Three-dimensional loop quantum gravity: towards a self-gravitating quantum field theory

Karim Noui

Laboratoire de Mathématiques et Physique Théorique, UMR-CNRS 60-83, Fédération Denis Poisson, Parc de Grandmont, 37200, Tours, France

E-mail: noui@phys.univ-tours.fr

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Abstract
In a companion paper, we have emphasized the role of the Drinfeld double $DSU(2)$ in the context of three-dimensional Riemannian loop quantum gravity coupled to massive spinless point particles. We make use of this result to propose a model for a self-gravitating quantum field theory (massive spinless non-causal scalar field) in three-dimensional Riemannian space. We start by constructing the Fock space of the free self-gravitating field: the vacuum is the unique $DSU(2)$ invariant state, one-particle states correspond to $DSU(2)$ unitary irreducible simple representations and any multi-particles states are obtained as the symmetrized tensor product between simple representations. The associated quantum field is defined by the usual requirement of covariance under $DSU(2)$. Then, we introduce a $DSU(2)$-invariant self-interacting potential (the obtained model is a group field theory) and explicitly compute the lowest order terms (in the self-interaction coupling constant $\lambda$) of the propagator and of the three-point function. Finally, we compute the lowest order quantum gravity corrections (in the Newton constant $G$) to the propagator and to the three-point function.

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1. Motivations
The quantization of a self-gravitating field theory is a difficult issue which is still open in loop quantum gravity (LQG). Nevertheless, this problem is essential and one has to solve it completely and clearly to claim that LQG is not only a candidate for pure quantum gravity but also provides a framework to unify the fundamental interactions. Paradoxically, it seems that we have all the ingredients to find a solution to that problem. On the one hand, LQG (resp. spin-foam models) offers a rigorous framework for a background-independent canonical (resp. covariant or path integral) quantization of general relativity (GR). On the other hand,
QFT is so successful to describe the physics of elementary particles. All the same, there seems to be a large incompatibility between the two approaches: GR does not fit in QFT and vice versa. It is customary to invoke the non-renormalizability of GR as the main reason of this disagreement and then usual perturbative QFT techniques are fundamentally inadapted to describe a quantum theory of GR. In QFT, the quantization of the self-gravitating field consists of quantizing first the field in a given (flat or curved) spacetime, and then quantizing the metric degrees of freedom by perturbations. If this method manifestly fails, one can try to attack the problem the other way around, i.e. one quantizes gravity before the matter degrees of freedom.

In fact, this idea is an old one but, to our knowledge, it was only recently that it was realized explicitly by Freidel and Livine in the framework of three-dimensional Riemannian spin-foam models [6]. In fact, they start with gravity coupled to classical particles, they quantize the gravitational degrees of freedom through the Ponzano–Regge spin-foam model and after that they quantize the matter field degrees of freedom allowing for the creation and annihilation of point particles. Thus, they have shown that 3D quantum gravity amplitude, in the context of the Ponzano–Regge model coupled to point particles [5], is actually the Feynman diagram evaluation of a braided non-commutative quantum field theory. This effective field theory describes the dynamics of a scalar field after integrating the gravitational degrees of freedom.

This paper is devoted to analysing this result in the Hamiltonian point of view. For that purpose, we use the results presented in a recent companion paper [11] which states that the Drinfeld double $DSU(2)$ appears to be the ‘quantum symmetry group’ of three-dimensional Riemannian loop quantum gravity coupled to massive spinless point particles once one imposes the Hamiltonian constraint. To be more precise, we have shown that any multi-particle physical states are defined by a tensor product of simple unitary irreducible representations (UIR) of $DSU(2)$ and the physical scalar product between two such states is given in terms of the symmetric (or Barrett–Crane) intertwiners (see [11] for a general definition and examples of the symmetric intertwiner). Therefore, we start from the simple UIR of the Drinfeld double and construct a Fock space as the direct sum of the symmetrized tensor product of these representations. Then, we define creation and annihilation operators acting on this Fock space and we construct a local self-gravitating quantum field $\phi$ by the usual requirement that it transforms covariantly under the Drinfeld double transformations. This defines the model we propose for a free self-gravitating scalar non-causal quantum field theory. By non-causal, we mean that Feynmann graphs’ amplitudes are those of a non-causal spin-foam models; therefore, the model admits no causal structure and no dynamics. In a sense, it behaves as a topological quantum scalar field theory. Finally, we introduce a (cubic) self-interaction in the model and show the effects of quantum gravity on some physical processes such as particle propagation or particle creation. In particular, the propagator of the self-gravitating free field, given by the two-points function $\Delta_G(x, y)$ ($G$ being the Newton constant), is computed and is manifestly different from the non-gravitating massive scalar field one $\Delta_0(x, y)$: we show that there are quantum gravity corrections to the classical propagator which are in total agreement with those obtained in the context of spin-foam models by Freidel and Livine. In the limit where $G$ tends to zero, the self-gravitating propagator $\Delta_G(x, y)$ tends to the non-gravitating propagator $\Delta_0(x, y)$. Then, we concentrate on the three-point function of the self-interacting theory: we compute lowest order terms (in the self-interaction coupling constant) in both self-gravitating and non-gravitating cases; thus we are able to compute explicitly the quantum gravity effects on the three-points function.

The paper is organized as follows. Section 2 is devoted to the construction (à la Wigner as recalled in Weinberg book) of the self-gravitating quantum field. We start by briefly recalling the construction in the case of a scalar field in three-dimensional Euclidean space: definitions
of one-particle states, of the Fock space and of creation and annihilation operators. Then, we adapt this construction to define our model for a self-gravitating non-causal quantum field theory on a sphere: we define one-particle and multi-particle states in terms of particles-spin-networks, construct naturally the Fock space and define the self-gravitating quantum field as an operator that transforms covariantly under $DSU(2)$ transformations. We show that our model defines in fact a non-commutative braided quantum field theory which is in total agreement with the results obtained by Freidel and Livine in the spin-foam context. We compute the self-gravitating propagator (two-points function) and exhibit the first quantum gravity corrections to the classical propagator.

In section 3, we give the Lagrangian formulation of our model. Then, we introduce a self-interaction (cubic term in the field) and compute lowest order (in the self-interaction coupling constant $\lambda$) in terms of the self-gravitating three-points function.

To conclude, we discuss the possibility of generalizing our construction to the Lorentzian case as well as to the case where there is a non-vanishing positive or negative cosmological constant.

2. A self-gravitating massive quantum field theory

This section aims at presenting the construction of the self-gravitating quantum field theory in the canonical framework. Following the Weinberg approach [13] based on the Wigner analysis, we first recall the construction of a bosonic scalar field in three-dimensional Euclidean space. Second, we adapt this method to construct and study the self-gravitating quantum scalar field.

2.1. A simple example: the scalar field in three-dimensional Euclidean space

We start by recalling the construction of a massive quantum field in the three-dimensional Euclidean space $E^3$ ($E^3$ is the space $\mathbb{R}^3$ endowed with the Euclidean metric $\text{diag}(+,-,+)$.)

The unitary irreducible representations of the universal covering of the isometry group of $E^3$, denoted by $ISU(2) = \mathbb{R}^3 \times SU(2)$, provide the one-particle space of states of the quantum field. These representations are classified by a mass $m \in \mathbb{R}$ and a spin $s \in \frac{1}{2}\mathbb{N}$. Among these representations, one distinguishes the simple ones, characterized by the fact that $s = 0$, whose associated vector space $H$ is the Hilbert space of states for the spinless massive particle on $E^3$. The one-particle Hilbert space is $H[m] \simeq (L^2(S^2), d\mu)$ where $L^2(S^2)$ is the set of functions on the 2-sphere $S^2 = SU(2)/U(1)$ which are squared integrable with respect to the normalized measure $d\mu$. In what follows, we will identify a point $\lambda \in S^2$ with an element $\lambda \in SU(2)/U(1)$.

The action of any element $(\vec{x}, g) \in \mathbb{R}^3 \times SU(2)$ on states $\psi \in H[m]$ associated with a particle of mass $m$ reads

$$\pi_m(\vec{x}, g)\psi(\lambda) = \exp(i\vec{x} \cdot \lambda \vec{m})\psi(g^{-1}\lambda),$$

where $\lambda \vec{m}$ denotes the action of $\lambda \in SU(2)$ on the ‘rest’ vector $\vec{m} = (m, 0, 0) \in \mathbb{R}^3$. It is customary to extend the space of states to the set of distributions on $S^2$ in order to include pure momentum states into $H[m]$. In that case, one introduces the usual bra-ket notation $|\lambda\rangle$ and the representation (1) is trivially rewritten as follows:

$$\pi_m(\vec{x}, g)|\lambda\rangle = \exp(i\vec{x} \cdot \lambda \vec{m})|g^{-1}\lambda\rangle.$$ (2)

In what follows, we will mostly use this notation which is more familiar in quantum field theory. Note that the duality bracket is given as usual by $\langle \lambda | \psi \rangle = \psi(\lambda)$. 

of one-particle states, of the Fock space and of creation and annihilation operators. Then, we adapt this construction to define our model for a self-gravitating non-causal quantum field theory on a sphere: we define one-particle and multi-particle states in terms of particles-spin-networks, construct naturally the Fock space and define the self-gravitating quantum field as an operator that transforms covariantly under $DSU(2)$ transformations. We show that our model defines in fact a non-commutative braided quantum field theory which is in total agreement with the results obtained by Freidel and Livine in the spin-foam context. We compute the self-gravitating propagator (two-points function) and exhibit the first quantum gravity corrections to the classical propagator.

