Discrete-to-Continuous Extensions: Lovász extension, optimizations and eigenvalue problems

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Abstract

In this paper, we use various versions of Lovász extension to systematically derive continuous formulations of problems from discrete mathematics. This will take place in the following context:

- For combinatorial optimization problems in quotient form, we systematically develop equivalent continuous versions, thereby making tools from convex optimization, fractional programming and more general continuous algorithms like the stochastic subgradient method available for such optimization problems. Among other applications, we present an iteration scheme combining the inverse power and the steepest descent method to relax a Dinkelbach-type scheme for solving the equivalent continuous optimization. These results are natural and nontrivial generalizations of the related works by Hein et al [42–44].

- For some combinatorial quantities like Cheeger-type constants, we suggest a nonlinear eigenvalue problem for a pair of Lovász extensions of certain functions, which encodes certain combinatorial structures. This helps us to understand the data generated by a pair of functions on a power set from a geometric point of view.

This theory has several applications to quantitative and combinatorial problems, including

1. The equivalent continuous representations for the max $k$-cut problem, various Cheeger sets and isoperimetric constants are constructed. This also initiates a study of Dirichlet and Neumann 1-Laplacians on graphs, in which the nodal domain property and Cheeger-type equalities are presented. Among them, some Cheeger constants using different versions of vertex-boundary introduced in expander graph theory [11], are transformed into continuous forms, which recover the inequalities and identities on graph Poincare profiles proposed by Hume et al [45–48]. Also, we find that the min-cut and max-cut problems are equivalent to solving the first nontrivial eigenvalue and the largest eigenvalue of a certain nonlinear eigenvalue problem provided by the Lovász extension, respectively. This leads to one of the best continuous algorithms for the max-cut problem [76], as recognized in the field of graph optimization.

2. Also, we derive a new equivalent continuous representation of the graph independence number, which can be compared with the Motzkin-Straus theorem. More importantly, an equivalent continuous optimization for the chromatic number is provided, which seems to be the first continuous representation of the graph vertex coloring number. We provide the first continuous reformulation of the frustration index in signed networks, and we find a connection to the so-called modularity measure. Graph matching numbers, submodular vertex covers and multiway partition problems can also be studied in our framework.

Keywords: Lovász extension; submodularity; combinatorial optimization; Cheeger inequalities & isoperimetric problems; chromatic number; frustration index; expanders

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1 Introduction and Background

As a fundamental tool in discrete mathematics, Lovász extension has been deeply connected to submodular analysis [10][59], and has been applied in many areas like combinatorial optimization, game theory, matroid theory, stochastic processes, electrical networks, computer vision and machine learning [38]. There are many generalizations, such as the disjoint-pair Lovász extension and the Lovász extension on distributive lattices [38][88]. Recent developments include quasi-Lovász extension learning [38], as well as Lovász-softmax loss in deep learning [18].

We shall start by looking at the original Lovász extension. For simplicity, we shall work through this paper with a finite and nonempty set \( V := \{0, 1, \ldots, n\} \) and its power set \( \mathcal{P}(V) \). Also, we shall sometimes work on \( \mathcal{P}(V)^k := \{(A_1, \ldots, A_k) : A_i \subset V, i = 1, \ldots, k\} \) and \( \mathcal{P}_k(V) := \{(A_1, \ldots, A_k) \in \mathcal{P}(V)^k : A_i \cap A_j = \emptyset, \forall i \neq j\} \), as well as some restricted family \( \mathcal{A} \subset \mathcal{P}(V)^k \). We denote the cardinality of a set \( A \) by \( |A| \), and identify every \( A \in \mathcal{P}(V) \setminus \{\emptyset\} \) with its indicator vector \( 1_A \in \mathbb{R}^V \). The Lovász extension extends the domain of \( f \) to the whole Euclidean space \( \mathbb{R}^V \). There are several equivalent expressions:

- For \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \), let \( \sigma : V \cup \{0\} \to V \cup \{0\} \) be a bijection such that \( x_{\sigma(1)} \leq x_{\sigma(2)} \leq \cdots \leq x_{\sigma(n)} \) and \( \sigma(0) = 0 \), where \( x_0 := 0 \). The Lovász extension of \( f \) is defined by

  \[
  f^L(x) = \sum_{i=0}^{n-1} (x_{\sigma(i+1)} - x_{\sigma(i)}) f(V_{\sigma(i)}(x)),
  \]

  where \( V^0(x) = V \) and \( V_{\sigma(i)}(x) := \{j \in V : x_j > x_{\sigma(i)}\}, \quad i = 1, \ldots, n-1 \). We can write (1) in an integral form as

  \[
  f^L(x) = \int_{\min_{1 \leq i \leq n} x_i}^{\max_{1 \leq i \leq n} x_i} f(V^t(x)) dt + f(V) \min_{1 \leq i \leq n} x_i
  \]

\[^{1}\text{Some other versions in the literature only extend the domain to the cube } [0, 1]^V \text{ or the nonnegative orthant } \mathbb{R}^V_{\geq 0}. \text{ In fact, many works on Boolean lattices identify } \mathcal{P}(V) \text{ with the discrete cube } \{0, 1\}^n.\]
where \( V^t(x) = \{ i \in V : x_i > t \} \). If we apply the Möbius transformation, this becomes
\[
f^L(x) = \sum_{A \subset V} \sum_{B \subset A} (-1)^{|A|-|B|} f(B) \bigwedge_{i \in A} x_i,
\] (3)
where \( \bigwedge_{i \in A} x_i \) is the minimum over \( \{ x_i : i \in A \} \).

It is easy to see that \( f^L \) is positively one-homogeneous, PL (piecewise linear) and Lipschitz continuous \cite{LP, LP2}. Also, \( f^L(x + t1_V) = f^L(x) + tf(V), \forall t \in \mathbb{R}, \forall x \in \mathbb{R}^V \), and \( f^L(1_A) = f(A) \) for any \( A \in \mathcal{P}(V) \setminus \{ \emptyset \} \). The definition of \( f^L \) does not involve the datum \( f(\emptyset) \), and thus by convention, it is natural to reset \( f(\emptyset) = 0 \) to match the equality \( f^L(0) = 0 \), unless stated otherwise. For convenience, we say that \( f : \mathcal{P}(V) \to \mathbb{R} \) is a constant (resp., positive) function if \( f \) is constant (resp., positive) on \( \mathcal{P}(V) \setminus \{ \emptyset \} \). Moreover, a continuous function \( F : \mathbb{R}^V \to \mathbb{R} \) is the Lovász extension of some \( f : \mathcal{P}(V) \to \mathbb{R} \) if and only if \( F(x + y) = F(x) + F(y) \) whenever \( (x_i - x_j)(y_i - y_j) \geq 0, \forall i, j \in V \).

In this paper, we shall use the Lovász extension and its variants to study the interplay between discrete and continuous aspects in topics such as convexity, optimization and spectral theory.

**Submodular and convex functions**

Submodular function have emerged as a powerful concept in discrete optimization, see Fujishige’s monograph \cite{Fujishige} and Bach’s works \cite{Bach2, Bach3}. We also refer the readers to some recent related works regarding submodular functions on hypergraphs \cite{Fujishige, Bach2, Bach3}. We recall that a discrete function \( f : A \to \mathbb{R} \) defined on an algebra \( A \subset \mathcal{P}(V) \) (i.e., \( A \) is closed under union and intersection) is submodular if \( f(A) + f(B) \geq f(A \cup B) + f(A \cap B), \forall A, B \in A \). The Lovász extension turns a submodular into a convex function, and we can hence minimize the former by minimizing the latter:

**Theorem 1.1** (Lovász \cite{Lovasz}). \( f : \mathcal{P}(V) \to \mathbb{R} \) is submodular if and only if \( f^L \) is convex.

![Submodularity → Lovász extension → Convexity](image)

**Theorem 1.2** (Lovász \cite{Lovasz}). If \( f : \mathcal{P}(V) \to \mathbb{R} \) is submodular with \( f(\emptyset) = 0 \), then
\[
\min_{ACV} f(A) = \min_{x \in [0,1]^V} f^L(x).
\]

![Submodular minimization → Lovász extension → Convex programming](image)

Thus, submodularity can be seen as some kind of ‘discrete convexity’, and that naturally lead to many generalizations, such as bisubmodular, \( k \)-submodular, \( L \)-convex and \( M \)-convex, see \cite{Fujishige, Bach2, Bach3}. Moreover, the following classical result characterizes the class of all functions which can be expressed as Lovász extensions of submodular functions.

**Theorem 1.3** (Theorem 7.40 in \cite{Fujishige}). A one-homogeneous function \( F : \mathbb{R}^V \to \mathbb{R} \) is a Lovász extension of some submodular function if and only if \( F(x + t1_V) = F(x) + tf(1_V), \forall t \in \mathbb{R}, \forall x \in \mathbb{R}^V \), and \( F(x) + F(y) \geq F(x \lor y) + F(x \land y) \), where the \( i \)-th components of \( x \lor y \) and \( x \land y \) are \( (x \lor y)_i = \max\{x_i, y_i\} \) and \( (x \land y)_i = \min\{x_i, y_i\} \).

One may want to extend such a result to the bisubmodular or more general cases. In that direction, we shall obtain some results such as Proposition 2.7 and Theorem 2.3 in Section 2.2. It is also worth noting that Bach investigated an interesting generalization of submodular functions by a generalized Lovász extension \cite{Bach2}.

So far, research has mainly focused on ‘discrete convex’ functions, leading to ‘Discrete Convex Analysis’ \cite{Bach2, Bach3}, whereas the discrete non-convex setting which is quite popular in modern sciences has not yet received that much attention.

**Non-submodular cases**
Obviously, the non-convex case is so diverse and general that it cannot be directly studied by standard submodular tools. Although some publications show several results on non-submodular (i.e., non-convex) minimization based on Lovász extension [43], so far, these only work for special minimizations over the whole power set. Here, we shall find applications for discrete optimization and nonlinear spectral graph theory by employing the multi-way Lovász extension on enlarged and restricted domains.

In summary, we are going to initiate the study of diverse continuous extensions in non-submodular settings. This paper develops a systematic framework for many aspects around the topic. We establish a universal discrete-to-continuous framework via multi-way extensions, by systematically utilizing integral representations. In [55], we establish the links between discrete Morse theory and continuous Morse theory via the original Lovász extension. We shall now discuss some connections with other various fields.

**Connections with combinatorial optimization**

Because of the wide range of applications of discrete mathematics in computer science, combinatorial optimization has been much studied from the mathematical perspective. It is known that any combinatorial optimization can be equivalently expressed as a continuous optimization via convex (or concave) extension, but often, there is the difficulty that one cannot write down an equivalent continuous object function in closed form. For practical purposes, it would be very helpful if one could transfer a combinatorial optimization problem to an explicit and simple equivalent continuous optimization problem in closed form. Formally, in many concrete situations, it would be useful if one could get an identity of the form

\[
\min_{(A_1, \ldots, A_k) \in \mathcal{A}^{\text{supp}(g)}} \frac{f(A_1, \ldots, A_k)}{g(A_1, \ldots, A_k)} = \inf_{\psi \in \mathcal{D}(A)} \frac{\tilde{f}(\psi)}{\tilde{g}(\psi)},
\]

where \(f, g : \mathcal{A} \to [0, \infty)\), \(\mathcal{D}(A)\) is a feasible domain determined by \(A\) only, \(\text{supp}(g)\) is the support of \(g\), and \(f\) and \(g\) are suitable continuous extensions of \(f\) and \(g\).

So far, only situations where \(f : \mathcal{P}(V) \to \mathbb{R}\) or \(f : \mathcal{P}_2(V) \to \mathbb{R}\) have been investigated systematically [24, 43], and what is lacking are situations with restrictions, that is, incomplete data.

Also, to the best of our knowledge, the known results in the literature do not work for combinatorial optimization directly on set-tuples. But most of combinatorial optimization problems should be formalized in the form of set-tuples, and only a few can be represented in set form or disjoint-pair form. Whenever one can find an equivalent Lipschitz function for a combinatorial problem in the field of discrete optimization, this makes useful tools available and leads to new connections. That is, one wishes to establish a *discrete-to-continuous transformation* like the operator \(\sim\) in (4). We will show in Section 3.4 that the Lovász extension and its variants are suitable choices for such a transformation (see Theorems A 3.1 and Proposition 3.1 for details).

To reach these goals, we need to systematically study various generalizations of the Lovász extension. More precisely, we shall work with the following two different multi-way forms:

1. **Disjoint-pair version:** for a function \(f : \mathcal{P}_2(V) \to \mathbb{R}\), its disjoint-pair Lovász extension is defined as

\[
f^L(x) = \int_0^{\|x\|} f(V^+_{\pm t}(x), V^-_{\pm t}(x)) dt,
\]

where \(V^+_{\pm t}(x) = \{i \in V : \pm x_i > t\}, \forall t \geq 0\). For \(A \subset \mathcal{P}_2(V)\) and \(f : A \to \mathbb{R}\), the feasible domain \(\mathcal{D}_A\) of the disjoint-pair Lovász extension is \(\{x \in \mathbb{R}^V : (V^+_{\pm t}(x), V^-_{\pm t}(x)) \in A, \forall t \geq 0\}\). We simply use \(1_{A,B}\) to represent the indicator vector \(1_A - 1_B \in \mathcal{D}_A\) of the disjoint set-pair \((A, B) \in A\).

It should be noted that the disjoint-pair Lovász extension introduced by Qi [73] has been systematically investigated by Fujishige [37, 38] and Murota [68] in the context of discrete convex analysis (or the theory of submodular functions). They defined and investigated the disjoint-pair Lovász extension in a summation form. The integral formulation (5), however, is more convenient to obtain a closed formula of the equivalent continuous optimization problem for a combinatorial optimization problem. Moreover, the references and the present paper focus on different aspects, with the exception of the submodularity theorem (i.e., \(f\) is bisubmodular iff \(f^L\) is convex).
(2) $k$-way version: for a function $f : \mathcal{P}(V)^k \to \mathbb{R}$, the simple $k$-way Lovász extension $f^L : \mathbb{R}^{kn} \to \mathbb{R}$ is defined as

$$f^L(x^1, \ldots, x^k) = \int_{\min x^i}^{\max x^i} f(V^t(x^1), \ldots, V^t(x^k)) dt + f(V, \ldots, V) \min x,$$  \hspace{1cm} (6)

where $V^t(x^i) = \{ j \in V : x^i_j > t \}$, $\min x = \min x^i_j$ and $\max x = \max x^i_j$. For $A \subset \mathcal{P}^k(V)$ with $(\emptyset, \ldots, \emptyset), (V, \ldots, V) \in A$ and $f : A \to \mathbb{R}$, we take $D_A = \{ x \in \mathbb{R}^{kn} : (V^t(x^1), \ldots, V^t(x^k)) \in A, \forall t \in \mathbb{R} \}$ as a feasible domain of the $k$-way Lovász extension $f^L$. For convenience, we will simply use $1_{A_1, \ldots, A_k}$ to represent the indicator vector $(1_{A_1}, \ldots, 1_{A_k}) \in D_A$ of the set-tuple $(A_1, \ldots, A_k) \in A$.

By the Lovász extension of submodular functions on distributive lattices [38, 68], our $k$-way version (6) can be reduced to the classical version on distributive lattices. Our main purposes and key results, however, are different from that approach. In fact, we mainly aim to deal with discrete fractional programming by the $k$-way Lovász extension, while those references concentrate on submodularity and convex optimization.

All these multi-way Lovász extensions satisfy the optimal identity Eq. (4):

**Theorem A** (Theorem 3.1 and Proposition 3.1). Given two functions $f, g : A \to [0, +\infty)$, let $\tilde{f}$ and $\tilde{g}$ be two real functions on $D_A$ satisfying $\tilde{f}(1_{A_1, \ldots, A_k}) = f(A_1, \ldots, A_k)$ and $\tilde{g}(1_{A_1, \ldots, A_k}) = g(A_1, \ldots, A_k)$, where $1_{A_1, \ldots, A_k} \in D_A$ is the indicator vector of the set-tuple $(A_1, \ldots, A_k) \in A$. Then Eq. (4) holds if $\tilde{f}$ and $\tilde{g}$ further possess (P1) or (P2) below. Correspondingly, if $\tilde{f}$ and $\tilde{g}$ fulfill (P1') or (P2'), there similarly holds

$$\max_{(A_1, \ldots, A_k) \in A^s} f(A_1, \ldots, A_k) = \sup_{\psi \in D_A^s} \tilde{f}(\psi).$$

Here the optional additional conditions of $\tilde{f}$ and $\tilde{g}$ are:

(P1) $\tilde{f} \geq f^L$ and $\tilde{g} \leq g^L$.

(P1') $\tilde{f} \leq f^L$ and $\tilde{g} \geq g^L$.

(P2) $\tilde{f} = ((f^\alpha)^L)^\frac{\alpha}{\alpha}$ and $\tilde{g} = ((g^\alpha)^L)^\frac{\alpha}{\alpha}$ for some $\alpha > 0$.

Here $f^L$ is either the original or the disjoint-pair or the $k$-way Lovász extension.

Theorem A shows that by the multi-way Lovász extension, the combinatorial optimization in quotient form can be transformed to fractional programming. And based on this fractional optimization, we propose an effective local convergence scheme, which relaxes the Dinkelbach-type iterative scheme and mixes the inverse power method and the steepest descent method. Furthermore, many other continuous iterations, such as Krasnoselski-Mann iteration, and the stochastic subgradient method, could be directly applied here. We refer the readers to [50] for another development on equalities between discrete and continuous optimization problems via various generalizations of Lovász extension.

The power of Theorem A is embodied in many new examples and applications including Cheeger-type problems, various isoperimetric constants and max $k$-cut problems (see Subsections 4.3, 4.4 and 4.6). And moreover, we find that not only combinatorial optimization, but also some combinatorial invariants like the independence number and the chromatic number, can be transformed into a continuous representation by this scheme.

**Theorem B** (Sections 4.5 and 4.6). For an unweighted and undirected simple graph $G = (V, E)$ with $\#V = n$, its independence number can be represented as

$$\alpha(G) = \max_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\sum_{\{i,j\} \in E} |x_i - x_j| + |x_i + x_j| - 2 \sum_{i \in V} (\deg_i - 1)|x_i|}{2 \|x\|_\infty},$$

where $\deg_i = \# \{ j \in V : \{j, i\} \in E \}$, $i \in V$, and its chromatic number is

$$\gamma(G) = n^2 \max_{x \in \mathbb{R}^{n^2} \setminus \{0\}} \frac{\sum_{k \in V} n \sum_{\{i,j\} \in E} (|x_{ik} - x_{jk}| + |x_{ik} + x_{jk}|) + 2n \|x^k\|_\infty - 2n \deg_k \|x^k\|_1}{2 \|x\|_\infty},$$

where $\|x\|_\infty = \max_{i \in V} |x_i|$.
where $x = (x_k)_{k \in V}$, $x^k = (x_{k1}, \ldots, x_{kn})$ and $x^k = (x_{1k}, \ldots, x_{nk})$. The maximum matching number of $G$ can be expressed as

$$\max_{y \in \mathbb{R}^{|E|} \setminus \{0\}} \frac{\|y\|_1^2}{\|y\|_1^2 - 2 \sum_{e \in \mathcal{E} = \emptyset} y_e y_e'}.$$

There are some equivalent continuous reformulations of the maxcut problem and the independence number of a graph in the literature. However, a continuous reformulation of the coloring number has not yet been proposed. The main reason seems to be the complexity of coloring a graph. Hence, it is very difficult to discover a continuous form of the coloring number by direct observation.

**Theorem C** (Theorem 3.1). Given functions $f_1, \ldots, f_n : A \to [0, +\infty)$, and $p$-homogeneous functions $P, Q : [0, +\infty)^n \to [0, +\infty)$, we have

$$\max_{A \in \mathcal{A}} \frac{P(f_1(A), \ldots, f_n(A))}{Q(f_1(A), \ldots, f_n(A))} = \sup_{x \in \mathcal{D}_A} \frac{P(f_1^L(x), \ldots, f_n^L(x))}{Q(f_1^L(x), \ldots, f_n^L(x))}$$

if $P^1$ is subadditive and $Q^1$ is superadditive. One can replace ‘max’ by ‘min’ if $P^1$ is superadditive and $Q^1$ is subadditive.

Theorems A, B, and C can be seen as natural and nontrivial generalizations of the related original works by Hein and Setzer [13].

**Connections with spectral graph theory**

Spectral graph theory aims to derive properties of a (hyper-)graph from its eigenvalues and eigenvectors. Going beyond the linear case, nonlinear spectral graph theory is developed in terms of discrete geometric analysis and difference equations on (hyper-)graphs. Every discrete eigenvalue problem is simply called the eigenvalue problem of the function pair $(P, Q)$.

Connections with spectral graph theory

| combinatorial quantities | Spectral graph theory | eigenvalues and eigenvectors |

We shall consider the following three concepts:

- **Eigenvalues and eigenvalues**: The set-valued eigenvalue problems above are usually written as $0 \in \nabla f^L(x) - \lambda \nabla g^L(x)$ by using the Minkowski summation of convex sets. We call $\lambda$ an eigenvalue and $x$ an eigenvector associated to $\lambda$.

- **Critical points and critical values**: The set of critical points $\{x \mid 0 \in \frac{\nabla f^L(x)}{g^L(x)}\}$ and the corresponding critical values.

- **Minimax critical values (i.e., variational eigenvalues in Rayleigh quotient form)**: The Lusternik-Schnirelman theory tells us that the min-max values

$$\lambda_m = \inf_{S \in \mathcal{I}_m} \sup_{x \in S} \frac{f^L(x)}{g^L(x)}, \quad m = 1, 2, \ldots, n, \quad (7)$$

where $\mathcal{I}_m$ is the set of $m$-element subsets of $V$. The function $f^L$ is subadditive and $g^L$ is superadditive. One can replace ‘max’ by ‘min’ if $f^L$ is superadditive and $g^L$ is subadditive.
are critical values of \( f^L(\cdot)/g^L(\cdot) \). Here \( \Gamma_m \) is a class of certain topological objects at level \( m \), e.g., the family of subsets with Krasnoselskii’s \( \mathbb{Z}_2 \)-genus (or Lusternik-Schnirelman category) not smaller than \( m \). Since this paper does not focus on the min-max critical values, we will not say more about Krasnoselskii’s \( \mathbb{Z}_2 \)-genus and the class \( \Gamma_m \). We refer the interested readers to \[56\] for systematic studies on this topic.

There are the following relations between these three classes:

\[
\{ \text{Eigenvalues in Rayleigh quotient} \} \subset \{ \text{Critical values} \} \subset \{ \text{Eigenvalues} \}.
\]

For linear spectral theory, the above three classes coincide. However, for the non-smooth spectral theory derived by Lovász extension, we only have the inclusion relations.

We have the following result on the eigenvalue problem for the disjoint-pair Lovász extension, while for the results on the original Lovász extension, we refer to Section 3.2 for details.

**Theorem D.** Given \( f, g : \mathcal{P}_2(V) \to \mathbb{R} \), then every eigenvalue of \( (f^L, g^L) \) has an eigenvector of the form \( 1_A - 1_B \). Moreover, we have the following claims:

- If \( 2f(A, B) = f(A, V \setminus A) + f(V \setminus B, B) \) and \( 2g(A, B) = g(A, V \setminus A) + g(V \setminus B, B) \) for any \( (A, B) \in \mathcal{P}_2(V) \setminus \{(\emptyset, \emptyset)\} \), then every eigenvalue of \( (f^L, g^L) \) has an eigenvector of the form \( 1_A - 1_{V\setminus A} \).

- If \( g = \text{Const} \), then for any \( A \subseteq V \), \( 1_A - 1_{V\setminus A} \) is an eigenvector.

- If \( f(A, B) = \hat{f}(A) + \hat{f}(B) \) and \( g(A, B) = \hat{g}(A) + \hat{g}(B) \) for some symmetric function \( \hat{f} : \mathcal{P}(V) \to \mathbb{R} \) (i.e., \( \hat{f}(A) = \hat{f}(V \setminus A) \), \( \forall A \)) and non-decreasing submodular function \( \hat{g} : \mathcal{P}(V) \to \mathbb{R}_+ \), then the second eigenvalue \( \lambda_2 \) of \( (f^L, g^L) \) equals

\[
\min_{1 \leq t \leq \infty} \min_{x \in \mathbb{R}} \frac{f_t^L(x)}{g^L(x - t1)} = \min_{A \in \mathcal{P}(V) \setminus \{\emptyset, V\}} \min_{(A, B) \in \mathcal{P}_2(V) \setminus \{(\emptyset, \emptyset)\}} \left\{ \frac{\hat{f}(A)}{\hat{g}(A)} \right\}.
\]

This generalizes recent results on the graph 1-Laplacian and Cheeger’s constant \[20,21,23,42,44\]. And as a new application, we show that the min-cut problem and the max-cut problem are equivalent to solving the smallest nontrivial (i.e., the second) eigenvalue and the largest eigenvalue of a certain nonlinear eigenvalue problem (see Theorem 1.1).

**Applications to frustration in signed network**

As a key measure for analyzing signed networks, the frustration index on a signed graph quantifies how far a signature is from being balanced (see Section 4.1). Computing the frustration index is NP-hard, and few algorithms have been proposed \[1,2\].

Considering a signed graph \((V, E_+ \cup E_-)\) with \(E_+\) (resp. \(E_-\)) the set of positive (resp. negative) edges, based on the disjoint-pair Lovász extension, we obtain an equivalent continuous optimization of the frustration index (or the line index of balance \[51\]):

\[
\# E_- + \min_{x \neq 0} \frac{\sum_{i,j} |x_i - x_j| - \sum_{i,j} |x_i - x_j|}{2\|x\|_{\infty}}.
\]

This new reformulation can be computed via typical algorithms in continuous optimization.

Also, we propose the eigenvalue problem

\[
\nabla \left( \sum_{\{i,j\} \in E_+} |x_i - x_j| + \sum_{\{i,j\} \in E_-} |x_i + x_j| \right) \cap \lambda \nabla \|x\|_{\infty} \neq \emptyset
\]

and we show an iterative scheme for searching the frustration index based on the smallest eigenvalue of the nonlinear eigenvalue problem \[3\]. See Section 4.7 for details and more results.

Since the transformation of a combinatorial optimization to a continuous optimization or a non-smooth eigenvalue problem usually leads to a quotient, the task for fractional programming then
becomes to compute an optimal value or an eigenvector. In Section 3.3, we present a general algorithm which is available to compute the resulting continuous reformulations arising in Theorems A, B, C and D.

In another paper [56], we present a systematic study of general function pairs \((F, G)\), in which \(F\) and \(G\) can be piecewise multilinear or other general extensions of certain discrete functions. The papers can be read independently of each other.

In summary, we present a systematic study for constructing nonlinear eigenvalue problems and equivalent continuous reformulations for combinatorial quantities, which capture the key properties of the original combinatorial problems. This is helpful to increase understanding of certain combinatorial problems by the corresponding eigenvalue problems and the equivalent continuous reformulations. The following picture summarizes the relations between the various concepts developed and studied in this paper.

![Diagram]

Figure 1: The relationship between the aspects studied in our work.

