R-robustly measure expansive homoclinic classes are hyperbolic

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Abstract

Let $f : M \to M$ be a diffeomorphism on a closed smooth $n(\geq 2)$-dimensional manifold $M$ and let $p$ be a hyperbolic periodic point of $f$. We show that if the homoclinic class $H_f(p)$ is $R$-robustly measure expansive then it is hyperbolic.

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1. Introduction

Roughly speaking, definition of expansiveness is, if two orbits are closed then they are one orbit which was introduced by Utz [22]. A main research is to study structure of the orbits in differentiable dynamical systems, and so a goal of differentiable dynamical system is to study stability properties (Anosov, Axiom A, hyperbolic, structurally stable, etc.). Therefore, expansiveness is an important notion to study stability properties. For instance, Mañé [11] proved that if a diffeomorphism is $C^1$ robustly expansive then it is quasi-Anosov. Arbieto [1] proved that for $C^1$ generic an expansive diffeomorphism is Axiom A without cycles. For expansivity, we can find various generalization notations, that is, continuum-wise expansive [5], $n$-expansive [13], and measure expansive [14]. Among that, we study measure expansiveness in the paper. Let $M$ be a closed smooth $n(\geq 2)$-dimensional Riemannian manifold, and let $\text{Diff}(M)$ be the space of diffeomorphisms of $M$ endowed with the $C^1$-topology. Denote by $d$ the distance on $M$ induced from a Riemannian metric $\| \cdot \|$ on the tangent bundle $TM$. Let $\Lambda$ be a closed $f$-invariant set. We say that $\Lambda$ is hyperbolic if the tangent bundle $T_\Lambda M$ has a $Df$-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E^s}\| \leq C\lambda^n \text{ and } \|D_x f^{-n}|_{E^u}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$. If $\Lambda = M$, then we say that $f$ is Anosov.

For any closed $f$-invariant set $\Lambda \subset M$, we say that $\Lambda$ is expansive for $f$, if there is $e > 0$ such that for any $x, y \in \Lambda$ if $d(f^n(x), f^n(y)) \leq e$ then $x = y$. Equivalently, $\Lambda$ is expansive for $f$ if there is $e > 0$ such that
transverse homoclinic point theorem, if of cycles. It is well known that if is said to be (see [14, Theorem 1.35]). We say that \( f \) is \( \Lambda \)-measure expansive if \( \mu(\Gamma^c(x)) = 0 \). \( \Lambda \) is said to be measure expansive for \( f \) if \( \Lambda \) is \( \mu \)-expansive for all \( \mu \in \mathcal{M}^*(M) \); that is, there is a constant \( e > 0 \) such that for any \( \mu \in \mathcal{M}^*(M) \) and \( x \in \Lambda \), \( \mu(\Gamma^c(x)) = 0 \). Here \( e \) is called a measure expansive constant of \( f \). Clearly, the expansiveness implies the measure expansiveness, but the converse does not hold in general (see [14, Theorem 1.35]). We say that \( f \) is quasi-Anosov if for any \( v \in TM \setminus \{0\} \), the set \( \{\|Df^n(v)\| : n \in \mathbb{Z}\} \) is unbounded. Sakai et al. [19] proved that if a diffeomorphism \( f \) is \( C^1 \) robustly measure expansive then it is quasi-Anosov. Lee [7] proved that for \( C^1 \) generic \( f \), if \( f \) is measure expansive then it is Axiom A without cycles. It is well known that if \( p \) is a hyperbolic periodic point of \( f \) with period \( \pi(p) \) then the sets

\[
W^s(p) = \{x \in M : f^n(x) \rightarrow p \text{ as } n \rightarrow \infty\}
\]

and

\[
W^u(p) = \{x \in M : f^{-n}(x) \rightarrow p \text{ as } n \rightarrow \infty\}
\]

are \( C^1 \)-injectively immersed submanifolds of \( M \). A point \( x \in W^s(p) \cap W^u(p) \) is called a homoclinic point of \( f \) associated to \( p \). The closure of the homoclinic points of \( f \) associated to \( p \) is called the homoclinic class of \( f \) associated to \( p \), and it is denoted by \( H_r(p) \). It is clear that \( H_r(p) \) is compact, transitive, and invariant.

