Local copying and local discrimination as a study for non-locality of a set

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We focus on the non-locality concerning local copying and local discrimination, especially for a set of orthogonal maximally entangled states in prime dimensional systems, as a study of non-locality of a set of states. As a result, for such a set, we completely characterize deterministic local copiability and show that local copying is more difficult than local discrimination. From these result, we can conclude that lack algebraic symmetry causes extra non-locality of a set.

I. INTRODUCTION

Non-locality is one of the oldest topics in quantum physics, and also is one of the most important topics in the newest fields, “quantum information”. The history of non-locality started with EPR’s discussion of local realism in 30’s\(^1\), and then, it was followed Bell’s formulation of local hidden variable theory and Bell inequality in 60’s\(^2\). In early 90’s, the development of quantum information shed a new light on this topic. The theory of non-locality was reformulated as entanglement theory, which is a useful formulation to treat entangled states as resources of quantum communication, like teleportation, dense-cording, key distribution, etc.\(^3\). Mathematically speaking, the study of conventional entanglement and local copying, the similar non-locality also appears commonly in various different fields of quantum information, e.g. quantum capacity, quantum estimation, etc.\(^6,7\). Since this type of non-locality does not depend only on entanglement convertibility, i.e., entanglement of each states, we call it Non-locality beyond individual entanglement.

Recently, as a similar problem to local discrimination, a new problem, “local copying”, was also raised\(^3,11\), as a problem to study a cloning of unknown entangled states under the LOCC restriction with only minimum entanglement resource.

The study of Bennett et. al suggests the new kind of non-locality, Non-locality of a set of states. At first, from analogy of the non-locality discussion in local discrimination, we can expand the concept of non-locality as follows. If the local (LOCC) restriction causes difficulty for a task concerning a set of states, e.g. discrimination, copying etc., then, we consider that this set has non-locality, and regard the degree of this difficulty as non-locality of the set. This concept of non-locality is not unnatural, since it is consistent with the conventional entanglement theory because of the following reasons. In entanglement theory, entanglement cost\(^15\) is one of the famous measures of entanglement, and can be regarded as a kind of difficulty of a task, i.e. the difficulty of entanglement dilution\(^15\). Moreover, if we consider the task to approximate a given state by separable states, we derive the relative entropy of entanglement\(^19\) by measuring this difficulty in terms of accuracy of the approximation, using relative entropy. These can be regarded as the degrees of difficulty of tasks with the local restrictions.

Indeed, local copying and local discrimination can be regarded as tasks for a set of states with the local restriction, because these problems are usually treated based on a set of candidates of the unknown states. Hence, we can measure “Non-locality of a set of states by the degree of their difficulty. We should note that this kind of difficulty cannot be often characterized only by entanglement of states of the given set. A typical example is the impossibility of local discrimination of the product basis of Bennett et.al. Actually, in addition to local discrimination and local copying, the similar non-locality also appears commonly in various different fields of quantum information, e.g. quantum capacity, quantum estimation, etc.\(^6,7\). Since this type of non-locality does not depend only on entanglement convertibility, i.e., entanglement of each states, we call it Non-locality beyond individual entanglement.

In this paper, as a study of non-locality beyond individual entanglement, we focus on non-locality of a set of states by means of local copying and local discrimination. Especially, we concentrate ourselves on a set of orthogonal maximally entangled states in a prime dimensional system for simplicity, and investigate the relationship between their local copiability and local distinguishability. As a result, we completely characterize the local copiability of such a set, that is, we prove that such a set is locally copiable, if and only if it has canonical Bell form and is simultaneous Schmidt decomposable. Using this result, we prove the following two facts. First, the maximal size of locally copiable sets is equal to the dimension of the local space as well as the maximal size of local dis-
t inguishable sets. Second, we also show that if such a set is locally copiable, then locally distinguishable by one-way communication. Thus, in this case, local copying is strictly more difficult than one-way local discrimination. The relationship of local copiability and distinguishability is summarized in FIG.1. From this relationship, we derive the conclusion related to the non-locality of a set concerning local copying and local discrimination: A simultaneous Schmidt decomposable state does not have non-locality beyond individual entanglement concerning local discrimination, since it is locally distinguishable. However, even if a set is simultaneous Schmidt decomposable, if such a set does not have a canonical Bell form, such a set still has non-locality concerning local copying. Therefore, lack of algebraic symmetry causes extra non-locality of a set.

Although we mainly concentrate ourselves on the aspect of local copying and local discrimination as the study of non-locality in this paper, local copying and local discrimination themselves are worth to investigate as basic protocols of quantum information processing with two-party. In the last part of this paper, we show that there are many important relationships between our local copying protocol and the other quantum information protocols. These results give many other interpretations for local copying.

This paper is organized as follows. In Section II as a preparation of our analysis, we review a necessary and sufficient condition for a locally copiable set as the preparation of our analysis, which is the main result of the paper [9]. In Section III we give an example of a locally copiable set of $D$ maximally entangled states, and then, prove that, in a prime-dimensional local system, the above example is the only case where local copying is possible. In Section IV we discuss the relation between local copying and LOCC discrimination by means of simultaneous Schmidt decomposition. In Section V we present the other protocols which are strongly related with our theory of local copying, i.e., channel copying, distillation protocol, error correction, and quantum key distribution. And then, we extend our results of local copying to these protocols. Finally, we summarize and discuss our results in Section VI.

II. THE LOCAL COPYING PROBLEM

In this section, as a preparation of our analysis, we introduce the formulation and the known results of local copying from [9].

Many researchers treated approximated cloning, for example, universal cloning [11], asymmetric cloning [12], tele-cloning [13]. This is because the perfect cloning, i.e., copying, is impossible without prior knowledge (no-cloning theorem) [8]. That is, the possibility of copying depends on the prior knowledge, or, in other words, the set of candidates for the unknown target state. If we know that the unknown state to be copied is contained by the set of orthogonal states, which is called the copied set, we can copy the given state. However, if the system to be copied has an entangled structure, and if our operation is restricted to local operations and classical communications (LOCC) [14], we cannot necessarily copy the given quantum state with the above orthogonal assumption, perfectly. Thus, it is interesting from both viewpoints of entanglement theory and cloning theory to extend cloning problem to the bipartite entangled setting. This is the original motivation of cloning problems with LOCC restriction [9, 10].

Recently, F. Anselmi, et. al. [9] focused on the perfect cloning of bipartite systems under the following assumptions:

1. Our operation is restricted to LOCC.
2. It is known that the unknown state to be copied is contained by the set of orthogonal entangled states, (the copied set).
3. An entangled state of the same size is shared.

They called this problem local copying, and they have characterized copied sets which we can locally copied for special cases. In the following, for simplicity, we say the set is locally copiable if local copying is possible with the prior knowledge to which the given state belongs. As is explained in Theorem [1] they showed that the possibility of local copying can be reduced to the simultaneous
transformation of unitary operators. That is, they derived a necessary and sufficient condition for a locally copiable set.

