Abstract. Let $R$ be a Noetherian commutative ring and $M$ a $R$-module with $\text{pd}_R M \leq 1$ that has rank. Necessary and sufficient conditions were provided in [8] for an exterior power $\wedge^k M$ to be torsion free. When $M$ is an ideal of $R$ similar necessary and sufficient conditions were provided in [12] for a symmetric power $S_k M$ to be torsion free. We extend these results to a broad class of Schur modules $L_{\lambda/\mu} M$. En route, for any map of finite free $R$ modules $\phi: F \to G$ we also study the general structure of the Schur complexes $L_{\lambda/\mu} \phi$, and provide necessary and sufficient conditions for the acyclicity of any given $L_{\lambda/\mu} \phi$ by computing explicitly the radicals of the ideals of maximal minors of all its differentials.

Introduction

Torsion freeness of symmetric powers of ideals, and more generally of modules, has been studied extensively in commutative algebra and algebraic geometry, see for example [14] and the references there. When $R$ is a commutative Noetherian ring and $k \geq 2$, equivalent conditions were provided in [12] for the $k$th symmetric power $S_k I$ of an ideal $I$ to be torsion free when the projective dimension of $I$ is less than or equal to 1. In a similar vein, due to their significance for the study of the structure of finite free resolutions, see [9], torsion freeness of exterior powers of $R$-modules was investigated in [8, 9, 13], and similar necessary and sufficient conditions for torsion freeness were provided in [8] for a module $M$ of projective dimension at most 1 that has rank.

Under the same assumptions on the module $M$, in this paper we investigate more generally the torsion freeness of its Schur modules $L_{\lambda/\mu} M$, see Section 3 for the definitions. In our second main result, Theorem 7.8, we provide, for a broad family of skew shapes $\lambda/\mu$, a necessary and sufficient condition for the torsion freeness of a given Schur module $L_{\lambda/\mu} M$ in terms of the grades of the Fitting ideals of $M$. This generalizes results from [8, 12] and also from the first author’s thesis [2].

To obtain this result it was necessary, for any given map $\phi: F \to G$ of finite free $R$-modules, to investigate in detail the structure of the Schur complexes $L_{\lambda/\mu} \phi$ introduced in [1]. In our first main result, Theorem 6.1 and its corollary Theorem 6.3, we compute explicitly the radicals of the ideals of maximal minors of the differentials of $L_{\lambda/\mu} \phi$ and provide a necessary and sufficient condition for its acyclicity. These results are of independent interest, and generalize work from [11] and [10, 11]. We also describe completely in Theorem 6.4 the skew shapes $\lambda/\mu$ for which $L_{\lambda/\mu}$ is acyclic in the generic case.

The paper is organized as follows. In Section 1 we recall basic facts from commutative algebra, and establish notation. In Section 2 we review the theory of
rigid functors from \[12\]. This will be one of the main tools we use to study torsion freeness. Section 3 is devoted to recalling definitions and basic properties of Schur and Weyl modules that we will need. In Section 4 we do the same for Schur complexes. In Section 5 we introduce the notion of \textit{threshold number} and describe some of its elementary properties. In Section 6 we study in detail the structure of the differentials of the Schur complexes. We state and prove our first main result, Theorem 6.1, and obtain important corollaries. Finally, in Section 7 we use these results to provide in Theorem 7.8 a necessary and sufficient condition for the torsion freeness of a Schur module.

1. Preliminaries

Throughout this paper rings are commutative Noetherian with unit, modules are unitary and finitely generated, and chain complexes are zero in negative homological degrees.

Let \( F = (F_i, \phi_i) \) be a finite (chain) complex of free \( R \)-modules of finite rank. If \( F \) is nonzero, we set

\[
\text{start} F = \min \{ n \mid F_n \neq 0 \}
\]

\[
\text{end} F = \max \{ n \mid F_n \neq 0 \}.
\]

The \textit{expected rank} of the differential \( \phi_n \) is the integer

\[
r_n = \sum_{i \geq n} (-1)^{i-n} \text{rank } F_i.
\]

We write \( I(\phi_n) \) for the ideal \( I_{r_n}(\phi_n) \) generated by all minors of size \( r_n \) of \( \phi_n \). We say that \( F \) is \textit{acyclic} if \( H_n F = 0 \) for \( n \neq 0 \), and call \( F \) \textit{exact} if in addition \( H_0 F = 0 \).

We also need the following basic consequence of the Buchsbaum-Eisenbud acyclicity criterion.

**Proposition 1.2.** Let \( F = (F_i, \phi_i) \) be a free resolution of a module \( M \). The following are equivalent:

1. \( M \) is torsion free;
2. \( F/rF \) is a resolution of \( M/rM \) over \( R/(r) \) for every nonzero divisor \( r \in R \);

If \( F \) is finite, conditions (1) and (2) are also equivalent to

\[
\text{grade } I(\phi_k) \geq k
\]

for all \( k \geq 1 \). In that case, we also have

\[
\sqrt{I(\phi_1)} \subseteq \cdots \subseteq \sqrt{I(\phi_i)} \subseteq \cdots
\]

We also need the following basic consequence of the Buchsbaum-Eisenbud acyclicity criterion.

**Theorem 1.1** (Buchsbaum-Eisenbud Acyclicity Criterion). Let \( R \) be a ring. A complex

\[
F : 0 \to F_n \xrightarrow{\phi_n} F_{n-1} \to \cdots \to F_1 \xrightarrow{\phi_1} F_0 \to 0
\]

of finite free \( R \)-modules is acyclic if and only if

\[
\text{grade } I(\phi_k) \geq k
\]

for all \( k \geq 1 \). In that case, we also have

\[
\sqrt{I(\phi_1)} \subseteq \cdots \subseteq \sqrt{I(\phi_i)} \subseteq \cdots
\]
(3) grade $I(\phi_i) \geq i + 1$ for each $i \geq 1$.

In the coming sections, for a map of finite free modules $\phi : F \rightarrow G$ with $f = \text{rank} F < \text{rank} G$, we will need to use a certain map $\epsilon$ coming from the so-called Buchsbaum-Rim complex of $\phi$, see [7]. The map

$$
\epsilon = \epsilon(\phi) : \wedge^{f+1}G^* \otimes \wedge^f F \rightarrow G^*
$$

is defined as the following composition:

$$
\wedge^{f+1}G^* \otimes \wedge^f F \xrightarrow{\Delta \otimes 1} G^* \otimes \wedge^f F \otimes \wedge^{1+1}F \xrightarrow{1 \otimes \phi^* \otimes 1} G^* \otimes \wedge^f F^* \otimes \wedge^{f+1}F \xrightarrow{\mu} G^* \otimes R = G^*,
$$

where $\Delta$ is the diagonal map, and $\mu$ is the evaluation map. In terms of elements, we have the following formula:

$$
\epsilon(e^*_I \otimes b) = \sum_{J \subset I, |J| = f} \text{sgn}(J \subset I) (\det \phi_J) e^*_I \setminus J
$$

where $I = \{i_1, \ldots, i_{f+1}\}$ is a subset of $\{1, \ldots, g\}$, the set $\{e_1, \ldots, e_g\}$ is a basis of $G$, the set $\{e^*_1, \ldots, e^*_g\}$ is the corresponding dual basis of $G^*$, the element $e_I = e_{i_1} \wedge \cdots \wedge e_{i_{f+1}}$ is a basis element of $\wedge^{f+1}G^*$, the set $\{b_1, \ldots, b_f\}$ is a basis of $F$, $b = b_1 \wedge \cdots \wedge b_f$ is the free generator of $\wedge^f F$, the matrix $\phi_J$ is obtained from the matrix of $\phi$ for the given bases of $F$ and $G$ by taking only the columns indexed by elements of $J$, and $\text{sgn}(J \subset I)$ is the sign of the permutation of $I$ that places the elements of $J$ in the first $f$ positions.

**Remarks 1.5.** (a) It is routine to check that the composition $\phi^* \circ \epsilon = 0$, and that, if $F = 0$, then $\epsilon = \text{id}_{G^*}$.

(b) Given a commutative diagram of free $R$-modules

$$
\begin{array}{ccc}
F & \xrightarrow{\phi} & G \\
\downarrow{\psi} & & \downarrow{\gamma} \\
F' & \xrightarrow{\phi'} & G'
\end{array}
$$

with $f = \text{rank} F < \text{rank} G = \text{rank} G'$, and with $\psi$ an isomorphism, it is a routine computation to verify that the diagram

$$
\begin{array}{ccc}
\wedge^f F \otimes \wedge^{f+1}G^* & \xrightarrow{\epsilon(\phi)} & G^* \\
\uparrow{\psi^{-1} \otimes \gamma} & & \uparrow{\gamma} \\
\wedge^f F' \otimes \wedge^{f+1}G'^* & \xrightarrow{\epsilon(\phi')} & G'^*
\end{array}
$$

is also commutative, where $f = \text{rank} F = \text{rank} F'$.

The following is the key to extending the results of [12] to the case of Schur modules.

**Lemma 1.6.** Let $R$ be a ring, let $M$ be a nonzero torsion-free $R$-module with a finite free resolution

$$
0 \rightarrow F \xrightarrow{\phi} G \rightarrow M \rightarrow 0,
$$

and let $f = \text{rank} F$. Then

$$
F \xrightarrow{\phi} G \xrightarrow{\epsilon} \wedge^f F^* \otimes \wedge^{f+1}G
$$
is exact at \( G \), where \( \epsilon^* = \epsilon(\phi)^* \) is the dual of the map \( \epsilon(\phi) \) from \([1.3]\).

We use the following basic fact, see \([3, \text{Theorem 1.3.4}]\), to prove the lemma and reduce to the case where \( R \) is a field.

