An analytic technique for the solutions of nonlinear oscillators with damping using the Abel Equation

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Abstract

Using the Chiellini condition for integrability we derive explicit solutions for a generalized system of Riccati equations $\ddot{x} + \alpha x^{2n+1} \dot{x} + x^{4n+3} = 0$ by reduction to the first-order Abel equation assuming the parameter $\alpha \geq 2\sqrt{2(n+1)}$. The technique, which was proposed by Harko et al, involves use of an auxiliary system of first-order differential equations sharing a common solution with the Abel equation. In the process analytical proofs of some of the conjectures made earlier on the basis of numerical investigations in [25] is provided.

Mathematics Classification (2010): 34C14, 34C20.

Keywords: Liénard equation, Abel equation, Chiellini integrability condition

1 Introduction

Second-order ordinary differential equations (ODEs) with linear damping are the most commonly studied extensions of undamped motion, the simplest example being the case of damped oscillations $\ddot{x} + \gamma \dot{x} + \omega^2 x = 0$ which admits a closed-form solution. In the case of nonlinear ODEs even with linear damping the construction of a closed form solution is often a nontrivial
task and such equations often display a variety of interesting phenomena such as chaos in the case of non-autonomous nonlinear terms, complex periodicity, limit cycles etc. An equation of the form \( \dot{x} + f(x)\dot{x} + g(x) = 0 \) where \( f(x) \) and \( g(x) \) are arbitrary \( C^\infty \) real-valued functions of \( x \) defined on a real interval \( I \subseteq \mathbb{R} \) is known as a Liénard equation [1]. There exists a vast literature on this equation alone as it is the favored equation for modelling several phenomena ranging from electrical circuits, heart beat activity, neuron activity, chemical kinetics to turbulence in fluid dynamics [2][3][4][5]. Mathematical techniques such as those of Lie symmetries [6][7] and Wierstrass integrability have been used to analyse the Liénard equation [8]. Its generalization the Levinson-Smith equation \( \ddot{x} + f(x, \dot{x})\dot{x} + g(x) = 0 \) [9] has found applications in astrophysics where for instance the time dependence of perturbations of the stationary solutions of spherically symmetric accretion processes is modelled by an equation of this form [10].

From a practical point of view the assumption of linear damping is often insufficient. Generally it is possible to divide oscillator systems into two broad categories, those with linear damping and nonlinear elasticity and those with nonlinear damping and linear elasticity. Extensive studies of both these categories may be found in [11][12][13][14]. Various quantitative methods have been employed for their analysis depending on the context and convenience such as the method of multiple scales, successive approximation, averaging method besides qualitative studies [15][16][17].

The following generalization of the Liénard equation involving a quadratic dependance on the velocity besides the usual linear damping term, viz

\[
\ddot{x} + g_2(x)\dot{x}^2 + g_1(x)\dot{x} + g_0(x) = 0, \tag{1}
\]

was studied by Bandic [18]. Special cases of this equation corresponding to \( g_1(x) = 0 \) naturally occur for oscillators involving a variable mass and are derivable from a Lagrangian of the form \( L(x, \dot{x}) = \frac{1}{2}m(x)\dot{x}^2 + V(x) \). Recently Kovacic and Rand [19] studied several examples of a position-dependent coefficient of the kinetic energy, which stem from a position-dependent mass or are the consequence of geometric/kinematic constraints. Some notable examples of position-dependent mass systems include the Mathews-Lakshmanan oscillator equation [20], which has also been studied in the quantum regime, the quadratic Loud systems [21] and the Cherkas system [22]. In [23] Cvetićanin analysed the case of strong quadratic damping with a model equation given by \( \ddot{x} + x + 2\delta\dot{x}|\dot{x}| = 0 \). The issue of isochronicity in equations of the Liénard type has also been extensively studied [24].

In [25] a variant of the generalized Riccati system of equations, viz

\[
\ddot{x} + \alpha x^{2n+1}\dot{x} + x^{4n+3} = 0, \tag{2}
\]

was considered. It was established on the basis of numerical studies that for \( \alpha \) much smaller than a critical value the dynamics is periodic, the origin being a centre. Furthermore the solution changes from being periodic to aperiodic at a critical value \( \alpha_c = 2\sqrt{2(n+1)} \), which is independent of the initial conditions. This behaviour is explained by finding a scaling
argument via which the phase trajectories corresponding to different initial conditions collapse onto a single universal orbit. Numerical evidence for the transition is shown. Further, using a perturbative renormalization group argument, it is conjectured that the oscillator,

\[ \ddot{x} + (2n + 3)x^{2n+1} + x^{4n+3} + w_0^2 x = 0, \quad (3) \]

exhibits isochronous oscillations. The correctness of the conjecture is established numerically. In this communication we provide analytical proofs for some of these assertions.

