ by construction. Thus, from (27), the edges $v_{j_1+aq}v_{j_1+(a+1)q+1}$ can not be mapped to $C_{2k+1}$. In the other direction, we have $v_{j_2}, v_{j_2+q} \in (v_{j_2}+(a'+1)q+1,v_{j_2}+(a'+1)q+1)r$, for $a' = 0, \ldots, 2k - 2$, but $v_{j_2} \in S$ and $v_{j_2+q} \notin S$ by construction. From this we get another $2k - 1$ edges which can not be mapped to $C_{2k+1}$. Finally, we note that since $S \subseteq [v_{j_1},v_{j_2}]$ and $|S| \geq 2k$, the edges $v_{j_1+aq}v_{j_1+(a+1)q+1}$ and $v_{j_2+(1-a')q+1}v_{j_2-a'}q$ are distinct. This proves that $g_2 \leq f_2$. \[\square\]
Example 2. With \( k = 3 \) and \( n = 5 \) the solution \( f = f(S) \) to \((K_{33/14}, \omega)\) of MAX \( C_7\)-COL created as in Lemma B.3 with Example 2.

The proof of Proposition 3.6 follows from a series of lemmas. The function \( \delta_r \) and how it is used for constructing solutions is presented in Appendix B.

Lemma C.1. Let \( k \geq 2 \) be an integer, and \( n \geq 3 \) be an odd integer. Then, there exists a solution \( f \) to \((K_{2kn+n-4}, \omega)\) with the following signature:

\[
\begin{align*}
f_1 &= |A_1|, \\
f_{2i} &= |A_{2i}| - \left(\frac{n-1}{2} - i\right)(2k+1) - (4k-2) \quad \text{for } i = 1, 2, \ldots, \frac{n-3}{4}, \\
f_{2i+1} &= |A_{2i+1}| - (i - 1)(2k+1) - (4k-2) \quad \text{for } i = 1, 2, \ldots, \frac{n-1}{4}.
\end{align*}
\]

Proof. Let \( G = K_{2kn+n-4} \), \( V(G) = \{v_0, \ldots, v_{2kn+n-5}\} \) and \( V(C_{2k+1}) = \{v_0, \ldots, v_{2k}\} \). A solution \( f = f(S) \) with this signature is obtained with \( S = [v_{r-1}(2kn-n-2k-2), v_0] \). We then have \(|S| = (2kn + n - 4) - (2kn + n - 2k - 2) + 1 = 2k - 1\), so \( f_1 = |A_1| \) by (24).

An orbit \( A_{2i} \) includes edges which connects vertices at a distance \( kn - 2 + i - 1 \). \( \delta_r(v_j, v_{j+k+n+2i-3}) = (\frac{n-1}{2} -(i-1))(2k+1) - 1 \). Lemma B.2 then says that \( f(v_j)f(v_{j+k+n+2i-3}) \in E(C_{2k+1}) \) if and only if \(|S \cap (v_j, v_{j+k+n+2i-3}]| \equiv 0 \) or \( 2k \) (mod \( 2k+1 \)). That is, \(|S \cap (v_j, v_{j+k+n+2i-3}]| \) must be 0. This is the case only when

\[
\delta_r(v_0, v_j) \leq \delta_r(v_0, v_{j+k+n+2i-3}) < \delta_r(v_0, v_{r-1}(2kn+n-2k-2)),
\]

which implies

\[
\delta_r(v_0, v_j) \leq \delta_r(v_0, v_{r-1}(2kn+n-2k-2)) - \delta_r(v_j, v_{j+k+n+2i-3}) - 1 \\
\leq 2kn + n - 2k - 2 - \left((\frac{n-1}{2} -(i-1))(2k+1) - 1\right) \\
= 2kn + n - 4 - \left((\frac{n-1}{2} -(i-1))(2k+1) - (4k-2) - 1\right),
\]

which holds for exactly \( 2kn + n - 4 - \left((\frac{n-1}{2} -(i-1))(2k+1) - (4k-2)\right) \) vertices \( v_j \).

