Total Edge Irregularity Strength for Graphs

Irwansyah
Department of Mathematics,
Faculty of Mathematics and Natural Sciences,
Universitas Mataram, Jl. Majapahit 62, Mataram, 83125
INDONESIA

Salman A.N.M.
Combinatorial Mathematics Research Group,
Faculty of Mathematics and Natural Sciences,
Institut Teknologi Bandung, Jl. Ganesha 10, Bandung, 40132,
INDONESIA

Abstract
An edge irregular total $k$-labelling $f : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ of a graph $G$ is a labelling of the vertices and the edges of $G$ in such a way that any two different edges have distinct weights. The weight of an edge $e$, denoted by $wt(e)$, is defined as the sum of the label of $e$ and the labels of two vertices which incident with $e$, i.e. if $e = vw$, then $wt(e) = f(e) + f(v) + f(w)$. The minimum $k$ for which $G$ has an edge irregular total $k$-labelling is called the total edge irregularity strength of $G$. In this paper, we determine total edge irregularity of connected and disconnected graphs.

1 Introduction

Let $G = (V, E)$ be a simple graph with the vertex set $V$ and the edge set $E$. An edge irregular total $k$-labelling of $G$ is a labelling $f : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ such that for any two different edges $e = vw$ and $g$ in $E$, $wt(e) \neq wt(g)$ where $wt(e) = f(e) + f(v) + f(w)$. The minimum $k$ for which $G$ has an edge irregular total $k$-labelling is called the total edge irregularity strength of $G$, denoted by $tes(G)$. Bača et al. \cite{1} proved that for any non trivial graph $G = (V, E)$, $\left\lceil \frac{|E|+2}{3} \right\rceil \leq tes(G) \leq |E|$, and if $G$ has the maximum degree $\Delta$, then $tes(G) \geq \lceil \frac{\Delta+1}{2} \rceil$. Furthermore, if $\Delta \geq \frac{|E|-1}{2}$, then $tes(G) \leq |E| - \Delta$.

Bača et al. \cite{1} determined the total edge irregularity strength of paths, cycles, stars,
wheels, and friendship graphs. Nurdin et al. [2] investigated the total edge irregularity strength of the corona product of paths with some graphs and in [3] they studied the total edge irregularity strength of a disjoint union of \( t \) copies of \( K_{2,n} \). Moreover, Ivančo and Jendrol’ [4] completely determined the total edge irregularity strength of trees, and they gave the following conjecture.

\[
\text{Conjecture 1. Let } G \text{ be an arbitrary graph different from } K_5. \text{ Then }
\]
\[
\text{tes}(G) = \max \left\{ \left\lceil \frac{|E_G| + 2}{3} \right\rceil, \left\lceil \frac{\Delta_G + 1}{2} \right\rceil \right\}.
\]

Conjecture 1 holds for all mentioned graphs above. Moreover, the conjecture is also true for dense graphs [10] and large graphs [9]. Some other papers about the edge irregularity strength of graphs can be seen in [6, 7, 8, 11].

In this paper we give a new a way to create the total labelling for trees and determine the edge irregularity of tree with some added edges. Using this result, we determine the edge irregularity of all connected and disconnected graphs.

## 2 The Total Labelling for Trees

Let \( T_n \) be a tree on \( n \) vertices. As shown in [4], \( \text{tes}(T_n) = \max \left\{ \left\lceil \frac{n+1}{3} \right\rceil, \left\lceil \frac{\Delta_{T_n} + 1}{2} \right\rceil \right\} \), where \( \Delta_{T_n} \) is the maximum degree of \( T_n \). In this part, we give another way to create a total labelling for \( T_n \) and then we determine the total edge irregularity for tree with some added edges.

