Schellbach-style Formulae for the
Derousseau-Pampuch Generalizations of the
Malfatti Circles

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Abstract

It is known that there exist 32 triplets of circles such that each circle is tangent to the other two circles and to two of the sides of the triangle or their extensions. We provide formulae to obtain the radii of the circles for each of the 32 triplets from the side lengths of the reference triangle by means of trigonometric or hyperbolic functions.

1 Introduction

The configuration of three circles inside a triangle such that each circle is tangent to the other two circles and to two of the sides of the triangle has been studied for more than two centuries. Today, such three circles are called the Malfatti circles of the triangle.

Sometime before 1773, Naonobu Ajima (1732?–1798), who was a samurai, or a member of the military class in old Japan, found a method to calculate the diameters of the Malfatti circles from the side lengths of an arbitrary triangle. The method was called Nanzan-shi san-sha naiyo san-en jutsu (“Nanzan’s

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method on a triangle that includes three circles”, as Nanzan is a pen name of Ajima’s) or San-sha san-en jutsu in short. A brief description of the method is found in [2, I ¶14]. A detailed description of the method including a proof is found in [1]. Unfortunately, Ajima’s method as well as any other results by Japanese mathematicians in those days was inaccessible from outside Japan until the Edo shogunate, the former government of Japan (1603–1868), abandoned the isolation policy in 1854.

In 1803, an Italian mathematician Gianfrancesco Malfatti (1731–1807) [10] gave a construction to draw the Malfatti circles for an arbitrary triangle. Despite Malfatti’s unawareness of Ajima’s works, Malfatti’s construction is considered identical in many parts to Ajima’s method.

In 1852, Schellbach [12][13] gave a set of formulae to obtain the distances between the vertices and the tangent points of the circles on the sides from the side lengths of an arbitrary triangle by using trigonometric functions. The same formulae with a proof essentially identical to Schellbach’s are described in English in [6, §30][7].

In 1895, Derousseau [4] generalized the Malfatti circles by removing the condition that the three circles are inside the triangle. Derousseau proved that there exist 32 triplets of circles such that each circle is tangent to the other two circles and to two of the sides of the reference triangle or their extensions. Some alternative proofs of the existence are known. In 1904, Pamhuch [11] gave another proof. In 1930, Lob and Richmond [9] gave yet another proof.

In this article, we provide formulae to obtain the radii of the circles for each of the 32 triplets from the side lengths by means of trigonometric or hyperbolic functions. In other words, we provide Schellbach-style formulae for all of the Derousseau-Pampuch generalizations.

2 Notation

Throughout this article, we use the following notation.

For a triangle $ABC$, let $a$, $b$, $c$ denote the lengths of the sides $BC$, $CA$, $AB$, $s$ the semiperimeter, $r$ the inradius, and $r_A$, $r_B$, $r_C$ the exradii as usual.

Let the incircle is tangent to the side $BC$ at $D$, to the side $CA$ at $E$, and to the side $AB$ at $F$. Let the excircle corresponding to the vertex $A$ is tangent to the side $BC$ at $D_A$, to the extension of the side $AC$ at $E_A$, and to the extension of the side $AB$ at $F_A$.

Suppose the circle $A'(r_1)$ is tangent to the line $CA$ at $E_1$ and to the line $AB$ at $F_1$, the circle $B'(r_2)$ is tangent to the line $AB$ at $F_2$ and to the line $BC$ at $D_2$, the circle $C'(r_3)$ is tangent to the line $BC$ at $D_3$ and to the line $CA$ at $E_3$, and the three circles are tangent to one another. Suppose the nine tangent points are distinct.

3 Classification

Since the center $A'$ does not locate neither on the line $AB$ nor on the line $AC$, it locates inside $\angle CAB$, inside $\angle C\overline{AB}$, inside $\angle CAB$ or inside $\angle C\overline{AB}$ where an overline indicates that the angle has, as one of its sides, the opposite ray instead of the ray including the triangle side. For example, $\angle C\overline{AB}$ denotes the
angle with the ray $\overrightarrow{AC}$ and the ray opposite to the ray $\overrightarrow{B}$. And $\angle \overrightarrow{A}B\overrightarrow{C}$ denotes the vertical angle of $\angle CAB$. Analogously, the center $B'$ locates inside $\angle ABC$, inside $\overrightarrow{ABC}$, inside $\angle A\overrightarrow{BC}$ or inside $\overrightarrow{A}B\overrightarrow{C}$ and the center $C'$ locates inside $\angle BCA$, inside $\overrightarrow{BCA}$, inside $\overrightarrow{B}CA$ or inside $\overrightarrow{B}CA$.

If the circles $A'(r_1), B'(r_2), C'(r_3)$ lie in $\Delta_1, \Delta_2, \Delta_3$, respectively, then $\Delta_1 \cap \Delta_2 \neq \emptyset, \Delta_1 \cap \Delta_3 \neq \emptyset$, and $\Delta_2 \cap \Delta_3 \neq \emptyset$ since the three circles are tangent to one another. Thus, for locations of the three centers $A', B', C'$, only 7 out of the 64 cases are consistent to the condition that the circles are tangent to one another. The following are the consistent cases.

| Case | $A'$ is inside $B'$ is inside $C'$ is inside |
|------|-----------------------------------------------|
| Case 1 | $\angle CAB$ | $\angle ABC$ | $\angle BCA$ |
| Case 2 | $\angle CAB$ | $\angle ABC$ | $\angle BCA$ |
| Case 3 | $\angle CAB$ | $\angle ABC$ | $\angle BCA$ |
| Case 4 | $\angle CAB$ | $\angle ABC$ | $\angle BCA$ |
| Case 5 | $\angle CAB$ | $\angle ABC$ | $\angle BCA$ |
| Case 6 | $\angle CAB$ | $\angle ABC$ | $\angle BCA$ |
| Case 7 | $\angle CAB$ | $\angle ABC$ | $\angle BCA$ |

4 Solution

4.1 Case 1

In Case 1, the following three conditions hold.

$BD_2 + D_3 C + D_2 D_3 = BD + DC$ or $BD_2 + D_3 C - D_2 D_3 = BD + DC$,

$AE_1 + E_3 C + E_1 E_3 = AE + EC$ or $AE_1 + E_3 C - E_1 E_3 = AE + EC$,

$AF_1 + F_2 B + F_1 F_2 = AF + FB$ or $AF_1 + F_2 B - F_1 F_2 = AF + FB$.
By expressing the lengths by the radii and the angle sizes, we obtain from the
first disjunction that
\[ r_2 \cot \frac{B}{2} + r_3 \cot \frac{C}{2} + 2\sqrt{r_2 r_3} = r \cot \frac{B}{2} + r \cot \frac{C}{2} \quad (1) \]
or
\[ r_2 \cot \frac{B}{2} + r_3 \cot \frac{C}{2} - 2\sqrt{r_2 r_3} = r \cot \frac{B}{2} + r \cot \frac{C}{2}, \quad (2) \]
we obtain from the second disjunction that
\[ r_1 \cot \frac{A}{2} + r_3 \cot \frac{C}{2} + 2\sqrt{r_1 r_3} = r \cot \frac{A}{2} + r \cot \frac{C}{2} \quad (3) \]
or
\[ r_1 \cot \frac{A}{2} + r_3 \cot \frac{C}{2} - 2\sqrt{r_1 r_3} = r \cot \frac{A}{2} + r \cot \frac{C}{2}, \quad (4) \]
and we obtain from the second disjunction that
\[ r_1 \cot \frac{A}{2} + r_2 \cot \frac{B}{2} + 2\sqrt{r_1 r_2} = r \cot \frac{A}{2} + r \cot \frac{B}{2} \quad (5) \]
or
\[ r_1 \cot \frac{A}{2} + r_2 \cot \frac{B}{2} - 2\sqrt{r_1 r_2} = r \cot \frac{A}{2} + r \cot \frac{B}{2}. \quad (6) \]