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In what follows, we will mostly use this notation which is more familiar in quantum field theory. Note that the duality bracket is given as usual by $\langle \lambda | \psi \rangle = \psi(\lambda)$.
To construct a bosonic field from these representations, one starts by defining the notation of a \( n \) particles state represented by elements \( |\lambda_1, \ldots, \lambda_n\rangle \) that belong to the symmetrized tensor product of one-particle state, i.e.,

\[
|\lambda_1, \ldots, \lambda_n\rangle = \frac{1}{n!} \sum_{\sigma \in P_n} |\lambda_{\sigma(1)}\rangle \otimes |\lambda_{\sigma(2)}\rangle \otimes \cdots \otimes |\lambda_{\sigma(n)}\rangle,
\]

where the sum runs over the permutation group \( P_n \). Note that there is no analogue of the well-known four-dimensional spin-statistic theorem for three-dimensional quantum field theories (see [14] for example). In particular, there exist in three-dimensional spacetime exotic statistics which interpolate between bosonic and fermionic statistics (as it is briefly explained in appendix A). Nevertheless, a spinless quantum field in flat Euclidean space cannot only be bosonic if one requires locality, covariance and causality; there is no restriction on the statistics if one considers a spinning quantum field as explained in appendix A. As we deal with spinless massive field, we will only consider the bosonic statistic here. In that case, the space of \( n \)-particles states is given by \( \mathcal{H}_n^m[m] = \mathcal{H}[m]^{\otimes n} \) where \( \otimes \) stands for the symmetrized tensor product. The bosonic Fock space \( \mathcal{F}[m] \) for a massive field of mass \( m \) is then defined as the tower of multi-particles states, i.e.,

\[
\mathcal{F}[m] \equiv \bigoplus_{n=0}^{\infty} \mathcal{H}_n^m[m] = \bigoplus_{n=0}^{\infty} \mathcal{H}[m]^{\otimes n}.
\]

The space \( \mathcal{H}_0[0] \approx \mathcal{H}[0] \) is the trivial representation Hilbert space of \( ISU(2) \). The Fock space carries in fact a reducible representation of \( ISU(2) \) whose action on \( \mathcal{F}[m] \) simply reads

\[
ISU(2) \times \mathcal{F}[G] \rightarrow \mathcal{F}[G]
\]

\[
(\tilde{x}, g) \times |\lambda_1, \ldots, \lambda_n\rangle \rightarrow U(\tilde{x}, g)|\lambda_1, \ldots, \lambda_n\rangle \equiv \pi^m_{\otimes n}(\Delta^{(n)}(\tilde{x}, g))|\lambda_1, \ldots, \lambda_n\rangle,
\]

where \( \Delta^{(n)}: ISU(2) \rightarrow ISU(2)^{\otimes n} \) is the iterated co-product defined by

\[
\Delta^{(1)}(\tilde{x}, g) = \Delta(\tilde{x}, g) = (\tilde{x}, g) \otimes (\tilde{x}, g) \quad \text{and} \quad \Delta^{(n+1)} = id \otimes \Delta^{(n)}.
\]

Note that \( \Delta \) is the usual co-commutative coproduct defined for groups.

Bosonic creation and annihilation operators are denoted as usual by \( a^\dagger(\lambda) \) and \( a(\lambda) \), respectively and satisfy the following commutation relations:

\[
[a^\dagger(\lambda_1), a(\lambda_2)] = 0 = [a(\lambda_1), a(\lambda_2)] \quad \text{and} \quad [a^\dagger(\lambda_1), a^\dagger(\lambda_2)] = \delta(\lambda_1^{-1} \lambda_2),
\]

and act on the Fock space by respectively raising or lowering the number of particles. There is a natural co-action of \( ISU(2) \) on the set of creation and annihilation operators defined by duality from the action (5). It is customary to describe this co-action with the following notations:

\[
a_{\pm}(\lambda) \equiv U(\tilde{x}, g) a_{\pm}(\lambda) U(\tilde{x}, g)^{-1} = \exp(\mp i \tilde{x} \cdot \lambda \tilde{m}) a_{\pm}(g^{-1} \lambda),
\]

where \( a_+(\lambda) = a^\dagger(\lambda) \) and \( a_-(\lambda) = a(\lambda) \) are the creation and annihilation operators.

We have now all the pieces to construct a free local quantum field. Following Weinberg [13], the quantum field \( \phi \) is defined as a Fock space operator valued function on the Euclidean space \( \mathbb{R}^3 \) that satisfies the fundamental properties of locality, covariance and causality. Locality is automatically satisfied as \( \phi \) is a function, i.e. is defined for each point \( \tilde{x} \) of \( \mathbb{R}^3 \) as follows:

\[
\phi(\tilde{x}) \equiv \int d\mu(\lambda)(c_+(\lambda, \tilde{x}) a^\dagger(\lambda) + c_-(\lambda, \tilde{x}) a(\lambda)).
\]

The covariance means that the field transforms in the same way as creation and annihilation operators under the action of the elements of \( ISU(2) \) (8). As an immediate consequence, the
quantum field is completely determined by its value at the origin $\phi(\vec{0})$ which is co-invariant under the action of the rotational subgroup $SU(2) \subset ISU(2)$. It follows that:

$$U(\vec{0}, g)\phi(\vec{0})U(\vec{0}, g)^{-1} = \phi(\vec{0}) \implies c_{\pm}(\lambda, \vec{x}) = A_{\pm} \exp(\mp i\vec{x} \cdot \lambda \vec{m}),$$

(10)

where $A_{\pm}$ can be chosen to be real numbers. Causality is meaningful only when the underlying base space admits a causal structure: in that case, a field is said causal if $[\phi(x), \phi(y)] = 0 = [\phi(x)^{\dagger}, \phi(y)]$ when $x$ and $y$ are causally disconnected points. In our case, $\mathbb{E}^{3}$ is an Euclidean space and we replace the causality property by the requirement that $[\phi(x), \phi(y)] = 0 = [\phi(x)^{\dagger}, \phi(y)]$ for each couple of points $(x, y)$. As a consequence, $A_{+} = A_{-} = A$ and the quantum field is finally given by

$$\phi(\vec{x}) = A \int d\mu(\lambda)(\exp(-i\vec{x} \cdot \lambda \vec{m})a(\lambda) + \exp(i\vec{x} \cdot \lambda \vec{m})a^{\dagger}(\lambda)).$$

(11)

Finally, the propagator of this free theory is given by the two-points function which reads

$$\Delta_{0}(\vec{x}, \vec{y}) \equiv \langle 0|\phi(\vec{x})\phi(\vec{y})|0 \rangle = A^{2}\sin m\|\vec{x} - \vec{y}\|/m\|\vec{x} - \vec{y}\|.$$  

(12)

The propagator is defined here up to a global constant $A$. In the following, we will take the value $A = 1$.

Note that the propagator is in fact the Hadamard function of the field: it is a solution of the Klein–Gordon equation but not a Green function. To recover the usual Feynman propagator, one has to choose a causal structure which means making a choice of a time variable and then defining the two-points function (12) as the expectation value of the chronological product of fields. There is a natural way to do so if we deal with a Lorentzian quantum field theory instead of a Euclidean one.

2.2. A quantum scalar field on a quantum Euclidean background

The idea is now to adapt this well-understood method to construct a self-gravitating three-dimensional massive quantum field theory. We will first present the construction of the Fock space of the theory starting with the description of the one-particle state and the multi-particles states. Then, we will construct the quantum fields requiring the basic properties of locality, covariance and causality. As we deal with Hamiltonian quantization, we will exclusively consider the case where the topology of three-dimensional spacetime $M$ is $M = \Sigma \times [t_{1}, t_{2}]$. Moreover, we concentrate on the spherical case only, i.e. $\Sigma = S^{3}$.

1 The change of variable $\vec{P} = \lambda \vec{m}$ and $d\mu(\lambda) = 1/(2\pi m)\delta(\vec{P} \cdot \vec{P} - m^{2})d^{3}\vec{P}$ in the previous expression allows us to re-express the field $\phi$ in the more familiar form involving the full momentum $\vec{P}$:

$$\phi(x) = \tilde{A} \int d^{3}\vec{P}(\exp(-i\vec{x} \cdot \vec{P})a(\vec{P}) + \exp(i\vec{x} \cdot \vec{P})a^{\dagger}(\vec{P})).$$

where $\tilde{A} = A/(2\pi m)$ and $a(\lambda \vec{m}) = a(\vec{P})$. Usually in QFT, one writes the quantum field as an integral involving only the space momentum $\vec{P}$ and not the ‘spacetime’ momentum $\vec{P}$. To do so, one integrates over the time component $P_{0}$ and imposes a causal structure by the constraint $P_{0} > 0$. However, not only there is no canonical way to exhibit a time component $P_{0}$ out of an Euclidean momentum $\vec{P}$ but also there is no consistent way to impose the constraint $P_{0} > 0$ when the space of Euclidean momentum of fixed mass $m$ is a sphere; however it is a double-connected hyperboloid in the Lorentzian case (which makes the positivity condition valuable). If we naively impose a causal structure anyway (with an Heaviside function $\Theta(P_{0})$), the scalar field would be defined as follows:

$$\phi(x) = \tilde{A} \int d^{3}\vec{P}(\exp(-i\vec{x} \cdot \vec{P})a(\vec{P}) + \exp(i\vec{x} \cdot \vec{P})a^{\dagger}(\vec{P})).$$

One could integrate over the variable $P_{0}$ to have an expression in terms of $\vec{P}$. Finally, the equal time canonical relation $[\phi(x_{0}, \vec{x}_{1}), \partial_{\vec{x}}\phi(x_{0}, \vec{y}_{1})] = i\hbar \Theta(\vec{x}_{1} - \vec{y}_{1})$ fixes the value of $\tilde{A}$. The Planck constant if fixed to $\hbar = 1$. 

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2.2.1. One-particle states. A system of gravitating particles is defined with respect to one observer $O$ whose mass is fixed to the value $m_0$. In fact, $m_0$ is the total mass of the system of particles. In the gravitational case, states of a spinless massive particle in the sphere are elements of $\mathcal{H}[m] \times \mathcal{L}(1)$ where $\mathcal{H}[m] = (L^2(S^2), d\mu)$ is the Hilbert space of one-particle state in $S^2$ and $\mathcal{L}(1)$ denotes the set of oriented links between one particle and the observer. As previously, we will prefer for clarity reasons to extend $\mathcal{H}[m]$ to distributional states and introduce $|\lambda\rangle$ that represents a pure momentum state written in the bra-ket notation. Thus, a one-particle state for a self-gravitating quantum field is denoted by $|\lambda, \ell\rangle$. The spin-network evaluation of this state of quantum gravity is rigorously given by $\langle \lambda, \ell | A | \lambda, \ell \rangle = \varphi(H_{\ell}(A)\lambda)\lambda$ where $A$ is a $SU(2)$ flat connection on $\Sigma$, $\varphi \in \mathcal{H}[m]$ characterizes the particle state and $H_{\ell}(A)$ is the holonomy of the connection along the oriented link $\ell$. As the gravitational degrees of freedom are pure gauge, we always omit to mention the flat connection that is implicitly gauge fixed to the trivial one. Because of diffeomorphisms invariance and the trivial topology of $S^2$, $|\lambda, \ell\rangle$ and $|\lambda, \ell'\rangle$ are physically equivalent whatever the links $\ell$ and $\ell'$ are.

