Masses, mixing angles and phases of general Majorana neutrino mass matrix

Biswajit Adhikary\textsuperscript{a}, Mainak Chakraborty\textsuperscript{b}, Ambar Ghosal\textsuperscript{b}
\textsuperscript{a})Department of Physics, Gurudas College, Narkeldanga, Kolkata-700054, India
\textsuperscript{b}) Saha Institute of Nuclear Physics, 1/AF Bidhannagar, Kolkata 700064, India

\textit{E-mail:} biswajitadhikary@gmail.com, mainak.chakraborty@saha.ac.in, ambar.ghosal@saha.ac.in

\textbf{KEYWORDS:} Neutrino Physics, Beyond Standard Model
General Majorana neutrino mass matrix is complex symmetric and for three generations of neutrinos it contains 12 real parameters. We diagonalize this general neutrino mass matrix and express the three neutrino masses, three mixing angles, one Dirac CP phase and two Majorana phases (removing three unphysical phases) in terms of the neutrino mass matrix elements. We apply the results in the context of a neutrino mass matrix derived from a broken cyclic symmetry invoking type-I seesaw mechanism. Phenomenological study of the above mass matrix allows enough parameter space to satisfy the neutrino oscillation data with only 10% breaking of this symmetry. In this model only normal mass hierarchy is allowed. In addition, the Dirac CP phase and the Majorana phases are numerically estimated. $\sum m_i$ and $|m_{\nu e}|$ are also calculated.
1 Introduction

It is very useful to have a straightforward framework to find the masses and mixing angles of a generalized neutrino mass matrix. In this work special emphasis is given on the diagonalization procedure of the most general $3 \times 3$ complex symmetric effective neutrino mass matrix ($m_\nu$). Starting from a most general $m_\nu$ we calculate three masses directly (without any approximation) in terms of the elements of $m_\nu$. Knowing the mass eigenvalues, three mixing angles and the Dirac CP phase are also obtained. Apart from the Dirac CP phase the total diagonalization matrix consists of three unphysical phases and two Majorana phases. Eliminating the unphysical phases, extraction of the Majorana phases (for generalized $m_\nu$) are also done. We would like to emphasize that those expressions are readily applicable in case of any symmetric or broken symmetric mass matrix. More importantly, the diagonalization is exact and the corresponding neutrino observables are calculated in an exact form without assuming any approximate procedure regarding diagonalization. To illustrate, we employ the obtained expressions in the context of a neutrino mass matrix derived from a broken symmetry.

In the field of neutrino physics, it is now a challenging task to build a suitable model which can accommodate neutrino oscillation experimental data comprising solar\cite{1, 2}, atmospheric\cite{3} and recent reactor neutrino \cite{4–7} experiments as well as the constraint on the sum of the three neutrino masses arising from cosmological data\cite{8, 9}. Furthermore, for Majorana type neutrino, an additional constraint on the $|m_{\nu_{ee}}|$ element of the neutrino mass matrix \cite{10–12} is also necessary to take into account. Popular paradigm is to invoke some symmetries or ansatz\cite{13–15}, viz. $A_4$\cite{16}, $\mu \tau$ symmetry\cite{17–44}, scaling ansatz\cite{45–54}, to generate nondegenerate mass eigenvalues\cite{55} and $\theta_{23}, \theta_{12} \neq 0$ with $\theta_{13} = 0$ at the leading order and nonzero $\theta_{13}$\cite{56–73} is generated by further breaking of such symmetries or ansatz. Contrary to the above idea, in the present work, we explore a typical symmetry, cyclic permutation symmetry\cite{74–76}, in which it is possible to generate all three mixing angles nonzero at the leading order, however, the mass eigenvalues become degenerate. To circumvent this loop hole, we break the symmetry in such a way that the degeneracy in mass eigenvalues is lifted but the mixing angles are still compatible with the extant data.

We consider standard $SU(2)_L \times U(1)_Y$ model with three right handed neutrinos $N_{eR}$, $N_{\mu R}$, $N_{\tau R}$ and invoke type-I seesaw mechanism to generate light neutrino masses. We further impose a cyclic permutation symmetry on both left and right chiral neutrino fields as

$$\nu_{eL} \rightarrow \nu_{\mu L} \rightarrow \nu_{\tau L} \rightarrow \nu_{eL},$$

$$N_{eR} \rightarrow N_{\mu R} \rightarrow N_{\tau R} \rightarrow N_{eR}. \tag{1.1}$$

Cyclic permutation symmetry is a subgroup of $S_3$ permutation symmetry\cite{77} with three of its elements as $\{P_0, P_{123}, P_{132}\}$\footnote{Permutation of three objects $\{a, b, c\}$ form $S_3$ group. There are six elements: $P_0, P_{12}, P_{13}, P_{23}, P_{123}, P_{132}$. Their operations are as follows: $P_0(a, b, c) \rightarrow (a, b, c), P_{12}(a, b, c) \rightarrow (b, a, c), P_{13}(a, b, c) \rightarrow (c, b, a), P_{23}(a, b, c) \rightarrow (a, c, b), P_{123}(a, b, c) \rightarrow (c, a, b), P_{132}(a, b, c) \rightarrow (b, c, a)$.}. One of the motivation to study the $S_3$ symmetry is to realize the well known Tribimaximal (TBM) mixing pattern.
The paper is organized as follows: In section 2, we present the most general solution of a complex $3 \times 3$ symmetric mass matrix to obtain three masses, three mixing angles and the Dirac CP phase. Expressions for the Majorana phases are given in section 3. Section 4 deals with a convenient parametrization and diagonalization of the proposed cyclic symmetry invariant Majorana neutrino mass matrix. Expression of $m_\nu$ in parametric form due to broken cyclic symmetry and corresponding numerical results and phenomenological discussions on the allowed parameter ranges are presented in Section 5. Section 6 contains a summary of the present work.