Define \( l, m, n \) by
\[ l = \cot \frac{A}{2}, \quad m = \cot \frac{B}{2}, \quad n = \cot \frac{C}{2}. \quad (7) \]

Define \( u, v, w, x, y, z \) by
\[
\begin{align*}
    u &= \begin{cases} 
    \frac{\sqrt{r_2 r_3}}{r} & \text{if (1) holds,} \\
    \frac{\sqrt{r_1 r_3}}{r} & \text{if (2) holds,}
    \end{cases} \\
    v &= \begin{cases} 
    \frac{\sqrt{r_1 r_2}}{r} & \text{if (3) holds,} \\
    \frac{\sqrt{r_1 r_2}}{r} & \text{if (4) holds,}
    \end{cases} \\
    w &= \begin{cases} 
    \frac{\sqrt{r_1 r_2}}{r} & \text{if (5) holds,} \\
    \frac{\sqrt{r_1 r_2}}{r} & \text{if (6) holds,}
    \end{cases} \\
    x &= \frac{r_1}{r}, \quad y = \frac{r_2}{r}, \quad z = \frac{r_3}{r}.
\end{align*}
\]

Then we have
\[
\begin{align*}
    my + nz + 2u &= m + n, \\
    lx + nz + 2v &= l + n, \\
    lx + my + 2w &= l + m, \\
    xy &= u^2, \\
    xz &= v^2, \\
    yz &= u^2.
\end{align*} \quad (8)
\]
For any triangle $ABC$, if $l, m, n$ are defined by (7), then $bmn = l + m + n$ holds. On the other hand, if positive reals $l, m, n$ satisfy $bmn = l + m + n$, then there exists a triangle $ABC$ that satisfies (7). Thus, Case 1 can be reduced into solving the system of equations (8) for $u, v, w, x, y, z$ with positive real parameters $l, m, n$ under the restriction $bmn = l + m + n$.

As we will show in Appendix A, the system of equations has the following 8 solutions.

For any triangle $ABC$, if $l, m, n$ are defined by (7), then $bmn = l + m + n$ holds. On the other hand, if positive reals $l, m, n$ satisfy $bmn = l + m + n$, then there exists a triangle $ABC$ that satisfies (7). Thus, Case 1 can be reduced into solving the system of equations (8) for $u, v, w, x, y, z$ with positive real parameters $l, m, n$ under the restriction $bmn = l + m + n$.

As we will show in Appendix A, the system of equations has the following 8 solutions.

\[
\begin{align*}
  u &= \frac{\sqrt{l^2 + 1} - l + 1}{2}, \\
  v &= \frac{\sqrt{m^2 + 1} - m + 1}{2}, \\
  w &= \frac{\sqrt{n^2 + 1} - n + 1}{2}, \\
  x &= \frac{l + m + n - 1 + \sqrt{l^2 + 1} - \sqrt{m^2 + 1} - \sqrt{n^2 + 1}}{2l}, \\
  y &= \frac{l + m + n - 1 - \sqrt{l^2 + 1} + \sqrt{m^2 + 1} - \sqrt{n^2 + 1}}{2m}, \\
  z &= \frac{l + m + n - 1 - \sqrt{l^2 + 1} - \sqrt{m^2 + 1} + \sqrt{n^2 + 1}}{2n}, \\
  u &= \frac{\sqrt{l^2 + 1} - l + 1}{2}, \\
  v &= \frac{\sqrt{m^2 + 1} + m - 1}{2}, \\
  w &= \frac{\sqrt{n^2 + 1} + n - 1}{2}, \\
  x &= \frac{l + m + n - 1 + \sqrt{l^2 + 1} + \sqrt{m^2 + 1} + \sqrt{n^2 + 1}}{2l}, \\
  y &= \frac{l + m + n - 1 - \sqrt{l^2 + 1} - \sqrt{m^2 + 1} + \sqrt{n^2 + 1}}{2m}, \\
  z &= \frac{l + m + n - 1 - \sqrt{l^2 + 1} + \sqrt{m^2 + 1} - \sqrt{n^2 + 1}}{2n}, \\
  u &= \frac{-\sqrt{l^2 + 1} + l - 1}{2}, \\
  v &= \frac{-\sqrt{m^2 + 1} + m - 1}{2}, \\
  w &= \frac{-\sqrt{n^2 + 1} + n - 1}{2}, \\
  x &= \frac{l + m + n - 1 - \sqrt{l^2 + 1} - \sqrt{m^2 + 1} + \sqrt{n^2 + 1}}{2l}, \\
  y &= \frac{l + m + n - 1 + \sqrt{l^2 + 1} + \sqrt{m^2 + 1} + \sqrt{n^2 + 1}}{2m}, \\
  z &= \frac{l + m + n - 1 + \sqrt{l^2 + 1} - \sqrt{m^2 + 1} - \sqrt{n^2 + 1}}{2n}.
\end{align*}
\]
\begin{align*}
    u &= -\frac{\sqrt{l^2 + 1} + l - 1}{2}, \\
    v &= -\frac{\sqrt{m^2 + 1} + m - 1}{2}, \\
    w &= \frac{\sqrt{n^2 + 1} - n + 1}{2}, \\
    x &= \frac{l + m + n - 1 - \sqrt{l^2 + 1} + \sqrt{m^2 + 1} - \sqrt{n^2 + 1}}{2l}, \\
    y &= \frac{l + m + n - 1 + \sqrt{l^2 + 1} - \sqrt{m^2 + 1} - \sqrt{n^2 + 1}}{2m}, \\
    z &= \frac{l + m + n - 1 + \sqrt{l^2 + 1} + \sqrt{m^2 + 1} + \sqrt{n^2 + 1}}{2n}. 
\end{align*} 

(12)

\begin{align*}
    u &= -\frac{\sqrt{l^2 + 1} + l + 1}{2}, \\
    v &= -\frac{\sqrt{m^2 + 1} + m + 1}{2}, \\
    w &= -\frac{\sqrt{n^2 + 1} + n + 1}{2}, \\
    x &= \frac{l + m + n + 1 - \sqrt{l^2 + 1} + \sqrt{m^2 + 1} + \sqrt{n^2 + 1}}{2l}, \\
    y &= \frac{l + m + n + 1 + \sqrt{l^2 + 1} - \sqrt{m^2 + 1} + \sqrt{n^2 + 1}}{2m}, \\
    z &= \frac{l + m + n + 1 + \sqrt{l^2 + 1} + \sqrt{m^2 + 1} - \sqrt{n^2 + 1}}{2n}. 
\end{align*} 

