ON GAUSS-BONNET AND POINCARÉ-HOPF TYPE THEOREMS FOR COMPLEX ∂-MANIFOLDS

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To Omegar Calvo-Andrade on the occasion of his 60th birthday

Abstract. We prove a Gauss-Bonnet and Poincaré-Hopf type theorems for complex ∂-manifold $\tilde{X} = X - D$, where $X$ is a complex compact manifold and $D$ is a reduced divisor. We will consider the cases such that $D$ has isolated singularities and also if $D$ has a (not necessarily irreducible) decomposition $D = D_1 \cup D_2$ such that $D_1$, $D_2$ have isolated singularities and $C = D_1 \cap D_2$ is a codimension 2 variety with isolated singularities. As application, we obtain a generalization for the Dimca-Papadima formula.

1. Introduction

Let $X$ be a compact complex manifold of dimension $n$. The classical Chern-Gauss-Bonnet theorem [7] say us that

$$\int_X c_n(\Omega^1_X) = (-1)^n \chi(X),$$

where $\chi(X)$ denotes the Euler characteristic of $X$. A complex ∂-manifold [23] is a complex manifold of the form $\tilde{X} = X - D$, where $X$ is an $n$-dimensional complex compact manifold and $D \subset X$ is a divisor which is called by boundary divisor. S. Iitaka in [18] proposed a version of Gauss-Bonnet theorem for ∂-manifold [23]. Such version was independently proved by Y. Norimatsu [23], R. Silvotti [26] and P. Aluffi [2]:

Theorem (Norimatsu-Silvotti-Aluffi). Let $\tilde{X}$ be a complex manifold such that $\tilde{X} = X - D$, where $X$ is an $n$-dimensional complex compact manifold and $D$ is a normally crossing hypersurface on $X$. Then

$$\int_X c_n(\Omega^1_X(\log D)) = (-1)^n \chi(\tilde{X}),$$

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where $\Omega^1_\log (\log D)$ denotes the sheaf of logarithmic 1-forms along $D$ and $\chi(X)$ denotes the Euler characteristic given by

$$\chi(X) = \sum_{i=1}^n \dim H^i_c(X, \mathbb{C}).$$

X. Liao has provided formulas in [21] in terms of Chern-Schwartz-MacPherson class.

On the other hand, the Poincaré-Hopf theorem applied for a compact complex manifold $X$ with a holomorphic vector field $v \in H^0(X, TX)$, with isolated singularities, give us the following

$$\chi(X) = \sum_{p \in \text{Sing}(v) \cap X} \text{PH}(v, p),$$

where $\text{PH}(v, p)$ denotes the Poincaré-Hopf index of $v$ on $p$. In [8] the first and third named authors have proved the following Poincaré-Hopf type theorem for $\partial$-manifolds with boundaries divisors having normal crossing singularities.

**Theorem.** Let $\tilde{X}$ be a complex manifold such that $\tilde{X} = X - D$, where $X$ is an $n$-dimensional complex compact manifold, $D$ is a reduced normal crossing hypersurface on $X$. Let $v$ be a holomorphic vector field on $X$, with isolated singularities (non-degenerate) and logarithmic along $D$. Then

$$\chi(\tilde{X}) = \sum_{x \in \text{Sing}(v) \cap \tilde{X}} \text{PH}(v, x),$$

where $\text{PH}(v, x)$ denotes the Poincaré-Hopf index of $v$ at $x$.

In this work we will provide Gauss-Bonnet and Poincaré-Hopf type theorems for $\partial$-manifolds of the form $\tilde{X} = X - D$, where $X$ is a complex compact manifold and $D$ is a reduced divisor. We will consider the case such that $D$ has isolated singularities and also if $D$ has a (not necessarily irreducible) decomposition $D = D_1 \cup D_2$ such that $D_1$, $D_2$ have isolated singularities and $C = D_1 \cap D_2$ is a codimension 2 variety with isolated singularities.

