HEAT-CONDUCTING FLUIDS IN DOMAINS WITH OPEN BOUNDARIES

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Abstract. We study a Boussinesq system with mixed boundary conditions in a bounded domain, for steady and non-steady cases. The special feature of the problem is that the domain boundary is assumed to have an “open part” where the fluid can leave or re-enter. At this outlet, the fluid flow is assumed to satisfy a classical do-nothing condition and a new artificial nonlinear condition that couples the fluid velocity and temperature is considered for the heat transfer. In particular, the latter depends on a bounded function \( \beta \) that may be chosen according to the problem of interest. For the stationary case, we prove the existence and uniqueness of a weak solution in two or three dimensions provided \( \beta \) is Lipschitz continuous, and show that existence is retained if \( \beta \) is only continuous. For the time evolutionary problem, we provide existence and uniqueness of a weak solution for \( \beta \) Lipschitz in the two-dimensional case via a specific choice of the state space. All results are obtained under a “small” data assumption and restrictions on Reynolds, Prandtl, and Grashof numbers. We further present numerical tests that show increased accuracy of the new boundary condition when compared to other standard ones.

Key words. Boussinesq system, artificial boundary conditions, mixed boundary conditions, do-nothing boundary condition, weak solutions.

AMS subject classifications. 34A34, 34B15, 76D05, 80A20.

1. Introduction. In this work, we consider a problem for non-isothermal incompressible Newtonian fluids in a domain with an outlet where the fluid is allowed to leave. In particular, we consider a recently introduced artificial nonlinear boundary condition, study the stationary and time evolutionary versions of the problem, and provide numerical tests that show more accurate performance than other standard choices. In order to motivate the issue, consider the problem to describe the temperature, velocity, and pressure of the air in a room with a heating device on some of its walls and a door or window which is always open. This may be a subtask of a larger problem where we aim to achieve both comfort and efficiency of a heating system. Figure 1.1 shows a possible configuration for the above situation and illustrates the effects of buoyancy on the fluid velocity due to temperature changes. A common approach to treat this problem numerically is to restrict the computational domain to the room itself and to include the effects of the surrounding environment through an appropriate selection of boundary conditions. While this is quite simple to do on the parts of the boundary that represent insulated walls or a heating device, it is a major problem on the open door. Particularly, the question that arises is: What conditions on the velocity and the temperature of the fluid would be appropriate at the place

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where the fluid is allowed to flow freely? A good perspective of such a question is given in the words of Gresho and Sani [43] when referring to boundary conditions:

“Nature is usually silent, or in fact perverse, in not communicating the appropriate ones.”

The problem of selecting artificial boundary conditions for isothermal incompressible Newtonian fluids took impulse in the early 1990s from the work [30] of Gresho (see also [31, 36]) and its high level of difficulty was noticed from the beginning. In [43], Gresho and Sani collected a set of benchmark problems treated with different artificial boundary conditions with the aim to find appropriate ones. In words of the authors, this goal was not met and since then many other works have been made guided by the same target (see, e.g., the surveys in [21] or [34]). However, the artificial boundary condition proposed by Gresho in his pioneering article [30] became quite popular. It is known as the “do-nothing condition” and, in dimensionless form, reads as follows:

\[- p n + \frac{1}{Re} \frac{\partial v}{\partial n} = h \quad \text{on} \quad \Gamma_o, \quad (1.1)\]

where \(p\) and \(v\) are the pressure and the velocity of the fluid, respectively, \(Re\) is the Reynolds number, \(n\) is the unit outer normal to the open boundary \(\Gamma_o\), and \(h\) is a given function. The success of the do-nothing condition (1.1) is probably due to its relative easy implementation for numerical methods based on variational formulations. Furthermore, even if the origin of the do-nothing condition relies on variational principles (see, e.g., [34]), it is verified by Poiseuille flows in circular pipes and thus it is physically plausible. The do-nothing condition (1.1) has been widely used in the last decade. Successful and insightful approaches can be found in the works of Beneš, Kučera and collaborators [1, 13, 35]. Nevertheless, the artificial condition (1.1) has two well-known drawbacks: One of them is that it cannot prevent an unbounded growth of the kinetic energy of the system since it allows the fluid to leave and re-enter the truncated domain at unrestricted rates. As a consequence, a priori energy estimates (and hence, existence of weak solutions) can be obtained only for “small” data. The other one concerns its numerical instability. A succinct discussion on both subjects can be found in [15] (see also [14]). We finally mention that some ad-hoc changes on the model proposed by Gresho have proven able to overcome some drawbacks of the do-nothing condition. Successful approaches have been obtained by adding an extra term to the left-hand-side of (1.1) (see [15, 17, 24, 26, 37, 39, 41, 42]) as well as by adding an extra condition on the open boundary (see [32, 34]). Other artificial boundary conditions for isothermal fluids can be found in [21] and the references therein.
The analogous problem concerning heat-conducting fluids has received less attention. Some authors have used a do-nothing condition for the velocity field together with a Neumann condition for the fluid temperature, see for example [3,7,9,10,12,13,41,42]. In other cases, the two ad-hoc changes on the Gresho’s model mentioned before, together with an artificial boundary condition that couples the temperature and the velocity of the fluid were used; see [17,39,41,42]. In [17], the authors introduced the condition

\[
\frac{1}{\text{Re} \Pr} \frac{\partial u}{\partial n} - u \beta (\mathbf{v} \cdot \mathbf{n}) (\mathbf{v} \cdot \mathbf{n}) = h \quad \text{on} \quad \Gamma_o, \tag{1.2}
\]

where \( h \) and \( \beta \) are given, \( u \) is the temperature of the fluid, and \( \text{Pr} \) is the Prandtl number. The function \( \beta : \mathbb{R} \to \mathbb{R} \) was originally conceived bounded, continuous everywhere except maybe at the origin, and such that \( \beta(s) \in [0, 1/2] \) if \( s \geq 0 \), \( \beta(s) \in [1/2, 1] \) otherwise. Boundary condition (1.2) arises consequently from the study of heat conduction at the outlet and through an appropriate selection of functions \( h \) and \( \beta \). For example, the homogeneous Neumann condition \( (h \equiv 0 \text{ and } \beta \equiv 0) \) may be suitable if advection dominates the heat transfer process at the open boundary. However, if conduction is not negligible, a complex heat transfer process takes place at the open boundary; for example, the outside temperature can not be longer considered impervious to the domain effects, and might be transported into the domain by incoming flows. These two mutually dependent processes contribute to the heat transfer at the outlet and the function \( \beta \neq 0 \) aims to capture this feature. The bounds of \( s \mapsto \beta(s) \), for \( s < 0 \) and \( s > 0 \), directly affect the influence of incoming and outgoing flows, respectively, on the heat transfer at the open boundary; see [17]. We refer to the numerical and experimental works on open heated cavities reported by Chan and Tien in [18,19] for insights on complex phenomena taking place at the open boundary. The coupled condition used in [39,41,42] is a special case of (1.2) corresponding to a function \( \beta \) given by \( \beta(s) = c/2 \) if \( s < 0 \) or by \( \beta(s) = 0 \) otherwise, where \( c > 0 \) is constant. To the best of our knowledge, there are no other approaches for dealing with artificial boundary conditions for non-isothermal fluids in open domains.

In this work we study stationary and evolutionary Boussinesq equations with mixed boundary conditions in a domain with an open boundary. The physical problem in focus throughout the article is related to the example given above: Describing the velocity field, the temperature, and the pressure distributions of a fluid inside a room with a heating device located on its walls (e.g., a radiator and/or a vent of a heating system) and an open window or door. At the open boundary, we assume that the velocity, the temperature, and the pressure of the fluid satisfy do-nothing condition (1.1) and coupled condition (1.2). In particular, we remove conditions on \( \beta \) according to the sign of its argument, so that Neumann conditions are included in our study. For simplicity, we consider \( h \equiv 0 \) and \( h \equiv 0 \) in (1.1) and (1.2), respectively. Existence of solutions for the stationary problem is obtained via two approaches and under a “small” data assumption. In particular, uniqueness of solutions is attained if \( \beta \) is considered to be Lipschitz continuous, while existence of solutions is retained for \( \beta \) continuous, bounded, and possibly discontinuous at the origin. The time evolutionary problem is significantly more complex. In fact, existence and uniqueness of solutions can be obtained as well for two-dimensional problems provided \( \beta \) is Lipschitz continuous. While on the surface, the approach seems analogous to the stationary case, the entire mathematical machinery only works by the careful selection of possible state spaces. The latter efforts lead additionally to an improved regularity result.

The outline of the article is as follows. In section 2 we consider the stationary
The stationary problem. Let $\Omega \subset \mathbb{R}^d$ be a simply connected bounded domain with Lipschitz boundary $\Gamma := \partial \Omega$, where $d = 2$ or $d = 3$. In addition, let $\Gamma_i, \Gamma_w, \Gamma_d, \Gamma_n, \Gamma_o$ be relative open subsets of $\Gamma$ with positive $(d-1)$-Lebesgue measure that satisfy

$$
\Gamma = \overline{\Gamma_i} \cup \overline{\Gamma_w} \cup \overline{\Gamma_o}, \quad \Gamma_i \cap \Gamma_w = \emptyset, \quad \Gamma_i \cap \Gamma_o = \emptyset, \quad \Gamma_o \cap \Gamma_w = \emptyset,
$$

$$
\Gamma = \overline{\Gamma_d} \cup \overline{\Gamma_n} \cup \overline{\Gamma_o}, \quad \Gamma_d \cap \Gamma_n = \emptyset, \quad \Gamma_d \cap \Gamma_o = \emptyset, \quad \Gamma_n \cap \Gamma_o = \emptyset.
$$

For the reference problem in which $\Omega$ represents a room with a heating device on some of its walls and an open window or door, the decompositions $\{\Gamma_i, \Gamma_w, \Gamma_o\}$ and $\{\Gamma_d, \Gamma_n, \Gamma_o\}$ are related to the behaviour on the boundary of the velocity and the temperature of the fluid, respectively; see Figure 2.1. The boundary partitions represent the following phenomena and elements:

- $\Gamma_o$. An open boundary, e.g., an open window or door, where fluid is allowed to abandon the domain.
- $\Gamma_i$. The inlet location and where fluid is allowed to enter the domain, e.g., a vent of a heating/cooling system.
- $\Gamma_w$. Parts of the boundary where fluid does not enter nor leave the domain and further does not slip, e.g., walls or floors.
- $\Gamma_d$. Heated regions on walls or floor, e.g., a radiating floor, or inlets where the temperature for the fluid is prescribed.
- $\Gamma_n$. Thermally insulated parts of the boundary.

Then, for example, we may have $\Gamma_i \subset \Gamma_d$ and $\Gamma_n \equiv \Gamma_w$. According to this reference problem, the subscripts “i”, “w”, and “o” refer to the inlet, the walls, and the open part of the room, and the subscripts “d” and “n” refer to the parts of the boundary on which Dirichlet and Neumann type conditions shall be respectively imposed. Furthermore, from now on and throughout the paper, $\mathbf{v}$, $u$, and $p$ denote the velocity, the temperature, and the pressure of the fluid, respectively.

The stationary problem of interest is the following.

**Problem ($\mathcal{P}$):** Find $\mathbf{v} : \Omega \to \mathbb{R}^d$, $u : \Omega \to \mathbb{R}$, and $p : \Omega \to \mathbb{R}$ that satisfy the stationary Boussinesq equations in $\Omega$,

$$
\mathbf{v} \cdot \nabla \mathbf{v} - \frac{1}{\text{Re}} \Delta \mathbf{v} + \nabla p = \frac{Gr}{\text{Re}^2} \mathbf{e} + \mathbf{g}_1, \quad (2.1)
$$

$$
div \mathbf{v} = 0, \quad (2.2)
$$

$$
\mathbf{v} \cdot \nabla u - \frac{1}{\text{Re} \text{Pr}} \Delta u = g_2, \quad (2.3)
$$
subject to the boundary conditions

\begin{align*}
\mathbf{v} &= \mathbf{v}_i \quad \text{on} \quad \Gamma_i, & \mathbf{v} &= 0 \quad \text{on} \quad \Gamma_w, \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on} \quad \Gamma_n, & u &= u_d \quad \text{on} \quad \Gamma_d, \\
\frac{1}{Re} \frac{\partial v}{\partial n} &= p \mathbf{n} \quad \text{on} \quad \Gamma_o, & \frac{1}{Re Pr} \frac{\partial u}{\partial n} &= \beta(v \cdot \mathbf{n}) (v \cdot \mathbf{n}) \quad \text{on} \quad \Gamma_o.
\end{align*}

Here, \( g_1 : \Omega \to \mathbb{R}^d \) and \( g_2 : \Omega \to \mathbb{R} \) represent an external force/perturbation and the intensity of external sources of heat, respectively. The non-negative constants \( Re, Pr, Gr \) are the Reynolds, Prandtl, and Grashof numbers. The vector \( \mathbf{e} \) represents the opposite direction of acceleration due to gravity, i.e., \( \mathbf{e} := (0, 1) \) if \( d = 2 \), or \( \mathbf{e} := (0, 0, 1) \) if \( d = 3 \). The function \( v_i : \Gamma_i \to \mathbb{R}^d \) is the velocity profile at the inlet, and \( u_d : \Gamma_d \to \mathbb{R} \), the temperature distribution at the inlet and/or the heated part of the boundary. The vector \( \mathbf{n} \) is the outer unit normal to the boundary \( \Gamma \), and \( \beta : \mathbb{R} \to \mathbb{R} \) is a given function that satisfies

\( (A1) \) \( \beta \) is bounded and continuous everywhere except maybe at the origin.

