Generalized Haldane-Shastry Models as
Supersymmetric Partners of the
Calogero-Sutherland Type Models

Avinash Khare++

Institute of Physics,
Sachivalaya Marg, Bhubaneswar-751 005, India

M. Sivakumar**

School of Physics, University of Hyderabad,
Hyderabad - 500 046, India

Abstract
We consider the supersymmetric Calogero-Sutherland type N-particle problems in
one dimension and show that the corresponding fermionic part can be identified
with the generalized X-Y models in the presence of an inhomogeneous magnetic
field. In particular we show that the generalized Haldane-Shastry models (with
magnetic field) are themselves the fermionic partners of the Calogero-Sutherland
type models. Several such models are discussed and a recipe is given for con-
structing spin models and finding their ground state energy from the corresponding
N-particle problems.

++ email: khare@iopb.ernet.in
** email: mssp@uohyd.ernet.in
In recent times, there has been a revival of interest in Calogero-Sutherland model (CSM) [1], which is a one-dimensional many body system with two-body inverse square interaction in a confining harmonic oscillator potential. Some remarkable connections have been found with various other areas of physics [2]. Further, it is an example of an ideal gas in one dimension with fractional exclusion statistics [3]. Besides, this model has several interesting mathematical structures like quantum integrability [4], W algebra [5] and is related with other subjects like random matrices, quantum chaos [2] and has found applications in diverse areas like quantum Hall effect [6] and mesoscopic systems [7]. Its spin generalization i.e. the Haldane-Shastry model[8] has similarly attracted a lot of attention specially because not only the static but even the dynamic correlation functions are already known for this barely nine year old model. This should be contrasted with more than sixty year old celebrated Bethe model for which even static correlation functions are not known as yet. Inspired by the success of the CSM, its several generalizations have also been introduced. For example, whereas the original CSM has a hidden $A_{N-1}$ Lie algebraic structure, generalization to other classical groups have been constructed and studied and are known as $BC_N$ models[9]. Further, One of the author has recently introduced a model having similar features as CSM which is also exactly solvable but where instead of the oscillator potential one has an N-body interaction [10].

The purpose of this letter is to introduce and study the supersymmetric versions of these models. Some time ago, Freedman and Mende[11] have in fact studied the supersymmetric generalization of the CSM but in their work the fermionic part has not been identified with any known model in statistical mechanics. In this note we follow the earlier work [12,13], and supersymmetrize the CSM and its other generalizations and by using the Jordan-Wigner transformation are able to identify the fermionic part with the spin models. In this way, in all the cases we are able to identify the fermionic
part with the generalized X-Y models in the presence of inhomogeneous magnetic field. Remarkably, we find that the asymmetric Haldane-Shastry model (HSM) plus magnetic field is the fermionic partner of the CSM. We believe that this connection has not been noticed before in the literature.

In view of this remarkable connection, the ground state energy and also the level structure (though not necessarily the energy eigenvalue spectrum itself since the degeneracy could be different in the fermionic and the bosonic spectrum) of the generalized HSM is immediately obtained. Knowing the ground state energy of the generalized X-Y model which otherwise is unknown, one can study the low temperature thermodynamic properties of the system. Further, we show that the square of the ground state wave function of the supersymmetric CSM is also related to the joint probability density function for the eigenvalues of random matrices from orthogonal, unitary or symplectic ensembles but for different values of coupling than in the bosonic case. As a result, we immediately know the static correlation functions for the supersymmetric model at these three values of the couplings. It may be noted here that for the CS model (which is the bosonic part of our SUSY model) Sutherland has already obtained the correlation functions but at different values of the coupling than in the SUSY case.

We also give a recipe for constructing the supersymmetric generalization of any N-particle problem in one dimension and as an illustration, we consider the supersymmetric partner of a N-body problem where the particles also interact through an N-body potential [9] and obtain the corresponding generalized X-Y model. Further, the SUSY variant of several generalized CSM are also given.

