On Pointed Hopf Algebras with Weyl Groups of Exceptional Type

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Abstract

All \(-1\)-type pointed Hopf algebras and central quantum linear spaces with Weyl groups of exceptional type are found. It is proved that every non \(-1\)-type pointed Hopf algebra with real \(G(H)\) is infinite dimensional and every central quantum linear space over finite group is finite dimensional. It is proved that except a few cases Nichols algebras of reducible Yetter-Drinfeld modules over Weyl groups of exceptional type are infinite dimensional.

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0 Introduction

This article is to contribute to the classification of finite-dimensional complex pointed Hopf algebras \(H\) with Weyl groups of exceptional type. The classification of finite dimensional pointed Hopf algebra with finite abelian groups has been completed (see \(\text{[AS98, AS02, AS00, AS05, He06]}\)). Papers \(\text{[AG03, Gr00, AZ07, Fa07, AF06, AF07]}\) considered some non-abelian cases, for example, symmetric group, dihedral group, alternating group and the Mathieu simple groups. It was shown in \(\text{[HS]}\) that every Nichols algebra of reducible Yetter-Drinfeld module over non-commutative finite simple group and symmetric group is infinite dimensional.

In this paper we find all \(-1\)-type pointed Hopf algebras and quantum linear spaces with Weyl groups of exceptional type. We show that every non \(-1\)-type pointed Hopf algebra is infinite dimensional and every quantum linear space is finite dimensional. It is desirable to do this in view of the importance of Weyl groups in the theories of Lie groups,
Lie algebras and algebraic groups. We first give the relation between the bi-one Nichols algebra $\mathcal{B}(\mathcal{O}_s, \rho)$ introduced in [Gr00, AZ07, AHS08, AFZ] and the arrow Nichols algebra introduced in [CR97, CR02, ZZC, ZCZ]. [ZWCYa, ZWCYb] applied the software GAP to compute the representatives of conjugacy classes, centralizers of these representatives and character tables of these centralizers in Weyl groups of exceptional type. Using the results in [ZWCYa, ZWCYb] and the classification theorem of quiver Hopf algebras and Nichols algebras in [ZCZ, Theorem 1] we find all $-1$-type pointed Hopf algebras and quantum linear spaces with Weyl groups of exceptional type. We prove that Nichols algebras of reducible Yetter-Drinfeld modules over Weyl groups of exceptional type are infinite dimensional except a few cases by applying [HS, Theorem 8.2, 8.6].

This paper is organized as follows. In section 1 it is shown that bi-one arrow Nichols algebras and $\mathcal{B}(\mathcal{O}_s, \rho)$ introduced in [DPR, Gr00, AZ07, AHS08, AFZ] are the same up to isomorphisms. In section 2 it is proved that every non $-1$-type pointed Hopf algebra with real $G(H)$ is infinite dimensional. In section 3 it is shown that every central quantum linear space is finite dimensional with an arrow PBW basis. In section 4 the programs to compute the representatives of conjugacy classes, centralizers of these representatives and character tables of these centralizers in Weyl groups of exceptional type are given. In section 5 all $-1$-type bi-one Nichols algebras over Weyl groups of exceptional type up to graded pull-push YD Hopf algebra isomorphisms are listed in tables. In section 6 all $-1$-type bi-one Nichols algebras over Weyl groups of exceptional type up to graded pull-push YD Hopf algebra isomorphisms are listed in tables. In section 7 all central quantum linear spaces over Weyl groups of exceptional type are found. In section 8 it is proved that except a few cases Nichols algebras of reducible Yetter-Drinfeld modules over Weyl groups of exceptional type are infinite dimensional.

### Preliminaries And Conventions

Throughout this paper let $k$ be the complex field; $G$ be a finite group; $\hat{G}$ denote the set of all isomorphic classes of irreducible representations of group $G$; $G^s$ denote the centralizer of $s$; $Z(G)$ denote the center of $G$. For $h \in G$ and an isomorphism $\phi$ from $G$ to $G'$, define a map $\phi_h$ from $G$ to $G'$ by sending $x$ to $\phi(h^{-1}xh)$ for any $x \in G$. Let $s^G$ or $\mathcal{O}_s$ denote the conjugacy class containing $s$ in $G$. The Weyl groups of $E_6$, $E_7$, $E_8$, $F_4$ and $G_2$ are called Weyl groups of exceptional type. Let $\deg \rho$ denote the dimension of the representation space $V$ for a representation $(V, \rho)$.

Let $\mathbb{N}$ and $\mathbb{Z}$ denote the sets of all positive integers and all integers, respectively. For a set $X$, we denote by $|X|$ the number of elements in $X$. If $X = \oplus_{i \in I} X_{(i)}$ as vector spaces, then we denote by $\iota_i$ the natural injection from $X_{(i)}$ to $X$ and by $\pi_i$ the corresponding projection from $X$ to $X_{(i)}$. We will use $\mu$ to denote the multiplication of an algebra and
use $\Delta$ to denote the comultiplication of a coalgebra. For a (left or right) module and a (left or right) comodule, denote by $\alpha^-$, $\alpha^+$, $\delta^-$ and $\delta^+$ the left module, right module, left comodule and right comodule structure maps, respectively. The Sweedler’s sigma notations for coalgebras and comodules are $\Delta(x) = \sum x_1 \otimes x_2$, $\delta^-(x) = \sum x_{(-1)} \otimes x_{(0)}$, $\delta^+(x) = \sum x_{(0)} \otimes x_{(1)}$.

A quiver $Q = (Q_0, Q_1, s, t)$ is an oriented graph, where $Q_0$ and $Q_1$ are the sets of vertices and arrows, respectively; $s$ and $t$ are two maps from $Q_1$ to $Q_0$. For any arrow $a \in Q_1$, $s(a)$ and $t(a)$ are called its start vertex and end vertex, respectively, and $a$ is called an arrow from $s(a)$ to $t(a)$. For any $n \geq 0$, an $n$-path or a path of length $n$ in the quiver $Q$ is an ordered sequence of arrows $p = a_n a_{n-1} \cdots a_1$ with $t(a_i) = s(a_{i+1})$ for all $1 \leq i \leq n - 1$. Note that a 0-path is exactly a vertex and a 1-path is exactly an arrow. In this case, we define $s(p) = s(a_1)$, the start vertex of $p$, and $t(p) = t(a_n)$, the end vertex of $p$. For a 0-path $x$, we have $s(x) = t(x) = x$. Let $Q_n$ be the set of $n$-paths. Let $yQ_n^x$ denote the set of all $n$-paths from $x$ to $y$, $x, y \in Q_0$. That is, $yQ_n^x = \{ p \in Q_n \mid s(p) = x, t(p) = y \}$.

A quiver $Q$ is finite if $Q_0$ and $Q_1$ are finite sets. A quiver $Q$ is locally finite if $yQ_1^x$ is a finite set for any $x, y \in Q_0$.

Let $K(G)$ denote the set of conjugate classes in $G$. A formal sum $r = \sum_{C \in K(G)} r_C C$ of conjugate classes of $G$ with cardinal number coefficients is called a ramification (or ramification data) of $G$, i.e. for any $C \in K(G)$, $r_C$ is a cardinal number. In particular, a formal sum $r = \sum_{C \in K(G)} r_C C$ of conjugate classes of $G$ with non-negative integer coefficients is a ramification of $G$.

For any ramification $r$ and $C \in K(G)$, since $r_C$ is a cardinal number, we can choose a set $I_C(r)$ such that its cardinal number is $r_C$ without loss of generality. Let $K_r(G) := \{ C \in K(G) \mid r_C \neq 0 \} = \{ C \in K(G) \mid I_C(r) \neq \emptyset \}$. If there exists a ramification $r$ of $G$ such that the cardinal number of $yQ_1^x$ is equal to $r_C$ for any $x, y \in G$ with $x^{-1}y \in C \in K(G)$, then $Q$ is called a Hopf quiver with respect to the ramification data $r$. In this case, there is a bijection from $I_C(r)$ to $yQ_1^x$, and hence we write $yQ_1^x = \{ a^{(i)}_{y,x} \mid i \in I_C(r) \}$ for any $x, y \in G$ with $x^{-1}y \in C \in K(G)$.

$(G, r, \overrightarrow{\rho}, u)$ is called a ramification system with irreducible representations (or RSR in short), if $r$ is a ramification of $G$; $u$ is a map from $K(G)$ to $G$ with $u(C) \in C$ for any $C \in K(G)$; $I_C(r, u)$ and $J_C(i)$ are sets with $| I_C(r, u) | = \deg(\rho^{(i)}_C)$ and $I_C(r) = \{(i, j) \mid i \in I_C(r, u), j \in J_C(i) \}$ for any $C \in K_r(G)$, $i \in I_C(r, u)$; $\overrightarrow{\rho} = \{ \rho^{(i)}_{C(u)} \}_{i \in I_C(r, u), C(u) \in K_r(G)} \in \prod_{C \in K_r(G)} G(u(C))[I_C(r, u)]$ with $\rho^{(i)}_C \in G(u(C))$ for any $i \in I_C(r, u), C \in K_r(G)$. In this paper we always assume that $I_C(r, u)$ is a finite set for any $C \in K_r(G)$. Furthermore, if $\rho^{(i)}_C$ is a one dimensional representation for any $C \in K_r(G)$, then $(G, r, \overrightarrow{\rho}, u)$ is called a ramification system with characters (or RSC $(G, r, \overrightarrow{\rho}, u)$ in short) (see [ZZC] Definition 1.8). In this case, $a^{(i,j)}_{y,x}$ is written as $a^{(i,j)}_{y,x}$ in short since $J_C(i)$ has only one element.

For RSR$(G, r, \overrightarrow{\rho}, u)$, let $\chi^{(i)}_C$ denote the character of $\rho^{(i)}_C$ for any $i \in I_C(r, u), C \in K_r(G)$. For any $(G, r, \overrightarrow{\rho}, u)$, let $\chi^{(i)}_C = \sum_{y \in G} a^{(i,j)}_{y,x} \chi_y^{(i)}(u(C)) = \sum_{y \in G} a^{(i,j)}_{y,x} \chi^{(i)}_{u(C)}(y)$.
\( K_r(C) \). If ramification \( r = r_C C \) and \( I_C(r,u) = \{ i \} \) then we say that \( \text{RSR}(G,r,\rightarrow \rho, u) \) is bi-one, written as \( \text{RSR}(G,\mathcal{O}_s,\rho) \) with \( s = u(C) \) and \( \rho = \rho_C^{(i)} \) in short, since \( r \) only has one conjugacy class \( C \) and \(| I_C(r,u) | = 1 \). Quiver Hopf algebras, Nichols algebras and Yetter-Drinfeld modules, corresponding to a bi-one \( \text{RSR}(G,r,\rightarrow \rho, u) \), are said to be bi-one.

If \((G,r,\rightarrow \rho, u)\) is an RSR, then it is clear that \( \text{RSR}(G,\mathcal{O}_{u(C)},\rho_C^{(i)}) \) is bi-one for any \( C \in K \) and \( i \in I_C(r,u) \), which is called a bi-one sub-RSR of \( \text{RSR}(G,r,\rightarrow \rho, u) \).

If \( \phi : A \to A' \) is an algebra homomorphism and \((M,\alpha^-)\) is a left \( A' \)-module, then \( M \) becomes a left \( A \)-module with the \( A \)-action given by \( a \cdot x = \phi(a) \cdot x \) for any \( a \in A \), \( x \in M \), called a pullback \( A \)-module through \( \phi \), written as \( \phi M \). Dually, if \( \phi : C \to C' \) is a coalgebra homomorphism and \((M,\delta^-)\) is a left \( C' \)-comodule, then \( M \) is a left \( C \)-comodule with the \( C' \)-comodule structure given by \( \delta'^- := (\phi \otimes \text{id})\delta^- \), called a push-out \( C' \)-comodule through \( \phi \), written as \( \phi M \).

If \( B \) is a Hopf algebra and \( M \) is a \( B \)-Hopf bimodule, then we say that \((B,M)\) is a Hopf bimodule. For any two Hopf bimodules \((B,M)\) and \((B',M')\), if \( \phi \) is a Hopf algebra homomorphism from \( B \) to \( B' \) and \( \psi \) is simultaneously a \( B \)-bimodule homomorphism from \( M \) to \( \phi M' \) and a \( B' \)-bimodule homomorphism from \( \phi M \) to \( M' \), then \((\phi,\psi)\) is called a pull-push Hopf bimodule homomorphism. Similarly, we say that \((B,M)\) and \((B,X)\) are a Yetter-Drinfeld (YD) module and YD Hopf algebra, respectively, if \( M \) is a YD \( B \)-module and \( X \) is a braided Hopf algebra in YD category \( B^B \text{YD} \). For any two YD modules \((B,M)\) and \((B',M')\), if \( \phi \) is a Hopf algebra homomorphism from \( B \) to \( B' \), and \( \psi \) is simultaneously a left \( B \)-module homomorphism from \( M \) to \( \phi M' \) and a left \( B' \)-module homomorphism from \( \phi M \) to \( M' \), then \((\phi,\psi)\) is called a pull-push YD module homomorphism. For any two YD Hopf algebras \((B,X)\) and \((B',X')\), if \( \phi \) is a Hopf algebra homomorphism from \( B \) to \( B' \), \( \psi \) is simultaneously a left \( B \)-module homomorphism from \( X \) to \( \phi X' \) and a left \( B' \)-module homomorphism from \( \phi X \) to \( X' \), meantime, \( \psi \) also is algebra and coalgebra homomorphism from \( X \) to \( X' \), then \((\phi,\psi)\) is called a pull-push YD Hopf algebra homomorphism (see \cite{ZLC}, the remark after Th.4).

For \( s \in G \) and \((\rho,V) \in \hat{G}^s \), here is a precise description of the YD module \( M(\mathcal{O}_s,\rho) \), introduced in \cite{Gr00,AZ07}. Let \( t_1 = s, \ldots, t_m \) be a numeration of \( \mathcal{O}_s \), which is a conjugacy class containing \( s \), and let \( g_i \in G \) such that \( g_i \triangleright s := g_i s g_i^{-1} = t_i \) for all \( 1 \leq i \leq m \). Then \( M(\mathcal{O}_s,\rho) = \bigoplus_{1 \leq i \leq m} g_i \otimes V \). Let \( g_i v := g_i \otimes v \in M(\mathcal{O}_s,\rho) \), \( 1 \leq i \leq m \), \( v \in V \). If \( v \in V \) and \( 1 \leq i \leq m \), then the action of \( h \in G \) and the coaction are given by

\[
\delta(g_i v) = t_i \otimes g_i v, \quad h \cdot (g_i v) = g_j (\gamma \cdot v), \tag{0.1}
\]

where \( h g_i = g_j \gamma \), for some \( 1 \leq j \leq m \) and \( \gamma \in G^s \). The explicit formula for the braiding is then given by

\[
c(g_i v \otimes g_j w) = t_i \cdot (g_j w) \otimes g_i v = g_j (\gamma \cdot v) \otimes g_i v \tag{0.2}
\]
for any $1 \leq i, j \leq m, v, w \in V$, where $t_i g_j = g_j \gamma$ for unique $j'$, $1 \leq j' \leq m$ and $\gamma \in G^s$. Let $\mathfrak{B}(O_s, \rho)$ denote $\mathfrak{B}(M(O_s, \rho))$. $M(O_s, \rho)$ is a simple YD module (see \cite[Section 1.2]{AZ07}). Furthermore, if $\chi$ is the character of $\rho$, then we also denote $\mathfrak{B}(O_s, \rho)$ by $\mathfrak{B}(O_s, \chi)$.

1 Relation between bi-one arrow Nichols algebras and $\mathfrak{B}(O_s, \rho)$

In this section it is shown that bi-one arrow Nichols algebras and $\mathfrak{B}(O_s, \rho)$ introduced in \cite{Gr00, AZ07, AHS08, AFZ} are the same up to isomorphisms.

For any RSR($G, r, \overrightarrow{\rho}, u$), we can construct an arrow Nichols algebra $\mathfrak{B}(kQ_1^1, ad(G, r, \overrightarrow{\rho}, u))$ (see \cite[Prop. 2.4]{ZCZ}), written as $\mathfrak{B}(G, r, \overrightarrow{\rho}, u)$ in short. Let us recall the precise description of arrow YD module. For an RSR($G, r, \overrightarrow{\rho}, u$) and a $kG$-Hopf bimodule $(kQ_1^1, G, r, \overrightarrow{\rho}, u)$ with the module operations $\alpha^{-}$ and $\alpha^{+}$, define a new left $kG$-action on $kQ_1$ by

$$g \triangleright x := g \cdot x \cdot g^{-1}, \quad g \in G, x \in kQ_1,$$

where $g \cdot x = \alpha^{-}(g \otimes x)$ and $x \cdot g = \alpha^{+}(x \otimes g)$ for any $g \in G$ and $x \in kQ_1$. With this left $kG$-action and the original left (arrow) $kG$-coaction $\delta^{-}$, $kQ_1$ is a Yetter-Drinfeld $kG$-module. Let $Q_1^1 := \{a \in Q_1 | s(a) = 1\}$, the set of all arrows with starting vertex 1. It is clear that $kQ_1^1$ is a Yetter-Drinfeld $kG$-submodule of $kQ_1$, denoted by $(kQ_1^1, ad(G, r, \overrightarrow{\rho}, u))$, called the arrow YD module.

**Lemma 1.1.** For any $s \in G$ and $\rho \in \hat{G}^s$, there exists a bi-one arrow Nichols algebra $\mathfrak{B}(G, r, \overrightarrow{\rho}, u)$ such that

$$\mathfrak{B}(O_s, \rho) \cong \mathfrak{B}(G, r, \overrightarrow{\rho}, u)$$

as graded braided Hopf algebras in $kG^G \mathcal{YD}$.

**Proof.** Assume that $V$ is the representation space of $\rho$ with $\rho(g)(v) = g \cdot v$ for any $g \in G, v \in V$. Let $C = O_s, r = r_C C$, $r_C = \deg \rho, u(C) = s$, $I_C(r, u) = \{1\}$ and $(v)\rho_C^{(1)}(h) = \rho(h^{-1})(v)$ for any $h \in G, v \in V$. We get a bi-one arrow Nichols algebra $\mathfrak{B}(G, r, \overrightarrow{\rho}, u)$.

