**Betti Numbers of Bresinsky’s Curves in** \( \mathbb{A}^d \)

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**Abstract.** Bresinsky defined a class of monomial curves in \( \mathbb{A}^d \) with the property that the minimal number of generators or the first Betti number of the defining ideal is unbounded above. We prove that the same behaviour of unboundedness is true for all the Betti numbers and construct an explicit minimal free resolution for this class.

1. **Bresinsky’s Examples**

Let \( r \geq 3 \) and \( n_1, \ldots, n_r \) be positive integers with \( \gcd(n_1, \ldots, n_r) = 1 \). Let us assume that the numbers \( n_1, \ldots, n_r \) generate the numerical semigroup

\[
\Gamma(n_1, \ldots, n_r) = \{ \sum_{j=1}^r z_j n_j \mid z_j \text{ nonnegative integers} \}
\]

minimally, that is if \( n_i = \sum_{j=1}^r z_j n_j \) for some non-negative integers \( z_j \), then \( z_j = 0 \) for all \( j \neq i \) and \( z_i = 1 \). Let \( \eta : k[x_1, \ldots, x_r] \to k[t] \) be the mapping defined by \( \eta(x_i) = t^{n_i} \), \( 1 \leq i \leq r \), where \( k \) is a field. Let \( p(n_1, \ldots, n_r) = \ker(\eta) \). Let \( \beta_1(p(n_1, \ldots, n_r)) \) denote the first Betti number of the ideal \( p(n_1, \ldots, n_r) \). Therefore, \( \beta_1(p(n_1, \ldots, n_r)) \) denotes the minimal number of generators \( p(n_1, \ldots, n_r) \). For a given \( r \geq 3 \), let \( \beta_i(r) = \sup(\beta_i(p(n_1, \ldots, n_r))) \), where \( \sup \) is taken over all the sequences of positive integers \( n_1, \ldots, n_r \). Herzog [8] proved that \( \beta_1(3) \) is 3 and it follows easily that \( \beta_2(3) \) is a finite integer as well. Bresinsky in [1], [2], [3], [4], [5], extensively studied relations among the generators \( n_1, \ldots, n_r \) of the numerical semigroup defined by these integers. It was proved in [2] and [3] respectively that, for \( r = 4 \) and for certain cases in \( r = 5 \), the symmetry condition on the semigroup generated by \( n_1, \ldots, n_r \) imposes an upper bound on the first Betti number \( \beta_1(p(n_1, \ldots, n_r)) \). This remains an open question in general whether symmetry condition on the numerical semigroup generated
by $n_1, \ldots, n_r$ imposes an upper bound on $\beta_1(p(n_1, \ldots, n_r))$. Bresinsky [1] constructed a class of monomial curves in $A^4$ to prove that $\beta_1(4) = \infty$. He used this observation to prove that $\beta_1(r) = \infty$, for every $r \geq 4$. Our aim in this article is to prove in Theorem 1.1 that for Bresinsky’s examples $\beta_i(4) = \infty$, for every $1 \leq i \leq 3$ and also describe all the syzygies explicitly in [3,2] and [4,1]. A similar study has been carried out by J. Herzog and D.I. Stamate in [9] and [12]. However, the objective and approach in our study are quite different. The main theorem and underlying objective of our work can be found after the description of Bresinsky’s examples.

Let us recall Bresinsky’s example of monomial curves in $A^4$, as defined in [1]. Let $q_2 \geq 4$ be even. $q_1 = q_2 + 1, d_1 = q_2 - 1$. Set $n_1 = q_1 q_2, n_2 = q_1 d_1, n_3 = q_1 q_2 + d_1, n_4 = q_2 d_1$. It is clear that $\gcd(n_1, n_2, n_3, n_4) = 1$. For the rest of the article let us use the shorthand $n$ to denote Bresinsky’s sequence of integers defined above. Bresinsky [1] proved that the set $A = A_1 \cup A_2 \cup \{g_1, g_2\}$ generates the ideal $p(n_1, \ldots, n_4)$, where $A_1 = \{f_\mu | f_\mu = x_1^{\mu - 1} x_3^{d_1 - \mu} - x_2^{q_2 - \mu} x_4^{q_1 + 1}, 1 \leq \mu \leq q_2\}$, $A_2 = \{f | f = x_1^{n_1} x_4^{n_4} - x_2^{n_2} x_3^{n_3}, n_1, n_3 < d_1\}$ and $g_1 = x_1^{d_1} - x_2^{q_2}, g_2 = x_3 x_4 - x_2 x_1$. Let us first state the main theorems proved in this paper:

**Theorem 1.1.** Let $S = A_1 \cup A_2' \cup \{g_1, g_2\}$, where $A_2' = \{h_m | x_1^{m} x_4^{(q_1 - m)} - x_2^{(q_2 - m)} x_3^{m}, 1 \leq m \leq q_2 - 2\}$.

(i) $S$ is a minimal generating set for the ideal $p(n)$;
(ii) $S$ is a Gröbner basis for $p(n)$ with respect to the lexicographic monomial order induced by $x_3 > x_2 > x_1 > x_4$ on $k[x_1, \ldots, x_4]$;
(iii) $\beta_1(p(n)) = |S| = 2q_2$;
(iv) $\beta_2(p(n)) = 4(q_2 - 1)$;
(v) $\beta_3(p(n)) = 2q_2 - 3$.
(vi) A minimal free resolution for the ideal $p(n)$ over the polynomial ring $R = K[x_1, x_2, x_3, x_4]$ is

$$
0 \rightarrow R^{2q_2 - 3} \xrightarrow{P} R^{4(q_2 - 1)} \xrightarrow{N} R^{2q_2} \rightarrow R \rightarrow R/p(n) \rightarrow 0,
$$

where

$$
P = [\delta_1 \ldots \delta_{q_2 - 2} | \xi | \zeta | \eta | \kappa_1 \ldots \kappa_{q_2 - 4}]_{(q_2 - 1) \times 2q_2 - 3}
$$

$$
N = [\beta_1 \ldots \beta_{q_2 - 1} | \gamma_1 \ldots \gamma_{q_2 - 3} | \alpha' | \beta' | \gamma' \alpha_1 \ldots \alpha_{q_2 - 1} | -\gamma | \beta' | \gamma']_{2q_2 \times 4(q_2 - 1)}.
$$