Denote by \( P(f) \) the set of all periodic points of \( f \). Let \( q \) be a hyperbolic periodic point of \( f \). We say that \( p \) and \( q \) are homoclinically related, and write \( p \sim q \) if

\[
W^s(p) \cap W^u(q) \neq \emptyset \quad \text{and} \quad W^u(p) \cap W^s(q) \neq \emptyset.
\]

It is clear that if \( p \sim q \) then \( \text{index}(p) = \text{index}(q) \), that is, \( \dim W^s(p) = \dim W^s(q) \). By the Smale’s transverse homoclinic point theorem, \( H_r(p) = \{q \in P(f) : q \sim p\} \), where \( A \) is the closure of the set \( A \) and \( P_h(f) \) is the set of all hyperbolic periodic points. Note that if \( p \) is a hyperbolic periodic point of \( f \) then there is a neighborhood \( U \) of \( p \) and a \( C^1 \)-neighborhood \( U(f) \) of \( f \) such that for any \( g \in U(f) \) there exists a unique hyperbolic periodic point \( p_g \) of \( g \) in \( U \) with the same period as \( p \) and \( \text{index}(p_g) = \text{index}(p) \). Such a point \( p_g \) is called the continuation of \( p \). We say that \( \Lambda \) is locally maximal if there is a neighborhood \( U \) of \( A \) such that \( \Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U) \).

In differentiable dynamical systems, a main research topic is to study that for a given system, if the system has a property then we consider that a system which is \( C^1 \)-nearby system has the same property. Then, we consider various type of \( C^1 \)-perturbation property on a closed invariant set which are the following.

(a) We say that \( H_r(p) \) is \( C^1 \) robustly \( \Psi \) property if there is a \( C^1 \)-neighborhood \( U(f) \) of \( f \) such that for any \( g \in U(f) \), \( H_g(p_g) \) is \( \Psi \) property. If \( \Psi \) is expansive then the expansive constant is uniform, which means that the constant only depends on \( f \) (see [15, 16]).

(b) We say that \( H_r(p) \) is \( C^1 \) persistently \( \Psi \) property if there is a \( C^1 \)-neighborhood \( U(f) \) of \( f \) such that for any \( g \in U(f) \), \( H_g(p_g) \) is \( \Psi \) property. If \( \Psi \) is expansive then the expansive constant is not uniform which means that the constant depends on \( g \in U(f) \) (see [20]).

(c) We say that \( H_r(p) \) is \( C^1 \) stably \( \Psi \) property if there are a \( C^1 \)-neighborhood \( U(f) \) of \( f \) and a neighborhood \( U \) of \( H_r(p) \) such that for any \( g \in U(f) \), \( \Lambda_g(U) \) is \( \Psi \) property, where \( \Lambda_g(U) = \bigcap_{n \in \mathbb{Z}} f^n(U) \) is the continuation of \( H_r(p) \). If \( \Psi \) is expansive, then the expansive constant is not uniform which means that the constant depends on \( g \in U(f) \) (see [8]).

In the item (c), we can also consider a closed invariant set. We say that a subset \( \mathcal{S} \subset \text{Diff}(M) \) is residual if \( \mathcal{S} \) contains the intersection of a countable family of open and dense subsets of \( \text{Diff}(M) \); in this case \( \mathcal{S} \) is dense in \( \text{Diff}(M) \). A property \( \mathcal{P} \) is said to be \( (C^1) \)-generic if \( \mathcal{P} \) holds for all diffeomorphisms which belong to some residual subset of \( \text{Diff}(M) \).

Li [10] introduced another \( C^1 \) robust property which is called \( R \)-robustly \( \Psi \) property. Using to the notion, we consider the following.
Definition 1.1. Let the homoclinic class \( H_f(p) \) associated to a hyperbolic periodic point \( p \). We say that \( H_f(p) \) is \( R \)-robustly measure expansive if there are a \( C^1 \)-neighborhood \( \mathcal{U}(f) \) of \( f \) and a residual set \( \mathcal{S} \) of \( \mathcal{U}(f) \) such that for any \( g \in \mathcal{S} \), \( H_g(p_g) \) is measure expansive, where \( p_g \) is the continuation of \( p \).

Recently, Pacifico and Vieites [17] proved that a diffeomorphism \( f \) in a residual subset far from homoclinic tangencies are measure expansive. Lee and Lee [9] proved that if the homoclinic class \( H_f(p) \) is \( C^1 \) stably measure expansive then it is hyperbolic. Koo et al. [6] proved that for \( C^1 \) generic \( f \), if a locally maximal homoclinic class \( H_f(p) \) is measure expansive, then it is hyperbolic. Owing to the result, we have the following which is a main theorem of the paper.