2.2.2. Multi-particle states: symmetrization versus diffeomorphisms invariance. Let us now describe multi-particle states. Any $n$-particle state on the surface $\Sigma$ is characterized by a set of $n$ points or circles $(x_1, \ldots, x_n)$ on $\Sigma$ corresponding to the 'locations' of the particles, a family of momenta $(\lambda_1, \ldots, \lambda_n)$ associated with each particle and a family of oriented links $(\ell_1, \ldots, \ell_n)$ between the observer $O$ and each particle (see figure 1). The links are oriented from the observer to the particles.

In fact, the set of links forms a minimal graph $\gamma_n$ which defines a $n$-particles-spin-network state, i.e. a quantum state of the coupled system $\{\text{gravity + particles}\}$. This structure has been introduced and studied in [10, 11]. Such a state is represented as a tensor product of one-particle states. The Hilbert space of $n$-particles states is a sub-space of $\mathcal{H}_n[m] = \mathcal{H}[m]^{\otimes n} \times \mathcal{L}(n)$ where $\mathcal{L}(n)$ denotes the set of links between $n$ particles and the observer.

Let us now adopt the following notation for elements of $\mathcal{H}_n[m]$:

$$|\lambda_1, \ell_1 \otimes \cdots \otimes \lambda_n, \ell_n\rangle \equiv |\lambda_1 \otimes \cdots \otimes \lambda_n; \ell_1 \otimes \cdots \otimes \ell_n\rangle \equiv |\otimes_1 \lambda_i; \gamma_n\rangle. \quad (13)$$

This notation makes a clear distinction between the $\mathcal{H}[m]^{\otimes n}$ part from the $\mathcal{L}(n)$ part in $\mathcal{H}_n[m]$ and means that: the point $x_i$ (ith element on the tensor product) is associated with a particle of momentum $\lambda_i$ and is linked to the observer by $\ell_i$. By convention, $\ell_i \otimes \ell_j$ means that $\ell_i < \ell_j$ for the order on the set of links on a given graph $\gamma_n$; it will be convenient for what follows to introduce the notation $\ell_1 \otimes^{op} \ell_2$ for a given graph $\gamma_2$ which means that $\ell_2 < \ell_1$. We will denote $\mathcal{H}_n[m, \gamma_n]$ the space of states defined on a given graph $\gamma_n$.

To evaluate such a spin-network, we fix a function $\varphi \in \mathcal{H}[m]^{\otimes n}$, a $SU(2)$ connection $A$ on $\Sigma$ and we have

$$\langle \psi, A \otimes_1 \lambda_i; \gamma_n\rangle = \varphi(H_{\ell_1}(A)\lambda_1 \otimes \cdots \otimes H_{\ell_n}(A)\lambda_n). \quad (14)$$

We have implicitly assumed that $\ell_1 < \cdots < \ell_n$ in the graph $\gamma_n$ (otherwise we have to perform a permutation in the evaluation of the spin-network). As for one-particle states, we omit the
where we have introduced the notation one inherited from the order on the links (case of the left picture of figure 1). We make this

\[ \ell \]

with

\[ m \]

gravitational part \( H_\ell(A) \) by gauge fixing the connection to the trivial one. This gauge fixing
does not affect, by definition, physical observations and predictions (see [11] for details).

What are the properties of multi-particle states under permutations of particles? To answer
this question, we first have to precise what we mean by permuting two particles as we needed
different Fock spaces but should be physically equivalent. This particle transposition is
do not affect, by definition, physical observations and predictions (see [11] for details).

The structure of the Fock space for a self-gravitating quantum field is similar to that of the space of cylindrical functions

\[ \mathcal{F}[m; \gamma] = \bigoplus_{n=0}^{\infty} \mathcal{H}_n[m; \gamma_n]. \]

The space of a bosonic self-gravitating \( n \)-particle states is denoted by \( \mathcal{H}_n[m; \gamma_n] \) and depends

\[ \mathcal{F}[m; \gamma] \]

on the given minimal graph \( \gamma_n \). We choose \( \gamma_n+1 \supset \gamma_n \) such that there exists a canonical inclusion \( \mathcal{H}_n[m; \gamma_n] \hookrightarrow \mathcal{H}_{n+1}[m; \gamma_{n+1}] \). Finally, the bosonic Fock space for a self-gravitating

quantum field of mass \( m \) is given as usual by the following infinite direct sum:

\[ \mathcal{F}[m; \gamma] = \bigoplus_{n=0}^{\infty} \mathcal{H}_n[m; \gamma_n]. \]

where we have introduced the notation \( \gamma = \bigotimes_{i=1}^{\infty} \ell_i = \bigcup_{n=1}^{\infty} \gamma_n \). The structure of the Fock space

for the self-gravitating quantum field is similar to that of the space of cylindrical functions

on \( \Sigma \) defined by means of projective limits. As we will show in what follows, the graph \( \gamma \)

is not physically relevant in the definition of the Fock space: once we construct the quantum

field, we will see that physical quantities as \( n \)-point functions do not depend on the choice of

\( \gamma \). This is directly linked to the diffeomorphism invariance of quantum gravity for any two

different \( \gamma \) and \( \gamma' \) on the sphere \( \Sigma \) related by a spatial diffeomorphism.

Let us see what happens in the simple example where one turns one particle around another

one without changing the homotopy class of the graph (figure 1). The two states belong to
different Fock spaces but should be physically equivalent. This particle transposition is

described in terms of a map \( \tau : \mathcal{H}_2[m, \gamma_2] \to \mathcal{H}_2[m, \gamma'_2] \) (its inverse \( \tau^{-1} \)) is given by

\[ \tau : [\ell_1 \otimes \ell_2; \tau_1 \otimes \tau_2] \longmapsto 1 \otimes 1 \]

\[ \tau^{-1} : [\ell_1 \otimes \ell_2; \tau'_1 \otimes \tau'_2] \longmapsto 1 \otimes 1 \]

where the product between the set of links is the usual composition and \( c_i^+ \) (resp. \( c_i^- \)) is

the anti-clockwise (resp. clockwise) loop around the particle located at the point \( x_i \). These

maps are trivially extended to the Fock space. For clarity reasons, we restrict ourselves to
the case \( n = 2 \): using the notations of figure 1, we have \( \gamma_2 = \ell_1 \otimes \ell_2 \) and \( \gamma'_2 = \ell'_1 \otimes \ell'_2 \)

with \( \ell'_1 = \ell_1 c_1^+ \ell_1' \ell_2 \) and \( \ell'_2 = \ell_1 \). Note that generically \( \tau^2 \neq 1 \) and therefore there are
In what follows, we will also need the expression of the antipode:
\[ \delta \]
Note that the Drinfeld double \( DSU(2) \) is the quantum symmetry group of a system of point massive particles whose symmetry group is \( ISU(2) \) instead of representations of \( ISU(2) \). As Hilbert spaces, representations of \( DSU(2) \) and \( ISU(2) \) are isometric. But the action of the group operators on the states will be different and that makes the essential difference in the construction of the self-gravitating quantum field compared to the usual quantum field in \( \mathbb{E}^3 \).

### 2.2.3. Gravitational deformation of translations and non-commutative spacetime

The Drinfeld double \( DSU(2) \) is clearly the quantum symmetry group of a system of point massive particles coupled to Riemannian three-dimensional quantum gravity. This has been shown in the context of combinatorial quantization [9], of Ponzano–Regge spin-foam models [5] and recently in the context of loop quantum gravity [11]. In fact, \( DSU(2) \) is a gravitational deformation of \( ISU(2) \) (appendix B) and can be viewed as the symmetry group of the system of self-gravitating particles once quantum gravity effects have been taken into account. Thus, quantizing a system of self-gravitating massive particles on \( \mathbb{E}^3 \) is equivalent to quantizing a system of non-gravitating massive point particles whose symmetry group is \( DSU(2) \) instead of \( ISU(2) \). As a consequence, \( n \)-particles states transform as representations of \( DSU(2) \) under rotations and translations; and these transformation laws are deformed compared to usual \( ISU(2) \) ones. Finally, the massive self-gravitating quantum field will naturally be defined by the requirement that it transforms covariantly under \( DSU(2) \).

Before going to the construction of the field operator, we present basic properties concerning the Drinfeld double. Following notations of appendix B, a (distributional) element of the Drinfeld double is given by a pair \( (g, u) \) of \( SU(2) \) elements. In this notation, the product and co-product of \( DSU(2) \) read
\[
(g_1, u_1) \cdot (g_2, u_2) = \delta(g_1^{-1}u_1g_2u_1^{-1})(g_1, u_1u_2) \tag{21}
\]
\[
\Delta(g, u) = \int dh(gh, u) \otimes (h^{-1}, u). \tag{22}
\]
Note that \( \delta \) is the delta distribution on the group \( SU(2) \) and \( \int dh \) is the \( SU(2) \) Haar measure. In what follows, we will also need the expression of the antipode: \( S(g, u) = (u^{-1}g^{-1}u, u^{-1}) \).
Comparing this structure with $ISU(2)$ one, it is clear that $DSU(2)$ is a deformation of $ISU(2)$. In fact, the rotational part remains the same as for $ISU(2)$ and is isomorphic as a Hopf algebra to $SU(2)$ whereas the translation part structure is deformed and is no longer isomorphic to $\mathbb{R}^3$.

The deformation of $ISU(2)$ into $DSU(2)$ is characterized by a group algebra morphism $\phi : ISU(2) \rightarrow DSU(2)$ which is obviously not a co-algebra morphism (appendix B). The map shows that $g$ has to be interpreted as a deformation of a momentum on $\mathbb{E}^3$ and not strictly as a deformation of a position in $\mathbb{E}^3$. In order to recover gravitational deformed analogues of the position variables, one has to introduce the following deformed Fourier transform:

$$\langle \tilde{x}, u \rangle \equiv \int dg \ e^{i\sigma(\vec{x})}(g, u),$$

(23)

where we make use of the identification $\mathbb{E}^3 \rightarrow su(2); \vec{x} \mapsto x = \vec{x} \cdot \vec{\sigma}$ with $\vec{\sigma} = (\sigma_0, \sigma_1, \sigma_2)$ the generators of the Lie algebra $su(2)$; $\text{tr}$ denotes the trace in the fundamental representation.