The problem of local copying can be arranged as the follows. We assume two players at a long distance, e.g., Alice and Bob in this protocol. They have two quantum systems $\mathcal{H}_A$ and $\mathcal{H}_B$ each of which is also composed by the same two $D$-dimensional systems, i.e., the systems $\mathcal{H}_A$ and $\mathcal{H}_B$ are described by $\mathcal{H}_A = \mathcal{H}_1 \otimes \mathcal{H}_3$, $\mathcal{H}_B = \mathcal{H}_2 \otimes \mathcal{H}_4$. In our problem, they try to copy an unknown state $|\Psi\rangle$ on the initial system $\mathcal{H}_3 \otimes \mathcal{H}_2$ to the target system $\mathcal{H}_3 \otimes \mathcal{H}_4$ with the prior knowledge that $|\Psi\rangle$ belongs to the copied set $\{|\Psi_j\rangle\}_{j=0}^{N-1}$. Moreover, we assume that they implement copying only by LOCC between them. Since LOCC operations do not increase the entanglement of whole states, they can copy no entangled state by LOCC without any entanglement resource. Thus, we also assume that they share a blank entangled state $|b\rangle$ in target systems $\mathcal{H}_3 \otimes \mathcal{H}_4$. Therefore, a set of states $\{|\Psi_j\rangle\}_{j=0}^{N-1}$ is called locally copiable with a blank state $|b\rangle$. Since LOCC operations do not increase the entanglement of whole states, they can copy any orthogonal set of maximally entangled states by use of quantum teleportation \[\text{4, (For the case when we share two entangled states as resources, see \[\text{10}].)}\]

III. LOCAL COPYING OF THE MAXIMALLY ENTANGLED STATES IN PRIME-DIMENSIONAL SYSTEMS

In this section, we solve Equation \[\text{2} \] and get the necessary and sufficient condition for all $N$ in the case of prime-$D$-dimensional local systems. That is, the form of $T$ is completely determined. As a consequence, we show that $D$ is the maximum size of a locally copiable set.

As the starting point of our analysis, we should remark that Equation \[\text{2} \] simultaneously presents $N^2$ matrix equations, but we may take care of only $N-1$ equations $A(T_{3j}\otimes I)A^\dagger = e^{i\theta_j-\theta_{j'}}(T_{3j'}\otimes T_{3j'})$, where $j = 0, \cdots, N-2$. This is because by multiplying the $j$ elements of the equation by the Hermitian conjugate of the $j'$ elements of the same equation, we can recover Equation \[\text{2} \]. Moreover, since $T_{3j} = U_j U_{j'}^\dagger$ and the coefficient $e^{i\theta_j}$ is only related to the unphysical global phase factor, we can treat only the following $N$ equations,

$$A(U_j \otimes I)A^\dagger = U_j \otimes U_j \quad (j = 0, \cdots, N-1). \quad \text{(4)}$$

Note that $|\Psi_j\rangle$ is represented as $|\Psi_j\rangle = U_j \otimes I |\Psi_0\rangle$.

At the first step, we construct an example of a locally copiable set of $D$ maximally entangled states.

Theorem 3 When the set of maximally entangled states $\{|\Psi_j\rangle\}_{j=0}^{N-1}$ is defined by

$$|\Psi_j\rangle = (U_j \otimes I) |\Psi_j\rangle \quad \text{(5)}$$

explicitly represented as a local unitary transformation $A^{13} \otimes A^{24}$. Finally, they solved Equation \[\text{2} \] for all $j, j'$ in the case of $N = 2$. In this case, there is only one independent equation, and these conditions are reduced to the condition $A(T \otimes I)A^\dagger = T \otimes T$, where the phase factor $e^{i\theta}$ is absorbed by $T$. The following theorem is the conclusion of their analysis of Equation \[\text{2} \] for $N = 2$.

**Theorem 2** There exists a unitary operator $A$ satisfying

$$A(T \otimes I)A^\dagger = T \otimes T, \quad \text{(3)}$$

if and only if a unitary operator $T$ satisfies the following two conditions:

1. The spectrum of $T$ is the set of power of $M$th roots of unity, where $M$ is a factor of $D$.

2. The distinct eigenvalues of $T$ have equal degeneracy.

Here, we should remark the number of maximally entangled states as the resource. If we allow to use three entangled states as a resource, we can always locally copy any orthogonal set of maximally entangled states by use of quantum teleportation \[\text{2, (For the case when we share two entangled states as resources, see \[\text{10}.)}\]
and
\[ U_j = \sum_{j=0}^{D-1} \omega^{jk} |k\rangle \langle k| , \]  
where \{ |k\rangle \}_{k=0}^{D-1} is an orthonormal basis of the \( \mathcal{H}_1 \), then the set \{ |\Psi_j\rangle \}_{j=0}^{N-1} can be locally copied.

**Proof** We define the unitary operator \( A \) by
\[ A = \text{CNOT} \equiv \sum_{a,b} |a \otimes b\rangle \langle a| \langle b| , \]  
where \( \text{CNOT} \) is an extension of Control-NOT gate represented in \{ |k\rangle \}_{k=0}^{D-1} for \( D \) dimensional systems. Then, we can easily verify Equation 4 as
\[ A(U_j \otimes I)A^\dagger = \text{CNOT}(U_j \otimes I)\text{CNOT}^\dagger \]  
where \( c = a \otimes b_1 \). Therefore, Theorem guarantees that the set \{ |\Psi_j\rangle \}_{j=0}^{D-1} can be locally copied. \( \square \)

This protocol of local copying used is the above proof is written as FIG 2. Here, we should remark that \( U_1 \) is the generalized Pauli’s Z operator which is one of the generators of the Weyl-Heisenberg Group, and another \( U_j \) is the \( j \)th power of \( U_1 = Z \). Hence, in the case of non-prime-dimensional local systems, the spectrum of \( U_1 \) is different from that of \( U_1 \) if \( j \) is a non-trivial factor of \( D \).

Moreover, the property of Weyl-Heisenberg Group not only guarantees that the above example satisfies 4, but also is essential for the condition 3. That is, as is proved below, any locally copiable set of maximally entangled states is restricted exclusively to the above example. Therefore, our main theorem can be written down as follows.

**Theorem 4** In prime-dimensional local systems, the set of maximally entangled states \{ \( U_j \otimes I |\Psi_0\rangle \}_{j=0}^{N-1} \) can be locally copied if and only if there exist an orthonormal basis \{ \( |a\rangle \) \}_{a=0}^{D-1} and a set of integers \{ \( n_j \) \}_{j=0}^{N-1} such that the unitary \( U_j \) can be written as
\[ U_j = \sum_{k=0}^{D-1} \omega^{n_j k} |k\rangle \langle k| , \]  
where \( \omega \) is the \( D \)th root of unity.

Since the size of the set \{ \( U_j \) \} is \( D \), \( D \), that is equal to the dimension of local space, is the maximum size of a locally copiable set of maximally entangled states with prime-dimensional local systems. In comparison with the case without LOCC restriction, this is actually the square root.

