We construct a field-theoretic description of spin waves in hexagonal antiferromagnets with three magnetic sublattices and coplanar 120° magnetic order. The three Goldstone modes can be separated by point-group symmetry into a singlet $a_0$ and a doublet $\beta$. The $a_0$ singlet is described by the standard theory of a free relativistic scalar field. The field theory of the $\beta$ doublet is analogous to the theory of elasticity of a two-dimensional isotropic solid with distinct longitudinal and transverse “speeds of sound.” The well-known Heisenberg models on the triangular and kagome lattices with nearest-neighbour exchange turn out to be special cases with accidental degeneracy of the spin-wave velocities. The speeds of sound can be readily calculated for any lattice model. We apply this approach to the compounds of the Mn$_3$X family with stacked kagome layers.

I. INTRODUCTION

The study of spin waves, gentle excitations around a magnetic ground state, in terms of a local, continuum field theory is well established [1]. The ordered moments are expressed in terms of classical vector fields:

$$S = S \left( \sin \theta \sin \phi, \sin \theta \cos \phi, \cos \theta \right)$$

(1)

where $S$ is the local spin/magnetization-length and $\theta(t, r)$ and $\phi(t, r)$ are slowly varying fields. Although this approach cannot be applied on the atomic scale, it has proved to be useful to study the slowly varying, in $(t, r)$, deviations from the ground state. These spin wave field theories were extensively studied in the case of the triangular lattice [13] and kagome [14] antiferromagnets. In these highly symmetric scenarios the emergent field theory is a non-linear $\sigma$ model for an appropriate order parameter ($O(3)$ for antiferromagnets and $O(2)$ for a planar ferromagnet). The field theory has also been utilised to study the combined interactions of spin waves (magnons) with solitons like domain walls [4], and magnetic vortices [5].

The spin waves are conveniently expressed in the basis of normal modes of the spin system. These modes form a symmetry governed irreducible representation for the spin degrees of freedom (rotational) in a magnetic unit cell [6]. The normal modes in the case of a system where exchange is the dominant interaction, provide an intuitive picture of the spin wave excitations. Indeed the calculation of the spin wave spectra using a field theory expressed in this basis avoids the heavy machinery of the Holstein-Primakoff mean field theory.

The theory is often stated as the Landau-Lifshitz equation for the magnetization field:

$$\partial_t \mathbf{s} = -\gamma H_{\text{eff}} \times \mathbf{s}, \quad H_{\text{eff}}(\mathbf{r}) = -\frac{1}{S} \frac{\delta U[\mathbf{s}(\mathbf{r})]}{\delta \mathbf{s}(\mathbf{r})}.$$  

(2)

Here $\mathbf{s}(t, \mathbf{r}) = \mathbf{S}(t, \mathbf{r})/S$ is the unit vector parallel to the spin and $\gamma$ is the gyromagnetic ratio. $H_{\text{eff}}$ is an effective magnetic field derived from the energy functional $U[\mathbf{s}]$ and is dominated by the exchange interactions.

The Landau-Lifshitz equation can be derived from the Lagrangian

$$\mathcal{L} = S \mathbf{a}(\mathbf{s}) \cdot \dot{\mathbf{s}} - U[\mathbf{s}],$$

(3)

where $\mathbf{a}(\mathbf{s})$ is a vector potential on the magnetization sphere such that $\nabla_s \times \mathbf{a} = -\mathbf{s}$. The standard choice for the vector potential, $\mathbf{a}(\mathbf{s}) \cdot \dot{\mathbf{s}} = (\cos \theta - 1)\dot{\phi}$, has a singularity (Dirac string) at the south pole, $\theta = \pi$ [7–9].

In antiferromagnets the exchange interaction enforces a zero net magnetization per unit cell, $\sum_i \mathbf{S}_i = 0$, where the summation is over sublattices. Normal modes that violate this condition are costly and will be referred to as ‘hard.’ We will be focused on soft modes that preserve the condition of zero net spin. They enter the energy density $U[\mathbf{s}]$ in the form of gradients.

In this paper we construct the spin-wave theory for generic hexagonal antiferromagnets with three magnetic sublattices. A field theory for an antiferromagnet on the triangular lattice was developed previously by Dombre and Read [10, 11] in search for a topological term in the quantum field theory as found in 1 dimension by Haldane [12]. Our primary motivation, however, is to obtain the spectrum of spin waves for a broad class of antiferromagnets with a hexagonal symmetry of the lattice and dominant Heisenberg exchange interactions. The triangular-lattice [13] and kagome [14] antiferromagnets are special cases with accidental degeneracy of the spin-wave spectra.

Some features unique to the three-sublattice antiferromagnet emerge through this theory. Firstly, there are now three Goldstone modes as compared to two for the two-sublattice case. This happens because the the Néel order parameter (staggered magnetization) for the two-sublattice case breaks the $SO(3)$ symmetry of the spin vectors only partially, down to $SO(2)$ rotations about the Néel vector. The three-sublattice magnetic order breaks the symmetry fully, resulting in three Goldstone modes. Secondly, from the perspective of point-group symmetry, the three Goldstone modes can be grouped into a singlet and a vector doublet. The field theory for this doublet is analogous to the continuum theory of elasticity in two dimensions.
We start by reviewing the familiar micromagnetic field theories of the easy-plane Heisenberg ferromagnet and the two-sublattice Heisenberg antiferromagnet in Sec. II. We highlight some technical features like generation of inertia from ‘hard’ modes in this familiar setting. We proceed to a study of the lattice geometry and normal mode structure in hexagonal antiferromagnets, Sec. III. We derive a field theory for the soft modes and test this theory on the familiar kagome and triangular lattice antiferromagnet in Sec. IV comparing our results with the Holstein Primakoff calculations on these models [11, 13, 14].

This is followed by a theory for the stacked kagome antiferromagnet Mn$_3$Ge, a representative of the Mn$_3$X family, Sec. V. In this system the calculated spin wave dispersions have been matched to the neutron-scattering data [15] to extract effective exchange constants for the compound. We end with a summary highlighting the broader applicability of the effective field theory in Sec. VI.

