EXISTENCE OF PERIODIC SOLUTIONS FOR THE PERIODICALLY FORCED SIR MODEL

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We prove that the seasonally-forced SIR model with a \( T \)-periodic forcing has a periodic solution with period \( T \) whenever the basic reproductive number \( R_0 > 1 \). The proof uses the Leray–Schauder degree theory.

1. Introduction

The periodically forced SIR model

\[
\begin{align*}
S' &= \mu(1 - S) - \beta(t)SI, \\
I' &= \beta(t)SI - (\gamma + \mu)I, \\
R' &= \gamma I - \mu R,
\end{align*}
\]

and its versions are extensively used to model seasonally recurrent diseases \([1, 2, 4, 5, 7–14, 16]\). Here, \( S, I, \) and \( R \) are the fractions of the population which are susceptible, infective, and recovered, respectively, \( \mu \) denotes the birth and death rate, \( \gamma \) is the recovery rate, and \( \beta(t) \), which is assumed to be a positive continuous \( T \)-periodic function, is the seasonally-dependent transmission rate (so that \( T \) is the annual period). Since \( S(t), I(t), \) and \( R(t) \) are fractions of the population, we require

\[
S(t) + I(t) + R(t) = 1.
\]

Note that, by adding the equations (1)–(3), we get \( (S(t) + I(t) + R(t))' = 0 \). Hence, if the initial conditions satisfy \( S(0) + I(0) + R(0) = 1 \), then relation (4) holds for all \( t \).

When simulating this model numerically, it is observed that:

(i) If \( R_0 \leq 1 \), where

\[
R_0 = \frac{\bar{\beta}}{\gamma + \mu},
\]

\[
\bar{\beta} = \frac{1}{T} \int_0^T \beta(t) \, dt,
\]

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then all solutions tend to the disease-free equilibrium $S_0 = 1$, $I_0 = 0$, and $R_0 = 0$. This fact can be rigorously proved; see [15].

(ii) If $R_0 > 1$, then depending on the values of the parameters, one observes either the convergence to $T$-periodic orbits, or the convergence to $nT$-periodic orbits with $n > 1$ (subharmonics), or the chaotic behavior.

A fundamental question that is addressed here is the existence of a $T$-periodic solution of the system. We demand, of course, that the components $S(t)$, $I(t)$, $R(t)$ of this solution must be positive. Obviously, for $R_0 \leq 1$, a positive periodic solution cannot exist because this solution would not converge to the disease-free equilibrium. However, we will prove the following theorem:

**Theorem 1.** Whenever $R_0 > 1$, there exists at least one $T$-periodic solution $(S(t), I(t), R(t))$ of (1)–(3), all components of which are positive and satisfy (4).

Thus, when $T$-periodic behavior is not observed in simulations, this is not due to the fact that this solution does not exist but rather due to the fact that all $T$-period solutions are unstable. These unstable periodic solutions cannot be found by direct simulation but they can be computed by numerical continuation techniques.

Despite the fact that the existence of a $T$-periodic solution of the $T$-periodically forced SIR model is a fundamental issue, the only paper in the literature of which we are aware to have dealt with this question is the recent paper of Jódar, Villanueva, and Arenas [9]. They treated a more general system than we do here (including loss of immunity and allowing other coefficients, in addition to $\beta(t)$, to be $T$-periodic). Restricting their existence result to the case of the SIR model (1)–(3), they proved, using Mawhin’s continuation theorem, that a $T$-periodic solution exists whenever the condition

$$\min_{t \in \mathbb{R}} \beta(t) > \gamma + \mu$$

holds. Note that the condition (5) implies that $\bar{\beta} > \gamma + \mu$, that is $R_0 > 1$, but it is a stronger condition. Theorem 1 uses only the condition $R_0 > 1$. Hence, together with the fact noted above that $T$-periodic solutions do not exist when $R_0 \leq 1$, we conclude that $R_0 > 1$ is a necessary and sufficient condition for the existence of a $T$-periodic solution with positive components.

Our technique for proving Theorem 1 is based on nonlinear functional analysis for which we refer the reader to the textbooks [3, 19]. Reformulating the problem as the problem of solving an equation in an infinite-dimensional space of periodic functions, we define a homotopy between the periodically forced problem and the autonomous problem in which $\beta(t)$ is replaced by the mean $\bar{\beta}$. The autonomous problem has an endemic equilibrium, which is a trivial periodic solution. We then employ the Leray–Schauder degree theory to continue this solution along the homotopy. The challenge here lies in the fact that we always have a trivial periodic solution given by the disease-free equilibrium, which lies on the boundary of the relevant domain $D$ in the functional space, and hence, it is necessary to construct a smaller domain $U \subset D$ excluding the trivial solution and to show that the conditions for applying the Leray–Schauder theory hold for the domain $U$.

We note that our proof of Theorem 1 is easily extended to give the same result for the SIR model, which includes the loss of immunity:

$$S' = \alpha S + \mu (1 - S) - \beta(t)SI,$$  

$$I' = \bar{\beta}(t)SI - (\gamma + \mu)I,$$  

$$R' = \gamma I - (\mu + \alpha)R.$$