An Additive Noise Approximation to Keller–Segel–Dean–Kawasaki Dynamics Part I: Local Well-Posedness of Paracontrolled Solutions

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Abstract

Using the method of paracontrolled distributions, we show the local well-posedness of an additive noise approximation to the fluctuating hydrodynamics of the Keller–Segel model on the two-dimensional torus. Our approximation is a non-linear, non-local, parabolic-elliptic stochastic PDE with an irregular, heterogeneous space-time noise. As a consequence of the irregularity and heterogeneity, solutions to this equation must be renormalised by a sequence of diverging fields. Using the symmetry of the elliptic Green’s function, which appears in our non-local term, we establish that the renormalisation diverges at most logarithmically, an improvement over the linear divergence one would expect by power counting. Similar cancellations also serve to reduce the number of diverging counterterms.

Keywords: singular stochastic partial differential equation, paracontrolled distributions, linear fluctuating hydrodynamics, parabolic-elliptic Keller–Segel model, Dean–Kawasaki equation;

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1 Introduction

In this work we are concerned with the local well-posedness of singular SPDE of the kind,
\[
\begin{cases}
    (\partial_t - \Delta)\rho = \nabla \cdot (\rho \nabla \Phi_{\rho}) + \nabla \cdot (\sigma \xi), & \text{in } [0, T) \times \mathbb{T}^2, \\
    -\Delta \Phi_{\rho} = \rho - \langle \rho, 1 \rangle_{L^2(\mathbb{T}^2)}, & \text{in } [0, T) \times \mathbb{T}^2, \\
    \rho|_{t=0} = \rho_0, & \text{on } \mathbb{T}^2,
\end{cases}
\]
where \( \xi = (\xi_1, \xi_2) \) is a two-dimensional vector of i.i.d. space-time white noises, \( \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 \) is the two-dimensional torus, \( T \in \mathbb{R}_+ \cup \{ +\infty \} \), \( \sigma \in C([0, T); \mathcal{H}^2(\mathbb{T}^2)) \) and \( \rho_0 \) is a suitable initial data which we specify later. The advection comes from the Keller–Segel model of chemotaxis [KS70]. Our interest in (1.1) stems from the theory of fluctuating hydrodynamics where one would ideally set \( \sigma = \sqrt{\rho} \) to obtain the Dean–Kawasaki noise [Dea96; Kaw94]. However, it was shown in [KLvR19; KLvR20] that the equation with smooth drift only admits solutions which are empirical measures of the underlying interacting particle system. Hence one does not expect (1.1) with \( \sigma = \sqrt{\rho} \) to admit non-atomic solutions.

Motivated by the theory of linear fluctuating hydrodynamics, our main motivating example is instead given by the choice \( \sigma = \sqrt{\rho_{\text{det}}} \) where \( \rho_{\text{det}} \) solves the deterministic PDE,
\[
\begin{cases}
    (\partial_t - \Delta)\rho_{\text{det}} = \nabla \cdot (\rho_{\text{det}} \nabla \Phi_{\rho_{\text{det}}}), & \text{in } \mathbb{R}_+ \times \mathbb{T}^2, \\
    -\Delta \Phi_{\rho_{\text{det}}} = \rho_{\text{det}} - \langle \rho_{\text{det}}, 1 \rangle_{L^2(\mathbb{T}^2)}, & \text{in } \mathbb{R}_+ \times \mathbb{T}^2, \\
    \rho_{\text{det}}|_{t=0} = \rho_0, & \text{on } \mathbb{T}^2,
\end{cases}
\]
with \( \rho_0 \) sufficiently regular. This choice will be applied in a follow-up paper, [MM22], to an additive-noise approximation to the Dean–Kawasaki equation associated to the Keller–Segel model. We also remark that this eventual application motivates us to consider mollified noise terms of the form \( \sigma(\psi \ast \xi) \), rather than mollifying the whole product, see [CSZ19, Sec. 3.2] and [FG19; DFG20].

Remark 1.1. We may also consider equations of the form
\[
\begin{cases}
    (\partial_t - \Delta)\rho_{\text{det}} = -\chi \nabla \cdot (\rho_{\text{det}} \nabla \Phi_{\rho_{\text{det}}}), & \text{in } [0, T) \times \mathbb{T}^2, \\
    -\Delta \Phi_{\rho_{\text{det}}} = \rho_{\text{det}} - \langle \rho_{\text{det}}, 1 \rangle_{L^2(\mathbb{T}^2)}, & \text{in } [0, T) \times \mathbb{T}^2, \\
    \rho_{\text{det}}|_{t=0} = \rho_0, & \text{on } \mathbb{T}^2,
\end{cases}
\]
where \( \chi \in \mathbb{R} \). In this setting, when one restricts \( \rho_0 \) to be non-negative and to integrate to 1 (i.e. the density of a probability measure) one recovers the usual parabolic-elliptic Keller–Segel equation, [KS70], the analysis of which has received much attention [Hor03; Hor04; HP09; Pai19]. The global existence of (1.3) in spatial dimensions two and higher depends on the size and sign of \( \chi \), [JL92; CPZ04; BDP06]. Since we are only concerned with local existence and all of our analysis is agnostic as to the size and sign of \( \chi \) we set it to be \(-1\) and work with equations of the form (1.1) and (1.2).

In this paper we will treat the general case where \( \sigma \) is an arbitrary function continuous in time and \( \mathcal{H}^2 \) in space. Due to the singularity of the noise, defining a suitable notion of solution to (1.1) is non-trivial and we will implement a paracontrolled approach to obtain local well-posedness, [GIP15]. To see why such an approach is necessary we consider the terms of (1.1) under a formal power counting argument. The proper definition of all function spaces used below can be found in Appendix A.
For any $T > 0$, the white noise takes values in $C_{par}^{−2}([0, T] \times \mathbb{T}^2)$.

Let us assume for now that we can define the product $\sigma \xi$ intrinsically and that it is no more regular than the white noise itself. Due to the regularising effect of the heat equation, the solution $1$, to the linear equation,

$$
\begin{cases}
(\partial_t - \Delta)1 = \nabla \cdot (\sigma \xi), & \text{in } \mathbb{R}_+ \times \mathbb{T}^2, \\
1|_{t=0} = 0, & \text{on } \mathbb{T}^2,
\end{cases}
$$

may be no more regular than $C_T C^{-1−}$. Here $C^\alpha(\mathbb{T}^2)$ denotes the Hölder–Besov space of regularity $\alpha \in \mathbb{R}$. Assuming that this regularity is passed to $\rho$ and applying the regularising effect of the elliptic equation we would have $\nabla \Psi_\rho \in C_T C^{0−}$. However, by Bony’s estimate the product $fg$, is only a priori well-defined for $f \in C^\alpha$ and $g \in C^2$ with $\alpha + \beta > 0$. Hence, the product $\rho \nabla \Phi_\rho$ is not a priori well-defined.

The theories of regularity structures, paracontrolled distributions, renormalisation groups and various recent extensions and adaptations thereof have revolutionised the study of singular SPDE [Hai14; GIP15; Kup16; Ott+21; Lin+21]. The common thread throughout these theories is to notice that the factors of the ill-defined products are not generic distributions but inherent structure from the noise. This inheritance allows one to define renormalised products, which excise the singular part, allowing one to give meaning to a renormalised equation which is continuous in a finite tuple of diagrams built from the noise. The noise, along with these diagrams, is referred to as an enhancement.

The theory of paracontrolled distributions was first developed by M. Gubinelli, P. Imkeller and N. Perkowski in [GIP15]. The central idea is to use harmonic analysis to construct regular commutators, which allow us to decompose the equation into exogenous noise terms and terms that can be constructed as fixed points. Paracontrolled distributions have been successfully applied to analyse a range of singular SPDE and operators including; the parabolic Anderson model (PAM) [GIP15; KPvZ20], the Anderson Hamiltonian [AC15; GUZ20; CvZ21], the $\Phi_3^4$ model [MW17b; CC18], the Kardar–Parisi–Zhang equation [GP17], the stochastic Burgers and Navier–Stokes equations [ZZ15; GP17] and the stochastic non-linear wave equation [GKO18].

In our case, we first define $\xi_{\delta} := (\psi_{\delta} * \xi^1, \psi_{\delta} * \xi^2)$, where $\psi_{\delta}$ denotes a standard, symmetric mollifier. We find that there exists a field $f_{\delta} : [0, T] \times \mathbb{T}^2 \to \mathbb{R}^2$ satisfying the bound,

$$
\|f_{\delta}(t)\|_{C^{−1−}} \lesssim \log(\delta^{-1})\|\sigma\|^2_{C_T H^2},
$$

and such that the sequence of solutions $(\rho_{\delta})_{\delta \in (0, 1)}$ each solving,

$$
\begin{cases}
(\partial_t - \Delta)\rho_{\delta} = \nabla \cdot (\rho_{\delta} \nabla \Phi_{\rho_{\delta}} - f_{\delta}) + \nabla \cdot (\sigma \xi_{\delta}), & \text{in } [0, T] \times \mathbb{T}^2, \\
-\Delta \Phi_{\rho_{\delta}} = \rho_{\delta} - \langle \rho_{\delta}, 1 \rangle_{L^2(\mathbb{T}^2)}, & \text{in } [0, T] \times \mathbb{T}^2, \\
\rho_{\delta}|_{t=0} = \rho_0, & \text{on } \mathbb{T}^2,
\end{cases}
$$

converge in $C_T C^{−1−}$ to a unique limit $\rho$, which we designate as the renormalised solution to (1.1).

Three points of interest arise from (1.4). Firstly, in the case where $\sigma$ is genuinely heterogeneous the field $f_{\delta}$ is in general also heterogeneous. This has been observed elsewhere, having been pointed out as a possibility in [Hai14] and seen explicitly in the renormalisation of singular SPDE on bounded domains, [GH19]. Secondly, if $\sigma$ is a constant, so that our noise agrees with that of the stochastic Burgers’ equation, then $f_{\delta}$ is zero. In this case the renormalised equation agrees exactly with the singular equation, i.e. the products are not explicitly renormalised when $\delta = 0$. This phenomenon has also been observed in [DPDT94; DPD02; ZZ15; GP17]. Thirdly, using the informal power counting described above, one might expect the singular product $\rho_{\delta} \nabla \Phi_{\rho_{\delta}}$ to diverge at the order of $\delta^{-1}$, since

---

1. Here $C_{par}^{−2}$ denotes the set of space-time Hölder-regular distributions of parameter $\alpha \in \mathbb{R}$ equipped with the parabolic scaling, i.e. regularity in time counts twice as much as regularity in space. These spaces are not used beyond the introduction and so we refer to [Hai14, Lem. 2.12 & Def. 3.7] for an example definition. The shorthand $\alpha \pm \varepsilon$ is used to denote $\alpha \pm \varepsilon$ for any $\varepsilon > 0$ but fixed.
this is the gap in regularity between the singular factors. However, (1.4) shows that this divergence is at most logarithmic. This improvement arises from symmetries in the fundamental solution of the elliptic problem, leading to non-trivial cancellations in our stochastic estimates.

To demonstrate the underlying principle, let us consider a one-dimensional example. We assume that $u^\delta \to u$ as $\delta \to 0$ in a space of regularity $-1/2-$. The product rule gives the identity,

$$u^\delta \partial_x \partial_x^{-2} u^\delta = \frac{1}{2}(\partial_x^2(\partial_x^{-1} u^\delta \partial_x^{-2} u^\delta) - \partial_x(u^\delta \partial_x^{-2} u^\delta)),$$

where we write $\partial_x^{-1}$ as a shorthand for integration in $x$, with the (arbitrary) normalization that the primitive is mean-free. While the product on the left hand side, between an object converging in $-1/2-$ and an object converging in $1/2-$ looks ill-posed, the right hand side is in fact classically well-posed: the first term is the second derivative of a product between objects in $1/2-$ and $3/2-$, while the second is the derivative of a product between an object in $-1/2-$ and one in $3/2-$. Hence the anticipated logarithmic divergence of the left hand side is removed by expanding as on the right hand side. This basic observation extends to our higher-dimensional case through the symmetry of the Green’s function for Poisson’s equation. We see that the symmetry alleviates divergences by one order. Linear divergences of $\delta^{-1}$ are improved to logarithmic, and logarithmic divergences are improved to well-posedness. The heterogeneity $\sigma$ makes these improvements visible, as when $\sigma$ is constant the same symmetries lead to perfect cancellations removing the need for renormalising counterterms all together. Similar observations have also been made in the context of the KPZ equation, [GP17, Lem. 9.5].

We observe that (1.1) is an example of a singular SPDE involving anisotropic regularity, the regularising effect of the elliptic equation only takes place in the spatial variable. In this regard the theory of paraccontrolled distributions proves especially convenient, since most of the analysis is conducted on the Fourier side so that space and time can be treated separately.

**Structure of the Paper:** In the rest of this section we first recall some basic notations and conventions which are used throughout the text. Some of these are accompanied by more rigorous presentations in the appendices. We then present an outline of the general strategy and our main result in Subsection 1.2. Section 2 contains a detailed proof of the existence and regularity of the various stochastic objects which we are required to construct and constitute our enhanced noise. The careful analysis of these stochastic objects and control over the diverging fields is the main contribution of this paper. In Section 3 we show the local well-posedness of paraccontrolled solutions given a suitable enhancement of the noise. Finally we include three appendices: Appendix A recalls some useful and well-known results concerning Besov spaces and paraproducts; Appendix B provides various estimates on the so-called shape coefficients which we introduce in Section 2 and Appendix C, contains a number of summation and discrete convolution estimates that we make repeated use of throughout the text.

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We define the action of the heat semigroup on \( T^2 \) by
\[
0 \mapsto \int_0^t P_s f(t) \, ds = \int_0^t \mathcal{H}_s f(t) \, ds.
\]
We define the solution to the elliptic equation by
\[
\mathcal{H}_t := \int_0^t P_s f(t) \, ds = \int_0^t \mathcal{H}_s f(t) \, ds.
\]
where for mean-free functions (resp. distributions),

\[
(\mathcal{G} \ast f)(x) = (-\Delta)^{-1} f(x) := \sum_{\omega \in \mathbb{Z}^2 \setminus \{0\}} e^{2\pi i (\omega, x)} \frac{1}{|2\pi \omega|^2} \hat{f}(\omega).
\]

Given a Banach space \( E \), a subset \( I \subseteq [0, \infty) \) and \( \kappa \in (0, 1) \), we write \( C_I E := C(I; E) \) (resp. \( C_I^\alpha E := C^\alpha(I; E) \)) for the space of continuous (resp. \( \kappa \)-Hölder continuous) maps \( f : I \to E \) equipped with the norm \( \|f\|_{C_I E} := \sup_{t \in I} \|f_t\|_E \) (resp. \( \|f\|_{C_I^\alpha E} := \|f\|_{C_I E} + \sup_{s \neq t \in I} \frac{\|f_t - f_s\|_E}{|t-s|^\kappa} \)). For \( T > 0 \), we use the shorthand \( C_T E = C_{[0,T]} E \) and \( C_T^\alpha E = C_{[0,T]}^\alpha E \). Note that the norm \( \|f\|_{C_T^\alpha E} \) is equivalent to \( \|f_0\|_E + \sup_{s \neq t \in [0,T]} \frac{\|f_t - f_s\|_E}{|t-s|^\kappa} \). For \( \kappa, \eta > 0 \) we let \( C_{\eta,T} E := C_{\eta}((0,T); E) \) and \( C_{\eta,T}^\alpha E := C_{\eta}((0,T); E) \) denote the Banach spaces of functions \( f : (0,T) \to E \) which are finite under the norms,

\[
\|f\|_{C_{\eta,T} E} := \sup_{t \in (0,T]} (1 + t)^{\eta} \|f_t\|_E, \quad \|f\|_{C_{\eta,T}^\alpha E} := \|f\|_{C_{\eta,T} E} + \sup_{s \neq t \in (0,T]} (1 + s \wedge t)^\eta \frac{\|f_t - f_s\|_E}{|t-s|^\kappa}.
\]

We also make use of the notation

\[
\|u\|_{\mathcal{L}^{\alpha}_{\eta,T} C^\alpha} := \max\{\|u\|_{C_{\eta,T}^\alpha C^{\alpha-2\kappa}}, \|u\|_{C_{\eta,T} C^\alpha}\},
\]

to denote a weighted interpolation space. We set \( \mathcal{L}^{\alpha}_{0,T} C^\alpha := \mathcal{L}^{\alpha}_{0,T} C^\alpha \) and understand \( \mathcal{L}^{\alpha}_{0,T} C^\alpha = \mathcal{L}_{0,T} C^\alpha \).

We write \( \lesssim \) to indicate that an inequality holds up to a constant depending on quantities that we do not keep track of or are fixed throughout. When we do wish to emphasise the dependence on certain quantities \( \alpha, p, d \), we either write \( \lesssim_{\alpha,p,d} \) or define \( C := C(\alpha, p, d) > 0 \) and write \( \leq C \).

Let \( u, v \in \mathcal{S}'(\mathbb{T}^2) \), we define the truncated sums

\[
\begin{align*}
\hat{u}(\omega_1) \hat{v}(\omega_2) &:= \sum_{\omega_1, \omega_2 \in \mathbb{Z}^2} \hat{u}(\omega_1) \hat{v}(\omega_2) \sum_{k,l \in \mathbb{N}} \tilde{g}_{k,l}(\omega_1, \omega_2) g_{k,l}\quad (1.6) \\
\hat{u}(\omega_1) \hat{v}(\omega_2) &:= \sum_{\omega_1, \omega_2 \in \mathbb{Z}^2} \hat{u}(\omega_1) \hat{v}(\omega_2) \sum_{k=1}^\infty \sum_{l=-1}^{k-2} g_l(\omega_1) g_{k,l},\quad (1.7)
\end{align*}
\]

where we implicitly assume that those sums converge absolutely. This is a discrete analogue of the usual paraproduct decomposition, cf. Appendix A.2. If \( \omega_1, \omega_2 \in \mathbb{Z}^2 \setminus \{0\} \), then \( \omega_1 \sim \omega_2 \) implies \( 9/64 \leq |\omega_1| \leq 64/9 \) and \( \omega_1 \sim \omega_2 \) implies \( |\omega_1| \leq 8/9 |\omega_2| \).

We also define a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) with a complete, right-continuous filtration, which we assume large enough to support a countable family of Brownian motions.

### 1.2 Strategy and Main Result

We first outline the paracontrolled approach to (1.1) in a relatively loose manner, identifying the main steps of the method and the diagrams that we will need to give meaning to. Recall that we wish to define a sufficiently robust notion of solution to (1.1) which in particular is stable under regular approximations to the noise. To do so we first write (1.1) in mild form, setting

\[
\rho = P \rho_0 + \nabla \cdot \mathcal{I}[\rho \nabla \Phi_p] + \nabla \cdot \mathcal{I}[\sigma \xi],
\]

In the remainder of this section we will assume that all terms on the right hand side of (1.1) are continuous in time while taking values in a Hölder–Besov space \( C^{\alpha}(\mathbb{T}^2) \), where \( \alpha \) is possibly negative.

Working, for now, with smooth initial data, we may assume that the final term on the right hand side is the least regular component of \( \rho \). Using the same stochastic estimates alluded to in the introduction, along with the regularising effect of the heat kernel and effect of the derivative, we will
work under the assumption that \( I := \nabla \cdot \mathcal{I}[\sigma \xi] \in C_T C^{-1} \). Passing this regularity to \( \rho \) and applying the regularising effect of the elliptic equation we expect to have \( \nabla \Phi_\rho \in C_T C^0 \). Therefore, as discussed in the introduction, the product \( \rho \nabla \Phi_\rho \) is not a priori well-defined. Our first step is to employ the so called Da Prato–Debussche trick, [DPD03], to remove the most singular term by defining \( u := \rho - I \) so that if \( \rho \) is a solution to (1.8),

\[
  u = P \rho_0 + \nabla \cdot \mathcal{I} [u \nabla \Phi_0] + \nabla \cdot \mathcal{I} [u \nabla \Phi_1] + \nabla \cdot \mathcal{I} [1 \nabla \Phi_0] + \nabla \cdot \mathcal{I} [1 \nabla \Phi_1].
\]

We notice that the product \( 1 \nabla \Phi_1 \) is not classically well-posed, however it can be renormalised and replaced with the symbol \( \Psi := \nabla \cdot \mathcal{I} [1 \nabla \Phi_1] - \Phi_\Psi \), where \( \Phi_\Psi := \mathbb{E}(\nabla \cdot \mathcal{I} [1 \nabla \Phi_1]) \) denotes the singular part of this product. The term \( \Psi \) will have the same regularity as \( \nabla \cdot \mathcal{I} [1 \nabla \Phi_1] \in C_T C^0 \), see Subsection 2.4. This will be shown rigorously using stochastic arguments, see Subsection 2.6. From now on we continue with our expansion, replacing the singular product by its renormalised counterpart \( \Psi \) so that we have in fact changed the equation solved by \( \rho \).

We are now in better shape, as we may now work with \( u \in C_T C^0 \), which renders the first product on the right hand side classically well-posed. However, the second and third products remain ill-defined. We may repeat the same trick, defining \( w := u - \Psi \), which should solve,

\[
  w = P \rho_0 + \nabla \cdot \mathcal{I} [w \nabla \Phi_1] + \nabla \cdot \mathcal{I} [1 \nabla \Phi_1] + \nabla \cdot \mathcal{I} [\Psi + \Psi] + Q(w, I, \Psi),
\]

where \( Q(w, I, \Psi) \) denotes a finite sum of classically well-posed terms involving \( w, I, \) and \( \Psi \). The formal definition of Bony’s decomposition into para and resonant products is given in Appendix A, however, for now we simply recall the rules that for \( f \in C^\alpha, g \in C^\beta \), one has

\[
  f \otimes g \in C^{\beta \wedge (\alpha + \beta)} \text{ for any } \alpha, \beta \in \mathbb{R} \quad \text{and} \quad f \otimes g \in C^{\alpha + \beta}, \text{ if } \alpha + \beta > 0.
\]

The new symbol appearing on the right hand side for \( w \) is a shorthand for \( \mathcal{V} := \Psi \otimes \nabla \Phi_1 + \nabla \Phi \otimes \Psi \). Although those resonant products are not classically well-defined, further stochastic arguments show that they can in fact be defined as objects finite in \( C_T C^0 \) without the subtraction of any infinite counterterms. The full definition of \( \mathcal{V} \), through stochastic calculus, is contained in Subsection 2.2. We show in Subsections 2.4 and 2.5 that \( \mathcal{V} \in C_T C^0 \) and so even though it requires significant work to define, it is not the least regular term on the right hand side.

Instead this is given by the paraproduct term, \( \nabla \cdot \mathcal{I} [\Psi \otimes \Phi_1] \), which using Bony’s estimate (Lemma A.4) is only finite in \( C_T C^0 \) and the formal product term \( \nabla \cdot \mathcal{I} [1 \nabla \Phi_1] \), which is not even a priori well-defined. Hence, as before we can only expect to find \( w \in C_T C^0 \) which is not regular enough to define the products \( w \nabla \Phi_1 \) and \( 1 \nabla \Phi \), a priori.

One sees that further applications of the Da Prato–Debussche trick will not improve the situation. Instead we employ the core idea that solutions should resemble the noise at small scales. This is formalised through the paracontrolled Ansatz, that is we only look for solutions such that,

\[
  w = \nabla \Phi_{w+} \Psi \otimes \mathcal{I} [I] + w^#, \quad \nabla \Phi_w = \nabla \Phi_{w+} \Psi \otimes \mathcal{I} [\Phi_1] + (\nabla \Phi_w)^#, \n\]

where \( w^# \) and \( (\nabla \Phi_w)^# \) are terms to be fixed by the equation which we stipulate must be finite in \( C_T C^0 \) and \( C_T C^1 \) respectively.\(^2\) This ensures that the products \( w^# \nabla \Phi_1 \) and \( 1 (\nabla \Phi_w)^# \) are classically well-defined. Rearranging, using the linearity of the map \( f \mapsto \nabla \Phi_f \) and applying Bony’s decomposition to the products \( w \nabla \Phi_1 \) and \( 1 \nabla \Phi_w \), we find the identity

\[
  w^# = P \rho_0 + \nabla \cdot \mathcal{I} [w \otimes \nabla \Phi_1] + \nabla \cdot \mathcal{I} [1 \otimes \nabla \Phi_w] + \nabla \cdot \mathcal{I} [\Psi \otimes 1] - \nabla \Phi_{w+} \Psi \otimes \mathcal{I} [I] \]
\[
  + \mathcal{Q}(w, 1, \Psi, \mathcal{V}),
\]

where \( \mathcal{Q}(w, 1, \Psi, \mathcal{V}) \) is a new polynomial of its arguments and can be expected to be of strictly positive regularity. Hence, the regularity of \( w^# \) is governed by that of the commutator and that of

\(^2\)Note that \( (\nabla \Phi_w)^# \) should be read strictly as a piece of notation and it is not equal to \( \nabla \Phi_{w, #} \). In fact in Section 3 we make use of an equivalent Ansatz which makes certain technical steps easier but is less clear to present - see Remark 3.3.
the resonant products $\nabla \cdot I[w \mathbin{\odot} \nabla \Phi_1]$ and $\nabla \cdot I[f \mathbin{\odot} \nabla \Phi_w]$. The commutator can be controlled by Lemmas A.9 and A.10, which show that

$$\nabla \cdot I[\nabla \Phi_{w+} \mathbin{\odot} \nabla \Phi_f] - \nabla \Phi_{w+} \mathbin{\odot} \nabla I[f] \in C_T C^1.$$ 

To treat the resonant products we make use of the Ansatz again, writing

$$w \mathbin{\odot} \nabla \Phi_f = (\nabla \Phi_{w+} \mathbin{\odot} \nabla I[f]) \mathbin{\odot} \nabla \Phi_f + w^# \mathbin{\odot} \nabla \Phi_f;$$

and

$$\mathbf{1} \mathbin{\odot} \nabla \Phi_w = \mathbf{1} \mathbin{\odot} (\nabla \Phi_{w+} \mathbin{\odot} \nabla^2 I[\Phi_f]) + \mathbf{1} \mathbin{\odot} (\nabla \Phi_w)^\#.$$ 

Under our stipulation that $w^# \in C_T C^{1+}$ and $(\nabla \Phi_w)^\# \in C_T C^{1+}$, the final two resonant products are classically well-defined and so it only remains to check that the first term of each expansion is finite. To achieve this last step we consider a commutator for the triple product,

$$(\nabla \Phi_{w+} \mathbin{\odot} \nabla I[f]) \mathbin{\odot} \nabla \Phi_f = (\nabla \Phi_{w+} \mathbin{\odot} \nabla I[f]) \mathbin{\odot} \nabla \Phi_f - \nabla \Phi_{w+} \mathbin{\odot} (\nabla I[f] \mathbin{\odot} \nabla \Phi_f) + \nabla \Phi_{w+} \mathbin{\odot} (\nabla I[f] \mathbin{\odot} \nabla \Phi_f).$$

Lemma A.11 shows that the commutator lies in $C_T C^{1-}$. We apply a similar trick to the resonant product $\mathbf{1} \mathbin{\odot} \nabla \Phi_w$, writing

$$\mathbf{1} \mathbin{\odot} (\nabla \Phi_{w+} \mathbin{\odot} \nabla^2 I[\Phi_f]) = \mathbf{1} \mathbin{\odot} (\nabla \Phi_{w+} \mathbin{\odot} \nabla^2 I[\Phi_f]) - \nabla \Phi_{w+} \mathbin{\odot} (\nabla^2 I[\Phi_f] \mathbin{\odot} \mathbf{1}) + \nabla \Phi_{w+} \mathbin{\odot} (\nabla^2 I[\Phi_f] \mathbin{\odot} \mathbf{1}).$$

Again, the regularity of the commutator follows from Lemma A.11. Taken together the last two exogenous terms produce the final diagram we are required to construct,

$$\nabla : = \nabla I[f] \mathbin{\odot} \nabla \Phi_f + \nabla^2 I[\Phi_f] \mathbin{\odot} \mathbf{1};$$

Note that the first resonant product above should be read as a vector outer product so that $\nabla$ is matrix valued. We would naively expect both summands of $\nabla$ to diverge logarithmically if we replace $\xi$ by $\xi^\delta$ and let $\delta \to 0$. However, the symmetry of the Green’s function allows us to show that after summing both terms, $\nabla$ is well-defined in a sufficiently strong topology even for $\delta = 0$.

