Exact phase transition of backtrack-free search with implications on the power of greedy algorithms

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Abstract
Backtracking is a basic strategy to solve constraint satisfaction problems (CSPs). A satisfiable CSP instance is backtrack-free if a solution can be found without encountering any dead-end during a backtracking search, implying that the instance is easy to solve. We prove an exact phase transition of backtrack-free search in some random CSPs, namely in Model RB and in Model RD. This is the first time an exact phase transition of backtrack-free search can be identified on some random CSPs. Our technical results also have interesting implications on the power of greedy algorithms, on the width of superlinear dense random hypergraphs and on the exact satisfiability threshold of random CSPs.

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1 Introduction

In constraint satisfaction problems (CSPs), values are assigned to variables to fulfil constraints among these variables [11, 43]. Backtracking is a basic strategy to solve CSPs [15, 14, 24, 7, 16, 32]. A CSP instance is called backtrack-free, if we can always extend from scratch a partial assignment to a solution without any reassignment (or backtracking) along a linear ordering on variables, and at each variable we only need to keep the extended partial assignment compatible with these constraints among assigned variables, implying that the instance is easy to solve [17]. In practice, backtrack-freeness is a very desirable property in many applications [12, 26, 5, 25, 45, 6]. In theory, sufficient conditions and on random instances for backtrack-freeness have been studied [17, 10, 46, 12, 47, 40, 26, 33, 44, 13]. Here, we study backtrack-freeness from a theoretical point of view along these two lines.

The sufficient conditions for backtrack-freeness on CSPs were given by Freuder in terms of strong consistency and the width of constraint graph [17, 18, 19], by van Beek and Dechter in terms of local and global consistency [10, 46], constraint tightness and looseness [17], by Dakic et al in terms of overlap of cliques in interval graph representation [12], by Jackson et al in terms of k-consistency and overlap in constraint graphs [26], by Pang and Goodwin in terms of ω-consistency and tree-structured ω-graph associated with constraint hypergraphs [40], and by Kolaitis and Vardi in terms of k-locality [33]. Here, yet another sufficient condition in terms of what we call vertex-centered consistency and the width of constraint hypergraph is given.

A non-zero probability of backtrack-freeness on random instances for a range of parameter values was used by Smith to lower bound the satisfiability threshold [44]. Dyer, Frieze and Molloy obtained a threshold for backtrack-freeness with respect to the parameter of the domain size of binary CSPs with a linear number of constraints [13]. Here we identify an exact threshold of backtrack-freeness with respect to the density parameter for non-binary CSPs with a superlinear number of constraints. This is the first time an exact phase transition of backtrack-freeness can be identified on random CSPs. Before, the exact phase transition results of algorithmic behaviors are rare and mainly about resolution [1, 36].

Our proofs work by first showing a phase transition result about variable-centered consistency and then estimating the width of a random hypergraph by determining the existence of specific k-cores. As far as we know, this is the first k-core result on k-uniform hypergraphs with rn ln n hyperedges and n vertices. In our case, the width increases smoothly with the density parameter, in sharp contrast to the earlier k-core threshold results in literatures for sparse hypergraphs [41, 44, 14, 13, 8, 35, 22, 23, 27, 28, 9, 42, 31].

Our results have implications on the power of greedy algorithms, since below the backtrack-freeness threshold we can find a solution in a greedy manner for almost all instances, while above the threshold we are forced to search with backtracking for almost all instances, even for satisfiable instances. To this end, we define the width of greedy algorithms. Also, our results show that for Model RB/RD, the satisfiability threshold and some local property threshold are linked tightly, so we suggest that a similar link might exist for random 3-SAT.

This paper is organized as follow. In Section 2 we fix our notations and give all necessary definitions and some known results. In Section 3 we show the exact phase transition
of backtrack-freeness. In Section 4 we show results about width and $k$-cores in random hypergraphs. In Section 5 we discuss some implications of our results.

2 Preliminaries

In constraint satisfaction problems (CSPs), a set of variables $\{u_1, u_2, \cdots, u_n\}$ and a set of constraints $\{C_1, C_2, \cdots, C_m\}$ are given for each instance. We call $n$ the input size and ratio $\frac{m}{n}$ the constraint density. Each variable can take a value from a finite domain $\{1, 2, \cdots, d\}$. We allow $d$ to increase with $n$, say $d = n^\alpha$, where $\alpha$ is a constant. An assignment is a mapping from the variable set to the domain and a partial assignment is a mapping from a variable subset to the domain. Each constraint involves a subset of variables and labels each partial assignment on these variables either as compatible or incompatible, but not both. In so called $k$-CSPs, each constraint involves $k$ variables. 2-CSPs are also called binary CSPs. An assignment compatible with all constraints is called a solution. Instances with at least one solution are called satisfiable, otherwise unsatisfiable.

In random CSPs, constraints are generated by a random process with a small number of control parameters, leading to a probabilistic distribution on all instances. In Model RB, given $n$ variables each with domain $\{1, 2, ..., d\}$, where $d = n^\alpha$ and $\alpha > 0$ is constant, select with repetition $m = r n \ln n$ random constraints, for each constraint select without repetition $k$ of $n$ variables, where $k = 2, 3, 4, ..., D$ and select uniformly at random without repetition $(1 - p) d^k$ compatible assignments for these $k$ variables, where $0 < p < 1$ is constant. If in the last step above, each assignment for the $k$ variables is selected with probability $1 - p$ as compatible independently, then it is called Model RD ([48]). Model RB is asymptotically similar to Model RD just as $G(n, M)$ is to $G(n, p)$, all asymptotic results should hold both for Model RB/RD ([48, 49]). For simplicity, here we only give proofs valid for Model RD and omit more complicated calculations for Model RB. For Model RB/RD, not only exact satisfiability thresholds can be identified [48] but also the existence of many hard instances around the thresholds can be demonstrated both theoretically [49] and experimentally [50].