We shall now briefly discuss how to apply this scheme. Our framework gives new continuous formulations and eigenvalue representations for certain combinatorial optimization and related discrete quantities. Compared to other formulations of those combinatorial problems, the main advantage of our formulation is that the critical data (including min-max data, saddle points, and optimal values) of the continuous representations incorporate all the key information of the original combinatorial problems (see Sections 3.1 and 3.2). For example, by the results in [35], the \(k\)-way Cheeger constant on a tree graph agrees with the \(k\)-th eigenvalue of the graph 1-Laplacian, which can be subsumed into the above framework.

Restricted onto optimization problems, the continuous representation obtained by Lovász-type extension leads to an iterative algorithm based on fractional programming, but we should point out that this is not the main focus of the present paper.

Although the associated algorithms are not the main contribution and focus of this work, in Section 3.3 we review fractional programming and explore more in this direction. A remarkable theoretical advantage we proved in this paper is that our scheme provides an iterative solution without rounding, and can be used to improve any initially given data. Moreover, just to explain the applicability, we should point out that this framework already performs well on the Cheeger cut problem (see Sections 4.3 and 4.6), and the maxcut problem (see Section 4.2 for details). One can expect a good performance of this framework also on other combinatorial problems, such as the frustration set problem, the independent set problem and the coloring problem.

**Related works.** The present paper is the second one in a series that develops a systematic bridge between constructions in discrete mathematics and the corresponding continuous analogs via Lovász type extensions, where the other two parts [55,56] are concerned with different aspects. Let us briefly describe their contents and put them into perspective. The series is motivated by recent developments...
on Cheeger inequalities, Lovász extensions, expander graphs, spectral graph theory and practical applications. We focus on the Lovász extension and introduce some useful generalizations, including the multi-way Lovász extension in this paper, and the piecewise multilinear extension in [56], which we simply call discrete-to-continuous extensions. Then, we investigate optimization and eigenvalue problems (see Section 3), Morse theory (see [35]), min-max theory, critical point theory, and spectral theory (see [56]) for the Lipschitz functions obtained by these discrete-to-continuous extensions. Thus, this series provides new perspectives for understanding certain relations and interactions between discrete and continuous worlds via Lovász-type extensions.

The present paper focuses on the aspect of eigenvalue problems and optimizations regarding Lovász extension, while in [35] we concentrate on the Morse and Lusternik-Schnirelman theoretical aspect involving Lovász extension. More generally, in [56], we further explore min-max relations, saddle point problems, spectral theory and critical point theory involving a more general class of discrete-to-continuous extensions (namely, the piecewise multilinear extensions). The mixed IP-SD algorithm proposed in Section 3.3 can also be applied in [56] for approximating the second eigenvalue.

Convention 1. Since this paper contains many interacting parts and relevant results, some notions and concepts may have slightly distinct meanings in different sections, but this will be stated at the beginning of each section.

2 A preliminary: Lovász extension and submodular functions

While most of the results on submodularity are known in the field of discrete convex analysis, we present some details in a simple manner, which should be helpful to understand our main results in Section 3.

We first formalize some important results about the original Lovász extension.

Definition 2.1. Two vectors $x$ and $y$ are comonotonic if $(x_i - x_j)(y_i - y_j) \geq 0$, $\forall i, j \in \{1, 2, \cdots, n\}$.

A function $F : \mathbb{R}^n \rightarrow \mathbb{R}$ is comonotonic additive if $F(x + y) = F(x) + F(y)$ for any comonotonic pair $x$ and $y$.

The following proposition shows that a function is comonotonic additive if and only if it can be expressed as the Lovász extension of some function.

Proposition 2.1. $F : \mathbb{R}^n \rightarrow \mathbb{R}$ is the Lovász extension $F = f^L$ of some function $f : \mathcal{P}(V) \rightarrow \mathbb{R}$ if and only if $F$ is comonotonic additive.

Recall the following known results:

Theorem 2.1 (Lovász). The following conditions are equivalent: (1) $f$ is submodular; (2) $f^L$ is convex; (3) $f^L$ is submodular.

Remark 1. The fact that $f$ is submodular if and only if $f^L$ is submodular is provided by Propositions 7.38 and 7.39 in [68]. We shall give a detailed proof for a generalized version of Theorem 2.1 (see Theorem 2.2).

Theorem 2.2 (Murota [68]). $F : \mathbb{R}^n \rightarrow \mathbb{R}$ is the Lovász extension $F = f^L$ of some submodular $f : \mathcal{P}(V) \rightarrow \mathbb{R}$ if and only if $F$ is positively one-homogeneous, submodular and $F(x + t1) = F(x) + tF(1)$.

Remark 2. Theorem 2.2 was originally proved by establishing a one-to-one correspondence between positively homogeneous L-convex functions and submodular functions (see Theorem 7.40 in Murota’s book [68]). An alternative proof is given in [17].

We shall establish these results for the disjoint-pair version and the $k$-way version of the Lovász extension.
2.1 Disjoint-pair and k-way Lovász extensions

Under the natural additional assumption that \( f(\emptyset, \emptyset) = 0 \), one can write (5) as

\[
f^L(x) = \int_0^\infty f(V_+^t(x), V_-^t(x))dt,
\]

where \( V_\pm^t(x) = \{ i \in V : \pm x_i > t \}, \forall t \geq 0 \). Another formulation of (9) (or (5)) is

\[
f^L(x) = \sum_{i=0}^{n-1} (|x_{\sigma(i)}| - |x_{\sigma(i)}|) f(V^\sigma(i)_+(x), V^\sigma(i)_-(x)),
\]

where \( \sigma : V \cup \{0\} \rightarrow V \cup \{0\} \) is a bijection such that \( |x_{\sigma(1)}| \leq |x_{\sigma(2)}| \leq \cdots \leq |x_{\sigma(n)}| \) and \( \sigma(0) = 0 \), where \( x_0 := 0 \), and

\[
V^\sigma(i)_\pm(x) := \{ j \in V : \pm x_j > |x_{\sigma(i)}| \}, \quad i = 0, 1, \ldots, n - 1.
\]

In fact, by \( f(\emptyset, \emptyset) = 0 \), \( \|x\|_\infty = |x(\sigma(i))| \), and \( f(V_+^t(x), V_-^t(x)) = f(V^\sigma(i)_+(x), V^\sigma(i)_-(x)) \) whenever \( |x(i)| \leq t < |x_{\sigma(i+1)}| \), we have

\[
\int_0^\infty f(V_+^t(x), V_-^t(x))dt = \int_0^\infty f(V_+^t(x), V_-^t(x))dt = \sum_{i=0}^{n-1} \int_{|x_{\sigma(i)}|}^{x_{\sigma(i+1)}} f(V^\sigma(i)_+(x), V^\sigma(i)_-(x))dt
\]

which deduces that (9), (5) and (10) are equivalent. We regard \( \mathcal{P}_2(V) = 3^V \) as \( \{-1, 0, 1\}^n \) by identifying the disjoint pair \((A, B)\) with the ternary (indicator) vector \(1_A - 1_B\).

One may compare the original and the disjoint-pair Lovász extensions by writing (5) as

\[
\int_{\min |x_i|}^{\max |x_i|} f(V_+^0(x), V_-^0(x))dt + \min_i |x_i| f(V_+^0(x), V_-^0(x))
\]

Note that (11) is very similar to (2). We say that \((A, B) \in \mathcal{P}_2(V)\) is an associate set-tuple of a given \(x \in \mathbb{R}^n\) if \((A, B) = (V_+^t(x), V_-^t(x))\) for some \(t \geq 0\). Of course, a vector \(x\) may have many associate set-tuples.

**Definition 2.2.** Given \(V_i = \{1, \ldots, n_i\}, i = 1, \ldots, k\), and a function \(f : \mathcal{P}(V_1) \times \cdots \times \mathcal{P}(V_k) \rightarrow \mathbb{R}\), the k-way Lovász extension \(f^L : \mathbb{R}^{V_1} \times \cdots \times \mathbb{R}^{V_k} \rightarrow \mathbb{R}\) can be written as

\[
f^L(x^1, \ldots, x^k) = \int_{\min x}^{\max x} f(V_1^t(x^1), \ldots, V_k^t(x^k))dt + f(V_1, \ldots, V_k) \min_{x} x
\]

where \(V_i^t(x^i) = \{ j \in V_i : x^i_j > t \}, \min_{x} x = \min_{x_{i,j}} x_{i,j} \) and \(\max_{x} x = \max_{x_{i,j}} x_{i,j}\).

We say that \((A_1, \ldots, A_k) \in \mathcal{P}(V_1) \times \cdots \times \mathcal{P}(V_k)\) is an associated set-tuple of a given vector \(x := (x^1, \ldots, x^k) \in \mathbb{R}^{V_1} \times \cdots \times \mathbb{R}^{V_k}\) if \((A_1, \ldots, A_k) = (V_1^t(x^1), \ldots, V_k^t(x^k))\) for some \(t \in \mathbb{R}\).

**Definition 2.3** (k-way analog for disjoint-pair Lovász extension). Given \(V_i = \{1, \ldots, n_i\}, i = 1, \ldots, k\), and a function \(f : \mathcal{P}_2(V_1) \times \cdots \times \mathcal{P}_2(V_k) \rightarrow \mathbb{R}\), define \(f^L : \mathbb{R}^{V_1} \times \cdots \times \mathbb{R}^{V_k} \rightarrow \mathbb{R}\) by

\[
f^L(x^1, \ldots, x^k) = \int_{0}^{\|x\|_\infty} f(V^t_+, (x^1), V^t_-, (x^1), \ldots, V^t_{k+}, (x^k), V^t_{k-}, (x^k))dt
\]

where \(V^t_{i,\pm}(x^i) = \{ j \in V_i : \pm x^i_j > t \}, \|x\|_\infty = \max_{i=1, \ldots, k} |x^i|_\infty\). We can replace \(\|x\|_\infty\) by \(+\infty\) if we set \(f(\emptyset, \ldots, \emptyset) = 0\). A set-tuple \((A^t_1, A^t_1, \ldots, A^t_k, A^t_k) \in \mathcal{P}_2(V_1) \times \cdots \times \mathcal{P}_2(V_k)\) is called an associated set-tuple of a given vector \(x := (x^1, \ldots, x^k) \in \mathbb{R}^{V_1} \times \cdots \times \mathbb{R}^{V_k}\) if \((A^t_1, A^t_1, \ldots, A^t_k, A^t_k) = (V^t_+, (x^1), V^t_-, (x^1), \ldots, V^t_{k+}, (x^k), V^t_{k-}, (x^k))\) for some \(t \geq 0\).
For convenience, we always use $f^L$ to express different variants of Lovász extensions of $f$. The reader can identify the version we are referring to by the domain of $f$.

Some basic properties of the multi-way Lovász extension are shown below.

**Proposition 2.2.** For the multi-way Lovász extension $f^L(x)$, we have

(a) $f^L(\cdot)$ is positively one-homogeneous, piecewise linear, and Lipschitz continuous.

(b) $(\lambda f)^L = \lambda f^L$, $\forall \lambda \in \mathbb{R}$.

**Proposition 2.3.** For the disjoint-pair Lovász extension $f^L(x)$, we have

(a) $f^L$ is Lipschitz continuous, and $|f^L(x) - f^L(y)| \leq 2 \max_{(A,B) \in \mathcal{P}_2(V)} f(A,B) \|x - y\|_1$, $\forall x, y \in \mathbb{R}^n$.

Also, $|f^L(x) - f^L(y)| \leq 2 \sum_{(A,B) \in \mathcal{P}_2(V)} f(A,B) \|x - y\|_\infty$, $\forall x, y \in \mathbb{R}^n$.

(b) $f^L(-x) = \pm f^L(x)$, $\forall x \in \mathbb{R}^V$ if and only if $f(A,B) = \pm f(B,A)$, $\forall (A,B) \in \mathcal{P}_2(V)$.

(c) $f^L(x+y) = f^L(x) + f^L(y)$ whenever $V^0_\pm(y) \subset V^0_\pm(x)$, where $x$ has components $\tilde{x}_i = \begin{cases} x_i, & \text{if } |x_i| = \|x\|_\infty, \\ 0, & \text{otherwise.} \end{cases}$

**Proof.** (a) and (b) are actually known results and their proofs are elementary. We refer to Theorem 2.2 and its proof in [24] for (a). While, for (b), see Proposition 2.5 in [24]. (c) can be derived from the definition [10].

Here we omit the proofs of Propositions 2.2 and 2.3 (c) because they are easy and similar to the case of the original Lovász extension.

**Definition 2.4.** Two vectors $x$ and $y$ are said to be absolutely comonotonic if $x_iy_i \geq 0$, $\forall i$, and $|x_i| - |x_j| - |y_i| - |y_j| \geq 0$, $\forall i, j$.

**Proposition 2.4.** A continuous function $F$ is a disjoint-pair Lovász extension of some function $f : \mathcal{P}_2(V) \to \mathbb{R}$, if and only if $F(x) + F(y) = F(x+y)$ whenever $x$ and $y$ are absolutely comonotonic.

**Proof.** By the definition of the disjoint-pair Lovász extension (see [13]), we know that $F$ is a disjoint-pair Lovász extension of some function $f : \mathcal{P}_2(V) \to \mathbb{R}$ if and only if $\lambda F(x) + (1-\lambda)F(y) = F(\lambda x + (1-\lambda)y)$ for all absolutely comonotonic vectors $x$ and $y$, $\forall \lambda \in [0,1]$. Therefore, we only need to prove the sufficiency part.

For $x \in \mathbb{R}^V$, since $sx$ and $tx$ with $s, t \geq 0$ are absolutely comonotonic, $F(sx) + F(tx) = F((s + t)x)$, which yields a Cauchy equation on the half-line. Thus the continuity assumption implies the linearity of $F$ on the ray $\mathbb{R}^+x$, which implies the property $F(tx) = tf(x)$, $\forall t \geq 0$, and hence $\lambda F(x) + (1-\lambda)F(y) = F(\lambda x + (1-\lambda)y)$ for any absolutely comonotonic vectors $x$ and $y$, $\forall \lambda \in [0,1]$. This completes the proof.

For relations between the original and the disjoint-pair Lovász extensions, we further have

**Proposition 2.5.** For $h : \mathcal{P}(V) \to [0, +\infty)$ with $h(\emptyset) = 0$, and $f : \mathcal{P}_2(V) \to [0, +\infty)$ with $f(\emptyset, \emptyset) = 0$ we have:

(a) If $f(A,B) = h(A) + h(V \setminus B) - h(V)$, $\forall (A,B) \in \mathcal{P}_2(V)$, then $f^L = h^L$.

(b) If $f(A,B) = h(A) + h(B)$ and $h(A) = h(V \setminus A)$, $\forall (A,B) \in \mathcal{P}_2(V)$, then $f^L = h^L$.

(c) If $f(A,B) = h(A)$, $\forall (A,B) \in \mathcal{P}_2(V)$, then $f^L(x) = h^L(x)$, $\forall x \in [0, \infty)^V$.

(d) If $f(A,B) = h(A \cup B)$, $\forall (A,B) \in \mathcal{P}_2(V)$, then $f^L(x) = h^L(x^+ + x^-)$.

(e) If $f(A,B) = h(A) \pm h(B)$, $\forall (A,B) \in \mathcal{P}_2(V)$, then $f^L(x) = h^L(x^+) \pm h^L(x^-)$.

Here $x^\pm := (\pm x) \vee 0$.

\footnote{In fact, if $h(\emptyset) \neq 0$ or $f(\emptyset, \emptyset) \neq 0$, one may change the value and it does not affect the related Lovász extension.}
Proof. (a) Note that
\[ f^L(x) = \int_0^{\|x\|_\infty} f(V^+_k(x), V^-_k(x))dt = \int_0^{\|x\|_\infty} (h(V^+_k(x)) + h(V^-_k(x)) - h(V))dt \
= \int_{-\|x\|_\infty}^{\|x\|_\infty} h(V^+_k(x))dt - \|x\|_\infty h(V) = \int_{x_\sigma(n)}^{x_\sigma(1)} h(V^+_k(x))dt + x_\sigma(1)h(V) = h^L(x), \]
where we use \(\|x\|_\infty = \max\{-x_\sigma(1), x_\sigma(n)\}\) and \(h(\emptyset) = 0.\)

(b) This is a direct consequence of (a) since \(h(V) = h(\emptyset) = 0\) and \(h(B) = h(V \setminus B).\)

(c) For any \(x \in \mathbb{R}^V\) with \(x_i \geq 0\), we note that \(f^L(x) = \int_{-\|x\|_\infty}^{\|x\|_\infty} h(V^+_k(x))dt = \int_{\min x_i}^{\max x_i} h(V^+_k(x))dt + \min x_i h(V) = h^L(x).\)

(d) Similar to (c), one can check that \(f^L(x) = h^L(x^+ + x^-).\)

(e) It is straightforward.

\[ \square \]

In the sequel, we will not distinguish the original and the disjoint-pair Lovász extensions, since the reader can infer it from the domains \((\mathcal{P}(V)\) or \(\mathcal{P}_2(V)\)). Sometimes we work on \(\mathcal{P}(V)\) only, and in this situation, the disjoint-pair Lovász extension acts on the redefined \(f(A, B) = h(A \cup B)\) as Proposition 2.3 states.

The next result is useful for the application on graph coloring.

**Proposition 2.6.** For the simple \(k\)-way Lovász extension of \(f : \mathcal{P}(V_1) \times \cdots \times \mathcal{P}(V_k) \to \mathbb{R}\) with the separable summation form \(f(A_1, \cdots, A_k) := \sum_{i=1}^k f_i(A_i), \forall (A_1, \cdots, A_k) \in \mathcal{P}(V)^k,\) we have \(f^L(x^1, \cdots, x^k) = \sum_{i=1}^k f_i^L(x^i), \forall (x^1, \cdots, x^k).\)

For \(f : \mathcal{P}_2(V_1) \times \cdots \times \mathcal{P}_2(V_k) \to \mathbb{R}\) with the form \(f(A_1, B_1 \cdots, A_k, B_k) := \sum_{i=1}^k f_i(A_i, B_i), \forall (A_1, B_1, \cdots, A_k, B_k) \in \mathcal{P}_2(V_1) \times \cdots \times \mathcal{P}_2(V_k),\) there similarly holds \(f^L(x^1, \cdots, x^k) = \sum_{i=1}^k f_i^L(x^i).\)

### 2.2 Submodularity and Convexity

In this subsection, we give new analogs of Theorems 2.1 and 2.2 for the disjoint-pair Lovász extension and the \(k\)-way Lovász extension. The major difference to existing results in the literature is that we work with the restricted or the enlarged domain of a function.

Let’s first recall the standard concepts of submodularity:

(S1) A discrete function \(f : A \to \mathbb{R}\) is submodular if \(f(A) + f(B) \geq f(A \cup B) + f(A \cap B), \forall A, B \in A,\) where \(A \subset \mathcal{P}(V)\) is an algebra (i.e., \(A\) is closed under union and intersection).

(S2) A continuous function \(F : \mathbb{R}^n \to \mathbb{R}\) is submodular if \(F(x) + F(y) \geq F(x \vee y) + F(x \wedge y),\) where \((x \vee y)_i = \max\{x_i, y_i\}\) and \((x \wedge y)_i = \min\{x_i, y_i\}, i = 1, \cdots, n.\) For a sublattice \(D \subset \mathbb{R}^n\) that is closed under \(\vee\) and \(\wedge,\) one can define submodularity in the same way.

**Convention 2.** The discussion about algebras of sets can be reduced to lattices. Classical submodular functions on a sublattice of the Boolean lattice \(\{0, 1\}^n\) and their continuous versions on \(\mathbb{R}^n\) are presented in (S1) and (S2), respectively. Bisubmodular functions on a graded sub-poset (partially ordered set) of \(\{-1, 0, 1\}^n\) are defined in [12] below.

Now, we recall the concept of bisubmodularity and introduce its continuous version.

(BS1) A discrete function \(f : \mathcal{P}_2(V) \to \mathbb{R}\) is bisubmodular if \(\forall (A, B), (C, D) \in \mathcal{P}_2(V)\)
\[
f(A, B) + f(C, D) \geq f((A \cup C) \setminus (B \cup D), (B \cup D) \setminus (A \cup C)) + f(A \cap C, B \cap D). \quad (12)\]
One can denote \(A \vee B = ((A_1 \cup B_1) \setminus (A_2 \cup B_2)), (A_2 \cup B_2) \setminus (A_1 \cup B_1))\) and \(A \wedge B = (A_1 \cap B_1, A_2 \cap B_2),\) where \(A = (A_1, A_2), B = (B_1, B_2).\) For a subset \(A \subset \mathcal{P}_2(V)\) that is closed under \(\vee\) and \(\wedge,\) the bisubmodularity of \(f : A \to \mathbb{R}\) can be expressed as \(f(A) + f(B) \geq f(A \vee B) + f(A \wedge B),\) \(\forall A, B \in A.\)
A continuous function $F : \mathbb{R}^n \to \mathbb{R}$ is bisubmodular if $F(x) + F(y) \geq F(x \vee y) + F(x \wedge y)$, where

$$(x \vee y)_i = \begin{cases} \max \{x_i, y_i\}, & \text{if } x_i, y_i \geq 0, \\ \min \{x_i, y_i\}, & \text{if } x_i, y_i \leq 0, \\ 0, & \text{if } x_i y_i < 0, \end{cases} \quad (x \wedge y)_i = \begin{cases} \min \{x_i, y_i\}, & \text{if } x_i, y_i \geq 0, \\ \max \{x_i, y_i\}, & \text{if } x_i, y_i \leq 0, \\ 0, & \text{if } x_i y_i < 0. \end{cases}$$

Henceforth, we simply use $1_{A,B}$ to denote the vector $1_A - 1_B$, where $A, B \subseteq V$.

**Proposition 2.7.** A function $F : \mathbb{R}^V \to \mathbb{R}$ is a disjoint-pair Lovász extension of a bisubmodular function if and only if $F$ is (continuously) bisubmodular (in the sense of (BS2)) and for any $x \in \mathbb{R}^V$, $t \geq 0$,

1. $F(tx) = tF(x)$ (positive homogeneity);
2. $F(x + t1_{V_+, V_-}) \geq F(x) + F(t1_{V_+, V_-})$ for some $V_+ \supset V^0_\pm(x)$ with $V_+ \cup V_- = V$.

The proof is a modification of the previous version on the original Lovász extension for submodular functions.

**Proof.** We focus on the “if” part. Take the discrete function $f$ defined as $f(A_1, A_2) = F(1_{A_1, A_2})$. One can check the bisubmodularity of $f$ directly. Fix an $x \in \mathbb{R}^n$ and let $\sigma : V \cup \{0\} \to V \cup \{0\}$ be a bijection such that $|x\sigma(1)| \leq |x\sigma(2)| \leq \cdots \leq |x\sigma(n)|$ and $\sigma(0) = 0$, where $x_0 := 0$, and

$$V^\sigma_+ = V^\sigma_-(x) := \{ j \in V : \pm x_j > |x\sigma(i)| \}, \quad i = 0, 1, \ldots, n - 1.$$

Also, we denote $x_{V^\sigma_+ \setminus V^\sigma_-} = x \ast 1_{V^\sigma_+ \cup V^\sigma_-}$ (i.e., the restriction of $x$ onto $V^\sigma_+ \cup V^\sigma_-$, with other components 0), where $x \ast y := (x_1y_1, \ldots, x_ny_n)$.

For simplicity, in the following formulas, we identify $\sigma(i)$ with $i$ for all $i = 0, \ldots, n$.

It follows from $|x_{i+1}|1_{V^i_+, V^i_-} \setminus x_{V^i_+ \setminus V^i_-} = |x_{i+1}|1_{V^i_+, V^i_-}$ and

$$|x_{i+1}|1_{V^i_+, V^i_-} \setminus x_{V^i_+ \setminus V^i_-} = |x_{i+1}|1_{V^i_+, V^i_-}$$

that

$$f^L(x) = \sum_{i=0}^{n-1} (|x_{i+1}|- |x_i|) f(V^i_+, V^i_-)$$

$$= \sum_{i=0}^{n-1} |x_{i+1}| (f(V^i_+, V^i_-) - f(V^i_+, V^{i+1}_-))$$

$$= \sum_{i=0}^{n-1} \left\{ F(|x_{i+1}|1_{V^i_+, V^i_-}) - F(|x_{i+1}|1_{V^i_+, V^{i+1}_-}) \right\}$$

$$\geq \sum_{i=0}^{n-1} \left\{ F(x_{V^i_+, V^i_-}) - F(x_{V^{i+1}_+, V^{i+1}_-}) \right\} = F(x).$$

On the other hand,

$$f^L(x) = \sum_{i=0}^{n-1} (|x_{i+1}|- |x_i|) f(V^i_+, V^i_-) = \sum_{i=0}^{n-1} F(|x_{i+1}|- |x_i|) 1_{V^i_+, V^i_-}$$

---

This is some kind of ‘translation linearity’ if we adopt the assumption $F(x + t1_{V_+, V_-}) = F(x) + F(t1_{V_+, V_-})$. 

13
\[
\begin{align*}
&= \sum_{i=0}^{n-2} \left\{ F(\mathbb{1}_{V^+_{i+1}}) - F(\mathbb{1}_{V^+_i}) \right\} \\
&\quad + \left\{ \sum_{i=0}^{n-2} F(\mathbb{1}_{V^+_i}) \right\} + F(\mathbb{1}_{V^+_n}) \\
\text{by (BS2)} &\leq \sum_{i=0}^{n-2} \left\{ F(\mathbb{1}_{V^+_{i+1}}) - F(\mathbb{1}_{V^+_i}) \right\} + F(\mathbb{1}_{V^+_n}) \\
\text{by (II)} &\leq \sum_{i=0}^{n-2} \left\{ F(\mathbb{1}_{V^+_{i+1}}) - F(\mathbb{1}_{V^+_i}) \right\} + F(\mathbb{1}_{V^+_n}) \\
&= F(\mathbb{1}_{V^+}) = F(x)
\end{align*}
\]
according to \((\mathbb{1}_{V^+_{i+1}} - \mathbb{1}_{V^+_i})\mathbb{1}_{V^+_i} = (\mathbb{1}_{V^+_{i+1}} - \mathbb{1}_{V^+_i})\mathbb{1}_{V^+_{i+1}}\) and 

\[ (\mathbb{1}_{V^+_{i+1}} - \mathbb{1}_{V^+_i})\mathbb{1}_{V^+_i} \cup (\mathbb{1}_{V^+_{i+1}} - \mathbb{1}_{V^+_i})\mathbb{1}_{V^+_{i+1}} = (\mathbb{1}_{V^+_{i+1}} - \mathbb{1}_{V^+_i})\mathbb{1}_{V^+_{i+1}} \]

for \(i = 0, \cdots, n - 2\), as well as \(\mathbb{1}_{V^+_{n-1}} \cup (\mathbb{1}_{V^+_{n-1}} - \mathbb{1}_{V^+_n}) = (\mathbb{1}_{V^+_{n-1}} - \mathbb{1}_{V^+_n})\mathbb{1}_{V^+_{n-1}}.\) Therefore, we have 

\[ F(x) = f^L(x). \]

The “only if” part is easy. We only need to prove that for a bisubmodular function \(f : \mathcal{P}_2(V) \to \mathbb{R}, \) 

\(f^L\) satisfies (BS2), (I) and (II). For convenience, the proof is provided below.

