Groups with undecidable word problem and almost quadratic Dehn function

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with an appendix written by M.V. Sapir

Abstract
We construct a finitely presented group with undecidable word problem and with Dehn function bounded by a quadratic function on an infinite set of positive integers.

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1. Introduction

1.1. Formulation of results
The minimal non-decreasing function $f(n) : \mathbb{N} \to \mathbb{N}$ such that every word $w$ vanishing in a group $G = \langle A \mid R \rangle$ and having length $\|w\| \leq n$, freely equal to a product of at most $f(n)$ conjugates of relators from $R$ is called the Dehn function of the presentation $G = \langle A \mid R \rangle$ (see [5]). By van Kampen’s Lemma, $f(n)$ is equal to the maximal area of minimal diagrams $\Delta$ with perimeter at most $n$. (See Section 5.1 for the definitions.) For finitely presented groups (that is, both sets $A$ and $R$ are finite), Dehn functions are usually taken up to equivalence to get rid of the dependence on a finite presentation for $G$ (see [8]). To introduce this equivalence $\sim$, we write $f \preceq g$ if there is a positive integer $c$ such that $f(n) \leq cg(cn) + cn$ for any $n \in \mathbb{N}$. Two non-decreasing functions $f$ and $g$ on $\mathbb{N}$ are called equivalent if $f \preceq g$ and $g \preceq f$.

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A function \( f : \mathbb{N} \to \mathbb{N} \) is called *almost quadratic* if there exists a constant \( C > 0 \) and an infinite set of integers \( B \), such that \( f(b) < Cb^2 \) for all \( b \in B \).

It is well known that a finitely presented group has undecidable word problem if and only if its Dehn function \( d(n) \) is not bounded by a recursive function (and if and only if \( d(n) \) is not recursive itself; see \([3, 4]\)) \(\), whence for every recursive function \( f(n) \), \( d(n) > f(n) \) for infinitely many values of \( n \). The main result of this paper shows that a non-recursive Dehn function can be almost quadratic at the same time.

**Theorem 1.1.** There exists a finitely presented group \( G \) with undecidable word problem and almost quadratic Dehn function.

Note that ‘almost quadratic’ is the smallest Dehn function one can obtain, because if the Dehn function of a finitely presented group is \( o(n^2) \) on some infinite set of integers, then the group is hyperbolic and its Dehn function is linear (this follows from Gromov \([5, 6.8.M]\) or Bowditch \([2]\)).

By Theorem 1.1, for some infinite set \( B \) of natural numbers \( b \), the Dehn function of \( G \) satisfies the condition \( f(b) < Cb^2 \) for some constant \( C \). Note that the set \( B \) is not recursive or even recursively enumerable (r.e.) although its complement is r.e. Indeed, if \( f(b) \geq Cb^2 \), then there exists a word \( w \) of length at most \( b \) which is equal to 1 in the group, but which is not the boundary label of any van Kampen diagram with less than \( Cb^2 \) cells; all diagrams with this number of cells and boundary length at most \( b \) can be enumerated; and all words that are equal to 1 in the group can be enumerated too. Moreover, \( B \) cannot contain any infinite r.e. subset (that is, it is *immune* in the terminology of \([9]\)). Indeed if \( B \) contains an infinite r.e. set enumerated by a Turing machine \( M \), then in order to check if a word \( w \) is 1 in \( G \) (and solve the word problem in \( G \)), we would do the following: wait till \( M \) produces a word \( w' \) longer than \( w \). Then the area of the minimal van Kampen diagram for \( w \) cannot exceed \( C\|w'\|^2 \) (here and below \( \|w\| \) denotes the length of the word \( w \)), and it would remain to check all diagrams of that area. Thus, although \( B \) exists and is infinite, there is no algorithm to find any infinite part of it.

As a corollary of Theorem 1.1 and the results of \([16]\) and \([6]\) we obtain:

**Corollary 1.2.** The group \( G \) from Theorem 1.1 has a simply connected and a non-simply connected asymptotic cones.

Indeed the asymptotic cone corresponding to the sequence \( B \) discussed in the previous paragraph is simply connected by \([16]\). On the other hand, all asymptotic cones of \( G \) cannot be simply connected because that would imply decidability of the word problem in \( G \) by \([6]\).

Undecidability of the conjugacy problem is easier to achieve than undecidability of the word problem.

**Theorem 1.3.** There exists a finitely presented (multiple) HNN extension \( M \) of a free group with finitely generated associated subgroups and with Dehn function \( f(n) \) such that the following hold.

1. The conjugacy problem is undecidable in \( M \).
2. There is an infinite set \( N_1 \subseteq \mathbb{N} \), such that for some constant \( C \) we have \( f(n) < Cn^2 \) for every \( n \in N_1 \).
3. For every \( n \), \( f(n) \leq Cn^3 \).

**Remark 1.4.** Probably the first example of an almost quadratic but not quadratic Dehn function of a finitely presented group was constructed in \([12]\). However, that function is
\[ O(n^2 \log n / \log \log n) \] which is not much bigger than a quadratic function. A slight modification of the proofs of the present paper provides us with a recursive almost quadratic Dehn function \( d(n) \) rapidly increasing on some infinite subset \( N_2 \) of \( \mathbb{N} \). (For example, almost quadratic \( d(n) \) is at least exponential on \( N_2 \) and at most exponential on the entire \( \mathbb{N} \); see Theorem 13.5 and Remark 13.6 for details.) The difference is that the proof of Theorem 1.1 uses Sapir’s Theorem A.1, but the recursive examples are independent of it.

**Remark 1.5.** Using [19] and Theorem A.1 one can easily obtain a weaker version of Theorem 1.1 replacing ‘almost quadratic’ by ‘almost polynomial’. Recall that a function \( f : \mathbb{N} \to \mathbb{N} \) is superadditive if \( f(m + n) \geq f(m) + f(n) \) for any \( m, n \in \mathbb{N} \). The superadditive closure \( \bar{f}(n) \) of a function \( f(n) \) is given by the formula \( \bar{f}(n) = \max(f(n_1) + \cdots + f(n_k)) \) over all non-negative partitions \( n = n_1 + \cdots + n_k \). If a Turing machine \( M \) accepts a language \( L \) with at least a linear time function \( T(n) \), and \( T(n)^4 \) is equivalent to a superadditive function, then by [19, Theorem 1.3], there is a finitely presented group \( G(M) \) with Dehn function \( d(n) \) equivalent to \( T(n)^4 \). But in fact, it is proved in [19] that omitting the assumption that \( T(n)^4 \) is superadditive, we have inequalities \( T(n)^4 \leq d(n) \leq \bar{T(n)^4} \). Thus, it suffices to construct a Turing machine \( M \) with non-recursive but ‘almost linear’ time function \( T(n) \). The existence of such a machine follows from Theorem A.1. (Moreover, one can derive from Theorem A.1 that \( \bar{T(n)^4} \) is ‘almost \( n^4 \).’)

Reducing to ‘almost quadratic’ (as in Theorem 1.1) requires a new approach. The \( S \)-machine we are going to use will be different from [19], and the analysis of diagrams will be much more delicate. The main reason for the difficulties arising here is that the ‘almost quadratic’ property is unimprovable. For example, the cubic upper bound of the Dehn function of the group \( M \) is obvious in [19] (see also Step 1 in the proof of Lemma 13.1), but the main contents of our paper focus on \( M \), starting with the properties of the machine defining \( M \) and ending with new quadratic invariants of the diagrams called mixture(s) on their boundaries. In the next subsection of the introduction, we discuss the outline of the proof of Theorem 1.1, and some ideas needed in its proof.

1.2. A short description of the proof of Theorem 1.1

Relations of a finitely presented group with undecidable word problem simulate the commands of a Turing machine \( M_0 \) with undecidable halting problem, and as in the works of P. Novikov, W. Boone and many other authors (see [17, 18]), one has to properly code the work of a Turing machine in terms of group relations. To obtain an almost quadratic Dehn function of a group \( G \), we must start with a machine having almost linear time function (but which is not bounded from above by any recursive function). Thus, we can just demand that the lengths of words accepted by \( M_0 \) form a very sparse subset of positive integers \( B \subset \mathbb{N} \). As a measure of how sparse \( B \) is, M.V. Sapir gives the following exact definition.

Let \( X \) be a r.e. language in the binary alphabet recognized by a Turing machine \( M \). If \( w \in X \), then the time of \( w \) (denoted \( \text{time}(w) \) or \( \text{time}_M(w) \)) is, by definition, the minimal time of an accepting computation of \( M \) with input \( w \). For an increasing function \( h : \mathbb{N} \to \mathbb{N} \), a number \( m \in \mathbb{N} \) is called \( h \)-good for \( M \) if for every word \( w \in X \) of length less than \( m \), we have \( h(\text{time}(w)) < m \).

For our estimates, it suffices to start with a Turing machine \( M_0 \) recognizing a r.e. non-recursive set \( X \) such that the set of all \( f \)-good numbers for \( M_0 \) is infinite, where \( f \) is arbitrary double exponential function. Such a machine is constructed in the appendix written by M.V. Sapir (see Theorem A.1).

Group relations always interpret the symmetrization of a machine. Thus, as a preliminary step, one has to add the inverse commands, in spite of the fact that the machine \( M_0 \) is replaced.
by a non-deterministic machine $M_1$. (Of course, one should be concerned that the symmetrization preserves some basic characteristics of the machine.) However, the interpretation problem for groups remains much harder than for semigroups even after modifying the machine because the group theoretic simulation can execute unforeseen computations with non-positive words. Boone and Novikov secured the positiveness of admissible configurations with the help of an additional ‘quadratic letter’ (see [17, Chapter 12]). However, this old trick implies that the constructed group $G$ contains Baumslag-Solitar groups $B_{1,2}$ and has an at least exponential Dehn function. Since we want to obtain almost quadratic Dehn function, we use a new approach suggested in [19]. Invented by Sapir, $S$-machines can work with non-positive words on the tapes.

Here, we use a composition $M_2$ of the symmetric machine $M_1$ with an ‘adding machine’ $Z(A)$ introduced in [15]. This $S$-machine is equivalent to $M_1$.

Here, we have to guarantee at least two important properties of the machine $M_2$ (since the violation of either of them makes the Dehn function of the group $M_2$ non-almost quadratic): a reduced computation of $M_2$ does not repeat the same configuration twice, and every accepting computation of $M_2$ is uniquely determined by the initial configuration (though $M_2$ is highly non-deterministic).

Every $S$-machine is, on the one hand, a rewriting system and, on the other hand, it can be treated as a multiple HNN-extension of a free group (see [15] or [18]). But when one takes an $S$-machine as an HNN-extension, then the number of working heads can arbitrarily be large and their order on the common tape can be non-standard. Therefore, as in [19] or [15], we have to extend the set of admissible words for the machine treated as a rewriting system. This makes the control of arbitrary computation difficult. (There was no need for such accurate control in [19] or [15].) Hence, we are forced to introduce auxiliary control heads which are called upon to examine the order of heads after and/or before the application of every rule of the machine $M_2$. The obtained machine $M_3$ is better than $M_2$ because it is able to accomplish only ‘simple’ computations with non-standard disposition of the heads.

When we introduce a new machine, then clearly, we should check that it inherits the important properties of the machines studied earlier. In particular, $M_3$ inherits the language accepted by $M_1$. The next modification is the machine $M_4$ which has two additional tapes with histories of what $M_3$ computes. For any computation with the standard order of heads of $M_4$ (the notion of standard base of $M_4$ is given in Section 4.3), we prove that either the time $T$ of this computation is ‘close’ to the time $T_i$ of some computation accepting a word from the sparse set provided by Theorem A.1, or the space on the ‘historical’ tapes at the beginning or at the end of the computation is bounded from below by a linear function of $T$. In the latter case, we have a quadratic upper estimate for the area of the trapezium corresponding to the computation. Here, the definition of trapezium as a special van Kampen diagram is borrowed from [15], and Lemma 5.10 translates the machine language to the diagram language.

Finally, the machine $M$ is a union of many copies and mirror copies of $M_4$ working in a parallel way. The corresponding HNN-extension $M$ is the group from Theorem 1.3. The accept word of the machine $M$ is called the hub. There is no algorithm deciding if a given word in the generators of $M$ is conjugate to the hub. The usual adding of the hub relation to the list of defining relations of $M$ (as in [17], and many papers) provides us with the group $G$ for Theorem 1.1. As in [19] or [11], the hub relation has many copies of the accept words of $M_4$. This makes the hub graph (with vertices in hub cells; see Section 5.3) associated with a van Kampen diagram, hyperbolic, and this is used in Lemmas 5.18 and 5.19. The mirror symmetry of the hub is used for the surgery removing a hub (see Section 12.2).

Unsolvability of the halting problem for the $S$-machine $M$ immediately implies that the Dehn function $d(n)$ of the group $G$ is not bounded from above by any recursive function. Other precautions used in the construction of the machines $M_0, \ldots, M_4$, $M$ eliminate a number of visual obstacles standing in the way of the almost quadratic property for the Dehn functions of $M$ and $G$. (For instance, if one uses only one historic tape for $M_4$ or the arrangement of the
associate a two-colored necklace with the boundary quadratic invariant of the boundaries of the diagrams, called a mixture. In Section 6, we nately, it is false for other combs whose areas must also be estimated. We have found another necklace correspond to different types of edges in the quadratic estimate Area(Δ₁) ⩽ C|y|(|z| − |y|), where the positive constant C does not depend on the diagrams. Then it is easy to see that the quadratic estimate Area(Δ₂) ⩽ C|yz|^2 for the subdiagram Δ₂ with perimeter at most n together with the estimate for Area(Δ₁) give

\[ \text{Area}(\Delta) \leq \text{Area}(\Delta_1) + \text{Area}(\Delta_2) \leq C|zz'|^2 = Cn^2 \]

as required. Thus, we are looking for pieces whose area can be estimated as for Δ₁.

First of all, among such ‘good’ pieces, we have so-called rim θ-bands with a restriction on the length (but here we have to change the usual combinatorial metric by the metric, where the generators from the tape alphabet of the machine M are much shorter than other generators of the groups M and G). The ‘good’ pieces of second type (again, under some restrictions) are combs defined in Section 7 (and introduced earlier in [15]).

The upper bound of the form C|y|(|z| − |y|) works for many types of combs but unfortunately, it is false for other combs whose areas must also be estimated. We have found another quadratic invariant of the boundaries of the diagrams, called a mixture. In Section 6, we associate a two-colored necklace with the boundary p of Δ. The black and white beads of this necklace correspond to different types of edges in p. To obtain the mixture μ(Δ) one calculates the number of pairs of white beads separated in p by black ones. (Another quadratic invariant, called dispersion, was introduced and applied earlier in [15], but the dispersion depends on the whole diagram and works for hub-free diagrams, while the mixture depends on the boundary label only and works for arbitrary diagram over G.)

The important observation is that for many types of subcombs Δ₁, we have inequalities Area(Δ₁) ⩽ C|y|(|z| − |y|) + μ(Δ₁) and μ(Δ₂) ⩽ μ(Δ) − μ(Δ₁). This was a breakthrough which inspired the confidence that the whole project would be completed. However, the original mixture cannot help in case of some special combs. Therefore, we have to consider boundary necklaces of three different types. The different mixtures help one to estimate the areas of different combs. But one of these mixtures helps in some cases and can be negative in some other cases, which causes a problem for our induction. Therefore, we use a weighted linear combination of three mixtures in Lemma 11.8 summarizing our estimates of comb areas. Hence, we have to estimate the behavior of these mixtures in different situations, which makes a number of comb lemmas complicated, and the comb part of the paper is the hardest one.

Then we consider a diagram Δ with hubs. Due to hyperbolicity of the hub structure mentioned above, there is a hub Π such that almost all ‘spokes’ starting on Π end on the boundary ∂Δ, and they bound (together with ∂Δ and ∂Π) a subdiagram Ψ without hubs. Now, we are able to remove redundant combs and rim bands from Ψ. The remaining crescent Ψ together with Π can be cut off by a relatively short cutting path. (Thus, one can also induct on the number of hubs in Δ.) As in [19], our surgery uses the mirror symmetry of the hub relation, but our inequalities are more delicate here than those used for the ‘snowball decomposition’ in [19] since we aim for almost quadratic bounds. Again, we estimate the area of the removed part in terms of the reduction of the perimeter, of the mixtures and more.
To complete the proof, we take into account that the auxiliary parameters are quadratically bounded with respect to the perimeter of a diagram.

The author is aware that such a long proof can be arduous to the reader. Making our apology, we collect all the definitions and terms at the end of the paper (see subject index) and insert many pictures and brief comments throughout the text. Besides, Lemma 5.17 reformulates all machine properties we need in terms of van Kampen diagrams so that the machine constructions can be forgotten after one has read that lemma.

2. A Turing machine

2.1. Definitions and notation related to Turing machines

In this section, we collect all information about Turing machines that we need in the proof of our main results.

As usual, we consider words as sequences of symbols from some alphabet $X$.

We shall use the following standard notation for Turing machines. A (multi-tape) Turing machine has $k$ tapes and $k$ heads observing the tapes. One can view it as a structure

$$M = \langle I, Y, Q, \Omega, \Theta \rangle,$$

where $I$ is the input alphabet, $Y$ is the tape alphabet ($I \subseteq Y$), $Q = \bigcup Q_i$, $i = 1, \ldots, k$ is the set of states of the heads of the machine (and $\bigcup$ denotes disjoint union), $\Omega = \{\alpha_1, \omega_1, \ldots, \alpha_k, \omega_k\}$ is the set of left and right markers of the tapes, and $\Theta$ is a set of commands.

The leftmost (the rightmost) square on the $i$th tape is always marked by $\alpha_i$ (by $\omega_i$). The head is placed between two consecutive squares on the tape. A configuration of the $i$th tape of a Turing machine is a word $\alpha_i u q v \omega_i$, where $q \in Q_i$ is the current state of the head of that tape, $u$ is the word in $Y$ to the left of the head and $v$ is the word in $Y$ to the right of the head, and so the word written on the entire tape is $uv$; so, we do not include $\alpha_i$ and $\omega_i$ and the state letter when we talk about the word written on the tape.

At every moment, the head of each tape observes two letters on that tape: the last letter of $u$ (or $\alpha_i$) and the first letter of $v$ (or $\omega_i$).

A configuration $U$ of a Turing machine is the word

$$U_1 U_2 \ldots U_k,$$

where $U_i$ is the configuration of tape $i$. We shall omit the indices $i$ of $\alpha_i$ and $\omega_i$ for the sake of brevity.

Assuming that the Turing machine is recognizing, we can define input configurations and accepted (stop) configurations. An input configuration is a configuration where the word written on the first tape is in $I$, all other tapes being empty, the head on the first tape observes the right marker $\omega$, and the states of all tapes form a special start $k$-vector $\vec{s}_1$. An accept (or stop) configuration is any configuration where the state vector for a special $k$-vector $\vec{s}_0$, the accept vector of the machine. We shall always assume (as can be easily achieved) that in the accept configuration of a Turing machine every tape is empty.

A transition (command) of a Turing machine is given by the states of the heads and some of the $2k$ letters observed by the heads. As a result of a transition, we replace some of these $2k$ letters by other letters, insert new squares in some of the tapes and may move the heads one square to the left (right) with respect to the corresponding tapes.

For example, in a one-tape machine, every transition is of the following form:

$$uqv \rightarrow u'q'v',$$

where $u, v, u'$, and $v'$ are letters (could be end markers) or empty words. The only constraint is that the result of applying the substitution $uqv \rightarrow u'q'v'$ to a configuration word must be a
configuration word again, in particular, the end markers cannot be deleted or inserted. This command means that if the state of the head is \( q \), \( u \) is written to the left of \( q \) and \( v \) is written to the right of \( q \), then the machine must replace \( u \) by \( u' \), \( q \) by \( q' \), and \( v \) by \( v' \).

For a general \( k \)-tape machine, a command is a vector

\[
[U_1 \rightarrow V_1, \ldots, U_k \rightarrow V_k],
\]

where \( U_i \rightarrow V_i \) is a command of a 1-tape machine, the elementary commands (also called parts of the command) \( U_i \rightarrow V_i \) are listed in the order of tape numbers. In order to execute this command, the machine checks if \( U_i \) is a subword of the configuration of tape \( i \) \( (i = 1, \ldots, k) \), and then replaces \( U_i \) by \( V_i \).

Note that for every command \( [U_1 \rightarrow V_1, \ldots, U_k \rightarrow V_k] \), the vector \( [V_1 \rightarrow U_1, \ldots, V_k \rightarrow U_k] \) is also a command of a Turing machine. These two commands are called mutually inverse. A Turing machine is called symmetric if for every command of that machine, the inverse is also a command of the machine. If a Turing machine is symmetric, we shall always consider a division of the set of its commands \( \Theta \) into two disjoint subsets, positive and negative commands: \( \Theta = \Theta^+ \sqcup \Theta^- \), so that the inverses of commands in \( \Theta^+ \) are in \( \Theta^- \) and vice versa.

We will assume that only input configurations of a Turing machine involve the state letters from \( \bar{s}_1 \) and only one (positive) command \( \theta_{\text{start}} \) is applicable to the input configurations. Similarly, we assume that there is a unique accept configuration \( s_0 \) and a unique (positive) accepting command \( \theta_{\text{accept}} \).

A computation is a sequence of configurations \( C_0 \rightarrow \cdots \rightarrow C_n \) such that for every \( 0 = 1, \ldots, n - 1 \) the machine passes from \( C_i \) to \( C_{i+1} \) by applying one of the commands from \( \Theta \). A configuration \( C \) is said to be accepted by a machine \( M \) if there exists at least one computation which starts with \( C \) and ends with the accept configuration.

A word \( u \) over \( I \) is said to be accepted by the machine if the corresponding input configuration is accepted. The set of all accepted words over the input alphabet \( I \) is called the language accepted (recognized) by the machine.

Let \( C = C_0 \rightarrow \cdots \rightarrow C_n \) be a computation of a machine \( M \) such that for every \( j = 0, \ldots, n - 1 \) the configuration \( C_{j+1} \) is obtained from \( C_j \) by a command \( \theta_{j+1} \) from \( \Theta \). Then we call the word \( \theta_1 \ldots \theta_n \) the history of this computation. The number \( n \) will be called the time (or length) of the computation.

**Remark 2.1.** Note that we can (and will) assume that in every command \( [u_1q_1v_1 \rightarrow u'_1q'_1v'_1, \ldots, u_kv_kv_k \rightarrow u'_kv'_k] \) the sum of numbers of letters from the tape alphabet \( Y \) in all \( u_i, u'_i, v_i, v'_i, i = 1, \ldots, k \), is at most 1. Indeed, this can be achieved by subdividing a command in the standard way. For example, a command \([aq \rightarrow bq']\) is replaced by two commands \([aq \rightarrow q'']\), \([q'' \rightarrow bq']\), where \( q'' \) is a new state letter.

It is convenient to consider empty computations consisting of one word \( W \). The history of an empty computation is the empty word, the start and end words of this computation are equal to \( W \). We do not only consider deterministic Turing machines, for example, we allow several transitions with the same left side. For example, most symmetric Turing machines are not deterministic.

### 2.2. A conversion of a deterministic Turing machine into a symmetric Turing machine

At first, let us add some useful properties to a machine.
Lemma 2.2. Let $M_0$ be a deterministic Turing machine recognizing a set of words $X$. Then there exists a deterministic Turing machine $M_1$ which recognizes $X$ and such that the following holds.

(a) If $W \equiv W'$ (that is, these two words are letter-for-letter equal) for a computation $W \rightarrow \cdots \rightarrow W'$, then this computation is empty.

(b) The property from Remark 2.1 (‘at most one tape letter’) holds for every command of $M_1$.

(c) The state letters from the start vector $\overline{s}_1$ (from the accept vector $\overline{s}_0$) occur on the left-hand side of a unique command $U \rightarrow V$ of $M_1$ and do not occur on the right-hand side of any command (respectively, occur on the right-hand side of a unique command $U' \rightarrow V'$ and do not occur on the left-hand side of any command).

(d) The letters used on different tapes are from disjoint alphabets. The letters to the left and to the right of the head of any tape are from disjoint alphabets.

Proof. Let $W$ be a configuration of $M_0$. The general form of a configuration of $M_1$ will be $W\tau \alpha q_{k+1} \omega$, that is, the machine $M_1$ has one more tape than $M_0$. The last tape contains a (non-negative) power of a special tape letter $\tau$. The set of state letters is then increased by one component \{$q_{k+1}$\}. At the beginning, the last tape is empty. The machine will execute $M_0$ on its tapes, adding new $\tau$ on the last tape after every step of computation. After $M_0$ accepts, $M_1$ erases the last tape and stops. The last tape guarantees Property (a). In order to obtain Property (b) of $M_1$, we apply the trick from Remark 2.1 since it does not violate Property (a).

Property (c) of $M_1$ for the start command follows from the same property of $M_0$, and we can define the accept command of $M_1$ so that Property (c) also holds for it.

In order to obtain Property (d), we use different copies of the tape alphabet for different tapes, and moreover, we use different copies from the left and from the right of each head. \(\Box\)

If $M_1$ is a deterministic Turing machine, satisfying the properties of Lemma 2.2, with the set of commands $\Theta$ such that $\Theta \cap \Theta^{-1} = \emptyset$, then let $\text{Sym}(M_1)$ be the Turing machine with the set of commands $\Theta \cup \Theta^{-1}$ and the same sets of state and tape letters. The division of the commands of $\text{Sym}(M_1)$ into positive and negative is natural: the commands of $M_1$ are positive, their inverses are negative. The computation of $\text{Sym}(M_1)$ is called reduced if its history is a reduced word. Clearly, every computation can be made reduced (without changing the start or end configurations of the computation) by removing consecutive mutually inverse commands.

Lemma 2.3. The Turing machine $\text{Sym}(M_1)$ satisfies the following properties.

(a) Every command of $\text{Sym}(M_1)$ satisfies Property (b) of Lemma 2.2.

(b) Every reduced history of computation of $\text{Sym}(M_1)$ has the form $H_1H_2^{-1}$, where $H_1$ and $H_2$ consist of positive commands.

(c) $\text{Sym}(M_1)$ satisfies Properties (a), (c) (for positive commands), and (d) of Lemma 2.2.

(d) The language recognized by $\text{Sym}(M_1)$ is $X$.

(e) For every $W \in X$, there exists only one accepting computation of $\text{Sym}(M_1)$. It is equal to the computation accepting $W$ by $M_1$, and if $M_1$ is given by Lemma 2.2, then the length of this computation is $\big-O$ of the length of the accepting computation of $M_0$ with input $W$.

Proof. Property (a) is obvious. Property (b) follows immediately from the fact that in a reduced computation, a command from $\Theta^{-1}$ cannot be followed by a command from $\Theta$ (since $M_1$ is deterministic). Properties (c)–(e) follow from (b). \(\Box\)
3. \textit{S-machines}

3.1. \textit{S-machines as rewriting systems}

There are several interpretations of \textit{S}-machines in groups, the most complicated so far is in [14]. Another interpretation is given in [15], and in fact it is probably the easiest way to view an \textit{S}-machine as a group that is a multiple HNN extension of a free group. Here, we use a definition which is close to the original one [19] and define \textit{S}-machines as rewriting systems, similar to Turing machines. \textit{S}-machines work with words in group alphabets and they are almost ‘blind’, that is, the heads do not observe the tape letters. But the heads can ‘see’ each other if there are no tape letters between them. We will use the following precise definition of an \textit{S}-machine \( S \).

A hardware of an \textit{S}-machine \( S \) is a pair \((Y, Q)\), where \( Q = \bigsqcup_{i=0}^{N} Q_{i} \) and \( Y = \bigsqcup_{i=1}^{N} Y_{i} \) (for convenience, we always set \( Y_{0} = Y_{N+1} = \emptyset \)). The elements from \( Q \) are called \textit{state letters}, the elements from \( Y \) are \textit{tape letters}. The sets \( Q_{i} \) and \( Y_{i} \) are called \textit{parts} of \( Q \) and \( Y \), respectively.

The \textit{language} of admissible words consists of reduced words of the form

\[ W \equiv q_{1}^{\pm 1}u_{1}q_{2}^{\pm 1} \cdots u_{k}q_{k+1}^{\pm 1}, \tag{3.1} \]

where every subword \( q_{i}^{\pm 1}u_{i}q_{i+1}^{\pm 1} \) either

1. belongs to \((Q_{j}, F(Y_{j}))q_{j+1}^{\pm 1}\) for some \( j \) and \( u_{i} \in F(Y_{j}) \), where \( F(Y_{j}) \) is the set of reduced group words in the alphabet \( Y_{j}^{\pm 1} \), or
2. has the form \( quq^{-1} \) for some \( q \in Q_{j} \) and \( u \in F(Y_{j+1}) \), or
3. is of the form \( q^{-1}uq \) for \( q \in Q_{j} \) and \( u \in F(Y_{j}) \).

We shall follow the tradition of calling state letters \( q \)-\textit{letters} and tape letters \( a \)-\textit{letters}, even though we shall use other letters as state or tape letters. Usually, parts of the set \( Q \) of state letters are denoted by capital letters. For example, a set \( K \) would consist of letters \( k \) with various indices. Then we shall say that letters in \( K \) are \( k \)-\textit{letters} or \( K \)-\textit{letters}.

If a group word \( W \) over \( Q \cup Y \) has the form \( q_{1}u_{1}q_{2}u_{2} \cdots q_{s} \), and \( q_{i} \in Q_{j}^{\pm 1} \), \( i = 1, \ldots, s, u_{i} \) are group words in \( Y \), then we shall say that the \textit{base} of \( W \) is \( \text{base}(W) \equiv Q_{j_{1}}^{\pm 1}Q_{j_{2}}^{\pm 1} \cdots Q_{j_{s}}^{\pm 1} \). Here, \( Q_{i} \) are just letters, denoting the parts of the set of state letters. Note that the base is not necessarily a reduced word, and the last equality is in the free semigroup. The subword of \( W \) between the \( Q_{j_{i}}^{\pm 1} \)-\textit{letter} and the \( Q_{j_{i+1}}^{\pm 1} \)-\textit{letter} will be called a \( Q_{j_{i}}^{\pm 1}Q_{j_{i+1}}^{\pm 1} \)-\textit{sector} of \( W \).

A word can certainly have many \( Q_{j_{i}}^{\pm 1}Q_{j_{i+1}}^{\pm 1} \)-\textit{sectors}.

For aesthetic reasons, we shall substitute the capital names of parts of \( Q \) by the corresponding small letters. For example, if \( i \in T, k \in K, \ldots \), we shall say that the base is \( tk \ldots \), that is, the state letters in \( W \) start with a \( t \)-\textit{letter}, followed by a \( k \)-\textit{letter}, and so on. Usually instead of specifying the names of the parts of \( Q \) and their order as in \( Q = Q_{0} \sqcup Q_{2} \sqcup \cdots \sqcup Q_{N} \), we say that the \textit{standard} base of the \( S \)-machine is \( Q_{0} \sqcup \cdots \sqcup Q_{N} \) or \( q_{0} \cdots q_{N} \).

The \textit{S}-machine also has a set of \textit{rules} \( \Theta \). Every \( \theta \in \Theta \) is assigned two sequences of reduced words: \([U_{0}, \ldots, U_{N}] \), \([V_{0}, \ldots, V_{N}] \), and a subset \( Y(\theta) = \bigcup Y_{i}(\theta) \) of \( Y \), where \( Y_{i}(\theta) \subseteq Y_{i} \).

The words \( U_{i} \) and \( V_{i} \) satisfy the following restriction.

\((**)\) For every \( i = 0, \ldots, N \), the words \( U_{i} \) and \( V_{i} \) have the form

\[ U_{i} \equiv v_{i}q_{i}u_{i+1}, \quad V_{i} \equiv v'_{i}q'_{i}u'_{i+1}, \]

where \( q_{i} \) and \( q'_{i} \) are state letters in \( Q_{i} \), \( u_{i+1} \) and \( u'_{i+1} \) are words in the alphabet \( Y_{i+1}(\theta)^{\pm 1} \), \( v_{i} \) and \( v'_{i} \) are words in the alphabet \( Y_{i}(\theta)^{\pm 1} \).

The pair of words \( U_{i} \) and \( V_{i} \) is called a \textit{part} of the rule, and is denoted \( U_{i} \rightarrow V_{i} \).
We shall need the following properties of our order to make \( \widetilde{Y} \) by \( \widetilde{q} \). Similarly, these words have no tape letters to the left of the state letters if the \( Q_{i-1}Q_{i} \)-sector is locked by the rule.

Every \( S \)-rule \( \theta = [U_1 \rightarrow V_1, \ldots, U_s \rightarrow V_s] \) has an inverse \( \theta^{-1} = [V_1 \rightarrow U_1, \ldots, V_s \rightarrow U_s] \) which is also a rule of \( S \); we set \( Y_i(\theta^{-1}) = Y_i(\theta) \). We always divide the set of rules \( \Theta \) of an \( S \)-machine into two disjoint parts, \( \Theta^+ \) and \( \Theta^- \) such that for every \( \theta \in \Theta^+ \), \( \theta^{-1} \in \Theta^- \) and for every \( \theta \in \Theta^- \), \( \theta^{-1} \in \Theta^+ \) (in particular, \( \Theta^{-1} = \Theta \), that is, any \( S \)-machine is symmetric). The rules from \( \Theta^+ \) and \( \Theta^- \) are called positive and negative, respectively.

To apply an \( S \)-rule \( \theta \) to an admissible word \( (3.1) \) means to check if all tape letters of \( W \) belong to the alphabet \( Y(\theta) \) and then, if \( W \) satisfies this condition, to replace simultaneously subwords \( U_i^{\pm 1} \) by subwords \( V_i^{\pm 1} \) \( (i = 1, \ldots, k + 1) \) and to trim a few first and last \( a \)-letters (to obtain an admissible word starting and ending with \( q \)-letters). This replacement is allowed to perform in the form \( q_i^{\pm 1} \rightarrow (v_{i-1}^{\pm 1}q_i^{\pm 1}u_i^{\pm 1})^{\pm 1} \) followed by the reducing of the resulting word. The following convention is important in the definition of \( S \)-machine: After every application of a rewriting rule, the word is automatically reduced. The reducing is not considered a separate step of an \( S \)-machine.

If a rule \( \theta \) is applicable to an admissible word \( W \) (that is, \( W \) belongs to the domain of \( \theta \)), then the word \( W \) is called \( \theta \)-admissible. The definitions of computation, its history, input admissible words, are similar to those of Turing machines. Similarly, we sometimes choose a distinguished stop word \( W_0 \) from the free group \( F(Q) \). It will always have the standard base (and no \( a \)-letters). We say that a word \( W \in F(Q \cup Y) \) is accepted if there exists a computation connecting this word and \( W_0 \).

3.2. Modifying the rules of \( S \)-machines

We shall need the following properties of our \( S \)-machines. All \( S \)-machines that appear in this paper, except for \( M_2 \), satisfy Property 3.1(1) below, and \( M_2 \) satisfies Property 3.1(2).

**Property 3.1.** (1) In every part \( v_iq_iu_{i+1} \rightarrow v_i'q_i'u_{i+1}', \) we have \( \|v_{i-1}\| + \|v_i'\| \leq 1 \) and \( \|u_i\| + \|u_i'\| \leq 1 \) (see the notation in (**)) above.

(2) For every rule, we have \( \sum_i (\|v_i\| + \|v_i'\| + \|u_i\| + \|u_i'\|) \leq 1 \).

Suppose that Property 3.1(1) is not satisfied. For example, suppose that a positive rule \( \theta \) of an \( S \)-machine \( S \) has the \( i \)th part of the form \( v_{i-1}aq_iu_i \rightarrow v_{i-1}'q_i'u_i' \), where \( u_{i-1}, v_i, u_i', \) and \( v_i' \) are words in the appropriate parts of the alphabet of \( a \)-letters, \( v_{i-1} \) is not empty, \( a \) is an \( a \)-letter, \( q_i \) and \( q_i' \) are \( q \)-letters (a very similar procedure can be done in all other cases). Then we create a new \( S \)-machine \( S' \) with the same standard base and the same \( a \)-letters as \( S \). In order to make \( S' \), we add one new (auxiliary) \( q \)-letter \( \tilde{q}_i \) to each part of the set of \( q \)-letters, and replace the rule \( \theta \) by two rules \( \theta' \) and \( \theta'' \). The rule \( \theta' \) is obtained from \( \theta \) by replacing the part \( v_{i-1}aq_iu_i \rightarrow v_{i-1}'q_i'u_i' \) by \( aq_iu_i \rightarrow \tilde{q}_iu_i' \), and all other parts \( U_j \rightarrow V_j \) by \( U_j \rightarrow \tilde{q}_j \) (here, \( \tilde{q}_j \) is the auxiliary \( q \)-letter in the corresponding part of the set of \( q \)-letters). The rule \( \theta'' \) is obtained from \( \theta \) by replacing the part \( v_{i-1}aq_iu_i \rightarrow v_{i-1}'q_i'u_i' \) by \( v_{i-1}u_i \rightarrow \tilde{q}_i \), and all other parts \( U_j \rightarrow V_j \) by \( \tilde{q}_j \rightarrow V_j \).

The key property of the new \( S \)-machine is in the following obvious lemma.
Lemma 3.2. There is a one-to-one correspondence between computations $w_0 \rightarrow \cdots \rightarrow w_n$ of $\mathcal{S}$ (with any base) such that $w_0$ and $w_n$ do not have auxiliary $q$-letters and computations of $\mathcal{S}$ connecting the same words. For every history $H$ of such computation of $\mathcal{S}$, the corresponding history of computation of $\tilde{\mathcal{S}}$ is obtained from $H$ by replacing every occurrence of the rule $\theta$ by the 2-letter word $\theta'\theta''$.

Note that the sum of lengths of words in all parts of $\theta'$ (respectively $\theta''$) in $\tilde{\mathcal{S}}$ is smaller than the similar sum for $\theta$. Clearly, applying this transformation to an $\mathcal{S}$-machine $\mathcal{S}$ several times, we obtain a new $\mathcal{S}$-machine satisfying Property 3.1(1). Similarly, one can obtain Property 3.1(2). Thus, Lemma 3.2 implies the following:

Lemma 3.3. For every $\mathcal{S}$-machine $\mathcal{S}$, there exists an $\mathcal{S}$-machine $\tilde{\mathcal{S}}$ with the same standard base, the same set of $a$-letters, and some new, auxiliary, $q$-letters such that $\mathcal{S}$ satisfies Property 3.1(2), and there exists a one-to-one correspondence between computations $w_0 \rightarrow \cdots \rightarrow w_n$ of $\tilde{\mathcal{S}}$ (with any base) such that $w_0$ and $w_n$ do not have auxiliary $q$-letters and computations of $\mathcal{S}$ connecting the same words. For every history $H$ of $\mathcal{S}$, the corresponding history of computation of $\tilde{\mathcal{S}}$ is obtained from $H$ by replacing every occurrence of the rule $\theta$ by the word $\phi(\theta)$ such that all rules in $\phi(\theta)$ are different, and $\phi(\theta)$ and $\phi(\theta')$ do not have common rules provided $\theta \neq \theta'$.

3.3. Some general properties of $\mathcal{S}$-machines

Note that the base of an admissible word is not always a reduced word. But, we have the following immediate corollary of the definition of admissible word.

Lemma 3.4. If the $i$th component of the rule $\theta$ has the form $q_i \rightarrow q_i'$, that is, $Y_{i+1}(\theta) = \emptyset$, then the base of any admissible for $\theta$ word cannot have subwords $Q_iQ_i^{-1}$ or $Q_i^{-1}Q_{i+1}$.

In this paper, we often use copies of words. If $W$ is a word and $A$ is an alphabet, then to obtain a copy of $W$ in the alphabet $A$, we substitute letters from $A$ for letters in $W$ so that different letters from $A$ substitute for different letters. Note that if $U'$ and $V'$ are copies of $U$ and $V$, respectively, corresponding to the same substitution, and $U' \equiv V'$, then $U \equiv V$.

The following lemma is obvious.

Lemma 3.5. Suppose that the base of an admissible word $W$ is $Q_iQ_{i+1}$. Suppose that each rule of a reduced computation starting with $W \equiv q_iu_{i+1}$ and ending with $W' \equiv q_i'u'q_{i+1}'$ multiplies the $Q_iQ_{i+1}$-sector by a letter on the left (respectively right). And suppose that different rules multiply that sector by different letters. Then the history of computation is a copy of the reduced form of the word $u'u^{-1}$ read from right to left (respectively of the word $u^{-1}u'$ read from left to right). In particular, if $u \equiv u'$, then the computation is empty.

Lemma 3.6. Let $W_0 \rightarrow \cdots \rightarrow W_t$ be a sequence of transformations of reduced words, where $W_i$ is a conjugate of $W_{i-1}$ ($i = 1, \ldots, t$) by a letter, and $H$ - a product of these letters, that is, $W_i = H^{-1}W_0H$. Then $H$ is equal to a reduced product $H_1H_2^kH_3$, where $k \geq 0$, $\|H_2\| \leq \min(\|W_0\|, \|W_t\|)$, $\|H_1\| \leq \|W_0\|/2$, and $\|H_3\| \leq \|W_t\|/2$.

Proof. One may assume that the consecutive transformations $W_{i-1} \rightarrow W_i$ are not mutually inverse. If $\|W_{i-1}\| < \|W_i\|$, then $\|W_i\| - \|W_{i-1}\| = 2$, and moreover,
∥W_i′∥ = ∥W_i∥, for every i′ ≥ i. Similar observation is true for inverse transformations W_1 → ⋯ → W_0. It follows that there exist subscripts i and j such that 0 ≤ i < j ≤ t, and each of the transformations W_0 → ⋯ → W_i decreases the length by 2, while ∥W_i∥ = ∥W_{i+1}∥ = ⋯ = ∥W_j∥, and each of the transformations W_j → ⋯ → W_1 increases the length by 2. Thus, we have W_i = (H^j_i)^{-1}W_0H^1_i, where ∥W_0∥ - ∥W_i∥ = 2∥H^1_i∥ and W_i = (H^k_i)^{-1}W_jH^3_i, where ∥W_i∥ - ∥W_j∥ = 2∥H^3_i∥.

We also have j - i one-letter cyclic shifts W_i → ⋯ → W_j, and this procedure is periodic with period ∥W_i∥, whence the middle part of the conjugating word H must be of the form H^k_i H, with k ≥ 0 and ∥H∥ < ∥H^k_i∥ = ∥W_i∥ ≤ min(∥W_0∥, ∥W_i∥), where H is a prefix of the word H^k_i. Replacing H^k_i by a cyclic permutation H^j_i one rewrites the same middle part as H' H^k_i H'', where ∥H^j_i∥ = ∥H^j_i∥ and ∥H'∥, ∥H''∥ ≤ 1/2(∥H∥ + 1) ≤ ∥W_i∥/2. Finally, we set H_1 = H^j_i H' and H_3 = H'' H^j_i, to obtain the required factorization of H with ∥H_1∥ ≤ 1/2(∥W_0∥ - ∥W_i∥) + ∥W_i∥/2 = ∥W_0∥/2 and also ∥H_3∥ ≤ ∥W_i∥/2.

Lemma 3.6 immediately implies:

**Lemma 3.7.** Suppose that the base of an admissible word W is Q_i Q_i^{-1} or Q_i^{-1} Q_i. Suppose that each rule θ of a reduced computation starting with W ≡ q_i u q_i^{-1} (respectively q_i^{-1} u q_i) where u ≠ 1, and ending with W' ≡ q_i' u q_i'^{-1} (respectively W' ≡ (q_i')^{-1} u q_i') has a component q_i → a q_i b q_i, where b_q (respectively a_b) is a letter, and for different θ-s the b_q-s (respectively a_b-s) are different. Then the history of the computation has the form H_1 H_2 H_3, where k ≥ 0, ∥H_2∥ ≤ min(∥u∥, ∥u'∥), ∥H_1∥ ≤ ∥u∥/2, and ∥H_3∥ ≤ ∥u'∥/2.

### 3.4. Turing machines as S-machines

Every symmetric Turing machine M satisfying Condition (d) of Lemma 2.2 can be viewed as an S-machine SM (see [19, p. 372]), such that the positive (negative) commands of M are the positive (negative) rules of SM. More precisely, we consider the α and ω symbols as state letters (hence the set of state letters has two more parts for each tape of the Turing machine).

A part of a command of the form u_q_i v → u_i' q_i' v_i', where u and v are tape letters or empty words, is replaced by

\[ \alpha_i \rightarrow \alpha_i, u_i q_i v_i \rightarrow u_i' q_i' v_i', \omega_i \rightarrow \omega_i \],

and a part of the form α_i q_i v_i → α_i q_i' v_i' is replaced by

\[ \alpha_i \rightarrow \alpha_i, q_i v_i \rightarrow q_i' v_i', \omega_i \rightarrow \omega_i \],

and a part of the form u_i q_i ω_i → u_i' q_i' ω_i is replaced by

\[ \alpha_i \rightarrow \alpha_i, u_i q_i \rightarrow u_i' q_i', \omega_i \rightarrow \omega_i \].

The language recognized by SM is in general much bigger than the language recognized by M since M works with a positive tape alphabet only. Nevertheless, the following property statement holds:

**Lemma 3.8** (compare with [19, Proposition 4.1]). Let M be a symmetric Turing machine satisfying the conditions of Lemma 2.3 (that is, the symmetrization of some deterministic Turing machine satisfying conditions of Lemma 2.2). Let W_1 → W_2 → ⋯ → W_k be a
computation of the $S$-machine $SM$ with the standard base consisting of positive words. Then it is a computation of the Turing machine $M$ (with the same history).

Proof. Indeed, every positive admissible word $W$ of $SM$ with the standard base is a configuration of the Turing machine $M$. If a rule $\theta$ of $SM$ satisfying Property (c) of Lemma 2.2 or its inverse applies to this $W$ and the word $W \cdot \theta$ is positive, then, obviously, the command $\theta$ of $M$ applies to $W$ and the result of the application is the same (here, we essentially use the fact that the rule $\theta$ or its inverse inserts (deletes) at most one letter). This immediately implies the statement of the lemma.

4. The $S$-machine

We turn to the proof of Theorem 1.1. From now, $M_0$ is the deterministic Turing machine recognizing language $X$ from Theorem A.1, the machine $M_1$ is constructed as in Lemma 2.2 and recognizes the language $X_1$, where $X_1 = X$. We keep the same notation $M_1$ for the symmetrization of $M_1$ given by Lemma 2.3. Note that by claim (e) of that lemma, the machine $M_1$ has infinitely many $h_\alpha$-good numbers for every $\alpha > 0$, where the functions $h_\alpha$ are defined in Theorem A.1. First, we need to construct a new $S$-machine which inherits important properties of the Turing machine $M_1$.

As in Section 3.3, we can view $M_1$ as an $S$-machine. We shall denote that $S$-machine by the same letter $M_1$.

Let $Q_0 \ldots Q_N$ be the standard base of $M_1$, let the components of the alphabet of $a$-letters be $Y_1, \ldots, Y_N$ (letters from $Y_i$ are in the $Q_{i-1}Q_i$-sectors of admissible words with the standard base).

4.1. The machine $M_1 \circ Z$

Let $A$ be a finite set of letters. Let the sets $A_1$ and $A_2$ be copies of $A$. It will be convenient to denote $A$ by $A_0$. For every letter $a_0 \in A_0$, let $a_1$ and $a_2$ denote its copies in $A_1$ and $A_2$.

As in [15], consider the following auxiliary ‘adding’ $S$-machine $Z(A)$.

Its set of state letters is $P_1 \cup P_2 \cup P_3$, where

$$P_1 = \{L\}, \quad P_2 = \{p(1), p(2), p(3)\}, \quad P_3 = \{R\}.$$ 

The set of tape letters is $Y_1 \cup Y_2$, where $Y_1 = A_0 \cup A_1$ and $Y_2 = A_2$.

The machine $Z(A)$ has the following positive rules (there $a$ is an arbitrary letter from $A$). The comments explain the meanings of these rules.

(i) $r_1(a) = [L \rightarrow L, p(1) \rightarrow a_1^{-1}p(1)a_2, R \rightarrow R]$.

Comment. The state letter $p(1)$ moves left searching for a letter from $A_0$ and replacing letters from $A_1$ by their copies in $A_2$.

(ii) $r_{12}(a) = [L \rightarrow L, p(1) \rightarrow a_0^{-1}a_1p(2), R \rightarrow R]$.

Comment. When the first letter $a_0$ of $A_0$ is found, it is replaced by $a_1$, and $p(1)$ turns into $p(2)$.

(iii) $r_2(a) = [L \rightarrow L, p(2) \rightarrow a_0p(2)a_2^{-1}, R \rightarrow R]$.

Comment. The state letter $p(2)$ moves toward $R$.

(iv) $r_{21} = [L \rightarrow L, p(2) \rightarrow p(1), R \rightarrow R], Y_1(r_{21}) = Y_1, Y_2(r_{21}) = \emptyset$.

Comment. $p(2)$ and $R$ meet, the cycle starts again.

(v) $r_{13} = [L \rightarrow L, p(1) \rightarrow p(3), R \rightarrow R], Y_1(r_{13}) = \emptyset, Y_2(r_{13}) = A_2$.

Comment. If $p(1)$ never finds a letter from $A_0$, the cycle ends, $p(1)$ turns into $p(3)$; $p$ and $L$ must stay next to each other in order for this rule to be executable.

(vi) $r_3(a) = [L \rightarrow L, p(3) \rightarrow a_0p(3)a_2^{-1}, R \rightarrow R], Y_1(r_3(a)) = A_0, Y_2(r_3(a)) = A_2$. 


Comment. The letter $p(3)$ returns to $R$.

For every letter $a \in A$, we set $r_i(a^{-1}) = r_i(a)^{-1}$ ($i = 1, 2, 3$).

The following lemmas from [15] contain the main properties of $Z(A)$ used later.

If $u \equiv a_1 \ldots a_m$ is a word, $a_i$ are letters, then we set $r_3(u) \equiv r_3(a_1)r_3(a_2) \ldots r_3(a_m)$, $r_2(u) \equiv r_2(a_1)r_2(a_2) \ldots r_2(a_m)$, $r_1(u) \equiv r_1(a_m) \ldots r_1(a_2)r_1(a_1)$.

Lemma 4.1 [15, Lemma 3.18]. Suppose that an admissible word $W$ of $Z(A)$ has the form $Lup\theta R$, where $u$ and $v$ are words in $(A_0 \cup A_1 \cup A_2)^{\pm 1}$. Let $W \cdot \theta \equiv Lu'p'v'R$. Then the projections of $uv$ and $u'v'$ onto $A$ are freely equal.

Lemma 4.2 Follows from the proof of [15, Lemma 3.25]. Let $W$ be an admissible word of $Z(A)$ with $\text{base}(W) \equiv LpR$. Then for every reduced computation $W \equiv W_0 \rightarrow W_1 \rightarrow \cdots \rightarrow W_t \equiv W \cdot H$ of the $S$-machine $Z(A)$, we have the following.

1. The inequality $||W_i|| \leq \max(||W_0||, ||W \cdot H||)$ holds for every $i = 0, \ldots, t$.

2. If $W \equiv LupR$, where $p = p(1)$ (respectively $p = p(3)$), $u$ is positive, then there exists a computation starting with $W$ and ending with $Lup(3)R$ (respectively $Lup(1)R$). Moreover if $u$ is any word in $a$-letters, and for some history of computation $H$, $W \cdot H$ contains $p(3)R$ (respectively $p(1)R$) and all $a$-letters in $W, W \cdot H$ are from $A_0^\pm 1$, then the length of $H$ is between $2||u||$ and $6 \cdot 2||u||$, $u$ and all words in the computation $W \rightarrow \cdots \rightarrow W \cdot H$ are positive, all words in that computation have the same length, and $H$ is uniquely determined by $u$. That computation (respectively its inverse) has the history of the following form:

$$D(u) \equiv E(u)r_{13}r_3(u),$$

where $E(u)$ is defined by induction: $E(\emptyset) \equiv \emptyset$ and if $u \equiv a_1u'$, then

$$E(u) \equiv E(u')r_{12}(a_1)r_2(u')E(u')r_1(a_1).$$

Lemma 4.3 The first part of the lemma is [15, Lemma 3.21]. For every admissible word $W$ of $Z(A)$ with $\text{base}(W) \equiv LpR$, every rule $\theta$ applicable to $W$, and every natural number $t > 1$, there is at most one reduced computation $W \rightarrow_\theta W_1 \rightarrow \cdots \rightarrow W_t$ of length $t$, where the lengths of the words are all the same. (In fact from the proof of [15, Lemma 3.21], it immediately follows that the history of that computation is a subword of $D(u)^{\pm 1}$ for some $u$).

Lemma 4.4 [15, Lemma 3.27]. Suppose $W$ is an admissible word of $Z(A)$ with $\text{base}(W) \equiv LpR$. Suppose that $W \cdot H$ exists for some reduced history $H$. Suppose that both $W$ and $W \cdot H$ contain $p(1)R$ (respectively $p(3)R$) and all $a$-letters in $W, W \cdot H$ are from $A_0$. Then $H$ is empty.

Lemma 4.5 [15, Lemma 3.24]. Let $W \equiv LpuvR$ and $\text{base}(W) \equiv LpR$. Suppose that $||W \cdot \theta|| > ||W||$. Then for every reduced computation $W \rightarrow_\theta W_1 \rightarrow W_2 \rightarrow \cdots \rightarrow W \cdot H$, we have $||W_i|| > ||W||$ for every $i \geq 1$.

We define the $S$-machine $M_2$ as the composition $M_1 \circ Z$ (this operation is defined in [15], see also below).

Essentially, we insert a $p$-letter between any two consecutive $q$-letters in admissible words of $M_1$ with standard base, and treat any subword $q_i \ldots q_{i+1} \ldots q_{i+1}$ as an admissible word for $Z(A)$ (that is $q_i$ plays the role of $L$ and $q_{i+1}$ plays the role of $R$). The only differences with the construction in [15] are that, for every state letter $q$, we keep the sets of $a$-letters that can
appear to the left and to the right of \( q \) disjoint, and after application of a main rule, not only the state letters of copies of \( Z(\mathcal{A}) \) remember the main rule but also the state letters coming from \( M_1 \) remember that rule. These changes do not affect the proofs of statements in [15] that we are going to use.

