Different Hamiltonians for the Painlevé P_{IV} equation and their identification using a geometric approach

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Abstract

It is well-known that differential Painlevé equations can be written in a Hamiltonian form. However, a coordinate form of such representation is far from unique – there are many very different Hamiltonians that result in the same differential Painlevé equation. In this paper we describe a systematic procedure of finding changes of coordinates transforming different Hamiltonian systems into some canonical form. Our approach is based on Sakai’s geometric theory of Painlevé equations. We explain our approach using the fourth differential P_{IV} equation as an example, but it can be easily adapted to other Painlevé equations as well.

1 Introduction

Painlevé equations play an increasingly important role in a wide range of nonlinear problems in Mathematics and Mathematical Physics [IKSY91, Con99]. It is well-known, since the foundational papers of K. Okamoto [Oka80a], that Painlevé equations can be written in a Hamiltonian form. However, there are many different Hamiltonian systems that reduce to the same differential Painlevé equation, and the relationship between such different systems is far from obvious. The main objective of this paper is to establish an algorithmic scheme for identifying different Hamiltonian systems related to the same Painlevé equation via an explicit birational change of variables. The approach that we use is motivated by the recently proposed identification scheme for discrete Painlevé equations [DFS20] and is based on the geometric theory of Painlevé equations initiated in the works of K. Okamoto [Oka79] and further developed in the seminal paper of H. Sakai [Sak01]. In this paper we focus on the differential P_{IV} equation, but the scheme is general and can be applied to other examples as well; this is a work in progress.

Thus, consider the differential P_{IV} equation

\[ P_{IV} = P_{IV_{\alpha, \beta}} : \frac{d^2 w}{dt^2} = \frac{1}{2w} \left( \frac{dw}{dt} \right)^2 + \frac{3}{2} w^3 + 4tw^2 + 2(t^2 - \alpha)w + \frac{\beta}{w}, \]  

(1.1)

where \( t \) is an independent variable, \( w = w(t) \) is a dependent variable, and \( \alpha, \beta \) are complex parameters. In what follows we consider a number of different Hamiltonian systems related to this equation: Okamoto, Jimbo-Miwa, Its-Prokhorov, Kecker, and Filipuk-Zołądek.
Its–Prokhorov and Jimbo-Miwa Hamiltonians  We begin by considering the Hamiltonian system obtained by A. Its and A. Prokhorov [IP18] as isomonodromic deformations of a $2 \times 2$ linear system with one irregular singular point at $z = \infty$ with the Poincare rank 2 and one Fuchsian singularity at $z = 0$,

$$\frac{d\Phi}{dz} = A(z; t)\Phi, \quad A(z; t) = \frac{A^{-1}(t)}{z} + A_0(t) + A_1(t)z$$

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_0 = \begin{bmatrix} -\frac{t}{2k} & -k \\ -q(4p-q-2t) + 4q_\infty & q^2(4p-q-2t)^2 - 16q_\infty^2 \end{bmatrix}, \quad A_1^{-1} = \frac{1}{2} \begin{bmatrix} q(4p-q-2t) & -kq \\ q^2(4p-q-2t)^2 - 16q_\infty^2 & -\frac{kq}{2} \end{bmatrix}.$$  \hspace{1cm} (1.2)

where $q = q(t)$, $p = p(t)$, $k = k(t)$ and $\Theta_0, \Theta_\infty$ are some (time independent) parameters (that are formal monodromy exponents at the corresponding singular points). The isomonodromy deformation is then given by the equation

$$\frac{d\Phi}{dt} = B(z; t)\Phi, \quad B(z) = B_1(z) + B_0(t), \quad B_1 = A_1, \quad B_0 = \begin{bmatrix} 0 & -kq \\ \frac{t}{2k} & q(4p-q-2t) + 4q_\infty \end{bmatrix}.$$  \hspace{1cm} (1.3)

The compatibility condition between (1.2) and (1.3) is the zero-curvature equation $A_t - B_z + [A, B] = 0$ that results in equations

$$\frac{dq}{dt} = 4pq,$$  \hspace{1cm} (1.4)

$$\frac{dp}{dt} = -2p^2 + 3q^2/8 + qt + t^2/2 - \Theta_\infty + 1/2 - 2\Theta_0^2/q^2,$$  \hspace{1cm} (1.5)

$$\frac{dk}{dt} = -(q + 2t)k.$$  \hspace{1cm} (1.6)

Equations (1.4) and (1.5) do not depend on the function $k$ and eliminating $p$ we obtain the fourth Painlevé equation (1.1) for the function $q(t)$ with parameters

$$\alpha = 2\Theta_\infty - 1, \quad \beta = -8\Theta_0^2.$$  \hspace{1cm} (1.7)

The system of equations (1.4) and (1.5) is a non-autonomous Hamiltonian system with the Its-Prokhorov Hamiltonian $H^{IP} = H^{IP}(q, p; t)$ (corresponding to symplectic form $\omega^{IP}$) given by

$$\begin{cases}
\frac{dq}{dt} = \partial H^{IP}/\partial p, \\
\frac{dp}{dt} = -\partial H^{IP}/\partial q,
\end{cases}$$

$$\omega^{IP} = dp \wedge dq.$$  \hspace{1cm} (1.8)

**Remark 1.** A different approach to Hamiltonian structures of Painlevé equations that starts directly with the equation itself and not the isomonodromy problem, and that leads to rational Hamiltonians, was suggested earlier in [ZF15]. For $P_{IV}$, this approach gives the following Hamiltonian system,

$$\begin{cases}
\frac{dx}{ds} = \partial H^{FZ}/\partial y = x^2, \\
\frac{dy}{ds} = -\partial H^{FZ}/\partial x = \frac{3x^2 - y^2}{2} + 4sx + 2(s^2 - \alpha) + \frac{\beta}{x^2},
\end{cases}$$

$$\omega^{FZ} = (1/4) dy \wedge dx.$$  \hspace{1cm} (1.9)

Eliminating the variable $y$, we get the standard fourth Painlevé equation (1.1) for the dependent variable $x = x(s)$ with parameters $\alpha$ and $\beta$. It turns out that for $P_{IV}$ this approach gives essentially the same Hamiltonian system as in [IP18],

$$H^{FZ}(x, y; t; \alpha, \beta) = 4H^{IP}\left( x, \frac{y}{4}; t; \Omega_0^2 = -\beta, \Omega_\infty = \frac{1 + \alpha}{2} \right) - 4t(1 + \alpha).$$  \hspace{1cm} (1.10)
Even though this change of variables can be observed directly, we want to emphasize that it also follows immediately from the geometric approach, as shown in Section 4.5.

In a much earlier paper [JM81] M. Jimbo and T. Miwa gave a different parameterization of the coefficient matrices of equations (1.4) – (1.5),

\[
A_0 = \begin{bmatrix} t & u \\ 2(z-\Theta_0-\Theta_\infty) & -t \end{bmatrix}, \quad A_{-1} = \begin{bmatrix} -z + \Theta_0 & -u y/2 \\ 2(z-\Theta_0-\Theta_\infty) y & z - \Theta_0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 & u \\ 2(z-\Theta_0-\Theta_\infty) & 0 \end{bmatrix}, \quad (1.10)
\]

where \( z = z(t), \ y = y(t), \ u = u(t), \) and \( \Theta_0 \) and \( \Theta_\infty \) are the same parameters as in [IP18]. The compatibility condition in this case leads to the following system of nonlinear differential equations for the functions \( y, z \) and \( u \):

\[
\begin{align*}
\frac{dy}{dt} &= -4z + y^2 + 2ty + 4\Theta_0, \\
\frac{dz}{dt} &= -\frac{2}{y}z^2 + \left( -y + \frac{4\Theta_0}{y} \right) z + (\Theta_0 + \Theta_\infty)y, \\
\frac{du}{dt} &= -(y - 2t)u.
\end{align*}
\]

Eliminating the function \( z \) from the first two equations (1.11) and (1.12) one can obtain the fourth Painlevé equation for the variable \( y \) with the same values of parameters \( \alpha \) and \( \beta \) as in (1.6) above. The system (1.11) and (1.12) is also Hamiltonian with the Jimbo-Miwa Hamiltonian \( H_{JM} = H_{JM}(y, z, t) \):

\[
\begin{align*}
-\frac{1}{y} \frac{dy}{dt} &= \frac{\partial H_{JM}}{\partial z}, \\
-\frac{1}{y} \frac{dz}{dt} &= -\frac{\partial H_{JM}}{\partial y},
\end{align*}
\text{where } H_{JM}(y, z; t) = \frac{2}{y}y^2 - \left( y + 2t + \frac{4\Theta_0}{y} \right) z + (\Theta_0 + \Theta_\infty)(y + 2t), \quad \omega_{JM} = (1/y) dy \wedge dz.
\]

Note that this time we need to use the logarithmic symplectic structure given by \( \omega_{JM} \); we explain the geometry behind this in Section 4.3.

What is the relationship between the Hamiltonian systems (1.7) and (1.13)? In this case we do not need any elaborate tools to match these two systems exactly. Indeed, comparing the matrix coefficients in (1.2) and (1.10) we immediately see that \( u = k \), parameters \( \Theta_0 \) and \( \Theta_\infty \) match, and the variables \( (q,p) \) and \( (y,z) \) are related by the following birational change of variables:

\[
\varphi : \begin{cases} y(q,p) = q, \\
z(q,p) = \frac{1}{4}(q^2 - 4pq + 2qt + 4\Theta_0), \end{cases} \quad \text{and conversely, } \varphi^{-1} : \begin{cases} q(x,y) = y, \\
p(x,y) = \frac{2ty + y^2 - 4z + 4\Theta_0}{4y}.
\end{cases}
\]

One can then check directly that this change of variables transforms system (1.7) to system (1.13). However, since the change of variables is time-dependent, we get some additional terms in the expression for the Hamiltonian. Specifically, the relationship between the Hamiltonians is obtained via the pull-back of a certain 2-form on the extended phase space,

\[
\Omega^H = dp \wedge dq - dH^H \wedge dt = \varphi^*(\Omega_{JM}), \quad \Omega_{JM} = \frac{1}{y} dy \wedge dz - dH_{JM} \wedge dt.
\]

Thus, using (1.14), we see that

\[
H^H(q(y,z,t),p(y,z,t);t) = H_{JM}(y,z; t) - \frac{y}{2} \quad \text{and} \quad H_{JM}(y(q,p,t), z(q,p,t); t) = H^H(q,p,t) + \frac{q}{2}.
\]

This remark is essential and so we explain it in more detail in Section 3.2.
Okamoto Hamiltonian In [IP18, JM81] the fourth Painlevé equation appeared from the monodromy preserving deformation of linear system (1.2). On the other hand, in the work of K. Okamoto the same equation appeared from the monodromy preserving deformation of a scalar second order linear differential equation, with coefficients depending on \( t \), that has certain singularities in the complex plane. This approach leads to a very different Hamiltonian system with a polynomial Hamiltonian [Oka80a, Oka80b, Oka86] given by

\[
\begin{align*}
\frac{df}{dt} &= \frac{\partial H_{Ok}}{\partial g} = 4fg - f^2 - 2tf - 2\kappa_0, \\
\frac{dg}{dt} &= -\frac{\partial H_{Ok}}{\partial f} = -2g^2 + (2f + 2t)g - \theta_\infty, \\
\end{align*}
\]

where \( H_{Ok}(f, g; t) = 2fg^2 - (f^2 + 2tf + 2\kappa_0)g + \theta_\infty f \).

Eliminating the function \( g = g(t) \) from these equations we get \( P_{IV} \) equation (1.1) for the function \( f = f(t) \) with parameters

\[
\alpha = 1 + 2\theta_\infty - \kappa_0, \quad \beta = -2\kappa_0^2. \tag{1.17}
\]

It turns out that the Okamoto space of initial conditions for this system has the simplest geometry, and so this system will be the reference example for the present paper.

**Remark 2.** We use the recent comprehensive survey paper [KNY17] (see also [Nou04]) as the main reference for our choice of the geometric data.

Kecker Hamiltonian Finally, in this paper we shall also deal with the Hamiltonian that appeared in [Kec16] and was further studied in [Kec19, Ste18]. This cubic Hamiltonian is given by

\[
\begin{align*}
\frac{dx}{dz} &= \frac{\partial H_{Kek}}{\partial y} = y^2 + xz + \tilde{\alpha}, \\
\frac{dy}{dz} &= -\frac{\partial H_{Kek}}{\partial x} = -x^2 - yz - \tilde{\beta}, \\
\end{align*}
\]

where \( H_{Kek}(x, y; z) = \frac{1}{3}(x^3 + y^3) + xzy + \tilde{\alpha}y + \tilde{\beta}x \).

