AN INTERESTING EXAMPLE OF A COMPACT NON-C-ANALYTIC REAL SUBVARIETY OF $\mathbb{R}^3$

JIŘÍ LEBL

Abstract. The purpose of this short note is to provide an interesting new example exhibiting some of the pathological properties of real-analytic subvarieties. We construct a compact irreducible real-analytic subvariety $S$ of $\mathbb{R}^3$ of pure dimension two such that 1) the only a real-analytic function is defined in a neighbourhood of $S$ and vanishing on $S$ is the zero function, 2) the singular set of $S$ is not a subvariety of $S$, nor is it contained in any one-dimensional subvariety of $S$, 3) the variety $S$ contains a proper subvariety of dimension two. The example shows how a badly behaved part of a subvariety can be hidden via a second well behaved component to create a subvariety of a larger set.

A closed set $S \subset \mathbb{R}^n$ is a real-analytic subvariety if for every $p \in S$, there exist real-analytic functions $\rho_1, \ldots, \rho_k$ defined in a neighbourhood $U$ of $p$, such that $S \cap U$ is equal to the set where all $\rho_1, \ldots, \rho_k$ vanish. A complex subvariety is precisely the same notion in $\mathbb{C}^n$, with real-analytic replaced by holomorphic (complex analytic). See [2,3] for more information.

Let us start with the notion of irreducibility. A real-analytic subvariety $X \subset \mathbb{R}^n$ is irreducible if whenever we write $X = X_1 \cup X_2$ for two subvarieties $X_1$ and $X_2$ of $\mathbb{R}^3$, then either $X_1 = X$ or $X_2 = X$. This notion is subtle. We will construct a set that is a union of two subvarieties, one of which is not a subvariety of $\mathbb{R}^3$ but of a strictly smaller domain, and the union is irreducible as a subvariety of $\mathbb{R}^3$.

Let us proceed in steps. Start with the sphere $z^2 = 1 - x^2 - y^2$, thinking of $z^2$ as a "graph". Pinch the sphere along the $y$-axis by multiplying by $x^2$ to obtain the subvariety $S_1$ given by

$$z^2 = (1 - x^2 - y^2)x^2.$$  (1)

The picture is the left hand side of Figure[1].
The subvariety $S_1$ is irreducible, and it contains the $y$-axis as a subvariety. This subvariety is already somewhat pathological. It has components of different dimensions, and the regular points of dimension two are not topologically connected.

We restrict our attention to the set where $-2 < y < 2$. We make a change of coordinates keeping $y$ and $z$ fixed, but sending $x$ to $x + \sin\left(\frac{1}{y+2}\right)$. The equation becomes

$$z^2 = \left(1 - \left(x - \sin\left(\frac{1}{y+2}\right)\right)^2 - y^2\right) \left(x - \sin\left(\frac{1}{y+2}\right)\right)^2.$$  

We call this set $S_2$, restricted to $-2 < y < 2$ to make it bounded. What we have done is made the $y$ axis appear like the graph of $\sin\left(\frac{1}{y+2}\right)$. The picture of the result is the right hand side of Figure 1. The set $S_2$ is a real-analytic subvariety of the set $\{ -2 < y < 2 \}$, and it becomes badly behaved as we approach $y = -2$.

**Proposition 1.** Let $\Omega \subset \mathbb{R}^3$ be a connected neighbourhood of the closure $\overline{S_2}$ and $r: \Omega \to \mathbb{R}$ a real-analytic function vanishing on $S_2$. Then $r$ is identically zero.

It would be very easy to prove that $r$ has to vanish on the $xy$-plane via essentially just staring at the picture and recalling basic properties of real-analytic functions. However, to prove that $r$ vanishes everywhere, we need to complexify $r$. The result is not just because the 1-dimensional component wiggles around, it is because the complexification of the 2-dimensional part gets dragged along the 1-dimensional component.

**Proof.** We consider $\mathbb{R}^3 \subset \mathbb{C}^3$. Then $S_2$ is also a subset of $\mathbb{C}^3$. Treating $(x, y, z)$ as complex variables, let us call $\mathcal{S}$ the complex subvariety of $\{ y \neq -2 \}$ set defined by (2). The subvariety $\mathcal{S}$ is locally irreducible at $(0, -1, 0)$. Indeed, think of $z^2$ as a graph over $(x, y)$, and so there can at most be “two sheets” in $\mathcal{S}$, for the two different square roots of the right hand side of (2). We can clearly move from one root to the other in an arbitrarily small neighbourhood of $(0, -1, 0)$.

Complexify $r$ to obtain a holomorphic function $\tilde{r}$ of a neighbourhood $U$ of $\overline{\mathcal{S}_2} \subset \mathbb{C}^3$. As $\tilde{r}$ vanishes on $\overline{\mathcal{S}_2}$, then by irreducibility of $\mathcal{S}$ at $(0, -1, 0)$, there exists a neighbourhood $W$ of $(0, -1, 0)$ in $\mathbb{C}^3$, such that $\tilde{r}$ vanishes on $W \cap \mathcal{S}$.