2 General Solution

In this section we calculate the exact algebraic expressions for the masses and mixing angles of the most general complex symmetric neutrino mass matrix ($m_\nu$) which is written in terms of real ($a_i$) and imaginary ($b_i$) parts as

$$m_\nu = \begin{pmatrix}
a_1 + ib_1 & a_2 + ib_2 & a_3 + ib_3 \\
a_2 + ib_2 & a_4 + ib_4 & a_5 + ib_5 \\
a_3 + ib_3 & a_5 + ib_5 & a_6 + ib_6
\end{pmatrix}. \tag{2.1}$$

2.1 Mass Eigenvalues

It is well known that any complex symmetric mass matrix can be diagonalized by a unitary transformation as

$$U^\dagger m_\nu U^* = \text{diag}(m_1, m_2, m_3) \tag{2.2}$$

where $U$ is a unitary matrix and $m_i$’s ($i = 1, 2, 3$) are real positive masses. However, the columns of $U$ can not be the eigenvectors of $m_\nu$ because

$$m_\nu U^* = U \text{diag}(m_1, m_2, m_3) \tag{2.3}$$

is essentially in the form

$$m_\nu |m_i\rangle^* = m_i |m_i\rangle \tag{2.4}$$

by considering $|m_i\rangle$ as columns of $U$. Since, the states in the l.h.s and r.h.s of eq.(2.4) are different, it is not possible to utilize the equation of the type $\text{Det}(m_\nu - \lambda I) = 0$ to obtain the masses $m_i$. It is therefore necessary to construct a hermitian matrix $h$ as $h = m_\nu m_\nu^\dagger$. Explicit expressions of the elements of $h$ matrix in terms of mass matrix parameters $a_i$ and $b_i$ are provided in Appendix A.1. The squared mass eigenvalues are obtained by direct diagonalization of $h$ matrix as

$$U^\dagger h U = \text{diag}(m_1^2, m_2^2, m_3^2) \tag{2.5}$$

where the matrix $U$ is constructed with the eigenvectors of $h$. It is now straightforward to write down the characteristic equation as $\text{Det}(h - \lambda I) = 0$ to find the eigenvalues. This gives a cubic equation

$$a\lambda^3 + b\lambda^2 + c\lambda + d = 0 \tag{2.6}$$
where the coefficients \( a, b, c, d \) are expressed in terms of the elements of \( h \) matrix and spelt out in Appendix A.2. The nature of the roots in eq.(2.6) depend on the sign of the discriminant \( \Delta \) where

\[
\Delta = 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2. \tag{2.7}
\]

Depending on the sign of \( \Delta \) two cases arise as

**Case I:** \( \Delta \geq 0 \) ⇒ All roots are real. The roots are distinct for \( \Delta > 0 \) and degenerate roots occur for \( \Delta = 0 \).

**Case II:** \( \Delta < 0 \) ⇒ One of the root is real and the other two are complex conjugate to each other.

Since hermitian matrix has real roots we stick to the condition \( \Delta \geq 0 \). The general expressions of the three roots of eq.(2.6) are given by

\[
\lambda_1 = -\frac{b}{3a} - \frac{1}{3a}\sqrt[3]{\frac{1}{2}(2b^3 - 9abc + 27a^2d + \sqrt{-27a^2\Delta})}
- \frac{1}{3a}\sqrt[3]{\frac{1}{2}(2b^3 - 9abc + 27a^2d - \sqrt{-27a^2\Delta})} \tag{2.8}
\]

\[
\lambda_2 = -\frac{b}{3a} - \frac{1 + i\sqrt{3}}{6a}\sqrt[3]{\frac{1}{2}(2b^3 - 9abc + 27a^2d + \sqrt{-27a^2\Delta})}
- \frac{1 - i\sqrt{3}}{6a}\sqrt[3]{\frac{1}{2}(2b^3 - 9abc + 27a^2d - \sqrt{-27a^2\Delta})} \tag{2.9}
\]

\[
\lambda_3 = -\frac{b}{3a} - \frac{1 - i\sqrt{3}}{6a}\sqrt[3]{\frac{1}{2}(2b^3 - 9abc + 27a^2d + \sqrt{-27a^2\Delta})}
- \frac{1 + i\sqrt{3}}{6a}\sqrt[3]{\frac{1}{2}(2b^3 - 9abc + 27a^2d - \sqrt{-27a^2\Delta})}. \tag{2.10}
\]

Subject to the condition \( \Delta \geq 0 \) eq.(2.8) is simplified as

\[
\lambda_1 = -\frac{b}{3a} - \frac{1}{3\sqrt{2a}}(\sqrt[3]{x + iy} + \sqrt[3]{x - iy}) \tag{2.11}
\]

where \( x = 2b^3 - 9abc + 27a^2d, ~ y = 3\sqrt{3a}\sqrt{\Delta} \). Substituting \( x = r \cos 3\theta, ~ y = r \sin 3\theta \) in eq.(2.11) the complex part cancels out and \( \lambda_1 \) is simplified to

\[
\lambda_1 = -\frac{b}{3a} - \frac{2\sqrt[3]{r}}{3\sqrt{2a}} \cos \theta. \tag{2.12}
\]

Following similar substitutions in eq.(2.9) and eq.(2.10) we get the simplified roots as

\[
\lambda_2 = -\frac{b}{3a} + \frac{\sqrt[3]{r}}{3\sqrt{2a}}(\cos \theta - \sqrt{3} \sin \theta) \tag{2.13}
\]
\[
\lambda_3 = -\frac{b}{3a} + \frac{\sqrt[3]{r}}{3\sqrt{2a}}(\cos \theta + \sqrt{3} \sin \theta). \tag{2.14}
\]

The mapping of \((\lambda_1, \lambda_2, \lambda_3)\) to \((m_1^2, m_2^2, m_3^2)\) is done through utilization of the experimental data.
2.2 Mixing Angles and Dirac CP phase

In the above section we have calculated the mass eigenvalues by directly solving the characteristic equation. In other words, the matrix $h$ is diagonalized through a rotation by a unitary matrix $U$, which is known as mixing matrix, as

$$U^\dagger h U = \text{diag}(m_1^2, m_2^2, m_3^2) = D$$  \hspace{1cm} (2.15)$$

or,

$$h U = U D.$$  \hspace{1cm} (2.16)$$

Eq.(2.16) is our key equation to get generalized expression of $U_{ij}$. Comparing l.h.s and r.h.s of eq.(2.16) we get 9 equations, and these 9 equations are clubbed in three equations in the following way

\begin{align*}
(h_{11} - m_i^2)U_{1i} + h_{12}U_{2i} + h_{13}U_{3i} &= 0 \\
{h'*}_1U_{1i} + (h_{22} - m_i^2)U_{2i} + h_{23}U_{3i} &= 0 \\
{h'*}_3U_{1i} + h_{23}U_{2i} + (h_{33} - m_i^2)U_{3i} &= 0
\end{align*}

