ON THE FITTING HEIGHT OF SOLUBLE GROUPS ADMITTING A COPRIME FACTORISATION

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Abstract. In this paper we are concerned with finite soluble groups G admitting a factorisation $G = AB$, with A and B proper subgroups having coprime order. We are interested in bounding the Fitting height of $G$ in terms of some group-invariants of $A$ and $B$: including the Fitting heights and the derived lengths.

1. Introduction

In this paper, all groups considered are finite and soluble, and hence the word “group” should always be understood as “finite soluble group”.

We investigate groups $G$ in which a factorisation

$G = AB = \{ab \mid a \in A, b \in B\}$

with A and B subgroups of G of coprime order is given. We are interested in obtaining some upper bounds on the Fitting height $h(G)$ of $G$, in terms of the Fitting heights ($h(A)$ and $h(B)$) and of the derived lengths ($d(A)$ and $d(B)$) of A and B. (Our notation is standard, see Section 2 for undefined terminology.)

Theorem 1.1. Let $G = AB$ be a finite soluble group factorised by its proper subgroups $A$ and $B$ with $\gcd(|A|, |B|) = 1$. If $|B|$ is odd, then

(1) $h(G) \leq h(A) + h(B) + 2d(B) - 1$.

If $B$ is nilpotent, then

(2) $h(G) \leq h(A) + 2d(B)$.

Before continuing with our discussion we need to introduce some notation. Given a group $G$, we write

$\delta(G) := \max\{d(S) \mid S \text{ Sylow subgroup of } G\}$,

that is, $\delta(G)$ is the maximal derived length of the Sylow subgroups of $G$. We also bound the Fitting height of $G$ in terms of the group-invariants $\delta(A)$ and $\delta(B)$.

Theorem 1.2. Let $G = AB$ be a finite soluble group factorised by its proper subgroups $A$ and $B$ with $\gcd(|A|, |B|) = 1$. Then

$h(G) \leq h(A) + (2\delta(B) + 1)h(B) - 1$.

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Both Theorems 1.1 and 1.2 extend and generalise some well-known results on groups admitting a factorisation with subgroups of coprime order, see for example the two monographs [1] Chapter 2 and [2] pages 133–135. Observe that when $A$ and $B$ are both nilpotent, we have $h(A) = h(B) = 1$ and the inequality in Theorem 1.2 specialises to the inequality of the main result in [7].

When $B$ is nilpotent, we have $\delta(B) = d(B)$ and $h(B) = 1$, and thus Theorem 1.1 (2) follows immediately from Theorem 1.2.

The hypothesis of $|B|$ being odd in Theorem 1.1 (1) is important in our proof because at a critical juncture we apply a remarkable theorem of Kazarin [6] (which requires $B$ having odd order). However, we believe that our hypothesis is only factitious and in fact we pose the following:

**Conjecture 1.3.** Let $G = AB$ be a finite soluble group factorised by its proper subgroups $A$ and $B$ with $\gcd(|A|, |B|) = 1$. Then

$$h(G) \leq h(A) + h(B) + 2d(B) - 1.$$ 

We also prove:

**Theorem 1.4.** Let $G = AB$ be a finite soluble group factorised by its proper subgroups $A$ and $B$ with $\gcd(|A|, |B|) = 1$. Then

$$h(G) \leq h(A)\delta(A) + h(B)\delta(B).$$

Finally, with an immediate application of Theorem 1.1 and of the machinery developed in Section 3 we prove:

**Corollary 1.5.** Let $G = AB$ be a finite soluble group factorised by its proper subgroups $A$ and $B$ with $\gcd(|A|, |B|) = 1$. For each $p \in \pi(B)$, let $B_p$ be a Sylow $p$-subgroup of $B$. Then

$$h(G) \leq h(A) + 2 \sum_{p \in \pi(B)} d(B_p).$$

In particular, $h(G) \leq h(A) + 2|\pi(B)|\delta(B)$.

In Section 2 we introduce some basic notation and some preliminary results that we use throughout the whole paper. In Section 3 we present our main tool (the towers as defined by Turull [9]) and we prove some auxiliary results. Section 4 is dedicated to the proof of Theorems 1.1 and 1.2 and of Corollary 1.5. The proof of Theorem 1.3 (which requires a slightly different machinery) is postponed to Section 5.

2. **Notation and Preliminary Results**

Given a group $G$, we denote by $F(G)$ the *Fitting subgroup* of $G$ (that is, the largest normal nilpotent subgroup of $G$). Moreover, the Fitting series of $G$ is defined inductively by $F_0(G) := 1$ and $F_{i+1}(G) := F_i(G)\langle F_i(G) \rangle$, for every $i \geq 0$. Clearly, $F_i(G) < F_{i+1}(G)$ when $F_i(G) < G$, and the minimum natural number $h$ with $F_h(G) = G$ is called the *Fitting height* (or Fitting length) of $G$ and is denoted by $h(G)$. Similarly, the *derived length* of $G$ is indicated by $d(G)$.

