2-REPRESENTATIONS OF SOERGEL BIMODULES

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Abstract. In this paper we study the graded 2-representation theory of Soergel bimodules for a finite Coxeter group. We establish a precise connection between the graded 2-representation theory of this non-semisimple 2-category and the 2-representation theory of the associated semisimple asymptotic bicategory. This allows us to formulate a conjectural classification of graded simple transitive 2-representations of Soergel bimodules, which we prove under certain assumptions.

Along the way we also show several results and provide examples which are interesting in their own right, e.g. we show that Duflo involutions have a Frobenius structure (in a certain quotient) and give an example of a left cell for which the underlying algebra of the cell 2-representation is not symmetric.

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1. Introduction

Classification problems are among the most important basic problems in mathematics. For example, classifying simple representations of Hecke algebras has played an important role in modern representation theory. The present paper is motivated by the problem of classifying graded simple transitive 2-representations of the 2-category \( \mathcal{F} = \mathcal{F}_C(W, S) \) of Soergel bimodules associated to a finite Coxeter group \( W = (W, S) \), which one can see as a categorification of the classification problem for Hecke algebras. We do not give a complete answer, but we propose a precise conjecture for this classification and prove this conjecture under a technical assumption on an invariant of a 2-representation, called the apex.

In case of finite Weyl groups, the 2-category of Soergel bimodules has Lie-theoretic origins as Soergel’s combinatorial description of the 2-category of projective functors on the integral blocks of the (thick) BGG category \( O \), see e.g. [BGG], [BG], [So1], [So3], [Ba]. The general case of a finitely generated Coxeter group is considered in [So2], [Wi]. In [So3] (for finite Weyl groups) and in [CW3] (in full generality) it was proved that the 2-category of Soergel bimodules categorifies the Hecke algebra of the Coxeter group in question, with respect to the Kazhdan–Lusztig basis [KL] of this Hecke algebra. The 2-category of Soergel bimodules and its various incarnations have many applications in...
representation theory, see e.g. [BFK], [Ab], [St], [Li], [AT], [RV], [KM], [EL], [CMZ], [CCM] and beyond, for instance in low-dimensional topology, mathematical physics and geometry, see e.g. [Kh], [Ro1], [KT], [GORS], [GS], [Ro2], [WW], [EH].

In this paper we consider the 2-category of Soergel bimodules for finite Coxeter groups and over the corresponding coinvariant algebra. Under these assumptions, the 2-category of Soergel bimodules is finitary and hence fits into the general framework of 2-representation theory developed in [MM1], [MM2], [MM3], [MM4], [MM5], [MM6]. In fact, the 2-category of Soergel bimodules has even more structure, it is flat in the terminology of [MM1], which implies the existence of left and right adjoints. Let us, however, emphasize that the 2-category of Soergel bimodules is not semisimple, nor abelian, and that, if we abelianize, we lose the existence of adjunctions. Hence, it does not fit into the general framework of 2-representation theory for the 2-categories known as multitensor categories, as developed in e.g. [Os3], [ENO], [EGNO].

Finitary 2-categories are natural 2-analogs of finite dimensional algebras. Just like finite dimensional algebras, finitary 2-categories have “simple” 2-representations called simple transitive 2-representations [MM5]. Given a finitary 2-category $C$, a natural basic problem is to classify equivalence classes of simple transitive 2-representations of $C$. This can be done for many 2-categories, see [Ma1] for a historical overview.

In particular, for the 2-category of Soergel bimodules in type $A$ this was done already in [MM5]. In this “easy” case, all simple transitive 2-representations are exhausted by the so-called cell 2-representations, which were originally defined in [MM1], [MM2] and which categorify Kazhdan–Lusztig cell modules [KL]. In [Zi], it was shown that the same result is also true in type $B_2$. The general case of dihedral groups, which are Coxeter groups of type $I_2(m)$, for $m \in \mathbb{N}_{\geq 3}$ (recall that Coxeter type $I_2(4)$ is equal to type $B_2$), was considered in [KMMZ], [MT]. As it turned out, for odd $m$, simple transitive 2-representations are again exhausted by cell 2-representations. However, for even $m > 4$, there are simple transitive 2-representations which are not cell 2-representations. The cases $m = 12, 18, 30$ turned out to be especially difficult and the classification problem in these cases was only solved in [MT] under the additional assumption of gradability. The answer for dihedral groups is rather nice and is given in terms of bicolored ADE Dynkin diagrams, with $m = 12, 18, 30$ being the Coxeter numbers of types $E_6$, $E_7$, $E_8$.

In the present paper, we propose a general approach for attacking the classification problem of graded simple transitive 2-representations of the 2-category $S$ of Soergel bimodules for an arbitrary finite Coxeter group $W$. Our approach is based on a connection between the 2-category of Soergel bimodules and the associated asymptotic bicategory $\mathcal{A}$ which categorifies the asymptotic Hecke algebra, also called the $J$-ring [Lu4], a (multi)fusion algebra. The bicategory $\mathcal{A}$ is no longer graded, but has the advantage of being semisimple, and even (multi)fusion. In particular, this implies that $\mathcal{A}$ has finitely many equivalence classes of simple transitive 2-representations and that these can be classified using the machinery developed in e.g. [Os3], [ENO], [EGNO].

To elaborate, we construct an oplax 2-functor from an appropriate part of $\mathcal{A}$ to a certain subquotient of $S$ and show that it can be used to “lift” simple transitive 2-representations. Our main conjecture, formulated in Subsection 4.6, is that every simple transitive 2-representation of $S$ can be obtained via such a “lift”, up to equivalence. Under some additional assumption we are able to prove this conjecture.

What is of crucial importance is that $\mathcal{A}$ is explicitly known and rather simple in all but a handful of cases, and so is the classification of its simple transitive 2-representations.
Thus, our conjecture, if true, would reduce the classification of simple transitive 2-representations of Soergel bimodules to a much easier problem. For example, in classical Weyl types the classification would boil down to computing the Schur multipliers of \((\mathbb{Z}/2\mathbb{Z})^k\), which is of course well-known. (In Weyl type A one always has \(k = 0\) and we recover the classification mentioned above without further work.) Another example, in dihedral types \(\mathcal{A}\) is (related to) the semisimplified quotient of quantum \(sl_2\)-modules. In this case, our conjecture holds and yields the above mentioned ADE classification via the work of Kirillov–Ostrik [KO]. Finally, we think that our methods are also applicable to the 2-categories in [MMMT2], where the asymptotic fusion 2-categories should be (related to) the semisimplified quotients of quantum \(sl_n\)-modules.

In order to explain the “additional assumptions”, we need to go into some technical detail. Each simple transitive 2-representation has an invariant, called the apex, introduced in [ChMa]. The general classification problem for simple transitive 2-representations splits naturally into disjoint subproblems, namely the classification of simple transitive 2-representations with a given apex. The apex of a simple transitive 2-representation is a two-sided cell in the sense of Kazhdan–Lusztig combinatorics, and, in the case of Soergel bimodules, any two-sided cell is the apex of some simple transitive 2-representation. After these brief explanations, here is the main statement:

**Theorem A.** Let \(\mathcal{J}\) be a two-sided cell in \(W\) containing the longest element of some parabolic subgroup of \(W\). Then all graded simple transitive 2-representations of \(\mathcal{J}\) with apex \(\mathcal{J}\) are lifts of simple transitive 2-representations of the associated asymptotic bicategory, up to equivalence.

To prove this result we work with the 2-category \(\mathcal{F} = \mathcal{F}(W, S)\) of singular Soergel bimodules, whose objects are indexed by the parabolic subsets \(I \subset S\), and e.g. we have \(\mathcal{F} = \mathcal{F}(\emptyset, \emptyset)\). Further, we crucially use the main result of [MMMZ] which restricts the classification of simple transitive 2-representations of \(\mathcal{F}\) with apex \(\mathcal{J}\) to the classification of simple transitive 2-representations of a very special subquotient \(\mathcal{F}_H\) of \(\mathcal{F}\) associated to the intersection of a fixed left cell \(L\) inside \(\mathcal{J}\) with a dual right cell (this intersection is called an \(H\)-cell, by an analogy with Green’s relations for semigroups [Gr], which also explains our notation). In case \(\mathcal{J}\) contains the longest element \(w_0^L\) of the parabolic subgroup of \(W\) associated to a set \(I\) of simple reflections, the main result of [MMMZ] allows us to relate \(\mathcal{F}_H\) to the corresponding subquotient of \(\mathcal{F}(I, I)\) containing the identity 1-morphism on \(I\). We show that the latter is biequivalent to the asymptotic bicategory associated to \(H\) and Theorem A follows.

The argument summarized in the previous paragraph requires a lot of preparation and technical work. After introducing the necessary preliminaries on 2-representation theory of finitary and fiat 2-categories in Section 2 a major part of this preparation is contained in Section 3. This section develops further the technique to study 2-representations using (co)algebra 1-morphisms which, for fiat 2-categories, was described in [MMMT1] and was based on the original ideas used in the case of abelian tensor categories in [Os3]. The 2-category \(\mathcal{F}\) is not abelian and, in order to use the technique of (co)algebra 1-morphisms, the paper [MMMT1] passes from \(\mathcal{F}\) to its abelianization \(\overline{\mathcal{F}}\). In particular, given a 2-representation of \(\mathcal{F}\), there exists a corresponding coalgebra 1-morphism in \(\overline{\mathcal{F}}\) which is unique up to Morita–Takeuchi equivalence. In Theorem 16 we prove the very surprising fact that, for a transitive 2-representation with apex \(\mathcal{J}\), this coalgebra 1-morphism can actually be chosen in the non-abelianized so-called \(\mathcal{J}\)-simple quotient of \(\mathcal{F}\). In fact, Theorem 16 is not specific for \(\mathcal{F}\) and is true for any fiat 2-category.

Section 4 is the heart of the paper. It develops the technicalities necessary to provide a connection between 2-representations of \(\mathcal{F}_H\) and \(\mathcal{A}_H\). It is in this section that we define...
\(A\), construct the oplax 2-functor linking \(A\) to \(S\) and use it to define the procedure of “going up” which allows us to lift 2-representations of \(A\) to 2-representations of \(S\). We also define the procedure of “going down” which associates a certain 2-representation of \(A\) to every 2-representation of \(S\). As already mentioned, the main conjecture is formulated in Subsection 4.6 and our main result, Theorem 34, which implies Theorem A, is proved in Subsection 4.7.

In Section 5, we take a closer look at the algebras underlying the cell 2-representations of \(S\). This section is, in part, motivated by [EH] Conjecture 4.40 which expects a Frobenius structure on the indecomposable 1-morphisms in \(S\) associated to the Duflo involutions (see also [Kl, Subsection 5.2]). To begin with, we show that each cell 2-representation of \(S\) is always “lifted” from the asymptotic bicategory and that the Morita–Takeuchi equivalence class of the associated cosimple coalgebra 1-morphism in \(S\) is that given by the Duflo involution in \(H\). Further, in Proposition 38 we prove that the algebra underlying the cell 2-representation is weakly symmetric and a Frobenius algebra of graded length \(2a\), where \(a\) is the value of Lusztig’s \(a\)-function on the cell in question. We give an explicit description of the projective bimodules over this algebra which represent the action of the indecomposable 1-morphisms indexed by the elements in \(H\). Consequently, in Subsection 5.4 we prove [EH] Conjecture 4.40 in the setup of the 2-category \(S\). Note that this is a weaker statement than the original [EH] Conjecture 4.40, which was formulated for the whole 2-category \(S\), and in Example 49 we provide evidence showing that our methods are not applicable in the case of \(S\).

In Subsection 5.5, we establish some necessary condition for the algebra underlying the cell 2-representation to be symmetric. Using this condition we provide, in Example 51, a very surprising example of an \(H\)-cell for which this underlying algebra is not symmetric, disproving the general expectation that this algebra should always be symmetric. This general expectation was based on the main result of [MS], proved in type \(A\). (We note that the proof given there extends to an arbitrary left cell of a finite Weyl group containing an element of the form \(w_0w_1\), for some set \(I\) of simple reflections.)

In Section 6, we provide a characterization of “lifted” simple transitive 2-representations, see Theorem 56, and extend to them the explicit descriptions from Section 5.

Finally, in Section 7, we list results of low rank computations (including all exceptional types) and show to which extent these low rank cases can be covered by our conjecture or Theorem A.

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2. Preliminaries

2.1. Categorical conventions. We use (small) categories $\mathcal{C}$ and 2-categories $\mathcal{C}$, bicategories or even 2-semicategories (we will stress when we work do not work with genuine 2-categories; silently adapting definitions etc. to the weaker setting if necessary) in this paper, where we view a monoidal category as a 2-category with one (possibly unspecified) object; a perspective which we will use throughout, e.g. for Soergel bimodules.

We will also use the following notation:

- objects in categories (which are not morphism categories in 2-categories) are denoted by letters such as $X \in \mathcal{C}$, and morphisms by $f \in \mathcal{C}$;
- objects in 2-categories are denoted by $i \in \mathcal{C}$, 1-morphisms by $F \in \mathcal{C}$ and 2-morphisms by greek letters such as $\alpha \in \mathcal{C}$;
- for $\mathcal{C}$ and $i, j \in \mathcal{C}$, we denote by $\mathcal{C}(i, j)$ the corresponding morphism category;
- identity 1-morphisms are denoted by $/BD_i$ and identity 2-morphisms by $id_F$, where the subscripts are sometimes omitted;
- we write $FG = F \circ G$ for composition of 1-morphisms, and $\circ_v$ and $\circ_h$ denote vertical and horizontal compositions, respectively.

The reader is referred to e.g. [ML], [Le] or [Be] for these and related notions.

2.2. Finitary and fiat 2-categories, and their 2-representations. Let $k$ be an algebraically closed field.

A category $\mathcal{C}$ is called finitary (over $k$) if it is equivalent to the category of finitely generated injective (or projective) modules over some associative, finite dimensional $k$-algebra. These categories assemble into a 2-category $\mathcal{A}^f = \mathcal{A}_k^f$ having additive, $k$-linear functors and natural transformations as 1- and 2-morphisms, respectively. Similarly, a 2-category $\mathcal{C}$ is finitary (over $k$) if it has finitely many objects, all identity 1-morphisms $/BD_i$ are indecomposable and each morphism category $\mathcal{C}(i, j)$ is finitary over $k$ with all compositions being (bi)additive and $k$-(bi)linear. We further say $\mathcal{C}$ is fiat if it has a weak antiinvolution $^*$ reversing the direction of both 1- and 2-morphisms and adjunction 2-morphisms associated to $^*$. If $^*$ is just a weak antiequivalence of finite order, then $\mathcal{C}$ is called weakly fiat.

Example 1. For a finite group $G$, the 2-category $\mathcal{R}ep(G, k)$ of finite dimensional representations of $G$ over $k$ is fiat if and only if the algebra $k[G]$ has finite representation type. This is true, for example, if $\text{char}(k) \nmid \#G$, in which case $\mathcal{R}ep(G, k)$ is semisimple.

Another example of a fiat 2-category is $\mathcal{S} = \mathcal{S}\mathcal{C}(W, S)$, the 2-category of Soergel bimodules over the coinvariant algebra of a finite Coxeter group, cf. Section 4.

A semisimple fiat 2-category is called a multifusion 2-category. A multifusion 2-category with one object is called a fusion 2-category.

Example 2. The 2-category $\mathcal{R}ep(G, k)$ is fusion unless $\text{char}(k) | \#G$. 
In contrast, the 2-category $\mathcal{S}$ is not fusion since it is not semisimple. However, for every parabolic subset $I \subseteq S$, a certain subquotient $\mathcal{S}_H$ of $\mathcal{S}$ associated to the $H$-cell $H$ containing $\mathbb{1}_I$ is semisimple, cf. the proof of Theorem [34].

For a finitary 2-category $\mathcal{C}$, a finitary 2-representation $M$ is an additive, $k$-linear 2-functor from $\mathcal{C}$ to $A$. Finitary 2-representations of $\mathcal{C}$ form a 2-category; in particular, there is an appropriate notion of equivalence. We set $M := \bigoplus_{i \in C} M(i)$.

The rank of $M$ is the number of isomorphism classes of indecomposable objects in $M$. Moreover, we will often use the action notation $FX := M(F)(X)$ for 2-representations.

Example 3. If $\mathcal{C}$ is finitary, then the so-called principal or Yoneda 2-representation $P_i := \mathcal{C}(i, -)$ is finitary, for all $i \in C$.

A 2-representation $M$ is called transitive if, for any $i \in C$ and any non-zero object $X \in M(i)$, the additive closure (in the sense of being closed under direct sums, direct summands and isomorphisms)

$$\text{add}\{FX \mid j \in C, F \in \mathcal{C}(i, j)\}$$

coincides with $M$. A transitive 2-representation $M$ is said to be simple transitive provided that $M$ has no non-trivial, $\mathcal{C}$-stable ideals. Moreover, every transitive 2-representation has a unique simple transitive quotient.

The importance of simple transitive 2-representations is explained, in particular, by the existence of a weak version of the Jordan–Hölder theorem. Namely, for any finitary 2-representation $M$ of $\mathcal{C}$, there is a finite filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$$

where every 2-representation $M_k$ generates a $\mathcal{C}$-stable ideal $I_k$ in $M_{k+1}$ such that $M_{k+1}/I_k$ is transitive, and thus, has a unique associated simple transitive quotient $L_{k+1}$. Up to equivalence and ordering, the set $\{L_k \mid 1 \leq k \leq m\}$ is an invariant of $M$.

See [MM1 Subsections 2.2, 2.3 and 2.4], [MM6 Subsection 2.5], [MM3 Subsection 2.3] and [MM5 Subsection 3.5] for details.

2.3. Abelianization. Finitary 2-categories can be injectively or projectively abelianized. The injective abelianization is denoted $\mathcal{C}$ and the projective abelianization is denoted $\overline{\mathcal{C}}$. Moreover, $\mathcal{C}$ embeds into $\mathcal{C}$ or into $\overline{\mathcal{C}}$, and the isomorphism closure of the image of this embedding is the 2-full 2-subcategory of injectives or projectives, respectively. In particular, each indecomposable 1-morphism $F \in \mathcal{C}$ has an associated simple socle, or head, in these abelianizations, respectively.

These abelianizations are rather technical and not all properties of $\mathcal{C}$ carry over to the abelianizations. In particular, the abelianizations of fiat 2-categories are usually not even finitary and the involution $^*$ only gives rise to an antiisomorphism between $\mathcal{C}$ and $\overline{\mathcal{C}}$.

The same abelianization process works, mutatis mutandis, for finitary 2-representations, where we use the same notation.

Example 4. When $\text{char}(k) \nmid \# G$, we have $\mathcal{R}ep(G,k) \cong \mathcal{R}ep(G,k) \cong \mathcal{R}ep(G,k)$, because $\mathcal{R}ep(G,k)$ is semisimple. In contrast, $\mathcal{S}$ is not abelian and neither $\mathcal{C}$ nor $\mathcal{S}$ are equivalent to it.
C-morphism
Given a left cell $L \geq L$, the and that, in fact, $D$ is a simple transitive left cell in $L$. Similarly one obtains right cells and two-sided cells. By [ChMa] Subsection 3.2, for any transitive 2-representation $M$ there is a unique two-sided cell $J$, an invariant of $M$ called the apex, which does not annihilate $M$ and is maximal, in the two-sided order, with respect to this property.

Example 5. When $\text{char}(k) \nmid \# G$, the 2-category $\mathcal{R}_{\text{rep}}(G, k)$ has only one cell, which is left, right and two-sided. When $\text{char}(k) \nmid \# G$, the 2-category $\mathcal{R}_{\text{rep}}(G, k)$ has more than one cell, for example, the projective modules form a two-sided cell.

The cells of $\mathcal{J}$ are given by the Kazhdan–Lusztig cells.

Fix a two-sided cell $J$. Then $\mathcal{C}$ has the associated $J$-simple quotient 2-category $\mathcal{C}_J$, whose simple transitive 2-representations with apex $J$ correspond bijectively to simple transitive 2-representations of $\mathcal{C}$ with apex $J$. Here, a 2-category is called $J$-simple if any non-zero 2-ideal contains the identity 2-morphisms of all 1-morphisms in $J$.

Each left cell $L$ can be used to define a cell 2-representation $C_L$ as follows. First we note that all 1-morphisms in $L$ have the same domain, say $1$. Define a 2-subrepresentation $M_{\geq L}$ of $P_1$ using the induced action of $\mathcal{C}$ on

$$\text{add}(\{F | F \geq L, L\}).$$

The 2-representation $M_{\geq L}$ has a unique maximal ideal $I$ and we define

$$C_L := M_{\geq L}/I,$$

which is always a simple transitive 2-representation. Note that there is an alternative construction of cell 2-representations via $P_1$ (or via $P_1$), respectively, described in [MM1] Subsection 4.5. It is proved in [MM2] Subsection 6.5 that these two constructions are equivalent.