(13)

\begin{align*}
    u &= -\frac{\sqrt{l^2 + 1} + l + 1}{2}, \\
    v &= -\frac{\sqrt{m^2 + 1} - m - 1}{2}, \\
    w &= \frac{\sqrt{n^2 + 1} - n - 1}{2}, \\
    x &= \frac{l + m + n + 1 - \sqrt{l^2 + 1} - \sqrt{m^2 + 1} - \sqrt{n^2 + 1}}{2l}, \\
    y &= \frac{l + m + n + 1 + \sqrt{l^2 + 1} + \sqrt{m^2 + 1} - \sqrt{n^2 + 1}}{2m}, \\
    z &= \frac{l + m + n + 1 + \sqrt{l^2 + 1} - \sqrt{m^2 + 1} + \sqrt{n^2 + 1}}{2n}. 
\end{align*} 

(14)
\[ u = \frac{\sqrt{l^2 + 1} - l - 1}{2}, \]
\[ v = -\frac{\sqrt{m^2 + 1} + m + 1}{2}, \]
\[ w = \frac{\sqrt{n^2 + 1} - n - 1}{2}, \]
\[ x = \frac{l + m + n + 1 + \sqrt{l^2 + 1} + \sqrt{m^2 + 1} - \sqrt{n^2 + 1}}{2l}, \]
\[ y = \frac{l + m + n + 1 - \sqrt{l^2 + 1} - \sqrt{m^2 + 1} - \sqrt{n^2 + 1}}{2m}, \]
\[ z = \frac{l + m + n + 1 - \sqrt{l^2 + 1} + \sqrt{m^2 + 1} + \sqrt{n^2 + 1}}{2n}. \]  

(15)

\[ u = \frac{\sqrt{l^2 + 1} - l - 1}{2}, \]
\[ v = \frac{\sqrt{m^2 + 1} - m - 1}{2}, \]
\[ w = -\frac{\sqrt{n^2 + 1} + n + 1}{2}, \]
\[ x = \frac{l + m + n + 1 + \sqrt{l^2 + 1} - \sqrt{m^2 + 1} + \sqrt{n^2 + 1}}{2l}, \]
\[ y = \frac{l + m + n + 1 - \sqrt{l^2 + 1} + \sqrt{m^2 + 1} + \sqrt{n^2 + 1}}{2m}, \]
\[ z = \frac{l + m + n + 1 - \sqrt{l^2 + 1} - \sqrt{m^2 + 1} - \sqrt{n^2 + 1}}{2n}. \]  

(16)

Define \( \alpha, \beta, \gamma \in (0, \pi/2) \) and \( \sigma \) by

\[ \sin^2 \alpha = \frac{a}{s}, \quad \sin^2 \beta = \frac{b}{s}, \quad \sin^2 \gamma = \frac{c}{s}, \quad \sigma = \frac{\alpha + \beta + \gamma}{2}. \]

The fourth equation in (9) corresponds to a value of \( r_1 \) as follows.

\[ r_1 = \frac{r(l + m + n - 1 + \sqrt{l^2 + 1} - \sqrt{m^2 + 1} - \sqrt{n^2 + 1})}{2l}. \]

Since

\[ r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}, \]
\[ l = \frac{s-a}{r} = \frac{s}{r_A}, \quad m = \frac{s-b}{r} = \frac{s}{r_B}, \quad n = \frac{s-c}{r} = \frac{s}{r_C}, \]
it holds that
\[
r(1 + m + n - 1 + \sqrt{l^2 + 1} - \sqrt{m^2 + 1} - \sqrt{n^2 + 1})
\]
\[
= \frac{r_A}{2} \left( 1 - \sqrt{\frac{(s-a)(s-b)(s-c)}{s^3}} + \sqrt{\frac{(s-a)bc}{s^3}} - \sqrt{\frac{a(s-b)c}{s^3}} - \sqrt{\frac{ab(s-c)}{s^3}} \right)
\]
\[
= \frac{r_A}{2}(1 - \cos \alpha \cos \beta \cos \gamma + \cos \alpha \sin \beta \sin \gamma - \sin \alpha \cos \beta \sin \gamma - \sin \alpha \sin \beta \cos \gamma)
\]
\[
= \frac{r_A}{2}(1 - \cos(\beta + \gamma - \alpha))
\]
\[
= r_A \sin^2(\sigma - \alpha).
\]

By making similar calculations on every last three equations in (9), (10), (11), (12), (13), (14), (15) and (16), we obtain the following respective solutions in Case 1.

\[
\begin{align*}
\begin{cases}
    r_1 &= r_A \sin^2(\sigma - \alpha), \\
    r_2 &= r_B \sin^2(\sigma - \beta), \\
    r_3 &= r_C \sin^2(\sigma - \gamma).
\end{cases} \\
\begin{cases}
    r_1 &= r_A \sin^2 \sigma, \\
    r_2 &= r_B \sin^2(\sigma - \gamma), \\
    r_3 &= r_C \sin^2(\sigma - \beta).
\end{cases} \\
\begin{cases}
    r_1 &= r_A \sin^2(\sigma - \gamma), \\
    r_2 &= r_B \sin^2 \sigma, \\
    r_3 &= r_C \sin^2(\sigma - \beta).
\end{cases} \\
\begin{cases}
    r_1 &= r_A \sin^2(\sigma - \beta), \\
    r_2 &= r_B \sin^2(\sigma - \alpha), \\
    r_3 &= r_C \sin^2 \sigma.
\end{cases} \\
\begin{cases}
    r_1 &= r_A \cos^2(\sigma - \alpha), \\
    r_2 &= r_B \cos^2(\sigma - \beta), \\
    r_3 &= r_C \cos^2(\sigma - \gamma).
\end{cases} \\
\begin{cases}
    r_1 &= r_A \cos^2 \sigma, \\
    r_2 &= r_B \cos^2(\sigma - \gamma), \\
    r_3 &= r_C \cos^2(\sigma - \beta).
\end{cases} \\
\begin{cases}
    r_1 &= r_A \cos^2(\sigma - \gamma), \\
    r_2 &= r_B \cos^2 \sigma, \\
    r_3 &= r_C \cos^2(\sigma - \alpha).
\end{cases} \\
\begin{cases}
    r_1 &= r_A \cos^2 \sigma, \\
    r_2 &= r_B \cos^2(\sigma - \alpha), \\
    r_3 &= r_C \cos^2 \sigma.
\end{cases}
\end{align*}
\]
4.2 Cases 2 & 3

In Case 2, we have that the following three disjunctions of equations hold.

