VARIABILITY REGIONS OF CLOSE-TO-CONVEX FUNCTIONS

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Abstract. M. Biernacki gave concrete forms of the variability regions of \( z/f(z) \) and \( zf'(z)/f(z) \) of close-to-convex functions \( f \) for a fixed \( z \) with \(|z| < 1\) in 1936. The forms are, however, not necessarily convenient to determine the shape of the full variability region of \( zf'(z)/f(z) \) over all close-to-convex functions \( f \) and all points \( z \) with \(|z| < 1\). We will propose a couple of other forms of the variability regions and see that the full variability region of \( zf'(z)/f(z) \) is indeed the complex plane minus the origin. We also apply them to study the variability regions of \( \log [z/f(z)] \) and \( \log [zf'(z)/f(z)] \).

1. Introduction

Let \( \mathcal{A} \) denote the class of analytic functions \( f \) on the unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) and let \( \mathcal{A}_0 \) and \( \mathcal{A}_1 \) be its subclasses described by the conditions \( f(0) = 1 \) and \( f(0) = f'(0) - 1 = 0 \), respectively. Traditionally, the subclass of \( \mathcal{A}_1 \) consisting of univalent functions is denoted by \( \mathcal{S} \). A function \( f \) in \( \mathcal{A}_1 \) is called starlike (resp. convex) if \( f \) is univalent and if \( f(\mathbb{D}) \) is starlike with respect to 0 (resp. convex). It is well known that \( f \in \mathcal{A}_1 \) is starlike (resp. convex) precisely when \( \Re \left[ \frac{zf'(z)}{f(z)} \right] > 0 \) (resp. \( \Re \left[ 1 + zf''(z)/f'(z) \right] > 0 \)) in \(|z| < 1\). The classes of starlike and convex functions in \( \mathcal{A}_1 \) will be denoted by \( \mathcal{S}^* \) and \( \mathcal{K} \) respectively.

A function \( f \in \mathcal{A}_1 \) is called close-to-convex if \( \Re \left[ e^{i\lambda}f'(z)/g'(z) \right] > 0 \) in \(|z| < 1\) for a convex function \( g \in \mathcal{K} \) and a real constant \( \lambda \) with \(|\lambda| < \pi/2\). The set of close-to-convex functions in \( \mathcal{A}_1 \) will be denoted by \( \mathcal{C} \). This class was first introduced and shown to be contained in \( \mathcal{S} \) by Kaplan [6]. A domain is called close-to-convex if it is expressed as the image of \( \mathbb{D} \) under the mapping \( af + b \) for some \( f \in \mathcal{C} \) and constants \( a, b \in \mathbb{C} \) with \( a \neq 0 \). He also gave a geometric characterization in terms of turning of the boundary of the domain. We recommend books [3] and [4] for general reference on these topics.

Prior to the work of Kaplan, Biernacki [2] introduced the notion of linearly accessible domains (in the strong sense). Here, a domain in \( \mathbb{C} \) is called linearly accessible if its complement is a union of half-lines which do not cross each other. Lewandowski [10], [11] proved that the class of close-to-convex domains is identical with that of linearly accessible domains (see also [1] and [8] for simpler proofs of this fact). Therefore, the work of Biernacki on linearly accessible domains and their mapping functions can now be interpreted as that on close-to-convex domains and functions.

For a non-vanishing function \( g \) in \( \mathcal{A}_0 \), unless otherwise stated, \( \log g \) will mean the continuous branch of \( \log g \) in \( \mathbb{D} \) determined by \( \log g(0) = 0 \). For instance, \( f(z)/z \) can be
regarded as a non-vanishing function in \( \mathcal{A}_0 \) for \( f \in \mathcal{S} \). Therefore, we can define \( \log f(z)/z \) in the above sense. In the present note, we are interested in the following variability regions for a fixed \( z \in \mathbb{D} \):

\[
U_z = \{ \frac{z}{f(z)} : f \in \mathcal{C} \}, \quad LU_z = \{ \log \frac{z}{f(z)} : f \in \mathcal{C} \}, \\
V_z = \{ f'(z) : f \in \mathcal{C} \}, \quad LV_z = \{ \log f'(z) : f \in \mathcal{C} \}, \\
W_z = \{ z f'(z) : f \in \mathcal{C} \}, \quad LW_z = \{ \log z f'(z) : f \in \mathcal{C} \}.
\]

We collect basic properties of these sets.

**Lemma 1.1.**

1. \( X_z \) is a compact subset of \( \mathbb{C} \) for each \( z \in \mathbb{D} \) and \( X = U, V, W, LU, LV, LW \).
2. \( X_z = \exp(LX_z) \) for each \( z \in \mathbb{D} \) and \( X = U, V, W \).
3. \( X_z = X_r \) for \( |z| = r < 1 \) and \( X = U, V, W, LU, LV, LW \).
4. \( X_r \subset X_s \) for \( 0 \leq r < s < 1 \) and \( X = U, V, W, LU, LV, LW \).

**Proof.** It is enough to outline the proof since the reader can reproduce the proof easily. Assertion (1) follows from compactness of the family \( \mathcal{C} \), whereas (2) is immediate by definition. To see (3) and (4), it is enough to show that \( X_z \subset X_w \) for \( |z| \leq |w| < 1 \). This follows from the fact that the function \( f_a(z) = f(az)/a \) belongs to \( \mathcal{C} \) again for \( f \in \mathcal{C} \) and \( a \in \mathbb{C} \) with \( 0 < |a| \leq 1 \).