We now only need to show that $M(O_s, \rho) \cong (kQ_1^1, ad(G, r, \overrightarrow{\rho}, u))$ in $kG^G \mathcal{YD}$. We recall the notation in \cite[Proposition 1.2]{ZCZ}. Assume $J_C(1) = \{1, 2, \cdots, n\}$ and $X_C^{(1)} = V$ with basis $\{x_C^{(1,j)} | j = 1, 2, \cdots, n\}$ without loss of generality. Let $v_j$ denote $x_C^{(1,j)}$ for convenience. In fact, the left and right coset decompositions of $G^s$ in $G$ are

$$G = \bigcup_{i=1}^{m} g_i G^s \quad \text{and} \quad G = \bigcup_{i=1}^{m} G^s g_i^{-1}, \quad (1.1)$$

respectively.
Let \( \psi \) be a map from \( M(O_s, \rho) \) to \( (kQ^1, \text{ad}(G, r, \frac{\delta_i}{\rho}, u)) \) by sending \( g_i v_j \) to \( a^{(1,j)}_{t_i,1} \) for any \( 1 \leq i \leq m, 1 \leq j \leq n \). Since the dimension is \( mn \), \( \psi \) is a bijective. See

\[
\delta^-(\psi(g_i v_j)) = \delta^-(a^{(1,j)}_{t_i,1}) = t_i \otimes a^{(1,j)}_{t_i,1} = (id \otimes \psi)\delta^-(g_i v_j).
\]

Thus \( \psi \) is a \( kG \)-comodule homomorphism. For any \( h \in G \), assume \( h g_i = g_{r} \gamma \) with \( \gamma \in G^s \). Thus \( g_i^{-1} h^{-1} = \gamma^{-1} g_{r}^{-1} \), i.e. \( \zeta_i(h^{-1}) = \gamma^{-1} \), where \( \zeta_i \) was defined in [ZZC (0.3)]. Since \( \gamma \cdot x^{(1,j)} \in V \), there exist \( k^{(1,j,p)}_{C,h^{-1}} \in k, 1 \leq p \leq n \), such that \( \gamma \cdot x^{(1,j)} = \sum_{p=1}^{n} k^{(1,j,p)}_{C,h^{-1}} x^{(1,p)} \).

Therefore \( x^{(1,j)} \cdot \zeta_i(h^{-1}) = \gamma \cdot x^{(1,j)} \) (by definition of \( \rho^{(1)}_C \))

\[
= \sum_{p=1}^{n} k^{(1,j,p)}_{C,h^{-1}} x^{(1,p)}.
\]

See

\[
\psi(h \cdot g_i v_j) = \psi(g_{r} \gamma v_j)
\]

\[
= \psi(g_{r} \left( \sum_{p=1}^{n} k^{(1,j,p)}_{C,h^{-1}} v_p \right))
\]

\[
= \sum_{p=1}^{n} k^{(1,j,p)}_{C,h^{-1}} a^{(1,p)}_{t_i,1}
\]

and

\[
h \triangleright (\psi(g_i v_j)) = h \triangleright (a^{(1,j)}_{t_i,1})
\]

\[
= a^{(1,j)}_{ht_i,h^{-1}} \cdot h^{-1}
\]

\[
= \sum_{p=1}^{n} k^{(1,j,p)}_{C,h^{-1}} a^{(1,p)}_{t_i,1} \quad \text{(by [ZCZ Pro.1.2] and (1.2)).}
\]

Therefore \( \psi \) is a \( kG \)-module homomorphism. \( \Box \)

Therefore we also say that \( \mathfrak{B}(O_s, \rho) \) is a bi-one Nichols Hopf algebra.

**Remark 1.2.** The representation \( \rho \) in \( \mathfrak{B}(O_s, \rho) \) introduced in [Gr00, AZ07] and \( \rho^{(i)}_C \) in RSR are different. \( \rho(g) \) acts on its representation space from the left and \( \rho^{(i)}_C(g) \) acts on its representation space from the right.

\( s \in G \) is real if \( s \) and \( s^{-1} \) are in the same conjugacy class. If every element in \( G \) is real, then \( G \) is real. Obviously, Weyl groups are real.

**Lemma 1.3.** Assume that \( s \in G \) is real and \( \chi \) is the character of \( \rho \in \tilde{G}^s \). If \( \chi(s) \neq -\deg(\rho) \) or the order of \( s \) is odd, then \( \dim \mathfrak{B}(O_s, \rho) = \infty \).
Proof. If the order of $s$ is odd, it follows from [AZ07, Lemma 2.2] and [AF07 Lemma 1.3]. Now assume that $\chi(s) \neq -\deg(\rho)$. Since $\rho(s) = q_{ss} id$, $\chi(s) = q_{ss}(\deg(\rho))$. Therefore $q_{ss} \neq -1$ and $\dim \mathfrak{B}(\mathcal{O}_s, \rho) = \infty$ by [AZ07, Lemma 2.2] and [AF07, Lemma 1.3]. □.

Lemma 1.4. $(kG, \mathfrak{B}(\mathcal{O}_s, \rho)) \cong (kG', \mathfrak{B}(\mathcal{O}_{s'}, \rho'))$ as graded pull-push YD Hopf algebras if and only if there exist $h \in G$ and a group isomorphism $\phi$ from $G$ to $G'$ such that $\phi(h^{-1}gh) = s'$ and $\rho' \phi_h \cong \rho$, where $\phi_h(g) = \phi(h^{-1}gh)$ for any $g \in G$.

Proof. Let $C$ and $C'$ be conjugacy classes of $G$ and $G'$, respectively; $r = r_{C'}C$ and $r' = r'_{C'}C'$ be ramifications of $G$ and $G'$, respectively. Applying Lemma [1.1, we only need show that $(kG, \mathfrak{B}(G, r, \rho, u)) \cong (kG', \mathfrak{B}(G', r', \rho', u'))$ as graded pull-push YD Hopf algebras if and only if there exist $h \in G$ and a group automorphism group isomorphism $\phi$ from $G$ to $G'$ such that $\phi(h^{-1}u(C)h) = u'(C')$ and $\rho'(\phi_h) \cong \rho'(\phi_{h})$. Applying [ZCZ, Theorem 4], we only need show that $\text{RSR}(G, r, \rho, u) \cong \text{RSR}(G', r', \rho', u')$ if and only if there exist $\phi$ and a group isomorphism $\phi$ from $G$ to $G'$ such that $\phi(h^{-1}u(C)h) = u'(C')$ and $\rho'\phi_h \cong \rho'\phi_h$. This is clear. □.

If we define a relation on group $G$ as follows: $x \sim y$ if and only if there exists a group automorphism $\phi$ of $G$ such that $\phi(x)$ and $y$ are contained in the same conjugacy class, then this is an equivalent relation. Let set $\{s_i \mid i \in \Omega\}$ denote all representatives of the equivalent classes, which is called the representative system of conjugacy classes of $G$ under isomorphism relations, or the representative system of iso-conjugacy classes of $G$ in short.

Proposition 1.5. Let $\{s_i \mid i \in \Omega\} \subseteq G$ be the representative system of iso-conjugacy classes of $G$. Then $\{\mathfrak{B}(\mathcal{O}_{s_i}, \rho) \mid i \in \Omega, \rho \in \hat{G}_{s_i}\}$ are all representatives of the bi-one Nichols algebra over $G$, up to graded pull-push YD Hopf algebra isomorphisms.

Proof. If $\mathfrak{B}(\mathcal{O}_s, \rho)$ is a bi-one Nichols Hopf algebra over $G$, then there exist $i \in \Omega$, $\phi \in \text{Aut}(G)$ and $h \in G$ such that $\phi_h(s) = s_i$. Let $\rho' = \rho(\phi_h)^{-1}$. By Lemma [1.4, $(kG, \mathfrak{B}(\mathcal{O}_s, \rho)) \cong (kG, \mathfrak{B}(\mathcal{O}_{s_i}, \rho'))$ as graded pull-push YD Hopf algebras.

It follows from Lemma [1.4 that $(kG, \mathfrak{B}(\mathcal{O}_{s_i}, \rho))$ and $(kG, \mathfrak{B}(\mathcal{O}_{s_j}, \rho'))$ are not graded pull-push YD Hopf algebra isomorphisms when $i \neq j$ and $i, j \in \Omega$. □

2 Diagram

In this section it is proved that every non $-1$-type pointed Hopf algebra with real $G(H)$ is infinite dimensional.

If $H$ is a graded Hopf algebra, then there exists the diagram of $H$, written diag($H$), (see [ZZC, Section 3.1] and [Ra]). If $H$ is a pointed Hopf algebra, then the coradical filtration Hopf algebra $gr(H)$ is a graded Hopf algebra. So $gr(H)$ has the diagram, written
diag_{filt}(H), called the filter diagram of $H$. diag_{filt}(H) is written as diag($H$) in short when it does not cause confusion (see [AS98, Introduction]).

A graded coalgebra $C = \oplus_{n=0}^{\infty} C(n)$ is strictly graded if $C(0) = k$ and $C(1) = P(C)$ (see [Sw, P232]).

**Proposition 2.1.** If $H = \oplus_{n=0}^{\infty} H(n)$ is a graded Hopf algebra and $R := \text{diag}(H)$ is strictly graded as coalgebras, then $H \cong \text{gr}H$ as gradedHopf algebras.

**Proof.** By [AS98, Lemma 2.5], $H$ is coradically graded, i.e. $H_m = \oplus_{i=0}^{m} H(i)$ for $m = 0, 1, 2, \cdots$, where $H_0 \subseteq H_1 \subseteq H_2 \subseteq H_3 \subseteq \cdots$ is the coradical filtration of $H$. Define a map $\psi$ from $H$ to $\text{gr}H$ by sending $a$ to $a + H_{m-1}$ for any $a \in H(m)$ and $m = 0, 1, 2, 3, \cdots$. Note $H_{-1} := 0$. Obviously, $\psi$ is bijective. If $a \in H(m)$, then there exist $a_{s}^{(j)}$, $b_{s}^{(j)} \in H(j)$ for $0 \leq j \leq m$, $1 \leq s \leq n_j$, such that $\Delta(a) = \sum_{i=0}^{m} \sum_{s=1}^{n_i} a_{s}^{(i)} \otimes b_{s}^{(m-i)}$. See

$$(\psi \otimes \psi)\Delta(a) = (\psi \otimes \psi) \sum_{i=0}^{m} \sum_{s=1}^{n_i} a_{s}^{(i)} \otimes b_{s}^{(m-i)}$$

$$= \sum_{i=0}^{m} \sum_{s=1}^{n_i} (a_{s}^{(i)} + H_{i-1}) \otimes (b_{s}^{(m-i)} + H_{m-i-1})$$

$$= \Delta(a + H_{m-1})$$

(by the definition of comultiplication of $\text{gr}H$ in [Sw, P229] )

$$= \psi(a).$$

Thus $\psi$ is a coalgebra homomorphism. Similarly, $\psi$ is a algebra homomorphism. □

Consequently, every pointed Hopf algebra of type one (since its diagram is Nichols algebra, see [ZCZ, Section 2]) is isomorphic to its filtration Hopf algebra as graded Hopf algebras.

**Lemma 2.2.** If $R$ is a graded braided Hopf algebra in $kG\cd YD$ and is strictly graded as coalgebra gradations, then the subalgebra $\check{R}$ generated by $R(1)$ as algebras is a Nichols algebra. Furthermore, $\check{R}$ generated by $R(1)$ as algebras in $R$ is a Nichols algebra when $R$ is the filter diagram of a pointed Hopf algebra $H$.

**Proof.** We show the first claim by the following steps. Let

$$x = x^{(1)}x^{(2)} \cdots x^{(n)}$$

with $x \in R$, $x^{(i)} \in R(1)$ for $i = 1, 2, \cdots, n$.

(i) $\check{R}$ is $kG$-submodule of $R$. In fact $h \cdot x = h \cdot x^{(1)}x^{(2)} \cdots x^{(n)} = (h \cdot x^{(1)})(h \cdot x^{(2)}) \cdots (h \cdot x^{(n)}) \in R(1)R(1) \cdots R(1) \subseteq \check{R}$ for any $h \in G$.

(ii) $\check{R}$ is $kG$-subcomodule of $R$. We use induction on $n$ to show $\delta^-(x) \in kG \otimes \check{R}$. When $n = 1$, it is clear. Assume $n > 1$. $\delta^-(x) = \delta^-(yz) = \sum y_{(-1)}z_{(-1)} \otimes y_{(0)}z_{(0)} \in kG \otimes \check{R}$. 

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(iii) \( \tilde{R} \) is a subcoalgebra of \( R \). We use induction on \( n \) to show \( \Delta(x) \in \tilde{R} \otimes \tilde{R} \). When \( n = 1 \) it is clear. Assume \( n > 1 \).

\[
\Delta_R(x) = \Delta_R(yz) = \sum_{(z)} yz_1 \otimes z_2 + \sum_{(z),(y)} y_{(-1)} \cdot z_1 \otimes y_{(0)}z_2,
\]

which implies \( \Delta_R(x) \in \tilde{R} \otimes \tilde{R} \).

For the second claim, since \( R \) is strictly graded as coalgebra gradations (see [AS02, Lemma 2.3 and Lemma 2.4]), \( \tilde{R} \) is a Nichols algebra by the first claim. □

Remark 2.3. By [AS02, Cor. 2.3] \( \tilde{R} \cong B(\text{diag}_{\text{filn}}(H)(1)) \) as graded braided Hopf algebra in \( kG \Delta D \), where \( \tilde{R} \) is the same as in Lemma 2.2. There exists an RSR\((G, r, \overrightarrow{\rho}, u)\) such that \( B(G, r, \overrightarrow{\rho}, u) \cong B(\text{diag}_{\text{filn}}(H)(1)) \) as graded braided Hopf algebra in \( kG \Delta D \), by [ZCZ, Pro. 2.4]. We call \( B(\text{diag}_{\text{filn}}(H)(1)) \) and RSR\((G, r, \overrightarrow{\rho}, u)\) the Nichols algebra and RSR of \( H \), respectively.

Definition 2.4. (i) RSR\((G, r, \overrightarrow{\rho}, u)\) is of \(-1\)-type, if \( u(C) \) is real and the order of \( u(C) \) is even with \( \chi_C^{(i)}(u(C)) = -\chi_C^{(i)}(1) \) (i.e. \( \chi_C^{(i)}(u(C)) = -\deg \rho_C^{(i)} \)) for any \( C \in K_r(G) \) and any \( i \in I_C(r, u) \).

(ii) Nichols algebra \( R \) over group \( G \) is of \(-1\)-type if there exists \(-1\)-type RSR\((G, r, \overrightarrow{\rho}, u)\) such that \( R \cong B(G, r, \overrightarrow{\rho}, u) \) as graded pull-push YD Hopf algebras.

(iii) Pointed Hopf algebra \( H \) with group \( G = G(H) \) is of \(-1\)-type if the Nichols algebra of \( H \) is of \(-1\)-type.

Proposition 2.5. (i) If RSR\((G, r, \overrightarrow{\rho}, u)\) \( \cong \) RSR\((G', r', \overrightarrow{\rho'}, u')\) and RSR\((G, r, \overrightarrow{\rho}, u)\) is of \(-1\)-type, then so is RSR\((G', r', \overrightarrow{\rho'}, u')\).

(ii) If \((kG, R) \cong (kG', R')\) as graded pull-push YD Hopf algebras and \( R \) is of \(-1\)-type, then so is \( R' \), where \( R \) and \( R' \) are Nichols algebras over group algebras \( kG \) and \( kG' \), respectively.

(iii) If pointed Hopf algebras \( H \) and \( H' \) are isomorphic as Hopf algebras and \( H \) is of \(-1\)-type, then so is \( H' \).

Proof. (i) There exist a group isomorphism \( \phi : G \rightarrow G' \), an element \( h_C \in G \) such that \( \phi(h_C^{-1}u(C)h_C) = u'(\phi(C)) \) for any \( C \in K(G) \) and a bijective map \( \phi_C : I_C(r, u) \rightarrow I_{\phi(C)}(r', u') \) such that \( \rho_C^{(i)} = \rho_{\phi(C)}^{(i)} \phi_{h_C} \) for any \( i \in I_C(r, u) \). Therefore

\[
\chi_{\phi(C)}^{(i)}(u'(\phi(C))) = \chi_{\phi(C)}^{(i)}(\phi(h_C^{-1}u(C)h_C)) = \chi_C^{(i)}(u(C)) \quad \text{(by the isomorphism)}
\]

\[
= -\chi_C^{(i)}(1) \quad \text{(by the definition of \(-1\)-type)}
\]

\[
= -\chi_{\phi(C)}^{(i)}(\phi_{h_C}(1)) = -\chi_{\phi(C)}^{(i)}(1),
\]

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which proves the claim.

(ii) By \([ZCZ]\) Pro.2.4, there exist two \(\text{RSR}(G, r, \overrightarrow{\rho}, u)\) and \(\text{RSR}(G', r', \overrightarrow{\rho}', u')\) such that \(R \cong \mathcal{B}(G, r, \overrightarrow{\rho}, u)\) and \(R' \cong \mathcal{B}(G', r', \overrightarrow{\rho}', u')\) as graded YD Hopf algebras. Thus \(\text{RSR}(G, r, \overrightarrow{\rho}, u) \cong \text{RSR}(G', r', \overrightarrow{\rho}', u')\) by \([ZCZ]\) Theorem 4. It follows from Definition 2.4 and Part (i) that \(\text{RSR}(G', r', \overrightarrow{\rho}', u')\) is of \(-1\)-type.

(iii) It is clear that \(\text{gr} H \cong \text{gr} H'\) as graded Hopf algebras. Thus \((kG, R) \cong (kG', R')\) as graded pull-push YD Hopf algebras by \([ZCZ]\) Lemma 3.1, where \(kG\) and \(kG'\) are the coradicals of \(H\) and \(H'\), respectively; \(R = \text{diag} H\) and \(R' = \text{diag} H'\). Let \(\bar{R}\) and \(\bar{R}'\) denote the subalgebras generated by \(R_{(1)}\) and \(R'_{(1)}\) as algebras in \(R\) and \(R'\), respectively. It is clear that \((kG, \bar{R}) \cong (kG', \bar{R}')\) as graded pull-push YD Hopf algebras. It follows from Part (ii) that \(H'\) is of \(-1\)-type. \(\square\)

In fact, the proof of Part (iii) above shows that if two pointed Hopf algebras are isomorphic, then their Nichols algebras are graded pull-push isomorphic. Similarly, we can prove that their RSR’s are isomorphic.

**Proposition 2.6.** If \(H\) is a pointed Hopf algebra with real \(G = G(H)\) and is not of \(-1\)-type, then \(H\) is infinite dimensional.