Let us describe \( M_2 \) in details. First, for every \( i = 1, \ldots, N \), we make three copies of the alphabet \( Y_i \) of \( M_1 \) \((i = 1, \ldots, N)\): \( Y_{i,0} = Y_i \), \( Y_{i,1} \), \( Y_{i,2} \). Let \( \Theta \) be the set of positive commands of \( M_1 \) (viewed as rules of an \( S \)-machine). The set of state letters of the new machine is

\[
Q'_i \cup P_1 \cup Q'_1 \cup P_2 \cup \cdots \cup P_N \cup Q'_N,
\]

where \( P_i = \{ p^{(1)}, p^{(1,i)}, p^{(0,i)}, p_1^{(\theta,i)}, p_2^{(\theta,i)}, p_3^{(\theta,i)} \mid \theta \in \Theta \} \), \( i = 1, \ldots, N \), \( Q'_i = Q_i \cup (Q_i \times \Theta) \), where \( Q_0 \cup Q_1 \cup \cdots Q_N \) is the set of state letters of \( M_1 \). We shall denote a pair \((q, \theta)\) from \( Q_i \times \Theta \) by \( q^{(\theta,i)} \). Thus, every ‘old’ state letter of \( M_1 \) obtains ‘multiple’ copies indexed by positive rules of \( M_1 \), and the state letters of various copies of \( Z \) have upper indices corresponding to the positive rules of \( M_1 \) and the number of sector where the machine is inserted. The \( Q'_i P_1 \)-sector of an admissible word with the standard base will be called the input sector of that word.

The set of tape letters is

\[
\bar{Y} = (Y_{1,0} \sqcup Y_{1,1}) \sqcup Y_{1,2} \sqcup (Y_{2,0} \sqcup Y_{2,1}) \sqcup Y_{2,2} \sqcup \cdots \sqcup (Y_{N,0} \sqcup Y_{N,1}) \sqcup Y_{N,2};
\]

the components of this union will be denoted by \( \bar{Y}_1, \ldots, \bar{Y}_{2N} \).

The set of positive rules \( \Theta \) of \( M_1 \circ Z \) is a union of the set of suitably modified positive rules of \( M_1 \) and \(|\Theta|N \) copies \( Z(\theta,i) \) \((\theta \in \Theta, i = 1, \ldots, N)\) of positive rules of the machine \( Z(Y_i) \) (also suitably modified).

More precisely, every rule \( \theta \in \Theta \) of the form

\[
[q_0 u_1 \rightarrow q'_0 u'_1, v_1 q_1 u_2 \rightarrow v'_1 q'_1 u'_2, \ldots, v_N q_N \rightarrow v'_N q'_N],
\]

where \( q_i, q'_i \in Q_i \), \( u_i \) and \( v_i \) are words in \( Y_i \), is replaced by

\[
\bar{\theta} = \begin{bmatrix}
[q_0 u_1 \rightarrow (q')^{(0,i)} u'_1, v_1 p^{(1)} \rightarrow v'_1 p_1^{(\theta,i)}, q_1 u_2 \rightarrow (q')^{(1,i)} u'_2, \ldots, \\
v_N p^{(N)} \rightarrow v'_N p_1^{(\theta,N)}, q_N \rightarrow (q')^{(\theta,N)}
\end{bmatrix}
\]

with \( \bar{Y}_{2i-1}(\bar{\theta}) = Y_{i,0}(\theta) \) and \( \bar{Y}_{2i}(\bar{\theta}) = \emptyset \). If \( \theta = \theta_{\text{start}} \) is the unique start rule of \( M_1 \), then all letters \( p^{(1,i)} \)-s in the above definition of \( \bar{\theta} \) must be replaced by the special start letters \( p^{(1,i)} \)-s.

Thus, each modified rule from \( \Theta \) turns on \( N \) copies of the machine \( Z(\mathcal{A}) \) (for different letters \( A \)'s). The rule \( \bar{\theta} \) will be called the rule of \( M_2 \) corresponding to the rule \( \theta \) of \( M_1 \).

Each machine \( Z(\theta,i) \) is a copy of the machine \( Z(Y_i(\theta)) \), where every rule \( \tau = [U_1 \rightarrow V_1, U_2 \rightarrow V_2, U_3 \rightarrow V_3] \) is replaced by the rule of the form

\[
\bar{\tau}_i(\theta) = \begin{bmatrix}
U_1 \rightarrow V_1, U_2 \rightarrow V_2, U_3 \rightarrow V_3, \\
(q')^{(0,j)} \rightarrow (q')^{(0,j)}, p_3^{(j)} \rightarrow p_3^{(j)}, j = 1, \ldots, i - 1, \\
(p_1^{(\theta,s)} \rightarrow P_1^{(\theta,s)}, (q')^{(\theta,s+1)} \rightarrow (q')^{(\theta,s+1)}, \quad s = i + 1, \ldots, N - 1
\end{bmatrix}
\]

where \( U_1, U_2, U_3, V_1, V_2 \) and \( V_3 \) are obtained from \( U_1, U_2, U_3, V_1, V_2 \) and \( V_3 \), respectively, by replacing \( p(j) \) with \( p_1^{(\theta,s)} \), \( \mathcal{L} \) with \( (q')^{(\theta,i)} \), and \( \mathcal{R} \) with \( (q')^{(\theta,i+1)} \), and for \( s \neq i, \bar{Y}_{s-1}(\bar{\tau}_i(\theta)) = Y_{s,0}(\theta) \). Thus, while the machine \( Z^{(\theta,i)} \) works all other machines \( Z^{(\theta,j)} \), \( j \neq i \) must stay idle (their state letters do not change and do not move away from the corresponding \( q \)-letters). After the machine \( Z^{(\theta,i)} \) finishes (that is, the state letter \( p_3^{(\theta,i)} \) appears next to \( (q')^{(\theta,i)} \)), the next machine \( Z^{(\theta,i+1)} \) starts working.
In addition, we need the following transition rule \( \zeta(\theta) \) that removes \( \theta \) from all state letters, and turns all \( p_3^{(\theta,j)} \) back into \( p^{(j)} \):
\[
[q_i^{(\theta,i)}] \rightarrow q_i^{(\theta,i)} p_3^{(\theta,j)} \overset{t}{\rightarrow} p^{(j)}, \quad i = 0, \ldots, N, j = 1, \ldots, N.
\]

If \( \theta \) is the unique accept rule of \( M_1 \), then all \( p^{(j)} \)-s in the above definition of \( \zeta(\theta) \) must be replaced by the special (accept) letters \( p^{(0,i)} \)-s.

**Lemma 4.6.** Let \( H \) be the history of a reduced computation \( W \rightarrow \cdots \rightarrow W \cdot H \) of \( M_2 \) with the standard base, of the form \( \delta H' \zeta(\theta) \), where \( \theta \) is a positive rule of the \( S \)-machine \( M_1 \), \( H' \) does not contain rules corresponding to rules of \( M_1 \) and occurrences of \( \zeta(\theta) \)\( ^{\pm 1} \). Let \( W \cdot \theta = q^{(0,0)} u_1 p^{(0,1)} q^{(0,1)} u_2 \ldots u_N p^{(0,N)} q^{(0,N)} \) for some words \( u_i \) in \( \Lambda_i,0 \). Then \( H' \equiv H_1 H_2 \ldots H_N \), where each \( H_s \) is the computation of the \( M \)-machine \( Z(\theta,s) \) whose history is a copy of \( D(u_s) \) (described in Lemma 4.2, part 2), and all words in the computation \( W \cdot \theta \rightarrow \cdots \rightarrow W \cdot H \) are positive. We shall denote \( H \) by \( \Pi_{1,2}(\theta,W) \). That computation is completely determined by its first (last) word and \( \theta \).

**Proof.** By assumption, \( H' \) consists of the rules of various \( S \)-machines \( Z(\theta,i) \) since only rules \( \bar{\delta}^{-1} \) and \( \zeta(\theta) \) can remove the \( \theta \)-index of state letters. Since the word \( W' = W \cdot (\bar{\delta} H') \) is in the domain of \( \zeta(\theta) \), all \( p \)-letters in \( W' \) have the form \( p_3^{(\theta,i)} \). Since rules from \( Z(\theta,i) \) can apply to an admissible word with the standard base only if \( p_j \)-letters (\( j \neq i \)) stay next to the left of the corresponding (copies of) state letters of \( M_1 \), and have the form \( p_3^{(\theta,j)} \), if \( j < i \), and the form \( p_3^{(\theta,j)} \) if \( j > i \), we can conclude that \( H' \equiv H_1 H_2 \ldots H_m \), where each \( H_s \) is a non-empty history of the computation of some \( Z(\theta,j(s)) \) such that \( j(s+1) = j(s) \pm 1, j(1) = 1 \), and each \( m \) between 1 and \( N \) occurs as \( j(s) \) for some \( s \). Note that if \( j(s+1) = j(s) - 1 \), then there must be \( s' > 1 \) such that \( j(s'-1) = j(s'-1) = j(s') - 1 \). But then the subcomputation of the \( S \)-machine \( Z(\theta,j(s)) \) with history \( H_s \) starts and ends with the \( p \)-letter of the form \( p_3^{(\theta,j(s))} \). By Lemma 4.4, then \( H_s \) is empty, a contradiction. Hence, \( j(s+1) = j(s) + 1 \) for every \( s \), which implies that \( m = N \), and \( j(s) = s \) for every \( s \). By Lemma 4.2, part 2, each \( H_s \) is uniquely determined by the word \( u_s \) (and \( \theta \)) and is equal to a copy of \( D(u_s) \) defined in Lemma 4.2, part 2. This implies the uniqueness of \( H' \). The fact that all words in the computation \( W \cdot \theta \rightarrow \cdots \rightarrow W \cdot H \) are positive follows from Lemma 4.2, part 2.

**Lemma 4.7.** Let \( H \) be the history of a reduced computation \( W \rightarrow \cdots \rightarrow W \cdot H \) of \( M_2 \) with the standard base, \( H'(\theta_1) \equiv H'(\theta_2) \equiv H' \equiv H_1 H_2 \ldots H_m \), where \( \theta_1 \) and \( \theta_2 \) are positive rules of the \( S \)-machine \( M_1 \), \( H' \) does not contain rules corresponding to rules of \( M_1 \) and occurrences of \( \zeta(\theta) \)\( ^{\pm 1} \), \( \epsilon_1, \epsilon_2 = \pm 1 \), \( i = 1 \) and \( \epsilon_2 = -1 \).

**Proof.** At first, let us prove that \( H' \) is empty. If \( \epsilon_1 = 1 \), then the \( p \)-letters in \( W \cdot \zeta(\theta_1) \) have the form \( p^{(j)} \), and no rules from any \( Z(\theta,s) \) apply to words with such \( p \)-letters, hence \( H' \) is empty (because by assumption it can contain only rules of various \( Z(\theta,j) \)). Thus, we can assume that \( \epsilon_1 = -1 \). Similarly \( \epsilon_2 = 1 \). As in the proof of Lemma 4.6, \( H' \equiv H_1 H_2 \ldots H_m \), where each \( H_i \) is a non-empty history of computation of some \( Z(\theta,j(s)) \) such that \( j(s+1) = j(s) \pm 1, j(1) = N \). Note that the \( p \)-letters in \( W \cdot \zeta(\theta_1)^{-1} \) and in \( W \cdot \zeta(\theta_1)^{-1} H' \) are of the form \( p_3^{(\theta,j)} \). Therefore, for some \( s \), we must have \( j(s+1) = j(s) \pm 1 \). As in the proof of Lemma 4.6, this implies that \( H_s \) is empty, a contradiction.

Since \( H' \) is empty, the \( p \)-letters in \( W \cdot \zeta(\theta_1) \) have no \( \theta \)-indices because otherwise they have to be equal to both \( \theta_1 \) and \( \theta_2 \), and \( H \equiv \zeta(\theta_1)^{-1} \zeta(\theta_1) \) would not be reduced. Therefore, \( \epsilon_1 = 1 \) and \( \epsilon_2 = -1 \).
Lemma 4.8. Let $H$ be the history of a reduced computation $W \to \cdots \to W \cdot H$ of $M_2$ with the standard base, $H \equiv \bar{\theta}_1^{-1} H' \bar{\theta}_2^{-1}$, where $\theta_1$ and $\theta_2$ are positive rules of the $S$-machine $M_1$, $H'$ does not contain rules corresponding to the rules of $M_1$. Then $H'$ is empty.

Proof. Indeed, if $H'$ is not empty, it must start with $\zeta(\theta_3)^{-1}$, and end with $\zeta(\theta_4)^{-1}$, for some positive $\theta_3, \theta_4$, and then $H'$ would have a subword of the form $\zeta(\theta')^{-1} H'' \zeta(\theta'')$ satisfying the assumptions of Lemma 4.7, which contradicts the statement of Lemma 4.7.

Lemma 4.9. Let $H$ be the history of a reduced non-empty computation $W \to \cdots \to W \cdot H$ of $M_2$ with the standard base, $H \equiv \bar{\theta}_1^{-1} H' \bar{\theta}_2^{-1}$, where $\theta_1$ and $\theta_2$ are positive rules of the $S$-machine $M_1$, and let $H'$ contain no rules corresponding to rules of $M_1$. Then $H \equiv \Pi_{1,2}(\theta_1, W) \Pi_{1,2}(\theta_2, W \cdot H)^{-1}$, all words in that computation except possibly the first and the last ones are positive, and $\theta_1 \neq \theta_2$.

Proof. If $H$ does not contain the rule $\zeta(\theta_1)$, we obtain that $H'$ is empty as in the proof of Lemma 4.7. Note that the $p$-letters of the admissible words in the domain of $\bar{\theta}_1^{-1}$ (of $\bar{\theta}_2^{-1}$) have the form $p_{1}^{(\theta_1,i)}$ (respectively, $p_{1}^{(\theta_2,i)}$). It follows that $\theta_1 \equiv \theta_2$, and so the history $H$ is not reduced, a contradiction.

Hence, we can assume that $H$ contains $\zeta(\theta_1)$. Then the next rule in $H$ must be $\zeta(\theta_3)^{-1}$ for some $\theta_3$ because only rules of the form $\bar{\theta}$ and $\zeta(\theta)^{-1}$ for positive $\theta$ are applicable to admissible words in the range of $\zeta(\theta_1)$ and $H'$ does not contain rules corresponding to rules of $M_1$. After application of $\zeta(\theta_3)^{-1}$, all $p$-letters in the admissible word have the form $p_{2}^{(\theta_3,i)}$. Recall that the $p$-letters of the admissible words in the domain of $\bar{\theta}_2^{-1}$ have the form $p_{1}^{(\theta_2,i)}$. Thus, if $\theta_3 \neq \theta_2$, the word $H'$ must contain a subword $\zeta(\theta_3)^{-1} H'' \zeta(\theta_3)$, where $H''$ does not contain rules of the form $\zeta(\theta)$ and their inverses. By Lemma 4.7, $H''$ is empty, and the computation is not reduced, a contradiction. Hence, $H'$ has the form $\zeta(\theta_3) \zeta(\theta_2)^{-1} H''$, where $H''$ and $H''$ do not contain rules of the form $\bar{\theta}^\pm 1$ and rules of the form $\zeta(\theta)^\pm 1$ by Lemma 4.7. Applying now Lemma 4.6 to the computation with history $\bar{\theta}_1 H'' \zeta(\theta_1)$ and the first word $W$, and to the computation with history $\bar{\theta}_2 (H'')^{-1} \zeta(\theta_2)$ and the first word $W \cdot H$, we obtain the desired equality $H \equiv \Pi_{1,2}(\theta_1, W) \Pi_{1,2}(\theta_2, W \cdot H)^{-1}$ and the fact that all words in that computation are positive except possibly the first and the last words. Now if $\theta_1 = \theta_2$, then $H'$ must be empty (since the computation is reduced and $H'$ is a product of two mutually inverse words in that case by the uniqueness statement of Lemma 4.6), and so the computation is empty, a contradiction.

For every admissible word $W$ of $M_2$ with the standard base, let $\pi_{2,1}(W)$ be the word obtained by removing state $p$-letters, removing $\theta$-indices (if any exist) of other state letters, and removing the $M_2$-indices of $a$-letters, and reducing the resulting word. We obtain a word in the alphabet of state and tape letters of $M_1$.

With every admissible word $W$ of the $S$-machine $M_1$ with the standard base, we associate the admissible word $\pi_{1,2}(W)$ of $M_2$ by inserting the state letters $p^{(1)}$-s next to the left of $q_{ij}$-s, and replacing every $a$-letter $a$ by $a_0$. (We insert the special letters $p^{(1,0)}$-s instead of $p^{(ij)}$-s if the word $W$ is admissible for the unique start command of $M_1$.) Let $W_0$ be the stop word of $M_1$ (considered as an $S$-machine). It exists because the Turing machine $M_1$ is recognizing. We call the word $\pi_{1,2}(W_0)$ the stop word of $M_2$.

Note that we have

$$\pi_{2,1}(\pi_{1,2}(W)) \equiv W. \quad (4.1)$$
For every input configuration $W$ of the $S$-machine $M_1$, we call $\pi_{1,2}(W)$ an input word of $M_2$. Note that an input word of $M_2$ has the standard base and all sectors except the $q_0p_1$-sector are empty.

For every rule $\theta'$ of $M_2$, if $(\theta')^{\pm 1}$ corresponds to a positive rule $\theta$ (that is, if $\theta' \equiv \bar{\theta}^{\pm 1}$) of $M_1$, we denote $\theta^{\pm 1} = \Pi_{2,1}(\theta')$. If $(\theta')^{\pm 1}$ does not correspond to a rule of $M_1$, we denote by $\Pi_{2,1}(\theta')$ the empty rule. The map $\Pi_{2,1}$ extends to histories of computations in the natural way.

**Lemma 4.10.** If $H$ is the reduced history of a computation of $M_2$ with the standard base and $W \cdot H = W'$, then $\Pi_{2,1}(H)$ is a reduced history of computation of the $S$-machine $M_1$. If $\pi_{2,1}(W)$ is an admissible word for the $S$-machine $M_1$, then

$$\pi_{2,1}(W) \cdot \Pi_{2,1}(H) = \pi_{2,1}(W'). \quad (4.2)$$

**Proof.** The fact that $\pi_{2,1}(H)$ is a history of computation of the $S$-machine $M_1$, and (4.2), immediately follows from the definition of $M_2$ and the fact that application of rules from $Z^{(\theta, i)}$ does not change the value of $\pi_{2,1}$ by Lemma 4.1. The fact that $\pi_{2,1}(H)$ is reduced follows from Lemmas 4.8 and 4.9. \qed

The next lemma-definition gives in a sense an inverse function of $\Pi_{2,1}$.

**Lemma 4.11.** For every positive $\bar{\theta}$-admissible word $W$ of $M_2$ with the standard base such that there exists a computation $w \to w_1 \to \cdots \to w \cdot H$ of the Turing machine $M_1$ with positive history $H$, starting with $w \equiv \pi_{2,1}(W)$ and having the first rule $\theta$, and all admissible words positive, there exists a unique reduced computation of $M_2$ starting with $W \to W \cdot \bar{\theta}$, whose history is $H'$ such that $\Pi_{2,1}(H') \equiv H$ and the last rule is of the form $\zeta(\theta)$. That history $H'$ will be denoted by $\Pi_{1,2}(H, W)$. This definition agrees with the notation $\Pi_{1,2}(\theta, W)$ of Lemma 4.6.

**Proof.** Indeed, if $H \equiv \theta \theta_1 \ldots \theta_n$, where all $\theta_i$ are positive rules of $M_1$, then we can define $\Pi_{1,2}(H, w)$ as $\Pi_{1,2}(\theta, W)\Pi_{1,2}(\theta_1, W_1) \ldots \Pi_{1,2}(\theta_n, W_n)$, where $W_i = \pi_{1,2}(w_i)$ ($i = 1, \ldots, n$).

For the uniqueness of $H'$, we note that every rule of the form $\bar{\theta}_i$ can follow only after a rule of the form $\zeta(*)$ since $H$ is positive. It follows from Lemma 4.7 that there is only one $\zeta(*)^{\pm 1}$-rule between $\bar{\theta}_i$ and a preceding rule of this form. Now the uniqueness of $H'$ follows from Lemma 4.6. \qed

Every time we use the notation $\Pi_{1,2}(H, W)$ below, the conditions of Lemma 4.11 will be assumed or clearly satisfied.

**Remark 4.12.** Note that if $\pi_{2,1}(W) \cdot H = W_0$, that is, the computation of the Turing machine $M_1$ is accepting, then the corresponding computation of $M_2$ with history $\Pi_{1,2}(H, W)$ is also accepting.

**Lemma 4.13.** There are no reduced computations $W \to W \cdot \bar{\theta}^{-1} \to W \cdot \bar{\theta}^{-1}\bar{\theta}'$ with the standard base, where the first and the third words are positive and $\theta$ and $\theta'$ are positive rules of $M_1$.

**Proof.** Assume that such a computation exists and $W \cdot \bar{\theta}^{-1}$ is positive too. Then by Lemma 3.8, this computation is a reduced computation of the symmetric Turing machine $M_1$ with history $\theta^{-1}\theta'$, contrary to the Property (b) of $M_1$ given by Lemma 2.3.
Now assume that the word $W \cdot \overline{\theta}^{-1}$ is not positive. By Property (b) from Lemma 2.2, each $\theta, \theta'$ inserts/deletes at most one tape letter. The only non-trivial case is when $\theta$ and $\theta'$ insert an $a$-letter: in other cases, the second word in the computation is obviously positive. But then $\overline{\theta}^{-1}$ must insert a letter $a^{-1}$ which is then removed by $\overline{\theta}'$ (in the same sector). Since both rules $\overline{\theta}$ and $\overline{\theta}'$ have word $W \cdot \overline{\theta}^{-1}$ in their domains, the left-hand sides in all parts of the rules of $\overline{\theta}$ and $\overline{\theta}'$ coincide. Since $M_1$ is the symmetrization of a deterministic Turing machine by construction, $\theta \equiv \theta'$ and our computation is not reduced, a contradiction.

LEMMA 4.14. Let $H$ be the history of a reduced computation $W \rightarrow \cdots \rightarrow W \cdot H$ of $M_2$ with the standard base.

(1) If $H \equiv \theta_1 H_1 \zeta' \zeta (\theta_2)$, where $\theta_1$ and $\theta_2$ are positive rules of the $S$-machine $M_1$. Then the word $H$ and all words in that computation except possibly the first one are positive.

(2) If $H \equiv \overline{\theta}_1 H_1 \overline{\theta}_2' \ldots \overline{\theta}_n H_n \overline{\theta}_{n+1}$, where $\theta_1, \ldots, \theta_{n+1}$ are positive rules of the $S$-machine $M_1$, $\epsilon_i = \pm 1$, $(\epsilon_n, \epsilon_{n+1}) \neq (-1, 1)$, and $H_1, \ldots, H_n$ have no rules corresponding to the rules of $M_1$, then all words in this computation except possibly the first one and the last one are positive.

Proof. (1) Induction on the length of $\Pi_1(H)$. Suppose $\Pi_1(H) \equiv \theta_1$. Then by Lemma 4.7, $H'$ does not contain rules of the form $\zeta(\theta) \pm 1$. Hence, we can apply Lemma 4.6 and conclude that all words in the computation except possibly the first one are positive.

Suppose that the length of $\Pi_1(H)$ is at least 2. Suppose further that the second letter of $\Pi_1(H)$ is positive, that is, $H \equiv \overline{\theta}_1 H_1 \overline{\theta}_3 H_2$ for some positive $\theta_3$ and $H_1$ not containing rules corresponding to the positive rules of $M_1$. Then $H_1$ must end with $\zeta(\theta')$ for some $\theta'$. Then we can apply the induction assumption to the computations with histories $\overline{\theta}_1 H_1$ and $\overline{\theta}_3 H_2$ and conclude that $H$ and all words, except for the first one, in the computation with history $H$ are positive as desired.

Now suppose that the second letter in $\Pi_1(H)$ is $\theta_3^{-1}$ for some positive $\theta_3$. Since $H$ ends with $\zeta(\theta_2)$, the last rule in $\Pi_1(H)$ is positive. Indeed the rule used in any reduced computation of $M_2$ immediately after a rule of the form $\overline{\theta}^{-1}$ for some positive $\theta$ is either $\zeta(\theta')^{-1}$ for some positive $\theta'$ or $\overline{\theta'}$ for some positive $\theta'$ (this can be seen by looking at the indices of $q$-letters of the admissible words). The first option is impossible by Lemma 4.7, the second option is impossible since we consider the last rule in $\Pi_1(H)$. Hence, $H \equiv \theta_1 H_1 \overline{\theta}_2^{-1} H_2 \ldots H_{m-1} \overline{\theta}_m^{-1} H_m \overline{\theta}_{m+1} H''$ for some positive rules $\theta_2, \ldots, \theta_{m+1}$, where $H_2, \ldots, H_m$ do not contain rules corresponding to rules of $M_1$ or their inverses, $\theta_{m+1}$ is the second positive rule in $\Pi_1(H)$. By Lemma 4.8, $H_m$ is empty. By Lemma 4.9, $\overline{\theta}_1 H_1 \overline{\theta}_2^{-1} \equiv \Pi_1(\theta_2, W)\Pi_1(\theta_3, W \cdot \overline{\theta}_1 H_1 \overline{\theta}_2^{-1})^{-1}$. Now, consider the computation of $M_2$ starting with the admissible word $W' = W \cdot \overline{\theta}_1 H_1 \overline{\theta}_2^{-1} H_2 \ldots H_{m-1} \overline{\theta}_m^{-1}$ and having the history $\overline{\theta}_m H_{m-1} \ldots H_2 \overline{\theta}_2^{-1} \Pi_1(\theta_2, W \cdot \overline{\theta}_1 H_1 \overline{\theta}_2^{-1})$. By the inductive hypothesis, all words in this computation, starting with the second one, and all words in the computation $W \cdot \overline{\theta}_1 \rightarrow \cdots \rightarrow W \cdot \Pi_1(\theta_1, W)$ are positive. By the induction assumption, the computation $W' \cdot \theta_{m+1} \rightarrow \cdots \rightarrow W \cdot H$ also consists of positive words. Therefore, the first and the third words in the subcomputation with history $\overline{\theta}_m^{-1} \theta_{m+1}$ are positive contrary to Lemma 4.13.

(2) There is nothing to prove if $n = 0$. The case $\epsilon_2 = 1$ was considered in the proof of claim (1). For the case $H \equiv \theta_3 H_1 \overline{\theta}_2^{-1}$, we also proved that the computation with subhistory $H \equiv \theta_1 H_1$ has all words positive except possibly the first one. Hence, we may assume that $n \geq 2$. If $H \equiv \theta_1 H_1 \overline{\theta}_2^{-1} H_2 \ldots H_{n-1} \overline{\theta}_m^{-1} H_m \overline{\theta}_{m+1}$, then we consider the computation with history $H^{-1}$ and again come to the case $\epsilon_2 = 1$. Therefore, we assume that $H \equiv \theta_1 H_1 \overline{\theta}_2^{-1} H_2 \ldots H_{m-1} \overline{\theta}_m^{-1} H_m \overline{\theta}_{m+1} H''$, where $m < n$ by the condition on $(\epsilon_n, \epsilon_{n+1})$. By Lemma 4.8, we have that $H_m$ is empty. By the inductive hypothesis, the computations with histories $\overline{\theta}_1 H_1 \overline{\theta}_2^{-1} H_2 \ldots H_{m-1} \overline{\theta}_m^{-1}$ and $\overline{\theta}_m H''$ have all words positive except possibly the first one and the last one. Therefore, we can apply Lemma 4.13 to the computation with history $\overline{\theta}_m^{-1} \theta_{m+1}$, a contradiction.
Lemma 4.15. For every reduced computation $w_0 \rightarrow \cdots \rightarrow w_t$ of $M_2$ with the standard base and a non-empty history $H$, we have $w_t \neq w_0$.

Proof. Assume that $w_t = w_0$, and $t > 0$ is minimal. Then the computation $w_1 \rightarrow \cdots \rightarrow w_{t-1}$ is not a counter-example, and so $H$ is a cyclically reduced word.

If $H' \equiv \Pi_{21}(H)$ is empty, we consider the computation $w_0 \rightarrow \cdots \rightarrow w_0 \circ H = w_0 \rightarrow \cdots \rightarrow w_0 \circ H^2 \rightarrow \cdots$ with history $H^k$, where $k$ is as large as we want. As in Lemmas 4.6 and 4.9, we have a decomposition $H \equiv H_1 \ldots H_s$, where $H_i$ corresponds to the work of some $Z^{(\theta,i)}$ or equal to some $\zeta(\theta)^{\pm 1}$ and $s$ is bounded by a constant independent of $k$. It follows that $H$ corresponds to only one $Z^{(\theta,i)}$, and then the equality $w_t = w_0$ and Lemma 4.5 imply $\|w_0\| = \|w_1\| = \cdots = \|w_t\|$. Now $\|H^k\|$ is uniformly bounded for all $k$-s, contrary to Lemma 4.3.

If $\|H'\| \geq 1$, then Lemma 4.10 gives a reduced computation $\pi_{21}(w_0) \rightarrow \cdots \rightarrow \pi_{21}(w_t)$ with history $H'$. As above we can obtain reduced computations $\pi_{21}(w_0) \rightarrow \cdots$ of the S-machine $M_1$ with histories $H^k$. For $k \geq 3$, Lemma 4.14(2) implies that all words in the computation with history $H$ are positive. Then the same property must be true for the computation with history $H'$, and by Lemma 3.8, it is also a computation of the Turing machine $M_1$ with the same history $H'$, contrary to Lemma 2.3(c). Thus, the lemma is proved by contradiction. \qed

Lemma 4.16. (a) Let $X_2$ be the set of all words $\pi_{1,2}(W)$ accepted by $M_2$, where $W$ is an input word of the Turing machine $M_1$. Then a word $W'$ belongs to $X_2$ if and only if $W' \equiv \pi_{1,2}(W)$, and $W$ is an input word of $M_1$ accepted by the Turing machine $M_1$. Hence, the set of words accepted by $M_2$ is not recursive.

(b) For every $W' \equiv \pi_{1,2}(W) \in X_2$, there exists only one reduced computation of $M_2$ accepting $W'$, the length of that computation is between the length $T$ of the reduced computation of $M_1$ accepting $W$ (this computation is unique by Lemma 2.3(e) and $\exp(O(T))$).

Proof. (a) Let $W' \equiv \pi_{1,2}(W)$, where $W$ is an input word of the Turing machine $M_1$. Suppose that $W$ is accepted by the (symmetric) Turing machine $M_1$. By part (b) of Lemma 2.3, the history $H$ of the accepting computation consists of positive commands only. Then the computation of $M_2$ with history $\Pi_{1,2}(H, W)$ accepts $W'$ by Remark 4.12.

Suppose that $W'$ is accepted by $M_2$ and $H'$ is the history of an accepting computation $C'$. By Lemma 4.10, $\Pi_{2,1}(H')$ is a reduced history of an accepting computation of the S-machine $M_1$ starting with the input admissible word $\pi_{2,1}(W')$. Therefore, by part (c) of Lemma 2.2, the first rule in $H'$ is $\theta$ for some positive rule $\theta$ of $M_1$. Again by Lemma 2.2(c), the last rule of $\Pi_{2,1}(H')$ is positive. It follows that the last rule of $H'$ must be $\zeta(\theta^k)$ for some positive rule $\theta^k$ since the accepted admissible word of $M_2$ has state letters having no $\theta$-indices. By Lemma 4.14, all words in computation $C'$ are positive because both $W$ and $W' \equiv \pi_{1,2}(W)$ are positive too. Therefore, all words in the accepting computation $W \rightarrow \cdots \rightarrow \pi_{2,1}(W' \cdot H')$ of the S-machine $M_1$ are positive. By Lemma 3.8, the latter computation is an accepting computation of the Turing machine $M_1$, whence $W \in X_1$.

(b) The existence and the uniqueness follow from part (a) of this lemma and Lemmas 4.6 and 4.11. The part about the length of computation follows from Lemma 4.2(2). \qed

Similarly to the case of Turing machines, for every function $f(n)$, we define $f$-good numbers for $M_2$. We call a number $b$ $f$-good provided for every input word $W$ from $X_2$, if the length of the input sector (that is, the $Q_0P_1$-sector) of $W$ is less than $b$, then $f(T) \leq b$, where $T$ is the time of accepting $W$ by $M_2$. Now Theorem A.1 and Lemma 4.16(b) imply the following lemma:

Lemma 4.17. For every $\alpha > 0$, the set of $\exp(\alpha n)$-good numbers of $M_2$ is infinite.
Remark 4.18. The machine $Z(A)$ and the copies of it $Z^{(θ,i)}$ do not satisfy Property 3.1 since two $a$-letters are involved in the (copies of) rules $r_{12}(a)$. Therefore, we will use the machine $M_2$ obtained from $M_2$ by the application of Lemma 3.3. Note that the claim of Lemma 4.15 is correct for $M_2$ as well. Indeed, if the words $w_0 \equiv w_i$ in a computation $w_0 \rightarrow \cdots \rightarrow w_t$ of $M_2$ with non-empty reduced history $H$ involve auxiliary state letters, then there is a computation of $M_2$ with a reduced history $H'$, where $H'$ is a free conjugate of $H$, which starts and ends with the words $w_0' \equiv w_t'$, having no special state letters. Then Lemmas 3.3 and 4.15 for $M_2$ lead to a contradiction. Since the modification of $M_2$ does not touch the rules $θ$ corresponding to the rules $θ$ of $M_1$, the statements of Lemmas 4.16 and 4.17 also remain valid for $M_2$. Thus Lemmas 4.15–4.17 will be applied to the modified machine $M_2$. Moreover, it follows from the definitions of $M_2$ and $M_2$ that these $S$-machines inherit the Property (c) from Lemma 2.2 of the Turing machine $M_1$ (for positive rules).

If the sum from Property 3.1(2) is positive for some rule of $M_2$, then this sum is 1, and we have $||v_i|| + ||v_i'|| = 1$ or $||u_{i+1}|| + ||u_{i+1}'|| = 1$ for a unique $i$. In the first case (in the second case), we say that the rule is left (is right).

Remark 4.19. Note that similarly, we can define the $o$-product $S \circ S'$ of any $S$-machine $S$ and $S'$-machine $S' = S'(A)$ depending on the tape alphabet $A$. Furthermore, one can replace the auxiliary machine $S'$ by several $S$-machines $S_1, \ldots, S_d$. Namely, one inserts a $p$-letter between two consecutive state letters in the standard base of $S$ and treats any subword $q_i \ldots q_{i+1}$ as an admissible subword for $S_i$-s. For each rule $θ$ of $S$, one has a modified rule $θ'$ of the composition. The application of the rule $θ'$ is normally framed by alternated works of the auxiliary machines $S_i^{(θ,i)}$, and the priorities of the work of these machines may depend on $θ$. We are not going to define this construction formally, leaving it to the reader. In the next subsection, we shall introduce the $o$-product of $M_2$ and two primitive $S$-machines.

4.2. The machine $M_3$

Let $M_3$ be the $S$-machine $M_1 \circ Z$ and $M_2$ be the modification from Remark 4.18. For every set of letters $A$, let $A'$ and $A''$ be disjoint copies of $A$, the maps $a \mapsto a'$ and $a \mapsto a''$ identify $A$ with $A'$ and $A''$, respectively. Let $Z = Z(A)$ and $Z = Z(A)$ be the $S$-machines with tape alphabet $A' \sqcup A''$, state alphabet $\{L\} \cup P \cup \{R\}$, where $P = \{p(1), p(2), p(3)\}$ and the following positive $S$-rules. For $Z$, we have the rules

\[ ξ_1(a) = [L \rightarrow L, p(1) \rightarrow a'p(1)(a'')^{-1}, R \rightarrow R], \quad a \in A. \]

Comment: The head moves from left to right, replacing the word on the tape by its copy in the alphabet $A'$.

\[ ξ_2 := [L \rightarrow L, p(1) \rightarrow p(2), R \rightarrow R]. \]

Comment: When the head meets $R$, it turns into $p(2)$.

For $Z$, we define the rules

\[ ξ_3(a) = [L \rightarrow L, p(2) \rightarrow (a')^{-1}p(2)a'', R \rightarrow R]. \]

Comment: The head $p(2)$ moves from right to left, replacing the word in $A'$ by its copy in $A''$.

\[ ξ_4 = [L \rightarrow L, p(2) \rightarrow p(3), R \rightarrow R]. \]

Comment: When the head reaches the left end of the tape, it turns into $p(3)$. 
Remark 4.20. For every \( a \in A, i = 1, 3 \), it will be convenient to denote \( \xi_i(a)^{-1} \) by \( \xi_i(a^{-1}) \).

It is clear from the definition \( \xi_i(a) \) that this does not lead to a confusion.

Remark 4.21. Note that if the machine \( \overline{Z} \) (the machine \( \overline{Z} \)) starts with the words \( L_p(1)u''R \) or \( Lu'p(2)R \) and ends with the word \( Lu'p(2)R \) or \( Lp(3)u''R \), respectively, where \( u'' \) is the word in \( A'' \)-letters, then the history \( H \) of the only reduced computation such that \( Lp(1)u''R \cdot H = Lu'p(2)R \) (such that \( Lu'p(2)R \cdot H = Lp(3)u''R \)) is \( \xi_1(a_1), \ldots, \xi_1(a_m) \xi_2 \) (is \( \xi_3(a_m), \ldots, \xi_3(a_1) \xi_4 \)) and its length is \( ||u|| + 1 \). Here \( u \equiv a_1 \ldots a_m \) is the copy of \( u' \) (of \( u'' \)) in the alphabet \( A \).

Similarly, any reduced computation of \( \overline{Z} \) or \( \overline{Z} \) ending with \( Lu'p(2)R \) or \( Lp(3)u''R \), respectively, is uniquely determined by its initial admissible word and has length at most \( ||u|| + 1 \).

Remark 4.21 implies, in particular, the following lemma:

Lemma 4.22. Suppose that \( W \rightarrow \cdots \rightarrow W \cdot H \) is a reduced computation of \( \overline{Z} \) (of \( \overline{Z} \)) with history \( H \) and the standard base. Suppose that both \( W \) and \( W \cdot H \) contain \( Lp(1) \) or both contain \( p(2)R \) (respectively, \( p(2)R \) or \( Lp(3) \)). Then \( H \) is empty.

Below, we define \( M_3 \) as \( M_2 \circ \{ \overline{Z}, \overline{Z} \} \) (see Remark 4.19), that is, we insert copies of \( \overline{Z} \) and \( \overline{Z} \) between every two consecutive state letters of \( M_2 \). We simplify and unify the notation by changing the value of \( N \) and renaming the parts of the state alphabet of \( M_3 \). In this section, we assume that \( M_2 \) has the standard base \( s_0s_1 \ldots s_N \) (and forget more detailed earlier notation).

For every \( i = 1, \ldots, N \), we make copies \( Y_i' \) and \( Y_i'' \) of the alphabet \( Y_i \) of \( M_2 \) (\( i = 1, \ldots, N \)).

Let \( \Theta \) be the set of positive commands of \( M_2 \). The set of state letters of \( M_3 \) is

\[
S_0 \cup P_1 \cup S_1 \cup P_2 \cup \cdots \cup P_N \cup S_N,
\]

where \( P_i = \{ p^{(i,0)}, p^{(i,1)}, p^{(\theta,i)(1)}, p^{(\theta,i)(2)}, p^{(\theta,i)(3)} \mid \theta \in \Theta \} \) \( i = 1, \ldots, N \), \( S_i = Q_i \uplus Q_i \times \Theta \) where \( Q_0 \uplus Q_1 \uplus \cdots \uplus Q_N \) is the set of state letters of \( M_2 \). Thus, the state letters \( L \) and \( R \) of the copies of machines \( \overline{Z} \) and \( \overline{Z} \) are identified with the corresponding \( S \)-letters as in the case of \( M_2 = M_1 \circ Z \). We shall call the state letters from \( P_i \)-s the control state letters or \( p \)-letters, and the other state letters (that is, the copies of the state letters of \( M_2 \)), the basic state letters.

The set of tape letters of \( M_3 \) is \( Y = Y_1 \uplus \cdots \uplus Y_{2N} = Y_1' \uplus Y_1'' \uplus Y_2' \uplus Y_2'' \uplus \cdots \uplus Y_N' \uplus Y_N'' \).

Let \( \theta \) be a positive \( M_2 \)-rule which is not a right rule. Assume that \( \bar{\theta} \) is of the form

\[
[s_0 \rightarrow s_0', v_1s_1 \rightarrow v_1's_1', \ldots, v_Ns_N \rightarrow v_N's_N'],
\]

where \( s_i, s_i' \in S_i \), and \( v_i \)-s are words in \( Y \). Then this rule is replaced in \( M_3 \) by positive

\[
\bar{\theta} = \left[ s^{(0,0)} f_{p^{(\theta,0)}(1)} (s')^{(\theta,0)}, p^{(\theta,1)}(1) \rightarrow p^{(\theta,1)}(1), v_1s^{(\theta,1)} f_{p^{(\theta,2)}(1)} (s')^{(\theta,1)}, \ldots, v_Ns^{(\theta,N)} \rightarrow v_N's^{(\theta,N)} \right],
\]

with \( Y_{2i-1}(\bar{\theta}) = \theta \) and \( Y_{2i}(\bar{\theta}) = Y_i''(\theta) \). As an exception, the left-hand sides of the parts of \( \bar{\theta} \) are of the form \( s_0 \rightarrow p^{(1,1)} \rightarrow s_1 \rightarrow p^{(2,1)} \rightarrow \ldots, s_N \rightarrow \) if \( \theta \) is the unique start rule, that is, they do not depend on the index \( \theta \).

A right positive rule \( \theta \) of the form

\[
[s_0u_1 \rightarrow s_0'u_1, s_1u_2 \rightarrow s_1'u_2, \ldots, s_N \rightarrow s_N'],
\]
is replaced by the right rule $\bar{\theta}$ of $M_3$:

$$\bar{\theta} = \left[ s(\theta,0)u_1 \rightarrow (s')^{(1)}(\theta)u_1', p^{(1)}(2) \leftarrow p^{(1)}(2), s(\theta,1)u_2 \rightarrow (s')^{(1)}(\theta)u_2', p^{(2)}(2) \leftarrow p^{(2)}(2), \ldots, s(\theta,N) \rightarrow (s')^{(1)}(\theta,N) \right]$$

with $Y_{2i-1}(\bar{\theta}) = Y'_{i}(\theta)$ and $Y_{2i}(\bar{\theta}) = \emptyset$.

Now we want to describe the alternating work of the auxiliary machines $\overline{Z}^{(\theta,i)}$ and $\overline{Z}^{(\theta,j)}$. Normally each of them is switched on exactly once in the frame of the rule $\theta$, but the sequence of their turning on depends on $\theta$.

First, we need the following transition rule $\zeta_-(\theta)$. This rule adds $\theta$ to all state letters and turns all $p^{(j)}$ into $p^{(\theta,j)}(1)$:

$$[s_i \leftarrow s(\theta,i), p^{(j)} \rightarrow p^{(\theta,j)}(1), i = 0, \ldots, N, j = 1, \ldots, N]$$

so that the rule $\theta$ becomes applicable if $\theta$ is not a right rule. Again, as an exception, we do not introduce $\zeta_-(\theta)$ for the start rule $\theta_{\text{start}}$ of $M_2$.

If $\theta$ is a right rule, then the rule $\zeta_-(\theta)$ successively switches on the machines $\overline{Z}^{(\theta,1)}, \ldots, \overline{Z}^{(\theta,N)}$. (We will not present formulas for the rules $\bar{\tau}_i(\theta)$ as in the definition of $M_2$, since the explicit forms of these rules are not necessary.) Then the state letters $p^{(\theta,j)}(1)$ ($j = 1, \ldots, N$) successively turn into $p^{(\theta,j)}(2)$, find themselves just before $s_i$-letters, and the rule $\theta$ can be applicable.

After an application of a non-right rule $\bar{\theta}$, the machines $\overline{Z}^{(\theta,j)}$ move the $p$-letters to the right, change $p^{(\theta,j)}(1)$ by $p^{(\theta,j)}(2)$, and then the machines $\overline{Z}^{(\theta,j)}$ move the $p$-letters to the left and change $p^{(\theta,j)}(2)$ by $p^{(\theta,j)}(3)$. After an application of a right rule $\bar{\theta}$, only machines $\overline{Z}^{(\theta,j)}$ work.

Finally, the transition rule $\zeta_+(\theta)$ removes index $\theta$ from all state letters, and turns all $p^{(\theta,j)}(3)$ into $p^{(j)}$:

$$[s(\theta,i) \leftarrow s_i, p^{(j)}(3) \rightarrow p^{(j)}, i = 0, \ldots, N, j = 1, \ldots, N].$$

If $\theta$ is the unique accept rule of $\bar{M}_2$, then all $p^{(j)}$-s in the above definition of $\zeta_+(\theta)$ must be replaced by special letters $p^{(\theta,0)}$-s.

An important specification is the following. If $\theta$ is a right rule and $\|u_i+1\| + \|u'_i+1\| = 1$ (see (*)), then the application of $\bar{\theta}$ always switches on the auxiliary machines in the order $\overline{Z}^{(\theta,i)}, \overline{Z}^{(\theta,i+1)}, \ldots, \overline{Z}^{(\theta,1)}, \overline{Z}^{(\theta,0)}, \ldots, \overline{Z}^{(\theta,i+1)}$. If $\theta$ is a left rule and $\|u_i\| + \|u'_i\| = 1$, then $\bar{\theta}$ must successively start up $\overline{Z}^{(\theta,i+1)}, \ldots, \overline{Z}^{(\theta,i)}, \overline{Z}^{(\theta,1)}, \overline{Z}^{(\theta,0)}$. If $\theta$ is neither right nor left, then the order for the first $N$ machines is $\overline{Z}^{(\theta,1)}, \ldots, \overline{Z}^{(\theta,N)}$.

For every admissible word $w$ of $M_2$ with standard base, let $\pi_{2,3}(w)$ be the admissible word of $M_3$ obtained by inserting control state letters $p^{(1)} \circ (p^{(1)}(1))$ or $p^{(1)}(0)$ if the word $w$ has state letters from the start vector $\vec{s}$, respectively, from the accept vector $\vec{s}_0$ of $\bar{M}_2$ next to the right of each $s_{i-1}, i \leq N$. The stop word of $M_3$ is $\pi_{2,3}(\pi_{1,2}(W_0))$, where $W_0$ is the stop word of $M_1$. For every input word $w$ of $M_2$ we call $\pi_{2,3}(w)$ an input word of $M_3$.

**Remark 4.23.** It follows from Remark 4.18 and the definition of $M_3$ that the $S$-machine $M_3$ inherits the Property (c) from Lemma 2.2 (for positive rules).

The $p^{(1)}s_1$-sector of an admissible word of $M_3$ is called the input sector of that word.

Assume that $w \rightarrow w \cdot \theta$ is a computation of the machine $M_3$ with standard base and a positive rule $\theta$. Then, by the definition of $M_3$, we have the canonically defined reduced computation $\cdots \rightarrow \pi_{2,3}(w) \rightarrow \pi_{2,3}(w) \cdot \bar{\theta} \rightarrow \cdots$ starting and ending with words whose state letters have no $\theta$-indices and all other words do have $\theta$-indices. The computation of $M_3$ with these properties
is unique since the base is standard. Indeed Remark 4.21 and the definition of $M_3$ uniquely determine the order of rules for each of the auxiliary machines $Z^{(\theta,j)}$ and $Z^{(\theta,j)}$. (For example, a machine $Z^{(\theta,j)}$ can start working only if the state letter $p^{(\theta,j)}(1)$ is the right neighbor of a letter $s^{(\theta,j)}$ since the $p^{(\theta,j)}(1)s^{(\theta,j)}$-sector is locked before the start, and $Z^{(\theta,j)}$ cannot finish its work until the $p$-letter becomes the left neighbor of $s^{(\theta,j)}$ and turns into $p^{(\theta,j)}(2)$, etc.) Thus, the following claim is true.

**Lemma 4.24.** For every computation $w \rightarrow w \cdot \theta$ of the machine $M_2$ with standard base and a positive rule $\theta$, there is a unique reduced $M_3$-computation $\cdots \rightarrow \pi_{2,3}(w) \rightarrow \pi_{2,3}(w) \cdot \theta \rightarrow \cdots$ such that it starts and ends with words whose state letters have no $\theta$-indices and all other words have $\theta$-indices. The history of this computation starts with $\zeta_-(\theta)$ and ends with $\zeta_+(\theta)$.

We denote the history of this computation by $\Pi_{2,3}(\theta, w)$. It follows from Remark 4.21 that the length of this history is $1 + 2(\lvert w \rvert_a + N)$, where $\lvert w \rvert_a$ is the number of $a$-letters in the word $w$. If $\theta$ is a negative rule of $M_2$, then we invert the computation constructed for $\theta^{-1}$, and so $\Pi_{2,3}(\theta, w) \equiv \Pi_{2,3}(\theta^{-1}, w \cdot \theta)^{-1}$.

Similarly, with arbitrary reduced computation $w \rightarrow \cdots \rightarrow w \cdot H$ with the standard base of $M_2$ and having a history $H \equiv \theta_1 \ldots \theta_t$, we associate the reduced computation of $M_3$ with history

$$\Pi_{2,3}(H, w) \equiv \Pi_{2,3}(\theta_1, w)\Pi_{2,3}(\theta_2, w \cdot \theta_1) \cdots \Pi_{2,3}(\theta_t, w \cdot \theta_1 \ldots \theta_{t-1}).$$

It follows from the previous paragraph that $\lVert H \rVert \leq \lVert \Pi_{2,3}(H, w) \rVert = O(\lVert H \rVert^2)$ for every accepted computation. Indeed $\lvert w \rvert_a = O(\lVert H \rVert)$ since the stop word has no tape letters.

Recall that only rules of the form $\zeta_+(\theta)$ and the start rule involve state letters without $\theta$-indices. Therefore, it follows from Lemma 4.24 that every reduced computation with the standard base of $M_3$ starting and ending with the admissible words without $\theta$-indices in their state letters, has history of the form $\Pi_{2,3}(H, w)$ for some reduced computation $w \rightarrow \cdots \rightarrow w \cdot H$ of $M_2$. In particular, our discussion and Lemma 4.16(b) imply the following lemma:

**Lemma 4.25.** (a) Let $X_3$ be the set of all words $\pi_{2,3}(w)$ accepted by $M_3$, where $w$ is an input word of $M_2$ (or $M_2$). Then a word $W$ belongs to $X_3$ if and only if $W = \pi_{2,3}(w)$, and $w \in X_2$. Hence the set of words accepted by $M_3$ is not recursive.

(b) For every $W \equiv \pi_{2,3}(w) \in X_3$ there exists only one reduced computation of $M_3$ accepting $W$, the length of that computation is between the length $T$ of the reduced computation of $M_2$ accepting $w$ and $O(T^2)$.

We can define $f$-good numbers of $M_3$ in a similar way as for $M_2$. Lemmas 4.17 and 4.25 imply the following lemma:

**Lemma 4.26.** For every constant $c > 0$, the set of $\exp(cn)$-good numbers of $M_3$ is infinite.

As in the previous section, we need to define more maps between $S$-machines $M_2$ and $M_3$.

For every admissible word $W$ of $M_3$ with the standard base, let $\pi_{3,2}(W)$ be the word obtained by removing state $p$-letters, $\theta$-indices of state letters, and the indices that distinguish $a$-letters from the left and from the right of $p$-letters. We obtain an admissible word of $M_2$. Note that we have

$$\pi_{3,2}(\pi_{2,3}(w)) \equiv w. \tag{4.3}$$
For every rule $\theta$ of $M_3$ corresponding to a rule $\theta$ of $\bar{M}_2$ we denote $\theta = \Pi_{3,2}(\theta)$. If $\theta$ does not correspond to a rule of $\bar{M}_2$, we denote by $\Pi_{3,2}(\theta)$ the empty rule. The map $\Pi_{3,2}$ extends to histories of computations in the natural way.

**Remark 4.27.** It can be proven similarly to Lemma 4.10, that if $H$ is a history of a computation of $M_3$ with standard base and $W \cdot H = W'$, then $\Pi_{3,2}(H)$ is reduced and

$$\pi_{3,2}(W) \cdot \Pi_{3,2}(H) \equiv \pi_{3,2}(W').$$

(4.4)

**Lemma 4.28.** Suppose a computation $W \rightarrow \cdots$ of $M_3$ with a base $B$ has a reduced history $H \equiv \ldots \theta_1H\theta_2^n \ldots$, where $\Pi_{3,2}(H) \equiv \theta_1\theta_2^n$ for some positive $\theta_1$ and $\theta_2$, and $n = \pm 1$.

(1) If $B$ is standard, then the word $W \cdot \bar{\theta}_1$ is completely determined by $H'$.

(2) If $\theta_2^n \neq \bar{\theta}_1^{-1}$, then $B$ or $B^{-1}$ is a subword of the standard base of the machine $M_3$.

(3) Let $\|u_j+1\| + \|u'_j+1\| = 1 (\|v_j\| + \|v'_j\| = 1)$ for the rule $\theta_1$, and $B$ be not a subword of the standard base or of its inverse. Then $\theta_2^n \equiv \bar{\theta}_1^{-1}$, and no rule from $H'$ locks the $s^{(\theta,j)}p^{(\theta,j+1)}$-sector (respectively, the $p^{(\theta,j)}s^{(\theta,j)}$-sector).

**Proof.** (1) The argument used for Lemma 4.24 shows that since the base is standard, each of the machines $\bar{Z}^{(\theta,j)}$ must accomplish its standard work after the application of the rule $\theta_1$. Therefore, the history of the work of $\bar{Z}^{(\theta,j)}$ completely determines the $p^{(j)}s^{j}$-sector subword of the word $W \cdot \bar{\theta}_1$. The $\theta$-indices of the state letters of this word are obviously determined by the histories of $\bar{Z}^{(\theta,j)}$ and Statement (1) is proved.

(2) The assumptions imply that the $\theta$-indices that the state letters have after the application of $\bar{\theta}_1$, must disappear earlier than one applies $\bar{\theta}_1^n$. Again by Remark 4.21, it follows that each of the machines $\bar{Z}^{(\theta_1,j)}$ must perform its standard work. Therefore, for every $j = 1, \ldots, N$, the history $H'$ has rules locking $s^{(\theta,j-1)}p^{(\theta,j)}(1)$-sectors and it has rules locking $p^{(\theta,j)}s^{(\theta,j)}$-sectors. Hence, by Lemma 3.4, the base $B$ has no subwords of the form $q^{\pm 1}q^{\mp 1}$, and so $B^{\pm 1}$ is a subword of the standard base by the definition of admissible word.

(3) First of all, we have $\theta_2^n \equiv \bar{\theta}_1^{-1}$ by Property (2). Then we assume that a right rule $\tau$ from $H'$ locks the $s^{(\theta,j)}p^{(\theta,j+1)}$-sector.

