THE GEOMETRY OF THE MODULI SPACE OF ODD SPIN CURVES

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The set of odd theta-characteristics on a general curve $C$ of genus $g$ is in bijection with the set $\theta(C)$ of theta hyperplanes $H \in (P^{g-1})^\vee$ everywhere tangent to the canonically embedded curve $C \to P^{g-1}$. Even though the geometry and the intricate combinatorics of $\theta(C)$ have been studied classically, see [Dol], [DK] for a modern account, it was only recently proved in [CS] that one can reconstruct a general curve $[C] \in M_g$ from the hyperplane configuration $\theta(C)$.

Odd theta-characteristics form a moduli space $\pi : S_g^- \to M_g$ which is an étale cover of degree $2g - 1$. The normalization of $M_g$ in the function field of $S_g^-$ gives rise to a finite covering $\pi : S_g^- \to M_g$. Furthermore, $S_g^-$ has a modular meaning being isomorphic to the coarse moduli space of the Deligne-Mumford stack of odd stable spin curves, cf. [C], [CCC], [AJ]. The map $\pi$ is branched along the boundary of $M_g$ and one expects $K_{S_g^-}$ to enjoy better positivity properties than $K_{M_g}$.

The aim of this paper is to describe the birational geometry of $S_g^-$ for all $g$. Our goals are (1) to understand the transition from rationality to maximal Kodaira dimension for $S_g^-$ as $g$ increases, and (2) to use the existence of Mukai models of $M_g$ in order to construct explicit unirational parameterizations of $S_g^-$. Remarkably, we end up having no gaps in the classification of $S_g^-$. First, we show that in the range where the general curve $[C] \in M_g$ lies on a $K3$ surface, the existence of special theta pencils on $K3$ surfaces, provides an explicit uniruled parameterization of $S_g^-:

Theorem 0.1. The odd spin moduli space $S_g^-$ is uniruled for $g \leq 11$.

When $g \leq 9$ or $g = 11$, a general spin curve $[C, \eta] \in S_g^-$ appears as a hyperplane section of a $K3$ surface $X \subset P^g$, such that if $d := \text{supp}(\eta)$ is the support of the theta-characteristic, then the linear span $\langle d \rangle \subset P^g$ is a codimension 2 linear subspace. A rational curve $P \subset S_g^-$ is induced by the pencil of hyperplanes $PH^0(X, I_d/X(C))$ containing $\langle d \rangle$. We show in Section 3 that $P \subset S_g^-$ is a covering rational curve, satisfying $P \cdot K_{S_g^-} = 2g - 24 < 0.$

Thus $P \cdot K_{S_g^-} < 0$ precisely when $g \leq 11$, which highlights the fact that the nature of $S_g^-$ is expected to change exactly when $g \geq 12$. This is something we shall achieve in the course of proving Theorem 0.3.

The previous argument no longer works for $S_{10}$, when the condition that a curve $[C] \in M_{10}$ lie on a $K3$ surface is divisorial [FP]. This case is in some sense a specialization of the genus 11 case. We use that a general 1-nodal irreducible curve $[C] \in \Delta_0 \subset \overline{M}_{11}$ of arithmetic genus 11, lies on a $K3$ surface $X \subset P^{11}$. By a degeneration argument,
we show that this construction can be also carried out in such a way, that if \( \nu : C' \to C \)
denotes the normalization of \( C \), then the points \( x, y \in C' \) with \( \nu(x) = \nu(y) \) (that is, mapping to the node of \( C \)), lie in the support of one of the odd-theta characteristics of \( [C'] \in M_{10} \). Ultimately, this produces a rational curve \( P \subset S_{10}^− \) through a general point, which shows that \( S_{10}^− \) is uniruled as well.

In the range in which a Mukai model of \( \overline{M}_g \) exists, our results are more precise:

**Theorem 0.2.** \( S_g^− \) is unirational for \( g \leq 8 \).

The proof relies on the existence, in this range, of Mukai varieties \( V_g \subset \mathbb{P}^{n_g + g - 2} \), where \( n_g = \dim(V_g) \), which have the property that general 1-dimensional linear sections of \( V_g \) are canonical curves \( [C] \in M_g \) with general moduli. We fix an integer \( 1 \leq \delta \leq g - 1 \) and consider the correspondence

\[ \mathcal{P}_{g,\delta}^o := \{(C, \Gamma, Z) : Z \subset C \cap \Gamma \subset V_g, |\text{sing}(\Gamma)| = \delta, \text{sing}(\Gamma) \subset Z, \} \]

where \( Z \subset V_g \) is a 0-dimensional subscheme of \( V_g \) of length \( 2g - 2 \), supported at \( g - 1 \) points and such that \( \dim(Z) = g - 2 \) (see Section 4 for a precise definition), \( \Gamma \subset V_g \) is an irreducible \( \delta \)-nodal curve section of \( V_g \) whose nodes are among the points in the support of \( Z \), and \( C \subset V_g \) is an arbitrary curve linear section of \( V_g \) containing \( Z \) as a subscheme. Thus if \( C \) is smooth, then \( Z \subset C \) is a divisor of even degree at each point in its support, and \( \mathcal{O}_C(Z/2) \) can be viewed as a theta-characteristic. The variety \( \mathcal{P}_{g,\delta}^o \) comes equipped with two projections

\[ S_g^− \leftarrow^\alpha \mathcal{P}_{g,\delta}^o \xrightarrow{\beta} B_g^−, \]

where \( B_g^− \subset S_g^− \) denotes the moduli space of irreducible \( \delta \)-nodal curves of arithmetic genus \( g \) together with an odd theta-characteristic on the normalization. It is easy to see that \( \mathcal{P}_{g,\delta}^o \) is birational to a projective bundle over the irreducible variety \( B_g^− \). Thus the unirationality of \( S_g^− \) follows once we prove that (i) \( \alpha \) is dominant, and (ii) \( B_g^− \), itself is unirational. We carry out this program when \( g \leq 8 \). In the process of proving Theorem 0.2 we establish some facts of independent interest concerning the Mukai models

\[ \mathbb{M}_g := G(g, n_g + g - 1)^{ss} // \text{Aut}(V_g). \]

These are birational models of \( \overline{M}_g \) having \( \text{Pic}(\mathbb{M}_g) = \mathbb{Z} \) and appearing as GIT quotients of Grassmannians; they can be viewed as log-minimal models of \( \overline{M}_g \) emerging from the constructions carried out in [M1], [M2], [M3].

Theorem 0.1 is sharp and the remaining moduli spaces \( S_g^− \) are of general type:

**Theorem 0.3.** The space \( S_g^− \) is a variety of general type for \( g > 11 \).

The border case of \( S_{12}^− \) is particularly challenging and takes up the entire Section 6. We remark that in the range \( 11 < g < 17 \), of the two moduli spaces \( S_g^− \) and \( \overline{M}_g \), one is of general type whereas the other has negative Kodaira dimension. More strikingly, Theorems 0.3 and 0.1 coupled with results from [FV], show that for \( 9 \leq g \leq 11 \), the space \( S_g^− \) is uniruled while \( S_g^+ \) is of general type! Finally, we note that \( S_8^− \) is unirational whereas \( S_8^+ \) is of Calabi-Yau type [FV].
We describe the main steps in the proof of Theorem 0.4. First, we use that for all \( g \geq 4 \) and \( l \geq 0 \), if \( \epsilon : \hat{S}_g \to \hat{S}_g^+ \) denotes a resolution of singularities, then there is an induced isomorphism, see [Lud]

\[
\epsilon^* : H^0(\hat{S}_g^+,-K_{\hat{S}_g}^l) \simeq H^0(\hat{S}_g,-K_{\hat{S}_g}^l).
\]

Thus to conclude that \( \hat{S}_g^+ \) is of general type, it suffices to exhibit an effective divisor \( D \) on \( \hat{S}_g^+ \) such that for appropriately chosen rational constants \( \alpha, \beta > 0 \), a relation of the type \( K_{\hat{S}_g^+} \equiv \alpha \lambda + \beta D + E \in \text{Pic}(\hat{S}_g^+) \) holds, where \( \lambda \in \text{Pic}(\hat{S}_g^+) \) is the pull-back to \( \hat{S}_g^+ \) of the Hodge class, and \( E \) is an effective \( \mathbb{Q} \)-class which is typically a combination of boundary divisors. It is essential to pick \( D \) so that (1) its class can be explicitly computed, that is, points in \( D \) have good geometric characterization, and (2) \( [D] \in \text{Pic}(\hat{S}_g^+) \) is in some way an extremal point of the effective cone of divisors so that the coefficients \( \alpha, \beta \) stand a chance of being positive. In the case of \( \hat{S}_g^+ \), the role of \( D \) is played by the divisor \( \mathcal{O}_\text{null} \) of vanishing theta-nulls, see [F3]. In the case of \( \hat{S}_g^+ \) we compute the class of degenerate theta-characteristics, that is, curves carrying a non-reduced odd theta-characteristic.

**Theorem 0.4.** We fix \( g \geq 3 \). The locus consisting of odd spin curves

\[ \mathcal{Z}_g := \{ [C, \eta] \in \mathcal{S}_g^- : \eta = \mathcal{O}_C(2x_1 + x_2 + \cdots + x_{g-2}) \text{ where } x_i \in C \text{ for } i = 1, \ldots, g-2 \} \]

is a divisor on \( \mathcal{S}_g^- \). The class of its compactification inside \( \mathcal{S}_g^- \) equals

\[ \bar{\mathcal{Z}}_g \equiv (g + 8) \lambda - \frac{g + 2}{4} \alpha_0 - 2 \beta_0 - \sum_{i=1}^{[g/2]} 2(g-i) \alpha_i - \sum_{i=1}^{[g/2]} 2i \beta_i \in \text{Pic}(\mathcal{S}_g^-), \]

where \( \lambda, \alpha_0, \beta_0, \ldots, \alpha_{[g/2]}, \beta_{[g/2]} \) are the standard generators of \( \text{Pic}(\mathcal{S}_g^-) \).

For low genus, \( \mathcal{Z}_g \) specializes to well-known geometric loci. For instance \( \bar{\mathcal{Z}}_3 \) is the divisor of hyperflexes on plane quartics. In particular, Theorem 0.4 yields the formula

\[ \pi_*([\bar{\mathcal{Z}}_3]) = 308 \lambda - 32 \delta_0 - 76 \delta_1 \in \text{Pic}(\mathcal{M}_3), \]

for the class of quartic curves having a hyperflex. This matches [Cu] formula (5.5). Moreover, one has the following relation in \( \text{Pic}(\mathcal{M}_3) \)

\[ \left\{ [C] \in \mathcal{M}_3 : \exists x \in C \text{ with } 4x \equiv K_C \right\} \equiv 8 \cdot \mathcal{M}_{3,2}^l + \pi_*([\bar{\mathcal{Z}}_3]), \]

where \( \mathcal{M}_{3,2}^l \equiv 9 \lambda - \delta_0 - 3 \delta_1 \) is the hyperelliptic class and the multiplicity 8 accounts for the number of hyperelliptic Weierstrass points.

We briefly explain how Theorem 0.4 implies that \( \hat{S}_g^+ \) is of general type for \( g \geq 11 \). We choose an effective divisor \( D \in \text{Eff}(\mathcal{M}_g) \) of small slope; for composite \( g + 1 \) one can take \( D = \mathcal{M}_{g,d}^r \) the closure of the Brill-Noether divisor of curves with a \( g_d^r \) where \( \rho(g, r, d) = -1 \); there exists a constant \( c_{g,d,r} > 0 \) such that [EH2],

\[ \mathcal{M}_{g,d}^r \equiv c_{g,d,r} \left( (g+3) \lambda - \frac{g+1}{6} \delta_0 - \sum_{i=1}^{[g/2]} i(g-i) \delta_i \right) \in \text{Pic}(\mathcal{M}_g). \]
We form the linear combination of divisors on $\mathfrak{S}_g$

$$\frac{2}{g-2} \mathcal{S}_g + \frac{3(3g-10)}{c_{g,d,r}(g-2)(g+1)} \pi^*(\mathcal{M}_{g,d}) = \frac{11g + 37}{g+1} \lambda - 2\alpha_0 - 3\beta_0 - \sum_{i=1}^{[g/2]} (a_i \cdot \alpha_i + b_i \cdot \beta_i),$$

where $a_i, b_i \geq 2$ for $i \neq 1$ and $a_1, b_1 > 3$ are explicitly known rational constants. The canonical class of $\mathfrak{S}_g$ is given by the Riemann-Hurwitz formula

$$K_{\mathfrak{S}_g} \equiv \pi^*(K_{\mathcal{M}_g}) + \beta_0 \equiv 13\lambda - 2\alpha_0 - 3\beta_0 - 2\sum_{i=1}^{[g/2]} (\alpha_i + \beta_i) - (\alpha_1 + \beta_1),$$

and by comparison, it follows that for $g > 12$ one can find a constant $\mu_g \in \mathbb{Q}_{>0}$ such that

$$K_{\mathfrak{S}_g} - \mu_g \cdot \lambda \in \mathbb{Q}_{\geq 0}(12, \alpha_1, \beta_1, \ldots, \alpha_{g/2}, \beta_{g/2}),$$

which shows that $K_{\mathfrak{S}_g}$ is big and thus proves Theorem 0.3.

For $g = 12$, there is no Brill-Noether divisor, and the reasoning above shows that in order to conclude that $\mathfrak{S}_{12}$ is of general type, one needs an effective divisor $\mathfrak{D}_{12}$ of slope $s(\mathfrak{D}_{12}) = 6 + 12/13$, that is, a counterexample to the Slope Conjecture. We define

$$\mathfrak{D}_{12} := \{ [C] \in \mathcal{M}_{12} : \exists L \in W_{14}(C) \text{ with } \text{Sym}^2 H^0(C, L) \xrightarrow{\mu_0(L)} H^0(C, L^\otimes 2) \text{ not injective} \},$$

that is, points in $\mathfrak{D}_{12}$ correspond to curves that admit an embedding $C \subset \mathbb{P}^4$ with $\deg(C) = 14$, such that $H^0(\mathbb{P}^4, \mathcal{I}_C/\mathcal{I}^4(2)) \neq 0$. The computation of the class of $\mathfrak{D}_{12} \subset \mathcal{M}_{12}$ is carried out in Section 6 and it turns out that $s(\mathfrak{D}_{12}) = \frac{445}{442} < 6 + \frac{12}{13}$. In particular $\mathfrak{D}_{12}$ violates the Slope Conjecture on $\mathcal{M}_{12}$, and as such, it contains the locus $\mathcal{K}_{12} := \{ [C] \in \mathcal{M}_{12} : C \text{ lies on a K3 surface} \}$.

1. FAMILIES OF STABLE SPIN CURVES

We briefly review some relevant facts about the moduli space $\mathfrak{S}_g$ that will be used throughout the paper, see also [C], [F3], [Lud] for details. As a matter of notation, we follow the convention set in [FL]; if $\mathcal{M}$ is a Deligne-Mumford stack, then we denote by $\mathcal{M}$ its associated coarse moduli space.

Following [C], a spin curve of genus $g$ consists of a triple $(X, \eta, \beta)$, where $X$ is a genus $g$ quasi-stable curve, $\eta \in \text{Pic}^{g-1}(X)$ is a line bundle of degree $g - 1$ such that $\eta_E = \mathcal{O}_E(1)$ for every exceptional component $E \subset X$, and $\beta : \eta^{\otimes 2} \to \omega_X$ is a sheaf homomorphism which is generically non-zero along each non-exceptional component of $X$.

It follows from the definition that if $(X, \eta, \beta)$ is a spin curve with exceptional components $E_1, \ldots, E_r$ and $\{ p_i, q_i \} = E_i \cap X - E_i$ for $i = 1, \ldots, r$, then $\beta_{| E_i } = 0$. Moreover, if $\tilde{X} := X - \bigcup_{i=1}^r E_i$ (viewed as a subcurve of $X$), then we have an isomorphism of sheaves $\eta_{\tilde{X}}^{\otimes 2} \simeq \omega_{\tilde{X}}$.

We denote by $\mathfrak{S}_g$ the non-singular Deligne-Mumford stack of spin curves of genus $g$, which obviously splits into two connected components $\mathfrak{S}_g^+$ and $\mathfrak{S}_g^-$ of relative degree $2g-1(2g+1)$ and $2g-1(2g-1)$ respectively. It is proved in [C] that the coarse moduli space of $\mathfrak{S}_g$ is isomorphic to the normalization of $\mathcal{M}_g$ in the function field of $S_g$. There
is a proper morphism $\pi : \overline{\mathcal{S}}_g \to \overline{\mathcal{M}}_g$ given by $\pi([X, \eta, \beta]) := [\text{st}(X)]$, where $\text{st}(X)$ denotes the stable model of the nodal curve $X$.

### 1.1. Spin curves of compact type.
We recall the description of the pull-back divisors $\pi^* (\Delta_i)$. We choose a spin curve $[X, \eta, \beta] \in \pi^{-1} ([C \cup_y D])$ where $[C, y] \in \mathcal{M}_{g-1,1}$ and $[D, y] \in \mathcal{M}_{g-1,1}$. Then necessarily $X := C \cup_{y_1} E \cup_{y_2} D$, where $E$ is an exceptional component such that $C \cap E = \{y_1\}$ and $D \cap E = \{y_2\}$. Moreover $\eta = (\eta_C, \eta_D, \eta_E = \mathcal{O}_E(1)) \in \text{Pic}^{g-1}(X)$, and since $\beta_E = 0$, it follows that $\eta_C^{\otimes 2} = K_C, \eta_D^{\otimes 2} = K_D$, that is, $\eta_C$ and $\eta_D$ are "honest" theta-characteristics on $C$ and $D$ respectively. The condition $h^0(X, \eta) \equiv 1 \text{ mod } 2$ implies that $\eta_C$ and $\eta_D$ must have opposite parities. We denote by $A_i \subset \overline{\mathcal{S}}_g$ the closure in $\overline{\mathcal{S}}_g$ of the locus corresponding to pairs $([C, \eta_C, y], [D, \eta_D, y]) \in S_{i,1}^{-} \times S_{g-i,1}^{+}$, and by $B_i \subset \overline{\mathcal{S}}_g$ the closure in $\overline{\mathcal{S}}_g$ of the locus corresponding to pairs $([C, \eta_C, y], [D, \eta_D, y]) \in S_{i,1}^{+} \times S_{g-i,1}^{-}$.

