ON THE COADJOINT ORBITS OF MAXIMAL UNIPOTENT SUBGROUPS OF REDUCTIVE GROUPS

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Abstract. Let $G$ be a simple algebraic group defined over an algebraically closed field of characteristic 0 or a good prime for $G$. Let $U$ be a maximal unipotent subgroup of $G$ and $u$ its Lie algebra. We prove the separability of orbit maps and the connectedness of centralizers for the coadjoint action of $U$ on (certain quotients of) the dual $u^*$ of $u$. This leads to a method to give a parametrization of the coadjoint orbits in terms of so-called minimal representatives which form a disjoint union of quasi-affine varieties. Moreover, we obtain an algorithm to explicitly calculate this parametrization which has been used for $G$ of rank at most 8, except $E_8$.

When $G$ is defined and split over the field of $q$ elements, for $q$ the power of a good prime for $G$, this algorithmic parametrization is used to calculate the number $k(U(q), u^*(q))$ of coadjoint orbits of $U(q)$ on $u^*(q)$. Since $k(U(q), u^*(q))$ coincides with the number $k(U(q))$ of conjugacy classes in $U(q)$, these calculations can be viewed as an extension of the results obtained in [8]. In each case considered here there is a polynomial $h(t)$ with integer coefficients such that for every such $q$ we have $k(U(q)) = h(q)$.

1. Introduction

Let $G$ be a simple algebraic group over an algebraically closed field $k$. We assume throughout that the characteristic $p$ of $k$ is either 0 or a good prime for $G$. Let $B$ be a Borel subgroup of $G$ and let $U$ be the unipotent radical of $B$. Further let $g$, $b$ and $u$ be the Lie algebras of $G$, $B$, and $U$ respectively.

We study the coadjoint action of $U$ on the dual $u^*$ of $u$. Our aim is to generalize the theory for the adjoint action of $U$ on $u$, developed in [7], to the coadjoint action.

Under some mild assumptions that do no harm, there is a nondegenerate, invariant, symmetric, bilinear form on $g$, see §2.2. Therefore, we can identify $g^*$ with $g$ as a $G$-module. Under this identification the annihilator of the $B$-submodule $u$ of $g$ is $b$. Thus, we have an isomorphism of $B$-modules $u^* \cong g/b$, and it is convenient for us to make this identification. More generally, we study quotients of the form $g/m$, where $m$ is a $B$-submodule of $g$ containing $b$; we restrict to submodules $m$ that are compatible with a certain representation of $G$, see Definition 2.3 for details. We note that for $G$ of classical type, one can show that all $B$-submodules $m$ of $g$ as above are compatible, and this is potentially also the case for $G$ of exceptional type, see Remark 2.5 for more details.

The following are the main results of this paper.

Theorem 1.1. Let $m \supseteq b$ be a compatible $B$-submodule of $g$ and let $X + m \in g/m$. Then the orbit map $U \to U \cdot X + m$ is separable.

Theorem 1.2. Let $m \supseteq b$ be a compatible $B$-submodule of $g$ and let $X + m \in g/m$. Then the centralizer $C_U(X + m)$ of $X + m$ in $U$ is connected.

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The counterparts to Theorems 1.1 and 1.2 are known to hold for the adjoint action of $U$, by previous work of the first author, see [7, Prop. 3.7, Cor. 4.3] for certain quotients of $u$ and [6] for generalization to all quotients; in fact, we remark that the methods in this paper could be used to give alternative proofs of those results.

Theorem 1.2 allows us to define minimal representatives of the $U$-orbits in $u^*$, in analogy with the case for the adjoint action of $U$, as defined in [7]. The minimal representatives form a disjoint union of quasi-affine subvarieties of $u^*$ and parameterize the $U$-orbits in $u^*$. As in the adjoint case (see [8]), this leads to an algorithm to calculate a parametrization of the $U$-orbits in $u^*$.

We have implemented this algorithm in GAP [5] including some calls to SINGULAR [10] and used it to calculate a parametrization of the $U$-orbits in $u^*$ for $G$ simple of rank at most 8, with the exception of $E_8$. In this latter case, not unexpectedly, the computations turn out to be too complex for the program to provide such a parametrization. It would be possible to present the parametrization of the orbits in detail. However, this would be rather complicated and would take up a lot of space. In fact, we can make a good estimate of the number of families necessary to parameterize the orbits, by the sum of the coefficients of the polynomials given in Table 1; the meaning of these polynomials is explained below. We can see that for the case $G$ of type $E_7$, tens of thousands of pages would be required.

We mention that Kirillov’s method of coadjoint orbits provides a correspondence between the coadjoint orbits of $U$ in $u^*$ and the unitary representations of $U$, [16]. Thus in the special case of the complex numbers, our algorithm yields a parametrization of the equivalence classes of irreducible unitary representations of $U$. This also holds over fields with prime characteristic $p$, where $p$ is large enough such that the Baker–Campbell–Hausdorff formula holds, see [15].

As mentioned above our algorithm is adapted from the algorithm we used to parameterize the adjoint $U$-orbits in $u$, see [8]. Note that this in turn is based on an algorithm due to Bürgstein and Hesselink which was used to calculate a parametrization of the $B$-orbits in $u$ and $u^*$ for some small rank cases, see [3]. We note however, the algorithm devised by Bürgstein and Hesselink is not able to calculate the parametrization of the $U$-orbits in $u$ or $u^*$ in general. Our procedure to tackle this problem is considerably more complex.

Now assume that $G$ is defined and split over the finite field $\mathbb{F}_q$ of $q$ elements. Consider the finite Chevalley group $G(q)$ of $\mathbb{F}_q$-rational points of $G$. In this case we also investigate the action of the Sylow $p$-subgroup $U(q)$ of $G(q)$ on $u^*(q)$.

In the special case when $G = \text{GL}_n(k)$, let $U_n(q)$ be the subgroup of $\text{GL}_n(q)$ consisting of upper unitriangular matrices, where $q$ is a power of a prime. A longstanding conjecture attributed to G. Higman (cf. [12]) states that the number of conjugacy classes of $U_n(q)$ for fixed $n$ is a polynomial in $q$ with integer coefficients. This conjecture has been verified for $n \leq 13$ by computer calculation in work of A. Vera-Lopez and J. M. Arregi, [27], and also by A. Evseev [4] with a different approach. There has been considerable interest in this conjecture, for example from G. R. Robinson [20] and J. Thompson [26]. We do remark here that a recent paper of Z. Halasi and P. P. Pálfy potentially casts some doubt on the conjecture, [11].