Theorem 1.2. Let the homoclinic class \( H_f(p) \) associated to a hyperbolic periodic point \( p \). If \( H_f(p) \) is \( R \)-robustly measure expansive then it is hyperbolic.

2. Dominated splitting and Hyperbolic periodic points in \( H_f(p) \)

Let \( M \) be as before, and let \( f \in \text{Diff}(M) \). A periodic point for \( f \) is a point \( p \in M \) such that \( f^{\pi(p)}(p) = p \), where \( \pi(p) \) is the minimum period of \( p \). Denote by \( P(f) \) the set of all periodic points of \( f \). For given \( x, y \in M \), we write \( x \to y \) if for any \( \delta > 0 \), there is a \( \delta \)-pseudo orbit \( \{x_i\}_{i=0}^{n} \) of \( f \) such that \( x_0 = x \) and \( x_n = y \). We write \( x \leftrightarrow y \) if \( x \to y \) and \( y \to x \). The set of points \( \{x \in M : x \leftrightarrow x \} \) is called the chain recurrent set of \( f \). It is clear that \( P(f) \subset \Omega(f) \subset \mathcal{R}(f) \). Here \( \Omega(f) \) is the non-wandering set of \( f \). Let \( p \) be a hyperbolic periodic point of \( f \). We say that the chain component if for any \( x \in M \), \( x \to p \) and \( p \to x \) and denote it by \( C_f(p) \). Note that the chain component \( C_f(p) \) of \( f \) is a class and it is a closed set and \( f \)-invariant set. The following was proved by Bonatti and Crovisier [2].

Remark 2.1. There is a residual set \( \mathcal{S}_1 \subset \text{Diff}(M) \) such that for any \( f \in \mathcal{S}_1 \), \( H_g(p) = C_f(p) \) for some hyperbolic periodic point \( p \).

Proposition 2.2. Let the homoclinic class \( H_f(p) \) be \( R \)-robustly measure expansive. If \( x \in W^s(p) \cap W^u(p) \), then \( x \in W^s(p) \cap W^u(p) \).

Proof. Since \( H_f(p) \) is \( R \)-robustly measure expansive, there exists a \( C^1 \)-neighborhood \( \mathcal{U}(f) \) and a residual set \( \mathcal{S} \subset \mathcal{U}(f) \) such that for any \( g \in \mathcal{S} \), \( H_g(p_g) \) is measure expansive. By [17, Proposition 2.6], there is \( g \in \mathcal{U}(f) \cap \mathcal{S} \) such that we can make a small arc \( \mathcal{J} \subset W^s(p_g) \cap W^u(p_g) \). Since \( H_g(p_g) = C_g(p_g) \), we know \( \mathcal{J} \subset C_g(p_g) \). Let \( \text{diam}(\mathcal{J}) = 1 \). We define a measure \( \mu \in \mathcal{M}^+(M) \) by \( \mu(C) = \nu(C \cap \mathcal{J}) \) for any Borel set \( C \subset M \), where \( \nu \) is a normalized Lebesgue measure on \( \mathcal{J} \). Let \( \epsilon = 1/4 \) be a measure expansive constant. Since \( \mathcal{J} \subset W^s(p_g) \cap W^u(p_g) \), there is \( N > 0 \) such that \( \text{diam}(g^i(\mathcal{J})) \leq \epsilon/4 \) for \( -N \leq i \leq N \), and \( g^i(\mathcal{J}) \subset W^s_{\epsilon/4}(p_g) \cap W^u_{\epsilon/4}(p_g) \). Thus for all \( i \in \mathbb{Z} \), we know that \( \text{diam}(g^i(\mathcal{J})) \leq \epsilon \). Recall that

\[ \Gamma_\epsilon(x) = \{y \in H_g(p_g) : d(g^i(x), g^i(y)) \leq \epsilon \text{ for } i \in \mathbb{Z} \} \]

We can construct the set

\[ A_\epsilon(x) = \{y \in \mathcal{J} : d(g^i(x), g^i(y)) \leq \epsilon \text{ for } i \in \mathbb{Z} \} \]

Then we know \( A_\epsilon(x) \subset \Gamma_\epsilon(x) \). Thus we have

\[ 0 < \mu(A_\epsilon(x)) \leq \mu(\Gamma_\epsilon(x)) \]

which is a contradiction to the measure expansivity of \( H_g(p_g) \).

\[ \square \]

