QFT and topology in two dimensions: 
SL(2, \mathbb{R})-symmetry and the de Sitter universe

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Abstract

We study bosonic Quantum Field Theory on the double covering \(\widetilde{dS}_2\) of the 2-dimensional de Sitter universe, identified to a coset space of the group \(SL(2,\mathbb{R})\). The latter acts effectively on \(\widetilde{dS}_2\) and can be interpreted as it relativity group. The manifold is locally identical to the standard the Sitter spacetime \(dS_2\); it is globally hyperbolic, geodesically complete and an inertial observer sees exactly the same bifurcate Killing horizons as in the standard one-sheeted case. The different global Lorentzian structure causes however drastic differences between the two models. We classify all the \(SL(2,\mathbb{R})\)-invariant two-point functions and show that: 1) there is no Hawking-Gibbons temperature; 2) there is no covariant field theory solving the Klein-Gordon equation with mass less than \(1/2R\), i.e. the complementary fields go away.

1 Introduction

Quantum field theory on curved spacetime provides to date the most reliable access to the study of quantum effects when gravity is present. The procedure amounts to finding a metric solving Einstein’s equations and then studying quantum field theory in that background, possibly considering also the back-reaction of the fields on the metric in a semiclassical approach to quantum gravity. The Hawking effect \[1\ 2\] and the spectrum of primordial perturbations \[3\] are found and characterized in this way.

There is however a caveat: by solving the Einstein equations one gets the local metric structure of the spacetime but the global topological properties remain inaccessible. This
problem is old and well-known (but nowadays a little under-appreciated) in cosmology: the question of the overall shape and size of the universe and of its global features dates back to the ancient philosophers and becomes a theme with Aristotle, Nicholas of Cusa and Giordano Bruno. This sort of study is today a research field that goes under the name of cosmic topology (see e.g. [4, 5]).

The global topological properties may also play an important role in the interplay between quantization and causality: they do and cannot be bypassed. Two very well known examples of this status of affairs are the Gödel universe and the anti-de Sitter spacetime which have closed timelike curves and the light-cone ordering is only local. In the anti de Sitter case a cheap way to ”treat” problem (which by the way is not so much challenging) consists in moving to the universal covering of the manifold: this allows for scalar theories of any mass to be considered. Another well known quantum topological effect is the Aharonov - Bohm phenomenon: the presence of a solenoid makes the space non-simply connected.

An example of a similar nature that we consider in this paper is based on the two-dimensional de Sitter spacetime which is topologically non trivial in the spacelike directions; here we study scalar quantum fields on its double covering.

Two-dimensional models of quantum field theory have played and still play an important role as theoretical laboratories where to explore quantum phenomena that have then been recognized to exist also in four dimensional realistic models, the best known being the Schwinger[6] and the Thirring[7, 8] models. By considering the same models formulated on the two-dimensional de Sitter universe $dS^2$ one may ask what are the features that survive on the curved manifold. This study may also throw new light on some of the difficulties encountered in perturbation theory.

To start this program we have recently reconsidered the free de Sitter Dirac field in two dimensions, founding interesting new features [9] which are related to the two inequivalent spin structures of the two-dimensional de Sitter manifold. Correspondingly, there are two distinct Dirac fields which may be either periodic (Ramond) or anti-periodic (Neveu-Schwarz) w.r.t. spatial translations (rotations in the ambient space) of an angle $2\pi$. A requirement of de Sitter covariance (in a certain generalized sense) may be implemented at the quantum level only in the anti-periodic case [9, 10]. As a consequence, the Thirring-de Sitter model admits de Sitter covariant solutions [11] only in the antiperiodic case. The double covering of the de Sitter manifold $\tilde{dS}^2$ naturally enters in the arena of soluble models of QFT through that door.

\footnote{For example the Schwinger model, which corresponds to two-dimensional quantum electrodynamics, has allowed for the pre-discovery of some of the most important phenomena expected from quantum chromodynamics such as asymptotic freedom and confinement.}
The manifold $\tilde{dS}_2$ is in itself a complete globally hyperbolic manifold. It carries a natural effective and transitive action of $SL(2, \mathbb{R})$, the double covering of $SO_0(1, 2)$, the pseudo-orthogonal de Sitter relativity group that acts on $dS_2$. The Lorentzian geometry of $\tilde{dS}_2$ is locally indistinguishable from that of $dS_2$ but the global properties are quite different. This fact has striking consequences at the quantum level; some of them have already been announced in [12]. In this paper we give a full characterization of the massive Klein-Gordon fields of $\tilde{dS}_2$ by constructing the most general local and $SL(2, \mathbb{R})$-invariant two-point functions. The most surprising physical consequences are the following: there exists no analogue of the so-called Bunch-Davies vacuum [13, 14, 15, 16, 17, 18] on the double covering of the de Sitter spacetime and therefore the Hawking-Gibbons [15, 16, 17, 18] temperature disappears; furthermore there are no nontrivial truly $SL(2, \mathbb{R})$-invariant (as opposed to $SO_0(1, 2)$-invariant) states whose mass is less than $m_{cr} = 1/2$.

It is known that thermal effects in QFT are related to the existence of a bifurcate Killing horizon. Examples of such spacetimes include Minkowski spacetime, the extended Schwarzschild spacetime and de Sitter spacetime. Kay and Wald [19] have proven a uniqueness theorem for such thermal states but they also provide counterexamples [19] where such a geometrical structure does not imply the existence of the corresponding thermal state as in the Schwarzschild-de Sitter and in the Kerr cases.

Our example is in a sense more peculiar: the double covering of the two-dimensional manifold is indistinguishable for a geodesic observer from the uncovered manifold. There is no classical experiment that can be done do to determine whether he or she lives in the de Sitter universe or its double covering. Yet at the quantum level things are different and the global geometric structure of the double covering makes $SL(2, \mathbb{R})$ invariance, locality and the relevant analyticity properties of the correlation functions incompatible and this forbids the existence of the thermal radiation from the horizons. One final lesson that may be drawn from our simple example is that we may perhaps probe the global shape of the universe by quantum experiments made in our laboratories.

2 The de Sitter universe as a coset space.

Let us consider the two-dimensional de Sitter group $G = SO_0(1, 2)$ which is the component connected to the identity of the pseudo-orthogonal Lorentz group acting on the three-dimensional Minkowski spacetime $M_3$ with metric $(+, -, -)$. The Iwasawa decomposition $KNA$ of a generic
element \( g \) of \( G \) is written as follows:

\[
g = k(\zeta)n(\lambda)a(\chi) = \exp(\zeta e_k) \exp(\lambda e_n) \exp(\chi e_a) =
\]

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \zeta & \sin \zeta \\
0 & -\sin \zeta & \cos \zeta
\end{pmatrix}
\begin{pmatrix}
1 + \frac{\lambda^2}{2} & -\frac{\lambda^2}{2} & \lambda \\
\frac{\lambda^2}{2} & 1 - \frac{\lambda^2}{2} & \lambda \\
\lambda & -\lambda & 1
\end{pmatrix}
\begin{pmatrix}
\cosh \chi & \sinh \chi & 0 \\
\sinh \chi & \cosh \chi & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

(1)

The above decomposition gives natural coordinates \((\lambda, \zeta)\) to points \( x = x(\lambda, \zeta) = k(\zeta)n(\lambda) \) of the coset space \( G/A \), which is seen to be topologically a cylinder. Here \( \zeta \) is a real number mod \( 2\pi \).

Once chosen the coset representatives \( x(\lambda, \zeta) \), the left action of the group \( G \) on \( G/A \) is explicitly written as follows:

\[
x' = x(\lambda', \zeta') = g x(\lambda, \zeta) a(g, x(\lambda, \zeta))
\]

(2)

The case of a rotation \( k(\beta) \in K \) is of course the easiest to account for and amounts simply to a shift of the angle \( \zeta \): \( \lambda'(\beta) = \lambda, \zeta'(\beta) = \zeta + \beta \), where both \( \zeta \) and \( \zeta' \) are real numbers mod \( 2\pi \).

The two other subgroups give rise to slightly more involved transformation rules.²

By introducing the variables

\[
u = \tan \left( \frac{\zeta}{2} + \arctan \lambda \right), \quad v = \cot \frac{\zeta}{2},
\]

(5)

the action of \( g(\alpha, \mu, \kappa) = k(\alpha)n(\mu)a(\kappa) \in G \) becomes interestingly simple:

\[
u \to u' = \frac{(e^\alpha u + \mu) \cos \frac{\zeta}{2} + \sin \frac{\zeta}{2}}{\cos \frac{\alpha}{2} - (e^\alpha u + \mu) \sin \frac{\zeta}{2}}, \quad v \to v' = \frac{(e^\alpha v - \mu) \cos \frac{\zeta}{2} - \sin \frac{\zeta}{2}}{\cos \frac{\alpha}{2} + (e^\alpha v - \mu) \sin \frac{\zeta}{2}}.
\]

(6)

The group action on the coset space transforms the variables \( u \) and \( v \) homographically.

The Maurer-Cartan 1-form \( g^{-1}dg \) provides a left invariant metric on \( G/H \) as follows:

\[
x^{-1}dx = d\zeta e_k + \left( \frac{1}{2} \lambda^2 d\zeta + d\lambda \right) e_n + \lambda d\zeta e_a =
\]

² A boost \( a(\kappa) \in A \) gives

\[
l'(\kappa) = \lambda \cosh \kappa + \sinh \kappa (\lambda \cos \zeta + \sin \zeta), \quad \sin \zeta'(\kappa) = \frac{\sin \zeta}{\cos \kappa \sin \kappa + \sin h \kappa}, \quad \cos \zeta'(\kappa) = \frac{\cos \zeta \cosh \kappa + \sinh \kappa}{\cos \kappa \sin \kappa + \sin h \kappa}.
\]

(3)

An element \( n(\mu) \in N \) gives

\[
l'(\mu) = \lambda (1 + \frac{\mu}{2} \lambda^2) - \mu (1 + \frac{\mu}{2} \lambda^2) \cos \zeta + \mu (1 - \frac{\mu}{2} \lambda^2) \sin \zeta, \quad \sin \zeta'(\mu) = \frac{2 \sin \zeta - 2 \mu (1 - \cos \zeta)}{\mu^2 (1 - \cos \zeta) + 2 (1 - \mu \sin \zeta)}, \quad \cos \zeta'(\mu) = \frac{\mu^2 (1 - \cos \zeta) - 2 \mu \sin \zeta + 2 \cos \zeta}{\mu^2 (1 - \cos \zeta) + 2 (1 - \mu \sin \zeta)}.
\]

(4)
Eqs. (2) and (7) then imply that

\[ x'^{-1}dx' = a^{-1}(x^{-1}dx) a + a^{-1}da = a^{-1} \omega \, a + (\lambda d\zeta) \, e_a + a^{-1}da \]  

(8)

Therefore

\[ ds^2 = \frac{1}{2} \text{Tr}(\omega^2) = -(\lambda^2 + 1)d\zeta^2 - 2d\lambda d\zeta. \]  

(9)

is invariant under the action (2). Note that the Iwasawa coordinate system \((\lambda, \zeta)\) is not orthogonal and that \(ds^2\) is not the restriction to the submanifold \(\chi = 0\) of the Maurer-Cartan metric

\[ \frac{1}{2} \text{Tr}(g^{-1}dg \, g^{-1}dg) = -d\zeta^2 - 2d\zeta d\lambda + 2d\zeta d\chi + d\chi^2. \]  

(10)

Changing to the variables \(u\) and \(v\) the metric takes the form

\[ ds^2 = \frac{4 \, du \, dv}{(u + v)^2}; \]  

(11)

\(u\) and \(v\) are light-cone coordinates and the metric is conformal to the Minkowski metric.

A simple geometrical interpretation of the above construction may be unveiled by introducing the standard representation of \(dS_2\) as a one-sheeted hyperboloid

\[ dS_2 = \left\{ x \in M_3 : \ x_0^2 - x_1^2 - x_2^2 = -1 \right\}. \]  

(12)

The coset space \(G/A\) can indeed be identified with \(dS_2\) as follows: given a point \(x \in G/A\) let us associate to it a vector in \(M_3\) whose components are the entries \(x_{02}, x_{12}\) and \(x_{22}\) of the third column of the matrix \(x(\lambda, \zeta)\) which is obviously invariant by the right action of the subgroup \(A\); one also has that \(x_{02}^2 - x_{12}^2 - x_{22}^2 = -1\). The so-defined map is a bijection between \(G/A\) and \(dS_2\) and gives natural global coordinates \((\lambda, \zeta)\) to points\(^3\) of the de Sitter hyperboloid:

\[ x(\lambda, \zeta) = \begin{cases} 
    x^0 = \lambda, \\
    x^1 = \lambda \cos \zeta + \sin \zeta, \\
    x^2 = \cos \zeta - \lambda \sin \zeta.
\end{cases} \]  

(13)

The left action of \(SO_0(1, 2)\) on the coset space \(G/A\) by construction coincides with the linear action of \(SO_0(1, 2)\) in \(M_3\) restricted to the manifold \(dS_2\):

\[ x(\lambda', \zeta') = gx(\lambda, \zeta) \]

\(^3\)We adopt the same letter \(x\) to denote points of the coset space \(G/A\) and of the de Sitter hyperboloid \(dS_2\), as they are identified.
and the metric coincides with the restriction of the ambient spacetime metric to de de Sitter manifold
\[ ds^2 = \left( dx^0 - dx^1 - dx^2 \right) \big|_{ds^2} = -2d\lambda d\zeta - (\lambda^2 + 1) d\zeta^2. \] (14)

The base point (origin) \( x(0,0) = (0,0,1) \) is invariant under the action of the subgroup \( A \). Of course (13) supposes that we have chosen a certain Lorentz frame in \( M_3 \), and this frame will remain fixed in the sequel.

The light-cone variables \( u \) and \( v \) have a simple geometric interpretation; they correspond to the two ratios that may be formed by factorizing the equation defining the de Sitter hyperboloid:
\[ u = \frac{1 - x^2}{x^1 - x^0} = \tg \left( \frac{\zeta}{2} + \arctan \lambda \right), \quad v = \frac{1 + x^2}{x^1 - x^0} = \cot \frac{\zeta}{2}. \] (15)

In term of the light-cone variables we get the following parametrization of the de Sitter manifold
\[
\begin{align*}
  x(u,v) &= \begin{cases} 
    x^0 &= \frac{u-1}{u+v}, \\
    x^1 &= \frac{u+1}{u+v}, \\
    x^2 &= \frac{v}{u+v}. 
  \end{cases}
\end{align*}
\] (16)

The complexification of the de Sitter manifold can be equivalently identified either with the coset space \( G^c/A^c \) of the corresponding complexified groups or with the complex de Sitter hyperboloid
\[ dS^c_2 = \{ z \in \mathbb{C}^3 : z^0^2 - z^1^2 - z^2^2 = -1 \}. \] (17)

As a complex 2-sphere, \( dS^c_2 \) is simply connected. Particularly important subsets of \( dS^c_2 \) are the forward and backward tuboids, defined as follows [16, 17, 18] :
\[
\begin{align*}
  T^+ &= \{ z \in dS^c_2 : (\text{Im } z^0)^2 - (\text{Im } z^1)^2 - (\text{Im } z^2)^2 > 0, \quad \text{Im } z^0 > 0 \}, \\
  T^- &= \{ z \in dS^c_2 : (\text{Im } z^0)^2 - (\text{Im } z^1)^2 - (\text{Im } z^2)^2 > 0, \quad \text{Im } z^0 < 0 \}. 
\end{align*}
\] (18)

These two domains can also be shown to be simply connected.

In the following we will make also use of the standard global orthogonal coordinate system:
\[
\begin{align*}
  x(t,\theta) &= \begin{cases} 
    x^0 &= \sh t, \\
    x^1 &= \ch t \sin \theta, \\
    x^2 &= \ch t \cos \theta. 
  \end{cases}
\end{align*}
\] (20)

Here \( \theta \) is a real number mod \( 2\pi \). The relation between the two above coordinate system is quite simple:
\[ \lambda = \sh t, \quad \tg \theta = \tg(\zeta + \arctan \lambda). \] (21)
3 The double covering of the 2-dim de Sitter manifold as a coset space.