It is natural to consider the analogue of Higman’s conjecture for other finite Chevalley groups. This investigation started in [9] and continued in [8]. Our new results in this
direction (see Section 4) can be viewed as evidence for this analogue of Higman's conjecture, as we are going to explain next.

By a trivial generalization of [17, Thm. 2.1.2], it is easy to see that the number \( k(U(q), u(q)) \) of \( U(q) \)-orbits in \( u(q) \) and the number \( k(U(q), u^*(q)) \) of \( U(q) \)-orbits in \( u^*(q) \) coincide.

The parametrization of the \( U \)-orbits in \( u^* \) can be used to adapt the algorithm employed in [8] to calculate the number \( k(U(q), u^*(q)) \) of \( U(q) \)-orbits in \( u^*(q) \) for \( G \) simple of rank at most 8, with the exception of \( E_8 \), see Section 4. Combining our results with [9, Thm. 1.1] and [8, Thm. 1.1], we obtain the following theorem.

**Theorem 1.3.** Let \( G \) be a split simple algebraic group defined over \( \mathbb{F}_q \) of rank at most 8, not of type \( E_8 \), where \( q \) is a power of a good prime. Let \( U \) be a maximal unipotent subgroup of \( G \) which is also defined over \( \mathbb{F}_q \). Then there is a polynomial \( h(t) \in \mathbb{Z}[t] \) which only depends on the Dynkin type of \( G \) such that the number \( k(U(q)) \) of conjugacy classes in \( U(q) \) is \( h(q) \). Furthermore, if one considers \( k(U(q)) \) as a polynomial in \( q - 1 \), then the coefficients are non-negative.

The polynomials giving \( k(U(q)) \) are presented in Table 1.

We mention that one can get various parabolic analogues of our results, and in particular of Theorem 1.3, see Remark 4.1 and [18, §5.1] for details. Also we remark that the restriction to good primes is necessary to get the same polynomial \( h(t) \) as the prime varies, as observed in [2]. Further, we note that there has been related recent work regarding the parametrization of the complex characters for \( U(q) \), see for example [13].

To end the introduction, we mention that in Section 5 we present some new results on the modality of the action of \( B \) on \( u \) which can be derived from our computations. More specifically, we are able to determine the modality of the action of \( B \) on \( u \) for instances where previously only lower bounds were known. This gives a significant extension of the results presented in [14, Tables II and III], and further demonstrates the strength of our algorithm.

### 2. Preliminaries

#### 2.1. Basic Notation.**

Let \( k \) be an algebraically closed field. Given an algebraic group \( H \) over \( k \), we write \( H^0 \) for the identity component of \( H \), and we denote the Lie algebra of \( H \) by \( \mathfrak{h} \). When \( H \) acts on an algebraic variety \( V \) and \( x \in V \), we write \( C_H(x) \) for the centralizer of \( H \) in \( V \), and \( H \cdot x \) for the \( H \)-orbit of \( x \). If \( W \) is a subset of \( V \), we write \( C_H(W) \) for the (pointwise) centralizer of \( W \) in \( H \). In the case, where \( V \) is a rational \( H \)-module, then \( V \) is also an \( \mathfrak{h} \)-module, and for \( x \in V \) we write \( \xi_\mathfrak{h}(x) \) for the centralizer of \( x \) in \( \mathfrak{h} \).

Throughout, \( G \) is a simple algebraic group over \( k \), where the characteristic \( p \) of \( k \) is either 0 or a good prime for \( G \); for convenience we also allow \( G = \text{GL}_n(k) \). Let \( B \) be a Borel subgroup of \( G \) and write \( U \) for the unipotent radical of \( B \). Let \( T \) be a maximal torus of \( G \) contained in \( B \) so that \( B = TU \).

#### 2.2. Compatible submodules.**

Let \( V \) be a faithful rational \( G \)-module. This allows us to identify \( G \) as a subgroup of \( \text{GL}(V) \) and \( \mathfrak{g} \) as a subalgebra of \( \text{gl}(V) \). We often want to have the following assumption.

**Assumption 2.1.** The restriction of the trace form on \( \text{gl}(V) \) to \( \mathfrak{g} \) is nondegenerate.

For \( G \) a classical group we note that Assumption 2.1 holds if we take \( V \) to be the natural module; except in the case \( G = \text{SL}_n(k) \) and \( p \mid n \), in which case we can “replace” \( \text{SL}_n(k) \) by
GL_n(k). For G of exceptional type, we can take V = g to be the adjoint module. See [19] or [25, Ch. I, Lem. 5.3] for details. As explained at the end of the proof of [7, Thm. 3.9], we have that U is independent up to isomorphism of the isogeny class of G, so our results on the coadjoint action of U do not depend on this assumption.

For the remainder of this subsection we make Assumption 2.1. We let ̃g be the orthogonal complement in gl(V) of g with respect to the trace form. Then ̃g is a G-submodule of gl(V) under the adjoint action, and moreover gl(V) admits a G-module decomposition

\[ gl(V) = g \oplus ̃g. \tag{2.2} \]

In other words this means that (GL(V), G) is a reductive pair, as defined in [19]. We use the notation ̃G = GL(V) and ̃g = gl(V).

We are interested in B-submodules m of g containing b that are compatible with the direct sum decomposition of (2.2) as set out in the next definition.

**Definition 2.3.** Make Assumption 2.1, and let ̃g = g ⊕ ̃g be the corresponding G-module decomposition from (2.2). Let m be a B-submodule of g containing b. We say that m is compatible (with V) provided there exist a Borel subgroup B of ̃G, a ̃B-submodule ̃m of ̃g, and a B-submodule m ⊆ ̃g such that

(i) \( B = ̃B \cap G; \)
(ii) \( m \supseteq ̃b; \) and
(iii) \( m = m \oplus ̃m \) as B-modules.

We note that a consequence of (iii) is that there is an isomorphism of B-modules

\[ ̃g/m \cong g/m \oplus ̃g/̃m. \tag{2.4} \]

**Remark 2.5.** It is quite an easy exercise to see that for G of classical type, all B-submodules m of g are compatible with the natural representation; a little care needs to be taken with regards to the graph automorphism in type D. Below we give a general method for constructing compatible B-submodules m of g; it seems plausible that all m can be obtained in this way, though we do not pursue this here.