Fix $z = ia$ for a small real $a$. Let $X_a$ be the set defined by

$$-a^2 = \left(1 - \left(x - \sin\left(\frac{1}{y+2}\right)\right)^2 - y^2\right) \left(x - \sin\left(\frac{1}{y+2}\right)\right)^2.$$  

**Figure 1.** The sets $S_1$ (left) and $S_2$ (right).
for real $x$ and $y$ with $-2 < y < -1$. The set $X_a$ is a connected smooth real-analytic curve, which is a subset of $S$. If $a$ is small enough, $X_a \subset S \cap U$ and $X_a \cap W$ is nonempty. The function $\tilde{r}$ then vanishes on $X_a \cap W$, an open set of $X_a$, and hence on all of $X_a$.

Next fix a small real $x$. The equation (3) is true for an infinite sequence of $y$ approaching $y = -2$ from above. Therefore, the holomorphic function of one complex variable $y \mapsto \tilde{r}(x, y, ia)$ defined in a neighbourhood of the origin vanishes identically. As this was true for all small enough $x$ and $a$, it is true for small enough complex $x$ and $a$, and $\tilde{r}$ vanishes in a neighbourhood of $(0, -2, 0)$ in $\mathbb{C}^3$. By analytic continuation $\tilde{r}$ is identically zero. \hfill \Box

To visualize how bad the complexification is as we approach $y = -2$, consider the set in the space $(x, y, a) \in \mathbb{R}^3$ given by (3). Looking at the set where $a \geq 0$, we have a “valley” whose bottom is the graph $x = \sin\left(\frac{1}{y+2}\right)$ with increasingly steep sides. See Figure 2.

![Figure 2. The trace of the complexification of $S_2$ in $(x, y, a)$-space for $a \geq 0$ and $y < -1$, approaching $y = -2$.](image1)

Let us “hide” the wild behavior near $y = -2$ and construct the purely 2-dimensional subvariety $S_3$ via

$$S_3 = S_2 \cup \{z = 0\}. \tag{4}$$

The picture is the left hand side of Figure 3. Suppose $\Omega \subset \mathbb{R}^3$ is a connected neighbourhood of $S_3$ and $r: \Omega \to \mathbb{R}$ a real-analytic function such that $r = 0$ on $S_3$. The set $\Omega$ is also a neighbourhood of $\overline{S_2}$ and $r = 0$ on $\overline{S_2}$. By the proposition, $r \equiv 0$. In the terminology of real-analytic varieties, $S_3$ is not $\mathbb{C}$-analytic.

![Figure 3. The sets $S_3$ (left) and $S_4$ (right).](image2)
The singular set of $S_3$ is the set
\[ z = 0, \]
\[ -1 \leq y \leq 1, \]
\[ 0 = \left( 1 - \left( x - \sin \left( \frac{1}{y+2} \right) \right)^2 - y^2 \right) \left( x - \sin \left( \frac{1}{y+2} \right) \right)^2. \] (5)
This singular set clearly is not a subvariety. In particular, it contains the set
\[ I = \left\{ (x, y, z) : x = \sin \left( \frac{1}{y+2} \right) \text{ and } -1 \leq y \leq 1 \right\}, \] (6)
and $I$ cannot be contained in any subvariety of $S_3$ of dimension 1. Any such subvariety would have to contain the entire set $x = \sin \left( \frac{1}{y+2} \right)$ for all $y > -2$, and it cannot possibly be a subvariety at points where $y = -2$.

The subvariety $S_3$ is irreducible. It contains a proper subvariety of dimension 2, namely the $xy$-plane. Any subvariety $S'$ that contains any open set of the regular points must contain $I$. Indeed, if $S'$ contains an open set of the $xy$-plane it must contain whole $xy$-plane. If $S'$ contains an open set of one of the smooth submanifolds outside of the $xy$-plane then $I$ is in the closure of this submanifold and hence in $S'$. Any subvariety that contains $I$ must contain the entire $xy$-plane.

We have demonstrated a subvariety with all the required properties but not a compact one. We make the subvariety compact by mapping the plane onto the sphere using spherical coordinates. For the picture on the right hand side of Figure 3 we used the map
\[(x, y, z) \mapsto \left( (z + 1) \sin(1 + y/2) \cos(x), (z + 1) \sin(1 + y/2) \sin(x), (z + 1) \cos(1 + y/2) \right). \] (7)
For $-\pi < x < \pi$, $0 < (1 + y/2) < \pi$, and $z + 1 > 0$, the mapping is a real-analytic diffeomorphism and we obtain the compact subvariety $S_4$ by taking the closure, which will fill in the missing meridian on the far side of the sphere.

This subvariety clearly has all the properties mentioned in the abstract; it is compact and inherits the rest of the properties from $S_3$.

The construction is easy to modify to show further strange behaviors. For example, if we start with $S_3$, but rescale the $z$ variable we obtain another irreducible subvariety that shares with $S_3$ a 2-dimensional component as a proper subvariety.

References
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Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, USA
E-mail address: lebl@math.okstate.edu