**Lemma 1.7.** Let \((R, m, k)\) be a local ring, and \( \psi : K \to L \) a homomorphism of finitely generated \( R \)-modules. Suppose that \( K \) is free, and let \( M \) be an \( R \)-module with \( m \) an associated prime of \( M \). Suppose that \( \psi \otimes M \) is injective. Then:

1. \( \psi \otimes k \) is injective;
2. if \( L \) is a free \( R \)-module, then \( \psi \) is injective, and \( \psi(K) \) is a free direct summand of \( L \).

**Proof of Lemma 1.7** We want to show that \( M = G/(\text{Im} \, \phi) \to \wedge^f 1 G \otimes \wedge^f F^* \) is injective, where \( \epsilon^* \) is the induced natural map.

Since \( M \) is torsion-free, it suffices to show that \( M_S \to (\wedge^f 1 G \otimes \wedge^f F^*)_S \) is injective where \( S = \{ \text{nonzero-divisors of } R \} \). Thus we may replace \( R \) with \( R_S \). Since \( \text{Ker}(\epsilon) = 0 \) and \( \text{Ker}(\epsilon^*) = 0 \) we may replace \( R \) with \( R_p \). Since \( M \) has rank, we know that \( M \) is free over \( R_p \), hence \( \phi \) is split. Therefore, we may use Lemma 1.7 and it suffices to prove the result with \( R \) replaced by the field \( R/pR \).

Since the dimension of \( \text{Im} \, \phi = f \), the dimension of \( \text{Ker} \epsilon^* \) is at least \( f \). We use the rank-nullity formula \( g = \text{dim} \, G = \text{dim} \, \text{Ker} \epsilon^* + \text{dim} \, \text{Im} \epsilon^* \) to show that it must be exactly \( f \).

Since rank \( \epsilon^* = \{ \text{determinantal rank of } \epsilon^* \} \), it remains to show a \((g - f) \times (g - f)\) submatrix of \( \epsilon^* \) with a nonzero determinant.

Since \( \phi \) is injective, it must have a nonzero \( f \times f \) minor. Without loss of generality, assume that \( d_1, \ldots, f \neq 0 \), where \( d_1, \ldots, f \) is the minor corresponding to rows \( 1, \ldots, f \). Then, with notation as in \([1.4]\), the map \( \epsilon^* \) has the matrix

\[
\begin{pmatrix}
\ldots & e_{f+1} & \ldots & e_{g} \\
\ldots & d_{1, \ldots, f} & 0 & \ldots & 0 \\
\ldots & \ast & \ast & \ast & \ast \\
\ldots & \vdots & \vdots & \vdots & \vdots \\
\ldots & \ast & \ast & \ast & \ast \\
\end{pmatrix}
\begin{pmatrix}
e_{1, \ldots, f, f+1} \otimes b^* \\
e_{1, \ldots, f, g-2} \otimes b^* \\
e_{1, \ldots, f, g-1} \otimes b^* \\
e_{1, \ldots, f, g} \otimes b^* \\
\end{pmatrix}
\]

and the result follows. \( \square \)

**Remark 1.8.** Suppose that \( \phi \) a matrix of indeterminates. In this case it follows from \([4]\) that \( \text{grade} \, I_f(\phi) \geq 2 \), so that \( M \) is torsion free by Proposition 1.2. Hence, the above lemma holds in this generic case.

2. **Rigidity**

We briefly review the notions and results on rigid functors from \([12]\) that will be used in this paper.

Let \( \text{Comp} \) be the category with objects \( \{ R, F \} \), where \( R \) is a ring and \( F \) a complex of free \( R \)-modules, and morphisms \( \{ \rho, \phi \} : \{ R, F \} \to \{ S, G \} \), where \( \rho : R \to S \) is a ring homomorphism and \( \phi : S \otimes_R F \to G \) is a morphism of complexes over \( S \). Here, \( S \otimes_R F = S \otimes_R F \) where we consider \( S \) as an \( R \)-module via \( \rho \).
For a given ring $R$, let $\text{Comp}(R)$ be the subcategory of $\text{Comp}$ with objects \{\{R, F\}\} and morphisms all morphisms of the form \{id_R, \psi\} where $\psi$ is a map of chain complexes over $R$. A ring homomorphism $\rho : R \to S$, naturally induces the base change functor $$\rho_* : \text{Comp}(R) \to \text{Comp}(S)$$ given by $$\rho_* \{R, F\} = \{S, S \otimes_R F\}$$ and $$\rho_* \{\text{id}_R, \psi\} = \{\text{id}_S, \text{id}_S \otimes \rho \psi\}.$$

**Definition 2.1.** Let $\chi$ be a subcategory of $\text{Comp}$.

(a) We say that an object $\{R, F\}$ of $\chi$ is an $R$-object of $\chi$.

(b) We write $\chi(R)$ for the subcategory of $\text{Comp}$ with objects the $R$-objects of $\chi$ and morphisms those morphisms of $\chi$ that are also morphisms of $\text{Comp}(R)$.

(c) A closed subcategory is a subcategory $\chi$ of $\text{Comp}$ which is closed under base change in the sense that for each ring homomorphism $\rho : R \to S$ the base change functor $\rho_* : \text{Comp}(R) \to \text{Comp}(S)$ induces by restriction a functor (also called a base change functor) $\rho_* : \chi(R) \to \chi(S)$.

**Definition 2.2.** Let $\chi$ be a closed subcategory, and let $\mathcal{F} : \chi \to \text{Comp}$ be a functor.

(a) We say that $\mathcal{F}$ is layered if the following conditions hold:

- For each $R$ the functor $\mathcal{F}$ restricts to a functor $\mathcal{F}_R : \chi(R) \to \text{Comp}(R)$.

In particular, for each complex $F$ of free $R$-modules we have $\mathcal{F}\{R, F\} = \{R, F_R(F)\}$, where $F_R(F)$ is again a chain complex of free $R$-modules.

- For every ring homomorphism $\rho : R \to S$ there exists an isomorphism of functors $\beta_\rho : \mathcal{F}_S \circ \rho_* \to \rho_* \circ \mathcal{F}_R$, i.e. $\mathcal{F}$ commutes with base change.

(b) We say that $\mathcal{F}$ is rigid if it is layered and also satisfies

- For any $i \geq 0$ and any object $\{R, F\}$ of $\chi$ one has $H_i \mathcal{F}_R(F) = 0$ if and only if $H_j \mathcal{F}_R(F) = 0$ for all $j \geq i$.

**Definition 2.3.** Let $\mathbb{K}$ be a ring, $R$ a polynomial ring over $\mathbb{K}$ in finitely many variables, $\chi$ a closed subcategory, and $A$ an object of $\chi(R)$.

(a) $A$ is said to be a $\chi$-generic object over $\mathbb{K}$ if for any commutative $\mathbb{K}$-algebra $S$ and any $S$-object $B$ of $\chi$ there is a homomorphism of $\mathbb{K}$-algebras $\rho : R \to S$ such that $\rho_*(A)$ and $B$ are isomorphic in $\chi(S)$.

(b) If for every ring $\mathbb{K}$ there exists a $\chi$-generic object over $\mathbb{K}$, the closed subcategory $\chi$ is said to be sufficiently generic.

The Rigidity Criterion below is used to prove the rigidity of certain complexes which are built from symmetric powers, exterior powers, or Schur functors of a module.

**Proposition 2.4** (Rigidity Criterion [12]). Let $\chi$ be a sufficiently generic closed subcategory, let $$\mathcal{F} : \chi \to \text{Comp}$$ be a layered functor, and assume that for every ring $\mathbb{K}$ there exists a $\chi$-generic over $\mathbb{K}$ object $\{R, F\}$ such that the complex $\mathcal{F}_R(F)$ is acyclic. Then the functor $\mathcal{F}$ is rigid.

The following example is essential to the proofs of our main results.
Example 2.5. (a) Let $t$ and $s$ be positive integers. Let $\mathcal{M}_{s,t}$ be the full subcategory of $\text{Comp}$ with objects $\{R,F\}$ such that rank $F_0 = s$, rank $F_1 = t$, and $F_i = 0$ for $i \geq 2$. In other words, $\mathcal{M}_{s,t}$ is the category of homomorphisms from free modules of rank $t$ to free modules of rank $s$. Let $R = \mathbb{K}[x_{i,j} | 1 \leq i \leq s, 1 \leq j \leq t]$ be the polynomial ring over $\mathbb{K}$ on the indicated set of variables. Let $\rho : R \to S$ by sending $x_{ij}$ to $y_{ij}$, where $Y = (y_{ij})$. The base change functor $\rho^*$ sends $0 \to R^t \xrightarrow{X} R^s \to 0$ to $0 \to S^t \xrightarrow{Y} S^s \to 0$.

Thus, $\{R,G\}$ is an $\mathcal{M}_{s,t}$-generic over $\mathbb{K}$ object and $\mathcal{M}_{s,t}$ is a sufficiently generic closed subcategory.

(b) Let $\tilde{\mathcal{M}}_{s,t}$ be the subcategory of $\mathcal{M}_{s,t}$ with same objects, and morphisms those morphisms $\{\rho, \phi\}$ such that $\phi_1$ is an isomorphism. In particular, isomorphisms in $\mathcal{M}_{s,t}$ are isomorphisms in $\tilde{\mathcal{M}}_{s,t}$. It is straightforward to verify that the $\mathcal{M}_{s,t}$-generic over $\mathbb{K}$ object from part (a) is also an $\tilde{\mathcal{M}}_{s,t}$-generic over $\mathbb{K}$ object, thus $\tilde{\mathcal{M}}_{s,t}$ is also a sufficiently generic closed subcategory.