Let us fix some notations before we state our main result: Let $W \subset X$ an analytic subspace and $v \in H^0(X, TX)$ a holomorphic vector field, we will denote by

$$\text{PH}(v, W) = \sum_{x \in W} \text{PH}(v, x),$$

$$\mu(D, W) = \sum_{x \in W} \mu_x(D),$$

$$\text{GSV}(v, D, W) = \sum_{x \in W} \text{GSV}(v, D, x),$$

where $\text{PH}(v, x)$ and $\text{GSV}(v, D, x)$ denote, respectively, the Poincaré-Hopf and GSV index of a vector field $v$ at $p$ and $\mu_x(D)$ is the Milnor number of $D$ at $x$. We also will denote $S(W) := \text{Sing}(W)$ and $S(v, W) := [\text{Sing}(v) \cap W] \cup \text{Sing}(W)$.

We prove the following results:
Theorem 1.1. Let $\tilde{X}$ be a complex manifold such that $\tilde{X} = X - D$, where $X$ is an $n$-dimensional ($n \geq 3$) complex compact manifold and $D$ is a reduced divisor on $X$. Given any (not necessarily irreducible) decomposition $D = D_1 \cup D_2$, where $D_1$, $D_2$ have isolated singularities and $C = D_1 \cap D_2$ is a codimension 2 variety and has isolated singularities,

(i) (Gauss-Bonnet type formula) the following formula holds

$$\int_X c_n(\Omega^1_X(\log D)) = (-1)^n \chi(\tilde{X}) + \mu(D_1, S(D_1)) + \mu(D_2, S(D_2)) - \mu(C, S(C)).$$

(ii) (Poincaré-Hopf type formula) if $v$ is a holomorphic vector field on $X$, with isolated singularities and logarithmic along $D$, we have that

$$\chi(\tilde{X}) = \sum_{x \in \text{Sing}(v)} PH(v, x) - \sum_{x \in S(v, D)} GSV(v, D, x) + (-1)^{n-1} \sum_{x \in \text{Sing}(D)} \mu_x(D).$$

Moreover, if the vector field $v$ has only non-degenerate singularities, then

$$\chi(\tilde{X}) = \sum_{x \in \text{Sing}(v) \cap [X - D_{\text{reg}}]} PH(v, x) - \sum_{x \in \text{Sing}(D)} [GSV(v, D, x) + (-1)^{n-1} \mu_x(D)].$$

Finally, we recover the Dimca-Papadima formula

$$\chi(\mathbb{P}^n \setminus D) = \sum_{i=0}^n (-1)^i (d-1)^i + (-1)^{n+1} \sum_{p \in \text{Sing}(D)} \mu_p(D),$$

see [11, Theorem 1] and [11, 14, 17]. In fact, we prove the following generalization.
Corollary 1.3. Given any (not necessarily irreducible) decomposition \( D = D_1 \cup D_2 \), where \( D_1, D_2 \) have isolated singularities and \( C = D_1 \cap D_2 \) is a codimension 2 variety and has isolated singularities. If \( \deg(D_i) = d_i \), for \( i = 1, 2 \), then
\[
\chi(\mathbb{P}^n \setminus D) = \sum_{i=0}^{n} \sigma_{n-i}(d_1-1, d_2-1) + (-1)^{n+1} \left[ \mu(D_1, S(D_1)) + \mu(D_2, S(D_2)) - \mu(C, S(C)) \right],
\]
where \( \sigma_{n-i} \) is the complete symmetric function of degree \( n-i \).

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2. Preliminaries

2.1. Logarithmic forms and logarithmic vector fields. Given \( X \) a complex manifold of dimension \( n \) and \( D \) a reduced hypersurface on \( X \). Let \( \Omega^q_X(D) \) be the sheaf of differential \( q \)-forms on \( X \) with at most simple poles along \( D \).

A logarithmic \( q \)-form along \( D \) on an open subset \( U \subset X \) is a meromorphic \( q \)-form \( \omega \) on \( U \), regular on \( U - D \) and such that both \( \omega \) and \( d\omega \) have at most simple poles along \( D \). Logarithmic \( q \)-forms along \( D \) form a coherent sheaf of \( \mathcal{O}_X \)-modules denoted by \( \Omega^q_X(\log D) \), in this case, for any open subset \( U \subset X \) we have
\[
\Gamma(U, \Omega^q_X(\log D)) = \{ \omega \in \Gamma(U, \Omega^q_X(D)) : d\omega \in \Gamma(U, \Omega^{q+1}_X(D)) \}.
\]
See for example [12], [19] and [24] for more details about the sheaf of logarithmic \( q \)-forms along \( D \).

