CATEGORICAL RESOLUTION OF SINGULARITIES

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Abstract. Building on the concept of a smooth DG algebra we define the notion of a smooth derived category. We then propose the definition of a categorical resolution of singularities. Our main example is the derived category $D(X)$ of quasi-coherent sheaves on a scheme $X$. We prove that $D(X)$ has a canonical categorical resolution if the base field is perfect and $X$ is a separated scheme of finite type with a dualizing complex.

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This research was supported in part by NSF grant 48-294-16.
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1. INTRODUCTION

There is a good notion of smoothness for DG algebras. Namely, a DG algebra $A$ is smooth if it is perfect as a DG $A^{op} \otimes A$-module. If $A$ is derived equivalent to a DG algebra $B$ then $A$ is smooth if and only if $B$ is such. Therefore it makes sense to define smoothness of the derived category $D(A)$ of DG $A$-modules. This also allows one to discuss smoothness of cocomplete triangulated categories $T$ which have a compact generator (and come from a DG category). For example $T$ may be the derived category of quasi-coherent sheaves on a quasi-compact separated scheme. If $k$ is a perfect field and $X$ is a separated $k$-scheme essentially of finite type, then $X$ is regular if and only if the category $D(X) = D(Qcoh_X)$ is smooth.

For any DG algebra $B$ one may view the full subcategory $\text{Perf}(B) \subset D(B)$ as a "dense smooth subcategory" of $D(B)$. So it is natural to define (Definition 4.1) a categorical resolution of $D(B)$ as a pair $(A, X)$, where $A$ is a smooth DG algebra and $X$ is a DG $B^{op} \otimes A$-module such that the restriction of the functor $L(\cdot) \otimes_B X : D(B) \to D(A)$

to the subcategory $\text{Perf}(B)$ is full and faithful.

In this paper we give examples of categorical resolutions. In particular we show that the Koszul duality functor is sometimes a categorical resolution (Proposition 5.6).

Our main example is the derived category $D(X)$ of quasi-coherent sheaves on a scheme $X$. If $\tilde{X} \to X$ is the usual resolution of singularities, then $L\pi^* : D(X) \to D(\tilde{X})$ is a categorical resolution if and only if $X$ has rational singularities. This may suggest that our definition of categorical resolution is not the right one. However we believe that this definition still makes sense and that a categorical resolution of $D(X)$ may in a sense be "better" than the usual $D(\tilde{X})$. (For example a categorical resolution of $D(X)$ exists for many nonreduced schemes $X$.)

We show that if $k$ is a perfect field, then for any separated $k$-scheme $X$ of finite type that has a dualizing complex there exists a categorical resolution (Theorem 6.3). The corresponding "resolving" smooth DG algebra $A$ is derived equivalent to $A^{op}$, but usually has unbounded cohomology. This is a canonical categorical resolution of $D(X)$; it has the flavor of Koszul duality. (After this paper was written we learned that the smoothness of
this DG algebra $A$ was conjectured by Kontsevich.) It was pointed to us by Van den Bergh that our result implies the smoothness of the unbounded homotopy category of injectives $K(\text{Inj} X)$ which was studied by Krause in [Kr]. We discuss this in the last section.

In a forthcoming paper [Lu2] we propose categorical resolutions of $D(X)$ of a different kind. Namely we construct new smooth categories by "glueing" smooth schemes. This is an extension of the work [Lu1].

It is our pleasure to thank Michel Van den Bergh, Mike Mandell, Bernhard Keller and Michael Artin for answering many question. We are also grateful to participants of the seminar on Algebraic Varieties at the Steklov Institute, where these ideas were presented. Dmitri Orlov pointed out to me the results in [Rou] and Dmitri Kaledin informed me of the paper [Ku] in which a similar notion appears but the approach is different. Alexander Kuznetsov drew my attention to the recent preprint [BuDr], where a categorical resolution is constructed for projective curves with only nodes and cusps as singularities. (As is pointed out in [BuDr], in some cases this resolution coincides with the one constructed in [Lu1].)

After our talk in Banff in October 2008 Osamu Iyama suggested a connection with Auslander algebras, but we did not work it out in this paper.

2. Triangulated categories, DG categories, compact object

This section contains some preliminaries.

Fix a field $k$. All categories are assumed to be $k$-linear and $\otimes$ means $\otimes_k$ unless mentioned otherwise.

2.1. Generation of triangulated categories. Fix a triangulated category $T$.

Let $I$ be a full subcategory of $T$. We denote by $\langle I \rangle$ the smallest strictly full subcategory of $T$ containing $I$ and closed under finite direct sums, direct summands and shifts. We denote by $\mathbf{T}$ the smallest strictly full subcategory of $T$ containing $I$ and closed under direct sums (existing in $T$) and shifts.

Let $I_1, I_2$ be two full subcategories of $T$. We denote by $I_1 \ast I_2$ the strictly full subcategory of objects $M$ such that there exists an exact triangle $M_1 \to M \to M_2$ with $M_i \in T_i$. Put $I_1 \diamond I_2 = \langle I_1 \ast I_2 \rangle$.

Define $\langle I \rangle_0 = 0$ and then define by induction $\langle I \rangle_i = \langle I \rangle_{i-1} \diamond \langle I \rangle$ for $i \geq 1$. Put $\langle I \rangle_\infty = \bigcup_{i \geq 0} \langle I \rangle_i$.

The objects of $\langle I \rangle_i$ are the direct summands of the objects obtains by taking an $i$-fold extension of finite direct sums of objects of $I$ ([BoVdB],2.2).

**Definition 2.1.** We say that

- $I$ generates $T$ if given $C \in T$ with $\text{Hom}(D[i], C) = 0$ for all $D \in I$ and all $i \in \mathbb{Z}$, then $C = 0$. 

• \(I\) classically generates \(T\) if \(T = \langle I \rangle_\infty\).
• An object \(D \in T\) is a strong classical generator for \(T\) if \(\langle I \rangle_d = T\) for some \(d \in \mathbb{N}\).

2.2. Cocomplete triangulated categories and compact objects. A triangulated category \(T\) is called cocomplete if it has arbitrary direct sums. An object \(C \in T\) is called compact if \(\text{Hom}(C, -)\) commutes with direct sums. Denote by \(T^c \subset T\) the full triangulated subcategory of compact objects. \(T\) is called compactly generated if \(T\) is generated by a set of compact objects. We say that \(T\) is Karoubian if every projector in \(T\) splits. The following theorem summarizes some known facts ([BoNe],[Ne],[Rou]).

**Theorem 2.2.** Let \(T\) be a cocomplete triangulated category.

a) Then \(T\) and \(T^c\) are Karoubian.

Assume in addition that \(T\) is compactly generated.

b) Then a set of objects \(E \subset T^c\) classically generates \(T^c\) if and only if it generates \(T\).

c) If a set of objects \(E \subset T^c\) generates \(T\) then \(T\) coincides with the smallest strictly full triangulated subcategory of \(T\) which contains \(E\) and is closed under direct sums.

2.3. DG algebras and their derived categories. A DG algebra is a graded unital associative (\(k\)-) algebra with a differential \(d\) of degree +1 satisfying the Leibnitz rule and such that \(d(1) = 0\). A homomorphism of DG algebras is a degree zero \(k\)-linear homomorphism (not necessarily unital) of graded associative rings which commutes with the differential. DG algebras \(A\) and \(B\) are quasi-isomorphic if there exist a diagram of DG algebras and homomorphisms

\[
A \leftarrow A_1 \rightarrow \ldots \leftarrow A_n \rightarrow B,
\]

where all arrows are quasi-isomorphisms.

Let \(A\) be a DG algebra. Denote by \(A\)-mod the DG category ([Ke1]) of unital right DG \(A\)-modules. For \(M, N \in A\)-mod we have the complex \(\text{Hom}(M, N) = \oplus_{n \in \mathbb{Z}} \text{Hom}^n(M, N)\), where \(\text{Hom}^n(M, N)\) consists of degree \(n\) homogeneous homomorphisms of graded modules over the graded algebra \(A\). Let \(Ho(A) = Ho(A\text{-mod})\) be the homotopy category of \(A\)-mod, in which we replace the \(\text{Hom}\)-complexes by the cohomology in degree zero. This is a triangulated category and we denote by \(D(A)\) the derived category of \(A\), which is the Verdier localization of \(Ho(A)\) with respect to quasi-isomorphisms. The categories \(Ho(A)\) and \(D(A)\) are cocomplete and the localization functor \(Ho(A) \rightarrow D(A)\) preserves direct sums.

A DG \(A\)-module \(S\) is called h-injective (resp. h-projective) if for every acyclic DG \(A\)-module \(M\) the complex \(\text{Hom}(M, S)\) is acyclic (resp. \(\text{Hom}(S, M)\) is acyclic). There are enough h-injectives and h-projectives in \(A\)-mod: for every \(M \in A\)-mod there exist quasi-isomorphisms \(M \rightarrow I, P \rightarrow M\), where \(I\) is h-injective and \(P\) is h-projective. Denote by
I(A), P(A) ⊂ A-mod the full DG subcategories consisting of h-injectives and h-projectives respectively. The induced triangulated functors Ho(I(A)) → D(A), Ho(P(A)) → D(A) are equivalences. One uses h-injectives and h-projectives to define right and left derived functors in the usual way.

Let φ : A → B be a homomorphism (not necessarily unital) of DG algebras. Denote φ(1_A) = e. We have the adjoint DG functors of extension and restriction of scalars

\[ \phi^*(-) = (-) \otimes_A B = (-) \otimes_A eB : A \text{-mod} \to B \text{-mod} \]
\[ \phi_*(-) = \text{Hom}(eB, -) : B \text{-mod} \to A \text{-mod} \]

and the induced triangulated functors \( \phi^* : Ho(A) \to Ho(B) \), \( \phi_* : Ho(B) \to Ho(A) \). Define the derived functor \( L\phi^* : D(A) \to D(B) \) using h-projectives. So \((L\phi^*, \phi_*)\) is an adjoint pair of functors between \( D(A) \) and \( D(B) \). If \( \phi \) is a quasi-isomorphism, then \((L\phi^*, \phi_*)\) is a pair of mutually inverse equivalences. Sometimes the functors \( \phi^* \) and \( \phi_* \) are denoted by Ind and Res respectively.

Denote by \( \text{Perf}(A) \subset D(A) \) the full triangulated subcategory which is classically generated by the DG \( A \)-module \( A \). We call objects of \( \text{Perf}(A) \) the perfect \( A \)-modules. Note that the functor \( L\phi^* \) as above preserves perfect modules (even though \( L\phi^*(A) \neq B \) when \( \phi \) is not unital).

For any \( M \in D(A) \) we have \( \text{Hom}_{Ho(A)}(A, M) = \text{Hom}_{D(A)}(A, M) = H^0(M) \). Thus \( A \) is a generator for \( D(A) \). Since \( H^0(-) \) commutes with direct sums, the object \( A \in D(A) \) is compact. Hence \( \text{Perf}(A) \subset D(A)^c \).

**Proposition 2.3** (Ke1). \( \text{Perf}(A) = D(A)^c \).

The following definition extends the notion of Morita equivalence to DG algebras.

**Definition 2.4.** DG algebras \( A \) and \( B \) are called derived equivalent if there exists a DG \( A^{op} \otimes B \)-module \( K \) such that the functor \( - \otimes_A K : D(A) \to D(B) \) is an equivalence of categories.

For example, if \( \phi : A \to B \) is a quasi-isomorphism of DG algebras then \( A \) and \( B \) are derived equivalent (\( K = B \)).

**2.4. Derived categories of abelian Grothendieck categories.** Let \( \mathcal{A} \) be an abelian category, \( C(\mathcal{A}) \) the abelian category of complexes over \( \mathcal{A} \), \( Ho(\mathcal{A}) \), \( D(\mathcal{A}) \) - the corresponding homotopy and derived categories. One can make \( C(\mathcal{A}) \) into a DG category \( C^{\text{dg}}(\mathcal{A}) \) in the usual way: given \( M, N \in C(\mathcal{A}) \) we get the complex \( \text{Hom}(M, N) = \oplus_{n \in \mathbb{Z}} \text{Hom}^n(M, N) \), where \( \text{Hom}^n(M, N) = \prod_{i \in \mathbb{Z}} \text{Hom}(M^i, N^{i+n}) \). Then \( Ho(C^{\text{dg}}(\mathcal{A})) = Ho(\mathcal{A}) \).
An object $I \in C(A)$ is called h-injective if for every acyclic $M \in C(A)$ the complex $\text{Hom}(M, I)$ is acyclic. Denote by $I(A) \subset C^{dg}(A)$ the full DG category of h-injectives.

Recall that an object $G \in A$ is called a g-object if the functor $X \mapsto \text{Hom}(G, X)$ is conservative, i.e. $X \to Y$ is an isomorphism as soon as $\text{Hom}(G, X) \to \text{Hom}(G, Y)$ is an isomorphism. Such an object $G$ is usually called a generator, but we already used this term in Definition 2.1 in a different context.