The locking rule $\tau$ must belong to the machine $\bar{Z}^{(\theta_1,j+1)}$ since other auxiliary machines working after the application of the right rule $\bar{\theta}_1$ do not lock this sector. Taking into account the order of the work of auxiliary machines after an application of a right rule, we conclude that the machines $\bar{Z}^{(\theta_1,j)}, \ldots, \bar{Z}^{(\theta_1,1)}, \bar{Z}^{(\theta_1,N)}, \ldots, \bar{Z}^{(\theta_1,j+2)}$ work before the machine $\bar{Z}^{(\theta_1,j+1)}$ starts working. Since the last rule of $\bar{Z}^{(\theta_1,j+2)}$ locks the $p^{(\theta,j+1)}s^{(\theta,j+1)}$-sector, $H'$ has a rule locking $p^{(\theta,j+1)}s^{(\theta,j+1)}$-sector. Proceeding in this manner, we then consider the work of the preceding machine $\bar{Z}^{(\theta_1,j+2)}$ and conclude that the $s^{(\theta,j+1)}p^{(\theta,j+2)}$-sectors was locked by some rules from $H'$. Finally, we see that every sector except for $p^{(\theta,j)}s^{(\theta,j)}$ was locked by some rule from $H'$. The $p^{(\theta,j)}s^{(\theta,j)}$-sector was locked by $\bar{\theta}_1$ since $\bar{\theta}_1$ is a right rule. By Lemma 3.4, $B$ is a subword of the standard base or of its inverse, a contradiction.

Similar argument works if $\bar{\theta}_1$ is a left rule. In this case, if $p^{(\theta,j)}s^{(\theta,j)}$-sector is locked by a rule from $H'$, then $\bar{\theta}_1$ switches on the machines $\bar{Z}^{(\theta_1,j+1)}, \ldots, \bar{Z}^{(\theta_1,N)}, \bar{Z}^{(\theta_1,1)}, \ldots, \bar{Z}^{(\theta_1,j)}$, and we again come to a contradiction.

**Lemma 4.29.** Suppose that the admissible word $W$ of $M_3$ has the standard base. Suppose that a reduced computation applicable to $W$ has history of the form $H^3$. Then $H$ does not contain rules $\theta$ corresponding to the rules $\theta$ of the $S$-machine $\bar{M}_2$. 

\[\square\]
Recall that the set of state letters of \( M \). The machine is determined by the same \( H \) where the parts of the rule (12) between \( t_k \) and \( p \) -letters or \( s_t \)-letters are divided into three steps. All sectors except for the start and \( t_k \)-sector to \( T \). The set of tape letters in the \( kS_0k \)-sector and in the \( sNk' \)-sector are empty, and the set of tape letters in the \( k'p' \)-sector is \( Y'(3) \). The positive rules of the machine \( M_4 \) are divided into three steps. Each rule below contains subrules \( s_0 \rightarrow s_t \), \( t \rightarrow t \) and \( t' \rightarrow t' \), so we sometimes omit these subrules.

**Step 1:**

\[
\rho_1(y) = \left[ k(1) \xrightarrow{\ell} yk(1), p^{(1)} \xrightarrow{\ell} p^{(1)}, s_i \xrightarrow{\ell} s_i(1 \leq i \leq N), \right. \\
\left. p^{(1)} \xrightarrow{\ell} p^{(1)}(2 \leq j \leq N), k'(1) \xrightarrow{\ell} k'(1) \right], \quad y \in Y(3),
\]

where the \( s \)- and \( p \)-letters form the start vector \( s_1 \) for the machine \( M_3 \).

**Comment:** The machine writes the \( Y(3) \)-copy of a (positive) history word in the \( tk \)-sector to the left of \( k_1 \). The word between \( k \) and \( k' \) is an input word of \( M_3 \). All sectors except for the \( tk \)-sector and the \( p^{(1)}s_1 \)-sector are locked by the rules of Step 1.

**Transition rule** (12) from Step 1 to Step 2 is the ‘extension’ \( \theta(M_4) \) of the unique start rule \( \theta = \theta_\text{start} \) of the machine \( M_3 \):

\[
(12) = [k(1) \xrightarrow{\ell} k(2), \ldots, s_N \xrightarrow{\ell} s_N, k'(1) \xrightarrow{\ell} k'(2)],
\]

where the parts of the rule (12) between \( k \)- and \( k' \)-letters are the parts of \( \theta_\text{start} \).

**Comment:** After this rule is applied, the machine is ready to execute copies of the machines \( \overline{Z}^{(\theta_\text{start}, 1)} \) and \( \overline{Z}^{(\theta_\text{start}, 1)} \), on tapes 1–\( N \). All sectors except the \( tk \)-sector and the \( p^{(1)}s_1 \)-sector are locked by this rule.

**Step 2:** For every \( \theta \in \Theta(3) \):

\[
\theta(M_4) = [k(2) \xrightarrow{\ell} y_0^{-1}k(2), \theta, k'(2) \rightarrow k'(2)y_0].
\]
Comment: On tapes 1–N, the machine executes (backwards) the history written in the tk-sector, erases the word in that sector, and copies it to the k't'-sector.

Transition rule (23) = \( \theta(M_4) \) from Step 2 to Step 3 ‘extends’ the accept rule \( \theta = \theta_{\text{accept}} \) of \( M_3 \):

\[
(23) = \left[ t \xrightarrow{k} t, k(2) \xrightarrow{k} k(3), \ldots, s_N \xrightarrow{k} s'_N, k'(2) \rightarrow k'(3), t \rightarrow t' \right],
\]
where the parts of the rule (23) between k- and k'-letters are the parts of \( \theta_{\text{accept}} \).

Comment. All sectors except for the k't'-sector are locked by this rule.

Step 3:

\[
\rho_3(\theta) = [t \xrightarrow{k} t, k(3) \xrightarrow{k} k(3), \ldots, s'_N \xrightarrow{k} s'_N, k'(3) \rightarrow k'(3)(y'_0)^{-1}, t' \rightarrow t'],
\]
where the state letters between k- and k'-letters form the accept vector \( \overline{s}_0 \) of \( M_3 \).

Comment: The machine erases the history from the k't'-sector. All other sectors are locked by the rules of this step.

For every admissible input word \( W \in X_3 \) of \( M_3 \), let \( \pi_{3,4}(W) \in X_4 \) be the admissible word of \( M_4 \) obtained by adding state letters \( k(1), k'(1), t, t' \), hence \( \pi_{3,4}(W) \equiv k(1)tW't'k'(1) \). For every input word \( W \) of \( M_3 \), we call the word \( \pi_{3,4}(W) \) an input word of \( M_4 \). The stop word of \( M_4 \), \( W_{M_4} \), is obtained from the stop word \( W_{M_3} \) of \( M_3 \) by adding state letters \( k(3), k'(3), t, t' \), that is, \( W_{M_4} \equiv k(3)tW_{M_3}t'k'(3) \).

Remark 4.30. From now on, we do not show the indices \( (i) \) \( (i = 1, 2, 3) \) of the letters \( k \) and \( k' \) assuming that the indices are appropriate for an admissible word.
For every accepting computation \( W \rightarrow W \cdot \theta_1 \rightarrow W \cdot \theta_1 \theta_2 \rightarrow \cdots \rightarrow W \cdot \theta_1 \ldots \theta_n \equiv W_{M_3} \) (where \( \theta_i \) are rules and \( W \) is an input word for \( M_3 \)) of \( M_3 \) with history \( H \equiv \theta_1 \theta_2 \ldots \theta_n \), \( W \in X_3 \), one canonically constructs a computation of \( M_4 \): \( \pi_{3,4}(W) \rightarrow \cdots \rightarrow W_{M_4} \). The history of that computation is denoted by \( \Pi_{3,4}(H) \). That computation first uses rules of Step 1 and writes a mirror copy of \( H' \) (that is, \( H \) without the start and the accept rules) in the alphabet \( Y(3) \) in the \( tk \)-sector, then executes rule (12), then executes the computation with history \( H' \) on the subword between \( k \) and \( k' \) while erasing the word in the \( tk \)-sector and moving it onto the \( k't' \)-sector (written in \( Y'(3) \)). After \( H' \) is completed, rule (23) is executed, then the \( k't' \)-sector is erased using rules of Step 3. Let \( X_4 = \pi_{3,4}(X_3) \). Every word from this set of input configurations is accepted by \( M_4 \). To simplify the notation, we can include the rules (12) and (23) to Step 2.

Suppose that a history of computation of \( M_4 \) has the form \( H \equiv H_1 H_2 \ldots \), where all rules of each \( H_i \) belong to the same step \( j_i \), and \( H_i \) is a maximal subword of \( H \) with this property. Then we say that the step history of that computation is \( (j_1)(j_2) \ldots \) (or that \( H \) is of type \( (j_1)(j_2) \ldots \)). The following lemma is a straightforward consequence of the definition of \( M_4 \) and will be used without reference throughout the paper.

**Lemma 4.31.** Every 2-letter subword of any step history of a computation of \( M_4 \) (with any base) is one of the following words: (1)(2), (2)(1), (2)(3) and (3)(2). Two consecutive steps are separated by \((12)^{\pm 1}\) or by \((23)^{\pm 1}\), respectively, and the letters of the history neighboring any \((12) \) (or \((23)\)) from the left and from the right belong to different steps.

**Lemma 4.32.** An admissible word of \( M_4 \) is not in the domain of the reduced histories of types:

(a) \((1)(2)(1)\) if the base of \( W \) has subword \( k't' \);
(b) \((3)(2)(3)\) if the base of \( W \) has subword \( tk \);
(c) \((3)(2)(1)(2)(3)\) if the base of \( W \) is standard.

**Proof.** Cases (a) and (b) are almost identical, so suppose that the history \( H \) contains a subword \((12)H(12)^{-1}\), where \( H \) is of type (2). The word \( V \equiv W \cdot (12) \) from the computation that is in the domain of \( H \) must have the subword between \( k' \) and \( t' \) empty (since it is in the domain of \((12)^{-1}\)). Similarly, the word \( V \cdot H \) must have the subword between \( k' \) and \( t' \) in \( V \cdot H \) equal to \((-y_{\theta_1}^{\pm 1}) \ldots (-y_{\theta_1}^{\pm 1}) \). Since this word is empty, we conclude that \( H \) is not reduced, a contradiction.

Suppose that the step history is of the form (c). Then the history \( H \) has the form \( H_3(23)^{-1}H_2(12)^{-1}H_1(12)H_2'(23)H_1' \), where \( H_i \) and \( H_i' \) contain rules from step \( i \) only. Restricting the computation to the subwords between \( k \) and \( k' \) of the admissible words, we obtain two reduced accepting computations of \( M_3 \) with the same initial word from \( X_3 \) and histories \( H_2^{-1} \) and \( H_2' \) (this follows from the definitions of the rules of Step 2). By Lemma 4.25(b), \( H_2^{-1} \equiv H_2' \). Since every rule of Step 1 multiplies the \( tk \)-sector of the admissible word by an \( a \)-letter uniquely determined by the rule, the \( tk \)-sectors \( A_{tk} \) and \( B_{tk} \) in the words \( W \cdot H_3(23)^{-1}H_2 \) and \( W \cdot H_3(23)^{-1}H_2'(12)^{-1}H_1(12) \), respectively, are the same. Since every rule of Step 1 multiplies that sector by a letter uniquely determined by that rule, we deduce that a copy of the word \( H_1 \) multiplied by \( A_{tk} \) is \( A_{tk} \). Hence, \( H_1 \) is empty, which contradicts the assumption that the computation is reduced.

**Lemma 4.33.** Suppose that \( W \equiv tW_1kW_2k'W_3t' \) is an admissible word of \( M_4 \) with the standard base. Suppose that \( W \) is in the domain of a reduced history of the form \((12)H(23)\).
Then we have the following:

1. $H$ contains only rules from Step 2, $(12)H(23) \equiv \theta_1(M_4)\ldots\theta_n(M_4)$ for some rules $	heta_1, \ldots, \theta_n$ of $M_3$;
2. the word $W_2$ is from $X_3$ and $\theta_1 \ldots \theta_n$ is a computation of $M_3$ accepting $W_2$.

**Proof.** Suppose that $H$ contains rules from Step 1 or 3. Then it contains a subword of one of two forms $(23)^{-1}H_1(23)$ or $(12)H_1(12)^{-1}$ with $H_1$ consisting of rules of Step 2 which contradicts Lemma 4.32. This implies part (1) of the lemma.

Since $W$ is in the domain of $(12)$, the subword $W_2$ is an admissible input word of $M_3$. Since $W \cdot (12)H(23)$ is in the domain of $(23)^{-1}$, the subword between $k$ and $k'$ is the stop word $W_{M_3}$ of $M_3$. This implies part (2) of the lemma.

**Lemma 4.34.** Suppose that $W$ is an admissible word of $M_4$ with the standard base. Then we have the following.

(a) The step history of any reduced computation starting with $W$ is a subword of $(2)(1)(2)(3)(2)(1)(2)$.
(b) The step history of any accepting reduced computations starting with $W$ is a suffix of the word $(2)(1)(2)(3)$.

**Proof.** Indeed, in every step history $(i_1)(i_2)\ldots(i_s)$ of a reduced computation of $M_4$, after (1) we should have (2), after (2) we should have (1) or (3), after (3) we should have (2). The statement then follows immediately from Lemma 4.32.

**Lemma 4.35.** Suppose that a history $H$ of a reduced computation of $M_4$ with standard base contains both $(12)^{\pm 1}$ and $(23)^{\pm 1}$.

(a) The number of occurrences of $(12)^{\pm 1}$ or $(23)^{\pm 1}$ in $H$ is at most 6.
(b) Suppose that the computation is accepting. Then the number of occurrences of $(12)^{\pm 1}$ or $(23)^{\pm 1}$ in $H$ is at most 3.

**Proof.** Immediately follows from Lemmas 4.34 and 4.31.

**Lemma 4.36.** Recall that $X_4$ is the set of all words of the form $\pi_{3,4}(W)$, $W \in X_3$. An input word $W' \equiv \pi_{3,4}(W)$ is accepted by $M_4$ if and only if $W \in X_3$. Hence, the language accepted by $M_4$ is not recursive.

**Proof.** If $W \in X_3$, then $\pi_{3,4}(W) \in X_4$ since the corresponding computation was constructed together with the definition of $M_4$. Let $W' \equiv \pi_{3,4}(W)$ for some admissible input word $W$ of $M_3$ and $H$ be the history of an accepting computation for $W'$. By Lemma 4.34 (b), $H \equiv H_1(12)H_2(23)H_3$, where $H_j$ contains only rules of step $j$ ($j = 1, 2, 3$). By Lemma 4.33, $W$ is in $X_3$, and $H_2$ corresponds to a computation of $M_3$ accepting $W$.

**Definition 4.37.** Let $T_1 < T_2 < \cdots$ be all the times of acceptance of acceptable input words of $M_3$.

We will call a computation of $M_4$ standard if it has the standard base and history of the form $(12)(2)(23)$. The following lemma gives (almost) linear upper bounds for the lengths of many computations with standard base.
LEMMA 4.38. (a) Suppose that an admissible word $W$ of $M_4$ is accepted by $M_4$. Suppose that the length of a reduced accepting computation of $W$ is not in $\bigcup_{i=1}^{\infty} (T_i, 9T_i)$. Then the length of this accepting computation of $W$ is at most $6|W|_a$.

(b) Let $b$ be an integer such that any standard computation starting with a word $W$ with $|W|_a \leq b$ has the history of length less than $\log b$. Suppose $W$ is any accepted admissible word for $M_4$ with $|W| < b$. Then the time of accepting $W$ by any reduced computation of $M_4$ is at most $4|W|_a + 3\log b$.

Proof. Let $H$ be the reduced history of an accepting computation of $M_4$ with the first word $W$. By Lemma 4.34, the step history of $H$ is a suffix of $(2)(1)(2)(3)$. Hence, the possible step histories are $(2)(1)(2)(3)$, $(1)(2)(3)$, $(2)(3)$ or $(3)$. We shall prove (a) and (b) in each of these cases.

Suppose that the step history of $H$ is $(2)(1)(2)(3)$. Then

$$H \equiv H_2(12)^{-1}H_1(12)H_2'(23)H_3,$$

where $H_2$ and $H_2'$ consist of rules of Step 2, $H_1$ and $H_3$ consists of rules of Step 1 and Step 3, respectively. By Lemma 4.33, the length of $(12)H_2'(23)$ is one of the $T_i$. Since every rule of Step 2 multiplies the $tk$-sector by a letter uniquely determined by that rule, and in any word in the domain of $(23)$ the $tk$-sector is empty, we conclude that the $tk$-sector of the word $W \cdot H_2(12)^{-1}H_1(12)$ is a copy of $H_2'$, hence its length is $T_i - 2$. The $k't'$-sector of that word is empty and every rule from $H_2'$ multiplies that sector by a letter uniquely determined by the rule. Hence, the $k't'$-sector of $W \cdot H_2(12)^{-1}H_1(12)H_2'$ has length $T_i - 2$. Since every rule of Step 3 multiplies that sector by a letter, and in the stop word of $M_4$ that sector is empty, we conclude that $|H_3| = T_i - 2$. Hence, $|(12)H_2'(23)H_3| \leq 2T_i - 2$.

Note that since every rule of Step 2 multiplies the $k't'$-sector by a letter uniquely determined by that rule, and in a word in the domain of $(12)^{-1}$ that sector is empty, we can conclude that $|H_2| \leq |W|_a$. Similarly, since the rules of Step 1 multiply the $tk$-sector by letters uniquely determined by these rules, we conclude that

$$|H_1| \leq |H_2| + |W|_a + |H_2'| \leq 2|W|_a + T_i - 2.$$

(We use that if a group word $U$ of length $l$ is obtained from a word $V$ of lengths $k$ after a series of one-side multiplications by one letter, and successive multiplications are not mutually inverse, then the number of multiplications does not exceed $k + l$.) Therefore, $|H| \leq 2T_i - 2 + |H_2| + |H_1| + 3T_i + 3|W|_a$, and since in the case under consideration, we have $|H| \geq 9T_i$ by the condition of the lemma, it follows that $|W|_a \geq (|H| - 3T_i)/3 > |H|/6$, as required for part (a).

Now assume that the assumption of (b) holds. Note that every rule of $H_2$ multiplies the $k't'$-sector by a letter and the input $p_1s_1$-sector also by at most one letter, the rules of $H_1$ do not touch the input sector. Therefore, the input sectors or $W \cdot H_2$ and $W \cdot H_2(12)^{-1}H_1(12)$ are the same and their lengths are at most the sum of lengths of the input sector of $W$ and the $k't'$-sector of $W$. Hence, the length of the input sector of $W \cdot H_2(12)^{-1}H_1(12)$ does not exceed $|W|_a \leq b$. By the condition of the lemma, we have that $|H_2'| = T_i - 2 \leq \log b - 2$ for some $i$. As before $|H| \leq 3T_i + 3|W|_a \leq 3|W|_a + 3\log b$. Suppose that the step history of $H$ is $(1)(2)(3)$, that is, $H = H_1(12)H_2(23)H_3$, where $H_i$ contains only rules of step $i$ ($i = 1, 2, 3$), and then again by Lemma 4.33 $|H_2'| = T_i - 2$ for some $i$, and the length of the $tk$-sector in $W \cdot H_1$ is $T_i - 2$. As in the previous paragraph, $|H_3| = T_i - 2$.

Under the assumptions of (a) then $|H_1| > 9T_i - 2T_i + 2 > 7T_i$. Since every rule of $H_1$ multiplies the $tk$-sector by a letter, we also have that $|H_1|$ does not exceed the sum of lengths
of $tk$-sectors in $W$ and in $W \cdot H_1$, whence $\|H_1\| \leq |W|_a + T_1 - 2$. Therefore, $|W|_a > 6T_i$ and

$$\|H\| = \|H_1\| + 2T_i - 2 < 2|W|_a.$$ 

Suppose that the assumptions of (b) hold. Then since the input sectors of $W$ and $W \cdot H_1(12)$ are the same, and their length is at most $|W|_a < b$, we conclude that $T_i \leq \log b$, and

$$\|H\| \leq \|H_1\| + \|H_2\| + \|H_3\| + 2 \leq |W|_a + T_i + T_i + T_i + 2 \leq 2|W|_a + 3\log b.$$ 

Suppose that the step history is (2)(3), that is, $H \equiv H_2(23)H_3$, and, again, $H_i$ has rules only from step $i$, where $i = 2, 3$. Note that every rule of $H_2$ multiplies the $tk$-sector by a letter, and that sector in any word which is a domain of (23) must be empty. Hence, $\|H_2\| \leq |W|_a$. Every rule in $H_2$ and $H_3$ multiplies the $k't'$-sector by a letter, hence $\|H_3\| \leq |W|_a + \|H_2\| \leq 2|W|_a$. Therefore, $H \leq \|H_2\| + \|H_3\| + 1 \leq 3|W|_a + 1 \leq 4|W|_a$. This implies both (a) and (b).

Finally, suppose that the step history is (3). Then clearly $\|H\| \leq |W|_a$ and both (a) and (b) follow.

We conclude that in every case both (a) and (b) hold. 

\begin{lemma}
Suppose that an admissible word of $M_4$ is accepted by $M_4$, $H$ is a history of an accepting computation. Then $\|W\|_a \leq 4\|H\|$.
\end{lemma}

\begin{proof}
Indeed, $W \cdot H$ does not contain $a$-letters, and each rule of $H$ decreases the number of $a$-letters in the admissible word by at most 4 (every rule of $M_4$ affects at most four $a$-letters: two letters in the subword between $k$ and $k'$, one letter in the subword between $t$ and $k$ and one letter in the subword between $k'$ and $t'$).
\end{proof}

\begin{definition}
Let $Q_i$ be a base letter. (Recall that usually we take a representative $q_i \in Q_i$.) We say that $Q_i$ (or $q_i$) is active from the left or from the right for a rule $\theta$ if in the corresponding component $v_{i-1}q_iu_i \rightarrow v'_{i-1}q'_iu'_i$ of $\theta$, the word $v_{i-1}^{-1}v_i^{-1}$ or $u_i^{-1}u'_i$, respectively, is not trivial (and so equal to a letter the free group by Property 3.1(1) of $M_4$). If $q_i$ is active from the left or from the right for $\theta$, then we say that $q_i^{-1}$ is active from the right or left, respectively, for $\theta$. We also say that $q_i$ is active for $\theta$ if it is active from the left or active from the right. Otherwise, $q_i$ is passive for $\theta$.
\end{definition}

\begin{lemma}
Let a reduced computation of $M_4$ have history (12)$H$ and have the base $s_0p_1$. Suppose that the letter $p_1$ is active in every rule $\theta$ of Step 2 from $H$. Then every rule of $H$ is of Step 2.
\end{lemma}

\begin{proof}
Recall, that the rule (12) extends the start rule $\theta = \theta_{\text{start}}$ of $M_3$. Therefore, the first rule of $H$ has $p^0,1(1)$ on the left-hand side. If no rule of $H$ changes $p^0,1(1)$, then every rule is (the extension of) a rule of the machine $Z^\theta$ with the $p_1$-part of the form $p^0,1(1) \rightarrow a'p^0,1(1)(a')^{-1}$, where $p_1$ is active from the both sides. Otherwise, the history has a subword of type either (12)$H'(12)^{-1}$ where the $p_1$-part of every rule of $H'$ is of form $p^0,1(1) \rightarrow a'p^0,1(1)(a')^{-1}$ because the (copy of the) rule $\xi_2$ of $Z^\theta$ belongs to Step 2 but it is passive. However, this case is impossible since then every rule of $H'$ inserts (or deletes) one letter $a'$ in the $s_0p_1$-sector from the right, different rules insert different letters, and the $s_0p_1$-sector is empty when the rule (12) or (12)$^{-1}$ is applicable.
\end{proof}
Every rule either makes a control state letter $p_t$ active from both sides or locks a neighbor sector. This property is useful for computations with non-standard bases as in the following:

**Lemma 4.42.** If the base of a reduced computation of $M_4$ contains a subword $(pp^{-1}p)^{\pm 1}$, where $p$ is a control state letter, then all rules of the computation correspond to the copy of the $S$-machine $\overrightarrow{Z}$ or of $\overleftarrow{Z}$ containing that state letter, and either every rule is a copy of some $\xi_1(a)^{\pm 1}$ (a depends on the rule) or every rule is a copy of some $\xi_3(a)^{\pm 1}$.

Proof. Indeed, suppose that $p \in P_t$. Then every rule not from the copy of $\overrightarrow{Z}$ or of $\overleftarrow{Z}$ containing that state letter, locks the sector $s_{t-1}p_t$ or the sector $p_ts_t$, and the copies of $\xi_2$ and $\xi_4$ lock either sector $s_{t-1}p_t$ or sector $p_ts_t$. Now we can apply Lemma 3.4.

Lemma 4.38 and the following lemma show the role of the ‘historical’ $tk$- and $k't'$-sectors.

**Lemma 4.43.** Suppose that a reduced computation of $M_4$ with the standard base has the history of the form $(12)H_2(23)H_3(23)^{-1}H'_2(12)^{-1}$, where $H_2$ and $H'_2$ contain rules from Step 2, and $H_3$ has rules of Step 3. Then $\|H_3\| \leq \|H_2\| + \|H'_2\|$.

Proof. Let $W$ be the initial word of the computation. Since every rule of $H_2$ and $H'_2$ multiplies the $k't'$-sector by one letter which determines the rule, and every word in the domain of $(12)^{\pm 1}$ has that sector empty, we conclude that $\|H_2\|$ is equal to the length of the $k't'$-sector $U$ of $W \circ (12)H_2$, and $\|H'_2\|$ is equal to the length of the $k't'$-sector $V$ in $W \circ (12)H_2H_3$. Similarly, every rule from $H_3$ multiplies the $k't'$-sector by a letter that determines the rule. Hence, $\|H_3\| \leq \|U\| + \|V\| = \|H_2\| + \|H'_2\|$ as required.

4.4. The machine $M$

Consider now $2L \gg 1$ copies of the machine $M_4$, denote them by $M_4(i)$, $i = 1, \ldots, 2L$. We denote the state and tape letter of $M_4(i)$ accordingly, by adding index $i$ to all letters, and all rules. Let $\Theta(M_4)$ be the set of positive rules of $M_4$. Let $B$ be the standard base of $M_3$, $B(i)$ be the copy word $B$ with new extra index $i$ added to all letters, and $B_i = k(i)B(i)k'(i)$. We now consider the $S$-machine $M$ with the rules

$$\theta(M) = [\theta(1), \ldots, \theta(2L)], \quad \theta \in \Theta(M_4)$$

(we shall denote $\theta(M)$ by $\theta$ also) and the standard base

$$t_1B_1t'_2B_2^{-1}t_3B_3t'_4B_4^{-1} \cdots t'_{2L}B_{2L}^{-1}t_{2L+1},$$

(4.5)

where we identify the state $t$-letters $t(1)$ and $t'(1)$ of $M_4(1)$ with $t_1$ and $t_2$, respectively, the $t$-letters $t(2)$ and $t'(2)$ of $M_4(2)$ with $t_3^{-1}$ and $(t'_2)^{-1}$, respectively an so on. Moreover, we identify $t_{2L+1}$ with $t_1$ and consider the standard base of $M$ up to cyclic permutations which may start with any $t$-letter and end with the same $t$-letter. The stop word is defined accordingly (every letter in the standard base is replaced by the corresponding letter in the stop word of $M_4(i)$). The stop word without the last letter $t_{2L+1}$ is called the hub. We may also take the hub up to cyclic permutations.

This construction is similar to the construction in [14, 19], though the application of mirror copies of machines goes back to the works of Boon and P.S. Novikov (see [17]). The condition $L \gg 1$ makes the hub graph hyperbolic (see Lemmas 5.18 and 5.19) and the mirror symmetry of the word (4.5) is used for the surgery that we define in Section 12.2.
For every admissible word \( W \) of \( M_4 \) with the standard base, we denote by \( W(M) \) the corresponding admissible word \( t_1k(1)W(1)k'(1)t'_2k'(2)^{-1}W(2)^{-1}k(2)^{-1}t_2 \ldots \) of \( M \) with the standard base (of \( M \)). By definition, \( W(M) \) is an input (the accept) word of \( M \) if \( W \) is an input (the accept) word of \( M_4 \).

The letters in the copy \( W(i) \) of the word \( W \) are equipped with the extra index \( i \). Thus, every \( a \)-letter and every \( q \)-letter (except for \( t \) and \( t' \)-letters), and every letter \( \theta(i) \) of the alphabets of \( M \) has this extra index. We call it the \( M \)-index of the letter and take it modulo \( 2L \).

**Remark 4.44.** (1) Note that for every rule \( \theta \) of \( M_4 \) and every admissible word \( W \) of \( M_4 \) with the standard base of \( M_4 \), we have \( W \cdot \theta = W' \) if and only if \( W(M) \cdot \theta = W'(M) \).

(2) Also note that \( \|W(M)\| < 2L\|W\| \) for every \( W \).

(3) Both machines \( M_4 \) and \( M \) enjoy Property 3.1(1) but not Property 3.1(2).

(4) The unique start and accept rules of the machines \( M_1, M_2 \) and \( M_3 \) are converted to the transition rules (12) and (34) of \( M_4 \) and \( M \). So, there are no specific start and accept rules of \( M_4 \) and \( M \). In particular, \( M \) accepts if it reaches the hub.

Remark 4.44(1) and Lemma 4.36 immediately imply the following lemma:

**Lemma 4.45.** Let \( X_5 \) be the set of all words of the form \( W(M) \), \( W \in X_4 \). Then for every input word \( W \) of \( M_4 \), \( W \) is accepted by \( M_4 \) if and only if \( W(M) \) is accepted by \( M \) if and only if \( W \in X_4 \). Hence, the set \( X_5 \) is not recursive.

**Remark 4.46.** Considered as a cyclic word, the hub has the following symmetries: it does not change if we reflect it about any \( t \)-letter or any \( t' \)-letter (with indices changing appropriately, and state letters, except for \( t \) and \( t' \)-letters, replaced by their inverses). From Remark 4.44(1), it follows, that every admissible accepted word of \( M \) has similar symmetries.

5. **Groups and diagrams**

Every \( S \)-machine can be considered as a finitely presented group (see \([19]\) and also \([13, 15]\)). Here, we apply the construction to the machine \( M \). To simplify formulas, it is convenient to redefine \( N \) once again. From now on, we shall denote by \( N + 1 \) the length of the smallest subword of the hub containing two \( t \)-letters. Thus, the length of the hub is \( LN \), \( Q = \bigcup_{i=0}^{LN} Q_i \) (where \( Q_{LN} = Q_0 \) \( Y = \bigcup_{i=1}^{LN} Y_i \), and \( \Theta \) is the set of rules of the \( S \)-machine \( M \). (But we will remember that, as for the machine \( M_4 \), the state letters of \( M \) are partitioned into the subsets of \( t \)-letters, \( t' \)-letters, \( k \)-letters, \( k' \)-letters, \( s \)-letters and \( p \)-letters.)

The finite set of generators of the group \( M \) (the same letter as for the machine) consists of \( q \)-letters corresponding to the states \( Q \), \( a \)-letters corresponding to the tape letters from \( Y \), and \( \theta \)-letters corresponding to the rules from the positive part \( \Theta^+ \) of \( \Theta \).

The relations of the group \( M \) correspond to the rules of the machine \( M \); for every \( \theta = [U_0 \rightarrow V_0, \ldots U_{LN} \rightarrow V_{LN}] \in \Theta^+ \), we have

\[
U_i \theta_{i+1} = \theta_i V_i, \quad \theta_j a = a \theta_j, \quad i, j = 0, \ldots, LN
\]

for all \( a \in \tilde{Y}_j(\theta) \). (Here, \( \theta_{LN} \equiv \theta_0 \).) The first type of relations will be called \((\theta, q)\)-relations, the second type \((\theta, a)\)-relations.

Finally, the required group \( G \) is given by the generators and relations of the group \( M \) and by one more additional relation, namely the hub-relation

\[
W_M = 1,
\]

where \( W_M \) is the hub, that is, the accept word (of length \( LN \)) of the machine \( M \).
Remark 5.1. The word $W_M$ has the symmetries mentioned in Remark 4.46. Since the machine $M$ is built of $2L$ copies $M_4(i)$, the set of relations is also symmetric in the following sense. Every relation $\theta_ja = a\theta_j$ from (5.1) has $2L$ copies (including itself) corresponding to different $M_4(i)$-s. If the relation $U_i\theta_{t+1} = \theta_iV_i$ from (5.1) involves neither $t$- nor $t'$-letters, then it has $L$ copies (including itself) and $L$ mirror copies. Every relation containing a $t$- or a $t'$-letter (denote this letter by $t$) has the form $\theta_{t+1} = \theta_t\tilde{t}$, that is, it contains no $a$-letters.

5.1. Minimal diagrams

Recall that a van Kampen diagram $\Delta$ over a presentation $P = \langle A \mid R \rangle$ (or just over the group $P$) is a finite oriented connected and simply connected planar 2-complex endowed with a labeling function $\text{Lab} : E(\Delta) \to A^{\pm 1}$, where $E(\Delta)$ denotes the set of oriented edges of $\Delta$, such that $\text{Lab}(e^{-1}) \equiv \text{Lab}(e)^{-1}$. Given a cell (that is a 2-cell) $\Pi$ of $\Delta$, we denote by $\partial\Pi$ the boundary of $\Pi$; similarly, $\partial\Delta$ denotes the boundary of $\Delta$. The labels of $\partial\Pi$ and $\partial\Delta$ are defined up to cyclic permutations. An additional requirement is that the label of any cell $\Pi$ of $\Delta$ is equal to $(\text{a cyclic permutation of})$ a word $R^{\pm 1}$, where $R \in R$. The label and the combinatorial length $|p|$ of a path $p$ are defined as for Cayley graphs.

The van Kampen Lemma states that a word $W$ over the alphabet $A^{\pm 1}$ represents the identity in the group $P$ if and only if there exists a diagram $\Delta$ over $P$ such that $\text{Lab}(\partial\Delta) \equiv W$, in particular, the combinatorial perimeter $|\partial\Delta|$ of $\Delta$ equals $|W|$ (see [7, Chapter 5, Theorem 1.1]). The word $W$ representing 1 in $P$ is freely equal to a product of conjugates to the words from $R^{\pm 1}$. The minimal number of factors in such products is called the area of the word $W$. The area of a diagram $\Delta$ is the number of cells in it. A diagram having the smallest number of cells among all diagrams with the same boundary label is called minimal. By van Kampen’s Lemma, $\text{Area}(W)$ is equal to the area of a minimal diagram $\Delta$ over $P$ with $\text{Lab}(\partial\Delta) \equiv W$. These definitions imply the following.

Lemma 5.2. Assume that a diagram $\Delta_0$ is divided into two subdiagrams $\Delta_1$ and $\Delta_2$ by a simple path $p$. Let a minimal diagram $\Delta$ have the same boundary label as $\Delta_0$. Then $\text{Area}(\Delta) \leq \text{Area}(\Delta_0) = \text{Area}(\Delta_1) + \text{Area}(\Delta_2)$.

We will study diagrams over the groups $M$ and $G$. The edges labeled by state letters (= $q$-letters) will be called $q$-edges, the edges labeled by tape letters (= $a$-letters) will be called $a$-edges and the edges labeled by $\theta$-letters are $\theta$-edges.

Remark 5.3. The symmetries of relations observed in Remark 5.1 make possible the following construction for given $i \leq 2L$ and a diagram $\Delta$ over $M$. Let $\nabla$ be a mirror copy of the map $\Delta$. For every edge $e$ of $\Delta$ whose label is equipped with an $M$-index ($j$) (that is, if $e$ is neither $t$- nor $t'$-edge), the mirror copy of $e$ in $\nabla$ is marked by the same letter but with $M$-index equal to $(2i - j - 1)$. The label $t_j$ of $e$ (the label $t'_j$) should be replaced for the mirror image by $t_{2i-j}^{-1}$ (by $(t_{2i-j})^{-1}$, respectively). It is easy to see that $\nabla$ is also a diagram over $M$. We say that $\nabla$ is obtained by $t_i$-reflection from $\Delta$. Similarly one can speak on $t_i$-reflections for paths of $\Delta$.

We denote by $|p|_a$ (by $|p|_\theta$, by $|p|_q$) the $a$-length (respectively, the $\theta$-length, the $q$-length) of a path/word $p$, that is, the number of $a$-edges/letters (the number of $\theta$-edges/letters, the number of $q$-edges/letters) in $p$. 

The cells corresponding to Relation (5.2) are called hubs, the cells corresponding to \((\theta, q)\)-relations are called \((\theta, q)\)-cells, and they are called \((\theta, a)\)-cells if they correspond to \((\theta, a)\)-relations.

Every minimal van Kampen diagram is reduced, that is, it does not contain two cells (= closed 2-cells) that have a common edge and are mirror images of each other (if such pairs of cells exist, they can be removed to obtain a diagram of smaller area and with the same boundary label). To study (van Kampen) diagrams over the group \(G\), we shall use their simpler subdiagrams such as bands and trapezia, as in [1, 11, 19], etc. Here, we repeat one more necessary definition.

**Definition 5.4.** Let \(Z\) be a subset of the set of generators \(X\) of the group \(M\). A \(Z\)-band \(B\) is a sequence of cells \(\pi_1, \ldots, \pi_n\) in a reduced van Kampen diagram \(\Delta\) such that we have the following:

(i) Every two consecutive cells \(\pi_i\) and \(\pi_{i+1}\) in this sequence have a common edge \(e_i\) labeled by a letter from \(Z\).

(ii) Each cell \(\pi_i\), where \(i = 1, \ldots, n\) has exactly two \(Z\)-edges, \(e_{i-1}\) and \(e_i\) (that is, edges labeled by a letter from \(Z\)).

(iii) If \(n = 0\), then \(B\) is just a \(Z\)-edge.

The counterclockwise boundary of the subdiagram formed by the cells \(\pi_1, \ldots, \pi_n\) of \(B\) has the factorization \(e^{-1}q_1f_1q_2^{-1}\) where \(e = e_0\) is a \(Z\)-edge of \(\pi_1\) and \(f = e_n\) is a \(Z\)-edge of \(\pi_n\). We call \(q_1\) the bottom of \(B\) and \(q_2\) the top of \(B\), denoted \(\text{bot}(B)\) and \(\text{top}(B)\), respectively. Top/bottom paths and their inverses are also called the sides of the band. The \(Z\)-edges \(e\) and \(f\) are called the start and end edges of the band. If \(n \geq 1\) but \(e = f\), then the \(Z\)-band is called a \(Z\)-annulus.

We will consider \(q\)-bands, where \(Z\) is one of the sets \(Q_i\) of state letters for the machine \(M\), \(\theta\)-bands for every \(\theta \in \Theta\), and \(a\)-bands, where \(Z = \{a\} \subseteq Y\). The convention is that \(a\)-bands do not contain \((\theta, q)\)-cells, and so they consist of \((\theta, a)\)-cells only.

**Remark 5.5.** To construct the top (or bottom) path of a band \(B\), at the beginning one can just form a product \(x_1 \ldots x_n\) of the top paths \(x_i\)-s of the cells \(\pi_1, \ldots, \pi_n\) (where each \(\pi_i\) is a \(Z\)-band of length 1). No \(\theta\)-letter is being canceled in the word \(W \equiv \text{Lab}(x_1), \ldots, \text{Lab}(x_n)\) if \(B\) is a \(q\)- or \(a\)-band since otherwise two neighbor cells of the band would be mirror copies of each other which is impossible in a reduced diagram.

Also there are no cancellations of \(a\)-letters if \(B\) is a \(q\)-band. Indeed, if both \(\pi_i\) and \(\pi_{i+1}\) have \(a\)-edges on their top, then the corresponding rules of \(M\) must belong to the same step since every cell is passive for (12)- and (23)-rules. Similarly, they correspond to the rule of the same machine \(\overline{Z}^{(\theta, i)}\) or \(\overline{Z}^{(\theta, i)}\) if \(B\) is a \(q\)-band for some control letter \(p_i\). Since the rules \(\xi_2\) and \(\xi_4\) provide no active cells. Then the rules are determined by the \(a\)-letters, and the cells should be mirror copies as in the previous paragraph. Similar argument works if \(q\) corresponds to any other letter of the standard base except for \(s_1\). But active \(s_j\)-cells cannot have a common edge too since this edge has a \(\theta\)-index in the label, and so the diagram is not reduced again.

Thus, if \(B\) is a \(q\)-band (or an \(a\)-band), then the top/bottom label is a product \(x_1 \ldots x_n\). If \(B\) is a \(\theta\)-band then a few cancellations of \(a\)-letters (but not \(\theta\)-letters) are possible in \(W\). (This can happen if one of \(\pi_i\), \(\pi_{i+1}\) is a \((\theta, q)\)-cell and another one is a \((\theta, a)\)-cell.) We will always assume that the top/bottom label of a \(\theta\)-band is a reduced form of the word \(W\). This property is easy to achieve: by folding edges with the same labels having the same initial vertex, one can make the boundary label of a subdiagram in a van Kampen diagram reduced (for example, see [19]).

If the path \((e^{-1}q_1f_1)^{\pm 1}\) or the path \((f_1q_2^{-1}e^{-1})^{\pm 1}\) is the subpath of the boundary path of \(\Delta\), then the band is called a rim band of \(\Delta\). We shall call a \(Z\)-band maximal if it is not contained...
in any other \( Z \)-band. Counting the number of maximal \( Z \)-bands in a diagram we will not distinguish the bands with boundaries \( e^{-1}q_1f q_2^{-1} \) and \( f q_2^{-1}e^{-1}q_1 \), and so every cell having two \( Z \)-edges belongs to a unique maximal \( \mathcal{Z} \)-band.

We say that a \( Z_1 \)-band and a \( Z_2 \)-band cross if they have a common cell and \( Z_1 \cap Z_2 = \emptyset \).

Sometimes, we specify the types of bands as follows. A \( \theta \)-band corresponding to the transition rule (12) (to (23)) is called a \((12)\)-band (\((23)\)-band), and it consists of \((12)\)-cells of (23)-cells). A \( q \)-band corresponding to one of the letters \( t_i \) of the base (4.5) (respectively, to \( t'_i \), \( k(i) \), \( k'(i) \)) is called a \( t \)-band (\( t' \)-band, \( k \)-band, \( k' \)-band) since the \( M \)-index \( i \) is, generally, not important for further considerations (but we may keep it if it is essential). Similarly, we can omit the \( M \)-index speaking on \( s \)- and \( p \)-bands, but we distinguish different letters of each particular \( B_i \) in the standard base, for example, the \( s_0 \)-letter follows after the \( k \)-letter in each subword \( B_i \); hence the standard base (4.5) has 2\( L \) different \( s_0 \)-letters (one in each subword \( B_i \)), 2\( L \) different \( p_1 \)-letters, and so on. Also this agreement allows us to speak on \((12)\)-letters and \((12)\)-edges, \ldots, \( p_r \)-letters, \( p_r \)-edges, and \( p_r \) (or \( s_r \)) bands.

The papers \([1, 12, 16]\) contain the proof of the following lemma in a more general setting. (In contrast to \([12, \text{Lemma 6.1}] \) and \([16, \text{3.11}] \), we have no \( x \)-cells here.)

**Lemma 5.6.** A reduced van Kampen diagram \( \Delta \) over \( M \) has no \( q \)-annuli, no \( \theta \)-annuli, and no \( a \)-annuli. Every \( \theta \)-band of \( \Delta \) shares at most one cell with any \( q \)-band and with any \( a \)-band.

If \( W \equiv x_1 \ldots x_n \) is a word in an alphabet \( X \), \( X' \) is another alphabet, and \( \phi : X \rightarrow X' \cup \{1\} \) (where 1 is the empty word) is a map, then \( \phi(W) \equiv \phi(x_1) \ldots , \phi(x_n) \) is called the projection of \( W \) onto \( X' \). We shall consider the projections of words in the generators of \( M \) onto \( \Theta \) (all \( \theta \)-letters map to the corresponding element of \( \Theta \), all other letters map to 1), and the projection onto the alphabet \( \{ Q_0 \cup \cdots \cup Q_{L N - 1} \} \) (every \( q \)-letter maps to the corresponding \( Q_i \), all other letters map to 1).

**Definition 5.7.** The projection of the label of a side of a \( q \)-band onto the alphabet \( \Theta \) is called the history of the band. The step history of this projection is the step history of the \( q \)-band. The projection of the label of a side of a \( \theta \)-band onto the alphabet \( \{ Q_0, \ldots, Q_{L N - 1} \} \) is called the base of the band; that is, the base of a \( \theta \)-band is equal to the base of the label of its top or bottom.

As for words, we will use representatives of \( Q_j \)-s in base words. (If \( p \in Q_4 \), \( s \in Q_5 \), we shall say that the word pas has base ks instead of \( Q_4Q_5 \), and so on.)

**Definition 5.8.** Let \( \Delta \) be a reduced diagram over \( M \) which has the boundary path of the form \( p_1^{-1}q_1p_2q_2^{-1} \), where \( p_1 \) and \( p_2 \) are sides of \( q \)-bands, and \( q_1 \), \( q_2 \) are maximal parts of the sides of \( \theta \)-bands such that \( \text{Lab}(q_1) \), \( \text{Lab}(q_2) \) start and end with \( q \)-letters.
Then $\Delta$ is called a trapezium. The path $q_1$ is called the bottom, the path $q_2$ is called the top of the trapezium, the paths $p_1$ and $p_2$ are called the left and right sides of the trapezium. The history (step history) of the $q$-band whose side is $p_2$ is called the history (respectively, step history) of the trapezium; the length of the history is called the height of the trapezium. The base of $\text{Lab}(q_1)$ is called the base of the trapezium.

Remark 5.9. Note that the top (bottom) side of a $\theta$-band $T$ does not necessarily coincide with the top (bottom) side $q_2$ (side $q_1$) of the corresponding trapezium of height 1, and $q_2$ ($q_1$) is obtained from $\text{top}(T)$ (respectively $\text{bot}(T)$) by trimming the first and the last $a$-edges if these paths start and/or end with $a$-edges. We shall denote the trimmed top and bottom sides of $T$ by $t\text{top}(T)$ and $t\text{bot}(T)$. By definition, for arbitrary $\theta$-band $T$, $t\text{top}(T)$ is obtained by such a trimming only if $T$ starts and/or ends with a $(\theta,q)$-cell; otherwise, $t\text{top}(T) = \text{top}(T)$. The definition of $t\text{bot}(T)$ is similar.

By Lemma 5.6, any trapezium $\Delta$ of height $h \geq 1$ can be decomposed into $\theta$-bands $T_1, \ldots, T_h$ connecting the left and the right sides of the trapezium. The word written on the trimmed top side of one of the bands $T_i$ is the same as the word written on the trimmed bottom side of $T_{i+1}$, where $i = 1, \ldots, h$. Moreover, the following lemma claims that every trapezium simulates the work of $M$. It summarizes the assertions of Lemmas 6.1, 6.3, 6.9 and 6.16 from [14]. For the formulation (1) below, it is important that $M$ is an $S$-machine. The analog of this statement is false for Turing machines. (See [13] for a discussion.)

Lemma 5.10. (1) Let $\Delta$ be a trapezium with history $\theta_1 \ldots \theta_d$ ($d \geq 1$). Assume that $\Delta$ has consecutive maximal $\theta$-bands $T_1, \ldots, T_d$, and the words $U_j$ and $V_j$ are the trimmed bottom and the trimmed top labels of $T_j$, where $j = 1, \ldots, d$. Then $U_j$ and $V_j$ are admissible words for $M$, and

$$V_1 \equiv U_1 \cdot \theta_1, U_2 \equiv V_1, \ldots, U_d \equiv V_{d-1}, V_d \equiv U_d \cdot \theta_d.$$ 

(2) For every reduced computation $U \rightarrow \cdots \rightarrow U \cdot H \equiv V$ of $M$ with $\|H\| \geq 1$ there exists a trapezium $\Delta$ with bottom label $U$, top label $V$, and with history $H$.

If $H' \equiv \theta_1, \ldots, \theta_j$ is a subword of the history $\theta_1 \ldots \theta_d$ from Lemma 5.10(1), then the bands $T_1, \ldots, T_j$ form a subtrapezium $\Delta'$ of the trapezium $\Delta$. This subtrapezium is uniquely defined by the subword $H'$ (more precisely, by the occurrence of $H'$ in the word $\theta_1 \ldots \theta_d$), and $\Delta'$ is called the $H'$-part of $\Delta$.

5.2. Properties of the group $M$

In this subsection, we want to translate the properties of the machine $M$ in the language of diagrams over the group $M$.

Recall that every $(\theta, q)$-cell $\pi$ has a boundary label of the form $U_i \theta_{i+1} V_i^{-1} \theta_i^{-1}$ (see Relations (5.1)), where the word $U_i$ (the word $V_i$) has exactly one positive $q$-letter $q_i \in Q_i$ ($q_i' \in Q_i$). Hence, the boundary label of $\pi$ is $q_i w_1 (q'_i)^{-1} w_2$ for some words $w_1$ and $w_2$.

Definition 5.11. The cell $\pi$ considered as a one-cell $q$-band with base $q_i$ is called active from the right (from the left) if the word $w_1$ (the word $w_2$) has at least one $a$-letter. If $\pi$ with base $q_i$ is active from the right (from the left) then, by definition, the same cell considered as a
A \( q \)-band with base \( q_i \) is active from the left (respectively, from the right). A \((\theta, q)\)-cell is called passive if it is not active either from the left or from the right.

The comparison with Definition 4.40 shows that the cell \( \pi \) with base \( q_i^{\pm 1} \) is active from the left (respectively, active from the right, passive) if and only if the base letter \( q_i^{\pm 1} \) is active on the left (respectively, active on the right, passive) for the rule corresponding to the \( \theta \)-edges of \( \pi \).

**Definition 5.12.** We say that a \( q \)-band with base \( q_i \) is **active from the left (from the right)** if every cell (with the same base) except for the first cell and the last one (if the first and/or the first cell corresponds to the rules \((12)^{\pm 1} \) or \((23)^{\pm 1} \)) is active from the left (from the right). A \( q \)-band is called passive if each of its cells is passive. Similarly, one can speak of a \( q \)-band with base \( q_i \) which is passive from the left or passive from the right.

**Remark 5.13.** The letter \( s_0 \) in the standard base of \( M_4 \) corresponds to the left-most \( \alpha \)-marker of the machines \( M_1-M_4 \), and so every \( s_0 \)-band is passive (from both sides).

**Definition 5.14.** We say that a \( q \)-band \( C \) with base \( q_i \) is **strongly active from the left (respectively right)** if every cell \( \pi \) is active from the left (from the right), \( \partial \pi \) has exactly one \( a \)-edge on the left side (right side) of \( C \), and these \( a \)-letters are different for the cells corresponding to different rules of the history of \( C \).

**Lemma 5.15.** Let \( \Delta \) be any reduced diagram over \( M \). Let \( C \) be a \( q \)-band, corresponding to the part \( Q_i \) of \( Q \). Suppose that \( C \) is strongly active from the left or from the right. Then \( \Delta \) does not have an \( a \)-band starting and ending on the left side (respectively, right side) of \( C \).

**Proof.** Suppose that an \( a \)-band \( A \) starts and ends on \( \text{bot}(C) \) which is the left side of \( C \). Let \( \Delta' \) be the subdiagram bounded by \( A \) and \( C \).

Then \( \Delta' \) has no other maximal \( q \)-bands except the part \( C' = C \cap \Delta' \) because \( a \)-bands and \( q \)-bands do not intersect and \( \Delta' \) has no \( q \)-annuli by Lemma 5.6. Since maximal \( a \)-bands do not intersect, we can assume, without loss of generality, that \( \Delta' \) does not have any other \( a \)-bands starting and ending on \( \text{bot}(C') \). Since \( C' \) is strongly active on the left, and the sides of \( A \) consist of \( \theta \)-edges, we conclude that \( C' \) consists of two cells having common \( q \)- and \( a \)-edges. A \( \theta \)-cell in \( C' \) is completely determined by its \( a \)-letter on its bottom side (see Remark 5.5 for the argument). Therefore, those two \( q \)-cells cancel, a contradiction with the assumption that \( \Delta \) is reduced.

The next proposition summarizes previously proved properties of the various submachines of the \( S \)-machine \( M \). We formulate these properties in the language of van Kampen diagrams, which makes it more convenient to apply these properties to the group \( G \).

Recall that the standard base of \( M_3 \) is denoted by \( B \). Note that the standard base of \( M \) contains \( L \) copies \( B(i) \) of \( B \) \((i = 1, 3, \ldots, 2L - 1) \) and \( L \) copies of \( B^{-1} \). We call the base of an admissible word of \( Maligned \), if every maximal subword of this base without letters \( t, k, k' \) and \( t' \) is a subword of a copy of \( B^{\pm 1} \).

**Remark 5.16.** Since \( \|B\| < N/2 \), Formula 4.5 and the definition of admissible words show that every aligned base of length at least \( N/2 \) must contain a \( k^{\pm 1} \)- or a \((k')^{\pm 1}\)-letter or entirely consists of \( t^{\pm 1} \)- or \((t')^{\pm 1}\)-letters.
A base of an admissible word of \( M \) is called normal if it is a subword of a power of the base of the hub. (Recall that \( t_1 \) and \( t_{2L+1} \) were identified in the definition of the machine \( M \).)

A base is called large if it contains a copy of \( B^{±1} \).

We shall say that a \((\theta,q)\)-cell \( \pi \) in a van Kampen diagram over \( M \) is odd if it contains exactly one \( a \)-edge on its boundary, and its base is not \( k \) or \( k' \). A \( \theta \)-band with \((1\text{-letter})\) history of type (2) is called odd if it contains odd cells.

A trapezium over \( M \) whose top label is one of \( 2L \) copies of the stop word of \( M_4 \) will be called a \( M_4 \)-accepting trapezium. (The trapezium pictured in Section 4.3 is \( M_4 \)-accepting and in addition, its bottom label is an input word of \( M_4 \).) A trapezium whose base is a copy of the standard base of \( M_4 \) is standard if its bottom label is in the domain of the rule (12) and its top label is in the domain of \((23)^{−1}\). By Lemmas 4.33 and 5.10, every standard trapezium has height \( T_i \) for some \( i \) and corresponds to a standard computation of \( M_4 \).

**Proposition 5.17.** The following properties of the group \( M \) hold. In all these properties, we assume that we are given a reduced van Kampen diagram \( \Delta \) over \( M \), all bands, cells, and edges are bands, cells, and edges of that diagram.

(i) A two letter base of a \( \theta \)-band is either a subword of the word \((4.5)\) or of the inverse word, or it has form \( q^{±1}q^{±1} \) for a base letter \( q \).

