Properties of a Single Photon Generated by a Solid-State Emitter:
Effects of Pure Dephasing

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We investigate the properties of a single photon generated by a solid-state emitter subject to strong pure dephasing. We employ a model in which all the elements of the system, including the radiating fields, are treated quantum mechanically. We analytically derive the density matrix of the emitted photon, which contains full information about the photon, such as its pulse profile, power spectrum, and purity. We visualize these analytical results using realistic parameters and reveal the conditions for maximizing the purity of generated photons.

KEYWORDS: single-photon source, pure dephasing, purity

1. Introduction

Cavity quantum electrodynamics (QED) is currently one of the hottest research topics in atomic physics. In particular, quantum–mechanical interactions between atoms and cavity photons have been intensively studied.1–3 Solid-state cavity QED systems composed of semiconductor quantum dots and cavities have recently been attracting much attention since they are suitable for creating compact optical devices.4–6 Strong coupling between a single dot and a cavity has been confirmed through a large vacuum Rabi splitting.3–5 This strong coupling has been applied to fabricate a single-dot laser6 and to generate nonclassical light including single photons.7–11 Excellent performances have been reported in generating indistinguishable photons12,13 and entangled photon pairs,14 both of which are useful for quantum information processing.15

In contrast to real atoms, semiconductor quantum dots are strongly influenced by environmental noise sources such as phonons and background carriers. The fluorescence spectra of solid-state and atomic cavity QED systems are qualitatively different. When a solid-state system is excited by pump light of the dot frequency, a spectral peak appears at the cavity frequency in spite of the large detuning between them.3–5 This phenomenon has not been reported for atomic cavity QED systems. Further experimental studies have characterized this peak more thoroughly16–22 and have revealed that the fluorescence at the cavity frequency is due to radiative decay of the dot. Subsequent theoretical studies accounted for the pure dephasing of the dot through the stochastic Schrödinger equation3,23 or the Master equation24–27 and successfully explained the peak at the cavity frequency. The influence of pure dephasing on the radiative decay of the dot can be understood in terms of the quantum Zeno and anti-Zeno effects.25,28

Therefore, when designing a single-photon source using solid-state cavity QED systems, it is crucial to quantitatively consider the pure dephasing of the dot. The performance of such photon sources should be evaluated from two aspects. One is the collection efficiency, namely, the probability that the emitted photon is transferred to the intended spatial mode (i.e., the radiation pattern of the cavity). This has been discussed in several studies in terms of the ratio of radiative decay rates.23,26,27 The other is the indistinguishability of generated photons, which can be measured by two-photon interference experiments and is evaluated by the purity. Single photons with high purity are required for quantum information processing, particularly for constructing scalable quantum circuits.

In this study, we investigate the properties of a single photon emitted by a solid-state cavity QED system and quantitatively observe the effects of pure dephasing. In order to obtain full information including the indistinguishability of generated single photons, we treat the five elements of the overall system (the dot, the cavity, radiation leaking from the cavity, non-cavity radiation modes, and the environment causing the pure dephasing of the dot) as active quantum–mechanical degrees of freedom, and analytically derive the density matrix of the emitted photon in the real-space representation. This density matrix contains full information about the emitted photon, including its pulse profile, frequency spectrum, and purity. These quantities are observed as functions of the pure dephasing rate of the dot. We reveal the optimum condition for maximizing the purity of the emitted photon.

2. Model

We investigate the radiative decay of an excited quantum dot placed inside a cavity, as illustrated in Fig. 1. This solid-state cavity QED system consists of the following five components: (i) a quantum dot, (ii) a cavity, (iii) a photon field leaking from the cavity (referred to as $b$ field hereafter), (iv) non-cavity radiation modes ($c$ field), and (v) a reservoir field, which causes the pure dephasing of the dot ($d$ field).29

The annihilation operators corresponding to these components are respectively denoted as $a$, $b$, $c$, and $d$, where $k$ is a one-dimensional wave number. Note that $\sigma$ is a Pauli operator, whereas the other operators are bosonic. Setting $\hbar = c = 1$, the Hamiltonian of the overall system is given by

\begin{equation}
\hat{H} = \hat{H}_{\text{dot}} + \hat{H}_{\text{cav}} + \hat{H}_{\text{dot-cav}} + \hat{H}_{\text{env}} + \hat{H}_{\text{diss}}.
\end{equation}
The parameters are defined as follows (see Fig. 1). \( \omega_d \) and \( \omega_c \) respectively denote the resonance frequencies of the dot and cavity, \( g \) represents the coupling between them, \( \kappa \) is the escape rate of cavity photons, \( \gamma \) is the radiative decay rate of the dot into non-cavity modes, and \( \gamma_p \) is the pure dephasing rate of the dot. \( \mathcal{H}_0 \) describes the Rabi oscillation between the dot and the cavity (Jaynes–Cummings Hamiltonian), \( \mathcal{H}_1 \) describes the leakage of a cavity photon to its radiation pattern, \( \mathcal{H}_2 \) describes the radiative decay of the dot in unintended directions, and \( \mathcal{H}_3 \) describes the pure dephasing of the dot. We can confirm that \( \mathcal{N} = \sigma^a + a^\dagger \sigma + \int d\mathbf{r} (\gamma_p/\pi \sigma^a (d^a_i + d_i)) \) commutes with the Hamiltonian. Therefore, the number of excitations is conserved in the dot, cavity, and b and c fields.

We assume that the dot is initially \( t = 0 \) in the excited state while the other fields are in their vacuum states. Then, denoting the overall vacuum state by \( |0\rangle \), the initial state vector is given by

\[
|\psi_i\rangle = \sigma^a |0\rangle. \tag{6}
\]

The Hamiltonian of Eq. (1) and the initial state vector of Eq. (6) form the basis of our analysis.

For later convenience, we introduce the real-space representation of the b field (cavity leakage). It is defined by

\[
\tilde{b}_r = (2\pi)^{-1/2} \int d\mathbf{r} \, e^{i\mathbf{kr}} b_\mathbf{r}. \tag{7}
\]

The \( r < 0 \) \((r > 0)\) region represents the incoming (outgoing) field. \( \tilde{c}_r \) and \( \tilde{d}_r \) can be formally defined in a similar manner. Our main concern lies in the properties of a single photon emitted in the b field.