**Proof.** Let \(R\) be the (filter) diagram of \(H\). By Lemma 2.2 \(R\) generated by \(R_{(1)}\) as algebras in \(R\) is a Nichols algebra. By \([ZCZ]\) Pro.2.4 (ii), there exists an \(\text{RSR}(G, r, \overrightarrow{\rho}, u)\) such that \(\bar{R} \cong \mathcal{B}(G, r, \overrightarrow{\rho}, u)\) is graded pull-push YD Hopf algebra isomorphism. By assumption, there exist \(C \in \mathcal{K}_r(G)\) and \(i \in I_C(r, u)\) such that \(\chi_C^{(i)}(u(C)) \neq -\text{deg}(\rho_C^{(i)})\) or the order of \(u(C)\) is odd. It follows from Lemma 1.3 that the bi-one Nichols algebra \(\mathcal{B}(G, r', \overrightarrow{\rho}', u')\) is infinite dimensional, where ramification \(r' = r'C\), \(\rho_C^{(i)} = \rho_C^{(i)}\), \(u'(C) = u(C), I_C(r', u') \subseteq I_C(r, u)\) with \(|I_C(r', u')| = 1\). Let \(Q'\) be a sub-quiver of \(Q\) with \(Q'_0 = Q_0\) and \(Q'_1 := \{a_{x,y}^{(i,j)} | x^{-1}y \in C, j \in J_C(i)\}\). Since \((k(Q')_1, \text{ad}(G, r', \overrightarrow{\rho}', u'))\) is a braided subspace of \((kQ_1, \text{ad}(G, r, \overrightarrow{\rho}, u))\), we have \(\dim \mathcal{B}(G, r, \overrightarrow{\rho}, u) = \infty\) and \(H\) is infinite dimensional. \(\square\)

\(\text{RSR}(G, r, \overrightarrow{\rho}, u)\) is said to be of infinite type if \(\mathcal{B}(G, r, \overrightarrow{\rho}, u)\) is infinite dimensional. Otherwise, it is said to be of finite type. For any \(\text{RSR}(G, r, \overrightarrow{\rho}, u)\), according to the proof above, if there exist \(C \in \mathcal{K}_r(G)\) and \(i \in I_C(r, u)\) such that \(\dim \mathcal{B}(O_{u(C)}, \rho_C^{(i)}) = \infty\), then \(\dim \mathcal{B}(G, r, \overrightarrow{\rho}, u) = \infty\). In this case \(\text{RSR}(G, r, \overrightarrow{\rho}, u)\) is said to be of essentially infinite type. Otherwise, it is said to be of non-essentially infinite type. For example, non \(-1\)-type RSR over real group is of essentially infinite type. However, it is an open problem whether \(\text{RSR}(G, r, \overrightarrow{\rho}, u)\) is of finite type when it is of non-essentially infinite type, although paper \([AHS08]\) gave a partial solution to this problem.
3 Generalized quantum linear spaces

In this section it is shown that every central quantum linear space is finite dimensional with an arrow PBW basis.

Let $\sigma$ denote the braiding of the braided tensor category $(\mathcal{C}, \sigma)$. If $A$ and $B$ are two objects of $\mathcal{C}$ and $\sigma_{A,B}\sigma_{B,A} = \text{id}_{B \otimes A}$ and $\sigma_{B,A}\sigma_{A,B} = \text{id}_{A \otimes B}$ then $\sigma$ is said to be symmetric on pair $(A, B)$. Furthermore, if $A = B$, then $\sigma$ is said to be symmetric on object $A$, in short, or $A$ is said to be quantum symmetric.

Every arrow YD module $(kQ_1^1, \text{ad}(G, r, \vec{\rho}, u))$ has a decomposition of simple YD modules:

$$(kQ_1^1, \text{ad}(G, r, \vec{\rho}, u)) = \bigoplus_{C \in \mathcal{K}_r(G), i \in I_C(r, u)} kQ_1^1(G, \mathcal{O}_{u(C)}, \rho_C^{(i)})$$

and $\sigma_{C^{(i)}, D^{(j)}}$ is a map from $kQ_1^1(G, \mathcal{O}_{u(C)}, \rho_C^{(i)}) \otimes kQ_1^1(G, \mathcal{O}_{u(D)}, \rho_D^{(j)})$ to $kQ_1^1(G, \mathcal{O}_{u(D)}, \rho_D^{(j)}) \otimes kQ_1^1(G, \mathcal{O}_{u(C)}, \rho_C^{(i)})$, where $kQ_1^1(G, \mathcal{O}_{u(C)}, \rho_C^{(i)}) := k\{a_{x,1}^{(i,j)} \mid x \in \mathcal{O}_{u(C)}, i \in I_C(r, u), j \in J_C(i)\}$ and $\sigma_{C^{(i)}, D^{(j)}}$ denotes $\sigma_{kQ_1^1(G, \mathcal{O}_{u(C)}, \sigma_C^{(i)}), kQ_1^1(G, \mathcal{O}_{u(D)}, \sigma_D^{(j)})}$ for any $i \in I_C(r, u), j \in I_D(r, u)$.

Every YD module over $kG$ has a decomposition [311] since every YD module is isomorphic to an arrow YD module by [ZCZ] Pro. 2.4, which shows every YD module over $kG$ is completely reducible (see [AZ07] Section 1.2).

**Definition 3.1.** An RSR$(G, r, \vec{\rho}, u)$ is said to be quantum symmetric if $\sigma_{C^{(i)}, D^{(j)}} = \sigma_{D^{(j)}, C^{(i)}}^{-1}$, i.e. $\sigma$ is symmetric on pair $(kQ_1^1(G, \mathcal{O}_{u(C)}, \rho_C^{(i)}), kQ_1^1(G, \mathcal{O}_{u(D)}, \rho_D^{(j)}))$, for any $C, D \in \mathcal{K}_r(G), i \in I_C(r, u)$ and $j \in I_D(r, u)$.

An RSR$(G, r, \vec{\rho}, u)$ is said to be quantum weakly symmetric if $\sigma_{C^{(i)}, D^{(j)}} = (\sigma_{D^{(j)}, C^{(i)}})^{-1}$ for any $C, D \in \mathcal{K}_r(G), i \in I_C(r, u)$ and $j \in I_D(r, u)$ with $(C, i) \neq (D, j)$ (i.e. either $C \neq D$ or $i \neq j$).

**Proposition 3.2.** If a non-essentially infinite RSR$(G, r, \vec{\rho}, u)$ is quantum weakly symmetric, then RSR$(G, r, \vec{\rho}, u)$ is a finite type.

**Proof.** It follows from [Gr00] Theorem 2.2. □

**Lemma 3.3.** (i) Assume that $H$ is a Hopf algebra with an invertible antipode and $M$ is a YD $H$-module, Then the braiding $\sigma$ of $H \mathcal{Y} D$ is symmetric on $M$ if and only if $\sigma$ is symmetric on $\mathfrak{B}(M)$.

(ii) The following conditions are equivalent.

1. RSR$(G, r, \vec{\rho}, u)$ is quantum symmetric.

2. The braiding $\sigma$ of $kQ_1^1 \mathcal{Y} D$ on the arrow YD module $(kQ_1^1, \text{ad}(G, r, \vec{\rho}, u))$ is symmetric.

3. The braiding $\sigma$ is symmetric on $\mathfrak{B}(G, r, \vec{\rho}, u)$. 
(4) \( \sigma^2(a^{(i,j)}_{x,1} \otimes a^{(i',j')}_{y,1}) = a^{(i,j)}_{x,1} \otimes a^{(i',j')}_{y,1} \) for any \( C := x^G, D := y^G \in \mathcal{K}_r(G) \), \( i \in I_C(r, u) \), \( i' \in I_D(r, u) \), \( j \in J_C(i) \), \( j' \in J_D(i') \).

(5) \( a^{(i,j)}_{x,1} \otimes a^{(i',j')}_{y,1} = (xy^{-1} \triangleright a^{(i,j)}_{x,1}) \otimes (x \triangleright a^{(i',j')}_{y,1}) \) for any \( C := x^G, D := y^G \in \mathcal{K}_r(G) \), \( i \in I_C(r, u) \), \( i' \in I_D(r, u) \), \( j \in J_C(i) \), \( j' \in J_D(i') \).

(iii) The following conditions are equivalent.

(1) \( \text{RSR}(G, r, \overrightarrow{p}, u) \) is quantum weakly symmetric.

(2) \( \sigma^2(a^{(i,j)}_{x,1} \otimes a^{(i',j')}_{y,1}) = a^{(i,j)}_{x,1} \otimes a^{(i',j')}_{y,1} \) for any \( C := x^G, D := y^G \in \mathcal{K}_r(G) \), \( i \in I_C(r, u) \), \( i' \in I_D(r, u) \), \( j \in J_C(i) \), \( j' \in J_D(i') \) with \( (C, i) \neq (D, i') \).

(3) \( a^{(i,j)}_{x,1} \otimes a^{(i',j')}_{y,1} = (xy^{-1} \triangleright a^{(i,j)}_{x,1}) \otimes (x \triangleright a^{(i',j')}_{y,1}) \) for any \( C := x^G, D := y^G \in \mathcal{K}_r(G) \), \( i \in I_C(r, u) \), \( i' \in I_D(r, u) \), \( j \in J_C(i) \), \( j' \in J_D(i') \) with \( (C, i) \neq (D, i') \).

**Proof.** (i) It is clear since \( M \) generates \( \mathfrak{B}(M) \) as algebras.

(ii) It follows from Definition 3.1 that (1) and (2) are equivalent. Part (i) implies that (2) and (3) are equivalent. Obviously, (4) and (2) are equivalent. Since \( (kQ^1_1, \text{ad}(G, r, \overrightarrow{p}, u)) \) is a YD module, we have

\[
\sigma^2(a^{(i,j)}_{x,1} \otimes a^{(i',j')}_{y,1}) = (xy^{-1} \triangleright a^{(i,j)}_{x,1}) \otimes (x \triangleright a^{(i',j')}_{y,1}).
\]

(3.2)

Therefore, (4) and (5) are equivalent.

(iii) It follows from (3.2) that (2) and (3) are equivalent. Obviously (1) and (2) according to the definition. \( \square \)

**Lemma 3.4.** For \( C := x^G, D := y^G \in \mathcal{K}_r(G) \), \( i \in I_C(r, u) \), \( i' \in I_D(r, u) \), \( j \in J_C(i) \), \( j' \in J_D(i') \), assume that \( \rho_C^{(i)} \) and \( \rho_D^{(j')} \) are one dimensional representations; The coset decomposition of \( G^{u(C)} \) and \( G^{u(D)} \) in \( G \) are

\[
G = \bigcup_{\theta \in \Theta_C} G^{u(C)} g_{\theta}, \quad \text{and} \quad G = \bigcup_{\eta \in \Theta_D} G^{u(D)} h_{\eta},
\]

respectively; \( x = g_{\theta}^{-1} u(C) g_{\theta} \) and \( y = h_{\eta}^{-1} u(D) h_{\eta} \); \( g_{\theta} y^{-1} = \zeta_{\theta}(y^{-1}) g_{\eta} \) and \( h_{\eta} x^{-1} = \zeta_{\eta}(x^{-1}) h_{\eta} \), with \( \zeta_{\theta}(y^{-1}) \in G^{u(C)} \) and \( \zeta_{\eta}(x^{-1}) \in G^{u(D)} \).

Then

\[
a^{(i,j)}_{x,1} \otimes a^{(i',j')}_{y,1} = (xy^{-1} \triangleright a^{(i,j)}_{x,1}) \otimes (x \triangleright a^{(i',j')}_{y,1}) \quad (3.3)
\]

if and only if

\[
xy = yx \quad \text{and} \quad \rho_C^{(i)}(\zeta_{\theta}(y^{-1})) \rho_D^{(j')}(\zeta_{\eta}(x^{-1})) = 1 \quad (3.4)
\]

**Proof.** By [ZCZ, Pro. 1.2] or [ZZC, Pro. 1.9], \( (xy^{-1} \triangleright a^{(i,j)}_{x,1}) \otimes (x \triangleright a^{(i',j')}_{y,1}) = \alpha a^{(i,j)}_{xyx^{-1}x^{-1},1} \otimes a^{(i',j')}_{xyx^{-1},1}, \) where \( \alpha \in k \). Thus (3.3) holds if and only if \( xy = yx \) and \( \alpha = 1 \).

By [ZCZ, Pro. 1.2], \( \alpha = 1 \) if and only if (3.4) holds. \( \square \)
Proposition 3.5. If \( \text{RSC}(G, r, \bar{\rho}, u) \) is non-essentially infinite and (3.3) holds for any \( C := x^G, D := y^G \in K_r(G), i \in I_C(r, u), i' \in I_D(r, u) \) with \( (C, i) \neq (D, i') \), then \( \text{RSC}(G, r, \bar{\rho}, u) \) is quantum weakly symmetric. Therefore \( \text{RSC}(G, r, \bar{\rho}, u) \) is a finite type.

Proof. It follows from Lemma 3.4, Lemma 3.3 and Proposition 3.2. \( \square \)

If \( 0 < q \in k \) and \( 0 \leq i < \text{ord}(q) \) (the order of \( q \)), we set \( (0)_q! = 1 \),
\[
\binom{n}{i}_q = \frac{(n)_q!}{(i)_q!(n-i)_q!}, \quad \text{where } (n)_q! = \prod_{1 \leq i \leq n} (i)_q, \quad (n)_q = \frac{q^n - 1}{q - 1}.
\]
In particular, \( (n)_q = n \) when \( q = 1 \).

Lemma 3.6. In \( kQ^c(G, r, \bar{\rho}, u) \), we have the following results.

(i) If \( C = \{g\} \in K_r(G) \) with \( i \in I_C(r, u) \), then there exists \( 0 \neq q \in k \) such that
\[
\rho^{(i)}_C(g) = q \text{ id and } a^{(i,j)}_{y,x} \cdot h = qa^{(i,j)}_{y,xh} \text{ for any } x^{-1}y \in C, \ for \ h \in G, \ j \in J_C(i).
\]

(ii) If \( a^{(i,j)}_{wv_0} \cdot h = qa^{(i,j)}_{wv_0v} \) for some \( v_0, w_0 \in G, h \in G^w(C), q \in k \) with \( v_0^{-1}w_0 \in C \in K_r(G) \), then \( a^{(i,j)}_{wv} \cdot h = qa^{(i,j)}_{wv} \) for any \( v, w \in G \) with \( v^{-1}w \in C \).

Proof. (i) It follows from [ZCZ] Pro. 1.2.

(ii) Let \( X_C^{(i)} \) be a representation space of \( \rho^{(i)}_C \) and \( \{x_C^{(i,j)} \mid j \in J_C(i)\} \) a k-basis of \( X_C^{(i)} \). By the proof of [ZCZ] Pro. 1.2, \( a^{(i,j)}_{wv} \cdot h = qa^{(i,j)}_{wv} \) since \( x_C^{(i,j)} \cdot \zeta(h) = qx_C^{(i,j)} \) by assumption. \( \square \)

Lemma 3.7. In co-path Hopf algebra \( kQ^c(G, r, \bar{\rho}, u) \), assume \( C := g^G \in K_r(G), i \in I_C(r, u), j \in J_C(i) \) and \( a^{(i,j)}_{g^i} \cdot g = qa^{(i,j)}_{g^i} \). If \( i_1, i_2, \cdots, i_m \) are non-negative integers, then
\[
a^{(i,j)}_{g^{i_1}g^{i_2} \cdots g^{i_m}} \cdot a^{(i,j)}_{g^{i_1}g^{i_2} \cdots g^{i_m}} \cdots a^{(i,j)}_{g^{i_1}} = q^{\beta_m(m)_q!} \rho^{(i,j)}_{g^{i_1}}(g^m, m)
\]
where \( \alpha_m = i_1 + i_2 + \cdots + i_m, \ P^{(i,j)}_{h}(g, m) := a^{(i,j)}_{g^m g^{m-1}h} \cdots a^{(i,j)}_{g^{i_2}g^{i_1}} \cdot \zeta(h), \beta_1 = 0 \) and \( \beta_m = \sum_{j=1}^{m-1} (i_1 + i_2 + \cdots + i_j) \) if \( m > 1 \).

Proof. We prove the equality by induction on \( m \). For \( m = 1 \), it is easy to see that the equality holds. Now suppose \( m > 1 \). We have
Recall that a braided algebra $A$ in braided tensor category $(\mathcal{C}, \sigma)$ with braiding $\sigma$ is said to be braided commutative or quantum commutative, if $ab = \mu \sigma(a \otimes b)$ for any $a, b \in A$, where $\mu$ is the multiplication of $A$.

By [CR02, Example 3.11], the multiplication of any two arrows $a_{g,x}^{(i,j)}$ and $a_{w,v}^{(m,n)}$ in co-path Hopf algebra $kQ^e(G, r, \overrightarrow{\rho}, u)$ is

$$a_{g,x}^{(i,j)} \cdot a_{w,v}^{(m,n)} = (y \cdot a_{w,v}^{(m,n)})(a_{g,x}^{(i,j)} \cdot v) + (a_{g,x}^{(i,j)} \cdot u)(x \cdot a_{w,v}^{(m,n)}). \quad (3.5)$$

**Lemma 3.8.** Let $C := x^G, D := y^G \in K_r(G), i \in I_C(r, u), j \in J_C(i), i' \in I_D(r, u), j' \in J_D(i')$, $\alpha, \beta \in k$ with $a_{g,x}^{(i',j')} \cdot x = \alpha a_{g,x}^{(i',j')} \cdot x$ and $a_{g,x}^{(i,j)} \cdot y = \beta a_{g,x}^{(i,j)}$ in co-path Hopf algebra $kQ^e(G, r, \overrightarrow{\rho}, u)$. If $xy = yx$ then $\alpha \beta = 1$ if and only if

$$a_{g,x}^{(i,j)} \cdot a_{y_1}^{(i',j')} = \alpha^{-1} a_{g,x}^{(i',j')} \cdot a_{y_1}^{(i,j)} \quad (3.6)$$

**Proof.** By (3.5) and [ZCZ, Pro. 1.2], we have

$$a_{g,x}^{(i,j)} \cdot a_{y_1}^{(i',j')} = a_{g,x}^{(i',j')} a_{g,x}^{(i,j)} + \beta a_{g,x}^{(i,j)} a_{y_1}^{(i',j')},
$$

$$a_{g,x}^{(i',j')} \cdot a_{x_1}^{(i,j)} = \alpha a_{g,x}^{(i',j')} a_{x_1}^{(i,j)} + \beta a_{g,x}^{(i,j)} a_{y_1}^{(i',j')} \cdot a_{x_1}^{(i,j)}. \quad (3.7)$$

Applying this we can complete the proof. □

**Lemma 3.9.** Assume that $RSR(G, r, \overrightarrow{\rho}, u)$ satisfies $C := \{g_C\} \subseteq Z(G)$ for any $C \in K_r(G)$. Let $p_C^{(i)}(g_D) = a^{(i)}_{C,D}$ id for any $C, D \in K_r(G), i \in I_C(r, u)$.

(i) The following conditions are equivalent:

1. $RSR(G, r, \overrightarrow{\rho}, u)$ is quantum symmetric
2. $q_{C,D}^{(i)}q_{D,C}^{(i')} = 1$ for any $C, D \in K_r(G), i \in I_C(r, u), i' \in I_D(r, u)$.
\(a_{g_{c,1}}^{(i,j)} \cdot a_{g_{d,1}}^{(i',j')} = (q_{D,C}^{(i)})^{-1} a_{g_{d,1}}^{(i',j')} \cdot a_{g_{c,1}}^{(i,j)}\) for any \(C, D \in \mathcal{K}_r(G), i \in I_C(r, u), i' \in I_D(r, u), j \in I_C(i), j' \in J_C(i')\).

(4) \(\mathfrak{B}(G, r, \rho, u)\) is quantum commutative in \(kG \mathfrak{Y}D\).

(5) \(\mathfrak{B}(G, r, \rho, u)\) is quantum symmetric.

(6) \((kQ_1, \text{ad}(G, r, \rho, u))\) is quantum symmetric.

