ON THE ALGEBRAIC $K$-THEORY OF TRUNCATED POLYNOMIAL ALGEBRAS IN SEVERAL VARIABLES

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ABSTRACT. We consider the algebraic $K$-theory of a truncated polynomial algebra in several commuting variables, $K(k[x_1, \ldots, x_n]/(x_1^{a_1}, \ldots, x_n^{a_n}))$. This naturally leads to a new generalization of the big Witt vectors. If $k$ is a perfect field of positive characteristic we describe the $K$-theory computation in terms of a cube of these Witt vectors on $\mathbb{N}^n$. If the characteristic of $k$ does not divide any of the $a_i$ we compute the $K$-groups explicitly. We also compute the $K$-groups modulo torsion for $k = \mathbb{Z}$.

To understand this $K$-theory spectrum we use the cyclotomic trace map to topological cyclic homology, and write $\text{TC}(k[x_1, \ldots, x_n]/(x_1^{a_1}, \ldots, x_n^{a_n}))$ as the iterated homotopy cofiber of an $n$-cube of spectra, each of which is easier to understand.

Key Words: Algebraic $K$-theory, trace map, truncated polynomial algebra.

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1. INTRODUCTION

About 15 years ago, Hesselholt and Madsen [10] computed the relative algebraic $K$-theory groups of $k[x]/(x^n)$ for a perfect field $k$ of positive characteristic. In this paper we extend their result to a truncated polynomial ring in multiple commuting variables. We study

$$K(k[x_1, \ldots, x_n]/(x_1^{a_1}, \ldots, x_n^{a_n}), (x_1), \ldots, (x_n)),$$

the appropriate multi-relative version of $K(k[x_1, \ldots, x_n]/(x_1^{a_1}, \ldots, x_n^{a_n}))$. For ease of exposition, let $A = k[x_1, \ldots, x_n]/(x_1^{a_1}, \ldots, x_n^{a_n})$, and let $\tilde{K}(A)$ denote the multi-relative version of $K(A)$.

Hesselholt and Madsen expressed $K_*(k[x]/(x^n), (x))$ in terms of big Witt vectors. Recall that given a truncation set $S \subset \mathbb{N}$, one can define the Witt ring $\mathbb{W}_S(k)$. With $S = \{1, 2, \ldots, m\}$ this gives the length $m$ big Witt vectors, and they proved that for a perfect field $k$ of positive characteristic

$$K_{2q-1}(k[x]/(x^n), (x)) \cong \mathbb{W}_{aq}(k)/\mathbb{W}_q(\mathbb{W}_q(k))$$

and

$$K_{2q}(k[x]/(x^n), (x)) = 0.$$
Here $V_a$ is the $a^{th}$ Verschiebung, one of the structure maps between Witt vectors. When $k = \mathbb{F}_p$, it is not much harder to write down the answer explicitly without referring to Witt vectors. However, this forces one to consider the cases $p | a$ and $p \nmid a$ separately, and the answer looks somewhat less elegant.

To express the answer in the $n$-variable case, we again take advantage of Witt vectors to organize our calculation. Our computations lead us naturally to an $n$-dimensional version of the big Witt vector construction. We will define these Witt vectors on $\mathbb{N}^n$ carefully in the next section. As far as we have been able to determine this construction is new; it is not equivalent to the definition given by Dress and Siebeneicher [3].

We say a set $S \subset \mathbb{N}^n$ is a truncation set in $\mathbb{N}^n$ if $(ds_1, \ldots, ds_n) \in S$ implies $(s_1, \ldots, s_n) \in S$, for $d \in \mathbb{N}$. Given such an $S$, we define the Witt vectors on $\mathbb{N}^n$, $W_S(k)$, in a way that generalizes the construction for $n = 1$. We then prove the following theorems computing the homotopy groups of the multi-relative $K$-theory.

Let $S_q(I) \subset \mathbb{N}^n$ be the truncation set in $\mathbb{N}^n$ given by

\begin{equation}
S_q(I) = \{ (s_1, \ldots, s_n) \in \mathbb{N}^n \mid \sum_{i=1}^{n} \frac{s_i - 1}{a \chi_I(i)} \leq q - 1 \}.
\end{equation}

Here $\chi_I$ is the characteristic function of $I$, evaluating to 1 if the argument is in $I$ and 0 otherwise.

**Theorem 1.2.** Suppose $k$ is a perfect field of positive characteristic and let

$$\hat{E}_1^{s,t} = \begin{cases} \bigoplus_{|I|=s} W_S_q(I)(k) & t = 2q - 1 \\ 0 & t = 2q, \end{cases}$$

Then there is a spectral sequence with

$$E_1^{s,*} \cong \hat{E}_1^{s,*} \otimes E(x_1, \ldots, x_{n-1}),$$

where $E(x_1, \ldots, x_{n-1})$ is an exterior algebra over $\mathbb{Z}$ on 1-dimensional classes. This spectral sequence collapses at $E_\infty$ and converges to $\tilde{K}_{t-s}(A)$.

Moreover, if $|k|$ is finite, then

$$|\hat{E}_1^{s,2q-1}| = \sum_{|I|=s} |k|^{n+q-1} \prod_{i \in I} a_i.$$

In Hesselholt and Madsen’s computation, there was an analogous spectra sequence which automatically collapsed at $E_2$, the unique differential given by the Verschiebung

$$V_a : W_q(k) \to W_{aq}(k).$$

For a perfect field of positive characteristic, the Verschiebung is injective, and this allows a simple determination of the relative algebraic $K$-groups as a quotient of the Witt vectors by the image of the Verschiebung. In the $n$-dimensional case the Verschiebungs do not capture the whole story. Instead we can say the following.

**Theorem 1.3.** Suppose $k$ is a perfect field of characteristic $p > 0$ and suppose $p$ does not divide any of the $a_i$. Then the $d_1$-differentials in the spectral sequence from Theorem 1.2 are split injective.

In this case, we can write the relative algebraic $K$-groups of $A$ as

$$\tilde{K}_*(A) \cong \hat{K}_*(A) \otimes E(x_1, \ldots, x_{n-1}),$$
where as above $E(x_1, \ldots, x_{n-1})$ is the exterior algebra over $\mathbb{Z}$ on 1-dimensional classes, and where $\hat{K}$ is concentrated in odd degrees with

$$
\hat{K}_{2q-1}(A) \cong \mathbb{W}_{S_q(\{1,\ldots,n\})}(k) / \left( \sum_{i=1}^{n} V_{a_i} \left( \mathbb{W}_{S_q(\{1,\ldots,i,\ldots,n\})}(k) \right) \right)
$$

(here $\hat{i}$ denotes skipping $i$).

If in addition $k$ is finite then

$$|\hat{K}_{2q-1}(A)| = |k|^{\left( n+q-1 \right)} \prod_{i=1}^{n} (a_i - 1).$$

This completely determines the algebraic $K$-theory groups in this case. We can be even more explicit. Given a truncation set $S$ in $\mathbb{N}^n$ and $(s_1, \ldots, s_n) \in S$ we can extract a 1-dimensional “$p$-typical” truncation set $p^{s_0} (s_1, \ldots, s_n) \cap S$ consisting of those $p$-power multiples of $(s_1, \ldots, s_n)$ which lie in $S$. Then the group $\hat{K}_{2q-1}(A)$ in Theorem 1.3 above is isomorphic to

$$\bigoplus_{p} \mathbb{W}_{p^{s_0} (s_1, \ldots, s_n) \cap S_q(\{1,\ldots,n\})}(k; p),$$

where the sum is over all $s_1, \ldots, s_n \geq 1$ with $p \nmid \gcd(s_i)$ and $a_i \nmid s_i$.