### 2.1 A New Technique for Total Labelling of Trees

Let \( v_0 \) be a vertex in \( T_n \), where \( \text{deg}(v_0) = \Delta_{T_n} \). Take \( v_0 \) as the root of \( T_n \). Let \( A \), \( B \), and \( C \) be the sets of vertices of \( T_n \) with cardinalities \( |A| = |C| = \left\lceil \frac{n+1}{3} \right\rceil \), and \( |B| = n - (2 \cdot \left\lceil \frac{n+1}{3} \right\rceil) \). Also, for sets of vertices \( X \) and \( Y \) of a graph \( G \) let us define \( E(X,Y) \) to be the set of all edges in \( G \) that have one end vertex in \( X \) and the other in \( Y \). First, we create partitions of vertices in \( T_n \) into the set \( A, B, \) and \( C \) as follows.

1. Take \( B \) such that \( v_0 \in B \) and take other \( |B| - 1 \) vertices including the neighbors of \( v_0 \) to be in \( B \).
2. Take \( A \) such that there are \( \left\lfloor \frac{L}{2} \right\rfloor \) leaves are in \( A \), where \( L \) is the number of leaves which are not in \( B \). Take the corresponding neighbors to be in \( A \) such that \( |A| = \left\lceil \frac{n+1}{3} \right\rceil \).
3. Take the rest of vertices to be in \( C \).

Here in an example on how to create partitions on \( T_n \).
Example 2. Consider the tree $T_{10}$ as in Figure 1. Using the above rule, we have the partition $A = \{v_4, v_5, v_7\}$, $B = \{v_0, v_1, v_2\}$, and $C = \{v_6, v_8, v_9\}$.

Let $\Lambda_{\alpha\beta} = |E(\alpha, \beta)|$, where $\alpha, \beta \in \{A, B, C\}$, and $\lambda = \max \left\{ \left\lceil \frac{n+1}{3} \right\rceil, \left\lceil \frac{\Delta T_n + 1}{2} \right\rceil \right\}$. Then we have the following algorithm to create a total labelling for $T_n$.

Algorithm 3. Given a tree $T_n$ and $\lambda$ as above.

1. We label all vertices in $A$ with 1.
2. We label all vertices in $C$ with $\lambda$.
3. We label the edges in $E(A, A)$ with consecutive integers from 1 to $\Lambda_{AA}$ to obtain the weights in the interval $[3, \Lambda_{AA} + 2]$.
4. We label the edges in $E(C, C)$ with consecutive integers from $\lambda$ down to $\lambda - \Lambda_{CC}$ to obtain the last $\Lambda_{CC} - 1$ weights, i.e. the weights in the interval $[3\lambda - \Lambda_{CC} + 2, 3\lambda]$.
5. We label the edges in $E(A, B)$ such that obtain the weights in the interval $[\Lambda_{AA} + 3, \Lambda_{AA} + 2 + \Lambda_{AB}]$. Note that, we label vertices in $B$ with $b_1 \leq b_2 \leq \cdots \leq b_{|B|}$. So, we need to do appropriate labeling for all edges in $E(A, B)$.
6. We create the weights interval $[3\lambda - \Lambda_{CC} + 2 - \Lambda_{CB}, 3\lambda - \Lambda_{CC} + 1]$ on the edges in $E(C, B)$.
7. We label the edges in $E(B, B)$ to obtain different weights from the interval $[\Lambda_{AA} + \Lambda_{AB} + 3, 3\lambda - \Lambda_{CC} + 1 - \Lambda_{BC}]$.
8. We label the edges in $E(A, C)$ to obtain weights in the interval $[\Lambda_{AA} + \Lambda_{AB} + 3, 3\lambda - \Lambda_{CC} + 1 - \Lambda_{BC}]$

which are different from all weights in step (5).

Figure 1: $T_{10}$
Here is an example of the labeling of $T_{10}$ in Figure 1 with partition as in Example 2.

**Example 4.** As we can check, $T_{10}$ in Figure 1 has $\lambda = 4$. Then we have the following labeling.

![Figure 2: Total edge irregular labelling for $T_{10}$](image)

**2.2 Tree with an Added Edge**

Let $T_{n,1}$ be a tree $T_n$ with 1-added edge. In this subsection, we will determine $\text{tes}(T_{n,1})$. Let $\lambda = \max\left\{\left\lceil \frac{n+1}{3} \right\rceil, \left\lceil \frac{\Delta T_{n,1}+1}{2} \right\rceil\right\}$. We have the following results.