(ii) The following conditions are equivalent:

(1) \(\text{RSR}(G, r, \rho, u)\) is quantum weakly symmetric

(2) \(q_{C, D}^{(i)} q_{D,C}^{(i')} = 1\) for any \(C, D \in \mathcal{K}_r(G), i \in I_C(r, u), i' \in I_D(r, u)\) with \((C, i) \neq (D, i').\)

(3) \(a_{g_{c,1}}^{(i,j)} \cdot a_{g_{d,1}}^{(i',j')} = (q_{D,C}^{(i)})^{-1} a_{g_{d,1}}^{(i',j')} \cdot a_{g_{c,1}}^{(i,j)}\) for any \(C, D \in \mathcal{K}_r(G), i \in I_C(r, u), i' \in I_D(r, u), j \in I_C(i), j' \in J_C(i')\) with \((C, i) \neq (D, i').\)

Proof. By [ZCZ] Lemma 2.2, \(\text{diag}(kG[kQ_1, r, \rho, u])\) is the Nichols algebra \(\mathfrak{B}(G, r, \rho, u)\) in \(kG \mathfrak{Y}D\). By [ZCZ] Pro. 1.2,

\[
\sigma^2(a_{g_{c,1}}^{(i,j)} \otimes a_{g_{d,1}}^{(i',j')}) = (q_{C,D}^{(i)} q_{D,C}^{(i')})^{-1} a_{g_{c,1}}^{(i,j)} \otimes a_{g_{d,1}}^{(i',j')}. \tag{3.8}
\]

(i) By Lemma 3.3. (ii), (1), (5) and (6) are equivalent. It follows from (3.8) that (6) and (2) are equivalent. By Lemma 3.3 (3) and (2) are equivalent. Obviously (3) and (6) are equivalent. (3) and (4) are equivalent since \(\mathfrak{B}(G, r, \rho, u)\) is generated by \(kQ_1\).

(ii) It follows from (3.8) that (1) and (2) are equivalent. (2) and (3) are equivalent according to (3.6). \(\square\)

Lemma 3.10. (See [AS98] Lemma 3.3) Let \(B\) be a Hopf algebra and \(R\) a braided Hopf algebra in \(kG \mathfrak{Y}D\) with a linearly independent set \(\{x_1, \ldots, x_t\} \subseteq P(R)\), the set of all primitive elements in \(R\). Assume that there exist \(g_j \in G(B)\) (the set of all group-like elements in \(B)\) and \(0 \neq k_{j,i} \in k\) such that

\[
\delta(x_i) = g_i \otimes x_i, \quad g_i \cdot x_j = k_{i,j} x_j, \text{ for all } i, j = 1, 2, \ldots, t.
\]

Then

\[
\{x_1^{m_1} x_2^{m_2} \cdots x_t^{m_t} \mid 0 \leq m_j < N_j, 1 \leq j \leq t\}
\]

is linearly independent, where \(N_i\) is the order of \(q_i := k_{i,i}\) (\(N_i = \infty\) when \(q_i\) is not a root of unit, or \(q_i = 1\) ) for \(1 \leq i \leq t\).

Proof. By the quantum binomial formula, if \(1 \leq n_j < N_j\), then

\[
\Delta(x_j^{n_j}) = \sum_{0 \leq i_j \leq n_j} \binom{n_j}{i_j} q_j^{i_j} x_j^{i_j} \otimes x_j^{n_j-i_j}.
\]

We use the following notation:

\[
n = (n_1, \ldots, n_j, \ldots, n_t), \quad x^n = x_1^{n_1} \cdots x_j^{n_j} \cdots x_t^{n_t}, \quad |n| = n_1 + \cdots + n_j + \cdots + n_t;
\]

\[
\Delta(x_j^{n_j}) = \sum_{0 \leq i_j \leq n_j} \binom{n_j}{i_j} q_j^{i_j} x_j^{i_j} \otimes x_j^{n_j-i_j}.
\]

We use the following notation:

\[
n = (n_1, \ldots, n_j, \ldots, n_t), \quad x^n = x_1^{n_1} \cdots x_j^{n_j} \cdots x_t^{n_t}, \quad |n| = n_1 + \cdots + n_j + \cdots + n_t;
\]
accordingly, \( N = (N_1, \ldots, N_t) \), \( \mathbf{1} = (1, \ldots, 1) \). Also, we set

\[
i \leq n \quad \text{if} \quad i_j \leq n_j, \quad j = 1, \ldots, t.
\]

Then, for \( n < N \), we deduce from the quantum binomial formula that

\[
\Delta(x^n) = x^n \otimes 1 + 1 \otimes x^n + \sum_{0 \leq i \leq n, \ 0 \neq i \neq n} c_{n,i} x^i \otimes x^{n-i},
\]

where \( c_{n,i} \neq 0 \) for all \( i \).

We shall prove by induction on \( r \) that the set

\[
\{ x^n \mid |n| \leq r, \ n < N \}
\]

is linearly independent.

Let \( r = 1 \) and let \( a_0 + \sum_{i=1}^t a_i x_i = 0 \), with \( a_j \in k, \ 0 \leq j \leq t \). Applying \( \epsilon \), we see that \( a_0 = 0 \); by hypothesis we conclude that the other \( a_j \)'s are also 0.

Now let \( r > 1 \) and suppose that \( z = \sum_{|n| \leq r, \ n < N} a_n x^n = 0 \). Applying \( \epsilon \), we see that \( a_0 = 0 \). Then

\[
0 = \Delta(z) = z \otimes 1 + 1 \otimes z + \sum_{1 \leq |n| \leq r, \ n < N} a_n \sum_{0 \leq i \leq n, \ 0 \neq i \neq n} c_{n,i} x^i \otimes x^{n-i}
\]

Now, if \( |n| \leq r, \ 0 \leq i \leq n \), and \( 0 \neq i \neq n \), then \( |i| < r \) and \( |n - i| < r \). By inductive hypothesis, the elements \( x^i \otimes x^{n-i} \) are linearly independent. Hence \( a_n c_{n,i} = 0 \) and \( a_n = 0 \) for all \( n, \ |n| \geq 1 \). Thus \( a_n = 0 \) for all \( n \). \( \square \)

The quantum linear space was defined in [AS98, Lemma 3.4] and now is generalized as follows.

**Definition 3.11.** Let \( 0 \neq k_{i,j} \in k \) and \( 1 < N_i := \text{ord}(k_{k_i,i}) < \infty \) for any \( i, j \in \Omega \), where \( \Omega \) is a finite set. If \( R \) is the algebra generated by set \( \{ x_j \mid j \in \Omega \} \) with relations

\[
x_i^{N_i} = 0, \ x_i x_j = k_{i,j} x_j x_i \quad \text{for any} \ i, j \in \Omega \ \text{with} \ i \neq j,
\]

then \( R \) is called the generalized quantum linear space generated by \( \{ x_j \mid j \in \Omega \} \).

**Definition 3.12.** (i) \( \text{RSR}(G, r, \overrightarrow{\rho}, u) \) is said to be a generalized quantum linear type if the following conditions are satisfied:

(GQL1) \( xy = yx \) for any \( C := x^G, D := y^G \in K_r(G) \).

(GQL2) there exists \( k_{x,y}^{(i,j)} \in k \) such that \( a_{x,1}^{(i,j)} \cdot y = k_{x,y}^{(i,j)} a_{x,y}^{(i,j)} \) for any \( C := x^G, D := y^G \in K_r(G) \), \( i \in I_C(r, u), \ j \in J_C(i) \).
(GQL3) \( k_{x,y}^{(i,j)} k_{y,x'}^{(i',j')} = 1 \) for any \( C := x^G, D := y^G \in \mathcal{K}_r(G), i \in \mathcal{I}_C(r, u), j \in \mathcal{J}_C(i), i' \in \mathcal{I}_D(r, u), j' \in \mathcal{J}_D(i') \) with \((x, i, j) \neq (y, i', j')\).

(GQL4) \( 1 < N_x^{(i,j)} := \text{ord}(k_{x,x}^{(i,j)}) < \infty \) for any \( C := x^G \in \mathcal{K}_r(G), i \in \mathcal{I}_C(r, u), j \in \mathcal{J}_C(i) \).

(ii) \( \text{RSR}(G, r, \overrightarrow{\rho}, u) \) is said to be a central quantum linear type if it is quantum symmetric and of the non-essentially infinite type with \( C \subseteq Z(G) \) for any \( C \in \mathcal{K}_r(G) \). In this case, \( \mathfrak{B}(G, r, \overrightarrow{\rho}, u) \) is called a central quantum linear space over \( G \).

Assume that \( A \) is an algebra with \( \{ b_\nu \mid \nu \in \Omega \} \subseteq A \) and \( \prec \) is a total order of \( \Omega \), \( N_\nu \in \mathbb{N} \) or \( \infty \) for any \( \nu \in \Omega \). If

\[
\{ b_{\nu_1}^{m_1} b_{\nu_2}^{m_2} \cdots b_{\nu_n}^{m_n} \mid \nu_1 \prec \nu_2, \cdots \prec \nu_n; 0 \leq m_s < N_{\nu_s}; 1 \leq s \leq n; n \in \mathbb{N} \}
\]  (3.11)

is a basis of \( A \), then the basis (3.11) is called a PBW basis generated by \( \{ b_\nu \mid \nu \in \Omega \} \). If \( \{ b_\nu \mid \nu \in \Omega \} \subseteq Q_1 \), then it is called an arrow PBW basis.

It is well-known that every quantum linear space is a braided Hopf algebra and has a BPW basis (see [AS98, Lemma 3.4]). Of course, every generalized quantum linear space is finite dimensional. However, it is not known whether every generalized quantum linear space has an PBW basis.

**Proposition 3.13.** If \( \text{RSR}(G, r, \overrightarrow{\rho}, u) \) is of the generalized quantum linear type, then \( \mathfrak{B}(G, r, \overrightarrow{\rho}, u) \) is a generalized quantum linear space with the arrow PBW basis

\[
\{ b_{\nu_1}^{m_1} b_{\nu_2}^{m_2} \cdots b_{\nu_n}^{m_n} \mid \nu_1 \prec \nu_2, \cdots \prec \nu_n; 0 \leq m_s < N_{\nu_s}; 1 \leq s \leq n; n \in \mathbb{N} \}
\]  (3.12)

and

\[
\dim(\mathfrak{B}(G, r, \overrightarrow{\rho}, u)) = \prod_{C := x^G \in \mathcal{K}_r(G), i \in \mathcal{I}_C(r, u), j \in \mathcal{J}_C(i)} N_x^{(i,j)},
\]  (3.13)

where \( \{ b_\nu \mid \nu \in \Omega \} := Q_1 \) with total order \( \prec \) and \( N_{\nu_s} = N_x^{(i,j)} := \text{ord}(k_{x,x}^{(i,j)}) \) if \( b_{\nu_s} = a_{x,1}^{(i,j)} \).

**Proof.** Since any two different arrows in \( \mathfrak{B}(G, r, \overrightarrow{\rho}, u) \) are quantum commutative (see Lemma 3.8) and \( (b_{\nu_s})^{N_{\nu_s}} = 0 \) (see Lemma 3.7), we have \( \mathfrak{B}(G, r, \overrightarrow{\rho}, u) \) is generated by (3.12).

For any \( \nu, \nu' \in \Omega \), \( b_\nu = a_{x,1}^{(i,j)} \) and \( b_{\nu'} = a_{y,1}^{(i',j')} \) with \( C := x^G, D := y^G \in \mathcal{K}_r(G), i \in \mathcal{I}_C(r, u), j \in \mathcal{J}_C(i), i' \in \mathcal{I}_D(r, u), j' \in \mathcal{J}_D(i') \), let \( g_{\nu} = x \) and \( k_{\nu,\nu'} = (k_{y,x}^{(i',j')})^{-1} \). By [ZCZ, Pro. 1.2] we have

\[
\delta^-(b_\nu) = \delta^-(a_{x,1}^{(i,j)}) = x \otimes a_{x,1}^{(i,j)} = g_{\nu} \otimes b_\nu \quad \text{and}
\]

\[
g_{\nu} \triangleright b_{\nu'} = x \cdot a_{y,1}^{(i',j')}. x^{-1}
\]

\[
= x \cdot (k_{y,x}^{(i',j')})^{-1} a_{y,x}^{(i',j')} \quad \text{(by (GQL2))}
\]

\[
= k_{\nu,\nu'} b_{\nu'} \quad \text{(by (GQL1)).}
\]
Therefore, by Lemma 3.10, 3.12 is linearly independent. Thus 3.12 is a basis of \( \mathfrak{B}(G, r, \overrightarrow{\rho}, u) \).

Let \( R \) be the generalized quantum linear space generated by \( \{ b\nu \mid \nu \in \Omega \} := kQ_1^1 \). It is clear that there exists an algebra map \( \psi \) from \( R \) to \( \mathfrak{B}(G, r, \overrightarrow{\rho}, u) \) by sending \( b\nu \) to \( b\nu \) for any \( \nu \in \Omega \). Since \( \mathfrak{B}(G, r, \overrightarrow{\rho}, u) \) has an arrow PBW basis (3.14), \( \psi \) is isomorphic. □

**Proposition 3.14.** Assume that \( C = \{ g_C \} \subseteq Z(G) \) and \( \rho_C^{(i)}(g_D) = q_{C,D}^{(i)} \) \( \text{id} \) for any \( C, D \in \mathcal{K}_r(G), i \in I_C(r,u) \). Then

(i) \( \text{RSR}(G, r, \overrightarrow{\rho}, u) \) is of the central quantum linear type if and only if \( q_{C,D}^{(i)} d_{D,C}^{(j)} = 1 \) and \( 1 < \text{ord}(q_{C,D}^{(i)}) < \infty \) for any \( C, D \in \mathcal{K}_r(G), i \in I_C(r,u), j \in I_D(r,u) \).

(ii) \( \text{RSR}(G, r, \overrightarrow{\rho}, u) \) is quantum weakly symmetric with non-essentially infinite type if and only if \( q_{C,D}^{(i)} d_{D,C}^{(j)} = 1 \) and \( 1 < \text{ord}(q_{C,D}^{(i)}) < \infty \) for any \( C, D \in \mathcal{K}_r(G), i \in I_C(r,u), j \in I_D(r,u) \) with \( (C,i) \neq (D,j) \).

**Proof.** (i) If \( \text{RSR}(G, r, \overrightarrow{\rho}, u) \) is of the central quantum linear type, then \( \dim \mathfrak{B}(G, O_{u(C)}, \rho_C^{(i)}) < \infty \) for any \( C \in \mathcal{K}_r(G), i \in I_C(r,u) \). Let \( N_C^{(i)} := \text{ord}(q_{C,C}^{(i)}) \) \( (N_C^{(i)} = \infty \) when \( q_{C,C}^{(i)} \) is not a root of unit or \( q_{C,C}^{(i)} = 1 \)). By Lemma 3.10 \( \{ (a_{g_C,1}^{(i,j)})^m \mid 0 \leq m < N_C^{(i)} \} \) is linearly independent. Thus \( 1 < \text{ord}(q_{C,C}^{(i)}) < \infty \). Since \( \text{RSR}(G, r, \overrightarrow{\rho}, u) \) is quantum symmetric, \( q_{C,D}^{(i)} d_{D,C}^{(j)} = 1 \) by Lemma 3.9.

Conversely, by Lemma 3.9 \( \text{RSR}(G, r, \overrightarrow{\rho}, u) \) is quantum symmetric. It is clear that \( \text{RSR}(G, r, \overrightarrow{\rho}, u) \) is of the generalized quantum linear type. Thus it is of the non-essentially infinite type by Proposition 3.13.

(ii) It is similar to (i). □

The following is the consequence of Proposition 3.13 and Proposition 3.15.

**Proposition 3.15.** If \( \text{RSR}(G, r, \overrightarrow{\rho}, u) \) is of the central quantum linear type, then \( \mathfrak{B}(G, r, \overrightarrow{\rho}, u) \) is a generalized quantum linear space with the arrow PBW basis

\[
\{ b^{\nu_1} b^{\nu_2} \cdots b^{\nu_n} \mid \nu_1 < \nu_2, \cdots < \nu_n; 0 \leq m_s < N_{\nu_s}; 1 \leq s \leq n; \ n \in \mathbb{N} \} \quad (3.14)
\]

and

\[
\dim(\mathfrak{B}(G, r, \overrightarrow{\rho}, u)) = \prod_{C \in \mathcal{K}_r(G), i \in I_C(r,u)} (N_C^{(i)})^{\text{deg}(\rho_C^{(i)})|C|}, \quad (3.15)
\]

where \( \{ b\nu \mid \nu \in \Omega \} := Q_1^1 \) with total order \( < \) and \( N_{\nu} = \text{ord}(q_{C,C}^{(i)}) \) if \( b_{\nu} = a_{g_C,1}^{(i,j)} \).

In particular, if \( \text{RSR}(G, r, \overrightarrow{\rho}, u) \) is quantum weakly commutative and of \(-1\)-type with \( C \subseteq Z(G) \) for any \( C \in \mathcal{K}_r(G) \), then it is of the central quantum linear type with \( N_C^{(i)} = 2 \) and

\[
\dim(\mathfrak{B}(G, r, \overrightarrow{\rho}, u)) = 2 \sum_{C \in \mathcal{K}_r(G), i \in I_C(r,u)} (N_C^{(i)})^{\text{deg}(\rho_C^{(i)})|C|}. \quad (3.16)
\]
Remark 3.16. RSR\((G, r, \overrightarrow{\rho}, u)\) is called a central ramification system with irreducible representations (or \text{CRSR} in short) if \(C \subseteq Z(G)\) for any \(C \in K_r(G)\). If \(G\) is a real group and \(r = r_C C\), then \text{CRSR}(G, r, \overrightarrow{\rho}, u)\) is of finite type if and only only if \text{CRSR}(G, r, \overrightarrow{\rho}, u)\) is \(-1\)-type. Indeed, The necessity follows from Proposition 2.6, the sufficiency follows from Proposition 3.14 (i) since \(\varphi_{C,C}^{(i)} = -1\) for any \(i \in I_C(u, r)\).

4 Program

In this section the programs to compute the representatives of conjugacy classes, centralizers of these representatives and character tables of these centralizers in Weyl groups of exceptional type are given.