Equation (1) may be reduced to a first-order ODE by means of the transformation \( \dot{x} = 1/v \), namely

\[ \frac{dv}{dx} = g_0(x)v^3 + g_1(x)v^2 + g_2(x)v = F(x,v). \quad (4) \]

This is an Abel equation of the first-order and first kind and may be viewed as a generalization of the Riccati equation. Such equations, which first appeared in course of Abel’s investigations of the theory of elliptic functions, usually arise in problems involving the reduction of order of second and higher-order equations and are frequently encountered in modelling of practical problems, e.g., the Emden equation, the van der Pol equation etc. They are also relevant in the study of quadratic systems in the plane \([26]\), the centre-focus problem \([27]\) and in certain cosmological models \([28]\).

1.1 Derivation of the Chiellini condition for integrability

Recently Harko et al \([29, 30]\) have considered certain exactly integrable cases of the Liénard equation by appealing to an integrability criterion known as the Chiellini condition and making use of the first-order Abel equation. Multiplying (4) by \( \exp(-\int g_2(x)dx) \) and setting \( u = v \exp(-\int g_2(x)dx) \) leads us to the standard form of the Abel equation of the first kind namely \([31]\)

\[ \frac{du}{dx} = A(x)u^2 + B(x)u^3, \quad (5) \]

where \( A(x) = g_1(x) \exp(\int g_2(x)dx) \) and \( B(x) = g_0(x) \exp(2\int g_2(x)dx) \) respectively. An exact solution of (5) can be constructed provided the Chiellini condition for integrability \([32, 33]\) for (5), given by

\[ \frac{d}{dx} \left( \frac{B}{A} \right) = sA(x), \quad (6) \]

is satisfied where \( s \) is a nonzero constant \([29]\). When \( g_1(x) \neq 0 \), the above condition becomes

\[ \frac{d}{dx} \left( \frac{g_0}{g_1} \right) = sg_1 - \frac{g_0 g_2}{g_1}. \quad (7) \]

In order to derive this condition let us consider the following generalized version of the Liénard equation, \( \text{viz} \)

\[ \ddot{x} + g_n(x)\dot{x}^n + g_0(x) = 0. \quad (8) \]
Set \( \dot{x} = \xi(x) \), so that (8) becomes
\[
\xi' + g_n(x)\xi^n + g_0(x) = 0, \quad \xi' = \frac{d\xi}{dx}. \tag{9}
\]

Suppose \( \xi = F(x)G(u(x)) \), where \( G \) is a function of \( u \). By differentiating \( \xi \) with respect to \( x \) and substituting it back into (9) we obtain
\[
u' = -\frac{FF'G^2 + g_n(x)F^nG^n + g_0(x)}{F^2G\frac{\partial G}{\partial u}}. \tag{10}
\]

In order to separate the variables and integrate equation (10), we observe that the function \( F \) should satisfy:
\[
\frac{F'}{F} = kg_n(x)F^{n-2} = \frac{l}{F^2}g_0(x),
\]
where \( k \) and \( l \) and constants or in other words
\[
k g_n(x) = F', \quad lg_0(x) = FF'.
\]

From these relations we obtain
\[
F^{2-n} = (2 - n) \int k g_n(x) dx \quad \text{and} \quad \frac{l}{k} g_n = F^n
\]
whence we have
\[
\frac{d}{dx} \left( \frac{g_0}{g_n} \right) = \frac{k^2}{l} (2 - n) g_n(x) \left( 2 - n \frac{k}{l} \int g_n(x) dx \right)^{(2(n-1)/(2-n))}. \tag{11}
\]

Now suppose \( G = u \), then (10) reduces to
\[
u' = -kg_n(x)F^{n-2}F^{n-2}u - g_n(x)F^{n-2}u^{n-1} - \frac{k}{lu}F^{n-2}g_n(x) = -g_n(x)F^{n-2} \left( ku + u^{n-1} + \frac{k}{lu} \right). \tag{12}
\]

This being separable it is solvable.