An orbit \( A_{2i+1} \) includes edges which connects vertices at a distance \( kn - 2 + 2i \). \( \delta_r(v_{j+k+n+2i-2}, v_j) = i(2k+1) - 1 \). Applying Lemma B.2 again asserts that \( S \cap (v_j, v_{j+k+n+2i-2}] \) must be empty. Thus,

\[
\delta_r(v_0, v_{j+k+n+2i-2}) \leq \delta_r(v_0, v_j) < \delta_r(v_0, v_{r-1}(2kn+n-2k-2)),
\]

which implies

\[
\delta_r(v_0, v_j) \leq \delta_r(v_0, v_{r-1}(2kn+n-2k-2)) - \delta_r(v_j, v_{j+k+n+2i-2}) - 1 \\
\leq 2kn + n - 2k - 2 - \left(i(2k+1) - 1\right) - \left((\frac{n-1}{2} -(i-1))(2k+1) - (4k-2) - 1\right),
\]

which holds for exactly \( 2kn + n - 4 - \left(i(2k+1) - 1\right)\) vertices \( v_j \).

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which holds for exactly $2kn + n - 4 - (i - 1)(2k + 1) - (4k - 2)$ vertices $v_j$.

Example 3. For $K_{31/13}$. The solution $f = f(S)$ as in Lemma C.1 with $k = 3$ and $n = 5$ has

$\begin{array}{cccccccc}
| v_0 | v_13 | v_26 | v_8 | v_21 | v_3 | v_16 |
|-----|-----|-----|-----|-----|-----|-----|
| v_29 | v_11 | v_24 | v_6 | v_19 | v_1 | v_14 |
| v_27 | v_9 | v_22 | v_4 | v_17 | v_30 | v_12 |
| v_25 | v_7 | v_20 | v_2 | v_15 | v_28 |
| v_18 | v_5 | v_23 | v_10 |
\end{array}$

The following technical lemma will prove useful in analysing the solutions in Lemma C.3. Some

Lemma C.2. Let $p, q, r, s$ be positive integers so that $r > 2p + q + s$ and $s \geq p$. Now consider
$r$ elements equidistantly placed on a circle, and select two sequences $P_1$ and $P_2$, each containing $p$
consecutive elements, with $q$ elements between them on one side and $r - 2p - q$ on the other side. Let
$\gamma(i)$ be the number of ways to select $s$ consecutive elements on the circle with exactly $i$ elements from
$P_1 \cup P_2$. Then, when $s \leq q$:

$$\gamma(i) = \begin{cases} 
r - 2p - 2s + 2 & \text{if } i = 0, \\
2s - 2p + 2 & \text{if } i = p, \\
0 & \text{if } i > p.
\end{cases}$$

when $s = q + p$:

$$\gamma(i) = \begin{cases} 
r - 3p - 2q + 1 & \text{if } i = 0, \\
2q + p + 1 & \text{if } i = p, \\
0 & \text{if } i > p.
\end{cases}$$

and when $s > q + p + 1$:

$$\gamma(i) = \begin{cases} 
r - 2p - q - s + 1 & \text{if } i = 0, \\
2q + 2 & \text{if } i = p, \\
2 & \text{if } i = p + 1, \\
0 & \text{if } i > 2p.
\end{cases}$$

Proof. Call the elements $\{0, \ldots, r - 1\}$. Suppose $P_1 = \{0, \ldots, p - 1\}$ and $P_2 = \{q + p, \ldots, q + 2p - 1\}$. For $s \leq q$. Then the sequences that starts with $p, \ldots, q + p - s$ as well as $q + 2p, \ldots, r - s$ are the
only ones that do not contain any element from $P_1 \cup P_2$ and that is $q + p + s + 1$ and $r - q - 2p - s + 1$
elements, and in total $r - 2p - 2s + 2$. To get $p$ elements we have the sequences that starts with $q - 2p - s, \ldots, q + p$ and $p - s, \ldots, 0$ as the only options, and that is $s - p + 1$ in both cases so $2s - 2p + 2$ in total. Also if $s \leq q$ clearly there is no sequence of length $s$ that includes element from
both $P_1$ and $P_2$.