The proof of Theorem 5 is as follows.

**Proof** (If part) We have already proven that \{ \( U_j \otimes I |\Psi_j\rangle \) \}_{j=0}^{D-1} can be copied by LOCC in Theorem 3. Therefore, the subset of them can be trivially copied by LOCC. (Only if part) Assume that a unitary operator \( A \) satisfies the condition 4 for all \( j \).

By applying Theorem 2 we can choose an orthonormal basis \{ \( |a\rangle \) \}_{a=0}^{D-1} such that
\[ U_1 = \sum_{a=0}^{D-1} \omega^a |a\rangle \langle a| , \]  
where \( \omega \) is \( D \)th root of unity. Moreover, Equation 4 implies that the unitary \( A \) should transform the subspace \( |a\rangle \otimes \mathcal{H} \) to subspace \( \text{span}\{ |k\rangle \otimes |l\rangle \}_{k,l=0}^{D-1} \). That is, \( A \) is expressed as
\[ A = \sum_{a,b,c} \xi_{b,c}^a |a \otimes c\rangle \langle a| \langle b| , \]  
where \( \xi_{b,c}^a \) is a unitary matrix for \( b,c \) for the same \( a \), that is, \( \sum_{c=0}^{D-1} \xi_{b,c}^a \xi_{b,c'}^a = \delta_{b,b}' \) and \( \sum_{b=0}^{D-1} \xi_{b,c}^a \xi_{b,c'}^a = \delta_{c,c'} \).

Thus, based on the basis \{ \( |a\rangle \) \}_{a=0}^{D-1}, Equation 4 for all \( a_1,a_2,b_1,b_2 \) is written down as
\[ \langle a_1| (b_1| A(U_j \otimes 1)A^\dagger |a_2|) |b_2\rangle = \langle a_1| U_j |a_2| \langle b_1| U_j |b_2\rangle . \]  
(11)

Therefore, substituting Equation 10 to Equation 11 for any integer \( j \), we obtain
\[ \sum_{k=0}^{D-1} \xi_{a_1,b_1}^k \xi_{a_2,b_2}^k \langle a_1 \oplus b_1| U_j |a_2 \oplus b_2\rangle = \langle a_1| U_j |a_2\rangle \langle b_1| U_j |b_2\rangle , \]  
(12)

for all \( a_1,a_2,b_1 \) and \( b_2 \).

To see that \( U_1 \) and \( U_j \) can be simultaneous orthogonalized, we need to prove the following lemma.
Lemma 1 A non-zero $D \times D$ matrix $U_{ab}$ satisfies the following equation,
\[
\Xi_{a_1 \otimes b_1, a_2 \otimes b_2} U_{a_1 \otimes b_1} a_2 \otimes b_2 = U_{a_1} a_2 U_{b_1} b_2,
\]
where $\Xi_{a_1} b_2 = \delta_{a_1} b_2$ and all indices have their value between 0 and $D - 1$, then $U_{ab}$ is a diagonal matrix.

Proof See Appendix A

We apply this Lemma 1 to the case when $U_{ab} = \langle a | U_j | b \rangle$ and $\Xi_{a_1 \otimes b_1, a_2 \otimes b_2} = \sum_{b=0}^{D-1} \delta_{a_1} b_2$. Then, this lemma shows that $U_j$ is orthogonal in the eigenbasis of $U_1$, therefore all unitaries $\{U_j\}_{j=0}^{N-1}$ are orthogonalized. Then, we can get the form of $\{U_j\}_{j=0}^{N-1}$ explicitly as follows. From the diagonal element of (12), we derive
\[
\langle a \otimes b | U_j | a \otimes b \rangle = \langle a | U_j | a \rangle \langle b | U_j | b \rangle.
\]
Since $\{\langle a |\}_{a=0}^{D-1}$ is also an eigenbasis of $U_j$, we can express $U_j$ as
\[
U_j = \sum_{a=0}^{D-1} \omega^{P_j(a)} |a\rangle \langle a|,
\]
where $P_j(a)$ is a bijection from $\{a\}_{a=0}^{D-1}$ to themselves. Then, Equation (13) guarantees that $P_j(a)$ is a self-isomorphism of the cyclic group $\{j\}_{a=0}^{D-1}$. Since a self-isomorphism of a cyclic group is identified by the image of the generator, we derive the formula (15) with $P_j(1) = n_j$.

We have solved the LOCC copying problem only for a prime-dimensional local space. In the case of a non-prime-dimensional local space, our proof of the “only if part” can be done in the same way. However, the “if part” is extended straightforwardly only for the case in which the set $\{U_j\}_{j=0}^{N-1}$ contains at least one unitary whose eigenvalues are generated by the $D$th root of unity. In this case, the proof is the following. By the same procedure of the prime-dimensional case, we obtain Equation (14). Then, Lemma 1 implies that all $U_j$ can be diagonalized, and also implies Equation 15 for all $U_j$. By writing $U_j$ as 15, we get the equation $P_j(a \otimes b) = P_j(a) \otimes P_j(b)$ and, so, $P_j(a) = aP_j(1)$. Hence, Theorem 2 guarantees the same representation of $U_j$ as 15. Therefore, we can solve the problem of local copying in non-prime-dimensional local spaces only in this special case as the direct extension of Theorem 3. On the other hand, if eigenvalues of all $U_j$ are degenerate, our proof of “if part” does not hold.

IV. RELATION WITH LOCC COPYING AND LOCC DISCRIMINATION

If we have no LOCC restriction, the possibility of the deterministic copying is equivalent with that of the perfect distinguishability. However, we can easily see that under the restriction of LOCC, this relation is non-trivial at all. As we have already mentioned in the introduction, these two problems share the common feature, that is, their difficulty can be regard as a non-locality of a set, and this non-locality can not be explained only by entanglement convertibility. Therefore, the study of this relationship is really important to understand the non-locality of a set. In this section, we compare the locally distinguishability and the locally copiability for a set of orthogonal maximally entangled states. Thus, by introducing Simultaneous Schmidt decomposition, we show the relationship between these two non-locality.

At first, we remind the definition of a locally distinguishable set, and then mention several known and new results of locally distinguishability. A set of states $\{\psi_j\}_{j=0}^{N-1}$ is called two-way (one-way) classical communication (c.c.) locally distinguishable, if there exists a POVM $\{M_j\}_{j=0}^{N-1}$ which can be constructed by two-way (one-way) LOCC and also satisfies the following conditions:
\[
\forall i, j, \quad \langle \psi_i | M_j | \psi_i \rangle = \delta_{ij}.
\]
In order to compare LOCC copying and LOCC discrimination, we should take care of the following point: We assume an extra maximally entangled state only in the LOCC copying case. This is because LOCC copying of a set of maximally entangled states is trivially impossible without a blank entangled state. This fact is contrary to LOCC discrimination since LOCC discrimination requires sharing no maximally entangled state.

