SOME CHARACTERIZATIONS OF COMPACT EINSTEIN-TYPE MANIFOLDS

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Abstract. In this work, we investigate the geometry and topology of compact Einstein-type manifolds with nonempty boundary. First, we prove a sharp boundary estimate, as consequence we obtain under certain hypotheses that the Hawking mass is bounded from below in terms of area. Then we give a topological classification for its boundary. Finally, we prove a gap result for a compact Einstein-type manifold with boundary.

1. Introduction

In the last decades, problems related to Einstein manifolds have become very present in theory, because there are many applications in mathematics and theoretical physics. We recall that a Riemannian manifold \((M^n, g)\) is said to be Einstein if \(\text{Ric} = \frac{R}{n} g\), where \(\text{Ric}\) and \(R\) are Ricci and scalar curvatures, respectively. Throughout the text the dimension will be considered \(n \geq 3\), unless explicitly mentioned. More recently, there have been some generalization about Einstein manifold (see e.g \([4]\) and \([10]\)). Here we use the following definition.

Definition 1. A Riemannian manifold \((M^n, g)\), is called an Einstein-type manifold if there are smooth functions \(f, h : M \rightarrow \mathbb{R}\) such that

\[
 f \text{Ric} = \nabla^2 f + hg,
\]

where \(f > 0\) in \(\text{int}(M)\), \(f = 0\) on \(\partial M\) and \(\nabla^2\) is the Hessian. We denote \((M^n, g, f, h)\) Einstein-type manifold.

Using the equation (1.1), it is easy to see that

\[
 f R = \Delta f + nh,
\]

here \(\Delta\) is the Laplacian operator.

We observe that if \((M^n, g, f, h)\) is an Einstein-type manifold, then it satisfies the following equation:

\[
 \tilde{\nabla}^2 f = f \tilde{\text{Ric}},
\]

where \(\tilde{\text{A}}\) is the traceless tensor, i.e, \(\tilde{\text{A}} = A - \frac{\text{trace}(A)}{n} g\).

It is possible to notice that the class of manifolds satisfying the Definition 1 generalizes several important examples:

Example 1. a) If we consider \(h = 0\) and \(R = 0\), we obtain the Static vacuum Einstein equation \([9]\).

b) If we take \(h = \frac{R f}{n - 1}\), we get the static vacuum equation with non null cosmological constant \([1]\).

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c) If $h = \frac{\mu - \rho}{n-1} f$, we have static perfect fluid, where $\mu = R/2$ and $\rho$ are, respectively, the density and the pressure smooth functions [15].

d) If we consider $h = \frac{R f + 1}{n-1}$, we get the called Miao-Tam equation [11].

e) If we consider $h = \lambda f$, where $\lambda$ is a constant we obtain $(\lambda, n + 1)$-Einstein manifold. Moreover, if we take $f = e^{-\phi}$, here $\phi$ is defined in the interior of $M$, then we obtain the well-known Bakry-Emery Ricci Tensor [7].

For instance, Shen [15] used some ideas from general relativity and proved a Robinson-type identity obtaining results about a Fisher-Marsden conjecture. Hwang et al. [9] proved that there are no multiple black holes in an $n$-dimensional static vacuum space-time having weakly harmonic curvature unless the Ricci curvature is trivial. In [1], Ambrozio proved, among others interesting results, some classification results for compact static three-manifolds with positive scalar curvature. Miao and Tam [11] classified all Einstein or conformally flat metrics which are critical points of volume in certain space. Barros and Gomes [2] proved that a compact gradient generalized $m$-quasi-Einstein metric with constant scalar curvature must be isometric to a standard Euclidean sphere $S^n$ with the potential $f$ well determined. Later, Coutinho et al. [6] studied the geometry of static perfect fluid space-time metrics on compact manifolds with boundary yielding a gap result for this space. Then, Freitas and Santos [7] investigated about generalized compact Einstein manifolds, giving some topological classifications for its boundary. Leandro [10] considered an Einstein-type equation which generalizes important geometric equations, he proved a result, among others, about the nonexistence of multiple black holes in static spacetimes.