By using the programs in GAP, papers [ZWCY, ZWCYb] obtained the representatives of conjugacy classes of Weyl groups of exceptional type and all character tables of centralizers of these representatives. We use the results in [ZWCY, ZWCYb] and the following program in GAP for Weyl group \(W(E_6)\).

```gap
gap> L:=SimpleLieAlgebra("E",6,Rationals);;
gap> R:=RootSystem(L);;
gap> W:=WeylGroup(R);Display(Order(W));
gap> ccl:=ConjugacyClasses(W);;
gap> q:=NrConjugacyClasses(W);; Display(q);
gap> for i in [1..q] do
> r:=Order(Representative(ccl[i]));Display(r);;
> od; gap
> s1:=Representative(ccl[1]);cen1:=Centralizer(W,s1);;
gap> cl1:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[2]);cen1:=Centralizer(W,s1);;
gap> cl2:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[3]);cen1:=Centralizer(W,s1);;
gap> cl3:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[4]);cen1:=Centralizer(W,s1);;
gap> cl4:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[5]);cen1:=Centralizer(W,s1);;
gap> cl5:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[6]);cen1:=Centralizer(W,s1);;
gap> cl6:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[7]);cen1:=Centralizer(W,s1);;
gap> cl7:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[8]);cen1:=Centralizer(W,s1);;
```
gap> cl8:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[9]);;cen1:=Centralizer(W,s1);
gap> cl9:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[10]);;cen1:=Centralizer(W,s1);
gap> cl10:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[11]);;cen1:=Centralizer(W,s1);
gap> cl11:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[12]);;cen1:=Centralizer(W,s1);
gap> cl2:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[13]);;cen1:=Centralizer(W,s1);
gap> cl13:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[14]);;cen1:=Centralizer(W,s1);
gap> cl14:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[15]);;cen1:=Centralizer(W,s1);
gap> cl15:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[16]);;cen1:=Centralizer(W,s1);
gap> cl16:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[17]);;cen1:=Centralizer(W,s1);
gap> cl17:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[18]);;cen1:=Centralizer(W,s1);
gap> cl18:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[19]);;cen1:=Centralizer(W,s1);
gap> cl19:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[20]);;cen1:=Centralizer(W,s1);
gap> cl20:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[21]);;cen1:=Centralizer(W,s1);
gap> cl21:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[22]);;cen1:=Centralizer(W,s1);
gap> cl22:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[23]);;cen1:=Centralizer(W,s1);
gap> cl23:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[24]);;cen1:=Centralizer(W,s1);
gap> cl24:=ConjugacyClasses(cen1);
gap> s1:=Representative(ccl[25]);;cen1:=Centralizer(W,s1);
gap> cl25:=ConjugacyClasses(cen1);
gap> for i in [1..q] do
  > s:=Representative(ccl[i]);;cen:=Centralizer(W,s);
  > char:=CharacterTable(cen);;Display (cen);Display(char);
> od; gap> for i in [1..q] do
> s:=Representative(ccl[i]);;cen:=Centralizer(W,s);;
> cl:=ConjugacyClasses(cen);;t:=NrConjugacyClasses(cen);;
> for j in [1..t] do
> if s=Representative(cl[j]) then
> Display(j);break; > fi;od;
> od;

The programs for Weyl groups of $E_7$, $E_8$, $F_4$ and $G_2$ are similar. It is possible that the order of representatives of conjugacy classes of $G$ changes when one uses the program.

5 Tables about $-1$- type

In this section all $-1$- type bi-one Nichols algebras over Weyl groups of exceptional type up to graded pull-push YD Hopf algebra isomorphisms, are listed in table 1–12.

Table 1 is about Weyl group $W(E_6)$; Tables 2–4 are about Weyl group $W(E_7)$; Tables 5–10 are about Weyl group $W(E_8)$; Table 11 is about Weyl group $W(F_4)$; Table 12 is about Weyl group $W(G_2)$. 

\[
\begin{array}{|c|c|c|c|c|}
\hline
E_6 & cl_i[p] & Order(s_i) & \text{the } j \text{ such that } \mathfrak{B}(\mathcal{O}_{s_i}, \chi_i^{(j)}) \text{ is of } -1\text{-type} & \nu_i^{(1)} & \nu_i^{(2)} \\
\hline
s_1 & cl_1[1] & 1 & & 25 & 25 \\
\hline
s_2 & cl_2[3] & 4 & 4,5,6,7,17 & & 20 \ 15 \\
\hline
s_3 & cl_3[24] & 2 & 13,14,23,24,25 & & 25 \ 20 \\
\hline
s_4 & cl_4[17] & 4 & 2,4,10,15,16 & & 20 \ 15 \\
\hline
s_5 & cl_5[7] & 4 & 3,4,7,8 & & 16 \ 12 \\
\hline
s_6 & cl_6[2] & 2 & 9,10,11,12,16,17,18,19,25 & & 25 \ 16 \\
\hline
s_7 & cl_7[19] & 2 & 2,3,6,7,10,12,15,16,19,20 & & 20 \ 10 \\
\hline
s_8 & cl_8[26] & 3 & & & 27 \ 27 \\
\hline
s_9 & cl_9[2] & 6 & 2,4,13 & & 18 \ 15 \\
\hline
s_{10} & cl_{10}[2] & 2 & 2,3,7,8,11,12,15,16,19,20,22 & & 22 \ 11 \\
\hline
s_{11} & cl_{11}[14] & 6 & 3,4,13 & & 18 \ 15 \\
\hline
s_{12} & cl_{12}[27] & 3 & & & 27 \ 27 \\
\hline
s_{13} & cl_{13}[4] & 10 & 2 & & 10 \ 9 \\
\hline
s_{14} & cl_{14}[9] & 5 & & & 10 \ 10 \\
\hline
s_{15} & cl_{15}[13] & 4 & 3,4,6,13,14 & & 16 \ 11 \\
\hline
s_{16} & cl_{16}[3] & 8 & 2 & & 8 \ 7 \\
\hline
s_{17} & cl_{17}[3] & 6 & 13 & & 15 \ 14 \\
\hline
s_{18} & cl_{18}[9] & 12 & 2 & & 12 \ 11 \\
\hline
s_{19} & cl_{19}[11] & 6 & 3,4 & & 12 \ 10 \\
\hline
s_{20} & cl_{20}[2] & 9 & & & 9 \ 9 \\
\hline
s_{21} & cl_{21}[2] & 3 & & & 24 \ 24 \\
\hline
s_{22} & cl_{22}[13] & 6 & 2,4,13 & & 18 \ 15 \\
\hline
s_{23} & cl_{23}[3] & 12 & 2 & & 12 \ 11 \\
\hline
s_{24} & cl_{24}[19] & 6 & 10,11,12 & & 21 \ 18 \\
\hline
s_{25} & cl_{25}[4] & 6 & 3,4,13 & & 18 \ 15 \\
\hline
\end{array}
\]

Table 1
| $E_7$ | $s_i$ | $cl_i[p]$ | $Order(s_i)$ | the $j$ such that $\mathcal{B}(O_{s_i}, \chi^{(j)}_i)$ is of $-1$-type | $\nu_i^{(1)}$ | $\nu_i^{(2)}$ |
|-------|------|----------|--------|---------------------------------|-----|-----|
|       | $s_1$ | $cl_1[1]$ | 1      | $2,3,4,8,11,12,14,40,43,44$    | 60  | 60  |
|       | $s_2$ | $cl_2[16]$ | 18     | 2                               | 18  | 17  |
|       | $s_3$ | $cl_3[15]$ | 9      | 2                               | 18  | 18  |
|       | $s_4$ | $cl_4[2]$  | 3      | 2                               | 48  | 48  |
|       | $s_5$ | $cl_5[4]$  | 6      | 2                               | 48  | 38  |
|       | $s_6$ | $cl_6[2]$  | 2      | $2,3,4,5,9,10,13,14,17,18,19,20,25,26,27,28,30,$ | 60  | 30  |
|       |       |           |        | $32,35,36,38,41,42,44,46,48,50,52,54,56,58,60$ |     |     |
|       | $s_7$ | $cl_7[23]$ | 6      | $2,3,6,7,25,26$                | 36  | 30  |
|       | $s_8$ | $cl_8[23]$ | 3      | $2,3,4,5,9,10,13,14,17,18,19,20,25,26,27,28,30,$ | 54  | 54  |
|       | $s_9$ | $cl_9[2]$  | 2      | $2,3,4,5,9,10,13,14,17,18,19,20,25,26,27,$ | 90  | 45  |
|       |       |           |        | $28,33,35,36,39,40,43,44,45,46,47,48,50,52,54,55,$ |     |     |
|       |       |           |        | $56,57,58,67,68,69,70,75,77,79,80,83,84,87,88$ |     |     |
|       | $s_{10}$ | $cl_{10}[21]$ | 6      | $2,3,4,5,26,28$              | 36  | 30  |
|       | $s_{11}$ | $cl_{11}[2]$ | 2      | $2,3,7,8,11,12,13,14,17,18,19,20,25,26,27,$ | 74  | 37  |
|       |       |           |        | $28,33,35,36,39,40,42,44,47,48,51,52,55,56,58,61,$ |     |     |
|       |       |           |        | $62,65,66,69,70,73,74$         |     |     |
|       | $s_{12}$ | $cl_{12}[24]$ | 6      | $3,4,7,8,26,28$              | 36  | 30  |
|       | $s_{13}$ | $cl_{13}[4]$ | 2      | $2,3,7,8,11,12,13,14,19,20,23,24,25,26,$ | 74  | 42  |
|       |       |           |        | $27,28,37,38,39,40,42,43,45,46,49,50,$ |     |     |
|       |       |           |        | $55,56,57,59,60,63,64,69,70,73,74$ |     |     |
|       | $s_{14}$ | $cl_{14}[14]$ | 6      | $2,4,6,8,26,27,39,40,41,42$  | 60  | 50  |
|       | $s_{15}$ | $cl_{15}[3]$ | 3      | $2,4,6,8$                      | 66  | 66  |
|       | $s_{16}$ | $cl_{16}[35]$ | 4      | $3,4,7,8,11,12,15,16,34,36,38,40,51,52,$ | 80  | 60  |
|       |       |           |        | $55,56,59,60,63,64$           |     |     |
|       | $s_{17}$ | $cl_{17}[2]$ | 2      | $25,26,27,28,61,62,71,72,73,74,75,76,77,$ | 106 | 80  |
|       |       |           |        | $78,79,80,81,82,99,100,101,102,103,104,105,106$ |     |     |
|       | $s_{18}$ | $cl_{18}[72]$ | 4      | $17,18,19,20,21,22,23,24,25,26,27,28,29,$ | 76  | 48  |
|       |       |           |        | $30,31,32,33,34,35,36,45,46,47,48,49,50,51,52$ |     |     |
|       | $s_{19}$ | $cl_{19}[2]$ | 2      | $25,26,27,28,29,30,31,32,43,44,45,46,47,48,$ | 90  | 60  |
|       |       |           |        | $49,50,67,68,69,70,71,72,73,74,77,78,87,88,89,90$ |     |     |
|       | $s_{20}$ | $cl_{20}[4]$ | 8      | $2,4,6,8$                      | 32  | 28  |
|       | $s_{21}$ | $cl_{21}[3]$ | 4      | $5,6,7,8,10,12,33,34,35,36,37,38,39,40,41,$ | 60  | 40  |
|       |       |           |        | $42,43,44,50,52$               |     |     |
|       | $s_{22}$ | $cl_{22}[8]$ | 12     | $2,5,7,8$                      | 48  | 44  |
|       | $s_{23}$ | $cl_{23}[34]$ | 6      | $49,50,51,52$                  | 60  | 56  |

Table 2
| $E_7$ | $s_i$ | $\text{cl}_i[p]$ | $\text{Order}(s_i)$ | $j$ such that $\mathfrak{N}(O_{s_i}, \chi_{s_i}^{(j)})$ is of $-1$-type | $\nu_i^{(1)}$ | $\nu_i^{(2)}$
|---|---|---|---|---|---|---|
| $s_{24}$ | $\text{cl}_{24}[47]$ | 4 | $2,5,7,8,10,13,15,16,34,35,38,39,51,52,53,54,59,60,61,62$ | 80 | 60 |
| $s_{25}$ | $\text{cl}_{25}[15]$ | 8 | $3,4,7,8$ | 32 | 28 |
| $s_{26}$ | $\text{cl}_{26}[2]$ | 2 | $2,3,4,5,6,7,8,9,21,22,23,24,27,28,37,38,39,40,41,42,43,44,45,53,54,55,56,57,58,59,60,62,67,68,69,70,73,74,79,80$ | 106 | 53 |
| $s_{27}$ | $\text{cl}_{27}[51]$ | 6 | $2,3,6,7,27,28,37,38,39,40$ | 60 | 50 |
| $s_{28}$ | $\text{cl}_{28}[36]$ | 4 | $2,3,6,7,10,11,14,15,34,36,37,39,49,50,53,54,55,60,63,64$ | 80 | 60 |
| $s_{29}$ | $\text{cl}_{29}[2]$ | 2 | $2,3,4,5,6,7,8,9,19,20,23,24,29,30,31,32,37,38,39,40,45,46,47,48,53,54,55,56,61,62,63,64,69,70,71,72,75,76,79,80$ | 80 | 40 |
| $s_{30}$ | $\text{cl}_{30}[2]$ | 2 | $2,3,4,5,6,7,8,9,21,22,23,24,33,34,35,36,37,38,39,40,49,50,51,52,53,54,55,56,65,66,67,68,69,70,71,72,77,78,79,80$ | 80 | 40 |
| $s_{31}$ | $\text{cl}_{31}[22]$ | 4 | $2,3,4,5,10,11,12,13,21,22,23,24,29,30,31,32,34,36,38,40,41,42,45,46,49,50,53,54$ | 76 | 48 |
| $s_{32}$ | $\text{cl}_{32}[30]$ | 6 | $2,7,8,20,29,30,38$ | 42 | 35 |
| $s_{33}$ | $\text{cl}_{33}[30]$ | 6 | $2,3,4,7,8,26,28$ | 36 | 30 |
| $s_{34}$ | $\text{cl}_{34}[80]$ | 4 | $2,3,4,5,10,11,12,13,34,35,38,39,51,52,53,54,59,60,61,62$ | 80 | 60 |
| $s_{35}$ | $\text{cl}_{35}[14]$ | 12 | $2,4,6,8$ | 48 | 44 |
| $s_{36}$ | $\text{cl}_{36}[40]$ | 6 | $2,3,4,6,8,11,13,14,16,49,52$ | 60 | 50 |
| $s_{37}$ | $\text{cl}_{37}[49]$ | 6 | $2,3,4,5,26,27,37,38,43,44$ | 60 | 50 |
| $s_{38}$ | $\text{cl}_{38}[50]$ | 6 | $2,5,7,8,27,28,31,32,50$ | 54 | 45 |
| $s_{39}$ | $\text{cl}_{39}[7]$ | 6 | $2,3,6,7$ | 24 | 20 |
| $s_{40}$ | $\text{cl}_{40}[2]$ | 10 | $2,3$ | 20 | 18 |
| $s_{41}$ | $\text{cl}_{41}[21]$ | 5 | $2,3$ | 20 | 18 |
| $s_{42}$ | $\text{cl}_{42}[33]$ | 12 | $2,3,4,7,8$ | 48 | 44 |
| $s_{43}$ | $\text{cl}_{43}[39]$ | 6 | $19,20,21,22,27,28$ | 42 | 36 |
| $s_{44}$ | $\text{cl}_{44}[5]$ | 4 | $2,3,5,7,9,10,12,14,16,19,21,23,25,26,28,30,32$ | 64 | 48 |
| $s_{45}$ | $\text{cl}_{45}[6]$ | 6 | $2,3,5,16,19,20,39,40,51,52,62$ | 66 | 55 |
| $s_{46}$ | $\text{cl}_{46}[6]$ | 6 | $2,3,6,7$ | 24 | 20 |
| $s_{47}$ | $\text{cl}_{47}[5]$ | 10 | $2,4$ | 20 | 18 |
| $s_{48}$ | $\text{cl}_{48}[12]$ | 10 | $2,4,21$ | 30 | 27 |

Table 3
| $s_i$ | $\text{cl}_i[p]$ | $\text{Order}(s_i)$ | the $j$ such that $\mathcal{B}(O_{s_i}, \chi_i^{(j)})$ is of $-1$-type | $\nu_i^{(1)}$ | $\nu_i^{(2)}$ |
|---|---|---|---|---|---|
| $s_{49}$ | $\text{cl}_{49}[5]$ | 30 | 2 | 30 | 29 |
| $s_{50}$ | $\text{cl}_{50}[15]$ | 15 | | 30 | 30 |
| $s_{51}$ | $\text{cl}_{51}[8]$ | 7 | | 14 | 14 |
| $s_{52}$ | $\text{cl}_{52}[2]$ | 14 | 2 | 14 | 13 |
| $s_{53}$ | $\text{cl}_{53}[53]$ | 6 | 3,4,7,8,26,28,39,40,43,44 | 60 | 50 |
| $s_{54}$ | $\text{cl}_{54}[5]$ | 8 | 3,5,6,8 | 32 | 28 |
| $s_{55}$ | $\text{cl}_{55}[10]$ | 12 | 3,4 | 24 | 22 |
| $s_{56}$ | $\text{cl}_{56}[14]$ | 8 | 3,5,6,8 | 32 | 28 |
| $s_{57}$ | $\text{cl}_{57}[3]$ | 4 | 2,3,5,6,10,12,21,22,23,24,31,32,37,38,39,40,41,42,50,52 | 60 | 40 |
| $s_{58}$ | $\text{cl}_{58}[15]$ | 12 | 2,4 | 24 | 22 |
| $s_{59}$ | $\text{cl}_{59}[38]$ | 12 | 3,5,6,8 | 48 | 44 |
| $s_{60}$ | $\text{cl}_{60}[9]$ | 4 | 2,4,6,8,10,12,14,16,18,20,22,24,26,28,30,32 | 64 | 48 |

Table 4
| $E_8$ | $s_i$ | $cl_i[p]$ | $Order(s_i)$ | the $j$ such that $\mathfrak{B}(O_{s_i},\chi_{i}^{(j)})$ is of $-1$-type | $\nu_{i}^{(1)}$ | $\nu_{i}^{(2)}$ |
|-------|-------|----------|-------------|-------------------------------------------------|--------------|--------------|
| $s_1$ | cl$_1[1]$ | 1 | | | 112 | 112 |
| $s_2$ | cl$_2[29]$ | 30 | 2 | | 30 | 29 |
| $s_3$ | cl$_3[23]$ | 15 | | | 30 | 30 |
| $s_4$ | cl$_4[2]$ | 5 | | | 45 | 45 |
| $s_5$ | cl$_5[3]$ | 3 | | | 102 | 102 |
| $s_6$ | cl$_6[6]$ | 10 | 6,7,27,41 | | 45 | 41 |
| $s_7$ | cl$_7[2]$ | 2 | 3,4,11,12,16,17,18,19,29,30,32,33,34, | | 112 | 67 |
|       |       | | 37,38,45,46,51,52,56,57,60,63,64,65,66,71,79, | |       |       |
|       |       | | 80,82,83,89,90,91,92,95,96,99,100,103,104, | |       |       |
|       |       | | 106,107,108,112 | |       |       |
| $s_8$ | cl$_8[4]$ | 6 | 4,5,31,33,34,35,36,61,62,79,81,82,92,97 | | 102 | 88 |
| $s_9$ | cl$_9[4]$ | 30 | 2,4 | | 60 | 58 |
| $s_{10}$ | cl$_{10}[13]$ | 15 | | | 60 | 60 |
| $s_{11}$ | cl$_{11}[5]$ | 5 | | | 70 | 70 |
| $s_{12}$ | cl$_{12}[2]$ | 3 | | | 150 | 150 |
| $s_{13}$ | cl$_{13}[46]$ | 10 | 2,4,6,8,41,44 | | 60 | 54 |
| $s_{14}$ | cl$_{14}[3]$ | 2 | 2,4,6,8,10,12,14,16,18,20,22,24,27,28,31,32, | | 120 | 60 |
|       |       | | 34,36,38,40,42,44,48,49,50,54,55,56,58,60,62,64, | |       |       |
|       |       | | 67,68,71,72,74,76,79,80,83,84,86,88,90,92,94,96, | |       |       |
|       |       | | 98,100,102,104,106,108,110,112,114,116,118,120 | |       |       |
| $s_{15}$ | cl$_{15}[8]$ | 6 | 2,3,6,7,29,30,31,32,37,38,39,40,77,78, | | 132 | 110 |
|       |       | | 79,80,101,102,103,104,123,124 | |       |       |
| $s_{16}$ | cl$_{16}[16]$ | 30 | 2,3 | | 60 | 58 |
| $s_{17}$ | cl$_{17}[9]$ | 10 | 2,3,23,24,43,44,62 | | 70 | 63 |
| $s_{18}$ | cl$_{18}[4]$ | 6 | 2,3,5,16,26,35,36,37,38,57,58,60,75,76, | | 150 | 125 |
|       |       | | 87,88,99,100,102,117,118,128,135,136,146 | |       |       |
| $s_{19}$ | cl$_{19}[23]$ | 20 | 3,4 | | 40 | 38 |
| $s_{20}$ | cl$_{20}[40]$ | 10 | 41,42 | | 50 | 48 |
| $s_{21}$ | cl$_{21}[8]$ | 2 | 9,10,43,44,53,54,55,56,77,78,87,88,89,90, | | 167 | 130 |
|       |       | | 91,92,109,110,111,112,121,134,151,152,153,154, | |       |       |
|       |       | | 155,156,157,158,159,162,163,164,165,166,167 | |       |       |
| $s_{22}$ | cl$_{22}[3]$ | 4 | 2,3,4,5,19,20,23,24,35,36,39,40,43,44,47, | | 144 | 108 |
|       |       | | 48,66,68,77,78,79,80,85,86,87,88,90,92,115, | |       |       |
|       |       | | 116,119,120,131,132,135,136 | |       |       |
| $s_{23}$ | cl$_{23}[39]$ | 10 | 3,4,7,8,42,44 | | 60 | 54 |

