Evolving network models under a dynamic growth rule

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Evolving network models under a dynamic growth rule which comprises the addition and deletion of nodes are investigated. By adding a node with a probability $P_a$ or deleting a node with the probability $P_d = 1 - P_a$ at each time step, where $P_a$ and $P_d$ are determined by the Logistic population equation, topological properties of networks are studied. All the fat-tailed degree distributions observed in real systems are obtained, giving the evidence that the mechanism of addition and deletion can lead to the diversity of degree distribution of real systems. Moreover, it is found that the networks exhibit nonstationary degree distributions, changing from the power-law to the exponential one or from the exponential to the Gaussian one. These results can be expected to shed some light on the formation and evolution of real complex real-world networks.

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I. INTRODUCTION

Considerable interest is focused on complex networks currently due to their potential to describe many systems in nature and society. Theoretical models are developed to reproduce topological properties of these systems. The recently observed scale-free (SF) property has made the scientists to realize that static network models do not provide appropriate description to real systems which essentially keep growing with time. Barabási and Albert proposed a simple evolving network model (BA model) to explain such SF property. In this model, the growing nature of real systems is captured by a BA-type growth rule. According to this rule, one node is added into the network at each time step, intending to mimic the growing process of real systems. Another ingredient in this model is the mechanism of preferential attachment (PA), which assumes that newly added nodes are attached preferentially to nodes with higher degrees. Based on the BA-type growth rule, many evolving network models are introduced.

The modeling framework of network evolution is formed. Among the many quantities proposed to characterize topological properties of networks, the degree distribution $p(k)$, which gives the probability that a node in the network possesses $k$ edges, is of particular importance. The study of networks has undergone a transition from the investigation of static network models with Poisson degree distribution to the explanation of real systems with various fat-tailed degree distributions (FTDDs). It was found that many real-world networks are characterized by the power-law degree distribution (PLDD) as well as the exponential degree distribution (EDD). Some more exhaustive experiment reveals that, in addition to the PLDD and the EDD, the truncated power-law degree distribution (TPLDD) and the truncated exponential degree distribution (TEDD) are also observed. Furthermore, in Ref. 8, the Gaussian degree distribution (GDD) was reported as well.

One theoretical challenge has been to explain the origin of these observed FTDDs. In the BA model, the fact is revealed that growing networks with PA yield the PLDD. In addition, growing networks without PA were also studied in the random evolving network model (REN model), in which networks grow by the growth rule of BA-type, while the newly added nodes connect to randomly chosen existing ones. In this model, the EDD was obtained. In order to explain those observed degree distributions which are neither strict power-law nor strict exponential, many other mechanisms were added into this two models, such as the mechanism of adding and rewiring edges between existing nodes, the mechanism of ageing and cost, as well as the mechanism of information filtering. These more specialized models indeed created the TPLDD or the EDD in some parameter regimes. However, relatively lesser attention is paid to systematical studies on the origin of the diversity of the observed FTDDs, with some recent exceptions.

The BA-type growth rule gives a somewhat simplified description to the evolution of real system. As a matter of fact, in real growing networks, there are constant addition of new elements, but accompanied by permanent removal of old elements (deletion of nodes). The scaling behavior of growing networks has already been shown to be strongly affected when the deletion of node is taken into account. In a recent work, we proposed a new type of network growth rule which comprises the addition and deletion (AD) of nodes. Based on such AD growth rule, by adding a node with a probability $P_a$ or deleting a node with probability $P_d = 1 - P_a$ at each time step, topological properties of growing networks with and without PA are studied, respectively. In ref. 17, we considered a simple case: $P_a$ and $P_d$ were treated as constants. ($P_a$ is an adjustable parameter in the model.) All the observed FTDDs of real systems were obtained in the model, indicating that the mechanism of AD can lead to the diversity of FTDD in real systems.

The AD growth rule introduced in 17 gives a more de-
tained description to the evolution of real systems. However, this evolution can be even more complex. For example, as real systems grow, limited resources will cause interelement competition which, in turn, has a tendency to retard the growth of these systems. Such competition is intensified when the number of element is increased. As a result, growth rate of real systems usually decreases with the increase of system size, exhibiting the so-called density-dependent growth [20]. In fact, competitive dynamics has been found to dominate the evolution of variety of social [24], biological [28] and economic [29] network systems. Thus, with respect to the AD growth rule, to be more realistic, $P_a$ should be a decreasing function of the number of nodes in the network, rather than a constant [17]. In this paper, we introduce the Logistic population equation [20] into the AD growth rule. As a result, the probabilities of addition (deletion) becomes a decreasing (increasing) function of the network size. Based on this dynamical AD growth rule, growing networks models with and without PA are investigated, respectively. In the present models, networks exhibit the density-dependent growth and all the observed FTDDs are created. Moreover, it is found that the networks exhibit nonstationary degree distributions, changing from power-law to exponential one or from exponential to Gaussian one. These results can be expected to shed some light on the formation and evolution of real complex real-world networks.