We remark that we can indeed show the stronger inclusion relation \( X_r \subset \text{Int} X_a \) for \( 0 \leq r < s < 1 \) by observing extremal functions corresponding to boundary points of \( X_r \). Here, \( \text{Int} E \) means the set of interior points of a subset \( E \) of \( \mathbb{C} \). However, we do not use this property in what follows.

Set \( X_{1-} = \bigcup_{0 < r < 1} X_r \) for \( X = U, V, W, LU, LV, LW \). In the sequel, \( \mathbb{D}(a, r) \) will stand for the open disk \( |z - a| < r \) in \( \mathbb{C} \) and \( \overline{\mathbb{D}}(a, r) \) will stand for its closure, namely, the closed disk \( |z - a| \leq r \).

Biernacki \([2]\) described \( U_z \) and \( W_z \) in his study on linearly accessible domains and their mapping functions. The results can be summarized as in the following.

**Lemma 1.2** (Biernacki (1936)). For \( 0 < r < 1 \), the following hold:

1. \( U_r = \{(1 + s)^2/(1 + s^2) : |s| \leq r, |t| \leq r \} = \{2u^2/(u + v) : |u - 1| \leq r, |v - 1| \leq r \} \).
2. \( W_r = (1 - r^2)^{-2}U_r \).
3. \( U_{1-} = \mathbb{D}(1, 3) \setminus \{0\} \) and \( LU_{1-} \subset \{ w \in \mathbb{C} : |\text{Im} \ w| < 3\pi/2 \} \).
4. \( LW_{1-} \subset \{ w \in \mathbb{C} : |\text{Im} \ w| < 3\pi/2 \} \).

The above expressions of \( U_r \) and \( W_r \) are simple but somewhat implicit. For instance, the parametrization of the boundary curve cannot be obtained immediately and the shape of the limit \( W_{1-} \) is not clear (as we will see below, this set is equal to \( \mathbb{C} \setminus \{0\} \)). Therefore, it would be nice to have more explicit or more convenient expressions of \( U_r \) and \( W_r \). We propose two such expressions in the present note.

**Theorem 1.3.** For \( 0 < r < 1 \), \( U_r = F(\overline{\mathbb{D}}(0, r)) \), where

\[
F(z) = \frac{(3 + \bar{z})(1 + z)^3}{3 + 3z + \bar{z} + z^2}, \quad z \in \mathbb{D}.
\]
We will prove the theorem by describing explicitly the envelope of the family of circles \( M_s(\partial \mathbb{D}(0,r)) \) for \( s = re^{i\theta}, \ 0 \leq \theta < 2\pi \), where \( M_s \) is the Möbius transformation \( t \mapsto (1 + s)^2/(1 + (s + t)/2) \). Lewandowski [11, p. 45] used the envelope to prove that the inclusion \( U_r \subset \{ w \in \mathbb{C} : \text{Re} \ w \geq 0 \} \) (equivalently, \( W_r \subset \{ w \in \mathbb{C} : \text{Re} \ w \geq 0 \} \)) is valid precisely when \( r \leq 4\sqrt{2} - 5 \). (This implies that the radius of starlikeness of close-to-convex functions is \( 4\sqrt{2} - 5 \).) However, any explicit form of the envelope was not given in [11] because it was not necessary for his results.

We note that \( F(e^{i\theta}) = 1 + 3e^{i\theta} \) for \( \theta \in \mathbb{R} \), which agrees with Lemma 1.2. But, this does not give enough information to determine the boundary curve of the domain \( LU_{1-} \). It turns out that \( LU_{1-} \) has relatively a simple description though \( LU_r \) does not have. We indeed derive the following result by making use of Theorem 1.3.

**Theorem 1.4.** The variability region \( LU_{1-} \) is an unbounded Jordan domain with the boundary curve \( \gamma(t), \ -2\pi < t < 2\pi \), given by

\[
\gamma(t) = \begin{cases} 
\log (1 + 3e^{it}) & \text{if } |t| < \pi \\
\log (1 - e^{it}) + \frac{t}{|t|} & \text{if } \pi \leq |t| < 2\pi.
\end{cases}
\]

Here and hereafter, \( \log w = \log |w| + i\text{Arg} \ w \) denotes the principal branch of \( \log w \) with \(-\pi < \text{Im} \log w = \text{Arg} \ w \leq \pi \).

As we will see in the next section, the function \( \log F \) is univalent in \( \mathbb{D} \). Therefore, the last theorem tells us that \( F : \mathbb{D} \to U_{1-} \) covers the disk \( \mathbb{D}(-1,1) \) bivalently whereas it covers \( \mathbb{D}(1,3) \setminus (\mathbb{D}(-1,1) \cup \{0\}) \) univalently.

The following expression of \( W_r \) is not very explicit but useful in some situation.

**Theorem 1.5.** For \( 0 < r < 1 \),

\[
W_r = \left\{ \frac{2u}{v(u + v)} : |u - 1| \leq r, |v - 1| \leq r \right\}.
\]

Indeed, as an application of the last theorem, we can show the following result.

**Theorem 1.6.** \( LW_{1-} = \{ w \in \mathbb{C} : |\text{Im} \ w| < 3\pi/2 \} \).

Since \( W_{1-} = \exp(LW_{1-}) \), we obtain the following corollary, which was used in [7].