Table 5

26
| $E_8$ | $s_i$ | cl$_i[p]$ | Order($s_i$) | the $j$ such that $\mathcal{B}(O_{s_i}, \chi^{(j)}_{s_i})$ is of $-1$-type | $\nu_i^{(1)}$ | $\nu_i^{(2)}$ |
|-------|-----|----------|-------------|---------------------------------------------|----------|----------|
| $s_{24}$ | cl$_{24}[2]$ | 2 | 2,3,7,8,11,12,15,16,19,20,23,24,29,30,31,32,35,36,39,40,43,44,51,52,53,54,55,56,59,60,63,64,69,70,71,72,75,76,81,82,83,84,87,88,91,92,95,96,99,100,103,104,107,108,111,112,115,116,119,120 | 120 | 60 |
| $s_{25}$ | cl$_{25}[44]$ | 10 | 3,5,6,8,41 | 50 | 45 |
| $s_{26}$ | cl$_{26}[2]$ | 2 | 27,28,29,30,31,32,33,34,55,56,79,80,81,82,83,84,87,88,91,92,95,96,99,100,101,102,103,104,105,106,107,108,122,123,124,125,126,127,128,130,131,132,133,134,151,152,153,154,157,158,159,160,161,162,163 | 167 | 105 |
| $s_{27}$ | cl$_{27}[28]$ | 20 | 2,3 | 40 | 38 |
| $s_{28}$ | cl$_{28}[4]$ | 4 | 2,3,4,5,19,20,23,24,37,38,39,40,45,46,47,48,66,68,78,79,80,81,82,83,84,89,90,91,92,115,116,119,120,131,132,135,136 | 144 | 108 |
| $s_{29}$ | cl$_{29}[3]$ | 4 | 2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,43,44 | 160 | 120 |
| $s_{30}$ | cl$_{30}[3]$ | 4 | 2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,27,28,29,30,33,34,35,36,57,58,59,60,73,76 | 80 | 60 |
| $s_{31}$ | cl$_{31}[72]$ | 6 | 25,26,31,32,39,40,45,46,68 | 72 | 63 |
| $s_{32}$ | cl$_{32}[2]$ | 3 | 135 | 135 |
| $s_{33}$ | cl$_{33}[58]$ | 8 | 3,4,7,8,34,35,49,50,55,56 | 80 | 70 |
| $s_{34}$ | cl$_{34}[3]$ | 4 | 5,6,7,8,11,12,15,16,29,30,31,32,38,39,40,41,42,55,56,57,58,63,64,65,66,85,86,87,88,89,90,91,92,98,99,100,101,102,108,119,120,131,132,135,136 | 140 | 95 |
| $s_{35}$ | cl$_{35}[2]$ | 2 | 31,32,33,34,35,36,37,38,63,64,65,66,67,68,69,70,71,72,73,74,104,105,106,107,108,109,110,111,112,113,114,115,118,119,130,131,132,133,146,147,148,149,150,151,152,153,154,155,156,157,159,168,169,170,171,172,173,174,175,176,178,182,183,198,199,200,201,202,203,208,209,210,211,213,215 | 215 | 140 |
| $s_{36}$ | cl$_{36}[64]$ | 8 | 2,4,6,8,34,36,51,52,55,56 | 80 | 70 |
| $s_{37}$ | cl$_{37}[44]$ | 4 | 9,10,11,12,13,14,15,16,17,18,21,22,23,43,44 | 44 | 30 |

Table 6
| $E_8$ | $s_i$ | $\text{cl}_i[p]$ | $\text{Order}(s_i)$ | the $j$ such that $\mathcal{B}(O_{s_i}, \chi_i^{(j)})$ is of $-1$-type | $\nu_i^{(1)}$ | $\nu_i^{(2)}$ |
|-------|-------|-----------------|-------------------|------------------------------------------|---------|---------|
|       | $s_{38}$ | $\text{cl}_{38}[2]$ | 2 | 21,22,23,24,25,26,27,28,33,34,35,36,41, 42,43,44,69,70,71,72,75,76,79,80,89,90,91,92, 93,94,95,96,97,98,99,100,101,102,103,104 | 105 | 65 |
|       | $s_{39}$ | $\text{cl}_{39}[73]$ | 4 | 4,5,8,9,10,11,14,15,20,21,24,25,26,27, 30,31,66,67,70,71,75,76,77,78,81,82,87,88 | 112 | 84 |
|       | $s_{40}$ | $\text{cl}_{40}[13]$ | 24 | 2,3 | 48 | 46 |
|       | $s_{41}$ | $\text{cl}_{41}[7]$ | 12 | 5,6,7,8,27,28,65,66,71,72 | 96 | 86 |
|       | $s_{42}$ | $\text{cl}_{42}[4]$ | 6 | 49,50,51,52,53,54,55,56,91,92,93,94,95, 96,97,98,145,146 | 150 | 132 |
|       | $s_{43}$ | $\text{cl}_{43}[98]$ | 12 | 4,5,6,7,12,13,14,15,98,100 | 120 | 110 |
|       | $s_{44}$ | $\text{cl}_{44}[4]$ | 6 | 57,58,59,60,133,134,139,140,141,142 | 150 | 140 |
|       | $s_{45}$ | $\text{cl}_{45}[3]$ | 4 | 5,6,7,8,11,12,15,16,29,30,31,32,33,34,35, 36,38,55,56,57,58,61,62,63,64,85,86,87,88,89,90, 91,92,93,94,95,96,98,108,119,120,131,132,133,134 | 140 | 95 |
|       | $s_{46}$ | $\text{cl}_{46}[19]$ | 8 | 2,4,6,8,9,10,12,14,16 | 64 | 56 |
|       | $s_{47}$ | $\text{cl}_{47}[3]$ | 4 | 33,34,35,36,37,38,39,40,41,42,43,44,45,46, 47,48,49,50,51,52,53,54,55,56,57,58,59,60,61,62, 63,64,69,70,71,72,77,78,79,80,81,82,83,84,85,86, 87,88,137,138,139,140,141,142 | 178 | 114 |
|       | $s_{48}$ | $\text{cl}_{48}[4]$ | 6 | 2,3,4,5,6,7,8,9,51,52,55,56,59,60,89,90,91, 92,97,98,99,100,134,141,142 | 150 | 125 |
|       | $s_{49}$ | $\text{cl}_{49}[4]$ | 8 | 4,5,6,7,33,34,35,36,66,67 | 80 | 70 |
|       | $s_{50}$ | $\text{cl}_{50}[3]$ | 4 | 2,3,4,5,9,10,11,12,18,20,22,24,41,42,43,44, 49,50,57,58,59,60,63,64,69,70,71,72,75,76,77,78, 79,80,85,86,98,100,102,104 | 120 | 80 |
|       | $s_{51}$ | $\text{cl}_{51}[3]$ | 4 | 2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,66, 67,70,71,73,76,77,80,97,98,103,104,105,106,111, 112,115,116,117,118,123,124,125,126 | 160 | 120 |
|       | $s_{52}$ | $\text{cl}_{52}[9]$ | 6 | 3,4,6,9,10,13,15,16,49,52,54,55 | 72 | 60 |

Table 7

| $s_i$ | $\text{cl}_i[p]$ | Order $s_i$ | the $j$ such that $\mathfrak{B}(O_{s_i}, \chi_i^{(j)})$ is of $-1$-type | $\nu_i^{(1)}$ | $\nu_i^{(2)}$ |
|------|------------------|-------------|-------------------------------------------------|-------------|-------------|
| $s_{53}$ | $\text{cl}_{53}[2]$ | 2 | 2,3,4,5,6,7,8,9,19,20,23,24,27,28,31,32, 37,38,39,40,45,46,47,48,53,54,55,56,61,62,63,64, 66,68,71,72,75,76,79,80,83,84,89,90,91,92,97,98, 99,100,105,106,107,108,113,114,115,116,121, 122,123,124,129,130,131,132,137,138,139,140, 145,146,147,148,150,152,154,156,159,160,163, 164,167,168,171,172,175,176,179,180 | 180 | 90 |
| $s_{54}$ | $\text{cl}_{54}[19]$ | 12 | 2,4,6,8 | 48 | 44 |
| $s_{55}$ | $\text{cl}_{55}[4]$ | 6 | 37,38,39,40,43,44,45,46,59,60,61,62,103, 104,105,106,113,114 | 126 | 108 |
| $s_{56}$ | $\text{cl}_{56}[2]$ | 3 |  | 144 | 144 |
| $s_{57}$ | $\text{cl}_{57}[4]$ | 6 | 2,3,5,6,7,8,15,16,18,23,24,25,26,29,30,33,34, 60,63,64,66,105,106,111,112,113,114,136,139,140 | 144 | 114 |
| $s_{58}$ | $\text{cl}_{58}[32]$ | 6 | 2,4,7,9,11,13,14,16,53,54,55,56,61,62, 63,64,99,100 | 108 | 90 |
| $s_{59}$ | $\text{cl}_{59}[42]$ | 8 | 3,5,6,8,11,13,14,16 | 64 | 56 |
| $s_{60}$ | $\text{cl}_{60}[10]$ | 6 | 6,7,8,9,10,11,12,13 | 48 | 40 |
| $s_{61}$ | $\text{cl}_{61}[70]$ | 6 | 2,3,4,5,10,11,12,13,50,52,54,56 | 72 | 60 |
| $s_{62}$ | $\text{cl}_{62}[2]$ | 2 | 2,3,4,5,6,7,8,9,21,22,23,24,29,30,31,32,41, 42,43,44,45,46,47,48,57,58,59,60,61,62,63,64,67, 68,73,74,75,76,81,82,83,84,93,94,95,96,97,98,99, 100,117,118,119,120,121,122,123,124,125,126, 127,128,129,130,131,132,141,142,143,144,145, 146,147,148,151,152,155,156,161,162,163, 164,169,170,171,172,177,178,179,180 | 180 | 90 |
| $s_{63}$ | $\text{cl}_{63}[4]$ | 6 | 25,26,33,34,51,52,53,54,68,77,78,79,80, 109,112,113,130,131 | 135 | 117 |
| $s_{64}$ | $\text{cl}_{64}[36]$ | 6 | 2,3,15,16,21,22,38,39,47,48,53,54,74,75 | 84 | 70 |
| $s_{65}$ | $\text{cl}_{65}[28]$ | 18 | 2,4,37 | 54 | 51 |
| $s_{66}$ | $\text{cl}_{66}[26]$ | 9 |  | 54 | 54 |
| $s_{67}$ | $\text{cl}_{67}[18]$ | 18 | 3,4 | 36 | 34 |
| $s_{68}$ | $\text{cl}_{68}[4]$ | 6 | 2,3,5,6,7,8,15,16,21,22,23,24,27,28,79, 80,85,86,87,88 | 96 | 76 |
| $s_{69}$ | $\text{cl}_{69}[12]$ | 18 | 2,4 | 36 | 34 |
| $s_{70}$ | $\text{cl}_{70}[4]$ | 6 | 2,3,5,6,7,8,14,15,19,20,21,22,26,27,78, 79,83,84,85,86 | 96 | 76 |

Table 8
| $E_8$ | $s_i$ | $\text{cl}_i[p]$ | $\text{Order}(s_i)$ | the $j$ such that $\mathcal{B}(O_{s_i}, \chi_j^{(i)})$ is of $-1$-type | $\nu_i^{(1)}$ | $\nu_i^{(2)}$ |
|-------|-------|------------------|---------------------|-------------------------------------------------|-----------|-----------|
|       | $s_{71}$ | $\text{cl}_{71}[18]$ | 9 | | 54 | 54 |
|       | $s_{72}$ | $\text{cl}_{72}[24]$ | 18 | 2,5,6 | 54 | 51 |
|       | $s_{73}$ | $\text{cl}_{73}[40]$ | 12 | 4,5,6,7 | 48 | 44 |
|       | $s_{74}$ | $\text{cl}_{74}[3]$ | 4 | 2,3,4,5,9,10,11,12,17,20,22,23,41,42, 43,44,49,50,57,58,59,60,63,64,69,70,71,72, 75,76,77,78,79,80,85,86,97,100,102,103 | 120 | 80 |
|       | $s_{75}$ | $\text{cl}_{75}[7]$ | 12 | 2,3,5,6,27,28,61,62,65,66 | 96 | 86 |
|       | $s_{76}$ | $\text{cl}_{76}[12]$ | 12 | 2,3,6,7,49,52 | 72 | 66 |
|       | $s_{77}$ | $\text{cl}_{77}[36]$ | 12 | 2,9,10,38,47,48,74 | 84 | 77 |
|       | $s_{78}$ | $\text{cl}_{78}[4]$ | 6 | 29,56,57,58,59,60,61,62,63,100,101, 102,103,104,105,106,107,108,127,129,130 | 147 | 126 |
|       | $s_{79}$ | $\text{cl}_{79}[43]$ | 12 | 7,8,14,37,38 | 48 | 43 |
|       | $s_{80}$ | $\text{cl}_{80}[4]$ | 4 | 11,12,17,18,19,20,26,35,36,39,40,41, 46,47,48,49 | 59 | 43 |
|       | $s_{81}$ | $\text{cl}_{81}[15]$ | 20 | 2 | 20 | 19 |
|       | $s_{82}$ | $\text{cl}_{82}[7]$ | 12 | 2,3,4,5,50,52,75,76,79,80 | 120 | 110 |
|       | $s_{83}$ | $\text{cl}_{83}[3]$ | 4 | 2,3,4,5,6,7,8,9,35,36,39,40,43,44,47, 48,69,70,71,72,77,78,79,80,85,86,87,88,93, 94,95,96,130,132,139,140,143,144,147,148, 151,152,173,174,175,176,181,182,183,184 | 200 | 150 |
|       | $s_{84}$ | $\text{cl}_{84}[107]$ | 12 | 4,5,6,7,12,13,14,15,98,100 | 120 | 110 |
|       | $s_{85}$ | $\text{cl}_{85}[11]$ | 14 | 2,3 | 28 | 26 |
|       | $s_{86}$ | $\text{cl}_{86}[26]$ | 7 | | 28 | 28 |
|       | $s_{87}$ | $\text{cl}_{87}[3]$ | 14 | 3,4 | 28 | 26 |
|       | $s_{88}$ | $\text{cl}_{88}[18]$ | 14 | 2,4 | 28 | 26 |
|       | $s_{89}$ | $\text{cl}_{89}[100]$ | 6 | 3,5,7,9,10,12,14,16,53,54,55,56,61,62, 63,64,99,100 | 108 | 90 |
|       | $s_{90}$ | $\text{cl}_{90}[56]$ | 6 | 3,4,23,24,29,30,38,39,43,44,57,58,73,76 | 84 | 70 |

Table 9
| \( E_8 \) | \( s_i \) | \( \text{cl}_i[p] \) | \( \text{Order}(s_i) \) | \( \text{the } j \text{ such that } \mathcal{B}(O_{s_i}, \chi_j^{(1)}) \text{ is of } -1\text{-type} \) | \( \nu_1^{(1)} \) | \( \nu_2^{(2)} \) |
|---|---|---|---|---|---|---|
| \( s_{91} \) | \( \text{cl}_{91}[29] \) | 8 | 2,3,10,23,24 | 32 | 27 |
| \( s_{92} \) | \( \text{cl}_{92}[8] \) | 24 | 2 | 24 | 23 |
| \( s_{93} \) | \( \text{cl}_{93}[9] \) | 12 | 2,4,6,8,50,52 | 72 | 66 |
| \( s_{94} \) | \( \text{cl}_{94}[4] \) | 6 | 2,3,5,6,7,8,38,41,42,51,52,75,76,77,78,93,94,104,113,114,122 | 126 | 105 |
| \( s_{95} \) | \( \text{cl}_{95}[35] \) | 12 | 2,4,17,18,29,30 | 72 | 66 |
| \( s_{96} \) | \( \text{cl}_{96}[60] \) | 6 | 31,32,33,34,67,68 | 72 | 66 |
| \( s_{97} \) | \( \text{cl}_{97}[7] \) | 12 | 2,3,4,5,50,52,75,76,79,80 | 120 | 110 |
| \( s_{98} \) | \( \text{cl}_{98}[19] \) | 12 | 2,4 | 24 | 22 |
| \( s_{99} \) | \( \text{cl}_{99}[59] \) | 12 | 6,7,8,9,10,11,12,13 | 96 | 88 |
| \( s_{100} \) | \( \text{cl}_{100}[4] \) | 6 | 2,3,4,5,6,7,8,9,50,52,54,56,75,76,79,80,83,84,87,88 | 120 | 100 |
| \( s_{101} \) | \( \text{cl}_{101}[80] \) | 12 | 4,5,8,9,10,11,14,15 | 96 | 88 |
| \( s_{102} \) | \( \text{cl}_{102}[84] \) | 6 | 2,3,6,7,10,11,14,15,51,52,53,54,59,60,61,62,98,99 | 108 | 90 |
| \( s_{103} \) | \( \text{cl}_{103}[82] \) | 12 | 2,9,10,37,39,40,74 | 84 | 77 |
| \( s_{104} \) | \( \text{cl}_{104}[54] \) | 8 | 3,5,6,8,17,18,25,26,65,68 | 80 | 70 |
| \( s_{105} \) | \( \text{cl}_{105}[35] \) | 30 | 2,4 | 60 | 58 |
| \( s_{106} \) | \( \text{cl}_{106}[6] \) | 6 | 2,3,4,5,27,28,31,32,35,36,39,40,75,76,79,80,99,100,103,104,122,124 | 132 | 110 |
| \( s_{107} \) | \( \text{cl}_{107}[11] \) | 8 | 3,4 | 16 | 14 |
| \( s_{108} \) | \( \text{cl}_{108}[4] \) | 6 | 2,3,4,5,27,28,31,32,35,36,54,55,56,58,81,82,95,96,97,98,119,120,121,122,140,146 | 150 | 125 |
| \( s_{109} \) | \( \text{cl}_{109}[4] \) | 6 | 2,3,4,5,6,7,8,9,49,50,51,52,73,74,75,76,77,78,79,80 | 120 | 100 |
| \( s_{110} \) | \( \text{cl}_{110}[2] \) | 24 | 3,4 | 48 | 46 |
| \( s_{111} \) | \( \text{cl}_{111}[23] \) | 12 | 2,4,26 | 36 | 33 |
| \( s_{112} \) | \( \text{cl}_{112}[92] \) | 6 | 3,5,6,8,23,24,35,36,49,50,61,62,73,76,77,78,101,102 | 108 | 90 |

