Semileptonic $B \to \rho$ and $B \to \pi$ Decays: Lattice and Dispersive Constraints

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Abstract

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Key-Words: Semileptonic Decays of $B$ Mesons, Determination of Kobayashi-Maskawa Matrix Elements ($V_{ub}$), Lattice QCD Calculation, Dispersion Relations, Heavy Quark Effective Theory.

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Semileptonic $B \to \rho$ and $B \to \pi$ Decays: Lattice and Dispersive Constraints

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I present two recent pieces of work on semileptonic $B \to \pi, \rho$ decays where it is shown how lattice QCD calculations can be used to test heavy-quark symmetry and determine phenomenologically relevant quantities despite the limits on these calculations’ kinematical reach. The study of semileptonic $B \to \rho$ decays was performed with the UKQCD Collaboration.

1. Motivations

The CLEO Collaboration has very recently presented measurements of the branching ratios for $B \to \rho \ell \bar{\nu}$ and $B \to \pi \ell \bar{\nu}$ decays ($\ell = e, \mu$) and has announced its intention to measure the corresponding differential decay rates. These various measurements represent an excellent opportunity to determine the poorly known CKM matrix element $|V_{ub}|$. Such determinations require understanding the non-perturbative, strong-interaction corrections to the elementary $b \to u + W$ coupling contained in the matrix elements of the weak currents $V^\mu = \bar{u} \gamma^\mu b$ and $A^\mu = \bar{u} \gamma^\mu \gamma^5 b$ between $B$ and $\pi$ or $\rho$ meson states. It is to calculate these matrix elements that we resort to the lattice.

heavy $\to$ light quark decays, such as the ones that concern us here, are also interesting because they enable one to test heavy-quark symmetry (HQS). For these decays, HQS is weaker than for heavy $\to$ heavy quark decays: it only applies in a limited region around the zero recoil point $q^2 = q_{\text{max}}^2$, where $q$ is the four-momentum transferred to the leptons, and imposes no normalization condition on the relevant form factors at $q_{\text{max}}^2$. Nevertheless, because both the mass and the spin of the heavy quark can be varied in lattice calculations, the deviations from the heavy-quark limit due to finite heavy-quark mass and spin effects can be measured.

2. Limitations

Current day lattice calculations, with lattice spacings on the order of $3 \text{ GeV}^{-1}$, do not permit one to simulate $b \to \text{light}$ quark decays over their full kinematical range. The problem is that the energies and momenta of the particles involved, whose orders of magnitude are set by the $b$ quark mass ($m_b \simeq 5 \text{ GeV}$), are large on the scale of the cutoff in much of phase space. To limit these energies in relativistic lattice quark calculations, one performs the simulation with heavy-quark mass values $m_Q$ around that of the charm ($m_c \simeq 1.5 \text{ GeV}$), where discretization errors remain under control. Then one extrapolates the results up to $m_b$ by fitting heavy-quark scaling relations (HQSR) with power corrections to the lattice results (see Sec. 3.1). Another approach is to work with discretized versions of effective theories such as Non-Relativistic QCD (NRQCD) or Heavy-Quark Effective Theory (HQET) in which the mass of the heavy quark is factored out of the dynamics. All approaches, however, are constrained to relatively small momentum transfers because of the limited applicability of HQS and because of momentum-dependent discretization errors. So one can only reconstruct the $q^2$ dependence of the relevant form factors in a limited region around $q_{\text{max}}^2$ and one is left with the problem of extrapolating these results to smaller $q^2$.

heavy $\to$ light quark decays are difficult in any theoretical approach. Indeed, they re-
qure understanding the underlying QCD dynamics over a large range of momentum transfers from $q_{\text{max}}^2=26.4$ GeV$^2$ (20.3 GeV$^2$) for semileptonic $B \to \pi$ ($B \to \rho$) decays, where the final state hadron is at rest in the frame of the $B$ meson, to $q^2=0$ where it recoils very strongly.

3. $B^0 \to \rho^+ \ell^- \bar{\nu}$ and a Model-Independent Determination of $|V_{ub}|$

One solution to the problem of the limited kinematical range of the lattice results is to ignore the problem or rather rely on the ingenuity of experimental groups to provide measurements of partial rates in the region where lattice results are available. Combined with lattice results for $B \to \rho \ell \bar{\nu}$ decays, such experimental measurements will enable a model-independent determination of $|V_{ub}|$. Rates should be sufficient since our lattice results span a range of $q^2$ from $\sim 14.4$ GeV$^2$ to $q_{\text{max}}^2$ over which the partially integrated is $4.6^{+4}_{-3}|V_{ub}|^2\text{ps}^{-1}$. This represents approximately 1/3 of the total rate obtained from light-cone sumrules (LCSR) in Ref. [3], whose results at large $q^2$ agree well with ours.