Long time ago, in an interesting paper, Crombrugghe and Rittenberg [12] have given the formalism for constructing supersymmetric Hamiltonians in the case of N-particles on a line. Subsequently, Maharana and one of us [13] had used their formalism and
studied several examples in case the bosonic Hamiltonian corresponds to the linear chain of atoms. We shall now use this approach to obtain the generalized X-Y models as the fermionic partners of the CSM and its generalizations.

The supersymmetric Hamiltonian for N-particles in 1-dimension is given by \([12,13]\) 
\( (\hbar = m = 1) \)
\[
H = H_B + H_F 
\]
where,
\[
2H_B = \sum_{i=1}^{N} \left[ (P_i + \frac{\partial F}{\partial x_i})^2 + (\frac{\partial G}{\partial x_i})^2 \right] 
\]
and
\[
2H_F = -2\sum_{k,l} \frac{\partial^2 G}{\partial x_k \partial x_l} F_{(1,k)(2,l)} 
\]
Here, F and G are arbitrary functions of N particle coordinates Since only momentum independent potentials are considered, hence, throughout this paper, we shall confine our discussion to the special case when \( F = 0 \). In the above equation, \( F_{(1,k)(2,l)} = \frac{i}{4}[C_1(k), C_2(l)] \) where \( C_i(k) \) are anti-commuting variables obeying \( \{C_i(k), C_j(l)\} = 2\delta_{ij}\delta_{kl} \).

We now make the key observation that by applying the Jordan-Wigner transformation, the fermionic Hamiltonian can be written as a generalized X – Y model in an external magnetic field \([12,13]\). In particular, on adopting the following representation
\[
F_{(1,k)(2,k)} = -\frac{1}{2}\sigma_z(k) 
\]
\[
F_{(1,k)(2,l)} + F_{(1,l)(2,k)} = \frac{1}{2}[\sigma_x(k) \otimes \sigma_x(l) + \sigma_y(k) \otimes \sigma_y(l)] 
\]
the fermionic Hamiltonian takes the form
\[
2H_F = \sum_{k=1}^{N} B_k \sigma_z(k) - \sum_{k \neq l} J_{kl}[\sigma_x(k) \otimes \sigma_x(l) + \sigma_y(k) \otimes \sigma_y(l)]. 
\]
where
\[
B_k = \frac{\partial^2 G}{\partial x_k^2} 
\]
\[ J_{kl} = \frac{\partial^2 G}{\partial x_k \partial x_l} \]  

(8)

By suitably choosing \( G(x_1, \ldots, x_n) \) (a prescription for it is given below) one can construct the supersymmetric models corresponding to the various CS-type models and in all the cases the corresponding fermionic part turns out to be some generalized X-Y model with an external magnetic field.

Let us illustrate this procedure for particles on a circle interacting with, inverse chord square distance, called as \( A_{N-1} \) trigonometric model. Let us choose

\[
G = -\lambda \sum_{k>j} \log[\sin(x_k - x_j)\frac{\pi}{L}]
\]  

(9)

On using eq. (9) in eq. (2) and the identity

\[
\sum_{i,j,k \atop i \neq j \neq k} \cot(\theta_k - \theta_i)\cot(\theta_k - \theta_j) = -N(N-1)(N-2)/3
\]

we get

\[
2H_B = \sum_k P_k^2 + \frac{\pi^2 \lambda^2}{L^2} \sum_{k,j} \frac{1}{\sin^2([x_k - x_j] \frac{\pi}{L})} - \frac{N(N^2-1)\pi^2 \lambda^2}{3L^2}
\]  

(10)

We now make the key observation that \( G \) chosen here is proportional to the logarithm of the ground state wave function for the bosonic Hamiltonian (except for a small but crucial difference in the form of the coupling constant \( \lambda \) as explained in Freedman and Mende [11], also see below). In fact, this is a general result which is valid in the case of all the models that we shall study. Thus, given any bosonic N-body problem, if one knows the ground state wave function \( \psi_0 \) of the system then one can easily supersymmetrize the problem by choosing \( G \propto \log \psi_0 \) and following the formalism as given above.