(2.17)  \hspace{1cm} (2.18)  \hspace{1cm} (2.19)

where $i = 1, 2, 3$. The unitary property of the $U$ matrix further constrains the elements as

$$|U_{1i}|^2 + |U_{2i}|^2 + |U_{3i}|^2 = 1.$$  \hspace{1cm} (2.20)$$

Thus utilizing eq.(2.17) to eq.(2.20) we get rowwise elements of $U$ as

\begin{align*}
U_{1i} &= \frac{(h_{22} - m_i^2)h_{13} - h_{12}h_{23}}{N_i} \\
U_{2i} &= \frac{(h_{11} - m_i^2)h_{23} - h_{12}'h_{13}}{N_i} \\
U_{3i} &= \frac{|h_{12}|^2 - (h_{11} - m_i^2)(h_{22} - m_i^2)}{N_i}
\end{align*}

(2.21)

where $N_i$ is the normalization constant given by

$$|N_i|^2 = |(h_{22} - m_i^2)h_{13} - h_{12}h_{23}|^2 + |(h_{11} - m_i^2)h_{23} - h_{12}'h_{13}|^2 + |(h_{12})^2 - (h_{11} - m_i^2)(h_{22} - m_i^2)|^2.$$  \hspace{1cm} (2.22)$$

The $U$ matrix obtained here in general can have three phases and three mixing angles. This can be understood easily by looking at the $h$ matrix. The $h$ matrix has six modulii and three phases in three off diagonal elements. After diagonalization we have three real positive eigenvalues and a unitary matrix $U$ in which remaining six parameters (three angles and three phases) are contained. Rotating the $h$ matrix by a diagonal phase matrix $P$: $h' = P^\dagger h P$ we can absorb atmost two phases from two off diagonal elements and the survived phase in rest off diagonal elements will be same as the phase of $h_{12}h_{23}h_{31}^2$, term.

\footnote{With $P = \text{diag}(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3})$ we have $h'_{12} = e^{i(\alpha_2 - \alpha_1)}h_{12}$, $h'_{13} = e^{i(\alpha_3 - \alpha_1)}h_{13}$, $h'_{23} = e^{i(\alpha_3 - \alpha_2)}h_{23}$. $h'_{12}$, $h'_{13}$ can be made real with the choice $\alpha_2 - \alpha_1 = -\text{arg}h_{12}$, $\alpha_3 - \alpha_1 = -\text{arg}h_{13}$ which in turn fixes $\alpha_3 - \alpha_2 = \text{arg}h_{12} - \text{arg}h_{13}$ and stops further absorption of phase. Hence survived phase in $h'_{23}$ will be $\text{arg}h_{12} + \text{arg}h_{23} - \text{arg}h_{13} \equiv \text{arg}h_{12}h_{23}h_{31}$.}
Phase of the quantity $h_{12}h_{23}h_{31}$ is independent of phase rotation i.e, phase of $h'_{12}h'_{23}h'_{31}$ is same as the phase of $h_{12}h_{23}h_{31}$. Now, unitary matrix with three angles and single phase in CKM type parametrization (following PDG[78] convention) is

$$U^{\text{CKM}} = \begin{pmatrix}
c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\
s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13}
\end{pmatrix}$$

(2.23)

with $c_{ij} = \cos \theta_{ij}$, $s_{ij} = \sin \theta_{ij}$ and $\delta$ is the Dirac CP phase. Obtained solution of $U_{ij}$ elements in (2.16) may contain unwanted phases which only can appear as the overall phase factor in elements of the $U$ matrix. Hence, we can directly compare their modulus with the modulus of $U^{\text{CKM}}_{ij}$: $|U^{\text{CKM}}_{ij}| = |U_{ij}|$. This gives the expressions of three mixing angles as

$$\tan \theta_{23} = \frac{|U_{23}|}{|U_{33}|}$$

(2.24)

$$\tan \theta_{12} = \frac{|U_{12}|}{|U_{11}|}$$

(2.25)

$$\sin \theta_{13} = \frac{|U_{13}|}{|U_{13}|}.$$  

(2.26)

To obtain the $\delta$ phase we utilize the phase rotation independent quantity $h_{12}h_{23}h_{31}$. Obviously, absence of phase factor in $h_{12}h_{23}h_{31}$ makes the $h$ matrix real symmetric under phase rotation. Therefore, $\text{Im}(h_{12}h_{23}h_{31})$ must be proportional to $\sin \delta$:

$$\text{Im}(h_{12}h_{23}h_{31}) = \frac{(m_2^2 - m_1^2)(m_3^2 - m_2^2)(m_3^2 - m_1^2) \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \cos \theta_{13} \sin \delta}{8}$$

(2.27)

which can be easily inverted to obtain the phase $\delta$. Thus, from $h$ we are able to find out three masses, three mixing angles and the Dirac CP phase in terms of the elements of neutrino mass matrix. Our next goal is to find out remaining two Majorana phases which we will explore in the next section.

### 3 Majorana Phases

In this section we explicitly calculate the Majorana phases assuming the three neutrino masses, three mixing angles and the Dirac CP phase are calculable in terms of the elements of neutrino mass matrix. For a complex symmetric $m_{\nu}$ matrix there are twelve independent parameters arising from six complex elements. These twelve parameters are counted as (i) three masses, (ii) three mixing angles, (iii) one Dirac CP phase, (iv) two Majorana phases and (v) three unphysical phases. These three unphysical phases take crucial part in diagonalization. Now, the unitary matrix with three angles and six phases can be parametrized as:

$$U_{\text{tot}} = P_\delta U^{\text{PMNS}}$$

(3.1)
where
\[ U^{\text{PMNS}} = U^{\text{CKM}} \left( \begin{array}{ccc} e^{i\alpha_M} & 0 & 0 \\ 0 & e^{i\beta_M} & 0 \\ 0 & 0 & 1 \end{array} \right) \]  
(3.2)

and
\[ P_\phi = \left( \begin{array}{ccc} e^{i\phi_1} & 0 & 0 \\ 0 & e^{i\phi_2} & 0 \\ 0 & 0 & e^{i\phi_3} \end{array} \right). \]  
(3.3)