Now, consider \( \Omega^1_X(\log D) \), the sheaf of logarithmic 1-form along \( D \), its dual sheaf is the sheaf of logarithmic vector fields along \( D \), denoted by \( \mathcal{T}_X(-\log D) \). We have an exact sequence
\[
\begin{array}{cccccc}
0 & \longrightarrow & T_X(-\log D) & \longrightarrow & T_X & \longrightarrow & \mathcal{J}_D & \longrightarrow & 0 \\
\end{array}
\]
where \( \mathcal{J}_D \) is the Jacobian ideal of \( D \) which is defined as the Fitting ideal
\[
\mathcal{J}_D := F^{n-1}(\Omega^1_X) \subset \mathcal{O}_D.
\]

Saito in [24] has showed that in general \( \Omega^1_X(\log D) \) and \( T_X(-\log D) \) are reflexive sheaves. If \( D \) is an analytic hypersurface with normal crossing singularities, the sheaves \( \Omega^1_X(\log D) \) and \( T_X(-\log D) \) are locally free, furthermore, the Poincaré residue map
\[
\text{Res} : \Omega^1_X(\log D) \longrightarrow \mathcal{O}_D \cong \bigoplus_{i=1}^N \mathcal{O}_{D_i}
\]
gives us the following exact sequence of sheaves on \( X \)
\[
\begin{array}{cccccc}
(2) & 0 & \longrightarrow & \Omega^1_X & \longrightarrow & \Omega^1_X(\log D) & \xrightarrow{\text{Res}} & \bigoplus_{i=1}^N \mathcal{O}_{D_i} & \longrightarrow & 0,
\end{array}
\]
where \( \Omega^1_X \) is the sheaf of holomorphics 1-forms on \( X \) and \( D_1, \ldots, D_N \) are the irreducible components of \( D \).
Now, if $D$ is such that $\text{codim}_X(\text{Sing}(D)) > 2$ then there exist the following exact sequence of sheaves on $X$ (see V. I. Dolgachev [13]):

$$
\begin{array}{cccccc}
0 & \longrightarrow & \Omega^1_X & \longrightarrow & \Omega^1_X(\log D) & \longrightarrow & \mathcal{O}_D & \longrightarrow & 0.
\end{array}
$$

Moreover, if $D = D_1 \cap D_2$, we have from [11] the following sequence

$$
\begin{array}{cccccc}
0 & \longrightarrow & \Omega^1_X & \longrightarrow & \Omega^1(\log D_1) \oplus \Omega^1(\log D_2) & \longrightarrow & \Omega^1(\log D_1) + \Omega^1(\log D_2) & \longrightarrow & 0,
\end{array}
$$

and since $\Omega^1(\log D_1) + \Omega^1(\log D_2) \cong \Omega^1(\log D)$ we obtain the exact sequence

$$
\begin{array}{cccccc}
0 & \longrightarrow & \Omega^1_X & \longrightarrow & \Omega^1(\log D_1) \oplus \Omega^1(\log D_2) & \longrightarrow & \Omega^1(\log D) & \longrightarrow & 0.
\end{array}
$$

2.2. **The GSV-Index.** X. Gomez-Mont, J. Sead and A. Verjovsky [16] introduced the GSV-index for a holomorphic vector field over an analytic hypersurface, with isolated singularities, on a complex manifold, generalizing the (classical) Poincaré-Hopf index. The GSV-index was extended for holomorphic vector fields on more general contexts. J. Seade e T. Suwa in [25], have defined the GSV-index for holomorphic vector fields on analytic subvarieties with isolated complete intersection singularity. J.-P. Brasselet, J. Seade and T. Suwa in [4], extended the notion of GSV-index for vector fields defined in certain types of analytical subvariety with non-isolated singularities.