Motivated by [6] and [10], we obtain some characterization results for the Einstein-type manifolds with dimension $n \geq 3$, compact and with boundary. Our first result is the following:

**Theorem 1.** Let $(M^n, g, f, h)$ be a compact Einstein-type manifold with boundary. If $(M^n, g)$ is Einstein, then $f$ satisfies the following differential equation

$$\nabla^2 f = \left( - \frac{R}{n(n-1)} f + C \right) g,$$

where $C$ is a constant. In particular, if $h$ or $f$ is not constant, then, for $R \geq 0$, $(M^n, g)$ is isometric to a geodesic ball on a sphere $S^n$.

An interesting result obtained by Boucher, Gibbons and Horowitz [3], and by Shen [15] showed that a boundary $\partial M$ of a compact 3-dimensional oriented static manifold with connected boundary and positive scalar curvature equals to 6 must be a 2-sphere whose area satisfies the inequality

$$|\partial M| \leq 4\pi.$$

The equality occurs if and only if $M^3$ is isometric to a standard hemisphere. Inspired by this result and its natural extension for static perfect fluid space-time with boundary $\partial M$ [6] and generalized $(\lambda, n + m)$-Einstein manifolds [7], we obtain the next result for Einstein-type manifold.

**Theorem 2.** Let $(M^n, g, f, h)$ be a compact, oriented Einstein-type manifold with boundary $\partial M$ such that $Ric^{\partial M} \geq \frac{n R^\partial M}{n-1} g|_{\partial M}$ with $\inf R^\partial M > 0$. Suppose that either:

(1) The scalar curvature $R$ is a positive constant, or

(2) $h \geq \frac{f}{n} R$ and $R_{\min} > 0$. 

Then,

$$|\partial M| \leq \left( \frac{n(n-1)}{R_{\min} + K(n, H)} \right)^{\frac{n-1}{2}} w_{n-1},$$
where $\text{Ric}^{\partial M}$ and $R^{\partial M}$ are Ricci and scalar curvatures on $\partial M$, respectively, $|\partial M|$ is the area of $\partial M$, $K(n, H) = \frac{(n-1)n}{|\partial M|} \int_{\partial M} H^2 dS$, $H$ is the mean curvature of $\partial M$, $R_{\min}$ is the minimum value of $R$ on $M^n$ and $w_{n-1}$ denotes the volume of the standard unit sphere. In particular,

$$|\partial M| \leq \left( \frac{n(n-1)}{R_{\min}} \right)^{\frac{1}{n-1}} w_{n-1}. \quad (1.4)$$

The equality holds in (1.4) if and only if $(M^n, g)$ is an Einstein manifold with totally geodesic boundary. In this case, $(M^n, g)$ is isometric to a geodesic ball on a sphere $S^n$.

We remember that in the proof of the positive mass theorem by Schoen and Yau [14] a crucial point is the study of minimal surfaces in a certain space $M^3$ that is important in general relativity. In 1973, motivated by a physical problem, Penrose conjectured that

$$16\pi m^2 \geq |\Sigma|,$$

where $\Sigma$ is a minimal surface in $M^3$, $m$ is the so-called ADM mass and $|\Sigma|$ is the area of $\Sigma$. In this way, Huisken and Ilmanen [8] proved this conjecture under certain hypotheses. To prove this result they used the Hawking quasi-local mass of a 2-surface defined by

$$m_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \left( 1 - \frac{1}{16\pi} \int_{\Sigma} H^2 dS \right). \quad (1.5)$$

This quantity has been proposed as a quasi-local measure for the strength of the gravitational field [5]. Hawking observed that it approaches the ADM mass for large coordinates spheres.

Inspired by these works and the Theorem 2, we obtain an interesting application, which shows under certain hypotheses that the Hawking mass is bounded from below in terms of area.