In the previous section, we have already proved that $D$ is the maximum size of locally copiable set of maximally entangled states. In the case of local discrimination, we can also prove that $D$ is the maximum size of a locally distinguishable set of maximally entangled states. This statement was proved by the paper 27 only when the set of maximally entangled states $\{\psi_j\}_{j=0}^{N-1}$ consists of canonical form Bell states, where a canonical form Bell state $\psi_{nm}$ is defined as
\[
\psi_{nm} \overset{\text{def}}{=} Z^n X^m \otimes I |\psi_{00}\rangle
\]
\[
|\psi_{00}\rangle \overset{\text{def}}{=} \sum_{k=0}^{d-1} |k\rangle \otimes |k\rangle
\]
\[
X \overset{\text{def}}{=} \sum_{k=1}^{d} |k\rangle \langle k \oplus 1|.
\]
Such a set is a special case of a set of maximally entangled states. Here, we give a simple proof of this statement for a general set of maximally entangled states by the same technique as 32.

Theorem 5 If an orthogonal set of maximally entangled states $\{\psi_j\}_{j=0}^{N-1}$ is locally distinguishable, then $N \leq D$.

5
Proof Suppose that $\{M_j\}_{j=0}^{N-1}$ is a separable POVM which distinguishes $\{|\Psi_j\rangle\}_{j=0}^{N-1}$, then they can be decomposed as $M_j = \sum_{k=0}^L p_{jk}|\psi_k\rangle\langle\psi_k| \otimes |\phi_k\rangle\langle\phi_k|$, where $p_{jk}$ is a positive coefficient. Then, we can derive an upper bound of $\langle\Psi_j|M_i|\Psi_j\rangle$ as follows,

\[
\langle\Psi_j|M_i|\Psi_j\rangle = \sum_{k=0}^L p_{jk} \langle\psi_k| \langle\phi_k| \langle\phi| \langle\psi| \langle\Psi_j \rangle
\]

\[
\leq \sum_{k=0}^L p_{jk} |\psi_k\rangle \left(\frac{1}{D}I\right)|\psi_k\rangle
\]

\[
\leq \frac{\text{Tr}M_i}{D},
\]

where the first inequality comes from the monotonity of the fidelity under partial trace operations concerning the system $B$. Since $\langle\Psi_j|M_i|\Psi_j\rangle = 1$, we have $1 \leq \text{Tr}(M_i)/D$. Finally, taking the summation of the inequality for $j$, we obtain $N \leq D^2/D = D$.

Therefore, in this case, the maximal size of both locally copiable and locally distinguishable sets is equal to the dimension of the local space.

When we consider the relationship between local discrimination and local copying of a set of maximally entangled states, it is quite useful to introduce “Simultaneous Schmidt Decomposition”. A set of states $\{|\Psi_\alpha\rangle\}_{\alpha \in \Gamma} \subset \mathcal{H}_1 \otimes \mathcal{H}_2$ is called simultaneously Schmidt decomposable, if they can be written down as

\[
|\Psi_\alpha\rangle = \sum_{k=0}^{d-1} k^{(\alpha)}|e_k\rangle |f_k\rangle,
\]

where $\Gamma$ is a parameter set, $\{|e_k\rangle\}_{k=0}^{d-1}$ and $\{|f_k\rangle\}_{k=0}^{d-1}$ are orthonormal bases of local spaces (simultaneous Schmidt basis) and $k^{(\alpha)}$ is a complex number coefficient. Actually, for a set of orthogonal maximally entangled states, simultaneous Schmidt decomposability is a sufficient condition for one-way local distinguishability and a necessary condition for local copiability of it. Moreover, simultaneous Schmidt decomposability is not a necessary and sufficient condition for the both cases. Therefore, a family of locally copiable sets of maximally entangled states is strictly included by a family of one-way locally distinguishable sets of maximally entangled states. In the following, we prove this relationship.

First, we explain the relationship between local discrimination and simultaneous Schmidt decomposition which has been already obtained by the paper [22]. If an unknown state $|\Psi_\alpha\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ is in a simultaneously Schmidt decomposable set of states $\{|\Psi_\alpha\rangle\}_{\alpha \in \Gamma}$, such a state can be transformed to a single local space $\mathcal{H}_A$ or $\mathcal{H}_B$ by LOCC. Rigorously speaking, there exists a LOCC $\Lambda$ on $\mathcal{H}_A \otimes \mathcal{H}_B$, which transforms $|\Psi_\alpha^{AB}\rangle \otimes |B_\alpha\rangle$ to $|\psi_\alpha A\rangle \otimes \sigma_\alpha B$, for all $\alpha \in \Gamma$, and also exists a LOCC $\Lambda$ on $\mathcal{H}_A \otimes \mathcal{H}_A \otimes \mathcal{H}_B$, which transforms $|0^{AB}\rangle \otimes |\Psi_\alpha^{AB}\rangle$ to $|\Psi_\alpha^{AB}\rangle \otimes \sigma_\alpha B$, for all $\alpha \in \Gamma$. Indeed, this LOCC transformation can be written down as the following Kraus representation [22].

\[
\rho \mapsto \sum_{k=0}^{d-1} F_k \rho F_k^*,
\]

where

\[
F_k \equiv (I_A \otimes \text{CNOT}) (U_k \otimes I_{A_2} \otimes B_2) (P_k \otimes I_{B_1} \otimes B_2)
\]

\[
P_k \equiv 1/D \left(\sum_i \omega^{ki} |e_i\rangle \langle e_i|\right)
\]

\[
U_k \equiv \sum_{kl} \omega^{ki} |f_k\rangle \langle f_l|
\]

\[
\text{CNOT} \equiv \sum_{k} |f_k\rangle \langle f_k| \otimes |\ell\rangle \langle \ell|
\]

In the above formula, both $\{|e_k\rangle\}_{k=0}^{D-1}$ and $\{|f_k\rangle\}_{k=0}^{D-1}$ are simultaneous Schmidt bases of $\{|\Psi_\alpha\rangle\}_{\alpha \in \Gamma}$, and $\{|\ell\rangle\}_{\ell=0}^{D-1}$ is the standard computational basis. This protocol can be written as FIG. 3 where $|G_k\rangle$ is a garbage state with no information. Using the above protocol, if a set $\{|\Psi_\alpha\rangle\}$ is simultaneously Schmidt decomposable, there exists a one-way-LOCC POVM $M' = \{M'_i\}$ for a given arbitrary POVM $M = \{M_i\}$ such that

\[
\langle\Psi_\alpha| M_i |\Psi_\alpha\rangle = \langle\Psi_\alpha| M'_i |\Psi_\alpha\rangle, \quad \forall i, \forall \alpha.
\]

That is, any POVM can be essentially realized by 1way LOCC. Therefore, “simultaneously Schmidt decomposable set of orthogonal maximally entangled states is one-way locally distinguishable.” On the other hand, the set of orthogonal maximally entangled states which is not simultaneously Schmidt decomposable was found by the paper [27]. Thus, a family of simultaneously Schmidt decomposable sets of maximally entangled states is strictly included by a family of locally distinguishable sets of maximally entangled states.