The easiest and most obvious way to describe the double covering $\tilde{dS}_2$ of the two-dimensional de Sitter universe $dS_2$ consists in unfolding the periodic coordinate $\theta$ in (20). More precisely, we may write the covering projection $pr : \tilde{dS}_2 \rightarrow dS_2$ as follows:

$$pr(\tilde{x}(t, \theta)) = x(t, \theta \mod 2\pi),$$

where we use the coordinates $(t, \theta)$ to parameterize also $\tilde{dS}_2$; at the lhs $\theta$ is a real number $\mod 4\pi$.

The double covering $\tilde{dS}_2$ arises also as a coset space of the double covering $\tilde{G} = SL(2, \mathbb{R})$ of $SO_0(1, 2)$. An element $\tilde{g} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (23) of $\tilde{G}$ is parametrised by four real numbers $a, b, c, d$ subject to the condition $\det \tilde{g} = ad - bc = 1$.

For $\tilde{g} \in \tilde{G}^c = SL(2, \mathbb{C})$ formulae are the same but all entries are complex. Let $\tilde{A}^c$ be the complex subgroup of all $2 \times 2$ matrices of the form

$$\tilde{h}(r) = \begin{pmatrix} r & 0 \\ 0 & \frac{1}{r} \end{pmatrix}, \quad r \in \mathbb{C}, \quad r \neq 0.$$

(24)

$\tilde{A}$ is the subgroup of $\tilde{A}^c$ in which $r > 0$. Note that $\tilde{A}^c \cap \tilde{G} = \tilde{A} \cup -\tilde{A}$ does not coincide with $\tilde{A}$. $Z_2 = \{1, -1\}$ is the common center of $\tilde{G}^c$ and $\tilde{G}$ and is contained in $\tilde{A}^c$ (but not in $\tilde{A}$).

$\tilde{G}$ (resp. $\tilde{G}^c$) operates on the real (resp. complex) 3-dimensional Minkowski space $M_3$ (resp. $M_3^c$) by congruence:

$$x \rightarrow X = \begin{pmatrix} x^0 + x^1 & x^2 \\ x^2 & x^0 - x^1 \end{pmatrix}, \quad X' = \begin{pmatrix} x^{0'} + x^{1'} & x^{2'} \\ x^{2'} & x^{0'} - x^{1'} \end{pmatrix} = \tilde{g}X\tilde{g}^T.$$

(25)

The real (resp. complex) de Sitter manifold is mapped into itself by the above action, which is transitive but not effective. In particular the subgroup $\tilde{A}^c$ is seen to be the stability subgroup of $(0, 0, 1)$ and the quotient $\tilde{G}^c/\tilde{A}^c$ can be identified with the complex de Sitter space $dS_2^c$. The real trace of the latter, i.e. $\tilde{G}/(\tilde{A} \cup -\tilde{A})$, can be identified to the real de Sitter space $dS_2$. On

\footnote{Let $\tilde{H}$ be a group, $\tilde{K}$ a subgroup of $\tilde{H}$, and $Z \subset \tilde{K}$ an invariant subgroup of $\tilde{H}$. Let $H = \tilde{H}/Z$ and $K = \tilde{K}/Z$. Then $\tilde{H}/\tilde{K} \simeq H/K$. It follows that $\tilde{G}^c/\tilde{A}^c \simeq G^c/A^c$.}
the other hand $\tilde{G}/\tilde{A}$ can be identified to the two-sheeted covering $\tilde{dS}_2$ of $dS_2$. If $\tilde{g} \in \tilde{G}^c$ and $r \neq 0$,

$$\tilde{g} \tilde{h}(r) = \begin{pmatrix} r & a & b & c \\ r & c & d & r \end{pmatrix}.$$  \hspace{1cm} (26)

In the real case when $\tilde{g} \in \tilde{G}$ and $r > 0$ we can take $r = (a^2 + c^2)^{-1/2}$ and get $a^2 + c^2 = 1$ (a and c cannot be both equal to 0 because $ad - bc = 1$). Thus every coset $\tilde{g} \tilde{A}$ contains exactly one element with this property; it can be parametrized by using the Iwasawa decomposition\footnote{The Iwasawa parametrization of $\tilde{G} = SL(2, \mathbb{R})$ is given by}

$$\tilde{x}(\lambda, \zeta) = \begin{pmatrix} \cos \frac{\zeta}{2} & \lambda \cos \frac{\zeta}{2} + \sin \frac{\zeta}{2} \\ -\sin \frac{\zeta}{2} \cos \frac{\zeta}{2} - \lambda \sin \frac{\zeta}{2} \end{pmatrix}.$$ \hspace{1cm} (29)

The double covering $\tilde{dS}_2$ can be thus represented as the following real algebraic manifold

$$\tilde{dS}_2 = \tilde{G}/\tilde{A} \simeq \{(a, b, c, d) \in \mathbb{R}^4 : ad - bc = 1, \ a^2 + c^2 = 1\}$$ \hspace{1cm} (30)

which is the intersection of two quadrics in $\mathbb{R}^4$ and can be verified to have no singular point.

$SL(2, \mathbb{R})$ acts on $\tilde{dS}_2$ (i.e. $\tilde{G}/\tilde{A}$) by left multiplication; the transformation rules formally coincide with the previous ones (see Eq. 2 and Footnote 2) with the only difference that the angular coordinates are now defined mod 4$\pi$. The action is effective and transitive and $SL(2, \mathbb{R})$ can be interpreted as the relativity group of $\tilde{dS}_2$.

The Maurer-Cartan form provides $\tilde{dS}_2$ with a natural Lorentzian metric that may be constructed precisely as in the previous section and one gets again Eq. \footnote{There is an obvious bijection between the group $SL(2, \mathbb{R})$ and the three-dimensional anti de Sitter manifold

$$AdS_3 = \left\{ x \in M_{2, 2} : x^{02} - x^{12} - x^{22} + x^{32} = 1 \right\}.$$ \hspace{1cm} (31)} \footnote{\text{3.1 $SL(2, \mathbb{R})$ and the three dimensional anti de Sitter universe}}.

$$\tilde{dS}_2$$ is a globally hyperbolic and geodesically complete spacetime.

3.1 $SL(2, \mathbb{R})$ and the three dimensional anti de Sitter universe

There is an obvious bijection between the group $SL(2, \mathbb{R})$ and the three-dimensional anti de Sitter manifold

$$AdS_3 = \left\{ x \in M_{2, 2} : x^{02} - x^{12} - x^{22} + x^{32} = 1 \right\}.$$ \hspace{1cm} (31)
given by the identification

\[
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}
= \begin{pmatrix}
  x^0 + x^1 & x^2 + x^3 \\
  x^2 - x^3 & x^0 - x^1
\end{pmatrix}.
\] (32)

It obviously follows that \( SL(2, \mathbb{R}) \) acts transitively on \( AdS_3 \). \( SL(2, \mathbb{R}) \) acting by left multiplication on the matrix in (32) leaves its determinant unchanged so that \( SL(2, \mathbb{R}) \) is a subgroup of \( SO(2, 2) \). This can also be seen by considering the mapping

\[
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}
\rightarrow \frac{1}{2} \begin{pmatrix}
  a + d & a - d & b + c & -b + c \\
  a - d & a + d & b - c & -b - c \\
  b + c & -b + c & a + d & a - d \\
  b - c & -b - c & a - d & a + d
\end{pmatrix}.
\] (33)

The Iwasawa decomposition (27) and Eq. (33) also provide an interesting global coordinate system for \( AdS_3 \). An easy calculation explicitly shows that the Maurer-Cartan metric and the \( AdS_3 \) metric coincide:

\[
ds^2 = \left( dx^0 \right)^2 - \left( dx^1 \right)^2 - \left( dx^2 \right)^2 + \left( dx^3 \right)^2 \bigg|_{AdS_3} = \frac{1}{2} \text{Tr} \left( \tilde{g}^{-1} d\tilde{g} \tilde{g}^{-1} d\tilde{g} \right) = \frac{1}{4} (d\zeta^2 + 2d\zeta d\lambda - 2\lambda d\zeta d\chi - d\chi^2).
\] (34)

(35)

(note that \( \zeta \) is a timelike variable in the \( AdS_3 \) manifold). Setting \( \chi = 0 \) we get a parametrization \( x(\lambda, \zeta, 0) \) of \( \tilde{dS}_2 \), represented here as the submanifold \( \tilde{M} \) of \( AdS_3 \):

\[
\tilde{M} = AdS_3 \cap \{ (x^0 + x^1)^2 + (x^2 - x^3)^2 = 1 \}, \quad \tilde{x}(\lambda, \zeta, 0) = \begin{cases}
  x^0 = \cos \frac{\zeta}{2} - \frac{1}{2} \lambda \sin \frac{\zeta}{2} \\
  x^1 = \frac{1}{2} \lambda \sin \frac{\zeta}{2} \\
  x^2 = \frac{1}{2} \lambda \cos \frac{\zeta}{2} \\
  x^3 = \frac{1}{2} \lambda \cos \frac{\zeta}{2} + \sin \frac{\zeta}{2}
\end{cases}
\] (36)

Of course \( \tilde{M} \) is only invariant under the linear action of the image of the one-parameter subgroup \( K \) of \( SL(2, \mathbb{R}) \), namely matrices of the form

\[
\begin{pmatrix}
  \cos \frac{\theta}{2} & 0 & 0 & -\sin \frac{\theta}{2} \\
  0 & \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} & 0 \\
  0 & -\sin \frac{\theta}{2} & \cos \frac{\theta}{2} & 0 \\
  \sin \frac{\theta}{2} & 0 & 0 & \cos \frac{\theta}{2}
\end{pmatrix}
\] (37)

and we cannot simply take the restriction to \( M \) of the \( AdS_3 \) metric to get an \( SL(2, \mathbb{R}) \) invariant metric.
On the other hand $SL(2, \mathbb{R})$ acts on $AdS_3$ by congruence
\[
X = \begin{pmatrix} x^0 + x^1 & x^2 + x^3 \\ x^2 - x^3 & x^0 - x^1 \end{pmatrix}, \quad X' = \begin{pmatrix} x^0' + x^1' & x^2' + x^3' \\ x^2' - x^3' & x^0' - x^1' \end{pmatrix} = \tilde{g}X\tilde{g}^T.
\]
This provides the map
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \frac{1}{2} \begin{pmatrix} (a^2 + b^2 + c^2 + d^2) & (a^2 - b^2 + c^2 - d^2) & (2ab + 2cd) & 0 \\ (a^2 + b^2 - c^2 - d^2) & (a^2 - b^2 - c^2 + d^2) & (2ab - 2cd) & 0 \\ 2ac + 2bd & 2ac - 2bd & 2bc + 2ad & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}
\] (39)
Note that $\tilde{g}$ and $-\tilde{g}$ are mapped into the same element of $SO(2, 2)$; indeed Eq. (39) is a covering projection of $SL(2, \mathbb{R})$ onto a $SO_0(1, 2)$ subgroup of $SO(2, 2)$. Using again the Iwasawa parameters we get a coordinate system for $dS_2$ represented here as a submanifold $\mathcal{M}$ of $AdS_3$
\[
\mathcal{M} = AdS_3 \cap \{x^3 = \sqrt{2}\}, \quad x(\lambda, \zeta, 0) = \begin{cases} 
\lambda \\
\lambda \cos(\zeta) + \sin(\zeta) \\
\cos(\zeta) - \lambda \sin(\zeta) \\
\sqrt{2} 
\end{cases}
\] (40)
Of course the restriction of the $AdS$ metric to $\mathcal{M}$ gives back Eq. (14).

3.2 Complexification

The algebraic manifold (30) can be complexified, i.e. we can define
\[
\mathcal{V} = \{(a, b, c, d) \in \mathbb{C}^4 : ad - bc = 1, \quad a^2 + c^2 = 1\}.
\] (41)
Let us again consider eq. (26) but now with complex $\tilde{g}$ and $r$. If $\tilde{g}$ is such that $a^2 + c^2 \neq 0$, we can choose $r = \pm (a^2 + c^2)^{-1/2}$ and thus the coset $\tilde{g}\tilde{A}^c$ contains two distinct (opposite) elements of $\mathcal{V}$. If $p, p'$ are points of $\mathcal{V}$ such that $p \neq \pm p'$ there is no $r \neq 0$ such that $p' = p\tilde{h}(r)$ hence $p$ and $p'$ belong to different elements of $\tilde{G}^c/\tilde{A}^c$. Conversely two opposite points $p$ and $-p$ of the manifold $\mathcal{V}$ belong to the same coset $p\tilde{A}^c$ i.e. determine a unique element of $\tilde{G}^c/\tilde{A}^c$. On the other hand if $\tilde{g}$ is such that $a^2 + c^2 = 0$, all elements of the coset $\tilde{g}\tilde{A}^c$ have the same property and none belongs to $\mathcal{V}$.

The cosets $\tilde{g}\tilde{A}^c$ such that $a^2 + c^2 = 0$ can be identified to certain points of the complex de Sitter space as follows. Let $g = (a, b, c, d) \in \tilde{G}^c$. Then $x = g(0, 0, 1)$ is given by
\[
\begin{pmatrix} x^0 + x^1 \\ x^2 \\ x^0 - x^1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 2ab & ad + bc \\ ad + bc & 2cd \end{pmatrix}
\] (42)
(the fact that the determinant of the lhs is equal to \(-1\) expresses \(x \in dS_2^{(c)}\). Since \(a\) and \(c\) cannot be both 0 we may suppose that \(c \neq 0\); we get
\[
\frac{a}{c} \frac{x^0 + x^1}{x^2 - 1} = \frac{x^2 + 1}{x^0 - x^1}.
\]
The condition \(a^2 + c^2 = 0\) is equivalent to \(a = \pm ic\) (thus excluding \(c = 0\)) and implies
\[
x^0 + x^1 = \pm i(x^2 - 1) \implies x^0 - x^1 = \mp i(x^2 + 1).
\]
Conversely one of these conditions implies \(a^2 + c^2 = 0\). Therefore
\[
a^2 + c^2 = 0 \iff (x^0 + x^1)^2 + (x^2 - 1)^2 = 0 \iff (x^0 - x^1)^2 + (x^2 + 1)^2 = 0.
\]
It follows that \(\mathcal{V}\) projects onto \(dS_2^{(c)}\) with the exception of the manifold \(\mathcal{N}\) defined by the above equations. In particular the points \((\pm i, 0, 0)\) belong to \(\mathcal{N}\). Of course the manifold \(\mathcal{N}\) is not invariant under the action of \(\widetilde{G}\) or \(\widetilde{G}^c\).

As a parenthesis let us take advantage of the preceding calculations to verify that \(\widetilde{G}^c\) acts transitively on \(dS_2^{(c)}\), i.e. given \((x^0, x^1, x^2) \in \mathbb{C}^3\) satisfying \(x^{02} - x^{12} - x^{22} = -1\), there is a \(g = (a, b, c, d)\) such that \(ad - bc = 1\) and eq. (43) holds.

There are other complex manifolds that contain \(\widetilde{dS}_2\) as a real form. For example let us represent \(\widetilde{dS}_2\) as a cylinder \(\mathbb{R} \times S^1\) as follows: a point is associated to a pair \((x^0, \theta)\) where \(x^0 \in \mathbb{R}\) and \(\theta \in \mathbb{R}/4\pi\mathbb{Z}\). It projects on the point
\[
\begin{pmatrix}
    x^0 \\
    \sqrt{x^{02} + 1}\cos \theta \\
    \sqrt{x^{02} + 1}\sin \theta
\end{pmatrix}
\in dS_2.
\]
The natural complexification of this is \(\Sigma \times \mathbb{C}/4\pi\mathbb{Z}\), where \(\Sigma\) is the Riemann surface of \(z \mapsto \sqrt{z^2 + 1}\), a two-sheeted covering of \(\mathbb{C}\setminus\{i, -i\}\). This complex manifold projects onto \(dS_2^{(c)}\) with the exception of the points such that \((x^{02} + 1) = 0\), or equivalently \((x^{12} + x^{22}) = 0\).