Reversing all of the above steps we find the modified equation solved by our paracontrolled object,

$$\rho = \mathbf{1} + \mathbf{\gamma} + \nabla \Phi_{w+} \mathbin{\odot} \nabla I[f] + w^#,$$

with $w^#$ a solution to

$$w^# = P \rho_0 + \nabla \cdot I[w^# \mathbin{\odot} \nabla \Phi_1] + \nabla \cdot I[\nabla \Phi_{w+} \mathbin{\odot} \nabla I[f]] + \nabla \cdot I[\mathbf{1} \mathbin{\odot} (\nabla \Phi_w)^\#] + \mathbb{Q}(w, \mathbf{1}, \mathbf{\gamma}, \mathbf{\nabla}),$$

for $\mathbb{Q}(w, \mathbf{1}, \mathbf{\gamma}, \mathbf{\nabla})$ a third polynomial of its arguments, their paraproducts and commutators.

In the paracontrolled decomposition of $\rho$, the first three terms lie in spaces of negative regularity. Hence, the singular parts of the product $\rho \nabla \Phi_\rho$ will be determined by non-linear combinations of the first three terms. Since $\mathbf{1}$ and $\mathbf{\gamma}$ will be supplied as data these terms can be handled directly. However, as $w$ also carries information from $\rho$, products involving $\nabla \Phi_{w+} \mathbin{\odot} \nabla I[f]$ cannot be handled in the same way. Instead we make use of the commutator estimates above. To see this in practice and to identify the possibly diverging field $f^\delta$ alluded to in the introduction, we recall our notion of a mollified noise, by setting, $\mathbf{1}^\delta := \nabla \cdot I[\sigma(\psi_\delta * \xi)]$, where $\psi_\delta$ is a standard mollifier. We use the notations $\mathbf{\gamma}^\delta$, $\mathbf{\nabla}^\delta$, $\mathbf{\nabla}^\delta$ to denote the same diagrams now constructed from $\mathbf{1}^\delta$ and define $\mathbf{\gamma}^\delta_{\text{can}} := \nabla \cdot I[t^\delta \nabla \Phi_1^\delta] = \mathbf{\gamma}^\delta + \mathbf{\nabla}^\delta$.

Let $\rho^\delta$ and $w^\delta$ be the associated solutions, we have the identity,

$$\rho^\delta \nabla \Phi_{\rho^\delta} = \mathbf{1}^\delta \nabla \Phi_{\mathbf{1}^\delta} + \mathbf{\gamma}^\delta_{\text{can}} \mathbin{\odot} \nabla \Phi_{\mathbf{1}^\delta} + \nabla \Phi_{\mathbf{\gamma}^\delta_{\text{can}}} \mathbin{\odot} \mathbf{1}^\delta - \mathbf{\gamma}^\delta \mathbin{\odot} \nabla \Phi_{\mathbf{1}^\delta} - \nabla \Phi_{\mathbf{\gamma}^\delta_{\text{can}}} \mathbin{\odot} \mathbf{1}^\delta + \nabla \Phi_{w^\delta_{\mathbf{\gamma}}} \mathbin{\odot} \mathbf{\nabla}^\delta + \ldots \quad (1.9)$$

Here we have only kept track of terms that are either not classically well-defined or contain stochastic diagrams which require construction. The final term involving $\mathbf{\nabla}^\delta$ arises from applying commutators to the paraproduct term in the expansion of $\rho^\delta$ where the more regular parts have been left implicit.
above. Since we only expect to have $\rho^\delta \to \rho$ in $C_T C^{-1}$ we do not expect (1.9) to converge directly. We have already identified the possibly diverging field which renormalises the first term, since

$$\nabla \cdot I[1^\delta \nabla \Phi^\delta] - Y^\delta = \psi^\delta \to Y \in C_T C^{0-}.$$  

As discussed in the introduction, formal power counting would lead one to expect $Y^\delta$ to diverge at order $\delta^{-1}$, however, exploiting the symmetry of the elliptic Green’s function we have that $\|Y^\delta\|_{C_T C^{0-}} \lesssim \log(\delta^{-1})\|\sigma\|_{C_T H^2}^2$.

The diverging diagram $Y^\delta$ is also contained in the terms $Y^\delta \circ \nabla \Phi_{1^\delta}$ and $\nabla \Phi_{Y^\delta \circ 1^\delta}$. However, since $Y^\delta$ is of regularity $0-$ and $1^\delta$ of regularity $-1-$, it is not directly clear how to make sense of those products. Note that if $Y^\delta$ were a diverging constant rather than a field this would simply be scalar multiplication and we would have no trouble. It turns out that the products $Y^\delta \circ \nabla \Phi_{1^\delta}$ and $\nabla \Phi_{Y^\delta \circ 1^\delta}$ can be defined directly as Itô objects and diverge at a rate no worse than $Y^\delta$. We refer to Section 2.6 for this argument.

Since $Y_{can}^\delta = Y^\delta + Y^\delta$, we may expand the product $Y_{can}^\delta \circ \nabla \Phi_{1^\delta} + \nabla \Phi_{Y_{can}^\delta} \circ 1^\delta$ to cancel the diverging terms $Y^\delta \circ \nabla \Phi_{1^\delta}$ and $\nabla \Phi_{Y^\delta \circ 1^\delta}$. Hence, we can construct the renormalized product $\rho^\delta \nabla \Phi_{\rho^\delta} - Y^\delta$ without further modifications.

We have therefore identified both the solution $\rho$ and the non-linear term in (1.8) as trilinear functions of a suitable enhancement of the noise. To conclude this section we paraphrase the main result of this paper. The complete statement and proof is split between Theorems 2.3 and 3.9.

**Theorem 1.2.** Let $\rho_0 \in B^{\beta_0}_{p,\infty}$ for any $p > 4$ and $\beta_0 > -1 + \frac{2}{p}$, $\xi = (\xi^1, \xi^2)$ be a two-dimensional, space-time white noise on $[0, \infty) \times \mathbb{T}^2$, $\sigma: [0, \infty) \times \mathbb{T}^2 \to \mathbb{R}$ be a map such that $\sigma \in C_T H^2$ for some $T > 0$ and $(\psi^\delta)_{\delta \in (0,1)}$ be a family of symmetric, compactly supported mollifiers. Then there exist enhancements $X = (1, Y, \psi, \nabla)$, $X^\delta = (1^\delta, Y^\delta, \psi^\delta, \nabla^\delta)$ as described above (in particular $X^\delta$ is built from $\sigma \xi^\delta$ with $\xi^\delta = \psi^\delta * \xi$) and for some $\delta \in [0, T]$ there exists a unique, paracontrolled solution $\rho$ to (1.1) in the sense that for any $t \in (0, T]$, $\rho(t)$ is the limit, in probability, in $C^{-1} - (\mathbb{T}^2)$ of solutions $\rho^\delta(t) \in C^{0-}$ to the mild equation,

$$\rho^\delta(t) = P_t \rho_0 + \nabla \cdot I[\rho^\delta \nabla \Phi_{\rho^\delta}] - Y^\delta(t) + 1^\delta(t).$$  

Furthermore $Y^\delta = \nabla \cdot I[1^\delta \nabla \Phi_{1^\delta}] - Y^\delta = \mathbb{E}(\nabla \cdot I[1^\delta \nabla \Phi_{1^\delta}])$. If $\sigma$ is a constant then $Y^\delta \equiv 0$ while in general one has the bound $\|Y^\delta\|_{C_T C^{0-}} \lesssim \log(\delta^{-1})\|\sigma\|_{C_T H^2}$ for all $\delta \leq 1 - \sqrt{2}/2$.

**Remark 1.3.** The requirement that $\delta \leq 1 - \sqrt{2}/2$ in the final claim serves merely to simplify some expressions, see Theorem 2.3 and Lemma C.2.

**Remark 1.4.** In the case of constant $\sigma$ it also holds that $\mathbb{E}(1^\delta \nabla \Phi_{1^\delta}) = 0$. This is due to the symmetry of the elliptic Green’s function, see the discussion of (1.5).

## 2 Noise Enhancement

In this section, we construct the enhancement required in Theorem 1.2 and establish its regularity.

### 2.1 Outline and Regularities

We begin by defining a vector $\xi = (\xi^1, \xi^2)$ of space-time white noises as in [MWX17]. Let $(W^j(\cdot, m))_{m \in \mathbb{Z}^2, j = 1, 2}$ be a family of complex-valued Brownian motions on $\mathbb{R}^+$ starting from 0 that satisfy $W^j(\cdot, m) = W^j(\cdot, -m)$ and are otherwise independent. We define for $j = 1, 2$, the space-time
white noise $\xi^j$ by setting for any $\phi \in L^2(\mathbb{R}_+ \times \mathbb{T}^2; C)$,

$$\xi^j(\phi) := \sum_{m_1 \in \mathbb{Z}^2} \int_0^\infty dW^j(u_1, m_1) \hat{\phi}(u_1, -m_1).$$

(2.1)

We define our choice of mollifiers.

**Definition 2.1.** Let $\varphi \in C^\infty(\mathbb{R}^2)$ be of compact support, supp$(\varphi) \subset B(0, 1)$, even and such that $\varphi(0) = 1$. Given $\varphi$, we define a sequence of mollifiers $(\psi_\delta)_{\delta > 0}$ by $\psi_\delta(x) := \sum_{\omega \in \mathbb{Z}^2} e^{2\pi i (\omega, x)} \varphi(\delta \omega)$.

We define a space of enhanced noises.

**Definition 2.2** (Enhanced rough noise). Let $T > 0$, $\alpha < -2$ and $\kappa \in (0, 1/2)$ and let the map

$$\Theta: (L^p_T C^{\alpha+2} \times L^p_T C^{2\alpha+5}) \to L^p_T C^{\alpha+1} \times L^p_T C^{2\alpha+4} \times L^p_T C^{3\alpha+6} \times L^p_T C^{2\alpha+4},$$

$$(v, f) \mapsto \Theta(v, f),$$

be given by

$$Y := \nabla \cdot [v \nabla \Phi_v] - f,$$

$$\Theta(v, f) := (v, Y \circ \nabla \Phi_v + \nabla \Phi v \circ v, \nabla I[v] \circ \nabla \Phi_v + \nabla^2 I[\Phi_v] \circ v).$$

We define the space $X^\alpha_{T, \kappa}$ to be the closure of the subset

$$\{\Theta(v, f) : (v, f) \in L^p_T C^{\alpha+2} \times L^p_T C^{2\alpha+5}\} \subset L^p_T C^{\alpha+1} \times L^p_T C^{2\alpha+4} \times L^p_T C^{3\alpha+6} \times L^p_T C^{2\alpha+4}.$$

We shall denote a generic element of this closure by $X = (1, Y, \varphi, \varphi)$ in $X^\alpha_{T, \kappa}$ and equip it with the norm

$$\|X\|_{X^\alpha_{T, \kappa}} := \max\{\|1\|_{L^p_T C^{\alpha+1}}, \|Y\|_{L^p_T C^{2\alpha+4}}, \|\varphi\|_{L^p_T C^{3\alpha+6}}, \|\varphi\|_{L^p_T C^{2\alpha+4}}\}.$$

Our main result is the following theorem, reminiscent of [GP17, Thm. 9.1].

**Theorem 2.3.** Let $T > 0$, $\alpha < -2$, $\kappa \in (0, 1/2)$ and $\sigma \in C_T \mathcal{H}^2$. Assume $\xi$ is a two-dimensional vector of space-time white noises, $(\psi_\delta)_{\delta > 0}$ a sequence of mollifiers as in Definition 2.1, $\xi^\delta := \psi_\delta \ast \xi := (\psi_\delta \ast \xi^1, \psi_\delta \ast \xi^2)$ and $X^\delta := (1, Y, \varphi, \varphi)$ is given by

$$1^\delta := \nabla \cdot [\sigma \xi^\delta], \quad Y^\delta := \nabla \cdot [1^\delta \nabla \Phi_v] - E(\nabla \cdot [1^\delta \nabla \Phi_v]),$$

$$\varphi^\delta := \nabla \circ \nabla \Phi_v + \nabla \Phi_v \circ v, \quad \varphi^\delta := \nabla I[1^\delta] \circ \nabla \Phi_v + \nabla^2 I[\Phi_v] \circ 1^\delta.$$

Then the following hold

1. Almost surely, $X^\delta \in L^p_T C^{\alpha+2} \times L^p_T C^{2\alpha+5} \times (L^p_T C^{3\alpha+8})^2 \times (L^p_T C^{2\alpha+6})^4.$

2. Almost surely, there exists some $X = (1, Y, \varphi, \varphi) \in X^\alpha_{T, \kappa}$, such that for any $p \in [1, \infty)$ we have $\lim_{\delta \to 0} E(||X - X^\delta||_{X^\alpha_{T, \kappa}})^{1/p} = 0$ and $E(||X||_{X^\alpha_{T, \kappa}})^{1/p} < \infty$.

3. Defining

$$\varphi^\delta := E(\nabla \cdot [1^\delta \nabla \Phi_v]), \quad Y^\delta_{\text{can}} := \nabla \cdot [1^\delta \nabla \Phi_v] = \varphi^\delta + \varphi^\delta,$$

$$\varphi^\delta_{\text{can}} := \varphi_{\text{can}} \circ \nabla \Phi_v + \nabla \Phi_{\text{can}} \circ 1^\delta,$$

it holds that almost surely

$$\max\{||\varphi^\delta||_{L^p_T C^{2\alpha+4}}, ||Y^\delta_{\text{can}}||_{L^p_T C^{2\alpha+4}}\} \lesssim \log(\delta^{-1})||\sigma||^3_{C_T \mathcal{H}^2},$$

and

$$||\varphi^\delta_{\text{can}}||_{L^p_T C^{3\alpha+6}} \lesssim \log(\delta^{-1})||\sigma||^3_{C_T \mathcal{H}^2}.$$
An explicit definition of the limit $X = (I, Y, \varrho, \nu)$ can be found in Subsection 2.2. We call $X$ the renormalized model and $X^\delta_{\text{can}} = (I^\delta, Y^\delta_{\text{can}}, \varrho^\delta_{\text{can}}, \nu^\delta)$ the canonical model.

The result will be shown in several parts, namely in Lemma 2.7 (I), Lemma 2.13 (Y), Lemma 2.16 and Lemma 2.21 (\varrho), Lemma 2.17 and Lemma 2.22 (\nu), and Lemma 2.23 (Y^\delta_{\text{can}}, \nu^\delta_{\text{can}}).

Remark 2.4. Different aspects of Theorem 2.3 require different assumptions on the heterogeneity $\sigma$. For example the regularity of $I$ and $Y$ only requires $\sigma \in C_T L^\infty$ (Lemma 2.7 and Lemma 2.13) while the regularities of the contractions contained in $\varrho$, $\nu$ and all diagrams built from the mollified noise require that uniformly over $t \in [0, T]$ one has $\sup_{\omega \in \mathbb{Z}^2} |\hat{\sigma}(t, \omega)|(1 + |\omega|^2) < \infty$. The assumption $\sigma \in C_T \mathcal{H}^2$ implies both of these conditions and provides a convenient norm and well-studied space that controls the latter quantity; hence we choose to work with this simpler, if sub-optimal restriction. Furthermore, with a view to setting $\sigma = \sqrt{\det}$ the condition $\sigma \in C_T \mathcal{H}^2$ is more straightforward to check.

Remark 2.5. As discussed in the introduction, we build our regular enhancement from $\sigma \xi^\delta$ instead of $(\sigma \xi)^\delta$ and with only a spatial convolution. In order to control these objects on the Fourier side we are required to have control on the second quantity described in the previous remark, namely $\sup_{\omega \in \mathbb{Z}^2} |\hat{\sigma}(t, \omega)|(1 + |\omega|^2)$. Therefore, even though $\mathcal{H}^2 \hookrightarrow C^1$ our estimates do not make use of this additional regularity. See the second half of the proof of Lemma 2.7 for an example.

Remark 2.6. Our methods also allow us to establish that $Y \in L_T^{-1} C^0$. However, since we do not make use of the additional time regularity, we omit the proof.

In the remainder of this section, we outline the basic arguments involved in proving Theorem 2.3. We motivate the definition of $I$ and establish its existence.

It is well-known that the Fourier frequencies of the stochastic heat equation are given by Ornstein-Uhlenbeck processes. Similarly, we can find an expression for the Fourier transform of $I = \nabla \cdot I[\sigma \xi]$. Let $H^I_t(\omega) := 2\pi i \omega \exp(-t|2\pi \omega|^2)1_{t \geq 0}, \omega \in \mathbb{Z}^2, t \in \mathbb{R}$, be the multiplier associated to $\partial_t I$. We define $I$ by applying the inverse Fourier transform to the sequence

$$\hat{I}(t, \omega) := \sum_{j, m \in \mathbb{Z}^2} \sum_{j = 1}^2 \int_0^t dW^{j_1}(u_1, m_1) \hat{\sigma}(u_1, \omega - m_1) H^I_{j-1} u_1(\omega).$$

We also introduce the Fourier transform of $\tau := \nabla \cdot I[\xi]$ by

$$\hat{\tau}(t, \omega) := \sum_{j = 1}^2 \int_0^t dW^{j_1}(u_1, \omega) H^I_{j-1} u_1(\omega).$$

Lemma 2.7. Let $T > 0$, $\alpha < -2$, $\kappa \in (0, 1/2)$ and $\sigma \in C_T L^\infty$. Then, for any $p \in [1, \infty]$ we have $E(\|I\|^p_{L_T^{p \kappa} C^{\alpha+1}})^{1/p} \lesssim \|\sigma\|_{C_T L^\infty}$ and in particular $I \in L_T^p C^{\alpha+1}$ a.s. Assume in addition $\sigma \in C_T \mathcal{H}^2$ and $\delta > 0$, then it holds that $E(\|I^\delta\|^p_{L_T^{p \kappa} C^{\alpha+2}})^{1/p} \lesssim (1 + \delta^{-2})^{1/2} \|\sigma\|_{C_T \mathcal{H}^2}$ and in particular $I^\delta \in L_T^p C^{\alpha+2}$ a.s.. What is more, $\lim_{\delta \to 0} E(\|I - I^\delta\|^p_{L_T^{p \kappa} C^{\alpha+1}})^{1/p} = 0$ for any $p \in [1, \infty]$.

Proof. Let $\gamma \in (0, 1], \varepsilon \in (0, \gamma/2)$ and $\max\{1/\varepsilon, 2\} < p < \infty$. To establish the existence and regularity of $I$ in a Besov space, we apply Nelson’s estimate (Lemma 2.11), Kolmogorov’s continuity criterion (Lemma 2.12) and the Besov embedding (A.1). Therefore, in order to establish that $I \in C_T^{1/2 - \varepsilon} C^{1-\gamma-3\varepsilon}$ almost surely, it suffices to control the quantity

$$\sum_{q \in \mathbb{N}_-} 2^{pqq} \sup_{t \in \mathbb{T}^2} \mathbb{E}(\|I - I^\delta(t, x)\|^p_{L_T^{p \kappa} C^{\alpha+1}})^{1/p} \lesssim (1 + \delta^{-2})^{1/2} \|\sigma\|_{C_T \mathcal{H}^2}.$$
uniformly in $s \neq t \in [0, T]$. Using that $\sigma \in C_T L^\infty$ is bounded, we can pass to real space to deduce

\[ \mathbb{E}(|\Delta_q \mathbf{1}(t, x) - \Delta_q \mathbf{1}(s, x)|^2) \leq \|\sigma\|_{C_T L^\infty}^2 \mathbb{E}(|\Delta_q \tau(t, x) - \Delta_q \tau(s, x)|^2), \]

see Lemma 2.10. Using that $\mathbb{E}(\tilde{\tau}(t, \omega)\tilde{\tau}(s, \omega')) = 0$ if $\omega \neq \omega' \in \mathbb{Z}^2$, we obtain

\[ \mathbb{E}(|\Delta_q \tau(t, x) - \Delta_q \tau(s, x)|^2) = \sum_{\omega \in \mathbb{Z}^2} \sum_{\omega' \in \mathbb{Z}^2} e^{2\pi i (\omega, \omega')} e^{-2\pi i (\omega', x)} \varrho_q(\omega) \varrho_q(\omega') \mathbb{E}((\tilde{\tau}(t, \omega) - \tilde{\tau}(s, \omega))(\tilde{\tau}(t, \omega') - \tilde{\tau}(s, \omega'))) \]

\[ = \sum_{\omega \in \mathbb{Z}^2} \varrho_q(\omega)^2 \mathbb{E}(|\tilde{\tau}(t, \omega) - \tilde{\tau}(s, \omega)|^2). \]

It follows by Itô’s isometry and interpolation (A.2),

\[ \mathbb{E}(|\tilde{\tau}(t, \omega) - \tilde{\tau}(s, \omega)|^2) \leq \sum_{j_1=1}^2 \int_{-\infty}^\infty du_1 |H^{j_1}_{t-u_1}(\omega) - H^{j_1}_{s-u_1}(\omega)|^2 \lesssim |t-s|^\gamma |\omega|^{2\gamma} \]

and therefore

\[ \mathbb{E}(|\Delta_q \tau(t, x) - \Delta_q \tau(s, x)|^2) \lesssim |t-s|^\gamma |\omega|^{2\gamma} \gamma(2+2\gamma). \]

We obtain by Lemma 2.11 and Lemma 2.12 for any $p \in [1, \infty)$,

\[ \mathbb{E}(\|\mathbf{1}\|_{C_T^{\gamma/2-\epsilon} C_{-1-\gamma-3\epsilon}}^p)^{1/p} \lesssim \|\sigma\|_{C_T L^\infty} \]

and therefore $\mathbf{1} \in C_T^{\gamma/2-\epsilon} C_{-1-\gamma-3\epsilon}$ a.s.

The approximating sequence $\mathbf{t}^\delta$, $\delta > 0$, corresponds in Fourier space to

\[ \tilde{\mathbf{t}}^\delta(t, \omega) = \sum_{j_1=1}^2 \sum_{m_1 \in \mathbb{Z}^2} \int_0^t dW^{j_1}(u_1, m_1) \tilde{\sigma}(u_1, \omega - m_1) \varphi(\delta m_1) H^{j_1}_{t-u_1}(\omega). \] (2.3)

We apply Itô’s isometry and the triangle inequality to estimate

\[ \mathbb{E}((\tilde{\mathbf{t}}^\delta(t, \omega) - \tilde{\mathbf{t}}^\delta(s, \omega))(\tilde{\mathbf{t}}^\delta(t, \omega') - \tilde{\mathbf{t}}^\delta(s, \omega'))) \lesssim |t-s|^\gamma |\omega||\omega'|\|\sigma\|_{C_T H^2}^2 \sum_{m_1 \in \mathbb{Z}^2} (1 + |\omega - m_1|^2)^{-1}(1 + |\omega' - m_1|^2)^{-1}(1 + |\delta m_1|^2)^{-1}, \]

where we used that $|\tilde{\sigma}(u, \omega)| \lesssim (1 + |\omega|^2)^{-1}\|\sigma\|_{C_T H^2}$, $u \in [0, T]$, $\omega \in \mathbb{Z}^2$, interpolation (A.2) and $(1 + x^2)^{1/2} |\varphi(x)| \lesssim 1$, $x \in \mathbb{R}^2$. We may assume $\omega, \omega' \in \mathbb{Z}^2 \setminus \{0\}$, since $\tilde{\mathbf{t}}^\delta(t, 0) = 0$. We decompose the sum over $m_1 \in \mathbb{Z}^2$ into the domains $m_1 = 0$, $m_1 = \omega$, $m_1 = \omega'$ and $m_1 \in \mathbb{Z}^2 \setminus \{0, \omega, \omega'\}$,

\[ \sum_{m_1 \in \mathbb{Z}^2} (1 + |\omega - m_1|^2)^{-1}(1 + |\omega' - m_1|^2)^{-1}(1 + |\delta m_1|^2)^{-1} \leq |\omega|^{-2} |\omega'|^{-2} + \delta^{-2} (1 + |\omega - \omega'|^{-2})^{-1} (|\omega|^2 + |\omega'|^{-2}) + \delta^{-2} \sum_{m_1 \in \mathbb{Z}^2 \setminus \{0, \omega, \omega'\}} |\omega - m_1|^{-2} |\omega' - m_1|^{-2} |m_1|^2. \]

We estimate the sum over $m_1 \in \mathbb{Z}^2 \setminus \{0, \omega, \omega'\}$. Assume $\omega = \omega'$, then by Lemma C.5 and (C.3),

\[ \sum_{m_1 \in \mathbb{Z}^2 \setminus \{0, \omega, \omega'\}} |\omega - m_1|^{-4} |m_1|^2 \lesssim |\omega|^{-2}. \]

Assume $\omega \neq \omega'$, we apply Lemma C.4 to estimate

\[ \sum_{m_1 \in \mathbb{Z}^2 \setminus \{0, \omega, \omega'\}} |\omega - m_1|^{-2} |\omega' - m_1|^{-2} |m_1|^2 \lesssim |\omega - \omega'|^{-2} (|\omega|^{-2+2\epsilon} + |\omega'|^{-2+2\epsilon}). \]
Having established the decay of the Fourier coefficients, we can bound the Littlewood–Paley blocks. Let \( q \in \mathbb{N}_{-1} \), we obtain
\[
\mathbb{E}(\mathbb{E}(\Delta_q \mathbf{I}_{\delta}(t, x) - \Delta_q \mathbf{I}_{\delta}(s, x))^2) \leq \sum_{0 \neq \omega, \omega' \in \mathbb{Z}^2} \theta_q(\omega) \theta_q(\omega') \mathbb{E}(\mathbb{E}(\mathbf{I}_{\delta}(t, \omega) - \mathbf{I}_{\delta}(s, \omega))(\mathbf{I}_{\delta}(t, \omega') - \mathbf{I}_{\delta}(s, \omega'))) \\
\leq (1 + \delta^{-2}) \|\sigma\|^2_{C_T \mathcal{H}^2} |t - s|^{2(q(3\varepsilon + 2\gamma)}.
\]
Consequently, by Lemma 2.11 and Lemma 2.12 for any \( p \in [1, \infty) \) and \( \varepsilon \in (0, \gamma/2) \),
\[
\mathbb{E}(\|\mathbf{I}_{\delta}\|_{C_T^{\gamma-\varepsilon} C_{\gamma-\gamma_0}^2})^{1/p} \leq (1 + \delta^{-2})^{1/2} \|\sigma\|_{C_T \mathcal{H}^2}
\]
and therefore \( \mathbf{I}_{\delta} \in \mathcal{L}_{C_T}^{p, C_{0}^0} \) a.s. for any \( \delta > 0 \) and \( \kappa \in (0, 1/2) \). An application of the dominated convergence theorem yields \( \lim_{\delta \to 0} \mathbb{E}(\|\mathbf{I} - \mathbf{I}_{\delta}\|_{\mathcal{L}_{C_T}^{p, C_{\alpha+1}^0}})^{1/p} = 0 \) for any \( p \in [1, \infty) \).

The convergence of the other approximations in \( \mathbb{X}_\delta \) is analogous, hence will be omitted.

### 2.2 Feynman Diagrams

As demonstrated by \( (2.2) \), we may construct our white-noise enhancement as (iterated) stochastic integrals. However, as we continue to multiply terms, we need to apply Itô’s product rule to increasingly complicated expressions. To implement this procedure efficiently, we use an extension of a graphical representation that was developed by [MWX17; GP17], which relates our stochastic objects to Feynman diagrams.

There are several types of vertices. A circle \( \circ \) denotes an instance of stochastic integration in time against a two-dimensional Brownian field with heterogeneity \( \sigma \). Graphically this integrator is given by
\[
(u_1, \omega_1, m_1, j_1) \quad \circ = \sum_{j_1=1}^{2} \sum_{m_1 \in \mathbb{Z}^2} \int_0^\infty dW_{j_1}(u_1, m_1) \tilde{\sigma}(u_1, \omega_1 - m_1) \ldots
\]
The placeholder \( \ldots \) stands for an integrand in \( j_1, m_1, u_1 \), which is to be determined from the remaining diagram.

