New predictions for generalized spin polarizabilities from heavy baryon chiral perturbation theory

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We extract the next-to-next-to-leading order results for spin-flip generalized polarizabilities (GPs) of
the nucleon from the spin-dependent amplitudes for virtual Compton scattering at $O(p^4)$ in heavy
baryon chiral perturbation theory. At this order, no unknown low-energy constants enter the theory,
allowing us to make absolute predictions for all spin-flip GPs. Furthermore, by using constraint
equations between the GPs due to nucleon crossing combined with charge conjugation symmetry of
the virtual Compton scattering amplitudes, we get a next-to-next-to-next-to-leading order prediction
for one of the GPs. We provide estimates for forthcoming double-polarization experiments which allow
one to access these spin-flip GPs of the nucleon.

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I. INTRODUCTION

Over the past decade, the virtual Compton scattering (VCS) process on the nucleon, accessed through the $ep \rightarrow epy$ reaction, has become a powerful new tool to study the internal structure of the nucleon both at low and high
energies (see [1] for a review). At low virtuality and energy, the outgoing real photon plays the role of an
applied quasistatic electromagnetic field, and the VCS process measures the response of the nucleon to this
applied field, which can be parametrized in nucleon structure quantities, termed generalized polarizabilities
(GPs) [2]. In this case, the virtuality of the initial photon can be dialed so as to map out the spatial distribution of
the electric polarization, the magnetization, or the spin densities of the nucleon [3]. Because the chiral dynamics
plays a dominant role in this regime, at low virtuality, Heavy Baryon Chiral Perturbation Theory (HBChPT)
provides a natural way to study the GPs of the nucleon.

The spin-dependent VCS amplitudes in HBChPT at order $O(p^4)$ have been calculated in Refs. [4,5]. The
HBChPT calculation for the spin-dependent VCS amplitudes has recently been extended to $O(p^4)$ in Ref. [6]. At
this order, no unknown low-energy constants enter the calculations so that one can extract the next-to-leading
order (NLO) results for the spin-flip GPs of the nucleon, without free parameters. In Ref. [6], it was possible to
extract the next-order corrections to three of the seven spin-flip GPs in HBChPT by identifying the terms linear
in the final photon energy in a low-energy expansion (LEX) of the spin-dependent VCS amplitudes.

In this paper, we demonstrate that by performing higher order expansions of the VCS amplitude with re-
spect to the final photon energy, i.e., by including the terms quadratic in the final photon energy, one can fur-
thermore extract the next-to-next-to-leading order (NNLO) results of three more spin-flip GPs and even
make a next-to-next-to-next-to-leading order (N 3LO) prediction for one particular GP. In this way we are
able to obtain the next-order corrections to all seven spin-flip GPs in HBChPT, without free parameters.

On the experimental side, the measurement of the VCS process became only possible in recent years, with the
advent of high precision electron accelerator facilities, as these experiments involve precision measurements of
small cross sections. At low photon energy and virtuality, first unpolarized VCS observables have been measured
at the MAMI accelerator [7] at a virtuality $Q^2 = 0.33 \ 	ext{GeV}^2$, and recently at JLab [8] at higher virtualities,
$1 < Q^2 < 2 \ 	ext{GeV}^2$, and data are under analysis at MIT-
Bates [9]. All those experiments measure two combina-
tions of GPs. To disentangle the four independent spin-flip GPs, requires us to also perform double-polarization
experiments. In this paper, we show predictions for such double-polarization asymmetries and demonstrate their
sensitivity to the spin-flip GPs.

The outline of this paper is as follows. We start in Sec. II by reviewing how the virtual Compton scattering
amplitude at low photon energies can be expanded in terms of GPs. We subsequently review the status of the
HBChPT calculations for the spin-flip GPs. In Sec. III, we outline how we can extract four spin-flip GPs at next-
order in HBChPT by performing a quadratic expansion in the final photon energy. In Sec. IV, we compare our
predictions for the four independent spin-flip GPs at next-order in HBChPT with the leading order (nonzero)
expressions. Furthermore, we demonstrate the sensitivity of double-polarization asymmetries to the spin-flip GPs.
Finally we give our conclusions in Sec. V. Technical de-
tails on crossing symmetry relations for the VCS amplitudes can be found in the appendix.

2. VIRTUAL COMPTON SCATTERING AND GENERALIZED POLARIZABILITIES

We begin by specifying our notation for the VCS process:
\[ \gamma'(\epsilon_1, q) + N(p_1) \rightarrow \gamma(\epsilon_2, q') + N(p_2). \] (1)

The VCS amplitudes are calculated in the c.m. frame and we choose the Coulomb gauge [6]. In the VCS process, the initial spacelike photon is characterized by its four momentum \( q = (\omega, \vec{q}) \), virtuality \( Q^2 \equiv -q^2 \), and polarization vector \( \epsilon_1 = (\vec{e}_1, 1) \). The outgoing real photon has four momentum \( q' = (\omega', |\vec{q'}|, \vec{q'}) \) and polarization vector \( \epsilon_2 = (0, \vec{e}_2) \). We define \( \vec{q} \equiv |\vec{q}| \), and denote \( \theta \) as the scattering angle between virtual and real photons, i.e., \( \cos \theta \equiv \vec{q} \cdot \vec{q}' \). The polarization vector \( \vec{e}_1 \) of the virtual photon can be decomposed into a longitudinal component \( \vec{e}_{1L} = (\vec{e}_1 \cdot \vec{q}) \vec{q} \) and a transverse component \( \vec{e}_{1T} \). For further use, we introduce the virtual photon energy in the limit \( \omega' = 0 \):

\[ \omega_0 \equiv \omega(\omega' = 0, \vec{q}) = M_N - \sqrt{M_N^2 + \vec{q}^2}, \]

(2)

with \( M_N \) the nucleon mass, and where the last line in Eq. (2) indicates the heavy baryon expansion.

The VCS amplitude \( M_{VCS} \) can be expressed in terms of 12 structure functions as [4]:

\[ \mathcal{M}_{VCS} = i e^2 \lambda^4_{11} \langle \vec{e}^*_1 \cdot \vec{e}_{1T} \rangle A_1 + \langle \vec{e}^*_2 \cdot \vec{q} \rangle (\vec{e}_{1T} \cdot \vec{q}') A_2 + i \vec{\sigma} \cdot (\vec{e}^*_2 \times \vec{e}_{1T}) A_3 + i \vec{\sigma} \cdot (\vec{q} \times \vec{q}') (\vec{e}^*_2 \cdot \vec{e}_{1T}) A_4 \]

\[ + \ i \vec{\sigma} \cdot (\vec{e}^*_2 \times \vec{q}) (\vec{e}_{1T} \cdot \vec{q}') A_5 + i \vec{\sigma} \cdot (\vec{e}^*_2 \times \vec{q}') (\vec{e}_{1T} \cdot \vec{q}) A_6 - i \vec{\sigma} \cdot (\vec{e}_{1T} \times \vec{q}) (\vec{e}^*_2 \cdot \vec{q}) A_7 - i \vec{\sigma} \cdot (\vec{e}_{1T} \times \vec{q}') (\vec{e}^*_2 \cdot \vec{q}) A_8 \]

\[ + \ i \vec{\sigma} \cdot (\vec{e}^*_2 \cdot \vec{q}) A_9 + i \vec{\sigma} \cdot (\vec{q} \times \vec{q}') (\vec{e}^*_2 \cdot \vec{q}) A_{10} + i \vec{\sigma} \cdot (\vec{e}^*_2 \cdot \vec{q}) A_{11} + i \vec{\sigma} \cdot (\vec{e}^*_2 \times \vec{q}) A_{12} \rangle \chi_1 \]

(3)

To extract the GPs, we first calculate the complete fourth order VCS amplitudes \( A_1 \), in HBChPT, and subsequently separate the amplitudes in a Born part \( A^{\text{Born}}_1 \), and a non-Born part \( \tilde{A}_1 \), as:

\[ A_1(\omega', \vec{q}, \theta) = A^{\text{Born}}_1(\omega', \vec{q}, \theta) + \tilde{A}_1(\omega', \vec{q}, \theta). \] (4)