First we introduce some more notation. Denote the character group of T by \( X(T) = \text{Hom}(T, k^{\times}). \) Let \( \Phi \subseteq X(T) \) be the root system of G with respect to T and let \( \Phi^+ \) be the set of positive roots determined by B, and \( \Phi^- = -\Phi^+. \) Let \( \Pi \) be the set of simple roots in \( \Phi^+. \)

We use certain preorders on \( X(T) \) to construct compatible B-submodules, as defined next.

**Definition 2.6.** We call a total preorder \( \preceq \) on \( X(T) \) a B-preorder if it satisfies the following three properties for all \( \beta, \gamma, \beta', \gamma' \in X(T). \)

(i) If \( \beta \in \Phi^+, \) then \( 0 \preceq \beta. \)
(ii) If \( \beta \preceq \gamma \) and \( \beta' \preceq \gamma', \) then \( \beta + \beta' \preceq \gamma + \gamma'. \)
(iii) If \( \beta \preceq \gamma, \) then \( -\gamma \preceq -\beta. \)

An example of a B-preorder, which is in fact a total order, can be given as follows. Fix an enumeration \( \Pi = \{\alpha_1, \ldots, \alpha_r\} \) of \( \Pi, \) and define a total order \( \preceq \) on \( X(T) \) as follows for \( \beta, \gamma \in X(T): \)

- If \( \text{ht} \beta < \text{ht} \gamma, \) then \( \beta < \gamma. \)
- If \( \text{ht} \beta = \text{ht} \gamma \) and \( \beta = \sum_{i=1}^{r} b_i \alpha_i, \ \gamma = \sum_{i=1}^{r} c_i \alpha_i \) where \( b_j < c_j \) for the highest \( j \) such that \( b_j \neq c_j, \) then \( \beta < \gamma. \)
Here we recall that $\text{ht}(\sum_{i=1}^r b_i \alpha_i) := \sum_{i=1}^r b_i$.

Also, we can obtain $B$-preorders from a function $d : X(T) \to \mathbb{R}$ such that $d(\alpha + \beta) = d(\alpha) + d(\beta)$ for all $\alpha, \beta \in X(T)$, and $d(\alpha) \geq 0$ for all $\alpha \in \Phi^+$. We note that such a function $d$ is uniquely determined from its restriction to $\Pi$, and that any function $d : \Pi \to \mathbb{R}_{\geq 0}$ can be obtained as such a restriction. Given such $d : X(T) \to \mathbb{R}$, we can define the $B$-preorder $\preceq$ by $\alpha \preceq \beta$ if and only if $d(\alpha) \leq d(\beta)$.

For the rest of this subsection we fix a $B$-preorder $\preceq$.

For $\beta \in \Phi$, let $g_\beta$ be the corresponding root space of $g$; Given $\gamma \in \Phi^-$ we define the subspaces $m_{\succ \gamma}$ of $g$ by

\begin{equation}
    m_{\succ \gamma} := \bigoplus_{\beta \succ \gamma} g_\beta.
\end{equation}

Our choice of order $\preceq$ ensures that each $m_{\succ \gamma}$ is a $B$-submodule of $g$ containing $b$.

Recall the faithful $G$-module $V$ from above. We continue to make the Assumption 2.1. We let $\Psi(V) \subseteq X(T)$ be the set of $T$-weights in $V$ and decompose

\begin{equation}
    V = \bigoplus_{\nu \in \Psi(V)} V_\nu
\end{equation}

as a direct sum of its $T$-weight spaces. We choose an (ordered) basis $\{v_1, \ldots, v_n\}$ of $V$ consisting of $T$-weight vectors, where the $T$-weight of $v_i$ is $\nu_i$, and we have $\nu_i \preceq \nu_j$, whenever $i \geq j$. Using this basis we can identify $\tilde{G}$ with $\text{GL}_n(k)$. We then let $B$ be the upper triangular matrices under this identification, and note that our choice of basis ensures that $B = \tilde{B} \cap G$.

Next we define $\Psi(\tilde{g}) \subseteq X(T)$ to be the set of weights of $T$ on $\tilde{g}$. We note that we have

\begin{equation}
    \Psi(\tilde{g}) = \{\nu - \mu \mid \nu, \mu \in \Psi(V)\}.
\end{equation}

Then we decompose $\tilde{g} = g\mathfrak{l}_n$ in to $T$-weight spaces

\begin{equation}
    \tilde{g} = \bigoplus_{\lambda \in \Psi(\tilde{g})} \tilde{g}_\lambda.
\end{equation}

Since $\tilde{g} = g \oplus \tilde{g}$ is a $G$-module direct sum decomposition, we obtain $\tilde{g}_\lambda = g_\lambda \oplus \tilde{g}_\lambda$ for all $\lambda \in \Psi(\tilde{g})$.

The following lemma ensures our supply of compatible $B$-submodules of $g$.

**Lemma 2.8.** Let $\gamma \in \Phi^-$, and define $m = m_{\succ \gamma}$ as in (2.7). Then $m$ is compatible. More precisely, if one chooses $\tilde{B}$ as above, and

\begin{equation}
    \tilde{m} = \tilde{m}_{\succ \gamma} = \bigoplus_{\lambda \succ \gamma} \tilde{g}_\lambda \quad \text{and} \quad \tilde{m} = \tilde{m}_{\succ \gamma} = \bigoplus_{\lambda \succ \gamma} \tilde{g}_\lambda
\end{equation}

then properties (i) – (iii) from Definition 2.3 hold for $\tilde{B}$, $\tilde{m}$ and $\tilde{m}$.

**Proof.** It is clear that $\tilde{m} \supseteq \tilde{b}$, and that $\tilde{m} = m \oplus \tilde{m}$. The main property to prove is that $\tilde{m}$ is a $B$-submodule of $\tilde{g}$; from this it follows easily that $\tilde{m}$ is a $B$-submodule of $\tilde{g}$.