We also compute the ranks of the multi-relative $K$-theory groups of truncated polynomials over $\mathbb{Z}$. We prove the following theorem:

**Theorem 1.4.** The Poincaré series of the rationalization of

$$K_*(\mathbb{Z}[x_1, \ldots, x_n]/(x_1^{a_1} \ldots x_n^{a_n}), (x_1), \ldots, (x_n))$$

is given by

$$\sum_{j \geq 1} t^{2j-1} (1+t)^{n-1} \binom{n+j-2}{n-1} \prod_{i=1}^{n} (a_i - 1).$$

This generalizes results of Soulé [14] and Staffeldt [15] in the single variable case. Note that the difference between $\hat{K}_*(A)$ for $k = \mathbb{Z}$ and $k = \mathbb{Q}$ is torsion, so Theorem 1.4 also gives the Poincaré series of

$$K_*(\mathbb{Q}[x_1, \ldots, x_n]/(x_1^{a_1} \ldots x_n^{a_n}), (x_1), \ldots, (x_n)).$$

1.1. **Organization.** In §2 we give a precise definition of the Witt vectors on $\mathbb{N}^n$. This section might be of independent interest. The later sections provide the proof of Theorems 1.2 and 1.3 computing these multi-relative $K$-groups via trace methods. In §3 we define cyclotomic spectra and give some relevant examples. In particular, we describe a cube in cyclotomic spectra, the homotopy pullback of which receives a map from the multi-relative $K$-theory in question. In §4 we review the cyclotomic trace map and define the multi-relative $K$-theory spectrum, before we prove the main theorems in §5 and §6.

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2. Witt vectors

At the heart of our computation is building an $n$-cube of $S^1$-equivariant spectra, analogous to a cofiber sequence of $S^1$-spectra that Hesselholt and Madsen used in the one variable case. The various vertices of this hypercube are underlain by $n$-fold smash powers of $S^1_+$, and the maps in the diagram arise from the Hesselholt-Madsen maps restricted to one of the factors. This suggests that a useful language will again be Witt vectors, but our Witt vectors will be modeled on truncation sets in $\mathbb{N}^n$, rather than those modeled on $\mathbb{N}$.

2.1. Classical Case. For the classical construction, recall (e.g. from [11]) that given a truncation set $S \subset \mathbb{N} = \{1, 2, \ldots \}$ and a commutative ring $R$ we can define a commutative ring $W_S(R)$. As a set, $W_S(R) = R^S$, and we define addition and multiplication in such a way that the ghost map $w : W_S(R) \to R^S$ that takes a vector $(a_s)_{s \in S}$ to the vector $(w_s)_{s \in S}$ with

$$w_s = \sum_{dt = s} ta_t^d$$

is a ring map. One can show that there is exactly one functorial way to do this.

We will be particularly interested in three families of truncation sets. If our set $S$ is $\{1, 2, \ldots, m\}$, we denote $W_S(R)$ by $W_m(R)$ and call it the big Witt vectors of length $m$. If $S = \{1, p, \ldots, p^{m-1}\}$, we denote $W_S(R)$ by $W_m(R;p)$ and call it the $p$-typical Witt vectors of length $m$. If $R$ is a $\mathbb{Z}(p)$-algebra, there is a splitting

$$W_m(R) \cong \prod_{p \nmid s} W_{\lfloor \log_p(m/s) + 1 \rfloor}(R;p).$$

Finally, for any $m$, there is a truncation set generated by $m$:

$$\langle m \rangle = \{d \in \mathbb{N}, d \mid m\}$$

and the associated Witt vectors $W_{\langle m \rangle}(R)$.

Note that associating Witt vectors to a truncation set is a special case of a more general construction. Consider $\mathbb{N}$ as a set with an action of the multiplicative monoid $\mathbb{N}$. A truncation set is then just a subposet of $\mathbb{N}$ which is closed under divisibility. The $\mathbb{N}$-action endows this with further structure, though: for any two elements $r$ and $s$ such that $r \leq s$, we have a “weight”: $s/r \in \mathbb{N}$. It is possible to build Witt vectors on other weighted posets of this form. For instance, Dress-Siebeneicher’s construction of Witt vectors relative to a profinite group [4] is built on the poset of finite index subgroups of a profinite group. Our new construction described below is built on $\mathbb{N}^n$.

2.2. Witt vectors on $\mathbb{N}^n$. Now we wish to replace $\mathbb{N}$ by $\mathbb{N}^n$. Coordinate multiplication gives an action by $\mathbb{N}$ and hence a weighted poset: $(t_1, \ldots, t_n)$ divides $(s_1, \ldots, s_n)$ if there is some $d \in \mathbb{N}$ such that $(dt_1, \ldots, dt_n) = (s_1, \ldots, s_n)$, and the number $d$ is the weight. We say that $S \subset \mathbb{N}^n$ is a truncation set in $\mathbb{N}^n$ if it is a closed subposet: if $(s_1, \ldots, s_n) \in S$ and $(t_1, \ldots, t_n)$ divides $(s_1, \ldots, s_n)$, then $(t_1, \ldots, t_n) \in S$. From now on we will use vector notation for $n$-tuples of natural numbers.
Just as in the 1-dimensional case, we can consider the truncation set generated by a single element:

\[(2.2) \quad \langle \overrightarrow{s} \rangle = \left\{ \overrightarrow{u} \in \mathbb{N}^n, \overrightarrow{u} \mid \overrightarrow{s} \right\}.\]

This is not the only way to associate 1-dimensional truncation sets to elements of \(\mathbb{N}^n\), however. As a weighted poset, \(\mathbb{N}^n\) splits into a disjoint union of countably infinitely many copies of \(\mathbb{N}\), indexed by those sequences \(\overrightarrow{s}\) with \(\gcd(\overrightarrow{s}) = 1\):

\[\mathbb{N}^n = \bigsqcup_{\gcd(\overrightarrow{s})=1} \mathbb{N} \cdot \overrightarrow{s}.\]

Thus our truncation sets in \(\mathbb{N}^n\) are simply disjoint unions of ordinary truncation sets. Given \(\overrightarrow{s} \in S\), we use the notation \(\mathbb{N} \overrightarrow{s} \cap S\) to denote all multiples of \(\overrightarrow{s}\) in \(S\). This can be thought of as a 1-dimensional truncation set by identifying \(\overrightarrow{s} \in S\) with 1 \(\in \mathbb{N}\).

We can generalize the classical construction of Witt vectors on \(\mathbb{N}\) to Witt vectors on \(\mathbb{N}^n\), and using the above decomposition of \(\mathbb{N}^n\) we can understand exactly what our generalization produces: Given a truncation set \(S\) in \(\mathbb{N}^n\), we construct the Witt vectors \(\mathbb{W}_S(R)\). As a set, \(\mathbb{W}_S(R) = R^S\). We define a ghost map

\[w : \mathbb{W}_S(R) \to R^S\]

as the map that takes a vector \((a_{\overrightarrow{s}})_{\overrightarrow{s} \in S}\) to the vector \((w_{\overrightarrow{s}})_{\overrightarrow{s} \in S}\) with

\[w_{\overrightarrow{s}} = \sum_{d \overrightarrow{u} = \overrightarrow{s}} \gcd(\overrightarrow{u})(a_{\overrightarrow{u}})^d.\]

**Lemma 2.3.** There is a unique functorial way to put a ring structure on \(\mathbb{W}_S(R)\) in such a way that the ghost map is a ring map.

Moreover, there is a canonical splitting

\[\mathbb{W}_S(R) \cong \prod_{\gcd(\overrightarrow{s})=1} \mathbb{W}_{\mathbb{N} \overrightarrow{s} \cap S}(R).\]

**Proof.** Since our truncation set splits as a disjoint union of classical truncation sets, this lemma is simply a restatement of two classical facts:

1. There is a unique functorial ring structure for a classical truncation set, and
2. the functor \(S \mapsto R^S\) takes disjoint unions to Cartesian products.

The classical Witt construction had a great many structure maps linking the Witt vectors for various \(S\). Here we have all of the classical ones and more. The real power of this construction is that we have various structure maps which mix the disjoint factors in \(\mathbb{N}^n\).