Recall that an abelian category $A$ is called a Grothendieck category if it has a g-object, small inductive limits and the filtered inductive limits are exact. In particular $A$ has arbitrary direct sums.

If $A$ is a Grothendieck category, then so is $C(A)$. Then the categories $Ho(A)$, $D(A)$ are cocomplete and the natural functors $C(A) \to Ho(A) \to D(A)$ preserve direct sums. The following proposition is proved for example in [Ka-Sch], Thm. 14.1.7.

**Proposition 2.5.** Let $A$ be a Grothendieck category. Then for every $M \in C(A)$ there exists a quasi-isomorphism $M \to I$, where $I \in C(A)$ is h-injective. Thus the triangulated category $Ho(I(A))$ is equivalent to $D(A)$. (Hence in particular the bi-functor $R\text{Hom}(-, -) : D(A)^{op} \times D(A) \to D(k)$ is defined.)

Derived categories (admitting a compact generator) of Grothendieck categories can be described using DG algebras. The proof of the following proposition is the same argument as in [Ke1], Lemma 4.2. We present in here because it will be used again later.

**Proposition 2.6.** Let $A$ be a Grothendieck category such that the triangulated category $D(A)$ has a compact generator $E$. Denote by $A$ the DG algebra $R\text{Hom}(E, E)$. Then the functor $R\text{Hom}(E, -) : D(A) \to D(A)$ is an equivalence of categories.

**Proof.** Since $Ho(I(A)) \simeq D(A)$ we may assume that $E$ is h-injective and hence $A = \text{Hom}(E, E)$. Define the DG functor

$I(A) \to A\text{-mod}, \quad M \mapsto \text{Hom}(E, M)$.

Let $\Psi_E : Ho(I(A)) \to D(A)$ be the composition of the induced functor $Ho(I(A)) \to Ho(A)$ with the localization $Ho(A) \to D(A)$.

Let us prove that $\Psi_E$ is full and faithful.

Let $T \subset Ho(I(A))$ be the full triangulated subcategory of objects $M$ such that the map

$\text{Hom}(E, M[n]) \to \text{Hom}(\Psi_E(E), \Psi_E(M[n]))$

is an isomorphism for all $n \in \mathbb{Z}$. Then $T$ contains $E$ and is closed under direct sums. Hence $T = Ho(I(A))$ by Theorem 2.2c). Similarly let $S \subset Ho(I(A))$ be the full triangulated
category consisting of objects \( N \) such that for each \( M \in Ho(I(A)) \) the map
\[
\text{Hom}(N, M) \to \text{Hom}(\Psi_E(N), \Psi_E(M))
\]
is an isomorphism. Then \( S \) contains \( E \) and is closed under direct sums. So \( S = Ho(I(A)) \).

The fully faithful triangulated functor \( \Psi_E \) preserves direct sums and takes the compact generator \( E \) to the compact generator \( A \). Since categories \( Ho(I(A)) \) and \( D(A) \) are cocomplete it follows from Theorem 2.2c) that \( \Psi_E \) is essentially surjective. \( \square \)

**Remark 2.7.** In the context of Proposition 2.6 let \( E' \) be another compact generator of \( D(A) \) with \( A' = R\text{Hom}(E', E') \). Then the DG algebras \( A \) and \( A' \) are derived equivalent. Indeed assume that \( E \) and \( E' \) are h-injective and consider the DG \( A^{op} \otimes A' \)-module \( \text{Hom}(E', E) \). Then using the notation in the proof of Proposition 2.6 we have the obvious morphism of functors
\[
\mu : \Psi_E(-) \otimes_A \text{Hom}(E', E) \to \Psi_{E'}(-).
\]
Both functors preserve direct sums and \( \mu(E) \) is an isomorphism. Hence \( \mu \) is an isomorphism (Theorem 2.2c). But \( \Psi_E \) and \( \Psi_{E'} \) are equivalences. Hence
\[
(-) \otimes_A \text{Hom}(E', E) : D(A) \to D(A')
\]
is also an equivalence. In fact it is easy to see (using Lemma 2.14) that the DG algebras \( A \) and \( A' \) are quasi-isomorphic.

Actually, Proposition 2.6 is a special case of the following general theorem of Keller ([Ke1],Thm.4.3).

**Theorem 2.8.** Let \( \mathcal{E} \) be a Frobenius exact category. Assume that the corresponding triangulated stable category \( \underline{\mathcal{E}} \) is cocomplete and has a compact generator. Then \( \underline{\mathcal{E}} \simeq D(A) \) for a DG algebra \( A \).

**Remark 2.9.** As in Remark 2.7 one can show that the DG algebra \( A \) in Theorem 2.8 is well defined up to a derived equivalence.

Triangulated categories which are equivalent to the stable category \( \underline{\mathcal{E}} \) of a Frobenius exact category are called algebraic in [Ke2]. For example derived categories of abelian categories are algebraic.

2.5. **Schemes.** Let \( X \) be a \( k \)-scheme. We denote by \( Qcoh.X \) the abelian category of quasi-coherent sheaves on \( X \). Put \( D(X) = D(Qcoh.X) \) and denote by \( \text{Perf}(X) \subset D(X) \) the full subcategory of perfect complexes (i.e. complexes which are locally quasi-isomorphic to a finite complex of free \( \mathcal{O}_X \)-modules of finite rank).
If $X$ is quasi-compact and quasi-separated, then $\text{Qcoh}X$ is a Grothendieck category [ThTr], Appendix B.

The first assertion in the next theorem is due to Neeman and the second is in [BoVdB] Theorem 2.10.

**Theorem 2.10.** Let $X$ be a quasi-compact and separated scheme. Then

a) $D(X)^c = \text{Perf}(X)$.

b) The category $D(X)$ has a compact generator.

**Corollary 2.11.** Let $X$ be a quasi-compact separated scheme. Then there exists a DG algebra $A$, such that $D(X) \simeq D(A)$.

**Proof.** Indeed, since $\text{Qcoh}X$ is a Grothendieck category the corollary follows from Proposition 2.6 and Theorem 2.10b). 

Thus many triangulated categories "in nature" look like $D(A)$ or $\text{Perf}(A)$ for a DG algebra $A$.

2.6. A few lemmas.

**Lemma 2.12.** Let $A$ and $B$ be DG algebras, $M \in A^{\text{op}} \otimes B\text{-mod}$ such that the functor

$$\Phi_M(-) := (-) \otimes_A M : D(A) \to D(B)$$

induces an equivalence of full subcategories $\text{Perf}(A) \sim \text{Perf}(B)$. Then $\Phi_M$ is an equivalence. In particular $A$ and $B$ are derived equivalent.

**Proof.** The DG $A$-module is a classical generator of $\text{Perf}(A)$. Hence the object $\Phi_M(A)$ is a classical generator for $\text{Perf}(B)$, and therefore by Proposition 2.3 and Theorem 2.2b) it is a compact generator for $D(B)$. Thus the functor $\Phi_M$ has the following three properties:

a) it preserves direct sums;

b) it maps a compact generator $A$ to a compact generator $\Phi_M(A)$;

c) it induces an isomorphism $\text{Ext}^\bullet(A, A) \sim \text{Ext}^\bullet(\Phi_M(A), \Phi_M(A))$.

Using the same argument as in the proof of Proposition 2.6 it follows easily from a),b),c) that $\Phi_M$ is an equivalence. 

**Lemma 2.13.** Let $A$ and $B$ be DG algebras and $F : D(A) \to D(B)$ be a triangulated functor with the following properties

a) $F(\text{Perf}(A)) \subset \text{Perf}(B)$.

b) The restriction of $F$ to $\text{Perf}(A)$ is full and faithful.

c) $F$ preserves direct sums.

Then $F$ is full and faithful.

**Proof.** Same argument as in the proof of Proposition 2.6 and Lemma 2.12. 

Let $\mathcal{A}$ be an abelian category, $X, Y \in C(\mathcal{A})$ and $f : X \to Y$ a morphism of complexes. Consider the cone $C_f \in C(\mathcal{A})$ of the morphism $f$ and the DG algebra $\text{End}(C_f)$. Let $\mathcal{C} \subseteq \text{End}(C_f)$ be the DG subalgebra which preserves the complex $Y$,

$$\mathcal{C} = \left( \begin{array}{cc} \text{End}(Y) & \text{Hom}(X[1], Y) \\ 0 & \text{End}(X[1]) \end{array} \right)$$

with the projections $p_X : \mathcal{C} \to \text{End}(X[1])$, $p_Y : \mathcal{C} \to \text{End}(Y)$. More generally, let $A \to \text{End}(X) = \text{End}(X[1])$ be a homomorphism of DG algebras. Then we can consider the corresponding DG algebra

$$\mathcal{C}_A = \left( \begin{array}{cc} \text{End}(Y) & \text{Hom}(X[1], Y) \\ 0 & A \end{array} \right)$$

with the projections $p_A : \mathcal{C}_A \to A$ and $p_Y : \mathcal{C}_A \to \text{End}(Y)$.

**Lemma 2.14.** Assume that the induced map $f^* : \text{End}(Y) \to \text{Hom}(X, Y)$ and the composition $A \to \text{End}(X) \xrightarrow{f} \text{Hom}(X, Y)$ are quasi-isomorphisms. Then $p_A$ and $p_Y$ are quasi-isomorphisms. In particular the DG algebras $A$ and $\text{End}(Y)$ are quasi-isomorphic.

**Proof.** Indeed, our assumptions imply that the kernels $\text{Ker} p_A = \text{End}(Y) \oplus \text{Hom}(X[1], Y)$ and $\text{Ker} p_Y = A \oplus \text{Hom}(X[1], Y)$ are acyclic. $\square$

## 3. Smooth DG algebras and smooth derived categories

**Definition 3.1. (Kontsevich).** A DG algebra $A$ is smooth if $A \in \text{Perf}(A^{\text{op}} \otimes A)$.

We thank Bernhard Keller for the following remark.

**Remark 3.2.** If $A$ is smooth, then so is $A^{\text{op}}$. Indeed, the isomorphism of DG algebras

$$A^{\text{op}} \otimes A \to A \otimes A^{\text{op}}, \quad a \otimes b \mapsto b \otimes a$$

induces an equivalence $D(A^{\text{op}} \otimes A) \simeq D(A \otimes A^{\text{op}})$ which preserves perfect DG modules and sends $A$ to $A^{\text{op}}$.

**Lemma 3.3.** Let $A$ and $B$ be smooth DG algebras. Then so is $A \otimes B$.

**Proof.** The bifunctor $\otimes : D(A^{\text{op}} \otimes A) \times D(B^{\text{op}} \otimes B) \to D((A \otimes B)^{\text{op}} \otimes A \otimes B)$ maps $\text{Perf}(A^{\text{op}} \otimes A) \times \text{Perf}(B^{\text{op}} \otimes B) \to \text{Perf}((A \otimes B)^{\text{op}} \otimes A \otimes B)$ and sends $(A, B)$ to $A \otimes B$. $\square$

The next definition is the analogue for DG algebras of the notion of finite global dimension for associative algebras.

**Definition 3.4.** We say that a DG algebra $A$ is weakly smooth if $D(A) = \langle A \rangle_d$ for some $d \in \mathbb{N}$ (Definition 2.1). That is every DG $A$-module is quasi-isomorphic to a direct summand of a $d$-fold extension of direct sums of shifts of $A$. 

Lemma 3.5. Assume that the DG algebra $A$ is weakly smooth, $D(A) = \langle A \rangle_d$. Then $\text{Perf}(A) = \langle A \rangle_d$. In particular $A$ is a strong generator for $\text{Perf}(A)$.

Proof. Recall that for any DG $A$-module $M$

$$\text{Hom}_{D(A)}(A, M) = H^0(M).$$

Since cohomology commutes with filtered inductive limits of complexes we have

$$\text{Hom}_{D(A)}(A, \lim_{\rightarrow} M_i) = \lim_{\rightarrow} \text{Hom}_{D(A)}(A, M_i)$$

for any filtered inductive system of DG $A$-modules $\{M_i\}$ (here the inductive limit is taken in the abelian category of DG $A$-modules with morphisms being closed morphisms of degree zero). Hence this holds also for any perfect DG $A$-module instead of $A$.

Fix $P \in \text{Perf}(A)$. By our assumption $P$ (as any DG $A$-module) is isomorphic to a direct summand of a $d$-fold extension $Q$ of direct sums of shifts of $A$. That is we have morphisms $P \xrightarrow{i} Q \xrightarrow{p} P$, such that $p \cdot i = \text{id}$. Notice that the DG module $Q$ is the union of its DG submodules $\{Q_j\}$ which are $d$-fold extensions of finite direct sums of shifts of $A$. Hence the morphism $i : P \to Q$ factors through some $Q_j \subset Q$, so that the composition $P \xrightarrow{i} Q_j \xrightarrow{p} P$ is the identity. Hence $P$ is isomorphic to a direct summand of $Q_j$, i.e. $P \in \langle A \rangle_d$. \qed

Lemma 3.6. a) Suppose $A$ is smooth. Then it is weakly smooth.

b) Assume that $A$ is smooth and is concentrated in degree zero. Then $A$ has finite global dimension.