One has the relation $\pi^*(\Delta_i) = A_i + B_i$ and clearly $\deg(A_i/\Delta_i) = 2g-2(2^i - 1)(2^{g-i} + 1)$ and $\deg(B_i/\Delta_i) = 2^{g-2}(2^i + 1)(2^{g-i} - 1)$. One denotes $\alpha_i := [A_i], \beta_i := [B_i] \in \text{Pic}(\overline{\mathcal{S}}_g)$.

### 1.2. Spin curves with an irreducible stable model.
In order to describe $\pi^*(\Delta_0)$ we pick a point $[X, \eta, \beta]$ such that $\text{st}(X) = C_{yq} := C/y \sim q$, where $[C, y, q] \in \mathcal{M}_{g-1,2}$ is a general point of $\Delta_0$. Unlike the case of curves of compact type, here there are two possibilities depending on whether $X$ possesses an exceptional component or not. If $X = C_{yq}$ and $\eta_C := \nu^*(\eta)$ where $\nu : C \to X$ denotes the normalization map, then $\eta_C^{\otimes 2} = K_C(y + q)$. For each choice of $\eta_C \in \text{Pic}^{g-1}(C)$ as above, there is precisely one choice of gluing the fibres $\eta_C(y)$ and $\eta_C(q)$ such that $h^0(X, \eta) \equiv 1 \text{ mod } 2$. We denote by $A_0$ the closure in $\overline{\mathcal{S}}_g$ of the locus of those points $[C_{yq}, \eta_C \in \sqrt{K_C(y + q)}]$ with $\eta_C(y)$ and $\eta_C(q)$ glued as above. One has that $\deg(A_0/\Delta_0) = 2^{2g-2}$.

If $X = C \cup_{(y, q)} E$ where $E$ is an exceptional component, then since $\beta|_E = 0$ it follows that $\beta_C \in H^0(C, \omega_X|_C \otimes \eta_C^{\otimes 2})$ must vanish at both $y$ and $q$ and then for degree reasons $\eta_C := \eta \otimes \mathcal{O}_C$ is a theta-characteristic on $C$. The condition $H^0(X, \omega) \cong H^0(C, \omega_C) \equiv 1 \text{ mod } 2$ implies that $[C, \eta_C] \in S_{g-1}^{-}$. In an étale neighborhood of a point $[X, \eta, \beta]$, the covering $\pi$ is given by

$$(\tau_1, \tau_2, \ldots, \tau_{3g-3}) \mapsto (\tau_1^2, \tau_2, \ldots, \tau_{3g-3}),$$

where one identifies $\mathbb{C}^{3g-3}$ with the versal deformation space of $(X, \eta, \beta)$ and the hyperplane $(\tau_1 = 0) \subset \mathbb{C}^{3g-3}$ denotes the locus of spin curves where the exceptional component $E$ persists. This discussion shows that $\pi$ is simply branched over $\Delta_0$ and we denote the ramification divisor by $B_0 \subset \overline{\mathcal{S}}_g$, that is, the closure of the locus of spin curves $[C \cup_{(y, q)} E, (C, \eta_C) \in S_{g-1}^{-}, \eta_E = \mathcal{O}_E(1)]$. If $\alpha_0 = [A_0] \in \text{Pic}(\overline{\mathcal{S}}_g)$ and $\beta_0 = [B_0] \in \text{Pic}(\overline{\mathcal{S}}_g)$, we then have the relation

$$\pi^*(\delta_0) = \alpha_0 + 2\beta_0. \quad (1)$$

We define several test curves in the boundary of $\overline{\mathcal{S}}_g$ which will be later used to compute divisor classes on the moduli space.
1.3. The family $F_i$. We fix $1 \leq i \leq [g/2]$ and construct a covering family for the boundary divisor $A_i$. We fix general curves $[C] \in \mathcal{M}_i$ and $[D, q] \in \mathcal{M}_{g-i,1}$ as well as an odd theta-characteristic $\eta_C^+$ on $C$ and an even theta-characteristic $\eta_D^+$ on $D$. If $E \cong \mathbb{P}^1$ is a fixed exceptional component, we define the family of spin curves

$$F_i := \{ [C \cup_y E \cup_q D], \eta : \eta_C^+, \eta_E = \mathcal{O}_E(1), \eta_D^+ = \mathcal{O}_D, E \cap C = \{y\}, E \cap D = \{q\} \}_{y \in C}.$$ 

One has that $F_i \cdot \beta_i = 0$ and then $F_i \cdot \alpha_i = -2i + 2$; furthermore $F_i$ has intersection number zero with the remaining generators of $\text{Pic}(\Sigma_g^+)$. 

1.4. The family $G_i$. As above, we fix $1 \leq i \leq [g/2]$ and curves $[C] \in \mathcal{M}_i, [D, q] \in \mathcal{M}_{g-i,1}$. This time we choose an even theta-characteristic $\eta_C^+$ on $C$ and an odd theta-characteristic $\eta_D^+$ on $D$. The following family covers the divisor $B_i$:

$$G_i := \{ [C \cup_y E \cup_q D], \eta : \eta_C^+, \eta_E = \mathcal{O}_E(1), \eta_D^+ = \mathcal{O}_D, E \cap C = \{y\}, E \cap D = \{q\} \}_{y \in C}.$$ 

Clearly $G_i \cdot \alpha_i = 0, G_i \cdot \beta_i = 2 - 2i$ and $G_i \cdot \lambda = G_i \cdot \alpha_j = G_i \cdot \beta_j = 0$ for $j \neq i$. 

1.5. Two elliptic pencils. The boundary divisor $\Delta_1 \subset \overline{\mathcal{M}}_g$ is covered by a standard elliptic pencil $R$ obtained by attaching to a fixed general pointed curve $[C, y] \in \mathcal{M}_{g-1,1}$ a pencil of plane cubic curves $\{E_\lambda = f^{-1}(\lambda)\}_{\lambda \in \mathbb{P}^1}$, where $f : \text{Bl}_0(\mathbb{P}^2) \rightarrow \mathbb{P}^1$. The points of attachment on the elliptic pencil are given by a section $\sigma : \mathbb{P}^1 \rightarrow \text{Bl}_0(\mathbb{P}^2)$ given by one of the base points of the pencil of cubics. We lift this pencil in two possible ways to the space $\Sigma_g^+$, depending on the parity of the theta-characteristic on the varying elliptic tail. We fix an even theta-characteristic $\eta_C^+ \in \text{Pic}^{g-2}(C)$ and $E \cong \mathbb{P}^1$ will again denote an exceptional component. We define the family

$$F_0 := \{ [C \cup_y E \cup \sigma(\lambda) f^{-1}(\lambda), \eta_C^+, \eta_E = \mathcal{O}_E(1), \eta_f^{-1}(\lambda) = \eta_f^{-1}(\lambda) : \lambda \in \mathbb{P}^1 \} \subset \Sigma_g^+.$$ 

Since $F_0 \cap B_1 = \emptyset$, we find that $F_0 \cdot \alpha_1 = \pi_s(F_0) \cdot \delta_1 = -1$. Similarly, $F_0 \cdot \lambda = \pi_s(F_0) \cdot \lambda = 1$ and obviously $F_0 \cdot \alpha_i = F_0 \cdot \beta_i = 0$ for $2 \leq i \leq [g/2]$. For each of the 12 points $\lambda_\infty \in \mathbb{P}^1$ corresponding to singular fibres of $R$, the associated $\eta_{\lambda_\infty} \in \text{Pic}^{g-1}(C \cup E \cup f^{-1}(\lambda_\infty))$ are actual line bundles on $C \cup E \cup f^{-1}(\lambda_\infty)$, that is, we do not have to blow-up the extra node. Thus we obtain that $F_0 \cdot \beta_0 = 0$ and then $F_0 \cdot \alpha_0 = \pi_s(F_0) \cdot \delta_0 = 12$. 

A second lift of the elliptic pencil to $\Sigma_g^+$ is obtained by choosing an odd theta-characteristic $\eta_C^- \in \text{Pic}^{g-2}(C)$ whereas on $E_\lambda$ one takes each of the 3 possible even theta-characteristics, that is,

$$G_0 := \{ [C \cup_y E \cup \sigma(\lambda) f^{-1}(\lambda), \eta_C^-, \eta_E = \mathcal{O}_E(1), \eta_f^{-1}(\lambda) = \eta_f^{-1}(\lambda) : \lambda \in \mathbb{P}^1 \} \subset \Sigma_g^+,$$

where $\gamma : \Sigma_{1,1}^+ \rightarrow \overline{\mathcal{M}}_{1,1}$ is the projection of degree 3. Since $\pi_s(G_0) = 3R \subset \Delta_1$, we obtain that $G_0 \cdot \lambda = 3$. Obviously $G_0 \cdot \alpha_1 = 0$, hence $G_0 \cdot \beta_1 = \pi_s(G_0) \cdot \delta_1 = -3$. The map $\gamma : \Sigma_{1,1}^+ \rightarrow \overline{\mathcal{M}}_{1,1}$ is simply ramified over the point corresponding to $j$-invariant $\infty$. Hence, $G_0 \cdot \alpha_0 = 12$ and $G_0 \cdot \beta_0 = 12$. 

1.6. A covering family in $B_0$. We start with a general pointed spin curve $[C, q, \eta_C^-] \in \Sigma_{g-1,1}^-$ and as usual $E \cong \mathbb{P}^1$ denotes an exceptional component. We construct a family of spin curves $H_0 \subset B_0$ with general member

$$[C \cup_{\{y, q\}} E, \eta_C = \eta_C^-, \eta_E = \mathcal{O}_E(1)] \subset \overline{\Sigma}_g^-.$$
consider the spin curve $t$

Theorem 2.1. For a general theta-characteristic $\Theta$ of genus $g$, denote by $G. Scorza \text{[Sc]}$ to provide a birational isomorphism between $M_{3g}$ and $S_g^+$ (see also [DK]), and recently in [TZ] where several conditional statements of Scorza’s have been rigourously established.

For a fixed theta-characteristic $[C, \eta] \in S_g^+ - \Theta_{null}$, we define the curve

$$T_\eta := \{(x, y) \in C \times C : H^0(C, \eta \otimes \mathcal{O}_C(x - y)) \neq 0\}.$$

By Riemann-Roch, it follows that $T_\eta$ is a symmetric correspondence which misses the diagonal $\Delta \subset C \times C$. The curve $T_\eta$ has a natural fixed point free involution and we denote by $f : T_\eta \rightarrow T_\eta$ the associated étale double covering. Under the assumption that $T_\eta$ is a reduced curve, its class is computed in [DK] Proposition 7.1.5:

$$T_\eta \equiv (g - 1)F_1 + (g - 1)F_2 + \Delta.$$

**Theorem 2.1.** For a general theta-characteristic $[C, \eta] \in S_g^+$, the Scorza curve $T_\eta$ is a smooth curve of genus $g(T_\eta) = 3g(g - 1) + 1$.

**Proof.** It is straightforward to show that a point $(x, y) \in T_\eta$ is singular if and only if

$$H^0(C, \eta \otimes \mathcal{O}_C(x - 2y)) \neq 0 \quad \text{and} \quad H^0(C, \eta \otimes \mathcal{O}_C(y - 2x)) \neq 0.$$

By induction on $g$, we show that for a general even spin curve such a pair $(x, y)$ cannot exist. We assume the result holds for a general $[C, \eta_C] \in S_{g-1}^+$. We fix a general point $q \in C$, an elliptic curve $D$ together with $\eta_D \in \text{Pic}^0(D) - \{O_D\}$ with $\eta_D^{\otimes 2} = O_D$ and consider the spin curve $t := [C \cup E \cup D, \eta_C = \eta_D, \eta_E = O_E(1), \eta_{\delta_0} = \eta_D] \in S_g^+$, obtained from $C \cup_q D$ by inserting an exceptional component $E$. Since the exceptional component plays no further role in the proof, we are going to suppress it.

We assume by contradiction that $t \in S_g^+$ lies in the closure of the locus of spin curves with singular Scorza curve. Then there exists a nodal curve $C \cup_q D'$ semistably equivalent to $C \cup_q D$ obtained by inserting a possibly empty chain on $\mathbb{P}^1$’s at the node $q$ (therefore, $p_a(D') = 1$ and we may regard $D$ as a subcurve of $D'$), as well as smooth points $x, y \in C \cup D'$ together with two limit linear series $\sigma = \{\sigma_C, \sigma_{D'}\}$ and $\tau = \{\tau_C, \tau_{D'}\}$ of type $g^0_{g-2}$ on $C \cup D'$ such that the underlying line bundles corresponding to $\sigma$ (resp. $\tau$) are uniquely determined twists at the nodes of the line bundle $\eta \otimes \mathcal{O}_{C \cup D'}(x - 2y)$ (resp. $\eta \otimes \mathcal{O}_{C \cup D'}(y - 2x)$). The precise twists are determined by the limit linear series
condition that each aspect of a limit $\eta_0$ have degree $g - 2$. We distinguish three cases depending on which components of $C \cup D'$ the points $x$ and $y$ specialize.

(i) $x, y \in C$. Then $\sigma_C \in H^0(C, \eta_C \otimes \mathcal{O}_C(x - 2y + q)), \tau_C \in H^0(C, \eta_C \otimes \mathcal{O}_C(y - 2x + q))$, while $\sigma_D, \tau_D \in H^0(D, \eta_D \otimes \mathcal{O}_D((g - 2q)))$. Denoting by $\{|q'\} \in D \cap (C \cup D') - D$ the point where $D$ meets the rest of the curve, one has the compatibility conditions

$\text{ord}_q(\sigma_C) + \text{ord}_q(\tau_D) \geq g - 2 \quad \text{and} \quad \text{ord}_q(\tau_C) + \text{ord}_q(\tau_D) \geq g - 2$,

which leads to $\text{ord}_q(\sigma_C) \geq 1$ and $\text{ord}_q(\tau_C) \geq 1$, that is, we have found two points $x, y \in C$ such that $H^0(C, \eta_C(x - 2y)) \neq 0$ and $H^0(C, \eta_C(y - 2x)) \neq 0$, which contradicts the inductive assumption on $C$.

(ii) $x, y \in D'$. This case does not appear if we choose $\eta_C$ such that $H^0(C, \eta_C) = 0$. Indeed, for degree reason, both non-zero sections $\sigma_C, \tau_C$ must lie in the space $H^0(C, \eta_C)$.

(iii) $x \in C, y \in D'$. For simplicity, we assume first that $y \in D$. We find that

$\sigma_C \in H^0(C, \eta_C \otimes \mathcal{O}_C(x - q)), \sigma_D \in H^0(D, \eta_D \otimes \mathcal{O}_D(g \cdot q' - 2y))$ and

$\tau_C \in H^0(C, \eta_C \otimes \mathcal{O}_C(2q - 2x)), \tau_D \in H^0(D, \eta_D \otimes \mathcal{O}_C(y + (g - 3) \cdot q'))$.

We claim that $\text{ord}_q(\sigma_C) = \text{ord}_q(\tau_C) = 0$ which can be achieved by a generic choice of $q \in C$. Then $\text{ord}_q(\sigma_D) \geq g - 2$, which implies that $\eta_D = \mathcal{O}_D(2y - 2q)$. Similarly, $\text{ord}_q(\tau_D) \geq g - 2$ which yields that $\eta_D = \mathcal{O}_D(q - y)$, that is, $\eta_D \otimes \mathcal{O}_D$. Since $\eta_D$ was assumed to be a non-trivial point of order 2 this leads to a contradiction. Finally, the case $y \in D' - D$, that is, when $y$ lies on an exceptional subcurve $E' \subset D'$ is dealt with similarly. Since $\text{ord}_q(\sigma_C) = \text{ord}_q(\tau_C) = 0$, by compatibility, after passing through the component $E'$, one obtains that $\text{ord}_q(\sigma_D) \geq g - 2$. Since $\sigma_D \in H^0(D, \eta_D \otimes \mathcal{O}_D((g - 2q'))) and \eta_D \neq \mathcal{O}_D, we obtain a contradiction.

[\square]

3. Theta pencils on K3 surfaces.

In this section we prove Theorem 0.1. As usual, we denote by $\mathcal{F}_g$ the moduli space of polarized $K3$ surfaces $[X, H]$, where $X$ is a K3 surface and $H \in \text{Pic}(X)$ is a (primitive) polarization of degree $H^2 = 2g - 2$. For integers $0 \leq \delta \leq g$, we introduce the universal Severi variety of pairs

$V_{g, \delta} := \{([X, H], C) : [X, H] \in \mathcal{F}_g and C \in |\mathcal{O}_X(H)| is an integral $\delta - \text{nodal curve}\}.$

If $\sigma : V_{g, \delta} \to \mathcal{F}_g$ is the obvious projection, we set $V_{g, \delta}(|H|) := \sigma^{-1}([X, H])$. It is known that every irreducible component of $V_{g, \delta}$ has dimension $19 + g - \delta$ and maps dominantly onto $\mathcal{F}_g$. It is in general not known whether $V_{g, \delta}$ is irreducible, see [De] for interesting work in this direction.

