THE SIGMA INVARIANTS OF THOMPSON’S GROUP $F$

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Abstract. Thompson’s group $F$ is the group of all increasing dyadic PL homeomorphisms of the closed unit interval. We compute $\Sigma^m(F)$ and $\Sigma^m(F;\mathbb{Z})$, the homotopical and homological Bieri-Neumann-Strebel-Renz invariants of $F$, and show that $\Sigma^m(F) = \Sigma^m(F;\mathbb{Z})$. As an application, we show that, for every $m$, $F$ has subgroups of type $F_{m-1}$ which are not of type $FP_m$ (thus certainly not of type $F_m$).

1. INTRODUCTION

1.1. The group $F$. Let $F$ denote the group of all increasing piecewise linear (PL) homeomorphisms whose points of non-differentiability $\in [0, 1]$ are dyadic rational numbers, and whose derivatives are integer powers of 2. This is known as Thompson’s Group $F$; it first appeared in [22].

The group $F$ has an infinite presentation

\[
\langle x_0, x_1, x_2, \ldots | x^{-1}_ix_{i+1} = x_{i+1} \text{ for } 0 \leq i < n \rangle
\]

Let $F(i)$ denote the subgroup $\langle x_i, x_{i+1}, \ldots \rangle$. The presentation (1.1) displays $F$ as an HNN extension with base group $F(1)$, associated subgroups $F(1)$ and $F(2)$, and stable letter $x_0$; see [17, Prop. 9.2.5] or [13] for a proof. Thus $F$ is an ascending HNN-extension whose base and associated subgroups are isomorphic to $F$.

The correspondence between the generators $x_i$ in the presentation (1.1) and PL homeomorphisms is as in [18]. For example, the generator $x_0$ corresponds to the PL homeomorphism with slope $\frac{1}{2}$ on $[0, \frac{1}{2}]$, slope 1 on $[\frac{1}{2}, \frac{3}{4}]$, and slope 2 on $[\frac{3}{4}, 1]$.

The group $F$ has type $F_\infty$ i.e. there is a $K(F, 1)$-complex with a finite number of cells in each dimension [13]. Therefore $F$ is finitely presented and has type $FP_\infty$. Furthermore, $F$ has infinite cohomological dimension [13], $H^*(F;\mathbb{Z})$ is trivial [14], $F$ does not contain a free subgroup of rank 2 [11], and the commutator subgroup $F'$ is simple [11, 15]. It is known that $F$ has quadratic Dehn function [18]. The group of automorphisms of $F$ was calculated in [9].

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\footnote{Here, PL homeomorphisms are understood to act on $[0, 1]$ on the left as in [15] rather than on the right as in [13].}

\footnote{See Subsection 2.1 for the definition.}

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1.2. The Sigma invariants of a group. By a (real) character on $G$ we mean a homomorphism $\chi : G \to \mathbb{R}$ to the additive group of real numbers. For a finitely generated group $G$ the character sphere $S(G)$ of $G$ is the set of equivalence classes of non-zero characters modulo positive multiplication. This is best thought of as the “sphere at infinity” of the real vector space $\text{Hom}(G, \mathbb{R})$. The dimension $d$ of that vector space is the torsion-free rank of $G/G'$, and the sphere at infinity has dimension $d - 1$. We denote by $[\chi]$ the point of $S(G)$ corresponding to $\chi$.

We recall the Bieri-Neumann-Strebel-Renz (or Sigma) invariants of a group $G$. Let $R$ denote a commutative ring\footnote{Only the rings $\mathbb{Z}$ and $\mathbb{Q}$ will play a role in this paper.} with $1 \neq 0$, and let $m \geq 0$ be an integer. When $G$ is of type $F_m$ (resp. $FP_m(R)$) the homotopical invariant $\Sigma^m(G)$ (resp. the homological invariant $\Sigma^m(G; R)$), is a subset of $S(G)$. In both cases we have $\Sigma^{m+1} \subseteq \Sigma^m$. We refer the reader to [7] for the precise definition, confining ourselves here to a brief recollection:

1.2.1. $m = 0$. All groups have type $F_0$ and type $FP_0(R)$. By definition $\Sigma^0(G) = \Sigma^0(G; R) = S(G)$. This will only be of interest when we consider subgroups of $F$ in Section 2.