(ii) Every cell with base \( k \) (every cell with base \( k' \)) corresponding to a rule of Step 1 or Step 2 except for the \((12)^{±1}\)-rule (corresponding to a rule of Step 2 or Step 3 except for the \((23)^{±1}\)-rule), is active from the left (respectively, from the right) and passive from the right (respectively, from the left). Every non-\( k \)-cell (non-\( k' \)-cell) corresponding to a rule of Step 1 (respectively, of Step 3) is passive. Every \( t \), \( t' \), or \( s_0 \)-band is passive.

(iii) (a) The boundary of every cell has at most two \( a \)-edges. It has either 0 or 2 \( a \)-edges if it is a \((\theta,q)\)-cell corresponding to a control letter \( p_i \), otherwise it has at most one \( a \)-edge.

(b) If there are two \( a \)-letters \( a \) and \( a' \) in the boundary label of a cell \( \pi \), then \( a' \) is a copy of \( a^{-1} \) and \( \pi \) is either a \((\theta,a)\)-cell or a \((\theta,q)\)-cell corresponding to a control letter \( p_i \), and the \( a \)-edges are separated by \( q \)-edges in \( \partial \pi \).

(c) A \((\theta,a)\)-cell has two mutually inverse \( \theta \)-letters in the boundary label.

(d) Two \((\theta,q)\)-cells corresponding to control letters \( p_i \) and \( p_j \) with \( i \neq j \) have no common \( a \)-letter in the boundary labels.

(iv) If a \( \theta \)-band \( T \) has three consecutive cells \( \pi_1, \pi_2 \) and \( \pi_3 \), where \( \pi_2 \) corresponds to a control letter \( p_i^{±1} \) and \( \pi_2 \) is not active from both sides, then one of the cells \( \pi_1, \pi_3 \) is a \((\theta,q)\)-cell whose base is an \( s \)-letter.

(v) Let a \((\theta,q)\)-cell \( \pi_i \) of a \( \theta \)-band \( T \) have base \( q \). If \( T \) corresponds to a rule of Step 1 or to (12) (respectively, Step 3 or to (23)), and the next cell, \( \pi_{i+1} \) in \( T \) is a \((\theta,a)\)-cell, then \( q \) can be only one of the following letters: \( t, p_i, k^{-1}, s_i^{-1} \) (respectively, \( k' \) or \((t')^{-1} \)) (with some indices). For other values of \( q \), the next letter after \( q \) in the base of \( T \) cannot be \( q^{-1} \).

(vi) If in the base of a \( \theta \)-band, there is a subword \( p_i^{±1}p_i^{±1}p_i^{±1} \) for some control letter \( p_i \) and there are neither \( k^{±1} \) nor \((k')^{±1} \)-letters, then the active cells in this band are precisely the \( p_i \)-cells, and these cells are active from both sides.

(vii) Suppose that \( C \) is a \( k^{±1} \), \((k')^{±1} \), or \( p_i^{±1} \)-band with top path \( y \). Suppose that each cell of \( C \) has a common \( a \)-edge with \( y \). Then no \( a \)-band of \( \Delta \) can start and end on \( y \).

In the remaining parts of the Proposition, \( \Delta \) is a trapezium.

(viii) If \( \Delta \) has base \( k't \) (base \( tk \)), then it cannot have step history \((12)(2)(12)^{−1} \) (respectively, \((23)^{−1}(2)(23) \)).
(ix) If \( p_1^{-1}s_0^{-1} \) is a subword of the base of \( \Delta \), and the history of \( \Delta \) has the form \((12)H\), then \( H \) is of type (2), it has no rules \((12)^{\pm1}\), \((23)^{\pm1}\), and in the \( H \)-part of \( \Delta \), all \( p_i \)-cells are active both from the left and from the right.

(x) Suppose that \( \Delta \) is \( M_4 \)-accepting. Then the step history of \( \Delta \) is a subword of \((2)(1)(2)(3)\). If \( \Delta \) is the label of the bottom path of \( \Delta \) and \( h \) is the height of \( \Delta \), then \( \|W\|_a \leq 4h \).

(xi) If the history of \( \Delta \) contains \((12)^{\pm1}\) and \((23)^{\pm1}\), then the base of \( \Delta \) is normal.

(xii) If (a) the length of the base of \( \Delta \) is at least \( N \) and its history contains both a rule \((12)^{\pm1}\) and a rule \((23)^{\pm1}\), or (b) the base of \( \Delta \) is standard, then the step history of \( \Delta \) is a subword of \((2)(1)(2)(3)(2)(1)(2)\).

(xiii) Suppose that the base of \( \Delta \) is not aligned and the history is of type (2). Then the label of the top (and of the bottom) of every maximal \( s_j \)-band of \( \Delta \) admits a factorization of the form \( u(b_1v_1b_1^{-1})\cdots(b_mv_mb_m^{-1})w \) where \( b_i^{\pm1} \) is an a-letter or 1 \((i = 1, \ldots, m)\), \( v_i \) is a group word in \( \theta \)-letters, \( b_i \) commutes with every letter of \( v_i \) by virtue of \((\theta, a)\)-relations, and each of \( u \) and \( w \) has at most one a-letter.

(xiv) If the base of \( \Delta \) is large, and its history has the form \( H^3 \), then \( \Delta \) does not have odd cells \( \pi \).

(xv) Suppose that the base of \( \Delta \) has the form \( k^{-1}k' \) or \( k'(k')^{-1} \), and all \( k \)-cells (respectively \( k' \)-cells) of \( \Delta \) are active. Let \( W \) and \( W' \) be the labels of the bottom and top of \( \Delta \), respectively. Then the history of \( \Delta \) has the form \( H_1H_2H_3 \), where \( k \geq 0 \), \( \|H_1\| \leq \|W\|_a/2 \), \( \|H_2\| \leq \min(\|W\|_a, \|W'|_a) \), \( \|H_3\| \leq \|W'|_a/2 \).

(xvi) If the base of \( \Delta \) is of length at least \( N \) and \( \Delta \) has the step history

\[
(12)(2)(23)(3)(23)^{-1}(2)(12)^{-1},
\]

then the height of the \((23)(3(23)^{-1})\)-part of \( \Delta \) is less than the sum of heights of the \((12)(2)-\) and \((2)(12)^{-1}\)-parts of it.

(xvii) Let \( m > 0 \) be an integer such that for every standard trapezium with a bottom label \( W \), inequality \( \|W\|_a \leq m \) implies \( \|H\| < \log m \). Suppose that \( \Delta \) is \( M_4 \)-accepting, the history of \( \Delta \) is \( H \), and the bottom label \( W' \) satisfies the inequality \( \|W'|_a \leq m \). Then we have

\[
\|H'\| \leq 4\|W'|_a + 3\log m.
\]

(xviii) The set of numbers \( m \) satisfying the assumption from (xvii) is infinite.

Proof. (i) This follows from the definition of admissible word and from Lemma 5.10.

(ii) and (iii) This follows from the definition of the rules of \( M \), the definitions of Relations (5.1), and from Remark 5.13.

(iv) Indeed, if a component \( p_i \to \cdots \to p_i \) of a rule from \( M \) is not active from both sides, then it locks either \( s_{i-1}p_i \)-sector or the \( p_is_i \)-sector, and we can apply Property (i) and Lemma 3.4.

(v) Indeed, the rules from Step 1 and the rule \((12)\) (respectively Step 3 and the rule \((23)\)) lock all sectors except the \( tk \)-sectors, and \( p_is_i \)-sectors (respectively the \( k't' \)-sectors) of the admissible words of \( M \). It remains to use Lemma 3.4.

(vi) This also follows from Lemma 3.4: if a rule of \( M \) does not lock the \( p_is_i \)-sectors or \( s_{i-1}p_i \)-sectors, then its component involving \( p_i \) has the form \( p_i \to ap'b \), where \( a \) and \( b \) are tape letters, and all other components, except for \( k \)- and \( k' \)-components, do not involve tape letters.

(vii) The condition means that the band \( C \) is active on the left and has no (passive) \((12)\)-or \((23)\)-cells. It follows from the definition of \( M \) that the \( k^{\pm1}, (k')^{\pm1} \), or \( p_i \)-band is strongly active on the left. It remains to apply Lemma 5.15.

(viii) This follows from Lemmas 4.32(a,b) and 5.10.
(ix) Let us apply Lemma 5.10 and consider the reduced computation corresponding to $\Delta$. The rule (12) switches on the copy of the machine $\bar{Z}^{\theta_1}$ where $\theta = \theta_{\text{start}}$ is the start rule of $M_3$. The (copies of the) rules of the form $\zeta_1(a)^{\pm 1}$ cannot follow by the (copy of the) rule $\zeta_2$ since $\zeta_2$ locks the $p_{1s1}$-sector. Also, by Lemma 3.5, it cannot follow by the rules (12)$^{\pm 1}$ or (23)$^{\pm 1}$ locking the $s_{0p1}$-sector. Therefore, each of the rules of $H$ is of the form $\zeta_1(a)^{\pm 1}$, and the statement follows.

(x) The first statement follows from Lemma 4.34(b) because $\Delta$ is the trapezium corresponding to an accepting computation of a copy of the machine $M_4$. The second property immediately follows from Lemma 4.39.

(xi) Indeed, every sector of the standard base of $M$ is locked by either (12) or (23). It remains to use (i) and Lemma 3.4.

(xii) Indeed, by Property (xi), the base of $\Delta$ is normal. Since its length is at least $N$, it must contain a copy of the base of $M_4$, and it remains to use Lemma 4.34(a).

(xiii) The base has non-aligned subword $B_0$ without $k$- and $t$-letters. Hence, the copy of $B_0^{\pm 1}$ is not a subword of the standard base of the machine $M_3$. If $H$ is the history of the corresponding computation of $M_3$, then by Lemma 4.28(2), we have $\Pi_{32}(H) \equiv (\theta^{-1})(\theta_1\theta_1^{-1})\ldots(\theta_m\theta_m^{-1})(\theta')$ for some positive rules $\theta, \theta_1, \ldots, \theta'$ of $M_2$ ($\theta$ and/or $\theta'$ may be absent).

Recall that a $(\theta, s_j)$-cell has at most one $a$-edge, and it has no $a$-edges, if it corresponds to a rule of one of the auxiliary machines $\bar{Z}$ or $\bar{Z}'$. Hence, the label of a side of the $s_j$-band has the form $u(b_1v_1b_1^{-1})\ldots(b_mv_m^{-1})w$ where $b_i^{\pm 1}$ is an $a$-letter or 1 ($i = 1, \ldots, m$), $v_i$ is a group word in $\theta$-letters, and each of $u, w$ has at most one $a$-letter; and we should prove that $b_i$ commutes with $v_i$ if $b_i$ is involved in the rule $\theta_i$.

Let us consider the right side of the $s_j$-band. (The ‘left’ case is similar.) Then $\theta_i$ is a right rule, and by Lemma 4.28(3), no rule of the subword $\theta_iH\theta_i^{-1}$ of $H$ locks the $s_jp_{1\pm 1}$-sector, and so the letter $b_i$ commutes with every $\theta$-letter of $v_i$ by the definition of relations for the machine $M$.

(xiv) Follows from Lemma 4.29 since the cells corresponding to $M_3$-rules can have exactly one $a$-edge in the boundary only if they correspond to the rules of $M_2$.

(xv) Follows from Lemma 3.7.

(xvi) Follows from Lemma 4.43 because by Property (xi) the base of the trapezium contains (as a subword) a copy of the base of $M_4$.

(xvii) Follows from Lemma 4.38(b).

(xviii) Follows from Lemmas 4.26 and 4.33.

(xix) This is a reformulation of Lemma 4.38(a). \qed

5.3. **Diagrams with hubs**

Given a reduced diagram $\Delta$ over the group $G$, one can construct a planar graph whose vertices are the hubs of this diagram plus one improper vertex outside $\Delta$, and the edges are maximal $t$-bands of $\Delta$.

Let us consider two hubs $\Pi_1$ and $\Pi_2$ in a minimal diagram, connected by a $t_i$-band $C_i$ and a $t_{i+1}$-band $C_{i+1}$, where there are no other hubs between these $t$-bands. These bands, together with parts of $\partial \Pi_1$ and $\partial \Pi_2$, bound either a subdiagram having no cells, or a trapezium $\Psi$ of height at least 1. The former is impossible since in this case the hubs have a common $t$-edge and they are mirror copies of each other contrary to the reducibility of the diagram. We want to show that the latter case is not possible either.

Indeed, in the latter case, both the top and the bottom of $\Psi$ are the subwords of the hub $W_M^{\pm 1}$, that is, the history $H$ of $\Psi$ and $H^{-1}$ are the histories of $M_1$-accepting subtrapezia of $\Psi$. Therefore, by Property (xii)/(b), the history $H$ is of type (3). We may assume that the base of $\Psi$ has a subword $(k't')^\epsilon$ with $\epsilon = 1$ since otherwise one can replace $\Psi$ by its mirror copy. Let $\Gamma$ be the maximal subtrapezium of $\Psi$ with base $k't'$. Then every cell of the maximal $k'$-band $C$
of $\Gamma$ is active from the right by Property (ii). But the $a$-bands starting on $C$ cannot end on the passive (see Property (ii)) $t'$-band of $\Gamma$. They also cannot end on $C$ by Property (vii). Hence, $\|H\| = 0$, a contradiction.

Thus, any two hubs of a reduced diagram cannot be connected by two $t$-bands, such that the subdiagram bounded by them contains no other hubs. This property makes the hub graph of a reduced diagram hyperbolic, in a sense, since the degree $L$ of every proper vertex (= hub) is high ($L \geq 40$). Below, we give a more precise formulation (proved for diagrams with such a hub graph, in particular, in [19, Lemma 11.4] and [11, Lemma 3.2]).

**Lemma 5.18.** If a reduced diagram over the group $G$ contains at least one hub, then there is a hub $\Pi$ in $\Delta$ such that $L - 3$ consecutive maximal $t$-bands $B_1, \ldots, B_{L-3}$ start on $\partial \Delta$, end on the boundary $\partial \Pi$, and for any $i \in [1, L - 4]$, there are no discs in the subdiagram $\Gamma_i$ bounded by $B_i, B_{i+1}, \partial \Pi$ and $\partial \Delta$.

A maximal $q$-band starting on a hub of a diagram is called a spoke.

**Lemma 5.18** implies the following lemma:

**Lemma 5.19.** If a reduced diagram $\Delta$ has $m \geq 1$ hubs, then the number of $q$-edges in the boundary path of $\Delta$ is greater than $mLN/2$.

**Proof.** $\Delta$ has a hub $\Pi$ satisfying the assumption of Lemma 5.18. Then we can separate a subdiagram with only one hub $\Pi$ from $\Delta$ by making cuts along the $t$-bands $B_1, B_{L-3}$, and along the part of $\partial \Pi$ having $3$ $t$-edges. Since, by Lemma 5.6, every spoke of $\Gamma_i$ ($i \in [1, L - 4]$) starting on $\Pi$ must end on $\partial \Pi$, the remaining diagram $\Delta'$ with $m - 1$ hubs has at most $|\partial \Delta'|_q - (L - 4)N + 4N$ $q$-edges in the boundary. Since $L - 8 > L/2$ the statement follows by induction on $m$. □

### 5.4. Parameters

The following constants will be used for the proofs in this paper.

$$L, N \ll J \ll \delta^{-1} \ll (\delta')^{-1} \ll c_0 \ll c_1 \ll \cdots \ll c_7.$$  \hspace{1cm} (5.3)

For each of the inequalities of this paper, one can find the highest constant (with respect to the order $\ll$) involved in the inequality and see that for fixed lower constants, the inequality is correct as soon as the value of the highest one is sufficiently large. This principle makes the system of all inequalities used in this paper consistent.

### 5.5. Modified length of words and paths

Recall that the standard length $\|w\|$ of a word (a path) is called the combinatorial length. To introduce new length function on the group words in the generators of the groups $M$ and $G$,
we first consider a word $w$ having no $q$-letters. We set the length $|a|$ of every $a$-letter a equal to $\delta$. We set the length of any $\theta$-letter equal to 1, but the length $|v|$ of any $\theta a$-syllable, that is, a 2-letter word $v$ with one $\theta$-letter and one $a$-letter, will be equal to $1 + \delta'$. The length of a decomposition of $w$ in a product of letters and $\theta a$-syllables is the sum of lengths of the factors of this decomposition. The length $|w|$ of $w$ is the smallest length of such decompositions. Finally, the length $|W|$ of arbitrary word $W = w_0u_1 \ldots u_n w_n$, where $u_i$-s are $q$-letters and the words $w_j$-s have no $q$-letters, is, by definition, $n + \sum_{i=1}^{n} |w_i|$. The length of a path in a diagram is the length of its label. The perimeter $|\partial \Delta|$ of a diagram is similarly defined by cyclic decompositions of its boundary $\partial \Delta$.

Why do we need such a modification? The assumption that $a$-edges are much shorter than other edges is used in Lemma 13.2 (Step (2)) and in other lemmas. The assumption that $\delta' \ll \delta$, and so the length of a $\theta a$-syllable is less than the sum of lengths of its letters, is used in Lemma 7.18 and in many other lemmas.

**Lemma 5.20.** Let $s$ be a path in a diagram $\Delta$, having $d$ $a$-edges and $e$ non-$a$-edges. Then

(a) $e + d\delta \geq |s| \geq e + d\delta' + \max(0,(d-e)(\delta - \delta')) \geq e + d\delta'$;

(b) if $s = s_1 s_2$, then $|s_1| + |s_2| \geq |s| \geq |s_1| + |s_2| - (\delta - \delta')$ and $|s| = |s_1| + |s_2|$ if $s_1$ ends or $s_2$ starts with a $q$-edge or if both these edges are not $a$-edges;

(c) if $s$ is a top or a bottom of a $q$-band having $h$ cells, then $h \leq |s| \leq h(1 + \delta')$; and $|s| = h$

if $s$ has no $a$-edges.

(d) $|\|s\|| \geq |s| \geq \delta |s|$.

**Proof.** (a) Since every path is a product of $q$, $\theta$, and $a$-edges, the first inequality follows. The second one is true because at most $e$ $a$-edges can be joined with $\theta$-letters to form 2-edge subpaths of $s$, and the remaining $a$-edges have to be taken alone with coefficient $\delta$ when one calculates $|s|$. To make the reader more familiar with the definition of $|s|$, we leave claims (b), (c), and (d) for exercises. \[\square\]

6. Mixture on the boundary of a diagram

Let $O$ be a circle with a two-colored finite set of points (or vertices) on it, more precisely, let any vertex of this finite set be either black or white. We call $O$ a necklace with black and white beads on it. We want to introduce the mixture of this finite set of beads.

Assume that there are $n$ white beads and $n'$ black ones on $O$. We define sets $P_j$ of ordered pairs of distinct white beads as follows. A pair $(o_1, o_2)$ ($o_1 \neq o_2$) belongs to the set $P_j$ if the simple arc of $O$ drawn from $o_1$ to $o_2$ in clockwise direction has at least $j$ black beads. We denote by $\mu_j(O)$ the sum $\sum_{j=1}^{J} \text{card } P_j$ (the $J$-mixture on $O$). Below, similar sets for another necklace $O'$ are denoted by $P'_{j,j}$. In this section, $J \geq 1$, but later on it will be a fixed large enough number $J$ from the list (5.3).

**Lemma 6.1.** (a) $\mu_j(O) \leq J(n^2 - n)$.

(b) Suppose a necklace $O'$ is obtained from $O$ after removal of a white bead $v$. Then $\text{card } P_{j} - n \leq \text{card } P'_{j} \leq \text{card } P_{j}$ for every $j$, and $\mu_j(O) - Jn \leq \mu_j(O') \leq \mu_j(O)$.

(c) Suppose a necklace $O'$ is obtained from $O$ after removal of a black bead $v$. Then $\text{card } P'_{j} \leq \text{card } P_{j}$ for every $j$, and $\mu_j(O') \leq \mu_j(O)$.

(d) Assume that there are three beads $v_1, v_2$, and $v_3$ of a necklace $O$, such that the clockwise arc $v_1 - v_3$ contains $v_2$ and has at most $J$ black beads (excluding $v_1$ and $v_3$), and the arcs $v_1 - v_2$ and $v_2 - v_3$ have $m_1$ and $m_2$ white beads, respectively. If $O'$ is obtained from $O$ by removal of $v_2$, then $\mu_j(O') \leq \mu_j(O) - m_1m_2$. 
Proof. (a) It is clear from the definition that $\text{card} P_j \leq n^2 - n$ and the statement (a) follows. The statements (b) and (c) are obvious.

(d) Let $o (o')$ be a white bead on $v_1 - v_2$ (on $v_2 - v_3$). Then for some $j \in \{1, \ldots, J\}$, the pair $(o, o')$ belongs to $P_j$ but does not belong to $P_{j+1}$. Now, on the one hand, the same pair $(o, o')$ considered on $O'$ does not belong to $P_j$. On the other hand, we clearly have $P'_{j} \subseteq P_j$. Therefore, $\mu_j(O) - \mu_j(O')$ is at least the number of such pairs $(o, o')$, which is equal to $m_1m_2$. The lemma is proved. \hfill\Box

We will also use the mixture of beads on a closed interval $x = [a, b]$ with real $a < b$. A string of beads is a finite set of white and black beads on $x$, but in the definition of mixture $\mu^c(x)$ we consider only pairs $(o, o')$ of white beads, where $o < o'$. This gives us the mixture $\mu^c_j(x)$ as above.

**Lemma 6.2.** Let $x$ be a string of beads and $J \geq 1$.

(a) We have $\mu^c_j(x) \leq J(n^2 - n)/2$.

(b) Suppose a string $x'$ is obtained from $x$ after removal of a white bead $v$. Then $\text{card} P'_{j} - n < \text{card} P_j$ for every $j$, and $\mu^c_j(x) - Jn < \mu^c_j(x') \leq \mu^c_j(x)$.

(c) Suppose a string $x'$ is obtained from $x$ after removal of a black bead $v$. Then $\text{card} P'_{j} \leq \text{card} P_j$ for every $j$, and $\mu^c_j(x') \leq \mu^c_j(x)$.

(d) Assume that there are three black beads $v_1 - v_2 - v_3$ on $x$ such that the interval $(v_1, v_3)$ has at most $J$ black beads, and the intervals $(v_1, v_2)$ and $(v_2, v_3)$ have $m_1$ and $m_2$ white beads, respectively. If $x'$ is obtained from $x$ after removal of the bead $v_2$, then $\mu^c_j(x') \leq \mu^c_j(x) - m_1m_2$.

(e) Assume that the set of black beads is non-empty. Then there is a black bead $v$, such that it divides $x$ into two subsegments with $m_1$ and $m_2$ white beads, respectively, $m_1 \geq m_2$ and $m_1m_2 \leq \mu^c_1(x) \leq (2m_1 - 1)m_2$.

Proof. The proof of statements (a)–(d) is similar to the proof of Lemma 6.1. To prove claim (e), we choose the black bead $v$ so that the difference $|m_1 - m_2|$ is minimal. We can assume that $m_1 \geq m_2$. Since $m_1$ white beads are separated by $v$ from $m_2$ white beads, we have $\mu^c_1(x) \geq m_1m_2$. On the other hand, there is a subsegment with $m_1 - m_2$ pairwise non-separated (by black beads) white beads. Therefore,

$$\mu^c_1(x) \leq \frac{1}{2}(m_1 + m_2)(m_1 + m_2 - 1) - \frac{1}{2}(m_1 - m_2)(m_1 - m_2 - 1) = (2m_1 - 1)m_2.$$

For any diagram $\Delta$, we introduce the following invariant $\kappa(\Delta) = \mu_1(\partial \Delta)$. To define it, we consider the boundary $\partial(\Delta)$, as a $\kappa$-necklace, that is, we consider a circle $O$ with $\|\partial \Delta\|$ edges labeled as the boundary path of $\Delta$. By definition, the white beads are the mid-points of the $\theta$-edges of $O$ and black beads are the mid-points of the $q$-edges $O$. Then, by definition, the $\kappa$-mixture on $\partial \Delta$ is $\kappa(\Delta) = \mu_1(O)$.

We will need an analogous parameter $\nu_j(\Delta)$. The definition of the $\nu$-necklace on $\partial \Delta$ is similar, but the black beads correspond to $t$- and $t'$-edges only while the set of white beads coincides with that for the $\kappa$-necklace. The $\nu$-necklace has a $\nu_j$-mixture for every $J \geq 1$, which is called the $\nu_j$-mixture on $\partial \Delta$ and denoted by $\nu_j(\Delta)$.

Recall that a $\theta$-letter is said to be (12)-letter ((23)-letter) if it corresponds to the rule (12) (to (23)). Such a letter is special if it is involved in a ((12), $t$)-relation or in a ((23), $t'$)-relation. An edge is a (12)-edge (a (23)-edge, a special edge) if it is labeled by a (12)-letter (by a (23)-letter, by a special $\theta$-letter, respectively). Note that if a $q$-band $C$ has a special (12)-edge (a special (23)-edge) on the left side, then the base of $C$ is either a $t^\pm 1$ or a $k$ (respectively, either a $(t')^\pm 1$ or a $(k')^{-1}$).

To define an auxiliary parameter $\lambda(\Delta)$ we consider the $\lambda$-necklace, where white beads are the middle points of all $\theta$-edges of $O$ which are neither (12)-edges nor (23)-edges, and the black
beads are the middle points of all non-special (12)- and (23)-edges and all \( q \)-edges of \( O \). The \( \lambda \)-necklace defines the \( \lambda_f \)-mixture on \( \partial \Delta \) for every \( J \), and for \( J = 1 \), we denote it by \( \lambda(\Delta) \).

By definition, \( \mu(\Delta) = c_0 \kappa(\Delta) + \lambda(\Delta) \). The \( \nu_f \)-mixtures on boundaries will be later applied for a large enough \( J \).

Similarly, we have \( \kappa(x) \), \( \lambda(x) \), \( \mu(x) \), and \( \nu_f(x) \) for any path \( x \) in a diagram. (Consider the strings of beads to define.) Clearly, each of these values remains unchanged if one replaces \( x \) by \( x^{-1} \).

7. General properties of combs

By Lemma 5.10, every property of a trapezium can be formulated as a property of a computation of the \( S \)-machine \( M \), and vice versa. Unfortunately, minimal diagrams can be much more complicated than trapezia. Now we define diagrams which are the main subject of our research in this paper.

As in [15], we say that a reduced diagram \( \Gamma \) over \( M \) with reduced boundary path (having no subpaths of the form \( ee^{-1} \)) is a comb if it has a rim \( q \)-band \( C \) (the handle of the comb), and every maximal \( \theta \)-band of \( \Gamma \) has a cell in \( C \). In particular, every trapezium is a comb.

Suppose that a maximal \( q \)-band \( C \) of a diagram \( \Delta \) starts and ends on \( \partial \Delta \). Then it divides \( \Delta \) into two subdiagrams \( \Gamma \) and \( \Gamma' \), where \( \Gamma' \) contains \( C \). Suppose \( \Gamma \) is a comb with handle \( C \). Then we call \( \Gamma \) a subcomb of \( \Delta \).

By Lemma 5.6, any maximal \( q \)-band \( C' \) of a comb \( \Gamma \) is itself a handle of a subcomb \( \Gamma' \) of \( \Gamma \) which does not contain (by definition of subcomb of a comb) cells from the handle \( C \) of \( \Gamma \) if \( C' \neq C \). In this case \( \Gamma' \) is a proper subcomb of the comb \( \Gamma \).

The base width of a comb is by definition the maximal number of letters in the bases of its \( \theta \)-bands. The history \( H \) and the step history of a comb are the history and step history of its handle. If \( H' \) is a subword of \( H \), then the \( H' \)-part of the comb is the union of all maximal \( \theta \)-bands corresponding to \( H' \).

It will be convenient to view a comb \( \Gamma \) with the handle on its right. Thus, the bottom of the handle \( C \) is the right side of \( C \), and it is the part of \( \partial \Gamma \). Respectively, every \( q \)-band of \( \Gamma \) has the right side and the left side. The words written on tops/bottoms of \( \theta \)-bands of \( \Gamma \) and their bases will be read from left to right, and so, for a base letter \( q \), one can distinguish \( q \)- and \( q^{-1} \)-bands of \( \Gamma \). In particular, a \( q \)-band of \( \Gamma \) can be active from the left, active from the right (or passive). If a \( q \)-band \( D \) is passive from the left (from the right), then \( h = |y'| \leq |y| \), where \( h \) is the number of cells in \( D \) (respectively, \( |y'| \geq |y| = h \)) by the definition of length and Lemma 5.20.

We introduce the following permanent notation for a comb \( \Gamma \) with a handle \( C \). Denote by \( H \) the history of \( C \) and set \( h = ||H|| \), that is, \( h \) is the length of \( C \), the number of \( q \)-cells in \( C \). The comb \( \Gamma \) is a one-step comb if the history \( H \) is one-step, that is, \( H \) has one of the types (1), (2), or (3).
The boundary of $C$ is $x_1y_2x_2$, where $x_1$ and $x_2$ are the boundary $q$-edges of the band $C$ and $yz$ is the boundary of $\Gamma$. (Thus, $y$ is the right side of $C$, and $(y')^{-1}$ is the left side.) Similarly, we have the decomposition $(y')^{-1}z$ for the boundary of $\Delta \setminus C$, where $z = x_2'x_1$. By definition, Area$(\Gamma') = \text{Area}(\Gamma \setminus C)$. Since $z$ starts (ends) with the $q$-edge $x_2$ (with $x_1$), we have $|\partial\Gamma| = \|y\| + |z|$ by Lemma 5.20(b). We also use $y^\Delta, z^\Delta, \ldots$ instead of $y, z, \ldots$ if we want to stress that the notation relates to a particular comb $\Delta$.

**Remark 7.1.** It follows from Lemma 5.6 that every maximal $\theta$-band crossing the handle of a comb $\Delta$ must end on $z^\Delta$. Therefore, $\|y^\Delta\|_\theta = \|y^\Delta\|_\theta = |z^\Delta|_\theta = h$.

For a comb $\Gamma$, we modify the notion of mixture. The *comb mixtures* are $\kappa^e(\Gamma) = \kappa(z) - \kappa(y)$, $\lambda^e(\Gamma) = \lambda(z) - \lambda(y)$, and similarly, $\nu^e_j(\Gamma) = \nu_j(z) - \nu_j(y)$ ($\lambda(\Gamma)$ can be negative if (12)- or (23)-cells separate other $\theta$-cells of the handle!). By definition $\mu^e(\Gamma) = c_0\kappa^e(\Gamma) + \lambda^e(\Gamma)$.

**Lemma 7.2.** In the above notation, we have (a) $\kappa^e(\Gamma) \geq 0$, (b) $\nu^e_j(\Gamma) = \nu_j(z) \geq 0$, (c) $\lambda^e(\Gamma) \geq 0$ if for every special edge $e$ of $z$, the edge $f$ of $y$ connected with $e$ by a $\theta$-band, is also special.

**Proof.** (a) and (b) Since the path $y$ has no $q$-edges, we have $\kappa(y) = \nu_j(y) = 0$, respectively, and so $\kappa^e(\Gamma) = \kappa(z) \geq 0$ ($\nu^e_j(\Gamma) = \nu_j(z) \geq 0$, respectively).

(c) Consider the strings of beads on $z$ and on $y$ used in the definitions of the comb mixture $\lambda^e(\Gamma)$. By Lemma 5.6, the maximal $\theta$-bands of $\Gamma$ establish a bijection between the white vertices of $z$ and white vertices of $y$, preserving the order of the beads on $z$ and $y^{-1}$, respectively. Every black bead on $y$ must belong to a non-special $\theta$-edge $f$. By the condition of the lemma, we have a black bead on the corresponding edge $e$ of $z$. Hence, one can apply Lemma 6.1(c) to the strings of beads on $y$ and $z$ several times to conclude that $\lambda(z) \geq \lambda(y)$, and so $\lambda^e(\Gamma) \geq 0$. □

**Lemma 7.3.** Let $\Gamma$ be a proper subcomb of a diagram (of a comb) $\Delta$. Let $\Delta \setminus \Gamma$ be the complement of $\Gamma$ in $\Delta$, whose handle is the handle of $\Delta$ if $\Delta$ is a comb. Then

(a) $\kappa(\Delta \setminus \Gamma) \leq \kappa(\Delta) - \kappa(\Gamma)$ and $\lambda(\Delta \setminus \Gamma) \leq \lambda(\Delta) - \lambda(\Gamma)$,

(b) $\nu_j(\Delta \setminus \Gamma) \leq \nu_j(\Delta) - \nu_j(\Gamma)$ for every $J \geq 1$,

(c) $\kappa^e(\Delta \setminus \Gamma) \leq \kappa^e(\Delta) - \kappa^e(\Gamma)$ and $\lambda^e(\Delta \setminus \Gamma) \leq \lambda^e(\Delta) - \lambda^e(\Gamma)$ if $\Delta$ is a comb,

(d) $\nu^e_j(\Delta \setminus \Gamma) \leq \nu^e_j(\Delta) - \nu^e_j(\Gamma)$ for every $J \geq 1$ if $\Delta$ is a comb,

(e) if $\Delta$ is a subcomb of a diagram $\Delta$ and $\Gamma$ is a subcomb of $\Delta$, then for every $J \geq 1$, $0 \leq \nu^e_j(\Delta \setminus \Gamma) - \nu^e_j(\Delta \setminus \Gamma) \leq \nu^e_j(\Delta \setminus \Gamma)$. (Also we have $\nu^e_j(\Delta) - \nu^e_j(\Delta \setminus \Gamma) \leq \nu^e_j(\Delta) - \nu^e_j(\Delta \setminus \Gamma)$ if $\Delta$ is a comb.)

**Proof.** (a) Let $y = y^\Gamma$, $z = z^\Gamma$, and $x$ the boundary path of $\Delta$. To obtain the necklace $O'$ corresponding to $\Delta \setminus \Gamma$, one replaces the subpath $z$ of the boundary by $y^{-1}$. Therefore, the pairs of white beads counted to obtain $\lambda(z)$ are replaced by pairs counted to obtain $\lambda(y)$. (Note that the white beads of $z$ are in bijective correspondence with white beads of $y$ by the definition of comb and Lemma 5.6.) Since every white bead of $z$ is separated from any white bead of $\partial \Delta \setminus \partial \Gamma$ by the black beads in the middle of the first and the last edges of $z$, inequality (a) is proved for $\lambda$-mixtures. The case of $\kappa$-mixtures is similar.

The proofs of claims (b)–(d) are also similar.

The path $y^\Gamma$ has no $t$-edges, and the first inequality of (e) follows. Similarly, every pair of white beads which makes a contribution to $\nu^e_j(\Delta)$ but not to $\nu^e_j(\Delta \setminus \Gamma)$ also contributes to
\[ \nu_J(\Delta) \text{ but not to } \nu_J(\Delta \setminus \Gamma), \text{ and the second inequality of (e) follows. The proof of the version in the parentheses is similar.} \]

Let \( \Gamma \) be a comb and \( z^1, \ldots, z^r \) the maximal subpaths of \( z = z^1 \) containing no \( q \)-edges. We denote by \( l^1, \ldots, l^r \) their \( \theta \)-lengths, and define \( l_- = l^1 \) to be \( h - \max_{i=1}^r l^i \). (Note that \( h = h^1 = \sum l^i \) by Lemma 5.6.)

A \( \theta \)-band which starts on the handle \( C \) of a comb \( \Gamma \) will be called simple if it has no \( (\theta, q) \)-cells except for the cell of \( C \), and is maximal with respect to this property.

We call a maximal \( q \)-band \( B \) a derivative band, if it is not \( C \) but it can be connected with \( C \) by a simple \( \theta \)-band. Throughout the paper, we will use notation \( C_1, \ldots, C_s \) for derivative bands of a comb \( \Gamma \). It is possible that \( s = 0 \), and every maximal \( \theta \)-band is simple in this case.

Every derivative band \( C_i \) is a handle of a subcomb \( \Gamma_i \) (which does not contain \( C \)). We will use this notation and call \( \Gamma_i \) a derivative subcomb of \( \Gamma \). It follows from the definitions that every cell of a comb belongs either to a derivative subcomb or to a simple band of \( \Gamma \).

Recall that every maximal \( \theta \)-band of a comb, in particular, a maximal \( \theta \)-band crossing a derivative band \( C_i \), must cross the handle \( C \). Therefore, every cell of \( C_i \) is connected with \( C \) by a \( \theta \)-band. Since there is a simple \( \theta \)-band among these \( \theta \)-bands, no other derivative \( C_j \) can intersect these connecting \( \theta \)-bands by Lemma 5.6, that is, all of them are simple. It follows that different derivative subcombs are disjoint, and if \( C_1, \ldots, C_s \) is the system of all derivative bands in \( \Gamma \) with histories \( H_1, \ldots, H_s \), then \( H_1, \ldots, H_s \) are pairwise disjoint subwords in the history \( H \) of \( \Gamma \). Therefore, \( \sum_{i=1}^s h_i \leq h \), where \( h_i = \|H_i\| \). We will also use \( h_- \) for \( \sum_{i=1}^s h_i - \max_{i=1}^s h_i \).

**Lemma 7.4.** In the above notation, \( l_- \geq \min(\sum_{i=1}^s h_i, h - \max_{i=1}^s h_i) \). In particular,

\[ h_- \leq l_- \tag{7.1} \]

**Proof.** Let \( |z^{i_0}|_\theta = l^{i_0} = \max_{i=1}^r l^i \). Then, either every maximal \( \theta \)-band ending on \( z^{i_0} \) crosses some derivative band \( C_j \), where \( j = j(i_0) \), or every maximal \( \theta \)-band crossing \( z^{i_0} \) crosses no derivative bands because otherwise a \( q \)-band would cross \( z^{i_0} \). (This follows from the definitions of comb, of \( z^i \)-s and from Lemma 5.6.) In the former case, \( l_- = h - l^{i_0} \geq h - \max_{i=1}^s h_i \geq h_- \), and in the latter case, \( l_- = h - l^{i_0} \geq \sum_{j=1}^s h_j \geq h_- \). \( \square \)

**Lemma 7.5.** In the above notation, we have \( hh_- \leq hl_- \leq 2\kappa^c(\Gamma) \).

**Proof.** By (7.1), it suffices to prove the second inequality. There are \( h \) white beads on \( z \). Every such bead \( o \) belongs to one of the paths \( z^i \) having \( \theta \)-length at most \( \max_{i=1}^r l^i \). Therefore, for every such \( o \), there are at least \( l_- \) white beads \( o' \) on \( z \) such that \( o \) and \( o' \) are separated on \( z \) by a black bead. Thus, we obtain at least \( hl_- \) pairs \( (o, o') \) of white beads on \( z \) separated by black beads. Since one of the pairs \( (o, o') \) and \( (o', o) \) contributes 1 to \( \kappa^c(\Gamma) \), the lemma is proved. \( \square \)

Let the handle \( C \) of a comb \( \Gamma \) be a \( t^{\pm 1} \)- or \( (t')^{\pm 1} \)-band with history having no \((23)\)-rules or no \((12)\)-rules, respectively; and every derivative \( C \) is a \( k^{\pm 1} \)- or a \( (k')^{\pm 1} \)-band such that there are no special \( \theta \)-edges (corresponding to the rules \((12)\) and \((23)\)) in the derivative subcomb \( \Gamma_i \). A subband \( B \) of some \( C \) which has neither \((12)\)-nor \((23)\)-edges and is maximal with respect to this property, is called a short derivative of \( C \). By Property (vii), there are no maximal \( a \)-bands starting and ending on the same short derivative band. Let \( h'_1, \ldots \) be the length of all short derivatives. Let \( h' \) be the number of maximal \( \theta \)-bands in \( \Gamma \), which do not correspond to the rules \((12)\) and \((23)\). Define \( h'' = h' - \max h'_j \).
Lemma 7.6. In the above notation, we have \(hh'_{-} \leq 6\lambda(z^\Gamma) = 6\lambda^c(\Gamma)\).

Proof. The sets of ends of the \(\theta\)-bands crossing two short derivatives are separated in \(z^\Gamma\) either by a \(q\)-edge or by a non-special \(\theta\)-edge. Therefore, arguing as in the proof of Lemma 7.5, we come to inequality \(h'h'_- \leq 2\lambda(z^\Gamma) = 2\lambda^c(\Gamma)\). (We note that under the assumption on the history, \(\lambda(y^t) = 0\) since \(C\) is a \((t')^\pm\)- or \((t')^\mp\)-band.) This implies the statement of the lemma if \(h'_- \neq 0\) since, in this case, we have \(h' \geq h/3\) because the handle \(C\), being a reduced diagram, cannot have two consecutive cells corresponding to the rules (12), (23) (or inverse). If \(h'_- = 0\) the claim of the lemma is obvious.

Lemma 7.7. Let \(\Gamma\) be a comb with a handle \(C\) of length \(h\). Then the number \(a_{ij}\) of all maximal \(a\)-bands of \(\Gamma\) starting on a derivative band \(C_i\) and ending on the bands \(C_j\) with \(j \neq i\) is at most \(h_-\). The total number of cells in these \(a\)-bands over all \(i < j\) does not exceed \(hh'_{-} \leq 2\kappa^c(\Gamma)\).

Proof. Recall that derivative subcombs with different handles \(C_i\) and \(C_j\) are disjoint and separated by these handles (which are \(q\)-bands) from the remaining part of \(\Gamma\). Therefore, every \(a\)-band \(A\) connecting some \(C_i\) and \(C_j\) \((i \neq j)\) connects a-edges of cells on the right sides of these derivative bands. But every \(q\)-cell of \(C_i\) has at most one a-edge on the right side of it by (iii)(b). Besides, a connecting \(a\)-band \(A\) under consideration either starts or ends on some \(C_j\), where \(j \neq i\). Thus, the total number of all connecting \(a\)-bands cannot exceed \(h - h_r\) for arbitrary \(r \leq s\). Now the first statement of the lemma follows from the definition of \(h_-\). Since the number of cells in \(A\) is at most \(h\) by Lemma 5.6, the second statement is also proved by Lemma 7.5.

Lemma 7.8. Let \(\Gamma\) be a comb and let its handle \(C\) be a \(t^\pm\)- or \((t')^\pm\)-band with history having no (23)-rules or (12)-rules, respectively, and every derivative \(C_i\) is a \(k^\pm\)- or \((k')^\pm\)-band such that there are no special \(\theta\)-edges in the derivative subcomb \(\Gamma_i\). Let \(B_1, \ldots, B_k\) be the system of all short derivative bands. Then the number of all the maximal \(a\)-bands of \(\Gamma\) starting on a short derivative \(B_i\) and ending on some \(B_j\) \((j \neq i)\), where \(B_i\) and \(B_j\) are subbands of the same derivative band, is at most \(h'_-\). The total number of cells in these \(a\)-bands over all \(i, j\) does not exceed \(hh'_{-} \leq 6\lambda(z^\Gamma) = 6\lambda^c(\Gamma)\).

Proof. The proof is similar to the proof of Lemma 7.7, but one should use Lemma 7.6 instead of Lemma 7.5.

Lemma 7.9. (a) Let \(\Gamma_1, \ldots, \Gamma_s\) be the derivative subcombs of a comb \(\Gamma\). Then \(\sum_{i=1}^s \kappa^c(\Gamma_i) \leq \kappa^c(\Gamma)\).

(b) If the history of a comb \(\Gamma\) is \(H \equiv H(1), \ldots, H(t)\), and \(\Gamma(1), \ldots, \Gamma(t)\) are \(H(1), \ldots, H(t)\)-parts of \(\Gamma\), respectively (\(\Gamma(i)\) is absent if \(H_i\) is empty), then \(\sum_{i=1}^t \kappa^c(\Gamma(i)) \leq \kappa^c(\Gamma)\).

Proof. (a) and (b) Note that every white bead of \(\Gamma_i\) (of \(\Gamma(i)\)) is placed on the boundary of \(\Gamma\), and two white beads of \(\partial\Gamma_i\) separated by a black bead are also separated by the same black bead on \(\partial\Gamma\). Since the sets of white beads of \(\Gamma_i\) and \(\Gamma_j\) (of \(\Gamma(i)\) and \(\Gamma(j)\)) are disjoint for \(i \neq j\), the statements (a) and (b) follow from the definition of \(\kappa^c(\cdot)\).
Lemma 7.10. (a) Let $\Gamma$ be a comb. Then the number $\alpha$ of $a$-edges in $z' = (z')^\Gamma$ does not exceed $(\delta')^{-1}(|z'| - h) = (\delta')^{-1}(|z| - h - 2)$.

(b) Assume, in addition, that the handle $C$ is passive from the left and there are no derivative bands $C_i$ such that some non-trivial $a$-band starts and ends on $\partial C_i$. Then the total area of all simple $\theta$-bands $S_1, \ldots, S_h$ of $\Gamma$ is at most

$$h(h_- + \alpha + 1) \leq h(h_- + (\delta')^{-1}(|z| - h - 1)).$$

Proof. (a) Note that $|z'| \geq h + \delta'\alpha$ by Lemmas 5.6 and 5.20(a), and so $\alpha \leq (\delta')^{-1}(|z'| - h) = (\delta')^{-1}(|z| - h - 2)$.

(b) The total number of cells in all $a$-bands connecting the derivative bands is at most $hh_-$ by Lemma 7.7. If a $(\theta, a)$-cell $\pi$ of a simple $\theta$-band $S_i$ does not belong to any such connecting $a$-bands, then one of the ends of the maximal $a$-band containing $\pi$ must belong to $\partial \Gamma$ because $C$ is passive from the left. The number of such $a$-bands is at most $a$, and the total number of their cells is at most $ah$, because their lengths do not exceed $h$ by Lemma 5.6. Since a simple band has one $q$-cell, the number of cells in all the simple $\theta$-bands is at most $h(h_- + \alpha + 1)$. □

Remark 7.11. If $C_i$ is a derivative band of a comb $\Gamma$, then every $a$-band connecting two cells from $C_i$ is of length at most $h_i$ and the total area of such bands crossing simple bands of $\Gamma$ at most $h_i^2/2$. Hence, if we omit the assumption that there are no derivative bands $C_i$ such that some non-trivial $a$-band starts and ends on $C_i$, then we may add $\sum_{i=1}^h h_i^2/2 \leq (h/2) \sum_{i=1}^h h_i$ to the estimate of Lemma 7.10(b), and in this case the total number of cells $n_s$ in all simple bands satisfies

$$n_s \leq h \left( h_- + \alpha + 1 + \sum_{i=1}^s h_i/2 \right) \leq h \left( (\delta')^{-1}(|z| - h - 1) + \frac{3}{2} \sum_{i=1}^h h_i \right). \quad (7.2)$$

If we use both Lemmas 7.7 and 7.8 instead of Lemma 7.7 in the proof of Lemma 7.10 we obtain the following:

Lemma 7.12. Let $\Gamma$ be a comb and let its handle $C$ be a $t^{\pm 1}$- or $(t')^{\pm 1}$-band with history having no (23)-rules or no (12)-rules, respectively, and every derivative $C_i$ in a $k^{\pm 1}$- or $(k')^{\pm 1}$-band such that there are no special $\theta$-edges in the derivative subcombs of $\Gamma_i$. Then the total area of all simple $\theta$-bands of $\Gamma$ is at most $h(h_- + h'_- + \alpha + 1) \leq h(h_- + h'_- + (\delta')^{-1}(|z| - h - 1))$, where $\alpha$ is the number of $a$-edges in $z' = (z')^\Gamma$.

The proof of the following lemma can be obtained from the proof of [15, Lemma 4.10] by replacing $|\Gamma|_a$ by $|z|_a$ and replacing the constant $C$ by 2 (since $C$ was the maximum of the numbers of $a$-letters in $(\theta, q)$-relations in [15]).

Lemma 7.13. Let $h$ and $b$ be the height and the base width of a comb $\Gamma$, respectively, and let $T_1, \ldots, T_h$ be consecutive $\theta$-bands of $\Gamma$. We can assume that $\text{bot}(T_1)$ and $\text{top}(T_h)$ are contained in $\partial \Gamma$. Let $\alpha = |\Gamma|_a$, and we denote by $\alpha_1$ the number of $a$-edges on $\text{bot}(T_1)$. Then $\alpha + 8bh \geq 2\alpha_1$, and the area of $\Gamma$ does not exceed $4bh^2 + 2\alpha h$.

Remark 7.14. For a comb $\Gamma$, we will use symbol $[\Gamma]$ to denote the product $h^{\Gamma}(|z'| - |y^{\Gamma}|)$. As we noted in Section 1, an estimate of the form $\text{Area}(\Gamma) \leq C[\Gamma]$ (where $C$ does not depend on $\Gamma$) would be perfect for the proof of the main theorem. It follows from the definition of
comb that every maximal \( \theta \)-band of \( \Gamma \) starting on \( y^\Gamma \) ends on \( z^\Gamma \) and vice versa, that is \(|z^\Gamma|_q = |y^\Gamma|_q = h^\Gamma \). Clearly, \(|z^\Gamma|_q - 2 \geq |y^\Gamma|_q = 0 \) since the path \( z^\Gamma \) contains at least 2 \( q \)-edges of the handle of \( \Gamma \), and therefore, when we estimate \(|z^\Gamma| - |y^\Gamma| \) from below in the proofs of several lemmas, our goal is to obtain a lower bound for the difference \(|z^\Gamma|_a - |y^\Gamma|_a|.

We observed earlier that if the handle of a comb \( \Gamma \) is passive from the right, then \(|y^\Gamma| = h^\Gamma \), and so \([\Gamma]\) is equal to \( h^\Gamma \Gamma^\Gamma \). If the height \( h \) of a comb \( \Gamma \) does not exceed \( (\delta')^{-1} \), then \(|z^\Gamma| - |y^\Gamma| > 0 \) and the area of \( \Gamma \) does not exceed \( 4(\delta')^{-1}[\Gamma] \).

**Lemma 7.15.** If the height \( h \) of a comb \( \Gamma \) does not exceed \( (\delta')^{-1} \), then \(|z^\Gamma| - |y^\Gamma| > 0 \) and the area of \( \Gamma \) does not exceed \( 4(\delta')^{-1}[\Gamma] \).

**Proof.** By Lemma 5.20(c), \(|y^\Gamma| \leq h^\Gamma + 1 \). If the base width of \( \Gamma \) is \( b \), then, by Lemma 5.6, \( z^\Gamma \) has at least \( 2b \) \( q \)-edges and at least \( h^\Gamma \theta \)-edges. Hence, by Lemma 5.20(a), \(|z^\Gamma| \geq 2b + h + \delta'|z^\Gamma|_a \). Therefore, \(|z^\Gamma| - |y^\Gamma| \geq (2b - 1) + \delta'|z^\Gamma|_a > 0 \) since \( b \geq 1 \). Then, by Lemma 7.13,

\[
\text{Area}(\Gamma) \leq 4b(h^\Gamma)^2 + 2|z^\Gamma|_a h^\Gamma \\
\leq h^\Gamma(\delta')^{-1}(4(2b - 1) + 2\delta'|z^\Gamma|_a) \\
\leq 4(\delta')^{-1}h^\Gamma(|z^\Gamma| - |y^\Gamma|). \tag*{□}
\]

**Remark 7.16.** Below, we are finding appropriate estimates for the areas of comb \( \Gamma \)-s or for the areas of some proper subcombs of them provided the base width \( b \) of \( \Gamma \) is not too small and not too large. It is not small in some lemmas because we need a choice to select a suitable subcomb of \( \Gamma \), and \( b \) is not too large since the estimates of Lemma 7.13 and of other lemmas depend on \( b \). The sufficiency of the upper bound \( b \leq 15N \) will be seen later.

**Lemma 7.17.** Let \( \Gamma \) be a comb with base width \( b \leq 15N \) and with passive handle \( \mathcal{C} \). Assume that \( \Gamma \) has a derivative band \( \mathcal{C}_n \) which contains an active from the right subband \( \tilde{\mathcal{C}} \) of length \( h_0 \). Assume also at most \( (1 - \delta)h_0 \) maximal \( a \)-bands starting on \( \tilde{\mathcal{C}} \) and ending on one of the bands \( \mathcal{C}_1, \ldots , \mathcal{C}_s \). Then (a) \( \text{Area}(\Gamma) \leq (\delta')^{-2}[\Gamma] \) if \( h_0 \geq \delta h \); (b) \( \sum_{i=1}^{s} \text{Area}(\Gamma_i) \leq (\delta')^{-2}[\Gamma] \) for the set of derivative subcombs \( \Gamma_1, \ldots , \Gamma_s \) if \( h_0 \geq \delta \sum h_i \).

**Proof.** To prove statement (a), we consider two cases.

**Case 1:** Assume that \( \alpha = |z|_a \geq \delta^2 h/2 \). Then \( |z| - |y| \geq \delta' \alpha \geq \delta' \delta^2 h/2 \) by Lemma 7.10(a) since \(|y| = h \). Hence, by Lemma 7.13 with \( b \leq 15N \), we have

\[
\text{Area}(\Gamma) \leq 60N^2h^2 + 2ah \leq h(|z| - |y|)(60N \times 2(\delta')^{-1} \delta^{-2} + 2(\delta')^{-1}) \leq (\delta')^{-2}[\Gamma]
\]

since \( (\delta')^{-1} > 120N \delta^{-2} + 2 \).

**Case 2:** Let \( \alpha = |z|_a < \delta^2 h/2 \). It follows from the condition of the lemma that at least \( h_0 - 2 \) maximal \( a \)-bands start on \( \tilde{\mathcal{C}} \) but at most \( (1 - \delta)h_0 \) and not on \( z \). Therefore \( \alpha \geq \delta h_0 - 2 \geq \delta^2 h - 2 \). The inequality \( \delta^2 h - 2 \leq \delta^2 h/2 \) arising in this case implies that \( h < 4\delta^{-2} < (\delta')^{-1} \) by the choice of \( \delta' \). Now Claim (a) follows from Lemma 7.15.