On using the \( G \) as given above, the fermionic part turns out to be

\[
2H_F = \frac{\pi^2 \lambda}{L^2} \sum_{k,j} \frac{\sigma_z(k)}{\sin^2([x_k - x_j] \frac{\pi}{L})} + \frac{\pi^2 \lambda}{2L^2} \sum_{k,j} \left[ \sigma_x(k) \otimes \sigma_x(j) + \sigma_y(k) \otimes \sigma_y(j) \right] \frac{1}{\sin^2([x_k - x_j] \frac{\pi}{L})}.
\]  

(11)
It is interesting to note that this $H_F$ is the asymmetric Haldane-Shastry model [8] with an external magnetic field in the z-direction. In view of the supersymmetry, the ground state energy and the level structure (though not necessarily the energy eigenvalue spectrum since the degeneracy could be different in the bosonic and fermionic spectrum) of the generalized HSM is immediately known since the bosonic spectrum is completely known.

Fermionic Ground State Energy: Following Sutherland [1], it is easily shown that the ground state energy and the eigenfunction of the $A_{N-1}$ bosonic Hamiltonian as given in eq. (10) is given by

$$E_0^B = \frac{\pi^2 N(N^2 - 1)}{12L^2} \left[1 + \sqrt{1 + 4\lambda^2}\right]$$ (12)

$$\psi_0^B = \prod'_{k,j} |\sin(x_k - x_j)|^\beta$$ (13)

where

$$\beta = \frac{1}{2} \left[1 + \sqrt{1 + 4\lambda^2}\right]$$ (14)

Now, for $\lambda > 0$ supersymmetry is unbroken and there is a unique zero energy ground state which is annihilated by the supersymmetry charges [10] and is given by

$$|\psi_0> = e^{-G} |0> = \prod'_{k,j} |\sin(x_k - x_j)|^\lambda |0>$$ (15)

where $|0>$ indicates the state in the $2^N$-dimensional fermion Fock space which is annihilated by all the fermionic operators. Since supersymmetry is unbroken in case $\lambda > 0$, hence it follows that the ground state energy of the asymmetric HSM with a magnetic field in the z-direction (eq. [11]) is given by

$$E_0^F = -E_0^B = -\frac{\pi^2 N(N^2 - 1)}{12L^2} \left[1 + \sqrt{1 + 4\lambda^2}\right]$$ (16)
So far as we are aware, the ground state energy of this X-Y model was not known before. Using this ground state energy, one can now proceed to calculate the low temperature thermodynamic properties of this X-Y model.

Further, the correlation functions of the supersymmetric theory can also be obtained at three values of the coupling constant $\lambda$ (which are different than the bosonic values). To this end notice that the exponent in the square of the bosonic and the supersymmetric ground state wave functions $|\psi_0|^2$ are not same but are $2\beta$ and $2\lambda$ respectively. Hence following Sutherland [1], it follows from eq. (15) that in the supersymmetric case, $|\psi_0|^2$ is identical to the joint probability distribution function from orthogonal, unitary or symplectic circular ensembles in case $\lambda^2 = 1/4, 1, 4$ respectively. As a result, the static correlation functions for the many-body theory corresponding to the supersymmetric Hamiltonian are immediately known at these three values of $\lambda^2$. On the other hand, since the exponent in the bosonic case is $2\beta$ (and not $2\lambda$), it follows that for the bosonic case the corresponding correlation functions are known in case $\beta$ is $1/2, 1$ or $2$ i.e. $\lambda^2$ is $-1/4, 0, 2$ respectively.