Another example will be used to discuss the analyticity of certain two-point functions in Sect. 10. The real space \(dS_2\) (resp. \(\widetilde{dS}_2\)) can be diffeomorphically mapped onto a slice of the cylinder \(\mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z}\) (resp. \(\mathbb{R} \times \mathbb{R}/4\pi\mathbb{Z}\)) as follows (using the notation of (20)):
\[
x(t, \theta) \mapsto (s, \theta),
\]
\[
sh t = \tan s, \quad s = \frac{1}{2i} \log \left(\frac{1 + i sh t}{1 - i sh t}\right) = \frac{1}{2i} \log \left(\frac{1 + ix^0}{1 - ix^0}\right),
\]
As \(t\) varies in \(\mathbb{R}\), \(s\) varies in \((-\pi/2, \pi/2)\), and as a consequence
\[
x^0 = \tan s, \quad \cosh t = \frac{1}{\cos s}, \quad \sin s = \sinh t, \quad t = \frac{1}{2} \log \left(\frac{1 + \sin s}{1 - \sin s}\right).
\]
This map is conformal:
\[
\frac{\cosh^2 t \, d\theta^2}{\cos^2 s}.
\] (49)
The cylinder \(\mathbb{R} \times \mathbb{R}/2\pi \mathbb{Z}\) (resp. \(\mathbb{R} \times \mathbb{R}/4\pi \mathbb{Z}\)) can be complexified as \(\mathbb{C} \times \mathbb{C}/2\pi \mathbb{Z}\) (resp. \(\mathbb{C} \times \mathbb{C}/4\pi \mathbb{Z}\)). However the map (47) becomes singular when \(x^0 = \pm i\), so that these complex cylinders project onto \(dS^c_2\) with the exception of the points where \(x^0 = \pm i\). Note that the complexified map, when restricted to the 'Euclidian' 2-sphere (minus the poles \(x^0 = \pm i\)), is the Mercator projection of that sphere.

The preceding examples are all of the following type. \(\tilde{dS}^2\) is imbedded in a connected complex manifold \(\tilde{M}\) equipped with a complex conjugation \(*\), and \(\tilde{dS}^2 \subset \tilde{M}^{(r)} = \{x \in \tilde{M} : x = x^*\}\). The projection \(\tilde{\text{pr}}\) of \(\tilde{dS}^2\) onto \(dS^c_2\) can be analytically continued to an analytic local homeomorphism of \(\tilde{M}\) onto an open subset \(M\) of \(dS^c_2\). But \(M\) can never be the whole \(dS^c_2\). Indeed the latter being simply connected, \(\tilde{\text{pr}}\) would be a homeomorphism of \(\tilde{M}\) onto \(dS^c_2\). This is not possible since the restriction of \(\tilde{\text{pr}}\) to \(\tilde{dS}^2\) is 2 to 1. \(M\) cannot be invariant under \(G^c\) since \(G^c\) acts transitively on \(dS^c_2\), and this would imply \(M = dS^c_2\). However the examples mentioned above are invariant under the subgroup of rotations. Note that in the case of the third example mentioned above, \(\tilde{M}\) is the cylinder \(\mathbb{C} \times \mathbb{C}/4\pi \mathbb{Z}\) minus the points where \(s \in \frac{\pi}{2} + \pi \mathbb{Z}\), and the projection \(\tilde{\text{pr}} : \tilde{M} \rightarrow dS^c_2\) is given by the formulae (48), i.e. \((s, \theta) \rightarrow (x^0 = \tan s, \theta \mod 2\pi)\).

4 More about the covering projection

We saw that the group \(\tilde{G} = SL(2, \mathbb{R})\) acts on the covering space \(\tilde{dS}^2\) as a group of spacetime transformations by left multiplication \(\tilde{x} \rightarrow \tilde{g}\tilde{x}\) and acts on \(dS^c_2\) by congruence. We denote both actions by the shortcut \((\cdot) \rightarrow \tilde{g}(\cdot)\). They commute with the covering projection:
\[
\text{pr} (\tilde{g}\tilde{x}) = \tilde{g} \text{pr} (\tilde{x}) \quad \forall \tilde{g} \in \tilde{G}, \quad \forall \tilde{x} \in \tilde{dS}^2. \quad (50)
\]

On the de Sitter manifold \(dS^c_2\) the antipodal map \(x \mapsto -x\) is expressed in the coordinates \((t, \theta)\) by \(x(t, \theta) \mapsto -x(t, \theta) = x(-t, \theta + \pi)\). Let \(\tau_1\) be the operation with the same expression on the covering manifold \(\tilde{dS}^2\), i.e.
\[
\tau_1 \tilde{x}(t, \theta) = \tilde{x}(-t, \theta + \pi), \quad \tilde{x} \in \tilde{dS}^2 \quad (51)
\]
and let
\[
\tau_2 \tilde{x}(t, \theta) = \tilde{x}(-t, \theta - \pi), \quad \tau \tilde{x}(t, \theta) = \tilde{x}(t, \theta + 2\pi), \quad \tilde{x} \in \tilde{dS}^2, \quad \tau = \tau_1^2 = \tau_2^2, \quad \tau_1 \tau_2 = 1. \quad (52)
\]
Obviously $\tau_1$ is a diffeomorphism of the covering manifold; we have
\[ \text{pr} (\tau_1 \tilde{x}) = -\text{pr} (\tilde{x}), \quad \text{pr} (\tau \tilde{x}) = \text{pr} (\tilde{x}). \] (53)

To prove that $\tau_1$ commutes with the action of the group $\tilde{G}$, it suffices to prove that it commutes with the elements of a basis of the linear differential operators associated with the action of the Lie algebra of $\tilde{G}$ on $\tilde{dS}_2$, e.g. the operators defined on $C^\infty(\tilde{dS}_2)$ by
\[ \mathcal{M}_j f(x) = \left. \frac{d}{dt} f(e^{\tau \mathcal{M}_j x}) \right|_{\tau = 0}, \quad j = 0, 1, 2, \]
\[ M_0 = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad M_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M_2 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \] (54)

In the coordinates $(t, \theta)$ defined in (20),
\[ \mathcal{M}_0 = -\frac{\partial}{\partial \theta}, \]
\[ \mathcal{M}_1 = \cos(\theta) \frac{\partial}{\partial t} - \sin(\theta) \text{th}(t) \frac{\partial}{\partial \theta}, \]
\[ \mathcal{M}_2 = \sin(\theta) \frac{\partial}{\partial t} + \cos(\theta) \text{th}(t) \frac{\partial}{\partial \theta}. \] (55) (56) (57)

It is easy to verify that
\[ [\mathcal{M}_1, \mathcal{M}_2] = -\mathcal{M}_0, \quad [\mathcal{M}_1, \mathcal{M}_0] = -\mathcal{M}_2, \quad [\mathcal{M}_2, \mathcal{M}_0] = \mathcal{M}_1. \] (58)

Consider now a smooth function $f(t, \theta)$ (not necessarily periodic in $\theta$) and let $f_t$ and $f_\theta$ denote its partial derivatives. We need to verify that
\[ \mathcal{M}_j (f \circ \tau_1) = (\mathcal{M}_j f) \circ \tau_1, \quad j = 0, 1, 2, \] (59)

This is obvious in the case of $\mathcal{M}_0$. In the case of $\mathcal{M}_1$
\[ \mathcal{M}_1 (f \circ \tau_1)(t, \theta) = -\cos(\theta) f_t(-t, \theta + \pi) - \sin(\theta) \text{th}(t) f_\theta(-t, \theta + \pi), \]
\[ (\mathcal{M}_1 f) \circ \tau_1(t, \theta) = \cos(\theta + \pi) f_t(-t, \theta + \pi) - \sin(\theta + \pi) \text{th}(-t) f_\theta(-t, \theta + \pi), \] (60)

and these two expressions are equal. The case of $\mathcal{M}_2$ is similar. It follows that $\tau$ and $\tau_2$ also commute with the action of $\tilde{G}$.

We also note that a necessary and sufficient condition for a smooth function or a distribution $f$ on $\tilde{dS}_2$ to be invariant under $\tilde{G}$ is that $\mathcal{M}_j f = 0$ for all $j = 0, 1, 2$, but in fact it suffices that this hold for $j = 0$ and $j = 1$ (or for $j = 0$ and $j = 2$) because of (58). Similarly a smooth function or distribution $f$ on $\tilde{dS}_2 \times \tilde{dS}_2$ is invariant if and only if $(\mathcal{M}_{j,\tilde{x}} + \mathcal{M}_{j,\tilde{x}'}) f(\tilde{x}, \tilde{x}') = 0$ for $j = 0, 1$ (or for $j = 0$ and $j = 2$).
5 Quantum field theory on the 2-dim dS universe vs its double covering

In the spherical coordinate system (both on $dS_2$ and on $\widehat{dS}_2$) the de Sitter Klein-Gordon equation takes the form

$$\Box \phi - \lambda(\lambda + 1)\phi = \frac{1}{\text{ch} t} \partial_t(\text{ch} t \partial_t \phi) - \frac{1}{\text{ch}^2 t} \partial^2_t \phi - \lambda(\lambda + 1)\phi = 0. \quad (61)$$

The parameter $\lambda$ and the squared mass

$$m^2_\lambda = -\lambda(\lambda + 1) \quad (62)$$

are complex numbers; $m^2_\lambda$ is real and positive in the following special cases:

- either $\lambda = -\frac{1}{2} + i\rho$, $\text{Im} \rho = 0$, $m = \sqrt{\frac{1}{4} + \rho^2} \geq \frac{1}{2}$, \quad (63)
- or $\text{Im} \lambda = 0$, $-1 < \text{Re} \lambda < 0$, $0 < m < \frac{1}{2}$. \quad (64)

Let us introduce the complex variable $z = i\text{sh} t$, so that $1 - z^2 = \text{ch}^2 t$, and separate the variables by posing $\phi = f(z)e^{i\theta}$. Eq. (61) implies that $f$ has to solve the Legendre differential equation:

$$(1 - z^2)f''(z) - 2zf'(z) + \lambda(\lambda + 1)f(z) - \frac{l^2}{(1 - z^2)}f(z) = 0. \quad (65)$$

The difference between $dS_2$ and its covering $\widehat{dS}_2$ is that in the first case $l$ is an integer number while in the second case $2l$ is integer. Enlarging the set of possible values of $l$ in this way will cause many unexpected new features. We will describe some of them below.

Two linearly independent solutions of the above equation are the Ferrers functions (also needed):

\begin{align*}
\mathbf{P}^\nu_\mu(0) &= \frac{2^{\nu+1} \sin \left(\frac{\pi}{2}(\mu + \nu)\right) \Gamma \left(\frac{1}{2}(\mu + \nu + 2)\right)}{\sqrt{\pi} \Gamma \left(\frac{1}{2}(\mu + \nu + 1)\right)}, \quad (66) \\
\mathbf{P}'_\nu^\nu(0) &= \frac{2^\nu \cos \left(\frac{\pi}{2}(\mu + \nu)\right) \Gamma \left(\frac{1}{2}(\mu + \nu + 1)\right)}{\sqrt{\pi} \Gamma \left(\frac{1}{2}(\mu + \nu + 2)\right)}. \quad (67)
\end{align*}

where $\mathbf{P}'(z) = \frac{d\mathbf{P}}{dz}$. We get

\begin{align*}
\mathcal{W}\{\mathbf{P}^\nu_\mu(i\text{sh} t), \mathbf{P}^\nu_{\mu'}(-i\text{sh} t)\} &= \frac{2}{\Gamma(-\mu - \nu)\Gamma(-\mu + \nu + 1)} (\text{ch} t)^{-2} \\
\mathcal{W}\{\mathbf{P}^\nu_\mu(i\text{sh} t), \mathbf{P}'^\nu_{\mu'}(i\text{sh} t)\} &= -\frac{2\sin(\pi \mu)}{\pi} (\text{ch} t)^{-2} \\
\mathcal{W}\{\mathbf{P}^\nu_{\mu'}(-i\text{sh} t), \mathbf{P}'^\nu_{\mu''}(-i\text{sh} t)\} &= \frac{2\sin(\pi \nu)}{\pi} (\text{ch} t)^{-2} \\
\mathcal{W}\{\mathbf{P}^\nu_\mu(i\text{sh} t), \mathbf{P}'^\nu_{\mu''}(-i\text{sh} t)\} &= -\frac{2\sin(\pi \nu)}{\pi} (\text{ch} t)^{-2} \quad (68)
\end{align*}
called “Legendre functions on the cut” \[20\]) \(P^\nu_\mu(z)\) and \(Q^\nu_\mu(z)\), where 
\[ \nu = \lambda, \mu = -l. \]

\(P^\nu_\mu(z)\) and \(Q^\nu_\mu(z)\) are holomorphic in the cut-plane
\[ \Delta_2 = \mathbb{C} \setminus (-\infty - 1) \cup [1, \infty). \]

and satisfy the reality conditions
\[ \overline{P^\nu_\mu(z)} = P^\nu_\mu(\bar{z}), \quad \overline{Q^\nu_\mu(z)} = Q^\nu_\mu(\bar{z}) \]
for all \(z \in \Delta_2\). \(P^\nu_\mu(z)\) respects the symmetry \([62]\) of the mass squared: for all \(z \in \Delta_2\) it satisfies
\[ P^\nu_{-l}(-z) = P^{\nu}_{-l}(z). \]

If \(\lambda - l\) and \(\lambda + l - 1\) are not non-negative integers, \(P^\nu_{-l}(z)\) and \(P^{\nu}_{-l}(-z)\) also constitute two linearly independent solutions of Eq. \([65]\). In this case the general solution has the form
\[ \phi_l(t, \theta) = [a_l P^\nu_{\lambda-l}(i \sin t) + b_l P^\nu_{\lambda-l}(-i \sin t)] e^{il\theta}. \]

In the following we will restrict our attention to values of \(\lambda\) such that
\[ -\frac{1}{2} < \text{Re} \lambda < 0. \]

In particular we do not consider here tachyonic fields.

## 6 Canonical commutation relations.

Let us focus on the modes
\[
\begin{align*}
\phi_l(t, \theta) &= [a_l P^\nu_{\lambda-l}(i \sin t) + b_l P^\nu_{\lambda-l}(-i \sin t)] e^{il\theta} \\
\phi^*_l(t, \theta) &= [a^*_l P^{-\nu}_{\lambda-l}(-i \sin t) + b^*_l P^{-\nu}_{\lambda-l}(i \sin t)] e^{-il\theta}
\end{align*}
\]

where either \(\lambda = -1/2 + iv\) or \(\lambda\) real. The KG product is defined as usual \([21]\):
\[ (f, g)_{KG} = i \int_\Sigma (f^* \partial_\mu g - g \partial_\mu f^*) d\Sigma^\mu(x) = i \int_\Sigma (f^* \partial_\theta g - g \partial_\theta f^*) d\theta \]

On \(dS^2\) the integral is over the interval \(\Sigma = [0, 2\pi]\) and \(l\) is integer. When we consider fields on the covering manifold \(\tilde{dS}^2\) the integral is over the interval \(\Sigma = [0, 4\pi]\) and \(2l\) is integer.
The first condition imposed by the canonical quantization procedure is the orthogonality 
\((\phi^*_l, \phi_{\nu})_{KG} = 0\) of the modes; it gives rise to the following conditions on the coefficients:

\[
a_l b_{-l} - b_l a_{-l} = 0 \quad \text{for } l \in \mathbb{Z} \text{ (i.e. on both } dS_2 \text{ and } \tilde{dS}_2), \quad (76)
\]

\[
a_l a_{-l} - b_l b_{-l} = c_l \sin(\pi \lambda), \quad a_l b_{-l} - b_l a_{-l} = c_l \sin(\pi l) \quad \text{for } l \in \frac{1}{2} + \mathbb{Z} \text{ (only on } \tilde{dS}_2). \quad (77)
\]

The constants \(c_l\) are unrestricted by the above conditions, which are summarized as follows:

\[
a_{-l} = c_l (a_l \sin(\pi \lambda) + b_l \sin(\pi l)), \quad b_{-l} = c_l (b_l \sin(\pi \lambda) + a_l \sin(\pi l)). \quad (78)
\]

The normalization condition is given by

\[
(\phi_l, \phi_{\nu})_{KG} = \frac{2k\pi}{\gamma_l} (|a_l|^2 - |b_l|^2) \delta_{\nu l} = \frac{1}{N_l} \delta_{\nu l}
\]

where \(k = 1\) for \(dS_2\) and \(k = 2\) for \(\tilde{dS}_2\) and

\[
\gamma_l = \frac{1}{2} \Gamma(l - \lambda) \Gamma(1 + \lambda + l)
\]

so that

\[
N_l = \frac{\gamma_l}{2k\pi(|a_l|^2 - |b_l|^2)} = 1.
\]

As a function of \(l\), the product \(\gamma_l\) is always positive for \(\lambda = -\frac{1}{2} + i\nu\). When \(-1 < \lambda < 0\) it takes negative values for negative half integer \(l\)'s.