The restriction map is the easiest to define. Given \(S' \subset S\) we get a restriction map

\[(2.4) \quad R^S_{S'} : \mathbb{W}_S(R) \to \mathbb{W}_{S'}(R)\]

by taking the obvious projection \(R^S \to R^{S'}\). Since a truncation set contains all the divisors of any of its elements, this projection commutes with applying the ghost maps, so it is in fact a ring homomorphism.
We have Frobenius maps in each of the \( n \) directions. For each \( i \in \{1, \ldots, n\} \) and \( r \geq 2 \), we have a truncation set
\[
S/(1, \ldots, r, \ldots, 1) = \{(t_1, \ldots, t_n), (t_1, \ldots, rt_i, \ldots, t_n) \in S\}.
\]
We can define a Frobenius map
\[
F^i_r : \mathcal{W}_S(R) \to \mathcal{W}_{S/(1, \ldots, r, \ldots, 1)}(R)
\]
by requiring that the diagram
\[
\begin{array}{ccc}
\mathcal{W}_S(R) & \xrightarrow{w} & R^S \\
\downarrow F^i_r & & \downarrow (F^i_r)^w \\
\mathcal{W}_{S/(1, \ldots, r, \ldots, 1)}(R) & \xrightarrow{w} & R^{S/(1, \ldots, r, \ldots, 1)}
\end{array}
\]
commutes. Here \((F^i_r)^w\) is defined by
\[
(F^i_r)^w(x_{t_1, \ldots, t_n}) = x_{(t_1, \ldots, rt_i, \ldots, t_n)}.
\]
We note that \( F^i_r \) and \( F^j_s \) commute if \( i \neq j \).

Associated to these Frobenius maps, we also have Verschiebung maps in each of the \( n \) directions. We can define a Verschiebung map
\[
V^i_r : \mathcal{W}_{S/(1, \ldots, r, \ldots, 1)}(R) \to \mathcal{W}_S(R)
\]
by
\[
V^i_r((x_{t_1, \ldots, t_n})) = \begin{cases} 
    x_{(t_1, \ldots, t_i/r, \ldots, t_n)} & \text{if } t_i/r \text{ is an integer} \\
    0 & \text{otherwise}
\end{cases}
\]
Once again we note that \( V^i_r \) and \( V^j_s \) commute if \( i \neq j \).

We need some basic computations with these Witt vectors on \( \mathbb{N}^n \). For the next three propositions, fix a truncation set \( S \), let \( \overrightarrow{s} \in S \) have \( \gcd(\overrightarrow{s}) = 1 \), and let \( S' = \mathbb{N} \cdot \overrightarrow{s} \cap S \). The truncation set \( S' \) can be considered as a truncation set in \( \mathbb{N}^n \) or as a classical one, via the identification of \( \mathbb{N} \cdot \overrightarrow{s} \) with \( \mathbb{N} \). The ghost maps and ring structure we get on \( \mathcal{W}_{S'}(R) \) are the same in either case, so we will not distinguish between the two. We can use the canonical splitting of \( \mathbb{N}^n \) into its disjoint factors to study the Frobenii and Verschiebungs on individual factors, each of which can be thought of as classical Witt vectors:

**Proposition 2.5.** The map \( V^i_r \) splits as a Cartesian product of maps
\[
V^i_r : \mathcal{W}_{S'/(1, \ldots, r, \ldots, 1)}(R) \to \mathcal{W}_{S'}(R),
\]
and similarly for \( F^i_r \).

This lets us focus attention on the simple factors. Fix an \( i \), and let \( d_i = \gcd(s_i, r) \) and let \( e_i = r/d_i \).

**Proposition 2.6.** We have an isomorphism between \( S'/(1, \ldots, r, \ldots, 1) \) considered as a truncation set in \( \mathbb{N}^n \) and \( S'/e_i \) considered as a 1-dimensional truncation set.

**Proof.** In the decomposition of \( \mathbb{N}^n \) into multiples of vectors with coordinates having a greatest common divisor of 1, \( S'/(1, \ldots, r, \ldots, 1) \) consists of multiples of \( r/d_i \cdot (s_1, \ldots, s_i/r, \ldots, s_n) \), where the multiplier ensures that the \( i \)’th coordinate is an integer. So when viewed as a 1-dimensional truncation set, \( S'/(1, \ldots, r, \ldots, 1) \) contains exactly those \( a \)’s for which \( ae_i(s_1, \ldots, s_n) \in S' \). \( \square \)
Since we are expressing our truncation set in terms of classical truncation sets, the classical results tell us the value of the Frobenius and Verschiebung.

**Proposition 2.7.** If we identify $\mathbb{W}_{S'/e_i'}(R)$ with $\mathbb{W}_{S'/e_i}(R)$, then $V^i_e$ is given by

$$\mathbb{W}_{S'/e_i}(R) \xrightarrow{V^i_e} \mathbb{W}_{S'}(R),$$

where $V^i_e$ is the classical Verschiebung.

Similarly, the Frobenius map $F^i_e$ is given by

$$\mathbb{W}_{S'}(R) \xrightarrow{F^i_e} \mathbb{W}_{S'/e_i}(R).$$

In particular, this means that the composite $F^i_e V^i_e$ is multiplication by $e_i$ on $\mathbb{W}_{S'/e_i'}(R)$.

**Proof.** Let $\overline{x} = (x_1)_{T \in S'/\{1,\ldots,r,\ldots,1\}}$ be an element of $\mathbb{W}_{S'/e_i}(R)$ (where $S'_{e_i}$ is considered as a one-dimensional truncation set) then the $a$'th coordinate of $V^i_e(\overline{x})$ in $\mathbb{W}_{S'}$ is the $a/e_i$'th coordinate of $\overline{x} \in \mathbb{W}_{S'/e_i}(R)$. Thus the map $V^i_e$ agrees with the classical Verschiebung $V^i_{e_i}$ under the identifications from Proposition 2.6.

In the case of the Frobenius, the ghost maps are the same as in the classical one-dimensional case, and the ghost version of $F^i_e$ is just the classical ghost version of $F^i_{e_i}$. Thus $F^i_e$ itself must be equal to the classical $F^i_{e_i}$.\hfill $\Box$

### 3. CYCLOTOMIC SPECTRA

Certain $S^1$-spectra, called **cyclotomic spectra**, are particularly important to computations of algebraic $K$-theory. We first recall the definition of a cyclotomic spectrum [11]. Let $G$ be a compact Lie group. Given a genuine $G$-spectrum $X$ indexed on a complete $G$-universe $U$, and a normal subgroup $H \triangleleft G$, there are two notions of $H$-fixed points [12]. Both notions of fixed points yield a $G/H$-(pre)spectrum. The first, denoted $X^H$, has $V$th space

$$X^H(V) = X(V)^H$$

for each $V \subset U^H$. The second notion, that of geometric fixed points, is defined as follows. Let $\mathcal{F}_H$ denote the family of subgroups of $G$ not containing $H$. Let $E\mathcal{F}_H$ denote the universal space of this family, and let $E\overline{\mathcal{F}}_H$ denote the cofiber of the map $E\mathcal{F}_H + \rightarrow S^0$ given by projection onto the non-basepoint. Then the geometric fixed point spectrum $X^{gH}$ is defined as

$$X^{gH} = (X \land E\overline{\mathcal{F}}_H)^H.$$
Thus there is always a map $X^H \to X^{gH}$ from the $H$-fixed points of $X$ to the $H$-geometric fixed points of $X$. If $X$ is a structured ring spectrum, this is a map of structured ring spectra.

We are now ready to recall the definition of cyclotomic spectra. Let $C_n \subset S^1$ denote the cyclic subgroup of order $n$, and let $\rho_n : S^1 \to S^1/C_n$ denote the isomorphism given by the $n$th root. Using this isomorphism, an $S^1/C_n$-spectrum $E$ indexed on $U^{C_n}$ determines an $S^1$-spectrum indexed on $U$. We write this spectrum as $\rho_n^*E$.