**Corollary 1.** Let $(M^3, g, f, h)$ be a compact, oriented Einstein-type manifold with boundary $\partial M$ closed, two-sided surface such that $\text{Ric}^{\partial M} \geq \frac{R^{\partial M}}{2} g^{\partial M}$ with $\inf R^{\partial M} > 0$. Suppose that either:

1. The scalar curvature $R$ is a positive constant, or
2. $h \geq \frac{f}{3} R$ and $R_{\min} > 0$.

Then,

$$m_H(\partial M) \geq \frac{1}{96\pi} \left( \sqrt{\frac{|\partial M|}{16\pi}} \right) (72\pi + R_{\min}|\partial M|). \quad (1.6)$$

The equality holds in (1.6) if and only if $(M^3, g)$ is an Einstein manifold with totally geodesic boundary and in particular the Hawking mass satisfies

$$m_H(\partial M) = \sqrt{\frac{|\partial M|}{16\pi}}.$$

In this case, $(M^3, g)$ is isometric to a geodesic ball on a sphere $S^3$.

Moreover, using the Gauss-Bonnet theorem we obtain the following topological characterization for the boundary in compact, oriented Einstein-type manifolds $(M^3, g, f, h)$.

**Corollary 2.** Let $(M^3, g, f, h)$ be a compact, oriented Einstein-type manifold with boundary. Suppose that either:

1. The scalar curvature $R$ is constant, or
2. $h \geq \frac{f}{3} R$.

In this case, $(M^3, g)$ is isometric to a geodesic ball on a sphere $S^3$. 
Then,
\[ \chi(\partial M) \geq \frac{R_{\min}}{12\pi} |\partial M|, \]
where \( \chi(\partial M) \) is the Euler characteristic of \( \partial M \). Moreover, the equality holds if and only if \((M^3, g)\) is an Einstein manifold with totally geodesic boundary. In particular, if \( R_{\min} > 0 \), then \( \partial M \) is topologically a 2-sphere.

Furthermore, we can prove that the geodesic ball on a sphere \( S^3 \) is the unique compact Einstein-type manifold with positive constant scalar curvature such that the norm of the without trace Einstein tensor \( |\tilde{Ric}| \) lies in the interval \( \left[ 0, \frac{\sqrt{6}}{12} \left( \frac{6h}{f} - R \right) \right] \). More precisely, we prove that following result.

**Theorem 3.** Let \((M^3, g, f, h)\) be a compact, oriented Einstein-type manifold with positive constant scalar curvature satisfying the gap condition
\[ |\tilde{Ric}| < \frac{\sqrt{6}}{12} \left( \frac{6h}{f} - R \right). \] (1.7)
Then \( M^3 \) is isometric to a geodesic ball on a sphere \( S^3 \).

### 2. Background and proofs

Now, we start this section with a Lemma for an Einstein-type manifold \((M^n, g, f, h)\) that shows a necessary and sufficient condition in terms of \( f \) and \( h \) for \((M^n, g)\) has constant scalar curvature. Then we present the proofs of our main results.

**Lemma 1.** Let \((M^n, g, f, h)\) be an Einstein-type manifold. The scalar curvature of \( M^n \) is constant if and only if \( Rf - (n - 1)h \) is constant.

**Proof.** To prove this result, first we use the second Bianchi identity given by \( \text{div} Ric = \frac{1}{2} \nabla R \) to obtain that
\[
\text{div}(f\tilde{Ric}) = \text{div} \left( f\tilde{Ric} - f \frac{R}{n}g \right)
\]
\[ = f \text{div}(Ric) + Ric(\nabla f) + n \frac{1}{n} \nabla(fR) \]
\[ = f \frac{2}{2n} \nabla R + Ric(\nabla f) - \frac{R}{n} \nabla(f) \]
\[ = \frac{n - 2}{2n} f \nabla R + Ric(\nabla f) - \frac{R}{n} \nabla f \] (2.1)
Second, using that \( \text{div} \nabla^2 f = Ric(\nabla f) + \nabla \Delta f \), we deduce
\[ \text{div}(\nabla^2 f) = Ric(\nabla f) + \frac{n - 1}{n} \nabla \Delta f. \] (2.2)
Next, by equations (1.2), (1.3), (2.1) and (2.2), we get
\[ \frac{n - 1}{n} \nabla \Delta f = \frac{n - 2}{2n} f \nabla R - \frac{R}{n} \nabla f. \] (2.3)
Finally, we deduce that
\[ \frac{1}{2} f \nabla R = \nabla (Rf - (n - 1)h). \]
The proof is finished. \( \square \)
To prove Theorem 1, we need the following Proposition from [6], which is a consequence of Obata’s work [12] and Reilly’s theorem [13].