1.2.2. $m = 1$. Let $X$ be a finite set of generators of $G$ and let $\Gamma^1$ be the corresponding Cayley graph, with $G$ acting freely on $\Gamma^1$ on the left. The vertices of $\Gamma^1$ are the elements of $G$ and there is an edge joining the vertex $g$ to the vertex $gx$ for each $x \in X$.

For any non-zero character $\chi : G \to \mathbb{R}$, and for any real number $i$ define $\Gamma^1_{\chi \geq i}$ to be the subgraph of $\Gamma$ spanned by the vertices

$$G_{\chi \geq i} = \{ g \in G \mid \chi(g) \geq i \}.$$  

By definition, $[\chi] \in \Sigma^1(G)$ if and only if $\Gamma^1_{\chi \geq 0}$ is connected. For a detailed treatment of $\Sigma^1$ from a topological point of view, see [17], Sec. 16.3.

1.2.3. $m = 2$. Let $\langle X \mid T \rangle$ be a finite presentation of $G$. Choose a $G$-invariant orientation for each edge of $\Gamma^1$ and then form the corresponding Cayley complex $\Gamma^2$ by attaching 2-cells equivariantly to $\Gamma^1$ using attaching maps indicated by the relations in $T$. Define $\Gamma^2_{\chi \geq i}$ to be the subcomplex of $\Gamma^2$ consisting of $\Gamma^1_{\chi \geq i}$ together with all the 2-cells which are attached to it.

By definition, $[\chi] \in \Sigma^2(G)$ if and only if $[\chi] \in \Sigma^1(G)$ and there is a nonpositive $d$ such that the map

$$(1.2) \quad \pi_1(\Gamma^2_{\chi \geq 0}) \to \pi_1(\Gamma^2_{\chi \geq d}),$$

induced by the inclusion of spaces $\Gamma^2_{\chi \geq 0} \subseteq \Gamma^2_{\chi \geq d}$ is zero (and $\Gamma^1_{\chi \geq 0}$ is connected). See, for example, [28]. Note that $\Gamma^2$ is the 2-skeleton of the universal cover of a $K(G, 1)$-complex which has finite 2-skeleton.

1.2.4. $m > 2$. The higher $\Sigma^m(G)$ are defined similarly, for groups of type $F_m$, using the $m$-skeleton, $\Gamma^m$, of the universal cover of a $K(G, 1)$-complex having finite $m$-skeleton. See [7].
1.2.5. The homological case. For a commutative ring $R$, the homological Sigma invariants $\Sigma^m(G; R)$ are defined similarly when the group $G$ is of type $FP_m(R)$, using a free resolution of the trivial (left) $RG$-module $R$ which is finitely generated in dimensions $\leq m$; see [7] for details. Among the basic facts to be used below, which hold for all rings $R$, are: $\Sigma^1(G) = \Sigma^1(G; R)$; and $\Sigma^m(G) \subseteq \Sigma^m(G; R)$ when both are defined (i.e. when $G$ has type $F_m$). If $G$ is finitely presented then “type $F_m$” and “type $FP_m(\mathbb{Z})$” coincide. In that case, $\Sigma^m(G; \mathbb{Z})$ can also be understood from the above topological definition of $\Sigma^m(G)$, replacing statements about homotopy groups by the analogous statements about reduced $\mathbb{Z}$-homology groups; more precisely, one requires
\begin{equation}
\hat{H}_{k-1}(\Gamma_{k \geq 0}) \to \hat{H}_{k-1}(\Gamma_{k \geq d}),
\end{equation}
to be trivial for all $k \leq m$.

Remark: The definition of $\Sigma^1$ given here agrees with the now-established conventions followed, for example, in [7] and in [2]. It differs by a sign from the $\Sigma^1$-invariant defined in [6]. This arises from our convention that $RG$-modules are left modules, while in [6] they are right modules.

1.3. Some facts about Sigma invariants. It is convenient to write $\{\chi \in \Sigma^\infty\}$ as an abbreviation for “$\{\chi \in \Sigma^m$ for all $m\}$.