The proof of the theorem is divided into various lemmas, theorems and corollaries in sections 2, 3 and 4. We first prove that a special subset of binomials form a minimal generating set as well as a Gröbner basis for the ideal $p(n)$ with respect to a suitable monomial order; see parts (i) and (ii).
of Theorem 1.1. We then compute the syzygy modules using this Gröbner basis explicitly and minimally in 3.2 and 4.1. We have not only computed all the total Betti numbers but also have written a minimal free resolution explicitly; see parts (iii) - (vi) in Theorem 1.1 and 4.2. However, in order to determine the minimality of the first syzygy module, we have used the second Betti number for these ideals calculated in [12]. It should be mentioned here that a minimal generating set of binomials for $p(n)$ has also been calculated in [9]. The authors have also computed a minimal standard basis in [9] and that has been used to calculate the Betti numbers in [12]. While the description of the tangent cone in [9] has been used to compute the Betti numbers in [12], we on the other hand have imitated Bresinsky’s approach and studied the generators of the ideal $p(n)$ and its syzygies, leading to a complete description of a minimal free resolution of the defining ideal.

This work grew out in an attempt to understand and generalize Bresinsky’s construction of the numerical semigroups in arbitrary embedding dimension. What is certainly interesting is that $n_1 + n_2 = n_3 + n_4$, for the sequence of integers $n = (n_1, n_2, n_3, n_4)$ given by Bresinsky. We have initiated a study of numerical semigroups defined by a sequence of integers formed by concatenation of two arithmetic sequences and we believe that such semigroups with correct conditions would finally give us a good model of numerical semigroups in arbitrary embedding dimension with unbounded Betti numbers; see [10], [11].

2. The first Betti number

We first construct the set $A_2' \subset A_2$, in order to extract a minimal generating set out of the generating set constructed by Bresinsky [11]. Let $q_2 \geq 4$ be an even integer. Considering the binomials occurring in the set $A_2$, we arrive at the equation

\[(2.1) \quad (q_2^2 + q_2)\nu_1 + (q_2^2 - q_2)\nu_4 + (1 - q_2^2)\mu_2 = (q_2^2 + 2q_2 - 1)\mu_3,\]

where $0 \leq \nu_1, \mu_3 \leq q_2 - 1$ and $\nu_4, \mu_2 \geq 0$. Simple divisibility conditions show that the only possibility is $\nu_1 = \mu_3 = \ell$, $0 \leq \ell \leq q_2 - 2$.

Consider the equations

\[(2.2) \quad (q_2^2 - q_2)\nu_4 + (1 - q_2^2)\mu_2 = (q_2^2 + 2q_2 - 1 - q_2^2 - q_2)\ell\]

\[(2.3) \quad q_2\nu_4 - (q_2 + 1)\mu_2 = \ell\]

The solution is given by $\nu_4 = (1 + q_2)t - \ell$, where $t \in \mathbb{Z}$, $\nu_4 \geq 0$ and $\mu_2 = q_2t - \ell$, where $t \in \mathbb{Z}$, $\mu_2 \geq 0$. Now $\nu_4 \geq 0$ implies that $(1 + q_2)t \geq \ell$. 
Let us consider the polynomials

\[ \nu_4 = (1 + q_2)t, \quad \mu_2 = q_2t, \quad \text{when} \quad \ell = 0 \quad \text{(and} \quad t \geq 0); \]

\[ \nu_4 = (1 + q_2)t - \ell, \quad \mu_2 = q_2t - \ell, \quad \text{when} \quad 1 \leq \ell \leq q_2 - 2 \quad \text{(and} \quad t \geq 1). \]

**Lemma 2.1.** Let us consider the polynomials \( \alpha_1 = x_4^{q_2+1} - x_2^{q_2}, \alpha_2 = x_4^{q_2+3} - x_2 x_3^{q_2-1} \). Then \( \alpha_1, \alpha_2 \in \langle S \rangle \), where \( S = A_1 \cup \{g_1, g_2\} \cup A'_2 \).

**Proof.** We have \( f_{q_2} = x_1^{q_2-1} - x_4^{q_2+1} \in A_1 \). It is clear that \( \alpha_1 = g_1 - f_{q_2} \) and \( \alpha_2 = x_4^2 \cdot f_{q_2} - x_2 \cdot f_1 + x_4^2 \cdot g_1 \). \( \Box \)

**Theorem 2.2.** The ideal \( p(\mu) \) is generated by the set \( S = A_1 \cup \{g_1, g_2\} \cup A'_2 \).

**Proof.** By Bresinsky’s result, the set \( A_1 \cup A_2 \cup \{g_1, g_2\} \) generates the ideal \( p(\mu) \). It is evident that \( A'_2 \subseteq A_2 \). Therefore it is enough to show that every element of \( A_2 \) is in the ideal generated by \( S \). Now, consider two cases: \textbf{Case 1:} \( \ell = 0 \); \textbf{Case 2:} \( 1 \leq \ell \leq q_2 - 2 \) and induct on \( t \) to show that \( H_t \in \langle S \rangle \) for all \( t \geq 0 \) in Case 1 and \( H_t \in \langle S \rangle \) for all \( t \geq 1 \) in Case 2, with the help of Lemma 2.1. \( \square \)

**Theorem 2.3.** The set \( S = A_1 \cup \{g_1, g_2\} \cup A'_2 \) is a minimal generating for the ideal \( p(\mu) \).

**Proof.** By lemma 4 of \( \mathbf{[1]} \) we have \( A_1 \) as a part of minimal generating set of ideal \( p(\mu) \). Now it is enough to prove that any element \( r \in T := \{g_1, g_2\} \cup A'_2 \) can not be expressible as the polynomial combination of \( S \setminus \{r\} \). Define maps \( \pi_1 : k[x_1, x_2, x_3, x_4] \to k[x_2, x_3, x_4] \), such that \( \pi_1(x_1) = 0 \) and \( \pi_1(x_i) = x_i \), for every \( i = 2, 3, 4 \). We also define \( \pi_2 : k[x_1, x_2, x_3, x_4] \to k[x_1, x_3, x_4] \), such that \( \pi_2(x_2) = 0 \) and \( \pi_1(x_i) = x_i \), for every \( i = 1, 3, 4 \).