**Theorem 2.1.** ([48], Theorem 1) Let $r_{cr} = \frac{\alpha}{\ln(1 - p)}$, where $\alpha > \frac{1}{k}$, $0 < p < 1$ are constants and $k \geq \frac{1}{1 - p}$. Then for a random instance $\phi$ in Model RB/RD,

$$
\lim_{n \to \infty} \Pr(\phi \text{ is satisfiable }) = \begin{cases} 
1 & r < r_{cr}, \\
0 & r > r_{cr}.
\end{cases}
$$

**Theorem 2.2.** ([49], Theorem 3) Almost all instances in Model RB/RD have no tree-like resolutions of length less than $2^{\Omega(n)}$ and no general resolutions of length less than $2^{\Omega(n/d)}$.

In graph theory, a hypergraph consists of some nodes and some hyperedges. Each hyperedge is a subset of nodes. A hypergraph is $k$-uniform if every hyperedge contains exact $k$ nodes. Every CSP has an underlying constraint (multi-)hypergraph: each variable corresponds to a node and each constraint corresponds to a hyperedge in a natural way. The constraint hypergraphs of random CSPs are random hypergraphs [29]. The constraint hypergraph of Model RB/RD, denoted by $HG(n, rn \ln n, k)$, is a random $k$-uniform multi-hypergraph with
n nodes and $r n \ln n$ hyperedges, where $r$ is constant and $k = 2, 3, 4, \ldots$. Denote by $HG$ a random hypergraph from $HG(n, r n \ln n, k)$.

Let $\phi$ be an instance of CSPs. Let $u$ be a variable. Let $C$ be a constraint involving $u$, where $C$ is called a $u$-constraint. For any $u$, the total number of $u$-constraints is called the degree of $u$ and denoted as $\text{deg}(u)$. Let $C_u$ be a set of $u$-constraints, where $C_u$ is called $u$-centered. Denote by $N_{C_u}$ the set of all variables involved in constraints in $C_u$. Denote by $C_{\setminus u}$ the set of all constraints among variables in $N_{C_u} \setminus \{u\}$. Denote by $T_{C_{\setminus u}}$ the set of all partial assignments each compatible with all constraints in $C_{\setminus u}$. Let $c$ be a partial assignment in $T_{C_{\setminus u}}$. Let $v$ be a value to $u$. Denote by $c'$ the partial assignment extending $c$ just with $u = v$.

Let $\pi$ be a linear ordering on variables in $\phi$, say $u_1 < u_2 < \cdots < u_n$. Denote by $C_{u_i}^\pi$ the set of all $u_i$-constraints such that all constraints in $C_{u_i}^\pi$ are among $\{u_1, u_2, \ldots, u_i\}$. The width of $u_i$ under $\pi$ is just $|C_{u_i}^\pi|$. The width of $\pi$ is max$_i$ width($u_i$), denoted by width($\pi$). The width of $\phi$ is min$_i$ width($\pi_i$), denoted by width($\phi$). For constraint hypergraphs, the degree and width can be defined in a similar way. The width and the associated optimal linear ordering can be found efficiently [17, 18, 19, 38]. Moreover, the linkage of a hypergraph $HG$ is the minimum degree of all its nodes, denoted by linkage($HG$). A $k$-core of a hypergraph is a nonempty maximal subgraph with minimum degree $k$. In [17], it was essentially proved that the width of a hypergraph is equal to the maximal linkage of its subgraphs.

Consider the following strategy to solve $\phi$. At step 1, we put an arbitrary value to $u_1$. Assume that after step $i - 1$, we have a partial assignment $c$ on $\{u_1, u_2, \ldots, u_{i-1}\}$ which is compatible with all constraints among $\{u_1, u_2, \ldots, u_{i-1}\}$. At step $i$, we find a value $v$ for $u_i$ such that, when $c$ is extended with $u_i = v$, the resulting assignment $c'$ is compatible to all constraints among $\{u_1, u_2, \ldots, u_i\}$. Such a $v$ is called available. When there are more than one available $v$‘s, we take an arbitrary one from them. Note that the only requirement to $v$ is that, when $c$ is extended with $u_i = v$, the resulting assignment $c'$ is compatible with all constraints among $\{u_1, u_2, \ldots, u_i\}$. In fact, the only requirement for $v$ is that $c'$ is compatible with all constraints in $C_{u_i}^\pi$. If at each step $i$ ($1 \leq i \leq n$), for every partial assignment $c$, we can always find such a value $v$ for $u_i$, then we say that $\phi$ is backtrack-free under $\pi$. Otherwise, we say that $\phi$ is not backtrack-free under $\pi$. If there is a $\pi$ such that $\phi$ is backtrack-free under $\pi$, then we say that $\phi$ is backtrack-free.