Proof. a) Any DG $A^{\text{op}} \otimes A$-module $M$ defines a functor $F_M : D(A) \to D(A)$, $F_M(-) = \text{L}(-) \otimes_A M$. We have $F_A \simeq \text{Id}_{D(A)}$. Thus if $A \in (A^{\text{op}} \otimes A)_d$, then for any $N \in D(A)$, we have $N \simeq F_A(N) \in \langle A \rangle_d$.

b) A perfect DG $A^{\text{op}} \otimes A$-module is a homotopy direct summand if a bounded complex of free $A^{\text{op}} \otimes A$-modules (of finite rank). Thus as in the proof of a) for any $A$-module $M$ the complex $F_A(M)$ (which is quasi-isomorphic to $M$) is a homotopy direct summand of a complex of free $A$-modules which is bounded independently of $M$. Hence $A$ has finite global dimension. \qed

Example 3.7. Let $A$ be a finite inseparable field extension of $k$. Then $A$ is weakly smooth (with $d = 1$), but not smooth.

Nevertheless one has the following result.

Proposition 3.8. Assume that the field $k$ is perfect. Let $A$ and $C$ be localizations of finitely generated commutative $k$-algebras.
a) Assume that the algebras $A, C$ have finite global dimension. Then the algebra $A \otimes C$ is also regular (hence so is $A \otimes A$) and $A$ is a perfect DG $A \otimes A$-module (i.e. the DG algebra $A$ is smooth).

b) Vice versa if $A$ has infinite global dimension, then $A$ is not a perfect DG $A \otimes A$-module (i.e. the DG algebra $A$ is not smooth).

Proof. a). Denote $B := A \otimes C$. Since $B$ is noetherian it suffices to prove that it is regular.

We need to prove that the localization $B_m$ of $B$ at every maximal ideal is a regular local ring. For this we may assume that $A$ and $C$ are finitely generated $k$-algebras. Put $K = B/m$. Then by Nullstellensatz $\dim_k K < \infty$. It follows that the ideal $n := m \cap (A \otimes 1) \subset A$ is also maximal. Put $L = A/nA$; this is a finite separable extension of $k$. Consider the obvious (flat) embedding of local rings $A_n \to B_m$. By Theorem 23.7 in [Ma] it suffices to prove that the ring $F := B_m/nB_m$ is regular.

Consider the embedding $A = A \otimes 1 \hookrightarrow B$ and the induced quotient $B/nB \cong L \otimes C$, which is an etale extension of $C$ (since the field $k$ is perfect). Thus $B/nB$ is a regular ring. But $F$ is a localization of $B/nB$ at (the image of) the ideal $m$. So $F$ is also regular.

b). Follows from Lemma 3.6b).

3.1. Derived invariance of smoothness. Let us show that smoothness is an invariant of the derived equivalence class of DG algebras.

**Lemma 3.9.** Assume that $A$ and $B$ are derived equivalent. Then $A$ is smooth if and only if $B$ is smooth.

Proof. For $M \in D(A^{op} \otimes B)$ denote by $\Phi_M(-) : D(A) \to D(B)$ the functor $(-) \otimes_A M$. It has the right adjoint functor $\Psi_M(-) := R \text{Hom}_B(M, -)$. Assume that $\Phi_M$ is an equivalence. Then so is $\Psi_M$, and hence in particular $\Psi_M$ preserves direct sums, i.e. $M$ is compact as a DG $B$-module. But then we claim that for any $T \in D(B)$ the canonical morphism of DG $A$-modules

$$T \otimes_B R \text{Hom}_B(M, B) \to R \text{Hom}_B(M, T)$$

is a quasi-isomorphism. Indeed, since $M$ is compact it suffices to check the claim for $T = B$ (Theorem 2.2c), where it is obvious. It follows that the functor $\Psi_M$ is isomorphic to the functor

$$\Phi_N(-) = (-) \otimes_B N, \text{ where } N = R \text{Hom}_B(M, B).$$

The isomorphisms of functors

$$\Phi_N \cdot \Phi_M \simeq \text{Id}, \quad \Phi_M \cdot \Phi_N \simeq \text{Id}$$
induce in particular the quasi-isomorphisms of DG $A^{\text{op}} \otimes A$ - and $B^{\text{op}} \otimes B$-modules respectively

$$M \overset{\mathcal{L}}{\otimes}_B N \simeq A, \quad N \overset{\mathcal{L}}{\otimes}_A M \simeq B.$$ 

Now consider the functors

$$N \Delta_M (-) := N \overset{\mathcal{L}}{\otimes}_A (-) \otimes_A M : D(A^{\text{op}} \otimes A) \to D(B^{\text{op}} \otimes B),$$

$$M \Delta_N (-) := M \overset{\mathcal{L}}{\otimes}_B (-) \otimes_B N : D(B^{\text{op}} \otimes B) \to D(A^{\text{op}} \otimes A).$$

The quasi-isomorphisms above imply the isomorphisms of functors

$$M \Delta_N \cdot N \Delta_M \simeq \text{Id}, \quad N \Delta_M \cdot M \Delta_N \simeq \text{Id}.$$ 

Hence $M \Delta_N$ and $N \Delta_M$ are mutually inverse equivalences. In particular they preserve compact objects, i.e. perfect complexes. But notice that $N \Delta_M(A) \simeq B$. This proves the lemma. □

**Corollary 3.10.** Assume that the DG algebras $A$ and $B$ are quasi-isomorphic. Then $A$ is smooth if and only if $B$ is smooth.

**Proof.** We may assume that there exists a quasi-isomorphism $\phi : A \to B$ of DG algebras. Then the functor

$$( - ) \overset{\mathcal{L}}{\otimes}_A B : D(A) \to D(B)$$

is an equivalence of categories. So we are done by Lemma 3.9. □

### 3.2. Gluing smooth DG algebras.

Let $A$ and $B$ be DG algebras and $N \in A^{\text{op}} \otimes B$-mod. Then we obtain a new DG algebra

$$C = \begin{pmatrix} B & 0 \\ N & A \end{pmatrix}.$$ 

**Proposition 3.11.** Assume that the DG algebras $A$ and $B$ are smooth. Also assume that $N \in \text{Perf}(A^{\text{op}} \otimes B)$. Then $C$ is smooth.

**Proof.** Since quasi-isomorphic DG algebras are derived equivalent we may assume that the DG $A^{\text{op}} \otimes B$-module $N$ is h-projective (hence it is also h-projective as DG $A^{\text{op}}$ - or $B$-module).

If $D$ and $E$ are DG algebras we will denote by $M_E$, $\mathcal{D}M$, $\mathcal{D}M_E$ respectively a DG $E$-, $D^{\text{op}}$-, $D^{\text{op}} \otimes E$-module.

It is easy to see that a DG $C$-module is the same as a triple $S = (S_A, S_B, \phi_S : S_A \otimes_A N \to S_B)$, where $S_A, S_B$ are DG $A$- and $B$-modules respectively and $\phi_S$ is a closed degree zero morphism of DG $B$-modules.
Similarly, a DG \( C^{\text{op}} \otimes C \)-module is given by the following data

\[
M = \{ BMA, AMA, BMB, AMB; \\
B\Theta_{AB} : (BMA) \otimes_A N \to BMB, \\
A\Theta_{AB} : (AMA) \otimes_A N \to AMB, \\
BA\Theta_A : N \otimes_B (BMA) \to AMA, \\
BA\Theta_B : N \otimes_B (BMB) \to AMB \}
\]

where all the \( \Theta \)'s are closed degree zero morphisms of the corresponding DG modules, such that the diagram

\[
\begin{array}{c}
N \otimes_B (BMA) \otimes_A N \xrightarrow{id \otimes (B\Theta_{AB})} N \otimes_B (BMB) \\
\downarrow_{BA\Theta_A \otimes id} \quad \downarrow_{BA\Theta_B} \\
AM_A \otimes_A N \xrightarrow{A\Theta_{AB}} AMB
\end{array}
\]

commutes. It is convenient to describe such DG \( C^{\text{op}} \otimes C \)-module \( M \) symbolically by a diagram

\[
\begin{array}{c}
BMA \xrightarrow{B\Theta_{AB}} BMB \\
\downarrow_{BA\Theta_A} \quad \downarrow_{BA\Theta_B} \\
AM_A \xrightarrow{A\Theta_{AB}} AMB
\end{array}
\]

Then the diagram corresponding to the diagonal DG module \( C \) is

\[
\begin{array}{c}
0 \to B \\
\downarrow \quad \downarrow id \\
A \xrightarrow{id} N
\end{array}
\]

We have the obvious (non-unital) inclusions of DG algebras \( A^{\text{op}} \otimes A \to C^{\text{op}} \otimes C, \ A^{\text{op}} \otimes B \to C^{\text{op}} \otimes C \), etc. Hence the corresponding DG functors of extension of scalars

\[
\text{Ind}_{A^{\text{op}} \otimes A} : A^{\text{op}} \otimes A\text{-mod} \to C^{\text{op}} \otimes C\text{-mod}, ...
\]

Consider the corresponding derived functors \( \text{LInd}_{A^{\text{op}} \otimes A} : D(A^{\text{op}} \otimes A) \to D(C^{\text{op}} \otimes C), ... \)

They preserve perfect DG modules.

Consider the diagonal DG \( A^{\text{op}} \otimes A \)-module \( A \). Then

\[
\begin{align*}
\text{LInd}_{A^{\text{op}} \otimes A}(A) &= A \otimes_{A^{\text{op}} \otimes A} (C^{\text{op}} \otimes C) \\
&= A \otimes_{A^{\text{op}} \otimes A} [(A^{\text{op}} \otimes A) \oplus (A^{\text{op}} \otimes N)] \\
&= A \oplus N.
\end{align*}
\]

Thus \( \text{LInd}_{A^{\text{op}} \otimes A}(A) \) is quasi-isomorphic to the DG \( C^{\text{op}} \otimes C \)-module

\[
\begin{array}{c}
0 \to 0 \\
\downarrow \quad \downarrow \\
A \xrightarrow{id} N
\end{array}
\]
Similarly, $\mathbf{L}\mathrm{Ind}_{B^{op} \otimes B}(B)$ is quasi-isomorphic to

$$
\begin{array}{ccc}
0 & \rightarrow & B \\
\downarrow & & \downarrow \mathrm{id} \\
0 & \rightarrow & N
\end{array}
$$

Also $\mathbf{L}\mathrm{Ind}_{A^{op} \otimes B}(N)$ is equal to

$$
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & N
\end{array}
$$

We conclude that the diagonal DG $C^{op} \otimes C$-module $C$ is quasi-isomorphic to the cone of the obvious morphism

$$\mathbf{L}\mathrm{Ind}_{A^{op} \otimes B}(N) \rightarrow \mathbf{L}\mathrm{Ind}_{A^{op} \otimes A}(A) \oplus \mathbf{L}\mathrm{Ind}_{B^{op} \otimes B}(B).$$

Thus our assumptions on $A, B,$ and $N$ imply that $C$ is perfect. □

3.3. Smoothness for schemes. Next we show that for nice schemes the two notions of smoothness coincide.

**Definition 3.12.** A (k-) scheme $Y$ is essentially of finite type if $Y$ is a separated scheme which admits a finite open covering by affine schemes $\text{Spec} C$, where $C$ is a localization of a finitely generated $k$-algebra. In particular it is quasi-compact.

**Proposition 3.13.** Assume that the field $k$ is perfect. Let $X$ be a scheme which is essentially of finite type. Let $E \in \text{Perf}(X)$ be a compact generator of $D(X)$, i.e. the functor $F : D(X) \rightarrow D(A), \quad F(M) = \mathbf{R}\mathrm{Hom}(E, M)$ is an equivalence, where $A = \mathbf{R}\mathrm{Hom}(E, E)$ (Proposition 2.6, Theorem 2.10, Corollary 2.11). Then $X$ is a regular scheme if and only if the DG algebra $A$ is smooth.

**Proof.** Note that Proposition 3.8 provides a local version of this proposition. Indeed, if $X = \text{Spec} C$ then $\mathcal{O}_X$ is a compact generator of $D(X)$, so that $D(X) = D(C)$ (Serre’s theorem).

Notice that the contravariant functor $M \mapsto M^* := \mathbf{R}\mathrm{Hom}(M, \mathcal{O}_X)$ is an auto-equivalence of the category $\text{Perf}(X)$. It follows that $E^*$ is also a generator of $D(X)$.

Moreover the following result implies that $E^* \boxtimes E \in \text{Perf}(X \times X)$ is a compact generator for $D(X \times X)$.