For $s = q + p$, the ones starting with $q + 2p, \ldots, r - s$ are the only ones that do not contains any
element from $P_1 \cup P_2$ and that is $r - q - 2p - s + 1 = r - 3p - 2q + 1$ elements. To get $p$ elements we have
that for any sequence of length $s$ starting with an element $i \in P_1$ contains $i$ number of elements from
$P_2$ so all sequences starting with $r - q, \ldots, q + p$ contains $p$ elements from $P_1 \cup P_2$, and that is
and due to (24).

Proof. Let \( f = f(S) \) be a solution to \((K_{2kn+n-2m}, \omega)\) of \( \text{MAX } C_{2k+1} \text{-COL} \) with \( f_1 = |A_1| \) and where \( S = P_1 \cup P_2, \) and \( P_1 \cap P_2 = \emptyset, \) where \( P_1 = [v_a, v_{a+(2k+1)-2}] \) and \( P_2 = [v_b, v_{b+(2k+1)-2}] \) such that \( \min_{v \in P_1, v \notin P_2} \delta(v, v) = (u-1)(2k+1). \) Let \( A_c \) be the orbit consisting of edges \( v_1v_h \) with \( \delta(v_1, v_h) = g(2k+1) - 1. \) Then,

\[
\begin{align*}
    f_c &= \begin{cases} 
        |A_c| - (8k - 4) & \text{if } g < u, \\
        |A_c| - (4k - 2) & \text{if } g = u, \\
        |A_c| - (g - u)(2k + 1) - (6k - 5) & \text{if } g > u.
    \end{cases}
\end{align*}
\]

Proof. We can apply Lemma C.2 since we according to Lemma B.2 must have \(|S \cap [v_{l+1}, v_h]| = 0, 2k \) or \( 2k + 1. \) So for Lemma C.2 we have \( r = |A_c|, \) \( p = 2k, \) \( q = (u - 1)(2k + 1) \) and \( s = g(2k + 1) - 1. \) We see that \( s = p + q \) when \( g = u \) and when \( g < u \) then \( s < q \) and when \( g > u \) then \( s > q + p + 1. \) So all we have to do is for each case count \( \gamma(0) + \gamma(p) + \gamma(p) + 1. \)

Now it is possible to construct a series of signatures with solutions \( f(S) \) where \( S \) will have the properties sought after by Lemma C.3.

Lemma C.4. There exists a set of solutions \( F = \{f^i\}, i = 2, \ldots, \frac{n+1}{4} \) to \((K_{2kn+n-2m}, \omega)\) of the problem \( \text{MAX } C_{2k+1} \text{-COL} \) with signatures:

\[
\begin{align*}
    f^1_i &= |A_1| \forall f^i \in F \\
    f^i_i &= |A_i| - (4k - 2) \forall f^i \in F \\
    f_{2j+1}^2 &= |A_{2j+1}| - (8k - 4) \forall f^{2i} \in F \text{ and } j = 1, 2, \ldots, \frac{n - 1}{4} \\
    f_{2j}^2 &= |A_{2j}| - (8k - 4) \forall f^{2i} \in F \text{ and } j = i + 1, i + 2, \ldots, \frac{n + 1}{4} \\
    f_{2j+1}^{2i+1} &= |A_{2j+1}| - (8k - 4) \forall f^{2i+1} \in F \text{ and } j = 1, 2, \ldots, i - 1 \\
    f_{2j}^{2i+1} &= |A_{2j}| - (i - j)(2k + 1) - (6k - 5) \forall f^{2i+1} \in F \text{ and } j = 1, 2, \ldots, i - 1 \\
    f_{2j+1}^{2i+1} &= |A_{2j+1}| - (j - i)(2k + 1) - (6k - 5) \forall f^{2i+1} \in F \text{ and } j = 1, 2, \ldots, \frac{n + 1}{4} \\
    f_{2j+1}^{2i+1} &= |A_{2j+1}| - (j - i)(2k + 1) - (6k - 5) \forall f^{2i+1} \in F \text{ and } j = i + 1, i + 2, \ldots, \frac{n - 1}{4} \\
\end{align*}
\]

Proof. Let \( f^{2i+1} = f(S) \) with \( S = P_1 \cup P_2, \) where \( P_1 = [v_{r-1((n-i-1)(2k+1)-2)}, v_{r-1((n-i)(2k+1)-3)}] \) and \( P_2 = [v_{r-1((n-i-1)(2k+1)-2)}, v_0] \). We have \(|P_1| = |P_2| = 2k \) so \(|S| = 4k, \) implying \( f_1^{2i+1} = |A_1| \) due to (24).