**Definition 3.1.** A cyclotomic spectrum is a genuine $S^1$-equivariant spectrum $X$ together with equivalences of $S^1$-spectra

$$r_n : \rho_n^*(X^{gC_n}) \cong X$$

for all $C_n \subset S^1$, such that for any $m, n > 0$ the following diagram commutes

\[
\begin{array}{ccc}
\rho_n^*(X^{gC_n}) & \xrightarrow{r_n} & X \\
\downarrow & & \downarrow & & \downarrow \\
\rho_n^*(X^{gC_m}) & \xrightarrow{r_m} & X
\end{array}
\]

We will also need the notion of a cyclotomic map between cyclotomic spectra.

**Definition 3.2.** A cyclotomic map $f : X \to Y$ between cyclotomic spectra is a map of genuine $S^1$-equivariant spectra such that the diagram

\[
\begin{array}{ccc}
\rho_n^*(X^{gC_n}) & \xrightarrow{r_n} & X \\
\downarrow & & \downarrow & & \downarrow \\
\rho_n^*(Y^{gC_n}) & \xrightarrow{r_n} & Y
\end{array}
\]

commutes for all $n$.

It is clear that cyclotomic spectra together with cyclotomic maps form a category. A cyclotomic spectrum can be built by taking the suspension spectrum of a cyclotomic space, as in Example 3.7 below, but cyclotomic spectra also arise in other ways.

**Example 3.3.** For a ring $A$, the topological Hochschild homology $S^1$-spectrum $\text{THH}(A)$ is a cyclotomic spectrum [11].

The importance of cyclotomic spectra stems from the fact that a cyclotomic spectrum $X$ comes with the structure required to define the topological cyclic homology of $X$, $\text{TC}(X)$, following [13]. This is done as follows. For any $S^1$-spectrum $X$, there are maps $F_n : X^{C_{mn}} \to X^{C_m}$ for all $n, m \geq 1$, given by inclusion of fixed points. These maps are called Frobenius maps. If $X$ is cyclotomic, there is a second class of maps: $R_n : X^{C_{mn}} \to X^{C_m}$, for all $n, m \geq 1$. These maps, called restriction maps, are given as composites

$$X^{C_{mn}} = (\rho_n^*(X^{C_n}))^{C_m} \to (\rho_n^*(X^{gC_n}))^{C_m} \xrightarrow{r_m} X^{C_m}$$

where the first map is the map from fixed points to geometric fixed points described earlier in the section. The topological cyclic homology of $X$ is then defined as

$$\text{TC}(X) = \text{holim}_{R,F} X^{C_n}.$$
As discussed in Example 3.3, for a ring $A$, the $S^1$-spectrum $\text{THH}(A)$ is cyclotomic. Hence we can take the topological cyclic homology of this spectrum. The resulting spectrum $\text{TC}(\text{THH}(A))$ is usually denoted $\text{TC}(A)$, and we will use this convention as well. This is the topological cyclic homology of the ring $A$. For our application to algebraic $K$-theory, three sources of cyclotomic spectra will be particularly important:

1. Topological Hochschild homology of rings
2. Homotopy fibers of cyclotomic maps
3. Suspension spectra of cyclotomic spaces

The first source of cyclotomic spectra, topological Hochschild homology, has been used extensively in algebraic $K$-theory computations. The consideration of homotopy fibers of cyclotomic maps as cyclotomic spectra is used implicitly in work of Blumberg and Mandell [2]. The third approach can be thought of as building cyclotomic spectra by hand. Below, we describe in detail the latter two approaches to generating cyclotomic spectra.

3.1. Cyclotomic spectra arising as homotopy fibers. For a cyclotomic map $f : X \to Y$, it follows immediately that the induced maps $X^C_n \to Y^C_n$ commute with $R$ and $F$. Hence we get an induced map $\text{TC}(X) \to \text{TC}(Y)$.

Proposition 3.4. Suppose $f : X \to Y$ is a cyclotomic map. Then

1. The homotopy fiber $\text{hofib}(f)$ is cyclotomic.
2. There is a natural equivalence $\text{TC}(\text{hofib}(f)) \simeq \text{hofib}(\text{TC}(X) \to \text{TC}(Y))$

Proof. The homotopy fiber is taken in the category of $S^1$-spectra, so it is an $S^1$-spectrum. Taking fixed points or geometric fixed points preserves homotopy (co)fiber sequences, so we get a diagram

$$
\begin{array}{ccc}
\rho_n^* (\text{hofib}(f)^C_n) & \longrightarrow & \rho_n^* (X^C_n) \\
\downarrow & & \downarrow \simeq \\
\text{hofib}(f) & \longrightarrow & X \\
\rho_n^* (Y^C_n) & \longrightarrow & \downarrow \simeq \\
& & \text{hofib}(X) \longrightarrow Y
\end{array}
$$

where the rows are homotopy (co)fiber sequences. It follows that we have an equivalence $\rho_n^* (\text{hofib}(f)^C_n) \to \text{hofib}(f)$ of $S^1$-spectra. For the second claim, note that both $\text{TC}$ and $\text{hofib}$ are homotopy limits, so they commute. Hence we have a chain of equivalences

$$
\text{TC}(\text{hofib}(f)) = \text{holim}_{R,F} \text{hofib}(X \to Y)^C_n \simeq \text{holim}_{R,F} \text{hofib}(X^C_n \to Y^C_n) \simeq \text{hofib}(\text{holim}_{R,F} X^C_n \to \text{holim}_{R,F} Y^C_n) = \text{hofib}(\text{TC}(X) \to \text{TC}(Y)).
$$

Example 3.5. Suppose we have a map $f : A \to B$ of rings. Then the induced map $\text{THH}(A) \to \text{THH}(B)$ is cyclotomic. The homotopy fiber is sometimes denoted $\text{THH}(f)$. If our map is the quotient $A \to A/J$, then the homotopy fiber is usually denoted $\text{THH}(A,J)$. It follows that the two possible definitions of $\text{TC}(f)$, as $\text{TC}(\text{THH}(f))$ or as $\text{hofib}(\text{TC}(A) \to \text{TC}(B))$, agree.

More generally, if $J_1, \ldots, J_n$ are ideals in $A$ we define the multi-relative cyclotomic spectrum $\text{THH}(A, J_1, \ldots, J_n)$ as the homotopy pullback, or iterated homotopy fiber, of the $n$-cube which in position $I$ for $I \subset \{1, 2, \ldots, n\}$ is $\text{THH}(A/J_I)$, where $J_I$ is the sum of all the $J_i$ with $i \in I$. It follows that the two possible definitions of
TC(A, J1, . . . , Jn) agree. When the ideals are clear from the context we will denote these multi-relative spectra by $\overline{THH}(A)$ and $\overline{TC}(A)$.

3.2. Suspension spectra of cyclotomic spaces. The category of cyclotomic spectra is tensored over a corresponding category of cyclotomic spaces.

Definition 3.6. A cyclotomic space $A$ is an $S^1$-equivariant space together with compatible equivalences

$$r_n : \rho^*_n(A^C_n) \xrightarrow{\cong} A$$

of $S^1$-spaces for all $n$.

A map of cyclotomic spaces is defined in the analogous way to that of a cyclotomic spectrum. The $S^1$-equivariant suspension spectrum of a cyclotomic space is a cyclotomic spectrum. Indeed, $\Sigma^\infty_{S^1}(-)$ is a functor from unbased cyclotomic spaces to cyclotomic spectra, and $\Sigma^\infty_{S^1}(-)$ is a functor from based cyclotomic spaces to cyclotomic spectra. Below we give several important examples of cyclotomic spaces.