For a point $[X, H] \in \mathcal{F}_g$, we consider a pencil of curves $P \subset |H|$, and denote by $Z$ the base locus of $P$. We assume that a general member $C \in P$ is a nodal integral curve. It follows that $C - Z$ is smooth and that $S := \text{sing}(C)$ is a, possibly empty, subset of $Z$. Let $\iota : X' := \text{Bl}_S(X) \to X$ be the blow-up of $X$ along the locus $S$ of nodes, and denote by $E$ the exceptional divisor of $\iota$. Let

$P' \subset |\iota^*H \otimes \mathcal{O}_{X'}(-2E)|$

be the strict transform of $P$ by $\iota$, and $Z'$ its base locus. Since a general member $C \in P$ is nodal precisely along $S$, a general curve $C' \in P'$ is smooth. We view $h' := Z' + E \cdot C'$ as a divisor on the smooth curve $C'$. By the adjunction formula, $h' \in |\omega_{C'}|$. 

**Theorem 3.4.** We say that $P$ is a theta pencil, if $h'$ has even multiplicity at each of its points, that is, $\mathcal{O}_{C'}(\frac{1}{2}h')$ is an odd theta-characteristic for every smooth curve $C' \in P'$.

The definition implies that the intersection multiplicity of two curves in $P$ is even at each point $p \in \text{supp}(Z)$. For every pair $[X, H] \in \mathcal{F}_g$ we have that:

**Proposition 3.2.** Every smooth curve $C \in |H|$ belongs to a theta pencil.

**Proof.** Let $d \in C_{g-1}$ be the support of a theta-characteristic on $C$ such that $h^0(C, \mathcal{O}_C(d)) = 1$. Then $\mathbb{P}H^0(X, I_d/X(H))$ is a theta pencil.

We can reverse the construction of a theta pencil, starting instead with the normalization of a nodal section of a $K3$ surface. Suppose

$$t := [C', x_1, y_1, \ldots, x_\delta, y_\delta, \eta] \in \mathcal{M}_{g-\delta, 2\delta} \times \mathcal{M}_{g-\delta} \mathcal{S}_{g-\delta}^-$$

is a $2\delta$-pointed curve together with an isolated odd theta-characteristic $\eta$, such that:

(i) $h^0(C', \eta \otimes \mathcal{O}_C(-\sum_{i=1}^\delta (x_i + y_i))) \geq 1$; we write $\text{supp}(\eta) = \sum_{i=1}^\delta (x_i + y_i) + d$, where $d \in C_{g-3\delta-1}$ is the residual divisor.

(ii) There exists a polarized $K3$ surface $[X, H] \in \mathcal{F}_g$ and a map $f : C' \to X$, such that $f(x_i) = f(y_i) = p_i$ for all $i = 1, \ldots, \delta$, $f_*(C') \in |H|$, and moreover $f : C' \to C$ is the normalization map of the $\delta$-nodal curve $C := f(C')$.

If $\varepsilon: X' \to X$ is the blow-up of $X$ at the points $p_1, \ldots, p_\delta$ and $E := \sum_{i=1}^\delta E_{p_i} \subset X'$ denotes the exceptional divisor, we may view $C' \subset X$, where $C' \equiv \varepsilon^*H - 2E$. Then

$$|I_d/X'(C')| = |I_{2d}/X'(C')| = |I_{2d+\sum_{i=1}^\delta (x_i+y_i)/X'}(C')|$$

is a theta pencil of $\delta$-nodal curves on $X$.

If $\mathcal{K}_{g-\delta, \delta}^- \subset \mathcal{M}_{g-\delta, 2\delta} \times \mathcal{M}_{g-\delta} \mathcal{S}_{g-\delta}^-$ is the locus of elements $[C, (x_i, y_i)_{i=1,\ldots,\delta}, \eta]$ satisfying conditions (i) and (ii), the previous discussion proves the following:

**Proposition 3.3.** Every irreducible component of $\mathcal{K}_{g-\delta, \delta}^-$ is uniruled.

This implies the following consequence of Proposition 4.4 to be established in the next section:

**Theorem 3.4.** We set $g \leq 9$ and $0 \leq \delta \leq (g + 1)/3$. Then the variety $\mathcal{K}_{g-\delta, \delta}^-$ is non-empty, uniruled and dominates the spin moduli space $\mathcal{S}_{g-\delta}^-$.

**Definition 3.5.** We say that a theta pencil $P$ is $\delta$-nodal if $|S| = \delta$. We say that $P$ is regular if $\text{supp}(Z)$ consists of $g - 1$ distinct points.

If $P$ is a $\delta$-nodal theta pencil, we have an induced map

$$m' : P' \cong \mathbb{P}^1 \to \overline{\mathcal{S}}_{g-\delta}^-,$$

obtained by sending a general $C' \in P'$ to the moduli point $[C', \mathcal{O}_{C'}(\frac{1}{2}h')] \in \overline{\mathcal{S}}_{g-\delta}^-$. We note in passing that a theta pencil also induces a map $m : P' \to \overline{\mathcal{S}}_g$ defined as follows. Consider the pencil $E + P'$ having fixed component $E$. The general member is a quasi-stable curve $D \in (E + P')$ of arithmetic genus $g$, with exceptional components $\{E_i\}_{i=1,\ldots,\delta}$ corresponding to the exceptional divisors of the blow-up $\varepsilon : X' \to X$. Then

$$m(C) := [C \cup \bigcup_{i=1}^\delta E_i, \eta_{E_i} = \mathcal{O}_{E_i}(1), \eta_{C'} = \mathcal{O}_{C'}(\frac{1}{2}h')] \in \overline{\mathcal{S}}_g.$$
These pencils will be used extensively in the proof of Theorem \ref{thm:main}.

Assume that \([X, H] \in F_g\) is a general point, in particular \(\text{Pic}(X) = \mathbb{Z} \cdot H\). Then every smooth curve \(C \in |H|\) is Brill-Noether general, \([La]\), which implies that \(h^0(C, \eta) = 1\), for every odd theta-characteristic \(\eta\) on \(C\). Theta pencils with smooth general member define a locally closed subset in the Grassmannian \(G(2, H^0(S, O_S(H)))\) of lines in \(|H|\). Let \(\Theta^-(X, H)\) be its Zariski closure in \(G(2, H^0(S, O_S(H)))\).

**Proposition 3.6.** \(\Theta^-(X, H)\) is pure of dimension \(g - 1\).

**Proof.** Let \(f : P^-(X, H) \to |H|\) be the projection map from the projectivized universal bundle over \(\Theta^-(X, H)\), and \(V_{g,0}(|H|) \subset |H|\) be the open locus of smooth curves. Under our assumptions \(f\) has finite fibres over \(V_{g,0}(|H|)\). Thus \(P^-(X, H)\) has pure dimension \(g\), and \(\Theta^-(X, H)\) has pure dimension \(g - 1\). \(\square\)

For a general (thus necessarily regular) theta pencil \(P \in \Theta^-(X, H)\), we study in more detail the map \(m : P' \to \overline{S}_g\). Let \(\Delta(X, H) \subset |H|\) be the discriminant locus. Since \([X, H] \in F_g\) is general, \(\Delta(X, H)\) is an integral hypersurface parameterizing the singular elements of \(|H|\). It is well-known that \(\text{deg} \Delta(X, H) = 6g + 18\).

**Proposition 3.7.** Let \(P \in \Theta^-(X, H)\) be a general theta pencil with base locus \(Z\). Then every singular curve \(C \subset P\) is nodal. Furthermore,

\[
P \cdot \Delta(X, H) = 2(a_1 + \cdots + a_{g-1}) + b_1 + \cdots + b_{4g+20},
\]

where \(a_i\) is the parameter point of a curve \(A_i \subset P\) having a point of \(Z\) as its only singularity, and \(b_j\) is the parameter point of a curve \(B_j \subset P\) such that \(\text{sing}(B_j) \subset X - Z\). Accordingly,

\[
P \cdot \alpha_0 = 4g + 20 \quad \text{and} \quad P \cdot \beta_0 = g - 1.
\]

**Proof.** We set \(\text{supp}(Z) = \{p_1, \ldots, p_{g-1}\}\). Since \(P\) is regular, for \(i = 1, \ldots, g - 1\), there exists a unique curve \(A_i \subset P\) singular at \(p_i\). Moreover, for degree reasons, \(p_i\) is the unique double point of \(A_i\). Each pencil \(T \subset |H|\) having \(p_i\) in its base locus is a tangent line to \(\Delta(X, H)\) at \(A_i\). Hence the intersection multiplicity \((P \cdot \Delta(X, H))_{A_i}\) is at least 2. It follows that the assertion to prove is open on any family of pairs \((P, [X, H])\) such that \(P \in \Theta^-(X, H)\). Since \(F_g\) is irreducible, it suffices to produce one polarized K3 surface \((X, H)\) satisfying this condition.

For this purpose, we use hyperelliptic polarized K3 surfaces \((X, H)\). Consider a rational normal scroll \(\mathbb{F} := \mathbb{F}_a \subset \mathbb{P}^g\), where \(a \in \{0, 1\}\) and \(g = 2n + 1 - a\). A general section \(R \in |O_{\mathbb{F}}(1)|\) is a rational normal curve of degree \(g - 1\). From the exact sequence

\[
0 \to O_{\mathbb{F}}(-2K_{\mathbb{F}} - R) \to O_{\mathbb{F}}(-2K_{\mathbb{F}}) \to O_R(-2K_{\mathbb{F}}) \to 0,
\]

one finds that there exist a smooth curve \(B \subset | - 2K_{\mathbb{F}}|\) and distinct points \(o_1, \ldots, o_{g-1} \in B\) such that the pencil \(Q \subset |O_{\mathbb{F}}(R)|\) of hyperplane sections through \(o_1, \ldots, o_{g-1}\) cuts out a pencil with simple ramification on \(B\).

Let \(\rho : X \to \mathbb{F}\) be the double covering of \(\mathbb{F}\) branched along \(B\). Then \(X\) is a K3 surface and \(|H| := |O_X(\rho^*R)|\) is a hyperelliptic linear system on \(X\) of genus \(g\). Then \(\rho^*(Q)\) is a regular theta pencil on \(X\) with the required properties. \(\square\)

Since theta pencils cover \(\overline{S}_g\) when \(g \leq 11\) and \(g \neq 10\), the following consequence of Proposition \ref{prop:main} is very suggestive concerning the variation of \(\kappa(\overline{S}_g)\) as \(g\) increases, in particular, in highlighting the significance of the case \(g = 12\).
Corollary 3.8. With the same notation as above, we have that $P \cdot K_{S^g} = 2g - 24$. In particular general theta pencils of genus $g < 12$ are $K_{S^g}$-negative.

Proof. Use that $(P \cdot \lambda)_{S^g} = (\pi_*(P) \cdot \lambda)_{\mathcal{M}_g} = g + 1$, $P \cdot \alpha_0 = 4g + 20$ and $P \cdot \beta_0 = g - 1$. \hfill $\square$

Proposition 3.9. The locally closed set of nodal theta pencils in $\Theta^-(X, H)$ is non empty. If $P$ is a general nodal theta pencil, then a general curve $C \in P$ has one node as its only singularity.

Proof. We keep the notation from the previous proof and construct a smooth curve $C \in |−2K_F|$ and choose general points $o, o_1, \ldots, o_{g−3} \in B$, such that the pencil $Q \subset |O_F(R)|$ of the hyperplane sections through $o_1 + \cdots + o_{g−3} + 2o$ cuts out a pencil with simple ramification on $B$. Then $\rho^*(Q)$ is a nodal theta pencil with the required properties. \hfill $\square$

Theorem 3.10. $S_g^−$ is uniruled for $g \leq 11$.

Proof. By [M1-4], a general curve $|C| \in \mathcal{M}_g$ is embedded in a K3 surface $X$ precisely when $g \leq 9$ or $g = 11$. By Proposition 3.7 $C$ belongs to a theta pencil $P \subset |O_X(C)|$ (which moreover, is $K_{\mathcal{M}_g}$-negative). Thus the statement follows for $g \leq 9$ and $g = 11$.

To settle the case of $S^g_{10}$, we show that $K_{10,1}$ is non-empty and irreducible. Indeed, then by Proposition 3.3 it follows that $K_{10,1}$ is uniruled, and since the projection map $K_{10,1} \to S^g_{10}$ is finite, $K_{10,1}$ dominates $S^g_{10}$. This implies that $S^g_{10}$ is uniruled.

The variety $K_{10,1}$ is an open subvariety of the irreducible locus

$$U := \{([C, x, y], \eta) \in \mathcal{M}_{10,2} \times \mathcal{M}_{10} S^g_{10} : h^0(C, \eta \otimes O_C(−x − y)) \geq 1\},$$

hence it is irreducible as well. To establish its non-emptiness, it suffices to produce an example of an element $([C, x, y], \eta) \in U$, such that the curve $C_{xy}$ can be embedded in a K3 surface. We specialize to the case when $C$ is hyperelliptic and $x, y, \in C$ are distinct Weierstrass points, in which case one can choose $\eta = O_C(x + y + w_1 + \cdots + w_7)$, where $w_i$ are distinct Weierstrass points in $C − \{x, y\}$. Again we let $\rho : X \to \mathbb{P} \subset \mathbb{P}^{11}$ be a hyperelliptic K3 surface branched along $B \subset |−2K_F|$, with polarization $H := \rho^*O_F(1)$, so that $[X, H] \in \mathcal{F}_{11}$. We set $C := \rho^*(R)$, where $R \subset |O_F(1)|$ is a rational normal curve of degree 10. We need to ensure that $C$ is 1-nodal, with its node $p \in C$ such that if $f : C' \to C$ denotes the normalization map, then both points in $f^{-1}(p)$ are Weierstrass points. This is satisfied once we choose $R$ in such a way that $B \cdot R \geq 2\rho(p)$. \hfill $\square$

4. UNIRATIONALITY OF $S_{g}^−$ FOR $g \leq 8$

To prove the claimed unirationality results, we use that a general curve $[C] \in \mathcal{M}_g$ has a sextic plane model when $g \leq 6$, or is a linear section of a Mukai variety, when $7 \leq g \leq 9$. We start with the easy case of small genus, before moving on to the more substantial study of Mukai models.

Theorem 4.1. $S_{g}^−$ is unirational for $g \leq 6$.

Proof. A general odd spin curve $[C, \eta] \in S_{g}^−$ of genus $3 \leq g \leq 6$, is birational to a pair $(\Gamma, \eta)$, where $\Gamma \subset \mathbb{P}^2$ is an integral nodal sextic. One can assume that $d := \text{supp}(\eta)$ is a reduced divisor contained in $\Gamma_{\text{reg}}$. Note that there exists a unique plane cubic $E$ such that $E \cdot \Gamma = 2e$, where $e$ is an effective divisor of degree 9 on $E$, supported on
sing(Γ) ∪ d. We denote by \( U \subset (\mathbb{P}^2)^9 \) the open set parameterizing general 9-tuples \((\bar{x}, \bar{y}) := (x_1, \ldots, x_\delta, y_1, \ldots, y_{g-1})\), where \( g = 10 - \delta \). Over \( U \) lies a projective bundle \( \mathcal{P} \) whose fibre at \((\bar{x}, \bar{y})\) is the linear system of plane sextics \( \Gamma \) which are singular along \( \bar{x} \) and totally tangent to \( E_{\bar{x}, \bar{y}} \) along \( \bar{y} \). Here \( E_{\bar{x}, \bar{y}} \in |\mathcal{O}_{\mathbb{P}^2}(3)| \) denotes the unique plane cubic through the points \( x_1, \ldots, x_\delta, y_1, \ldots, y_{g-1} \). Then \( \mathcal{P} \) is a rational variety, and by the previous remark, it dominates \( \overline{S}_g \). Thus \( \overline{S}_g \) is unirational. \( \square \)

We assume now that \( 7 \leq g \leq 10 \) and denote by \( V_g \subset \mathbb{P}^{N_g} \) the rational homogeneous space \( V_g \) defined as follows [M1], [M2], [M3]:

- \( V_{10} \): the 5-dimensional variety \( G_2/P \subset \mathbb{P}^{17} \) corresponding to the Lie group \( G_2 \),
- \( V_9 \): the Plücker embedding of the symplectic Grassmannian \( \text{SG}(3, 6) \subset \mathbb{P}^{13} \),
- \( V_8 \): the Plücker embedding of the Grassmannian \( G(2, 6) \subset \mathbb{P}^{14} \),
- \( V_7 \): the Plücker embedding of the orthogonal Grassmannian \( \text{OG}(5, 10) \subset \mathbb{P}^{15} \).

Note that \( N_g = g + \dim(V_g) - 2 \). Inside the Hilbert scheme \( \text{Hilb}(V_g) \) of curvilinear sections of \( V_g \), we consider the open set \( U_g \) classifying curves \( C \subset V_g \) such that

- \( C \) is a nodal integral section of \( V_g \) by a linear space of dimension \( g - 1 \),
- the residue map \( \rho : H^0(C, \omega_C) \to H^0(C, \omega_C \otimes \text{O}_{\text{sing}(C)}) \) is surjective.