**Proposition 1** ([6], Proposition 1). Let \((M^n, g, f)\) be a compact Einstein manifold with positive scalar curvature and \(f\) a smooth function on \(M^n\) satisfying (1.3).

1. If \(\partial M\) is empty, then \(M^n\) is isometric to a round sphere \(\mathbb{S}^n\).
2. If \(\partial M\) is connected non-empty and \(f|_{\partial M}\) is constant, then \(M^n\) is isometric to a geodesic ball on a sphere \(\mathbb{S}^n\).

**Proof of Theorem 1.** If \((M^n, g)\) is Einstein, \(n \geq 3\), then \(R\) is constant. So, by Lemma 1, we obtain that

\[
Rf - (n - 1)h = c, \tag{2.4}
\]

where \(c\) is a constant. Since \((M^n, g)\) is Einstein and \(f\) satisfies (1.3), we deduce that \(\nabla^2 f = \frac{\Delta f}{n} g\). Using (1.2) and (2.4), we obtain that \(f\) satisfies the following differential equation

\[
\nabla^2 f = \left( -\frac{R}{n(n-1)} f + C \right) g,
\]

where \(C = c/(n-1)\).

Now, from (2.4) we infer that \(f\) is constant if and only if \(h\) is constant. If \(R = 0\), then \(f\) is constant, but this does not occurs. Thus, the scalar curvature is positive, i.e., \(R > 0\). Since \(f = 0\), on \(\partial M\), and \(f\) satisfies the equation (1.3), then by Proposition 1 we conclude that \(M^n\) is isometric to a geodesic ball on a sphere \(\mathbb{S}^n\). \(\square\)

2.1. Einstein-type manifolds with boundary. In this subsection, we study compact Einstein-type manifold with boundary. By definition \(f = 0\) on \(\partial M\), then \(f\) does not change of sign on \(\partial M\). In particular, \(|\nabla f| \neq 0\) on \(\partial M\) and we can consider the normal vector on \(\partial M\) defined by \(\nu = -\frac{\nabla f}{|\nabla f|}\). Since \(f = 0\) on \(\partial M\), then by equation (1.3), we obtain that \(\nabla^2 f = \frac{\Delta f}{n} g\), on \(\partial M\). This implies that

\[
X(|\nabla f|^2) = 2\langle \nabla_X \nabla f, \nabla f \rangle = 2\nabla^2 f(X, \nabla f) = 2\frac{\Delta f}{n} g(X, \nabla f) = 0,
\]

where \(X \in \mathfrak{X}(\partial M)\). This proves that \(|\nabla f|\) is constant and non-null on \(\partial M\). Now, we consider an orthonormal frame \(\{e_1, \cdots, e_{n-1}, e_n = \nu\}\) on \(\partial M\). From equations (1.2), (1.3) and \(f = 0\) on \(\partial M\), we infer that

\[
\nabla^2 f = \frac{\Delta f}{n} g = -hg. \tag{2.5}
\]

We denote by \(\alpha_{ab}\) the second fundamental form, where \(1 \leq a, b \leq n - 1\). Then by definition of \(\alpha_{ab}\) and using (2.5), we obtain

\[
\alpha_{ab} = \langle \nabla_{e_a} \nu, e_b \rangle = -\frac{1}{|\nabla f|} \langle \nabla_{e_a} \nabla f, e_b \rangle = -\frac{1}{|\nabla f|} \nabla_a \nabla_b f = -\frac{\Delta f}{n|\nabla f|} g_{ab} = \frac{h}{|\nabla f|} g_{ab}.
\]

This shows that \(\partial M\) is totally umbilical with mean curvature \(H = \frac{h}{|\nabla f|}\). From Gauss equation, we deduce

\[
R^\partial_M_{abcd} = R_{abcd} - \alpha_{ad}\alpha_{bc} + \alpha_{ac}\alpha_{bd}.
\]

This implies that

\[
R^\partial_M_{ac} = R_{ac} - R_{ancn} + \frac{h^2}{|\nabla f|^2} (n - 2),
\]
and finally, we obtain that the scalar curvature on \( \partial M \) is given by

\[
R^{\partial M} = R - 2R_{nn} + \frac{h^2}{|\nabla f|^2}(n-1)(n-2).
\]

Which is equivalent to

\[
R_{nn} = \frac{R - R^{\partial M}}{2} + \frac{(n-1)(n-2)}{2}H^2.
\]  (2.6)

In this way, we obtain the following Lemma.