\[ BD_2 + D_3 C + D_2 D_3 = BD_A + D_A C \]
\[ AE_1 - CE_3 + E_1 E_3 = AE_A - CE_A \]
\[ AF_1 - BF_2 + F_1 F_2 = AF_A - BF_A \]

or

\[ BD_2 + D_3 C - D_2 D_3 = BD_A + D_A C, \]
\[ AE_1 - CE_3 - E_1 E_3 = AE_A - CE_A, \]
\[ AF_1 - BF_2 - F_1 F_2 = AF_A - BF_A. \]

By expressing the lengths by the radii and the angle sizes, we obtain from the first disjunction that

\[ r_2 \tan \frac{B}{2} + r_3 \tan \frac{C}{2} + 2 \sqrt{r_2 r_3} = r_A \tan \frac{B}{2} + r_A \tan \frac{C}{2} \tag{25} \]

or

\[ r_2 \tan \frac{B}{2} + r_3 \tan \frac{C}{2} - 2 \sqrt{r_2 r_3} = r_A \tan \frac{B}{2} + r_A \tan \frac{C}{2}, \tag{26} \]

we obtain from the second disjunction that

\[ r_1 \cot \frac{A}{2} - r_3 \tan \frac{C}{2} + 2 \sqrt{r_1 r_3} = r_A \cot \frac{A}{2} - r_A \tan \frac{C}{2} \tag{27} \]

or

\[ r_1 \cot \frac{A}{2} - r_3 \tan \frac{C}{2} - 2 \sqrt{r_1 r_3} = r_A \cot \frac{A}{2} - r_A \tan \frac{C}{2}, \tag{28} \]

and we obtain from the third disjunction that

\[ r_1 \cot \frac{A}{2} - r_2 \tan \frac{B}{2} + 2 \sqrt{r_1 r_2} = r_A \cot \frac{A}{2} - r_A \tan \frac{B}{2} \tag{29} \]

or

\[ r_1 \cot \frac{A}{2} - r_2 \tan \frac{B}{2} - 2 \sqrt{r_1 r_2} = r_A \cot \frac{A}{2} - r_A \tan \frac{B}{2}. \tag{30} \]

Define \( l, \bar{m}, \bar{n} \) by

\[ l = \cot \frac{A}{2}, \quad \bar{m} = \tan \frac{B}{2}, \quad \bar{n} = \tan \frac{C}{2}. \tag{31} \]

Define \( u, v, w, x, y, z \) by

\[ u = \begin{cases} -\frac{\sqrt{r_2 r_3}}{r_A} & \text{if (25) holds,} \\ \frac{\sqrt{r_2 r_3}}{r_A} & \text{if (26) holds,} \end{cases} \]

\[ v = \begin{cases} \frac{\sqrt{r_1 r_3}}{r_A} & \text{if (27) holds,} \\ -\frac{\sqrt{r_1 r_3}}{r_A} & \text{if (28) holds,} \end{cases} \]

\[ w = \begin{cases} \frac{\sqrt{r_1 r_2}}{r_A} & \text{if (29) holds,} \\ -\frac{\sqrt{r_1 r_2}}{r_A} & \text{if (30) holds,} \end{cases} \]

\[ x = \frac{r_1}{r_A}, \quad y = \frac{r_2}{r_A}, \quad z = \frac{r_3}{r_A}. \]
Then we have
\[
\begin{align*}
\bar{m}y + \bar{n}z - 2u &= \bar{m} + \bar{n}, \\
xz &= v^2, \\
yz &= u^2.
\end{align*}
\] (32)

In Case 3, we have
\[
-BD_2 - D_3 C + D_2 D_3 = BD_A + D_A C, \\
-\bar{A}E_1 + \bar{C}E_3 + \bar{E}_1 \bar{E}_3 = \bar{A}E_A - \bar{C}E_A \text{ or } -\bar{A}E_1 + \bar{C}E_3 - \bar{E}_1 \bar{E}_3 = \bar{A}E_A - \bar{C}E_A, \\
-\bar{A}F_1 + \bar{B}F_2 + \bar{F}_1 \bar{F}_2 = \bar{A}F_A - \bar{B}F_A \text{ or } -\bar{A}F_1 + \bar{B}F_2 - \bar{F}_1 \bar{F}_2 = \bar{A}F_A - \bar{B}F_A.
\]

By expressing the lengths by the radii and the angle sizes, we obtain from the first equation that
\[
- r_2 \tan \frac{B}{2} - r_3 \tan \frac{C}{2} + 2\sqrt{r_2 r_3} = r_A \tan \frac{B}{2} + r_A \tan \frac{C}{2},
\] (33)
we obtain from the second conjunction that
\[
- r_1 \cot \frac{A}{2} + r_3 \tan \frac{C}{2} + 2\sqrt{r_1 r_3} = r_A \cot \frac{A}{2} - r_A \tan \frac{C}{2},
\] (34)
or
\[
- r_1 \cot \frac{A}{2} + r_3 \tan \frac{C}{2} - 2\sqrt{r_1 r_3} = r_A \cot \frac{A}{2} - r_A \tan \frac{C}{2},
\] (35)
and we obtain from the third conjunction that
\[
- r_1 \cot \frac{A}{2} + r_2 \tan \frac{B}{2} + 2\sqrt{r_1 r_2} = r_A \cot \frac{A}{2} - r_A \tan \frac{B}{2},
\] (36)
or
\[
- r_1 \cot \frac{A}{2} + r_2 \tan \frac{B}{2} - 2\sqrt{r_1 r_2} = r_A \cot \frac{A}{2} - r_A \tan \frac{B}{2}.
\] (37)

Define \( l, \bar{m}, \bar{n} \) by (31). Define \( u, v, w, x, y, z \) by
\[
u = -\frac{\sqrt{r_2 r_3}}{r_A},
\]
\[
w = \begin{cases} \frac{\sqrt{r_1 r_3}}{r_A} & \text{if (34) holds,} \\
-\frac{\sqrt{r_1 r_3}}{r_A} & \text{if (35) holds,} \end{cases}
\]
\[
w = \begin{cases} \frac{\sqrt{r_1 r_2}}{r_A} & \text{if (36) holds,} \\
-\frac{\sqrt{r_1 r_2}}{r_A} & \text{if (37) holds,} \end{cases}
\]
\[
x = -\frac{r_1}{r_A}, \quad y = -\frac{r_2}{r_A}, \quad z = -\frac{r_3}{r_A}.
\]
Then we have the same system of equations as (32).
For any triangle $ABC$, if $l, m, n$ are defined by (31), then $l\overline{m}\overline{n} = l - \overline{m} - \overline{n}$ holds. On the other hand, if positive reals $l, m, n$ satisfy $l\overline{m}\overline{n} = l - \overline{m} - \overline{n}$, then there exists a triangle $ABC$ that satisfies (31). Thus, Case 2 and Case 3 can be unified and reduced into solving the system of equations (32) for $u, v, w, x, y, z$ with positive real parameters $l, m, n$ under the restriction $l\overline{m}\overline{n} = l - \overline{m} - \overline{n}$.