Through the basis $\{v_1, \ldots, v_n\}$ of $V$ we have our identification of $\tilde{g}$ with $g\mathfrak{l}_n(k)$ and we write $\{e_{ij} \mid 1 \leq i, j \leq n\}$ for its standard basis. For $\lambda \in \Psi(\tilde{g})$, we have $\tilde{g}_\lambda = \langle e_{ij} \mid \lambda = \nu_i - \nu_j \rangle$. By considering how matrices are multiplied together, we see that it suffices to prove the following: Let $1 \leq i, j, k, l \leq n$ and suppose that $k \leq i$ and $j \leq l$, then $\nu_i - \nu_j \preceq \nu_k - \nu_l$. Well, since $k \leq i$, we have $\nu_i \preceq \nu_k$, and since $j \leq l$, we have $\nu_j \preceq \nu_l$. Therefore, $-\nu_j \preceq -\nu_i$, and so $\nu_i - \nu_j \preceq \nu_k - \nu_l$, which is what we require, so the proof is complete. $\square$
2.3. **Springer isomorphisms.** Denote by \( \mathcal{U} \) the unipotent variety of \( G \) and by \( \mathcal{N} \) the nilpotent variety of \( \mathfrak{g} \). In \([24]\), Springer showed that, if \( G \) is simply connected, then there is a \( G \)-equivariant isomorphism of varieties

\[
\phi : \mathcal{U} \to \mathcal{N},
\]

see also \([25, \text{Thm. 3.12}]\). Such an isomorphism is called a *Springer isomorphism*. We note that for \( G \) not necessarily simply connected, we still obtain a \( G \)-equivariant bijective morphism of varieties \( \phi : \mathcal{U} \to \mathcal{N} \), which is an isomorphism if the covering map from the simply connected cover of \( G \) is separable. See, for example \([7, \S 2]\), for an overview of the theory of Springer isomorphisms.

We recall the construction of a Springer isomorphism due to Bardsley and Richardson \([1, \S 9]\), in the case where Assumption 2.1 holds. For that recall the \( G \)-module decomposition of \( \mathfrak{g} = \mathfrak{gl}(V) \) from (2.2). Let \( \iota : G \to \text{GL}(V) \) be the embedding of \( G \) into \( \text{GL}(V) \subseteq \mathfrak{gl}(V) \) and \( \pi : \mathfrak{gl}(V) \to \mathfrak{g} \) be the canonical projection. It follows from the proof of \([1, \text{Lem. 9.3.1}]\) that under the assumptions made, we obtain a Springer isomorphism by restricting \( \pi \circ \iota \) to the unipotent variety \( \mathcal{U} \) of \( G \), i.e. the image of \( \pi \circ \iota|_{\mathcal{U}} \) is precisely \( \mathcal{N} \), and this restriction is an isomorphism of varieties onto \( \mathcal{N} \).

3. **The coadjoint action of \( U \) on \( u^* \)**

We continue to make Assumption 2.1, so that there is a nondegenerate, symmetric, invariant, bilinear form on \( \mathfrak{g} \). As explained in the introduction this allows us to identify \( u^* \) with \( \mathfrak{g}/\mathfrak{b} \) as \( U \)-modules. Let \( \mathfrak{m} \supseteq \mathfrak{b} \) be a compatible \( B \)-submodule of \( \mathfrak{g} \), and let \( \bar{\mathfrak{m}} \), \( \bar{\mathfrak{m}} \) and \( \bar{B} \) be as in Definition 2.3, and write \( \bar{U} \) for the unipotent radical of \( \bar{B} \). Then we have an isomorphism of \( B \)-modules from (2.4), which we use throughout this section.

In this section we prove Theorems 1.1 and 1.2 regarding the action of \( U \) on \( \mathfrak{g}/\mathfrak{m} \). Our strategy is to adapt and extend some arguments of Springer and Steinberg from \([25, \text{Ch. I \S 5}]\).

Let \( X + \mathfrak{m} \in \mathfrak{g}/\mathfrak{m} \). The \( U \)-orbit of \( X + \mathfrak{m} \) in \( \mathfrak{g}/\mathfrak{m} \) is denoted by \( U \cdot X + \mathfrak{m} \) and we write \( \bar{U} \cdot X + \bar{\mathfrak{m}} \) for the \( \bar{U} \)-orbit of \( X + \bar{\mathfrak{m}} \) in \( \bar{\mathfrak{g}}/\bar{\mathfrak{m}} \). We define

\[
V(X, \bar{\mathfrak{m}}) := \{ u \cdot X - X + \bar{\mathfrak{m}} \in \bar{\mathfrak{g}}/\bar{\mathfrak{m}} \mid u \in \bar{U} \}.
\]

**Lemma 3.1.** Define the map \( f : \bar{U} \to V(X, \bar{\mathfrak{m}}) \) by

\[
 f(u) = (u \cdot X) - X + \bar{\mathfrak{m}}.
\]

Then \( f \) is separable.

**Proof.** We show that the differential

\[
(df)_1 : \bar{u} \to T_0(V(X, \bar{\mathfrak{m}}))
\]

of \( f \) is surjective. A direct calculation shows that \((df)_1 \) maps \( Y \in \bar{u} \) to \([Y, X] + \bar{\mathfrak{m}} \). We have

\[
\dim T_0(V(X, \bar{\mathfrak{m}})) = \dim(\bar{U} \cdot X + \bar{\mathfrak{m}}) = \dim \bar{U} - \dim C_{\bar{U}}(X + \bar{\mathfrak{m}}),
\]

while

\[
\dim \text{im}(df)_1 = \dim \bar{u} - \dim \ker(df)_1 = \dim \bar{U} - \dim c_u(X + \bar{\mathfrak{m}}).
\]

Thus, it suffices to show that \( \dim C_{\bar{U}}(X + \bar{\mathfrak{m}}) = \dim c_u(X + \bar{\mathfrak{m}}) \). We claim that in fact

\[
C_{\bar{U}}(X + \bar{\mathfrak{m}}) = 1 + c_u(X + \bar{\mathfrak{m}}).
\]
If \( u \in C_U(X+m) \), then \( uXu^{-1} + \bar{m} = X + \bar{m} \). Since \( \bar{m} \) is a \( B \)-submodule of \( \bar{g} \), we have that \( \bar{m} \) is stable under right multiplication by elements of \( \bar{B} \). Thus right multiplication by \( u \) and subtraction of \( Xu \) on both sides yields
\[
\bar{m} = uX - Xu + \bar{m} = (u - 1)X - X(u - 1) + \bar{m} = [u - 1, X] + \bar{m}.
\]
So \( C_U(X + \bar{m}) \subseteq 1 + C_u(X + \bar{m}) \). The reverse inclusion can be proved similarly, and so \((3.2)\) follows.

