THE THEORY OF COMPLEX PROBABILITY AND THE FIRST ORDER RELIABILITY METHOD

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ABSTRACT

The Kolmogorov’s system of axioms can be extended to encompass the imaginary set of numbers and this by adding to the original five axioms an additional three axioms. Hence, any experiment can thus be executed in what is now the complex set C (Real set R with real probability + Imaginary set M with imaginary probability). The objective here is to evaluate the complex probabilities by considering supplementary new imaginary dimensions to the event occurring in the “real” laboratory. Whatever the probability distribution of the input random variable in R is, the corresponding probability in the whole set C is always one, so the outcome of the random experiment in C can be predicted totally. The result indicates that chance and luck in R is replaced now by total determinism in C. This new complex probability model will be applied to the concepts of degradation and the Remaining Useful Lifetime (RUL), thus to the field of prognostic based on reliability. Therefore, an example of Young modulus will be applied and the First Order Reliability Method (FORM) analysis will be used for this purpose.

Keywords: Complex Probability, Prognostic, Degradation, Remaining Useful Lifetime, Young Modulus, First Order Reliability Method (FORM), Failure Probability

1. INTRODUCTION

Abou Jaoude et al. (2010); Abou Jaoude (2013a; 2013b; 2005; 2007; 2012); Bell (1992); Benton (1996); Boursin (1986); Chan Man Fong et al. (1997); Cheney and Kincaid (2004); Dacunha-Castelle (1996); Dalmedico Dahan et al. (1992); Dalmedico Dahan and Peiffer (1986); Ekeland (1991); Feller (1968); Finney et al. (2004); Gentle (2003); Gerald and Wheatley (1999); Gleick (1997) and Greene (2000) firstly, the Extended Kolmogorov’s Axioms (EKA for short) paradigm can be illustrated by the following figure (Fig. 1).

In engineering systems, the remaining useful lifetime prediction is related deeply to many factors that generally have a chaotic behavior which decreases the degree of our knowledge of the system.

As the Degree of Our Knowledge (DOK for short) in the real universe R is unfortunately incomplete, the extension to the complex universe C includes the contributions of both the real universe R and the imaginary universe M. Consequently, this will result in a complete and perfect degree of knowledge in C = R+M (P_c = 1). In fact, in order to have a certain prediction of any event it is necessary to work in the complex universe C in which the chaotic factor is quantified and subtracted from the Degree of Our Knowledge to lead to a probability in C equal to one (P_c^2 = DOK-Chf = 1).

Thus, the study in the complex universe results in replacing the phenomena that used to be random in R by deterministic and totally predictable ones in C.

This hypothesis is verified in a previous study and paper by the mean of many examples encompassing both discrete and continuous distributions.

From the Extended Kolmogorov’s Axioms (EKA), we can deduce that if we add to an event probability in the real set R the imaginary part M (like the lifetime variables) then we can predict the exact probability of the remaining lifetime with certainty in C (P_c = 1).

We can apply this idea to prognostic analysis through the degradation evolution of a system. As a matter of fact, prognostic analysis consists in the prediction of the remaining useful lifetime of a system at any instant t_0 and during the system functioning.
Let us consider a degradation trajectory $D(t)$ of a system where a specific instant $t_0$ is studied. The instant $t_0$ means here the time or age that can be measured also by the cycle number $N$.

Referring to the figure above (Fig. 2), the previous statement means that at the system age $t_0$, the prognostic study must give the prediction of the failure instant $t_N$. Therefore, the RUL predicted here at instant $t_0$ is the following interval: $RUL(t_0) = t_N - t_0$.

In fact, at the beginning ($t_0 = 0$) (point J), the failure probability $P_r = 0$ and the chaotic factor in our prediction is zero ($Chf = 0$). Therefore, $RUL(t_0 = 0) = t_N - t_0 = t_N$.

If $t_0 = t_N$ (point L) then the RUL ($t_N$) = $t_N - t_N = 0$ and the failure probability is one ($P_r = 1$).