In the Born process, the virtual photon is absorbed on a nucleon and the intermediate state remains a nucleon. We calculate them, following the definition of Ref. [2], from direct and crossed Born diagrams with the electromagnetic vertex given by:

\[ \Gamma^\mu(q^2) = F_1(q^2) \gamma^\mu + F_2(q^2) i \sigma^{\mu\nu} \frac{q_\nu}{2M_N}, \]

(5)

where \( F_1(F_2) \) are the nucleon Dirac (Pauli) form factors, respectively. The non-Born terms contain the information on the internal structure of the nucleon. In particular, we are interested in this work in the response of the nucleon to an applied quasistatic electromagnetic dipole field. This is accessed by performing a low-energy expansion (LEX), in the outgoing photon energy of the non-Born VCS amplitudes, and by selecting the term which is linear in the outgoing photon energy, given by:

\[ \left[ \frac{\partial \tilde{A}}{\partial \omega'} (\omega', q, \theta) \right]_{\omega' \rightarrow 0} = S_0(\vec{q}) + P(q) \cos \theta. \] (6)

In Eq. (6), the nucleon structure quantities \( S_0 \) and \( P \) can be expressed in terms of the GPs of the nucleon, as introduced in [2], which are functions of \( \vec{q} \) and which are denoted by \( P^{(p'L', pL')}(\vec{q}) \). In this notation, \( p (p') \) refers to the electric \( (E) \), magnetic \( (M) \), or longitudinal \( (L) \) nature of the initial (final) photon, \( L (L') \) represents the angular momentum of the initial (final) photon, and \( S \) differentiates between the spin-flip \( (S = 1) \) and non-spin-flip \( (S = 0) \) character of the transition at the nucleon side. Restricting oneself to a dipole transition for the final photon \( (i.e., L' = 1) \), angular momentum and parity conservation leads to three scalar and seven spin GPs [2]. The seven spin GPs are obtained as [10]:

\[ p^{(M1,L2)} = -\frac{2\sqrt{2}}{3\sqrt{3}} \frac{1}{\omega_0} \sqrt{\frac{M_N}{E_N}} S_{10}, \]

\[ p^{(L1,L1)} = -\frac{2\sqrt{2}}{3\sqrt{3}} \frac{1}{\omega_0} \sqrt{\frac{M_N}{E_N}} S_{11}, \]

\[ p^{(M1,L0)} = -\frac{2\sqrt{2}}{3\sqrt{3}} \frac{1}{\omega_0} \sqrt{\frac{M_N}{E_N}} S_{12} - \frac{2}{3} S_{10}, \]

\[ p^{(L1,M2)} = -\frac{2\sqrt{2}}{3\sqrt{3}} \frac{1}{\omega_0} \sqrt{\frac{M_N}{E_N}} S_{8}, \]

\[ p^{(M1,L1)} = -\frac{2\sqrt{2}}{3\sqrt{3}} \frac{1}{\omega_0} \sqrt{\frac{M_N}{E_N}} S_{7} - S_{4}, \]

\[ \tilde{p}^{(L1,L1)} = -\frac{2\sqrt{2}}{3\sqrt{3}} \frac{1}{\omega_0} \sqrt{\frac{M_N}{E_N}} S_{3} + \frac{1}{2} S_{8} - S_{11}, \]

(7)

where \( E_N = \sqrt{M_N^2 + \vec{q}^2} \), and where the GPs denoted by \( \tilde{P} \) correspond with mixed electric and longitudinal multipoles, as introduced in [2]. It has been shown [10] that nucleon crossing combined with charge conjugation symmetry of the VCS amplitudes provides three constraints among the seven spin GPs:
leaving four independent spin GPs. By expanding $\omega_0$ as in (2), it is obvious that the relations (8) connect quantities of different order in the heavy baryon expansion. These constraints have been verified in HBChPT in Ref. [6].

By calculating the VCS amplitudes in HBChPT at $O(p^3)$, one extracts from Eq. (7) the following expressions for the GPs at LO (see Ref. [5]):

\begin{align}
[P^{(M,1,2)}](\vec{q})]^{\text{LO}} & = [P^{(M,1,0)}](\vec{q})]^{\text{LO}} = [\tilde{P}^{(M,1,2)}](\vec{q})]^{\text{LO}} = 0, \\
[P^{(M,1,1)}](\vec{q})]^{\text{LO}} & = \frac{-g_A^2}{24\sqrt{2}\pi^2F_{\pi}^2\vec{q}^2} \left[ 1 - \frac{4}{w\sqrt{w^2 + 4}}\sinh^{-1}\left(\frac{w}{2}\right) \right], \\
[\tilde{P}^{(L,1,1)}](\vec{q})]^{\text{LO}} & = \frac{g_A^2}{24\sqrt{6}\pi^2F_{\pi}^2\vec{q}^2} \left[ 3 - \frac{4w^2 + 12}{w\sqrt{w^2 + 4}}\sinh^{-1}\left(\frac{w}{2}\right) \right].
\end{align}

where $w = \vec{q}/m_{\pi}$, with $m_{\pi}$ the pion mass. Furthermore, throughout this paper we use the values: $g_A = 1.267$, $F_{\pi} = 0.0924$ GeV, and $m_{\pi} = 0.14$ GeV.

The VCS amplitudes in HBChPT at $O(p^4)$, yield the GPs at NLO, from Eq. (7) (see Ref. [6]):

\begin{align}
[P^{(M,1,2)}](\vec{q})]^{\text{NLO}} & = \frac{-g_A^2}{12\sqrt{6}\pi^2F_{\pi}^2\vec{q}^2} \left[ 1 - \frac{4}{w\sqrt{w^2 + 4}}\sinh^{-1}\left(\frac{w}{2}\right) \right], \\
[P^{(M,1,0)}](\vec{q})]^{\text{NLO}} & = \frac{g_A^2}{12\sqrt{3}\pi^2F_{\pi}^2} \left[ 2 - \frac{3w^2 + 8}{w\sqrt{w^2 + 4}}\sinh^{-1}\left(\frac{w}{2}\right) \right], \\
[P^{(M,1,1)}](\vec{q})]^{\text{NLO}} & = \frac{g_A^2}{24\pi^2 F_{\pi}^2 M_N} \left[ 1 - \frac{w^2 + 4}{w\sqrt{w^2 + 4}}\sinh^{-1}\left(\frac{w}{2}\right) \right], \\
[\tilde{P}^{(M,2)}](\vec{q})]^{\text{NLO}} & = \frac{-g_A^2}{24\sqrt{10}\pi^2F_{\pi}^2 M_N \vec{q}^2} \left[ 3 - \frac{2w^2 + 12}{w\sqrt{w^2 + 4}}\sinh^{-1}\left(\frac{w}{2}\right) \right], \\
[\tilde{P}^{(L,1,1)}](\vec{q})]^{\text{NLO}} & = 0.
\end{align}

One can get more predictions by use of the crossing relations (8), which hold in general in a relativistic quantum field theory, and which were calculated in HBChPT to $O(p^4)$ in Ref. [6]. By plugging in the fourth order amplitudes in (8), we can extract fifth order predictions from $S_5^{(4)} = -2M_N/\vec{q}S_7^{(5)}$, and $S_5^{(4)} - S_5^{(6)} = -2M_N/\vec{q}S_7^{(5)}$. In this way, these relations allow us to extract two spin GPs at next-to-next-to-leading order (NNLO):

\begin{align}
[P^{(M,1,1)}](\vec{q})]^{\text{NNLO}} & = \frac{-g_A^2}{192\pi^2 F_{\pi}^2 M_N^2} \left[ \frac{3}{w} - \frac{5}{2} + \frac{6}{w^2} \right] \tan^{-1}\left(\frac{w}{2}\right) + \tau_3 \left[ \frac{1}{2w} + \frac{1}{4} - \frac{1}{w^2} \right] \tan^{-1}\left(\frac{w}{2}\right), \\
[\tilde{P}^{(L,1,1)}](\vec{q})]^{\text{NNLO}} & = \frac{g_A^2}{48\pi^2 F_{\pi}^2 M_N} \left[ -1 + \frac{2w^2 + 4}{w\sqrt{w^2 + 4}}\sinh^{-1}\left(\frac{w}{2}\right) \right].
\end{align}