Example 6. The (unique) cell 2-representation of $\mathcal{R}_{\text{rep}}(G, \mathbb{C})$ coincides with its unique principal 2-representation.

We define $H$-cells as the intersection of left and right cells. If $\mathcal{C}$ is fiat, then, for every left cell $L$, we define the associated $H$-cell

$$H(L) := L \cap L^*.$$

By construction, $H = H(L)$ lies in the same two-sided cell $J$ as $L$. Further, recall that each $L$ contains a unique distinguished 1-morphism $D = D(L)$ called Duflo involution and that, in fact, $D \in H(L)$.

Given a left cell $L$ in some two-sided cell $J$, we define $\mathcal{C}_H$ to be the 2-full 2-subcategory of $\mathcal{C}_J$ generated by all 1-morphisms in $H := H(L)$ together with the identity 1-morphism $1_L$, where $L$ is the unique domain and codomain of all 1-morphisms in $H$. The 2-category $\mathcal{C}_H$ is fiat, has $H$ as its maximal two-sided cell and is $H$-simple, as follows from the lemma below.

We refer to [MM1] Subsection 4.5], [ChMa] Subsection 3.2], [MM5] Section 3 and [MMMTZ] Subsection 4.2] for further details.

Lemma 7. Let $\mathcal{C}$ be a $J$-simple fiat 2-category and $L$ a left cell in $J$. Then the 2-category $\mathcal{C}_H$, where $H = H(L)$, constructed as above, is $H$-simple.
Proof. Consider the cell 2-representation \( C_L \) of \( \mathcal{C} \) and note that it is 2-faithful by \( \mathcal{J} \)-simplicity of \( \mathcal{C} \). By [KMMZ, Theorem 2], the action of each \( F \in \mathcal{H} \) is represented via \( C_L \) by a projective bimodule over the underlying algebra of \( C_L \). Let \( D \) be the Duflo involution in \( \mathcal{H} \). Then, by [MM1, Lemma 12], in the abelianization of the cell 2-representation, \( D \) does not annihilate any simples indexed by elements of \( \mathcal{H} \). Therefore, given \( F, G \in \mathcal{H} \) and a non-zero \( \alpha : F \to G \), the 2-morphism

\[
\text{id}_D \circ h \circ \text{id}_D : D \text{FD} \to \text{DGD}
\]

is non-zero. As \( C_L(D) \) is a projective bimodule, the morphism \( C_L(\text{id}_D \circ h \circ \text{id}_D) \) is not a radical morphism in the category of bimodules. Thus, \( \text{id}_D \circ h \circ \text{id}_D \) contains, as a direct summand, an isomorphism from some non-zero summand of \( \text{DGD} \) to some summand of \( \text{DGD} \). The claim follows.

\[
\square
\]

2.5. Coalgebra and algebra 1-morphisms. Recall that a coalgebra 1-morphism \( C := (C, \Delta_C, \varepsilon_C) \) in \( \mathcal{C} \) is a 1-morphism equipped with 2-morphisms \( \Delta_C : C \to CC \), called comultiplication, and \( \varepsilon_C : C \to \mathbb{1} \), called counit, satisfying the usual conditions. A right comodule \( M = (M, \rho_M) \) of such a coalgebra 1-morphism is a 1-morphism in \( \mathcal{C} \) together with a coaction \( \rho_M : M \to M \mathcal{C} \), again satisfying the usual coherence conditions. These assemble into a category

\[
\text{comod}_\mathcal{C}(C) := \{ (M, \rho_M) \mid M \in \mathcal{C}, M \text{ is a right } C\text{-comodule} \}.
\]

of right comodules. All of these can be defined for left actions as well, of course, and the same notions in the \( \mathbb{k} \)-linear setting or for algebra 1-morphisms are defined, mutatis mutandis. Let

\[
\text{inj}_\mathcal{C}(C) := \{ (M, \rho_M) \mid M \in \text{comod}_\mathcal{C}(C) \text{ injective} \}
\]

denote the full subcategory of \( \text{comod}_\mathcal{C}(C) \) consisting of all injective objects.

Similarly, for an algebra 1-morphism \( A := (A, \mu_A, \iota_A) \) in \( \mathcal{C} \), that is a 1-morphism \( A \) together with multiplication \( \mu_A : AA \to A \) and unit \( \iota_A : \mathbb{1} \to A \), satisfying the usual conditions, one can define the category \( \text{mod}_\mathcal{C}(A) \) of right \( A \)-modules in \( \mathcal{C} \) and its subcategory \( \text{proj}_\mathcal{C}(A) \) of projectives.

Note that \( \text{comod}_\mathcal{C}(C) \), \( \text{inj}_\mathcal{C}(C) \), \( \text{mod}_\mathcal{C}(A) \) and \( \text{proj}_\mathcal{C}(A) \) have a \( \mathcal{C} \)-action given by left multiplication. Moreover, any finitary 2-representation of a fiat 2-category \( \mathcal{C} \) arises in this way. Simple transitive 2-representations correspond to cosimple coalgebra 1-morphisms (or, dually, to simple algebra 1-morphisms).

Example 8. The identity 1-morphism \( \mathbb{1} \) has the natural structure of both a coalgebra and an algebra 1-morphism, given by the identity 2-morphisms. The 2-representation \( \text{comod}_\mathcal{C}(\mathbb{1}) \) of \( \mathcal{C} \) is equivalent to \( \text{P}_\mathbb{1} \) and the 2-representation \( \text{inj}_\mathcal{C}(\mathbb{1}) \) of \( \mathcal{C} \) is equivalent to \( \text{P}_\mathbb{1} \). Similarly, the 2-representation \( \text{mod}_\mathcal{C}(\mathbb{1}) \) of \( \mathcal{C} \) is equivalent to \( \text{P}_\mathbb{1} \) and the 2-representation \( \text{proj}_\mathcal{C}(\mathbb{1}) \) of \( \mathcal{C} \) is equivalent to \( \text{P}_\mathbb{1} \).

See also e.g. [EGNO, Chapter 7], [MMMT1, Section 4] and [MMMZ, Section 3.6].

3. Avoiding abelianization

Our first main result will be that, under certain circumstances, we can avoid abelianization altogether.
3.1. **Framing coalgebra 1-morphisms.** Let \( \mathcal{C} \) be a fiat 2-category. Recall that, for all 1-morphisms \( F \in \mathcal{C}, (F, F^*) \) forms an adjoint pair in \( \mathcal{C} \), and denote by \( \eta_F : \mathbb{1} \to F^*F \) and \( \epsilon_F : F^*F \to \mathbb{1} \) the unit and counit for this adjoint pair.

**Lemma 9.** If \( C := (C, \delta_C, \epsilon_C) \) is a coalgebra 1-morphism in \( \mathcal{C} \), then the 1-morphism \( 0 \neq FCF^* \in \mathcal{C} \) has a coalgebra structure with comultiplication

\[
\delta_{FCF^*} := (\text{id}_F \circ \eta_F \circ \text{id}_{CF^*}) \circ_v (\text{id}_F \circ \delta_C \circ \text{id}_F)
\]

and counit

\[
\epsilon_{FCF^*} := \epsilon_F \circ_v (\text{id}_F \circ \epsilon_C \circ \text{id}_F).
\]

**Proof.** Using straight black lines for \( C \) and dotted blue lines for \( F \) and \( F^* \), let us denote the structure 2-morphisms by

\[
\delta_C = \begin{array}{c}
\circ
\end{array}, \quad \epsilon_C = \begin{array}{c}
1
\end{array}, \quad \epsilon_F = \begin{array}{c}
\circ
\end{array}, \quad \eta_F = \begin{array}{c}
\circ
\end{array}.
\]

In this diagrammatic notation, the comultiplication and counit are

\[
\delta_{FCF^*} = \begin{array}{c}
\circ
\end{array}, \quad \epsilon_{FCF^*} = \begin{array}{c}
\circ
\end{array}.
\]

Moreover, the coassociativity and counitality of \( C \), and adjunction of \( (F, F^*) \) become

\[
(1) \quad \begin{array}{c}
\circ
\end{array} = \begin{array}{c}
\circ
\end{array}, \quad \begin{array}{c}
\circ
\end{array} = \begin{array}{c}
\circ
\end{array}, \quad \begin{array}{c}
\circ
\end{array} = \begin{array}{c}
\circ
\end{array}, \quad \begin{array}{c}
\circ
\end{array} = \begin{array}{c}
\circ
\end{array},
\]

while the ones for \( FCF^* \) are

\[
(2) \quad \begin{array}{c}
\circ
\end{array} = \begin{array}{c}
\circ
\end{array}, \quad \begin{array}{c}
\circ
\end{array} = \begin{array}{c}
\circ
\end{array}, \quad \begin{array}{c}
\circ
\end{array} = \begin{array}{c}
\circ
\end{array}, \quad \begin{array}{c}
\circ
\end{array} = \begin{array}{c}
\circ
\end{array}.
\]

Thus, the claim follows by using (1) on the diagrams in (2). \( \square \)

Clearly, we also obtain the dual statement of the above Lemma 9.

**Lemma 10.** If \( A := (A, \mu_A, \iota_A) \) is an algebra 1-morphism in \( \mathcal{C} \), then the 1-morphism \( F^*AF \in \mathcal{C} \) has an algebra structure with multiplication

\[
\mu_{F^*AF} := (\text{id}_F \circ \mu_A \circ \text{id}_F) \circ_v (\text{id}_F \circ \iota_A \circ \text{id}_F)
\]

and unit

\[
\iota_{F^*AF} := (\text{id}_{F^*} \circ \iota_A \circ \text{id}_F) \circ_v \eta_F.
\]

Let \( J \) be a two-sided cell in \( \mathcal{C} \) and \( M \) a transitive 2-representation of \( \mathcal{C} \) with apex \( J \). Denote by \( C \) the coalgebra 1-morphism in \( \mathcal{C} \) associated to a non-zero object \( X \in M \) which is uniquely (up to isomorphism) determined by the natural isomorphism

\[
\text{Hom}_M(X, GX) \cong \text{Hom}_C(C, G)
\]

for all 1-morphisms \( G \in \mathcal{C} \), see [MMMT1, Subsection 4.3]. The 1-morphism \( C \) is also called the *internal hom* and denoted by \( [X, X] \). We denote by \( \text{coev}_M^{X, X} \in \text{Hom}_M(X, CX) \) the image of \( \text{id}_C \) under the isomorphism (3) when \( G = C \).
Proposition 11. If $F$ is a 1-morphism in $\mathcal{C}$ such that $FX \neq 0$, then the 1-morphism $\text{FCF}^* \in \mathcal{C}$ with coalgebra structure defined in Lemma 2 is the coalgebra 1-morphism associated to $FX$.

Proof. By adjunction and (3) we have the following natural isomorphisms

$$
\text{Hom}_\mathcal{M}(FX, GF X) \cong \text{Hom}_\mathcal{M}(X, F^* GF X)
$$

(4)

$$
\cong \text{Hom}_\mathcal{C}(C, F^* GF X)
$$

$$
\cong \text{Hom}_\mathcal{C}(\text{FCF}^*, F^*)
$$

$$
\cong \text{Hom}_\mathcal{C}(\text{FCF}^*, G)
$$

for all 1-morphisms $G \in \mathcal{C}$.

Considering $G = 1$, we now prove that $\varepsilon_{\text{FCF}^*}$ is the image of $\text{id}_{FX}$ under the isomorphisms in (4). It suffices to show that the image of $\varepsilon_{\text{FCF}^*}$ and the image of $\text{id}_{FX}$ coincide whenever compared in any $\text{Hom}$-spaces appearing in (4) via those isomorphisms. On one hand, the first isomorphism $\text{Hom}_\mathcal{M}(FX, FX) \cong \text{Hom}_\mathcal{M}(X, F^* FX)$ sends $\text{id}_{FX}$ to $(\eta_f)_X$. On the other hand, chosing the image of $\varepsilon_{\text{FCF}^*}$, the last isomorphism $\text{Hom}_\mathcal{C}(\text{FCF}^*, 1) \cong \text{Hom}_\mathcal{C}(\text{FCF}^*, 1)$ sends $\varepsilon_{\text{FCF}^*}$ to

$$(\text{id}_{FX} \circ h \varepsilon_{\text{FCF}^*}) \circ v (\eta_f \circ h \text{id}_{FX})$$

Going right and then down, we obtain that the composite sends $\text{id}_{\mathcal{C}}$ to $(\varepsilon_C)_X \circ v \text{coev}_X^M$, while going down and then right, the composite sends $\text{id}_{\mathcal{C}}$ to $\text{id}_{FX}$. Commutativity implies that $(\varepsilon_C)_X \circ v \text{coev}_X^M = \text{id}_X$. Hence, $\varepsilon_{\text{FCF}^*} \in \text{Hom}_\mathcal{C}(\text{FCF}^*, 1)$ is the image of $\text{id}_{FX}$ under the isomorphisms in (4).

Considering $G = \text{FCF}^*$, we denote by $\text{coev}_X^M \in \text{Hom}_\mathcal{M}(FX, F^* FX)$ the image of $\text{id}_{\text{FCF}^*}$ under the isomorphisms in (4). In detail, the last isomorphism sends $\text{id}_{\text{FCF}^*}$ to $\eta_f \circ h \text{id}_{\text{FCF}^*} \in \text{Hom}_\mathcal{C}(\text{FCF}^*, F^* \text{FCF}^*)$.

The third isomorphism $\text{Hom}_\mathcal{C}(\text{FCF}^*, F^* \text{FCF}^*) \cong \text{Hom}_\mathcal{C}(\text{FCF}^*, F^* \text{FCF}^*)$ sends $\eta_f \circ h \text{id}_{\text{FCF}^*}$ to

$$(\eta_f \circ h \text{id}_{\text{FCF}^*}) \circ v (\text{id}_C \circ h \eta_f) = \eta_f \circ h \text{id}_C \circ h \eta_f.$$
And the second isomorphism \( \text{Hom}(C, F^*FC^*F) \cong \text{Hom}(X, F^*FC^*F X) \) sends \( \eta \circ h \text{id}_{C} \circ h \eta \) to \( (\eta \circ h \text{id}_{C} \circ h \eta) X \circ \text{coev}_{X, X}^{M} \). The latter, under the first isomorphism \( \text{Hom}(X, F^*FC^*F X) \cong \text{Hom}(F X, F^*FC^*F X) \), is sent to

\[
(\varepsilon \circ F CF^*F X) \circ v \left( (\text{id}_F \circ h \left( (\eta \circ h \text{id}_{C} \circ h \eta) X \circ \text{coev}_{X, X}^{M} \right) \right)
\]

\[
= \left( (\varepsilon \circ F CF^*F X) \circ v \left( (\text{id}_F \circ h \eta \eta) X \circ \text{id}_F \circ h \text{coev}_{X, X}^{M} \right) \right) \circ \left( (\text{id}_F \circ h \eta \eta) X \circ v \left( (\text{id}_F \circ h \text{coev}_{X, X}^{M} \right) \right)
\]

By the construction in [MMMT1, Section 4], we know that the element

\[
f := (\text{id}_F \circ h \text{coev}_{X, X}^{M}) \circ \text{coev}_{X, X}^{M} \in \text{Hom}(F X, F^*FC^*F X)
\]

gives rise to the comultiplication of \( FCF^* \) via the isomorphisms in [4], where \( G = FCF^* \). Now we prove that the comultiplication is exactly \( \delta_{FCF^*} \) defined in Lemma [3]. On the one hand, by definition, we have

\[
f = (\text{id}_F \circ h \eta \eta) X \circ v \left( (\text{id}_F \circ h \text{id}_{C} \circ h \eta \eta) X \circ \text{id}_F \circ h \text{coev}_{X, X}^{M} \right) \circ v \left( (\text{id}_F \circ h \text{id}_{C} \circ h \eta \eta) X \circ \text{id}_F \circ h \text{coev}_{X, X}^{M} \right) \circ v \left( (\text{id}_F \circ h \text{id}_{C} \circ h \eta \eta) X \circ \text{id}_F \circ h \text{coev}_{X, X}^{M} \right) \circ v \left( (\text{id}_F \circ h \text{id}_{C} \circ h \eta \eta) X \circ \text{id}_F \circ h \text{coev}_{X, X}^{M} \right)
\]

The first isomorphism

\[
\text{Hom}(F X, F^*FC^*F X) \cong \text{Hom}(X, F^*FC^*F X)
\]

sends \( f \) to

\[
(\text{id}_F \circ h \circ f) \circ v \left( (\text{id}_F \circ h \text{id}_{C} \circ h \eta \eta) X \circ \text{id}_F \circ h \text{coev}_{X, X}^{M} \right) \circ v \left( (\text{id}_F \circ h \text{id}_{C} \circ h \eta \eta) X \circ \text{id}_F \circ h \text{coev}_{X, X}^{M} \right)
\]

where the second equality uses the interchange law twice. Considering the 2-morphism

\[
\beta := \eta \circ h \text{id}_{C} \circ h \eta \text{id}_{C} \circ h \eta : CC \rightarrow F^*FC^*F, \text{ via the naturality of the isomorphism [2]}, we have the commutative diagram
\]

\[
\text{Hom}(C, CC) \xrightarrow{\beta_{v}} \text{Hom}(X, CC X) \xrightarrow{\beta_{v}} \text{Hom}(C, F^*FC^*F) \cong \text{Hom}(X, F^*FC^*F X).
\]

Commutativity of the diagram implies that

\[
\beta_{v} \circ v \left( \text{id}_{C} \circ h \text{coev}_{X, X}^{M} \right) \circ v \text{coev}_{X, X}^{M} = \beta_{v} \circ v \left( \delta_{C} \right) \circ v \text{coev}_{X, X}^{M}.
\]

Therefore, we obtain \( (\text{id}_F \circ h \circ f) \circ v \left( (\eta \eta) X = \beta_{v} \circ v \left( (\delta_{C}) \circ v \text{coev}_{X, X}^{M}. \right) \right) \]. On the other hand, chasing the image of \( \delta_{FCF^*} \), the last isomorphism

\[
\text{Hom}(FCF^*, F^*FC^*F) \cong \text{Hom}(CF^*, F^*FC^*F^*)
\]
Then the third isomorphism
\[ \text{Hom}_\mathcal{M}(\text{FC}, \text{FC}^* \text{FC} \text{FC}^*) \cong \text{Hom}_\mathcal{M}(\text{C}, \text{FC}^* \text{FC}^*) \]
sends \( (\eta_F \circ h \circ \text{id}_{\text{FC}^*}) \circ_v (\delta \circ h \circ \text{id}_{\text{FC}^*}) \) to
\[ (\eta_F \circ h \circ \text{id}_{\text{FC}^*} \circ h \circ \text{id}_{\text{FC}^*}) \circ_v (\delta \circ h \circ \text{id}_{\text{FC}^*}) \]
\[ = (\eta_F \circ h \circ \text{id}_{\text{FC}^*} \circ h \circ \text{id}_{\text{FC}^*}) \circ_v (\delta \circ h \circ \text{id}_{\text{FC}^*}) \]
\[ = (\eta_F \circ h \circ \text{id}_{\text{FC}^*} \circ h \circ \text{id}_{\text{FC}^*}) \circ_v (\delta \circ h \circ \text{id}_{\text{FC}^*}) \]
\[ = (\eta_F \circ h \circ \text{id}_{\text{FC}^*} \circ h \circ \text{id}_{\text{FC}^*}) \circ_v (\delta \circ h \circ \text{id}_{\text{FC}^*}). \]

where the latter, under the second isomorphism
\[ \text{Hom}_\mathcal{M}(\text{C}, \text{FC}^* \text{FC}^* \text{FC}^*) \cong \text{Hom}_\mathcal{M}(X, \text{FC}^* \text{FC}^* \text{FC}^* X), \]
is sent to \( (\beta \circ \delta)_C \circ_v \text{coev}^M_{X,X}. \) This completes the proof. \( \square \)

3.2. Framed Morita–Takeuchi equivalence. A combination of Proposition 11 and [MMMT1, Theorem 9] yields:

Corollary 12. The coalgebra 1-morphisms \( C \) and \( \text{FC}^* \) are Morita–Takeuchi equivalent.

Remark 13. Note that Corollary 12 does not necessarily hold if we would drop the assumption on \( M \) of being transitive.