3.1. Form Factors and Heavy-Quark Extrapolation

To describe $B^0 \to \rho^+ \ell^- \bar{\nu}$ decays, we must evaluate the matrix element $\langle \rho^+(p',\eta)|V^\mu - A^\mu|B^0(p) \rangle$, traditionally decomposed in terms of four form factors $A_1$, $A_2$, $A$ and $V$ which are functions of $q^2$, where $q=p-p'$. We calculate this matrix element for four values of the heavy-quark mass around that of the charm. Then, to obtain $A_1$ at the scale of the $B$ meson we fit, to the lattice results, the HQSR

$$A_1(\omega,M)\alpha_s(M)^{2/\beta_0}\sqrt{M} = c(\omega) \left(1 + \frac{d(\omega)}{M} \right) + \mathcal{O}(\Lambda^2/M^2),$$

(1)

where $M$ is the mass of the decaying meson, $\beta_0=11-2n_f/3$ and $\Lambda$ is an energy characteristic of the light degrees of freedom. This scaling relation holds for $\omega=(M^2+m^2-q^2)/2Mm$ close

$\text{to 1 and the fit parameters } c \text{ and } d \text{ are independent of } M$. Here $m$ is the mass of the final state meson. Once $c$ and $d$ are fixed, it is trivial to obtain $A_1(M=m_B)$ at the corresponding value of $\omega$. Furthermore, $d$ determines the size of corrections to the heavy-quark limit. Repeating this procedure for many values of $\omega$, one obtains the $q^2$ dependence of the desired form factor around $q_{\text{max}}^2$. The resulting $A_1(q^2,m_B)$ is plotted together with the LCSR result of Ref. [3] and the lattice results of APE [4] and ELC [5]. Agreement amongst these four calculations is excellent as it for $A_2$ and $V$ [6], which are obtained in an entirely analogous way.

3.2. Rates

Having determined $A_1$, $A_2$ and $V$, we can compute $(1/|V_{ub}|^2)d\Gamma/dq^2$. Our results are plotted in Fig. 2 (squares). In the region of $q^2$ accessed, we can legitimately expand, around $q_{\text{max}}^2$, the helicity amplitudes that appear in the rate. Thus we fit, to the lattice points, the parametrization

$$\frac{10^{12}}{|V_{ub}|^2}\frac{d\Gamma}{dq^2} \simeq \frac{G_F^2}{192\pi^3m_B^3}q^2\lambda(q^2)^{1/2} \times a^2 \left(1 + b(q^2 - q_{\text{max}}^2) \right),$$

(2)

where $\lambda(q^2)=(M^2+m^2-q^2)-4M^2m^2$. We find $a=4.6^{+3}_{-2}$ GeV and $b=-(8^{+6}_{-0})10^{-2}$ GeV$^{-2}$ where the second order on $a$ is systematic.
other errors being statistical. With $a$ and $b$ determined, the only unknown in Eq. (2) is $|V_{ub}|$. Therefore, a fit of the parametrization of Eq. (2) to an experimental measurement of the differential decay rate around $q^2_{\text{max}}$ determines $|V_{ub}|$. In this determination, $a$ plays the role of $\mathcal{F}(1)$ in the extraction of $|V_{cb}|$ from semileptonic $B \to D$ or $D^*$ decays [7] and $b$ the role of $\mathcal{F}'(1)$. The difference, here, is that $a$ is not determined by HQS up to small radiative and power corrections. It is a genuinely non-perturbative quantity. Another way of determining $|V_{ub}|$ from the lattice results is to compare partially integrated rates from $q^2 \geq 14 \text{ GeV}^2$ to $q^2_{\text{max}}$ given by Eq. (2) to the corresponding experimental measurements. Both these methods yield $|V_{ub}|$ with approximately 10% statistical and 12% theoretical uncertainties.

3.3. A Test of HQS

In Fig. 3 we compare semileptonic $B \to \rho$ form factors with those governing the short distance contribution to radiative $B \to K^*\gamma$ decays for which the relevant hadronic matrix element is $\langle K^*(p', \eta)|\bar{s}\sigma^{\mu\nu}q'\gamma_\nu b_R|B(p)\rangle$, with $q=p-p'$. This matrix element is traditionally decomposed in terms of three form factors, $T_1$, $T_2$ and $T_3$. The comparison is made for three initial meson masses: $M=m_D$, $M=m_B$ and $M \to \infty$. For identical final-state vector mesons (in Fig. 3 all light-quarks involved have the same mass, slightly larger than that of the strange), HQS predicts

$$V(q^2) = 2T_1(q^2), \quad A_1 = 2iT_2(q^2), \quad \omega$$

for $q^2$ around $q^2_{\text{max}}$ or, equivalently, $\omega$ close to 1. While $V/2T_1$ exhibits large $1/M$ corrections at the $D$ and even $B$ meson scale, $A_1/2iT_2$ exhibits no such corrections even at the $D$ scale. Both ratios, however, converge to 1 in the heavy-quark limit which gives us confidence that we control the heavy-quark-mass dependence of the various form factors. Furthermore, these ratios can help constrain the possible $q^2$ dependences of the var-

Figure 2. The data points are our lattice results and the solid curve, the fit to Eq. (2).