Among the principal results of $\Sigma$-theory for a group $G$ of type $F_m$ (resp. type $FP_m(R)$) are: (1) $\Sigma^m(G)$ (resp. $\Sigma^m(G; R)$) is an open subset of the character sphere $S(G)$, and (2) $\Sigma^m(G)$ (resp. $\Sigma^m(G; R)$) classifies all normal subgroups $N$ of $G$ containing the commutator subgroup $G'$ by their finiteness properties in the following sense:

**Theorem 1.1.** [7], [27], [28] Let $G$ be a group of type $F_m$ (resp. type $FP_m(R)$) with a normal subgroup $N$ such that $G/N$ is abelian. Then $N$ is of type $F_m$ (resp. $FP_m$) if and only if for every non-zero character $\chi$ of $G$ such that $\chi(N) = 0$ we have $[\chi] \in \Sigma^m(G)$ (resp. $[\chi] \in \Sigma^m(G; R)$).

A non-zero character is discrete if its image in $\mathbb{R}$ is an infinite cyclic subgroup. A special case of Theorem 1.1 (the only one we will use) is:

**Corollary 1.2.** If the non-zero character $\chi$ is discrete then its kernel has type $F_m$ (resp. type $FP_m(R)$) if and only if $[\chi]$ and $[-\chi]$ lie in $\Sigma^m(G)$ (resp. $\Sigma^m(G; R)$).

The invariants $\Sigma^m(G)$ and $\Sigma^m(G; R)$ have been calculated for only a few families of groups $G$, even fewer when $m > 1$. For metabelian groups $G$ of type $F_m$ there is the still-open $\Sigma^m$-Conjecture: $\Sigma^m(G)^c = \Sigma^m(G; \mathbb{Z})^c = conv_{\leq m} \Sigma^1(G)^c$, where $conv_{\leq m}$ denotes the union of the (spherical) convex hulls of all $\leq m$-tuples; this is known for $m = 2$ [20] but only for larger $m$ under strong restrictions on $G$ [21], [24]. A complete description of $\Sigma^m(G)$ and $\Sigma^m(G; \mathbb{Z})$ for any right angled Artin group $G$ is given in [23]. Recently the homotopical invariant $\Sigma^m(G)$ has been generalized to an invariant of group actions on proper CAT(0) metric spaces [2]; the corresponding invariants for the natural action of $SL_n(\mathbb{R})$ on its symmetric space have been calculated: for $n = 2$ (action by Möbius transformations on the hyperbolic plane) in [3], and for $n > 2$ in [26]. A similar generalization of the homological case, $\Sigma^m(G; R)$, to the CAT(0) setting will appear in [4].

\footnote{It is customary to use the notation $A^c$ for the complement of the set $A$ in a character sphere; e.g. $\Sigma^m(G)^c$ or $\Sigma^m(G; R)^c$.}
1.4. Sigma invariants of $F$. In this paper we calculate the Sigma invariants \( \Sigma^m(F) \) and \( \Sigma^m(F; R) \) of the group $F$. For $x \in F$ and $i = 0$ or $1$ let $\chi_i(x) := \log_2 x^i(1)$, i.e. the (right) derivative of the map $x$ at $0$ is $2^\chi_0(x)$ and the (left) derivative of $x$ at $1$ is $2^\chi_1(x)$. In terms of the presentation (1.1) $\chi_0(x_0) = -1$ and $\chi_0(x_i) = 0$ for $i \geq 1$, while $\chi_1(x_i) = 1$ for all $i \geq 0$. These two characters are linearly independent. Thus $[\chi_0]$ and $[\chi_1]$ are not antipodal points of the circle $S(F)$. From (1.1) we see that the real vector space $\text{Hom}(F, \mathbb{R})$ has dimension 2, so these two characters span $\text{Hom}(F, \mathbb{R})$. It follows that the convex sum of $[\chi_0]$ and $[\chi_1]$ is a well-defined interval in the circle $S(F)$; it members are the points \( \{ [a\chi_0 + b\chi_1] | a, b > 0 \} \). We call it the “shorter interval”. We call $\chi_0$ and $\chi_1$ the “special” characters.