It is surprising that the third-order calculation [5] was able to obtain some of these NLO and one NNLO results. Actually, the NLO results for $P^{(M,1,2)}$1 and $P^{(M,1,0)}$1 were obtained in Refs. [4,5], by performing the LEX of the amplitudes $\tilde{A}_i$ to second order in $\omega$, and, by isolating two terms at order $\omega^2$ which depend on those spin GPs. Furthermore, they used the crossing relations (8) to obtain the NLO results for $P^{(M,1,1)}$1 and $\tilde{P}^{(M,1,2)}$1, as well as the NNLO result for $P^{(L,1,1)}$1.
In the present work, we generalize this method and obtain the NNLO result for $p(M^{1,2})^2$, $p(M^{1,0})^2$, and $\tilde{p}(M^{1,2})^2$, and even the N3LO result for $p(L^1,L^1)$. The first step is to define the following quantities:

$$\left[\frac{\partial^2 \tilde{A}_i(\omega', \bar{q}, \theta)}{\partial \omega'^2}\right]_{\omega' = 0} = 2\alpha_i(\bar{q}) + 2\beta_i(\bar{q}) \cos \theta + 2\gamma_i(\bar{q}) \cos^2 \theta. \quad (12)$$

Our goal is to express $\alpha_i$, $\beta_i$, and $\gamma_i$ in terms of GPs. To achieve this goal, one needs to start from the covariant virtual Compton scattering tensor. The VCS amplitudes are obtained as the contraction of the VCS tensor $M_{\mu\nu}$ with the polarization vectors of the photons, evaluated between the nucleon spinors in the initial and the final states,

$$\mathcal{M}_{\text{VCS}} = -ie^2 \bar{u}(p_1) \sum_{i=1}^{12} e_{1\mu} e_{2\nu}^* \rho_i^{\mu\nu} f_i(q^2, q \cdot q', q' \cdot P) u(p_2), \quad (13)$$

where $P = p_1 + p_2$, $f_i$ are the VCS amplitudes, and $\rho_i^{\mu\nu}$ are the gauge-invariant independent tensor structures for VCS, as given by [10]. The key point of our method is to connect the covariant amplitudes $f_i$ and the c.m. amplitudes $A_i$ defined in Eq. (3). Subsequently, we will perform a low-energy expansion of the amplitudes $A_i$ and sort out the relations between $f_i$ and the $\alpha_i$, $\beta_i$, and $\gamma_i$ defined through Eq. (12). In addition, the GPs can also be expressed in terms of the covariant VCS amplitudes $f_i$. Comparing both results will enable us to express the higher order coefficients of the LEX in terms of GPs. The advantage of working with the covariant Compton tensors is that symmetry properties due to photon crossing as well as nucleon crossing combined with charge conjugation, as derived in Ref. [10] and detailed in the appendix, are immediately manifest.

### III. CALCULATION OF GENERALIZED POLARIZABILITIES AT NNLO AND N3LO IN HBCHPT

To extract the GPs at NNLO, we start by connecting both sets of VCS amplitudes $A_i$ and $f_i$ of Eqs. (3) and (13). To this goal, one has to expand $e_{1\mu} e_{2\nu}^* \rho_i^{\mu\nu}$ in Eq. (13) in $\omega'$ in the c.m. frame and also expand the non-Born parts of the amplitudes $f_i$ in $\omega'$:

$$f_i^{\text{non-Born}} = \hat{f}_i(\bar{q}) + \omega' \left[ g_i(\bar{q}) + \bar{h}_i(\bar{q}) \cos \theta \right] + \mathcal{O}(\omega'^2). \quad (14)$$

For the new predictions of three NNLO GPs and one N3LO GP which we will derive, we only need the amplitudes $A_{10}$, $A_{11}$ and $A_{12}$. The expressions for these amplitudes including terms up to order $\omega'^2$ can be found, after some algebra, as

$$A_{10} = \omega' \left[ 2\omega_0 \bar{q} \hat{f}_4 - \frac{\omega_0^3}{2q} \hat{f}_5 - \frac{\omega_0 q}{2} \hat{f}_7 - 2\omega_0 \bar{q} \hat{f}_{10} - 2\omega_0 \bar{q} \hat{f}_{11} - M_N \omega_0 \hat{f}_{12} \right] + \omega'^2 \left[ 2\omega_0 \bar{q} g_4 - \frac{\omega_0^3}{2q} g_5 - \frac{\omega_0 q}{2} g_7 - \frac{\omega_0^2}{q} g_{10} \right. \right.$$

$$- 2\omega_0 g_{11} - M_N \omega_0 g_{12} \right] + \omega'^2 \cos \theta \left[ 2\omega_0 \bar{q} h_4 - \frac{\omega_0^3}{2q} h_5 - \frac{\omega_0 q}{2} h_7 - \frac{\omega_0^2}{q} h_{10} - 2\omega_0 \bar{q} h_{11} - M_N \omega_0 h_{12} \right] + \omega'^2 \left[ \frac{\omega_0^2}{2M_N q} \hat{f}_1 + \omega_0 \bar{q} \hat{f}_2 + \left( 2q + \frac{\omega_0 q}{M_N} \right) \hat{f}_4 + \left( -2q - \frac{\omega_0 q}{M_N} \right) \hat{f}_5 + \left( \frac{4M_N \omega_0^2}{q} + \omega_0^2 \cos \theta \right) \hat{f}_6 \right. \right.$$

$$+ \left( -\frac{\omega_0 \bar{q}}{4M_N - \frac{q}{2}} \right) \hat{f}_7 + \omega_0 M_N \hat{f}_8 + \omega_0 \bar{q} \hat{f}_9 + \frac{\bar{q}}{M_N} \hat{f}_{10} + \left( -2q - \frac{\omega_0 q}{M_N} - 4\omega_0^2 \right) \hat{f}_{11} + \left( -\frac{3\omega_0^3}{2q} + \omega_0 \bar{q} \right) \hat{f}_{12} \right. \right.$$

$$+ \mathcal{O}(\omega'^3), \quad (15)$$

1Note that we omit the global normalization factor $\sqrt{\frac{E_p + M_0}{2M_p}}$ on the right hand side of Eqs. (15)–(17).
We can simplify these expressions by using the crossing symmetry relations for the amplitudes \( f_i \), as given in the appendix, which lead to \( f_4 = f_8 = f_{10} = 0 \). It is now straightforward to read off from Eqs. (15)–(17) the coefficients \( S_i \) of Eq. (6), which appear in the LEX of the c.m. amplitudes \( \tilde{A}_i \), and express them as combinations of the \( \tilde{f}_i \):

\[
S_{i0} = \left[ -\frac{\omega_0^3}{2q} \tilde{f}_5 - \frac{\omega_0 \bar{q}}{2} \tilde{f}_7 - 2\omega_0 \bar{q} \tilde{f}_{11} - \frac{M_N \omega_0^3}{q} \tilde{f}_{12} \right].
\]

\[
S_{i1} = \left[ -\frac{\omega_0^2}{2} \tilde{f}_5 - \frac{\omega_0 \bar{q}}{2} \tilde{f}_7 - 2\omega_0 \bar{q} \tilde{f}_{11} - 2M_N \omega_0^2 \tilde{f}_{12} \right].
\]

\[
S_{i2} = \left[ -\frac{M_N \omega_0^2}{q} \tilde{f}_5 + \left( -\frac{2M_N \omega_0^2}{q} + M_N \omega_0 \bar{q} \right) \tilde{f}_{12} \right].
\]
By applying the defining relations of Eq. (7) for the GPs, one then obtains:

\[ p^{(M1L2)\dagger}(\bar{q}) = \frac{2\sqrt{2}}{3\sqrt{3}} \sqrt{\frac{E_N + M_N}{2E_N}} \left[ \frac{\omega_0^2}{2\bar{q}^2} \bar{f}_5 + \frac{1}{2} \bar{f}_7 + 2\bar{f}_{11} + \frac{M_N \omega_0^2}{\bar{q}^2} \bar{f}_{12} \right]. \]

\[ p^{(M1L0)\dagger}(\bar{q}) = \frac{2}{\sqrt{3}} \sqrt{\frac{E_N + M_N}{2E_N}} \left[ \left( \frac{1}{3} \omega_0^2 - M_N \omega_0 \right) \bar{f}_5 + \frac{1}{3} \bar{q}^2 \bar{f}_7 + \frac{4}{3} \bar{q}^2 \bar{f}_{11} + (M_N \bar{q}^2 - 2M_N^2 \omega_0 + \frac{2}{3} M_N \omega_0^2) \bar{f}_{12} \right]. \]  

\[ p^{(L1L1)\dagger}(\bar{q}) = \frac{2}{3} \sqrt{\frac{E_N + M_N}{2E_N}} \left[ -\frac{\omega_0}{2} \bar{f}_5 - \frac{\omega_0}{2} \bar{f}_7 - 2\omega_0 \bar{f}_{11} - 2M_N \omega_0 \bar{f}_{12} \right]. \]  