If not (i.e., $0 < t_0 < t_N$) (point K), the probability of the occurrence of this instant and the prediction probability of RUL are both less than one (not certain) due to non-zero chaotic factors. The degree of our knowledge (DOK for short) is consequently less than 1. Thus, by applying here the EKA method, we can determine the system RUL with certainty in $C = R + M$ where $P_c = 1$ always.

Furthermore, we need in our current study the absolute value of the chaotic factor that will give us the magnitude of the chaotic and random effects on the...
studied system. This new term will be denoted accordingly MChf or Magnitude of the Chaotic Factor. Hence, we can deduce the following:

\[ \text{MChf}(t_0) = |\text{Chf}(t_0)| \geq 0 \quad \text{and} \quad \text{Pc}^2(t_0) = \text{DOK}(t_0) - \text{Chf}(t_0) = \text{DOK}(t_0) + |\text{Chf}(t_0)| \]

since \(-0.5 \leq \text{Chf}(t_0) \leq 0\)

\[ = \text{DOK}(t_0) + \text{MChf}(t_0) = 1, \quad \forall \quad 0 \leq t_0 \leq t_N \]

\[ \Leftrightarrow 0 \leq \text{MChf}(t_0) \leq 0.5 \quad \text{where} \quad 0.5 \leq \text{DOK}(t_0) \leq 1 \]

Moreover, we can define two complementary events E and \(\overline{E}\) with their respective probabilities:

\[ \text{P}_{\text{rob}}(E) = p \quad \text{and} \quad \text{P}_{\text{rob}}(\overline{E}) = q = 1 - p \]

Then \(\text{P}_{\text{rob}}(E)\) in terms of the instant \(t_0\) is given by:

\[ \text{P}_{\text{rob}}(E) = \text{P}_r = \text{P}_{\text{rob}}(t \leq t_0) = \text{F}(t_0) \]

where \(\text{F}(t_0)\) is the cumulative probability distribution function of the random variable \(t\). Since:

\[ \text{P}_{\text{rob}}(E) + \text{P}_{\text{rob}}(\overline{E}) = 1 \]

Therefore:

\[ \text{P}_{\text{rob}}(\overline{E}) = 1 - \text{P}_{\text{rob}}(E) = 1 - \text{P}_r = \]

\[ 1 - \text{P}_{\text{rob}}(1 \leq t_0) = \text{P}_{\text{rob}}(t > t_0) \]

Let us define the two particular instants: \(t_0 = 0\) assumed as the initial time of functioning (raw state) corresponding to \(D = D_0 = 0\) and \(t_N = \) the failure instant (wear out state) corresponding to the degradation \(D = 1\). The boundary conditions are:

For \(t_0 = 0\) then \(D = D_0\) (initial damage that may be zero or not) and:

\[ \text{F}(t_0) = \text{P}_{\text{rob}}(t \leq 0) = 0 \]

For \(t_0 = t_N\) then \(D = 1\) and \(\text{F}(t_0) = \text{F}(t_0) = \text{P}_{\text{rob}}(t \leq t_N) = 1\). Also \(\text{F}(t_0)\) is a non-decreasing function that varies between 0 and 1. In fact, \(\text{F}(t_0)\) is a cumulative function (Fig. 3). In addition, since \(\text{RUL}(t_0) = t_N - t_0\) and \(0 \leq t_0 \leq t_N\) then \(\text{RUL}(t_0)\) is a non-increasing remaining useful lifetime function (Fig. 4).

Referring to Fig. 5 below, we can infer the following.