**Lemma 2.** Let \( (M^n, g, f, h) \) be a compact, oriented Einstein-type manifold. Then

\[
\int_{\partial M} |\nabla f| R^{\partial M} dS - (n-1)(n-2) \int_{\partial M} |\nabla f| H^2 dS = 2 \int_M f|Ric|^2 - \frac{(n-2)}{n} \int_M R \Delta f dV.
\]

**Proof.** It is well known that \( \text{div}(\tilde{Ric}(\nabla f)) = \text{div} \tilde{Ric}(\nabla f) + \langle \tilde{Ric}, \nabla^2 f \rangle \). Using that \( \nabla^2 f = fRic - hg \) and the second identity of Bianchi \( \text{div} \tilde{Ric} = \frac{1}{2} \nabla R \), we deduce

\[
\text{div}(\tilde{Ric}(\nabla f)) = \frac{n-2}{2n} \langle \nabla R, \nabla f \rangle + f|Ric|^2.
\]

Integrating the last equation over \( M \) and using the Stokes’s theorem, we obtain

\[
\int_M \text{div}(\tilde{Ric}(\nabla f)) dV = \int_M f|Ric|^2 dV + \frac{n-2}{2n} \int_M \langle \nabla R, \nabla f \rangle dV
\]

\[
= \int_M f|Ric|^2 dV + \frac{n-2}{2n} \left( \int_{\partial M} R(\nabla f, \nu) dS - \int_M R \Delta f dV \right)
\]

\[
= \int_M f|Ric|^2 dV - \frac{n-2}{2n} \left( \int_{\partial M} R(\nabla f) dS + \int_M R \Delta f dV \right). \tag{2.7}
\]

By one hand, we infer

\[
\int_M \text{div}(\tilde{Ric}(\nabla f)) dV = \int_{\partial M} \langle \tilde{Ric}(\nabla f), \nu \rangle dS
\]

\[
= - \int_{\partial M} |\nabla f| \tilde{Ric}(\nu, \nu) dS
\]

\[
= - \int_{\partial M} |\nabla f| \left( R_{nn} - \frac{R}{n} \right) dS. \tag{2.8}
\]

By other hand, combining Eqs. (2.6), (2.7) and (2.8), we get

\[
\int_M f|Ric|^2 dV - \frac{n-2}{2n} \int_M R \Delta f dV = - \int_{\partial M} |\nabla f| \left( R_{nn} - \frac{R}{n} \right) dS
\]

\[
+ \frac{n-2}{2n} \int_{\partial M} R|\nabla f| dS
\]

\[
= \frac{1}{2} \int_{\partial M} |\nabla f| R^{\partial M} dS - \frac{(n-1)(n-2)}{2} \int_{\partial M} |\nabla f| H^2 dS.
\]

□
After, motivated by Proposition 3 in [6] in the context of static perfect fluid space-time, we prove a result in an Einstein-type manifold. More precisely, we obtain the following.

**Proposition 2.** Let \((M^n, g, f, h)\) be a compact, oriented Einstein-type manifold. Suppose that either:

1. The scalar curvature \(R\) is constant, or
2. \(h \geq \frac{f}{n}R\).

Then,

\[
\int_{\partial M} R\partial M dS \geq \frac{2}{k} \int_M f|Ric|^2 dV + \frac{n-2}{n} R_{\min}|\partial M| + (n-1)(n-2) \int_{\partial M} H^2 dS,
\]

where \(k = |\nabla f|\) on \(\partial M\), \(R_{\min}\) is the minimum value of \(R\) on \(M^n\) and \(|\partial M|\) is the area of \(\partial M\). In particular, we obtain

\[
\int_{\partial M} R\partial M dS \geq \frac{n-2}{n} R_{\min}|\partial M|,
\]

and the equality occurs if and only if \((M^n, g)\) is Einstein and \(\partial M\) is totally geodesic.