**II. ONE AND TWO SUBLATTICE FIELD THEORIES IN 2D**

**A. Easy-plane ferromagnet**

In this section we start with the field theory for an easy-plane ferromagnet [16, 17]. The dynamical term in the Lagrangian is purely geometric and measures the Berry phase picked up by an individual moment at a location in space over time [18, 19].

\[
\mathcal{L} = S \mathbf{a}(s) \cdot \dot{s} = S(\cos \theta - 1) \dot{\phi},
\]

where $\mathbf{a}$ has been written using the standard gauge choice, see Eq. (3), and the spin vector has been expressed using Eq. (1).

The Heisenberg exchange expressed in terms of gradients of the unit vector spin field $s(t, \mathbf{r})$ producing a potential energy density:

\[
U[\mathbf{S}] = -\mathcal{J} \sum_{<i,j>} \mathbf{S}_i \cdot \mathbf{S}_j + \frac{S^2 \delta}{2} \sum_i s_i^2
\]

\[
= S^2 \int dV \left( \frac{\mathcal{J}}{2} (\nabla s)^2 + \frac{S^2 \delta}{2} s_i^2 \right),
\]

where we have added an easy plane anisotropy of strength $\delta$ and $\mathcal{J}$ is the strength of the antiferromagnetic exchange. The anisotropy $\phi$ prefers the spins to lie in the easy plane, $\theta \simeq \pi/2$. For this reason the coordinate $\theta$ is a hard coordinate while $\phi$ is a soft mode. In polar angles, the full continuum theory for the simple ferromagnet has the Lagrangian

\[
\mathcal{L} = S(\cos \theta - 1) \dot{\phi} - \frac{\mathcal{J} S^2}{2} \left[ (\nabla \theta)^2 + (\sin \theta \nabla \phi)^2 \right] - \frac{S^2 \delta}{2} \cos^2 \theta.
\]

The Heisenberg ferromagnet in 2d has regions of order, where the spins are mostly aligned, alongside gentle spin waves. The order spontaneously breaks the $O(2)$ symmetry in plane, leading to a single Goldstone mode (the spin wave in the $xy$ plane).

To obtain the dispersion for the Goldstone mode we drop the gradient of the hard mode. The assumption is that the hard field dynamics occurs on a far shorter timescale than the soft mode dynamics and is essentially guided by the soft mode. This allows us to integrate out the hard mode $\theta$ using its equation of motion: $\cos \theta = \phi/S\delta$. The theory for the $\phi(t, \mathbf{r})$ field now reads:

\[
\mathcal{L}(\phi) = \frac{\rho}{2} \dot{\phi}^2 - \frac{\mathcal{J} S^2}{2} (\nabla \phi)^2,
\]

where $\rho = 1/\delta$ quantifies the inertia of the $\phi$ field. This procedure of generating an inertia for the soft mode is ubiquitous and this ‘mass’ is referred to as a Döring mass [20]. From this it is easy to see that the single Goldstone mode has a linear dispersion $\omega = c k$ with the spin-wave velocity $c^2 = \mathcal{J} S^2/\rho$.

**B. Two-sublattice antiferromagnet**

The field theory for the two sublattice antiferromagnet is constructed from the underlying ferromagnetic field theories. The exchange energy for the system is:

\[
U[\mathbf{S}] = \mathcal{J} \sum_{<i,j>} \mathbf{S}_i \cdot \mathbf{S}_j + \frac{\mathcal{J} S^2}{2} \sum_\alpha (s_1 + s_2)^2,
\]

where the second expression is a sum over the unit cells formed from the two sites. The energy is minimized if $s_1 + s_2 = 0$ for each unit cell.

The spins on each sublattice, $s_1$ and $s_2$, can be rewritten in terms of the staggered magnetization $\mathbf{n} = (s_1 - s_2)/2$ and the uniform magnetization $\mathbf{m} = s_1 + s_2$. Both $s_1, s_2$ are unit vectors and this imposes $\mathbf{n}^2 = 1$ and $\mathbf{n} \cdot \mathbf{m} = 0$ [21].

The Lagrangian density is given by [22]:

\[
\mathcal{L} = S[a_1(s_1) \cdot \dot{s}_1 + a_2(s_2) \cdot \dot{s}_2] = \mathbf{S} \cdot (\mathbf{n} \times \mathbf{m}).
\]

The exchange energy in terms of the hard field $\mathbf{m}$ and the gradients of the soft field (lattice constant set to one):

\[
U = \frac{\mathcal{J} S^2}{2} \sum_\alpha (s_1 + s_2)^2 = \int dV \frac{\mathcal{J} S^2}{2} \mathbf{m}^2.
\]

Note that unlike the XY ferromagnet here the energy of the hard mode $\mathcal{J} S^2 \mathbf{m}^2/2$ is due to exchange. Nevertheless, the procedure to obtain the effective field theory is similar: we integrate out the hard field and express the theory in terms of the soft field and this process generates an inertia for the soft mode.

\[
\mathcal{L} = \mathbf{S} \cdot (\mathbf{n} \times \mathbf{m}) - \frac{\mathcal{J} S^2}{2} \left[ (\nabla \mathbf{n})^2 + (\mathbf{m}^2) \right].
\]
where the gradient term $(\nabla n)^2$ is from a gradient expansion of the exchange in terms of the staggered magnetization field. Integrating out the hard field $m = (\mathbf{n} \times \mathbf{n})/JS$, we get a field theory for the soft Neel field $n$:

$$L = \frac{\rho}{2} n^2 - \frac{JS^2}{2} (\nabla n)^2,$$

with $\rho = 1/\mathcal{J}$.

The ordered state spontaneously breaks the degrees of freedom associated with the staggered magnetization vector $n[\theta, \phi]$. Hence in this case there are two Goldstone modes, one for each continuous degree of freedom, dispersing linearly according to $\omega = \pm (\mathcal{J}S)$.

III. THREE-SUBLATTICE FIELD THEORY IN 2D

In this section, we review the geometry of the spins and their normal modes in three-sublattice antiferromagnets, below their ordering temperatures. This forms the first step towards the construction of a general theory of spin waves for these lattices, which we shall extend to the more complex situation of $\text{Mn}_3\text{X}$.

The simplest examples of this class of magnets are the Heisenberg model on a triangular lattice or on a kagome network of corner sharing triangles, Fig. 1. Although their spectra differ significantly—the kagome antiferromagnet with nearest-neighbor interactions has many spin waves with zero frequency—there are features common to many models.