We let $|G|$ denote the order of $G$ and we let $\pi(G)$ denote the set of prime divisors of $|G|$. Given a prime number $p$, we write $G_p$ for a Sylow $p$-subgroup of $G$. A *Sylow basis of $G$* is a family $\{G_p\}_{p \in \pi(G)}$ of Sylow subgroups of $G$ such that $G_pG_q = G_qG_p$ for any $p, q \in \pi(G)$. By a pioneering result of Philip Hall [8] 9.1.7, 9.1.8 and
Given a set \( \pi \) of prime numbers, we set \( \pi':=\{ p \text{ prime} \mid p \notin \pi \} \). Moreover, when \( \pi=\{ p \} \), for simplicity we write \( p' \) for \( \pi' \). As usual, \( O_\pi(G) \) is the largest normal \( \pi \)-subgroup of \( G \) and the upper \( \pi'\pi \)-series of \( G \) is generated by applying \( O_{\pi'} \) and \( O_\pi \) (in this order) repeatedly to \( G \), that is, the series \( 1 = P_0 \leq N_0 \leq P_1 \leq N_1 \leq \cdots \leq P_i \leq N_i \leq \cdots \) defined by
\[
N_i/P_i:=O_{\pi'}(G/P_i) \quad \text{and} \quad P_{i+1}/N_i:=O_\pi(G/N_i).
\]
This is a series of characteristic subgroups having factor groups \( \pi' \)- and \( \pi \)-groups, alternately. The minimum natural number \( \ell \) such that the \( \pi'\pi \)-series terminates is named the \( \pi \)-length of \( G \) and denoted by \( \ell_\pi(G) \). When \( \pi=\{ p \} \), we write simply \( O_p(G) \) and \( \ell_p(G) \).

We first state a basic elementary result which will be used repeatedly and without comment.

**Lemma 2.1.** Let \( G=AB \) be a group factorised by \( A \) and \( B \) with \( \gcd(|A|,|B|)=1 \). Then there exists a Sylow basis \( \{G_p\}_{p\in\pi(G)} \) with \( A=\prod_{p\in\pi(A)}G_p \) and \( B=\prod_{p\in\pi(B)}G_p \).

**Proof.** From [1, Lemma 1.3.2], we see that for every \( p \in \pi(G) \) there exists a Hall \( p' \)-subgroup \( A_p \) of \( A \) and a Hall \( p' \)-subgroup \( B_p \) of \( B \) such that \( A_pB_p \) is a Hall \( p' \)-subgroup of \( G \). Now, for each \( p \in \pi(G) \), define \( G_p:=\bigcap_{q\in\pi(G)\setminus\{p\}}A_qB_q \). A computation shows that \( \{G_p\}_{p\in\pi(G)} \) is a Sylow basis of \( G \) (see for example [8, 9.2.1]). Moreover, \( A=\prod_{p\in\pi(A)}G_p \) and \( B=\prod_{p\in\pi(B)}G_p \). \( \square \)

The next two results are crucial for our proofs of Theorems [1.1] and [1.2]

**Theorem 2.2.** Let \( G \) be a group and let \( p \) be a prime. Then \( \ell_p(G) \leq d(G_p) \).

**Proof.** When \( p \) is odd, this is [3, Theorem A (i)]. The analogous result for \( p=2 \) is proved in [3]. \( \square \)

Kazarin [6] has proved Theorem 2.2 for arbitrary sets of primes \( \pi \) with \( 2 \notin \pi \). We state this generalisation in a form tailored to our needs.

**Theorem 2.3.** Let \( G \) be a group and let \( \pi \) be a set of primes. If \( 2 \notin \pi \) or if \( G_\pi \) is nilpotent, then \( \ell_\pi(G) \leq d(G_\pi) \).

**Proof.** When \( 2 \notin \pi \), this is the main result of [6] (see also [2, Theorem 1.7.20]). When \( G_\pi \) is nilpotent, the proof follows from Theorem 2.2. \( \square \)

3. OUR TOOLKIT: TOWERS

We start this section with a pivotal definition introduced by Turull [9]. (The definition of \( B \)-tower in [9, Definition 1.1] is actually more general than the one we give here and coincides with ours when \( B=1 \).)

**Definition 3.1.** Let \( G \) be a group. A family \( \mathcal{F}:=(P_i \mid i \in \{1,\ldots,h\}) \) is said to be a tower of length \( h \) of \( G \) if the following are satisfied.

1. \( P_i \) is a \( p_i \)-subgroup of \( G \) and \( p_i \in \pi(G) \).
2. If \( 1 \leq i \leq j \leq h \), then \( P_i \) normalises \( P_j \).
3. Define inductively \( \overline{P_h}:=P_h \), and \( \overline{P_i}:=P_i/C_{P_i}(\overline{P_{i+1}}) \) for \( i \in \{1,\ldots,h-1\} \).

Then \( \overline{P_i} \neq 1 \), for every \( i \in \{1,\ldots,h\} \).
p_i \neq p_{i+1}, \text{ for every } i \in \{1, \ldots, h-1\}.