**Proposition 5.** If $T_n$ is a tree with $|E_{T_n}| = 3\lambda - 2$ and $T_{n,1} = T_n \cup \{e\}$, then

$$\text{tes}(T_{n,1}) = \max\left\{\lambda + 1, \left\lceil \frac{\Delta T_{n,1}+1}{2} \right\rceil\right\}.$$  

**Proof.** Take $v_0 \in V(T_{n,1})$ as the root for $T_{n,1}$, where $\deg(v_0) = \Delta T_{n,1}$, create partition $A, B, C$, and apply Algorithm 3 to create total labelling for $T_n$. Without loss of generality, assume

$$\max\left\{\lambda + 1, \left\lceil \frac{\Delta T_{n,1}+1}{2} \right\rceil\right\} = \lambda + 1.$$  

Consider the following three cases.

1. Edge $e \in \{E(A, A), E(C, C)\}$.

   Without loss of generality, assume $e \in E(A, A)$. Apply Algorithm 3 where the weights interval for edges in $E(A, A)$ will become $[3, \Lambda'_AA+3]$, where $\Lambda'_AA = |E(A, A)| - 1$. The labelling for all vertices and edges in $A$ exists because $E(A, A) \leq \lambda + 1$. The labelling for other edges follows by shifting the weights intervals and suitable labels of $T_n$.

2. Edge $e \in \{E(A, B), E(C, B)\}$.
Without loss of generality, assume $e \in E(A, B)$. Let the labelling for edges in $E(A, A)$ and its corresponding vertices stay the same as in the labelling for $T_n$. Create labelling for edges in $E(A, B)$ such that its weights interval become

$$[\Lambda_{AA} + 3, \Lambda_{AA} + 3 + \Lambda'_{AB}],$$

where $\Lambda'_{AB} = |E(A, B)| - 1$. The labelling exists because $|E(A, B)| \leq \lambda + 1$. The rest of the labelling done by shifting the weights intervals and suitable labels of $T_n$.

(3) Edge $e \in \{E(B, B), E(A, C)\}$.

We let the labelling in $E(A, A)$ unchanged. We change the label of all vertices in $C$ to be $\lambda + 1$ and create labelling in $E(C, C)$ such that its weight interval become

$$[3(\lambda + 1) - \Lambda_{CC} + 2, 3(\lambda + 1)].$$

Now, if $e \in E(A, C)$, then the labelling must exists because, as we know, the weights in $E(A, C)$ are of the form $1+i+(\lambda+1)$, where $i \in \{1, 2, \ldots, \lambda+1\}$, and $|E(A, C)| \leq \lambda + 1$. The rest of the labelling done by shifting the weights intervals for the rest of the edges sets.

Otherwise, if $e \in E(B, B)$, then consider the labelling for $T_n$. The existence of the labelling for $T_n$ guarantee the existence of matching between the available weights with the edges in $E(B, B)$. Now, by adding edge $e$ to $E(B, B)$, we have an extra label, i.e. the label $\lambda + 1$, and two extra weights. So, at least can be paired with one of available weights. Therefore, by Marriage Theorem, we have the labelling for all edges in $E(B, B)$ and for $T_{n,1}$.

**Proposition 6.** If $T_n$ is a tree with $|E_{T_n}| + \delta = 3\lambda - 2$, where $\delta \in \{1, 2\}$, and $T_{n,1} = T_n \cup \{e\}$, then

$$\text{tes}(T_{n,1}) = \max \left\{ \lambda, \left\lceil \frac{\Delta_{T_{n,1}} + 1}{2} \right\rceil \right\}.$$  

**Proof.** We have to note that, in this case, the number of available weights is more than the number of edges in $T_n$. Take $v_0 \in V(T_{n,1})$ as the root for $T_{n,1}$, where $\deg(v_0) = \Delta_{T_{n,1}}$, create partition $A, B, C$, and apply Algorithm 3 to create total labelling for $T_n$. Without loss of generality, assume

$$\max \left\{ \lambda, \left\lceil \frac{\Delta_{T_{n,1}} + 1}{2} \right\rceil \right\} = \lambda.$$  

Consider the following three cases.