Generally, vertices are equipped with tuples of internal variables \( (u_k, \omega_k, m_k, j_k) \), where \( u_k \in (0, \infty) \), \( \omega_k \in \mathbb{Z}^2 \), \( m_k \in \mathbb{Z}^2 \) and \( j_k = 1, 2 \). Those usually hold a subscript \( k \in \mathbb{N} \) and are integrated or summed over. We refer to the \( \omega \), \( \omega_k \) as the frequencies and the \( m_k \) as the modes. We denote coordinates of those by \( \omega_k^j, j_k = 1, 2 \).

A root \( \blackdot (t, \omega) \) represents the argument \( (t, \omega) \). Vertices are connected by different types of directed edges, i.e., arrows, representing integrands. Black arrows \( (t, \omega) \rightarrow (u_k, \omega_k, m_k, j_k) \) are associated to the integrand \( H_{\xi_{u_k}}^{j_k}(\omega_k) \), which is the Fourier multiplier appearing in \( \nabla \cdot \mathcal{I} \). Highlighted arrows \( (t, \omega) \xrightarrow{j}(u_k, \omega_k, m_k, j_k) \) for \( j = 1, 2 \), are associated to \( G^j(\omega_k)H_{\xi_{u_k}}^{j_k}(\omega_k) \), where
\[
G^j(\omega_k) := 2\pi i \omega_k^j |2\pi \omega_k|^{-2} \mathbb{1}_{\omega_k \neq 0}
\]
is the multiplier for \( \partial_j \Phi \).

Integrators are then determined by the vertices at the arrowheads. The direction of an arrow indicates the smaller time variable \( u_k \) in the integration. For example, applying those rules, we can represent \( (2.2) \) as
\[
\hat{\mathbf{I}}(t, \omega) = \sum_{j_1=1}^{2} \sum_{m_1 \in \mathbb{Z}^2} \int_0^t dW_{j_1}(u_1, m_1) \tilde{\sigma}(u_1, \omega - m_1) H_{\xi_{u_1}}^{j_1}(\omega),
\]
and \( \nabla \Phi_1 \) as

\[
\widehat{\partial_j \Phi_1}(t, \omega) = \sum_{(u_1, \omega, m_1, j_1)} (u_1, \omega, m_1, j_1) \quad \Rightarrow \quad \frac{dW^{j_1}(u_1, m_1)}{dt} \nabla \Phi_1(t, \omega) = \sum_{j_1, m_1 \in \mathbb{Z}^2} \int_0^t dW^{j_1}(u_1, m_1) \delta(u_1, \omega - m_1)G^j(\omega)H^{j_1}_{t-u_1}(\omega).
\]

In particular, if \( \omega = 0 \), then \( H^j_t(\omega) = G^j(\omega) = 0 \), hence we may assume \( \omega \neq 0 \) whenever it appears in either multiplier.

As a general rule, in our diagrams **black** objects are associated to scalars and **highlighted** objects are associated to vectors. Our arrows have highlighted arrowheads, indicating that they act on vector-valued objects. On the other hand, the type of object they return is determined by the arrow shaft. Note that \((t, \omega) \rightarrow (u_k, \omega_k, m_k, j_k)\) produces a scalar and \((t, \omega) \rightarrow (u_k, \omega_k, m_k, j_k)\) a vector.

The existence of \( \nabla \cdot \mathcal{I}[1\nabla \Phi_1] \) is not guaranteed by Lemma A.4, since \( \mathcal{I} \in C_T \mathcal{C}^{−1} \) and hence \( \nabla \Phi_1 \in C_T \mathcal{C}^{0} \). In order to construct such non-linear objects, we formally apply Itô's product rule to identify the candidate Fourier transform. Let \( n \in \mathbb{N} \) and assume \( a_1, \ldots, a_n \in \mathbb{N} \) are distinct. We denote by \( \Sigma(a_1, \ldots, a_n) \) the permutation group of \( \{a_1, \ldots, a_n\} \).

Let \( \omega_1, \omega_2 \in \mathbb{Z}^2 \), we compute

\[
\mathcal{F}(1\nabla \Phi_1)(t, \omega, j) = \sum_{\omega_1, \omega_2 \in \mathbb{Z}^2} \hat{f}(t, \omega_1) \hat{\partial_j \Phi_1}(t, \omega_2)
\]

\[
= \sum_{\omega_1, \omega_2 \in \mathbb{Z}^2} \sum_{j_1, j_2 = 1}^2 \sum_{m_1, m_2 \in \mathbb{Z}^2} \int_0^t dW^{j_2}(u_2, m_2) \int_0^{u_2} dW^{j_1}(u_1, m_1)
\]

\[
\sum_{\omega_1, \omega_2 \in \mathbb{Z}^2} \sum_{j_1, j_2 = 1}^2 \sum_{m_1, m_2 \in \mathbb{Z}^2} \int_0^t dW^{j_2}(u_2, m_2) \int_0^{u_2} dW^{j_1}(u_1, m_1)
\]

\[
\sum_{j_1, j_2 = 1}^2 \sum_{m_1, m_2 \in \mathbb{Z}^2} \int_0^t dW^{j_2}(u_2, m_2) \int_0^{u_2} dW^{j_1}(u_1, m_1)
\]

\[
\sum_{j_1, j_2 = 1}^2 \sum_{m_1, m_2 \in \mathbb{Z}^2} \int_0^t dW^{j_2}(u_2, m_2) \int_0^{u_2} dW^{j_1}(u_1, m_1)
\]

\[
\sum_{j_1, j_2 = 1}^2 \sum_{m_1, m_2 \in \mathbb{Z}^2} \int_0^t dW^{j_2}(u_2, m_2) \int_0^{u_2} dW^{j_1}(u_1, m_1)
\]

\[
= \nabla \Phi (t, \omega, j) + \mathcal{O}(t, \omega, j).
\]

The symmetrization of the first integrand is a direct consequence of the Itô product rule. Note that in the second term, \( j_1 = j_2, u_1 = u_2 \) but \( m_1 = -m_2 \), which is a consequence of the Hermitian structure of complex Brownian motion. Such a decomposition of stochastic products into iterated (stochastic) integrals is often called a **Wiener chaos decomposition** after [Wie38], see [Hai16; MWX17] for more details.

To represent \( \nabla \rho \), let us extend our graphical rules. Two arrows emerging from a common vertex represent a convolution in Fourier space. Their integrands are multiplied, but are related by the **Kirchhoff rule** [MWX17]: each vertex \( v \) has a frequency \( \omega \) or \( \omega_k \) which is part of its internal variables. This frequency will be called **ingoing** at the vertex \( v \). An ingoing frequency at a vertex \( v \) is **outgoing** for the vertex \( w \), if there exists an arrow pointing from \( w \) to \( v \). A vertex is called **internal**, if there exists an arrow emerging from it. The rule states that at each internal vertex, the ingoing frequency (e.g., \( \omega \) above) equals the sum of the outgoing frequencies (e.g., \( \omega_1, \omega_2 \) above). In graphical notation,

\[
\nabla \rho (u_1, \omega_1, m_1, j_1) (u_2, \omega_2, m_2, j_2) \quad \Rightarrow \quad \mathcal{O}(t, \omega) := \mathcal{I}_{\omega=\omega_1+\omega_2} H^{j_1}_{t-u_1}(\omega_1)G^j(\omega_2)H^{j_2}_{t-u_2}(\omega_2).
\]

Those arrows will target the integrators \( \mathcal{O}(u_1, \omega_1, m_1, j_1) \) and \( \mathcal{O}(u_2, \omega_2, m_2, j_2) \) which will be multiplied and integrated over. The integral is then restricted to the simplex \( u_1 < u_2 \) to ensure that the integrand is
adapted. To obtain the integral over the full domain, we symmetrize the integrand by permuting the indices that appear in the simplex. For example,

\[(u_1, \omega_1, m_1, j_1)(u_2, \omega_2, m_2, j_2)\]

\[\rightarrow \hat{\sigma}(t, \omega, j)\]

is the first object in the decomposition (2.4).

Next, let us discuss \(\odot\). As can be seen in (2.4), instances of Lebesgue integration arise through Itô correction terms. Itô corrections will be denoted by contractions, i.e. two arrows pointing at different vertices \(\odot\) and \(\circ\) are merged at a common vertex \(\ast\). Graphically,

\[\begin{array}{c}
(u_1, \omega_1, m_1, j_1)
\end{array}\]

\[\begin{array}{c}
(u_2, \omega_2, m_2, j_2)
\end{array}\]

are contracted to

\[\begin{array}{c}
(u_1, \omega_1, m_1, j_1)
\end{array}\]

\[\begin{array}{c}
(u_2, \omega_2, m_2, j_2)
\end{array}\]

Using the orthogonality of \((W^j(u, m))_{u \geq 0, m \in \mathbb{Z}^2})\), we can identify some of the internal variables of the two vertices that are being merged. Indeed, as in (2.4), we set \(j_1 = j_2, u_1 = u_2, m_1 = -m_2\), but leave \(\omega_1, \omega_2\) as is. To make it easier for the reader to discern the multipliers attached to each arrow, we give both tuples of internal variables, even after the contraction. Graphically,

\[\begin{array}{c}
(u_1, \omega_1, m_1, j_1)
\end{array}\]

\[\begin{array}{c}
(u_2, \omega_2, m_2, j_2)
\end{array}\]

are contracted to

\[\begin{array}{c}
(u_1, \omega_1, m_1, j_1)
\end{array}\]

\[\begin{array}{c}
(u_2, \omega_2, m_2, j_2)
\end{array}\]

This results in the diagram

\[\begin{array}{c}
(u_1, \omega_1, m_1, j_1)
\end{array}\]

\[\begin{array}{c}
(u_2, \omega_2, m_2, j_2)
\end{array}\]

\[\rightarrow \odot(t, \omega, j)\]

which is the Itô correction in the decomposition (2.4) and coincides with the mean, \(\odot = E(\nabla \Phi)\). Diagrams carrying contractions are often called Wick contractions after [Wic50], see [GP17] for more details.

Depending on \(\sigma\), \(\odot\) may be infinite. Hence, we consider the renormalized model, where we only keep the first term \(\nabla \Phi\) of the decomposition (2.4) to define \(\varphi\). Formally, this is equivalent to subtracting the mean as a counterterm,

\[\varphi = \nabla \Phi - E(\nabla \Phi)\]

This identity can be made rigorous with suitable regularization and limit procedures, see the discussion of the canonical model at the end of this section.

Another source of Lebesgue integrals are concatenations of the \(I\) operation. We obtain arrows pointing at other arrows, connected through a vertex \(\ast\). We multiply their multipliers and make sure to respect the Kirchhoff rule. The multiplier of the incoming arrow will be determined by a tuple of internal variables \((u_k, \omega_k, j_k)\) at the connecting vertex. For example, \(\nabla \cdot I(\nabla \Phi)\) can be expressed as

\[\nabla \cdot I(\nabla \Phi) = \sum_{j_1, j_2 = 1}^2 \sum_{m_1 \in \mathbb{Z}^2} \int_0^t du \int_0^{u_2} dW^{j_1}(u_1, m_1) \delta(u_1, \omega - m_1) H_{t-u_2}^j(\omega) G_{t}^{j_2}(\omega) H_{u_2-u_1}^j(\omega).\]
The renormalised stochastic object \( \mathcal{Y} = \nabla \cdot \mathcal{I}[\nabla \Phi_1] - \mathbb{E}(\nabla \cdot \mathcal{I}[\nabla \Phi_1]) \) can then be expressed as

\[
(\mathcal{Y}(t, \omega)) = \sum_{\omega_1, \omega_2 \in \mathbb{Z}^2} \int_0^t du_3 \int_0^{u_3} dW^{j_1}(u_2, m_2) \int_0^{u_2} dW^{j_1}(u_1, m_1)
\]

\[
\mathcal{Y}(t, \omega)
\]

\[
\hat{\mathcal{Y}}(t, \omega)
\]

\[
\hat{\mathcal{Y}}(t, \omega, j) = \sum_{\omega_1, \omega_2, \omega_4 \in \mathbb{Z}^2} \int_0^t du_3 \int_0^{u_3} dW^{j_2}(u_4, m_4) \int_0^{u_4} dW^{j_1}(u_1, m_1) \int_0^{u_1} dW^{j_2}(u_2, m_2)
\]

\[
\hat{\sigma}(u_1, \omega_1 - m_1) \hat{\sigma}(u_2, \omega_2 - m_2) \hat{\sigma}(u_4, \omega_4 - m_4) \sum_{\varsigma \in \Sigma(1,2)} H_{t-u_4}^{j_4}(\omega_4(1) + \omega_4(2)) H_{u_4-u_1}^{j_3}(\omega_4(1)) G^{j_2}(\omega_4(2)) H_{u_1-u_1}^{j_2}(\omega_4(2))
\]

\[
\times H_{u_3-u_2}^{j_1}(\omega_1(1) + \omega_1(2)) G^{j_1}(\omega_1(2)) H_{u_1-u_1}^{j_1}(\omega_1(2)) \sum_{k,l \in \mathbb{N}-1} |k-l| \leq 1
\]

In fact, we will not construct \( \mathcal{Y} \) in itself. As we will see in Lemma 2.13, \( \mathcal{Y} \) can be constructed as a continuous function in time that takes values in a space of distributions. On the other hand, we do not expect \( \mathcal{Y} \) to admit pointwise-in-time values. Instead, we expect it to exist as a proper space-time distribution. This resembles the situation discussed in [MWX17, pp. 23–24 & pp. 32–33] and [CC18; HM18].

A particular variant of the root \( \bullet \) is the vertex \( \circ \) which arises through applications of the resonant product. The vertex \( \circ \) relates the frequencies of the arrows that it joins through the \( \sim \)-relation defined in (1.6), see also [MWX17, (64)].

Let us consider more complicated objects. We have the Wiener chaos decomposition

\[
\mathcal{V} = \mathcal{V}_0 + \mathcal{V}_1 + \mathcal{V}_2 + (\mathcal{V}_3 + \mathcal{V}_4) + \mathcal{V}_5.
\]

The contractions \( \mathcal{V}_0 \) and \( \mathcal{V}_5 \) that one might expect are absent in the renormalized model due to our definition of \( \mathcal{Y} = \nabla \cdot \mathcal{I}[\nabla \Phi_1] - \mathcal{Y} \). We express the third-order Wiener chaos term \( \mathcal{V}_5 \) as follows. For \( j = 1, 2 \),

\[
(\mathcal{V}_5(t, \omega, j)) = \sum_{\omega_1, \omega_2 \in \mathbb{Z}^2} \int_0^t du_3 \int_0^{u_3} dW^{j_1}(u_2, m_2) \int_0^{u_2} dW^{j_1}(u_1, m_1)
\]

\[
\hat{\sigma}(u_1, \omega_1 - m_1) \hat{\sigma}(u_2, \omega_2 - m_2) \hat{\sigma}(u_4, \omega_4 - m_4) \sum_{\varsigma \in \Sigma(1,2)} H_{t-u_4}^{j_4}(\omega_4(1) + \omega_4(2)) H_{u_4-u_1}^{j_3}(\omega_4(1)) G^{j_2}(\omega_4(2)) H_{u_1-u_1}^{j_1}(\omega_4(2))
\]

\[
\times H_{u_3-u_2}^{j_1}(\omega_1(1) + \omega_1(2)) G^{j_1}(\omega_1(2)) H_{u_1-u_1}^{j_1}(\omega_1(2)) \sum_{k,l \in \mathbb{N}-1} |k-l| \leq 1
\]
The diagrams \( \rho \) and \( \sigma \), may not exist in themselves, but the summed object \( \tilde{\rho} \) := \( \rho + \sigma \) does. Its iterated integral representation is given by

\[
\tilde{\rho}(t, \omega, j) = (u_1, \omega_1, m_1, j_1) + (u_2, \omega_2, m_2, j_2) + (u_3, \omega_3, m_3, j_3)
\]

\[
:= \sum_{\omega_1, \omega_2, \omega_3} \sum_{j_1, j_2, j_3} \int_0^t du_1 \int_0^{u_1} du_2 \int_0^{u_2} d\omega_1 \rho_1(u_1, m_1)
\]

\[
\tilde{\sigma}(u_1, \omega_1 - m_1) \tilde{\rho}(u_2, \omega_2 - m_2) \tilde{\rho}(u_3, \omega_3 + m_3)(G^1(\omega_1) + G^1(\omega_2))
\]

\[\times H_{t-u_1}^{j_1}(\omega_1 + \omega_2) H_{t-u_2}^{j_2}(\omega_3) G_{u_3-u_1}(\omega_1) H_{u_3-u_2}^{j_2}(\omega_2).
\]

The remaining diagrams \( \rho \), \( \sigma \) and \( \tilde{\rho} \) are similar to the ones given above.

We will show in Lemma C.1 that the resulting factor \( G^1(\omega_1) + G^1(\omega_2) \) has better decay in \( \omega_4 \), than \( G^1(\omega_4) \). This is due to the symmetry of \( G^1 \), which allows us to write

\[
G^1(\omega_1 + \omega_2) + G^1(\omega_1 + \omega_2) = G^1(\omega - \omega_4) - G^1(-\omega_4).
\]

The improved decay leads to the well-posedness of \( \tilde{\rho} \) and is a higher-dimensional analogue of the product rule discussed in (1.5).

**Remark 2.8.** One could simplify the contraction by using the identity

\[
\sum_{m_2 \in \mathbb{Z}^2} \tilde{\sigma}(u_2, \omega_2 - m_2) \tilde{\rho}(u_2, \omega_4 + m_2) = \tilde{\sigma}^2(u_2, \omega_2 + \omega_4).
\]

This idea would allow us to derive bounds in terms of \( \|\tilde{\sigma}\|_{C_T^2} \) rather than \( \|\sigma\|_{C_T^2} \). However, (2.5) is no longer applicable in our preliminaring model due to the cut-off \( \varphi(\delta m_2) \). Instead we use direct estimates that do not rely on (2.5).

We extend our graphical rules to incorporate the operator \( \partial \mathcal{I}, l = 1, 2 \). An indexed black arrow pointing at a scalar object \((t, \omega) \xrightarrow{l} (u_k, \omega_k)\) is associated to the multiplier \( \xi_{t-u_k}(\omega_k) \). On the other hand, a doubly-indexed, highlighted arrow pointing at a scalar object \((t, \omega) \xrightarrow{l,j} (u_k, \omega_k)\) is associated to \( G^1(\omega_k) H_{t-u_k}(\omega_k) \).

The remaining object in the enhancement is \( \tilde{\nu}_{k,j} \), \( k, j = 1, 2 \). We consider the Wiener chaos decomposition

\[
\tilde{\nu}_{k,j} = \rho_{(k,j)} + \sigma_{(k,j)} + (\tilde{\nu}_{(k,j)} + \tilde{\nu}_{(k,j)}).
\]

The first term is given by

\[
(t, \omega, k, j) = (u_1, \omega_1, m_1, j_1) + (u_2, \omega_2, m_2, j_2) + (u_3, \omega_3, m_3, j_3)
\]

\[
:= \sum_{\omega_1, \omega_2, \omega_3} \sum_{j_1, j_2, j_3} \int_0^t du_1 \int_0^{u_1} du_2 \int_0^{u_2} d\omega_1 \rho_1(u_1, m_1)
\]

\[
\tilde{\sigma}(u_1, \omega_1 - m_1) \tilde{\rho}(u_2, \omega_2 - m_2) \tilde{\rho}(u_3, \omega_3 + m_3)(G^1(\omega_1) + G^1(\omega_2))
\]

\[\times H_{t-u_1}^{j_1}(\omega_1 + \omega_2) H_{t-u_2}^{j_2}(\omega_3) G_{u_3-u_1}(\omega_1) H_{u_3-u_2}^{j_2}(\omega_2).
\]
and the second term $\Phi$ is again similar. We consider the contractions as a summed object $\mathcal{X} := \mathcal{Y} + \Phi$. We obtain

$$
\mathcal{X}(t, \omega, k, j) = \sum_{\omega_1, \omega_2 \in \mathbb{Z}^d \mid \omega_1 + \omega_2 = \omega} \sum_{j_1 = 1}^2 \int_0^t \int_0^{u_3} \int_0^{u_1} \int_0^{u_2} d\omega_1 \sigma(u_1, \omega_1 + m_1) \sigma(u_2, \omega_2) \phi(\delta m_1)^2 \times H^{j_3}_{t-u_3}(\omega_1) H^{j_1}_{t-u_1}(\omega_2).
$$

We can define approximate diagrams as in (2.3) by multiplying the cut-off $\phi(\delta m_1)$ to each instance of the noise $\phi(u_k, \omega_k, m_k, j_k)$. In general, we denote the regularization of a diagram by a superscript $\delta$. The canonical model $\mathcal{X}^{\delta \text{can}} = (\mathcal{Y}^{\delta \text{can}}, \mathcal{X}^{\delta \text{can}}, \mathcal{Y}^{\delta \text{can}}, \mathcal{\Phi}^{\delta \text{can}})$ is then built from regularized noise terms, but retains the diverging sequences that are removed in the renormalized model $\mathcal{X}$. Repeating (2.4), we may consider the decomposition of the diagram with cut-off,

$$
\mathcal{X}^{\delta \text{can}} = \nabla \cdot \mathcal{I} [\mathcal{Y}^{\delta}] = \mathcal{Y}^{\delta} + \Phi^{\delta},
$$

where $\mathcal{Y}^{\delta} = \mathbb{E}(\nabla \cdot \mathcal{I} [\mathcal{Y}^{\delta} \Phi^{\delta}])$. In addition to $\mathcal{Y}^{\delta}$, we also have to control the mean,

$$
\mathcal{Y}^{\delta}(t, \omega) = \mathcal{X}^{\delta} + \Phi^{\delta}.
$$

The diagram $\mathcal{Y}^{\delta}$ is given by

$$
\mathcal{Y}^{\delta}(t, \omega, j) = \sum_{\omega_1, \omega_2 \in \mathbb{Z}^d \mid \omega_1 + \omega_2 = \omega} \sum_{\omega_4, \omega_3 \in \mathbb{Z}^d \mid \omega_4 + \omega_3 = \omega} \int_0^t \int_0^{u_3} \int_0^{u_1} \int_0^{u_2} d\omega_1 \sigma(u_1, \omega_1 + m_1) \sigma(u_2, \omega_2) \phi(\delta m_1)^2 \times H^{j_3}_{t-u_3}(\omega_1) H^{j_1}_{t-u_1}(\omega_2).$$

Including $\Phi^{\delta}$ in $\mathcal{Y}^{\delta \text{can}}$ generates additional terms in the decomposition of $\mathcal{X}^{\delta \text{can}}$.

$$
\mathcal{X}^{\delta \text{can}} = \mathcal{Y}^{\delta} + \mathcal{X}^{\delta \text{can}} + \Phi^{\delta}.
$$

and $\mathcal{Y}^{\delta}$ is again similar. Here, we have implicitly changed our graphical rules to include the cut-off.
2.3 Existence and Regularity of Stochastic Objects

In this section, we define the notion of an iterated Itô integral with heterogeneity $\sigma$ and discuss in Lemma 2.10 how it can be controlled by passing to real space. Subsequently, we introduce Nelson’s estimate (Lemma 2.11) and derive a general criterion of existence for stochastic objects taking values in Besov spaces, Lemma 2.12. See also [MWX17; GP17] for different instances of the same arguments.

Let $n \in \mathbb{N}$, $D \subset \mathbb{R}^n_+ = (0, \infty)^n$ and $\phi \in L^2(D \times \mathbb{T}^{2n} \times \{1, 2\}^n; \mathbb{C})$. We define the spatial Fourier transform of $\phi$,

$$\hat{\phi}(u_1, \omega_1, j_1, \ldots, u_n, \omega_n, j_n) := \int_{(\mathbb{T}^2)^n} e^{-2\pi i (\omega_1 x_1 + \ldots + \omega_n x_n)} \phi(u_1, x_1, j_1, \ldots, u_n, x_n, j_n) \, dx_1 \ldots dx_n.$$

**Definition 2.9 (Iterated Itô integral).** Let $n \in \mathbb{N}$ and let

$$(0, \infty)^n \gamma := \{(u_1, \ldots, u_n) \in \mathbb{R}^n_+ : u_n < u_{n-1} < \ldots < u_1\}.$$

We define the iterated Itô integral acting on $\phi \in L^2((0, \infty)^n \gamma \times \mathbb{T}^{2n} \times \{1, 2\}^n; \mathbb{C})$ by

$$I^n(\phi) := \sum_{\omega_1, \ldots, \omega_n \in \mathbb{Z}^2} \sum_{j_1, \ldots, j_n = 1}^2 \int_0^\infty \int_0^{u_n-1} \int_0^{u_{n-1}} \int_0^{u_1} \, dW^{j_1}(u_1, \omega_1) \ldots dW^{j_n}(u_n, \omega_n) \hat{\phi}(u_1, -\omega_1, j_1, \ldots, u_n, -\omega_n, j_n).$$

Next we show how one may control the influence of the heterogeneity $\sigma$.

**Lemma 2.10.** Let $n \in \mathbb{N}$, $T > 0$ and $\sigma \in C_T L^\infty$. Let

$$(0, T)^n \gamma := \{(u_1, \ldots, u_n) \in \mathbb{R}^n_+ : u_n < u_{n-1} < \ldots < u_1 < T\}.$$

We define the heterogeneous iterated Itô integral acting on $\phi \in L^2((0, T)^n \gamma \times \mathbb{T}^{2n} \times \{1, 2\}^n; \mathbb{C})$ by

$$I^n_\sigma(\phi) := \sum_{\omega_1, \ldots, \omega_n \in \mathbb{Z}^2} \sum_{j_1, \ldots, j_n = 1}^2 \sum_{m_1, \ldots, m_n} \int_0^T \int_0^{u_n-1} \int_0^{u_{n-1}} \int_0^{u_1} \, dW^{j_1}(u_1, m_1) \int_0^{u_n-1} \int_0^{u_{n-1}} \int_0^{u_1} \, dW^{j_2}(u_2, m_2) \ldots dW^{j_n}(u_n, m_n)$$

$$\hat{\sigma}(u_1, \omega_1 - m_1) \ldots \hat{\sigma}(u_n, \omega_n - m_n) \hat{\phi}(u_1, -\omega_1, j_1, \ldots, u_n, -\omega_n, j_n).$$

Then

$$\mathbb{E}(\|I^n(\phi)\|^2) \leq \|\sigma\|_{C_T L^\infty} \mathbb{E}(\|I^n_\sigma(\phi)\|^2).$$

**Proof.** Let $\omega = (\omega_1, \ldots, \omega_n)$, $m = (m_1, \ldots, m_n)$, $u = (u_1, \ldots, u_n)$ and $j = (j_1, \ldots, j_n)$. We represent

$$\hat{\sigma}(u_1, \omega_1 - m_1) \ldots \hat{\sigma}(u_n, \omega_n - m_n) \hat{\phi}(u_1, -\omega_1, j_1, \ldots, u_n, -\omega_n, j_n) = \hat{\sigma}^\otimes(u, \omega - m) \hat{\phi}(u, -\omega, j).$$

Using that the Fourier transform turns products of $L^2(\mathbb{T}^{2n})$-functions into convolutions, we obtain

$$\sum_{\omega \in (\mathbb{Z}^2)^n} \hat{\sigma}^\otimes(u, \omega - m) \hat{\phi}(u, -\omega, j) = \mathcal{F}(\hat{\sigma}^\otimes)_{\hat{\phi}}(u, -m, j)$$

and consequently, $I^n_\sigma(\phi) = I^n(\sigma^\otimes \phi)$. We apply Itô’s isometry, Parseval’s theorem and the uniform boundedness of $\hat{\sigma}$ to bound

$$\mathbb{E}(\|I^n_\sigma(\phi)\|^2) = \sum_{j \in \{1, 2\}^n} \sum_{m \in (\mathbb{Z}^2)^n} \int_0^T \int_0^{u_n-1} \int_0^{u_{n-1}} \int_0^{u_1} \, du_1 \ldots du_n \mathcal{F}(\hat{\sigma}^\otimes \phi)(u, -m, j) \overline{\mathcal{F}(\hat{\sigma}^\otimes \phi)(u, -m, j)}$$

$$= \sum_{j \in \{1, 2\}^n} \int_0^T \int_0^{u_n-1} \int_0^{u_{n-1}} \int_0^{u_1} \, du_1 \ldots du_n \int_{(\mathbb{T}^2)^n} \, dx |\sigma^\otimes(u, x)|^2 |\phi(u, x, j)|^2$$

$$\leq \|\sigma\|_{C_T L^\infty}^{2n} \sum_{j \in \{1, 2\}^n} \int_0^T \int_0^{u_n-1} \int_0^{u_{n-1}} \int_0^{u_1} \, du_1 \ldots du_n \int_{(\mathbb{T}^2)^n} \, dx |\phi(u, x, j)|^2$$

$$= \|\sigma\|^{2n}_{C_T L^\infty} \mathbb{E}(\|I^n(\phi)\|^2).$$

This yields the claim. □
The following result, Nelson’s estimate, allows us to bound $p$-moments of iterated Itô integrals by their second moments.