Now the relationship with the standard \( P_{IV} \) equation is less straightforward. First, note that now we have \( z \) as an independent variable, \( x = x(z) \) and \( y = y(z) \) are dependent variables, and \( \tilde{\alpha}, \tilde{\beta} \) (complex) parameters. Let us introduce a new variable \( \tilde{w} = \tilde{w}(z) := x(z) + y(z) - z \). Then, as shown in [Kec16, Kec19, Ste18], (1.18) gives the following equation on \( \tilde{w}(z) \):

\[
\frac{d^2 \tilde{w}}{dz^2} = \frac{1}{2\tilde{w}} \left( \left( \frac{d\tilde{w}}{dz} \right)^2 - \frac{1}{2} \tilde{w}^3 - 2z\tilde{w}^2 - \frac{\tilde{w}(3\tilde{w}^2 + 2(\tilde{\alpha} + \tilde{\beta}))}{\tilde{w}} - \frac{(1 - \tilde{\alpha} + \tilde{\beta})^2}{\tilde{w}} \right),
\]

which then reduces to the standard equation (1.1) for parameters

\[
\alpha = \frac{i}{\sqrt{3}} (\tilde{\alpha} + \tilde{\beta}), \quad \beta = -\frac{2}{9}(1 - \tilde{\alpha} + \tilde{\beta})^2. \tag{1.19}
\]

if we put \( z = (-\frac{i}{3})^\frac{1}{4} t \) and \( \tilde{w}(z) = \frac{3}{2} (-\frac{i}{3})^\frac{1}{4} w(t) \).

**Remark 3.** We need to mention that not every known example of Hamiltonian systems that reduce to Painlevé equations fits into our approach. As an example, consider the following system studies in [Tak01]

\[
\begin{align*}
\frac{dq}{dt} &= \frac{\partial H_{Tak}}{\partial p} = p, \\
\frac{dp}{dt} &= -\frac{\partial H_{Tak}}{\partial q} = -\frac{\partial V(q, t)}{\partial q}, \\
\end{align*}
\]

where \( H_{Tak}(q, p; t) = \frac{p^2}{2} + V(q, t) \),

\[
\omega_{Tak} = dp \wedge dq, \tag{1.20}
\]

and the potential \( V(q, t) \) is given by

\[
V(q, t) = \frac{1}{2} \left( \frac{q}{2} \right)^6 - 2t \left( \frac{q}{2} \right)^4 - 2(t^2 - \alpha) \left( \frac{q}{2} \right)^2 + \beta \left( \frac{q}{2} \right)^{-2}.
\]
By taking \( w = (q/2)^2 \) one can easily show that \( w \) satisfies (1.1). Moreover, in [Tak01] it was shown that if one takes

\[
\begin{align*}
f(q, p) &= \left(\frac{q}{2}\right)^2, \\
g(q, p) &= \frac{t}{2} + \frac{2\kappa_0}{q^2} + \frac{p}{2q} + \frac{q^2}{16},
\end{align*}
\]  

(1.21)

then (1.16) transforms to (1.20) with \( \alpha = 1 + 2\theta_\infty - \kappa_0 \) and \( \beta = -2\kappa_0^2 \), which is exactly (1.17). Further, the Liouville form giving the canonical transformation is

\[
\theta = gdf - H^\text{Ok} dt = \frac{1}{4}(p dq - H^\text{Tak} dt) + d\phi(q, p, t),
\]

where \( d\phi(q, p, t) \) is an exact form which can be calculated explicitly.

However, the change of variables (1.21) is only algebraic, but not birational. In this case it is impossible to construct the space of initial conditions for the system (1.20) in the sense of [Oka79] by resolving singularities, and so our approach can not be used.

We summarize the relationship between different systems considered in this paper in Figure 1.

![Diagram showing the relationship between different Hamiltonian Systems for Painlevé IV](image)

Figure 1: Relationship between different Hamiltonian Systems for Painlevé IV

The paper is organized as follows. In Section 2 we summarize our main results and give explicit change of coordinates between different Hamiltonian systems that we discuss in this paper, and we also show how to explicitly match the Hamiltonian functions. In Section 3 we carefully review the construction of the space of initial conditions for a Painlevé Hamiltonian system and then discuss the Hamiltonian formalism for time-dependent Hamiltonian functions. In Section 4 we describe an algorithmic procedure on constructing a birational change of coordinates matching two different Hamiltonian systems based on identifying the geometry of the space of initial conditions. Some standard facts about the geometry of the standard \( E_6^{(1)} \) surface, including the choice of root bases for the surface and symmetry sub-lattices in its Picard lattice, are in the Appendix.

## 2 Main Results

In this section we summarize the relationship between different Hamiltonian systems considered in this paper, on the level of the Hamiltonians, by providing explicit coordinate transformations and parameter matching.
Its-Prokhorov and Jimbo-Miwa systems

**Theorem 1.** The Its-Prokhorov Hamiltonian system (1.7) and the Okamoto Hamiltonian system (1.16) are related by the following change of variables and parameter correspondence:

\[
\begin{aligned}
q(f,g,t) &= f, \\
p(f,g,t) &= g - \frac{f}{2} - t - \frac{\kappa_0}{2f}, \\
\Theta_0 &= \frac{\kappa_0}{2}, \\
\Theta_\infty &= 1 + \theta_\infty - \frac{\kappa_0}{2}, \\
\end{aligned}
\]

and conversely

\[
\begin{aligned}
f(q,p,t) &= q, \\
g(q,p,t) &= p + \frac{q}{4} + \frac{t + \Theta_0}{q}, \\
\kappa_0 &= 2\Theta_0, \\
\theta_\infty &= -1 + \Theta_0 + \Theta_\infty.
\end{aligned}
\]

The Hamiltonians are then related by

\[
H^{IP}(q,p;\Theta_0,\Theta_\infty) = H^{Ok}(f(q,p),g(q,p);t;\kappa_0,\theta_\infty) + \frac{q}{2}.
\]

**Proof.** The change of variables (2.1) was established in Lemma 6. To relate the Hamiltonians, as explained in Section 3.2 we need to match the 2-forms \(\Omega^{Ok} = \omega^{Ok} - dH^{Ok} \wedge dt = \omega^{IP} - dH^{IP} \wedge dt = \Omega^{IP}\). We get

\[
\begin{aligned}
\Omega^{Ok} &= \omega^{Ok} - dH^{Ok} \wedge dt = dg \wedge df - d\left(2fg^2 - (f^2 + 2tf + 2\kappa_0)g + \theta_\infty f\right) \wedge dt \\
&= dp \wedge dq + \frac{1}{2} dt \wedge dq - d\left(2p^2q - \frac{q^3}{8} - \frac{tq^2 - t^2q}{2} + \Theta_\infty q - \frac{2\Theta_0^2}{q} - q - 2t\Theta_0\right) \wedge dt \\
&= \omega^{IP} - d\left(H^{IP} - 2t(\Theta_0 + \Theta_\infty)\right) \wedge dt,
\end{aligned}
\]

and the purely \(t\)-dependent terms can then be ignored.

Similar result holds for the Jimbo-Miwa system (1.13).

**Theorem 2.** The change of coordinates and parameter matching between the Jimbo-Miwa and Okamoto Hamiltonian systems is given by

\[
\begin{aligned}
y(f,g,t) &= f, \\
z(f,g,t) &= \frac{f^2}{2} - fg + ft + \kappa_0, \\
\Theta_0 &= \frac{\kappa_0}{2}, \\
\Theta_\infty &= 1 + \theta_\infty - \frac{\kappa_0}{2}, \\
\end{aligned}
\]

and conversely

\[
\begin{aligned}
f(y,z,t) &= y, \\
g(y,z,t) &= \frac{y}{2} + \frac{z}{y} + t + \frac{2\Theta_0}{y}, \\
\kappa_0 &= 2\Theta_0, \\
\theta_\infty &= -1 + \Theta_0 + \Theta_\infty.
\end{aligned}
\]

The Hamiltonians are then related by

\[
H^{JM}(y,z,t;\Theta_0,\Theta_\infty) = H^{Ok}(f(y,z,t),g(y,z,t);t;\kappa_0,\theta_\infty) = -1 + \Theta_0 + \Theta_\infty + y.
\]

**Proof.** The change of variables (2.3) was established in Lemma 8. To relate the Hamiltonians, note that

\[
\begin{aligned}
\Omega^{Ok} &= \omega^{Ok} - dH^{Ok} \wedge dt = dg \wedge df - d\left(2fg^2 - (f^2 + 2tf + 2\kappa_0)g + \theta_\infty f\right) \wedge dt \\
&= -\frac{dz}{y} \wedge dy + dt \wedge dy - d\left(\frac{2z^2}{y} - zy - 2tz - \frac{4\Theta_0 z}{y} - y + (\Theta_0 + \Theta_\infty) y\right) \wedge dt \\
&= \frac{1}{y} dy \wedge dz + d\left(H^{JM} - 2t(\Theta_0 + \Theta_\infty)\right) \wedge dt = \omega^{JM} - dH^{JM} \wedge dt,
\end{aligned}
\]

since the \(t\)-dependent terms can be ignored, as usual.

Note that combining these change of variables immediately gives us the change of variables between the Its-Prokhorov coordinates \((q,p)\) and the Jimbo-Miwa coordinates \((y,z)\) that we already obtained in equation (1.14).
**The Kecker system**  The identification between the Kecker and the Okamoto systems is more complicated. In particular, the geometric normalization of the symplectic form differs from the original normalization of the symplectic form \( \Omega_{\text{Kek}} \) in (1.18); \( \Omega_{\text{Kek}} = (1/3)\omega_{\text{Kek}} \).

**Theorem 3.** The Kecker Hamiltonian system (1.18) and the Okamoto Hamiltonian system (1.16) are related by the following change of variables and parameter correspondence:

\[
\begin{aligned}
\left\{ \begin{array}{l}
x(f, g, t) = \frac{(1 + i)}{4(3)^{3/4}} \left( 3(\sqrt{3} - i)f + 12i g + 2(\sqrt{3} - 3i)t \right), \\
y(f, g, t) = \frac{(1 + i)}{4(3)^{3/4}} \left( 3(\sqrt{3} + i)f - 12i g + 2(\sqrt{3} + 3i)t \right), \\
z(t) = \left( -\frac{3}{4} \right)^{1/4} t,
\end{array} \right.
\end{aligned}
\]

and conversely,

\[
\begin{aligned}
\left\{ \begin{array}{l}
f(x, y, z) = \frac{1 - i}{3^{3/4}} (x + y - z), \\
g(x, y, z) = -\frac{(1 + i)}{4(3)^{3/4}} \left( (\sqrt{3} + i)x - (\sqrt{3} - i)y + 2iz \right), \\
t(z) = \left( -\frac{3}{4} \right)^{1/4} z,
\end{array} \right.
\end{aligned}
\]

The Hamiltonians are then related by

\[
H_{\text{Kek}}(x, y, z; \tilde{\alpha}, \tilde{\beta}) = 3 \left( \frac{-3}{4} \right)^{1/4} H_{\text{Ok}}(f(x, y, z), g(x, y, z); t(z); \kappa_0(\tilde{\alpha}, \tilde{\beta}), \theta(\tilde{\alpha}, \tilde{\beta}))
\]

\[
- \frac{1 + i\sqrt{3}}{6} x + \frac{1 - i\sqrt{3}}{6} y + \frac{z^3}{9} + \frac{1 + i\sqrt{3}}{6} z\tilde{\alpha} + \frac{1 - i\sqrt{3}}{6} z\tilde{\beta}
\]

**Proof.** The change of variables (2.5–2.6) was established in Lemma 10. For the Hamiltonians, given the difference in normalizations, we should have

\[
\Omega_{\text{Ok}} = \omega_{\text{Ok}} - dH_{\text{Ok}} \wedge dt = \frac{1}{3} \cdot (\omega_{\text{Kek}} - dH_{\text{Kek}} \wedge dz) = \frac{1}{3} \Omega_{\text{Kek}}.
\]