In [15] X. Gomez-Mont introduced the homological index of holomorphic vector field on an analytic hypersurface with isolated singularities, which coincides with GSV-index. There is also the virtual index, introduced by D. Lehmann, M. Soares and T. Suwa [20], that via Chern-Weil theory can be interpreted as the GSV-index. M. Brunella [6] also present the GSV-index for foliations on complex surfaces by a different approach and in [9] the authors have introduced a GSV type index for varieties invariant by holomorphic Pfaff systems.

Let us recall the definition of the GSV-index ([5], Ch.3, 3.2). Let $D$ be a hypersurface with isolated singularities on an $n$-dimensional complex manifold $X$ and let $v$ be a holomorphic vector fields on $X$, with isolated singularities, and logarithmic along $D$. Given a singular point $x_0 \in \text{Sing}(D)$, let $h$ be an analytic function defining $D$ on a neighborhood $U_0$ of $x_0$. The gradient vector field $\text{grad}(h)$ is nowhere vanishing away from $x_0$ (because $x_0$ is an isolated singularity) and it is normal to $D$.

Denote by $v_\ast$ the restriction of $v$ to the regular part $D_{\text{reg}} = D - \text{Sing}(D)$ of $D$. On the neighborhood $U_0$, suppose that the vector fields $v$ is non-singular away from $x_0$. Since $v$ is logarithmic along $D$, we have that $\text{grad}(h)(z)$ and $v_\ast(z)$ are linearly independent at each point $z \in U_0 \cap (D - \{x_0\})$. Assume that $(z_1, \ldots, z_n)$ is a system of complex coordinates on $U_0$ and consider

$$
S_\varepsilon = \{ z = (z_1, \ldots, z_n) : \| z - x_0 \| = \varepsilon \}.
$$
the sphere sufficiently small so that \( K = D \cap S_\varepsilon \) is the link of the singularity of \( D \) at \( x_0 \) (see, for example, [22]). It is an \((2n - 1)\)-dimensional real oriented manifold. By using the Gram-Schmidt process, if necessary, the vector fields \( v_* \) and \( \operatorname{grad}(h) \) define a continuous map
\[
\phi_v := (v_*, \operatorname{grad}(h)) : K \rightarrow W_{2,n+1}
\]
where \( W_{2,n+1} \) is the Stiefel manifold of complex 2-frames in \( \mathbb{C}^{n+1} \).

**Definition 2.1.** The GSV-index of \( v \) in \( x_0 \in D \), denoted by \( \operatorname{GSV}(v, D, x_0) \), is defined as the degree of map \( \phi_v \).

**Remark 2.2.** In the definition 2.1 the vector fields \( v \) can be considered continuous rather than holomorphic vector fields. For more details see [5], [16], [27].

**Remark 2.3.** If \( x_0 \in D_{\text{reg}} \) is a regular point of \( D \), since \( v \) logarithmic along \( D \), we have that the Poincaré-Hopf index of \( v|_D \) in \( x_0 \) is defined and it coincides with the GSV-index. In this case, if \( x_0 \) is a non-degenerate singularity of \( v \), then \( \operatorname{PH}(v, x_0) = \operatorname{PH}(v|_D, x_0) \) and we have
\[
\operatorname{GSV}(v, D, x_0) = \operatorname{PH}(v, x_0).
\]

3. Proof of Theorems

In order to prove the Theorem 1.1 and the Theorem 1.2 we will prove the following preliminary result:

**Theorem 3.1.** Let \( X \) be an \( n \)-dimensional \((n \geq 3)\) complex compact manifold and \( D \) a reduce divisor on \( X \).

(i) If \( D = D_1 \cup D_2 \) is any (not necessarily irreducible) decomposition, where \( D_1, D_2 \) is isolated singularities and \( C = D_1 \cap D_2 \) is a codimension 2 variety and has isolated singularities, then
\[
\int_X c_n(\Omega^1_1(\log D)) = (-1)^n \left[ \int_X c_n(TX) - \int_{D_1} c_{n-1}(TX - [D_1]) - \int_{D_2} c_{n-1}(TX - [D_2]) \right] + \\
+ \left[ \int_C c_{n-2}(TX - [D_1] \oplus [D_2]) \right].
\]

(ii) If \( D \) is an isolated singularity, then
\[
\int_X c_n(\Omega^1_1(\log D)) = (-1)^n \left[ \int_X c_n(TX) - \int_D c_{n-1}(TX - [D]) \right].
\]

