Beam propagation factor of Hollow higher-order Cosh-Gaussian beams

Faroq Saad¹,³ · Ahmed Abdulrab Ali Ebrahim² · Abdelmajid Belafhal⁴

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Abstract
Based on the second-order moments definition, the beam propagation factor of a new mathematical model of Hollow higher-order Cosh-Gaussian (HhCG) beams is investigated. Analytical formula for the M²-factor of HhCG beams is derived, which depends on the hollowness parameter \( l \), the beam order \( n \), the parameter of the cosh part \( \delta \) and the beam waist \( \omega_0 \). The influences of these parameters are illustrated with numerical examples. The result provides more general characteristics because the higher-order Cosh-Gaussian, Cosh-Gaussian and Gaussian beams are obtained as special cases of HhCG beams.

Keywords Hollow higher-order Cosh-Gaussian · Second-order moment · M²-factor

1 Introduction

The parametric characterization of the laser beam and beam quality have received considerable interest. Early, the beam-propagation factor has proposed by Siegman (1990, 1992) as a better parameter for characterizing various laser beam quality and their mode structures. Also, the M²-factor plays an important role in the framework of the paraxial approximation (Borghi et al. 1997). On the basis of higher-order irradiance moments, the symmetry and degree of flatness of laser beams can be obtained (Martinez-Herrero et al. 1992; Weber 1992). On the other hand, Hermite–Sinusoidal–Gaussian (HSG) beams has been introduced as exact solutions of the paraxial wave equation (Caperson and Tover 1997, 1998), Gaussian, cosh-Gaussian and higher order cosh-Gaussian beams can be regarded as special cases of HSG beams. The propagation properties of these beams family have been the subject of many papers in several optical systems (Lü et al. 1999; Lü and Luo 2000; Yu et al. 2002; Belafhal and Ibnchaikh 2000; Wang and Lü 2001; Ibnchaikh et al. 2001; Hricha and Belafhal 2005; Mei et al. 2004;

* Abdelmajid Belafhal
belafhal@gmail.com

1 High Institute for Training & Qualifying Teachers, Ministry of Education, Taiz, Yemen
2 Ministry of Education, Taiz, Yemen
3 Faculty of Engineering, Al-Janad University for Science & Technology, Taiz, Yemen
4 Laboratory LPNAMME, Laser Physics Group, Department of Physics, Faculty of Sciences, Chouaib Doukkali University, P. B 20, 24000 El Jadida, Morocco
Tang et al. 2007; Zhou 2009, 2011; Li et al. 2010a, 2010b; Zhou and Zheng 2009). The beam propagation factor of hard-edge diffracted cosh-Gaussian beams and their mode-coherence coefficients have been examined (Li et al. 1999; Lü and Luo 2000). The propagation characteristic of Hermite-cosh-Gaussian beams and elegant Hermite-cosh-Gaussian beams have been studied, and their propagation factor has been given and expressed in a concise form (Yu et al. 2002; Belafhal and Ibnchaikh 2000; Wang and Lü 2001; Ibnchaikh et al. 2001; Hricha and Belafhal 2005; Mei et al. 2004). A new method has been proposed to generate flattened Gaussian beam by incoherent combination of cosh Gaussian beams (Tang et al. 2007). Recently, the propagation properties of the higher-order cosh-Gaussian beam model have been investigated in detailed, and the corresponding beam propagation factor has also been studied (Zhou 2009, 2011; Li et al. 2010a, b; Zhou and Zheng 2009). More recently, Saad and Belafhal have investigated a novel hollow beam model of HhCG, as a generalized beam of cosh Gaussian (Saad and Belafhal 2021). Similarly, the $M^2$-factor of various dark hollow beams in free space has been studied (Deng 2005; Mei and Zhao 2004, 2006; Deng et al. 2005; Ebrahim et al. 2014; Zeng et al. 2018; Zhou et al. 2019). Generalized $M^2$-factor of hard-edged diffracted Hypergeometric-Gaussian type-II beams has been carried out (Ebrahim et al. 2014). The $M^2$-factor of controllable dark-hollow beams through a multi-apertured ABCD optical system has also been presented (Zeng et al. 2018). Zhou et al. have investigated the propagation factor and kurtosis parameter of hollow vortex Gaussian beams (Zhou et al. 2019). Up to now, to our knowledge, the $M^2$-factor of HhCG beams hasn’t been reported. In this paper, our aim is to investigate the $M^2$-factor of HhCG beams. In Sect. 2, analytical formula for the $M^2$-factor of HhCG beams on propagation is derived. Some numerical examples are given and discussed in Sect. 3. The main results are outlined in Sect. 4.