The proof of statement (b) is similar, but now two cases appear due to the comparison of \( \alpha \) with \( \delta^2(\sum h_i)/2 \), which leads to inequality \( \sum h_i < 4\delta^{-2} \) in the second case. Also one takes into account inequalities \( \sum h_i \leq h \) and \( \sum(|z_i| - |y_i|) \leq |z| - |y| \) in both cases. \( \tag*{□} \)

**Lemma 7.18.** Assume that a comb \( \Gamma \) has no maximal \( q \)-bands except for its handle \( \mathcal{C} \), and there are no non-trivial \( a \)-bands both starting and terminating on \( y^\Gamma \).
(a) If $\mathcal{C}$ is active from the left or passive from the left, then
\[
\text{Area}'(\Gamma) \leq (\delta')^{-1} h(|z'| - |y'| + 1).
\]

(b) If $\mathcal{C}$ is active from the left or $\mathcal{C}$ is passive (from both sides), then
\[
\text{Area}(\Gamma) \leq (\delta')^{-1} |\Gamma|.
\]

**Proof.** (a) Let $\mathcal{T}$ be (one of) the longest $\theta$-band in $\Gamma$, $d$ the number of $a$-cells in $\mathcal{T}$. Denote by $T_1$ and $T_2$ the top and the bottom of $\mathcal{T}$. Consider the families $\mathbf{S}_1$ and $\mathbf{S}_2$ of $a$-bands starting on $T_1$ and $T_2$, respectively, which are maximal with respect to the requirement that these bands do not contain cells from $\mathcal{T}$. Observe that the cardinalities $|\mathbf{S}_1|$ and $|\mathbf{S}_2|$ of these sets satisfy inequality
\[
-1 \leq |\mathbf{S}_1| - |\mathbf{S}_2| \leq 1 \tag{7.3}
\]
since every maximal $a$-band of $\Gamma$ crossing $T_1$ has to cross $T_2$ (and vice versa), with at most one exception for the $a$-band starting on the $a$-edge of the unique (see (iii)(b)) $(\theta, q)$-cell of $\mathcal{T}$.

If there were a non-trivial $a$-band from $\mathbf{S}_1$ and a non-trivial $a$-band from $\mathbf{S}_2$ both ending on the path $y'$, then the maximal extension of one of them would connect different edges of $y'$ contrary to the assumption of the lemma. Therefore, either no non-trivial band from $\mathbf{S}_1$ ends on $y'$ or no non-trivial band from $\mathbf{S}_2$ ends on $y'$.

We may consider the former case only.

![Diagram](image)

The path $T_2$ cuts $\Gamma$ into two subdiagrams. We denote by $\Gamma(1)$ the subdiagram of $\Gamma$ containing the bands from $\mathbf{S}_1$ and the $\theta$-band $T$. It has boundary $y(1)z(1)T_2$, where $y(1)$ and $z(1)$ are subpaths of $y = y(2)y(1)$ and $z = z(1)z(2)$, respectively. Similarly, $y' = y'(1)y'(2)$, $z' = z'(2)z'(1)$, and we define $\Gamma(2)$ as a subdiagram bounded by $z(2)y(2)T_2^{-1}$.

Denote by $\mathbf{S}$ the family of maximal $a$-bands of $\Gamma(1)$ starting on $\partial \mathcal{C}$. It follows from the choice of $\Gamma(1)$ that the families $\mathbf{S}_1$ and $\mathbf{S}$ have at most one common band starting on the intersection of $\mathcal{C}$ and $\mathcal{T}$, and every $a$-band from these families must end on $z'(1)$. Thus,
\[
|z'(1)|_a \geq |y'(1)|_a + |\mathbf{S}_1| - 1. \tag{7.4}
\]

**Case 1:** If $\mathcal{C}$ is active from the left, then $|y'(1)|_a = |y'(1)|_\theta - 2$ by the definition. Note that $|y'(1)|_a = |z'(1)|_\theta$ by Lemma 5.6, and so $|y'(1)|_a \geq |z'(1)|_\theta - 2$. This inequality together with (7.4) imply $|z'(1)|_a \geq |z(1)|_\theta + |\mathbf{S}_1| - 3$, and by Lemma 5.20(a),
\[
|z'(1)| \geq |z'(1)|_\theta + \delta'(|z'(1)|_\theta - 3) + \delta(|\mathbf{S}_1| - 3) \\
= |y'(1)|_a(1 + \delta') + \delta(|\mathbf{S}_1| - 3) - 3\delta' \\
\geq |y'(1)| + \delta(|\mathbf{S}_1| - 3) - 3\delta' = |y'(1)| + \delta|\mathbf{S}_1| - 3(\delta + \delta'). \tag{7.5}
\]
Since at most $|S_2|$ maximal $a$-bands of $\Gamma(2)$ starting on $C$ terminate on $T$, Lemma 5.20 and (7.3) give inequality $|y'(2)| \leq |z'(2)| + \delta'|S_2| \leq |z'(2)| + \delta'(|S_1| + 1)$. Therefore, by (7.5),

$$|y'| \leq |y'(1)| + |y'(2)| \leq |z'(1)| + |z'(2)| - \delta|S_1| + 3(\delta + \delta') + \delta'(|S_1| + 1).$$

Since by Lemma 5.20(b), $|z'(1)| + |z'(2)| \leq |z'| + \delta - \delta'$ and also $|S_1| \geq d - 1$, it follows that

$$|y'| \leq |z'| - (\delta - \delta'|S_1| + 4\delta \leq |z'| - (d - 1)(\delta - \delta') + 4\delta,$$

whence $\delta \leq (\delta - \delta')^{-1}(|z'| - |y'| + 5\delta)$.

Case 2: If $C$ is passive from the left, we have $|y'(1)| = |y'(1)|_\theta$, and therefore

$$|z'(1)| \geq |y'(1)| + \delta'(|S_1| - 1)$$

by (7.4) and by Lemma 5.20(a). Also we have $|z'(2)| \geq |y'(2)|$ by Lemmas 5.6 and 5.20(c), and so

$$|y'| \leq |y'(1)| + |y'(2)| \leq |z'(1)| + |z'(2)| - \delta'(|S_1| - 1) < |z'| - \delta'(d - 1) + \delta.$$

Hence, $\delta \leq (\delta')^{-1}(|z'| - |y'|) + 1 + \delta(\delta')^{-1}$.

Thus, in any case $\delta < (\delta')^{-1}(|z'| - |y'|) + 1$ because $5\delta' < \delta < 1/5$.

The number of maximal $\theta$-bands of $\Gamma$ is $h$, whence $\text{Area}'(\Gamma) \leq dh < (\delta')^{-1}h(|z'| - |y'| + 1)$, as required.

(b) Now it follows from the assumption of the lemma and the definition of length, that $|y| \leq |y'| + 2\delta'$ because if $C$ is active from the left, then it has at most two passive from the left cells. However, $|z| = |z'| + 2$, and so $|z| - |y| \geq |z'| - |y'| + 2 - 2\delta'$. Now by (a), and inequality $3\delta' < 1$,

$$\text{Area}(\Gamma) = \text{Area}'(\Gamma) + h \leq (\delta')^{-1}h(|z'| - |y'| + 1 + \delta')$$

$$\leq (\delta')^{-1}h(|z| - |y|) = (\delta')^{-1}[\Gamma].$$

We say that a $q$-band $C'$ is close to a $q$-band $C$ in a diagram $\Delta$ without hubs (that is, over the group $M$) if every maximal $\theta$-band crossing $C'$ also crosses $C$.

Observe that a derivative band of a comb is close to the handle of this comb.

**Definition 7.19.** If $C'$ is close to $C$, then there is a unique minimal subtrapezium in $\Delta$ containing both $C'$ and a subband $B$ of $C$, where the numbers of $(\theta, q)$-cells in $C'$ and in $B$ are equal. We will denote this filling subtrapezium by $Tp(C', C)$.

**Lemma 7.20.** Assume that a comb $\Delta$ has no active $k$- or $k'$-cells and contains a $\theta$-band $T$ having a subword $(pp^{-1}p)^{\pm 1}$ in the base, where $p$ is a control letter. Then $\Delta$ has one-step subcomb $\Gamma$ such that $\text{Area}(\Gamma) \leq (\delta')^{-1}[\Gamma]$.

**Proof.** Consider the maximal $p$-bands $C'$ and $C''$ of $\Delta$ crossing $T$ at the $(q, \theta)$-cells corresponding to the first and the third letters of the subword $(pp^{-1}p)^{\pm 1}$, respectively. Then $C'$ is a handle of a subcomb $\Delta'$ of $\Delta$, and the filling trapezium $Tp(C', C'')$ has base $(pp^{-1}p)^{\pm 1}$.

By (v) and (vi), all the $q$-cells of $Tp(C', C'')$, in particular, of $C'$, are active from both sides and have one-step history. It follows from Property (vi) applied to the comb $\Delta' \cup Tp(C', C'')$, that every $q$-band of $\Delta'$ is either active from both sides or passive, and no non-trivial $a$-band of $\Delta'$ can start and end on the same $q$-band by (vii). Hence, there is a subcomb $\Gamma$ of $\Delta'$ satisfying the condition of Lemma 7.18(b), and the statement is proved. \qed
8. Chains and quasicoms

8.1. Intersections of chains and \( \theta \)-bands

Let the boundary \( \partial \pi \) of a \((\theta,q)\)-cell \( \pi \) have a \( p_i \)-edge for a control state letter \( p_i \). Then by Property (iii)(a), \( \partial \pi \) either has no \( a \)-edges or contains two \( a \)-edges. Below, we utilize this property in the definition of a chain.

Assume that \( \mathcal{A}_1, \ldots, \mathcal{A}_m \) are maximal \( a \)-bands such that \( \mathcal{A}_i \) terminates on an \( a \)-edge of an active \( p \)-cell \( \pi_i \) and \( \mathcal{A}_{i+1} \) starts with a different \( a \)-edge of \( \pi_i \) (\( i = 1, \ldots, m-1 \)). Then we say that \( \mathcal{A}_1, \pi_1, \mathcal{A}_2, \pi_2, \ldots, \mathcal{A}_m \) form a chain with \( m-1 \) links \( \pi_1, \ldots, \pi_{m-1} \). By Property (iii), all links are \( p \)-cells for the same control base letter \( p = p_j \). A chain \( \mathcal{A} \) is called a \textit{chain-annulus} if the first \( a \)-edge of \( \mathcal{A}_1 \) coincides with the last one of \( \mathcal{A}_m \).

It also follows that if \( \mathcal{A}_1 \) starts with an edge \( e_1 \) and \( \mathcal{A}_m \) ends with \( f_m \), then the letters \( \text{Lab}(f_m) \) and \( \text{Lab}(e_1) \) are either equal or they are copies of each other in different subalphabets \( Y_i \) and \( Y_{i+1} \) (corresponding, respectively, to some parts \( Y'_j \) and \( Y''_j \) of the tape alphabet of the machine \( M \)).

A chain \( \mathcal{A} \) is \textit{non-trivial} if it has at least one cell.

\textbf{Lemma 8.1.} Let \( x \) be a subpath of the boundary of a reduced diagram \( \Delta \) without hubs, and \( \text{Lab}(x) \) a reduced word in a tape subalphabet \( Y_i \) of the machine \( M \). Then no chain \( \mathcal{A} \) can start and end on \( x \).

\textit{Proof.} Proving by contradiction, we assume that a non-trivial chain \( \mathcal{A} \) starts and ends on \( x \). Note that it cannot cross itself since every cell has at most two \( a \)-edges by (iii)(a). Thus, it starts with an edge \( e \) of \( x \) and ends at an edge \( f \) of \( x \), where \( e \neq f \), but \( \text{Lab}(e) = (\text{Lab}(f))^{-1} \) since these two letters belong to the same subalphabet \( Y_j \). We may assume that \( \mathcal{A} \) is chosen so that \( ex'f \) is a subpath of \( x \), where the subpath \( x' \) is of minimal possible length.

Now the chain \( \mathcal{A} \) and the path \( x' \) bound a subdiagram \( \Delta' \) all of whose boundary \( a \)-edges belong to \( x' \). By our observation, every \( q \)-edge on the boundary \( \partial \Delta' \) is a \( p_i^{-1} \)-edge for the same control base letter \( p = p_j \). Also every \( p \)-cell \( \pi \) of \( \Delta' \) is active from both sides. Indeed if a cell \( \pi \) of \( \Delta' \) is not active from both sides it does not belong to the chain \( \mathcal{A} \) by definition. Therefore, \( \pi \) must have a neighbor \((\theta,s_i)\)-cell in \( \Delta' \) by (iv) (consider the maximal \( \theta \)-band of \( \Delta' \) containing \( \pi \)). Then we may apply Lemma 5.6 to the maximal \( s_i \)-band of \( \Delta' \) containing the \((\theta,s_i)\)-cell and obtain a letter of the form \( s_i \) in the boundary label of \( \mathcal{A} \) or in \( \text{Lab}(x) \), a contradiction.

It follows that every \((\theta,q)\)-cell in \( \Delta' \) is a \( p \)-cell (corresponds to a control letter) active from both sides. Since a maximal chain cannot end on a \( p \)-cell, every maximal chain of \( \Delta' \) must start and end on \( x' \).

This property and the minimality of choice for \( x' \) imply \( |x'| = 0 \), and so \( \text{Lab}(x) \) has a non-empty freely trivial subword \( \text{Lab}(ef) \); a contradiction. The lemma is proved. \( \square \)

\textbf{Lemma 8.2.} Let a non-trivial chain \( \mathcal{A} \) cross a maximal \( p \)-band \( \mathcal{C} \) of a reduced diagram \( \Delta \) over \( M \) twice at cells \( \pi' \) and \( \pi'' \). Then there is a \( p \)-cell \( \pi_0 \) in \( \mathcal{C} \) between \( \pi' \) and \( \pi'' \), which is not active from both sides.

\textit{Proof.} As in the proof of Lemma 8.1, there is a subdiagram \( \Delta' \) bounded by the portion of \( \mathcal{A} \) and a segment \( x \) of the side of \( \mathcal{C} \) between \( \pi \) and \( \pi' \). As there, arguing by contradiction, we may assume that \( x \) has no \( a \)-edges. Hence, if there is no cell \( \pi_0 \) lying between \( \pi' \) and \( \pi'' \) in \( \mathcal{C} \) which is not active from both sides, then the cells \( \pi \) and \( \pi' \) must share a \( p \)-edge. Since the chain \( \mathcal{A} \) connects \( \pi \) and \( \pi' \), we obtain that these two cells must have the same \( a \)-letter in the
boundary labels (being letters from the same subalphabet \(Y_i\)), and they are mirror copies of each other as it follows from the defining relations of the group \(M\), having two \(a\)-letters.

We come to a contradiction because \(\Delta\) is a reduced diagram. The lemma is proved.

**Lemma 8.3.** Assume that no \(\theta\)-band of a comb \(\Delta\) has a base with a subword \((pp^{-1}p)\pm 1\), where \(p\) is a control letter. Then every chain \(A\) of \(\Delta\) has at most 9 common \((\theta,a)\)-cells with any \(\theta\)-band \(T\) of \(\Delta\).

**Proof.** Let \(\pi_1, \ldots, \pi_m\) be the common cells of \(A\) and \(T\) counted on \(T\) from left to right. Denote by \(T'\) the subband of \(T\) starting with \(\pi_1\) and ending by \(\pi_m\). Every \((\theta,q)\)-cell of \(T'\) must be a \(p\)-cell since every maximal \(q\)-band of \(\Delta\) crossing \(T'\), has to cross the chain \(A\) too by Lemma 5.6. Since by the assumption of the lemma and by Property (i), the base of \(T\) has no triples of consecutive \(p\pm 1\)-letters, the subband \(T'\) can have at most two \((\theta,q)\)-cells.

Assume that \(A\) crosses \(T\) consequently at 3 \((\theta,a)\)-cells \(\pi_{i_1}, \pi_{i_2}, \pi_{i_3}\), and \(i_1 < i_3 < i_2\), that is, \(A\) has a ‘convolution of a spiral’ \(A'\), starting at \(\pi_{i_1}\) and ending at \(\pi_{i_3}\).

By Lemma 8.1 for the part \(\Delta'\) of \(\Delta\) bounded by the subchain \(A_{23}\) of \(A'\) connecting \(\pi_{i_2}\) and \(\pi_{i_3}\), and the subband of \(T\) connecting the same cells, \(T\) has a \(p\)-cell \(\pi\) between \(\pi_{i_2}\) and \(\pi_{i_3}\). The maximal \(p\)-band \(C\) crossing \(T\) at \(\pi\) must cross \(A'\) at least twice (above and below \(T\)). Hence, there is a part of \(A'\) satisfying, together with \(C\), the condition of Lemma 8.2, and so there is a cell \(\pi_0\) given by that lemma, inside the part \(\Delta''\) of \(\Delta\) bounded by \(A'\) and the part \(T_{13}\) of \(T\) connected \(\pi_{i_2}\) and \(\pi_{i_3}\). But then, according to Property (iv), \(\Delta''\) has to contain some \(s_i\)-cell neighboring \(\pi_0\), contrary to Lemma 5.6 since neither \(A'\) nor \(T_{13}\) have \(s_i\)-cells. Thus, our assumption on the existence of \(A'\) leads to a contradiction, and so \(i_3 > i_2\) if \(i_2 > i_1\).

Assume that \(A\) crosses \(T\) consequently at 4 \((\theta,a)\)-cells \(\pi_{i_1}, \pi_{i_2}, \pi_{i_3}\), and \(\pi_{i_4}\). We may assume that \(i_1 < i_2\), and so \(i_1 < i_2 < i_3 < i_4\). Then again by Lemma 8.1, we have at least 3 \(p\)-cells on \(T'\) (between \(\pi_{i_u}\) and \(\pi_{i_{u+1}}\) for \(u = 1, 2, 3\)), a contradiction. Hence, such a series of common \((\theta,a)\)-cells is impossible, and since \(A\) has at most 2 common \((\theta,q)\)-cells with \(T\), we conclude that traveling along \(A\) one meets \(m \leq 3 + 1 + 3 + 1 + 3 = 11\) cells of \(T\) (at most 3 \((\theta,a)\)-cells, then a \((\theta,q)\)-cell, and so on), and the number of \(a\)-cells among them does not exceed 9.

**Lemma 8.4.** A comb \(\Delta\) has no chain-annuli.

**Proof.** Assume that \(A\) is a chain-annulus. By Lemma 5.6, it must have a link \(\pi_i\). Then there is a left-most maximal \(p^\pm 1\)-band \(C'\) of the comb \(\Delta\) containing a link \(\pi_i\) from
A, where \( p_i \) is a control letter, that is, the subcomb \( \Gamma \) of \( \Delta \) with handle \( C' \) has no links of \( A \) except for those on \( C' \). But \( \partial \pi_i \) has an a-edge from the left by (iii), and so two different a-edges of \( C' \) are connected by an a-band \( A_j \) from the left of \( C' \). We will assume that \( A_j \) is the shortest a-band with this property, and so the \((\theta,q)\)-cells \( \pi^1, \ldots, \pi^m \) situated on \( C' \) between \( \pi_{j-1} \) and \( \pi_j \) (if any) are inside the subdiagram \( \Delta' \) bounded by the chain \( A \). By Lemma 5.6 for \( \Delta' \), these cells cannot have common edges with \((\theta,q)\)-cells which do not correspond to \( p_i^{±1} \). Now it follows from (iv) that the cells \( \pi^1, \ldots, \pi^m \) are active from both sides. Since the cells \( \pi_{j-1} \) and \( \pi_j \) of the chain-annulus are also active from both sides, the existence of the a-band \( A_j \) contradicts Property (vii). The lemma is proved.

\[ \square \]

8.2. An application of quasicomb

The surgery we use in Lemma 8.6 requires a slight modification of the notion of comb.

Let \( \Delta \) be a reduced diagram over \( M \) with boundary \( yz \) such that every maximal \( \theta \)-band of \( \Delta \) has exactly one \( \theta \)-edge from \( y \). Assume that one can construct a \( q \)-band \( Q \) with a top or bottom path \( y_0 \), and \( \text{Lab}(y) \) can be obtained from \( \text{Lab}(y_0) \) after deleting of some a-letters. Then we say that \( \Delta \) is a quasicomb with the support \( y \). As for combs, we use the standard factorization \( yz = y^\alpha z^\delta \) of the boundary path of a quasicomb. The number \( h = h^\delta \) is the \( \theta \)-length \( |y_0| \); the notation \( [\Delta] \) is also extended to quasicombs as well as \( h_- \), and the \( \kappa-, \lambda-, \mu-, \) and \( \nu \)-mixtures.

In particular, every comb is a quasicomb. (Take \( Q \) to be the handle of the comb.) It follows from the definition that the history \( H \) of the quasicomb \( \Delta \) is the history of \( Q \), and by Lemma 5.6, the boundary label of \( \Delta \) uniquely determines the end edges of all maximal \( \theta \)-bands \( T_1, \ldots, T_h \) starting on \( y \). It is easy to see that the set of cells of \( T_1 \) is also uniquely determined by the boundary of \( \Delta \), and by induction, the same is true for \( T_2, \ldots, T_h \). Hence, every quasicomb (in particular, every comb) is a minimal diagram.

**Remark 8.5.** The statements and the proofs of Lemmas 8.3 and 8.4 remain valid if \( \Delta \) is a quasicomb.

We say that a diagram \( \Delta \) admits a (proper) quasicomb \( \Gamma \) if it has a (proper) subcomb \( \Gamma_0 \) such that \( \text{Lab}(z^\Gamma) \equiv \text{Lab}(z^{\Gamma_0}) \), the handle of \( \Gamma_0 \) serves for \( \Gamma \) as \( Q \) in the definition of quasicomb, and the words \( \text{Lab}(y^\Gamma) \) and \( \text{Lab}(y^{\Gamma_0}) \) are equal modulo \((\theta,a)\)-relations. In particular, every subcomb of \( \Delta \) is admitted.

**Lemma 8.6.** Let \( \Delta \) be a comb with history of type (2). Assume that \( \Delta \) has a subcomb \( \Gamma_0 \) of base width at most 3 with a \( s^{±1} \)-handle \( C' \) such that the filling trapezium \( Tp(C',C) \) is not aligned. Also assume that all maximal \( q \)-bands of \( \Gamma_0 \) except for \( C' \), are \( p^{±1} \)-bands. Then \( \Delta \) admits a proper quasicomb \( \Gamma \) such that \( \text{Area}(\Gamma) \leq 5(\delta')^{-1}[\Gamma] \).

**Proof.** By Property (xiii), \( \text{Lab}(y^{\Gamma_0}) \) has a factorization of the form \( u(b_1v_1b_1^{-1}) \ldots (b_nv_mv_n^{-1})w \), where \( b_i^{±1} \) is an a-letter or 1 \((i = 1, \ldots, m)\), \( v_i \) is a group word in \( \theta \)-letters, \( b_i \) commutes with every letter of \( v_i \) by virtue of \((\theta,a)\)-relations, and each of \( u \) and \( w \) has at most one a-letter. Similar property holds for \( \text{Lab}(y^{\Gamma_0}) \). Hence, one can separate the band \( C' \) from the diagram \( \Gamma_0 \) and attach several \((\theta,a)\)-cells to the left and right sides of it, and obtain an auxiliary subdiagram \( E \) with the boundary \( exfY' \) whose label 'almost' equal to the boundary label of \( C' \), but \( \text{Lab}(x) \) (respectively, \( \text{Lab}(x') \)) are obtained from \( \text{Lab}(y^{\Gamma_0}) \) (from \( \text{Lab}(y^{\Gamma_0}) \)).
by deleting the $a$-letters $b_i^{±1}$-s, and therefore each of $x$ and $x'$ has at most 2 $a$-edges.

Then we continue the surgery as follows. We can construct the mirror copies of the $(\theta, a)$-cells attached to $y'^{\Gamma_0}$ in $E$, and attach these copies to the diagram $\Gamma_0 \setminus C'$ to obtain (after possible cancellations of some $(\theta, a)$-cells) a reduced quasicomb $E'$ whose support can be denoted by $x'$ since its label is $\text{Lab}(x')$, and the boundary of $E'$ is $x'(z')^{\Gamma_0}$. Finally, we identify $E'$ and $E$ along $x'$ and (after possible cancellations of $(\theta, a)$-cells) we have a desired quasicomb $\Gamma$ with support $x$. Now, to estimate the area of the minimal diagram $\Gamma$ from above it suffices to estimate $\text{Area}(E')$ (and use Lemma 5.2) since obviously $\text{Area}(E) \leq 3(\text{Area}(C')) \leq 3h$.

Since the path $x$ has at most 2 $a$-edges we obtain

$$h^{\Gamma_0} \leq |x| \leq h^{\Gamma_0} + 2\delta'$$

(8.1)

by the definition of the length of a path.

Note that at most two non-trivial chains of $E'$ can start/end on $x'$ since $|x'|_a \leq 2$. Consequently, by Lemma 8.3 (and Remark 8.5), every $\theta$-band of $E'$ has at most 18 cells belonging to such chains.

Let $d$ be the maximal number of $a$-cells in $\theta$-bands of $E'$. Taking into account the argument of the previous paragraph and the lack of chain-annuli in $E'$, we conclude that there are at least $(d - 18)/9$ maximal chains having both ends on $(z')^{\Gamma_0}$. Now it follows from Lemma 5.20 and inequality (8.1) that

$$|z^\Gamma| \geq |(z')^{\Gamma_0}| + 2 \geq 2 + h^{\Gamma_0} + 2\delta'(d - 18)/9$$

$$\geq 2 + |x| - 2\delta' + 2\delta'(d - 18)/9 \geq |x| + 1 + 2\delta'(d + 2)/9$$

because $\delta' \leq 1/7$. This implies $d + 2 \leq 5(\delta')^{-1}(|z^\Gamma| - |x| - 1)$, and since every $\theta$-band of $E'$ has at most 2 $p$-cells, we have, as required:

$$\text{Area}(\Gamma) \leq \text{Area}(E') + \text{Area}(E) \leq (2 + d)h^\Gamma + 3h^\Gamma < 5h^\Gamma(\delta')^{-1}(|z^\Gamma| - |x|).$$

\[\square\]

**Lemma 8.7.** Let $\Delta$ be a comb with base width $b \geq N$. Assume that $\Delta$ has a one-step history and has no active $k$- or $k'$-cells. Then $\Delta$ admits a proper quasicomb $\Gamma$ such that $\text{Area}(\Gamma) \leq 5(\delta')^{-1} [\Gamma]$.

**Proof.** We first assume that the history of $\Delta$ is of type (2). It follows from the conditions of the lemma and (ii) that $\Delta$ cannot have $k$- or $k'$-cells at all, with the possible exception of the cells of the first and the last $(12)^{±1}$ or $(23)^{±1}$-bands. By Lemma 7.20, we may also assume that the base of any $\theta$-band of $\Delta$ does not have subword $(pp^{-1}p)^{±1}$, where $p^{±1}$ is any control letter.

Therefore, a left-most maximal $s^{±1}$-, $k^{±1}$-, $(k')^{±1}$-, $t^{±1}$-, or $(t')^{±1}$-band $C'$ of $\Delta$ is a handle of a subcomb $\Gamma$ of base width $b^\Gamma \leq 3$ and all other $q$-bands of $\Gamma$ (if any exist) correspond to the same control letter $p^{±1}$ by (i). One may assume that $C'$ has $h' > (\delta')^{-1}$ cells because otherwise Lemma 7.15 is applicable to the subcomb with handle $C'$. 


The filling trapezium $\Delta' = Tp(C', C)$ has height $h' > (\delta')^{-1} > 2$, and therefore it has a maximal $\theta$-band without $k$- or $k'$-cells. Hence, the base of $\Delta'$ has no $k$- or $k'$-letters. Since $b \geq N$, $\Delta'$ is of base width $b' \geq N - 2$, but having no $k$- or $k'$-letters, the base of $\Delta'$ is not aligned or has only $t^\pm 1$ or $(t')^\pm 1$-letters by Remark 5.16. In the former case, we are done by Lemma 8.6. In the latter case, we may refer to Lemma 7.18 since there are no derivative $p^\pm 1$-bands for a $t$-band by Property (i).

If the history of $\Delta$ is of type (1) or (3), then it follows from the absence of active $k$- and $k'$-cells in $\Delta$ that all $q$-cells of $\Delta$ are passive by Property (ii), and the statement follows from Lemma 7.18.

□

9. Combs with one-step histories

In this section, we obtain the estimates of the areas of one-step combs. Lemmas 9.2, 9.6, and 9.7 will be used in the next sections.

**Lemma 9.1.** Let $\Gamma$ be a comb whose proper subcombs have no active $k$- or $k'$-cells and the handle $C$ of $\Gamma$ is passive from the left. Assume that $\Gamma$ has one-step history and admits no proper quasicombs $\Delta$ such that $\text{Area}(\Delta) \leq 5(\delta')^{-1}|\Delta|$. Assume that $\Gamma$ has at most $L_0$ odd maximal $\theta$-bands for some $L_0$. Then (a) $\text{Area}(\Gamma) \leq 5(\delta')^{-1}h(|z'| - |y'| + L_0)$; (b) if $C$ is also passive from the right, then $\text{Area}(\Gamma) < 5(\delta')^{-1}h(|z| - |y| + L_0)$.

**Proof.** (a) By Lemma 8.7, we may assume that the base width $b$ of $\Gamma$ does not exceed $N$. By Lemma 7.20, we may assume that $\Delta$ has no bands with $p^\pm 1p^\mp 1$ in the base or arbitrary control letter $p$. By Lemma 8.3, every chain of $\Gamma$ and every $\theta$-band of $\Gamma$ have at most 9 $a$-cells in common. We also use below that $\Gamma$ has no chain-annuli by Lemma 8.4.

At most $2(b - 1)L_0$ non-trivial maximal chains can start/terminate on the odd $\theta$-bands of $\Gamma$ by (iii)(a)–(c). Let $T$ be a $\theta$-band of $\Gamma$ having maximal number of $a$-cells $d$. Then among maximal chains crossing $T$, we have at least $(d - 18(b - 1)L_0)/9$ chains with both ends on $z'$, and so $|z'|_a \geq (d - 18(b - 1)L_0)/9$. Therefore, by Lemma 5.20,

$$|z'| - |y'| \geq 2(b - 1) + 2\delta'(d - 18(b - 1)L_0)/9 > 2\delta'(d + b - 1)/9 - 4\delta'NL_0$$

since $b \leq N$. Hence, $d + b - 1 \leq (9/2)(\delta')^{-1}(|z'| - |y'|) + 18NL_0$, and therefore

$$\text{Area}(\Gamma) \leq (b - 1 + d)h \leq 5h(\delta')^{-1}(|z'| - |y'| + L_0)$$

since $18N \leq (\delta')^{-1}/2$.

(b) Since $C$ is passive from the right, we have $|y| \leq |y'| + 2\delta'$ and $|z| = |z'| + 2$. Therefore, $|z| - |y| > |z'| - |y'| + 1$, and so statement (a) implies (b) because $\text{Area}(C) \leq h$.

□

**Lemma 9.2.** Let $\Gamma$ be a one-step comb of base width $b \leq 15N$. Assume that its handle $C$ is a $t^\pm 1$- or $(t')^\pm 1$-band. Then

(1) $\Gamma$ has a subcomb $\Delta$ such that its handle $C^\Delta$ is passive from the right, and

$$\text{Area}(\Delta) \leq c_1\left(|\Delta| + \frac{1}{2}h^\Delta h^\Delta\right). \quad (9.1)$$

(2) $\text{Area}(\Gamma) \leq c_1(|\Gamma| + \kappa^c(\Gamma))$. 

Proof. (1) We may assume that \( \Gamma \) has no \( t^{+1} \) or \( (t')^{+1} \)-bands except for \( C \), since otherwise we obtain a smaller subcomb \( \Delta \), and \( \Delta \) also satisfies the assumptions of the lemma. In particular, the derivative bands \( C_1, \ldots, C_s \) are all \( k^{-1} \)- or \( k' \)-bands by Property (i).

Assume first that the history of \( \Gamma \) is of Step (1) or (3), and the derivative bands are not active from the right. Then either \( \Gamma \) has a maximal \( k' \)- or \( (k')^{-1} \)-band \( C' \) active from the left and passive from the right (and there exists the left-most band with this property), or all its \( q \)-bands are passive by Properties (i) and (ii). In any case, Lemma 7.18(b) is applicable to a subcomb of \( \Gamma \). Thus, we may further assume that the history of \( \Gamma \) is of Step 2. Property (v) implies that the derivative bands of \( \Gamma \) are active from the right.

The sum of areas of derivative subcombs \( \Gamma_1, \ldots, \Gamma_s \) is at most \( \sum(60Nh_i^2 + 2\alpha h_i) \) by Lemma 7.13, where \( \alpha = |z|i \). Hence, by Property (vii) and Lemma 7.10,

\[
\text{Area}(\Gamma) \leq h(h_- + 3\alpha + 1) + 60N \sum h_i^2. \tag{9.2}
\]

We also recall that by Lemma 7.10, \( \alpha \leq (\delta')^{-1}(|z| - |y| - 1) \) since \(|y| = h\).

We assume first that \( h_i \leq 2 \sum_{j \neq i} h_j \) for every \( i \). Then \( h_- \geq \frac{1}{3} \sum h_i \), and therefore \( \sum h_i^2 \leq h \sum h_i \leq 3hh_- \). Now it follows from (9.2) and the subsequent estimate for \( \alpha \) that

\[
\text{Area}(\Gamma) \leq (60N)3hh_- + hh_- + 3(\delta')^{-1}[\Gamma] \leq c_1[\Gamma] + c_1hh_-/2,
\]

by the choice of \( c_1 \), as required.

Now assume that there is \( i_0 \) such that \( h_{i_0} > 2 \sum_{j \neq i_0} h_j \). Then less than \( h_{i_0}/2 \) maximal \( a \)-bands starting on \( C_{i_0} \) end on other derivative bands, and we can refer to Lemma 7.17(b).

Therefore, \( \sum \text{Area}(\Gamma_i) \leq (\delta')^{-2}[\Gamma] \). Hence,

\[
\text{Area}(\Gamma) \leq (\delta')^{-2}[\Gamma] + h(h_- + 3\alpha + 1) \leq (\delta')^{-2}[\Gamma] + hh_- + 3(\delta')^{-1}[\Gamma] \\
\leq c_1[\Gamma] + hh_-/2,
\]

as required, since \( c_1 > (\delta')^{-2} + 3(\delta')^{-1} \). Thus the desired inequality is true in any case.

(2) We will induct on the number of maximal \( q \)-bands in \( \Gamma \). By (1), we have a subcomb \( \Delta \) of \( \Gamma \) satisfying (9.1), and therefore

\[
\text{Area}(\Delta) \leq c_1[\Delta] + c_1\kappa^c(\Delta) \tag{9.3}
\]

by Lemma 7.5.

One may assume that the subcomb \( \Delta \) of \( \Gamma \) is proper since otherwise there is nothing to prove. Now \( \Gamma \) is a union of \( \Delta \) and the remaining comb \( \Delta' \) with the handle \( C \). We observe that

\[
y^{\Delta'} = y \quad \text{and} \quad |z^{\Delta'}| = |z| - |z^{\Delta}| + |y^{\Delta}| \tag{9.4}
\]

by Lemma 5.20(b). By the inductive hypothesis,

\[
\text{Area}(\Delta') \leq c_1[\Delta'] + c_1\kappa^c(\Delta'). \tag{9.5}
\]

Now the sum of the first summands on the right-hand sides of (9.3) and (9.5) does not exceed \( c_1h(|z| - |y|) \) because \((|z^{\Delta'}| - |y^{\Delta'}|) + (|z^{\Delta}| - |y^{\Delta}|) = |z| - |y| \) by (9.4). The sum of the second summands of (9.3) and (9.5) does not exceed \( c_1\kappa^c(\Gamma) \) by Lemma 7.3(c), and the lemma is proved since \( \text{Area}(\Gamma) = \text{Area}(\Delta) + \text{Area}(\Delta') \).

Let \( \Gamma \) be a comb with a handle \( C \). Assume that \( \Gamma \) is a subcomb of (or can be embedded as a subcomb in) a larger comb \( \bar{\Gamma} \) with a handle \( \bar{C} \), and the filling trapezium is \( Tp(C, \bar{C}) \). Then the comb \( \bar{\Gamma} \) is called an extension of \( \Gamma \). The extension is called regular if the base width of \( Tp(C, \bar{C}) \) is at least \( N \). A comb \( \Gamma \) is called regular if there exists a regular extension of \( \Gamma \). Recall that by definition of comb, every cell (of the handle) of a subcomb of \( \Gamma \) is connected with \( C \) by a \( \theta \)-band. Therefore, a subcomb of a regular comb is regular itself.
Remark 9.3. A regular comb is organized better than a random one because its history
coincides with the history of the filling trapezium $Tp(C, \hat{C})$ having a sufficiently long base.
Therefore, this history is a subject of some restrictions imposed on trapezia by Proposition 5.17.
Recall that by Lemma 5.10, the properties of trapezia reflect all the features of the computations
executed by the machine $M$. In particular, the next lemma uses Property (xiv) based on the
aperiodicity of the histories formulated in Lemma 4.29, which, in turn goes back to Lemmas 4.15
and 2.2(a).

Lemma 9.4. Let $\Gamma$ be a one-step regular comb, and the handle $C$ of $\Gamma$ is active from the
left $k$- or $(k')^{-1}$-band. Assume that $\Gamma$ has neither $t^{\pm 1}$- nor $(t')^{\pm 1}$-bands and has no active from
the left maximal $k$- or $(k')^{-1}$-bands except for $C$, and $\Gamma$ admits no proper quasitcomb $\Delta$ such
that $\text{Area}(\Delta) \leq 5(\delta')^{-1}|\Delta|$. Then $\text{Area}(\Gamma) \leq 16(\delta')^{-2}|\Gamma|$.

Proof. Let $C_1, C_2, \ldots, C_s$ be the set of derivative subbands (connected with $C$ by simple
$\theta$-bands). Since none of them are a $t^{\pm 1}$- or $(t')^{\pm 1}$-band, they must be $k^{-1}$- or $k'$-bands active
from the right by (i). Moreover, since the derivative subcombs $\Gamma_1, \ldots, \Gamma_s$ has no $k$- or $k'$-
'cells active from the left, it is easy to see from Property (i) that $\Gamma$ has no active $k^{-1}$- or $(k')^{\pm 1}$-
cells except for those belonging to $C, C_1, \ldots, C_s$. By (vii), every maximal $a$-band
starting on $\Gamma$ ends either on some $C_i$ or on $z = z^{k'}$. Besides, if $T$ is a simple $\theta$-band starting
on $C$, then every maximal $a$-band starting on $T$ ends either on one of the bands $C, C_1, \ldots, C_s$, or
on $z$.

Assume that $T$ is a simple $\theta$-band of maximal length $d$ among the simple $\theta$-bands starting
on $C$. Then the total number of cells in all simple $\theta$-bands is at most $dh$.

Let $T_1$ and $T_2$ be top and bottom of $T$. Since no non-trivial $a$-band starts and ends on $C$, then
no $a$-band $A$ starting on $T_1$ (and having no cell from $T$) ends on $C$ or no $a$-band $A$
starting on $T_2$ (and having no cells from $T$) ends on $C$. We consider only the first variant.

If there are $m$ maximal $\theta$-bands above $T$, then at least $d - 1$ maximal $a$-bands starting on
$T$ and $m - 1$ maximal $a$-bands starting on $C$ on at most $m (q, \theta)$-cells of the derivative
bands and on $z$. Therefore, at least $d - 2$ of them end on $z$, and so $|z| - |y| \geq 2 + (d - 2)\delta'$
by Lemma 5.20(a). We set $\Delta_i = Tp(C_i, C) (i = 1, \ldots, s)$ and $d \geq 2$ if $s > 0$ since otherwise two
neighbor $k$- or $k'$-cells of $T$ form a non-reduced subdiagram). By Property (xv), $H_i = u_iw_i^{k_i},$
where $|u_i|, |v_i| \leq (d - 1)/2, |w_i| \leq d - 1$.

Consider a regular extension $\Gamma$ of the comb $C$. Let $\hat{C}$ be a handle of $\Gamma$ and $\Pi_i = Tp(C_i, \hat{C})$.

If the base of $\Pi_i$ is not aligned, and $\Gamma_i$ has an $s_j^{\pm 1}$-band for some $j$, then it follows
from Property (i) and Lemma 7.20 that $\Gamma_i$ satisfies the assumptions of Lemma 8.6, and we have a contradiction with the hypothesis of Lemma 9.4. Therefore, $\Gamma_i$ has no $s_j^{\pm 1}$-
'bands, and so, by Property (i), it has no maximal $q$-bands at all except for $C_i$. Hence, by
Lemma 7.20, $\text{Area}(\Gamma_i) \leq (\delta')^{-1}h_i(|z_i' | - |y_i' | + 1)$ in standard notation, if the base of $\Pi_i$ is not
aligned.

Assume that the base of $\Pi_i$ is aligned. It starts with $k$ (or with $(k')^{-1}$ and the second letter
of the base is not $k^{-1}$ (not $k'$) by Lemma 3.4. Therefore, by Property (i), the second letter is the
copy of the first letter (or of the inverse of the last letter) of the standard base $B$ of $M_3$. Since
the base of $\Pi$ is aligned and its base width at least $N > 2\|B\|$, this base is large. Therefore, if $k_i \geq 3$, then the $w_k$-part of $\Pi$ has no odd $\theta$-bands by (xv), and so the number of maximal odd
$\theta$-bands is at most $|u_i| + |v_i| \leq d - 1$. If $k_i < 2$, then $h_i \leq |u_i| + |w_i| + |v_i| \leq 3(d - 1)$. Thus, in
d any case, $L_i$ of odd maximal $\theta$-bands in $\Pi_i$ (and in $\Gamma_i$) is at most $3(d - 1)$.

Using $z_i, z_i', y_i$, and $y_i'$ in the standard way for the subcombs $\Gamma_i$-s, we have $\text{Area}'(\Gamma_i) \leq 5(\delta')^{-1}h_i(|z_i' | - |y_i' | + 1)$ by Lemma 9.1, if the base of $\Pi_i$ is aligned. Thus, in any case,
$\text{Area}'(\Gamma_i) \leq 5(\delta')^{-1}h_i(|z_i' | - |y_i' | + 3(d - 1))$. Since $|y_i' | = h_i$ and $|y| = h$, the sum of these areas
is at most $5(\delta')^{-1}h(|z| - |y| + 3(d - 1))$. Finally,

$$\begin{align*}
\text{Area}(\Gamma) &\leq dh + 5(\delta')^{-1}h(|z| - |y| + 3(d - 1)) \\
&\leq h(d + 5(\delta')^{-1}(|z| - |y| + 3(d - 1))).
\end{align*}$$

Recall that $d - 2 \leq (\delta')^{-1}(|z| - |y|)$ and $|z| - |y| \geq 2$ since $y$ has no $a$-edges. Therefore,

$$d + 5(\delta')^{-1}(|z| - |y| + 3(d - 1)) \leq 16(\delta')^{-2}(|z| - |y|),$$

and so by (9.6), $\text{Area}(\Gamma) \leq 16(\delta')^{-2}[\Gamma]$, as required.

We call a (quasi)comb $\Gamma$ long if $|z| = |z^\Gamma| > |y| = |y^\Gamma|$. If the handle of a comb $\Gamma$ is passive from the right, then $\Gamma$ is long since $|z| \geq 2 + h$ and $|y| = h$. Obviously, a (quasi)comb $\Gamma$ is long if $\text{Area}(\Gamma) \leq c[\Gamma]$ for some $c > 0$.

**Remark 9.5.** Observe that if $\Gamma$ is a long subcomb of a diagram $\Delta$, then for the complementary subdiagram $\Delta' = \Delta \setminus \Gamma$ cut from $\Delta$ along the path $y$, we have $|\partial \Delta'| < |\partial \Delta|$ by Lemma 5.20.

If $\Gamma$ is a long quasicomb admitted by a minimal diagram $\Delta$ with boundary path $z z'$ (where $z = z^\Gamma$), then there exists a minimal diagram $\Delta'$ with boundary label $\text{Lab}(y^{-1}) \text{Lab}(z')$. It follows from Lemma 5.6 that if $\Delta$ is a comb with handle $C$, and $\Delta$ admits a proper quasicomb $\Gamma$, then $\Delta'$ is also a comb with the same handle $C$ but with fewer number of maximal $q$-bands than in $\Delta$. It is clear that $\text{Area}(\Delta) \leq \text{Area}(\Gamma) + \text{Area}(\Delta')$ (or see Lemma 5.2). We also use notation $\Delta \setminus \Gamma$ for such a ‘complement’ $\Delta'$ of $\Gamma$. Then all the statements of Lemma 7.3 hold if one replaces ‘subcomb $\Gamma$’ by ‘admitted quasicomb $\Gamma'$ in their formulations because the quasicomb $\Gamma$ and the subcomb $\Gamma_0$ from the definition of admitted quasicomb have equal $\kappa$, $\lambda$, $\mu$, and $\nu$-necklaces.

**Lemma 9.6.** Let $\Gamma$ be a one-step regular comb of base width $b \in [2N, 15N]$. Then $\Gamma$ admits a long quasisubcomb $\Delta$ such that

$$\text{Area}(\Delta) \leq c_1([\Delta] + \frac{1}{2}h^2h[\Delta]) \leq c_1([\Delta] + \kappa^\kappa(\Delta)).$$

**Proof.** For the beginning, we recall that every subcomb of a regular comb is regular. By Lemma 9.2 (and Lemma 7.5), one may assume that $\Gamma$ has neither $t$- nor $t'$-cells. If $\Gamma$ has a subcomb $\Delta$ of base width at least $N$ without active $k^\pm 1$- or $(k')^\pm 1$-cells, then one can apply Lemma 8.7 since $c_1 \geq 5(\delta')^{-1}$. Otherwise, by (ii), there is a maximal, active from the left or from the right $q$-band $C'$ corresponding to a $k^\pm 1$- or $(k')^\pm 1$-letter, such that the subcomb $\Delta$ with handle $C'$ has no other maximal active $k^\pm 1$- or $(k')^\pm 1$-bands and the base width of the filling trapezium $T p(C', C)$, is at least $N$ since $b \geq 2N$. If $C'$ is active from the left, then the statement follows from Lemma 9.4. Otherwise, $C'$ is active from the right, and from the right of $C'$, there must be a maximal band $C'' \neq C$ corresponding to $k$- or $(k')^{-1}$-letter, which therefore is active from the left. (Recall that $\Gamma$ has neither $t$- nor $t'$-cells, and so such $C''$ exists by Property (i).) Lemma 9.4 is now applicable to the subcomb with handle $C''$, and so Lemma 9.6 is proved in any case.

**Lemma 9.7.** Let $\Gamma$ be a regular comb having history of type (1) or (3) and base width at most $15N$. Assume that the handle $C$ is a passive from the left $k^\pm 1$- or $(k')^\pm 1$-, or $s_0$-band. Then

$$\text{Area}'(\Gamma) \leq c_1h(|z'| - |y'| + 1) + c_1\kappa^\kappa(\Gamma).$$

\[\square\]
Proof. At first, we will prove that either $\Gamma$ admits a proper quasicomb $\Delta$ such that
\[
\text{Area}(\Delta) \leq c_1|\Delta| + c_1\kappa^c(\Delta)
\]  
(9.7)
or $\text{Area}'(\Gamma) \leq c_1 h(|z'| - |y'| + 1) + c_1\kappa^c(\Gamma)$.

By Lemma 9.2, we may assume that $\Gamma$ has neither $t^{\pm 1}$-bands nor $(t')^{\pm 1}$-bands, and it also has neither $k$-bands nor $(k')^{-1}$-bands active from the left by Lemma 9.4. Therefore, by Property (i), there are no active $k^{\pm 1}$- or $(k')^{\pm 1}$-cells except for those in $C$. Since the step history of $\Gamma$ is (1) or (3), every other $q$-band of $\Gamma$ is passive by (ii), and if there exist other $q$-bands, then we can find the desired subcomb $\Delta$ by Lemma 7.18(b). If $\Gamma$ has no maximal $q$-bands except for the handle $C$, then the statement follows from Lemma 7.18(a).

To complete the proof, we will induct on the number of maximal $q$-bands in $\Gamma$, as in Lemma 9.2(b). By the previous argument, we may assume that $\Gamma$ admits a proper (quasi)subcomb $\Delta$ satisfying (9.7), since otherwise there is nothing to prove. Now $\Gamma$ is a union of $\Delta$ and the ‘complement’ $\bar{\Delta} = \Gamma \setminus \Delta$ (see Remark 9.5) with the handle $C$. By the inductive hypothesis,
\[
\text{Area}'(\Delta) \leq c_1 h\bar{\Delta}(|(z')\bar{\Delta}| - |(y')\bar{\Delta}| + 1) + c_1\kappa^c_{C, 1}(\bar{\Delta}).
\]  
(9.8)

Note that $h\bar{\Delta} \leq h$. It follows that the sum of the first summands of (9.7) and (9.8) does not exceed $c_1 h(|z'| - |y'| + 1)$. The sum of second summands of (9.7) and (9.8) does not exceed $c_1\kappa^c_{C, 1}(\Gamma)$ by Lemma 7.3(c), and the lemma is proved since $\text{Area}'(\Gamma) = \text{Area}(\Delta) + \text{Area}'(\bar{\Delta})$. \(\square\)

10. Combs with incomplete sets of steps

In this section, we analyze combs whose histories have no rules of one of the Steps (1) and (3), and the main goal is Lemma 10.6. Lemmas 10.5 and 10.6 utilize the $\lambda$-mixture, but unfortunately, this parameter can be negative for some other combs. Therefore, first of all, we have to bound it from below in terms of other ‘quadratic’ parameters of combs.

Lemma 10.1. Let $\Gamma$ be a comb whose handle $C$ is either (a) a $t^{\pm 1}$-band or (b) a $(t')^{\pm 1}$-band. Respectively, let the history $H$ of $\Gamma$, either (a) have rules (23)$^{\pm 1}$ but no rules (12)$^{\pm 1}$ or (b) have rules (12)$^{\pm 1}$ but no rules (23)$^{\pm 1}$. Then $\lambda^c(\Gamma) \geq -8(\delta')^{-1}[\Gamma] - 36\kappa^c(\Gamma)$.

Proof. We shall prove the variant (a) only. For this goal, we consider the set $T$ of maximal (23)$^{\pm 1}$-bands of $\Gamma$, which do not cross derivative bands of $\Gamma$, and so both their edges on $y^d$ and on $z^d$ are labeled by the same $\theta$-letter, and therefore they are non-special edges by the definition of $\lambda^c$. (The set $T$ may be empty.) Consider all (non-empty) maximal combs $\Delta_j$ ($j = 1, \ldots, r$) in which $\Gamma$ is separated by the bands of $T$. (The combs $\Delta_j$-s do not contain the separating $\theta$-bands from $T$.) Then
\[
\lambda^c(\Gamma) = \sum_j \lambda^c(\Delta_j)
\]  
(10.1)
because two arbitrary white beads which are not separated by a black one in $z = z^\Gamma$ or in $y = y^d$ belong to the boundary of some $\Delta_j$ since black beads are placed on $q$-edges and also on non-special $\theta$-edges (and also, every bead of $\partial \Delta_j$ is on $z$). Below we call an edge black (white) if its middle point is a black (white) bead.

Therefore, we will estimate $\lambda^c(\Delta_j)$ for every $j$ from below and then will use (10.1). Clearly, this number is at least $-\lambda^c(y(j))$ ($j = 1, \ldots, r$), where $y(j) = y^\Delta_j$. To give an upper bound for $\lambda^c(y(j))$, we apply Lemma 6.2(e) to $y_j$ and select an appropriate ‘black’ edge $e_j$ (if any exists) on $y(j)$ such that $y(j) = y_2(j)e_jy_1(j)$ with $m_1(j)$ and $m_2(j)$ white beads on $y_1(j)$ and
\[ y_2(j), \text{ respectively, and} \]
\[
\lambda^c(y(j)) \leq 2m_1(j)m_2(j). \tag{10.2}
\]

Since the path \( y \) has no \( q \)-edges, \( e_j \) is a (23)-edge. Let \( T_j \) be the maximal (23)-band of \( \Delta_j \) containing the edge \( e_j \). By the choice of \( \Delta_j \), the \( \theta \)-band \( T_j \) crosses a derivative band \( C_{i(j)} \) of \( \Gamma \), and this derivative band is a \( k^{-1} \)-band by Properties (i) and (v). The step history of \( C_{i(j)} \) has no subwords \((23)^{-1}(2)(23)\) by (viii) applied to the filling trapezium \( Tp(C_{i(j)}, C) \), and so the top or the bottom path of \( T_j \) cuts the derivative band in two parts such that one of them is a \( k^{-1} \)-subband \( D_j \) with step history \( (2) \) (without the cell from \( T_j \)). We denote by \( d(j) \) the length of this subband. Every maximal \( \theta \)-band crossing a derivative band, also crosses the handle of \( \Delta_j \), so we may assume that \( D_j \) belongs to the union \( E_j \) of the maximal \( \theta \)-bands of \( \Gamma \) ending on the \( m_1(j) \) white edges of \( \mathbf{z}^{\Delta_j} \), because one can interchange \( m_1(j) \) and \( m_2(j) \) in (10.2), in particular, \( d(j) \leq m_1(j) \). We consider three cases.

1. If \( d(j) \leq m_1(j)/2 \), then we say that \( j \in J_1 \subset \{1, \ldots, r\} \).
2. If \( d(j) > m_1(j)/2 \) and at least \( d(j)/2 \) maximal \( a \)-bands starting on \( D_j \) end on derivative bands of \( \Gamma \) non-equal to \( C_{i(j)} \), then \( j \in J_2 \).
3. If \( d(j) > m_1(j)/2 \) and less than \( d(j)/2 \) maximal \( a \)-bands starting on \( D_j \) end on derivative bands non-equal to \( C_{i(j)} \), then \( j \in J_3 \).