**Proof.** From exact sequence [4] we obtain
\[
c(\Omega^1_1)c(\Omega^1_1(\log D)) = c(\Omega^1_1(\log D_1))c(\Omega^1_1(\log D_2)) = c(\Omega^1_1)c(\mathcal{O}_{D_1})c(\Omega^1_1)c(\mathcal{O}_{D_2}),
\]
where in last equality we use the following relations

\begin{equation}
    c(\Omega^1_X(\log D_i)) = c(\Omega^1_X)c(O_{D_i}), \quad i = 1, 2,
\end{equation}

which can be obtained from the exact sequence [3]. Thus, we get

\begin{equation}
    c(\Omega^1_X(\log D)) = c(O_{D_1})c(O_{D_2})c(\Omega^1_X),
\end{equation}

and, consequently,

\[
\int_X c_n(\Omega^1_X(\log D)) = \int_X \sum_{i_1+i_2+i_3=n} c_{i_1}(O_{D_1})c_{i_2}(O_{D_2})c_{i_3}(\Omega^1) = \\
= \int_X c_n(\Omega^1_X) + \sum_{i_2+i_3=n} \int_X c_{i_2}(O_{D_2})c_{i_3}(\Omega^1_X) + \sum_{i_1+i_3=n} \int_X c_{i_1}(O_{D_1})c_{i_3}(\Omega^1_X) + \\
+ \sum_{i_1+i_2+i_3=n} \int_X c_{i_1}(O_{D_1})c_{i_2}(O_{D_2})c_{i_3}(\Omega^1_X) = \\
\quad (9)
\]

\[
= (-1)^n \int_X c_n(T_X) + \sum_{i_2+i_3=n} \int_X c_1([D_2])^{i_2}c_{i_3}(\Omega^1_X) + \sum_{i_1+i_3=n} \int_X c_1([D_1])^{i_1}c_{i_3}(\Omega^1_X) + \\
+ \sum_{i_1+i_2+i_3=n} \int_X c_1([D_1])^{i_1}c_1([D_2])^{i_2}c_{i_3}(\Omega^1_X),
\]

where in the last step we are using that \( c_{i_j}(O_{D_j}) = c_1([D_j])^{i_j} \), since \( c(O_{D_j}) = c([D_j])^{-1} \), for \( j = 1, 2 \).

The proof will be finalized by calculating each sum on the right hand side. Indeed, in the first one, by using that \( c_1([D_2]) \) is Poincaré dual to the fundamental class of \( D_2 \), we obtain

\[
\sum_{i_2+i_3=n} \int_X c_1([D_2])^{i_2}c_{i_3}(\Omega^1_X) = \int_{D_2} \sum_{i_2+i_3=n} c_1([D_2])^{i_2-1}c_{i_3}(\Omega^1_X) \\
= \int_{D_2} c_{n-1}(\Omega^1_X - [D_2]^*) \\
= (-1)^{n-1} \int_{D_2} c_{n-1}(T_X - [D_2]),
\]

where in the last step we are using the relation between the Chern classes of a vector bundle and of its dual. Similarly, we obtain

\[
\sum_{i_1+i_3=n} \int_X c_1([D_1])^{i_1}c_{i_3}(\Omega^1_X) = (-1)^{n-1} \int_{D_1} c_{n-1}(T_X - [D_1]).
\]
Finally, the last sum can be calculated by using that \( c_1([D_1])c_1([D_2]) \) is Poincaré dual to the fundamental class of \( C = D_1 \cap D_2 \). Thus,

\[
\sum_{i_1 + i_2 + i_3 = n} \int_X c_1([D_1])^{i_1} c_1([D_2])^{i_2} c_3(\Omega_X) = \sum_{i_1 + i_2 + i_3 = n} \int_C c_1([D_1])^{i_1} c_1([D_2])^{i_2} c_3(\Omega_X)
\]

\[
= \int_C c_{n-2}(\Omega_X^1 - [D_2]^* - [D_1]^*)
\]

\[
= (-1)^{n-2} \int_C c_{n-2}(TX - [D_1] \oplus [D_2]).
\]