Among these robust universal features are three Goldstone modes: spin waves with a linear dispersion, $\omega \sim \pm k$, in the long-wavelength limit. Their existence is related to the spontaneous breaking of the spin-rotation symmetry. The are affected by the presence of anisotropic spin interactions. However, because Heisenberg exchange is typically the dominant form of interactions for spins, this symmetry exists in at least an approximate form and the picture of three Goldstone modes with a linear dispersion is a good starting point.

A. Lattice and geometry

The general setting is an antiferromagnet with Heisenberg exchange interactions on a two-dimensional lattice with a triangle as a building block, see Fig. 2(a). We assume that classical ground states have a magnetic unit cell with three coplanar spins $S_1$, $S_2$, and $S_3$ such that

$$S_1 + S_2 + S_3 = 0.$$

This provides the triangle inequality between the three spin vectors of equal magnitude which will be crucial in establishing our analogy between the spin wave theory of these magnetic states and the continuum theory of elasticity.

1. Local geometry of the normal modes

The geometry of the ground state and the lattice is shown in Fig. 2(a). Spatial rotations through the angle $+2\pi/3$ in the $x-y$ plane produce a cyclic exchange of the spin variables:

$$
\begin{pmatrix}
S'_1 \\
S'_2 \\
S'_3
\end{pmatrix}
= 
\begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
S_1 \\
S_2 \\
S_3
\end{pmatrix}.
\tag{14}
$$

Note that the spins $S_i$ are permuted but there is no rotation in spin space. A mirror reflection $x \rightarrow -x$, $y \rightarrow y$ exchanges spins 1 and 2.
\[
\begin{pmatrix}
S'_1 \\
S'_2 \\
S'_3
\end{pmatrix} =
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
S_1 \\
S_2 \\
S_3
\end{pmatrix}.
\]

(15)

Because we are not interested in spatial variations of magnetic order, it will suffice to consider the three spins on a triangle,

\[S_i = S(\sin \theta_i \cos \phi_i, \sin \theta_i \sin \phi_i, \cos \theta_i)\]

with \(i = 1, 2, 3\) representing the three sublattices. It is convenient to express these angles in terms of six normal modes \(\alpha_0, \alpha_x, \alpha_y, \beta_0, \beta_x, \) and \(\beta_y\), see Fig. 2(b):

\[
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\phi_3
\end{pmatrix} = q \begin{pmatrix}
\frac{2\pi}{3} \\
\frac{\pi}{3} \\
0
\end{pmatrix} - qR \begin{pmatrix}
\alpha_x \\
\alpha_y \\
\alpha_0
\end{pmatrix},
\]

(16)

\[
\begin{pmatrix}
\theta_1 \\
\theta_2 \\
\theta_3
\end{pmatrix} = \begin{pmatrix}
\frac{\pi}{3} \\
\frac{\pi}{2} \\
\frac{2\pi}{3}
\end{pmatrix} + R \begin{pmatrix}
\beta_x \\
\beta_y \\
\beta_0
\end{pmatrix},
\]

(17)

where \(R\) is the orthogonal matrix.

\[
R = \begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\
0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}}
\end{pmatrix}.
\]

(18)

We measure the ground state spin angles from \(S_3\) as the reference. Here \(q = \pm 1\) accounts for the chirality of the ground state: \(q = 1\) is the chiral ground state while \(q = -1\) is the anti-chiral ground state. Note that the ground state retains a \(O(2)\) degree of freedom in the \(xy\) plane captured by the \(\alpha_0\) mode.

Under spatial transformations preserving the equilateral triangle (rotations by \(\pm 2\pi/3\) and mirror reflections), \(\alpha_0\) and \(\beta_0\) stay unchanged. We therefore call them scalar modes.

Modes \(\alpha_x\) and \(\alpha_y\) form a doublet transforming as 2 components of a polar vector. Under the \(+2\pi/3\) rotation (14),

\[
\begin{pmatrix}
\alpha_x' \\
\alpha_y'
\end{pmatrix} = \begin{pmatrix}
-\frac{1}{\sqrt{3}} & -\frac{\sqrt{3}}{2} \\
\frac{\sqrt{3}}{2} & -\frac{1}{\sqrt{3}}
\end{pmatrix}\begin{pmatrix}
\alpha_x \\
\alpha_y
\end{pmatrix}.
\]

(19)

Under the reflection (15),

\[
\begin{pmatrix}
\alpha_x' \\
\alpha_y'
\end{pmatrix} = \begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix}\begin{pmatrix}
\alpha_x \\
\alpha_y
\end{pmatrix}.
\]

(20)

The same applies to the modes \((\beta_x, \beta_y)\) which also form a doublet.

Thus we can group together the soft modes into two singlets \(\alpha_0\) and \(\beta_0\) which are scalar fields and two doublets \(\alpha = (\alpha_x, \alpha_y)\) and \(\beta = (\beta_x, \beta_y)\) which are fields that transform as vectors in the \(xy\) plane.

2. Hard and soft modes

Modes \(\alpha_x, \alpha_y,\) and \(\beta_0\) create a net spin on a triangle. They allow us to define three mutually orthogonal spin axes \(\xi, \eta,\) and \(\zeta\), respectively, Fig. 2(b). The other three modes, \(\beta_x, \beta_y,\) and \(\alpha_0\), generate rotations about the spin directions \(-\xi, -\eta,\) and \(\zeta\), respectively.

By creating a net spin on a triangle, modes \(\alpha_x, \alpha_y,\) and \(\beta_0\) increase its exchange energy \(J(S_1 + S_2 + S_3)/2\). These modes are therefore hard. The remaining fields \(\beta_x, \beta_y,\) and \(\alpha_0\) are soft.

The addition of anisotropies do ‘harden’ the soft modes [15], especially the \(\beta\) doublet which is gapped by a combination of the DM interaction and the easy-plane anisotropy. The local six-fold anisotropy term \(\delta\) gaps the \(\alpha_0\) singlet \((\sim \delta^3/J^2)\) and splits the \(\beta\) doublet \((\sim \delta^2/J)\).

However, since \(\delta/J \ll 1\) we can safely drop this effect in our theory. At the level of the exchange interaction, we integrate out the hard modes to obtain a description in terms of soft modes only.