That is,

\[
\Omega_{\text{Ok}} = dg \wedge df - d(2fg^2 - f^2g - 2 tfg - 2\kappa g + \theta f) \wedge dt
\]

\[
= -2\sqrt{3} dx \wedge dy - \sqrt{3}(1 + i\sqrt{3}) dx \wedge dz + \sqrt{3}(1 - i\sqrt{3}) dy \wedge dz
\]

\[
- d \frac{(1 + i) \left( 2\sqrt{3}(x^3 + y^3 - z^3 + 3xyz) + 3(3i + \sqrt{3} + 2\sqrt{3}\tilde{\beta})x + 3(3i - \sqrt{3} + 2\sqrt{3}\tilde{\alpha})y \right)}{18(3^{3/4})} \wedge \left( -\frac{3}{4} \right)^{1/4} dz
\]

\[
= \frac{1}{3} \left( dy \wedge dx + \frac{1 + i\sqrt{3}}{2} dx \wedge dz - \frac{1 - i\sqrt{3}}{2} dy \wedge dz
\]

\[
- d\left( x^3 + y^3 - z^3 + \tilde{\beta}x + \frac{1 + i\sqrt{3}}{2} x + \tilde{\alpha}y + \frac{-1 + i\sqrt{3}}{2} y \right) \wedge dz \right)
\]

\[
= \frac{1}{3} \left( dy \wedge dx - dH_{\text{Kek}}(x, y, z; \tilde{\alpha}, \tilde{\beta}) \wedge dz \right),
\]
which gives us the needed rescaling coefficients, as well as the \( x \) and \( y \)-dependent corrections to the Hamiltonian \( H^{\text{Ok}}(f(x, y), g(x, y); t(z); \kappa_0(\alpha, \beta), \theta_\infty(\alpha, \beta)) \). The remaining terms are purely time \( z \)-dependent and can be ignored.

\[ \Box \]

3 Preliminaries

3.1 The Okamoto Space of Initial Conditions

**Notation.** For this system we use the following notation: coordinates \((f, g)\), parameters \(\kappa_0\) and \(\theta_\infty\); time variable \(t\); base points \(q_i\), exceptional divisors \(F_i\).

The foundations of the geometric analysis of Painlevé equations were developed by K. Okamoto [Oka79]. To make this paper self-contained, in this section we briefly explain how to construct the space of initial conditions for the Hamiltonian system (1.16). This will also allow us to introduce various notational conventions. This section is closely related to [KNY17, Section 2.6] that we recommend for details.

Recall that the Painlevé property essentially requires that the general solution of an ODE has no movable (i.e., dependent of initial conditions) singularities other than poles. If we think about parameterizing solutions via initial conditions at some time \(t_0\), we then need to allow infinities as initial conditions, i.e., we need to change from \(\mathbb{C}\) to \(\mathbb{P}^1\). Thus, we consider the pair of dependent variables \((f, g)\) as affine coordinates on the complex projective plane \(\mathbb{P}^1 \times \mathbb{P}^1\). We then introduce three more charts \((F, g)\), \((f, G)\), and \((F, G)\), where \(F = 1/f\) and \(G = 1/g\) are coordinates in the neighborhood of infinity, and via direct substitution we can easily rewrite our system in those charts:

\[
(f, G) : \begin{cases}
\frac{df}{dt} = -f^2 + \frac{4f}{G} - 2tf - 2\kappa_0, \\
\frac{dG}{dt} = \theta_\infty G^2 - 2fg - 2tG + 2,
\end{cases}
\]

\[
(f, g) : \begin{cases}
\frac{df}{dt} = -f^2 + 4fg - 2tf - 2\kappa_0, \\
\frac{dg}{dt} = -2g^2 + 2fg + 2tg - \theta_\infty,
\end{cases}
\]

\[
(F, G) : \begin{cases}
\frac{dF}{dt} = 2\kappa_0 F^2 - \frac{4F}{G} + 2tF + 1, \\
\frac{dG}{dt} = \theta_\infty G^2 - \frac{2G}{F} - 2tG + 2,
\end{cases}
\]

\[
(F, g) : \begin{cases}
\frac{dF}{dt} = 2F^2 \kappa_0 - 4Fg + 2tF + 1, \\
\frac{dg}{dt} = -2g^2 + \frac{2g}{F} + 2tg - \theta_\infty.
\end{cases}
\]

Consider, for example, our system in the \((f, g)\)-chart. Solutions correspond to the flowlines of the vector field \(V(F, g) = (2F^2 \kappa_0 - 4Fg + 2Ft + 1)\partial_F + (-2g^2 + \frac{2g}{F} + 2tg - \theta_\infty)\partial_g\) that becomes undefined when \(F = 0\). Rescaling, \(FV(F, g) = (2F^3 \kappa_0 - 4F^2g + 2F^2t + F)\partial_F + (-2Fg^2 + 2g + 2tg - \theta_\infty)\partial_g + \partial_t\), we see that at the points \((F = 0, g \neq 0)\) the field becomes \(2g\partial_g\), and so this flow is “vertical” (has a zero \(\partial_t\)-component) and hence such points do not parametrize solutions of the Painlevé equations (that are functions of \(t\)). Thus, we call the curve given by \(F = 0\) (in this chart) a *vertical leaf* (or an *inaccessible divisor*). However, at the point \((0, 0)\) the rescaled field also vanishes (and we see the indeterminacy \(F/g = 0/0\) in the original field \(V(F, g)\)) and so a further adjustment is needed. This indeterminacy is resolved by the standard blowup procedure from the algebraic geometry, see, e.g., [Sha13] for details.

In the two-dimensional case the blowup procedure is particularly simple and can be thought of as follows [DFS20]. Geometrically, the blowup procedure “separates” the lines passing through the point \(q_i\) (the *center* of the blowup) by “lifting” them according to their “slopes” (see the left picture on Figure 2 for the local illustration of a blowup in the real-variable case). Topologically, for complex surfaces, blowup is a surgery that creates a Riemann sphere “bubble” (projectivized tangent space) \(S^2 \simeq \mathbb{P}^1\) in place of the center of the blowup \(q_i\), thus adding a new spherical class to homology (and, via the Poincaré duality, cohomology) of the surface. Algebraically, the blowup procedure is an introduction of two new charts \((u_i, v_i)\) and \((U_i, V_i)\) in the neighborhood of the blowup point \(q_i(x_i, y_i)\), where the change of variables is given by \(x = x_i + u_i = x_i + U_i V_i\) and \(y = y_i + u_i v_i = y_i + V_i\). This change of variables is a bijection away from \(q_i\), but the point \(q_i\) is replaced by the \(\mathbb{P}^1\)-line of all possible slopes, called the *central fiber* or the *exceptional divisor* of the blowup. We denote this central fiber by \(F_i\) (and sometimes by \(E_i\)), it is given in the blowup charts by local equations
$u_i = 0$ and $V_i = 0$. For these charts the upper/lower-case naming convention is only for convenience and, in contrast to the naming of affine charts, it does not hold that $U_i = 1/u_i$. However, it is true that $v_i = 1/U_i$ — these local coordinates on $\mathbb{P}^1$ represent all possible slopes of lines passing through the point $q_i$, and so this variable change “separates” all curves passing through $q_i$ based on their slopes. Schematically, it is convenient to illustrate the blowup on a diagram as shown on the right on Figure 2. The notation $L - F_i$ denotes the proper transform $\pi^{-1}(L - (x_i, y_i))$, that needs to be distinguished from the total transform $\pi^{-1}(L) = (L - F_i) + F_i$. Note that, despite the presence of the negative sign, $L - F_i$ is an actual geometric curve, i.e., an effective divisor.

![Blowup Procedure Diagram](image)

Figure 2: The Blowup Procedure

Thus, blowing up the point $q_1 (F = 0, g = 0)$ amounts to introducing coordinate charts $(u_1, v_1)$ and $(U_1, V_1)$ via $F = u_1 = U_1 V_1$, $g = u_1 v_1 = V_1$. Since the change of variables is algebraic, it is easy to rewrite our system in those charts:

$$(u_1 v_1): \begin{align*}
\frac{du_1}{dt} &= -4u_1^2v_1 + 2\kappa_0 u_1^2 + 2u_1 + 1, \\
\frac{dv_1}{dt} &= 2u_1 v_1^2 + \frac{v_1 - \theta_{\infty}}{u_1} - 2\kappa_0 u_1 v_1, \\
\end{align*}$$

$$(U_1, V_1): \begin{align*}
\frac{dU_1}{dt} &= 2\kappa_0 U_1^2 V_1 + \frac{\theta_{\infty} U_1 - 1}{V_1} - 2U_1 V_1, \\
\frac{dV_1}{dt} &= -2V_1^2 + \frac{2}{U_1} + 2t V_1 - \theta_{\infty}. \\
\end{align*}$$

Performing the same kind of analysis we see that there is another vertical leaf given by the equations $u_1 = V_1 = 0$ (that corresponds to the central fiber) that contains a new indeterminate point $q_2 (u_1 = 0, v_1 = \theta_{\infty})$ (equivalently, $U_1 = 1/\theta_{\infty}, V_1 = 0$): since all new base points appear on exceptional divisors, we omit the coordinates and write them, e.g., as $q_2(0, \theta_{\infty})$ or $q_2(1/\theta_{\infty}, 0)$. Introducing the new charts $(u_2, v_2)$ and $(U_2, V_2)$ via $u_1 = u_2 = U_2 V_2$, $v_1 = \theta_{\infty} + u_2 v_2 = \theta_{\infty} + V_2$ and repeating this process we see that there are no new base points in these charts and this cascade is resolved.

![Initial Conditions Diagram](image)

Figure 3: The Space of Initial Conditions for the Okamoto Hamiltonian System (1.16)
We now do the same procedure in the remaining charts to get the following cascades of base points:

\[
\begin{align*}
q_1(\infty, 0) \leftarrow q_2(0, \theta_\infty), & \quad q_3(0, \infty) \leftarrow q_4(\kappa_0, 0), \\
q_5(\infty, \infty) \leftarrow q_6(0, 2) \leftarrow q_7(0, -4t) \leftarrow q_8(0, 4(1 + 2t^2 + \theta_\infty - \kappa_0)).
\end{align*}
\]  

(3.1)

Remark 4. Since the blowup points that appear in the cascades are on exceptional divisors, when we write the coordinates of these points as \(q_i(0, a)\), we work in the \((u, v)\) coordinate system, and points \(q_i(a, 0)\) are in the \((U, V)\) coordinate system; for points \(q_i(0, 0)\) we would always specify which coordinate system is used.

The resulting configuration of the base points and vertical leaves is shown on Figure 3. The Okamoto space of initial conditions is then the resulting surface with the configuration of vertical leaves removed. For the geometric analysis it is, however, more convenient to consider the compact surface and just realize that points on vertical leaves do not correspond to initial conditions of our equation.

Note that, modulo the coordinates of the base points, we get exactly the standard \(E_6^{(1)}\) surface as given in [KNY17], see Figure 8. All of the standard geometric data, such as the choice of surface and root bases, for this surface is summarized in Appendix A (note that we use the notation \(F_i\) for exceptional divisors in the Okamoto case, and \(E_i\) in the KNY case). However, the normalization of points \(q_6(0, 2)\) in the Okamoto case and \(p_6(0, 1)\) in the standard case is slightly different, which results in the need for some scaling adjustments in the variables. Specifically, the Hamiltonian system

\[
\begin{align*}
\frac{dq}{dt} = \frac{\partial H_{\text{KNY}}}{\partial p} = 2qp - q^2 - \hat{t}q - a_1, & \quad \text{where} \quad H_{\text{KNY}}(q, p, \hat{t}) = qp^2 - q^2p - \hat{t}qp - a_2q - a_3p, \\
\frac{dp}{dt} = -\frac{\partial H_{\text{KNY}}}{\partial q} = 2qp - p^2 + \hat{t}q - a_2,
\end{align*}
\]

(3.2)

considered in [KNY17, Section 2.6] results in the following form of \(P_{IV}\) equation:

\[
\frac{d^2 q}{dt^2} = \frac{1}{2q} \left( \frac{dq}{dt} \right)^2 + \frac{3}{2} q^2 \hat{t}^2 q + \left( a_2 - a_0 + \frac{\hat{t}^2}{2} \right) q - a_3^2 \left( \sqrt{2} \right).
\]

By the rescaling change of variables \(\hat{t} = \sqrt{2t}, q(\hat{t}) = f(t(\hat{t}))/\sqrt{2}\) this equation can be transformed to the standard form (1.1) with parameters \(\alpha = 1 - a_1 - 2a_2, \beta = -2a_1^2\). The parameters \(a_1\) and \(a_2\) here are canonical geometric parameters called the root variables, as explained in Section A.2. For future computations, it is important to relate these parameters to the parameters \(\kappa_0\) and \(\theta_\infty\) of the Okamoto Hamiltonian. Comparing

\[
\alpha = 1 - a_1 - 2a_2 = 1 + 2\theta_\infty - \kappa_0, \quad \beta = -2a_1^2 = -2\kappa_0^2
\]

(3.3)

we immediately see that

\[
a_1 = \kappa_0, \quad a_2 = -\theta_\infty, \quad \text{and also,} \quad a_0 = 1 - a_2 - a_2 = 1 + \theta_\infty - \kappa_0.
\]

(3.4)

Note that we could have also computed the root variables directly from the geometric data in the same way as outlined in Lemma 11 using the symplectic form \(\omega_{\text{Ok}} = dq \wedge df\).

