ON THE EXPONENT CONJECTURE OF SCHUR

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Abstract
It is a longstanding conjecture that for a finite group \( G \), the exponent of the second homology group \( H_2(G, \mathbb{Z}) \) divides the exponent of \( G \). In this paper, we prove this conjecture for \( p \)-groups of class at most \( p \), finite nilpotent groups of odd exponent and of nilpotency class 5, \( p \)-central metabelian \( p \)-groups, and groups considered by L. E. Wilson in [34]. Moreover, we improve several bounds given by various authors. We achieve most of our results using an induction argument.

Keywords: Schur multiplier, regular \( p \)-groups, powerful \( p \)-groups, Schur cover, group actions.

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1. Introduction

The Schur multiplier of a group \( G \), denoted by \( M(G) \) is the second homology group of \( G \) with coefficients in \( \mathbb{Z} \), i.e \( M(G) = H_2(G, \mathbb{Z}) \). A longstanding conjecture attributed to I. Schur says that for a finite group \( G \),

\[
\exp(M(G)) \mid \exp(G).
\] (1)

To prove (1), it is enough to restrict ourselves to \( p \)-groups using a standard argument given in Theorem 4, Chapter IX of [31]. A. Lubotzky and A. Mann showed that (1) holds for powerful \( p \)-groups([16]), M. R. Jones in [14] proved

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that (1) holds for groups of class 2. P. Moravec showed that the conjecture holds for groups of nilpotency class at most 3, groups of nilpotency class 4 and of odd order, potent $p$-groups, metabelian $p$-groups of exponent $p$, $p$-groups of class at most $p-2$ ([22], [23], [24], and [25]) and some other classes of groups. The general validity of (1) was disproved by A. J. Bayes, J. Kautsky and J. W. Wamsley in [3]. Their counterexample involved a 2-group of order $2^{68}$ with $\exp(G) = 4$ and $\exp(M(G)) = 8$. Nevertheless for finite groups of odd exponent, this conjecture remains open till date. This problem has remained open even for finite $p$-groups of class 5 having odd exponent. The purpose of this paper is to prove (1) for finite $p$-groups of class at most $p$ and finite $p$-groups of class 5 with odd exponent. Moreover we also prove the above mentioned results of [16], [22], [23], [24] and [25] for odd primes and hence proving all these results using a common technique and bringing them under one umbrella. We briefly describe the organization of the paper by listing the main results according to their sections.

In [30], I. Schur proves that if $Z(G)$ has finite index in $G$, then the commutator subgroup $\gamma_2(G)$ is finite. In the next Theorem, we generalize this classical theorem of Schur for a $p$-group $G$ of class at most $p+1$. We use this generalization to prove the conjecture for $p$-groups of class at most $p$.

**Theorem 3.9.** Let $p$ be an odd prime and $G$ be a $p$-group of nilpotency class $p+1$. If $G$ is $p^n$-central, then $\exp(\gamma_2(G)) \mid p^n$.

In [24], the author shows that (1) holds for $p$-groups of class less than or equal to $p-2$. Authors of [21] prove the same for groups of class less than or equal to $p-1$. In the next Theorem, we generalize both the above results by proving

**Theorem 3.11.** Let $p$ be an odd prime and $G$ be a finite $p$-group. If the nilpotency class of $G$ is at most $p$, then $\exp(G \wedge G) \mid \exp(G)$. In particular, $\exp(M(G)) \mid \exp(G)$.

In [26], the author proves that if $G$ is a group of nilpotency class 5, then $\exp(M(G)) \mid (\exp(G))^2$. In the next Theorem, we improve this bound and in fact prove the conjecture for groups with odd exponent.

**Theorem 4.6.** Let $G$ be a finite $p$-group of nilpotency class 5. If $p$ is odd, then $\exp(G \wedge G) \mid \exp(G)$. In particular, $\exp(M(G)) \mid \exp(G)$.
In Section 5, we prove two main theorems which we list below. The second condition in the next Theorem generalizes the definition of powerful 2-groups for odd primes and were considered by L. E. Wilson in [34], and the first condition includes the class of groups considered by D. Arganbright in [2], and it also includes the class of potent \( p \)-groups considered by J. Gonzalez-Sanchez and A. Jaikin-Zapirain in [12]. Thus as a corollary of the next Theorem, we obtain the well-known result that (1) holds for powerful \( p \)-groups ([16]) and potent \( p \)-groups ([25]).

**Theorem 5.5.** Let \( p \) be an odd prime and \( G \) be a finite \( p \)-group satisfying either of the conditions below,

(i) \( \gamma_m(G) \subset G^p \) for some \( m \) with \( 2 \leq m \leq p - 1 \).

(ii) \( \gamma_p(G) \subset G^{p^2} \).

Then \( \exp(G \land G) \mid \exp(G) \). In particular, \( \exp(M(G)) \mid \exp(G) \).

In [29], the author has shown that proving the conjecture for regular \( p \)-groups is equivalent to proving it for groups with exponent \( p \). As a corollary to the next Theorem, we obtain the same.

**Theorem 5.6.** The following statements are equivalent:

(i) \( \exp(G \land G) \mid \exp(G) \) for all regular \( p \)-groups \( G \).

(ii) \( \exp(G \land G) \mid \exp(G) \) for all groups \( G \) of exponent \( p \).

In section 6, we give bounds on the exponent of \( M(G) \) that depend on nilpotency class. G. Ellis in [10] showed that if \( G \) is a group with nilpotency class \( c \), then \( \exp(M(G)) \mid (\exp(G))^\lceil \frac{3}{2} \rceil \). P. Moravec in [22] improved this bound by showing that \( \exp(M(G)) \mid (\exp(G))^{2(\lceil \log_2 c \rceil)} \). In the next Theorem, we improve the bounds given in [10] and [22].

**Theorem 6.1.** Let \( G \) be a finite group with nilpotency class \( c > 1 \) and set \( n = \lceil \log_3(\frac{c+1}{2}) \rceil \). If \( \exp(G) \) is odd, then \( \exp(G \land G) \mid (\exp(G))^n \). In particular, \( \exp(M(G)) \mid (\exp(G))^n \).

In Theorem 1.1 of [29], N. Sambonet improved all the bounds obtained by various authors by proving that \( \exp(M(G)) \mid (\exp(G))^m \), where \( m = \lceil \log_p -1 c \rceil + 1 \). We improve the bound given in [29] by proving,
Theorem 6.5. Let $p$ be an odd prime and $G$ be a finite $p$-group of nilpotency class $c \geq p$. Then $\exp(G \wedge G) \mid (\exp(G))^n$, where $n = 1 + \lceil \log_{p-1}(c+1) \rceil$. In particular, $\exp(M(G)) \mid (\exp(G))^n$.

For a solvable $p$-group of derived length $d$, the author of [28] proves that if $p$ is odd, then $\exp(M(G)) \mid (\exp(G))^d$, and if $p = 2$, then $\exp(M(G)) \mid 2^{d-1}(\exp(G))^d$. Using our techniques, we obtain the following generalization of Theorem A of [28], which is one of their main results.

Theorem 7.3. Let $G$ be a solvable group of derived length $d$.

(i) If $\exp(G)$ is odd, then $\exp(G \otimes G) \mid (\exp(G))^d$. In particular, $\exp(M(G)) \mid (\exp(G))^d$.

(ii) If $\exp(G)$ is even, then $\exp(G \otimes G) \mid 2^{d-1}(\exp(G))^d$. In particular, $\exp(M(G)) \mid 2^{d-1}(\exp(G))^d$.

2. Preparatory Results

R. Brown and J.-L. Loday introduced the nonabelian tensor product $G \otimes H$ for a pair of groups $G$ and $H$ in [5] and [6] in the context of an application in homotopy theory, extending the ideas of J.H.C. Whitehead in [33]. A special case, the nonabelian tensor square, already appeared in the work of R.K. Dennis in [7]. The nonabelian tensor product of groups is defined for a pair of groups that act on each other provided the actions satisfy the compatibility conditions of Definition 2.1 below. Note that we write conjugation on the left, so $^gg' = gg'g^{-1}$ for $g,g' \in G$ and $^gg'g^{-1} = [g,g']$ for the commutator of $g$ and $g'$. Moreover, our commutators are right normed, i.e $[a,b,c] = [a,[b,c]]$.

Definition 2.1. Let $G$ and $H$ be groups that act on themselves by conjugation and each of which acts on the other. The mutual actions are said to be compatible if

$$^hgh' = hgh^{-1}h'$$

and

$$^hb' = h^{-1}b'$$

for all $g,g' \in G$ and $h,h' \in H$. (2.1.1)

Definition 2.2. Let $G$ be a group that acts on itself by conjugation, then the nonabelian tensor square $G \otimes G$ is the group generated by the symbols $g \otimes h$ for $g,h \in G$ with relations

$$gg' \otimes h = (^g g' \otimes ^g h)(g \otimes h),$$

(2.2.1)
\[ g \otimes hh' = (g \otimes h)(h g \otimes h' h), \quad (2.2.2) \]

for all \( g, g', h, h' \in G \).

There exists a homomorphism \( \kappa : G \otimes G \to G' \) sending \( g \otimes h \) to \([g, h]\). Let \( \nabla(G) \) denote the subgroup of \( G \otimes G \) generated by the elements \( x \otimes x \) for \( x \in G \). The exterior square of \( G \) is defined as \( G \wedge G = (G \otimes G)/\nabla(G) \). We get an induced homomorphism \( G \wedge G \to G' \), which we also denote as \( \kappa \).

We can find the following results in [4] and Proposition 3 of [32].

**Proposition 2.3.** (i) There are homomorphisms of groups \( \lambda : G \otimes H \to G \), \( \lambda' : G \otimes H \to H \) such that
\[
\lambda(g \otimes h) = g h g^{-1}, \quad \lambda'(g \otimes h) = g h h^{-1},
\]
for all \( t, t_1 \in G \otimes H, g \in G \) (and similarly for \( \lambda' \)).

(ii) The crossed module rules hold for \( \lambda \) and \( \lambda' \), that is,
\[
\lambda(g t) = g(\lambda(t))g^{-1},
\]
\[
\lambda'(g t) = g(\lambda'(t))g^{-1},
\]
\[
\lambda'(t)\lambda'(t_1) = \lambda'(t)\lambda'(t_1),
\]
for all \( t, t_1 \in G \otimes H, g \in G \) and \( h \in H \). Hence \( G \) acts trivially on \( \ker \lambda' \) and \( H \) acts trivially on \( \ker \lambda \).