If whenever $|C_u| \leq t$, then for every $c \in T_{C_{\setminus u}}$, we can always find a $v$ such that $c'$ is compatible with all constraints in $C_u$ (that is, for all $C \in C_u$, $c'$ is compatible with $C$), then we say that $u$ is variable-centered $t$-consistent. If every $u$ in an instance is variable-centered $t$-consistent, then we call this instance variable-centered $t$-consistent and $t$ is called the critical size of this variable-centered consistency.

Denote by $E(X)$ the expectation of a random variable $X$, $B(n, p)$ the binomial distribution, $\Pr(\mathcal{E})$ the probability of event $\mathcal{E}$. An event $\mathcal{E}$ occurs with high probability, or whp, if $\lim_{n \to \infty} \Pr(\mathcal{E}) = 1$.

**Lemma 2.3.** (Chernoff Bound) [3] [37] [29] [39] For a random variable $X$ with distribution $B(n, x/n)$ and $0 < \epsilon < 1$, we have $\Pr(X \leq (1-\epsilon)\mu) \leq e^{-\mu\epsilon^2/2}$ and $\Pr(X \geq (1+\epsilon)\mu) \leq e^{-\mu\epsilon^2/3}$, and for any $\mu_h > \mu$, $\Pr(X \geq (1+\epsilon)\mu_h) \leq e^{-\mu_h\epsilon^2/3}$.

Finally, $f \ll g$ means $f = o(g)$ or $\lim_{n \to \infty} \frac{f}{g} = 0$. A useful inequality is $1 - x < e^{-x} <
$1 - x + o(x)$ for small $x > 0$.

## 3 The exact threshold of backtrack-freeness

In this section we give the exact threshold of backtrack-freeness for the Model RB and Model RD. We first give a sufficient condition for backtrack-freeness.

**Note:** In this section, when we use $\phi$, $\pi$, $C_u^\pi$, $u$, $C_u$, $N_{C_u}$, $C_{\setminus u}$, $T_{C_{\setminus u}}$, $c$, $v$, $c'$ and $C$, we implicitly assume that they adhere to the descriptions in Section 2.

**Theorem 3.1.** If $\phi$ is vertex-centered $\text{width}(\phi)$-consistent, then $\phi$ is backtrack-free.

**Proof.** By definition of backtrack-freeness, clearly

$$\phi \text{ is backtrack-free } \iff \exists \pi, \forall u, \forall c \in T_{C_{\setminus u}} \exists v, \forall C \in C_u, c' \text{ is compatible with } C.$$

By definition of width, there is a $\pi$ such that $\text{width}(\phi) = \text{width}(\pi)$. Under $\pi$, for all $u_i$, $\text{width}(u_i) \leq \text{width}(\pi) = \text{width}(\phi)$. Then the vertex-centered $\text{width}(\phi)$-consistency guarantees that at each $u_i$, the partial assignment can be extended as desired by backtrack-free search. \hfill \Box

As a warm up, we upper bound the number of $u$-constraints for any $u$ as $O(\ln n)$.

**Lemma 3.2.** $\max_u \deg(u) < (1 + \sqrt{\frac{6}{kn}})kr \ln n$ whp.

**Proof.** Since the total number of constraints is $rn \ln n$, every constraint involves exactly $k$ vertices, and a given vertex appears in a constraint with probability $\frac{k}{n}$, $\deg(u)$ is a random variable with binomial distribution $B(rn \ln n, \frac{k}{n})$. By Chernoff bound, for any $u$ we have $\Pr(\deg(u) \geq (1 + \sqrt{\frac{6}{kn}})kr \ln n) \leq \frac{1}{n^2}$. By Union bound, we have $\Pr(\exists u, \deg(u) \geq (1 + \sqrt{\frac{6}{kn}})kr \ln n) \leq n \cdot \frac{1}{n^2} = \frac{1}{n}$, so $\Pr(\forall u, \deg(u) < (1 + \sqrt{\frac{6}{kn}})kr \ln n) \geq 1 - \frac{1}{n}$, that is, $\max_u \deg(u) < (1 + \sqrt{\frac{6}{kn}})kr \ln n$ whp. \hfill \Box

Our main observation is that there is a threshold for density parameter $r$ in Model RB/RD, such that below this threshold, almost all instances are variable-centered consistent for some critical size, while above this threshold, almost all instance are not variable-centered consistent for another critical size. Happily, the two critical sizes can be very close!

**Lemma 3.3.** Let $r_{bf} = -\frac{\alpha}{k \ln(1-p)}$, where $\alpha > 0$, $0 < p < 1$, $k = 2, 3, 4, ...$ are constants. If $r < r_{bf}$, $0 < \epsilon < \min(\frac{k\epsilon}{r}, \frac{1}{2})$ and $t = (1 + \epsilon)kr \ln n$, then $\Pr(\forall u, u \text{ is vertex-centered } t\text{-consistent}) \geq 1 - e^{-n^{O(1)}}$.

**Proof.** Given $u$, $C_u$, $c$, $v$, $C$ and $c'$ as described in Section 2 and only consider $C_u$’s with $|C_u| \leq t$, $u$ is vertex-centered $t$-consistent $\iff \forall C_u, \forall c, \exists v, \forall C, c'$ is compatible with $C$. 