A general point \([C \hookrightarrow \mathbb{P}^{g-1}] \in U_g \) is a smooth, canonical curve of genus \( g \). Moreover \( C \) has general moduli if \( g \leq 9 \). For each \( 0 \leq \delta \leq g - 1 \), we define the locally closed sets of \( \delta \)-nodal curvilinear sections of \( V_g \)

\[
U_{g, \delta} := \{ [C \hookrightarrow \mathbb{P}^{g-1}] \in U_g : |\text{sing}(C)| = \delta \}.
\]

**Proposition 4.2.** \( U_{g, \delta} \) is smooth of pure codimension \( \delta \) in \( U_g \).

**Proof.** A general 2-dimensional linear section of \( V_g \) is a polarized K3 surface \((S, H) \in \mathcal{F}_g \) with general moduli. It is known [13], that \( \delta \)-nodal hyperplane sections of \( S \) form a pure \((g - \delta)\)-dimensional family \( V_g, \delta(H) \subset [H] \). In particular \( U_{g, \delta} \neq \emptyset \) and \( \text{codim}(U_{g, \delta}, U_g) \leq \delta \). We fix a curve \([C] \in U_{g, \delta} \), then consider the normal bundle \( N_C \) of \( C \) in \( V_g \) and the map \( r : H^0(C, N_C) \to \text{O}_{\text{sing}(C)} \) induced by the exact sequence

\[
0 \to T_C \to T_{V_g} \otimes \text{O}_C \to N_C \xrightarrow{r} T_C^1 \to 0,
\]

where \( T_C^1 = \text{O}_{\text{sing}(C)} \) is the Lichtenbaum-Schlessinger sheaf of \( C \). Using the identification \( T_{[C]}(U_g) = H^0(C, N_C) \), it is known that \( \text{Ker}(r) \) is isomorphic to \( T_{[C]}(U_{g, \delta}) \). We have that \( N_C \cong \omega_C^{\oplus (N_9-g+1)} \) and \( r = \rho^{\oplus (N_9-g+1)} \), where \( \rho : H^0(C, \omega_C) \to H^0(C, \text{O}_{\text{sing}(C)}) \) is the map given by the residues at the nodes. Since \( \rho \) is surjective, \( \text{Ker}(r) \) has codimension \( \delta \) inside \( T_{[C]}(U_g) \) and the statement follows. \( \square \)

The automorphism group \( \text{Aut}(V_g) \) acts in the natural way on \( \text{Hilb}(V_g) \). Since the locus of singular curvilinear sections \([C] \in U_g \) is an \( \text{Aut}(V_g) \)-invariant divisor which misses a general point of \( U_g \), it follows that \( U_{g, \delta}^{\text{ss}} := U_g \cap \text{Hilb}(V_g)^{\text{ss}} \neq \emptyset \). Note that since \( \rho(V_g) = 1 \), the notion of stability is independent of the polarization. The (quasi-projective) GIT-quotient

\[
\mathcal{M}_g := U_{g, \delta}^{\text{ss}} \sslash \text{Aut}(V_g)
\]
is said to be the Mukai model of \( \overline{M}_g \). We have the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{U}_g^{ss} & \longrightarrow & \mathcal{U}_g \\
\downarrow u_g & & \downarrow m_g \\
\mathfrak{M}_g & \longrightarrow & \overline{\mathfrak{M}}_g
\end{array}
\]

where \( u_g : \mathcal{U}_g^{ss} \to \mathfrak{M}_g \) is the quotient map and \( m_g : \mathcal{U}_g \to \overline{\mathfrak{M}}_g \) is the moduli map. The general fibre of \( m_g \) is an \( \text{Aut}(V_g) \)-orbit. Summarizing results from [M1], [M2], [M3], we state the following:

**Theorem 4.3.** For \( 7 \leq g \leq 9 \), the map \( \phi_g : \mathfrak{M}_g \to \overline{\mathfrak{M}}_g \) is a birational isomorphism. The inverse map \( \phi_g^{-1} \) contracts the (unique) Brill-Noether divisor \( \overline{M}_{g,d} \subset \overline{M}_g \) of curves with a \( g_r^\nu \) as well as the boundary divisors \( \Delta_i \), with \( 1 \leq i \leq [g/2] \).

Next, let \( \Delta_g^\delta \subset \Delta_0 \subset \overline{\mathfrak{M}}_d \) be the locus of integral stable curves of arithmetic genus \( g \) with \( \delta \) nodes. Then \( \Delta_g^\delta \) is irreducible of codimension \( \delta \) in \( \overline{\mathfrak{M}}_g \).

**Lemma 4.4.** Set \( g \leq 9 \) and let \( D \) be any irreducible component of \( \mathcal{U}_g,\delta \). Then the restriction morphism \( m_{g|D} : D \to \Delta_g^\delta \) is dominant. In particular, a general \( \delta \)-nodal curve \([C] \in \Delta_g^\delta \) lies on a smooth \( K3 \) surface.

**Proof.** Since \( \mathcal{U}_g,\delta \) is smooth, \( D \) is a connected component of \( \mathcal{U}_g,\delta \), that is, for \([C] \in D \), the tangent spaces to \( D \) and to \( \mathcal{U}_g,\delta \) coincide. We consider again the sequence (3):

\[
0 \to T_C \to T_{V_g} \otimes \mathcal{O}_C \to N_C^\nu \to 0,
\]

where \( N_C^\nu := \text{Im} \{ T_{V_g} \otimes \mathcal{O}_C \to N_C \} \) is the equisingular sheaf of \( C \). We have that \( H^0(C, N_C^\nu) = \text{Ker}(r) \). As remarked in the proof of Proposition 4.2, \( H^0(C, N_C^\nu) \) is the tangent space \( T_{[C]}(\mathcal{U}_g,\delta) \) and its codimension in \( H^0(C, N_C) \) equals \( \delta \). Consider the coboundary map \( \partial : H^0(C, N_C^\nu) \to H^1(C, T_C) \). Since \( H^1(C, T_C) \) classifies topologically trivial deformations of the nodal curve \( C \), the image \( \text{Im}(\partial) \) is isomorphic to the image of the tangent map \( dm_{g|D,\delta} \) at \([C] \). On the other hand \( H^0(C, T_{V_g} \otimes \mathcal{O}_C) \) is the tangent space to the orbit of \( C \) under the action of \( \text{Aut}(V_g) \). This is reduced and the stabilizer of \( C \), being a subgroup of \( \text{Aut}(C) \), is finite, hence we obtain:

\[
\dim \text{Im}(\partial) = h^0(C, N_C) - \delta - \dim \text{Aut}(V_g) = 3g - 3 - \delta.
\]

Since \( \Delta_g^\delta \) has codimension \( \delta \) in \( \overline{\mathfrak{M}}_g \), it follows that \( m_{g|D} \) is dominant. \( \square \)

**Proposition 4.5.** Fix \( 0 \leq \delta \leq g - 1 \) and \( D \) an irreducible component of \( \mathcal{U}_g,\delta \). Then \( D^{ss} \neq \emptyset \).

**Proof.** It suffices to construct an \( \text{Aut}(V_g) \)-invariant divisor which does not contain \( D \). We carry out the construction when \( g = 8 \), the remaining cases being largely similar.

We fix a complex vector space \( V \cong \mathbb{C}^6 \), and then \( V_8 := G(2, V) \subset \mathbb{P}(\wedge^2 V) \) and \( \mathcal{U}_8 \subset G(8, \wedge^2 V) \). For a projective 7-plane \( \Lambda \in G(8, \wedge^2 V) \), we denote the set of containing hyperplanes \( F_\Lambda := \{ H \in \mathbb{P}(\wedge^2 V)^\vee : H \supset \Lambda \} \), and define the \( \text{Aut}(V_8) \)-invariant divisor

\[
Z := \{ \Lambda \in \mathcal{U}_8 : F_\Lambda \cap G(2, V)^\vee \subset \mathbb{P}(\wedge^2 V)^\vee \text{ is not a transverse intersection} \}.
\]

We claim that \( D \not\subseteq Z \). Indeed, let us fix a general point \([C \hookrightarrow \Lambda] \in D \), where \( \Lambda = \langle C \rangle \), corresponding to a general curve \([C] \in \Delta_g^\delta \). In particular, we may assume that \( C \) lies outside the closure in \( \overline{\mathfrak{M}}_g \) of curves violating the Petri theorem. Thus \( C \) possesses no
It is clear that for every $A \in \overline{W}_5^1(C)$, the intersection $F_A = \Phi \left( \ker \left( \Lambda^2 H^0(C, E) \rightarrow H^0(C, \omega_C) \right) \right)$. Moreover, $F_A = \mathbf{P} \{ \ker \left( \Lambda^2 H^0(C, E) \rightarrow H^0(C, \omega_C) \right) \}$. In particular, the intersection $F_A \cap G(2, H^0(C, E))$ corresponds to the pencils $A \in \overline{W}_5^1(C)$.

Since $C$ is Petri general, $\overline{W}_5^1(C)$ is a smooth scheme, thus $|C \hookrightarrow \Lambda| \notin Z$. □

We consider the quotient $\mathcal{M}_{g, \delta} := \mathcal{U}_{g, \delta}^{ss} / \text{Aut}(V_g)$ and the induced map $\phi_{g, \delta} : \mathcal{M}_{g, \delta} \to \Delta^\delta_g$.

**Theorem 4.6.** The variety $\mathcal{M}_{g, \delta}$ is irreducible and $\phi_{g, \delta}$ is a birational isomorphism.

**Proof.** By Lemma 4.4, any irreducible component $Y$ of $\mathcal{M}_{g, \delta}$ dominates $\Delta^\delta_g$. On the other hand, $\phi_g : \mathcal{M}_g \to \overline{M}_g$ is a birational morphism and $\phi_{g, \delta} = \phi_g |_{\mathcal{M}_{g, \delta}}$. Since $\overline{M}_g$ is normal, each fibre of $\phi_g$ is connected, thus $\mathcal{M}_{g, \delta}$ is irreducible and $\text{deg}(\phi_{g, \delta}) = 1$. □

We lift our construction to the space of odd spin curves. Keeping $g \leq 9$, we consider the Hilbert scheme $\text{Hilb}_{2g-2}(V_g)$ of 0-dimensional subschemes of $V_g$ having length $2g-2$.

**Definition 4.7.** Let $Z_{g-1} \subset \text{Hilb}_{2g-2}(V_g)$ be the parameter space of those 0-dimensional schemes $Z \subset V_g$ such that:

1. $Z$ is a hyperplane section of a smooth curve section $[C] \in \mathcal{U}_g$.
2. $Z$ has multiplicity two at each point of its support.
3. $\text{supp}(Z)$ consists of $g-1$ linearly independent points.

One thinks of $Z_{g-1}$ as classifying length $g-1$ clusters on $V_g$. A general point of $Z_{g-1}$ corresponds to a 0-cycle $x_1 + \cdots + x_{g-1} \in \text{Sym}^{g-1}(V_g)$ satisfying

$$\dim \langle x_1, \ldots, x_{g-1} \rangle \cap \mathbb{T}_{x_i}(V_g) \geq 1, \text{ for } i = 1, \ldots, g-1.$$  

Clearly $\dim(Z_{g-1}) = (g-1)(N_g - g+1)$. Then we consider the incidence correspondence $\mathcal{U}_g^- := \{(C, Z) \in \mathcal{U}_g \times Z_{g-1} : Z \subset C \}$. The first projection map $\pi_1 : \mathcal{U}_g^- \to \mathcal{U}_g$ is finite of degree $2^{g-1}(2^g - 1)$; its fibre at a general point $[C] \in \mathcal{U}_g$ is in bijective correspondence with the set of odd theta-characteristics of $C$. In particular, both $\mathcal{U}_g^-$ and $Z_{g-1}$ are irreducible varieties. The spin moduli map

$$m_g^- : \mathcal{U}_g^- \dashrightarrow \overline{\mathcal{S}}_g$$

is defined by $m_g^-(C, Z) := [C, \mathcal{O}_C(Z/2)]$, for each point $(C, Z) \in \mathcal{U}_g^-$ corresponding to a smooth curve $C$. Later we shall extend the rational map $m_g^-$ to a regular map over $\mathcal{U}_g^-$.

It is clear that $m_g^-$ induces a map $\phi_g^- : Q_g^- \dashrightarrow \overline{\mathcal{S}}_g$ from the quotient

$$Q_g^- := \pi_1^{-1}(\mathcal{U}_g^{ss}) / \text{Aut}(V_g).$$
We may think of $Q_g^-$ as being the Mukai model of $\mathcal{S}_g^-$. If $\pi^- : Q_g^- \to \mathcal{M}_g$ is the map induced by $\pi$ at the level of Mukai models, we have a commutative diagram:

$$
\begin{array}{ccc}
Q_g^- & \xrightarrow{\phi_g^-} & \mathcal{S}_g^- \\
\pi^- & \downarrow & \pi \\
\mathcal{M}_g & \xrightarrow{\phi_g} & \mathcal{M}_g
\end{array}
$$

**Proposition 4.8.** The spin Mukai model $Q_g^-$ is irreducible and $\phi_g^- : Q_g^- \to \mathcal{S}_g^-$ is a birational isomorphism.

One extends the rational map $m_g^-$ (therefore $\phi_g^-$ as well) to a regular morphism as follows. Let $(C, Z) \in U_g^-$ be an arbitrary point, and set $\text{supp}(Z) := \{p_1, \ldots, p_{g-1}\}$. Assume that $\text{sing}(C) \cap \text{supp}(Z) = \{p_1, \ldots, p_b\}$, where $b \leq g-1$. Consider the partial normalization $\nu : N \to C$ at the points $p_1, \ldots, p_b$. In particular, there exists an effective Cartier divisor $e$ on $C$ of degree $g - b - 1$, such that $2e = Z \cap (C - \text{sing}(C))$, and set $\epsilon := O_N(\nu^*e)$. Then $m_g^-(C, Z)$ is the spin curve $[X, \eta] \in \mathcal{S}_g^-$ defined as follows:

**Definition 4.9.**

1. $X := N \cup E_1 \cup \cdots \cup E_b$, where $E_i = \mathbb{P}^1$ for $i = 1, \ldots, b$.
2. $\eta \cap N = \nu^{-1}(p_1)$, for every node $p_i \in \text{sing}(C) \cap \text{supp}(Z)$.
3. $\eta \otimes O_{E_i} \cong \epsilon$ and $\eta \otimes O_E \cong O_{\mathbb{P}^1}(1)$.

We note that $N$ is smooth of genus $g - b$, precisely when $\text{sing}(C) \subset \text{supp}(Z)$. In this case $\epsilon \in \text{Pic}^{g-1-b}(N)$ is a theta characteristic and $h^0(N, \epsilon) = 1$. Since we are specially interested in this case, for $1 \leq b \leq g-1$ we introduce the locally closed sets

$$
U_{g, b}^- := \{(C, Z) \in U_g^- : \text{sing}(C) \subset \text{supp}(Z), |\text{sing}(C)| = b\}.
$$

We denote by $B_{g, b}^-$ the closure of $m_g^-(U_{g, b}^-)$ inside $\mathcal{S}_g^-$; this is the closure in $\mathcal{S}_g^-$ of the locus of $b$-nodal spin curves having $b$ exceptional components. Clearly $B_{g, b}^-$ is an irreducible component of $\pi^{-1}(\Delta_b^-)$. We set

$$
Q_{g, b}^- := U_{g, b}^- \cap \pi^{-1}(U_g^{ss})/\text{Aut}(V_g),
$$

and let $u_g^- : U_{g, b}^- \dasharrow Q_{g, b}^-$ denote the quotient map. Keeping all previous notation, we have a further commutative diagram

$$
\begin{array}{ccc}
U_{g, b}^- & \xrightarrow{u_g^-} & Q_{g, b}^- \\
\downarrow & & \pi^- \downarrow \\
U_{g, b}^- & \xrightarrow{u_g} & \mathcal{M}_{g, b} \\
\downarrow & & \phi_g^- \downarrow \\
U_{g, b}^- & \xrightarrow{u_g} & \mathcal{M}_{g, b} \xrightarrow{\phi_g} \Delta_g
\end{array}
$$

where $\phi_g^-$ is the morphism induced on $Q_{g, b}^-$ by $m_g^-$. 

**Theorem 4.10.** We fix $7 \leq g \leq 9$ and $1 \leq b \leq g - 1$. Then $\phi_g^- : Q_{g, b}^- \to B_{g, b}^-$ is a birational isomorphism.

**Proof.** It suffices to note that $\phi_g^-$ is birational, and the vertical arrows of the diagram are finite morphisms of the same degree, namely the number of odd theta-characteristics on a curve of genus $g - b$. 

\[\square\]
We construct a projective bundle over $B_{g, \delta}$, then show that for certain values $\delta \leq g - 1$, the locus $B_{g, \delta}$ itself is unirational, whereas the above mentioned bundle dominates $S_{g}^-$. Let $C_{g, \delta} \subset \mathcal{U}_{g, \delta}^{-} \times V_g$ be the universal curve, endowed with its two projection maps

$$\mathcal{U}_{g, \delta}^{-} \xleftarrow{p} C_{g, \delta} \xrightarrow{q} V_g.$$  

We fix an arbitrary point $(\Gamma, Z) \in \mathcal{U}_{g, \delta}^{-}$ and let $\nu : N \to \Gamma$ be the normalization map. Recall that $\text{sing}(\Gamma)$ consists of $\delta$ linearly independent points and that $h^0(N, \mathcal{O}_N(\nu^*e)) = 1$, where $e$ is the effective divisor on $\Gamma$ characterized by $Z|_{\Gamma_{	ext{reg}}} = 2e$. Thus the restriction map $H^0(\Gamma, \omega_\Gamma) \to H^0(\omega_T \otimes \mathcal{O}_Z)$ has 1-dimensional kernel. In particular the relative cotangent sheaf $\omega_p$ admits a global section $s$ inducing an exact sequence

$$0 \to \mathcal{O}_{\mathcal{U}_{g, \delta}} \to \omega_p \to \mathcal{O}_W \otimes \omega_p \to 0,$$  

which defines a subscheme $W \subset C_{g, \delta}$, whose fibre at the point $(\Gamma, Z) \in \mathcal{U}_{g, \delta}$ is $Z$ itself. We set

$$A := p_* \left( \mathcal{I}_{W/C_{g, \delta}} \otimes q^* \mathcal{O}_V(1) \right),$$  

which is a vector bundle on $\mathcal{U}_{g, \delta}^-$ of rank $N_g - g + 2$. The fibre of $A(\Gamma, Z)$ is identified with $H^0(V_g, \mathcal{I}_{Z/V_g}(1))$. One has a natural identification

$$\text{P}H^0(\mathcal{I}_{Z/V_g}(1))^\vee = \{1\text{-dimensional linear sections of } V_g \text{ containing } Z\}.$$  

**Definition 4.11.** $\mathcal{P}_{g, \delta}$ is the projectivized dual of $A$.

From the definitions and the previous remark it follows:

**Proposition 4.12.** $\mathcal{P}_{g, \delta}$ is the Zariski closure of the incidence correspondence

$$\mathcal{P}_{g, \delta}^0 := \{(C, (\Gamma, Z)) \in \mathcal{U}_g \times \mathcal{U}_{g, \delta}^- : Z \subset C\}.$$  

Consider the projection maps

$$\mathcal{U}_g \xleftarrow{\alpha} \mathcal{P}_{g, \delta}^0 \xrightarrow{\beta} \mathcal{U}_{g, \delta}^-.$$  

We wish to know when $\alpha$ is a dominant map. For $1 \leq \delta < g \leq 9$, we have the following:

**Proposition 4.13.** The map $\alpha$ is dominant if and only if $\delta \leq N_g + 1 - g = \dim(V_g) - 1$.

**Proof.** By definition, the morphism $\beta$ is surjective. Let $(\Gamma, Z) \in \mathcal{U}_{g, \delta}^-$ be an arbitrary point, and set $\text{sing}(\Gamma) := \{p_1, \ldots, p_\delta\} \subset Z$. We define $P_Z$ to be the locus of 1-dimensional linear sections of $V_g$ containing $Z$. Inside $P_Z$ we consider the space

$$P_{\Gamma,Z} = \{\Gamma' \in P_Z : \text{sing}(\Gamma') \cap Z \supseteq \text{sing}(\Gamma) \cap Z\},$$  

First note that for $p \in \text{sing}(\Gamma)$, the locus $H_p := \{\Gamma' \in P_Z : p \in \text{sing}(\Gamma')\}$ is a hyperplane in $P_Z$. Indeed, we identify $P_Z$ with the family of linear spaces $L \subset G(g, N_g + 1)$ such that $\langle Z \rangle \subset L$. By the definition of the cluster $Z$, it follows that $l := \mathbb{T}_p(V_g) \cap \langle Z \rangle$ is a line. For $L \in P_Z$, the intersection $L \cap V_g$ is singular at $p$ if and only if $\dim(L \cap \mathbb{T}_p(V_g)) \geq 2$. This is obviously a codimension 1 condition in $P_Z$. Therefore, if for $1 \leq i \leq \delta$ we define the hyperplane $H_i := \{L = \langle \Gamma' \rangle \in P_Z : \dim(L \cap \mathbb{T}_{p_i}(V_g)) \geq 2\}$, then

$$P_{\Gamma,Z} = H_1 \cap \cdots \cap H_\delta.$$  

This shows that the general point in $\beta^{-1}(C, Z)$ corresponds to a smooth curve $C \supseteq Z$. We now fix a general point $(\Gamma, Z) \in \mathcal{U}_{g, \delta}$, corresponding to a general cluster $Z \in S_{g-1}$.
Claim: $P_{g, \delta}$ has codimension $\delta$ in $P_g$; its general element is a nodal curve with $\delta$ nodes.