Let \( r = g_1 = \sum_{\mu=1}^{q_2} A_\mu f_\mu + \alpha_2 g_2 + \sum_{m=1}^{q_2-2} B_m h_m \), where \( A_\mu, \alpha_2, B_m \in k[x_1, x_2, x_3, x_4] \), for \( 1 \leq \mu \leq q_2 \) and \( 1 \leq m \leq q_2 - 2 \). Application of \( \pi_1 \) on both side of the above equation gives rise to an absurd equality. Now let \( r = g_2 = \sum_{\mu=1}^{q_2} A_\mu f_\mu + \alpha_1 g_1 + \sum_{m=1}^{q_2-2} B_m h_m \), where \( A_\mu, \alpha_1, B_m \in k[x_1, x_2, x_3, x_4] \) for \( 1 \leq \mu \leq q_2 \) and \( 1 \leq m \leq q_2 - 2 \). Application of \( \pi_2 \) on the above equation gives us an equality which is absurd. Finally, let \( r \in A'_2 \), given by \( r = h_m = \sum_{\mu=1}^{q_2} A_\mu f_\mu + \alpha_1 g_1 + \sum_{m=1, m \neq m'}^{q_2-2} B_m h_m \). Application of \( \pi_2 \) on the above expression leads to an absurd equality. \( \square \)

**Lemma 2.4.** \( \beta_1(p(\mu)) = |S| = 2q_2 \).
Proof. It is very easy to observe that the sets $A_1$, $\{g_1, g_2\}$ and $A'_2$ are mutually disjoint. Therefore $|S| = |A_1| + |\{g_1, g_2\}| + |A'_2| = q_2 + 2 + (q_2 - 2) = 2q_2$. □

3. THE SECOND BETTI NUMBER

Theorem 3.1. Consider the lexicographic monomial order induced by $x_3 > x_2 > x_1 > x_4$ in $k[x_1, x_2, x_3, x_4]$. Then,

(i) The set $S$ forms Gröbner basis for the ideal $p(n)$ with respect to the above monomial order.

(ii) Let $D$ denotes the set of all Schreyer tuples obtained from Gröbner basis $S$ which generate the first syzygy module (see Theorem 1.43 [7]). Then each entry in the elements of $D$ is either a non-constant polynomial or zero.

Proof. At first we order the set $S = A_1 \cup \{g_1, g_2\} \cup A'_2$ as follows,

$$(f_1, \ldots, f_{q_2}, g_1, g_2, h_1, \ldots, h_{q_2-2}).$$

Let $f, g \in S$. We consider the $S$-polynomials $S(f, g)$ and divide the proof into cases based on the sets $f$ and $g$ belonging to.

Case 1. $f, g \in A_1$.

1(a). $f = f_\mu$, $g = f_{\mu+1}$ where, $1 \leq \mu \leq q_2 - 1$. We have

$$S(f_\mu, f_{\mu+1}) = x_1 \cdot f_\mu - x_3 \cdot f_{\mu+1}$$

$$= (x_2^{q_2-\mu} x_4^{\mu+1}) \cdot g_2 \rightarrow_S 0.$$

Therefore, the set

$$T_1 = \{\beta_\mu = (\beta_{(\mu, 1)}, \ldots, \beta_{(\mu, 2q_2)}) \mid 1 \leq \mu \leq q_2 - 1\}$$

gives the Schreyer tuples, where

$$\beta_{(\mu, \mu)} = -x_1$$

$$\beta_{(\mu, \mu+1)} = x_3$$

$$\beta_{(\mu, q_2+2)} = x_2^{q_2-(\mu+1)} x_4^{\mu+1}$$

$$\beta_{(\mu, i)} = 0, \text{ for } i \notin \{\mu, \mu + 1, q_2 + 2\}.$$

1(b). $f = f_\mu$, $g = f_{\mu'}$ where, $\mu' > \mu + 1$. We have

$$S(f_\mu, f_{\mu'}) = x_1^{\mu'-\mu} \cdot f_\mu - x_3^{\mu'-\mu} \cdot f_{\mu'}$$

$$= x_2^{q_2-\mu} x_4^{\mu'+1} \cdot (x_3 x_4)^{\mu'-\mu-1} + (x_3 x_4)^{\mu'-\mu-2} (x_1 x_2) + \cdots + (x_1 x_2)^{\mu'-\mu-1} \cdot g_2$$

$$\rightarrow_S 0.$$
Therefore, the set 

\[ \mathbb{T}_2 = \{ \gamma_{\mu, \mu'} = (\gamma(\mu', 1), \ldots, \gamma(\mu', i), \ldots, \gamma(\mu', q_2)) \mid 1 \leq \mu \leq q_2 - 2, \mu + 2 \leq \mu' \leq q_2 \} \]

gives the Schreyer tuples, where

\[
\begin{align*}
\gamma(\mu', \mu) &= -x_1^{\mu'-\mu} \\
\gamma(\mu', \mu') &= x_3^{\mu'-\mu} \\
\gamma(\mu', q_2+2) &= x_2^{q_2-\mu} x_4^{\mu'+1} ((x_3 x_4)^{\mu'-\mu-1} + (x_3 x_4)^{\mu'-\mu-2} (x_1 x_2) + \cdots + (x_1 x_2)^{\mu'-\mu-1}) \\
\gamma(\mu', i) &= 0 \quad \text{for} \quad i \notin \{\mu, \mu', (q_2 + 2)\}.
\end{align*}
\]