For the rest of this paper, the phrase “generic case” refers to the case where $F : 0 \to F \xrightarrow{\phi} G \to 0$ is a complex of free $R$-modules with $f = \text{rank } F$ and $g = \text{rank } G$, and the pair $\{R,F\}$ is the $\mathcal{M}_{g,f}$-generic over $\mathbb{K}$ object from Example 2.5(a) for some $\mathbb{K}$.

3. Schur and Weyl Modules

We review basic facts about Schur and Weyl modules. For more details and proofs the reader is referred to [1] and to the excellent exposition in [15].

Recall that a partition is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of nonnegative integers such that $\lambda_1 \geq \lambda_2 \geq \cdots$ and only a finite number of elements in the sequence are nonzero. The weight of the partition $\lambda$ is the integer $|\lambda| = \sum_i \lambda_i$. We denote by $\bar{\lambda} = (\lambda_1^{\geq}, \lambda_2^{\geq}, \ldots)$ the conjugate to $\lambda$ partition, where $\lambda_i^{\geq}$ is the number of $\lambda_j$’s that are $\geq i$.

A skew partition $\lambda/\mu$ is a pair of partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ such that $\mu_i \leq \lambda_i$ for all $i$. In this case, we also write $\mu \subseteq \lambda$. We set $|\lambda/\mu| = \sum_i \lambda_i - \mu_i$. Let $(a_{ij})$ be the matrix defined as

$$a_{ij} = \begin{cases} 1 & \text{if } \mu_i + 1 \leq j \leq \lambda_i; \\ 0 & \text{otherwise.} \end{cases}$$
**Definition 3.1.** Let $F$ be a free $R$-module. The Schur module $L_{\lambda/\mu} F$ is the image of the following composition $d_{\lambda/\mu}(F)$ of maps:

\[
\begin{align*}
\wedge^{\lambda/\mu} &= \wedge^{\lambda/\mu} F := \wedge^{\lambda_1-\mu_1} F \otimes \wedge^{\lambda_2-\mu_2} F \otimes \cdots \\
&\quad \downarrow \Delta \otimes \Delta \otimes \cdots \\
(S_{a_{11}} F \otimes S_{a_{12}} F \otimes \cdots) \otimes (S_{a_{21}} F \otimes S_{a_{22}} F \otimes \cdots) \otimes \cdots \\
&\quad \downarrow \text{Rearrange Factors} \\
(S_{a_{11}} F \otimes S_{a_{12}} F \otimes \cdots) \otimes (S_{a_{21}} F \otimes S_{a_{22}} F \otimes \cdots) \otimes \cdots \\
&\quad \downarrow \text{Multiplication} \\
S_{\lambda/\mu} &= S_{\lambda/\mu} F := S_{\lambda_1-\mu_1} F \otimes S_{\lambda_2-\mu_2} F \otimes \cdots.
\end{align*}
\]

**Example 3.2.** Let $F$ be a free $R$-module, let $\lambda = (2, 1)$, and take $\mu = 0$. Then we have $\tilde{\lambda} = (2, 1)$ and

\[
(a_{ij}) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.
\]

The map $d_{\lambda/\mu}(F)$ is the composition

\[
\begin{align*}
\wedge^2 F \otimes F \\
&\quad \downarrow \Delta \otimes \Delta \\
(F \otimes F) \otimes F \\
&\quad \downarrow \text{Rearrange Terms} \\
(F \otimes F) \otimes F \\
&\quad \downarrow \text{Multiplication} \\
S_2(F) \otimes F
\end{align*}
\]

Given $a, b, c \in F$, under this composition we have

\[
(a \wedge b) \otimes c \mapsto (a \otimes b - b \otimes a) \otimes c = a \otimes b \otimes c - b \otimes a \otimes c \\
\mapsto a \otimes c \otimes b - b \otimes c \otimes a \\
\mapsto ac \otimes b - bc \otimes a.
\]

Note that when $\mu = 0$ and $\lambda$ is a partition of the form $\lambda = (1, 1, \ldots, 1)$ with $k$ number of 1’s, the Schur module $L_{\lambda} F = L_{\lambda/0} F$ is equal to the $k$th symmetric power $S_k F$; while for a partition $\lambda$ of the form $\lambda = (k)$, the Schur module $L_{\lambda} F$ is equal to the $k$th exterior power $\wedge^k F$.

**Definition 3.3.** The *skew-shape* $\Delta_{\lambda/\mu}$ associated with $\mu \subseteq \lambda$ is the set of integer pairs $\Delta_{\lambda/\mu} = \{(i, j) \mid a_{ij} = 1\}$. It is often visualized by replacing the nonzero entries in the matrix $(a_{ij})$ with rectangular boxes, and removing all zero entries.
For example, for $\lambda = (4, 3, 2)$ and $\mu = (3, 1)$, the skew shape looks like this:

\[
\begin{array}{ccc}
& & \\
& & \\
& & \\
\end{array}
\]

**Definition 3.4.** Let $S$ be a set, let $\lambda$ and $\mu$ be partitions with $\mu \subseteq \lambda$, and let $\Delta_{\lambda/\mu}$ be the skew-shape associated to this pair. A **tableau** of shape $\lambda/\mu$ with values in the set $S$ is a function from $\Delta_{\lambda/\mu}$ to $S$. The set of all such tableaux is denoted by $\text{Tab}_{\lambda/\mu}(S)$.

**Example 3.5.** Let $\lambda = (4, 3, 2)$, $\mu = (3, 1)$, and $S = \{a, b, c, d\}$. Then $\Delta_{\lambda/\mu}$ looks like

\[
\begin{array}{ccc}
& & \\
& & \\
& & \\
\end{array}
\]

In terms of coordinates we have $\Delta_{\lambda/\mu} = \{(1, 4), (2, 2), (2, 3), (3, 1), (3, 2)\}$. Now define a map to $S$ by

\[
\begin{align*}
(1, 4) & \mapsto a \\
(2, 2) & \mapsto b \\
(2, 3) & \mapsto c \\
(3, 1) & \mapsto c \\
(3, 2) & \mapsto d
\end{align*}
\]

This function is a tableau of shape $\lambda/\mu$ with values in $S$. We may think of it as a way to fill the tableau with values in $S$ as seen below

\[
\begin{array}{ccc}
& & a \\
& b & c \\
& c & d \\
\end{array}
\]

**Definition 3.6.** Let $F$ be a free $R$-module, and let $S$ be a subset of $F$. Given an element $T \in \text{Tab}_{\lambda/\mu} S$, we associate to it a simple tensor $Z_T \in \wedge^{\lambda/\mu} F$, whose component in $\wedge^{\lambda/\mu} F$ is the exterior product of the elements in the $j$th row of $T$. For example, the tableau above corresponds to the simple tensor $a \otimes (b \wedge c) \otimes (c \wedge d)$ in $\wedge^{\lambda/\mu} F = \wedge^1 \otimes \wedge^2 \otimes \wedge^2$.

The Weyl module $K_{\lambda/\mu} F$ of a free module $F$ is defined in a dual manner to that of the Schur module, and is the image of a map $d'_{\lambda/\mu} : D_{\lambda/\mu} F \to \wedge^{\lambda/\mu} F$. Explicitly,
the Weyl module is defined as the image of the following composition of maps:

\[
D_{\lambda_1 - \mu_1} F \otimes D_{\lambda_2 - \mu_2} F \otimes \cdots \\
\downarrow \Delta \otimes \Delta \otimes \cdots \\
(D_{a_{11}} F \otimes D_{a_{12}} F \otimes \cdots) \otimes (D_{a_{21}} F \otimes D_{a_{22}} F \otimes \cdots) \otimes \cdots \\
\| \\
(\wedge^{a_{11}} F \otimes \wedge^{a_{12}} F \otimes \cdots) \otimes (\wedge^{a_{21}} F \otimes \wedge^{a_{22}} F \otimes \cdots) \otimes \cdots \\
\downarrow \text{Rearrange Factors} \\
(\wedge^{a_{11}} F \otimes \wedge^{a_{12}} F \otimes \cdots) \otimes (\wedge^{a_{21}} F \otimes \wedge^{a_{22}} F \otimes \cdots) \otimes \cdots \\
\downarrow \text{Multiplication} \\
\wedge^{\lambda_1 - \mu_1} F \otimes \wedge^{\lambda_2 - \mu_2} F \otimes \cdots.
\]

Note that the Schur module of a free module \( F \) is zero if and only if a row in the skew partition \( \lambda/\mu \) is longer than the rank of \( F \), whereas the Weyl module is zero if and only if a column is longer than the rank.

To conclude this section, we recall how to extend the definition of the Schur module \( L_{\lambda/\mu} M \) to the case when \( M \) which is not necessarily free.