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Under the distribution on random instances of Model RD, we have
\[
\begin{align*}
\Pr(c' \text{ is compatible with } C) &= 1 - p, \\
\Pr(\forall C, c' \text{ is compatible with } C) &= (1 - p)^{|C_u|}, \\
\Pr(\exists C, c' \text{ is incompatible with } C) &= 1 - (1 - p)^{|C_u|}, \\
\Pr(\forall v, \exists C, c' \text{ is incompatible with } C) &= (1 - (1 - p)^{|C_u|})^d.
\end{align*}
\]
To apply the Union bound on \( u, C_u \) and \( c \), we only need to upper bound \((1 - (1 - p)^{|C_u|})^d\) and the number of choices of \( u, C_u \) and \( c \) respectively. To upper bound \((1 - (1 - p)^{|C_u|})^d\), recall that \( \epsilon < \frac{r_{bf} - r}{r} \), denote \( \delta = r_{bf} - (1 + \epsilon) r > 0 \) and \( \gamma = -\delta k \ln(1 - p) > 0 \), then \(|C_u| \leq t = (1 + \epsilon) kr \ln n = (r_{bf} - \delta) k \ln n = (\frac{\alpha}{\ln(1 - p)} - \delta k) \ln n = -\frac{\alpha - \gamma}{\ln(1 - p)} \ln n \), so we have \((1 - (1 - p)^{|C_u|})^d \leq (1 - (1 - p)^{-\frac{\alpha - \gamma}{\ln(1 - p)} \ln n}) n^\alpha = (1 - n^{-\alpha + \gamma})^n \leq (e^{-n^{-\alpha + \gamma}})^n = e^{-n^{-\alpha + \gamma}} = e^{-n^{O(1)}} \), the last inequality is by \( 1 - x < e^{-x} \) for \( x \neq 0 \). The number of possible choices of \( u \) is no greater than \( n = e^{\ln n} \). By Lemma 3.2 for any \( u \), the total number of \( u \)-constraints is \( \text{deg}(u) = O(\ln n) \) \text{ whp}, so the number of possible choices of \( C_u \) is no more than \( 2^{\text{deg}(u)} = e^{O(\ln n)} \) \text{ whp}. For any \( C_u \), the number of variables in \( N_{C_u} \) is no more than \( k|C_u| \), since each constraint includes exactly \( k \) variables. Each variable can take at most \( d = n^\alpha \) different values, so the number of possible choice of \( c \) is \(|T_{C_u}| \leq d^{n|C_u|} \leq d^{n|C_u|} \leq (n^\alpha)^k|C_u| \leq n^{kt} = n^{O(\ln n)} = e^{O(\ln n)^2}\). By Union bound, we have \( \Pr(\exists u, \exists C_u, \exists c, \forall v, \exists C, c' \text{ is incompatible with } C) \leq e^{\ln n \cdot e^{O(\ln n)}} \cdot e^{O(\ln n)^2} \cdot e^{-n^{O(1)}} = e^{-n^{O(1)}} \). By taking complement, we have \( \Pr(u \text{ is vertex-centered } t\text{-consistent}) = \Pr(\forall u, \exists C_u, \exists c, \forall v, \exists C, c' \text{ is compatible with } C) \geq 1 - e^{-n^{O(1)}} \).

Lemma 3.4. Let \( r_{bf} = \frac{\alpha}{\ln(1 - p)} \), where \( \alpha > 0, 0 < p < 1, k = 2, 3, 4, \ldots \) are constants. If \( r > r_{bf} \), \( 0 < \epsilon < \min\left(\frac{r - r_{bf}}{r}, \frac{1}{2}\right) \), \( \delta = (1 - \epsilon)r - r_{bf} > 0 \), \( \gamma = -\delta k \ln(1 - p) > 0 \) and \( t = (1 - \epsilon) kr \ln n \), then for all \( u \) and for all \( C_u \) with \(|C_u| \geq t \), \( \Pr(\forall v, \exists C_u, \forall c, \exists v, \forall C, c' \text{ is compatible with } C) \leq n^{-\frac{\alpha}{\ln(1 - p)}} \).