Proof of claim: Indeed $P_g$ is a general fibre of the projective bundle $\mathcal{U}_g^- \to \mathfrak{Z}_{g-1}$. The claim follows since $\text{codim}(\mathcal{U}_g^-, \mathcal{U}_g^-) = \delta$.

The fibre $\alpha^{-1}((C, Z))$ over a general point $(C, Z) \in \mathcal{U}_g^-$, is the union of $(g-1)$ linear spaces $H_1 \cap \cdots \cap H_\delta \subset P_Z$ as above. By the claim above, when $Z \in \mathfrak{Z}_{g-1}$ is a general cluster, this is a union of linear spaces $P_{g, \delta}$ as before, having codimension $\delta$ in $P_Z$. Hence $\alpha^{-1}((C, Z))$ is not empty if and only if $\delta \leq \dim P_Z$, that is, $\delta \leq N_g - g + 1$. \qed

Let us fix the following notation:

**Definition 4.14.**

1. $\overline{\mathcal{P}}_{g, \delta} := (\mathcal{P}_{g, \delta}^\circ)_{// \text{Aut}(V)}$.
2. $\overline{\beta} : \overline{\mathcal{P}}_{g, \delta} \to \overline{\mathcal{S}}_g$ is the morphism induced by $\beta$ at the level of quotients.

Note that $\beta : \mathcal{P}_{g, \delta} \to \mathcal{U}_g^-$ is a projective bundle and $\text{Aut}(V)$ acts linearly on its fibres, therefore $\beta$ descends to a projective bundle on $B_{g, \delta}$. Then it follows from the previous remark that $\mathcal{P}_{g, \delta}$ is birationally isomorphic to $\mathbb{P}^{N_g - g + 1} \times B_{g, \delta}$.

To finish the proof of the unirationality of $\mathcal{S}_g^-$, we proceed as follows:

**Theorem 4.15.** Let $7 \leq g \leq 9$ and assume that (1) $B_{g, \delta}^- \text{ is unirational}$ and (ii) $\delta \leq N_g - g + 1$. Then $\mathcal{S}_g^-$ is unirational.

Proof. By assumption (ii), $\beta : \mathcal{P}_{g, \delta}^\circ \to \mathcal{U}_g^-$ is dominant, Hence the same is true for the induced morphism $\overline{\beta} : \overline{\mathcal{P}}_{g, \delta} \to \overline{\mathcal{S}}_g$. By (i) and the above remark, $\overline{\mathcal{P}}_{g, \delta}$ is unirational. Therefore $\overline{\mathcal{S}}_g^-$ is unirational as well. \qed

Theorem 4.15 has some straightforward applications. The case $\delta = g - 1$ is particularly convenient, since $B_{g, g-1}$ is isomorphic to the moduli space of integral curves of geometric genus 1 with $g - 1$ nodes. For $\delta = g - 1$, the assumptions of Theorem 4.15 hold when $g \leq 8$. In this range, the unirationality of $\mathcal{S}_g^-$ follows from that of $B_{g, g-1}$.

**Theorem 4.16.** $B_{g, g-1}^- \text{ is unirational for } g \leq 10$.

Proof. Let $I \subset \mathbb{P}^2 \times (\mathbb{P}^2)^\vee$ be the natural incidence correspondence consisting of pairs $(x, l)$ such that $x$ is a point on the line $l$. For $\delta \leq 9$, we define

$$\Pi_\delta := \{(x_1, l_1, \ldots, x_\delta, l_\delta, E) \in I^\delta \times \mathbb{P} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3)) : x_1, \ldots, x_\delta \in E\}.$$ 

Then there exists a rational map $f_\delta : \Pi_\delta \dashrightarrow B_{\delta+1, \delta}^-$ sending $(x_1, l_1, \ldots, x_\delta, l_\delta, E)$ to the moduli point of the $\delta$-nodal, integral curve $C$ obtained from the elliptic curve $E$, by identifying the pairs of points in $E \cap l_i - \{x_i\}$ for $1 \leq i \leq \delta$. It is easy to see that $\Pi_\delta$ is rational if $\delta \leq 9$. Clearly $f_\delta$ is dominant, just because every elliptic curve can be realized as a plane cubic. It follows that $B_{\delta+1, \delta}^-$ is unirational when $\delta \leq 9$. \qed

Unfortunately one cannot apply Theorem 4.16 to the case $g = 9$, since the assumptions of Theorem 4.15 are satisfied only if $\delta \leq 5$. 
5. The stack of degenerate odd theta-characteristics

In this section we define a Deligne-Mumford stack $X_g \rightarrow \overline{S}_g$ parameterizing limit linear series $g^0_{g-1}$ which appear as limits of degenerate theta-characteristics on smooth curves. The push-forward of $[X_g]$ is going to be precisely our divisor $\overline{Z}_g$. Having a good description of $X_g$ over the boundary will enable us to determine all the coefficients in the expression of $[\overline{Z}_g]$ in $\text{Pic}(\overline{S}_g)$.

We first define a partial compactification $\tilde{M}_g := M_g \cup \Delta_0 \cup \ldots \cup \Delta_{[g/2]}$ of $M_g$, obtaining by adding to $M_g$ the open sub-stack $\Delta_0 \subset \Delta_0$ of one-nodal irreducible curves $[C_{yq} := C/y \sim q]$, where $[C, y, q] \in M_{g-1,2}$ is a Brill-Noether general curve together with their degenerations $[C \cup D_{\infty}]$ where $D_{\infty}$ is an elliptic curve with $j(D_{\infty}) = \infty$, as well as the open substacks $\Delta_j \subset \Delta_j$ for $1 \leq j \leq [g/2]$ classifying curves $[C \cup D]$ where $[C] \in M_j$ and $[D] \in M_{g-j}$ are Brill-Noether general curves in the respective moduli spaces. Let $p : \tilde{M}_{g,1} = \tilde{M}_g$ be the universal curve. We denote $\tilde{S}_g := \pi^*(-M_g) \subset S_g$ and note that for all $0 \leq j \leq [g/2]$ the boundary divisors $A_j := A_j \cap \tilde{S}_g$, $B_j := B_j \cap \tilde{S}_g$ are mutually disjoint inside $\tilde{S}_g$. Finally, we consider $Z := \tilde{S}_g \times_{\tilde{M}_g} \tilde{M}_{g,1}$ and denote by $p_1 : Z \rightarrow \tilde{S}_g$ the projection.

Following the local description of the projection $S_g \rightarrow M_g$ carried out in [C], in order to obtain the universal spin curve over $\tilde{S}_g$, one has to blow-up the codimension 2 locus $V \subset Z$ corresponding to points

$$v = \left( [C \cup_{(y,q)} E], \eta_C^{\otimes 2} = K_C, \eta_C^0(\eta_C) \equiv 1 \text{ mod } 2, \eta_E = O_E(1), \nu(y) = \nu(q) \right) \in B_0 \times \tilde{M}_g \tilde{M}_{g,1}$$

(recall that $\nu : C \rightarrow C_{yq}$ denotes the normalization map, so $v$ corresponds to the marked point specializing to the node of the curve $C_{yq}$). Suppose that $(\tau_0, \ldots, \tau_{3g-3})$ are local coordinates in an étale neighbourhood of $[C \cup_{(y,q)} E, \eta_C, \eta_E] \in \tilde{S}_g$ such that the local equation of the divisor $B_0$ is $(\tau_0 = 0)$. Then $Z$ around $v$ admits local coordinates $(x, y, \tau_1, \ldots, \tau_{3g-3})$ verifying the equation $xy = \tau_1^2$, in particular, $Z$ is singular along $V$. Next, for $1 \leq j \leq [g/2]$ one blows-up the codimension 2 loci $V_j \subset Z$ consisting of points

$$\left( [C \cup D, \eta_C, \eta_D], q \in C \cap D \right) \in (A_j \cup B_j) \times \tilde{M}_g \tilde{M}_{g,1}.$$  

This corresponds to inserting an exceptional component in each spin curve in $\pi^*(\Delta_j)$. We denote by

$$\mathcal{C} := \text{Bl}_{V \cup V_1 \cup \ldots \cup V_{[g/2]}}(Z)$$

and by $f : C \rightarrow \tilde{S}_g$ the induced family of spin curves. Then for every $[X, \eta, \beta] \in \tilde{S}_g$ we have an isomorphism between $f^{-1}([X, \eta, \beta])$ and the quasi-stable curve $X$.

There exists a spin line bundle $P \in \text{Pic}(\mathcal{C})$ of relative degree $g-1$ as well as a morphism of $\mathcal{O}_C$-modules $B : P^{\otimes 2} \rightarrow \omega_f$ having the property that $P_{f^{-1}([X, \eta, \beta])} = \eta$ and $B_{f^{-1}([X, \eta, \beta])} = \beta : \eta^{\otimes 2} \rightarrow \omega_X$, for all spin curves $[X, \eta, \beta] \in \tilde{S}_g$. We note that for the even moduli space $\tilde{S}_g^+$ one has an analogous construction of the universal spin curve.

Next we define the stack $\tau : X_g \rightarrow \tilde{S}_g$ classifying limit $g^0_{g-1}$ which are twists of degenerate odd-spin curves. For a tree-like curve $X$ we denote by $\overline{G}_{d}(X)$ the scheme of limit linear series $g^0_{d}$. The fibres of the morphism $\tau$ have the following description:
• $\tau^{-1}(S^{-}_g)$ parameterizes triples $([C, \eta], \sigma, x)$, where $[C, \eta] \in S^{-}_g$, $x \in C$ is a point and $\sigma \in \text{Pic}^0(C, \eta)$ is a section such that $\text{div}(\sigma) \geq 2x$.

• For $1 \leq j \leq \lfloor g/2 \rfloor$ the inverse image $\tau^{-1}(B'_{j})$ parameterizes elements of the form

$$(X, \sigma \in \mathcal{G}_{g-1}(X), x \in \mathcal{X}_{\text{reg}}),$$

where $(X, x)$ is a 1-pointed quasi-stable curve semistably equivalent to the underlying curve of a spin curve $[C \cup q E \cup q' D, \eta_C, \eta_E, \eta_D] \in A'_j \cup B'_j$, with $E$ denoting the exceptional component, $g(C) = j$, $g(D) = g - j$, $\{q\} = C \cap E$, $\{q'\} = E \cap D$ and $\sigma_C \in \text{Pic}^0(C, \eta_C \otimes \mathcal{O}_C((g-j)q))$, $\sigma_D \in \text{Pic}^0(D, \eta_D \otimes \mathcal{O}_D(jq'))$, $\sigma_E \in \text{Pic}^0(E, \mathcal{O}_E(g-1))$ are aspects of the limit linear series $\sigma$ on $X$. Moreover, we require that $\text{ord}_x(\sigma) \geq 2$.

• $\tau^{-1}(B'_{0})$ parameterizes elements $(X, \eta \in \text{Pic}^{g-1}(X), \sigma \in \text{Pic}^0(X, \eta), x \in \mathcal{X}_{\text{reg}})$, where $(X, x)$ is a 1-pointed quasi-stable curve equivalent to the curve underlying a point $[C \cup (yq) E, \eta_C, \eta_E] \in B'_{0}$, the line bundle $\eta$ on $X$ satisfies $\eta|_C = \eta_C$ and $\eta|_E = \eta_E$ and $\eta|_Z = \mathcal{O}_Z$ for the remaining components of $X$. Finally, we require $\text{ord}_x(\sigma) \geq 2$.

• $\tau^{-1}(A'_{0})$ corresponds to points $(X, \eta \in \text{Pic}^{g-1}(X), \sigma \in \text{Pic}^0(X, \eta), x \in \mathcal{X}_{\text{reg}})$, where $(X, x)$ is a 1-pointed quasi-stable curve equivalent to the curve underlying a point $[C_{yq}, \eta_{C_{yq}}] \in A'_{0}$, and if $\mu : X \to C_{yq}$ is the map contracting all exceptional components, then $\mu^*(\eta_{C_{yq}}) = \eta$ (in particular $\eta$ is trivial along exceptional components), and finally $\text{ord}_x(\sigma) \geq 2$.

Using general constructions of stacks of limit linear series cf. [EH1], [F2], it is clear that $X_g$ is a Deligne-Mumford stack. There exists a proper morphism

$$\tau : X_g \to \mathcal{S}^{-}_g$$

that factors through the universal curve and we denote by $\chi : X_g \to \mathcal{C}$ the induced morphism, hence $\tau = f \circ \chi$. The push-forward of the coarse moduli space $\tau_*([X_g])$ equals scheme-theoretically $\mathcal{Z}_g \cap \mathcal{S}^{-}_g$. It appears possible to extend $X_g$ over the entire $\mathcal{S}^{-}_g$ but this is not necessary in order to prove Theorem [13] and we skip the details.

We are now in a position to calculate the class of the divisor $\mathcal{Z}_g$ and we expand its class in the Picard group of $\mathcal{S}^{-}_g$

$$(4) \quad \mathcal{Z}_g \equiv \lambda \cdot \lambda - \alpha_0 \cdot \alpha_0 - \beta_0 \cdot \beta_0 - \sum_{i=1}^{\lfloor g/2 \rfloor} \alpha_i \cdot \alpha_i - \sum_{i=1}^{\lfloor g/2 \rfloor} \beta_i \cdot \beta_i \in \text{Pic}(\mathcal{S}^{-}_g),$$

where $\lambda, \alpha_i, \beta_i \in \mathbb{Q}$ for $i = 0, \ldots, \lfloor g/2 \rfloor$. We start by determining the coefficients of the divisors $\alpha_i$ and $\beta_i$ for $1 \leq i \leq \lfloor g/2 \rfloor$.

**Proposition 5.1.** For $1 \leq i \leq \lfloor g/2 \rfloor$ we have that $F_i \cdot \mathcal{Z}_g = 4(g - i)(i - 1)$ and the intersection is everywhere transverse. It follows that $\alpha_i = 2(g - i)$.

**Proof.** We recall from the definition of $F_i$ that we have fixed theta-characteristics of opposite parity $\eta^{-}_C \in \text{Pic}^{i-1}(C)$ and $\eta^{-}_D \in \text{Pic}^{g-i-1}(D)$. We choose a point $t = (X, \eta, \sigma, x) \in \tau^{-1}(F_i)$. It is a simple exercise to show that the "double" point $x$ of $\sigma \in \mathcal{G}_{g-1}(X)$ cannot specialize to the exceptional component, therefore one has only two cases to consider depending on whether $x$ lies on $C$ or on $D$. Assume first that $x \in C$ and then $\sigma_C \in \text{Pic}^0(C, \eta^{-}_C \otimes \mathcal{O}_C((g-i)q))$ and $\sigma_D \in \text{Pic}^0(D, \eta^+_D \otimes \mathcal{O}_D(iq))$, where $\{q\} = C \cap D$ is a point which moves on $C$ but is fixed on $D$. Then $\text{ord}_q(\sigma_D) \leq i - 1$, therefore
ord_q(σ_C) ≥ g − i and then σ_C(−(g − i)q) ∈ PH^0(C, η_C^−). In particular, if we choose [C, η_C] ∈ S_i − Z_i, then the section σ_C(−(g − i)q) has only simple zeros, which shows that x cannot lie on C, so this case does not occur.