Example 3.7. A free loop space $LX$ is a cyclotomic space, where $S^1$ acts by rotation of loops. There is an equivariant homeomorphism

$$r^{-1} : LX \rightarrow \rho^*_n((LX)^{C_n})$$

given by $r(\lambda)(z) = \lambda(z^n)$. The $S^1$-equivariant suspension spectrum of this space, $\Sigma^\infty_{S^1}LX_+$, is a cyclotomic spectrum. If we have a map of spaces $X \rightarrow Y$, then the induced map $LX \rightarrow LY$ is a map of cyclotomic spaces, and by taking suspension spectra we get a map $\Sigma^\infty_{S^1}LX_+ \rightarrow \Sigma^\infty_{S^1}LY_+$ of cyclotomic spectra.

Example 3.8. If $\Pi$ is a pointed monoid, the cyclic bar construction $B^{cy}(\Pi)$ is a based cyclotomic space. This is relevant to our computation because of the splitting:

$$THH(k(\Pi)) \simeq THH(k) \wedge B^{cy}(\Pi).$$

of $[\Pi]$. If we can describe how $B^{cy}(\Pi)$ is built out of cyclotomic spaces then we get a corresponding description of $THH(k(\Pi))$, which can then be used to calculate $TC(k(\Pi))$.

Example 3.9. Suppose $\{A(s)\}_{s \geq 1}$ is a family of $S^1$-equivariant based spaces with compatible homeomorphisms

$$\rho^*_n(A(s)^{C_n}) \cong \begin{cases} A(s/n) & \text{if } n \mid s \\ \ast & \text{if } n \nmid s \end{cases}$$

Then

$$A = \bigvee_{s \geq 1} A(s)$$

is a based cyclotomic space.

Example 3.10. As a concrete example of a family as described in Example 3.9, suppose we have real $S^1$-representations $\lambda(s)$ for $s \geq 1$ with the property that $\rho^*_n(\lambda(s)^{C_n}) \cong \lambda(s/n)$ whenever $n \mid s$. Then $\{S^{\lambda(s)}\}_{s \geq 1}$ is such a family. Further, if we let $A(s) = S^1/C_{s+} \wedge S^{\lambda(s)}$, where $S^1$ acts on the first coordinate by left-multiplication and on the second by the obvious action (fixing the point at infinity) then $\{A(s)\}_{s \geq 1}$ is also such a family. Therefore

$$A = \bigvee_{s \geq 1} S^1/C_{s+} \wedge S^{\lambda(s)}$$
is a based cyclotomic space.

4. Algebraic K-theory Calculations

For a ring $A$, there is a map relating the algebraic K-theory of $A$ and the topological cyclic homology of $A$:

$$\text{trc} : K(A) \to TC(A).$$

This map, called the cyclotomic trace, is due to Bökstedt, Hsiang, and Madsen [3]. This map is often close to an equivalence [13, 11, 7]. Indeed, in the case of $A = k[x_1, \ldots, x_n]/(x_1^{a_1}, \ldots, x_n^{a_n})$, the cyclotomic trace map on the multi-relative K-theory group $\tilde{K}(A)$:

$$\text{trc} : \tilde{K}(A) \to \tilde{TC}(A)$$

is an equivalence. Here the multi-relative K-theory spectrum $\tilde{K}(A)$ is defined as a homotopy pullback in the same way as the multi-relative spectrum $\tilde{TC}(A)$. It follows from [5] that this map is an equivalence after $p$-completion. The proof that this can be extended to an integral statement can be found in [6].

To compute the multi-relative K-theory, we will compute the multi-relative topological cyclic homology. As noted above, $TC(A) = TC(\text{THH}(A))$, so the first step is understanding the topological Hochschild homology of $A$. For any pointed monoid algebra $k(\Pi)$ there is splitting [11]:

$$\text{THH}(k(\Pi)) \simeq \text{THH}(k) \wedge B^{cy}(\Pi).$$

Note that in the single variable case, $k[x]/(x^a)$ is a pointed monoid algebra $k(\Pi_a)$ where $\Pi_a$ is the monoid $\Pi_a = \{0, 1, x, \ldots, x^{a-1}\}, x^a = 0$, so we have the splitting

$$\text{THH}(k[x]/(x^a)) \simeq \text{THH}(k) \wedge B^{cy}(\Pi_a).$$

To proceed with the K-theory computation, one needs to compute the fixed points of topological Hochschild homology. The $S^1$-equivariant homotopy type of $B^{cy}(\Pi)$ for a pointed monoid $\Pi$ has only been described in a small number of cases: [9] for the pointed monoids $\{0, 1, z, z^2, \ldots\}$ and $\{0, 1, x, x^2, \ldots, y, y^2, \ldots\}$ (where in the latter case, $xy = yx = 0$), and [14] for the pointed monoid $\Pi_a$. Let $\tilde{B}^{cy}(\Pi_a)$ denote the part of the cyclic bar construction involving at least one positive power of $x$ in one of the coordinates. Then

$$\text{THH}(k) \wedge \tilde{B}^{cy}(\Pi_a) \simeq \text{THH}(k[x]/(x^a), (x)) = \text{hofib}(\text{THH}(k[x]/(x^a)) \to \text{THH}(k))$$

is the relative topological Hochschild homology, which we will use to calculate the relative algebraic K-theory.

Hesselholt and Madsen wrote $\tilde{B}^{cy}(\Pi_a)$ as the homotopy cofiber of a map of $S^1$-spaces:

$$\bigvee_{s \geq 1} S^1/C_s \wedge S^{\lambda(s-1)} \to \bigvee_{s \geq 1} S^1/C_s \wedge S^{\lambda(1 + \frac{s-1}{s-1})}.$$  \hspace{1cm} (4.1)

Here $\lambda(s)$ denotes the $S^1$-representation $\mathbb{C}(1) \oplus \ldots \oplus \mathbb{C}(s)$, where $\mathbb{C}(i)$ is $\mathbb{C}$ with $S^1$ acting by $z \cdot w = z^i w$. The wedge summand indexed by $s$ maps by a multiplication by a map to the wedge summand indexed by $as$. This can be found as Theorem B in [10], or Theorem 5 in [3].

Lemma 4.2. The map in Equation (4.1) is a map of cyclotomic spaces.
Proof. Following Example 3.10, both of the spaces in Equation 4.1 are cyclotomic spaces. If \( r \mid s \), we have the following commutative diagram:

\[
\begin{array}{ccc}
\rho^*_r \left( \left( S^1 / C_s + S^{\lambda(s-1)} \right)^{C_r} \right) & \xrightarrow{a} & \rho^*_r \left( \left( S^1 / C_{as} + S^{\lambda(s-1)} \right)^{C_r} \right) \\
\approx & & \approx \\
S^1 / C_{s/r} + S^{\lambda(s/r-1)} & \xrightarrow{a} & S^1 / C_{as/r} + S^{\lambda(s/r-1)}. \\
\end{array}
\]

So the map in Equation 4.1 preserves the cyclotomic structure. \( \square \)

To proceed with the calculation, we establish the following notation: Let

\[
X = \bigvee_{s \geq 1} S^1 / C_s + S^{\lambda(s-1)}
\]

and let

\[
X(I) = \bigvee_{s \geq 1} S^1 / C_s + S^{\lambda\left(\frac{s}{|I|}\right)}
\]

so that we have a homotopy cofiber sequence

\[
X \longrightarrow X(I) \longrightarrow \widetilde{B}_{cy}(\Pi_{a_i}).
\]

We can now generalize this approach to studying the multi-relative topological cyclic homology of truncated polynomials in multiple variables. The ring \( A \) is a pointed monoid algebra:

\[
k[x_1, \ldots, x_n]/(x_1^{a_1}, \ldots, x_n^{a_n}) \cong k(\Pi_{a_1} \wedge \ldots \wedge \Pi_{a_n})
\]

where \( \Pi_i \) is defined as above. Thus we can write

\[
\text{THH}(A) \cong \text{THH}(k) \wedge B^{cy}(\Pi_{a_1} \wedge \ldots \wedge \Pi_{a_n}),
\]

and the multi-relative THH-spectrum, \( \widehat{\text{THH}}(A) \) then splits as

\[
\widehat{\text{THH}}(A) \cong \text{THH}(k) \wedge \widetilde{B}^{cy}(\Pi_{a_1} \wedge \ldots \wedge \Pi_{a_n}).
\]

Here \( \widetilde{B}^{cy}(\Pi_{a_1} \wedge \ldots \wedge \Pi_{a_n}) \) denotes the piece of the cyclic bar construction where the total exponent of all of the \( x_i \) combined is required to be positive. We can then use the cofiber description of \( \widetilde{B}^{cy}(\Pi_{a_i}) \) to get a description of the equivariant homotopy type of this multi-relative one.