To model the pure dephasing of the dot, the d field interacts with the dot so as to conserve the dot excitation. Using Eqs. (2) and (5), the dot Hamiltonian can be rewritten as \( [\alpha_d(t) + f(t)]\sigma^a \), where \( f(t) = \sqrt{2\gamma_p} (d^a_i + d_i) \) is the fluctuation of the dot resonance frequency induced by the d field. Using Eqs. (6) and (10), we can confirm that \( \langle f(t)f(t') \rangle = 2\gamma_p \delta(t - t') \), where \( \langle \cdots \rangle = \langle \psi_i | \cdots | \psi_i \rangle \). Therefore, the present model assumes a white noise spectrum for the fluctuation of the dot resonance.

### 3. Analysis

In this section, we present analytical results that are rigorously derivable from the model described in Sect. 2. We solve the time evolution of the overall system within the input–output formalism and derive several formulae to characterize the emitted single photon (density matrix, pulse shape, spectrum, and purity). These analytical results are visualized in the next section for specific parameters.

#### 3.1 Heisenberg equations

Here, we present the Heisenberg equations for the system \((\sigma, a)\) and field \((b_r, c_r, d_r)\) operators. Deriving the raw Heisenberg equations for the field operators from Eq. (1) and transforming them into real-space representations, we obtain the following relations that connect the incoming \((r < 0)\) and outgoing \((r > 0)\) fields:

\[
\tilde{b}_r(t) = \tilde{b}_{r-}(0) - i\sqrt{\kappa} \theta(r) \delta(t - t) a(t - r), \tag{8}
\]

\[
\tilde{c}_r(t) = \tilde{c}_{r-}(0) - i\gamma_p(r) \delta(t - t) c(t - r), \tag{9}
\]

\[
\tilde{d}_r(t) = \tilde{d}_{r-}(0) - i\sqrt{2\gamma_p} \theta(r) \delta(t - t) a(t - r), \tag{10}
\]

where \( \theta(x) \) is the Heaviside step function. From the raw Heisenberg equations for the system operators and the above input–output relations, the Heisenberg equations for system operators are given by

\[
\frac{d}{dt} a = -i\omega_d a + ig(1 - 2\sigma^a) a - i(1 - 2\sigma^a) N_c(t), \tag{11}
\]

\[
\frac{d}{dt} \sigma = -i\omega_c \sigma - ig(1 - 2\sigma^a) a - i(1 - 2\sigma^a) N_c(t), \tag{12}
\]

where \( \omega_d = \omega_d - i(\gamma_p/2 + \gamma_p) \) and \( \omega_c = \omega_c - i\kappa/2 \) respectively are the complex frequencies of the dot and cavity, and the noise operators are defined by \( N_c(t) = \sqrt{\kappa} \tilde{b}_{r-}(0) \), \( N_c(t) = \sqrt{\gamma_p} \tilde{c}_{r-}(0) \), and \( N_d(t) = \sqrt{2\gamma_p} \tilde{d}_{r-}(0) \). Note that the noise operators are the initial-time operators and, consequently, \( N_c(0) = 0 \) \((j = b, c, d)\).

#### 3.2 State vector

The state vector of the overall system at an arbitrary time \( t \) is determined by \( \langle \psi(t) | = e^{-iH t} | \psi(0) \rangle \). Since the initial dot excitation is conserved in the dot, cavity, and b and c fields, the state vector can be written as

\[
|\psi(t)\rangle = \left[ \alpha_d(t)\sigma^a + \beta_d(t)a^\dagger + \int dr \gamma^\prime(r, t) \tilde{b}^\dagger_r \right] |0\rangle + \sum_{m=1}^{\infty} \int d^m x \left[ \alpha_m(x, t)\sigma^a + \beta_m(x, t)a^\dagger \right] ^\dagger + \int dr \gamma_m(r, x, t) \tilde{c}^\dagger_r + \int dr \delta_m(r, x, t) \tilde{d}^\dagger_r |0\rangle, \tag{13}
\]

where \( m \) denotes the number of excitations in the d field and \( \int d^m x \) denotes a multi-dimensional integral with respect to
We can set $x_1 \leq \cdots \leq x_m$ without loss of generality. As we show later, these coefficients are nonzero only when $0 \leq t \leq x_1 \leq \cdots \leq x_m \leq t$.

First, we discuss $\alpha_0$ and $\beta_0$. From Eq. (13), we can confirm that $\alpha_0(t) = \langle \sigma(t) \sigma^\dagger(0) \rangle$ and $\beta_0(t) = \langle \bar{\sigma}(t) \sigma^\dagger(0) \rangle$, where $\langle \cdots \rangle = \langle 0 \rangle\langle \cdots \rangle$. From Eqs. (11) and (12), their equations of motion are given by

$$\frac{d}{dt} \left( \begin{array}{c} \alpha_0(t) \\ \beta_0(t) \end{array} \right) = \left( \begin{array}{cc} -i\omega_d & -ig \\ -i\omega_d & -i\omega_c \end{array} \right) \left( \begin{array}{c} \alpha_0(t) \\ \beta_0(t) \end{array} \right),$$

with the initial conditions $\alpha_0(0) = 1$ and $\beta_0(0) = 0$. The solutions are given by

$$\alpha_0(t) = A_1 e^{i\omega t} + A_2 e^{i\omega t},$$
$$\beta_0(t) = B_1 e^{i\omega t} + B_2 e^{i\omega t},$$

where $\lambda_1$ and $\lambda_2$ are the two eigenvalues of the $2 \times 2$ matrix in Eq. (14) (see Fig. 2), $A_1 = (\lambda_1 + i\omega_c)/(\lambda_1 - \lambda_2)$, $A_2 = (\lambda_2 + i\omega_c)/(\lambda_2 - \lambda_1)$, and $B_1 = -B_2 = -ig/(\lambda_1 - \lambda_2)$. The real parts of $\lambda_1$ and $\lambda_2$ are always negative and, consequently, $\alpha_0$ and $\beta_0$ vanish as $t \to \infty$. Equation (13) also implies that $\gamma_0(r,t) = \langle \bar{\sigma}(t) \sigma(0) \rangle$ and $\delta_0(r,t) = \langle \bar{\sigma}(t) \sigma^\dagger(0) \rangle$. From Eqs. (8) and (9), we have

$$\gamma_0(r,t) = -i\sqrt{2\gamma_p} \delta_0(t - r),$$
$$\delta_0(r,t) = -i\gamma_0\delta_0(t - r).$$

Rigorously, $\theta(r)\theta(t - r)$ should appear on the right-hand sides of these equations. However, below, we implicitly assume $0 \leq r \leq x_1 \leq \cdots \leq x_m \leq t$ and omit the Heaviside functions.