The orbits \( A_{2j+1} \) include edges which connects vertices at a distance \( kn - 2 + 2j \) and \( \delta(v_{l+kn+2j-2}, v_l) = j(2k + 1) - 1. \) We now have the situation in Lemma C.3 with \( u = i \) and \( g = j. \) When \( i < j \) then
Example 4. With \(k = 3\) and \(n = 5\) the solution \(f = f(S)\) to \(K_{31/13}\) of \(\text{MAX } C_7\)-COL from Lemma [C.4] has

\[S = \{v_{14}, v_{27}, v_9, v_{22}, v_4, v_{17}\} \cup \{v_{28}, v_{10}, v_{23}, v_5, v_{18}, v_0\},\]

and looks like:

\[
\begin{array}{cccccccc}
  f^{-1}(w_0) & f^{-1}(w_1) & f^{-1}(w_2) & f^{-1}(w_3) & f^{-1}(w_4) & f^{-1}(w_5) & f^{-1}(w_6) \\
  v_0 & v_{13} & v_{26} & v_8 & v_{21} & v_3 & v_{16} \\
  v_{29} & v_{11} & v_{24} & v_6 & v_{19} & v_1 & v_{17} \\
  v_4 & v_{22} & v_9 & v_{27} & v_{14} & v_{15} \\
  v_{30} & v_{12} & v_{25} & v_7 & v_{20} & v_2 & v_{18} \\
  v_{19} & v_5 & v_{23} & v_{10} & v_{28} & & \\
\end{array}
\]

We will now prove Proposition [3.6] using the solutions from Lemma [B.1], Lemma [C.1] and Lemma [C.4].

Proposition [3.6]

Proof. We get \((n + 1)/2\) inequalities from Lemma [B.1, C.1] and [C.4], where as noted above, we have removed the inequality generated by \(f^3\). As variables we have \(s\) and \(\omega_i\) for \(i = 1, \ldots, (n + 1)/2\). To solve the relaxation of \((LP)\), we solve the corresponding system with equalities. A similar treatment of the dual confirms that the obtained solution is indeed the optimum.

We start by reducing our \(\frac{n+1}{2} \times \frac{n+1}{2}\) system to a \(4 \times 4\) system. However we need to rearrange the orbits to conveniently describe how they depend on each other. Let \(A_1 = A_1, A_1' = A_2, \ldots, A_l = A_2, A_l' = A_{2l-2}, \ldots, A_{l/2} = A_{2l-1}, A_{l/2}' = A_2, A_{l/2} + 1 = A_{2l-1}, A_{l/2}' = A_{2l-2}, \ldots, A_{l/2 + 1} = A_{2l-1}\). Furthermore introduce new solutions \(h\) so that \(h_i\) denotes the solution that maximises \(h_i\). This rearrangement
makes sense, as it puts the orbits and solution in such an order that for all solutions \( h^\tau \) we have \( h^\tau_i > h^\tau_i+1 > h^\tau_i+2 > \cdots > h^\tau_i+\frac{n}{2} \).

