Dynamical equilibrium states of a class of irrotational non-orthogonally transitive $G_2$ cosmologies I: The conjecture of chaotic cosmological inhomogeneity

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Abstract

The Einstein field equations for a class of irrotational non-orthogonally transitive $G_2$ cosmologies are written down as a system of partial differential equations. The equilibrium points are self-similar and can be written as a one-parameter, five-dimensional, ordinary differential equation (ODE). The corresponding cosmological models both evolve and have one-dimension of inhomogeneity. The major mathematical features of this ODE are derived, and a cosmological interpretation is given. The relationship to the exceptional Bianchi models is explained and exploited to provide a conjecture about future generalizations. It is assumed throughout that the source is a perfect fluid with a linear equation of state.

Keywords: G2 cosmologies, dynamical systems, conjecture of chaotic cosmological inhomogeneity, non orthogonally transitive

1. Introduction

In order to obtain cosmological models that have spatial structure, one is forced to examine inhomogeneous cosmologies. The simplest evolving inhomogeneous cosmologies admit two spacelike isometries. In a previous work, [1], we defined the $G_2$ cosmologies to be the spacetimes admitting two commuting spacelike Killing vector fields (KVFs) and satisfying the Einstein field equations (EFEs) with a perfect fluid source. The EFEs for that subclass in which the $G_2$ acts orthogonally transitively (OT), were written as a first order system of quasi-linear partial differential equations (PDEs). The equilibrium points of that system were defined
to be those models whose dimensionless variables do not evolve, and were labelled ‘dynamical equilibrium states’, (DES). The corresponding cosmological models are self-similar, admitting an $H_3$ acting on the timelike three spaces containing the orbits of the $G_2$ and the fluid flow vector. The EFEs for these models reduce to a system of ordinary differential equations (ODEs), which describes the spatial structure of these models. A qualitative analysis of the dynamical equilibrium states of the orthogonally transitive $G_2$ cosmologies was given in [2], and their asymptotic spatial structure was determined. The DES of the OT $G_2$ cosmologies are interesting cosmological models in their own right: open sets of these models are well behaved, having a big-bang initial singularity and they are asymptotically spatially homogeneous at large spatial distance, with their inhomogeneity only being significant over a finite spatial region. In addition, it was conjectured in [1] that these DES may act as asymptotic states for less symmetric cosmological models at either early or late times, or both, and explicit examples were provided with the exact solutions of [3, 4], their generalisations [1], and [5]. The main goal of the current paper is to extend this approach to the $G_2$ cosmologies in which the Abelian $G_2$ does not necessarily act OT, and for which the fluid flow vector is orthogonal to the group orbits. We refer to these models as the irrotational N-OT $G_2$ cosmologies. We complete this goal in three major stages: first of all we write down the EFEs for the irrotational N-OT $G_2$ cosmologies, then we write down the EFEs for the DES of the irrotational N-OT $G_2$ cosmologies, and in the final stage we provide a qualitative analysis of the resulting ODE and make some conclusions about these cosmological models. We make use of local behaviour and the approximation property of orbits to prove the existence of an open set of models which are well-behaved. They are asymptotically vacuum at large spatial distance, where they may be approximated by a plane wave model. A typical density profile for one of these models appears in figure 8 of the companion paper by Rashidi, Hewitt and Charbonneau [6]. Furthermore, these models possess a big-bang singularity at a finite time into the past, which is a consequence of their self-similarity and the alignment of the fluid relative to the group orbits.

A secondary goal is to provide comparison between the structure and features of both the differential equations and the corresponding cosmological models for the DES of the irrotational N-OT $G_2$ cosmologies and the exceptional orthogonal spatially homogeneous (OSH) cosmologies. The third goal is to extend these comparisons further by introducing the conjecture of chaotic spatial inhomogeneity. In the companion paper [6], we provide a complete mathematical analysis of the DES of the subclass in which at least one of the KVF is hypersurface orthogonal (HO).

We assume throughout that:

- H1: spacetime admits a two dimensional Abelian isometry subgroup, $G_2$, and its orbits are spacelike.
- H2: the source is a perfect fluid with a linear equation of state, $p = (\gamma - 1)\mu$. Here $p$ is the pressure, $\mu$ is the energy density and $\gamma$ is the equation of state parameter ($1 < \gamma < 2$). These are the $G_2$ cosmologies.
- H3: the isometry subgroup, $G_2$, does not (necessarily) act OT.

These are the N-OT $G_2$ cosmologies. Just for clarification, note that OT $G_2$ cosmologies are also N-OT $G_2$ cosmologies.

The fluid four-velocity for the OT $G_2$ cosmologies is forced to be orthogonal to the $G_2$ orbits. If we do not restrict the action of the isometry group then the fluid four-velocity does not have any further constraints. For simplicity we assume that:

- H4: the fluid velocity, $u$ is orthogonal to the orbits of the isometry subgroup.
An immediate consequence of this restriction is that the fluid is forced to be irrotational. This is the class of irrotational N-OT $G_2$ cosmologies, that will be studied in this paper and in [6].

The first stage of the main goal of this paper is completed in appendix A by writing the EFEs for the irrotational N-OT $G_2$ cosmologies as a first order system of quasilinear PDEs in terms of dimensionless dynamical variables. Dynamical equilibrium states are also defined there, and it is explained that the corresponding cosmological models are self-similar, admitting a three parameter similarity group, $H_3$, acting on the three spaces containing the $G_2$ orbits and $u^1$.

Consequently, in the main part of the paper (i.e., all except the appendices), we assume that:

- H5: spacetime admits a three-dimensional similarity group ($H_3$) having timelike orbits, and
- H6: the fluid velocity $u$ is tangential to the orbits of the similarity group.

In section 2 we provide further motivation for studying cosmological models which adhere to H1–H6. We introduce a subclassification scheme which shows the connection of the DES of the irrotational N-OT $G_2$ cosmologies to previously studied self-similar and spatially homogeneous cosmologies. This both partially addresses the second goal and leads naturally to the completion of the third goal.

The EFEs for the DES of the OT $G_2$ cosmologies may be written as a four-dimensional system of ODEs with two parameters. One of the variables is a shear variable and one of the parameters is a shear parameter. In section 3, we complete the second stage of our primary goal, by writing the EFEs for the DES of the irrotational N-OT $G_2$ cosmologies as an autonomous system of ODEs, with no constraints. The particular system is five-dimensional and has one parameter: the above-mentioned shear parameter is fixed to a specific value, and, in addition, the models allow an additional (off-diagonal) shear variable.

The symmetries and invariant sets of the ODE are also listed in section 3. In section 4, we analyse the local stability of the equilibrium points; outline the main features of this ODE; deduce some of the properties of the solutions; make statements about open sets of trajectories, and complete the third stage of our primary goal by giving some cosmological interpretations. In section 5, we conclude by completing our second goal of discussing the analogies between the analyses of the exceptional OSH cosmologies and the DES of the N-OT irrotational $G_2$ cosmologies.

It is assumed that the reader is familiar with the orthonormal frame formalism of Ellis and MacCallum [8]. The reader is referred to the book [9] for a thorough treatment of the techniques and methods of dynamical systems, especially when applied to cosmology. We use geometrised units with $c = 1$, $8\pi G = 1$, and the sign conventions of [8]. Throughout this paper, lower case indices $a, b, c$ assume the values $0, 1, 2, 3$ while upper case indices $A, B, C$ assume the values $2, 3$.

2. Motivation

We do not intend to attempt to convince the reader that the Universe is spatially inhomogeneous, nor do we wish to debate whether the Universe may be considered to be inhomogeneous on various scales, see, for example, [10]. Rather, we consider this work herein and in [6] to be a further investigation of the rich diversity of the types of models permitted by the EFEs.

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1 A three-parameter similarity group must admit a two-parameter isometry subgroup and in all but one case (see [7]) this subgroup is Abelian.
2 Note that it is shown in appendix A that there are no dust models in this class.
Table 1. Wainwright Subclassification for models satisfying H1, H2 and H4–H6.

| Class | Geometric restriction | Variables | Parameters | Reference |
|-------|-----------------------|------------|------------|-----------|
| A(i)  | Not OT no HO KVFs     | 5          | 1          | Sections 3, 4 |
| A(ii) | Not OT 1 HO KVF       | 3          | 1          | [6]       |
| B(i)  | OT no HO KVFs         | 4          | 2          | [2]       |
| B(ii) | OT 2 orthogonal HO KVFs| 2          | 3          | [12, 13]  |

Furthermore, even if one considers the Universe to be spatially homogeneous on all observable scales, it is of interest to determine whether or not there are exact solutions of the EFEs which both possess inhomogeneities and yet are also close to being considered to be spatially homogeneous, by all metrics currently available to us, almost everywhere.

In this section we provide reasons why cosmological models satisfying H1–H6 are worth examination. First of all,

- Many of these models are well-behaved inhomogeneous cosmological models themselves.
- Inhomogeneous self-similar models have been shown to act as asymptotic states, at both early and/or late times, for less symmetric (non-self-similar) cosmological models. Thus the spatial structure of these models may provide insight into the possible spatial structure of generic cosmological solutions of the EFEs.
- These models generalise well-known spatially homogeneous models which appear as equilibrium points in phase space. These are some plane wave models; the Collins VIh models, the Wainwright VI−19 model and a Robinson–Trautman solution.
- The additional variables which arise when moving from the OT models to the N-OT models are shear variables. This work can be interpreted as an examination of the effects that additional shear has on the inhomogeneity of cosmological models.