Therefore, we conclude that

\[
\int_X c_n(\Omega^1(\log D)) = (-1)^n \left[ \int_X c_n(TX) - \int_{D_1} c_{n-1}(TX - [D_1]) - \int_{D_2} c_{n-1}(TX - [D_2]) \right] +
\]

\[
+ \int_C c_{n-2}(TX - [D_1] \oplus [D_2]).
\]

Now, suppose that \( D \) is an isolated singularity. Using (7) we obtain

\[
\int_X c_n(\Omega_X^1(\log D)) = \sum_{i=0}^n \int_X c_{n-i}(\Omega_X^1) c_i(\mathcal{O}_D)
\]

\[
= \int_X c_n(\Omega_X^1) + \sum_{i \geq 1} \int_X c_{n-i}(\Omega_X^1) c_i(\mathcal{O}_D).
\]

By the exact sequence

\[
0 \longrightarrow \mathcal{O}(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0.
\]

we obtain that

\[
(10) \quad c_i(\mathcal{O}_D) = c_1([D])^i, \quad i = 1, \ldots, n,
\]

Thus, we can compute the last sum on right side:

\[
\sum_{i \geq 1} \int_X c_{n-i}(\Omega_X^1) c_i(\mathcal{O}_D) = \sum_{i \geq 1} \int_X c_{n-i}(\Omega_X^1) c_1([D])^i
\]

\[
= \sum_{i \geq 1} \int_D c_{n-i}(\Omega_X^1) c_1([D])^{i-1}
\]

\[
= (-1)^n \sum_{i \geq 1} \int_D c_{n-i}(TX) c_1([D]^*)^{i-1}
\]

\[
= (-1)^n \int_X c_n(TX - [D]),
\]
where in the second step we have used the fact that $c_1([D])$ is Poincaré dual of the fundamental class of $D$. Now, replacing the last identity in the initial equality, we obtain
\[
\int_X c_n(\Omega^1_X(\log D)) = \int_X c_n(T_X) + (-1)^n \int_X c_n(T_X - [D]) \\
= (-1)^n \int_X c_n(T_X) + (-1)^n \int_D c_{n-1}(T_X - [D])
\]
\[
= (-1)^n \left[ \int_X c_n(T_X) - \int_D c_{n-1}(T_X - [D]) \right].
\]

\[\square\]

**Proof of Theorem 1.1:** By using the classical Chern-Gauss-Bonnet formula (1), we obtain
\[
\int_X c_n(T_X) = (-1)^n \int_X c_n(\Omega^1_X) = \chi(X).
\]
From [27, Theorem 3.9], we have that
\[
\int_{D_i} c_{n-1}(T_X - [D_i]) = \chi(D_i) + (-1)^{n-1} \sum_{p \in \text{Sing}(D_i)} \mu_p(D_i), \quad i = 1, 2.
\]
Moreover, since the complete intersection $C = D_1 \cap D_2$, its normal bundle is $(\{D_1 \oplus [D_2]\})_C$, and once again from [27, Theorem 3.9] we have that
\[
\int_C c_{n-2}(T_X - [D_1 \oplus D_2]) = \chi(C) + (-1)^{n-2} \sum_{p \in \text{Sing}(C)} \mu_p(C).
\]
Now, substituting (11), (12) and (13) in the formula of item (i) of Theorem 3.1, we get the desired formula
\[
\int_X c_n(\Omega^1_X(\log D)) = \left[ (-1)^n \chi(\tilde{X}) + \sum_{p \in \text{Sing}(D_1)} \mu_p(D_1) + \sum_{p \in \text{Sing}(D_2)} \mu_p(D_2) + \sum_{p \in \text{Sing}(C)} \mu_p(C) \right].
\]

On the other hand, if $v$ is a holomorphic vector field on $X$, with isolated singularities and logarithmic along $D_1, D_2$ and $C$, it follows from [27, Theorem 7.16] that for each $i = 1, 2$
\[
\int_{D_i} c_{n-1}(T_X - [D_i]) = \sum_{x \in S(v, D_i)} \text{GSV}(v, D_i, x)
\]
and
\[
\int_C c_{n-2}(T_X - [D_1 \oplus D_2]) = \sum_{x \in S(v, C)} \text{GSV}(v, C, x),
\]
Thus, using the Theorem 3.1, we obtain

\[ D \text{ and Poincaré-Hopf theorems, we have} \]