Setting \( n = 1 \), (11) reduces to
\[
\frac{d}{dx} \left( \frac{g_0}{g_1} \right) = \frac{k^2}{l} g_1(x), \tag{13}
\]
while from (1) it follows that when \( g_2 = 0 \),
\[
\frac{dv}{dx} = g_1 v^2 + g_0 v^3
\]
which is to be compared with (5). It is now obvious that the criterion stated in (6) is identical to (13) with \( s = k^2/l \).
1.2 Construction of an implicit solution

As explained in [29] an implicit solution of (5) can be accomplished by defining a new variable \( w = uB/A \) and using the Chiellini condition such that (5) is transformed to

\[
\frac{dw}{dx} = \frac{A^2}{B} w(w^2 + w + s).
\]  

(14)

This leads to a separation of the variables, namely

\[
F(w, s) := \int \frac{dw}{w(w^2 + w + s)} = \int \frac{A^2}{B} dx = \frac{1}{s} \int d\ln(B/A)
\]  

(15)

where the Chiellini condition has been used once again and finally allows us to express the solution of (14) in the implicit form

\[
\left| \frac{B}{A} \right| = K^{-1} e^{sF(w, s)}
\]  

(16)

where \( K^{-1} \) is an arbitrary constant of integration. It follows that

\[
\dot{x} = \frac{1}{v} = \frac{1}{ue^{\int g_2 dx}} = \frac{B}{Ae^{\int g_2 dx} w(x)} = \frac{g_0(x)}{g_1(x) w(x)}
\]

and hence

\[
t - t_0 = \int \frac{w(x) g_1(x)}{g_0(x)} dx.
\]  

(17)

The form of the right-hand-side of (16) depends on the value of the parameter \( s \) and

\[
e^{sF(w, s)} = \begin{cases} 
\frac{w}{\sqrt{w^2 + w + s}} \exp\left(-\frac{1}{\sqrt{4s-1}} \arctan\left(\frac{2w+1}{\sqrt{4s-1}}\right)\right) & s > \frac{1}{4} \\
\frac{w}{\sqrt{w^2 + w + s}} \exp\left(\frac{1}{2w+1}\right) & s = \frac{1}{4} \\
\frac{w}{\sqrt{w^2 + w + s}} \left| 1 - \frac{1+2w}{\sqrt{1-4s}} \right| - \frac{1}{\sqrt{4s-1}} \left| 1 + \frac{1+2w}{\sqrt{1-4s}} \right| & s < \frac{1}{4}
\end{cases}
\]  

(18)

2 Solution of first-order ODEs via an auxiliary system of ODEs

In [25] the behaviour of the dynamical system described by the equation

\[
\ddot{x} + \alpha x^{2n+1} \dot{x} + x^{4n+3} = 0,
\]  

(19)

was analysed and it was conjectured that there exists a critical value of the parameter \( \alpha_c \) below which the system admits closed orbits. Extensive numerical computations indicated that the critical value was \( \alpha_c = 2\sqrt{2(n+1)} \). In view of the method described above one can obtain analytically this critical value by reducing the equation to a first-order Abel equation:

\[
\frac{dv}{dx} = \alpha x^{2n+1} v^2 + x^{4n+3} v^3.
\]  

(20)
It is observed that the Cheillini integrability condition is satisfied with the constant \( s = 2(n + 1)/\alpha^2 \). From (18) it is seen that the nature of the solution for \( w \) changes as \( s \) varies from less than \( 1/4 \) to greater than \( 1/4 \). The critical value corresponding to \( s = 1/4 \) implies that the parameter \( \alpha_c = 2\sqrt{2(n + 1)} \). This provides a proof of the validity of the conjecture made in [25].

A useful method of solving a first-order ordinary differential equation (FOODE) is by the introduction of an auxiliary system of first-order ODEs which have a common solution with the given equation [29, 30]. To explain how this is achieved consider a first-order ODE given by

\[
\frac{dv}{dx} = F(x, v),
\]

and introduce an auxiliary system of first-order ODEs

\[
\frac{dv}{dx} = -F_1(x, v) + G(x)f(v),
\]

\[
\frac{dv}{dx} = \frac{1}{2}F_2(x, v) + \frac{1}{2}G(x)f(v),
\]

subject to the constraint \( F_1 + F_2 = F \), where \( G(x) \) is a function to be determined. If a function \( G(x) \) exists such that (22) and (23) have a common solution then it is easy to show that this solution satisfies the equation (21). The above technique can be adapted to deal with second-order ODEs which frequently arise in physical applications.