**Corollary 1.7.** The full variability region \( \{ zf'(z)/f(z) : z \in \mathbb{D}, f \in \mathbb{C} \} \) is equal to \( \mathbb{C} \setminus \{0\} \).

The corollary means that \( W_{1-} \subset \mathbb{C} \setminus \{0\} \). We note here that this does not seem to follow immediately from Lemma 1.2.

Krzysi [9] showed that \( LV_r \) is convex and determined its shape for \( 0 < r < 1 \).

**Proposition 1.8** (Krzysi). For \( 0 < r < 1 \), the variability region \( LV_r \) is convex and its boundary is described by the curve \( \gamma_r(t) = \log (1 - re^{i\theta_1(t)})/(1 - re^{i\theta_2(t)})^3 \), \( -\pi \leq t \leq \pi \).

Here,

\[
\theta_1(t) = t - \arcsin(r \sin t), \quad \theta_2(t) = \pi + t + \arcsin(r \sin t).
\]

He also proved that \( LV_r \) is contained in the domain \( |\text{Im} \ w| < 4 \arcsin r \) for each \( 0 < r < 1 \) and that this bound is sharp. (See also [4, Chap. 11].) In particular, \( LV_{1-} \subset \{ w : |\text{Im} \ w| < 2\pi \} \). Since \( \text{Re} \ \sigma_r(t) \to +\infty \) as \( r \to 1^+ \) for \( |t| < \pi/2 \) and \( \text{Re} \ \sigma_r(t) \to -\infty \) as \( r \to 1^- \) for \( \pi/2 < |t| \leq \pi \), it is not very easy to determine the limiting shape of \( LV_r \) as \( r \to 1^- \) by the above proposition. We thus complement his results by showing the following.
Theorem 1.9. \[ V_r = \{(1 + s)/(1 + t)^3 : |s| \leq r, |t| \leq r\} = \{u/v^3 : |u - 1| \leq r, |v - 1| \leq r\} \]

for \(0 < r < 1\). Moreover, \(LV_1^- = \{w : |\text{Im} w| < 2\pi\}\) and \(V_1^- = \mathbb{C} \setminus \{0\}\).

One might expect that \(LU_r\) and \(LW_r\) would also be convex for each \(0 < r < 1\). This is, however, not true unlike \(LV_r\).

Theorem 1.10. The variability regions \(LU_r\) and \(LW_r\) are closed Jordan domains for each \(0 < r < 1\). Moreover, there exists a number \(0 < r_0 < 1\) such that both \(LU_r\) and \(LW_r\) are not convex for every \(r\) with \(r_0 < r < 1\).

We prove the above results in Section 3. Section 2 will be devoted to the study of mapping properties of the function \(G = \log F\) that are necessary to show our results.

2. Univalence of the function \(G = \log F\)

In order to analyze the shape of \(LU_r\) or \(LW_r\), we need to investigate mapping properties of the functions \(F\) and \(G = \log F\), where \(F\) is given in Theorem 1.3. Therefore, before showing the main results in Introduction, we see basic properties of the functions \(F\) and \(G\). Here, we remark that \(F\) can be expressed in the form \[ F(z) = \frac{(1 + z)^3}{1 + z^{\frac{3 + 2i\phi}{3 + z}}} \]

Therefore, the continuous branch \(G\) of \(\log F\) with \(G(0) = 0\) is represented by

\[ G(z) = 3 \log (1 + z) - \log (1 + ze^{2i\phi}), \quad \phi = \text{Arg} (3 + z). \]

The goal in this section is to prove the following:

**Theorem 2.1.** The function \(G = \log F\) is a homeomorphism of the unit disk \(\mathbb{D}\) onto the domain \(LU_{1^-}\).

For \(r \in (0, 1)\) and \(x \in (0, \pi)\), we set \[ \Phi_r(x) = \text{Arg} (1 + re^{ix}). \]

We will use the following elementary properties of the function \(\Phi_r\).

**Lemma 2.2.** Let \(r \in (0, 1)\). Then

\[ \Phi_r'(x) = \frac{r(r + \cos x)}{1 + 2r \cos x + r^2}, \quad x \in (0, \pi). \]

In particular, \(\Phi_r(x)\) is increasing in \(0 < x < x_r\) and decreasing in \(x_r < x < \pi\), where \(x_r = \pi - \text{arccos} r\). Furthermore, \(\Phi_r'(x)\) is decreasing in \(0 < x < \pi\) and therefore \(\Phi_r\) is concave in \((0, \pi)\).

We also need the following information.

**Lemma 2.3.** Let \(0 < r < 1\). Then the inequalities \(0 < \theta + 2\phi < \pi\) hold for \(0 < \theta < \pi\) and \(\phi = \text{Arg} (3 + re^{i\theta}).\)
Proof. Set \( h_r(\theta) = \theta + 2\phi = \theta + 2\text{Arg}(3 + re^{i\theta}), \quad 0 \leq \theta \leq \pi. \) Then
\[
h'_r(\theta) = 1 + 2\frac{\partial \phi}{\partial \theta} = \frac{3(3 + 4r \cos \theta + r^2)}{9 + 6r \cos \theta + r^2} \geq \frac{3(3 - 4r + r^2)}{9 + 6r \cos \theta + r^2} = \frac{3(1 - r)(3 - r)}{9 + 6r \cos \theta + r^2} > 0.
\]
Therefore, \( h_r(\theta) \) is increasing in \( 0 < \theta < \pi \), which implies that \( 0 = h_r(0) < h_r(\theta) < h_r(\pi) = \pi \) for \( 0 < \theta < \pi \). □

As for the function \( G \), we first show its local univalence.