Given \( I \subset \{1, \ldots, n\} \), let

\[
X_I = \bigwedge_{i=1}^n X_{I \cap \{i\}}.
\]

Then smashing together all of the cofiber sequences used to describe \( B^{cy}(\Pi_{a_i}) \) gives us an \( n \)-cube \( \mathcal{X} = \{X_I\}_{I \subset \{1, \ldots, n\}} \), the maps of which are the obvious maps from the Hesselholt-Madsen 1-variable case. The following is immediate from the defining cofiber sequences.

**Proposition 4.3.** The space \( \widetilde{B}^{cy}(\Pi_{a_1} \wedge \ldots \wedge \Pi_{a_n}) \) is the total homotopy cofiber of the \( n \)-cube \( \mathcal{X} \).
For computations, it is helpful to provide another description of $X_I$. Let $\chi_I$ be the characteristic function of $I$, evaluating to 1 if the argument is in $I$ and 0 otherwise. Then let

$$\lambda_I(\mathcal{I}) = \bigoplus_{i=1}^{n} \chi_i \left( \left[ \frac{s_i - 1}{a_i^{\chi_i(i)}} \right] \right)$$

with $\lambda(s) = \mathbb{C}(1) \oplus \ldots \oplus \mathbb{C}(s)$ as before. Then distributing the smash products over the wedges gives an equivalence of cyclotomic spectra:

$$X_I \simeq \bigvee_{s_1, \ldots, s_n \geq 1} S^1/C_{s_1+} \wedge \ldots \wedge S^1/C_{s_n+} \wedge S^{\lambda_I(\mathcal{I})}.$$  

Smashing the $n$-cube with $T = \text{THH}(k)$ we find that $\widetilde{\text{THH}}(A)$ is the iterated homotopy cofiber of the cube $T \wedge X = \{ T \wedge X_I \}_{I \subseteq \{1, \ldots, n\}}$. Checking the condition of Definition 3.1 directly, the smash product of a cyclotomic spectrum with a cyclotomic space is again cyclotomic. It follows from Example 3.10 and Lemma 4.2 that the spectra $T \wedge X_I$ and maps in the cube $T \wedge X$ are cyclotomic. Proposition 3.3 then gives the following proposition.

**Proposition 4.6.** The multi-relative topological cyclic homology is given as an iterated homotopy cofiber

$$\widetilde{\text{TC}}(A) = \text{hocofib}\left( \{ \text{TC}(T \wedge X_I) \}_{I \subseteq \{1, \ldots, n\}} \right).$$

Applying homotopy groups then gives us a Mayer-Vietoris spectral sequence, and this is the spectral sequence in Theorem 1.2.

**Corollary 4.7.** There is a spectral sequence with $E_1$-page given by

$$E_1^{s,t} = \bigoplus_{|I|=s} \text{TC}_t(T \wedge X_I)$$

and converging to $\text{TC}_{t-s}(A)$. This spectral sequence collapses at $E_n$.

Each $\text{TC}(T \wedge X_I)$ for $I \subseteq \{1, \ldots, n\}$ will be computed in two stages: taking the homotopy limit of $\text{THH}(T \wedge X_I)^{C_n}$ over the Frobenius maps to get the spectrum $\text{TF}(T \wedge X_I)$, and then taking the homotopy equalizer of the identity and the restriction maps on $\text{TF}(T \wedge X_I)$. To find $\text{THH}(T \wedge X_I)^{C_n}$, it will be convenient to write the structure of $X_I$ as an $S^1$-space in a different way.

Let $S^1(s)$ denote $S^1$ with an accelerated $S^1$-action. This is $S^1$-equivariantly homeomorphic to $S^1/C_s$. We take a matrix $A = [a_{ij}] \in M_n(\mathbb{Z})$ to represent the map $(S^1)^n \to (S^1)^n$ given by

$$(z_1, \ldots, z_n) = (z_1^{a_{11}}, \ldots, z_1^{a_{1n}}, \ldots, z_n^{a_{n1}}, \ldots, z_n^{a_{nn}}).$$

If $A$ is invertible then this map is a homeomorphism, with inverse represented by $A^{-1}$.

**Lemma 4.8.** Given integers $s_1$, $s_2$, and $a$, let $g = \gcd(s_1, s_2)$. Write $a = de$ with $eg = \gcd(s_1, as_2)$. The Euclidean algorithm shows that there are matrices $A, B, B', C \in GL_2(\mathbb{Z})$ such that

$$A \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} g \\ 0 \end{bmatrix}, \quad B \begin{bmatrix} s_1 \\ c_{s_2} \end{bmatrix} = B' \begin{bmatrix} s_1 \\ c_{s_2} \end{bmatrix} = \begin{bmatrix} eg \\ 0 \end{bmatrix}, \quad C \begin{bmatrix} s_1 \\ as_2 \end{bmatrix} = \begin{bmatrix} eg \\ 0 \end{bmatrix},$$

with $c_{s_2} = \gcd(c_{s_2}, e)$. If $d = 1$, then

$$A \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} g \\ 0 \end{bmatrix},$$

and $B$ and $B'$ are inverses of $A$.
\[
\begin{bmatrix}
0 & 1 \\
1 & 0 \\
e & 0
\end{bmatrix} =
\begin{bmatrix}
1 & 0 \\
0 & e
\end{bmatrix},
\]
and
\[
\begin{bmatrix}
1 & 0 \\
0 & d
\end{bmatrix} =
\begin{bmatrix}
1 & 0 \\
0 & d
\end{bmatrix}.
\]

Hence the diagram of \(S^1\)-equivariant maps
\[
\begin{array}{ccc}
S^1(s_1) \times S^1(s_2) & \xrightarrow{A} & S^1(g) \times S^1(0) \\
\xrightarrow{[1 \ 0 \ e \ 0]} & & \xrightarrow{B' B^{-1}} \\
S^1(s_1) \times S^1(as_2) & \xrightarrow{C} & S^1(eg) \times S^1(0)
\end{array}
\]
commutes.

The proof is straightforward, with the algorithm for constructing \(B\) based on \(A\) and the algorithm for constructing \(B'\) based on \(C\).

The map \(A\) in Lemma 4.8 provides \(S^1\)-equivariant homeomorphisms
\[
S^1(s_1) \times S^1(s_2) \cong S^1(\gcd(s_1, s_2)) \times S^1(0),
\]
and Lemma 4.8 is saying additionally that we can compatibly choose these homeomorphisms to understand the effect of the \(a^{th}\) power map on one of the factors.

**Remark 4.9.** By fixing a homeomorphism
\[
S^1(s_1) \times \ldots \times S^1(s_{n-1}) \to S^1(\gcd(s_1, \ldots, s_{n-1})) \times \mathbb{T}^{n-2}
\]
we can reduce the case \(n > 2\) to the case \(n = 2\). In particular the multiplication by \(d\) map in Lemma 4.8 is concentrated in a single torus factor.