3.2 Non-autonomous Hamiltonian systems, symplectic transformations and 2-forms

Suppose we have two Hamiltonian systems, with time-dependent Hamiltonians, and we found a birational change of coordinates that transforms one system into the other. In this section we address the question of how, given this change of coordinates, to find the relationship between the Hamiltonians themselves. The difficulty here stems from the fact that for time-dependent coordinate systems the above change of variables is in general also time-dependent. As a result, the direct change of coordinates in the Hamiltonian for one system does not, as a rule, give the Hamiltonian for another system – some additional terms appear. To understand the nature of these terms we need some facts about non-autonomous Hamiltonian systems that are relevant to global Hamiltonian structures of Painlevé equations on Okamoto’s spaces [ST97, MMT99, Mat97]. This
point is explained in detail in [DFS21], so here we only briefly illustrate it, locally, using the Its-Prokhorov Hamiltonian system (1.7) and the Jimbo-Miwa Hamiltonian system (1.13), since the change of coordinates (1.14) from $(q,p)\to (y,z)$ is $t$-dependent, and we can clearly see the additional correction terms.

Consider two copies of $\mathbb{C}^2$ with coordinates $(x,y)$ and $(X,Y)$ equipped with rational symplectic forms $\omega = F(x,y)dx \wedge dy$ and $\tilde{\omega} = G(X,Y)dx \wedge dy$, i.e., $F(x,y)$ and $G(X,Y)$ are rational functions of their arguments. Suppose that we have a time-dependent birational change of variables $\varphi$ that we can consider as a birational transformation on the extended phase space $\mathbb{C}^3$,

$$\varphi : \mathbb{C}^3 \ni (x,y,t) \mapsto (X(x,y,t),Y(x,y,t),t) \in \mathbb{C}^3.$$  

Then, for each fixed $t$, we have a birational transformation $\varphi_t : (x,y) \mapsto (X(x,y;t),Y(x,y;t))$ and we say that $\varphi_t$ is symplectic with respect to $\omega$ and $\tilde{\omega}$ if, under $\varphi_t$, we have

$$\omega = F(x,y)dx \wedge dy = \varphi_t^* (\tilde{\omega}) = G(X(x,y;t),Y(x,y;t))dt \wedge dX,$$

where $d_t$ indicates the exterior derivative on $\mathbb{C}^2$, so $t$ is treated as a constant in the calculation. We usually omit the pull-back symbol and simply write $\omega = \tilde{\omega}$. Suppose that we now have two time-dependent Hamiltonian functions $H(x,y,t)$, $K(X,Y,t)$. Then we can define 2-forms $\Omega = \omega - dH \wedge dt$ and $\tilde{\Omega} = \tilde{\omega} - dK \wedge dt$ on the two copies of the extended phase space $\mathbb{C}^3$. Then, if under the transformation $\varphi$ we have the equality of 2-forms

$$\Omega = F(x,y)dy \wedge dx - dH \wedge dt = G(X,Y)dY \wedge dX - dK \wedge dt = \tilde{\Omega},$$

where $d$ here is the exterior derivative on the extended phase space $\mathbb{C}^3$ and so $t$ is treated as a variable in this calculation, then the Hamiltonian system

$$\begin{cases}
F(x,y) \frac{dx}{dt} = \partial H / \partial y, \\
F(x,y) \frac{dy}{dt} = -\partial H / \partial x,
\end{cases}
$$

is transformed into

$$\begin{cases}
G(X,Y) \frac{dX}{dt} = \partial K / \partial Y, \\
G(X,Y) \frac{dY}{dt} = -\partial K / \partial X.
\end{cases}
$$

This equality of 2-forms dictates the correction between the Hamiltonians modulo purely $t$-dependent functions, with

$$d(H - K) \wedge dt = G(X,Y) \left( \frac{\partial X}{\partial t} \frac{\partial Y}{\partial y} - \frac{\partial Y}{\partial t} \frac{\partial X}{\partial x} \right) dy \wedge dt + G(X,Y) \left( \frac{\partial X}{\partial t} \frac{\partial Y}{\partial x} - \frac{\partial Y}{\partial t} \frac{\partial X}{\partial y} \right) dx \wedge dt. \quad (3.7)$$

Returning now to our example, note that the change of variables (1.14) is symplectic with respect to the symplectic forms $\omega^{IP}$ and $\omega^{JM}$, and so the equality of 2-forms

$$\Omega^{IP} = dp \wedge dq - dH^{IP} \wedge dt = \frac{1}{2y} dy \wedge dz - dH^{JM} \wedge dt = \Omega^{JM}$$

gives

$$d(H^{IP} - H^{JM}) \wedge dt = -\frac{1}{2} dy \wedge dt = \frac{1}{2} dq \wedge dt. \quad (3.8)$$

Thus, modulo purely $t$-dependent terms, we get

$$H^{IP}(q(y,z,t),p(y,z,t);t) = H^{JM}(y,z,t) - \frac{y}{2} \quad \text{and} \quad H^{JM}(y(q,p,t),z(q,p,t);t) = H^{IP}(q,p,t) + \frac{q}{2}.$$

4 Reducing Hamiltonian Systems to the Canonical Form

In this chapter we construct spaces of initial conditions for each of the Hamiltonian systems described in the Introduction. We then follow the identification procedure, described below, to match the spaces of initial conditions to some reference case, that we take to be the Okamoto Hamiltonian system (1.16), both on the level of the Picard lattice, and on the level of birational change of coordinates.
4.1 The Identification Procedure

(Step 1) **Construct the space of initial conditions for the system.** We carefully review such construction in Section 3.1. We need to remark here that this step is potentially more involved than in the discrete case [DFS20], since indeterminacies of the vector field do not necessarily constitute singularities which need to be resolved through blowups. In particular one must identify when a singularity is inaccessible and should be removed along with the vertical leaves rather than blown up — see [DFS21] for an example where this is required. There is also the possibility that a system is indeterminate at a point, but it does not require blowing up since there is not a family of local solutions passing through, but rather a single one. This can occur when a system in coordinates is related by a birational transformation to a Painlevé equation in a way such that the symplectic form in these coordinates is t-dependent. In such cases more detailed inspection of solutions is required along the lines of classical Painlevé analysis, but we do not encounter such examples here.

(Step 2) **Determine the surface type, according to Sakai’s classification scheme.** Recall that almost all surfaces in the Sakai classification scheme are obtained as blowups of \( \mathbb{P}^1 \times \mathbb{P}^1 \) at eight base points (the only exception of the \( E_8^{(1)} \) surface corresponding to \( P_1 \) that is obtained as a blowup of \( \mathbb{P}^2 \) at nine points, and we do not consider that special case). These base points lie on a biquadratic curve on \( \mathbb{P}^1 \times \mathbb{P}^1 \) that is a polar divisor of the symplectic form \( \omega \) defining the Hamiltonian structure. The surface type of the system is determined by the configuration of the irreducible components of this curve. Each such component should have self-intersection index \(-2\) and is associated with a node of an affine Dynkin diagram; nodes are connected when the corresponding components intersect. The type of this Dynkin diagram is called the *surface type* of the equation. This description assumes that the surfaces \( X_n \) are *minimal*, and this may not be the case in general. In that case some of the \(-1\) curves, that should be inaccessible divisors for the system, need to be blown down, as in example in Sections 4.2 and 4.4.

(Step 3) **Find a preliminary change of basis of** \( \text{Pic}(X) \). At this step, we only need to ensure that this change of basis identifies the *surface roots* (or nodes of the Dynkin diagrams of our surface) with the standard example.

(Step 4) **Adjust this change of basis using root variables and parameter matching.** Using a preliminary change of basis we can define the *symmetry roots* for our surface that match the standard example. We can then compute the *canonical parameters* for the surface, known as the *root variables*, using these symmetry roots. Those root variables can be expressed in terms of various parameters for the Hamiltonian systems. Since we consider systems that are known to reduce to some differential Painlevé equation, we have correspondence between their parameters. From that, we can see the correspondence between the root variables, which would either match or differ by some symmetry. In that case, we need to act by that symmetry transformation on our preliminary choice of basis to ensure that the final change of basis will result in matching Hamiltonian systems on the level of parameters as well.

(Step 5) **Find the change of variables reducing the applied problem to the standard example.** Adjusting the change of bases in \( \text{Pic}(X) \) if necessary, we now have the identification on the level of the Picard lattice. Next, we need to find the actual change of variables that induces that linear change of basis. For that, identify the curves that form the basis in the corresponding coordinate pencils. Those curves then are our projective coordinates, up to a Möbius transformation. To fix the Möbius transformations, use the mapping of coordinate divisors. An important part of this computation is the identification of the parameters between the two systems.

4.2 The Its-Prokhorov Hamiltonian System

*Notation.* For this system we use the following notation: coordinates \((q,p)\), parameters \(\Theta_0\) and \(\Theta_\infty\); time variable \(s\); base points \(y_i\), exceptional divisors \(K_i\).
In this section we construct the space of initial conditions for the Hamiltonian system (1.7). Note that, if we only look at the Hamiltonians, we can not see the correspondence between the Okamoto parameters \( \{ \theta_\infty, \kappa_0 \} \) in (1.7) and the Its-Prokhorov parameters \( \{ \Theta_0, \Theta_\infty \} \) in (1.7). However, since both equations reduce to the same standard form (1.1) with \[
\alpha = 2\Theta_\infty - 1 = 1 + 2\theta_\infty - \kappa_0, \quad \beta = -8\Theta_0^2 = -2\kappa_0^2,
\] we see that the relationship between parameters is given by \[
\Theta_0^2 = \frac{\kappa_0^2}{4}, \quad \Theta_\infty = \theta_\infty - \frac{\kappa_0}{2} + 1.
\] (4.1)

The time variable \( t \) does not have to be the same either, so for now we denote it by \( s \), but from \( \text{P}_{IV} \) we know that we can take \( s = t \).

The space of initial conditions for this system is constructed in the same way as in Section 3.1. However, this time we get ten base points (and so the space of initial conditions is not minimal),

\[
\begin{align*}
&y_1(0, \infty) \quad y_2(-\Theta_0, 0) \\
&y_3(\Theta_0, 0) \quad y_4(\infty, \infty) \\
&y_5(0, -4) \quad y_6(0, 8s) \quad y_7(0, 16(1 - s^2 - \Theta_\infty)) \\
&y_8(0, 4) \quad y_9(0, -8s) \quad y_{10}(0, 16(s^2 + \Theta_\infty))
\end{align*}
\] (4.2)

whose configuration is shown on Figure 4 (left). After blowing them up and denoting the exceptional divisors for the blowup points \( y_i \) by \( K_i \), we get the configuration of vertical leaves that includes two curves with self-intersection index \(-3\), \( K_1 - K_2 - K_3 \) and \( K_4 - K_5 - K_8 \). These points lie on the polar divisor of a symplectic form \( \omega = kdq \wedge dp \).