The complex probability \(Z(t_0) = \text{P}_r(t_0) + \text{P}_{\text{rob}}(t_0) = \text{P}_r(t_0) + i(1 - \text{P}(t_0))\). The square of the norm of \(Z(t_0)\) is:

\[ |Z(t_0)|^2 = \text{DOK}(t_0) = 1 + 2\text{P}_r(t_0)\text{P}_{\text{rob}}(t_0) \]

\[ = 1 - 2\text{P}_r(t_0)[1 - \text{P}(t_0)] \]

\[ = 1 - 2\text{P}_r(t_0) + 2\text{P}_r^2(t_0) \]

The Chaotic Factor and the Magnitude of the Chaotic Factor are:

\[ \text{Chf}(t_0) = -2\text{P}_r(t_0)[1 - \text{P}(t_0)] - 2\text{P}_r^2(t_0) \]

is null when \(\text{P}_r(t_0) = \text{P}(t_0) = 0\) (point J) or when \(\text{P}_r(t_0) = \text{P}(t_0) = 1\) (point L) and \(\text{MChf}(t_0) = |\text{Chf}(t_0)| = 2\text{P}_r(t_0)[1 - \text{P}(t_0)] = 2\text{P}_r(t_0) - 2\text{P}_r^2(t_0) \)

is null when \(\text{P}_r(t_0) = \text{P}(t_0) = 0\) (point J) or when \(\text{P}_r(t_0) = \text{P}(t_0) = 1\) (point L)

At any instant \(t_0\) (point K), the probability expressed in the complex set C is:

\[ \text{Pc}(t_0) = \text{P}_r(t_0) + \text{P}_{\text{rob}}(t_0)i = \text{P}_r(t_0) + [1 - \text{P}(t_0)] = 1 \]

Always.

Hence, the prediction of \(\text{RUL}(t_0)\) of the system degradation in C is permanently certain.

### 2. APPLICATION OF EXTENDED KOLMOGOROV’S AXIOMS (EKA) TO DEGRADATION PROGNOSTIC BASED ON RELIABILITY

#### 2.1. Review of Reliability Theory (Greene, 2004; Guillen, 1995; Gullberg, 1997; Kuhn, 1996; Liu, 2001; Mandelbrot, 1997; Montgomery and Runger, 2005; Müller, 2005; Orluc and Poirier, 2005; Poincaré, 1968; Prigogine and Stengers, 1992; Prigogine, 1997; Christian and Casella, 2005; Srinivasan and Mehata, 1978; Stewart, 1996; 2002; Van Kampen, 2007; Walpole, 2002; Warusfel and Ducrocq, 2004; Weinberg, 1992)

The reliability is the probabilistic evaluation of a limit state of performance on a domain of basic variables. In other words, it is obtained by the computation of the failure probability toward a criterion or a limit state.

#### 2.1.1. Methodology

- Identify the limit states that govern the lifetime of the structure
- Identify the basic parameters intervening in the limit state
- Deduce their probability density functions
- Compute the failure probability that expresses the risk when the limit states are not satisfied
Fig. 3. Occurrence probability

Fig. 4. RUL prognostic model

Fig. 5. Degradation prognostic model
Two types of methods exist: The Monte Carlo simulation and the approximate method First Order Reliability Method (FORM). The Monte Carlo simulation method is based on a large number of simulations and we must use $N$ simulations when we want to evaluate a probability of order of $10^{-(N+4)}$.

The approximate method FORM is an iterative procedure that allows to calculate an index of reliability (denoted $\beta$).

The index $\beta$ is the distance between the origin and the limit state function $G(t)$ in a standard space. Once we have calculated $\beta$ we can deduce the failure probability $P_f = \Phi(-\beta)$.

In FORM approximation the real (usually nonlinear) limit state is replaced by its tangent plane at a specific point called the Most Probable Failure Point (MPFP). This point is the closest point on $G(t)$ to the origin.

The limit state $G(t)$ divides the space into two regions:
- First region where $G(t) > 0$ called safe region
- And the second region where $G(t) \leq 0$ called failure region

2.1.2. Work Plan

We choose, in a general case, $N$ random variables, correlated and of any density functions, as well as a nonlinear limit state function. This method is based on the following iterative algorithm:

- Transforming of basic random variables into standard normal variables $N(0,1)$
- Transforming the limit state from the original space to the standard normal space
- Search of the MPFP point by replacing the limit state surface by its tangent hyper-plane at the same point
- Calculate the index $\beta$ and the probability of failure $P_f$

2.1.3. Description of the Algorithm

The transformation from the basic state to the normalized state is implicit in the algorithm. The steps are the following (Fig. 6):

Let the limit state equation be: $g(z)$

where, $z = z_1,z_2,z_3,\ldots,z_n$ is the random vector of the limit state, therefore.