The group law satisfied by the elements $(\tilde{x}, u)$ is trivially obtained from relation (21)

$$(\tilde{x}_1, u_1) \cdot (\tilde{x}_2, u_2) = (\tilde{x}_1 + u_1 \tilde{x}_2, u_1 u_2)$$

(24)

and it becomes clear that $\tilde{x}$ is the very analogue of the position. The difference with position variables in $\mathbb{E}^3$ is that the variables (23) are in fact position variables on a non-commutative spacetime.

Let us be more precise. First, we introduce the notation $\mathbb{E}^3_G \equiv DSU(2)/SU(2)$. $\mathbb{E}^3_G$ admits a Hopf algebra structure. The space of functions on $\mathbb{E}^3_G$, denoted by $\text{Fun}(\mathbb{E}^3_G)$ inherits by duality a Hopf-algebra structure. Of particular interest are the plane waves on $\mathbb{E}^3_G$ defined for any $g \in SU(2)$ by

$$w_g : \mathbb{E}^3_G \longrightarrow \mathbb{C} \quad \vec{x} \mapsto w_g(\vec{x}) = e^{i\sigma(\vec{x})g},$$

(25)

where we identify $\mathbb{E}^3_G$ and $su(2)$ as in (23). In terms of these deformed plane waves, the product and co-product $\Delta$ of $\text{Fun}(\mathbb{E}^3_G)$ respectively read

$$(w_g \cdot w_h)(\vec{x}) \equiv (w_g \otimes w_h)(\Delta(\vec{x})) = w_{gh}(\vec{x})$$

(26)

$$\Delta_F(w_g)(\vec{x} \otimes \vec{y}) \equiv w_g((\vec{x}, 1) \cdot (\vec{y}, 1)) = w_g(\vec{x} + \vec{y}).$$

(27)

Due to the non-commutativity of $SU(2)$, the product $\ast$ on $\text{Fun}(\mathbb{E}^3_G)$ is non-commutative and therefore the space $\mathbb{E}^3_G$ is a non-commutative spacetime. In the non-gravitational limit, $\mathbb{E}^3_G$ tends to the usual Euclidean spacetime $\mathbb{E}^3$ and therefore becomes commutative as expected. Note that the non-commutativity is a direct consequence of the non-co-commutativity of the Drinfeld double. The product on $\text{Fun}(\mathbb{E}^3_G)$ is closely related to the convolution product $\circ$ on the space of functions on $SU(2)$ denoted by $\text{Fun}(SU(2))$. Indeed, the following map:

$$(\text{Fun}(SU(2)), \circ) \rightarrow (\text{Fun}(\mathbb{E}^3_G), \ast) \quad f \mapsto \tilde{f} : \vec{x} \mapsto f(\vec{x}) = \int dg \tilde{f}(g) e^{i\sigma(\vec{x})}$$

(28)

is an algebra morphism. To be more precise, the map is a surjection whose kernel $\text{Ker} \simeq \mathbb{Z}_2$ is given by the set of ‘odd’ functions satisfying the relation $\tilde{f}(g) + \tilde{f}(g^{-1}h(\pi))) = 0$. Therefore, restricted to the subspace $\text{Fun}(SU(2))/\text{Ker} \simeq \text{Fun}(SO(3))$ the map is an isomorphism and its inverse is given by [6]

$$f \mapsto \tilde{f} : g \mapsto \tilde{f}(g) = \frac{1}{8\pi} \int d^3\tilde{x}(f \ast w_{g^{-1}})(\tilde{x}).$$

(29)
Symmetry transformations of the bosonic states. Before going to the construction of the quantum field, we need to write the action of elements $\tilde{T}(\vec{x}, u) \in DSU(2)$ on bosonic states. To do so, we first compute the following coproduct:

$$
\Delta^{(n)}(\tilde{x}, u) = \int \prod_{i=1}^{n} dg_i w_{\vec{g}_{i-1} u_i}(\tilde{x}) \otimes (g_i, u)
$$

which is an immediate consequence of expression (22). Then, the action of elements $\tilde{T}(\vec{x}, u) \in DSU(2)$ on a (non-symmetrized) one-particle states’ tensor product $|\otimes_i \lambda_i; \gamma_n \rangle \in \mathcal{H}_a[m; \gamma_n]$ reads

$$
U(\tilde{x}, u)|\otimes_i \lambda_i; \gamma_n \rangle = \pi_m^{\otimes n} \Delta^{(n)}(\tilde{x}, u)|\otimes_i \lambda_i; \gamma_n \rangle
$$

$$
= w_{\lambda_1 h(m) \lambda_1^{-1}} \cdots w_{\lambda_n h(m) \lambda_n^{-1}}(\vec{x})|\otimes_i u^{-1} \lambda_i; \gamma_n \rangle
$$

$$
= \exp \left[ i \text{tr} \left( \chi \prod_{i=1}^{n} \lambda_i h(m) \lambda_i^{-1} \right) \right] |\otimes_i u^{-1} \lambda_i; \gamma_n \rangle. \tag{31}
$$

We used the same notations as those introduced for the non-gravitational case. These transformation laws extend trivially to the bosonic Fock space $\mathcal{F}[m, \gamma]$ by symmetrization and $U(\tilde{x}, u)$ are still unitary operators. It will be useful in what follows to distinguish deformed translations from rotations and we will use the notations $T(\tilde{x}) \equiv U(\tilde{x}, 1)$ and $R(u) = U(0, u)$.

Due to the non-commutativity nature of the spacetime $\mathbb{E}_G$, it is convenient to dualize the translation operators and to define the translation co-actions $T^*$ on multi-particles states as follows:

$$
T^*: \mathcal{H}_a[m; \gamma_n] \rightarrow \mathcal{H}_a[m; \gamma_n]^* \otimes \text{Fun}(\mathbb{E}_G^1)
$$

$$
\langle \lambda_1, \ldots, \lambda_n; \gamma_n | \rightarrow \langle T^*(\lambda_1, \ldots, \lambda_n; \gamma_n) | \equiv \langle w_{\vec{g}_1}, \ldots, w_{\vec{g}_m} \lambda_i; \gamma_n |, \tag{32}
$$

where $g_i = \lambda_i h(m) \lambda_i^{-1}$ and

$$
\langle T^*(\lambda_1, \ldots, \lambda_n); \gamma_n |\tilde{x}\lambda_1', \ldots, \lambda_n'; \gamma_n \rangle \equiv \langle \lambda_1, \ldots, \lambda_n; \gamma_n | T(\tilde{x})|\lambda_1', \ldots, \lambda_n'; \gamma_n \rangle.
$$

We have introduced the notation $\mathcal{H}_a[m; \gamma_n]^*$ for the dual Hilbert space of $\mathcal{H}_a[m; \gamma_n]$ and used bra-ket notations for vectors and linear forms. The image of $T^*$ is canonically identified with a subspace $\mathcal{W}_a[m; \gamma_n] \subset \mathcal{H}_a[m; \gamma_n] \otimes \text{Fun}(\mathbb{E}_G^1)$ which naturally generalizes the space of $n$-particles states. The space $\mathcal{W}_a[m; \gamma_n]$ puts forward the phases induced by translations on $n$-particles states: in the deformed case, usual phases are replaced by functions on the $\mathbb{E}_G^1$. The definition of $T^*$ trivially extends to the space of symmetric states and we will denote the corresponding image by $\mathcal{W}_a^s[m; \gamma_n]$. By convention, we will adopt the notation $|w_{\vec{g}_1} \lambda_1 \otimes w_{\vec{g}_2} \lambda_2 = |w_{\vec{g}_1} \lambda_1 \otimes \lambda_2$ for elements of $\mathcal{W}_a^s[m; \gamma_n)$ and similarly for elements of $\mathcal{W}_a^s[m; \gamma_n]$.

We finish this section with a remark. Pure momenta states are no longer (compared to the non-gravitational case) eigenvectors of the translation operators $U(\tilde{x}) \equiv U(\tilde{x}, 1)$ due to the non-co-commutativity of $DSU(2)$. For instance, a two-particle bosonic state transforms as

$$
T(\tilde{x})|\lambda_1 \otimes \lambda_2 \rangle = \exp \left( i \text{tr}(x \lambda_1 h(m) \lambda_1^{-1} \lambda_2 h(m) \lambda_2^{-1}) \right) |\lambda_1 \otimes \lambda_2 \rangle
$$

$$
+ \exp \left( i \text{tr}(x \lambda_2 h(m) \lambda_2^{-1} \lambda_1 h(m) \lambda_1^{-1}) \right) |\lambda_2 \otimes \lambda_1 \rangle.
$$

The resulting state is still symmetric but there is an interference term that prevents it from being a pure momentum state.
2.2.5. The quantum field operator. Bosonic creation and annihilation operators for the self-gravitating field are denoted as usual by $a^\dagger(\lambda)$ and $a(\lambda)$ respectively and satisfy the same commutation relations as the non-gravitating field (7). They act on the bosonic Fock space (17) by raising and lowering the number of particles according to the following maps:

$$a^\dagger(\lambda) : \mathcal{H}_m^0[\gamma_n] \rightarrow \mathcal{H}_{m+1}^0[\gamma_{n+1}], \quad \bigotimes_i^f \lambda_i; \gamma_n \mapsto \bigotimes_i^f \lambda_i \otimes \gamma_{n+1}$$

$$a(\lambda) : \mathcal{H}_m^0[\gamma_n] \rightarrow \mathcal{H}_{m-1}^0[\gamma_{n-1}], \quad \bigotimes_i^f \lambda_i; \gamma_n \mapsto \sum_{r=1}^n \delta(\lambda^{-1} \lambda_r) \bigotimes_i^{\prime \prime} \lambda_i; \gamma_{n-1}.$$  

(34)

Note that, whatever the value of $\lambda$, $a(\lambda)$ deletes always the link $\ell_n$ of $\gamma_n$ and then there is a redistribution of the momenta on the remaining links. It will be convenient to introduce the notations $a_+ (\lambda)$ and $a_- (\lambda)$ respectively for creation and annihilation operators.

Symmetries on the states induce symmetries on the operators $a_{\pm} (\lambda)$. Rotations work in the same way as for the classical case and we have

$$R(u) \triangleright a_{\pm} (\lambda) = a_{\pm} (u^{-1} \lambda).$$  

(35)

This action satisfies $R(u)(a_{\pm} (\lambda) \bigotimes_i^f \lambda_i; \gamma_n) = (R(u) \triangleright a_{\pm} (\lambda))(R(u) \bigotimes_i^f \lambda; \gamma_n)$ and therefore it is customary to write $R(u) \triangleright a_{\pm} (\lambda) = R(u)a_{\pm} (\lambda)R(u)^{-1}$.