B. Field theory for the soft modes

The kinetic term originates from the local Berry phase Eq. (4). For a spin confined to the \(xy\) plane \(\theta \approx \pi/2,\) this reduces to: \((\pi/2 - \theta_i)\dot{\phi}_i\) at each lattice site, \(i \in (1, 2, 3)\). The \(\beta\) modes are angular deviations from the planar state, hence \((\pi/2 - \theta_i) = \beta_i\). For the triangle this leads to a Lagrangian density, expressed in terms of the normal modes:

\[
\mathcal{L}_B = S \sum_{i=1}^{3} \beta_i \dot{\alpha}_i = S(\dot{\alpha}_0 \beta_0 - \alpha \cdot \dot{\beta}),
\]

(21)

where \(S\) is the spin density at a site. The potential energy density from the dominant exchange interaction is

\[
\mathcal{U} = \frac{A}{2}(\alpha_x^2 + \alpha_y^2 + 2\beta_0^2),
\]

(22)

where \(A\) is a lattice-dependent constant. We can now integrate out the hard modes, using their equations of motion:

\[
S \dot{\beta}_x = -A \alpha_x, \quad S \dot{\beta}_y = -A \alpha_y, \quad S \dot{\alpha}_0 = A \beta_0.
\]

(23)

This leads to a kinetic energy density for the soft modes:

\[
\mathcal{K} = \frac{\rho_x}{2} \dot{\alpha}_0^2 + \frac{\rho_\beta}{2}(\beta_x^2 + \beta_y^2), \quad \rho_x = \frac{S^2}{2A}, \quad \rho_\beta = \frac{S^2}{A}.
\]

(24)

We can now obtain the potential energy density \(\mathcal{U}\) by expanding the exchange interaction in terms of the soft mode gradients. There are restrictions imposed on the kinds of terms generated, namely the \(\alpha_0\) transforms as a scalar and the \(\beta\) doublet like a vector. We also generate six-fold terms allowed by the hexagonal symmetry of the lattice.
What emerges from this is a theory analogous to a continuum theory of elasticity in 2-d for the $\beta$ doublet. The gradients of the displacement fields, $u(r)$, in elasticity $(\partial_i u_j)$ are replaced by the gradients of the $\beta$ doublet.

1. Singlet

The singlet mode $\alpha_0$ has simple dynamics. Its Lagrangian density consists of a kinetic energy with mass density $\rho_\alpha$ and a potential energy quadratic in the gradients of $\alpha_0$:

$$\mathcal{L} = \frac{\rho_\alpha}{2} \dot{\alpha}_0^2 - \frac{\kappa}{2} \partial_i \alpha_0 \partial_i \alpha_0. \quad (25)$$

Summation is assumed over doubly repeated Cartesian indices $i = x, y$.

As often happens in highly symmetric solids, the effective Lagrangian (25) obeys not just the discrete symmetries of the point group $D_3$ but also the full rotational symmetry $SO(2)$. Spin waves have a linear dispersion $\omega = ck$ with the speed $c = \sqrt{\kappa/\rho_\alpha}$.

2. Doublet

The continuum theory for the doublet is more involved as the doublet field $\beta$ itself transforms like a vector under rotations. The Lagrangian of this field has the following form:

$$\mathcal{L} = \frac{\rho_\beta}{2} \dot{\beta}_i \dot{\beta}_i - \frac{C_{ijkl}}{2} \beta_{ij} \beta_{kl} - \frac{\tilde{C}_{ijkl}}{2} \tilde{\beta}_{ij} \tilde{\beta}_{kl}. \quad (26)$$

Here we have introduced symmetrized and antisymmetrized gradients,

$$\beta_{ij} \equiv \frac{1}{2} (\partial_i \beta_j + \partial_j \beta_i), \quad \tilde{\beta}_{ij} \equiv \frac{1}{2} (\partial_i \beta_j - \partial_j \beta_i). \quad (27)$$

The inertia density $\rho_\beta$ is generally different from its counterpart $\rho_\alpha$ for the singlet mode. The stiffness coefficients are four-rank tensors with the following symmetry properties: $C_{ijkl}$ is symmetric and $\tilde{C}_{ijkl}$ is antisymmetric under the exchanges $i \leftrightarrow j$ and $k \leftrightarrow l$; both tensors are symmetric under the exchange $(ij) \leftrightarrow (kl)$.

The structure of the Lagrangian (26) is highly reminiscent of the theory of elasticity in two dimensions. Here $\beta_i$ identifies with the lattice displacement, $\beta_{ij}$ with strain, and $\tilde{\beta}_{ij}$ with rotation of the lattice. In a solid, rotations do not increase the elastic energy, so $\tilde{C}_{ijkl} = 0$ for lattice vibrations. For spin waves, $\tilde{C}_{ijkl} \neq 0$ in general.

As with the elastic constants, the highly symmetric hexagonal environment drastically reduces the number of independent potential coefficients. Both four-rank tensors can be expressed in $SO(2)$-invariant forms:

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}),$$

$$\tilde{C}_{ijkl} = \tilde{\mu} \epsilon_{ij} \epsilon_{kl} = \tilde{\mu} (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}). \quad (28)$$

Here $\delta_{ij}$ is the Kronecker delta and $\epsilon_{ij}$ is the antisymmetric Levi-Civita symbol, $\epsilon_{xy} = -\epsilon_{yx} = +1$. The Lamé parameters $\lambda$ and $\mu$ determine the bulk modulus $\lambda + \mu$ (in 2 dimensions) and the shear modulus $\mu$. To continue the analogy with a solid, we will refer to $\tilde{\mu}$ as the rotation modulus. The explicit form of the Lagrangian for the $\beta$ modes is

$$\mathcal{L} = \frac{\rho_\beta}{2} \dot{\beta}_i \dot{\beta}_i - \frac{\lambda}{2} \partial_i \beta_i \partial_j \beta_j - \frac{\mu + \tilde{\mu}}{2} \partial_i \beta_j \partial_j \beta_i - \frac{\mu - \tilde{\mu}}{2} \partial_i \beta_j \partial_j \beta_i. \quad (29)$$

Spin waves for the $\beta$ modes with longitudinal and transverse polarizations have the propagation speeds

$$c_{||} = \sqrt{\frac{\lambda + 2\mu}{\rho_\beta}}, \quad c_{\perp} = \sqrt{\frac{\mu - \tilde{\mu}}{\rho_\beta}}. \quad (30)$$

3. Six-fold symmetric gradient

The continuum spin-wave Lagrangians (25) and (29) exhibit full $SO(2)$ rotational invariance. In a hexagonal solid, this symmetry is only approximate and is explicitly broken if we include terms of higher orders in the gradients.