1) Initialization of the coordinates of MPFP. The mean value of each variable is a good choice:

$$Z_m = \mu_1, \mu_2, \ldots, \mu_m$$

2) Calculate the following parameters: (m is the number of the iteration).

The value of the limit state at MPFP:

$$g_m = g(z_m)$$

![Fig. 6. The First Order Reliability Method (FORM)](image-url)
The gradient at MPFP is assumed to be:

$$g_i^n = \frac{\partial g}{\partial z_i}(z_i^n, ..., z_n^n)$$

The equivalent normal standard deviation and mean value of non-normal variables:

$$\sigma_i^n = \frac{\Phi^{-1}(F_i(z_i^n))}{f_i(z_i^n)}$$
$$\mu_i^n = z_i^n - \sigma_i^n \Phi^{-1}(F_i(z_i^n))$$

3) Calculate the intermediate parameters:

$$z^n = \sum_{i=1}^{m} b_i^n z_i^n$$
$$\mu_i^n = \sum_{i=1}^{m} b_i^n \mu_i^n$$
$$\sigma_i^n = \sqrt{\sum_{i=1}^{m} (g_i^n)^2 (\sigma_i^n)^2}$$

4) Calculate:

The directive cosine:

$$\alpha_i = \frac{g_i^n \sigma_i^n}{\sigma_z^n}$$

The reliability index:

$$\beta^n = -\frac{z^n - \mu^n - \mu_i^n}{\sigma_z^n}$$

The new coordinates of MPFP:

$$z_i^n = \mu_i^n + \alpha_i^n \beta^n \sigma_i^n$$

5) Verify the convergence criterion:

$$\|z^{n+1} - z^n\| < 1 \text{ and } \|\beta^{n+1} - \beta^n\| < 1$$

6) Repeat the steps from 2 till 5 until convergence.

7) Calculate the failure probability:

$$P_f = \Phi(-\beta)$$

2.2. Application of FORM to Prognostic

In this part, we study the extended Kolmogorov axioms in the context of reliability by defining a limit state G that describes the lifetime margin of the system. For each value of an instant \(t_0\) we determine its corresponding probability of survival or of the Remaining Useful Lifetime (RUL).

We have:

$$G(t_0) = \text{RUL}(t_0) = t_N - t_0$$

Where:

$$G(t_0) = \text{The limit state of lifetime.}$$
$$t_N = \text{The fixed lifetime of the system which follows a normal distribution } \mathcal{N}(0.0006, 1)$$
$$t_0 = \text{An arbitrary instant that varies from 0 to } t_N \text{ and which follows a normal distribution } \mathcal{N}(t_0; 0.1 \times t_0)$$

When RUL \((t_0)\) is zero or negative then we have a case of \(t_0 \geq t_N\) that means that we have a system failure that cannot live until the instant \(t_0\). In the other case where \(t_0 < t_N\), the system can live above the instant \(t_0\) and we have a case of success.

The probability:

$$P_r(t_0) = P_{\text{rob}} \{ G(t_0) \leq 0 \} = P_{\text{rob}} \{ \text{RUL}(t_0) \leq 0 \}$$

is computed by the FORM (First Order Reliability Method) procedure that uses a reliability index \(\beta\).

$$\beta = -\Phi^{-1}[P_r(t_0)] \text{ where } P_r(t_0) \text{ is the cumulative probability and } \Phi \text{ is the normal cumulative distribution function. Hence, } \Phi^{-1} \text{ is the inverse of } \Phi \text{ and } P_r(t_0) = \Phi(-\beta).$$

In the extended Kolmogorov’s axioms, the real part of probability is taken here \(P_r(t_0)\). As we make the instant \(t_0\) vary between 0 and \(t_N\), then \(P_r(t_0)\) varies between 0 and 1 (Fig. 7).