Translations are a bit more involved to define creation and annihilation operators. We start by denoting $\mathcal{O}$ the set of operators on the Fock space defined as a finite product of creation and annihilation operators. The co-action $T^*$ extends to the space $\mathcal{O}$ and we have

$$T^* \triangleright a^\dagger (\lambda) \equiv a^\dagger (w_{\gamma-1} \lambda) \quad \text{with} \quad g = \lambda h(m)\lambda^{-1} \quad \text{and s.t.} \quad (36)$$

$$a^\dagger (w_{\gamma} \lambda) \bigotimes_i^f \lambda_i; \gamma_n = \bigotimes_i^f \lambda_i \bigotimes w_{\gamma} \lambda; \gamma_n \in \mathcal{W}_{n+1}[m; \gamma_{n+1}].$$  

(37)

The action on $a(\lambda)$ is obtained by adjointness and extends to the whole set $\mathcal{O}$ by linearity and morphism, i.e. $T^* \triangleright (a^\dagger (\lambda)a^\dagger (\lambda_2)) = T^* \triangleright (a^\dagger (\lambda))T^* \triangleright (a^\dagger (\lambda_2))$. Then, we see immediately that we have the consistency relation:

$$\langle T^* \bigotimes_i^f \lambda_i; \gamma_n | 0 \rangle = \left( T^* \triangleright \left( \prod_{i=1}^n a^\dagger (\lambda_i) \right) \right) 0.$$  

(38)

where $| 0 \rangle$ is the vacuum state. Moreover, given an element $A \in \mathcal{O}$ and a state $\langle s |$, we have the property:

$$\langle T^* (As) | = \sum_{T^*} \langle (T^* \triangleright A)(T^* s) | = \langle (T^* \triangleright A)(T^* s) |$$

(39)

because $\Delta_F (T^*) = T^* \otimes T^*$. Therefore, as for the classical case, it is consistent to write formally the action of $T^*$ on $A$ as $T^* \triangleright A = T^* AS(T^*)$ which means that

$$T^* \triangleright A)(\tilde{x}) = \sum_{(\tilde{x})} (T^*(\tilde{x}_{(1)}))A(S(T^*)(\tilde{x}_{(2)})).$$  

(40)

This identity is the deformed analogue of the classical relation $T(\tilde{x}) \triangleright A = T(\tilde{x})AT(-\tilde{x})$. In that sense, we say that creation and annihilation operators transform covariantly under translations.

Now, we have all the ingredients to construct the quantum field operator for a massive self-gravitating quantum field theory. Such a field $\phi$ is defined as a Fock space operator valued
function on $\mathbb{R}_G^3$ that satisfies the properties of locality, covariance and causality. We consider the self-adjoint state at the origin defined as

$$\phi(\tilde{0}) = \int d\mu(\lambda) (c_+(\lambda)a^\dagger(\lambda) + c_-(\lambda)a(\lambda)),$$

where $c_\pm$ are complex-valued functions on $S^2$. If we ask that $\phi(\tilde{0})$ is invariant under rotations $R(u)$, then the functions $c_\pm$ are in fact fixed to a constant $A_\pm$ that can be chosen real. The field $\phi(\tilde{x})$ at any point $\tilde{x} \in \mathbb{R}_G^3$ is obtained by covariance, i.e. $\phi \equiv T^*\phi(\tilde{0})$ and then

$$\phi(\tilde{x}) = \int d\mu(\lambda) (A_+ \exp(i \text{tr}(x\lambda h(m)\lambda^{-1}))a^\dagger(\lambda) + A_- \exp(-i \text{tr}(x\lambda h(m)\lambda^{-1}))a(\lambda).$$

As for the classical case, the causality requirement is meaningless in an Euclidean theory. All the same, we say that the field is causal if $\omega(\lambda) = 0 = [\phi(\tilde{x}), \phi(\tilde{y})]$, which implies that $A_+ = A_- = A_G$. The constant $A_G$ will be fixed later on.

At this point, we can compute the propagator of the self-gravitating quantum field theory given by the two-points function:

$$\Delta_G(\tilde{x}, \tilde{y}) = A_G^2 \int d\mu(\lambda) \exp(i \text{tr}((x-y)\lambda h(m)\lambda^{-1})) = A_G \frac{\sin(\|\tilde{x} - \tilde{y}\| \sin m)}{\|\tilde{x} - \tilde{y}\| \sin m}.$$

Quantum gravity is responsible for the renormalization of the mass of the quantum field which is no longer given by $m$ but by $\sin m$. Note that the mass is expressed in terms of the Planck mass $m_p = 1/G$ and the distances are expressed in terms of the Planck length $l_p = G (\hbar$ and $c$ are set to 1).

At the non-gravity limit $G \mapsto 0$, $\Delta_G$ tends to the classical propagator $\Delta$ according to the following limit:

$$\Delta_G(\tilde{x}) = A_G^2 \frac{\sin \left(\frac{m}{\|\tilde{x}\|} \sin m \right)}{\frac{m}{\|\tilde{x}\|} \sin m} \simeq A_G^2 A^2 \Delta(\tilde{x}) \left(1 - \frac{m^2 G^2}{6} (\|\tilde{x}\| \cotan(m\|\tilde{x}\|) - 1) \right) + O(G^3).$$

The choice $A_G = A = 1$ is consistent with the classical limit.

Note that $\Delta_G$ is in fact the symmetric propagator associated with $DSU(2)$ as defined in [7]. A symmetric propagator is a function on $\mathbb{R}_G^3 = DSU(2)/SU(2)$ defined by a simple representation labelled by a mass $m$ as follows:

$$K_m(\tilde{x}) = \int d\mu(\lambda) \omega(\lambda)(\pi_m(\tilde{x})\omega(\lambda)).$$

where $\omega \in \text{Fun}(S^2)$ is the unique $SU(2)$ co-invariant function. A direct comparison of the previous formula and the formula defining $\Delta_G$ shows the equality between the two functions.

It may be interesting to show this equality starting from the QFT two-points function as follows:

$$\Delta_G(\tilde{x}) = \langle 0 | \phi(\tilde{0}) \phi(\tilde{x}) | 0 \rangle = \sum_{\tilde{x}_1} \langle 0 | \phi(\tilde{0}) (T^*(\tilde{x}_1)) \phi(\tilde{0}) (S(T^*)(\tilde{x}_2)) | 0 \rangle = \langle \omega | T^*(\tilde{x}) \omega \rangle = \langle \omega | \pi_m(\tilde{x}) \omega \rangle = K_m(\tilde{x}).$$

We have successively used relation (40) for symmetry transformations of the quantum fields, the fact that $\omega(\lambda) = |\lambda(\phi(\tilde{0})0)\rangle$ is the normalized $SU(2)$ invariant function on $DSU(2)$ and finally the invariance of the vacuum state $|0\rangle$ under translations. Moreover, the representation $m$ appears in the last line of previous calculations for the field is implicitly of mass $m$. 
Finally, the free self-gravitating field in not so different from the free non-gravitating scalar field theory: the main difference is the mass term that becomes bounded for the usual mass $m$ is replaced by $\sin(m/m_p)$ where $m_p$ is the Planck mass. This result is consistent with those obtained in different contexts like spin-foam [6] or combinatorial quantization [9]. The theory becomes more interesting when one introduces self-interaction in the model. This is what we are going to do in the following section.

3. Self-interacting, self-gravitating quantum field

The previous section was devoted to the construction of a free self-gravitating quantum scalar field theory. By free, we mean that the field is only subjected to gravitational interactions but not to self-interaction. This section aims to introduce self-interactions into the theory and compute transition amplitudes. To do so, we leave the Hamiltonian formulation for a moment and switch into the Lagrangian formulation.

3.1. The Lagrangian formulation

We want to construct the free Lagrangian theory whose propagator gives back the Hadamard propagator we found in the Hamiltonian framework. As we showed in the previous section, quantum states of a self-gravitating spinless scalar field of mass $m$ are described in terms of functions $\phi$ on the quantum group $\text{DSU}(2)$ which are co-invariant under the action of $\text{SU}(2)$. Moreover, the propagator is defined as the symmetric intertwiner $K_m$ and therefore it is natural to view the action defining the Lagrangian of the free field as the following integral on $\text{DSU}(2)^{\otimes 2}$:

$$S_0[\phi] \equiv h^{\otimes 2} \left( \phi_1 K^{-1}_{12} \phi_2 \right),$$

where $h : \text{DSU}(2) \rightarrow \mathbb{C}$ is the Haar measure on $\text{DSU}(2)$ and we use the standard universal notations $\phi_1 = \phi \otimes \text{id}$ and so on. The inverse kernel $K^{-1}$ is defined such that

$$(\text{id} \otimes h \otimes \text{id}) K^{-1}_{12} K_{23} = (\text{id} \otimes h \otimes \text{id}) K_{12} K^{-1}_{23} = \text{id}_{13}.$$

When restricted on mass $m$ particles states, $K$ is in fact the identity and therefore is its own inverse ($K$ can be viewed as a projector [8]). One can immediately write this action as an integral on $\left( \mathbb{W}_G^3 \right)^{\otimes 2}$: it is also interesting to view the action as an integral on $C_m^{\otimes 2}$ where $C_m$ is the conjugacy class defined by $C_m \equiv \{ g \in \text{SU}(2) | \exists \lambda \in S^2, g = \lambda h(m) \lambda^{-1} \}$ and physically represents the space of momenta of one-particle states. The normalized measure on $C_m$ will be denoted by $d\mu_m(g)$. Therefore, in terms of positions or momenta variables, the free-field action reads

$$S_0[\phi] = \int d^3 \tilde{x} (\phi \ast \phi)(\tilde{x}) = \int d^3 \tilde{x} d^3 \tilde{y} \phi(\tilde{x}) \Delta_G(\tilde{x} - \tilde{y}) \phi(\tilde{y})$$

$$= \int d\mu_m(g) \tilde{\phi}(g) \tilde{\phi}(g^{-1}).$$

where $\phi$ and $\tilde{\phi}$ are related by the Fourier transform $\phi = \int d\mu_m(g) \tilde{\phi}(g) w_g$. Thus, the action of the self-gravitating quantum field theory can be viewed as a non-commutative quantum field theory, a non-local quantum field theory or a group field theory. These three pictures are completely equivalent and one can see the non-commutativity (or the non-locality) as the result of quantum gravity effects. Whatever the formulation we consider, it is immediate to show the identity:

$$\int [\mathcal{D} \phi] \phi(\tilde{x}) \phi(\tilde{y}) e^{i \mathcal{H}_0[\phi]} = \langle 0 | \phi(\tilde{x}) \phi(\tilde{y}) | 0 \rangle$$
which proves the equivalence between the Lagrangian and Hamiltonian descriptions. As usual, $[D\phi]$ is the normalized path integral measure.