2 Theoretical models

The optical field distribution of HhCG beams at the source plane ($z=0$) in Cartesian coordinate system is evaluated as (Saad and Belafhal 2021)

$$E_{n,l}(x,y,0) = E_{n,l}(x,0)E_{n,l}(y,0),$$

where $E_{n,l}(x,0)$ and $E_{n,l}(y,0)$ are given by

$$E_{n,l}(j,0) = A_0 \left( \frac{j}{\omega_0} \right)^j \exp \left( -\frac{j^2}{\omega_0^2} \right) \cosh^n(\Omega j),$$

with $j=x$ or $y$, $A_0$ is the constant amplitude, $\omega_0$ is the Gaussian waist width, $\Omega$ is the parameter associated with the cosh part, $l$ is the hollowness parameter and $n$ is the beam order. $E_{n,l}(j,0)$ can also be written in the form as follows:

$$E_{n,l}(j,0) = \frac{A_0}{2^n} \left( \frac{j}{\omega_0} \right)^l \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} \exp \left( a_{mn}^2 \delta \right) \exp \left[ -\left( \frac{j}{\omega_0} - a_{mn} \sqrt{\delta} \right)^2 \right],$$

where

$$a_{mn} = \left(m - \frac{n}{2}\right) \text{ and } \delta = \omega_0^2 \Omega^2.$$

Our goal is to evaluate the $M^2$-factor, which is defined by
\[ M^2 = \sqrt{M_x^2 M_y^2} \]  

(4)

Because their symmetry, the two components \( M_x \) and \( M_y \) are equivalents and consequently \( M_x = M_y \) and the \( M^2 \)-factor of the considered beams family is evaluated as

\[ M^2 = M_x^2 \left( = M_y^2 \right) \]  

(5)

In the following, we will evaluate only the component \( M_x \) which is defined by (Siegman 1990)

\[ M_x^2 = 4\pi \sigma_{0x} \cdot \sigma_{\infty x}, \]

(6)

where \( \sigma_{0x} \) and \( \sigma_{\infty x} \) are the second-order moments of the intensity distribution in the spatial domain and in the spatial frequency domain, respectively. In accordance to the second order moments definition of \( \sigma_{0x} \) in the waist plane, we have

\[ \sigma_{0x}^2 = \frac{I_{v=2}}{I_{v=0}}, \]

(7)

where

\[ I_v = \int_{-\infty}^{+\infty} x^n |E_{n,l}(x,0)|^2 \, dx. \]

(8)

If we substitute Eq. (3) into Eq. (8), and recalling the following integral formula (Belafhal et al 2020)

\[ \int_{-\infty}^{+\infty} x^n e^{-px^2 + 2qxt} \, dx = e^{q^2/p} \sqrt{\frac{\pi}{p}} \left( \frac{1}{2i \sqrt{p}} \right)^m H_m \left( i \frac{q}{\sqrt{p}} \right), \quad \text{with } p > 0, \]

(9)

after tedious integral calculations, the result of Eq. (8) can be obtained as

\[ I_v = \frac{A_0^2}{4^n} \sqrt{\frac{\pi}{2}} \frac{\omega_0^{2v+1}}{2i \sqrt{2}} Q^0_{n,l}(v), \]

(10)

where

\[ Q^0_{n,l}(v) = \sum_{m=0}^{n} \sum_{m'=0}^{n} \frac{(n!)^2}{m! m'!(n-m)!(n-m')!} e^{i\frac{\delta}{2}(a_{mn} + a_{m'n})^2} H_{v+2l} \left( i \sqrt{\frac{\delta}{2}} (a_{mn} + a_{m'n}) \right). \]

(11)

By substituting Eqs. (10) and (11) into Eq. (7), the final result of \( \sigma_{0x} \) in the waist plane can be written as

\[ \sigma_{0x} = \sqrt{\frac{I_2}{I_0}} i \omega_0 \frac{Q^0_{n,l}(2)}{2 \sqrt{2} Q^0_{n,l}(0)} \]