Let \( p_i = \lambda_i - \mu_i \), let \( q = \max \{ i \mid p_i \neq 0 \} \), and define the functor \( \wedge(\lambda/\mu)^+ \) to be the direct sum

\[
\sum_{i=1}^{q-1} \sum_{t=p_i-\mu_i+1}^{\lambda_i+1-\mu_i+1} \wedge^{p_1} \otimes \cdots \otimes \wedge^{p_i-1} \otimes \wedge^{p_i+t} \otimes \wedge^{p_{i+1}-t} \otimes \wedge^{p_{i+2}} \otimes \cdots \otimes \wedge^{p_q} 
\]

Define a map

\[
\square_{\lambda/\mu} M : \wedge(\lambda/\mu)^+ M \to \wedge^{\lambda/\mu} M
\]

as follows. If \( q < 2 \), set \( \wedge(\lambda/\mu)^+ = 0 \). If \( q = 2 \), the map

\[
\square_{\lambda/\mu} M : \sum_{t=\mu_1-\mu_2+1}^{\lambda_2-\mu_2} \wedge^{p_1+t} M \otimes \wedge^{p_2-t} M \to \wedge^{p_1} M \otimes \wedge^{p_2} M
\]

is defined on the component \( \wedge^{p_1+t} M \otimes \wedge^{p_2-t} M \) as the composition

\[
\wedge^{p_1+t} M \otimes \wedge^{p_2-t} M \xrightarrow{\Delta \otimes 1} \wedge^{p_1} M \otimes \wedge^{t} M \otimes \wedge^{p_2-t} M \xrightarrow{1 \otimes \wedge} \wedge^{p_1} M \otimes \wedge^{p_2} M
\]

where \( \Delta \) and \( \wedge \) are the diagonal and exterior multiplication maps, respectively. If \( q > 2 \), set \( \lambda^i = (\lambda_i, \lambda_{i+1}) \), \( \mu^i = (\mu_i, \mu_{i+1}) \) and define \( \square_{\lambda/\mu} M \), to be

\[
\sum_{i=1}^{q-1} 1_1 \otimes \cdots \otimes 1_{i-1} \otimes \square_{\lambda^i/\mu^i} M \otimes 1_{i+2} \otimes \cdots \otimes 1_q.
\]

Now we have

**Definition 3.7 (II).** Let \( M \) be an \( R \)-module, and let \( \mu \subseteq \lambda \) be partitions. The \( R \)-module \( L_{\lambda/\mu} M = \text{Coker}(\square_{\lambda/\mu}) \) is called the **Schur module** of \( M \).
4. Schur Complexes

In this section we recall some basic properties of Schur complexes. All definitions and results presented here are due to [1], to which we refer the reader for more details and proofs.

Let \( M \) be a module over \( R \) and let \( k \geq 0 \). Let \( \phi : F \rightarrow G \) be a homomorphism of finite free \( R \)-modules with cokernel \( M \), let \( f = \text{rank} \ F \), and \( g = \text{rank} \ G \). From the definition, it is straightforward that the maps below have cokernels \( S^k M \) and \( \wedge^k M \) respectively:

\[
\begin{align*}
(4.1) \quad F \otimes S^{k-1}G & \rightarrow S^k G, \quad f \otimes u \mapsto \phi(f)u, \\
(4.2) \quad F \otimes \wedge^{k-1}G & \rightarrow \wedge^k G, \quad f \otimes v \mapsto \phi(f) \wedge v,
\end{align*}
\]

where \( f \in F, u \in S^{k-1}G \), and \( v \in \wedge^{k-1}G \).

For each \( t \), we have a complex \( S^t \phi \), referred to by [7] as the \( t \)th graded strand of the Koszul complex of \( \phi \):

\[
\begin{align*}
0 \rightarrow \wedge^n F \otimes S_{t-n}G & \xrightarrow{d_n} \wedge^{n-1} F \otimes S_{t-n+1}G \xrightarrow{d_{n-1}} \ldots \xrightarrow{d_2} F \otimes S_{t-1}G \xrightarrow{d_1} S_t G \rightarrow 0
\end{align*}
\]

with \( n = \min \{ \text{rank} F, t \} \) where

\[
d_j(f_1 \wedge f_2 \wedge \cdots \wedge f_j \otimes m) = \sum_{r=1}^{j} (-1)^{r-1} f_1 \wedge \cdots \hat{f}_r \wedge \cdots \wedge f_j \otimes \phi(f_r)m.
\]

We have by (4.1) that \( H_0(S_t \phi) = S_t M \). Similarly, Lebelt [9] studied a complex

\[
\begin{align*}
\wedge^t \phi : \quad 0 \rightarrow D_t F \rightarrow D_{t-1} F \otimes G \rightarrow \cdots \rightarrow \wedge^t G \rightarrow 0
\end{align*}
\]

where for \( 0 \leq i \leq t \) the free module \( D_i F = S_i (F^*)^\ast \) is the \( i \)th divided power of \( F \), see [7] Section A2.4], and the differential is given by

\[
b_1^{(a_1)} b_2^{(a_2)} \ldots b_f^{(a_f)} \otimes v \mapsto \sum_{r=1}^{f} b_1^{(a_1)} \ldots b_r^{(a_r-1)} \ldots b_f^{(a_f)} \otimes (\phi(b_r) \wedge v).
\]

By [12], this complex has \( \wedge^t M \) as its zeroth homology.

Set

\[
\wedge^{\lambda/\mu} \phi = \wedge^{\lambda_1-\mu_1} \phi \otimes \wedge^{\lambda_2-\mu_2} \phi \otimes \ldots
\]

and

\[
S_{\lambda/\mu} \phi = S_{\lambda_1-\mu_1} \phi \otimes S_{\lambda_2-\mu_2} \phi \otimes \ldots.
\]

**Definition 4.3.** The Schur complex \( L_{\lambda/\mu} \phi \) is the image of the map

\[
d_{\lambda/\mu} \phi : \wedge^{\lambda/\mu} \phi \rightarrow S_{\lambda/\mu} \phi
\]
defined as the composition

\[
\begin{align*}
\Lambda^{\lambda_1-\mu_1} \phi \otimes \Lambda^{\lambda_2-\mu_2} \phi \otimes \cdots \\
\downarrow \Delta \otimes \Delta \otimes \cdots \\
(\Lambda^{a_{11}} \phi \otimes \Lambda^{a_{12}} \phi \otimes \cdots) \otimes (\Lambda^{a_{21}} \phi \otimes \Lambda^{a_{22}} \phi \otimes \cdots) \otimes \cdots \\
\downarrow \\
(S_{a_{11}} \phi \otimes S_{a_{12}} \phi \otimes \cdots) \otimes (S_{a_{21}} \phi \otimes S_{a_{22}} \phi \otimes \cdots) \otimes \cdots \\
\downarrow \text{Rearrange Factors} \\
(S_{a_{11}} \phi \otimes S_{a_{21}} \phi \otimes \cdots) \otimes (S_{a_{12}} \phi \otimes S_{a_{22}} \phi \otimes \cdots) \otimes \cdots \\
\downarrow \text{Multiplication} \\
S_{\lambda_1-\mu_1} \phi \otimes S_{\lambda_2-\mu_2} \phi \otimes \cdots
\end{align*}
\]

where \(\Delta\) is the diagonal map.

**Remark 4.4.** From the definition of the complex \(L_{\lambda/\mu} \phi\) it is straightforward that it commutes with base change, hence induces a layered functor \(L_{\lambda/\mu} : M_{g,f} \to \text{Comp}\).

**Definition 4.5.** Let \(S\) be a totally ordered set, let \(X\) be a subset of \(S\), and let \(T \in \text{Tab}_{\lambda/\mu}(S)\).

(a) The tableau \(T\) is said to be row-standard mod \(X\) if each row of \(T\) is non-decreasing, and if, when repeats occur in a row, they occur only among elements of \(X\). Call \(T\) column-standard mod \(X\) if each column is non-decreasing, and if, when repeats occur in a column, they occur only among elements in the complement of \(X\). Call \(T\) standard mod \(X\) if \(T\) is both row- and column-standard mod \(X\).

(b) Let \(\phi : F \to G\) be a map of finite free \(R\)-modules, and suppose that \(X\) is a subset of \(F\) and \(S \setminus X\) is a subset of \(G\). When \(T\) is standard mod \(X\), write \(Z_T\) for the simple tensor in \(\Lambda^{\lambda/\mu} \phi\) whose component in \(\Lambda^{\lambda_i-\mu_i} \phi\) is given by the product in the divided powers algebra of \(F\) of the elements of \(X\) in row \(j\) (a repeated element is given its corresponding divided power) tensored by the exterior product of the elements from \(S \setminus X\) that are in row \(j\) of \(T\); see the examples below.

**Example 4.6.** Let \(F\) and \(G\) be free modules with bases \(X = \{a,b\}\) and \(Y = \{x,y,x\}\), respectively. Let \(S = \{a,b,x,y,z\}\) be ordered by \(a < b < x < y < z\), let \(\lambda = (4,3)\) and let \(\mu = (2,1)\). The following are examples of standard tableaux \(T\) mod \(X\) and their corresponding simple tensors \(Z_T\) in \(\Lambda^{\lambda/\mu} \phi = \Lambda^2 \phi \otimes \Lambda^2 \phi\):

\[
T = \begin{array}{c}
a \\
a \\
b \\
a \\
b \end{array}
\]

\[
Z_T = (a^2 \otimes 1) \otimes (ab \otimes 1) \in (D_2 F \otimes \Lambda^0 G) \otimes (D_2 F \otimes \Lambda^0 G) \subset \Lambda^2 \phi \otimes \Lambda^2 \phi;
\]

\[
T = \begin{array}{c}
x \\
y \\
a \\
x \\
\end{array}
\]

\[
Z_T = (1 \otimes x \otimes y) \otimes (a \otimes x) \in (D_0 F \otimes \Lambda^2 G) \otimes (D_1 F \otimes \Lambda^1 G) \subset \Lambda^2 \phi \otimes \Lambda^2 \phi;
\]
Theorem 4.7. Let $\mu$ and $\lambda$ be partitions with $\mu \subseteq \lambda$, and let $\phi : F \to G$ be a map of free $R$-modules. Let $X$ and $Y$ be bases for $F$ and $G$ respectively, and let $S = X \sqcup Y$ be totally ordered. Then the set

$$\{ \lambda/\mu \mu(Z_T) \mid T \in \text{Tab}_{\lambda/\mu}(S) \text{ is standard mod } X \}$$

is a basis for $L_{\lambda/\mu}$. 

We will also need the following results.

Theorem 4.8. If $\phi = \phi_1 \oplus \phi_2$, where $\phi_1$ is an isomorphism, then the natural inclusion map $L_{\lambda/\mu} \phi_2 \to L_{\lambda/\mu} \phi$ splits and yields an isomorphism $L_{\lambda/\mu} \phi \cong L_{\lambda/\mu} \phi_2 \oplus E$ where $E$ is a contractible chain complex.