**Lemma 3.3.** \( \text{Lie} C_U(X + m) = c_u(X + m) \).

**Proof.** Since \( \text{Lie} C_U(X + m) \subseteq c_u(X + m) \), it suffices to show that the dimensions coincide. This is equivalent to showing that \( \dim(U \cdot X + m) = \dim([u, X] + m) \), or equivalently
\[
\dim(U \cdot X + m) = \dim(\text{ad}(X)(u) + m).
\]
Consider the restriction of \( f \) from Lemma 3.1 to \( U \). Through \((2.4)\), we have that the image of \( f|_U \) is contained in \( g/m \), and thus also \( T_0(f(U)) \subseteq g/m \). We prove \((3.4)\) by verifying the following sequence of inclusions:
\[
T_0(f(U)) \subseteq T_0(f(\bar{U})) \cap T_0(g/m)
= (df)_1(\bar{u}) \cap g/m
= (\text{ad}(X)(\bar{u}) + \bar{m}) \cap g/m
= \text{ad}(X)(u) + m
\subseteq T_0(f(U)).
\]
The initial inclusion is immediate, whilst the first equality is just Lemma 3.1 and the second is the definition of \((df)_1\).

To see why the third equality holds, consider \( \bar{Y} = Y + \bar{Y} \in \bar{u} \) with \( Y \in u \) and \( \bar{Y} \in u \cap \bar{g} \). Then
\[
[X, \bar{Y}] + \bar{m} = [X, Y + \bar{Y}] + \bar{m} = [X, Y] + [X, \bar{Y}] + \bar{m},
\]
where, thanks to the \( G \)-module decomposition of \( \bar{g} \) from \((2.2)\), we have that \( [X, Y] \) belongs to \( \bar{g} \) and \( [X, \bar{Y}] \) lies in \( \bar{g} \). Thus through \((2.4)\), we have that \( [X, Y] + m \in g/m \) and \( [X, \bar{Y}] + \bar{m} \in \bar{g}/\bar{m} \). In particular, if \( [\bar{Y}, X] + \bar{m} \in g/m \), then \( [\bar{Y}, X] + \bar{m} = [Y, X] + \bar{m} \). This shows that \( (\text{ad}(X)(\bar{u}) + \bar{m}) \cap g/m \subseteq \text{ad}(X)(u) + m \), and the reverse inclusion is clear.

The last inclusion holds because \( \text{ad}(X)(u) + m = (df)_1(u) \).

Finally, \((3.5)\) yields that \( T_0(f(U)) = \text{ad}(X)(u) + m \), and, clearly, we also have that \( \dim f(U) = \dim(U \cdot X + m) \), which together imply \((3.4)\). \(\square\)

We are now in a position to prove our main results.

**Proof of Theorem 1.1.** By a standard argument this is equivalent to Lemma 3.3. \(\square\)

**Proof of Theorem 1.2.** Let \( \phi : U \to N \) be the Springer isomorphism for \( G \) obtained from the Bardsley–Richardson construction, as in Section 2.3. We prove the statement by showing that \( C_U(X + m) = \phi^{-1}(c_u(X + m)) \). Since \( c_u(X + m) \) is a vector space and thus is irreducible as a variety, so is \( \phi^{-1}(c_u(X + m)) \). Then it follows that \( C_U(X + m) \) is connected.

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We identify $U$ as a subgroup of $\bar{U}$. Thus, if $u \in C_U(X + m)$, then $u \in C_U(X + \bar{m})$, so $uXu^{-1} + \bar{m} = X + \bar{m}$. Since $\bar{m}$ is a $B$-submodule of $\bar{g}$, it is invariant under right multiplication by elements in $\bar{U}$. It follows that $uX + \bar{m} = Xu + \bar{m}$. Therefore, $[u, X] + \bar{m} = \bar{m}$. The direct sum decomposition (2.2) of $\bar{g}$ and the definition of $\phi$ gives

$$u = \phi(u) + \tilde{Y}$$

with $\tilde{Y} \in \bar{g}$. So we have

$$[u, X] + \bar{m} = [\phi(u), X] + \tilde{Y}, X + \bar{m}$$

where $[\phi(u), X] \in g$ and $[\tilde{Y}, X] \in \bar{g}$. Thus, using (2.4), we obtain $[\phi(u), X] \in m$ and $[\tilde{Y}, X] \in \bar{m}$. Together with the fact that $\phi$ restricts to an isomorphism $U \cong u$ (see for example [7, Cor. 2.2]), this implies that $\phi(u) \in c_u(X + m)$.

Consequently, $\phi(C_U(X + m)) \subseteq c_u(X + m)$. Since $\phi(C_U(X + m))$ is a closed subvariety of $c_u(X + m)$, both have the same dimension, by Lemma 3.3, and $c_u(X + m)$ is irreducible, we must have that $\phi(C_U(X + m)) = c_u(X + m)$. Finally, as $\phi$ is an isomorphism, $C_U(X + m) = \phi^{-1}(c_u(X + m))$, as desired.

We now explain how Theorems 1.1 and 1.2 allow us to give a parametrization of the coadjoint $U$-orbits in $u^*$; this is an analogue of the parametrization of the adjoint $U$-orbits in $u$ described in [9]. We first fix a $B$-preorder $\preceq$ on $X(T)$, as defined in Definition 2.6, and assume that $\preceq$ restricts to a total order on $\Phi$. Now we enumerate $\Phi^+ = \{\beta_1, \ldots, \beta_N\}$ so that $\beta_i \prec \beta_j$, whenever $i < j$. To ease notation, we also define $\gamma_i = -\beta_{N+1-i}$, so that $\Phi^- = \{\gamma_1, \ldots, \gamma_N\}$.

We define subspaces $m_i$ of $g$, for each $0 \leq i \leq N$, by setting

$$(3.6) \quad m_i := m_{\gamma_i} = \left( \bigoplus_{j=i+1}^{N} g_{\gamma_j} \right) \oplus b \subseteq g.$$ 

Then each $m_i$ contains $b$ and is a compatible $B$-submodule of $g$, by Lemma 2.8. Thus the quotient $g/m_i$ is a $B$-module as well. Furthermore, we have

$$b = m_N \subset \ldots \subset m_i \subset \ldots \subset m_0 = g$$

and $\dim m_i/m_{i-1} = 1$ for $i = 1, \ldots, N$.

Using Theorem 1.2 one can prove the following lemma, which is an analogue of [7, Lem. 5.1]. In the statement $e_\beta$ denotes a generator of the root space $g_\beta$, for $\beta \in \Phi$.