The commutator finally takes the following form:

\[
C(t, \theta, t', \theta') = \sum_{k,l \in \mathbb{Z}} N_l[\phi_l(t, \theta)\phi^*_l(t', \theta') - \phi_l(t', \theta')\phi^*_l(t, \theta)] =
\]

\[
= \sum_{k,l \in \mathbb{Z}} N_l(|a_l|^2 - |b_l|^2)[P_{\lambda}^{-l}(i \sh t)P_{\lambda}^{-l}(-i \sh t') - P_{\lambda}^{-l}(-i \sh t)P_{\lambda}^{-l}(i \sh t')] \cos(l\theta - l't')
\]

\[
+ \sum_{k,l \in \mathbb{Z}} iN_l(|a_l|^2 + |b_l|^2)[P_{\lambda}^{-l}(i \sh t)P_{\lambda}^{-l}(-i \sh t') + P_{\lambda}^{-l}(-i \sh t)P_{\lambda}^{-l}(i \sh t')] \sin(l\theta - l't')
\]

\[
+ \sum_{k,l \in \mathbb{Z}} [2iN_l a_l b_l P_{\lambda}^{-l}(i \sh t)P_{\lambda}^{-l}(i \sh t') - 2iN_l a^*_l b^*_l P_{\lambda}^{-l}(-i \sh t)P_{\lambda}^{-l}(-i \sh t')] \sin(l\theta - l't').
\]

where, again, \(k = 1\) for \(dS_2\) and \(k = 2\) for \(\tilde{dS}_2\). We left in this expression explicitly indicated \(N_l\) as a function of \(a_l\) and \(b_l\), as in Eq. (81), even though the normalization condition imposes \(N_l = 1\). This allows to verify the locality property of the above expression more easily. Let us indeed prove that the equal time commutator

\[
C(0, \theta, 0, \theta') = 2i \sum_{k,l \in \mathbb{Z}} \frac{\gamma_l(|a_l|^2 + |b_l|^2)}{2k\pi(|a_l|^2 - |b_l|^2)}[P_{\lambda}^{-l}(0)]^2 \sin(l\theta - l't'). \quad (83)
\]
vanishes. The terms contributing to \( C(0, \theta, 0, \theta') \) are the ones antisymmetric in the exchange of \( \theta \) and \( \theta' \) (the second and third line in Eq. (82)). By using Eq. (78) we have that

\[
\frac{|a_l + b_l|^2}{|a_l|^2 - |b_l|^2} = \cot \left( \frac{1}{2} \pi (l + \lambda) \right) \tan \left( \frac{1}{2} \pi (\lambda - l) \right) \frac{|a_{-l} + b_{-l}|^2}{|a_{-l}|^2 - |b_{-l}|^2}
\] (84)

In deriving the above identity we took into account the hypothesis \( \lambda = -1/2 + i \nu \) or \( \lambda \) real, which implies that \( \sin \pi \lambda \) is a real number.

On the other hand formula (66) gives

\[
\gamma_l P_{\lambda}^{-l}(0)^2 \gamma_{-l} P_{\lambda}^{l}(0)^2 = \tan \left( \frac{1}{2} \pi (l + \lambda) \right) \cot \left( \frac{1}{2} \pi (\lambda - l) \right) .
\] (85)

Therefore the coefficients of \( \sin[l(\theta - \theta')] \) and of \( \sin[-l(\theta - \theta')] \) are equal and the equal time commutator \( C(0, \theta, 0, \theta') \) vanishes.

Let us verify now the CCR’s:

\[
\partial_t C(t, \theta, t, \theta')|_{t=t'=0} = -2i \sum_{kl \in \mathbb{Z}} \frac{\gamma_l}{2k\pi} [P_{\lambda}^{-l}(0)P_{\lambda}^{l}(0)] \cos (l \theta - l \theta') +
\]

\[
+2i \sum_{kl \in \mathbb{Z}} \frac{\gamma_l}{2k\pi} (|a_l|^2 - |b_l|^2) (a_l^* b_{-l} - a_{-l}^* b_l) [P_{\lambda}^{-l}(0)P_{\lambda}^{l}(0)] \sin (l \theta - l \theta') =
\]

\[
= i \sum_{kl \in \mathbb{Z}} \frac{1}{2k\pi} \cos (l \theta - l \theta') = i \delta(\theta - \theta' \mod 2\pi)
\] (86)

where we used again Eq. (78) As a byproduct we deduce that the second and third line in Eq. (82) vanish identically and the covariant commutator may be re-expressed as follows:

\[
C(t, \theta, t', \theta') = \sum_{kl \in \mathbb{Z}} \frac{\gamma_l}{2k\pi} [P_{\lambda}^{-l}(i \cosh t)P_{\lambda}^{l}(-i \cosh t') - P_{\lambda}^{-l}(-i \cosh t)P_{\lambda}^{l}(i \cosh t')] \cos (l \theta - l \theta')
\]

\[
= \sum_{kl \in \mathbb{Z}} \frac{\gamma_l}{2k\pi} [P_{\lambda}^{-l}(i \cosh t)P_{\lambda}^{l}(-i \cosh t') - P_{\lambda}^{-l}(-i \cosh t)P_{\lambda}^{l}(i \cosh t')] \exp (i l \theta - i l \theta') .
\] (87)

The second step follows from the symmetry of the generic term of the series at the right hand side of Eq. (87) under the change \( l \rightarrow -l \).

7 \( SL(2, \mathbb{R}) \)-invariance of the commutator

While the \( SL(2, \mathbb{R}) \) invariance of the commutator is a priori guaranteed by the vanishing of the equal time commutator and by the CCR’s (86), it is instructive for what follows to give a direct proof based on the recurrence relations satisfied by the Legendre functions on the cut [20]. This will prepare the task of Section 8 where the question of finding the more general invariant two-point function will be addressed.
To this aim, let us start by considering the first term at the RHS of Eq. (87):

$$W_0(\tilde{x}, \tilde{x}') = \frac{1}{4\pi} \sum_{2l \in \mathbb{Z}} \gamma_l \left[ P_{-l}^{-1}(i \sin t)P_{l}^{-1}(-i \sin t') \right] \exp(i(\theta - \theta')).$$

(88)

This kernel is non-local but it turns out to be $SL(2, \mathbb{R})$-invariant. The proof of this statement amounts to checking that the following infinitesimal condition holds:

$$\delta W = \sin \theta \partial_\theta W + \theta t \cos \theta \partial_{\theta'} W + \sin \theta' \partial_\theta W + \theta t' \cos \theta' \partial_{\theta'} W = 0$$

(89)

namely

$$\delta W_0 = 0 = \sum_l \gamma_l \left[ i \sin \theta \partial_\theta P_{-l}^{-1}(z) + il \theta t \cos \theta P_{-l}^{-1}(z) \right] P_{l}^{-1}(-z') e^{i(\theta - \theta')}
- \sum_l \gamma_l P_{-l}^{-1}(z) \left[ i \sin \theta' \partial_{\theta'} P_{-l}^{-1}(z') + il \theta t' \cos \theta' P_{-l}^{-1}(z') \right] e^{i(\theta - \theta')}.
$$

(90)

Singling out the Fourier coefficient of $\exp(i\theta)$, the above condition translates into the following requirement:

$$\gamma_{-1} P_{-1}^{-1}(-z') e^{i\theta'} \left[ \partial_\theta P_{-1}^{-1}(z) + i(l - 1) \theta t P_{-1}^{-1}(z) \right] + \gamma_{l+1} P_{l+1}^{-1}(-z') e^{-i\theta} \left[ \partial_{\theta'} P_{l+1}^{-1}(z) + i(l + 1) \theta t P_{l+1}^{-1}(z) \right] + \gamma_l P_{l}^{-1}(z) \left[ i \sin \theta' \partial_{\theta'} P_{l}^{-1}(z') + il \theta t' \cos \theta' P_{l}^{-1}(z') \right] e^{i\theta} = 0.$$

(91)

This expression may be simplified by using the following crucial identities:

$$\text{ch} \, t \, P_{l}^{1-l}(z) + i(l - 1) \theta t \, P_{l}^{-1-l}(z) = (\lambda - l + 1) \theta t \, P_{l}^{-1-l}(z)
= (\lambda - l + 1)(\lambda + l) \left[ -\text{ch} \, t \, P_{l}^{-1-l}(z) + i(l + 1) \theta t \, P_{l}^{-1-l}(z) \right]$$

(92)

$$= (\lambda - l + 1)(\lambda + l) P_{l}^{-1-l}(z).$$

(93)

It takes a little work to verify that the above formulae are nothing but a rewriting of known relations among the Legendre functions. To prove Eq. (92), one first removes the derivative $P' = \frac{dP}{dz}$ by using Eq. 3.8.19 from Bateman’s book [20] and get

$$i(\lambda - l + 1) \sin t \, P_{l}^{1-l}(z) + (\lambda - l + 1)P_{l}^{1-l}(z)
-(\lambda - l + 1)(\lambda + l) \left[ -i \sin t \, (\lambda + l + 1)P_{l}^{1-l}(z) + (\lambda - l - 1)P_{l-1}^{-1-l}(z) \right] = 0.$$

(94)

Eqs. 3.8.11 and 3.8.15 from allow to show that (94) is equivalent to

$$P_{l}^{1-l}(z) - 2il \sin t \, P_{l}^{-1-l}(z) + (\lambda - l)(\lambda + l + 1)P_{l}^{-1-l}(z) = 0$$

(95)

\footnote{The word local here and everywhere refers to \textit{local commutativity}. We stress again that locality on the de Sitter manifold and on its covering are two distinct notions.}
which in turn coincides with Eq. 3.8.11 from [20]. To prove the second equality (93) one invokes Eqs. 3.8.17 and 3.8.19 [20].

Now we are ready to show the \( SL(2, \mathbb{R}) \)-invariance of the kernel (88). Let us insert Eqs. (92) and (93) in Eq. (91) and divide by \( \gamma_{l-1} \); we get the following equivalent expression

\[
\left[ e^{i\theta'} P_\lambda^{-l}(-z') - e^{-i\theta'} (l - \lambda)(\lambda + l + 1) P_\lambda^{-l}(-z') \right] + \\
2 \left[ i \sin \theta' \cosh t' P_\lambda'^{-l}(-z') + il \cos \theta' P_\lambda^{-l}(-z') \right] = 0. \tag{96}
\]

Here the variable \( z \) has disappeared and the condition (91) is now tractable. By singling out the coefficients of \( \cos \theta' \) and \( \sin \theta' \) we are led to examine the validity of the following identities:

\[
P_\lambda'^{-l}(-z') - (l - \lambda)(\lambda + l + 1) P_\lambda'^{-l}(-z') + 2il \cos \theta' P_\lambda^{-l}(-z') = 0, \tag{97}
\]

\[
P_\lambda'^{-l}(-z') + (l - \lambda)(\lambda + l + 1) P_\lambda'^{-l}(-z') + 2 \sin \theta' P_\lambda'^{-l}(-z') = 0. \tag{98}
\]

Eq. (97) it is once more a known relationship among Legendre functions on the cut, namely Eq. 3.8.11 of [20]. As regards the second identity, it can be proven by observing that the difference of the above two equations coincides with the relation given in Eq. (92). The \( SL(2, \mathbb{R}) \)-invariance of the kernel (88) is proven.

An immediate corollary that follows from Eqs. (97) and (98) is that the kernels obtained by taking the even and the odd parts of \( W_0(\tilde{x}, \tilde{x}') \), namely

\[
W_{0, \text{even}}(\tilde{x}, \tilde{x}') = \frac{1}{4\pi} \sum_{l \in \mathbb{Z}} \gamma_l \left[ P_\lambda'^{-l}(z)P_\lambda'^{-l}(-z') \right] \exp(il(\theta - \theta')) \tag{99}
\]

and

\[
W_{0, \text{odd}}(\tilde{x}, \tilde{x}') = \frac{1}{4\pi} \sum_{\frac{1}{2} + l \in \mathbb{Z}} \gamma_l \left[ P_\lambda'^{-l}(z)P_\lambda'^{-l}(-z') \right] \exp(il(\theta - \theta')) \tag{100}
\]

are separately invariant. The even part coincides with the so-called Bunch-Davies vacuum [13, 14, 15, 16, 17, 18].

The second corollary is the invariance of the commutator. Let us consider indeed the map \( \tau_1 \) given in Eq. (51). Since it commutes with the action of \( SL(2, \mathbb{R}) \) on the covering manifold \( dS_2 \) we immediately get that also the kernel

\[
W_1(\tilde{x}, \tilde{x}') = \sum_{2l \in \mathbb{Z}} \gamma_l \left[ P_\lambda'^{-l}(i \sin t)P_\lambda'^{-l}(-i \sin t') \right] \exp(-il(\theta - \theta')) \tag{101}
\]

is \( SL(2, \mathbb{R}) \) invariant. The invariance of the commutator follows.
8 Invariance under $SL(2, \mathbb{R})$ and other properties of general two-point functions

Once given the commutator, the crucial step to get a physical model is to represent the field $\phi$ as an operator-valued distribution in a Hilbert space $\mathcal{H}$. This can be done by finding a positive-semidefinite bivariate distribution $W(\tilde{x}, \tilde{y})$ solving the KG equation and the functional equation \[ C(\tilde{x}, \tilde{x}') = W(\tilde{x}, \tilde{x}') - W(\tilde{x}', \tilde{x}). \] (102)

Actually, $C$ and $W$ are not functions but distributions so the above equation must be understood in the sense of distributions. Given a solution $W$ of Eq. (102) an analogue of Wightman’s reconstruction theorem [24] (in the simplest case of generalised free fields) provides the Fock space of the theory and a representation of the field as a local operator-valued distribution.

There are of course infinitely many inequivalent solution of Eq. (102). Here we will characterize the most general $SL(2, \mathbb{R})$-invariant solution. To this aim let us first consider a general two-point function, i.e. a distribution $W$ on $\tilde{dS}_2 \times \tilde{dS}_2$. There are several conditions that we may (or may not) want to impose on such a function.

(C1) Local commutativity (locality): Let
\[ \mathcal{R} = \{(\tilde{x}, \tilde{x}') \in \tilde{dS}_2 \times \tilde{dS}_2 : \tilde{x} \text{ and } \tilde{x}' \text{ are spacelike separated}\}. \] (103)

$W(\tilde{x}, \tilde{x}')$ has the property of local commutativity (or locality) if
\[ W(\tilde{x}, \tilde{x}') - W(\tilde{x}', \tilde{x}) = 0 \quad \forall (\tilde{x}, \tilde{x}') \in \mathcal{R}. \] (104)

(C2) Symmetry or anti-symmetry:
\[ W(\tilde{x}, \tilde{x}') = \pm W(\tilde{x}', \tilde{x}) \quad \forall (\tilde{x}, \tilde{x}') \in \tilde{dS}_2 \times \tilde{dS}_2. \] (105)

(C3) Invariance under the group $SL(2, \mathbb{R})$:
\[ W(\tilde{g}\tilde{x}, \tilde{g}\tilde{x}') = W(\tilde{x}, \tilde{x}') \quad \forall (\tilde{x}, \tilde{x}') \in \tilde{dS}_2 \times \tilde{dS}_2. \forall \tilde{g} \in SL(2, \mathbb{R}). \] (106)

(C4) Hermiticity:
\[ W(\tilde{x}, \tilde{x}') = \overline{W(\tilde{x}', \tilde{x})} \quad \forall (\tilde{x}, \tilde{x}') \in \tilde{dS}_2 \times \tilde{dS}_2. \] (107)

(C5) Positive definiteness:
\[ \int \int W(\tilde{x}, \tilde{x}') \bar{f}(\tilde{x}) f(\tilde{x}') d\tilde{x} d\tilde{x}' \geq 0 \quad \forall f \in C_0^\infty(\tilde{dS}_2). \] (108)
(C6) Klein-Gordon equation in $\tilde{x}$ and $\tilde{x}'$ with a “mass” $\lambda$.

(C7) Canonical Commutation Relations (102).