In particular, the following relations hold for \( g, g_1 \in G \) and \( h, h_1 \in H \):

(iv) \[
g(g^{-1} \otimes h) = (g \otimes h)^{-1} = h(g \otimes h^{-1}). \quad (2.3.1)\]

(v) \[
(g \otimes h)(g_1 \otimes h_1)(g \otimes h)^{-1} = ([g, h] g_1 \otimes [g, h] h_1). \quad (2.3.2)\]

(vi) \[
[g, h] \otimes h_1 = (g \otimes h) h_1 (g \otimes h)^{-1}. \quad (2.3.3)\]

(vii) \[
g_1 \otimes [g, h] = g_1 (g \otimes h)(g \otimes h)^{-1}. \quad (2.3.4)\]

(viii) \[
[g \otimes h, g_1 \otimes h_1] = [g, h] [g_1, h_1]. \quad (2.3.5)\]
Lemma 2.5. Let $N \trianglelefteq G$ and $[n, n, g] = 1$ for all $g \in G, n \in N$. In [10], G. Ellis proves that for all integers $t \geq 2$, $n^t \otimes g = (n \otimes g)^t(n \otimes [n, g])^{(t)}$. In the next lemma, we will generalize this identity.

Lemma 2.4. Let $N, M \trianglelefteq G$. If $n \otimes [n, n, m] = 1$ and $n \otimes m, n \otimes [n, m]$ commute for all $n \in N, m \in M$, then $n^t \otimes m = (n \otimes [n, m])^{(t)}(n \otimes m)^t = (n^{(t)} \otimes [n, m])(n \otimes m)^t$, for every $t \geq 2$.

Proof. We first prove that for every $t$, $n^t \otimes m = (n \otimes [n, m])^{(t)}(n \otimes m)^t$. Note that $n^2 \otimes m = n(n \otimes m)(n \otimes m)$ and using (2.3.4), we obtain $n(n \otimes m) = (n \otimes [n, m])(n \otimes m)$. Thus $n^2 \otimes m = (n \otimes [n, m])(n \otimes m)^2$, and now we proceed by induction. Let $t > 2$ and assume the statement for $t - 1$. We have $n^t \otimes m = n(n^{t-1} \otimes m)(n \otimes m) = n(n \otimes [n, m])^{(t-1)} n(n \otimes m)^{t-1}(n \otimes m)$. Since $n \otimes [n, n, m] = 1$, applying (2.3.4) yields $n(n \otimes [n, m]) = n \otimes [n, m]$. Thus we obtain

$$n^t \otimes m = (n \otimes [n, m])^{(t-1)}((n \otimes [n, m])(n \otimes m))^{t-1}(n \otimes m)$$

$$= (n \otimes [n, m])^{(t-1)}(n \otimes [n, m])^{t-1}(n \otimes m)^{t-1}(n \otimes m)$$

$$= (n \otimes [n, m])^{(t)}(n \otimes m)^t,$$

which completes the inductive step. Since $n(n \otimes [n, m]) = n \otimes [n, m]$, it follows that $n^{(t)} \otimes [n, m] = (n \otimes [n, m])^{(t)}$, which in turn yields $n^t \otimes m = (n^{(t)} \otimes [n, m])(n \otimes m)^t$. \hfill \qed

Lemma 2.5. Let $N, M$ be normal subgroups of a group $G$. If $N$ is abelian, then the following holds:

(i) For all $n_i \in N, m_i \in M, i = 1, 2$ and for all $t \geq 2$,

$$((n_1 \otimes m_1)(n_2 \otimes m_2))^t = ([n_2, m_2] \otimes [n_1, m_1])^{(t)}(n_1 \otimes m_1)^t(n_2 \otimes m_2)^t$$

$$= ([n_2, m_2]^{(t)} \otimes [n_1, m_1])(n_1 \otimes m_1)^t(n_2 \otimes m_2)^t.$$

(ii) If $\exp(N)$ is odd, then $\exp(N \otimes M) \mid \exp(N)$.

(iii) If $\exp(N)$ is even, then $\exp(N \otimes M) \mid 2 \exp(N)$.

Proof. Let $\exp(N) = e$. 

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(i) Since \( N \) is abelian, it follows from \([8]\) that the nilpotency class of \( N \otimes M \) is at most 2. Hence,

\[
((n_1 \otimes m_1)(n_2 \otimes m_2))^t = [(n_2 \otimes m_2), (n_1 \otimes m_1)](n_1 \otimes m_1)^t(n_2 \otimes m_2)^t = ([n_2, m_2] \otimes [n_1, m_1])(n_1 \otimes m_1)^t(n_2 \otimes m_2)^t.
\]

As \( N \) is abelian \([n_2, m_2]([n_2, m_2] \otimes [n_1, m_1]) = [n_2, m_2] \otimes [n_1, m_1] \), yielding \([n_2, m_2](n_1) \otimes [n_1, m_1]) = ([n_2, m_2] \otimes [n_1, m_1])(n_1 \otimes m_1)^t(n_2 \otimes m_2)^t.

(ii) Because \( N \) is abelian, we have \([n, n, m] = 1 \). Hence \( n \otimes [n, n, m] = 1 \) and \([n \otimes m], (n \otimes [n, m]) = [n, m] \otimes [n, n, m] = 1 \). Thus by Lemma \( 2.4 \) we obtain that \( n_1^e \otimes m_1 = (n_1(n_1) \otimes [n_1, m_1])^e(n_1 \otimes m_1)^e. \) Since \( e \) is odd, it follows that \( e \mid (\frac{2e}{3}) \), whence \( (n_1 \otimes m_1)^e = 1 \). By (i), we have that \((n_1 \otimes m_1)(n_2 \otimes m_2)^e = ([n_2, m_2](n_1) \otimes [n_1, m_1])^e(n_2 \otimes m_2)^e = 1 \), which proves \( \exp(N \otimes M) \mid e \).

(iii) By Lemma \( 2.4 \) we obtain \( n_1^{2e} \otimes m_1 = (n_1(n_1) \otimes [n_1, m_1])^{2e}(n_1 \otimes m_1)^{2e}. \) Since \( e \mid (\frac{2e}{3}) \), it follows that \( (n_1 \otimes m_1)^{2e} = 1 \). By (i) we obtain \((n_1 \otimes m_1)(n_2 \otimes m_2)^{2e} = ([n_2, m_2](n_1) \otimes [n_1, m_1])^{2e}(n_2 \otimes m_2)^{2e}, \) which proves \( \exp(N \otimes M) \mid 2 \exp(N) \).

In \([27]\) and \([11]\), the authors give an isomorphism between the nonabelian tensor square of \( G \) and the subgroup \([G, G^\phi] \) of \( \eta(G) \). We use this isomorphism in the proof of the next lemma.

**Lemma 2.6.** Let \( G \) be a nilpotent group of class \( c \) and set \( m = \lceil \frac{e+1}{3} \rceil \). If \( \gamma_m(G) \) has odd exponent, then \( \exp(\text{im}(\gamma_m(G) \otimes G))) \mid \exp(\gamma_m(G)) \).

**Proof.** Let \( \exp(\gamma_m(G)) = e \). Consider the isomorphism \( \psi : G \otimes G \to [G, G^\phi] \) defined by \( \psi(g \otimes h) = [g, h^\phi] \), where \( G^\phi \) is an isomorphic copy of \( G \). Recall that \([G, G^\phi]\) is a subgroup of \( \eta(G) \). By Theorem A of \([27]\), \( \eta(G) \) is a group of nilpotency class at most \( c+1 \). Note that for \( n \in \gamma_m(G), g \in G, \psi(n \otimes n, n, g) \in \gamma_{3m+1}(\eta(G)) \). Since \( 3m+1 \geq c+2, \gamma_{3m+1}(\eta(G)) \subset \gamma_{c+2}(\eta(G)) = 1 \). Thus \( n \otimes n, n, g = 1 \). Similarly \( \psi([n, g] \otimes [n, n, g]) \subset \gamma_{3m+2}(\eta(G)) \), giving \([n \otimes g, n \otimes [n, g]] = [n, g] \otimes [n, n, g] = 1 \). Thus by Lemma \( 2.4 \).
we obtain \( n^e \otimes g = (n(g^2) \otimes [n, g])(n \otimes g)^e \). Since \( e \) is odd, it follows that \( e \mid \binom{e}{2} \), whence \( (n \otimes g)^e = 1 \). Moreover for \( n_1, n_2 \in \gamma_m(G) \) and \( g_1, g_2 \in G \), we have \( \psi(\gamma_3((n_1 \otimes g_1, n_2 \otimes g_2))) \subseteq \gamma_{3m+3}(\eta(G)) \). But \( 3m + 3 \geq c+2 \), so \( \gamma_{3m+3}(\eta(G)) = 1 \). Hence \( (n_1 \otimes g_1, n_2 \otimes g_2) \) is a group of class at most 2. Thus \( ((n_1 \otimes g_1)(n_2 \otimes g_2))^e = ([n_2, g_2] \otimes [n_1, g_1])^e(n_1 \otimes g_1)^e(n_2 \otimes g_2)^e = 1 \), proving the result. \( \square \)

In the next lemma, we collect some commutator identities.

**Lemma 2.7.**  
(i) Let \( G \) be a group and \( g_i, h_i \in G \) for \( i = 1, 2 \). Then

\[
[g g_1, h] = \eta[g_1, h][g, h] = [g, g_1, h][g_1, h][g, h], \tag{2.7.1}
\]

\[
[g, h h_1] = [g, h^2][g, h_1] = [g, h][h, g_1, h][g, h_1]. \tag{2.7.2}
\]

(ii) Let \( G \) be a group of nilpotency class \( c \), \( x \in \gamma_i(G) \), \( y \in \gamma_j(G) \) and \( z \in \gamma_k(G) \). If \( i + j + k \geq c+1 \), then

\[
[x, y][z, u] = [z, u][x, y]. \tag{2.7.3}
\]

(iii) Let \( G \) be a group of nilpotency class \( c \), \( x \in \gamma_i(G) \), \( y \in \gamma_j(G) \), \( z \in \gamma_k(G) \) and \( u \in \gamma_l(G) \). If \( i + j + k + l \geq c+1 \), then

\[
[x, y][z, u] = [z, u][x, y]. \tag{2.7.4}
\]

**Proof.** Identities in (i) can be found in any standard book on group theory and (ii), (iii) follows from the identities \( a b = [a, b]b \), \( a b = [a, b]b a \) respectively. \( \square \)

The following lemma might be standard, we include it here for the sake of completeness.