The six-fold symmetric terms can be constructed as follows. Take three unit vectors $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ making angles of 120° with one another. For arbitrary vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, the sum

$$\sum_{i=1}^{3} (\mathbf{a} \cdot \mathbf{n}_i)(\mathbf{b} \cdot \mathbf{n}_i)(\mathbf{c} \cdot \mathbf{n}_i) \quad (31)$$

is invariant under 120° rotations. Furthermore, the square of this quantity is invariant under 60° rotations. For the $\alpha_0$ mode, the only vector available is the gradient $\nabla$ (or the wave-vector $\mathbf{k}$), so we take $\mathbf{a} = \mathbf{b} = \mathbf{c} = \nabla$. A quantity invariant under 60° rotations is

$$\mathcal{L}_6 = -\frac{\sigma_0}{8} \left[ (\partial_x^2 - 3\partial_y \partial_y^2) \alpha_0 \right]^2. \quad (32)$$

Adding this term to the Lagrangian of the $\alpha_0$ mode alters the magnon dispersion, warping the cone $\omega = ck$ as follows:

$$\omega^2 = c^2 k^2 + \frac{\sigma_0}{\rho_\alpha} k^6 \cos^2 3\phi, \quad (33)$$

where $\phi$ is the angle at which the magnon propagates in the $xy$ plane, $\mathbf{k} = (k \cos \phi, k \sin \phi)$. The warping is strongly suppressed near the center of the Brillouin zone.

For the $\beta$ modes, we have two vectors to play with, $\nabla$ and $\mathbf{B}$. The relevant invariant is

$$\mathcal{L}_6 = -\frac{\sigma_6}{2} \left[ (\partial_x^2 - \partial_y^2) \beta_x - 2\partial_x \partial_y \beta_y \right]^2. \quad (34)$$

For nondegenerate longitudinal and transverse modes ($c_{||} \neq c_{\perp}$), the magnon dispersions are warped as follows:

$$\omega^2 = c_{||}^2 k^2 + \frac{\sigma_6}{\rho_\beta} k^4 \cos^2 3\phi,$$

$$\omega^2 = c_{\perp}^2 k^2 + \frac{\sigma_6}{\rho_\beta} k^4 \sin^2 3\phi. \quad (35)$$
The inertia for the $\beta$ modes comes at a lower order in the gradient expansion and is therefore more pronounced than for the $\alpha_0$ mode. Note that if either of the velocities ($c_{\parallel}, c_\perp$) are zero this makes the six-fold pattern very prominent.

IV. FAMILIAR EXAMPLES

Let us now explicitly construct the field theory for the cases of the nearest neighbour triangular antiferromagnet and the kagome antiferromagnet, see Fig. 1. The starting point for the triangular antiferromagnet is a hexagonal unit and for the kagome is a double triangle.

The kinetic term is constructed from the local spin Berry phase, see Eq. (21). To generate the inertia we expand the nearest neighbour exchange, characterized by $J$, to quadratic order in field strength.

\[
U = \frac{3JN S^2}{4} (\alpha \cdot \alpha + 2 \beta_0^2),
\]

where $N = 3$ for the triangular lattice and $N = 2$ for the kagome ($N$ counts the number of triangles per unit cell). Using this energy density we can integrate out the hard modes ($\beta_0, \alpha_x, \alpha_y$) and generate inertia for the soft modes, see Eq. (25,26), giving us a kinetic energy density for the soft modes,

\[
L_{\text{kin}} = \frac{\rho_\alpha}{2} \alpha_0^2 + \frac{\rho_\beta}{2} \beta \cdot \beta.
\]

For the potential energy density $U$ we do a gradient expansion in the soft fields with the amplitudes of the hard modes set to zero.

A. Triangular antiferromagnet

In the nearest-neighbor Heisenberg model on the triangular lattice [11, 13], the gradient expansion yields the energy density:

\[
U = \frac{3JS^2a^2}{8} [ (\nabla \alpha_0)^2 + (\nabla \beta_x)^2 + (\nabla \beta_y)^2 ].
\]

The inertia for the soft modes are $\rho_\beta = \frac{2}{\sqrt{3}} = 2\rho_\alpha$. The $\alpha_0$ mode has the speed $c = \frac{2S^2}{\sqrt{3}} JSa$, where $S$ is the spin length and $a$ is the nearest-neighbor distance. The $\beta$ modes are degenerate and have the speed $c_\parallel = c_\perp = c/\sqrt{2}$, see Fig. 3(c). The degeneracy is associated with the special values of the Lamé coefficients, $\lambda = 0$ and $\mu = \mu_1$, and reflects a higher, $SO(2) \times SO(2)$ symmetry of the Lagrangian,

\[
\mathcal{L} = \frac{1}{2} \rho_\beta \dot{\beta}_i \dot{\beta}_i - \mu \partial_i \beta_j \partial_i \beta_j,
\]

where one $SO(2)$ rotates spatial coordinates and the other transforms components of the $\beta$ doublet.

B. Kagome antiferromagnet

In the nearest-neighbor kagome antiferromagnet [14] the corresponding energy density is:

\[
U = \frac{JS^2a^2}{16} [ (\nabla \alpha_0)^2 + 2(\nabla \cdot \beta)^2 ],
\]

where $a$ is the lattice parameter, which is twice the side of the triangle. The inertia for the soft modes are $\rho_\beta = \frac{1}{\sqrt{3}} = 2\rho_\alpha$. Here the $\alpha_0$ mode and the longitudinal $\beta$ mode have the speed $c_\alpha = c_\parallel = \frac{S^2}{\sqrt{3}} JSa$, whereas the transverse $\beta$ mode has $c_\perp = 0$, see Fig. 3(a). The zero transverse speed is associated with the vanishing shear and rotation moduli, $\mu = \mu_1 = 0$. In this sense, the nearest-neighbor kagome antiferromagnet resembles a fluid.