Knowing that we take \(t_0\) and \(t_N\) as two normal random variables where the value of \(t_N\) corresponds to nearly 5798 number of cycles (critical value: \(N_c\)). After a reliability calculation using a Matlab program, we deduce a value of \(P_r(t_0)\) for each value of instant \(t_0\). For this set of \(P_r(t_0)\) we have computed and plotted the extended Kolmogorov’s parameters and components \(\text{Chf}(t_0), \text{MChf}(t_0), \text{DOK}(t_0), \text{Pc}(t_0), \text{Pm}(t_0)/i\).

Therefore, we get the following figures (Fig. 8 and 9).
Fig. 8. DOK, Chf and Pc as functions of the probability of failure

Fig. 9. DOK, MChf and Pc as functions of the probability of failure
We note from the figure that the DOK is maximum 
\( \text{DOK} = 1 \) when absolute value of Chf which is MChf 
is minimum (MChf = 0) (points J & L), that means 
when the magnitude of the chaotic factor (MChf) 
decreases our certain knowledge (DOK) increases. 
Afterward, MChf starts to increase during the 
functioning due to the environment and intrinsic 
conditions thus leading to a decrease in DOK until they 
both reach 0.5 at \( t_0 = 1500 \) (point K). The real 
probability \( P_r \) and the complementary probability \( P_{com} \) will intersect with DOK also at the point (1500, 0.5) 
(point K). With the increase of \( t_0 \), the Chf and MChf 
return to zero and the DOK returns to 1 where we reach 
total damage (\( D = 1 \)) and hence the total certain failure 
(\( P_c(t) = 1 \)) of the system (point L). At this last point the 
failure here is definite, \( P_r(t_0) = 1 \) and RUL(\( t_0 \)) = \( t_N - t_0 \) 
= 0 with \( P_c(t_0) = 1 \), so the logical explanation of the 
value DOK = 1 follows.

We note that the point K is not at the middle of DOK 
since the probability of failure distribution evaluated by 
FORM is not symmetric.

Furthermore, at each instant \( t_0 \), the remaining useful 
lifetime RUL(\( t_0 \)) is certainly predicted in the complex set 
C with \( P_c \) maintained as equal to one through continuous 
compensation between DOK and Chf. This 
compensation is from instant \( t_0 = 0 \) where D(\( t_0 \)) = 0 until the failure instant \( t_N \) where D(\( t_N \)) = 1.

### 2.2.2. The Cube of Probability Components

In the following figure, we represent the extended 
Kolmogorov’s probability components \( P_r \) and \( P_{com} \) in a 
three dimensional graph in terms of \( t \) and of each other 
(Fig. 10).

It is important to mention that if we rescale the time 
axis to an interval \([0,1]\) so the minimal value of DOK is 
at the instant \( t_0 = \frac{1500}{5798} = 0.2587 \) where \( N = 5798 \) cycles 
corresponds to \( t_N \). This last important point is clearly 
shown in this cube and in the following one.

From the cube below, we can notice that the 
probability \( P_c \) in the complex set C = \( R + M \) is obtained 
at each instant \( t_0 \) as the sum of \( P_r \) and \( P_{com} \) and is always 
equal to one.

### 2.2.2. The Cube of Probability Parameters

In the following figure, we represent the extended 
Kolmogorov’s probability parameters DOK and Chf in a 
three dimensional graph in terms of \( t \) and of each other 
(Fig. 11).

#### 2.3. Example: Application to Young Modulus

We consider once again the Young modulus example 
previously treated in the first paper on extended 
kolmogorov’s axioms and “Complex Probability Theory”.

Let \( E \) be the Young modulus in a material bar 
domain (Fig. 12) and we assume that it follows a Normal 
Gaussian distribution.

The limit state considered here for FORM analysis is:

\[
G(E_0) = \text{RUL}(E_0) = E_N - E_0
\]

When RUL (\( E_0 \)) is zero or negative then we have the 
case of \( E_0 \geq E_N \) that means that we have a system failure 
that cannot live until \( E_N \). In the other case where \( E_0 < E_N \), 
the system can live above \( E_0 \) and we have a case of success.