(C8) Analyticity. By this we mean that there is an open tuboid $\mathcal{U}_+$ in a complexified version of $d\tilde{S}_2 \times d\tilde{S}_2$ (see Sect. 3) such that, in a neighborhood of any real point $(x, x') \in d\tilde{S}_2 \times d\tilde{S}_2$ we have, in the sense of distributions,

$$F(x, x') = \lim_{(w, w') \in \mathcal{U}_+, (w, w') \to (x, x')} F_+(w, w'),$$

(109)

where $F_+$ is holomorphic with locally polynomial behavior in $\mathcal{U}_+$. $F_-(w, w') \overset{\text{def}}{=} F_+(w', w)$ is analytic in $\mathcal{U}_- = \{ (w, w') : (w', w) \in \mathcal{U}_+ \}$ (110) and we suppose $\mathcal{U}_- = \overline{\mathcal{U}_+}$ (111).

(C9) Local analyticity:

By this we mean that there is a complex open connected neighborhood $\mathcal{N}$ of $\mathcal{R}$ such that, in $\mathcal{R}$, both $F(\tilde{x}, \tilde{x}')$ and $F(\tilde{x}', \tilde{x})$ are restrictions of the same function holomorphic in $\mathcal{N}$. If a two-point function $F$ has the two properties of locality and analyticity as defined above, then it also has the property of local analyticity by the edge-of-the-wedge theorem.

If $W$ is any two-point function, it can be written as $W = W_r + iW_i$, where

$$W_r(\tilde{x}, \tilde{x}') = \frac{1}{2} W(\tilde{x}, \tilde{x}') + \frac{1}{2} W(\tilde{x}', \tilde{x}), \quad W_i(\tilde{x}, \tilde{x}') = \frac{1}{2i} W(\tilde{x}, \tilde{x}') - \frac{1}{2i} W(\tilde{x}', \tilde{x}).$$

(112)

$W_r$ and $W_i$ are hermitic, and if $W$ satisfies any one of the conditions (C2), (C3), or (C6), so do $W_r$ and $W_i$. If $W$ is any two-point function, it can be written as $W = W_{\text{even}} + W_{\text{odd}}$ where

$$W_{\text{even}}(\tilde{x}, \tilde{x}') = \frac{1}{2} W(\tilde{x}, \tilde{x}') + \frac{1}{2} W(\tilde{x}, \tau \tilde{x}'), \quad W_{\text{odd}}(\tilde{x}, \tilde{x}') = \frac{1}{2} W(\tilde{x}, \tilde{x}') - \frac{1}{2} W(\tilde{x}, \tau \tilde{x}').$$

(113)

Those two-point functions which are rotation-invariant in our fixed frame, i.e. such that

$$W((t, \theta), (t', \theta')) = W((t, \theta + a), (t', \theta' + a)) \quad \forall a \in \mathbb{R}$$

(114)

(recall that $\tau$ is such a rotation for $a = 2\pi$) can be expanded in a Fourier series as follows:

$$W(\tilde{x}, \tilde{x}') = \sum_{l \in \mathcal{L}} c_l(\tau) e^{i(l \theta - \theta')}.$$
As before $z = i\text{sh} t$ and $z' = i\text{sh} t'$. The set $\mathcal{L}$ can be $\mathbb{Z}$, $\frac{1}{2}\mathbb{Z}$, or $\frac{1}{2} + \mathbb{Z}$. If $\mathcal{L} = \frac{1}{2}\mathbb{Z}$ then $W_{\text{even}}$ (resp. $W_{\text{odd}}$), as defined in (113), is the sum over $\mathbb{Z}$ (resp. $\frac{1}{2} + \mathbb{Z}$).

The following is the main result of the present paper:

**Theorem 1** The most general $SL(2, \mathbb{R})$-invariant hermitic two-point function satisfying the Klein-Gordon equation in each variable for a positive squared mass $m_\lambda^2 = -\lambda(\lambda + 1) > 0$ (i.e., for the principal and the complementary values) is characterized by four independent real constants $A_0, B_0, A_{\frac{1}{2}}, B_{\frac{1}{2}}$ and two independent complex constants $C_0, C_{\frac{1}{2}}$ by the following Fourier expansion

$$W(\tilde{x}, \tilde{x}') = \sum_{2l \in \mathbb{Z}} \gamma_l [A_l \mathcal{P}_\lambda^{-l}(z)\mathcal{P}_\lambda^{-l}(-z') + B_l \mathcal{P}_\lambda^{-l}(-z)\mathcal{P}_\lambda^{-l}(z')] + e^{i\pi l} C_l \mathcal{P}_\lambda^{-l}(z)\mathcal{P}_\lambda^{-l}(z') + e^{-i\pi l} C_l^* \mathcal{P}_\lambda^{-l}(-z)\mathcal{P}_\lambda^{-l}(-z')] e^{i \theta(\theta - \theta')}.$$  (116)

where

$$A_l = A_0, \quad B_l = B_0, \quad C_l = C_0 \quad \text{for } l \in \mathbb{Z},$$

$$A_l = A_{\frac{1}{2}}, \quad B_l = B_{\frac{1}{2}}, \quad C_l = C_{\frac{1}{2}} \quad \text{for } l \in \frac{1}{2} + \mathbb{Z}.$$

*Here we neither impose the locality property nor the positive definiteness.* We have put $\gamma_l = \frac{1}{2} \Gamma(l - \lambda) \Gamma(1 + \lambda + l)$ in evidence for convenience, by taking inspiration from the previous section.

Let us therefore consider the kernel given in Eq. (116). For the chosen mass parameters $\lambda$ and $z \in i\mathbb{R}$ it happens that $\mathcal{P}_\lambda^{-l}(z) = \mathcal{P}_\lambda^{-l}(-z)$. With this restriction, $W$ is hermitic iff $A_l = A_l^*$, $B_l = B_l^*$.

The two-point function (116) is $SL(2, \mathbb{R})$-invariant if and only if condition (89) holds. This amounts to

$$\delta_A + \delta_B + \delta_C + \delta_{C^*} = \sum_{2l \in \mathbb{Z}} i \gamma_l \sin \theta t \left[ (A_l \mathcal{P}_\lambda^{-l}(-z') + C_l e^{i\pi l} \mathcal{P}_\lambda^{-l}(z')) \mathcal{P}_\lambda^{-l}(z) - (B_l \mathcal{P}_\lambda^{-l}(z') + C_l^* e^{-i\pi l} \mathcal{P}_\lambda^{-l}(-z')) \mathcal{P}_\lambda^{-l}(-z) \right] e^{i \theta(\theta - \theta')}$$

$$+ \sum_{2l \in \mathbb{Z}} i \gamma_l \cos \theta t \left[ (A_l \mathcal{P}_\lambda^{-l}(z') + C_l e^{i\pi l} \mathcal{P}_\lambda^{-l}(z')) \mathcal{P}_\lambda^{-l}(z) + (B_l \mathcal{P}_\lambda^{-l}(z') + C_l^* e^{-i\pi l} \mathcal{P}_\lambda^{-l}(-z')) \mathcal{P}_\lambda^{-l}(-z) \right] e^{i \theta(\theta - \theta')}$$

$$+ \sum_{2l \in \mathbb{Z}} i \gamma_l \sin \theta t' \left[ (B_l \mathcal{P}_\lambda^{-l}(-z) + C_l e^{i\pi l} \mathcal{P}_\lambda^{-l}(z)) \mathcal{P}_\lambda^{-l}(z') - (A_l \mathcal{P}_\lambda^{-l}(z) + C_l^* e^{-i\pi l} \mathcal{P}_\lambda^{-l}(-z)) \mathcal{P}_\lambda^{-l}(-z') \right] e^{i \theta(\theta - \theta')}$$

$$- \sum_{2l \in \mathbb{Z}} i \gamma_l \cos \theta t' \left[ (B_l \mathcal{P}_\lambda^{-l}(-z) + C_l e^{i\pi l} \mathcal{P}_\lambda^{-l}(z)) \mathcal{P}_\lambda^{-l}(z') + (A_l \mathcal{P}_\lambda^{-l}(z) + C_l^* e^{-i\pi l} \mathcal{P}_\lambda^{-l}(-z)) \mathcal{P}_\lambda^{-l}(-z') \right] e^{i \theta(\theta - \theta')} = 0$$

(117)
where $\delta_A$ includes all the terms containing $A$ and so on. Singling out the Fourier coefficient of $\exp il\theta$ we get

$$
\delta_A(l) = \frac{1}{2} e^{-il\theta} \gamma_l e^{i\theta} A_{l-1} \mathbf{P}^{1-l}(z) + i(l - 1) \sin \theta' \mathbf{P}^{1-l}(z) + i(l + 1) \cos \theta' \mathbf{P}^{1-l}(z) + \mathbf{P}^{1-l}(z) \frac{z}{2} \frac{l}{2} (\lambda + 1) + 1) e^{-i\theta} A_{l+1} \mathbf{P}^{1-l}(z) + e^{-il\theta} \gamma_l A_l \mathbf{P}^{1-l}(z) \left[ i \sin \theta' \mathbf{P}^{1-l}(z) + \mathbf{P}^{1-l}(z) \right].
$$

By taking into account the crucial identities (92) and (93) and also Eq. (80) this expression includes all the terms containing $e^{-l\theta} \gamma_l e^{i\theta} A_{l-1} \mathbf{P}^{1-l}(z)$.

By using the operator $\gamma_1$ and $\gamma_2$ we also immediately get that

$$
\delta_B(l) = \frac{1}{2} e^{-il\theta} \gamma_l e^{i\theta} B_{l-1} \mathbf{P}^{1-l}(z) + i(l - 1) \sin \theta' \mathbf{P}^{1-l}(z) + i(l + 1) \cos \theta' \mathbf{P}^{1-l}(z) + \mathbf{P}^{1-l}(z) \frac{z}{2} \frac{l}{2} (\lambda + 1) + 1) e^{-i\theta} B_{l+1} \mathbf{P}^{1-l}(z) + e^{-il\theta} \gamma_l B_l \mathbf{P}^{1-l}(z) \left[ i \sin \theta' \mathbf{P}^{1-l}(z) + \mathbf{P}^{1-l}(z) \right],
$$

$$
\delta_C(l) = \frac{1}{2} e^{-il\theta} e^{i\theta} C_{l-1} \mathbf{P}^{1-l}(z) + i(l - 1) \sin \theta' \mathbf{P}^{1-l}(z) + i(l + 1) \cos \theta' \mathbf{P}^{1-l}(z) + \mathbf{P}^{1-l}(z) \frac{z}{2} \frac{l}{2} (\lambda + 1) + 1) e^{-i\theta} C_{l+1} \mathbf{P}^{1-l}(z) + e^{-il\theta} \gamma_l C_l \mathbf{P}^{1-l}(z) \left[ i \sin \theta' \mathbf{P}^{1-l}(z) + \mathbf{P}^{1-l}(z) \right],
$$

$$
\delta_{C^*}(l) = -\frac{1}{2} e^{-il\theta} e^{i\theta} C^*_{l-1} \mathbf{P}^{1-l}(z) + i(l - 1) \sin \theta' \mathbf{P}^{1-l}(z) + i(l + 1) \cos \theta' \mathbf{P}^{1-l}(z) + \mathbf{P}^{1-l}(z) \frac{z}{2} \frac{l}{2} (\lambda + 1) + 1) e^{-i\theta} C^*_{l+1} \mathbf{P}^{1-l}(z) + e^{-il\theta} \gamma_l C^*_l \mathbf{P}^{1-l}(z) \left[ i \sin \theta' \mathbf{P}^{1-l}(z) + \mathbf{P}^{1-l}(z) \right].
$$

Let us begin by discussing the simpler case where $B = C = 0$ which reduces to $\delta_A(l) = 0$. Singling out as before the coefficients of $\sin \theta'$ and $\cos \theta'$ we get

$$
\left( \left( \mathbf{P}^{1-l}(z') - (l - \lambda)(\lambda + l + 1) \frac{A_{l-1}}{2} \mathbf{P}^{1-l}(z') + iA_l \mathbf{P}^{1-l}(z') \right) \mathbf{P}^{1-l}(z) = 0
$$

$$
\left( -\frac{A_{l-1}}{2} \mathbf{P}^{1-l}(z') + (l - \lambda)(\lambda + l + 1) \frac{A_{l+1}}{2} \mathbf{P}^{1-l}(z') - iA_l \mathbf{P}^{1-l}(z') \right) \mathbf{P}^{1-l}(z) = 0.
$$

23
Taking their sum we get
\[(A_l - A_{l-1}) P^{1-l}_\lambda(-z') P^{-l}_\lambda(z) = 0.\]  
(120)
This shows that \(A_l = A_0\) for \(l \in \mathbb{Z}\) and \(A_l = A_{\frac{1}{2}}\) for \(l \in \frac{1}{2} + \mathbb{Z}\). The above two equations now reduce to
\[
P^{1-l}_\lambda(-z') - (l - \lambda)(\lambda + l + 1) P^{-1-l}_\lambda(-z') + 2i l \text{th} t' P^{-l}_\lambda(-z') = 0 \]  
(121)
and this is a known relation between contiguous Legendre functions (Bateman Eq. 3.8.11). In the general case, proceeding in the same way, we get that condition (117) implies
\[(A_l - A_{l-1}) P^{1-l}_\lambda(-z') P^{-l}_\lambda(z) - (B_l - B_{l-1}) P^{1-l}_\lambda(-z') + e^{i\pi l} (C_l - C_{l-1}) P^{1-l}_\lambda(-z') P^{-l}_\lambda(z) = 0. \]  
(122)
Therefore also in the general case the possible values for \(A_l, B_l\) and \(C_l\) are characterized by only 8 arbitrary real constants:
\[
A_l = A_0, \quad B_l = B_0, \quad C_l = C_0 \quad \text{for} \quad l \in \mathbb{Z} \\
A_l = A_{\frac{1}{2}}, \quad B_l = B_{\frac{1}{2}}, \quad C_l = C_{\frac{1}{2}} \quad \text{for} \quad l \in \frac{1}{2} + \mathbb{Z}.
\]  
(123, 124)
The verification that these conditions indeed guarantee that \(\delta_A(l) + \delta_B(l) + \delta_C(l) + \delta_{C^*}(l) = 0\) proceeds as before.

### 8.1 Canonicity

If we impose that an invariant two-point function as in the previous theorem satisfies also the canonical commutation relation (102), a simple calculation shows that
\[
A_0 - B_0 = \frac{1}{2k\pi}, \quad A_{\frac{1}{2}} = \frac{1}{4k\pi} - e^{i\pi l} C_{\frac{1}{2}}, \quad B_{\frac{1}{2}} = -\frac{1}{4k\pi} - e^{i\pi l} C_{\frac{1}{2}}, \quad C_{\frac{1}{2}} = -C^*_{\frac{1}{2}}
\]  
(125)
while \(C_0\) is unrestricted (\(A_{\frac{1}{2}}, B_{\frac{1}{2}}\) and \(C_{\frac{1}{2}}\) are present only if \(L = \frac{1}{2} \mathbb{Z}\)).

### 8.2 Positivity

Let us again consider a hermitic 2-point function \(W\) as in Eq. (115). A necessary and sufficient condition for (C5) (Eq. (108)) to hold is that, for every \(l\) and every test function \(f\) on \(\mathbb{R}\),
\[
\int_{\mathbb{R} \times \mathbb{R}} \overline{f(t)} u_l(z, z') f(t') \, dt \, dt' \geq 0,
\]  
(126)

24
where \( z = i \text{sh} t, \ z' = i \text{sh} t' \). Suppose that \( W \) is of the form (116) with \( A_l = A_l^*, \ B_l = B_l^* \). If \( f \) is a test-function on \( \mathbb{R} \), let
\[
f_1 = \int_{\mathbb{R}} f(t) P_{-l}^{-l}(-i \text{sh} t) \, dt, \quad f_2 = \int_{\mathbb{R}} f(t) P_{-l}^{-l}(i \text{sh} t) \, dt.
\]
(127)

Then
\[
\int_{\mathbb{R} \times \mathbb{R}} \overline{f(t)} u_l(z, z') f(t') \, dt \, dt' = \gamma_l \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right) \left( \begin{array}{cc} A_l & e^{i \pi l} C_l^* \\ e^{-i \pi l} C_l & B_l \end{array} \right) \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right).
\]
(128)

Therefore \( W \) is of positive type if and only if
\[
\gamma_l A_l \geq 0, \ \gamma_l B_l \geq 0, \ \gamma_l^2 (A_l B_l - C_l C_l^*) \geq 0 \quad \forall l \in \mathcal{L}.
\]
(129)

Let us now suppose that \( W \) is invariant, i.e. the conditions (123) and (124) are satisfied. If \( l \in \mathbb{Z} \), with our choices of \( \lambda, \ \gamma_l > 0 \) for all \( l \in \mathbb{Z} \), hence \( W_{\text{even}} \) is of positive type iff
\[
A_0 > 0, \quad B_0 > 0, \quad A_0 B_0 - C_0 C_0^* > 0.
\]
(130)

If \( l \in \frac{1}{2} + \mathbb{Z} \) and \( \lambda = -\frac{1}{2} + i \rho, \ \rho \neq 0 \), then \( \gamma_l \) is always \( > 0 \) and \( W_{\text{odd}} \) is of positive type iff
\[
A_{\frac{1}{2}} > 0, \quad B_{\frac{1}{2}} > 0, \quad A_{\frac{1}{2}} B_{\frac{1}{2}} - C_{\frac{1}{2}} C_{\frac{1}{2}}^* > 0.
\]
(131)

If \( l \in \frac{1}{2} + \mathbb{Z} \) and \( -1 < \lambda < 0 \), then \( \gamma_l \) has the sign of \( l \) and \( W_{\text{odd}} \) is never of positive type.

