Finite-Dimensional Bicomplex Hilbert Spaces

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Abstract. This paper is a detailed study of finite-dimensional modules defined on bicomplex numbers. A number of results are proved on bicomplex square matrices, linear operators, orthogonal bases, self-adjoint operators and Hilbert spaces, including the spectral decomposition theorem. Applications to concepts relevant to quantum mechanics, like the evolution operator, are pointed out.

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1. Introduction

Bicomplex numbers [1], just like quaternions, are a generalization of complex numbers by means of entities specified by four real numbers. These two number systems, however, are different in two important ways: quaternions, which form a division algebra, are noncommutative, whereas bicomplex numbers are commutative but do not form a division algebra.

Division algebras do not have zero divisors, that is, nonzero elements whose product is zero. Many believe that any attempt to generalize quantum mechanics to number systems other than complex numbers should retain the division algebra property. Indeed considerable work has been done over the years on quaternionic quantum mechanics [2].

In the past few years, however, it was pointed out that several features of quantum mechanics can be generalized to bicomplex numbers. A generalization of Schrödinger’s equation for a particle in one dimension was

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proposed [3], and self-adjoint operators were defined on finite-dimensional bi-complex Hilbert spaces [4]. Eigenvalues and eigenfunctions of the bicomplex analogue of the quantum harmonic oscillator Hamiltonian were obtained in full generality [5].

The perspective of generalizing quantum mechanics to bicomplex numbers motivates us in developing further mathematical tools related to finite-dimensional bicomplex Hilbert spaces and operators acting on them. After a brief review of bicomplex numbers and modules in Section 2, we devote Section 3 to a number of results in linear algebra that do not depend on the introduction of a scalar product. Basic properties of bicomplex square matrices are obtained and theorems are proved on bases, idempotent projections and the representation of linear operators. In Section 4 we define the bicomplex scalar product and derive a number of results on Hilbert spaces, orthogonalization and self-adjoint operators, including the spectral decomposition theorem. Section 5 is devoted to applications to unitary operators, functions of operators and the quantum evolution operator. We conclude in Section 6.

2. Basic Notions

This section summarizes known properties of bicomplex numbers and modules, on which the bulk of this paper is based. Proofs and additional results can be found in [1, 3, 4, 6].

2.1. Bicomplex Numbers

2.1.1. Definition. The set \( \mathbb{T} \) of bi-complex numbers is defined as
\[
\mathbb{T} := \{ w = z_1 + z_2i_2 | z_1, z_2 \in \mathbb{C}(i_1) \},
\]

(2.1)

where \( i_1, i_2 \) and \( j \) are imaginary and hyperbolic units such that \( i_1^2 = -1 = i_2^2 \) and \( j^2 = 1 \). The product of units is commutative and is defined as
\[
i_1i_2 = j, \quad i_1j = -i_2, \quad i_2j = -i_1.
\]

(2.2)

With the addition and multiplication of two bicomplex numbers defined in the obvious way, the set \( \mathbb{T} \) makes up a commutative ring.

Three important subsets of \( \mathbb{T} \) can be specified as
\[
\mathbb{C}(i_k) := \{ x + yi_k | x, y \in \mathbb{R} \}, \quad k = 1, 2;
\]

(2.3)

\[
\mathbb{D} := \{ x + yj | x, y \in \mathbb{R} \}.
\]

(2.4)

Each of the sets \( \mathbb{C}(i_k) \) is isomorphic to the field of complex numbers, while \( \mathbb{D} \) is the set of the so-called hyperbolic numbers.

2.1.2. Conjugation and Moduli. Three kinds of conjugation can be defined on bicomplex numbers. With \( w \) specified as in (2.1) and the bar (\( \bar{\cdot} \)) denoting complex conjugation in \( \mathbb{C}(i_1) \), we define
\[
w^{\dagger 1} := \bar{z}_1 + z_2i_2, \quad w^{\dagger 2} := z_1 - z_2i_2, \quad w^{\dagger 3} := \bar{z}_1 - \bar{z}_2i_2.
\]

(2.5)