In general, the coefficients of order \( \omega^2 \) are combinations of \( \bar{f}, \bar{g}_i, \) and \( h_i \). On the other hand, the dipole GPs defined through Eq. (7) are all combinations of \( \hat{f} \). Actually \( g_i \) and \( h_i \) are related to higher order polarizabilities (\( L' > 1 \)), and they can be defined in a similar way as in Ref. [2]. We leave the study of such higher order GPs to a future work and concentrate in this work on new predictions for dipole \( (L' = 1) \) GPs.

To compare with the amplitudes obtained from HBCBPT, it is necessary to make the heavy baryon expansions on Eqs. (15)–(17). Note that not only \( \omega_0 \) has to be expanded in \( \bar{q}/M_N \) as in Eq. (2), but also \( \hat{f}, \bar{g}_i, \) and \( h_i \). Denoting any quantity proportional to \((1/M_N)^n\) by a superscript \((n)\), we can express \( \hat{f}, \) as

\[ \hat{f}_i = f_i^{(0)} + f_i^{(1)} + \mathcal{O}(1/M_N^2), \quad i = 1, 5, 6, 7, 9, 11. \]  

\[ \hat{f}_i = f_i^{(0)} + f_i^{(1)} + \mathcal{O}(1/M_N^2), \quad i = 2, 12. \]  

The coefficients in the heavy baryon expansion are furthermore constrained by the crossing symmetry relations for the amplitudes \( f_i \) (for details, see the appendix). We can then separate the leading and subleading terms in the heavy baryon expansion for the amplitudes \( A_{10}, A_{11}, \) and \( A_{12} \) as:

\[ A_{10}^{(L0)} = \omega^2 \left[ -\frac{\bar{q}}{2} \bar{f}_7 + 2\bar{q} \bar{f}_{11} \right]. \]

\[ A_{11}^{(L0)} = \omega^2 \left[ 2\bar{q}^2 \bar{g}_4^{(0)} + \omega^2 \cos \left[ \frac{\bar{q}}{2} \bar{f}_7 + 2\bar{q} \bar{f}_{11} \right] \right]. \]

\[ A_{12}^{(L0)} = \omega^2 \left[ 2\bar{q}^2 \bar{f}_5^{(0)} + 2M_N \bar{q} \bar{f}_{12}^{(0)} \right] + \omega^2 \cos \left[ -2\bar{q}^2 \bar{f}_6^{(0)} + \bar{q}^2 \bar{f}_{12}^{(0)} \right]. \]

\[ \begin{align*}
A_{10}^{(L1)} & = \omega^2 \left[ -\frac{\bar{q}}{2} \bar{f}_7^{(0)} - 2\bar{q} \bar{f}_{11}^{(0)} \right] + \frac{1}{M_N} \omega^2 \left[ \frac{\bar{q}^2}{4} \bar{f}_7^{(0)} + \bar{q} \bar{f}_{11}^{(0)} \right] + \frac{1}{M_N} \omega^2 \cos \left[ -\frac{\bar{q}^2}{4} \bar{f}_6^{(0)} + \bar{q} \bar{f}_{12}^{(0)} \right] \right] \\
& + \frac{1}{M_N} \omega^2 \cos \theta \left[ -\frac{\bar{q}^2}{4} \bar{g}_5^{(0)} - \bar{q} \bar{g}_{10}^{(0)} \right], \end{align*} \]

\[ A_{11}^{(L2)} = \omega^2 \left[ 2\bar{q}^2 \bar{f}_{12}^{(0)} + 2\bar{q}^2 \bar{g}_4^{(0)} \right] + \omega^2 \cos \left[ \frac{\bar{q}}{2} \bar{f}_7^{(0)} + 2\bar{q} \bar{f}_{11}^{(0)} \right] + \frac{1}{M_N} \omega^2 \cos \left[ -\frac{\bar{q}^2}{4} \bar{f}_6^{(0)} + \frac{3}{2} \bar{f}_9^{(0)} + 2\bar{q}^2 \bar{g}_4^{(0)} \right]. \]

\[ \begin{align*}
A_{12}^{(L0)} & = \omega^2 \left[ \frac{\bar{q}}{2} \bar{f}_5^{(0)} + 2M_N \bar{q} \bar{f}_{12}^{(0)} \right] + \frac{1}{M_N} \omega^2 \left[ -\frac{\bar{q}^2}{4} \bar{f}_5^{(0)} - \bar{q} \bar{f}_{12}^{(0)} \right] + \frac{1}{M_N} \omega^2 \cos \left[ -2M_N \bar{q}^2 \bar{f}_6^{(0)} + M_N \bar{q}^2 \bar{f}_{12}^{(0)} \right] \\
& + \frac{1}{M_N} \omega^2 \cos \left[ \frac{\bar{q}^2}{4} \bar{f}_5^{(0)} + 2\bar{q} \bar{f}_7^{(0)} - \bar{q} \bar{f}_{11}^{(0)} - \frac{\bar{q}^3}{4} h_7^{(0)} - M_N \bar{q}^3 h_{12}^{(0)} \right]. \end{align*} \]

From Eq. (22) one can now identify the coefficients \( \alpha_i, \beta_i, \) and \( \gamma_i \) appearing in the LEX of Eq. (12) as follows:
The GPs of Eq. (19) can now be expressed in terms of the above LEX coefficients. The power counting of the GPs needs some word of explanation though. From HBChPT, the leading order amplitudes \( S_0^{(3)} \sim \mathcal{O}(1) \) and \( S_0^{(4)} \sim \mathcal{O}(1/M_N) \). Because \( \omega_0 \) is a quantity of order \( \mathcal{O}(1/M_N) \), one remarks from Eq. (7) that the three GPs \( p^{(M,L,2)} \), \( p^{(L,L,1)} \), and \( p^{(M,L,0)} \) should start at \( \mathcal{O}(M_N) \). Actually \( S_0^{(3)} = S_0^{(3)} = S_0^{(3)} = 0 \), and consequently these three GPs are nonzero only at NLO:

\[
\begin{align*}
\alpha^{NLO}_{10} &= -\frac{\bar{q}}{2} q^{(0)} f_7^{(0)} - 2 q^{2} f_{11}^{(0)}, \\
\alpha^{NLO}_{10} &= -\frac{\bar{q}}{2} q^{(0)} f_7^{(0)} - 2 q^{2} f_{11}^{(0)} + \frac{1}{M_N} \left[ -\bar{q} g^{(0)} \right], \\
\beta^{NLO}_{10} &= \frac{1}{M_N} \left[ \frac{\bar{q}^3}{4} h_7^{(0)} + q^3 h_7^{(0)} \right], \\
\gamma^{NLO}_{10} &= 0, \\
\gamma^{NLO}_{11} &= 0, \\
\alpha^{NLO}_{11} &= 2 \bar{q}^2 g_4^{(0)}, \\
\beta^{NLO}_{11} &= \frac{1}{M_N} \left[ -\bar{q}^3 h_7^{(0)} - q^3 h_7^{(0)} \right], \\
\alpha^{NLO}_{12} &= \frac{\bar{q}^2}{2} f_s^{(0)} + 2 \bar{q}^2 g_{12}^{(0)} + \frac{1}{M_N} \left[ \frac{\bar{q}^2}{2} f_s^{(0)} + \frac{3}{2} \bar{q}^2 f_{11}^{(0)} + \frac{2}{3} \bar{q} g_4^{(0)} \right], \\
\beta^{NLO}_{12} &= \frac{1}{M_N} \left[ -\bar{q}^3 f_6^{(0)} - q^3 f_6^{(0)} \right], \\
\alpha^{NLO}_{12} &= \frac{\bar{q}^2}{2} f_s^{(0)} + 2 \bar{q}^2 g_{12}^{(0)} + \frac{1}{M_N} \left[ \frac{\bar{q}^2}{2} f_s^{(0)} + \frac{3}{2} \bar{q}^2 f_{11}^{(0)} + \frac{2}{3} \bar{q} g_4^{(0)} \right], \\
\beta^{NLO}_{12} &= \frac{1}{M_N} \left[ -\bar{q}^3 f_6^{(0)} - q^3 f_6^{(0)} \right].
\end{align*}
\]