A concept that resembles the definition of tower was originally introduced by Dade in [4] for investigating the Fitting height of a group. The relationship between Fitting height and towers was uncovered by Turull.

**Lemma 3.2** ([9, Lemma 1.9]). Let \( G \) be a group. Then

\[
    h(G) = \max\{h \mid G \text{ admits a tower of length } h\}.
\]

In view of Lemma 3.2 we give the following:

**Definition 3.3.** We say that the tower \( \Sigma \) of \( G \) a Fitting tower if \( \Sigma \) has length \( h(G) \).

The following is an easy consequence of [9, Lemma 1.5]. For simplifying the notation, given a \( p \)-group \( P \), we write \( \pi^*(P) = p \) when \( P \neq 1 \), and \( \pi^*(P) = 1 \) when \( P = 1 \). Observe that when \( P \neq 1 \) we have \( \pi(P) = \{\pi^*(P)\} \).

**Lemma 3.4.** Let \( G \) be a group, let \( \Sigma = (P_i \mid i \in \{1, \ldots, h\}) \) be a tower of \( G \), let \( j \in \{1, \ldots, h\} \), let \( s \geq 0 \) be an integer and let \( \Sigma' = (P_i \mid i \in \{1, \ldots, h\} \setminus \{j, j + 1, \ldots, j + s - 1, j + s\}) \). Then either \( \Sigma' \) is a tower of \( G \), or \( 1 < j \leq j + s < h \) and \( \pi^*(P_{j-1}) = \pi^*(P_{j+s+1}) \).

**Proof.** Lemma 1.5 in [9] says that, for every \( h_0 \) with \( 1 \leq h_0 \leq h \) and for every increasing function \( f : \{1, \ldots, h\} \to \{1, \ldots, h\} \), the family \( (P_{f(i)} \mid i \in \{1, \ldots, h_0\}) \) satisfies the conditions (1), (2) and (3) in Definition 3.1. Applying this with \( h_0 := h - s - 1 \) and with \( f : \{1, \ldots, h_0\} \to \{1, \ldots, h\} \) defined by

\[
    f(i) = \begin{cases} 
        i & \text{ if } 1 \leq i < j, \\
        i + s + 1 & \text{ if } j \leq i \leq h_0, 
    \end{cases}
\]

we obtain that \( \Sigma' \) satisfies the conditions (1), (2) and (3) of Definition 3.1. As \( \Sigma \) satisfies Definition 3.1 (4), we immediately get that either \( \Sigma' \) satisfies also (4) (and hence is a tower of \( G \)), or \( 1 < j \leq j + s < h \) and \( \pi^*(P_{j-1}) = \pi^*(P_{j+s+1}) \). \( \square \)

**Definition 3.5.** Let \( G \) be a group, let \( \Sigma = (P_i \mid i \in \{1, \ldots, h\}) \) be a tower of \( G \) and let \( \sigma \) be a set of primes. We set

\[
    \nu_\sigma(\Sigma) := |\{i \in \{1, \ldots, h\} \mid \pi^*(P_i) \in \sigma\}|.
\]

Clearly, \( \nu_\sigma(\Sigma) = 0 \) when \( \sigma \) has no element in common with \( \{\pi^*(P_1), \ldots, \pi^*(P_h)\} \).

Now, set \( P_0 := 1 \) and \( P_{h+1} := 1 \). For \( i, j \in \{1, \ldots, h\} \) with \( i \leq j \), the sequence \( (P_{i+\ell} \mid i \leq \ell \leq j) \) of consecutive elements of \( \Sigma \) is said to be a \( \sigma \)-block if

- \( \pi^*(P_{i+s}) \in \sigma \) for every \( s \) with \( 0 \leq s \leq j - i \), and
- \( \pi^*(P_{i-1}) \notin \sigma \), \( \pi^*(P_{j+1}) \notin \sigma \).

Moreover, we denote by \( \beta_\sigma(\Sigma) \) the number of \( \sigma \)-blocks of \( \Sigma \).

The main result of this section is Lemma 3.8 before proceeding to its proof we single out two basic observations.

**Lemma 3.6.** Let \( \Sigma = (P_i \mid i \in \{1, \ldots, h\}) \) be a tower of \( G \). Then, for \( j \in \{1, \ldots, h-1\} \), we have \( C_{P_{j+1}}(P_h) \leq C_{P_{j}}(P_{j+1}) \).

**Proof.** We argue by induction on \( h-j \). If \( j = h-1 \), then \( P_h = P_h \) and hence there is nothing to prove. Suppose \( h-j > 1 \) and set \( R := C_{P_{j}}(P_h) \). We have \( |R, P_h, P_{j+1}| = 1 \), and also \( |P_h, P_{j+1}, R| \leq |P_h, R| = 1 \) by Definition 3.1 (2). Thus the Three
Subgroups Lemma yields \([P_{j+1}, R, P_h] = 1\), that is, \([P_{j+1}, R] \leq C_{P_{j+1}}(P_h)\). Now the inductive hypothesis gives \([P_{j+1}, R] \leq C_{P_{j+1}}(P_{j+2})\), and hence \([P_{j+1}, R] = 1\). Therefore \(C_{P_j}(P_h) = R \leq C_{P_j}(P_{j+1})\).