Proof. As in proof of Lemma 3.3 but only consider \( C_u \)'s with \(|C_u| \geq t \),
\[
\begin{align*}
\Pr(\forall v, \exists C, c' \text{ is incompatible with } C) &= (1 - (1 - p)^{|C_u|})^d, \\
\Pr(\exists v, \forall C, c' \text{ is compatible with } C) &= 1 - (1 - (1 - p)^{|C_u|})^d, \\
\Pr(\forall v, \exists C, c' \text{ is compatible with } C) &= (1 - (1 - (1 - p)^{|C_u|})^d)^{|T_{C_u}|}.
\end{align*}
\]
This time we only need to lower bound \((1 - (1 - p)^{|C_u|})^d\) and \(|T_{C_u}|\). To lower bound \((1 - (1 - p)^{|C_u|})^d\), recall that \( \epsilon < \frac{r - r_{bf}}{r} \), \( \delta = (1 - \epsilon)r - r_{bf} > 0 \) and \( \gamma = -\delta k \ln(1 - p) > 0 \), then \(|C_u| \geq t = (1 - \epsilon) kr \ln n = (\delta + r_{bf}) k \ln n = (\delta k - \frac{\alpha}{\ln(1 - p)} k) \ln n = -\frac{\alpha - \gamma}{\ln(1 - p)} \ln n \), so \((1 - (1 - p)^{|C_u|})^d \geq (1 - (1 - p)^{-\frac{\alpha - \gamma}{\ln(1 - p)} \ln n}) n^\alpha = (1 - n^{-\alpha - \gamma})^n n^\alpha \approx e^{-n^{-\alpha + \gamma}} \), the last approximation is by \( (1 - \frac{1}{n})^n \approx \frac{1}{e} \). To lower bound \(|T_{C_u}|\), recall that \( C_{\setminus u} \) denote the set of all constraints among variables in \( N_{C_u} \setminus \{u\} \) and
\[
\text{E}(|T_{C_u}|) = (1 - p)^{|C_u|} \cdot d^{n|N_{C_u} \setminus \{u\}|} = (1 - p)^{|C_u|} \cdot d^{n|N_{C_u}| - 1},
\]
so we only need to upper bound \(|C_{\setminus u}|\) and to lower bound \(|N_{C_u}|\).
To upper bound \(|C_u|\), we only need to upper bound \(|N_{C_u}|\), since each constraint in \(|C_u|\) is among variables in \(N_{C_u} \setminus \{u\}\). In turn, we only need to upper bound \(|C_u|\), since each variable in \(N_{C_u}\) is contained in some constraint in \(C_u\) and each constraint contains exactly \(k\) variables. By Lemma 3.2, \(|C_u| = O(ln n) \text{ whp.}\) so \(|N_{C_u}| \leq k|C_u| = O(ln n) \text{ whp.}\) Since each constraint contains exactly \(k\) variables, the probability that a given constraint is among \(N_{C_u} \setminus \{u\}\) is \(\left(\frac{k}{n}\right)^k \leq \frac{k^k}{n^k} \leq \left(\frac{O(ln n)}{n}\right)^k = \left(\frac{O(ln n)}{n}\right)^k\). Since the total number of constraints is \(rn \ln n = O(ln n)\), we have \(E(|C_u|) \leq \left(\frac{O(ln n)}{n}\right)^k \cdot O(ln n) = \frac{O(ln n)^2}{n^{k-1}} = o(1)\) for \(k \geq 2\). By Markov inequality, \(\Pr(|C_u| \geq 1) \leq E(|C_u|) = o(1)\), so \(|C_u| = 0 \text{ whp.}\)

To lower bound \(|N_{C_u}|\), the number of variables involved in constraints in \(C_u\), we only need to upper bound the probability that a variable does not appear in any constraint in \(C_u\). Since each constraint includes exactly \(k\) variables, a variable appears in a constraint with probability \(\frac{k}{n}\), not appears in a constraint with probability \(1 - \frac{k}{n}\), and not appears in all constraints in \(C_u\) with probability \((1 - \frac{k}{n})^{C_u} = \left(e^{-\frac{k}{n}}\right)^C = e^{-\frac{kC}{n}} < 1 - \frac{kt}{n} + o\left(\frac{kt}{n}\right)\), using \(1 - e^{-x} < 1 - x + o(x)\) for \(x \neq 0\) and \(|C_u| \geq t\). So \(E(|N_{C_u}|) = n[1 - (1 - \frac{k}{n})^{C_u}] = n \cdot \left(\frac{kt}{n} - o\left(\frac{kt}{n}\right)\right) = kt - o(n)\), since \(t = O(ln n)\). By Chernoff bound, \(\Pr(|N_{C_u}| \leq (1 - \epsilon)kt) = o(1)\), so \(|N_{C_u}| > (1 - \epsilon)kt \text{ whp.}\)

Now we have
\[
E(|T_{C_u}|) = (1 - p)^{|C_u|}d^{N_{C_u} - 1} > (1 - p)^0 \cdot (n^\alpha(1 - \epsilon)kt - 1) = n^{O(ln n)} \text{ whp.}
\]
By the second moment method similar to that in [RS], we can prove that \(|T_{C_u}| \geq n^{O(ln n)} \text{ whp.}\) So \(\Pr(\forall c, \exists v, \forall C, c' \text{ is compatible with } C) = (1 - (1 - (1 - p)^{|C_u|}d)^{|T_{C_u}|} < (1 - e^{-\alpha})(\gamma n)|^{O(ln n)} = n^{-\gamma n^{O(ln n)}}\).

Finally, we can prove the exact phase transition of backtrack-freeness on Model RB/RD.

**Theorem 3.5.** Let \(r_{bf} = \frac{\alpha^\alpha \gamma^{\alpha^\alpha}}{\ln(1-p)}\), where \(\alpha > 0\), \(0 < p < 1\), \(k = 2, 3, 4, \ldots\) are constants. Then
\[
\lim_{n \to \infty} \Pr(\phi \text{ is backtrack-free}) = \begin{cases} \frac{1}{2} & r < r_{bf}, \\ 0 & r > r_{bf}. \end{cases}
\]

**Proof.** If \(r < r_{bf}\), let \(0 < \epsilon < \min\left(\frac{r_{bf}}{r}, \frac{1}{2}\right)\). From Lemma 3.3, \(\phi\) is vertex-centered \((1 + \epsilon)kr \ln n\)-consistent \(\text{whp.}\). From Lemma 3.1, \(\text{width}(\phi) < (1 + \epsilon)kr \ln n \text{ whp.}\). By definition, for \(t' < t\), vertex-centered \(t\)-consistency implies vertex-centered \(t'\)-consistency, so \(\phi\) is vertex-centered \(\text{width}(\phi)\)-consistent \(\text{whp.}\). By Theorem 3.1, \(\phi\) is backtrack-free \(\text{whp.}\). This completes the first half of our proof.