Moreover, by [MMMT1, Corollary 18], the internal hom \([X, F X]\), respectively \([F X, X]\), is the bijective \( \text{FC}^*\text{-C-bicomodule}, \) respectively \( \text{C-FC}^*\text{-bicomodule, inducing this Morita–Takeuchi equivalence.} \) In fact, noting that \( C = [X, X] \) and \( \text{FC}^* = [F X, F X], \) we have

\[ \text{Hom}_\mathcal{M}([X, F X], G) \cong \text{Hom}_\mathcal{M}(F X, G X) \]
\[ \cong \text{Hom}_\mathcal{M}(X, \text{FC}^* G X) \]
\[ \cong \text{Hom}_\mathcal{M}(C, \text{FC}^* G) \]
\[ \cong \text{Hom}_\mathcal{M}(\text{FC}^*, G) \]

and

\[ \text{Hom}_\mathcal{M}([F X, X], G) \cong \text{Hom}_\mathcal{M}(X, G F X) \]
\[ \cong \text{Hom}_\mathcal{M}(C, GF) \]
\[ \cong \text{Hom}_\mathcal{M}(\text{FC}^*, G) \]

for all 1-morphisms \( G \in \mathcal{M} \). Therefore, we have \([X, F X] \cong \text{FC} \) and \([F X, X] \cong \text{FC}^* \), cf. also [EGNO, Lemma 7.9.4].

Corollary 14. The Morita–Takeuchi equivalence between the coalgebra 1-morphisms \( C \) and \( \text{FC}^* \) is realized by the bicomodules \( \text{FC} \) and \( \text{FC}^* \), whose right and left \( C\)-comodule structures, respectively, are the canonical one and whose left and right \( \text{FC}^*\text{-comodule structures, respectively, are given by} (\text{id}_{\text{FC}} \circ h \circ \text{id}_{\text{FC}^*}) \circ_v (\text{id} \circ h \circ \delta) \) and \( (\text{id}_{\text{FC}} \circ h \circ \text{id}_{\text{FC}^*}) \circ_v (\text{id} \circ h \circ \delta). \)
The left comodule structure on $FC$ and the right comodule structure on $CF^*$ can also be illustrated as

\[
\begin{array}{cccc}
FC & FCF & FC & CF^* \\
\downarrow & \downarrow & \downarrow & \downarrow \\
FC & FC & CF^* & CF^*
\end{array}
\]

\[
\begin{array}{cccc}
FC & FCF & FC & CF^* \\
\downarrow & \downarrow & \downarrow & \downarrow \\
FC & FC & CF^* & CF^*
\end{array}
\]

\[
\begin{array}{cccc}
FC & FCF & FC & CF^* \\
\downarrow & \downarrow & \downarrow & \downarrow \\
FC & FC & CF^* & CF^*
\end{array}
\]

\[
\begin{array}{cccc}
FC & FCF & FC & CF^* \\
\downarrow & \downarrow & \downarrow & \downarrow \\
FC & FC & CF^* & CF^*
\end{array}
\]

Proof. Before we start, recall that the cotensor product over $C$ of a right $C$-comodule $M$ with a left $C$-comodule $N$ is the kernel of the map $\rho_M \circ h \cdot \text{id}_N - \text{id}_M \circ h \cdot \lambda_N : MN \rightarrow MCN$.

Now, on the one hand, we have that $FCF^* \cong FC \square CF^*$, since $\square C C$ is isomorphic to the identity functor on $\comod_C(C)$, see [MMMZ, Lemma 5].

On the other hand, to prove $CF^* \square FCF^* \cong C$, we consider the following diagram

\[
(C, FCF^*) \xrightarrow{\alpha} (C^*, FCF^*) \xrightarrow{\beta_1 - \beta_2} (C, FCF^*) \xrightarrow{\gamma} (C, C)
\]

where

\[
\beta_1 = (\text{id}_C \circ h \cdot \eta_F \circ h \cdot \text{id}_{CF^*FC}) \circ (\delta_C \circ h \cdot \text{id}_{FC^*FC}),
\]

\[
\beta_2 = (\text{id}_{CF^*FC} \circ h \cdot \eta_F \circ h \cdot \text{id}_C) \circ (\text{id}_{CF^*FC} \circ h \cdot \delta_C).
\]

Coassociativity of $\delta_C$ and the interchange law imply that the composite $(\text{id}_C \circ h \cdot \eta_F \circ h \cdot \text{id}_C) \circ (\text{id}_{CF^*FC} \circ h \cdot \delta_C)$ equals $\beta_1$ and $\beta_2$. Note that $\alpha$ is the kernel of $\beta_1 - \beta_2$. Hence, there exists a unique 2-morphism $\gamma$ such that the diagram (5) commutes.

Applying $F$ to the diagram (5) from the left, we obtain the following commutative diagram

\[
\begin{array}{cccc}
FCF^* \square FCF^* & FCF^* & FCF^* \square FCF^* & FCF^* \square FCF^* \\
\downarrow & \downarrow & \downarrow & \downarrow \\
FCF^* & FCF^* & FCF^* & FCF^*
\end{array}
\]

where the upper row is still exact, as the functor $F$ is left exact when acting on $\comod_C(C)$. By the universal property of kernels, the map $\text{id}_{F \circ h} \cdot \gamma$ provides an isomorphism $FCF^* \square FCF^* \cong FC$, see the proof of [MMMZ, Lemma 5]. Analogously, for any $H \in \mathcal{C}$, we have the isomorphisms

\[
\begin{array}{cccc}
HFCF^* \square FCF^* FC & HFCF^* \square FCF^* FC & HFCF^* \square FCF^* FC \\
\downarrow & \downarrow & \downarrow \\
HFCF^* & HFCF^* & HFCF^*
\end{array}
\]

where the first isomorphism is induced by the natural isomorphism between $\square C C$ and the identity functor on $\inj_{\mathcal{C}}(C)$ and the second isomorphism is induced by $\gamma$. Therefore, we have $KFCF^* \square FCF^* FC \cong FC$ for any direct summand $K$ of $HF$ with some $H \in \mathcal{C}$ and this is functorial. Combining this with the fact that $FC$ generates $\inj_{\mathcal{C}}(C)$, the natural transformation $\square C C : \square C C \rightarrow \square C C \circ \square FCF^* FC$ is an isomorphism. This implies that $\gamma$ is an isomorphism.

\[
3.3. \ \text{Internal hom and abelianization.}
\]

Proposition 15. Let $\mathcal{C}$ be a $\mathcal{J}$-simple fiat 2-category and $F \in \mathcal{J}$. The functor

\[
F \cdot \bullet^* : \mathcal{C} \rightarrow \mathcal{C}, \quad G \mapsto FG^*
\]

takes values in $\inj_{\mathcal{C}}(C) \cong \mathcal{C}$. 

Proof. Let us consider the fiat and $\mathcal{J} \boxtimes \mathcal{J}^{\text{op}}$-simple 2-category
\[ \mathcal{C}^0 = \mathcal{C} \boxtimes \mathcal{C}^{\text{op}}, \]
cf. [MM6 Section 6 and Proposition 21]. Note that $\mathcal{C}$ is a 2-representation of $\mathcal{C}^0$, and thus, by $\mathcal{J}$-simplicity, $\text{add}(\mathcal{J})$ is a simple transitive 2-representation of $\mathcal{C}^0$. By construction, $\text{add}(\mathcal{J})$ has apex $\mathcal{J} \boxtimes \mathcal{J}^{\text{op}}$ in $\mathcal{C}^0$. From [KMMZ Theorem 2], we know that $F \cdot X F^*$ is injective in $\text{add}(\mathcal{J})$ for any $X \in \text{add}(\mathcal{J})$. Finally, since for any simple 1-morphism $\mathbf{L}$ in $\mathcal{C}$ we have
\[ FLF^* = 0 \iff \mathbf{L} \text{ is not supported in } \mathcal{J}, \]
cf. [MM6 Proposition 24], the claim follows. \hfill \Box

Theorem 16. Let $\mathcal{C}$ be a $\mathcal{J}$-simple fiat 2-category and $\mathcal{M}$ a transitive 2-representation of $\mathcal{C}$ with apex $\mathcal{J}$. Then, for any $X, Y \in \mathcal{M}$, the 1-morphism $[X, Y]$ is in $\mathcal{C}$ (rather than in $\mathcal{C}^0$).

Proof. First, recall from Subsection 3.2 that
\[ [F X, F Y] \cong F[X, Y]F^*. \]
By Proposition 15 we know that $[F X, F Y] \cong F[X, Y]F^*$ belongs to $\mathcal{C}$ for all 1-morphisms $F$. Hence, by transitivity we can assume that $F$ satisfies
\[ F X \cong X \oplus X', \quad F Y \cong Y \oplus Y', \]
for some $X'$ and $Y'$ in $\mathcal{M}$. Since the internal hom is additive in both entries, we see that $[X, Y]$ is a direct summand of $[F X, F Y]$ and therefore, it belongs to $\mathcal{C}$ as well. \hfill \Box

Example 17. For any coalgebra 1-morphism $C$ in $\mathcal{C}$ we have
\[ C \cong [C, C] \]
as follows e.g. from [ChMi Lemma 3]. However, this does not contradict Theorem 16 since a coalgebra 1-morphism $C$ which is strictly in $\mathcal{C}$ will correspond to a 2-representation $\mathcal{M}$ that is neither transitive or has smaller apex.

4. Soergel bimodules and the asymptotic bicategory

4.1. Grading conventions. Recall that $k$ is a fixed algebraically closed field.

Let $k\text{-grmod}$ denote the category of finite dimensional, $(\mathbb{Z})$-graded $k$-vector spaces. An object in $k\text{-grmod}$ has the form $V = \bigoplus_{d \in \mathbb{Z}} V_d$, where $V_d$ denotes the elements of $V$ which are homogeneous of degree $d$. Morphisms in $k\text{-grmod}$ are $k$-linear maps (not necessarily homogeneous, but each morphism is a linear combination of homogeneous morphisms). The group $\mathbb{Z}$ acts on $k\text{-grmod}$ by grading shift $(\_)_d$ via the rule $(V(k))_d = V_{d+k}$, for all $k, d \in \mathbb{Z}$. From now on we use the notation that, if $p(v) = d_k v^k + \cdots + d_l v^l \in \mathbb{N}_0[v, v^{-1}]$ and $V$ is a graded vector space, then we let
\[ V^{(p)} := V(k)_{d_k} \oplus \cdots \oplus V(-l)_{d_l}. \]
With this notation, we have e.g. $V(-d) = V^{(d)}$. Further, if $V$ is a graded, finite dimensional vector space, then its graded dimension $\text{grdim}(V) \in \mathbb{N}_0 \langle v, v^{-1} \rangle$ is uniquely defined by the property that the graded vector spaces $V'$ and $k^{\langle \text{grdim}(V) \rangle}$ are isomorphic, where $k$ is concentrated in degree zero. A finite dimensional algebra $A$ is called positively graded if it is non-negatively graded, i.e. $\text{grdim}(A) \in \mathbb{N}_0 \langle v \rangle$, and its degree 0 component $A_0$ is semisimple. A finite dimensional, non-negatively graded algebra $A$ is called a
A graded, $k$-linear category $C$ is a category enriched over $k$-grmod. This means, in particular, that $\text{Hom}_C(X,Y) = \bigoplus_{d_1,d_2 \in \mathbb{Z}} \text{Hom}_C(X,Y)_{d_1+id_2}$ and the composite of homogeneous elements of degrees $d_1$ and $d_2$ is homogeneous of degree $d_1 + d_2$. We let $\text{Hom}_C(X,Y) := \text{Hom}_C(X,Y)_0$ denote the degree 0 morphisms. For such a category $C$, we denote by $C^{(0)}$ the (non-full) subcategory of $C$ given by taking the degree zero morphisms between all objects of $C$. A similar notation is also used for the corresponding notion for a finitary 2-category.

Given a graded, $k$-linear category $C$, we can define a new graded, $k$-linear category $C'$ with objects $(X,s)$, where $X \in C$ and $s \in \mathbb{Z}$, and

$$\text{Hom}_{C'}((X,s),(Y,t)) := \text{Hom}_C(X,Y)^{s+t},$$

with the evident composition. Then $C'$ admits translations in the sense that $(\_)$ gives rise to a strict action of $\mathbb{Z}$ on $C'$. Moreover, the natural embedding of $C$ into $C'$ which sends $X$ to $(X,0)$ is an equivalence. Hence, without loss of generality, we may always assume that the graded $k$-linear categories we work with admit translations, cf. [Ho, Section 1].

There are natural graded analogues of the notions from Section 2 e.g. graded finitary 2-categories and graded finitary 2-representations. See e.g. [MM3, Section 7], [M1, Definition 3.4] or [MMMT2, Section S] for details. For example, in the graded setting, the isomorphism classes of indecomposable 1-morphisms defining the rank have to be considered up to grading shifts.

### 4.2. Soergel bimodules

From now on we set $k = \mathbb{C}$ (or any other algebraically closed field of characteristic zero), and all categories and 2-categories etc. are over $\mathbb{C}$ if not indicated otherwise.

Let $W = (W,S)$ be a finite Coxeter group. For $I \subset S$, we denote by $W_I$ the corresponding parabolic subgroup of $W$ and by $w_0^I$ the longest element in $W_I$. We also set $w_0 := w_0^\varnothing$, which is the longest word in $W$.

We fix a reflection representation of $W$ and let $C$ be the corresponding coinvariant algebra with the usual ($\mathbb{Z}$-)grading. For $I \subset S$, we denote by $C^I$ the subalgebra of $C$ consisting of all elements that are invariant under all $s \in I$. Then $C^I$ inherits a grading from $C$. We have $C^\varnothing = \mathbb{C}$ and, for $I \subset J$, we have $C^I \supset C^J$.

The bicategory of associated singular Soergel bimodules is the 2-full, isomorphism closed and idempotent split subbicategory of the bicategory of bimodules over finite dimensional, associative algebras, which has objects $I \subset S$ (corresponding to $C^I$) and which is generated by all bimodules representing restriction and induction functors between all possible pairs of subalgebras $C^I \subset C^J$, where $I \subset J$, cf. [W]. This bicategory is graded (with the grading coming from the above mentioned grading on $C^I$), by construction. We denote by $\mathcal{S} = \mathcal{S}(W,S)$ a strict version of this bicategory which also has the induced grading.

The endomorphism category of $\varnothing$ in singular Soergel bimodules is the one-object 2-category, or monoidal category, of (regular) Soergel bimodules, which we denote by $\mathcal{S} = \mathcal{S}(W,S):= \mathcal{S}(\varnothing,\varnothing)$. By [So3], see also [EW2, Theorem 3.5], the split, graded Grothendieck ring of $\mathcal{S}$ is isomorphic to the Hecke algebra $H := H_{\mathbb{Z}[v,v^{-1}]}(W,S)$ of $W$. We refer to [So3] and [EW2] for more details on Soergel bimodules and we
adapt the conventions from [EW2] for these and the underlying Hecke algebra. Note that both 2-categories, $\mathcal{S}$ and $\mathcal{J}$, are fiat.

Furthermore, by [So3] and [EW2], for each $w \in W$ there is a unique (up to homogeneous isomorphism of degree zero) indecomposable Soergel bimodule, which we denote by $B_w$, that is sent to the Kazhdan–Lusztig basis element $b_w$ corresponding to $w$ under the character isomorphism [EW2, Theorem 1.1]. We call this fact the Soergel–Elias–Williamson categorification theorem. As a consequence, the cell structure of $\mathcal{S}$ is given by the Kazhdan–Lusztig combinatorics. In particular, Lusztig’s conjectures [Lu3, Conjecture 14.2], which we will use several times, hold in our case, see e.g. [Lu3, Subsection 15.1] or [duCl, Corollary 1.4].

We set
\begin{equation}
C_w := B_w^{\otimes a(w)} \quad \text{and} \quad c_w = v^{a(w)} b_w,
\end{equation}
where we recall that, to each $w \in W$, Lusztig [Lu1] assigns a number $a(w)$, called its $a$-value, such that the function $a$ is constant on two-sided cells. In $\mathcal{S}$ the projective 1-morphism $B_w$ has a simple head, and dually in $\mathcal{J}$ the injective 1-morphism $B_w$ has a simple socle.

**Example 18.** For $s \in S$ we have $B_s B_s \cong B_s^{\otimes (v^{-1} + v)}$ but $C_s C_s \cong C_s^{\otimes (1 + v^2)}$. The heads of $B_s$ and $C_s$, seen as projective 1-morphisms in $\mathcal{S}$, are concentrated in degrees $-1$ and $0$, respectively.

For a fixed two-sided cell $\mathcal{J}$, we have the corresponding $\mathcal{J}$-simple quotient $\mathcal{S}_{\mathcal{J}}$, and, for a fixed left cell $\mathcal{C}$ in $\mathcal{J}$ and $\mathcal{H} = \mathcal{L} \cap \mathcal{C}^*$, we have the corresponding 2-subcategory $\mathcal{S}_{\mathcal{H}}$ in $\mathcal{S}_{\mathcal{J}}$ as defined in Subsection 2.4. Each such $\mathcal{H}$ contains a unique Duflo involution $d = d_{\mathcal{H}}$. The corresponding element $b_d$ is the image of $D$ from Subsection 2.4 under the character isomorphism.

For $x, y, z \in W$, we let $h_{x,y,z} \in \mathbb{N}_0[v, v^{-1}]$ and $\gamma_{x,y,z^{-1}} \in \mathbb{N}_0$ be given by:
\begin{equation}
B_x B_y \cong \bigoplus_{z \in W} B_z^{\delta_{h_{x,y,z^{-1}}}}, \quad h_{x,y,z} = v^{a(z)} \gamma_{x,y,z^{-1}} \pmod{v^{a(z)-1}\mathbb{N}_0[v^{-1}]}. \tag{7}
\end{equation}

By the Soergel–Elias–Williamson categorification theorem, the $h$ are also the structure constants of $H$ with respect to the basis $\{b_w \mid w \in W\}$, while the $\gamma$ are the structure constants of Lusztig’s asymptotic Hecke algebra $A = A_{Z}(H)$ with respect to the basis $\{b_w \mid w \in W\}$ (denoted $t_w$ in [Lu4]). Since the $h$ are bar invariant, i.e. invariant with respect to the symmetry $v \leftrightarrow v^{-1}$, we also have
\begin{equation}
h_{x,y,z} = v^{-a(z)} \gamma_{x,y,z^{-1}} \pmod{v^{a(z)+1}\mathbb{N}_0[v]}. \tag{8}
\end{equation}

From Soergel’s hom formula, cf. [EW3, Theorem 3.6], we obtain
\[\text{grdim} \left( \text{hom}_{\mathcal{S}}(B_v, B_w^{\otimes k}) \right) = \delta_{v, w} \delta_{0, k}, \quad v, w \in W, k \in \mathbb{Z}_{\geq 0}.
\]

Consequently, the 2-endomorphism algebra of Soergel bimodules is positively graded. This property is inherited by $\mathcal{S}_{\mathcal{H}}$.

Finally, the following lemmas are evident and we state them for later use.

**Lemma 19.** In $\mathcal{S}_{\mathcal{H}}$, for all $x, y, z \in \mathcal{H}$, we have that
\[C_x C_y \cong \bigoplus_{z \in \mathcal{H}} C_z^{\otimes a(h_{x,y,z})}.
\]

In particular, if $C_z^{\otimes k}$ is isomorphic to a direct summand of $C_x C_y$, then $k \geq 0$. 

Lemma 19 has the following “negative” counterpart.

**Lemma 20.** For $x \in H$, set $\tilde{C}_x := C_{\tilde{H}}^{\oplus a_k(H)} = B_{\tilde{H}}^{\oplus a_k(H)}$. Then, in $\mathcal{H}$, for all $x, y, z \in H$, we have

$$\tilde{C}_x \tilde{C}_y \cong \bigoplus_{z \in H} C_{\tilde{H}}^{\oplus a_k(H)} h_{x, y, z}.$$ 

In particular, if $\tilde{C}_{\tilde{x}}$ is isomorphic to a direct summand of $\tilde{C}_x \tilde{C}_y$, then $k \leq 0$.

### 4.3. The asymptotic bicategory.

We define two 2-semicategories inside $\mathcal{H}$:

$$\mathcal{X} := \text{add}\{(C_{\tilde{H}}^{\oplus k}) \mid w \in H, k \geq 0\},$$

$$\tilde{\mathcal{X}} := \text{add}\{(C_{\tilde{H}}^{\oplus k}) \mid w \in H, k > 0\}.$$ 

The 2-semicategory $\mathcal{X}$ is, in fact, a lax monoidal category with lax identity 1-morphism $C_d$ and strict associators. Let us explain this in detail. From [MM3 Subsection 7.6] and positivity of the grading on Soergel bimodules it follows that there is a unique, up to a non-zero scalar, map $\varepsilon_d : C_d \to \mathbb{1}_H$. The lax structure of the identity 1-morphism of $\mathcal{X}$ on $C_d$ is now defined, for $X \in \mathcal{X}$, by the two 2-morphisms

$$\ell_X : C_d X \xrightarrow{\varepsilon_d \circ_{\mathcal{X}} \text{id}_X} X,$$

$$r_X : X C_d \xrightarrow{\text{id}_X \circ_{\mathcal{X}} \varepsilon_d} X,$$

with the unitality condition expressed, for $X, Y \in \mathcal{X}$, by the diagram

\[ (XC_d)Y \xrightarrow{r_X \circ_{\mathcal{X}} \text{id}_Y} X(C_d)Y \]

which commutes by associativity and the interchange law.

