The Property of Frequency Shift in 2D-FRFT Domain with Application to Image Encryption

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Abstract—The Fractional Fourier Transform (FRFT) has been playing a unique and increasingly important role in signal and image processing. In this paper, we investigate the property of frequency shift in two-dimensional FRFT (2D-FRFT) domain. It is shown that the magnitude of image reconstruction from phase information is frequency shift-invariant in 2D-FRFT domain, enhancing the robustness of image encryption, an important multimedia security task. Experiments are conducted to demonstrate the effectiveness of this property against the frequency shift attack, improving the robustness of image encryption.

Index Terms—Frequency Shift, 2D-FRFT, image encryption.

I. INTRODUCTION

THE Fourier Transform (FT) is one of the most important analysis tools used in physical optics and signal processing [1-3]. As a generalization of the FT, Fractional Fourier Transform (FRFT) was introduced in 1980 [4-5]. Different from the FT, the FRFT of a signal is flexibly operated at any angle with respect to the time axis on the time-frequency plane, generating a versatile representation for time-frequency distributions (TFDS) of the Cohen class. In fact, the conventional FT is a special case of the FRFT, when the operation angle is 90 degree with respect to the time axis. FRFT provides a powerful tool to analyze signals in the time-frequency domain. Nevertheless, as a standing problem, frequency shift can introduce interference into the phase information, leading to poor performance on related applications [22-23].

To address the aforementioned issues, in this letter, we present a study of the properties of frequency shift in 2D-FRFT from amplitude and phase information with mathematical verification and computer simulations. The main contributions are summarized as follows.

1. It is demonstrated that the magnitude of image reconstruction from phase information is frequency shift-invariant in 2D-FRFT domain while the magnitude of reconstruction from amplitude information does not possess this property.

2. In application, we show that the utilization of this property improves robustness of image encryption.

The remainder of this letter is organized as follows: Section II reviews related work. Section III introduces and verifies the property of frequency shift in 2D-FRFT domain. Section IV presents application examples and Section V draws conclusions.

II. RELATED WORK

In this section, we will briefly present the existing fundamentals of FRFT and 2D-FRFT, respectively.

A. FRFT

The transform of a 1D signal $h(t)$ by FRFT is written as

$$H_\alpha (u) = \{F_\alpha [h(t)]\}(u) = \int_{-\infty}^{\infty} h(t) K_\alpha (t, u) dt,$$  \hspace{1cm} (1)

with the transform kernel $K_\alpha (t, u)$, in the following form

$$K_\alpha (t, u) = \begin{cases} k_\alpha \cdot \exp \left( \frac{i^2 u^2}{2} \cot \alpha - itu \csc \alpha \right), & \alpha \neq n\pi \\ \delta(t - u), & \alpha = 2n\pi \\ \delta(t + u), & \alpha = (2n \pm 1)\pi \end{cases}$$  \hspace{1cm} (2)

where $k_\alpha = \sqrt{1 - i \cot \alpha / 2\pi}$ ($i = \sqrt{-1}$) and $\alpha$ is the rotation angle in FRFT.
B. 2D-FRFT

With two rotation angles $\alpha$ and $\beta$, 2D-FRFT provides two degrees of freedom coping with signal and image processing problems. Analytically, the definition of 2D-FRFT to a 2D signal $d(s, t)$ is given as

$$D_{\alpha,\beta}(u,v) = \{F_{\beta}\{F_{\alpha}[d(s, t)]\}(u,t)\}(u,v).$$

Let the size of a discrete 2D signal $g(p,q)$ be $(P,Q)$. The forward and inverse 2D-FRFT to a 2D discrete signal $g(p,q)$ are expressed as in [19]:

$$G_{\alpha,\beta}(m,n) = \sum_{p=0}^{P-1} \sum_{q=0}^{Q-1} g(p,q)K_{\alpha,\beta}(p,q,m,n),$$

$$g(p,q) = \sum_{m=0}^{P-1} \sum_{n=0}^{Q-1} G_{\alpha,\beta}(m,n)K_{-\alpha,-\beta}(p,q,m,n),$$

where $K_{\alpha,\beta}(p,q,m,n)$ and $K_{-\alpha,-\beta}(p,q,m,n)$ are the forward and inverse 2D discrete transform kernels, respectively.