II. NETWORK MODELS WITH LOGISTIC AD GROWTH RULE

The Logistic equation [20] was proposed by Pierre Verhulst for the analysis of population competitive dynamics. Verhulst assumed that during the population growth, as a result of the competition for the limited resources, the death rate per individual is a linearly increasing function of the number of population. Then the Logistic equation can be written as

$$\frac{dN(t)}{dt} = [a - cN(t)]N(t) = a[1 - \frac{N(t)}{K}]N(t),$$

where $N(t)$ denotes the number of population at time $t$; $a$ and $c$ are constants; $a$ is the birth rate per individual and $cN(t)$ stands for the death rate per individual at time $t$; $K = a/c$ is called carrying capacity [30,31], which represents the largest population the environment can support due to limited resources.

A more realistic description to the real-network’s evolution can be achieved by the introduction of such Logistic dynamics into the AD growth rule. To do this, we obtain the probability of addition $P_a$ and the probability of deletion $P_d$ by the following two equations

$$P_a = \frac{aN(t)}{cN(t)^2},$$

and

$$P_a + P_d = 1,$$

which yield

$$P_a(t) = \frac{a}{a + cN(t)} = \frac{K}{K + N(t)}$$

and

$$P_d(t) = \frac{cN(t)}{a + cN(t)} = \frac{N(t)}{K + N(t)}.$$  

Where $N(t)$ stands for the number of nodes at time $t$ and $K$ is the parameter which denotes the maximum nodes in the network.

Then the Logistic AD model can be defined as follows: We start from $m_0$ isolated nodes, which act as the nuclei of a growing network. At each time step, either a new node is added to the network with probability $P_a$ or a randomly selected old node is removed from the network with probability $P_d$. The newborn and the destroyed nodes are replaced with the probability $P_a$ and $P_d$, respectively.

When a new node is added to the network, there are still two ways for it to attach to the existing nodes in the network. One way is to randomly choose $m$ nodes to set up connections (growing network without PA) [11,12], and the other way is to preferentially select $m$ nodes to connect by means of preferential probability in the BA model [3], which reads

$$\Pi_i = \frac{k_i + 1}{\sum_j (k_j + 1)},$$

where $k_i$ is the degree of the $i$th node (growing network with PA). Here, we should note that in order to give chance for isolated nodes to receive a new edge we choose $\Pi_i$ proportional to $k + 1$ [13].

One can find from Eq. (4) and Eq. (5) that when $N(t) \ll K$, $P_a(t) \gg P_d(t)$. This means that the networks grow rapidly at the initial stages of their evolution. In fact, in a Logistic model [20],

$$\frac{dN(t)}{dt} = a\left[\frac{K - N(t)}{K}\right]N(t),$$

when $N(t) \ll K$,

$$\frac{dN(t)}{dt} \approx aN(t),$$

the solution is:

$$N(t) = C_0 \exp(at),$$

where $C_0$ is a integral constant. Indeed, this kind of exponential growth has been observed in many newly emerged real-world networks, such as the World-Wide-Web (WWW) and the Internet [19,20]. This indicates that the rapid growth of some real systems in their young
age is well described by the Logistic AD model. As \( N(t) \) increase, \( P_a(t) \) decrease while \( P_d(t) \) increase, the growth of network is slowed down. In the limit of large \( t \),

\[
\lim_{t \to \infty} P_a(t) = \lim_{t \to \infty} P_d(t) = \frac{1}{2},
\]

and

\[
\lim_{t \to \infty} N(t) = K,
\]

i.e., the network reach a steady state and the number of nodes has the upper limit \( K \). The above analysis imply that networks in the Logistic AD model exhibit density-dependent growth characterizing the evolution of many real systems, not captured in most previous network models.