In just a few words, \( \beta \) is a function associated to the thermal conductive phenomena at the outlet. In particular, for slow moving fluids, \( \beta \) allows to account for heat contributions outside the domain. We refer to \[30\] and \[17\] for details about the boundary conditions on \( \Gamma_o \).

2.1. Weak formulation. In this section we establish the weak formulation of the stationary problem together with the function spaces of interest and generalities about involved operators. Let

\begin{align*}
E_1(\Omega) := \{ y \in C^\infty(\Omega)^d : \text{div } y = 0, \text{ supp } y \cap (\Gamma_i \cup \Gamma_w) = \emptyset \}, \\
E_2(\Omega) := \{ y \in C^\infty(\Omega) : \text{ supp } y \cap \Gamma_d = \emptyset \},
\end{align*}
where \( C^\infty(\Omega) \) is the set of restrictions to \( \Omega \) of the real-valued infinitely differentiable functions in \( \mathbb{R}^d \). We define the spaces

\[
V_1 : \text{ closure of } E_1(\Omega) \text{ with respect to the norm } \| \cdot \|_{W^{1,2}(\Omega)^d},
V_2 : \text{ closure of } E_2(\Omega) \text{ with respect to the norm } \| \cdot \|_{W^{1,2}(\Omega)},
H_1 : \text{ closure of } E_1(\Omega) \text{ with respect to the norm } \| \cdot \|_{L^2(\Omega)^d},
H_2 : \text{ closure of } E_2(\Omega) \text{ with respect to the norm } \| \cdot \|_{L^2(\Omega)}.
\]

Notice that \( \| \nabla \cdot \|_{L^2(\Omega)^d} \) is equivalent to \( \| \cdot \|_{W^{1,2}(\Omega)^d} \) in \( V_1 \) and, analogously, \( \| \nabla \cdot \|_{L^2(\Omega)} \) is equivalent to \( \| \cdot \|_{W^{1,2}(\Omega)} \) in \( V_2 \). From now on, we consider \( V_1 \) and \( V_2 \) endowed with the norms

\[
\| \cdot \|_{V_1} := \| \nabla \cdot \|_{L^2(\Omega)^d}, \quad \| \cdot \|_{V_2} := \| \nabla \cdot \|_{L^2(\Omega)}.
\]

In addition, we identify the Hilbert spaces \( H_1 \) and \( H_2 \) with their topological duals \( H'_1 \) and \( H'_2 \) by the Riesz map. Observe that the embedding \( V_i \hookrightarrow H_i \) is dense and continuous, and hence \( H_i \equiv H'_i \hookrightarrow V'_i \) for \( i = 1, 2 \) is also as well; see [28].

We assume that

(A2) \( \Omega \) belongs to the domain class \( \hat{C}^{0,1} \) and the boundary parts \( \Gamma_d \) and \( \Gamma_i \cup \Gamma_w \) consist of a finite number of relative open sets in \( \Gamma \), which possess a projective boundary Lipschitz regularity condition if \( d = 3 \) (see [20]),

where \( \hat{C}^{0,1} \) is the subset of the Lipschitz domain class \( C^{0,1} \) formed by domains with a Lipschitz boundary consisting of a finite number of smooth parts with a finite number of relative maxima, minima, and inflexion points, and in three dimensions, also a finite number of saddle points (see [20]). Hence, the spaces \( V_1 \) and \( V_2 \) admit the characterizations

\[
V_1 = \{ y \in W^{1,2}(\Omega)^d : \text{div} \ y = 0 \text{ in } \Omega, \ y|_{\Gamma_i \cup \Gamma_w} = 0 \text{ in the trace sense} \},
V_2 = \{ y \in W^{1,2}(\Omega) : y|_{\Gamma_d} = 0 \text{ in the trace sense} \}.
\]

The equivalence between the two definitions of \( V_2 \) was proved in [20]. The analogous result for \( V_1 \) is obtained similarly.

We further assume that

(A3) \( v_i \in W^{1/2,2}(\Gamma_i)^d \), \quad and \quad \( u_d \in W^{1/2,2}(\Gamma_d) \).

This allows us to look for weak solutions to problem \((P)\) in the form

\[
v = \bar{v} + \mathbf{V}, \quad u = \bar{u} + U,
\]

with \( \bar{v} \in V_1 \) and \( \bar{u} \in V_2 \), where \( \mathbf{V} \in W^{1,2}(\Omega)^d \) and \( U \in W^{1,2}(\Omega) \) satisfy

\[
\mathbf{V} = v_i \quad \text{on} \quad \Gamma_i, \quad \text{V} = 0 \quad \text{on} \quad \Gamma_w, \quad \text{div} \mathbf{V} = 0 \quad \text{in} \quad \Omega, \quad U = u_d \quad \text{on} \quad \Gamma_d,
\]

and the estimates

\[
\| \mathbf{V} \|_{W^{1,2}(\Omega)^d} \leq C \| v_i \|_{W^{1/2,2}(\Gamma_i)^d}, \quad \| U \|_{W^{1,2}(\Omega)} \leq C \| u_d \|_{W^{1/2,2}(\Gamma_d)}, \quad (2.7)
\]
(see, e.g., [27]). Here and in what follows, $C$ denotes a positive constant taking various non-essential values that may depend on the domain $\Omega$, the dimension $d$, and the sets involved in the decompositions $\{\Gamma_i, \Gamma_w, \Gamma_o\}$ or $\{\Gamma_d, \Gamma_n, \Gamma_o\}$.

We shall study a weak formulation of $(P)$ that is obtained from the usual approach: We formally multiply equations (2.1) and (2.3) by test functions in $V_1$ and $V_2$, respectively. Then, we integrate over $\Omega$, and finally use the boundary conditions on $\Gamma_n$ and $\Gamma_o$ to rewrite the integrals with second order derivatives using Green’s theorem.

In order to introduce the weak formulation, we introduce the operators below. In particular, the estimations given for each of them will be useful to prove the existence of weak solutions in the next section 2.2. In what follows, $\langle \cdot, \cdot \rangle_2$ denotes the usual inner product in either $L^2(\Omega)$ or in any finite copies of it.

Let $w_1, w_2, w \in W^{1,2}(\Omega)^d$ and $\tau \in W^{1,2}(\Omega)^d$. We define $B_1(w_1, w_2) \in V_1'$ and $B_2(w, w) \in V_2'$ by
\[
\langle B_1(w_1, w_2), y \rangle := \int_{\Omega} (w_1 \cdot \nabla w_2) \cdot y \, dx, \quad \text{and} \quad \langle B_2(w, w), y \rangle := \int_{\Omega} (w \cdot \nabla w) y \, dx.
\]
By Hölder’s inequality and the embedding $W^{1,2}(\Omega) \hookrightarrow L^4(\Omega)$, we find
\[
|\langle B_1(w_1, w_2), y \rangle| \leq ||w_1||_{L^4(\Omega)^d} ||\nabla w_2||_{L^4(\Omega)^d} ||y||_{L^4(\Omega)^d},
\]
\[
|\langle B_2(w, w), y \rangle| \leq ||w||_{L^4(\Omega)^d} ||\nabla w||_{L^4(\Omega)^d} ||y||_{L^4(\Omega)}.
\]

Hence, we observe the estimates
\[
||B_1(w_1, w_2)||_{V_1'} \leq C ||w_1||_{W^{1,2}(\Omega)^d} ||w_2||_{W^{1,2}(\Omega)^d},
\]
\[
||B_2(w, w)||_{V_2'} \leq C ||w||_{W^{1,2}(\Omega)^d} ||w||_{W^{1,2}(\Omega)}.
\]

In addition, we define $H(w, w)$ by
\[
\langle H(w, w), y \rangle := \int_{\Gamma_o} w \beta(w \cdot n)(w \cdot n) y \, d\sigma,
\]
for $y \in V_2$. Notice that Hölder’s inequality and the embedding $W^{1,2}(\Omega) \hookrightarrow L^4(\Gamma_o)$ yield
\[
|\langle H(w, w), y \rangle| \leq ||\beta||_{L^\infty(\Omega)} ||w||_{L^4(\Gamma_n)^d} ||w||_{L^4(\Gamma_n)^d} ||y||_{L^2(\Gamma_n)}.
\]

Therefore, $H(w, w) \in V_2'$ and satisfies
\[
||H(w, w)||_{V_2'} \leq C ||\beta||_{L^\infty(\Omega)} ||w||_{W^{1,2}(\Omega)^d} ||w||_{W^{1,2}(\Omega)}.
\]

Finally, we define $A_1(w, w) \in V_1'$ and $A_2(w) \in V_2'$ by
\[
\langle A_1(w, w), y \rangle := \frac{1}{\text{Re}} (\nabla w, \nabla y)_2 - \frac{\text{Gr}}{\text{Re}^2} (w e, y)_2,
\]
\[
\langle A_2(w), y \rangle := \frac{1}{\text{Re} \text{Pr}} (\nabla w, \nabla y)_2,
\]
and observe the estimates
\[
||A_1(w, w)||_{V_1'} \leq C \left( \frac{1}{\text{Re}} ||w||_{W^{1,2}(\Omega)^d} + \frac{\text{Gr}}{\text{Re}^2} ||w||_{W^{1,2}(\Omega)} \right),
\]
\[
||A_2(w)||_{V_2'} \leq \frac{C}{\text{Re} \text{Pr}} ||w||_{W^{1,2}(\Omega)}.
\]
We are now in a position to write the weak formulation of (P):

Problem \((P_w)\): Find \((\bar{v}, \bar{u}) \in V_1 \times V_2\) such that

\[
A_1(\bar{v} + V, \bar{u} + U) + B_1(\bar{v} + V, \bar{v} + V) = g_1 \quad \text{in } V'_1,
\]

\[
A_2(\bar{u} + U) + B_2(\bar{v} + V, \bar{u} + U) - H(\bar{v} + V, \bar{u} + U) = g_2 \quad \text{in } V'_2,
\]

where \(g_1 \in V'_1\), \(g_2 \in V'_2\).

A weak solution to \((P)\) is then defined as \((\bar{v} + V, \bar{u} + U)\) where \((\bar{v}, \bar{u})\) solves \((P_w)\).

### 2.2. Existence of weak solutions

We provide two different proofs for the existence of solutions to \((P_w)\) associated to different assumptions on \(\beta\). Initially, we consider that \(\beta\) is Lipschitz continuous in Theorem 2.1, where we prove also uniqueness of the solution. The argument of the proof relies on a fixed point strategy that uses Banach’s fixed point theorem. Then, in the case when \(\beta\) is continuous (but not Lipschitz), we prove existence in Theorem 2.3 by a perturbation approach combined with Leray-Schauder’s fixed point theorem. Both theorems are proved under the assumption of “small” boundary data, small Re and Gr numbers, and (possibly) large Pr in the sense given in Theorem 2.1.

In the following, we consider the product space \(V_1 \times V_2\) endowed with the norm

\[
\| (w, w) \|_{V_1 \times V_2} := \| w \|_{V_1} + \| w \|_{V_2},
\]

and denote by \(B_\tau\) the closed ball in \(V_1 \times V_2\) with center at the origin and radius \(\tau > 0\).

#### 2.2.1. Existence and uniqueness for \(\beta\) Lipschitz continuous

In this section, we provide an existence and uniqueness result built on the assumption of smallness of the data \(v_1\) and \(u_4\), and of enough regularity of \(\beta\). The proof is based on re-writing the weak formulation as a fixed point equation and is given next.