- By the definition of \(f^L\), it is positively homogeneous. Thus, (I) holds.
- Again, by the definition of \(f^L\), it is easy to check that \(f^L(x + t\mathbb{1}_{V^+_{n-1}}) = f^L(x) + tf(V^+_{n-1})\) for any \(V^+_{n-1} \supset V^+_{n-2}(x)\), and any \(t \geq 0.\) So, (II) is proved.
- We use the formulation (II) of \(f^L\). It is easy to check that 

\[ (V^+_1(x), V^+_1(x)) \cup (V^+_1(y), V^+_1(y)) = (V^+_1(x \cup y), V^+_1(x \cup y)), \]

\[ (V^+_1(x), V^+_1(x)) \cap (V^+_1(y), V^+_1(y)) = (V^+_1(x \cap y), V^+_1(x \cap y)). \]

By the bisubmodularity of \(f\), and the above equalities, we have 

\[ f^L(x) + f^L(y) = \int_0^\infty \left( f(V^+_1(x), V^+_1(x)) + f(V^+_1(y), V^+_1(y)) \right) dt \]

\[ \geq \int_0^\infty \left( f(V^+_1(x \cup y), V^+_1(x \cup y)) + f(V^+_1(x \cap y), V^+_1(x \cap y)) \right) dt \]

\[ = f^L(x \cup y) + f^L(x \cap y). \]

The proof is completed. \(\square\)

**Proposition 2.8.** A continuous function \(F\) is a disjoint-pair Lovász extension of some function \(f : \mathcal{P}_2(V) \to \mathbb{R}\) if and only if for any \(x \in \mathbb{R}^n\), there exists \((V^+_i, V^-_i) \in \mathcal{P}_2(V)\) with \(V^+_i \supset cV^+_i(x)\), such that \(F(x \cap c1_{V^+_i, V^-_i}) + F(x - x \cap c1_{V^+_i, V^-_i}) = F(x)\), \(\forall c \geq 0.\)

**Proof.** Let \(F\) be a continuous function such that for any \(x \in \mathbb{R}^n\), there exists \((V^+_i, V^-_i) \in \mathcal{P}_2(V)\) with \(V^+_i \supset cV^+_i(x)\) satisfying \(F(x \cap c1_{V^+_i, V^-_i}) + F(x - x \cap c1_{V^+_i, V^-_i}) = F(x)\), \(\forall c \geq 0.\) Define the function \(f : \mathcal{P}_2(V) \to \mathbb{R}\) by \(f(A, B) = F(1_{A,B}).\) By induction, the property \(F(x \cap c1_{V^+_i, V^-_i}) + F(x - x \cap c1_{V^+_i, V^-_i}) = F(x)\) implies a summation form of \(F\), i.e., 

\[ F(x) = \sum_{i=0}^{n-1} \left\{ F(\mathbb{1}_{V^+_{i+1}} - \mathbb{1}_{V^+_i}) \right\}. \]
Also, for any \((A,B) \in \mathcal{P}_2(V), t > 0\) and \(c \geq 0\), taking \(x = (t + c)1_{A,B}\) and \((V_+, V_-) = (A,B)\), we obtain \(F(c1_{A,B}) + F(t1_{A,B}) = F((t + c)1_{A,B})\). By Cauchy’s functional equation, this implies that for any \(t \geq 0\) and \((A,B) \in \mathcal{P}_2(V)\), \(F(t1_{A,B}) = tF(1_{A,B}) = tf(A,B)\), and together with (13) and the summation form (see (10)) of the disjoint-pair Lovász extension, we further derive

\[
F(x) = \sum_{i=0}^{n-1} (|x_{\sigma(i+1)}| - |x_{\sigma(i)}|) f(V^\sigma_+(x), V^\sigma_-(x)) = f^L(x).
\]

On the other hand, based on (10), it is easy to check that for any \(x \in \mathbb{R}^n\), for any \((V_+, V_-) \in \mathcal{P}_2(V)\) with \(V_+ \supseteq V_0^2(x), f^L(x \land c1_{V_+, V_-}) + f^L(x - x \land c1_{V_+, V_-}) = f^L(x), \forall c \geq 0\). The proof is then completed.

The \(k\)-way submodularity can be naturally defined as (S1) and (S2):

(KS) Given a tuple \(V = (V_1, \cdots, V_k)\) of finite sets and \(A \subseteq \{A_1, \cdots, A_k\} : A_i \subset V_i, i = 1, \cdots, k\), a discrete function \(f : A \rightarrow \mathbb{R}\) is \(k\)-way submodular if \(f(A) + f(B) \geq f(A \lor B) + f(A \land B)\), \(\forall A, B \in A\), where \(A\) is a lattice under the corresponding lattice operations \(\lor\) and \(\land\). Defined by \(A \lor B = (A_1 \cup B_1, \cdots, A_k \cup B_k)\) and \(A \land B = (A_1 \cap B_1, \cdots, A_k \cap B_k)\).

**Theorem 2.3.** Under the assumptions and notations in (KS) above, \(D_A\) is also closed under \(\land\) and \(\lor\), with \(\land\) and \(\lor\) as in (S2). Moreover, the following statements are equivalent:

a) \(f\) is \(k\)-way submodular on \(A\);

b) the \(k\)-way Lovász extension \(f^L\) is convex on each convex subset of \(D_A\);

c) the \(k\)-way Lovász extension \(f^L\) is submodular on \(D_A\).

If one replaces (KS) and (S2) by (BS1) and (BS2) respectively for the bisubmodular setting, then all the above results hold analogously.

The proof is a slight variation of the original version by Lovász, and is provided for convenience.

**Proof.** Note that \(V^t(x) \lor V^t(y) = V^t(x \lor y)\) and \(V^t(x) \land V^t(y) = V^t(x \land y)\), where \(V^t(x) := (V^t(x^1), \cdots, V^t(x^k)), \forall t \in \mathbb{R}\). Since \(x \in D_A\) if and only if \(V^t(x) \in A, \forall t \in \mathbb{R}\), and \(A\) is a lattice, \(D_A\) must be a lattice that is closed under the operations \(\land\) and \(\lor\). According to the \(k\)-way Lovász extension (10), we may write

\[
f^L(x) = \int_{-N}^{N} f(V^t(x)) dt - N f(V)
\]

where \(N > \|x\|_\infty\) is a sufficiently large number\(^4\). Note that \(1_A \lor 1_B = 1_{A \lor B}\) and \(1_A \land 1_B = 1_{A \land B}\). Combining the above results, we immediately get

\[
f(A) + f(B) \geq f(A \lor B) + f(A \land B) \iff f^L(x) + f^L(y) \geq f^L(x \lor y) + f^L(x \land y),
\]

which proves (a) \(\iff\) (c). Note that for \(x \in D_A\), \(f^L(x) = \sum_{A \in C(x)} \lambda_A f(A)\) for a unique chain \(C(x) \subset A\) that is determined by \(x\) only, and the extension \(f^{\text{convex}}(x) := \inf_{\lambda_a \neq 0} \sum_{A \in A} \lambda_A f(A)\) is convex on each convex subset of \(D_A\), where \(A(x) := \{\lambda_A\}_{A \in A} \in \mathbb{R}^A : \sum_{A \in A} \lambda_A 1_A = x, \lambda_A \geq 0 \text{ whenever } A \neq V\}. \) We only need to prove \(f^L(x) = f^{\text{convex}}(x)\) if and only if \(f\) is submodular. In fact, along a standard idea proposed in Lovász’s original paper [59], one could prove that for a (strictly) submodular function, the set \(\{A : \lambda^*_A = 0\}\) must be a chain, where \(\sum_{A \in A} \lambda^*_A f(A) = f^{\text{convex}}(x)\) achieves the minimum over \(A(x)\), and one can then easily check that it agrees with \(f^L\). The converse can be proved in a standard way: \(f(A) + f(B) = f^L(1_A) + f^L(1_B) \geq f^L(1_A + 1_B) = f(1_{A \lor B} + 1_{A \land B}) = f(1_{A \lor B}) + f(1_{A \land B}) = f(A \lor B) + f(A \land B)\). Now, the proof is completed.

For the bisubmodular case, the above reasoning can be repeated with minor differences. \(\square\)

\(^4\)Here we set \(f(\varnothing, \cdots, \varnothing) = 0\)
Remark 3. We show some examples about how both convexity and continuous submodularity can be satisfied. In fact, it is easy to see that the \( l^p \)-norm \( \|x\|_p \) is both convex and continuously submodular on \( \mathbb{R}^n_+ \), while the \( l^1 \)-norm \( \|x\|_1 \) is convex and continuously submodular on the whole \( \mathbb{R}^n \). Besides, an elementary proof shows that a one-homogeneous continuously submodular function on \( \mathbb{R}^n_+ \) must be convex.

3 Main results on optimization and eigenvalue problems

We uncover the links between combinatorial optimization and continuous programming as well as eigenvalue problems in a general setting.

3.1 Combinatorial and continuous optimization

As we have told in the introduction, the application of the Lovász extension to non-submodular optimization meets with several difficulties, and in this section, we start attacking those. First, we set up some useful results.

Convention 3. In this section, \( \mathbb{R}_{\geq 0} := [0, \infty) \) is the set of all non-negative numbers. We use \( f^L \) to denote the multi-way Lovász extension which can be either the original or the disjoint-pair or the \( k \)-way Lovász extension.

We show some examples about how both convexity and continuous submodularity can be satisfied. In fact, it is easy to see that the \( l^p \)-norm \( \|x\|_p \) is both convex and continuously submodular on \( \mathbb{R}^n_+ \), while the \( l^1 \)-norm \( \|x\|_1 \) is convex and continuously submodular on the whole \( \mathbb{R}^n \). Besides, an elementary proof shows that a one-homogeneous continuously submodular function on \( \mathbb{R}^n_+ \) must be convex.

Theorem 3.1. Given set functions \( f_1, \ldots, f_n : A \to \mathbb{R}_{\geq 0} \), and a zero-homogeneous function \( H : \mathbb{R}^n_{\geq 0} \setminus \{0\} \to \mathbb{R} \cup \{+\infty\} \) with \( H(a+b) \geq \min\{H(a),H(b)\} \), \( \forall a,b \in \mathbb{R}^n_{\geq 0} \setminus \{0\} \), we have

\[
\min_{A' \in \mathcal{A}'} H(f_1(A), \ldots, f_n(A)) = \inf_{x \in \mathcal{D}'} H(f^L_1(x), \ldots, f^L_n(x)),
\]

where \( \mathcal{A}' = \{A \in A : (f_1(A), \ldots, f_n(A)) \in \text{Dom}(H)\} \), \( \mathcal{D}' = \{x \in \mathcal{D} : (f^L_1(x), \ldots, f^L_n(x)) \in \text{Dom}(H)\} \) and \( \text{Dom}(H) = \{a \in \mathbb{R}^n_{\geq 0} \setminus \{0\} : H(a) \in \mathbb{R}\} \).

Proof. By the property of \( H \), \( \forall t_i \geq 0, n \in \mathbb{N}^+, a_{i,j} \geq 0, i = 1, \ldots, m, j = 1, \ldots, n, \)

\[
H \left( \sum_{i=1}^{m} t_i a_{i,1}, \ldots, \sum_{i=1}^{m} t_i a_{i,n} \right) = H \left( \sum_{i=1}^{m} t_i a^i \right) \geq \min_{i=1, \ldots, m} H(t_i a^i) = \min_{i=1, \ldots, m} H(a^i) = \min_{i=1, \ldots, m} H(a_{i,1}, \ldots, a_{i,n}).
\]

Therefore, in the case of the original Lovász extension, for any \( x \in \mathcal{D}' \),

\[
H \left( f^L_1(x), \ldots, f^L_n(x) \right) = H \left( \max_{x} f_1(V^l(x)) \right) = f_1(V(x)) \min x, \ldots, \max_{x} f_n(V^l(x)) \min x \)
\]

\[
= H \left( \sum_{i=1}^{m} (t_i - t_{i-1}) f_1(V^{i-1}(x)), \ldots, \sum_{i=1}^{m} (t_i - t_{i-1}) f_n(V^{i-1}(x)) \right)
\]

\[
= H \left( \sum_{i=1}^{m} (t_i - t_{i-1}) f_1(V^{i-1}(x)), \ldots, \sum_{i=1}^{m} (t_i - t_{i-1}) f_n(V^{i-1}(x)) \right)
\]

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\[ \sum_{i=1}^{m} t_i a_i \geq \min_{i=1,\ldots,m} H(f_1(V^{t_i-1}(x)),\ldots,f_n(V^{t_n-1}(x))) \]

\[ \geq \min_{A \in \mathcal{A}'} H(f_1(A),\ldots,f_n(A)) \]

\[ = \min_{A \in \mathcal{A}'} H(f_1^1(A),\ldots,f_n^1(A)) \]

\[ \geq \inf_{x' \in \mathcal{D}'} H(f_1^1(x'),\ldots,f_n^1(x')). \]

Combining (15) with (16), we have \( \inf_{x' \in \mathcal{D}'} H(f_1^1(x'),\ldots,f_n^1(x')) \geq \min_{A \in \mathcal{A}'} H(f_1(A),\ldots,f_n(A)) \), and then together with (16) and (17), we get the reverse inequality. Hence, (14) is proved for the original Lovász extension \( f^L \). For the multi-way settings, the proof is similar and thus we omit them. \( \square \)

**Remark 4.** Duality: If one replaces \( H(a+b) \geq \min\{H(a), H(b)\} \) by \( H(a+b) \leq \max\{H(a), H(b)\} \), then

\[ \max_{A \in \mathcal{A}'} H(f_1(A),\ldots,f_n(A)) = \sup_{x' \in \mathcal{D}'} H(f_1^1(x'),\ldots,f_n^1(x')). \]

The proof of the identity (18) is similar to that of (13), and thus we omit it.

**Remark 5.** A function \( H : [0, +\infty)^n \rightarrow \mathbb{R} \) has the (MIN) property if

\[ H\left( \sum_{i=1}^{m} t_i a_i \right) \geq \min_{i=1,\ldots,m} H(a^i), \forall t_i > 0, m \in \mathbb{N}^+, a^i \in [0, +\infty)^n. \]

The (MAX) property is formulated analogously.

We can verify that the (MIN) property is equivalent to the zero-homogeneity and \( H(x+y) \geq \min\{H(x), H(y)\} \). A similar correspondence holds for the (MAX) property.

**Remark 6.** Theorem 3.1 shows that if \( H \) has the (MIN) or (MAX) property, then the corresponding combinatorial optimization is equivalent to a continuous optimization by means of the multi-way Lovász extension. Here are some examples:

Given \( c, c_i \geq 0 \) with \( \sum_i c_i > 0 \), let \( H(f_1,\ldots,f_n) = \frac{c_1 f_1^{p_1} + \cdots + c_n f_n^{p_n}}{f_1^{p_1} + \cdots + f_n^{p_n}} \). Then \( H \) satisfies the (MIN) property, and by Theorem 3.1, we have

\[ \min_{A \in \mathcal{A}'} \frac{\sum_i c_i f_i(A) - c \sqrt{\sum_i f_i^2(A)}}{\sum_i f_i(A)} = \inf_{\psi \in \mathcal{D}'} \frac{\sum_i c_i f_i^1(\psi) - c \sqrt{\sum_i (f_i^1(\psi))^2}}{\sum_i f_i^1(\psi)}. \]

Taking \( f_1,\ldots,f_n = (c_1 f_1^{p_1} + \cdots + c_n f_n^{p_n})^{1/p} \) for some \( p > 1 \), then \( H \) satisfies the (MAX) property, and by Theorem 3.1, there holds

\[ \max_{A \in \mathcal{A}'} \frac{(\sum_i c_i f_i(A)^p)^{1/p}}{\sum_i f_i(A)} = \sup_{\psi \in \mathcal{D}'} \frac{(\sum_i c_i f_i^1(\psi)^p)^{1/p}}{\sum_i f_i^1(\psi)}. \]

**Proof of Theorem 3.1.** Without loss of generality, we may assume that \( P(f_1,\ldots,f_n) \) is one-homogeneous and subadditive, while \( Q(f_1,\ldots,f_n) \) is one-homogeneous and superadditive on \( (f_1,\ldots,f_n) \in \mathbb{R}^n_0 \). Then \( H(f_1,\ldots,f_n) = \frac{P(f_1,\ldots,f_n)}{Q(f_1,\ldots,f_n)} \) is zero-homogeneous on \([0, +\infty)^n\), and

\[ H(f + g) = \frac{P(f + g)}{Q(f + g)} \leq \frac{P(f) + P(g)}{Q(f) + Q(g)} \leq \max\left\{ \frac{P(f)}{Q(f)}, \frac{P(g)}{Q(g)} \right\} = \max\{H(f), H(g)\} \]

where \( f = (f_1,\ldots,f_n) \) and \( g = (g_1,\ldots,g_n) \).

Then the proof is completed by Theorem 3.1 (and Remark 4). \( \square \)

**Example 3.1.** Given a finite graph \((V, E)\), for \( \{i, j\} \in E \), let \( f_{\{i,j\}}(A) = 1 \) if \( A \cap \{i, j\} = \{i\} \) or \( \{j\} \), and \( f_{\{i,j\}}(A) = 0 \) otherwise. Let \( g(A) = |A| \) for \( A \subseteq V \). It is clear that

\[ \frac{1}{g(A)} \left( \sum_{(i,j) \in E} |f_{\{i,j\}}(A)| \right)^{1/p} \]

satisfies the condition of Theorem 3. Thus, we derive that

\[ \max_{A \neq \emptyset} \frac{|\partial A|^p}{|A|} = \frac{\left( \sum_{(i,j) \in E} P_{f_{\{i,j\}}(A)}(A) \right)^{1/p}}{g(A)} = \max_{x \in \mathbb{R}^n_0} \frac{\left( \sum_{(i,j) \in E} |x_i - x_j|^p \right)^{1/p}}{\sum_{i \in V} x_i} = \max_{x \neq 0} \frac{\left( \sum_{(i,j) \in E} |x_i - x_j|^p \right)^{1/p}}{\sum_{i \in V} |x_i|}. \]
Similarly, we have
\[
\max_{A \neq \emptyset} |\partial A|_\infty^{\frac{1}{p}} = \max_{x \in \mathbb{R}_+^V} \frac{\sum_{\{i,j\} \in E} \max_{i \in V} x_i}{\sum_{\{i,j\} \in E} (|x_i - x_j|^p)^{\frac{1}{p}}} = \max_{x \neq 0} \frac{\sum_{\{i,j\} \in E} (x_i - x_j)^p}{\sum_{i \in V} x_i}.
\]
which gives a continuous representation of the Max-Cut problem. The last equality holds due to the following reason: letting \( F(x) = (\sum_{\{i,j\} \in E} |x_i - x_j|^p)^{\frac{1}{p}} \), we can check that \( \max_{x \in \mathbb{R}_+^V} \frac{F(x)}{\|x\|_\infty} \) achieves its maximum at some characteristic vector \( 1_A \), and then \( 1_A - 1_{V \setminus A} \) is a maximizer of \( \frac{F(x)}{\|x\|_\infty} \) on \( \mathbb{R}_+^V \setminus \{0\} \).

Similarly, \( \max_{x \neq 0} \frac{F(x)}{\|x\|_\infty} \) achieves its maximum at \( 1_A - 1_{V \setminus A} \) for some \( A \), and then \( 1_A \) indicates a maximizer of \( \frac{F(x)}{\max_{i \in V} x_i} \) on the first orthant \( \mathbb{R}_+^V \). We need the factor 2 because \( F(1_A - 1_{V \setminus A}) = 2F(1_A) \).

It should be noted that the two equivalent continuous reformulations are derived by the original and disjoint-pair Lovász extensions in the following two ways:
\[
\max_{A \neq \emptyset} |\partial A|_\infty^{\frac{1}{p}} = \max_{A \in \mathcal{P}(V) \setminus \{\emptyset\}} \frac{\sum_{\{i,j\} \in E} f_{\{i,j\}}(A) + f_{\{i,j\}}(B))}{\sum_{\{i,j\} \in E} (|x_i - x_j|^p)^{\frac{1}{p}}} = \max_{x \neq 0} \frac{\sum_{\{i,j\} \in E} (x_i - x_j)^p}{\sum_{i \in V} x_i}
\]
where we use \( f_{\{i,j\}}(x) = |x_i - x_j| \) and \( 1_L = \max_{i \in V} x_i \);
\[
\max_{A \neq \emptyset} |\partial A|_\infty^{\frac{1}{p}} = \max_{(A,B) \in \mathcal{P}_2(V) \setminus \{(\emptyset,\emptyset)\}} \frac{\sum_{\{i,j\} \in E} f_{\{i,j\}}(A) + f_{\{i,j\}}(B))}{\sum_{\{i,j\} \in E} (|x_i - x_j|^p)^{\frac{1}{p}}} = \max_{x \neq 0} \frac{\sum_{\{i,j\} \in E} (x_i - x_j)^p}{\sum_{i \in V} x_i}
\]
where we use the fact that the disjoint-pair Lovász extension of \( (A,B) \mapsto f_{\{i,j\}}(A) + f_{\{i,j\}}(B) \) is \( |x_i - x_j| \) and the disjoint-pair Lovász extension of \( (A,B) \mapsto 1 \) is \( \|x\|_\infty \).

**Example 3.2.** There are many other equalities that can be obtained by Theorem [C] such as:
\[
\min_{A \neq \emptyset} \frac{|\partial A|}{|A|^{\frac{1}{p}}} = \min_{x \in \mathbb{R}_+^V : \min_{x=0} \frac{\sum_{\{i,j\} \in E} |x_i - x_j|}{\sum_{i \in V} x_i}^{\frac{1}{p}}
\]
and
\[
\max_{(A,B) \in \mathcal{P}_2(V) \setminus \{(\emptyset,\emptyset)\}} \frac{2|E(A,B)|}{\text{vol}(A \cup B)} = \max_{x \neq 0} \frac{\sum_{\{i,j\} \in E} (\|x_i + x_j - |x_i + x_j|^p\|_{\frac{1}{p}}}{\sum_{i \in V} \deg_i |x_i|}
\]
whenever \( p \geq 1 \). Here, \( \text{vol} A = \sum_{i \in A} \deg_i \).

The last equality shows a variant of the dual Cheeger constant. A slight modification gives
\[
\max_{A \in \mathcal{P}(V)} \frac{2|\partial A|}{|A|^{\frac{1}{p}}} = \max_{(A,B) \in \mathcal{P}_2(V)} \frac{2|E(A,B)|}{|A \cup B|} = \max_{x \neq 0} \frac{\sum_{\{i,j\} \in E} (\|x_i + x_j - |x_i + x_j|^p\|_{\frac{1}{p}}}{\|x\|_\infty}
\]
showing a new continuous formulation of the Max-Cut problem.

Taking \( n = 2 \) and \( H(f_1, f_2) = \frac{f_1}{f_2} \) in Theorem 3.1, then such an \( H \) satisfies both (MIN) and (MAX) properties. So, we get
\[
\min_{A \in \mathcal{A}} \frac{f_1(A)}{f_2(A)} = \inf_{\psi \in \mathcal{D}'} \frac{f_{T\psi}(\psi)}{f_{T\psi}(\psi)} \quad \text{and} \quad \max_{A \in \mathcal{A}} \frac{f_1(A)}{f_2(A)} = \sup_{\psi \in \mathcal{D}'} \frac{f_{T\psi}(\psi)}{f_{T\psi}(\psi)}.
\]
In fact, we can get more:

**Proposition 3.1.** Given two functions \( f, g : A \to [0, +\infty) \), let \( \tilde{f}, \tilde{g} : \mathcal{D}_A \to \mathbb{R} \) satisfy \( \tilde{f} \geq f_L \), \( \tilde{g} \leq g_L \), \( \tilde{f}(1_A) = f(A) \) and \( \tilde{g}(1_A) = g(A) \) for any \( A \in \mathcal{A} \). Then
\[
\min_{A \in \mathcal{A} \setminus \text{supp}(g)} \frac{f(A)}{g(A)} = \inf_{\psi \in \mathcal{D}_A \setminus \text{supp}(\tilde{g})} \frac{\tilde{f}(\psi)}{\tilde{g}(\psi)}.
\]
If we replace the condition $\tilde{f} \geq f^L$ and $\tilde{g} \leq g^L$ by $\tilde{f} \leq f^L$ and $\tilde{g} \geq g^L$, then

$$\max_{A \in A^\prime \supp(g)} \frac{f(A)}{g(A)} = \sup_{\psi \in D_A \supp(g)} \frac{\tilde{f}(\psi)}{\tilde{g}(\psi)}.$$ 

For any $\alpha \neq 0$, then $\tilde{f} = ((f^\alpha)^L)^{\frac{1}{\alpha}}$ and $\tilde{g} = ((g^\alpha)^L)^{\frac{1}{\alpha}}$ satisfy the above two identities.

**Proof.** It is obvious that

$$\inf_{\psi \in D_A \supp(g)} \frac{\tilde{f}(\psi)}{\tilde{g}(\psi)} \leq \min_{A \in A^\prime \supp(g)} \frac{\tilde{f}(1_A)}{\tilde{g}(1_A)} = \min_{A \in A^\prime \supp(g)} \frac{f(A)}{g(A)}.$$ 

On the other hand, for any $\psi \in D_A \supp(\tilde{g})$, $g^L(\psi) \geq \tilde{g}(\psi) > 0$. Hence, there exists $t \in (\min \tilde{\beta} - 1, \max \tilde{\beta} + 1)$ satisfying $g(V^t(\psi)) > 0$. Here $\tilde{\beta} \psi = \psi$ (resp., $|\psi|$), if $f^L$ represents either the original or the $k$-way Lovász extension of $f$ (resp., either the disjoint-pair or the $k$-way disjoint-pair Lovász extension). So, the set $W(\psi) := \{t \in \mathbb{R} : g(V^t(\psi)) > 0\}$ is nonempty. Since $\{V^t(\psi) : t \in W(\psi)\}$ is finite, there exists $t_0 \in W(\psi)$ such that $f(V^t_0(\psi)) = \min_{t \in W(\psi)} f(V^t(\psi))$. Accordingly, $f(V^t(\psi)) \geq \frac{f(V^t_0(\psi))}{g(V^t_0(\psi))} g(V^t(\psi))$ for any $t \in W(\psi)$, and thus

$$f(V^t(\psi)) \geq C g(V^t(\psi)),$$

holds for any $t \in \mathbb{R}$ (because $g(V^t(\psi)) = 0$ for $t \in \mathbb{R} \setminus W(\psi)$ which means that the above inequality automatically holds). Consequently,

$$\tilde{f}(\psi) \geq f^L(\psi)$$

$$= \int_{\min \tilde{\beta} \psi}^{\max \tilde{\beta} \psi} f(V^t(\psi)) dt + f(V(\psi)) \min \tilde{\beta} \psi$$

$$\geq C \left( \int_{\min \tilde{\beta} \psi}^{\max \tilde{\beta} \psi} g(V^t(\psi)) dt + g(V(\psi)) \min \tilde{\beta} \psi \right).$$

where we used $\min \tilde{\beta} \psi \geq 0$. The proof of $\min \tilde{\beta} \psi \geq 0$ is straightforward: in fact, by $\psi \in D_A$, if we use the original or the $k$-way Lovász extension introduced in Definition 2.2 then $\tilde{\beta} \psi = \psi$, and $D_A$ lies in the closure of the first orthant of the Euclidean space, meaning that $\tilde{\beta} \psi = \psi \geq 0$; and if we use the disjoint-pair or the $k$-way disjoint-pair Lovász extension in Definition 2.3 then $\tilde{\beta} \psi = |\psi| \geq 0$.