Proceeding in the same way, one can obtain the $X - Y$ models corresponding to the other $N$-body problems by choosing $G$ to be proportional to the logarithm of the ground state wave function of that particular $N$-body problem. In Table 1, we tabulate the results for the other supersymmetric many body systems. It is worth noting that in all the cases the bosonic spectrum is already known in the literature. Further, since supersymmetry is unbroken in all the cases that we consider hence the ground state energy and the level structure of the corresponding X-Y model is immediately known (though not necessarily the full spectrum since the degeneracy could be different in the fermionic and the bosonic spectrum). It is worth emphasizing that so far the ground state energy is not known in the literature for any one of these X-Y models. It may also
be noted here that apart from the circular $A_{N-1}$ case, the static correlation functions are also known in the supersymmetric rational $A_{N-1}$ case when $\lambda^2 = 1/4, 1, 4$ since in these cases one can map the problem to that of Gaussian orthogonal, hermitian or symplectic ensembles respectively.

Summarizing, we have shown that the fermionic partners of the CS-type N-body problems in one dimension are the generalized X-Y models which so far have not been studied in the literature. Using supersymmetry we have been able to obtain the ground state energy for these models which would be useful in studying the low temperature thermodynamic properties of these systems. Further, in few cases we also have been able to obtain the static correlation functions of the supersymmetric theory at some special values of the inverse square coupling.

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Table 1: Generalized X-Y models in Nonuniform Magnetic Field and their Ground State Energy $E_0^F$. Note that once $G$ is specified then $H_B$ and $H_F$ are uniquely specified by eqs. (2) and (6) to (8) respectively. In the following, $\lambda'$ is the positive root of $\lambda'(\lambda' - 1) = \lambda^2$. Similarly, $\beta'(\delta')$ is the positive root of $\beta'^2(\delta'^2) - \beta'(\delta') = \beta^2(\delta^2)$. The results for the bosonic models corresponding to the $BC_N$ trigonometric, and AK model are from references [14], [10] and respectively.

| Model  | $G(x_1, \cdots, x_N)$ | $E_0^F$ |
|--------|------------------------|---------|
| Trigonometric $A_{N-1}$ CSM | $-\lambda \sum_{i \neq j} \ln \sin(x_i - x_j)$ | $\frac{\pi^2}{6L^2}N(N^2 - 1)(\lambda'^2 - \lambda^2)$ |
| Rational $A_{N-1}$ CSM | $-\lambda \sum_{i \neq j} \ln(x_i - y_j) + \frac{\omega}{2} \sum x_i^2$ | $\frac{N\omega}{2}[1 + (N - 1)(\lambda' - \lambda)]$ |
| Trigonometric $BC_N$ CSM | $-\gamma \sum_i \ln \sin x_i - \delta \sum_i \ln 2x_i$ | $\frac{2\omega^2}{L^2} \sum_{i=1}^N \left[ \left( \frac{\gamma'}{2} + \delta' + (N - i)\beta' \right)^2 - \left( \frac{\gamma'}{2} + \delta + (N - i)\beta \right)^2 \right]$ |
| $BC_N$ CSM | $-\beta \sum_{i \neq j} \ln \sin(x_i - x_j) \sin(x_i + x_j)$ | where $\gamma'(\gamma' + 2\delta' - 1) = \gamma^2$ |
| Rational $B_N$ CSM | $-\gamma \sum_i x_i - \beta \sum_{i \neq j} \ln(x_i + x_j)(x_i - x_j)$ | $N\omega[\gamma' - \gamma + (N - 1)(\beta' - \beta)]$ |
| $B_N$ CSM | where $\gamma'(\gamma' - 1) = \gamma$ |
| AK Model | $-\lambda \sum_{i \neq j} \ln(x_i - x_j) + \frac{\alpha}{\lambda N(N - 1)} \left( \sum_{i=1}^N x_i^2 \right)$ | $\frac{\alpha^2}{2(N-1)^2} \left[ \frac{1}{N^2} - \frac{1}{(1 + \lambda^2)^2} \right]$ |

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