Case (1): In this case, the end \( f \) of \( D_j \) is a \( q \)-edge factorizing the path \( \mathbf{z} = \mathbf{z}^{\Delta_j} \) in a product \( \mathbf{z}' \mathbf{z}'' \), where \( \mathbf{z}' \) has at least \( m_1(j)/2 \) white \( \theta \)-edges (the ends of maximal \( \theta \)-bands from \( E_j \) non-crossing \( D_j \)) and \( \mathbf{z}'' \) has at least \( m_2(j) \) white edges. Therefore, \( m_2(j)m_1(j)/2 \leq \kappa(\mathbf{z}^{\Delta_j}) = \kappa^c(\Delta_j) \). Hence, by inequality (10.2) and by Lemma 7.9(a), we have

\[
\sum_{j \in J_1} \lambda^c(y(j)) \leq 2 \sum_{j \in J_1} m_1(j)m_2(j) \leq 4 \sum_{j \in J_1} \kappa^c(\Delta_j) \leq 4\kappa^c(\Gamma). \tag{10.3}
\]

Case (2): In this case, at least \( d(j)/2 \) \( a \)-bands connect \( C_{i(j)} \) with a different derivative band. Therefore, the number of \( a \)-bands connecting pairwise different derivative bands of \( \Gamma \) is at least \( \frac{1}{2} \sum_{j \in J_2} d(j) \). On the other hand, the same number does not exceed \( h_- = h_-^\Gamma \) by Lemma 7.7, whence \( \sum_{j \in J_2} d(j) \leq 4h_- \). Since \( m_1(j) \leq 2d(j) \), this inequality implies \( \sum_{j \in J_2} m_1(j) \leq 8h_- \). Therefore, by Lemma 7.7 and (10.2), we have

\[
\sum_{j \in J_2} \lambda^c(y(j)) \leq 2(8h_-) \sum_{j \in J_2} m_2(j) \leq 16h_- h \leq 32\kappa^c(\Gamma). \tag{10.4}
\]
Case (3): Since the step history of the $k^{-1}$-band $D_j$ is (2), every cell of this band is active from the right by (ii), and so there are no $a$-bands starting and ending on the same $D_j$ by (vii). No $a$-band starting on $D_j$ can cross $T_j$ since there are no $(\theta, a)$-cells between the intersection cells of $T_j$ with $C_{j(i)}$ and with $C$ by (v). Thus in Case (3), more than $d(j)/2$ $a$-bands starting on $D_j$ end on $\partial \Gamma$. Therefore, $|z|_a \geq \sum_{j \in J_3} d(j)/2 \geq \sum_{j \in J_3} m_1(j)/4$. Hence, by Lemma 7.10(a),

$$\sum_{j \in J_3} \lambda^c(y(j)) \leq 2 \sum_{j \in J_3} m_1(j)m_2(j) \leq 8|z|_a h \leq 8(\delta')^{-1}[\Gamma]. \quad (10.5)$$

Altogether, inequalities (10.1) and (10.3)–(10.5) imply the inequality

$$\lambda^c(\Gamma) \geq -\sum_{j \in J_1 \cup J_2 \cup J_3} \lambda^c(y(j)) \geq -36\kappa^c(\Gamma) - 8(\delta')^{-1}[\Gamma]. \quad \square$$

Lemma 10.2. Let $\Delta$ be a regular comb with step history (2)(1)(2) or (2)(3)(2), where the first or the last (2) can be absent. Let $H \equiv H(1)H(2)H(3)$ be the corresponding step decomposition of the history $H$. Assume that the handle $C$ is $k^{-1}$- or $k^{-1}$-$\epsilon$, or $s_0$-band and the base width of $\Delta$ is at most $15N$. Let $l = \|H(1)\| + \|H(3)\|$, then $\text{Area}(\Delta) \leq c_2(h(|z'| - h + l + 1) + \kappa^c(\Delta))$, where $h = \|H\|$ and $z' = z'\Delta$.

Proof. Let $\Delta(i)$ be the $H(i)$-part of $\Delta$, $i = 1, 2, 3$. We will abbreviate $h(1) = h^{\Delta(1)}$, and so on. By (ii), the handle $C$ is passive from the left, and so by Lemma 9.7,

$$\text{Area}'(\Delta(2)) \leq c_1 h(2)(|z'(2)| - h(2) + 1) + c_1\kappa^c(\Delta(2)) \quad (10.6)$$

and therefore the statement is true if $l = 0$ since $c_2 > c_1 + 1$. Below we assume that $l \geq 1$.

Since the base width of $\Delta$ does not exceed $15N$, we have by Lemma 7.13,

$$\text{Area}(\Delta(1)) + \text{Area}(\Delta(3)) \leq 60N(h(1)^2 + h(3)^2) + 2(h(1)\alpha(1) + h(3)\alpha(3)), \quad (10.7)$$

where $\alpha(1) = |z(1)|_a, \alpha(3) = |z(3)|_a$.

Let $x_{12}$ be the common part of the boundaries of $\Delta(1)$ and $\Delta(2)$. By Lemma 7.13,

$$|x_{12}|_a \leq (\alpha(1) + 120Nh(1))/2 \leq (\alpha + |x_{12}|_a + 120Nh(1))/2, \quad (10.8)$$

where $\alpha = |z|_a = |z\Delta|_a$. From (10.8) and similar inequality with $x_{23}$, we have

$$|x_{12}|_a \leq \alpha + 120Nh(1) \quad \text{and} \quad |x_{23}|_a \leq \alpha + 120Nh(3). \quad (10.9)$$

Therefore

$$\alpha(1) \leq |x_{12}|_a + \alpha \leq 2\alpha + 120Nh(1) \quad \text{and} \quad \alpha(3) \leq 2\alpha + 120Nh(3). \quad (10.10)$$

Now we use inequalities (10.7), (10.10), equality $h(1) + h(3) = l$, and Lemma 7.10(a) to obtain inequality

$$\text{Area}(\Delta(1)) + \text{Area}(\Delta(3)) \leq 60Nl^2 + 2l(2\alpha + 120Nl)$$

$$\leq 60Nl^2 + 2l(2(\delta')^{-1}(|z'| - h) + 120Nl). \quad (10.11)$$

We also have $|z'(2)| \leq |z'| + |x_{12}| + |x_{23}| \leq |z'| + 2\alpha + 120Nl$ by (10.9). Therefore,

$$|z'(2)| - h(2) \leq |z'| + 2\alpha + 120Nl - (h - l) \leq |z'| - h + 2\alpha + 121Nl.$$ 

Hence, $|z'(2)| - h(2) \leq (|z'| - h)(2(\delta')^{-1} + 1) + 121Nl$ by Lemma 7.10(a). Using this estimate we deduce from (10.6) and Lemma 7.9(b) that

$$\text{Area}'(\Delta(2)) \leq c_1 h(2)(|z'(2)| - h(2) + 1) + c_1\kappa^c(\Delta(2))$$

$$\leq c_1 h(|z'| - h)(2(\delta')^{-1} + 1) + 122Nl) + c_1\kappa^c(\Delta).$$
In turn, this inequality and (10.11) show that
\[
\text{Area}(\Delta) \leq \text{Area}(\Delta(1)) + \text{Area}(\Delta(3)) + \text{Area}'(\Delta(2)) + h
\]
\[
\leq 60Nh_1 + 2h(2(\delta')^{-1}|z'| - h) + 120Nl
\]
\[
+ c_1 h((|z'| - h)(2(\delta')^{-1} + 1) + 122Nl) + c_1 \kappa c(\Delta) + h
\]
\[
\leq c_2 h(|z'| - h + l) + c_2 \kappa c(\Delta)
\]
because \(l \geq 1, |z'| - h \geq 0\) and \(c_2 \geq c_1 \max(3(\delta')^{-1}, 123N)\). The lemma is proved. \(\square\)

**Lemma 10.3.** Let \(\Gamma\) be a regular comb of base width \(b \leq 15N\). Assume that it has no one-step long subcombs \(\Delta\) with \(\text{Area}(\Delta) \leq c_1 \kappa c(\Delta)\). Let the history \(H\) of \(\Gamma\) either (a) have rules (23)\(\pm^2\) but no rules (12)\(\pm^1\) or (b) have rules (12)\(\pm^1\) but no rules (23)\(\pm^1\). Also assume that in case (a), \(C\) is a \((t')\pm^1\)-band, and in case (b), \(C\) is a \((t')\pm^1\)-band. Then \(\text{Area}(\Gamma) \leq c_2(2(\delta')^{-1}|\Gamma| + 6\kappa c(\Gamma)) \leq c_3|\Gamma| + c_2 \mu c(\Gamma)\).

**Proof.** Case (b): We consider the system of derivative bands \(C_1, \ldots, C_s\). As usual, \(\Gamma_i\) is the derivative subcomb with handle \(C_i\). If a derivative band is a \((t')\pm^1\)-band, then it must have a one-step history since the base of a \((12)\)-band cannot contain subwords \((t')\pm^1(t')\pm^1\) by (vi). By Lemma 9.2(2), this would contradict our assumption on long subcombs. Therefore, all the derivative bands are \(k\)'-bands by (i).

Let them have histories \(H_1, \ldots, H_s\), respectively. Then \(H_i\) \((i = 1, \ldots, s)\) has no subwords of type (12)(2)(12)\(\pm^1\) by (viii) applied to the filling trapezium \(T_p(\mathcal{C}, C)\) (a similar argument was used in Lemma 10.1). Therefore, \(H_i = H_i(1)H_i(2)H_i(3)\) \((i \leq s)\), where \(H_i(1)\) and \(H_i(3)\) are of Step (2) and \(H_i(2)\) is of Step (1), where some of the factors can be empty. The lengths of the histories are denoted by \(h_i\) and \(h_i(j), j = 1, 2, 3\).

If \(H_i(2)\) is non-empty, then by Lemma 10.2 for \(\Gamma_i\) and equality \(|z|_i = |z'|_i + 2, \text{Area}(\Gamma_i) \leq \text{Area}'(\Gamma) + h_i \leq c_2(h_i(|z|_i - h) + h_i(1) + h_i(3)) + \kappa c(\Gamma_i))\). If \(H_i = H_i(1)\), then \(\text{Area}(\Gamma_i) \leq 60Nh_i(1)^2 + 2h_i(1)|z|_a\) by Lemma 7.13.

Since \(c_2 \geq 2(\delta')^{-1}\) and \(|z|_a \leq (\delta')^{-1}(|z| - h)\) by Lemma 5.20(a), we derive from the inequalities of the previous paragraph and Lemma 7.9(b) that
\[
\sum_{i=1}^s \text{Area}(\Gamma_i) \leq c_2 h(|z| - h + \sum(h_i(1) + h_i(3))) + c_2 \kappa c(\Gamma) + 60N \sum h_i(1)^2. \quad (10.12)
\]

Let \(h_{i_0} = \max h_i\). Then \(\sum_{i \neq i_0} h_i = h_\cdot\) by the definition of \(h_\cdot\). By (vi), every maximal \(a\)-band starting on the \(H_{i_0}(1)\)- or \(H_{i_0}(3)\)-part of \(C_{i_0}\) must end either on the \(H_i(1)\)- or \(H_i(3)\)-part of some \(C_i\), \(i \neq i_0\), or on the path \(z\). (Indeed, the \(H_i(1)\)-part of a band \(C_i\) cannot be connected with the \(H_i(3)\)-part by an \(a\)-band since the \(H_i(2)\)-part of \(C_i\) has common \(\theta\)-edges with \(C\) by (vi).) Therefore, \(h_{i_0}(1) + h_{i_0}(3) \leq h_\cdot + |z|_a \leq h_\cdot + (\delta')^{-1}(|z| - h)\) by Lemma 5.20(a), and so,
\[
\sum_{i=1}^s (h_i(1) + h_i(3)) \leq 2h_\cdot + (\delta')^{-1}(|z| - h). \quad (10.13)
\]

It follows from this estimate, inequalities (10.12) and \(hh_\cdot \leq 2\kappa c(\Gamma)\) (see Lemma 7.5) that
\[
\sum_{i=1}^s \text{Area}(\Gamma_i) \leq c_2 h((|z| - h)(1 + (\delta')^{-1} + 2h_\cdot) + c_2 \kappa c(\Gamma)
\]
\[
+ 60Nh(2h_\cdot + (\delta')^{-1}(|z| - h))
\]
\[
\leq (c_2 + c_2(\delta')^{-1} + 60N(\delta')^{-1}|\Gamma| + (4c_2 + c_2 + 240N)\kappa c(\Gamma)).
\]

By Lemmas 7.10 and 7.5, the total area of all simple bands in \(\Gamma\) does not exceed
\[
h(h_\cdot + |z|_a + 1) \leq h(h_\cdot + (\delta')^{-1}(|z| - h - 1)) < (\delta')^{-1}|\Gamma| + 2\kappa c(\Gamma).
\]
This inequality and (10.13) imply

\[ \text{Area}(\Gamma) \leq \left( \frac{3}{2} (\delta')^{-1} c_2 + 60N(\delta')^{-1} \right) [\Gamma] + (5c_2 + 240N + 2)\kappa^c(\Gamma), \]

and to obtain inequality \( \text{Area}(\Gamma) \leq c_2(2(\delta')^{-1}[\Gamma] + 6\kappa^c(\Gamma)) \) it remains to use that \( c_2 > 240N + 2 \).

To complete the proof, we observe that by Lemma 10.1,

\[ c_3[\Gamma] + c_2\mu^c(\Gamma) = c_3[\Gamma] + c_2(c_0\kappa^c(\Gamma) + \lambda^c(\Gamma)) \]
\[ \geq c_3[\Gamma] + c_2c_0\kappa^c(\Gamma) - c_2(36\kappa^c(\Gamma) + 8(\delta')^{-1}[\Gamma]) \]
\[ = (c_3 - 8(\delta')^{-1}c_2)[\Gamma] + c_2(c_0 - 36\kappa^c(\Gamma)) \geq c_2(2[\Gamma] + 6\kappa^c(\Gamma)) \]

since \( c_3 \geq 9(\delta')^{-1}c_2 \), and \( c_0 \geq 42 \).

Case (a) of the lemma is completely analogous. \( \Box \)

**Lemma 10.4.** Let \( \Delta \) be a regular comb of base widths at most \( 15N \), where the history \( H^\Delta \) of \( \Delta \) has no rules (23)\( ^\pm1 \). Assume that \( \Delta \) has neither \( t^- \) nor \( t^+ \)-cells and has no one-step long subcombs \( \Gamma' \) such that \( \text{Area}(\Gamma') \leq c_1[\Gamma'] + c_1\kappa^c(\Gamma') \). If \( \Delta \) has a (12)-band \( T \) with base of the form \( \ldots p_1p_1^{-1}s_0^{-1} \ldots \), then \( \Delta \) has a long subcomb \( \Gamma \) with \( \text{Area}(\Gamma) \leq c_3[\Gamma] + c_2\mu^c(\Gamma) \).

**Proof.** Let us consider the maximal \( q \)-bands \( C \) and \( C_0 \) of \( \Delta \) corresponding to the distinguished letters \( p_1 \) and \( s_0^{-1} \), respectively, in the base of \( T \). By Property (ix) for the trapezium \( Tp(C,C_0) \), the history \( H \) of \( C \) has no subhistories of the form (12)(2)(12)-\( ^-1 \). It follows that \( H \) has type (2)(1)(2) (or a subword of (2)(1)(2)), in particular, \( H \) has at most two rules (12)\( ^\pm1 \), and by (ix), all \( p_1 \)-cells corresponding to the rules of Step 2 are active from both sides in the subcomb \( \Gamma \) with handle \( C \) as well. Below the usual notation \( h, y, z, y', \) and \( z' \) will be used for \( \Gamma \).

We consider the system of derivative bands \( C_1, \ldots, C_s \) of \( \Gamma \). Let them have histories \( H_1, \ldots, H_s \), respectively. Then \( H_i = H_0(1)H_i(2)H_i(3) (i \leq s) \), where \( H_0(1) \) and \( H_i(3) \) are of type (2) and \( H_i(2) \) is of type (1), and some of the factors can be empty. The lengths of the histories are denoted by \( h_i \) and \( h_i(j), j = 1, 2, 3 \). As usual, \( \Gamma_i \) is the derivative subcomb with handle \( C_i \).

Assume that \( C_i \) is a derivative \( p_i^{-1} \)-band. Then \( H_i \) has neither rules (12)\( ^\pm1 \) nor rules of Step 1 by (v). Hence, the history \( H_i \) is one-step (of Step 2) and moreover all \( p_i \)-bands of \( \Gamma_i \) are active from both sides and all other \( q \)-bands of \( \Gamma_i \) are passive by (vi). Therefore, one can apply Lemma 7.18(b) to the subcomb whose handle is a left-most maximal \( q \)-band of \( \Gamma_i \) which contradicts the assumption of the lemma about long subcombs. Hence, by (i), all the derivative bands are \( s_0 \)-bands. By the same argument and by Lemma 9.7, derivative subcombs cannot have one-step histories. (Indeed, since the \( s_0 \)-band is passive by (ii), we have \( |z^{F_1}| - |y^{F_1}| = |z^{F_1}| - |y^{F_1}| + 2 \), and Lemma 9.7 would imply \( \text{Area}(\Gamma_i) = \text{Area}'(\Gamma_i) + h^{F_1} \leq c_1[\Gamma_i] + c_1\kappa^c(\Gamma_i) \) contrary to the condition of the lemma.) Therefore, each of the derivative bands crosses a (12)-band, and so \( s \leq 2 \).

We factorize the history \( H \) as \( H' H(3) H'' \), where \( H(3) \) is of Step 1 including the (12)\( ^\pm1 \)-rules, and \( H' \) and \( H'' \) are of Step 2. If the \( H' \)-part (\( H'' \)-part) of \( \Gamma \) is non-empty and is crossed by a derivative band of \( \Gamma \), then this unique band has to cross the (12)\( ^\pm1 \)-band separating the \( H(3) \)-part from the \( H' \)-part (from the \( H'' \)-part). Using this observation, we further factorize \( H' \equiv H(1)H(2) \) so that every maximal \( \theta \)-band of the \( H(2) \)-part \( \Gamma(2) \) of \( \Gamma \) crosses a derivative \( s_0 \)-band (\( H(2) \) can be empty) and the \( H(1) \)-part \( \Gamma(1) \) has no derivative bands. Similarly, we have \( H'' \equiv H(4)H(5) \).

Thus, \( H \equiv H(1), \ldots, H(5) \). Let \( h(1), \ldots, h(5) \) be corresponding lengths of the histories (some of them may be zero), \( \Gamma(i) (i = 1, \ldots, 5) \) be the corresponding parts of \( \Gamma \) (some of them may
be empty) with handles $C(i)$, and $z = z(5), \ldots, z(1)$, where $z(i)$ is the common part of $z$ and $\partial \Gamma(i)$. Note that if the maximal (12)-band $T$ separating $\Gamma(3)$ and $\Gamma(2)$ is not crossed by a derivative band, then by (v), $T$ has no cells except for the intersection cell with the handle, and so the boundaries of $\Gamma(3)$ and $\Gamma(2)$ have no common edges except for that in $C$. Similar note is true for the combs $\Gamma(3)$ and $\Gamma(4)$.

Let $x$ be the extension of the path $z(1)^{-1}$ along $z^{-1}$, such that $|x| = |z(1)|$ and the last edge of $x$ is the $s_0$-edge of $z(2)$. Then every maximal $a$-band starting on $y(2)y'(1)$ must end on $x$ by Lemma 5.6 because (a) the derivative $s_0$-band is passive, (b) it has a common (12)-edges with $C$ by (v), and (c) every maximal $a$-band of $\Gamma(2)$ starting on the side $y'(2)y'(1)$ of $C(1)C(2)$ cannot end on $y'(2)y'(1)$ by (vii). Since all cells of $C(1)C(2)$ are active, the path $x$ has at least $h(1) + h(2)$ $a$-edges and only $h(1)$ $\theta$-edges. It follows therefore from Lemma 5.20(a) that

$$|x| - h(1) \geq 1 + \delta h(2) + \delta' h(1). \quad (10.15)$$

(Here 1 is added for the $q$-edge of the handle $C$.) Since the path $z(2)$ has $h(2)$ $\theta$-edges which do not belong to $x$, we obtain from 10.15:

$$|z(2)z(1)| - h(1) - h(2) \geq 1 + \delta h(2) + \delta' h(1)$$

and

$$|z(5)z(4)| - h(4) - h(5) \geq 1 + \delta h(4) + \delta' h(5). \quad (10.16)$$

The band $C(3)$ is passive by (ii); therefore $y(3) = h(3)$ and so $|y| = |y'| = h + \delta'(h(1) + h(2) + h(4) + h(5))$. This together with equality $|z(i)| \geq |z(i)|_\theta = h(i)$, Lemma 5.20(b), and (10.16) give rise to inequality

$$|z| - |y| \geq |z(1)| + \cdots + |z(5)| - |y(1)| - \cdots - |y(5)| - 4(\delta - \delta')$$

$$\geq 2 + (\delta - \delta')(h(2) + h(4) - 4) > \max(1, (\delta - \delta')(h(2) + h(4))). \quad (10.17)$$

In particular, $\Gamma$ is a long subcomb.

Let $p_{1,2}$ be the common segment of $\partial \Gamma(2)$ and $\partial \Gamma(1)$; it is a top/bottom path of a $\theta$-band, and it consists of a $p_1$-edge and of $a$-edges. The path $p_{4,5}$ is defined similarly. Since every
maximal \( \alpha \)-band of \( \Gamma(2) \) starting on \( p_{1,2} \) must end on \( y'(2) = (y')^{(2)} \), we have

\[
|p_{1,2}| \leq |y'(2)| = h(2) \quad \text{and also} \quad |p_{4,5}| \leq |y'(4)| = h(4).
\] (10.18)

Now we set \( E = \Gamma(2) \cup \Gamma(3) \cup \Gamma(4) \). Then \( z^E = p_{1,2}z(2)z(3)z(4)p_{45} \) and

\[
|z^E| - |y^E| \leq |z| - |y| + \delta(|p_{1,2}| + |p_{4,5}|) = |z| - |y| + \delta(|h(2)| + |h(4)|)
\] (10.19)

by (10.18), since the maximal \( \theta \)- and \( \alpha \)-bands starting on \( y'(1) \) and \( y'(5) \) end on \( z(1) \) and \( z(5) \), respectively.

The comb \( E \) has \( s \leq 2 \) derivative subcombs \( E_j = \Gamma_j \) \((j \leq s \leq 2)\) whose handles are \( C_j \)-s. By Lemma 10.2,

\[
\sum_j \text{Area}(E_j) \leq c_2 \left( \left( \sum_j |(z')^{E_j}| - h^{E_j} \right) + h(2) + h(4) + 1 \right) \times \left( h(2) + h(3) + h(4) \right) + \sum_j \kappa^c(E_j).
\] (10.20)

Since \( \sum_j |(z')^{E_j}| - h^{E_j} \)| \( \leq |z'| - |z(1)| - |z(5)| - h(2) - h(3) - h(4) \) and \( h(2) + h(3) + h(4) \)
\( \leq h \), the inequalities (10.17) and (10.20) imply

\[
\sum_j \text{Area}(E_j) \leq c_2 h(|z'| - |z(1)| - |z(5)| - h(2) - h(3)) - h(4) + (\delta - \delta')^{-1}(|z| - |y|) + 1 + c_2 \sum_j \kappa^c(E_j).
\] (10.21)

Observe that any simple band of \( \Gamma(3) \) has no cells except for one cell \( \pi \) of the handle \( C(3) \)
because there are no \( (\theta, \alpha) \)-cells from the left of \( \pi \) by (v). Hence, the comb \( E \) consists of the cells of \( E_j \)-s, the cells of the handle of \( E \) and the cells of maximal \( \alpha \)-bands connecting this handle with \( p_{1,2} \) and \( p_{4,5} \) and intersecting at most \( h(2) \) and \( h(4) \) \( \theta \)-bands, respectively, by Lemma 5.6. Therefore, we obtain from (10.21), (10.18), and Lemma 7.9(a) that

\[
\text{Area}(E) \leq c_2 h(|z'| - |z(1)| - |z(5)| - h(2) - h(3) - h(4)) + (\delta - \delta')^{-1}(|z| - |y|) + 1 + c_2 \kappa^c(E) + h(2) + h(4)
\] (10.22)

because \( |z| = |z'| + 2 \). Since \( |y(2)| + |y(3)| + |y(4)| = h(2) + h(3) + h(4) + \delta'(h(2) + h(4)) \leq h(2) + h(3) + h(4) + \delta'(\delta - \delta')^{-1}(|z| - |y|) \) by (10.17), inequality (10.22) yields

\[
\text{Area}(E) \leq c_2 h(|z'| - |z(1)| - |z(5)|) - y(2) - y(3) - y(4) + (\delta' + 1)(\delta - \delta')^{-1}(|z| - |y|)) + c_2 \kappa^c(E) + h(2) + h(4). \quad (10.23)
\]

Since the combs \( \Gamma(1) \) and \( \Gamma(5) \) have no derivative bands, applying Lemma 7.18(b), we obtain the upper estimates

\[
\text{Area}(\Gamma(1)) \leq (\delta')^{-1}h(|z(1)| + |p_{1,2}| - |y(1)|) \leq c_2 h(|z(1)| + 1 + \delta h(2) - |y(1)|)
\] (10.24)

as \( |p_{1,2}| \leq 1 + \delta h(2) \) by (10.18) and Lemma 5.20; and similarly,

\[
\text{Area}(\Gamma(5)) \leq c_2 h(|z(5)| + 1 + \delta h(4) - |y(5)|).
\] (10.25)

Summing inequalities (10.23)–(10.25) and using (10.17), we obtain

\[
\text{Area}(\Gamma) \leq c_2 h(|z| - |y|)(2 + (\delta' + 1 + \delta)(\delta - \delta')^{-1}) + c_2 \kappa^c(E) 
\leq 2\delta^{-1}c_2 [\Gamma] + c_2 \kappa^c(E).
\] (10.26)
The handle $C$ has (at most) two $(12)$-cells (see the first paragraph in the proof of the lemma), and so it has at most three maximal subbands without $(12)$-cells. It follows from the definition of the $\lambda$-mixture that

$$\lambda(y) \leq (h(1) + h(2))(h(3) + h(4) + h(5)) + (h(4) + h(5))h(3). \quad (10.27)$$

If $h(1) + h(5) \geq h(2) + h(4)$, then $2h(1) + 2h(5) \geq (h(1) + h(2)) + (h(4) + h(5))$, and so the right-hand side of (10.27) does not exceed $2h(1)(h - h(1)) + 2h(5)(h - h(5))$, which, in turn, does not exceed $4\kappa^c(\Gamma)$ since the ends of $s_0$-bands separate the $\theta$-edges of $z$ in parts with $h(1), h(5)$, and $h - h(1) - h(5)$ $\theta$-edges. If $h(1) + h(5) < h(2) + h(4)$, then the right-hand side of (10.27) does not exceed $2(h(2) + h(4))h \leq 2(\delta - \delta')^{-1}h(|z| - |y|)$ by (10.17). Thus, in any case $\lambda^c(\Gamma) \geq -\lambda(y) \geq -4\kappa^c(\Gamma) - 2(\delta - \delta')^{-1}[\Gamma]$. Therefore, by this inequality and (10.26),

$$\text{Area}(\Gamma) \leq 2\delta^{-1}c_2[\Gamma] + c_2\kappa^c(\Gamma) \leq (c_3 - 2(\delta - \delta')^{-1}c_2)[\Gamma] + c_2(c_0 - 4)\kappa^c(\Gamma)$$

$$= c_3[\Gamma] + c_2(c_0\kappa^c(\Gamma) - 4\kappa^c(\Gamma) - 2(\delta - \delta')^{-1}[\Gamma])$$

$$\leq c_3[\Gamma] + c_2(c_0\kappa^c(\Gamma) + \lambda^c(\Gamma)) = c_3[\Gamma] + c_2(\mu^c(\Gamma))$$

since $c_0 \geq 5$ and $c_3 > 5\delta^{-1}c_2$. The lemma is proved.

\begin{lemma}
Let $\Gamma$ be a regular comb of base width $b \leq 15N$. Assume that it has no maximal $t^{\pm 1}$-bands except for the handle $C$, and $C$ is either (a) a $t'$-$t^1$-band or (b) a $t^1$-band. Also assume that there are no special $\theta$-edges in any derivative subcomb. Let $\Gamma$ in case (a), have $(23)^{\pm 1}$-bands but have no $(12)^{\pm 1}$-bands, and in case (b), have $(12)^{\pm 1}$-bands but no $(23)^{\pm 1}$-bands. Then $\text{Area}(\Gamma) \leq (\delta')^{-2}[\Gamma] + c_2\mu^c(\Gamma)$.
\end{lemma}

\textbf{Proof.} We will consider the case (b) only. Observe that the $(12)$-cells of the $t^1$-band $C$ are special, and so $\lambda(y) = 0$. Let $C_1, \ldots, C_s$ be the system of derivative bands of $\Gamma$. It follows from (i) and the assumptions of the lemma that all of them must be $k^{-1}$-bands. First assume that a derivative band $C_i$ is of length $h_i = 1$. Then $\text{Area}(\Gamma_i) \leq 4(\delta')^{-1}[\Gamma_i]$ by Lemma 7.15. Since the statement of the lemma can by induction be assumed true for the comb $\Gamma \setminus \Gamma_i$, this implies that the statement is true for $\Gamma$ since $\mu^c(\Gamma) \geq \mu^c(\Gamma \setminus \Gamma_i) \geq 0$ and $[\Gamma_i = [\Gamma_i] + [\Gamma_i \setminus \Gamma_i]$. Thus, we may assume further that $h_i > 1$ for every $i$.

\textbf{Case 1:} There is no derivative band whose length $h_i \leq 0.8 \sum_{i=1}^s h_i$. Then $\sum h_i^2 \leq 4h_\infty \leq 8\kappa^c(\Gamma_i)$ by Lemma 7.5. By Lemmas 7.13 and 7.10(a), $\text{Area}(\Gamma_i) \leq 60Nh_i^2 + 2(\delta')^{-1}h_i(|z_i| - |h_i|)$, and therefore by Lemma 7.9, $\sum \text{Area}(\Gamma_i) \leq 480N\kappa^c(\Gamma) + 2(\delta')^{-1}[\Gamma]$.

\textbf{Case 2:} There is a derivative band $C_i$ with $h_i \geq 0.8 \sum_{i=1}^s h_i$, and there is a short derivative $C_j$ with $h_j > 0.6h' \geq 0.2 \sum h_i$. (Here, we assume that the total sum of lengths of short derivatives $h'$ is at least $\sum h_i/3$ because $h_i > 1$ for every $i$.) Then, by (ii) and (vii), at least $h_j/3$ $a$-bands starting on $C_j$ must end on $z$. Therefore, by Lemmas 7.13 and 7.10(a),

$$\sum \text{Area}(\Gamma_i) \leq \sum 60Nh_i^2 + 2(\delta')^{-1}[\Gamma] \leq 60Nh_i^2(0.2)^{-1}(1/3)^{-1} + 2(\delta')^{-1}[\Gamma]$$

$$\leq 900N(\delta')^{-1}[\Gamma] + 2(\delta')^{-1}[\Gamma] \leq 0.5(\delta')^{-2}[\Gamma]$$

since $(\delta')^{-1} \geq 2000N$.

\textbf{Case 3:} There is a derivative $C_i$ with $h_i \geq 0.8 \sum_{i=1}^s h_i$, and there are no short derivatives $C_j$ of length $h_j' > 0.6h'$. Then $h' \geq h' - 0.6h' \geq 0.4 \sum h_i/3 = \frac{2}{15} \sum h_i$. Hence, by Lemmas 7.13 and 7.6,

$$\sum \text{Area}(\Gamma_i) \leq 60Nh_i^2 + 2(\delta')^{-1}[\Gamma] \leq \frac{15}{2} 60Nh_i^2 + 2(\delta')^{-1}[\Gamma]$$

$$\leq 2700N\lambda^c(\Gamma) + 2(\delta')^{-1}[\Gamma].$$
Thus, in any case
\[ \sum \text{Area}(\Gamma_i) \leq 480N\kappa^c(\Gamma) + 2700N\lambda^c(\Gamma) + 0.5(\delta')^{-2}[\Gamma]. \] (10.28)

By Lemmas 7.12, 7.5, and 7.6, the number of cells in all simple bands of \( \Gamma \) does not exceed
\[ h(h_- + h'_- + (\delta')^{-1}(|z| - h - 1) \leq 2\kappa^c(\Gamma) + 6\lambda^c(\Gamma) + (\delta')^{-1}[\Gamma]. \] (10.29)

The two upper bounds (10.28) and (10.29) together prove the lemma since \( c_0 \geq 1 \) and \( c_2 > 2800N \).

**Lemma 10.6.** Let \( \Gamma \) be a regular comb of base width \( b \), where \( 3N \leq b \leq 15N \). Assume that its history either (a) contains \((12)\)-rules but does not contain \((23)\)-rules or (b) vice versa. Then it admits a long quasicomb \( \Delta \) such that \( \text{Area}(\Delta) \leq c_3[\Delta] + c_2\mu^c(\Delta) \).

**Proof.** Here, we consider case (a) only. If \( \Gamma \) has a \((t')^{\pm 1}\)-band, the statement follows from Lemma 10.3 provided this band crosses a maximal \((12)\)-band, and it follows from Lemma 9.2(2) otherwise since \( \lambda \) takes non-negative values on one-step combs and every subcomb with (passive) \((t')^{\pm 1}\)-handle is long. We may therefore assume that \( \Gamma \) has no such bands.

Assume \( \Gamma \) has a left-most maximal \( t^{\pm 1}\)-band \( B \). If \( B \) does not have \((12)\)-cells, then again the statement follows from Lemma 9.2(2). So we assume further that \( B \) has \((12)\)-cells.

Let \( \Delta \) be the subcomb with the handle \( B \). If there are no special \( \theta \)-edges from the left of each derivative band of \( \Delta \), then the statement follows from Lemma 10.5 since \( c_3 > (\delta')^{-2} \).

Therefore, we may assume that \( \Delta \) has a maximal \((12)\)-band \( T \) crossing a derivative band \( B_i \) of \( B \) and having a special \( \theta \)-edge from the left of \( B_i \).

Note that \( B_i \) is a \( k^{-1}\)-band by the choice of \( B \) and by (i). Since only \( t \)-bands and \( k^{\pm 1}\)-bands can have special \((12)\)-edges (and \( k^{-1}\)-bands can have them from the right only), it follows that \( T \) has a \( k \)-cell from the left of \( B_i \). Since \( \Gamma \) has no \((t')\)-cells, the base of \( T \) is not aligned between the letter \( k \) and the next letter \( k^{-1} \) by (i). Moreover, it has a subword \( p_1p_1^{-1}s_0^{-1} \) between the \( k \)- and \( k^{-1} \)-letters by (v). So the statement of the lemma follows from Lemma 10.4. (Again, we take into account that \( \lambda \) is non-negative on one-step combs.) Thus, we may further assume that \( \Gamma \) has no \( t \)-cells.

By Lemma 9.6, one may assume that \( \Gamma \) has no one-step subcombs of base width more than \( 2N \). Since \( b \geq 3N \), this implies the existence of a \((12)\)-band \( T \) with base \( B_0 \) of length at least \( N \). Since \( N > 2||B|| \) and \( B_0 \) has neither \( t \)-nor \( t' \)-letters, it must have at least two subwords of the form \( q^{-1}q^{\mp 1} \) for some base letter \( q \) (see (i)). But the existence of \( s_1^{-1}s_1 \) excludes the possibility of all other subwords \( q^{\pm 1}q^{\mp 1} \) by (v), and also by (v), the existence of \( k^{-1}k \) implies the existence of at least one subword \( s_0p_1p_1^{-1}s_0^{-1} \). Thus, in any case, \( B \) must have a subword \( p_1p_1^{-1}s_0^{-1} \), which finishes the proof as in the previous paragraph. \( \square \)

11. Combs with multi-step histories

In this section, we allow all three steps in comb histories. Although Lemma 11.8 gives no estimate of the area if the size of a comb is close, in a sense, to one of the numbers \( T_i \)-s, this lemma (together with the lemmas of the next section) will imply that the Dehn functions of the groups \( M \) and \( G \) are almost quadratic because the set of \( T_i \)-s has infinitely many very long gaps. Again, to obtain upper estimates of areas for various combs one should apply a skillful combination of a number of quadratic parameters. For example, Lemma 11.3 (and also Lemma 12.9 in the next section) shows the use of the \( \nu \)-mixture.

Let \( H \) be the history of a comb \( \Gamma \). Consider a factorization \( H = H(1), \ldots, H(m) \), where no two non-empty factors are separated by empty ones. We say that this factorization is firm if, for every \( i = 1, \ldots, m - 1 \),
(a) for non-empty \( H_i \) and \( H_{i+1} \), the last letter of \( H_i \) and the first letter of \( H_{i+1} \) must belong to different steps; so one of these two letters is \((12)^{±1}\) or \((23)^{±1}\)-letter calling \((i) − (i + 1)\) transition letters; the maximal \( \theta \)-band of \( \Gamma \) corresponding to the \((i) − (i + 1)\) transition letter is an \((i) − (i + 1)\)-transition band;
(b) the transition \( \theta \)-bands of \( \Gamma \) are not simple.

There might be many firm factorizations of \( \Gamma \). Observe that if a factorization \( H ≡ H(1), \ldots, H(m) \) is firm, then \( H^{-1} ≡ H(m)^{-1}, \ldots, H(1)^{-1} \) is a firm factorization for the history of the mirror copy \( \Gamma^{-1} \) of the comb \( \Gamma \).

**Lemma 11.1.** Let \( \Gamma \) be a comb of base width \( b \leq 15N \) with a firm factorization of the history \( H ≡ H(1)H(2)H(3) \), where \( H(2) \) and \( H(3) \) are one-step histories, and \( h(2) \geq 3h(3) \) (or \( h(3) \geq 3h(2) \)). Assume that the handle \( C \) of \( \Gamma \) is a \( t^{±1} \)- or \((t')^{±1} \)-band, and the \((1) − (2)\) transition band has no \((\theta, a)\)-cells between \( C \) and the derivative band crossing this transition band, and the \( H(2)\)-part (respectively, the \( H(3)\)-part) of \( \Gamma \) has passive \( k \)- or \( k' \)-cells only in the \((12)\)- or in the \((23)\)-bands. Then provided \( h(2) \geq 0.01h \) (respectively, \( h(3) \geq 0.01h \)), there is a long subcomb \( \Delta \) in \( \Gamma \) with \( \text{Area}(\Delta) \leq c_3|\Delta| + c_2\mu(\Delta) \).

**Proof.** We will prove the lemma assuming that \( h(2) \geq 3h(3) \) and \( h(2) \geq 0.01h \) (> 0) since the proof of the second version of the lemma is similar.

Consider the system of derivative bands \( C_1, \ldots, C_u \) of \( \Gamma \). Let \( C_1, \ldots, C_u \) have histories \( H_1, \ldots, H_u \) which are subwords of \( H(1) \), \( C_{u+1} \) have history \( H_{u+1} ≡ H_{u+1}(1)H_{u+1}(2) \), where \( H_{u+1}(1) \) and \( H_{u+1}(2) \) are a suffix and a prefix of \( H(1) \) and \( H(2) \), respectively: \( C_{u+1} \) is a union of bands \( C_{u+1}(1) \) and \( C_{u+1}(2) \) having these two histories, and \( h_{u+1} = h_{u+1}(1) + h_{u+1}(2) > 0 \). Similarly we define subwords \( H_{u+2}, \ldots, H_v \) of \( H(2) \) and \( H_{v+1} ≡ H_{v+1}(2)H_{v+1}(3) \), while \( H_{v+2}, \ldots, H_s \) are subwords of \( H(3) \). (It is also possible that \( C_{u+1} \) has history \( H_{u+1}(1)H_{u+1}(2)H_{u+1}(3) \), where \( H_{u+1}(2) ≡ H(2) \), and we will come back to this case later on.)

Proving by contradiction, we assume that \( \Gamma \) has no subcombs \( \Delta \) with area satisfying the statement of the lemma.

The band \( C_{u+1}(2) \), if it has non-zero length, is not a \( t^{±1} \)- or \((t')^{±1} \)-band by the condition on the \((1)−(2)\)-transition band and by (v). If some \( C_i \) is a \( t^{±1} \)- or \((t')^{±1} \)-band, for \( i \in [u + 2, v + 1] \), then the derivative subcomb \( \Gamma_i \) satisfies the condition of Lemma 9.2, a contradiction since \( c_3 > c_1 \). Therefore, all the derivative bands of the system \( C_{u+1}(2), C_{u+2}, \ldots, C_{v+1}(2) \) are \( k^{-1} \)- or \( k' \)-bands by (v).

Assume that one of the numbers \( h_{u+1}(2), h_{u+2}, \ldots, h_{v+1}(2) \), redenote it by \( g \), is at least \( 0.9h(2) \geq 0.009h > \delta h \).
Denote by $\mathcal{G}$ the corresponding derivative (sub)band of length $g$. Since $h(2) \geq 3h(3)$, we see that at most $h(2)/3 + 0.1h(2) \leq h(2)/2 < 2g/3$ maximal $a$-bands starting on $\mathcal{G}$ end on the other derivative bands of $\Gamma$. So we may apply Lemma 7.17(a) to $\mathcal{G}$ and obtain inequality
\[
\text{Area}(\Gamma) \leq (\delta')^{-2}|\Gamma|.
\]
(11.1)

Now assume that each of $h_{u+1}(2), h_{u+2}(2), \ldots, h_{v+1}(2)$ is less than $0.9h(2)$. It follows from this assumption that $\max_{i=1}^{s} h_i < h - h(2) + 0.9h(2) \leq (1 - 0.001)h$ since $h(2) \geq 0.01h$. Therefore, by Lemma 7.4, $l_2 \geq \min\{\sum_{i=1}^{s} h_i, h - \max_{i=1}^{s} h_i\} \geq 0.001 \sum_{i=1}^{s} h_i$. Hence, by Lemmas 7.13, 7.10(1), and by inequality $\delta'^{-1} > \max(40N, 4000)$, we have
\[
\sum_{i=1}^{s} \text{Area}(\Gamma_i) \leq 60N \sum_{i=1}^{s} h_i^2 + \sum_{i=1}^{s} 2\alpha_i h_i \leq 60Nh \times 1000l_- + 2h(\delta')^{-1}(|z| - h)
\]
\[
\leq 2(\delta')^{-1}|\Gamma| + (\delta')^{-2}hl_-,
\]
(11.2)
where $\alpha_i = |z^i{|}_a$.

By the inequalities (7.2) and $l_- \geq 0.001 \sum_{i=1}^{s} h_i$, the number $n_s$ of cells in all the simple $\theta$-bands of $\Gamma$ satisfies inequality
\[
n_s \leq h \left(\frac{3}{2} \sum_{i=1}^{s} h_i + \delta'^{-1}(|z| - |y|)\right) \leq h(1500l_- + \delta'^{-1}(|z| - |y|)).
\]
From this inequality, (11.2), and by Lemma 7.5, we have
\[
\text{Area}(\Gamma) \leq c_1[\Gamma] + c_1hl_-/2 \leq c_1[\Gamma] + \kappa^c(\Gamma)
\]
(11.3)
since $c_1 \geq 3(\delta')^{-2}$.

If $\lambda^c(\Gamma) \geq 0$, the statement of the lemma follows from inequalities (11.1) and (11.3). Then we will assume that $-\lambda(y) \leq \lambda^c(\Gamma) < 0$.

To estimate $\lambda^c(\Gamma)$ from below, we again start with the assumption that $g \geq 0.9h(2)$, and so at least $g/2$ maximal $a$-bands end on $z$. We have $|z|_a \geq g/2 \geq 0.4h(2) \geq 0.004h$. Hence, $|z| - h \geq \delta'h/250$ by Lemma 7.10(a). Then, by Lemma 6.2(a),
\[
\lambda^c(\Gamma) \geq -\lambda(y) \geq -h^2 \geq -250(\delta')^{-1}|\Gamma|.
\]
Since $c_1 \geq 2(\delta')^{-2}$, this estimate together with (11.1) yields $\text{Area}(\Gamma) \leq c_1[\Gamma] + \lambda^c(\Gamma) \leq c_1[\Gamma] + \mu^c(\Gamma)$. Here, the right-hand side does not exceed $c_3[\Gamma] + c_2\mu^c(\Gamma)$ because $c_3 > c_1c_2$, and so the lemma proved in case $g \geq 0.9h(2)$.

Now let $g < 0.9h(2)$. Since $\lambda(y) > 0$, by Lemma 6.2(e), there is a maximal (12)- or (23)-band $T$ such that there are $m_1$ $\theta$-bands crossing $C$ below $T$, $m_2$ $\theta$-bands crossing $C$ above $T$, and $\lambda(y) \leq 2m_1m_2$.

Assume first that $T$ belongs to the $H(1)$-part of $\Gamma$. Then one of the two ends of the derivative band $\mathcal{C}_{u+1}$ lies above $T$ but there are at least $0.1h(2)$ $\theta$-bands above this end since $g \leq 0.9h(2)$. Therefore, by Lemma 6.2(d),
\[
\kappa^c(\Gamma) \geq 0.1m_1h(2) \geq 0.001m_1h \geq 0.001m_1m_2 \geq \lambda(y)/2000.
\]
Then assume that $T$ belongs to the $H(2)H(3)$-part of $\Gamma$. Then $m_2 \leq h(3) \leq h(2)/3$, since the one-step history $H(2)$ has no (12)$^{\pm 1}$- or (23)$^{\pm 1}$-rules. On the other hand, there is an (lower) end of the derivative $\mathcal{C}_{v+1}$ such that there are at least $h(3)$ maximal $\theta$-bands above it and at least $0.1h(2)$ below it since $g \leq 0.9h(2)$. Hence,
\[
\kappa^c(\Gamma) \geq 0.1h(3)h(2) \geq 0.1m_2(0.01h) \geq 0.001m_1m_2 \geq \lambda(y)/2000.
\]
Thus, $\lambda^c(\Gamma) \geq 2000\kappa^c(\Gamma)$ if $g \leq 0.9h(2)$. This inequality and (11.3) imply $\text{Area}(\Gamma) \leq c_1[\Gamma] + \mu^c(\Gamma)$ because $c_0 \geq 2001$. This leads to a contradiction since $c_3 > c_2 > c_1$.

The case where $\mathcal{C}_{u+1}$ had history $H_{u+1}(1)H_{u+1}(2)H_{u+1}(3)$ with $H_{u+1}(2) \equiv H(2)$ can be treated as the above subcase with $g \geq 0.9h(2)$, since now $g = h(2)$.
We omit the proof of the following lemma since the argument would just be a simplified version of the proof given above for Lemma 11.1: instead of the inequalities \( h(3) \geq 0.01h \) and \( h(3) \geq 3h(2) \), below we have that \( h'' \geq 0.7h \) (and so \( h'' \geq \frac{7}{8}h' \)) and \( H(1) \) is empty.

**Lemma 11.2.** Let \( \Gamma \) be a comb of basic width \( b \leq 15N \) with a \( t^{\pm 1} \)- or \( (t')^{\pm 1} \)-handle \( C \), and let the history of \( \Gamma \) have a firm factorization \( H \equiv H'H'' \), where \( h'' \geq 0.7h \) and \( H'' \) is of Step (2). Let the derivative bands \( C \) be all \( k^{\pm 1} \)-or \( (k')^{\pm 1} \)-bands. Then \( \Gamma \) has a long subcomb \( \Delta \) with Area\((\Delta) \leq c_3[\Delta] + c_2\mu^c(\Delta) \).

**Lemma 11.3.** Let \( \Gamma' \) be a subcomb of a comb \( \Gamma \) of base width \( b \leq 15N \), \( C' \) and \( C \) their handles with histories \( H' \) and \( H \), respectively, and each of these handles a \( t^{\pm 1} \)- or \( (t')^{\pm 1} \)-band. Assume that \( h' < h/2 \), and \( H \) has at most six letters \((12)^{\pm 1} \) and \((23)^{\pm 1} \). Then either \( \Gamma \) has a long subcomb \( \Delta \) with Area\((\Delta) \leq c_3[\Delta] + c_2\mu^c(\Delta) \) or \( \text{Area}(\Gamma') \leq c_1([\Gamma'] + \lambda^c(\Gamma') + \nu^\prime_j(\Gamma) - \nu^\prime_j(\Delta')) \), where \( \Delta' = \Gamma \setminus \Gamma' \).

**Proof.** If there is a maximal \( t \)- or \( t' \)-band in \( \Gamma \), having no \((12)\)- or \((23)\)-cells, then by Lemma 9.2(b), it is a handle of a long subcomb \( \Delta \) with

\[
\text{Area}(\Delta) \leq c_1[\Delta] + c_1\kappa^c(\Delta) \leq c_3[\Delta] + c_2\mu^c(\Delta)
\]

because \( \lambda^c(\Delta) \geq -\lambda(J^{\Delta}) = 0 \) for a one-step \( \Delta \), and any subcomb with handle passive from the right is long.

Therefore, we may assume that every \( t \)- or \( t' \)-band of \( \Gamma \) intersects a \( \theta \)-band corresponding to one of the \( \theta \)-letters \((12)^{\pm 1}, (23)^{\pm 1} \). Since their base widths are at most \( 15N \), the number of maximal \( t \)- and \( t' \)-bands in \( \Delta \) does not exceed \( 90N < J/2 \).

Now, we will prove that

\[
\nu^\prime_j(\Delta') < \nu^\prime_j(\Gamma) - (h')^2. \quad (11.4)
\]

Recall that \( \nu^\prime_j(\Gamma) = \nu_j(z^\Gamma) \) by Lemma 7.2(b). So we consider the two-colored string of beads responsible for the \( \nu_j \)-mixture of \( z^\Gamma \). Denote by \( o_1 \) and \( o_3 \) the black beads on the two ends of \( C \) and by \( o_2 \) and \( o'_2 \) the black beads on the two ends of \( C' \). We have \( h' \) white beads between \( o_2 \) and \( o'_2 \), \( a \) white beads between \( o_1 \) and \( o_2 \) and \( b \) white beads between \( o'_2 \) and \( o_3 \) for some \( a, b \geq 0 \). Thus, \( a + h' + b = h \). When we pass from \( z^\Gamma \) to \( z^{\Delta'} \), we delete at least two black beads \( o_2 \) and \( o'_2 \). But the number of black beads between the vertices \( o_1 \) and \( o_3 \) is less than \( J \). Hence, we may apply Lemma 6.2, parts (d,c), and obtain that \( \nu^\prime_j(\Delta') \leq \nu^\prime_j(\Gamma) - a(h' + b) - b(h' + a) \). But here \( a(h' + b) + b(h' + a) > (h')^2 \) since \( a + b > h' \). So the inequality 11.4 is obtained.

Now by Lemmas 7.13 and 7.10,

\[
\text{Area}(\Gamma') \leq 60N(h')^2 + 2(\delta')^{-1}[\Gamma']. \quad (11.5)
\]
Since by Lemma 6.2(a) and (11.4),
\[ \lambda^\ell(\Gamma') \geq -\lambda^\ell(y^{\Gamma'}) \geq -(h')^2/2 \geq (-\nu_j^\ell(\Gamma) + \nu_j^\ell(\Delta'))/2 \] (11.6)
we deduce from (11.5) and (11.6) that
\[
\text{Area}(\Gamma') \leq (60N + c_1/2 - c_1/2)(\nu_j^\ell(\Gamma) - \nu_j^\ell(\Delta')) + 2(\delta')^{-1}[\Gamma']
\leq \left( 60N + \frac{c_1}{2} \right) (\nu_j^\ell(\Gamma) - \nu_j^\ell(\Delta')) + c_1 \lambda^\ell(\Gamma') + 2(\delta')^{-1}[\Gamma']
\leq c_1(\nu_j^\ell(\Gamma) - \nu_j^\ell(\Delta') + \lambda^\ell(\Gamma') + [\Gamma']),
\]
because \( c_1 \geq 61N + c_1/2 \) and \( c_1 \geq 2(\delta')^{-1} \), and the lemma is proved.

**Lemma 11.4.** Let \( \Delta \) be a comb with history \( H^\Delta \) of type (1)(12)(2)(23)(3), where the (1)-part and the (3)-part of \( \Delta \) can be empty. Assume that the base width \( b \) of \( \Delta \) satisfies inequalities \( 4N < b \leq 15N \). Then either (a) \( \Gamma \) admits a long quasiconic with
\[
\text{Area}(\Gamma) \leq c_3[\Gamma] + c_2\mu^\ell(\Gamma) + c_3(\nu_j^\ell(\Delta) - \nu_j^\ell(\Delta \setminus \Gamma))
\]
(11.7)
or (b) \( \Delta \) has a maximal \( t^{\pm 1} \) or \( (t')^{\pm 1} \)-band of length \( l \), where \( T_i \leq l < 10T_i \) for some \( i \).

**Proof.** \( \Delta \) has a regular subcomb \( \Delta_1 \) of base width \( b_1 > 3N \) such that the base width of the trapezium \( T = T_p(D_1, D) \) are at least \( N + 1 \), where \( D_1 \) and \( D \) are the handles of \( \Delta_1 \) and \( \Delta \), respectively. If the history of \( \Delta_1 \) has one step, then the Property (a) of the lemma is a consequence of Lemma 9.6 since in this case \( \lambda^\ell(\Delta) \geq -\lambda(y^\Delta) = 0 \), and \( \nu_j^\ell(\Delta) - \nu_j^\ell(\Delta \setminus \Gamma) \geq 0 \) by Lemmas 7.2 and 7.3(d). It is a consequence of Lemma 10.6 if the history of \( \Delta_1 \) has no (12)- or (23)-rules. Hence, we may assume that the history of \( \Delta_1 \) is of type (1)(12)(2)(23)(3) as well.

Since the history of \( T \) has both rules (12)\( ^{\pm 1} \) and (23)\( ^{\pm 1} \), the base of \( T \) is normal by (xi), and since the base of \( T \) has at least \( N + 1 \) letters, \( T \) contains a standard subtrapezium, and so the subtrapezium of \( T \) bounded by the (12)- and (23)-bands has height \( T_i \) for some \( i \).

Since \( T \) has a normal base of length at least \( N + 1 \), its base must contain a letter \( t^{\pm 1} \) and a letter \( (t')^{\pm 1} \). Denote by \( C (C') \) a maximal \( t^{\pm 1} \)-band \( ((t')^{\pm 1} \)-band) of \( \Delta \) crossing \( T \). We may assume that neither of them corresponds to the first letter of the base of \( T \) since otherwise this normal base of length \( N + 1 \) has one more \( t^{\pm 1} \) or \( (t')^{\pm 1} \)-letter, respectively, and one can select one of the bands \( C, C' \) closer to \( D \).

By \( \Gamma \) and \( \Gamma' \), we denote the subcombs with handles \( C \) and \( C' \), respectively. Let the histories of these handles be \( H \) and \( H' \). Without loss of the generality of our further proof, we assume that \( \Gamma' \) is contained in \( \Gamma \). Since \( 0 \leq \nu_j^\ell(\Gamma) - \nu_j^\ell(\Gamma \setminus \Gamma') \leq \nu_j^\ell(\Delta) - \nu_j^\ell(\Delta \setminus \Gamma') \) by Lemmas 7.2(b) and 7.3(d,e), we may assume by Lemma 11.3, that \( h' = h^{\Gamma'} \geq h/2 \).

Let \( H \equiv H(1)H(2)H(3) \) and \( H' \equiv H'(1)H'(2)H'(3) \) be the step factorizations. Since the left-most \( q \)-band of \( T \) is not a subband of \( C' \) or \( C \), the maximal (12)- and (23)-bands of \( \Gamma' \) and \( \Gamma \) are not simple, and so the factorizations \( H \equiv H(1)H(2)H(3) \) and \( H' \equiv H'(1)H'(2)H'(3) \) are firm. Recall that by (ii), every \( k \)-cell of the \( H(1)- \) and \( H(2)- \)parts of \( \Gamma \) (every \( k'- \)cell of the \( H(2)- \) and \( H(3)- \)parts of \( \Gamma \)) is not passive unless it belongs to a (12)- or (23)-band.

One may assume that \( 10h'^{(2)} \leq h' \) because \( h'^{(2)} = T_i \) and if \( 10h'^{(2)} > h' \), then the length of \( C' \) belongs to the segment \([T_i, 10T_i] \), and we obtain Property (b). Similarly we may assume that \( 10h(2) \leq h \).