The bicategory $\mathcal{H}$ is defined as the quotient $\mathcal{X}/(\tilde{\mathcal{X})) \cong \mathcal{X}^{(0)}/(\tilde{\mathcal{X}}^{(0)})$, where $(\tilde{\mathcal{X}})$ and $(\mathcal{X}^{(0)})$ denote the 2-ideals of $\mathcal{X}$ and $\tilde{\mathcal{X}}^{(0)}$, respectively, generated by $\tilde{\mathcal{X}}$ and $\mathcal{X}^{(0)}$. For $X \in \mathcal{X}$, we denote by $[X]$ the image of $X$ in $\mathcal{H}$.

Note that $\mathcal{H}$ is only a bicategory as it does not contain any strict identity 1-morphism, but, at the same time, the composition in $\mathcal{H}$ is strictly associative. Up to isomorphism, the identity 1-morphism in $\mathcal{H}$ is the image $\Lambda_d = [C_d]$ of the oplax identity 1-morphism $C_d$ of $\mathcal{X}$.

By the Soergel–Elias–Williamson categorification theorem and [7], $\mathcal{H}$ categorifies the asymptotic Hecke algebra $\Lambda_H$ associated to $H$ in the sense that $\Lambda_w := [C_w]$ categorifies the basis element $a_w$ for $w \in H$. The algebra $\Lambda_H$ is a fusion algebra and the following categorifies this fact.

**Proposition 21.** $\mathcal{H}$ is a fusion bicategory. In particular, it is fiat and semisimple. Moreover, if $W$ is a Weyl group, then $\mathcal{H}$ is biequivalent to $\mathcal{C}_k^H$ from [BFU Section 5] and [Os4].

Note that this implies that $\mathcal{H} = \mathcal{H}^* = \mathcal{H}$.

**Proof.** The bicategory $\mathcal{H}$ is fiat by [EWH Theorem 5.2] (see also [Lu3 Subsection 18.19], [Lu5 Subsection 9.3] and [BFU Subsection 4.3]). Moreover, [KMMZ Theorem 2] and $\mathcal{X}$ being fiat imply that the Jacobson radical of $\mathcal{X}$ is a 2-ideal. This 2-ideal has to be zero in the quotient since $\mathcal{H}$ is $H$-simple, showing that $\mathcal{H}$ is semisimple. Finally, the connection to [BFU Section 5] and [Os4] follows from Soergel’s identification of
and the category of semisimple perverse sheaves on the associated flag variety, cf. [So1, Erweiterungssatz 5]. □

Combined with Conjecture 33 stated in Subsection 4.6, Proposition 21 significantly reduces the classification problem of graded simple transitive 2-representations for Soergel bimodules.

4.4. Going up. Consider the natural projection

$$\Pi : \mathcal{X}^{(0)} \rightarrow \mathcal{A}_H.$$ 

By the definition of $\mathcal{A}_H$, the projection $\Pi$ is the identity on both 1- and 2-morphisms. In fact, $\Pi$ is a genuine 2-semifunctor. It sends the lax identity $C_d$ of $\mathcal{X}^{(0)}$ to the (honest) identity $A_d$ of $\mathcal{A}_H$ and, for any object $X, Y \in \mathcal{X}^{(0)}$, we have $\Pi(X)\Pi(Y) = \Pi(XY)$, by definition.

Indeed, for any $X = \Pi(F) \in \mathcal{A}_H$, we can define the 2-morphism

$$\lambda_X : A_d X \rightarrow \Pi(C_d) X = \Pi(C_d) \Pi(F) = \Pi(C_d) \Pi(\ell_F) \Pi(F) = \Pi(\ell_F) \Pi(F) = X.$$

Then $\lambda$ is a natural transformation from $A_d \circ \pi$ to the identity 2-functor on $\mathcal{A}_H$ and we have $\lambda_{\Pi(F)} = \Pi(\ell_F)$. Similarly, one can define a natural transformation $\rho$ from $\pi \circ A_d$ to the identity 2-functor on $\mathcal{A}_H$ via $\rho_{\Pi(F)} = \Pi(r_F)$. Here $\lambda$ and $\rho$ are the left and right unitors for the bicategory $\mathcal{A}_H$. In details, by applying $\Pi$ to (9), we obtain the commutative diagram

$$\begin{array}{ccc}
(XA_d)Y & \to & X(A_d Y) \\
\downarrow & & \downarrow \\
(X\Pi(C_d))Y & \rightarrow & X(Y) \\
\rho_X \circ \Pi(\ell_F) & \rightarrow & \Pi(\ell_F) \Pi(\Pi(F)) Y,
\end{array}$$

for all $X, Y \in \mathcal{A}_H$.

Observing that $\lambda_{\Pi(F)} = \Pi(\ell_F)$ and $\rho_{\Pi(F)} = \Pi(r_F)$, we obtain the commutative diagrams

$$\begin{array}{ccc}
A_d \Pi(F) & \rightarrow & \Pi(C_d) \Pi(F) \\
\downarrow & & \downarrow \\
\Pi(F) & \rightarrow & \Pi(C_d) \Pi(F),
\end{array}$$

$$\begin{array}{ccc}
\Pi(\ell_F) & \rightarrow & \Pi(C_d) \Pi(F) \\
\downarrow & & \downarrow \\
\Pi(F) & \rightarrow & \Pi(C_d) \Pi(F).
\end{array}$$

Together with associativity (where all maps are identity), implying that $\Pi$ is a lax bifunctor.

Positivity of the grading on Soergel bimodules shows that the functor underlying $\Pi$ has a left adjoint

$$\Theta : \mathcal{A}_H \rightarrow \mathcal{X}^{(0)}, \quad \text{Hom}_{\mathcal{X}^{(0)}}(\Theta(F), G) \cong \text{Hom}_{\mathcal{A}_H}(F, \Pi(G)),$$

which is unique up to natural 2-isomorphisms. We emphasize that it is important here to work with $\mathcal{X}^{(0)}$ and not with $\mathcal{X}$. Up to isomorphism, $\Theta$ is determined by $\Theta(A_w) \cong C_w$, in particular, $\Theta$ is an embedding.

Since $\Pi$ is lax, the doctrinal adjunction [Ke] (see also [SS, Formula (3.5)]) implies that $\Theta$ is an oplax bifunctor. Note that, for each $X \in \mathcal{X}$, the object $\Pi(X)$ is isomorphic to $\Pi(Y)$, where $Y$ is the subobject of $X$ generated in degree 0. As $Y \cong \Theta(F)$, for some $F \in \mathcal{A}_H$, the adjunction morphisms guarantee that $\Pi \Theta \cong \text{id}_{\mathcal{A}_H}$. 
We can further use $\epsilon_d: C_d \to \mathbb{I}_{\mathcal{H}}$ to extend $\Theta$ to an oplax bifunctor from $\mathcal{A}_H$ to $\mathcal{H}$ via the embeddings $X^{(0)} \hookrightarrow X \hookrightarrow \mathcal{H}$.

We use the notation $\mathbb{I}_{\mathcal{A}_H} := A_d$, where $d$ is the Duflo involution in $\mathcal{H}$. Note that $\mathbb{I}_{\mathcal{A}_H}$ is only a weak identity 1-morphism. Then $\Theta(\mathbb{I}_{\mathcal{A}_H}) = C_d$.

**Lemma 22.** For any coalgebra 1-morphism $\Lambda$ in $\mathcal{A}_H$, the oplax 2-functor $\Theta$ induces the structure of a coalgebra 1-morphism on $\Theta(\Lambda)$. Moreover, a (left or right) $\Lambda$-comodule $X$ is sent to a (left or right) $\Theta(\Lambda)$-comodule $\Theta(X)$.

**Proof.** Mutatis mutandis as in [JS, Proposition 5.5]. □

**Proposition 23.** If $A$ as in Lemma 22 is cosimple in $\mathcal{A}_H$, then so is $\Theta(A)$ in $\mathcal{H}$.

**Proof.** By construction, all simples in the socle of $\Theta(A)$ (considered as a 1-morphism of $\mathcal{A}_H$) correspond to 1-morphisms in $\mathcal{H}$. Let $B$ be a cosimple, subcoalgebra 1-morphism of $\Theta(A)$. By [MMMZ, Corollary 12], the corresponding 2-representation $\text{inj}_{\mathcal{A}_H}(B)$ is simple transitive. By Example 17, we have $B \cong [B, B]$, which, by Theorem 16, implies that $B$ is a direct summand of $\Theta(A)$ in $\mathcal{H}$. Hence, $B$ is of the form $\Theta(\Lambda)$, for some 1-morphism $\Lambda$ in $\mathcal{A}_H$. It is easy to check that $\Lambda$ is a subcoalgebra 1-morphism of $A$ and is thus isomorphic to $A$. The claim follows. □

**Example 24.** Being an identity 1-morphism, $A_d$ is a cosimple coalgebra 1-morphism in $\mathcal{A}_H$. By Lemma 22 and Proposition 23, this implies that $C_d$ has the structure of a cosimple coalgebra 1-morphism in $\mathcal{H}$. By duality, this implies that $\tilde{C}_d$ has the structure of a simple algebra 1-morphism in $\mathcal{H}$.

In Subsection 5.4, we will additionally see that $B_d = C_d^\text{op} = \tilde{C}_d^\text{op}$ has the structure of a Frobenius algebra 1-morphism.

**Lemma 25.** Let $\Lambda$ be a coalgebra 1-morphism in $\mathcal{A}_H$, $X$ a right $\Lambda$-comodule and $Y$ a left $\Lambda$-comodule. Then

$$\Theta(X \square_{\Lambda} Y) \cong \Theta(X) \square_{\Theta(\Lambda)} \Theta(Y).$$

**Proof.** Consider the commutative diagram

$$\begin{array}{c}
\Theta(X \square_{\Lambda} Y) \\
\downarrow \\
\Theta(X) \square_{\Theta(\Lambda)} \Theta(Y)
\end{array} \quad \begin{array}{c}
\Theta(XY) \\
\downarrow \\
\Theta(X) \Theta(Y)
\end{array} \quad \begin{array}{c}
\Theta(X \Lambda Y) \\
\downarrow \\
\Theta(X) \Theta(Y)
\end{array} \quad \begin{array}{c}
\varphi \\
\downarrow \\
\Theta(X) \Theta(\Lambda) \Theta(Y)
\end{array}
$$

where the top row is given by applying $\Theta$ to the definition of $X \square_{\Lambda} Y$, the bottom row is the definition of $\Theta(X) \square_{\Theta(\Lambda)} \Theta(Y)$ and the two solid inclusions are given by the oplax structure of $\Theta$. The dashed inclusion is induced by commutativity of the solid square.

Note that the vertical arrows in the solid square restrict to isomorphisms in degree 0. Consequently, the dashed arrow is an isomorphism in degree 0. To prove the lemma, we need to show that $\varphi$ is injective when restricted to summands of $\Theta(X) \Theta(Y)$ generated in positive degrees (as 1-morphisms in $\mathcal{A}_H$). We will verify this by passing to the cell 2-representation $C_{\mathcal{H}}$.

Let $B$ be the underlying algebra of $C_{\mathcal{H}}$. Then $B$ is naturally a positively graded algebra. The bimodules representing $\Theta(A)$, $\Theta(X)$ and $\Theta(Y)$ are all projective by [KMMZ, Theorem 2] and generated in degree 0, by construction. Let $M$ be an indecomposable summand of $C_{\mathcal{H}}(\Theta(X) \Theta(Y))$ not generated in degree 0. Writing any non-zero element
Proposition 26. Let $A$ and $B$ be Morita–Takeuchi equivalent coalgebra 1-morphisms in $\mathcal{H}$. Then $\Theta(A)$ and $\Theta(B)$ are Morita–Takeuchi equivalent.

Proof. By [MMMT1] Theorem 5.1, there exist a $B$-$A$-bicomodule $X$ and an $A$-$B$-bicomodule $Y$ such that $B X \Box_A Y_B \cong B B$ and $A Y \Box_B X_A \cong A A_A$.

Applying Lemmas 22 and 25 we obtain

$$e_{\Theta(B)} \Theta(X) \Box_{\Theta(A)} \Theta(Y) \Theta_{\Theta(B)} \cong e_{\Theta(B)} \Theta(B) \Theta_{\Theta(B)},$$

$$e_{\Theta(A)} \Theta(Y) \Box_{\Theta(B)} \Theta(X) \Theta_{\Theta(A)} \cong e_{\Theta(A)} \Theta(A) \Theta_{\Theta(A)}.$$

The claim follows.

Proposition 27. If $A$ as in Lemma 24 is cosimple and $X \in \text{inj}_{\mathcal{H}}(A)$, then $\Theta(X)$ is in $\overline{\text{inj}_{\mathcal{H}}}(\Theta(A))$.

Proof. By additivity, it suffices to prove that $\Theta(A_w A)$ is in $\overline{\text{inj}_{\mathcal{H}}}(\Theta(A))$, for any $w \in \mathcal{H}$.

Set $B = [A_w A, A_w A]$. By Corollary 14, the Morita–Takeuchi equivalence between $A$ and $B$ is given by the $A$-$B$-bicomodule $AA^*_w$ and the $B$-$A$-bicomodule $A_w A$. By Lemma 25 and Proposition 25 the $\Theta(A)$-$\Theta(B)$-bicomodule $\Theta(AA^*_w)$ and the $\Theta(B)$-$\Theta(A)$-bicomodule $\Theta(A_w A)$ provide a Morita–Takeuchi equivalence between $\Theta(A)$ and $\Theta(B)$. In particular, $\Theta(A_w A)$ is in $\overline{\text{inj}_{\mathcal{H}}}(\Theta(A))$.

Corollary 28. The ranks of $\text{inj}_{\mathcal{H}}(A)$ and $\overline{\text{inj}_{\mathcal{H}}}(\Theta(A))$ are equal.

Proof. The fact that $\Pi \Theta \cong \text{id}_{\mathcal{H}}$ and Proposition 27 imply that application of $\Theta$ induces an injection from the set of isomorphism classes of indecomposable objects in $\text{inj}_{\mathcal{H}}(A)$ to the set of isomorphism classes of indecomposable objects in $\overline{\text{inj}_{\mathcal{H}}}(\Theta(A))$.

Let $M \in \text{inj}_{\mathcal{H}}(\Theta(A))$ be indecomposable. Up to grading shift, we may assume that, as a 1-morphism of $\mathcal{H}$, we have a decomposition

$$M \cong \bigoplus_{w \in \mathcal{H}} C_{w}^{\oplus p_w(0)},$$

where the $p_w \in \mathbb{N}[v]$. Let

$$M^{(0)} := \bigoplus_{w \in \mathcal{H}} C_{w}^{\oplus p_w(0)},$$

where $p_w(0)$ means evaluation. Thanks to positivity of the grading on $\mathcal{H}$ and Lemma 19 $M^{(0)}$ is a $\Theta(A)$-subcomodule of $M$. 

Note that \([\Theta(A)] \cong A\) by construction. Similarly, \([M^{(0)}\Theta(A)] \cong [M^{(0)}][\Theta(A)]\). Hence, there is an induced \(A\)-comodule structure on \([M^{(0)}]\). Now, \(\Theta([M^{(0)}]) \cong M^{(0)}\) is an injective \(\Theta(A)\)-comodule. Consequently, \(M^{(0)}\) is isomorphic to \(M\). \(\Box\)

4.5. Going down. Let \(M\) be a graded simple transitive 2-representation of \(\mathcal{S}_H\) with apex \(H\). Let \(C\) be a graded coalgebra 1-morphism in \(\mathcal{S}_H\) such that \(M\) is equivalent to the 2-action of \(\mathcal{S}_H\) on \(\text{inj} \mathcal{S}_H\)(C). Let \(X_1, \ldots, X_n\) be a complete and irredundant list of representatives of isomorphism classes (up to grading shift) of indecomposable objects in \(\text{inj} \mathcal{S}_H\)(C), normalized such that as objects in \(\mathcal{S}_H\) they are concentrated in non-negative degrees with nonzero degree zero part. Set \(X := X_1 \oplus X_2 \oplus \cdots \oplus X_n\).

Lemma 29. The quotient
\[
\text{add}\{X^\oplus v^k \mid k \geq 0\}/\text{add}\{X^\oplus v^k \mid k > 0\}
\]
carries an induced action of \(\mathcal{S}_H\).

Proof. Lemma 19 implies that, if \(X_j^\oplus v^k\) is isomorphic to a direct summand of \(C_w X_j\), where \(w \in H\), then \(k \geq 0\). Therefore \(\text{add}\{X^\oplus v^k \mid k \geq 0\}\) is stable under the action of \(\text{add}\{C_w^\oplus v^l \mid w \in W, l \geq 0\}\). Moreover, \(\text{add}\{C_w^\oplus v^l \mid w \in W, l > 0\}\) maps \(\text{add}\{X^\oplus v^k \mid k \geq 0\}\) to \(\text{add}\{X^\oplus v^k \mid k > 0\}\). The claim follows. \(\Box\)

Lemma 30. Given any 2-representation \(M\) of a (weakly) fiat 2-category \(\mathcal{C}\), there exists a coalgebra 1-morphism \(C\) such that \(\text{inj} \mathcal{S}_E\)(C) is equivalent to \(M\) and \(C\) is the image of a multiplicity free direct sum of representatives of isomorphism classes of indecomposable objects in \(\text{inj} \mathcal{S}_E\)(C) under the forgetful functor to \(\mathcal{C}\).

(We stress that Lemma 30 actually holds for any weakly fiat 2-category \(\mathcal{C}\) over any algebraically closed field \(k\), not just for \(\mathcal{S}\) in its various incarnations.)

Proof. Let \(C'\) be any coalgebra 1-morphism in \(\mathcal{S}\) and \(Y\) a multiplicity free direct sum of representatives of isomorphism classes of indecomposable objects in \(\text{inj} \mathcal{S}_E\)(C'). Then the coalgebra 1-morphism \(C = [Y, Y]\) is Morita–Takeuchi equivalent to \(C'\), and the equivalence between \(\text{inj} \mathcal{S}_E\)(C') and \(\text{inj} \mathcal{S}_E\)(C) identifies \(Y\) with \(C\), so \(C\) is the image of a multiplicity free direct sum of representatives of isomorphism classes of indecomposable objects in \(\text{inj} \mathcal{S}_E\)(C) under the forgetful functor to \(\mathcal{C}\), as required. \(\Box\)

Theorem 31. There is an injective map \(\hat{\Theta}\) from the set of equivalence classes of simple transitive 2-representations of \(\mathcal{S}_H\) to the set of equivalence classes of graded simple transitive 2-representations of \(\mathcal{S}_H\) with apex \(H\).

Proof. Let \(N\) be a simple transitive 2-representation of \(\mathcal{S}_H\). Let \(A\) be a coalgebra 1-morphism in \(\mathcal{S}_H\) such that \(N\) is equivalent to \(\text{inj} \mathcal{S}_H\)(A). Then \(\text{inj} \mathcal{S}_H\)(\(\Theta(A)\)) is a simple transitive 2-representation of \(\mathcal{S}_H\) (with apex \(H\)) due to Proposition 23 and [MMMZ, Corollary 12]. By Proposition 26 this yields a well-defined map from the set of equivalence classes of simple transitive 2-representations of \(\mathcal{S}_H\) to the set of equivalence classes of graded simple transitive 2-representations of \(\mathcal{S}_H\) with apex \(H\).

Now assume \(A\) is chosen such that \(A\) is the image of a multiplicity free direct sum of representatives of isomorphism classes of indecomposable objects in \(\text{inj} \mathcal{S}_H\)(A) under the forgetful functor to \(\mathcal{S}_H\), cf. Lemma 30. Set \(C := \Theta(A)\). Then, by Proposition 27 and Corollary 28, the object \(X\) defined in the going down procedure is isomorphic to \(C\). The 2-representation of \(\mathcal{S}_H\) obtained by going down is now, clearly, equivalent.
to \( \text{inj}_{\mathcal{A}_H}(\Lambda) \). Therefore, the map defined in the previous paragraph is injective, as claimed. \( \square \)

Remark 32. We do not know what kind of 2-representation of \( \mathcal{A}_H \) we can obtain by applying the going down procedure to a general graded simple transitive 2-representation of \( \mathcal{A}_H \). For example, at this stage we do not even know whether such a 2-representation of \( \mathcal{A}_H \) is always transitive, let alone simple transitive. Of course, the validity of Conjecture 33 below would guarantee simple transitivity.