Thus, we obtain

Using that \( \chi \) of the restriction \( (T_{\chi}) \) in the item (i) of Theorem 3.1 we get

\[ (-1)^n \int_X c_n(\Omega^1_X (\log D)) = \]

\[ = \sum_{x \in \text{Sing}(v)} PH(v, x) - \sum_{x \in \mathcal{S}(v, D_1)} GSV(v, D_1, x) - \sum_{x \in \mathcal{S}(v, D_2)} GSV(v, D_2, x) + \sum_{x \in \mathcal{S}(v, C)} GSV(v, C, x). \]

Now, replacing it in the formula (14), we get

\[ \chi(\tilde{X}) = \sum_{x \in \text{Sing}(v)} PH(v, x) - \sum_{x \in \mathcal{S}(v, D_1)} GSV(v, D_1, x) - \sum_{x \in \mathcal{S}(v, D_2)} GSV(v, D_2, x) + \]

\[ + \sum_{x \in \mathcal{S}(v, C)} GSV(v, C, x) + (-1)^{n-1} \left[ \sum_{p \in \text{Sing}(D_1)} \mu_p(D_1) + \sum_{p \in \text{Sing}(D_2)} \mu_p(D_2) + \sum_{p \in \text{Sing}(C)} \mu_p(C) \right]. \]

\[ \Box \]

**Proof of Theorem 1.2:** Using (11) in the formula (6), we obtain the following

\[ \int_X c_n(\Omega^1_X (\log D)) = (-1)^n \left[ \chi(X) - \int_D c_{n-1}(T_X - [D]) \right]. \]

On the other hand, it follows from [27, Theorem 3.9] that the top Chern number of the restriction \( (T_X - [D]) \) is given by

\[ \int_D c_{n-1}(T_X - [D]) = \chi(D) + (-1)^{n-1} \sum_{p \in \text{Sing}(D)} \mu_p(D). \]

Thus, we obtain

\[ \int_X c_n(\Omega^1_X (\log D)) = (-1)^n \left[ \chi(X) - \chi(D) + (-1)^n \sum_{p \in \text{Sing}(D)} \mu_p(D) \right]. \]

Using that \( \chi(\tilde{X}) = \chi(X) - \chi(D) \), we obtain the desired formula of item (i).

Let \( v \) be a holomorphic vector fields as described in item (ii). By Gauss-Bonnet and Poincaré-Hopf theorems, we have

\[ \int_X c_n(T_X) = \chi(X) = \sum_{x \in \text{Sing}(v)} PH(v, x). \]

Since \( D \) is compact, it follows from [27, Theorem 7.16] that

\[ \int_D c_{n-1}(T_X - [D]) = \sum_{x \in \mathcal{S}(v, D)} GSV(v, D, x). \]

Thus, using the Theorem 3.4 we obtain

\[ \int_X c_n(T_X (-\log D)) = (-1)^n \left[ \sum_{x \in \text{Sing}(v)} PH(v, x) - \sum_{x \in \mathcal{S}(v, D)} GSV(v, D, x) \right]. \]
Now, replacing it in the equality of item (i), we get

\[ (-1)^n \chi(\tilde{X}) + \sum_{p \in \text{Sing}(D)} \mu_p(D) = (-1)^n \left[ \sum_{x \in \text{Sing}(v)} PH(v, x) - \sum_{x \in S(v, D)} GSV(v, D, x) \right]. \]

Hence,

\[ \chi(\tilde{X}) = \sum_{x \in \text{Sing}(v)} PH(v, x) - \sum_{x \in S(v, D)} GSV(v, D, x) + (-1)^{n-1} \sum_{x \in \text{Sing}(D)} \mu_p(D), \]

and the first formula of item (ii) is proved.