Figure 3. Ratios $V/2T_1$ and $A_1/2iT_2$ for 5 values of $\omega$ and three initial meson masses. The solid lines are the HQS predictions.
ious form factors around \( q_{\text{max}}^2 \) at \( M=m_B \).

4. \( B^0 \to \pi^+ \ell^- \bar{\nu} \) and Dispersive Constraints

A second solution to the problem of the limited kinematical reach of lattice simulations of \( \text{heavy} \to \text{light} \) quark decays is to combine lattice results for the relevant form factors around \( q_{\text{max}}^2 \) with dispersive bound techniques to obtain improved, model-independent bounds for the form factors for all \( q^2 \). For the case of \( B^0 \to \pi^+ \ell^- \bar{\nu} \) decays, whose hadronic matrix element, \( \langle \pi^+(p') | V^{\mu} | B^0(p) \rangle \), is traditionally decomposed in terms of two form factors \( f^+(q^2) \) and \( f^0(q^2) \), one can use the kinematical constraint, \( f^+(0)=f^0(0) \), to further constrain the bounds.

4.1. Dispersive Bounds

The subject of dispersive bounds in semileptonic decays has a long history going back to S. Okubo et al. who applied them to semileptonic \( K \to \pi \) decays. C. Bourrely et al. first combined these techniques with QCD and applied them to semileptonic \( D \to K \) decays. Very recently, C.G. Boyd et al. applied them to \( B \to \pi \ell \bar{\nu} \) decays.

The starting point for \( B \to \pi \ell \bar{\nu} \) decays is to combine lattice simulations of \( B \pi \) decays, whose hadronic matrix element, \( \langle \pi^+(p') | V^{\mu} | B^0(p) \rangle \), is traditionally decomposed into two form factors \( f^+(q^2) \) and \( f^0(q^2) \), one can use the kinematical constraint, \( f^+(0)=f^0(0) \), to further constrain the bounds.

4.2. Imposing the Kinematical Constraint

The first problem is that Eq. (6) and the equivalent constraint for \( f^+ \) yield independent bounds on the form factors which do not satisfy the kinematical constraint \( f^+(0)=f^0(0) \). The bounds on \( f^+ \) require \( f^+(0) \) to lie within an interval of values \( I_+ \) and those on \( f^0 \), within an interval \( I_0 \). Together with these bounds, however, the kinematical constraint requires \( f^+(0)=f^0(0) \) to lie somewhere within \( I_+ \cap I_0 \). Thus, we seek bounds on the form factors which are consistent with this new constraint.

A natural definition is to require these new bounds to be the envelope of the set of pairs of bounds obtained by allowing \( f^+(0) \) and \( f^0(0) \) to take all possible values within the interval \( I_+ \cap I_0 \). In Ref. [8], I show how this envelope can be constructed efficiently and that the additional constraint can only improve the bounds on the form factors for all \( q^2 \). Also, as by product, one obtains a formalism which enables one to constrain bounds on a form factor with the knowledge that it must lie within an interval of values at one or
more values of $q^2$.

4.3. Taking Errors into Account

As they stand, the methods of Ref. [10] can only accommodate exact values of the form factors at given kinematical points and contain no provisions for taking errors on these values into account. Of course, the results given by the lattice do carry error bars. More precisely, the lattice provides a probability distribution for the value of the form factors at various kinematical points. What must be done, then, is to translate this distribution into some sort of probability statement on the bounds. The conservative solution is to consider the probability that complete pairs of bounds lie within a given finite interval at each value of $q^2$. Then, using this new probability, one can define upper and lower $p\%$ bounds at each $q^2$ as the upper and lower boundaries of the interval that contains the central $p\%$ of this probability.

These bounds indicate that there is at least a $p\%$ probability that the form factors lie within them at each $q^2$.

4.4. Lattice-Constrained Bounds

To constrain the bounds on $f^+$ and $f^0$, I use the lattice results of the UKQCD Collaboration [12], to which I add a large range of systematic errors to ensure that the bounds obtained are conservative. Because of these systematic errors, the probability distribution of the lattice results is not known. I make the simplifying and rather conservative assumption that the results are uncorrelated and gaussian distributed. I construct the required probability by generating 4000 pairs of bounds from a Monte-Carlo on the distribution of the lattice results. My results for the bounds on the form factors are shown in Fig. 3. I have plotted the two form factors back-to-back to show the effect of the kinematical constraint. Without it, the bounds on $f^+$ would be looser, especially around $q^2=0$, where phase space is large. Since $f^+$ determines the rate, the kinematical constraint and the bounds on $f^0$ are important.