Secondly, it is useful to be able to visualise where the models considered in this paper lie relative to previously considered models. In order to make this comparison, it is instructive to recall, and utilise the work of Wainwright [11]. In that work, spacetimes that admit an Abelian $G_2$ were classified according to the nature of the action of the $G_2$. In addition to whether or not the Abelian $G_2$ acts OT, this classification considers whether or not there are HO KVFs. In order to clearly illustrate how the models considered in this paper are related to previously considered inhomogeneous models, we superimpose the Wainwright classification onto cosmologies which satisfy restrictions H1, H2 and H4–H6 in table 1.

- It is of interest to complete the examination of the inhomogeneous cosmological models of table 1.
- Most studies on inhomogeneous cosmology consider models which admit an OT $G_2$ (see, for example the book, [14]). The current work considers a large class of models in which there is a N-OT $G_2$, and thus progress is being made towards the possibility of understanding the most general cosmological model, i.e. one which admits no symmetries.

Thirdly, the models considered in this paper are very closely related to (some) OSH models. In fact, the EFEs for both classes can be obtained by applying additional restrictions to the PDE in appendix A for the EFEs of the irrotational N-OT $G_2$ cosmologies. In order to make these similarities explicit, we have found it useful to introduce table 2 for the OSH models. We first need to refine the hypotheses H5 and H6 by, H∗5 and H∗6:

- $H^*$5: spacetime admits a three-parameter isometry group ($G_3$) having spacelike orbits, and
- $H^*$6: the fluid velocity $u$ is orthogonal to the orbits of the $G_3$. 


Table 2. Wainwright Subclassification for models satisfying H1, H2 and H4, H∗5 and H∗6.

| Class | Geometric restriction | Variables | Parameters | Reference |
|-------|-----------------------|-----------|------------|-----------|
| A(i)  | Not OT no HO KVFs     | 5         | 1          | [15]      |
| A(ii) | Not OT 1 HO KVFs      | 3         | 1          | [16]      |
| B(i)  | OT no HO KVFs         | 4         | 2          | [17, 18]  |
| B(ii) | OT 2 orthogonal HO KVFs | 2       | 3          | [19]      |

- On comparing these two tables, it is clear that the subdivision into the various subclasses is identical.

In the conclusion we emphasise the similarities and differences between the N-OT subclasses in the self-similar and spatially homogeneous cases. Examination of the DES of the irrotational N-OT G2 cosmologies provides a first step in the consideration of the Chaotic Inhomogeneity Conjecture, which we now outline.

In [1, 2] the role played by the spatially homogeneous plane wave solutions as asymptotic states for spatially inhomogeneous models at large distance has been revealed. There are several analogies with the role played by the Kasner models as asymptotic states at early times for some OSH models (i.e. those which are also OT G2 cosmologies). However, it is also well known that for the most general OSH models and generic tilted spatially homogeneous models (i.e. those containing the general tilted Bianchi II model as a subclass) the extra variables involved have the effect of destabilising the Kasner equilibrium points and turning them into saddle points. The dynamical effect of this is rather drastic: a typical trajectory is no longer past asymptotic to a single equilibrium point, but bounces around the phase space as it is directed from one saddle point to another. The corresponding cosmological model has an oscillatory initial singularity, and the word ‘chaos’ is often associated with the dynamical behaviour. One of the goals of this paper is to consider whether the analogy between Kasner models and plane wave models extends further, that is we consider the conjecture:

**The conjecture of chaotic cosmological inhomogeneity**.

The spatial structure of a generic spatially inhomogeneous cosmological model is chaotic. Generic spatially inhomogeneous cosmological models possess an oscillatory nature: as one moves in any given spatial direction the model is vacuum dominated and may be approximated by a plane wave model over a finite volume and then further on it may be approximated by another plane wave model. This sequence extents indefinitely, and between any two plane wave regimes there is, possibly, a smaller matter dominated region. The differential equation which determines the sequence of oscillations between the intermediate plane waves states is chaotic.

The observational consequence for inhomogeneous cosmological models which satisfy this conjecture is rather simple: a typical cosmological model is approximately vacuum (i.e. very low energy density) almost everywhere, but not the same type of vacuum solution. Between different vacuum regions there is a comparatively small region where the energy density may be significant, and matter close to the boundary between two regions would be accelerated towards the smaller matter dominated region. On a large scale, these predictions are similar to actual observations of the large scale structure of the Universe as voids (lower energy density regions) and filaments (smaller regions of higher energy density), see for example [21]. In

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3 This conjecture was briefly discussed at both the 13th Canadian conference on general relativity and the 12th Marcel Grossmann Meeting [20].
In figure 1 we provide a schematic projected phase portrait to illustrate this conjecture, and in figure 2 we provide a possible density profile (plot of the dimensionless energy density vs distance) for a typical trajectory as given in figure 1.

The research which has been carried out to produce this paper can be considered as a first attempt to verify the chaotic inhomogeneity conjecture. On considering the analogy from the spatially homogeneous models, we expect that it is necessary to examine inhomogeneous cosmological models in which any $G_2$ does not act OT. Models which satisfy assumptions H1–H6 provide a relatively simple and accessible class, however their analysis is complicated as the phase space is non-compact. We show that the effect of the additional shear variable is indeed to destabilise most points on the curve of the plane-wave models, however there still remains segments which act as a source or as a sink. We conclude that an examination of an even more general class is required; in that case, there is the possibility that the additional variable(s) would further destabilise the plane wave arcs and change the stability of equilibrium points to saddles. We remark that Coley and Hervik have shown in [22] that for a specific class of plane wave models, each plane wave equilibrium point possesses an unstable manifold of dimension at least one, when the phase space is large enough.

3. EFEs for the DES of the irrotational N-OT $G_2$ cosmologies

The dimensionless form of the EFEs for the DES of the irrotational N-OT $G_2$ cosmologies have been developed in appendix A. There is rotational freedom in the group invariant orthonormal frame and there are constraints on the system of ODEs. When examining the OT models, [1].
we found it convenient to use group-invariant variables, that is variables that are not affected by the rotational freedom in the orthonormal frame. The same approach is not suitable for the N-OT models due to the large number of additional constraint equations which are introduced. Instead, we choose to make use of the rotational freedom of the orthonormal frame to align $e_3$ with a KVF and to set the dimensionless variables $\Sigma_{12}$ to zero. The details are provided in appendix B. The same approach has been employed in other studies, including the examination of the exceptional orthogonal Bianchi models in [15]. The consequences on the dimensionless scalars are $R = 3\tilde{\Sigma}_{23}, N_+ = 3\tilde{N}_{22}$, and $\Sigma_{22} = \frac{1}{4}(6 - 5\gamma)$. From this point on we will only consider values of the equation of state parameter $\gamma$ which satisfy $1 < \gamma \leq \frac{2}{3}$. The reasons for this are explained in appendix A (see (A.71)).

Once this has been done then the non-zero independent dimensionless variables are, the three spatial curvature variables: $A, \tilde{N}_{23}, \tilde{N}_{22}$, and the two shear variables: $\tilde{\Sigma}_{23}$ and $\Sigma_{13}$. The acceleration and the energy density may be expressed, as quadratic and cubic polynomials respectively, in terms of the other variables.

The EFEs reduce to: spatial structure equations:

$$\theta_1(A) = 2[A^2 - (1 + \Sigma_+)^2] + \frac{3}{2}(2 - \gamma)\Omega + 9\Sigma_{13}^2,$$

$$\theta_1(\tilde{N}_{22}) = 2(A + 3\tilde{N}_{23})\tilde{N}_{22} + (6\tilde{\Sigma}_{22}^2 + 2 - q)\tilde{\Sigma}_{23},$$

$$\theta_1(\tilde{N}_{23}) = 2A\tilde{N}_{23} + 6\tilde{\Sigma}_{23}^2 - 6\tilde{N}_{22}^2 + (q - 2)\tilde{\Sigma}_{22} - 3\Sigma_{13}^2,$$

$$\theta_1(\tilde{\Sigma}_{23}) = -6(\tilde{\Sigma}_{23}\tilde{\Sigma}_{23} + \tilde{N}_{22}\tilde{\Sigma}_{22}),$$

$$\theta_1(\Sigma_{13}) = 3(A + \tilde{U} - \tilde{N}_{23})\Sigma_{13}. \quad (1)$$

Defining equations for $\tilde{U}$ and $\Omega$:

$$3(2 - \gamma)\tilde{U} = (3\gamma - 2)A + 12\tilde{\Sigma}_{23}\tilde{N}_{22} - 12\tilde{\Sigma}_{22}\tilde{N}_{23}, \quad (2a)$$

$$3(\gamma - 1)\Omega = A(A - 6\tilde{U}) - 9\tilde{\Sigma}_{22}^2 - 9\tilde{\Sigma}_{23}^2 - 9\tilde{N}_{22}^2 - 9\tilde{N}_{23}^2 - (1 + \Sigma_+)(1 + 7\Sigma_+) + 9\Sigma_{13}^2, \quad (2b)$$

where

$$\Sigma_+ = \frac{-3(\gamma - 2)}{4}, \quad \tilde{\Sigma}_{22} = \frac{6 - 5\gamma}{4}, \quad q = \frac{(3\gamma - 2)}{2}. \quad (3)$$