**Case 2.** \( f \in A_1, \ g \in \{g_1, g_2\} \)

**2(a).** Let \( f = f_\mu, \ g = g_1 \), where \( 1 \leq \mu \leq q_2 \). We have

\[
S(f_\mu, g_1) = x_2^{q_2} \cdot f_\mu + x_1^{\mu-1} x_3^{q_2-\mu} \cdot g_1
\]

\[
= x_1^{q_2-1} \cdot f_\mu + x_2^{q_2-\mu} x_4^{\mu'+1} \cdot g_1 \quad \longrightarrow \mathbb{S} 0
\]

Therefore, the set 

\[ \mathbb{T}_3 = \{ \gamma_\mu \mid \gamma_\mu = (\gamma(\mu, 1), \ldots, \gamma(\mu, i), \ldots, \gamma(\mu, q_2)) \mid 1 \leq \mu \leq q_2 - 1 \} \]

gives us the Schreyer tuples, where

\[
\begin{align*}
\gamma(\mu, \mu) &= x_1^{q_2-1} - x_2^{q_2} \\
\gamma(\mu, q_2+1) &= x_2^{q_2-\mu} x_4^{\mu'+1} - x_1^{\mu-1} x_3^{q_2-\mu} \\
\gamma(\mu, i) &= 0, \quad \text{for} \quad i \notin \{\mu, (q_2 + 1)\}, \ 1 \leq i \leq 2q_2.
\end{align*}
\]

**2(b).** Let \( f = f_\mu, \ g = g_2 \), where \( 1 \leq \mu \leq q_2 - 1 \). We have

\[
S(f_\mu, g_2) = x_4 \cdot f_\mu - x_1^{\mu-1} x_3^{q_2-(\mu+1)} \cdot g_2
\]

\[
= x_2 \cdot f_{\mu+1} \quad \longrightarrow \mathbb{S} 0
\]

Therefore, the set 

\[ \mathbb{T}_4 = \{ \alpha_\mu \mid \alpha_\mu = (\alpha(\mu, 1), \ldots, \alpha(\mu, i), \ldots, \alpha(\mu, q_2)) \mid 1 \leq \mu \leq q_2 - 1 \} \]

gives the Schreyer tuples, where

\[
\begin{align*}
\alpha(\mu, \mu) &= -x_4 \\
\alpha(\mu, \mu+1) &= x_2 \\
\alpha(\mu, q_2+2) &= x_1^{\mu-1} x_3^{q_2-(\mu+1)} \\
\alpha(\mu, i) &= 0, \quad \text{for} \quad i \notin \{\mu, \mu + 1, (q_2 + 2)\}, \ 1 \leq i \leq 2q_2.
\end{align*}
\]
2(c). Let \( f = f_{q_2}, g = g_2 \). We have
\[
S(f_{q_2}, g_2) = x_3x_4 \cdot f_{q_2} - x_1^{q_2-1} \cdot g_2 \\
= x_1x_2 \cdot f_{q_2} - x_4^{q_2+1} \cdot g_2 \quad \rightarrow S 0
\]
Therefore, the set
\[
T_5 = \{ \alpha = (\alpha_1, \ldots, \alpha_i, \ldots, \alpha_{2q_2}) \}
\]
gives the Schreyer tuples, where
\[
\alpha_{q_2} = x_1x_2 - x_3x_4 \\
\alpha_{(q_2+2)} = x_1^{q_2-1} - x_4^{q_2+1} \\
\alpha_i = 0, \quad \text{for} \quad i \notin \{q_2, (q_2 + 2)\}, \; 1 \leq i \leq 2q_2.
\]

Case 3. Let \( f = g_1, g = g_2 \). We have
\[
S(g_1, g_2) = -x_3x_4 \cdot g_1 - x_2^{q_2} \cdot g_2 \\
= -x_1x_2 \cdot g_1 - x_4^{q_2-1} \cdot g_2 \quad \rightarrow S 0
\]
Therefore, the set
\[
T_6 = \{ \beta = (\beta_1, \ldots, \beta_i, \ldots, \beta_{2q_2}) \}
\]
gives the Schreyer tuples, where
\[
\beta_{q_2} = x_3x_4 - x_1x_2 \\
\beta_{(q_2+2)} = x_2^{q_2} - x_1^{q_2-1} \\
\beta_i = 0, \quad \text{for} \quad i \notin \{(q_2 + 1), (q_2 + 2)\}, \; 1 \leq i \leq 2q_2.
\]

Case 4. \( f \in \{g_1, g_2\}, g \in A'_2 \).
4(a). Let \( f = g_1, g = h_1 \). We have
\[
S(g_1, h_1) = -x_3 \cdot g_1 + x_2 \cdot h_1 \\
= -x_1 \cdot f_{q_2-1} \quad \rightarrow S 0
\]
Therefore, the set
\[
T_7 = \{ \gamma = (\gamma_1, \ldots, \gamma_i, \ldots, \gamma_{2q_2}) \}
\]
gives us the Schreyer tuples, where
\[
\gamma_{q_2-1} = -x_1 \\
\gamma_{q_2+1} = x_3 \\
\gamma_{q_2+3} = -x_2 \\
\gamma_i = 0, \quad \text{for} \quad i \notin \{(q_2 - 1), (q_2 + 1), (q_2 + 3)\}.
4(b). Let $f = g_1, g = h_m$, with $1 < m \leq (q_2 - 2)$. We have

$$S(g_1, h_m) = -x_3^m \cdot g_1 + x_2^m \cdot h_m$$
$$= -x_1^m \cdot f_{q_2 - m} \quad \rightarrow_S 0$$