It follows that

$$\frac{\tilde{f}(\psi)}{\tilde{g}(\psi)} \geq C \min_{A \in A^{\prime \supp(g)}} \frac{f(A)}{g(A)}$$

and thus the proof is completed. The dual case is similar.

For $\alpha > 0$, we can simply suppose $\supp(g) = A$. Then

$$\min_{A \in A} \frac{f(A)}{g(A)} = \min_{A \in A} \frac{(f^\alpha)^{\frac{1}{\alpha}}(A)}{(g^\alpha)^{\frac{1}{\alpha}}(A)} = \left( \min_{A \in A} f^\alpha(A) \right)^{\frac{1}{\alpha}} = \left( \inf_{\psi \in D_A} \frac{(f^\alpha)^L(\psi)}{(g^\alpha)^L(\psi)} \right)^{\frac{1}{\alpha}} \geq \inf_{\psi \in D_A} \left( \frac{(f^\alpha)^L(\psi)}{(g^\alpha)^L(\psi)} \right)^{\frac{1}{\alpha}}.$$ 

For $\alpha < 0$, we may suppose without loss of generality that $g(A) > 0$ and $f(A) > 0$ for any $A \in A$. Then, in this case,

$$\min_{A \in A} \frac{f(A)}{g(A)} = \min_{A \in A} \frac{(f^\alpha)^{\frac{1}{\alpha}}(A)}{(g^\alpha)^{\frac{1}{\alpha}}(A)} = \left( \max_{A \in A} f^\alpha(A) \right)^{\frac{1}{\alpha}} = \left( \sup_{\psi \in D_A} \frac{(f^\alpha)^L(\psi)}{(g^\alpha)^L(\psi)} \right)^{\frac{1}{\alpha}} = \inf_{\psi \in D_A} \left( \frac{(f^\alpha)^L(\psi)}{(g^\alpha)^L(\psi)} \right)^{\frac{1}{\alpha}}.$$ 

This completes the proof.\[\square\]
Table 1: Original Lovász extension of some objective functions.

| Set function $f(A)$ | Lovász extension $f^L(x)$ |
|---------------------|-------------------------|
| $\#E(A, V \setminus A)$ | $\sum_{i,j \in E} |x_i - x_j|$ (see [43]) |
| $C$ | $C \max_i x_i$ (see [43]) |
| $\text{vol}(A)$ | $\sum_i \deg_i x_i$ (see [43]) |
| $\min\{\text{vol}(A), \text{vol}(V \setminus A)\}$ | $\min_{t \in \mathbb{R}} \|x - t\|_1$ (see [43]) |
| $\#A \cdot \#(V \setminus A)$ | $\sum_{i,j \in V} |x_i - x_j|$ |
| $(\#A)^k$ | $\sum_{i_1, \ldots, i_k \in V} \min\{x_{i_1}, \ldots, x_{i_k}\}$ |
| $\text{vol}(A)^k$ | $\sum_{i_1, \ldots, i_k \in V} \deg_{i_1} \cdots \deg_{i_k} \min\{x_{i_1}, \ldots, x_{i_k}\}$ |
| $\#V(E(A, V \setminus A))$ | $\sum_{i=1}^n \left( \max_{j \in N(i)} x_j - \min_{j \in N(i)} x_j \right)$ |

It is worth noting that in Proposition 3.1, $A$ can be a family of some set-tuples, and $f^L$ is the multi-way Lovász extension of the corresponding $f$. We point out that we can replace the Lovász extension $f^L$ by any other extension $f^E$ with the property that $f^E/g^E$ achieves its minimum and maximum at some 0-1 vector $1_A$ for some $A \in A$. Similarly, we have:

**Proposition 3.2.** Let $f, g : A \to [0, +\infty)$ be two set functions and $f := f_1 - f_2$ and $g := g_1 - g_2$ be decompositions of differences of submodular functions.

Let $f_1, g_1$ be the restriction of positively one-homogeneous convex functions onto $D_A$, with $f_1(A) = f_1(1_A)$ and $g_2(A) = g_2(1_A)$ for any $A \in A$. Define $\tilde{f} = f_1 - \tilde{f}_2$ and $\tilde{g} = \tilde{g}_1 - g_2$. Then,

$$\min_{A \in \mathcal{X}_{\text{vol}(g)}} \frac{f(A)}{g(A)} = \min_{x \in \mathcal{X}_{\text{vol}(g)}} \frac{\tilde{f}(x)}{\tilde{g}(x)}$$

**Remark 7.** Hirai et al introduce the generalized Lovász extension of $f : \mathcal{L} \to \mathbb{R}$ on a graded poset $\mathcal{L}$ (see [22, 30]). Since $f^L(x) = \sum_i \lambda_i f(p_i)$ for $x = \sum_i \lambda_i p_i$ lying in the orthoscheme complex $K(\mathcal{L})$, the same results as stated in Theorem 3.1 and Proposition 3.1 hold for such a generalized Lovász extension $f^L$. Propositions 3.1 and 3.2 are also generalizations of Theorem 3.1 in [43] and Theorem 1 (b) in [7].

Although the continuous representations translate the original problems into equivalently difficult optimization problems, we should point out that the continuous formulations ensure that many fast algorithms in continuous programming can be applied directly to certain combinatorial optimization problems. For example, the fractional form of the equivalent continuous optimizations shown in Theorem 3.1 as well as Propositions 3.1 and 3.2 implies that we can directly adopt the Dinkelbach iteration in Fractional Programming [75] to solve them. In addition, since the equivalent continuous formulation is Lipschitz, we can also adopt the stochastic subgradient method [31] to solve certain discrete optimization problems directly.

Tables 1 and 2 and Propositions 3.3, 3.4 and 3.5 present a general correspondence between set or set-pair functions and their Lovász extensions. We shall make use of several of those in Section 4. Note that the first four lines in Table 1 for the original Lovász extension, and the first five lines in Table 2 for the disjoint-pair Lovász extension are known (see [24, 43]).

**Proposition 3.3.** Suppose $f, g : \mathcal{P}(V) \to [0, +\infty)$ are two set functions with $g(A) > 0$ for any $A \in \mathcal{P}(V) \setminus \{\emptyset\}$. Then

$$\min_{A \in \mathcal{P}(V) \setminus \{\emptyset\}} \frac{f(A)}{g(A)} = \min_{(A,B) \in \mathcal{P}(V) \setminus \{(\emptyset, \emptyset)\}} \frac{f(A) + f(B)}{g(A) + g(B)} = \min_{(A,B) \in \mathcal{P}(V) \setminus \{(\emptyset, \emptyset)\}} \frac{f(A) + f(B)}{g(A) + g(B)}$$

where the right identity needs additional assumptions like $f(\emptyset) = g(\emptyset) = 0$ or that $f$ and $g$ are

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5 This setting is natural, as the Lovász extension doesn’t use the datum on $\emptyset$. 
Disjoint-pair Lovász extension

For any functions can be transformed to the original and the disjoint-pair versions.

Similarly, by employing Propositions 2.6, 3.4 and 3.5, the k-way Lovász extension of some special functions can be transformed to the original and the disjoint-pair versions.

Proposition 3.4. Suppose \( f, g : \mathcal{P}(V) \to [0, +\infty) \) are two set functions with \( g(A) > 0 \) for any \( A \in \mathcal{P}(V) \setminus \{\emptyset\} \). Then

\[
\min_{A \in \mathcal{P}(V)} \frac{f(A)}{g(A)} = \min_{(A_1, \ldots, A_k) \in \mathcal{P}(V)^k} \frac{\sum_{i=1}^k f(A_i)}{\sum_{i=1}^k g(A_i)} = \min_{(A_1, \ldots, A_k) \in \mathcal{P}(V)^k} \sqrt[k]{\prod_{i=1}^k \frac{f(A_i)}{g(A_i)}} = \min_{(A_1, \ldots, A_k) \in \mathcal{P}_k(V)} \frac{\sum_{i=1}^k f(A_i)}{\sum_{i=1}^k g(A_i)},
\]

where the last identity needs additional assumptions like \( f(\emptyset) = g(\emptyset) = 0 \).

Proposition 3.5. Suppose \( f, g : \mathcal{P}_2(V) \to [0, +\infty) \) are two set functions with \( g(A, B) > 0 \) for any \( (A, B) \in \mathcal{P}_2(V) \setminus \{(\emptyset, \emptyset)\} \). Then

\[
\min_{(A, B) \in \mathcal{P}_2(V)} \frac{f(A, B)}{g(A, B)} = \min_{(A_1, B_1, \ldots, A_k, B_k) \in \mathcal{P}_2(V)^k} \frac{\sum_{i=1}^k f(A_i, B_i)}{\sum_{i=1}^k g(A_i, B_i)} = \min_{(A_1, B_1, \ldots, A_k, B_k) \in \mathcal{P}_{2k}(V)} \frac{\sum_{i=1}^k f(A_i, B_i)}{\sum_{i=1}^k g(A_i, B_i)},
\]

where the last identity needs additional assumptions like \( f(\emptyset, \emptyset) = g(\emptyset, \emptyset) = 0 \).

Together with Propositions 2.3 and 3.3, one may directly transfer the data from Table 1 to Table 2. Similarly, by employing Propositions 2.6, 3.4 and 3.5 the k-way Lovász extension of some special functions can be transformed to the original and the disjoint-pair versions.

Proposition 3.6. For any \( a < b \), and for any \( f \),

\[
\min_{x \in [a, b]^n} f^L(x) = \min_{x \in [a, b]^n} f^L(x) = a f(V) + (b - a) \min_{A \subset V} f(A).
\]

Clearly, we can replace all ‘min’ by ‘max’.

Proof. Since \( f^L \) is linear on each piece \( \Delta_\sigma \cap [a, b]^n := \{ x \in \mathbb{R}^n : a \leq x_i \leq b, x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)} \} \), where \( \sigma : \{1, \ldots, n\} \to \{1, \ldots, n\} \) is a permutation, the maximum and minimum of \( f^L \) on \( \Delta_\sigma \cap [a, b]^n \) can be reached at some vertices of the simplex \( \Delta_\sigma \cap [a, b]^n \). Note that the vertices of \( \Delta_\sigma \cap [a, b]^n \) are included in \( [a, b]^n \) (i.e., the vertices of the hypercube \( [a, b]^n \)). Thus, the maximum and minimum of \( f^L \) on \( [a, b]^n \) can be attained at some points in \( [a, b]^n \). By the definition of Lovász extension, it is easy to check that \( f^L(b1_B + a1_{V \setminus B}) = a f(V) + (b - a) f(B) \). The proof is completed.

Discrete vs. continuous optimization and our approach: The general framework based on multi-way Lovász extensions is universal and fundamental with potential to design some simple iterative algorithms using equivalent continuous optimization to approach and solve discrete optimization

\(^6\)A function \( f : \mathcal{P}(V) \to \mathbb{R} \) is symmetric if \( f(A) = f(V \setminus A), \forall A \subset V \).

\(^7\)This setting is natural, as the disjoint-pair Lovász extension doesn’t use the information on \( (\emptyset, \emptyset) \).
problems. To illustrate this point, we present in the last paragraph of this section some advantages of our equivalent continuous formulations for certain combinatorial optimization problems, and we present in Section 3.3 an algorithm for solving such combinatorial optimization problems.

It is a fundamental aspect of our scheme that the critical data (including min-max data, saddle points, and optimal values) of the continuous representations that we develop cover all the key information of the original combinatorial problems (see Section 3.2).

From the viewpoint of applied and computational mathematics, a significant advantage of the formulations obtained by multi-way Lovász extensions in this paper is that the approach can be generally applied to many combinatorial optimization problems, and compared to other formulations, our formulation is in quotient form, with both numerator and denominator expressed as the difference of convex functions, which allows us to directly use techniques from DC (Difference of Convex functions) programming and fractional programming. Perhaps even more importantly, our approach doesn’t need any additional rounding techniques. We refer to Section 3.3 for a detailed explanation for such advantages. As can be seen in the next two subsections, these formulations provide new insight into structure and properties of certain combinatorial problems, and allow one to develop more efficient algorithms for computing optimal and approximate solutions.

### 3.2 Eigenvalue problems for Lovász extension

For convenience, we shall work in a normed space $X$, and we will take $X$ as the usual Euclidean space $\mathbb{R}^n$ in this subsection, and a general normed space $X$ will be used in Section 3.3.

For a convex function $F : X \to \mathbb{R}$, its sub-gradient (or sub-derivative) $\nabla F(x)$ is defined as the collection of $u \in X^*$ satisfying $F(y) - F(x) \geq \langle u, y - x \rangle$, $\forall y \in X$, where $X^*$ is the dual of $X$ and $\langle u, y - x \rangle$ is the action of $u$ on $y - x$. The concept of a sub-gradient has been extended to Lipschitz functions. This is called the Clarke derivative [19]:

$$
\nabla F(x) = \left\{ u \in X^* \mid \limsup_{y \to x, t \to 0^+} \frac{F(y + th) - F(y)}{t} \geq \langle u, h \rangle, \forall h \in X \right\}.
$$

And it can even be generalized to the class of lower semi-continuous functions [32, 33].

In this section, we give the proof of Theorem 1 by establishing some properties on the nonlinear eigenvalue problem of the function pair $(f^L, g^L)$.

**Definition 3.1.** $\lambda \in \mathbb{R}$ is called an eigenvalue, and $x \in \mathbb{R}^n \setminus \{0\}$ a corresponding eigenvector of the function pair $(f^L, g^L)$ if

$$0 \in \nabla f^L(x) - \lambda \nabla g^L(x).$$

We then also call $(\lambda, x)$ an eigapair.

**Proposition 3.7.** In the setting of the original Lovász extension, for any eigenvalue $\lambda$ of $(f^L, g^L)$, there exists $A \in \mathcal{P}(V) \setminus \{\emptyset\}$ such that $\lambda = f(A)/g(A)$ and $1_A$ is a corresponding eigenvector. Indeed, every eigenvalue has an eigenvector in $\{a, b\}^n$, for given distinct real numbers $a$ and $b$. Moreover, we have:

- if $g(V) \neq 0$, then $\lambda = f(V)/g(V)$ is the only possible eigenvalue of $(f^L, g^L)$;
- if $g(V) = 0$ and $(f^L, g^L)$ has at least one eigenvalue, then $f(V) = 0$ and in this case, $(f^L, g^L)$ may have many distinct eigenvalues.

**Proof.** We need the following basic statement.

**Argument.** For a piecewise linear function $F : \mathbb{R}^n \to \mathbb{R}$ with finite pieces, if $F$ is linear on a convex subset $\Omega$, then $\nabla F(x) \subset \nabla F(y)$ for any relative interior point $x$ of $\Omega$ and any relative boundary point $y$ in $\Omega$.

Suppose that $(\lambda, x)$ is an eigapair of $(f^L, g^L)$. Since $f^L$ is one-homogeneous and linear along the direction $1$, we can assume without loss of generality that $x$ lies in the interior of the simplex $\Delta$ with vertices $1_{A_1}, \ldots, 1_{A_k}$, where $A_1, \ldots, A_k$ are upper level sets of $x$. Applying the above argument to the piecewise linear functions $f^L$ and $g^L$, we immediately get $0 \in \nabla f^L(x) - \lambda \nabla g^L(x) \subset \nabla f^L(1_{A_i}) - \lambda \nabla g^L(1_{A_i})$ for any $i = 1, \ldots, k$, meaning that each $1_{A_i}$ is an eigenvector of $(f^L, g^L)$.
In this case, \( f(V) = 0 \), we may take a look at the example \( f(A) = |\partial A| \) and \( g(A) = \min\{\mathrm{vol}(A), \mathrm{vol}(V \setminus A)\} \) defined on a simple graph \( G = (V, E) \). Then the eigenvalue problem of \((f^L, g^L)\) reduces to the problem for the graph 1-Laplacian. And it is known that the 1-Laplacian may have many different eigenvalues [23].

**Proposition 3.8.** In the setting of the disjoint-pair Lovász extension, for any eigenvalue \( \lambda \) of \((f^L, g^L)\), there exists \((A, B) \in \mathcal{P}_2(V) \setminus \{(\emptyset, \emptyset)\} \) such that \( \lambda = f(A, B)/g(A, B) \) and \( 1_A - 1_B \) is a corresponding eigenvector. If we further assume that \( 2f(A, B) = f(A, V \setminus A) + f(V \setminus B, B) \) and \( 2g(A, B) = g(A, V \setminus A) + g(V \setminus B, B) \) for any \((A, B) \in \mathcal{P}_2(V) \setminus \{(\emptyset, \emptyset)\}\), then for any eigenvalue \( \lambda \) of \((f^L, g^L)\), there exists \( A \subset V \) such that \( \lambda = f(A, V \setminus A)/g(A, V \setminus A) \) and \( 1_A - 1_V\setminus A \) is a corresponding eigenvector.

**Proof.** The general result on the disjoint-pair Lovász extension is similar to Proposition 3.7 and thus we omit the proof.

Let’s focus on the special case that \( 2f(A, B) = f(A, V \setminus A) + f(V \setminus B, B) \) and \( 2g(A, B) = g(A, V \setminus A) + g(V \setminus B, B) \) for any \((A, B) \in \mathcal{P}_2(V) \setminus \{(\emptyset, \emptyset)\}\). We shall prove that in this case, typical eigenvectors can be taken from \( \{-1, 1\}^n \).

**Claim.** For any \( A_1 \subset \cdots \subset A_k \) with \( (A_1, A_k) \neq (\emptyset, V) \), \( f^L \) is linear on \( \mathcal{P}(1_A - 1_V \setminus A_i : i = 1, \ldots, k) \).

**Proof.** The absolute comonotonicity of the disjoint-pair Lovász extension implies that \( f^L \) is linear on \( \mathcal{P}(\{1_A - 1_B : i = 1, \ldots, k\}) \) whenever \( A_1 \subset \cdots \subset A_k \) and \( B_1 \subset \cdots \subset B_k \). Thus, for any \( \sigma, \tau : \{1, \ldots, k\} \rightarrow \{1, \ldots, k\} \) with \( \sigma(1) \leq \cdots \leq \sigma(k) \leq \tau(1) \leq \cdots \leq \tau(1) \), \( f^L \) is linear on \( \mathcal{P}(1_{A_{\sigma(i)} - 1_V \setminus A_{\tau(i)} : i = 1, \ldots, k} \).

Let \( \sigma, \tau : \{1, \ldots, k\} \rightarrow \{1, \ldots, k\} \) be bijections. Let \( \mathcal{P}(1_A - 1_V \setminus A_i : i = 1, \ldots, k) \) be a \((k-1)\)-dim simplex, and there are exactly \( \sum_{i=1}^{k-1} \binom{k-1}{i-1} = 2^{k-1} \) simplexes of dimension \((k-1)\). Since

\[
2f^L(1_{A_{\sigma(i)} - 1_V \setminus A_{\tau(i)}}) = 2f(A_{\sigma(i)}, V \setminus A_{\tau(i)}) = f(A_{\sigma(i)}, V \setminus A_{\tau(i)}) + f(A_{\tau(i)}, V \setminus A_{\tau(i)}) = f^L(1_{A_{\sigma(i)} - 1_V \setminus A_{\tau(i)}}) + f^L(1_{A_{\tau(i)} - 1_V \setminus A_{\tau(i)}})
\]

and \( 1_{A_{\sigma(i)} - 1_V \setminus A_{\tau(i)}} + 1_{A_{\tau(i)} - 1_V \setminus A_{\tau(i)}} = 2(1_{A_{\tau(i)} - 1_V \setminus A_{\tau(i)}}) \), \( f^L \) must be linear on the segment \( \mathcal{P}(1_{A_{\sigma(i)} - 1_V \setminus A_{\tau(i)}}, 1_{A_{\tau(i)} - 1_V \setminus A_{\tau(i)}}) \). Therefore, one can check that \( f^L \) is linear on the simplex \( \mathcal{P}(1_A - 1_V \setminus A_i : i = 1, \ldots, k) \).

Given an eigenvalue \( \lambda \), let \( 1_A - 1_B \) be a corresponding eigenvector for some \((A, B) \in \mathcal{P}_2(V) \setminus \{(\emptyset, \emptyset)\}\). By the above claim and argument, it can be verified that both \( 1_A - 1_V \setminus A \) and \( 1_V \setminus B - 1_B \) are eigenvectors w.r.t. \( \lambda \).

**Remark 8.** In Proposition 3.8, the condition \( 2f(A, B) = f(A, V \setminus A) + f(V \setminus B, B) \) for any \((A, B) \in \mathcal{P}_2(V) \) is natural and easy to satisfy. Below, we provide two examples satisfying the condition.

**Example 1.** \( f(A) = \hat{f}(A) + \check{f}(B) \) for some \( \hat{f} : \mathcal{P}(V) \rightarrow \mathbb{R} \) with \( \hat{f}(A) = f(V \setminus A) \) for any \( A \subset V \).

In this case, \( f^L(x) = \hat{f}^L(x) \).

**Example 2.** \( f(A, B) = 1 \) whenever \( A \cup B \neq \emptyset \). In this case, \( f^L(x) = \|x\|_{\infty} \).

One may observe that in the above examples, \( f \) is symmetric, i.e., \( f(A, B) = f(B, A) \), but it is not a sufficient condition for Proposition 3.8. In fact, for symmetric functions \( f \) and \( g \) on \( \mathcal{P}_2(V) \),
not every eigenvalue has an eigenvector possessing the form of $1_A - 1_{V \setminus A}$. In fact, taking $f(A, B) = \#E(A, V \setminus A) + \#E(B, V \setminus B)$ and $g(A, B) = \vol(A \cup B)$, we have $f^L(x) = \sum_{i,j} e_i \cdot e_j$ and $g^L(x) = \sum_{i \in V} \deg(i)x_i$. Letting $V = \{1, 2, 3\}$ and $E = \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$, it is known in [20] that $1_A - 1_{V \setminus A}$ cannot be an eigenvector w.r.t. the largest eigenvalue of the 1-Laplacian, $\forall A \subset V$.

We conclude the following result, which asserts that every vector in $\{-1, 1\}^n$ is an eigenvector of $(f^L, || \cdot ||_\infty)$ if $f^L$ is nonnegative, where $f^L$ indicates the disjoint-pair Lovász extension of $f$.

**Proposition 3.9.** Let $f : \mathcal{P}_2(V) \to [0, +\infty)$ be nonnegative. Then, for any $A \subset V$, $1_A - 1_{V \setminus A}$ is an eigenvector of $(f^L, || \cdot ||_\infty)$.

**Proof.** By the definition of the eigenvalue problem of $(f^L, || \cdot ||_\infty)$, we only need to prove that $\nabla f^L(x) \cap ((-R^n_{sign(x)} \cup R^n_{sign(x)})) \neq \emptyset$ for any $x \in \{-1, 1\}^n$, where $R^n_{sign(x)} := \{y \in \mathbb{R}^n : y_i x_i \geq 0, \forall i\} = \cone(\nabla ||x||_\infty)$. We first assume that $f$ is positive-definite, i.e., $f(A, B) > 0$ whenever $(A, B) \neq (\emptyset, \emptyset)$, and we shall apply the following argument about polar cones to this case.

**Argument.** Let $C$ and $\Omega$ be two convex cones in $\mathbb{R}^n$ such that $\Omega \cap ((-C) \cup C) = \{0\}$. Then $\Omega^* \cap C^* \neq \{0\}$ and $\Omega^* \cap (C^*)^* \neq \{0\}$, where $C^*$ indicates the polar cone of $C$.

Proof. Indeed, $\Omega^* \cap C^* = (\Omega \cup C)^* \supset (\Omega + C)^*$, where $\Omega + C$ is the Minkowski summation of $C$ and $\Omega$. If $\Omega + C = \mathbb{R}^n$, then for any $c \in (-C) \setminus \{0\}$, there exist $a \in \Omega \setminus \{0\}$ and $c' \in C \setminus \{0\}$ such that $a + c' = c$. This implies $a = -c' - c \in (-C) \setminus \{0\}$, which contradicts the condition that $\Omega \cap ((-C) \cup C) = \{0\}$. Therefore, the convex cone $\Omega + C$ is not the whole space $\mathbb{R}^n$, which implies that $(\Omega + C)^* \neq \{0\}$. Consequently, $\Omega^* \cap C^* \neq \{0\}$ and similarly, $\Omega^* \cap (-C^*) \neq \{0\}$. The proof is completed.

Suppose on the contrary, that $\nabla f^L(x) \cap ((-R^n_{sign(x)} \cup R^n_{sign(x)})) = \emptyset$ for some $x \in \{-1, 1\}^n$. Fixing such an $x$, then cone$(\nabla f^L(x)) \cap ((-R^n_{sign(x)} \cup R^n_{sign(x)})) = \{0\}$, and by the above argument, we have cone$(\nabla f^L(x)) \cap R^n_{sign(x)} = 0$ and cone$(\nabla f^L(x)) \cap (-R^n_{sign(x)}) \neq \{0\}$.

However, since $f$ is positive-definite, it is known that cone$(\nabla f^L(x)) \subset T_x(\{y : f^L(y) \leq f^L(x)\})$, meaning that $T_x(\{y : f^L(y) \leq f^L(x)\}) \subset R^n_{sign(x)}$, where $T_x$ represents the tangent cone at $x$.

Now, suppose $x = 1_A - 1_B$ with $A_n \cup B_n = V$. Every permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ determines a sequence $(\{A_i, B_i\} : i = 1, \ldots, n) \subset P_2(V) \setminus \{\emptyset, \Omega\}$ by the iterative construction: $A_1 \cup B_1 = \{\sigma(1)\}$ and $A_{i+1} \cup B_{i+1} = A_i \cup B_i \cup \{\sigma(i+1)\}, i = 1, \ldots, n - 1$.

Since $f(A_i, B_i) > 0$, $f^L(x^i) = 1$ where $x^i := (1_A - 1_B)/f(A_i, B_i), i = 1, \ldots, n$. Also, $T_\infty(\{y : f^L(y) \leq 1\}) = T_x(\{y : f^L(y) \leq f^L(x)\})$. Without loss of generality, we may assume that $x = x^\sigma$.

The definition of $f^L$ yields that $\text{conv}(0, x^1, \ldots, x^\sigma) \subset \{y : f^L(y) \leq 1\}$. We denote by $\triangle_\sigma = \text{conv}(0, x^1, \ldots, x^\sigma)$ since the construction of $x^1, \ldots, x^\sigma$ depends on the permutation $\sigma$. For any $y = \sum_{i=1}^n t_i x_i \in \text{conv}(0, x^1, \ldots, x^\sigma) \setminus \{x\}$, $y - x_{\sigma(n)} x_{\sigma(n)} = -(1-t_n) x_{\sigma(n)} < 0$, and thus $y - x \notin R^n_{sign(x)}$.

Hence, $T_x(\triangle_\sigma) \cap R^n_{sign(x)} = \{0\}$. It follows from the fact $T_x(\{y : f^L(y) \leq 1\}) = \bigcup T_x(\triangle_\sigma)$ that $T_x(\{y : f^L(y) \leq 1\}) \cap R^n_{sign(x)} = \{0\}$. This is a contradiction.