**Lemma 2.4.** The function \( G \) is orientation-preserving and locally univalent in \( \mathbb{D} \).

Proof. The partial derivatives of \( G \) are given by
\[
G_z(z) = \frac{6 + 4z + 3\bar{z} + z^2}{(1 + z)(3 + 3z + \bar{z} + z^2)},
\]
\[
G_{\bar{z}}(z) = \frac{z(3 + z)}{(3 + \bar{z})(3 + 3z + \bar{z} + z^2)}.
\]
It suffices to show that the Jacobian \( J_G = |G_z|^2 - |G_{\bar{z}}|^2 \) is positive in \( \mathbb{D} \), which is equivalent to the condition \(|6 + 4z + 3\bar{z} + z^2| > |z(1 + z)| \) in \(|z| < 1\). If we write \( z = re^{i\theta} \), then
\[
|6 + 4z + 3\bar{z} + z^2|^2 - |z(1 + z)|^2
= 6(6 + 14r \cos \theta + 4r^2 + 6r^2 \cos 2\theta + r^3 \cos \theta + r^3 \cos 3\theta)
= 12(1 + r \cos \theta)(2(1 + r \cos \theta)^2 + 1 - r^2) > 0.
\]
Thus we are done. □

**Lemma 2.5.** For a fixed \( 0 < r < 1 \), the real part of \( G(re^{i\theta}) \) is a decreasing function in \( 0 \leq \theta \leq \pi \).

Proof. By differentiating both sides of
\[
\text{Re } G(re^{i\theta}) = \frac{3}{2} \log(1 + 2r \cos \theta + r^2) - \frac{1}{2} \log(1 + 2r \cos(\theta + 2\phi) + r^2)
\]
with respect to \( \theta \), we obtain
\[
\frac{\partial}{\partial \theta} \text{Re } G(re^{i\theta}) = \frac{-3r \sin \theta}{1 + 2r \cos \theta + r^2} + \frac{r \sin(\theta + 2\phi)}{1 + 2r \cos(\theta + 2\phi) + r^2} \left(1 + \frac{\partial \phi}{\partial \theta}\right).
\]
Since \( \tan \phi = r \sin \theta / (3 + r \cos \theta) \), we have the relations
\[
\frac{\partial \phi}{\partial \theta} = \frac{r(3 \cos \theta + r)}{9 + 6r \cos \theta + r^2},
\]
\[
\cos 2\phi = \frac{1 - \tan^2 \phi}{1 + \tan^2 \phi} = 1 - \frac{2r^2 \sin^2 \theta}{9 + 6r \cos \theta + r^2},
\]
\[
\sin 2\phi = \frac{2 \tan \phi}{1 + \tan^2 \phi} = \frac{2r(3 + r \cos \theta) \sin \theta}{9 + 6r \cos \theta + r^2}.
\]
We substitute them into the above expression of $\partial(\text{Re } G)/\partial \theta$ and make some simplifications to obtain

$$\frac{\partial}{\partial \theta} \text{Re } G(re^{i\theta}) = \frac{-6r \sin \theta(3 + 4r \cos \theta + r^2 \cos 2\theta)H(r, \theta)}{(1 + 2r \cos \theta + r^2)(9 + 6r \cos \theta + r^2)K(r, \theta)},$$

where

$$H(r, \theta) = 9 + 12r \cos \theta - 4r^2 \sin^2 \theta - 4r^3 \cos \theta - r^4,$$

$$K(r, \theta) = 9 - 2r^2 + r^4 + 24r \cos \theta + 24r^2 \cos^2 \theta + 8r^3 \cos^3 \theta.$$

Note first that

$$3 + 4r \cos \theta + r^2 \cos 2\theta = 1 - r^2 + 2(1 + r \cos \theta)^2 > 0.$$

Secondly, we compute

$$\frac{\partial H(r, \theta)}{\partial \theta} = -4r(3 + 2r \cos \theta - r^2) \sin \theta.$$

In particular, $H(r, \theta)$ is decreasing in $0 < \theta < \pi$ for a fixed $0 < r < 1$. Hence,

$$H(r, \theta) \geq H(r, \pi) = (1 - r)(3 - r)(3 - r^2) > 0.$$

Thirdly, we see that the function $p(x) = 24x + 24x^2 + 8x^3$ is increasing because $p'(x) = 24(1 + x)^2 \geq 0$. Therefore,

$$K(r, \theta) \geq K(r, \pi) = 9 - 24r + 22r^2 - 8r^3 + r^4 = (3 - r)^2(1 - r)^2 > 0.$$

We summarize the above observations to conclude that $\partial(\text{Re } G)/\partial \theta < 0$ for $0 < \theta < \pi$.