The real Probability of failure is given by:

\[
P_r(E_0) = P_{rob}(G(E_0) \leq 0) = P_{rob}(\text{RUL}(E_0) \leq 0) = P_{rob}[E_N \leq E_0]
\]

The reliability index \( \beta = -\Phi^{-1}(P_r) \).

In the extended Kolmogorov’s axioms, the real part 
of probability is taken here as \( P_r \).

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probability \( P_c \) in the complex set C = \( R + M \) is obtained 
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at each instant \( t_0 \) as the sum of \( P_r \) and \( P_{com} \) and is always 
equal to one.
Fig. 10. The probabilities $P_r$ and $P_{m/i}$ in terms of $t$ and of each other

Fig. 11. DOK and Chf in terms of $t$ and of each other
Fig. 12. The Young modulus E in a material domain

![Diagram of Young modulus E in a material domain]

Fig. 13. Probability of failure

And:

\[
\int_{-\infty}^{\infty} dF = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du = \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2} \frac{(E-E_0)^2}{\sigma}\right) dE = 1
\]

Now:

\[
P_m = \int_{-\infty}^{29575} \frac{1}{\sqrt{2\pi} \cdot 1507} \exp\left(-\frac{1}{2} \frac{(E-29575)^2}{1507}\right) dE = 0.5
\]

The correspondent probability in the imaginary domain M is:

\[
P_m = i(1 - P_m) = i P_m, \text{ where } E > 29575 = i \cdot [1 - F(29575)]
\]

\[
= i \int_{29575}^{\infty} \frac{1}{\sqrt{2\pi} \cdot 1507} \exp\left(-\frac{1}{2} \frac{(E-29575)^2}{1507}\right) dE = i \cdot 0.5
\]

If we compute the norm of the complex number:

\[
Z_0 = P_0 + P_m \]

we have:

\[
|Z_0|^2 = P_0^2 + (P_m/i)^2 = P_0^2 + (1 - P_0)^2 = 1 + 2P_0(1 - P_0)
\]

This implies that:

\[
1 = |Z_0|^2 + 2P_0(1 - P_0)
\]

\[
= [Z_0]_1^2 - 2i^2 P_0(1 - P_0)
\]

\[
= [Z_0]_1^2 - 2i P_0 P_m
\]

\[
= P_0^2 + (P_m/i)^2 - 2i P_0 P_m
\]

\[
= (P_0 + P_m/i)^2 = P_{c_0}^2 \Rightarrow P_{c_0} = 1
\]

We note that:

\[
Z_0 = P_0 + P_m = \int_{-\infty}^{E_0} f_u(u) du + i \int_{-\infty}^{E_0} f_u(u) du = 0.5 + i \times 0.5
\]

We have also:

\[
P_{c_0}^2 = (P_0 + P_m/i)^2 = \left(\int_{-\infty}^{E_0} f_u(u) du + i \int_{-\infty}^{E_0} f_u(u) du\right)^2 = 1^2 = 1
\]

and the chaotic factor is:

\[
Chf_0 = 2i P_0 P_m = 2i \times \int_{-\infty}^{E_0} f_u(u) du \times \int_{-\infty}^{E_0} f_u(u) du = -2 \times \int_{-\infty}^{E_0} \times \left(1 - \int_{-\infty}^{E_0} f_u(u) du\right)
\]

Where:

\[
Chf_0 = 0 \text{ if } \begin{cases} E_0 \to -\infty, & \text{hence } P_0 = 0 \\ E_0 \to +\infty, & \text{hence } P_0 = 1 \end{cases}
\]

Moreover, the Magnitude of the chaotic factor is:

\[
MChf_0 = |2i P_0 P_m| = \sqrt{2i \times \int_{-\infty}^{E_0} \times \int_{-\infty}^{E_0} f_u(u) du \times \left(1 - \int_{-\infty}^{E_0} f_u(u) du\right) = 2 \times \int_{-\infty}^{E_0} \times \left(1 - \int_{-\infty}^{E_0} f_u(u) du\right)\}
\]