If \( h(1) \geq 0.3h \), then \( h(1) \geq 3h(2) \), and one can apply Lemma 11.1 to \( \Gamma^{-1} \) and obtain the desired estimate (11.7) for \( \text{Area}(\Gamma) \). If \( h(1) < 0.3h \), then \( h(1) \leq h(1) < 0.6h' \) since \( h' \geq h/2 \). It
follows that $h'(3) = h' - h'(1) - h'(2) \geq (1 - 0.6 - 0.1)h' \geq 0.3h' \geq 3h'(2)$. Now one can apply Lemma 11.1 to $\Gamma'$ and obtain the required estimate (11.7) for $\text{Area}(\Gamma')$.

**Lemma 11.5.** Let $\Gamma$ be a regular comb of width $b \leq 15N$ with a handle $C$ containing both $(12)$- and $(23)$-cells, and the history $H$ of $\Gamma$ contains, in its step factorization, a product $H(1)H(2)H(3)$, where $h(2) \geq h/30$ and $h(1) + h(3) < h(2)/2$. Let one of the derivative bands $C_i$ be a $k^\pm 1$- or $(k')^\pm 1$-band which crosses all the maximal $\theta$-bands of the $H(2)$-part of $\Gamma$. Assume also that either

(a) $C$ is a $t^\pm 1$-band, and $H(1)H(2)H(3)$ is of the form $(2)(1)(2)$ or
(b) $C$ is a $(t')^\pm 1$-band, and $H(1)H(2)H(3)$ is of the form $(2)(3)(2)$.

Then $\text{Area}(\Gamma) \leq c_3[\Gamma] + c_2\mu^c(\Gamma)$.

**Proof.** It follows from (xii) that $H$ has no subwords of type $(1)(2)(1)$ or $(3)(2)(3)$ because $\Gamma$ is regular and has both rules $(12)^\pm 1$ and $(23)^\pm 1$ in its history. An $a$-band starting on the $H(2)$-part $C_i(2)$ of $C_i$ cannot cross a $(23)$-band in case (a) or $(12)$-band in case (b) by (v). Also it cannot end on $C_i(2)$ by (vi). Hence, every maximal $a$-band starting on $C_i(2)$ must end either on the parts $C_i(1), C_i(3)$ or on the path $z = z^\Gamma$. Now inequalities $h(1) + h(3) < h(2)/2$ and $h(2) \geq h/30 \geq \delta h$ make it possible to apply Lemma 7.17 to $\Gamma$. Hence,

$$\text{Area}(\Gamma) \leq (\delta')^{-2}[\Gamma].$$

(11.8)

Since more than $\frac{1}{2} \frac{h(2)}{h} - 2 \geq \frac{1}{60} h - 2$ maximal $a$-bands end on $z$, we have $|z| - |y| > \delta'(\frac{60}{h} h - 2) + 1 > \frac{\delta'}{60} h$ by Lemma 7.10(a), that is, $h < 60(\delta')^{-1}(|z| - |y|)$. Hence, by Lemma 6.2(a),

$$\lambda^c(\Gamma) \geq -\lambda^c(y) \geq -h^2/2 \geq -30(\delta')^{-1}h(|z| - |y|) = -30(\delta')^{-1}[\Gamma].$$

This inequality and (11.8) complete the proof of the lemma since

$$(\delta')^{-2}[\Gamma] = ((\delta')^{-2} + 30(\delta')^{-1}c_2)[\Gamma] - 30(\delta')^{-1}c_2[\Gamma] \leq c_3[\Gamma] + c_2\lambda^c(\Gamma) \leq c_3[\Gamma] + c_2\mu^c(\Gamma)$$

by the choice of $c_3$, the definition of $\mu^c(\Gamma)$, and by Lemma 7.2(a).

**Lemma 11.6.** Let $\Gamma$ be a regular comb of width $b \leq 15N$ whose handle is a $t^\pm 1$-band (or $(t')^\pm 1$-band) with firm factorization of the history $H \equiv H(1), \ldots, H(5)$, where $H(2)$ and $H(4)$ are both of type (1) (or both of type (3), respectively), and $h(2) + h(4) \geq 0.9h$. Assume that $H(3)$ contains both $(12)^\pm 1$ and $(23)^\pm 1$, and one of the derivative bands $C_i$ crosses all the maximal $(12)$- and $(23)$-bands of $\Gamma$. Then $\Gamma$ has a long subcomb $\Delta$ with $\text{Area}(\Delta) \leq c_3[\Delta] + c_2\mu^c(\Delta)$.

**Proof.** We will prove only the first version of the lemma. The history $H_i$ of $C_i$ contains $H(3)$, and so it contains the rule $(23)^\pm 1$, and therefore the band $C_i$ cannot be a $t^\pm 1$-band; it is a $k^{-1}$-band by (i). It follows from the assumption of the lemma that the derivative $C_i$ must cross $H(2)$-, $H(3)$-, and $H(4)$-parts of the comb $\Gamma$, and therefore $h_i > 0.9h$. Moreover, the sum of lengths of $H(2)$- and $H(4)$-parts of $C_i$ is at least $0.9h$, and so $\text{max}(h(2), h(4)) \geq 0.4h$.

There are at most $0.1h$ maximal $a$-bands starting on $C_i$ and ending on other derivative bands. There are no $a$-bands starting on the $H(2)$-part and ending on the $H(4)$-part of $C_i$ by the condition on $H(3)$, because an $a$-band cannot cross both $(12)$- and $(23)$-bands by (v). Besides, both the $H(2)$- and the $H(4)$-part of $C_i$ are active from the right by (ii). Therefore,
we can apply Lemma 7.17 to $\Gamma$:

$$\text{Area}(\Gamma) \leq (\delta')^{-2}[\Gamma].$$

(11.9)

Note that at least $0.9h - 4 - 0.1h = 0.8h - 4$ maximal $a$-bands end on $z$, and so $|z| - |y| - 2 = |z'| - |y'| \geq (0.8h - 4)\delta'$ by Lemma 7.10(1), whence

$$h \leq \frac{5}{4}(\delta')^{-1}(|z| - |y|).$$

(11.10)

Hence by Lemma 6.2(a) and inequality (11.10), we obtain

$$\lambda^c(\Gamma) \geq -\lambda(y) > -h^2/2 > -h(\delta')^{-1}(|z| - |y|) = -(\delta')^{-1}[\Gamma].$$

This inequality and (11.9) complete the proof as in Lemma 11.5 because

$$(\delta')^{-2}[\Gamma] = ((\delta')^{-2} + (\delta')^{-1}c_2)[\Gamma] - (\delta')^{-1}c_2[\Gamma]$$

$$\leq c_3[\Gamma] + c_2\lambda^c(\Gamma)$$

$$\leq c_3[\Gamma] + c_2\mu^c(\Gamma).$$

\[\square\]

**Lemma 11.7.** Let $\Delta$ be a regular comb. Assume that history $H^\Delta$ has $m \leq 6$ letters $(12)_{\pm 1}$ and $(23)_{\pm 1}$, and $\max(4, 2m)N < b \leq 15N$ for the base width $b$ of $\Delta$. Then either $\Delta$ admits a long quasicomb $\Delta'$ with

$$\text{Area}(\Delta') \leq c_3[\Delta'] + c_2\mu^c(\Delta') + c_3(\nu_j^c(\Delta) - \nu_j^c(\Delta \setminus \Delta'))$$

or $\Delta$ has a maximal $t^\pm$ or $(t')^\pm$-band of length $l$, where $T_i \leq l < 200T_i$ for some $T_i$.

**Proof.** If $m \leq 2$ or the handle of $\Delta$ does not contain either $(12)$-cells or $(23)$-cells, then the statement follows from Lemmas 9.6, 10.6, and 11.4 because $\lambda^c$ is non-negative for one step (quasi)combs, the third summand in the above inequality is positive by Lemmas 7.2(a) and 7.3(d), and $c_3 > c_2 > c_1$. Then we will induct on $m$ assuming that $3 \leq m \leq 6$ and that the history $H^\Delta$ of $\Delta$ contains both rules $(12)_{\pm 1}$ and $(23)_{\pm 1}$.

The comb $\Delta$ has a regular subcomb $\Delta_1$ of base width $b_1 > N(2m - 1)$ such that the base width of the filling trapezium $T = Tp(D_1, D)$ is $N + 1$, where $D_1$ and $D$ are the handles of $\Delta_1$ and $\Delta$, respectively. If the history of $\Delta_1$ has $m_1$ letters $(12)_{\pm 1}$ and $(23)_{\pm 1}$, and $m_1 < m$, then the statement of the lemma is a consequence of the inductive hypothesis since $2m_1 \leq 2m - 1$. Hence, we may assume that $m_1 = m$. Similarly, $\Delta_1$ has a regular subcomb $\Delta_2$ of width $b_2 > (2m - 2)N$ with handle $D_2$ and the filling trapezium $T' = Tp(D_2, D_1)$ of width $N + 1$, and we may assume that $m_2 = m_1 = m$ since otherwise $2m_2 \leq 2m - 2$ and one may apply the inductive conjecture to $\Delta_2$.

Thus, both $D_1$ and $D_2$ have $(12)$- and $(23)$-cells. Hence, the base of $T$ is normal by (xi), and so it contains a letter $t^\pm$ and a letter $(t')^\pm$, neither of which are the first letter in this base.
The same is true for $T'$. Denote by $C$ ($C'$) a maximal $t^{\pm 1}$-band or $(t')^{\pm 1}$-band of $\Delta$ crossing $T$ (crossing $T'$) and corresponding to this letter of the base. By $\Gamma$ and $\Gamma'$, we denote the subcombs with handles $C$ and $C'$, respectively. The histories of these handles are $H$ and $H'$. We will assume that $h' > h/2$ for their length, because otherwise one can apply Lemma 11.3 to $H$ since $c_1 < c_2 < c_3$.

Observe that there are derivative bands in both $\Gamma$ and $\Gamma'$ corresponding to the rules $(12)^{\pm 1}$ and $(23)^{\pm 1}$. This follows from the equality of the numbers of $(12)$- and $(23)$-cells in $D$ and $D'$. Hence, such a derivative band is a $k^{-1}$ or $k'$-band by (i) and (v), and there exist firm factorizations $H = H^{(1)}, \ldots, H^{(m+1)}$ and $H' = (H')^{(1)}, \ldots, (H')^{(m+1)}$ for $\Gamma$ and $\Gamma'$, where $(H')^i \equiv H^i$ for $i = 2, \ldots, m - 1$.

Besides one may assume that all other derivative bands of $\Gamma$ and $\Gamma'$ (if any) are also either $k^{-1}$- or $k'$-bands. Indeed, they do not cross $(12)$- and $(23)$-bands and so if a derivative band is a $t^{\pm 1}$- or $(t')^{-1}$-band, one can apply Lemma 9.2 to a derivative diagram $\Delta'$, and the statement of our lemma follows. (Similarly, the comb $\Lambda$ from Case 4, also enjoys this property of $\Gamma$ and $\Gamma'$ by the same reason.)

By Property (xii), the step history of $\Delta$ is a subword of $(2)(1)(2)(3)(2)(1)(2)$. Since $m > 3$ and one always can replace $\Delta$ by $\Delta^{-1}$ (and $H$ by $H^{-1}$), we have to consider the following six step histories: $(1)(2)(3)(2)(1)(2)$, $(1)(2)(3)(2)(1)(2)$, $(1)(2)(3)(2)(1)(2)$, $(2)(1)(2)(3)(2)$, $(2)(1)(2)(3)(2)$, and $(2)(1)(2)(1)(2)(1)(2)(2)(3)(2)$.

Case 1: The history $H$ is of type $(1)(2)(3)(2)$, and $H^{(1)}H^{(2)}H^{(3)}H^{(4)}$ is the corresponding firm factorization. In this case, we select $C$ to be a $t^{\pm 1}$-band and $C'$ a $(t')^{\pm 1}$-band.

If $h^{(4)} > 0.01u$, then one can apply Lemma 11.1 to $\Gamma$ with $H(1) \equiv H^{(1)}H^{(2)}H^{(3)}H^{(4)}$, $H(2) \equiv H^{(4)}$, and $(3) \equiv \emptyset$, since the condition on the $(1)-(2)$-transition holds by (v), and the passive cells of the $H(2)$-part of $C$ must be $(12)$- or $(23)$-cells by (ii). Hence, we obtain a required subcomb. Therefore, we may further assume that $h^{(4)} < 0.01u$.

Since the base of $T$ is normal and has at least $N$ letters, this trapezium contains a standard subtrapezium with history $H^{(2)}$, and so $h^{(2)} = T_i$ for some $i$. We may assume that $T_i \leq h/200$ because otherwise $h < 200T_i$, and the lemma is true. Thus, $h^{(2)} + h^{(4)} < h/60$.

Assume that $h^{(3)} \geq h/30$. Then $(h')^{(3)} = h^{(3)} \geq h'/30$ and $(h')^{(4)} \leq h^{(2)} + h^{(4)} < h/60 \leq h^{(3)}/2 = (h')^{(3)}/2$. Hence, one can apply Lemma 11.5(b) to $\Gamma$. (Here $H(1) \equiv (H')^{(2)}$, $H(2) \equiv (H')^{(4)}$, and $H(3) \equiv \emptyset$. Therefore, we can further assume that $h^{(3)} < h/30$.

Now, $h^{(1)} > h - h^{(2)} - h^{(3)} - h^{(4)} > h(1 - 1/60 - 1/30) = 0.95h$. Therefore, Lemma 11.6 is applicable to $\Gamma$ with $H(1) \equiv \emptyset$, $H(2) \equiv H^{(1)}$, $H(3) \equiv H^{(2)}H^{(3)}H^{(4)}$, $H(4) \equiv H(5) \equiv \emptyset$. This completes Case 1.

Case 2: The history $H$ is of type $(3)(2)(1)(2)$, and $H^{(1)}H^{(2)}H^{(3)}H^{(4)}$ is the corresponding firm factorization. In this case, we will assume that $C$ is a $(t')^{\pm 1}$-band and $C'$ a $t^{\pm 1}$-band. Then the proof coincides with that in Case 1.

Case 3: The history $H$ is of type $(1)(2)(3)(2)(1)$ and $H^{(1)}H^{(2)}H^{(3)}H^{(4)}H^{(5)}$ is the corresponding firm factorization. In this case, we will assume that $C$ is a $t^{\pm 1}$-band.

As in Case 1, one may assume that $\max(h^{(2)}, h^{(4)}) \leq h/200$. Then $h^{(3)} < h/100$ by (xvi). Therefore, $h^{(1)} + h^{(5)} > h - 2h/100 = 0.98h$. Therefore, one can apply Lemma 11.6 to $\Gamma$ with $H(1) \equiv \emptyset$, $H(2) \equiv H^{(1)}$, $H(3) \equiv H^{(2)}H^{(3)}H^{(4)}$, $H(4) \equiv H^{(5)}$, and $H(5) \equiv \emptyset$.

Case 4: The history $H$ is of type $(2)(1)(2)(3)(2)$ and $H^{(1)}H^{(2)}H^{(3)}H^{(4)}H^{(5)}$ is the corresponding firm factorization. In this case, we will assume that both $C$ and $C'$ are $t^{\pm 1}$-bands and consider an auxiliary maximal $(t')^{\pm 1}$-band $B$ between them. It exists since the base of $T_{\rho}(C', C)$ is normal, and determines a subcomb $\Delta$ of $\Delta$ whose history $G$ has a firm factorization $G^{(1)}, \ldots, G^{(5)}$.

If $h^{(5)} \geq 0.01h$, then one can apply Lemma 11.1 to $\Gamma$ with $H(1) \equiv H^{(1)}H^{(2)}H^{(3)}H^{(4)}$, $H(2) = H^{(5)}$ and $H(3) = \emptyset$. Hence, we may assume that $h^{(5)} < 0.01h$. 


Since $\Delta$ is regular, $h^{(3)} = T_i$ for some $i$. We may assume that $T_i \leq h/200$ because otherwise $h < 200T_i$, as desired. Thus, $h^{(3)} + h^{(5)} < h/60$.

Assume that $h^{(4)} \geq h/30$. Then $g^{(4)} = h^{(4)} \geq g/30$ and $g^{(3)} + g^{(5)} \leq h^{(3)} + h^{(5)} < h/60 \leq h^{(4)}/2 = g^{(4)}/2$. Hence, one can apply Lemma 11.5 to $\Lambda$. (Here, $H(1) \equiv G^{(3)}$, $H(2) \equiv G^{(4)}$ and $H(3) \equiv G^{(5)}$.) Therefore, we can further assume that $h^{(4)} < h/30$.

Suppose $h^{(1)} \geq 0.7h$. Then Lemma 11.2 can be applied to $\Gamma^{-1}$ with $H'' \equiv (H^{(1)})^{-1}$. Hence, we may assume that $h^{(1)} < 0.7h$, and therefore $h^{(2)} > h(1 - 0.7 - 1/30 - 1/60) = h/4$.

If $g^{(1)} > 0.01g$, then Lemma 11.1 is applicable to $\Lambda^{-1}$ with $H(2) \equiv (G^{(1)})^{-1}$ and $H(3) \equiv \emptyset$. Therefore, we may assume that $g^{(1)} < 0.01g$.

Now $(\mu^{(2)}) = h^{(2)} > h/4 \geq h'/4$ and $(\mu^{(1)} + (\mu^{(3)} - g^{(1)}) + h^{(3)} < 0.01g + h/200 \leq 0.015h < (h'/2)/2$. Hence, Lemma 11.5 is applicable to $\Gamma'$, and the lemma is proved in Case 4.

Case 5. The history $H$ is of type $(1)(2)(3)(1)(2)$, and $H^{(1)}H^{(2)}H^{(3)}H^{(4)}H^{(5)}$ is the corresponding firm factorization. In this case we will assume that $H$ is a $(t')^{\pm 1}$-band and $C'$ is a $(t')^{\pm 1}$-band.

If $h^{(6)} \geq 0.01h$, then one can apply Lemma 11.1 to $\Gamma$ with $H(1) \equiv H^{(1)}(H^{(2)}H^{(3)}H^{(4)}H^{(5)}$, $H(2) \equiv H^{(6)}$, and $H(3) \equiv \emptyset$. Hence, we may assume that $h^{(6)} < 0.01h$. Then, as in Case 3, we may assume that max$(h^{(2)}, h^{(4)}) \leq h/200$ and $h^{(3)} < h/100$. Since $h' > h/2$, it follows that

$$(h')^{(1)} + (h')^{(5)} = h' - (h')^{(2)} - (h')^{(3)} - (h')^{(4)} - (h')^{(6)} \geq h' - h(2) - h^{(3)} - h^{(4)} - h^{(6)} > h' - 0.03h$$

$$(h')^{(1)} + (h')^{(5)} \geq h' - h^{(2)} - h^{(3)} - h^{(4)} - h^{(6)} > h' - 0.06h' = 0.94h'.$$

Therefore, one can apply Lemma 11.6 to $\Gamma'$ with $H(1) \equiv \emptyset$, $H(2) \equiv (H')^{(1)}$, $H(3) \equiv (H')^{(2)}(H')^{(3)}(H')^{(4)}$, $H(4) \equiv (H')^{(5)}$, and $H(5) \equiv (H')^{(6)}$.

Case 6. The history $H$ is of type $(2)(1)(2)(3)(2)(1)(2)$ and $H^{(1)}H^{(2)}H^{(3)}H^{(4)}H^{(5)}H^{(6)}H^{(7)}$ is the corresponding firm factorization. In this case, we will assume that $C$ is a $(t')^{\pm 1}$-band and $C'$ is a $(t')^{\pm 1}$-band.

If $h^{(7)} \geq 0.01h$, then one can apply Lemma 11.1 to $\Gamma$ with $H(1) \equiv H^{(1)}H^{(2)}H^{(3)}H^{(4)}H^{(5)}H^{(6)}$, $H(2) \equiv H^{(7)}$ and $H(3) \equiv \emptyset$. Hence, we may assume that $h^{(7)} < 0.01h$. Similarly, $h^{(1)} < 0.01h$. Then, as in Cases 3 and 5, we may assume that max$(h^{(3)}, h^{(5)}) < h/200$ and $h^{(4)} < h/100$. Therefore,

$$(h')^{(2)} + (h')^{(6)} = h' - (h')^{(1)} - (h')^{(3)} - (h')^{(4)} - (h')^{(5)} - (h')^{(7)} \geq h' - h(1) - h^{(3)} - h^{(4)} - h^{(5)} - h^{(7)} > h' - 0.04h$$

$$(h')^{(2)} + (h')^{(6)} \geq h' - h(1) - h^{(3)} - h^{(4)} - h^{(5)} - h^{(7)} > h' - 0.08h' = 0.92h'.$$

Therefore one can apply Lemma 11.6 to $\Gamma'$ with $H(1) \equiv (H')^{(1)}$, $H(2) \equiv (H')^{(2)}$, $H(3) \equiv (H')^{(3)}(H')^{(4)}(H')^{(5)}$, $H(4) \equiv (H')^{(6)}$, and $H(5) \equiv (H')^{(7)}$.

The lemma is proved in any case.

\[ \square \]

**Lemma 11.8.** Let $\Delta$ be a comb of base width $b > 13N$. Then either $\Delta$ admits a long quasiconvex $\Gamma'$ with

$$\text{Area}(\Gamma') \leq c_3[I\Gamma'] + c_2\mu \varepsilon(\Gamma') + c_3(\nu_f^{(\Delta)} - \nu_f^{(\Delta \setminus \Gamma')})$$

or $\Delta$ has a maximal $(t')^{\pm 1}$ or $(t')^{\pm 1}$-band of length $l$, where $T_i \leq l < 200T_i$ for some $T_i$, and this band is a handle of a subcomb of base width at most $14N$.

**Proof.** Recall that the third term on the right-hand side of (11.11) is positive for every sub(quasi)comb $\Gamma'$ by Lemma 7.3(e) and Remark 9.5.

Then we observe that $\Delta$ has a subcomb $\Delta_0$ of base withs $b^{(0)} \in (13N, 15N]$, and in turn, $\Delta_0$ has a regular subcomb $\Gamma$ with $12N < b^{(0)} \leq 14N$. If $\Gamma$ is a one-step comb, then by Lemma 9.6,
it admits a long quasicomb $\Gamma'$ with $\text{Area}(\Gamma') \leq c_1([\Gamma'] + \kappa(\Gamma'))$. Here, the right-hand side does not exceed $c_3[\Gamma'] + c_2\mu(\Gamma')$ since $c_1 < c_2 < c_3$ and $\lambda(\Gamma') \geq -\lambda(y^{\Gamma'}) = 0$ for one-step comb $\Gamma'$. Inequality (11.11) follows in this case.

Then we may assume that the history $H$ of $\Gamma$ has one of the rules (12) or (23). If $H$ has no (12) or (23), then the statement of the lemma follows from Lemma 10.6. Otherwise, $H$ has at most six letters (12) and (23) by Properties (xii) and (viii), since $\Gamma$ is a regular comb (and so there exists a trapezium of width at least $N$ with history $H$). Now the application of Lemmas 11.7 and 7.3(e) completes the proof.

12. Separation of a hub

In this section, we consider minimal diagrams over the group $G$ with cyclically reduced boundary paths. Thus, in contrast to previous sections, we study diagrams with hubs.

12.1. Solid diagrams

Let $\pi$ be a hub in a diagram $\Delta$, connected with the boundary $\partial\Delta$ by $t$-spokes $B$ and $B'$. We denote by $\text{cl}(\pi, B, B')$ the subdiagram bounded by these spokes (and including them) and by subpaths of the boundaries of $\Delta$ and $\pi$, and call this subdiagram a clove if it has no hubs.

**Lemma 12.1.** (a) Let $\Psi = \text{cl}(\pi, B, B')$ be a clove in a reduced diagram $\Delta$. Assume that $\Psi$ contains a rim $\theta$-band $T$, which crosses neither $B$ nor $B'$, and every rim $\theta$-band of $\Psi$ with this property has at least $2LN$ $q$-cells. Then there is a maximal $q$-band $C$ in $\Psi$ and a subcomb $\Gamma$ with handle $C$ such that the base width of $\Gamma$ is $15N$ and no $q$-band of $\Gamma$ is a subband of a spoke of $\Delta$.

(b) Assume that a reduced diagram $\Delta$ contains cells but has no hubs. Then either it has a rim band of base width less than $2LN$ or it has a subcomb of base width $15N$.

**Proof.** (a) Since (1) a hub has $LN$ spokes, (2) no $q$-band of $\Psi$ intersects $T$ twice by Lemma 5.6, (3) $T$ has at least $2LN$ $q$-cells, and (4) $L > 30$, there exists a maximal $q$-band $C'$ such that a subdiagram $\Gamma'$ separated from $\Psi$ by $C'$ contains no edges of the spokes of $\pi$ and the part of $T$ belonging to $\Gamma'$ has at least $15N$ $q$-cells.

If $\Gamma'$ is not a comb, and so a maximal $\theta$-band of it does not cross $C'$, then $\Gamma'$ must contain another rim band $T'$ having at least $2LN$ $q$-cells by the assumption of the lemma. This makes it possible to find a subdiagram $\Gamma''$ of $\Gamma'$ such that a part of $T'$ is a rim band of $\Gamma''$ containing at least $LN > 15N$ $q$-cells, and $\Gamma'''$ does not contain $C'$. Since $\text{Area}(\Gamma') > \text{Area}(\Gamma'') > \cdots$, such a procedure must stop. Hence, for some $i$, we obtain a subcomb $\Gamma^{(i)}$ of width $b > 15N$ intersected by no spokes. If $b > 15N$, then a derived subcomb of it has width $b - 1 \geq 15N$. Finally we obtain the desired $\Gamma$. 

(b) The proof is easier than that for (a): one should just ignore the hub.

We call a minimal diagram solid if it has no rim \(\theta\)-bands of base width at most \(2LN\), no subcombs of base width \(15N\) and no one-step subcombs whose handles are \(t^{\pm 1}\) or \((t')^{\pm 1}\)-bands.

Here, we focus on solid diagrams since the proof of Theorem 1.1 will be reduced to them in the next section.

For a clove \(\Psi = \text{cl}(\pi, B, B')\) in a diagram \(\Delta\), we denote by \(p(\Psi)\) the common subpath of \(\partial\Psi\) and \(\partial\Delta\) starting with the \(q\)-edge of \(B\) and ending with the \(q\)-edge of \(B'\).

**Lemma 12.2.** Let \(\Psi = \text{cl}(\pi, B, B')\) be a clove in a solid diagram \(\Delta\). Then every maximal \(\theta\)-band of \(\Psi\) crosses either \(B\) or \(B'\); the base width of any \(\theta\)-band of \(\Psi\) is less than \(2LN\), and \(\text{Area}(\Psi) \leq (2LN(h + h') + \delta^{-1}|p(\Psi)|)(h + h')\), where \(h\) and \(h'\) are the lengths of the bands \(B\) and \(B'\), respectively.

**Proof.** If the first claim were wrong, then one could find a rim \(\theta\)-band which crosses neither \(B\) nor \(B'\). Then by Lemma 12.1(a), either the first or the second condition in the definition of solid diagram would be violated, a contradiction. Thus, the first statement of the lemma is proved, and \(\Psi\) has at most \(h + h'\) maximal \(\theta\)-bands.

Now we consider a maximal \(\theta\)-band \(T\) in \(\Psi\) (\(T\) is not an annulus by Lemma 5.6). If its base width is at least \(2LN\), then there is a maximal \(q\)-band \(C\) intersecting \(T\) which does not start/end on the hub \(\pi\) because the number of spokes starting on the same hub cell is \(LN\). Moreover, as in the proof of Lemma 12.1, one can select \(C\) so that \(C\) separates a comb of base width more than \((2LN - LN)/2 \geq 15N\) from \(\Psi\), contrary to the assumption that \(\Delta\) is solid. Thus, the base width of \(T\) is less than \(2LN\). Therefore, the number of \((\theta, q)\)-cells in \(\Psi\) is at most \(2LN \times (h + h')\). Every \((\theta, q)\)-cell has at most two \(a\)-edges by (iii). Hence, the number of maximal \(a\)-bands starting and ending on the \((\theta, q)\)-cells of \(\Psi\) (but not on \(\partial\Delta\)) is at most \(2LN(h + h')\). Their lengths do not exceed \(h + h'\) by Lemma 5.6 since the number of maximal \(\theta\)-bands in \(\Psi\) is at most \(h + h'\). Thus, the total area of these \(a\)-bands does not exceed \(2LN(h + h')^2\). By the same reason, all other maximal \(a\)- and \(q\)-bands of \(\Psi\) have length at most \(h + h'\), but each of them has at least one edge on \(p(\Psi)\). Therefore, their total area is at most \((h + h')(|p(\Psi)|_a + |p(\Psi)|_q) \leq \delta^{-1}|p(\Psi)|\) by Lemma 5.20(d). Since every cell of \(\Psi\) belongs to a \(\theta\)-band, every \((\theta, a)\)-cell belongs to an \(a\)-band and every maximal \(a\)-band starts and ends on a \((\theta, q)\)-cell or on \(\partial\Delta\), the sum of these two inequalities gives the inequality from the lemma.

Let \(\Psi\) be a clove at a hub \(\pi\) in a solid diagram \(\Delta\). Assume that it has more than \(L\) \(t\)- and \(t'\)-spokes. (Recall that \(\partial\pi\) has \(2L\) \(t\) - and \(t'\)-edges.) Then we denote by \(\Delta\) the subdiagram formed by \(\pi\) and \(\Psi\), and denote by \(p\) the path \(\text{top}(B)u^{-1}\text{bot}(B')^{-1}\), where \(u\) is a subpath on \(\partial\pi\), such that \(p\) separate \(\Delta\) from the remaining subdiagram \(\Psi'\) of \(\Delta\). It follows that the total number of \(t\)- and \(t'\)-edges in \(u\) is less than \(L\), \(|u| < LN\), and the number of \(t\)- and \(t'\)-edges in \(p(\Psi)\) is at least \(L + 1\).
Lemma 12.3. If $|\partial \Delta| = n$ and $|p(\Psi)| \geq 2LN \max(h, h')$, then, in the preceding notation, $|p(\Psi)| - |\bar{p}| > 0$ and

$$\text{Area}(\Delta) \leq c_4(\mu(\Delta) - \mu(\Psi')) + c_5(\nu(\Delta) - \nu(\Psi')) + c_6n(n - |\partial \Psi'|).$$

Proof. Let us present $\partial \Psi'$ in the form $\bar{p}$. By Lemma 5.20(b), $|\partial \Psi'| \leq |p| + |\bar{p}|$ and $n = |p(\Psi)| + |\bar{p}|$ since the first and the last edges of $p(\Psi)$ are $q$-edges. Hence,

$$n - |\partial \Psi'| \geq |p(\Psi)| - |\bar{p}|. \tag{12.1}$$

Note that by the definition of $\Psi$, we have $|\bar{p}| \leq h + h' + |u| \leq h + h' + LN - 1$, and $|p(\Psi)| \geq LN + 1$. Therefore, in case $h = h' = 0$, we have $|p(\Psi)| - |\bar{p}| \geq \max(2, \delta|p(\Psi)|)$ by Lemma 5.20(d), since $\delta^{-1} \geq LN$. If $\max(h, h') > 1$, then by the second assumption of the lemma $3|\Psi| - |\bar{p}| > 1.5LN \max(h, h') - 2 \max(h, h') - LN \geq 2$, and since $\delta < 1/4$, in any case we obtain

$$|p(\Psi)| - |\bar{p}| \geq \max(\delta|p(\Psi)|, 2). \tag{12.2}$$

Inequalities (12.1), (12.2), and the assumption of the lemma on $|p(\Psi)|$ imply

$$|p(\Psi)| \leq \delta^{-1}(n - |\partial \Psi'|), \quad n - |\partial \Psi'| \geq 2, \quad \text{and} \quad h + h' \leq n/LN. \tag{12.3}$$

Now by Lemma 12.2 and inequalities (12.3), we have

$$\text{Area}(\Delta) = \text{Area}(\Psi) + 1 \leq (2LN(h + h') + \delta^{-1}|p(\Psi)|)(h + h') + 1$$

$$\leq (2|p(\Psi)| + \delta^{-1}|p(\Psi)|)(h + h') + 1$$

$$\leq (2 + \delta^{-1})\delta^{-1}(n - |\partial \Psi'|)(h + h') + 1$$

$$\leq 3(\delta^2LN)^{-1}(n - |\partial \Psi'|)n \leq c_6n(n - |\partial \Psi'|)/2. \tag{12.4}$$

Recall now that in the definition of $\kappa$- and $\lambda$-mixtures, the middle point of every boundary $q$-edge is a black bead of the necklace on the boundary of diagram, and every white bead is a middle point of a boundary $\theta$-edge (see Section 6 for details). It follows that $\kappa(\Delta) - \kappa(\Psi') \geq -|\bar{p}|n$ and $\lambda(\Delta) - \lambda(\Psi') \geq -|\bar{p}|n$ because new pairs of white beads $(o, o')$ separated by black beads can appear in the necklace on $\partial \Psi'$ (in comparison with the necklace on $\partial \Delta$) only if one of the beads $o$ and $o'$ belongs to $\bar{p}$. Hence, by (12.2),

$$\min(\kappa(\Delta) - \kappa(\Psi'), \lambda(\Delta) - \lambda(\Psi')) \geq -\delta^{-1}(|p(\Psi)| - |\bar{p}|)n,$$

and therefore by (12.1),

$$c_4(\mu(\Delta) - \mu(\Psi')) = c_4((c_0\kappa(\Delta) + \lambda(\Delta)) - (c_0\kappa(\Psi') + \lambda(\Psi'))$$

$$= c_4(c_0(\kappa(\Delta) - \kappa(\Psi'))) + (\lambda(\Delta) - \lambda(\Psi'))$$

$$\geq -c_4(c_0 + 1)\delta^{-1}(|p(\Psi)| - |\bar{p}|)n \geq -c_6n(n - |\partial \Psi'|)/4. \tag{12.5}$$

Recall also that the number of $t$- and $t'$-edges in the path $\bar{p}$ (or in the path $u$) does not exceed the similar number for $p(\Psi)$. Therefore, any two white beads $o$ and $o'$ of the $\nu$-necklace on $\partial \Delta$, provided they both belong to $p(\Psi)$, are separated by at least the same number of black beads in the $\nu$-necklace for $\Delta$ as in the $\nu$-necklace for $\Psi$ (either the clockwise arc $o-o'$ includes $p(\Psi)$ or not). So such a pair contributes to $\nu(\Delta)$ at least the amount it contributes to $\nu(\Psi')$. Thus, to estimate $\nu(\Delta) - \nu(\Psi')$ from below, it suffices to consider the contribution to $\nu(\Psi')$ for the pairs $o, o'$, where one of the two beads lies on $\bar{p}$. Then the argument we used above for $\kappa$- and $\lambda$-mixtures, yields $\nu(\Delta) - \nu(\Psi') \geq -J\delta^{-1}(|p(\Psi)| - |\bar{p}|)n$. Hence, $c_5(\nu(\Delta) - \nu(\Psi')) \geq -c_6n(n - |\partial \Psi'|)/4$ by (12.1) since $c_6 > 4J\delta^{-1}c_5$. This inequality together with (12.4) and (12.5) prove the lemma. □
A clove $\text{cl}(\pi, B, B')$ will be called a crescent if

1. It contains $\ell \geq L - 20 > L/2$ consecutive $t^\pm 1$-spokes $C_1 = B$, $C_2, \ldots, C_\ell = B'$ connecting $\partial \Delta$ and $\partial \pi$;
2. Every maximal $\theta$-band of this clove crosses either $C_1$ or $C_\ell$: moreover, either all maximal $\theta$-bands of $\Psi$ cross $C_1$, or all of them cross $C_\ell$, or there exists $i$, $2 \leq i \leq \ell - 2$ such that the $\theta$-bands crossing $C_1$ but not $C_\ell$, do not cross $C_1$, and the $\theta$-bands crossing $C_\ell$ but not $C_1$, do not cross $C_\ell$;
3. Every maximal $(12)$- or $(23)$-band of $\Psi$ crossing $C_1$ (crossing $C_\ell$) also crosses $C_2$ (crosses $C_{\ell - 1}$), and every spoke of the clove is crossed by at most 3 $(12)$- or $(23)$-bands.

**Lemma 12.4.** Assume a solid diagram $\Delta$ has a hub. Then it contains a crescent $\Psi = \text{cl}(\pi, C_1, C_2)$ such that the cloves $\text{cl}(\pi, C_2, C_1)$ and $\text{cl}(\pi, C_1, C_{\ell - 1})$ are also crescents.

**Proof.** We consider a hub $\pi$ provided by Lemma 5.18. There are consecutive maximal $t^\pm 1$-bands $B_1, \ldots, B_{L-3}$ connecting (counter-clockwise) $\partial \Delta$ and $\partial \pi$, such that the subdiagram $\bar{\Psi}$ bounded by $B_1$, $B_{L-3}$, $\partial \Delta$, and $\partial \pi$ contains all these $t^\pm 1$-bands but does not contain hubs. Observe that by Lemma 12.1, every maximal $\theta$-band of $\bar{\Psi} = \text{cl}(\pi, B_1, B_{L-3})$ crosses either $B_1$ or $B_{L-3}$ because $\Delta$ is solid.

Consider a subdiagram $\Psi(0) = \text{cl}(\pi, B_u, B_v)$ of $\bar{\Psi}$ with $u - v = L - k$ for some $k$, $6 \leq k < L$, $u > 1$, $v < L - 3$. Every maximal $\theta$-band of $\Psi(0)$ crosses either $B_u$ or $B_v$.

Let $\Lambda_{u-1,u}$ (let $\Lambda_{u+1,u}$) be the trapezium formed by all $\theta$-bands starting on $B_{u-1}$ (starting on $B_{u+1}$, respectively) and ending on $B_u$. It contains $M_4$-accepting subtrapezia with the same histories, and so there are at most 3 $(12)$- and $(23)$-bands among them by (x) and (viii); similarly for $\Lambda_{v,v+1}$ and $\Lambda_{v,v-1}$. Since $B_u$ and $B_v$ must belong to one of trapezia $\Lambda_{u-1,u}$ and $\Lambda_{u+1,u}$, and to one of $\Lambda_{v,v+1}$ and $\Lambda_{v,v-1}$, respectively, the number of maximal $(12)$- and $(23)$-cells in $B_u$ (in $B_v$), and therefore in any spoke of $\Psi(0)$, is at most 3, and the number of maximal $(12)$- and $(23)$-bands in $\Psi(0)$ is at most 6. We want to obtain a clove $\Psi(1) = \text{cl}(\pi, B_{u'}, B_{v'})$ ($u' \geq u, v' \leq v$) applying one of the following transitions changing the pair $(u, v)$.

(a) If a $(12)$- or $(23)$-band crosses $B_u$ but not $B_{u+2}$ ($B_v$ but not $B_{v-2}$), then we set $u' = u + 2, v' = v (u' = u, v' = v - 2)$.

(b) Note that either all maximal $\theta$-bands of $\Psi(0)$ cross $B_u$ or all of them cross $B_v$, or there exists $i$ ($u \leq i < v$) such that the $\theta$-bands crossing $B_u'$ but not $B_{u+2}$ do not cross $B_u$, and the $\theta$-bands crossing $B_v$ but not $B_{v-2}$ do not cross $B_v$. If in the latter case $i \geq v - 2$, then we set $u' = u, v' = v - 2$. Similarly we set $u' = u + 2, v' = v$ if $i \leq u + 1$.

After a transition (a) or (b), we obtain a clove $\Psi(1) = \text{cl}(\pi, B_{u'}, B_{v'})$ with $v' - u' \geq L - k'$, where $k' = k + 2$. Let us start with $u = 2$ and $v = L - 4$ (that is, $k = 6$) and apply a maximal series of transitions of type (a). Since every transition of type (a) removes a maximal $(12)$- or $(23)$-band and the number of such bands in $\Psi(0)$ is at most 6, the length of the series is also at most 6. Then, if possible, we apply a transition of type (b). Note that no transition of types (a) and (b) is applicable to a clove $\Psi(m)$ with $m \leq 7$. We have $k(m) \leq 6 + 2 \times 7 = 20$. 


It remains to set \( C_1 = B_{\alpha(m)} \), and \( C_t = B_{\alpha(t)} \). Then \( \Psi = \Psi(m) = \text{cl}(\pi, C_1, C_t) \) satisfies the conditions (1)–(3) from the definition of crescent. Indeed condition (1) holds since \( k^{(m)} - 1 \leq 19 \) and \( L > 40 \), condition (2) and condition (3) hold since no transition of type (b) or of type (a), respectively, is applicable to \( \Psi \). The cloves \( \text{cl}(\pi, C_2, C_t) \) and \( \text{cl}(\pi, C_1, C_{t-1}) \) are also crescents since (1) \( 20 \leq 41/2 \) and (2), (3) no transitions of type (b) and (a) are applicable to \( \Psi \).

**Lemma 12.5.** The number of maximal \( t \)- and \( t' \)-bands in a crescent \( \Psi \) is less than \( 13LN < J/2 \).

**Proof.** The number of maximal \( t \)- or \( t' \)-band \( C \) starts and ends on the subpath \( p(\Psi) \) of the boundary of \( \Delta \) and separates a subdiagram \( \Phi \) from the crescent \( \Psi = \text{cl}(\pi, B, B') \) such that \( \Phi \) contains \( C \) but has no cells from \( B \) or \( B' \). However, by the definition of crescent, every cell from \( \Phi \) is connected with either \( B \) or \( B' \) by a \( \theta \)-band. It follows that every maximal \( \theta \)-band of \( \Phi \) has to cross \( C \), that is, \( \Phi \) is a subcomb of \( \Delta \) with handle \( C \). This subcomb is not one step since \( \Delta \) is a solid diagram, and therefore \( C \) has either (12)- or (23)-cells. In other words, it crosses one of the maximal \( \theta \)-bands crossing \( C_1 \) or \( C_t \) and corresponding to one of the rules (12), (13). The number of such \( \theta \)-bands is at most 6 by Properties (3) and (2) from the definition of crescent. Their base widths are less than \( 2LN \) by Lemma 12.2, and so the number of maximal \( t \)- and \( t' \)-bands \( C \) which have no ends on \( \pi \), is less than \( 12LN \). The lemma is proved because \( 12LN + 2L < 13LN \). \( \square \)

12.2. Surgery removing a hub

When we induct on the number of hubs, we want to cut up a subdiagram \( \Delta_1 \) with one hub so that \( \text{Area}(\Delta_1) \) is bounded in ‘quadratic terms’ (as we did earlier for subcombs). The estimates of Lemmas 12.8 and 12.9 will be applied in the final section 13.

In this subsection we consider a solid diagram \( \Delta \) with the hub \( \pi \) and the crescent \( \Psi = \text{cl}(\pi, C_1, C_t) \) provided by Lemma 12.4. We will assume that the \( t \)-spokes \( C_1 \) and \( C_t \) are enumerated counter-clockwise with respect to the hub \( \pi \), and the histories \( H_1, \ldots, H_l \) of \( C_1, \ldots, C_t \) are read towards \( \pi \). We have \( \partial \Psi = \text{top}(C_1)^{-1}p\text{bot}(C_t)\text{top}(C_1)s^{-1} \), where \( p = p(\Psi) \) and \( s \) is a subpath in \( \partial \pi \). Then \( \partial \Delta = pp_2 \). The diagram \( \Delta \) is the union of \( \Psi, \pi \), and the remaining subdiagram \( \Psi' \). Let now \( h_i = ||H_i|| \), where \( i = 1, \ldots, l \).

![Diagram of a subdiagram with hubs and spokes](image)

**Without loss of generality, we will assume further that** \( h_1 \geq h_i \) **for** \( \Psi \). **Under this assumption, we will use the following special surgery for** \( \Delta \). **Denote by** \( e_j \) **the common** \( t \)-edge of \( \partial C_j \) **and** \( \partial \pi \). **Consider the reduced subpath** \( e_{2l-L-1}u e_l \) **of** \( s \). **Denote by** \( \Gamma \) **the subdiagram without hubs bounded by** \( \text{bot}(C_{2l-L-1})^{-1}v\text{top}(C_t)(u')^{-1} \), **where** \( v \) **is a subpath of** \( \partial \Delta \). **We have** \( p(\Psi) = p = p_1vf \) **for some** \( p_1 \), **where** \( f \) **is the common edge of** \( \partial C_t \) **end** \( \partial \Delta \).
There is a reduced path \( e_i^{-1} \bar{u} e_i^{-1} \), where \( (\bar{u})^{-1} \) is a subpath of \( \partial \pi \). Then the path \( w_i = \text{top}(C_i) (u')^{-1} \) is obtained from \( w = \text{bot}(C_i) \bar{u}^{-1} \) by a \( t_i \)-reflection since \( C_i \) is a \( t_i \)-band for some \( i \) (see definitions in Remark 5.3). Therefore, the following surgery is possible.

(1) Cut \( \Delta \) along \( w \).

(2) Construct a diagram \( \Gamma_1 \) obtained from \( \Gamma \) by the \( t_i \)-reflection (see Remark 5.3) and take a standard mirror copy \( \Gamma_2 \) of \( \Gamma_1 \) (where the mirror edges have equal labels). Glue \( \Gamma_1 \) and \( \Gamma_2 \) together along the path \( r \) obtained by the \( t_i \)-reflection from \( \text{bot}(C_{2i-L-1})^{-1} v \), and obtain a diagram \( \Pi \) with boundary \( w'w'' \), where \( \text{Lab}(w') \equiv \text{Lab}((w'')^{-1}) \equiv \text{Lab}(w^{-1}) \).

(3) Insert \( \Pi \) in to the hole of \( \Delta \) obtained after step (1).

(4) Cut up the obtained disc diagram along \( \text{top}(C_1) r \), and obtain two diagrams \( \Delta_1 \) and \( \Delta_2 \), where \( \Delta_1 \) is a minimal diagram with the same boundary label as the union of \( \Psi, \pi \) and \( \Gamma_1 \), and \( \Delta_2 \) is a union of \( \Psi' \) and \( \Gamma_2 \).

(5) Let \( H_0 \) be the history of the maximal trapezium bounded by \( C_1 \) and \( C_{2i-L-1} \) in \( \Psi \) (it is the filling trapezium \( Tp(C_{2i-L-1}, C_1) \) if every maximal \( \theta \)-band crossing \( C_{2i-L-1} \) also crosses \( C_1 \), and so \( H_0 \) is a suffix of both \( H_1 \) and \( H_{2i-L-1} \). Therefore, \( 2h_0 = 2\|H_0\| \) letters can be canceled in the product \( \text{Lab}((\text{top}(C_1)))\text{Lab}(r) \). And so we shorten the corresponding part of the boundary of \( \Delta_2 \) by \( 2h_0 \) edges and replace the obtained diagram by a minimal diagram \( \Delta' \).

Thus, the boundary of \( \Delta' \) is \( p_3 p_2 \), where \( p_3 = x'v' \), \( v' \) is obtained by the \( t_i \)-reflection of \( v \), and \( |x| = |x|_\theta = h_1 + h_{2i-L-1} - 2h_0 \). Since the path \( p_1 \) has at least \((h_1 - h_0) + (h_{2i-L-1} - h_0)\) \( \theta \)-edges (the ends of maximal \( \theta \)-bands cross \( C_1 \) but do not cross \( C_{2i-L-1} \), and vice versa) and also has \( q \)-edges, we have

\[
|x| = |x|_\theta \leq |p_1|_\theta \leq |p_1| - 1. \tag{12.6}
\]

Moreover using the maximal \( \theta \)-bands crossing \( C_1 \) and \( C_l \) in the crescent \( \Psi \), one can for every \( \theta \)-edge of \( x \), find a \( \theta \)-edge of \( p_1 \) corresponding to the same rule \( \theta^\pm \), and the obtaining mapping from the set of (non-oriented) edges of \( x \) to the set of edges of \( p_1 \) is injective.

**Lemma 12.6.** With the preceding notation, we have (a) \( |\partial \Delta'| \leq |\partial \Delta| - 1 \); (b) \( \kappa(\Delta') \leq \kappa(\Delta) \); (c) \( \lambda(\Delta') \leq \lambda(\Delta) + 2h_0^2 \); (d) \( \nu_f(\Delta') \leq \nu_f(\Delta) \).

**Proof.** (a) Since \( |v'| = |v| \), the statement (a) follows from inequality (12.6) and Lemma 5.20(b).

(b) Recall that \( v' \) is constructed as the \( t_i \)-reflection of \( v \). Thus, when passing from the boundary label of \( \Delta \) to the boundary label of \( \Delta' \), we, in essence, just replace \( \text{Lab}(p_1) \) by \( \text{Lab}(x) \). But \( x \) has no \( q \)-edges, and so it has no black beads (see the definition of the \( \kappa \)-mixture of a diagram), and the number of white beads of \( x \) is at most the number of white beads on \( p_1 \) by (12.6). Therefore, \( \kappa(\Delta') \leq \kappa(\Delta) \) by Lemma 6.1 (Parts b,c).
(d) Similarly, using the fact that the path $x$ has no $t$- or $t'$-edges, one concludes that $\nu_j(\Delta') \leq \nu_j(\Delta)$.

(e) The remark made before the formulation of the lemma, allows us to obtain an injective mapping from the set of white beads of the $\lambda$-necklace $O'$ for $\partial\Delta'$ to the set of white beads of the $\lambda$-necklace $O$ on $\partial\Delta$, so that the beads from $x$ map to the beads on $p_1$. It follows that if $o, o'$ are two white beads from $O'$, but not both on $x$, and they are separated by a black bead in $O'$, then the corresponding white beads of $O$ are also separated by a black bead. (We take into account that $p_1$ starts and ends with $q$-edges having black beads by the definition of the necklace $O$.) Therefore, to estimate the difference $\lambda(\Delta) - \lambda(\Delta')$ from below, we may consider only the pairs of white beads of $O'$, where both $o$ and $o'$ belong to $x$, whence $\lambda(\Delta) - \lambda(\Delta') \geq -\lambda(x)$. By Lemma 6.2(a), $\lambda(x) < |x|^2/2 \leq (2h_1)^2/2$ and claim (c) is proved.

Remark 12.7. (1) The surgery described before the formulation of Lemma 12.6 can be also done for the original clove $\text{cl}(\pi,B_1,B_{L-3})$ even if one does not assume that $\Delta$ is a solid diagram. In this case again, exactly as in the proof of Lemma 12.6, we obtain the inequality $|\partial\Delta'| \leq |\partial\Delta| - 1$.

(2) Assume that $\Gamma$ is a subcomb of a diagram $\Delta$, and the handle of $\Gamma$ is a $t^{\pm 1}$ or $(t')^{\pm 1}$-band, $y = y^r$, and $\Delta' = \Delta \setminus \Gamma$. Then by Lemma 5.6, we have a preserving order bijective mapping from the set of $\theta$-edges of $y^{-1}$ to the set of the $\theta$-edges of $z = z^r$. Then arguing exactly as in the proof of Part (c) of Lemma 12.6, we obtain $\lambda(\Delta) - \lambda(\Delta') \geq -\lambda(y) > -|y|^2/2$. Hence,

$$\mu(\Delta) - \mu(\Delta') > -|y|^2/2,$$

by the definition of $\mu(\cdot)$ and Lemmas 7.2(a) and 7.3(a).

Lemma 12.8. Assume that $n = |\partial\Delta|$, $h_1$ does not belong to any interval $(T_i, 9T_i)$ ($i = 1, 2, \ldots$) and $h_2 > (1 - 1/30N)h_1$. Then, with the notation of Lemma 12.6, we have

$$\text{Area}(\Delta_1) \leq c_4(\mu(\Delta) - \mu(\Delta')) + c_5(\nu_j(\Delta) - \nu_j(\Delta')) + c_6 n - |\partial\Delta'|.$$

Proof. Since $h_1 = \max_{i=1}^l h_i$, the condition (2) from the definition of crescent implies that every maximal $\theta$-band crossing the $t^{\pm 1}$-band $C_2$ in the crescent $\Psi$, has to cross $C_1$ as well. Therefore, we can consider the trapezium $\Lambda_{12} = Tp(C_2,C_1)$ of height $h_2$ between $C_1$ and $C_2$. The bottom path $z_{12}$ of $\Lambda_{12}$, must be of $a$-length at least $h_2/6$ by (ix), since $h_2$ does not belong to any interval $(T_i, 9T_i)$.

Recall that the diagram $\Delta$ is solid, and therefore the clove $\text{cl}(\pi,C_1,C_2)$ has $h_1 - h_2 < h_1/30N$ maximal $\theta$-bands outside $\Lambda_{12}$. Hence, the maximal $a$-bands starting on $z_{12}$ can end outside of $\Lambda_{1,2}$ on at most $N(\theta,q)$-cells of each less than $h_1/30N \theta$-bands. Hence, by (iii)(a), at least $|z_{12}|_a - h_1/15$ $a$-bands starting on $z_{12}$ end on $p_1$, and so $|p_1|_a > h_1(1/6 - 1/15) = h_1/10$. 

[Diagram of $\Lambda_{12}$ and $C_1$, $C_2$, $\pi$, and $\Gamma$]
Therefore, by Lemma 5.20(a) and almost quadratic inequality (12.6), we obtain
\[ |p| = |p_1| + |v| > |p_1| + \delta'h_1/10 + |v| \geq |x| + |v| + \delta'h_1/10 \geq |p_3| + \delta'h_1/10, \]
and so \(|p| - |p_3| > \delta'h_1/10\). Thus,
\[
h_1 < 10(\delta')^{-1}(|p| - |p_3|). \tag{12.7}
\]
We have, by (12.7) and Lemma 12.2,
\[
\text{Area}(\Delta_1) \leq 2\text{Area}(\Psi) + 1 \leq 4(2LN(h_1 + h_l) + \delta^{-1}|p|)h_1 + 1
\]
\[
\leq 16LNh_1^2 + 5\delta^{-1}|p|h_1 < 16LN \times 100(\delta')^{-2}(|p| - |p_3|)^2
\]
\[
+ 5\delta^{-1}|p| \times 10(\delta')^{-1}(|p| - |p_3|) < c_6|p||(p| - |p_3|)/2 \tag{12.8}
\]
since \(|p_3| \leq |p|\). By Lemma 12.6, \(\lambda(\Delta) - \lambda(\Delta') \geq -2h_2^2\), and therefore by (12.7),
\[
c_4(\lambda(\Delta) - \lambda(\Delta')) \geq -2c_4h_1^2
\]
\[
\geq -2c_4(10)(\delta')^{-2}(|p| - |p_3|)^2
\]
\[
\geq -c_6|p||(|p| - |p_3|)/2 \tag{12.9}
\]
since \(c_6 > 400(\delta')^{-2}c_4\). Hence, by (12.8) and (12.9),
\[
\text{Area}(\Delta_1) \leq c_6|p||(p| - |p_3|) + c_4(\lambda(\Delta) - \lambda(\Delta'))
\]
\[
\leq c_6\varnothing n - |\partial\Delta'| + c_4(\lambda(\Delta) - \lambda(\Delta')) \]
as \(|p| - |p_3| \leq n - |\partial\Delta'|\). Now the statement follows from Lemma 12.6(b,d).