On the other hand, the relationship between simultaneous Schmidt decomposability and local copiability can be derived as the following theorem.

**Theorem 6** In prime-dimensional local systems, an orthogonal set of maximally entangled states $\{|\Psi_j\rangle\}_{j=0}^{N-1}$
is locally copiable, if and only if it is a simultaneously Schmidt decomposable subset of canonical form Bell states under the same local unitary operation.

**Proof** We can easily see the “only if” part of the above statement from Theorem 3. The “if” part can be showed as follows. The paper [28] shows that the states \(|\Psi_{n,a,m,a}\rangle \ (a = 1,2, \cdots, l)\) are simultaneously Schmidt decomposable, if and only if there exist integers \(p, q\) and \(r\) \((p \neq 0 \text{ or } q \neq 0)\) satisfying \(pn_a \otimes qm_a = r\) for all \(a\). Since the ring \(Z_p\) is a field in the prime number \(p\) case, the above condition is reduced to the existence of \(f\) and \(g\) such that \(m_a = fn_a + g\). Then, we get

\[
|\Psi_{n,a,m,a}\rangle = |\Psi_{n,a}(fn_a+g)\rangle = C_a(\mathcal{Z} X^f)^{n_a} X^g \otimes I \Psi_00 \rangle.
\]

(18)

Since \(\mathcal{Z} X^f\) is unitary equivalent to \(\mathcal{Z}\), the state \(|\Psi_{n,a,m,a}\rangle\) is locally unitary equivalent with \(U_j \otimes I \Psi_00\rangle\) in Theorem 6.

We add a remark here. Under the assumption of simultaneous Schmidt decomposition, a set has canonical Bell form, and satisfies the following condition as the follows, the set is simultaneous Schmidt decomposable and satisfies the following condition by a renumbering,

\[
U_1^D = I, \quad U_k = U_1 \cdots U_1.
\]

We finally derive FIG.4 and, therefore, for maximally entangled states, a family of locally copiable sets is strictly included by a family of simultaneously Schmidt decomposable sets. In other words, local copying is more difficult than local discrimination.

At this last part of this section, we discuss our main results in FIG.4 in the view point of non-locality beyond individual entanglement.

In the case of bipartite pure states, all information of a bipartite state \(|\Psi\rangle = \sum_{i=0}^{D-1} \lambda_i |e_i\rangle \otimes |f_i\rangle\) can be separated to two parts, that is, Schmidt coefficients \(\lambda_i\) and Schmidt basis \(|\{e_i\}, |f_i\}\rangle\), where \(\lambda_i \geq 0\). Because of local unitary equivalence, Schmidt coefficients completely determine entanglement convertibility [14]. Therefore, conversely, we can regard the non-locality coming from interrelationship among Schmidt basis as non-locality purely beyond individual entanglement. In the following discussion, we try to separate non-locality which depends on Schmidt coefficients and Schmidt basis.

At first, since all sets in FIG.4 have the same Schmidt coefficients, the structure of non-locality in FIG.4 is determined only by the interrelationship of Schmidt basis, and the effect of Schmidt coefficients do not appear directly in this figure. On the other hand, since Schmidt basis do not concern the definition of the maximal sizes of local distinguishable and copiable sets, the maximal sizes depend only on Schmidt coefficients. Therefore, Schmidt coefficients may affect only the maximal size of local distinguishable and copiable sets.

The interrelationship of Schmidt basis like is determined by the unitary operator \(U = \sum_{i=0}^{D-1} |e_i\rangle \langle f_i|\). In FIG.4, the two properties of the interrelationship of Schmidt basis, that is, such unitary operators, are related to non-locality of a set. That is, simultaneous Schmidt decomposability and canonical Bell form seems to reduce non-locality of a set. For simultaneous Schmidt decomposable sets, we can explain their lack of non-locality as follows. As we well know, in the case of pure bipartite states, one person can always apply the local operation which causes the same transformation for a given state as another person’s local operation causes (Lo-Popescu’s theorem [29]). The simple structure of entanglement convertibility originates in the above symmetry between local systems. This symmetry is caused by the existence of Schmidt decomposition. Similarly, in the case of local discrimination, the protocol FIG.4 seems to utilize this kind of symmetry between local systems. Therefore, the existence of simultaneous Schmidt decomposable basis gives the symmetry between the local systems, and this fact may decreases the non-locality of the sets of states.

In the case of canonical Bell form, the interesting fact is that this algebraic property is related to local copiability, and not to local distinguishability. As we have already seen, since a simultaneous Schmidt decomposable set can be transformed to a single local space by LOCC, we can use any global discrimination protocols to such a set by only LOCC. Therefore, concerning local discrimination, the sets of simultaneous Schmidt decomposable states seem not to possess any non-locality which originates in interrelationship between their Schmidt basis. However, if such a set does not have a canonical Bell form, it is not locally copiable. That is, a set has extra non-locality beyond individual entanglement concerning local copying, if it has no algebraic structure given in (19), even if it is simultaneous Schmidt decomposable. Finally, we can conclude that, in the view point of problems of non-locality beyond individual entanglement, the above algebraic non-locality is most remarkable difference between local copying and local discrimination.

**V. APPLICATION TO CHANNEL, DISTILLATION, AND ERROR CORRECTION**

So far, we have treated local copying mainly in the context of non-locality of a set. On the other hand, since local copying of maximally entangled states is one of fundamental two-party protocol, this problem itself is worth to investigate. In this last section, we apply our results, especially Theorem 4 for different contexts from local copying, and give several other interpretations for our results, like channel copying, entanglement distillation, error correction, and QKD. Thus, these many connections imply the fruitfulness of local copying problem as
a fundamental two-party protocol. Moreover, seeing local copying problem from these various points of view, we may also derive some clue which help us to construct further development of understanding of non-locality beyond entanglement convertibility.

A. channel copying

In section [1] in the analysis of local copying, we treated not directly maximally entangled states, but unitary operators which represent the maximally entangled states based on a some standard maximally entangled states. This method is a kind of operator algebraic method, or physically speaking equal to Heisenberg picture. Therefore, we can interpret our results as directly the results for these unitary operators themselves. Then, as a result, we can derive the result for the problem of “unitary channel copying”.