Note that there is no dynamics (i.e. no kinetic term) in the theory for we deal with the Euclidean theory without causal structure. Indeed, from the very construction of our theory, the two-points function is in fact the deformed analogue of the Hadamard propagator which is a solution and not a Green function of the equations of motion. To recover a dynamic, one can replace the propagator $\Delta G$ by its associated Green function as it was done in [6]. In that case, one obtains the following dynamical Lagrangian (up to a global constant):

$$S_0[\phi] = \int d^3x \left[ \partial_i \phi \star \partial_i \phi - \sin^2(m)(\phi \star \phi)(\vec{x}) \right].$$

Recently, it was nicely shown [12] that the emergence of the dynamical part of the action can be viewed as a consequence of the implementation of the causality in the context of spin-foam models. This is the most natural way to have a dynamical theory but the quantum evolution becomes non-unitary which is a recurrent problem of non-commutative quantum field theories [6]. Apparently, the problem of unitarity is solved when one deals with Lorentzian theory instead. As we deal with Euclidean theory, we keep studying the non-dynamical theory that reproduces in fact the non-causal spin-foam models. This is the reason why we call our theory non-causal.

Up to now, we have concentrated only on the case of the free-field theory. Therefore, we do not have non-trivial transition amplitudes involving particles creation and annihilation processes in our model. In order to include such transition amplitudes, we need to add an interaction term $S_{int}$ to the action $S_0$. The only physical requirement we ask is that $S_{int}$ is invariant under the action of $DSU(2)$ as $DSU(2)$ appears to be the symmetry quantum group of our field theory. Furthermore, we restrict ourselves to a tri-valent interaction. As a consequence, the interaction term $S_{int}$ is a three-valent intertwining coefficient given by

$$S_{int}[\phi] = \frac{\lambda}{3!} \epsilon((\omega^\otimes 3)|\phi \otimes \phi \otimes \phi) \quad \text{with} \quad \epsilon(\vec{x}) = (\pi_m \otimes \pi_m \otimes \pi_m)\Delta^{(3)}(\vec{x}).$$

Note that $(\omega^3)$ is the symmetric intertwiner between three same simple representations labelled by $m$ that takes value in the space of functions on $E_3^G \subset DSU(2)$; recall that $\omega$ is the normalized $SU(2)$ invariant vector and $h$ is the Haar measure on $DSU(2)$. To be more concrete, one can write the interacting term in positions or momenta variables as follows:

$$S_{int} = \frac{\lambda}{3!} \int d^3x d^3y d^3z V_{int}(\vec{x}, \vec{y}, \vec{z}) \phi(\vec{x})\phi(\vec{y})\phi(\vec{z})$$

As for the free action $S_0[\phi]$, the interacting term can be written as a non-local interaction as shown by the following expression:

$$S_{int} = \int d^3x d^3y d^3z V_{int}(\vec{x}, \vec{y}, \vec{z}) \phi(\vec{x})\phi(\vec{y})\phi(\vec{z})$$

with

$$V_{int}(\vec{x}, \vec{y}, \vec{z}) = \frac{\lambda}{3!} \int d\mu_m(g_1) d\mu_m(g_2) w_{g_1}(\vec{x}) w_{g_1 g_2}(\vec{y}) w_{g_1 g_2}(-\vec{z}).$$

At the non-gravity limit ($G \rightarrow 0$), it is clear that the interaction becomes commutative.

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2 Private communication with E Livine.
The theory defined by the total action $S[\phi] = S_0[\phi] + S_{\text{int}}[\phi]$ has been introduced and studied in the context of spin-foam models in [6]. From our point of view, $S[\phi]$ is clearly invariant under the action of $DSU(2)$ because it is constructed from $DSU(2)$ intertwiners: in fact, the quadratic term $S_0$ is a two-valent intertwiner and the cubic term $S_{\text{int}}$ is obviously, by construction, a three-valent intertwiner. In order to show the invariance explicitly, we compute $U(\vec{x}, u)S[\phi]$, i.e. how the action $S[\phi]$ transforms under the action of an element $(\vec{x}, u) \in DSU(2)$. For that purpose, it is more convenient to work in the momenta representation (the group field version of the action) and we have

$$U(\vec{x}, u)S_0[\phi] = \int d\mu_m(g) \left( \pi_m^{\otimes 2} \Delta(\vec{x}, u) \phi \otimes \phi \right) (g \otimes g^{-1}) = \epsilon(\vec{x}) S_0[\phi]$$

$$U(\vec{x}, u)S_{\text{int}}[\phi] = \int d\mu_m(g_1) \, d\mu_m(g_2) \, d\mu_m(g_3) \, \delta(g_1 g_2 g_3)$$

$$\times \left( \pi_m^{\otimes 3} \Delta^{(3)}(\vec{x}, u) \phi \otimes \phi \otimes \phi \right) (g_1 \otimes g_2 \otimes g_3)$$

$$= \epsilon(\vec{x}) S_{\text{int}}[\phi],$$

where $\epsilon(\vec{x}) = 1$ is the co-unit on $\mathbb{B}_G^3$. The action transforms under the trivial representation of $DSU(2)$ and is, as a consequence, $DSU(2)$-invariant. The action $S[\phi]$ is also invariant under braiding. Indeed, the braiding action on the space of multi-particles states induces a braiding map at the level of the action defined by the following:

$$S_0[\phi] \equiv S_0[\phi \otimes \phi] \mapsto \tilde{S}_0[R(\phi \otimes \phi)]$$

$$S_{\text{int}}[\phi] \equiv S_{\text{int}}[\phi \otimes \phi \otimes \phi] \mapsto \tilde{S}_{\text{int}}[R(\phi \otimes \phi \otimes \phi)],$$

where we have emphasized the fact that $S_0$ is quadratic whereas $S_{\text{int}}$ is cubic in the field and $R$ denotes the action of the $DSU(2)$ $R$-matrix. This invariance is in fact a direct consequence of the fact that symmetric intertwining coefficients are left invariant under braiding (see [11] for example). Note that there is an apparent ambiguity in the definition of the braiding action on $S_{\text{int}}$ for one can act on any pair of the three arguments defining $S_{\text{int}}$. In fact, one can easily show that the braiding action is unchanged whatever the choice of the pair of arguments. A corollary of these invariances is that transition amplitudes are invariant under $DSU(2)$ and under braiding as expected from the beginning.

Before computing examples of transition amplitudes, we generalize our model to the case where the mass $m$ of the field is not fixed. Such a generalization is immediate for one only has to relax the condition that the field $\phi$ is a function on the conjugacy class $C_m$ and to allow the field to be a function on the whole group $SU(2)$. The ‘dynamics’ of this theory is governed by the following group field action (in momenta variables):

$$S_f[\phi] = \int dg_1 \, dg_2 \, \phi(g_1) \delta(g_1 g_2) \phi(g_2) + \lambda \int \frac{d^3 g_1}{3!} \, dg_2 \, dg_3 \, \delta(g_1 g_2 g_3) \phi(g_1) \phi(g_2) \phi(g_3).$$

3.2. Example of transition amplitudes

This section is devoted to studying some properties of our self-gravitating, self-interacting quantum field theory through some concrete examples. In particular, we will focus on the computation of the propagator (two-points functions) and of the vertex (three-points functions) first-order terms (in the coupling $\lambda$). One could see quantum gravity effects on the field propagator and on some physical processes involving particles creations and annihilations.
Before going to the details, we start by giving the Feynmann rules of the free theory in the picture of figure 2. It will be convenient to compare the results of Feynmann graph evaluations with the classical case. For that purpose, we give the Feynmann rules for the non-gravitating quantum field theory that can be obtained by the non-gravitational limit of the previous rules (figure 2). In particular, the normalization factors for the vertex and for the propagator are chosen such that classical Feynmann graph evaluations fit correctly with the no-gravity limit of gravitational Feynmann graph evaluations: $N = 2\pi^2$ and $P(m) = 4\pi m^2$.

Given a graph $\Gamma$ we will note $I_0(\Gamma)$ and $I_G(\Gamma)$ the classical and gravitational (deformed) evaluations. Note that there is an overall factor $N^{-1}$ in the gravitational evaluation of any graph $\Gamma$ in the sense that $I_G(\Gamma) = N^{-1} I_0(\Gamma)$ where $I_0(\Gamma)$ is the very Feynmann graph evaluation (according to the gravitational Feynmann rules).

3.2.1. Corrections to the propagator. Quantum gravity corrections to the free propagator have already been computed in the previous section (44). This section aims at illustrating quantum gravity effects on the propagator of the self-interacting theory. Formally, the propagator of the self-interacting theory is given by a power series in the coupling constant $\lambda$ as follows:

$$\langle \phi(\vec{x}_1)\phi(\vec{x}_2) \rangle \equiv \sum_{\Gamma} \lambda^{v(\Gamma)} \frac{1}{\text{Sym}(\Gamma)} F_{\langle\Gamma\rangle}(\vec{x}_1,\vec{x}_2).$$

The sum runs over all Feynmann graph $\Gamma$ with two open edges, $v(\Gamma)$ the number of vertices, $\text{Sym}(\Gamma)$ the symmetry factor and $F_{\langle\Gamma\rangle}$ is the evaluation of the graph viewed as a function of the position variables $\vec{x}_1$ and $\vec{x}_2$. Here we are not interested in convergence and renormalization issues and concentrate only on lowest order terms in the series. The lowest term $F_P$ ($P$ being the trivial graph that contains only one edge) is the free propagator and is given by the function $\Delta_f(\vec{x}_1,\vec{x}_2)$ computed in previous sections ($I$ is 0 or $G$ if we are dealing with the non-gravitating or the self-gravitating theory).

Quantum gravity corrections to higher order terms in the series (56) are a bit more involved to compute. We want to illustrate them with the non-trivial example of the Feynmann graph $\Lambda$ drawn in figure 3.