**Lemma 2.8.** Let \( G \) be a group of nilpotency class \( c > 1 \), \( r \) be a positive integer and \( [\ldots, \cdot, \ldots, \cdot] \) be a commutator of weight \( r+1 \). If \( a \in \gamma_{n_1}(G) \), \( b \in \gamma_{m_2}(G) \) and \( g_i \in \gamma_{m_i}(G) \) for \( 1 \leq i \leq r \), satisfies \( n_1 + n_2 + m_1 + \cdots + m_r \geq c+1 \), then the following holds:

(i) \([a, g_r, \ldots, g_1] = [a, g_r, \ldots, g_1][b, g_r, \ldots, g_1].\)

(ii) \([g_r, \ldots, g_1, a] = [g_r, \ldots, g_1, a][g_r, \ldots, g_1, b].\)
Proof. We prove (i), (ii) and (iii) by induction on $r$.

(i) Let $r = 1$, then \((2.7.1)\) yields $[ab, g_1] = [a, b, g_1][b, g_1][a, g_1] = [a, g_1][b, g_1]$. Let $r > 1$, and $[ab, g_r, \ldots, g_1]$ be of the form $[[ab, g_r], g_r-1, \ldots, g_1]$. Then applying \((2.7.1)\) to $[ab, g_r]$ in $[[ab, g_r], g_r-1, \ldots, g_1]$, and using induction hypothesis we obtain

\[
[[ab, g_r], g_r-1, \ldots, g_1] = [[a, b, g_r][b, g_r][a, g_r], g_r-1, \ldots, g_1] = [[a, b, g_r], g_r-1, \ldots, g_1] [[b, g_r], g_r-1, \ldots, g_1] [[a, g_r], g_r-1, \ldots, g_1].
\]

The first term vanishes and the remaining terms commute by \((2.7.4)\), yielding the required result. In case $g_r$ is not next to a right bracket, then observe that there exist a $j \in \{1, \ldots, r-1\}$ such that $[ab, g_r, \ldots, g_j]$ in $[[ab, g_r, \ldots, g_j], \ldots, g_1]$ can be written as $[ab, h_t, \ldots, h_1]$, $h_k \in G$, $t < r - (j - 1)$, giving the result we seek by induction hypothesis.

(ii) Now using \((2.7.2)\), the proof follows mutatis mutandis the proof of (i).

(iii) Note that $[g_2, ab, g_1]$ can be bracketed only in 2 ways, either $[g_2, [ab, g_1]]$ or $[[g_2, ab], g_1]$. Expanding $[ab, g_1]$ in $[g_2, [ab, g_1]]$ by \((2.7.1)\), then using (i) we obtain

\[
[g_2, [ab, g_1]] = [g_2, [a, b, g_1][b, g_1][a, g_1]] = [g_2, [a, b, g_1]][g_2, [b, g_1]][g_2, [a, g_1]].
\]

The first term vanishes and $[g_2, [b, g_1]], [g_2, [a, g_1]]$ commute by \((2.7.3)\) yielding $[g_2, [ab, g_1]] = [g_2, [a, g_1]][g_2, [b, g_1]]$. Similarly we get

\[
[[g_2, ab], g_1] = [[g_2, a][a, g_2, b][g_2, b], g_1] \quad \text{(by (2.7.2))}
\]

\[
= [[g_2, a], g_1][[a, g_2, b], g_1][[g_2, b], g_1] \quad \text{(by (ii))}.
\]

The second term vanishes yielding $[[g_2, ab], g_1] = [[g_2, a], g_1][[g_2, b], g_1]$.

Let $r > 2$, we proceed by considering the following cases:

(a) Let a left bracket be next to $ab$, and $[g_r, \ldots, g_{i+1}, ab, g_i, \ldots, g_1]$ be of the form $[g_r, \ldots, g_{i+1}, [ab, g_i], \ldots, g_1]$. Then expanding $[ab, g_i]$ in $[g_r, \ldots, g_{i+1}, [ab, g_i], \ldots, g_1]$ by \((2.7.1)\), we obtain $[g_r, \ldots, g_{i+1}, [ab, g_i], \ldots, g_1] = \ldots
Corollary 2.9. Let $G$ be a group of nilpotency class $c > 1$, and $[\ldots, \ldots]$ be a commutator of weight $c$. Then the map $[\ldots, \ldots]_{c \text{ times}} : G \times G \times \cdots \times G \rightarrow \gamma_c(G)$ given by $(g_c, \ldots, g_1) \mapsto [g_c, \ldots, g_1]$ is multiplicative in each coordinate.

Lemma 2.10. Let $G$ be a nilpotent group of class $c$ and $a \in \gamma_i(G)$, $b \in \gamma_j(G)$.

(i) If $2i+3j \geq c+1$, then $[b^n, a] = \prod_{t=1}^{i} [b, a]^{(t)} [b, a]^n$ for all $n \in \mathbb{N}$. Moreover, if $i+j \geq c+1$ and $2i+3j \geq c+1$, then $[b^n, a] = \prod_{t=1}^{i} [b, a]^{(t)} [b, a]^n$ for all $n \in \mathbb{N}$. 

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(ii) If $3i + 2j \geq c+1$, then $[b, a^n] = \prod_{i=n-1}^1 [a, b, a]^{(i+1)}[b, a]^n$ for all $n \in \mathbb{N}$. Moreover, if $ri + j \geq c+1$ and $3i + 2j \geq c+1$, then $[b, a^n] = \prod_{i=r-2}^1 [a, b, a]^{(i+1)}[b, a]^n$ for all $n \in \mathbb{N}$.

(iii) If $i + 3j \geq c+1$ and the order of $b$ is $n$ modulo the center, then the order of $[b, a]$ is at most $n$ if $n$ is odd, and at most $2n$ if $n$ is even.

(iv) If $3i + j \geq c+1$ and the order of $a$ is $n$ modulo the center, then the order of $[b, a]$ is at most $n$ if $n$ is odd, and at most $2n$ if $n$ is even.

**Proof.** The first two parts can be proved using the commutator identities in Lemma 2.7 and by induction on $n$. To prove (iii) we set $m = n$ in case $n$ is odd and $m = 2n$ in case $n$ is even. Using (i), we have $1 = [b^m, a] = [b, b, a]^{(m)}[b, a]^m$. Applying Lemma 2.8 yields $[b^{(m)}, b, a] = [b, b, a]^{(m)}$. Since $n \mid (m)$, we obtain $[b^{(m)}, b, a] = 1$, proving $[b, a]^m = 1$. Similarly (iv) can be proven using (ii).

In the next Theorem, we recall the collection formula given in Theorem 12.3.1 of [13].

**Theorem 2.11.** We may collect the product $(x_1 x_2 \ldots x_j)^n$ in the form $(x_1 x_2 \ldots x_j)^n = x_1^n x_2^n \cdots x_j^n C_{j+1}^{(n)} \cdots C_k^{(n)} R_1 R_2 \cdots R_a$, where $C_{j+1}, \ldots, C_k$ are the basic commutators on $x_1, x_2, \ldots, x_j$ in order, and $R_1, \ldots, R_a$ are basic commutators later than $C_i$ in the ordering. For $j + 1 \leq i \leq k$, the exponent $e_i$ is of the form $e_i = a_1 n + a_2 n^{(2)} + \cdots + a_m n^{(m)}$, where $m$ is the weight of $C_j$, the $a$’s are non-negative integers and do not depend on $n$ but only on $C_j$. Here $n^{(i)} = \binom{n}{i}$.

Let $\mathcal{A}(\{a, b\})$ denote the set of all basic commutators of $a, b$. Set $\mathcal{A}_i(\{a, b\}) = \{C \in \mathcal{A}(\{a, b\}) : w(C) \leq i\}$, in which $w(C)$ denote the weight of $C$ in $a, b$. We restate Theorem 2.11 in the following way.

**Lemma 2.12.** Let $G$ be a group, $a, b \in G$ and $n, i \in \mathbb{N}$. Then $(ab)^n = \prod_{C \in \mathcal{A}_i(\{a, b\})} C^{f_C(n)} a^n b^n \mod \gamma_i + 1(\langle a, b \rangle)$. For $C \in \mathcal{A}(\{a, b\})$, $f_C(n) = a_1 \binom{n}{1} + a_2 \binom{n}{2} + \cdots + a_{r_C} \binom{n}{r_C}$, where $r_C$ is the largest $r$ with $a_r \neq 0$ and $a_1, a_2, \ldots, a_{r_C}$ are non-negative integers depending only on $C$.

The following remark can be found in Lemma 3.2.5 of [19].

**Remark 2.13.** When applying the commutator collection process to $(ab)^n$, if $C_i = [b, i a]$, then $e_i = \binom{n}{i+1}$.
Remark 2.14. Since we follow left notations, \([b, a]\) in Remark 2.13 corresponds to the commutator \([a, b, a]\).

The next lemma can be found in [19].

Lemma 2.15. Let \(G\) be a regular \(p\)-group and \(n\) be a positive integer. Then the following holds:

(i) For all \(a, b \in G\), the following are equivalent: \([b, a]^{p^n} = 1; [b, a^{p^n}] = 1; [b^{p^n}, a] = 1\).

(ii) For all \(a \in G\), any commutator \(C\) of weight at least 1 in \(a\) has order at most the order of a modulo the center.

(iii) For all \(g_1, g_2, \ldots, g_r \in G\), the order of the product \(g_1g_2\ldots g_r\) is at most the maximum of the orders of \(g_1, g_2, \ldots, g_r\).

(iv) For all \(a, b \in G\), \(a^{p^n} = b^{p^n}\) if and only if \((ab^{-1})^{p^n} = 1\).

3. On a Theorem of Schur and solving the Conjecture for \(p\)-groups of class at most \(p\)

For an odd prime \(p\), the conjecture was proved for \(p\)-groups of class at most \(p - 2\) by the author of [24] and for \(p\)-groups of class at most \(p - 1\) by authors of [21]. We prove this result for \(p\)-groups of class at most \(p\), by showing that the exponent of the commutator subgroup of a Schur cover of \(G\) divides exponent of \(G\). We refer the reader to [15] for an account on central extensions and Schur covers. In this section we also prove the generalization of the theorem by I. Schur ([30]) mentioned in the introduction.

A group \(G\) is said to be \(n\)-central if exponent of \(G/Z(G)\) divides \(n\) and \(G\) is said to be \(n\)-abelian if for all \(a, b \in G\), \((ab)^n = a^nb^n\). In [22], the author has proved that regular \(p\)-groups have zero exponential rank. Thus for a regular \(p\)-group \(G\), \(p^n\)-central implies \(p^n\)-abelian, which can also be seen using Lemma 3.1. Moreover Lemma 3.2 for the case \(n = 1\) was proved by A. Mann in Lemma 9 of [17]. The next two lemmas follow from Lemma 2.15. We include them for the sake of completeness.