We next prove the following:

**Lemma 2.6.** $\text{Im } G(z) > 0$ for $z \in \mathbb{D}$ with $\text{Im } z > 0$.

**Proof.** Fix $r \in (0, 1)$. Let $\theta \in (0, \pi)$ and let $\phi$ be given in (2.1). Note that $0 < \phi < \theta$. We need to show that

$$g_r(\theta) = \text{Im } G(re^{i\theta}) = 3\Phi_r(\theta) - \Phi_r(\theta + 2\phi)$$

is positive. Note that $0 < \theta < \theta + 2\phi < \pi$ by Lemma 2.3. Lemma 2.2 implies that $\Phi_r$ takes its maximum value $\Phi_r(x_r) = \arcsin r$ in $(0, \pi)$. In particular, we have $\Phi_r(\theta + 2\phi) \leq \arcsin r$. Therefore, $g_r(\theta) > 0$ for $x_r^- < \theta < x_r^+$, where $x_r^-$ and $x_r^+$ are the solutions to the equation $3\Phi_r(x) = \arcsin r$ with $0 < x_r^- < x_r < x_r^+ < \pi$.

We next assume that $x_r^+ \leq \theta < \pi$. Since $\Phi_r$ is decreasing in $x_r^+ < x < \pi$ by Lemma 2.2, we see that $g_r(\theta) > \Phi_r(\theta) - \Phi_r(\theta + 2\phi) > 0$. Finally, we assume that $0 < \theta \leq x_r^-$. In view of concavity of $\Phi_r$ (see Lemma 2.2) together with $\Phi_r(0) = 0$, we have $\Phi_r(x_r^-) = \Phi_r(x_r)/3 \leq \Phi_r(x_r/3)$. Hence, $x_r^- \leq x_r/3$. In particular, we have $\theta + 2\phi \leq 3\theta \leq 3x_r^- \leq x_r$. Since $\Phi_r$ is increasing and concave in $0 < x < x_r$, the inequalities $\Phi_r(\theta + 2\phi) < \Phi_r(3\theta) \leq 3\Phi_r(\theta)$ follow. Thus we have shown that $g_r(\theta) > 0$ in this case, too.

The following property will be used for the proof of Theorem 1.4.

**Lemma 2.7.** The function $g_r$ defined in (2.2) satisfies $g_r'(\theta) > 0$ for $0 < \theta < x_r = \pi - \arccos r$. 

Proof. By definition, we have
\[ g_r'(\theta) = 3\Phi_r'(\theta) - \left(1 + 2\frac{\partial \phi}{\partial \theta}\right) \Phi_r'(\theta + 2\phi). \]
Since \(1 + 2\partial \phi/\partial \theta > 0\) (see the proof of Lemma 2.3), Lemma 2.2 implies
\[ g_r'(\theta) \geq 3\Phi_r'(\theta) - \left(1 + 2\frac{\partial \phi}{\partial \theta}\right) \Phi_r'(\theta) = 2 \left(1 - \frac{\partial \phi}{\partial \theta}\right) \Phi_r'(\theta). \]
We note here that
\[ 1 - \frac{\partial \phi}{\partial \theta} = \frac{3(3 + r \cos \theta)}{9 + 6r \cos \theta + r^2} > 0. \]
Since \(\Phi_r'(\theta) > 0\) for \(0 < \theta < x_r\), by Lemma 2.2 the required assertion follows.

We are now ready to prove the theorem.

Proof of Theorem 2.1. Since \(G\) is orientation preserving and locally univalent by Lemma 2.1 it is enough to show that \(G\) is injective on the circle \(|z| = r\) for each \(r \in (0, 1)\). We note here that \(G\) is symmetric in the real axis, in other words, \(G(z) = \overline{G(z)}\) for \(z \in \mathbb{D}\). By Lemmas 2.5 and 2.6, \(G\) maps the upper half of the circle \(|z| = r\) univalently onto a Jordan arc in the upper half plane. Taking into account the symmetry, we have confirmed that \(G\) maps the circle \(|z| = r\) univalently onto a Jordan curve which is symmetric in the real axis. Thus the proof is complete.

3. Proof of main results

In this section, we show the main results presented in Section 1. We begin with the proof of Theorem 1.3.

Proof of Theorem 1.3. Fix \(0 < r < 1\). To simplify computations, we consider the set \(1/U_r = \{1/w : w \in U_r\}\) instead of \(U_r\). For \(s \in \mathbb{D}\), we denote by \(\Delta_s\) the image of the disk \(|t| \leq r\) under the mapping \(t \mapsto (1 - \frac{s + 1}{2})/(1 - s)^2\). Note that \(\Delta_s\) is the closed disk with center \((1 - s/2)/(1 - s)^2\) and radius \(r|1 - s|^{-2}/2\). It is easily verified that the function \((1 - s/2)/(1 - s)^2\) is univalent on the unit disk \(|s| < 1\). For \(0 \leq q \leq r\), let \(E_q\) be the union of \(\Delta_s\) over \(s \in \mathbb{C}\) with \(|s| = q\). Then Lemma 1.2 (1) implies that \(E_q\) sweeps \(1/U_r\) when \(q\) moves from 0 to \(r\). Note here that \(E_0\) is the disk \(\overline{B}(1, r/2)\). By these observations, we see that the boundary of \(1/U_r\) is contained in the outer envelope of the family of circles \(\partial \Delta_s\) over \(|s| = r\).