Consider a second-order ODE of the form (1), viz

\[
\frac{d^2x}{dt^2} + g_1(x)\frac{dx}{dt} + g_2(x)\left(\frac{dx}{dt}\right)^2 + g_0(x) = 0.
\]

Typically if \( g_2 = 0 \) then we have an equation of the Liénard type, and if \( g_1 = 0 \) we obtain an equation with a quadratic dependance on the velocity which, from a Newtonian point of view, may be interpreted as arising from the dependance of the mass of a particle on its position coordinate. Both types of equations having either a linear or a quadratic dependance on the velocity have been extensively studied [22, 34, 35, 24]. The transformation \( dx/dt = 1/v(x) \) causes (24) to become

\[
\frac{dv}{dx} = g_0v^3 + g_1v^2 + g_2v := F(x, v).
\]

Demanding \( F_1 = F(x, v), F_2 = 0 \) and \( f(v) = v^3 \) the analogs of (22) and (23) then have the following forms, in terms of the transformed variables, namely:

\[
\frac{dv}{dx} = (G(x) - g_0)v^3 - g_1v^2 - g_2v,
\]

\[
\frac{dv}{dx} = \frac{1}{2}G(x)v^3.
\]
The use of the Chiellini condition for (26) allows us to express $G(x)$ as

$$G(x) = g_0 + g_1 \exp\left(\int g_2 dx\right) \left[\Gamma + s' \int g_1 \exp\left(-\int g_2 dx\right) dx\right], \quad (28)$$

where $\Gamma$ is a constant of integration with the constant $s'$ appearing as a result of the use of the Chiellini condition. Notice that owing to the convenient choices made for the functions $F_1$ and $F_2$, (27) is separable and its solution is given by

$$\frac{1}{v} = \frac{dx}{dt} = \pm \sqrt{B - \int G(x) dy}, \quad (29)$$

with $B$ being a constant of integration.

Now the existence of a common solution means that

$$\frac{dv}{dx} = \frac{1}{2} G(x) v^3 = (G(x) - g_0) v^3 - g_1 v^2 - g_2 v,$$

which implies upon using (29)

$$\frac{G(x) - 2g_0}{g_1} = \pm 2 \sqrt{B - \int G(x) dx + \frac{2g_2}{g_1} \left( B - \int G(x) dx \right)}. \quad (30)$$

Eqn (30) may be used to determine the values of the parameters $s', \Gamma$ and $B$ after substituting the value of $G(x)$ from (28). Knowledge of $G(x)$ then allows us to obtain the common solution from (29) in the form

$$\pm t - t_0 = \int \frac{dx}{\sqrt{B - \int G(x) dx}}, \quad (31)$$

with $t_0$ being a parameter which defines the families of solutions. The procedure is illustrated below.

**Example 3.1:** $\ddot{x} + \alpha x^{2n+1} \dot{x} + x^{4n+3} + w_0^2 x^{2n+1} = 0$

Under the transformation $\dot{x} = 1/v$ this equation becomes

$$\frac{dv}{dx} = (\alpha x^{2n+1}) v^2 + (x^{4n+3} + w_0^2 x^{2n+1}) v^3 := F(x,v)$$

Choose the auxiliary system of FOODEs to be the following:

$$\frac{dv}{dx} = -F(x,v) + G(x) v^3 = (G(x) - x^{4n+3} - w_0^2 x^{2n+1}) v^3 - \alpha x^{2n+1} v^2$$

Choose the auxiliary system of FOODEs to be the following:

$$\frac{dv}{dx} = \frac{1}{2} G(x) v^3$$

Applying the Cheillini integrability condition to the first of these equations we have

$$\frac{d}{dx} \left( \frac{G(x) - x^{4n+3} - w_0^2 x^{2n+1}}{-\alpha x^{2n+1}} \right) = s'(-\alpha x^{2n+1})$$
We solve this for \( G(x) \) to get

\[
G(x) = x^{4n+3} \left( \frac{s' \alpha^2}{2(n+1)} + 1 \right) + (\alpha \Gamma + w_0^2) x^{2n+1}
\]  \hspace{1cm} (32)