4.6. Main conjecture. Theorem 31 shows that \( \hat{\Theta} \) is injective. There is plenty of numerical evidence that \( \hat{\Theta} \) is also surjective, cf. Section 7, which motivates our main conjecture:

Conjecture 33. The map \( \hat{\Theta} \) is bijective.

4.7. Proof of the main conjecture given the longest element of a parabolic.

Theorem 34. Assume that \( \mathcal{H} \) contains an element \( w_0^1 \). Then the statement of Conjecture 33 is true.

Proof. Consider the following:

- the \( \mathcal{H} \)-cell \( \tilde{\mathcal{H}} \) of \( \mathbb{I}_1 \);
- the \( \mathcal{H} \)-simple 2-category \( \mathcal{A}_{\tilde{\mathcal{H}}} \) associated to \( \tilde{\mathcal{H}} \) which we define as the \( \tilde{\mathcal{H}} \)-simple quotient of \( \mathcal{A}(\mathbb{I}, \mathbb{I}) \);
- the asymptotic bicategory \( \mathcal{A}_{\tilde{\mathcal{H}}} \) associated to \( \tilde{\mathcal{H}} \);
- the algebra \( C \) of \( W_1 \)-invariants in \( C \).

From the Chevalley–Shephard–Todd Theorem and positivity of the grading on \( C \), we see that

\[ C_C \cong (C^1)^{\oplus p}, \quad \text{where} \quad p \in \mathbb{N}_0[v] \quad \text{and} \quad p(0) = 1. \]

Therefore, by the definition of the asymptotic bicategory, mapping

\[ \tilde{\mathcal{H}} \ni F \mapsto C \otimes C \otimes C \]

induces a 2-functor from \( \mathcal{A}_{\tilde{\mathcal{H}}} \) to \( \mathcal{A}_H \).

By [Wi, Proposition 7.4.3], (10) sends indecomposable 1-morphisms in \( \tilde{\mathcal{H}} \) to indecomposable 1-morphisms in \( \mathcal{H} \) inducing a bijection between these two sets. Therefore, the 2-functor from \( \mathcal{A}_{\tilde{\mathcal{H}}} \) to \( \mathcal{A}_H \) induced by (10) is an equivalence.

Next we claim that the (ungraded version of the) 2-category \( \mathcal{A}_{\tilde{\mathcal{H}}} \) is monoidally equivalent to \( \mathcal{A}_{\tilde{\mathcal{H}}} \). To establish this, it is sufficient to show that the 2-category \( \mathcal{A}_{\tilde{\mathcal{H}}} \) is semisimple, as then the degree 0 part of \( \mathcal{A}_{\tilde{\mathcal{H}}} \) is equivalent to the ungraded version of \( \mathcal{A}_{\tilde{\mathcal{H}}} \) and the necessary equivalence follows directly from the definition of \( \mathcal{A}_{\tilde{\mathcal{H}}} \).

From [KMMZ, Theorem 2], it follows that \( \mathbb{I}_1 \) acts as a projective functor for the algebra underlying the cell 2-representation \( C_{\tilde{\mathcal{H}}} \) of \( \mathcal{A}_{\tilde{\mathcal{H}}} \). In particular, the latter algebra is semisimple and thus, the Jacobson radical of \( \mathcal{A}_{\tilde{\mathcal{H}}} \) is a left 2-ideal. Applying the involution from the fiat structure on \( \mathcal{A}_{\tilde{\mathcal{H}}} \) maps the Jacobson radical bijectively to the Jacobson radical and shows that it is also a right 2-ideal. Hence, it is a 2-ideal and thus zero by the \( \mathcal{H} \)-simplicity of \( \mathcal{A}_{\tilde{\mathcal{H}}} \).

Summing up the above, we see that \( \mathcal{A}_{\tilde{\mathcal{H}}} \) is biequivalent to the ungraded version of \( \mathcal{A}_{\tilde{\mathcal{H}}} \). The claim of the theorem now follows from [MMMZ, Theorem 15]. \( \square \)
Remark 35. For fixed $W$ of rank $n$, let $\#J_n$ and $\#J(w^1_0)_n$ be the number of all two-sided cells and the number of those two-sided cells containing a $w^1_0$, respectively. For such cells Conjecture 33 holds by Theorem 34. Thus, it would be interesting to calculate the ratio $\#J(w^1_0)_n/\#J_n$.

- In types $A$ or $I_2(m)$ this ratio is always 1;
- In exceptional types, this ratio is always $\geq \frac{1}{2}$, due to the fact that either $J$ or its “dual” $J':= Jw_0$ contain a $w^1_0$, cf. Section 7
- For $W$ of type $B$, it might happen that neither $J$ nor $J'$ contain a $w^1_0$. Moreover, we expect that $\lim_{n \to \infty} \#J(w^1_0)_n/\#J_n = 0$ as a heuristic indicates:
  
  First, in low ranks we obtain the following (decreasing) ratios.

| $n$ | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  | 13  | 14  | 15  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| ratio | 1.0000 | 1.0000 | 1.0000 | 0.9570 | 0.9252 | 0.9060 | 0.8867 | 0.8684 | 0.8524 | 0.8757 | 0.7770 | 0.7566 | 0.7432 | 0.7163 |

Second, in type $B$ the number of two-sided cells and the number of conjugacy classes of parabolic subgroups are easy to compute as they are counting problems, see e.g. [Lu2] (4.5.6) for the first. (Note that it follows from [Ga], Theorem 1.2.13] or [Ia], Section 2.1] that $w^1_0$ of conjugate parabolic subgroups are all in the same two-sided cell.) The ratio of these numbers tends to zero as $n$ grows, e.g. in case $n=100$ one already has $\#J(w^1_0)_n/\#J_n \leq 0.25$;

- Finally, we expect type $D$ to be similar to type $B$.

5. The role of the Duflo involution

Throughout this section we set $a:=a(\mathcal{H})$ for our fixed $\mathcal{H}$-cell, and let $d$ be the Duflo involution in $\mathcal{H}$.

5.1. Cell 2-representations and Duflo involutions. Let $w \in \mathcal{H}$ and $\tilde{L}_w$ be the corresponding simple object in $\mathbb{C}_\mathcal{H}(\emptyset)$, concentrated in degree zero. By [MM3] Section 7] (see also [MM1] Subsection 4.5]), $\tilde{C}_w \tilde{L}_d$ is an indecomposable injective object in $\mathbb{C}_\mathcal{H}(\emptyset)$ with simple socle $L_w$ concentrated in degree 0.

Dually, let $L_w$ be the simple object in $\mathbb{C}_\mathcal{H}$, corresponding to $w$, concentrated in degree zero. Then $C_w L_d$ is an indecomposable projective object in $\mathbb{C}_\mathcal{H}(\emptyset)$ with simple head $L_w$ concentrated in degree 0.

Lemma 36. For any $x, w \in \mathcal{H}$,

(i) the injective objects $\tilde{C}_x \tilde{L}_w$ and $C_x \tilde{L}_w$ in $\mathbb{C}_\mathcal{H}(\emptyset)$ are concentrated between the degrees $-2a$ and 0 and between the degrees 0 and $2a$, respectively;

(ii) the projective objects $C_x L_w$ and $C_x \tilde{L}_w$ in $\mathbb{C}_\mathcal{H}(\emptyset)$ are concentrated between the degrees $-2a$ and 0 and between the degrees 0 and $2a$, respectively.

Proof. Let us prove the first statement in (i). As the 2-category of Soergel bimodules is positively graded, the fact that $\tilde{C}_x \tilde{C}_w L_d$ is concentrated in non-positive degrees follows from Lemma 20. This in turn implies that $\tilde{C}_x \tilde{L}_w$ is concentrated in non-positive degrees as well.

By adjunction, we have $\text{hom}_{\mathbb{C}_\mathcal{H}}(\tilde{C}_x \tilde{L}_w, \tilde{L}_y^{\oplus v}) \cong \text{hom}_{\mathcal{H}}(\tilde{L}_w, \tilde{C}_x \tilde{L}_y^{\oplus (2a+k)})$. 

As the right-hand side of the above isomorphism is zero for $2a + k < 0$, we deduce that 
\[ \hom_{\mathcal{C}_H}(\tilde{C}_x \tilde{L}_w, \tilde{L}_y^{\oplus k}) \] is also zero if $k < -2a$.

The second statement in (i) follows from the first one and the fact that \( \tilde{C}_x = C_x^{\oplus 2a} \).

The dual statements in (ii) are proved in exactly the same way, using Lemma 19 instead of Lemma 20.

Let \( P := P_w \) be the principal 2-representation of \( \mathcal{S}_H \) and \( P \) its injective abelianization, cf. Example 3. Denote by \( I_w \) and \( I_e \) the corresponding injective object in \( \mathcal{P}(\mathcal{O}) \) with respect to \( C_w \) and \( C_e = 1_{\mathcal{O}} \), respectively, see [MMMT1, Subsection 3.1] for details. Note that \( I_w \) has simple socle \( L_w \) concentrated in degree 0 and \( I_e \) has simple socle \( L_e \) concentrated in degree \( a \).

**Lemma 37.** For any \( x \in \mathcal{H} \), the following hold.

(i) The injective object \( \tilde{C}_x \tilde{L}_d \) in \( \mathcal{C}_H(\mathcal{O}) \) has simple head \( \tilde{L}_x \) concentrated in degree \(-2a\).

(ii) The projective object \( C_x \tilde{L}_d \) in \( \mathcal{C}_H(\mathcal{O}) \) has simple socle \( L_x \) concentrated in degree \( 2a \).

**Proof.** We prove the statement (i) since the dual statement (ii) follows by similar arguments. As \( \mathcal{S}_H \) is flat, the underlying algebra of \( \mathcal{C}_H \) is self-injective, which implies that the indecomposable injective object \( \tilde{C}_x \tilde{L}_d \) is also projective and thus, has a simple head. Therefore, it suffices to prove that \( \hom_{\mathcal{C}_H}(\tilde{C}_x \tilde{L}_d, \tilde{L}_x) \) is non-zero. By adjunction, this is equivalent to proving that
\[ \hom_{\mathcal{C}_H}(\tilde{L}_d, \tilde{C}_x \tilde{L}_x) \neq 0, \]
which holds if and only if \( \tilde{C}_d \tilde{L}_d \) appears as a direct summand of \( \tilde{C}_x \tilde{L}_x \) in \( \mathcal{C}_H(\mathcal{O}) \).

Note that the latter holds if and only if the same is true in \( \mathcal{P}(\mathcal{O}) \). (Recall the two equivalent constructions of cell 2-representations, cf. Subsection 2.4 and [MM1, Subsection 4.5].) In the principal 2-representation, we can use the following fact. Since \( \tilde{C}_x \tilde{L}_d \cong I_d \), we have
\[ \hom_{\mathcal{C}_H}(\tilde{C}_x \tilde{L}_d, I_e^{\oplus k}) \cong \hom_{\mathcal{P}(\mathcal{O})}(\tilde{L}_d, I_e^{\oplus 2a+k}), \]
whose right-hand side is zero unless \( x = d \) and \( k = -2a \). In other words, the only \( x \in \mathcal{H} \) such that \( \tilde{C}_x \tilde{L}_d \) has a composition factor isomorphic to \( \tilde{L}_e \), up to a shift, is \( x = d \). Therefore, to prove (i), it is enough to show that \( \hom_{\mathcal{P}(\mathcal{O})}(\tilde{C}_x \tilde{L}_d, I_e^{\oplus 2a}) \) is non-zero. By adjunction, we have
\[ \hom_{\mathcal{C}_H}(\tilde{C}_x \tilde{L}_d, I_e^{\oplus 2a}) \cong \hom_{\mathcal{C}_H}(\tilde{L}_d, I_x), \]
where the right-hand side is non-zero. Hence, \( \tilde{L}_x^{\oplus 2a} \) appears in the head of \( \tilde{C}_x \tilde{L}_d \) in \( \mathcal{C}_H(\mathcal{O}) \). \( \square \)

Assume that \( B \) is the underlying basic algebra of \( \mathcal{C}_H \). Then the indecomposable objects in \( \mathcal{C}_H(\mathcal{O}) \) are identified with indecomposable injective \( B \)-modules.

**Proposition 38.** The algebra \( B \) is a finite dimensional, positively graded, weakly symmetric Frobenius algebra of graded length \( 2a \).

**Proof.** By construction, the algebra \( B \) is non-negatively graded and its degree 0 part, which is isomorphic to \( \mathcal{C}_H^\# \), is semisimple, so \( B \) is positively graded.
Since \( \mathcal{A} \) is fiat, the algebra \( B \) is self-injective and thus Frobenius as it is basic.

The fact that \( B \) is weakly symmetric follows from Lemma 37. Together with Lemma 30, this implies \( B \) is of graded length 2a. \( \Box \)

Since \( \mathcal{A} \) is fiat, the action of \( C_w \) is exact on the category of \( B \)-modules. Further, by [KMMZ, Theorem 2], we know that the action of \( C_w \) via \( \mathcal{A} \) is given by tensoring with a projective-injective bimodule. It follows from Lemma 30 that the bimodule representing \( C_w \) is isomorphic to a direct sum of bimodules of the form \( B_{e_u} \otimes_C e_v B \), possibly with multiplicities but without grading shifts, where \( e_u, e_v \) are some primitive idempotents of \( B \). By Proposition 38 and the fact that \( \mathcal{A} \) is a faithful 2-functor which is degree-preserving on 2-morphisms, this implies that the 1-morphism \( C_w \) in \( \mathcal{A} \) has graded length at most 4a.

Recall from Example 24 that \( C_d \) is a cosimple coalgebra 1-morphism in \( \mathcal{A} \). By [MMMZ, Corollary 12], \( M := \text{inj}_{\mathcal{A}}(C_d) \) is a graded simple transitive 2-representation of \( \mathcal{A} \) with apex \( H \). We denote by \( B^M \) the basic algebra underlying \( M \).

**Proposition 39.** The algebra \( B^M \) is a positively graded Frobenius algebra of graded length 2a.

**Proof.** The algebra \( B^M \) is graded, by definition, and self-injective and Frobenius by the same arguments as in Proposition 38.

By Lemma 30 we can choose a coalgebra 1-morphism \( \Lambda \) such that \( \text{inj}_{\mathcal{A}}(\Lambda_d) \) is equivalent to \( \text{inj}_{\mathcal{A}}(\Lambda_d) \) and \( \Lambda \) is the image of a multiplicity free direct sum of representatives of isomorphism classes of indecomposable objects in \( \text{inj}_{\mathcal{A}}(\Lambda_d) \) under the forgetful functor to \( \mathcal{A} \). By Corollary 28, the object \( C = \Theta(\Lambda) \) in \( \text{inj}_{\mathcal{A}}(\Lambda_d) \) is a multiplicity free direct sum of representatives, up to grading shift, of isomorphism classes of indecomposable objects in \( \text{inj}_{\mathcal{A}}(\Lambda_d) \) (and \( \text{inj}_{\mathcal{A}}(\Lambda_d) \) is equivalent to \( \mathcal{M} \), cf. Proposition 26). From the definition of \( \mathcal{M} \), for all \( k \in \mathbb{Z} \), we have

\[
\text{hom}_{\mathcal{A}}(C, \mathbb{I}^k_{\mathcal{A}}) \cong \text{hom}_{\mathcal{M}}(C, \mathbb{I}^k_{\mathcal{M}}).
\]

If \( k > 0 \), then the fact that Soergel bimodules are positively graded implies that the left-hand side of (12) is zero. Consequently, \( B^M \) is positively graded.

Recall that \( C \) is isomorphic to a direct sum of \( C_w \), where \( w \in H \), possibly with some multiplicities but without shifts. This immediately implies that the left-hand side of (12) is zero if \( k < -2a \) and hence, the graded length of \( B^M \) is at most 2a. Furthermore, since \( \mathcal{M} \) is transitive, every \( C_w \) appears in \( C \) with a non-zero multiplicity, for \( w \in H \). As the algebra underlying the cell 2-representation has graded length 2a, cf. Proposition 39, we know that there exists a \( C_w \) such that \( \text{hom}_{\mathcal{A}}(C_w, \mathbb{I}^k_{\mathcal{A}}) \neq 0 \), which implies that the left-hand side of (12) is non-zero for \( k = -2a \). Thus, the graded length of \( B^M \) is exactly 2a. This completes the proof. \( \Box \)

**Lemma 40.** For all \( w \in H \), the 1-morphism \( C_w \) in \( \mathcal{A} \) is of graded length 4a.

**Proof.** Recall from Example 17 that \( C_d \cong [C_d, C_d] \). We first claim that

\[
\text{hom}_{\mathcal{A}}(C_d, C_d^{\otimes^4}) \cong \text{hom}_{\mathcal{M}}(C_d, C_d^{\otimes^4})\]

is non-zero, where \( C \) is as in the proof of Proposition 39. By Lemma 19 we have

\[
C_w C_d \cong \bigoplus_{w \in H} C_d^{\otimes^4}.
\]
Noting that [7], [3] and [Lu3] Conjecture 14.2.P2, P5 and P7] imply that
\[ \gamma_{w,d,z^{-1}} = \gamma_{z^{-1},w,d} = \delta_{z,w}, \]
we deduce that
\[ \sqrt[n]{A_{w,d,z}} \in \begin{cases} 1 + \cdots + \sqrt[n]{2^a} & \text{if } z = w, \\ \sqrt[n]{N_0[v]} \cap \sqrt[n]{2^{a-1}}N_0[v^{-1}] & \text{if } z \neq w. \end{cases} \]

Therefore, the object \( C^{\oplus v^{-2a}} \) is isomorphic to a direct summand of \( C^{\oplus v^{-2a}} \). The head of the indecomposable injective object \( C_d \) in \( \mathbf{M} \) is isomorphic to a direct summand of the socle of \( C^{\oplus v^{-2a}} \), cf. Proposition [39]. Hence, the right-hand side of (13) is non-zero, which implies that the left-hand side is non-zero. This shows that \( C \) has a direct summand isomorphic to \( C_v \) with graded length at least \( 4a \). Therefore \( C_v \) must have graded length exactly \( 4a \), as we already know that it is at most \( 4a \). By adjunction, we have
\[ 0 \neq \text{hom}_{\mathcal{H}}(C_d, C^{\oplus v^{-2a}}) \cong \text{hom}_{\mathcal{H}}(\tilde{C}_v^{-1}, C^{\oplus v^{-2a}}) \cong \text{hom}_{\mathcal{H}}(C_v^{-1}, C^{\oplus v^{-2a}}), \]
yielding that the graded length of \( C_d \) is at least \( 4a \). Thus, as above, it must be equal to \( 4a \). Note that for any \( w \in \mathcal{H} \) we have \( \sqrt[n]{A_{w,w^{-1},d}} \in 1 + \cdots + \sqrt[n]{2^a} \). Therefore each \( \tilde{C}_w^{-1}C^{\oplus v^{-2a}} \) contains a direct summand \( C^{\oplus v^{-2a}} \). By (14), we have
\[ 0 \neq \text{hom}_{\mathcal{H}}(C_v^{-1}, \tilde{C}_w^{-1}C^{\oplus v^{-2a}}) \cong \text{hom}_{\mathcal{H}}(C_wC_v^{-1}, C^{\oplus v^{-2a}}). \]
By Lemma [19] the direct summands appearing in \( C_wC_v^{-1} \) have non-negative shift. Again as before, this shows that the graded length of \( C_w \) is at least \( 4a \), which implies that it must be equal to \( 4a \). \( \Box \)

Recall from Example [24] that \( \tilde{C}_d \) is a simple algebra 1-morphism in \( \mathcal{H} \).

**Proposition 41.**
(i) The 2-representation \( \text{proj}_{\mathcal{H}}(C_d) \) is equivalent to the cell 2-representation \( C_{\mathcal{H}} \).
(ii) The 2-representation \( \text{proj}_{\mathcal{H}}(\tilde{C}_d) \) is also equivalent to the cell 2-representation \( C_{\mathcal{H}} \).