(1) Edge $e \in \{E(A, A), E(C, C)\}$.

Without loss of generality, assume $e \in E(C, C)$. Since we still able to add an edge $e$ to $E(C, C)$, we have that $\lambda > \Lambda_{CC}'$, where $\Lambda_{CC}' = |E(C, C)| - 1$. Therefore, we can label edge $e$ with $i \in \{1, 2, \ldots, \lambda - \Lambda_{CC}' - 1\}$. The labelling in the other set of edges
follows by shifting the labels to get the desired weights.

(2) Edge \( e \in \{E(A, B), E(C, B)\} \).

Let the labelling in \( E(A, A) \) and \( E(C, C) \) unchanged as in \( T_n \). Without lost of generality, assume \( e \in E(C, B) \). We know that, \( \Lambda'_{CB} \leq 0 \), where \( \Lambda_{CB} = |E(C, B)| - 1 \). Therefore, we still have available label \( i \) for edge \( e \), where \( \{1, 2, \ldots, \lambda\} \). The labelling in the other set of edges follows by shifting the labels to get the desired weights.

(3) Edge \( e \in \{E(B, B), E(A, C)\} \).

Let the labelling in \( E(A, A) \) and \( E(C, C) \) unchanged as in \( T_n \). If \( e \in E(A, C) \), then we have \( \lambda > \Lambda'_{AC} \), where \( \Lambda'_{AC} = |E(A, C)| - 1 \). So, we can label edge \( e \) with label \( i \in \{1, 2, \ldots, \lambda\} \). The labelling in the other set of edges follows by shifting the labels to get the desired weights.

By similar argument, we can have the labelling for \( T_{n,1} \) where \( e \in E(B, B) \).

Here are two examples of the total labelling for \( T_{n,1} \).

**Example 7.** Let \( T_{10} \) as in Figure 1 and its total labelling as in Example 2. We can see that, \( |E_{T_{10}}| + 1 = 2\lambda - 2 \), where \( \lambda = 4 \). The following figure shows the total labelling for \( T_{10,1} = T_{10} \cup \{e\} \) where \( e \in E(B, B) \), i.e. the red colored edge.

![Figure 3: Total edge irregular labelling for \( T_{10,1} \)](image)

**Example 8.** Given the labelling for \( T_8 \) as shown in Figure 4. As we can see that, \( |E_{T_8}| = 3\lambda - 2 \), where \( \lambda = 3 \). If we consider \( T_{8,1} = T_8 \cup \{e\} \), where \( e \in E(A, A) \), i.e. the orange colored edge, then we have the total labelling as shown in Figure 5.

### 2.3 Total Edge Irregularity of Connected Graphs

Let \( T_{n,j} \) be a tree \( T_n \) with \( j \)-added edges. In this subsection, we will determine \( \text{tes}(T_{n,j}) \), where \( T_{5,3} \neq K_5 \), and determine the edge irregularity of connected graphs.

Let \( \lambda_i = \max \left\{ \left\lfloor \frac{|E_{T_{n,i}}|+2}{3} \right\rfloor, \left\lceil \frac{|\Delta_{T_{n,i}}|+1}{2} \right\rceil \right\} \). The following result determines the edge irregularity of \( T_{n,j} \).
Theorem 9. If $T(n, j)$ is a tree with $j$-added edges which is different from $K_5$, then $\text{tes}(T_{n,j}) = \lambda_j$.