**Lemma 3.14.** Let $Y$ and $Z$ be quasi-compact separated schemes. Assume that $S \in \text{Perf}(Y), \ T \in \text{Perf}(Z)$ are the compact generators of $D(Y)$ and $D(Z)$ respectively. Then $S \boxtimes T$ is a compact generator of $D(Y \times Z)$

**Proof.** It is [BoVdB], Lemma 3.4.1. □
Lemma 3.15. There exist canonical quasi-isomorphisms of DG algebras

a) \( R \text{Hom}(E^*, E^*) \simeq A^{\text{op}}, \)

b) \( R \text{Hom}(E^* \boxtimes E, E^* \boxtimes E) \simeq A^{\text{op}} \otimes A. \)

Let \( \Delta : X \to X \times X \) be the diagonal closed embedding.

c) There exists a canonical equivalence of categories \( D(X \times X) \to D(A^{\text{op}} \otimes A) \) which takes the object \( \Delta_\ast \mathcal{O}_X \) to the diagonal DG \( A^{\text{op}} \otimes A \)-module \( A. \)

Proof. The proof is essentially the same as that of Proposition 6.17 below. We omit it. \( \square \)

It follows from part c) of Lemma 3.15 that \( \Delta_\ast \mathcal{O}_X \in \text{Perf}(X \times X) = D(X \times X)^c \) if and only if \( A \in \text{Perf}(A^{\text{op}} \otimes A) = D(A^{\text{op}} \otimes A)^c. \) If \( X \) is regular, then \( X \times X \) is also regular by Proposition 3.8a) hence \( D^b(\text{coh}(X \times X)) = \text{Perf}(X \times X) \), so in this case \( A \) is smooth.

Vice versa, assume that \( X \) is not regular. It suffices to prove that \( \Delta_\ast \mathcal{O}_X \) is not in \( \text{Perf}(X \times X) \). The question is local, so we may assume that \( X = \text{Spec}C \), where \( C \) is a localization of a finitely generated \( k \)-algebra. Then \( C \) has infinite global dimension and by Proposition 3.8b) we know that \( C \) is not a perfect DG \( C \otimes C \)-module. \( \square \)

3.4. Smooth triangulated categories. Let \( T \) be a cocomplete triangulated category with a compact generator. We would like to say that \( T \) is smooth if there exists an equivalence of triangulated categories \( T \simeq D(A) \), where \( A \) is a smooth DG algebra. However, we don’t know if this is well defined, because there exist DG algebras which are not derived equivalent, but their derived category are equivalent as triangulated categories. So the triangulated category \( T \) should come with an enhancement, i.e. some DG category. For example, \( T \) maybe the derived category of an abelian Grothendieck category or the stable category of a Frobenius exact category. Then using Proposition 2.6, Theorem 2.8 and Remarks 2.7, 2.9 we may define the notion of smoothness for \( T \).

Definition 3.16. a) Let \( A \) be a DG algebra. We call its derived category \( D(A) \) smooth if \( A \) is smooth.

b) Let \( \mathcal{A} \) be an abelian Grothendieck category such that the derived category \( D(A) \) has a compact generator \( K \). Denote \( A = R \text{Hom}(K, K) \), so that \( D(A) \simeq D(A) \) (Proposition 2.6). Then \( D(A) \) is called smooth if \( A \) is smooth.

c) Let \( \mathcal{E} \) be an exact Frobenius category such that the stable category \( \mathcal{E} \) is cocomplete and has a compact generator. Then \( \mathcal{E} \simeq D(A) \) for a DG algebra \( A \) (Theorem 2.8). We call \( \mathcal{E} \) smooth if \( A \) is smooth.

Note that b) and c) are well defined by Remarks 2.7, 2.9.

Note that we have defined smoothness only for ”big”, i.e. cocomplete categories.
4. Definition of a categorical resolution of singularities

**Definition 4.1.** Let $A$ be a DG algebra. A categorical resolution of $D(A)$ (or of $A$) is a pair $(B, X)$, where $B$ is a smooth DG algebra and $X \in D(A^{\text{op}} \otimes B)$ is such that the restriction of the functor

$$\theta(-) := (-) \otimes_A X : D(A) \to D(B)$$

to the subcategory $\text{Perf}(A)$ is full and faithful. We also call a categorical resolution of $D(A)$ a pair $(B, E)$, where $B$ is a smooth DG algebra and $E \in D(A \otimes B^{\text{op}})$ is such that the restriction of the functor

$$\theta(-) := R\text{Hom}(E, -) : D(A) \to D(B)$$

to the subcategory $\text{Perf}(A)$ is full and faithful.

Sometimes we will say that the pair $(D(B), \theta)$, or simply $D(B)$ or $\theta$ is a resolution of $D(A)$.

Let us try to explain this definition. For any DG algebra $A$ the perfect DG $A$-modules form (in our opinion) a "smooth dense subcategory" of $D(A)$. Hence a categorical resolution of $D(A)$ should not change the subcategory $\text{Perf}(A)$.

**Remark 4.2.** Let $A$ be a DG algebra and $B$ be a smooth DG algebra. Let $E$ be a DG $A \otimes B^{\text{op}}$-module such that the functor $R\text{Hom}(E, -) : D(A) \to D(B)$ is full and faithful on the subcategory $\text{Perf}(A)$. Then the functor $(-) \otimes_A L_{\text{RHom}}(E, A) : D(A) \to D(B)$ is also a categorical resolution of singularities. Indeed, there is a natural isomorphism of functors from $\text{Perf}(A)$ to $D(B)$

$$(-) \otimes_A L_{\text{RHom}}(E, A) \to R\text{Hom}(E, -).$$

So the existence of two possibilities in Definition 4.1 is only for convenience.

**Definition 4.3.** Let $A$ be a DG algebra and $(B, \theta)$, $(B', \theta')$ two categorical resolutions of $D(A)$. We say that these resolutions are equivalent if there exists a DG $B^{\text{op}} \otimes B'$-module $S$ such that the functor $\Phi_Y(-) := (-) \otimes_B S : D(B) \to D(B')$ is an equivalence and the functors $\Phi_Y \cdot \theta$ and $\theta'$ are isomorphic.

In the rest of the paper we will discuss some examples of categorical resolutions.

5. Miscellaneous examples of categorical resolutions

**Example 5.1.** Assume that $k$ is a perfect field. Let $X$ be an algebraic variety over $k$ and $\pi : \tilde{X} \to X$ its resolution of singularities. Then by Proposition 3.13 the category $D(\tilde{X})$
is smooth. The pair \((D(\tilde{X}), L\pi^*)\) is a categorical resolution of \(D(X)\) if and only if the adjunction morphism

\[ \phi(M) : M \to R\pi_*L\pi^*(M) \]

is a quasi-isomorphism for every \(M \in \text{Perf}(X)\). This question is local on \(X\), so it suffices to check if the morphism \(\phi(O_X)\) is a quasi-isomorphism. We conclude that \((D(\tilde{X}), L\pi^*)\) is a categorical resolution of \(D(X)\) if and only if \(X\) has rational singularities.

The above example may suggest that our definition of categorical resolution of singularities is not the right one because it is consistent with the usual geometric resolution only in the case of rational singularities. To make things even worse let us note that if a morphism of varieties \(Y \to X\) defines a categorical resolution of \(D(X)\), then so does the morphism \(\mathbb{P}^n \times Y \to X\). Nevertheless, in this paper we want to argue that our definition makes sense. In particular, we will show that even if \(X\) has nonrational singularities (and the field \(k\) has positive characteristic!) there exists a categorical resolution of \(D(X)\).

**Example 5.2.** Assume that \(\text{char}(k) = 0\). Let \(R\) be a commutative finitely generated \(k\)-algebra, such that \(Y = \text{Spec}R\) is smooth. Let \(G\) be a finite group acting on \(Y\) and denote by \(R \ast G\) the corresponding crossed product algebra. It is smooth. Consider the possibly singular scheme \(Y//G := \text{Spec}R^G\). Then the functor

\[ R_! \otimes_{R^G} (-) : D(R^G) \to D(R \ast G) \]

is a categorical resolution of singularities. Note that \(D(R^G) = D(Y//G)\) and \(D(R \ast G)\) is equivalent to the derived category of \(G\)-equivariant quasi-coherent sheaves on \(Y\).

**Example 5.3** (VdB). Let \(k\) be algebraically closed and \(R\) be a an integral commutative Gorenstein \(k\)-algebra. Let \(M\) be a reflexive \(R\)-module such that the algebra \(A = \text{End}_R(M)\) has finite global dimension and is a maximal Cohen-Macaulay \(R\)-module. Van den Bergh informs us that if \(R\) is a localization of a finitely generated \(k\)-algebra, then the DG algebra \(A\) is smooth and so the functor

\[ M^! \otimes_R (-) : D(R) \to D(A^{op}) \]

is a categorical resolution of \(D(R)\).

**Remark 5.4.** Note that in the last two examples the singular varieties \((Y//G\) and \(\text{Spec}R\) respectively) have rational singularities [StVdB].
5.1. **Resolution by Koszul duality.** Let $A$ be an augmented DG algebra with the augmentation ideal $A^+$. Consider the shifted complex $A^+[1]$ and the corresponding DG tensor coalgebra $BA := T(A^+[1])$. The differential in $BA$ depends on the differential in $A$ and the multiplication in $A$. It is called the bar construction of $A$. Its graded linear dual $(BA)^*$ is again an augmented DG algebra called the *Koszul dual* of $A$ and denoted $\tilde{A}$. The map $\sigma : BA \to (BA)^*$ is an isomorphism of DG coalgebras. (Here $\bar{b}$ is the degree of $b$). Therefore the Koszul dual of $A^*$ is $(\tilde{A})^*$. Since $A$ is a DG algebra and $BA$ is a DG coalgebra the complex $\text{Hom}(BA, A)$ is naturally a DG algebra. An element $\alpha \in \text{Hom}_1(BA, A)$ is called a *twisting cochain* if it satisfies the Maurer-Cartan equation $d\alpha + \alpha^2 = 0$. The projection of $TA^+[1]$ onto its first component $A^+[1]$ followed by the (shifted) identity map $A^+[1] \to A^+$ is the universal twisting cochain which we denote by $\tau$. Consider the tensor product $BA \otimes A$ with the differential $d = d_{BA} \otimes 1 + 1 \otimes d_A + t_\tau$ where $t_\tau(b \otimes a) = b(1) \otimes \tau(b(2))a$ (here $b \mapsto b(1) \otimes b(2)$ is the symbolic notation for the comultiplication map $BA \to BA \otimes BA$). Then indeed $d^2 = 0$ and we denote the corresponding complex by $BA \otimes_{\tau} A$. It is quasi-isomorphic to $k$ and is called the bar complex of $A$. This bar complex is naturally a right DG $A$-module. It is also a left DG $BA$-comodule in the obvious way. Hence in particular $BA \otimes_{\tau} A$ is a DG $A \otimes \tilde{A}$-module. Define the Koszul functor $K_A(-) := (-) \otimes_A (A \otimes_{\tau} BA) : D(A) \to D(\tilde{A})$. This functor is often full and faithful on the subcategory $\text{Perf}(A)$. Hence it defines a categorical resolution of $D(A)$ in case the DG algebra $\tilde{A}$ is smooth. The following lemma is proved in [ELOII].

**Lemma 5.5.** Assume that an augmented DG algebra $A$ satisfies the following properties.

i) $A^{<0} = 0$;

ii) $A^0 = k$;

iii) $\dim A^i < \infty$ for every $i$. Then the Koszul functor $K_A$ is full and faithful on the subcategory $\text{Perf}(A)$.

Here we consider another example.
Proposition 5.6. Let $A$ be an augmented finite dimensional DG algebra concentrated in nonpositive degrees. Assume in addition that the augmentation ideal $A^+$ is nilpotent. Then the Koszul functor $K_A : D(A) \to D(\check{A}^{\text{op}})$ is a categorical resolution.

The proposition is equivalent to the following two lemmas.

Lemma 5.7. Let $A$ be as in Proposition 5.6. Then the DG algebras $\check{A}$ and $\check{A}^{\text{op}}$ are smooth.

Proof. It suffices to prove that the DG algebra $\check{A}$ is smooth. Indeed, replace $A$ by $A^{\text{op}}$.

Let us combine the two versions of the bar complex in one. Consider the tensor product $BA \otimes A \otimes BA$ with the differential

$$d = d_{BA} \otimes 1 \otimes 1 + 1 \otimes d_A \otimes 1 + 1 \otimes 1 \otimes d_{BA} + t_\tau \otimes 1 + 1 \otimes s_{-\tau}.$$ 

Then $d^2 = 0$ and $BA \otimes A \otimes BA$ is a DG $(BA)^{\text{op}} \otimes BA$-comodule in the obvious way. We denote it by $BA \otimes_\tau A \otimes_\tau BA$. The map $\nu : BA \to BA \otimes A \otimes BA$, $\nu(b) = b(1) \otimes 1 \otimes b(2)$ is a morphism of DG $(BA)^{\text{op}} \otimes BA$-comodules. Our assumption on $A$ implies that $BA \otimes A \otimes BA$ is finite dimensional in each degree. Hence its graded dual is $\check{A} \otimes A^* \otimes \check{A}$. It is a DG $\check{A}^{\text{op}} \otimes \check{A}$-module which we denote by $\check{A} \otimes_{\tau^*} A^* \otimes_{\tau^*} \check{A}$.