We are left with the possibility x ∈ D − {q}. One quickly concludes that the only possibility is ord_q(σ_C) = g − i + 1 and ord_q(σ_D) = i − 2. In particular, q ∈ supp(η_C^−) which gives i − 1 choices for the moving point q ∈ C. Furthermore σ_D(−(i − 2)q) ∈ H^0(D, η_D^+ ⊗ O_D(2q − 2x)), that is, x specializes to one of the ramification points of the pencil η_D^+ ⊗ O_D(2q) ∈ W_{q,i−1}^+(D). We note that because of the generality of [D, η_D^+] ∈ S_{g−i}^+ as well as that of q ∈ D, the pencil is base point free and complete. From the Hurwitz-Zeuthen formula one finds 4(g − i) ramification points of ∣η_D^+ ⊗ O_D(2q)∣, which leads to the formula F_i · Z_q = 4(g − i)(i − 1). The fact that τ_s(X_q) is transverse to F_i follows because the formation of X_q commutes with restriction to B'_0 and then one can easily show in a way similar to [EH2] Lemma 3.4 or by direct calculation that X_q × S− B'_0 is smooth at any of the points in τ^{-1}(F_i).

\[\square\]

**Proposition 5.2.** For 1 ≤ i ≤ [g/2] we have that G_i · Z_q = 4i(i − 1) and the intersection is transversal. In particular β̃_i = 2i.

**Proof.** This time we fix general points [C, η_C] ∈ S_i^+ and [D, η_D] ∈ S_{g−i}^− and q ∈ C ∩ D which is a fixed general point on D but an arbitrary point on C. Again, it is easy to see that if t = (X, σ, x) ∈ τ_−1(G_i) then x must lie either on C or on D. Assume first that x ∈ C − {q}. Then the aspects of σ are described as follows

\[σ_C ∈ PH^0(C, η_C^+ ⊗ O_C((g − i)q)), \quad σ_D ∈ PH^0(D, η_D^− ⊗ O_D(iq))\]

and moreover ord_x(σ_C) ≥ 2. The point q ∈ D can be chosen so that it does not lie in supp(η_D^−), hence ord_q(σ_D) ≤ i and then ord_q(σ_C) ≥ g − i − 1. This leads to the conclusion H^0(C, η_C^+ ⊗ O_C(y − 2x)) ≠ 0, or equivalently (x, y) ∈ C × C is a ramification point of the degree i covering p_1 : T_{η_C}^+ → C from the associated Scorza curve. We have shown that T_{η_C}^+ is smooth of genus 1 + 3i(i − 1) (cf. Theorem 2.1) and moreover all the ramification points of p_1 are ordinary, therefore we find

\[\deg \text{Ram}(p_1) = 2g(T_{η_C}^+) − 2 − \deg(p_1)(2i − 2) = 4i(i − 1)\]

choices when x ∈ C. Next possibility is x ∈ D − {q}. The same reasoning as above shows that ord_q(σ_C) ≤ g − i − 1, therefore ord_q(σ_D) ≥ i as well as ord_x(σ_D) ≥ 2. Since σ_D(−iq) ∈ PH^0(D, η_D), this case does not occur if [D, η_D] ∈ S_{g−i}^− − Z_{g−i}.

\[\square\]

Next we prove that Z_q is disjoint from both elliptic pencils F_0 and G_0:

**Proposition 5.3.** We have that F_0 · Z_q = 0 and G_0 · Z_q = 0. The equalities α̃ − 12α̃_0 + α̃_1 = 0 and 3β̃ − 12β̃_0 − 12β̃_1 + 3β̃_1 = 0 follow.

**Proof.** We first show that F_0 ∩ Z_q = ∅ and we assume by contradiction that there exists t = (X, σ, x) ∈ τ_−1(F_0). Let us deal first with the case when st(X) = C ∩ E_λ, with E_λ being a smooth curve of genus 1. The key point is that the point of attachment q ∈ C ∩ E_λ being general, we can assume that (x, q) /∈ Ram{p_1 : T_{η_C}^− → C}, for all x ∈ C. This implies that H^0(C, η_C^+ ⊗ O_C(q − 2x)) = 0 for all x ∈ C, therefore a section σ_C ∈ PH^0(C, η_C^+ ⊗ O_C(q)) cannot vanish twice anywhere. Thus either x ∈ E_λ − {q}
or $x$ lies on some exceptional component of $X$. In the former case, since $\text{ord}_q(\sigma_C) = 0$, it follows that $\text{ord}_q(\sigma_{E_0}) \geq g - 1$, that is, $\sigma_{E_0}$ has no zeroes other than $q$ (simple or otherwise). In the latter case, when $x \in E$, with $E$ being an exceptional component, we denote by $q' \in E$ the point of intersection of $E$ with the connected subcurve of $X$ containing $C$ as a subcomponent. Since as above, $\text{ord}_q(\sigma_C) = 0$, by compatibility it follows that $\text{ord}_{q'}(\sigma_E) = g - 1$. But $\sigma_E \in \text{PH}^0(E, O_E(g - 1))$, that is, $\sigma_E$ does not vanish at $x$, a contradiction. The proof that $G_0 \cap \bar{Z}_g = \emptyset$ is similar and we omit the details. \hfill $\square$

The trickiest part in the calculation of $[\bar{Z}_f]$ is the computation of the following intersection number:

**Proposition 5.4.** If $H_0 \subset B_0$ is the covering family lying in the ramification divisor of $S_g$, then one has that $H_0 \cdot \bar{Z}_g = 2(g - 2)$ and the intersection consists of $g - 2$ points each counted with multiplicity 2. Therefore the relation $(g - 1)\beta_0 - \beta_1 = 2(g - 2)$ holds.

**Proof.** We first determine the set-theoretic intersection $\tau_* (\mathcal{X}_g) \cap H_0$. We recall that we have fixed $[C, q, \eta_C] \in S_{g-1,1}$ and start by choosing an arbitrary point $t = (X, \eta, \sigma, x) \in \tau^{-1}(H_0)$. Assume first that $X = C \cup \{ y, q \} E$, where $y \in C$, that is, $x$ does not specialize to one of the nodes of $C \cup E$. Suppose first that $x \in C - \{ y, q \}$. From the Mayer-Vietoris sequence on $X$ we write

$$0 \neq \sigma \in H^0(X, \eta \otimes O_X(-2x)) = \text{Ker}\{ H^0(C, \eta_C \otimes O_C(-2x)) \oplus H^0(E, O_E(1)) \xrightarrow{\text{ev}_{y, q}} C^2_{(y, q)} \},$$

we obtain that $H^0(C, \eta_C \otimes O_C(-2x)) \neq 0$. This case can be avoided by choosing $[C, \eta_C] \in S_{g-1} - \bar{Z}_{g-1}$.

Next we consider the possibility $x \in E - \{ y, q \}$. The same Mayer-Vietoris argument reads in this case $0 \neq \text{Ker}\{ H^0(C, \eta_C) \oplus H^0(E, O_E(-1)) \xrightarrow{\text{ev}_{y, q}} C^2_{(y, q)} \}$, that is, $y + q \in \text{supp}(\eta_C)$. This case can be avoided as well by starting with a general point $q \in C - \text{supp}(\eta_C)$. Thus the only possibility is that $x$ specializes to one of the nodes $y$ or $q$.

We deal first with the case when $x$ and $q$ coalesce and there is no loss of generality in assuming that $X = C \cup E \cup E'$, where both $E$ and $E'$ are copies of $\mathbb{P}^1$ and $C \cap E = \{ y \}$, $C \cap E' = \{ y \}$, $E \cap E' = \{ y' \}$ and moreover $x \in E' - \{ y', q \}$. The restrictions of the line bundle $\eta \in \text{Pic}^{g-1}(X)$ are such that $\eta_C = \eta_C, \eta_E = O_E(1)$ and $\eta_E' = O_{E'}$. We write

$$0 \neq \sigma = (\sigma_C, \sigma_E, \sigma_E) \in \text{Ker}\{ H^0(C, \eta_C) \oplus H^0(E, O_E(1)) \oplus H^0(E', O_{E'}(1)) \xrightarrow{\text{ev}_{y, y', q}} C_{y, y', q} \},$$

hence $\sigma_{E'} = 0$, and then by compatibility $\sigma_C(q) = 0$, that is, $q \in \text{supp}(\eta_{C})$ and again this case can be ruled out by a suitable choice of $q$. The last possible situation is when $x$ and the moving point $y \in C$ coalesce, in which case $X = C \cup E \cup E'$, where this time $C \cap E = \{ q \}, C \cap E' = \{ q \}, E \cap E' = \{ y' \}$ and again $x \in E' - \{ y', q \}$. Writing one last time the Mayer-Vietoris sequence we find that $\sigma_{E'} = 0$ and then $\sigma_E(y') = 0$ and $\sigma_{C}(y) = 0$, that is, $y \in \text{supp}(\eta_{C})$ and then $\sigma_{C}$ is uniquely determined up to a constant. Finally $\sigma_E \in H^0(E, O_E(1)(-y'))$ is also uniquely specified by the gluing condition $\sigma_{E}(q) = \sigma_C(q)$. All in all, $H_0 \cap \bar{Z}_g = \# \text{supp}(\eta_C) = g - 2$.

This discussion singles out an irreducible component $\Xi \subset \chi_{s}(\mathcal{X}_g) \subset \mathcal{C}$ of the intersection $\chi_{s}(\mathcal{X}_g) \cap f^{-1}(B'_0)$, namely

$$\Xi = \{ [(C \cup \{ y, q \} E, \eta_C, \eta_E), x] : y \in \text{supp}(\eta_{C}), x = y \in X_{\text{sing}} \},$$
where we recall that \( f : C \to \mathcal{S}_g^- \) is the universal spin curve. Since \( \Xi \subset \text{Sing}(\chi_*(\mathcal{X}_g)) \), it follows after a simple local analysis that each point in \( \tau^{-1}(H_0) \) should be counted with multiplicity 2.

**Remark 5.5.** An partial independent check of Theorem 0.4 is obtained by computing using the Porteous formula the coefficient \( \lambda \) in the expression of \( [Z_g] \). By an abuse of notation we still denote by \( f : C \to \mathcal{S}_g^- \) the restriction of the universal spin curve to the locus of smooth curves and \( \eta \in \text{Pic}(C) \) the spin bundle of relative degree \( g - 1 \). Then \( Z_g \) is the push-forward via \( f : C \to \mathcal{S}_g^- \) of the degeneration locus of the sheaf morphism \( \phi : f_* (\eta) \to J_1 (\eta) \) (both these sheaves are locally free away a subset of codimension 3 in \( \mathcal{S}_g^- \) and throwing away this locus has no influence on divisor class calculations). Since \( \det (f_* \eta) = (f_* \eta)^{\otimes 2} \), it follows that \( c_1 (f_* (\eta)) = -\lambda / 4 \), whereas the Chern classes of the first jet bundle \( J_1 (\eta) \) are calculated using the standard exact sequence on \( C \)

\[
0 \to \eta \otimes \omega_f \to J_1 (\eta) \to \eta \to 0.
\]

Remembering Mumford’s formula \( f_* (c_1^2 (\omega_f)) = 12 \lambda \), one finally writes that

\[
[Z_g] = f_* c_2 \left( J_1 (\eta) - f_* (\eta) \right) = f_* \left( \frac{3}{4} c_1 (\omega_f)^2 - 2 c_1 (\omega_f) \cdot c_1 (f_* (\eta)) \right) = (g + 8) \lambda \in \text{Pic}(\mathcal{S}_g^-).
\]

6. A DIVISOR OF SMALL SLOPE ON \( \overline{M}_{12} \)

The aim of this section is to construct an effective divisor \( D \in \text{Eff}(\overline{M}_{12}) \) of slope \( s(D) < 6 + 12 / 13 \), that is, violating the Slope Conjecture. As pointed out in the proof of Theorem 0.3 this is precisely what is needed to show that \( \overline{S}_{12} \) is of general type.

**Theorem 6.1.** The following locus consisting of curves of genus 12

\( \mathcal{D}_{12} := \{ [C] \in \mathcal{M}_{12} : \exists L \in W_1^1 (C) \text{ with } \mu_0 (L) : \text{Sym}^2 H^0 (C, L) \to H^0 (C, L^{\otimes 2}) \text{ not injective} \} \)

is a divisor on \( \mathcal{M}_{12} \). The class of its compactification inside \( \overline{M}_{12} \) equals

\[
\overline{\mathcal{D}}_{12} \equiv 13245 \lambda - 1926 \delta_0 - 9867 \delta_1 - \sum_{j=2}^{6} b_j \delta_j \in \text{Pic}(\overline{M}_{12}),
\]

where \( b_j \geq b_1 \) for \( j \geq 2 \). In particular, \( s(\overline{\mathcal{D}}_{12}) = \frac{4415}{642} < 6 + \frac{12}{13} \).

This implies the following upper bound for the slope \( s(\overline{M}_{12}) \) of the moduli space:

**Corollary 6.2.**

\[
6 + \frac{10}{12} \leq s(\overline{M}_{12}) := \inf_{D \in \text{Eff}(\overline{M}_{12})} s(D) \leq \frac{4415}{642} \left( = 6 + \frac{10}{12} + \frac{14}{321} \right).
\]

Another immediate application, via [Log], [F1], concerns the birational type of \( \overline{M}_{g,n} \):

**Theorem 6.3.** The moduli space of \( n \)-pointed curves \( \overline{M}_{12,n} \) is of general type for \( n \geq 11 \).

The divisor \( \mathcal{D}_{12} \) is constructed as the push-forward of a codimension 3 cycle in the stack \( \mathcal{G}_{14} \to \mathcal{M}_{12} \) classifying linear series \( \mathcal{G}_{14} \). We describe the construction of this cycle, then extend this determinantal structure over a partial compactification of \( \mathcal{M}_{12} \). This will be essential to understand the intersection of \( \mathcal{D}_{12} \) with the boundary divisors \( \Delta_0 \) and \( \Delta_1 \) of \( \overline{M}_{12} \). We denote by \( \mathcal{M}_{12}^p \) the open substack of \( \mathcal{M}_{12} \) consisting of curves...
Let \( \pi: \mathcal{M}_{12,1} \to \mathcal{M}_{12} \) be the universal curve and then \( p_2: \mathcal{M}_{12,1}^{\circ} \times_{\mathcal{M}_{12}} \mathcal{G}_{14}^4 \to \mathcal{G}_{14}^4 \) denotes the natural projection. If \( L \) is a Poincaré bundle over \( \mathcal{M}_{12,1}^{\circ} \times_{\mathcal{M}_{12}} \mathcal{G}_{14}^4 \) (or over an étale cover), then by Grauert’s Theorem, both \( \mathcal{E} := (p_2)_*(\mathcal{L}) \) and \( \mathcal{F} := (p_2)_*(\mathcal{L}^\otimes 2) \) are vector bundles over \( \mathcal{G}_{14}^4 \), with \( \text{rank}(\mathcal{E}) = 5 \) and \( \text{rank}(\mathcal{F}) = h^0(C,L^\otimes 2) = 17 \) respectively. There is a natural vector bundle morphism over \( \mathcal{G}_{14}^4 \) given by multiplication of sections,

\[
\phi: \text{Sym}^2(\mathcal{E}) \to \mathcal{F},
\]

and we denote by \( \mathcal{U}_{12} \subset \mathcal{G}_{14}^4 \) its first degeneracy locus. We set \( \mathcal{D}_{12} := \sigma_*(\mathcal{U}_{12}) \). Since the degeneracy locus \( \mathcal{U}_{12} \) has expected codimension 3 inside \( \mathcal{G}_{14}^4 \), the locus \( \mathcal{D}_{12} \) is a virtual divisor on \( \mathcal{M}_{12}^{\circ} \).

We extend the vector bundles \( \mathcal{E} \) and \( \mathcal{F} \) over a partial compactification of \( \mathcal{G}_{14}^4 \) given by limit \( g_{14}^4 \). We denote by \( \Delta^p_C \subset \Delta_1 \subset \mathcal{M}_{12} \) the locus of curves \( [C \cup y E] \), where \( E \) is an arbitrary elliptic curve, \( [C] \in \mathcal{M}_{11} \) is a Brill-Noether general curve and \( y \in C \) is an arbitrary point. We then denote by \( \Delta^p_C \subset \Delta_0 \subset \mathcal{M}_{12} \) the locus consisting of curves \( [C_{y\eta}] \in \Delta_0 \), where \( [C,q] \in \mathcal{M}_{11,1} \) is Brill-Noether general and \( y \in C \) is arbitrary, as well as their degenerations \( [C \cup q E_{\infty}] \) where \( E_{\infty} \) is a rational nodal curve. Once we set

\[
\mathcal{M}_{12}^p := \mathcal{M}_{12} \cup \Delta^p_0 \cup \Delta^p_C \subset \mathcal{M}_{12},
\]

can extend the morphism \( \sigma \) to a proper morphism

\[
\sigma: \mathcal{G}_{14}^4 \to \mathcal{M}_{12}^p,
\]

from the stack \( \mathcal{G}_{14}^4 \) of limit linear series \( g_{14}^4 \) over the partial compactification \( \mathcal{M}_{12}^p \) of \( \mathcal{M}_{12} \).