Proof. We prove this by induction. As we can see, Proposition 5 and Proposition 6 give us the induction base. Now, assume that $\text{tes}(T_{n,j-1}) = \lambda_{j-1}$, i.e. we have created the total labelling for $T_{n,j-1}$ using Algorithm 3. Using the similar analysis as in Proposition 5 and Proposition 6 we can have the labelling for $T_{n,j}$ with

$$\lambda_j = \max \left\{ \left\lceil \frac{|E_{T_{n,j}}| + 2}{3} \right\rceil, \left\lceil \frac{|\Delta_{T_{n,j}}| + 1}{2} \right\rceil \right\}.$$

Therefore, $\text{tes}(T_{n,j}) = \lambda_j$. \hfill \Box

As a consequence, we have the following theorem.

Theorem 10. If $G$ is a connected graph, where $G \neq K_5$, then

$$\text{tes}(G) = \max \left\{ \left\lceil \frac{|E_G| + 2}{3} \right\rceil, \left\lceil \frac{|\Delta_G| + 1}{2} \right\rceil \right\}.$$ 

Proof. We can see that every connected graph can be constructed from some tree by adding some edges. Therefore, $G = T_{n,j}$, for some $T_n$ and $j$. By Theorem 9 we have the result. \hfill \Box
Here is an example.

**Example 11.** The total labelling of a connected graph constructed from $T_8$ from Example 8 by adding 4 edges, i.e. red colored edges, is shown in Figure 6.

![Figure 6: Total edge irregular labelling for a connected graph constructed from $T_8$ by adding 4 edges](image)

**3 Total Edge Irregularity of Disconnected Graphs**

As we know, every disconnected graph is a union of connected graphs, i.e. every component of disconnected graph is connected graph. The following results determine the edge irregularity of disconnected graphs with two components.

**Proposition 12.** If $G = G_1 \cup G_2$, where $G_i$ is a connected graph and $G_i \neq K_5$, for all $i = 1, 2$, then

$$\text{tes}(G) = \max \left\{ \left\lceil \frac{|E_G| + 2}{3} \right\rceil, \left\lceil \frac{\Delta_G + 1}{2} \right\rceil \right\}.$$ 

**Proof.** First, consider the case when

$$\max \left\{ \left\lceil \frac{|E_{G_1}| + 2}{3} \right\rceil, \left\lceil \frac{\Delta_{G_1} + 1}{2} \right\rceil \right\} = \left\lceil \frac{|E_G| + 2}{3} \right\rceil.$$ 

We create total labelling for $G_1$ using Algorithm 3 with minimum label in its vertices is 1 and the maximum one is $\lambda_1 = \max \left\{ \left\lceil \frac{|E_{G_1}| + 2}{3} \right\rceil, \left\lceil \frac{\Delta_{G_1} + 1}{2} \right\rceil \right\}$. Then we continue to create labelling for $G_2$ using Algorithm 3 starting from the smallest available weight, with minimum label in its vertices is $\lambda_1 + 1$ and maximum one is $\max \left\{ \left\lceil \frac{|E_{G_2}| + 2}{3} \right\rceil, \left\lceil \frac{\Delta_{G_2} + 1}{2} \right\rceil \right\}$. Based on the total irregularity for connected graphs, we have the conclusion.
Second, consider the case when

\[ \max \left\{ \left\lceil \frac{|E_G| + 2}{3} \right\rceil, \left\lceil \frac{\Delta_G + 1}{2} \right\rceil \right\} = \left\lceil \frac{\Delta_G + 1}{2} \right\rceil. \]

Without loss of generality, assume \( G_2 \) contains vertex \( v \) with \( \deg(v) = \Delta_G \). Then proceed as the first case by creating the labelling for \( G_2 \) and then for \( G_1 \).

**Proposition 13.** If \( G = K_5 \cup G_1 \), where \( G_1 \) is a connected graph with at least one edge, then

\[ \text{tes}(G) = \max \left\{ \left\lceil \frac{|E_G| + 2}{3} \right\rceil, \left\lceil \frac{\Delta_G + 1}{2} \right\rceil \right\}. \]

**Proof.** First, we create the total labelling for \( K_5 \) as in [11, Proposition 2.2]. Then we proceed as in the proof of Proposition 12.

Here are two examples correspond to the above results.