**Proof.** Again, we will prove the statement (i) and the dual statement (ii) verbatim. Consider \( C_d \) as an object of \( C_{\mathcal{H}}(\mathbb{C}) \) and set \( C := [C_d, C_d] \). As a 1-morphism in \( \mathcal{H} \) we have
\[ C \cong \bigoplus_{w \in \mathcal{H}} C^{\oplus p_w}, \]
with \( p_w \in \mathbb{N}_0[v, v^{-1}] \). Furthermore,
\[ \text{hom}_{\mathcal{H}}(C, C^{\oplus v^{-k}}) = \text{hom}_{\mathcal{H}}([C_d, C_d], C^{\oplus v^{-k}}) \cong \text{hom}_{C_{\mathcal{H}}}(C_d, C^{\oplus v^{-k}}C_d). \]

Thanks to the positivity of the grading on \( \mathcal{H} \) and Lemma [19] we see that \( p_w \in \mathbb{N}_0[v^{-1}] \).

If \( k < -4a \), then the right-hand side of (15) is zero, because \( C_d \) is an indecomposable injective object of graded length \( 2a \) by Lemma [36] and the action of \( C_w \) increases the graded length by at most \( 2a \) by Lemma [19]. On the left-hand side, the indecomposable injective object \( C_w \) has graded length \( 4a \), cf. Lemma [40] which implies that \( C \) lives in non-negative degrees, that is, \( p_w = p_w(0) \).

For \( k = 0 \), the right-hand side of (15) is one dimensional if \( w = d \), and zero otherwise. This implies \( C \cong C_d \in \mathcal{H} \). Since the degree 0 maps \( C_d \to 1_{\mathcal{H}} \) and \( C_d \to C_dC_d \) are unique up to scalar, it follows that \( C \cong \Theta(\mathbb{1}_{\mathcal{H}}) \) as coalgebra 1-morphisms in \( \mathcal{H} \). \( \Box \)
5.2. The categorified bar involution.

**Proposition 42.** There exists a functorial involution $\vee : \mathcal{S} \rightarrow \mathcal{S}$, which is covariant on 1-morphisms and contravariant and degree-preserving on 2-morphisms, such that
\[(B^\oplus_{w^k})^\vee \cong B^\oplus_{w^\vee^k}\]
for all $w \in W$ and $k \in \mathbb{Z}$.

**Proof.** Let $\mathcal{S}$ be the diagrammatic Soergel category as in [EW2]. By [EW2] Theorem 6.28, we can identify $\mathcal{S}$ with $\text{add}(\mathcal{S})$. (Strictly speaking, we have to quotient $\text{add}(\mathcal{S})$ by the 2-ideal generated by the totally invariant polynomials with no constant term in the base ring $R$ in that paper, because $\mathcal{S}$ was defined over the coinvariant algebra. This is a technical detail, which we will suppress from now on.) Let $w$ be an arbitrary word in the simple reflections of $W$ and $BS(w)$ the corresponding Bott–Samelson bimodule. Then $\vee : \mathcal{S} \rightarrow \mathcal{S}$ is defined by $(BS(w)^{\oplus^k})^\vee := BS(w)^{\oplus^\vee^k}$, for all words $w$ and $k \in \mathbb{Z}$, and by flipping the Soergel diagrams upside-down. By definition, $\vee$ is covariant on 1-morphisms, contravariant and degree-preserving on 2-morphisms. Note that $(BS(w)^{\oplus^k})^\vee \cong BS(w)^{\oplus^\vee^k}$ under the identification $\mathcal{S} \cong \text{add}(\mathcal{S})$.

Extend $\vee$ to $\text{add}(\mathcal{S})$. To show (16), we use induction on the length $\ell(w)$ of $w$, the case $\ell(w) = 0$ being immediate. Assume that $\ell(w) > 0$ and that (16) holds for all $v \in W$ with $\ell(v) < \ell(w)$ and all $k \in \mathbb{Z}$. By [So1] Satz 6.24 (see also [EW2] Theorem 3.14 and the text around it) and [EW2] Corollary 6.26, we have

$$BS(w) \cong B_w \oplus \bigoplus_{v \prec w} B_{v^\oplus^p^w}^{\oplus^v^w}\vspace{12pt}$$

for every $w \in W$, where $w$ is an arbitrary reduced expression for $w$, the $p_{w,v} \in \mathbb{N}[v,v^{-1}]$ are invariant under the bar involution (which follows from [Lu3] Chapter 4) and the Soergel–Elias–Williamson categorification theorem) and $\prec$ is the Bruhat order. As noted above, we have $BS(w)^\vee \cong BS(w)$. By induction, we also have $(B_{p^w v}^{\oplus^p^w,v})^\vee \cong B_{v^\oplus^p^w,v}$ for all $v \prec w$, using the bar invariance of $p_{w,v}$. Since $\mathcal{S}$ is Krull–Schmidt, we deduce that $B_w^\vee \cong B_w$. This implies that (16) holds for all $k \in \mathbb{Z}$. \hfill $\square$

Recall that the bar involution on the Hecke algebra is uniquely determined by the fact that it is $\mathbb{Z}$-linear, sends $v^k \mapsto v^{-k}$ for all $k \in \mathbb{Z}$ and fixes the Kazhdan–Lusztig basis elements, see e.g. [Lu3] Chapters 4 and 5. By Proposition 42 and [EW2] Theorem 6.28 and Corollary 6.26, the duality $\vee$ thus categorifies the bar involution on the Hecke algebra. We will therefore refer to it as the "categorified bar involution".

**Remark 43.** Note that $\vee$ also appears in [EW2] Definition 6.22, where it is denoted $\iota$ and gives an antiinvolution on double light leaves.

**Proposition 44.** The categorified bar involution on Soergel bimodules defines a functorial involution $\vee$ on $\mathcal{S}_H$, which is covariant on 1-morphisms and contravariant and degree-preserving on 2-morphisms, such that $(B_{x^k}^\oplus)^\vee \cong B_{x^\vee^k}^\oplus$ for all $x \in H$ and $k \in \mathbb{Z}$. This functorial involution extends to an equivalence between $\mathcal{S}_H$ and $\mathcal{S}_H^\vee$, which sends injective 1-morphisms in the first 2-category to projective 1-morphisms in the second.

**Proof.** Because $\vee$ is covariant on 1-morphisms and $(B_{w^k}^\oplus)^\vee \cong B_{w^\vee^k}^\oplus$ for $w \in W$ and $k \in \mathbb{Z}$, it preserves left, right and two-sided cells.

Let $\mathcal{S}_H$ be the graded 2-full 2-subcategory of $\mathcal{S}$ generated by the $B_{x^k}^\oplus$, for $x \in H$ and $k \in \mathbb{Z}$. Then $\vee$ restricts to a functorial involution on $\mathcal{S}_H$. As $\vee$ sends identity
2-morphisms to identity 2-morphisms, it also preserves the maximal 2-ideal \( I_H \) in \( \mathcal{F}_H \) which does not contain any identity 2-morphism on \( B_x^{\oplus v} \), for \( x \in H \) and \( k \in \mathbb{Z} \). Since \( \mathcal{F}_H = \mathcal{F}_H / I_H \), the first claim follows.

Finally, since \( \vee \) is contravariant on 2-morphisms, it extends to an equivalence between \( \mathcal{F}_H \) and \( \mathcal{F}_H \).

**Corollary 45.** The functorial involution \( \vee \) induces a functorial involution on \( C_H \), also denoted \( \vee \), which is contravariant and degree-preserving on morphisms and satisfies \( (B_x^{\oplus v})^\vee \cong B_x^{\oplus v} \) for all \( x \in H \) and \( k \in \mathbb{Z} \). This functorial involution extends to an equivalence between \( C_H \) and \( C_H \) which sends injective objects in the category underlying the first to projective objects in the category underlying the second.

**Proof.** As already remarked, the functorial involution \( \vee \) also preserves left cells. The rest now follows as in the proof of Proposition 44. \( \square \)

**Remark 46.** The existence of \( \vee \) implies that any statement in Subsection 5.1 has a dual counterpart. In particular, the equivalence \( C_H \cong C_H \) gives \( B_\vee \cong B \), cf. Proposition 38.

### 5.3. Explicit bimodules for the cell 2-representation.

In the following, we will use projective abelianizations instead of injective ones. As we are in the fiat setup, the difference does not play an essential role on an abstract level, but with this choice we describe the action of \( C_w \) by projective bimodules and their composition by tensoring over the underlying algebra, which is very convenient.

Denote by

\[
B := \text{End}_{C_H}(\bigoplus_{w \in H} C_w)
\]

the algebra underlying the cell 2-representation. Fix a set of primitive idempotents \( e_w \in B \), for \( w \in H \), corresponding to the indecomposable projective objects \( C_w \in C_H(\varnothing) \). Set \( Q_w := B e_w \) and let \( L_d \) (recall the notion in Subsection 5.1) be the simple head of \( Q_w \) in \( C_H \). Note that \( C_w L_d \cong Q_w \) for \( w \in H \), cf. Subsection 2.4 and [MM3, Section 7]. Lemma 37(ii) implies that the socle of \( Q_w \) is isomorphic to \( L_d^{\oplus v} \).

For every pair \( x, y \in H \), we have

\[
\text{grdim} (\text{Hom}_B(Q_x, Q_y)) = \text{grdim} (\text{Hom}_B(C_x L_d, C_y L_d))
= \text{grdim} (\text{Hom}_B(C_y, C_x L_d^{\oplus v} L_d))
= \sum_{z \in H} h_{y^{-1}, x, z} \text{grdim} (\text{Hom}_B(C_z L_d^{\oplus v} L_d))
= \sum_{z \in H} h_{y^{-1}, x, z} \text{grdim} (\text{Hom}_B(Q_z^{\oplus v} L_d))
= v^a h_{y^{-1}, x, d}.
\]

(17)

By [Lu3 Subsection 13.6], we know that

\[
v^a h_{y^{-1}, x, d} \in \begin{cases} 1 + \cdots + v^{2a} & \text{if } x = y; \\ v^{N_0}[v] \cap v^{2a-1}N_0[v^{-1}] & \text{if } x \neq y. \end{cases}
\]

Recall that, by definition of the Kazhdan–Lusztig basis, the \( h_{y^{-1}, x, d} \) are invariant under the bar involution.
Remark 47. By \[Lu3\] 13.1(e), we have $h_{v^{-1}, x, d} = h_{x^{-1}, v, d}$, which corresponds to the fact that

$$\text{grdim}(\text{Hom}_B(Q_x, Q_v)) = \text{grdim}(\text{Hom}_B(Q_v, Q_x))$$

for all $x, v \in \mathcal{H}$.

**Proposition 48.** For any $w \in \mathcal{H}$, the action of $C_w$ on the category of finite dimensional, graded $B$-modules is isomorphic to tensoring with the graded projective $B$-$B$-bimodule

$$\bigoplus_{u, v \in \mathcal{H}} (B e_u \otimes_C e_v B) \otimes \gamma_{w, v, u}^{-1}.$$

**Proof.** By \[KMMZ\] Theorem 2, we know that the action of $C_w$ is given by tensoring with a $B$-$B$-bimodule of the form

$$\bigoplus_{u, v \in \mathcal{H}} (B e_u \otimes_C e_v B) \otimes c_{w, v, u},$$

for certain $c_{w, v, u} \in \mathbb{N}_0[v, v^{-1}]$. We also know that, for any $x \in \mathcal{H}$, we must have

$$C_w Q_x \cong C_w C_x L_d \cong \bigoplus_{u \in \mathcal{H}} Q_u B e_{x - 1, v, d} c_{w, v, u}.$$

On the other hand, using (17), we obtain

$$\bigoplus_{u, v \in \mathcal{H}} (B e_u \otimes_C e_v B) \otimes \gamma_{w, v, u}^{-1} \cong \bigoplus_{u, v \in \mathcal{H}} (B e_u \otimes_C e_v B e_x) \otimes \gamma_{w, v, u}^{-1} \cong \bigoplus_{u, v \in \mathcal{H}} B e_v \otimes \text{grdim}(e_u B e_x) c_{w, v, u} \cong \bigoplus_{u, v \in \mathcal{H}} B e_u \otimes \gamma_{w, v, u}^{-1} \otimes \gamma_{w, v, u}^{-1} c_{w, v, u},$$

and hence, deduce that the $c_{w, v, u}$ have to satisfy

(19) $\sum_{v \in \mathcal{H}} h_{x^{-1}, v, d} c_{w, v, u} = h_{w, x, u}$

for all $w, x, u, v \in \mathcal{H}$.

For every fixed pair $w, u \in \mathcal{H}$, this is a system of $\# \mathcal{H}$ linear equations, indexed by $x \in \mathcal{H}$, in $\# \mathcal{H}$ variables, indexed by $v \in \mathcal{H}$. We claim that $c_{w, v, u} = \gamma_{w, v, u^{-1}}$ for $v \in \mathcal{H}$, is the unique solution of (19).

Let us first show that $c_{w, v, u} = \gamma_{w, v, u^{-1}}$ is a solution of (19), i.e. that we have

$$\sum_{v \in \mathcal{H}} h_{x^{-1}, v, d} \gamma_{w, v, u^{-1}} = h_{w, x, u}.$$

This equation is similar to one in \[Lu3\] Subsection 18.8 and can be proved in the same way, using:

- the equation at the beginning of the proof of Theorem 18.9(b) in \[Lu3\], i.e.

(20) $\sum_{z \in W} h_{x_1, z, y} \gamma_{z, x_3, y^{-1}} = \sum_{z \in W} h_{x_1, z, y} \gamma_{z, x_3, y^{-1}}$;

- the symmetries in \[Lu3\] 13.1(e)];

- \[Lu3\] Proposition 13.9(b) and Conjecture 14.2.P7, i.e.

(21) $h_{a, b, c} = h_{b, a^{-1}, c^{-1}}$, $\gamma_{a, b, c} = \gamma_{b, a^{-1}, c^{-1}}$, $\gamma_{a, b, c} = \gamma_{c, a, b}$;
By (20), (21) and (22), we have
\[
\sum_{v \in H} h^{v-1, w, v} \gamma_{v, w, u} = \sum_{v \in H} h^{v-1, w, v} \gamma_{v, w, u} = h^{w, x, u}.
\]

Finally, note
\[
h_x^{v-1, w, u, v} = v^{-a} \gamma_x^{v-1, w, u} \quad \text{(mod } v^{-a} + 1\text{N}_0[v]),
\]
so the determinant of the matrix
\[
(h_x^{v, w, u})_{v, w \in H}
\]
belongs to \( v^{-a} \#H (1 + v\text{N}_0[v]) \), and the matrix is hence invertible over \( \mathbb{C}(v) \). Our system of linear equations in (19) therefore has a unique solution and the statement of the proposition follows.

5.4. The Frobenius structure on the Duflo involution. In this subsection, we describe the structure of a Frobenius algebra 1-morphism on the Duflo involution in \( \mathcal{H}_0 \) explicitly. As \( C_d \cong [C_d, C_d] \) is a graded coalgebra 1-morphism, the structure maps involved in the comultiplication have degree \( \pm 2a \).

More precisely, Proposition 48 implies that \( C_d \) acts via the \( B \)-\( B \)-bimodule
\[
\bigoplus_{u \in H} B e_u \otimes_C e_u B.
\]

The comultiplication on this bimodule is given by
\[
\delta_d : \bigoplus_{u \in H} B e_u \otimes_C e_u B \to \bigoplus_{u, v \in H} B e_u \otimes_C e_u B e_v \otimes_C e_v B, \quad e_u \otimes e_u \mapsto e_u \otimes e_u \otimes e_u
\]
and the counit by
\[
\varepsilon_d : \bigoplus_{u \in H} B e_u \otimes_C e_u B \to B, \quad ae_u \otimes e_u b \mapsto ae_u b.
\]

To describe the algebra structure, consider the Frobenius trace \( tr_B : B \to C \) and note that \( tr_B(e_u be_v) = 0 \) for a homogeneous element \( e_u be_v \) unless \( u = v \) (because \( B \) is weakly symmetric) and the degree of \( e_u be_v \) is \( 2a \).

Let \( B_0 \) be a homogeneous basis of \( e_u B e_v \) and set \( B = \bigcup_{u, v \in H} v B, v \), which is a homogeneous idempotent-adapted basis of \( B \). For \( a \in B \), denote by \( a^* \) the dual basis element such that \( tr_B(a^* a) = 1 \) and \( tr_B(a^* b) = 0 \) for all other \( b \in B \).

The algebra structure is then given by the multiplication
\[
\mu_d : \bigoplus_{u, v \in H} (B e_u \otimes_C e_u B e_v \otimes_C e_v B)^{\otimes 2a} \to \bigoplus_{u \in H} B e_u \otimes_C e_u B,
\]
In particular, this implies that the dimension of becomes:

\[ \text{Example} \]

need not be one, in general. Let us give one simple example.

For rank \( h \) frequently does. For an explicit and minimal example, let

\[ \text{Note that, in terms of } C = 1 \]

\[ \text{Proposition 50.} \]

\( \text{If } B \text{ is symmetric. For every } \lambda \text{, the Soergel's Hom formula (cf. [EW3] Theorem 3.6)} \]

\[ \text{becomes:} \]

\[ \text{In particular, this implies that the dimension of} \]

\[ \text{need not be one, in general. Let us give one simple example.} \]

\[ \text{Example 49. For rank } 2 \text{ or lower } C_d C_d \text{ never contains such direct summands, but for} \]

\[ \text{higher ranks it frequently does. For an explicit and minimal example, let } W \text{ be of type } A_1 \text{ with simple reflections } s_j, \]

\[ \text{where we write } i = s_j \text{ for short, and Coxeter diagram} \]

\[ \begin{array}{ccc}
1 & 2 & 3
\end{array} \]

\[ \text{Set } d = 12321. \text{ Then } a = 3. \text{ Consider also the longest element } w_0 = 12321 \text{ of } W \]

\( \text{(whose } a\text{-value is 6), which is strictly greater than } d \text{ in the two-sided order. We have} \]

\[ C_d C_d \cong C_d^{\oplus 131 \oplus 33^2 \oplus 4^2} \oplus C_{w_0}^{\oplus 3^2 \oplus 4^2 \oplus 6^2 \oplus 8^2 \oplus 9^2}, \]

\[ \text{where the minimal shift of } C_d \text{ is strictly smaller than } a - a(w_0) = -3 \text{ and the maximal shift is strictly bigger than } 3a - a(w_0) = 3. \]

\[ 5.5. \text{A necessary numerical condition for } B \text{ to be symmetric.} \]

\[ \text{The explicit description of the bimodules appearing in the cell 2-representation can be used to prove a necessary numerical condition for } B \text{ to be symmetric. For every } u \in H, \text{ define} \]

\[ \lambda_u := \sum_{w \in H} h_{u^{-1}, w, d}(1) \in \mathbb{N}_0, \]

\[ \text{where by } h_{u^{-1}, w, d}(1) \text{ we mean the evaluation of the Laurent polynomial } h_{u^{-1}, w, d} \text{ at } v = 1. \text{ By (17), we have } \lambda_u = \dim_C(B^u) = \dim_C(Q^u). \]

\[ \text{Proposition 50. If } B \text{ is symmetric, then} \]

\[ \lambda_u = \lambda_w, \text{ for all } u, w \in H. \]

\[ (23) \]

\[ \text{Proof. Assume that } B \text{ is symmetric.} \]

\[ \text{Define } r_d \in \text{Hom}_H(C_d, C_d) \text{ as the composite} \]

\[ C_d \xrightarrow{\delta_d} C_d C_d \cong C_d \oplus C_d \xrightarrow{id_C \oplus \eta(u \circ v, u + \delta)} \text{id}_C C_d \oplus \text{id}_C C_d \cong (C_d C_d)^{\oplus 2a} \xrightarrow{\mu_d} C_d. \]

\[ \text{As } \text{Hom}_H(C_d, C_d) \cong \mathbb{C}, \text{ the map } r_d \text{ is equal to } \lambda \cdot \text{id}_C, \text{ for some } \lambda \in \mathbb{C} \text{.} \]

\[ \text{Computing this in the cell 2-representation, we obtain, for any } u \in H, \text{ the equality} \]

\[ r_d(e_u \otimes e_u) = \sum_{w \in H} \sum_{a \in \mathbb{C}} \text{tr}(aa^*)(e_u \otimes e_u) = \lambda_u(e_u \otimes e_u). \]

\[ (24) \]
The first equality in (24) is a direct consequence of the definition of $\mu_d, \iota_d,\delta_d$ and $\epsilon_d$ in Subsection 5.4. The last equality in (24) follows from the fact that $\text{tr}(aa^*) = \text{tr}(a^*a) = 1$ for all $w$ and $a$, since $B$ is assumed to be symmetric.

This shows that $\lambda = \lambda_u$. Since $u$ is arbitrary, the proposition follows.

Let $L(H)$ be the left cell for $H$. From [MS, Theorem 4.6] we know that $B$ is symmetric if $W$ is a Weyl group and

$$\text{there exists } I \subset S \text{ such that } w_0w_I0 \in L(H).$$

And, indeed, (23) holds in these cases. However, let us give some examples in Weyl and in non-Weyl types where (23) does not hold.