This implies that we can write the \(X_I\) defined in Equation 4.5 as
\[
X_I \cong \bigvee_{\overline{s} \in \mathbb{N}^n} \mathbb{T}^{n-1} \wedge \mathcal{S}^1/\mathcal{C}(\mathcal{S})^+ \wedge S^{\lambda_I(\overline{s})}.
\]

Now we define
\[
\hat{X}_I = \bigvee_{\overline{s} \in \mathbb{N}^n} \mathcal{S}^1/\mathcal{C}(\mathcal{S})^+ \wedge S^{\lambda_I(\overline{s})},
\]
i.e., \(X_I\) without the \((n-1)\)-torus.

To understand \(\text{TC}(T \wedge \hat{X}_I)\) we use the following result (see Equations 4.10 and 4.4 for the notation):

**Lemma 4.11.** Up to profinite completion we have an equivalence
\[
\text{TF}(T \wedge \hat{X}_I) \cong \bigvee_{\overline{s} \in \mathbb{N}^n} \Sigma[T \wedge S^{\lambda_I(\overline{s})}]^{\mathcal{C}(\mathcal{S})}.\]

The restriction maps \(R_d\) mapping the wedge summand corresponding to \(d \cdot \overline{s}\) to that of \(\overline{s}\) is the smash product of restriction map \(R_d\) of \(T\) and of the homeomorphism
\[
r_d: (S^{\lambda_I(d \overline{s})})_d \to S^{\lambda_I(\overline{s})}.
\]
Proof. This is by \cite[Lemma 8.2]{11}, which says that the inclusion of the $S^1$-fixedpoints of an $S^1$-spectrum into the homotopy inverse limit with respect to inclusions of the $C_n$ fixedpoints over all $n \in \mathbb{N}$ becomes an equivalence after profinite completion. In our case $S^1$ acts on each wedge summand corresponding to $\pi \in \mathbb{N}^n$ separately, so $TF(T \wedge \tilde{X}_I)$ splits as a wedge over $\pi$. For each $\pi$, the Wirthmüller isomorphism gives an equivalence

$$\Sigma[F(S^1/C_{\gcd(\pi)}+, T \wedge S^{\lambda_1(\pi)})] \simeq S^1/C_{\gcd(\pi)}+ \wedge T \wedge S^{\lambda_1(\pi)}$$

(with trivial $S^1$-action on the suspension of the function space), meaning that the $S^1$-fixedpoints of the spectrum on the right are the same as those of the spectrum on the left. Since the function spectrum is conduced, the fixed points are exactly $\Sigma[T \wedge S^{\lambda_1(\pi)}]C_{\gcd(\pi)}$.

The restriction maps $R_d$ are induced by those of the original cyclotomic spectrum $T \wedge \tilde{X}_I$. \hfill \Box

It follows that up to profinite completion,

$$TC(T \wedge \tilde{X}_I) = \bigvee_{\gcd(\pi)=1} \holim_n \Sigma[T \wedge S^{\lambda_1(d\pi)}]C_d.$$ 

Observe that the indexing set for this wedge is the same as the indexing set that identifies $\mathbb{N}^n$ as a disjoint union of copies of $\mathbb{N}$. This plays an essential role in our computation.

Let $TF(T \wedge \tilde{X}_I; \pi)$ (for $\pi \in \mathbb{N}^n$) and $TC(T \wedge \tilde{X}_I; \pi)$ (when $\gcd(\pi) = 1$) denote the summands corresponding to $\pi$ in the respective wedge sums.

5. Calculations for perfect fields

For any ring $R$, there is a close connection between $\text{THH}(R)$ and Witt vectors. In fact, we have an isomorphism

$$\pi_0 \text{THH}(R)^{C_m} \cong W_{(m)}(R),$$

where the Witt vectors defined by the truncation set $\langle m \rangle$ from Equation 2.1 \cite[Addendum 3.3]{11}.

Let $k$ be a perfect field of characteristic $p > 0$. Recall that in this case, by \cite{11},

$$\text{THH}_*(k) \cong k[\mu_0]$$

with $|\mu_0| = 2$, and

$$\pi_*(\text{THH}(k)^{C_{p^m-1}}) \cong \text{TR}_*(k; p) \cong W_m(k; p)[\mu_0].$$

It follows that

$$\pi_*(\text{THH}(k)^{C_m}) \cong W_{(m)}(k)[\mu_0]$$

for all $m$.

From this, we can recover the big Witt vectors by taking the limit

$$\lim_{R, m \leq n} \pi_*(\text{THH}(k)^{C_m}) \cong W_n(k)[\mu_0].$$

We can now prove the main computational result (recalling Definitions \cite{11} and \cite{11}). Suggestively mirroring the notation used for the wedge summands above, let

$$S_q(I; \pi) = S_q(I) \cap (\pi).$$
From the calculation above, and it follows that

$$\pi_{2q-2}((T \wedge S^{\lambda_1}(\mathbb{F}))^{C_{\text{odd}()}})$$

We consider the more general situation where $\lambda$ is any complex $S^1$-representation, and we will compute

$$\pi_n((T \wedge S^\lambda)^{C_{p^d}}),$$

for $(p, d) = 1$. Since $k$ is a $\mathbb{Z}(p)$-algebra, there is a splitting

$$(T \wedge S^\lambda)^{C_{p^d}} \overset{\cong}{\to} \prod_{e \mid d} (T \wedge S^{\lambda_{d/e}})^{C_{p^e}},$$

where the map to the $e$th factor is $R_{d/e} \circ F_e$ [11 Proof of Proposition 4.2.5]. By [11 Proposition 9.1], the homotopy groups are concentrated in even degrees and

$$\pi_{2q}((T \wedge S^{\lambda_{d/e}})^{C_{p^e}}) \cong \begin{cases} \mathbb{W}_S(k; p) & \text{dim}_C(\lambda_{C^{d/e}(p^{r+s+1})}) \leq q < \text{dim}_C(\lambda_{C^{d/e}(p^{r-s})}) \\ \mathbb{W}_{r+1}(k; p) & q \geq \text{dim}_C(\lambda_{C^{d/e}}) \end{cases}$$

We recast this last isomorphism in a form more amenable to comparing with the Witt vectors on $\mathbb{N}^n$. Recall that for any truncation set $S$, there is a splitting

$$\mathbb{W}_S(k) \cong \prod_{p \mid e} \mathbb{W}_{p^{n_0} \cap S/(e)}(k),$$

where the projection onto the $e$th factor is given by $R_{d/e} \circ F_e$. Hence $\mathbb{W}_S(k)$ splits as a product of $p$-typical Witt vectors. Let $S_q(p^d)$ be the truncation set of all divisors $m$ of $p^d$ such that $\text{dim}_C(\lambda_{C^{d/m}}) \leq q$. If $s$ is the unique integer such that

$$\text{dim}_C(\lambda_{C^{d/e}(p^{r+s+1})}) \leq q < \text{dim}_C(\lambda_{C^{d/e}(p^{r-s})}),$$

and $s = r + 1$ if $q \geq \text{dim}_C(\lambda_{C^{d/e}})$, then

$$p^{n_0} \cap S/e = \{1, p, p^2, \ldots, p^{s-1}\}.$$
Associate to each \( m \mid \gcd(\bar{s}) \) a vector \( \bar{u} \mid \bar{s} \) via
\[
\bar{u} = \frac{m}{\gcd(\bar{s})} \bar{s}.
\]
Then \( \dim_{C}(\lambda_{I}(\bar{s})^{C_{\gcd(\pi)/m}}) \) is
\[
\sum_{i=1}^{n} \left\lfloor \frac{s_{i} - 1}{a_{i}^{\chi(i)} \gcd(\bar{s})} \right\rfloor m = \sum_{i=1}^{n} \left\lfloor \frac{u_{i} - \frac{m}{\gcd(\bar{s})}}{a_{i}^{\chi(i)}} \right\rfloor = \sum_{i=1}^{n} \left\lfloor \frac{u_{i} - 1}{a_{i}^{\chi(i)}} \right\rfloor
\]
because \( 0 < \frac{m}{\gcd(\bar{s})} \leq 1 \) and \( u_{i} \) and \( a_{i}^{\chi(i)} \) are integers. Thus the truncation set \( S \) is isomorphic to the truncation set \( S_{q}(I; \bar{s}) \).