\( P_\phi \) is the unphysical phase matrix with unphysical phases \( \phi_{1,2,3} \). Phase matrix in extreme right of the \( U^{\text{PMNS}} \) matrix contains two Majorana phases \( \alpha_M \) and \( \beta_M \). Now \( m_\nu \) can be diagonalized as
\[ U^{\dagger}_{\text{tot}} m_\nu U^{\dagger}_{\text{tot}} = \text{diag}(m_1, m_2, m_3) \]  
(3.4)

which can be inverted as
\[ m_\nu = U^{\dagger}_{\text{tot}} \text{diag}(m_1, m_2, m_3) U^{\dagger}_{\text{tot}}. \]  
(3.5)

Equating both sides of eq.(3.5) elements of \( m_\nu \) matrix can be written in terms of masses, mixing angles and phases as
\[ (m_\nu)_{11} = e^{2i\phi_1} (c_{12}^2 c_{13}^2 m_1 e^{i\alpha_M} + s_{12}^2 c_{13}^2 m_2 e^{i\beta_M} + m_3 s_{13}^2 e^{-2i\delta}) \]  
(3.6)
\[ (m_\nu)_{12} = e^{i(\phi_1 + \phi_2)} c_{13} \left( -m_1 e^{i\alpha_M} (c_{12}s_{12}c_{23} + c_{12}s_{13}s_{23}e^{i\delta}) + m_2 e^{i\beta_M} (c_{12}s_{12}c_{23} - s_{12}^2 s_{13}s_{23}e^{i\delta}) \right) + m_3 s_{13} c_{23} e^{-i\delta} \]  
(3.7)
\[ (m_\nu)_{13} = e^{i(\phi_1 + \phi_3)} c_{13} \left( m_1 e^{i\alpha_M} (-c_{12}^2 c_{23}s_{13}e^{i\delta} + c_{12}s_{12}s_{23}) - m_2 e^{i\beta_M} (c_{12}s_{12}s_{23} + s_{12}^2 s_{13}c_{23}e^{i\delta}) \right) + m_3 s_{13} c_{23} e^{-i\delta} \]  
(3.8)
\[ (m_\nu)_{22} = e^{2i\phi_2} \left( m_1 e^{i\alpha_M} (s_{12} c_{23} + c_{23}s_{13}e^{i\delta})^2 + m_2 e^{i\beta_M} (c_{12} c_{23} - s_{12}^2 s_{13}s_{23}e^{i\delta})^2 \right) + m_3 c_{13}^2 c_{23} \]  
(3.9)
\[ (m_\nu)_{23} = e^{i(\phi_2 + \phi_3)} m_1 e^{i\alpha_M} \left\{ c_{12} s_{12}s_{13}(c_{23}^2 - s_{23}^2)e^{i\delta} + c_{12}^2 c_{23}s_{23}s_{13}^2 e^{2i\delta} - s_{12}^2 s_{23}c_{23} \right\} - m_2 e^{i\beta_M} \left\{ c_{12} s_{12}(c_{23}^2 - s_{23}^2)s_{13}e^{i\delta} + c_{12}^2 s_{23}c_{23} - s_{12}^2 s_{13}s_{23}e^{2i\delta} \right\} + m_3 c_{13}^2 c_{23}^2 \]  
(3.10)

We now extract \( \alpha_M \) and \( \beta_M \) eliminating unwanted \( \phi_i \) phases. Modulus \( |(m_\nu)_{ij}| \) of all elements are free from \( \phi_i \) phases. The combinations such as \( \frac{|(m_\nu)_{ij}|^2}{(m_\nu)_{ii} (m_\nu)_{jj}} \) \( (i \neq j) \) are also independent of those \( \phi_i \) phases. Neglecting terms of \( O(s_{13}^2) \) and higher order, we find, among all the \( |(m_\nu)_{ij}| \) terms, the term \( |(m_\nu)_{11}| \) has the simplest structure and independent of \( \phi_i \). We can easily extract \( \beta_M - \alpha_M \) from this term as
\[ \cos(\beta_M - \alpha_M) = \frac{|(m_\nu)_{11}|^2 - c_{12}^2 m_1^4 - s_{12}^2 m_2^2}{2 c_{12}^2 s_{12}^2 m_1 m_2}, \]  
(3.11)
To find the individual value of Majorana phases, we consider the term $\frac{[m_{\nu}]_{33}^2}{(m_{\nu})_{22}(m_{\nu})_{33}}$ which looks simpler by neglecting terms like $(c_{23}^2-s_{23}^2)s_{13}$, $s_{13}^2$ and their higher power. Substituting Majorana phase difference $\beta_M - \alpha_M$ in the term $\frac{[m_{\nu}]_{23}^2}{(m_{\nu})_{22}(m_{\nu})_{33}}$ we can construct two different complex equations only with $\alpha_M$ and $\beta_M$ respectively. It is straightforward to find out two Majorana phases with the chain of expressions in a generic form as

$$\tan \theta_j = \frac{Y_j W_j - W_j Y_j}{X_j W_j - W_j X_j}$$

(3.12)

where $j = 1, 2$ and $\theta_1 = \alpha_M$, $\theta_2 = \beta_M$ with

$$X_1 = A_i - \{D_r \sin(\beta_M - \alpha_M) + D_i \cos(\beta_M - \alpha_M) + F_i \sin 2(\beta_M - \alpha_M) +$$

$$F_i \cos 2(\beta_M - \alpha_M) + E_i\}$$

$$X_1' = \{D_r \sin(\beta_M - \alpha_M) - D_i \sin(\beta_M - \alpha_M) + F_r \cos 2(\beta_M - \alpha_M) -$$

$$F_i \sin 2(\beta_M - \alpha_M) + E_r\} - A_r$$

$$Y_1 = A_r + \{D_r \cos(\beta_M - \alpha_M) - D_i \sin(\beta_M - \alpha_M) + F_r \cos 2(\beta_M - \alpha_M) -$$

$$F_i \sin 2(\beta_M - \alpha_M) + E_r\}$$

$$Y_1' = A_i + \{D_r \sin(\beta_M - \alpha_M) + D_i \cos(\beta_M - \alpha_M) + F_r \sin 2(\beta_M - \alpha_M) +$$

$$F_i \cos 2(\beta_M - \alpha_M) + E_i\}$$

$$W_1 = B_r + C_r \cos(\beta_M - \alpha_M) - C_i \sin(\beta_M - \alpha_M)$$

$$W_1' = B_i + C_r \sin(\beta_M - \alpha_M) + C_i \cos(\beta_M - \alpha_M)$$

(3.13)