Next, we proceed to investigate higher-order quantities including reservoir (d field) excitations. We consider $\alpha_1$ and $\beta_1$ as examples. Applying the same reasoning as that for $\gamma_0$ and $\delta_0$, we have $\alpha_1(x,t) = -i\sqrt{2\gamma_p} \langle \sigma(t) \sigma^\dagger(t - x) \sigma(0) \rangle$ and $\beta_1(x,t) = -i\sqrt{2\gamma_p} \langle \bar{\sigma}(t) \sigma^\dagger(t - x) \sigma(t)(t - x) \sigma(0) \rangle$.

Thus, we must evaluate the three-time correlation functions $\langle \sigma(t_2) \sigma^\dagger(t_1) \sigma(t) \sigma^\dagger(0) \rangle$ and $\langle \sigma(t_2) \sigma^\dagger(t_1) \sigma(t) \sigma^\dagger(0) \rangle$ with $t_2 > t_1 > 0$. Their equations of motion with respect to $t_2$ have the same form as Eq. (14) and their initial values $(t_2 \to t_1)$ are $\alpha_0(t_1)$ and 0, respectively. This implies that the two-time correlation functions can be factorized as $\langle \sigma(t_2) \sigma^\dagger(t_1) \sigma(t) \sigma^\dagger(0) \rangle = \alpha_0(t_2 - t_1) \alpha_0(t_1)$ and $\langle \sigma(t_2) \sigma^\dagger(t_1) \sigma(t) \sigma^\dagger(0) \rangle = \beta(t_2 - t_1) \alpha_0(t_1)$. Thus, we have

$$\alpha_1(x,t) = -i\sqrt{2\gamma_p} \alpha_0(t - x) \alpha_0(x),$$
$$\beta_1(x,t) = -i\sqrt{2\gamma_p} \alpha_0(t - x) \beta_0(x).$$

Repeating the same arguments, all coefficients can be written as products of $\alpha_0$ and $\beta_0$, as follows:

$$\alpha_n(x,t) = (-i\sqrt{2\gamma_p})^n K_n(x,t) \alpha_0(x_0(x_1)),$$
$$\beta_n(x,t) = (-i\sqrt{2\gamma_p})^n K_n(x,t) \beta_0(x_0(x_1)),$$

$$\gamma_n(r,t) = (-i\sqrt{2\gamma_p})^n K_n(x,t) \delta_0(x_0(x_1)),$$
$$\delta_n(r,t) = (-i\sqrt{2\gamma_p})^n K_n(x,t) \alpha_0(x_0(x_1)).$$

We here introduce the Laplace transform of $|\alpha_0|^2$, which is defined as $L_{|\alpha_0|^2}(z) = \int_0^\infty dt e^{-zt} |\alpha_0(t)|^2$. It is given by

$$L_{|\alpha_0|^2}(z) = \sum_{m,n=1,2} \frac{A_m A_n^*}{z - \lambda_m - \lambda_n^*},$$

where $\lambda_{1,2}$ and $A_{1,2}$ are defined in Sect. 3.2. The Laplace transform of $P(t)$ is then given by

$$L_P(z) = \frac{L_{|\alpha_0|^2}(z)}{1 - 2\gamma_p L_{|\alpha_0|^2}(z)}.$$

$P(t)$ is obtained by analyzing the poles of this function in the $z$-plane. We denote the four roots of the equation $1 - 2\gamma_p L_{|\alpha_0|^2}(z) = 0$ by $\mu_j$ ($j = 1, \ldots, 4$) (see Fig. 3). $P(t)$ is then given by

$$P(t) = \sum_{j=1}^4 P_j e^{\mu_j t},$$

where

$$E_j = \frac{\prod_{m,n=1,2} (\mu_j - \lambda_m - \lambda_n^*)}{\prod_{m,n=1,2} (\mu_j - \lambda_m - \lambda_n^*)} \sum_{m,n=1,2} A_m A_n^*.$$
Note that the real parts of \( \mu_j \) are always negative and that the survival probability \( P(t) \) vanishes in the \( t \to \infty \) limit, as expected.

### 3.4 Density matrix of emitted photon

In the \( t \to \infty \) limit, the initial dot excitation is completely transformed into a photon propagating in the intended mode (\( b \) field) or in unintended directions (\( c \) field). In the following subsections, we analyze the photon emitted in the \( b \) field. It is fully characterized by its density matrix \( \hat{\rho}(t) \). In the real-space representation, the matrix element \( \rho(r, r', t) \) is given by

\[
\rho(r, r', t) = \langle \psi(t) | \hat{b}_r \hat{b}_r^\dagger | \psi(t) \rangle.
\]

(31)

We make the following three comments regarding this quantity: (i) \( \hat{\rho}(t) \) is Hermitian, namely, \( \rho(r', r, t) = \rho^\ast(r, r', t) \). Therefore, we need consider only the \( r < r' \) region in this subsection. (ii) As we will see later, \( \rho(r, r', t) = \rho(r-t, r'-t) \) in the \( t \to \infty \) limit. This reflects the translational motion of the emitted photon. (iii) \( \text{Tr} \hat{\rho}(t) = \int \text{d}r \rho(r, r, t) \) represents the probability of finding the emitted photon in the \( b \) field. This is unity when \( \gamma = 0 \).

Using Eqs. (13) and (23), the matrix element can be rewritten as

\[
\rho(r, r', t) = \kappa \beta_0^2(t-r) \beta_0^2(r-t) + 2 \gamma \kappa \int \text{d}x \beta_0(x-r) \beta_0^2(x-r') |\alpha_0(t-x)|^2 + \cdots. \tag{32}
\]

We here introduce the Laplace transform of \( \beta_0^2(t) \), which is defined by \( \mathcal{L}_{\beta_0^2}(r, z) = \int_0^\infty \text{d}r \beta_0^2(t-r) \beta_0^2(r-t) \). It is given by

\[
\mathcal{L}_{\beta_0^2}(r, z) = \sum_{m,n=1,2} \frac{B_m B_n^\ast}{2} e^{\lambda_m (r-t) - \lambda_n^\ast (r-t)} \tag{33}
\]

where \( \lambda_{1,2} \) and \( B_{1,2} \) are defined in Sect. 3.2. The Laplace transform of \( \rho(r, r', t) \) is then given by

\[
\mathcal{L}_\rho(r, r', z) = \frac{\kappa \mathcal{L}_{\beta_0^2}(r, r', z)}{1 - 2 \gamma \kappa \mathcal{L}_{\alpha_0^2}(z)}. \tag{34}
\]

By analyzing the poles of this function in the \( z \)-plane, \( \rho(r, r', t) \) is obtained as follows:

\[
\rho(r, r', t) = \sum_{j=1}^4 \sum_{m=1,2} \rho_{jm} e^{\lambda_j (r-t) - \lambda_m (r-t)}, \tag{35}
\]

\[
\rho_{jm} = \frac{\sum_{i=1,2} \mu_i^2 \sum_{i=1,2} \mu_i - \lambda_m - \lambda_n^\ast}{} \tag{36}
\]

where \( 0 < r < r' < t \) and \( \mu_j (j = 1, \ldots, 4) \) are defined in Sect. 3.3. We can check that this quantity depends on only two variables, \( r - t \) and \( r' - t \).