Finally, we observe that if \( x_0 \in \text{Sing}(D) \) is a non-degenerate singularity of \( D \) such that \( x_0 \) belongs to regular part of \( D \), then (see remark 2.3)

\[ PH(v, x_0) - GSV(v, D, x_0) = 0 \]

and we obtain the formula. \( \square \)

4. Proof Corollary

First of all, let us show how recover the Dimca-Papadima formula: Let \( D \) be a reduced divisor on \( \mathbb{P}^n \) with isolated singularities and degree \( d \). The Dimca-Papadima formula say us that

\[ \chi(\mathbb{P}^n \setminus D) = (-1)^n d_t(D) + \sum_{i=0}^{n-1} (-1)^i (d-1)^i, \]

where \( d_t(D) \) is the polar degree of \( D \). We can write

\[ (-1)^n d_t(D) = (-1)^n (d-1)^n - (-1)^n \sum_{p \in \text{Sing}(D)} \mu_p(D). \]

Thus

\[ (-1)^n (d-1)^n = (-1)^n d_t(D) + (-1)^n \sum_{p \in \text{Sing}(D)} \mu_p(D). \]

Let us first prove the formula

\[ \chi(\mathbb{P}^n \setminus D) = (-1)^n (d-1)^n + \sum_{i=0}^{n-1} (-1)^i (d-1)^i - (-1)^n \sum_{p \in \text{Sing}(D)} \mu_p(D). \]

Indeed, it follows for Theorem 1.1 that

\[ (-1)^n \chi(\mathbb{P}^n \setminus D) = \int_{\mathbb{P}^n} c_t(\Omega^1_{2\mathbb{P}^n}(\log D)) - \sum_{p \in \text{Sing}(D)} \mu_p(D). \]

The total Chern classe of \( c(\Omega^1_{2\mathbb{P}^n}(\log D)) \) is

\[ c(\Omega^1_{2\mathbb{P}^n}(\log D)) = \frac{\big[c(\Omega^1_{2\mathbb{P}^n})\big]}{\big[c(\mathcal{O}_{\mathbb{P}^n}(-d))\big]} = c(\Omega^1_{\mathbb{P}^n} - \mathcal{O}_{\mathbb{P}^n}(-d)). \]
In particular, $c_n(\Omega^1_{\mathbb{P}^n}(\log D)) = c_n(\Omega^1_{\mathbb{P}^n} - \mathcal{O}_{\mathbb{P}^n}(-d)) = c_n(\Omega^1_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(d))$. Since $c_n(\Omega^1_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(d))) = (-1)^n c_n(T_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(-d)))$, we conclude that $c_n(\Omega^1_{\mathbb{P}^n}(\log D)) = (-1)^n c_n(T_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(-d)))$.

We have

$$c_n(T_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(-d))) = \sum_{i=0}^{n} (1-d)^i h^n = \sum_{i=0}^{n} (-1)^i(d-1)^i h^n.$$

Thus, we get

$$(-1)^n \chi(\mathbb{P}^n \setminus D) = (-1)^n \sum_{i=0}^{n} (-1)^i(d-1)^i - \sum_{p \in \text{Sing}(D)} \mu_p(D).$$

Now, we have

$$\chi(\mathbb{P}^n \setminus D) = (-1)^n(d-1)^n + \sum_{i=0}^{n-1} (-1)^i(d-1)^i - (-1)^n \sum_{p \in \text{Sing}(D)} \mu_p(D).$$

Then

$$\chi(\mathbb{P}^n \setminus D) = (-1)^n d_1(D) + \sum_{i=0}^{n-1} (-1)^i(d-1)^i.$$

The general formula can be computed by using (8) which give us

$$c_n(\Omega^1_{\mathbb{P}^n}(\log D)) = \left[\frac{c(\Omega^1_{\mathbb{P}^n})}{c(\mathcal{O}_{\mathbb{P}^n}(-d_1))c(\mathcal{O}_{\mathbb{P}^n}(-d_2))}\right]_n = \left[\frac{(1-h)^{n+1}}{(1-d_1 h)(1-d_2 h)}\right]_n.$$

It follows from [10] Proposition 4.4 that

$$\left[\frac{(1-h)^{n+1}}{(1-d_1 h)(1-d_2 h)}\right]_n = \sum_{i=0}^{n} \sigma_{n-i}(d_1-1,d_2-1) h^n,$$

where $\sigma_{n-i}$ is the complete symmetric function of degree $n-i$.

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