There is a useful automorphism $\nu$ of $F$ which is most easily expressed when $F$ is regarded as a group of PL homeomorphisms as above: it is conjugation by the linearly independent. Thus $\Sigma^m(F; R)$ and $\Sigma^m(F)$ are in $S(F)$; it members are the points \( \{ [a\chi_0 + b\chi_1] | a, b > 0 \} \). We call it the “shorter interval”. We call $\chi_0$ and $\chi_1$ the “special” characters.

The Theorems of this paper can now be stated:

**Theorem A.** $\Sigma^1(F)$ consists of all points of $S(F)$ except $[\chi_0]$ and $[\chi_1]$. The points of $S(F)$ lying in the open convex hull of $[\chi_0]$ and $[\chi_1]$, i.e. in the shorter interval, are in $\Sigma^1(F)$ but are not in $\Sigma^2(F)$. The other (longer) open interval between $[\chi_0]$ and $[\chi_1]$ is the set $\Sigma^\infty(F)$. The sets $\Sigma^m(F; R)$ and $\Sigma^m(F)$ coincide for all $m$ and any ring $R$.

One part of this is not new: $\Sigma^1(F)$ was computed in [6].

**Theorem B.** For every $m \geq 1$, $F$ contains subgroups of type $F_{m-1}$ which are not of type $FP_m(\mathbb{Z})$ (thus certainly not of type $F_m$).

Theorem A is proved in Section 2 and Theorem B is proved (using [5]) in Section 3.

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2. Proof of Theorem A

2.1. $\Sigma_0$ and $\Sigma_1$. By an ascending HNN extension we mean a group presented by\( \langle H, t | t^{-1}ht = \phi(h) \rangle \) for $h \in H$ where $\phi : H \to H$ is a monomorphism. Such a group is denoted by $H_{*\phi,t}$.

We begin by citing:

**Theorem 2.1.** Let $G$ decompose as an ascending HNN extension $H_{*\phi,t}$. Let $\chi : G \to \mathbb{R}$ be the character given by $\chi(H) = 0$ and $\chi(t) = 1$.

1. If $H$ is of type $F_m$ (resp. $FP_m(R)$) then $[\chi] \in \Sigma^m(G)$ (resp. $[\chi] \in \Sigma^m(G; R)$).
(2) If $H$ is finitely generated and $\phi$ is not onto $H$ then $[-\chi] \in \Sigma^1(G)^c$.

Proof. The homological case of (1) for all $m$ is [24 Prop. 4.2] and the homotopical case for $m = 2$ is a special case of [25 Thm. 4.3]. The homotopical case of (1) for all $m$ then follows.

(2) is elementary: we recall the argument. Let $N$ be the kernel of $\chi$. By (1) and Corollary 1.2 (2) is equivalent to claiming that the group $N$ is not finitely generated. The hypothesis that $\phi$ is not onto implies $t^{-1}Ht$ is a proper subgroup of $H$. Thus $N = \cup_{n \geq 1}t^nHt^{-n}$ is a proper ascending union, so it cannot be finitely generated. \hfill \Box

Applying Theorem 2.1 together with “$\nu$-symmetry” to the group $F$, i.e. $G = F$, $t = x_0$, $H = F(1)$, and $\chi = -\chi_0$, we get part of Theorem A:

Corollary 2.2. $\{[-\chi_0], [-\chi_1]\} \subseteq \Sigma^\infty(F)$ and $\{[\chi_0], [\chi_1]\} \subseteq \Sigma^1(F)^c$.

Theorem 8.1 of [6] is the assertion that the complement of the two-point set $\{[\chi_0], [\chi_1]\}$ is precisely $\Sigma^1(F)$.

2.2. The “longer” interval. The following is proved by combining two theorems of H. Meinert, namely [24 Prop. 4.1] and [25 Thm. B]:

Theorem 2.3. Let $G$ decompose as an ascending HNN extension $H*_{\phi,t}$. Let $\chi : G \to \mathbb{R}$ be a character such that $\chi|H \neq 0$. If $H$ is of type $F_\infty$ and if $[\chi|H] \in \Sigma^\infty(H)$ then $[\chi] \in \Sigma^\infty(G)$.