For \( f \in \text{Diff}(M) \), we say that a compact \( f \)-invariant set \( \Lambda \) admits a dominated splitting if the tangent bundle \( T\Lambda M \) has a continuous \( Df \)-invariant splitting \( E \oplus F \) and there exist \( C > 0, 0 < \lambda < 1 \) such that for all \( x \in \Lambda \) and \( n \geq 0 \), we have

\[ \|Df^n|_{E(x)}\| \cdot \|Df^{-n}|_{F(f^n(x))}\| \leq C\lambda^n \.]
Theorem 2.3. Let \( H_f(p) \) be the homoclinic class containing a hyperbolic periodic point \( p \). Suppose that \( H_f(p) \) is R-robustly measure expansive. Then there exist a \( C^1 \)-neighborhood \( \mathcal{U}(f) \) of \( f \) and a residual set \( \mathcal{S} \subset \mathcal{U}(f) \) such that for any \( g \in \mathcal{S} \), \( H_g(p_g) \) admits a dominated splitting \( \mathcal{T}_{H_g(p_g)}M = E(g) \oplus F(g) \) with \( \text{index}(p_g) = \dim E(g) \).

Proof. Suppose that \( H_f(p) \) is R-robustly measure expansive. Then as in the proof of [20, Theorem 1], there is \( m > 0 \) such that for every \( x \in W^s(p) \cap W^u(p) \) there exists \( m_1 \in [90, m] \) such that \( \|Df^{m_1}|_{E(x)}\cdot\|Df^{-m_1}|_{E(f^{-m_1}(x))}\| < 1/2 \). Since the dominated splitting can be extended by continuity to the

\[
W^s(p) \cap W^u(p) = H_f(p),
\]

we have that \( H_f(p) \) has a dominated splitting \( E \oplus F \).

\[
\square
\]

Theorem 2.4. Let the homoclinic class \( H_f(p) \) be R-robustly measure expansive. Then there exist \( C > 0, 0 < \lambda < 1 \) and \( m > 0 \) such that \( q \) is a hyperbolic periodic point of period \( \pi(q) \) and \( q \sim p \), then

\[
\prod_{l=0}^{k-1} \|Df^m|_{E_x(f^m(q))}\| < C\lambda^k \quad \text{and} \quad \prod_{l=0}^{k-1} \|Df^{-m}|_{E_u(f^{-m}(q))}\| < C\lambda^k,
\]

where \( k = \lfloor \pi(q)/m \rfloor \) (\( \lfloor \cdot \rfloor \) represents the integer part).

Proof. Since \( H_f(p) \) is R-robustly measure expansive, there are a \( C^1 \)-neighborhood \( \mathcal{U}(f) \) and a residual set \( \mathcal{S} \subset \mathcal{U}(f) \) such that for any \( g \in \mathcal{S} \), \( H_g(p_g) \) is measure expansive. Let \( \mathcal{G} = \mathcal{G}_1 \). Since \( q \in H_f(p) \) and \( p \sim q \), as in the proof of [20], it is enough to show that the family of periodic sequences of linear isomorphisms of \( \mathbb{R}^n \) generated by \( Df \) along the hyperbolic periodic points \( q \in H_f(p), p \sim q \) and \( \text{index}(p) = \text{index}(q) \) is uniformly hyperbolic. Suppose, by contradiction, that the assume does not hold. Then as in the proof of [18, Theorem B], we may assume that a hyperbolic periodic point \( q \in H_f(p) \) such that the weakest normalized eigenvalue \( \lambda \) is close to 1. Then by Franks lemma, there is \( g \in \mathcal{G} \) such that for any small \( \gamma > 0 \) we can construct a closed small curve \( \mathcal{J}_q \) containing \( q \) or a closed small circle \( \mathcal{E}_q \) centered at \( q \) such that \( \mathcal{J}_q \subset C_g(p_g) \) and two endpoints are related to \( p_g \) and \( \mathcal{E}_q \subset C_g(p_g) \). Note that \( \mathcal{J}_q \) and \( \mathcal{E}_q \) are \( g^\pi(q) \)-invariant, normally hyperbolic, and \( g^\pi(q)|_{\mathcal{J}_q} \) is the identity map for some \( l > 0 \) (see [18]). For \( \mathcal{J}_q \), we define a measure \( \mu \in \mathcal{M}^s(M) \) by

\[
\mu(C) = \frac{1}{\pi(q)} \sum_{i=0}^{l\pi(q)-1} \nu(g^{-i}(C \cap g^i(\mathcal{J}_q)))
\]