This situation is in fact a direct analogy to the continuum elasticity theory of nearest neighbour kagome lattice, which is critical according to the Maxwell criteria for stability [24, 25]. The n.n kagome lattice is unstable to shear distortions with floppy modes which are lifted by an addition of a next nearest neighbour elastic coupling [26].

Similarly, for the spin system the addition of further neighbour exchanges, lifts the degeneracy between the $\alpha_0$ and the longitudinal $\beta$ mode and generates a finite velocity for the transverse $\beta$ mode [14], see Fig. 3(b).

V. STACKED KAGOME

The construction of the field theory for Mn$_3$X is a slightly more involved endeavour. Here we show how to do so using Mn$_3$Ge as our prototype with the ground
state configuration presented in Chen et al. [15] as our starting point.

The compound is a layered AB stacked kagome system. In each layer the lattice parameter is given by the constant $a$, the separation in the $z$-direction between two planes of the same type is $l$. The ground state has spins confined to the corresponding plane and there is a slight deviation from the $2\pi/3$ anti-chiral state, see Eq. (17), due to local Ge site induced anisotropy [27–29]. We shall ignore this deviation in our development of the field theory.

An effective description of the system requires two sets of modes: $(\alpha_0, \alpha, \beta)$ for the A layer and $(\alpha_0', \alpha', \beta', \beta')$ for the B layer. The theory is better expressed in terms of symmetric and antisymmetric combinations of the two sets, $\zeta^{+} = \frac{\alpha^{+} + \beta^{+}}{\sqrt{2}}$ and $\zeta^{-} = \frac{\alpha^{+} - \beta^{+}}{\sqrt{2}}$, where $\zeta$ stands for any of the $\alpha$ or $\beta$ fields.

The primary unit is the David-star motif consisting of an up triangle in the lower (blue) layer and a down triangle of the upper (red) layer, see Fig. 4(a). The net Berry phase can be expressed in terms of the symmetric and antisymmetric fields:

$$\mathcal{L} = S (\dot{\alpha}_0^+ \dot{\beta}_0^+ - \alpha^+ \cdot \dot{\beta}^+) + S (\dot{\alpha}_0^- \dot{\beta}_0^- - \alpha^- \cdot \dot{\beta}^-).$$

To obtain the potential energy density at the $k = 0$ point we have to consider three types of exchange interactions.

The dominant exchange is the intralayer nearest neighbour antiferromagnetic Heisenberg exchange characterized by the strength $J_2$. The interlayer exchange interactions $J_1$ and $J_4$ respect the $D_3$ symmetry of the triangle and the inversion symmetry with respect to the center of the star. $J_4$ connects sites with the same sublattice index, whereas $J_1$ connects different sublattices, see Fig. 4.

The index $i$ on the exchange strength $J_i$ represents the actual distance between two neighbours. The layer separation is small and hence the first neighbour is indeed interlayer ($J_1$). However, our fits to data [15] show that this exchange is much smaller compared to the intralayer nearest neighbour Heisenberg exchange, $J_2$. The third nearest neighbour exchange, $J_3$ which is also interlayer, is excluded as it along with $J_1$ couples $S_i$ of one layer to the net spin, $S_1 + S_2 + S_3$, of the plaquette above or below. Since the net spin per plaquette is zero to first order in the soft fields, we can exclude either one of $J_1$ or $J_3$. The data fits bear this out and we pick up almost no strength for $J_3$ within error bars. This prompts us to keep the simplest possible model: $(J_1, J_2, J_4)$.

We can write down the energy functional at the $\Gamma$ point by expanding the three exchange interactions in terms of the twelve fields:

$$\mathcal{U} = C_1 \left[ (\alpha^+)^2 + 2 (\beta^+)^2 \right] + C_2 (\alpha^-)^2 + C_3 (\beta^-)^2$$

$$+ C_4 \left[ (\beta^+)^2 + (\alpha_0^-)^2 \right].$$

The constants $C_n$ are:

$$C_1 = \left( \frac{3}{2} J_2 + \frac{3}{2} J_1 \right) S^2.$$

We now calculate the interaction energy generated by the Heisenberg exchanges from the gradient expansion of the symmetric and antisymmetric normal modes. Although the interactions $(J_1, J_4)$ harden the antisymmetric modes [see the last term in Eq. (42)], we include them in the theory. We proceed one exchange interaction at a time, highlighting the features in each case.
A. Intralayer interactions

Heisenberg antiferromagnetic exchange between nearest neighbour sites confined to a single kagome plane, \( J_2 \) (see Fig. 4(a)) reproduces the kagome lattice example worked out earlier. This is the dominant exchange term in this compound. The energy density of the soft modes is:

\[
\mathcal{U} = \frac{J_2}{16} a^2 S^2 \left[ (\nabla \alpha_0)^2 + (\nabla \alpha_0^*)^2 \right] + \frac{J_2}{8} a^2 S^2 \left[ (\nabla \cdot \beta^a)^2 + (\nabla \cdot \beta^a)^2 \right].
\] (46)

In the absence of interlayer coupling, the symmetric and antisymmetric fields are degenerate. This implies that the inertia for the symmetric and antisymmetric modes is the same, \( \rho^*_s = \rho^*_a = \rho_s \) and \( \rho^*_a = \rho^*_s = \rho_a \).

We can read off the velocity of the \( \alpha_0^a \) mode from Eq. (25), identifying \( \kappa = \frac{J_2}{8} a^2 S^2 \), \( c_\alpha = \sqrt{\kappa/\rho_a a S} = \sqrt{J_2/(8\rho_a a S)} \).

For the \( \beta^a \) elasticity theory we can read off the elasticity moduli: \( \lambda = \frac{J_2}{4} a^2 S^2 \), \( \mu = \mu_1 = 0 \) and hence the velocities:

\[
\frac{\mathcal{U}}{S^2} = (2J_1 - 4J_4) \left[ (\alpha_0^2)^2 + (\beta^a)^2 \right] + \frac{a^2}{6} \left( \frac{J_1}{8} - J_4 \right) (\nabla \alpha_0)^2
\]

\[
- \frac{a^2}{8} \beta^a \nabla \beta^a + \frac{a^2}{24} (J_1 - 5J_4) \beta^a \nabla \beta^a
\]

\[
- \frac{a}{\sqrt{3}} \left( J_1 + J_4 \right) \left[ \beta^a \nabla \beta^a + \beta^a \nabla \beta^a \right].
\] (48)

where we have dropped the gradients of the massive antisymmetric fields, \( \alpha_0^a \) and \( \beta^a \).