As we will show in Appendix A, the system of equations has the following 8 solutions.

\[
\begin{align*}
\begin{cases}
    u = \frac{\sqrt{l^2 + 1 - l - 1}}{2}, \\
    v = \frac{\sqrt{m^2 + 1 + \overline{m} - 1}}{2}, \\
    w = \frac{\sqrt{n^2 + 1 + \overline{n} - 1}}{2}, \\
    x = \frac{\sqrt{l^2 + 1 - \sqrt{m^2 + 1} - \sqrt{n^2 + 1 + l - \overline{m} - \overline{n} + 1}}}{2l}, \\
    y = \frac{\sqrt{l^2 + 1 - \sqrt{m^2 + 1} + \sqrt{n^2 + 1 + l + \overline{m} + \overline{n} - 1}}}{2\overline{m}}, \\
    z = \frac{\sqrt{l^2 + 1 + \sqrt{m^2 + 1} - \sqrt{n^2 + 1 + l + \overline{m} + \overline{n} - 1}}}{2\overline{n}}, \\
    x = \frac{\sqrt{l^2 + 1 + \sqrt{m^2 + 1} + \sqrt{n^2 + 1 + l - \overline{m} - \overline{n} + 1}}}{2l}, \\
    y = \frac{\sqrt{l^2 + 1 + \sqrt{m^2 + 1} - \sqrt{n^2 + 1 + l + \overline{m} + \overline{n} - 1}}}{2\overline{m}}, \\
    z = \frac{\sqrt{l^2 + 1 - \sqrt{m^2 + 1} + \sqrt{n^2 + 1 + l + \overline{m} + \overline{n} - 1}}}{2\overline{n}}, \\
    x = \frac{\sqrt{l^2 + 1 + \sqrt{m^2 + 1} - \sqrt{n^2 + 1 + l - \overline{m} - \overline{n} + 1}}}{2l}, \\
    y = \frac{\sqrt{l^2 + 1 + \sqrt{m^2 + 1} + \sqrt{n^2 + 1 + l + \overline{m} + \overline{n} + 1}}}{2\overline{m}}, \\
    z = \frac{\sqrt{l^2 + 1 - \sqrt{m^2 + 1} - \sqrt{n^2 + 1 + l + \overline{m} + \overline{n} + 1}}}{2\overline{n}}.
\end{cases}
\end{align*}
\]
\[
\begin{align*}
\begin{cases}
u &= \sqrt{l^2 + 1 - l + 1}, \\
v &= \frac{\sqrt{\bar{m}^2 + 1 + \bar{n} + 1}}{2}, \\
w &= \frac{\sqrt{\bar{n}^2 + 1 - \bar{n} - 1}}{2}, \\
x &= \frac{\sqrt{l^2 + 1 - \sqrt{\bar{m}^2 + 1} + \sqrt{\bar{n}^2 + 1} + l - \bar{m} - \bar{n} - 1}}{2l}, \\
y &= \frac{\sqrt{l^2 + 1 - \sqrt{\bar{m}^2 + 1} - \sqrt{\bar{n}^2 + 1} - l + \bar{m} + \bar{n} + 1}}{2\bar{m}}, \\
z &= \frac{\sqrt{l^2 + 1 + \sqrt{\bar{m}^2 + 1} + \sqrt{\bar{n}^2 + 1} - l + \bar{m} + \bar{n} + 1}}{2\bar{n}}.
\end{cases}
\end{align*}
\]
\[
\begin{align*}
\mathbb U &= -\frac{\sqrt{l^2 + 1 + l + 1}}{2}, \\
\mathbb V &= \frac{\sqrt{\bar{m}^2 + 1 + \bar{m} - 1}}{2}, \\
\mathbb W &= \frac{\sqrt{\bar{n}^2 + 1 - \bar{n} + 1}}{2}, \\
\mathbb X &= -\frac{\sqrt{l^2 + 1 + \sqrt{\bar{m}^2 + 1 - \sqrt{\bar{n}^2 + 1 - l + \bar{m} + \bar{n} + 1}}}}{2l}, \\
\mathbb Y &= \frac{\sqrt{l^2 + 1 + \sqrt{\bar{m}^2 + 1 + \sqrt{\bar{n}^2 + 1 + l - \bar{m} - \bar{n} + 1}}}}{2\bar{m}}, \\
\mathbb Z &= \frac{\sqrt{l^2 + 1 - \sqrt{\bar{m}^2 + 1 - \sqrt{\bar{n}^2 + 1 - l - \bar{m} - \bar{n} + 1}}}}{2\bar{n}}.
\end{align*}
\]

(44)

\[
\begin{align*}
\mathbb U &= \frac{\sqrt{l^2 + 1 + l + 1}}{2}, \\
\mathbb V &= \frac{\sqrt{\bar{m}^2 + 1 - \bar{m} + 1}}{2}, \\
\mathbb W &= \frac{\sqrt{\bar{n}^2 + 1 + \bar{n} - 1}}{2}, \\
\mathbb X &= -\frac{\sqrt{l^2 + 1 - \sqrt{\bar{m}^2 + 1 + \sqrt{\bar{n}^2 + 1 + l - \bar{m} - \bar{n} + 1}}}}{2l}, \\
\mathbb Y &= \frac{\sqrt{l^2 + 1 - \sqrt{\bar{m}^2 + 1 - \sqrt{\bar{n}^2 + 1 + l - \bar{m} - \bar{n} + 1}}}}{2\bar{m}}, \\
\mathbb Z &= \frac{\sqrt{l^2 + 1 + \sqrt{\bar{m}^2 + 1 + \sqrt{\bar{n}^2 + 1 + l - \bar{m} - \bar{n} + 1}}}}{2\bar{n}}.
\end{align*}
\]

(45)

Define \(\alpha_A, \beta_A, \gamma_A \in (0, +\infty)\) and \(\sigma_A\) by

\[
\begin{align*}
\sinh^2 \alpha_A &= \frac{a}{s - a}, & \sinh^2 \beta_A &= \frac{s - c}{s - a}, & \sinh^2 \gamma_A &= \frac{s - b}{s - a}, \\
\sigma_A &= \frac{\alpha_A + \beta_A + \gamma_A}{2}.
\end{align*}
\]

The fourth equation in (38) corresponds to a value of \(r_1\) as follows.

\[
r_1 = \frac{r_A(\sqrt{l^2 + 1 - \sqrt{\bar{m}^2 + 1} - \sqrt{\bar{n}^2 + 1 + l - \bar{m} - \bar{n} + 1}})}{2l}
\]

Since

\[
\begin{align*}
l &= -\frac{s}{r_A} = \frac{s - a}{r}, & \bar{m} &= \frac{s - c}{r_A} = \frac{s - a}{rC}, & \bar{n} &= \frac{s - b}{r_A} = \frac{s - a}{rB},
\end{align*}
\]
it holds that
\[ r_A \left( \sqrt{t^2 + 1} - \sqrt{m^2 + 1} - \sqrt{n^2 + 1} + l - m - n + 1 \right) \]
\[ \frac{2l}{r^2} \left( \sqrt{\frac{bc}{(s-a)^3}} - \sqrt{\frac{ab(s-b)}{(s-a)^3}} - \sqrt{\frac{ac(s-c)}{(s-a)^3}} + \sqrt{\frac{s(s-b)(s-c)}{(s-a)^3} + 1} \right) \]
\[ = \frac{r}{2} (\cosh \alpha_A \cosh \beta_A \cosh \gamma_A - \sinh \alpha_A \cosh \beta_A \sinh \gamma_A \]
\[ - \sinh \alpha_A \sinh \beta_A \cosh \gamma_A + \cosh \alpha_A \sinh \beta_A \sinh \gamma_A + 1) \]
\[ = \frac{r_A (\cosh(\beta_A + \gamma_A - \alpha_A) + 1)}{2} \]
\[ = r_A \sinh^2(\sigma_A - \alpha_A). \]