**Lemma 2.11** (Nelson’s estimate). Let $n \in \mathbb{N}$ and $p \in [2, \infty)$. Then, there exists a $C > 0$ such that for any $\phi \in L^2((0, \infty)^n_+ \times T^{2n} \times \{1, 2\}^n; \mathbb{C})$,

$$E(|I^n(\phi)|^p)^{1/p} \leq CE(|I^n(\phi)|^2)^{1/2}.$$  

**Proof.** For a proof, see [Nua06; MWX17].

The following Kolmogorov criterion provides an efficient method for establishing regularity of stochastic processes in Hölder–Besov spaces. The presentation of this lemma is reminiscent of [Per20, Prop. 4.1].

**Lemma 2.12.** Let $X : [0, T] \to \mathcal{S}'(T^2)$, $X(0) = 0$, be a stochastic process and let $\alpha \in \mathbb{R}$, $p \in (1, \infty)$, $\gamma \in (1/p, 1]$. Assume there exists some $K > 0$ such that uniformly in $0 \leq s < t \leq T$,

$$\sum_{q \in \mathbb{N}_1} 2^{pq\alpha} \sup_{x \in T^2} \frac{E(|\Delta_qX(t, x) - \Delta_qX(s, x)|^p)}{|t - s|^p \gamma} \leq K < \infty.$$  

Then there exists a modification of $X$ (which we do not relabel) such that for any $\gamma' \in (0, \gamma - 1/p)$,

$$E(\|X\|^p_{C_T^\gamma C^{\alpha-2/p}}) \lesssim E(\|X\|^p_{C_T^{\gamma', \alpha} B_{p,p}}) \lesssim \gamma. p, \gamma', T K.$$  

In particular $\mathbb{P}$-a.s. $X \in C_T^{\gamma', \alpha-2/p}$.

**Proof.** The bound follows by the definition of $B^{\alpha}_{p,p}$ (Definition A.1), the Kolmogorov continuity criterion [FV10, Thm. A.10] and the Besov embedding (A.1). To show this, it suffices to exhibit a smooth, approximating sequence. This can be achieved by combining the Besov embedding (A.1) with the fact that the smooth functions are dense in $B^{\alpha}_{p,p}$.

### 2.4 Diagrams of Order 2 and 3

In this section, we construct the second-order diagrams $\Upsilon$, $\Theta$ and $\Theta'$ and the third-order diagrams $\Upsilon''$, $\Theta''$ and $\Theta'''$.  

**Lemma 2.13.** Let $T > 0$, $\alpha < -2$, $\kappa \in (0, 1/2)$ and $\sigma \in C_T L^\infty$. Then, for any $p \in [1, \infty)$ we have

$$E(\|Y\|^p_{C_T^{\kappa, 2\alpha+4}})^{1/p} \lesssim \|\sigma\|^2_{C_T L^\infty}$$  

and in particular $Y \in L^p_T C^{2\alpha+4}$ a.s..

From now on we denote $\Upsilon' := \Upsilon$ to emphasize the separate rôles of colour and shape. We first derive a useful upper bound on the second moments of $\Upsilon'$ in terms of an explicit, time-dependent function $S_{s,t} \Upsilon$. We call this function the shape coefficient.

**Definition 2.14.** Let $s, t \geq 0$ and $\omega_1, \omega_2 \in 2\pi \mathbb{Z}^2 \setminus \{0\}$. We define the shape coefficient

$$S_{s,t} \Upsilon(\omega_1, \omega_2) := \int_0^t \int_0^s du_3 \int_{-\infty}^{u_3 \wedge u'_3} \int_{-\infty}^{u_3 \wedge u'_3} du_2 \int_{-\infty}^{u_3 \wedge u'_3} du_1 \ e^{-|t+s-(u_3+u'_3)||\omega_1+\omega_2|^2} e^{-|u_3+u'_3-2u_1||\omega_1|^2} e^{-|u_3+u'_3-2u_2||\omega_2|^2},$$  

and the increment shape coefficient

$$D_{s,t} \Upsilon := S_{t,t} \Upsilon + S_{s,s} \Upsilon - S_{s,t} \Upsilon - S_{t,s} \Upsilon.$$  

(2.7)
Here, the letter $S$ stands for shape and $D$ for difference. It will be clear from the proof of Lemma 2.15 that $D_{s,t} Y \geq 0$.

Shape coefficients play a central rôle in our bounds, as they capture the iterated applications of $\nabla \cdot Z$; they fundamentally depend on the shape of the diagram, as opposed to the additional colouring induced by $\nabla \Phi$. Using this notation, we obtain the following bound.

**Lemma 2.15.** Let $s, t \in [0, T]$, $x \in T^2$ and $q \in \mathbb{N}_{-1}$. It holds that

$$E(|\Delta_q Y(t, x) - \Delta_q Y(s, x)|^2) \leq \|\sigma\|_{C^T L^\infty}^4 2!(2\pi)^3 \sum_{\omega \in \mathbb{Z}^2} \theta_q(\omega)^2 \sum_{\omega_1, \omega_2 \in \mathbb{Z}^2 \setminus \{0\}} |\omega_1|^2 |\omega_2|^{-2} |(\omega_2, \omega_1 + \omega_2)|^2 D_{s,t} Y(2\pi\omega_1, 2\pi\omega_2).$$

We refer to the prefactor $|\omega_1|^2 |\omega_2|^{-2} |(\omega_2, \omega_1 + \omega_2)|^2$ as the *colouring* of $Y$.

Before we give the proof of Lemma 2.15, let us comment on its general strategy. In $Y$, $\Theta$ and $\Phi$, it does not suffice to apply the triangle inequality to push the absolute value past the integral sign. This is related to the appearance of the sub-diagram $\Theta\Phi$, which we do not expect to be pointwise evaluable. Instead, we rely on bilinearity and expand the integrand

$$(f(t) - f(s))(g(t) - g(s)) = f(t)g(t) + f(s)g(s) - f(s)g(t) - f(t)g(s),$$

which leads to the common equation for this type of shape coefficient,

$$D_{s,t} = S_{t,t} + S_{s,s} - S_{s,t} - S_{t,s}.$$ We refer to Lemma 2.19, Lemma 2.21, Lemma 2.22 and Lemma 2.23 for instances where we can simplify our calculations by applying the triangle inequality.

**Proof of Lemma 2.15.** Let $s, t \in [0, T]$, $x \in T^2$ and $q \in \mathbb{N}_{-1}$. An application of Lemma 2.10 yields

$$E(|\Delta_q Y(t, x) - \Delta_q Y(s, x)|^2) \leq \|\sigma\|_{C^T L^\infty}^4 E(|\Delta_q Y(t, x) - \Delta_q Y(s, x)|^2)$$

where $Y$ is defined by

$$\tilde{Y}(t, \omega) := \sum_{\omega_1, \omega_2 \in \mathbb{Z}^2} \sum_{j_1, j_2, k_3 = 1}^2 \int_0^t du_3 \int_0^u \int_0^2 dW^{j_2}(u_2, \omega_2) \int_0^{u_2} dW^{j_1}(u_1, \omega_1) \times \sum_{c \in \Sigma(1,2)} H^{j_3}_{t-u_3}(\omega_1(1) + \omega_2(2))(H^{j_1}_{u_3-u_1}(\omega_1(2))G^{j_3}(\omega_1(2) \omega_3(2)) H^{j_2}_{u_3-u_2}(\omega_2(2)).$$

Using that $E(\tilde{Y}(t, \omega) \tilde{Y}(s, \omega')) = 0$ if $\omega \neq \omega' \in \mathbb{Z}^2$, we obtain

$$E(|\Delta_q Y(t, x) - \Delta_q Y(s, x)|^2) = \sum_{\omega \in \mathbb{Z}^2} \theta_q(\omega)^2 E(\tilde{Y}(t, \omega) - \tilde{Y}(s, \omega))^2).$$

It follows by an application of Itô’s isometry and Jensen’s inequality, using that for $z \in \mathbb{C}$, $|z|^2 = z\bar{z}$,

$$E(|\tilde{Y}(t, \omega) - \tilde{Y}(s, \omega)|^2) \leq \sum_{\omega_1, \omega_2 \in \mathbb{Z}^2} \sum_{j_1, j_2, k_3 = 1}^2 2! \int_{-\infty}^\infty du_2 \int_{-\infty}^\infty du_1 \int_0^\infty du_3 \int_0^\infty du_3' \times (H^{j_3}_{t-u_3}(\omega_1 + \omega_2) - H^{j_3}_{s-u_3}(\omega_1 + \omega_2)) \times (H^{j_1}_{u_3-u_1}(\omega_1) G^{j_3}(\omega_2) H^{j_2}_{u_3-u_2}(\omega_2)).$$
Recalling the definition of $D_{s,t}\mathcal{Y}$ from (2.7), we obtain
\[
E(|\mathcal{Y}(t,\omega) - \mathcal{Y}(s,\omega)|^2) \leq 2!(2\pi)^4 \sum_{\omega_1,\omega_2 \in \mathbb{Z}^2 \setminus \{0\}} |\omega_1|^2|\omega_2|^2|\omega_1 + \omega_2|^2 D_{s,t}\mathcal{Y}(2\pi\omega_1, 2\pi\omega_2).
\]
This yields the claim.

To evaluate the integrals in $S_{s,t}\mathcal{Y}$, we use a case distinction over $(\omega_1 \perp \omega_2)$ and $-(\omega_1 \perp \omega_2)$. We can then find explicit expressions for $D_{s,t}\mathcal{Y}$ via (2.7), which can be used to derive bounds. This is the content of Lemma B.1. We can now give the proof of Lemma 2.13.

**Proof of Lemma 2.13.** Let $T > 0$ and $\gamma \in [0, 1]$. To decompose the right hand side of (2.8), we introduce the orthogonal sum
\[
E^+(|\mathcal{Y}(t,\omega) - \mathcal{Y}(s,\omega)|^2) := \sum_{\omega_1,\omega_2 \in \mathbb{Z}^2 \setminus \{0\}} |\omega_1|^2|\omega_2|^2|\omega_1 + \omega_2|^2 D_{s,t}\mathcal{Y}(2\pi\omega_1, 2\pi\omega_2)
\]
and the non-orthogonal sum
\[
E^-(|\mathcal{Y}(t,\omega) - \mathcal{Y}(s,\omega)|^2) := \sum_{\omega_1,\omega_2 \in \mathbb{Z}^2 \setminus \{0\}} |\omega_1|^2|\omega_2|^2|\omega_1 + \omega_2|^2 D_{s,t}\mathcal{Y}(2\pi\omega_1, 2\pi\omega_2).
\]
We obtain the decomposition
\[
E(|\Delta_q\mathcal{Y}(t,x) - \Delta_q\mathcal{Y}(s,x)|^2) \leq ||\sigma||_{C_t-L^\infty}^4 2!(2\pi)^4 \sum_{\omega \in \mathbb{Z}^2 \setminus \{0\}} \sigma_q(\omega)^2 (E^+(|\mathcal{Y}(t,\omega) - \mathcal{Y}(s,\omega)|^2) + E^-(|\mathcal{Y}(t,\omega) - \mathcal{Y}(s,\omega)|^2)),
\]
where we used that $\mathcal{Y}(t,0) = 0$. For the orthogonal sum $E^+$, we obtain by Lemma B.1,
\[
D_{s,t}\mathcal{Y}(2\pi\omega_1, 2\pi\omega_2) \lesssim |t-s|^\gamma |\omega_1|^{-2}|\omega_2|^{-2} |\omega_1 + \omega_2|^{-4+2\gamma},
\]
so that
\[
E^+(|\mathcal{Y}(t,\omega) - \mathcal{Y}(s,\omega)|^2) \lesssim |t-s|^\gamma |\omega|^{-4+2\gamma} \sum_{\omega_1,\omega_2 \in \mathbb{Z}^2 \setminus \{0\}} 1,
\]
where we used the orthogonality $(\omega_1 \perp \omega_2)$ to identify $|\langle \omega_2, \omega_1 + \omega_2 \rangle|^2 = |\omega_2|^4$. Using the orthogonality again, we obtain the bound $|\omega_1|^2 \leq |\omega_1|^2 + |\omega_2|^2 = |\omega|^2$. By applying (C.1) to the finite sum over $\omega_1 \in \mathbb{Z}^2 \setminus \{0\}$, $|\omega_1| \leq |\omega|$, we arrive at
\[
E^+(|\mathcal{Y}(t,\omega) - \mathcal{Y}(s,\omega)|^2) \lesssim |t-s|^\gamma |\omega|^{-2+2\gamma}.
\]
Next we consider the non-orthogonal sum $E^-$. Lemma B.1 yields
\[
D_{s,t}\mathcal{Y}(2\pi\omega_1, 2\pi\omega_2) \lesssim |t-s|^\gamma |\omega_1|^{-4+2\gamma} |\omega_2|^{-2} |\omega_1 + \omega_2|^{-2} + |t-s|^\gamma |\omega_1|^{-4} |\omega_2|^{-2} |\omega_1 + \omega_2|^{-2+2\gamma},
\]
so that by Lemma C.3 for any $\gamma \in (0, 1)$ and $\varepsilon \in (0, (2-2\gamma) \land 1)$,
\[
E^-(|\mathcal{Y}(t,\omega) - \mathcal{Y}(s,\omega)|^2) \lesssim |t-s|^\gamma \sum_{\omega_1,\omega_2 \in \mathbb{Z}^2 \setminus \{0\}} (|\omega_1|^{-2+2\gamma} |\omega_2|^{-2} + |\omega|^2 |\omega_1|^{-2} |\omega_2|^{-2})
\].
\[
\lesssim |t-s|^\gamma |\omega|^{-2+2\gamma+2\varepsilon},
\]
where we used the Cauchy–Schwarz inequality to control $|⟨ω_2, ω_1 + ω_2⟩|^2 \lesssim |ω_2|^2|ω_1 + ω_2|^2$. Applying these results to the bound (2.8), we arrive at

$$\mathbb{E}(\|Δ_q \mathcal{V}(t, x) - Δ_q \mathcal{V}(s, x)\|^2) \lesssim |t - s|^\gamma \|σ\|^4_{C_T L^\infty} 2^{q(2γ + 2ε)}.$$ 

Assume in addition $ε < γ/2$. Using that $Δ_q \mathcal{V}$ denotes an iterated Itô integral, we obtain by Lemma 2.11 and Lemma 2.12 for any $p \in [1, \infty)$,

$$\mathbb{E}(\|\mathcal{V}\|^p_{C_T^{3α + 6} C_γ L^\infty})^{1/p} \lesssim \|σ\|^p_{C_T L^\infty}$$

and therefore $\mathcal{V} \in \mathcal{L}_T^p C^{0−}$ a.s. for any $κ \in (0, 1/2)$.

Next we consider the third-order diagrams $\mathcal{V}$ and $\mathcal{W}$.

**Lemma 2.16.** Let $T > 0$, $α < -2$, $κ \in (0, 1/2)$ and $σ \in C_T L^\infty$. Then, for any $p \in [1, \infty)$ we have

$$\mathbb{E}(\|\mathcal{V}\|^p_{\mathcal{L}_T^q C^{3α + 6} C_γ L^\infty})^{1/p} \lesssim \|σ\|^p_{C_T L^\infty}$$

and in particular $\mathcal{V}, \mathcal{W} \in \mathcal{L}_T^p C^{2α + 4}$ a.s.

**Proof.** The proof of this lemma is similar to the one for Lemma 2.13, so we only provide a sketch. The key idea is to consider the shape coefficient

$$S_{s,t} \mathcal{V}(ω_1, ω_2, ω_4) := S_{s,t} \mathcal{V}(ω_1, ω_2) S_{s,t} \mathcal{W}(ω_4), \quad s, t \geq 0, \quad ω_1, ω_2, ω_4 \in 2\pi \mathbb{Z}^2 \setminus \{0\},$$

where the factor $S_{s,t} \mathcal{V}$ was already defined in (2.6), and $S_{s,t} \mathcal{W}$ is given by

$$S_{s,t} \mathcal{W}(ω_4) := \int_{-∞}^{s∧t} du_4 e^{-|t - u_4||ω_4|^2} e^{-|s - u_4||ω_4|^2} = \frac{1}{2} |ω_4|^2 e^{-|t - s||ω_4|^2}.$$

The factorization $S_{s,t} \mathcal{V} = S_{s,t} \mathcal{V} S_{s,t} \mathcal{W}$ follows since there is no arrow pointing at the common root between the vertices labelled by $u_3$ and $u_4$.

We can then find explicit expressions for $D_{s,t} \mathcal{V}$, which we use to bound the second moments of $\mathcal{V}$ and $\mathcal{W}$. The claim then follows by Lemma 2.10, Lemma 2.11 and Lemma 2.12. \(\square\)

We can also show the existence of the diagrams $\mathcal{V}$ and $\mathcal{W}$.

**Lemma 2.17.** Let $T > 0$, $α < -2$, $κ \in (0, 1/2)$ and $σ \in C_T L^\infty$. Then, for any $p \in [1, \infty)$ we have

$$\mathbb{E}(\|\mathcal{V}\|^p_{\mathcal{L}_T^q C^{2α + 4} C_γ L^\infty})^{1/p} \lesssim \|σ\|^p_{C_T L^\infty}$$

and in particular $\mathcal{V}, \mathcal{W} \in \mathcal{L}_T^p C^{2α + 4}$ a.s.

We define a shape coefficient for $\mathcal{V}$ and $\mathcal{W}$. Since those do not contain the problematic sub-diagram $\mathcal{V}$, it suffices to push the absolute value past the integral sign. We denote this fact by the letter $A$ for absolute value. In particular, we may bound any integral over $[0, \infty)$ by $(-\infty, \infty)$, which simplifies our calculations.

**Definition 2.18.** Let $s, t \geq 0$, $ω_1, ω_1', ω_2 \in \mathbb{Z}^2 \setminus \{0\}$ and $k, k' = 1, 2$. We set

$$A_{s,t}^{k,k'} \mathcal{V}(ω_1, ω_1', ω_2) := \sum_{j=1}^{2} \int_{-∞}^{∞} du' \int_{-∞}^{∞} dω_1'' \int_{-∞}^{∞} dω_2'' |H^k_{t-u_1}(ω_1)H^{j}_{t-u_2}(ω_2) - H^k_{s-u_1}(ω_1)H^{j}_{s-u_2}(ω_2)|$$

$$\times |H^{k'}_{t-u_1'}(ω_1')H^{j}_{t-u_2}(ω_2) - H^{k'}_{s-u_1'}(ω_1')H^{j}_{s-u_2}(ω_2)|.$$
We can then bound the second moment of \( \mathcal{Q} \) in terms of this object.

**Lemma 2.19.** Let \( s, t \in [0, T], x \in \mathbb{T}^2, k, j = 1, 2 \) and \( q \in \mathbb{N}_{-1} \). It holds that

\[
\mathbb{E}(\Delta_q \mathcal{Q}(t, x, j) - \Delta_q \mathcal{Q}(s, x, k, j))^2 \lesssim \|\sigma\|^4_{C_T L^\infty} \sum_{\omega \in \mathbb{Z}^2} q_\omega(\omega)^2 \sum_{\omega_1, \omega_2 \in \mathbb{Z}^2 \setminus \{0\}} |\omega_2|^{-2} A_{s, t}^{k, k} \mathcal{Q}(\omega_1, \omega_1, \omega_2).
\]

**Proof of Lemma 2.17.** We provide a sketch. The shape coefficient is controlled in Lemma B.3, we can then use fairly direct estimates and apply Lemma 2.11 and Lemma 2.12 as before. We observe that \( \mathcal{Q} \) differs from \( \mathcal{Q} \) only in its colouring of the \( \omega_1 \) and \( \omega_2 \) arrows, with the sum of the exponents preserved. Consequently by the same arguments as for \( \mathcal{Q} \), we can also construct \( \mathcal{Q} \). \( \square \)

### 2.5 Wick Contractions

In this section, we construct the contractions \( \mathcal{Q}, \mathcal{Q}, \mathcal{Q} = \mathcal{Q} + \mathcal{Q} \) and \( \mathcal{Q} = \mathcal{Q} + \mathcal{Q} \). The diagrams \( \mathcal{Q}, \mathcal{Q} \) differ from \( \mathcal{Q}, \mathcal{Q} \) and \( \mathcal{Q}, \mathcal{Q} \), despite their similarity in structure. The first two are well-defined, since two applications of \( \nabla \Phi \) appear inside the \( \mathcal{Q} \)-shaped sub-diagram. This is not the case for \( \mathcal{Q}, \mathcal{Q} \) and \( \mathcal{Q}, \mathcal{Q} \) as one may tell by the distribution of highlighted arrows.

However, by adding the problematic diagrams, \( \mathcal{Q} = \mathcal{Q} + \mathcal{Q} \) and \( \mathcal{Q} = \mathcal{Q} + \mathcal{Q} \) we can make use of the symmetry of the multiplier \( G^j \) to establish the existence of the summed objects.

We define a shape coefficient for those diagrams.

**Definition 2.20.** Let \( s, t \geq 0, \omega_1, \omega_2, \omega_3 \in \mathbb{Z}^2 \setminus \{0\} \) and \( k = 1, 2 \). We set

\[
A_{s, t}^{k, k} \mathcal{Q}(\omega_1, \omega_2, \omega_3) := 2 \int_{-\infty}^{t} du_1 \int_{-\infty}^{u_1} du_2 \left| \left( H_{t-u_1}^k(\omega_1)H_{s-u_2}^j(\omega_2) - H_{s-u_1}^k(\omega_1)H_{s-u_2}^j(\omega_2) \right) \right|.
\]

In Lemma 2.21 we establish the existence of \( \mathcal{Q}, \mathcal{Q}, \mathcal{Q} \) and in Lemma 2.22, we establish the existence of \( \mathcal{Q} \).

**Lemma 2.21.** Let \( T > 0, \alpha < -2, \kappa \in (0, 1/2) \) and \( \sigma \in C_T H^2 \). Then, for any \( p \in [1, \infty) \) we have

\[
\mathbb{E}(\|\mathcal{Q}\|_{L^p_T C^{3a+6}}) \lesssim \|\sigma\|_{C_T L^\infty} \|\sigma\|_{C_T H^2}
\]

and in particular \( \mathcal{Q}, \mathcal{Q}, \mathcal{Q} \in L^p_T C^{3a+6} \ a.s. \).

**Lemma 2.22.** Let \( T > 0, \alpha < -2, \kappa \in (0, 1/2) \) and \( \sigma \in C_T H^2 \). Then, it holds that

\[
\|\mathcal{Q}\|_{L^p_T C^{2a+4}} \lesssim \|\sigma\|_{C_T H^2}^2.
\]

We first show Lemma 2.21. Let us focus on \( \mathcal{Q} \), the derivation for \( \mathcal{Q} \) and \( \mathcal{Q} \) is similar, but easier.

**Proof of Lemma 2.21.** Let \( s, t \in [0, T], x \in \mathbb{T}^2, j = 1, 2 \) and \( q \in \mathbb{N}_{-1} \). An application of Lemma 2.10 yields

\[
\mathbb{E}(\Delta_q \mathcal{Q}(t, x, j) - \Delta_q \mathcal{Q}(s, x, j))^2 \leq \|\sigma\|_{C_T L^{\infty}}^2 \mathbb{E}(\Delta_q \mathcal{Q}(t, x, j) - \Delta_q \mathcal{Q}(s, x, j))^2).
\]
where is defined by

\[ \sum_{\omega = \omega' + \omega'' + \omega'''} \int_{0}^{t} du_{3} \int_{0}^{u_{3}} du_{2} \int_{0}^{u_{2}} dW_{j_{3}}(u_{1}, \omega_{1}) \tilde{\sigma}(u_{2}, \omega_{2} - m_{2}) \tilde{\sigma}(u_{2}, \omega_{1} + m_{2}) \]

By the definition of the Littlewood–Paley block \( \Delta_{q} \),

\[ \mathbb{E}(|\Delta_{q} \mathcal{O}_{\xi}(t, x, j) - \Delta_{q} \mathcal{O}_{\xi}(s, x, j)|^{2}) \]

\[ \leq \sum_{\omega' \in \mathbb{Z}^{2}} \sum_{\omega'' \in \mathbb{Z}^{2}} \sigma(t) \sigma(\omega') |E(\tilde{\sigma}(t, \omega, j) - \tilde{\sigma}(s, \omega', j))| \]

We apply Itô’s isometry and a decay estimate (Lemma C.1) for the symmetrized elliptic multiplier \( G^{j}(\omega_{1} + \omega_{2}) + G^{j}(\omega_{4}) = G^{j}(\omega_{1} + \omega_{2} - \omega_{4}) - G^{j}(-\omega_{1}) \). We obtain the bound

\[ \mathbb{E}(|\Delta_{q} \mathcal{O}_{\xi}(t, x, j) - \Delta_{q} \mathcal{O}_{\xi}(s, x, j)|^{2}) \]

\[ \lesssim \|\sigma\|_{C^{1}T, L_{\infty}} \|\sigma\|_{C^{1}T, H^{2}} \sum_{\omega' \in \mathbb{Z}^{2}} \sum_{\omega'' \in \mathbb{Z}^{2}} \sigma(t) \sigma(\omega') |E(\tilde{\sigma}(t, \omega, j) - \tilde{\sigma}(s, \omega', j))| \]

We control the shape coefficient with Lemma B.2 and obtain for \( \gamma \in [0, 1] \),

\[ A_{k,t}^{j} \mathcal{V}(\omega_{1} + \omega_{2}, \omega_{4}, \omega_{2}) \lesssim |t - s| \gamma |\omega_{4}|^{2\gamma} |\omega_{2}|^{-1}. \]

We can then plug this expression into (2.9) and apply Lemma C.6 to control the sums over \( \omega_{4} \) and \( m_{2} \in \mathbb{Z}^{2} \). Let \( \gamma \in (0, 1/2) \) and \( \varepsilon \in (0, 1 - 2\gamma) \), it follows that

\[ \mathbb{E}(|\Delta_{q} \mathcal{O}_{\xi}(t, x, j) - \Delta_{q} \mathcal{O}_{\xi}(s, x, j)|^{2}) \]

\[ \lesssim \|\sigma\|_{C^{1}T, L_{\infty}} \|\sigma\|_{C^{1}T, H^{2}} \sum_{\omega' \in \mathbb{Z}^{2}} \sum_{\omega'' \in \mathbb{Z}^{2}} \sigma(t) \sigma(\omega') |E(\tilde{\sigma}(t, \omega, j) - \tilde{\sigma}(s, \omega', j))| \]

\[ \times \sum_{\omega_{1} \in \mathbb{Z}^{2}\backslash\{0\}} |\omega_{1}|^{-2} (1 \lor |\omega - \omega_{1}|)^{-2+\varepsilon} (1 \lor |\omega' - \omega_{1}|)^{-2+\varepsilon}. \]

Hence it suffices to control the remaining sum over \( \omega_{1} \in \mathbb{Z}^{2}\backslash\{0\} \),

\[ \sum_{\omega_{1} \in \mathbb{Z}^{2}\backslash\{0\}} |\omega_{1}|^{-2} (1 \lor |\omega - \omega_{1}|)^{-2+\varepsilon} (1 \lor |\omega' - \omega_{1}|)^{-2+\varepsilon}. \]

We distinguish the cases \( \omega = \omega' \) and \( \omega \neq \omega' \). In the case \( \omega = \omega' \), we decompose the sum into the regions \( \omega_{1} = \omega \) and \( \omega_{1} \in \mathbb{Z}^{2}\backslash\{0, \omega\} \). We then estimate by Lemma C.3,

\[ \sum_{\omega_{1} \in \mathbb{Z}^{2}\backslash\{0, \omega\}} |\omega_{1}|^{-2} (1 \lor |\omega - \omega_{1}|)^{-2+\varepsilon} \leq |\omega|^{-2+2\varepsilon}. \]

In the case \( \omega \neq \omega' \), we decompose the sum into the regions \( \omega_{1} = \omega \) and \( \omega_{1} \in \mathbb{Z}^{2}\backslash\{0, \omega, \omega'\} \),

\[ \sum_{\omega_{1} \in \mathbb{Z}^{2}\backslash\{0, \omega, \omega'\}} |\omega_{1}|^{-2} (1 \lor |\omega - \omega_{1}|)^{-2+\varepsilon} (1 \lor |\omega' - \omega_{1}|)^{-2+\varepsilon} \]

\[ = (|\omega|^{-2} + |\omega'|^{-2}) |\omega - \omega'|^{-2+\varepsilon} + \sum_{\omega_{1} \in \mathbb{Z}^{2}\backslash\{0, \omega, \omega'\}} |\omega_{1}|^{-2} |\omega - \omega_{1}|^{-2+\varepsilon} |\omega' - \omega_{1}|^{-2+\varepsilon}. \]
and apply Lemma C.4 to bound
\[ \sum_{\omega_1 \in \mathbb{Z}^2 \setminus \{0, \omega, \omega'\}} |\omega_1|^{-2} |\omega - \omega_1|^{-2\varepsilon} |\omega' - \omega_1|^{-2\varepsilon} \lesssim |\omega - \omega'|^{-2+\varepsilon} |\omega|^{-2+2\varepsilon} + |\omega - \omega'|^{-2+\varepsilon} |\omega'|^{-2+2\varepsilon}. \]

Assume \( \omega, \omega' \in \text{supp}(g_q) \), \( q \in \mathbb{N}_0 \). It follows that \( 2^q \lesssim |\omega|, |\omega'| \lesssim 2^q \) and if \( \omega \neq \omega' \) then \( 2^q \lesssim |\omega - \omega'| \lesssim 2^q \) as well. We obtain by (2.10),
\[ \mathbb{E}(|\Delta_q^0 (t, x, j) - \Delta_q^c (s, x, j)|^2) \lesssim |\sigma|^{2}_{C_T L^\infty} |\sigma|^{4}_{C_T H^2} \|t - s\|^{2\gamma} 2^{\gamma(4\gamma+5\varepsilon)}. \]

Assume in addition \( \varepsilon < \gamma \). We obtain by Lemma 2.11 and Lemma 2.12 for any \( p \in [1, \infty) \),
\[ \mathbb{E}(\| \sigma^{p}_{C_T^{1-\varepsilon} C_{-\gamma-6\varepsilon}} \|^{1/p} \lesssim |\sigma|^{2}_{C_T L^\infty} |\sigma|^{2}_{C_T H^2} \]
and therefore \( \sigma^{p}_{C_T^{1-\varepsilon} C_{-\gamma-6\varepsilon}} \) a.s. for any \( \kappa \in (0, 1/2) \).