and

$$X_2 = A_i - \{D_i \cos(\beta_M - \alpha_M) - D_r \sin(\beta_M - \alpha_M) + E_i \cos 2(\beta_M - \alpha_M) -$$

$$E_r \sin 2(\beta_M - \alpha_M) + F_i\}$$

$$X_2' = \{D_i \cos(\beta_M - \alpha_M) + D_r \sin(\beta_M - \alpha_M) + E_r \cos 2(\beta_M - \alpha_M) +$$

$$E_i \sin 2(\beta_M - \alpha_M) + F_r\} - A_r$$

$$Y_2 = A_r + \{D_i \cos(\beta_M - \alpha_M) + D_r \sin(\beta_M - \alpha_M) + E_r \cos 2(\beta_M - \alpha_M) +$$

$$E_i \sin 2(\beta_M - \alpha_M) + F_r\}$$

$$Y_2' = A_i + \{D_i \cos(\beta_M - \alpha_M) - D_r \sin(\beta_M - \alpha_M) + E_i \cos 2(\beta_M - \alpha_M) -$$

$$E_r \sin 2(\beta_M - \alpha_M) + F_i\}$$

$$W_2 = C_r + B_r \cos(\beta_M - \alpha_M) + B_i \sin(\beta_M - \alpha_M)$$

$$W_2' = C_i + B_r \cos(\beta_M - \alpha_M) - B_i \sin(\beta_M - \alpha_M)$$

(3.14)
where suffix $i$ and $r$ stand for imaginary and real part respectively. The complex quantities $A$, $B$, $C$, $D$, $E$ and $F$ are defined as follows

\[
A = m_3^2 |Z - 1| \\
B = m_3 m_1 \left[ Z s_{12}^2 \frac{1 + t_{23}^4}{t_{23}^2} - Z \sin 2\theta_{12} s_{13} e^{i\delta} \frac{t_{23}^2 - 1}{t_{23}} + 2 s_{12}^2 \right] \\
C = m_3 m_2 \left[ Z c_{12}^2 \frac{1 + t_{23}^4}{t_{23}^2} + Z \sin 2\theta_{12} s_{13} e^{i\delta} \frac{t_{23}^2 - 1}{t_{23}} + 2 c_{12}^2 \right] \\
D = m_1 m_2 \left[ 2 Z c_{12}^2 s_{12}^2 + \sin 2\theta_{12} \cos 2\theta_{12} s_{13} e^{i\delta} \frac{t_{23}^2 - 1}{t_{23}} - 2 s_{12}^2 c_{12}^2 \right] \\
E = m_1^2 \left[ Z s_{12}^4 + Z s_{12}^2 \sin 2\theta_{12} s_{13} e^{i\delta} \frac{t_{23}^2 - 1}{t_{23}} - s_{12}^4 \right] \\
F = m_2^2 \left[ Z c_{12}^4 - Z c_{12}^2 \sin 2\theta_{12} s_{13} e^{i\delta} \frac{t_{23}^2 - 1}{t_{23}} - c_{12}^4 \right] \tag{3.15}
\]

with $t_{23} = \tan \theta_{23}$ and $Z = \frac{[(M_\nu)_{23}]^2}{(M_\nu)_{22}(M_\nu)_{33}}$. Again in the expressions of $B$, $C$, $D$, $E$ and $F$ terms containing $s_{13}(t_{23}^2 - 1)e^{i\delta}$ is proportional to $s_{13}(c_{23}^2 - s_{23}^2)$. Dropping those terms one can further simplify the expressions of $B$, $C$, $D$, $E$ and $F$ keeping other dominating terms. This simplification makes expressions of $A$ to $F$ free from the Dirac phase and their complex nature is only due to $Z$ parameter.

Thus, apart from the masses, finally, we gather complete information of the $U^{PMNS}$ matrix containing mixing angles and physical phases from a general three generation Majorana neutrino mass matrix.

4 Cyclic Symmetry

4.1 Basic Formalism

The most general leptonic mass term of the Lagrangian in the present model is

\[
- \mathcal{L}_{\text{mass}} = (m_\ell)_{ll'} l^L_{l'} + m_{D_{ll'}} l^L N^R_{l'} + M_{R_{ll'}} N^L_{l} N^R_{l'} \tag{4.1}
\]

where $l$, $l' = e$, $\mu$, $\tau$. We demand that the neutrino part of the Lagrangian is invariant under the cyclic permutation symmetry as given in eq.(1.1). The symmetry invariant Dirac neutrino mass matrix $m_D$ takes the form

\[
m_D = \begin{pmatrix} y_1 & y_2 & y_3 \\ y_3 & y_1 & y_2 \\ y_2 & y_3 & y_1 \end{pmatrix} \tag{4.2}
\]

where in general all the entries are complex. Without loss of generality, we consider a basis in which the right handed neutrino mass matrix $M_R$ and charged lepton mass matrix $m_\ell$ are diagonal. Further, imposition of cyclic symmetry dictates the texture of $M_R$ as

\[
M_R = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix} \tag{4.3}
\]
Now, within the framework of type-I seesaw mechanism the effective neutrino mass matrix \( m_\nu \),

\[
m_\nu = -m_D M_R^{-1} m_D^T
\]

(4.4)
takes the following form with cyclic symmetric \( m_D \) (eq.(4.2)) and \( M_R \) (eq.(4.3)) as

\[
m_\nu = -\frac{1}{m} \begin{pmatrix}
  y_1^2 + y_2^2 + y_3^2 & y_1 y_2 + y_2 y_3 + y_3 y_1 & y_1 y_2 + y_2 y_3 + y_3 y_1 \\
  y_1 y_2 + y_2 y_3 + y_3 y_1 & y_1^2 + y_2^2 + y_3^2 & y_1 y_2 + y_2 y_3 + y_3 y_1 \\
  y_1 y_2 + y_2 y_3 + y_3 y_1 & y_1 y_2 + y_2 y_3 + y_3 y_1 & y_1^2 + y_2^2 + y_3^2
\end{pmatrix}.
\]

(4.5)