**Lemma 3.7.** Let \(\mathcal{T} = (P_i \mid i \in \{1, \ldots, h\})\) be a tower of \(G\) and let \(N\) be a normal subgroup of \(G\) with

\[
P_j \cap N \leq C_{P_j}(P_h),
\]

for every \(j \in \{1, \ldots, h - 1\}\). Then \(\mathcal{T}' := (P_iN/N \mid i \in \{1, \ldots, h - 1\})\) is a tower of \(G/N\).

**Proof.** From (3) and Lemma 3.6, we have \(R_h := C_{P_h}(P_{j+1})\) for \(j < h\). Set \(R_h := 1\), and set \(R_j := C_{P_j}(P_{j+1})\) for \(j < h\). Thus \(P_j = P_j/R_j\), for every \(j\).

Now, for \(j < h\), we have

\[
P_j \cap N = R_j \cap N
\]

and hence

\[
\frac{P_j N}{R_j N} = \frac{P_j (R_j N)}{R_j N} \cong \frac{P_j}{R_j} \frac{P_j \cap R_j N}{R_j (P_j \cap N)} \frac{P_j}{R_j} \frac{P_j}{R_j} = \frac{P_j}{R_j} = \frac{P_j}{R_j} = \frac{P_j}{R_j}.
\]

For each \(j \in \{1, \ldots, h - 1\}\), set \(Q_j := P_j N/N\), and define \(Q_{h-1} := Q_{h-1}\) and \(\overline{Q_j} := Q_j/C_{Q_j}(\overline{Q_{j+1}})\) for \(j < h - 1\). In particular, for each \(j \in \{1, \ldots, h - 2\}\), there exists \(L_j \leq P_j\) with \(C_{Q_j}(\overline{Q_{j+1}}) = L_j N/N\). Moreover, set \(L_{h-1} := 1\).

We show (by induction on \(h - j\)) that \(L_j \leq R_j\), for each \(j \in \{1, \ldots, h - 1\}\). If \(h - j = 1\), then \(L_j = L_{h-1} = 1 \leq R_{h-1} = R_j\). Assume then that \(h - j > 1\) and let \(x \in L_j\). As \([xN, Q_{j+1}] = 1\), we get \([xN, Q_{j+1}] \leq C_{Q_j}(\overline{Q_{j+1}}) = L_j N/N\) when \(h - j > 2\), and \([xN, Q_{j+1}] = 1\) when \(h - j = 2\). In both cases, applying the inductive hypothesis, we obtain

\[
[x, Q_{j+1}] \leq \frac{L_{j+1} N}{N} = \frac{R_{j+1} N}{N}.
\]

This gives

\[
[x, P_{j+1}] \leq P_{j+1} \cap R_{j+1} N = R_{j+1}(P_{j+1} \cap N).
\]

Combining (3), Lemma 3.6 and the definition of \(R_{j+1}\), we have \(P_{j+1} \cap N \leq C_{P_{j+1}}(P_h) \leq C_{P_{j+1}}(P_{j+2}) = R_{j+1}\). Therefore \([x, P_{j+1}] \leq R_{j+1}\) and hence \(x \in C_{P_j}(P_{j+1}/R_{j+1}) \leq C_{P_j}(\overline{P_{j+1}}) = R_j\). Thus \(L_j \leq R_j\) and the induction is proved.

Observe that

\[
\overline{Q_j} = \frac{P_j N/N}{L_j N/N} \cong \frac{P_j N}{L_j N}.
\]

As \(L_j N \leq R_j N \leq P_j N\), from (4) and (5), we see that \(\overline{Q_j}\) is an epimorphic image of \(\overline{Q_j}\). Finally, since \(\mathcal{T}\) is a tower of \(G\), it follows immediately that \(\mathcal{T}'\) is a tower of \(G/N\).

Given a tower \(\mathcal{T} = (P_i \mid i \in \{1, \ldots, h\})\) and \(j \in \{1, \ldots, h\}\), we set \(T_j := P_h P_{h-1} \cdots P_j\). Observe that from Definition 3.1 (2), we have \(T_j \leq T_i\).

We are now ready to prove one of the main tools of our paper.