Here, we consider a problem “channel copying”, that is, a problem in which we ask a question as follows, in the case we do not have complete description of a channel, “Can we simulate two copies of the channel by using the channel only once?” As we will see in the following discussion, channel copying with help of one-way classical communication (c.c.) is equivalent to local copying of corresponding entangled states with help of one-way c.c.. The problem setting of channel copying can be written down as follows,

**Definition 1** We call that a set of channel \( \{ \Lambda_i \}_{i=1}^N; \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B) \) is copiable with one way c.c. and a blank channel \( \Lambda_b \), if for all \( i \), there exists sets of Kraus’s operators \( \{ A_k \}_{k=1}^K \subset \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_A), \{ B_k^l \}_{l=1}^L \subset \mathcal{B}(\mathcal{H}_B \otimes \mathcal{H}_B) \) such that \( \sum_{k=1}^K A_k^l A_k = I_A, \sum_{l=1}^L B_k^l B_k^l = I_B \) for all \( l \), and for all \( i \) and \( \rho \) on \( \mathcal{H}_A \otimes \mathcal{H}_A \),

\[
\sum_{kl} B_k^l [\Lambda_i \otimes \Lambda_b (A_k \rho A_k^\dagger)] B_k^l = \Lambda_i \otimes \Lambda_l(\rho). \tag{20}
\]

The meaning of the above definition can be sketched as the FIG.4 that is, by an encoding operation \( \{ A_k \}_{k=1}^K \) and a decoding operation \( \{ B_k^l \}_{l=1}^L \), \( \Lambda_i \) with a blank channel (may be noisy) \( \Lambda_b \) works as \( \Lambda_i \otimes \Lambda_l \).

Then, we can easily show that the channel copying problem with one way c.c. is exactly same as the local copying of corresponding entangled states with one way c.c..

**Theorem 7** A set of channel \( \{ \Lambda_i \}_{i=0}^{N-1} \) is copiable with one way c.c. and a blank channel \( \Lambda_b \), if and only if a set of entangled states \( \{ \Lambda_i \otimes I(|\Psi\rangle \langle \Psi|) \}_{i=0}^{N-1} \) is locally copiable with one way c.c. and a blank states \( \Lambda_b \otimes I(|\Psi\rangle \langle \Psi|) \), where \( |\Psi\rangle \) is an arbitrarily fixed maximally entangled states.

**Proof** Suppose \( \{ \Lambda_i \}_{i=0}^{N-1} \) is copiable with one way c.c. and a blank channel \( \Lambda_b \). Consider four systems \( \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_4 \), and prepare two copies of maximally entangled states \( |\Psi\rangle \) on \( \mathcal{H}_1 \otimes \mathcal{H}_3 \) and \( \mathcal{H}_2 \otimes \mathcal{H}_4 \), respectively. Then, by applying channel copying protocol for \( \mathcal{H}_1 \otimes \mathcal{H}_2 \), we derive the following calculations.

\[
\sum_{kl} B_k^{l12} \otimes I^{34}[\Lambda_i^1 \otimes \Lambda_i^2 \otimes I^{34}(A_k^2 \otimes I^{34})|\Psi^{13}\rangle \langle \Psi^{13}|] \\
\otimes |\Psi^{24}\rangle \langle \Psi^{24}| A_k^{12} \otimes I^{34})B_k^{l12} \otimes I^{34} \\
= \sum_{kl} B_k^{l12} \otimes I^{34}[\Lambda_i^1 \otimes \Lambda_i^2 \otimes I^{34}(A_k^2 \otimes I^{34})|\Psi^{13}\rangle \langle \Psi^{13}|] \\
\otimes |\Psi^{24}\rangle \langle \Psi^{24}| I^{12} \otimes A_k^{24}B_k^{l12} \otimes I^{34} \\
= \sum_{kl} B_k^{l12} \otimes A_k^{34}[(\Lambda_i^1 \otimes I^{24})|\Psi^{13}\rangle \langle \Psi^{13}|] \\
\otimes (\Lambda_i^2 \otimes I^{12}|\Psi^{24}\rangle \langle \Psi^{24}|)]B_k^{l12} \otimes A_k^{34} \\
= \Lambda_i^1 \otimes I^{34}(|\Psi^{13}\rangle \langle \Psi^{13}|) \otimes \Lambda_i^2 \otimes I^{24}(|\Psi^{24}\rangle \langle \Psi^{24}|),
\]

where the last equality come from Eq. [1]. Therefore, \( \Lambda_i \otimes I(|\Psi\rangle \langle \Psi|) \) is locally copiable with one-way classical communication. We can also easily check the opposite direction of the proof.

The correspondence between channels \( \Lambda \) and entangled states \( \Lambda \otimes I(|\Psi\rangle \langle \Psi|) \) is called Jamilowski’s isomorphism [30]. The above theorem shows that the channel copying problems can be always identified to corresponding local copying problems of entangled states in the case of one-way classical communication. On the other hands, since not all states can be written down as \( \Lambda \otimes I(|\Psi\rangle \langle \Psi|) \) for some maximally entangled state \( |\Psi\rangle \), not all local discrimination problems can be considered as a channel copying problem.

Choosing all \( \Lambda_i \) as unitary channels, we derive maximally entangled states for corresponding entangled states. Therefore, our results in Section [11] and [12] give also results for unitary channel copying as follows.

**Corollary 1** In prime-dimensional systems, a set of Unitary channels \( \{ \Lambda_i \}_{i=0}^{N-1} \) with \( \Lambda_i(\rho) = U_i \rho U_i^\dagger \) is copiable with blank noiseless channel \( \Lambda_b = I \) and one way classical communication, if and only if \( \{ U_i \}_{i=0}^{N-1} \) is simultaneously diagonalizable subset of Weil-Heisenberg (Generalized Pauli) Group.
Proof. We can easily see from Theorem 6 and 7.

The above correspondence between local copying and channel copying come from the fact that interrelationship of Schmidt basis can be always represented by actions of unitary operators. Hence, we can regard this mathematical correspondence as the one method to represent the non-locality of sets of Schmidt basis in the operational form for the corresponding unitary operations.

B. Local copying of mixed states and Entanglement distillation

As the next example of applications of our results, we consider local copying of mixed states and entanglement distillation. Although we have only considered the local copying of pure states so far, we apply our protocol for mixed states in forward and also backward directions in this subsection.

At first, if we apply our local copying protocol FIG.2 of sets of maximally entangled states \( \{ |\Psi_i\rangle\}_{i=1}^{N-1} \) to the mixed states \( \rho_m = \sum_{i=0}^{N-1} p_i |\Psi_i^1\rangle \langle \Psi_i^1| \), then we derive states \( \rho_{\text{out}} = \sum_{i=0}^{N-1} p_i (|\Psi_i^1\rangle \langle \Psi_i^1|) \otimes (|\Psi_i^2\rangle \langle \Psi_i^2|) + \sum_{i,j} a_{ij} |\Psi_i\rangle \langle \Psi_j| \otimes |\Psi_j\rangle \langle \Psi_i| \) as a result. Since the output state \( \rho_{\text{out}} \) is equivalent to \( \text{Tr}_{12}\rho_{\text{out}} \), and also to \( \sum_{i,j} a_{ij} \rho_{\text{out}} \), we seem to succeed in copying these mixed states. However, if we take account of the optimality of entanglement resource, there would be a more efficient protocol for each choice of individual probability distribution \( \{ p_i \} \). Therefore, if we try to understand the correspondence between local copying of a set of maximally entangled states and local copying of mixed states, it would be better to define the local copiability of mixed states for a set of states as well as that of pure states.