Theorem 4.9. Let $\phi : F \to G$ be a map of free $R$-modules and let $\mu \subseteq \lambda$ be partitions. Define $(L_{\lambda/\mu} \phi)_j$ to be the component in homological degree $j$ of the complex $L_{\lambda/\mu} \phi$. There is a natural filtration on $(L_{\lambda/\mu} \phi)_j$ whose associated graded module is

$$\sum_{\mu \leq \gamma \leq \lambda/\mu \lambda - |\gamma| = j} L_{\gamma/\mu} G \otimes K_{\lambda/\gamma} F.$$ 

In particular, $(L_{\lambda/\mu})_0 = L_{\lambda/\mu} G$.

Corollary 4.10. Let $\nu''$ be the partition with $\nu'' = \min\{\lambda_i, \mu_i + \text{rank } G\}$, and let $\nu'$ be the partition with $\nu' = \max\{\tilde{\lambda}_i, \tilde{\lambda}_i - \text{rank } F\}$.

(a) $(L_{\lambda/\mu} \phi)_j \neq 0$ if and only if for some $\gamma$ with $\mu \subseteq \gamma \subseteq \lambda$ and $|\lambda| - |\gamma| = j$ we have $\tilde{\lambda}_i - \tilde{\gamma}_i \leq \text{rank } F$ for all $i$, and $\tilde{\gamma}_i - \tilde{\mu}_i \leq \text{rank } G$ for all $t$, or equivalently, $\nu' \subseteq \gamma \subseteq \nu''$.

(b) In particular, $L_{\lambda/\mu} \phi \neq 0$ if and only if $\nu' \subseteq \nu''$. In that case:

(c) The unique maximal $j$ with $(L_{\lambda/\mu} \phi)_j \neq 0$ is obtained when taking $\gamma = \nu'$, hence equals

$$|\lambda| - |\nu'| = |\lambda| - \sum_i \max\{\tilde{\mu}_i, \tilde{\lambda}_i - \text{rank } F\}.$$ 

The unique minimal $j$ with $(L_{\lambda/\mu} \phi)_j \neq 0$ is obtained when taking $\gamma = \nu''$, hence equals

$$|\lambda| - |\nu''| = |\lambda| - \sum_i \min\{\lambda_i, \mu_i + \text{rank } G\}.$$ 

(d) $(L_{\lambda/\mu} \phi)_j \neq 0$ if and only if

$$|\lambda| - \sum_t \min\{\lambda_t, \mu_t + \text{rank } G\} \leq j \leq |\lambda| - \sum_i \max\{\tilde{\mu}_i, \tilde{\lambda}_i - \text{rank } F\}.$$ 

Proposition 4.11. Let $\pi : G \to M = G/\text{Im } \phi$ be the canonical projection. Then

$$(L_{\lambda/\mu} \phi)_1 \xrightarrow{d_1} L_{\lambda/\mu} G \xrightarrow{L_{\lambda/\mu}(\pi)} L_{\lambda/\mu} M \to 0$$

is exact, where $d_1$ is the differential of $L_{\lambda/\mu} \phi$. In particular, $H_0(L_{\lambda/\mu} \phi) = L_{\lambda/\mu} M$. 

\[
T = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\] 

\[
Z_T = (ab \otimes 1) \otimes (1 \otimes y \wedge z) \in (D_2 F \otimes \wedge^0 G) \otimes (D_0 F \otimes \wedge^2 G) \subset \wedge^2 \phi \otimes \wedge^2 \phi.
\]
5. More on Schur Complexes

The following notation and terminology will be useful for us in the sequel. Our main point here is the introduction of the \textit{threshold number} in Definition 5.3.

\textbf{Definition 5.1.} \((a)\) We call the integer \(W(\lambda/\mu) = \max\{\lambda_i - \mu_i \mid i \geq 1\}\) the width of \(\lambda/\mu\), and we call \(H(\lambda/\mu) = \max\{\lambda_i - \mu_i \mid i \geq 1\}\) the height of \(\lambda/\mu\).

\((b)\) For each integer \(n\) define the partition \(\nu'' = \nu''(\lambda, \mu, n)\) by setting

\[\nu_i'' = \min\{\lambda_i, \max\{\mu_i, \mu_i + n\}\},\]

and define the partition \(\nu' = \nu'(\lambda, \mu, n)\) by setting

\[\nu_i' = \max\{\tilde{\lambda}_i, \tilde{\lambda}_i - n\}\].

\textbf{Remark 5.2.} It is immediate from the definition that we have the following inclusions and equalities:

\[\lambda = \cdots = \nu'(\lambda, \mu, -1) = \nu'(\lambda, \mu, 0) \geq \cdots \geq \nu'(\lambda, \mu, H) = \nu'(\lambda, \mu, H + 1) = \cdots = \mu;\]

\[\mu = \cdots = \nu''(\lambda, \mu, -1) = \nu''(\lambda, \mu, 0) \leq \cdots \leq \nu''(\lambda, \mu, W) = \nu''(\lambda, \mu, W + 1) = \cdots = \lambda;\]

where \(H = H(\lambda/\mu)\) and \(W = W(\lambda/\mu)\). Furthermore, if \(\mu \neq \lambda\) then all inequalities above are strict.

\textbf{Definition 5.3.} \((a)\) For any integers \(f\) and \(g\) we set

\[T_{\lambda/\mu}(f, g) = \max\{t \mid \nu'(\lambda, \mu, f - t) \subseteq \nu''(\lambda, \nu, g - t)\},\]

and call this the \textit{threshold number} of \(\lambda/\mu\) with respect to the pair \((f, g)\).

\((b)\) For any integer \(n\) we set

\[l_n = |\nu'(\lambda, \mu, n - 1)| - |\nu'(\lambda, \mu, n)| \quad \text{and} \quad k_n = |\nu''(\lambda, \mu, n)| - |\nu''(\lambda, \mu, n - 1)|.\]

As a straightforward consequence of the definitions we have the following basic properties:

\textbf{Remarks 5.4.} Let \(H = H(\lambda/\mu)\), let \(W = W(\lambda/\mu)\), and let \(T = T_{\lambda/\mu}(f, g)\).

\((a)\) \(T_{\lambda/\mu}(f, g) \geq 0\) if and only if \(\nu'(\lambda, \mu, f) \subseteq \nu''(\lambda, \mu, g)\).

\((b)\) \(T_{\lambda/\mu}(f - 1, g - 1) = T_{\lambda/\mu}(f, g) - 1\).

\((c)\) \(W \geq g - T\). Indeed, by the definition of \(T\) we have \(\nu'(\lambda, \mu, f - T) \subseteq \nu''(\lambda, \mu, g - T)\), and \(\nu'(\lambda, \mu, f - T - 1) \nsubseteq \nu''(\lambda, \mu, g - T - 1)\). If \(W < g - T\) then \(\nu''(\lambda, \mu, g - T - 1) = \nu''(\lambda, \mu, g - T) = \lambda\) and so \(\nu'(\lambda, \mu, f - T - 1) \subseteq \nu''(\lambda, \mu, g - T - 1)\), a contradiction.

\((d)\) Either \(T_{\lambda/\mu}(0, g) < 0\) or \(T_{\lambda/\mu}(0, g) = g - W \geq 0\).

\((e)\) Either \(T_{\lambda/\mu}(f, 0) < 0\) or \(T_{\lambda/\mu}(f, 0) = f - H \geq 0\).

\((f)\) If \(n \geq 1\) then \(k_n\) is the number of \(t\)’s such that \(\mu_k + n \leq \mu_t\), and \(l_n\) is the number of \(t\)'s such that \(\tilde{\mu}_t \leq \tilde{\lambda}_t - n\). In particular, \(\lambda\) differs from \(\mu\) in exactly \(k_1\) of its rows and in exactly \(l_1\) of its columns.

\((g)\) \(k_n = 0\) for \(n \leq 0\) and \(\mu_0 \geq W\), and \(l_n = 0\) for \(n \leq 0\) and \(n \leq H\).

\((h)\) We have \(l_1 \geq \cdots \geq l_{H-1}\) and \(k_1 \geq \cdots \geq k_{W-1}\).

\((i)\) For each \(n\) we have

\[\sum_{t \geq n + 1} k_t = |\lambda| - |\nu''(\lambda, \mu, n)| \quad \text{and} \quad \sum_{t \leq n} l_t = |\lambda| - |\nu'(\lambda, \mu, n)|.\]
In particular,
\[ |\lambda| - |\mu| = \sum_{n=1}^{W} k_n = \sum_{n=1}^{H} l_n. \]

We can now reformulate Corollary 4.10 as follows.

**Corollary 5.5.** The complex \( L_{\lambda/\mu}\phi \) is nonzero if and only if
\[ \nu'(\lambda, \mu, \text{rank } F) \subseteq \nu''(\lambda, \mu, \text{rank } G), \]
which is if and only if \( T_{\lambda/\mu}(\text{rank } F, \text{rank } G) \geq 0 \). In that case the component \( (L_{\lambda/\mu}\phi)_j \) is nonzero if and only if
\[ \sum_{t\geq 1+\text{rank } G} k_t \leq j \leq \sum_{1\leq t \leq \text{rank } F} l_t. \]

We will need the following related basic observation.

**Proposition 5.6.** Suppose \( \mu \subseteq \lambda \) and let \( \phi : F \rightarrow G \) be a generic map. The \( j \)-th differential of \( L_{\lambda/\mu}\phi \) satisfies \( \delta_j \neq 0 \) whenever \( \text{start } L_{\lambda/\mu}\phi < j \leq \text{end } L_{\lambda/\mu}\phi \).