**Example 51.** In type $F_4$, there is a two-sided cell containing $H$-cells of size 5, see (42). For those $H$-cells which do not satisfy (25), condition (23) does not hold either. Explicitly, let

$$1 \quad 2 \quad 3 \quad 4$$

$$H = \{d = d^{-1} = 213432, u = u^{-1} = 2134321324, v = v^{-1} = 213234321324, w = w^{-1} = 213234321324321324, x = x^{-1} = 213234321324321324\}.$$

As before, we write $i = s_i$ for short. In this case, we have

$\lambda_d = \lambda_x = 20, \quad \lambda_u = \lambda_w = 24, \quad \lambda_v = 32.$

Thus, (23) does not hold and $B$ is not symmetric.

We also checked all $H$-cells for Weyl groups with fewer elements than the one of type $F_4$, but all of them satisfy (23). From this perspective, the $F_4$ example is the smallest counterexample.

Finally, we found various examples of higher rank where (23) fails, e.g. $H$-cells of size 3 (see (44)) in type $E_6$.

**Example 52.** If $W$ is not a Weyl group, then (23) might fail. This never happens in dihedral type, where (23) is always satisfied, but it does happen for Coxeter types $H_3$ and $H_4$, where it is not hard to find $H$-cells for which (23) does not hold. For example, this happens repeatedly in the $H$-cells which are not group-like, i.e. in case (b) in these types, see Section 7.

6. Lifted simple transitive 2-representations

6.1. The underlying algebra. Suppose $(A, \delta_A, \epsilon_A)$ is a cosimple coalgebra 1-morphism in $\mathcal{A}_H$. By [MMMZ, Corollary 12], $N := \text{inj}_{\mathcal{A}_H}(A)$ is a simple transitive 2-representation of $\mathcal{A}_H$. By Lemma 22 and Proposition 23, $\Theta(A)$ is also a cosimple coalgebra 1-morphism, which implies that $M := \text{inj}_{\mathcal{A}_H}(\Theta(A))$ is a graded simple transitive 2-representation of $\mathcal{A}_H$ with apex $H$, using [MMMZ, Corollary 12] again.

Since $\mathcal{A}_H$ is semisimple, see Proposition 21 $M$ must contain a direct summand isomorphic to $A_H$, which is the identity 1-morphism in $\mathcal{A}_H$, and $\tilde{\epsilon}_A : A \rightarrow A_H$ is the projection, which is a morphism of coalgebra 1-morphisms. Hence, we obtain a faithful morphism of 2-representations of $\mathcal{A}_H$,

$$\Phi_{\mathcal{A}_H} : \text{inj}_{\mathcal{A}_H}(A) \rightarrow \text{inj}_{\mathcal{A}_H}(A_H) \cong A_H,$$

which is the identity on morphisms and sends $(N, \rho_N)$ in $N$ to $(N, (\text{id}_N \circ h, \tilde{\epsilon}_A) \circ \rho_N)$. Here $A_H$ denotes the cell 2-representation of $\mathcal{A}_H$. 

Since \( \Theta \) is linear, the above implies that \( \Theta(\Lambda) \) contains a direct summand isomorphic to \( \Theta(A_d) = C_d \) and the counit \( \varepsilon_{\Theta(\Lambda)} : \Theta(\Lambda) \to 1_\mathcal{R} \) is the composite of the projection \( \pi_d : \Theta(\Lambda) \to C_d \), i.e. \( \Theta(\varepsilon_{\Lambda}) \), and \( \varepsilon_d : C_d \to 1_\mathcal{R} \). In particular, we obtain a faithful, degree-preserving morphism of 2-representations of \( \mathcal{F}_H \)

\[
\Phi_{\mathcal{F}_H} : \text{inj}_{\mathcal{H}}(\Theta(\Lambda)) \to \text{inj}_{\mathcal{H}}(C_d) \cong C_H,
\]

which is the identity on morphisms and sends \((M, \rho_M)\) in \(\text{M} \) to \((M, (\text{id}_M \circ \pi_d) \circ \rho_M)\).

Altogether, this yields a commuting square

\[
\begin{array}{ccc}
\text{M} & \xrightarrow{\Phi_{\mathcal{F}_H}} & C_H \\
\downarrow & & \downarrow \\
\text{N} & \xrightarrow{\Phi_{\mathcal{F}_H}} & A_H
\end{array}
\]

Let \( N_1, \ldots, N_r \) be a complete set of pairwise non-isomorphic, simple objects in \( \mathcal{N} \). For every \( i = 1, \ldots, r \), we have

\[
\Phi_{\mathcal{F}_H}(N_i) \cong \bigoplus_{w \in H} A_w^\otimes p_{i,w},
\]

for certain \( p_{i,w} \in \mathbb{N}_0 \).

Let \( M_i := \Theta(N_i) \), for \( i = 1, \ldots, r \). Then \( M_1, \ldots, M_r \) is a complete and irredundant set of indecomposable objects of \( \mathcal{M} \) up to isomorphism and grading shift, and

\[
\Phi_{\mathcal{F}_H}(M_i) \cong \bigoplus_{w \in H} C_w^\otimes p_{i,w},
\]

for every \( i = 1, \ldots, r \).

**Lemma 53.** For \( w \in \mathcal{H} \) and \( i, j = 1, \ldots, r \), define \( \tilde{h}_{w,i,j} \in \mathbb{N}[v,v^{-1}] \) by

\[
C_w M_i \cong \bigoplus_{j=1}^r M_j^\otimes \tilde{h}_{w,i,j}.
\]

Then

\[
\tilde{h}_{w,i,j} \in v^2 \mathbb{N}_0[v^{-1}] \cap \mathbb{N}_0[v].
\]

**Proof.** On one hand, we have

\[
\Phi_{\mathcal{F}_H}(C_w M_i) \cong \bigoplus_{j=1}^r \Phi_{\mathcal{F}_H}(M_j^\otimes \tilde{h}_{w,i,j}) \cong \bigoplus_{j=1}^r C_v^\otimes p_{w,i,j} \tilde{h}_{w,i,j}.
\]

On the other hand, by (25) and the fact that \( \Phi_{\mathcal{F}_H} \) is a morphism of 2-representations, we have

\[
\Phi_{\mathcal{F}_H}(C_w M_i) \cong C_w \Phi_{\mathcal{F}_H}(M_i) \cong \bigoplus_{u \in \mathcal{H}} C_w C_u^\otimes p_{i,u} \cong \bigoplus_{u,v \in \mathcal{H}} C_v^\otimes \tilde{h}_{w,u,v}.
\]

Comparing (28) and (29) for a fixed \( v \), we obtain the equation

\[
\sum_{j=1}^r \tilde{h}_{w,i,j} p_{j,v} = v^a \sum_{u \in \mathcal{H}} p_{u,w} \tilde{h}_{w,u,v}.
\]

The result now follows from the fact that \( p_{i,u}, p_{j,v} \in \mathbb{N}_0 \) and \( v^a \tilde{h}_{w,u,v} \in v^2 \mathbb{N}_0[v^{-1}] \cap \mathbb{N}_0[v] \), as for every \( j = 1, \ldots, r \) there exists at least one \( v \in \mathcal{H} \) such that \( p_{j,v} \neq 0 \).
Define
\[ B^M := \text{End}_M \left( \bigoplus_{i=1}^r M_i \right). \]

Then \( M \) is equivalent to the category of finite dimensional graded injective \( B^M \)-modules.

**Proposition 54.** The algebra \( B^M \) is a positively graded Frobenius algebra of graded length \( 2a \).

**Proof.** The case of \( M \) being the cell 2-representation \( C_H \) is discussed in Proposition 38 (or in Proposition 39 because of Proposition 41). In the case when \( M \) is not necessarily the cell 2-representation, the proposition follows from similar arguments as in the proof of Proposition 59. □

**Remark 55.** In contrast to the situation in Subsection 5.1, we do not know a priori that \( B^M \) is weakly symmetric. Therefore, we have to include a possible Nakayama permutation in Subsection 6.3 below. Only at the end of that section, we will be able to show that it is trivial.

### 6.2. A characterization of 2-representations in the image of \( \hat{\Theta} \).

We note the following classification result regarding the image of the map \( \hat{\Theta} \) from Theorem 31.

**Theorem 56.** Let \( M \) be a graded simple transitive 2-representation of \( S_H \) with apex \( H \). Then \( M \) is in the image of \( \hat{\Theta} \) if and only if the following conditions are satisfied:

(i) there is a choice \( \{ M_i \mid i \in I \} \) of representatives of isomorphism classes, up to grading shift, of indecomposable objects in \( M(\emptyset) \) such that the endomorphism algebra \( B \) of \( M := \bigoplus_{i \in I} M_i \) is positively graded, and, additionally;

(ii) for every \( w \in H \) and \( i \in I \), the object \( C_w M_i \) decomposes into a direct sum whose summands (up to isomorphisms) are of the form \( M_j^{\oplus l_j} \), where \( j \in I \) and \( 0 \leq l \leq 2a \);

(iii) the graded length of \( B \) is not greater than \( 2a \).

**Proof.** For the “only if” part observe that, by Proposition 27, we can pick a choice of representatives of isomorphism classes, up to grading shift, of indecomposable objects in \( M \), which are in the image of \( \Theta \). The condition in (ii) then follows from Lemma 19. Conditions (i) and (iii) hold by Proposition 54.

The “if” direction follows the proof of Proposition 41 closely. Set \( C = [M, M] \). Then

\[ \text{hom}_{S_H}(C, C_w^{\oplus k}) \cong \text{hom}_M(M, C_w^{\oplus k} M). \]

Conditions (i) and (iii) imply that the right-hand side is zero if \( k \geq 0 \). Hence, writing

\[ C \cong \bigoplus_{w \in H} C_w^{\oplus p_w}, \]

we obtain \( p_w \in \mathbb{N}[v^{-1}] \).

Next we want to establish an analogue of Lemma 36. Namely, we claim that, for any simple object \( L \) in \( M(\emptyset) \) concentrated in degree 0, and for any \( w \in H \), the injective object \( C_w L \) is concentrated between the degrees 0 and \( 2a \). Similarly to the proof of Lemma 36, the fact that \( C_w L \) is concentrated in positive degrees follows from...
Conditions (1) and (3). The fact that $C_w L$ is concentrated in degrees below $2a$ follows from conditions (1) and (3).

Now, if $k < -4a$, then the right-hand side of (30) is zero since, by condition (3), $M$ is a projective-injective object of graded length at most $2a$ and the action of $C_w$ is given by projective functors which increase the graded length by at most $2a$ (see the previous paragraph). Given the graded length of $C_w$ in $\mathcal{H}$, cf. Lemma 40 this again shows that $p_w = p_w(0) \in \mathbb{N}$ for all $w \in \mathcal{H}$, so $C$ is in the image of $\Theta$, as claimed. \hfill \Box

6.3. Explicit bimodules for the 2-action. The degree-zero part of $B^M$ is isomorphic to $\bigoplus_{i=1}^r C_{e_i}$, where $e_1, \ldots, e_r$ is a complete and irredundant set of primitive, orthogonal idempotents corresponding to $M_1, \ldots, M_r$ respectively. Due to (26), every $M_i$ is concentrated between degrees 0 and $2a$, whence

$$M_i \cong \text{Hom}_C(e_i B^M, C)^{\oplus v^{2a}} \cong B^M e_{\sigma(i)},$$

where $\sigma$ is the Nakayama permutation of $B^M$. By [KMMZ Theorem 2], the action of $C_w$ on the category of finite dimensional, graded injective $B^M$-modules, for $w \in \mathcal{H}$, is given by tensoring over $B$ with a $B^M$-$B^M$ bimodule of the form

$$\bigoplus_{i,j=1}^r (B^M e_{\sigma(j)} \otimes \mathbb{C} e_{\sigma(i)} B^M)^{\oplus \tilde{\gamma}_{w,i,j}},$$

for certain $\tilde{\gamma}_{w,i,j} \in \mathbb{N}_[v, v^{-1}]$.

**Proposition 57.** We have

$$\tilde{\gamma}_{w,k,j} = \tilde{h}_{w,k,j}(0) \in \mathbb{N}_0,$$

for all $w \in \mathcal{H}$ and $j, k = 1, \ldots r$.

**Proof.** For $w \in \mathcal{H}$ and $1 \leq k \leq r$, we obtain two different expressions for $C_w B^M e_{\sigma(k)}$. On one hand, by (27) and the fact that $M_i \cong B^M e_{\sigma(i)}$, we have

$$C_w B^M e_{\sigma(k)} \cong \bigoplus_{j=1}^r B^M e_{\tilde{h}_{w,k,j}}^{\oplus \tilde{\gamma}_{w,i,j}}.$$

On the other hand, we have

$$C_w B^M e_{\sigma(k)} \cong \bigoplus_{i,j=1}^r B^M e_{\tilde{\gamma}_{w,i,j}}^{\oplus \text{grdim}(e_{\sigma(i)} B^M e_{\sigma(k)})}. $$

Comparing the terms in (31) and (32) for a fixed $j$, shows that

$$\sum_{i=1}^r \tilde{\gamma}_{w,i,j} \text{grdim}(e_{\sigma(i)} B^M e_{\sigma(k)}) = \tilde{h}_{w,k,j}.$$

Suppose that $\tilde{\gamma}_{w,\sigma^{-1}(k),j}$ has a non-zero term belonging to $v\mathbb{N}_0[v]$ for some $w, i, j$. By (33) and the fact that $\text{grdim}(e_{\sigma(i)} B^M e_{\sigma(k)})$ has highest term $v^{2a}$, see Proposition 54 this implies that $\tilde{h}_{w,k,j}$ has a non-zero term belonging to $v^{2a+1} \mathbb{N}_0[v]$. However, this contradicts Lemma 53.

Since $\text{grdim}(e_{\sigma(j)} B^M e_{\sigma(k)}) \in \delta_{i,k} + v\mathbb{N}_0[v]$, the equation in (33) implies that $\tilde{\gamma}_{w,k,j}$ cannot have non-zero terms belonging to $v^{-1} \mathbb{N}_0[v^{-1}]$ either, whence

$$\tilde{\gamma}_{w,k,j} = \tilde{h}_{w,k,j}(0) \in \mathbb{N}_0,$$

for all $w \in \mathcal{H}$ and $i, j = 1, \ldots, r$. \hfill \Box
In particular, note that the fact that the constant term in \( \text{grdim}(e_{\sigma(i)}BMe_{\sigma(k)}) \) is 1 if \( i = k \), and 0 otherwise, implies that
\[
C_d BMe_{\sigma(k)} \cong \bigoplus_{j=1}^{r} BMe_{\sigma(j)} \oplus R,
\]
where all summands of \( R \) have coefficients in \( \mathbb{V}[v] \). Since the first summand descends to the action of \( A_d \), which is the identity 1-morphism in \( \mathcal{A}_H \) on \( N \), by Lemma 29, we see that \( \tilde{h}_{d,i,j} = \delta_{i,j} \). By Proposition 57 and equation (33), this shows that the action of \( C_d = \Theta(A_d) \) is given by tensoring with the bimodule
\[
\bigoplus_{i=1}^{r} BMe_i \otimes \mathbb{V} e_i BMe
\]
and that
\[
\text{grdim}(\text{Hom}_B(Me_i, Me_k)) = \text{grdim}(e_i BMe_k) = \tilde{h}_{d,\sigma^{-1}(k),\sigma^{-1}(i)}.
\]
We also obtain an analog of Proposition 50. For every \( i = 1, \ldots, k \), define
\[
\lambda_i := \sum_{j=1}^{r} \tilde{h}_{d,i,j}(1).
\]
Proposition 58. If \( BMe \) is symmetric, then
\[
\lambda_i = \lambda_j, \quad \text{for all } i, j = 1, \ldots, r.
\]
In \cite{Lu3} Theorem 18.9, Lusztig defined a homomorphism \( \phi: H \to A \otimes \mathbb{Z}[v, v^{-1}] \) of \( \mathbb{Z}[v, v^{-1}] \)-algebras. Its restriction to \( \mathcal{H} \) is given by
\[
\phi_H(c_w) = \sum_{u \in \mathcal{H}} v^a h_{w,d,u} a_u,
\]
where \( c_w := [C_w] \) in the split Grothendieck group \( [\mathcal{A}_H]_{\oplus} \) (which should not be confused with Lusztig’s \( c_w \)) and \( a_u := [A_u] \) in \( [\mathcal{A}_H]_{\oplus} \). Let \( \phi_H^! \) denote the pullback of \( \phi \).

Proposition 59. We have
\[
[M]_{\oplus} \cong \phi_H^!(N)_{\oplus}.
\]
Proof. By (34), we have
\[
C_dCe_{\sigma(i)} \cong BMe_{\sigma(i)}.
\]
Using this, we obtain two expressions for \( C_w BMe_{\sigma(i)} \). On one hand,
\[
C_w BMe_{\sigma(i)} \cong \bigoplus_{j=1}^{r} BMe_{\sigma(j)} \otimes h_{w,i,j}.
\]
On the other hand,
\[
C_w BMe_{\sigma(i)} \cong C_w C_d Ce_{\sigma(i)} \cong \bigoplus_{u \in \mathcal{H}} C_u \otimes^a h_{w,d,u} \otimes^a C e_{\sigma(i)}
\]
\[
\cong \bigoplus_{u \in \mathcal{H}} \bigoplus_{j=1}^{r} BMe_{\sigma(j)} \otimes^a h_{w,d,u} \tilde{h}_{u,i,j}.
\]
Comparing (36) and (37) for a fixed \( j \) yields
\[
\tilde{h}_{w,i,j} = \sum_{u \in \mathcal{H}} v^a h_{w,d,u} \tilde{h}_{u,i,j},
\]
which is precisely what we had to prove. □

**Corollary 60.** $v^a\tilde{h}_{w,i,j}$ is bar invariant.

**Proof.** Equation (33) implies that $v^a\tilde{h}_{w,i,j}$ is bar invariant, since the $h_{w,d,u}$ and $\tilde{h}_{w,i,j}$ are bar invariant. This completes the proof. □

**Proposition 61.** The algebra $B^M$ is weakly symmetric.

**Proof.** Recalling (7), (8) and (22), we know that $\nu^a h_{d,d,u} \in \begin{cases} 1 + \cdots + v^{2a} & \text{if } u = d; \\ vN_0[v] \cup v^{2a-1}N_0[v^{-1}] & \text{if } u \neq d. \end{cases}$

By (33) and the equality $\tilde{\gamma}_{d,i,j} = \delta_{i,j}$, (35) then shows that $\operatorname{grdim}(\iota_i B^M G_1) = \tilde{h}_{d,\sigma^{-1}(i),\sigma^{-1}(i)} = 1 + \cdots + v^{2a}$. Therefore, we obtain $\sigma(i) = i$ for all $i$ and the claim follows. □

7. Classification results

**The asymptotic bicategory and its 2-representations.** Recall that, by Proposition [22], $\mathcal{A}_H$ is a fusion bicategory and thus, all of its simple transitive 2-representations are semisimple. Moreover, up to a handful of exceptions, the asymptotic bicategory $\mathcal{A}_H$ comes in three flavors and for all of them a classification of simple transitive 2-representations is known, as we will summarize now (giving more details below). Recall that $k = \mathbb{C}$ and $\mathfrak{R}ep(G)$ is the fusion bicategory of finite dimensional $G$-modules. Let $\mathcal{V}ect(G)$ denote the fusion bicategory of $G$-graded, finite dimensional vector spaces ($\mathcal{V}ect = \mathcal{V}ect(1)$ are plain finite dimensional vector spaces), and $\mathcal{F}O(3)_k$ the fusion bicategory of complex, finite dimensional representations of quantum $so_3$ semisimplified at level $k$, see e.g. [EGNO, Examples 2.3.3 and 8.18.5].

(A) **Weyl type (excluding $G_2$): generic case.** Up to three exceptions in types $E_7$ and $E_8$, explained in (B), for each two-sided cell $J$ there exists an $H$-cell $\mathcal{H}$ and a finite group $G = G(\mathcal{H})$ such that $\mathcal{A}_H \cong \mathcal{V}ect(G)$ for $G = (\mathbb{Z}/2\mathbb{Z})^k$, or $\mathcal{A}_H \cong \mathfrak{R}ep(G)$ for $G$ being $S_3$, $S_4$ or $S_5$, see [BFO, Theorem 4].