**Theorem 2.1.** Let \(\beta: \mathbb{R} \to \mathbb{R}\) be Lipschitz continuous. Then, provided that

\[
\| v_1 \|_{W^{1,2}(\Gamma_1)}, \quad \| u_4 \|_{W^{1,2}(\Gamma_4)},
\]

are sufficiently small, there exist \(\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0\) such that if

\[
\text{Re} \in (0, \varepsilon_1), \quad \text{Pr} \in \left(0, \frac{\varepsilon_2}{\text{Re}}\right), \quad \text{Gr} \in (0, \varepsilon_3 \text{Re}),
\]

(2.20)

there exists a unique solution \((\bar{v}, \bar{u})\) to problem \((P_w)\) in some closed ball \(B_\tau\).

**Proof.** For convenience, we write the equations in problem \((P_w)\) as follows:

\[
\frac{1}{\text{Re}} (\nabla \bar{v}, \nabla y)_2 = \langle g_1, y \rangle - \langle B_1(\bar{v} + V, \bar{v} + V), y \rangle - \langle A_1(\bar{v} + V, \bar{v} + V), y \rangle \quad \forall y \in V_1,
\]

\[
\frac{1}{\text{Re} \text{Pr}} (\nabla \bar{u}, \nabla y)_2 = \langle g_2, y \rangle - \langle B_2(\bar{v} + V, \bar{u} + U), y \rangle - \langle A_2(U), y \rangle + \langle H(\bar{v} + V, \bar{u} + U), y \rangle \quad \forall y \in V_2.
\]

From this, we observe that a necessary and sufficient condition for \((\bar{v}, \bar{u})\) to be a solution to \((P_w)\) is that

\[
F(\bar{v}, \bar{u}) = (\bar{v}, \bar{u}),
\]

where
for the map $F := (S_1^{-1}P_1, S_2^{-1}P_2)$, where $P_1 : V_1 \times V_2 \to V_1'$, and $P_2 : V_1 \times V_2 \to V_2'$ are defined as

$$P_1(w, w) := g_1 - B_1(w + V, w + V) - A_1(V, w + U), \quad (2.21)$$

$$P_2(w, w) := g_2 - B_2(w + V, w + U) - A_2(U) + H(w + V, w + U), \quad (2.22)$$

and $S_1 : V_1 \to V_1'$, $S_2 : V_2 \to V_2'$ are given by

$$\langle S_1(w), y \rangle = \frac{1}{\text{Re}}(\nabla w, \nabla y)_2 \quad \text{and} \quad \langle S_2(w), y \rangle = \frac{1}{\text{Pr Re}}(\nabla w, \nabla y)_2. \quad (2.23)$$

We notice that $V_1$ and $V_2$ are Hilbert spaces with the inner product $(\nabla, \nabla)_2$ and hence the inverse maps $S_1^{-1} : V_1' \to V_1$ and $S_2^{-1} : V_2' \to V_2$ exist by virtue of Riesz’s theorem. Therefore, $F$ is well-defined.

The rest of the proof consists of proving that $F$ is a contraction from some closed ball $B_r$ into itself. We introduce the following constants to simply notation:

$$\ell_{v_1} := \|V_1\|_{W^{1/2, 2}(Y)}^d, \quad \ell_{u_1} := \|u_1\|_{W^{1/2, 2}(Y)}^d, \quad \ell_{\beta} := \|\beta\|_{L^\infty(R)},$$

$$\ell_{g_1} := \|g_1\|_{V_1'}, \quad \ell_{g_2} := \|g_2\|_{V_2'}.$$

Let $\tau > 0$, and consider $(w_1, w_1), (w_2, w_2) \in B_r$. From the linearity of $S_1$ and $S_2$, and the identities

$$\|S_1^{-1}(h_1)\|_{V_1} = \text{Re} \|h_1\|_{V_1'} \quad \forall h_1 \in V_1', \quad \|S_2^{-1}(h_2)\|_{V_2} = \text{Pr Re} \|h_2\|_{V_2'} \quad \forall h_2 \in V_2',$$

we find

$$\|F(w_1, w_1) - F(w_2, w_2)\|_{V_1 \times V_2} \quad (2.24)$$

$$= \text{Re} \|P_1(w_1, w_1) - P_1(w_2, w_2)\|_{V_1'} + \text{Pr Re} \|P_2(w_1, w_1) - P_2(w_2, w_2)\|_{V_2'}.$$

Further, we observe that

$$P_1(w_1, w_1) - P_1(w_2, w_2)$$

$$= B_1(w_2 + V, w_1 - w_2) + B_1(w_1 - w_2, w_1 + V) - A_1(0, w_1 - w_2).$$

Then, using the estimates (2.11) and (2.18) we obtain

$$\|P_1(w_1, w_1) - P_1(w_2, w_2)\|_{V_1'}$$

$$\leq C(\|w_2 + V\|_{W^{1/2, 2}(\Omega)}^d + \|w_1 + V\|_{W^{1/2, 2}(\Omega)}^d) \|w_1 - w_2\|_{V_1} + C \frac{G}{R^2} \|w_1 - w_2\|_{V_2}$$

$$\leq C \left( \tau + \ell_{v_1} + \frac{G}{R^2} \right) \| (w_1, w_1) - (w_2, w_2) \|_{V_1 \times V_2}.$$  \quad (2.25)

We shall now estimate $\|P_2(w_1, w_1) - P_2(w_2, w_2)\|_{V_2'}$. We write

$$P_2(w_1, w_1) - P_2(w_2, w_2) = I + J,$$
where
\[ I := B_2(w_2 + V, w_1 - w_2) + B_2(w_1 - w_2, w_1 + U), \]
\[ J := H(w_1 + V, w_1 + U) - H(w_2 + V, w_2 + U). \]

Similarly as we estimated \( \|P_1(w_1, w_1) - P_1(w_2, w_2)\|_{V_1'} \), we have
\[ \|I\|_{V_2'} \leq C (\tau + \ell_{\nu} + \ell_{\alpha}) \|(w_1, w_1) - (w_2, w_2)\|_{V_1 \times V_2}. \]

Let \( y \in V_2 \). We consider
\[ \langle H(w_1 + V, w_1 + U), y \rangle - \langle H(w_2 + V, w_2 + U), y \rangle = J_1 + J_2, \]
where
\[ J_1 := \langle H(w_1 + V, w_1 + U), y \rangle - \langle H(w_1 + V, w_2 + U), y \rangle, \]
\[ J_2 := \langle H(w_1 + V, w_2 + U), y \rangle - \langle H(w_2 + V, w_2 + U), y \rangle. \]

From Hölder’s inequality and the embeddings \( W^{1,2}(\Omega) \hookrightarrow L^p(\Gamma_o) \) for \( p = 2 \) and \( p = 4 \), we obtain
\[
|J_1| \leq \ell_{\beta} \int_{\Gamma_o} |w_1 - w_2||w_1 + V||y| \, d\sigma \\
\leq \ell_{\beta} \|w_1 - w_2\|_{L^4(\Gamma_o)}\|w_1 + V\|_{L^4(\Gamma_o)}\|y\|_{L^2(\Gamma_o)} \\
\leq C \ell_{\beta} \|w_1 - w_2\|_{V_2} (\|w_1 + \ell_{\nu}\|_V \|y\|_{V_2}).
\]

Analogously, defining \( F_{\beta}(w_i, w_j) := \beta((w_i + V) \cdot n)((w_j + V) \cdot n) \) for \( i, j = 1, 2 \), we have
\[
|J_2| \leq \int_{\Gamma_o} |w_2 + U||F_{\beta}(w_1, w_1) - F_{\beta}(w_1, w_2)||y| \, d\sigma \\
+ \int_{\Gamma_o} |w_2 + U||F_{\beta}(w_1, w_2) - F_{\beta}(w_2, w_2)||y| \, d\sigma \\
\leq \ell_{\beta} \int_{\Gamma_o} |w_2 + U||w_1 - w_2||y| \, d\sigma + L \int_{\Gamma_o} |w_2 + U||w_2 + V||w_1 - w_2||y| \, d\sigma \\
\leq C \ell_{\beta} (\|w_2\|_{V_2} + \ell_{\alpha})\|w_1 - w_2\|_V \|y\|_{V_2} \\
+ CL (\|w_2\|_{V_2} + \ell_{\alpha})(\|w_2\|_V + \ell_{\nu})\|w_1 - w_2\|_V \|y\|_{V_2},
\]

where \( L > 0 \) is the Lipschitz constant of \( \beta \).

From the bounds on \( J_1 \) and \( J_2 \), it follows that
\[ \|J\|_{V_2'} \leq C (Lr^2 + (\ell_{\beta} + L(\ell_{\nu} + \ell_{\alpha})))\tau + \ell_{\beta}(\ell_{\nu} + \ell_{\alpha}) + L\ell_{\nu}\ell_{\alpha}) \]
\[ \times \|(w_1, w_1) - (w_2, w_2)\|_{V_1 \times V_2}, \]
Furthermore, stability properties of solutions in the case \( \beta \) relax hypothesis on \( \beta \) for any \( (w_1, w_2) \) and thus \( (\bar{w}_1, \bar{w}_2) \) existence by Leray-Schauder’s theorem via compactness properties for the case \( \beta \neq 0 \) in Theorem 2.3. We start with the aforementioned lemma.

Let \( w \) be a sequence in \( V_1 \) associated to \( (\bar{w}_1, \bar{w}_2) \), we observe that we can always find \( \tau > 0 \) and \( \varepsilon_1, \varepsilon_2, \varepsilon_3 > 0 \) such that if (2.20) holds true then

\[
K_1 := C \left( \frac{\text{Gr}}{\text{Re}} + \text{Re Pr} \left( \frac{\text{Gr}}{\text{Re}} + \text{Re} \left( 1 + \ell_{v_1} + \ell_{u_1} \right) \right) \right) + C \left( \ell_{v_1} + \ell_{u_1} \right).
\]

Analogous computations yield

\[
\|F(w, w)\|_{V_1 \times V_2} \leq K_2,
\]

for any \( (w, w) \in B_\tau \), where

\[
K_2 := C \left( \frac{\text{Gr}}{\text{Re}} + \text{Re Pr} \left( \ell_{g_1} + \ell_{u_1} \right) \right) + C \left( \ell_{g_1} + \ell_{u_1} \right).
\]

From the definitions of \( K_1 \) and \( K_2 \), we observe that we can always find \( \tau > 0 \) and \( \varepsilon_1, \varepsilon_2, \varepsilon_3 > 0 \) such that if (2.20) holds true then

\[
K_1 < 1, \quad \text{and} \quad K_2 \leq \tau,
\]

provided that \( \ell_{v_1} \) and \( \ell_{u_1} \) are sufficiently small. Hence, \( F \) is a contraction from \( B_\tau \) into itself. Then, by Banach’s fixed point theorem there is a unique solution to (\( \mathbb{P}_w \)) in the closed ball \( B_\tau \).

### 2.2.2. Existence of solutions for \( \beta \) satisfying only (A1)

We now aim to relax hypothesis on \( \beta \) and still retain existence of solutions. The proof plan considers stability properties of solutions in the case \( \beta \equiv 0 \) in Lemma 2.2 and then provides existence by Leray-Schauder’s theorem via compactness properties for the case \( \beta \neq 0 \) in Theorem 2.3. We start with the aforementioned lemma.

**Lemma 2.2.** Let \( \beta \equiv 0 \), and \( \{g_n^2\} \) be a sequence in \( V_1' \) such that \( g_n^2 \to g_2 \) in \( V_1' \). Then, provided that \( \|v_1\|_{W^{1,2}(\Gamma_v, v_1)} \) and \( \|u_4\|_{W^{1,2}(\Gamma_n, u_1)} \) are small enough, there exist \( \varepsilon_1, \varepsilon_2, \varepsilon_3 > 0 \) such that if (2.20) holds true, problems \( \mathbb{P}_w \) associated to \( g_2 \) or \( g_n^2 \), \( n \in \mathbb{N} \), have unique solutions \( (\bar{v}, \bar{u}) \) and \( (\bar{v}_n, \bar{u}_n) \), respectively, in some closed ball \( B_\tau \).

Furthermore,

\[
(\bar{v}_n, \bar{u}_n) \to (\bar{v}, \bar{u}) \quad \text{in} \quad V_1 \times V_2.
\]

(2.29)
Proof. The existence and uniqueness of solutions is proved exactly as in Theorem 2.1, but replacing $\ell_{g_2}$ by $\sup_n \|g_2^n\|_{V_2}$ in the definition of $K_2$.

The proof of (2.29) is split into two steps. First, we show that the sequence of solutions $\{(\bar{v}_n, \bar{u}_n)\}$ converges weakly in $V_1 \times V_2$ to $(\bar{v}, \bar{u})$, and then we prove that the sequence of norms $\{\|\bar{v}_n, \bar{u}_n\|_{V_1 \times V_2}\}$ converges to $\|\bar{v}, \bar{u}\|_{V_1 \times V_2}$.