III. FREQUENCY SHIFT IN 2D-FRFT DOMAIN

A. Mathematical Derivation

Again, the 2D-FRFT to a 2D signal $f(x,y)$ is expressed as

$$F_{\alpha,\beta}(u,v) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} K_{\alpha}(x,u) \cdot K_{\beta}(y,v) \cdot f(x,y) \cdot dx \cdot dy,$$

where $K_{\alpha}(x,u)$ and $K_{\beta}(y,v)$ are transform kernel functions defined in equation (2). Equation (6) is rewritten equivalently as follows

$$F_{\alpha,\beta}(u,v) = A_{\alpha,\beta}(u,v) \cdot \exp(i2\pi \varphi_{\alpha}(u) + i2\pi \varphi_{\beta}(v)),$$

where $A_{\alpha,\beta}(u,v)$ and $\exp(i2\pi \varphi_{\alpha}(u) + i2\pi \varphi_{\beta}(v))$ represent the amplitude and phase components of equation (6).

Then, the amplitude part $f_{A}(x,y)$ and the phase part $f_{\varphi}(x,y)$ in the space domain are reconstructed from equation (7) by inverse 2D-FRFT transform [19], and are defined in equations (8) and (9), respectively.

$$f_{A}(x,y) = F_{-\alpha,-\beta}(A_{\alpha,\beta}(u,v)),$$

$$f_{\varphi}(x,y) = F_{-\alpha,-\beta}(\exp(i2\pi \varphi_{\alpha}(u) + i2\pi \varphi_{\beta}(v))$$

where $F_{-\alpha,-\beta}$ is the inverse 2D-FRFT transform with rotation angles $-\alpha$ and $-\beta$.

Set the horizontal and vertical frequency shifts as $\delta$ and $\varepsilon$, expressing in the forms of $\exp(i2\pi x \delta)$ and $\exp(i2\pi y \varepsilon)$ in 2D-FRFT. The frequency shift operation $F_{\alpha,\beta}(u,v)$ in 2D-FRFT is in the form of:

$$F_{\alpha,\beta}(u,v) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Gamma_{\alpha,\beta} \cdot \exp(i2\pi x \delta + i2\pi y \varepsilon) dx \cdot dy,$$

where $\Gamma_{\alpha,\beta} = K_{\alpha}(x,u) \cdot K_{\beta}(y,v) \cdot f(x,y)$. Then, equation (10) can be equivalently written as follows

$$F_{\alpha,\beta}(u,v) = A_{\alpha,\beta}(u,v) \cdot \exp(i2\pi \varphi_{\alpha}(u) + i2\pi \varphi_{\beta}(v)),$$

where $A_{\alpha,\beta}(u,v)$ and $\exp(i2\pi \varphi_{\alpha}(u) + i2\pi \varphi_{\beta}(v))$ represent the amplitude component and phase component of equation (10).

$A_{\alpha,\beta}(u,v)$ is equivalently expressed as follows

$$A_{\alpha,\beta}(u,v) = |F_{\alpha,\beta}(u,v)|.$$
Using an algebraic operation, equation (12) is further written as shown below:

\[
A_{\alpha,\beta}(u, v) = \left[ +\infty +\infty \right] \int \int \kappa_{\alpha,\beta}(x, y, u, v) \cdot f(x - \rho, y - \lambda) dx dy,
\]

where \( \kappa_{\alpha,\beta}(x, y, u, v) = K_{\alpha}(x, u) \cdot K_{\beta}(y, v) \). Equation (16) is equivalently given in the form of amplitude \( A_{\alpha,\beta}(u, v) \) and phase \( \exp(i2\pi\varphi_{\alpha}(u) + i2\pi\varphi_{\beta}(v)) \) in equation (17),

\[
F_{\alpha,\beta}(u, v) = A_{\alpha,\beta}(u, v) \cdot \exp(i2\pi\varphi_{\alpha}(u) + i2\pi\varphi_{\beta}(v)).
\]

Using an algebraic method, \( A_{\alpha,\beta}(u, v) \) is equivalently written as follows:

\[
A_{\alpha,\beta}(u, v) = \left[ +\infty +\infty \right] \int \int \kappa_{\alpha,\beta}(x, y, u, v) \cdot f(x - \rho, y - \lambda) dx dy,
\]

where \( \kappa_{\alpha,\beta}(x, y, u, v) = K_{\alpha}(x, u) \cdot K_{\beta}(y, v) \). Equation (16) is equivalently given in the form of amplitude \( A_{\alpha,\beta}(u, v) \) and phase \( \exp(i2\pi\varphi_{\alpha}(u) + i2\pi\varphi_{\beta}(v)) \) in equation (17),

\[
F_{\alpha,\beta}(u, v) = A_{\alpha,\beta}(u, v) \cdot \exp(i2\pi\varphi_{\alpha}(u) + i2\pi\varphi_{\beta}(v)).
\]