**Proof.** If \(R\) is constant, then

\[
\int_M R\Delta f dV = R \int_M \Delta f dV = R \int_{\partial M} \langle \nabla f, \nu \rangle = -R \int_{\partial M} |\nabla f| dS = -Rk|\partial M|.
\]

Now using the Lemma 2, we obtain

\[
k \int_{\partial M} R\partial M dS = (n-1)(n-2)k \int_{\partial M} H^2 dS + 2 \int_M f|Ric|^2 dV + \frac{n-2}{n} Rk|\partial M| \geq 2 \int_M f|Ric|^2 dV + \frac{n-2}{n} Rk|\partial M| \geq \frac{n-2}{n} Rk|\partial M|.
\]

By hypothesis \(h \geq \frac{f}{n}R\), this implies that \(R\Delta f \leq R_{\min}\Delta f\). We deduce

\[
\int_M R\Delta f \leq R_{\min} \int_M \Delta f dV = -R_{\min}k|\partial M|.
\]

Again, by Lemma 2, we infer

\[
k \int_{\partial M} R\partial M dS \geq (n-1)(n-2)k \int_{\partial M} H^2 dS + 2 \int_M f|Ric|^2 dV + \frac{n-2}{n} R_{\min}k|\partial M|.
\]

In particular,

\[
\int_{\partial M} R\partial M \geq \frac{n-2}{n} R_{\min}|\partial M|.
\]

(2.10)

If occurs the equality in (2.9) or in (2.10), then \(Ric = 0\), i.e \((M^n, g)\) is an Einstein manifold and \(H = 0\) and since \(\partial M\) is totally umbilical, then the result follows.

If \(h = fR/n\), then \(\Delta f = 0\) and because \(M\) is compact, we conclude that \(f\) is constant. This implies that

\[
f \int_M |Ric|^2 dV + (n-1)(n-2) \int_{\partial M} H^2 dS = 0.
\]

By definition \(f > 0\) in \(\text{int}(M)\), then \(Ric = 0\) and \(H = 0\), this finishes the proof. \(\square\)
Proof of Corollary 2. From Proposition 2, we obtain
\[ \int_{\partial M} R^\partial M dS \geq \frac{1}{3} R_{\text{min}} |\partial M|. \]

Now, using the Gauss-Bonnet theorem, we infer
\[ 4\pi \chi(\partial M) = 2 \int_{\partial M} K dS = \int_{\partial M} R^\partial M dS \geq \frac{1}{3} R_{\text{min}} |\partial M|, \]
where \( K \) is the Gaussian curvature of \( \partial M \). Thus,
\[ \chi(\partial M) \geq \frac{1}{12\pi} R_{\text{min}} |\partial M|. \]

In particular, if \( R_{\text{min}} > 0 \), then \( \chi(\partial M) > 0 \). So, in this case \( \partial M \) is topologically a 2-sphere. \( \square \)

Now we are ready to prove the Theorem 2.

Proof of Theorem 2. Since \( \inf R^\partial M > 0 \) and \( \text{Ric}^\partial M \geq \frac{R^\partial M}{n-1} g_{\partial M} \), then there exists \( \delta > 0 \) such that
\[ \inf \{ \text{Ric}^\partial M(V, V); V \in T\partial M, |V| = 1 \} = (n-2)\delta. \]
So,
\[ \text{Ric}^\partial M \geq (n-2)\delta. \]

Using Bonnet-Myers theorem, we deduce that the diameter of \( \partial M \) satisfies \( \text{diam}(\partial M) \leq \frac{\pi}{\sqrt{\delta}} \).