### 4.2 Parametrization and Diagonalization

With a suitable choice of parametrization the effective neutrino mass matrix given in eq. (4.5) can be rewritten as

\[
m_\nu = m_0 \begin{pmatrix}
  1 + p^2 e^{2i\alpha} + q^2 e^{2i\beta} & pe^{i\alpha} + qe^{i(\alpha+\beta)} & pe^{i\alpha} + qe^{i(\alpha+\beta)} \\
  pe^{i\alpha} + qe^{i\beta} + pqe^{i(\alpha+\beta)} & 1 + p^2 e^{2i\alpha} + q^2 e^{2i\beta} & pe^{i\alpha} + qe^{i(\alpha+\beta)} \\
  pe^{i\alpha} + qe^{i\beta} + pqe^{i(\alpha+\beta)} & pe^{i\alpha} + qe^{i(\alpha+\beta)} & 1 + p^2 e^{2i\alpha} + q^2 e^{2i\beta}
\end{pmatrix}
\]

(4.6)

where we have parametrized the different elements \((y_1, y_2, y_3)\) of \( m_\nu \) in terms of \( p, q \) and two phases \( \alpha, \beta \) accordingly

\[
m_0 = -\frac{y_3^2}{m}, \quad pe^{i\alpha} = \frac{y_1}{y_3}, \quad qe^{i\beta} = \frac{y_2}{y_3}.
\]

(4.7)

Denoting

\[
P = 1 + p^2 e^{2i\alpha} + q^2 e^{2i\beta}
\]

\[
Q = pe^{i\alpha} + qe^{i(\alpha+\beta)}
\]

(4.8)

\( m_\nu \) is written in a convenient form as

\[
m_\nu = m_0 \begin{pmatrix}
  P & Q & Q \\
  Q & P & Q \\
  Q & Q & P
\end{pmatrix}.
\]

(4.9)

We construct the matrix \( h = m_\nu m_\nu^\dagger \) to calculate the mixing angles and mass eigenvalues. Expression of \( h \) obtained as

\[
h = m_\nu m_\nu^\dagger = m_0^2 \begin{pmatrix}
  A & B & B \\
  B & A & B \\
  B & B & A
\end{pmatrix}
\]

(4.10)

where

\[
A = |P|^2 + 2|Q|^2
\]

\[
B = |Q|^2 + PQ^* + P^*Q.
\]

(4.11)
Diagonalizing the matrix $h$ given in eq.(4.10) through

\[ U^\dagger hU = \text{diag}(m_1^2, m_2^2, m_3^2) \]

we get the mass squared eigenvalues as

\[ m_1^2 = m_3^2(A - B) \]
\[ m_2^2 = m_3^2(A + 2B) \]
\[ m_3^2 = m_3^2(A - B). \]

However, there is a problem of unique determination of the diagonalization matrix $U$ due to the degeneracy in the eigenvalues ($m_1^2 = m_3^2 \neq m_2^2$). Any vector in the plane orthogonal to the unique eigenvector of eigenvalue $m_3^2$ can be an eigenvector of $m_1^2$ or $m_2^2$. One can choose two mutually orthogonal eigenvectors on that plane for the eigenvalues $m_1^2$ and $m_3^2$. So, in effect, we can have the $U$ matrix of the above case with these three eigenvectors. But, choice of eigenvectors for $m_1^2$ and $m_3^2$ on the degenerate plane is arbitrary. Any other two orthogonal combinations of these two eigenvectors are equally good for construction of the $U$ matrix for the same eigenvalues. So, the diagonalization matrix can not be unique and hence the derived mixing angles are also not unique.

Here, one observation is that the eigenvector of $m_2^2$: $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ coincides with the 2nd column of TBM mixing matrix. Due to degeneracy $m_1^2 = m_3^2$, one of the possible choice of diagonalization matrix could be the well known TBM mixing matrix. However, it is also possible to generate all three mixing angles nonzero by proper combination of eigenvectors corresponding to the degenerate eigenvalues. Furthermore, in order to accommodate solar and atmospheric neutrino mass squared differences it is necessary to break the symmetry to remove the degeneracy between the mass eigenvalues.

5 Breaking of cyclic symmetry

In this scheme, we break the cyclic symmetry in the right chiral neutrino sector only. Retaining the flavour diagonal texture of $M_R$, we introduce only two symmetry breaking parameters $\epsilon_1$ and $\epsilon_2$ in any two diagonal entries. (It is sufficient to incorporate two symmetry breaking parameters to achieve all the eigenvalues of $M_R$ different). This can be done in three ways as

(i) $M_R = \text{diag} \left( m, m + \epsilon_1, m + \epsilon_2 \right)$,

(ii) $M_R = \text{diag} \left( m + \epsilon_1, m + \epsilon_2, m \right)$,

(iii) $M_R = \text{diag} \left( m + \epsilon_1, m, m + \epsilon_2 \right)$.

It is to be noted that instead of perturbative approach, we directly diagonalize the broken symmetric mass matrix with the help of the results obtained in section 2. Let us first consider case (i) where symmetry breaking occurs at ‘22’ and ‘33’ elements. Using the expression of $M_R$ given in (i) and $M_D$ as given in eq.(4.2), the effective neutrino mass matrix is obtained due to type-I seesaw mechanism as

\[
m_{\nu} = \frac{y_2^2}{m} \left( \begin{array}{ccc}
\frac{y_1^2}{y_2^2} + \frac{y_1^2}{y_3^2} \frac{1}{(1+\epsilon_1)} & \frac{y_1}{y_2} & \frac{y_1}{y_3} \\
\frac{y_1}{y_2} & \frac{y_1}{y_2} \frac{1}{(1+\epsilon_2)} + \frac{1}{(1+\epsilon_2)} & \frac{y_1}{y_3} \\
\frac{y_1}{y_3} & \frac{y_1}{y_3} \frac{1}{(1+\epsilon_2)} & \frac{1}{(1+\epsilon_2)} \end{array} \right)
\]

\[ (5.1) \]
where we have defined \( \epsilon_1' = \frac{\Omega_1}{m}, \epsilon_2' = \frac{\Omega_2}{m} \). We rewrite \( m_\nu \) as