(23)

To evaluate the GPs \( \alpha^{NLO}_{10} \) and \( \alpha^{NLO}_{12} \) at NNLO, requires the knowledge of the VCS amplitudes at \( \mathcal{O}(p^5) \) in HBChPT through the coefficients \( \alpha^{NLO}_{10} \) and \( \alpha^{NLO}_{12} \). In general, the amplitudes at \( \mathcal{O}(p^5) \) in HBChPT contain two parts. The first one consists of two-loop diagrams and one-loop diagrams with one vertex from the third-order Lagrangian \( \mathcal{L}^{(3)} \). The second one consists of one-loop diagrams with vertices from the heavy baryon expansion. The main difference between both contributions is that the former ones contain no \( 1/M_N \) factor whereas the latter ones contain a \( 1/M_N^2 \) prefactor. We will present here the results due to the one-loop diagrams with vertices from the heavy baryon expansion. Taking into account these contributions will precisely restore the symmetry property due to nucleon crossing combined with charge conjugation, which relates the VCS amplitudes at different orders in the heavy baryon expansion. The new analytic structures which one can expect to arise at \( \mathcal{O}(p^5) \) due to two-loop effects have to then satisfy the crossing relations separately.

We find for \( \alpha^{NLO}_{10} \) and \( \alpha^{NLO}_{12} \) by explicit calculation in HBChPT the following results:
\[\alpha_{10}^{\text{NLO}} = \frac{g_A^2}{64 M_N \pi F_\pi^2} \left[ \left( \frac{2}{w} - \frac{4}{w} \cdot \frac{6 - 6 w^2}{4 w + w^3} - \frac{6 + 15 w^2 + 2 w^4}{4 w + w^3} + \left( 12 + 6 \frac{w}{w^2} \right) \tan^{-1} \left( \frac{w}{2} \right) \right) + \frac{1}{2} (1 + \kappa_v) \right] - \left( 1 + \kappa_v \right) \tau_3 \left[ - \frac{8}{w} + \frac{w}{3} + \left( 2 + \frac{16}{w^2} \right) \tan^{-1} \left( \frac{w}{2} \right) \right], \]

\[\alpha_{12}^{\text{NLO}} = \frac{g_A^2}{64 M_N \pi F_\pi^2} \left[ \left( \frac{7}{w} + \frac{2 w}{3} + \left( 2 + \frac{14}{w^2} \right) \tan^{-1} \left( \frac{w}{2} \right) \right) + \frac{1}{2} \left( 1 + \kappa_v \right) \left[ - \frac{3}{w} + \frac{w}{6} + \left( \frac{1}{2} + \frac{2}{w^2} \right) \tan^{-1} \left( \frac{w}{2} \right) \right] \right] + \frac{1}{2} \left( 1 + \kappa_v \right) \tau_3 \left[ - \frac{1}{w} - \frac{w}{6} + \left( \frac{5}{2} + \frac{2}{w^2} \right) \tan^{-1} \left( \frac{w}{2} \right) \right], \]  

(25)

where \( \kappa_v = 3.70 \) ( \( \kappa_s = -0.12 \) ) are the nucleon isovector (isoscalar) anomalous magnetic moments, respectively. Combining Eqs. (24) and (25), allows us to obtain the NNLO results for two GPs:

\[ [p^{(M_{1,0})}(\bar{q})]^{\text{NNLO}} = \frac{g_A^2}{96 \sqrt{3} M_N \pi F_\pi^2 \rho \bar{q}} \left[ \left( 17 \frac{w}{w} - 2 w - 8 - 12 \frac{w^2}{4 w + w^3} + \frac{8}{3} \frac{6 + 15 w^2 + 2 w^4}{4 w + w^3} + \left( 18 + 30 \frac{w}{w^2} \right) \tan^{-1} \left( \frac{w}{2} \right) \right) + 1 \right] + \frac{1}{4} \left( 1 + \kappa_v \right) \tau_3 \left[ - \frac{7}{w} - \frac{w}{12} + \left( \frac{19}{4} - 1 \right) \tan^{-1} \left( \frac{w}{2} \right) \right] + ([1 + \kappa_v] \tau_3 \left[ - \frac{19}{2 w} + \frac{w}{12} + \left( \frac{23}{4} + \frac{19}{w^2} \right) \tan^{-1} \left( \frac{w}{2} \right) \right] + 1), \]

\[ [p^{(M_{1,2})}(\bar{q})]^{\text{NNLO}} = \frac{-g_A^2}{48 \sqrt{6} M_N \pi F_\pi^2 \rho \bar{q}} \left[ \left( - \frac{2}{w} + \frac{4 - 6 w^2}{4 w + w^3} - \frac{4}{3} \frac{6 + 15 w^2 + 2 w^4}{4 w + w^3} + \left( 12 + 6 \frac{w}{w^2} \right) \tan^{-1} \left( \frac{w}{2} \right) \right) + 1 \right] + \frac{1}{4} \left( 1 + \kappa_v \right) \tau_3 \left[ - \frac{4}{w} + \frac{w}{6} + \left( \frac{1 + \frac{8}{w^2}}{2} \right) \tan^{-1} \left( \frac{w}{2} \right) \right]. \]

\[ (26) \]

A third NNLO prediction follows by applying the crossing symmetry constraint of Eq. (8), i.e., \( S_{10} - S_{12} = \frac{\hat{q}}{w_0} S_{11} \), which yields the relation between three spin GPs:

\[ 3 \frac{g_A^2}{\omega_0} p^{(L,1,1)} = \sqrt{3} p^{(M_{1,0})} + \sqrt{3} \hat{q}^2 p^{(M_{1,2})}. \]

\[ (28) \]

After the heavy baryon expansion, this becomes:

\[ -6 M_N [p^{(L,1,1)}]^{\text{NLO}} = \sqrt{3} [p^{(M_{1,0})}]^{\text{NLO}} + \sqrt{3} \hat{q}^2 [p^{(M_{1,2})}]^{\text{NLO}}, \]

from which we obtain:

\[ [p^{(L,1,1)}(\bar{q})]^{\text{NLO}} = \frac{g_A^2}{576 \sqrt{6} M_N \pi F_\pi^2 \rho \bar{q}} \left[ \left( \frac{15}{w} - 2 w + \frac{4 - 6 w^2}{4 w + w^3} - \frac{4}{3} \frac{6 + 15 w^2 + 2 w^4}{4 w + w^3} + \left( 30 - \frac{24}{w^2} \right) \tan^{-1} \left( \frac{w}{2} \right) \right) + 1 \right] + \frac{1}{4} \left( 1 + \kappa_v \right) \tau_3 \left[ \frac{27}{2 w} - \frac{w}{4} + \left( \frac{27}{4} - \frac{27}{w^2} \right) \tan^{-1} \left( \frac{w}{2} \right) \right]. \]

\[ (30) \]