for any Borel set \( C \) of \( M \), where \( \nu \) is a normalized Lebesgue measure on \( \mathcal{J}_q \). Let \( \gamma \leq \varepsilon \) be a measure expansive constant of \( g|_{H_g(p_g)} \). By [14, Proposition], \( g \) is measure expansive if and only if \( g^n \) is measure expansive for \( n \in \mathbb{Z} \setminus \{0\} \). Let \( \Gamma^g_{\varepsilon}(x) = \{ y \in H_g(p_g) : d(g^{l\pi(q)i}(x), g^{l\pi(q)i}(y)) \leq \varepsilon, \text{ for all } i \in \mathbb{Z} \} \). Then we have

\[
\{ y \in \mathcal{J}_q : d(g^{l\pi(q)i}(x), g^{l\pi(q)i}(y)) \leq \varepsilon \text{ for all } i \in \mathbb{Z} \} = \{ y \in \mathcal{J}_q : d(g^i(x), g^i(y)) \leq \varepsilon \text{ for all } i \in \mathbb{Z} \}.
\]

Thus we know

\[
0 < \mu(\{ y \in \mathcal{J}_q : d(g^i(x), g^i(y)) \leq \varepsilon \text{ for all } i \in \mathbb{Z} \}) \leq \mu(\Gamma^g_{\varepsilon}(x)).
\]

Since \( H_g(p_g) \) is measure expansive for \( g \), we know \( \mu(\Gamma^g_{\varepsilon}(x)) = 0 \). Thus we have

\[
\mu(\{ y \in \mathcal{J}_q : d(g^i(x), g^i(y)) \leq \varepsilon \text{ for all } i \in \mathbb{Z} \}) = 0.
\]

This is a contradiction.

For \( \mathcal{E}_q \) case, if \( \mathcal{E}_q \) is irrational rotation then using the Franks’ lemma, there is \( h \in \mathcal{U}(g) \cap \mathcal{G} \) such that \( \mathcal{E}_{q_h} \) is rational rotation which is centered at \( q_{h\nu} \), where \( \mathcal{U}(g) \) is a \( C^1 \)-neighborhood of \( g \), and \( q_{h\nu} \) is the
Proposition 3.1. Let \( q \) be a hyperbolic periodic point of period \( \pi(q) \) with \( L > \pi(q) \) and \( q \sim p \), then
\[
\prod_{i=0}^{\pi(q)-1} \| Df|_{E^s(f^i(q))} \| < \lambda^{\pi(q)} \quad \text{and} \quad \prod_{i=0}^{\pi(q)-1} \| Df|_{E^u(f^{-i}(q))} \| < \lambda^{\pi(q)}.
\]

3. Local product structure

Let \( \Lambda \) be a closed, \( f \)-invariant set. We say that \( \Lambda \) has a local product structure if for given \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that if \( d(x, y) < \delta \) and \( x, y \in \Lambda \), then
\[
\emptyset \neq W^s_\epsilon(x) \cap W^u_\epsilon(y) \subset \Lambda.
\]

By the uniqueness of the dominated splitting, if \( q \in H_f(p) \) is a periodic point with \( q \sim p \) then we have \( E(q) = E^s(q) \) and \( F(q) = E^u(q) \). Let \( \dim E = s \) and \( \dim F = u \), and put \( D^r_f = \{ x \in \mathbb{R}^1 : \| x \| \leq r \} \) \( (r > 0) \), for \( j = s, u \). Let \( \text{Emb}_\Lambda(D^r_f, M) \) be the space of \( \mathcal{C}^1 \) embeddings \( \beta : D^r_f \rightarrow M \) such that \( \beta(0) \in \Lambda \) endowed with the \( \mathcal{C}^1 \) topology. Then we have the following.

Proposition 3.1 ([4, 12]). Let \( H_f(p) \) be the homoclinic class of \( f \) associated to a hyperbolic periodic point \( p \), and let \( \Lambda = H_f(p) \). Suppose that \( \Lambda \) has a dominated splitting \( E \oplus F \). Then there exist sections \( \phi^s : \Lambda \rightarrow \text{Emb}_\Lambda(D^s_f, M) \) and \( \phi^u : \Lambda \rightarrow \text{Emb}_\Lambda(D^u_f, M) \) such that by defining \( W^s_\epsilon(x) = \phi^s(x)D^s_\epsilon \) and \( W^u_\epsilon(x) = \phi^u(x)D^u_\epsilon \), for each \( x \in \Lambda \), we have