### 3.5 Pulse shape

The pulse shape of the emitted photon is characterized by the intensity distribution \( f(r, t) = \langle \psi(t) | \hat{b}_r \hat{b}_r^\dagger | \psi(t) \rangle = \rho(r, r, t) \), which is the diagonal element of the density matrix. This quantity is real and positive for \( 0 < r < t \). By setting \( r' = r \) in Eq. (35), we have

\[
f(r, t) = \sum_{j=1}^4 f_j e^{\mu_j (t-r)}, \tag{37}
\]

### 3.6 Frequency spectrum

The frequency spectrum of the emitted photon is defined by \( S(k, t) = \langle \psi(t) | \hat{b}_k^\dagger \hat{b}_k | \psi(t) \rangle \). Apparently, \( S(k, t) \) becomes independent of \( t \) in the \( t \to \infty \) limit, and we are interested in \( S(k) = \lim_{t \to \infty} S(k, t) \). By definition, \( S(k, t) \) is the Fourier transform of the density matrix element:

\[
S(k, t) = \frac{1}{2\pi} \int \text{d}r' e^{ik(r-r')} \rho(r, r', t). \tag{39}
\]

We consider the Laplace transform of \( S(k, t) \) defined by \( \mathcal{L}_S(k, z) = \int_0^\infty \text{d}r e^{-z} S(k, r) \). Using Eqs. (34) and (39), it is given by

\[
\mathcal{L}_S(k, z) = \kappa \mathcal{L}_E'_{\beta_0^2}(k, z), \tag{40}
\]

where

\[
\mathcal{L}_E'_{\beta_0^2}(k, z) = \int_0^\infty \text{d}r \int \text{d}r' e^{ik(r-r')-z} \beta_0^2(t-r) \beta_0^2(r-t) = \int_0^\infty \text{d}r e^{-z} \sum_{j=1}^4 \frac{B_m B_n^\ast}{2} \left| e^{ik(r-t) - \lambda_m (r-t)} \right|^2 \tag{41}
\]

Note that \( \mathcal{L}_E'_{\beta_0^2} \) has a pole at \( z = 0 \). Since our interest lies in the \( t \to \infty \) limit of \( S(k, t) \), we must investigate the pole of \( \mathcal{L}_S(k, z) \) at \( z = 0 \) only. Therefore, \( S(k) = (k/2\pi)[1 - 2 \gamma \kappa \mathcal{L}_{\alpha_0^2}(0)^{-1}] \text{Res}_{z=0}[\mathcal{L}_E'_{\beta_0^2}(k, z)] \). After some calculations, we obtain

\[
S(k) = \frac{N}{(k - \tilde{\omega}_d)(k - \tilde{\omega}_a) - \gamma^2 k^2}, \tag{42}
\]

where \( N = (\kappa \gamma^2 / 2\pi)[1 - 2 \gamma \kappa \mathcal{L}_{\alpha_0^2}(0)]^{-1} \) is a factor that is independent of \( k \). This spectral shape was predicted by Glauber,\(^{32}\) and it was confirmed in recent theoretical studies based on the quantum Langevin equations\(^{33,24}\) and the Master equations.\(^{25,27}\) The relation to the Master equation approach is summarized in Appendix A.

### 3.7 Purity

Quantum information processing requires high indistinguishability between single photons. A popular measure of the indistinguishability of photons is the coincidence probability \( P_{\text{co}} \) in two-photon interference experiments (see Fig. 4). Two solid-state emitters simultaneously emit single photons into two input ports of a beam splitter. These two photons are mixed by the beam splitter and are counted by photon detectors. When two indistinguishable photons are simultaneously input to a beam splitter, they always appear at the same output port (Hong–Ou–Mandel interference), namely, \( P_{\text{co}} = 0 \). However, pure dephasing generates entanglement between an emitted photon and the environment of its source, making two-photon interference imperfect (\( P_{\text{co}} > 0 \)). The coincidence probability is related to the purity \( P \) of a photon by \( P_{\text{co}} = (1 - P)/2 \) (see Appendix C for the derivation).

The purity is defined in terms of the density matrix \( \hat{\rho}(t) \) by \( P = \text{Tr}[\hat{\rho}^2(t)] \). As expected, this quantity becomes indepen-
dent of $t$ when $t$ is sufficiently large. Using the real-space matrix element, the purity can be rewritten as

$$P = \lim_{t \to \infty} \int dr \int dr' \rho(r, r', t) \rho(r', r, t)$$

$$= \lim_{t \to \infty} \int dr \int dr' |\rho(r, r', t)|^2.$$

Using Eq. (36), we have

$$P = \sum_{j=1}^{4} \sum_{m=1}^{2} \sum_{m'=1}^{2} \frac{2 \rho_{jm} \rho_{j'm'}^{*}}{(\mu_j + \mu_j') (\Lambda_m + \Lambda_{m'})}.$$

4. Numerical Results

The analytical results derived in the previous section are rigorous and applicable to any set of parameters, $(\omega_d - \omega_c, g, \kappa, \gamma, \gamma_p)$. In this section, we visualize these results by employing specific parameters. Throughout this section, we assume, for simplicity, that radiative decay to unintended modes does not occur ($\gamma = 0$).

4.1 Zeno and anti-Zeno effects

First, we observe the effects of pure dephasing on the decay of dot excitation. We focus on the weak-coupling regime ($\kappa = 6g$) in this subsection, where the dot decays monotonically without revival and obeys the exponential decay law with high accuracy. The decay rate of the dot is well defined in this case and is given by $\Gamma = \lim_{t \to \infty} [- \log P(t)/t]$. This reduces to $\min_j |\text{Re} \mu_j|$, where $\mu_j$ is defined in Sect. 3.3. Figure 5 shows the temporal behavior of the survival probability $P(t)$. In Fig. 5(a), where the dot is in resonance with the cavity $(\omega_d = \omega_c)$, the decay becomes slower as pure dephasing increases. In contrast, in Fig. 5(b), where the dot is detuned from the cavity $(\omega_d - \omega_c = 600 \mu eV)$, the decay becomes faster under small pure dephasing ($\gamma_p = 200 \mu eV$), whereas the decay becomes slower under larger pure dephasing ($\gamma_p = 12800 \mu eV$).