\[
m_\nu = m_0 \begin{pmatrix}
p e^{2i\alpha} + \frac{q e^{i\beta}}{(1+\epsilon_1')} + \frac{1}{(1+\epsilon_2')}
p e^{i\alpha} + \frac{q e^{i\beta}}{(1+\epsilon_1')} + \frac{pe^{i\alpha}}{(1+\epsilon_1')} + \frac{q e^{i(\alpha+\beta)}}{(1+\epsilon_1')} + \frac{pe^{i(\alpha+\beta)}}{(1+\epsilon_1')} + \frac{q e^{i}\lambda}{(1+\epsilon_1')} + \frac{pq e^{i\lambda}}{(1+\epsilon_1')}
q e^{i\beta} + \frac{pe^{i\alpha}}{(1+\epsilon_2')} + \frac{q e^{i(\alpha+\beta)}}{(1+\epsilon_2')} + \frac{pe^{i(\alpha+\beta)}}{(1+\epsilon_2')} + \frac{q e^{i}\lambda}{(1+\epsilon_2')} + \frac{pq e^{i\lambda}}{(1+\epsilon_2')}
q e^{i\beta} + \frac{pe^{i\alpha}}{(1+\epsilon_2')} + \frac{q e^{i(\alpha+\beta)}}{(1+\epsilon_2')} + \frac{pe^{i(\alpha+\beta)}}{(1+\epsilon_2')} + \frac{q e^{i}\lambda}{(1+\epsilon_2')} + \frac{pq e^{i\lambda}}{(1+\epsilon_2')}
\end{pmatrix},
\]

where we mimic the parametrization previously shown in eq.(4.7). The other two cases, case (ii) and (iii) also produce the same form of \( m_\nu \) given in eq.(5.2) with a different set of parametrizations given by

- **Case (ii)**
  \[
m_0 = \frac{y_1^2}{m}, \quad p e^{i\alpha} = \frac{y_2}{y_1}, \quad q e^{i\beta} = \frac{y_3}{y_1}.
\]

- **Case (iii)**
  \[
m_0 = -\frac{y_2^2}{m}, \quad p e^{i\alpha} = \frac{y_3}{y_2}, \quad q e^{i\beta} = \frac{y_1}{y_2}.
\]

### 5.1 Numerical results and phenomenological discussions

It is now straightforward to calculate the eigenvalues and mixing angles of the above mass matrix \( m_\nu \). The coefficients \( a, b, c \) and \( d \) of the general characteristic equation (eq.(2.6)) can be written in terms of Lagrangian parameters \( (p, q, \alpha, \beta) \) through the substitution of elements of general \( m_\nu \) (eq.(2.1)) by the corresponding elements of broken symmetric \( m_\nu \) (eq.(5.2)). Substituting these values in eq.(2.8), (2.9) and (2.10) it is possible to calculate three eigenvalues. The mapping of \( (\lambda_1, \lambda_2, \lambda_3) \) to \( (m_1^2, m_2^2, m_3^2) \) is done by utilizing neutrino oscillation experimental data shown in Table 5.1.

| Quantity                        | 3σ ranges/other constraint         |
|---------------------------------|-----------------------------------|
| \( \Delta m_{21}^2 \)          | \( 7.12 < \Delta m_{21}^2(10^5 \text{ eV}^{-2}) < 8.20 \) |
| \( |\Delta m_{31}^2| (N) \)      | \( 2.31 < |\Delta m_{31}^2(10^3 \text{ eV}^{-2}) < 2.74 \) |
| \( |\Delta m_{31}^2| (I) \)      | \( 2.21 < |\Delta m_{31}^2(10^3 \text{ eV}^{-2}) < 2.64 \) |
| \( \theta_{12} \)              | \( 31.30^\circ < \theta_{12} < 37.46^\circ \) |
| \( \theta_{23} \)              | \( 36.86^\circ < \theta_{23} < 55.55^\circ \) |
| \( \theta_{13} \)              | \( 7.49^\circ < \theta_{13} < 10.46^\circ \) |
| \( \delta \)                   | \( 0 - 2\pi \)                     |

Before proceeding to carry out the numerical analysis few remarks are in order:

i) Taking into account different cosmological experiments with recent PLANCK satellite experimental results [8] the upper limit of the sum of the three neutrino masses can vary mostly within the range as \( \Sigma m_i (= m_1 + m_2 + m_3) < (0.23 – 1.11) \text{eV} \) [79]. A combined analysis of PLANCK, WMAP low \( l \) polarization, gravitational lensing and results of prior
on the Hubble constant $H_0$ from Hubble space telescope data corresponds to the higher value of $\Sigma m_i$ whereas inclusion of SDSS DR8 result with the above combination sharply reduce the upper limit of $\Sigma m_i$ at the above mentioned lower edge. However, in our set up individual masses of the neutrinos and sum of the neutrino masses are considered as predictions of this model. We investigate to check the viability of the sum of the three neutrino masses in view of the upper bound provided by the extant cosmological data.

ii) Another constrain arises from $\beta\beta_0\nu$ decay experiments [10–12] on the matrix element $|m_{\nu e}| (= m_{\nu e1})$. At present lots of experiments are running/proposed among them EXO-200 Collaboration [80] has quoted a range on the upper limit of $|m_{\nu e}|$ as $|m_{\nu e}| < (0.14 - 0.38)$eV. In the present work, we are not restricting the value of $|m_{\nu e}|$ rather treat it also as a prediction to testify the present model in the foreseeable future.