First step. Since $\{(\bar{v}_n, \bar{u}_n)\} \subset B_r$, we have that $\{(\bar{v}_n, \bar{u}_n)\}$ is bounded in $V_1 \times V_2$ and hence it admits a weakly convergent subsequence to some $(\bar{v}, \bar{u}) \in B_r$. We now prove that $(\bar{v}, \bar{u})$ solves problem $(P_w)$ associated with $g_2$ by taking the limit as $n \to \infty$ on

$$
\langle A_1(\bar{v}_n + V, \bar{u}_n + U), y \rangle + \langle B_1(\bar{v}_n + V, \bar{v}_n + V), y \rangle = \langle g_1, y \rangle, \quad (2.30)
$$

$$
\langle A_2(\bar{u}_n + U), y \rangle + \langle B_2(\bar{v}_n + V, \bar{u}_n + U, y \rangle = \langle g_2^n, y \rangle, \quad (2.31)
$$

for each $y \in V_1$ and $y \in V_2$. Thus, by the uniqueness of solutions in $B_r$, we will have that $(\bar{v}, \bar{u}) = (\bar{v}, \bar{u})$ and hence $(\bar{v}_n, \bar{u}_n) \rightharpoonup (\bar{v}, \bar{u})$ in $V_1 \times V_2$ along the entire sequence (and not only a subsequence).

Let $y \in V_1$ and $y \in V_2$. From the weak convergences $\bar{v}_n \rightharpoonup \bar{v}$ in $V_1$, $\bar{u}_n \rightharpoonup \bar{u}$ in $V_2$, and the strong convergence $g_2^n \to g_2$ in $V_2'$, we obtain

$$
\langle A_1(\bar{v}_n + V, \bar{u}_n + U), y \rangle \to \langle A_1(\bar{v} + V, \bar{u} + U), y \rangle,
$$

$$
\langle A_2(\bar{u}_n + U), y \rangle \to \langle A_2(\bar{u} + U), y \rangle,
$$

$$
\langle g_2^n, y \rangle \to \langle g_2, y \rangle.
$$

We notice that

$$
\langle B_1(\bar{v}_n + V, \bar{v}_n + V, y \rangle = \sum_{i,j=1}^d \int_{\Omega} y_j (\bar{v}_n + V)_i \frac{\partial(\bar{v}_n + V)_j}{\partial x_i} \, dx.
$$

Let $i, j \in \{1, \ldots, d\}$. Since $\bar{v}_n \rightharpoonup \bar{v}$ in $V_1$, we have

$$
\frac{\partial(\bar{v}_n + V)_j}{\partial x_i} \to \frac{\partial(\bar{v} + V)_j}{\partial x_i} \quad \text{in} \quad L^2(\Omega).
$$

In addition,

$$
y_j (\bar{v}_n + V)_i \to y_j (\bar{v} + V)_i \quad \text{in} \quad L^2(\Omega),
$$

given that $(\bar{v}_n + V)_i \rightharpoonup (\bar{v} + V)_i$ in $L^4(\Omega)$ by virtue of the compact embedding $W^{1,2}(\Omega) \hookrightarrow L^4(\Omega)$. Therefore,

$$
\langle B_1(\bar{v}_n + V, \bar{v}_n + V, y \rangle \to \langle B_1(\bar{v} + V, \bar{v} + V, y \rangle. \quad (2.32)
$$

From an identical argument, we find

$$
\langle B_2(\bar{v}_n + V, \bar{u}_n + U), y \rangle \to \langle B_2(\bar{v} + V, \bar{u} + U), y \rangle, \quad (2.33)
$$

which completes the proof of the first step, i.e., $(\bar{v}_n, \bar{u}_n) \rightharpoonup (\bar{v}, \bar{u})$ in $V_1 \times V_2$.

Second step. Taking $y = \bar{v}_n$ and $y = \bar{u}_n$ in (2.30) and (2.31), respectively, we find

$$
\frac{1}{\text{Re}} \|\bar{v}_n\|_{V_1}^2 = \langle g_1, \bar{v}_n \rangle - \langle B_1(\bar{v}_n + V, \bar{v}_n + V), \bar{v}_n \rangle - \langle A_1(V, \bar{u}_n + U), \bar{v}_n \rangle, \quad (2.34)
$$

$$
\frac{1}{\text{RePr}} \|\bar{u}_n\|_{V_2}^2 = \langle g_2, \bar{u}_n \rangle - \langle B_2(\bar{v}_n + V, \bar{u}_n + U), \bar{u}_n \rangle - \langle A_2(U), \bar{u}_n \rangle. \quad (2.35)
$$
Similarly, we have
\[
\frac{1}{\text{Re}} \| \tilde{v} \|_{V_2}^2 = (g_1, \tilde{v}) - (B_1(\tilde{v} + \mathbf{V}, \tilde{v} + \mathbf{V}), \tilde{v}) - (A_1(\mathbf{V}, \bar{u} + \mathbf{U}), \tilde{v}),
\]
(2.36)
\[
\frac{1}{\text{Re} \text{Pr}} \| \bar{u} \|_{V_2}^2 = (g_2, \bar{u}) - (B_2(\bar{v} + \mathbf{V}, \bar{u} + \mathbf{U}), \bar{u}) - (A_2(\mathbf{U}), \bar{u}).
\]
(2.37)
We shall prove the convergence \( \| (\tilde{v}_n, \bar{u}_n) \|_{V_1 \times V_2} \rightarrow \| (\tilde{v}, \bar{u}) \|_{V_1 \times V_2} \) by showing that the right-hand sides of (2.34) and (2.35) converge to the right-hand sides of (2.36) and (2.37), respectively.

The weak convergences \( \tilde{v}_n \rightharpoonup \tilde{v} \) in \( V_1, \bar{u}_n \rightharpoonup \bar{u} \) in \( V_2 \), and the strong convergence \( g_2^n \rightarrow g_2 \) in \( V_2 \) imply
\[
\langle A_1(\mathbf{V}, \bar{u}_n + \mathbf{U}), \tilde{v}_n \rangle \rightarrow \langle A_1(\mathbf{V}, \bar{u} + \mathbf{U}), \tilde{v} \rangle,
\]
\[
\langle A_2(\mathbf{U}), \bar{u}_n \rangle \rightarrow \langle A_2(\mathbf{U}), \bar{u} \rangle,
\]
\[
\langle g_1, \tilde{v}_n \rangle \rightarrow \langle g_1, \tilde{v} \rangle,
\]
\[
\langle g_2^n, \bar{u}_n \rangle \rightarrow \langle g_2, \bar{u} \rangle.
\]
To obtain the first convergence, we have used that \( \bar{u}_n \rightarrow \bar{u} \) by virtue of the compact embedding \( W^{1,2}(\Omega) \hookrightarrow L^2(\Omega) \). Analogously as we proved (2.32) and (2.33), we obtain
\[
\langle B_1(\tilde{v}_n + \mathbf{V}, \tilde{v}_n + \mathbf{V}), \tilde{v}_n \rangle \rightarrow \langle B_1(\tilde{v} + \mathbf{V}, \tilde{v} + \mathbf{V}), \tilde{v} \rangle,
\]
\[
\langle B_2(\tilde{v}_n + \mathbf{V}, \bar{u}_n + \mathbf{U}), \bar{u}_n \rangle \rightarrow \langle B_2(\tilde{v} + \mathbf{V}, \bar{u} + \mathbf{U}), \bar{u} \rangle,
\]
and hence \( \| (\tilde{v}_n, \bar{u}_n) \|_{V_1 \times V_2} \rightarrow \| (\tilde{v}, \bar{u}) \|_{V_1 \times V_2} \), which completes the proof. \( \square \)

We are now in shape to prove the existence result for the case when \( \beta \) only satisfies (A1) but it is not necessarily Lipschitz by making use on the above lemma.

**Theorem 2.3.** Provided that \( \| v_1 \|_{W^{1,2}(\Gamma_i)} \) and \( \| u_d \|_{W^{1,2}(\Gamma_i)} \) are sufficiently small, there exist \( \varepsilon_1, \varepsilon_2, \varepsilon_3 > 0 \) such that if (2.20) holds true then problem \((\mathcal{P}_w)\) admits at least one solution.

**Proof.** Let \( \tau > 0 \) and \( (\tilde{v}, \bar{u}) \in B_\tau \). From the estimate (2.15) we find
\[
\| H(\tilde{v} + \mathbf{V}, \bar{u} + \mathbf{U}) \|_{V_2'} \leq C \| \beta \|_{L^\infty(\mathbb{R})} (\tau + \| v_1 \|_{W^{1,2}(\Gamma_i)} \varepsilon) (\tau + \| u_d \|_{W^{1,2}(\Gamma_i)} \varepsilon).
\]

Thus, we replace \( \ell_{g_2} \) by \( C \| \beta \|_{L^\infty(\mathbb{R})} (\tau + \| v_1 \|_{W^{1,2}(\Gamma_i)} \varepsilon) (\tau + \| u_d \|_{W^{1,2}(\Gamma_i)} \varepsilon) \) in the definition of \( K_2 \) in the proof of Theorem 2.1 and observe that the same arguments yield the existence and uniqueness of a solution \( (\tilde{v}, \bar{u}) \in B_\tau \) to
\[
A_1(\tilde{v} + \mathbf{V}, \bar{u} + \mathbf{U}) + B_1(\tilde{v} + \mathbf{V}, \bar{u} + \mathbf{U}) = g_1 \quad \text{in} \quad V_1',
\]
(2.38)
\[
A_2(\bar{u} + \mathbf{U}) + B_2(\tilde{v} + \mathbf{V}, \bar{u} + \mathbf{U}) = g_2 + H(\tilde{v} + \mathbf{V}, \bar{u} + \mathbf{U}) \quad \text{in} \quad V_2',
\]
(2.39)
for some \( \tau > 0 \). Therefore, the map \( G : B_\tau \rightarrow B_\tau \) given by \( G(\tilde{v}, \bar{u}) = (\tilde{v}, \bar{u}) \), where \( (\tilde{v}, \bar{u}) \in B_\tau \) satisfies (2.38) and (2.39), is well-defined. From (2.38) and (2.39), we observe that any \( (\tilde{v}, \bar{u}) \in B_\tau \) such that \( G(\tilde{v}, \bar{u}) = (\tilde{v}, \bar{u}) \) is a solution to \((\mathcal{P}_w)\). The rest of the proof is devoted to show that \( G \) is a compact map, which will give the existence of solutions to \((\mathcal{P}_w)\) by Leray-Schauder’s fixed point theorem.
Let \( \{(w_n, w_n)\} \subset B_r \) be a weakly convergent sequence in \( V_1 \times V_2 \) to some \((w, w) \in B_r \). We observe that, by virtue of Lemma 2.2 to prove the convergence \( G(w_n, w_n) \to G(w, w) \), it suffices to show that

\[
H(w_n, w_n) \to H(w, w) \quad \text{in} \quad V'_2.
\]  

(2.40)

Since the embedding \( W^{1,2}(\Omega) \hookrightarrow L^p(\Gamma_o) \) is compact for any \( 1 \leq p < 4 \) (see Theorem 6.2 of [40]), we find

\[
w_n \to w \quad \text{in} \quad L^p(\Gamma_o)^d, \quad \text{and} \quad w_n \to w \quad \text{in} \quad L^p(\Gamma_o) \quad \text{for} \quad 1 \leq p < 4.
\]  

(2.41)

We notice that the map \( \vartheta : \Gamma_o \times \mathbb{R}^d \to \mathbb{R} \) given by \( \vartheta(x, z) := \beta(z \cdot n(x)) (z \cdot n(x)) \) is continuous with respect to the second argument for a.e. \( x \in \Gamma_o \) since the only point where \( \beta \) is allowed to be discontinuous is the origin. Furthermore, \( \vartheta \) satisfies the growth condition \( |\vartheta(x, z)| \leq ||\beta||_{L^\infty(\mathbb{R})} |z| \). Therefore, the Nemytskii operator

\[
\tilde{w} \mapsto F_\beta(\tilde{w}) := \beta(\tilde{w} \cdot n)(\tilde{w} \cdot n),
\]

generated by \( \vartheta \) is continuous from \( L^p(\Gamma_o)^d \) into \( L^p(\Gamma_o) \) (see Theorems 1 and 4 of [29]). Combining this with the first convergence in (2.41), we obtain

\[
F_\beta(w_n + V) \to F_\beta(w + V) \quad \text{in} \quad L^p(\Gamma_o) \quad 1 \leq p < 4.
\]

From this and the second convergence in (2.41), we find

\[
(w_n + U) F_\beta(w_n + V) \to (w + U) F_\beta(w + V) \quad \text{in} \quad L^p(\Gamma_o) \quad 1 \leq p < 2.
\]  

(2.42)

Let \( \xi \in (0, 2/3) \) and \( y \in V_2 \). We observe that \( p := 2 - \xi \) and its H"older’s conjugate, \( p' = \frac{p}{p-1} \), satisfy \( 1 \leq p < 2 \) and \( 1 \leq p' < 4 \). Then, by H"older’s inequality we obtain

\[
\int_{\Gamma_o} |(w_n + U) F_\beta(w_n + V) - (w + U) F_\beta(w + V)| \, |y| \, d\sigma \\
\leq ||(w_n + U) F_\beta(w_n + V) - (w + U) F_\beta(w + V)||_{L^p(\Gamma_o)} \, ||y||_{L^{p'}(\Gamma_o)},
\]

which, together with (2.42) gives (2.40). \( \square \)

### 3. The evolutionary problem.

In this section we consider the same geometry and boundary conditions established in section 2 and extend results into the time evolutionary setting. The task is significantly more complex than in the stationary case: In this case an appropriate state space for solutions is tailored specifically for our special case, and a restrictive partition of the boundary is required. The problem of interest is the following time evolutionary one.