Figure 4: The Space of Initial Conditions for the Its-Prokhorov Hamiltonian System (1.7)

To match this space with the space of initial conditions for the Okamoto system shown on Figure 3, we need to blow down two \(-1\) curves, and we should choose them among the curves intersecting with above \(-3\) curves to reduce the index: to avoid going back we should not choose the exceptional divisors. Thus, we choose the curves \( H_q - K_1 \) and \( H_p - K_4 \). Note that these curves are also \textit{inaccessible divisors}, and so blowing them down does not change the space of initial conditions. It is also more convenient to, rather than blowing down those two curves, to blow up two points on the Okamoto surface. Looking at the intersection diagram (7) and Figure 3, we see that one point should be on the curve \( d_3 = F_5 - F_6 \) corresponding to the central node of the Dynkin diagram \( E_6^{(1)} \), and the other one should be taken on a divisor corresponding to one of the boundary nodes. At this point we do not have enough information to make the right choice, so we just choose one of them, say \( d_1 = F_1 - F_2 \). Using the \textit{root variables} we can later see whether this choice is correct and make adjustments, if necessary. The same applies to the matching of the remaining two
legs of the $E_6^{(1)}$ diagram. Thus, we want to find an identification of two bases of the Picard lattice of two non-minimal surfaces

$$\text{Pic}^\text{IP}(X) = \text{Span}_\mathbb{Z}\{\mathcal{H}_q, \mathcal{H}_p, \mathcal{K}_1, \ldots, \mathcal{K}_{10}\} \simeq \text{Pic}^\text{Ok}(X) = \text{Span}_\mathbb{Z}\{\mathcal{H}_f, \mathcal{H}_g, \mathcal{F}_1, \ldots, \mathcal{F}_{10}\},$$

that matches the irreducible components of the anti-canonical divisor as follows:

$$\begin{align*}
\delta_0 &= \mathcal{F}_7 - \mathcal{F}_8 = \mathcal{K}_9 - \mathcal{K}_{10}, \\
\delta_1 &= \mathcal{F}_1 - \mathcal{F}_2 - \mathcal{F}_9 = \mathcal{K}_1 - \mathcal{K}_2 - \mathcal{K}_3, \\
\delta_2 &= \mathcal{H}_f - \mathcal{F}_1 - \mathcal{F}_5 = \mathcal{H}_p - \mathcal{K}_1 - \mathcal{K}_4, \\
\delta_3 &= \mathcal{F}_5 - \mathcal{F}_6 - \mathcal{F}_{10} = \mathcal{K}_4 - \mathcal{K}_5 - \mathcal{K}_8,
\end{align*}$$

We summarize such preliminary identification in the following Lemma.

**Lemma 4.** A preliminary change of basis resulting in the identification (4.3) is given by

$$\begin{align*}
\mathcal{H}_f &= \mathcal{H}_p + 2\mathcal{H}_q - \mathcal{K}_1 - \mathcal{K}_3 - \mathcal{K}_4 - \mathcal{K}_5, \\
\mathcal{H}_g &= \mathcal{H}_q, \\
\mathcal{H}_p &= \mathcal{H}_f + 2\mathcal{H}_q - \mathcal{F}_1 - \mathcal{F}_5 - \mathcal{F}_9 - \mathcal{F}_{10}, \\
\mathcal{F}_1 &= \mathcal{H}_q - \mathcal{K}_3, \\
\mathcal{F}_2 &= \mathcal{H}_p - \mathcal{K}_2, \\
\mathcal{F}_3 &= \mathcal{H}_p - \mathcal{K}_6, \\
\mathcal{F}_4 &= \mathcal{H}_p - \mathcal{K}_7, \\
\mathcal{F}_5 &= \mathcal{H}_q - \mathcal{K}_5, \\
\mathcal{F}_6 &= \mathcal{K}_8, \\
\mathcal{F}_7 &= \mathcal{K}_9, \\
\mathcal{F}_8 &= \mathcal{K}_{10}, \\
\mathcal{F}_9 &= \mathcal{H}_q - \mathcal{K}_1, \\
\mathcal{F}_{10} &= \mathcal{H}_q - \mathcal{K}_4,
\end{align*}$$

The corresponding symmetry roots then become

$$\begin{align*}
\alpha_0 &= 2\mathcal{H}_q + \mathcal{H}_p - \mathcal{K}_1 - \mathcal{K}_3 - \mathcal{K}_4 - \mathcal{K}_8 - \mathcal{K}_9 - \mathcal{K}_{10}, \\
\alpha_1 &= 2\mathcal{H}_q + \mathcal{H}_p - \mathcal{K}_1 - \mathcal{K}_3 - \mathcal{K}_4 - \mathcal{K}_5 - \mathcal{K}_6 - \mathcal{K}_7, \\
\alpha_2 &= \mathcal{K}_3 - \mathcal{K}_2,
\end{align*}$$

and using the Period Map for the symplectic form $\omega = kdq \land dp$ we get the following root variables:

$$\begin{align*}
a_0 &= -k(\Theta_0 + \Theta_{\infty}), \\
a_1 &= -k(1 + \Theta_0 - \Theta_{\infty}), \\
a_2 &= 2k\Theta_0.
\end{align*}$$

Imposing the normalization condition $a_0 + a_1 + a_2 = 1$ gives $k = -1$, and so we get back the standard symplectic form $\omega^{\text{IP}} = dp \land dq$.

Now we can use the root variables to check whether we got the correct bases identification. We know the correspondence between the Okamoto parameters $\kappa_0$ and $\theta_{\infty}$, Painlevé parameters $\alpha$, $\beta$, and root variables $a_i$ from (3.3) and (3.4). We also have the correspondence between the Its-Prokhorov parameters $\Theta_0$ and $\Theta_{\infty}$ and Painlevé parameters $\alpha$, $\beta$ from (1.6), and between $\Theta_0$, $\Theta_{\infty}$ and $\kappa_0$ and $\theta_{\infty}$ from (4.1). In particular, we have $\Theta_0^2 = \frac{\alpha_0^2}{4}$, and so we need to choose the correct sign, which turns out to be a bit delicate, since the Its-Prokhorov system only depends on $\Theta^2$. However, the same parameters are also used in the Jimbo-Miwa system (1.13) considered in the next section, and from that parameter matching we see that we need to take $\kappa_0 = 2\Theta_0$. Then we get

$$\begin{align*}
a_0^{\text{IP}} &= \Theta_0 + \Theta_{\infty} = 1 + \theta_{\infty} = a_0^{\text{Ok}} + a_1^{\text{Ok}}, \\
a_1^{\text{IP}} &= 1 + \Theta_0 - \Theta_{\infty} = \kappa_0 - \theta_{\infty} = a_1^{\text{Ok}} + a_2^{\text{Ok}}, \\
a_2^{\text{IP}} &= -2\Theta_0 = -\kappa_0 = -a_1^{\text{Ok}},
\end{align*}$$

(4.5)
where we chose $\sqrt{\kappa_0} = -\kappa_0$. Thus, we see that our root variables (and hence the bases identification) differ by a composition of a reflection $w_2$ and an automorphism $\sigma_1$ described in Theorem 12.

Remark 5. We are off by a sign in the action of $\sigma_1$ on $a_i$, but that’s due to some normalization choices and can be ignored at this point.

Acting by $\sigma_1 \circ w_2 = w_2 w_{\kappa_1} w w_{\kappa_2} w_{\kappa_3} w_{\kappa_4} w_{\kappa_5}$ on the identification (4.4) we arrive at the final bases identification.

Lemma 5. The change of bases for Picard lattices between the Its-Prokhorov and the Okamoto (with two additional blowup points) surfaces is given by

$$
\begin{align*}
\mathcal{H}_f &= \mathcal{H}_q, \\
\mathcal{H}_g &= 2\mathcal{H}_q + \mathcal{H}_p - \mathcal{K}_1 - \mathcal{K}_2 - \mathcal{K}_4 - \mathcal{K}_5, \\
\mathcal{F}_1 &= \mathcal{K}_6, \\
\mathcal{F}_2 &= \mathcal{K}_7, \\
\mathcal{F}_3 &= \mathcal{H}_q - \mathcal{K}_2, \\
\mathcal{F}_4 &= \mathcal{K}_3, \\
\mathcal{F}_5 &= \mathcal{H}_q - \mathcal{K}_5, \\
\mathcal{F}_6 &= \mathcal{K}_8, \\
\mathcal{F}_7 &= \mathcal{K}_9, \\
\mathcal{F}_8 &= \mathcal{K}_{10}, \\
\mathcal{F}_9 &= \mathcal{H}_q - \mathcal{K}_1, \\
\mathcal{F}_{10} &= \mathcal{H}_q - \mathcal{K}_4.
\end{align*}
$$

This results in the following correspondences between the surface roots (note that we need to move an additional blowup point from the divisor $d_1$ to the divisor $d_5 = F_3 + F_4$),

$$
\begin{align*}
\delta_0 &= \mathcal{F}_7 - \mathcal{F}_8 = \mathcal{K}_9 - \mathcal{K}_{10}, \\
\delta_1 &= \mathcal{F}_1 - \mathcal{F}_2 = \mathcal{K}_5 - \mathcal{K}_7, \\
\delta_2 &= \mathcal{H}_f - \mathcal{F}_1 - \mathcal{F}_5 = \mathcal{K}_5 - \mathcal{K}_6, \\
\delta_3 &= \mathcal{F}_5 - \mathcal{F}_6 - \mathcal{F}_{10} = \mathcal{K}_4 - \mathcal{K}_5 - \mathcal{K}_8, \\
\delta_4 &= \mathcal{H}_g - \mathcal{F}_3 - \mathcal{F}_5 = \mathcal{H}_p - \mathcal{K}_1 - \mathcal{K}_4, \\
\delta_5 &= \mathcal{F}_3 - \mathcal{F}_4 - \mathcal{F}_9 = \mathcal{K}_1 - \mathcal{K}_2 - \mathcal{K}_3, \\
\delta_6 &= \mathcal{F}_6 - \mathcal{F}_7 = \mathcal{K}_8 - \mathcal{K}_9.
\end{align*}
$$

and the symmetry roots,

$$
\begin{align*}
\alpha_0 &= \mathcal{H}_f + \mathcal{H}_g - \mathcal{F}_5 - \mathcal{F}_6 - \mathcal{F}_7 - \mathcal{F}_8 = 2\mathcal{H}_q + \mathcal{H}_p - \mathcal{K}_1 - \mathcal{K}_2 - \mathcal{K}_4 - \mathcal{K}_8 - \mathcal{K}_9 - \mathcal{K}_{10}, \\
\alpha_1 &= \mathcal{H}_f - \mathcal{F}_3 - \mathcal{F}_4 = \mathcal{K}_2 - \mathcal{K}_3, \\
\alpha_2 &= \mathcal{H}_g - \mathcal{F}_1 - \mathcal{F}_2 = 2\mathcal{H}_q + \mathcal{H}_p - \mathcal{K}_1 - \mathcal{K}_2 - \mathcal{K}_4 - \mathcal{K}_5 - \mathcal{K}_6 - \mathcal{K}_7.
\end{align*}
$$

The symplectic form is the standard one, $\omega^{IP} = dp \wedge dq$, and the root variables match,

$$
a_0^{IP} = \Theta_\infty - \Theta_0 = 1 + \theta_\infty - \kappa_0 = a_0^{Ok}, \quad a_1^{IP} = 2\Theta_0 = 2\kappa_0 = a_1^{Ok}, \quad a_2^{IP} = 1 - \Theta_0 - \Theta_\infty = -\theta_\infty = a_2^{Ok}.
$$

It remains to find the birational change of variables that corresponds to this change of bases of the Picard lattice. It is given in the following Lemma.