1. \( T_xW^s_\epsilon(x) = E(x) \) and \( T_xW^u_\epsilon(x) = F(x) \);
2. for every \( 0 < \epsilon_1 < 1 \) there exists \( 0 < \epsilon_2 < 1 \) such that \( f(W^s_\epsilon^1(x)) \subset W^s_{\epsilon_2}(f(x)) \) and \( f^{-1}(W^u_\epsilon^1(x)) \subset W^u_{\epsilon_2}(f^{-1}(x)) \);
3. for every \( 0 < \epsilon_1 < 1 \) there exists \( 0 < \delta < 1 \) such that if \( d(x, y) < \delta \) \( (x, y \in \Lambda) \) then \( W^s_{\epsilon_1}(x) \cap W^u_{\epsilon_1}(y) \neq \emptyset \), and this intersection is transverse.

The sets \( W^s_\epsilon(x) \) and \( W^u_\epsilon(x) \) are called the local center stable and local unstable manifolds of \( x \), respectively. The following lemma can be proved similarly to that of Lemma 4 in [20].

Lemma 3.2. Let \( H_f(p) \) be the homoclinic class of \( f \) associated to a hyperbolic periodic point \( p \), and suppose that \( H_f(p) \) is \( R \)-robustly measure expansive. Then for \( C, \lambda \) as in Theorem 3.1 and \( \delta > 0 \) satisfying \( \lambda' = \lambda(1 + \delta) < 1 \) and \( q \sim p \), there exists \( 0 < \epsilon_1 < 1 \) such that if for all \( 0 \leq n \leq \pi(q) \) it holds that for some \( \epsilon_2 > 0 \), \( f^n(W^s_\epsilon^1(q)) \subset W^s_{\epsilon_2}(f^n(q)) \), then
\[
f^n(q)(W^s_\epsilon^1(q)) \subset W^s_{C\lambda^m(q)\epsilon_2}(q).
\]

Similarly, if \( f^{-n}(W^u_\epsilon^1(q)) \subset W^u_{\epsilon_1}(f^{-n}(q)) \), then
\[
f^{-n}(q)(W^u_\epsilon^1(q)) \subset W^u_{C\lambda^m(q)\epsilon_2}(q).
\]
Recall that by using the Smale’s transverse theorem, we have $H_f(p) = \overline{\text{homo}_p}$, where $\text{homo}_p = \{q \in P_h(f) : q \sim p\}$.

**Lemma 3.3.** Let $H_f(p)$ be the homoclinic class of $f$ associated to a hyperbolic periodic point $p$, and let $\epsilon > 0$ be a measure expansive constant. Suppose that $H_f(p)$ is $R$-robustly measure expansive. Then

(a) for any hyperbolic periodic point $q \in \text{homo}_p$ and $0 < \epsilon_1 < \epsilon$, there is $\epsilon_2 > 0$ such that

$$f^n(W^{cs}_{\epsilon_2}(q)) \subset W^{cs}_{\epsilon_1}(f^n(q)) \text{ and } f^{-n}(W^{cs}_{\epsilon_2}(q)) \subset W^{cs}_{\epsilon_1}(f^{-n}(q)) \text{ for all } n \geq 0.$$  

(b) for any $y \in W^{cs}_{\epsilon_2}(q)$ and $q \in \text{homo}_p$ we have

$$\lim_{n \to \infty} d(f^n(q), f^n(y)) = 0.$$  

**Proof.** Let $f \in \mathcal{S} = S_1$ and let $H_f(p)$ be a measure expansive class. To prove (a), it is enough to show that $f^n(W^{cs}_{\epsilon_2}(q)) \subset W^{cs}_{\epsilon_1}(f^n(q))$. Let $\sup(\text{dim} W^{cs}_{\epsilon_2}(q) : q \in \text{homo}_p) < \epsilon$. Since $q \in \text{homo}_p$, we define

$$\epsilon(q) = \sup(\epsilon > 0 : f^n(W^{cs}_{\epsilon_2}(q)) \subset W^{cs}_{\epsilon_1}(f^n(q)) \text{ for all } n \geq 0).$$

By Proposition 3.1 and Lemma 3.2, $\epsilon(q) > 0$. Let $\epsilon_0 = \inf(\epsilon(q) : q \in \text{homo}_p)$. If $\epsilon_0 > 0$ then it is a proof of (a). Suppose, by contradiction, that there is a sequence $\{q_n\} \subset \text{homo}_p$ such that $\epsilon(q_n) \to 0$ as $n \to \infty$. Then we have $0 < m_n < \pi(q_n)$ and $y_n \in W^{cs}_{\epsilon(q_n)}(q_n)$ such that $d(f^m(q_n), f^m(y_n)) = \epsilon_1$ for $f^m(q_n), f^m(y_n) \in W^{cs}_{\epsilon(q_n)}(q_n)$. Let $I_n$ be a closed connected arc joining $f^m(q_n)$ with $f^m(y_n)$. Then we know that

(i) $I_n \subset W^{cs}_{\epsilon(q_n)}(q_n)$;
(ii) $f^i(I_n) \subset W^{cs}_{\epsilon_1}(f^i(q_n))$ for $0 \leq i \leq \pi(q_n)$;
(iii) $\text{diam}(I_n) = \epsilon_1$.