Now we turn to the general case that $f \geq 0$. Take a sequence $(f_n)_{n \geq 1}$ of positive-definite functions on $P_2(V)$ such that $f_n \to f$ as $n$ tends to $+\infty$. Then it can be verified that for any $v_n \in \nabla f_n^L(x)$, all limit points of $(v_n)_{n \geq 1}$ belong to $\nabla f^L(x)$. Now, there exist $u_n \in \nabla ||x||_\infty$ and $\lambda_n = \frac{f_n^L(x)}{||x||_\infty} > 0$ such that $\lambda_n u_n \in \nabla f_n^L(x)$. Then for any limit point $u$ of $(u_n)_{n \geq 1}$, $u \in \nabla ||x||_\infty$ and $\lambda u \in \nabla f^L(x)$ where $\lambda = \lim_{n \to +\infty} \lambda_n$. Therefore, $(x, \lambda)$ is an eigenepair of $(f^L, || \cdot ||_\infty)$.

The proof is completed. □

By Propositions 3.8 and 3.9, we have

**Corollary 3.1.** If $2f(A, B) = f(A, V \setminus A) + f(V \setminus B, B)$ for any $(A, B) \in P_2(V)$, then the set of eigenvalues of $(f^L, || \cdot ||_\infty)$ coincides with $\{f(A, V \setminus A) : A \subset V\}$, and every vector in $\{-1, 1\}^n$ is an eigenvector.

**Remark 9.** The proof of Proposition 3.9 heavily depends on the property that $N_v(X) = -T_v(X)$ for any vertex $v$ of the hypercube $X := \{x : ||x||_\infty \leq 1\}$. Characterizing the class of polytopes satisfying $N_v = -T_v$ for any vertex $v$ remains an open problem, where $N_v$ is the normal cone at $v$ and $T_v$ is the tangent cone at $v$. 
Motivated by Propositions 3.8 and 3.9 we suggest a combinatorial eigenvalue problem for \((f, g)\) as follows:

Given \(A \subset V\) and a permutation \(\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}\), there exists a unique sequence \((\{A_i^\sigma, B_i^\sigma\} : i = 1, \ldots, n) \in \mathcal{P}_2(V) \setminus \{(\emptyset, \emptyset)\}\) satisfying \(A_1^\sigma \subset \cdots \subset A_n^\sigma = A, B_1^\sigma \subset \cdots \subset B_n^\sigma = V \setminus A, A_i^\sigma \cup B_i^\sigma = \{\sigma(1)\}\) and \(A_{i+1}^\sigma \cup B_{i+1}^\sigma = A_i^\sigma \cup B_i^\sigma \cup \{\sigma(i+1)\},\ i = 1, \ldots, n - 1\). Let \(u^{A, \sigma} \in \mathbb{R}^n\) be defined by

\[
u_i^{A, \sigma} = \begin{cases} f(A_{\sigma^{-1}(i)}^\sigma, B_{\sigma^{-1}(i)}^\sigma) - f(A_{\sigma^{-1}(i)-1}^\sigma, B_{\sigma^{-1}(i)-1}^\sigma), & \text{if } i \in A, \\ f(A_{\sigma^{-1}(i)-1}^\sigma, B_{\sigma^{-1}(i)-1}^\sigma) - f(A_{\sigma^{-1}(i)}^\sigma, B_{\sigma^{-1}(i)}^\sigma), & \text{if } i \notin A. \end{cases}\]

Denote by \(S(f, A) = \{u^{A, \sigma} : \sigma \in S_n\}\) and

\[
\nabla f(A, B) := \text{conv} \left( \bigcup_{A : A \subseteq A \subseteq V \setminus B} S(f, A) \right), \ \forall (A, B) \in \mathcal{P}_2(V)
\]

where \(S_n\) is the permutation group over \(\{1, \ldots, n\}\).

**Definition 3.2** (Combinatorial eigenvalue problem). Given \(f, g : \mathcal{P}_2(V) \to \mathbb{R}\), the combinatorial eigenvalue problem of \((f, g)\) is to find \(\lambda \in \mathbb{R}\) and \((A, B) \in \mathcal{P}_2(V) \setminus \{(\emptyset, \emptyset)\}\) such that \(\nabla f(A, B) \cap \lambda \nabla g(A, B) \neq \emptyset\), in which \(\lambda\) is called an eigenvalue, and \((A, B)\) is called an eigenset.

Since it can be verified that \(\nabla f^L(1_A - 1_B) = \nabla f(A, B)\), Proposition 3.8 (or Theorem D) implies that the combinatorial eigenvalue problem for \((f, g)\) is equivalent to the nonlinear eigenvalue problem of \((f^L, g^L)\).

By Propositions 3.7 and 3.8 for a pair of functions \(f\) and \(g\) on \(\mathcal{P}(V)\) (resp., \(\mathcal{P}_2(V)\)), every eigenvalue of the function pair \((f^L, g^L)\) generated by Lovász extension has an eigenset of the form \(1_A\) (resp., \(1_A - 1_B\)) for some \(A \in \mathcal{P}(V) \setminus \{\emptyset\}\) (resp., \((A, B) \in \mathcal{P}_2(V) \setminus \{(\emptyset, \emptyset)\}\)). We call such a set \(A\) (resp., \((A, B)\)) an eigen-set of \((f, g)\). And, we are interested in the eigen-sets and the corresponding eigenvalues, which encode the key information about the data structure generated by the function pair \((f, g)\). The spectrum of \((f^L, g^L)\) provides a way to understand the interaction between data on \(f\) and data on \(g\).

Next, we study the second eigenvalue of the function pair \((f^L, g^L)\), which is closely related to a combinatorial Cheeger-type constant of the form

\[
\text{Ch}(f, g) := \min_{A \in \mathcal{P}(V) \setminus \{\emptyset, V\}} \frac{f(A)}{\min \{g(A), g(V \setminus A)\}}
\]

where \(f : \mathcal{P}(V) \to \mathbb{R}\) is symmetric, i.e., \(f(A) = f(V \setminus A), \forall A\), and \(g : \mathcal{P}(V) \to \mathbb{R}_+\) is submodular and non-decreasing.

**Proposition 3.10.** Let \(f, g : \mathcal{P}_2(V) \to \mathbb{R}\) be defined by \(f_s(A, B) = f(A) + f(B)\) and \(g_s(A, B) = g(A) + g(B)\). Then

\[
\text{Ch}(f, g) = \text{the second eigenvalue of the function pair } (f^L_s, g^L_s) \text{ (or equivalently } (f_s, g_s))\).
\]

We need the following auxiliary proposition.

**Proposition 3.11.** Suppose that \(g : \mathcal{P}(V) \to \mathbb{R}_+\) is non-decreasing, i.e., \(g(A) \leq g(B)\) whenever \(A \subset B\). Let \(G : \mathbb{R}^n \to \mathbb{R}\) be the disjoint-pair Lovász extension of the function \((A, B) \to g(A) + g(B)\). Then the Lovász extension of the function \(A \mapsto \min \{g(A), g(V \setminus A)\}\) is \(\min_{t \in \mathbb{R}} G(x - t1)\).

**Proof.** We put \(g_m(A) = \min \{g(A), g(V \setminus A)\}\) and \(g_s(A, B) = g(A) + g(B)\), where \(g^L_m\) is the original Lovász extension of \(g_m\), and \(g^L_s\) is the disjoint-pair Lovász extension of \(g_s\). Since \(g\) is non-decreasing, \(g(V^t(x))\) must be non-increasing on \(t \in \mathbb{R}\), i.e., \(g(V^{t_1}(x)) \geq g(V^{t_2}(x))\) whenever \(t_1 \leq t_2\). Hence, there exists \(t_0 \in \mathbb{R}\) such that \(g(V^t(x)) \geq g(V \setminus V^t(x)), \forall t \leq t_0\); and \(g(V^t(x)) \leq g(V \setminus V^t(x)), \forall t \geq t_0\). Then

\[
g^L_m(x) = \int_{\min x}^{\max x} g_m(V^t(x)) dt + \min x g_m(V)
\]
we have

Thus, by the equivalence of submodularity and convexity, where the first equality is based on the fact that $x \perp \perp 1$. Since $x$ is actually the second eigenvalue of the function pair $(f, g)$, which is a slightly extended version of the combinatorial eigenvalue problem for $(f, g)$ (see Definition 3.2). We shall note that (19) is equivalent in some sense to the nonlinear eigenvalue problem:

$$0 \in \sum_i \nabla f_i(A) - \lambda \sum_j \nabla g_j(A).$$

In fact, similar to Propositions 3.7 and 3.8 we have:

The proof is completed.

**Proof of Proposition 3.11.** Since $f$ is symmetric, by Proposition 2.4, $f^L(x) = f^L(x) = f^L_m(x)$, where $f_m$ is defined by $f_m(A) := \min\{f(A), f(V \setminus A)\}$, and $f^L_m$ is the original Lovász extension of $f_m$.

Since $g$ is positive, submodular and non-decreasing, it is not difficult to check that $g$ is bisubmodular. Thus, by the equivalence of submodularity and convexity, $g^L$ is a convex function. Therefore, we have

$$\min_{x:1} \min_{t \in \mathbb{R}} \frac{f^L_s(x)}{\min_{t \in \mathbb{R}} g^L_s(x - t1)} = \min_{x:1} \min_{t \in \mathbb{R}} \frac{f^L_t(x)}{\min_{t \in \mathbb{R}} g^L_t(x - t1)}$$

is actually the second eigenvalue of the function pair $(f^L_s, g^L_s)$. 

Finally, we prove that for any $A, B \neq \emptyset$ with $A \cap B = \emptyset$,

$$\max\left\{ \frac{f(A)}{g(A)}, \frac{f(B)}{g(B)} \right\} \geq \min\left\{ \frac{f(A)}{\min\{g(A), g(V \setminus A)\}}, \frac{f(B)}{\min\{g(B), g(V \setminus B)\}} \right\}.$$

Suppose the contrary, and keep $f(A) = f(V \setminus A)$ in mind. Then, we have $g(A) > g(V \setminus A)$ and $g(B) > g(V \setminus B)$, implying $g(A) + g(B) > g(V \setminus A) + g(V \setminus B)$. Since $A \subset V \setminus B$ and $g$ is non-decreasing, one has $g(A) \leq g(V \setminus B)$. Similarly, $g(B) \leq g(V \setminus A)$, which leads to a contradiction.

Combining all the results and discussions in this section, we complete the proof of Theorem D.
Proposition 3.12. Any associate set-tuple of any eigenvector $x$ of the nonlinear eigenvalue problem \((20)\) is an eigenset of the combinatorial eigenvalue problem \((19)\). Conversely, for any eigenset $A$ satisfying \((19)\), its indicator vector $1_A$ is an eigenvector $x$ satisfying \((20)\). Moreover, the minimum and maximum eigenvalues of \((20)\) and their corresponding eigenvectors are also that of \((f^L,g^L)\).

3.3 Dinkelbach-type schemes and mixed IP-SD algorithms

We would like to establish an iteration framework for finding minimal and maximal eigenvalues. These extremal eigenvalues play significant roles in optimization theory. They can be found via the so-called Dinkelbach iterative scheme \([31]\). This will provide a good starting point for an appropriate iterative algorithm for the resulting fractional programming. Actually, the equivalent continuous optimization has a fractional form, but such kind of fractions have been hardly touched in the field of fractional programming \([75]\), where optimizing the ratio of a concave function to a convex one is usually considered.

Theorem 3.2 (Global convergence of a Dinkelbach-type scheme \([31]\)). Let $S$ be a compact set and let $F,G:S \to \mathbb{R}$ be two continuous functions with $G(x) > 0$, $\forall x \in S$. Then the sequence $\{r^k\}$ generated by the two-step iterative scheme
\[
\begin{align*}
\begin{cases}
x^{k+1} = \arg \text{opti}_{x \in S} \{F(x) - r^kG(x)\}, \\
r^{k+1} = \frac{F(x^{k+1})}{G(x^{k+1})},
\end{cases}
\end{align*}
\]
from any initial point $x^0 \in S$, converges monotonically to a global optimum of $F(\cdot)/G(\cdot)$, where ‘opti’ is ‘min’ or ‘max’.

Corollary 3.2. If $F/G$ is a zero-homogeneous continuous function, then the iterative scheme \((21) (22)\) from any initial point $x^0$ converges monotonically to a global optimum on the cone spanned by $S$ (i.e., $\{tx : t > 0, x \in S\}$).

We note that Theorem 3.2 generalizes Theorem 3.1 in \([21]\) and Theorem 2 in \([24]\). Since it is a Dinkelbach-type iterative algorithm in the field of fractional programming, we omit the proof.

Many minimization problems in the field of fractional programming possess the form
\[
\min \frac{\text{convex} \ F}{\text{concave} \ G},
\]
which is not necessary for a convex programming problem. The original Dinkelbach iterative scheme turns the ratio form to the inner problem \((21)\) with the form like
\[
\min (\text{convex} \ F - \text{concave} \ G),
\]
which is indeed a convex programming problem. However, most of our examples are in the form
\[
\min \frac{\text{convex} \ F}{\text{convex} \ G},
\]
i.e., both the numerator and the denominator of the fractional object function are convex. Since the difference of two convex functions may not be convex, the inner problem \((21)\) is no longer a convex optimization problem and hence might be very difficult to solve.

In other practical applications, we may encounter optimization problems of the form
\[
\min \frac{\text{convex} \ F_1 - \text{convex} \ F_2}{\text{convex} \ G_1 - \text{convex} \ G_2}.
\]
This is NP-hard in general. Fortunately, we can construct an effective relaxation of \((21)\).

The starting point of the relaxation step is the following classical fact:
Proposition 3.13. For any function \( f : A \to \mathbb{R} \), there are two submodular functions \( f_1 \) and \( f_2 \) on \( A \) such that \( f = f_1 - f_2 \).

Although this is an old result, for readers' convenience, we present a short proof below.

Proof. We put
\[
\delta(g) := \min_{A \neq A' \in A} \left( g(A) + g(A') - g(A \cup A') - g(A \cap A') \right).
\]

Recall that a function \( g : A \to \mathbb{R} \) is strictly submodular if \( g(A) + g(A') > g(A \cup A') + g(A \cap A') \) whenever \( A \neq A' \in A \). Since \( A \) has finitely many elements, it is known that there always exists a strict submodular function on \( A \). Clearly, \( g \) is submodular if and only if \( \delta(g) \geq 0 \), while \( g \) is strictly submodular if and only if \( \delta(g) > 0 \). Let \( g : A \to \mathbb{R} \) be strictly submodular, and pick \( C > \max \{ \delta(A) \}_A \). Take \( f_2 = Cg \) and \( f_1 = f + f_2 \). It is clear that \( \delta(f_2) = C\delta(g) > 0 \) and \( \delta(f_1) \geq \delta(f) + \delta(f_2) = \delta(f) + C\delta(g) \geq \delta(f) + \frac{\delta(f)}{\delta(g)}\delta(g) = 0 \). Therefore, we have the decomposition \( f = f_1 - f_2 \), where \( f_2 \) is strictly submodular and \( f_1 \) is submodular. \( \square \)

Thanks to Proposition 3.13, any discrete function can be expressed as the difference of two submodular functions. Since the Lovász extension of a submodular function is convex, every Lovász extension function is the difference of two convex functions.

Then, for the fractional programming derived by Theorem A (or Propositions 3.1 and 3.2), both the numerator and denominator can be rewritten as the differences of two convex functions. This implies that a simple iterative algorithm can be obtained via further relaxing the Dinkelbach iteration by techniques in DC Programming [53]. It should be noted that the following recent works (especially the papers by Hein et al [42–44, 81]) motivated us to investigate more on this direction:

1. The efficient generalization of the inverse power method proposed by Hein et al [42] and the extended steepest descent method by Bresson et al [10] deal with fractional programming in the same spirit. For more relevant papers, we refer to [43] for the RatioDCA method, and [81] for the generalized RatioDCA technique.

2. In [65, 66], the authors address difference convex programming (DC programming) for discrete convex functions, in which an algorithm and a convergence result similar to Theorem 3.3 are presented.

3. A simple iterative algorithm based on the continuous reformulation by the disjoint-pair Lovász extension provides the best cut values for maxcut on a G-set among all existing continuous algorithms [76].

In view of these recent developments, and in order to enlarge the scope of fractional programming and RatioDCA method, it is helpful to study this aspect by general formulations (see also Remark 10 for the most general form). Thus, we begin to establish a method based on convex programming for solving \( \min \frac{F(x)}{G(x)} \) with \( F = F_1 - F_2 \) and \( G = G_1 - G_2 \) being two nonnegative functions, where \( F_1, F_2, G_1, G_2 \) are four nonnegative convex functions on \( X \). For any \( y \in X \), let \( H_y : X \to \mathbb{R} \) be a convex differentiable function such that \( y \) is a minimizer of \( H_y \). For example, we may simply take \( H_y(x) = \| x - y \|^2_2 \). Consider the following three-step iterative scheme

\[
\begin{align*}
\begin{cases}
    x^{k+1} & \in \arg\min_{x \in \mathbb{B}} \{ F_1(x) + r^kG_2(x) - (\langle u^k, x \rangle + r^k(v^k, x)) + H_{x^k}(x) \}, \\
    r^{k+1} & = F(x^{k+1})/G(x^{k+1}), \\
    u^{k+1} & \in \nabla F_2(x^{k+1}), \quad v^{k+1} \in \nabla G_1(x^{k+1}).
\end{cases}
\end{align*}
\]

where \( \mathbb{B} \) is a convex body containing \( 0 \) as its inner point. The following slight modification

\[
\begin{align*}
\begin{cases}
    y^{k+1} & \in \arg\min_{x \in X} \{ F_1(x) + r^kG_2(x) - (\langle u^k, x \rangle + r^k(v^k, x)) + H_{x^k}(x) \}, \\
    r^{k+1} & = F(y^{k+1})/G(y^{k+1}), \quad x^{k+1} = \partial \mathbb{B} \cap \{ ty^{k+1} : t \geq 0 \}, \\
    u^{k+1} & \in \nabla F_2(x^{k+1}), \quad v^{k+1} \in \nabla G_1(x^{k+1}).
\end{cases}
\end{align*}
\]
Case 2.2. For the scheme (25), \(x^{k+1}\) indicates the normalization of \(y^{k+1}\) w.r.t. the convex body \(B\); in particular, \(x^{k+1} := y^{k+1}/\|y^{k+1}\|_2\) if we let \(B\) be the unit ball. These schemes mixing the inverse power (IP) method and steepest descent (SD) method can be well used in computing special eigenpairs of \((F,G)\). Note that the inner problem (24a) (resp. (25a)) is a convex optimization and thus many algorithms in convex programming are applicable. We should note that the above schemes provide a generalization of the RatioDCA technique in [43], and we establish our proof by revising the technique in [12,43].

Theorem 3.3 (Local convergence for the mixed IP-SD scheme). The sequence \(\{r^k\}\) generated by the iterative scheme (24) (resp. (25)) from any initial point \(x^0 \in \text{supp}(G) \cap B\) (resp. \(x^0 \in \text{supp}(G)\)) converges monotonically, where \(\text{supp}(G)\) is the support of \(G\).

Next we further assume that \(X\) is of finite dimension. If one of the following additional conditions holds, then \(\lim_{k \to +\infty} r^k = r^*\) is an eigenvalue of the function pair \((F,G)\) in the sense that it fulfills \(0 \in \nabla F_1(x^*) - \nabla F_2(x^*) - r^* (\nabla G_1(x^*) - \nabla G_2(x^*))\), where \(x^*\) is a cluster point of \(\{x^k\}\).

Case 1. For the scheme (24), \(F_2\) and \(G_1\) are one-homogeneous, and \(F_1\) and \(G_2\) are \(p\)-homogeneous with \(p \geq 1\), and \(H_x\) is constant, \(\forall x \in B\).

Case 2.1. For the scheme (25), \(F_1, F_2, G_1\) and \(G_2\) are one-homogeneous, and \(H_x(x)\) is a continuous function of \(x \in B\) and \(\forall M > 0, \exists C > 0\) such that \(H_x(y) > M\|y\|_2\) whenever \(x \in B\) and \(\|y\|_2 \geq C\).

Theorem 3.3 partially generalizes Theorem 3.4 in [21], and it is indeed an extension of both the IP and the SD method [10,18,42,69].

Proof of Theorem 3.3. It will be helpful to divide this proof into several parts and steps.

Step 1. We may assume \(G(x^k) > 0\) for any \(k\). In fact, the initial point \(x^0\) satisfies \(G(x^0) > 0\). We will show \(F(x^1) = 0\) if \(G(x^1) = 0\) and thus the iteration should be terminated at \(x^1\). This tells us that we may assume \(G(x^k) > 0\) for all \(k\) before the termination of the iteration.

Note that

\[
F_1(x^1) + r^0 G_2(x^1) - \langle u^0, x^1 \rangle + r^0 \langle v^0, x^1 \rangle + H_{x^0}(x^1) \\
\leq F_1(x^0) + r^0 G_2(x^0) - \langle u^0, x^0 \rangle + r^0 \langle v^0, x^0 \rangle + H_{x^0}(x^0),
\]

which implies

\[
F_1(x^1) - F_1(x^0) + r^0 (G_2(x^1) - G_2(x^0)) + H_{x^0}(x^1) - H_{x^0}(x^0) \\
\leq \langle u^0, x^1 - x^0 \rangle + r^0 \langle v^0, x^1 - x^0 \rangle \leq F_2(x^1) - F_2(x^0) + r^0 (G_1(x^1) - G_1(x^0)),
\]

i.e.,

\[
F(x^1) - F(x^0) + H_{x^0}(x^1) - H_{x^0}(x^0) \leq r^0 (G(x^1) - G(x^0)) \\
= -r^0 G(x^0) = -F(x^0).
\]

Since the equality holds, we have \(F(x^1) = 0\), \(H_{x^0}(x^1) = H_{x^0}(x^0)\), \(\langle u^0, x^1 - x^0 \rangle = F_2(x^1) - F_2(x^0)\) and \(\langle v^0, x^1 - x^0 \rangle = G_1(x^1) - G_1(x^0)\). So this step is finished.

Step 2. \(\{r^k\}_{k=1}^{\infty}\) is monotonically decreasing and hence convergent.

Similar to (26) in Step 1, we can arrive at

\[
F(x^{k+1}) - F(x^k) + H_{x^k}(x^{k+1}) - H_{x^k}(x^k) \leq r^k (G(x^{k+1}) - G(x^k)),
\]

which leads to

\[
F(x^{k+1}) \leq r^k G(x^{k+1}).
\]

Since \(G(x^{k+1})\) is assumed to be positive, \(r^{k+1} = F(x^{k+1})/G(x^{k+1}) \leq r^k\). Thus, there exists \(r^* \in [r_{\min}, r^0]\) such that \(\lim_{k \to +\infty} r^k = r^*\), where \(r_{\min} := \min_{x \neq 0} F(x)/G(x)\).
In the sequel, we assume that the dimension of $X$ is finite.

Step 3. $\{x^k\}, \{u^k\}$ and $\{v^k\}$ are sequentially compact.

In this setting, $\mathbb{B}$ must be compact. In consequence, there exist $k_i, r^*, x^*, x^{**}, u^*$ and $v^*$ such that $x^{k_i} \to x^*, x^{k_i+1} \to x^{**}, u^{k_i} \to u^*$ and $v^{k_i} \to v^*$, as $i \to +\infty$.

Clearly, the statements in Steps 1, 2 and 3 are also available for the scheme (25).

Step 4. For the scheme (24), $x^*$ is a minimizer of $F_1(x) + r^*G_2(x) - ((u^*, x) + r^*(v^*, x)) + H_{x^*}(x)$ on $\mathbb{B}$. For the scheme (25), under the additional assumptions introduced in Case 2.1 or 2.2, $x^*$ is a minimum of $F_1(x) + r^*G_2(x) - ((u^*, x) + r^*(v^*, x)) + H_{x^*}(x)$ on $X$.

Let $g(r, y, u, v) = \min_{x \in \mathbb{B}} [F_1(x) + rG_2(x) - ((u, x) + r(v, x)) + H_y(x)]$. It is standard to verify that $g(r, y, u, v)$ is continuous on $\mathbb{R}^1 \times X \times \mathbb{R}^n \times \mathbb{R}^n$ according to the compactness of $\mathbb{B}$.

Suppose the contrary, that $x^*$ is not a minimum of $F_1(x) + r^*G_2(x) - ((u^*, x) + r^*(v^*, x)) + H_{x^*}(x)$ on $\mathbb{B}$. Then

$$F_1(x^*) + r^*G_2(x^*) - ((u^*, x^{**}) + r^*(v^*, x^{**})) + H_{x^*}(x^*)$$

and thus $F(x^*) < r^*G(x^*)$ (similar to Step 1), which implies $G(x^*) > 0$ and $F(x^*)/G(x^*) < r^*$. This is a contradiction. Consequently, $x^*$ is a minimizer of $F_1(x) + r^*G_2(x) - ((u^*, x) + r^*(v^*, x)) + H_{x^*}(x)$ on $\mathbb{B}$.

On the scheme (25), we refer to Cases 2.1 and 2.2 below for details.

Next, we will verify that $(r^*, x^*)$ is an eigenpair under certain additional conditions.

Case 1. On the scheme (24), $F_2$ and $G_1$ are one-homogeneous, and $F_1$ and $G_2$ are $p$-homogeneous with $p \geq 1$, and $H_x = \text{Const}$, $\forall x \in \mathbb{B}$.

Since $H_{x^k} = \text{Const}$, the above claim shows that $x^*$ is a minimizer of $F_1(x) + r^*G_2(x) - ((u^*, x) + r^*(v^*, x))$ on $\mathbb{B}$. Also, since $F_2$ and $G_1$ are one-homogeneous, the Euler identity on homogeneous functions gives $F_1(x^*) + r^*G_2(x^*) - ((u^*, x^*) + r^*(v^*, x^*)) = F_1(x^*) + r^*G_2(x^*) - (F_2(x^*) + r^*G_1(x^*)) = F(x^*) - r^*G(x^*) = 0$. Thus, $F_1(x) + r^*G_2(x) - ((u^*, x) + r^*(v^*, x)) \geq 0, \forall x \in \mathbb{B}$, and the equality holds when $x = x^*$.

Since $\mathbb{B}$ contains 0 as its inner point, we have $\{ax : x \in \mathbb{B}, a \geq 1\} = X$. Keeping $a \geq 1$ and $p \geq 1$ in mind, for any $a \geq 1$ and $x \in \mathbb{B}$,

$$F_1(ax) + r^*G_2(ax) - ((u^*, ax) + r^*(v^*, ax))$$

$$\geq (\alpha^p - \alpha)(F_1(x) + r^*G_2(x))$$

(by Step 4)

Consequently, $x^*$ is a minimizer of $F_1(x) + r^*G_2(x) - ((u^*, x) + r^*(v^*, x))$ on $X$, and thus

$$0 \in \nabla_{x=x^*} (F_1(x) + r^*G_2(x) - ((u^*, x) + r^*(v^*, x)))$$

$$= \nabla F_1(x^*) + r^*\nabla G_2(x^*) - u^* - r^*v^*$$

$$\subset \nabla F_1(x^*) - \nabla F_2(x^*) + r^*\nabla G_2(x^*) - r^*\nabla G_1(x^*)$$.

Case 2.1. On the scheme (25), $F_1, F_2, G_1$ and $G_2$ are $p$-homogeneous with $p > 1$.

Denote by $B : X \to [0, +\infty)$ the unique convex and one-homogeneous function satisfying $B(\partial \mathbb{B}) = 1$. Then the normalization of $x$ in (25) can be expressed as $x/B(x)$.