The dual of the morphism $\nu$ is the morphism of DG $\check{A}^{\text{op}} \otimes \check{A}$-modules

$$\nu^* : \check{A} \otimes_{\tau^*} A^* \otimes_{\tau^*} \check{A} \to \check{A},$$

where $\check{A}$ is the diagonal DG $\check{A}^{\text{op}} \otimes \check{A}$-module.

Notice that $\nu^*$ is a quasi-isomorphism. Indeed, it suffices to show that $\nu$ is such. Let $\epsilon : A \to k$ and $\eta : BA \to k$ be the augmentation and the counit respectively. Then the map $\eta \otimes \epsilon : BA \otimes_\tau A \to k$ is a quasi-isomorphism. Thus the morphism of complexes

$$\eta \otimes \epsilon \otimes 1 : BA \otimes_\tau A \otimes_\tau BA \to k \otimes BA = BA$$

is a quasi-isomorphism. But the composition $\eta \otimes \epsilon \otimes 1 \cdot \nu : BA \to BA$ is the identity. Hence $\nu$ is a quasi-isomorphism.

We claim that $\check{A} \otimes_{\tau^*} A^* \otimes_{\tau^*} \check{A}$ is a perfect DG $\check{A}^{\text{op}} \otimes \check{A}$-module. Indeed consider the finite filtration of $A$ by powers of the augmentation ideal and refine this filtration by the image of the differential. (Note that $\cap_n (A^+)^n = 0$ since $A^+$ is nilpotent.) This induces a filtration of the DG $(BA)^{\text{op}} \otimes BA$-comodule $BA \otimes_\tau A \otimes_\tau BA$ with the subquotients being isomorphic to a direct sum of shifted copies of $(BA)^{\text{op}} \otimes BA$. This implies that the subquotient of the dual filtration of $\check{A} \otimes_{\tau^*} A^* \otimes_{\tau^*} \check{A}$ are finite sums of free shifted DG $\check{A}^{\text{op}} \otimes \check{A}$-modules. That is $\check{A} \otimes_{\tau^*} A^* \otimes_{\tau^*} \check{A}$ is a perfect DG $\check{A}^{\text{op}} \otimes \check{A}$-module. This proves the lemma. □
Lemma 5.8. Let $A$ be as in Proposition 5.6. Then the Koszul functor $K_A$ is full and faithful on the subcategory $\text{Perf}(A)$.

Proof. Notice that $K_A(A) = k$, hence it suffices to prove that the natural map $A \to R\text{Hom}_{A^{op}}(k,k)$ is a quasi-isomorphism.

As in the proof of Lemma 5.7 consider the filtration of $A$ by the powers of the augmentation ideal $A^+$ refined by the image of the differential. Then the induced filtration of the DG $BA$-comodule $A \otimes BA$ has subquotients which are finite sums of shifted copies of $BA$. Notice that the DG $\tilde{A}^{op}$-module $BA$ is $h$-injective. (Indeed, $BA = (\tilde{A})^*$ since $BA$ is finite dimensional in each degree.) Hence the DG $\tilde{A}^{op}$-module $A \otimes BA$ is $h$-injective so that

$$R\text{Hom}_{A^{op}}(k,k) = \text{Hom}_{A^{op}}(k, A \otimes BA).$$

But $\text{Hom}_{A^{op}}(k, A \otimes BA) = A$. This proves the lemma and finishes the proof of Proposition 5.6. $\Box$

Here are some examples illustrating Proposition 5.6.

Example 5.9. Let $V$ be a finite dimensional (graded) vector space concentrated in degree zero. Consider the DG algebra $A = TV/V^{\otimes 2}$ - the truncated tensor algebra on $V$. This DG algebra is not smooth if $\dim V > 0$. The Koszul dual DG algebra $\tilde{A}$ has zero differential and is isomorphic to the tensor algebra $T(V^*[-1])$, where $V^*[-1]$ is the dual space to $V$ placed in degree 1. This is a smooth DG algebra and the Koszul functor $K_A$ is a categorical resolution of $D(A)$.

Example 5.10. Let $A$ be a finite dimensional augmented algebra (concentrated in degree zero) with the nilpotent augmentation ideal. For example we can take the group algebra $k[G]$ of a finite $p$-group $G$ in case the field $k$ is algebraically closed and has characteristic $p$. Then again the Koszul functor $K_A$ is a categorical resolution of $D(A)$.

6. Categorical resolution for schemes

The following theorem was proved in [Rou].

Theorem 6.1. Let $X$ be a separated scheme of finite type over a perfect field. Then there exists $E \in D^b(\text{coh}X)$ and $d \in \mathbb{N}$ such that $D^b(\text{coh}X) = \langle E \rangle_d$.

Denote $A = R\text{Hom}(E,E)$. The theorem implies that the functor

$$R\text{Hom}(E,-) : D(X) \to D(A)$$

induces an equivalence of subcategories $D^b(\text{coh}X) \simeq \text{Perf}(A)$. Consequently $\text{Perf}(A) = \langle A \rangle_d$, i.e. $A$ is a strong generator for $\text{Perf}(A)$. 
Remark 6.2. Unlike in [Rou] we do not regard the equivalence $D^b(\text{coh} X) \simeq \text{Perf}(A)$ with $A$ weakly smooth (or even smooth) as saying that "going to the DG world, $X$ becomes regular". Indeed, according to our definition only the "big" category $D(X)$ can be smooth or not.

We are going to strengthen Rouquier’s result.

Theorem 6.3. Let $X$ be a separated scheme of finite type over a perfect field $k$. Then

a) There exists a classical generator $E \in D^b(\text{coh} X)$, such that the DG algebra $A = R\text{Hom}(E, E)$ is smooth and hence the functor

$$R\text{Hom}(E, -) : D(X) \to D(A)$$

is a categorical resolution.

b) Given any other classical generator $E' \in D^b(\text{coh} X)$ with $A' = R\text{Hom}(E', E')$, the DG algebras $A$ and $A'$ are derived equivalent (hence $A'$ is also smooth) and the categorical resolutions $D(A)$ and $D(A')$ of $D(X)$ are equivalent.

Proof. Let us first prove b) assuming a):

The functors $R\text{Hom}(E, -)$, $R\text{Hom}(E', -)$ induce respective equivalences $D^b(\text{coh} X) \simeq \text{Perf}(A)$, $D^b(\text{coh} X) \simeq \text{Perf}(A')$. Consider the DG $A' \otimes A'^{\text{op}}$-module $R\text{Hom}(E', E)$ and the obvious morphism of functors from $D^b(\text{coh} X)$ to $\text{Perf}(A')$

$$\mu : R\text{Hom}(E, -) \otimes_A R\text{Hom}(E', E) \to R\text{Hom}(E', -).$$

Then $\mu(E)$ is an isomorphism, hence $\mu$ is an isomorphism. This implies that the functor

$$(-) \otimes_A R\text{Hom}(E', E) : D(A) \to D(A')$$

induces an equivalence $\text{Perf}(A) \overset{\sim}{\to} \text{Perf}(A')$. Thus it is an equivalence by Lemma 2.12, so that $A$ and $A'$ are derived equivalent and the categorical resolutions $D(A)$ and $D(A')$ of $D(X)$ are equivalent (Definition 4.3).

The proof of part a) requires some preparation. All schemes are assumed to be $k$-schemes.

For a scheme of finite type $Z$ we denote by $Z^{\text{red}}$ (resp. $Z^{\text{ns}}$, resp. $Z^{\text{sg}}$) the scheme $Z$ with the reduced structure (resp. the open subscheme of regular points, resp. the closed subscheme of singular points).

Definition 6.4. Let $Y$ be a scheme of finite type. An admissible covering of $Y$ is a finite collection of closed reduced subschemes $\{Z_j\}$ such that the following set theoretical conditions hold

a) $Y = \cup Z_j$, 

b) for every $j$

$$Z_j^{sg} \subset \bigcup_{s \mid Z_s \subset Z_j} Z_s^{ns}.$$  

**Example 6.5.** For each scheme of finite type $Y$ there exists a canonical admissible covering: $Z_1 = Y^{\text{red}}$, $Z_{j+1} = (Z_j^{sg})^{\text{red}}$.

**Definition 6.6.** Let $Z$ be a reduced scheme of finite type. We call $F \in D^b(\text{coh}Z)$ a quasi-generator for $D(Z)$ if $F|_{Z^{ns}}$ is a compact generator for $D(Z^{ns})$.

For example if $Z$ is a reduced separated scheme of finite type and $F \in \text{Perf}(Z)$ is a generator for $D(Z)$ (Theorem 2.10b)), then it is a quasi-generator.

**Definition 6.7.** A generating data on a scheme of finite type $Y$ is a collection $\{Z_j, E_j\}$, where $\{Z_j\}$ is an admissible covering of $Y$ and $E_j \in D^b(\text{coh}Z_j)$ is a quasi-generator for $D(Z_j)$ for each $j$.

If $Y$ is a separated scheme of finite type, then it admits a generating data. Indeed, we can take the canonical admissible covering $\{Z_j\}$ as in Example 6.5 above, with $E_j \in \text{Perf}(Z_j)$ being a compact generator for $D(Z_j)$.

**Proposition 6.8.** Let $Y$ be a separated scheme of finite type with a generating data $\{Z_j, E_j\}$. Let $i_j : Z_j \to Y$ be the corresponding closed embedding. Then

$$E := \bigoplus_j i_j^* E_j$$

is a classical generator for $D^b(\text{coh}X)$.

**Proof.** For a noetherian scheme $S$ and a closed subset $W \subset S$ we denote as usual by $D^b_W(\text{coh}S)$ the full subcategory of $D^b(\text{coh}S)$ consisting of complexes whose cohomology sheaves are supported on $W$.

We may assume that $Z_i \subset Z_j$ implies that $i < j$. Define the closed subsets $W_j := \cup_{s \leq j} Z_s$. It suffices to prove for each $j$ the following assertion

\((*)_j\): The object $\bigoplus_{s \leq j} i_{s*} E_s$ is a classical generator for the category $D^b_{W_j}(\text{coh}Y)$.

Let us prove these assertions $(*)_j$ by induction on $j$.

$j = 1$. We have $Z_1^{ns} = Z_1$, hence $E_1$ is a classical generator for $D^b(\text{coh}Z_1) = \text{Perf}(Z_1) = D(Z_1)^c$ (Theorem 2.2 b), Theorem 2.10a)).

**Lemma 6.9.** Let $T$ be a separated noetherian scheme and $i : Z \to T$ be the embedding of a reduced closed subscheme. Let $F \in D^b(\text{coh}Z)$ be a classical generator. Then $i_* F$ is a classical generator for the category $D^b_T(\text{coh}T)$.
Proof. This follows from Lemmas 7.37, 7.41 in [Rou]. □

Thus $i_1^* E_1$ is a classical generator of $D^b_{Z_1}(\text{coh} Y) = D^b_{W_1}(\text{coh} Y)$.

$j - 1 \Rightarrow j$. Consider the following localization sequence of triangulated categories

$$D^b_{W_{j-1}}(\text{coh} Y) \rightarrow D^b_{W_j}(\text{coh} Y) \rightarrow D^b_{W_{j-1}}(\text{coh}(Y - W_{j-1})).$$

By our assumption $W_j - W_{j-1} \subset Z_j^{\text{ns}}$ and $E_j|Z_j^{\text{ns}}$ is a compact generator for $D(Z_j^{\text{ns}})$, hence a classical generator for $D^b(\text{coh}Z_j^{\text{ns}}) = \text{Perf}(Z_j)$. Since $W_j - W_{j-1}$ is an open subset of the scheme $Z_j^{\text{ns}}$, we may consider it with the induced (reduced) scheme structure. Then $E_j|_{W_j - W_{j-1}}$ is a classical generator for $D^b(\text{coh}(W_j - W_{j-1})) = \text{Perf}(W_j - W_{j-1})$. So by Lemma 6.9 $(i_j^*E_j)|_{Y - W_{j-1}}$ is a classical generator for $D^b_{W_{j-1}}(\text{coh}(Y - W_{j-1}))$. Now the next Lemma 6.10 and the induction hypothesis imply that

$$D^b_{W_j}(\text{coh} Y) = \bigoplus_{s \leq j} i_s^* E_j,$$

which completes the induction step and proves the proposition. □

Lemma 6.10. Let $S \rightarrow T \xrightarrow{\pi} T/S$ be a localization sequence of triangulated categories. Let $G_1 \subset S$ and $G_2 \subset T$ be subsets of objects such that $S = \langle G_1 \rangle$ and $T/S = \langle \pi(G_2) \rangle$. Then $T = \langle G_1 \cup G_2 \rangle$.