**Problem (\( \bar{P} \)):** Let \( T > 0 \) be given. Find \( v : (0, T) \times \Omega \to \mathbb{R}^d, u : (0, T) \times \Omega \to \mathbb{R}, \) and \( p : (0, T) \times \Omega \to \mathbb{R} \) that satisfy the evolutionary Boussinesq equations in \( (0, T) \times \Omega, \)

\[
\frac{\partial v}{\partial t} + v \cdot \nabla v - \frac{1}{Re} \Delta v + \nabla p = \frac{Gr}{Re^2} u e + g_1, \tag{3.1}
\]

\[
\text{div} \, v = 0, \tag{3.2}
\]

\[
\frac{\partial u}{\partial t} + v \cdot \nabla u - \frac{1}{Re \, Pr} \Delta u = g_2, \tag{3.3}
\]
subject to the boundary conditions
\[ v = v_i \quad \text{on} \quad (0, T) \times \Gamma_i, \quad v = 0 \quad \text{on} \quad (0, T) \times \Gamma_w, \]  
\[ \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad (0, T) \times \Gamma_n, \quad u = u_i \quad \text{on} \quad (0, T) \times \Gamma_i, \]  
\[ \frac{1}{\text{Re}} \frac{\partial v}{\partial n} = p \mathbf{n} \quad \text{on} \quad (0, T) \times \Gamma_o, \quad \frac{1}{\text{Re Pr}} \frac{\partial u}{\partial n} = u \beta (v \cdot \mathbf{n}) \quad \text{on} \quad (0, T) \times \Gamma_o, \]  
and the initial conditions
\[ v = v_o, \quad u = u_o, \quad p = p_o \quad \text{on} \quad \{0\} \times \Omega. \]  

Here \( g_1 : (0, T) \times \Omega \to \mathbb{R}^d, g_2 : (0, T) \times \Omega \to \mathbb{R} \) are given together with \( v_o : \Omega \to \mathbb{R}^d, u_o : \Omega \to \mathbb{R}, \) and \( p_o : \Omega \to \mathbb{R}, \) which represent the initial velocity, temperature, and pressure distributions, respectively.

3.1. Weak formulation. In order to address the time evolutionary problem, we define the spaces
\[ W_i(0, T) := \{ \varphi \in L^2(0, T; V_i) : \partial_t \varphi \in L^2(0, T; V'_i) \}, \]
for \( i = 1, 2, \) endowed with the norms
\[ \| \varphi \|_{W_i(0, T)} := \| \varphi \|_{L^2(0, T; V_i)} + \| \partial_t \varphi \|_{L^2(0, T; V'_i)}. \]
The weak formulation of problem \((\tilde{P})\) is determined following analogous steps as for the stationary case leading to:

**Problem** \((\tilde{P}_w)\): Find \((\bar{v}, \bar{u}) \in W_1(0, T) \times W_2(0, T)\) such that \(\bar{v}(0) = v_o - V, \bar{u}(0) = u_o - U, \) and
\[ \partial_t \bar{v}(t) + A_1(\bar{v}(t) + V, \bar{u}(t) + U) + B_1(\bar{v}(t) + V, \bar{v}(t) + V) = g_1(t) \quad \text{in} \quad V'_1, \]
\[ \partial_t \bar{u}(t) + A_2(\bar{v}(t) + V, \bar{u}(t) + U) + B_2(\bar{v}(t) + V, \bar{u}(t) + U) \]
\[ -H(\bar{v}(t) + V, \bar{u}(t) + U) = g_2(t) \quad \text{in} \quad V'_2, \]
for a.e. \( t \in (0, T), \) where \( g_1 \in L^2(0, T; V'_1), \) \( g_2 \in L^2(0, T; V'_2), \) \( v_o \in L^2(\Omega)^d, \) and \( u_o \in L^2(\Omega). \)

Then, a weak solution to \((\tilde{P})\) is defined as \((\bar{v} + V, \bar{u} + U)\) where \((\bar{v}, \bar{u})\) solves \((\tilde{P}_w).\)

Further notice that the initial conditions on \( \bar{v} \) and \( \bar{u} \) in problem \((\tilde{P}_w)\) are meaningful since \( W_i(0, T) \hookrightarrow C([0, T]; H_i) \) for \( i = 1, 2 \) (see, e.g., [14]).

3.2. Existence of weak solutions. The existence and uniqueness of a weak solution to \((\tilde{P})\) will be proven in Theorem 3.2 in the case that \( \beta \) is a Lipschitz continuous function and \( d = 2. \) The proof is analogous to the one of section 2.2 but where state spaces are defined in a non-trivial fashion, and we restrict boundary parts to have specific geometrical properties. In particular, we do not require working with Sobolev spaces of non-integer order, which are sometimes used to show existence of weak
solutions to Boussinesq systems with mixed boundary conditions in open domains; see [9,13,15]. Like in the stationary case, the proof is built upon the assumption of small data, and provides existence and uniqueness for Re and Gr sufficiently small, and for Pr possibly large. The sense in which all of this is considered is given on the statement of Theorem [3,2].

We start with the following lemma which provides continuity properties on some specific Bochner spaces of $B_1$ and $B_2$ in the time dependent case.

**Lemma 3.1.** Let $\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2 \in L^2(0,T;W^{2,2}(\Omega)^d) \cap L^\infty(0,T;W^{1,2}(\Omega)^d)$ and $\mathbf{w} \in L^2(0,T;W^{1,2}(\Omega))$ be arbitrary. Then, we observe:

1. The maps $B_1(\mathbf{w}_1, \mathbf{w}_2) : (0,T) \to V'_1$ and $B_2(\mathbf{w}, \mathbf{w}) : (0,T) \to V'_2$ given by

$$B_1(\mathbf{w}_1, \mathbf{w}_2)(t) := B_1(\mathbf{w}_1(t), \mathbf{w}_2(t)), \quad B_2(\mathbf{w}, \mathbf{w})(t) := B_2(\mathbf{w}(t), \mathbf{w}(t)),$$

belong to $L^2(0,T;L^2(\Omega)^d)$ and $L^2(0,T;V'_2)$, respectively, and satisfy the estimates

$$\|B_1(\mathbf{w}_1, \mathbf{w}_2)\|_{L^2(0,T;L^2(\Omega)^d)} \leq C\|\mathbf{w}_1\|_{L^2(0,T;W^{2,2}(\Omega)^d)}\|\mathbf{w}_2\|_{L^\infty(0,T;W^{1,2}(\Omega)^d)}, \quad (3.8)$$

and

$$\|B_2(\mathbf{w}, \mathbf{w})\|_{L^2(0,T;V'_2)} \leq C\|\mathbf{w}\|_{L^\infty(0,T;W^{1,2}(\Omega)^d)}\|\mathbf{w}\|_{L^2(0,T;W^{1,2}(\Omega))}. \quad (3.9)$$

2. The map $H(\mathbf{w}, \mathbf{w}) : (0,T) \to V'_2$ given by

$$H(\mathbf{w}, \mathbf{w})(t) := H(\mathbf{w}(t), \mathbf{w}(t)),$$

belongs to $L^2(0,T;V'_2)$ and satisfies the estimate

$$\|H(\mathbf{w}, \mathbf{w})\|_{L^2(0,T;V'_2)} \leq C\|\beta\|_{L^\infty(\Omega)}\|\mathbf{w}\|_{L^\infty(0,T;W^{1,2}(\Omega)^d)}\|\mathbf{w}\|_{L^2(0,T;W^{1,2}(\Omega))}.$$

**Proof.** 1. Let $t \in (0,T)$. From the embedding $W^{2,2}(\Omega) \hookrightarrow L^\infty(\Omega)$, we find

$$\int_O |\mathbf{w}_1(t) \cdot \nabla \mathbf{w}_2(t)|^2 \, dx \leq \|\mathbf{w}_1(t)\|_{L^\infty(\Omega)^d}^2 \|\nabla \mathbf{w}_2(t)\|_{L^2(\Omega)^{d \times d}}^2$$

$$\leq C\|\mathbf{w}_1(t)\|_{W^{2,2}(\Omega)^d}^2 \|\mathbf{w}_2(t)\|_{W^{1,2}(\Omega)^d}^2. \quad (3.10)$$

Hence, $\mathbf{w}_1(t) \cdot \nabla \mathbf{w}_2(t) \in L^2(\Omega)$, and integrating with respect to $t$ from 0 to $T$ the above expression, we observe

$$\int_0^T \|\mathbf{w}_1(t) \cdot \nabla \mathbf{w}_2(t)\|_{L^2(\Omega)^d}^2 \, dt \leq C \int_0^T \|\mathbf{w}_1(t)\|_{W^{2,2}(\Omega)^d}^2 \|\mathbf{w}_2(t)\|_{W^{1,2}(\Omega)^d}^2 \, dt$$

$$\leq C\|\mathbf{w}_1\|_{L^2(0,T;W^{2,2}(\Omega)^d)}^2 \|\mathbf{w}_2\|_{L^\infty(0,T;W^{1,2}(\Omega))}^2.$$ 

Therefore, $B_1(\mathbf{w}_1, \mathbf{w}_2) \in L^2(0,T;L^2(\Omega)^d)$ with the desired estimate. The result concerning the map $B_2(\mathbf{w}, \mathbf{w})$ is obtained similarly from the estimate (2.12).

2. Integrating the square of (2.15) with respect to $t$ between 0 and $T$, we obtain

$$\int_0^T \|H(\mathbf{w}, \mathbf{w})(t)\|_{V'_2}^2 \, dt \leq C\|\beta\|_{L^\infty(\Omega)}^2 \|\mathbf{w}\|_{L^\infty(0,T;W^{1,2}(\Omega)^d)}^2 \|\mathbf{w}\|_{L^2(0,T;W^{1,2}(\Omega))}^2.$$ 

Hence, $H(\mathbf{w}, \mathbf{w}) \in L^2(0,T;V'_2)$ with the desired estimate. □
Next we provide the result of existence and uniqueness of solutions to the Boussinesq system of interest. The proof is given for the case when \( d = 2 \) and the boundary subparts \( \Gamma_i, \Gamma_w, \Gamma_o \) satisfy additional regularity and geometrical properties, which allow to obtain weak solutions with increased spatial regularity.