Where:
Therefore, we say that:

\[ \text{DOK} + \text{MChf} = 0.5 + 0.5 = 1 = \text{Pe}_0 \]

Conversely, if we assume that:

\[ \text{Chf}_0 = 0 \implies \text{MChf}_0 = 0 \implies |Z_0|^2 = 1 \implies \text{Pe}_0 + (\text{Pm}_0/i)^2 = 1 \]

If \( \text{Chf}_0 = -\frac{1}{2} \implies \text{MChf}_0 = \frac{1}{2} \implies E_0 = E \text{ and } |Z_0|^2 = \frac{1}{2} \)

\[ \implies 2\text{Pm}(1-\text{Pe}_0) = 0 \implies \begin{cases} \text{Pe}_0 = 0 & \implies E_0 \to -\infty \\ \text{Pm} = 1 & \implies E_0 \to +\infty \end{cases} \]

If \( E_0 \) increases to become \( = 4000 \) then both \( |Z_0|^2 \) and \( \text{Chf}_0 \) increase and \( \text{MChf}_0 \) decreases.

Therefore we can infer that:

\[ \lim_{E_0 \to +\infty} (\text{Chf}_0) = 0, \quad \lim_{E_0 \to +\infty} (\text{MChf}_0) = 0 \quad \text{and} \quad \lim_{E_0 \to +\infty} \left\lfloor \frac{|Z_0|^2}{2} \right\rfloor = 1 \]

Where:

\[ \text{DOK} = \text{Pe}_0 - \text{Chf}_0 \]

\[ = \text{DOK}_0 - \text{Chf}_0 \]

\[ = |Z_0|^2 + \text{MChf}_0 \]

\[ = \text{DOK}_0 + \text{MChf}_0 = 1, \]

for every \( E_0 \) in the real set \( R \).

We note from the figure below that the DOK is maximum (\( \text{DOK} = 1 \)) when absolute value of \( \text{Chf} \) which is \( \text{MChf} \) is minimum (\( \text{MChf} = 0 \)) (points J & L), that means when the magnitude of the chaotic factor (MChf) decreases, our certain knowledge (DOK) increases. Afterward, MChf starts to increase during the functioning due to the environment and intrinsic conditions thus leading to a decrease in DOK until they both reach 0.5 at \( E_0 = 29575 \) (point K). The real probability \( P_r \) and the complementary probability \( \text{Pm}/i \) will intersect with DOK also at the point (29575, 0.5) (point K). With the increase of \( E_0 \), the Chf and MChf return to zero and the DOK returns to 1 where we reach total damage (\( D = 1 \)) and hence the total certain failure (\( P_r = 1 \)) of the system (point L). At this last point the failure here is definite, \( P_i(E_N) = 1 \) and RUL(\( E_N \)) = \( E_N - E_N = 0 \) with \( \text{Pe}(E_N) = 1 \), so the logical explanation of the value \( \text{DOK} = 1 \) follows (Fig. 14 and 15).

We note that the point K is not at the middle of DOK since the probability of failure distribution evaluated by FORM is not symmetric.
Fig. 14. DOK, Chf and Pc as functions of the probability of failure

Fig. 15. DOK, MChf and Pc as functions of the probability of failure
Fig. 16. The probabilities $P_i$ and $P_m/i$ in terms of $E$ and of each other

Fig. 17. DOK and Chf in terms of $E$ and of each other
which seems to be random and stochastic in R is now

\[ DOK + MChf = 1, \]

that means that the phenomenon process of degradation we have

\[ Pc = DOK - Chf < 0 \quad \text{and} \quad 0 < MChf < 0.5. \]

Notice that during the whole degradation \( 0 < D < 1 \) we have:

\[ 0.5 < DOK < 1, \]

\[ -0.5 < Chf < 0, \]

\[ 0 < MChf < 0.5. \]