Proof. Denote $T' \coloneqq \langle G_1 \cup G_2 \rangle \subset T$. Then $T'$ is by definition closed under direct summands. It suffices to prove that $T/T' = 0$. But $S \subset T' \subset T$. Hence $T/T' \simeq (T/S)/(T'/S)$, and $T/S = \langle \pi(G_2) \rangle \subset T'/S$. Thus $T/T' = 0$. □

In Proposition 6.8 above we have constructed a special classical generator $E$ for the category $D^b(\text{coh} Y)$. We will show that the DG algebra $\textbf{R}\text{Hom}(E, E)$ is smooth (if $k$ is perfect). This will complete the proof of Theorem 6.3.

For a scheme of finite type $Y$ denote by $D_Y \in D^b(\text{coh} Y)$ a dualizing complex on $Y$ (which exists and is unique up to a shift and a twist by a line bundle on each connected component of $Y$, [Ha2],VI, Thm.3.1.,§10.), so that the functor

$$D(-) := \textbf{R}\text{Hom}(-, D_Y) : D^b(\text{coh} Y) \rightarrow D^b(\text{coh} Y)$$

is an anti-involution. Clearly, if $E$ is a classical generator for $D^b(\text{coh} Y)$, then so is $D(E)$. Recall that the duality commutes with direct image functors under proper morphisms. In particular, if $i : Z \rightarrow Y$ is a closed embedding and $F \in D^b(\text{coh} Z)$, then

$$i_*D(F) \simeq D(i_*F).$$

(Here one should take $D_Z = i^!D_Y$, [Ha2],III, Thm.6.7;V, Prop.2.4.)
Lemma 6.11. Let \( \{Z_j, E_j\} \) be a generating data on a scheme of finite type \( Y \). Then so is \( \{Z_j, D(E_j)\} \).

Proof. Fix \( Z_j \). We need to show that \( D(E_j)|_{Z_{ns}^j} \) is a compact generator of \( D(Z_{ns}^j) \). We have \( D(E_j)|_{Z_{ns}^j} = D(E_j|_{Z_{ns}^j}) \), hence the assertion follows from the next lemma. \( \square \)

Lemma 6.12. Assume that \( W \) is a smooth scheme of finite type and \( F \in \text{Perf}(W) \) is a compact generator for \( D(W) \). Then so is \( D(F) \).

Proof. Since \( W \) is regular, \( \mathcal{O}_W \) is a dualizing complex on \( W \). The functor \( R\text{Hom}(-, \mathcal{O}_W) \) induces an anti-involution of the subcategory \( \text{Perf}(W) \). The lemma follows. \( \square \)

Definition 6.13. Let \( Y \) be a separated scheme of finite type with a generating data \( \{Z_j, E_j\} \). We call \( \{Z_j, D(E_j)\} \) the dual generating data. We have \( \bigoplus_{j,s} D(E_j) = D(\bigoplus_{j,s} E_j) \), hence the dual generating data produces the dual generator of \( D^b(\text{coh}Y) \).

Proposition 6.14. Assume that the field \( k \) is perfect. Let \( S, Y \) be separated schemes of finite type. Let \( \{Z_j, E_j\} \) (resp. \( \{W_s, F_s\} \)) be a generating data on \( S \) (resp. on \( Y \)). Then \( \{Z_j \times W_s, E_j \boxtimes F_s\} \) is a generating data for \( S \times Y \).

Proof. We need a lemma.

Lemma 6.15. Let \( k \) be a perfect field, \( A, B \) - noetherian \( k \)-algebras. Assume that \( A \) and \( B \) are reduced. Then so is \( A \otimes B \).

Proof. Let \( p_1, \ldots, p_n \subset A \) (resp. \( q_1, \ldots, q_m \subset B \)) be the minimal primes. Then by our assumption \( A \subset \prod A/p_i \), \( B \subset \prod B/q_j \). Hence also \( A \otimes B \subset \prod(A/p_i \otimes B/q_j) \). Therefore we may assume that \( A \) and \( B \) are integral domains.

The algebra \( A \) is the union of its finitely generated \( k \)-subalgebras \( A = \bigcup A_i \), and \( A \otimes B = \bigcup(A_i \otimes B) \). So we may assume that \( A \) is finitely generated. Also, replacing \( B \) by its fraction field, we may assume that \( B \) is a field. Then by Exercise II, 3.14 in [Ha1] it suffices to prove that the algebra \( A \otimes \overline{k} \) is reduced. But this algebra is the union of its subalgebras which are étale over \( A \) (since the field \( k \) is perfect). Therefore it is reduced. This proves the lemma. \( \square \)

The lemma implies that for each \( j, s \) the scheme \( Z_j \times W_s \) is a closed reduced subscheme of \( S \times Y \). Clearly \( S \times Y = \bigcup_{j,s} Z_j \times W_s \).

By Proposition 3.8a) for each \( j, s \) \( Z_{ns}^j \times W_{ns}^s \subset (Z_j \times W_s)_{ns} \). Actually the two schemes are equal. Indeed, let \( x \in Z_j \) be a point and \( B \) the corresponding local ring. Let \( y \in Z_j \times W_s \),
be a nonsingular point lying over \( x \) with the corresponding local ring \( C \). Then \( C \) is a flat over \( B \). Hence by [Gro],Prop.17.3.3 or by [Ma],Thm.23.7i) \( B \) is also regular.

Therefore

\[(Z_j \times W_s)^{sg} = (Z_j^{sg} \times W_s) \cup (Z_j \times W_s^{sg}).\]

This implies that \( \{Z_j \times W_s\} \) is an admissible covering of \( X \times Y \).

We have

\[(E_j \boxtimes F_s)|_{(Z_j \times W_s)^{ns}} = (E_j|_{Z_j^{ns}}) \boxtimes (F_s|_{W_s^{ns}}).\]

Since \( E_j|_{Z_j^{ns}} \) and \( F_s|_{W_s^{ns}} \) are compact generators of \( D(Z_j^{ns}) \) and \( D(W_s^{ns}) \) respectively, then \( (E_j \boxtimes F_s)|_{(Z_j \times W_s)^{ns}} \) is a compact generator by Lemma 3.14. This proves the proposition. \( \Box \)

**Corollary 6.16.** Let \( \{Z_j, E_j\} \) be a generating data on a separated scheme of finite type \( X \). Let \( i_j : Z_j \to X \) denote the corresponding closed embedding. Then \( \{Z_j \times Z_s, E_j \boxtimes D(E_s)\} \) is a generating data on \( X \times X \). In particular, if \( E = \bigoplus j i_j^* E_j \), then \( E \boxtimes D(E) \) is a classical generator for \( D^b(\text{coh}(X \times X)) \).

**Proof.** Follows from Lemma 6.11 and Proposition 6.14. \( \Box \)

**Proposition 6.17.** Let \( Y \) be a separated scheme of finite type over a perfect field \( k \). Choose a classical generator \( E \) of \( D^b(\text{coh}Y) \) as in Proposition 6.8 above and denote \( A = \mathbb{R}\text{Hom}(E, E) \). Let \( D(E) \) be the dual generator. Then there exist canonical quasi-isomorphisms of DG algebras

a) \( \mathbb{R}\text{Hom}(D(E), D(E)) \simeq A^{op} \),

b) \( \mathbb{R}\text{Hom}(D(E) \boxtimes E, D(E) \boxtimes E) \simeq A^{op} \otimes A \).

Let \( \Delta : Y \to Y \times Y \) be the diagonal closed embedding.

c) There exists a canonical equivalence of categories \( D^b(\text{coh}(Y \times Y)) \simeq \text{Perf}(A^{op} \otimes A) \) which takes the object \( \Delta_*(D_Y) \) to the diagonal DG \( A^{op} \otimes A \)-module \( A \). In particular the DG algebra \( A \) is smooth.

We prove this proposition in Subsection 6.1 below.

Part a) of Theorem 6.3 now follows. Indeed, let \( E \) be a classical generator for \( D^b(\text{coh}X) \) as in Proposition 6.8, then by Proposition 6.17 the DG algebra \( A = \mathbb{R}\text{Hom}(E, E) \) is smooth. \( \Box \)

6.1. **Proof of Proposition 6.17.** a). Since \( D : D^b(\text{coh}Y) \to D^b(\text{coh}Y) \) is an anti-involution the map

\[ D : \text{Ext}(E, E) \to \text{Ext}(D(E), D(E)) \]

is an isomorphism. Choose h-injective resolutions \( E \to I, \ D_Y \to J, \) so that \( A = \text{Hom}(I, I) \) and \( D(E) = \mathcal{H}\text{om}(I, J) \). Let \( \rho : \mathcal{H}\text{om}(I, J) \to K \) be an h-injective resolution, so that
\[ B := \text{Hom}(K, K) = \mathbf{R}\text{Hom}(D(E), D(E)). \] We have the natural homomorphism of DG algebras
\[ \epsilon : A^{\text{op}} \to \text{Hom}(\mathcal{H}\text{om}(I, J), \mathcal{H}\text{om}(I, J)) \]
such that the composition of \( \epsilon \) with the map
\[ \text{Hom}(\mathcal{H}\text{om}(I, J), \mathcal{H}\text{om}(I, J)) \to \text{Hom}(\mathcal{H}\text{om}(I, J), K) \]
is a quasi-isomorphism (since this composition induces the map \( D \) above between the Ext-groups). Notice also that the map \( \rho^* : B \to \text{Hom}(\mathcal{H}\text{om}(I, J), K) \) is a quasi-isomorphism. It follows from Lemma 2.14 that the DG algebra
\[ \begin{pmatrix} B & \text{Hom}(\mathcal{H}\text{om}(I, J)[1], K) \\ 0 & A^{\text{op}} \end{pmatrix} \]
(where the differential is defined using the above maps) is quasi-isomorphic to DG algebras \( B \) and \( A^{\text{op}} \) by the obvious projections. This proves a).

b). The proof is similar and we will use the same notation. In addition to resolutions \( E \to I, \ D(E) \to K \) choose an h-injective resolution \( \sigma : D(E) \boxtimes E \to L \), so that \( \mathbf{R}\text{Hom}(D(E) \boxtimes E, D(E) \boxtimes E) = \text{Hom}(L, L) \). We need a couple of lemmas.

**Lemma 6.18.** The obvious morphism of sheaves of DG algebras on \( Y \times Y \)
\[ \mathcal{H}\text{om}(K, K) \boxtimes \mathcal{H}\text{om}(I, I) \to \mathcal{H}\text{om}(K \boxtimes I, K \boxtimes I) \]
is a quasi-isomorphism.

**Proof.** The question is local so we may assume that \( Y = \text{Spec}B \) for some noetherian \( k \)-algebra \( B \). Then we can find bounded above complexes \( P, Q \) of free \( B \)-modules of finite rank which are quasi-isomorphic to \( D(E) \) and \( E \) respectively. Similarly, we can find bounded below complexes \( M, N \) of injective \( B \)-modules which are quasi-isomorphic to \( D(E) \) and \( E \) respectively. It suffices to prove that the corresponding map
\[ \text{Hom}_B(P, M) \otimes \text{Hom}_B(Q, N) \to \text{Hom}_{B \otimes B}(P \otimes Q, M \otimes N) \]
is an isomorphism. This follows from the formula
\[ \text{Hom}_B(B, S) \otimes \text{Hom}_B(B, T) = S \otimes T = \text{Hom}_{B \otimes B}(B \otimes B, S \otimes T) \]
for any \( B \)-modules \( S, T \). \( \square \)

**Lemma 6.19.** \( \mathbf{R}\Gamma(\mathcal{H}\text{om}(I, I)) = \Gamma(\mathcal{H}\text{om}(I, I)), \mathbf{R}\Gamma(\mathcal{H}\text{om}(K, K)) = \Gamma(\mathcal{H}\text{om}(K, K)). \)
Proof. It suffices to prove the first assertion. Since \( I \) is quasi-isomorphic to a bounded complex we can find a quasi-isomorphism \( \theta : I \to I' \), where \( I' \) is a bounded below complex of injective quasi-coherent sheaves which are also injective in the category \( \text{Mod}_{\mathcal{O}_Y} \) of all \( \mathcal{O}_Y \)-modules [Ha],II,Thm.7.18. Both \( I \) and \( I' \) are h-injective in \( D(Y) \), so the map \( \theta \) is a homotopy equivalence. Hence also \( \theta_* : \mathcal{H}om(I, I) \to \mathcal{H}om(I, I') \) is a homotopy equivalence. So it suffices to prove that \( R\Gamma(\mathcal{H}om(I, I')) = \Gamma(\mathcal{H}om(I, I')) \). The complex \( I' \) is h-injective in the category \( C(\text{Mod}_{\mathcal{O}_Y}) \), hence \( \mathcal{H}om(I, I') \) is weakly injective in this category in the terminology of [Sp],Prop.5.14. Hence \( R\Gamma(\mathcal{H}om(I, I')) = \Gamma(\mathcal{H}om(I, I')) \) by Proposition 6.7 in [Sp]. \( \square \)