We investigated the degree distribution of the networks by extensive computer simulations. In the simulation, we set \( m_0 = m = 5 \) and \( K = 100000 \). Network is left to evolve until the steady state is reached. Cumulative degree distributions of growing networks with and without PA at different time step are given in Fig. 1(a) and Fig. 1(b), respectively. We found that in the Logistic AD model, the networks exhibit nonstationary degree distribution before the steady state is reached. This is illustrated in Fig. 1. For the growing network with PA, as time goes on, \( P(k) \) of the network undergoes a process of transition, which can be roughly separated into several stages: (1) At the earlier stage of network evolution, i.e., when \( t \leq 72000 \), the network exhibits various PLDDs with different power-law exponents. In addition, the exponent increases with time. Particularly, in the asymptotic cases of \( P_a \) being close to 1, the network is almost equivalent to the well-known BA model, thus it have PLDD with power-law exponents \( \gamma = 3 \). (2) 72000 < \( t < 1000000 \). \( P(k) \) of the network is truncated by an exponential cutoff and the network exhibits the TPLDD during this stage. Generally, if a node needs to obtain the high degree, it must exists in network with enough long time. While, longer do the nodes live, higher is the probability that they are deleted. Moreover, with \( t \) increasing, \( P_a \) increases. Thus the surviving probability of the nodes with high degrees is greatly reduced, and the degree distribution evolves with time, gradually changing from a power-law form to a power-law with a exponential cutoff, then to a exponential one. (3) In the limit of large \( t \), e.g., when \( t \geq 1000000 \), the network shows a well shaped stationary EDD [see Fig. 1(a)]. Speciously, this asymptotic case seems to be a non-growing network (or a very slowly growing one). The case of the preferential attachment on a non-growing network was considered in Ref. [19] where it was found that \( P(k) \) is not stationary, changing from a power-law type directly to is a Gaussian one. On the other hand, for the growing network without PA, as time goes on, \( P(k) \) of the network exhibits a con- tinuous transition from the EDD to a variety of TEDDs which prove to be a series of intermediate states, and in the limit of large \( t \), e.g., when \( t \geq 1045700 \), the network exhibits a well shaped stationary GDD [see Fig. 1(b)].

The results of Fig. 1 reveal a nonstationary behavior for the degree distribution of real systems. During the evolution of these systems, various events take place on different timescales. These events include, for instance, the addition and deletion of nodes, the creating and rewiring of edges between existing nodes (internal edges), and so on. In general, the timescale on which a node join or leave the network (the addition and deletion of nodes) may be much longer than the timescale on which other events take place. For example, in a social network, creating internal edges means that the individuals make new friends, which happens on the timescale that can be as short as hours or days. While the timescale on which individuals are born or die is typically some years or decades. So that our model, which is based on the mechanism of addition and deletion, gives the prediction of a long-run behavior of real systems. For this reason, this kind of nonstationary FTDD of real systems, which can be difficult to observe in a relatively short time interval, has been neglected in most previous studies [11-19]. However, in recent years, the Internet and the WWW is
in the initial and rapid growth stage of their evolutions. Perhaps, the observations for their degree distributions might provide some clues for our results. The WWW at the document level had grown at least five times larger during the two years time delay between the first and last web crawl, and the degree distributions obtained at different time are reported to be of a power-law form [32–34]. The power-law exponent, though seems to be invariable for the in-degree distributions, has an increasing tendency with the sample size or time for the out-degree distribution, changing from 2.45 in 1999 to 2.72 in 2000 [32–34]. Another clue is referred to the Internet on the inter-domain level. The degree distribution of the Internet is also of power-law form. The exponent, however, suffer a little change from 2.15 in November 1997, to 2.16 in April 1998, then to 2.20 in December 1998 [33]. In fact, it is hard to achieve this precision. One may estimate the value of the exponent using the highest degrees and the system’s size [30]. Such estimations also confirm the reported values. For November 1977, we gets $\gamma \simeq 2.22$, for April 1998, $\gamma \simeq 2.24$, and for December 1998, $\gamma \simeq 2.26$. Of course, exist additional factors (nonlinear PA [36] and accelerating growth of edges, etc.) that may change these values. Finally, we are glad to point out that in a recent work [14], such nonstationary degree distribution was also reported when the creation of internal edges was considered in growing and static networks.

It is well known that the structure of complex networks has strong effects on their function. Therefor such neglect, at least in some case, may be dangerous because networks with different degree distribution act differently if we consider the dynamic processes taking place on them. This is especially the case, for example, in the strategy designing for the prevention of virus spreading or for the defense of network attacking [38–40]. Provided that, after some years, the WWW has gradually changed into an exponential network, but is still treated as a scale-free one, what will happen to this network of high technological importance if it runs under the antivirus strategy designed for a scale-free network?

### III. CONCLUSION

In summary, we have proposed a dynamic growth rule which comprises the addition and deletion of nodes in the network. Based on this rule, by adding a node with a probability $P_a$ or deleting a node with probability $P_d = 1 - P_a$ at each time step, topological properties of growing networks with and without PA are studied respectively. The probabilities of addition and deletion are assumed to be determined by the Logistic population equation. In our model, networks exhibit the density-dependent growth which characterizes the evolution of various real systems, and all the observed degree distributions of real systems are created. Moreover, the networks exhibit nonstationary degree distributions, changing from power-law to exponential one or from exponential to Gaussian one. It is discussed that this kind of nonstationary behavior can be unconscious in real-world networks. The result indicates that the degree distribution of real systems, which has been believed to be stationary in most previous studies, may in fact be nonstationary.

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