**Lemma 3.8.** Let \(G\) be a group, let \(\sigma\) be a non-empty subset of \(\pi(G)\), let \(A\) be a Hall \(\sigma\)-subgroup of \(G\) and let \(\mathcal{T} := (P_i \mid i \in \{1, \ldots, h\})\) be a tower of \(G\). Then

1. \(h(A) \geq \nu_\sigma(\mathcal{T}) - \beta_\sigma(\mathcal{T}) + 1\), and
2. \(\ell_\sigma(G) \geq \beta_\sigma(\mathcal{T})\).
Proof. Observe that $h(A), \ell_\sigma(G) \geq 1$ because $\emptyset \neq \sigma \subseteq \pi(G)$. In particular, we may assume that $\nu_\sigma(\Sigma), \beta_\sigma(\Sigma) \neq 0$ and hence $\sigma_0 := \sigma \cap \{\pi(P_i) \mid 1 \leq i \leq h\} \neq \emptyset$. Let $A_0$ be a Hall $\sigma_0$-subgroup of $T_1$. Observe that $\Sigma$ is a tower of $T_1$ and that the hypothesis of this lemma are satisfied with $(G, \sigma, A)$ replaced by $(T_1, \sigma_0, A_0)$. As $h(A_0) \leq h(A)$ and $\ell_\sigma(T_1) \leq \ell_\sigma(G)$, for proving parts (1) and (2) we may assume that $G = T_1, \sigma = \sigma_0$ and $A = A_0$.

Part (1): We argue by induction on $h + |G|$. If $h = 1$, then $\nu_\sigma(\Sigma) = \beta_\sigma(\Sigma) = 1$ and the proof follows.

Assume that $\pi^*(P_h) \notin \sigma$. Write $\Sigma' := (P_i \mid i \in \{1, \ldots, h-1\})$. From Lemma 3.4 the family $\Sigma'$ is a tower of $G$. As $\nu_\sigma(\Sigma') = \nu_\Sigma(\Sigma), \beta_\sigma(\Sigma') = \beta_\Sigma(\Sigma)$ and $\Sigma'$ has length $h-1$, the proof follows by induction.

Assume that $\pi^*(P_h) \in \sigma$. Let $t \in \{1, \ldots, h\}$ with $T_t = P_tP_{t-1} \cdots P_1$ a $\sigma$-block of $\Sigma$. Suppose that $T_t$ is the only $\sigma$-block of $\Sigma$. Thus $\nu_\sigma(\Sigma) = h - t + 1, \beta_\sigma(\Sigma) = 1$ and $T_t$ is a Hall $\sigma$-subgroup of $G$. Moreover, since $T_t \subseteq T_1 = G$, we have $A = T_t$. Write $\Sigma' := (P_i \mid i \in \{t, \ldots, h\})$. From Lemma 3.4 the family $\Sigma'$ is a tower of $G$ and hence a tower of $A$. As $\Sigma'$ has length $h - t + 1$, from Lemma 3.2 we get $h(A) \geq h - t + 1$ and the proof follows.

Suppose that $T_t$ is not the only $\sigma$-block of $G$, and let $j \in \{1, \ldots, t-1\}$ be maximal with $\pi^*(P_j) \in \sigma$. Suppose $\pi^*(P_j) \neq \pi^*(P_t)$. Then Lemma 3.4 yields that $\Sigma' := (P_i \mid i \in \{1, \ldots, h\} \setminus \{j+1, \ldots, t-1\})$ is a tower of $G$. Since $\Sigma'$ has length $h - (t - j - 1) < h$, from our induction we deduce

$$h(A) \geq \nu_\sigma(\Sigma') - \beta_\sigma(\Sigma') + 1 = \nu_\sigma(\Sigma) - (\beta_\sigma(\Sigma) - 1) + 1 = \nu_\sigma(\Sigma) - \beta_\sigma(\Sigma) + 2.$$  

Finally, suppose that $\pi^*(P_j) = \pi^*(P_t)$. In particular, either $\pi^*(P_{j-1}) \neq \pi^*(P_t)$ or $j = 1$. Now, Lemma 3.4 gives that $\Sigma' := (P_i \mid i \in \{1, \ldots, h\} \setminus \{j+1, \ldots, t-1\})$ is a tower of $G$. As $\Sigma'$ has length $h - (t - j) < h$, the inductive hypothesis yields

$$h(A) \geq \nu_\sigma(\Sigma') - \beta_\sigma(\Sigma') + 1 = (\nu_\sigma(\Sigma) - 1) - (\beta_\sigma(\Sigma) - 1) + 1 = \nu_\sigma(\Sigma) - \beta_\sigma(\Sigma) + 1.$$  

Part (2): As in Part (1), we proceed by induction on $h + |G|$. Assume $\pi^*(P_h) \notin \sigma$. Then $\Sigma' := (P_i \mid i \in \{1, \ldots, h\})$ is a tower of $G$ of length $h - 1$ with $\beta_\sigma(\Sigma') = \beta_\sigma(\Sigma)$. Thus the proof follows by induction.

Assume that $\pi^*(P_h) \in \sigma$. Write $N := O_{\sigma}(G)$ and assume first that $N \neq 1$. For $j \in \{1, \ldots, \, h-1\}$, we have $[P_j \cap N, P_h] \leq N \cap P_h = 1$ and hence $P_j \cap N \leq C_{P_h}(P_h)$. In particular, by Lemma 3.7 $\Sigma' := (P_iN/N \mid i \in \{1, \ldots, h-1\})$ is a tower of $G/N$ and, by induction, $\beta_\sigma(\Sigma') \leq \ell_{\sigma}(G/N)$. Since $\beta_\sigma(\Sigma) = \beta_\sigma(\Sigma')$ and $\ell_{\sigma}(G) \geq \ell_{\sigma}(G/N)$, we get $\beta_\sigma(\Sigma) \leq \ell_{\sigma}(G)$. Assume then that $N = 1$.