Similarly, by applying the crossing symmetry constraint of Eq. (8), i.e., \( S_4 = 0 \), which yields the relation between three spin GPs:

\[ \sqrt{3} \frac{g_A^2}{\omega_0} p^{(M_{1,2})}(\bar{q}) + p^{(M_{1,1})}(\bar{q}) + \sqrt{3} \hat{q}^2 \hat{p}^{(M_{1,2})} = 0, \]

\[ (31) \]
\[
[\hat{\rho}^{(M,1,2)}(\vec{q})]^{\text{NNLO}} = \frac{-s_{A}^{2}}{96\sqrt{10} M_{N}^{2} \pi F_{2}^{2} q} \left[\frac{-5}{w} + \frac{4 - 6w^{2}}{4w + w^{3}} - \frac{4}{3} \frac{6 + 15w^{2} + 2w^{4}}{4w + w^{3}} + \frac{29}{2} \frac{12}{w^{2}} \tan^{-1}\left(\frac{w}{2}\right)\right] \\
+ \left[(1 + \kappa_{\pi}) - (1 + \kappa_{e})\tau_{3}\right] \left[-\frac{4}{w} + \frac{1}{6} + \frac{8}{w^{3}} \tan^{-1}\left(\frac{w}{2}\right)\right] + \tau_{3}\left[-\frac{1}{w} + \frac{1}{2} + \frac{2}{w^{2}} \tan^{-1}\left(\frac{w}{2}\right)\right].
\]

(32)

At the real photon point (\( \vec{q} = 0 \)), we can express the GP \( P^{(M,1,2)}(0) \) in terms of a sum of two spin polarizabilities \((\gamma_{2} + \gamma_{4})\), introduced by Ragusa [11], as:

\[
\gamma_{2} + \gamma_{4} = -\alpha_{em}^{2} \frac{3\sqrt{3}}{2\sqrt{2}} P^{(M,1,2)}(0),
\]

where \( \alpha_{em}^{2} = e^{2}/(4\pi) = 1/137 \). The result of Eq. (27) then allows one to extract the result for \( \gamma_{2} + \gamma_{4} \) at NLO as:

\[
(\gamma_{2} + \gamma_{4})^{\text{NLO}} = \frac{\alpha_{em}^{2} s_{A}^{2}}{384 \pi F_{2}^{2} m_{\pi} M_{N}} \left[-3 + 2(1 + \kappa_{\pi}) \right] \\
- 2(1 + \kappa_{e})\tau_{3},
\]

(33)

which exactly reproduces the NLO calculations for \( \gamma_{2} + \gamma_{4} \) of Refs. [12,13].

From our results, it is also worth noting that the crossing relations of Eq. (8) allow one to classify the GPs into two groups:

- group 1 = \( \{P^{(M,1,2)}(0)\}, P^{(M,1,1)}(0), P^{(M,1,1)}(1), P^{(M,1,2)}(1)\} \)
- group 2 = \( \{P^{(L,1,1)}(0), P^{(L,1,1)}(1), P^{(L,1,1)}(2)\} \).

It is interesting to observe that their analytical forms alternate from one order to the next, as noticed already in Ref. [6]. At LO one has:

- group 1: \( \frac{1}{\pi} \tan^{-1}\left(\frac{\vec{q}}{2m_{\pi}}\right) \)
- group 2: \( \frac{1}{\pi} \sinh^{-1}\left(\frac{\vec{q}}{2m_{\pi}}\right) \)

At NLO, one has:

- group 1: \( \frac{1}{\pi} \sinh^{-1}\left(\frac{\vec{q}}{2m_{\pi}}\right) \)
- group 2: \( \frac{1}{\pi} \tan^{-1}\left(\frac{\vec{q}}{2m_{\pi}}\right) \)

At NNLO one again has the same analytical structure as at LO, etc. One therefore observes that the alternating analytical forms teach us how the crossing relations (8), connecting the GPs between two different orders in HBChPT, work. The same alternating analytical forms were observed when comparing leading and next-to-leading order HBChPT results for the forward spin polarizabilities in Ref. [14].

IV. RESULTS AND DISCUSSION

The main results of this work are the new NNLO predictions of Eqs. (26), (27), and (32) for the spin GPs \( P^{(M,1,0)}(0), P^{(M,1,2)}(0), P^{(M,1,2)}(1) \), as well as the \( N^{3}\)LO prediction of Eq. (30) for the spin GP \( P^{(L,1,1)}(1) \).

In Fig. 1, we compare these results for the GPs \( P^{(M,1,2)}(0) \) and \( P^{(L,1,1)}(1) \) with their lowest order nonzero expressions, and display as well the next-order results for the GPs \( P^{(L,1,2)}(0) \) and \( P^{(M,1,1)}(1) \), which were calculated before in Ref. [6]. Note that the three remaining spin GPs of Eq. (7) can be expressed in terms of these four independent spin GPs by making use of the crossing symmetry relations of Eq. (8). By comparing the lowest order with next-order results in HBChPT for each of the GPs, one observes large corrections at next-order. We also compare these results in Fig. 1 with the phenomenological estimate of Refs. [15,16] using dispersion relations. One sees that for the GPs \( P^{(L,1,2)}(0), P^{(M,1,1)}(1), \) and \( P^{(L,1,1)}(1) \) the corrections to the leading order bring the HBChPT results toward the phenomenological estimates, though they overshoot it for...
For the GP $P^{(L,1,L\parallel)}$, the sizeable large correction at the real photon point brings the HBCpt results close to the dispersive results but shows a much slower falloff with $Q^2$ than the dispersive estimate. The comparison in Fig. 1 clearly indicates that a satisfactory description of spin GPs is still a challenging task. To completely disentangle the four spin GPs experimentally requires four independent observables, which we discuss in the following.

It was shown in Ref. [2] that GPs can be accessed experimentally by measuring the $ep \rightarrow ep\gamma$ reaction and performing a low-energy expansion in the outgoing photon energy $\omega'$. In particular, the VCS unpolarized squared amplitude $M^{\text{exp}}$ takes on the form [2]:

$$M^{\text{exp}} = \frac{M^{\text{exp}}}{\omega'} = \frac{M^{\text{exp}}}{\omega'} + M^{\text{exp}}_0 + O(\omega').$$

Because of the low-energy theorem (LET), the threshold coefficients $M^{\text{exp}}_0$ and $M^{\text{exp}}_1$ are known [2], and are fully determined from the Bethe-Heitler+Born (BH + B) amplitudes. The information on the GPs is contained in $M^{\text{exp}}_0$, which contains a part originating from the BH + B amplitudes and another one which is a linear combination of the GPs, with coefficients determined by the kinematics. The unpolarized observable $M^{\text{exp}}_0$ can be expressed in terms of three structure functions $P_{LL}(\hat{q})$, $P_{TT}(\hat{q})$, and $P_{LT}(\hat{q})$ as [2]:

$$M^{\text{exp}}_0 - M^{\text{exp}}_{0,\text{BH}+\text{B}} = 2K_2 \left[ v_1 [e P_{LL}(\hat{q}) - P_{TT}(\hat{q})] + \left( v_2 - \frac{\omega_0}{\hat{q}} v_3 \right) \sqrt{2 \epsilon (1 + \epsilon)} P_{LT}(\hat{q}) \right],$$

where $K_2$ is a kinematical factor, $e$ is the virtual photon polarization (in the standard notation used in electron scattering), and $v_1, v_2, v_3$ are kinematical quantities depending on $e$ and $\hat{q}$ as well as on the c.m. polar and azimuthal angles ($\theta, \gamma$, and $\phi$, respectively) of the produced real photon (for details see Ref. [1]). The three unpolarized observables of Eq. (35) can be expressed in terms of the independent GPs as [1,2]:

$$P_{LL} = -2\sqrt{6} M_N G_E(Q^2) P^{(L,1,L\parallel)}_0,$$

$$P_{TT} = -3 G_M(Q^2) \frac{Q^2}{\omega_0} \left( P^{(M,1,M\parallel)}_0 - \sqrt{2} \omega_0 P^{(L,1,M\parallel)}_0 \right),$$

$$P_{LT} = \sqrt{\frac{3}{2}} \frac{M_N \hat{q}}{Q} G_E(Q^2) P^{(M,1,M\parallel)}_0 + \frac{3 Q \hat{q}}{2 \omega_0} G_M(Q^2) P^{(L,1,L\parallel)}_0,$$

where $Q^2 = Q^2 - \omega^2_0 = -2 M_N \omega_0$, and $G_E(Q^2)$ and $G_M(Q^2)$ stand for the electric and magnetic nucleon form factors, respectively. The spin-independent GPs $P^{(L,1,L\parallel)}_0$ and $P^{(M,1,M\parallel)}_0$ are directly proportional to the electric and magnetic polarizabilities of the nucleon, respectively. A Fourier transform of their $\hat{q}$-dependence allows us to map out the spatial distribution of the electric polarization and magnetization of the nucleon, as discussed in Ref. [3].