That is, we define local copiability as follows, “a sets of mixed states \( \{ \rho^1_{\xi} \}_{\xi \in \Xi} \) is locally copiable with blank states \( \sigma_{34} \) if for all \( \xi \in \Xi \), \( \text{Tr}_{12} \rho_{\xi} \otimes \rho_{\xi} = \rho_{\xi} \otimes \rho^i_{\xi} \). Then, easily, we can translate the local copiability of a set of pure states to that of mixed states as follows, “a set of maximally entangled states \( \{ |\Psi_i\rangle\}_{i=0}^{N-1} \) is locally copiable, if and only if a set of mixed states \( \{ \rho_p \}_{p \in P} \) is locally copiable with \( |\Psi_0\rangle \), where \( \rho_p = \sum_{i=0}^{N-1} p_i |\Psi_i\rangle \langle \Psi_i| \) and \( P \) is a set of all probability distribution on a set \( \{ 0, \cdots, N - 1 \} \).”

Since our local copying protocol consists of local unitary operations, we can also consider the opposite direction of our protocol. This inverse of local copying protocol is actually entanglement distillation protocol by local unitary. As we can see in FIG.2, the inverse of our protocol transforms \( |\Psi_i\rangle \otimes |\Psi_j\rangle \) to \( |\Psi_i\rangle \otimes |\Psi_0\rangle \) by a local unitary operation. Therefore, if we consider mixed states like

\[
\rho = \sum_{ij} a_{ij} |\Psi_i\rangle \otimes |\Psi_j\rangle \langle \Psi_j| \langle \Psi_i| ,
\]

and apply our local copying protocol, where \( \{ |\Psi_i\rangle\}_{i=0}^{D-1} \) is a set of simultaneous Schmidt decomposable subset of canonical Bell states, then we derive

\[
A^\dagger \otimes A^\dagger \rho A \otimes A = (\sum_{ij} a_{ij} |\Psi_i\rangle \langle \Psi_j| \otimes |\Psi_j\rangle \langle \Psi_i| \otimes |\Psi_j\rangle \langle \Psi_j| \otimes |\Psi_j\rangle \langle \Psi_i|) ,
\]

where \( A \) is a local unitary operator defined at (10). This protocol is actually entanglement distillation protocol deriving one e-bits for all mixed state. Moreover, in the case \( a_{ij} = \delta_{ij}/D \), since \( \sum_{i=0}^{D-1} \frac{1}{D} |\Psi_i\rangle \langle \Psi_i| \) is a separable state, this distillation protocol by the local unitary is optimal. Actually, the states (21) belong to a class of states called “maximally correlated states”, and the simple formula of distillable entanglement for maximally correlated states has been already known [17]. However the above protocol is deterministic and moreover unitary, this is actually important point. Generally speaking, deterministic distillable entanglement is strictly less than usual asymptotic one [18]. Therefore, this is a very rare case where the meaningful lower bound of deterministic entanglement distillation can be derived for mixed states.

C. Error correction and QKD

As another application, we can apply our result to error correction and quantum key distribution with the following specific noisy channel. Now, we consider the inverse of channel copying protocols in subsection 4.A and we derive the error correcting protocol which corresponds to the above distillation protocol. Consider a channel \( \Lambda (\rho) = \sum_{k=1}^{N} E_k \rho E_k^\dagger \) on \( \mathcal{H}_1 \otimes \mathcal{H}_2 \), where \( E_k \) satisfies \( \sum_{k=1}^{N} E_k = I \) and can be written down as \( E_k = \sum_{i=0}^{D-1} c_{ik} U_i \otimes U_i \) by a simultaneous diagonalized subset \( \{ U_i \}_{i=0}^{D-1} \) of Generalized Pauli’s Group. In particular, when a channel can be decomposed by a set of Kraus operators which have a form \( E_k = p_k U_k \otimes U_k \), the channel is called collective noise. Such a noize may occur, for example, in the case when we send two potonic qubits simultaneously through optical fibre or free space [31]. Since whole dimension of operator space \( \mathfrak{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \) is \( D^4 \), these error operators have very limited forms. However, in this case, the inverse of our copying protocol gives one noiseless channel as follows. If the channel \( \Lambda \) satisfies the above condition, then the channel \( \Lambda \) can be written down as \( \Lambda (\rho) = \sum_{ij} a_{ij} U_i \otimes U_i \rho U_j^\dagger \otimes U_j^\dagger \). Then, by the inverse of channel copying operation, we have the following relation.

\[
A^\dagger [\Lambda (A \rho A^\dagger)] A = \sum_{ij} a_{ij} A^\dagger (U_i \otimes U_i) A \rho A^\dagger (U_j^\dagger \otimes U_j^\dagger) A = \sum_{ij} a_{ij} (U_i \otimes I) \rho (U_j \otimes I).
\]

Thus, using an ancilla \( \sigma_0 \), encoding operation \( A \) and decoding operation \( A^\dagger \), we derive a noiseless channel in \( \mathcal{H}_2 \) as follows,

\[
\text{Tr}_1 A^\dagger [\Lambda (A \rho_0 \otimes \sigma) A^\dagger] A = \sigma.
\]
Similarly to the distillation case, as is shown later, when $a_{ij} = \delta_{ij}/D$, this error correcting protocol attains the asymptotic optimal rate of transmitting the quantum state through the channel $\Lambda$. That is, the transmission rate of this protocol is equal to the quantum capacity of this rate.

This fact can be seen by the correspondence between quantum capacity and distillable entanglement given in [5]. Thus, for generalized Pauli’s channel, quantum capacity coincides with distillable entanglement of the corresponding state, which is the state derived as the output state when inputting a part of a maximally entangled state, i.e., $\sum_{i=0}^{D-1} \frac{1}{D} |\Psi_i\rangle \otimes |\Psi_i\rangle \otimes \langle \Psi_i|$. Since our protocol is the optimal distillation protocol for these states, this channel coding protocol is also optimal.

Next, we apply this error correcting protocol to QKD. In the two-dimensional case, applying the above encoding and decoding protocol for usual BB84 protocol, we derive the following protocol. Alice randomly chooses a basis from $\{ |0\rangle, |1\rangle \}$ and $\{ \sqrt{\frac{1}{2}} (|0\rangle + |1\rangle) \}$. Bob performs the measurement $\{ I \otimes |0\rangle, I \otimes |1\rangle \}$ or $\{ \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) |0\rangle, \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) |1\rangle \} = \{ |0\rangle |0\rangle, |1\rangle |1\rangle \}$. When Alice uses the former basis and Bob uses the former measurement, Bob’s measured data coincides with Alice’s bit in this channel. Similarly, when Alice uses the latter basis and Bob uses the latter measurement, Bob’s measured data coincides with Alice’s bit in this channel. Therefore, if noise satisfies our assumption, we can realize the noiseless QKD by the above protocol. Even if Bob cannot perform the later, if he can realize the two-valued Bell measurement $\{ \frac{1}{\sqrt{2}} (|0\rangle |0\rangle + |1\rangle |1\rangle) |(0\rangle |0\rangle + |1\rangle |1\rangle), \frac{1}{\sqrt{2}} (|1\rangle |1\rangle - |0\rangle |0\rangle), |1\rangle |0\rangle + |0\rangle |1\rangle |0\rangle |1\rangle \}$, the noiseless QKD is available by the postselection.