Measurements to check the survival of dot excitation destroy the quantum coherence between the excited and ground states that preserves the population of these two states. Therefore, pure dephasing has the same effect on the dot as continuous measurements, if the measurement results are unquestioned. The changes in the decay rates induced by pure dephasing are often interpreted as the quantum Zeno and anti-Zeno effects.25,33,34 Previous analyses indicated that the anti-Zeno effect can be observed only when the dot–cavity detuning is large and pure dephasing is small.28 This agrees with the present numerical results. Figure 6 shows the dependence of the radiative decay rate $\Gamma$ on the pure dephasing rate $\gamma_p$. To clearly observe the Zeno and anti-Zeno effects, $\Gamma(\gamma_p)$ is normalized by the free decay rate of $\Gamma(0)$; $\Gamma(\gamma_p)/\Gamma(0) < 1$ indicates the Zeno effect, whereas $\Gamma(\gamma_p)/\Gamma(0) > 1$ exhibits the anti-Zeno effect. Non monotonic behavior of $\Gamma(\gamma_p)$ is clearly observed for large detuning.

4.2 Pulse shape and spectrum

In this subsection, we examine the pulse shape and
frequency spectrum of the emitted photon using the same parameters as those used in the previous subsection. First, we observe the results for the resonant \((\omega_d = \omega_c)\) case. The pulse shapes \(f(r, t)\) of the emitted photon are shown in Fig. 7(a) for three pure dephasing rates \(\gamma_p\). Each pulse shape is normalized \[ \int_{-\infty}^{\infty} f(r, t) \, dr = 1 \] since \(\gamma \equiv 0\) is assumed here and a single photon is necessarily generated in the \(b\) field. The pulse becomes longer as pure dephasing increases. This is consistent with the quantum Zeno effect discussed in the previous subsection: in the resonant case, the decay of the dot becomes monotonically slower with increasing pure dephasing. Figure 7(b) shows the spectra \(S(k)\) of the photon for the same parameters as those in Fig. 7(a). These spectra have a single peak at \(k = \omega_d \equiv \omega_c\) and are normalized \[ \int_{-\infty}^{\infty} dk \, S(k) = 1. \] The spectrum broadens with increasing pure dephasing. This is confirmed by Fig. 7(c), in which the spectral width (defined as \(\Delta = \int \sqrt{dk(k - \omega_d)^2 S(k)} \) is plotted as a function of the pure dephasing rate. Thus, the pulse broadens in both the real and frequency spaces with increasing \(\gamma_p\) and it is thus not Fourier-limited. This implies that the emitted photon is in a mixed state when \(\gamma_p \neq 0\). The purity of the photon is discussed later in Sect. 4.3.

Next, we observe the results for the detuned case. Figure 8(a) shows the pulse shapes of the photon. The pulse shape is approximately exponential, except for the oscillatory behavior at the very initial stage. The pulse length is inversely proportional to the decay rate of the dot. Figure 8(b) shows the photon spectra for the same parameters as those in Fig. 8(a). A notable difference from the resonant case is that the spectra are doubly peaked with peaks at both the dot frequency \(\omega_d\) and the cavity frequency \(\omega_c\). The widths of these peaks are determined by \[ |\text{Im}\, \omega_d| = \gamma/2 + \gamma_p \] and \[ |\text{Im}\, \omega_c| = \kappa/2. \] Therefore, the width of the peak at \(\omega_d\) is sensitive to pure dephasing. When pure dephasing is weak \((\gamma_p \ll \kappa)\), as in atomic cavity QED systems, the dominant peak of \(S(k)\) appears at \(\omega_d\). In contrast, when pure dephasing is strong \((\gamma_p \gg \kappa)\), as in solid-state systems, the dominant peak of \(S(k)\) appears at \(\omega_c\). This behavior can be seen more clearly in an approximate form of the spectrum,

\[
S(k) = \frac{2\gamma_p}{2\gamma_p + \kappa} S_{\text{cov}}(k) + \frac{\kappa}{2\gamma_p + \kappa} S_{\text{dot}}(k),
\]

\[
S_{\text{cov}}(k) = \frac{\gamma_p/\pi}{(k - \omega_d)^2 + (\kappa/2)^2},
\]

\[
S_{\text{dot}}(k) = \frac{\gamma_p/\pi}{(k - \omega_c)^2 + \gamma_p^2},
\]

which is valid for \(|\omega_d - \omega_c| \gg \gamma\). From this expression, it is proved that the cavity spectrum \(S_{\text{cov}}\) is dominant for \(\gamma_p \ll \kappa\), whereas the dot spectrum \(S_{\text{dot}}\) is dominant for \(\gamma_p \gg \kappa\). This result partly explains the detuned peaks observed in the resonance fluorescence spectrum in solid-state cavity QED systems, as discussed in previous theoretical works.23–25)
The mean energy of the emitted photon is evaluated by $E_p = \int dk \hat{k} \hat{s}(k)$. Owing to energy conservation, the mean photon energy is expected to always be identical to the dot frequency (i.e., $E_p = \omega_d$). However, Fig. 8(b) clearly shows that the mean photon energy is sensitive to pure dephasing and may deviate from the dot frequency when pure dephasing is present. This discrepancy can be resolved by considering the energy released to the environment during decay, which is given by $E_e = \int dk k (d^\dagger d)$. It can be shown that (see Appendix D for derivation)

$$E_p = \frac{2\gamma_p}{\kappa + 2\gamma_p} \omega_d + \frac{\kappa}{\kappa + 2\gamma_p} \omega_d,$$  \hspace{1cm} (48)

$$E_e = \frac{2\gamma_p}{\kappa + 2\gamma_p} (\omega_d - \omega_c).$$  \hspace{1cm} (49)

Thus, energy conservation is satisfied when the environmental energy is included ($E_p + E_e = \omega_d$). It should be noted that while pure dephasing coupling never induces a transition in the dot, this coupling enables energy exchange between the dot and the environment. When the cavity frequency exceeds the dot frequency, Eq. (49) indicates that the dot may absorb environmental energy during decay.$^{20}$

### 4.3 Purity

As discussed in Sect. 3.7, the coincidence probability in two-photon interference $P_{co}$, which is a popular measure of indistinguishability, is related to the purity as $P_{co} = (1 - P)/2$. In this subsection, we show numerical results for the purity $P$ for various parameter regions. Qualitative features in some limiting cases are discussed in Appendix B on the basis of perturbation calculation.