Therefore, the set

$$T_8 = \{ \alpha'_m = (\alpha'_{(m,1)}, \ldots, \alpha'_{(m,i)}, \ldots, \alpha'_{(m,2q_2)}) | 1 < m \leq q_2 - 2 \}$$

gives us the Schreyer tuples, where

$$\alpha'_{(m,q_2-m)} = -x_1^m$$
$$\alpha'_{(m,q_2+2)} = x_3^m$$
$$\alpha'_{(m,q_2+2+m)} = -x_2^m$$
$$\alpha'_{(m,i)} = 0, \text{ for } i \notin \{q_2 - m, q_2 + 1, q_2 + 2 + m\}.$$

4(c). Let $f = g_2, g = h_1$. We have

$$S(g_2, h_1) = x_2^{q_2-1} \cdot g_2 + x_4 \cdot h_1$$
$$= x_1 \cdot g_1 - x_1 \cdot f_{q_2} \quad \rightarrow_S 0$$

Therefore, the set

$$T_9 = \{ \alpha' = (\alpha'_1, \ldots, \alpha'_i, \ldots, \alpha'_{2q_2}) \}$$

gives us the Schreyer tuples, where

$$\alpha'_{q_2} = -x_1$$
$$\alpha'_{(q_2+1)} = x_1$$
$$\alpha'_{(q_2+2)} = -x_2^{(q_2-1)}$$
$$\alpha'_{(q_2+3)} = -x_4$$
$$\alpha'_i = 0, \text{ for } i \notin \{q_2, q_2 + 1, q_2 + 2, q_2 + 3\}.$$

4(d). Let $f = g_2, g = h_m$, where $1 < m \leq q_2 - 2$. We have

$$S(g_2, h_m) = x_2^{q_2-m} \cdot x_3^{m-1} \cdot g_2 - x_4 \cdot h_m$$
$$= -x_1 \cdot h_{m-1} \quad \rightarrow_S 0$$

Therefore, the set

$$T_{10} = \{ \beta'_m = (\beta'_{(m,1)}, \ldots, \beta'_{(m,i)}, \ldots, \beta'_{(m,2q_2)}) | 1 < m \leq q_2 - 2 \}$$
gives us the Schreyer tuples, where
\[ \beta'_{(m,q_2+2)} = x_2^{q_2-m}x_3^{-1} \]
\[ \beta'_{(m,q_2+m+1)} = -x_1 \]
\[ \beta'_{(m,q_2+2+m)} = x_4 \]
\[ \beta'_{(m,i)} = 0, \quad \text{for } i \notin \{q_2 + 2, (q_2 + m + 1), (q_2 + 2 + m)\}. \]

**Case 5.** \( f, g \in A_2' \)

5(a) Let \( f = h_m \) and \( g = h_{m+1} \), where \( 1 \leq m \leq q_2 - 3 \). We have
\[ S(h_m, h_{m+1}) = -x_3 \cdot h_m + x_2 \cdot h_{m+1} = x_1^m x_4^{q_2-m} \cdot g_2 \quad \rightarrow_{S} 0 \]

Therefore the set
\[ T_{11} = \{ \gamma'_m = (\gamma'_{(m,1)}, \ldots, \gamma'_{(m,i)}, \ldots, \gamma'_{(m,2q_2)}) \mid 1 \leq m \leq q_2 - 3 \} \]
gives us the Schreyer tuples, where
\[ \gamma'_{(m,q_2+2)} = -x_1^m x_4^{-m} \]
\[ \gamma'_{(m,q_2+2+m)} = x_3 \]
\[ \gamma'_{(m,q_2+3+m)} = -x_2 \]
\[ \gamma'_{(m,i)} = 0, \quad \text{for } i \notin \{q_2 + 2, q_2 + m + 2, q_2 + m + 3\}. \]

5(b). Let \( f = h_m \) and \( g = h_{m'} \), where \( m' > m + 1 \). We have
\[ S(h_m, h_{m'}) = -x_3^{m'-m} \cdot h_m + x_3^{m'-m} \cdot h_{m'} \]
\[ = x_1^m x_4^{q_2+1-m'}((x_3 x_4)^{m'-m-1} + (x_3 x_4)^{m'-m-2}(x_1 x_2) + \ldots + (x_1 x_2)^{m'-m'-1}) \cdot g_2 \]
\[ \rightarrow_{S} 0 \]

Therefore, the set
\[ T_{12} = \{ \alpha'_{mm'} = (\alpha'_{(mm',1)}, \ldots, \alpha'_{(mm',i)}, \ldots, \alpha'_{(mm',2q_2)}) \mid 1 \leq m \leq q_2 - 4, \]
\[ m + 2 \leq m' \leq q_2 - 2 \} \]
gives the Schreyer tuples, where
\[ \alpha'_{(mm',m)} = x_3^{m'-m} \]
\[ \alpha'_{(mm',m')} = -x_2^{m'} \]
\[ \alpha'_{(mm',q_2+2)} = x_1^m x_4^{q_2+1-m'}((x_3 x_4)^{m'-m-1} + (x_3 x_4)^{m'-m-2}(x_1 x_2) + \ldots + (x_1 x_2)^{m'-m'-1}) \]
\[ \alpha'_{(mm',i)} = 0, \quad \text{for } i \notin \{m, m', q_2 + 2\}. \]
Case 6. $f \in A$ and $g \in A'_2$

6(a). Let $f = f_1$, $g = h_{q_2-2}$. We have

$$S(f_1, h_{q_2-2}) = x_2^2 \cdot f_1 + x_3 \cdot h_{q_2-2} = x_1^{q_2-2} \cdot x_4^2 + x_2 x_4^2 \cdot g_1 \rightarrow S \ 0$$

Therefore, the set

$$T_{13} = \{ \beta' = (\beta'_1, \ldots, \beta'_{i}, \ldots, \beta'_{2q_2}) \}$$

gives the Schreyer tuples, where

$$\beta'_1 = -x_2^2$$
$$\beta'_{(q_2+1)} = x_2 x_4^2$$
$$\beta'_{(q_2+2)} = x_1^{q_2-2} x_4^2$$
$$\beta'_{2q_2} = -x_3$$
$$\beta'_i = 0, \text{ for } i \notin \{1, (q_2 + 1), (q_2 + 2), 2q_2\}, 1 \leq i \leq 2q_2.$$