In general, we assume that incoming and outgoing particles have the same mass $m$ whereas the particles inside the loop have different masses fixed to the values $m_1$ and $m_2$. An immediate calculation shows that $F_{\langle\Lambda\rangle} \equiv f_{\langle\Lambda\rangle}(m, m_1, m_2) \Delta_f(\vec{x}_1,\vec{x}_2)$. In the gravitational
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Figure 3. One-loop corrections of the propagator and of the vertex: the graphs are respectively denoted by $\Lambda$ and $T$. Incoming and outgoing particles are pure momenta states whose momenta are the group elements $g_1, g_2$ and $g_3$.

case, the coefficient $f^G_{(\Lambda)}$ reads

$$ f^G_{(\Lambda)}(m, m_1, m_2) \equiv \int d\mu_{m_1}(g_1) d\mu_{m_2}(g_2) \delta(g^{-1} g_1 g_2) $$

$$ = \sum_{\ell=0}^{\infty} \frac{1}{\ell} \sin(\ell m) \sin(\ell m_1) \sin(\ell m_2) = \frac{\pi}{4} \frac{Y(m_1, m_2, m_3)}{\sin m_1 \sin m_2 \sin m_3}. $$

(57)

$Y(m_1, m_2, m_3)$ is equal to 1 if $m_1, m_2, m_3$ satisfy triangular inequalities and is equal to 0 otherwise. It is interesting to compare this expression to the classical expression $f^0_{(\Lambda)}$. A similar calculation shows that

$$ f^0_{(\Lambda)}(m, m_1, m_2) \equiv N^2 \int \frac{d^3 \vec{p}_1}{P(m_1)} \frac{d^3 \vec{p}_2}{P(m_2)} \delta(3)(\vec{p}_1 + \vec{p}_2 + \vec{p}) $$

$$ = \frac{N^2}{2\pi^2} \int \frac{dx \sin(xm) \sin(xm_1) \sin(xm_2)}{m_1m_2m_3} = \frac{\pi}{4} \frac{Y(m_1, m_2, m_3)}{m_1m_2m_3}. $$

(58)

We have denoted by $\vec{p}$ the incoming (and outgoing) momentum whose mass is $m$. Note that the expressions of $f^G_{(\Lambda)}$ and $f^0_{(\Lambda)}$ are very similar: the first is expressed as a discrete series whereas the second is an integral over the space variable $x$. This suggests, as expected, that spacetime becomes discrete at the Planck scale (at least for Euclidean gravity) and the spectrum of the length variable $x$ is given by $(2I + 1)\ell_p$.

For the massive quantum field, we are exclusively interested in the case where all the masses are fixed to the same value $m$. In that case, the first-order gravitational corrections to the Feynmann graph $\Lambda$ are trivially obtained from quantum gravity corrections of the free propagator and from the following corrections:

$$ f^G_{(\Lambda)}(m, m, m) \equiv f^G_{(\Lambda)}(m, m, m) = \frac{\pi}{4 \sin^3 m} = \frac{1}{G^3} \left(1 + \frac{m^2 G^2}{2} + O(G^4)\right) f^0_{(\Lambda)}(m, m, m). $$

(59)

Note the presence of the overall factor $G^{-3}$ in the expansion of $f^G_{(\Lambda)}(m)$; then the coefficient that admits the right classical limit is $G^3 f^G_{(\Lambda)}(m)$. In the Hamiltonian point of view, this means that the two-point function for the interacting quantum field theory is modified due to the quantum gravity effects and is given (at the lowest order in the couplings $\lambda$ and $G$) by

$$ \langle 0| \phi(\vec{0}) H_{\text{int}} \phi(\vec{x}) \rangle = \left(1 + G^3 \frac{\lambda^2}{2} f^G_{(\Lambda)}(m) + O(\lambda^3)\right) \Delta_G(\vec{x}). $$

(60)

The formal notation $H_{\text{int}}$ is for the self-interaction of the self-gravitating quantum field. Quantum gravity effects appear in the expression of the free propagator $\Delta_G(\vec{x})$ and also in the expression of $\Delta_G^{(1)}$, i.e. the perturbative corrections due to self-interaction.
3.2.2. Corrections to the three-point function. The three-point function \( \langle \phi(\vec{x}_1)\phi(\vec{x}_2)\phi(\vec{x}_3) \rangle \) is defined as a series in the coupling constant \( \lambda \) which can be formally written as

\[
\langle \phi(\vec{x}_1)\phi(\vec{x}_2)\phi(\vec{x}_3) \rangle \equiv \sum_{\Gamma} \lambda^{\nu(\Gamma)} \frac{1}{\text{Sym}(\Gamma)} H_{(\Gamma)}(\vec{x}_1, \vec{x}_2, \vec{x}_3),
\]

where the sum runs over Feynman graph \( \Gamma \) with three open edges. As for the previous section, we are particularly interested in computing some lowest order terms \( H_{(\Gamma)} \) in the series: the evaluation of the graph \( V \) is the vertex of the free theory in the position variables, the evaluation of the tetrahedron graph \( T \) is a lowest order correction to the self-interacting theory vertex. We will introduce the index \( I \in \{0, G\} \) in the functions \( H_{(\Gamma)} \) of the series (61) to specify the ‘classical’ and the self-interacting theories.

Let us start by computing \( H_{(V)}(\vec{x}_1, \vec{x}_2, \vec{x}_3) \). A direct application of Feynmann rules leads to the following expression:

\[
H_{(V)}^0(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \int \frac{d^3 p_1}{\text{Sym}(\Gamma_1)} \frac{\delta(p_1 - m_1)}{P(m_1)} e^{i\vec{p}_1 \cdot \vec{x}_1} g^{(3)}(\vec{p}_1 + \vec{p}_2 + \vec{p}_3).
\]

This property is easy to see from expression (62) but it is more transparent in the following expression that we obtain by decomposing the delta function \( \delta^{(3)}(\vec{p}) \) in Fourier modes:

\[
H_{(V)}^0(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \frac{1}{(2\pi)^3} \frac{1}{m_1 m_2 m_3} \int \frac{d^3 \vec{x}_0}{\sin(m_1 \vec{x}_{10}) \sin(m_2 \vec{x}_{20}) \sin(m_3 \vec{x}_{30})} \sin(\vec{x}_{10} \cdot \vec{x}_{20} \cdot \vec{x}_{30}).
\]

The above expression is interesting for it explicitly exhibits the symmetry between the position variables \( \vec{x}_i \) but it is an indefinite three-dimensional integral. There exist other equivalent expressions that apparently break this symmetry but may be more useful because they are expressed as a single definite integrals. To obtain such an expression, one starts by integrating over the momentum variable \( \vec{p}_3 \) in expression (62) and after some simple calculations, one shows that

\[
H_{(V)}^0(\vec{x}_1, \vec{x}_2, \vec{x}_3) = N_0 \int d^2 \vec{n}_1 d^2 \vec{n}_2 \delta(\vec{n}_1 \cdot \vec{n}_2 - \cos S_0) e^{i(\vec{A}_1 \cdot \vec{n}_1 + \vec{A}_2 \cdot \vec{n}_2)},
\]

where \( \vec{A}_1 \equiv m_1(\vec{x}_1 - \vec{x}_3) \) and \( \cos S_0 \equiv (m_3^2 - m_1^2 + m_2^2) / (2m_1 m_2) \) with \( S_0 > 0 \) and the normalization factor is given by \( N_0^{-1} = 4\pi m_1 m_2 m_3 \). As shown in appendix C, the function \( H_{(V)}^0 \) can be simplified to the following expression:

\[
H_{(V)}^0(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \frac{N_0}{4} \int_0^{\pi} d\theta \sin \theta e^{iA_1 \cos U \cos \theta} J_0(A_1 \sin U \sin \theta) e^{iA_2 \cos S \cos \theta} J_0(A_2 \sin S \sin \theta)
\]

with \( A_i = \| \vec{A}_i \| \), the angle \( U > 0 \) is defined by the relation \( \vec{A}_1 \cdot \vec{A}_2 = A_1 A_2 \cos U \) and \( J_0 \) is the Bessel function of the first kind (see appendix C for details). This expression will appear more convenient when we compare to the self-gravitating analogue. In some particular cases, one can perform the previous integral explicitly:
(i) If $S_0 = 0 \iff m_1^2 = (m_1 + m_2)^2 \iff m_3 = m_1 + m_2 (m_3 > 0)$ then

$$H^0_{(V)}(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \frac{1}{8\pi m_1 m_2 m_3} \sin\left(\frac{\|m_1 \vec{x}_1 + m_2 \vec{x}_2 - m_3 \vec{x}_3\|}{\|m_1 \vec{x}_1 + m_2 \vec{x}_2 - m_3 \vec{x}_3\|}\right).$$

Note however that this case is not of a particular physical interest for we will be interested to the cases where the masses are all fixed to the same value (which is obviously incompatible with the condition $S_0 = 0$).

(ii) If $U = 0 \iff \vec{A}_1$ and $\vec{A}_2$ are colinear then

$$H^0_{(V)}(\vec{x}, \vec{x}_2, \vec{x}_3) = \frac{1}{8\pi m^2} \sin\left(\frac{m \|\vec{x} - \vec{x}_2\|}{\|\vec{x} - \vec{x}_2\|}\right).$$

In particular, we see that $H^0_{(V)}(\vec{x}, \vec{x}_2, \vec{x}_3) = H^0_{(V)}(0, \vec{0}, \vec{0})$ as expected.

Before studying the gravitational case, let us mention that one can re-express the function $H^0_{(V)}$ as a series involving Gegenbauer polynomials and Bessel functions of half-integer order as shown in appendix C.