Table 10
| $F_4$ | $s_i$ | $\text{cl}_i[p]$ | $\text{Order}(s_i)$ | $j$ such that $\mathcal{B}(\mathcal{O}_{s_i}, \chi^{(j)}_i)$ is of $-1$-type | $\nu_i^{(1)}$ | $\nu_i^{(2)}$ |
|------|------|------------------|-----------------|---------------------------------------|-----------|-----------|
|      | $s_1$ | $\text{cl}_1[1]$ | 1               |                                        | 25        | 25        |
|      | $s_2$ | $\text{cl}_2[2]$ | 2               | 9,10,11,12,16,17,18,19,25              | 25        | 16        |
|      | $s_3$ | $\text{cl}_3[25]$ | 2               | 17,18,19,20,25                         | 25        | 20        |
|      | $s_4$ | $\text{cl}_4[16]$ | 4               | 3,4,6,13,14                            | 16        | 11        |
|      | $s_5$ | $\text{cl}_5[5]$  | 3               |                                        | 18        | 18        |
|      | $s_6$ | $\text{cl}_6[15]$ | 6               | 3,4,13                                 | 18        | 15        |
|      | $s_7$ | $\text{cl}_7[10]$ | 2               | 2,3,4,5,10,12,15,16,19,20              | 20        | 10        |
|      | $s_8$ | $\text{cl}_8[19]$ | 2               | 2,4,6,8,9,10,13,14,15,16               | 20        | 10        |
|      | $s_9$ | $\text{cl}_9[16]$ | 4               | 3,5,6,8                                | 16        | 12        |
|      | $s_{10}$ | $\text{cl}_{10}[16]$ | 3             |                                        | 18        | 18        |
|      | $s_{11}$ | $\text{cl}_{11}[12]$ | 6             | 3,4,13                                  | 18        | 15        |
|      | $s_{12}$ | $\text{cl}_{12}[9]$ | 3             |                                        | 21        | 21        |
|      | $s_{13}$ | $\text{cl}_{13}[21]$ | 6             | 10,11,12                                | 21        | 18        |
|      | $s_{14}$ | $\text{cl}_{14}[11]$ | 12            | 2                                       | 12        | 11        |
|      | $s_{15}$ | $\text{cl}_{15}[10]$ | 6             | 2,3                                     | 12        | 10        |
|      | $s_{16}$ | $\text{cl}_{16}[12]$ | 6             | 2,4                                     | 12        | 10        |
|      | $s_{17}$ | $\text{cl}_{17}[19]$ | 2             | 2,3,4,6,7,10,12,15,16,19,20            | 20        | 10        |
|      | $s_{18}$ | $\text{cl}_{18}[18]$ | 2             | 2,4,6,8,9,12,13,14,15,16,19,20         | 20        | 10        |
|      | $s_{19}$ | $\text{cl}_{19}[6]$  | 4             | 2,4,6,8                                | 16        | 12        |
|      | $s_{20}$ | $\text{cl}_{20}[11]$ | 6             | 2,3                                     | 12        | 10        |
|      | $s_{21}$ | $\text{cl}_{21}[12]$ | 6             | 2,4                                     | 12        | 10        |
|      | $s_{22}$ | $\text{cl}_{22}[2]$  | 2             | 6,7,8,9,10,11,12,13                    | 16        | 8         |
|      | $s_{23}$ | $\text{cl}_{23}[4]$  | 8             | 2                                       | 8         | 7         |
|      | $s_{24}$ | $\text{cl}_{24}[17]$ | 4             | 2,4,6,8,18                             | 20        | 15        |
|      | $s_{25}$ | $\text{cl}_{25}[9]$  | 4             | 2,4,6,8,17                             | 20        | 15        |

Table 11
In this section all $-1$-type bi-one Nichols algebra over Weyl groups $G$ of exceptional type up to graded pull-push YD Hopf algebra isomorphisms are given.

In Table 1–12, we use the following notations. $s_i$ denotes the representative of $i$-th conjugacy class of $G$ ($G$ is the Weyl group of exceptional type); $\chi_{i}^{(j)}$ denotes the $j$-th character of $G_{s_i}$ for any $i$; $\nu_{i}^{(1)}$ denotes the number of conjugacy classes of the centralizer $G_{s_i}$; $\nu_{i}^{(2)}$ denote the number of character $\chi_{i}^{(j)}$ of $G_{s_i}$ with non $-1$-type $\mathfrak{B}(O_{s_i},\chi_{i}^{(j)})$; cl$_i[j]$ denote that $s_i$ is in $j$-th conjugacy class of $G_{s_i}$.

We give one of the main results.

**Theorem 1.** Let $G$ be a Weyl group of exceptional type. Then

(i) For any bi-one Nichols algebra $\mathfrak{B}(O_{s},\chi)$ over Weyl group $G$, there exist $s_i$ in the first column of the table of $G$ and $j$ with $1 \leq j \leq \nu_{i}^{(1)}$ such that $(kG, \mathfrak{B}(O_{s_i},\chi_{i}^{(j)})) \cong (kG, \mathfrak{B}(O_{s_i},\chi_{i}^{(j)}))$ as graded pull-push YD Hopf algebras;

(ii) $\mathfrak{B}(O_{s_i},\chi_{i}^{(j)})$ is of $-1$-type if and only if $j$ appears in the fourth column of the table of $G$;

(iii) dim($\mathfrak{B}(O_{s_i},\chi_{i}^{(j)})) = \infty$ if $j$ does not appears in the fourth column of the table of $G$.

**Proof.** (i) We assume that $G$ is the Weyl group of $E_6$ without loss of generality. There exists $s_i$ such that $s_i$ and $s$ are in the same conjugacy class since $s_1, s_2, \cdots, s_{25}$ are the representatives of all conjugacy classes of $G$. Lemma 1.1 and [ZCZ] The remark of Pro. 1.5] or Proposition 1.5 yield that there exists $j$ such that $(kG, \mathfrak{B}(O_{s_i},\chi_{i}^{(j)})) \cong (kG, \mathfrak{B}(O_{s_i},\chi_{i}^{(j)}))$ as graded pull-push YD Hopf algebras, since $\chi_{i}^{(1)}, \chi_{i}^{(2)}, \cdots, \chi_{i}^{(\nu_{i}^{(1)})}$ are all characters of all irreducible representations of $G_{s_i}$.

(ii) It follows from the program.

| $G_2$ | $s_i$ | Order($s_i$) | $j$ such that $\mathfrak{B}(O_{s_i},\chi_{i}^{(j)})$ is of $-1$-type | $\nu_{i}^{(1)}$ | $\nu_{i}^{(2)}$ |
|-------|------|-------------|-------------------------------------------------|-------------|-------------|
| $s_1$ | cl$_1[1]$ | 1 | | 6 | 6 |
| $s_2$ | cl$_2[3]$ | 2 | 2,4 | 4 | 2 |
| $s_3$ | cl$_3[3]$ | 2 | 2,4 | 4 | 2 |
| $s_4$ | cl$_4[4]$ | 2 | 3,4,5 | 6 | 3 |
| $s_5$ | cl$_5[3]$ | 6 | 2 | 6 | 5 |
| $s_6$ | cl$_6[5]$ | 3 | | 6 | 6 |

Table 12

6 Bi-one Nichols algebras over Weyl groups of exceptional type
(iii) It follows from Lemma 7.3 \(\square\)

By [Ca72], \(W(G_2)\) is isomorphic to dihedral group \(D_6\). Set \(y = s_5\) and \(x = s_3\). It is clear that \(xyx = y^{-1}\) with \(\text{ord}(y) = 6\) and \(\text{ord}(x) = 2\). Thus it follows from [AF07] Table 2 that \(\dim(\mathfrak{B}(C_{s_5}, \chi_5^{(2)})) = 4 < \infty\).

It is clear that if there exists \(\phi \in \text{Aut}(G)\) such that \(\phi(s_i) = s_j\) then \(\text{ord}(s_i) = \text{ord}(s_j), \nu_1^{(1)} = \nu_2^{(1)}, \nu_1^{(2)} = \nu_2^{(2)}\) for Weyl group \(G\) of exceptional type. Consequently, the representative system of iso-conjugacy classes of \(W(E_6)\) is \(\{s_i \mid 1 \leq i \leq 25\}\). The representative system of iso-conjugacy classes of \(W(F_4)\) is \(\{s_i \mid 1 \leq i \leq 25, i \neq 8, 10, 11, 16, 17, 18, 19, 20, 21, 25\}\). The representative system of iso-conjugacy classes of \(W(G_2)\) is \(\{s_1, s_2, s_4, s_5, s_6\}\).

7 Pointed Hopf algebras over Weyl groups of exceptional type

In this section all central quantum linear spaces over Weyl groups of exceptional type are found.

**Lemma 7.1.** \(Z(W(E_6)) = \{1\}; Z(W(E_7)) = \{1, s_6\}; Z(W(E_8)) = \{1, s_7\}; Z(W(F_4)) = \{1, s_2\}; Z(W(G_2)) = \{1, s_4\}.\)

**Proof.** If \(s_i \in Z(G)\), then \(G^{s_i} = G\).

(i) Let \(G = W(W_6)\). The number of conjugacy classes of \(G\) is 25 by table 1. The numbers of conjugacy classes of both \(G^{s_3}\) and \(G^{s_6}\) also are 25. \(G, G^{s_3}\), and \(G^{s_6}\) have 16, 8 and 4 one dimensional representations, respectively, according to the character tables in [ZWCYA]. Thus \(s_3\) and \(s_6\) do not belong to the center of \(G\).

(ii) Let \(G = W(W_7)\). The number of conjugacy classes of \(G, G^{s_6}, G^{s_{14}}, G^{s_{21}}, G^{s_{23}}, G^{s_{27}}, G^{s_{36}}, G^{s_{37}}, G^{s_{53}}, G^{s_{57}}\) is 60 by table 1–4. They have 2, 2, 24, 8, 3, 24, 48, 24, 24 and 8 one dimensional representations, respectively, according to the character tables in [ZWCYA]. Thus they do not belong to the center of \(G\) but \(s_6\). Obviously \(s_6 \in Z(G)\).

(iii) Let \(G = W(W_8)\). The number of conjugacy classes of \(G, G^{s_7}\), and \(G^{s_{39}}\) is 112 by table 5–10. They have 2, 2 and 64 one dimensional representations, respectively, according to the character tables in [ZWCYA]. Thus \(s_{39}\) does not belong to the center of \(G\). Obviously \(s_7 \in Z(G)\).

(iv) Let \(G = W(F_4)\). The number of conjugacy classes of \(G, G^{s_2}\), and \(G^{s_{3}}\) is 25 by table 11. They have 4, 4 and 16 one dimensional representations, respectively, according to the character tables in [ZWCYA]. Thus \(s_3\) does not belong to the center of \(G\). Obviously \(s_2 \in Z(G)\).

(v) Let \(G = W(G_2)\). The number of conjugacy classes of \(G, G^{s_4}, G^{s_5}\), and \(G^{s_6}\) is 6 by table 12. They have 4, 4, 6 and 6 one dimensional representations, respectively, according
to the character tables in [ZWCYa]. Thus \( s_5 \) and \( s_6 \) do not belong to the center of \( G \). Obviously \( s_4 \in \mathbb{Z}(G) \). □

We give the other main result.

**Theorem 2.** Every central quantum linear space \( \mathcal{B}(G, r, \overrightarrow{\rho}, u) \) over Weyl Groups of exceptional type is one case in the following:

(i) \( G = W(E_7), C = \{ s_6 \}, r = r_C C \) and \( \chi_C^{(i)} \in \{ \chi_6^{(j)} | j = 2, 4, 6, 8, 10, 12, 15, 16, 18, 20, 22, 26, 27, 28, 30, 32, 35, 36, 38, 41, 42, 44, 46, 48, 50, 52, 54, 56, 58, 60 \} \) for any \( i \in I_C(r, u) \).

(ii) Let \( G = W(E_8), C = \{ s_7 \}, r = r_C C \) and \( \chi_C^{(i)} \in \{ \chi_7^{(j)} | j = 3, 4, 11, 12, 16, 17, 18, 19, 29, 30, 32, 33, 34, 37, 38, 45, 46, 51, 52, 56, 57, 60, 63, 64, 65, 66, 71, 79, 80, 82, 83, 89, 90, 91, 92, 95, 96, 99, 100, 103, 104, 106, 107, 110, 112 \} \) for any \( i \in I_C(r, u) \).

(iii) Let \( G = W(F_4), C = \{ s_2 \}, r = r_C C \) and \( \chi_C^{(i)} \in \{ \chi_2^{(j)} | j = 9, 10, 11, 12, 16, 17, 18, 19, 25 \} \) for any \( i \in I_C(r, u) \).

(iv) Let \( G = W(G_2), C = \{ s_4 \}, r = r_C C \) and \( \chi_C^{(i)} \in \{ \chi_4^{(3)}, \chi_4^{(4)}, \chi_4^{(5)} \} \) for any \( i \in I_C(r, u) \).

**Proof.** Let us first consider the case of (i). By Theorem II and Table 2, \( \text{RSR}(G, r, \overrightarrow{\rho}, u) \) is of \(-1\)-type. Applying Lemma 7.1 we have that \( \mathcal{B}(G, r, \overrightarrow{\rho}, u) \) is a central quantum linear space. Similarly, \( \mathcal{B}(G, r, \overrightarrow{\rho}, u) \) is a central quantum linear space under the other case.

Conversely, if \( \mathcal{B}(G, r, \overrightarrow{\rho}, u) \) is a central quantum linear space over Weyl Group \( G \) of exceptional type, then for any \( C \in \mathcal{K}_r(G), C \) has to be \( \{ s_6 \} \) with \( G = W(E_7) \) or \( \{ s_8 \} \) with \( G = W(E_8) \) or \( \{ s_2 \} \) with \( G = W(F_4) \) or \( \{ s_4 \} \) with \( G = W(G_2) \) by Lemma 7.1. This implies \( r = r_C C \) and \( C \) is one case in this theorem. Furthermore, every bi-one type \( \text{RSR}(G, \mathcal{O}_u(C), \rho^{(i)}_C) \) for any \( i \in I_C(r, u) \) is of \(-1\)-type by Proposition 2.3. Applying Theorem II and Table 2, Table 5, Table 11 and Table 12, we have that \( \chi^{(i)}_C \) has to be one case in this theorem for any \( i \in I_C(r, u) \). □

In other words we have

**Remark 7.2.** Let \( G \) be a Weyl Group of exceptional type and \( M = M(\mathcal{O}_a, \rho^{(1)}) \oplus M(\mathcal{O}_a, \rho^{(2)}) \oplus \cdots \oplus M(\mathcal{O}_a, \rho^{(m)}) \) is a YD module over \( kG \). Then \( \mathcal{B}(M) \) is finite dimensional in the following cases:

(i) \( G = W(E_7), a = s_6 \) and the characters of \( \rho^{(1)}, \rho^{(2)}, \cdots, \rho^{(m)} \) are in \( \{ \chi_6^{(j)} | j = 2, 4, 6, 8, 10, 12, 15, 16, 18, 20, 22, 26, 27, 28, 30, 32, 35, 36, 38, 41, 42, 44, 46, 48, 50, 52, 54, 56, 58, 60 \} \).

(ii) \( G = W(E_8), a = s_7 \) and the characters of \( \rho^{(1)}, \rho^{(2)}, \cdots, \rho^{(m)} \) are in \( \{ \chi_7^{(j)} | j = 3, 4, 11, 12, 16, 17, 18, 19, 29, 30, 32, 33, 34, 37, 38, 45, 46, 51, 52, 56, 57, 60, 63, 64, 65, 66, 71, 79, 80, 82, 83, 89, 90, 91, 92, 95, 96, 99, 100, 103, 104, 106, 107, 110, 112 \} \).

(iii) \( G = W(F_4), a = s_2 \) and the characters of \( \rho^{(1)}, \rho^{(2)}, \cdots, \rho^{(m)} \) are in \( \{ \chi_2^{(j)} | j = 9, 10, 11, 12, 16, 17, 18, 19, 25 \} \).
(iv) $G = W(G_2)$, as $a_4$ and the characters of $\rho^{(1)}$, $\rho^{(2)}$, \ldots, $\rho^{(m)}$ are in $\{\chi_4^{(3)}, \chi_4^{(4)}, \chi_4^{(5)}\}$.

8 Nichols algebras of reducible \text{YD} modules

In this section it is proved that except a few cases Nichols algebras of reducible Yetter-Drinfeld modules over Weyl groups of exceptional type are infinite dimensional.

$\mathcal{O}_{s_i}$ and $\mathcal{O}_{s_j}$ are said to be square-commutative if $stst = tsts$ for any $s \in \mathcal{O}_{s_i}$, $t \in \mathcal{O}_{s_j}$. $a$ and $b$ are said to be square-commutative if $abab = baba$.

**Lemma 8.1.** Let $G$ be a Weyl group of Exceptional Type.

(i) $\mathcal{O}_{s_i}$ and $\mathcal{O}_{s_j}$ are not commutative for any $i$ and $j$ with $i, j \neq 1$ when $G = W(E_6)$.

(ii) $\mathcal{O}_{s_i}$ and $\mathcal{O}_{s_j}$ are not square-commutative when $G = W(E_7)$ and $(i, j) \neq (9, 11), (9, 13), (11, 19), (13, 19)$ with $i, j \neq 1, 6$.

(iii) $\mathcal{O}_{s_i}$ and $\mathcal{O}_{s_j}$ are not square-commutative when $G = W(E_8)$ and $(i, j) \neq (5, 14), (5, 24), (8, 14), (8, 24), (14, 35), (14, 80), (24, 35), (24, 80)$ with $i, j \neq 1, 7$.

(iv) $\mathcal{O}_{s_i}$ and $\mathcal{O}_{s_j}$ are not square-commutative when $G = W(F_4)$ and $(i, j) \neq (3, 3), (3, 4), (3, 7), (3, 8), (3, 17), (3, 18), (3, 24), (3, 25), (4, 4), (4, 7), (4, 8), (4, 17), (4, 18), (4, 24), (4, 25), (7, 12), (7, 13), (7, 17), (7, 18), (8, 12), (8, 13), (8, 17), (8, 18), (12, 17), (12, 18), (13, 17), (13, 18)$ with $i, j \neq 1, 2$.