The consequence of the last statement is the disappearance of the complementary series on the double covering of the de Sitter manifold. In other words there exists no local and \( SL(2, \mathbb{R}) \)-covariant scalar free field on \( \tilde{dS}_2 \) with mass
\[
0 < m^2 < \frac{1}{4}.
\]
(132)

We will clarify further this point by studying the vacuum representations in the following section.

9 Study of local ”vacuum” states invariant under \( SL(2, \mathbb{R}) \)

One particular instance of Eq. (116) is the two-point function constructed in terms of the system of modes \( \phi_l \) (74) as their ”vacuum” expectation value:
\[
W(x, x') = \sum_l \phi_l(x) \phi_l^*(x') = \sum_l [a_l \mathbf{P}_l^{-l}(i \text{sh} t) + b_l \mathbf{P}_l^{-l}(-i \text{sh} t)][a_l^* \mathbf{P}_l^{-l}(-i \text{sh} t') + b_l^* \mathbf{P}_l^{-l}(i \text{sh} t')] e^{il\theta - il\theta'}
\]
(133)
For any possible choice of $a_l$ and $b_l$ the "vacuum states" given by Eq. (133) are "pure states" i.e. they provide through the GNS construction irreducible representations of the field algebra.

Let us now single out among them those states who are $SL(2,\mathbb{R})$–invariant. This is an easy corollary of the theorem of the previous section. Eqs. (123) and (124) imply the following relations:

\[
|a_l|^2 = c_1(\epsilon)\gamma_l, \quad |b_l|^2 = c_2(\epsilon)\gamma_l, \quad a_l b_l^* = c_3(\epsilon)\gamma_l e^{i\phi}
\]

where $\epsilon = 0$ for $l \in \mathbb{Z}$ and $\epsilon = 1$ for $l \in \mathbb{Z} + \frac{1}{2}$ (i.e. there are six independent constants). The above equations, together with the normalization condition (81), can be solved as follows:

\[
a_l = \sqrt{\frac{2\gamma}{\pi k}} \text{ch} \alpha_{\epsilon}, \quad b_l = \sqrt{\frac{2\gamma}{\pi k}} \text{sh} \alpha_{\epsilon} e^{i\phi} e^{-i\phi}, \quad \epsilon = 0, 1.
\]

Here we took $a_l$ real without loss of generality.

9.1 Canonicity: pure de Sitter

Let us examine whether the above equations are compatible with the requirements imposed by canonicity. In the pure de Sitter case (as opposed to its covering) $l$ is integer and the CCR’s amount to the condition (76) which imposes no further restriction and any choice of $\alpha_0$ and $\phi_0$ gives rise to a de Sitter invariant state which has the right commutator (relatively to the de Sitter manifold). These states are well-known: they are the so-called alpha vacua [25, 26, 27, 28].

Among them, there is a particularly important state corresponding to the choice $\alpha_0 = 0$: this is the so-called Bunch-Davies vacuum \[13, 14, 15, 16, 17, 18\]

\[
W_{BD}(x, x') = W_{\alpha_0 = 0}(x, x') = \sum_{l \in \mathbb{Z}} \frac{2\gamma}{2\pi} P_{-l}(i \text{sh} t) P_{-l}(-i \text{sh} t') e^{i\theta - i\theta'} = \frac{\Gamma(-\lambda)\Gamma(\lambda + 1)}{4\pi} P_{\lambda}(\zeta), \quad (136)
\]

where $P_{\lambda}(\zeta)$ is the associated Legendre function of the first kind [20] and the de Sitter invariant variable $\zeta$ is the scalar product $\zeta = x(t - i\epsilon, \theta) \cdot x'(t' + i\epsilon, \theta)$ in the ambient space sense. Actually, $W_{BD}(x, x')$ admits an extension to the complex de Sitter manifold and satisfies there the maximal analyticity property \[16, 17, 18\]: it is holomorphic for all $\zeta \in \mathbb{C} \setminus (-\infty, -1]$ i.e. everywhere except on the locality cut. This crucial property singles the Bunch-Davies vacuum out of all the other invariant vacua and has a very well known thermal interpretation \[15, 16, 17, 18\]: the restriction of the Bunch-Davies state to a wedge-like region is a thermal state at temperature $T = 1/2\pi$. A similar property is expected in interacting theories based on an analogue of the Bisognano-Wichmann theorem \[18\]. We will come back on this point later.
9.2 Covering

In the antiperiodic case the CCR’s

\[
\begin{align*}
  a_l a_{-l} - b_l b_{-l} &= c_l \sin(\pi l) \\
  a_l b_{-l} - b_l a_{-l} &= c_l \sin(\pi l)
\end{align*}
\]

(137)

imply the following relation between the constants \( \alpha \) and \( \phi \) and the mass parameter \( \lambda \) of the field:

\[
e^{i \phi} \sin(\pi l) (-i \sinh(2\alpha) \sin(\pi \lambda) - i \cosh(2\alpha) \sin \phi + \cos \phi) = 0.
\]

(138)

For \( \lambda = -1/2 + i\nu \) there is only one possible solution given by

\[
\coth 2\alpha = \cosh \pi \nu, \quad \phi = \frac{\pi}{2}.
\]

(139)

We denote the corresponding two-point function \( W^{(1/2)}_{\nu}(x, x') \). Note that the value \( \alpha = 0 \), that would correspond to the above-mentioned maximal analyticity property, is excluded: it would be attained only for an infinite value of the mass. On the other hand Equation (138) has no solution at all when \( \lambda \) is real: there is no invariant vacuum of the complementary series.

In conclusion, for \( \lambda = -1/2 + i\nu \) the most general invariant vacuum state is the superposition of an arbitrary alpha vacuum (the even part) plus a fixed odd part \( W^{(1/2)}_{\nu}(x, x') \) as follows

\[
W(x, x') = W^{(0)}_{\alpha_0, \phi_0}(x, x') + W^{(1/2)}_{\nu}(x, x').
\]

(140)

For \( \lambda = -1/2 + \nu \) there is no \( SL(2, \mathbb{R}) \) local invariant vacuum state i.e. there is no field of a would-be complementary series.
10 Convergence and (lack of) analyticity

We again consider a series of the form (116). In the preceding sections such a series was regarded as the Fourier expansion of some two-point function (or rather distribution). Here we suppose the series given and ask about its convergence. No generality is lost by the restriction to a hermitic series. However we will consider only a particular example from which the general case can be understood. Let

\[ F_0(x, x') = \sum_{l \in \mathcal{L}} c_l(z, z') e^{il(\theta - \theta')}, \quad c_l(z, z') = \gamma_l \mathbf{P}_\lambda^{-l}(z) \mathbf{P}_\lambda^{-l}(-z'), \]

where \( z = i \text{sh} t \) and \( z' = i \text{sh} t' \). The dependence of \( c_l \) on \( \lambda \) has been omitted for simplicity, and \( \mathcal{L} = \frac{1}{2} \mathbb{Z} \). This series is the simplest example of the \( SL(2, \mathbb{R}) \) invariant series discussed in Theorem (1). We also set

\[ z = ix^0 = i \text{sh} t = i \text{tg} s, \quad z' = ix'^0 = i \text{sh} t' = i \text{tg} s' \]

\[ u = s + \theta, \quad v = s - \theta, \quad u' = s' + \theta', \quad v' = s' - \theta'. \]

Note that the variables \( u, v, u', v' \) defined here and used in this section are not those used in Sect. (2) \( c_l(z, z') \) is holomorphic in \( z \) (resp. \( z' \)) in the cut-plane \( \Delta_2 \) (see (69)). Values such that \( \text{Re} z > 0, \text{Re} z' < 0 \) correspond to \( x \in T_- \), \( x' \in T_+ \) while \( \text{Re} z < 0, \text{Re} z' > 0 \) correspond to \( x \in T_-, \) \( x' \in T_- \). The convergence of the series can be studied separately for \( l \in \mathbb{Z} \) and \( l \in \frac{1}{2} + \mathbb{Z} \).

10.1 Case of integer \( l \)

We wish to investigate the convergence of the series (141) in the case when \( l \in \mathbb{Z} \), \( \text{Re} z > 0 \) and \( \text{Re} z' < 0 \). One may check that if \( l \in \mathbb{Z} \), then \( c_l(z, z') = c_{-l}(z, z') \). The proof is based on the following relation \[ \mathbf{P}_l^\lambda(z) = \frac{\cos(l\pi)\Gamma(\lambda + l + 1)}{\Gamma(\lambda - l + 1)} \mathbf{P}_\lambda^{-l}(z), \]

(143)

Therefore it suffices to examine the half series \( l \geq 0 \). We use \[ \mathbf{P}_l^{-l}(z) = \frac{1}{\Gamma(1 + l)} \left( \frac{1 - z}{1 + z} \right)^{\frac{l}{2}} F_l \left( \frac{1 - z}{2} \right) \]

(144)

where we defined

\[ F_l(z) = F(-\lambda, 1 + \lambda; 1 + l; z) \]

(145)
Thus
\[ c_l(z, z') = \frac{\gamma_l(\lambda)}{\Gamma(1 + l)^2} \left( \frac{1 - z}{1 + z} \right)^{\frac{l}{2}} \left( \frac{1 + z'}{1 - z'} \right)^{\frac{l}{2}} F_l\left( \frac{1 - z}{2} \right) F_l\left( \frac{1 + z'}{2} \right). \] (146)

Simple geometry shows that
\[ \pm \operatorname{Re} z > 0 \iff \left| \frac{1 \mp z}{1 \pm z} \right| < 1. \] (147)

Using the discussion in Appendix A (eqs (208-214)), we make estimates of all the factors occurring in \( c_l(z, z') \) which will be valid even if \( l \) is not an integer, provided \( l \geq l_0 \) for some \( l_0 > 0 \).

We first let \( N \) be the smallest integer \( \geq |\operatorname{Re} \lambda| + 1 \). Then for \( l > N \)
\[ \left| \frac{\Gamma(l - \lambda)}{\Gamma(l + 1)} \right| \leq \frac{\Gamma(l + N)}{\Gamma(l + 1)} \leq (l + N)^{N-1} \leq (2l)^{N-1}, \]
\[ \left| \frac{\Gamma(l + 1 + \lambda)}{\Gamma(l + 1)} \right| \leq \frac{\Gamma(l + 1 + N)}{\Gamma(l + 1)} \leq (2l)^N. \] (148)

We now set \( z = i \tan s \) with \( \text{Im} s < 0 \) and \( z' = i \tan s' \) with \( \text{Im} s' > 0 \). Then
\[ \frac{1 - z}{1 + z} = e^{-2is}, \quad \left| \frac{1 - z}{1 + z} \right| < 1, \quad \operatorname{Re} z > 0, \quad \frac{1 - z'}{1 + z'} = e^{-2is'}, \quad \left| \frac{1 + z'}{1 - z'} \right| < 1, \quad \operatorname{Re} z' < 0. \] (149)

To discuss the first hypergeometric function appearing in (146) we temporarily denote \( w = (1 - z)/2 \) which satisfies
\[ \operatorname{Re} w < \frac{1}{2}, \quad w = \frac{e^{-is}}{2\cos(s)}, \quad |w| \leq \frac{1}{2|\text{Im} s|}. \] (150)

According to (208-214)
\[ |F_l(w) - 1| \leq \frac{1}{l+1} |w| M(w)|1 + \operatorname{Re} \lambda| \text{ch}(\pi \text{Im} \lambda), \]
\[ M(w) = \sup_{0 \leq u \leq 1} |(1 - uw)^{l-1}|. \] (151)

We have, for \( 0 \leq u \leq 1, \operatorname{Re} uw < \frac{1}{2} \), \( |\text{Arg}(1 - uw)| < \pi \),
\[ |(1 - uw)^{l-1}| \leq |1 - uw|^{|\text{Re} \lambda - 1}|e^{|\text{Im} \lambda|}. \] (152)

with our choices of \( \lambda, \operatorname{Re} \lambda - 1 < -1 \). Since \( \operatorname{Re} uw < \frac{1}{2}, |1 - uw| > \frac{1}{2} \), so that
\[ M(w) \leq 2^{1-\operatorname{Re} \lambda} e^{\pi |\text{Im} \lambda|}, \]
\[ |F_l\left( \frac{1 - z}{2} \right) - 1| \leq \frac{1}{(l+1)|\text{Im} s|} 2^{1-\operatorname{Re} \lambda} e^{2|\text{Im} \lambda|}. \] (153)
An analogous bound, with $s'$ instead of $s$, holds for the second hypergeometric function occurring in (146). Gathering all this shows that there are positive constants $E$ and $Q$ depending only on $\lambda$ such that

$$|c_l(z, z')| \leq E \left(1 + \frac{1}{|\text{Im } s|}\right) \left(1 + \frac{1}{|\text{Im } s'|}\right)(l + 1)^Q e^{l(\text{Im } s - \text{Im } s')}.$$  \hfill (155)

Recall again that here $\text{Im } s < 0$ and $\text{Im } s' > 0$, and that the bound (155) only requires $l \geq l_0$ for some $l_0 > 0$, and the genericity of $\lambda$.

Returning to the case of integer $l$, we see that the two series

$$\sum_{l \in \mathbb{Z}, l \geq 0} c_l(z, z') e^{il(\theta - \theta')} \quad \text{and} \quad \sum_{l \in \mathbb{Z}, l > 0} c_{-l}(z, z') e^{il(\theta' - \theta)} = \sum_{l \in \mathbb{Z}, l > 0} c_l(z, z') e^{il(\theta' - \theta)}$$  \hfill (156)

converge absolutely and uniformly on any compact subset of the tubes

$$\{(s, s', \theta, \theta') : \text{Im } s < 0, \text{ Im } s' > 0, \text{ Im}(s' - s - \theta' + \theta) > 0\}$$  \hfill (157)

and

$$\{(s, s', \theta, \theta') : \text{Im } s < 0, \text{ Im } s' > 0, \text{ Im}(s' - s + \theta' - \theta) > 0\}$$  \hfill (158)

respectively, and that the limits are holomorphic functions having boundary values in the sense of tempered distributions at the real values of $(s, s', \theta, \theta')$.

Hence $\sum_{l \in \mathbb{Z}} c_l(z, z', \lambda) e^{il(\theta - \theta')}$ converges to a function holomorphic in the tube

$$T_{-,+} = \{(s, s', \theta, \theta') : \text{Im } s < 0, \text{ Im } s' > 0, \text{ Im}(s' - s) - |\text{Im}(\theta' - \theta)| > 0\}$$  \hfill (159)

which has a tempered boundary value at the real values of $(s, s', \theta, \theta')$. Denoting $u = s + \theta$, $v = s - \theta$, $u' = s' + \theta'$, $v = s' - \theta'$, the tube (159) contains the tube

$$T_{-,+} = \{(u, v, u', v') : \text{Im } u < 0, \text{ Im } v < 0, \text{ Im } u' > 0, \text{ Im } v' > 0\}.$$  \hfill (160)

10.2 Case of half-odd-integers

We still consider the series (141) now assuming that $l \in \frac{1}{2} + \mathbb{Z}$.