We first consider the resonant case ($\omega_d = \omega_c$). In Fig. 9(a), the purity $P$ is plotted as a function of $\kappa$ and $\gamma_p$. The purity becomes large for $\gamma_p < \min(\kappa/2, 2\gamma_c/\kappa)$ (the region below “Line 1” in the figure). From the figure, we can see that for the generation of coherent photons, pure dephasing should be reduced as much as possible, and that for fixed pure dephasing, $\kappa$ should be taken near the point of critical damping ($\kappa = 2\gamma_c$). In addition to this main region appropriate for highly coherent photon generation, there is a line on which the purity has a small peak (denoted as “Line 2” in the figure) in the region of $\gamma_p \gg g$. By perturbation calculation (see Appendix B), we can show that the purity takes a maximum value of $P_{max} = 3 - 2\sqrt{2} \simeq 0.17$ at $\gamma_p = \kappa/(2\sqrt{2}\gamma_c)$. The improvement of the purity is also possible in the detuned case (see the vertical dashed line in Fig. 9(a)).

Next, we consider the strongly detuned case ($|\omega_d - \omega_c| \gg g$). In Fig. 9(b), the purity is shown as a function of $\kappa$ and $\gamma_p$ for $\omega_d - \omega_c = 8g$. The purity has a large value below the line $\gamma_p < \min(\kappa/2, 2\gamma_c/\kappa)$ (the region below “Line 1” in the figure), where $\kappa = 2\gamma_c/|\omega_d - \omega_c|^2$ is a small parameter dependent on detuning. In Fig. 9(b), we can see that the purity for the detuned case is suppressed compared with the resonant case, and that the high-purity region is narrowed. From the figure, we can see that for coherent photon emission, dephasing should be reduced as much as possible, and that for fixed pure dephasing, $\kappa$ should be taken to be around $\kappa = (2\sqrt{2}\gamma_c/\gamma_p)^{1/2}$. In addition to the main high-purity region, the purity has a small peak (denoted as “Line 2”) similarly to the resonant case. By perturbation calculation, we can show that the purity has a maximum value of $P_{max} = 3 - 2\sqrt{2} \simeq 0.17$ at $\gamma_p = \kappa/(2\sqrt{2}\gamma_c)$. The purity and efficiency of filtered photons are given as

$$P(T) = \frac{\text{Tr}(\rho_f^2)}{\mathcal{P}_d(T)} = \frac{1}{\mathcal{P}_d} \int_{-T}^{T} dr \int_{-T}^{T} dr' |\rho(r, r', t)^2|,$$  \hspace{1cm} (50)

$$P_d(T) = [\text{Tr}(\rho_f)]^2 = \int_{-T}^{T} dr |\rho(r, r, t)|^2,$$  \hspace{1cm} (51)

where $P_d(T)$ is the square of the probability that a single photon is emitted during $0 \leq t < T$. $P_d(T)$ is also the square of the probability that a single photon is obtained after time filtering. By time filtering, the effective purity $P(T)$ is improved at the expense of the square of the probability $P_d(T)$.

We first discuss the resonant ($\omega_d = \omega_c$) case. In Fig. 10(a), we show the $T$-dependence of $P(T)$ and $P_d(T)$ for $\kappa/g = 2$ and $\gamma_p/g = 0.5$. In the $T \to \infty$ limit, $P(T)$ approaches 0.61, which is the purity without time filtering. As $T$ is shortened, the purity $P(T)$ increases, whereas the square of the probability $P_d(T)$ decreases. If we allow the reduction of the square of the probability to 1/2, one can set the filtering time at $T = 2\gamma_p$ (see the vertical dashed line in Fig. 10(a)), and obtain an improved purity of 0.85. Improvement of the purity is also possible in the detuned case. In Fig. 10(b), we show the $T$-dependence of $P(T)$ and $P_d(T)$ for $\kappa/g = \sqrt{2}/50$, $\gamma_p/g = 100$, and $\omega_d - \omega_c = 8g$, which is on the upper line of “Line 2” in Fig. 9(b). If we...
allow the reduction of the square of the probability to 1/2, the purity is improved from 0.17 to 0.28. We can improve the purity further by choosing a shorter filtering time. The efficiency, however, decreases exponentially in the limit $T \to 0$.

5. Summary

We analyzed the radiative decay of an excited dot in a solid-state cavity QED system and investigated the quantum–mechanical properties of the emitted photon, stressing the effects of pure dephasing. Our analysis is based on a model in which all the elements of the system (including environmental ones) are treated as active quantum–mechanical degrees of freedom. We rigorously solved the time evolution of the overall system and derived analytical expressions for the density matrix, pulse shape, spectrum, and purity of the emitted photon. These analytical results were visualized under realistic parameters. The main results are summarized as follows. (i) Changes in the dot decay rate owing to pure dephasing can be explained in terms of the quantum Zeno and anti-Zeno effects. The emitted photon pulse length is approximately given by the inverse of the dot decay rate. (ii) The emitted photon spectrum agrees with the Glauber formula. The mean energy of an emitted photon is not necessarily identical to that of the dot and energy conservation is seemingly broken. However, the present analysis revealed that the dot can exchange energy with the environment through pure dephasing coupling. Energy conservation holds when the environmental energy is included. (iii) The purity of the emitted photon is calculated as a measure of indistinguishability. To generate pure single photons, dephasing should be reduced as much as possible and the escape rate of cavity photons $\kappa$ should be taken near the optimal value ($\kappa = 2g$ for the resonant case and $\kappa = (2\sqrt{2g^2 + \gamma}^1/2$ for the strongly detuned case). (iv) Time filtering improves the purity of the emitted photon at the expense of a lower probability of single photon generation.

In our approach, we have assumed white noise for the level fluctuation of the QD. For a detailed comparison with results of experiments, we must consider more realistic models taking into account phonons and background carriers. The extension of the present approach to these extended models is an important problem left for future study.