Auxiliary equation:

$$\theta_1(\Omega) = -3\frac{(2 - \gamma)}{(\gamma - 1)}\tilde{U}\Omega. \quad (4)$$

The system admits the two discrete symmetries:

$$(\bar{x}, A, \tilde{N}_{23}, \tilde{N}_{22}, \tilde{\Sigma}_{23}, \Sigma_{13}) \mapsto (\bar{x}, A, \tilde{N}_{23}, \tilde{N}_{22}, \tilde{\Sigma}_{23}, -\Sigma_{13}) \quad (5)$$

$$(\bar{x}, A, \tilde{N}_{23}, \tilde{N}_{22}, \tilde{\Sigma}_{23}, \Sigma_{13}) \mapsto (-\bar{x}, -A, -\tilde{N}_{23}, -\tilde{N}_{22}, -\tilde{\Sigma}_{23}, \Sigma_{13}) \quad (6)$$

As a consequence of the first symmetry and the fact that $\Sigma_{13} = 0$ is an invariant set, we need to only consider models with $\Sigma_{13} \geq 0$ from this point onward.

Since the expression for the energy density $\Omega$ is cubic, the vacuum part of the boundary of the phase space is not compact and consequently the physically relevant region of phase space
is also non-compact. We have been unable to find a particularly elegant and natural way of compactifying the complete system, however, in section 4, we obtain a number of important deductions about the corresponding cosmological models.

We refer to the physical region of phase space to mean the subset of $\mathbb{R}^5$ with $\Sigma_{13} \geq 0$ and $\Omega \geq 0$.

The vector $Z = (A, \tilde{N}_{23}, \tilde{N}_{22}, \tilde{\Sigma}_{23}, \Sigma_{13})$ determines the dynamical state of the cosmological model. The EFEs for the models under consideration have been written as a five-dimensional differential equation (1) with a single parameter, $\gamma$, for the dependent variable $Z$. In appendix B we introduced local co-ordinates $\{\tau, \tilde{x}, y, z\}$ through which we can provide some cosmological interpretation of this work. We have $\partial_0 = \frac{3\gamma}{2} \partial_\tau$, and $\partial_1 = \frac{3\gamma}{2} \partial_\tau + P(\tau, \tilde{x}) \frac{\partial}{\partial \tilde{x}} + Q(\tau, \tilde{x}) \frac{\partial}{\partial y}$. The orbits of the $H_3$ are labelled by the co-ordinate $\tilde{x}$, and $Z = Z(\tilde{x})$, so that $\partial_1(Z(\tilde{x})) = \frac{3\gamma}{2} \frac{\partial}{\partial \tilde{x}}(Z(\tilde{x}))$. Thus the DE (1) determines the changes in the spatial structure of the models as we move from one $H_3$ orbit to another. The orbit of the Abelian $G_2$ are labelled by $\tilde{x} = \text{constant}, \tau = \text{constant}$.

In order to evaluate the physical variables, we must first obtain the expansion scalar $\theta$ by examination of the decoupled equations derived from (A.52), (A.54), and (A.44):

$$\frac{1}{\theta} \frac{\partial}{\partial \tau} \theta = \frac{3\gamma}{2} \frac{\partial}{\partial \tau} \theta, \quad \partial_1(\theta) = -3 \dot{\theta} \theta.$$  \hfill (7)

Combining these with the auxiliary equation (4), it is clear that we have,

$$\theta = \theta_0 \Omega(\tilde{x})^{\frac{1}{2\mu_0}} e^{-\tau}, \quad \mu = \mu_0 \Omega(\tilde{x})^{\frac{1}{2\mu_0}} e^{-2\tau},$$  \hfill (8)

where $\theta_0$ and $\mu_0$ are constants. It follows that the energy density $\mu$ is bounded on any spatial slice $\tau = \text{constant}$ if and only if the dimensionless density parameter, $\Omega$, is bounded on that slice.

There are a number of invariant sets contained within this system of equations, these are given in table 3.

### 4. Stability of the equilibrium points and cosmological consequences

In table 4 we list the equilibrium points of system (1), and then we proceed to discuss their local stability and role in phase space, by examining the eigenvalues of the linearization at each equilibrium point. Each of the possible types of behaviour is illustrated in [6].

We introduce the non-negative quantity, $s$, given by $s^2 = (3\gamma - 2)(2 - \gamma)$, and we define the quadratics $Q_1(\gamma), Q_2(\gamma)$ and $Q_3(\Sigma_{13})$ by

$$Q_1(\gamma) = 36\gamma - 20 - 15\gamma^2, \text{ which is non-negative for } \gamma \in (1, \frac{3}{2}).$$  \hfill (9)

$$Q_2(\gamma) = (10 - 7\gamma)(\gamma - 1), \text{ and } Q_3(\Sigma_{13}) = 4 + 81\Sigma_{13}^2.$$  \hfill (10)

| Table 3. Invariant sets. |
|-------------------------|
| Restriction            | Restriction on the variable | Reference |
|-------------------------|
| $G_2$ acts OT           | $\Sigma_{13} = 0$          | [2]       |
| $G_2$ has one HO KVF    | $\tilde{\Sigma}_{23} = 0$ | $\tilde{N}_{22} = 0$ | [6] |
| $G_2$ has two HO KVFs   | $\Sigma_{13} = 0$          | $\tilde{\Sigma}_{23} = 0$ | $\tilde{N}_{22} = 0$ | [12, 13] |
| Vacuum                  | $\Omega = 0$              |           |
4.1. The Collins equilibrium points

These equilibrium points exist in the physical region of phase space for values of the equation of state parameter satisfying \( 1 < \gamma \leq \frac{10}{3} \). For values of \( \gamma \) in this interval, but \( \gamma \neq \frac{5}{3} \), it is found that the real part of three of the eigenvalues take one sign while the real part of the fourth eigenvalue takes the opposite sign, and the fifth eigenvalue is \( \frac{3(10 - 9\gamma)}{2(2 - 3\gamma)} \). We conclude that the Collins equilibrium points are saddle points when they lie in the physical region of phase space.

There are three values of \( \gamma \) which are bifurcation values for the Collins equilibrium points. First of all, when \( \gamma = \frac{10}{3} \) the fifth eigenvalue changes sign as a line bifurcation transfers one of the eigenvalues to the opposite value. Thus the Wainwright curves of equilibrium points are saddles in phase space.

4.2. The Wainwright curves of equilibrium points \( (\gamma = \frac{10}{3} \text{ only}) \)

When the equation of state parameter is equal to \( \frac{10}{3} \), then there is a curve of equilibrium points in phase space, given by:

\[
\Sigma_{13} \in \mathbb{R}, \quad 54A^2 - 4 - 81\Sigma_{13}^2 = 0, \quad \bar{N}_{23} = A, \quad \bar{N}_{22} = \bar{N}_{23} = 0. \tag{11}
\]

The energy density along this curve is given by

\[
\Omega = \frac{5 - 81\Sigma_{13}^2}{9}, \tag{12}
\]

and is non-negative iff \( 0 \leq \Sigma_{13}^2 \leq \frac{5}{81} \). We have two finite hyperbolic arcs of Wainwright equilibrium points in the physical region of phase space.

One of the five eigenvalues is zero due to the fact that there is a curve of equilibrium points. The other four are given by \([A \pm \frac{5}{3}\sqrt{441A^2 - 72}], [A \pm \sqrt{1521A^2 - 192}] \). When \( A \) is positive then the real parts of these eigenvalues are \((+ , - , +, + )\) for \( 0 \leq A^2 < \frac{12}{25} \), \((+ , - , + , + )\) for \( \frac{12}{25} = A^2 \), and \((+ , - , + , - )\) for \( \frac{12}{25} < A^2 \leq \frac{1}{5} \). When \( A \) is negative then all these signs switch to the opposite value. Thus the Wainwright curves of equilibrium points are saddles in phase space. The change in sign of one of the eigenvalues arises as there is a saddle-node bifurcation occurring on the one parameter family of invariant two-spaces, see [6].
4.3. The Robinson–Trautman equilibrium points

At each of these equilibrium points, two of the eigenvalues are positive and two of the eigenvalues are negative. The fifth eigenvalue is proportional to the factor \((10 - 9\gamma)\). This is expected as each one of them lies on one of the Wainwright curves of equilibrium points when \(\gamma = \frac{40}{9}\), and so it must have a zero eigenvalue at \(\gamma = \frac{40}{9}\). These equilibrium points are saddles in phase space for all values of \(\gamma\) in the range \((1, \frac{3}{2})\).