By making similar calculations on every last three equations in \((48), (49), (50), (51), (52), (53), (54)\) and \((55)\), we obtain the following respective solutions in Cases 2 and 3.

\[
\begin{align*}
 r_1 &= r \cosh^2(\sigma_A - \alpha_A), \\
 r_2 &= r_C \sinh^2(\sigma_A - \beta_A), \\
 r_3 &= r_B \sinh^2(\sigma_A - \gamma_A),
\end{align*}
\]
\[(46)\]

\[
\begin{align*}
 r_1 &= r \cosh^2 \sigma_A, \\
 r_2 &= r_C \sinh^2(\sigma_A - \gamma_A), \\
 r_3 &= r_B \sinh^2(\sigma_A - \beta_A),
\end{align*}
\]
\[(47)\]

\[
\begin{align*}
 r_1 &= r \cosh^2(\sigma_A - \gamma_A), \\
 r_2 &= r_C \sinh^2 \sigma_A, \\
 r_3 &= r_B \sinh^2(\sigma_A - \alpha_A),
\end{align*}
\]
\[(48)\]

\[
\begin{align*}
 r_1 &= r \cosh^2(\sigma_A - \beta_A), \\
 r_2 &= r_C \sinh^2(\sigma_A - \alpha_A), \\
 r_3 &= r_B \sinh^2 \sigma_A,
\end{align*}
\]
\[(49)\]

\[
\begin{align*}
 r_1 &= r \sinh^2(\sigma_A - \alpha_A), \\
 r_2 &= r_C \cosh^2(\sigma_A - \beta_A), \\
 r_3 &= r_B \cosh^2(\sigma_A - \gamma_A),
\end{align*}
\]
\[(50)\]

\[
\begin{align*}
 r_1 &= r \sinh^2 \sigma_A, \\
 r_2 &= r_C \cosh^2(\sigma_A - \gamma_A), \\
 r_3 &= r_B \cosh^2(\sigma_A - \beta_A),
\end{align*}
\]
\[(51)\]

\[
\begin{align*}
 r_1 &= r \sinh^2(\sigma_A - \beta_A), \\
 r_2 &= r_C \cosh^2 \sigma_A, \\
 r_3 &= r_B \cosh^2(\sigma_A - \alpha_A),
\end{align*}
\]
\[(52)\]

\[
\begin{align*}
 r_1 &= r \sinh^2(\sigma_A - \alpha_A), \\
 r_2 &= r_C \cosh^2(\sigma_A - \alpha_A), \\
 r_3 &= r_B \cosh^2 \sigma_A.
\end{align*}
\]
\[(53)\]
4.3 Cases 4 & 5

Define $\alpha_B, \beta_B, \gamma_B \in (0, +\infty)$ and $\sigma_B$ by

\[
\sinh^2 \alpha_B = \frac{s - c}{s - b}, \quad \sinh^2 \beta_B = \frac{b}{s - b}, \quad \sinh^2 \gamma_B = \frac{s - a}{s - b}, \quad \sigma_B = \frac{\alpha_B + \beta_B + \gamma_B}{2}.
\]

Analogously to 4.2 we obtain the following solutions in Cases 4 and 5.

\[
\begin{align*}
\left\{ 
\begin{array}{l}
\alpha = \sinh \left( \frac{s - c}{s - b} \right), \\
\beta = \sinh \left( \frac{b}{s - b} \right), \\
\gamma = \sinh \left( \frac{s - a}{s - b} \right), \\
\sigma = \frac{\alpha + \beta + \gamma}{2}.
\end{array}
\right.
\end{align*}
\]

4.4 Cases 6 & 7

Define $\alpha_C, \beta_C, \gamma_C \in (0, +\infty)$ and $\sigma_C$ by

\[
\sinh^2 \alpha_C = \frac{s - b}{s - c}, \quad \sinh^2 \beta_C = \frac{s - a}{s - c}, \quad \sinh^2 \gamma_C = \frac{c}{s - c}, \quad \sigma_C = \frac{\alpha_C + \beta_C + \gamma_C}{2}.
\]
Analogously to 4.2, we obtain the following solutions in Cases 6 and 7.

\[
\begin{align*}
\begin{cases}
    r_1 &= r_B \sinh^2(\sigma_C - \alpha C), \\
    r_2 &= r_A \sinh^2(\sigma_C - \beta C), \\
    r_3 &= r \cosh^2(\sigma C - \gamma C).
\end{cases} & (62) \\
\begin{cases}
    r_1 &= r_B \sinh^2 \sigma C, \\
    r_2 &= r_A \sinh^2(\sigma C - \gamma C), \\
    r_3 &= r \cosh^2(\sigma C - \beta C).
\end{cases} & (63) \\
\begin{cases}
    r_1 &= r_B \sinh^2(\sigma C - \gamma C), \\
    r_2 &= r_A \sinh^2(\sigma C - \beta C), \\
    r_3 &= r \cosh^2(\sigma C - \beta C).
\end{cases} & (64) \\
\begin{cases}
    r_1 &= r_B \sinh^2(\sigma C - \alpha C), \\
    r_2 &= r_A \sinh^2(\sigma C - \alpha C), \\
    r_3 &= r \cosh^2 \sigma C.
\end{cases} & (65) \\
\begin{cases}
    r_1 &= r_B \cosh^2(\sigma C - \alpha C), \\
    r_2 &= r_A \cosh^2(\sigma C - \beta C), \\
    r_3 &= r \sinh^2(\sigma C - \gamma C).
\end{cases} & (66) \\
\begin{cases}
    r_1 &= r_B \cosh^2 \sigma C, \\
    r_2 &= r_A \cosh^2(\sigma C - \gamma C), \\
    r_3 &= r \sinh^2(\sigma C - \beta C).
\end{cases} & (67) \\
\begin{cases}
    r_1 &= r_B \cosh^2(\sigma C - \beta C), \\
    r_2 &= r_A \cosh^2(\sigma C - \beta C), \\
    r_3 &= r \sinh^2(\sigma C - \beta C).
\end{cases} & (68) \\
\begin{cases}
    r_1 &= r_B \cosh^2(\sigma C - \beta C), \\
    r_2 &= r_A \cosh^2(\sigma C - \alpha C), \\
    r_3 &= r \sinh^2 \sigma C.
\end{cases} & (69)
\end{align*}
\]

5 Conclusion

Theorem 1. For any triangle, there exist 32 triplets of circles such that each circle is tangent to the other two circles and to two of the sides of the reference triangle or their extensions. The radii can be expressed by (17) – (24), (46) – (53), (54) – (61), (62) – (69).