6(b). Let $f = f_2$, $g = h_{q_2-2}$. We have

$$S(f_2, h_{q_2-2}) = x_2^2 \cdot f_2 + x_1 \cdot h_{q_2-2} = x_4^3 \cdot g_1 \rightarrow S \ 0$$

Therefore, the set

$$T_{14} = \{ \gamma' = (\gamma'_1, \ldots, \gamma'_i, \ldots, \gamma'_{2q_2}) \}$$

gives the Schreyer tuples, where

$$\gamma'_2 = -x_2^2$$
$$\gamma'_{(q_2+1)} = x_4^3$$
$$\gamma'_{(2q_2)} = -x_1$$
$$\gamma'_i = 0, \text{ for } i \notin \{2, (q_2 + 1), (2q_2)\}, 1 \leq i \leq 2q_2.$$

6(c). Let $f = f_\mu$ and $g = h_m$, $(\mu, m) \neq (1, q_2-2)$ and $(\mu, m) \neq (2, q_2-2)$.

(i) $\mu + m < q_2$. We have

$$S(f_\mu, h_m) = x_2^{q_2-m} \cdot f_\mu + x_1^{\mu-1} x_4^{q_2-\mu-m} \cdot h_m = x_4^{q_2+1-m} \cdot f_{\mu+m} - x_2^{q_2-\mu-m} x_4^{\mu+1} \cdot (f_{q_2} - g_1) \rightarrow S \ 0$$
Therefore, the set
\[ T_{15} = \{ \beta'_{\mu m} = (\beta'_{(\mu m, 1)}, \ldots, \beta'_{(\mu m, i)}, \ldots, \beta'_{(\mu m, 2q_2)}) \mid 1 \leq \mu \leq q_2 - 1, \]
\[ 1 \leq m < q_2 - \mu \} \]
gives the Schreyer tuples, where
\[
\begin{align*}
\beta'_{(\mu m, \mu)} &= -x_2^{q_2 - m} \\
\beta'_{(\mu m, \mu + m)} &= x_4^{q_2 + 1 - m} \\
\beta'_{(\mu m, q_2)} &= -x_2^{q_2 - m - \mu} x_4^{\mu + 1} \\
\beta'_{(\mu m, q_2 + 1)} &= x_2^{q_2 - m - \mu} x_4^{\mu + 1} \\
\beta'_{(\mu m, q_2 + 2 - m)} &= -x_1^{\mu - 1} x_3^{q_2 - m} \\
\beta'_{(m, i)} &= 0, \text{ for } i \notin \{ \mu, \mu + m, q_2, (q_2 + 1), (q_2 + 2 + m) \}.
\end{align*}
\]

(ii) Let \( \mu + m = q_2 \). We have
\[
S(f_{\mu}, h_m) = x_2^{q_2 - m} \cdot f_{\mu} + x_1^{\mu - 1} \cdot h_m = x_4^{\mu + 1} \cdot g_1 \rightarrow S 0
\]
Therefore, the set
\[ T_{16} = \{ \gamma'_{\mu m} = (\gamma'_{(\mu m, 1)}, \ldots, \gamma'_{(\mu m, i)}, \ldots, \gamma'_{(\mu m, 2q_2)}) \mid 1 \leq \mu \leq q_2 - 1, \]
\[ m = q_2 - \mu \} \]
gives us the Schreyer tuples, where
\[
\begin{align*}
\gamma'_{(\mu m, \mu)} &= -x_2^{q_2 - m} \\
\gamma'_{(\mu m, q_2 + 1)} &= x_4^{\mu + 1} \\
\gamma'_{(\mu m, q_2 + 2 - \mu)} &= -x_1^{\mu - 1} \\
\gamma'_{(\mu m, i)} &= 0, \text{ for } i \notin \{ \mu, (q_2 + 1), (2q_2 + 2 - \mu) \}.
\end{align*}
\]

(iii) Let \( \mu + m > q_2 \). We have
\[
S(f_{\mu}, h_m) = x_2^{q_2 - m} x_3^{\mu + m - q_2} \cdot f_{\mu} + x_1^{\mu - 1} \cdot h_m = x_4^{\mu + 1} \cdot h_{\mu + m - q_2} + x_1^{\mu + m - q_2} x_4^{q_2 + 1 - m} \cdot f_{q_2} \rightarrow S 0
\]
Therefore, the set
\[ T_{17} = \{ \alpha''_{\mu m} = (\alpha''_{(\mu m, 1)}, \ldots, \alpha''_{(\mu m, i)}, \ldots, \alpha''_{(\mu m, 2q_2)}) \mid 1 \leq \mu \leq q_2 - 1, \]
\[ 1 \leq m < q_2 - \mu \} \]
The module $M$ gives us the Schreyer tuples, where
\begin{align*}
\alpha''_{\mu(m, \mu)} &= -x_2^{q_2-m} x_3^{\mu+m-q_2} \\
\alpha''_{\mu(m, q_2)} &= x_1^{\mu+m-q_2} x_3^{q_2+1-m} \\
\alpha''_{\mu(m, \mu+m+2)} &= x_4^{q_2+1}, \quad \alpha''_{\mu(m, q_2+2+m)} = -x_1^{\mu-1} \\
\alpha''_{(m, i)} &= 0, \quad \text{for} \quad i \notin \{\mu, q_2, (q_2 + 2 + m), (\mu + m + 2)\}.
\end{align*}

The set $\mathbb{D} = \bigcup_{i=1}^{17} T_i$ gives us all the Schreyer tuples which form the generating set for the first syzygy module $\text{Syz}(p(n))$. \square

**Theorem 3.2.** Let $T = T_1 \cup T_4 \cup T_5 \cup T_8 \cup T_{10} \cup T_{11} \cup T_{13} \cup T_{14}$. Let $M_1$ denote the first syzygy module $\text{Syz}(p(n))$. Then, the set $\mathbb{D} \subset M_1/mM_1$ is linearly independent over the field $R/m = k$, where $m = \langle x_1, x_2, x_3, x_4 \rangle$. The set $\mathbb{D}$ is a minimal generating set for the first syzygy module and $\beta_2(p(n)) = 4(q_2 - 1)$.