4.4. The plane wave equilibrium points

The eigenvalues on the lines of plane wave equilibrium points are:

\[
0, \quad \pm \frac{3}{(\gamma - 1)}[4\Sigma_{23}^2 - Q_2(\gamma)], \quad \pm \frac{3}{2}(2 - \gamma) \pm 8\Sigma_{23}i,
\]

where the leading sign is chosen to be the same sign as the sign of the variable \(A\) at the equilibrium point. The zero eigenvalue arises because the plane waves form a line of equilibrium points. The last eigenvalue determines the stability of the plane waves relative to the invariant set \(\Sigma_{13} = 0\). For values of \(\gamma\) in the interval \((1, \frac{4}{3})\), each plane wave line possesses two segments given by

\[
-\sqrt{Q_1(\gamma)/8} < \Sigma_{23} < -\sqrt{Q_2(\gamma)/4} \quad \text{and} \quad \sqrt{Q_2(\gamma)/4} < \Sigma_{23} < \sqrt{Q_1(\gamma)/8},
\] (13)

on which the (real parts of the) other four eigenvalues have the same sign. For values of \(\gamma\) in the interval \((\frac{4}{3}, \frac{3}{2}]\) each plane wave line possesses a single segment given by

\[
-\sqrt{Q_1(\gamma)/8} < \Sigma_{23} < \sqrt{Q_1(\gamma)/8},
\] (14)

on which the (real parts of the) other four eigenvalues have the same sign.

4.5. Cosmological interpretation

Although it is not currently possible to provide a complete examination of the full five-dimensional system, we can still provide concrete statements about the possible behaviours of some of the trajectories, and their corresponding cosmological models. We have shown that the Collins, Wainwright and Robinson–Trautman equilibrium points are saddle points in phase space, and so, none of these equilibrium points can act as a source or sink for generic trajectories. In particular, the physical features of matter domination together with asymptotic spatial homogeneity at large spatial distance are not generic features of the models, as they can be in some of the subclasses of lower dimension. However, these equilibrium points may act as intermediate asymptotes for typical trajectories, see [6, figures 11–14] for explicit examples.

On the other hand, the plane waves play a much more important role in phase space. Although most of the plane wave equilibrium points are saddles, for all values of the equation of state parameter in the interval \((1, \frac{4}{3})\), there is always at least one segment on the line with positive \(A\) value which is a source, and there is at least one corresponding segment on the line with negative \(A\) value which is a sink. In addition, we are aware of trajectories in the invariant set \(\Sigma_{13} = 0\) which flow from the source to the sink.
It now follows, from the approximation property of orbits [23], that, for all values of the equation of state parameter in the interval \((1, \frac{3}{2})\) there is an open set of trajectories in the five-dimensional phase space, with the property that each one of them flows from one plane wave equilibrium point to another plane wave equilibrium point. Corresponding to each of these trajectories there is a well behaved cosmological model. The cosmological model is asymptotic to a plane wave model at large spatial distance, that is as \(x \to +\infty\) and as \(x \to -\infty\). We expect the asymptote to be a different plane wave model in the two different spatial directions. The cosmological model is vacuum dominated and acceleration dominated, that is to say, the energy density becomes insignificant at large spatial distance, in the \(\tilde{x}\) direction, with the acceleration of the \(e_0\) congruence persisting. There is only a small volume of space in which there is non-insignificant spatial inhomogeneity and energy density. A typical density profile appears in figure 8 of [6].

5. Conclusions

In this work we have considered a class of inhomogeneous cosmological models which possess an Abelian \(G_2\). This class has been ignored to a large extent due to the complication that this Abelian \(G_2\) does not act OT. It is necessary to examine such models in order to gain insight into the properties of the general cosmological solutions of the EFEs.

Due to the complexity of the general class of such models we have imposed two additional restrictions. The first is that the fluid flow is orthogonal to the orbits of the Abelian \(G_2\), a consequence of which is that the fluid is irrotational. The second is that there is an additional symmetry of a homothety aligned in such a way that the fluid flow is contained in the resulting \(H_3\) orbits. A consequence of this is that the EFEs reduce to ODEs describing the spatial structure, while their temporal evolution is determined by the self-similarity. The resulting models are of interest both as examples of inhomogeneous cosmological models which do not have an OT acting \(G_2\), and as possible asymptotic states at late times for non-self-similar models.

We have proved the existence of a well-behaved five dimensional subset of these inhomogeneous self-similar cosmological models. These models possess a big bang singularity at a finite time in the past. They are asymptotically vacuum at large spatial distance where they may be approximated by a plane wave model, however we refer to them as being acceleration dominated since the acceleration does not tend to zero at large spatial distance.

The models considered here may be called \textit{exceptional} in analogy with the exceptional orthogonal Bianchi models and it is useful to compare their similarities and differences.

Similarities:

- Both classes possess an Abelian \(G_2\) which does not act OT leading to off diagonal shear terms.
- The fluid flow is orthogonal to the \(G_2\) orbits and thus it is irrotational.
- The models possess an additional symmetry, either an isometry or a homothety. The EFEs reduce to an ODE, describing evolution in the spatially homogeneous case, and describing spatial structure in the self-similar case.
- In the OT case there are two parameters, the equations of state parameter \(\gamma\) and the group parameter \(h\). In the non OT case these two parameters are restricted: in the spatially homogeneous case we have \(h = -\frac{4}{9}\), and in the self-similar case the restriction is the relation, \(h = \frac{1}{2}\left(\frac{33}{15}\gamma - 2\right)\).
- The class of models is five-dimensional and thus has the same size as the non-exceptional class, which is four-dimensional, with an additional free parameter.
- The following transitively self-similar solutions appear as equilibrium points: the Collins Bianchi VI, solution; the Wainwright Bianchi VI \( \gamma = \frac{10}{9} \) solution; the Robinson–Trautman solution; some spatially homogeneous plane wave solutions.

- Each phase space contains a one-parameter family of equilibrium points which play an important role for the trajectories corresponding to the other models.

Differences:

- For spatially homogeneous models, the effect of more variables is to reduce the region of the Kasner ring which acts as a source. For example in Bianchi I models the whole of the Kasner ring is a source; for the non-exceptional Bianchi VI, models, only a part of the Kasner ring is a source; for the exceptional Bianchi V, models the additional shear variable has such a dramatic destabilising effect on the Kasner ring that no single equilibrium point is a source. All the points on the Kasner ring are saddle points, there are oscillations between these Kasner saddle points at early times and this results in a chaotic initial singularity (see, for example, [9]).

- For the self-similar models, it is the curves of plane wave equilibrium points which are important as asymptotes for the other models, at large distance. The effect of the additional shear variable is to destabilise the plane wave curves, and most of the equilibrium points are saddle points. However, the destabilisation is not exhaustive and there is a small segment which is either a source or sink. Thus chaotic and oscillatory behaviour is not necessarily the fate of these models.

- The plane waves also appear as equilibrium points for the spatially homogeneous models. In the non-exceptional case, plane-wave arcs are late-time asymptotes for generic models for certain ranges of the equation of state parameter. However, in the exceptional case the additional shear variable completely destabilises these arcs and the late time asymptote becomes the Robinson–Trautman equilibrium point. Thus the Robinson–Trautman model plays a very important role as the only possible late-time asymptote for the generic exceptional models. However in the analysis of the self-similar models, the Robinson–Trautman equilibrium point is only a saddle point. It acts as an intermediate spatial asymptote for some trajectories, but most trajectories are unaffected by its existence: it only plays a minor role in phase space.

As far as the chaotic inhomogeneity conjecture is concerned, we have taken a first step in proving this result by providing a class of inhomogeneous cosmological models in which the plane wave curves are partially destabilised. It is likely that by examining a slightly broader class of related models, e.g. by introducing one-dimension of tilt, it will be possible to complete this destabilisation.

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Data availability statement

No new data were created or analysed in this study.
Appendix A. The EFEs for the irrotational N-OT G₂ cosmologies

In this appendix we derive the EFEs for perfect fluid spacetimes with a linear equation of state, and which admit an Abelian two dimensional isometry group with space-like orbits and where the fluid flow is orthogonal to these orbits. We then dimensionalize the equations. The dynamical equilibrium states are defined to be the equilibrium points of the PDE. These equilibrium points correspond to spacetimes which admit, in addition, a homothetic vector field, aligned so that fluid flow is tangential to the resulting $H_3$ orbits. The ODE for the dynamical equilibrium states are also derived here.