VI. DISCUSSION

In this paper, we focus on a set consisting of several maximally entangled states in a prime-dimensional system. In this case, we completely characterized locally copiability and showed the relationship between locally copiability and local distinguishability. In sections [11] and [14] we proved that such a set is locally copiable, if a word, at least in prime-dimensional systems, local copying is more difficult than one-way local discrimination for a set of maximally entangled states.

In the case of local discrimination, a simultaneous Schmidt decomposable set is locally distinguishable. However, if such a set of states does not have canonical Bell form, the set is not locally copiable. We can interpret the above fact as follows. A simultaneous Schmidt decomposable set does not possess non-locality beyond individual entanglement concerning local discrimination. On the other hand, if such a set does not have canonical Bell form, such a set still has non-locality concerning local copying. In other words, we can conclude that the lack of algebraic symmetry causes extra non-locality of a set concerning local copying.

Although we only treated orthogonal sets of maximally entangled states in this paper, our result of FIG [11] also regard as the classification of sets of Schmidt basis by their non-locality. Therefore, in the case of a set of general entangled states, the structure of non-locality of sets of Schmidt basis may be similar to FIG [11] though its possesses additional non-locality which originates in various Schmidt coefficients. Therefore, our result may be useful as the base for more general discussion of non-locality problems of Schmidt basis, especially for general discussion of the local copying problems.

In section [17] we showed that our results and protocol of local copying can be interpret as results of several different and closely related quantum information processing, that is, local copying of mixed states, entanglement distillation, channel copying, error correction, and quantum key distribution. These close relation with many other protocols suggests the importance of local copying as a fundamental protocol of non-local quantum information processing.

Finally, we should mention a remained open problem. In this paper, we showed the necessity of the form of states [8] for LOCC copying only in prime-dimensional local systems. However, we restrict this dimensionality only by the technical reason, and this restriction has no physical meaning. Thus, the validity of Theorem [10] for non-prime-dimensional systems still remains as an open question.

After finishing the first draft [34], the authors found a related paper [32] which contains a different approach to Theorem [3].

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APPENDIX A: PROOF OF LEMMA 1

First, by choosing \( c = a_1 \oplus b_1 = a_2 \oplus b_2 \), we have

\[
\delta_{b_1 b_2} U_{c c} = U_{c \oplus b_1} \oplus b_2 U_{b_1 b_2}.
\]

(A1)

In addition, choosing \( b_1 \neq b_2 \), we derive

\[
U_{c \oplus b_1} \oplus b_2 U_{b_1 b_2} = 0
\]

(A2)

for all \( c \). The above equation means,

\[
b_1 \neq b_2 \Rightarrow U_{b_1 b_2} = 0 \quad \text{or} \quad \forall c, U_{c \oplus b_1} \oplus b_2 = 0.
\]

(A3)

By means of the above fact, we prove that \( U_{b b \oplus n} = U_{b \oplus b \oplus n} = 0 \) for all \( b \) by induction concerning the integer \( n \).

At first, we prove \( U_{b \oplus b \oplus 0} = 0 \) for all \( b \) by contradiction. We assume there exists \( b_1 \) such that \( U_{b_1 b_1 \oplus 1} \neq 0 \), then Equation (A3) implies \( U_{b_1 b_1 \oplus 1} = 0 \) for all \( b_1 \). On the other hand, Equation (A3) guarantees that

\[
\Xi_{b_1 b_1 \oplus 1} U_{b_1 b_1 \oplus 1} = U_{b_1 b_1 \oplus b_1 b_1 \oplus 2} U_{b_1 b_1 \oplus 1}.
\]

(A4)

Therefore, \( U_{b \oplus b \oplus 0} = 0 \) for all \( b \). Thus, repeatedly using (A4), we have \( U_{b \oplus b \oplus n} = 0 \) for all \( a \) and \( b \). This is a contradiction for the fact that \( U_{ab} \) is a non-zero matrix. So, we have \( U_{b \oplus b \oplus 0} = 0 \) for all \( b \). Similarly, we can prove \( U_{b \oplus b \oplus n} = 0 \) for all \( b \) by a contradiction as follows. Suppose there exists \( b_1 \) such that \( U_{b_1 b_1 \oplus 1} = 0 \), then Equation (A3) implies \( U_{b_1 b_1 \oplus 1} U_{b_1 b_1 \oplus b_1 b_1 \oplus (n+1)} = U_{b_1 b_1 \oplus b_1 b_1 \oplus 1} = 0 \). Therefore, \( U_{b \oplus b \oplus n} = 0 \) for all \( b \) and repeating this procedure, we have \( U_{b \oplus b \oplus n} = 0 \) for all \( a \) and \( b \). This is a contradiction. Therefore, \( U_{b \oplus b \oplus 0} = 0 \) for all \( b \).

At the next step, we assume \( U_{b \oplus b \oplus k} = U_{b \oplus b \oplus k} = 0 \) for all \( k \leq n - 1 \) and show \( U_{b \oplus b \oplus n} = 0 \) for any \( b \) by contradiction. Assume that there exists \( b_1 \) such that \( U_{b_1 b_1 \oplus n} \neq 0 \), then Equation (A4) implies \( U_{b_1 b_1 \oplus n} = 0 \) for all \( b_1 \). This is a contradiction as follows. Suppose that there exists \( b_1 \) such that \( U_{b_1 b_1 \oplus n} = 0 \), then Equation (A4) implies \( U_{b_1 b_1 \oplus n} = 0 \) for all \( b_1 \). Finally, by the mathematical induction, we prove \( U_{b \oplus b \oplus n} = 0 \) for all \( b \). Therefore, \( U_{ab} \) is a diagonal matrix. □
The paper [9] showed that if a copied set \( \{ |\psi_j\rangle\}_{j=0}^{N-1} \) has at least one maximally entangled state and is locally copiable, then all of states \( |\psi_j\rangle \) in the copied set must be maximally entangled.

[28] T. Hiroshima and M. Hayashi, *Phys. Rev. A* **70**, 030302(R) (2004).
[29] H.-K. Lo and S. Popescu, *Phys. Rev. A* **63**, 022301 (2001).
[30] A. Jamilkowski, *Rep. Math. Phys.* **3**, 275 (1972).
[31] X.B. Wang, quant-ph/0406100.
[32] M. Nathanson, quant-ph/0411110.
[33] Y. Tsuda, K. Matsumoto, M. Hayashi, quant-ph/0504203 (2005).
[34] M. Owari, M. Hayashi, quant-ph/0411143.
[35] The paper [9] showed that if a copied set \( \{ |\psi_j\rangle\}_{j=0}^{N-1} \) has at least one maximally entangled state and is locally copiable, then all of states \( |\psi_j\rangle \) in the copied set must be maximally entangled.