**Lemma 3.7.** For $0 \leq i \leq N$, let $m_i$ be defined as in (3.6). Let $X + m_{i-1}$ be an element in $g/m_{i-1}$. Denote $X + ke_{\gamma_i} + m_i = \{X + \lambda e_{\gamma_i} + m_i \mid \lambda \in k\} \subseteq g/m_i$. Then either

(I) all elements of $X + ke_{\gamma_i} + m_i$ are $U$-conjugate, or

(R) no two elements of $X + ke_{\gamma_i} + m_i$ are $U$-conjugate.

In case (I) of the above lemma we say that $i$ is an **inert point** of $X + m_i$, whereas in case (R) we call $i$ a **ramification point** of $X + m_i$. We say that $X + m_i = \sum_{j=1}^{i} a_j e_{\gamma_j} + m_i \in g/m_i$ is the **minimal representative** of its $U$-orbit in $g/m_i$ provided $a_j = 0$ whenever $j \leq i$ is an inert point of $X$. It follows from the counterparts to [7, Prop. 5.4 and Lem. 5.5] in our setting that each $U$-orbit in $g/m_i$ contains a unique minimal representative. Now we can adapt the
arguments in [8, §2], to show that the minimal representatives are partitioned in to families $\mathcal{X}_c$, where $c$ runs over the indexing set $C = \{I, R_0, R_n\}^N$. For $c \in \{I, R_0, R_n\}^N$, we define

$$(g/b)_c = \{X + b \in g/b \mid j \text{ is an inert point of } X \text{ if and only if } c_j = I\}$$

and

$$\mathcal{X}_c = \left\{ \sum_{j=1}^{i} a_j e_{c_j} + b \in (g/b)_c \left| a_j = 0 \text{ if and only if } c_j \in \{I, R_0\} \right. \right\}.$$

Then the quasi-affine varieties $\mathcal{X}_c$ give a parametrization of the $U$-orbits in $g/b \cong u^*$.

The discussion above, along with Theorem 1.1, implies that the algorithm from [8, §3] to calculate this parametrization in case of the adjoint action of $U$ on $u$ can be adapted to the coadjoint action of $U$ on $u^*$. As explained in the introduction, we have implemented this adaptation using GAP [5], and used it to calculate the $U$-orbits in $u^*$, when $G$ is of rank at most 8, with the exception of $E_8$. As noted in the introduction this parametrization becomes rather intricate as the rank grows, because the equations defining the quasi affine varieties $\mathcal{X}_c$ become more complicated.

4. Counting conjugacy classes in $U(q)$

Now assume that $G$ is defined and split over the field $\mathbb{F}_q$ where $q$ is some power of a good prime $p$ for $G$. We denote by $G(q)$ the group of $\mathbb{F}_q$-rational points of $G$. We also assume that $B$ and $T$ are chosen to be defined over $\mathbb{F}_q$. Then $U$ is defined over $\mathbb{F}_q$ and $U(q)$ is a Sylow $p$-subgroup of $G(q)$.

We continue to use the notation in the previous section, so we have a fixed $B$-preorder $\preceq$ on $X(T)$ which restricts to a total order on $\Phi$, and enumerate $\Phi^+$ and $\Phi^-$ accordingly. Then the $m_i$ defined in (3.6) are $F$-stable $B$-submodules. Thanks to Theorem 1.2, there are analogues of [7, Prop. 6.2 and Lem. 6.3] for the coadjoint action. From this, we deduce that the $U(q)$-orbits in $u^*(q)$ are in bijection with the minimal representatives with coefficients in $\mathbb{F}_q^*$.

The algorithm explained in [8, §3] can be adapted to calculate the number $k(U(q), u^*(q))$ of coadjoint orbits of $U(q)$ on $u^*(q)$ from the parametrization of the coadjoint orbits. We have done this in GAP and run it for all simple groups $G$ of rank at most 8, except $E_8$. The results of our calculations are presented in Table 1 below. As we have already observed in the introduction, the values $k(U(q), u^*(q))$ and the number $k(U(q))$ of $U(q)$-conjugacy classes in $U(q)$ coincide.

Table 1 contains the values of $k(U(q), u^*(q))$ for $G$ of types $E_7$, $B_8$, $C_8$ and $D_8$ as polynomials in $v = q - 1$. Combining these with our previous results [9, Thm. 1.1] and [8, Thm. 1.1], we obtain Theorem 1.3. We recall also that for $G$ of type $A_r$, the polynomials have been calculated up to $r = 12$ and are given in [27], see also [4].

We note that the values of $k(U(q))$ are always polynomials in $v = q - 1$ with non-negative integer coefficients. This is also the case for the polynomials obtained in [8]. It follows from [8, Thm. 5.4] that the coefficient of $k(U(q))$ (as a polynomial in $v$) of degree 1 equals $|\Phi^+|$ and that of degree 2 is also given by combinatorial data only depending on $\Phi^+$.

It is worth noting that the algorithm is able to compute $k(U(q), u^*(q))$ for groups for which the algorithm employed in [8] is not able to calculate $k(U(q), u(q))$. That is, the coadjoint action seems to be more favourable for computational purposes as opposed to the adjoint
action. A similar observation was already made by Bürgstein and Hesselink in [3, §1.8] in their study of the $B$-orbits in $u$ and $u^*$.

| $G(q)$ | $k(U(q), u^*(q))$ |
|--------|------------------|
| $E_7$  | $v^{17} + 18v^{16} + 154v^{15} + 839v^{14} + 3298v^{13} + 10104v^{12} + 25702v^{11} + 57351v^{10}$ $+ 114413v^9 + 194330v^8 + 255908v^7 + 238441v^6 + 145845v^5 + 54705v^4 + 11655v^3$ $+ 1281v^2 + 63v + 1$ |
| $D_8$  | $v^{16} + 19v^{15} + 175v^{14} + 1057v^{13} + 4770v^{12} + 17192v^{11} + 50639v^{10} + 119410v^9$ $+ 213853v^8 + 274244v^7 + 239428v^6 + 136164v^5 + 48090v^4 + 9912v^3 + 1092v^2$ $+ 56v + 1$ |
| $B_8/C_8$ | $2v^{17} + 40v^{16} + 387v^{15} + 2422v^{14} + 11077v^{13} + 39613v^{12} + 115125v^{11} + 274653v^{10}$ $+ 525983v^9 + 772250v^8 + 824340v^7 + 607950v^6 + 294658v^5 + 88816v^4 + 15568v^3$ $+ 1456v^2 + 64v + 1$ |

Table 1: $k(u(q), u^*(q))$ as polynomial in $v = q - 1$

As noted in [9] and [8] for the smaller rank instances, the polynomials in Theorem 1.3 coincide for types $B_n$ and $C_n$.