Lemma 3.1. Let \(G\) be a regular \(p\)-group and \(a, b \in G\). If \([b, a^{p^n}] = 1\), then \((ab)^{p^n} = a^{p^n}b^{p^n}\) for all \(n \in \mathbb{N}\).
Proof. Since $G$ is regular $(ab)^{p^n} = a^{p^n} b^{p^n} s^{p^n}$, where $s \in \gamma_2(a, b)$. Using Lemma \ref{2.15}(i), we get $[b, a]^{p^n} = 1$. Note that $\gamma_2(a, b) = \langle \{g[b, a] : g \in \langle a, b \rangle \} \rangle$. Hence by Lemma \ref{2.15}(iii), we obtain $s^{p^n} = 1$ completing the proof. \hfill \qed

Lemma 3.2. Let $G$ be a $p$-group of nilpotency class $p$ and $a, b \in G$. Then the following are equivalent: $[b, a]^{p^n} = 1; [b, a^{p^n}] = 1; [b^{p^n}, a] = 1$ for all $n \in \mathbb{N}$.

Proof. By symmetry it is enough to show $[b, a]^{p^n} = 1$ if and only if $[b, a^{p^n}] = 1$. We have $[b, a^{p^n}] = ba^{p^n}b^{-1}(a^{-1})^{p^n} = (b)_{p^n}(a^{-1})^{p^n}$ and $[b, a]^{p^n} = (baa^{-1})^{p^n}$.

Note that the group $\langle b, a, a \rangle = \langle [b, a], a \rangle$ has nilpotency class $\leq p - 1$. Hence using Lemma \ref{2.15}(iv), we obtain $(b)_{p^n} = (a)_{p^n}$ if and only if $(baa^{-1})^{p^n} = 1$. Thus $[b, a^{p^n}] = 1$ if and only if $[b, a]^{p^n} = 1$. \hfill \qed

Corollary 3.3. Let $G$ be a $p$-group of nilpotency class $p$. Then $\exp(\gamma_2(G)) \mid \exp(G / Z(G))$.

Corollary 3.3 follows from Lemma 3.2 and Lemma 3.1.

Corollary 3.4. Let $G$ be a $p$-group of nilpotency class at most $p - 1$. Then $\exp(M(G)) \mid \exp(G)$.

Proof. Applying Corollary 3.3 to a Schur cover of $G$ yields the proof. \hfill \qed

The next lemma is a generalization of Lemma \ref{2.15}(ii) for groups of class $p$.

Lemma 3.5. Let $G$ be a $p$-group of nilpotency class $p$. Then for all $a \in G$, any commutator $C$ of weight at least 1 in $a$ has order at most the order of $a$ modulo the center.

Proof. Let the order of $a$ modulo the center be $p^n, n \geq 1$. If $C = [g_1, a]$ or $C = [a, g_2], g_1, g_2 \in G$, then applying Lemma 3.2 yields $C^{p^n} = 1$. We proceed by induction on the weight of $C$. Let $w(C) > 2$ and $C = [C_2, C_1]$, where both $C_1, C_2$ are having lesser weight than $C$. If either $C_1$ or $C_2$ is $a$, then using Lemma 3.2 we obtain $C^{p^n} = 1$. Otherwise, suppose $C_1$ has weight $\geq 1$ in $a$. Then by induction hypothesis, $C_1^{p^n} = 1$ and thus $[C_2, C_1^{p^n}] = 1$. Hence Lemma 3.2 implies that $C^{p^n} = 1$. \hfill \qed

Observe that if $G$ is a group of nilpotency class 3, then for every $a, b \in G$ $[b, a^n] = [a, b, a] \langle g \rangle [b, a]^n$ for all $n \in \mathbb{N}$. So Lemma 3.3 clearly holds for the case $p = 2$. 

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Lemma 3.6. Let $p$ be an odd prime and $n$ be a positive integer. Suppose $G$ is a $p$-group and $H = \langle a, b \rangle$ has nilpotency class at most $p + 1$ for some $a, b \in G$. If $[b, a^p^n] = 1$, then \[ [p^{-1}a, [h, a]]^{(p^n)}[h, a]^{p^n} = 1 \text{ for all } h \in H. \]

Proof. Since $H = \langle a, b \rangle$, $a^p^n \in Z(H)$. So $1 = [h, a^p^n] = h a^{p^n} h^{-1} a^{-p^n} = (h a)^{p^n} a^{-p^n} = ([h, a])^{p^n} a^{-p^n}$. Note that the group $\langle [h, a], a \rangle$ has class $\leq p$. Applying Lemma 2.12 to $([h, a], a)$, we have

\[ 1 = \prod_{C \in A_p([h, a], a)} C^{f_C(p^n)}[h, a]^{p^n}. \]

(3.6.1)

Note that Lemma 3.5 implies that $C^{p^n} = 1$ for every $C \in A_p([h, a], a)$. Consider a $C \in A_p([h, a], a)$ and let $n_1, n_2$ be weights of $C$ in $[h, a], a$ respectively. Then $n_1 + n_2 \leq p$ and since $C \in \gamma_{2n_1+n_2}(H)$, $2n_1 + n_2 \leq p + 1$. From Lemma 2.12 recall that $r_C \leq n_1 + n_2$. So if $n_1 + n_2 < p$, then $r_C < p$. Moreover if $n_1 + n_2 = p$, then $2n_1 + n_2 \leq p + 1$ yielding $n_1 = 1$ and $n_2 = p - 1$. Therefore $r_C < p$ for every $C \in A_p([h, a], a)$, hence $p^n | f_C(p^n)$, except for $C = [p^{-1}a, [h, a]]$. Also $f_{[p^{-1}a, [h, a]]}(p^n) = (p^n) / p$ by Remark 2.14. Thus (3.6.1) gives the result we seek. \hfill \square

Definition 3.7. Define $\alpha_m(n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_{m-1} < n} \binom{n}{i_{m-1}} \binom{m-1}{i_{m-2}} \cdots \binom{2}{i_1}$, where $1 < m \leq n \in \mathbb{N}$.

Lemma 3.8. For $2 < m \leq n$, $\alpha_m(n) = \sum_{k=m-1}^{n-1} \binom{n}{k} \alpha_{m-1}(k)$.

Proof. Evaluating the sum on the right

\[
\sum_{k=m-1}^{n-1} \binom{n}{k} \alpha_{m-1}(k) = \sum_{k=m-1}^{n-1} \left( \binom{n}{k} \times \sum_{1 \leq i_1 < i_2 < \cdots < i_{m-2} < k} \binom{k}{i_{m-2}} \binom{i_{m-2}}{i_{m-3}} \cdots \binom{2}{i_1} \right)
\]

\[ = \sum_{1 \leq i_1 < i_2 < \cdots < i_{m-2} < k < n} \binom{n}{k} \binom{k}{i_{m-2}} \cdots \binom{2}{i_1}
\]

\[ = \alpha_m(n). \]

\hfill \square

If $G$ is a $2$-group of class $3$ and $exp(G / Z(G)) = 2^n$, then $[b, a^{2^n}] = [a, b, a]^{(2^n)}[b, a]^{2^n} = 1$ for all $a, b \in G$. Since $a, b \in G$ are arbitrary, replacing $a$ with $ab$ in $[a, b, a]^{(2^n)}[b, a]^{2^n} = 1$ yields $[ab, b, ab]^{(2^n)}[a, ab]^{2n} = 1$. 

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\[ [a, b, a]^{(2n)} [b, b, a]^{(2n)} [b, a]^{2n} = 1. \] Thus we obtain \([b, b, a]^{(2n)} = 1.\) Moreover \([b, b, a] = [b, a, b]^{-1},\) yielding \([b, a, b]^{(2n)} = 1.\) Now interchanging the roles of \(a\) and \(b\) gives \([a, b, a]^{(2n)} = 1.\) Thus we obtain \([b, a]^{2n} = 1,\) and hence \(\exp(\gamma_2(G)) \mid 2n.\) In Theorem 3.9 we ignore the case \(p = 2\) since the conjecture was proved for groups of class 2 in [14].

**Theorem 3.9.** Let \(p\) be an odd prime and \(G\) be a \(p\)-group of nilpotency class \(p + 1.\) If \(G\) is \(p^n\)-central, then \(\exp(\gamma_2(G)) \mid p^n.\)

**Proof.** Note that \(\gamma_p(\gamma_2(G)) \subset \gamma_{2p}(G) \subset \gamma_{p+2}(G) = 1.\) Thus \(\gamma_2(G)\) is \(p^n\)-abelian by Lemma 3.1. Hence it is enough to show \([b, a]^{p^n} = 1\) for every \(a, b \in G.\) Let \(a, b \in G,\) applying Lemma 3.6 we have

\[
[p-1a, [b, a]]^{(p^n)} [b, a]^{p^n} = 1. \tag{3.9.1}
\]

In the remaining proof we proceed to show \([p-1a, [b, a]]^{(p^n)} = 1.\) Towards that end, since \(a, b \in G\) are arbitrary, we keep replacing \(a\) with \(ab\) starting from (3.9.1). Let \(S = \{[x_0, x_1, x_2, \ldots, x_p, [b, a]] : x_0, x_1, x_2, \ldots, x_p \in \{a, b\}\},\) where the commutators in \(S\) are right normed. For \(1 \leq i \leq p-1,\) define \(\sigma_i : S \rightarrow \{a, b\}\) as \(\sigma_i([x_0, x_1, x_2, \ldots, x_p, [b, a]]) = x_i.\) For \(C \in S,\) define \(S_C = \{D \in S : \forall 1 \leq i \leq p - 1, \text{ if } \sigma_i(C) = b, \text{ then } \sigma_i(D) = b\}.\) Observe that all the elements of \(S\) commute with one another. Then using Corollary 2.9, replacing \(a\) in \(C\) with \(ab\) gives \(\prod_{D \in S_C} D.\) For \(0 \leq r \leq p - 1,\) let \(S_r = \{C \in S : \sigma_i(C) = b \text{ for exactly } r \text{ } \sigma_i, \quad 1 \leq i \leq p - 1\}\) and \(E_r = \prod_{C \in S_r} C.\) Consider \(C \in S_r\) and \(D \in S_t.\) Note that \(D \in S_C\) implies \(t \geq r.\) Moreover for every \(D \in S_t,\) the number of \(C \in S_r\) having \(D \in S_C\) is \(\binom{t}{r}.\) Thus replacing \(a\) with \(ab\) in \(E_r\) gives \(\prod_{C \in S_r} (\prod_{D \in S_C} D) = \prod_{r=0}^{p-1} E_r^{(t)}.\) Now we replace \(a\) in (3.9.1) with \(ab\) and observe that \([p-1a, [b, a]] = E_0\) and \([b, ab] = [b, a].\) After this replacement (3.9.1) becomes

\[
\left( \prod_{r=0}^{p-1} E_r^{(r^n)} [b, a]^{p^n} \right) = 1. \tag{3.9.2}
\]