When spacetime admits a G₂, with infinitesimal generators $\xi$ and $\eta$, it is possible [24] to select a group-invariant orthonormal frame:

$$[e_\alpha, \xi] = 0, \quad [e_\alpha, \eta] = 0. \quad (A.1)$$

It is also possible to further align the frame so that $e_2$ and $e_3$ are tangential to the group orbits (orbit-aligned), and then the commutation functions ($\gamma_{bc}^a$) of this group-invariant orbit-aligned frame are restricted by

$$e_a(\gamma_{bc}^a) = 0. \quad (A.2)$$

Furthermore, we can, in addition, align the frame so that $e_0 = u$, and refer to the frame as fluid aligned. Any unit vector which is both invariant under the action of the $G_2$, and is orthogonal to the group orbits, is HO [11, p 1134]. That is both $e_0$ and $e_1$ are HO, and the commutation functions are also restricted by:

$$\gamma_{0a}^0 = 0, \quad \gamma_{1a}^0 = 0, \quad \gamma_{0a}^1 = 0, \quad \gamma_{1a}^1 = 0. \quad (A.3)$$

Since we have an Abelian $G_2$, that is $\xi$ and $\eta$ commute, it follows that the two vectors that are aligned with the $G_2$ orbits, $e_2$ and $e_3$, commute:

$$\gamma_{23}^a = 0. \quad (A.4)$$

The remaining non-discrete frame-freedom is a rotation in the $G_2$ orbits. The non-zero commutation functions may be expressed in terms of the kinematic and spatial curvature variables of [8] and are:

$$\theta, \quad \sigma_{ab}, \quad \dot{u}_1, \quad a_1, \quad n_{AB}, \quad \Omega_a, \quad (A.5)$$

where,

$$\Omega_2 = \sigma_{13}, \quad \Omega_3 = -\sigma_{12}. \quad (A.6)$$

At this point we follow [1] and introduce a two-dimensional tensor formalism to distinguish between quantities which transform as scalars, vectors or rank 2 tensors under the rotational freedom remaining. We define:

$$n_+ = \frac{3}{2} n^C, \quad \tilde{n}_{AB} = n_{AB} - \frac{1}{3} n_+ \delta_{AB}, \quad \tilde{n}_{AB}^C = \tilde{n}_{AB} \epsilon_{BC}, \quad (A.7)$$

$$\sigma_+ = \frac{3}{2} \sigma^C, \quad \tilde{\sigma}_{AB} = \sigma_{AB} - \frac{1}{3} \sigma_+ \delta_{AB}, \quad \tilde{\sigma}_{AB}^C = \tilde{\sigma}_{AB} \epsilon_{BC}, \quad (A.8)$$

$$\sigma_{1A} = \varepsilon_{AB} \sigma^B_1 = \Omega_A. \quad (A.9)$$
where $\varepsilon_{AB}$ is the two-dimensional permutation symbol. Under the rotational frame-freedom, the variables $\theta, \sigma_+, u_1,$ and $a_1$ transform as scalars, the variables $\sigma_{1A}$ ($\Omega_1$) transform as the components of a vector and the trace-free symmetric quantities $\tilde{\sigma}_{AB}$ and $\tilde{n}_{AB}$ transform as rank two tensors. The quantities $n_+$ and $\Omega_1$ are non-tensorial and transform according to

$$n_+ = n_+ + 3e_1(\phi), \quad \Omega_1 = \Omega_1 + e_0(\phi),$$  \hspace{1cm} (A.10)

where $\phi$ is the angle of rotation of the frame. A $1 + 1 + 2$ decomposition of the EFEs can now be performed in an analogous manner to that done in [1]. The evolution equation for $u_1$, valid for $\mu \neq 0$, is obtained by applying the commutator $[e_0, e_1]$ to $\mu$.

System of PDEs in the physical variables

Evolution equations:

$$e_0(\theta) = -\frac{1}{3} \theta^2 - \frac{2}{3} \sigma_+^2 - \tilde{\sigma}^{AB} \tilde{\sigma}_{AB} + (\dot{u}_1 - 2a_1) \ddot{u}_1 + e_1(\ddot{u}_1)$$

$$- \frac{1}{2} (3\gamma - 2) \mu - 2\sigma_+ \sigma_+^{LC},$$ \hspace{1cm} (A.11)

$$e_0(\sigma_+) = -\theta \sigma_+ - \tilde{n}^{AB} \tilde{n}_{AB} - (\dot{u}_1 + a_1) \ddot{u}_1 - e_1(\ddot{u}_1) + e_0(a_1) + 3\sigma_+ \sigma_+^{LC},$$ \hspace{1cm} (A.12)

$$e_0(\tilde{\sigma}_{AB}) = -\theta \tilde{\sigma}_{AB} + 2\Omega_1 \tilde{\sigma}_{AB} - \frac{2}{3} n_+ \tilde{n}_{AB} + (2\sigma_1 - \dot{u}_1) \tilde{\sigma}_{AB}$$

$$- e_1(\tilde{n}_{AB}) + 2\sigma_{1A} \tilde{\sigma}_{1B} - \sigma_+ \sigma_+^{IC} \delta_{AB},$$ \hspace{1cm} (A.13)

$$e_0(\sigma_{1A}) = \frac{1}{3} (2\sigma_+ + \theta) a_1 - \frac{1}{3} (\sigma_+ + \theta) \dot{u}_1 - \frac{1}{3} e_1(\sigma_+ + \theta),$$ \hspace{1cm} (A.14)

$$e_0(\dot{n}_+) = \frac{1}{3} (2\sigma_+ - \theta) n_+ + 3\tilde{\sigma}^{AB} \tilde{\sigma}_{AB} + 3\dot{u}_1 \Omega_1 + 3e_1(\Omega_1),$$ \hspace{1cm} (A.15)

$$e_0(\tilde{n}_{AB}) = \frac{1}{3} (2\sigma_+ - \theta) n_+ \tilde{n}_{AB} + 2\Omega_1 \tilde{n}_{AB} + \frac{2}{3} n_+ \tilde{\sigma}_{AB}$$

$$+ \dot{u}_1 \tilde{\sigma}_{AB} + e_1(\tilde{\sigma}_{AB}),$$ \hspace{1cm} (A.16)

$$e_0(\ddot{u}_1) = \frac{1}{3} (2\sigma_+ - \theta) \dot{u}_1 + (\gamma - 1)(\dot{u}_1 + e_1(\theta)),$$ \hspace{1cm} (A.17)

$$e_0(\sigma_{1A}) = - (\theta + \sigma_+) \sigma_+^{1A} + \Omega_1 \sigma_+^{1A} - \tilde{\sigma}_A^{LC} \sigma_1 C.$$ \hspace{1cm} (A.18)

Constraint equations:

$$e_1(\sigma_+ + \theta) = 3a_1 \sigma_+ + \frac{3}{2} \tilde{n}^{AB} \tilde{n}_{AB},$$ \hspace{1cm} (A.19)

$$e_1(\sigma_{1A}) = 3a_1 \sigma_{1A} + \tilde{n}^{B}_{A} \sigma_{1B} + \frac{1}{3} n_+ \sigma_1^{1A}.$$ \hspace{1cm} (A.20)

Defining equation for $\mu$:

$$e_1(a_1) = \frac{3}{2} a_1^2 - \frac{1}{6} \theta^2 + \frac{1}{6} \sigma_+^2 + \frac{1}{4} \tilde{\sigma}^{AB} \tilde{\sigma}_{AB} + \frac{1}{4} \tilde{n}^{AB} \tilde{n}_{AB} + \frac{1}{2} \mu + \frac{1}{2} \sigma_+^{1A} \sigma_1 A.$$ \hspace{1cm} (A.21)
Auxiliary equations:
\[ e_0(\mu) = -\gamma \mu \theta, \quad \text{(A.22)} \]
\[ (\gamma - 1)e_1(\mu) = -\gamma \mu \dot{u}_1. \quad \text{(A.23)} \]

Commutator of \( e_0 \) and \( e_1 \):
\[ [e_0, e_1] = \dot{u}_1 e_0 + \frac{1}{3} (2\sigma_+ - \theta) e_1 - 2\sigma_1 e_2 - 2\sigma_3 e_3. \quad \text{(A.24)} \]

Dimensionless variables are now defined by normalising with the rate of expansion scalar, \( \theta \):
\[ \Sigma_+ = \frac{\sigma_+}{\theta}, \quad \Sigma_{AB} = \frac{\sigma_{AB}}{\theta}, \quad \Sigma_{1A} = \frac{\sigma_{1A}}{\theta}, \quad \dot{U} = \frac{\dot{u}_1}{\theta}, \quad R = \frac{3\Omega_1}{\theta^2}, \quad \text{(A.25)} \]

We must also introduce two dimensionless scalars, \( q \) and \( r \), which are formed from the derivatives of the expansion scalar:
\[ e_0(\theta) = -\frac{1}{3} (1 + q) \theta^2, \quad e_1(\theta) = -\frac{1}{3} r \theta^2, \quad \text{(A.27)} \]
and the dimensionless differential operators:
\[ \partial_a = \frac{3}{\theta} e_a. \quad \text{(A.28)} \]