The first VCS experiment was performed at MAMI [7] and the response functions $P_{LT}$ and $P_{LL} - P_{TT}/\epsilon$ were extracted at $Q^2 = 0.33 \text{ GeV}^2$. Going to higher $Q^2$, the VCS process has also been measured at JLab and data have been obtained at $Q^2 = 1 \text{ GeV}^2$ and $Q^2 = 1.9 \text{ GeV}^2$ [8]. To unambiguously extract the spin-independent GPs from experiment requires one to separate $P_{LL}$ from $P_{TT}$.

This can be done by performing two unpolarized VCS experiments at a fixed value of $\hat{q}$ (or equivalently $Q^2$) and by varying $\epsilon$ (or equivalently the beam energy). Such experiments are planned at MAMI in the near future, making use of the beam energy upgrade. Besides extracting the electric GP $P^{(L,1,L\parallel)}_0$ such an experiment will also allow for a measurement of the combination of the two spin GPs $P^{(M,1,M\parallel)}_0$ and $P^{(L,1,M\parallel)}_0$ of Eq. (37).

Until now, we discussed only unpolarized VCS observables. An unpolarized VCS experiment gives access to only three combinations of the six GPs, as given by Eqs. (36)–(38). It was shown in Ref. [17] that VCS double-polarization observables with a polarized lepton beam and a polarized target (or recoil) nucleon, will allow us to measure three more combinations of spin-flip GPs. Therefore a measurement of unpolarized VCS observables (at different values of $\epsilon$) and of three double-polarization observables will enable us to disentangle all six GPs. The VCS double-polarization observables, which are denoted by $\Delta M(h, i)$ for an electron of helicity $h$, are defined as the difference of the squared amplitudes for recoil (or target) proton spin orientation in the direction and opposite to the axis $i$ ($i = x, y, z$), where the $z$-direction is chosen along the virtual photon momentum (see Ref. [17] for details). In a LEX, this polarized squared amplitude yields:

$$\Delta M^{\text{exp}} = \frac{\Delta M^{\text{exp}}_0}{\omega^2} + \frac{\Delta M^{\text{exp}}_1}{\omega'} + \Delta M^{\text{exp}}_0 + O(\omega').$$

Analogous to the unpolarized squared amplitude of Eq. (34), the threshold coefficients $\Delta M^{\text{exp}}_0, \Delta M^{\text{exp}}_1, \Delta M^{\text{exp}}_2$ are known due to the LET. It was found in Ref. [17] that the polarized squared amplitude $\Delta M^{\text{exp}}_0$ can be expressed in terms of three new structure functions $P^{(h, L\parallel)}_0(\hat{q})$, $P^{(h, L\parallel)}_1(\hat{q})$, and $P^{(h, L\parallel)}_2(\hat{q})$. These new structure functions are related to the spin GPs as [1,17]:

$$P^{(h, L\parallel)}_0 = \frac{3 Q \hat{q}}{2 \omega_0} G_M(Q^2) P^{(L,1,L\parallel)}_0 - \frac{3 M_N \hat{q}}{Q} G_E(Q^2) P^{(M,1,M\parallel)}_0,$$

$$P^{(h, L\parallel)}_1 = \frac{6 M_N \hat{q}}{Q} G_M(Q^2) P^{(L,1,M\parallel)}_0 + \frac{3 Q \hat{q}}{2 \omega_0} G_M(Q^2) P^{(L,1,L\parallel)}_0,$$

$$P^{(h, L\parallel)}_2 = \frac{3 Q \hat{q}}{2 \omega_0} G_M(Q^2) P^{(L,1,L\parallel)}_0 - \frac{3 M_N \hat{q}}{Q} G_E(Q^2) P^{(M,1,M\parallel)}_0,$$
NEW PREDICTIONS FOR GENERALIZED SPIN...

\[ P_{LT}^i = -\frac{3}{2} G_M(Q^2) P_{L,L,L}^{(1,1,1)} + \frac{3M_N\tilde{q}^2}{Q\omega_0} G_F(Q^2) P_{M,L,M}^{(1,1,1)}, \]

(41)

\[ P_{LT}^f = \frac{3\tilde{q}Q}{2\omega_0} G_M(Q^2)(P_{L,L,L}^{(1,1,1)} - \sqrt{\frac{3}{2}} \omega_0 P_{M,L,L}^{(1,1,1)}). \]

(42)

While \( P_{LT}^i \) and \( P_{LT}^f \) can be accessed by in-plane kinematics (\( \phi = 0^{\circ} \)), the measurement of \( P_{LT}^f \) requires an out-of-plane experiment.

In Fig. 2, we show the results for the double-polarization observables, with polarized electron and by measuring the recoil proton polarization either along the virtual photon direction (z-direction) or parallel to the reaction plane and perpendicular to the virtual photon (x-direction). These asymmetries directly depend on the response functions of Eqs. (40)–(42). One sees from Fig. 2 that the double-polarization asymmetries are quite large (due to a nonvanishing asymmetry for the BH + B mechanism).

In Fig. 3, we show the results of the dispersion relation calculation from Refs. [16,18]. Because of the smaller spin GPs in the dispersion calculation, the relative effect on the asymmetries is smaller in the dispersion relation approach.

To quantify the relative effect on the asymmetries due to the spin GPs, we display in Fig. 3 the double-polarization asymmetries with the well-known Bethe-Heitler+Born contribution subtracted. Another contribution which is well-known is the anomaly contribution (due to a t-channel \( \pi^0 \) pole) to the spin GPs, which is shown in Fig. 3 (lower panels). One furthermore sees from Fig. 3 (upper panels) that the effect of the spin GPs, excluding the anomaly contribution, on the double-polarization asymmetries is of the order of 5%. To

FIG. 2. VCS double-polarization asymmetry (polarized electron, recoil proton polarization along either the z- or x-directions in the c.m. frame) in MAMI kinematics as a function of the photon scattering angle. The dashed-dotted curves correspond to the BH + B contribution. The dotted curves are the lowest order HBChPT predictions from Refs. [4,5]. The dashed curves are obtained by including the next-order HBChPT correction for all spin-flip GPs, as calculated in this work, while using the lowest order predictions for the spin-independent GPs. For comparison, we also show the results of a phenomenological dispersion relation calculation [16,18] (solid curves).

FIG. 3. Deviation of the double-polarization VCS asymmetry from the Bethe-Heitler+Born result, calculated within the LEX formalism in MAMI kinematics as a function of the photon scattering angle. Upper panels: lowest order HBChPT predictions from Refs. [4,5] (dotted curves); results including the next-order HBChPT corrections for the spin-flip GPs, as calculated in this work, and using the leading order predictions for the spin-independent GPs (dashed curves); dispersion relation results [16,18] (solid curves). The effect of the anomaly contribution (i.e., t-channel \( \pi^0 \) pole) is neglected in all the three model-predictions. In the lower panels, a comparison is shown between the dispersive predictions without the anomaly contribution (dashed curves) and with the anomaly contribution (solid curves).
distinguish between the different predictions, a double-polarization VCS experiment is clearly called for. Although these double-polarization observables are tough to measure, a first test experiment is already underway at MAMI [19].

V. CONCLUSIONS

In this work, we calculated the spin-dependent VCS amplitude in HBChPT at $\mathcal{O}(p^3)$. In a low-energy expansion of the outgoing photon energy, the spin-dependent VCS amplitude can be parametrized in terms of seven generalized spin polarizabilities which characterize the response of the nucleon. Nucleon crossing symmetry combined with charge conjugation invariance leads to relations among these GPs, leaving four independent spin-flip GPs. We calculated all seven spin-flip GPs in HBChPT at $\mathcal{O}(p^3)$ and checked the three crossing symmetry relations to this order. At order $\mathcal{O}(p^4)$, no unknown low-energy constants enter the theory, allowing us to make absolute predictions for all seven spin-flip GPs to this order. Three of the spin-flip GPs at next-order have been extracted before in Ref. [6] by identifying the linear term in the outgoing photon energy in the LEX of the spin-dependent VCS amplitudes. In the present work, we calculated within HBChPT at $\mathcal{O}(p^3)$ the quadratic terms in the outgoing photon energy, in which the dipole GPs also appear. From these quadratic terms, we were able to extract the remaining four spin-flip GPs at next-order in HBChPT. A noteworthy feature of our results is that we provide analytical expressions for all spin-flip GPs at next-order, analogously as has been done at lowest order in Ref. [5]. Another interesting feature arises from the crossing symmetry relations which connect the HBChPT amplitudes at different orders. As a result, our prediction for the GP $P^{(1,1,1)}$ is formally a N$^3$LO result, and could only be obtained from a HBChPT calculation at $\mathcal{O}(p^3)$, if one were not to use the crossing relations.