Comparing (3.9.1) with (3.9.2) yields

\[
\prod_{r=1}^{p-1} E_r^{(r^n)} = 1. \tag{3.9.3}
\]
By replacing \( a \) in (3.9.3) with \( ab \), we obtain
\[
E_{p-2}^p \prod_{r=1}^{p-2} \left( \prod_{t=r}^{p-1} E_t^{(i)} \right)^{p^n} = 1. 
\] (3.9.4)

By comparing (3.9.3) and (3.9.4), we obtain
\[
\prod_{p=1}^{p-2} \left( \prod_{t=r+1}^{p-1} E_t^{(i)} \right)^{p^n} = 1. 
\] (3.9.5)

Now rearranging \( \prod_{p=1}^{p-2} \left( \prod_{t=r+1}^{p-1} E_t^{(i)} \right)^{p^n} \) as
\[
\prod_{p=1}^{p-2} \left( \prod_{t=r+1}^{p-1} E_t^{(i)} \right)^{p^n} = 1. 
\] (3.9.5)

We claim that for \( 2 \leq m \leq p-1 \),
\[
\prod_{k=m}^{p-1} \left( E_k^{\alpha_m(k)} \right)^{p^n} = 1. 
\] (3.9.6)

Observe that \( \sum_{t=1}^{m-1} (i_t) = \alpha_2(t) \), and hence (3.9.5) becomes \( \prod_{t=2}^{p-1} \left( E_t^{\alpha_2(t)} \right)^{p^n} = 1 \). Now we proceed by induction on \( m \). For \( 2 < m \leq p-1 \), assuming (3.9.6) for \( m-1 \) yields
\[
\prod_{k=m-1}^{p-1} \left( E_k^{\alpha_{m-1}(k)} \right)^{p^n} = 1. 
\] (3.9.7)

By replacing \( a \) with \( ab \) in (3.9.7), we get
\[
\left( E_{p-1}^{\alpha_{m-1}(p-1)} \right)^{p^n} \prod_{k=m-1}^{p-1} \left( \prod_{j=k}^{p-1} E_j^{(i)} \right)^{\alpha_{m-1}(k)}{p^n} = 1. 
\] (3.9.8)

Now comparison of (3.9.7) and (3.9.8) gives
\[
\prod_{k=m-1}^{p-1} \left( \prod_{j=k+1}^{p-1} E_j^{(i)} \right)^{\alpha_{m-1}(k)}{p^n} = 1. 
\] (3.9.8)

Rearranging
\[
\prod_{k=m-1}^{p-1} \left( \prod_{j=k+1}^{p-1} E_j^{(i)} \right)^{\alpha_{m-1}(k)}{p^n} = 1 
\]
and using Lemma 3.8, we obtain
\[
\prod_{j=m}^{p-1} \left( E_j^{\alpha_m(j)} \right)^{p^n} = 1. 
\] (3.9.9)

This proves the claim. Setting \( m = p-1 \) in (3.9.6) yields \( E_{p-1}^{\alpha_{p-1}(p-1)}{p^n} = 1 \). Note that \( \alpha_{p-1}(p-1) = (p-1)! \) is relatively prime to \( p \), so...
We have \( E_{p-1} = [\ p-1b, [b, a]] \), and interchanging \( a \) and \( b \) in (3.9.10) gives \([ p-1a, [a, b]]^{(n_p)} = 1. \) Since \([ p-1a, [b, a]] = [ p-1a, [a, b]]^{-1} \), we obtain \([ p-1a, [b, a]]^{(n_p)} = 1 \), proving the theorem.

Keeping the notations as in the proof of Theorem 3.9, we give an illustration of Theorem 3.9 by taking the special case \( p = 5 \).

Example 3.10.

\[ E_0 = [\ 4a, [b, a]], \]
\[ E_1 = [b, a, a, a, [b, a]] [a, b, a, [b, a]] [a, a, a, [b, a]] [a, a, a, [b, a]], \]
\[ E_2 = [b, a, a, a, [b, a]] [a, b, a, [b, a]] [a, b, a, [b, a]] [a, b, a, [b, a]], \]
\[ E_3 = [b, a, a, a, [b, a]] [a, b, a, [b, a]] [a, b, a, [b, a]], \]
\[ E_4 = [\ 4b, [b, a]]. \]

Replacing \( a \) in \( E_0 \) with \( ab \) gives \( E_0 E_1 E_2 E_3 E_4 \). Similarly replacing \( a \) in \( E_1, E_2, E_3, E_4 \) with \( ab \) gives \( E_1 E_2^2 E_3^3 E_4^4, E_2 E_3^3 E_4^6, E_3 E_4^4, E_4 \) respectively.

By (3.9.1) we have

\[ E_0^{(n_p)} [b, a]^{5^n} = 1. \] (3.10.1)

Replacement of \( a \) in (3.10.1) with \( ab \) gives

\[ (E_0 E_1 E_2 E_3 E_4)^{(n_p)} [b, a]^{5^n} = 1. \] (3.10.2)

Comparing (3.10.1) and (3.10.2) yields

\[ (E_1 E_2 E_3 E_4)^{(n_p)} = 1. \] (3.10.3)

Again replacement of \( a \) in (3.10.3) with \( ab \) gives

\[ ((E_1 E_2^2 E_3^3 E_4^4)(E_2 E_3^3 E_4^6)(E_3 E_4^4)(E_4))^ {(n_p)} = 1. \] (3.10.4)

By comparing (3.10.3) and (3.10.4), we obtain \((E_2^2 E_3^3 E_4^4)(E_3 E_4^4)(E_4))^ {(n_p)} = 1. \) Thus

\[ (E_2^2 E_3^3 E_4^{14})^{(n_p)} = 1. \] (3.10.5)
Again replacement of $a$ in (3.10.3) with $ab$ yields,

$$
((E_2 E_3^3 E_4^6)^2 (E_3 E_4^4 E_6^{14}))^{(5^n)} = 1. \tag{3.10.6}
$$

Now by comparing (3.10.5) and (3.10.6) we get

$$
((E_3^3 E_4^6)^2 (E_4^4)^6)^{\binom{n}{5}} = 1.
$$

This gives

$$
((E_3^3 E_4^6)^2 (E_4^4)^6)^{\binom{n}{5}} = 1. \tag{3.10.7}
$$

After replacing $a$ in (3.10.7) with $ab$, (3.10.7) becomes

$$
((E_3^3 E_4^6)^2 (E_4^4)^6)^{\binom{n}{5}} = 1. \tag{3.10.8}
$$

Finally comparing (3.10.7) and (3.10.8) yields

$$
(E_4^{24})^{\binom{n}{5}} = 1. \tag{3.10.9}
$$

The values of $\alpha_m(n)$ for $1 \leq m < n \leq 4$ are listed below:

$$
\alpha_2(2) = \binom{2}{1} = 2, \quad \alpha_2(3) = \binom{3}{1} + \binom{3}{2} = 6, \quad \alpha_2(4) = \binom{4}{1} + \binom{4}{2} + \binom{4}{3} = 14,
$$

$$
\alpha_3(3) = \binom{3}{2} \binom{2}{1} = 6, \quad \alpha_3(4) = \binom{4}{2} \binom{2}{1} + \binom{4}{3} \binom{3}{1} + \binom{4}{3} \binom{3}{2} = 36,
$$

$$
\alpha_4(4) = \binom{4}{3} \binom{3}{2} \binom{2}{1} = 24.
$$

From (3.10.5), (3.10.7) and (3.10.9), we can see that

$$\prod_{k=m} E_k^{\alpha_m(k)} \binom{n}{5} = 1 \text{ for } m = 2, 3, 4.$$

**Theorem 3.11.** Let $p$ be an odd prime and $G$ be a finite $p$-group. If the nilpotency class of $G$ is at most $p$, then $\exp(G \wedge G) \mid exp(G)$. In particular, $\exp(M(G)) \mid exp(G)$.

**Proof.** Let $H$ be a Schur cover for $G$. Since $H$ is a central extension of $G$, the nilpotency class of $H$ is at most $p + 1$. Thus by Theorem 3.9, $exp(\gamma_2(H)) \mid \exp(H / Z(H)) \mid \exp(G)$. The theorem now follows by noting that $G \wedge G \cong \gamma_2(H)$. \qed
4. Validity of the Conjecture for $p$-groups of nilpotency class at most 5 and of odd exponent.

The author of [26] has listed basic commutators in $a, b$ of weight at most 6 and their respective powers arising in the collection formula. The same collection formula, when we follow left notations is given in (i) of the next lemma.

**Lemma 4.1.** (i) Let $G$ be a group of nilpotency class 5, $a, b \in G$. Then for all $n \in \mathbb{N}$,

$$(ab)^n = [b, a, b, a]_6^{(n)} + 18^{(n)} + 12^{(n)} [b, a, b, a]_3^{(n)} + 7^{(n)} + 6^{(n)} [a, b, b, a]_3^{(n) + 4^{(n)}}$$

$$(b,a)_3^{(n) + 6^{(n)} + 6^{(n)} [a, b, b, a]_3^{(n) + 4^{(n)}}}$$

$$(b,a)_3^{(n) + 2^{(n)} + 2^{(n)} [b, b, a]_3^{(n) + 2^{(n)} [b, b, a]_3^{(n) + 2^{(n)} [b, b, a]_3^{(n) + 2^{(n)} b^n}.$$

(ii) Let $G$ be a group of nilpotency class 6, $a, b \in G$. Suppose $a \in \gamma_2(G)$, then for all $n \in \mathbb{N}$,

$$(ab)^n = [b, b, b, b, a]_6^{(n)} + 18^{(n)} + 12^{(n)} [b, b, b, a]_3^{(n)} + 7^{(n)} + 6^{(n)} [a, b, b, a]_3^{(n) + 4^{(n)}}$$

$$(b,a)_3^{(n) + 6^{(n)} + 6^{(n)} [a, b, b, a]_3^{(n) + 4^{(n)}}}$$

$$(b,a)_3^{(n) + 2^{(n)} + 2^{(n)} [b, b, a]_3^{(n) + 2^{(n)} [b, b, a]_3^{(n) + 2^{(n)} [b, b, a]_3^{(n) + 2^{(n)} b^n}.$$

(iii) Let $G$ be a group of nilpotency class 6, $a, b \in G$. Then for all $n \in \mathbb{N}$,

$$[b, a^n] = [a, a, a, a, b, a]_3^{(n)} [b, a, b, a]_3^{(n) + 3^{(n)} [a, a, a, a, b, a]_3^{(n)}}$$

$$[[b, a, a, a, b, a]_3^{(n) + 2^{(n)} [a, a, a, a, b, a]_3^{(n)}} [a, a, a, a, b, a]_3^{(n) + 2^{(n)} b^n}. $$

**Proof.** Note that (ii) follows from (i) and for proving (iii), we begin by writing $[b, a^n]$ as $([b, a]a)^n a^{-n}$. Now applying (ii) to $([b, a]a)^n$ yields (iii). □

**Lemma 4.2.** Let $G$ be a 3-group of class 6 and $a, b \in G$. If $n > 1$ and the order of a modulo the center is $3^n$, then

(i) $[a^3, a^3, a^3, a^3, b, a^3]^{3^{n-2}} = 1.$

(ii) $[a^3, a^3, a^3, b, a^3]^{3^{n-2}} = 1.$

(iii) $[a^3, a^3, b, a^3]^{3^{n-1}} = 1.$

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Theorem 4.3. Let $G$ be a 3-group of class 6, $a, b \in G$ and $c \in \gamma_2(G)$. 