In the presence of these interlayer interactions our ‘elastic’ theory analogy seems to fail due to terms, linear in field derivatives \( \beta^a \nabla \beta^a \). Inversion transformations about the common triangle center (center of the David’s star motif) reduces the form of allowed terms to:

\[
\beta^a \nabla \beta^a \rightarrow -\beta^a \nabla \beta^a \rightarrow -\nabla \beta^a
\]

which are invariant under inversions, since \( \beta^a \rightarrow -\beta^a \), \( \beta^a \rightarrow -\beta^a \) and \( \nabla \rightarrow -\nabla \) leaving the combination unchanged, see Fig. 4.

These terms do not fit directly into the mould of an elasticity theory, and the kinetic term \( K \propto (\beta^a)^2 \), keeps us from integrating out the massive antisymmetric modes to re-obtain an elastic theory. However, with the assumption that the antisymmetric modes are sufficiently gapped by an ferromagnetic \( J_4 \) and an antiferromagnetic \( J_1 \), we can do a perturbation theory in these linear terms for their contribution to the velocities of the symmetric modes. In this limit we return to an elasticity theory involving the \( \beta^a \) fields.

The linear gradient interactions are also responsible for inducing a 6-fold pattern in the dispersions at the quartic level \( (\sim k^4) \), see Eq. (52). This is especially apparent for an antiferromagnetic \( J_1 \) interaction. However, in the presence of a ferromagnetic \( J_4 \) this six-fold behaviour is alleviated by the more isotropic nature of \( J_4 \). This is important as the dispersion in the sample data is isotropic around the \( \Gamma \) point, see Fig. 5.

1. In-plane velocities

We can now list the velocities of all the gapless modes in the presence of both in-plane and out-of-plane inter-
actions. In the presence of \((J_1, J_2, J_4)\) the velocities are:

\[
c_\alpha^* = \frac{1}{\rho_\alpha^*} \left( \frac{J_2}{8} - \frac{J_4}{3} + \frac{J_1}{24} \right) \alpha S
\]

\[
c_\parallel^* = \frac{1}{\rho_\parallel^*} \left( \frac{J_2}{4} - \frac{5J_4}{24} - \frac{J_1}{12} - \frac{3J_1J_4}{8(J_1 - 2J_4)} \right) \alpha S
\]

\[
c_\perp^* = \frac{1}{\rho_\perp^*} \left( -\frac{3J_4}{8} - \frac{3J_1J_4}{8(J_1 - 2J_4)} \right) \alpha S,
\]

where \(a\) is the nearest neighbour distance in the kagome plane. Note that with just a \(J_1\) out of plane interaction \((J_4 = 0)\) the perpendicular mode \((c_\perp^*)\) that was flat under \(J_2\) remains flat to linear order and develops flat directions in q-space at the quadratic level, see Fig. 5(a). The situation with \(J_4\) as the out of plane interaction is isotropic.

2. Out-of-plane velocities

For the \(\omega_0\) mode the dispersion is given by \(\rho_\omega^2 = (\frac{J_1}{4} - \frac{J_2}{4})(kS)^2\). For the \(\beta_x, \beta_y\) modes the \(c\)-dispersion is \(\rho_{\beta x}^2 = (\frac{J_4}{4} - \frac{J_2}{4})(kS)^2\). These lead to the out of plane velocities:

\[
c_\alpha = \sqrt{\frac{J_1 - 2J_4}{4\rho_\alpha^*}} l S
\]

\[
c_\beta = \sqrt{\frac{J_1 - 2J_4}{4\rho_\beta^*}} l S,
\]

where \(l\) is the separation between unit cells in the \(c\)-direction. Now since \(\rho_\beta^* = 2\rho_\alpha^*\) the relation between the velocity of the two types of modes is \(c_\alpha = \sqrt{2}c_\beta^*\) in the \(k_z\)-direction.

C. Symmetry features of the interplane interactions

The interplane interactions expressed using the symmetric vector field \(\beta^*\) and the antisymmetric vector field \(\beta^\alpha\) contain the following terms:

1. A mass term for the field \(\beta^\alpha\).

2. Direct quadratic interactions: \(\partial_i \beta^\alpha \cdot \partial_j \beta^\alpha\) and \(\partial_i \beta^\beta \cdot \partial_j \beta^\beta\) (‘elasticity’ theory).

3. Crossed interaction terms between \(\beta^\alpha\) and \(\beta^\beta\) which are linear in derivatives \(\beta^\alpha_i \partial_j \beta^\beta_k\). The cross terms have to follow the inversion symmetry criteria for the exchanges.

The interlayer exchanges are shown in Fig. 4 and their gradient expanded forms are shown in Eq. (48). Let us take a closer look at the linear term which is common to both expressions:

\[
U_{linear} \propto \left[ \beta^\alpha_x (\partial_y \beta^\beta_x + \partial_x \beta^\beta_y) + \beta^\beta_x (\partial_y \beta^\alpha_x - \partial_x \beta^\alpha_y) \right].
\]

We motivated a generic construction of a six-fold term in Eq. (31). In that construction if we take the vectors \(a = (-\beta^\beta_y, \beta^\alpha_x), b = \nabla,\) and \(c = (\beta^\beta_x, \beta^\alpha_y)\) we generate the cross term in Eq. (51).

In section III, we noted that such a structure has a 120° symmetry. For the case of the interlayer coupling this turns into a 60° symmetry. This happens because in Eq. (51), a 60° degree rotation interchanges the three unit vectors \(e_i\) with a flipped sign and flips the primed and unprimed fields, which leads to \(\beta^\alpha \rightarrow -\beta^\beta\) and \(\beta^\beta \rightarrow \beta^\alpha\). The two flips of sign cancel to produce a 60° symmetry, see Fig. 6.