Recall the Kunneth formula [Lip],Th.3.10.3: the natural map

\[
R\Gamma(S) \otimes R\Gamma(T) \to R\Gamma(S \boxtimes T)
\]

is a quasi-isomorphism for all \( S, T \in D(Y) \). Applying this to \( S = \mathcal{H}om(K, K), T = \mathcal{H}om(I, I) \) and using Lemmas 6.18 and 6.19 we conclude that the composition of the homomorphism of DG algebras \( B \otimes A \to \mathcal{H}om(K \boxtimes I, K \boxtimes I) \) with the map \( \sigma_* : \mathcal{H}om(K \boxtimes I, K \boxtimes I) \to \mathcal{H}om(K \boxtimes I, L) \) is a quasi-isomorphism. Now as in the proof of part a) we conclude that the DG algebra

\[
\left( \begin{array}{cc}
\mathcal{H}om(L, L) & \mathcal{H}om(\mathcal{H}om(K \boxtimes I)[1], L) \\
0 & B \otimes A
\end{array} \right)
\]

is quasi-isomorphic to both \( \mathcal{H}om(L, L) \) and \( B \otimes A \). But \( B \simeq A^{\text{op}} \) by a), which proves b).

c). We still use the same notation. By definition \( I \) is a DG \( A^{\text{op}} \)-module (more precisely, a sheaf of DG \( A^{\text{op}} \)-modules), hence \( \mathcal{H}om(I, I) \) is a DG \( A \)-module via the action on \( I \). It follows that

\[
\Psi(-) := R\text{Hom}(\mathcal{H}om(I, J) \boxtimes I, -)
\]

is a functor from \( D(Y \times Y) \) to \( D(A^{\text{op}} \otimes A) \). We claim that \( \Psi \) induces an equivalence between \( D^b(\text{coh}(Y \times Y)) \) and \( \text{Perf}(A^{\text{op}} \otimes A) \). Indeed, by Corollary 6.16 \( L \) is a classical generator for \( D^b(\text{coh}(Y \times Y)) \). Hence it suffices to show that \( \Psi(L) = A^{\text{op}} \otimes A \). Consider the commutative diagram

\[
\begin{array}{ccc}
B \otimes A & \to & \mathcal{H}om(K \boxtimes I, K \boxtimes I) \\
\downarrow & & \downarrow \sigma_* \\
\mathcal{H}om(\mathcal{H}om(I, J), K) \otimes A & \to & \mathcal{H}om(\mathcal{H}om(I, J) \boxtimes I, K \boxtimes I) \to \mathcal{H}om(\mathcal{H}om(I, J) \boxtimes I, L)
\end{array}
\]

where the maps in the top row were considered in the proof of b) (and the composition is a quasi-isomorphism), and the vertical arrows are induced by the quasi-isomorphism \( \mathcal{H}om(I, J) \to K \). At least the left and right vertical arrows are quasi-isomorphisms. Thus the composition of arrows in the bottom row (which are maps of DG \( A^{\text{op}} \otimes A \)-modules)
is a quasi-isomorphism. Now recall the quasi-isomorphism of DG $A^{\text{op}}$-modules $A^{\text{op}} \to \text{Hom}(\text{Hom}(I, J), K)$ from the proof of a). As a result we obtain a quasi-isomorphism of DG $A^{\text{op}} \otimes A$-modules

$$A^{\text{op}} \otimes A \to \text{Hom}(\text{Hom}(I, J), K) \otimes A \to \text{Hom}(\text{Hom}(I, J) \otimes I, L) = \Psi(L)$$

as required.

Now it is easy to see that $\Psi(\Delta_* D_Y) = A$ (with the diagonal DG $A^{\text{op}} \otimes A$-module structure). Namely, denote by $Y \xleftarrow{p} Y \times Y \xrightarrow{q} Y$ the two projections. Then

$$\Psi(\Delta_* D_Y) = \mathcal{R} \text{Hom}(\mathcal{H}om(I, J) \otimes I, \Delta_* D_Y)$$

$$= \mathcal{R} \text{Hom}(p^* I, \mathcal{R} \text{Hom}(q^* \mathcal{H}om(I, J), \Delta_* D_Y))$$

$$= \mathcal{R} \text{Hom}(p^* I, \Delta_* \mathcal{H}om(\mathcal{H}om(I, J), J))$$

$$= \mathcal{R} \text{Hom}(p^* I, \Delta_* \mathcal{H}om(\mathcal{H}om(I, J), J))$$

$$= \mathcal{R} \text{Hom}(\mathcal{L} \Delta^* p^* I, \mathcal{H}om(\mathcal{H}om(I, J), J))$$

$$= \mathcal{R} \text{Hom}(\mathcal{L} \Delta^* p^* I, \mathcal{H}om(\mathcal{H}om(I, J), J))$$

Note that all these equalities are quasi-isomorphisms of DG $A^{\text{op}} \otimes A$-modules. Note also that the natural map $I \to \mathcal{H}om(\mathcal{H}om(I, J), J)$ is a quasi-isomorphism of DG $A^{\text{op}}$-modules. Hence we obtain a quasi-isomorphism of DG $A^{\text{op}} \otimes A$-modules

$$\mathcal{R} \text{Hom}(I, \mathcal{H}om(\mathcal{H}om(I, J), J)) = \text{Hom}(I, I) = A$$

as required. This proves c) and the proposition.

The proof of Proposition 6.17 gives more than stated. Namely, using similar arguments we obtain the following result.

**Proposition 6.20.** Let $Y, Z$ be noetherian $k$-schemes, $F_1, F_2 \in D^b(\text{coh} Y)$, $G_1, G_2 \in D^b(\text{coh} Z)$.

a) There exists a natural quasi-isomorphism of complexes

$$\mathcal{R} \text{Hom}(F_1, F_2) \otimes \mathcal{R} \text{Hom}(G_1, G_2) \simeq \mathcal{R} \text{Hom}(F_1 \otimes G_1, F_2 \otimes G_2).$$

b) There exists a natural quasi-isomorphism of DG algebras

$$\mathcal{R} \text{Hom}(F_1, F_1) \otimes \mathcal{R} \text{Hom}(G_1, G_1) \simeq \mathcal{R} \text{Hom}(F_1 \otimes G_1, F_1 \otimes G_1).$$

6.2. **Concluding remarks on Theorem 6.3.** Assume that the field $k$ is perfect. By Theorem 6.3 for a separated scheme of finite type $X$ there exists a canonical (up to equivalence) categorical resolution of singularities $D(X) \to D(A)$. It has the flavor of Koszul duality (Subsection 5.1) and may be called the "inner" resolution. It has two notable properties: 1) The DG algebra $A$ is derived equivalent to $A^{\text{op}}$ (indeed, we can use a classical generator $E$ for $D^b(\text{coh} X)$ or its dual $D(E)$); 2) $A$ usually has unbounded cohomology. In the forthcoming paper [Lu2] we suggest a different type of a categorical resolution
of $D(X)$: the resolving smooth DG algebra has bounded cohomology, but is usually not derived equivalent to its opposite.

6.3. Some remarks on duality for noetherian schemes.

**Definition 6.21.** Let $\mathcal{D}$ be a triangulated category. An object $M \in \mathcal{D}$ is called homologically (resp. cohomologically) finite if for every $N \in \mathcal{D}$, $\text{Hom}(M, N[i]) = 0$ for $|i| >> 0$ (resp. $\text{Hom}(N, M[i]) = 0$ for $|i| >> 0$). Denote by $\mathcal{D}_{\text{hf}}$ (resp. $\mathcal{D}_{\text{chf}}$) the full triangulated subcategory of $\mathcal{D}$ consisting of homologically (resp. cohomologically) finite objects.

**Definition 6.22.** For a noetherian scheme $Y$ consider the bifunctor $\mathbb{R}\text{Hom}(-, -) : \mathcal{D}^b(\text{coh}\, Y)^{\text{op}} \times \mathcal{D}^b(\text{coh}\, Y) \to \mathcal{D}^+(\text{coh}\, Y)$.

We say that $F \in \mathcal{D}^b(\text{coh}\, Y)$ is locally homologically (resp. locally cohomologically) finite if $\mathbb{R}\text{Hom}(F, G) \in \mathcal{D}^b(\text{coh}\, Y)$ (resp. $\mathbb{R}\text{Hom}(G, F) \in \mathcal{D}^b(\text{coh}\, Y)$) for all $G \in \mathcal{D}^b(\text{coh}\, Y)$. Let $\mathcal{D}^b(\text{coh}\, Y)_{\text{hf}}$ (resp. $\mathcal{D}^b(\text{coh}\, Y)_{\text{chf}}$) be the full subcategory of $\mathcal{D}^b(\text{coh}\, Y)$ consisting of locally homologically (resp. locally cohomologically) finite objects.

Let $Y$ be a noetherian scheme with a dualizing complex $D_Y \in \mathcal{D}^b(\text{coh}\, Y)$. The duality equivalence

$$D(-) = \mathbb{R}\text{Hom}(-, D_Y) : \mathcal{D}^b(\text{coh}\, Y)^{\text{op}} \to \mathcal{D}^b(\text{coh}\, Y)$$

induces equivalences

$$D : \mathcal{D}^b(\text{coh}\, Y)_{\text{hf}}^{\text{op}} \to \mathcal{D}^b(\text{coh}\, Y)_{\text{chf}},$$

$$D : \mathcal{D}^b(\text{coh}\, Y)_{\text{chf}}^{\text{op}} \to \mathcal{D}^b(\text{coh}\, Y)_{\text{hf}}.$$  

Denote by $\text{Fid}(Y) \subset \mathcal{D}^b(\text{coh}\, Y)$ the full subcategory consisting of complexes which are quasi-isomorphic to a finite complex of injectives in $\text{Qcoh}\, X$.

**Lemma 6.23.** Let $Y$ be a noetherian scheme with a dualizing complex, $F \in \mathcal{D}^b(\text{coh}\, Y)$. Then the conditions a), b), c) are equivalent

a) $F \in \text{Perf}(Y)$,
b) $F \in \mathcal{D}^b(\text{coh}\, Y)_{\text{hf}}$,
c) $F \in \mathcal{D}^b(\text{coh}\, Y)_{\text{hf}}$.

Also the dual conditions d), e), f) are equivalent

d) $F \in \text{Fid}(Y)$,
e) $F \in \mathcal{D}^b(\text{coh}\, Y)_{\text{chf}}$,
f) $F \in \mathcal{D}^b(\text{coh}\, Y)_{\text{chf}}$. 

Definition 6.26. A DG $A$-module $M$ is called bounded if $H^i(M) = 0$ for $|i| > 0$. Denote by $D^b(A) \subset D(A)$ the full subcategory consisting of bounded DG modules. Put $\text{Perf}(A)^b = \text{Perf}(A) \cap D^b(A)$. 

Proof. It is obvious that $a) \Rightarrow b) \Rightarrow c)$. 

Assume that $F \in D^b(\text{coh} Y_{hf})$. Let $U = \text{Spec} C$ be an open affine subscheme of $Y$. Then $C$ is a noetherian $k$-algebra. Choose a bounded above complex $P = \ldots \rightarrow P^n \rightarrow P^{n+1} \rightarrow \ldots$ of free $C$-modules of finite rank which is quasi-isomorphic to $F|_U$. Then for $n << 0$ the truncation $\tau_{\geq n} P = 0 \rightarrow \text{Ker} d^n \rightarrow P^n \rightarrow P^{n+1} \rightarrow \ldots$ is also quasi-isomorphic to $F|_U$. Let $x \in U$ be a closed point with the residue field $k(x)$. Since $\text{Ext}^n(F, k(x)) = 0$ for $m >> 0$, this implies that $\text{Ext}_C^>(\text{Ker} d^n, k(x)) = 0$. Hence the $C$-module $\text{Ker} d^n$ is free at $x$ for $n >> 0$. Hence it is free in an open neighborhood of $x$. So $F \in \text{Perf}(Y)$. 

Again the implications $d) \Rightarrow e) \Rightarrow f)$ are clear. Actually $d) \Leftrightarrow e)$ by [Ha2],II,Prop.7.20. It remains to prove that $f) \Rightarrow e)$. Let $F \in D^b(\text{coh} Y_{chf})$. Then $D(F) \in D^b(\text{coh} Y_{hf})$, so also $D(F) \in D^b(\text{coh} Y_{chf})$ by $c) \Rightarrow b)$. But then $D(D(F)) = F \in D^b(\text{coh} Y_{chf})$. 

Corollary 6.24. In the above notation the duality functor induces an equivalence $D : \text{Perf}(Y)^{\text{op}} \sim \text{Fid}(Y)$. 