Write $\Sigma' := (P_i \mid i \in \{1, \ldots, h-1\})$. By Lemma 3.1 $\Sigma'$ is a tower of $G$ of length $h - 1$. If $P_h$ is a not $\sigma$-block, then $\beta_\sigma(\Sigma') = \beta_\sigma(\Sigma)$ and, by induction, $\beta_\sigma(\Sigma') \leq \ell_{\sigma}(G)$.

Suppose that $P_h$ is a $\sigma$-block, that is, $\pi^*(P_{h-1}) \notin \sigma$. Clearly, $\beta_\sigma(\Sigma) = \beta_\sigma(\Sigma') + 1$.

Write $M := O_{\sigma}(G)$ and observe that $M \neq 1$ and $\ell_{\sigma}(G) = \ell_{\sigma}(G/M) + 1$ because $O_{\sigma}(G) = N = 1$. For $j \in \{1, \ldots, h-2\}$, we have $[P_j \cap M, P_{h-1}] \leq M \cap P_{h-1} = 1$ and hence $P_j \cap M \leq C_{P_h}(P_{h-1})$. In particular, by Lemma 3.7 (applied to $\Sigma'$), $\Sigma'' := (P_jM/M \mid j \in \{1, \ldots, h-2\})$ is a tower of $G/M$. Now, by induction, $\ell_{\sigma}(G/M) \geq \beta_\sigma(\Sigma'') = \beta_\sigma(\Sigma')$ from which it follows that $\ell_{\sigma}(G) \geq \beta_\sigma(\Sigma)$. 

\[\square\]

4. Factorisations: Proofs of Theorems 1.1 and 1.2 and Corollary 1.5

We start by proving the following.
Lemma 4.1. Let $G$ be a group, let $\sigma$ be a non-empty proper subset of $\pi(G)$ and let $G = AB$ be a factorisation, with $A$ a $\sigma$-subgroup of $G$ and $B$ a $\sigma'$-subgroup of $G$. Then

$$h(G) \leq h(A) + h(B) + \ell_\sigma(G) + \ell_{\sigma'}(G) - 2$$

and

$$h(G) \leq h(A) + h(B) + 2 \min\{\ell_\sigma(G), \ell_{\sigma'}(G)\} - 1.$$  

Proof. Let $\mathcal{T}$ be a Fitting tower of $G$ (see Definition 3.3). Using first Lemma 3.8 part (1) and then part (2), we have

$$\ell_\sigma(G) \leq h(A) + h(B) + \ell_\sigma(G) - 2.$$  

Observe that, for each set of prime numbers $\pi$, from the definition of $\pi'$-$\pi$-series we have $\ell_\pi(G) \leq \ell_\pi(G) + 1$. Applying this remark with $\pi = \sigma'$ and with $\pi = \sigma''$, from (†) we get $h(G) \leq h(A) + h(B) + 2 \min\{\ell_\sigma(G), \ell_{\sigma'}(G)\} - 1$. □

Proof of Theorem 1.1. Write $\sigma := \pi(A)$ and $\sigma' := \pi(B)$. If $|B|$ is odd or if $B$ is nilpotent, then Theorem 2.3 yields $\ell_{\sigma'}(G) \leq d(B)$. In the first case, Eq. (1) follows directly from Lemma 4.1. In the second case, $h(B) = 1$ and now Eq. (2) follows again from Lemma 4.1. □

We now show that the bounds in Theorem 1.1 are (in some cases) best possible. (We denote by $C_n$ a cyclic group of order $n$.)

Example 4.2. Let $p, q, r$ and $t$ be distinct primes and let $n \geq 1$. Define $H_0 := C_p \wr C_q$ and $H_1 := (H_0 \wr C_r) \wr (C_q \wr C_p)$. Now, for each $i \geq 1$, define inductively $H_{2i} := (H_{2i-1} \wr C_r) \wr (C_q \wr C_p)$ and $H_{2i+1} := (H_{2i} \wr C_r) \wr (C_q \wr C_p)$.

We let $H := H_n$ and $G := C_t \wr H$. Let $A$ be a Hall $\{p, q\}$-subgroup of $G$ and let $B$ be a Hall $\{r, t\}$-subgroup of $G$. A computation shows that $h(A) = n + 2$, $h(B) = 2$, $h(G) = 3n + 3$ and $d(B) = n + 1$. Theorem 1.1 (1) predicts $h(G) \leq h(A) + h(B) + 2d(B) - 1$, and in fact in this example the equality is met.

Example 4.3. Let $p$ and $q$ be distinct primes and let $n \geq 0$. Define $G_0 := C_p$ and $G_1 := G_0 \wr C_q$. Now, for each $i \geq 1$, define inductively $G_{2i} := G_{2i-1} \wr C_p$ and $G_{2i+1} := G_{2i} \wr C_q$.