The compactness of $\{x : B(x) \leq 1\}$ and the upper semi-continuity and compactness of subderivatives imply that $\bigcup_{x : B(x) \leq 1} \nabla F_2(x)$ and $\bigcup_{x : B(x) \leq 1} \nabla G_1(x)$ are bounded sets. So, we
have a uniform constant $C_1 > 0$ such that $\|u\|_2 + r^* \|v\|_2 \leq C_1$, $\forall u \in \nabla F_2(x), v \in \nabla G_1(x)$, $\forall x \in \mathbb{B}$. Let $C_2 > 0$ be such that $\|x\|_2 \leq C_2 B(x)$, and $C_3 = \min_{B(x)=1} F_1(x) > 0$ (here we assume without loss of generality that $F_1(x) > 0$ whenever $x \neq 0$). For any $x$ with $B(x) \geq \max\{2, (2C_1C_2/C_3)^{\frac{1}{p-1}}\}$, and for any $x^* \in \mathbb{B}$, $u^* \in \nabla F_2(x^*)$, $v^* \in \nabla G_1(x^*)$,

$$F_1(x) + r^* G_2(x) - (\langle u^*, x \rangle + r^* \langle v^*, x \rangle) + H_{x^*}(x)$$

$$= B(x)^p F_1\left(\frac{x}{B(x)}\right) + r^* B(x)^p G_2\left(\frac{x}{B(x)}\right) - (\|x\|_2^2 \langle u^*, \frac{x}{\|x\|_2} \rangle + r^* \|x\|_2 \langle v^*, \frac{x}{\|x\|_2} \rangle) + H_{x^*}(x)$$

$$\geq B(x)^p F_1\left(\frac{x}{B(x)}\right) - \|x\|_2^2 \|u^*\|_2^2 + r^* \|v^*\|_2^2 + H_{x^*}(x)$$

$$\geq B(x)^p C_3 - C_2 C_1 B(x) + H_{x^*}(x^*) = B(x)(B(x)^{p-1} C_3 - C_2 C_1) + H_{x^*}(x^*) > H_{x^*}(x*)$$

$$= F_1(x^*) + r^* G_2(x^*) - (\langle u^*, x^* \rangle + r^* \langle v^*, x^* \rangle) + H_{x^*}(x^*)$$

which means that the minimizers of $F_1(x) + r^* G_2(x) - (\langle u^*, x \rangle + r^* \langle v^*, x \rangle) + H_{x^*}(x)$ exist and they always lie in the bounded set $\{x : B(x) < \max\{2, (2C_1C_2/C_3)^{\frac{1}{p-1}}\}\}$. Since $B(x^k) = 1$, $\{y^{k_i}\}$ must be a bounded sequence. There exists $\{k_i \in k\}$ such that $x^{k_i} \to x^*$, $y^{k_i+1} \to y^*$, $x^{k_i+1} \to x^*$ for some $x^*$, $y^*$ and $x^* = y^*/F(y^*)$. Similar to Step 4 and Case 1, $x^*$ is a minimizer of $F_1(x) + r^* G_2(x) - (\langle u^*, x \rangle + r^* \langle v^*, x \rangle) + H_{x^*}(x)$ on $X$, and thus

$$0 \in \nabla F_1(x^*) + r^* \nabla G_2(x^*) - u^* - r^* v^* \subset \nabla F_1(x^*) - \nabla F_2(x^*) + r^* \nabla G_1(x^*) = F_1(x^*) + r^* G_2(x^*) - (\langle u^*, x^* \rangle + r^* \langle v^*, x^* \rangle) + H_{x^*}(x^*).$$

Case 2.2. On the scheme (25), $F_1$, $F_2$, $G_1$ and $G_2$ are one-homogeneous; $H_x(x)$ is continuous of $x \in \mathbb{B}$ and for any $M > 0$, there exists $C > 0$ such that $H_{x}(y) > M \cdot B(y)$ whenever $x \in \mathbb{B}$ and $B(y) \geq C$.

Taking $M = C_1 C_2 + 2$ in which the constants $C_1$ and $C_2$ are introduced in Case 2.1, there exists $C > \max\{\max_{x \in \mathbb{B}} H_x(x), 1\}$ such that $H_{x^*}(x) \geq M \cdot B(x)$ whenever $x^* \in \mathbb{B}$ and $B(x) \geq C$.

Similar to Case 2.1, for any $x^* \in \mathbb{B}$, $x \in X$ with $B(x) \geq C$, and $\forall u^* \in \nabla F_2(x^*)$, $v^* \in \nabla G_1(x^*)$,

$$F_1(x) + r^* G_2(x) - (\langle u^*, x \rangle + r^* \langle v^*, x \rangle) + H_{x^*}(x)$$

$$> B(x)(C_3 - C_2 C_1 + (1/C_1 C_2 + 2) \cdot B(x)) \geq 2 B(x) > H_{x^*}(x^*)$$

$$= F_1(x^*) + r^* G_2(x^*) - (\langle u^*, x^* \rangle + r^* \langle v^*, x^* \rangle) + H_{x^*}(x^*).$$

The remaining part can refer to Case 2.1.

Remark 10. As some direct extensions of the so-called generalized RatioDCA in [81], we have the following modified schemes:

$$\left\{ \begin{array}{l}
x^{k+1} \in \arg \min_{x \in \mathbb{B}} F_1(x) + r^k G_2(x) - (\langle u^k, x \rangle + r^k \langle v^k, x \rangle) + H_{x^*}(x) \quad \text{if} \ r^k \geq 0, \quad (27a) \\
x^{k+1} \in \arg \min_{x \in \mathbb{B}} G_1(x) - \langle w^k, x \rangle - \frac{1}{r^k} (F_1(x) - \langle u^k, x \rangle) + H_{x^*}(x) \quad \text{if} \ r^k < 0, \quad (27b) \\
r^{k+1} = F(x^{k+1})/G(x^{k+1}), \quad (27c) \\
u^{k+1} \in \nabla F_2(x^{k+1}), \quad v^{k+1} \in \nabla G_1(x^{k+1}), \quad w^{k+1} \in \nabla G_2(x^{k+1}) \quad (27d) \end{array} \right.$$
in which the previous assumption $F_1 - F_2 \geq 0$ in [24] and [25] has been removed. For these modifications, a convergence property like Theorem 3.3 still holds.

Remark 11. Theorem 3.3 shows the local convergence of a general relaxation of Dinkelbach’s algorithm in the spirit of DC programming. The DC programming consists in minimizing $F - G$ where $F$ and $G$ are convex functions. As described in [63.60], both the original DC algorithm and its discrete version can be written as the simple iteration: $u^k \in \nabla G(x^k)$, $x^{k+1} \in \nabla F^*(u^k)$, where $F^*$ is the Fenchel conjugate of $F$. It is known that such an iteration is equivalent to the following scheme

$$
\begin{align*}
  x^{k+1} &\in \arg \min_{x} F(x) - \langle u^k, x \rangle, \\
  u^{k+1} &\in \nabla G(x^{k+1}).
\end{align*}
$$

Moreover, a slight variation of the above scheme by adding a normalization step

$$
\begin{align*}
  x^{k+1} &\in \arg \min_{x \in \mathbb{R}^n} F(x) - \langle u^k, x \rangle, \\
  x^{k+1} &\in \frac{1}{G(x^{k+1})} \nabla G(x^{k+1}), \\
  u^{k+1} &\in \nabla G(x^{k+1}).
\end{align*}
$$

can be used to solve the fractional programming $\min F/G$, where $F$ and $G$ are convex and $p$-homogeneous with $p > 1$. This scheme is nothing but Algorithm 2 in [42]. In fact, we can say more about it.

Proposition 3.14. Let $F$ and $G$ be convex, $p$-homogeneous and positive-definite functions on $\mathbb{R}^n$, where $p > 1$. Then, for any initial point $x^0$, the sequence of the pairs $\{(r^k, x^k)\}_{k \geq 1}$ produced by the following scheme

$$
\begin{align*}
  x^{k+1} &\in \arg \min_{x \in \mathbb{R}^n} F(x) - a_k \langle u^k, x \rangle, \\
  x^{k+1} &\in \frac{1}{G(x^{k+1})} \nabla G(x^{k+1}), \\
  u^{k+1} &\in \nabla G(x^{k+1}),
\end{align*}
$$

converges to an eigenpair $(r^*, x^*)$ of $(F, G)$ in the sense that $\lim_{k \to +\infty} r^k = r^*$ and $x^*$ is a limit point of $\{x^k\}_{k \geq 1}$, whenever $a_k, b_k > 0$ as well as both $\{a_k\}_{k \geq 1}$ and $\{b_k x^k\}_{k \geq 1}$ are bounded away from 0 and $\infty$.

The proof is very similar to the original proof of Theorem 3.1 in [42], with an additional trick like the proof of Case 2.1 in Theorem 3.3. It can be regarded as a supplement of both Theorem 3.1 in [42] and Theorem 3.3. It is also interesting that the scheme is stable under perturbations of $a_k$ and $b_k$. Besides, it can be seen that the resulting eigenvalue $r^*$ should be independent of the choice of $a_k$ and $b_k$. In fact, $r^*$ only depends on the initial data and the choice of subgradient $u^k$. The assumption that $F$ is positive-definite can be removed in some sense. Indeed, if $r^k \leq 0$ for some $k$, we can modify (31a) as $x^{k+1} \in \arg \min_{x \in \mathbb{R}^n} F(x) - r^k G(x)$ or $\dot{x}^{k+1} \in \arg \min_{x \in \mathbb{R}^n} G(x) - \frac{1}{r^k} F(x)$ when $r^k < 0$.

Then $\{x^k\}$ converges to the global minimum of $F/G$.

Now, we apply the above mixed IP-SD scheme to fractional combinatorial optimization problems. By the results in Section 3.4, any combinatorial optimization in ratio form can be translated to fractional programming of the form [23] via multi-way Lovász extensions. Then, applying the mixed IP-SD scheme to the resulting optimization, we get a solution of the equivalent continuous optimization. And it is surprising that such a continuous solution can produce a combinatorial solution of the original problem directly, as precisely described in the following proposition.

Proposition 3.15. Given $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}_+$, where $A = \mathcal{P}(V)$ or $\mathcal{P}_2(V)$ or $\mathcal{P}_k(V)$ or $\mathcal{P}_k^d(V)$, let $F = f^L$, $G = g^L$, and take $x^0 = 1_A$ in the iteration scheme (21) or (22) for some $A \in A$. Suppose that $x^*$ is a limit point of the iterative sequence $\{x^k\}$ obtained by (21) or (22). Then, any associated set-tuple $A^*$ of $x^*$ is an eigen-set of the corresponding combinatorial eigenvalue problem [32], and there holds $f(A^*)/g(A^*) \leq f(A)/g(A)$. 32
Proof. It suffices to consider the case that \( F = f_1 - f_2 \) and \( G = g_1 - g_2 \) are one-homogeneous in Theorem 3.3 where \( f = f_1 - f_2 \) and \( g = g_1 - g_2 \) are submodular decompositions. Then, from any initial point \( x^0 := 1_A \), either (24) or (25) provides a solution \( x^* \) which must be an eigenvector of the nonlinear eigenvalue problem

\[
0 \in \nabla f_1(x^*) - \nabla f_2(x^*) - r^*(\nabla g_1(x^*) - \nabla g_2(x^*)).
\]

Similar to the proof of Proposition 3.7 or simply using Proposition 3.12 for any associate set-tuple \( A^* \) of \( x^* \), the indicator vector \( 1_{A^*} \) also satisfies

\[
0 \in \nabla f_1(1_{A^*}) - \nabla f_2(1_{A^*}) - r^*(\nabla g_1(1_{A^*}) - \nabla g_2(1_{A^*})).
\]

which can be rewritten in the form of the combinatorial eigenvalue problem

\[
0 \in \nabla f_1(A^*) - \nabla f_2(A^*) - r^*(\nabla g_1(A^*) - \nabla g_2(A^*)).
\] (32)

Moreover,

\[
\frac{f(A^*)}{g(A^*)} = \frac{f^L(1_{A^*})}{g^L(1_{A^*})} = \frac{f^L(x^*)}{g^L(x^*)} \leq \frac{f^L(x^0)}{g^L(x^0)} = \frac{f^L(1_A)}{g^L(1_A)} = \frac{f(A)}{g(A)}
\]

where the inequality is due to Theorem 3.3.

**Advantages of the mixed IP-SD algorithm.** By Proposition 3.15, the advantage of the mixed IP-SD scheme over existing continuous algorithms for solving combinatorial optimization in fractional form is that it provides an iterative solution without rounding, and can be used to improve initially given data. In fact, it should be noted that these two advantages, namely, an iterative solution without rounding, and usage to improve initially given data, do not apply to other continuous algorithms, like semi-definite relaxations \([40,41]\) and its variants \([8]\), spectral cut method \([36,72]\) and its recursive implementations \([80]\), as well as polynomial programming \([79]\).

A special version of the previous mixed IP-SD algorithm has been actually used in some classic graph cut problems \([21,22,42,76]\). Although we have not yet systematically investigated the solution quality and the computational complexity in general, some good numerical simulations have been reported for certain problems, including the Cheeger cut \([15,21,42]\), the dual Cheeger problem \([22]\), and the maxcut problem \([76]\). In particular, in Section 4.2 we will discuss the maxcut problem in detail to illustrate the performance, solution quality and numerical simulations. Successful numerical experiments have shown that the mixed IP-SD iterative algorithm is likely to be efficient, and converges in polynomial time. We propose to investigate the computation time or convergence rates required to obtain the solution in future work.

Furthermore, the mixed IP-SD scheme proposed in this section can be generalized slightly to compute the second eigenvalue of the function pair obtained by the Lovász extension. We refer the reader to Section 2.1 in \([56]\) for a more general description.

Another solver for the continuous optimization \(\min F(x)/G(x)\) is the stochastic subgradient method:

\[
x^{k+1} = x^k - \alpha_k(y^k + \xi^k),\quad y^k \in \nabla F(x^k)/G(x^k),
\]

where \(\{\alpha_k\}_{k \geq 1}\) is a step-size sequence and \(\{\xi^k\}_{k \geq 1}\) is now a sequence of random variables (the “noise”) on some probability space. Theorem 4.2 in \([34]\) shows that under some natural assumptions, almost surely, every limit point of the stochastic subgradient iterates \(\{x^k\}_{k \geq 1}\) is critical for \(F/G\), and the function values \(\{F(x^k)/G(x^k)\}_{k \geq 1}\) converge. Of course, many other continuous optimization algorithms can be applied, and the mixed IP-SD scheme is just one suitable option. It is expected that better algorithms can be designed based on the obtained equivalent continuous optimization problem via our multi-way Lovász extensions.
4 Examples and Applications

4.1 Submodular vertex cover and multiway partition problems

As a first immediate application of Theorem [A] we obtain an easy way to rediscover the famous identity by Lovász, and the two typical submodular optimizations – submodular vertex cover and multiway partition problems.

Example 4.1. The identity \( \min_{A \in \mathcal{P}(V)} f(A) = \min_{x \in [0,1]^V} f^L(x) \) discovered by Lovász in his original paper [L] can be obtained by our result. In fact, \( \min_{A \in \mathcal{P}(V)} f(A) = \min_{x \in [0,1]^V} f^L(x) = \min_{x \in [0,1]^V, \max_{i \in V} x_i = 1} f^L(x) \). Checking this is easy: if \( f \geq 0 \), then \( \min_{x \in [0,1]^V, \max_{i \in V} x_i = 1} f^L(x) = 0 \); if \( f(A) < 0 \) for some \( A \subset V \), then \( \min_{x \in [0,1]^V, \max_{i \in V} x_i = 1} f^L(x) = \min_{x \in [0,1]^V} f^L(x) \).

Vertex cover number A vertex cover (or node cover) of a graph is a set of vertices such that each edge of the graph is incident to at least one vertex of the set. The vertex cover number is the minimal cardinality of a vertex cover. Similarly, the independence number of a graph is the maximal number of vertices not connected by edges. The sum of the vertex cover number and the independence number is the cardinality of the vertex set.

By a variation of the Motzkin-Straus theorem and Theorem [B], the vertex cover number thus has at least two equivalent continuous representations similar to the independence number.

Submodular vertex cover problem Given a graph \( G = (V,E) \), and a submodular function \( f : \mathcal{P}(V) \to [0, \infty) \), find a vertex cover \( S \subset V \) minimizing \( f(S) \).

By Theorem [A]

\[
\min \{ f(S) : S \subset V, S \text{ is a vertex cover} \} = \min_{x \in D} \frac{f^L(x)}{\|x\|_\infty} = \min_{x \in \overline{D}} f^L(x)
\]

where \( D = \{ x \in [0, \infty)^V : V^t(x) \text{ vertex cover}, \forall t \geq 0 \} = \{ x \in [0, \infty)^V : x_i + x_j > 0, \forall \{i,j\} \in E, \{i : x_i = \max_j x_j \} \text{ vertex cover} \} \), and \( \overline{D} = \{ x \in D : \|x\|_\infty = 1 \} = \{ x \geq 0 : x_i + x_j \geq 1, \forall \{i,j\} \in E, \{i : x_i = \max_j x_j \} \text{ vertex cover} \} \). Note that

\[
\text{conv}(\overline{D}) = \{ x : x_i + x_j \geq 1, \forall \{i,j\} \in E, x_i \geq 0, \forall i \in V \}.
\]

Therefore, \( \min_{x \in \text{conv}(\overline{D})} f^L(x) \leq \min \{ f(S) : \text{ vertex cover } S \subset V \} \), which rediscovering the convex programming relaxation.

Submodular multiway partition problem This problem is about to minimize \( \sum_{i=1}^{k} f(V_i) \) subject to \( V = V_1 \cup \cdots \cup V_k \), \( V_i \cap V_j = \emptyset \), \( i \neq j \), \( v_i \in V_i \), \( i = 1, \ldots, k \), where \( f : \mathcal{P}(V) \to \mathbb{R} \) is a submodular function.

Letting \( \mathcal{A} = \{ \text{ partition } (A_1, \ldots, A_k) \text{ of } V : A_i \ni a_i, i = 1, \ldots, k \} \), by Theorem [A]

\[
\min_{(A_1, \ldots, A_k) \in \mathcal{A}} \sum_{i=1}^{k} f(A_i) = \inf_{x \in \mathcal{D}_A} \frac{\sum_{i=1}^{k} f^L(x^{i})}{\|x\|_\infty} = \inf_{x \in \mathcal{D}'} \sum_{i=1}^{k} f^L(x^{i}),
\]

where \( \mathcal{D}_A = \{ x \in [0, \infty)^{kn} : (V^t(x^1), \ldots, V^t(x^k)) \text{ is a partition}, V^t(x^i) \ni a_i, \forall t \geq 0 \} = \{ x \in [0, \infty)^{kn} : x^i = t1_{A_i}, A_i \ni a_i, \forall t \geq 0 \} \), and \( \mathcal{D}' = \{ (x^1, \ldots, x^k) : x^i \in [0, \infty)^V, x^i = 1_{A_i}, A_i \ni a_i \} \). Note that

\[
\text{conv}(\mathcal{D}') = \{ (x^1, \ldots, x^k) : \sum_{v \in V} x^i_v = 1, x^i_{a_i} = 1, x^i_v \geq 0 \}.
\]

So one rediscover the corresponding convex programming relaxation

\[
\min_{x \in \text{conv}(\mathcal{D}')} \sum_{i=1}^{k} f^L(x^i).
\]

34
4.2 Min-cut and Max-cut

Given an undirected weighted graph \((V, E, w)\), the min-cut problem

\[
\min_{S \neq \emptyset, V} |\partial S| := \min_{S \neq \emptyset, V} |E(S, V \setminus S)| = \min_{S \neq \emptyset, V} \sum_{i \in S, j \in V \setminus S} w_{ij}
\]

and the max-cut problem

\[
\max_{S \neq \emptyset, V} |\partial S| := \max_{S \neq \emptyset, V} |E(S, V \setminus S)| = \max_{S \neq \emptyset, V} \sum_{i \in S, j \in V \setminus S} w_{ij}
\]

have been investigated systematically.

**Theorem 4.1.** Let \((V, E, w)\) be a weighted undirected graph. Then, we have the equivalent continuous optimization formulations for the min-cut and max-cut problems:

\[
\begin{align*}
\min_{S \neq \emptyset, V} |\partial S| &= \min_{\lambda_2} \frac{\sum_{ij \in E} w_{ij}|x_i - x_j|}{2\|x\|_\infty} = \tilde{\lambda}_2, \\
\max_{S \neq \emptyset, V} |\partial S| &= \max_{x \neq 0} \frac{\sum_{ij \in E} w_{ij}|x_i - x_j|}{2\|x\|_\infty} = \tilde{\lambda}_{\text{max}},
\end{align*}
\]

where \(\tilde{\lambda}_2\) and \(\tilde{\lambda}_{\text{max}}\) are the second (i.e., the smallest nontrivial) eigenvalue and the largest eigenvalue of the nonlinear eigenvalue problem:

\[
0 \in \nabla \sum_{ij \in E} w_{ij}x_i - x_j - \lambda \nabla 2\|x\|_\infty. \tag{33}
\]

**Proof.** We only prove the min-cut case. It is clear that

\[
\min_{S \neq \emptyset, V} |\partial S| = \min_{A, B \neq \emptyset, A \cap B = \emptyset} \frac{|\partial A| + |\partial B|}{2}
\]

Let \(A = \{(A, B) \in \mathcal{P}_2(V) : A, B \neq \emptyset\}\). Then \(\mathcal{D}_A = \{x \in \mathbb{R}^n : \max_i x_i = -\min_i x_i > 0\}\), and by Theorem A

\[
\min_{S \neq \emptyset, V} |\partial S| = \min_{(A, B) \in \mathcal{D}_A} \frac{|\partial A| + |\partial B|}{2} = \min_{x \in \mathcal{D}_A} \frac{\sum_{ij \in E} w_{ij}|x_i - x_j|}{2\|x\|_\infty}.
\]

In addition, according to Theorem B the set of the eigenvalues of \((f^L, g^L)\) coincides with

\[
\left\{ \begin{array}{l}
 f^L(1_A - 1_{V \setminus A}) \\
 g^L(1_A - 1_{V \setminus A})
\end{array} : A \subset V \right\} = \left\{ \begin{array}{l}
 f(A, V \setminus A) \\
 g(A, V \setminus A)
\end{array} : A \subset V \right\} = \left\{ \frac{|\partial A| + |\partial (V \setminus A)|}{2} : A \subset V \right\} = \{|\partial A| : A \subset V\},
\]

where \(f(A, B) = |\partial A| + |\partial B|\) and \(g(A, B) = 2\). In consequence, \(\min_{S \neq \emptyset, V} |\partial S|\) is the second eigenvalue of \((f^L, g^L)\). The proof is completed.

Eq. (33) shows the first nonlinear eigenvalue problem which possesses two nontrivial eigenvalues that are equivalent to two important graph optimization problems, respectively.

In addition, by our results, we present a lot of equivalent continuous optimizations for the maxcut problem (see Examples 3.1 and 3.2):

\[
\max_{S \subset V} |\partial S| = \max_{x \neq 0} \frac{\sum_{ij \in E} w_{ij}|x_i + x_j - x_i + x_j|^p}{(2\|x\|_\infty)^p}
\]

\[
= \max_{x \neq 0} \frac{\sum_{ij \in E} w_{ij}|x_i - x_j|^p}{(2\|x\|_\infty)^p} = \max_{\|x\|_\infty \leq \frac{1}{2}} \sum_{ij \in E} w_{ij}|x_i - x_j|^p \tag{34}
\]

for any \(p \geq 1\). We shall now show three applications of the above formulation.
CirCut algorithm (by Burer et al [8]). From the equality (34), it is easy to see
\[ \max_{S \subseteq V} |\partial S| = \frac{1}{2^p} \max_{\theta \in \mathbb{R}^n} \sum_{(i,j) \in E} w_{ij} |\cos \theta_i - \cos \theta_j|^p, \]
and taking \( p = 2 \), we have the equivalent formulation of the maxcut problem
\[ \max_{S \subseteq V} |\partial S| = \frac{1}{4} \max_{\theta \in \mathbb{R}^n} \sum_{(i,j) \in E} w_{ij} (\cos \theta_i - \cos \theta_j)^2 = \max_{\theta \in \mathbb{R}^n} \sum_{(i,j) \in E} w_{ij} \sin^2 \frac{\theta_i - \theta_j}{2} \sin^2 \frac{\theta_i + \theta_j}{2}. \tag{35} \]
If we remove the term \( \sin^2 \frac{\theta_i + \theta_j}{2} \) on the right-hand-side of (35), that is, consider instead the continuous relaxation
\[ \max_{\theta \in \mathbb{R}^n} \sum_{(i,j) \in E} w_{ij} \sin^2 \frac{\theta_i - \theta_j}{2} = \frac{1}{2} \sum_{(i,j) \in E} w_{ij} - \min_{\theta \in \mathbb{R}^n} \sum_{(i,j) \in E} w_{ij} \cos(\theta_i - \theta_j) \underset{\theta \in \mathbb{R}^n \text{ to } \mathbb{R}^n}{\implies} \min_{\theta \in \mathbb{R}^n} \sum_{(i,j) \in E} w_{ij} \cos(\theta_i - \theta_j) \]
we immediately recover the CirCut algorithm proposed by Burer, Monteiro and Zhang [8], which is a smart relaxation of the maxcut problem. Until now, it is still one of the best algorithms for solving maxcut in terms of numerical experiments. Burer et al [8] consider their method as a rank-two relaxation of the Goemans-Williamson algorithm, where the latter is a semi-definite relaxation of the maxcut problem. Thus, our new formulation (35) indeed provides an alternative perspective to the Burer-Monteiro-Zhang’s CirCut algorithm.

A simple iterative algorithm. Based on the case of taking \( p = 1 \) in (34), there is a previous work on computing the maxcut problem by the second author and his collaborators [76], in which the mixed IP-SD algorithm is essentially used. Specifically, the simple iterative algorithm in [76] is indeed an implementation of the mixed IP-SD scheme in Section 3.3 by taking \( H = 0 \), \( F(x) = F_1(x) = \sum_{(i,j) \in E} w_{ij} |x_i - x_j| \) and \( G(x) = G_1(x) = \|x\|_\infty \) in the iterative scheme (24).

We will briefly report the performance of the mixed IP-SD algorithm applied to the maxcut problem, which is presented in [76]. As discussed in Section 5.3, our algorithm is completely rounding-free, whereas almost all other algorithms require additional explicit or implicit rounding operations; for example, the Procedure-CUT operation in the CirCut algorithm of Burer et al [8] can be seen as an implicit rounding technique. More importantly, the iterative values obtained by our mixed IP-SD algorithm are monotonic to the equivalent continuous objective function of the maxcut problem and can be used for post-processing to improve the quality of the solution obtained by any other algorithms. In particular, we would like to point out that our algorithm does improve the cuts obtained by the CirCut algorithm, while conversely the CirCut algorithm cannot improve the quality of the solutions produced by our mixed IP-SD algorithm (see Section 4.4 in [76] for a detailed comparison and illustration). These numerical experiments in [76] show that the mixed IP-SD algorithm is efficient, and converges in polynomial time, and is one of the best continuous iterative algorithms for the maxcut problem.