Proof. (i) Applying Lemma 2.8 twice yields $[a^3, a^3, a^3, b, a^3]^{3n-2} = ([a^3, a^3, a^3, b, a^3]^{3n-2} = [a^3, a^3, a^3, b, a^3] = 1.$

(ii) Expanding $[b, a^3]$ in $[a^3, a^3, a^3, b, a^3]$ by Lemma 4.1(iii), and then using Lemma 2.8, we obtain $[a^3, a^3, a^3, b, a^3] = [a^3, a^3, a^3, [a, b, a]]$. Since $a^{3n} \in Z(G)$, Lemma 2.10(iii) yields $[a^3, a^3, a^3, [a, b, a]]^{3n-1} = [a^3, a^3, a^3, [a, b, a]]^{3n-1} = 1.$ Hence by (2.7.3), we have $[a^3, a^3, a^3, b, a^3]^{3n-2} = 1.$

(iii) We have $[a^3, a^3, a^3, b, a^3] = 1$. Now expanding $([a^3]^{3n-1}, a^3, b, a^3]$ by Lemma 2.10(i) yields $[a^3, a^3, a^3, b, a^3]^{(3n-1)}[a^3, a^3, b, a^3]^{(3n-1)} = 1.$$ Hence it is clear from (i), (ii) and (iii).

(iv) Since $[a^3, b, a^3] = 1$, expanding $([a^3]^{3n-1}, b, a^3]$ by Lemma 2.10(i) gives $[a^3, a^3, a^3, b, a^3]^{(3n-1)}[a^3, a^3, b, a^3]^{(3n-1)} = 1$. Now the result follows from (i), (ii) and (iii).

(v) Expanding the left most $[b, a^3]$ in $[[b, a^3], a^3, b, a^3]$ by Lemma 4.1(iii), and then applying Lemma 2.8 yields $[[b, a^3], a^3, b, a^3] = [[a, b, a], a^3, b, a^3]^{3n-1}$, $[a, b, a]^3 = 1$, by Lemma 2.10(iv), $[a, b, a, a^3, b, a^3]$ and $[b, a, a^3, b, a^3]$ have orders at most $3^n$. Now (2.7.4) yields $[[b, a^3], a^3, b, a^3]^{3n-2} = 1.$

(vi) Applying Lemma 2.10(i) to $[a^3, a^3, b, a^3]$ gives $[a^3, a^3, b, a^3] = [a, a, a, a^3, b, a^3]^{3n-1} = 1$, by Lemma 2.10(iv), $[a, a, a^3, b, a^3]$ and $[a, a, a^3, b, a^3]$ have orders at most $3^n$. Further using Lemma 2.8, we have $[a, a, a^3, b, a^3]^{3n-2} = ([a, a, a, a, b, a]^9)^{3n-2} = [a^3, a, a, a, b, a] = 1.$ Hence (2.7.4) yields $[a^3, a^3, b, a^3]^{3n-2} = 1.$

(vii) Since $[a^3, a^3, b, a^3]^{3n-2} = 1$, Lemma 2.10(iv) gives $[[b, a^3], a^3, b, a^3]^{3n-2} = 1.$

$\square$

Theorem 4.3. Let $G$ be a 3-group of class 6, $a, b \in G$ and $c \in \gamma_2(G)$. 

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(i) If \( a^{3n} \in Z(G) \), then \([b, a^3]^{3^{n-1}} = 1\).

(ii) If \( a^{3n} \in Z(G) \) and \( c^{3^{n-1}} = 1 \), then \(([b, a^3]c)^{3^{n-1}} = 1\).

**Proof.** (i) Since \( a^{3n} \in Z(G) \), applying Lemma 4.1(iii) to \([b, (a^3)^{3^{n-1}}]\) yields

\[
1 = [a^3, a^3, a^3, b, a^3]^{3^{n-1}} [b, a^3, a^3, b, a^3]^{2(3^{n-1})} [3^{n-1}] + [3^{n-1}]
\]

\[
[a^3, a^3, a^3, b, a^3]^{(3^{n-1})} [b, a^3, a^3, b, a^3]^{2(3^{n-1})} + [3^{n-1}]
\]

\[
[a^3, a^3, b, a^3]^{(3^{n-1})} [a^3, b, a^3]^{(3^{n-1})} [b, a^3]^{n-1}.
\]

Now by Lemma 4.2 we obtain that \([b, a^3]^{3^{n-1}} = 1\).

(ii) Expanding \(([b, a^3]c)^{3^{n-1}} \) by Lemma 4.1(ii) gives

\[
([b, a^3]c)^{3^{n-1}} = [b, a^3, c, b, a^3]^{(3^{n-1})} + [3^{n-1}]
\]

\[
[c, c, b, a^3]^{(3^{n-1})} [c, b, a^3]^{(3^{n-1})} [b, a^3]^{3^{n-1}} c^{3^{n-1}}.
\]

Since \([b, a^3]^{3^{n-1}} = 1\), Lemma 2.10(iv) yields \([c, b, a^3]^{3^{n-1}} = 1\). Now by applying Lemma 2.8 we have that \([b, a^3, c, b, a^3]^{3^{n-2}} = ([b, a^3, c, b, a^3]^{3^{n-2}} = [b, a^3]^{3^{n-2}} = [b, a^3]^{3^{n-2}} = [c, b, a^3]^{3^{n-2}} = 1\). Therefore \([b, a^3]c)^{3^{n-1}} = 1\). \(\square\)

Recall that if \( G \) has nilpotency class 5, then \( \eta(G) \) will have nilpotency class at most 6. In the proof of next Lemma we use the isomorphism of \( G \otimes G \) with the subgroup \([G, G^\phi]\) of \( \eta(G) \).

**Lemma 4.4.** Let \( G \) be a 3-group of class less than or equal to 5 and exponent \( 3^n \). Then the exponent of the image of \( G^3 \wedge G \) in \( G \wedge G \) divides \( 3^{n-1} \).

**Proof.** Let \( g, h, a_i, b_i \in G \) for \( i = 1, 2 \). Consider the isomorphism \( \psi : G \otimes G \rightarrow [G, G^\phi] \) defined by \( \psi(g \otimes h) = [g, h^\phi] \), where \( G^\phi \) is an isomorphic copy of \( G \). Applying Theorem 4.3 we obtain that \([a_i^3, b_i^3]^{3^{n-1}} = 1\) for \( i = 1, 2 \), and \(([a_i^3, b_i^3][a_i^3, b_i^3])^{3^{n-1}} = 1\). Thus \((a_i^3 \wedge b_i)^{3^{n-1}} = 1\) and \((a_i^3 \wedge b_1)(a_i^3 \wedge b_2))^{3^{n-1}} = 1\), proving the result we seek. \(\square\)

The following Lemma can be found in \([4]\) which is used for the proof of the main theorem of this section.
Lemma 4.5. Let $N, M$ be normal subgroups of a group $G$. If $M \subset N$, then we have the exact sequence $M \wedge G \to N \wedge G \to \frac{N}{M} \wedge \frac{G}{M} \to 1$.

Now we come to the main theorem of this section.

Theorem 4.6. Let $G$ be a finite $p$-group of nilpotency class 5. If $p$ is odd, then $\exp(G \wedge G) | \exp(G)$. In particular, $\exp(M(G)) | \exp(G)$.

Proof. The claim holds when $p \geq 5$ by Theorem 3.11. Now we proceed to prove for $p = 3$. Consider the exact sequence $G^3 \wedge G \to G \wedge G \to \frac{G}{G^2} \to 1$. Thus $\exp(G \wedge G) | \exp(im(G^3 \wedge G)) \exp(\frac{G}{G^2} \wedge \frac{G}{G^3})$. Note that $\exp(\frac{G}{G^2}) = 3$ and hence $\exp(\frac{G}{G^2} \wedge \frac{G}{G^3}) | 3$ (cf. [22]). By Lemma 4.4 we have $\exp(im(G^3 \wedge G)) | 3^{n-1}$ and the result follows.

5. Regular groups, Powerful groups and groups with power-commutator structure

In this section, we prove that $\exp(M(G)) | \exp(G)$ for powerful $p$-groups, the class of groups considered by D. Arganbright in [2], which includes the class of potent $p$-groups, and the class of groups considered by L.E. Wilson in [34], which includes $p$-groups of class at most $p - 1$.

Recall that a $p$-group $G$ is said to be powerful if $\gamma_2(G) \subset G^p$, when $p$ is odd and $\gamma_2(G) \subset G^4$, for $p = 2$. The next lemma can be found in [19].

Lemma 5.1. Let $G$ be a finite $p$-group. If $G$ is powerful, then the subgroup $G^p$ of $G$ is the set of all $p$th powers of elements of $G$ and $G^p$ is powerful.

In [16], the authors prove the conjecture for powerful groups. The authors of [18] prove that if $N$ is powerfully embedded in $G$, then the $\exp(M(G, N)) | \exp(N)$. In the theorem below, we generalize both these results.

Theorem 5.2. Let $p$ be an odd prime and $N$ be a normal subgroup of a finite group $G$. If $N$ is a powerful $p$-group, then $\exp(N \wedge G) | \exp(N)$. In particular, $\exp(M(G, N)) | \exp(N)$.