Upon solving the second auxiliary FOODE, which is separable, we obtain

\[
\frac{1}{v} = \pm \sqrt{B - \int G(x) dx}
\]  \hspace{1cm} (33)

where \( B \) and \( \Gamma \) are arbitrary constants of integration. If a common solution exists for the two auxiliary FOODEs then we must have

\[
\frac{1}{2} G(x)v^3 = (G(x) - x^{4n+3} - w_0^2 x^{2n+1})v^3 + (-\alpha x^{2n+1})v^2
\]

which leads

\[
G(x) = 2x^{2n+1} \left[ (x^{2(n+1)} + w_0^2) \pm \alpha \sqrt{B - \int G(x) dx} \right]
\]  \hspace{1cm} (34)

Equating (32) and (34) we have upon equating coefficients of different powers of \( x \) (with \( \xi = \alpha^2 s'/2(n+1) - 1 \)),

\[
\xi^2 = -\frac{\alpha^2}{n+1} (\xi + 2)
\]  \hspace{1cm} (35)

\[
\xi (\alpha \Gamma - w_0^2) = -\frac{\alpha^2}{n+1} (\alpha \Gamma + w_0^2), \quad (\alpha \Gamma - w_0^2)^2 = 4\alpha^2 B
\]  \hspace{1cm} (36)

We can solve for the constants of integration and \( \xi \) to obtain

\[
\xi = \frac{1}{2} \left[ -\left( \frac{\alpha^2}{n+1} \right) \pm \sqrt{ \left( \frac{\alpha^2}{n+1} \right)^2 - 8 \left( \frac{\alpha^2}{n+1} \right) } \right]
\]  \hspace{1cm} (37)

\[
\Gamma = \frac{w_0^2}{\alpha} \left[ \frac{(n+1)\xi - \alpha^2}{(n+1)\xi + \alpha^2} \right], \quad B = \frac{\alpha^2 w_0^4}{[(n+1)\xi + \alpha^2]^2}
\]  \hspace{1cm} (38)

Knowing the constants of integration, the solution may be reduced to quadrature using (32) and (33), i.e.,

\[
\pm t - t_0 = \int dx \frac{\sqrt{B - \frac{\xi + 2}{4(n+1)} x^{4(n+1)} - \frac{w_0^2 \xi}{[(n+1)\xi + \alpha^2]^2} x^{2(n+1)}}}{x^{2(n+1)}}.
\]  \hspace{1cm} (39)

**Case A:** \( \alpha = 2n + 3 \) and \( w_0 \neq 0 \)

For this choice of the parameter \( \alpha \) we find from (37) since \( s' = 2(n+1)(\xi + 1)/\alpha^2 \)

\[
(s'_+, s'_-) = \left( -\frac{2(n+2)}{(2n+3)^2}, -\frac{2(n+1)(4n+5)}{(2n+3)^2} \right)
\]
\[(B_+, B_-) = \left( \frac{w_0^4}{4(n+1)^2}, w_0^4 \right)\]
\[(\Gamma_+, \Gamma_-) = \left( -\frac{(n+2)}{(n+1)(2n+3)} w_0^2, \frac{(4n+5)}{(2n+3)} w_0^2 \right)\]

These values lead to the following expressions for the unknown function \(G(x)\), viz,
\[G_+(x) = -\frac{1}{n+1} \left[ x^{4n+3} + w_0^2 x^{2n+1} \right] \]
\[G_-(x) = -4(n+1) \left[ x^{4n+3} + w_0^2 x^{2n+1} \right] \]

which in turn yield the solutions
\[\pm t - t_0^+ = 2(n+1) \int \frac{dx}{x^{2(n+1)} + w_0^2},\]
\[\pm t - t_0^- = \int \frac{dx}{x^{2(n+1)} + w_0^2} \]
respectively.