Now we compare the equations in (LP) from the signatures \( h_{\frac{n}{2}}^i \) and \( h_{\frac{n+1}{2}}^i \). Note that these are the signatures \( f^2 \) and \( f^4 \) from Lemma C.1. We see then that we have

\[
\sum_j h_j^{\frac{n}{2}-1} \omega_j = \sum_j h_j^{\frac{n}{2}} \omega_j + (4k-2) \cdot \omega_{\frac{n}{2}+1} - (4k-2) \cdot \omega_{\frac{n}{2}},
\]

since we assume \( \sum_j h_j^{\frac{n}{2}-1} \omega_j = \sum_j h_j^{\frac{n}{2}} \omega_j = s \), we get \( \omega_{\frac{n}{2}} = \omega_{\frac{n}{2}+1} \). For the general case we have

\[
\sum_j h_j^i \omega_j = \sum_j h_j^{i+1} \omega_j + (4k-2) \cdot \omega_i - (4k-2) \cdot \omega_{i+1} - (2k+1) \cdot \sum_{j=i+2}^{i+n+1} \omega_j,
\]

for \( i = 3, 4, \ldots, \frac{n+1}{2} - 1 \). Since again we assume \( \sum_j h_j^i \omega_j = \sum_j h_j^{i+1} \omega_j = s \), we get

\[
\omega_i = \omega_{i+1} + \frac{2k+1}{4k-2} \cdot \sum_{j=i+2}^{i+n+1} \omega_j,
\]

for \( i = 3, 4, \ldots, \frac{n+1}{2} - 1 \). For \( i = \frac{n+1}{2} - 1 \) this means \( \omega_i = \omega_{i+1} \). For all other \( i \), we use the fact that (30) also holds for \( \omega_{i+1} \) and thus have:

\[
\omega_{i+1} = \omega_{i+2} + \frac{2k+1}{4k-2} \cdot \sum_{j=i+3}^{i+n+1} \omega_j,
\]

for \( i = 3, 4, \ldots, \frac{n+1}{2} - 2 \). From (31) we get,

\[
(2k+1) \sum_{j=i+5}^{i+n+1} \omega_j = (4k-2) \cdot (\omega_{i+1} - \omega_i+2).
\]

We then insert (32) into (30) to express \( \omega_i \) in terms of \( \omega_{i+1} \) and \( \omega_{i+2} \) only:

\[
\omega_i = \omega_{i+1} + \frac{2k+1}{4k-2} \cdot \omega_{i+2} + (\omega_{i+1} - \omega_{i+2}) = 2 \cdot \omega_{i+1} - \frac{2k-3}{4k-2} \cdot \omega_{i+2},
\]

for \( i = 3, 4, \ldots, \frac{n+1}{2} - 2 \). We now define \( \omega_i = g_{\frac{n+1}{2}-i} \cdot \omega_{\frac{n+1}{2}} \) with

\[
g_i = \begin{cases} 
1 & i = 0, 1, \\
2 \cdot g_{i-1} - \frac{2k-3}{4k-2} \cdot g_{i-2} & i = 2, 3, \ldots, \frac{n+1}{2} - 3.
\end{cases}
\]

Thus, we can express \( \omega_3, \ldots, \omega_{\frac{n+1}{2}-1} \) in terms of \( \omega_{\frac{n+1}{2}} \). However, to proceed we need to express the coefficients \( g_i \) in terms of \( k \) and \( n \). Define \( G(z) = \sum_{g \geq 0} g_n z^n \). We do not have to worry about the upper limit, since as far as we are concerned the recursion could go on towards infinity, without affecting the values we are interested in. After multiplying with \( z^n \) and summing up from \( n \geq 2 \) we get

\[
g_2 z^2 + g_3 z^3 + \cdots = 2 \{ g_1 z^2 + g_2 z^3 + \cdots \} - \frac{2k-3}{4k-2} \{ g_0 z^2 + g_1 z^3 + \cdots \},
\]