Our predictions show sizeable corrections to the leading order (nonzero) results for the spin-flip GPs. The sign of these corrections is such as to move the leading order results in the direction of a phenomenological estimate of GPs based on dispersion relations. Our calculations signal a lack of convergence of the heavy baryon expansion for these spin-flip GPs. As the spin-flip GPs allow for absolute predictions at two successive orders in HBChPT without new low-energy constants entering, they may also be useful as a testing ground of chiral theories. It would therefore be desirable to have direct experimental information on the spin-flip GPs to test the different predictions.

One combination of the four spin-flip GPs can be extracted by doing an unpolarized VCS experiment at a fixed value of $Q^2$ and by varying the beam energy. One can extract the other three spin-flip GPs through VCS double-polarization observables with polarized lepton beam and polarized recoil nucleon. We gave predictions of these double-polarization asymmetries for experimental conditions accessible at MAMI. We found that the relative effect on these asymmetries due to the spin-flip GPs is of the order of 5%. Although the measurement of these asymmetries requires dedicated experiments, such a measurement would challenge the parameter-free chiral predictions at next-order for the spin-flip GPs presented in this work.

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APPENDIX: CROSSING SYMMETRY

PROPERTIES OF THE VCS AMPLITUDES IN THE COVARIANT BASIS

In this appendix, we study the properties of the amplitudes $f_i$ under nucleon crossing combined with charge conjugation, which transforms $f_i(q^2, q \cdot q', q \cdot P)$ into $f_i(q^2, q \cdot q', -q \cdot P)$. First we expand the arguments of the amplitudes $f_i$ as:

\[ f_i(q^2, q \cdot q', q \cdot P) = \sum_{a,b,c} C^i_{abc} \cdot (q^2)^a \cdot (q \cdot q')^b \cdot (q \cdot P)^c, \]

\[ q^2 = 2M_N \omega_0 + 2\omega_0 \cdot \omega' + \left( \frac{\omega_0}{M_N} + 1 \right) \cdot \omega'^2 + \mathcal{O}(\omega^3), \]

\[ q \cdot q' = 0 + (\omega_0 - \bar{q} \cos \theta) \cdot \omega' + 1 \cdot \omega'^2 + \mathcal{O}(\omega^3), \]

\[ q \cdot P = 0 + \left( \frac{-\bar{q}^2}{\omega_0} + \bar{q} \cos \theta \right) \cdot \omega' + 1 \cdot \omega'^2 + \mathcal{O}(\omega^3). \]

Now we obtain:
Relations at the real photon point: Furthermore, photon crossing leads to the symmetry properties of the amplitudes \( f_i \):

\[
f_i(q^2, q' \cdot q', q \cdot P) = f_i(q^2, q' \cdot q', -q \cdot P), \quad i = 1, 2, 5, 6, 7, 9, 11, 12,
\]

\[
f_i(q^2, q' \cdot q', q \cdot P) = -f_i(q^2, q' \cdot q', -q \cdot P), \quad i = 3, 4, 8, 10.
\]

(A3)

Furthermore, photon crossing leads to the symmetry relations at the real photon point:

\[
f_i(0, q \cdot q', q \cdot P) = 0, \quad i = 7, 9. \tag{A4}
\]

From these symmetry relations, we obtain for the coefficients entering the expansion of Eq. (A1):

\[
C_{a01}^i = 0, \quad i = 1, 2, 5, 6, 11, 12,
\]

\[
C_{a10}^i = C_{a00}^i = 0, \quad i = 3, 4, 8, 10,
\]

\[
C_{a01}^i = C_{a01}^0 = 0, \quad i = 7, 9. \tag{A5}
\]

As a result, we obtain for the LEX of Eq. (14):

\[
g_i = \sum_{a=0}^{\infty} C_{a10}^i (2M_N\omega_0)^a\omega_0 + \sum_{a=1}^{\infty} C_{a01}^i \cdot a(2M_N\omega_0)^a \cdot 2\omega_0,
\]

\[
h_i = \sum_{a=0}^{\infty} -C_{a10}^i (2M_N\omega_0)^a \bar{q}, \quad \text{for } i = 1, 2, 5, 6, 11, 12,
\]

\[
h_i = \sum_{a=0}^{\infty} +C_{a10}^i (2M_N\omega_0)^a \bar{q}, \quad \text{for } i = 3, 4, 8, 10.
\]

(A6)

furthermore

\[
f_i = 0, \quad g_i = \sum_{a=0}^{\infty} C_{a01}^i (2M_N\omega_0)^a(2M_N - \omega_0),
\]

\[
h_i = \sum_{a=0}^{\infty} C_{a01}^i (2M_N\omega_0)^a \bar{q}, \quad \text{for } i = 3, 4, 8, 10,
\]

and

\[
g_i = \sum_{a=0}^{\infty} C_{a10}^i (2M_N\omega_0)^a\omega_0 + \sum_{a=1}^{\infty} C_{a00}^i \cdot a(2M_N\omega_0)^a \cdot 2\omega_0,
\]

\[
h_i = \sum_{a=0}^{\infty} -C_{a10}^i (2M_N\omega_0)^a \bar{q}, \quad \text{for } i = 7, 9.
\]

(A8)

The above results in turn yield relations for the coefficients in the heavy baryon expansion as in Eqs. (20) and (21). Since in the heavy baryon expansion, \( \omega_0 \) is counted as \( n = 1 \) and \( 2M_N\omega_0 \) as \( n = 0 \), we obtain:

\[
\delta_1^{(0)} = \delta_2^{(0)} = \delta_3^{(0)} = \delta_4^{(0)} = \delta_5^{(0)} = \delta_6^{(0)} = \delta_7^{(0)} = \delta_8^{(0)} = \delta_9^{(0)} = \delta_{10}^{(0)} = \delta_{11}^{(0)} = \delta_{12}^{(0)} = 0. \tag{A9}
\]

Moreover we know

\[
2f_6^{(0)} + f_9^{(0)} = f_{12}^{(0)} = 0,
\]

because otherwise the amplitudes \( \tilde{A}_6 \) and \( \tilde{A}_{12} \) would start at \( n = -1 \) which is impossible. Therefore

\[
2\delta_6^{(0)} + \delta_9^{(0)} = \delta_{12}^{(0)} = 0, \tag{A10}
\]

and analogously

\[
2\delta_6^{(1)} + \delta_9^{(1)} = 0, \quad h_3^{(0)} = h_4^{(0)} = h_8^{(0)} = h_{10}^{(0)} = 0. \tag{A11}
\]

From \( h_{12}^{(0)} = 0 \) it follows that \( C_{a10}^{12(n-0)} = 0 \), and from \( f_{12}^{(0)} = 0 \) it follows that \( C_{a00}^{12(n-0)} = 0 \). Therefore we obtain \( \delta_{12}^{(1)} = 0 \). Similarly, one can show that:

\[
2\delta_6^{(1)} + \delta_9^{(1)} = 0, \quad h_3^{(0)} = h_4^{(0)} = h_8^{(0)} = h_{10}^{(0)} = 0, \tag{A12}
\]

otherwise \( \delta_3, \delta_4, \delta_8, \delta_{10} \) will start at \( n = -1 \). Lastly, Eq. (A7) yields the relations:

\[
h_i^{(1)} = \frac{\delta_i^{(0)}}{2M_N}, \quad \text{for } i = 3, 4, 8, 10. \tag{A13}
\]
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