We use this to show that whenever $\chi : F \to \mathbb{R}$ is such that $\chi(x_1) < 0$ we always have $[\chi] \in \Sigma^\infty(F)$. Recall that $F$ is an HNN extension with base group $F(1) = \langle x_1, x_2, \ldots \rangle$, associated subgroups $F(1)$ and $F(2)$ and with stable letter $x_0$, where $F(i) = \langle x_i, x_{i+1}, \ldots \rangle$. As $\{x_i\}_{i \geq 1}$ are conjugate in $F$ we see that $\chi(x_1) = \chi(x_i) < 0$ for all $i \geq 1$. Let $\tilde{\chi}$ be the restriction of $\chi$ to $F(1)$. If we identify $F(1)$ with $F$ via the isomorphism that sends $x_i$ to $x_{i-1}$ for $i \geq 1$, then $\tilde{\chi}$ gets identified with $-\chi_1$ and, by Corollary 2.2 $[-\chi_1] \in \Sigma^\infty(F)$. Thus we have:

Corollary 2.4. (2.1) $\{[\chi] \in S(F) \mid \chi(x_1) < 0\} \subseteq \Sigma^\infty(F)$.

This shows that the open interval in the circle $S(F)$ from $[\chi_0]$ to $[-\chi_0]$ which contains $[-\chi_1]$ lies in $\Sigma^\infty(F)$. By $\nu$-symmetry its image under $\nu$ has the same property, and this enlarges the interval in question to cover the whole “long” open interval between $[\chi_0]$ and $[\chi_1]$. In summary:

Proposition 2.5. All of $S(F)$ except possibly the closed convex sum of the points $[\chi_0]$ and $[\chi_1]$ lies in $\Sigma^\infty(F)$.

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5But note the change of conventions explained in the Remark at the end of Subsection 1.2.
2.3. The “shorter” interval. For the homotopical version of Theorem A we could simply apply the following:

**Theorem 2.6.** [19] Let $G$ be a finitely presented group which has no free non-abelian subgroup. Then $\text{conv}_{\leq 2} \Sigma^2(G)^c \subseteq \Sigma^2(G)^c$.

However, the homological version of Theorem 2.6 is only known under restrictive conditions, so we proceed in a manner which handles the homotopical and homological versions at the same time. We begin by citing:

**Theorem 2.7.** Let $G$ have no non-abelian free subgroups and have type $FP_2(R)$. Let $\tilde{\chi}: G \to \mathbb{R}$ be a non-zero discrete character. Then $G$ decomposes as an ascending HNN extension $H_{*p,t}$ where $H$ is a finitely generated subgroup of $\ker(\tilde{\chi})$, and $\tilde{\chi}(t)$ generates the image of $\tilde{\chi}$.

This is an immediate consequence of [8, Thm. A]. That theorem yields an HNN extension and the hypothesis about free subgroups ensures it is an ascending HNN extension.6

We apply Theorem 2.7 to understand $\Sigma^2(F;R)$. Consider the non-zero character $a\chi_0 + b\chi_1$ where $a,b \in \mathbb{Q}$. Let $G := \ker(a\chi_0 + b\chi_1)$. Since $F/F'$ is a free abelian group of rank 2, it is not hard to see that $G = \langle F', t \rangle$ for some $t \in F$. For the same reason, there is a non-zero discrete character $\tilde{\chi}: G \to \mathbb{R}$ whose kernel is $F'$ such that $\tilde{\chi}(t)$ generates $\text{im}(\tilde{\chi})$. We assume that $G$ has type $FP_2(R)$ and we consider what this implies. By Theorem 2.7 the existence of $\tilde{\chi}$ implies that $G$ decomposes as $H_{*p,t}$ where $H$ is a finitely generated subgroup of $F'$. The group $F'$ consists of all PL homeomorphisms whose left and right slopes are 1. Since $H$ is finitely generated, there must exist $\epsilon > 0$ such that all elements of $H$ are supported in the interval $[\epsilon, 1 - \epsilon]$. We may assume $\epsilon$ is so small that the PL homeomorphism $t$ is linear on $[0, \epsilon]$ and on $[1 - \epsilon, 1]$.