**Theorem 3.2.** Let \( d = 2 \) and \( \beta : \mathbb{R} \to \mathbb{R} \) be Lipschitz continuous. Assume that \( \Gamma_i \cup \Gamma_w \neq \emptyset \), the set \( A := \Gamma \setminus (\Gamma_i \cup \Gamma_w \cup \Gamma_o) \) is finite, any portion of \( \Gamma_o \) is flat and forms a right angle with \( \Gamma_i \cup \Gamma_w \) at each contact point in \( A \), and that any boundary subpart of \( \Gamma_i \) or \( \Gamma_w \) is of class \( C^\infty \). Further, suppose that

\[
g_1 \in L^2(0,T; L^2(\Omega)^2), \quad \mathbf{v}_o \in V_1, \quad \text{and} \quad \mathbf{v}_i \equiv 0.
\]

Then, provided that

\[
\|g_1\|_{L^2(0,T;L^2(\Omega)^2)}, \quad \|g_2\|_{H^1(0,T;V_2^2)}, \quad \|\mathbf{v}_o\|_{V_1}, \quad \|u_o\|_{L^2(\Omega)}, \quad \|u_d\|_{W^{1,2}(\Gamma_\alpha)},
\]

are sufficiently small, there exist \( \varepsilon_1, \varepsilon_2, \varepsilon_3 > 0 \) such that if

\[
\Re \in (0, \varepsilon_1), \quad \Pr \in \left( 0, \frac{\varepsilon_2}{\Re} \right), \quad \Gr \in (0, \varepsilon_3 \min(\Re, \Re^2)),
\]

(3.11)

there exists a unique solution \((\mathbf{v}, \mathbf{u}) \in W_1(0,T) \times W_2(0,T)\) to problem \((\hat{P}_w)\) satisfying \(\mathbf{v} \in \hat{W}_1(0,T)\), where \(\hat{W}_1(0,T) \subset W_1(0,T)\) is given by

\[
\hat{W}_1(0,T) := \{ \mathbf{w} \in L^2(0,T; W^{2,2}(\Omega)^2) \cap L^\infty(0,T; V_1) : \partial_t \mathbf{w} \in L^2(0,T; L^2(\Omega)^2) \}.
\]

**Proof.** First note that since \( \mathbf{v}_i \equiv 0 \) by assumption, then we can select \( \mathbf{v} \equiv 0 \) also. Hence, we can write the equations in problem \((\hat{P}_w)\) as follows: For a.e. \( t \in (0,T) \), we have

\[
\langle \partial_t \mathbf{v}(t), \mathbf{w} \rangle + \frac{1}{\Re} \langle \nabla \mathbf{v}(t), \nabla \mathbf{w} \rangle_2 = \langle g_1(t), \mathbf{w} \rangle_2 - \langle B_1(\mathbf{v}(t), \mathbf{v}(t)), \mathbf{w} \rangle

- \langle A_1(0, \mathbf{u}(t) + U), \mathbf{w} \rangle \quad \forall \mathbf{w} \in V_1,
\]

\[
\langle \partial_t \mathbf{u}(t), \mathbf{w} \rangle + \frac{1}{\Re \Pr} \langle \nabla \mathbf{u}(t), \nabla \mathbf{w} \rangle_2 = \langle g_2(t), \mathbf{w} \rangle - \langle B_2(\mathbf{v}(t), \mathbf{u}(t) + U), \mathbf{w} \rangle - \langle A_2(U), \mathbf{w} \rangle

+ \langle H(\mathbf{v}(t), \mathbf{u}(t) + U), \mathbf{w} \rangle \quad \forall \mathbf{w} \in V_2.
\]

From this and in what follows, we formulate the existence and uniqueness of a solution to \((\hat{P}_w)\) as a fixed point problem by choosing the state space as \(\hat{W}_1(0,T) \times W_2(0,T)\).

We endow \(\hat{W}_1(0,T)\) with the norm

\[
\|\mathbf{w}\|_{\hat{W}_1(0,T)} := \|\mathbf{w}\|_{L^2(0,T; W^{2,2}(\Omega)^2)} + \|\mathbf{w}\|_{L^\infty(0,T; V_1)} + \|\partial_t \mathbf{w}\|_{L^2(0,T; L^2(\Omega)^2)}.
\]

and the product space \(\hat{W}_1(0,T) \times W_2(0,T)\) is considered with the usual norm

\[
\|(\mathbf{w}, \mathbf{w})\|_{\hat{W}_1(0,T) \times W_2(0,T)} := \|\mathbf{w}\|_{\hat{W}_1(0,T)} + \|\mathbf{w}\|_{W_2(0,T)}.
\]

We define the maps \( P_1 : \hat{W}_1(0,T) \times W_2(0,T) \to L^2(0,T; L^2(\Omega)^2) \) and \( P_2 : \hat{W}_1(0,T) \times W_2(0,T) \to L^2(0,T; V_2^2) \) by

\[
\langle P_1(\mathbf{w}, \mathbf{w})(t), \mathbf{y} \rangle := (g_1(t), \mathbf{y})_2 - \langle B_1(\mathbf{w}(t), \mathbf{w}(t)), \mathbf{y} \rangle - \langle A_1(0, \mathbf{w}(t) + U), \mathbf{y} \rangle,
\]

\[
\langle P_2(\mathbf{w}, \mathbf{w})(t), \mathbf{x} \rangle := (g_2(t), \mathbf{x})_2 - \langle B_2(\mathbf{v}(t), \mathbf{u}(t) + U), \mathbf{x} \rangle - \langle A_2(U), \mathbf{x} \rangle - \langle H(\mathbf{v}(t), \mathbf{u}(t) + U), \mathbf{x} \rangle,
\]

where \( \mathbf{y} \in V_1 \) and \( \mathbf{x} \in V_2 \).
and
\[ \langle P_2(w, w)(t), y \rangle := \langle g_2(t), y \rangle - \langle B_2(w(t), w(t) + U), y \rangle - \langle A_2(U), y \rangle + \langle H(w(t), w(t) + U), y \rangle. \]

Note that \( P_1 \) and \( P_2 \) are well-defined by virtue of Lemma 3.1 and initial assumptions.

In addition, by analogy with the stationary case we define \( S^{-1}_1 \) as the linear operator that maps any \( h \in L^2(0, T; L^2(\Omega)^2) \) into the solution \( w \in \dot{W}^1(0, T) \) to the abstract evolutionary Stokes problem
\[ \langle \partial_t w(t), y \rangle + \frac{1}{\text{Re}} \langle \nabla w(t), \nabla y \rangle_2 = \langle h(t), y \rangle_2 \quad \text{for all } y \in V_1 \text{ and a.e. } t \in (0, T), \]
\[ w(0) = v_0. \]

Similarly, \( S^{-1}_2 \) maps \( h \in L^2(0, T; V_2') \) into the solution \( w \in W_2(0, T) \) to
\[ \langle \partial_t w(t), y \rangle + \frac{1}{\text{Re Pr}} \langle \nabla w(t), \nabla y \rangle_2 = \langle h(t), y \rangle \quad \text{for all } y \in V_2 \text{ and a.e. } t \in (0, T), \]
\[ w(0) = u_0 - U. \]

The existence and uniqueness result that yields the well-defined property of \( S^{-1}_1 \) can be obtained by analogous arguments of those in Theorem 5 of section 7.1.3 of [23] together with the regularity result on solutions to steady Stokes problems given in Theorem A.1 of [11] (see Theorem A.1 in the Appendix). The analogous result for \( S^{-1}_2 \) follows from Proposition III.2.3 of [44].

Then, a necessary and sufficient condition for an element \( (\bar{v}, \bar{u}) \in \dot{W}^1(0, T) \times W_2(0, T) \) to be a solution to \( \dot{P}_w \) is that
\[ F(\bar{v}, \bar{u}) = (\bar{v}, \bar{u}), \]
for \( F := (S^{-1}_1 P_1, S^{-1}_2 P_2) \).

We denote by \( B_\tau \) the closed ball in \( \dot{W}^1(0, T) \times W_2(0, T) \) with radius \( \tau > 0 \) and center at the origin. In what follows, we shall prove that \( F \) is a contraction from some closed ball \( B_\tau \) into itself and hence the existence and uniqueness of a solution to problem \( \dot{P}_w \) in \( B_\tau \) will follow by Banach fixed point theorem.

Initially, we observe that
\[ \| S^{-1}_1 h \|_{\dot{W}^1(0, T)} \leq c_1(\text{Re}) \| h \|_{L^2(0, T; L^2(\Omega)^2)} + c_2(\text{Re}) \| v_0 \|_{V_1}, \quad (3.12) \]
for all \( h \in L^2(0, T; L^2(\Omega)^2) \), and that
\[ \| S^{-1}_2 h \|_{W_2(0, T)} \leq C(1 + \text{Re Pr}) (\| h \|_{L^2(0, T; V_2')} + \| u_0 - U \|_{L^2(\Omega)}), \quad (3.13) \]
for all \( h \in L^2(0, T; V_2') \), where
\[ c_1(\text{Re}) := C(1 + \text{Re}^{1/2} + \text{Re}) \quad \text{and} \quad c_2(\text{Re}) := C(1 + \text{Re}^{-1/2} + \text{Re}^{1/2}). \]

The first estimate is proved in Theorem A.1 of [44]. The second one is a direct consequence of Proposition III.2.3 of [44].
Let \( \tau > 0 \) and consider arbitrary \((w_1, w_1), (w_2, w_2) \in B_\tau\). Estimates \([3.12]-[3.13]\) together with the linearity of \( S_1^{-1} \) and \( S_2^{-1} \), allow us to obtain

\[
\|F(w_1, w_1) - F(w_2, w_2)\|_{\bar{W}_1(0,T) \times W_2(0,T)}
\]

\[
= \|S_1^{-1}(P_1(w_1, w_1) - P_1(w_2, w_2))\|_{\bar{W}_1(0,T)}
\]

\[
+ \|S_2^{-1}(P_2(w_1, w_1) - P_2(w_2, w_2))\|_{W_2(0,T)}
\]

\[
\leq c_1(\text{Re})\|P_1(w_1, w_1) - P_1(w_2, w_2)\|_{L^2(0,T; L^2(\Omega)^2)}
\]

\[
+ C(1 + \text{Re} \text{Pr})\|P_2(w_1, w_1) - P_2(w_2, w_2)\|_{L^2(0,T; V'_2)}.
\]

From the identity

\[
P_1(w_1, w_1) - P_1(w_2, \tilde{w}_2) = B_1(w_2, w_1 - w_2) + B_1(w_1 - w_2, w_1) - A_1(0, w_1 - w_2),
\]

where \( A_1(0, w_1 - w_2)(t) = A_1(0, w_1(t) - w_2(t)) \), and estimates \([2.18]\) and \([3.8]\), we have

\[
\|P_1(w_1, w_1) - P_1(w_2, w_2)\|_{L^2(0,T; L^2(\Omega)^2)}
\]

\[
\leq C(\|w_1\|_{\bar{W}_1(0,T)} + \|w_2\|_{\bar{W}_1(0,T)})\|w_1 - w_2\|_{\bar{W}_1(0,T)} + C\frac{Gr}{\text{Re}^2}\|w_1 - w_2\|_{W_2(0,T)}
\]

\[
\leq C\left(\tau + \frac{Gr}{\text{Re}^2}\right)\|(w_1, w_1) - (w_2, w_2)\|_{\bar{W}_1(0,T) \times W_2(0,T)}.
\]

Next, we estimate \( \|P_2(w_1, w_1) - P_2(w_2, w_2)\|_{L^2(0,T; V'_2)} \). In order to simplify notation, we define the following constants:

\[
\ell_{u_d} := \|u_d\|_{W^{1/2, 2}(\Gamma_d)}, \quad \ell_{v_0} := \|v_0\|_{V_1}, \quad \ell_{u_0} := \|u_0\|_{L^2(\Omega)},
\]

\[
\ell_{g_1} := \|g_1\|_{L^2(0,T; L^2(\Omega)^2)}, \quad \ell_{g_2} := \|g_2\|_{L^2(0,T; V'_2)}, \quad \ell_\beta := \|\beta\|_{L^\infty(\mathbb{R})}.
\]

First, we consider

\[
P_2(w_1, w_1) - P_2(w_2, w_2) = I + J,
\]

where

\[
I := B_2(w_2, w_1 - w_2) + B_2(w_1 - w_2, w_1 + U),
\]

\[
J := H(w_1, w_1 + U) - H(w_2, w_2 + U).
\]

Similarly as we estimated \( \|P_1(w_1, w_1) - P_1(w_2, w_2)\|_{L^2(0,T; L^2(\Omega)^2)} \), we find

\[
\|I\|_{L^2(0,T; V'_2)} \leq C(\tau + T^{1/2}\ell_{u_d})\|(w_1, w_1) - (w_2, w_2)\|_{\bar{W}_1(0,T) \times W_2(0,T)}.
\]

In addition, we have (see the proof of Theorem \([2.1]\))

\[
\|J(t)\|_{V'_2} \leq C\ell_\beta\|w_1(t) - w_2(t)\|_{V_2}\|w_1(t)\|_{V_1}
\]

\[
+ C\ell_\beta(\|w_2(t)\|_{V_2} + \ell_{u_d})\|w_1(t) - w_2(t)\|_{V_1}
\]

\[
+ CL(\|w_2(t)\|_{V_2} + \ell_{u_d})\|w_1(t) - w_2(t)\|_{V_1}\|w_2(t)\|_{V_1},
\]

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for almost every $t \in (0, T)$, and where $L > 0$ is the Lipschitz constant of $\beta$. From this, we find

$$\|J\|_{L^2(0, T; V')} \leq C(LT^2 + (\ell_\beta + LT^{-1/2} \ell_u) \tau + T^{1/2} \ell_u \ell_\beta) \|\tilde{w}_1 \times w_2\|_{\tilde{W}_1(0, T) \times W_2(0, T)}.$$ 