Let $G := G_{2n}$, let $A$ be a Sylow $p$-subgroup of $G$ and let $B$ be a Sylow $q$-subgroup of $G$. A computation shows that $h(A) = 1$, $d(B) = n$ and $h(G) = 2n + 1$. Theorem 1.1 (2) predicts $h(G) \leq h(A) + 2d(B)$, and in fact in this example the equality is met.

Proof of Corollary 1.3. From Lemma 2.1 there exists a Sylow basis $\{G_p\}_{p \in \pi(G)}$ of $G$ with $A = \prod_{p \in \pi(A)} G_p$ and $B = \prod_{p \in \pi(B)} G_p$.

Now, we argue by induction on $|\pi(B)|$. If $|\pi(B)| = 1$, then $B$ is nilpotent and hence the proof follows from Theorem 1.1 (2). Suppose that $|\pi(B)| > 1$. Fix $q \in \pi(B)$ and write $B_{q'} := \prod_{p \in \pi(B) \setminus \{q\}} G_p$. Clearly, $G = AB = (AG_q)B_{q'}$ and hence, by induction,

$$h(G) \leq h(AG_q) + 2 \sum_{p \in \pi(B_{q'})} d(G_p) \leq (h(A) + 2d(G_q)) + 2 \sum_{p \in \pi(B_{q'})} d(G_p)$$

$$= h(A) + 2 \sum_{p \in \pi(B)} d(G_p).$$
The proof of Theorem 1.2 will follow at once from the following lemma, which (in our opinion) is of independent interest.

**Lemma 4.4.** Let $G$ be a group, let $\sigma$ be a non-empty subset of $\pi(G)$ and let $H$ be a Hall $\sigma$-subgroup of $G$. Then $\ell_\sigma(G) \leq \delta(H)h(H)$.

**Proof.** When $|\sigma| = 1$, the proof follows immediately from Theorem 2.2 In particular, we may assume that $|\sigma| > 1$. Now we proceed by induction on $|G| + |\sigma|$.

Clearly, $\ell_\sigma(G) = \ell_\sigma(G/O_{\sigma'}(G))$ and $HO_{\sigma'}(G)/O_{\sigma'}(G) \cong H$ is a Hall $\sigma$-subgroup of $G/O_{\sigma'}(G)$. When $O_{\sigma'}(G) \neq 1$, the proof follows by induction, and hence we may assume that $O_{\sigma'}(G) = 1$.

Suppose that $G$ contains two distinct minimal normal subgroups $N$ and $M$. Clearly, $\pi(N), \pi(M) \subseteq \sigma$. As $O_{\sigma'}(G) = 1$, we deduce that $\ell_\sigma(G/N) = \ell_\sigma(G) = \ell_\sigma(G/M)$. Moreover, by induction, $\ell_\sigma(G/N) \leq \delta(H/N)h(H/N) \leq \delta(H)h(H)$.

This gives $\ell_\sigma(G) \leq \delta(H)h(H)$, and hence we may assume that $G$ contains a unique minimal normal subgroup. This yields $F(G) = O_p(G)$, for some $p \in \sigma$. As $C_G(O_p(G)) \leq O_p(G)$ and $O_p(G) \leq H$, we have $F(H) = O_p(H)$.

Write $\tau := \sigma \setminus \{p\}$. Observe that $\ell_\sigma(G) \leq \ell_p(G) + \ell_\tau(G)$. As $G_\tau$ is isomorphic to a subgroup of $H/F(H)$, we get $h(G_\tau) \leq h(H/F(H)) = h(H) - 1$. Now, from the inductive hypothesis, we obtain

$$
\ell_\sigma(G) \leq \ell_p(G) + \ell_\tau(G) \leq \delta(G_p)h(G_p) + \delta(G_\tau)h(G_\tau) \leq \delta(H) + \delta(H)(h(H) - 1) \leq \delta(H)h(H).
$$

**Proof of Theorem 1.2.** Write $\sigma := \pi(A)$ and $\sigma' := \pi(B)$. From Lemma 4.4 we get $\ell_\sigma(G) \leq \delta(A)h(A)$ and $\ell_{\sigma'}(G) \leq \delta(B)h(B)$. Now the proof follows from the second inequality in Lemma 4.1. 

5. Factorisations: Proof of Theorem 1.4

Before proceeding with the proof of Theorem 1.4 we need to introduce some auxiliary notation.

Given a group $G$, we denote with $R(G)$ the nilpotent residual of $G$, that is, the smallest (with respect to inclusion) normal subgroup $N$ of $G$ with $G/N$ nilpotent. Then, we define inductively the descending normal series $\{R_i(G)\}$, by $R_0(G) := G$ and $R_{i+1}(G) := R_i(R_i(G))$, for every $i \geq 0$. Observe that if $h = h(G)$, then for every $i \in \{0, \ldots, h\}$ we have $R_{h-i}(G) \leq F_i(G)$.