**Proof.** Let \( f = \text{rank } F \) and \( g = \text{rank } G \), let \( X = \{x_1, \ldots, x_f\} \) be a basis for \( F \), and let \( Y = \{y_1, \ldots, y_g\} \) be a basis for \( G \). We order the disjoint union \( S = X \sqcup Y \) by setting \( y_1 < \cdots < y_g < x_1 < \cdots < x_f \).

If \( L_{\lambda/\mu}\phi = 0 \) the proposition is trivially true. Suppose \( (L_{\lambda/\mu}\phi)_j \neq 0 \). Thus there is a partition \( \gamma \) such that \( |\lambda| - |\gamma| = j \) and
\[ \nu' = \nu'(\lambda, \mu, f) \subseteq \gamma \subseteq \nu''(\lambda, \mu, g) = \nu''. \]

It suffices to show that if \( j \neq \text{start } L_{\lambda/\mu}\phi \) then we can specialize \( \phi \) to a map \( \psi : F \rightarrow G \) so that for the \( j \)-th differential \( \delta_j \) of \( L_{\lambda/\mu}\psi \) we have \( \delta_j \neq 0 \). So, assume \( \gamma < \nu'' \), and take \( \psi \) to be the map that sends \( x_1 \) to \( y_g \), and all the other \( x_i \) to 0. Consider the basis element \( d_{\lambda/\mu}(Z_{\Gamma}) \) of \( L_{\lambda/\mu}(\psi)_j \), where \( \Gamma \) is the standard mod \( X \) tableau of shape \( \lambda/\mu \) given by
\[ \Gamma(i,j) = \begin{cases} x_{i-g_i} & \text{if } (i,j) \in \lambda/\gamma; \\ y_{g_i-m_i} & \text{if } (i,j) \in \gamma/\mu, \end{cases} \]
and \( Z_{\Gamma} \) is the corresponding basis element of \( \wedge^{\lambda/\mu}\psi \). Let \( J = \{\tilde{g}_j + 1 \mid j \geq 1\} \).

Note that \( 1 \in J \), and that \( 2 \leq i \in J \) exactly if \( \gamma_i < \gamma_{i-1} \). Let
\[ A = \{i \in J \mid \gamma_i < \min(\lambda_i, \mu_i + g)\}. \]

Note that, as \( \gamma < \nu'' \), the set \( A \) is nonempty. Therefore, for each \( i \in A \), by adding one box to the \( i \)-th row of \( \gamma \) we obtain a new partition \( \gamma^{(i)} \). As \( \gamma < \gamma^{(i)} \leq \nu'' \), we obtain a corresponding standard mod \( X \) tableau \( \Gamma^{(i)}(p,q) \) of shape \( \lambda/\mu \) by setting
\[ \Gamma^{(i)}(p,q) = \begin{cases} y_g & \text{if } (p,q) = (i, \gamma_i + 1); \\ \Gamma(p,q) & \text{otherwise}. \end{cases} \]

Now it is straightforward to check that the differential \( d_j \) of \( \wedge^{\lambda/\mu}\psi \) sends the basis element \( Z_{\Gamma} \) to
\[ d_j(Z_{\Gamma}) = \sum_{i \in A} (-1)^{\gamma_i} Z_{\Gamma^{(i)}}, \]
hence \( \delta_j(d_{\lambda/\mu}(Z_{\Gamma})) = d_{\lambda/\mu}(d_j(Z_{\Gamma})) = \sum_{i \in A} (-1)^{\gamma_i} d_{\lambda/\mu}(Z_{\Gamma^{(i)}}) \neq 0 \) as the elements \( d_{\lambda/\mu}(Z_{\Gamma^{(i)}}) \) are part of a basis for \( L_{\lambda/\mu}\psi \). Thus \( \delta_j \neq 0 \) as desired. \( \Box \)
6. Acyclicity of Schur complexes

Here is the first main result of the paper.

**Theorem 6.1.** Let \( \mu \subseteq \lambda \), let \( \phi : F \rightarrow G \) be a map of finite free \( R \)-modules, let \( f = \text{rank } F \), \( g = \text{rank } G \), and let \( T = T_{\lambda/\mu}(f,g) \). 

1. Suppose that \( f - g < H(\lambda/\mu) \). Then \( r_n \geq 0 \) for each \( n \geq 1 \), and:
   
   (a) For \( \sum_{i \geq 1+g-T} k_i < n \leq \sum_{i \leq f-T} l_i \) we have
   
   \[
   \sqrt{I(d_n)} = \sqrt{I_{1+T}(\phi)};
   \]

   (b) For \( \sum_{i \leq j-1} l_i < n \leq \sum_{i \leq j} l_i \) with \( j \geq f - T + 1 \) we have
   
   \[
   \sqrt{I(d_n)} = \sqrt{I_{-j+1}(\phi)};
   \]

   (c) For \( \sum_{i \geq 2+g-j} k_i < n \leq \sum_{i \geq 1+g-j} k_i \) with \( j \leq T \) we have
   
   \[
   \sqrt{I(d_n)} = \sqrt{I_j(\phi)};
   \]

2. Suppose that \( f - g \geq H(\lambda/\mu) \). Then \( \text{end } L_{\lambda/\mu} \phi = |\lambda/\mu| \), and:

   (a) For \( 1 \leq n \leq |\lambda/\mu| \) with \( |\lambda/\mu| - n \) even we have \( r_n > 0 \) and
   
   \[
   I(d_n) = 0;
   \]

   (b) For \( 1 \leq n \leq |\lambda/\mu| \) with \( |\lambda/\mu| - n \) odd we have
   
   \[
   \sqrt{I(d_n)} \geq \sqrt{I_0(\phi)};
   \]

where \( r_n \) are the expected ranks of the differentials \( d_n \) in \( L_{\lambda/\mu}(\phi) \).

**Proof.** We proceed by induction on \( m = \min(f,g) \). When \( m = 0 \) we have either \( f = 0 \) or \( g = 0 \). In the former case \( L_{\lambda/\mu} \phi \) is concentrated in homological degree 0 and by Remark 5.4(d) either \( T < 0 \) or \( T = g - W \geq 0 \), thus the theorem is trivially true. In the latter case, \( L_{\lambda/\mu} \phi \) is concentrated in homological degree \( |\lambda/\mu| \) and by Remark 5.3(e) either \( T < 0 \) or \( T = f - H \geq 0 \), thus, in view of Remark 5.3(a) and Corollary 6.2, the theorem is again trivially true.

Now assume \( m \geq 1 \). By a standard argument, we may assume that the ring \( R \) is a polynomial ring over \( \mathbb{Z} \) with indeterminates the entries of \( \phi \). Note that, by Theorem 1.3 after inverting any \( m \) by \( m \) minor of \( \phi \) we obtain

\[
L_{\lambda/\mu} \phi \cong L_{\lambda/\mu} \phi' \oplus E
\]

where \( E \) is a contractible complex of free modules, hence split exact, and \( \phi' : F' \rightarrow G' \) is a map of free modules with \( f' = \text{rank } F' = f - m \) and \( g' = \text{rank } G' = g - m \). Since \( f' - g' = f - g \) and the expected rank of \( d_n \) is at least the same as the expected rank \( r_n' \) for the differential \( d_n' \) of \( L_{\lambda/\mu} \phi' \), all expected rank inequalities claimed in the theorem follow from our induction hypothesis. Since \( R \) is a domain and \( I(d_n') \) is the localization of \( I(d_n) \), our induction hypothesis also yields (2) part (a). When in case (2) we have \( g = m \), and \( L_{\lambda/\mu} \phi' \) is nonzero and concentrated in homological degree \( |\lambda/\mu| \), in particular end \( L_{\lambda/\mu} \phi = |\lambda/\mu| \). Thus in (2) part (b) we have \( r_n' < 0 \) and so \( I(d'_n) \) equals the whole ring; therefore the radical of \( I(d_n) \) contains \( I_g(\phi) \), completing the proof of (2) part (b).

It remains to prove the radical equalities claimed in case (1). We note that they are trivially satisfied when \( T < 0 \) because then \( L_{\lambda/\mu} \phi = 0 \) and so \( I(d_n) = R \) for all \( n \), and we assume for the rest of this proof that \( T \geq 0 \) and that we are
in case (1). Thus, we already know that the expected rank $r_n$ for the differential $d_n$ of $L_{\lambda/\mu}\phi$ is nonnegative for each $n \geq 1$. Furthermore, $L_{\lambda/\mu}\phi'$ is concentrated in homological degree 0, hence $L_{\lambda/\mu}\phi$ becomes acyclic after localization and thus for each $n \geq 1$ the expected rank $r_n$ is the same as the determinantal rank of $d_n$. Therefore from Proposition 5.6 we obtain that $r_n = 0$ for $1 \leq n \leq \text{start } L_{\lambda/\mu}\phi$ and $n > \text{end } L_{\lambda/\mu}\phi$, and $r_n > 0$ otherwise. Now Corollary 5.5 yields $r_n = 0$ and $I(d_n) = R$ for $1 \leq n \leq \sum_{1+g \leq i} k_i$ and for $n > \sum_{1 \leq t \leq \lambda} l_t$; hence (1) part (b) is true for $j \geq f + 1$ and (1) part (c) is true for $j \leq 0$. Thus for the rest of this proof we will also assume that $\sum_{1+g \leq i} k_i < n \leq \sum_{1 \leq t \leq \lambda} l_t$. In particular, we also have $r_n > 0$, and by construction $I(d_n) \subseteq I(\phi)$.

Now let $P$ be a prime ideal of $R$. If $I(\phi) \subseteq P$ then $P$ contains the radicals of both $I(d_n)$ and the corresponding ideal of minors of $\phi$ in each of the cases (a), (b), and (c). Therefore, we may assume without loss of generality that $I(\phi)$ is not contained in $P$, and it suffices to prove our statements after localization at $P$.