We next describe the outer envelope. Let \(c(\alpha)\) and \(\rho(\alpha)\) be the center and the radius of the disk \(\Delta_s\) for \(s = re^{i\alpha}\). Explicitly,
\[ c(\alpha) = \frac{2 - s}{2(1 - s)^2} \quad \text{and} \quad \rho(\alpha) = \frac{r}{2|1 - s|^2} = \frac{r}{2(1 - 2r \cos \alpha + r^2)}. \]

Note that
\[ c'(\alpha) = \frac{is(3 - s)}{2(1 - s)^3} \quad \text{and} \quad \rho'(\alpha) = -\frac{r^2 \sin \alpha}{|1 - s|^3}. \]

We can parametrize the outer envelope in the form
\[ \zeta(\alpha) = c(\alpha) + \rho(\alpha)e^{i\beta(\alpha)}, \quad -\pi < \alpha \leq \pi. \]
By the symmetry in the real axis and the fact that \( \text{Im} \ c'(0) > 0 \), we can take \( \beta(0) \) so that \( \beta(0) = 0 \). Here, \( \beta = \beta(\alpha) \) is a real-valued differentiable function of \( \alpha \) satisfying the condition that the tangent vector \( \zeta'(\alpha) \) is tangent to the circle \( |w - c(\alpha)| = \rho(\alpha) \) at \( \zeta(\alpha) \). In other words, \( \zeta'(\alpha) = kie^{i\beta} \) for a real number \( k \). Taking the real part of the relation

\[
\zeta'(\alpha)e^{-i\beta} = c'(\alpha)e^{-i\beta} + \rho'(\alpha) + i\beta'(\alpha)\rho(\alpha) = ki,
\]

we obtain

\[
\text{Re}[c'(\alpha)e^{-i\beta}] + \rho'(\alpha) = 0,
\]

which implies that

\[
\cos(\beta - \text{arg} \ c'(\alpha)) = \cos(\text{arg} \ c'(\alpha) - \beta) = -\frac{\rho'(\alpha)}{|c'(\alpha)|} = \frac{2r \sin \alpha}{|1 - s||3 - s|}.
\]

Hence,

\[
\frac{|c'(\alpha)|}{c'(\alpha)}e^{i\beta} = e^{i(\beta - \text{arg} c'(\alpha))} = \frac{2r \sin \alpha}{|1 - s||3 - s|} \pm i \sqrt{1 - \left(\frac{2r \sin \alpha}{|1 - s||3 - s|}\right)^2} = \frac{2r \sin \alpha \pm i(3 - 4r \cos \alpha + r^2)}{|1 - s||3 - s|}.
\]

We recall that \( \beta(0) = 0 \) and substitute \( \alpha = 0 \) into this relation in order to eliminate ambiguity of the sign. We then see that the minus sign should be taken there. Hence, \( e^{i\beta} = \frac{-i(1 - s)(3 - s)}{|1 - s||3 - s|} \), \( \frac{c'(\alpha)}{|c'(\alpha)|} = \frac{s(3 - s)(1 - s)^2}{r(3 - s)(1 - s)^2} \).

We now get the form of \( \zeta(\alpha) \):

\[
\zeta(\alpha) = \frac{2 - s}{2(1 - s)^2} + \frac{r}{2|1 - s|^2} \cdot s(3 - s)(1 - \bar{s})^2 = \frac{3 - 3s - \bar{s} + s^2}{(3 - s)(1 - s)^3} = \frac{1}{F(-s)}.
\]

Thus the assertion follows.

\[\square\]

**Proof of Theorem 1.4.** As we saw, the mapping \( G = \log F \) is a homeomorphism of \( \mathbb{D} \) onto \( LU_1 \). We now observe that \( G \) extends continuously to \( \overline{\mathbb{D}} \setminus \{-1\} \). Since \( G(e^{it}) = \gamma(t) \) for \( |t| < \pi \), the boundary of \( LU_1 \) contains the arc \( \gamma([-\pi, \pi]) \).

We next investigate the limit points of \( G(z) \) as \( z \to -1 \). Let \( \alpha(\delta) = (a\delta)^{1/3} \) and put \( z = (1 - \delta)e^{i(\pi - \alpha(\delta))} \) for \( 0 < \delta < 1 \) and a positive constant \( a \). Then

\[
z = -(1 - \delta) \left(1 - i\alpha(\delta) - \frac{\alpha(\delta)^2}{2} + \frac{i\alpha(\delta)^3}{6} + O(\delta^{4/3}) \right)
= -1 + i(a\delta)^{1/3} + \frac{(a\delta)^{2/3}}{2} + \left(1 - \frac{ia}{6}\right)\delta + O(\delta^{4/3})
\]
as \( \delta \to 0^+ \). Therefore,

\[
\frac{3 + z}{3 + \bar{z}} = 1 + i(a\delta)^{1/3} - \frac{(a\delta)^{2/3}}{2} - \frac{2ia\delta}{3} + O(\delta^{4/3})
\]
and
\[ 1 + \frac{3 + z}{3 + \bar{z}} = \left(1 + \frac{ia}{2}\right) \delta + O(\delta^{4/3}). \]
Since \((1 + z)^3 = -ia\delta + O(\delta^{4/3})\), we have
\[ \frac{(1 + z)^3}{1 + z \frac{3 + \bar{z}}{3 + z}} = -\left(1 + \frac{a + 2i}{a - 2i}\right) + O(\delta^{4/3}) \]
as \(\delta \to 0^+\). Thus
\[ \lim_{\delta \to 0^+} G((1 - \delta)e^{i(\pi - \alpha(\delta)))} = \pi i + \log \left(1 + \frac{a + 2i}{a - 2i}\right). \]
Since \(a\) is an arbitrary positive real number, the boundary of \(LU_{1-} = G(D)\) contains the curve \(\gamma(t) : \pi < t < 2\pi\). The same is true for \(-2\pi < t < -\pi\) by the symmetry of the function \(G\).