Proof. Let $\exp(N) = p^e$. We will proceed by induction on $e$. If $\exp(N) = p$, then $\gamma_2(N) \subset N^p = 1$. So by Lemma 2.3(ii), $\exp(N \wedge G) | \exp(N)$. Now let $e > 1$. Using Lemma 4.5, consider the exact sequence $N^p \wedge G \to N \wedge G \to \frac{N^p}{N} \wedge \frac{G}{N^p} \to 1$. Then $\exp(N \wedge G) | \exp(im(N^p \wedge G)) \exp(\frac{N^p}{N} \wedge \frac{G}{N^p})$. By Lemma 5.1 $N^p$ is powerful and $\exp(N^p) = p^{e-1}$. Thus by induction hypothesis, we
have \( \exp(N^p \land G) \mid \exp(N^p) \), and hence \( \exp(im(N^p \land G)) \mid \exp(N^p) \). Note that \( \frac{N}{N^p} \) is a powerful group of exponent \( p \), and we have showed above that the theorem holds for powerful groups of exponent \( p \). Therefore \( \exp(\frac{N}{N^p} \land \frac{G}{N^p}) \mid p \), whence the result.

**Definition 5.3.** Let \( G \) be a finite \( p \)-group.

(i) We say \( G \) satisfies condition (1) if \( \gamma_m(G) \subset G^p \) for some \( m = 2, 3, \ldots, p - 1 \), and it is said to be a potent \( p \)-group if \( m = p - 1 \).

(ii) We say \( G \) satisfies condition (2) if \( \gamma_p(G) \subset G^{p^2} \).

The next Theorem can be found in [19], [12] and [34].

**Theorem 5.4.** Let \( G \) be a finite \( p \)-group.

(i) If \( G \) regular, then \( G^p \) is the set of all \( p \)th powers of elements of \( G \) and \( G^p \) is powerful.

(ii) If \( G \) satisfies condition (1), then \( G^p \) is the set of all \( p \)th powers of elements of \( G \) and \( G^p \) is powerful.

(iii) If \( G \) satisfies condition (2), \( G^p \) is the set of all \( p \)th powers of elements of \( G \) and \( G^p \) is powerful.

Groups satisfying condition (1) were studied by D. E. Arganbright in [2]. This class includes the class of powerful \( p \)-groups for \( p \) odd and potent \( p \)-groups. The groups satisfying condition (2) were studied by L. E. Wilson in [34]. In the theorem below, we show that the conjecture is true for all these classes of groups. The first part of the Theorem below can be found in [25], but we give a different proof here. Since the proofs of both the parts of the next theorem are similar, we only prove the second part.

**Theorem 5.5.** Let \( p \) be an odd prime and \( G \) be a finite \( p \)-group satisfying either of the conditions below,

(i) \( \gamma_m(G) \subset G^p \) for some \( m \) with \( 2 \leq m \leq p - 1 \).

(ii) \( \gamma_p(G) \subset G^{p^2} \).

Then \( \exp(G \land G) \mid \exp(G) \). In particular, \( \exp(M(G)) \mid \exp(G) \).
Proof. (ii) Let \( \exp(G) = p^n \). If \( n = 2 \), then \( \gamma_p(G) \subset G^{p^2} = 1 \). Therefore by Theorem 3.11 \( \exp(G \wedge G) \mid \exp(G) \). For \( n > 2 \), consider the exact sequence \( G^p \wedge G \rightarrow G \wedge G \rightarrow G^p \wedge G^p \rightarrow 1 \), which implies \( \exp(G \wedge G) \mid \exp(G^p \wedge G^p) \exp(G^p \wedge G^p) \). By Theorem 3.4 \( G^p \) is powerful and \( \exp(G^p) = p^{n-1} \). By Theorem 5.2 \( \exp(G^p \wedge G) \mid p^{n-1} \), and hence \( \exp(G^p \wedge G) \mid p^{n-1} \). Since \( G^p \subset G^{p^2} \subset G^p \), the group \( G^p \) has nilpotency class \( \leq p-1 \). Now applying Theorem 3.11 we obtain \( \exp(G \wedge G) \mid p^n \). Therefore the result.

In the next Theorem, we show that to prove the conjecture for regular \( p \)-groups, it is enough to prove it for groups of exponent \( p \). We prove it more generally for the exterior square.

Theorem 5.6. The following statements are equivalent:

(i) \( \exp(G \wedge G) \mid \exp(G) \) for all regular \( p \)-groups \( G \).

(ii) \( \exp(G \wedge G) \mid \exp(G) \) for all groups \( G \) of exponent \( p \).

Proof. Since groups of exponent \( p \) are regular, one direction of the proof is trivial. To see the other direction, let \( G \) be a regular \( p \)-group. Suppose \( \exp(G) = p^n, n > 1 \). Consider the exact sequence \( G^p \wedge G \rightarrow G \wedge G \rightarrow G^p \wedge G^p \rightarrow 1 \), which implies \( \exp(G \wedge G) \mid \exp(G^p \wedge G^p) \exp(G^p \wedge G^p) \). By Theorem 3.4 \( G^p \) is powerful and \( \exp(G^p) = p^{n-1} \). Hence Theorem 5.2 yields \( \exp(G^p \wedge G) \mid p^{n-1} \), whence \( \exp(G^p \wedge G) \mid p^{n-1} \). Since \( G^p \) has exponent \( p \), our hypothesis implies that \( \exp(G^p \wedge G^p) \mid p^n \). Therefore \( \exp(G \wedge G) \mid p^n \).

6. Bounds depending on the nilpotency class

In [3], G. Ellis proves that if the nilpotency class of \( G \) is \( c \), then \( \exp(M(G)) \mid (\exp(G))^{c+1} \). In [22], P. Moravec improves this bound by showing that \( \exp(M(G)) \mid (\exp(G))^{2[\log_2 c]} \). In the next Theorem, we improve both these bounds. The case \( c = 1 \) has been excluded as the conjecture is known to be true in that case.

Theorem 6.1. Let \( G \) be a finite group with nilpotency class \( c > 1 \) and set \( n = \lceil \log_3 (c+1) \rceil \). If \( \exp(G) \) is odd, then \( \exp(G \wedge G) \mid (\exp(G))^n \). In particular, \( \exp(M(G)) \mid (\exp(G))^n \).
Furthermore, he has given an isomorphism between \( \mathbb{N} \). Theorem 3.11, we have
\[ \exp G. \] Ellis has proved the existence of a covering pair for a pair of groups \( (G, N) \). Now we proceed to prove it for \( \exp G \). Let \( p \) be an odd prime and \( G \) be a finite \( p \)-group. If \( N \) is a group of nilpotency class at most \( p - 2 \). The proof is by induction on \( n \). Note that \( n \geq \log_3(\frac{c+1}{3}) \) if and only if \( c \leq (3^n \times 2) - 1 \). When \( n = 1 \), the statement follows by Theorem 4.6. Now we proceed to prove it for \( n \). Set \( m = \lceil \frac{c+1}{3} \rceil \) and consider the exact sequence \( \gamma_m(G) \triangleleft G \rightarrow G \triangleleft \mathbb{G} \rightarrow \frac{G}{\gamma_m(G)} \triangleleft \frac{G}{\gamma_m(G)} \rightarrow 1 \), which can be obtained from Theorem 3.1 in [1]. Thus \( \exp(G \triangleleft G) \mid \exp(\gamma_m(G) \triangleleft G) \bigg| \exp\left(\frac{G}{\gamma_m(G)} \triangleleft \frac{G}{\gamma_m(G)}\right)\). By Lemma 2.6, we obtain that \( \exp(\gamma_m(G) \triangleleft G) \mid \exp(G) \). Now \( \frac{G}{\gamma_m(G)} \) is a group of nilpotency class \( m - 1 \). Since \( c + 1 \leq (3^n \times 2) \), \( \frac{c+1}{3} \leq (3^{n-1} \times 2) \) giving \( m \leq (3^{n-1} \times 2) \). Now applying induction hypothesis to the group \( \frac{G}{\gamma_m(G)} \), we obtain \( \exp\left(\frac{G}{\gamma_m(G)} \triangleleft \frac{G}{\gamma_m(G)}\right) \mid (\exp(G))^n - 1 \), which proves the theorem.

By the previous theorem \( \exp(M(G)) \mid (\exp(G))^2 \) for a group \( G \) with odd exponent and nilpotency class \( c \) satisfying \( 6 \leq c \leq 17 \). Moreover, if \( G \) is a \( p \)-group with \( p \geq c \), then \( \exp(M(G)) \mid \exp(G) \). In the following theorem, we show that if \( p + 1 \leq c \leq 3p + 2 \), then \( \exp(M(G)) \mid (\exp(G))^2 \).

**Theorem 6.2.** Let \( p \) be an odd prime and \( G \) be a finite \( p \)-group of nilpotency class \( c \). If \( m := \lceil \frac{c+1}{3} \rceil \leq p + 1 \), then \( \exp(G \triangleleft G) \mid \exp(\gamma_m(G) \triangleleft G) \bigg| \exp\left(\frac{G}{\gamma_m(G)} \triangleleft \frac{G}{\gamma_m(G)}\right)\). In particular, \( \exp(M(G)) \mid (\exp(G))^2 \).

**Proof.** Consider the exact sequence, \( \gamma_m(G) \triangleleft G \rightarrow G \triangleleft \mathbb{G} \rightarrow \frac{G}{\gamma_m(G)} \triangleleft \frac{G}{\gamma_m(G)} \rightarrow 1 \). Thus we have \( \exp(G \triangleleft G) \mid \exp(\gamma_m(G) \triangleleft G) \bigg| \exp\left(\frac{G}{\gamma_m(G)} \triangleleft \frac{G}{\gamma_m(G)}\right)\). Now applying Lemma 2.6 yields \( \exp(\gamma_m(G) \triangleleft G) \mid \exp(\gamma_m(G)) \). Further by Theorem 3.11, we have \( \exp\left(\frac{G}{\gamma_m(G)} \triangleleft \frac{G}{\gamma_m(G)}\right) \mid \exp\left(\frac{G}{\gamma_m(G)}\right)\), and the result follows.

The next lemma is crucially used in the proof of theorem 6.4. In [9], G.Ellis has proved the existence of a covering pair for a pair of groups \( (G, N) \). Furthermore, he has given an isomorphism between \( [N^*, G] \) and \( N \triangleleft G \), where \( N^* \) is a covering pair of the pair \( (G, N) \). We use this isomorphism in the following theorem.

**Lemma 6.3.** Let \( p \) be an odd prime and \( G \) be a finite \( p \)-group. If \( N \trianglelefteq G \) of nilpotency class at most \( p - 2 \), then \( \exp(N \triangleleft G) \mid \exp(N) \).

**Proof.** Consider a projective relative central extension \( \delta : N^* \rightarrow G \) associated with the pair \( (G, N) \). Therefore \( \delta(N^*) = N \) and \( G \) acts trivially on \( \ker(\delta) \). We know from [3] that \( [N^*, G] \cong N \triangleleft G \). Since \( N \) is of class atmost \( p - 2 \),
Let $p$ be an odd prime and $N$ be a normal subgroup of a finite $p$-group $G$. If the nilpotency class of $N$ is $c$, then $\exp(N \wedge G) | (\exp(N))^n$, where $n = \lceil \log_p(c+1) \rceil$.