Also shown in Fig. 3 is the LCSR result of $f^0(|q^2|)$ and $f^+(q^2)$ versus $q^2$. The data points are the lattice results of UKQCD [12] with added systematic errors. The pairs of fine curves are, from the outermost to the innermost, the 95%, 70% and 30% bounds. The shaded curve is the LCSR result of Ref. [13].

Ref. [13], which has two components: for $q^2$ below 15 GeV$^2$, the $q^2$ dependence of $f^+$ is determined directly from the sumrule; for larger $q^2$, pole dominance is assumed with a residue determined from the same correlator. Agreement with the bounds is excellent. In Ref. [3], the bounds are compared with the predictions of more authors as well as with direct fits of various parametrizations to the lattice results. Though certain predictions are strongly disfavored, the lattice results and bounds will have to improve before a firm conclusion can be drawn as to the precise $q^2$ dependence of the form factors.

The bounds on $f^+$ also enable one to constrain the $B^* B \pi$ coupling $g_{B^* B \pi}$ which determines the residue of the $B^*$ pole contribution to $f^+$. The constraints obtained are poor because $f^+$ is weakly bounded at large $q^2$. Fitting the lattice results for $f^0$ and $f^+$ to a parametrization which assumes $B^*$ pole dominance for $f^+$ and which is consistent with HQS and the kinematical constraint gives the more precise result $g_{B^* B \pi} = 28 \pm 1$. However, because this result is model-dependent, it should be taken with

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\[ \text{Figure 4.} \quad f^0(|q^2|) \text{ and } f^+(q^2) \text{ versus } q^2. \]
Table 1: Bounds on rate in units of $|V_{ub}|^2 \times 10^{-8}$ and on $f^+(0)$. 

| $\Gamma (B^+ \to \pi^+ \ell^- \bar{\nu})$ | $f^+(0)$ | CL  |
|-----------------------------------|----------|-----|
| $2.4 \to 28$                     | $-0.26 \to 0.92$ | 95% |
| $2.8 \to 24$                     | $-0.18 \to 0.85$ | 90% |
| $3.6 \to 17$                     | $0.00 \to 0.68$  | 70% |
| $4.4 \to 18$                     | $0.10 \to 0.57$  | 50% |
| $4.8 \to 10$                     | $0.18 \to 0.49$  | 30% |

4.5. Bounds on the rate and $|V_{ub}|$

As was done for the form factors, one can define the probability of finding a complete pair of bounds on the rate in a given interval and from that probability determine confidence level (CL) intervals for the rate. The resulting bounds are summarized in Table 1. They were obtained by appropriately integrating the 4000 bounds generated for $f^+(q^2)$, taking the skewness of the resulting “distribution” of bounds on the rate into account. The CL bounds obtained can be used, in conjunction with the branching ratio measurement of CLEO [1], to determine $|V_{ub}|$. One finds $|V_{ub}| 10^4 \sqrt{\Gamma_{B^0}/1.56 \times 10^{-8}} = (34 \div 49) \pm 8 \pm 6 \, , \qquad (7)$

where the range given in parentheses is that obtained from the 30% CL bounds on the rate and represents the most probable range of values for $|V_{ub}|$. The first set of errors is obtained from the 70% CL bounds and the second is obtained by combining all experimental uncertainties in quadrature and applying them to the average value of $|V_{ub}|$ given by the 30% CL results. This determination of $|V_{ub}|$ has a theoretical error of approximately 37%. Though non-negligible, this error is quite reasonable given that the bounds on the rate are completely model-independent and are obtained from lattice data which lie in a limited kinematical domain and include a conservative range of systematic errors.

5. Conclusion and Outlook

Because HQS applies to heavy → light quark decays in a rather limited way, it is not possible to determine the full $q^2$ dependence of the relevant form factors from the lattice alone. The flip side of the coin is that the model-independent information provided by lattice calculations about these decays, though limited, is still very important, because the relevant matrix elements are not anchored at zero recoil by HQS, up to small radiative and power corrections, as they are in heavy → heavy quark decays.

I have presented two approaches by which the information provided by the lattice on exclusive semileptonic $b \to u$ decays can be used to extract $|V_{ub}|$. Both approaches will benefit from forthcoming, improved lattice results. The lattice-constrained bounds would also benefit enormously from an increase in the range of accessible $q^2$.

Finally, the techniques developed in Ref. [3] to construct lattice-improved bounds for $B \to \pi \ell \bar{\nu}$ decays are in principle applicable to limited results obtained by non-lattice means and to other processes such as $B \to \rho \ell \bar{\nu}$ and $B \to K^* \gamma$ decays.

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