**Example 14.** The graph in Figure 7 consists of connected graph with 8 and 7 edges. So, we must have the total edge irregularity strength equals to 6.

![Figure 7: Labelling of disconnected graph with two connected components](image)

**Example 15.** The graph in Figure 8 is a union of two \( K_5 \). So, we must have the total edge irregularity strength equals to 8.

The following theorem gives the total edge irregularity for disconnected graphs.

**Theorem 16.** If \( G = \bigcup_{i=1}^{n} G_i \), where \( G_i \) is a connected graph with at least one edge, then

\[ \text{tes}(G) = \max \left\{ \left\lceil \frac{|E_G| + 2}{3} \right\rceil, \left\lceil \frac{\Delta_G + 1}{2} \right\rceil \right\}. \]
Proof. We will prove the theorem using induction. As we can see, Proposition 12 and Proposition 13 give us the induction base. Now, assume that the conclusion is true for \( n - 1 \) union of connected graphs. We would like to prove for \( n \) union of graphs. We can see that, by similar method as in the proof of Proposition 12 and Proposition 13, we can create the labelling for \( n \) union of connected graphs.

Here is an example for the above theorem.

**Example 17.** The graph in Figure 9 is a union of \( S_4, C_4, \) and \( P_4 \). The total edge irregularity strength of the graph is 5.

![Figure 8: Labelling for union of \( K_5 \) with other graph](image1)

![Figure 9: Labelling for disconnected graph with three connected components](image2)

The following remark gives some trivial examples of graphs where Conjecture 1 do not hold.

**Remark 18.** If \( G = K_5 \cup_{i=1}^{m} G_i \), where \( G_i \) is a graph with only one vertex, for all \( i = 1, 2, \ldots, m \), then \( \text{tes}(G) = 5 \).

**Proof.** Based on Conjecture 1, \( \text{tes}(G) \) is supposed to be 4. By creating the total labelling for \( K_5 \), we will have that \( \text{tes}(G) \) must be 5.
References

[1] M. Bača, S. Jendrol’, M. Miller and J. Ryan, On irregular total labellings, *Discrete Math.* **307** (2007) 1378-1388.

[2] Nurdin, A.N.M. Salman and E.T. Baskoro, The total edge-irregular strengths of the corona product of paths with some graphs, *J. Combin. Math. Combin. Comput.* **65** (2008) 163-175.

[3] Nurdin, E.T. Baskoro and A.N.M. Salman, The total edge irregular strengths of union graphs of $K_{2,n}$, *J. Mat. Sains 11 3* (2006) 105-109.

[4] J. Ivančo and S. Jendrol’, Total edge irregularity strength of trees, *Discuss. Math. Graph Theory* **26** (2006) 449-456.

[5] Y. Shibata, Y. Kikuchi, Graph products based on the distance in graphs, *IEICE Trans. Fundam. 3 E83-A* (2000) 459–464.

[6] Nurdin, E.T. Baskoro, A.N.M. Salman and N.N. Gaos, On total edge-irregularity strength of the corona product of paths with stars, *Proceedings of ICORAFSS III* (2009) 114-116.

[7] A.N.M. Salman, T. Agustina and Nurdin, On the total edge-irregular strengths of the grids, *Proceedings of ICMNS II* (2008) 1405-1408.

[8] Nurdin, E.T. Baskoro and A.N.M. Salman, The edge total irregular strength on the corona product of cycles with the complement of complete graphs, *Proceedings of ICMNS I* (2006) 781-784.

[9] F. Pfender, Total edge irregularity strength for large graphs, *Discrete Mathematics* **312** (2009) 229-237.

[10] S. Brandt, J. Miškuf, and D. Rautenbach, On a conjecture about edge irregular total labellings, *Journal of Graph Theory* (2008) 333-343, DOI: 10.1002/jgt.20287.

[11] S. Jendrol’, J. Miškuf, and R. Soták, Total edge irregularity strength of complete graphs and complete bipartite graphs, *Discrete Mathematics* **310** (2010) 400-407.