![Figure 1: Graph for \(n = 0 - 10\), for \(n = 0\) we obtain \text{arctan} and for large \(n\) curves are dense.](image)

It is evident from these solutions that they are equivalent up to a scaling. Indeed setting \(w_0 = 1\) one may explicitly express the solution in terms of the hypergeometric function \(\, _2F_1(a, b; c; x)\) because
\[
\int \frac{dx}{x^{2(n+1)} + 1} = x_2F_1\left(1, \frac{1}{2(n+1)}; 1 + \frac{1}{2(n+1)}; -x^{2(n+1)}\right)
\]
From (37) it is evident that \( \alpha^2 = (2n + 3)^2 > 8(n + 1) \). The critical value of \( \alpha \) corresponding to the vanishing of the discriminant is \( \alpha_c = 2\sqrt{2(n + 1)} \). Thus when \( \alpha > \alpha_c \) and \( n = 0 \) we obtain the case of periodic motion. Incidentally this corresponds to isochronous motion, in which the period function is independent of the initial condition. This is easily verified from the corresponding criterion given by Sabatini in [24]. There it is shown that for a Liénard equation \( \ddot{x} + f(x)\dot{x} + g(x) = 0 \) having an isochronous center at the origin with \( f, g \in C^1(J, R), f(0) = g(0) = 0, g'(0) > 0 \) the forcing term \( g(x) \) must be of the form

\[
g(x) = g'(0)x + \frac{1}{x^3} \left( \int_0^x sf(s)ds \right)^2
\]

It is straightforward to verify that these conditions are satisfied by the equation \( \ddot{x} + (2n + 3)x^2n+1\dot{x} + x^{4n+3} + w_0^2x = 0 \), and hence by the equation \( \ddot{x} + 3x\dot{x} + x^3 + w_0^2x = 0 \), which corresponds to \( n = 0 \).

**Case B:** \( \alpha = 2n + 3 \) and \( w_0 = 0 \)

When \( w_0 = 0 \) we have from (38) that \( \Gamma = B = 0 \) and the solutions are

\[
x = \left[ \frac{2(n + 1)}{2n + 1} \left( \frac{1}{t_0^+ + t} \right) \right]^{1/(2n+1)} \quad \text{and} \quad x = \left[ \frac{1}{2n + 1} \left( \frac{1}{t_0^- + t} \right) \right]^{1/(2n+1)}
\]

respectively and are singular.

![Figure 2: Graph for n = 1 − 10, where t₀⁻ and t₀⁺ are approaching from lhs and rhs of t = 0](image-url)
3 Generalizations of the Chiellini condition

The Chiellini integrability condition has been used in a number of works (see [30] and references therein). Its generalization to the case when higher powers of $u$ appear in the right hand side of (5) has also been studied. In view of its efficacy in deriving solutions of the first-order Abel equation we consider below higher-order generalizations of the Liénard equation.

3.1 Higher-order Liénard equation

Consider the higher-order Liénard equation
\[ \ddot{x} + f(x)\dot{x}^{n+1} + g(x)\dot{x}^n = 0. \] (40)
Suppose $\dot{x} = \xi(x)$, so that (40) becomes
\[ \xi' + f(x)\xi^n + g(x)\xi^{n-1} = 0. \] (41)
Once again we assume $\xi = F(x)G(u(x))$, where $G$ is a function of $u$. Following the procedure outlined in Section 4.1 we obtain
\[ u' = -\frac{F'F^{n-3}G^n + f(x)F^{2n-3}G^{2n-1} + g(x)}{F^{n-2}G^{n-1}\frac{\partial G}{\partial u}}. \] (42)
After separating the variables we have
\[ \frac{F'}{F} = kf(x)F^{m-1} = \frac{g(x)}{F^{n-2}}, \]
and this leads to the generalized Chiellini condition [30]
\[ \left( \frac{g}{f} \right)' = \frac{l^{n-1}}{k^{n-2}} \left( \frac{g^n}{f^{n-1}} \right) \equiv K \left( \frac{g^n}{f^{n-1}} \right), \] (43)
which for $n = 0$ reduces to the usual Chiellini condition stated in [13]. Upon introducing the transformation
\[ \xi = \left( \frac{g(x)}{f(x)} \right) \eta(x), \]
(41) becomes
\[ \frac{d\eta(x)}{dx} = \frac{g^{n-1}(x)}{f^{n-2}(x)} \left( \eta^n + \eta^{n-1} + K\eta \right), \] (44)
which is clearly separable.

Acknowledgement

The authors wish to thank Professors J. K Bhattacharjee and A. Mallik for their interest and encouragement. One of us (PG) wishes to acknowledge Professor Tudor Ratiu for his gracious hospitality at the Bernoulli Centre, EPFL during the fall semester of 2014, where part of this work was done.
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