System of PDEs in dimensionless variables evolution equations
\[ \partial_0(\Sigma_+) = (q - 2)\Sigma_+ - 3\bar{N}_{AB}\bar{N}^{AB} - (3\dot{U} - r + A)\dot{U} \]
\[ - \frac{1}{3} rA - \partial_1(\dot{U}) + \frac{1}{3} \partial_1(A) + 9\Sigma_{1A}\Sigma^{1A}, \quad \text{(A.29)} \]
\[ \partial_0(\bar{N}_{AB}) = (q - 2)\bar{N}_{AB} + 2R^* \bar{N}_{AB} - 2N_+\bar{N}_{AB} - \partial_1(\bar{N}_{AB}) \]
\[ - (3\dot{U} - r - 2A)^* \bar{N}_{AB} + 6\Sigma_{1A}\Sigma^{1A} - 3\Sigma_{1C}\Sigma^{1C} \delta_{AB}, \quad \text{(A.30)} \]
\[ \partial_0(\dot{A}) = (q + 2\Sigma_+)\dot{A} - (3\dot{U} - r)(1 + \Sigma_+) - \partial_1(\Sigma_+), \quad \text{(A.31)} \]
\[ \partial_0(N_+) = (q + 2\Sigma_+)N_+ + 9\bar{N}_{AB}\bar{N}_{AB} + (3\dot{U} - r)R + \partial_1(R), \quad \text{(A.32)} \]
\[ \partial_0(\bar{N}_{1A}) = (q + 2\Sigma_+)\bar{N}_{1A} + 2R^* \bar{N}_{1A} + 2N_+\bar{N}_{1A} \]
\[ + (3U - r)^* \bar{N}_{1A} + \partial_1(\bar{N}_{1A}), \quad \text{(A.33)} \]
\[ \partial_0(\dot{U}) = (q + 2\Sigma_+)\dot{U} + (\gamma - 1)(3\dot{U} - r), \quad \text{(A.34)} \]
\[ \partial_0(\Sigma_{1A}) = (q - 2 - 3\Sigma_+ )\Sigma_{1A} + R^* \Sigma_{1A} - 3\Sigma_{2C} \Sigma^{1C}. \quad \text{(A.35)} \]

Defining equations for \( r, \Omega \) and \( q \):
\[ \partial_1(\Sigma_+) = 3A\Sigma_+ - \frac{9}{2} R^* \bar{N}_{AB} + r(1 + \Sigma_+). \quad \text{(A.36)} \]
\[
\partial_1(A) = \left( \frac{3}{2} A + r \right) A + \frac{9}{4} (\Sigma^{AB} \Sigma_{AB} + \tilde{N}^{AB} \tilde{N}_{AB}) + \frac{3}{2} \Sigma^2_+ - \frac{3}{2} + \frac{3}{2} \Omega + \frac{9}{2} \Sigma_{1A} \Sigma^{1A}, \tag{A.37}
\]
\[
\partial_1(\dot{U}) = -(3\dot{U} - r - 2A)\dot{U} + 2\Sigma^2_+ + 3\Sigma^{AB} \tilde{\Sigma}_{AB} + \frac{1}{2} (3\gamma - 2)\Omega - q + 6\Sigma_{1A} \Sigma^{1A}. \tag{A.38}
\]

Constraint equation:
\[
\partial_1(\Sigma_{1A}) = (3A + r) \Sigma_{1A} + 3^* \tilde{N}_A^B \Sigma_{1B} + N_+ \Sigma_{1A}. \tag{A.39}
\]

Auxiliary equations:
\[
\partial_0(r) - \partial_1(q) = (3\dot{U} - r)(1 + q) + (q + 2\Sigma_+)r, \tag{A.40}
\]
\[
\partial_0(\Omega) = [2q - (3\gamma - 2)]\Omega, \tag{A.41}
\]
\[
(\gamma - 1)\partial_1(\Omega) = [2(\gamma - 1)r - 3\gamma \dot{U}]\Omega. \tag{A.42}
\]

Commutator of \(\partial_0\) and \(\partial_1\):
\[
[\partial_0, \partial_1] = (3\dot{U} - r)\partial_0 + (q + 2\Sigma_+)\partial_1 - 6\Sigma_{12}\partial_2 - 6\Sigma_{13}\partial_3. \tag{A.43}
\]

Decoupled equations:
\[
\partial_0(\theta) = -(1 + q)\theta, \quad \partial_1(\theta) = -r\theta. \tag{A.44}
\]

We consider the vector
\[
X = (\Sigma_+, \Sigma_{AB}, \Sigma_{1A}, \dot{U}) \tag{A.45}
\]
as providing the dynamical state of a model.

It is pertinent at this point to note that were we to restrict the equations by \(\partial_1 X = 0\), then it follows from (A.43) and (A.40)–(A.42) that the corresponding models must satisfy one of the following:

(a) \(\partial_0 X = 0\), in this case the models are irrotational transversely self-similar. These are all well-known exact solutions (see [25]), and they arise as equilibrium points in the models which we examine.

(b) \(\gamma = 2\), in this case the models are spatially self-similar and stiff–these have been discussed at length by various authors (see [26, 27] section 23.1).

(c) \(\Omega = 0\), in this case the models are both spatially self-similar and vacuum.

(d) \(\dot{U} = 0\), in this case the models are orthogonally spatially homogeneous of Bianchi type I–VII. These models are separated into non-exceptional (orthogonally-transitive), and exceptional (non-orthogonally-transitive), according to the vanishing, or non-vanishing, of \(\Sigma_{1A}\). In the latter case it is the constraint equation for \(\Sigma_{1A}\) (A.39) which forces the well known restriction on the group parameter, namely \(h = \frac{1}{\sqrt{q}}\). In the models considered below it is the evolution equation for \(\Sigma_{1A}\) (A.35) which yields the analogous restriction discussed after equation (A.66).

The dynamical equilibrium states of the \(G_2\) cosmologies are defined (see [1]) by imposing the additional condition of
\[
\partial_0 X = 0. \tag{A.46}
\]
It follows from the EFEs that
\[ \partial_0 \partial_1 X = 0, \] (A.47)
amost everywhere. The defining equations for \( r, q \) and \( \Omega \) now yield
\[ \partial_0 r = 0, \quad \partial_0 \Omega = 0, \quad \partial_0 q = 0, \] (A.48)
and the commutator of \( \partial_0 \) and \( \partial_1 \) now implies that
\[ (q + 2\Sigma_+) \partial_1 X = 0, \] (A.49)
and so there are the two possibilities: either
\[ \partial_1 X = 0, \] (A.50)
or \[ \partial_1 X \neq 0, \quad (q + 2\Sigma_+) = 0. \] (A.51)

Models which satisfy the former condition have been discussed above, see the section after equation (A.45). We now consider those models which satisfy the latter conditions, which is our primary intent, these are referred to as the spatially inhomogeneous dynamical equilibrium states of the \( G_2 \) cosmologies. For non-vacuum models we can use the \( \partial_0 \dot{U} \) equation to deduce that
\[ 3\dot{U} - r = 0, \] (A.52)
and the \( \partial_0 A \) equation now yields:
\[ \partial_1 \Sigma_+ = 0. \] (A.53)
Furthermore, the \( \partial_0 \Omega \) equation now implies that:
\[ q = \frac{1}{2}(3\gamma - 2), \quad \Sigma_+ = -\frac{1}{4}(3\gamma - 2). \] (A.54)

The following proposition characterises the spatially inhomogeneous dynamical equilibrium states as being self-similar:

**Proposition 1.** An irrotational N-OTG \( G_2 \) cosmology is a spatially inhomogeneous dynamical equilibrium state if and only if the spacetime is self-similar, admitting a maximal \( H_3 \) acting on the hypersurfaces generated by the two Killing vector fields and the fluid four-velocity.

**Proof.** This result have been proven for the Orthogonally Transitive \( G_2 \) Cosmologies in [1], and the same proof may be used in the non-orthogonally transitive case. \( \square \)

Although vacuum models are not our primary concern, they are of interest both in their own right and due to the fact that they arise as the boundary of the non-vacuum models. For the vacuum models which admit just an Abelian \( G_2 \) isometry group, there does not exist a unique timelike congruence which is orthogonal to the orbits of the \( G_2 \) orbits. Thus multiple representations of these models will occur, unless additional restrictions are imposed. This unavoidable phenomenon occurs for example when one examines tilted spatially homogeneous cosmologies, see, for example, the Kasner cylinder in [28]. However, our primary focus in this paper, and its companion, is on models which admit a \( H_3 \) acting transitively on timelike hypersurfaces. We choose to align our orthonormal frame so that \( e_2 \) and \( e_3 \) are tangential to the orbits of the Abelian \( G_2 \) and with \( e_0 \) tangential to the orbits of the \( H_3 \). When there is a
perfect fluid flowing orthogonal to the orbits of the Abelian $G_2$ and tangential to the orbits of the $H_3$, we can align $e_0$ with $u$. In this case the physical variables are related to the kinematical properties of the fluid through the decomposition of the commutation functions, as performed earlier in this appendix. On the other hand, if we are dealing with vacuum models then this alignment of $e_0$, tangential to the orbits of the $H_3$ and orthogonal to the orbits of the Abelian $G_2$, is still unique (up to a reflection). Hence, we may still examine properties of the vacuum models by making use of the kinematical properties of the $e_0$ congruence. These vacuum models admit a parameter, which is still labelled by $\gamma$, and this parameter is related to the deceleration parameter of the timelike $e_0$ congruence through the expression $q = \frac{1}{2}(3\gamma - 2)$.

The third set of equations which we provide is for the spatially inhomogeneous dynamical equilibrium states of the $G_2$ cosmologies. These equations consist of spatial structure equations, which involve the spatial derivative operator $\partial_1$, and constraint equations.