The character $\tilde{\chi}$ expresses $G$ as a semidirect product of $F'$ and $\mathbb{Z}$. Thus we have $F' = \bigcup_{n \geq 1} t^n H t^{-n}$. So for each $x \in F'$ there is some $n > 0$ such that $t^{-n} x t^n \in H$, and hence the support of $t^{-n} x t^n$ lies in $[\epsilon, 1 - \epsilon]$.

This implies that the support of $x$ lies in $[t^n(\epsilon), t^n(1 - \epsilon)]$, and hence these end points have subsequences converging to 0 and 1 respectively as $x$ varies in $F'$. If $t$ has slope $\geq 1$ on $[0, \epsilon]$ then $t(\epsilon) \geq \epsilon$ so $t^n(\epsilon) \geq \epsilon$ for all $n > 0$. Therefore $t$ must have slope $< 1$ near 0. Similarly $t$ must have slope $< 1$ near 1. Since $a\chi_0(t) + b\chi_1(t) = 0$ it follows that (still assuming $G$ has type $FP_2(R)$) $ab < 0$.

Expressing the contrapositive, we have

**Proposition 2.8.** If $ab > 0$ then $\ker(a\chi_0 + b\chi_1)$ does not have type $FP_2(R)$. □

Now assume $a$ and $b$ are positive and rational. Write $\chi = a\chi_0 + b\chi_1$; thus $\chi$ is discrete. By Corollary 1.2 $\ker(\chi)$ has type $FP_2(R)$ if and only if both $[\chi]$ and $[-\chi]$ lie in $\Sigma^2(F;R)$. But by Proposition 2.5 $[-\chi] \notin \Sigma^2(F;R)$. So $[\chi]$ cannot lie in $\Sigma^2(F;R)$.

**Proposition 2.9.** No point in the open convex sum of $[\chi_0]$ and $[\chi_1]$ (i.e. the shorter open interval) lies in $\Sigma^2(F;R)$.

6See Sec. 1.3 for the definition of $\text{conv}_{\leq 2}$.
7The equivalence of “almost finitely presented” with respect to $R$, the term actually used in [9], and $FP_2(R)$ is well-known: see, for example, Exercise 3 of [14] VIII 5.
Proof. We have just shown that a dense subset of the open convex sum lies in $$\Sigma^2(F; R)^c$$, and since $$\Sigma^2(F; R)$$ is open in $$S(F)$$ this is enough. □

The proof of Theorem A is completed by recalling that for any ring $$R$$

1. $$\Sigma^1(F; R) = \Sigma^1(F)$$, and
2. $$\Sigma^m(F) \subseteq \Sigma^m(F; R)$$.

3. Subgroups of $$F$$ with different finiteness properties

As before, we denote the complement of any subset $$A$$ of a sphere by $$A^c$$. The Direct Product Formula for homological Sigma invariants (which is not always true) reads as follows:

$$\Sigma^n(G \times H; R)^c = \bigcup_{p=0}^n \Sigma^p(G; R)^c \ast \Sigma^{n-p}(H; R)^c$$

Here, * refers to “join” of subsets of the spheres $$S(G)$$ and $$S(H)$$ which are considered to be subspheres of the sphere $$S(G \times H)$$. In particular, when $$p = 0$$ or $$n$$ one of these sets is empty, and then the join is treated in the usual way: e.g., $$A \ast \emptyset = A$$.

It has been known for many years that one inclusion of the Direct Product Formula is always true:

**Theorem 3.1.** (Meinert’s Inequality)

$$\Sigma^n(G \times H; R)^c \subseteq \bigcup_{p=0}^n \Sigma^p(G; R)^c \ast \Sigma^{n-p}(H; R)^c$$

and

$$\Sigma^n(G \times H)^c \subseteq \bigcup_{p=0}^n \Sigma^p(G)^c \ast \Sigma^{n-p}(H)^c$$

Meinert did not publish this, but a proof can be found in [16, Section 9]. The paper [11] also contains a proof of the homotopy version.