This 6-fold symmetry is explicit in the dispersions. Keeping only two antiferromangetic interactions \(J_1\) and \(J_2\) with \(k = k (\cos \phi_k, \sin \phi_k)\) the two gapless modes have dispersions:

\[
\rho_{\beta_1}^* = \left( \frac{J_1 + J_2}{384} \right) (1 + \cos(6\phi_k))^2 k^4
\]

\[
\rho_{\beta_2}^* = \left( \frac{J_2}{4} - \frac{J_1}{12} \right) k^2
- \frac{1}{1152}(3J_2 - 5J_1 + 3(J_2 + J_1) \cos(6\phi_k))k^4
\]

\[
\begin{align*}
\text{FIG. 5. Color plots for the dispersions of the } \beta \text{ doublet with an antiferromagnetic } (J_2, J_1) \text{ and ferromagnetic } J_4. \text{ Upper panels: show the dispersions for the } \beta^* \text{ modes with } J_1 = 2.5J_2, \text{ and } J_4 = 0. \text{ We can clearly see the six fold pattern in both cases. Left: } \text{The dispersion for the } c_\parallel^* \text{ mode the flat lines represent directions in the k-space for which } c_\parallel^* = 0. \text{ Right: } \text{The dispersion for the } c_\perp^* \text{ mode. There are no flat directions but a six fold pattern is prominent. Lower panels: show the dispersions for } \beta^\alpha \text{ modes with } J_2 = 4|J_4|, \text{ and } J_1 = 0 \text{ with } c_\perp^* \text{ mode on the left and } c_\parallel^* \text{ mode on the right. Both modes are isotropic and there are no flat directions.}
\end{align*}
\]
Here we choose the three unit vectors $e_1$, $e_2$, $e_3$ along highly symmetric directions for purposes of illustration. It is clear that after a $\pi/3$ rotation the blue and red (up and down) fields are interchanged and the unit vector axes are reversed. Note that the cyclic permutation of labels caused by the rotation is absorbed into the summation over the labels in Eq. (31).

Both the gapless $\beta$ modes display a six fold feature at the quartic level, see Fig. 5(a),(b). One of these dispersions (the $\beta_2^s$ mode) is modified by $J_2$ at the quadratic level making it isotropic near the $\Gamma$ point. However note that as $J_1 \rightarrow 3J_2$ the quadratic part goes to zero and six-fold features will become prominent, Fig. 5(b).

The other mode is the ‘six-fold flat mode’ which results from the interlayer interaction $J_1$ lifting the flat mode associated with the frustrated $J_3$-only kagome lattice, in a non-isotropic fashion at the quartic order in $k$.

In contrast and as apparent in Eq. (49), $J_4$ has quadratic contributions to both the gapless $\beta^s$ modes resulting in an isotropic dispersion of the former flat mode, see Fig. 5(c),(d).

VI. CONCLUSION

We have presented a field theory for spin waves in a hexagonal antiferromagnet with three magnetic sublattices in terms of normal modes of a spin triangle. The zero net spin condition imposed on each triangular plaquette leads to a spin wave theory which has three Goldstone modes each with a different velocity, in the generic case. The theory decomposes into a field theory for a singlet $0_S$ and a doublet $\beta$. The theory for the doublet maps to a continuum theory for elasticity with the spin wave velocities as ‘sound’ velocities.

We use the familiar settings of the Heisenberg antiferromagnet on the triangular and kagome lattice to demonstrate the features of the field theory. In this case, the two examples are slight outliers because of their highly symmetric lattice environment.

The triangular lattice has the $\beta$ modes as degenerate, and in the kagome we have a degeneracy between the $0_S$ singlet and one of the $\beta$ modes while the other one is zero throughout the Brillouin Zone, see Fig. 3. We show that the flat mode of the kagome can be anticipated from the elasticity analogy: the mechanical kagome lattice (phonons) with nearest neighbour interaction has zero shear and this property is manifest in our spin wave analog as the flat mode.

Although the spin wave analyses around the 120° ground state of both the triangular Heisenberg antiferromagnet and the kagome antiferromagnet are well documented [11, 13] their description in terms of three sublattice field theory is absent from the literature to the best of our knowledge. This analysis also removes the need for the involved manipulations of the Holstein-Primakoff decomposition.

Additionally, in the case of a local $D_3$ symmetric environment we provide a generic construction scheme for six fold symmetric terms. This is particularly useful in presence of local anisotropies which break the $O(2)$ symmetry in the plane but keep the six fold symmetry intact. We extend our theory to the study of the stacked kagome Mn$_3$X.

The study of the normal modes and their natures reveal effective ways of coupling to the magnetic order. External probes like magnetic fields couple to the spins locally, or the net spin of the plaquette and engender terms which are $D_3$ symmetric. These couplings are expressed in the basis of the normal modes, which represent the spin degrees of freedom. Given that the normal modes are $D_3$ symmetric by construction and decouple into a pair of singlets and a pair of doublets we can limit the terms that can be produced based on symmetry properties alone.

For instance, for an external magnetic field the Zeeman coupling is between two time reversal odd vectors: the magnetic field $B_{\text{ext}}$ and a net spin per plaquette. The only vectors available at the linear order in fields, which are also time reversal odd are, $B_{\text{ext}}$, and $\alpha$. Hence the Zeeman term will be of the form $B_{\text{ext}} \cdot (R\alpha)$ where $R$ is a 2-d rotation matrix, which accounts for the global $O(2)$ freedom of the spins in the xy plane.

Since the magnetism in these materials is intricately linked to the conduction bands of the electrons, through an $s$-$d$ coupling [30], certain features like the location of the Weyl points and, the magnitude of the anomalous Hall response [29, 30] can be manipulated through the local magnetic order. This is a promising avenue of future work in these materials.

The emergent elasticity theory is also interesting from a more general point of view than just the present scenario, allowing a comparison of this case with other emergent elasticity theories like in Skyrmion crystals[31]. It also leaves open avenues of investigation along the lines of the duality theory developed in [32] and [33], especially since in Mn$_3$Ge the non collinear ground state allows a spin-phonon coupling, which might make a melting transition particularly interesting.

A detailed study of the soft modes, as provided here, is of use in spintronics where they can couple to external perturbations [34]. In the effective theory for a two
sublattice antiferromagnet presented in [35], it was noted that space-time dependent external perturbations introduce gauge fields which can be used to interact with and drive solitons. A similar construction can be envisioned for the three-sublattice case where the solitons in question can be domain walls between the six-fold ground states [36].

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