(12)
Similar calculations can be performed in the spatial-frequency domain. As well-known, this moment is given for the intensity distribution of the far-field and is defined as $\theta/2\pi$ with $\theta$ is the far-field during angle. In the spatial-frequency domain, the considered optical field reads as (Goodman 1996)

$$E_{nl}(f_x, 0) = \int_{-\infty}^{+\infty} E_{nl}(x, 0) \exp \left(-i2\pi f_x x\right) dx,$$

where $f_x$ is the transverse spatial frequency. Inserting Eq. (3) into Eq. (13) and integrating in the use of Eq. (9), we obtain

$$E_{nl}(f_x, 0) = \frac{\sqrt{\pi} \omega_0 A_0}{i 2^{n+1}} \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} e^{X^2} H_l(iX_{mn}),$$

with

$$X_{mn} = a_{mn} \sqrt{\delta - i\pi f_x \omega_0}.$$  

According to the second-order moment definition of $\sigma_{\omega x}$ in spatial frequency, one obtains

$$\sigma_{\omega x} = \frac{K_2}{K_0}$$

where

$$K_v = \int_{-\infty}^{+\infty} \left| E_{nl}(f_x, 0) \right|^2 df_x.$$  

On substituting from Eq. (14) into Eq. (17) and used the integral formula of Eq. (9), and after some calculations, the result of Eq. (17) can be arranged as

$$K_v = \frac{\pi A_0^2 \omega_0^2}{2^{n+2l}} \sum_{m=0}^{n} \sum_{m'=0}^{n} \frac{(n!)^2}{m!m'!(n-m)!(n-m')!} J_{v, mn},$$

where $J_{v, mn}$ is given by

$$J_{v, mn} = \int_{-\infty}^{+\infty} f_x^n e^{\left(\frac{x^2}{\omega_0^2} + \frac{x'^2}{\omega_0'^2}\right)} H_l(iX_{mn})H_l(-iX_{m'n}) df_x.$$  

By using the following formula (Erdelyi and Magnus 1954; Abramowitz and Stegun 1970)

$$H_n(x+y) = \frac{1}{2^{n/2}} \sum_{k=0}^{n} \binom{n}{k} H_k\left(\sqrt{2x}\right) H_{n-k}\left(\sqrt{2y}\right),$$

Equation (19) becomes
Beam propagation factor of Hollow higher-order Cosh-Gaussian beams is given by the integral formula of Eq. (9), and inserted Eq. (21) into Eq. (18), we then obtain

\[ K_v = A_{n,l}(v)Q_{n,l}^\infty(v), \]  

(22)

where

\[ Q_{n,l}^\infty(v) = \sum_{n=0}^{\infty} \sum_{m'=0}^{\infty} \frac{(n!)^2}{m!m'!(n-m)!(n-m')!} e^{\frac{i}{2}(a_{mn}+a_{m'n})^2} \]

\[ \times \sum_{r=0}^{l} \sum_{p=0}^{l} \left( \frac{1}{r} \right)^{l-r} \left( \frac{1}{p} \right)^{l-p} H_{l-r}(i\sqrt{2}\delta a_{mn}) H_{l-p}(-i\sqrt{2}\delta a_{m'n}) \]

\[ \times \sum_{k=0}^{[r/2]} \sum_{k'=0}^{[p/2]} \frac{(-1)^{k+k'} r!p!}{k!(r-2k)!k!(p-2k')!} \left( \frac{1}{i^{r+p-2k-2k'}} H_{v+r+p-2k-2k'} \sqrt{\frac{2}{\delta}} (a_{mn} - a_{m'n}) \right), \]

(23)

and

\[ A_{n,l}(v) = \frac{A_0^2}{\sqrt{2\pi}} \frac{\omega_0}{2^{2n+2l}} \frac{1}{(2i\sqrt{2\pi}\omega_0)^v}. \]

(24)

Finally, the second moment of the intensity distribution in the spatial frequency domain \( \sigma_\infty \) can be written as follows

\[ \sigma_\infty = K_2 \]

\[ \frac{K_0}{K_0} = \frac{A_{n,l}(2)Q_{n,l}^\infty(2)}{A_{n,l}(0)Q_{n,l}^\infty(0)} = \frac{1}{2i\pi \sqrt{2\omega_0}} \sqrt{Q_{n,l}^\infty(2)\over Q_{n,l}^\infty(0)}, \]

(25)

Thus, from Eq. (6) and by using Eqs. (12) and (25), the \( M^2 \)-factor of HhCG beams turns out to be

\[ M_x^2 = 4\pi \sigma_0 \sigma_\infty = \frac{1}{2} \sqrt{Q_{n,l}^0(2)Q_{n,l}^\infty(2) \over Q_{n,l}^0(0)Q_{n,l}^\infty(0)}. \]

(26)

This last expression is the closed form propagation formula of the considered beam which describes the \( M^2 \)-factor of HhCG beams.
3 Numerical results and discussions