It is proved in [5] that the Direct Product Formula holds when $$R$$ is a field. On the other hand, an example in [29] shows that the Formula does not always hold when $$R = \mathbb{Z}$$. However, it is shown in [5] that when $$\Sigma^n(G; \mathbb{Z}) = \Sigma^n(G; \mathbb{Q})$$ for all $$n$$ then the Direct Product Formula does hold when $$R = \mathbb{Z}$$. Writing $$F^r$$ for the $$r$$-fold direct product of copies of $$F$$, one concludes (by induction on $$r$$) that the Formula holds for $$F^r$$ when $$R = \mathbb{Z}$$. More precisely, we have:

**Theorem 3.2.** Let $$r \geq 2$$. Then for all $$n$$

$$\Sigma^n(F^r; \mathbb{Z})^c = \bigcup_{p=0}^n \Sigma^p(F; \mathbb{Z})^c \ast \Sigma^{n-p}(F^{r-1}; \mathbb{Z})^c$$

and $$\Sigma^n(F^r) = \Sigma^n(F^r; \mathbb{Z})$$.

Proof. Only the last sentence requires some explanation. It follows from Meinert’s Inequality (Theorem 3.1) together with the fact that for any group $$G$$ we have $$\Sigma^m(G) \subseteq \Sigma^m(G; R)$$. □
Theorem A implies that $\Sigma^m(F)^c$ is a (spherical) 1-simplex if $m \geq 2$, is the 0-skeleton of that 1-simplex when $m = 1$, and is empty (i.e., the (-1)-skeleton of the 1-simplex) when $m = 0$. And that 1-simplex has the property that it is disjoint from its negative. It follows from Theorem 3.2 that $\Sigma^m(F)^c$ is the $(m-1)$-skeleton of a spherical $(2r-1)$-simplex in the $(2r-1)$-sphere $S(F^r)$, a simplex which is disjoint from its negative.

We now prove Theorem B. Consider $[\chi]$ in $S(F^r)$ which lies in the $(m-1)$-skeleton but not in the $(m-2)$-skeleton of the $(2r-1)$-simplex. Since the discrete characters are dense we can always choose $\chi$ discrete. Then $[\chi]$ lies in $\Sigma^m(F)^c \cap \Sigma^{m-1}(F^r)$ while $[-\chi]$ lies in $\Sigma^m(F^r)$. Thus, by Corollary 1.2 the kernel of $\chi$ has type $F_{m-1}$ but not type $FP_m(\mathbb{Z})$ when $m < 2r - 1$. Now, $F$ contains copies of $F^r$ for all $r$; for example, let $0 < t_1 < \cdots < t_r < 1$ be a subdivision of $[0,1]$ into $r$ segments where the subdivision points are dyadic rationals. The subgroup of $F$ which fixes all the points $t_i$ is a copy of $F^r$. Thus Theorem B is proved.

**Example:** Here is an explicitly described subgroup $G_r \leq F$ which has type $F_{2r-1}$ but does not type $FP_{2r}(\mathbb{Z})$. Fix a dyadic subdivision of $[0,1]$ into $r$ subintervals as above. Let $G_r$ denote the subgroup of $F$ consisting of all elements $x$ for which the product of the numbers in the following set $D_r$ equals 1. The members of $D_r$ are: the left and right derivatives of $x$ at the $(r-1)$ subdivision points $t_i$, the right derivative of $x$ at 0, and the left derivative of $x$ at 1. This subgroup of $F$ (we consider $F^r$ embedded in $F$ as above) corresponds to the barycenter of the $(2r-1)$-simplex, and thus has the claimed properties.

**Remark 3.3.** This example is “structurally stable” in the following sense: The interior of the $(2r-1)$-simplex is open in the sphere $S(F^r)$. Thus all the points in that open set which correspond to discrete characters on $F^r$ (they are dense) give rise to groups $\hat{G}_r$ with exactly the finiteness properties possessed by $G_r$. These groups $\hat{G}_r$ should be thought of as all the normal subgroups of $F^r$ “near” $G_r$ which have infinite cyclic quotients.

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