The remaining thing is to prove that the boundary of \(LU_{1-}\) in \(C\) contains no other points than the curve \(\Gamma = \{\gamma(t) : |t| < 2\pi\}\). We note here that \(LU_r\) is convex in the direction of imaginary axis for each \(0 < r < 1\) by Lemma 2.5. Therefore, the same is true for the limit \(LU_{1-}\). We observe also that \(\Gamma\) encloses an unbounded Jordan domain convex in the direction of imaginary axis.

Suppose that there is a boundary point \(p_0\) of \(LU_{1-}\) with \(p_0 \notin \Gamma\). We may assume that \(\text{Im} p_0 > 0\). Let \(p_1\) be the point in \(\Gamma\) with \(\text{Im} p_1 > 0\) and \(\text{Re} p_1 = \text{Re} p_0\). Then the convexity of \(LU_{1-}\) in the direction of imaginary axis implies that the segment \([p_0, p_1]\) is contained in \(\partial LU_{1-}\). We can choose \(p_0\) so that the segment is maximal. Since the family of smooth Jordan domains \(LU_r\), \(0 < r < 1\), exhausts the domain \(LU_{1-}\), for a small enough \(\delta > 0\) there exist three points \(z_1^- (\delta), z_0 (\delta), z_1^+ (\delta)\) on the circle \(|z| = 1 - \delta\) with \(0 < \text{Arg} z_1^- (\delta) < \text{Arg} z_0 (\delta) < \text{Arg} z_1^+ (\delta)\) such that \(G(z_1^- (\delta)) \to p_1\), \(G(z_0 (\delta)) \to p_0\), \(G(z_1^+ (\delta)) \to p_1\) as \(\delta \to 0^+\). In particular, \(\text{Im} G(z_0 (\delta)) < \text{Im} G(z_1^+ (\delta))\) for sufficiently small \(\delta > 0\). Therefore, \(g_{1-\delta}(\theta_0) = \text{Im} G((1 - \delta)e^{i\theta})\) takes a local minimum at a point \(\theta_0\) with \(\text{Arg} z_1^- (\delta) < \theta_0 < \text{Arg} z_1^+ (\delta)\). In particular, \(g'_{1-\delta}(\theta_0) = 0\). Note here that
\[ \text{Re} G((1 - \delta)e^{i\theta}) \to \text{Re} p_0 \quad (\delta \to 0^+). \]

We write \(\theta_0 = \pi - \beta(\delta)\). Then, by Lemma 2.7, we see that \(\beta_0 \geq x_{1-\delta}\), equivalently, \(\beta(\delta) \leq \arccos(1 - \delta)\). This implies that \(\beta(\delta) = O(\delta^{1/2})\) as \(\delta \to 0^+\). Therefore, \(z = (1 - \delta)e^{i(\pi - \beta(\delta))} = -1 + i\beta(\delta) + O(\delta)\), \((3 + z)/(3 + \bar{z}) = 1 + i\beta(\delta) + O(\delta)\) and thus \(1 + z(3 + z)/(3 + \bar{z}) = O(\delta)\) as \(\delta \to 0^+.\) In particular,
\[ \text{Re} G((1 - \delta)e^{i\theta_0}) \to -\infty \quad (\delta \to 0^+), \]
which is a contradiction.

We now conclude that \(\partial LU_{1-} = \Gamma\). \(\square\)

In order to prove Theorem 3.5, we will make use of a weakened version of Lemma 5.1 in Greiner and Roth [5], which is an outcome of the duality methods developed by Ruscheweyh and Sheil-Small.

For \(|a| \leq 1, |b| \leq 1\), define a function \(f_{a,b} \in A_1\) by
\[ f_{a,b}(z) = z \frac{1 + (a + b)z/2}{(1 + bz)^2}. \]
It is easy to see that \( f_{a,b} \) belongs to the class \( \mathcal{C} \) of close-to-convex functions. The linear space \( \mathcal{A} \) is naturally equipped with the topology of uniform convergence on compact subsets in \( \mathbb{D} \).

**Lemma 3.1** ([5] Lemma 5.1]). Let \( \lambda_1 \) and \( \lambda_2 \) be continuous linear functionals on \( \mathcal{A} \) such that \( \lambda_2 \) does not vanish on \( \mathcal{C} \). Then for every \( f \in \mathcal{C} \) there exist complex numbers \( a, b \) with \( |a| \leq 1, |b| \leq 1 \) such that

\[
\frac{\lambda_1(f)}{\lambda_2(f)} = \frac{\lambda_1(f_{a,b})}{\lambda_2(f_{a,b})}.
\]

**Proof of Theorem 1.6.** Fix \( 0 < r < 1 \). Let \( f \in \mathcal{C} \). We now apply Lemma 3.1 to the choice \( \lambda_1(f) = rf'(r) \) and \( \lambda_2(f) = f(r) \) to see that

\[
\frac{rf'(r)}{f(r)} = \frac{\lambda_1(f)}{\lambda_2(f)} = \frac{\lambda_1(f_{a,b})}{\lambda_2(f_{a,b})} = \frac{2(1 + ar)}{(1 + br)(2 + (a + b)r)}
\]

for some \( a, b \in \overline{\mathbb{D}} \). The proof is complete by letting \( u = 1 + ar \) and \( v = 1 + br \). \( \Box \)