The result follows.

The claim about the restriction maps follows from the one-dimensional case, since each restriction map
\[
R_{d} : TF(T \wedge \tilde{X}_{I}; a^{\bar{s}}) \to TF(T \wedge \tilde{X}_{I}; \bar{s})
\]
is happening in the one-dimensional truncation set \( \mathbb{N} \bar{s} \).

This allows us to prove Theorem 1.2.

**Proof of Theorem 1.2.** By Theorem 5.1, if we take the limit
\[
\text{holim}_{R} TF_{S_{q} - 1}(T \wedge \tilde{X}_{I}; \bar{s})
\]
over all \( \bar{s} \in S_{q}(I) \), then we get \( \mathbb{W}_{S_{q}(I)}(k) \). There is an equivalence between the limit over \( R \) of this system and the equalizer of \( R \) and the identity function over all \( \bar{s} \), so the result follows. The calculation of the order when \( k \) is finite follows from the formula
\[
|S_{q}(I)| = \binom{n + q - 1}{n} \prod_{i \in I} a_{i}.
\]

We have now computed the homotopy groups at each vertex of our cube. The maps in the \( n \)-cube are constructed from the \( a_{i}^{th} \) power maps on the various factors. In the 1-variable case studied by Hesselholt and Madsen \[10\] that yielded the ordinary Verschiebung map \( V_{a} \). In this case the situation is a little bit more complicated. On the cube \( TF(THH(k) \wedge \mathcal{X}) \) we find the following.

**Lemma 5.2.** Let \( k \) be any ring. For each \( \bar{s} = (s_{1}, \ldots, s_{n}) \), let
\[
\bar{u} = (s_{1}, \ldots, a_{i} s_{i}, \ldots, s_{n}).
\]
Let \( e_{i} = \gcd(u)/\gcd(s) \) and let \( \bar{t} = (s_{1}, \ldots, e_{i} s_{i}, \ldots, s_{n}) \). Up to profinite completion, and using the factorization in Lemma 4.8, we can factor the \( a_{i}^{th} \) power map
\[
TF(THH(k) \wedge \mathcal{X}_{I - (i)}; \bar{s}) \to TF(THH(k) \wedge \mathcal{X}_{I}; \bar{u})
\]
as the composite of
\[
\Sigma [THH(k) \wedge S^{\lambda}]^{C_{\gcd(\tau)}} \wedge T_{+}^{n-1} \xrightarrow{V_{e_{i}^{\lambda}} \wedge 1} \Sigma [THH(k) \wedge S^{\lambda}]^{C_{\gcd(\tau)}} \wedge T_{+}^{n-1},
\]
an equivalence on \( \Sigma[\text{THH}(k) \wedge S^\lambda]^{C_{\gcd(T)}} \wedge T_n \wedge T_{n-1} \), and

\[
\Sigma[\text{THH}(k) \wedge S^\lambda]^{C_{\gcd(T)}} \wedge T_n + 1 \rightarrow \Sigma[\text{THH}(k) \wedge S^\lambda]^{C_{\gcd(T)}} \wedge T_{n-1}
\]

Here \( d_i \) denotes multiplication by \( d_i \) on the last torus factor.

**Proof.** This follows from Remark 4.9 by smashing with \( \text{THH}(k) \wedge S^\lambda \) and then applying Lemma 4.11. \( \square \)

**Proof of Theorem 1.3.** Suppose \( p \) does not divide any of the \( a_i \). Then both the Verschiebung \( V_{a_i} \) and a map of degree \( d_i \) on the last torus factor as in Lemma 5.2 above induce isomorphisms. Hence \( T \mathcal{C}(A) \) splits as a wedge over those \( \Sigma \) for which \( a_i \nmid s_i \) for all \( i \). The result follows. The calculation of the order follows from that in Theorem 1.2. \( \square \)

### 6. Calculations for \( \mathbb{Z} \)

In the single variable case, the algebraic \( K \)-theory groups \( K_q(\mathbb{Z}[x]/x^n, (x)) \) have been studied by the first two authors and Hesselholt \[1\]. They compute these groups completely for \( q \) odd and up to extensions for \( q \) even. We now consider the \( n \)-variable case, proving Theorem 1.4 stated in the introduction. To prove the theorem we compute the Poincaré series of \( T \mathcal{C}(T \wedge X_I) \) at each vertex of our \( n \)-cube \( T \mathcal{C}(T \wedge X_I) \).

**Proposition 6.1.** For \( k = \mathbb{Z} \) the Poincaré series of \( T \mathcal{C}(T \wedge X_I) \) is given by

\[
\sum_{j \geq 1} j^{2j-1} \left( \frac{n + j - 2}{n - 1} \right) \prod_{i \in I} a_i.
\]

**Proof.** We use the formula from Lemma 4.11 up to profinite completion,

\[
\text{TF}(T \wedge X_I) = \bigvee_{\Sigma(T \wedge S^\lambda(\overline{\gamma}))} \Sigma(T \wedge S^\lambda(\overline{\gamma}))^{C_{\gcd(\overline{\gamma})}}.
\]

Using the computations of \( RO(S^1) \)-graded \( TR \)-groups of \( \mathbb{Z} \) found in \[1\], we know that

\[
\Sigma(\text{THH}(\mathbb{Z}) \wedge S^\lambda(\overline{\gamma}))^{C_{\gcd(\overline{\gamma})}},
\]

contributes a \( \mathbb{Z} \) in degree \( (2 \dim_S(S^\lambda(\overline{\gamma})) + 1) \) for each \( d | \gcd(\overline{\gamma}) \). Each such \( \mathbb{Z} \) comes from \( \pi_0(\text{THH}(\mathbb{Z})) \cong \mathbb{Z} \) and the sphere \( (S^\lambda(\overline{\gamma}))^{C_d} \). Note that the restriction maps of the cyclotomic spectrum \( \text{THH}(\mathbb{Z}) \) induce the identity on \( \pi_0 \), and that the restriction maps \( r_m : (S^\lambda(m \overline{\gamma}))^{C_m} \rightarrow S^\lambda(\overline{\gamma}) \) exactly send \( (S^\lambda(m \overline{\gamma}))^{C_m} \) homeomorphically onto \( (S^\lambda(\overline{\gamma}))^{C_d} \). So in \( T \mathcal{C}_*(T \wedge X_I) \), we will get a \( \mathbb{Z} \) in degree \( (2 \dim_S(S^\lambda(\overline{\gamma})) + 1) \) for every \( \overline{\gamma} \in \mathbb{N}^n \).

Therefore, to compute the free rank of \( T \mathcal{C}_{2j-1}(T \wedge X_I) \), we need to count the number of vectors \( \overline{\gamma} \) such that \( 2 \dim_S(S^\lambda(\overline{\gamma})) + 1 = 2j - 1 \). Hence we want to count the number of \( \overline{\gamma} = (s_1, \ldots, s_n) \) such that \( \dim_S(S^\lambda(\overline{\gamma})) = j - 1 \). There are \( \binom{n + j - 2}{n - 1} \) vectors \((b_1, \ldots, b_n)\) with \( b_i \geq 0 \) for all \( 1 \leq i \leq n \) and \( b_1 + \ldots + b_n = j - 1 \), and there are

\[
\begin{align*}
1 \quad & \text{if } i \notin I \\
1 \quad & \text{if } i \in I
\end{align*}
\]
positive integers \( s_i \) with
\[
\dim_{\mathbb{C}} \lambda \left( \left\lfloor \frac{s_i - 1}{a_i^{\chi_i(n)}} \right\rfloor \right) = b_i.
\]
The result follows. \( \square \)

Theorem 1.4 follows directly from Proposition 6.1 and the computation of Verschiebung maps in [1].

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