Similarly, the magnitude of the reconstructed phase component \( f_{\varphi}(x, y) \) from \( \exp(i2\pi\varphi_{\alpha}(u) + i2\pi\varphi_{\beta}(v)) \) yields the following expression,

\[
|f_{\varphi}(x, y)| = |F_{\alpha,\beta}(exp(i2\pi\varphi_{\alpha}(u) + i2\pi\varphi_{\beta}(v)))| = |F_{\alpha,\beta}(exp(i2\pi\varphi_{\alpha}(u)) \cdot exp(i2\pi\varphi_{\beta}(v)))|.
\]

Since 2D-FRFT is equivalent to apply FRFT on the two variables successively, mathematical manipulation of (22) yields

\[
|f_{\varphi}(x, y)| = |F_{\alpha,\beta}(exp(i2\pi\varphi_{\alpha}(u)) \cdot exp(i2\pi\varphi_{\beta}(v)))|.
\]

From equation (23), \( |F_{\alpha}(exp(i2\pi\varphi_{\alpha}(u)))| \) can be rewritten as follows:

\[
|F_{\alpha}(exp(i2\pi\varphi_{\alpha}(u)))| = |F_{\alpha}(exp(i2\pi\varphi_{\alpha}(u) + i2\pi\varphi_{\beta}(v)))| = |F_{\alpha}(exp(i2\pi\varphi_{\alpha}(u)) \cdot exp(i2\pi\varphi_{\beta}(v)))|.
\]

By definition,

\[
f_{\varphi}(x) = F_{\alpha}(exp(i2\pi\varphi_{\alpha}(u))).
\]

According to the separability of 2D-FRFT and equation (18), \( |F_{\alpha,\beta}(exp(i2\pi\varphi_{\alpha}(u)) \cdot exp(i2\pi\varphi_{\beta}(v)))| \) is further written as

\[
|F_{\alpha}(exp(i2\pi\varphi_{\alpha}(u)))| = |F_{\alpha}(exp(i2\pi\varphi_{\alpha}(u) + i2\pi\varphi_{\beta}(v)))| = |F_{\alpha}(exp(i2\pi\varphi_{\alpha}(u)) \cdot exp(i2\pi\varphi_{\beta}(v)))|.
\]

Employing equation (13) and substituting (26) into (24) yields equation (27)

\[
|F_{\alpha}(exp(i2\pi\varphi_{\alpha}(u)))| = |F_{\alpha}(exp(i2\pi\varphi_{\alpha}(u) + i2\pi\varphi_{\beta}(v)))| = |F_{\alpha}(exp(i2\pi\varphi_{\alpha}(u)) \cdot exp(i2\pi\varphi_{\beta}(v)))| = |f_{\varphi}(y)|,
\]

The equivalence between \( |F_{\alpha,\beta}(exp(i2\pi\varphi_{\beta}(v)))| \) and \( |f_{\varphi}(y)| \) can be similarly verified, thus

\[
|F_{\alpha,\beta}(exp(i2\pi\varphi_{\beta}(v)))| = |F_{\alpha,\beta}(exp(i2\pi\varphi_{\beta}(v) + i2\pi\varphi_{\beta}(v)))| = |F_{\alpha,\beta}(exp(i2\pi\varphi_{\beta}(v) \cdot exp(i2\pi\varphi_{\beta}(v)))| = |f_{\varphi}(y)|.
\]

Equations (21) and (29) demonstrate that the magnitude of reconstruction from amplitude-only information in 2D-FRFT domain is due to the corresponding shift with respect to the frequency shift operations. Nevertheless, the magnitude of reconstruction from phase-only information in 2D-FRFT domain has no shift at all.