Let $\Omega(G)$ denote the set of subgroups of $G$ up to conjugacy, $K$ a choice of representative of $[K] \in \Omega(G)$, and $H^2(K, \mathbb{C}^\times)$ the second group cohomology of $K$ with values in $\mathbb{C}^\times$, whose non-trivial generators are called Schur multipliers. By e.g. [EGNO] Example 7.4.10 and Corollary 7.12.20, we have

$$\begin{cases} \text{equivalence classes of simple transitive} \\
\text{2-representations of } \mathcal{V}ect(G) \text{ or } \mathfrak{R}ep(G) \end{cases} \xrightarrow{\cong} \left\{ ([K], \varpi) \mid [K] \in \Omega(G), \varpi \in H^2(K, \mathbb{C}^\times) \right\}.$$  

The simple transitive 2-representations of $\mathcal{V}ect(G)$ have rank $\#G/\#K$ and the ones for $\mathfrak{R}ep(G)$ are the $\varpi$-twisted representation categories $\mathfrak{R}ep^\varpi(K)$ (in particular, their rank is equal to the rank of the character ring of $K$ for trivial $\varpi$).

(B) **Weyl type: exceptional case.** Type $E_7$ contains one and type $E_8$ two so-called exceptional cells. For these, by [Os4] Theorem 1.1, we have $\mathcal{A}_H \cong \mathcal{V}ect(Z/2Z)$, having its 2-structure twisted by the non-trivial element $\varsigma$ in the third group cohomology $H^3(Z/2Z, \mathbb{C}^\times) \cong Z/2Z$.

In this case $\mathcal{A}_H$ has only one associated simple transitive 2-representation, which is of rank 2, see e.g. [Os2, Theorem 3.1].
(C) **Dihedral type (including $G_2$).** We have either $A_H \cong \text{Vect}$ for the cells containing the identity element or the longest element, or $A_H \cong \mathcal{O}(3)_k$ for the middle cell, by [MMMZ, Theorem 2.15].

By e.g. [KO, Theorem 6.1] and [Os3, Theorem 6.1], we have

\[
\begin{align*}
\{ \text{equivalence classes of simple transitive} & \quad 2\text{-representations of } \mathcal{O}(3)_k \} \\
& \quad \leftrightarrow \\
\{ \text{bicolored ADE diagrams} & \quad \text{with Coxeter number } k+2 \}.
\end{align*}
\]

The corresponding simple transitive 2-representations have rank equal to the number of vertices of the associated ADE diagram.

(D) **Types $H_3$ and $H_4$.** We do not know what $A_H$ is in general. For details see below.

---

**What the conjecture covers.** Recall that any graded simple transitive 2-representation of $\mathcal{S}$ has an apex $J$ in $W$. Assume that Conjecture 33 holds. Then, together with [MMMZ, Theorem 15], it implies that we can chose $H \subset J$ such that

\[
\begin{align*}
\{ \text{equivalence classes of graded} & \quad \text{simple transitive 2-representations} \\
& \quad \text{of } \mathcal{S} \text{ with apex } J \} \quad \leftrightarrow \\
\{ \text{equivalence classes of} & \quad \text{simple transitive 2-representations} \\
& \quad \text{of } A_H \}
\end{align*}
\]

(Note that Corollary 28 also gives us also ranks of the simple transitive 2-representations of $A_H$ associated to the ones from $A_H$. However, the corresponding simple transitive 2-representations for $\mathcal{S}$ might have bigger ranks.) Thus, assuming Conjecture 33 the above shows that only certain cells in Coxeter types $H_3$ and $H_4$ – most prominently, the cell (43) in type $H_4$ given below – would remain open with respect to a complete classification of graded simple transitive 2-representations of $\mathcal{S}$.

For all other cases, the conjecture would give a complete classification and parametrization of the graded simple transitive 2-representations of $\mathcal{S}$, as we will summarize now. In the dihedral case (including $G_2$), this follows from the above, while in Weyl types, up to three exceptional cells where we have one associated simple transitive 2-representation of rank 2, we need to analyze the simple transitive 2-representations of $\text{Vect}(G)$ or $\text{Rep}(G)$, which are given by (conjugacy classes of) subgroups of $K \subset G$, their numbers $\#$, and Schur multipliers in $H^2$ of these subgroups. We additionally list their ranks $\text{rk}$.

Listing the data that we need is easy (calculating the subgroups and their numbers for $(\mathbb{Z}/2\mathbb{Z})^k$ is a pleasant exercise, while the Schur multipliers of these subgroups were already determined by Schur, see e.g. [Ber, Theorem 4] for a more modern reference;
the data for the other three cases, $S_3$, $S_4$ and $S_5$, can be calculated by computer): 

\[
\begin{array}{c|cccc}
K & (\mathbb{Z}/2\mathbb{Z})^I & \mathbb{F}_{\text{ct}}(\mathbb{Z}/2\mathbb{Z})^I & \mathbb{F}_{\text{ct}}(\mathbb{Z}/2\mathbb{Z})^J & \mathbb{F}_{\text{ct}}(\mathbb{Z}/2\mathbb{Z})^J/I \\
\# & (t^I) & (t^J) & (t^J) & (t^J) \\
H^2 & (\mathbb{Z}/2\mathbb{Z})^{(I-1)/2} & (\mathbb{Z}/2\mathbb{Z})^{(J-1)/2} & (\mathbb{Z}/2\mathbb{Z})^{(J-1)/2} & (\mathbb{Z}/2\mathbb{Z})^{(J-1)/2} \\
rk & k/l & k/l & k/l & k/l \\
\end{array}
\]

\[
\begin{array}{c|cccccccc}
K & 1 & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/3\mathbb{Z} & \mathbb{Z}/4\mathbb{Z} & (\mathbb{Z}/2\mathbb{Z})^2 & S_3 & D_4 & A_4 & S_4 \\
\# & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 1 & 1 \\
H^2 & 1 & 1 & 1 & 1 & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \\
rk & 1 & 2 & 3 & 4 & 4,1 & 3 & 5,2 & 4,3 & 5,3 \\
\end{array}
\]

\[
\begin{array}{c|cccccccc}
K & 1 & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & (\mathbb{Z}/2\mathbb{Z})^2 & S_3 & D_4 & A_4 & S_4 \\
\# & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 1 & 1 \\
H^2 & 1 & 1 & 1 & 1 & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \\
rk & 1 & 2 & 3 & 4 & 4,1 & 3 & 5,2 & 4,3 & 5,3 \\
\end{array}
\]

(Here $GA(1, 5)$ is the general affine group of rank one over $\mathbb{F}_5$. ) Thus, taking everything together gives a complete answer regarding the classification in Weyl types.

**What we cover.** Let us give some details of what is covered by the results in this paper, i.e. what does not depend on Conjecture 33.

First, by Theorem 33, we need to identify those two-sided cells $J$ which contain a $w_0^J$. Second, we have to identify at least one $H \subset J$ for which we know $\mathfrak{H}_H$ and the classification of its simple transitive 2-representations. (Here, we use the classification of simple transitive 2-representations mentioned for the main cases above, cf. \[39\].) We say that a type is “done” if all $J$ contain a $w_0^J$ and if we can identify $\mathfrak{H}_H$ and classify its simple transitive 2-representations for at least one $H \subset J$.

For this purpose, we use what we call a cell matrix:

\[
\begin{array}{cccccccc}
3_{4,5} & 1_{5,5} & 1_{3,20} & 2_{5,25} & 2_{5,25} \\
1_{5,5} & 3_{5,5} & 1_{3,20} & 2_{5,25} & 2_{5,25} \\
1_{20,5} & 1_{20,5} & 3_{20,20} & 2_{20,25} & 2_{20,25} \\
2_{20,5} & 2_{20,5} & 2_{20,20} & 4_{20,25} & 1_{20,25} \\
2_{20,5} & 2_{20,5} & 2_{20,20} & 1_{20,25} & 4_{20,25} \\
\end{array}
\]

Here we indicate the number of elements in left or right cells, where e.g. $2_{20,25}$ is to be understood as a 20-by-25 matrix containing only the entry 2 (thus, having 1000 elements). The shaded boxes are (matrices of) $H$-cells.
Of special interest will be

| strongly regular: | \( \mathcal{A}_H \cong \text{Vect} \) |
|-------------------|-----------------------------------|
| nice:             | \( \mathcal{A}_H \cong \text{Vect}(\mathbb{Z}/2\mathbb{Z}) \) |
| exceptional:      | \( \mathcal{A}_H \cong \text{Vect}^n(\mathbb{Z}/2\mathbb{Z}) \) |
| dihedral:         | \( \mathcal{A}_H \cong \mathcal{O}(3)_{m-2} \) |

where \( a^2, 2(b^2+c^2+bc), 2d^2 \) or \( 2(m-1) \), respectively, is the size of the cell in question. (Note that, knowing the size of the cells, one can recover \( a, b, c, d \) since there is always a unique solution in positive integers.) The first case is \( k = 0 \) below, for which we always get a full classification, cf. [MM5, Theorem 18], the second case is \( k = 1 \) below. In all these cases we have a complete classification of simple transitive 2-representations of \( \mathcal{A}_H \), see above.

Type \( A_n \). This type is done for all \( n \):

(a) Every \( \mathcal{J} \) contains a \( w_0 \).
(b) All cells are strongly regular.

Type \( B_n \). This type is done up to rank 4:

(a) The first example where some \( \mathcal{J} \) does not contain a \( w_0 \) is \( B_5 \).
(b) For all \( \mathcal{H} \), we have \( G = (\mathbb{Z}/2\mathbb{Z})^k \) for some \( k \in \mathbb{N} \) with \( k(k+1) \leq n \).
(c) The diagonal of the cell matrix is \( 2^k \), all other entries are \( 2^l \) for \( l < k \).
(d) For \( B_2, B_3 \) and \( B_4 \), (b) and (c) imply that all cells are strongly regular or nice.
(e) \( B_5 \) is the smallest case where we do not have a classification; see below.
(f) \( B_6 \) is the smallest example in classical type where we have a non-cell, simple transitive 2-representation; see below.

Type \( B_5 \) is:

| cell | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 7' | 8 | 8' | 9 | 2' | 1 | 0' |
|------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| size | 1 | 42 | 150 | 100 | 225 | 152 | 600 | 650 | 650 | 600 | 152 | 225 | 100 | 150 | 42 | 1 |
| m   | 0 | 1 | 2 | 3 | 3 | 3 | 4 | 4 | 4 | 5 | 6 | 7 | 9 | 10 | 10 | 11 | 16 | 25 |
| w_0 | y | y | y | y | y | y | y | y | y | y | y | y | n | y | y | y | y | y |
| k   | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 |

Here and throughout: from left to right, we have listed the numbered cells, paired \( \mathcal{J} \sim \mathcal{J}' = \mathcal{J}w_0 \) (with 0 being the minimal containing 1 and 0' the maximal cell containing \( w_0 \)). From top to bottom, we have listed their sizes, the \( m \)-values, whether they contain a \( w_0 \) (yes or no) and the number \( k \) recording the diagonal, respectively.

Type \( B_6 \) is:
The cell 12 is displayed in (40). In this case, we have \( G = (\mathbb{Z}/2\mathbb{Z})^2 \), which has the (non-conjugate) subgroups \( 1, K_1, K_2, K_3, G \). The subgroups 1 and \( K_1 \cong K_2 \cong K_3 \cong \mathbb{Z}/2\mathbb{Z} \) all have trivial second group cohomology, but \( H^2(G, \mathbb{C}) \cong \mathbb{Z}/2\mathbb{Z} \). Thus, we have six equivalence classes of graded simple transitive 2-representations of ranks 1, 1, 2, 2, 2 and 4, respectively, cf. (39). It follows from Theorem 31 that this case gives a non-cell, simple transitive 2-representation. The same happens repeatedly for higher ranks.

**Type \( D_n \).** This type is done **up to rank 6:**

(a) The first example, where some \( J \) does not contain a \( u_0^1 \) is \( D_4 \). However, the lowest rank where some \( J \) does not contain a \( u_0^1 \) and this two-sided cell is not strongly regular, is \( D_7 \).

(b), (c) As for type \( B_n \), but with \((k + 1)^2 \leq n\).

**Type \( I_2(m) \).** This type is done for all \( m > 2 \):

(a) Every \( J \) contains a \( u_0^1 \); there are only three two-sided cells.

(b) The bottom and top cell are strongly regular.

(c) The middle cell has a dihedral cell matrix, cf. (41). This is the smallest example with non-cell, simple transitive 2-representations, starting from type \( I_2(6) = G_2 \).

**Type \( H_3 \).** This type needs **more** work:

\[
\begin{array}{cccccccc}
\text{cell} & 0 & 1 & 2 & 3 = 3' & 2' & 1' & 0' \\
\text{size} & 1 & 18 & 25 & 32 & 25 & 18 & 1 \\
a & 0 & 1 & 2 & 3 & 5 & 6 & 15 \\
w_1^a & y & y & y & y & y & y & y \\
\mathcal{A}_H \sim (a) & (b) & (c) & (a) & (b) & (a) \\
\end{array}
\]

(a) These cases are strongly regular two-sided cells.

(b) In these cases, the cell is \( 2,3,3 \), and the Grothendieck rings of \( \mathcal{A}_H \) and \( \mathcal{O}(3)_3 \) coincide.

(c) Cell 3 is \( 2,4,4 \), and the Grothendieck rings of \( \mathcal{A}_H \) and \( \text{Vect}(\mathbb{Z}/2\mathbb{Z}) \) coincide.

(d) By [Ost1, Section 2.5], the only two possibilities for (b) are \( \mathcal{A}_H \cong \mathcal{O}(3)_3 \) or \( \mathcal{A}_H \cong M(2, 5) \) (in the notation of Ostrik). Similarly, by [Ost1 Section 2.4], the only two possibilities for (c) are \( \mathcal{A}_H \cong \text{Vect}(\mathbb{Z}/2\mathbb{Z}) \) or \( \mathcal{A}_H \cong \text{Vect}^\varsigma(\mathbb{Z}/2\mathbb{Z}) \). However, only in the case of the cell 1 do we know which option it is, namely \( \mathcal{A}_H \cong \mathcal{O}(3)_3 \), since this case is covered by [KMMZ Theorem 28].

**Type \( F_4 \).** This type needs **a bit more** work:

\[
\begin{array}{cccccccc}
\text{cell} & 0 & 1 & 2 & 3 & 4 & 5 = 5' & 4' & 3' & 2' & 1' & 0' \\
\text{size} & 1 & 24 & 81 & 64 & 684 & 64 & 64 & 81 & 24 & 1 \\
a & 0 & 1 & 2 & 3 & 3 & 4 & 9 & 9 & 10 & 13 & 24 \\
w_1^a & y & y & y & y & y & y & y & y & y & y & y \\
\mathcal{A}_H \sim (a) & (b) & (c) & (a) & (b) & (a) & (a) & (b) & (a) \\
\end{array}
\]
where we write $k$ for $G = (\mathbb{Z}/2\mathbb{Z})^k$, with the cell matrices as in (41). In the remaining case we have (for appropriate $\mathcal{H}$):

$$
\begin{array}{cccccccc}
5_{3,3} & 5_{3,3} & 4_{3,3} & 5_{1,3} & 2_{1,3} \\
3_{3,3} & 5_{3,3} & 4_{3,3} & 2_{1,3} & 5_{1,3} \\
4_{3,3} & 4_{3,3} & 9_{4,4} & 6_{4,4} & 6_{4,4} \\
5_{1,3} & 2_{1,3} & 6_{1,4} & 9_{1,4} & 3_{1,1} \\
2_{1,3} & 5_{1,3} & 6_{1,4} & 3_{1,1} & 9_{1,1}
\end{array}
\begin{array}{c}
5_{3,3} : \mathcal{A}_\mathcal{H} \cong \text{Rep}(S_4).
\end{array}
$$

For the list of equivalence classes of simple transitive 2-representations of $\text{Rep}(S_4)$, see (39). This is the second smallest example in Weyl type with a non-cell, simple transitive 2-representation.

**Type $H_4$.** This type needs much more work:

$$
\begin{array}{cccccccccc}
\text{cell} & 0 & 1 & 2 & 3 & 4 & 5 & 6=6' & 5' & 4' & 3' & 2' & 1' & 0'
\end{array}
\begin{array}{cccccccccccccccc}
\text{size} & 1 & 32 & 162 & 512 & 625 & 1296 & 9144 & 1296 & 625 & 512 & 162 & 32 & 1
\end{array}
\begin{array}{ccccc}
\mathfrak{a} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 15 & 16 & 18 & 22 & 31 & 60
\end{array}
\begin{array}{cccccccccccccccc}
\mathfrak{a}' & (a) & (b) & (c) & (d) & (e) & (f) & (g) & (h) & (i) & (j) & (k)
\end{array}
\begin{array}{cccccccccccccccc}
\mathfrak{a}' & (a) & (b) & (c) & (d) & (e) & (f) & (g) & (h) & (i) & (j) & (k)
\end{array}
$$

(a),(b),(c) are similar to (a),(b),(c) in type $H_3$, and the same remark as in (d) holds. In the remaining case we have:

$$
\begin{array}{cccc}
14_{1,1} & 13_{1,10} & 14_{0,8} \\
13_{1,10} & 18_{10,10} & 18_{6,6} \\
14_{1,1} & 18_{10,6} & 24_{6,6}
\end{array}
$$

We were not able to find $\mathcal{A}_\mathcal{H}$ in the literature. In fact, for none of the $\mathcal{H}$-cells do we know what $\mathcal{A}_\mathcal{H}$ is; we only know the multiplication tables of their Grothendieck rings with respect to the asymptotic Kazhdan–Lusztig bases $\{a_w : w \in \mathcal{H}\}$, see also [AI]. For example, if $\mathcal{H}$ is in the 14-14 block, then the Grothendieck ring of $\mathcal{A}_\mathcal{H}$ is not commutative, $\mathcal{A}_\mathcal{H}$ has Perron–Frobenius dimension $120(9 + 4\sqrt{5})$ and a simple generating 1-morphism of Perron–Frobenius dimension $1 + \sqrt{5}$ and fusion graph

```
*------------------*
|                   |
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```

**Type $E_6$.** This type is done, i.e. we have:

$$
\begin{array}{cccccccccccccccc}
\text{cell} & 0 & 1 & 2 & 3 & 4 & 5 & 6=6' & 5' & 4' & 3' & 2' & 1' & 0'
\end{array}
\begin{array}{cccccccccccccccc}
\text{size} & 1 & 2 & 3 & 4 & 5 & 6 & 15 & 16 & 18 & 22 & 31 & 60 & 1
\end{array}
\begin{array}{cccccccccccccccc}
\mathfrak{a} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 15 & 16 & 18 & 22 & 31 & 60
\end{array}
\begin{array}{cccccccccccccccc}
\mathfrak{a}' & (a) & (b) & (c) & (d) & (e) & (f) & (g) & (h) & (i) & (j) & (k)
\end{array}
\begin{array}{cccccccccccccccc}
\mathfrak{a}' & (a) & (b) & (c) & (d) & (e) & (f) & (g) & (h) & (i) & (j) & (k)
\end{array}
$$

where we write $k$ in case $G = (\mathbb{Z}/2\mathbb{Z})^k$. The corresponding cells are strongly regular or nice. The remaining case is:

$$
\begin{array}{cccc}
3_{10,10} & 2_{20,10} & 1_{20,10} \\
2_{10,10} & 3_{30,50} & 3_{30,50} \\
1_{30,10} & 3_{60,20} & 6_{60,20}
\end{array}
\begin{array}{c}
\mathcal{A}_\mathcal{H} \cong \text{Rep}(S_3).
\end{array}
$$

Again, we get non-cell simple transitive 2-representations, cf. (39).
Type $E_7$. This type needs a bit more work, and it is quite similar to type $E_6$:

| Cell | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|------|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|
| val  | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
| G    | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Cell 17 is exceptional with $\mathfrak{H} \cong \mathcal{H}^{\text{ect}}(\mathbb{Z}/2\mathbb{Z})$, see [Ox4, Theorem 1.1]. The remaining cells 11 and 11' are as in (44) (with diagonals $3^{70,70}, 3^{210,210}$ and $6^{35,35}$), giving non-cell, simple transitive 2-representations.

Type $E_8$. This type needs much more work, and it is similar to type $E_7$:

| Cell | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|------|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|
| val  | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
| G    | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

As before, we have exceptional cells with $\mathfrak{H} \cong \mathcal{H}^{\text{ect}}(\mathbb{Z}/2\mathbb{Z})$, see [Ox4, Theorem 1.1], and also cells as in (44) (with diagonals $3^{148,448}, 3^{996,896}$ and $6^{30,56}$, or $3^{175,175}, 3^{575,875}$ and $6^{150,350}$), giving non-cell, simple transitive 2-representations. There is one remaining cell with $\mathfrak{H} \cong \mathbb{H}(\mathcal{S}_5)$, giving again non-cell simple transitive 2-representations. For the list of equivalence classes of simple transitive 2-representations of $\mathbb{H}(\mathcal{S}_5)$, see (39).

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