By Lemma 3.2, we know $f^\pi(q_n)(W^{cs}_{\epsilon(q_n)}(q_n)) \subset W^{cs}_{\epsilon(q_n)}(q_n)$. Observe that if $n \to \infty$ then $m_n \to \infty$ and $\pi(q_n) - m_n \to \infty$. Suppose that $f^m(q_n) \to x$ and $f^m(y_n) \to y$ as $n \to \infty$. Then $I_n \to I$, where $I$ is a close connected arc joining $x$ with $y$. It means that $\text{diam}(I) < \epsilon_1$ for all $j \in \mathbb{Z}$, and $x \in \overline{\text{homo}_p} = H_f(p)$. We show that the closed connected arc $I \subset H_f(p)$. Since $f \in \mathcal{S}$, $H_f(p) = C_f(p)$. For any $a \in I$, take $a_n \in W^{cs}_{\epsilon(q_n)}(q_n)$ such that $f^m(a_n) \to a$ as $n \to \infty$. As in the proof of [21, Lemma 2.6], let $\epsilon > 0$ be arbitrary. Let $n \in \mathbb{N}$ be such that $\epsilon(q_n) < \epsilon$. Then for $n$ sufficiently large, $\{q_n, f(a_n), \ldots, f^{m-1}(a_n), a, f^{m+1}(a_n), \ldots, f^{\pi(a_n)-1}(a_n), q_n\}$ is a periodic $\epsilon$-chain through $a$ and having a point in $H_f(p)$. Since $q_n \in \text{homo}_p$, $H_f(q_n) = H_f(p) = C_f(q_n) = C_f(p)$ and so the closed connected arc $I \subset H_f(p)$. We define a measure $\mu \in M^s(M)$ by $\mu(C) = \mu_1(C \cap I)$ for any Borel set $C$ of $M$, where $\mu_1$ is a normalized Lebesgue measure on $I$. Let

$$\Gamma_{\epsilon}(x) = \{y \in H_f(p) : d(f^i(x), f^i(y)) \leq \epsilon \text{ for } i \in \mathbb{Z}\}.$$  

Since for all $i \in \mathbb{Z}$, $\text{diam}(f^i(I)) \leq \epsilon$, we can construct the set

$$\{y \in I : d(f^i(x), f^i(y)) \leq \epsilon \text{ for } i \in \mathbb{Z}\}.$$  

Then we know $\{y \in I : d(f^i(x), f^i(y)) \leq \epsilon \text{ for } i \in \mathbb{Z}\} \subset \Gamma_{\epsilon}(x)$. Thus we have

$$0 < \mu(\{y \in I : d(f^i(x), f^i(y)) \leq \epsilon \text{ for } i \in \mathbb{Z}\}) \leq \mu(\Gamma_{\epsilon}(x)).$$

Since $H_f(p)$ is measure expansive, $\mu(\Gamma_{\epsilon}(x)) = 0$. Thus $\mu(\{y \in I : d(f^i(x), f^i(y)) \leq \epsilon \text{ for } i \in \mathbb{Z}\}) = 0$ which is a contradiction.

The proof of (b) is similar as in the proof of item (b) of [21, Lemma 2.6].
Remark 3.4. In the Lemma 3.3, we consider \( q \in \text{homo}_p \). Then we can extend \( x \in H_f(p) \), that is, for any \( x \in H_f(p) \) and \( \epsilon_1 > 0 \) there exists \( \epsilon_2 > 0 \) such that \( f^n(W_{\epsilon_2}^s(x)) \subset W_{\epsilon_1}^s(f^n(x)) \) for all \( n \geq 0 \). And if \( z \in W_{\epsilon_2}^s(x) \) and \( z \in H_f(p) \), then \( d(f^k(z), f^k(x)) \to 0 \) as \( k \to \infty \).