We observe that there exists an unit vector field \( V \) such that \( \text{Ric}^\partial M(V, V) = (n-2)\delta \). We deduce that \( (n-2)\delta \geq \frac{R^\partial M}{n-1} \). This implies that
\[ (n-1)(n-2)w_{n-1}^2 \geq R^\partial M |\partial M| \frac{2}{n-1}. \]

Integrating this expression over \( \partial M \), we obtain
\[ (n-1)(n-2)w_{n-1}^2 \geq |\partial M| \frac{(n-1)n}{n-1} \int_{\partial M} R^\partial M dS. \quad (2.11) \]

From Proposition 2 and (2.11), we deduce that
\[ (n-1)w_{n-1}^2 \geq |\partial M| \frac{(n-1)n}{n} \left( R_{\text{min}} + \frac{(n-1)n}{|\partial M|} \int_{\partial M} H^2 dS \right) . \]

Thus,
\[ |\partial M| \leq \left( \frac{n(n-1)}{R_{\text{min}} + K(H, n)} \right)^{\frac{n-1}{2}} w_{n-1} \leq \left( \frac{n(n-1)}{R_{\text{min}}} \right)^{\frac{n-1}{2}} w_{n-1}, \quad (2.12) \]
where \( K(H, n) = \frac{n(n-1)}{|\partial M|} \int_{\partial M} H^2 dS. \)

Moreover, the equality holds in (2.12) if and only if \((M^n, g)\) is Einstein with totally geodesic boundary. Finally, using the Proposition 1, we conclude that \((M^n, g)\) is isometric to a geodesic ball on a sphere \( S^n \). \( \square \)

Proof of Corollary 1. By Theorem 2 and definition of Hawking quasi-local mass (1.5), we obtain the result. \( \square \)
In [6] was obtained a Böchner type formula for Riemannian manifolds that satisfies the equation (1.3), was proved the following result.

**Lemma 3 ([6], Theorem 2).** Let \((M^3, g)\) be a Riemannian manifold and \(f\) a smooth function on \(M^3\) satisfying \(f \nabla |\tilde{R}|^2 - \frac{1}{2} f \tilde{R} c(\nabla R)\). Then it holds
\[
\frac{1}{2} \text{div} \left( f \nabla |\tilde{R}|^2 - \frac{1}{2} f \tilde{R} c(\nabla R) \right) = f |\nabla \tilde{R}|^2 - \frac{1}{24} f |\nabla R|^2 + f |C|^2 + \left( \frac{R f}{2} - \Delta f \right) |\tilde{R}|^2 + 6 f \text{tr}(\tilde{R}^3),
\]
where \(|C|^2 = C_{ijk} C^{ijk}\) and \(C_{ijk}\) is the Cotton tensor.

In particular, we can show that the geodesic ball on a sphere \(S^3\) is the unique compact Einstein-type manifold with positive constant scalar curvature such that the norm of the without trace Einstein tensor \(|\tilde{R}|\) lies in the interval \([0, \frac{\sqrt{6}}{12} (\frac{6h}{1} - R)]\). More precisely, we prove the Theorem 3.

**Proof of Theorem 3.** Since the scalar curvature \(R\) is constant, then by Lemma 3, we obtain
\[
\frac{1}{2} \text{div} \left( f \nabla |\tilde{R}|^2 \right) = f |\nabla \tilde{R}|^2 + f |C|^2 + \left( \frac{R f}{2} - \Delta f \right) |\tilde{R}|^2 + 6 f \text{tr}(\tilde{R}^3). \tag{2.13}
\]
Substituting (1.2) and the Okumura’s inequality \(6 \text{tr}(\tilde{R}^3) \geq -\sqrt{6} |\tilde{R}|^3\) in (2.13), we infer that
\[
\frac{1}{2} \text{div} \left( f \nabla |\tilde{R}|^2 \right) \geq f |\nabla \tilde{R}|^2 + f |C|^2 + \left( - \frac{R f}{2} + 3h - \sqrt{6} f |\tilde{R}| \right) |\tilde{R}|^2.
\]
Integrating over \(M\), using that \(f = 0\) on \(\partial M\) and the gap condition (1.7), we obtain
\[
0 \geq \int_M \left( f |\nabla \tilde{R}|^2 + f |C|^2 + \left( - \frac{R f}{2} + 3h - \sqrt{6} f |\tilde{R}| \right) |\tilde{R}|^2 \right) dS \geq 0.
\]
Again using (1.7), we deduce that \(\tilde{R} = 0\), i.e., \((M^3, g)\) is an Einstein manifold. Finally, by Proposition 1 we conclude the proof.

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