Lemma 6. The change of coordinates and parameter matching between the Its-Prokhorov and Okamoto Hamiltonian systems is given by

$$
\begin{align*}
g(f,g,t) &= f, \\
p(f,g,t) &= g - \frac{f}{4} - \frac{t}{2} - \frac{\kappa_0}{2f}, \\
\Theta_0 &= \frac{\kappa_0}{2}, \\
\Theta_\infty &= 1 + \theta_\infty - \frac{\kappa_0}{2}, \\
f(q,p,t) &= q, \\
g(q,p,t) &= p + \frac{q}{4} + \frac{t}{2} + \frac{\Theta_0}{q},
\end{align*}
$$

and conversely

$$
\begin{align*}
\kappa_0 &= 2\Theta_0, \\
\theta_\infty &= -1 + \Theta_0 + \Theta_\infty.
\end{align*}
$$
Proof. The proof here is a standard computation, see detailed examples in [DT18, DFS20], but to make this paper self-contained, we briefly outline the argument here as well. From the change of basis we see that the coordinate classes are

\[ \mathcal{H}_f = \mathcal{H}_q, \quad \mathcal{H}_q = 2\mathcal{H}_q + \mathcal{H}_p - \mathcal{K}_1 - \mathcal{K}_2 - \mathcal{K}_4 - \mathcal{K}_5. \]

Thus, up to a Möbius transformation, the coordinates \( f \) and \( q \) coincide,

\[ f = \frac{Aq + B}{Cq + D}, \]

where \( A, \ldots, D \) are the parameters of the Möbius transformation, defined up to a common multiple, that we still need to find. The \( g \)-coordinate is more interesting — it is a (projective) coordinate on a pencil of \((2,1)\)-curves passing through the points \( y_1, y_2, y_4, \) and \( y_5 \). A generic \((2,1)\)-curve, written in the affine \((q,p)\) chart, has the equation

\[ a_{21}q^2p + a_{20}q^2 + a_{11}qp + a_{10}q + a_{01}p + a_{00} = 0, \]

and imposing the conditions given by points \( y_1, y_2, y_4, y_5 \) reduces this equation to

\[ a_{20}(q^2 + 4qp + 4\Theta_0) + a_{10}q = 0. \]

Thus, equations \( q = 0 \) and \( q^2 + 4qp + 4\Theta_0 = 0 \) define two basis curves in this pencil, and so

\[ g(q,p) = \frac{Kq + L(q^2 + 4qp + 4\Theta_0)}{M + N(q^2 + 4qp + 4\Theta_0)}, \]

where \( K, L, M, N \) are again some parameters to be determined. Let now \( \varphi : (q,p) \to (f,g) \) be our change of variables, and consider the induced forward mapping on the (unique irreducible) divisors corresponding to classes of \(-3, -2, \) and \(-1\) curves. For example, \( \varphi_*(\mathcal{H}_p - \mathcal{K}_4) = \mathcal{H}_g - \mathcal{F}_3 - \mathcal{F}_5 \) means that the \(-2\) curve \( \mathcal{H}_p - \mathcal{K}_1 - \mathcal{K}_4 \), whose projection down to \( \mathbb{P}^1 \times \mathbb{P}^1 \) is given in the \((q,f)\)-chart by the equation \( P = 0 \) (and parameterized by \( g \)) should map on the \(-2\) curve \( \mathcal{H}_g - \mathcal{F}_3 - \mathcal{F}_5 \) whose projection in the \((f,G)\)-chart is given by the equation \( G = 0 \) (and parameterized by \( f \)). Thus,

\[ (f,G)(q,P = 0) = \left( \frac{Aq + B}{Cq + D}, \frac{P(Mq + N(q^2 + 4\Theta_0)) + 4Nq}{P(Kq + L(q^2 + 4\Theta_0)) + 4Lq} \right) \bigg|_{P = 0} = \left( \frac{Aq + B}{Cq + D}, \frac{N}{L} \right) \]

implies that \( N = 0 \) (and hence \( M \neq 0 \) and we can take \( M = 1 \)). So \( g(q,p) = K + L \left( \frac{q + 4p + 4\Theta_0}{q} \right) \).

Similarly, \( \varphi_*(\mathcal{K}_0 - \mathcal{K}_10) = \mathcal{F}_7 - \mathcal{F}_8 \) means that the \(-2\)-curve \( \mathcal{K}_9 - \mathcal{K}_{10} \), given in the \((u_9,v_9)\) chart in the domain by the equation \( u_9 = 0 \) and parameterized by \( v_9 \), should collapse on the \( q_5 \leftarrow q_6 \leftarrow q_7 \) cascade. This results in \( C = 0, D = 1, A = 4L, B = 2(K + 2Ls - t) \) and so \( f(q,p) = 2(K + 2L(q + s) - t) \). Similar computations for \( \varphi_*(\mathcal{K}_5 - \mathcal{K}_7) = \mathcal{F}_1 - \mathcal{F}_2 \) imply that \( K = 2Ls \), and \( \varphi_*(\mathcal{K}_1 - \mathcal{K}_2 - \mathcal{K}_3) = \mathcal{F}_3 - \mathcal{F}_4 - \mathcal{F}_9 \) gives \( L = t/(4s) \). Finally, \( \varphi_*(\mathcal{K}_0) = \mathcal{K}_8 \) gives \( s = \pm t \), as expected, and we take \( s = t \) (we also expect this from the Painlevé equation) to get the final change of variables. The inverse change of variables is then immediate. \( \square \)

### 4.3 The Jimbo-Miwa Hamiltonian System

**Notation.** For this system we use the following notation: coordinates \((y,z)\), parameters \( \Theta_0 \) and \( \Theta_\infty \) (same as in Section 4.2); time variable \( s \); base points \( x_i \), exceptional divisors \( M_i \).

Consider now the system (1.13). It has eight base points,

\[ x_1(0,0), \quad x_2(0,2\Theta_0), \quad x_3(\infty, \Theta_0 + \Theta_\infty), \quad x_4(\infty, \infty) \leftarrow x_2(u_4 = 0, u_4 = 0) \leftarrow x_6(0, 2) \leftarrow y_2(0, -4s) \leftarrow x_8(0, 4(-1 + 2s^2 - \Theta_0 + \Theta_\infty)) \]

that lie on the polar divisor of a symplectic form \( \omega = k \frac{dz}{dy} \). Blowing them up we obtain the space of initial conditions for the Jimbo-Miwa system shown on Figure 5. Proceeding in the same way as in Section 4.2, but omitting the preliminary basis identification, we get the following result.
Lemma 7. The change of bases for Picard lattices between the Jimbo-Miwa and the Okamoto surfaces is given by
\[
\begin{align*}
\mathcal{H}_f &= \mathcal{H}_y, \\
\mathcal{H}_g &= 2\mathcal{H}_y + \mathcal{H}_z - \mathcal{M}_2 - \mathcal{M}_4 - \mathcal{M}_5 - \mathcal{M}_6, \\
\mathcal{J}_1 &= \mathcal{M}_7, \\
\mathcal{J}_2 &= \mathcal{M}_8, \\
\mathcal{J}_3 &= \mathcal{H}_y - \mathcal{M}_2, \\
\mathcal{J}_4 &= \mathcal{M}_1, \\
\mathcal{J}_5 &= \mathcal{H}_y - \mathcal{M}_6, \\
\mathcal{J}_6 &= \mathcal{H}_y - \mathcal{M}_5, \\
\mathcal{J}_7 &= \mathcal{H}_f - \mathcal{M}_4, \\
\mathcal{J}_8 &= \mathcal{M}_3.
\end{align*}
\]

This results in the following correspondences between the surface roots,
\[
\begin{align*}
\delta_0 &= \mathcal{J}_7 - \mathcal{J}_8 = \mathcal{H}_y - \mathcal{M}_3 - \mathcal{M}_4, \\
\delta_1 &= \mathcal{J}_1 - \mathcal{J}_2 = \mathcal{M}_7 - \mathcal{M}_8, \\
\delta_2 &= \mathcal{H}_f - \mathcal{J}_1 - \mathcal{J}_5 = \mathcal{M}_6 - \mathcal{M}_7, \\
\delta_3 &= \mathcal{J}_5 - \mathcal{J}_6 = \mathcal{M}_5 - \mathcal{M}_6, \\
\delta_4 &= \mathcal{H}_g - \mathcal{J}_3 - \mathcal{J}_5 = \mathcal{H}_z - \mathcal{M}_4 - \mathcal{M}_5, \\
\delta_5 &= \mathcal{J}_3 - \mathcal{J}_4 = \mathcal{H}_y - \mathcal{M}_1 - \mathcal{M}_2, \\
\delta_6 &= \mathcal{J}_6 - \mathcal{J}_7 = \mathcal{M}_4 - \mathcal{M}_5; \\
\delta_7 &= \mathcal{H}_f - \mathcal{J}_3 - \mathcal{J}_5 - \mathcal{J}_6 = \mathcal{M}_2 - \mathcal{M}_3 - \mathcal{M}_4 - \mathcal{M}_5 - \mathcal{M}_6 - \mathcal{M}_7 - \mathcal{M}_8.
\end{align*}
\]

and the symmetry roots,
\[
\begin{align*}
\alpha_0 &= \mathcal{H}_f + \mathcal{H}_g - \mathcal{J}_5 - \mathcal{J}_6 - \mathcal{J}_7 - \mathcal{J}_8 = \mathcal{H}_z - \mathcal{M}_2 - \mathcal{M}_3, \\
\alpha_1 &= \mathcal{H}_f - \mathcal{J}_3 - \mathcal{J}_4 = \mathcal{M}_2 - \mathcal{M}_1, \\
\alpha_2 &= \mathcal{H}_g - \mathcal{J}_1 - \mathcal{J}_2 = 2\mathcal{H}_y + \mathcal{H}_z - \mathcal{M}_2 - \mathcal{M}_4 - \mathcal{M}_5 - \mathcal{M}_6 - \mathcal{M}_7 - \mathcal{M}_8.
\end{align*}
\]

The normalization \(a_0 + a_1 + a_2 = 1\) then results in \(k = 1\) for the symplectic form, and so we recover the symplectic form \(\omega^{\text{JM}} = (1/y)dy \wedge dz\). The root variables then match the Okamoto (and the Its-Prokhorov) ones,
\[
\begin{align*}
a_0^{\text{JM}} &= \Theta_\infty - \Theta_0 = a_0^{\text{IP}} = a_0^{\text{Ok}}, \\
a_1^{\text{JM}} &= 2\Theta_0 = a_1^{\text{IP}} = a_1^{\text{Ok}}, \\
a_2^{\text{JM}} &= 1 - \Theta_0 - \Theta_\infty = a_2^{\text{IP}} = a_2^{\text{Ok}}.
\end{align*}
\]

The corresponding birational change of variables is given in the following Lemma.
Lemma 8. The change of coordinates and parameter matching between the Jimbo-Miwa and Okamoto Hamiltonian systems is given by

\[
\begin{align*}
    \begin{cases}
        y(f, g, t) = f, \\
        z(f, g, t) = \frac{f^2}{2} - fg + ft + \kappa_0, \\
        \Theta_0 = \frac{\kappa_0}{2}, \\
        \Theta_\infty = 1 + \theta_\infty - \frac{\kappa_0}{2},
    \end{cases}
\end{align*}
\]

and conversely

\[
\begin{align*}
    \begin{cases}
        f(y, z, t) = y, \\
        g(y, z, t) = \frac{y}{2} - \frac{z}{y} + t + \frac{2\Theta_0}{y},
    \end{cases}
\end{align*}
\]

(4.13)

Remark 6. We also note that knowing the configuration of the base points amounts to knowing the correct symplectic structure, from which we can then directly obtain the Hamiltonian, as was observed in [DFLS21]. Indeed, equations (1.11 – 1.11) are clearly not Hamiltonian w.r.t. the “standard” symplectic form \(dz \wedge dy\). However, using the correct form obtained in Lemma 7, we get

\[
\begin{align*}
    \frac{\partial H}{\partial z} &= \frac{dy}{ydt} = \frac{4z}{y} - y - 2t - \frac{4\Theta_0}{y}, \\
    \frac{\partial H}{\partial y} &= \frac{dz}{ydt} = -\frac{2z^2}{y^2} - z + \frac{4\Theta_0 z}{y^2} + (\Theta_0 + \Theta_\infty),
\end{align*}
\]

and therefore

\[
H(y, z; t) = \frac{2z^2}{y} - yz - 2tz - \frac{4\Theta_0 z}{y} + (\Theta_0 + \Theta_\infty)y = H^{JM}(y, z; t) - (\Theta_0 + \Theta_\infty)t,
\]

and so we recovered the Jimbo-Miwa Hamiltonian, up to \(t\)-dependent terms.

4.4 The Kecker Hamiltonian System

Notation. For this system we use the following notation: coordinates \((x, y)\), parameters \(\alpha\) and \(\beta\); time variable \(z\); base points \(w_i\), exceptional divisors \(N_i\). The space of initial conditions for this system is again not minimal and has ten base points, but it looks quite different from the other examples. The base points come in three cascades originating from a single point \(w_1(\infty, \infty)\), reflecting the cubic nature of \(H^{Kek}(x, y; z)\). Because of that, it is convenient to introduce the cubic roots of unity,

\[
\xi_0 = 1, \quad \xi_1 = e^{\frac{2\pi i}{3}} = -1 + i\sqrt{3}, \quad \xi_2 = e^{\frac{4\pi i}{3}} = -1 - i\sqrt{3}.
\]

(4.14)

Then the base point cascades are

\[
\begin{align*}
    w_2(-\xi_0, 0) &\quad w_3(-z, 0) &\quad w_4(-\beta - (1 + z^2)\xi_0 + \alpha\xi_0, 0) \\
    w_1(\infty, \infty) &\quad w_5(-\xi_2, 0) &\quad w_6(-z, 0) &\quad w_7(-\beta - (1 + z^2)\xi_1 + \alpha\xi_2, 0) \\
    w_8(-\xi_1, 0) &\quad w_9(-z, 0) &\quad w_{10}(-\beta - (1 + z^2)\xi_2 + \alpha\xi_1, 0)
\end{align*}
\]

that lie on the polar divisor of a symplectic form \(\omega = kdx \wedge dy\). The space of initial conditions is shown on Figure 6. Note that this time we have a curve \(N_1 - N_2 - N_5 - N_8\) with self-intersection index \(-4\), and it is clear that we should blow down two \(-1\) curves intersecting with it, with the obvious choice being the inaccessible divisors \(H_x - N_1\) and \(H_y - N_2\).
The process here is analogous to the one described in Section 4.2. We add two additional blowup points to the divisor $d_4 = F_5 - F_6$ corresponding to the central node of the surface Dynkin diagram $E_6^{(1)}$ and then find the change of basis of the Picard lattices that matches the irreducible components of the anti-canonical divisor, using the root variables and the period map to find the correct identification between the three branches of the $E_6^{(1)}$ diagrams. We only state the result.