As mentioned in [8, §3], there are situations where the algorithm might carry out implicit divisions by certain primes, and then the results of the calculation are not necessarily valid for these primes. Previously this did not cause any problems, as the only primes occurring were bad primes. However, for the rank 8 cases the prime 3 showed up. An alternative run of the program where division by 3 was not allowed verified that the polynomials are also valid for the prime 3.

E. A. O’Brien and W. Unger have calculated the values of $k(U(q))$ for $q = p = 3$ or 5 for $B_8$, $C_8$ and $D_8$, and for $q = p = 5$ for $E_7$. For this they used an improved implementation of an algorithm by M. C. Slattery which calculates the number of irreducible characters of each degree for a $p$-group, [23]. Their calculations verified our results in these cases.

For $G$ of type $E_7$ we also used the modified program for the coadjoint action to calculate $k(U(q), u^{(1)*}(q))$, where $u^{(1)}$ is the Lie algebra of the commutator subgroup $U^{(1)}$ of $U$. For that we used $u^{(1)*} \cong g/(b \oplus g_{-\alpha_1} \oplus \ldots \oplus g_{-\alpha_r})$, where $\Pi = \{\alpha_1, \ldots, \alpha_r\}$. The resulting polynomial is

$$v^{16} + 16v^{15} + 121v^{14} + 579v^{13} + 1983v^{12} + 5232v^{11} + 11268v^{10} + 21028v^9$$ $+ 36019v^8 + 55895v^7 + 68476v^6 + 56196v^5 + 27571v^4 + 7393v^3 + 980v^2 + 56v + 1.$

This extends the results in Table 2 from [8].

Remark 4.1. Let $P$ be a parabolic subgroup of $G$ with unipotent radical $U_P$ and Lie algebra $u_P$ of $U_P$. Without loss, we may assume that $P$ contains $B$. It is possible to define a suitable filtration of $u_P$ and $u^*_P$ so that we obtain separability and connectivity results analogous
to Theorems 1.1 and 1.2 for the actions of $U$ on $u_p$ and $u_p^*$; and we also get these for the actions of $U_p$ on $u_p$ and $u_p^*$. Then a result analogous to Lemma 3.7 allows one to define inert and ramification points for these actions. In turn this leads to the notion of minimal representatives for each of these actions.

This allows one to generalize our algorithm to obtain a parameterization of each of: the $U$-orbits in $u_p$; the $U$-orbits in $u_p^*$; the $U_p$-orbits in $u_p$; and the $U_p$-orbits in $u_p^*$.

Analogous to our previous results one gets that the number $k(U(q), U_P(q))$ of $U(q)$-conjugacy classes in $U_P(q)$ coincides with $k(U(q), u_P(q)) = k(U(q), u_P^*(q))$, the number of $U(q)$-conjugacy classes in $u_P(q)$. Likewise for the action of $U_P(q)$, we obtain that $k(U_P(q)) = k(U_P(q), u_P(q)) = k(U_P(q), u_P^*(q))$.

To calculate the number of $U_P(q)$-classes in $U_P(q)$, first the algorithm for $k(U(q), u_P(q))$ is used to determine the minimal representatives of $U(q)$-orbits in $u_P(q)$. Then, for each minimal representative $X \in u_P(q)$, the number $k(U_P(q), U(q) \cdot X)$ of all $U_P(q)$-orbits in the $U(q)$-orbit of $X$ is calculated using the formula given in [18, Lem. 5.2.1].

We refer to [18, Ch. 5] for further details of these parabolic analogues and for a list of explicit results obtained using the parabolic version of the algorithm.

5. The modality of the action of $B$ on $u$

In this final section, we record another application of the calculations to parameterize the $U$-orbits in $u$ respectively $u^*$. Namely to determine the modality of the action $B$ in $u$ in the cases we have considered in Section 4. This is particularly meaningful since, at present, there appears to be no effective method to calculate a good upper bound for $\text{mod}(B : u)$. Our results allow us to extend the list of known values of $\text{mod}(B : u)$, which are given in [14, Tables II and III].

Recall that the modality of the action of the Borel subgroup $B$ on $u$ is defined by

$$\text{mod}(B : u) := \max_{i \in \mathbb{Z}_{\geq 0}} (\text{dim} u_i - i),$$

where $u_i := \{ X \in u \mid \text{dim} B \cdot X = i \}$. Likewise, the modality of the action of $U$ on $u$ is defined analogously by replacing the adjoint $B$-action on $u$ by the $U$-action, i.e.,

$$\text{mod}(U : u) := \max_{i \in \mathbb{Z}_{\geq 0}} (\text{dim} \{ X \in u \mid \text{dim} U \cdot X = i \} - i).$$

Each $U$-orbit in $u$ admits a minimal representative, and as explained in [7, §2], the minimal representatives are partitioned into families $\mathcal{X}_c$, where $c$ runs over $C = \{I, R_0, R_n\}^N$. Then $\text{mod}(U : u)$ is just the largest dimension of a family of minimal representatives. We also note that $\text{mod}(U : u) = \text{mod}(U : u^*)$, [22, Thm. 1.4], so we can determine this modality by considering the coadjoint action of $U$ on $u^*$.

In the following theorem, we show that $\text{mod}(B : u)$ can easily be determined from $\text{mod}(U : u)$; the parametrization by minimal representatives is required for the proof.

**Theorem 5.1.** Let $G$ be a simple algebraic group and suppose that $p$ is either zero or a good prime for $G$. Let $B$ be a Borel subgroup of $G$ with unipotent radical $U$. Then

$$\text{mod}(U : u) - \text{rank} G = \text{mod}(B : u).$$

**Proof.** By [7, Prop. 7.7], we see that for any minimal representative $X \in u$ we have

$$C_B(X) = C_T(X)C_U(X)$$

(5.2)
and we note that the definition of the families $X_c$ ensures that $\dim T \cdot X$ is independent of the choice of $X \in X_c$. Each family $X_c$ of $U$-orbits in $u$ gives rise to a family of $B$-orbits of dimension

$$\dim X_c - \dim T \cdot X = \dim X_c - \text{rank } G + \dim C_T(X),$$

where $X \in X_c$.