Spatial structure equations:

$$
\partial_1(\dot{U}) = \frac{1}{3}[A^2 - (1 + \Sigma_+)^2] + \frac{3}{2}\Sigma AB \Sigma AB - \frac{3}{2}\Sigma AB \Sigma AB + \frac{1}{2}\gamma \Omega + 9\Sigma_{1A}\Sigma^{1A},
$$

(A.55)

$$
\partial_1(A) = 2[A^2 - (1 + \Sigma_+)^2] + \frac{3}{2}(2 - \gamma)\Omega + 9\Sigma_{1A}\Sigma^{1A},
$$

(A.56)

$$
\partial_1(\tilde{N}_{AB}) = (q-2)\Sigma AB + 2R^C \Sigma AB - 2N+ \tilde{N}_{AB} + 2A^C \tilde{N}_{AB} + 6\Sigma_{1A}\Sigma_{1B} - 3\Sigma_{1C}\Sigma^{1C}\delta_{AB},
$$

(A.57)

$$
\partial_1(R) = -9\tilde{\Sigma}^{AB}\tilde{N}_{AB},
$$

(A.58)

$$
\partial_1(\ast \tilde{\Sigma}_{AB}) = -2R^C \tilde{N}_{AB} - 2N+ \tilde{\Sigma}_{AB},
$$

(A.59)

$$
\partial_1(\Sigma_{1A}) = 3(A + \dot{U})\Sigma_{1A} + 3^* \tilde{N}^B_A \Sigma_{1B} + N+ \Sigma_{1A}.
$$

(A.60)

Defining equations for $\Omega$:

$$
3(\gamma - 1)\Omega = A(A - 6\dot{U}) - \frac{9}{2}\tilde{N}^{AB}\tilde{N}_{AB} - \frac{9}{2}\tilde{\Sigma}^{AB}\Sigma_{AB} - (1 + \Sigma_+)(1 + 7\Sigma_+) + 9\Sigma_{1A}\Sigma^{1A}.
$$

(A.61)

Constraint equations:

$$
0 = (q - 2 - 3\Sigma_+)\Sigma_{1A} + R^C \Sigma_{1A} - 3\Sigma^{C} A \Sigma_{1C},
$$

(A.62)

$$
0 = A\Sigma_+ - \frac{3}{2}\Sigma^{AB} \Sigma_{AB} + \dot{U}(1 + \Sigma_+).
$$

(A.63)

Auxiliary equations:

$$
(\gamma - 1)\partial_1(\Omega) = 3\dot{U}(\gamma - 2)\Omega.
$$

(A.64)

The system admits a first integral, $k$, given by:

$$
k = \frac{9}{2}\Sigma^{AB} \Sigma_{AB} - R^2.
$$

(A.65)

Constraint equation (A.62) leads to two possibilities. The first of these is that

$$
\Sigma_{12} = \Sigma_{13} = 0,
$$

(A.66)
in which case the $G_2$ acts OT (these models have been considered before [2]). The second possibility is that

$$k = \frac{9}{16} (6 - 5\gamma)^2,$$  \hfill (A.67)

$$0 = \left[ \frac{3}{4} (5\gamma - 6) - 3\Sigma_{22} \right] \Sigma_{12} + (R - 3\Sigma_{23})\Sigma_{13}. \hfill (A.68)$$

We concentrate on this second class of models in this paper. For these models the Abelian $G_2$ does not necessarily act OT, equivalently equation (A.66) does not necessarily hold. Instead there is a restriction on the parameters in the model, namely $k$ and $\gamma$ are related by expression (A.67), and, in addition, the two additional shear variables are related by expression (A.68). Since their EFEs reduce to a five-dimensional ODE with a single parameter ($\gamma$), they are as general as the DES for the OT $G_2$ cosmologies, whose EFEs reduce to a four-dimensional ODE with two parameters ($k$ and $\gamma$) see [2] for more details. There is exactly the same relationship between the exceptional and the non-exceptional orthogonal Bianchi cosmologies, including the origin and imposition of the constraint, which is, of course, $h = -\frac{1}{9}$, in the exceptional case.

The Bianchi type of the $H_3$ is given by the table A1. The value of the group parameter $h$ is related to the first integral $k$ by $h = -\frac{3\Sigma^2}{4}$.

Since we impose the restriction (A.67), and the value of $\Sigma_+$ is given in (A.54), it follows that this exceptional class of models is of Bianchi type VI$^{-1}$ with

$$h = -\left[ \frac{2(3\gamma - 2)}{5\gamma - 6} \right]^2. \hfill (A.69)$$

(n.b. this is the Bianchi type of the $H_3$). Thus, the exceptional class of self-similar models is forced to admit a restriction which connects the equation of state parameter and the Bianchi type of the $H_3$. The existence of such a restriction is an analogous situation that occurs for the exceptional Bianchi models, however in that case the restriction is much simpler ($h = -\frac{1}{9}$). In the paper of [2], studying the OT acting subclass, a parameter $k^*$ was defined as

$$k^* = -(1 + \Sigma_+)(1 + 7\Sigma_+) = \frac{9}{16} (2 - \gamma)(7\gamma - 6), \hfill (A.70)$$

and it was explained that well behaved models arise when the combination $k - k^*$ is non-negative. We note that this combination takes the following value here,

$$k^* - k = 9(\gamma - 1)(3 - 2\gamma), \hfill (A.71)$$

and we assume that this quantity is non-negative throughout, leading to $1 \leq \gamma \leq \frac{3}{2}$.

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**Table A1. The Bianchi type of the models.**

| \(\Sigma_+\) | I | II | V| VI | VII |
|--------------|---|----|---|----|-----|
| \(\Sigma_{AB}\) | 0 | 0 | 0 | 0 | 0 |
| \(\Sigma_{AB}\) | 0 | 0 | 0 | 0 | 0 |
| \(R\) | 0 | 0 | < 0 | > 0 | 0 |
| \(k\) | 0 | 0 | < 0 | > 0 | 0 |

*It follows from (A.54) that \(\Sigma_+ = 0\) iff \(\gamma = \frac{2}{3}\).*
We conclude with the observation that there are no irrotational dust N-OT $G_2$ cosmologies which admit a $H_3$ acting transitively on the timelike three spaces spanned by an Abelian $G_2$ and fluid four velocity. This may be proven by combining equations (A.54), (A.55), (A.61), (A.65) and (A.67) when $γ = 1$, and $U = 0$, and deducing that $3Ω + 4R^2 + 36Σ_{iA}Σ^{iA} = 0$. Thus, we have $1 < γ ≤ \frac{3}{2}$, in this work.

**Appendix B. Local co-ordinates**

The formulation of the field equations in appendix A is independent of a choice of a co-ordinates system. In appendix B we introduce local co-ordinates and link them with a class of group invariant orthonormal frames. In particular we show that there exists a scalar field, $x$, which labels the orbits of the $H_3$, and that there exists a cosmological time, $t$, with the models possessing a big-bang singularity at $t = 0$.

Since we use dimensionless variables in our formulation, the time variable does not play a major role in this paper, however the scalar field $x$ (or a function thereof) is the independent variable in the EFEs.

In a $G_2$ cosmology, the two KVFs and the fluid four-velocity generate a family of timelike hypersurfaces. Proposition 2 guides us on the introduction of co-ordinates which are adapted to these hypersurfaces when these hypersurfaces admit a three-dimensional similarity group.

**Proposition 2.** A $G_2$ cosmology admits a three-dimensional similarity group, $H_3$, acting transitively if there exist scalar fields $x$ and $t$, functions $N_3(x)$ and $M_3(x)$, and an orthonormal frame $\{e_a\}$ such that:

(a) $[e_a, e_b] = F_{ab}^c(x)Γ^{-1} e_c$.

(b) $e_a(x) = N_3(x), e_a(t) = M_3(x)Γ^{-1}$.

If, in addition, the $H_3$ has generators $\{η, ξ, H\}$, where $η$ and $ξ$ are KVFs then

(a) $η(x) = ξ(x) = H(x) = 0, η(t) = ξ(t) = 0$, and $H(t) = k t$, with $k$ a non-zero constant.

(b) $e_a, η] = 0, [e_a, ξ] = 0, [e_a, H] = k e_a$.

The freedom in the scalar fields is:

$$x \rightarrow f_1(x), \quad t \rightarrow f_2(x)t,$$

for suitable functions $f_1$ and $f_2$, and the orthonormal frame is unique up to an $x$-dependent Lorentz transformation.

For a proof of this result see, for example [29, appendix 1].

Clearly, from (a), the orbits of the $H_3$ are labelled by the scalar field $x$, so that $x = c$, a constant, selects a particular orbit of the $H_3$, and choosing a different constant, results in the selection of a different orbit of the $H_3$.

It follows from (b) that if we align the orthonormal frame so that $e_3$ and $e_1$ are tangential to the orbits of the Abelian $G_2$, with $e_0 = u$, at one point in an $H_3$ orbit, then this will hold at all points in that $H_3$ orbit. And since we have the freedom of an $x$-dependent Lorentz transformation, we can thus arrange for $e_3$ and $e_1$ to be tangential to the orbits of the Abelian $G_2$, with $e_0 = u$, everywhere.