**Proof of Theorem 1.9.** Let \( \Omega = \{(r, s, t) \in \mathbb{R}^3 : 0 < s < 2, 0 < rs^2 < 2, -\pi/2 < t < \pi/2\} \). Then \( u = rs^2e^{it}\cos t \) and \( v = se^{-it}\cos t \) satisfy \( |u - 1| < 1 \) and \( |v - 1| < 1 \), whence the point

\[
w(r, s, t) = \log \frac{2u}{v(u + v)} = \log(2r) + 3it - \log(1 + rs e^{2it} \cos t)
\]

belongs to the region \( LW_1 \) for \( (r, s, t) \in \Omega \) by Theorem 1.5.

For a given point \( z_0 = x_0 + iy_0 \) with \( |y_0| < 3\pi/2 \), we now look for \( (r, s, t) \in \Omega \) such that \( w(r, s, t) = z_0 \).

Let \( r_0 = e^{y_0}/2 \) and take small enough \( 0 < s_0 < 1 \) so that \( r_0s_0 < 1/2 \). Then \( r_0s_0^2 < s_0 < 2 \) and \( x_0 \pm 3\pi i/2 \) are the endpoints of the curve \( \alpha(t) = w(r_0, s_0, t) \). We now take a \( t_0 \in (-\pi/2, \pi/2) \) such that \( \Im \alpha(t_0) = y_0 \) and let \( x_1 = x_0 - \Re \alpha(t_0) \). Since the function \( -\log(1 - x) \) is convex, we have the inequality \( -\log(1 - x) \leq 2x \log 2 \) for \( 0 \leq x < 1/2 \). We now estimate \( -x_1 \) in the following way:

\[
x_1 = -\log|1 + r_0s_0e^{2i\alpha(t_0)}| \leq -\log(1 - r_0s_0) \leq 2r_0s_0 \log 2,
\]

which implies

\[
r_0s_0^2e^{-x_1} < s_0e^{-x_1} \leq s_0e^{2r_0s_0 \log 2} < s_0e^{\log 2} = 2s_0 < 2.
\]

Therefore \( (r_0e^{x_1}, s_0e^{-x_1}, t_0) \in \Omega \) and

\[
w(r_0e^{x_1}, s_0e^{-x_1}, t_0) = x_1 + w(r_0, s_0, t_0) = x_0 + iy_0 = z_0
\]

as desired. \( \Box \)

**Proof of Theorem 1.9.** For a fixed \( 0 < r < 1 \), we consider the continuous linear functionals \( \lambda_1 \) and \( \lambda_2 \) on \( \mathcal{A} \) defined by \( \lambda_1(f) = f'(r) \) and \( \lambda_2(f) = f'(0) \) for \( f \in \mathcal{A} \). Then Lemma 3.1 implies that for any \( f \in \mathcal{C} \),

\[
f'(r) = \frac{\lambda_1(f)}{\lambda_2(f)} = \frac{\lambda_1(f_{a,b})}{\lambda_2(f_{a,b})} = \frac{1 + ar}{(1 + br)^2}.
\]
for some \( a, b \in \mathbb{D} \). Thus the first part of the theorem has been proved.

By the first part, we have the expression \( LV_{1-} = \{ \log (1+z) - 3 \log (1+w) : z, w \in \mathbb{D} \} \). Let \( a \) and \( b \) be real numbers with \( |b| < \pi/2 \). We shall show that \( a + 4bi \in LV_{1-} \). It is easy to observe that the domain \( \{ \log (1+z) : z \in \mathbb{D} \} \) is convex and its boundary curve

\[
\tau(t) = \log (1 + e^{it}) = \log (2 \cos^2 t) + \frac{t}{2}i \quad (-\pi < t < \pi)
\]
satisfies \( \text{Re} \tau(t) \to -\infty \) and \( \text{Im} \tau(t) \to \pm \pi/2 \) as \( t \to \pm \pi \). Therefore, there are \( z, w \in \mathbb{D} \) such that \( a - 3c + bi = \log (1 + z) \) and \( -c - bi = \log (1 + w) \) for a sufficiently large \( c > 0 \). In particular, \( a + 4bi = \log (1 + z) - 3 \log (1 + w) \in LV_{1-} \). \( \square \)

**Proof of Theorem 1.** Since \( LU_r = LW_r + \log(1 - r^2) \) by Lemma 1.2 (2), it suffices to prove the assertion for \( LU_r \). If there is no such an \( r_0 \) as in the assertion, then the limiting domain \( LU_{1-} \) must be convex. Note that \( LU_{1-} \) is convex if and only if \( \frac{d}{dt} \text{arg} \gamma'(t) \geq 0 \), where \( \gamma \) is given in Theorem 1.4. A simple computation gives us

\[
\frac{d}{dt} \text{arg} \gamma'(t) = \text{Im} \frac{d}{dt} \log \gamma'(t)
= \text{Re} \frac{1}{1 + 3e^{it}} = \frac{1 + 3 \cos t}{|1 + 3e^{it}|^2}
\]
for \( |t| < \pi \). This is negative when \( \cos t < -1/3 \) and thus we get a contradiction. The proof is now complete. \( \square \)

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