If \(r > r_{bf}\), let \(\epsilon < \min\left(\frac{r_{bf}}{r}, \frac{1}{2}\right)\). By Lemma 3.2, for any \(\pi\), \(\text{width}(\pi) \geq (1 - \epsilon)kr \ln n \text{ whp.}\) so exists a \(u\) such that \(|C_u^n| \geq (1 - \epsilon)kr \ln n\). By Lemma 3.1, for any \(u\),
\[
\Pr(\forall c \in T_{C_u^n}, \exists v, \forall C \in C_u, c' \text{ is compatible with } C) = n^{-\gamma n^{O(ln n)}}.
\]
Since the number of choices of $\pi$ is $n!$, by Union bound,
\[
\Pr(\phi \text{ is backtrack-free}) \leq n! \Pr(\forall u, \forall e \in T_{\pi_u}^\pi, \exists v, \forall C \in C_{\pi_u}^\pi, e' \text{ is compatible with } C) \\
\leq n! \Pr(\forall e \in T_{\pi_u}^\pi, \exists v, \forall C \in C_{\pi_u}^\pi, e' \text{ is compatible with } C) \\
\leq n! \cdot n^{-\gamma n^{\Omega(\ln n)}} \approx \left(\frac{n}{e}\right)^n \cdot n^{-\gamma n^{\Omega(\ln n)}} = o(1).
\]

This completes our proof. \hfill \Box

4 Width of random hypergraphs

In this section we determine the width of some random hypergraphs with a superlinear number of hyperedges. We apply a probabilistic method mainly inspired by [13, 35] to detect the existence of $k$-cores. Denote by $HG$ a random hypergraph from $HG(n, rn \ln n, k)$. We show that whp the width of $HG$, denoted as $\text{width}(HG)$, is asymptotically equal to average degree $kr \ln n$, due to high concentration of distribution of node degree in $HG$.

Lemma 4.1. For any $0 < \epsilon < 1$, $\text{width}(HG) \leq (1 + \epsilon)kr \ln n$ whp.

Proof. The number of hyperedges in a subgraph $G' \subseteq HG$ is a random variable $X_{G'}$. If $G'$ has $f(n)$ nodes, when adding a hyperedge to $HG$ with repetition, the value of $X_{G'}$ increases by 1 with probability $\binom{f(n)}{k}$, so $X_{G'}$ distributes as $\text{B}(rn \ln n, \binom{f(n)}{k})$, and
\[
\mathbb{E}(X_{G'}) = rn \ln n \cdot \frac{f(n)!}{k!(f(n) - k)!} \leq r \ln n \cdot f(n) < (1 + \epsilon) r \ln n \cdot f(n). \tag{1}
\]

Let $\text{avd}(G')$ denote the average degree of $G'$. By [11] and Chernoff Bound, we have
\[
\Pr(\text{avd}(G') > (1 + \epsilon)kr \ln n) = \Pr(X_{G'} > (1 + \epsilon) r \ln n \cdot f(n)) \leq e^{-r \ln n \cdot f(n) \epsilon^2/3} = n^{-r \epsilon^2/3 \cdot f(n)}, \tag{2}
\]

Let random variable $N_i = |\{G' \mid \text{subgraph } G' \text{ has } i \text{ nodes } \wedge \text{avd}(G') > (1 + \epsilon)kr \ln n \geq 1\}|$ and $N = N_1 + N_2 + \ldots + N_n$. Since the width of a hypergraph is equal to the maximal linkage of its subgraphs [17], we have
\[
\Pr(\text{width}(HG) > (1 + \epsilon)kr \ln n) = \Pr(\exists G' \subseteq HG, \text{linkage}(G') > (1 + \epsilon)kr \ln n) \\
\leq \Pr(\exists G' \subseteq HG, \text{avd}(G') > (1 + \epsilon)kr \ln n) \leq \Pr(N_1 + N_2 + \ldots + N_n \geq 1) \leq \mathbb{E}(N). \tag{3}
\]

Below we show that $\mathbb{E}(N)$ tends to 0 by showing that $\mathbb{E}(N_{f(n)}) = o(1/n)$.

Case 1. When $f(n)$ is large, namely $n^{1 - r \epsilon^2/3} \ll f(n) \leq n$, since by [2], we have
\[
\mathbb{E}(N_{f(n)}) \leq \left(\frac{n}{f(n)}\right)^n \cdot n^{-r \epsilon^2/3 \cdot f(n)} \leq \left(\frac{en}{f(n)}\right)^{f(n)} \cdot n^{-r \epsilon^2/3 \cdot f(n)} = \left(\frac{en^{1 - r \epsilon^2/3}}{f(n)}\right)^{f(n)} = o(1/n).
\]
Case 2. When \( f(n) \) is small, that is \( f(n) \ll n \), since by \( \mathbf{[1]} \), for all \( i > (1 + \epsilon)r \ln n \cdot f(n) \), we have \( \Pr(\text{avd}(G') = i) \leq \Pr(\text{avd}(G') = (1 + \epsilon)kr \ln n) \), so

\[
\Pr(\text{avd}(G') > (1 + \epsilon)kr \ln n) \leq n \Pr(\text{avd}(G') = (1 + \epsilon)kr \ln n) = n \Pr(X_{G'} = (1 + \epsilon)kr \ln n \cdot f(n)) \leq n \left( \frac{rn \ln n}{(1 + \epsilon)r \ln n \cdot f(n)} \right)^{(1 + \epsilon)r \ln n \cdot f(n)} \leq n \left( \frac{rn \ln n}{(1 + \epsilon)r \ln n \cdot f(n)} \right)^{(1 + \epsilon)r \ln n \cdot f(n)} \leq n \left( \frac{en}{f(n)} \right)^{n(C_1 \cdot \frac{f(n)}{n})} = o(1/n),
\]

where \( C_1 > 0 \) and \( C_2 > 0 \) are two constants. Then,

\[
E(N_{f(n)}) \leq \sum_{f(n) \ll n} E(N_{f(n)}) + \sum_{f(n) \gg n^{1 - \epsilon^2/3}} E(N_{f(n)}) \leq 2n \cdot o(1/n) = o(1).
\]