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Appendix A: Relation to the Master Equation and Perturbation Method

To analyze the properties of emitted photons, one can employ the Master equation approach as an alternative method. The Master equation is derived for the present model as

$$
\frac{d}{dt} \begin{pmatrix}
\rho_{\sigma\sigma} \\
\rho_{\sigma a} \\
\rho_{a\sigma} \\
\rho_{aa}
\end{pmatrix} = -i \begin{pmatrix}
\gamma + \frac{\kappa}{2} & \gamma & -i \gamma & 0 \\
\gamma & \gamma + \frac{\kappa}{2} + \gamma_p + i \Delta \omega & 0 & -i \gamma \\
-i \gamma & 0 & \gamma + \frac{\kappa}{2} - i \Delta \omega & i \gamma \\
0 & -i \gamma & i \gamma & \kappa
\end{pmatrix} \begin{pmatrix}
\rho_{\sigma\sigma} \\
\rho_{\sigma a} \\
\rho_{a\sigma} \\
\rho_{aa}
\end{pmatrix},
$$

(A-1)

where $\rho_{\sigma\sigma} = |\sigma\rangle \langle \sigma|$, $\rho_{\sigma a} = |\sigma\rangle \langle a|$, $\rho_{a\sigma} = |a\rangle \langle \sigma|$, $\rho_{aa} = |a\rangle \langle a|$, and $\Delta \omega = \omega_d - \omega_e$. Using the solution of the master equation, the density matrix $\rho(r, r', t)$ of emitted photons can be written as

$$
\rho(r, r', t) = \begin{pmatrix}
\rho_{\sigma\sigma}(t - r' - r) \rho_{\sigma\sigma}(r' - r) + \rho_{\sigma a}(t - r') \rho_{a\sigma}(r' - r) & (r < r' < t), \\
\rho_{\sigma a}(t - r) \rho_{\sigma\sigma}(r' - r) + \rho_{\sigma a}(t - r') \rho_{a\sigma}(r' - r) & (r' < r < t)
\end{pmatrix},
$$

(A-2)

Here, we have defined $(\tilde{\sigma}_{0}(t), \tilde{\sigma}_{0}(t))$ as the solutions of Eq. (14) under the initial condition $(\tilde{\sigma}_{0}(0), \tilde{\sigma}_{0}(0)) = (0, 1)$.
The four eigenvalues of the matrix in Eq. (A-1) correspond to the values of \( \mu_i \). Therefore, numerical calculation of the eigenvalues of this matrix is a convenient method for obtaining the values of \( \mu_i \). It is also useful to treat the matrix in Eq. (A-1) in the perturbation calculation given in Appendix B. Because the matrix in Eq. (A-1) is a normal matrix, it is diagonalized by a unitary matrix. Therefore, the perturbation method used in quantum mechanics can be applied to obtain the small shift of the values of \( \mu_i \) with respect to small parameters.

**Appendix B: Expressions for Purity in Limiting Cases**

A simple expression for \( \mathcal{P} \) can be obtained in some limiting cases. We first discuss the strongly detuned case \((|\omega_d - \omega_c| \gg \gamma_p)\). The purity can then be expressed by a sum of two contributions as \( \mathcal{P} = \mathcal{P}_{\text{det}} + \mathcal{P}_{\text{cav}} \), where \( \mathcal{P}_{\text{det}} \) and \( \mathcal{P}_{\text{cav}} \) are the purities of photons whose frequencies are \( \omega_{\text{det}} \) and \( \omega_{\text{cav}} \), respectively. The former contribution \( \mathcal{P}_{\text{det}} \) becomes large only below “Line 1” in Fig. 9(b), whereas \( \mathcal{P}_{\text{cav}} \) has a maximum on “Line 2” in the figure. For \( \gamma_p \ll \gamma \ll \kappa \), perturbation calculation gives

\[
\mathcal{P}_{\text{det}} = \frac{\kappa}{2\gamma_p + \kappa} \times \frac{\kappa \epsilon}{2\gamma_p + \kappa \epsilon}.
\]

(B-1)

From this expression, it is proved that the purity takes a maximum value of \( 3 - 2\sqrt{2} / \gamma_p \) at \( \gamma_p = \sqrt{2\kappa / \kappa} \), which corresponds to the upper line of “Line 2” in Fig. 9(b).

Next, we consider the resonant case \((\omega_c = \omega_d)\). Some of the features are common to the strongly detuned case; the purity is reduced for \( \kappa \approx 2\gamma_p^2 / \gamma_p \) [the right-side line of “Line 1” in Fig. 9(a)] and takes a maximum value of \( 3 - \sqrt{2} / \gamma_p \) at \( \gamma_p = \sqrt{2\kappa / \kappa} \) “Line 2” in Fig. 9(a). For \( \gamma_p \ll \kappa \), the purity \( \mathcal{P} \) is approximately calculated as

\[
\mathcal{P} = \frac{\kappa^2 + \gamma_p}{2(\kappa + \gamma_p)(\kappa + \gamma_p)}.
\]

(B-2)

From this expression, it is proved that \( \mathcal{P} \geq 1/2 \) is realized for the condition \( \kappa \geq (1 + \sqrt{3}) \gamma_p \approx 2.73 \gamma_p \), which corresponds to the left line of “Line 1” in Fig. 9(a).

**Appendix C: Relation between Coincidence Probability and Purity**

Here, we derive the relation \( \mathcal{P}_{\text{co}} = (1 - \mathcal{P}) / 2 \) between the coincidence probability \( \mathcal{P}_{\text{co}} \) and the purity \( \mathcal{P} \). We consider the following situation (see Fig. 4). Two solid-state emitters (S1 and S2) simultaneously and deterministically emit single photons (assuming \( \gamma = 0 \) for simplicity) into two input ports (P1 and P2) of a beam splitter. These two photons are mixed by the beam splitter and are output to ports P3 and P4. We denote the photon field operators for port Pj by \( \tilde{a}_j(t) \) (j = 1, …, 4), and the pure dephasing reservoir operators for emitter Sj by \( \tilde{d}_j(t) \) (j = 1, 2, 3). Assuming \( t \rightarrow \infty \) and \( \gamma = 0 \) (and thus \( \delta_m = 0 \)) in Eq. (13), the state vector of the photon at P3 is given by

\[
|\psi_3\rangle = \frac{1}{2^\infty} \int \mathrm{d}t \int \mathrm{d}x \int \mathrm{d}x' \gamma_3(t, r, x) \tilde{b}_3(t, r, x) |\psi_3\rangle,
\]

(C-1)

where \( |\psi_3\rangle = \tilde{d}_1(t_1) \cdots \tilde{d}_4(t_4) |0\rangle \). Thus, the emitted photon is entangled with the environment of its source. The input state vector including the photons at both P1 and P2 is then given by

\[
|\psi_{12}\rangle = \frac{1}{2^\infty} \int \mathrm{d}t \int \mathrm{d}t' \int \mathrm{d}x \int \mathrm{d}x' \gamma_{12}(t, r, x) \gamma_{12}(t', x') \times \tilde{b}_1(t, r, x) \tilde{b}_2(t, r, x') |\psi_{12}\rangle,
\]