**Proof.** We proceed by induction on $c$. If $c \leq p - 2$, then by Lemma 6.3, $\exp(N \wedge G) | \exp(N)$. Let $c > p - 2$ and set $m = \lceil \frac{c+1}{p-1} \rceil$. By Lemma 4.5, we have an exact sequence $\gamma_m(N) \wedge G \to N \wedge G \to \frac{N}{\gamma_m(N)} \wedge \frac{G}{\gamma_m(N)} \to 1$ yielding $\exp(N \wedge G) | \exp(\im(\gamma_m(N) \wedge G)) \exp(\frac{N}{\gamma_m(N)} \wedge \frac{G}{\gamma_m(N)})$. Since $m(p-1) \geq c+1$, we get $\gamma_{p-1}(\gamma_m(N)) = 1$. Now using Lemma 6.3, we obtain $\exp(\gamma_m(N) \wedge G) | \exp(\gamma_m(N))$, and hence $\exp(\im(\gamma_m(N) \wedge G)) | \exp(N)$. Since $c+1 \leq (p-1)^n$, we have $\frac{c+1}{p-1} \leq (p-1)^{n-1}$, and hence $m \leq (p-1)^{n-1}$. Now by induction hypothesis, $\exp(\frac{N}{\gamma_m(N)} \wedge \frac{G}{\gamma_m(N)}) | (\exp(\frac{N}{\gamma_m(N)}))^{n-1}$. Therefore $\exp(N \wedge G) | (\exp(N))^n$. \qed

For a finite $p$-group $G$ with $p$ odd, the author of [29] proves that $\exp(M(G)) | (\exp(G))^m$, where $m = \lceil \log_p(c+1) \rceil$. We improve his bound in the next theorem.

**Theorem 6.5.** Let $p$ be an odd prime and $G$ be a finite $p$-group of nilpotency class $c \geq p$. Then $\exp(G \wedge G) | (\exp(G))^n$, where $n = 1 + \lceil \log_p(c+1) \rceil$. In particular, $\exp(M(G)) | (\exp(G))^n$.

**Proof.** For $c = p$, we have $\exp(G \wedge G) | \exp(G)$ by Theorem 3.11. Let $c > p$, and consider the exact sequence $\gamma_{p+1}(G) \wedge G \to G \wedge G \to \frac{G}{\gamma_{p+1}(G)} \wedge \frac{G}{\gamma_{p+1}(G)} \to 1$, which yields $\exp(G \wedge G) | \exp(\im(\gamma_{p+1}(G) \wedge G)) \exp(\frac{G}{\gamma_{p+1}(G)} \wedge \frac{G}{\gamma_{p+1}(G)})$. By Theorem 3.11, $\exp(\frac{G}{\gamma_{p+1}(G)} \wedge \frac{G}{\gamma_{p+1}(G)}) | \exp(\frac{G}{\gamma_{p+1}(G)})$. Let $k$ be the nilpotency class of $\gamma_{p+1}(G)$. Since $\lceil \frac{c+1}{p+1} \rceil (p+1) \geq c+1$, we obtain $k+1 \leq \lceil \frac{c+1}{p+1} \rceil$. Observe that $\frac{c+1}{p+1} \leq (p-1)^{n-1}$ gives $k+1 \leq (p-1)^{n-1}$. Now applying Theorem 6.3, we get $\exp(\gamma_{p+1}(G) \wedge G) | (\exp(\gamma_{p+1}(G)))^{n-1}$. Hence $\exp(\im(\gamma_{p+1}(G) \wedge G)) | (\exp(G))^{n-1}$, and the result follows. \qed
7. Bounds depending on the derived length

In [22], the author proved that the conjecture is true for metabelian $p$-groups of exponent $p$. In the theorem below, we prove it for $p$-central metabelian $p$-groups.

**Theorem 7.1.** Let $G$ be a $p$-central metabelian $p$-group. Then $\exp(M(G)) | \exp(G)$.

**Proof.** For groups of exponent $p$, the theorem holds by [22]. Now we consider groups of exponent $p^n$ with $n > 1$. Since $G$ is $p$-central, we have the commutative diagram

$$
\begin{array}{ccc}
G^p \cap G & \rightarrow & G \cap G \\
\downarrow \alpha & & \downarrow \beta \\
1 & \rightarrow & \gamma_2(G) \\
\end{array}
$$

where $\alpha$ and $\beta$ are the natural commutator maps.

Now Snake Lemma yields the exact sequence $\ker(\alpha) \rightarrow M(G) \rightarrow \ker(\beta) \rightarrow 1$. Since $\ker(\beta) \leq M(\frac{G}{G^p})$, we have $\exp(M(G)) | \exp(\ker(\beta))$.

Observe that $\exp(\ker(\beta)) | p^{n-1}$ because $G$ is $p$-central. Furthermore $\frac{G}{G^p}$ being a metabelian $p$-group of exponent $p$, $\exp(M(\frac{G}{G^p})) | p$, and the result follows.

The following Lemma can be deduced from Lemma 3 and the proof of Proposition 5 in [10].

**Lemma 7.2.** Let $N$ be a normal subgroup of a group $G$. If $N \subset \gamma_2(G)$, then the sequence $N \otimes G \rightarrow G \otimes G \rightarrow \frac{G}{G^p} \otimes \frac{G}{G^p} \rightarrow 1$ is exact.

In [22], the author showed that if $d$ is the derived length of $G$, then $\exp(M(G)) | (\exp(G))^{2(d-1)}$. The author of [23] improved this bound for $p$-groups by proving that $\exp(M(G)) | (\exp(G))^d$ when $p$ is odd, and $\exp(M(G)) | 2^{d-1}(\exp(G))^d$, when $p = 2$. Using our techniques, we obtain the following generalization of Theorem A of [28].

**Theorem 7.3.** Let $G$ be a solvable group of derived length $d$. 

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(i) If $\exp(G)$ is odd, then $\exp(G \otimes G) \mid (\exp(G))^d$. In particular, $\exp(M(G)) \mid (\exp(G))^d$.

(ii) If $\exp(G)$ is even, then $\exp(G \otimes G) \mid 2^{d-1}(\exp(G))^d$. In particular, $\exp(M(G)) \mid 2^{d-1}(\exp(G))^d$.

Proof. The proof proceeds by induction on $d$. When $d = 1$, $G$ is abelian and hence $\exp(G \otimes G) \mid \exp(G)$. Let $d > 1$ and $G^{(d-1)}$ denote the $d$th term of the derived series. Now consider the exact sequence $G^{(d-1)} \otimes G \to G \otimes G \to G^{(d-1)} \otimes G \to 1$ obtained from Lemma 7.2. Thus we have $\exp(G \otimes G) \mid \exp(\text{im}(G^{(d-1)} \otimes G)) \exp(\frac{G}{G^{(d-1)}} \otimes \frac{G}{G^{(d-1)}})$.

(i) Let $\exp(G)$ be odd. Since $G^{(d-1)}$ is abelian, Lemma 2.5(ii) implies $\exp(G^{(d-1)} \otimes G) \mid \exp(G^{(d-1)})$. Now induction hypothesis yields $\exp(\frac{G}{G^{(d-1)}} \otimes (\exp(G^{(d-1)}))^d) \mid (\exp(G^{(d-1)}))^d-1$, the result follows.

(ii) Now using Lemma 2.5(iii), the proof follows mutatis mutandis the proof of (i).

In the next lemma, we consider $N \trianglelefteq G$ instead of $G \otimes G$.

Lemma 7.4. Let $N \trianglelefteq G$. Suppose $N$ is solvable of derived length $d$.

(i) If $\exp(N)$ is odd, then $\exp(N \trianglelefteq G) \mid (\exp(N))^d$. In particular, $\exp(M(G, N)) \mid (\exp(N))^d$.

(ii) If $\exp(N)$ is even, then $\exp(N \trianglelefteq G) \mid 2^d(\exp(N))^d$. In particular, $\exp(M(G, N)) \mid 2^d(\exp(N))^d$.

Proof. We prove the Lemma by induction on $d$. For $d = 1$, the theorem follows from Lemma 2.5(ii) and (iii). Let $d > 1$, and consider the exact sequence $N^{(d-1)} \trianglelefteq G \to N \trianglelefteq G \to \frac{N}{N^{(d-1)}} \meq 1$ obtained from Lemma 4.5. Now the proof follows mutatis mutandis the proof of Theorem 7.3.

In the following table, we list and compare the values of $m$ for which $\exp(M(G)) \mid \exp(G)^m$, obtained by G. Ellis, P. Moravec and Theorem 6.1 of this paper.
Table I

| c   | G. Ellis [10] | P. Moravec [22] |
|-----|--------------|----------------|
| 3   | 2            | 2              |
| 4   | 2            | 4              |
| 5   | 3            | 4              |
| 6   | 3            | 4              |
| 17  | 9            | 8              |
| 53  | 27           | 10             |
| 161 | 81           | 14             |

P. Moravec improves the bound given by G. Ellis for \( c > 11 \). It can be seen that the bound obtained in Theorem 6.1 improves the other bounds. In the following table, we consider \( p \)-groups of nilpotency class \( c \) and exponent \( p^n \). The bounds \( p^m \), where \( \exp(M(G)) \mid p^m \), obtained by P. Moravec, N. Sambonet and Theorem 6.5 are listed.

Table II

| c   | p  | n  | P. Moravec [22] | N. Sambonet [29] |
|-----|----|----|----------------|-----------------|
| 5   | 3  | 1  | 3^2            | 3^4             |
| 5   | 3  | 2  | 3^8            | 3^6             |
| 7   | 7  | 1  | 7^2            | 7^2             |
| 15  | 13 | 2  | 13^9           | 13^4            |
| 24  | 5  | 1  | 5^4            | 5^3             |
| 168 | 13 | 1  | 13^{14}        | 13^4            |

where \( k \) is defined in [22].

We end this paper by making the following conjecture for which there is no counterexample so far. The two counterexamples to Schur’s conjecture found for 2-groups is not a counterexample to the following conjecture.

**Conjecture.** Let \( G \) be a finite \( p \)-group. Then \( \exp(M(G)) \mid p \exp(G) \), i.e. exponent of the Schur multiplier divides \( p \) times the exponent of the group.
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In a private communication with the third author, P. Moravec had mentioned that for groups with nilpotency class 5, he could prove \( \exp(M(G)) \mid (\exp(G))^3 \), while the computer evidence indicated that the bound should be 2 instead of 3. Later he himself proved the bound to be 2 in [26]. We thank P. Moravec for sharing this insight with us.

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