The lemma follows from \( \mathbf{[3]} \) and \( \mathbf{[4]} \).

\[\square\]

**Lemma 4.2.** For any \( 0 < \epsilon < 1 \), \( \text{width}(HG) \geq (1 - \epsilon)kr \ln n \) whp.

**Proof.** Let \( m = (1 - \epsilon)kr \ln n \). Since the width of a hypergraph is equal to the maximal linkage of its subgraphs \( \mathbf{[17]} \), we need to prove the existence of a subgraph of \( HG \) whose minimum degree is at least \( m \) whp, or the existence of an \( m \)-core whp, which can be achieved by an analysis of the following standard \( m \)-core detecting algorithm: while there exists any node with degree less than \( m \), randomly select such a node and delete it together with all hyperedges containing it, if there is no node left then output No, otherwise output the remaining subgraph.

Let \( X_i \) denotes the number of nodes whose degree are less than \( m \) after deleting the \( i \)th node. Let \( W_{i,j} = \{ u | u \text{ has degree } j \text{ after deleting the } i \text{th node} \} \), then \( X_i = |W_{i,1}| + |W_{i,2}| + \ldots + |W_{i,m-1}| \). Obviously, an \( m \)-core exists if and only if the node-hyperedge deletion process cannot delete all nodes, and if and only if there exists a \( j < n \), such that \( X_j = 0 \). Since

\[
\Pr(\text{width}(HG) \geq m) = \Pr(\exists j < n, X_j = 0) \geq \Pr(X_0 + |W_{0,m}| < n^\delta \wedge \exists j < n, X_j = 0) = \Pr(X_0 + |W_{0,m}| < n^\delta \cdot \Pr(\exists j < n, X_j = 0 | X_0 + |W_{0,m}| < n^\delta),
\]

where \( \delta \in (0, 1) \) will be determined later, we only need to estimate the last two probabilities.

Whenever we add a hyperedge to \( HG \) with repetition, a node’s degree increases by 1 with a probability of \( k/n \). So the degree of each node in \( HG \) is a random variable with distribution \( B(rn \ln n, k/n) \). By Chernoff bound, for a specific node \( u \), we have

\[
\Pr(\text{u’s degree is not more than } m) \leq n^{-kr^2}.
\]
Since \( X \) random variables

Let \( B \) be determined by (\( \cdot \)). Clearly, for all \( i < n \), the probability that one deleted hyperedge containing an

so the probability that one deleted hyperedge containing an \( m \)-degree node is

Let \( T_i \) be a random variable with distribution \( B((m - 1)(k - 1), q_i) \), then the sequence of random variables \( X_0, X_1, \ldots \) can be described as

Since \( |W_{0,m}| \leq X_0 + |W_{0,m}| < n^\delta \) and \( \sum_{j \geq 1} j|W_{0,j}| = krn\ln n \), we have

After deleting the \( i \)th node, comparing with the beginning of the node-hyperedge deletion process, the number of \( m \)-degree node increases by at most \( (m - 1)(k - 1)i \), and the sum \( \sum_{j \geq 1} j|W_{i,j}| \) decreases by at most \( (m - 1)i \). So for all \( i < n^{\delta'} \), where \( \delta' \in (\delta, 1) \), we have

Thus, \( \mathbf{E}(T_i) = (m - 1)(k - 1)q_i \) can be arbitrary small. Without loss of generality, let \( q \) be determined by \( (m - 1)(k - 1)q = 1/2 \). Let \( D_i \) be a random variable with distribution \( B((m - 1)(k - 1), q) \). We now define a new sequence of random variables \( Y_0, Y_1, \ldots \) by

Clearly, for all \( i < n^{\delta'} \), \( X_i \) is statistically dominated by \( Y_i \), and \( \sum_{i=1}^{n^{\delta'}} D_i \) distributes as \( B(n^{\delta'}(m - 1)(k - 1), q) \). Therefore,

So \( \mathbf{E}(X_0 + |W_{0,m}|) \leq n \cdot n^{-kr^2/2} = n^{1-kr^2/2} \). Then by Markov inequality, we have \( \Pr(X_0 + |W_{0,m}| \geq n^\delta) \leq \mathbf{E}(X_0 + |W_{0,m}|)/n^\delta \leq n^{1-kr^2/2-\delta} \), so for \( \delta \in (1-kr^2/2, 1) \), we have

Now assume that \( X_0 + |W_{0,m}| < n^\delta \), where \( 1 - kr^2/2 < \delta < 1 \). When deleting the \((i + 1)\)th node, at most \((m - 1)\) hyperedges are deleted together, which contain at most \((m - 1)(k - 1)\) other nodes, among which only the \( m \)-degree nodes will count for \( X_{j+1} \). Since any subhypergraph with a given degree sequence is uniformly random, see for example [29], such a subhypergraph can be generated according to the configuration model [29], so the probability that one deleted hyperedge containing an \( m \)-degree node is

\[
q_i = m|W_{i,m}|/\sum_{j \geq 1} j|W_{i,j}|
\]