**Lemma 9.** The change of bases for Picard lattices between the Kecker and the Okamoto (with two additional blowup points) surfaces is given by

$$
\begin{align*}
\mathcal{H}_f &= \mathcal{H}_x + \mathcal{H}_y - N_1 - N_2, & \mathcal{H}_x &= \mathcal{H}_f + \mathcal{H}_g - F_9 - F_{10}, \\
\mathcal{H}_g &= \mathcal{H}_x + \mathcal{H}_y - N_1 - N_5, & \mathcal{H}_y &= \mathcal{H}_f + \mathcal{H}_g - F_5 - F_9, \\
\mathcal{F}_1 &= N_6, & N_1 &= \mathcal{H}_f + \mathcal{H}_g - F_5 - F_9 - F_{10}, \\
\mathcal{F}_2 &= N_7, & N_2 &= \mathcal{H}_g - F_5, \\
\mathcal{F}_3 &= N_3, & N_3 &= \mathcal{F}_3, \\
\mathcal{F}_4 &= N_4, & N_4 &= \mathcal{F}_4, \\
\mathcal{F}_5 &= \mathcal{H}_x + \mathcal{H}_y - N_1 - N_2 - N_5, & N_5 &= \mathcal{H}_f - F_5, \\
\mathcal{F}_6 &= N_8, & N_6 &= \mathcal{F}_1, \\
\mathcal{F}_7 &= N_9, & N_7 &= \mathcal{F}_2, \\
\mathcal{F}_8 &= N_{10}, & N_8 &= \mathcal{F}_6, \\
\mathcal{F}_9 &= \mathcal{H}_x - N_1, & N_9 &= \mathcal{F}_7, \\
\mathcal{F}_{10} &= \mathcal{H}_y - N_1, & N_{10} &= \mathcal{F}_8.
\end{align*}
$$

This results in the following correspondences between the surface roots,

$$
\begin{align*}
\delta_0 &= \mathcal{F}_7 - \mathcal{F}_8 = N_9 - N_{10}, & \delta_4 &= \mathcal{H}_g - \mathcal{F}_3 - \mathcal{F}_5 = N_2 - N_3, \\
\delta_1 &= \mathcal{F}_7 - \mathcal{F}_2 = N_6 - N_7, & \delta_5 &= \mathcal{F}_3 - \mathcal{F}_4 = N_3 - N_4, \\
\delta_2 &= \mathcal{H}_f - \mathcal{F}_1 - \mathcal{F}_5 = N_5 - N_6, & \delta_6 &= \mathcal{F}_6 - \mathcal{F}_7 = N_8 - N_9; \\
\delta_3 &= \mathcal{F}_5 - \mathcal{F}_6 - \mathcal{F}_9 - \mathcal{F}_{10} = N_1 - N_2 - N_5 - N_8,
\end{align*}
$$

and the symmetry roots,

$$
\begin{align*}
\alpha_0 &= \mathcal{H}_f + \mathcal{H}_g - \mathcal{F}_5 - \mathcal{F}_6 - \mathcal{F}_7 - \mathcal{F}_8 = \mathcal{H}_x + \mathcal{H}_y - N_1 - N_8 - N_9 - N_{10}, \\
\alpha_1 &= \mathcal{H}_f - \mathcal{F}_3 - \mathcal{F}_4 = \mathcal{H}_x + \mathcal{H}_y - N_1 - N_2 - N_3 - N_4, \\
\alpha_2 &= \mathcal{H}_g - \mathcal{F}_1 - \mathcal{F}_2 = \mathcal{H}_x + \mathcal{H}_y - N_1 - N_5 - N_6 - N_7.
\end{align*}
$$
The symplectic form here is \( \tilde{\omega}^{\text{Ke}} = (1/3)dy \wedge dx \), and the root variables match, \( a_0^{\text{Ke}} = \frac{\xi_0 + \beta \xi_1 - \alpha \xi_2}{3} = \frac{1}{3} + \frac{\alpha - \beta}{6} + \frac{i(\alpha + \beta)}{2\sqrt{3}} = 1 + \theta_\infty - \kappa_0 = a_0^{\text{Ok}}, \)
\( a_1^{\text{Ke}} = \frac{\xi_0 + \beta \xi_0 - \alpha \xi_0}{3} = \frac{1 - \alpha + \beta}{3} = 2\kappa_0 = a_1^{\text{Ok}}, \)
\( a_2^{\text{Ke}} = \frac{\xi_0 + \beta \xi_2 - \alpha \xi_1}{3} = \frac{1}{3} + \frac{\alpha - \beta}{6} - \frac{i(\alpha + \beta)}{2\sqrt{3}} = -\theta_\infty = a_2^{\text{Ok}}. \)

To find the corresponding birational change of variables, we need to keep in mind that we also have to find the relationship between the independent variables that enter geometrically as coordinates of the blowup points, so we include some details in the proof.

**Lemma 10.** The change of coordinates and parameter matching between the Kecker and Okamoto Hamiltonian systems is given by

\[
\begin{align*}
x(f, g, t) &= \frac{(1 + i)}{4(3)^{3/4}} (3(\sqrt{3} - i)f + 12h + 2(\sqrt{3} - 3i)t), \\
y(f, g, t) &= \frac{(1 + i)}{4(3)^{3/4}} (3(\sqrt{3} + i)f - 12h + 2(\sqrt{3} + 3i)t), \\
z(t) &= \left(-\frac{3}{4}\right)^{1/4} t,
\end{align*}
\]

and conversely,

\[
\begin{align*}
f(x, y, z) &= \frac{1}{3^{1/4}} (x + y - z), \\
g(x, y, z) &= \frac{(1 + i)}{4(3)^{3/4}} \left((\sqrt{3} + i)x - (\sqrt{3} - i)y + 2iz\right), \\
t(z) &= \left(-\frac{3}{4}\right)^{1/4} z,
\end{align*}
\]

\[
\begin{align*}
\alpha &= \frac{3(\sqrt{3} - 3i) - 6i\theta_\infty - 3(\sqrt{3} - i)\kappa_0}{2\sqrt{3}}, \\
\beta &= \frac{(-\sqrt{3} - 3i) - 6i\theta_\infty + 3(\sqrt{3} + i)\kappa_0}{2\sqrt{3}}, \\
\kappa_0 &= \frac{1 - \alpha + \beta}{3}, \\
\theta_\infty &= \frac{-2 + (\sqrt{3}i - 1)\alpha + (\sqrt{3}i + 1)\beta}{6}.
\end{align*}
\]

**Proof.** Let \( \varphi : (g, p) \rightarrow (f, g) \) be the change of variables that induces the above change of bases of the Picard lattice. In the \((x, y)\)-chart, the equations of the base curves for the pencil \(|\mathcal{H}_f| = |\mathcal{H}_x + \mathcal{H}_y - N_1 - N_2| \) can be taken to be \( y + \xi_0x \) and 1, and for the pencil \(|\mathcal{H}_g| = |\mathcal{H}_x + \mathcal{H}_y - N_1 - N_5| \) we can take \( y + \xi_2x \) and 1. Thus, up to the Möbius transformations, we get

\[
\begin{align*}
f(x, y) &= \frac{A(y + \xi_0x) + B}{C(y + \xi_0x) + D}, \\
g(x, y) &= \frac{K(y + \xi_2x) + L}{M(y + \xi_2x) + N}.
\end{align*}
\]

Using the correspondence between the \(-4\) and \(-2\) curves \( \delta \) given by (4.16) we can evaluate the coefficients \( A, \ldots, N \) to get

\[
\begin{align*}
f(x, y) &= \frac{-2t\xi_1\xi_2(x + \xi_0^2y - \xi_0z)}{\xi_0z(\xi_0 - \xi_2)(\xi_1 - \xi_0)}, \\
g(x, y) &= \frac{-2t\xi_1\xi_2(x + \xi_0^2y - \xi_0z)}{\xi_0z(\xi_0 - \xi_2)(\xi_1 - \xi_0)},
\end{align*}
\]

or, after some simplification with \( \xi_i \),

\[
\begin{align*}
f(x, y) &= \frac{2t(x + y - z)}{z(\xi_1 - 1)(\xi_2 - 1)}, \\
g(x, y) &= \frac{-t\xi_1(x + \xi_1y - \xi_2z)}{z(\xi_1 - 1)(\xi_2 - 1)}.
\end{align*}
\]

Next, from the divisor matching \( \varphi_*(N_7) = F_2, \varphi_*(N_4) = F_4, \) and \( \varphi_*(N_{10}) = F_8 \) we get a correspondence between the coordinates of the blowup points, which in turn gives us the following relationship between
parameters:

$$\theta_\infty = \frac{2t^2(\beta + \xi_1 - \xi_2 \alpha)}{3z^2(\xi_2 - 1)}, \quad \kappa_0 = -\frac{2t^2 \xi_1 (\beta + 1 - \alpha)}{3z^2(\xi_2 - 1)}, \quad 1 + 2t^2 + \theta_\infty - \kappa_0 = \frac{2t^2 (3z^2 (\xi_2 - 1) + \alpha - \xi_1 - \xi_2 \beta)}{3z^2(\xi_2 - 1)}.$$  

These equations then give us the required parameter matching, confirming the root variable matching in Lemma 4.15 and the parameter matching in (1.19), as well as the relation between the different time variables \( t \) (for the Okamoto System) and \( z \) (for the Kecker system), \( \sqrt{3}z^2 = 2t^2 \), i.e., \( z = \left(-\frac{4}{3}\right)^{\frac{1}{2}} t \), as expected.

Substituting expressions (4.14) for \( \xi \), then gives (4.19) and (4.18) is established directly by inverting the change of coordinates.

4.5 The Filipuk-Żołądek Hamiltonian System

_Notation._ For this system we use the following notation: coordinates \((x, y)\), parameters \(\alpha\) and \(\beta\) (same as in the standard PIV equation (1.1)); time variable \(s\); base points \(z_i\). For the construction of the space of initial conditions, it is convenient to put \(\beta = -2 \gamma^2\).