We extend the vector bundles \( \mathcal{E} \) and \( \mathcal{F} \) over the stack \( \mathcal{G}_{14}^4 \). The proof of the following result proceeds along the lines of the proof of Proposition 3.9 in [F1]:

**Proposition 6.4.** There exist two vector bundles \( \mathcal{E} \) and \( \mathcal{F} \) defined over \( \mathcal{G}_{14}^4 \) with \( \text{rank}(\mathcal{E}) = 5 \) and \( \text{rank}(\mathcal{F}) = 17 \), together with a vector bundle morphism \( \phi: \text{Sym}^2(\mathcal{E}) \to \mathcal{F} \), such that the following statements hold:

- For \( [C,L] \in \mathcal{G}_{14}^4 \), with \( [C] \in \mathcal{M}_{12}^p \), we have that

\[
\mathcal{E}(C,L) = H^0(C,L) \quad \text{and} \quad \mathcal{F}(C,L) = H^0(C,L^\otimes 2).
\]

- For \( t = (C \cup_y E, l_C, l_E) \in \sigma^{-1}(\Delta^p_C) \), where \( g(C) = 11, g(E) = 1 \) and \( l_C = |L_C| \) is such that \( L_C \in W^1_{14}(C) \) has a cusp at \( y \in C \), then \( \mathcal{E}(t) = H^0(C,L_C) \) and

\[
\mathcal{F}(t) = H^0(C,L_C^\otimes 2(-2y)) \oplus \mathbb{C} \cdot u^2,
\]

where \( u \in H^0(C,L_C) \) is any section such that \( \text{ord}_y(u) = 0 \). If \( L_C \) has a base point at \( y \), then \( \mathcal{E}(t) = H^0(C,L_C) = H^0(C,L_C \otimes \mathcal{O}_C(-y)) \) and the image of a natural map \( \mathcal{F}(t) \to H^0(C,L_C^\otimes 2) \) is the subspace \( H^0(C,L_C^\otimes 2 \otimes \mathcal{O}_C(-2y)) \).
• Fix \( t = [C_{yy} := C/y \sim q, L] \in \sigma^{-1}(\Delta_0^p) \), with \( q, y \in C \) and \( L \in \overline{W}_{14}(C_{yy}) \) such that \( h^0(C, \nu^*L \otimes \mathcal{O}_C(-y - q)) = 4 \), where \( \nu : C \to C_{yy} \) is the normalization map. In the case when \( L \) is locally free we have that \( \mathcal{E}(t) = H^0(C, \nu^*L) \) and \( \mathcal{F}(t) = H^0(C, \nu^*L \otimes \mathcal{O}_C(-y - q)) \oplus \mathcal{C} \cdot u^2 \), where \( u \in H^0(C, \nu^*L) \) is any section not vanishing at \( y \) and \( q \). In the case when \( L \) is not locally free, that is, \( L \in \overline{W}_{14}(C_{yy}) - W_{14}^A(C_{yy}) \), then \( L = \nu_*(A) \), where \( A \in W_{13}(C) \) and the image of the natural map \( \mathcal{F}(t) \to H^0(C, \nu^*L \otimes \mathcal{O}_C) \) is the subspace \( H^0(C, A^{\otimes 2}) \).

To determine the push-forward \( \overline{\mathcal{G}_{12}}^{\text{virt}} = \sigma_*(c_3(F - \text{Sym}^2(E)) \in A^4(M_{12}^\circ) \), we study the restriction of the morphism \( \mathcal{G}_4 \) along the pull-backs of two curves sitting in the boundary of \( \overline{M}_{12} \) and which are defined as follows: We fix a general pointed curve \( [C, q] \in M_{11,1} \) and a general elliptic curve \( [E, y] \in M_{11,1} \). Then we consider the families

\[ C_0 := \{ C/y \sim q : y \in C \} \subset \Delta_0^p \subset \overline{M}_{12} \text{ and } C_1 := \{ C \cup_\pi E : y \in C \} \subset \Delta_1^p \subset \overline{M}_{12}. \]

These curves intersect the generators of \( \text{Pic}(\overline{M}_{12}) \) as follows:

\[ C_0 \cdot \lambda = 0, \quad C_0 \cdot \delta_0 = \deg(\omega_{C_{yy}}) = 22, \quad C_0 \cdot \delta_1 = 1 \text{ and } C_0 \cdot \delta_j = 0 \text{ for } 2 \leq j \leq 6, \text{ and} \]
\[ C_1 \cdot \lambda = 0, \quad C_1 \cdot \delta_0 = 0, \quad C_1 \cdot \delta_1 = -\deg(K_C) = -20 \text{ and } C_1 \cdot \delta_j = 0 \text{ for } 2 \leq j \leq 6. \]

Next, we fix a general pointed curve \( [C, q] \in M_{11,1} \) and describe the geometry of the pull-back \( \sigma^*(C_0) \subset \mathcal{G}_{12}^4 \). We consider the determinantal 3-fold

\[ Y := \{ (y, L) \in C \times W_{13}^A(C) : h^0(C, L \otimes \mathcal{O}_C(-y - q)) = 4 \} \]

together with the projection \( \pi_1 : Y \to C \). Inside \( Y \) we consider the following divisors

\[ \Gamma_1 := \{ (y, A \otimes \mathcal{O}_C(y)) : y \in C, \ A \in W_{13}^A(C) \} \text{ and} \]
\[ \Gamma_2 := \{ (y, A \otimes \mathcal{O}_C(q)) : y \in C, \ A \in W_{13}^A(C) \} \]

intersecting transversally along the curve \( \Gamma := \{ (q, A \otimes \mathcal{O}_C(q)) : A \in W_{13}^A(C) \} \cong W_{13}^A(C) \). We introduce the blow-up \( Y' \to Y \) of \( Y \) along \( \Gamma \) and denote by \( E_\Gamma \subset Y' \) the exceptional divisor and by \( \Gamma_1, \Gamma_2 \subset Y' \) the strict transforms of \( \Gamma_1 \) and \( \Gamma_2 \) respectively. We then define \( \tilde{Y} := Y'/\Gamma_1 \cong \Gamma_2 \), to be the variety obtained from \( Y' \) by identifying the divisors \( \Gamma_1 \) and \( \Gamma_2 \) over each \( (y, A) \in C \times W_{13}^A(C) \). Let \( \epsilon : \tilde{Y} \to Y \) be the projection map.

**Proposition 6.5.** With notation as above, one has a birational morphism of 3-folds

\[ f : \sigma^*(C_0) \to \tilde{Y}, \]

which is an isomorphism outside a curve contained in \( \epsilon^{-1}(\pi_1^{-1}(q)) \). The map \( f_{|\epsilon^{-1}(\pi_1^{-1}(q))} \) corresponds to forgetting the \( E_\infty \)-aspect of each limit linear series. Accordingly, the vector bundles \( \mathcal{E}_1^\sigma(C_0) \) and \( \mathcal{F}_1^\sigma(C_0) \) are pull-backs under \( \epsilon \circ f \) of vector bundles on \( Y \).

**Proof.** We fix a point \( y \in C - \{ q \} \) and denote by \( \nu : C \to C_{yy} \) the normalization map, with \( \nu(y) = \nu(q) \). We investigate the variety \( \overline{W}_{14}(C_{yy}) \subset \overline{\text{Pic}}^4(C_{yy}) \) of torsion-free sheaves \( L \) on \( C_{yy} \) with \( \deg(L) = 14 \) and \( h^0(C_{yy}, L) \geq 5 \). A locally free \( L \in \overline{W}_{14}(C_{yy}) \) is determined by \( \nu^*(L) \in W_{14}(C) \), which has the property \( h^0(C, \nu^*L \otimes \mathcal{O}_C(-y - q)) = 4 \) (use that since \( W_{12}^A(C) = \emptyset \), there exists a section of \( L \) that does not vanish simultaneously at both \( y \) and \( q \)). However, the line bundles of type \( A \otimes \mathcal{O}_C(y) \) or \( A \otimes \mathcal{O}_C(q) \) with \( A \in W_{13}^A(C) \), do not appear in this association, though \( (y, A \otimes \mathcal{O}_C(y)), (y, A \otimes \mathcal{O}_C(q)) \in \)
Y. In fact, they correspond to the situation when $L \in \overline{W}_4^{14}(C_{yy})$ is not locally free, in which case necessarily $L = \nu_*(A)$ for some $A \in W^{14}_4(C)$. Thus, for a point $y \in C - \{q\}$, there is a birational morphism $\pi^{-1}_1(y) : \overline{W}_4^{14}(C_{yy})$ which is an isomorphism over the locus of locally free sheaves. More precisely, $\overline{W}_4^{14}(C_{yy})$ is obtained from $\pi^{-1}_1(y)$ by identifying the disjoint divisors $\Gamma_1 \cap \pi^{-1}_1(y)$ and $\Gamma_2 \cap \pi^{-1}_1(y)$.

A special analysis is required when $y = q$, when $C_{yy}$ degenerates to $C \cup q E_\infty$, where $E_\infty$ is a rational nodal cubic. If $\{l_C, l_{E_\infty}\} \in \sigma^{-1}([C \cup q E_\infty])$, then the corresponding Brill-Noether numbers with respect to $q$ satisfy $\rho(l_C, q) \geq 0$ and $\rho(l_{E_\infty}, q) \leq 2$. The statement about the restrictions $\mathcal{E}|_{\sigma^*(C_0)}$ and $\mathcal{F}|_{\sigma^*(C_0)}$ follows, because both restrictions are defined by dropping the information coming from the elliptic tail.

To describe $\sigma^*(C_1) \subseteq \overline{\Theta}_4^{14}$, where $[C] \in M_{11}$, we define the determinantal 3-fold

$$X := \{(y, L) \in C \times W^{14}_4(C) : h^0(L \otimes \mathcal{O}_C(-2y)) = 4\}.$$ 

In what follows we use notation from [EH1], to denote vanishing sequences of limit linear series:

**Proposition 6.6.** With notation as above, the 3-fold $X$ is an irreducible component of $\sigma^*(C_1)$. Moreover one has that $c_3((\mathcal{F} - \text{Sym}^2 \mathcal{E})|_{\sigma^*(C_1)}) = c_3((\mathcal{F} - \text{Sym}^2 \mathcal{E})|_{X})$.

**Proof.** By the additivity of the Brill-Noether number, if $\{l_C, l_E\} \in \sigma^{-1}([C \cup q E])$, we have that $2 = \rho(12, 4, 14) \geq \rho(l_C, y) + \rho(l_E, y)$. Since $\rho(l_C, y) \geq 0$, we obtain that $\rho(l_C, y) \leq 2$. If $\rho(l_E, y) = 0$, then $l_E = 9y + |\mathcal{O}_E(5y)|$, that is, $l_E$ is uniquely determined, while the aspect $l_C \in G^{4}_{14}(C)$ is a complete $g_{14}^1$ with a cusp at the variable point $y \in C$. This gives rise to an element from $X$. The remaining components of $\sigma^*(C_1)$ are indexed by Schubert indices $\tilde{\alpha} := (0 \leq \alpha_0 \leq \cdots \leq \alpha_4 \leq 10)$ such that $\tilde{\alpha} > (0, 1, 1, 1, 1)$ and $5 \leq \sum_{j=0}^4 \alpha_j \leq 7$. For such $\tilde{\alpha}$, we set $\tilde{\alpha}^c := (10 - \alpha_4, \ldots, 10 - \alpha_0)$ to be the complementary Schubert index, then define

$$X_{\tilde{\alpha}} := \{(y, l_C) \in C \times G^{4}_{14}(C) : \rho_{l_C}(y) \geq \tilde{\alpha}\} \text{ and } Z_{\tilde{\alpha}} := \{l_E \in G^{4}_{14}(E) : \rho_{l_E}(y) \geq \tilde{\alpha}^c\}.$$ 

Then $\sigma^*(C_1) = X + \sum_{\tilde{\alpha}} X_{\tilde{\alpha}} \times Z_{\tilde{\alpha}}$. The last claim follows by dimension reasons. Since $\dim X_{\tilde{\alpha}} = 1 + \rho(11, 4, 14) - \sum_{j=0}^4 \alpha_j < 3$, for every $\tilde{\alpha} > (0, 1, 1, 1, 1)$ and the restrictions of both $\mathcal{E}$ and $\mathcal{F}$ are pulled-back from $X_{\tilde{\alpha}}$, one obtains that $c_3(\mathcal{F} - \text{Sym}^2 \mathcal{E})|_{X_{\tilde{\alpha}} \times Z_{\tilde{\alpha}}} = 0$. □

We also recall standard facts about intersection theory on Jacobians. For a Brill-Noether general curve $[C] \in \mathcal{M}_g$, we denote by $\mathcal{P}$ a Poincaré bundle on $C \times \text{Pic}^d(C)$ and by $\pi_1 : C \times \text{Pic}^d(C) \to C$ and $\pi_2 : C \times \text{Pic}^d(C) \to \text{Pic}^d(C)$ the projections. We define the cohomology class $\eta = \pi_1^!([\text{point}]) \in H^2(C \times \text{Pic}^d(C))$, and if $\delta_1, \ldots, \delta_2g \in H^1(C, \mathbb{Z}) \cong H^1(\text{Pic}^d(C), \mathbb{Z})$ is a symplectic basis, then we set

$$\gamma := - \sum_{\alpha=1}^{2^g} \left(\pi_1^!(\delta_{\alpha}) \cdot \pi_2^!(\delta_{\eta + \alpha}) - \pi_1^!(\delta_{\eta + \alpha}) \cdot \pi_2^!(\delta_{\alpha})\right) \in H^2(C \times \text{Pic}^d(C)).$$

One has the formula $c_1(\mathcal{P}) = d\eta + \gamma$, corresponding to the Hodge decomposition of $c_1(\mathcal{P})$, as well as the relations $\gamma^2 = 0$, $\eta \gamma = 0$, $\eta^2 = 0$ and $\gamma^2 = -2\eta \pi_2^!(\theta)$. On $W^r_4(C)$ there is a tautological rank $r + 1$ vector bundle $\mathcal{M} := (\pi_2)_* (\mathcal{P}|_{C \times W^r_4(C)})$. To compute the Chern numbers of $\mathcal{M}$ we employ the Harris-Tu formula [HT]. We write $\sum_{i=0}^r c_i(\mathcal{M}^i) =$
(1 + x_1) \cdots (1 + x_{r+1}), and then for every class \( \zeta \in H^*(\text{Pic}^d(C), \mathbb{Z}) \) one has the following formula:

\[
x^{i_1} \cdots x^{i_{r+1}} \zeta = \det \left( \frac{g_{g-r+i_j-j+t}}{(g + r - d + i_j - j + t)} \right)_{1 \leq j, t \leq r+1}.
\]

We compute the classes of the 3-folds that appear in Propositions 6.5 and 6.6.

**Proposition 6.7.** Let \([C, q] \in M_{11, 1}\) be a Brill-Noether general pointed curve. If \( \mathcal{M} \) denotes the tautological rank 5 vector bundle over \( W^4_{14}(C) \) and \( c_i := c_i(\mathcal{M}) \in H^{2i}(W^4_{14}(C), \mathbb{C}) \), then one has the following relations:

(i) \( [X] = \pi_2^*(c_4) - 6\eta\theta \pi_2^*(c_2) + (48\eta + 2\gamma)\pi_2^*(c_3) \in H^{8}(C \times W^4_{14}(C), \mathbb{C}) \).

(ii) \( [Y] = \pi_2^*(c_4) - 2\eta\theta \pi_2^*(c_2) + (13\eta + \gamma)\pi_2^*(c_3) \in H^{8}(C \times W^4_{14}(C), \mathbb{C}) \).

**Proof.** We start by noting that \( W^4_{14}(C) \) is a smooth 6-fold isomorphic to the symmetric product \( C_6 \). We realize \( X \) as the degeneracy locus of a vector bundle morphism defined over \( C \times W^4_{14}(C) \). For each pair \((y, L) \in C \times W^4_{14}(C)\), there is a natural map

\[ H^0(C, L \otimes \mathcal{O}_2) \to H^0(C, L)^{\nu} \]

which globalizes to a vector bundle morphism \( \zeta : \mathcal{J}(\mathcal{P})^\nu \to \pi_2^*(\mathcal{M})^\nu \) over \( C \times W^4_{14}(C) \). Then we have the identification \( X = Z_1(\zeta) \) and the Thom-Porteous formula gives that \([X] = c_4(\pi_2^*(\mathcal{M}) - J_1(\mathcal{P}^\nu))\). From the usual exact sequence over \( C \times \text{Pic}^{14}(C) \)

\[ 0 \to \pi_1^*(K_C) \otimes \mathcal{P} \to J_1(\mathcal{P}) \to \mathcal{P} \to 0, \]

we can compute the total Chern class of the jet bundle

\[ c_t(J_1(\mathcal{P})^\nu)^{-1} = \left( \sum_{j \geq 0} (d(L)\eta + \gamma)^j \right) \cdot \left( \sum_{j \geq 0} ((2g(C) - 2 + d(L))\eta + \gamma)^j \right) = 1 - 6\eta\theta + 48\eta + 2\gamma, \]

which quickly leads to the formula for \([X]\). To compute \([Y]\) we proceed in a similar way. We denote by \( \mu, \nu : C \times C \times \text{Pic}^{14}(C) \to C \times \text{Pic}^{14}(C) \) the two projections, by \( \Delta \subset C \times C \times \text{Pic}^{14}(C) \) the diagonal and we set \( \Gamma_q := \{ q \} \times \text{Pic}^{14}(C) \). We introduce the rank 2 vector bundle \( \mathcal{B} := (\mu)_*(\nu^*(\mathcal{P}) \otimes \mathcal{O}_{\Delta \times \nu^*(\mathcal{V}_q)}) \) defined over \( C \times W^4_{14}(C) \). We note that there is a bundle morphism \( \chi : \mathcal{B}^\nu \to (\pi_2)^*(\mathcal{M})^\nu \), such that \( Y = Z_1(\chi) \). Since we also have that

\[ c_t(\mathcal{B}^\nu)^{-1} = (1 + (d(L)\eta + \gamma) + (d(L)\eta + \gamma)^2 + \cdots) (1 - \eta), \]

we immediately obtained the stated expression for \([Y]\). \( \square \)

**Proposition 6.8.** Let \([C] \in M_{11} \) and denote by \( \mu, \nu : C \times C \times \text{Pic}^{14}(C) \to C \times \text{Pic}^{14}(C) \) the natural projections. We define the vector bundles \( \mathcal{A}_2 \) and \( \mathcal{B}_2 \) on \( C \times \text{Pic}^{14}(C) \) having fibres

\[ \mathcal{A}_2(y, L) = H^0(C, L^\otimes 2 \otimes \mathcal{O}_C(-2y)) \text{ and } \mathcal{B}_2(y, L) = H^0(C, L^\otimes 2 \otimes \mathcal{O}_C(-y - q)), \]

respectively. One has the following formulas:

\[ c_1(\mathcal{A}_2) = -4\theta - 4\gamma - 76\eta, \quad c_1(\mathcal{B}_2) = -4\theta - 2\gamma - 27\eta, \]

\[ c_2(\mathcal{A}_2) = 8\theta^2 + 280\eta\theta + 16\gamma\theta, \quad c_2(\mathcal{B}_2) = 8\theta^2 + 100\eta\theta + 8\theta\gamma, \]

\[ c_3(\mathcal{A}_2) = -\frac{32}{3}\theta^3 - 512\eta\theta^2 - 32\theta^2\gamma \quad \text{and} \quad c_3(\mathcal{B}_2) = -\frac{32}{3}\theta^3 - 184\eta\theta^2 - 16\theta^2\gamma. \]

**Proof.** Immediate application of Grothendieck-Riemann-Roch with respect to \( \nu \). \( \square \)
Before our next result, we recall that if $\mathcal{V}$ is a vector bundle of rank $r + 1$ on a variety $X$, we have the formulas:

(i) $c_1(\text{Sym}^2(\mathcal{V})) = (r + 2)c_1(\mathcal{V})$.
(ii) $c_2(\text{Sym}^2(\mathcal{V})) = \frac{r(r+3)}{6}c_1^2(\mathcal{V}) + (r+3)c_2(\mathcal{V})$.
(iii) $c_3(\text{Sym}^2(\mathcal{V})) = \frac{r(r+4)(r-1)}{6}c_1^3(\mathcal{V}) + (r+5)c_3(\mathcal{V}) + (r^2 + 4r - 1)c_1(\mathcal{V})c_2(\mathcal{V})$.