After deleting the \( i \)th node, the number of \( m \)-degree node increases by at most \((m - 1)(k - 1)i\), and the sum \( \sum_{j \geq 1} j|W_{i,j}| \) decreases by at most \((m - 1)i\). So for all \( i < n^{\delta'} \), where \( \delta' \in (\delta, 1) \), we have

\[
(m - 1)(k - 1)q_i < \frac{(m - 1)(k - 1)m(|W_{0,m}| + (m - 1)(k - 1)n^{\delta'})}{krn\ln n - (m - 1)n^{\delta'}} < \frac{(m - 1)(k - 1)m(n^{\delta} + (m - 1)(k - 1)n^{\delta'})}{krn\ln n - (m - 1)n^{\delta'}} = o(1).
\]

Thus, \( \mathbf{E}(T_i) = (m - 1)(k - 1)q_i \) can be arbitrary small. Without loss of generality, let \( q \) be determined by \( (m - 1)(k - 1)q = 1/2 \). Let \( D_i \) be a random variable with distribution \( B((m - 1)(k - 1), q) \). We now define a new sequence of random variables \( Y_0, Y_1, \ldots \) by

\[
Y_0 = n^{\delta} \text{ and } Y_{i+1} = Y_i - 1 + D_i.
\]

the last second step above is by Chernoff bound. The lemma follows from (5), (6) and (7). \( \square \)
5 Discussions

We have proved that in some random CSP models (Model RB/RD), the backtrack-freeness threshold \( r_{bf} \) in Theorem 3.5 not only exists, but also has a fixed ratio to the satisfiability threshold \( r_{cr} \) in Theorem 2.1, that is, \( r_{bf} = \frac{r_{cr}}{k} \), where \( k \) is the number of variables in each constraint.

The first implication is on the power of greedy algorithms. A CSP algorithm is called greedy, if at each step we choose an unassigned variable by some rule and assign an available value for it, here by availability we mean that the extended partial assignment is compatible with all constraints among all assigned variables. The availability is a natural feature in common greedy algorithms. A greedy algorithm succeeds on an instance if all variables can be assigned in this way, fails otherwise. To specify a greedy algorithm, we need to specify the rule to choose the next variable from unassigned variables and the rule to choose an available value for the variable. In turn, every greedy algorithm specifies a linear ordering, called induced ordering, on all variables in an instance, and the width of the induced ordering on constraint graph can be called the width of the greedy algorithm on this instance. Note that some greedy algorithms have a fixed linear ordering not depending on instances thus a fixed width. For others, we can define the width of the greedy algorithm as the maximum width over all instances.

If an instance is backtrack-free under an ordering \( \pi \), then every greedy algorithm as described above with induced ordering \( \pi \) will succeeds on this instance, no matter how to choose an available value for each variable. Moreover, if an instance is vertex-centered \( t \)-consistent, then every greedy algorithm as described above with induced width no greater than \( t \) will succeed on this instance, no matter how to choose an available value for each variable. As far as we know, this is the first time to define explicitly the width of a greedy CSP algorithm and relate it to the power of greedy algorithms on CSPs.

As a concrete example to the above discussion, let us consider Model RB/RD. On the one hand, Model RB/RD is \( NP \)-complete for all positive values for the density parameter \( r \).

On the other hand, at least in a constant portion to the satisfiable range of values for parameter \( r \) (that is, \( r < r_{bf} = \frac{r_{cr}}{k} \)), there is an easily determined ordering of variables such that almost surely, every greedy algorithm following that ordering will succeed on almost all instances of Model RB/RD, in sharp contrast to its worst-case complexity. When \( k = 2 \), at least in half portion to the satisfiable range of values for parameter \( r \) (that is, \( r < r_{bf} = \frac{r_{cr}}{2} \)), almost all instances can be easily solved by greedy algorithms. While for instances above \( r_{bf} \), with high probability, there does not exist such an ordering to guarantee the success of every greedy algorithm. This implies that the exact threshold of backtrack-freeness obtained in this paper can also be viewed as a threshold for the power of greedy algorithms.

The second implication is about the satisfiability threshold for random CSPs. For Model RB/RD, the exact threshold of satisfiability is \( r_{cr} = -\frac{\alpha_0}{\ln(1-p)} \) (Theorem 1 in [LS]), which is independent of \( k \), the number of variables in each constraint, while the exact threshold of backtrack-freeness is \( r_{bf} = -\frac{\alpha_0}{k\ln(1-p)} = \frac{r_{cr}}{k} \), which decreases with \( k \). For fixed \( k \), these two thresholds have a fixed ratio \( k \), so an exact link between them exists. Note that the backtrack-freeness threshold also coincides with the threshold of vertex-centered consistency.
a local property. So our results show an evidence that for random CSPs, the exact threshold of satisfiability might has links to thresholds of some local properties, say local consistency. Based on this evidence, we propose the following two steps to attack the notorious problem of determining the satisfiability threshold for random 3-SAT.

- Step 1: reduce the satisfiability threshold to some local property (say local consistency) threshold.
- Step 2: determine the local property threshold.

Since reductions are commonly used in computer science and local properties are usually easier to handle than global properties, hopefully the two steps each will be easier than directly attacking the original satisfiability threshold problem.

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