Next we prove the existence of \( \sigma^{c}_{c_{-\gamma}} \).

**Proof of Lemma 2.22.** Let \( 0 \leq s \leq t \leq T \), \( \omega \in \mathbb{Z}^2 \) and \( k, j = 1, 2 \). We can bound the increment
\[ |\sigma^{c}_{c_{-\gamma}} (t, \omega, k, j) - \sigma^{c}_{c_{-\gamma}} (s, \omega, k, j)| \lesssim |\sigma|^{2}_{C_T H^2} \|t - s\|^{\kappa} |\omega| \sum_{\omega_1, \omega_2 \in \mathbb{Z}^2 \setminus \{0\}} \sum_{m_2 \in \mathbb{Z}^2} \frac{|G^j(\omega_2) + G^j(\omega_1)|}{1 + |\omega_1 + m_2|^2} A_{k, t}^{\omega}(\omega_1, \omega_2, \omega_1, \omega_2). \]
We control the shape coefficient with Lemma B.2 and the elliptic multiplier with Lemma C.1. Let \( \kappa \in [0, 1] \), we arrive at
\[ |\sigma^{c}_{c_{-\gamma}} (t, \omega, k, j) - \sigma^{c}_{c_{-\gamma}} (s, \omega, k, j)| \lesssim |\sigma|^{2}_{C_T H^2} \|t - s\|^{\kappa} |\omega| \sum_{\omega_1, \omega_2 \in \mathbb{Z}^2 \setminus \{0\}} \sum_{m_2 \in \mathbb{Z}^2} \frac{(1 + |\omega||\omega_2|^{-1})|\omega_1|^{-3+2\varepsilon}}{(1 + |\omega_1 + m_2|^2)(1 + |\omega_2 - m_2|^2)}. \]
Let \( \varepsilon \in (0, 1) \). We bound by Lemma C.3,
\[ \sum_{m_2 \in \mathbb{Z}^2} \frac{(1 + |\omega_1 + m_2|^2)^{-1}(1 + |\omega_2 - m_2|^2)^{-1}}{(1 + |\omega|^2)^{-1} + \sum_{m_2 \in \mathbb{Z}^2 \setminus \{-\omega_2\}} (1 + |\omega_1 + m_2|^2)^{-1}(1 + |\omega_2 - m_2|^2)^{-1}} \lesssim |\omega|^{-2+2\varepsilon} \]
and obtain
\[ |\sigma^{c}_{c_{-\gamma}} (t, \omega, k, j) - \sigma^{c}_{c_{-\gamma}} (s, \omega, k, j)| \lesssim |\sigma|^{2}_{C_T H^2} \|t - s\|^{\kappa} |\omega|^{-1+2\varepsilon} \sum_{\omega_1, \omega_2 \in \mathbb{Z}^2 \setminus \{0\}} \sum_{m_2 \in \mathbb{Z}^2} \frac{(1 + |\omega||\omega_2|^{-1})|\omega_1|^{-3+2\varepsilon}}{(1 + |\omega_1 + m_2|^2)(1 + |\omega_2 - m_2|^2)}. \]
For any \( \kappa \in (0, 1/2) \), we bound by Lemma C.3,
\[ |\sigma^{c}_{c_{-\gamma}} (t, \omega, k, j) - \sigma^{c}_{c_{-\gamma}} (s, \omega, k, j)| \lesssim |\sigma|^{2}_{C_T H^2} \|t - s\|^{\kappa} |\omega|^{-2+2\kappa+2\varepsilon}. \]
It follows directly that
\[ \| \sigma^{c}_{c_{-\gamma}} \|_{C_{-\gamma}^{2-2\varepsilon}} \lesssim |\sigma|^{2}_{C_T H^2} \]
and we obtain \( \sigma^{c}_{c_{-\gamma}} \in \mathcal{L}^p_T c_{0}^{-} \) for any \( \kappa \in (0, 1/2) \).
2.6 Construction of the Canonical Model

In this section, we construct $\delta^\phi$ and $Q^\delta_\phi$, $Q^\delta_\phi$ for $\delta > 0$. Additionally, we bound the speed of divergence as $\delta \to 0$ by a logarithmic rate using the symmetry of the elliptic equation.

**Lemma 2.23.** Let $T > 0$, $\alpha < -2$, $\kappa \in (0, 1)$, $\delta \in (0, 1 - \sqrt{2}/2]$ and $\sigma \in C_T H^2$. Then, it holds that

$$\| \delta^\phi \|_{L^T_T C^{2n+4}} \lesssim \log(\delta^{-1})\| \sigma \|_{C_T H^2}^2, \quad \| \delta^\phi \|_{L^T_T C^{2n+5}} \lesssim \delta^{-1}\log(\delta^{-1})\| \sigma \|_{C_T H^2}^2,$$

and for any $\kappa \in (0, 1/2)$, $p \in [1, \infty)$,

$$\mathbb{E}(\| Q^\delta_\phi \|_{L^T_T C^{2n+6}}^p)^{1/p} \lesssim \log(\delta^{-1})\| \sigma \|_{C_T L^\infty} \| \sigma \|_{C_T H^2}^2,$$

and

$$\mathbb{E}(\| Q^\delta_\phi \|_{L^T_T C^{2n+6}}^p)^{1/p} \lesssim \log(\delta^{-1})\| \sigma \|_{C_T L^\infty} \| \sigma \|_{C_T H^2}^2.$$
Assume \( s \leq t \) and \( \gamma \in [0, 1] \). We apply Lemma B.3 to control the shape coefficient. We can then apply Lemma C.7 and obtain for \( \varepsilon \in (0, 1/2) \), \( \delta \in (0, 1 - \sqrt{2}/2) \),

\[
E(\Delta_{\gamma} \mathcal{Q}^\delta(t, x, j) - \Delta_{\gamma} \mathcal{Q}^\delta(s, x, j))^2) \\ \lesssim \log(\delta^{-1})^2 \|\sigma\|^2_{C_T L^\infty} \|\sigma\|^4_{C_T H^2} |t - s|^\gamma \sum_{\omega, \omega' \in \mathbb{Z}^2} \phi(\omega)\phi(\omega') \\
\times \sum_{\omega_4 \in \mathbb{Z}^2 \setminus \{(0, \omega, \omega') \}} |\omega_4|^{-2} |\omega - \omega_4|^{-2} |\varepsilon^{\gamma + 3\varepsilon} |\omega' - \omega_4|^{-2} |\varepsilon^{\gamma + 3\varepsilon}.
\]

Assume \( 6\varepsilon < 4 - 2\gamma \). We apply Hölder’s inequality,

\[
\sum_{\omega_4 \in \mathbb{Z}^2 \setminus \{(0, \omega, \omega') \}} |\omega_4|^{-2} |\omega - \omega_4|^{-2} |\varepsilon^{\gamma + 3\varepsilon} |\omega' - \omega_4|^{-2} |\varepsilon^{\gamma + 3\varepsilon} \\
\leq \left( \sum_{\omega_4 \in \mathbb{Z}^2 \setminus \{(0, \omega, \omega') \}} |\omega_4|^{-2} |\omega - \omega_4|^{-4 + 2\gamma + 6\varepsilon} \right)^{1/2} \left( \sum_{\omega_4 \in \mathbb{Z}^2 \setminus \{(0, \omega, \omega') \}} |\omega_4|^{-2} |\omega' - \omega_4|^{-4 + 2\gamma + 6\varepsilon} \right)^{1/2} \\
\lesssim (1 \vee |\omega|)^{-2 + \gamma + 3\varepsilon} (1 \vee |\omega'|)^{-2 + \gamma + 3\varepsilon},
\]

which implies

\[
E(\Delta_{\gamma} \mathcal{Q}^\delta(t, x, j) - \Delta_{\gamma} \mathcal{Q}^\delta(s, x, j))^2) \\ \lesssim \log(\delta^{-1})^2 \|\sigma\|^2_{C_T L^\infty} \|\sigma\|^4_{C_T H^2} |t - s|^\gamma \sum_{\omega, \omega' \in \mathbb{Z}^2} \phi(\omega)\phi(\omega') (1 \vee |\omega|)^{-2 + \gamma + 3\varepsilon} (1 \vee |\omega'|)^{-2 + \gamma + 3\varepsilon}.
\]

Assume in addition \( \gamma \in (0, 1) \) and \( \varepsilon \in (0, \gamma/2) \), we obtain by Lemma 2.11 and Lemma 2.12 for any \( p \in [1, \infty) \),

\[
E(\| \mathcal{Q}^\delta \|_{C_T^{\gamma/2 - \varepsilon, \gamma - 6\varepsilon}}^p)^{1/p} \lesssim \log(\delta^{-1}) \|\sigma\|_{C_T L^\infty} \|\sigma\|_{C_T H^2}^2
\]

and therefore \( \mathcal{Q}^\delta \in L_T^p L^{\infty}_{x,j} \) a.s. for any \( \kappa \in (0, 1/2) \).

The only difference between \( \mathcal{Q}^\delta \) and \( \mathcal{Q}^\delta \) is that the factor \( G^\delta(\omega_1 + \omega_2) \) replaces \( G^\delta(\omega_4) \). Instead of (2.11), we estimate by Hölder’s inequality,

\[
\sum_{\omega_4 \in \mathbb{Z}^2 \setminus \{(0, \omega, \omega') \}} |\omega - \omega_4|^{-3 + \gamma + 3\varepsilon} |\omega' - \omega_4|^{-3 + \gamma + 3\varepsilon} \\
\leq \left( \sum_{\omega_4 \in \mathbb{Z}^2 \setminus \{(0, \omega, \omega') \}} |\omega - \omega_4|^{-6 + 2\gamma + 6\varepsilon} \right)^{1/2} \left( \sum_{\omega_4 \in \mathbb{Z}^2 \setminus \{(0, \omega, \omega') \}} |\omega' - \omega_4|^{-6 + 2\gamma + 6\varepsilon} \right)^{1/2} \\
\lesssim (1 \vee |\omega|)^{-2 + \gamma + 3\varepsilon} (1 \vee |\omega'|)^{-2 + \gamma + 3\varepsilon}.
\]

As before, we obtain by Lemma 2.10, Lemma 2.11 and Lemma 2.12 for any \( \gamma \in (0, 1) \), \( 6\varepsilon < 4 - 2\gamma \), \( \varepsilon \ll \gamma/2 \), \( \delta \in (0, 1 - \sqrt{2}/2) \) and \( p \in [1, \infty) \),

\[
E(\| \mathcal{Q}^\delta \|_{C_T^{\gamma/2 - \varepsilon, \gamma - 6\varepsilon}}^p)^{1/p} \lesssim \log(\delta^{-1}) \|\sigma\|_{C_T L^\infty} \|\sigma\|_{C_T H^2}^2
\]

and therefore \( \mathcal{Q}^\delta \in L_T^p L^{\infty}_{x,j} \) a.s. for any \( \kappa \in (0, 1/2) \).
Remark 2.24. We may also construct $\mathcal{O}^\delta = \mathbb{E}(t^\delta \nabla \Phi_t^\delta)$, which similar to $\mathcal{G}^\delta$ but without the lower stem. However, to obtain $\kappa$-time regularity, we need to trade $2\kappa$-space regularity in the parabolic multipliers $H_{\kappa}^{\alpha_1}(\omega_1)H_{\kappa}^{\beta_1}(\omega_2)$. We then sum over $\omega_1, \omega_2 \in \mathbb{Z}^2$, hence we will be stuck with a divergence of $\delta^{-2\kappa}$, where $\kappa$ is arbitrarily small but positive.

Remark 2.25. We can show that $\mathcal{O}^\delta \equiv 0$, if $\sigma \equiv 1$. Indeed, if we choose $\sigma \equiv 1$, then $\delta(u, \omega) = 1_{\omega=0}$. Consequently on the right hand side of (2.4), $\omega_1 = m_1$ and $\omega_2 = -m_1$. By the symmetrization, we obtain the factor $G^\delta(\omega_1) + G^\delta(\omega_2)$. Using that $G^\delta$ is odd and that $\omega_1 = -\omega_2$, we see that this term is zero, so that $\mathcal{O}^\delta \equiv 0$. Similarly, $\mathcal{G}^\delta = \mathcal{G}^\delta = \mathcal{G}^\delta \equiv 0$, if $\sigma \equiv 1$.

3 Existence of Paracontrolled Solutions

In this section we employ the enhancement constructed in Section 2 and show existence and uniqueness of solutions to equation (1.10) along with existence and uniqueness of a limit point, as $\delta \to 0$, of these solutions in a space of negative regularity that we dub the paracontrolled solution to (1.1).

Throughout we fix exponents satisfying the assumptions below. To explain their usage: $p$ and $\beta_0$ will be the integrability and regularity exponents of the admissible initial condition in the Besov scale $\mathcal{B}_{p,q}^{\beta_0}$, where $q$ is the microscopic parameter; $\alpha$ will be the regularity of the space-time white noise, so that almost surely $1 \in C_T C^{\alpha+1}$; $\beta$ measures the regularity of the second Da Prato–Debussche remainder, $w$, in the Hölder scale and $\eta$ the allowed blow-up of $w$ at $t = 0$; $\beta^\#$ measures the regularity of the paracontrolled remainder in the same scale and finally $\kappa$ will be used to denote time regularity.

From now on we fix $p$, $q$, $\alpha$, $\beta$, $\beta^\#$, $\beta_0$, $\kappa$ and $\eta$ satisfying

$$
\begin{align*}
p &\in (4, \infty], \\
q &\in [1, \infty], \\
\alpha &\in (-9/4 + 1/p, -2), \\
\beta &\in (-1/2, 2\alpha + 4), \\
\beta^\# &\in (-\alpha - 2, \alpha + \beta + 3 - 2/p), \\
\beta_0 &\in (2\beta^\# - \alpha - \beta - 3 + 2/p, \beta^\#], \\
\kappa &\in ((\beta^\# - \alpha - 2)/2, 1), \\
\eta &\in [(\beta^\# - \beta_0)/2 + 1/p, 1 - (\beta^\# - \alpha - \beta - 1)/2).
\end{align*}
$$

Remark 3.1. By taking $\alpha \approx -2$, $\beta \approx 0$ and $\beta^\# \approx 0$, we can choose any $\beta_0 > -1 + 2/p$. Using the embedding $L^p(\mathbb{T}^2) \subset \mathcal{B}_{p,\infty}^{\beta_0}$, we can then choose $\rho_0 \in L^p(\mathbb{T}^2)$ for any $p > 4$.

Let us fix a $T > 0$. We define the space of paracontrolled distributions.

Definition 3.2. Assume $X \in \mathcal{X}_T^{\alpha,\kappa}$ and $\rho_0 \in \mathcal{B}_{p,q}^{\beta_0}$. Then we define the space

$$
\mathcal{D}_T \subset \mathcal{L}_{n,T}^\kappa C^\beta(\mathbb{T}^2; \mathbb{R}) \times \mathcal{L}_{n,T}^\kappa C^{\beta+1}(\mathbb{T}^2; \mathbb{R}) \times \mathcal{L}_{n,T}^\kappa C^{\beta^\#}(\mathbb{T}^2; \mathbb{R})
$$

of distributions paracontrolled by $X$ as those $w := (w, w', w^\#)$, such that

$$
w = \nabla \cdot I[w' \otimes 1] + w^\# \quad (3.1)
$$

and $w|_{t=0} = \rho_0$, $w'|_{t=0} = \nabla \Phi_{\rho_0}$, $w^\#|_{t=0} = \rho_0$. We equip this space with the norm

$$
||w||_{\mathcal{D}_T} := \max\{||w||_{\mathcal{L}_{n,T}^\kappa C^\beta}, ||w'||_{\mathcal{L}_{n,T}^\kappa C^{\beta+1}}, ||w^\#||_{\mathcal{L}_{n,T}^\kappa C^{\beta^\#}}\}.
$$

Remark 3.3. The Ansatz (3.1) allows us to write the mild equation for $w^\#$ as $w^\# = P\rho_0 + \nabla \cdot I[\Omega^\#(w)]$, with some $\Omega^\#(w)$ determined by (1.10). We can hence simplify our estimates by using the space-time regularization of $I$. Note that (3.1) is equivalent to the Ansatz discussed in the introduction, up to commutators.

Lemma 3.4. Given $X \in \mathcal{X}_T^{\alpha,\kappa}$ and $\rho_0 \in \mathcal{B}_{p,q}^{\beta_0}$, the space $\mathcal{D}_T$ is a non-empty, complete metric space.
Proof. To show that $\mathcal{D}_T$ is non-empty, we can first choose $w' = \nabla \Phi_{P\rho}$ and $w^\# = P\rho_0$. The initial condition is satisfied, since $w'|_{t=0} = \nabla \Phi_{P\rho_0}$ and $w^\#|_{t=0} = \rho_0$. By Lemma A.6 and Lemma A.8, using that $\beta \vee \beta_0 \leq \beta^\#$ and $(\beta^\# - \beta_0)/2 + 1/p \leq \eta$, 
\[
\|\nabla \Phi_{P\rho}\|_{L^2_t\mathcal{C}^{\beta+1}} \lesssim \|P\rho_0\|_{L^2_t\mathcal{C}^{\beta^\#}} \lesssim T \|\rho_0\|_{E^\rho_0} \, ,
\]
so that $w' \in L^\infty_t\mathcal{C}^{\beta+1}$ and $w^\# \in L^\infty_t\mathcal{C}^{\beta^\#}$. Setting $w := \nabla \cdot I[w' \odot 1] + w^\#$, we find by an application of the triangle inequality, Lemma A.6 and Lemma A.4, using that $\beta < \alpha + 2$ and $\beta + 1 > 0$,
\[
\|w\|_{L^\infty_t\mathcal{C}^{\beta}} \lesssim T \|w'\|_{L^\infty_t\mathcal{C}^{\beta+1}} \|1\|_{C^\alpha+1} + \|w^\#\|_{L^\infty_t\mathcal{C}^{\beta^\#}} .
\]
To show completeness, let $(w_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $\mathcal{D}_T$. By the completeness of $L^\infty_t\mathcal{C}^\beta \times L^\infty_t\mathcal{C}^{\beta+1} \times L^\infty_t\mathcal{C}^{\beta^\#}$, we obtain $w_n \to w \in L^\infty_t\mathcal{C}^\beta$, $w'_n \to w' \in L^\infty_t\mathcal{C}^{\beta+1}$ and $w^\#_n \to w^\# \in L^\infty_t\mathcal{C}^{\beta^\#}$ all with the correct initial conditions. It suffices to show $w = \nabla \cdot I[w' \odot 1] + w^\#$ so that $w \in \mathcal{D}_T$. Then again by Lemma A.6 and Lemma A.4,
\[
w_n - \nabla \cdot I[w'_n \odot 1] - w^\#_n = \nabla \cdot I[(w'_n - w') \odot 1] + w^\#_n - w^\# \to 0 \quad \text{in} \quad L^\infty_t\mathcal{C}^\beta,
\]
which combined with $w_n \to w \in L^\infty_t\mathcal{C}^\beta$ yields $w = \nabla \cdot I[w' \odot 1] + w^\#$. \qed

In the next lemma we show that $w \circ \nabla \Phi + \mathfrak{l} \circ \nabla \Phi_w$ is well-defined on $\mathcal{D}_T \times \mathcal{X}^\alpha_{T'}$.

**Lemma 3.5.** There exists a continuous operator $\mathcal{P} : \mathcal{D}_T \times \mathcal{X}^\alpha_{T'} \to C^\alpha_T C^{2\alpha+4}$, such that when all objects are smooth,
\[
\mathcal{P}(w, X) = w \circ \nabla \Phi + \mathfrak{l} \circ \nabla \Phi_w .
\]

Proof. Using the notation $\mathcal{C}(f, g, h) = (f \odot g) \circ h - f(g \circ h)$ (see Lemma A.11) and recalling that $\mathfrak{v} = \nabla I[1] \circ \nabla \Phi + \nabla^2 I[\Phi] \circ \mathfrak{l}$ we can expand the product into
\[
\mathcal{P}(w, X) := \mathcal{C}(w', \nabla I[1], \nabla \Phi) + \mathcal{C}(w', \nabla^2 I[\Phi], \mathfrak{l}) + (w^\# + \nabla \cdot I[w' \odot 1] - w' \circ \nabla I[1]) \circ \nabla \Phi + (\nabla \Phi_w \circ \nabla \cdot I[w' \odot 1] - w' \circ \nabla^2 I[\Phi]) \circ \mathfrak{l} + w' \mathfrak{v},
\]
where $w \in \mathcal{D}_T$ and $X \in \mathcal{X}^\alpha_{T'}$. In what follows, we establish the bound
\[
\|\mathcal{P}(w, X)\|_{C^\alpha_T C^{2\alpha+4}} \lesssim T (\|w'\|_{L^\infty_t\mathcal{C}^{\beta+1}} + \|w^\#\|_{L^\infty_t\mathcal{C}^{\beta^\#}})(1 + \|1\|_{C^\alpha T^{\alpha+1}} + \|\mathfrak{v}\|_{C^\alpha_T C^{2\alpha+4}})^2
\]
by various applications of our commutator results. Using Lemma A.11 and that $\beta + 1 \in (0, 1)$, $2\alpha + 4 < 0 < 2\alpha + \beta + 5$, we obtain
\[
\|\mathcal{C}(w', \nabla I[1], \nabla \Phi)\|_{C^\alpha_T C^{2\alpha+4}} \lesssim \|w'\|_{L^\infty_t\mathcal{C}^{\beta+1}} \|\nabla I[1]\|_{C^\alpha T^{\alpha+2}} \|\nabla \Phi\|_{C^\alpha T^{\alpha+2}}
\]
and
\[
\|\mathcal{C}(w', \nabla^2 I[\Phi], \mathfrak{l})\|_{C^\alpha_T C^{2\alpha+4}} \lesssim \|w'\|_{L^\infty_t\mathcal{C}^{\beta+1}} \|\nabla^2 I[\Phi]\|_{C^\alpha T^{\alpha+3}} \|\mathfrak{l}\|_{C^\alpha_T T^{\alpha+1}}.
\]
Further, by Lemma A.4, using that $2\alpha + 4 < 0 < \beta^\# + \alpha + 2$,
\[
\|(w^\# + \nabla \cdot I[w' \odot 1] - w' \circ \nabla I[1]) \circ \nabla \Phi\|_{C^\alpha_T C^{2\alpha+4}} \lesssim \|w^\# + \nabla \cdot I[w' \odot 1] - w' \circ \nabla I[1]\|_{C^\alpha_T C^{\beta^\#}} \|\nabla \Phi\|_{C^\alpha_T C^{\alpha+2}}.
\]
To control the remainder, we apply Lemma A.9 and Lemma A.10, using that $\kappa \in ((\beta^\# - \alpha - 2)/2, 1/2)$ and $\beta^\# < \alpha + \beta + 3$,
\[
\|w^\# + \nabla \cdot I[w' \odot 1] - w' \circ \nabla I[1]\|_{C^\alpha_T C^{\beta^\#}} \lesssim T \|w^\#\|_{C^\alpha_T C^{\beta^\#}} + \|w'\|_{L^\infty_t\mathcal{C}^{\beta+1}} \|1\|_{C^\alpha T^{\alpha+1}}.
\]
Similarly, by Lemma A.4, Lemma A.9 and Lemma A.10, using that $2\alpha + 4 < 0 < \beta^\# + \alpha + 2$,
\[
\|\nabla \Phi_w \circ \nabla \cdot I[w' \odot 1] - w' \circ \nabla^2 I[\Phi] \|_{C^\alpha_T C^{2\alpha+4}} \lesssim T (\|w^\#\|_{C^\alpha_T C^{\beta^\#}} + \|w'\|_{L^\infty_t\mathcal{C}^{\beta+1}} \|1\|_{C^\alpha T^{\alpha+1}}) \|1\|_{C^\alpha T^{\alpha+1}}.
\]
Finally, by Lemma A.4, using that $2\alpha + 4 < 2\alpha + \beta + 5$,
\[ \| w' \nabla \|_{C_{\eta,T}\mathcal{C}^{2\alpha + 1}} \lesssim \| w' \|_{C_{\eta,T}\mathcal{C}^{\beta + 1}} \| \nabla \|_{C_{\eta,T}\mathcal{C}^{2\alpha + 4}}. \]
This yields the claim. \qed

We can now derive a priori bounds for our solution map.