A new geometric perspective for the Goemans-Williamson algorithm. In addition, our equivalent continuous reformulation of the maxcut problem also provides a new geometric perspective for Goemans-Williamson’s SDP algorithm, via the following relations:
\[ \max_{\|x\|_\infty \leq 1} \sum_{(i,j) \in E} w_{ij} |x_i - x_j|^2 = \frac{1}{n} \max_{\|x\|_\infty \leq 1} \sum_{(i,j) \in E} w_{ij} \|x^i - x^j\|^2_2 = \max_{\|x\|_\infty \leq 1/\sqrt{n}} \sum_{(i,j) \in E} w_{ij} \|x^i - x^j\|^2_2 \]
\[ \leq \max_{\|x\|_2 \leq 1} \sum_{(i,j) \in E} w_{ij} \|x^i - x^j\|^2_2 = \max_{\|x\|_2 = 1} \sum_{(i,j) \in E} w_{ij} \|x^i - x^j\|^2_2 \]
\[ = 2 \sum_{(i,j) \in E} w_{ij} - 2 \min_{\|x\|_2 = 1} \langle x^i, x^j \rangle \]
where the inequality is based on the fact that \( \|x\|_\infty \leq 1/\sqrt{n} \) implies \( \|x\|_2 \leq 1 \), and the second-to-last equality is due to the convexity of the relaxed objective function. According to the above inequality, we can see that the famous Goemans-Williamson algorithm for the maxcut problem actually relaxes the constraint domain from the \( l^\infty \)-ball (i.e., a hypercube) to its circumscribed sphere.
4.3 Max k-cut problem

The max k-cut problem is to determine a graph k-cut by solving

\[
\text{MaxC}_k(G) = \max_{\text{partition } (A_1,A_2, \ldots, A_k)\text{ of } V} \sum_{i \neq j} |E(A_i, A_j)| = \max_{(A_1,A_2, \ldots, A_k)\in \mathcal{P}(V)} \sum_{i=1}^{k} |\partial A_i|, \tag{36}
\]

where \( C_k(V) = \{(A_1, \ldots, A_k) | A_i \cap A_j = \emptyset, \bigcup_{i=1}^{k} A_i = V\} \), and \( \partial A_i := E(A_i, V \setminus A_i) \). We may write (36) as

\[
\text{MaxC}_k(G) = \max_{(A_1,A_2, \ldots, A_k)\in \mathcal{P}(V)} \sum_{i=1}^{k-1} |\partial A_i| + |\partial (A_1 \cup \cdots \cup A_{k-1})|.
\]

Taking \( f_k(A_1, \ldots, A_k) = \sum_{i=1}^{k} |\partial A_i| + |\partial (A_1 \cup \cdots \cup A_k)| \), the k-way Lovász extension is

\[
f_k^k(x^1, \ldots, x^k) = \sum_{i=1}^{k} \sum_{i \sim j} |x^i_j - x^j_i| + \sum_{j=1}^{k} \max_{i=1, \ldots, k} x^i_j - \max_{i=1, \ldots, k} x^j_i.
\]

Applying Theorem A we have

\[
\text{MaxC}_{k+1}(G) = \max_{x^1 \in \mathbb{R}^n_{\geq 0} \setminus \{0\}, \text{supp}(x^1) \cap \text{supp}(x^i) = \emptyset} \frac{\sum_{i=1}^{k} \sum_{i \sim j} |x^i_j - x^j_i| + \sum_{j=1}^{k} \max_{i=1, \ldots, k} x^i_j - \max_{i=1, \ldots, k} x^j_i}{\max_{i,j} x^i_j}
\]

Also, it is clear that

\[
\text{MaxC}_k(G) = \max_{(A_1,A_2, \ldots, A_k)\in \mathcal{P}(V)} \sum_{i \neq j} |E(A_i, A_j)| = \max_{(A_1,A_2, \ldots, A_k)\in \mathcal{P}(V)} \sum_{i=1}^{k} |\partial A_i|,
\]

and by employing Theorem A the max k-cut constant \( \text{MaxC}_k(G) \) has the following equivalent continuous reformulations:

\[
\max_{x^1 \in \mathbb{R}^n_{\geq 0} \setminus \{0\}, \text{supp}(x^1) \cap \text{supp}(x^i) = \emptyset} \frac{\sum_{i=1}^{k} \sum_{i \sim j} |x^i_j - x^j_i|}{\max_{i,j} x^i_j} = \max_{x^1 \in \mathbb{R}^n_{\geq 0} \setminus \{0\}, \text{supp}(x^1) \cap \text{supp}(x^i) = \emptyset} \sum_{i=1}^{k} \sum_{i \sim j} |x^i_j - x^j_i| \max_{i,j} x^i_j
\]

\[
= \max_{x^1 \in \mathbb{R}^n_{\geq 0} \setminus \{0\}, \text{supp}(x^1) \cap \text{supp}(x^i) = \emptyset} \frac{\sum_{i=1}^{k} \sum_{i \sim j} |x^i_j - x^j_i|}{2 \max_{i} \|x^i\|_{\infty}} = \max_{x^1 \in \mathbb{R}^n_{\geq 0} \setminus \{0\}, \text{supp}(x^1) \cap \text{supp}(x^i) = \emptyset} \sum_{i=1}^{k} \sum_{i \sim j} |x^i_j - x^j_i| \frac{1}{2 \|x^i\|_{\infty}}
\]

4.4 Relative isoperimetric constants on a subgraph with boundary

Given a finite graph \( G = (V, E) \) and a subgraph, we consider the Dirichlet and Neumann eigenvalue problems for the corresponding 1-Laplacian. For \( A \subseteq V \), put \( \overline{A} = A \cup \partial A \), where \( \partial A \) is the set of points in \( A^c \) that are adjacent to some points in \( A \) (see Fig. 2).

Given \( S \subseteq \overline{A} \), denote the boundary of \( S \) relative to \( A \) by

\[
\partial_A S = \{(u, v) \in E : u \in S \cap A, v \in \partial A \setminus S \text{ or } u \in S, v \in A \setminus S\}.
\]

If \( S \subseteq A \), then \( \partial_A S = \{(u, v) \in E : u \in S, v \in \overline{A} \setminus S\} \).

The Cheeger (cut) constant of the subgraph \( A \) of \( G \) is defined as

\[
h(A) = \min_{S \subseteq \overline{A}} \frac{|\partial_A S|}{\min\{\text{vol}(A \cap S), \text{vol}(A \setminus S)\}}.
\]

A set pair \((S, \overline{A} \setminus S)\) that achieves the Cheeger constant is called a Cheeger cut.
The Cheeger isoperimetric constant of \( A \) is defined as
\[
h_1(A) = \min_{S \subset A} \frac{\| \partial_A S \|}{\text{vol}(S)},
\]
where a set \( S \) achieving the Cheeger isoperimetric constant is called a Cheeger set. In the sequel, we fix \( A \subset V \), and we write \( h(G) \) and \( h_1(G) \) instead of \( h(A) \) and \( h_1(A) \), respectively.

According to our generalized Lovász extension, we have
\[
h_1(G) = \inf_{x \in \mathbb{R}^n \setminus \{0\}, \text{supp}(x) \subset A} \frac{\sum_{i \sim j} |x_i - x_j| + \sum_{i \in A} p_i |x_i|}{\sum_{i \in A} d_i |x_i|},
\]
and
\[
h(G) = \inf_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\sum_{i \sim j, i,j \in A} |x_i - x_j| + \sum_{i \sim j, i \in A, j \in \delta A} |x_i - x_j|}{\inf_{c \in \mathbb{R}} \sum_{i \in A} d_i |x_i - c|}.
\]

Note that the term on the right hand side of (37) can be written as
\[
\inf_{x \in \partial(V \setminus A) \setminus \{0\}} \mathcal{R}_1(x)
\]
which is called the Dirichlet 1-Poincare constant (see [70]) over \( S \), where
\[
\mathcal{R}_1(x) := \frac{\sum_{\{i,j\} \in E} |x_i - x_j|}{\sum_i d_i |x_i|}
\]
is called the 1-Rayleigh quotient of \( x \).

We can consider the corresponding spectral problems.

- Dirichlet eigenvalue problem:
\[
\begin{align*}
\Delta_1 x \cap \mu D \text{ Sgn } x &\neq \emptyset, \quad \text{in } A \\
x &\equiv 0, \quad \text{on } \delta A
\end{align*}
\]
where \( D \) is the diagonal matrix of the vertex degrees, that is,
\[
\begin{align*}
(\Delta_1 x)_i - \mu d_i \text{ Sgn } x_i &\geq 0, \quad i \in A \\
x_i &\equiv 0, \quad i \in \delta A
\end{align*}
\]
whose component form is: \( \exists \ c_i \in \text{ Sgn}(x_i), \ z_{ij} \in \text{ Sgn}(x_i - x_j) \) satisfying \( z_{ji} = -z_{ij} \) and
\[
\sum_{j \sim i} z_{ij} + p_i c_i \in \mu d_i \text{ Sgn}(x_i), \ i \in A,
\]
in which \( p_i \) is the number of neighbors of \( i \) in \( \delta A \).

\footnote{Some authors call it the Dirichlet isoperimetric constant.}
Figure 3: In this example, there are 3 nodal domains of an eigenvector corresponding to the first Dirichlet eigenvalue of the graph 1-Laplacian. Each nodal domain is the vertex set of the 4-order complete subgraph shown in the figure.

Figure 4: In this example, there are 4 nodal domains of an eigenvector corresponding to the second Neumann eigenvalue of the graph 1-Laplacian. Each nodal domain is the vertex set of the 3-order subgraph after removing the center vertex and its edges.

- Neumann eigenvalue problem: There exists $c_i \in \text{Sgn}(x_i)$, $z_{ij} \in \text{Sgn}(x_i - x_j)$ with $z_{ji} = -z_{ij}$ such that
  \[
  \begin{aligned}
  \sum_{j \sim i, j \in A} z_{ij} - \mu d_i c_i &= 0, & i & \in A \\
  \sum_{j \sim i, j \in A} z_{ij} &= 0, & i & \in \delta A.
  \end{aligned}
  \]
  
  For a graph $G$ with boundary, we use $\Delta_D^1(G)$ and $\Delta_N^1(G)$ to denote the Dirichlet 1-Laplacian and the Neumann 1-Laplacian, respectively. Then

**Proposition 4.1.**

\[
h_1(G) = \lambda_1(\Delta_D^1(G)) \quad \text{and} \quad h(G) = \lambda_2(\Delta_N^1(G)).
\]

For a connected graph, the first eigenvector of $\Delta_N^1(G)$ is constant and it has only one nodal domain while the first eigenvector of $\Delta_D^1(G)$ may have any number of nodal domains. In fact, we have:

**Proposition 4.2.** For any $k \in \mathbb{N}^+$, there exists a connected graph $G$ with boundary such that its Dirichlet 1-Laplacian $\Delta_D^1(G)$ has an eigenvector corresponding to $\lambda_1(\Delta_D^1(G))$ with exactly $k$ nodal domains; and there exists a connected graph $G'$ with boundary such that its Neumann 1-Laplacian $\Delta_N^1(G')$ possesses an eigenvector corresponding to $\lambda_2(\Delta_N^1(G'))$ with exactly $k$ nodal domains.

We provide a final comment on the computational aspects for Cheeger constants and 1-Laplacians on graphs. The work [15] shows that the spectral clustering based on the graph $p$-Laplacian for $p \to 1$ generally has a superior performance compared to the standard linear spectral clustering. In their subsequent work [42], the authors also developed an improved method based on the eigenvectors of the graph 1-Laplacian, which can be computed using their nonlinear inverse power method. This method runs faster and produces better cuts, and in fact, this process achieved state-of-the-art results of its time in terms of solution quality and runtime [42]. Their nonlinear inverse power algorithms [15, 42] have been subsumed into our mixed IP-SD scheme in Section 3.3.
4.5 Independence number

The independence number \( \alpha(G) \) of an unweighted and undirected simple graph \( G \) is the largest cardinality of a subset of vertices in \( G \), no two of which are adjacent. It can be seen as an optimization problem \( \max_{S \subseteq V} \#S \) \( \text{s.t.} \ E(S) = \emptyset \). However, such a graph optimization is not global, and the feasible domain seems to be very complicated. But we may simply multiply by a truncated term \( (1 - \#E(S)) \).

The independence number can then be expressed as a global optimization on the power set of vertices:

\[
\alpha(G) = \max_{S \subseteq V} \#S (1 - \#E(S)),
\]

and thus the Lovász extension can be applied.

**Proof of Eq. (38).** Since \( G \) is simple, \( \#S \) and \( \#E(S) \) take values in the natural numbers. Therefore,

\[
\#S (1 - \#E(S)) \begin{cases} 
\leq 0, & \text{if } E(S) \neq \emptyset \text{ or } S = \emptyset, \\
\geq 1, & \text{if } E(S) = \emptyset \text{ and } S \neq \emptyset.
\end{cases}
\]

Thus, \( \max_{S \subseteq V} \#S(1 - \#E(S)) = \max_{S \subseteq V \text{ s.t. } E(S) = \emptyset} \#S = \alpha(G). \)

However, Eq. (38) is difficult to calculate. By the disjoint-pair Lovász extension, it equals

\[
\alpha(G) = \max_{x \neq 0} \left\| x \right\|_1 - \sum_{k \in V, i \neq j} \min \{|x_k|, |x_i|, |x_j|\}
\]

but we don’t know how to further simplify it.

Fortunately, there is a known representation of the independence number as follows, and we present a proof for convenience.

**Proposition 4.3.** The independence number \( \alpha(G) \) of a finite simple graph \( G = (V, E) \) satisfies

\[
\alpha(G) = \max_{S \subseteq V} (\#S - \#E(S)).
\]

**Proof.** Let \( A \) be an independent set of \( G \), then \( \alpha(G) = \#A = \#A - \#E(A) \leq \max_{S \subseteq V} (\#S - \#E(S)) \) because there is no edge connecting points in \( A \).

Let \( B \subset V \) satisfy \( \#B - \#E(B) = \max_{S \subseteq V} (\#S - \#E(S)) \). Assume the induced subgraph \( (B, E(B)) \) has \( k \) connected components, \( (B_i, E(B_i)), i = 1, \ldots, k \). Then \( B = \sqcup_{i=1}^k B_i \) and \( E(B) = \sqcup_{i=1}^k E(B_i). \) Since \( (B_i, E(B_i)) \) is connected, \( \#B_i \leq \#E(B_i) + 1 \) and equality holds if and only if \( (B_i, E(B_i)) \) is a tree. Now taking \( B' \subset B \) such that \( \#(B' \cap B_i) = 1, i = 1, \ldots, k \), then \( B' \) is an independent set and thus

\[
\alpha(G) \geq \#B' = k \sum_{i=1}^k \max (\#B_i - \#E(B_i)) = \sum_{i=1}^k \#B_i - \sum_{i=1}^k \#E(B_i)
\]

\[
= \#(\sqcup_{i=1}^k B_i) - \#(\sqcup_{i=1}^k E(B_i)) = \#B - \#E(B) = \max_{S \subseteq V} (\#S - \#E(S)).
\]

As a result, Eq. (39) is proved.

According to Lovász extension, we get

\[
\alpha(G) = \max_{x \neq 0} \frac{||x||_1 - \sum_{i \neq j} \min \{|x_i|, |x_j|\}}{||x||_\infty}.
\]

By the elementary identities: \( \sum_{i,j} |x_i + x_j| + \sum_{i,j} |x_i - x_j| = 2 \sum_{i \neq j} \max \{|x_i|, |x_j|\} = \sum_{i \neq j} |x_i| - |x_j| + \sum_i \deg_i |x_i| \) and \( \sum_i \deg_i |x_i| = \sum_{i \neq j} \max \{|x_i|, |x_j|\} + \sum_{i \neq j} \min \{|x_i|, |x_j|\}, \) Eq. (40) can be reduced to

\[
\alpha(G) = \max_{x \neq 0} \frac{2||x||_1 + I^-(x) + I^+(x) - 2||x||_{1,\deg}}{2||x||_\infty},
\]

40
where $I^+(x) = \sum_{i<j} |x_i \pm x_j|$ and $\|x\|_{1,\text{deg}} = \sum_{i} \deg_i |x_i|$. One would like to write Eq. (11) as

$$\alpha(G) = \max_{x \neq \emptyset} \frac{I^-(x) + I^+(x) - 2\|x\|_{1,\text{deg}^c}}{2\|x\|_{\infty}},$$

(42)

where $\|x\|_{1,\text{deg}^c} = \sum_{i \in V} (\deg_i - 1)|x_i|$.

**Remark 12.** The maximum clique number can be reformulated in a similar way. In addition, we refer to [124, 14, 78] for some other continuous formulations of the independence number.

**Chromatic number of a perfect graph**  Berge’s strong perfect graph conjecture has been proved in [25]. A graph $G$ is perfect if for every induced subgraph $H$ of $G$, the chromatic number of $H$ equals the size of the largest clique of $H$. The complement of every perfect graph is perfect.

So for a perfect graph, we have an easy way to calculate the chromatic number. In a general simple graph, we refer to Section 4.3 for transforming the chromatic number.

**Maximum matching**  A matching $M$ in $G$ is a set of pairwise non-adjacent edges, none of which are loops; that is, no two edges share a common vertex. A maximal matching is one with the largest possible number of edges.

Consider the line graph $(E, R)$ whose vertex set $E$ is the edge set of $G$, and whose edge set is $R = \{\{e, e’\} : e \cap e’ \neq \emptyset, e, e’ \in E\}$. Then the maximum matching number of $(V, E)$ coincides with the independence number of $(E, R)$. So, we have an equivalent continuous optimization for a maximum matching problem.

Hall’s Marriage Theorem provides a characterization of bipartite graphs which have a perfect matching and the Tutte theorem provides a characterization for arbitrary graphs.

The Tutte-Berge formula says that the size of a maximum matching in a graph is

$$\frac{1}{2} \min_{U \subset V} (\#V + \#U - \# \text{ odd connected components of } G|_{V \cup U}).$$

Can one transform the above discrete optimization problem into an explicit continuous optimization via some extension?

**k-independence number**  The independence number admits several generalizations: the maximum size of a set of vertices in a graph whose induced subgraph has maximum degree $(k - 1)$ [20]; the size of the largest $k$-colourable subgraph [74]; the size of the largest set of vertices such that any two vertices in the set are at short-path distance larger than $k$ (see [39]). For the $k$-independence number involving short-path distance, one can easily transform it into the following two continuous representations:

$$\alpha_k = \max_{x \in \mathbb{R}^n \backslash \{0\}} \frac{\|x\|_1^2}{\|x\|_2^2 - 2 \sum_{\text{dist}(i,j) \geq k + 1} x_i x_j} = \max_{x \in \mathbb{R}^n \backslash \{0\}} \frac{\sum_{\text{dist}(i,j) \leq k} (|x_i - x_j| + |x_i + x_j|) - 2 \sum_{i \in V} (\deg_k(i) - 1)|x_i|}{2\|x\|_{\infty}},$$

where $\deg_k(i) = \# \{j \in V : \text{dist}(j,i) \leq k\}$, $i = 1, \cdots, n$.

**4.6 Various and variant Cheeger problems**

In [43], the equality relating the Cheeger constant on graphs and the second eigenvalue of the graph 1-Laplacian was reproved via Lovász extension. Moreover, an equality relating the dual Cheeger constant on graphs and the first eigenvalue of the signless 1-Laplacian has been obtained by the second author and his coauthors via the disjoint-pair Lovász extension [22, 24]. As the reported results on both analytical properties and numerical experiments are very satisfactory, we believe that the multi-way Lovász extension in our general framework should be useful to obtain more results on other types of discrete Cheeger constants, from which the mixed IP-SD iterative algorithm is expected to be efficient. In this section, several Cheeger-type constants on graphs have been proposed that are different from the classical one. And based on our general multi-way Lovász extension framework, we establish some equivalent continuous representations of these Cheeger-type constants.
Multiplicative Cheeger constant  

For instance

\[ h = \min_{\delta \neq A \subseteq V} \frac{\#E(A, V \setminus A)}{\#A \cdot \#(V \setminus A)}. \]

It is called the normalized cut problem which has many applications in image segmentation and spectral clustering \([3][4][63]\). By Proposition 0.1, the sparsest cut problem is equivalent to solve

\[ \min_{\langle x, 1 \rangle = 0, x \neq 0} \sum_{i,j} |x_i - x_j|. \]

(Weighted) sparest cut problem  

Given non-negative weights \(w_{ij}\) and \(\mu_{ij}\) for \(i, j \in V\), the weighted sparest cut problem is to solve

\[ \min_{\delta \neq A \subseteq V} \frac{\sum_{i \in A, j \in V \setminus A} w_{ij}}{\sum_{i \in A, j \in V \setminus A} \mu_{ij}} \]

which is related to some famous open problems in theoretical computer science such as the Unique Games Conjecture \([3][10][63]\]. By Proposition 0.1, the sparsest cut problem is equivalent to solve

\[ \min \{ \sum_{i,j \in V} w_{ij} |x_i - x_j| : \text{denominator nonzero} \} = \min \{ \sum_{i,j \in V} \mu_{ij} |y^i - y^j|_1 : \text{denominator nonzero} \} \]

which provides a direct way to get the \(l^1\)-metric tight relaxation, and if we replace the \(l^2\)-norm by the squared \(l^2\)-norm with the additional constraint \(\|y^i - y^j\|_1^2 \leq \|y^i - y^j\|_2^2 + \|y^k - y^j\|_2^2\) for all \(i, j, k\), we immediately get the relaxed sparest cut problem.

Isoperimetric profile  

The isoperimetric profile \(IP : \mathbb{N} \rightarrow [0, \infty)\) is defined by

\[ IP(k) = \inf_{A \subseteq V, \#A \leq k} \frac{\#E(A, V \setminus A)}{\#A}. \]

Then by Lovász extension, it is equal to

\[ \inf_{x \in \mathbb{R}^V, 1 \leq \#(supp(x)) \leq k} \frac{\sum_{\{i,j\} \subseteq E} |x_i - x_j|}{\|x\|_1} = \min_{x \in CH_k(\mathbb{R}^V)} \frac{\sum_{\{i,j\} \subseteq E} |x_i - x_j|}{\|x\|_1}, \]

where \(CH_n := \{x \in \mathbb{R}^V, \#(supp(x)) \leq k\}\) is the union of all \(k\)-dimensional coordinate hyperplanes in \(\mathbb{R}^V\).

Modified Cheeger constant  

On a graph \(G = (V, E)\), there are three definitions of the vertex-boundary of a subset \(A \subseteq V\):

\[ \partial_{\text{ext}} A := \{ j \in V \setminus A \mid \{ j, i \} \in E \text{ for some } i \in A \} \]

\[ (43) \]

\[ \partial_{\text{int}} A := \{ i \in A \mid \{ i, j \} \in E \text{ for some } j \in V \setminus A \} \]

\[ (44) \]

\[ \partial_{\text{ver}} A := \partial_{\text{ext}} A \cup \partial_{\text{int}} A = V(E(A, V \setminus A)) = V(\partial_{\text{edge}} A) \]

\[ (45) \]

The external vertex boundary \([43]\) and the internal vertex boundary \([44]\) are introduced and studied recently in \([82][83]\). Research on metric measure space \([46]\) suggests to consider the vertex boundary \([45]\).

Denote by \(N(i) = \{i\} \cup \{ j \in V : \{ i, j \} \in E \}\) the 1-neighborhood of \(i\). Then the Lovász extensions of \(#\partial_{\text{ext}} A\), \(#\partial_{\text{int}} A\) and \(#\partial_{\text{ver}} A\) are

\[ \sum_{i=1}^n \max_{j \in N(i)} x_j - x_i, \sum_{i=1}^n \min_{j \in N(i)} x_j \text{ and } \sum_{i=1}^n \max_{j \in N(i)} x_j - \min_{j \in N(i)} x_j, \]

42
respectively. They can be seen as the ‘total variation’ of \( x \) with respect to \( V \) in \( G \), while the usual edge boundary leads to \( \sum_{(i,j) \in E} |x_i - x_j| \) which is regarded as the total variation of \( x \) with respect to \( E \) in \( G \). Their disjoint-pair Lovász extensions are

\[
\sum_{i=1}^{n} \max_{j \in N(i)} |x_j| - \|x\|_1, \quad \|x\|_1 - \sum_{i=1}^{n} \min_{j \in N(i)} |x_j|, \quad \max_{j \in N(i)} \left( \sum_{i=1}^{n} \left( \max_{j \in N(i)} |x_j| - \min_{j \in N(i)} |x_j| \right) \right).
\]

Comparing with the graph 1-Poincare profile (see [45-47])

\[
P^1(G) := \inf_{(x,1) = 0, x \neq 0} \frac{\sum_{i \in V} \max_{j \sim i} |x_i - x_j|}{\|x\|_1},
\]

we easily get the following

**Proposition 4.4.**

\[
\frac{1}{2} \max \{ h_{\text{int}}(G), h_{\text{ext}}(G) \} \leq P^1(G) \leq h_{\text{ver}}(G) := \min_{A \in \mathcal{P}(V) \setminus \{\emptyset, V\}} \frac{\# \partial_{\text{ver}} A}{\min \{\#(A), \#(V \setminus A)\}}.
\]

where \( h_{\text{int}}(G) \), \( h_{\text{ext}}(G) \) and \( h_{\text{ver}}(G) \) are modified Cheeger constants w.r.t. the type of vertex-boundary.

**Proof.** By Theorem 13

\[
h_{\text{ver}}(G) = \min_{A \in \mathcal{P}(V) \setminus \{\emptyset, V\}} \frac{\# \partial_{\text{ver}} A}{\min \{\#(A), \#(V \setminus A)\}} = \inf_{(x,1) = 0, x \neq 0} \frac{\sum_{i \in V} \max_{j \sim i} |x_i - x_j|}{\min_{t \in \mathbb{R}} \|x - t \cdot 1\|_1} \geq P^1(G).
\]

On the other hand, it is easy to check that \( \min \|x - t \cdot 1\|_1 \geq \frac{1}{2} \|x\|_1 \) whenever \( (x,1) = 0 \). Thus, \( h_{\text{ver}}(G) \leq 2P^1(G) \). The proof is then completed by noting that \( \max \{ h_{\text{int}}(G), h_{\text{ext}}(G) \} \leq h_{\text{ver}}(G) \). \( \square \)

**Remark 13.** We remark here that the numerator term \( \sum_{i \in V} \max_{j \sim i} |x_i - x_j| \) in general is neither the Lovász extension of any discrete function \( f : \mathcal{P}(V) \to \mathbb{R} \) nor the disjoint-pair Lovász extension of any discrete function \( f : \mathcal{P}_2(V) \to \mathbb{R} \).

**Cheeger-like constant** Some further recent results [54] can be also rediscovered via Lovász extension.

A main equality in [54] can be absorbed into the following identities:

\[
\max_{\text{edges } (v,w)} \left( \frac{1}{\deg v} + \frac{1}{\deg w} \right) = \max_{\gamma : E \to \mathbb{R}} \frac{\sum_{v \in V} \frac{1}{\deg v} \left| \sum_{e_{\text{in: input}}} \gamma(e_{\text{in}}) \right| - \sum_{e_{\text{out: output}}} \gamma(e_{\text{out}}) \right|}{\sum_{e \in E} \|\gamma(e)\|_1} = \max_{\Gamma \subseteq \Gamma_{\text{bipartite}}} \frac{\sum_{v \in V} \frac{\deg_{\Gamma}(v)}{\deg_{\Gamma}(v)}}{|E(\Gamma)|}, \quad (46)
\]

where the left quantity is called a Cheeger-like constant [54].