Now, let $A$ be a Hall subgroup of $G$ and, for $i \in \{1, \ldots, h\}$, define

$$
\ell^i(A, G) := \max\{\ell_p(G) \mid p \in \pi(R_{i-1}(A)/R_i(A))\} \quad \text{and} \quad \Lambda^i_G(A) := \sum_{i=1}^{h(A)} \ell^i(A, G).
$$

It is clear that, for every normal subgroup $N$ of $G$, $\Lambda_{G/N}(AN/N) \leq \Lambda_G(A)$. 

**Lemma 5.1.** Let $G = AB$ be a finite soluble group factorised by its proper subgroups $A$ and $B$ with $\gcd(|A|, |B|) = 1$. Then $h(G) \leq \Lambda_G(A) + \Lambda_G(B)$.

**Proof.** We argue by induction on $|G|$. Suppose that $G$ contains two distinct minimal normal subgroups $N$ and $M$. Clearly, $h(G/N) = h(G) = h(G/M)$ and hence by induction $h(G) \leq \Lambda_{G/N}(AN/N) + \Lambda_{G/N}(BN/N) \leq \Lambda_G(A) + \Lambda_G(B)$. In particular,
we may assume that $G$ contains a unique minimal normal subgroup $N$ and, replacing $A$ by $B$ if necessary, that $\{p\} = \pi(N) \subseteq \pi(A)$. This yields $F(G) = O_p(G)$. As $C_G(O_p(G)) \leq O_p(G)$ and $O_p(G) \leq A$, we have $F(A) = O_p(A)$.

Write $h := h(A)$. Now, $R_{h-1}(A) \leq F_1(A) = F(A)$ and hence $R_{h-1}(A)$ is a $p$-group. Thus

$$\Lambda_G(A) = \ell_p(G) + \sum_{i=1}^{h-1} \ell^i(G, A).$$

Since $\ell_p(G/F(G)) = \ell_p(G) - 1$, we get $\Lambda_{G/F(G)}(A/F(G)) \leq \Lambda_G(A) - 1$. Moreover, since $p \notin \pi(B)$, we have $\Lambda_{G/F(G)}(B/F(G)) = \Lambda_G(B)$. Therefore the inductive hypothesis gives

$$h(G) = h(G/F(G)) + 1$$
$$\leq \Lambda_{G/F(G)}(A/F(G)) + \Lambda_{G/F(G)}(B/F(G)) + 1$$
$$\leq \Lambda_G(A) + \Lambda_G(B),$$

and the proof is complete. $\square$

**Proof of Theorem 1.4.** For each $p \in \pi(A)$, Theorem 2.2 yields $\ell_p(G) \leq d(G_p)$ and hence $\ell_p(G) \leq \delta(A)$. It follows that $\Lambda_G(A) \leq \delta(A)h(A)$. The same argument applied to $B$ gives $\Lambda_G(B) \leq \delta(B)h(B)$. Now the proof follows from Lemma 5.1. $\square$

A weaker form of Theorem 1.4 can be deduced from the results in Section 4. Indeed, from the first inequality in Lemma 4.1 and from Lemma 1.4, we get

$$h(G) \leq h(A) + h(B) + \ell_\sigma(G) + \ell_\sigma'(G) - 2$$
$$\leq h(A) + h(B) + \delta(A)h(A) + \delta(B)h(B) - 2$$
$$= (\delta(A) + 1)h(A) + (\delta(B) + 1)h(B) - 2.$$

Clearly Theorem 1.4 always offer a better estimate on $h(G)$.

**References**

[1] B. Amberg, S. Franciosi, F. de Giovanni, *Products of groups*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1992.

[2] A. Ballester-Bolinches, R. Esteban-Romero, M. Asaad, *Products of finite groups*, Expositions in Mathematics 53, Walter de Gruyter, Berlin, 2010.

[3] E. G. Brjuhanova, The relation between 2-length and derived length of a Sylow 2-subgroup. (Russian), *Mat. Zametki* 29 (1981), 161–170, 316.

[4] E. C. Dade, Carter subgroups and Fitting heights of finite solvable groups, *Illinois J. Math.* 13 (1969), 449–514.

[5] P. Hall, G. Higman, On the $p$-length of $p$-soluble groups and reduction theorems for Burnside’s problem, *Proc. London Math. Soc.* 6 (1956), 1–42.

[6] L. S. Kazarin, Soluble products of groups, *Infinite groups 1994*, Proceedings of the International Conference held in Ravello, May 23–27 1994, (F. de Giovanni, and M. L. Newell eds.), Walter de Gruyter, Berlin, 1996, 111–123.

[7] G. Parmeggiani, The Fitting series of the product of two finite nilpotent groups, *Rend. Sem. Mat. Univ. Padova* 91 (1994), 273–278.

[8] D. J. S. Robinson, *A Course in the Theory of Groups*, Springer-Verlag, New York, 1982.
[9] A. Turull, Fitting height of groups and of fixed points, *J. Algebra* 86 (1984), 555–566.

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