In that case $\phi_P = \phi_1 \oplus \phi''$ with $\phi_1$ an isomorphism, and $\phi'' : F'' \rightarrow G''$ with $f'' = \text{rank } F'' = f - 1$ and $g'' = \text{rank } G'' = g - 1$. By Theorem 6.3 we have for each $n$ that $I(d_n) \subseteq I(d''_n)$ where $d''_n$ are the maps in $L_{\lambda/\mu}\phi''$. In view of Remark 5.4(b) the desired conclusion is now immediate from our induction hypothesis applied to $L_{\lambda/\mu}\phi''$.

The following theorem generalizes [1] Theorem 5.1.17, [10] Theorem 2.1, and [2] Theorem 5.0.6.

**Theorem 6.3.** Let $\mu \subseteq \lambda$, let $\phi : F \rightarrow G$ be a map of finite free $R$-modules, let $f = \text{rank } F$, let $g = \text{rank } G$, and let $T = T_{\lambda/\mu}(f,g)$. The following are equivalent:

1. The Schur complex $L_{\lambda/\mu}\phi$ is acyclic;
2. grade $I_{f-1}(\phi) \geq \sum_{1 \leq j \leq \lambda} l_j$ for each $j \geq \max(1, f - T)$.

**Proof.** When $T < 0$ we have $L_{\lambda/\mu}\phi = 0$ and both conditions of the theorem are trivially satisfied. When $T \geq 0$ the desired conclusion is immediate from Theorem 6.1 and the Buchsbaum-Eisenbud acyclicity criterion. We say that $\lambda/\mu$ is a shift by $(s,t) \in \mathbb{N}^2$ of a partition $\gamma$ if we have $\mu_j = t$ and $\lambda_j = t + \gamma_{j-s}$ for $s < j \leq s + \gamma_1$, and $\mu_j = \lambda_j$ otherwise. The following acyclicity criterion holds in the generic case and generalizes in a different way [1] Theorem 5.1.17.

**Theorem 6.4.** Let $\mu \subseteq \lambda$, let $\phi : F \rightarrow G$ be a generic map with $f = \text{rank } F \geq 1$ and $g = \text{rank } G \geq 1$. The following are equivalent:

1. The Schur complex $L_{\lambda/\mu}\phi$ is acyclic and nonzero.
2. Either
   (a) $\lambda$ differs from $\mu$ in at most $g - f + 1$ columns;
   or
   (b) $\lambda/\mu$ is the shift of a partition $\gamma$ with $g - f + 1 < \gamma_1 \leq g$ and $\tilde{\gamma}_{\gamma_1} > \gamma_1 - (g - f + 1)$.

**Proof.** First we prove that (1) implies (2). The acyclicity and nonzero assumption, together with Proposition 5.6 and the fact that by construction $I_1(d_n) \subseteq I_1(\phi)$ for each $n$, imply that $L_{\lambda/\mu}\phi \neq 0$. Also, $T = T_{\lambda/\mu}(f,g) \geq 0$. Furthermore, by Theorem 6.1 we must have $\sqrt{I(d_n)} \subseteq \sqrt{I(d_2)} \subseteq \ldots$. Thus Theorem 6.3(a,c) yields $W = W(\lambda/\mu) \leq g - T$, whence $W = g - T$ by Remark 5.4(c), and so
\( \nu''(\lambda, \mu, g-T) = \lambda \). Consider a box in the earliest possible row \( i \) of \( \nu'(\lambda, \mu, f-T-1) \) that is not in \( \nu''(\lambda, \mu, W-1) \). Thus \( \lambda_i - \mu_i = W \), and our box is in column \( \lambda_i \). Also, our box is not in \( \mu \). Now, we either have \( f-T \leq 1 \), or there are at least \( f-T-1 \geq 1 \) boxes below our box in column \( \lambda_i \). In the former case we obtain \( f-T \leq 1 \leq W = g-T \), hence \( f \leq g \), and our acyclicity assumption and Theorem 6.3 yield that \( g - f + 1 = \text{grade} I_f(\phi) \geq l_1 \) which implies (2) part (a). In the latter case for every box in column \( \lambda_i \) and row \( j > i \) we must have \( \lambda_j = \lambda_i \) and so \( \lambda_j - \mu_j \geq \lambda_i - \mu_i = W \), hence \( \mu_j = \mu_i \). Therefore we have \( l_t \geq W \) for \( t = 1, \ldots, f-T \), which yields

\[ l_1 + \cdots + l_{f-T} \geq (f-T)W = (f-T)(g-T) = \text{grade} I_{1+T}(\phi). \]

Our acyclicity assumption and Theorem 6.3 now imply grade \( I_{1+T}(\phi) = l_1 + \cdots + l_{f-T} \). But this equality is possible if and only if \( l_t = W \) for \( t = 1, \ldots, f-T \), hence in this case \( \lambda/\mu \) is the shift by \((i, \mu_i)\) of a partition \( \gamma \) with \( g \geq \gamma_1 = g-T > g-f+1 \) and \( \gamma_{i-g-T} \geq f-T = \gamma_1 - g + f \), which yields (2) part (b). This completes the proof of (1) \( \iff \) (2).

Next, we show (2) implies (1). Suppose first that (2) part (a) holds. Then

\[ g - f + 1 \geq l_1 \geq \cdots \geq l_f, \]

and therefore grade \( I_{f-k+1}(\phi) = k(g - f + k) \geq k(g - f + 1) \geq l_1 + \cdots + l_k \) for all \( k \geq 1 \). Thus Theorem 6.3 yields the desired acyclicity. Furthermore, \( g-T \leq W \) by Remark 6.4. Thus \( g-T \leq W \leq g - f + 1 \leq g \) hence \( T \geq 0 \), yielding that \( L_{\lambda/\mu}\phi \) is nonzero and completing the proof that (2) part (a) implies (1).

Finally, suppose (2) part (b) holds, i.e. that \( \lambda/\mu \) is the shift by \((s, t)\) of a partition \( \gamma \) with \( g \geq \gamma_1 > g - f + 1 \) and \( \gamma_{i-g-T} > \gamma_1 - (g - f + 1) \). We want to show that grade \( I_{f-k+1}(\phi) \geq l_1 + \cdots + l_k \) for \( k \geq f-T \) and use Theorem 6.3.

Note that in this case we must have \( T = g - \gamma_1 \). Indeed,

\[ \nu''(\lambda, \mu, g - (g - \gamma_1)) = \nu''(\lambda, \mu, \gamma_1) = \lambda \]

hence it contains \( \nu'(\lambda, \mu, f - (g - \gamma_1)) = \nu'(\lambda, \mu, \gamma_1 - (g - f)) \). Also, row \( s+1 \) of \( \nu''(\lambda, \mu, \gamma_1 - 1) \) has length \( \mu_{s+1} + \gamma_1 - 1 \), while row \( s+1 \) of \( \nu'(\lambda, \mu, \gamma_1 - (g - f + 1)) \) has, in view of our assumption on \( \gamma_1 \), length \( \mu_{s+1} + \gamma_1 = \lambda_{s+1} \). Thus \( T = g - \gamma_1 \) as claimed. In particular, \( T \geq 0 \) and so \( L_{\lambda/\mu}\phi \) is nonzero.

Now we have \( f - T = f - (g - \gamma_1) = \gamma_1 - (g - f + 1) + 1 \geq 2 \). Also, \( l_i \leq \gamma_1 \) for each \( i \). Thus when \( k \geq f - T \) we obtain \( k \geq f - (g - \gamma_1) \) hence \( 1 \leq \gamma_1 \leq g - f + k \) and therefore for \( f - T \leq k \leq f \) we get

\[ l_1 + \cdots + l_k \leq k\gamma_1 \leq k(g - f + k) = \text{grade} I_{f-k+1}(\phi). \]

The desired acyclicity now follows from Theorem 6.3.

**Example 6.5.** Let \( \phi : F \to G \) be the generic map with \( f = 2, \ g = 2, \ \lambda = (3, 2, 1), \) and \( \mu = (2, 2, 1) \). Then \( T = 1 \) and

\[
\begin{align*}
grade I_2(\phi) &= 1 \geq l_1 = 1 \\
grade I_1(\phi) &= 4 \geq l_1 + l_2 = 1 \\
grade I_{f-k+1}(\phi) &= \infty \geq 1 \text{ for all } k \geq 3
\end{align*}
\]

so that \( L_{\lambda/\mu}\phi \) is acyclic by Theorem 6.4.
Example 6.6. Let $\phi : F \to G$ be the generic map with $f = 4$, $g = 2$, and $\lambda = (1,1,1)$. Then $\gamma_1 = 1$ and $\tilde{\gamma}_1 = 3$ so that condition (2)(b) is satisfied in Theorem 6.4 and it follows that $L_{\lambda/\mu} \phi$ is acyclic.

Example 6.7. Let $\phi : F \to G$ be generic with $f = 2$, $g = 4$, $\lambda = (4,4,1)$, and $\mu = (0)$. Since $3 < \gamma_1 \leq 4$ and $\tilde{\gamma}_1 > 1$, Theorem 6.4 insures that the complex $L_{\lambda/\mu} \phi$ is acyclic.

7. Torsion freeness of Schur modules

Everything is now in place to begin our study of the torsion freeness of Schur modules. First, we introduce a chain complex with good rigidity properties.