Proof. This follows from Lemma 6.23. 

Recall that a noetherian scheme $Y$ is called Gorenstein, if all its local rings are Gorenstein local rings. Then $Y$ is Gorenstein if and only if $\mathcal{O}_Y$ is a dualizing complex on $Y$ [Ha2]. 

Lemma 6.25. A noetherian scheme $Y$ is Gorenstein if and only if $\text{Perf}(Y) = \text{Fid}(Y)$. 

Proof. The functor $\mathbf{R}\text{Hom}(-, \mathcal{O}_Y) : D^b(\text{coh} Y)^{\text{op}} \rightarrow D^+(\text{coh} Y)$ induces an equivalence $\text{Perf}(Y)^{\text{op}} \sim \text{Perf}(Y)$. So if $Y$ is Gorenstein then $\text{Perf}(Y) = \text{Fid}(Y)$ by Corollary 6.24. 

Conversely if $\text{Perf}(Y) = \text{Fid}(Y)$ then in particular $\mathcal{O}_Y \in \text{Fid}(Y)$. In any case $\mathbf{R}\text{Hom}(\mathcal{O}_Y, \mathcal{O}_Y) = \mathcal{O}_Y$, so $\mathcal{O}_Y$ is a dualizing complex on $Y$ by [Ha2],Ch.V,Prop.2.1. 

6.4. Canonical categorical resolution as a mirror which switches ”perfect” and ”bounded”. Let the field $k$ be perfect and $X$ be a separated $k$-scheme of finite type with a dualizing complex $D_X \in D^b(\text{coh} X)$. 

Choose a classical generator $E \in D^b(\text{coh} X)$ and denote the corresponding equivalence 

$$\Psi(-) := \mathbf{R}\text{Hom}(E, -) : D(\text{coh} X) \rightarrow \text{Perf}(A),$$

where $A = \mathbf{R}\text{Hom}(E, E)$ (Theorem 6.3). Consider also the equivalence 

$$\Psi \cdot D(-) = \mathbf{R}\text{Hom}(E, \mathbf{R}\text{Hom}(-, D_X)) : D(\text{coh} X)^{\text{op}} \rightarrow \text{Perf}(A).$$

Definition 6.26. A DG $A$-module $M$ is called bounded if $H^i(M) = 0$ for $|i| > 0$. Denote by $D^b(A) \subset D(A)$ the full subcategory consisting of bounded DG modules. Put $\text{Perf}(A)^b = \text{Perf}(A) \cap D^b(A)$. 

Proposition 6.27. a) The functor $\Psi$ induces an equivalence $\text{Fid}(X) \cong \text{Perf}(A)^b$;  
b) The composition $\Psi \cdot D$ induces an equivalence $\text{Perf}(X)^{\text{op}} \simeq \text{Perf}(A)^b$.

Proof. a). Clearly $\Psi(\text{Fid}(X)) \subset \text{Perf}(A)^b$. Vice versa, assume that $\Psi(G) \in \text{Perf}(A)^b$ for some $G \in D^b(\text{coh}X).$ Since $E$ is a classical generator for $D^b(\text{coh}X)$ the complex $R\text{Hom}(F,G)$ has bounded cohomology for all $F \in D^b(\text{coh}X)$. That is $G \in D^b(\text{coh}X)_{\text{chf}}$. But then $F \in \text{Fid}(X)$ by Lemma 6.23.

b). Follows from a) and Corollary 6.24. □

Recall the triangulated category of singularities $D_{\text{sg}}(X) = D^b(\text{coh}X)/\text{Perf}(X)$ ([Or]).

Corollary 6.28. The functor $\Psi \cdot D$ induces an equivalence

$$D_{\text{sg}}(X)^{\text{op}} \simeq \text{Perf}(A)/\text{Perf}(A)^b.$$  

Corollary 6.29. Assume that $X$ Gorenstein. Then in the context of Proposition 6.27 the functor $\Psi$ induces an equivalence $\text{Perf}(X) \rightarrow \text{Perf}(A)^b$. Hence in particular $D_{\text{sg}}(X) \simeq \text{Perf}(A)/\text{Perf}(A)^b$.

Proof. Since $X$ is Gorenstein $\text{Perf}(X) = \text{Fid}(X)$. Hence the corollary follows from Proposition 6.27a). □

6.5. Connection with the stable derived category of a locally noetherian Grothendieck category. Let $\mathcal{A}$ be a locally noetherian Grothendieck category such that its derived category $D(\mathcal{A})$ is compactly generated. Denote by $\text{noeth} \mathcal{A} \subset \mathcal{A}$ the full subcategory of noetherian objects. Let $\text{Inj} \mathcal{A} \subset \mathcal{A}$ be the full subcategory of injective objects and consider its homotopy category $K(\text{Inj} \mathcal{A}) := Ho(\text{Inj} \mathcal{A}).$ Let $S(\mathcal{A}) \subset K(\text{Inj} \mathcal{A})$ be the full triangulated category of acyclic complexes. In [Kr] the following assertions were proved:

1) The natural diagram of triangulated categories and exact functors

$$S(\mathcal{A}) \overset{I}{\longrightarrow} K(\text{Inj} \mathcal{A}) \overset{Q}{\longrightarrow} D(\mathcal{A})$$

is a localization sequence, in particular $D(\mathcal{A}) \simeq K(\text{Inj} \mathcal{A})/S(\mathcal{A})$.

2) The functors $I, Q$ have left adjoints $I_\lambda, Q_\lambda$ and right adjoints $I_\rho, Q_\rho$ respectively.

3) The category $K(\text{Inj} \mathcal{A})$ is cocomplete and compactly generated.

4) The functor $Q$ induces an equivalence of categories

$$K(\text{Inj} \mathcal{A})^c \overset{\sim}{\longrightarrow} D^b(\text{noeth} \mathcal{A})$$

with the quasi-inverse being induced by $Q_\rho$.

In [Kr] the category $S(\mathcal{A})$ is called the stable derived category of $\mathcal{A}$ and Krause suggests a deeper study of the category $K(\text{Inj} \mathcal{A})$.  

Let \( k \) be a perfect field and \( X \) be a separated \( k \)-scheme of finite type. The Grothendieck category \( \mathcal{A} = Qcoh X \) is locally noetherian with \( \text{noeth} \mathcal{A} = coh X \). The derived category \( D(X) = D(\mathcal{A}) \) is compactly generated (Theorem 2.10). We denote \( \text{Inj} X = \text{Inj} \mathcal{A}, \text{K}(\text{Inj} X) = \text{K}(\text{Inj} \mathcal{A}), \text{S}(X) = \text{S}(\mathcal{A}) \). So we obtain the localization sequence

\[
\text{S}(X) \rightarrow \text{K}(\text{Inj} X) \rightarrow D(X).
\]

**Proposition 6.30.** The pair \((\text{K}(\text{Inj} X), Q)\) is a categorical resolution of \( D(X) \) which is equivalent to the canonical resolution constructed in Theorem 6.3. In particular the category \( \text{K}(\text{Inj} X) \) is smooth.

**Proof.** Let \( \text{K}_{\text{Inj}}(X) \subset \text{K}(\text{Inj} X) \) be the full subcategory consisting of h-injective complexes (with injective components). The functor \( Q \) induces an equivalence \( \text{K}_{\text{Inj}}(X) \rightarrow D(X) \).

Its quasi-inverse (=”taking h-injective resolution”) composed with the inclusion \( \text{K}_{\text{Inj}}(X) \subset \text{K}(\text{Inj} X) \) is the functor \( Q_{\rho} \). Thus we may identify \( D(X) \) with \( \text{K}_{\text{Inj}}(X) \).

Hence by 4) above the category \( D^b(coh X) \) is identified with \( \text{K}(\text{Inj} X)^c \).

Let \( E \in \text{K}_{\text{Inj}}(X) \) be a classical generator of \( D^b(coh X) \), hence also of \( \text{K}(\text{Inj} X)^c \); put \( A = \text{Hom}(E, E) \). By Theorem 6.3 the functor

\[
\Psi_E : \text{K}_{\text{Inj}}(X) \rightarrow D(A), \quad \Psi_E(-) = \text{Hom}(E, -)
\]

is a categorical resolution. So it suffices to prove that the functor \( \Psi_E' : \text{K}(\text{Inj} X) \rightarrow D(A) \) defined by the same formula is an equivalence.

We know that the category \( \text{K}(\text{Inj} X) \) is cocomplete. Hence by Theorem 2.2 b) \( E \in \text{K}(\text{Inj} X)^c \) is a compact generator for \( \text{K}(\text{Inj} X) \). Now one shows that \( \Psi_E' \) is an equivalence by copying the proof of Proposition 2.6.

We thank Michel Van den Bergh for pointing to us the connection between our categorical resolution of \( D(X) \) and the category \( \text{K}(\text{Inj} X) \).

**Question.** For which locally noetherian Grothendieck categories \( \mathcal{A} \) (such that \( D(\mathcal{A}) \) is compactly generated) the category \( \text{K}(\text{Inj} \mathcal{A}) \) is smooth (hence a categorical resolution of \( D(\mathcal{A}) \))?

**References**

[AlJeLi] L. Alonso, A. Jeremias, J. Lipman, Local homology and cohomology on schemes, Ann. Sci Ecole Norm. Sup. (4) 30 (1997), no.1, 1-39.

[BoNe] M. Bokstedt, A. Neeman, Homotopy limits in triangulated categories, Compositio Math. tome 86, no. 2 (1993), p.209-234.

[BoKa1] A. I. Bondal, M. M. Kapranov, Enhanced triangulated categories, *Math. USSR - Sbornik* 70 (1991), No. 1, 93-107.
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[BoKa2] A. I. Bondal, M. M. Kapranov, Representable functors, Serre functors, and reconstructions, Math USSR-Izv. 35 (1990), no. 3, 519-541.

[BoVdB] A. Bondal, M. Van den Bergh, Generators and representability of functors in commutative and noncommutative geometry, Mosc. Math. J. 3 (2003), no.1, 1-36.

[BuDr] I. Burban, Yu. Drozd, Tilting on noncommutative rational projective curves, arXiv:0905.1231

[ELOII] A. I. Efimov, V. A. Lunts, D. O. Orlov, Deformation theory of objects in homotopy and derived categories II: pro-representability of the deformation functor, arXiv:math/0702839.

[GiSou] Gillet, Soule, Descent, motives and K-theory, J. Reine Angew. Math. 478 (1996), 127-176.

[Ha1] R. Hartshorn, Algebraic Geometry, Springer 1977.

[Ha2] R. Hartshorn, Residues and Duality, LNM 20.

[KaSch] M. Kashiwara, P. Schapira, Categories and sheaves, Springer-Verlag, 2006.

[Ke1] B. Keller, Deriving DG categories, Ann. scient. Éc. Norm. Sup., 4e série, t. 27, (1994), 63-102.

[Ke2] B. Keller, On differential graded categories, International Congress of Mathematicians. Vol. II, 151–190, Eur. Math. Soc., Zrich, 2006

[Kr] H. Krause, The stable derived category of a noetherian scheme, arXiv:0403526.

[Ku] A. Kuznetsov, Lefschetz decompositions and categorical resolutions of singularities, Selecta Math. (N.S.) 13 (2008), no. 4 661-696.

[Li] J. Lipman, Foundations of Grothendieck duality for diagrams of schemes, LNM 1960.

[Lu1] V. A. Lunts, Coherent sheaves on configuration schemes, Journal of Algebra 244, 379-406 (2001).

[Lu2] V. A. Lunts, Coherent sheaves on poset schemes and categorical resolutions of singularities, (in preparation).

[Ma] A. Neeman, Commutative ring theory, Cambridge Univ. Press (2006).

[Ne] A. Neeman, The connection between the K-theory localization theorem of Thomason, Trobaugh and Yao and the smashing subcategories of Bousfield and Ravenel, Annales sci. de l'E.N.S. 4 serie, tome 25, no. 5 (1992), p. 547-566.

[Or] D. Orlov, Triangulated categories of singularities and D-branes in Landau-Ginzburg models, Proc. Steklov Inst. Math. 2004, no.3 (246) 227-248.

[Rou] R. Rouquier, Dimensions of triangulated categories. J. K-Theory 1 (2008), no. 2, 193-256.

[Sp] N. Spaltenstein, Resolutions of unbounded complexes, Compositio Math. 65:121-154 (1988).

[StVdB] T. Stafford, M. Van deh Bergh, Noncommutative resolutions and rational singularities, Michigan Math. J. 57 (2008) 659-674.

[ThTr] R. W. Thomason, T. Trobaugh, Higher algebraic K-theory of schemes and of derived categories, The Grothendieck Festschrift, Vol. III, Birkhauser 1990.

[VdB] M. Van den Bergh, Non-commutative crepant resolutions, The legacy of Niels Henrik Abel, 749-770, Springer, Berlin, 2004.

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