Let $u_{\text{dist}}$ be the union of all $U$-orbits in $u$ such that $\dim C_T(X) = 0$ for $X$ the minimal representative of the orbit; we note that this is equivalent to $C_T(X)^o = 1$. Note that thanks to (5.2), we have that $u_{\text{dist}}$ is the $B$-stable subvariety of $u$ consisting of all $X \in u$ such that $C_B(X)^o$ is unipotent. Then by (5.3), we have $\mod(B : u_{\text{dist}}) = \mod(U : u_{\text{dist}}) - \text{rank } G$.

To complete the proof, we show by induction on the rank of $G$ that

$$\mod(B : u) = \mod(B : u_{\text{dist}})$$

and

$$\mod(U : u) = \mod(U : u_{\text{dist}}).$$

The base case where $G$ has rank 0 is trivial, so we assume inductively that (5.4) and (5.5) are true in lower rank.

Let $u_{\text{dec}} := u \setminus u_{\text{dist}}$ be the complement of $u_{\text{dist}}$ in $u$. We show that

$$\mod(U : u_{\text{dec}}) \leq \mod(U : u_{\text{dist}}) \quad \text{and} \quad \mod(B : u_{\text{dec}}) \leq \mod(B : u_{\text{dist}}),$$

which gives the inductive step.

Let $X \in u_{\text{dec}}$. Then there is a torus $S \subseteq T$ of positive dimension contained in $C_B(X)$. We have that $L = C_G(S)$ is a proper Levi subgroup of $G$, and moreover, $L \cap B$ is a Borel subgroup of $L$ and $L \cap u$ is the Lie algebra of the unipotent radical of $L \cap B$. Then the intersection of a $B$-orbit in $u$ with $L \cap u$ is a union of $(L \cap B)$-orbits; and also the intersection of a $U$-orbit in $u$ with $L \cap u$ is a union of $(L \cap U)$-orbits. Also, up to conjugacy by $B$, there are only finitely many possibilities for $L$, (each such is determined by its root system which is a subsystem of the root system of $G$).

Therefore, we have that

$$\mod(B : u_{\text{dec}}) \leq \max_L \mod(L \cap B : L \cap u),$$

where the maximum is taken over all Levi subgroups $L$ of $G$ containing $T$. Similarly,

$$\mod(U : u_{\text{dec}}) \leq \max_L \mod(L \cap U : L \cap u).$$

We proceed to show that

$$\mod(L \cap B : L \cap u) \leq \mod(B : u_{\text{dist}})$$

and that

$$\mod(L \cap U : L \cap u) \leq \mod(U : u_{\text{dist}}),$$

when $L$ is a Levi subgroup of $G$ containing $T$. For this we may assume that the simple roots of $L$ determined by $L \cap B$ are a subset of the simple roots of $G$ determined by $B$.

By the inductive hypothesis, there is a family $X_c(L)$ of minimal representatives of $(L \cap U)$-orbits in $L \cap u$ such that:

1. $C_T(X)^o = Z(L)^o$ for all $X \in X_c(L)$;
2. $\dim X_c(L) = \mod(L \cap U : L \cap u)$; and
3. $\dim X_c(L) - \text{rank } G + \dim Z(L) = \mod(L \cap B : L \cap u)$. 

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Let $\alpha_1, \ldots, \alpha_h$ be the simple roots of $G$ that are not simple roots of $L$. We consider

$$\mathcal{Y}_c = \left\{ \sum_{i=1}^h a_i e_{\alpha_i} + X \mid a_i \in k^* \text{ and } X \in \mathcal{X}_c(l) \right\}.$$  

Clearly, the elements of $\mathcal{Y}_c$ lie in distinct $U$-orbits in $u$. Moreover, $\mathcal{Y}_c \subseteq u_{\text{dist}}$, and $\dim \mathcal{Y}_c = \dim \mathcal{X}_c(l) + h$, so we deduce that $\text{mod}(L \cap U : \mathfrak{l} \cap u) \leq \text{mod}(U : u_{\text{dist}})$.

We have that $T$ acts on $\mathcal{Y}_c$, and the dimension of each $T$-orbit is $\dim T$. Therefore, $\mathcal{Y}_c$ gives a family of $B$-orbits in $u_{\text{dist}}$ of dimension $\dim \mathcal{X}_c(l) - \text{rank } G + \dim Z(L)$. From this it follows that $\text{mod}(L \cap B : \mathfrak{l} \cap u) \leq \text{mod}(B : u_{\text{dist}})$. This completes the proof. $\Box$

In the cases in which we have calculated $k(U(q))$, the modality $\text{mod}(U : u)$ is simply the degree of the polynomial giving $k(U(q))$. Then we can calculate $\text{mod}(B : u)$ using Theorem 5.1. It is also possible to determine $\text{mod}(B : u^*)$ directly from our parametrization of the $U$-orbits in $u^*$ and use that $\text{mod}(B : u) = \text{mod}(B : u^*)$, by [22, Thm. 1.4].

This allows us to extend the list of values for $\text{mod}(B : u)$ as follows. The list of previously known values for $\text{mod}(B : u)$ for smaller rank groups is given in [14, Tables II and III]. Below for completeness we also include the values of this modality for $G$ of type $A_n$, which we obtain via the polynomials calculated in [27, §5].

| Type of $G$ | $A_{11}$ | $A_{12}$ | $B_7$ | $B_8$ | $C_8$ | $D_8$ | $E_7$ |
|------------|---------|---------|------|------|------|------|------|
| $\text{mod}(B : u)$ | 7      | 8      | 7    | 9    | 9    | 8    | 10   |

Table 2. Modality of the action of $B$ on $u$

Note that the values given in Table 2 were previously only known to be lower bounds for $\text{mod}(B : u)$ and were obtained by means of a rather coarse dimension estimate, see [21, Prop. 3.3, Table 4]. The construction of the lower bounds for $\text{mod}(B : u)$ is given in the following way. There exists a normal subgroup $A$ of $B$ contained in $U$ with Lie algebra $\mathfrak{a}$, such that $\dim \mathfrak{a}/[\mathfrak{a}, \mathfrak{a}] - \dim B/A$ is quite large, and this is the lower bound given. To explain this further, we note that the adjoint action of $B$ on $\mathfrak{u}$ restricts to an action on $\mathfrak{a}$, which in turn induces an action of $B/A$ on $\mathfrak{a}/[\mathfrak{a}, \mathfrak{a}]$. Then clearly, we have that $\text{mod}(B : u) \geq \dim \mathfrak{a}/[\mathfrak{a}, \mathfrak{a}] - \dim B/A$. Our results show that in fact in the cases where the modality is known, it is actually obtained in this way.

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