Moreover, since \( x \) and \( y \) are integers in the field of digital image processing, \( \exp(i2\pi x \delta) \) and \( \exp(i2\pi y \varepsilon) \) change into
periodic functions and the period is 1 for \( \delta \) and \( \varepsilon \), respectively. Therefore, during the following computer simulations, \( \delta \) and \( \varepsilon \) satisfy the relation (30)

\[
\begin{align*}
\exp(i2\pi x\delta) &= \exp(i2\pi x(\delta + 1)), \\
\exp(i2\pi x\varepsilon) &= \exp(i2\pi x(\varepsilon + 1)).
\end{align*}
\]

(30)

B. Simulations

In this subsection, the impact of frequency shift on the amplitude and phase components in 2D-FRFT is shown by the following computer simulations. During the simulations, the rotation angles are selected as \( \alpha=36^\circ \) and \( \varepsilon \) randomly. The simulation results on image ‘Lena’ are illustrated in Fig. 1. From the simulation results, it is observed the phase information is frequency shift-invariant for image reconstruction in 2D-FRFT domain while the amplitude information does not possess this property. Moreover, experimental results on the periodic characteristics of \( \exp(i2\pi x\delta) \) \((\delta = 0.2, \delta = 10.2) \) in 2D-FRFT domain \((\alpha=36^\circ)\) are shown in Fig. 1 (d) to Fig. 1 (i).

IV. Applications

As two degrees of freedom are provided in 2D-FRFT, raising the potential to generate more security [24]. 2D-FRFT has been widely applied in the field of image encryption. In this section, we present utilization of the property of frequency shift in 2D-FRFT, which is expected to find applications in the aforementioned fields to improve the robustness.

Information processing in the encrypted domain has attracted considerable research interests [25-26]. In [27], a double random phase fractional order Fourier domain encoding scheme is proposed for image encryption to enhance the level of security. It demonstrated that the double random phase method is robust against attacks such as occlusion, crop, and so forth [27]. However, there is a chronic issue [28-29] that frequency shift can introduce interference into phase information and decrease the robustness of the double random phase encoding scheme. Since the image reconstruction from phase information satisfies the frequency shift-invariance property, it has potential to extract encryption information/data even when frequency shift attacks exist. In the following experiments, we will select the method of double random phase encoding used in [27] to demonstrate the effectiveness of the frequency shift-invariant property in image encryption.

In the double random phase encoding method, an independent random function \( r(x, y) \) is uniformly distributed in the interval \([0, 2\pi]\) and the rotation angles are set as \( \alpha=\beta=9^\circ \) randomly. Then, the method of random phase encoding on a two dimensional signal \( I(x, y) \) is written as follows

\[
g(\varsigma, \eta) = \int \int I(x, y) \cdot \exp(2\pi i \cdot r(x, y)) \cdot \Phi \cdot dx dy,
\]

\[
i = (-1)^{1/2},
\]

where \( \Phi = K_{\alpha=9^\circ, \beta=9^\circ}I(\varsigma, \eta, x, y) \) is the transform kernel function in 2D-FRFT, and the function \( g(\varsigma, \eta) \) is the encrypted signal.

Since it satisfies the property of inverse in 2D-FRFT domain, the original signal \( I(x, y) \) can be recovered with the correct independent random functions and rotation angles. Nevertheless, when the frequency shifts \( \exp(i2\pi x\delta) \) and \( \exp(i2\pi y\varepsilon) \) are introduced, \( I(x, y) \cdot \exp(i2\pi x\delta) \cdot \exp(i2\pi y\varepsilon) \cdot \exp(2\pi i r(x, y)) \) will replace \( I(x, y) \cdot \exp(2\pi i r(x, y)) \) in equation (31), resulting in failures of the encrypted information/data recovery even with the correct independent random functions and rotation angles. However, we can recover the encryption information/data successfully benefiting from the property of frequency shift-invariance from phase information in 2D-FRFT domain.

Experiments are provided to demonstrate the effectiveness of this property against the frequency shift attack with rotation angles \((\alpha=\beta=9^\circ)\) in 2D-FRFT domain shown in Fig. 2. In Fig. 2, when there is no frequency shift in Fig. 2(b), the key image is successfully recovered straightforwardly in Fig. 2(c). When there is frequency shift existing in Fig. 2(d), we see the failure without using the frequency shift-invariant property shown in Fig. 2(e), and the success using the property shown in Fig. 2(f).

![Fig. 2 Experiments on image encryption with the frequency shift-invariance property](image)

V. Conclusions

In this letter, the property of frequency shift operation, from the amplitude and phase information in 2D-FRFT domain, has been studied. It is demonstrated that the magnitude of image reconstruction from phase information is frequency shift-invariant while the property does not hold for the amplitude information. Experiments are provided, illustrating the
effectiveness of the property in improving the robustness of image encryption.

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