The system is totally known. During the process of chaotic factor (Chf and MChf) is 0 since the state of the system is deterministic and certain in \( C = R + M \) and this after adding to R the contributions of M and hence after subtracting the chaotic factor from the degree of our knowledge. Moreover, for each value of an instant \( t_0 \) or \( E_0 \), I have determined their corresponding probability of survival or of the remaining useful lifetime \( RUL(t_0) = t_0 - t, \) or \( RUL(E_0) = E_N - E_0. \) In other words, at each instant \( t_0 \) or \( E_0, \) \( RUL(t_0) \) or \( RUL(E_0) \) are certainly predicted in the complex set C with Pc maintained as equal to one through a continuous compensation between DOK and Chf. This compensation is from \( E_0 = 0 \) where \( D(E_0) = 0 \) until failure at \( E_N \) where \( D(E_N) = 1. \)

### 2.3.1. The Cube of Probability Components

In the figure above, we represent the extended Kolmogorov’s probability components \( P_i \) and \( P_d/i \) in a three dimensional graph in terms of \( E \) and of each other (Fig. 16).

It is important to mention that if we rescale the E axis to an interval \([0,1]\) so the minimal value of DOK is at the instant where \( E_0 = \frac{29575}{114317} \approx 0.2587 \) knowing that \( E = 114317 \) corresponds to \( E_N. \) This last important point is clearly shown in this cube and in the following one.

From the cube above, we can notice that the probability \( Pc \) in complex set \( C = R + M \) is obtained at each value \( E_0 \) as the sum of \( P_i \) and \( P_d/i \) and is always equal to one.

### 2.3.2. The Cube of Probability Parameters

In the figure above, we represent the extended Kolmogorov’s probability parameters DOK and Chf in a three dimensional graph in terms of \( E \) and of each other (Fig. 17).

### 3. CONCLUSION

In this study I applied the theory of Extended Kolmogorov Axioms to Prognostic based on Reliability. I used for this purpose the very well known First Order Reliability Method or FORM analysis for short. Consequently, I established a tight link between the new theory and degradation or the remaining useful lifetime and reliability. Hence, I developed the theory of “Complex Probability” beyond the scope of the previous three papers on this topic. As it was proved and illustrated, when the degradation index is 0 or 1 and correspondingly the RUL is \( t_N \) or 0 then the Degree of Our Knowledge (DOK) is one and the chaotic factor (Chf and MChf) is 0 since the state of the system is totally known. During the process of degradation \( 0 < D < 1 \) we have:

\[ 0.5 < DOK < 1, \]  
\[ -0.5 < Chf < 0, \]  
\[ 0 < MChf < 0.5. \]

Notice that during the whole process of degradation we have \( Pc = DOK - Chf = DOK + MChf = 1, \) that means that the phenomenon which seems to be random and stochastic in R is now deterministic and certain in \( C = R + M \) and this after adding to R the contributions of M and hence after subtracting the chaotic factor from the degree of our knowledge. Moreover, for each value of an instant \( t_0 \) or \( E_0 \), I have determined their corresponding probability of survival or of the remaining useful lifetime \( RUL(t_0) = t_N - t, \) or \( RUL(E_0) = E_N - E_0. \) In other words, at each instant \( t_0 \) or \( E_0, \) \( RUL(t_0) \) or \( RUL(E_0) \) are certainly predicted in the complex set C with Pc maintained as equal to one through a continuous compensation between DOK and Chf. This compensation is from \( E_0 = 0 \) where \( D(t_N) = 0 \) until the failure instant \( t_N \) where \( D(t_N) = 1 \) and this compensation is also from \( E_0 = 0 \) where \( D(E_0) = 0 \) until failure at \( E_N \) where \( D(E_N) = 1. \) Furthermore, using all these graphs illustrated throughout the whole paper, we can visualize and quantify both the system chaos (Chf and MChf) and the system certain knowledge (DOK and Pc). Additionally, an application to Young modulus was successfully done here. This is certainly very interesting and fruitful and shows once again the benefits of extending Kolmogorov’s axioms and thus the originality and usefulness of this new field in mathematics that can be called verily: “The Complex Probability and Statistics Paradigm”.

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