**Lemma 3.6.** Assume $X \in \mathcal{X}_{\eta}^{\alpha,\kappa}$ and $\rho_0 \in B_{p,q}^{\beta_0}$. Let $\Psi$, acting on $u = (u,u',u^\#) \in \mathcal{D}_T$, be given by $\Psi(u) := (w,w',w^\#)$, where
\[
\begin{aligned}
w &:= \nabla \cdot I[w' \otimes \mathbf{I}] + w^#, \\
w' &:= \nabla \Phi u + \nabla \Phi \psi, \\
w^# &:= \rho_0 + \nabla \cdot I[\Omega^#(u)],
\end{aligned}
\]
with
\[
\Omega^#(u) := u \nabla \Phi u + u \nabla \Phi \psi + \psi \nabla \Phi u + \psi \nabla \Phi \psi + \nabla \Phi \psi \otimes \psi + \psi \otimes \nabla \Phi \psi + I \otimes \nabla \Phi \psi + u \otimes \nabla \Phi \psi + \nabla \Phi \psi \otimes u + I \otimes \nabla \Phi \psi + \mathcal{D}(u,X).
\]
Then there exists some $\theta > 0$ depending only on the chosen parameters and the dimension, such that for $T \leq 1$,
\[
\max\{\|w\|_{L_{\eta,T}^{\kappa}\mathcal{C}^{\beta}}, \|w^\#\|_{L_{\eta,T}^{\kappa}\mathcal{C}^{\beta + 1}}\} \lesssim (1 + T^\theta \|u\|_{\mathcal{D}_T})^2 (1 + \|X\|_{\mathcal{X}_{\eta}^{\alpha,\kappa}} + \|\rho_0\|_{B_{p,q}^{\beta_0}})^2, \tag{3.3}
\]
\[
\|w'\|_{L_{\eta,T}^{\kappa}\mathcal{C}^{\beta + 1}} \lesssim \|u\|_{L_{\eta,T}^{\kappa}\mathcal{C}^{\beta}} + \|X\|_{\mathcal{X}_{\eta}^{\alpha,\kappa}}. \tag{3.4}
\]
In particular, $\Psi(u) \in \mathcal{D}_T$.

**Proof.** We derive bounds for our solution map in several steps.

*Step 1.* The $L_{\eta,T}^{\kappa}\mathcal{C}^{\beta}$-regularity of $w$. As in the proof of Lemma 3.4, but this time keeping track of the dependency on $T$, we see that
\[
\|w\|_{L_{\eta,T}^{\kappa}\mathcal{C}^{\beta}} \lesssim (T^{1 - \frac{\beta - \alpha}{2}} \vee T^{1 - \kappa}) \|w'\|_{C_{\eta,T}\mathcal{C}^{\beta + 1}} \|T\|_{C_{\eta,T}^{\alpha + 1}} + \|w^\#\|_{L_{\eta,T}^{\kappa}\mathcal{C}^{\beta}}. \]

*Step 2.* The $L_{\eta,T}^{\kappa}\mathcal{C}^{\beta + \#}$-regularity of $w^\#$. By Lemma A.6, using that $\beta_0 \leq \beta^\#, (\beta^\# - \beta_0)/2 + 1/p \leq \eta$, $\eta < 1/2$, $\beta^\# + 1 < \alpha + \beta + 4$ and $(\beta^\# - \alpha - \beta - 1) / 2 \vee \kappa < 1 - \eta$,
\[
\|w^\#\|_{L_{\eta,T}^{\kappa}\mathcal{C}^{\beta + \#}} \lesssim T^{\eta - \frac{\beta - \alpha}{2} + \frac{\eta}{2}} \|\rho_0\|_{B_{p,q}^{\beta_0}} + (T^{1 - \frac{\beta^\# - \alpha - \beta - 1}{2} - \eta} \vee T^{1 - \kappa - \eta}) \|\Omega^#(u)\|_{C_{2\eta,T}^{\alpha + \beta + 2}}.
\]

*Step 3.* The $C_{2\eta,T}^{\alpha + \beta + 2}$-regularity of $\Omega^#(u)$. We obtain by various applications of Lemma A.4, using in particular that $\beta > -1/2$,
\[
\|u \nabla \Phi u\|_{C_{2\eta,T}^{\alpha + \beta + 2}} \lesssim \|u\|^2_{C_{\eta,T}^{\kappa}},
\]
\[
\max\{\|u \nabla \Phi \psi\|_{C_{2\eta,T}^{\alpha + \beta + 2}}, \|\nabla \Phi u\|_{C_{2\eta,T}^{\alpha + \beta + 2}}, \|u \otimes \nabla \Phi \psi\|_{C_{2\eta,T}^{\alpha + \beta + 2}}, \|\nabla \Phi \psi \otimes u\|_{C_{2\eta,T}^{\alpha + \beta + 2}}\} \lesssim T^n \|u\|_{C_{\eta,T}^{\kappa}} \|X\|_{\mathcal{X}_{\eta}^{\alpha,\kappa}},
\]
and
\[
\max\{\|\nabla \Phi \psi\|_{C_{2\eta,T}^{\alpha + \beta + 2}}, \|\nabla \Phi \psi \|_{C_{2\eta,T}^{\alpha + \beta + 2}}, \|\nabla \Phi \psi \otimes \psi\|_{C_{2\eta,T}^{\alpha + \beta + 2}}, \|\nabla \Phi \psi \|_{C_{2\eta,T}^{\alpha + \beta + 2}}\} \lesssim T^{2n} \|u\|^2_{C_{\eta,T}^{\kappa}}. \tag{3.2}
\]
By (3.2) of Lemma 3.5, using that $\alpha + \beta + 2 \leq 2\alpha + 4$,
\[
\|\mathcal{D}(u,X)\|_{C_{2\eta,T}^{\alpha + \beta + 2}} \lesssim T^n (\|u'\|_{L_{\eta,T}^{\kappa}\mathcal{C}^{\beta + 1}} + \|u^\#\|_{C_{\eta,T}^{\kappa}\mathcal{C}^{\beta}})(1 + \|T\|_{C_{\eta,T}^{\alpha + 1}} + \|\nabla\|_{C_{\eta,T}^{2\alpha + 4}})^2.
\]
Step 4. The $\mathcal{L}^{\kappa}_{\eta,T}C^{\beta+1}$-regularity of $w'$. By definition, $w' = \nabla \Phi_u + \nabla \Phi \varphi$. Using that $\beta + 1 \leq 2\alpha + 5$, we obtain

$$||w'||_{\mathcal{L}^{\kappa}_{\eta,T}C^{\beta+1}} \lesssim ||u||_{\mathcal{L}^{\kappa}_{\eta,T}C^{\beta}} + T^\eta ||\gamma||_{\mathcal{L}^{\kappa}_{\eta,T}C^{2\alpha+4}}.$$  

Step 5. Closing the bounds. Using that $T \leq 1$, we can collect all of the terms above and cast them in the form (3.3)–(3.4). This yields the claim. □

While Lemma 3.6 shows that $\Psi$ is a map from $\mathcal{D}_T$ to itself, it is not a contraction for small $T$, since there is no small time parameter on the right hand side of (3.4). The remedy is to apply $\Psi$ twice and argue that a fixed point of $\Psi^{o2}$ is also a fixed point of $\Psi$ itself.

**Proposition 3.7.** Assume $\chi \in \mathcal{X}^{\alpha,\kappa}_T$ and $\rho_0 \in \mathcal{B}^{\beta_0}_{p,q}$. Then there exists some $\bar{T} \in (0, T]$ such that there is a unique solution $w \in \mathcal{D}_T$ to the equation

$$\begin{align*}
\begin{cases}
  w := \nabla \cdot \mathcal{I}[(\nabla \Phi_u + \nabla \Phi \varphi) \otimes \mathbf{1}] + w^#, \\
  w^# := P\rho_0 + \nabla \cdot \mathcal{I}[\Omega^#(w)].
\end{cases}
\end{align*}$$

Proof. We let $\bar{T} \in (0, T]$, $\bar{T} \leq 1$, $\chi \in \mathcal{X}^{\alpha,\kappa}_T$, $u \in \mathcal{D}_T$ and define

$$(\Psi(u), \Psi(u)^#, \Psi(u)^{##}) := \Psi(u).$$

By Lemma 3.6 there exists some $\theta > 0$ such that

$$\max\{||\Psi(u)||_{\mathcal{L}^{\kappa}_{\eta,T}C^{\beta}}, ||\Psi(u)^#||_{\mathcal{L}^{\kappa}_{\eta,T}C^{\beta}}\} \lesssim (1 + \bar{T}^{\theta}||u||_{\mathcal{D}_T})^2(1 + ||\chi||_{\mathcal{X}^{\alpha,\kappa}_T} + ||\rho_0||_{\mathcal{B}^{\beta_0}_{p,q}})^2, \quad (3.5)$$

and

$$||\Psi(u)^#||_{\mathcal{L}^{\kappa}_{\eta,T}C^{\beta+1}} \lesssim ||u||_{\mathcal{L}^{\kappa}_{\eta,T}C^{\beta}} + ||\chi||_{\mathcal{X}^{\alpha,\kappa}_T}.$$  \hspace{1cm} (3.6)

Now denote

$$(\Psi^{o2}(u), \Psi^{o2}(u)^#, \Psi^{o2}(u)^{##}) := \Psi^{o2}(u).$$

By iterating the bounds (3.5)–(3.6), using $\bar{T} \leq 1$ to streamline exponents, we obtain

$$||\Psi^{o2}(u)||_{\mathcal{D}_T} \lesssim (1 + \bar{T}^{\theta}||u||_{\mathcal{D}_T})^4(1 + ||\chi||_{\mathcal{X}^{\alpha,\kappa}_T} + ||\rho_0||_{\mathcal{B}^{\beta_0}_{p,q}})^6. \quad (3.7)$$

Let $C > 0$ be larger than the implicit constants of the inequalities (3.5) and (3.7) above. Assume that $M, R > 0$ are sufficiently large that

$$(1 + ||\chi||_{\mathcal{X}^{\alpha,\kappa}_T} + ||\rho_0||_{\mathcal{B}^{\beta_0}_{p,q}}) < M, \quad 2CM^6 < R.$$  

Assume further that $||u||_{\mathcal{D}_T} < R$. Using the bound (3.7), we can choose $\bar{T} = \bar{T}(M, R) \leq 1$ smaller, if necessary, such that

$$||\Psi^{o2}(u)||_{\mathcal{D}_T} \leq 2CM^6 < R.$$

Consequently, $\Psi^{o2}$ is a self-mapping on the ball

$$\mathfrak{B}_{R;T} := \{u \in \mathcal{D}_T : ||u||_{\mathcal{D}_T} < R\}.$$  

Upon choosing $R > 0$ sufficiently large, we can ensure that $\mathfrak{B}_{R;T} \subset \mathcal{D}_T$ is non-empty. To achieve contractivity, we use the bilinearity of the equation. Let $v = (v, v^#), w = (w, w^#, w^{##}) \in \mathcal{D}_T$ and denote

$$(\Psi(v), \Psi(v)^#, \Psi(v)^{##}) := \Psi(v), \quad (\Psi(w), \Psi(w)^#, \Psi(w)^{##}) := \Psi(w).$$

We see that

$$\begin{align*}
\Psi(v) - \Psi(w) &= \nabla \cdot \mathcal{I}[(\Psi(v)^# - \Psi(w)^#) \otimes \mathbf{1}] + \Psi(v)^# - \Psi(w)^#,
\Psi(v)^# - \Psi(w)^# &= \nabla \Phi_{v-w},
\Psi(v)^{##} - \Psi(w)^{##} &= \nabla \cdot \mathcal{I}[\Omega^#(v) - \Omega^#(w)].
\end{align*}$$
where
\[ \Omega^\#(v) - \Omega^\#(w) = v \nabla \Phi_{v - w} + (v - w) \nabla \Phi_w + (v - w) \nabla \Phi_Y + \Psi \nabla \Phi_{v - w} + (v - w) \odot \nabla \Phi_f + \nabla \Phi_{v - w} + \nabla \Phi_f \odot (v - w) + 1 \odot \nabla \Phi_{v - w} + \mathcal{P}(v - w, x). \]

The difference of the renormalized products is given by
\[
\mathcal{P}(v - w, x) = \mathcal{C}(v' - w', \nabla I[1], \nabla \Phi_f) + \mathcal{C}(v' - w', \nabla^2 \Phi_f[1], 1) + (v' - w') \odot \nabla \Phi_f + \nabla \Phi_{v' - w'} \odot \nabla \Phi_{v - w} + (v' - w') \odot \nabla^2 \Phi_f[1] \odot 1 + (v' - w') \mathcal{N}.
\]

Using the same bounds as before, we obtain, for some \( \theta > 0 \),
\[
\|\Psi(v) - \Psi(w)\|_{\mathcal{L}_{n,T}^\alpha C^\beta} \lesssim T_\theta^\delta \|\Psi(v') - \Psi(w')\|_{C_{n,T}C^{\alpha+1}} + \|\Omega^\#(v) - \Omega^\#(w)\|_{\mathcal{L}_{n,T}^\alpha C^\beta},
\]
and
\[
\|\Psi(v') - \Psi(w')\|_{\mathcal{L}_{n,T}^\alpha C^{\beta+1}} \lesssim T_\theta^\delta \Omega^\#(v) - \Omega^\#(w),
\]
as well as
\[
\|\Psi(v') - \Psi(w')\|_{\mathcal{L}_{n,T}^\alpha C^{\beta+1}} \lesssim \|v - w\|_{\mathcal{L}_{n,T}^\alpha C^\beta}.
\]

For the right hand side,
\[
\begin{align*}
\|\Omega^\#(v) - \Omega^\#(w)\|_{C_{n,T}C^{\alpha+\beta+2}} & \lesssim \|v\|_{C_{n,T}C^\beta} \|w\|_{C_{n,T}C^\beta} \|v - w\|_{C_{n,T}C^\beta} \|v - w\|_{C_{n,T}C^\beta} \|v - w\|_{C_{n,T}C^\beta} \|v - w\|_{C_{n,T}C^\beta} \|\Psi\|_{C_{n,T}C^{\alpha+4}} \\
& \quad + \|v - w\|_{C_{n,T}C^\beta} \|C_{n,T}C^{\alpha+1} + \|\mathcal{P}(v - w, x)\|_{C_{n,T}C^{\alpha+4}}.
\end{align*}
\]

By the same arguments as in the proof of Lemma 3.5 we have,
\[
\|\mathcal{P}(v - w, x)\|_{C_{n,T}C^{\alpha+4}} \lesssim \|v\|_{\mathcal{L}_{n,T}^\alpha C^{\beta+1}} + \|w\|_{\mathcal{L}_{n,T}^\alpha C^{\beta+1}} + \|v - w\|_{\mathcal{L}_{n,T}^\alpha C^{\beta+1}} + \|\Psi\|_{\mathcal{L}_{n,T}^\alpha C^{\beta+1}}^2 + \|\Psi\|_{\mathcal{L}_{n,T}^\alpha C^{\beta+1}}^2 + \|\Psi\|_{\mathcal{L}_{n,T}^\alpha C^{\beta+1}}^2.
\]

Combining the bounds above, we obtain
\[
\begin{align*}
\max\{\|\Psi(v) - \Psi(w)\|_{\mathcal{L}_{n,T}^\alpha C^\beta}, \|\Psi(v') - \Psi(w')\|_{\mathcal{L}_{n,T}^\alpha C^\beta}\} & \lesssim T_\theta^\delta \|v - w\|_{\mathcal{L}_{n,T}^\alpha C^\beta} (1 + \|\Psi\|_{\mathcal{L}_{n,T}^\alpha C^\beta} + \|w\|_{\mathcal{L}_{n,T}^\alpha C^\beta} + \|v\|_{\mathcal{L}_{n,T}^\alpha C^\beta}), \\
\|\Psi(v) - \Psi(w)\|_{\mathcal{L}_{n,T}^\alpha C^{\beta+1}} & \lesssim \|v - w\|_{\mathcal{L}_{n,T}^\alpha C^\beta}. \quad (3.8)
\end{align*}
\]

Next we consider
\[
(\Psi^{\odot 2}(v), \Psi^{\odot 2}(w), \Psi^{\odot 2}(v')) := \Psi^{\odot 2}(v), \quad (\Psi^{\odot 2}(w), \Psi^{\odot 2}(w'), \Psi^{\odot 2}(v')) := \Psi^{\odot 2}(w).
\]

Iterating (3.8)-(3.9), we arrive at
\[
\|\Psi^{\odot 2}(v) - \Psi^{\odot 2}(w)\|_{\mathcal{L}_{n,T}} \lesssim T_\theta^\delta \|v - w\|_{\mathcal{L}_{n,T}} (1 + \|\Psi\|_{\mathcal{L}_{n,T}^\alpha C^\beta} + \|w\|_{\mathcal{L}_{n,T}^\alpha C^\beta} + \|\Psi\|_{\mathcal{L}_{n,T}^\alpha C^\beta} + \|\Psi\|_{\mathcal{L}_{n,T}^\alpha C^\beta})^4.
\]

Assume \( v, w \in \mathcal{B}_{R,T} \). It follows by the definition of \( \mathcal{B}_{R,T} \) and (3.5),
\[
\|v\|_{\mathcal{L}_{n,T}^\alpha C^\beta} < R, \quad \|w\|_{\mathcal{L}_{n,T}^\alpha C^\beta} < R, \quad \|\Psi(v)\|_{\mathcal{L}_{n,T}^\alpha C^\beta} < R, \quad \|\Psi(w)\|_{\mathcal{L}_{n,T}^\alpha C^\beta} < R.
\]

Choosing \( \bar{T} \) still smaller, if necessary, we can arrange that for some \( c < 1 \),
\[
\|\Psi^{\odot 2}(v) - \Psi^{\odot 2}(w)\|_{\mathcal{L}_{n,T}} < c\|v - w\|_{\mathcal{L}_{n,T}},
\]
showing that \( \Psi^{\odot 2} \) is a contraction on \( \mathcal{B}_{R,T} \).
By Banach’s fixed-point theorem there exists a unique fixed point for $\Psi^\circ\omega$ in $\mathcal{B}_{R;\bar{T}}$. It suffices to argue that a fixed point to $\Psi^\circ\omega$ is also a fixed point to $\Psi$. The following is due to [Per20, Thm. 5.15]. Denote for the sake of notation, $w = \Psi^\circ\omega(w)$ and $v := \Psi(w)$. We have $\Psi^\circ\omega(v) = \Psi(w) = v$, hence $v$ is itself a fixed point to $\Psi^\circ\omega$, yielding by uniqueness that $v = w$. One can furthermore show that this fixed point is in fact unique in all of $\mathcal{D}_\bar{T}$; it suffices to compare two putative solutions in $\mathcal{D}_\bar{T}$ and similar estimates to those above show that they must be equal on a small time interval. Continuity then gives equality on all of $[0,\bar{T}]$. □

The utility of Proposition 3.7 is that it allows us to show existence and uniqueness of a suitable renormalised notion of solution to (1.1) by setting $\rho := \mathbb{1} + \mathcal{Y} + w$. The next lemma shows that this solution is locally Lipschitz continuous in the noise enhancement and the initial condition.

**Lemma 3.8.** Let $R > 0$, $\mathcal{X} = (\mathcal{X}_X, \mathcal{Y}_X, \mathcal{V}_X, \mathcal{W}_X), \mathcal{Y} = (\mathcal{Y}_Y, \mathcal{V}_Y, \mathcal{W}_Y) \in \mathcal{X}_T^{\alpha,\kappa}$ and $\rho^X_0, \rho^Y_0 \in \mathcal{B}_{p,q}^{\beta_0}$ be such that

$$\max\{||\mathcal{X}||_{\mathcal{X}_T^{\alpha,\kappa}}, ||\mathcal{Y}||_{\mathcal{X}_T^{\alpha,\kappa}}, ||\rho^X_0||_{\mathcal{B}_{p,q}^{\beta_0}}, ||\rho^Y_0||_{\mathcal{B}_{p,q}^{\beta_0}}\} < R.$$  

Then there exists some $\bar{T} = \bar{T}(R) \in (0, T]$ with the following properties.

1. There exist solutions $w^X := (w^X_x, w^\delta_x, \bar{w}^\delta_x)$ and $w^Y := (w^Y_y, w^\delta_y, \bar{w}^\delta_y)$ to

$$\begin{cases}
\frac{d}{dt} w^X_x = \nabla \cdot [\mathcal{I}[w^X_x] \otimes 1] + w^\delta_x, \quad w^X_x := \nabla \Phi w^X_x + \nabla \Phi \mathcal{X}_x, \\
\frac{d}{dt} w^\delta_x := P \rho^X_0 + \nabla \cdot [\mathcal{I}[\Omega^X_{\delta} (w^X_x)]],
\end{cases}$$

and

$$\begin{cases}
\frac{d}{dt} w^Y_y = \nabla \cdot [\mathcal{I}[w^Y_y] \otimes 1] + w^\delta_y, \quad w^Y_y := \nabla \Phi w^Y_y + \nabla \Phi \mathcal{Y}_y, \\
\frac{d}{dt} w^\delta_y := P \rho^Y_0 + \nabla \cdot [\mathcal{I}[\Omega^Y_{\delta} (w^Y_y)]],
\end{cases}$$

on $[0, \bar{T}]$ by an application of Proposition 3.7. Here, $\Omega^X_{\delta}(w^X_x)$ and $\Omega^Y_{\delta}(w^Y_y)$ are defined as in Lemma 3.6 with noises $\mathcal{X}, \mathcal{Y}$ respectively.

2. Setting

$$\rho^X := \mathbb{1} + \mathcal{X} + w^X, \quad \rho^Y := \mathbb{1} + \mathcal{Y} + w^Y,$$

one has

$$||\rho^X - \rho^Y||_{\mathcal{S}^{\alpha,\kappa}_{p,T} \mathcal{C}^{\alpha,\kappa} + 1} \lesssim ||\rho^X_0 - \rho^Y_0||_{\mathcal{B}_{p,q}^{\beta_0}} + ||\mathcal{X} - \mathcal{Y}||_{\mathcal{X}_T^{\alpha,\kappa}} (||w^X||_{\mathcal{B}_{p,q}^{\beta_0}} + ||w^Y||_{\mathcal{B}_{p,q}^{\beta_0}} + ||\mathcal{X}||_{\mathcal{X}_T^{\alpha,\kappa}} + ||\mathcal{Y}||_{\mathcal{X}_T^{\alpha,\kappa}} + 1)^2.$$  

**Proof.** The claim follows as in Lemma 3.6, using the trilinearity of the equation. □

Finally, we can combine the deterministic solution theory given above with the stochastic existence of an enhancement $\mathcal{X} \in \mathcal{X}_T^{\alpha,\kappa}$. This is the main result of this section and is similar to results in [GIP15] and [CC18, Cor. 3.13].

We collect the necessary set-up. Let $\xi$ be a two-dimensional space-time white-noise vector, $\sigma \in C_T \mathcal{H}^2$ for some $T > 0$ and $\mathcal{X} \in \mathcal{X}_T^{\alpha,\kappa}$ be as constructed in Theorem 2.3. We then let $(\psi_\delta)_{\delta > 0}$ be a family of mollifiers as defined by Definition 2.1 and $\mathcal{X}^\delta$ be the enhancement built from $\sigma(\psi_\delta \ast \xi)$ such that $\mathcal{X}^\delta \to \mathcal{X} \in \mathcal{X}_T^{\alpha,\kappa}$ in probability. Theorem 2.3 shows the validity of this set-up.

**Theorem 3.9.** Given $\rho_0 \in \mathcal{B}_{p,q}^{\beta_0}$ and the above set-up, there exists some $\bar{T} \in (0, T]$ with the following properties.

1. There exist solutions $\rho = \mathbb{1} + \mathcal{Y} + w$ and $\rho^\delta = \mathbb{1} + \mathcal{Y}^\delta + w^\delta$ in $\mathcal{D}_T$ to the equation (1.1) with noise given by $\sigma \xi$ and $\sigma(\psi_\delta \ast \xi)$ respectively.

2. For any $\lambda > 0$ one has that

$$\mathbb{P}(||\rho - \rho^\delta||_{\mathcal{S}^{\alpha,\kappa}_{p,T} \mathcal{C}^{\alpha,\kappa} + 1} > \lambda) \to 0 \quad \text{as} \quad \delta \to 0.$$
Proof. The existence of the paracontrolled solutions $\rho$, $\rho^\delta$ follows from Proposition 3.7 and the subsequent discussion. Applying the local Lipschitz continuity given by Lemma 3.8, it follows that for any $\lambda > 0$ there exists some $0 < \varepsilon < \lambda/3$ such that $\|X - X^\delta\|_{\mathcal{C}^{1,q}} \leq \varepsilon$ implies $\|w - w^\delta\|_{\mathcal{C}^{0,1}} \leq \lambda/3$. Consequently,

$$\mathbb{P}(\|\rho - \rho^\delta\|_{\mathcal{C}^{1,q}} > \lambda) \leq \mathbb{P}(\|X - X^\delta\|_{\mathcal{C}^{1,q}} > \varepsilon) \to 0 \quad \text{as} \quad \delta \to 0,$$

which yields the claimed convergence. 

\[\square\]

A Besov and Hölder–Besov Spaces

Throughout the following section $d, n \in \mathbb{N}$ and all properties are given for mappings or distributions on $\mathbb{T}^d$ taking values in $\mathbb{R}^n$.

A.1 Besov Spaces

Applying essentially the same arguments as in the proof of [BCD11, Prop. 2.10] there exist a dyadic partition of unity, i.e. a pair of non-negative, radially symmetric and compactly supported smooth functions $\varrho_{-1}, \varrho_0 \in C^\infty(\mathbb{R}^d; [0, 1])$ such that $\text{supp}(\varrho_{-1}) \subset B(0, 1/2)$ and $\text{supp}(\varrho_0) \subset \{x \in \mathbb{R}^d : 9/32 \leq |x| \leq 1\}$. Furthermore, defining for $k \in \mathbb{N}$, $\varrho_k(x) := \varrho_0(2^{-k}x)$, we assume it holds that

1. $\text{supp}(\varrho_k) \cap \text{supp}(\varrho_l) = \emptyset$ if $|k - l| \geq 2$,
2. $\sum_{k = -1}^\infty \varrho_k(x) = 1$ for any $x \in \mathbb{R}^d$.

For $k \geq -1$ we define the Littlewood–Paley blocks to be the Fourier multipliers $\Delta_k u := \mathcal{F}^{-1}(\varrho_k \mathcal{F} u)$ and set

$$\Delta_{<k} u := \sum_{l = -1}^{k-1} \Delta_l u.$$

As with the Fourier transform, we initially define these operators on smooth functions and then extend them by duality to $\mathcal{S}'(\mathbb{T}^d; \mathbb{R}^n)$.

Definition A.1 (Besov spaces). Let $\alpha \in \mathbb{R}$ and $p, q \in [1, \infty]$. We define the nonhomogeneous Besov space $\mathcal{B}^\alpha_{p,q}(\mathbb{T}^d; \mathbb{R})$ to be the completion of the smooth functions $C^\infty(\mathbb{T}^d; \mathbb{R})$ under the norm

$$\|u\|_{\mathcal{B}^\alpha_{p,q}(\mathbb{T}^d; \mathbb{R})} := \left\| (2^{k\alpha} \|\Delta_k u\|_{L^p(\mathbb{T}^d; \mathbb{R})})_{k \in \mathbb{N}_0} \right\|_q,$$

which is extended to vector resp. matrix-valued functions in a natural componentwise manner. Here $\ell^q$ denotes the usual space of $q$-summable sequences (or bounded when $q = \infty$). When $p = q = \infty$ we recall the shorthand $C^\alpha(\mathbb{T}^d; \mathbb{R}^n) := \mathcal{B}^\alpha_{\infty,\infty}(\mathbb{T}^d; \mathbb{R}^n)$ and call $C^\alpha(\mathbb{T}^d; \mathbb{R}^n)$ the Hölder–Besov space.

Remark A.2. Note that the dyadic partition of unity obtained by [BCD11, Prop. 2.10] is built from $\tilde{\varrho}_{-1}, \tilde{\varrho}_0$ with $\text{supp}(\tilde{\varrho}_{-1}) \subset B(0, 4/3)$ and $\text{supp}(\tilde{\varrho}_0) \subset \{x \in \mathbb{R}^d : 3/4 \leq |x| \leq 8/3\}$. However, for our purposes it is convenient to rescale these functions by a factor of $3/8$ so that the only integer in the support of $\tilde{\varrho}_{-1}$ is 0. Since the Besov spaces are independent of the chosen dyadic partition of unity [BCD11, Cor. 2.70] this change is harmless.