10.3 Positive $l$

Here $l = n + \frac{1}{2}$, with integer $n \geq 0$. Eqs. (144) and (146) remain valid. With $\text{Im } s < 0$ and $\text{Im } s' > 0$, the estimate (155) still holds and therefore the series

$$\sum_{l \in \frac{1}{2} + \mathbb{Z}, l > 0} c_l(z, z') e^{il(\theta - \theta')}$$  \hfill (161)
converges uniformly on every compact of the tube \((157)\) (hence also of \(T_{-+}\) or \(T_{-+}\) (see \((158,160)\)) to a holomorphic function that has a boundary value in the sense of tempered distributions at real values of \(s, s', \theta, \theta'\) (or \(u, v, u', v'\)).

10.4 Negative \(l\)

Taking again \(l = \frac{1}{2} + n\) with integer \(n \geq 0\) we consider

\[
c_{-l}(z, z') = \gamma(-l)P^l_{\lambda}(z)P^l_{\lambda}(-z')
\]

and use the formula (obtainable from \([20, 3.3.2 (17) p. 141]\))

\[
P^l_{\lambda}(z) = \frac{\Gamma(l + \lambda + 1)\Gamma(l - \lambda)}{\pi \Gamma(1 + l)} \left[ -\sin(\lambda \pi) \left( \frac{1 - z}{1 + z} \right)^{\frac{1}{2}} F_1 \left( \frac{1 - z}{2} \right) 
+ \sin(l \pi) \left( \frac{1 + z}{1 - z} \right)^{\frac{1}{2}} F_1 \left( \frac{1 + z}{2} \right) \right].
\]

and the identity

\[
\frac{1}{2} \Gamma(-l - \lambda)\Gamma(\lambda - l + 1) \left( \frac{\Gamma(\lambda + l + 1)^2 \Gamma(l - \lambda)^2}{\pi^2 \Gamma(l + 1)^2} \right) = \frac{-\gamma l}{\cos^2(\pi \lambda)\Gamma(l + 1)^2}.
\]

We can rewrite

\[
c_{-l}(z, z') = \sum_{\varepsilon, \varepsilon' = \pm} c_{-l, \varepsilon, \varepsilon'}(z, z')
\]
where \( \varepsilon = \mp \) (resp. \( \varepsilon' = \pm \)) denotes the choice of the first or second term in the bracket of equation (163). Thus

\[
c_{-l, -, -}(z, z') = \frac{\Gamma(l - \lambda)\Gamma(l + \lambda + 1)\sin(\pi l)\sin(\pi \lambda)}{2\Gamma(l + 1)^2\cos(\pi \lambda)^2} \left(\frac{1 - z}{1 + z}\right)^{\frac{l}{2}} \left(\frac{1 - z'}{1 + z'}\right)^{\frac{l}{2}} \times \frac{1 - z}{2} \frac{1 + z}{2} F_{l} \left(\frac{1 - z}{2}\right) F_{l} \left(\frac{1 - z'}{2}\right), \quad (166)
\]

\[
c_{-l, +, +}(z, z') = \frac{\Gamma(l - \lambda)\Gamma(l + \lambda + 1)\sin(\pi l)\sin(\pi \lambda)}{2\Gamma(l + 1)^2\cos(\pi \lambda)^2} \left(\frac{1 + z}{1 - z}\right)^{\frac{l}{2}} \left(\frac{1 + z'}{1 - z'}\right)^{\frac{l}{2}} \times \frac{1 + z}{2} \frac{1 + z'}{2} F_{l} \left(\frac{1 + z}{2}\right) F_{l} \left(\frac{1 + z'}{2}\right), \quad (167)
\]

\[
c_{-l, -, +}(z, \zeta') = -\frac{\Gamma(l - \lambda)\Gamma(l + \lambda + 1)\sin^2(\pi l)}{2\Gamma(l + 1)^2\cos(\pi \lambda)^2} \left(\frac{1 - z}{1 + z}\right)^{\frac{l}{2}} \left(\frac{1 + z'}{1 - z'}\right)^{\frac{l}{2}} \times \frac{1 - z}{2} \frac{1 + z}{2} F_{l} \left(\frac{1 - z}{2}\right) F_{l} \left(\frac{1 + z'}{2}\right), \quad (168)
\]

\[
c_{-l, +, -}(z, z') = -\frac{\Gamma(l - \lambda)\Gamma(l + \lambda + 1)\sin^2(\pi l)}{2\Gamma(l + 1)^2\cos(\pi \lambda)^2} \left(\frac{1 + z}{1 - z}\right)^{\frac{l}{2}} \left(\frac{1 - z'}{1 + z'}\right)^{\frac{l}{2}} \times \frac{1 + z}{2} \frac{1 + z'}{2} F_{l} \left(\frac{1 + z}{2}\right) F_{l} \left(\frac{1 + z'}{2}\right). \quad (169)
\]

For a given choice of \( \varepsilon \) and \( \varepsilon' \) the estimates (148-155) are readily adapted so that the series

\[
\sum_{l=\frac{1}{2}+n, n \geq 0} c_{-l, \varepsilon, \varepsilon'}(z, z', \lambda) e^{-il(\theta - \theta')}
\]

converges absolutely to a holomorphic function of \((s, s', \theta, \theta')\) in the tube

\[
T_{\varepsilon, \varepsilon'} = \{(s, s', \theta, \theta') : \varepsilon \text{ Im } s > 0, \varepsilon' \text{ Im } s' > 0, \text{ Im}(\varepsilon' s' + \varepsilon s) - \text{ Im}(\theta' - \theta) > 0\} \quad (171)
\]

as well as in

\[
T_{\varepsilon, \varepsilon'} = \{(u, v, u', v') : \varepsilon \text{ Im } u > 0, \varepsilon' \text{ Im } v > 0, \varepsilon' \text{ Im } u' > 0, \varepsilon' \text{ Im } v' > 0\}. \quad (172)
\]

This function has a boundary value at real values of these variables in the sense of tempered distributions.

\section*{10.5 Conclusion}

The series (141) converges to a distribution \( F_0 \) which is a finite sum of boundary values of functions holomorphic in several non-intersecting open tuboids. Thus \( F_0 \) is not the boundary value of a function holomorphic in a single open tuboid. It is possible to verify that this is also
true for the invariant functions of the type given by \((123)\) and \((124)\). It might be asked if \(F_0\) (or one of its siblings) could not still have a tuboid of analyticity beyond what follows from the above proofs of convergence. However in the following Sect. \(11\) a general lemma will show that analyticity is incompatible with the simultaneous requirements of locality \((C1)\), invariance \((C3)\) and Klein-Gordon equation \((C6)\). The proof of convergence given in this subsection also works for more general (non-invariant) series of the form \((116)\) provided the \(A_l, \ldots C_l^*\) are polynomially bounded in \(l\).
11 Incompatibility of analyticity with some other requirements

In this section the following lemma will be proved:

**Lemma 1** A two-point function $F$ on $dS_2 \times dS_2$ that simultaneously satisfies (C3) Invariance under $SL(2, \mathbb{R})$, (C6) Klein-Gordon equation, and (C9) Local analyticity (see Sect. [8]), vanishes on $\mathcal{R}$ (i.e. $F(x, x') = 0$ whenever $x$ and $x'$ are space-like separated).

An example of a non-zero two-point function satisfying these requirements is the canonical commutator ([87]). As an obvious corollary of this lemma,

**Lemma 2** A two-point function $F$ on $dS_2 \times dS_2$ that simultaneously satisfies (C1) Locality, (C3) Invariance under $SL(2, \mathbb{R})$, (C6) Klein-Gordon equation, and (C8) Analyticity (see Sect. [8]), is equal to 0.

**Proof.** Suppose $F$ satisfies the conditions (C3), (C6), and (C9). As in Sect. [8] let $\mathcal{R}$ denote the (open, connected) set of space-like separated points in $dS_2 \times dS_2$, and $\mathcal{N}$ a complex open connected neighborhood of $\mathcal{R}$ in which $F(\tilde{x}, \tilde{x}')$ and $F(\tilde{x}', \tilde{x})$ have a common analytic continuation. We denote $F_{\pm}$ this analytic continuation. For any pair $k = (\tilde{w}, \tilde{w}')$ we denote, by abuse of notation, $\tilde{w} \cdot \tilde{w}'$ or also $\psi(k)$ the scalar product of the projections $w$ and $w'$ of $\tilde{w}$ and $\tilde{w}'$ into the complex Minkowski space $M_3^{(c)}$ (i.e. with an abuse of notation, $\psi(k) = -1 - \frac{1}{2}(w - w')^2$).

In particular $\mathcal{N}$ contains the subset

$$E_{\varepsilon, \eta} = \{(t, \theta + iy), (t', \theta') : t = t' = \theta' = 0, \varepsilon < \theta < 4\pi - \varepsilon, |y| < \eta\} , \quad (173)$$

where $\varepsilon > 0$ and $\eta > 0$ must be chosen small enough. Note that for points of the form (173),

$$\zeta = -\cos(\theta + iy) = -\cos(\theta) \text{ch}(y) + i\sin(\theta) \text{sh}(y) . \quad (174)$$

Let $k_0 = (\tilde{w}_0, \tilde{w}'_0) \in \mathcal{N}$ be such that $\zeta_0 = \psi(k_0) \neq \pm 1$. There exist open neighborhoods $U_1 \subset \subset U_2 \subset \subset \mathcal{N}$ of $k_0$, and an open neighborhood $W_0$ of the identity in the group $SL(2, \mathbb{C})$ such that, for all $g \in W_0$ and $k \in U_1$, $gk \in U_2$ and $F_{+}(gk) = F_{+}(k)$ (since this holds for real $g$). Moreover we suppose $U_2$ small enough that the restriction to $U_2$ of the projection $\text{pr} \times \text{pr}$ is an isomorphism, and also that $k \mapsto \sqrt{1 - \psi(k)^2}$ can be defined as a holomorphic function on $U_2$ (in particular $\psi(k) \neq \pm 1$ for all $k \in U_2$).

We will prove\footnote{These arguments are special cases of more general well-known facts. See e.g. [30], [31].} that there is an open neighborhood $V_0 \subset \subset U_1$ of $k_0$, and a function $f_0$ holomorphic on $\psi(V_0)$ such that $f_0(\psi(k)) = F_{+}(k)$ for all $k \in V_0$. To do this we adopt the simplifying notation whereby if $\tilde{t} \in dS_2^{(c)}$ then $t$ denotes $\text{pr} \tilde{t}$ and conversely if $t \in dS_2^{(c)}$
then \( \tilde{t} \) denotes \( \text{pr}^{-1}t \). For any \( k = (\tilde{w}, \tilde{w}') \in U_2 \) we construct a complex Lorentz frame \((e_0(k), e_1(k), e_2(k))\) as follows:

\[
\begin{align*}
e_0(k) &= i\tilde{w}, \\
e_1(k) &= \alpha\tilde{w} + \beta\tilde{w}', \quad e_0(k) \cdot e_1(k) = 0, \quad e_1(k)^2 = -1, \\
e_2(k) &= e_0(k) \times e_1(k), \quad \text{i.e. } e_2(k)^\mu = -\varepsilon^{\mu
u\rho}e_0(k)_\nu e_1(k)_\rho.
\end{align*}
\]

(175)

Denoting \( \zeta = w \cdot w' = \psi(k) \), this implies

\[
\alpha = \beta\zeta, \quad \beta = \frac{\pm 1}{\sqrt{1 - \zeta^2}}, \quad \text{and we choose } \beta = \frac{1}{\sqrt{1 - \zeta^2}},
\]

(176)

where \( \sqrt{1 - \zeta^2} \) denotes the determination of \( \sqrt{1 - \psi(k)^2} \) mentioned above. This implies

\[
w = -i\epsilon_0(k), \quad w' = \sqrt{1 - \zeta^2}e_1(k) + i\zeta e_0(k).
\]

(177)

For any \( \zeta \) sufficiently close to \( \zeta_0 \) let \( h(\zeta) = (\sqrt{\zeta}, \sqrt{\zeta}) \) be defined by

\[
v(\zeta) = -i\epsilon_0(k_0), \quad v'(\zeta) = \sqrt{1 - \zeta^2}e_1(k_0) + i\zeta e_0(k_0).
\]

(178)

It is clear that \( v(\zeta)^2 = v'(\zeta)^2 = -1, v(\zeta) \cdot v'(\zeta) = \zeta, \) and \( e_j(h(\zeta)) = e_j(k_0) \) for \( j = 0, 1, 2 \). If \( k = (\tilde{w}, \tilde{w}') \) is close to \( k_0 \) and \( \psi(k) = \zeta \), there exists an element \( g(k) \) of \( SL(2, \mathbb{C}) \) close to the identity, such that \( k = g(k)h(\zeta) \). This element projects onto the unique Lorentz transformation \( \Lambda(k) \) such that \( e_j(k) = \Lambda(k)e_j(h(\zeta)) = \Lambda(k)e_j(k_0) \) for \( j = 0, 1, 2 \). We have therefore \( F_+(k) = F_+(h(\zeta)) \), i.e. a holomorphic function of \( \zeta \). We have now shown that every \( k_0 \in \mathcal{N} \) such that \( \psi(k_0) \neq \pm 1 \) has an open neighborhood \( V_0 \) such that \( F_+(k) = f_0(\psi(k)) \) for all \( k \in V_0 \), where \( f_0 \) is holomorphic in \( \psi(V_0) \).

If \( k_1 \) is another point of \( \mathcal{N} \) such that \( \psi(k_1) \neq \pm 1 \), and \( V_1, f_1 \) are the analogous objects, and if \( V_0 \) and \( V_1 \) overlap, it is clear that \( f_1 \) is an analytic continuation of \( f_0 \). Thus for any compact arc contained in \( \mathcal{N} \) from \( k_0 \) to \( k_2 \), \( f_0 \) can be analytically continued along the image under \( \psi \) of that arc, provided this image avoids the points \( \pm 1 \).

Since \( F \) satisfies the Klein-Gordon equation in \( x \) and in \( x' \), it follows, by a well-known calculation, that the \( f = f_0 \) obtained by the above procedure at a point \( k_0 \) (with \( \psi(k_0) \neq \pm 1 \)) must be a solution of the Legendre equation \( (65) \) with \( l = 0 \). By the general theory of such equations, \( f \) may be analytically continued along any arc in the complex plane which avoids the points \( \pm 1 \). As before, two linearly independent solutions of the equation are

\[
P_\lambda(\zeta) = P_\lambda^0(\zeta) = F \left( -\lambda, \lambda + 1; 1; \frac{1 - \zeta}{2} \right).
\]

(179)
and \( \hat{P}_\lambda(\zeta) = \hat{P}_\lambda^0(\zeta) = P_\lambda^0(-\zeta) \). Recall that \( P_\lambda \) is holomorphic in the cut-plane with a cut along \((-\infty, -1]\) and is singular at \(-1\); according to [20, p. 164] it has a logarithmic singularity at \(-1\). Hence \( \hat{P}_\lambda \) is holomorphic in the cut-plane with a cut along \([1, \infty)\) and has a logarithmic singularity at \(1\). It follows from [20, (36) pp. 132-133] that \( Q_{\mu}^\nu \) is holomorphic in \( \mathbb{C} \setminus (-\infty, 1) \), in particular on \((1, \infty)\). By [20, (10) p. 140], for real \( t > 1 \),

\[
\hat{P}_\lambda(t + i0) - \hat{P}_\lambda(t - i0) = -2i \sin(\lambda\pi) P_\lambda(t), \quad t > 1.
\]

We must have \( f(\zeta) = aP_\lambda(\zeta) + b\hat{P}_\lambda(\zeta) \), where \( a \) and \( b \) are constants. We specialize \( k_0 \) as

\[
k_0 = ((t_0 = 0, \theta_0 = 2\varepsilon), (t'_0 = 0, \theta'_0 = 0)),
\]

where \( 0 < \varepsilon \) is as in (173), and we suppose \( \varepsilon < \pi/8 \). We denote

\[
G(\theta + iy) = F_+((t = 0, \theta + iy), (t' = 0, \theta' = 0)).
\]

As the restriction of \( F_+ \) to the set \( E_\varepsilon, \eta \), \( G \) is holomorphic in the rectangle \( \varepsilon < \theta < 4\pi - \varepsilon, \) \( |y| < \eta \). We consider an arc (actually a straight line) \( \gamma_y \) lying in \( E_\varepsilon, \eta \), given by

\[
\theta \mapsto \gamma_y(\theta) = ((t = 0, \theta + iy), (t' = 0, \theta' = 0)).
\]

Here \( y \) is real with \( |y| \leq \tau \) and \( \theta \) varies in the interval \([2\varepsilon, 2\pi] \). We also require \( 0 < \tau < \eta \) to be small enough that the starting point \( \gamma_y(2\varepsilon) \) be always contained in the neighborhood \( V_0 \) of \( k_0 \) where the function \( f = f_0 \) is initially defined. Let \( y \) be fixed with \( 0 < y < \tau \). As \( \theta \) varies in \([2\varepsilon, 2\pi] \), \( \zeta = \psi(\gamma_y(\theta)) = -\cos(\theta + iy) \) runs along an arc of an ellipse with foci at \( \pm 1 \), starting in the upper half-plane, crossing the real axis at \( t = \cosh(y) \), and returning through the lower half-plane to \(-t - i0\) (see Fig. 1). Along this arc, \( f \) can be analytically continued; starting as \( f(\zeta) = aP_\lambda(\zeta) + b\hat{P}_\lambda(\zeta) \) it becomes, after crossing the real axis at \( t \), equal to \([a - 2ib\sin(\lambda\pi)]P_\lambda(-t - i0) + b\hat{P}_\lambda(-t - i0) \) (as a consequence of (180)). Thus at the end point,

\[
f(-t - i0) = G(2\pi + iy) = [a - 2ib\sin(\lambda\pi)]P_\lambda(-t - i0) + b\hat{P}_\lambda(-t - i0) \tag{184}
\]

Since \( |G(2\pi + iy)| \) is bounded uniformly in \( y \), while \( |P_\lambda(-t - i0)| \to \infty \) as \( t \to 1 \), we must have

\[
a - 2ib\sin(\lambda\pi) = 0. \tag{185}
\]

Repeating the argument with \( y < 0 \) (the arc of ellipse starts in the lower half-plane and finishes in the upper half-plane) we now get

\[
a + 2ib\sin(\lambda\pi) = 0. \tag{186}
\]
This implies \( f = 0 \), and therefore that \( F_+ \) vanishes in an open complex neighborhood of \( k_0 \). Hence \( F_+ = 0 \), hence \( F \) vanishes on \( \mathcal{R} \).