Let now \(\Psi_{2,l}\) be the part of the crescent \(\Psi\) between \(C_2\) and \(C_l\). By Lemma 12.4, \(\Psi_{2,l}\) is a crescent too. For the crescent \(\Psi_{2,l}\), one can define the analogs of \(p, p_1, p_3, v, v', \Delta_1, \) and \(\Delta'\) introduced earlier for the crescent \(\Psi\). We denote them by \(p(0), p_1(0), p_3(0), v(0), v'(0), \Delta_1(0)\) and \(\Delta'(0)\), respectively.

The substitution of \(\Psi\) by \(\Psi_{2,l}\) in Lemma 12.2, gives us
\[
\text{Area}(\Psi_{2,l}) \leq (h_2 + h_l)(2LN(h_2 + h_l) + \delta^{-1}|p(0)|). \tag{12.10}
\]

**Lemma 12.9.** Assume that \(|p(0)| \leq 2LN\max(h_2, h_l), h_2 < (1 - \frac{1}{20LN})h_3,\) and \(\max(h_2, h_l)\) does not belong to any interval \((T_i, 9T_i)\). Then the following inequality holds:
\[
\text{Area}(\Delta_1(0)) \leq c_4(\mu(\Delta) - \mu(\Delta'(0))) + c_5(\nu_J(\Delta) - \nu_J(\Delta'(0)))
\]
\[
+ c_6|\partial\Delta|(|\partial\Delta'| - |\partial\Delta'(0)|). \tag{12.11}
\]

**Proof.** Assume that \(h_0 \geq (1 - \frac{1}{30LN})h_{2,l}\), where \(h_{2,l} = \max(h_2, h_l)\). Since \(h_i \geq h_0\) for any \(i = 1, \ldots, l\), to complete the proof, it suffices to apply Lemma 12.8 to \(\Delta_1(0)\). Then we assume that \(h_0 < (1 - \frac{1}{30LN})h_{2,l}\).

By Lemma 12.2 and the restriction on \(|p(0)|\),
\[
\text{Area}(\Delta_1(0)) \leq 2\text{Area}(\Psi_{2,l}) + 1 \leq 4(2LN(h_2 + h_l) + \delta^{-1}|p(0)|)h_{2,l} + 1
\]
\[
\leq 16LNh_{2,l}^2 + 10\delta^{-1}(2LN)^2h_{2,l}^2 \leq (\delta')^{-1}h_{2,l}^2 \tag{12.12}
\]
since \((\delta')^{-1} > 20\delta^{-1}L^2N^2\).

Now, we want to estimate \(\nu_J(\Delta) - \nu_J(\Delta'(0))\). For this aid, we observe that the common \(t\)-edge \(f_2\) of the spoke \(C_2\) and \(\partial\Delta\) separates at least \(h_1 - h_2 = m_1\) \(\theta\)-edges placed on \(p\) between \(C_1\) and \(C_2\) and \(m_2\) ones placed between \(C_2\) and \(C_l\), where \(m_2 \geq \max(h_2 - h_0, h_1 - h_0) \geq \frac{1}{30LN}h_{2,l}\). Lemmas 12.5 and 6.1(d) imply that one decreases \(\nu_J(\Delta)\) at least by \(m_1m_2\) when erasing the
black bead on $f_2$ in the $\nu$-necklace on $\partial \Delta$.

Nevertheless, we do such erasing while passing from $\Delta$ to $\Delta'(0)$ since the path $x(0)$ (replacing the path $p_1(0)$ with edge $f_2$) has no $t$- or $t'$-edges and $v'(0)$ is a copy of $v(0)$. (We might erase some other black and white beads). Hence,

$$\nu_J(\Delta) - \nu_J(\Delta'(0)) \geq m_1 m_2 \geq \frac{1}{30N} h_1 \left( \frac{1}{30N} \right) h_{2,1} \geq \frac{1}{(30N)^2} h_{2,1}^2. \quad (12.13)$$

The inequalities (12.13) and (12.12) imply

$$\text{Area}(\Delta(0)) \leq c_5 (\nu_J(\Delta) - \nu_J(\Delta'(0)))/2 \quad (12.14)$$

since $c_5 > 2000 N^2 (\delta')^{-1}$.

By Lemma 12.6(b,c) for the diagrams $\Delta$ and $\Delta'(0)$, we have $\mu(\Delta) - \mu(\Delta'(0)) \geq \lambda(\Delta) - \lambda(\Delta'(0)) \geq -2h_{2,1}^2$, whence by (12.13), $0 \leq c_5 (\nu_J(\Delta) - \nu_J(\Delta'(0)))/2 + c_4 (\mu(\Delta) - \mu(\Delta'(0)))$ since $c_5 > 2000 N^2 c_4$. Adding this inequality with (12.14), we have

$$\text{Area}(\Delta(0)) \leq c_5 (\nu_J(\Delta) - \nu_J(\Delta'(0))) + c_4 (\mu(\Delta) - \mu(\Delta'(0))).$$

This implies inequality (12.11) since the third summand at the right-hand side of (12.11) is positive by Lemma 12.6(a) applied to the diagrams $\Delta$ and $\Delta'(0)$.

The notation of this subsection will also be used in the next section.

13. Almost quadratic upper bound

We denote by $g(n)$ the minimal function such that the height of any $M_4$-accepting trapezium is at most $g(n)$ if the $a$-length of its bottom label does not exceed $n$. (Such upper bounds exist for every $n$ by Properties (xvii) and (xviii).) Since $g(n)$ is non-decreasing, the auxiliary function $f(n) = n (8g(\delta^{-1} n^2)^2 + \delta^{-1} n^3)$ used in this section is also non-decreasing. To begin this section, we need a crude upper bound for areas of diagrams.

**Lemma 13.1.** Let $\Delta$ be a minimal diagram of perimeter at most $n$. Then \(\text{Area}(\Delta) \leq f(n)\).

**Proof.** Step 1: Assume that $\Delta$ has no hubs. Then we can use Lemma 5.6, and the total number of maximal $q$-bands and maximal $\theta$-bands of $\Delta$ is at most $n/2$. Hence, the number of $(q, \theta)$-cells is at most $n^2/16$. Since every maximal $a$-band ends either on the boundary $\partial \Delta$ or on a $(q, \theta)$-cell, the number of such bands is at most $(2 \times n^2/16 + \delta^{-1} n)/2$ by Lemma 5.20(d) and (iii)(a). Each of these $a$-bands crosses at most $n/2 \theta$-bands, and so their total area is at most $n^3/16 + \delta^{-1} n^2/4$. Therefore, $\text{Area}(\Delta) \leq n^3/16 + \delta^{-1} n^2/4 + n^2/16 \leq \delta^{-1} n^3/3$. 
Step 2: In any case, the number of hubs $n_{hub}$ in $\Delta$ is at most $2n/LN$ by Lemma 5.19. To complete the proof of the lemma, it suffices to assume that $n_{hub} \geq 1$ and to prove by induction on $n_{hub}$ that

$$\text{Area}(\Delta) \leq n_{hub}(4LN\delta^{-1}n^2 + 6\delta^{-1}n^3) + \delta^{-1}n^3/3.$$ 

There are a hub $\pi$ and a clove $\Psi = cl(\pi, B, B')$ given by Lemma 5.18. Let $\Lambda$ be the subdiagram of $\Psi$ formed by the $\theta$-bands of $\Psi$ crossing both $t$-bands $B = B_1$ and $B' = B_{L-3}$. Let the remaining part $\Lambda' = \Psi \setminus \Lambda$ be separated from $\Lambda$ by a path $p(\Lambda)$.

It follows from the choice of $\Lambda$ that every maximal $\theta$-band $T$ of $\Lambda'$ starts or ends on $\partial \Delta$. Hence, the number of such $\theta$-bands is at most $n$. In the diagram $\Lambda'$, a $\theta$-band of $\Delta$ and a $q$-band have at most one common $(q, \theta)$-cells by Lemma 5.6. Since the number of maximal $q$-bands of $\Lambda'$ is at most $|p(\Psi)| \leq n$, the number of $(q, \theta)$-cells in $\Lambda'$ is not greater than $n^2$. Since the number of $\theta$-bands of $\Lambda'$ starting on the path $p(\Lambda)$ must not exceed one of these $(q, \theta)$-cells or on $\partial \Delta$, the number of $a$-edges in $p(\Lambda)$ is at most $2n^2 + \delta^{-1}n^3$ by Lemma 5.20(d). Since by Lemma 5.6, a maximal $a$-band intersects a $\theta$-band of the diagram $\Lambda'$ at most once, there are at most $2\delta^{-1}n^3$ $(\theta, a)$-cells in $\Lambda'$. Thus, $\text{Area}(\Lambda') \leq 2\delta^{-1}n^3 + n^2 \leq 2.5\delta^{-1}n^3$.

Since $|p(\Lambda)|_a \leq 2\delta^{-1}n^2$ and $\Lambda$ has $2(L - 4)$ $M_4$-accepting trapezia whose top labels are just copies of one of them (see Remark 5.3 and Lemma 5.10), the number of maximal $\theta$-bands in $\Lambda$ is at most $g((2\delta^{-1}n^2)(2L - 8)^{-1}) \leq g(\delta^{-1}n^2)$. By (x) applied to $2L - 8$ $M_4$-accepting trapezia, the number of cells in any maximal $\theta$-band of $\Lambda$ does not exceed $LN + (2L - 8)g(\delta^{-1}n^2)$. Multiplying this number by the height of $\Lambda$, we obtain

$$\text{Area}(\Lambda) \leq ((8L - 32)g(\delta^{-1}n^2) + LN)g(\delta^{-1}n^2) \leq 2LN\delta^{-1}n^2.$$ 

Therefore, $\text{Area}(\Psi) = \text{Area}(\Lambda) + \text{Area}(\Lambda') \leq 2LN\delta^{-1}n^2 + 3\delta^{-1}n^3 - 1$.

Now we use the surgery from Remark 12.7(1) and have $\text{Area}(\Delta_1) \leq 2\text{Area}(\Psi) + 1 \leq 4LN\delta^{-1}n^2 + 6\delta^{-1}n^3$ and $|\partial \Delta'| \leq |\partial \Delta| - 1$. Since the number of hubs of $\Delta'$ is strictly less than this number for $\Delta$, we have by the inductive hypothesis, $\text{Area}(\Delta') \leq (n_{hub} - 1)(4LN\delta^{-1}n^2 + 6\delta^{-1}n^3) + (\delta')^{-1}n^3/3$, and therefore, by Lemma 5.2, as required,

$$\text{Area}(\Delta) \leq \text{Area}(\Delta_1) + \text{Area}(\Delta') \leq n_{hub}(4LN\delta^{-1}n^2 + 6\delta^{-1}n^3) + (\delta')^{-1}n^3/3.$$ 

The following lemma summarizes our efforts and ensures the main result.

**Lemma 13.2.** Let the perimeter $n = |\partial \Delta|$ of a minimal diagram $\Delta$ satisfy inequality $f(n) \leq \tau_i$ for some $i$, where $f$ is the function from Lemma 13.1. Then Area($\Delta$) $\leq F(\Delta)$ for $F(\Delta) = F'(\delta \Delta) = c_4\mu(\Delta) + c_5\nu_J(\Delta) + c_6n^2 + c_7nQf(T_{i-1})$, where $nQ$ is the number of $q$-edges in $\partial \Delta$.

**Proof.** (1) If $\partial \Delta$ has no $q$-edges, then $\Delta$ has no $q$-edges by Lemmas 5.18 and 5.6. Then Area($\Delta$) $< \delta^{-1}n^2 \leq c_6n^2$ because (1) a maximal $\theta$-band and a maximal $a$-band have at least one common $(\theta, a)$-cell, (2) $\Delta$ has no $\theta$- and $a$-annuli and so every maximal $\theta$- and $a$-band starts and ends on $\partial \Delta$ by Lemma 5.6, and (3) $|\partial \Delta|_a \leq |\partial \Delta| \leq \delta^{-1}n$ by Lemma 5.20(a,d).
Thus, we suppose $n_Q \geq 1$. By Lemma 13.1, $\text{Area}(\Delta) \leq f(T_{i-1})$ if $n \leq T_{i-1}$. Then we may suppose $n > T_{i-1}$, $n \geq 1$ and prove the lemma by contradiction assuming further that $\Delta$ is a counter-example with minimal perimeter $n$.

(2) If $\Delta$ is a union of a subdiagram $\Delta'$ and a rim $\theta$-band $T$ of base width at most $2LN$, then there are at most $4LN$ $\alpha$-edges on the boundaries of $(\theta,q)$-cells of $T$ by (iii)(a), and so $|\text{top}(T)| - |\text{bot}(T)| \leq 4LN\delta$ by Lemma 5.20(a). Therefore, $|\partial\Delta'|_\theta \leq 4LN\delta + |\partial\Delta|_\theta$ but $|\partial\Delta'|_\theta = |\partial\Delta|_\theta - 2$. Hence, $|\partial\Delta'| \leq |\partial\Delta| - 1$ by Lemma 5.20(b) since $5LN < \delta^{-1}$.

If $T$ has $m$ $q$-cells, then $n \geq m$, and so, by Lemma 5.20(d), the number of cells in $T$ is at most $\delta^{-1}n + 2m \leq 2\delta^{-1}n$ by Lemma 5.20(d). Also we have $\mu(\Delta') \leq \mu(\Delta)$ and $\nu_j(\Delta') \leq \nu_j(\Delta)$ by Lemma 6.1(b), and $n'_Q \leq n_Q$ by Lemma 5.6. Therefore, by the inductive hypothesis for $\Delta'$,

$$\text{Area}(\Delta) \leq F(\Delta') + 2\delta^{-1}n \leq F(\Delta) - c_6(2n - 1) + 2\delta^{-1}n \leq F(\Delta)$$

since $c_6 > 2\delta^{-1}$; a contradiction. Therefore, $\Delta$ has no rim $\theta$-bands of base width at most $2LN$.

(3) Assume that $\Delta$ has a subcomb $\Delta$ of base width $15N$. Hence, we can apply Lemma 11.8 to the comb $\bar{\Delta}$ and consider two arising cases.

(a) $\Delta$ admits a long quasicomb $\Gamma$ such that

$$\text{Area}(\Gamma) \leq c_3[\Gamma] + c_4\mu^c(\Gamma) + c_5(\nu_j(\Delta) - \nu_j^c(\Delta \setminus \Gamma)).$$

We multiply the right-hand side by the number $c_3c_4^{-1} > 1$ and then replace the two coefficients $c_3c_4c_2^{-1}$ by bigger coefficients $c_0$ and $c_5$, respectively; this is legal since $\nu_j(\Delta) \geq \nu_j(\Delta \setminus \Gamma) \geq 0$ by Lemma 7.3(e) and Remark 9.5, and $[\Gamma] \geq 0$ since $\Gamma$ is a long subcomb. Hence,

$$\text{Area}(\Gamma) \leq c_6[\Gamma] + c_4\mu^c(\Gamma) + c_5(\nu_j(\Delta) - \nu_j^c(\Delta \setminus \Gamma)). \quad (13.1)$$

Let $y = y^\Gamma$ and $z = z^\Gamma$. Since $\Gamma$ is long, the complement diagram $\Delta' = \Delta \setminus \Gamma$ satisfies $|\partial\Delta'| \leq |\partial\Delta| - |\Gamma| < |\partial\Delta|$. By Lemma 7.3(a,e) and Remark 9.5, we also have $\mu(\Delta) \geq \mu(\Delta') + \mu^c(\Gamma)$ and $\nu_j(\Delta) \geq \nu_j(\Delta \setminus \Gamma) \leq \nu_j(\Delta) - \nu_j(\Delta')$. Since $\text{Area}(\Delta') \leq F(\Delta')$ by the inductive hypothesis, it follows from (13.1) that

$$\text{Area}(\Delta) \leq \text{Area}(\Delta') + c_6[\Gamma] + c_4\mu^c(\Gamma) + c_5(\nu_j(\Delta) - \nu_j(\Delta'))$$

$$\leq c_6(n - (|z| - |y|))^2 + c_4\mu(\Delta') + c_5(\nu(\Delta') + c_T n_Q f(T_{i-1}) + c_6[\Gamma])$$

$$+ c_4\mu^c(\Gamma) + c_5(\nu_j(\Delta) - \nu_j(\Delta'))$$

$$\leq c_6n^2 - c_6n(|z| - |y|) + c_6n(|z| - |y|) + c_4\mu(\Delta) + c_5\nu(\Delta)$$

$$+ c_T n_Q f(T_{i-1}) \leq F(\Delta)$$

since $|z| \leq |y| \leq n, c_3 < c_6$, and $c_5 < c_5$, and $c_2 < c_4$. Therefore, $\Delta$ is not a counter-example, a contradiction.

(b) $\Delta$ has a subcomb whose handle $C$ is a $t$- or $t'$-band with length $l$ satisfying $T_j \leq l < 200T_j$ for some $j$, and $C$ separates a subcomb $\Gamma$ of base width at most $14N$ from $\Delta$. By Remark 7.1 applied to $\Gamma$, we have $n > l$. Now since $T_j \leq l < n < f(n) \leq T_i$, we have $j \leq i - 1$. Again let $\Delta'$ be the diagram $\Delta \setminus \Gamma$. Let $y = y^\Gamma$ and $z = z^\Gamma$. Then we have

$$|\partial\Delta'| \leq |\partial\Delta| - (|z| - |y|) \leq |\partial\Delta| - 2 \quad (13.2)$$

since the handle $C$ of $\Gamma$ is passive, and so $\Gamma$ is a long subcomb. Since $\Gamma$ has a $q$-band $C$, we immediately obtain

$$n_Q - n'_Q \geq 2 \quad (13.3)$$
for the numbers of \( q \)-edges in \( \partial \Delta \) and in \( \partial \Delta' \), and
\[
\nu_f(\Delta) - \nu_f(\Delta') \geq 0 \tag{13.4}
\]
by Lemmas 7.3(b) and 7.2(b).

By Remark 12.7(2),
\[
\mu(\Delta') \leq \mu(\Delta) + l^2/2 < \mu(\Delta) + (200T_{i-1})^2/2 \leq \mu(\Delta) + f(T_{i-1}) \tag{13.5}
\]
since \((\delta')^{-1} > 2 \times 10^4\). Now, from the definition of the function \( F \) and inequalities (13.2), (13.4), and (13.3), we obtain
\[
F(\Delta) \geq F(\Delta') + c_6(n^2 - (n - (|z| - |y|))^2) + c_5(\nu(\Delta) - \nu(\Delta'))
\]
\[
+ c_4(\mu(\Delta) - \mu(\Delta')) + c_7(n_Q - n'_Q) f(T_{i-1})
\]
\[
\geq F(\Delta') + c_6(|z| - |y|) + c_4(\mu(\Delta) - \mu(\Delta')) + 2c_7 f(T_{i-1})
\]
which together with inequality (13.5) implies
\[
F(\Delta) \geq F(\Delta') + c_6(|z| - |y|) - c_4 f(T_{i-1}) + 2c_7 f(T_{i-1}) \tag{13.6}
\]
because \( \text{Area}(\Delta') \leq F(\Delta') \) by (13.2) and the minimality of the counter-example \( \Delta \).

On the other hand, by Lemmas 7.13, 5.20(d), and inequality \(|y| = t < 200T_{i-1} \), we obtain
\[
\text{Area}(\Gamma) \leq 60N|y|^2 + 2\delta^{-1}|z||y| = (60N + 2\delta^{-1})|y|^2 + 2\delta^{-1}|y||(|z| - |y|)
\]
\[
\leq (60N + 2\delta^{-1})|y|^2 + 2\delta^{-1}|\Gamma| \leq 200^2 \times (60N + 2\delta^{-1})T_{i-1}^2 + 2\delta^{-1}|\Gamma|
\]
\[
\leq c_6|\Gamma| + c_7 f(T_{i-1}) \tag{13.7}
\]
because \( c_6 > 2\delta^{-1}, c_7 \geq 10^6, \) and \( f(T_{i-1}) \geq \delta^{-1}T_{i-1}^2 \). Now inequalities (13.6, 13.7) yield
\[
\text{Area}(\Delta) = \text{Area}(\Delta') + \text{Area}(\Gamma) \leq F(\Delta) - c_6|\Gamma| - c_7 f(T_{i-1}) + \text{Area}(\Gamma) \leq F(\Delta),
\]
a contradiction. Hence \( \Delta \) has no subcombs of base width \( 15N \).

(4) Assume that \( \Delta \) has a one-step subcomb \( \Gamma \) whose handle is \( t \) or \( t' \)-band. By (3), we may assume that its base width is less than \( 15N \). Then we can use Lemma 9.2(2) and come to a contradiction as in (3(a)) above.

By (2)–(4), the diagram \( \Delta \) is solid. By Lemma 12.1(b), it has a hub.

(5) Suppose we have a hub \( \pi \) and a crescent \( \Psi = \text{cl}(\pi, C_1, \ldots, C_l) \) given by Lemma 12.4. Assume that \( |p(\Psi)| > 2LN \max_{i=1}^l h_i, \) where \( h_i \) is the length of \( C_i \). Then by Lemma 12.3, \( |\partial \Psi| < |\partial \Delta| \) for the subdiagram \( \partial \Psi' = \Delta \setminus (\pi \cup \Psi) \). Besides, it follows from the definition of crescent that \( n'_Q < n_Q \), where \( n'_Q \) is the number of \( q \)-edges in \( \partial \Psi' \). Since by the inductive hypothesis \( \text{Area}(\Psi') \leq F(\Psi') \), we obtain by Lemma 12.3 that
\[
\text{Area}(\Delta) \leq F(\Psi') + c_4(\mu(\Delta) - \mu(\Psi')) + c_5(\nu_f(\Delta) - \nu_f(\Psi')) + c_6(n - |\partial \Psi'|) \leq F(\Delta),
\]
and so \( \Delta \) is not a counter-example.

(6) Now we assume that \( \Delta \) has a crescent \( \Psi \) and a hub as in (5), but now \( |p(\Psi)| \leq 2LN \max_{i=1}^l h_i \). If the conditions of Lemma 12.8 are satisfied, then that lemma leads to a contradiction as in case (5) above since \( |\partial \Delta'| < |\partial \Delta| \) by Lemma 12.6(a), \( n'_Q < n_Q \), and the diagram \( \Delta_1 \cup \Delta' \) (with notation of Section 12) has the same boundary label as \( \Delta \) (see Lemma 5.2). Similarly, we obtain a contradiction under assumptions of Lemma 12.9, if we cut off the subdiagram \( \Delta_1(0) \) with the spokes \( C_2, \ldots, C_l \) since these spokes also bound a crescent \( \Psi_{2,l} \) by Lemma 12.4.

(7) Thus, it remains to assume that the maximal \( h_i \) for the crescent, say \( h_1 \) (since the case with \( \Psi_{2,l} \) from Lemma 12.9 is absolutely similar) satisfies inequalities \( T_j \leq h_1 < 9T_j \) for some \( j \) and \( |p| = |p(\Psi)| \leq 2LN h_1 \). Note that \( j \leq i - 1 \) because, by Lemma 13.1 and the assumption \( f(n) \leq T_i \), we have \( T_j \leq h_1 < \text{Area}(\Delta) \leq f(n) \leq T_i \).
Now we will use the notation of Lemma 12.6. By Lemma 12.2, we have
\[
\text{Area}(\Delta_1) \leq 2\text{Area}(\Psi) + 1 \leq 4(2LN(2h_1) + \delta^{-1}(2LNh_1))h_1 + 1
\leq (16LN + 8\delta^{-1}LN)h_1^2 + 1 < 9\delta^{-1}LNh_1^2 < 800\delta^{-1}LNT_{i-1}^2,
\]
(13.8)

By Lemma 12.6 (c),
\[
\mu(\Delta') \leq \mu(\Delta) + 2h_1^2 < \mu(\Delta) + 200T_{i-1}^2 \leq \mu(\Delta) + f(T_{i-1}),
\]
and so by Lemma 12.6(b) and the definition of \(\mu(\ast)\),
\[
\mu(\Delta) - \mu(\Delta') = c_0(\kappa(\Delta) - \kappa(\Delta')) + (\lambda(\Delta) - \lambda(\Delta')) \geq -f(T_{i-1}).
\]
(13.9)

Now using Lemma 12.6(a,d), inequalities \(n_Q \geq n_Q' + 2\) and (13.9) we have
\[
F(\Delta) - F(\Delta') > c_6 \times 0 - c_4f(T_{i-1}) + c_5 \times 0 + 2c_7f(T_{i-1}) \geq c_7f(T_{i-1}).
\]
This inequality, (13.8), the inductive hypothesis (valid by Lemma 12.6(a)), and Lemma 5.2 imply
\[
\text{Area}(\Delta) \leq \text{Area}(\Delta') + \text{Area}(\Delta_1) < F(\Delta') + 800LNf(T_{i-1})
\leq F(\Delta') + c_7f(T_{i-1}) < F(\Delta),
\]
and so \(\Delta\) is not a counter-example in this case too.

The proof is complete. \(\square\)

Now we go back to the combinatorial length \(\| \cdot \|\) and make use of the obvious quadratic upper bounds for the mixtures.

**Lemma 13.3.** There is a constant \(C\) such that for every \(i = 2, 3, \ldots\) and arbitrary minimal diagram \(\Delta\) with \(\|\partial\Delta\| = r\), we have \(\text{Area}(\Delta) \leq C(r^2 + rT_{i-1}g(CT_{i-1}^2)^2 + T_{i-1}^4)\) provided \(Cr(gCr^2)^2 + Cr^3) < T_i\).

**Proof.** By Lemma 5.20, \(\|\partial\Delta\| = n \leq r \leq \delta^{-1}n\). Recall also that by Lemma 6.1(a) and the definition of \(\mu\)- and \(\nu\)-mixtures, \(\mu(\Delta) \leq (c_0 + 1)r^2\), \(\nu_\ast(\Delta) \leq Jr^2\), and also \(n_Q \leq n\). Now the statement of the lemma follows with a constant \(C \geq 2\delta^{-1}c_7\) from Lemma 13.2 and from the definitions of the function \(f\). \(\square\)

Finally, we apply Theorem A.1 converted into Property (xviii) of trapezia.

**Lemma 13.4.** The Dehn functions of the groups \(G\) and \(M\) are almost quadratic.

**Proof.** We consider only the Dehn function \(d(r)\) of the group \(G\) since a simpler proof works for \(M\). (One considers only diagrams having no hubs in the later case.)

Assume that an integer \(m\) satisfies the hypothesis of (xvii) and \(m\) is large enough, say \(m > \max(T_i, C^{20}(\log m)^{40})\), where \(C\) is provided by Lemma 13.3. There is a maximal \(i\) such that \(T_{i-1} \leq m\). Since \(T_{i-1}\) is the height of a standard trapezium with some bottom \(W\) and any rule corresponding to this trapezium can decrease the length of the input sector at most by 1, we have \(|W|_a \leq T_{i-1} \leq m\), and so by (xvii) and the choice of \(i\), we obtain
\[
\exp T_{i-1} < m < T_i.
\]
(13.10)
It follows from (13.10) and the choice of \( m \) that
\[
CT_{i-1}^2 < m^{1/20}.
\]
(13.11)
Now, on the one hand, inequality \( n < m/7 \) implies
\[
g(n) \leq \frac{4}{7}m + 3 \log m < T_i
\]
by (xvii). On the other hand, by Property (xix), any value \( g(n) \) of the function \( g \) either belongs to some interval \( (T_j, 9T_j) \) or \( g(n) \leq 6n \). By (13.12), we have \( j < i \) in the former case. Therefore, if \( n < m/7 \), then in any case
\[
g(n) \leq \max(9T_{i-1}, 6n) \leq \max(9 \log m, 6n).
\]
(13.13)
Hence, there is a constant \( D \) such that for every integer \( r \) such that \( r > 9 \log m \) and \( Dr^5 \leq m \), we have
\[
Cr(g(Cr)^2 + Cr^3) < Dr^5 \leq m < T_i.
\]
(13.14)
Now by inequalities (13.14), (13.10), (13.13), (13.11), and by Lemma 13.3, we have
\[
d(r) \leq C(r^2 + rT_{i-1}g(CT_{i-1}^2)^2 + T_{i-1}^4) \leq C(r^2 + r(\log m)g(CT_{i-1}^2)^2 + T_{i-1}^4)
\]
\[
\leq C(r^2 + r(\log m)(\max(9 \log m, 6m^{1/20}))^2 + (\log m)^4) \leq 2Cr^2
\]
if \( Dr^5 \leq m \), and \( r > m^{1/6} \).

Since the set of integers \( m \) satisfying the hypothesis of (xvii), is infinite by (xviii), we can find for almost every such \( m \), an integer \( r \) satisfying inequalities \( Dr^5 \leq m < r^6 \), and so the inequality \( d(r) \leq 2Cr^2 \) holds on an infinite set of integers \( r \).

End of proofs of Theorems 1.1 and 1.3. Using the notation of Lemma 4.45, we consider a word \( V \equiv W(M) \) for an arbitrary admissible input word \( W \) of the machine \( M_4 \). Assume that \( V = 1 \) in the group \( G \). Then there is a minimal diagram \( \Delta \) whose boundary path is labeled by \( V \). Since every state letter from the vector of start states of \( M \) occurs in \( V \) exactly once, every maximal \( q \)-band of \( \Delta \) must end on a hub, and \( \Delta \) has \( m \geq 1 \) hubs. On the other hand, \( m \leq 1 \) by Lemma 5.19, since \( |V|_q = LN \) by the definition of the standard base for the machine \( M \). Thus, \( \Delta \) has exactly one hub \( \Pi \), and so every maximal \( q \)-band of \( \Delta \) connects the boundaries of \( \Pi \) and \( \Delta \).

Since \( V \) has no \( \theta \)-edges, by Lemma 5.6, every non-hub cell of \( \Delta \) belongs to a \( \theta \)-annulus surrounding the hub \( \Pi \). (The set of these annuli is not empty since \( V \) has no state letters of the hub relation.) Hence, one can remove \( \Pi \), make a radial cut, and construct a trapezium with base (4.5). By Lemma 5.10(1), the computation of \( M \) corresponding to this trapezium accepts the word \( V \), and therefore \( V \in X_5 \) by Lemma 4.45.

Conversely, assume that \( V \equiv W(M) \in X_5 \). Then, by Lemma 5.10(2), there is a trapezium with base (4.5) corresponding to an accepting computation \( W(M) \to \cdots \) of \( M \). Now one may identify the left-most and the right-most maximal \( t \)-bands of this trapezium and paste up the hole of the obtained annular diagram by a hub. Hence, \( V \) is a boundary label of a disc van Kampen diagram, and therefore \( V = 1 \) in \( G \).

The obtained criterion shows that the word problem is undecidable for \( G \) since the set \( X_5 \) is not recursive by Lemma 4.45. By Lemma 13.4, the proof of Theorem 1.1 is complete.
Relations (5.1) of the group \( M \) define the structure of a (multiple) HNN-extension on the group \( M \) whose base is the free subgroup generated by all \( a \)- and \( q \)-letters, and for every rule, one has a stable \( \theta \)-letter. (See the presentation of every \( S \)-machine as an HNN-extension in [15].) The statements 2 and 3 of Theorem 1.3 hold for \( M \) by Lemma 13.4 and by Step 1 of the proof of Lemma 13.1. Finally, a word \( V \equiv W(M) \) is conjugate to the hub in \( M \) if and only if \( V \in X_5 \). (The proof is similar to the criterion obtained above for the equality \( W(M) = 1 \) in \( G \), but now one considers annular diagrams over \( M \) instead of disc diagrams over \( G \).) Now the statement 1 of Theorem 1.3 follows from Lemma 4.45, and the proof is complete.

**Theorem 13.5.** There exists a finitely presented group \( G \) with almost quadratic Dehn function \( d(n) \) such that \( d(n) \geq \exp n \) for infinitely many \( n \)-s, and \( d(n) \) is bounded from above on the entire \( \mathbb{N} \) by an exponential function.

**Proof.** We will make a few alternations in the proof of Theorem 1.1.

Given a word \( a^n \), it is easy to check in linear time whether \( n = 2^m \) for some natural \( m \) or not and to compute \( m = \log_2 n \) if \( m \in \mathbb{N} \). Therefore, there is a deterministic Turing machine \( M_0 \) with linear time complexity which accepts a word \( a^n \) if and only if \( n \) belongs to the sequence \( n_1 = 1, n_i = 2^{2^{n_{i-1}}} \) for \( i > 1 \). Clearly, almost every \( n_i \) is an \( h_{\alpha_i} \)-good number for any function \( h_{\alpha_i}(x) = 2^{2^{x^{\alpha_i}}} \), and we can use this property instead of Theorem A.1.

Starting with \( M_0 \), we construct the machines \( M_1, \ldots, M_4, M \) and define the group \( G \) as in the paper. Then we obtain, as in Theorem 1.1, that \( d(n) \) is almost quadratic (since the non-recursiveness from Theorem A.1 has never been used for this goal).

For some positive constants \( c' \) and \( c'' \), Lemmas 4.16(b) and 4.25(b) give the estimates \( \exp(c'n_i) < T_i < \exp(c''n_i) \) for the time of acceptance \( T_i \) of the word \( a^{n_i} \) by the machine \( M_3 \). As in the above ‘End of proofs’, it follows that the length of the corresponding to \( a^{n_i} \) accepted input word \( V_i \) of the machine \( M \) is \( O(n_i) \), while the area is at least \( T_i > \exp(c'n_i) \). Thus, \( d(n) \) is bounded from below on the infinite sequence of \( n_i \)-s by an exponent. It remains to obtain an exponential upper bound \( d(n) < \exp(Cn) \) on the entire \( \mathbb{N} \). (We do not need any mixtures for this goal.)

Assume that \( C \) is large enough, and a minimal diagram \( \Delta \) over \( G \) has area at least \( \exp(Cn) \).

Then \( \Delta \) has no rim-\( \theta \)-bands \( T \) of base width at most \( 2LN \) because \( |\partial \Delta'| < n \) for the subdiagram \( \Delta' = \Delta \setminus T \) in Case (2) of the proof of Lemma 13.2. Similarly, \( \Delta \) has no long subcombs (or subquasicomsbs) since Lemma 13.1 (Step 1 of the proof) provides us with a cubic upper bound of the area of any subcomb (as function of \( n \)). Therefore, the diagram \( \Delta \) is solid, and therefore Lemma 12.6(a) reduces our task to diagrams having at most one hub. Indeed, by Lemma 5.19, the number of hubs in \( \Delta \) does not exceed \( n \), and the functions \( \exp(Cn) \) and \( n \exp(Cn) \) are equivalent.

We may assume that \( \Delta \) has exactly one hub since otherwise its area is bounded by a cubic function of the perimeter. Now, by Lemma 12.2 applied to the whole \( \Delta \), we conclude that every maximal \( \theta \)-band of \( \Delta \) is an annulus, and so, as at the ‘End of proofs’ above, the boundary label of \( \Delta \) is of the form \( V \equiv W(M) \) for some admissible input word of the machine \( M_4 \). Therefore, it suffices (by Lemma 5.10) to find an exponential upper bound for the accepting computations of \( M_4 \) with respect to the length \( \|W\| \) of an input admissible word \( W \). Such an upper bound (even a linear bound) is given by Lemma 4.38 if the length of the reduced computation of \( W \) does not belong to any interval \((T_i, 9T_i)\). The argument of that lemma works in other cases if the computation does not contain the standard computation of length \( T_i \) (that is, the computation of \( n_i \) in our situation). However, the proof of Lemma 4.38 also shows that \( \|W\| \geq n_i \) in the remaining cases. Therefore, the length of the computation has the exponential upper bound \( 9T_i \leq 9\exp(c''\|W\|) \), and the proof is complete since \( C \gg c'' \).
Remark 13.6. One can replace the exponential function by a multieponential one or by many other functions with at least exponential growth in the formulation and in the proof of Theorem 13.5.

Appendix. A very sparse immune set
by M.V. Sapir†

Let \( X \) be a recursively enumerable (r.e.) language in the binary alphabet recognized by a Turing machine \( M \). If \( x \in X \), then the time of \( x \) (denoted \( \text{time}(x) \) or \( \text{time}_M(x) \)) is, by definition, the minimal time of an accepting computation of \( M \) with input \( x \). For any increasing function \( h : \mathbb{N} \to \mathbb{N} \), a real number \( m \) is called \( h \)-good for \( M \) if for every \( w \in X \), \( \|w\| < m \) implies \( h(\text{time}(w)) < m \).

For every number \( n \), the number of digits in \( n \) is denoted by \( \|n\|_2 \). This number is roughly \( \log_2 n \). Since we are not using any other logarithms in the appendix, we shall omit 2 in \( \log_2 \).

Similarly, we shall write \( \exp x \) for \( 2^x \).

The proof of the following theorem uses an idea communicated to the author by S.Yu. Podzorov. For every \( \alpha > 0 \), let \( h_\alpha(n) = \exp \exp(\alpha n) \).

Theorem A.1. There exists a Turing machine \( M_0 \) recognizing a r.e. non-recursive set \( X \) such that the set of all \( h_\alpha \)-good numbers for \( M_0 \) is infinite for all \( \alpha > 0 \).

Proof. We use a recursive enumeration of all Turing machines from [9]. By Matiyasevich’s solution of the 10th Hilbert problem [10], there exists a polynomial \( F(a, b, x_1, \ldots, x_s) \) with integer coefficients such that \( a \) is recognized by the Turing machine number \( b \) if and only if \( F(a, b, x_1, \ldots, x_s) = 0 \) for some natural numbers \( x_1, \ldots, x_s \). We are going to use Gödel numbering of \( s \)-tuples of natural numbers. For every natural \( m \), let \( g(m) \) be the \( s \)-tuple having Gödel number \( m \). Note that all coordinates of this tuple do not exceed \( m \) and the time to compute \( g(m) \) is linear in \( \|m\|_2 \).

Note also that if \( \|a\|_2, \|b\|_2, \|x\|_2 \leq n \) \((i = 1, \ldots, s)\), then the time needed to compute \( F(a, b, x_1, \ldots, x_s) \) is bounded by a polynomial in \( n \) depending only on \( F \). Also, the time to compute binary value of the exponent \( \exp n \) (given \( n \) written in binary) is linear in \( n \) (and exponential in \( \|n\|_2 \)).

The algorithm of enumerating elements in \( X \) involves auxiliary formulas for functions \( f_m : \{1, 2, \ldots\} \to \{1, 2, \ldots\} \), and two sequences of numbers \( b(m), x(m), m = 0, 1, 2, \ldots \).

Step 0: Set \( f_0(i) = i \) (that is by definition \( f_0 \) is the identity function), \( b(0) = 0 \), \( x(0) = 0 \), \( X = \emptyset \).

Step \( m \geq 1 \): Search for the minimal \( i = i(m) \leq m \) such that for some \( m' \leq m \), \( F(f_{m-1}(i), i, g(m')) = 0 \) and \( f_{m-1}(i) \) is not already in \( X \). If such an \( i \) exists, add \( f_{m-1}(i) \) to \( X \), compute the new numbers \( b(m) = \exp \exp \exp \exp (m + \|b(m-1)\|_2 + x(m-1)) \) (four exponents), \( x(m) = \max(x(m-1), f_{m-1}(i)) \), and define the function \( f_m \) by adding in the definition of \( f_m \) that \( f_m(j) = j + b(m) \) for every \( j > i \). (Note that \( f_m(j) = f_{m-1}(j) \) for every \( j \leq i \).) In that case we say that the step \( m \) was successful, and \( i \) is responsible for counting \( f_{m-1}(i) \) into \( X \). Otherwise (if either \( i(m) \) does not exist or \( f_{m-1}(i) \) is already in \( X \)), let \( f_m = f_{m-1}, b(m) = b(m-1), x(m) = x(m-1) \). Then go to the next step. Note that

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for every \( i, m \) we have \( f_m(i) \leq i + b(m) \). Therefore, if step \( m \) is successful, then we have \( x(m) \leq x(m - 1) + m + b(m - 1) \). Hence,

\[
x(m) < b(m). \tag{A.1}
\]

Inequality (A.1) holds also for unsuccessful \( m \) if \( m \) is larger than the number of the first successful step by induction, because in that case \( x(m) = x(m - 1), b(m) = b(m - 1) \).

We claim that every number \( i \geq 1 \) is responsible for at most \( 2^i \) members of \( X \). Indeed, every \( i \) can be responsible only for numbers of the form \( f_m(i) \). The value \( f_m(i) \) can differ from \( f_{m-1}(i) \) only if some number \( i' < i \) is responsible for counting some number \( f_{m-1}(i') \) into \( X \) at some step \( m' > m \geq i \). Therefore, we have \( f_1(1) = f_2(1) = \cdots \), so \( 1 \) can be responsible only for at most one number in \( X \). This implies that \( 2 \) can be responsible for at most two numbers only: the value \( f_0(2) \) is 2, and the value \( f_m(2) \) can differ from \( f_{m-1}(2) \) only if \( 1 \) is responsible for some number in \( X \). Similarly, the value \( f_m(i + 1) \) can differ from \( f_{m-1}(i + 1) \) only when a number \( j \leq i \) becomes responsible for counting a number into \( X \). By induction it can happen at most \( 1 + 2 + 2^2 + \cdots + 2^i = 2^{i+1} - 1 \) times. Therefore, \( i + 1 \) can be responsible for at most \( 2^{i+1} \) numbers in \( X \) as claimed.

Let us prove now that the set \( X \) is what we need. It is clear that \( X \) is r.e.: the machine \( M \) enumerating this set is described in the definition of \( X \). (Recall, that a Turing machine enumerating a set of words \( X \) in a finite alphabet differs from a Turing machine recognizing it: it does not have the input sector and the accept configuration. It starts working with all tapes empty, and writes words from \( X \) in the first tape one by one, separated by a special symbol. After a new word is written in tape 1 (that is, when the machine \textit{counts a new word into} \( X \)), the machine puts the separating symbol next to that word and continues working. If \( X \) is infinite, the machine works infinitely long. For every \( u \in X \), we can talk about the \textit{time to count it into} \( X \), that is, the shortest length of the computation after which the word first appears in the first tape.)

Let us prove that \( X \) is not recursive. Suppose the contrary (that \( X \) is recursive). Then its complement is r.e. Therefore, there exists a natural number \( b \) such that

\[
(*) \quad F(a, b, x_1, \ldots, x_s) = 0 \text{ for some } x_1, \ldots, x_s \text{ if and only if } a \text{ is not in } X.
\]

Let \( b \) be the number from (\( * \)). There exists \( m \geq 1 \) which is bigger than the number of the first successful step, and such that for every \( m' \geq m \) either \( i(m') > b \) or the step number \( m' \) is not successful (this follows from the fact that each \( i \) is responsible for finitely many members of \( X \) only). Let \( m \) be one of the numbers in this property.

By definition,

\[
f_{m'}(j) = f_m(j) \tag{A.2}
\]

for every \( m' > m, j \leq b \).

Claim. No \( j \neq b \) can be responsible for counting \( r = f_{m-1}(b) \) into \( X \).

Suppose that \( j < b \) is responsible for counting \( r \) into \( X \). That cannot happen at step \( m' > m \) because \( i(m') > b \) since \( r = f_{m-1}(j) \leq f_{m'-1}(j) < f_{m'-1}(i(m')) \) by definition of \( m \). If that happens at step \( m \), then \( f_{m-1}(j) = r = f_{m-1}(b) \) which is impossible since \( f_{m-1} \) is strictly increasing. If that happens at step number \( m' \leq m \), then, since \( j < b, f_{m'}(b) \geq b + b(m') \geq b(m') \), and we would have

\[
f_{m-1}(b) \geq f_{m'}(b) \geq b(m') > x(m') \geq r
\]

by (A.1) and because \( x(m') \) is the maximum of all numbers counted into \( X \) at steps at most \( m' \), including \( r \), a contradiction.

Suppose that \( j > b \) is responsible for counting \( r \) into \( X \) at some step \( m' \). Suppose that \( m' < m \). We have \( f_{m'-1}(j) = r \). Since \( j \neq b, f_{m'-1}(b) \neq r \). Therefore, \( f_k(b) \) has changed at some step \( k \) such that \( m' \leq k \leq m - 1 \). Hence, there exists a successful step number \( k, m' \leq k \leq m - 1 \).
such that \( i(k) < b. \) But then
\[
f_{m-1}(b) > f_k(b) > b(k) > (m') > r,
\]
a contradiction.

It remains to consider the case when \( j > b, m' \geq m. \) But in that case (since \( f_{m'-1} \) is strictly increasing)
\[
f_{m'-1}(j) > f_{m'-1}(b) \geq f_{m-1}(b) = r,
\]
a contradiction. This completes the proof of our claim.

Now if \( r \) is in \( X \), then for some \( m' \), \( F(r, b, g(m')) = 0 \) (since by the Claim only \( b \) can be responsible for counting \( r \) into \( X \)). But this would mean, by the choice of \( b \) (see (*)), that \( r \) is not in \( X \), a contradiction. On the other hand, if \( r \) is not in \( X \), then \( F(r, b, g(j)) = 0 \) for some \( j \), therefore at some step \( m' \), \( b \) would be responsible for counting \( r \) into \( X \), so \( r \in X \), a contradiction. This shows that \( X \) is not recursive. In particular, \( X \) is infinite.

Note that there exists a deterministic Turing machine \( M_0 \) which recognizes \( X \) and such that for every \( m \in X \), the time to recognize it by \( M_0 \) is linearly bounded in terms of the time to count it into \( X \) by \( M \). Indeed let us add the input tape to the tapes of \( M \). The machine \( M_0 \) will execute \( M \) on its tapes. Every time there is a new word counted into \( X \), the machine \( M_0 \) checks whether this word coincides with the input word. After the match is found, \( M_0 \) erases all tapes and stops.

Now let us determine the \( h_\alpha \)-good numbers of the machine \( M_0 \). We say that a number \( n \) is appropriate if \( i(n) \) exists, \( f_{n-1}(i(n)) \) is counted into \( X \) at step \( n \) and none of \( i(n') \) with \( n' > n \) is smaller than \( i(n) \). Clearly, the set of appropriate numbers is infinite (since every number is responsible only for finitely many members of \( X \), see above). Let \( B \) be the set of numbers \([\log b(n)] \) for appropriate \( n \). Let us show that almost all numbers in \( B \) are \( h_\alpha \)-good.

Indeed, let us estimate the time of a number \( r = f_{n-1}(i(n)) \) counted into \( X \) at an appropriate step number \( n \) by \( M_0 \). The total number of evaluations of \( F \) needed for this is at most \( n^3 \) (at most \( n \) steps, at most \( n \times n \) evaluations of \( F(f_{n-1}(i), i, g(t)) \) at each step where \( 1 \leq i \leq n, 1 \leq t \leq n \)). We can estimate the time of each evaluation of \( F \) as a polynomial in \( \|n\|_2 + \|b(n-1)\|_2 \).

In addition to computing values of \( F \), we also have to compute the numbers \( b(n') \) and the formulas for \( f_{n'} \) (at most \( n \) times). The time of computing \( b(n') \) and \( f_{n'} \) does not exceed the time of computing \( b(n-1) \) and \( f_{n-1} \). And those times can be bounded by a polynomial in \( \|b(n-1)\|_2 \).

Recall that the time of recognizing \( r \) by \( M_0 \) is bounded by a constant times the time of counting \( r \) into \( X \) by \( M \). Thus, the total time of accepting \( r \) by the machine \( M_0 \) is bounded by \( cn^\alpha \|b(n-1)\|_2 + c \) for some constant \( c \).

Note that
\[
\exp \exp(cn^\alpha \|b(n-1)\|_2 + c) < \exp \exp(\exp(\|b(n-1)\|_2) < \log b(n) \tag{A.3}
\]
for almost all \( n \). Also note that since \( n \) is appropriate, by the definition of \( f_n \), there are no numbers \( r' \in X \) between \( r+1 \) and \( b(n) \). Hence, if \( r \in X \) and \( \|r\|_2 < \log b(n) \), then \( \exp(\alpha \operatorname{time}(r)) < \log b(n) \) by (A.3) (for all but finitely many \( n \) and some \( \alpha \)). Hence, \( \log b(n) \) is an \( h_\alpha \)-good number of \( M_0 \) for almost all \( n \). Therefore, the set of \( h_\alpha \)-good numbers for \( M_0 \) is infinite.

\[\square\]
Active from the left/right band 822
Active from the left/right cell 821
Active from the left/right letter 815
Admissible word 793
a-edge 818
a-length \(|a|\) 808,818
a-letter 817
Aligned base 822
Almost quadratic function 785
Annulus 819
Application of a rewriting rule 794
Appropriate number 879
Area of a diagram 818
Area of a word 818
Band 819
Base of a band 820
Base of a trapezium 821
Base width of a comb 829
Base of a word 793
Bead black or white 827
Bottom of a band 819
Bottom of a trapezium 821
Cell 818
Chain 837
Chain-annulus 837
Close q-band (to) 836
Clove 862
Comb 833
Combinatorial length of a path 818
Comb mixture 830
Command of a Turing machine 790
Computation 791
Configuration of a Turing machine 790
Copy of a word 795
Control state letter 806
Crescent 865
Crossing bands 820
Dehn function 785
Derivative subcomb 831
Derivative q-band 831
Diagram 818
Diagram admitting a quasicomb 839
Domain of a rule 794
Enumerating Turing machine 878
Equivalence of functions 785
f-good number 787, 808
Filling subtrapezium 836
Firm factorization 853
Group $G$ 817
Group $M$ 817
Handle of a comb
Height of a trapezium
History of a band
History/step history of a comb
History of a computation
History of a trapezium
History of type . . .
Hub
$H'$-part of a comb
$H'$-part of a trapezium
Input configuration of a Turing machine
Input sector of $M_4$
Inverse command
$(i) - (i + 1)$ transition band
$k$-band
$k'$-band
$k$-letter
$k'$-letter
Language accepted by a machine
Large base
Left and right rules
Left/right sides of a $q$-band in a comb
Length (time) of a computation
Length $|\cdot|$ of a word
Length $\|\cdot\|$ of a word (combinatorial)
Length $|\cdot|$ of a path
Link of a chain
Long (quasi)comb
Maximal band
Minimal diagram
$M$-index
$M_2$-index
$M_4$-accepting trapezium
Necklace
Normal base
Odd cell
Odd $\theta$-band
One-step comb
Parameters
Part of a rule
Passive band (from the left/right)
Passive cell
Passive $q$-letter
$p$-band
Perimeter $|\cdot|$ of a diagram
$p$-letter
$p_i$-letter, edge, band
Positive and negative commands
Positive and negative rules
Projection of a word
Proper subcomb

$q$-band

$q$-edge

$q$-length $| \cdot |_q$

$q$-letter

Quasicomb

Reduced computation

Reduced diagram

Regular comb

Regular extension of a comb

Relations of the group $M$

Responsible for counting number

Rim band

Rule

Rule locks a sector

Sector of a word

Short derivative

Side of a band

Sides of a trapezium

$s$-band

$s_i$-band

Simple $\theta$-band

$s$-letter

$S$-machine

$S$-machine $M$

$S$-machine $M_2$

$S$-machine $M_3$

$S$-machine $M_4$

$S$-machine $M_4(i)$

$S$-machine $S_1$

$S$-machine $Z(A)$

$S$-machines $\overline{Z}(A)$ and $\overline{Z}(A)$

Solid diagram

Special $\theta$-edge

Spoke

Standard base

Standard computation of $M_4$

Standard trapezium

Start and end edges of a band

Start vector $s_1$

State letter

Step history of a band

Step history of a computation

Step history of a trapezium

Steps of $M_4$

String of beads

Strongly active from the left/right band

Subcomb of a diagram/comb

Successful step
Symmetric Turing machine
Tape letter
t-band
t'-band
t-letter	
	't'-letter
ti-reflection
Top of a band
Top of a trapezium
Transition rule (12)
Transition rule (23)
Trapezium
Trimmed sides of a trapezium
Turing machine
Turing machine $M_0$
Turing machine $M_1$
$B$
$B(i)$
$\text{bot}(\cdot)$
$C_1, \ldots, C_s$
$\text{cl}(\pi, B, B')$
$\partial$
$F(\Delta)$
h
$h'$
h_i
$h'_i, \ldots$
h_-
$H_1, \ldots, H_s$
l_-
$\text{Lab}(\cdot)$
$P_k$
$p(\Psi)$
$\text{Sym}(M_1)$
$T_i$
time($x$)
$\text{tbot}(\cdot)$
$\text{ttop}(\cdot)$
$\text{top}(\cdot)$
$T_\rho(\cdot, \cdot)$
$W_0$
$W(i)$
$W(M)$
$X_1$
$X_2$
$X_3$
$X_4$
$X_5$
y
y′
Y(3) and Y′(3)
Y_i(θ)
\(z\)
\(z'\)
\(Z^{(θ,i)}\)
\(\bar{Z}^{(θ,i)}\) and \(\bar{Z}^{(θ,i)}\)
\(Γ_i\)
\([Γ] = h(z - y)\)
\(ζ(θ)\)
\(ζ_{-}(θ)\) and \(ζ_{+}(θ)\)
\(κ(·)\)
\(κ\)-mixture
\(κ\)-necklace
\(κ^c(·)\)
\(λ(·)\)
\(λ\)-mixture
\(λ\)-necklace
\(λ^c(·)\)
\(μ(·)\)
\(μ_K(·)\)
\(μ^c(·)\)
\(ν_K\)-mixture
\(ν\)-necklace
\(ν^c_K(·)\)
\(π_{12}(·)\)
\(π_{21}(·)\)
\(π_{23}(·)\)
\(π_{32}(·)\)
\(π_{34}(·)\)
\(Π_{12}(·)\)
\(Π_{21}(·)\)
\(Π_{23}(·)\)
\(Π_{32}(·)\)
\(Π_{34}(·)\)
\(θ_{accept}\)
\(θ_{start}\)
\(θ(M_4)\)
\(θ\)-band
\(θ\)-edge
\(θ\)-length \(|·|_θ\)
\(θ\)-letter
\((θ,a)\)-cell
\((θ,a)\)-relation
\((θ,q)\)-cell
\((θ,q)\)-relation
\(Θ(M_4)\)
\(Θ(3)\)
\(Θ^±\)
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