**Proof.** The module $M_1$ is generated by $\mathbb{D}$, therefore $mM_1$ must be generated by $L = x_1D \cup x_2D \cup x_3D \cup x_4D$. It follows from the construction of $\mathbb{D}$ that each component of elements of $L$ is either a monomial of total degree greater than one or zero. Now, to show that $\mathbb{D}$ is linearly independent in the vector space $M_1/mM_1$ over the field $R/m = k$, we consider the element $v \in mM_1$ given by
\begin{align*}
v = \sum_{\mu=1}^{q_2-1} a_\mu \beta_\mu + \sum_{m=1}^{q_2-3} b_m \gamma'_m + c_1 \alpha' + \sum_{m=2}^{q_2-2} c_m \beta'_m \\
+ \sum_{\mu=1}^{q_2-1} d_\mu \alpha_\mu - l_1 \gamma + l_2 \beta' + l_3 \gamma'.
\end{align*}
where $a_\mu, b_m, c_m, d_\mu, l_m \in k$. The linear term in the first coordinate $v_1$ is $a_1(-x_1) + d_1(-x_4)$. Each component of elements of $L$ being a monomial of total degree greater than one or zero, we have $a_1(-x_1) + d_1(-x_4) = 0$, that is $a_1 = d_1 = 0$. Now, the linear term in the $i$-th coordinate $v_i$, for $1 < i < q_2 - 1$ is $a_{i-1}(x_3) + a_i(-x_1) + d_{i-1}(x_2) + d_i(-x_4)$. Once again, by the same argument we must have $a_i = d_i = 0$ for $1 \leq i < q_2 - 1$. Similarly, the linear term in $v_{q_2+1}$ is $c_1(x_1) + l_1(-x_3)$. Therefore, for similar reasons we must have $c_1 = l_1 = 0$. We now compute linear term in $v_{q_2-1}$ and obtain $a_{q_2-1}(-x_1) + d_{q_2-1}(-x_4)$. Equating this linear term to zero we obtain $a_{q_2-1} = d_{q_2-1} = 0$. It turns out that, $v = \sum_{m=1}^{q_2-3} b_m \gamma'_m + \sum_{m=2}^{q_2-2} c_m \beta'_m + l_2 \beta' + l_3 \gamma'$.

We now compute the linear term in $v_{q_2+3}$ and obtain $b_1 x_3 + c_2(-x_1)$. Equating this to zero we get $b_1 = c_2 = 0$. The linear term in $v_{q_2+2+i}$, for
1 < i ≤ q_2 − 3 is b_{i−1} (−x_2) + b_i x_3 + c_i x_4 + c_{i+1} (−x_1). Therefore, b_i = c_i = 0, for 1 < i < q_2 − 2. Finally computing linear term in v_{2q_2} and equating it to zero we obtain b_{q_2−3} (−x_2) + c_{q_2−2} x_4 + l_2 (−x_3) + l_3 (−x_1) = 0. Therefore, b_{q_2−3} = c_{q_2−2} = l_2 = l_3 = 0. This proves that the set \( \mathbb{T} \subset M_1/mM_1 \) is linearly independent.

We have \( \mathbb{T} \) as a part of a minimal generating set for the first syzygies module, and cardinality of \( \mathbb{T} \) is

\[
(q_2 - 1) + (q_2 - 3) + (q_2 - 2) + (q_2 - 1) + 3 = 4(q_2 - 1).
\]

By Theorem 8.1 [12], the second betti number in the resolution of \( R/p(n) \) is \( 4(q_2 - 1) \), hence \( \mathbb{T} \) is a minimal generating set for first syzygy module. Therefore, \( \beta_2(p(n)) = |\mathbb{T}| = 4(q_2 - 1) \).

\[\square\]

**Theorem 3.3.** \( \beta_3(p(n)) = 2q_2 - 3 \).

**Proof.** We know that \( R/p(n) \cong k[t^{n_1}, t^{n_2}, t^{n_3}, t^{n_4}] \) is a one dimensional integral domain and therefore \( \text{depth}(R/p(n)) = 1 \). By the Auslander Buchsbaum theorem, \( \text{projdim}_R(R/p(n)) = 3 \) and we have

\[
1 - 2q_2 + \beta_2(p(n)) - \beta_3(p(n)) = 0.
\]

Therefore, by 3.2, we have \( \beta_3(p(n)) = 2q_2 - 3 \). \[\square\]

### 4. The Second Syzygy and a Minimal Free Resolution

Let us order the generating vectors of second syzygy, and consider the matrix

\[
N = \begin{bmatrix}
\beta_1 \ldots \beta_{q_2−1} & \gamma_1' \ldots \gamma_{q_2−3}' & \alpha' & \beta_2' \ldots \beta'_{q_2−2} & \alpha_1' \ldots \alpha_{q_2−1}' & −\gamma & \beta' & \gamma'
\end{bmatrix}_{2q_2 \times 4(q_2−1)}
\]

We consider the following sets of vectors.