There are examples of 2-point functions that satisfy any three out of the four conditions (C1), (C3), (C6), and (C8). For instance let

\[
F_2(x, x', \lambda) = c_0(z, z', \lambda) + \sum_{l \in \mathcal{L}, l > 0} c_l(z, z', \lambda) [e^{il(\theta - \theta')} + e^{-il(\theta - \theta')}],
\]

where \( z = i \text{sh} t, \ z' = i \text{sh} t' \), and

\[
c_l(z, z', \lambda) = \gamma_l(\lambda) \mathbf{P}_\lambda^{-l}(z)(\mathbf{P}_\lambda^{-l}(-z')).
\]

Then \( F_2 \) satisfies (C1), (C6) as well as canonicity (up to a constant factor) and positivity. It also satisfies (C8) and is the boundary value of a function holomorphic in the tuboid \( T_{-,+} \) (see (159)) (or \( T_{-,+} \) (see (160))). But \( F_2 \) is not invariant under \( SL(2, \mathbb{R}) \).

12 Concluding remarks: absence of the Gibbons-Hawking temperature

We have seen that an invariant local two-point function satisfying the Klein-Gordon equation cannot have the property of analyticity. As a consequence, if such a function is used, a geodesic observer on the double covering of the two-dimensional de Sitter space-time will never detect a thermal bath of particles, even though he or she cannot distinguish the global topological
structure of the space time manifold in any other way! Let us summarize some well-known facts in the case of the two-dimensional de Sitter spacetime. The wedge

\[ U = \{ x \in dS_2, \ |x^1| \leq 1, \ x^2 \geq 0 \}. \]  

is invariant under the one-parameter subgroup of \( SL(2, \mathbb{R}) \) given by

\[ t \mapsto g(t) = \begin{pmatrix} \text{ch} t & 0 & \text{sh} t \\ 0 & 1 & 0 \\ \text{sh} t & 0 & \text{ch} t \end{pmatrix}. \]

As \( t \) varies in \( \mathbb{R} \) the point

\[ x(t, r) = \begin{pmatrix} \sqrt{1 - r^2} \text{sh} t \\ r \\ \sqrt{1 - r^2} \text{ch} t \end{pmatrix} = g(t)x(0, r), \quad \text{where} \quad -1 < r < 1, \]

describes an orbit of this subgroup. This is a branch of a hyperbola, and, in the case \( r = 0 \), it is a geodesic of \( dS_2 \), which can be regarded as the world-line of a “geodesic observer”. The orbits are restrictions of complex curves in \( dS_2^{(c)} \) obtained by letting \( t \) vary in \( \mathbb{C} \). If \( t = a + ib \) with \( a \in \mathbb{R} \) and \( b = \pi \mod 2\pi \), \( x(t, r) \) describes another real branch of the complex hyperbola.

Suppose now that \( W(z_1, z_2) \) is a maximally analytic two-point function in the complex de Sitter hyperboloid; let the first point \( z_1 = x(t_1, r_1) \) be fixed in the wedge \( U \) and let the second point vary on the complex hyperbola \( z_2 = x(t_2, r_2) \) (\( t_2 \) is a complex time variable). It easy to check \([16]\) that the two-point function,

\[ W(z_1, z_2) = W(-\sqrt{1 - r_1^2} \sqrt{1 - r_2^2} \text{ch}(t_1 - t_2) - r_1r_2) \]  

as a function of the complex variable \( t_2 \) is holomorphic on the strip \( \{0 < \text{Im}(t_2) < 2\pi\} \) and that the boundary values satisfy the KMS condition

\[ \lim_{\epsilon \to 0} W(x_1(t_1, r_1), x_2(t_2 + i\epsilon, r_2)) = W(x_1, x_2), \]

\[ \lim_{\epsilon \to 0} W(x_1(t_1, r_1), x_2(t_2 + 2\pi i - i\epsilon, r_2)) = W(x_2, x_1), \]

These relations are usually interpreted by saying that the geodesic observer perceives a thermal bath of particles at temperature \( T = 1/2\pi \). Analyticity and locality actually suffice for these relations to hold, without supposing invariance \([18]\). But these properties disappear in the case of the two-sheet covering \( dS_2 \) of the two-dimensional de Sitter space.
To understand this in more detail, let us again consider $F_2(x, x', \lambda)$ given by (187) and (188), and

$$F_0(x, x', \lambda) = \sum_{L} c_l(z, z', \lambda) e^{il(\theta - \theta')}, \quad (195)$$

$$F_1(x, x', \lambda) = \frac{1}{2} F_0(x, x', \lambda) + \frac{1}{2} F_0(\tau_1 x, \tau_1 x, \lambda) = \sum_{l \in L} \frac{1}{2} c_l(z, z', \lambda) \left[ e^{il(\theta - \theta')} + e^{-il(\theta - \theta')} \right]. \quad (196)$$

Here $L = \frac{1}{2} \mathbb{Z}$. We use the same notations as in Sect. 10. Recall that $F_0$ and $F_1$ are invariant but not analytic, and $F_2$ is analytic but not invariant. $F_1$ and $F_2$ are local, $F_0$ is not. Let $F_{j,\text{even}}$ (resp. $F_{j,\text{odd}}$) denote (for $j = 0, 1, 2$) the sum of the corresponding series over integer (resp. non-integer) values of $l$. Since for integer $l$, $c_l(z, z', \lambda) = c_{-l}(z, z', \lambda)$ (see Subsect. 10.1), we have $F_{2,\text{even}} = F_{0,\text{even}}$. This is an invariant and analytic 2-point function, local in the sense of $dS_2$, and in fact (see Sect. 4) it coincides with the Bunch-Davies function. Hence it possesses the analyticity along complex hyperbolae discussed at the beginning of this section.

To understand the behavior of the $F_{j,\text{odd}}$, we study their limits as $\lambda$ tends to 0. If $\lambda \to 0$, $c_l(z, z', \lambda)$ tends to a well-defined limit provided $l \neq 0$. If $l \in \frac{1}{2} + \mathbb{Z}$ and $l > 0$,

$$c_l(z, z', 0) \overset{\text{def}}{=} \lim_{\lambda \to 0} c_l(z, z', \lambda) = \frac{1}{2l} \left( \frac{1}{1 + z} \right)^{\frac{1}{2}} \left( \frac{1 + z'}{1 - z'} \right)^{\frac{1}{2}} = \frac{1}{2l} e^{il(s'-s+ic)}, \quad (197)$$

$$c_l(z, z', 0)e^{il(\theta - \theta')} = \frac{1}{2l} e^{il(v'-v)}, \quad c_{-l}(z, z', 0)e^{-il(\theta - \theta')} = \frac{1}{2l} e^{il(u'-u)}. \quad (198)$$

For the same $l \geq \frac{1}{2}$, we find

$$c_{-l}(z, z', 0)e^{-il(\theta - \theta')} = -\frac{1}{2l} e^{il(v'-v)}, \quad c_{-l}(z, z', 0)e^{il(\theta - \theta')} = -\frac{1}{2l} e^{il(u'-u)}. \quad (199)$$

$$F_{0,\text{odd}}(x, x', 0) = \frac{1}{2} \log \left( \frac{1 + e^{i(v'-v+ic)/2}}{1 - e^{i(v'-v+ic)/2}} \right) - \frac{1}{2} \log \left( \frac{1 + e^{i(v'-v+ic)/2}}{1 - e^{i(v'-v+ic)/2}} \right). \quad (200)$$

The first (resp. second) term is the boundary value term of a function holomorphic in the tuboid $\{\text{Im}(v'-v) > 0\}$ (resp $\{\text{Im}(v'-v) < 0\}$). $F_1$ is the sum of the boundary values of four functions analytic in two pairs of opposite tuboids:

$$F_{1,\text{odd}}(x, x', 0) = \frac{1}{4} \log \left( \frac{1 + e^{i(u'-u+ic)/2}}{1 - e^{i(u'-u+ic)/2}} \right) - \frac{1}{4} \log \left( \frac{1 + e^{i(u'-u+ic)/2}}{1 - e^{i(u'-u+ic)/2}} \right)$$

$$\frac{1}{4} \log \left( \frac{1 + e^{i(v'-v+ic)/2}}{1 - e^{i(v'-v+ic)/2}} \right) - \frac{1}{4} \log \left( \frac{1 + e^{i(v'-v+ic)/2}}{1 - e^{i(v'-v+ic)/2}} \right). \quad (201)$$

If $x' = (t', \theta')$ and $\theta$ are fixed real, then $t \mapsto F_{0,\text{odd}}((t, \theta), x')$ is not the boundary value of a function holomorphic in a strip of the upper half-plane. The same is true for $F_{1,\text{odd}}((t, \theta), x')$. 39
On the other hand

\[
F_{2,\text{odd}}(x, x', 0) = \frac{1}{2} \log \left( \frac{1 + e^{i(v' - v + i\epsilon)/2}}{1 - e^{i(v' - v + i\epsilon)/2}} \right) + \frac{1}{2} \log \left( \frac{1 + e^{i(u' - u + i\epsilon)/2}}{1 - e^{i(u' - u + i\epsilon)/2}} \right)
\]

\[
= -\frac{1}{2} \log \left( \frac{-\sin \left( \frac{v' - v + i\epsilon}{4} \right) \sin \left( \frac{u' - u + i\epsilon}{4} \right)}{-\cos \left( \frac{v' - v + i\epsilon}{4} \right) \cos \left( \frac{u' - u + i\epsilon}{4} \right)} \right)
\]

is the boundary value of a function of \(u, v, u', v'\), holomorphic in the tube

\[
\{(u, v, u', v') : \text{Im}(u' - u) > 0, \text{Im}(v' - v) > 0\},
\]

which, of course, can be continued in a larger domain. In fact starting from any point in the tube (203), the function can be analytically continued along any arc which does not contain any point such that \(u - u' \in 2\pi\mathbb{Z}\) or \(v - v' \in 2\pi\mathbb{Z}\). In particular \(F_{2,\text{odd}}(x, x', 0)\) is analytic at all real points such that \(u - u' \in 2\pi\mathbb{Z}\) and \(v - v' \in 2\pi\mathbb{Z}\). However we have from (202)

\[
G(x, x') \overset{\text{def}}{=} \exp(-2F_{2,\text{odd}}(x, x', 0)) = \cos \left( \frac{s' - s}{2} \right) - \cos \left( \frac{\theta' - \theta}{2} \right) \cos \left( \frac{s' + s}{2} \right) + \cos \left( \frac{\theta' - \theta}{2} \right).
\]

Fixing \(\theta\) and \(\theta'\) real and \(s' = 0\), and substituting

\[
\cos(s/2) = \left[ \frac{1}{2}(1 + \cos(s)) \right]^\frac{1}{2} = \left[ \frac{1}{2} \left( 1 + \frac{1}{\text{ch}(t)} \right) \right]^\frac{1}{2},
\]

(204) becomes

\[
G(x, x') = \sqrt{\frac{1}{2}(1 + \text{ch}(t)) - \sqrt{\text{ch}(t)} \cos \left( \frac{\theta' - \theta}{2} \right)} \sqrt{\frac{1}{2}(1 + \text{ch}(t)) + \sqrt{\text{ch}(t)} \cos \left( \frac{\theta' - \theta}{2} \right)}.
\]

If \(\cos((\theta' - \theta)/2) \neq 0\), this is singular in \(t\) when \(\text{ch}(t)\) vanishes, i.e. \(it \rightarrow \pi/2 \mod \pi\), so that \(F_2\) is not analytic in \(t\) in the strip \(0 < \text{Im} t < \pi\). This does not prove, but makes it very likely that the same holds for \(F_2(x, x', \lambda)\) with \(\lambda \neq 0\).

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A Appendix. Estimations for hypergeometric functions

We reproduce here for completeness some estimates from [20] 2.3.2 pp 76-77. Recall (see e.g. [32] 5.6(ii))

\[
|\Gamma(x + iy)| \leq |\Gamma(x)|, \quad |\Gamma(x + iy)| \geq \frac{1}{\sqrt{\text{ch} y}} |\Gamma(x)|, \quad x \geq \frac{1}{2}.
\]

(207)
Define

\[ \rho_{n+1}(a, b, ; c ; z) = F(a, b, ; c ; z) - 1 - \frac{ab}{c}z - \ldots - \frac{(a)n(b)n}{(c)n!}z^n \]

\[ = \frac{\Gamma(c)\Gamma(a+n)z^{n+1}}{\Gamma(b)\Gamma(c-b)\Gamma(a)n!} \int_0^1 ds \int_0^1 dt \; t^{b+n}(1-t)^{c-b-1}(1-s)^n(1-stz)^{-a-n-1}. \] (208)

Let \( a = \alpha + i\alpha', \ b = \beta + i\beta', \ c = \gamma + i\gamma' \). Then

\[ |\rho_{n+1}| \leq \mu(n) |z|^{n+1} |c|^{-\beta} \gamma^{-\beta-n-1}, \] (209)

where it is assumed that \( |\arg(1-z)| < \pi, \gamma > \beta, \ n + \beta > 0, \ | \arg c | < \pi - \varepsilon, \gamma > 0 \) sufficiently large, \( n \) sufficiently large. \( \mu(n) \) depends on \( n, a, b, z \).

Example: \( n = 0 \)

\[ \rho_1(a, b, ; c ; z) = \frac{\Gamma(c)z}{\Gamma(b)\Gamma(c-b)} I, \]

\[ I = \int_0^1 ds \int_0^1 dt \; t^{b}(1-t)^{\gamma-\beta-1}(1-stz)^{-a-1}. \] (210)

We assume \( \beta + 1 > 0, \gamma - \beta > 0, \gamma > 0, \) and that \( |1-z| > \varepsilon, |\arg(1-z)| < \pi - \varepsilon \) for some \( \varepsilon > 0 \). In this case the modulus \( |I| \) of the integral is bounded by

\[ |I| \leq M(z) \int_0^1 t^\beta(1-t)^{\gamma-\beta-1} dt \] (211)

with

\[ M(z) = \sup_{0 \leq u \leq 1} |(1-uz)^{-a-1}|. \] (212)

\[ |I| \leq M(z) \frac{\Gamma(\beta+1)\Gamma(\gamma-\beta)}{\Gamma(\gamma+1)}. \] (213)

Hence using (207)

\[ |\rho_1(a, b, ; c ; z)| \leq \frac{|z|M(z)|\beta|\sqrt{\text{ch}(\pi\beta')}\text{ch}\pi(\gamma' - \beta')}{\gamma}. \] (214)

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