We expand $\sigma_*(c_3(\mathcal{F} - \text{Sym}^2\mathcal{E})) \equiv a\lambda - b_0\delta_0 - b_1\delta_1 \in A^1(\mathcal{M}_{12})$ and determine the coefficients $a, b_0$ and $b_1$. This will suffice in order to compute $s(\mathcal{D}_{12})$.

**Theorem 6.9.** Let $[C] \in \mathcal{M}_{11}$ be a Brill-Noether general curve and denote by $C_1 \subset \Delta_1 \subset \mathcal{M}_{12}$ the associated test curve. Then the coefficient of $\delta_1$ in the expansion of $\mathcal{D}_{22}$ is equal to

$$b_1 = \frac{1}{2g(C) - 2} \sigma^*(C_1) \cdot c_3(\mathcal{F} - \text{Sym}^2\mathcal{E}) = 9867.$$

**Proof.** We intersect the degeneracy locus of the map $\phi : \text{Sym}^2(\mathcal{E}) \to \mathcal{F}$ with the 3-fold $\sigma^*(C_1) = X + \sum_\alpha X_\alpha \times Z_\alpha$. As already explained in Proposition 6.6, it is enough to estimate the contribution coming from $X$ and we can write

$$\sigma^*(C_1) \cdot c_3(\mathcal{F} - \text{Sym}^2\mathcal{E}) = c_3(\mathcal{F}|_X) - c_3(\text{Sym}^2\mathcal{E}|_X) - c_1(\mathcal{F}|_X)c_2(\text{Sym}^2\mathcal{E}|_X) + 2c_1(\text{Sym}^2\mathcal{E}|_X)c_2(\text{Sym}^2\mathcal{E}|_X) - c_1(\text{Sym}^2\mathcal{E}|_X)c_2(\mathcal{F}|_X) + c_1(\text{Sym}^2\mathcal{E}|_X)c_1(\mathcal{F}|_X) - c_3(\text{Sym}^2\mathcal{E}|_X).$$

We are going to compute each term in the right-hand-side of this expression.

Recall that we have constructed in Proposition 6.7 a vector bundle morphism $\zeta : J_1(\mathcal{P})^\vee \to \pi_5^*(\mathcal{M})^{\vee}$. We consider the kernel line bundle $\text{Ker}(\zeta)$. If $U$ is the line bundle on $X$ with fibre

$$U(y, L) = \frac{H^0(C, L)}{H^0(C, L \otimes \mathcal{O}_C(-2y))} \hookrightarrow H^0(C, L \otimes \mathcal{O}_{2y})$$

over a point $(y, L) \in X$, then one has an exact sequence over $X$

$$0 \to U \to J_1(\mathcal{P}) \to \text{Ker}(\zeta)^\vee \to 0.$$ 

In particular, $c_1(U) = 2\gamma + 48\eta - c_1(\text{Ker}(\zeta))$. The products of the Chern class of $\text{Ker}(\zeta)^\vee$ with other classes on $C \times W_{14}^4(C)$ can be computed from the Harris-Tu formula [HT]:

$$c_1(\text{Ker}(\zeta)^\vee) \cdot \xi|_X = -c_5(\pi_5^*(\mathcal{M})^{\vee} - J_1(\mathcal{P})^\vee) \cdot \xi|_X = -(\pi_5^*(c_5) - 6\eta \pi_5^*(c_3) + (48\eta + 2\gamma) \pi_5^*(c_4)) \cdot \xi|_X,$$

for any class $\xi \in H^2(C \times W_{14}^4(C), \mathcal{C})$.

If $A_3$ denotes the rank 18 vector bundle on $X$ having fibres $A_3(y, L) = H^0(C, L^{\otimes 2})$, then there is an injective morphism $U^{\otimes 2} \hookrightarrow A_3/A_2$, and we consider the quotient sheaf

$$\mathcal{G} := \frac{A_3/A_2}{U^{\otimes 2}}.$$ 

Since the morphism $U^{\otimes 2} \twoheadrightarrow A_3/A_2$ vanishes along the locus of pairs $(y, L)$ where $L$ has a base point, $\mathcal{G}$ has torsion along $\Gamma \subset X$. A straightforward local analysis now shows that $\mathcal{F}|_X$ can be identified as a subsheaf of $A_3$ with the kernel of the map $A_3 \to \mathcal{G}$. Therefore, there is an exact sequence of vector bundles on $X$

$$0 \to A_2|_X \to \mathcal{F}|_X \to U^{\otimes 2} \to 0,$$

which over a general point of $X$ corresponds to the decomposition

$$\mathcal{F}(y, L) = H^0(C, L^{\otimes 2} \otimes \mathcal{O}_C(-2y)) \oplus \mathcal{C} \cdot u^2,$$
where \( u \in H^0(C, L) \) is such that \( \text{ord}_y(u) = 1 \). The analysis above, shows that the sequence stays exact over the curve \( \Gamma \) as well. Hence
\[
\begin{align*}
c_1(F|_X) &= c_1(A_2|_X) + 2c_1(U), \quad c_2(F|_X) = c_2(A_2|_X) + 2c_1(A_2|_X)c_1(U) \quad \text{and} \\
c_3(F|_X) &= c_3(A_2) + 2c_2(A_2|_X)c_1(U).
\end{align*}
\]
Furthermore, since \( E|_X = \pi_2^*(M)|_X \), we obtain that:
\[
\begin{align*}
\sigma^*(C_1) \cdot c_3(F - \text{Sym}^2E) &= c_3(A_2|_X) + c_2(A_2|_X)c_1(U^\otimes 2) - c_3(\text{Sym}^2\pi_2^*M|_X) - \\
&\left( \frac{r(r + 3)}{2} c_1(\pi_2^*M|_X) + (r + 3)c_2(\pi_2^*M|_X) \right) \left( c_1(A_2|_X) + c_1(U^\otimes 2) - 2(r + 2)c_1(\pi_2^*M|_X) \right) - \\
&-(r + 2)c_1(\pi_2^*M|_X)c_2(A_2|_X) - (r + 2)c_1(\pi_2^*M|_X)c_1(A_2|_X)c_1(U^\otimes 2) + \\
&(r + 2)^2c_1(\pi_2^*M|_X)c_1(A_2|_X) + (r + 2)^2c_1^2(\pi_2^*M|_X)c_1(U^\otimes 2) - (r + 2)^3c_1^3(\pi_2^*M|_X).
\end{align*}
\]
As before, \( c_1(\pi_2^*M|_X) = \pi_2^*(c_1) \in H^2_1(X, \mathbb{C}) \). The coefficient of \( c_1(\text{Ker}(\zeta)) \) in the product \( \sigma^*(C_1) \cdot c_3(F - \text{Sym}^2E) \) is evaluated via \( \mathcal{L} \). The part of this product that does not contain \( c_1(\text{Ker}(\zeta)) \) equals
\[
\begin{align*}
28\pi_2^2(c_2)\theta - 88\pi_2^2(c_1^2)\theta + 440\eta\pi_2^2(c_1^2) - 53\pi_2^2(c_1c_2) - \frac{32}{3}\theta^3 + 128\eta\theta^2 - 432\eta\theta\pi_2^2(c_1) \\
+ 64\pi_2^2(c_1^3) - 140\eta\pi_2^2(c_2) + 48\theta^2\pi_2^2(c_1) + 9\pi_2^2(c_3) \in H^0_0(C \times W^3_{14}(C), \mathbb{C}).
\end{align*}
\]
Multiplying this quantity by the class \([X]\) obtained in Proposition \( \ref{Proposition6.7} \) and then adding to it the contribution coming from \( c_1(\text{Ker}(\zeta)) \), one obtains a homogeneous polynomial of degree 7 in \( \eta, \theta \) and \( \pi_2^2(\pi_1) \) for \( 1 \leq i \leq 4 \). The only non-zero monomials are those containing \( \eta \). After retaining only these monomials, the resulting degree 6 polynomial in \( \theta, c_1 \in H^0_1(W_{14}^3(C), \mathbb{Z}) \) can be brought to a manageable form, by noting that, since \( h^1(C, L) = 1 \), the classes \( c_i \) are independent. Precisely, if one fixes a divisor \( D \in C_e \) of large degree, there is an exact sequence
\[
0 \to M \to (\pi_2)_*(\mathcal{P} \otimes \mathcal{O}(\pi^*D)) \to (\pi_2)_*(\mathcal{P} \otimes \mathcal{O}(\pi_1^*D)|\pi_1^*D) \to R^1\pi_{2*}(\mathcal{P}|_{C \times W^3_{14}(C)}) \to 0,
\]
from which, via the well-known fact \( c_i((\pi_2)_*(\mathcal{P} \otimes \mathcal{O}(\pi_1^*D))) = \theta^i \), it follows that
\[
c_i R^1\pi_{2*}(\mathcal{P}|_{C \times W^3_{14}(C)}) \cdot e^{-\theta} = \sum_{i=0}^{4}(-1)^i c_i.
\]
Hence \( c_{i+1} = \theta^i c_i / i! - i\theta^{i+1} / (i+1)! \), for all \( i \geq 2 \). After routine manipulations, one finds that \( b_1 = \sigma^*(C_1) \cdot c_3(F - \text{Sym}^2E) / 20 = 9867 \).

**Theorem 6.10.** Let \([C, q] \in M_{1,1,1}\) be a Brill-Noether general pointed curve and we denote by \( C_0 \subset \Delta_0 \subset \overline{M}_{12} \) the associated test curve. Then \( \sigma^*(C_0) \cdot c_3(F - \text{Sym}^2E) = 22b_0 - b_1 = 32505 \). It follows that \( b_0 = 1926 \).

**Proof.** As already noted in Proposition \( \ref{Proposition6.5} \) the vector bundles \( \mathcal{E}_{[\sigma^*(C_0)]} \) and \( \mathcal{F}_{[\sigma^*(C_0)]} \) are both pull-backs of vector bundles on \( Y \) and we denote these vector bundles \( \mathcal{E} \) and \( \mathcal{F} \) as well, that is, \( \mathcal{E}_{[\sigma^*(C_0)]} = (e \circ f)^*(\mathcal{E}_Y) \) and \( \mathcal{F}_{[\sigma^*(C_0)]} = (e \circ f)^*(\mathcal{F}_Y) \). Like in the proof of Theorem \( \ref{Theorem6.9} \) we evaluate each term appearing in \( \sigma^*(C_0) \cdot c_3(F - \text{Sym}^2E) \).

Let \( V \) be the line bundle on \( Y \) with fibre
\[
V(y, L) = \frac{H^0(C, L)}{H^0(C, L \otimes \mathcal{O}_C(-y - q))} \cong H^0(C, L \otimes \mathcal{O}_{y+q})
\]
over a point \((y, L) \in Y\). There is an exact sequence of vector bundles over \(Y\)
\[
0 \to V \to B \to \text{Ker}(\chi) \to 0,
\]
where \(\chi : B^\vee \to \pi_2^*(M)^\vee\) is the bundle morphism defined in the second part of Proposition 6.7. In particular, \(c_1(V) = 13\eta + \gamma - c_1(\text{Ker}(\chi))\). By using again [HT], we find the following formulas for the Chern numbers of \(\text{Ker}(\chi)\):
\[
c_1(\text{Ker}(\chi)) \cdot \xi_Y = -c_5(\pi_2^*(M)^\vee - B^\vee) \cdot \xi_Y = -\pi_2^2(c_5) + \pi_2^2(c_4)(13\eta + \gamma) - 2\pi_2^2(c_3)\eta\theta \cdot \xi_Y,
\]
for any class \(\xi \in H^2(C \times W_{14}^4(C), \mathbb{C})\). Recall that we introduced the vector bundle \(B_2\) over \(C \times W_{14}^4(C)\) with fibre \(B_2(y, L) = H^0(C, L^\otimes 2 \otimes O_C(-y - q))\). We claim that one has an exact sequence of bundles over \(Y\)
\[
0 \to B_{2|Y} \to \mathcal{F}_{|Y} \to V^\otimes 2 \to 0.
\]
If \(B_3\) is the vector bundle on \(Y\) with fibres \(B_3(y, L) = H^0(C, L^\otimes 2)\), we have an injective morphism of sheaves \(V^\otimes 2 \hookrightarrow B_3/B_2\) locally given by
\[
v^\otimes 2 \mapsto v^2 \mod H^0(C, L^\otimes 2 \otimes O_C(-y - q)),
\]
where \(v \in H^0(C, L)\) is any section not vanishing at \(q\) and \(y\). Then \(\mathcal{F}_{|Y}\) is canonically identified with the kernel of the projection morphism
\[
B_3 \to B_3/B_2 \overset{\otimes 2}{\to}
\]
and the exact sequence (7) now becomes clear. Therefore \(c_1(\mathcal{F}_{|Y}) = c_1(B_{2|Y}) + 2c_1(V)\), \(c_2(\mathcal{F}_{|Y}) = c_2(B_{2|Y}) + 2c_1(B_{2|Y})c_1(V)\) and \(c_3(\mathcal{F}_{|Y}) = c_3(B_{2|Y}) + 2c_2(B_{2|Y})c_1(V)\). The part of the total intersection number \(\sigma^*(C_0) \cdot c_3(\mathcal{F} - \text{Sym}^2\mathcal{E})\) that does not contain \(c_1(\text{Ker}(\chi))\) equals
\[
28\pi_2^2(c_2)\theta - 88\pi_2^2(c_1^2)\theta - 22\eta\pi_2^2(c_1^2) - 53\pi_2^2(c_1c_2) - \frac{32}{3}\theta^3 +
\]
\[-8\eta\theta^2 + 24\eta\theta \pi_2^2(c_1) + 64\pi_2^2(c_1^3) + 7\eta\pi_2^2(c_2) + 48\theta^2 \pi_2^2(c_1) + 9\pi_2^2(c_3) \in H^6(C \times W_{14}^4(C), \mathbb{C})\]
and this gets multiplied with the class \([Y]\) from Proposition 6.7. The coefficient of \(c_1(\text{Ker}(\chi))\) in \(\sigma^*(C_0) \cdot c_3(\mathcal{F} - \text{Sym}^2\mathcal{E})\) equals
\[
-2c_2(B_{2|Y}) - 2(r + 2)^2\pi_2^2(c_1^2) - 2(r + 2)c_1(B_{2|Y})\pi_2^2(c_1) + r(r + 3)\pi_2^2(c_1^2) + 2(r + 3)\pi_2^2(c_2).
\]
All in all, \(22b_0 - b_1 = \sigma^*(C_0) \cdot c_3(\mathcal{F} - \text{Sym}^2\mathcal{E})\) and we evaluate this using (6).

The following result follows from the definition of the vector bundles \(\mathcal{E}\) and \(\mathcal{F}\) given in Proposition 6.4.

**Theorem 6.11.** Let \([C, q] \in M_{11,1}\) be a Brill-Noether general pointed curve and \(R \subset \overline{M}_{12}\) the pencil obtained by attaching at the fixed point \(q \in C\) a pencil of plane cubics. Then
\[
a - 12b_0 + b_1 = \sigma_+ c_3(\mathcal{F} - \text{Sym}^2\mathcal{E}) \cdot R = 0.
\]

**End of the proof of Theorem 6.7.** The fact that the virtual divisor \(\mathcal{O}_{12}\) is a genuine divisor on \(M_{12}\) follows from [11]. Assuming by contradiction that for every curve \([C] \in M_{12}\), there exists \(L \in W_{14}^4(C)\) such that \(\mu_0(L)\) is not-injective, one can construct a stable vector bundle \(E\) of rank 2 sitting in an extension
\[
0 \to K_C \otimes L^\vee \to E \to L \to 0,
\]
such that \( h^0(C, E) = h^0(C, L) + h^1(C, L) = 7 \), and for which the Mukai-Petri map
\[ \text{Sym}^2 H^0(C, E) \to H^0(C, \text{Sym}^2 E) \]
is not injective. This is a contradiction. To determine the slope of the divisor \( D_{12} \), we write \( D_{12} \equiv a\lambda - \sum_{j=0}^{6} b_j \delta_j \in \text{Pic}(\mathcal{M}_{12}) \). Since \( a/b_0 = 4415/642 \leq 71/10 \), we are in a position to apply Corollary 1.2 from \([FP]\), which gives the inequalities \( b_j \geq b_0 \) for \( 1 \leq j \leq 6 \). Therefore \( s(D_{12}) = a/b_0 < 13/2 \).

\[ \square \]

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