**Proposition 3.5.** Suppose that the homoclinic class \( H_f(p) \) is R-robustly measure expansive. Then \( H_f(p) \) has a local product structure.

**Proof.** By Lemma 3.3, there is \( \epsilon_2 > 0 \) such that for any \( q \in \text{homo}_p \)

\[
W_{\epsilon_2}^c(q) = W_{\epsilon_2}^s(q) \text{ and } W_{\epsilon_2}^{cu}(q) = W_{\epsilon_2}^u(q).
\]

By Proposition 3.1 (3), there is \( \delta > 0 \) such that for any \( q, r \in \text{homo}_p \),

\[
W_{\epsilon_2}^c(q) \cap W_{\epsilon_2}^r(r) \neq \emptyset.
\]

By \( \lambda \)-lemma, \( W_{\epsilon_2}^c(q) \subset W^c(p) \) and \( W_{\epsilon_2}^{cu}(r) \subset W^u(p) \). Thus we know that \( W_{\epsilon_2}^c(q) \cap W_{\epsilon_2}^{cu}(r) \subset H_f(p) \). This means that \( H_f(p) \) has a local product structure. \( \square \)

**Corollary 3.6.** Suppose that the homoclinic class \( H_f(p) \) is R-robustly measure expansive. Then for any hyperbolic periodic point \( q \in H_f(p) \), \( \text{index}(p) = \text{index}(q) \).

**Proof.** The proof is directly obtained by Proposition 3.1 (3), Lemma 3.3, and Proposition 3.5. Thus for any hyperbolic periodic point \( q \in H_f(p) \),

\[
W^s(p) \cap W^u(q) \neq \emptyset \text{ and } W^u(p) \cap W^s(q) \neq \emptyset.
\]

Thus we have \( \text{index}(p) = \text{index}(q) \). \( \square \)

4. Proof of Theorem 1.2

For any \( \delta > 0 \), a sequence \( \{x_i\}_{i \in \mathbb{Z}} \) is a \( \delta \)-pseudo orbit of \( f \) if \( d(f(x_i), x_{i+1}) < \delta \) for all \( i \in \mathbb{Z} \). Let \( \Lambda \) be a closed \( f \)-invariant set. We say that \( f \) has the shadowing property on \( \Lambda \) such that for any \( \epsilon > 0 \) there is \( \delta > 0 \) such that for any \( \delta \)-pseudo orbit \( \{x_i\}_{i \in \mathbb{Z}} \subset \Lambda \) there is \( z \in M \) such that \( d(f^{i}(z), x_i) < \epsilon \) for all \( i \in \mathbb{Z} \). The following proposition is a very useful result for proving of Theorem 1.2.

**Proposition 4.1** ([23, Proposition 3.3]). Let \( p \) be a hyperbolic periodic point, and let \( H_f(p) \) be the homoclinic class of \( f \) containing \( p \). Let \( 0 < \lambda < 1 \) and \( L \geq 1 \) be given. Assume that \( H_f(p) \) satisfies the following properties.

(1) There is a continuous \( Df \)-invariant splitting \( T_{H_f(p)}M = E \oplus F \) with \( \dim E = \text{index}(p) \) such that for any \( x \in H_f(p) \),

\[
\|Df|_{E(x)}\|/m(Df|_{F(x)}) < \lambda^2,
\]

where \( m(A) = \inf \|A\| : \|v\| = 1 \) denotes the miniminorm of a linear map \( A \).

(2) For any \( q \in H_f(p) \cap P(f) \), if \( q \) is hyperbolic and \( \pi(q) > L \), then \( \text{index}(p) = \text{index}(q) \) and

\[
\prod_{i=0}^{\pi(q)-1} \|Df|_{E(f^i(q))}\| < \lambda^{\pi(q)}, \quad \prod_{i=0}^{\pi(q)-1} \|Df^{-1}|_{E(f^{-i}(q))}\| < \lambda^{\pi(q)}.
\]

(3) \( f \) has the shadowing property on \( H_f(p) \).

Then \( H_f(p) \) is hyperbolic.

**End of the Proof of Theorem 1.2.** Since \( H_f(p) \) is R-robustly measure expansive, by Theorems 2.3 and 2.5, propositions (1) and (2) hold. By Proposition 3.5 and Bowen’s result [3, Proposition 3.6], if the homoclinic class \( H_f(p) \) is R-robustly measure expansive then \( f \) has the shadowing property on \( H_f(p) \), and so, proposition (3) also holds. Thus if \( H_f(p) \) is R-robustly measure expansive then it is hyperbolic. \( \square \)
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