In fact, given \( c_i \geq 0, i \in V \),

\[
\max_{(i,j) \in E} (c_i + c_j) = \max_{E \subseteq E'} \frac{\sum_{(i,j) \in E'} (c_i + c_j)}{\# E'},
\]

and then via Lovász extension, one immediately gets that the above constant equals to

\[
\max_{x \in [0,\infty)^E \setminus \{0\}} \frac{\sum_{(i,j) \in E} x_e (c_i + c_j)}{\sum_{e \in E} x_e}, \quad \max_{x \in [0,\infty)^E \setminus \{0\}} \frac{\sum_{e \in E} x_e}{\sum_{e \in E} x_e}, \quad \max_{x \in [\mathbb{R}^E \setminus \{0\}]} \frac{\sum_{e \in E} x_e}{\sum_{e \in E} x_e}, \quad \max_{x \in [\mathbb{R}^{E'} \setminus \{0\}]} \frac{\sum_{e \in E} x_e}{\sum_{e \in E} x_e}.
\]
Thus, for any family $E \subset \mathcal{P}(E)$ such that $E' \in E \Rightarrow E' \supset \{\{e\} : e \in E\}$, we have

$$\max_{(i,j) \in E} (c_i + c_j) = \max_{x \in \mathbb{R}^{E \setminus \{\emptyset\}}} \frac{\sum_{i \in V} c_i \left| \sum_{e \ni i} x_e \right|}{\sum_{e \in E} |x_e|} = \max_{E' \in E} \frac{\sum_{(i,j) \in E'} (c_i + c_j)}{\# E'},$$

which recovers the interesting equality (46) by taking $c_i = \frac{1}{\deg i}$ and $E$ the collections of all edge sets of bipartite subgraphs.

A similar simple trick gives

$$\min_{(v,w)} \left| N(v) \cap N(w) \right| = \min_{x \in \mathbb{R}^{E \setminus \{\emptyset\}}} \sum_{i \in V} \sum_{e \ni i} |x_e| \cdot \# \text{ triangles containing } e \sum_{e = \{i,j\} \in E} |x_e| \max \{\deg i, \deg j\}.$$

By our general spectral theory for discrete structures [56], we immediately obtain the $k$-way Cheeger inequality and the $k$-way dual Cheeger inequality involving the graph 1-Laplacian [23, 56]. For more results on various types of Cheeger constants on hypergraphs, we refer the reader to [56].

### 4.7 Frustration in signed networks

In this section, we apply our theory to signed graphs, a concept first introduced by Harary [52].

**Definition 4.1.** A signed graph $\Gamma$ consists of a vertex set $V$ and a set $E$ of undirected edges with a sign function

$$s : E \rightarrow \{+1, -1\}. \quad (47)$$

The adjacency matrix of $(\Gamma, s)$, is denoted by $A^s := (s_{ij})_{i,j \in V}$, where $s_{ij} := s(e)$ if $e = \{i, j\} \in E$, and $s_{ij} := 0$ otherwise.

When we replace the sign function $s$ by $-s$, we shall call the resulting graph antisigned.

**Definition 4.2.** The signed cycle $C_m$ (consisting of $m$ vertices that are cyclically connected by $m$ edges) is balanced if

$$\prod_{i=1}^{m} s(e_i) = 1. \quad (48)$$

A signed graph $(\Gamma, s)$ is balanced if every cycle contained in it is balanced. $(\Gamma, s)$ is antibalanced if $(\Gamma, -s)$ is balanced.

The frustration index of a signed graph $\Gamma = (V, E)$ is

$$\min_{x_i \in \{-1, 1\}, \forall i} \sum_{(i,j) \in E} |x_i - s_{ij} x_j|, \quad (49)$$

where $s_{ij} \in \{-1, 1\}$ indicates the sign of the edge $(i, j)$.

The frustration index then vanishes iff the graph is balanced.

**Definition 4.3.** The (normalized) Laplacian $\Delta^s$ of a signed graph is defined by

$$(\Delta^s x)_i := x_i - \frac{1}{\deg i} \sum_{j \sim i} s_{ij} x_j = \frac{1}{\deg i} \sum_{j \sim i} (x_i - s_{ij} x_j) \quad (50)$$

for a vector $x \in \mathbb{R}^V$.

**Remark 14.** The Laplacian thus is of the form $\Delta^s = \text{id} - A^s$, and when we change the signs of all the edges, that is, go from a signed graph to the corresponding antisigned graph, the operator becomes $\Delta^{-s} = \text{id} + A^s$. Therefore, the eigenvalues simply change from $\lambda$ to $2 - \lambda$ (and therefore, also the ordering gets reversed).
By Proposition 3.3, it is easy to verify that every eigenvalue of the function pair \((F, G)\) has an eigenvector in \([-1, 0, 1]^n\), where \(F(x) = \sum_{(i,j) \in E} |x_i - s_{ij}x_j|\) and \(G(x) = \|x\|_{\infty}\). One may relax (49) as

\[
\min_{x \in \{-1,0,1\}^n \setminus \{0\}} \sum_{(i,j) \in E} |x_i - s_{ij}x_j|.
\]

This suggests the eigenvalue problem of \((F(x), \|x\|_{\infty})\) on a signed graph, where \(F(x) = \sum_{(i,j) \in E} |x_i - s_{ij}x_j|\). Below, we show some key properties.

- The coordinate form of the eigenvalue problem \(\nabla \sum_{(i,j) \in E} |x_i - s_{ij}x_j| \cap \lambda \nabla \|x\|_{\infty} \neq \emptyset\) reads as
  \[
  \sum_{j \sim i} z_{ij} = 0, \quad i \in D_0(x),
  \]
  \[
  \sum_{j \sim i} z_{ij} \in \lambda \text{ sign}(x_i) \cdot [0, 1], \quad i \in D_{\pm}(x),
  \]
  \[
  \sum_i |\sum_{j \sim i} z_{ij}| = \lambda,
  \]

  where \(D_{\pm}(x) = \{i \in V | \pm x_i = \|x\|\}\), and \(D_0(x) = \{i \in V | |x_i| < \|x\|\}\).

- All eigenvalues are integers in \(\{0, 1, \cdots, \text{vol}(V)\}\). And each eigenvalue has an eigenvector in \([-1, 0, 1]^n\).

  Proof: This is a direct consequence of Proposition 3.8.

- The largest eigenvalue has an eigenvector in \([-1, 1]^n\).

  Proof: Let \(1_A - 1_B\) be an eigenvector w.r.t. the largest eigenvalue. Note that \(1_A - 1_B = \frac{1}{2}(1_A - 1_{V \setminus A} + 1_{V \setminus B} - 1_B)\). By the convexity of \(F\), we have \(F(1_A - 1_B) \leq \max\{F(1_A - 1_{V \setminus A}), F(1_{V \setminus B} - 1_B)\}\). Hence, either \(1_A - 1_{V \setminus A}\) or \(1_{V \setminus B} - 1_B\) is an eigenvector w.r.t. the largest eigenvalue.

**The frustration index is an eigenvalue.** However, in general, we don’t know which eigenvalue the frustration index is.

Proof: We shall check that for any \(A \subset V\), the binary vector \(x := 1_A - 1_{V \setminus A}\) is an eigenvector w.r.t. the eigenvalue \(\lambda := 2[|E_+(A, V \setminus A)| + |E_-(A)|] + |E_-(V \setminus A)|\), where \(|E_+(A, V \setminus A)|\) indicates the number of positive edges lying between \(A\) and \(V \setminus A\), while \(|E_-(A)|\) denotes the number of negative edges lying in \(A\). Indeed, \(D_+(x) = A\) and \(D_-(x) = V \setminus A\). For \(i \in A\), taking \(z_{ij} = 1\) if \(s_{ij}x_j < 0\); and \(z_{ij} = 0\) if \(s_{ij}x_j > 0\). Similarly, for \(i \in V \setminus A\), letting \(z_{ij} = 0\) if \(s_{ij}x_j < 0\); and \(z_{ij} = -1\) if \(s_{ij}x_j > 0\). It is easy to see that \(z_{ij} \in \text{Sgn}(x_i - s_{ij}x_j)\) and \(z_{ij} + s_{ij}z_{ji} = 0\) for any edge \(ij\). Next, we verify the conditions (53) and (54).

Note that \(\sum_{j \sim i} z_{ij} = \#\{j \in A : ij\ \text{is negative}\} \cup \{j \in V \setminus A : ij\ \text{is positive}\} \in [0, \lambda]\) for \(i \in A\), and \(\sum_{j \sim i} z_{ij} = -\#\{j \in A : ij\ \text{is positive}\} \cup \{j \in V \setminus A : ij\ \text{is negative}\} \in [-\lambda, 0]\) for \(i \in V \setminus A\). Therefore, \(\sum_{i \in V} \sum_{j \sim i} z_{ij} = 2[|E_+(A, V \setminus A)| + |E_-(A)|] + |E_-(V \setminus A)|\).

In particular, for \(x \in \{-1, 1\}^n\) that realizes the frustration index, \(x\) must be an eigenvector, and the frustration index is the corresponding eigenvalue. This fact can also be derived by Proposition 3.9.

- We can use the Dinkelbach-type scheme in Section 3.3 directly to calculate the smallest eigenvalue. When we get an eigenvector \(x\), we can take \(1_{D_+(x)} - 1_{D_-(x)}\) instead of \(x\).

- We construct a recursive method to approximate the frustration index:

  - Input a signed graph \(G\), and use the Dinkelbach-type algorithm to get a subpartition \((U_+, U_-)\) where \(U_+ = D_+(x)\) and \(U_- = D_-(x)\) with \(x\) being an eigenvector w.r.t. the smallest eigenvalue.
Let \( G \) be the signed graph induced by \( V \setminus (U_+ \cup U_-) \), and let \( (U'_+, U'_-) \) be the subpartition found by the Dinkelbach-type algorithm; return \((U_+ \cup U'_+, U_- \cup U'_-)\) or \((U_+ \cup U'_-, U_- \cup U'_+)\), whichever is better.

Repeat the above process, until we get a partition \((V_+, V_-)\) of \( V \), which derives an approximate solution of the frustration index. There are at most \( n \) iterations.

In other words, the relaxation problem (51) can approximate the frustration index (49) in a recursive way. This is inspired by the recursive spectral cut algorithm for the maxcut problem proposed by Trevisan [80].

Next, we show some equivalent continuous representations of the frustration index. Let \( E_+ \) (resp. \( E_- \)) collect all the positive (resp. negative) edges of \((V, E)\). Note that up to a scale factor, (49) is equivalent to solve \( \min_{A \subset V} |E_+(A, V \setminus A)| + |E_-(A)| + |E_-(V \setminus A)| \), where \(|E_+(A, V \setminus A)|\) denotes the number of positive edges between \( A \) and \( V \setminus A \), while \(|E_-(A)|\) indicates the number of negative edges in \( A \). By Lovász extension, the frustration index is equivalent to

\[
|E_-| + \min_{x \neq 0} \frac{\sum_{\{i,j\} \in E_+} |x_i - x_j| + \sum_{i \in V} \deg_i |x_i| - \sum_{\{i,j\} \in E_-} (|x_i - x_j| + |x_i + x_j|)}{\|x\|_{\infty}}.
\]

Also, (49) is equivalent to \( |E_-| + \min_{A \subset V} (|E_+(A, V \setminus A)| - |E_-(A, V \setminus A)|) \), and by Lovász extension, the frustration index equals

\[
|E_-| + \min_{x \neq 0} \frac{\sum_{\{i,j\} \in E_+} |x_i - x_j| - \sum_{\{i,j\} \in E_-} |x_i - x_j|}{2\|x\|_{\infty}}.
\]

One can then apply the Dinkelbach-type scheme in Section 3.3 straightforwardly to compute the frustration index.

**Remark 15.** We should point out that the notion \(|E_+(A)|\) (resp. \(|E_-(A)|\)) indicates the number of positive (resp. negative) edges (unordered pairs) whose vertices are in \( A \). Therefore, in our paper, the values of \(|E_+(A)|\) and \(|E_-(A)|\) are half of those of Atay-Liu [4], in which they count the ordered pairs.

Besides, by Theorem C (or Theorem 3.1), we can derive another continuous formulation of the frustration index:

\[
\min_{A \subset V} |E_+(A, V \setminus A)| + |E_-(A)| + |E_-(V \setminus A)| = \min_{x \neq 0} \frac{\sum_{\{i,j\} \in E_+} |x_i - x_j|^\alpha + \sum_{\{i,j\} \in E_-} (2\|x\|_{\infty} - |x_i - x_j|)^\alpha}{(2\|x\|_{\infty})^\alpha}
\]

whenever \( 0 < \alpha \leq 1 \). It is interesting that by taking \( \alpha \to 0^+ \), we immediately get

\[
\min_{A \subset V} |E_+(A, V \setminus A)| + |E_-(A)| + |E_-(V \setminus A)| = \min_{x \neq 0} \sum_{\{i,j\} \in E_+} \text{sign}(x_i - x_j) + \sum_{\{i,j\} \in E_-} \text{sign}(2\|x\|_{\infty} - |x_i - x_j|).
\]

### 4.8 Modularity measure

For a weighted graph \((V, (w_{ij})_{i,j \in V})\), the modularity measure [81] is defined as

\[
Q(A) = \sum_{i,j \in A} w_{ij} - \frac{\text{vol}(A)^2}{\text{vol}(V)}, \quad \text{where } A \subset V,
\]

and it satisfies the following equalities (see Theorem 3.7 and Theorem 3.9 in [81], respectively)

\[
\max_{A \subset V} Q(A) = \max_{x \neq 0} \frac{\sum_{i,j \in V} (\frac{\text{deg}(i) \text{deg}(j)}{\text{vol}(V)} - w_{ij}) |x_i - x_j|}{4\|x\|_{\infty}^2}
\]
Proposition 4.5. For a weighted graph \((V, w_{ij})_{i,j \in V}\), let \(\tilde{w}_{ij} = w_{ij} - \frac{\deg(i) \deg(j)}{\text{vol}(V)}\). In the signed weighted graph \((V, (\tilde{w}_{ij})_{i,j \in V})\), \(\{i, j\}\) is a positive (resp. negative) edge if \(\tilde{w}_{ij} > 0\) (resp. \(\tilde{w}_{ij} < 0\)). Then, the frustration index of \((V, (\tilde{w}_{ij})_{i,j \in V})\) equals \(2 \left( \sum_{\{i,j\} : \tilde{w}_{ij} < 0} |\tilde{w}_{ij}| - \max_{A \subseteq V} Q(A) \right)\).

Proof. We know from Section 4.7 (or by Theorem [A]) that the frustration index of \((V, (\tilde{w}_{ij})_{i,j \in V})\) equals

\[
2 \left( \sum_{\{i,j\} : \tilde{w}_{ij} < 0} |\tilde{w}_{ij}| + \min_{x \neq 0} \frac{\sum_{i,j \in V} \tilde{w}_{ij} |x_i - x_j|}{\|x\|_\infty} \right).
\]

The proof is then completed by (55). \(\square\)

4.9 Chromatic number

The chromatic number (i.e., the smallest vertex coloring number) of a graph is the smallest number of colors needed to color the vertices so that no two adjacent vertices share the same color. Given a simple connected graph \(G = (V, E)\) with \(#V = n\), its chromatic number \(\gamma(G)\) can be expressed as a global optimization on the \(n\)-power set of vertices:

\[
\gamma(G) = \min_{(A_1, \ldots, A_n) \in \mathcal{P}_n(V)} \left\{ n \sum_{i=1}^{n} \#E(A_i) + \sum_{i=1}^{n} \text{sign}(\#A_i) + n \left( n - \sum_{i=1}^{n} \#A_i \right)^2 \right\}
\]

and similarly, we get the following

Proposition 4.6. The chromatic number \(\gamma(G)\) of a finite simple graph \(G = (V, E)\) satisfies

\[
\gamma(G) = \min_{(A_1, \ldots, A_n) \in \mathcal{P}(V)^n} \left\{ n \sum_{i=1}^{n} \#E(A_i) + \sum_{i=1}^{n} \text{sign}(\#A_i) + n \left( n - \# \bigcup_{i=1}^{n} A_i \right) \right\}
\]
Proof. Let \( f : \mathcal{P}(V)^n \to \mathbb{R} \) be defined by

\[
 f(A_1, \ldots, A_n) = n \sum_{i=1}^{n} \#E(A_i) + \sum_{i=1}^{n} \text{sign}(\#A_i) + n \left( n - \# \bigcup_{i=1}^{n} A_i \right).
\]

Let \( \{C_1, \ldots, C_{\gamma(G)}\} \) be a proper coloring class of \( G \), and set \( C_{\gamma(G)+1} = \cdots = C_n = \emptyset \). Then we have \( E(C_i) = \emptyset \), \( \# \bigcup_{i=1}^{n} C_i = n \), \( \#C_i \geq 1 \) for \( 1 \leq i \leq \gamma(G) \), and \( \#C_i = 0 \) for \( i > \gamma(G) \). In consequence, \( f(C_1, \ldots, C_n) = \gamma(G) \). Thus, it suffices to prove \( f(A_1, \ldots, A_n) \geq \gamma(G) \) for any \( (A_1, \ldots, A_n) \in \mathcal{P}(V)^n \).

If \( \bigcup_{i=1}^{n} A_i \neq V \), then \( f(A_1, \ldots, A_n) \geq n + 1 > \gamma(G) \).

If there exist at least \( \gamma(G) + 1 \) nonempty sets \( A_1, \ldots, A_{\gamma(G)+1} \), then \( f(A_1, \ldots, A_n) \geq \gamma(G) + 1 > \gamma(G) \).

So we focus on the case that \( \bigcup_{i=1}^{n} A_i = V \) and \( A_{\gamma(G)+1} = \cdots = A_n = \emptyset \). If there further exists \( i \in \{1, \ldots, \gamma(G)\} \) such that \( A_i = \emptyset \), then by the definition of the chromatic number, there is \( j \in \{1, \ldots, \gamma(G)\} \setminus \{i\} \) with \( E(A_j) \neq \emptyset \). So \( f(A_1, \ldots, A_n) \geq n + 1 > \gamma(G) \). Accordingly, each of \( A_1, \ldots, A_{\gamma(G)} \) must be nonempty, and thus \( f(A_1, \ldots, A_n) \geq \gamma(G) \).

Also, when the equality \( f(A_1, \ldots, A_n) = \gamma(G) \) holds, one may see from the above discussion that \( A_1, \ldots, A_{\gamma(G)} \) are all independent sets of \( G \) with \( \bigcup_{i=1}^{n} A_i \neq V \). 

Let \( \hat{f} : \mathcal{P}_2(V)^n \to \mathbb{R} \) be defined by

\[
 \hat{f}(A_1^+, A_1^-, \ldots, A_n^+, A_n^-) = \sum_{i=1}^{n} (\text{sign}(\#(A_i^+ \cup A_i^-)) + n\#E(A_i^+ \cup A_i^-)) + n \left( n - \# \bigcup_{i=1}^{n} A_i^+ \cup A_i^- \right).
\]

and based on \([60]\), it is clear that \( \gamma(G) = \min_{(A_1^+, A_1^-, \ldots, A_n^+, A_n^-) \in \mathcal{P}_2(V)^n} \hat{f}(A_1^+, A_1^-, \ldots, A_n^+, A_n^-) \). Note that

\[
 \# \bigcup_{j=1}^{n} V^t(x^j) = \# \{ j \in V : \exists i \text{ s.t. } x_{i,j} > t \} = \sum_{j=1}^{n} \max_{i=1, \ldots, n} x_{i,j} > t = \sum_{i=1}^{n} \max_{j=1, \ldots, n} x_{i,j} > t
\]

So the \( n \)-way Lovász extension of \( \# \bigcup_{i=1}^{n} A_i \) is

\[
 \int_{\min x}^{\max x} \sum_{i=1}^{n} V^t(x^i) dt + \min x \# \bigcup_{i=1}^{n} V(x^i) = \sum_{j=1}^{n} \int_{\min x}^{\max x} 1_{x_{i,j} > t} dt + \min x \# V
\]

\[
 = \sum_{j=1}^{n} (\max_{i=1, \ldots, n} x_{i,j} - \min x) + n \min x
\]

And the \( n \)-way disjoint-pair Lovász extension of \( \# \bigcup_{i=1}^{n} A_i^+ \cup A_i^- \) is \( \sum_{j=1}^{n} \max_{i=1, \ldots, n} |x_{i,j}| = \sum_{j=1}^{n} \|x^j\|_{\infty} \).

The \( n \)-way Lovász extension of \( \text{sign}(\#A_i) \) is

\[
 \int_{\min x}^{\max x} \text{sign}(\#V^t(x^i)) dt + \min x \text{sign}(\#V(x^i)) = \int_{\min x}^{\max x} 1 dt + \min x \text{sign}(\#V)
\]

\[
 = \max_{j=1, \ldots, n} x_{i,j} - \min x + \min x = \max_{j=1, \ldots, n} x_{i,j}
\]

and the \( n \)-way disjoint-pair Lovász extension of \( \text{sign}(\#(A_i^+ \cup A_i^-)) \) is \( \|x^i\|_{\infty} \). Similarly, the \( n \)-way disjoint-pair Lovász extension of \( \#E(A_i^+ \cup A_i^-) \) is \( \sum_{j\sim j'} \min\{|x_{i,j}|, |x_{i,j'}|\} \).

Thus,

\[
 \hat{f}^t(x) = n \sum_{i=1}^{n} \sum_{j \sim j'} \min\{|x_{i,j}|, |x_{i,j'}|\} + \sum_{i=1}^{n} \|x^i\|_{\infty} + n \left( n\|x\|_{\infty} - \sum_{j=1}^{n} \|x^j\|_{\infty} \right)
\]

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According to Proposition 3.1 in the context of the multi-way disjoint-pair Lovász extension, we obtain

\[ \gamma(G) = \min_{(A^+_1, A^-_1, \cdots, A^+_n, A^-_n) \in \mathcal{P}_n(V)} \frac{f(A^+_1, A^-_1, \cdots, A^+_n, A^-_n)}{1} = \min_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\hat{f}(x)}{\|x\|_\infty} \]

\[ = n^2 - \max_{x \in \mathbb{R}^n \setminus \{0\}} \sum_{\{i,j\} \in E} \frac{\sum_{k \in V} (|x_{ik} - x_{jk}| + |x_{ik} + x_{jk}|) + 2n\|x^k\|_\infty - 2n \deg_k \|x^k\|_1 - 2\|x^k\|_\infty}{2\|x\|_\infty}. \]

Clique covering number The clique covering number of a graph $G$ is the minimal number of cliques in $G$ needed to cover the vertex set. It is equal to the chromatic number of the graph complement of $G$. Consequently, we can explicitly write down the continuous representation of a clique covering number by employing Theorem 3.

5 Conclusions and Discussion

The firm bridge between the discrete data world and the continuous mathematical field with well-established mathematics such as analytic techniques, topological schemes and algebraic structures should be tremendously helpful. In [55, 56] and in this paper, we build these fruitful connections in a variety of areas through Lovász extension and some more general discrete-to-continuous extensions. Our contribution is two-fold: the theoretical framework for Lovász-type extensions and the corresponding spectral theory; and their practical applications to the computation of the resulting optimization and eigenvalue problems. Let us describe the contributions of this paper in more specific terms.

Contributions to optimization. Continuous approaches for solving combinatorial optimization problems have been widely used in practice, such as spectral clustering and its recursive versions, SDP-type techniques, and polynomial methods. Overall, continuous approaches can be roughly classified into continuous relaxations and continuous reformulations, where the continuous reformulations are also called the equivalent continuous representations (or tight relaxations) of the original combinatorial problems. However, most of these approaches require certain additional rounding techniques, even for many equivalent continuous formulations. In addition, although some tight relaxations (i.e., equivalent continuous formulation) have been constructed for certain combinatorial optimization problems, many of the constructions are specific and not general enough to be applied to a wider range of combinatorial optimization problems.

Our constructions based on the multi-way Lovász extensions overcome these inconveniences. In fact, the equivalent continuous optimization problem we obtained fully inherits all the local optimal data of the original combinatorial objective function, and therefore fits better with the original combinatorial optimization problem. Therefore, our discrete-to-continuous framework is more convenient and appropriate for obtaining new relaxations and reformulations than many other approaches. Also, the reformulations obtained by multi-way Lovász extension are of simple ratio form, which offer new possibilities for designing continuous optimization algorithms for combinatorial problems in practical terms. In particular, we provide the mixed IP-SD scheme to confirm the effectiveness of our discrete-to-continuous framework, which has worked well in many practical combinatorial optimization problems. For example, in [76] we proposed a simple iterative algorithm for maxcut, which is based on a previous specific version of the mixed IP-SD scheme and which performs very well in numerical experiments. This method can also be used to find specific eigenvalues of a function pair (see [54]). We believe that the mixed IP-SD algorithm is one of the best continuous iterative schemes for solving certain combinatorial optimization problems like independence number, coloring number.
and frustration index, because it fully exploits some new equivalent continuous formulations. It is expected that further efficient continuous optimization algorithms will be designed for combinatorial optimization problems based on our discrete-to-continuous framework.

**Contributions to nonlinear eigenvalue problems.** Nonlinear eigenvalue problems arise in many contexts, including quantum chemistry, physics, engineering, and image processing. Recently, nonlinear operators and the associated spectral theories have allowed for more general, accurate and efficient models and techniques for handling network problems. For example, the 1-Laplace operator on graphs has been successfully applied to spectral clustering with a spectrum that has many good properties and is closely related to multi-way Cheeger constants. However, it is not entirely clear why the 1-Laplacian is good, and whether similar properties can be generalized to other nonlinear operators.

Our framework on multi-way Lovász extensions establishes a systematic and deep spectral theory for a class of nonlinear operators. We prove that the spectrum of the function pair obtained by the multi-way Lovász extension encodes all the key data of the original combinatorial functions, and we particularly characterize the second eigenvalue in terms of combinatorial quantities. This generalizes the important fact that the second eigenvalue of the graph 1-Laplacian equals the graph Cheeger constant. We also provide many other applications, for example, we found that the min-cut and max-cut problems are equivalent to solving the first nontrivial eigenvalue and the largest eigenvalue of a certain nonlinear eigenvalue problem provided by the Lovász extension, respectively. Further progress is collected in [56], and based on these fundamental results, we can analyze the structure of eigenspaces in depth.

**Relations to other works and further remarks.** There are many other applications of Lovász extension beyond this paper, for example, critical point theory for combinatorial functions can be studied with the help of Lovász extension. In [55], we build the relationship between the Morse theory of a discrete Morse function and its Lovász extension. We also propose a combinatorial version of the Lusternik-Schnirelman category on abstract simplicial complexes to bridge the classical Lusternik-Schnirelman theorem and its discrete analog on finite simplicial complexes.

For further applications, we introduce the piecewise multilinear extension in [56], and we provide several min-max relations based on such general extension. The mountain pass characterizations, linking theorems, nodal domain inequalities, inertia bounds, duality theorems and distribution of eigenvalues for pairs of \( p \)-homogeneous functions are derived. In particular, we show a simple one-to-one correspondence between the nonzero eigenvalues of the vertex \( p \)-Laplacian and the edge \( p^* \)-Laplacian of a graph. We also apply the extension theory to Cheeger inequalities and \( p \)-Laplacians on oriented hypergraphs and simplicial complexes, which contribute to the field of expander graph and spectral graph theory. In addition, these results have some applications on tensor eigenvalues, providing a strong spectral estimate for the adjacency tensor of a hypergraph.

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