(v) $\mathcal{O}_{s_i}$ and $\mathcal{O}_{s_j}$ are not square-commutative when $G = W(G_2)$ and $(i, j) \neq (2, 5), (2, 6), (3, 5), (3, 6), (5, 5), (5, 6), (6, 6)$ with $i, j \neq 1, 4$.

**Proof.** Let $A := \{(i, j) \mid (i, j) = (9, 11), (9, 13), (11, 19), (13, 19), \text{or } i, j = 1, 6 \}$, $B := \{(i, j) \mid (i, j) = (5, 14), (5, 24), (8, 14), (8, 24), (14, 35), (14, 80), (24, 35), (24, 80), \text{or } i, j = 1, 7 \}$, $C := \{(i, j) \mid (i, j) = (3, 3), (3, 4), (3, 7), (3, 8), (3, 17), (3, 18), (3, 24), (3, 25), (4, 4), (4, 7), (4, 8), (4, 17), (4, 18), (4, 24), (4, 25), (7, 12), (7, 13), (7, 17), (7, 18), (8, 12), (8, 13), (8, 17), (8, 18), (12, 17), (12, 18), (13, 17), (13, 18), \text{or } i, j = 1, 2 \}$.

(i) It follows from Table 13.

(ii) $\mathcal{O}_{s_i}$ and $\mathcal{O}_{s_j}$ are square-commutative in $W(E_7)$ for $(i, j) \in A$. $s_i$ and $s_j$ are not square-commutative if $(i, j) \not\in A$ and there does not exist $t$ such that $s_i$ and $s_is_js_i^{-1}$ are in table 14–16.

(iii) $\mathcal{O}_{s_i}$ and $\mathcal{O}_{s_j}$ are square-commutative in $W(E_8)$ for $(i, j) \in B$. $s_i$ and $s_{110}s_j s_{110}^{-1}$ in $W(E_8)$ are not square-commutative if $(i, j) \not\in B$ and there does not exist $t$ such that $s_i$ and $s_is_js_i^{-1}$ are in table 17.

(iv) $\mathcal{O}_{s_i}$ and $\mathcal{O}_{s_j}$ are square-commutative in $W(F_4)$ for $(i, j) \in C$. $s_i$ and $s_3s_js_3^{-1}$ are not square-commutative in $W(F_4)$ if $(i, j) \not\in C$ and there does not exist $t$ such that $s_i$ and $s_is_js_i^{-1}$ are in table 18.

(v) $\mathcal{O}_{s_i}$ and $\mathcal{O}_{s_j}$ are square-commutative in $W(G_2)$ for any $(i, j)$ but $(i, j) = (2, 3), (2, 2), (3, 3)$. $s_2$ and $s_5s_3s_5^{-1}$, $s_2$ and $s_6s_2s_6^{-1}$, $s_3$ and $s_5s_3s_5^{-1}$ are not square-commutative, respectively. □
Note that we have proved that $O_{s_i}$ and $O_{s_j}$ are square-commutative in $G = (W(E_7))$, $G = (W(E_8))$ and $G = (W(F_4))$ if and only if $(i, j) \in A, B, C$, respectively. The programs to prove that $O_{s_i}$ and $O_{s_j}$ in $W(E_7)$ are square-commutative are the following:

```gap
L:=SimpleLieAlgebra("E",7,Rationals);;
R:=RootSystem(L);;
W:=WeylGroup(R);;
ccl:=ConjugacyClasses(W);
qu:=NrConjugacyClasses(W);;Display (qu);
con1:=Elements(ccl[11]);;m:=Size(con1);
for k in [1..m] do
  s:=con1[k];
  con2:=Elements(ccl[19]);n:=Size(con2);
  for l in [1..n] do
    t:=con2[l];
    if (s * t)^2 = (t * s)^2 then
      Print( " k=",k," AND l=",l," \n" );
    fi;
  od;
end;
```

For any reducible YD module $M$ over $kG$, there are at least two irreducible YD sub-modules of $M$. Therefore we only consider the direct sum of two irreducible YD modules.

We give the final main result.

**Theorem 3.** Let $G$ be a Weyl group of Exceptional Type. Then $\dim(\mathfrak{B}(M(O_{s_i}, \rho^{(1)}) \oplus M(O_{s_j}, \rho^{(2)})) = \infty$ in the following cases:

(i) $G = W(E_6)$.
(ii) $G = W(E_7)$ and $(i, j) \neq (9, 11), (9, 13), (11, 19), (13, 19)$ and $i, j \neq 6$.
(iii) $G = W(E_8)$ and $(i, j) \neq (8, 14), (8, 24), (14, 35), (14, 80), (24, 35), (24, 80)$ and $i, j \neq 7$.
(iv) $G = W(F_4)$ and $(i, j) \neq (3, 3), (3, 4), (3, 7), (3, 8), (3, 17), (3, 18), (3, 24), (3, 25), (4, 4), (4, 7), (4, 8), (4, 17), (4, 18), (4, 24), (4, 25), (7, 13), (7, 17), (7, 18), (8, 13), (8, 17), (8, 18), (13, 17), (13, 18)$ and $i, j \neq 2$.
(v) $G = W(G_2)$ and $(i, j) \neq (2, 5), (3, 5), (5, 5)$ and $i, j \neq 4$.

**Proof.** It follows from [HS, Theorem 8.2, Theorem 8.6] and Lemma 8.1. Note that the orders of $s_{12}$ in $W(F_4)$, $s_5$ in $W(E_8)$ and $s_6$ in $W(G_2)$ are odd. □
\[ E_6 \]

| \( s_i \) | \( s_i \) and \( s_i s_j s_i^{-1} \) are not commutative |
|---------|--------------------------------------------------|
| \( s_2 \) | \( s_7 s_2 s_7^{-1}, s_7 s_3 s_7^{-1}, s_7 s_4 s_7^{-1}, s_5, s_6, s_7, s_8, s_9, s_{10}, s_{11}, s_{12}, s_{13}, s_{14}, s_5 s_1 s_5^{-1}, s_{16}, s_{17}, s_{18}, s_{19}, s_{20}, s_{21}, s_{22}, s_{23}, s_{24}, s_{25} \) |
| \( s_3 \) | \( s_7 s_3 s_7^{-1}, s_7 s_4 s_7^{-1}, s_8 s_8 s_8^{-1}, s_7, s_8, s_9, s_{10}, s_{11}, s_{12}, s_{13} \) |
| \( s_4 \) | \( s_7 s_4 s_7^{-1}, s_7 s_5 s_7^{-1}, s_8 s_6 s_8^{-1}, s_7, s_8, s_9, s_{10}, s_{11}, s_{12}, s_{13} \) |
| \( s_5 \) | \( s_2 s_5 s_2^{-1}, s_2 s_6 s_2^{-1}, s_2 s_7 s_2^{-1}, s_2 s_8 s_2^{-1}, s_2 s_9 s_2^{-1}, s_2 s_{10} s_2^{-1}, s_{11}, s_{12}, s_{13} \) |
| \( s_6 \) | \( s_2 s_6 s_2^{-1}, s_2 s_7 s_2^{-1}, s_2 s_8 s_2^{-1}, s_2 s_9 s_2^{-1}, s_2 s_{10} s_2^{-1}, s_{11}, s_{12}, s_{13} \) |
| \( s_7 \) | \( s_{14}, s_{21} s_{21}^{-1}, s_{16}, s_{17}, s_{18}, s_{19}, s_{20}, s_{21}, s_{22}, s_{23}, s_{24}, s_{25} \) |
| \( s_8 \) | \( s_{22}, s_{21} s_{21}^{-1}, s_{23}, s_{24}, s_{25} \) |
| \( s_9 \) | \( s_{24}, s_{23} s_{23}^{-1}, s_{25} \) |
| \( s_{10} \) | \( s_{25} \) |
| \( s_{11} \) | \( s_{25} \) |
| \( s_{12} \) | \( s_{25} \) |
| \( s_{13} \) | \( s_{25} \) |
| \( s_{14} \) | \( s_{25} \) |
| \( s_{15} \) | \( s_{25} \) |
| \( s_{16} \) | \( s_{25} \) |
| \( s_{17} \) | \( s_{25} \) |
| \( s_{18} \) | \( s_{25} \) |
| \( s_{19} \) | \( s_{25} \) |
| \( s_{20} \) | \( s_{25} \) |
| \( s_{21} \) | \( s_{25} \) |
| \( s_{22} \) | \( s_{25} \) |
| \( s_{23} \) | \( s_{25} \) |
| \( s_{24} \) | \( s_{25} \) |
| \( s_{25} \) | \( s_{25} \) |

Table 13
| $E_7$ | $s_i$ and $s_is_j^{-1}$ are not square-commutative |
|-------|--------------------------------------------------|
| $s_1$ |  $s_6s_5s_6^{-1}, s_6s_5s_6^{-1}, s_6s_4s_6^{-1}, s_6s_5s_6^{-1}$, |
| $s_2$ |  $s_6s_5s_6^{-1}, s_6s_4s_6^{-1}, s_6s_5s_6^{-1}$, |
| $s_3$ |  $s_6s_5s_6^{-1}, s_6s_4s_6^{-1}, s_6s_5s_6^{-1}$, |
| $s_4$ |  $s_6s_5s_6^{-1}, s_6s_5s_6^{-1}, s_6s_9s_5^{-1}, s_4s_1s_4^{-1}, s_4s_1s_4^{-1}$, |
| $s_5$ |  $s_6s_5s_6^{-1}, s_4s_9s_4^{-1}, s_4s_1s_4^{-1}, s_4s_8s_4^{-1}, s_4s_5s_7^{-1}$, |
| $s_6$ |  $s_6s_7s_6^{-1}, s_4s_8s_4^{-1}, s_4s_10s_4^{-1}, s_4s_1s_4^{-1}, s_4s_1s_4^{-1}, s_4s_1s_4^{-1}$, |
| $s_7$ |  $s_4s_14s_4^{-1}, s_4s_15s_4^{-1}, s_4s_21s_4^{-1}$, |
| $s_8$ |  $s_6s_8s_6^{-1}, s_4s_9s_4^{-1}, s_4s_10s_4^{-1}, s_4s_11s_4^{-1}, s_4s_12s_4^{-1}, s_4s_13s_4^{-1}, s_4s_14s_4^{-1}$, |
| $s_9$ |  $s_6s_8s_6^{-1}, s_4s_9s_4^{-1}, s_4s_10s_4^{-1}, s_4s_11s_4^{-1}, s_4s_12s_4^{-1}, s_4s_13s_4^{-1}, s_4s_14s_4^{-1}$, |
| $s_{10}$ |  $s_2s_8s_2^{-1}, s_4s_10s_4^{-1}, s_4s_12s_4^{-1}, s_2s_15s_2^{-1}, s_2s_15s_2^{-1}$, |
| $s_{11}$ |  $s_2s_17s_2^{-1}, s_2s_18s_2^{-1}, s_2s_19s_2^{-1}, s_2s_21s_2^{-1}, s_2s_24s_2^{-1}, s_2s_26s_2^{-1}, s_2s_45s_2^{-1}$, |
| $s_{12}$ |  $s_2s_10s_2^{-1}, s_2s_11s_2^{-1}, s_2s_12s_2^{-1}, s_2s_13s_2^{-1}, s_2s_14s_2^{-1}, s_2s_15s_2^{-1}, s_2s_21s_2^{-1}, s_2s_45s_2^{-1}$, |
| $s_{13}$ |  $s_2s_10s_2^{-1}, s_2s_11s_2^{-1}, s_2s_12s_2^{-1}, s_3s_13s_3^{-1}, s_3s_14s_3^{-1}, s_3s_15s_2^{-1}, s_3s_16s_2^{-1}, s_3s_17s_3^{-1}, s_3s_18s_3^{-1}$, |
| $s_{14}$ |  $s_3s_19s_3^{-1}, s_3s_20s_2^{-1}, s_3s_21s_3^{-1}, s_3s_22s_2^{-1}, s_3s_23s_3^{-1}, s_3s_24s_2^{-1}, s_3s_25s_3^{-1}, s_3s_26s_2^{-1}$, |
| $s_{15}$ |  $s_3s_19s_3^{-1}, s_3s_20s_2^{-1}, s_3s_21s_3^{-1}, s_3s_22s_2^{-1}, s_3s_23s_3^{-1}, s_3s_24s_2^{-1}, s_3s_25s_3^{-1}, s_3s_26s_2^{-1}$, |
| $s_{16}$ |  $s_3s_19s_3^{-1}, s_3s_20s_2^{-1}, s_3s_21s_3^{-1}, s_3s_22s_2^{-1}, s_3s_23s_3^{-1}, s_3s_24s_2^{-1}, s_3s_25s_3^{-1}, s_3s_26s_2^{-1}$, |
| $s_{17}$ |  $s_3s_19s_3^{-1}, s_3s_20s_2^{-1}, s_3s_21s_3^{-1}, s_3s_22s_2^{-1}, s_3s_23s_3^{-1}, s_3s_24s_2^{-1}, s_3s_25s_3^{-1}, s_3s_26s_2^{-1}$, |
| $s_{18}$ |  $s_3s_19s_3^{-1}, s_3s_20s_2^{-1}, s_3s_21s_3^{-1}, s_3s_22s_2^{-1}, s_3s_23s_3^{-1}, s_3s_24s_2^{-1}, s_3s_25s_3^{-1}, s_3s_26s_2^{-1}$, |

Table 14
| \( E_7 \) | \( s_{20} \) | \( s_{21} \) | \( s_{22} \) | \( s_{23} \) | \( s_{24} \) | \( s_{25} \) | \( s_{26} \) | \( s_{27} \) | \( s_{28} \) | \( s_{29} \) | \( s_{30} \) | \( s_{31} \) | \( s_{32} \) | \( s_{33} \) | \( s_{34} \) | \( s_{35} \) | \( s_{36} \) | \( s_{37} \) | \( s_{38} \) | \( s_{39} \) | \( s_{40} \) | \( s_{41} \) | \( s_{42} \) | \( s_{43} \) | \( s_{44} \) | \( s_{45} \) | \( s_{46} \) | \( s_{47} \) | \( s_{48} \) | \( s_{49} \) | \( s_{50} \) | \( s_{51} \) | \( s_{52} \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( s_{20} \) | \( s_{21} \) | \( s_{22} \) | \( s_{23} \) | \( s_{24} \) | \( s_{25} \) | \( s_{26} \) | \( s_{27} \) | \( s_{28} \) | \( s_{29} \) | \( s_{30} \) | \( s_{31} \) | \( s_{32} \) | \( s_{33} \) | \( s_{34} \) | \( s_{35} \) | \( s_{36} \) | \( s_{37} \) | \( s_{38} \) | \( s_{39} \) | \( s_{40} \) | \( s_{41} \) | \( s_{42} \) | \( s_{43} \) | \( s_{44} \) | \( s_{45} \) | \( s_{46} \) | \( s_{47} \) | \( s_{48} \) | \( s_{49} \) | \( s_{50} \) | \( s_{51} \) | \( s_{52} \) |

Table 15
| $E_7$ |
|---|
| $s_5$ | $s_2 s_{53} s_2^{-1}$ |
| $s_5$ | $s_2 s_{54} s_2^{-1}$ |
| $s_5$ | $s_2 s_{55} s_2^{-1}$, $s_2 s_{58} s_2^{-1}$ |
| $s_5$ | $s_2 s_{56} s_2^{-1}$, $s_2 s_{57} s_2^{-1}$ |
| $s_5$ | $s_2 s_{57} s_2^{-1}$ |
| $s_5$ | $s_2 s_{58} s_2^{-1}$ |
| $s_5$ | $s_2 s_{59} s_2^{-1}$ |
| $s_5$ | $s_2 s_{60} s_2^{-1}$ |

Table 16

| $E_8$ |
|---|
| $s_i$ | $s_i$ and $s_i s_j s_i$ are not square-commutative |
| $s_5$ | $s_4 s_5 s_4^{-1}$, $s_5 s_{15} s_5^{-1}$, $s_{112} s_{18} s_{112}^{-1}$, $s_9 s_{26} s_9^{-1}$, $s_2 s_{38} s_2^{-1}$, $s_2 s_{106} s_2^{-1}$, $s_4 s_4 s_4^{-1}$ |
| $s_6$ | $s_2 s_{12} s_2^{-1}$, $s_2 s_{44} s_2^{-1}$ |
| $s_7$ | $s_4 s_8 s_4^{-1}$, $s_{112} s_{12} s_{112}^{-1}$, $s_2 s_{22} s_2^{-1}$, $s_9 s_{26} s_9^{-1}$, $s_2 s_{38} s_2^{-1}$ |
| $s_8$ | $s_2 s_{24} s_2^{-1}$, $s_2 s_{26} s_2^{-1}$, $s_2 s_{50} s_2^{-1}$, $s_2 s_{51} s_2^{-1}$, $s_2 s_{62} s_2^{-1}$, $s_4 s_{18} s_{41}^{-1}$ |
| $s_9$ | $s_2 s_{14} s_2^{-1}$, $s_4 s_{21} s_4^{-1}$, $s_9 s_{26} s_9^{-1}$, $s_2 s_{32} s_2^{-1}$, $s_2 s_{38} s_2^{-1}$, $s_2 s_{39} s_2^{-1}$, $s_2 s_{53} s_2^{-1}$, $s_9 s_{56} s_9^{-1}$, $s_9 s_{57} s_9^{-1}$ |
| $s_{10}$ | $s_2 s_{58} s_2^{-1}$, $s_9 s_{68} s_9^{-1}$, $s_{70}$, $s_9 s_{108} s_9^{-1}$ |
| $s_{11}$ | $s_2 s_{26} s_2^{-1}$ |
| $s_{12}$ | $s_2 s_{24} s_2^{-1}$, $s_2 s_{26} s_2^{-1}$, $s_2 s_{62} s_2^{-1}$, $s_2 s_{74} s_2^{-1}$, $s_4 s_{18} s_{41}^{-1}$ |
| $s_{13}$ | $s_4 s_{21} s_4^{-1}$, $s_2 s_{53} s_2^{-1}$ |
| $s_{14}$ | $s_2 s_{56} s_2^{-1}$, $s_2 s_{57} s_2^{-1}$, $s_2 s_{70} s_2^{-1}$ |
| $s_{15}$ | $s_2 s_{38} s_2^{-1}, s_2 s_{39} s_2^{-1}$, $s_4 s_{42} s_4^{-1}$, $s_2 s_{51} s_2^{-1}$, $s_3 s_{53} s_3^{-1}$, $s_2 s_{105} s_2^{-1}$, $s_9 s_{106} s_9^{-1}$ |
| $s_{16}$ | $s_2 s_{35} s_2^{-1}$, $s_2 s_{42} s_2^{-1}$, $s_2 s_{51} s_2^{-1}$, $s_2 s_{106} s_2^{-1}$ |
| $s_{17}$ | $s_2 s_{48} s_2^{-1}$ |
| $s_{18}$ | $s_2 s_{75} s_2^{-1}$, $s_2 s_{80} s_2^{-1}$, $s_2 s_{108} s_2^{-1}$ |
| $s_{19}$ | $s_2 s_{106} s_2^{-1}$ |
| $s_{20}$ | $s_2 s_{110} s_2^{-1}$ |

Table 17
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