Definition 7.1. Let $\phi : F \to G$ be a map of free $R$-modules with $f = \text{rank } F < \text{rank } G$. We write $\tilde{L}_{\lambda/\mu} \phi$ for the chain complex

$$0 \to \Omega_n \to \Omega_{n-1} \to \cdots \to \Omega \frac{L_{\lambda/\mu}(\epsilon)}{L_{\lambda/\mu}(\wedge^f G \otimes \wedge^F F)} \to 0,$$

where $\epsilon = \epsilon(\phi)$ is the map from (1.3), and the piece

$$0 \to \Omega_n \to \Omega_{n-1} \to \cdots \to L_{\lambda/\mu} G$$

is just the Schur complex $L_{\lambda/\mu} \phi$ shifted so that $L_{\lambda/\mu} G$ is in homological degree 1.

Remarks 7.2. (a) Since by Remark 1.5(a) the map $\epsilon^* \phi$ factors through the canonical projection $\pi : G \to M = G/\text{Im } \phi$, it follows from Proposition 4.11 that $\tilde{L}_{\lambda/\mu} \phi$ is indeed a chain complex.

(b) It is straightforward from the definition that the chain complex $\tilde{L}_{\lambda/\mu} \phi$ commutes with base change, and therefore, in view of Remark 1.5(b), it induces a layered functor $\tilde{L}_{\lambda/\mu} : M_{g,f} \to \text{Comp}$.

The first homology of $\tilde{L}_{\lambda/\mu} \phi$ has the following key property.

Lemma 7.3. Let $\phi : F \to G$ be an injective map of free $R$-modules with $f = \text{rank } F < \text{rank } G$, let $M = \text{Coker } \phi$, and let $\epsilon^* : M \to \wedge^f G \otimes \wedge^F F^*$ be the induced by $\epsilon^*$ map. The following are equivalent:

1. $L_{\lambda/\mu} M$ is torsion free;
2. The map $L_{\lambda/\mu}(\epsilon^*) : L_{\lambda/\mu} M \to L_{\lambda/\mu}(\wedge^f G \otimes \wedge^F F)$ is injective.
3. $H_1(\tilde{L}_{\lambda/\mu} \phi) = 0$.

Proof. Let $\pi : G \to M = G/\text{Im } \phi$ be the canonical projection, so that $\epsilon^* = \epsilon^* \pi$. The equivalence of (2) and (3) is immediate from Proposition 4.11. As (2) implies (1) trivially, we proceed to show that (1) implies (2). Thus, we may assume all non-zerodivisors are units, hence $M$ is free and $\phi$ splits. We may therefore further assume, as in the proof of Lemma 1.6, that $R$ is a field. But then by Lemma 1.6 we have that $\epsilon^*$ is injective, hence split. Since split injections are functorial, $L_{\lambda/\mu}(\epsilon^*)$ must also be split injective.

We also need the following immediate consequence of Theorem 6.3.

Corollary 7.4. Let $\mu \subseteq \lambda$, let $\phi : F \to G$ be a map of finite free $R$-modules, let $f = \text{rank } F$, let $g = \text{rank } G$, let $T = T_{\lambda/\mu}(f,g)$, and let $M = \text{Coker } \phi$.

The following are equivalent:

1. The complex $L_{\lambda/\mu}(\phi)$ is acyclic and $L_{\lambda/\mu} M$ is torsion-free.
(2) \( \text{grade} I_{f-j+1}(\phi) \geq 1 + \sum_{i \leq j} l_i \) for each \( j \geq \max(1, f - T) \).

This allows us to obtain the following result in the generic case.

**Theorem 7.5.** Let \( \mu \subsetneq \lambda \), and let \( \phi : F \to G \) be generic with \( f = \text{rank} F \geq 1 \) and \( g = \text{rank} G \geq 1 \). Let \( M = \text{Coker} \phi \). The following are equivalent:

1. The complex \( L_{\lambda//\mu} \phi \) is acyclic, and \( L_{\lambda//\mu} M \) is nonzero and torsion-free.
2. The partition \( \lambda \) differs from \( \mu \) in at most \( g - f \) columns.

**Proof.** We show that (2) implies (1). Indeed, by (2) we have \( g - f \geq l_1 \geq \cdots \geq l_f \), therefore for all \( 1 \leq k \leq \min(f,g) \) we have

\[
\text{grade} I_{f-k+1}(\phi) = k(g - f + k) = k^2 + k(g - f) \geq 1 + k(g - f) \geq 1 + \sum_{i \leq k} l_i,
\]

and the desired acyclicity and torsion freeness follow from Corollary 7.4. Since by construction of the differentials \( d_i \) of \( L_{\lambda//\mu} \phi \) one has \( \text{Im} d_i \subseteq I_1(\phi)(L_{\lambda//\mu} \phi)_{i-1} \) for each \( i \), acyclicity together with the fact that \( \phi \) is generic imply that \( L_{\lambda//\mu} \phi \), and hence \( L_{\lambda//\mu} M \), are nonzero.

For (1) implies (2), we must rule out (b) in Theorem 6.4 (2). Assume that \( \gamma_1 = g - f + s \) where \( s \geq 1 \) and that \( \bar{\gamma}_1 = \gamma_1 - (g-f+1) = (g-f+s) - (g-f+1) = s-1 \). We note that \( \nu'' \) starts getting smaller when \( g-t < \gamma_1 = g-f+s \Rightarrow f-s < t \).

In particular \( \nu_1 \) changes from \( \lambda_1 \) to \( \lambda_1 - 1 \) when \( t = f-s+1 \). Also, \( \nu'_1 = \lambda_1 \) when \( f-t < \bar{\gamma}_1 \Rightarrow f - \bar{\gamma}_1 < t \).

By assumption, we have \( \bar{\gamma}_1 \geq s \) so that \( f - \bar{\gamma}_1 \leq f - s \). It follows that \( f - T = f - (f - s) = s \). But then \( \text{grade} I_{f-s+1}(\phi) = (f - f + s - 1 + 1)(g - f + s - 1 + 1) = (s)(g - f + s) \geq s \bar{\gamma}_1 + 1 = s(g - f + s) + 1 \). Thus we have ruled out (b) and the case in (a) when \( \lambda \) differs from \( \mu \) in \( g-f+1 \) columns and the result follows. \( \square \)

We will need the following key consequence of Theorem 7.5.

**Corollary 7.6.** Suppose \( \mu \subsetneq \lambda \). The functor \( L_{\lambda//\mu} : \widetilde{\mathcal{M}}_{g,f} \to \text{Comp} \) is rigid when \( \lambda \) differs from \( \mu \) in at most \( g - f \) columns.

**Proof.** Since \( \mu \neq \lambda \) and they differ in at most \( g-f \) columns, we must have \( f < g \). By the rigidity criterion, we only need to show that \( L_{\lambda//\mu} \phi \) is acyclic in the generic case. Note that if \( f = 0 \) then \( L_{\lambda//\mu} \phi \) has the form \( 0 \to L_{\lambda//\mu} G \to L_{\lambda//\mu} G \to 0 \), hence is acyclic. Thus we may assume \( f \geq 1 \). But then by Theorem 7.5 the complex \( L_{\lambda//\mu} \phi \) is acyclic, and \( L_{\lambda//\mu} M \) is torsion free. The desired acyclicity is now immediate from Lemma 7.3. \( \square \)

The following is the “local” version of our second main result:

**Theorem 7.7.** Let \( \phi : F \to G \) be an injective map of free \( R \)-modules with \( f = \text{rank} F < \text{rank} G = g \). Let \( M \) be the cokernel of \( \phi \). Let \( \mu \subsetneq \lambda \) be partitions such that \( \lambda \) differs from \( \mu \) in at most \( g-f \) columns, and let \( T = T_{\lambda//\mu}(f,g) \).

Then \( L_{\lambda//\mu} M \) is torsion free if and only if \( \text{grade} I_{f-k+1}(\phi) \geq 1 + \sum_{i \leq k} l_i \) for all \( k \geq \max(1, f - T) \).

**Proof.** If \( \text{grade} I_{f-k+1}(\phi) \geq 1 + \sum_{i \leq k} l_i \) for all \( k \geq \max(1, f - T) \), then by Corollary 7.4 the complex \( L_{\lambda//\mu} \phi \) is acyclic and \( L_{\lambda//\mu} M \) is torsion free.

If \( L_{\lambda//\mu} M \) is torsion free, then by Lemma 7.3 we have \( H_1(L_{\lambda//\mu} \phi) = 0 \). Since \( L_{\lambda//\mu} \) is rigid by Corollary 7.4, the complex \( L_{\lambda//\mu} \phi \) is acyclic and the result follows from Corollary 7.4. \( \square \)
Now we are ready to prove our second main result.

**Theorem 7.8.** Let $M$ be a finitely generated $R$-module with \( \text{pd}_RM \leq 1 \) and \( \text{rank}M = r \). Let $\mu \subseteq \lambda$ be partitions such that $\lambda$ differs from $\mu$ in at most $r$ columns, and let $T = T_{\lambda/\mu}(0, r)$.

Then $L_{\lambda/\mu}M$ is torsion free if and only if \( \text{gradeFitt}_{r+k-1}(M) \geq 1 + \sum_{i \leq k} l_i \) for all $k \geq \max(1, -T)$.

**Proof.** Since $L_{\lambda/\mu}M$ is finitely generated and has rank, the module $L_{\lambda/\mu}M$ is torsion free if and only if $L_{\lambda/\mu}M_p$ is torsion free for all $p \in \text{Spec} R$. Thus we may assume that $R$ is local, in particular $M$ has a minimal free resolution $0 \rightarrow F \rightarrow G \rightarrow 0$. Let $f = \text{rank} F$ and $g = \text{rank} G$. Since $\mu \neq \lambda$ and $\lambda$ differs from $\mu$ in at most $r$ columns we get $1 \leq r = g - f$, hence $f < g$. Also, by Remark 5.4(b) we have $T_{\lambda/\mu}(f, g) = T_{\lambda/\mu}(0, r)$, and the desired result follows from Theorem 7.7. $\square$

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