The space of initial conditions for the system is essentially the same as the one for the Its-Prokhorov system constructed in Section 4.2 and shown on Figure 4, the only difference is the coordinates of the base points. We get the following ten points:

\[
\begin{align*}
z_2(-2\gamma, 0) & \quad z_5(0, -1) & \quad z_6(0, 2s) & \quad z_7(0, 2(1 - 2s^2 - \alpha)) \\
z_1(0, \infty) & \quad z_3(2\gamma, 0) & \quad z_4(\infty, \infty) & \quad z_8(0, 1) & \quad z_9(0, -2s) & \quad z_{10}(0, 2(1 + 2s^2 + \alpha))
\end{align*}
\]

(4.20)

The issue here is a different scaling normalization of the coordinates. Let us introduce the scaling gauge action \(\tilde{x} = \lambda x, \, \tilde{y} = \mu y\) on the coordinates \((x, y)\) for the Filipuk-Żołądek system. Then for the point \(z_8\) in (4.20) we get

\[
\dot{v}_4^{\text{FZ}} = \frac{\dot{Y}}{X} = \frac{\lambda x}{\mu y} = \frac{\lambda}{\mu}v_4^{\text{FZ}}, \quad \dot{v}_4^{\text{FZ}}(z_8) = \frac{\lambda}{\mu}v_4^{\text{FZ}}(z_8) = \frac{\lambda}{\mu}.
\]

Thus, to match it with \(v_4^{\text{IP}}(y_8) = 4\) in (4.20) we should put \(\lambda/\mu = 4\). Proceeding in a same way, we get \(\dot{v}_8^{\text{FZ}}(z_9) = 4\lambda v_8^{\text{FZ}}(z_9) = -8s\), and so if we want to match with \(v_8^{\text{IP}}(y_9) = -8s\), as well have the same time variable \(s\), we should put \(\lambda = 1\) and \(\mu = 1/4\). Thus, we get

\[
q = \tilde{x} = x, \quad p = \tilde{y} = \frac{y}{4}, \quad \omega^{\text{IP}} = dp \land dq = (1/4)dy \land dx = \omega^{\text{FZ}}.
\]

To match the parameters, note that

\[
2\gamma = U_1^{\text{FZ}}(z_3) = x(z_3)y(z_3) = 4\tilde{x}(z_3)\tilde{y}(z_3) = 4\tilde{U}_1^{\text{FZ}}(z_3) = 4U_1^{\text{IP}}(y_3) = 4\theta_0 \quad \implies \quad \Theta_0 = \frac{\gamma}{4} = \frac{-\beta}{8},
\]

and

\[
2(1 + 2s^2 + \alpha) = v_9^{\text{FZ}}(z_{10}) = \frac{\mu}{\lambda^3}v_9^{\text{FZ}}(z_{10}) = \frac{\mu}{\lambda^3}v_9^{\text{IP}}(y_{10}) = \frac{1}{4} \cdot 16(s^2 + \Theta_\infty) \quad \implies \quad \Theta_\infty = \frac{1 + \alpha}{2},
\]

which is exactly the relationship between parameters in (1.9). Note that an additional coefficient of 4 in front of \(H^{\text{IP}}\) is related to the normalization of the symplectic form, similar to the Kecker case that is explained in detail in the proof of Theorem 3, and the remaining term is purely time-dependent.
5 Conclusion

In this paper we showed that Sakai’s geometric theory of Painlevé equations is an effective tool in studying the relationship between different Hamiltonian systems related to the same differential Painlevé equation, if the systems are related by a birational change of variables. Specifically, we formulated an essentially algorithmic procedure on how to obtain this change of variables explicitly via the geometry identification between the corresponding spaces of initial conditions. Although the procedure is illustrated for the systems related to $P_{IV}$, it is very general and can be used for other systems as well.

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A The Geometry of the Standard $E_6^{(1)}$ Surface

In this section we describe the geometry of the model example of $E_6^{(1)}$ Sakai surface, following the standard reference [KNY17]. This surface is the Okamoto space of initial conditions for $P_{IV}$. Its symmetry group is the extended affine Weyl group $A^{(1)}_2$. Here we describe the standard configuration of the blowup points, the choice of the surface and the symmetry root bases in the Picard lattice, and the birational representation of the symmetry group $\tilde{W}(A^{(1)}_2)$. Recall that each nontrivial family of Sakai surfaces is obtained by blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ at 8 points lying on the polar divisor of some symplectic form. The Picard lattice of the resulting rational algebraic surface $\mathcal{X}$ has rank 10 and is $\text{Pic}(\mathcal{X}) = \text{Span}_\mathbb{Z} \{H_1, H_2, E_1, \ldots, E_8\}$, where $H_i$ stand for the classes of coordinate divisors and $E_i$ are the exceptional divisors, or the classes of central fibers of the blowup points. The anti-canonical divisor class then is $-K_{\mathcal{X}} = 2H_1 + 2H_2 - E_1 - \cdots - E_8$, and surfaces of different types correspond to different configurations of irreducible components of the anti-canonical divisor [Sak01].

A.1 The Point Configuration

Consider the following decomposition of the anti-canonical divisor class $\delta = -K_{\mathcal{X}}$ into classes $\delta_i$ (surface roots) of the irreducible components $d_i$ of the anti-canonical divisor $-K_{\mathcal{X}}$:

$$
\delta = \delta_0 + \delta_1 + 2\delta_2 + 3\delta_3 + 2\delta_4 + \delta_5 + 2\delta_6
$$

$$
= (E_7 - E_8) + (E_1 - E_2) + 2(H_q - E_1 - E_5) + 3(E_5 - E_6) + 2(H_p - E_3 - E_5) + (E_3 - E_4) + 2(E_6 - E_7).
$$

The intersection configuration of those roots is given by the Dynkin diagram of type $E_6^{(1)}$, as shown on Figure 7.

![Figure 7: The Surface Root Basis for the standard $E_6^{(1)}$ Sakai surface](image)

\begin{align}
\delta_0 &= E_7 - E_8, & \delta_4 &= H_p - E_3 - E_5, \\
\delta_1 &= E_1 - E_2, & \delta_5 &= E_3 - E_4, \\
\delta_2 &= H_q - E_1 - E_5, & \delta_6 &= E_6 - E_7, \\
\delta_3 &= E_5 - E_6, & \delta_3 &= E_5 - E_6,
\end{align}
Consider the complex projective plane \( \mathbb{P}^1 \times \mathbb{P}^1 \) covered by four coordinate charts \((q,p), (Q,p), (q,P),\) and \((Q,P),\) where \(Q = 1/q\) and \(P = 1/p.\) Using the action of the gauge group \( \text{PGL}_2(\mathbb{C}) \times \text{PGL}_2(\mathbb{C}) \) of Möbius transformations, we can, without loss of generality, put the divisors \(d_2\) and \(d_4,\) where \( \delta_i = [d_i],\) to be the lines at infinity, \(d_2 = V(Q) = \{q = \infty\}\) and \(d_4 = V(P) = \{p = \infty\}\) and put the base points \(p_1, p_3,\) and \(p_5\) to be on the intersection of coordinate lines: \(p_1(\infty, 0), p_2(0, \infty),\) and \(p_5(\infty, \infty).\) We then get the blowup diagram for the standard example of Sakai surface of type \(E_6^{(1)}\) as shown on Figure 8.

**Remark 7.** Here we use the same coordinates \((q, p)\) as \([\text{KNY17}],\) but these coordinates should not be confused with the different coordinates \((q, p)\) for the Its-Prokhorov Hamiltonian system in (1.7) and Section 4.2.

![Figure 8: The standard \(E_6^{(1)}\) Sakai surface](image)

This point configuration can be parameterized by five parameters \(b_2, b_4, b_6, b_7,\) and \(b_8\) as follows:

\[
\begin{align*}
p_1(\infty, 0) &\leftrightarrow p_2(\infty, 0; qp = 2b_2), & p_3(0, \infty) &\leftrightarrow p_4(\infty, 0; qp = 2b_4), \\
p_5(\infty, \infty) &\leftrightarrow p_6(\infty, \infty; \frac{q}{p} = b_6) &\leftrightarrow p_7(\infty, \infty; \frac{q}{p} = b_6; \frac{q(q - b_6p)}{p} = b_7) \\
&\leftrightarrow p_8(\infty, \infty; \frac{q}{p} = b_6; \frac{q(q - b_6p) - b_7p}{p} = b_8). \\
\end{align*}
\]

(A.2)

The residual two-parameter gauge group acts on these configurations via rescaling,

\[
\begin{pmatrix}
b_2 \\
b_4 \\
b_6 \\
b_7 \\
b_8 \\
f \\
g
\end{pmatrix}
\sim
\begin{pmatrix}
\lambda & \mu & b_2 \\
\frac{\lambda}{\mu} & b_4 \\
\frac{\lambda}{\mu} & b_6 \\
\frac{\lambda^2}{\mu} & b_7 \\
\frac{\lambda^3}{\mu} & b_8 \\
\lambda & \mu & f \\
\lambda & \mu & g
\end{pmatrix}, \quad \lambda, \mu \neq 0,
\]

(A.3)

and so the true number of parameters is three. A canonical choice of such parameters is known as the root variables, as we explain next.

### A.2 The Period Map and the Root Variables

To define the root variables we need to choose a root basis in the symmetry sub-lattice \(Q = \Pi(R^1) \triangleleft \text{Pic}(\mathcal{X})\) and find a symplectic form \(\omega\) whose polar divisor \(-K_{\mathcal{X}}\) is the configuration of \(-2\)-curves shown on Figure 8. As usual, we take the same basis as in [\text{KNY17}], see Figure 9.

A symplectic form \(\omega \in -K_{\mathcal{X}}\) such that \(|\omega| = \delta_0 + \delta_1 + 2\delta_2 + 3\delta_3 + 2\delta_4 + \delta_5 + 2\delta_6\) can be written in the main affine \((q, p)-\)chart as \(\omega = kdq \wedge dp.\) Then a standard computation described in detail in [\text{DFS20}] gives us the following result.

**Lemma 11.** The root variables \(a_i\) are given by

\[
a_0 = k \frac{b_2^2 - b_6 b_8}{b_6}, \quad a_1 = -kb_4, \quad a_2 = kb_2.
\]

(A.5)
Reflections isomorphisms which can be extended to an action on point configurations by elementary birational maps (which lifts to Weyl symmetry group $\tilde{\mathfrak{W}}$).

For completeness, we also include here the description of the birational representation of the extended affine A.3 The Extended Affine Weyl Symmetry Group

As in [KNY17] or in [DFS20] and are omitted, we only state the final result.

The natural action of this group on $\operatorname{Pic}(X)$ is given by reflections in the roots $\alpha_i$,

$$w_i(\mathcal{C}) = w_{\alpha_i}(\mathcal{C}) = \mathcal{C} - 2 \frac{\mathcal{C} \cdot \alpha_i}{\alpha_i \cdot \alpha_i} \alpha_i = \mathcal{C} + (\mathcal{C} \cdot \alpha_i) \alpha_i, \quad \mathcal{C} \in \operatorname{Pic}(X),$$

which can be extended to an action on point configurations by elementary birational maps (which lifts to isomorphisms $w_i$ on the family of Sakai’s surfaces), this is known as a birational representation of $W(A_2^{(1)})$.

The group of Dynkin diagram automorphisms $\operatorname{Aut}(D_4^{(1)}) \simeq \mathbb{D}_3$, where $\mathbb{D}_3$ is the usual dihedral group. Thus we only describe two transpositions $\sigma_1, \sigma_2$ that generate the whole group.

**Theorem 12.** Reflections $w_i$ on $\operatorname{Pic}(X)$ are induced by the elementary birational mappings

$$w_0: \begin{pmatrix} a_0 & a_1 & q \\ a_2 & t & p \end{pmatrix} \mapsto \begin{pmatrix} -a_0 & a_0 + a_1 & q - \frac{a_0}{q - p + t} \\ a_0 + a_2 & t & p - \frac{a_0}{q - p + t} \end{pmatrix},$$

$$w_1: \begin{pmatrix} a_0 & a_1 & q \\ a_2 & t & p \end{pmatrix} \mapsto \begin{pmatrix} a_0 + a_1 & -a_1 & q - \frac{a_1}{q} \\ a_1 + a_2 & t & p - \frac{a_1}{q} \end{pmatrix},$$

$$w_2: \begin{pmatrix} a_0 & a_1 & q \\ a_2 & t & p \end{pmatrix} \mapsto \begin{pmatrix} a_0 + a_2 & a_1 + a_2 & q + \frac{a_2}{p} \\ -a_2 & a_1 + a_2 & q + \frac{a_2}{p} \end{pmatrix}.$$
For the automorphisms $\sigma_i$ we choose the following two generators whose induced action on $\text{Pic}(X)$ can be represented as a composition of reflections in roots (but no longer symmetry roots), and that act on the surface and symmetry root bases as follows:

$$\sigma_1 = w_{\varepsilon_2 - \varepsilon_3}w_{\varepsilon_1 - \varepsilon_4} \sim (\alpha_1 \alpha_2) \sim (\delta_1 \delta_2),$$

$$\sigma_2 = w_{\varepsilon_4 - \varepsilon_5}w_{\varepsilon_1 - \varepsilon_6} \sim (\alpha_0 \alpha_2) \sim (\delta_0 \delta_2).$$

The corresponding birational mappings then are

$$\sigma_1:\begin{pmatrix} a_0 & a_1 & q \\ a_2 & t & p \end{pmatrix} \mapsto \begin{pmatrix} -a_0 & -a_2 & -p \\ -a_1 & t & -q \end{pmatrix},$$

$$\sigma_2:\begin{pmatrix} a_0 & a_1 & q \\ a_2 & t & p \end{pmatrix} \mapsto \begin{pmatrix} -a_2 & -a_1 & q \\ -a_0 & t & q - p + t \end{pmatrix}.$$

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