In this Section, some numerical calculations were performed using the expression derived in the above Section. The influence of the parameters of $l$, $n$, $\delta$ and $\omega_0$ on the beam propagation factor of HhCG beams at the source plane ($z=0$), are illustrated. As shown in Fig. 1a–b, where the beam propagation factor $M^2$ for HhCG beams is plotted as a function of $\delta$ and $l$ for different values of $n$. One can notice that when $n=0$ and $l=0$, $M^2$-factor of HhCG beams reduced to that of the fundamental Gaussian beams. The variation of $M^2$ is not obvious for all values of $\delta$ and equal to one, but its value increases on propagation factor as its hollowness parameter $l$ increases (see Fig. 1a). For nonzero beam order $n$, the $M^2$-factor of HhCG beams varies with the parameters $\delta$ and $l$ (see Fig. 1b–c), from which it follows that the $M^2$-factor of HhCG beams increases with increasing $n$ and $l$. Also, it does vary monotonically with $\delta$, which means that, the $M^2$-factor of HhCG beams increases more rapidly with larger value of $\delta$ ($\delta > 0$) and it still maintains the same minimum values at $\delta=0$, for all values of $l$ and $n$. Noteworthily, when $l=0$, 1 and 5, the minimum values of $M^2$-factor always equals to 1, 3 and 4.819, respectively, (i.e., the central dark size increases). One can also note that with larger hollowness parameter $l$, is large affected by the $M^2$-factor. This characteristic may be considered to describe highly divergent HhCG beams. In addition, our result agrees well with Fig. 1 ($\alpha=\infty$) of Ref. (Lü and Luo 2000), as $l=0$, $n=1$, and as $l=0$, $n>1$, it is consistent with Ref. (Zhou, and Zheng 2009).

![Fig. 1 The $M^2$-factor of HHCGs as a function of parameter $\delta$ and with different values of $l$: a $n=0$, b $n=1$, c $n=5$ and d $n=8$.](image_url)
Figure 2 shows the variation of the $M^2$-factor of HhCG beams as a function of beam order $n$ for a fixed value of hollowness parameter $l$ and different values of parameter $\delta$. One finds from this figure that, the $M^2$-factor increases more rapidly as $\delta$ increases, while, its minimum value remains invariant at about 3, for any value of $\delta$ and zero beam order $n$, which indicates that the hollowness parameter plays an important role in beam propagation control. Moreover, as expected the $M^2$-factor of HhCG beams also increases with increasing order $n$.

Figure 3 gives the variation of the $M^2$-factor of HhCG beams versus $\omega_0$ for different values of $n$, with $l = 3$ and $\Omega = 10 \text{ m}^{-1}$, from which it turns out that the $M^2$-factor of the HhCG beams varies with the beam waist $\omega_0$ and both parameters of $l$ and $n$. It’s clear from this figure that, the $\omega_0$ value of $M^2$ first remains invariant about at $\omega_0 \leq 0.04$. With further increases of $\omega_0$, $M^2$ increases rapidly and reaches its maximum value with larger beam order $n$.

Finally, from the above numerical results, it follows that the $M^2$-factor of HhCG beams closely related to the parameters of $l$, $n$, $\omega_0$ and $\delta$. Moreover, When $l > 0$, changes in the minimum value of $M^2$ are obvious compared with the special case of $l = 0$ for beam propagation higher order cosh Gaussian which has always only one minimum value at $M^2 = 1$.

![Figure 2](image.png)

**Fig. 2** The $M^2$-factor of HhCG beams as a function of the beam order $n$ and with the hollowness order $l = 1$: 
(a) $\delta = 0.1$, (b) $\delta = 1$, (c) $\delta = 3$ and (d) $\delta = 10$. 

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4 Conclusion

In summary, the propagation factor of HhCG beams has been investigated. Based on the second-order moment, an analytical expression for the $M^2$-factor of HhCG beams has been derived. The $M^2$-factor is determined by the hollowness parameter $l$, the beam order $n$, the parameter of the cosh part $\delta$ and the beam waist $\omega_0$. The $M^2$-factor is illustrated in terms of these parameters with numerical examples. Our obtained results have also shown that, the hollowness parameter $l$ has an important influence on the $M^2$-factor. Also, we have shown that the variation of $M^2$ value presents always different minima for each parameter of $l$, while we have always only one minima at $M^2 = 1$, for all order of $n$ with $l = 0$. Moreover, the finding of this research is helpful to the practical applications of a hollow higher-order cosh–Gaussian beam, such as in fiber communication and laser optics.

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Declarations

Conflict of interest The authors have not disclosed any competing interests.
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