Once these alignments have been performed we have:

$$e_a(x) = δ_a^3 N(x), \quad e_a(t) = 0, \quad A = 2, 3.$$  \hspace{1cm} (B.2)

Furthermore, it is possible to align $e_3$ with the KV $\eta$. Once again, if we do this at one point in an $H_3$ orbit then it follows from (b), (c) and (d) that this will hold at all points in that $H_3$.
orbit. This latter alignment restricts the commutation functions as follows:

\[ R = 3\tilde{\Sigma}_{23}, \quad \text{and} \quad N_+ = 3\tilde{N}_{22}, \quad (B.3) \]

and the first integral of the EFEs (A.65) now reads: \( k = 9\tilde{\Sigma}_{22}^2 = \frac{9}{16}(6 - 5\gamma)^2 \).

The operator \( e_1 \) is a spatial differential operator: it determines changes in any variable as the scalar \( x \) changes. The operator \( e_0 \) is a differential operator along the fluid: it determines changes in any variable as one moves along the fluid, and, equivalently as the scalar \( t \) changes. We can interpret \( t \) as a time variable.

From the form of the commutation functions, \( \gamma_{\alpha \beta}^c = F_{\alpha \beta}(x)t^{-1} \), it is clear that these self-similar cosmological models possess a big-bang singularity at \( t = 0 \), when all the non-zero physical variables diverge. Due to this power law dependence of the physical variables in the scalar \( t \), the evolution of these models back in time towards this initial singularity is rather straightforward, although that does not mean that there could not be some interesting structure near the initial singularity (see, for example, [30]). However, there cannot be any unusual behaviour arising from evolution alone such as oscillatory temporal behaviour (see, for example, [9]) or spikes (see, for example, [31]).

Furthermore, once we know the spatial structure of one of these models on any spatial slice, \( t = c > 0 \) (with \( c \) constant) then the spatial structure on any other slice, \( t = d > 0 \) (with \( d \neq c \) constant) is almost identical due to the self similarity: the physical variables are just scaled by the appropriate factor.

One goal of this paper, and its companion, is to examine that spatial structure of these models on one spatial slice, \( t = c > 0 \), and thus unravel the spatial structure of these models on every spatial slice, \( t = d > 0 \).

We may also introduce scalar fields \( y \) and \( z \) so that the generators of the Abelian \( G_2 \) are:

\[ \eta = \frac{\partial}{\partial y}, \quad \text{and} \quad \xi = \frac{\partial}{\partial z}, \quad (B.4) \]

and these equations are preserved under the co-ordinate changes:

\[ y \to y + g_1(x, t), \quad z \to z + g_2(x, t)t, \quad (B.5) \]

for suitable functions \( g_1 \) and \( g_2 \).

It is possible to make use of this co-ordinate freedom in \( y \) and \( z \), to arrange for \( e_0 \) to not contain terms in either \( \frac{\partial}{\partial y} \) or \( \frac{\partial}{\partial z} \):

\[ e_0 = N(x)\frac{\partial}{\partial t}. \quad (B.6) \]

It is also possible to make use of the co-ordinate freedom (B.1) in \( t \) to arrange \( e_1 \) to not contain a term in \( \frac{\partial}{\partial t} \):

\[ e_1 = M(x)\frac{\partial}{\partial x} + P(t, x)\frac{\partial}{\partial y} + Q(t, x)\frac{\partial}{\partial z}, \quad (B.7) \]

for some functions \( P(t, x) \) and \( Q(t, x) \). We include the form of \( e_2 \) and \( e_3 \) here for completeness:

\[ e_2 = A(t, x)\frac{\partial}{\partial y}, \quad \text{and} \quad e_3 = B(t, x)\frac{\partial}{\partial y} + C(t, x)\frac{\partial}{\partial z}, \quad (B.8) \]

for some functions \( A(t, x) \), \( B(t, x) \), \( C(t, x) \). These choices are often made for the canonical form of the line element for the irrotational \( G_2 \) cosmologies (see for example, [11] theorem 3.1) and equation (3) in the companion paper [6].
We will now make use of the generic form of the commutation functions to fine-tune the functions, $A(t, x), B(t, x), C(t, x), P(t, x)$ and $Q(t, x)$ appearing in the orthonormal frame. Taking the commutator of $e_0$ with each one of $e_1, e_2, e_3$, in turn, and insisting that the commutation functions satisfy $\gamma'_{ab} = F^c_{\, ab}(x)e^{-\tau}$, yields the restrictions:

$$e_1 = M(x)\tau^{-1} \frac{\partial}{\partial x} + P(x)\tau^{\rho} \frac{\partial}{\partial y} + Q(x)\tau^{\rho} \frac{\partial}{\partial z}, \quad (B.9)$$

$$e_2 = A(x)\tau^{\rho} \frac{\partial}{\partial y}, \quad (B.10)$$

and

$$e_3 = B(x)\tau^{\rho} \frac{\partial}{\partial y} + C(x)\tau^{\rho} \frac{\partial}{\partial z}, \quad (B.11)$$

for some functions $P(x), Q(x), A(x), B(x)$ and $C(x)$, and constants $a$ and $c$.

We will be examining the spatial structure of the EFEs in the bulk of the paper, however we may determine the constants $a$ and $c$ from some of the equations in appendix A.

We make a final rotational choice of orthonormal frame so that the variable $\Sigma_{12}$ is zero in a single $H_3$ group orbit $\bar{x} = \text{constant}$. It follows from the EFE (A.60) with $\lambda = 2$ that $\Sigma_{12}$ is zero everywhere. It also follows from the constraint equation (A.63) that $\Sigma_{12} = \frac{1}{2}(6 - 5\gamma)$. A comparison of this expression with the commutators and the expression $\Sigma_{12} = \frac{1}{2}(3\gamma - 2)$, lead to the equations:

$$a = \frac{3\gamma - 4}{\gamma}, \quad c = \frac{-2(\gamma - 1)}{\gamma}, \quad (B.12)$$

We also make the following c-ordinate choices:

(a) Introduce a logarithmic time $\tau$, $t = e^{\tau}$, so the big bang is approached as $\tau \to -\infty$, and $\gamma^a_{\, ab} = F^c_{\, ab}(x)e^{-\tau}$. This is usually done when examining spatially homogeneous models, see [9] for example.

(b) Introduce a new spatial variable $\bar{x}$, defined by $\frac{dx}{dt} = \frac{\bar{x}}{M(x)}$, so that $e_0$ and $e_1$ have a common leading factor of $K(x)e^{-\tau} = N(x(\bar{x}))e^{-\tau}$,

$$e_0 = K(\bar{x})e^{-\tau} \frac{\partial}{\partial \tau}, \quad (B.13)$$

$$e_1 = K(\bar{x})e^{-\tau} \frac{\partial}{\partial \bar{x}} + P(x(\bar{x}))e^{\frac{4(\gamma - 1)}{\gamma}} \frac{\partial}{\partial y} + Q(x(\bar{x}))e^{\frac{4(\gamma - 1)}{\gamma}} \frac{\partial}{\partial z}. \quad (B.14)$$

The expansion scalar of the fluid congruence is $\theta = K(\bar{x})(1 - a - c)e^{-\tau} = \frac{1}{\gamma}K(\bar{x})e^{-\tau}$, and so we now have that:

$$\partial_0 = \frac{3e_0}{\theta} = \frac{3\gamma}{2} \frac{\partial}{\partial \tau}, \quad (B.15)$$

$$\partial_1 = \frac{3e_1}{\theta} = \frac{3\gamma}{2} \frac{\partial}{\partial \bar{x}} + P(\bar{x})e^{\frac{4(\gamma - 1)}{\gamma}} \frac{\partial}{\partial y} + Q(\bar{x})e^{\frac{4(\gamma - 1)}{\gamma}} \frac{\partial}{\partial z}. \quad (B.16)$$

with $\bar{P}(\bar{x}) = \frac{3}{2} \frac{P(x(\bar{x}))}{K(\bar{x})}$ and $\bar{Q}(\bar{x}) = \frac{3}{2} \frac{Q(x(\bar{x}))}{K(\bar{x})}$, and the dimensionless commutation functions have the form:

$$\Gamma^c_{\, ab} = \frac{\gamma^c_{\, ab}}{\theta} = \frac{\gamma}{2} \frac{F^c_{\, ab}(x(\bar{x}))}{K(\bar{x})}. \quad (B.17)$$
Since $\Gamma^c_{ab} = \Gamma^c_{ab}(\tilde{x})$, it now follows that
$$\partial_1(\Gamma^c_{ab}) = \frac{3\gamma d}{d\tilde{x}}(\Gamma^c_{ab}(\tilde{x})),$$
so that the EFEs are a system of ODEs which determine how the dimensionless dynamical variables, obtained by scaling the basic physical variables with the expansion scalar, change as we move in space between the different orbits of the similarity group, as determined by the spatial variable $\tilde{x}$.

Remark: in this work it is assumed the Abelian $G_2$ does not necessarily act O'T, and so, once we have aligned $e_2$ and $e_3$ with the orbits of the Abelian $G_2$, the two vector fields $e_0$ and $e_1$ do not form a closed Lie algebra. As a consequence, the two dimensionless operators $\partial_0$ and $\partial_1$ do not commute, and thus we cannot obtain local co-ordinates, $\tau$ and $\tilde{x}$ so that $\partial_0 = \frac{\partial}{\partial \tau}$ and $\partial_1 = \frac{\partial}{\partial \tilde{x}}$.

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