The Besov spaces enjoy a number of useful properties which we list below. Proofs of the following statements can be found in [BCD11; GIP15].

1. Embeddings: there exists a constant $C > 0$ such that for any $\alpha \in \mathbb{R}$, $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$,

$$\|u\|_{\mathcal{B}^{\alpha - d(1/p_1 - 1/p_2)}_{p_2,q_2}} \leq C\|u\|_{\mathcal{B}^{\alpha}_{p_1,q_1}}. \tag{A.1}$$

We also have the following, continuous embeddings,

$$\|u\|_{\mathcal{B}^\alpha_{p,q}} \lesssim \|u\|_{\mathcal{B}^{\alpha'}_{p',q'}} \quad \alpha < \alpha' \in \mathbb{R},$$

$$\|u\|_{\mathcal{B}^\alpha_{p,q}} \lesssim \|u\|_{\mathcal{B}^{\alpha'}_{p,q'}} \quad \alpha < \alpha' \in \mathbb{R}, q < q' \in [1, \infty].$$
2. Relations to $L^p$-spaces: for $p \in [1, \infty]$, one has,
\[
\|f\|_{g^p_{\infty}} \lesssim \|f\|_{L^p} \lesssim \|f\|_{g^1_{1}}.
\]
We regularly work in a scale of interpolation spaces which relate temporal and spatial regularity and are suitable for solutions to parabolic PDE.

**Definition A.3** (Interpolation spaces). Let $T > 0$, $\eta \geq 0$, $\alpha \in \mathbb{R}$ and $\kappa \in (0, 1)$. We define the norm
\[
\|u\|_{\mathcal{L}^\kappa_{\eta,T}C^\alpha} := \max\{\|u\|_{C^{\alpha-2\kappa}_T}, \|u\|_{C^\alpha_T}\},
\]
and the spaces $\mathcal{L}^\kappa_{\eta}C^\alpha = C^{\alpha-2\kappa}_\eta \cap C^\alpha_{\eta,T}$. We set $\mathcal{L}^\kappa_T C^\alpha := \mathcal{L}^\kappa_{0,T} C^\alpha$ and by an abuse of notation understand $\mathcal{L}^\kappa_{0,T} C^\alpha = C^\alpha_{0,T}$.

**A.2 Paraproducts**

For $u, v \in \mathcal{S}(\mathbb{T}^d; \mathbb{R})$ we define the paraproduct $\otimes$ and resonant product $\circ$ by
\[
u \otimes v := \sum_{k \geq 1} \Delta_{<k-1}u \Delta_k v, \quad u \circ v := \sum_{|k-l| \leq 1} \Delta_k \Delta_l v.
\]
Formally one has the decomposition $uv = u \otimes v + v \otimes u + u \circ v$. Conditions under which this decomposition is valid for (time-dependent) distributions $u$ and $v$ are given by Bony’s estimates in the following lemma. These operators naturally extend to vector-valued and matrix-valued objects as either inner or outer products. Where the precise meaning is not clear from context it will be specified in the text.

**Lemma A.4** (Bony’s estimates). Let $T > 0$ and $\eta, \eta_1, \eta_2 \geq 0$ be such that $\eta = \eta_1 + \eta_2$. If $\beta \in \mathbb{R}$, then
\[
\|u \otimes v\|_{C^\alpha_T \mathcal{L}^\beta\mathbb{T}^d, \mathbb{R}} \lesssim \beta \|u\|_{C^{\alpha-2\beta}_T L^\infty(\mathbb{T}^d, \mathbb{R})} \|v\|_{C^{\eta_2}_T C^\beta(\mathbb{T}^d, \mathbb{R})}.
\]
If $\beta \in \mathbb{R}$ and $\alpha < 0$, then
\[
\|u \otimes v\|_{C^\alpha_T C^{\alpha+\beta}(\mathbb{T}^d, \mathbb{R})} \lesssim \alpha, \beta \|u\|_{C^{\alpha}_T L^\infty(\mathbb{T}^d, \mathbb{R})} \|v\|_{C^{\eta_2}_T C^\beta(\mathbb{T}^d, \mathbb{R})}.
\]
Finally for $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta > 0$,
\[
\|u \circ v\|_{C^\alpha_T C^{\alpha+\beta}(\mathbb{T}^d, \mathbb{R})} \lesssim \alpha, \beta \|u\|_{C^{\alpha}_T L^\infty(\mathbb{T}^d, \mathbb{R})} \|v\|_{C^{\eta_2}_T C^\beta(\mathbb{T}^d, \mathbb{R})}.
\]

**Proof.** The result is a direct consequence of [GIP15, Lem. 2.1].

**A.3 Parabolic and Elliptic Regularity Estimates**

We will make use of the following interpolation inequality. Let $x \geq 0$ and $\gamma \in [0, 1]$, then
\[
0 \leq 1 - e^{-x} \leq x^\gamma.
\]  
(A.2)

We also apply the following rapid decay inequality. For any $r > 0$, uniformly in $x \geq 0$,
\[
x^r e^{-x} \lesssim 1.
\]  
(A.3)

The operators $P$, $I$ and $\Phi$ introduced in Subsection 1.1 and their accompanying kernels $\mathcal{K}$ and $\mathcal{G}$ can be generalized to $\mathbb{T}^d$ mutatis mutandis. We also apply $I$ to $f = (f_1, \ldots, f_n): [0, T] \times \mathbb{T}^d \to \mathbb{R}^n, n \in \mathbb{N}$, by setting $I[f] = (I[f_1], \ldots, I[f_n]).$
Lemma A.5. Let $\alpha \leq \beta \in \mathbb{R}$ and $p, q, p', q' \in [1, \infty]$ be such that $p \geq p'$ and $q \geq q'$. Then for any $t > 0$,
\[
\|P_t f\|_{B^\alpha_{p,q}} \lesssim (1 + t^{-\frac{\alpha}{p'}})(1 + t^{-\frac{\beta}{p'}})\|f\|_{B^\alpha_{p',q'}}. 
\] (A.4)
Secondly, if $\alpha \leq \beta \leq \alpha + 2$ then for any $t > 0$,
\[
\|(P_t - 1)f\|_{B^\alpha_{p,q}} \lesssim t^{\frac{\alpha}{p'}}\|f\|_{B^\beta_{p',q'}}.
\] (A.5)

Proof. We first show (A.4), which is an easy consequence of the Besov embedding and the regularising effect of the heat flow. By (A.1) for any $\beta \in \mathbb{R}$ and $p, q, p', q' \in [1, \infty]$ with $p \geq p'$ and $q \geq q'$, it holds that,
\[
\|P_t f\|_{B^\alpha_{p,q}} \lesssim \|P_t f\|_{B^{\alpha+d(1/p'-1)/p}_{p',q'}}.
\]

We now apply the semigroup property and the regularizing effect of the heat flow [BCD11, Lem. 2.4],
\[
\|P_t f\|_{B^{\alpha+d(1/p'-1)/p}_{p',q'}} \lesssim (1 + t^{-\frac{\alpha}{p'}}(\frac{1}{p'} - \frac{1}{p}))\|P_t f\|_{B^\beta_{p',q'}} \lesssim (1 + t^{-\frac{\beta}{p'}}(\frac{1}{p'} - \frac{1}{p}))\|f\|_{B^\beta_{p',q'}}.
\]
This yields (A.4). The bound (A.5) can be found in [MW17b, Prop. A.13], who cite [MW17a, Prop. 6] for a proof in the full space. We provide a short argument. We consider the Littlewood–Paley blocks $\Delta_k(P_t - 1)u = (P_t - 1)\Delta_k u$, $k \in \mathbb{N}_{-1}$. Since $(P_t - 1)\Delta_{-1} u = 0$, we may assume $k \in \mathbb{N}$. We apply [BCD11, Lem. 2.4 & Lem. 2.1] to obtain the existence of some $c > 0$ such that
\[
\|(P_t - 1)\Delta_k f\|_{L^p} \leq \int_0^t \|\partial_x P_s \Delta_k f\|_{L^p} ds = \int_0^t \|P_s \Delta_k f\|_{L^p} ds \\
\lesssim \|\Delta_k f\|_{L^p} \int_0^t e^{-cs2^2k} ds \lesssim \|\Delta_k f\|_{L^p}2^{2k} \int_0^t e^{-cs2^2k} ds \\
\lesssim \|\Delta_k f\|_{L^p}(1 - e^{-ct2^{2k}}) \lesssim \|\Delta_k f\|_{L^p}(\frac{\alpha}{2} + 2k(\beta - \alpha)).
\]
In the last inequality, we applied (A.2) with $(\beta - \alpha)/2 \in [0, 1]$. This yields the claim using the definition of the $B^\alpha_{p,q}$-norm.

We can now establish Schauder estimates similar to [GIP15, Lem. A.9] and [CC18, Prop. 2.7].

Lemma A.6. Let $T > 0$, $\eta \geq 0$, $\alpha \leq \beta \in \mathbb{R}$, $\kappa \in [0, 1]$ and $p, q \in [1, \infty]$. Then the following hold

1. If $\frac{\beta - \alpha}{2} + \frac{d}{2p} \leq \eta$, then
\[
\|F\|_{L^p_{\eta,T}C^\beta} \lesssim (1 + T^{-\frac{\beta - \alpha}{2}})(1 + T^{-\frac{\beta}{p}})(1 + T)^\eta\|f\|_{B^\alpha_{p,q}}. 
\] (A.6)

2. If $\eta' \in [0, 1)$, $\eta \leq \eta'$ and $\beta < \alpha + 2$ are such that $\frac{\beta - \alpha}{2} \leq \kappa \leq 1 - (\eta' - \eta)$, then
\[
\|\mathcal{I}[f]\|_{L^p_{\eta',T}C^\beta} \lesssim T \|f\|_{C^{\eta' + \kappa}_{T}}.
\] (A.7)

In particular if $T \leq 1$, then
\[
\|\mathcal{I}[f]\|_{L^p_{\eta',T}C^\beta} \lesssim (T^{1-\frac{\beta - \alpha}{2}-(\eta' - \eta)} \lor T^{1-\kappa-(\eta' - \eta)})\|f\|_{C^{\eta' + \kappa}_{T}}.
\]

3. Furthermore, both $\mathcal{P} : C^\alpha \to L^p_{\infty}C^\alpha$ and $\mathcal{I} : C_T C^\alpha \to L^p_{T}C^{\alpha+2}$ are continuous maps.

Proof. The proofs of (A.6) and (A.7) are simple consequences of Lemma A.5, the semigroup property and the definition of the interpolation spaces, Definition A.3. The continuity of $t \mapsto P_t u_0$ in $C^\alpha$ and $t \mapsto \mathcal{I}[f]|_t$ in $C^{\alpha+2}$ follows from the fact that the same statement is true for smooth functions and then by taking limits along a smooth approximating sequence.
Assume $\theta: \mathbb{R}^d \to \mathbb{C}$ is smooth such that for all multi-indices $\nu \in \mathbb{N}_0^d$, $\partial^\nu \theta$ is of at most polynomial growth. Additionally assume that $\theta$ satisfies the reality condition

$$\overline{\theta(\omega)} = \theta(-\omega), \quad \omega \in \mathbb{Z}^d. \quad (A.8)$$

We define the Fourier multiplier acting on $u \in S'(\mathbb{T}^d; \mathbb{R})$ by the expression

$$\theta(D)u := \mathcal{F}^{-1}(\theta(\omega)\hat{u}(\omega)).$$

**Lemma A.7.** Let $u \in \mathcal{B}_{p,q}^0(\mathbb{T}^d; \mathbb{R})$, $\alpha \in \mathbb{R}$, $p, q \in [1, \infty]$ and let $\theta: \mathbb{R}^d \to \mathbb{C}$, $\theta \in C^k(\mathbb{R}^d \setminus \{0\}; \mathbb{C})$, $k = 2\lfloor 1 + d/2 \rfloor$, satisfy $\theta(0) = 0$ and the reality condition (A.8). Assume there exists some $m \in \mathbb{R}$ and $C > 0$, such that for any multi-index $\nu \in \mathbb{N}_0^d$, $|\nu| \leq k$,

$$|\partial^\nu \theta(x)| \leq C|x|^{m-|\nu|}, \quad x \in \mathbb{R}^d \setminus \{0\}.$$

Then,

$$\|\theta(D)u\|_{\mathcal{B}_{p,q}^{m-m}} \lesssim \|u\|_{\mathcal{B}_{p,q}^0}.$$

**Proof.** Since $u$ is periodic the only frequency contained in the support of $\rho_{-1}$ is $\omega = 0$, hence $\theta(D)\Delta_{-1}u = 0$. The remaining Littlewood–Paley blocks can then be addressed directly with [BCD11, Lem. 2.2].

Lemma A.7 leads directly to a control on solutions to Poisson’s equation and their derivatives.

**Lemma A.8.** Let $u \in S'(\mathbb{T}^d; \mathbb{R})$ be such that $\langle u, 1 \rangle_{L^2(\mathbb{T}^d)} = 0$. Then for any $\alpha \in \mathbb{R}$ and $p, q \in [1, \infty]$,

$$\|\mathcal{G} \ast u\|_{\mathcal{B}_{p,q}^\alpha} \lesssim \|u\|_{\mathcal{B}_{p,q}^{-2}}, \quad \|\nabla \mathcal{G} \ast u\|_{\mathcal{B}_{p,q}^\alpha} \lesssim \|u\|_{\mathcal{B}_{p,q}^{\alpha-1}}.$$

**Proof.** Simply apply Lemma A.7 to the multipliers $\theta_1(\omega) = \frac{1}{(2\pi \omega)^2}1_{\omega \neq 0}$ and $\theta_2(\omega) = \frac{2\pi \omega}{(2\pi \omega)^2}1_{\omega = 0}$. 

### A.4 Commutator Results

Many of the results presented below are analogues and simple extensions of similar results found in [Per13; CC18] to time-weighted spaces, $C_{\eta, \tau}^\alpha$.

**Lemma A.9.** Let $T > 0$, $\eta, \eta_1, \eta_2 \geq 0$, $\eta = \eta_1 + \eta_2$, $\alpha \in (-\infty, 1)$, $\beta \in \mathbb{R}$ and $k = 2\lfloor 1 + d/2 \rfloor$. Assume $\theta: \mathbb{R}^d \to \mathbb{C}$, $\theta \in C^{k+1}(\mathbb{R}^d \setminus \{0\}; \mathbb{C})$, satisfies $\theta(0) = 0$ and (A.8). Assume there exists some $m \in \mathbb{R}$ and $C > 0$, such that for any multi-index $\nu \in \mathbb{N}_0^d$, $|\nu| \leq k + 1$,

$$|\partial^\nu \theta(x)| \leq C|x|^{m-|\nu|}, \quad x \in \mathbb{R}^d \setminus \{0\}.$$

Then,

$$\|\theta(D)(u \ast v) - u \ast \theta(D)v\|_{C_{\eta, \tau}^{\alpha+\beta-m}} \lesssim \|u\|_{C_{\eta_1, \tau}^\alpha} \|v\|_{C_{\eta_2, \tau}^\beta}.$$

**Proof.** The result is a simple extension of [Per13, Lem. 5.3.20] and [CC18, Lem. A.1] to functions with prescribed blow-up in $C^\alpha$ at $t = 0$.

Next, we consider the commutator between the operators $\mathcal{I}$ and $\ast$, a result reminiscent of [CC18, Prop. 2.7].

**Lemma A.10.** Let $T > 0$, $\eta, \eta_1, \eta_2 \in (0, 1)$, $\eta = \eta_1 + \eta_2$, $\beta > 0$, $\alpha \in (-\infty, (1 \land 2\kappa))$, $\beta \in \mathbb{R}$ and $m \in (0, 2)$. Then,

$$\|\mathcal{I}[u \ast v] - u \ast \mathcal{I}[v]\|_{C_{\eta, \tau}^{\alpha+\beta+m}} \lesssim_T \|u\|_{C_{\eta_1, \tau}^{\alpha}} \|v\|_{C_{\eta_2, \tau}^\beta}.$$
Proof. By definition,

\[ \mathcal{I}[u \otimes v]_t - (u \otimes \mathcal{I}[v])_t = \int_0^t P_{t-s}(u(s) \otimes v(s)) - u(t) \otimes P_{t-s}v(s) \, ds \]

\[ = \int_0^t P_{t-s}(u(s) \otimes v(s)) - u(s) \otimes P_{t-s}v(s) \, ds + \int_0^t (u(s) - u(t)) \otimes P_{t-s}v(s) \, ds. \]

To bound the first summand, we apply [Per13, Lem. 5.3.20], and use that \( \alpha < 1 \), to estimate

\[ \|P_{t-s}(u(s) \otimes v(s)) - u(s) \otimes P_{t-s}v(s)\|_{C^{\alpha+\beta+m}} \lesssim |t-s|^{-m/2(1 \wedge \delta)} \|u\|_{C^{\alpha} \cap C^0} \|v\|_{C^{\alpha+\beta} \cap C^0}. \]

Taking the supremum, we obtain

\[ \sup_{t \in [0,T]} (1 \wedge t)^{\eta} \left\| \int_0^t P_{t-s}(u(s) \otimes v(s)) - u(s) \otimes P_{t-s}v(s) \, ds \right\|_{C^{\alpha+\beta+m}} \lesssim \left( \sup_{t \in [0,T]} (1 \wedge t)^{\eta} \int_0^t |t-s|^{-m/2(1 \wedge \delta)} \, ds \right) \|u\|_{C^{\alpha} \cap C^0} \|v\|_{C^{\alpha+\beta} \cap C^0}. \]

We can use a case distinction \( t \in [0,1], \ t \in (1, \infty) \) and the assumptions \( \eta < 1, \ m < 2 \), to bound

\[ \sup_{t \in [0,T]} (1 \wedge t)^{\eta} \int_0^t |t-s|^{-m/2(1 \wedge \delta)} \, ds \lesssim_{T} 1. \]

To bound the second summand, we apply Lemma A.5, and use that \( \alpha - 2\kappa < 0 \), to estimate

\[ \|u(s) - u(t)\|_{C^{\alpha+\beta+m}} \lesssim \|u(s) - u(t)\|_{C^{\alpha-2\kappa}} \|P_{t-s}v(s)\|_{C^{\beta+2\kappa+m}} \]

\[ \lesssim |t-s|^\kappa (1 \wedge |t-s|^{-\kappa-m/2})(1 \wedge \delta) \|u\|_{C^{\alpha} \cap C^0} \|v\|_{C^{\beta} \cap C^0}. \]

Taking the supremum, we obtain

\[ \sup_{t \in [0,T]} (1 \wedge t)^{\eta} \left\| \int_0^t (u(s) - u(t)) \otimes P_{t-s}v(s) \, ds \right\|_{C^{\alpha+\beta+m}} \lesssim \left( \sup_{t \in [0,T]} (1 \wedge t)^{\eta} \int_0^t |t-s|^\kappa (1 \wedge |t-s|^{-\kappa-m/2})(1 \wedge \delta) \, ds \right) \|u\|_{C^{\alpha} \cap C^0} \|v\|_{C^{\beta} \cap C^0}. \]

We can use a case distinction \( t \in [0,1], \ t \in (1, 2], \ t \in (2, \infty) \) and the assumptions \( \eta < 1, \ m < 2 \), to bound

\[ \sup_{t \in [0,T]} (1 \wedge t)^{\eta} \int_0^t |t-s|^\kappa (1 \wedge |t-s|^{-\kappa-m/2})(1 \wedge \delta) \, ds \lesssim_{T} 1. \]

It follows that

\[ \|\mathcal{I}[u \otimes v] - u \otimes \mathcal{I}[v]\|_{C^{\alpha+\beta+m}} \lesssim_T \|u\|_{C^{\alpha} \cap C^0} \|v\|_{C^{\beta} \cap C^0}. \]

This yields the claim. \( \square \)

Finally, we present a commutator result between the operators \( \otimes \) and \( \odot \).

Lemma A.11. Assume \( T > 0, \ \eta, \eta_1, \eta_2, \eta_3 \geq 0, \ \eta = \eta_1 + \eta_2 + \eta_3 \) and \( \alpha \in (0,1), \ \beta, \gamma \in \mathbb{R} \) such that \( \beta + \gamma < 0 \) and \( \alpha + \beta + \gamma > 0 \). We define

\[ \mathcal{C}(f, g, h) = (f \otimes g) \odot h - f(g \odot h), \quad (f, g, h) \in C^{\eta_1}_T \cap C^{\eta_2}_T \cap C^{\eta_3}_T. \]

Then \( \mathcal{C} \) extends to a bounded, trilinear operator \( \mathcal{C} : C^{\eta_1}_T \cap C^{\eta_2}_T \cap C^{\eta_3}_T \to C^{\eta_1+\beta+\gamma}_T. \)

Proof. The result is a direct consequence of [GIP15, Lem. 2.4]. \( \square \)
B  Shape Coefficient Estimates

In this section, we bound our shape coefficients in terms of the time regularity \(|t-s|^\gamma, \gamma \in [0, 1]\), and the space regularity \(|\omega_k|^{\beta}, \beta \in \mathbb{R}, \omega_k \in \mathbb{Z}^2\). We first consider the shape coefficient for \(\Psi\).

**Lemma B.1.** Let \(s, t \geq 0, \gamma \in [0, 1]\) and \(\omega_1, \omega_2 \in 2\pi\mathbb{Z}^2 \setminus \{0\}\) be such that \(\omega_1 + \omega_2 \neq 0\).

1. In the case \((\omega_1 \perp \omega_2)\), we obtain
   \[
   D_{s,t}\Psi(\omega_1, \omega_2) \lesssim |t-s|^\gamma |\omega_1|^{-2} |\omega_2|^{-2} |\omega_1 + \omega_2|^{-4+2\gamma}.
   \]

2. In the case \(-(\omega_1 \perp \omega_2)\), we obtain
   \[
   D_{s,t}\Psi(\omega_1, \omega_2) \lesssim |t-s|^\gamma |\omega_1|^{-4+2\gamma} |\omega_2|^{-2} |\omega_1 + \omega_2|^{-2} + |t-s|^\gamma |\omega_1|^{-4} |\omega_2|^{-2} |\omega_1 + \omega_2|^{-2+2\gamma}.
   \]

The implied constants are uniform in \(\omega_1, \omega_2\).

**Proof.** By evaluating the exponential integrals in (2.6) and computing \(D_{s,t}\Psi(\omega_1, \omega_2)\) through (2.7), we obtain explicit expressions. We can then further decompose
   \[
   D_{s,t}\Psi(\omega_1, \omega_2) = DL_{s,t}\Psi(\omega_1, \omega_2) + DE_{s,t}\Psi(\omega_1, \omega_2),
   \]
   into the leading term \(DL_{s,t}\Psi(\omega_1, \omega_2)\) and the error term \(DE_{s,t}\Psi(\omega_1, \omega_2)\). The error term is generated by the zero initial condition of the noise, i.e. the remaining restriction \(u_3, u'_3 \geq 0\) in (2.6).

Assume \((\omega_1 \perp \omega_2)\), then
   \[
   DL_{s,t}\Psi(\omega_1, \omega_2) = \frac{1}{2^2} |\omega_1|^{-2} |\omega_2|^{-2} |\omega_1 + \omega_2|^{-4} \left(1 - e^{-|t-s||\omega_1 + \omega_2|^2} - |t-s||\omega_1 + \omega_2|^2 e^{-|t-s||\omega_1 + \omega_2|^2}\right)
   \]
   and
   \[
   DE_{s,t}\Psi(\omega_1, \omega_2) = \frac{1}{2^2} |\omega_1|^{-2} |\omega_2|^{-2} |\omega_1 + \omega_2|^{-4} \left(2t|\omega_1 + \omega_2|^2 (e^{-(t+s)||\omega_1 + \omega_2|^2} - e^{-2t||\omega_1 + \omega_2|^2}) + 2s|\omega_1 + \omega_2|^2 (e^{-(t+s)||\omega_1 + \omega_2|^2} - e^{-2s||\omega_1 + \omega_2|^2})\right)
   \]
   \[
   - (e^{t||\omega_1 + \omega_2|^2} - e^{-s||\omega_1 + \omega_2|^2})^2).\]

We first consider \(DL_{s,t}\Psi(\omega_1, \omega_2)\) and estimate by (A.2) for \(\gamma \in [0, 1]\),
   \[
   DL_{s,t}\Psi(\omega_1, \omega_2) \lesssim |t-s|^\gamma |\omega_1|^{-2} |\omega_2|^{-2} |\omega_1 + \omega_2|^{-4+2\gamma}.
   \]

Next we estimate the error term \(DE_{s,t}\Psi(\omega_1, \omega_2)\). By symmetry it is enough to consider \(s \leq t\), for which
   \[
   e^{-(t+s)||\omega_1 + \omega_2|^2} - e^{-2s||\omega_1 + \omega_2|^2} \leq 0.
   \]

We can then proceed to bound the remaining non-negative term of \(DE_{s,t}\Psi(\omega_1, \omega_2)\) by (A.2) and (A.3), giving the result in the case \((\omega_1 \perp \omega_2)\).

Next assume \(-(\omega_1 \perp \omega_2)\). We obtain the expressions
   \[
   DL_{s,t}\Psi(\omega_1, \omega_2) = \frac{1}{2^2} |\omega_1|^{-2} |\omega_2|^{-2} |\omega_1 + \omega_2|^{-2} \left(\frac{1}{|\omega_1|^2 + |\omega_2|^2} (e^{-|t-s||\omega_1|^2 + |\omega_2|^2} - e^{-|t-s||\omega_1 + \omega_2|^2})
   \]
   \[
   + \frac{1}{|\omega_1|^2 + |\omega_2|^2 + |\omega_1 + \omega_2|^2} (2 - e^{-|t-s||\omega_1 + \omega_2|^2} - e^{-|t-s||(|\omega_1|^2 + |\omega_2|^2)|})\right)
   \]
   (B.1)
and

\[
\text{DE}_{s,t} \mathcal{Y}(\omega_1, \omega_2) = \frac{1}{2^2} \left| \omega_1 \right|^{-2} \left| \omega_2 \right|^{-2} \left( \frac{1}{\left| \omega_1 \right|^2 + \left| \omega_2 \right|^2 - \left| \omega_1 + \omega_2 \right|^2} - \frac{1}{\left| \omega_1 \right|^2 + \left| \omega_2 \right|^2 + \left| \omega_1 + \omega_2 \right|^2} \right) \times \left( 2(e^{t\left| \omega_1 + \omega_2 \right|^2 - \left| \omega_1 - \omega_2 \right|^2} - 1) - e^{-t\left| \omega_1 + \omega_2 \right|^2} \right)
\]

(B.2)

We first consider (B.1). The first term

\[
\frac{1}{\left| \omega_1 \right|^2 + \left| \omega_2 \right|^2 - \left| \omega_1 + \omega_2 \right|^2} \left( e^{t\left| \omega_1 + \omega_2 \right|^2 - \left| \omega_1 - \omega_2 \right|^2} - 1 \right) - e^{-t\left| \omega_1 + \omega_2 \right|^2} \left| \omega_1 + \omega_2 \right|^2
\]

is non-positive so that by (A.2),

\[
\text{DE}_{s,t} \mathcal{Y}(\omega_1, \omega_2) \lesssim |t - s| |\omega_1|^{-2} \left| \omega_2 \right|^{-2} |\omega_1 + \omega_2|^{-2}.
\]

In (B.2), we bound the second, non-positive term by 0. In the first term, we distinguish cases to fix the sign of the prefactor \((\left| \omega_1 \right|^2 + \left| \omega_2 \right|^2 - \left| \omega_1 + \omega_2 \right|^2)^{-1}\). Assume first \(|\omega_1|^2 + |\omega_2|^2 < |\omega_1 + \omega_2|^2\) and by symmetry \(s \leq t\). We obtain by (A.2) and (A.3),

\[
\frac{1}{\left| \omega_1 \right|^2 + \left| \omega_2 \right|^2 - \left| \omega_1 + \omega_2 \right|^2} \left( e^{t\left| \omega_1 + \omega_2 \right|^2 - \left| \omega_1 - \omega_2 \right|^2} - 1 \right) - e^{-t\left| \omega_1 + \omega_2 \right|^2} (1 - e^{-t\left| \omega_1 + \omega_2 \right|^2}) \left| \omega_1 + \omega_2 \right|^2
\]

(B.3)

\[
\lesssim |t - s| \left| \frac{\left| \omega_1 + \omega_2 \right|^2}{\left| \omega_1 \right|^2 + \left| \omega_2 \right|^2} \right|^\gamma |\omega_1 + \omega_2|^{-2+2\gamma} |\omega_1|^{-2}.
\]

If instead \(|\omega_1 + \omega_2|^2 < |\omega_1|^2 + |\omega_2|^2\), \(s \leq t\), then

\[
\frac{1}{\left| \omega_1 \right|^2 + \left| \omega_2 \right|^2 - \left| \omega_1 + \omega_2 \right|^2} \left( e^{t\left| \omega_1 + \omega_2 \right|^2 - \left| \omega_1 - \omega_2 \right|^2} - 1 \right) - e^{-t\left| \omega_1 + \omega_2 \right|^2} (1 - e^{-t\left| \omega_1 + \omega_2 \right|^2}) \left| \omega_1 + \omega_2 \right|^2
\]

(B.4)

\[
\lesssim |t - s| \left| \frac{\left| \omega_1 + \omega_2 \right|^2}{\left| \omega_1 \right|^2 + \left| \omega_2 \right|^2} \right|^\gamma |\omega_1 + \omega_2|^{-2+2\gamma} |\omega_1|^2 t + s
\]

\[
\lesssim |t - s| \left| \frac{\left| \omega_1 + \omega_2 \right|^2}{\left| \omega_1 \right|^2 + \left| \omega_2 \right|^2} \right|^\gamma |\omega_1 + \omega_2|^{-2+2\gamma} |\omega_1|^2
\]

By combining (B.2) with (B.3) and (B.4), we arrive at

\[
\text{DE}_{s,t} \mathcal{Y}(\omega_1, \omega_2) \lesssim |t - s| \left| \frac{\left| \omega_1 + \omega_2 \right|^2}{\left| \omega_1 \right|^2 + \left| \omega_2 \right|^2} \right|^\gamma |\omega_1 + \omega_2|^{-2+2\gamma} |\omega_1|^2 t + s
\]

This yields the claim. □
Next we bound the shape coefficient $A \nabla$, which appears in $\mathcal{O}, \mathcal{O}, \mathcal{O}$ and $\mathcal{O}$.