Figures 3–34 illustrate the 32 triplets of circles for a triangle \(ABC\) such that \(A = 45^\circ, B = 54^\circ, C = 81^\circ\).

A Solutions of the systems of equations

In this appendix, we will solve some systems of equations by computing Gröbner bases. Although it is difficult to compute the Gröbner bases by hand, any
computer algebra system that can compute Gröbner bases should work.

**Proposition 1.** The system of equations [8] for the variables $u, v, w, x, y, z$ with the positive real parameters $l, m, n$ under the restriction $lmn = l + m + n$ has 8 solutions [9]–[16].

Proof. Counting $l, m, n$ among the variables in addition to $u, v, w, x, y, z$, we compute the reduced Gröbner basis of \{my + nz + 2u - m - n, lx + nz + 2v - l - n, lx + my + 2w - l - m, xy - w^2, xz - v^2, yz - u^2, lmn - l - m - n\} with the degree reverse lexicographical ordering $x > y > z > u > v > w > l > m > n$. The reduced Gröbner basis consists of 67 polynomials including

$$f_1 = (2u^2 + 2lu - 2u - l)(2u^2 + 2lu + 2u + l),$$

$$f_2 = (2v^2 + 2nv - 2v - m)(2v^2 + 2nv + 2v + m),$$

$$f_3 = (2w^2 + 2nw - 2w - n)(2w^2 + 2nw + 2w + n),$$

$$f_4 = lx - u + v + w - l,$$

$$f_5 = my - u + v + w - m,$$

$$f_6 = nz - u + v + w - n.$$

By solving \{f_1 = 0, f_2 = 0, f_3 = 0, f_4 = 0, f_5 = 0, f_6 = 0\} for $u, v, w, x, y, z$, we obtain 128 solutions. Note that $lmn = l + m + n$ is equivalent to

$$l = \frac{m + n}{mn - 1}. \quad (70)$$

By assigning each of the 128 solutions together with [70] to \{my + nz + 2u - m - n, lx + nz + 2v - l - n, lx + my + 2w - l - m, xy - w^2, xz - v^2, yz - u^2\} and then picking out the solutions such that the assignment makes all of the polynomials equal 0, we still have 8 solutions [9]–[16], which are the solutions of [8].

**Proposition 2.** The system of equations [32] for the variables $u, v, w, x, y, z$ with the positive real parameters $l, \bar{m}, \bar{n}$ under the restriction $l\bar{m}\bar{n} = l - \bar{m} - \bar{n}$ has 8 solutions [33]–[45].

Proof. Counting $l, \bar{m}, \bar{n}$ among the variables in addition to $u, v, w, x, y, z$, we compute the reduced Gröbner basis of \{my + nz + 2u - m - n, lx + nz + 2v - l - n, lx - my + 2w - l + \bar{m}, xy - w^2, xz - v^2, yz - u^2, l\bar{m}\bar{n} - l + \bar{m} + \bar{n}\} with the degree reverse lexicographical ordering $x > y > z > u > v > w > l > \bar{m} > \bar{n}$. The reduced Gröbner basis consists of 67 polynomials including

$$f_1 = (2u^2 + 2lu - 2u - l)(2u^2 + 2lu + 2u + l),$$

$$f_2 = (2v^2 - 2nv - 2v + \bar{m})(2v^2 - 2nv + 2v - \bar{m}),$$

$$f_3 = (2w^2 - 2\bar{w}w - 2w + \bar{n})(2w^2 - 2\bar{w}w + 2w - \bar{n}),$$

$$f_4 = lx - u + v + w - l,$$

$$f_5 = \bar{m}y - u + v + w - \bar{m},$$

$$f_6 = \bar{n}z - u + v + w - \bar{n}.$$

By solving \{f_1 = 0, f_2 = 0, f_3 = 0, f_4 = 0, f_5 = 0, f_6 = 0\} for $u, v, w, x, y, z$, we obtain 128 solutions. Note that $l\bar{m}\bar{n} = l - \bar{m} - \bar{n}$ is equivalent to

$$l = -\frac{\bar{m} + \bar{n}}{\bar{m}\bar{n} - 1}. \quad (71)$$
By assigning each of the 128 solutions together with (71) to \(\{\bar{m}y + \bar{n}z - 2u - \bar{m} - \bar{n}, lx - \bar{n}z + 2v - l + \bar{n}, lx - \bar{m}y + 2w - l + \bar{m}, xy - w^2, xz - v^2, yz - u^2\}\) and then picking out the solutions such that the assignment makes all of the polynomials equal 0, we still have 8 solutions (38)–(45), which are the solutions of (32).

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Figure 3: \[
\begin{align*}
    r_1 &= r_A \sin^2(\sigma - \alpha), \\
    r_2 &= r_B \sin^2(\sigma - \beta), \\
    r_3 &= r_C \sin^2(\sigma - \gamma).
\end{align*}
\]

Figure 4: \[
\begin{align*}
    r_1 &= r_A \sin^2 \sigma, \\
    r_2 &= r_B \sin^2(\sigma - \gamma), \\
    r_3 &= r_C \sin^2(\sigma - \beta).
\end{align*}
\]
Figure 5: \[
\begin{align*}
    r_1 &= r_A \sin^2(\sigma - \gamma), \\
    r_2 &= r_B \sin^2 \sigma, \\
    r_3 &= r_C \sin^2(\sigma - \alpha).
\end{align*}
\]
Figure 6: \[
\begin{align*}
    r_1 &= r_A \sin^2(\sigma - \beta), \\
    r_2 &= r_B \sin^2(\sigma - \alpha), \\
    r_3 &= r_C \sin^2 \sigma.
\end{align*}
\]
Figure 7: \[
\begin{align*}
    r_1 &= r_A \cos^2(\sigma - \alpha), \\
    r_2 &= r_B \cos^2(\sigma - \beta), \\
    r_3 &= r_C \cos^2(\sigma - \gamma).
\end{align*}
\]
Figure 8:
\[
\begin{align*}
  r_1 &= r_A \cos^2 \sigma, \\
  r_2 &= r_B \cos^2(\sigma - \gamma), \\
  r_3 &= r_C \cos^2(\sigma - \beta).
\end{align*}
\]
Figure 9: 
\[ \begin{align*} 
    r_1 &= r_A \cos^2(\sigma - \gamma), \\
    r_2 &= r_B \cos^2 \sigma, \\
    r_3 &= r_C \cos^2(\sigma - \alpha). 
\end{align*} \]
Figure 10: \[
\begin{align*}
    r_1 &= r_A \cos^2(\sigma - \beta), \\
    r_2 &= r_B \cos^2(\sigma - \alpha), \\
    r_3 &= r_C \cos^2 \sigma.
\end{align*}
\]