(C-2)

where \( |\psi_{12}\rangle = \tilde{d}_1(t_1) \cdots \tilde{d}_4(t_4) |0\rangle \). The beam splitter mixes the two photons as \( \tilde{b}_3(t, r, x) \rightarrow \tilde{b}_1(t, r, x) / \sqrt{2} \) and \( \tilde{b}_2(t, r, x) \rightarrow \tilde{b}_1(t, r, x) / \sqrt{2} \), but it obviously does not affect the environmental degrees of freedom. The output state vector is then given by \( |\psi_{\text{out}}\rangle = |\psi_{33}\rangle + |\psi_{44}\rangle + |\psi_{34}\rangle \), where

\[
|\psi_{33}\rangle = \frac{1}{2^\infty} \int \mathrm{d}t \int \mathrm{d}t' \int \mathrm{d}x \int \mathrm{d}x' \gamma_{33}(t, r, x) \gamma_{33}(t', r', x') \times \tilde{b}_3(t, r, x) \tilde{b}_3(t', r', x') |\psi_{33}\rangle,
\]

(C-3)

\[
|\psi_{44}\rangle = - \frac{1}{2^\infty} \int \mathrm{d}t \int \mathrm{d}t' \int \mathrm{d}x \int \mathrm{d}x' \gamma_{44}(t, r, x) \gamma_{44}(t', r', x') \times \tilde{b}_3(t, r, x) \tilde{b}_3(t', r', x') |\psi_{44}\rangle,
\]

(C-4)
It is readily confirmed that \(|\psi_{33}\rangle\langle\psi_{33}| + |\psi_{44}\rangle\langle\psi_{44}| + |\psi_{34}\rangle\langle\psi_{34}| = 1\) and \(|\psi_{33}\rangle\langle\psi_{33}| = |\psi_{44}\rangle\langle\psi_{44}|\). The coincidence probability \(P_{\text{co}}\) (i.e., the probability of finding single photons at both \(P_1\) and \(P_2\)), is given by \(P_{\text{co}} = |\langle\psi_{34}\rangle|\). Since the density matrix element is given by \(\rho(r,r') = \sum_{m=0}^{\infty} \int d^3 x \gamma_m^*(r',x)\gamma_m(r,x)\), \(P_{\text{co}}\) is recast in the following form:

\[
P_{11} = \frac{1}{2} - \frac{1}{2} \int dr \rho(r,r')\rho(r',r) = \frac{1}{2} - \frac{P_{\text{co}}}{2}.
\]  

(C-6)

When pure dephasing is present, the purity of the emitted photon will be less than unity and the coincidence probability will be nonzero.

### Appendix D: Proof of Eqs. (48) and (49)

Here, we analytically evaluate \(E_{\text{p}} = \int d\mathbf{k} |b_{\mathbf{k}}(t)b_{-\mathbf{k}}(t)|\) and \(E_{\text{c}} = \int d\mathbf{k} k|d_{\mathbf{k}}(t)d_{-\mathbf{k}}(t)|\) in the \(t \rightarrow \infty\) limit, where \((\cdots)_t = \langle \psi_1 | \cdots | \psi_t \rangle\). Switching to real-space representations, we have

\[
E_{\text{p}} = \frac{i}{2} \int dr \langle (\partial_t \hat{b}_r) \hat{b}_r - \hat{b}_r (\partial_t \hat{b}_r) \rangle,
\]  

(D-1)

\[
E_{\text{c}} = \frac{i}{2} \int dr \langle (\partial_t \hat{d}_r) \hat{d}_r - \hat{d}_r (\partial_t \hat{d}_r) \rangle.
\]  

(D-2)

From Eq. (8), we obtain \((\partial_t \hat{b}_r) \hat{b}_r = -\kappa \theta(t-r) \times \hat{b}_r \theta(t-r)\), where \(r = t-r\). Using similar equations, we have

\[
E_{\text{p}} = -\frac{\xi}{2} \int \frac{dr}{dr} \left( \frac{d}{dr} \right) a - a \left( \frac{d}{dr} \right) a, \quad \frac{\xi}{2}
\]  

(D-3)

\[
E_{\text{c}} = -2i\gamma \int \left( \frac{d}{dr} \right) a \left( \frac{d}{dr} \right) a - \sigma^a a \left( \frac{d}{dr} \right) \sigma^a a,
\]  

(D-4)

Using Eqs. (11) and (12) and their conjugates, we have

\[
E_{\text{p}} = \kappa a + \frac{g}{2} \int dr (a \sigma^a a + \langle \sigma^a a \rangle),
\]  

(D-5)

\[
E_{\text{c}} = g \gamma \int dr (\sigma^a a + \langle \sigma^a a \rangle).
\]  

(D-6)

Thus, we must evaluate \(I_1 = \int_0^\infty dr (a \sigma^a a), \quad I_2 = \int_0^\infty dr (a \sigma^a a), \quad I_3 = \int_0^\infty dr (a \sigma^a a).\)

The equations of motion for \(\langle \sigma^a a \rangle, \quad \langle a \sigma^a a \rangle, \quad \langle a \sigma^a a \rangle\) are given by

\[
\frac{d}{dt} (\sigma^a a) = -\gamma (\sigma^a a) - ig(\sigma^a a) - c.c.,
\]  

(D-7)

\[
\frac{d}{dt} (a \sigma^a a) = -\kappa (a \sigma^a a) + ig(\sigma^a a) - c.c.,
\]  

(D-8)

\[
\frac{d}{dt} (a \sigma^a a) = i(\sigma^a a + \sigma^a a) + ig(\sigma^a a - \langle \sigma^a a \rangle).
\]  

(D-9)

Integrating these equations with respect to \(r\), we obtain \(1 = \frac{dI_1}{dt} + ig(I_3 - I_1), \quad 0 = \kappa I_2 - ig(I_3 - I_1), \quad 0 = i(\sigma_3 - \sigma_3) I_3 + ig(I_3 - I_1).\) When \(g = 0\), these solutions are satisfied to yield \(I_2 = \frac{1}{\kappa} I_2 + \frac{1}{g} I_3 + g I_1\). Integrating these equations with respect to \(r\), we obtain \(E_{\text{p}} = \kappa I_2 + \frac{g}{2} (I_3 + I_1)\) and \(E_{\text{c}} = \gamma (I_3 + I_1)\), we obtain Eqs. (48) and (49).