Therefore,

$$\|P_2(w_1, w_1) - P_2(w_2, w_2)\|_{L^2(0, T; V')} \leq C(LT^2 + (\ell_\beta + LT^{-1/2} \ell_u) \tau + T^{1/2} \ell_u \ell_\beta) \|\tilde{w}_1 \times W_2(0, T).$$

Hence,

$$\|F(w_1, w_1) - F(w_2, w_2)\|_{\tilde{W}_1(0, T) \times W_2(0, T)} \leq K_1 \|\tilde{w}_1 \times W_2(0, T),$$

where

$$K_1 := CL(1 + Re Pr) \tau^2 + (c_1(Re) + C(1 + Re Pr)(1 + \ell_\beta + LT^{-1/2} \ell_u)) \tau$$

$$+ c_1(Re) \frac{Gr}{Re^2} + C T^{1/2} \ell_u \ell_\beta.$$

From analogous arguments, we find

$$\|F(w, w)\|_{\tilde{W}_1(0, T) \times W_2(0, T)} \leq K_2,$$

for any $(w, w) \in B_T$, where

$$K_2 := (c_1(Re) + C(1 + Re Pr)(1 + \ell_\beta)) \tau^2$$

$$+ \left( c_1(Re) \frac{Gr}{Re^2} + C T^{1/2} \ell_u \ell_\beta \right) \tau$$

$$+ c_1(Re) \frac{Gr}{Re^2} \ell_u \ell_\beta$$

$$+ C \left( T^{1/2} \frac{1 + Re^{-1} Pr^{-1}}{\ell_u \ell_\beta} \right).$$

Looking at the definitions of $K_1$ and $K_2$, we notice that there exist $\tau > 0$ and $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ such that if (3.11) holds true then

$$K_1 > 1, \quad \text{and} \quad K_2 \leq \tau,$$

provided that $\ell_{\ell_1}, \ell_u, \ell_\beta,$ and $\ell_{\ell_\beta}$ are sufficiently small. Therefore, $F$ is a contraction from $B_T$ into itself. \(\square\)
4. Numerical results. This section is devoted to analyze the performance of the artificial boundary condition proposed in this paper:

\[
\frac{1}{\text{Re} \text{Pr}} \frac{\partial u}{\partial n} - u \beta (v \cdot n)(v \cdot n) = 0 \quad \text{on} \quad \Gamma_o, \quad (4.1)
\]

where \(\Gamma_o\) represents an open/artificial boundary of a truncated domain \(\Omega\), and \(\beta\) satisfies (A1). We perform a variety of tests to compare solutions on \(\Omega\) using (4.1) with respect to the restriction to \(\Omega\) of the solution to a problem on a larger domain \(\Omega^{\text{ext}}\) that contains \(\Omega\). In all our tests we consider \(g_1 \equiv 0\) and \(g_2 \equiv 0\) in order to fully concentrate on buoyancy effects.

Geometrical setup and boundary conditions. We consider a geometry that represents a 2-dimensional open cavity, similar to that considered by Chan and Tien in [18]. Let

\[
\Omega := (0,1)^2, \quad \text{and} \quad \Omega^{\text{ext}} := (0,1)^2 \cup ((1,2) \times (-1,2)).
\]

The domain \(\Omega\) represents the cavity and, in what follows, we refer to it as the truncation of \(\Omega^{\text{ext}}\) at the line

\[
\Gamma_o := \{(1,x_2) : x_2 \in (0,1)\},
\]

which is the open boundary for \(\Omega\); see Figure 4.1. Further, we refer to \(\Omega^{\text{ext}}\) as the extension of \(\Omega\) and define \(\Gamma^{\text{ext}} := \partial \Omega^{\text{ext}}\).

![Fig. 4.1: Geometry of the setup for \(\Omega\) and \(\Omega^{\text{ext}}\).](image)

On the extended domain \(\Omega^{\text{ext}}\), we consider the following boundary decomposition and conditions:

\[
v = 0 \quad \text{on} \quad \Gamma^{\text{ext}}_w, \quad u = 1 \quad \text{on} \quad \Gamma^{\text{ext}}_d, \quad \frac{1}{\text{Re} \text{Pr}} \frac{\partial v}{\partial n} = p n \quad \text{on} \quad \Gamma^{\text{ext}}_o, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \Gamma^{\text{ext}}_n \cup \Gamma^{\text{ext}}_o,
\]

where

\[
\Gamma^{\text{ext}}_d := \{(0,x_2) : x_2 \in (0,1)\}, \quad \Gamma^{\text{ext}}_o := \{(2,x_2) : x_2 \in (-1,2)\},
\]

\[
\Gamma^{\text{ext}}_n := \Gamma^{\text{ext}} \setminus (\Gamma^{\text{ext}}_o \cup \Gamma^{\text{ext}}_d), \quad \Gamma^{\text{ext}}_w := \Gamma^{\text{ext}} \setminus (\Gamma^{\text{ext}}_o).
\]
Specifically, the above conditions determine that we consider a heating element on the left vertical wall $\Gamma_{ext}^d$, and the rest of the boundary of $\Omega_{ext}$ is insulated. Furthermore, flow is allowed to leave the extended domain on the right vertical part of the boundary, $\Gamma_{ext}^o$, and in the rest of the boundary a no-slip condition is imposed. See Figure 4.1 for clarification. The solution on this domain is called reference solution.

For the domain of interest, $\Omega \subset \Omega_{ext}$, we consider:

\[
\begin{align*}
  u &= 1 \quad \text{on } \Gamma_d, & \frac{\partial u}{\partial n} &= 0 \quad \text{on } \Gamma_n, & v &= 0 \quad \text{on } \Gamma_w,
\end{align*}
\]

where

\[
\begin{align*}
  \Gamma_w &:= \Gamma \setminus \Gamma_o, & \Gamma_d &:= \{(0, x_2) : x_2 \in (0, 1)\}, \\
  \Gamma_n &:= \{(x_1, x_2) : x_1 \in (0, 1), x_2 = 0 \text{ or } x_2 = 1\}.
\end{align*}
\]

See Figure 4.1 for the graphic description of boundary parts. In order to show the performance of the boundary condition proposed in this paper, we consider several choices of artificial conditions on $\Gamma_o$. We assume that the fluid flow satisfies either a do-nothing or a “directional” do-nothing condition, which consists in adding the extra term $-\frac{1}{2} v (v \cdot n)_-$ on the left-hand side of the do-nothing condition. This modification was recently introduced for isothermal fluids in [15], where the authors obtain accurate results and enhance stability properties of the system with respect to the standard do-nothing condition. We denote do-nothing and directional do-nothing conditions by (DN) and (DDN), respectively, that is:

\[
\begin{align*}
  (\text{DN}) & \quad \frac{1}{Re} \frac{\partial v}{\partial n} = p n, \\
  (\text{DDN}) & \quad \frac{1}{Re} \frac{\partial v}{\partial n} - \frac{1}{2} v (v \cdot n)_- = p n.
\end{align*}
\]

In addition, we assume that the artificial boundary condition (4.1) holds true at the line of truncation. The case when $\beta \equiv 0$ corresponds to a homogeneous Neumann condition. Then, we denote condition (4.1) differently according to $\beta \equiv 0$ or $\beta \neq 0$, as follows:

\[
\begin{align*}
  (\text{N}) & \quad \frac{\partial u}{\partial n} = 0, & (\text{N}_\beta) & \quad \frac{1}{Re Pr} \frac{\partial u}{\partial n} = u \beta (v \cdot n) (v \cdot n).
\end{align*}
\]

In particular, we consider the following two choices for the function $\beta$:

\[
\beta_1(s) = 1/2 - 1/\pi \cdot \arctan(100s), \quad \beta_2(s) = \begin{cases} 
  1/2 & \text{if } s < 0, \\
  0 & \text{otherwise},
\end{cases}
\]

and observe that $\beta_2$ corresponds to the boundary condition considered in [41,42] as an ad-hoc modification on the homogeneous Neumann condition for analysis purposes.

**Discretization and solver details.** For discretization we follow Elman et al. [22]. We subdivide the time interval $[0, T]$ into $N$ sub-intervals of length $k$ and semi-discretize in time (3.1), (3.2), and (3.3) in Problem (\tilde{P}) by applying the trapezoid rule. In this manner, we obtain

\[
\begin{align*}
  \frac{2}{k} v^{n+1} + v^{n+1} \cdot \nabla v^{n+1} = & \frac{1}{Re} \Delta v^{n+1} + \nabla p^{n+1} - \frac{Gr}{Re^2 u e} = \frac{2}{k} v^n + D_t v^n, \quad (4.2) \\
  \text{div } v^{n+1} = & \quad 0, \quad (4.3) \\
  \frac{2}{k} u^{n+1} + v^{n+1} \cdot \nabla u^{n+1} = & \frac{1}{Re Pr} \Delta u^{n+1} = \frac{2}{k} u^n + D_t u^n, \quad (4.4)
\end{align*}
\]
for \( n = 0, 1, \ldots, N - 1 \), where
\[
D_t v^n := -v^n \cdot \nabla v^n + \frac{1}{Re} \Delta v^n - \nabla p^n + \frac{Gr}{Re} u^n e,
\]
\[
D_t u^n := -v^n \cdot \nabla u^n + \frac{1}{Re Pr} \Delta u^n,
\]
and \( v^0, u^0, p^0 \) are assumed to be known. We notice that boundary conditions in problem (\( \tilde{P} \)) remain the same for the semi-discretized problem since they do not depend on the time variable.

Then, we consider a linearization of a standard weak form of (4.2)-(4.4), on the base of approximating \( v^{n+1} \) and \( u^{n+1} \) by the linear extrapolations \( \tilde{v}^{n+1} := 2v^n - v^{n-1} \) and \( \tilde{u}^{n+1} := 2u^n - u^{n-1} \). Thus, for a test triplet \((y, q, y)\) the linearized weak formulation of (4.2) is given by
\[
\frac{2}{k} \int_\Omega v^{n+1} \cdot y \, dx + \int_\Omega (\tilde{v}^{n+1} \cdot \nabla v^{n+1}) \cdot y \, dx + \frac{1}{Re} \int_\Omega \nabla v^{n+1} \cdot \nabla y \, dx
\]
\[- \int_\Omega p^{n+1} \cdot \nabla y \, dx - \frac{Gr}{Re^2} \int_\Omega u^{n+1} e \cdot y \, dx - C_v(v^{n+1}, \tilde{v}^{n+1}, y)\]
\[
= \frac{2}{k} \int_\Omega v^n \cdot y \, dx - \int_\Omega (\tilde{v}^{n+1} \cdot \nabla v^n) \cdot y \, dx - \frac{1}{Re} \int_\Omega \nabla v^n \cdot \nabla y \, dx
\]
\[+ \int_\Omega p^{n+1} \cdot \nabla y \, dx + \frac{Gr}{Re^2} \int_\Omega u^n e \cdot y \, dx + C_v(v^n, v^n, y),\]
where \( C_v \) depends on the choice of the boundary condition: For the directional do-nothing condition (DDN), the linearized boundary term is
\[
C_v(v_1, v_2, y) = \int_{\Gamma_u} \frac{1}{2} (v_1 \cdot y)(v_2 \cdot n)_- \, d\sigma,
\]
and for the do-nothing condition (DN), is \( C_v \equiv 0 \).

The weak formulation of the divergence free condition (4.3) is given by
\[
\int_\Omega \text{div} \ v \cdot q \, dx = 0,\]
and the one for (4.4) is obtained as
\[
\frac{2}{k} \int_\Omega u^{n+1} \cdot y \, dx + \int_\Omega (\tilde{v}^{n+1} \cdot \nabla u^{n+1}) \cdot y \, dx + \frac{1}{Re Pr} \int_\Omega \nabla u^{n+1} \nabla y \, dx
\]
\[- C_u(u^{n+1}, \tilde{v}^{n+1}, y)\]
\[
= \frac{2}{k} \int_\Omega u^n \cdot y \, dx - \int_\Omega (\tilde{v}^n \cdot \nabla u^n) \cdot y \, dx - \frac{1}{Re Pr} \int_\Omega \nabla u^n \nabla y \, dx + C_u(u^n, v^n, y),\]
(4.7)
where for (N\( \beta \)), we have
\[
C_u(v, u, y) = \int_{\Gamma_u} u\beta_i(v \cdot n)(v \cdot n)y \, d\sigma,
\]
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and in case of the homogeneous Neumann condition (N), we have \( C_u \equiv 0 \).

Discretization in space is done by the application of finite elements method on a regular triangular mesh. The discrete approximations for velocity and pressure are computed by piecewise quadratic, respectively linear, functions on mixed \( P_2-P_1 \) Taylor-Hood triangular elements, and temperature is approximated by piecewise quadratic functions on \( P_2 \) elements as well. The variational form is directly obtained from (4.5)-(4.7) by replacing the functions by their respective finite-dimensional approximations.