We have varied the symmetry breaking parameters $\epsilon'_1$, $\epsilon'_2$ in the range $-0.1 < \epsilon'_1$, $\epsilon'_2 < 0.1$ to keep the symmetry breaking effect small. With such values of $\epsilon'_{1,2}$ and taking neutrino experimental data[10, 81, 82] given in Table 5.1 as input, we find admissible parameter space of the model. The allowed region of the $p$ vs $q$ parametric plane is shown in left panel of figure 1, wherefrom the allowed ranges of $p$ and $q$ can be read as $0.27 < p < 2.09$, $0.44 < q < 2.21$. The two phase parameters $\alpha$ and $\beta$ are varied as $-180^\circ < \alpha < 180^\circ$ and the allowed parameter space in $\alpha$ vs $\beta$ plane is shown in right panel of figure 1. Two tiny disconnected patches are allowed and one is mirror image to the other. The allowed ranges of $\alpha$, $\beta$ obtained as $-161.12^\circ < \alpha < -89.35^\circ$ with $91.09^\circ < \beta < 166.53^\circ$ and $90.80^\circ < \alpha < 161.02^\circ$ with $-166.35^\circ < \beta < -92.11^\circ$. Next in figure 2, in the left panel we plot $\Sigma m_i$ vs $|m_{\nu e}|$ and the ranges obtained as $0.076eV < \Sigma m_i < 0.23eV$, $0.002eV < |m_{\nu e}| < 0.009eV$. The upper limit of $\Sigma m_i$ obtained from figure 2 marginally touches the most optimistic cosmological upper bound 0.23 eV, however, the lower limit is very far to probe in the near future. On the otherhand, both the higher and lower values $|m_{\nu e}|$ is well within the upper bound of running/proposed experiments (for example KamLAND+Zen, EXO). In the right panel of figure 2, $m_1$ vs $m_{2,3}$ plot is given and it is clear from the plot that the mass ordering is normal ($m_1 < m_2 < m_3$). The ranges of individual mass eigenvalues obtained as $0.0122eV < m_1 < 0.0720eV$, $0.0143eV < m_2 < 0.0730eV$, $0.0495eV < m_3 < 0.09eV$. Thus, the testability of the present model crucially relies upon the determination of the neutrino mass hierarchy by future neutrino experiments. We have successively plotted the variation of Jarlskog invariant $J_{CP}$ \(^3\) with the Dirac CP phase ($\delta$) in the left panel of figure 3 and Majorana phases $\alpha_M$ vs $\beta_M$ in the right panel of figure 3. We see that $-0.044 < J_{CP} < 0.044$ and all values of $\delta$ lies within the range $-90^\circ$ to $90^\circ$ whereas Majorana phases admit almost all values in the range $-90^\circ < \alpha_M$, $\beta_M < 90^\circ$. Before concluding this section we like to comment on the necessity of the two breaking parameters $\epsilon'_1$ and $\epsilon'_2$. It is seen from the present analysis that in the present model it is possible to explain the neutrino oscillation data with either of the $\epsilon'_i$ parameter equal to zero.

\(^3\) $J_{CP} = \frac{\text{Im}(h_{12}h_{32}h_{31})}{(m_2^2 - m_1^2)(m_3^2 - m_1^2)(m_3^2 - m_2^2)} = \frac{\sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \cos \theta_{13} \sin \delta}{8}$
Figure 1. (colour online) Plot of the allowed parameter space in \( p, q \) (left) plane and \( \alpha, \beta \) (right) plane satisfying input data shown in Table 5.1

Figure 2. (colour online) Plot of \( \Sigma m_i \) vs \( |m_{\nu e}| \) (left), \( m_1 \) vs \( m_{2,3} \) (right) satisfying input data shown in Table 5.1
Summary

The main objective of this paper is to develop a simple methodology to obtain exact mass eigenvalues, mixing angles, the Majorana phases and the Dirac CP phase of a general complex symmetric Majorana neutrino mass matrix without any approximation. The hermitian matrix \( h \) constructed from \( m_\nu \) (\( h = m_\nu m_\nu^\dagger \)) is solved to get the squared mass eigenvalues. The elements of the diagonalization matrix \( U \) and hence, three mixing angles and the Dirac CP phase \( \delta \) are calculated by solving the set of eigenvalue equations. Since \( m_\nu \) has twelve independent parameters, the total diagonalization matrix which diagonalizes \( m_\nu \), should contain five more phase parameters apart from the Dirac CP phase (The other six parameters are three mass squared values and three mixing angles.). The above mentioned five phase parameters contain three unphysical phases and two Majorana phases. General expressions for the Majorana phases are obtained by eliminating those unphysical ones.

We demonstrate this general and exact methodology in the context of a neutrino mass matrix obtained from a cyclic symmetry transformation invoking type-I seesaw mechanism. The symmetry invariant structure of the effective neutrino mass matrix leads to degeneracy in the mass eigenvalues and thereby, prohibited by the experimental data. The symmetry is broken in the right handed neutrino sector only in order to fulfill the phenomenological demands of nonzero mass squared differences and mixing angles. All the physical parameters (three mixing angles, one Dirac CP phase, two Majorana phases) of the total diagonalization matrix \( (U_{\text{tot}}) \) and the mass eigenvalues of the broken symmetric mass matrix are readily expressed in terms of the Lagrangian parameters through the utilization of the results obtained from general diagonalization procedure. For completeness of the analysis, we explore the parameter space and it is revealed that the mass hierarchy of the neutrinos is normal and inverted hierarchy is completely ruled out. Plots of the allowed parameter space show that this model is capable of producing those observables (mixing angles, solar and atmospheric mass squared differences) within experimentally constrained
ranges. Finally, the exact expressions obtained for physical parameters can be directly applicable in any (symmetry invariant or broken) neutrino mass matrix.

A Appendix

A.1 Elements of $h$ in terms of elements of $m_\nu (a_i, b_i)$

\[
\begin{align*}
h_{11} &= a_1^2 + b_1^2 + a_3^2 + b_3^2 + a_5^2 + b_5^2 \\
h_{22} &= a_2^2 + b_2^2 + a_4^2 + b_4^2 + a_7^2 + b_7^2 \\
h_{33} &= a_3^2 + b_3^2 + a_6^2 + b_6^2 + a_8^2 + b_8^2 \\
h_{12} &= (a_1a_2 + b_1b_2 + a_2a_4 + b_2b_4 + a_3a_5 + b_3b_5) \\
&\quad + i(b_1a_2 - a_1b_2 + b_2a_4 - a_2b_4 + a_3a_5 - a_2b_5) \\
h_{13} &= (a_1a_3 + b_1b_3 + a_2a_5 + b_2b_5 + a_3a_6 + b_3b_6) \\
&\quad + i(b_1a_3 - a_1b_3 + b_2a_5 - a_2b_5 + a_3a_6 - a_2b_6) \\
h_{23} &= (a_2a_3 + b_2b_3 + a_4a_5 + b_4b_5 + a_5a_6 + b_5b_6) \\
&\quad + i(b_2a_3 - a_2b_3 + b_4a_5 - a_4b_5 + a_5a_6 - a_4b_6)
\end{align*}
\]

A.2 Coefficients of the cubic equation in terms of elements of $h$

\[
\begin{align*}
a &= 1 \\
b &= -(h_{11} + h_{22} + h_{33}) \\
c &= h_{33}h_{11} + h_{33}h_{22} + h_{11}h_{22} - |h_{12}|^2 - |h_{13}|^2 - |h_{23}|^2 \\
d &= h_{11}|h_{23}|^2 + h_{33}|h_{12}|^2 + h_{22}|h_{13}|^2 - h_{11}h_{22}h_{33} - 2\text{Re}(h_{12}h_{23}h_{13})
\end{align*}
\]

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