Figure 11: \[
\begin{align*}
    r_1 &= r_c \cosh^2(\sigma_A - \alpha_A), \\
    r_2 &= r_c \sinh^2(\sigma_A - \beta_A), \\
    r_3 &= r_B \sinh^2(\sigma_A - \gamma_A).
\end{align*}
\]
Figure 12: \[ r_1 = r \cosh^2 \sigma_A, \]
\[ r_2 = r_C \sinh^2(\sigma_A - \gamma_A), \]
\[ r_3 = r_B \sinh^2(\sigma_A - \beta_A). \]
Figure 13: \[
\begin{align*}
    r_1 &= r \cosh^2(\sigma_A - \gamma_A), \\
    r_2 &= r_C \sinh^2 \sigma_A, \\
    r_3 &= r_B \sinh^2 (\sigma_A - \alpha_A).
\end{align*}
\]
Figure 14: \[
\begin{aligned}
\begin{cases}
    r_1 &= r \cosh^2(\sigma_A - \beta_A), \\
    r_2 &= r_C \sinh^2(\sigma_A - \alpha_A), \\
    r_3 &= r_B \sinh^2 \sigma_A.
\end{cases}
\end{aligned}
\]
\[ r_1 = r \sinh^2(\sigma_A - \alpha_A), \]
\[ r_2 = r_C \cosh^2(\sigma_A - \beta_A), \]
\[ r_3 = r_B \cosh^2(\sigma_A - \gamma_A). \]
\[
\begin{align*}
  r_1 &= r \sinh^2 \sigma_A, \\
  r_2 &= r_C \cosh^2 (\sigma_A - \gamma_A), \\
  r_3 &= r_B \cosh^2 (\sigma_A - \beta_A).
\end{align*}
\]
Figure 17: \[
\begin{align*}
r_1 &= r \sinh^2 (\sigma_A - \gamma_A), \\
r_2 &= r_C \cosh^2 \sigma_A, \\
r_3 &= r_B \cosh^2 (\sigma_A - \alpha_A).
\end{align*}
\]
\begin{align*}
  r_1 &= r \sinh^2(\sigma_A - \beta_A), \\
  r_2 &= r_C \cosh^2(\sigma_A - \alpha_A), \\
  r_3 &= r_B \cosh^2 \sigma_A.
\end{align*}

Figure 18: \begin{align*}
  r_1 &= r \sinh^2(\sigma_A - \beta_A), \\
  r_2 &= r_C \cosh^2(\sigma_A - \alpha_A), \\
  r_3 &= r_B \cosh^2 \sigma_A.
\end{align*}
Figure 19: \[
\begin{align*}
    r_1 &= r_C \sinh^2(\sigma_B - \alpha_B), \\
    r_2 &= r \cosh^2(\sigma_B - \beta_B), \\
    r_3 &= r_A \sinh^2(\sigma_B - \gamma_B).
\end{align*}
\]
Figure 20: \[
\begin{align*}
  r_1 &= r_C \sinh^2 \sigma_B, \\
  r_2 &= r \cosh^2(\sigma_B - \gamma_B), \\
  r_3 &= r_A \sinh^2(\sigma_B - \beta_B).
\end{align*}
\]
Figure 21: \[
\begin{align*}
    r_1 &= r_C \sinh^2(\sigma_B - \gamma_B), \\
    r_2 &= r \cosh^2 \sigma_B, \\
    r_3 &= r_A \sinh^2(\sigma_B - \alpha_B).
\end{align*}
\]
Figure 22: \[
\begin{align*}
    r_1 &= r_C \sinh^2(\sigma_B - \beta_B), \\
    r_2 &= r \cosh^2(\sigma_B - \alpha_B), \\
    r_3 &= r_A \sinh^2 \sigma_B.
\end{align*}
\]
\[
\begin{align*}
  r_1 &= r_C \cosh^2(\sigma_B - \alpha_B), \\
  r_2 &= r \sinh^2(\sigma_B - \beta_B), \\
  r_3 &= r_A \cosh^2(\sigma_B - \gamma_B).
\end{align*}
\]

Figure 23: \[
\begin{align*}
  r_1 &= r_C \cosh^2(\sigma_B - \alpha_B), \\
  r_2 &= r \sinh^2(\sigma_B - \beta_B), \\
  r_3 &= r_A \cosh^2(\sigma_B - \gamma_B).
\end{align*}
\]
Figure 24: \[
\begin{align*}
    r_1 &= r_C \cosh^2 \sigma_B, \\
    r_2 &= r \sinh^2 (\sigma_B - \gamma_B), \\
    r_3 &= r_B \cosh^2 (\sigma_B - \beta_B).
\end{align*}
\]
\[ r_1 = r_C \cosh^2(\sigma_B - \gamma_B), \]
\[ r_2 = r \sinh^2 \sigma_B, \]
\[ r_3 = r_A \cosh^2(\sigma_B - \alpha_B). \]
Figure 26: \[
\begin{cases}
    r_1 = r_C \cosh^2(\sigma_B - \beta_B), \\
    r_2 = r \sinh^2(\sigma_B - \alpha_B), \\
    r_3 = r_A \cosh^2 \sigma_B.
\end{cases}
\]
Figure 27: \[
\begin{align*}
  r_1 &= r_B \sinh^2(\sigma_C - \alpha_C), \\
  r_2 &= r_A \sinh^2(\sigma_C - \beta_C), \\
  r_3 &= r \cosh^2(\sigma_C - \gamma_C).
\end{align*}
\]
Figure 28: \[
\begin{align*}
    r_1 &= r_B \sinh^2 \sigma_C, \\
    r_2 &= r_A \sinh^2(\sigma_C - \gamma_C), \\
    r_3 &= r \cosh^2(\sigma_C - \beta_C).
\end{align*}
\]
$r_1 = r_B \sinh^2(\sigma_C - \gamma_C),$

$r_2 = r_A \sinh^2 \sigma_C,$

$r_3 = r \cosh^2(\sigma_C - \alpha_C).$
Figure 30: \[
\begin{align*}
    r_1 &= r_B \sinh^2(\sigma_C - \beta_C), \\
    r_2 &= r_A \sinh^2(\sigma_C - \alpha_C), \\
    r_3 &= r \cosh^2 \sigma_C.
\end{align*}
\]
Figure 31: \[
\begin{align*}
    r_1 &= r_B \cosh^2(\sigma_C - \alpha_C), \\
    r_2 &= r_A \cosh^2(\sigma_C - \beta_C), \\
    r_3 &= r \sinh^2(\sigma_C - \gamma_C).
\end{align*}
\]
Figure 32: \[
\begin{align*}
    r_1 &= r_B \cosh^2 \sigma_C, \\
    r_2 &= r_A \cosh^2(\sigma_C - \gamma C), \\
    r_3 &= r \sinh^2(\sigma_C - \beta C).
\end{align*}
\]
Figure 33: \[ \begin{align*}
    r_1 &= r_B \cosh^2(\sigma_C - \gamma_C), \\
    r_2 &= r_A \cosh^2 \sigma_C, \\
    r_3 &= r \sinh^2(\sigma_C - \beta_C). 
\end{align*} \]
$$
\begin{align*}
  r_1 &= r_B \cosh^2(\sigma C - \beta C), \\
  r_2 &= r_A \cosh^2(\sigma C - \alpha C), \\
  r_3 &= r \sinh^2 \sigma C.
\end{align*}
$$