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which we identify as
\[ G(z) - z - 1 = 2z(G(z) - 1) - \frac{2k - 3}{4k - 2} z^2 G(z). \]
Solving for \( G(z) \) gives
\[ G(z) = \frac{1 - z}{\frac{2k - 3}{4k - 2} z^2 - 2z + 1}. \]
The denominator has two distinct roots whose reciprocals are:
\[ \alpha_1 = 1 + \sqrt{1 - \frac{2k - 3}{4k - 2}} \quad \text{and} \quad \alpha_2 = 1 - \sqrt{1 - \frac{2k - 3}{4k - 2}}. \]
Hence, we can express the \( n \)th coefficient of \( G(z) \) as
\[ z[n]G(z) = \frac{(1 - \frac{1}{\alpha_1}) \cdot \alpha_1^{n+1}}{2 - \frac{2k - 3}{4k - 2} \cdot \frac{2}{\alpha_1}} + \frac{(1 - \frac{1}{\alpha_2}) \cdot \alpha_2^{n+1}}{2 - \frac{2k - 3}{4k - 2} \cdot \frac{2}{\alpha_2}}. \]
We can now write down the smaller \( 4 \times 4 \) system of equations. Let \( |V| = |A_1| = |A_2| = \cdots = |A_{n+1}| \).
From the equations of the signatures \( h_1 \) (from Lemma B.1), \( h_2 \) (from Lemma C.1), and \( h_3 \) (\( f_2 \) from Lemma C.4), we get
\[
\begin{align*}
(|V| - 2) \cdot \omega_1 + |V| \cdot \omega_2 + |V| \cdot \sum_{i=0}^{n+1-3} g_i \cdot \omega_{n+1}^{i+1} &= s \\
|V| \cdot \omega_1 + (|V| - (4k - 2)) \cdot \omega_2 + \sum_{i=0}^{n+1-3} (2k + 1) \frac{n - 1 + 2i}{2} \cdot g_i \cdot \omega_{n+1}^{i+1} &= s \\
|V| \cdot \omega_1 + (|V| - (8k - 4)) \cdot (\omega_2 + \left( \sum_{i=0}^{n+1-3} g_i - (4k - 2) \right) \cdot \omega_{n+1}^{i+1}) &= s \\
|V| \cdot (\omega_1 + \omega_2 + \sum_{i=0}^{n+1-3} g_i \cdot \omega_{n+1}^{i+1}) &= 1.
\end{align*}
\]
Solving this gives
\[ s = \frac{(2kn + n - 4)(\xi_n(4k - 1) + (2k - 1))}{(2kn + n - 4)(\xi_n(4k - 1) + (2k - 1)) + (4k - 2)(1 - \xi_n)}, \]
where
\[ \xi_n = \left( \frac{\alpha_1^{(n-1)/2} + \alpha_2^{(n-1)/2}}{4}. \right) \]
\[ \square \]
D Proofs of Results from Section 3.3

Lemma 3.7

Proof. Since $\chi_c(G) \leq r$ means there exist one $r' \leq r$ such that $G \to K_{r'}$, and since $K'_{r'} \to K_r$ we have $G \to K_r$ and $K_2$ has a homomorphism to every graph that contains at least one edge so $K_2 \to G \to K_r$ and we can apply Lemma 1.1. We also have that $C_{2k+1}$ has a homomorphism to each graph which contains an odd cycle with length at most $2k + 1$. It is obvious that $C_{2k+1}$ has an homomorphism into a graph containing a cycle of length exactly $2k + 1$. But we also know that $C_{2k+1} \to C_{2m+1}$ if $m \leq k$ so if $G$ contains an odd cycle of length at most $2k + 1$ then we have $C_{2k+1} \to G \to K_r$. \qed

Proposition 3.8

Proof. By Pan and Zhu we know the following for graphs $G$ that are $K_4$-minor-free and integers $k \geq 1$:

- If $G$ has odd girth at least $6k - 1$ then $\chi_c(G) \leq 8k/(4k - 1)$;
- If $G$ has odd girth at least $6k + 1$ then $\chi_c(G) \leq (4k + 1)/2k$;
- If $G$ has odd girth at least $6k + 3$ then $\chi_c(G) \leq (4k + 3)/(2k + 1)$.

The above, combined with Proposition 3.1, can be used to specify values on $s(K_2, G)$. We get that when the odd girth is at least $6k - 1$ then $s(K_2, G) \geq \frac{4k}{4k + 1}$ and when the odd girth is at least $6k + 3$ then $s(K_2, G) \geq \frac{4k + 2}{4k + 3}$. For graphs with odd girth $6k + 1$ the result of Pan and Zhu give no other guarantee than that a homomorphism exists to the cycle $C_{4k+1}$, which gives us no better bound than for graphs with girth $6k - 1$. \qed