The discrete problem is then solved with FEniCS, under the use of LU-decomposition for solving the linear systems. For each time-step, we utilize that \( v^{n+1} \) does not occur in the linearization of the temperature equation, which allows (4.7) to be solved separately.

**Test Results.** Computations are produced for

\[
\text{Re} \in \{2, 3, 4, 5\}, \quad \text{Gr} \in \{500, 1000, 2000\}, \quad \text{and} \quad \text{Pr} = 1,
\]

in the time interval \([0, T]\) for \( T = 1 \).

In order to provide a qualitative idea of solutions, the reference velocity field and temperature distribution on \( \Omega^\text{ext} \) are depicted in Figure 4.2 for \( \text{Re} = 3 \), \( \text{Gr} = 10^3 \) and final time \( t = T = 1 \). Furthermore, a solution on \( \Omega \) is shown in Figure 4.3 for boundary conditions (DN) – (N\( \beta_1 \)). All numerical solutions are computed on a regular triangulation with 6489 nodes for a time-discretization parameter of \( k = 10^{-2} \).

![Fig. 4.2: Velocity and temperature of the reference solution on \( \Omega^\text{ext} \) at \( t = T = 1 \) for \( \text{Re} = 3.0 \) and \( \text{Gr} = 10^3 \).](image)
Fig. 4.3: Velocity and temperature of the solution on the truncated domain Ω at \( t = T = 1 \)
 for the artificial boundary conditions (DN) \(- (N\beta_1)\) for \( Re = 3.0 \) and \( Gr = 10^3 \).

or \( n = N \) to the reference solution \((v^N_{\text{ref}}, u^N_{\text{ref}})\) by

\[
\text{res}_\Omega = \| \nabla v^N - \nabla v^N_{\text{ref}} \|^2_2 + \| \nabla u^N - \nabla u^N_{\text{ref}} \|^2_2,
\]

where \( \| \cdot \|^2_2 \) denotes the norm in either \( L^2(\Omega)^2 \times 2 \) or \( L^2(\Omega)^2 \). The values are displayed in Table 4.1a. Additionally, we define residuals over the artificial boundary in the trace sense by

\[
\text{res}_{\Gamma_o} = \| v^N - v^N_{\text{ref}} \|^2_{L^2(\Gamma_o)^2} + \| u^N - u^N_{\text{ref}} \|^2_{L^2(\Gamma_o)},
\]

for which the values are given in Table 4.1b. It is clearly seen, that both indicators, \( \text{res}_\Omega \) and \( \text{res}_{\Gamma_o} \), are consistently improved by the use of the boundary condition proposed in this paper.

For \( Re = 3 \) and \( Gr = 10^3 \), the profiles of temperature and the horizontal component of velocity field are shown in Figure 4.4. One can observe the positive effect of the proposed boundary condition almost on every point of \( \Gamma_o \). For the velocity profile the advantage of the directional do-nothing over the do-nothing condition is significant as well.
(a) Normal velocity $\mathbf{v} \cdot \mathbf{n}$ at $x_1 = 1$ and time $t = T = 1$.

(b) Temperature $u$ at $x_1 = 1$ and time $t = T = 1$.

Fig. 4.4: Comparison of velocity and temperature profiles for $x_1 = 1$ at $t = T = 1$ for $Re = 3.0$ and $Gr = 10^3$. 
Table 4.1: Comparison of $\text{res}^\Omega_N$ and $\text{res}_\Gamma^N$ for solutions on the truncated domain $\Omega$ with the regarded open boundary conditions.
Appendix A. Regularity of solutions to the evolutionary Stokes problem with mixed boundary conditions. Although the following result may be obtained from classical ones, the specific structure of the constants obtained within the bounds is hard to be inferred from known theorems. In order to keep the paper self-contained, the proof is given.

In what follows we assume that $\Omega$ is a subset of $\mathbb{R}^2$ that satisfies condition (A2) in page 6 and the boundary decomposition $\{\Gamma_i, \Gamma_w, \Gamma_o\}$ satisfies the regularity and geometrical conditions assumed in Theorem 3.2.

**Theorem A.1.** Let $h \in L^2(0,T;L^2(\Omega)^2)$ and $w_o \in V_1$. Then, the abstract evolutionary Stokes problem

$$
\langle \partial_t w(t), y \rangle + \frac{1}{Re} \langle \nabla w(t), \nabla y \rangle_2 = \langle h(t), y \rangle_2 \quad \text{for all } y \in V_1 \text{ and a.e. } t \in (0,T),
$$

$$
w(0) = w_o,
$$

has a unique solution $w$ which belongs to

$$
\bar{W}_1(0,T) = \{ w \in L^2(0,T;W^{2,2}(\Omega)^2) \cap L^\infty(0,T;V_1), \partial_t w \in L^2(0,T;L^2(\Omega)^2) \},
$$

and satisfies the estimate

$$
\| w \|_{\bar{W}_1(0,T)} := \| w \|_{L^2(0,T;W^{2,2}(\Omega)^2)} + \| w \|_{L^\infty(0,T;V_1)} + \| \partial_t w \|_{L^2(0,T;L^2(\Omega)^2)} \leq c_1(Re) \| h \|_{L^2(0,T;L^2(\Omega)^2)} + c_2(Re) \| w_o \|_{V_1},
$$

where

$$
c_1(Re) := C(1 + Re^{1/2} + Re), \quad c_2(Re) := C(1 + Re^{-1/2} + Re^{1/2}),
$$

and $C$ is a positive constant that depends only on $\Omega$.

**Proof.** From Proposition III.2.3 of [44], we know that there exists a unique $w \in W_1(0,T)$ that solves problem (A.1). The increased regularity of $w$, i.e., that $w \in \bar{W}_1(0,T)$, can be obtained by analogous arguments of those leading to Theorem 5 of section 7.1.3 of [23] in combination with the regularity result for steady Stokes problems given in Theorem A.1 of [11]. We describe them below.

Let $\{y_1, y_2, \ldots\}$ be an orthogonal basis of $V_1$ that is also an orthonormal basis of $H_1$, and let $m \in \mathbb{N}$. The same techniques for proving the existence and uniqueness of Galerkin approximations for parabolic problems (see, e.g., Theorem 1 of section 7.1.2 of [23]) yield the existence and uniqueness of an element $w_m$ of the form

$$
w_m(t) := \sum_{k=1}^{m} d_m^k(t) y_k \quad \text{for a.e. } t \in (0,T),
$$

that satisfies

$$
\langle \partial_t w_m(t), y_k \rangle_2 + \frac{1}{Re} \langle \nabla w_m(t), \nabla y_k \rangle_2 = \langle h(t), y_k \rangle_2,
$$

$$
d_m^k(0) = \langle w_o, y_k \rangle_2,
$$

for all $y \in V_1$ and a.e. $t \in (0,T)$, where $k = 1, \ldots, m$. 

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Thus, taking into account that \( \| \) and that by Young’s inequality we have
\[
(\nabla w_m(t), \nabla \partial_t w_m(t))_2 = \frac{1}{2} \partial_t \| w_m(t) \|_{V_1}^2,
\]
and that by Young’s inequality we have
\[
|(h(t), \partial_t w_m(t))_2| \leq \frac{1}{2\varepsilon} \| \partial_t w_m(t) \|_{L^2(\Omega)}^2 + \frac{\varepsilon}{2} \| h(t) \|_{L^2(\Omega)}^2,
\]
for any \( \varepsilon > 0 \), we obtain
\[
\| \partial_t w_m(t) \|_{L^2(\Omega)}^2 + \frac{1}{2 \text{Re}} \| \partial_t \| w_m(t) \|_{V_1}^2 \leq \frac{1}{2\varepsilon} \| \partial_t w_m(t) \|_{L^2(\Omega)}^2 + \frac{\varepsilon}{2} \| h(t) \|_{L^2(\Omega)}^2.
\]
Selecting \( \varepsilon = 1 \), we find
\[
\| \partial_t w_m(t) \|_{L^2(\Omega)}^2 + \frac{1}{\text{Re}} \| \partial_t \| w_m(t) \|_{V_1}^2 \leq \| h(t) \|_{L^2(\Omega)}^2. \tag{A.6}
\]

Integration of (A.6) from 0 to \( t \) yields
\[
\int_0^t \| \partial_s w_m(s) \|_{L^2(\Omega)}^2 \, ds + \frac{1}{\text{Re}} \| w_m(t) \|_{V_1}^2 \leq \int_0^t \| h(s) \|_{L^2(\Omega)}^2 \, ds + \frac{1}{\text{Re}} \| w_m(0) \|_{V_1}^2.
\]

Thus, taking into account that \( \| w_m(0) \|_{V_1} \leq \| w_0 \|_{V_1} \) by (A.5), we have
\[
\int_0^t \| \partial_s w_m(s) \|_{L^2(\Omega)}^2 \, ds + \frac{1}{\text{Re}} \| w_m(t) \|_{V_1}^2 \leq \int_0^t \| h(s) \|_{L^2(\Omega)}^2 \, ds + \frac{1}{\text{Re}} \| w_0 \|_{V_1}^2,
\]
and since \( t \in (0, T) \) was arbitrary, we observe
\[
\int_0^T \| \partial_t w_m(t) \|_{L^2(\Omega)}^2 \, dt + \frac{1}{\text{Re}} \sup \{ \| w_m(t) \|_{V_1}^2 : t \in (0, T) \} \leq \| h \|_{L^2(0, T; L^2(\Omega))^2}^2 + \frac{1}{\text{Re}} \| w_0 \|_{V_1}^2. \tag{A.7}
\]

Passing to the limit as \( m \to \infty \) in (A.7) we obtain \( \partial_t w \in L^2(0, T; L^2(\Omega)^2) \), \( w(t) \in L^\infty(0, T; V_1) \), and that the following estimates hold true:
\[
\| \partial_t w \|_{L^2(0, T; L^2(\Omega)^2)} \leq \| h \|_{L^2(0, T; L^2(\Omega)^2)} + \frac{1}{\text{Re}} \| w_0 \|_{V_1}^2, \tag{A.8}
\]
\[
\| w \|_{L^\infty(0, T; V_1)} \leq \text{Re} \| h \|_{L^2(0, T; L^2(\Omega)^2)} + \| w_0 \|_{V_1}^2. \tag{A.9}
\]

Let \( t \in (0, T) \). Notice that \( w \) satisfies
\[
\frac{1}{\text{Re}} (\nabla w(t), \nabla y)_2 = (\tilde{h}(t), y)_2,
\]

where \( \tilde{h}(t) \) is defined by (A.4).
for all $y \in V_1$, where $\tilde{h}(t) := h(t) - \partial_t w(t) \in L^2(\Omega)^2$. Therefore, it follows from Theorem A.1 of \cite{11} and assumptions on the boundary decomposition $\{\Gamma_1, \Gamma_w, \Gamma_o\}$ that $w(t) \in W^{2,2}(\Omega)^2$ and satisfies
\[
\|w(t)\|_{W^{2,2}(\Omega)^2} \leq C \text{Re} \|\tilde{h}(t)\|_{L^2(\Omega)},
\]
and thus,
\[
\|w(t)\|_{W^{2,2}(\Omega)^2} \leq C \text{Re} \left(\|h(t)\|_{L^2(\Omega)^2} + \|\partial_t w(t)\|_{L^2(\Omega)^2}\right).
\]
Integrating the square of the above inequality with respect to $t$ from 0 to $T$, and using \ref{A.8}, we obtain
\[
\int_0^T \|w(t)\|_{W^{2,2}(\Omega)^2}^2 \, dt \leq C \text{Re}^2 \|h\|_{L^2(0,T;L^2(\Omega)^2)}^2 + C \text{Re} \|w_0\|_{V_1}^2,
\]
that is, $w \in L^2(0,T;W^{2,2}(\Omega)^2)$ and
\[
\|w\|_{L^2(0,T;W^{2,2}(\Omega)^2)}^2 \leq C \text{Re}^2 \|h\|_{L^2(0,T;L^2(\Omega)^2)}^2 + C \text{Re} \|w_0\|_{V_1}^2. \tag{A.10}
\]
From \ref{A.8}, \ref{A.9}, and \ref{A.10} we find that $w \in \tilde{W}_1(0,T)$, and in addition, we have
\[
\|w\|_{W^1(0,T)}^2 \leq C(1 + \text{Re} + \text{Re}^2) \|h\|_{L^2(0,T;L^2(\Omega)^2)}^2 + C(1 + \text{Re}^{-1} + \text{Re}) \|w_0\|_{V_1}^2,
\]
which yields the desired estimate \ref{A.2}.

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