(i) \( \mathbb{H}_1 = \{ \delta_{\mu} = (\delta_{(\mu,1)}, \ldots, \delta_{(\mu,4(q_2−1))}) \mid 1 \leq \mu \leq q_2 − 2 \} \), where

\[
\delta_{(\mu,\mu)} = x_4
\]

\[
\delta_{(\mu,\mu+1)} = -x_2
\]

\[
\delta_{(\mu,(3q_2−6+\mu))} = -x_1
\]

\[
\delta_{(\mu,(3q_2−5+\mu))} = x_3
\]

\[
\delta_{(\mu,i)} = 0, \quad \text{for} \quad i \notin \{ \mu, \mu + 1, (3q_2 − 6 + \mu), (3q_2 − 5 + \mu) \}.
\]
(ii) $\mathbb{H}_2 = \{ \xi = (\xi_1, \ldots, \xi_{4(q_2 - 1)}) \}$, where

$$
\begin{align*}
\xi_1 &= x_2^2 \\
\xi_{2q_2-3} &= x_2x_4^2 \\
\xi_{4q_2-7} &= x_1x_4^2 \\
\xi_{4q_2-6} &= x_4^3 \\
\xi_{4q_2-5} &= -x_1 \\
\xi_{4q_2-4} &= x_3 \\
\xi_i &= 0, \text{ for } i \notin \{1, (2q_2 - 3), (4q_2 - 7), (4q_2 - 6), (4q_2 - 5), (4q_2 - 4)\}.
\end{align*}
$$

(iii) $\mathbb{H}_3 = \{ \zeta = (\zeta_1, \ldots, \zeta_{4(q_2 - 1)}) \}$, where

$$
\begin{align*}
\zeta_{q_2-1} &= x_1 \\
\zeta_{q_2} &= x_4 \\
\zeta_{2q_2-3} &= x_3 \\
\zeta_{2q_2-2} &= x_2 \\
\zeta_{4q_2-6} &= x_1 \\
\zeta_i &= 0, \text{ for } i \notin \{(q_2 - 1), q_2, (2q_2 - 3), (2q_2 - 2), (4q_2 - 6)\}.
\end{align*}
$$

(iv) $\mathbb{H}_3 = \{ \eta = (\eta_1, \ldots, \eta_{4(q_2 - 1)}) \}$, where

$$
\begin{align*}
\eta_{2q_2-4} &= -x_1 \\
\eta_{3q_2-6} &= -x_3 \\
\eta_{3q_2-5} &= x_2^2 \\
\eta_{4q_2-5} &= -x_4 \\
\eta_{4q_2-4} &= x_2 \\
\eta_i &= 0, \text{ for } i \notin \{(2q_2 - 4), (3q_2 - 6), (3q_2 - 5), (4q_2 - 5), (4q_2 - 4)\}.
\end{align*}
$$

(v) $\mathbb{H}_3 = \{ \kappa_\mu = (\kappa_{(\mu,1)}, \ldots, \kappa_{(\mu,4(q_2 - 1))}) \mid 1 \leq \mu \leq q_2 - 4 \}$, where

$$
\begin{align*}
\eta_{(\mu,q_2-1+\mu)} &= -x_1 \\
\eta_{(\mu,4q_2-5+\mu)} &= x_4 \\
\eta_{(\mu,2q_2-3+\mu)} &= -x_3 \\
\eta_{(\mu,2q_2-2+\mu)} &= x_2 \\
\eta_{(\mu,i)} &= 0, \text{ for } i \notin \{(q_2 - 1 + \mu), (q_2 + \mu), (2q_2 - 3 + \mu), (2q_2 - 2 + \mu)\}.
\end{align*}
$$

Theorem 4.1. Suppose $\mathbb{H} = \mathbb{H}_1 \cup \mathbb{H}_2 \cup \mathbb{H}_3 \cup \mathbb{H}_4$. Then $\mathbb{H}$ is a minimal generating set of the second syzygy module of $p(\mathbb{H})$.

Proof. We consider the matrix,

$$
P = [\delta_1 \ldots \delta_{q_2-2} \mid \xi \mid \zeta \mid \eta \mid \kappa_1 \ldots \kappa_{4(q_2-1)(4q_2-1)}]_{4(q_2-1)x2q_3-3}.
$$
It is easy to check that \( N \cdot P = 0 \). Therefore, elements of \( \mathcal{H} \) are elements of the second syzygy module. Let \( M_2 \) denote the second syzygy module of \( p(n) \). We claim that, \( \mathcal{H} \subset M_2 / mM_2 \) is linearly independent set, where \( m = \langle x_1, x_2, x_3, x_4 \rangle \). We proceed in the same way as in [3, 2] considering the expression

\[
\sum_{\mu=1}^{q_2-2} p_\mu \delta_\mu + q \xi + w \zeta + s \eta + \sum_{j=1}^{q_2-4} t_j \kappa_j = u \text{ (say)},
\]

where \( p_\mu, q, w, s, t_j \in k \) for \( 1 \leq \mu \leq q_2 - 2, 1 \leq j \leq q_2 - 4 \) and we compute linear terms in each coordinate of \( u \). If we compute linear term in \( u_1 \) we get, \( p_1 x_4 \), hence \( p_1 = 0 \). Next we compute linear term in \( u_2 \) and we get \( -p_1 x_2 + p_2 x_4 \). We have \( -p_1 x_2 + p_2 x_4 = 0 \), hence \( p_2 = 0 \). Proceeding like this, we observe that all coefficients are zero and \( H \) is a part of minimal generating set of \( M_2 \). Since the third betti number is \( 2q_2 - 3 \) and \( | \mathcal{H} | = 2q_2 - 3 \), therefore \( \mathcal{H} \) is a minimal generating set of the second syzygy module. □

**Corollary 4.2.** A minimal free resolution for the ideal \( p(n) \) in the polynomial ring \( R = K[x_1, x_2, x_3, x_4] \) is

\[
0 \longrightarrow R^{2q_2-3} \longrightarrow P \longrightarrow R^{4(q_2-1)} \longrightarrow R^{2q_2-3} \longrightarrow R \longrightarrow R/p(n) \longrightarrow 0,
\]

where

\[
P = [\delta_1 \ldots \delta_{q_2-2} | \xi | \zeta | \eta | \kappa_1 \ldots \kappa_{q_2-4}].\]

\[
N = \left[ \beta_1 \ldots \beta_{q_2-1} | \gamma_1 \ldots \gamma_{q_2-3} | \alpha_1 \ldots \alpha_{q_2-1} | -\gamma | \beta' | \gamma' \right].\]
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