**Lemma B.2.** Let $s, t \geq 0$, $k = 1, 2$, $\gamma \in [0, 1]$ and $C \geq 1$. Then uniformly in $\omega_1, \omega_2, \omega_3 \in \mathbb{Z}^2 \setminus \{0\}$ such that $C^{-1}|\omega_1| \leq |\omega_2| \leq C|\omega_1|$, it holds that

$$A_{s,t}^k (\omega_1, \omega_2, \omega_3) \lesssim |t - s|^\gamma |\omega_2|^{2\gamma} |\omega_3|^{-1}.$$  

**Proof.** The claim follows by the triangle inequality and repeated applications of (A.2). \qed

The following lemma controls the shape coefficient $A \nabla$, which appears in $\mathcal{O}, \mathcal{O}$ and $\mathcal{O}, \mathcal{O}$.

**Lemma B.3.** Let $s, t \geq 0$, $k, k' = 1, 2$, $\gamma \in [0, 1]$ and $C, C' \geq 1$. Then uniformly in $\omega_1, \omega'_1, \omega_2, \omega'_2 \in \mathbb{Z}^2 \setminus \{0\}$ such that $C^{-1}|\omega_1| \leq |\omega_2| \leq C|\omega_1|$ and $(C')^{-1}|\omega'_1| \leq |\omega_2| \leq C'|\omega'_1|$, it holds that

$$A_{s,t}^{k,k'} (\omega_1, \omega'_1, \omega_2) \lesssim |t - s|^\gamma |\omega_1|^{-1+\gamma} |\omega'_1|^{-1+\gamma}.$$  

**Proof.** The claim follows by the triangle inequality and repeated applications of (A.2). \qed

# C Summation Estimates

## C.1 Basic Estimates

We prove a number of summation and discrete convolution estimates that are central to our bounds.

Recall that $G_j^j(\omega) = 2\pi i \omega |2\pi i |\omega|^{-2} \mathbf{1}_{\omega \neq 0}, \omega \in \mathbb{Z}^2, j = 1, 2$. The following lemma shows that $|G_j^j(\omega + \omega_1) - G_j^j(\omega_1)|$ has better decay in $\omega + \omega_1$ than $|G_j^j(\omega + \omega_1)|$.

**Lemma C.1.** Let $j = 1, 2$. Then uniformly in $\omega, \omega_1 \in \mathbb{Z}^2$ such that $\omega_1, \omega + \omega_1 \neq 0$, it holds that

$$|G_j^j(\omega + \omega_1) - G_j^j(\omega_1)| \lesssim |\omega||\omega + \omega_1|^{-2}(1 + |\omega||\omega_1|^{-1}).$$  

**Proof.** We compute

$$G_j^j(\omega + \omega_1) - G_j^j(\omega_1) = \frac{i}{2\pi} \left( \frac{\omega^j + \omega_1^j}{|\omega + \omega_1|^2} - \frac{\omega_1^j}{|\omega_1|^2} \right),$$

which can be bounded in absolute value by

$$\left| \frac{\omega^j + \omega_1^j}{|\omega + \omega_1|^2} - \frac{\omega_1^j}{|\omega_1|^2} \right| \frac{|\omega_1|^2 |\omega|^2 - |\omega|^2 |\omega + \omega_1|^2}{|\omega + \omega_1|^2 |\omega_{1,j}|^2} \lesssim |\omega||\omega + \omega_1|^{-2} + |\omega|^2 |\omega_1|^{-1} |\omega + \omega_1|^{-2}. $$

This yields the claim. \qed

We apply the following summation estimates repeatedly to establish the regularities of our diagrams.

**Lemma C.2.** It holds that uniformly in $\delta > 0$,

$$\sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \frac{1}{|k|} \lesssim \delta^{-2} \quad \text{and} \quad \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} |k|^{-2} \lesssim \log(\delta^{-1} + \sqrt{2}/2) + \log((1 - \sqrt{2}/2)^{-1}). \quad \text{(C.1)}$$

In particular if $\delta \leq 1 - \sqrt{2}/2$,

$$\sum_{k \in \mathbb{Z}^2 \setminus \{0\}} |k|^{-2} \lesssim \log(\delta^{-1}). \quad \text{(C.2)}$$

What is more, for any $\alpha > 2$,

$$\sum_{k \in \mathbb{Z}^2 \setminus \{0\}} |k|^{-\alpha} \lesssim 1. \quad \text{(C.3)}$$
We make repeated use of the following convolution estimate to construct non-linear objects.

**Lemma C.3** ([ZZ15, Lem. 3.10], [MWX17, Lem. 5 & Lem. 6]). Let \( \alpha, \beta \in \mathbb{R} \) such that \( \alpha + \beta > 2 \). We have uniformly in \( \omega \in \mathbb{Z}^2 \),

\[
\sum_{\omega_1 \in \mathbb{Z}^2 \setminus \{0, \omega\}} |\omega_1|^{-\alpha} |\omega - \omega_1|^{-\beta} \lesssim_{\alpha + \beta} (1 \vee |\omega|)^{-\alpha - \beta + 2}.
\]

If in addition \( \alpha, \beta < 2 \), then we have uniformly in \( \omega \in \mathbb{Z}^2 \),

\[
\sum_{\omega_1 \in \mathbb{Z}^2 \setminus \{0, \omega\}} |\omega_1|^{-\alpha} |\omega - \omega_1|^{-\beta} \lesssim_{\alpha, \beta, \alpha + \beta} (1 \vee |\omega|)^{-\alpha - \beta + 2}.
\]

The next convolution result is useful for estimating correlated frequencies \( \omega \neq \omega' \).

**Lemma C.4.** Let \( \alpha, \beta, \gamma \in (0, 2) \) such that \( \alpha + \gamma > 2 \) and \( \beta + \gamma > 2 \). We have uniformly in \( \omega, \omega' \in \mathbb{Z}^2 \setminus \{0\} \) such that \( \omega \neq \omega' \),

\[
\sum_{\omega_1 \in \mathbb{Z}^2 \setminus \{0, \omega, \omega'\}} |\omega - \omega_1|^{-\alpha} |\omega' - \omega_1|^{-\beta} |\omega_1|^{-\gamma} \lesssim |\omega - \omega'|^{-\beta} |\omega|^{-\alpha - \gamma + 2} + |\omega - \omega'|^{-\alpha} |\omega'|^{-\beta - \gamma + 2}.
\]

**Proof.** The proof follows by two applications of Lemma C.3, one in the case \( |\omega - \omega_1| \leq |\omega - \omega'|/2 \) and the other in the complement. \( \Box \)

To derive finer estimates, it is useful to introduce a discrete paraproduct analogue to extend Lemma C.3.

**Lemma C.5.** Let \( \alpha, \beta \in \mathbb{R} \) such that \( \alpha > 2 \) and \( \beta \geq 0 \). We have uniformly in \( \omega \in \mathbb{Z}^2 \setminus \{0\} \),

\[
\sum_{\omega_1 \in \mathbb{Z}^2 \setminus \{0, \omega\}} |\omega_1|^{-\alpha} |\omega - \omega_1|^{-\beta} \lesssim_{\alpha} |\omega|^{-\beta}.
\]

The proof is immediate by (C.3) and the bound induced by (1.7).

**C.2 Double Sum Estimates**

When we take the Fourier transform of the noise \( \sigma \xi \), it generates convolutions of \( \hat{\sigma}(t, \omega - m_1) \) against \( dW^{j_1}(t, m_1) \) in \( m_1 \in \mathbb{Z}^2 \). We also generate convolutions in \( \omega_k \in \mathbb{Z}^2 \) by constructing non-linear objects such as \( \nabla \cdot Z [IV \Phi] \). Those steps lead to double sums over \( \omega_k \) and \( m_k \) that do not factorize. In this section, we estimate those sums.

We apply the following estimate in Subsection 2.5 to construct \( \hat{\sigma} \).

**Lemma C.6.** Let \( \gamma \in [0, 1/2) \) and \( \varepsilon \in (0, 1 - 2\gamma) \). Then uniformly in \( \omega, \omega_1 \in \mathbb{Z}^2 \), it holds that

\[
\sum_{\omega_4 \in \mathbb{Z}^2 \setminus \{0, \omega, \omega - \omega_1\}} |\omega - \omega_4|^{-2} (1 + |\omega||\omega_4|^{-1}) |\omega_4|^{2\gamma} |\omega - \omega_1 - \omega_4|^{-1} \\
\times \sum_{m_2 \in \mathbb{Z}^2} (1 + |\omega - \omega_1 - \omega_4 - m_2|^{-1}) (1 + |\omega_4 + m_2|^{-1})^{-1} \lesssim (1 \vee |\omega - \omega_1|)^{-2 + \varepsilon} (1 \vee |\omega|)^{-1 + 2\gamma + \varepsilon}.
\]
Proof. We decompose the sum over $m_2 \in \mathbb{Z}^2$ into the regions $m_2 = \omega - \omega_1 - \omega_4$, $m_2 = -\omega_4$ and $m_2 \in \mathbb{Z}^2 \setminus \{\omega - \omega_1 - \omega_4, -\omega_4\}$. We only give the bound that involves the sum over $m_2 \in \mathbb{Z}^2 \setminus \{\omega - \omega_1 - \omega_4, -\omega_4\}$. We estimate

$$
\sum_{m_2 \in \mathbb{Z}^2 \setminus \{\omega - \omega_1 - \omega_4, -\omega_4\}} (1 + |\omega - \omega_1 - \omega_4 - m_2|^2)^{-1}(1 + |\omega_4 + m_2|^2)^{-1} \leq \sum_{m_2 \in \mathbb{Z}^2 \setminus \{\omega - \omega_1 - \omega_4, -\omega_4\}} |\omega - \omega_1 - \omega_4 - m_2|^{-2}|\omega_4 + m_2|^{-2}.
$$

Let $\varepsilon \in (0, 1)$. We can then apply Lemma C.3,

$$
\sum_{\omega_4 \in \mathbb{Z}^2 \setminus \{0, \omega, \omega_1\}} \sum_{(\omega - \omega_4)\sim \omega_4} |\omega - \omega_4|^{-2}(1 + |\omega||\omega_4|^{-1})|\omega_4|^{2\gamma}|\omega - \omega_1 - \omega_4|^{-1} \leq (1 \vee |\omega - \omega_4|)^{-2+2\varepsilon} \sum_{\omega_4 \in \mathbb{Z}^2 \setminus \{0, \omega, \omega_1\}} |\omega - \omega_4|^{-2}(1 + |\omega||\omega_4|^{-1})|\omega_4|^{2\gamma}|\omega - \omega_1 - \omega_4|^{-1}.
$$

Let $p \in (1, \infty)$, $q = p/(p - 1)$, $\delta > 0$. By Hölder’s inequality,

$$
\sum_{\omega_4 \in \mathbb{Z}^2 \setminus \{0, \omega, \omega_1\}} |\omega - \omega_4|^{-2}(1 + |\omega||\omega_4|^{-1})|\omega_4|^{2\gamma}|\omega - \omega_1 - \omega_4|^{-1} \leq \left( \sum_{\omega_4 \in \mathbb{Z}^2 \setminus \{0, \omega, \omega_1\}} |\omega - \omega_4|^{-2p}|\omega_4|^\delta p(1 + |\omega||\omega_4|^{-1})^p \right)^{1/p} \times \left( \sum_{\omega_4 \in \mathbb{Z}^2 \setminus \{0, \omega, \omega_1\}} |\omega_4|^{-\delta q}|\omega_4|^{2\gamma q}|\omega - \omega_1 - \omega_4|^{-q} \right)^{1/q}.
$$

We assume $\gamma \in [0, 1/2)$, $2 < p$ and $1 - 2/p + 2\gamma < \delta < 2 - 2/p$. It follows

$$
p(2 - \delta) > 2, \quad q(\delta - 2\gamma) < 2, \quad q < 2, \quad q(\delta - 2\gamma + 1) > 2.
$$

Consequently, by Lemma C.3,

$$
\left( \sum_{\omega_4 \in \mathbb{Z}^2 \setminus \{0, \omega, \omega_1\}} |\omega - \omega_4|^{-2p}|\omega_4|^\delta p(1 + |\omega||\omega_4|^{-1})^p \right)^{1/p} \times \left( \sum_{\omega_4 \in \mathbb{Z}^2 \setminus \{0, \omega, \omega_1\}} |\omega_4|^{-\delta q}|\omega_4|^{2\gamma q}|\omega - \omega_1 - \omega_4|^{-q} \right)^{1/q} \leq (1 \vee |\omega|)^{-2+\delta+2/p(1 \vee |\omega - \omega_1|)^{-\delta+2\gamma-1}2/q}.
$$

Let $\varepsilon \in (0, 1 - 2\gamma)$. We can now let $\delta = 1 - 2/p + 2\gamma + \varepsilon$ to conclude

$$
\sum_{\omega_4 \in \mathbb{Z}^2 \setminus \{0, \omega, \omega_1\}} |\omega - \omega_4|^{-2(1 + |\omega||\omega_4|^{-1})|\omega_4|^{2\gamma}|\omega - \omega_1 - \omega_4|^{-1} \leq (1 \vee |\omega|)^{-1+2\gamma+\varepsilon(1 \vee |\omega - \omega_1|)^{-\varepsilon}}.
$$

This yields the claim. \qed

We apply the following estimate in Subsection 2.6 to construct $\varphi^\delta$, $\varphi^\delta$ and $\varphi^\delta$. We use the restriction $|m_1| \leq \delta^{-1}$ induced by the cut-off $\varphi(\delta m_1)$ to establish a bound in terms of $\log(\delta^{-1})$. 
Lemma C.7. Let \( \varepsilon \in (0, 1/2) \) and \( \delta \in (0, 1 - \sqrt{2}/2) \). Then uniformly in \( \omega \in \mathbb{Z}^2 \setminus \{0\} \) it holds that
\[
\sum_{m_1 \in \mathbb{Z}^2} \sum_{\omega_1 \in \mathbb{Z}^2 \setminus \{0, \omega\}} (1 + |\omega_1 - m_1|^2)^{-1}(1 + |\omega - \omega_1 + m_1|^2)^{-1}|\omega_1|^{-2}(1 + |\omega||\omega - \omega_1|^{-1}) \lesssim |\omega|^{-2+3\varepsilon} \log(\delta^{-1}). \tag{C.4}
\]

Proof. To bound \( \text{(C.4)} \) it suffices to estimate the two parts
\[
\sum_{m_1 \in \mathbb{Z}^2} \sum_{\omega_1 \in \mathbb{Z}^2 \setminus \{0, \omega\}} (1 + |\omega_1 - m_1|^2)^{-1}(1 + |\omega - \omega_1 + m_1|^2)^{-1}|\omega_1|^{-2} \tag{C.5}
\]
and
\[
\sum_{m_1 \in \mathbb{Z}^2} \sum_{\omega_1 \in \mathbb{Z}^2 \setminus \{0, \omega\}} (1 + |\omega_1 - m_1|^2)^{-1}(1 + |\omega - \omega_1 + m_1|^2)^{-1}|\omega_1|^{-2}|\omega - \omega_1|^{-1}. \tag{C.6}
\]

Let us consider \( \text{(C.5)} \). We decompose the sum over \( m_1 \) into the regions \( m_1 = 0 \), \( m_1 = -\omega \) and \( m_1 \in \mathbb{Z}^2 \setminus \{0, -\omega\} \). The sum over \( m_1 \in \mathbb{Z}^2 \setminus \{0, -\omega\} \) is given by
\[
\sum_{m_1 \in \mathbb{Z}^2 \setminus \{0, -\omega\}} \sum_{\omega_1 \in \mathbb{Z}^2 \setminus \{0, \omega\}} (1 + |\omega_1 - m_1|^2)^{-1}(1 + |\omega - \omega_1 + m_1|^2)^{-1}|\omega_1|^{-2}.
\]

The sum over \( \omega_1 \in \mathbb{Z}^2 \setminus \{0, \omega\} \) can then be further decomposed into the regions \( \omega_1 = m_1 \), \( \omega_1 = \omega + m_1 \) and \( \omega_1 \in \mathbb{Z}^2 \setminus \{0, \omega, m_1, \omega + m_1\} \). We only give the bound that involves the sums over \( m_1 \) and \( \omega_1 \). Using that \( \omega_1 \in \mathbb{Z}^2 \setminus \{0, \omega, m_1, \omega + m_1\} \), we may estimate
\[
\sum_{m_1 \in \mathbb{Z}^2 \setminus \{0, -\omega\}} \sum_{\omega_1 \in \mathbb{Z}^2 \setminus \{0, \omega, m_1, \omega + m_1\}} (1 + |\omega_1 - m_1|^2)^{-1}(1 + |\omega - \omega_1 + m_1|^2)^{-1}|\omega_1|^{-2} \lesssim \sum_{m_1 \in \mathbb{Z}^2 \setminus \{0, -\omega\}} \sum_{\omega_1 \in \mathbb{Z}^2 \setminus \{0, \omega, m_1, \omega + m_1\}} |\omega_1 - m_1|^{-2}|\omega - \omega_1 + m_1|^{-2}|\omega_1|^{-2}.
\]

Introducing the dyadic partition of unity \( (\varrho_q)_{q \in \mathbb{N}_+} \), we decompose this sum into
\[
\sum_{m_1 \in \mathbb{Z}^2 \setminus \{0, -\omega\}} \sum_{\omega_1 \in \mathbb{Z}^2 \setminus \{0, \omega, m_1, \omega + m_1\}} |\omega_1 - m_1|^{-2}|\omega - \omega_1 + m_1|^{-2}|\omega_1|^{-2} = \sum_{m_1 \in \mathbb{Z}^2 \setminus \{0, -\omega\}} \sum_{\omega_1 \in \mathbb{Z}^2 \setminus \{0, \omega, m_1, \omega + m_1\}} |\omega_1 - m_1|^{-2}|\omega - \omega_1 + m_1|^{-2}|\omega_1|^{-2} \tag{C.7}
\]
\[
+ \sum_{m_1 \in \mathbb{Z}^2 \setminus \{0, -\omega\}} \sum_{\omega_1 \in \mathbb{Z}^2 \setminus \{0, \omega, m_1, \omega + m_1\}} |\omega_1 - m_1|^{-2}|\omega - \omega_1 + m_1|^{-2}|\omega_1|^{-2} \tag{C.8}
\]
\[
+ \sum_{m_1 \in \mathbb{Z}^2 \setminus \{0, -\omega\}} \sum_{\omega_1 \in \mathbb{Z}^2 \setminus \{0, \omega, m_1, \omega + m_1\}} |\omega_1 - m_1|^{-2}|\omega - \omega_1 + m_1|^{-2} \tag{C.9}
\]
Assume \( \varepsilon < 2/3 \). The terms \( \text{(C.7)} \) and \( \text{(C.9)} \) can be estimated by two applications of Lemma C.3,
\[
\sum_{m_1 \in \mathbb{Z}^2 \setminus \{0, -\omega\}} \sum_{\omega_1 \in \mathbb{Z}^2 \setminus \{0, \omega, m_1, \omega + m_1\}} |\omega + m_1|^{-2} \lesssim \sum_{m_1 \in \mathbb{Z}^2 \setminus \{0, -\omega\}} \sum_{\omega_1 \in \mathbb{Z}^2 \setminus \{0, \omega, m_1, \omega + m_1\}} |\omega + m_1|^{-2+2\varepsilon} \lesssim |\omega|^{-2+3\varepsilon}.
\]
The second term (C.8) can be estimated by Lemma C.3 and (C.2),

\[
\sum_{m_1 \in \mathbb{Z}^2 \setminus \{0, \omega\}} \sum_{\substack{\omega_1 \in \mathbb{Z}^2 \setminus \{0, \omega_1 + m_1\} \\ \omega_1 \not\supseteq (\omega - \omega_1 + m_1)}} |\omega_1 - m_1|^{-2} |\omega - \omega_1 + m_1|^{-2} |\omega_1|^{-2}
\lesssim \sum_{m_1 \in \mathbb{Z}^2 \setminus \{0, \omega\}} \sum_{\substack{\omega_1 \in \mathbb{Z}^2 \setminus \{0, \omega_1 + m_1\} \\ \omega_1 \not\supseteq (\omega - \omega_1 + m_1)}} |\omega + m_1|^{-2} \sum_{\omega_1 \in \mathbb{Z}^2 \setminus \{0, \omega_1 + m_1\}} |\omega_1 - m_1|^{-2} |\omega - \omega_1 + m_1|^{-2}
\lesssim |\omega|^{-2 + 2\epsilon} \sum_{m_1 \in \mathbb{Z}^2 \setminus \{0, \omega\}} |\omega + m_1|^{-2} \lesssim |\omega|^{-2 + 3\epsilon} \log(\delta^{-1}).
\]

Decomposing (C.5) as discussed and bounding the resulting terms yields

\[
\sum_{m_1 \in \mathbb{Z}^2} \sum_{\omega_1 \in \mathbb{Z}^2 \setminus \{0, \omega\}} (1 + |\omega_1 - m_1|^2)^{-1} (1 + |\omega - \omega_1 + m_1|^2)^{-1} |\omega_1|^{-2} \lesssim |\omega|^{-2 + 3\epsilon} \log(\delta^{-1}).
\]

Assume \(\epsilon \in (0, 1/2)\). The bound on (C.6) then follows in a similar manner. \(\square\)
In this glossary we collect our frequently-used symbols.

### Table of distribution spaces

| Space   | Description                                                                 | Reference         |
|---------|-----------------------------------------------------------------------------|-------------------|
| $C^\infty$ | The smooth, periodic functions on $\mathbb{T}^2$.                         | Subsection 1.1    |
| $S'$    | The periodic distributions on $\mathbb{T}^2$.                             | Subsection 1.1    |
| $\mathcal{B}^{\alpha}_{p,q}$ | The completion of $C^\infty$ under the Besov-norm $\| \cdot \|_{\mathcal{B}^{\alpha}_{p,q}}$. | Definition A.1    |
| $C^\alpha$ | The Hölder–Besov space $C^\alpha = \mathcal{B}^{\alpha}_{\infty,\infty}$. | Definition A.1    |
| $\mathcal{H}^2$ | The Sobolev space $\mathcal{H}^2 = \mathcal{B}^{2}_{2,2}$. | Subsection 1.1    |
| $C_{T,E}$ | The continuous functions $f : [0,T] \to E$.                              | Subsection 1.1    |
| $C_{\kappa,T,E}$ | The $\kappa$-Hölder continuous functions $f : [0,T] \to E$. | Subsection 1.1    |
| $C_{\kappa,T,E}$ | The $\kappa$-Hölder functions $f : (0,T] \to E$ with blow-up of at most $t^{-\eta}$. | Subsection 1.1    |
| $L^\kappa_{\eta,T}C^\alpha$ | The weighted interpolation space $L^\kappa_{\eta,T}C^\alpha = C^\kappa_{\eta,T}C^{\alpha-2\kappa} \cap C_{\eta,T}C^\alpha$. | Definition A.3    |
| $\mathcal{X}^{\alpha,\kappa}_T$ | The space of enhanced rough noises.                                      | Definition 2.2    |
| $\mathcal{D}_T$ | The space of paracontrolled distributions.                               | Definition 3.2    |

### Table of noise objects and Feynman diagrams

| Diagram | Description                                                                 | Reference |
|---------|-----------------------------------------------------------------------------|-----------|
| $\xi$   | The space-time white-noise vector $\xi = (\xi^1, \xi^2)$.                    | (2.1)     |
| $\mathfrak{t}$ | $= \nabla \cdot I[\sigma \xi]$.                                           | (2.2)     |
| $\mathfrak{y}$ | $= \mathbb{E}(\nabla \cdot I[\mathfrak{t} \nabla \Phi \mathfrak{t}])$.  | Subsection 2.2 |
| $\mathfrak{y}$ | $= \nabla \cdot I[\mathfrak{t} \nabla \Phi \mathfrak{t}] - \varphi = \mathfrak{y}$. | Subsection 2.2 |
| $\mathfrak{y}$ | $= \mathfrak{y} \odot \nabla \Phi \mathfrak{t} + \nabla \Phi \mathfrak{y} \odot \mathfrak{t} = \mathfrak{y} \mathfrak{t} + \mathfrak{t} \mathfrak{y} + \mathfrak{y} \mathfrak{t} + \mathfrak{t} \mathfrak{y}$. | Subsection 2.2 |
| $\mathfrak{y}$ | $= \nabla I[\mathfrak{t}] \odot \nabla \Phi \mathfrak{t} + \nabla^2 I[\Phi \mathfrak{t}] \odot \mathfrak{t} = \mathfrak{y} \mathfrak{t} + \mathfrak{t} \mathfrak{y} + \mathfrak{y} \mathfrak{t}$. | Subsection 2.2 |
| $\mathfrak{y}$ | $= \mathfrak{y} + \mathfrak{y}$.                                           | Subsection 2.2 |
| $\mathfrak{t}^\delta$ | $= \nabla \cdot I[\sigma (\psi^* \Phi)]$.                                  | (2.3)     |
| $\mathfrak{y}^\delta_{\text{can}}$ | $= \nabla \cdot I[\mathfrak{t}^\delta \nabla \mathfrak{y} \mathfrak{t}^\delta] = \mathfrak{y}^\delta + \mathfrak{y}^\delta$. | Subsection 2.2 |
| $\mathfrak{y}^\delta_{\text{can}}$ | $= \mathfrak{y}^\delta_{\text{can}} \odot \nabla \Phi \mathfrak{t}^\delta + \nabla \Phi \mathfrak{y}^\delta_{\text{can}} \odot \mathfrak{t}^\delta = \mathfrak{y}^\delta + \mathfrak{y}^\delta + \mathfrak{y}^\delta$. | Subsection 2.2 |
| $\mathfrak{y}$ | Lemma 2.10 applied to $\mathfrak{y}^\delta_{\text{can}}$.                  | Subsection 2.4 |
| $\mathfrak{y}$ | Lemma 2.10 applied to $\mathfrak{y}^\delta$.                             | Subsection 2.5 |
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