Let us now compute the function $H^G_{(V)}(\vec{x}_1, \vec{x}_2, \vec{x}_3)$. By definition, it is given by

$$H^G_{(V)}(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \frac{1}{N} \int \prod_{i=1}^3 dg_i \delta_m(g) e^{i\pi(g, g_1)} \delta(g_1, g_2, g_3).$$

(70)

We recall that $N = 2\pi^2$. In order to make a clear comparison between $H^G_{(V)}$ and $H^0_{(V)}$, it is convenient to decompose the group elements $g_i = \cos m_1 \vec{\ell} + \sin m_1 \vec{n}_i \cdot \vec{\sigma}$ where $\vec{\ell}$ and $\vec{\sigma}$ are respectively the identity and Pauli matrices; $\vec{n}_i$ is an unit vector. With this parametrization, the measure reduces to the following form:

$$\delta_m(g) \ dg_i = d^2\vec{n}_i \quad \text{with} \quad \delta_m(g) = \pi / \sin m \delta(\text{tr}(g) - 2 \cos m).$$

(71)

Using these results, one can show after some calculations that $H^G_{(V)}$ can be written in a form similar to the classical expression (62) and we have the following formula:

$$H^G_{(V)}(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \frac{1}{\cos m_3} \int \prod_{i=1}^3 d^3p_i \frac{\delta(p_i - \sin m_1)}{P(\sin m_1)} \cdot \delta(3) \times \left(\vec{p}_3 + \vec{p}_1 \cos m_2 + \vec{p}_2 \cos m_1 + \vec{p}_1 \wedge \vec{p}_2\right).$$

(72)

Details can be found in appendix C. It is clear that $H^G_{(V)}$ tends to $H^0_{(V)}$ at the no-gravity limit $G \to 0$ (up to some powers of $G$ we will precise later on). Another immediate constatation is that $H^G_{(V)}$ does no longer satisfy the property (63): this can be interpreted as a consequence of the non-commutativity of the space coordinates. In order to make a quantitative comparison with the classical counterpart $H^0_{(V)}$, it is convenient to write $H^G_{(V)}$ as a form similar to (65) as follows:

$$H^G_{(V)}(\vec{x}_1, \vec{x}_2, \vec{x}_3) = N_G \int d^2\vec{n}_1 d^2\vec{n}_2 \delta(\vec{n}_1 \cdot \vec{n}_2 - \cos S_G) \exp(i(\vec{B}_1 \cdot \vec{n}_1 + \vec{B}_2 \cdot \vec{n}_2 + C \cdot \vec{n}_1 \wedge \vec{n}_2)),$$

(73)

where we have introduced the notations:

$$\vec{B}_1 \equiv \sin m_1 (\vec{x}_1 - \cos m_2 \vec{x}_3), \quad \vec{B}_2 \equiv \sin m_2 (\vec{x}_2 - \cos m_1 \vec{x}_3), \quad C \equiv -\sin m_1 \sin m_2 \vec{x}_1.$$
The normalization factor is given by $N_G^{-1} = 4\pi \sin m_1 \sin m_2 \sin m_3$ and the angle $S_G > 0$ is fixed by the relation $\cos S_G = (\cos m_1 \cos m_2 - \cos m_3)/(\sin m_1 \sin m_2)$. One can make the integration over the variable $\vec{n}$ and one shows that $H_{(V)}^G$ reduces to the following form:

$$H_{(V)}^G(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \frac{N_G}{2} \int d^2\vec{n} \ e^{i(\vec{B}_1 \cdot \vec{n})} \sqrt{\frac{A_2^2 - (\vec{A}_2 \cdot \vec{n})^2}{A_2^2 - (\vec{A}_2 \cdot \vec{n})^2}} J_0(\sin S_G R(\vec{n})), \quad (74)$$

where $R(\vec{n}) = \sqrt{B_3^2 + C^2 - (\vec{B}_2 \cdot \vec{n})^2 - (\vec{C} \cdot \vec{n})^2 + 2\vec{B}_2 \cdot \vec{C} \cdot \vec{n}}$. It is clear that (74) simplifies, when one takes the no-gravitational limit, and gives back expression (66) of $H_{(V)}^0$.

In order to compute the classical limit and the quantum gravity corrections of $H_{(V)}^G$, we start by recalling that the masses and the positions are given in terms of the Planck mass and the Planck length: therefore, we replace in the expression of $H_{(V)}^G$ the dimensionless variables $\vec{x}_i$ and the masses $m_i$ respectively by $\vec{x}_i/l_p$ and $m_i/m_2$, where $m_p = G^{-1}$ and $l_p = G$. In that case, $\vec{x}_i$ and $m_i$ become dimensionful variables. After some calculations (whose details are presented in appendix C), one shows that the development of $H_{(V)}^G$ around $G = 0$ is given by

$$G^3 H_{(V)}^G(\vec{x}_1, \vec{x}_2, \vec{x}_3) = H_{(V)}^0 + G N_0 m_1 m_2^2 H_{(V)}^{G(1)}(\vec{x}_1, \vec{x}_2, \vec{x}_3) + O(G^2), \quad (75)$$

where the first-order correcting term $H_{(V)}^{G(1)}$ reads

$$H_{(V)}^{G(1)}(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \int d^2\vec{n} \ e^{i(\vec{A}_1 \cdot \vec{n})} \sqrt{\frac{A_2^2 - (\vec{A}_2 \cdot \vec{n})^2}{A_2^2 - (\vec{A}_2 \cdot \vec{n})^2}} J_1(\sin S_G R(\vec{n})) \frac{\vec{x}_2 \wedge \vec{x}_3 \cdot \vec{n}}{A_2 \cdot (\vec{A}_2 \cdot \vec{n})^2}. \quad (76)$$

The variables have been introduced previously and $J_1$ is the first-order Bessel function. Note that one has to rescale $H_{(V)}^0$ with a factor $G^3$ (as in the propagator case) in order to have a good classical limit. In that case, we remark that the lowest correcting term is generically proportional to $G$ whereas the lowest correction to the propagator is proportional to $G^2$. In order to simplify the previous expression, we choose a tri-dimensional basis where $A_2 = A_2(1, 0, 0)$ and, if we still denote by $U$ the positive angle between $\vec{A}_1$ and $\vec{A}_2$, we obtain that

$$H_{(V)}^{G(1)}(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \frac{1}{A_2} \vec{x}_2 \wedge \vec{x}_3 \cdot \vec{\nabla}_\vec{x} I. \quad (77)$$

where $\vec{\nabla}_\vec{x}$ is the gradient with respect to the coordinates $\vec{x}$ and $I$ is an integral viewed as a function of the vector $\vec{A}_1$. After some calculations (see appendix C for details), we show that $I$ is given by

$$I = \int_0^1 \frac{dx}{\sqrt{1 - x^2}} e^{i(\vec{A}_1 \cos T + A_2 \cos S_G x)} J_0(\vec{A}_1 \sin T \sqrt{1 - x^2}) J_1(\vec{A}_2 \sin S_G \sqrt{1 - x^2}). \quad (78)$$

This is the more general expression of the lowest order correcting term due to quantum gravity to the classical three-point function. One immediately remarks that the term $H_{(V)}^{G(1)}$ vanishes in some particular cases: if $\vec{x}_2$ and $\vec{x}_3$ are colinear or null and also if $\vec{A}_1$ and $\vec{A}_2$ are colinear (one can see this property from expression (77)). In such cases, one has to go further in the computation of the correcting terms and it is easy to see that the corrections are of order $G^2$. We would not give the expression of correcting term of order $G^2$. We will instead give the expression of $H_{(V)}^{G(1)}$ in some simple cases from which it is easier to compute quantum gravity corrections.

(i) If $S_G = 0 \iff \cos(m_1 + m_2) = \cos m_3 \iff m_3 = m_1 + m_2 (m_3 > 0)$ then

$$H_{(V)}^{G(1)}(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \frac{1}{8\pi \sin m_1 \sin m_2 \sin m_3} \frac{\sin \|m_1 \vec{x}_1 + \sin m_2 \vec{x}_2 - \sin m_3 \vec{x}_3\|}{\sin \|m_1 \vec{x}_1 + \sin m_2 \vec{x}_2 - \sin m_3 \vec{x}_3\|}. \quad (79)$$

This case has no physical interest.
(ii) If the coupling vanishes i.e. $\tilde{C} = 0 \iff \tilde{x}_3 = 0$ (as $m_1 \neq 0 \neq m_2$) then $H_{(V)}^G$ takes exactly the same form as $H_{(V)}^G$:

$$H_{(V)}^G(x_1, x_2, x_3) = \frac{N_G}{4} \int_0^\pi d\theta \sin \theta e^{iB_1 \cos U \cos \theta} J_0(B_1 \sin U \sin \theta)$$

$$\times e^{iB_2 \cos S_G \cos \theta} J_0(B_2 \sin S_G \sin \theta),$$

(80)

where $U > 0$ is the angle between $\mathbf{B}_1$ and $\mathbf{B}_2$. Therefore, this integral can be performed explicitly in the particular case where $\mathbf{B}_1$ and $\mathbf{B}_2$ are colinear and one obtains for $H_{(V)}^G$ the following expression:

$$\frac{N_G}{2} \sin \sqrt{\sum_{i=1}^2 x_i^2 \sin^2 m_i + \alpha x_1^2 + 2 \sum_{i<j}^2 \mathbf{x}_i \cdot \mathbf{x}_j (\cos m_i \cos m_j - \cos m_2)}$$

$$\sqrt{\sum_{i=1}^2 x_i^2 \sin^2 m_i + \alpha x_1^2 + 2 \sum_{i<j}^2 \mathbf{x}_i \cdot \mathbf{x}_j (\cos m_i \cos m_j - \cos m_2)},$$

(81)

where $\alpha = \cos^2 m_1 + \cos^2 m_2 - 2 \cos m_1 \cos m_2 \cos m_3$. This expression simplifies in the particular case where the masses are fixed to the same value $m$. It is important to remark that $H_{(V)}^G(\tilde{x}, \tilde{x}, \tilde{x})$ does depend on $\tilde{x}$ contrary to the classical case and explicitly reads

$$H_{(V)}^G(\tilde{x}, \tilde{x}, \tilde{x}) = \frac{1}{8\pi \sin^2 m} \frac{\sin(\beta(m)x)}{\beta(m)x}$$

with $\beta(m) = \sqrt{2 - 2 \cos m (\cos^2 m + 3 \cos m - 3)}$.

This fact can be interpreted as a consequence of the non-commutativity of the space at the Planck length.

We finish this section by computing the vertex one-loop correction contribution to the vertex associated with the tetrahedron graph $T$ represented in figure 3. An immediate calculation shows that $H_T = h_T^G(m_1, \ldots, m_6) H_{(V)}^G(x_1, x_2, x_3)$: $m_1, m_2, m_3$ are the masses of outgoing particles and $m_4, m_5, m_6$ those of the three particles in the loop. The coefficients $h_T^G(m_1, \ldots, m_6)$ are in fact $(6j)$ symbols between simple representations of $DSU(2)$ ($I = G$) or $ISU(2)$ ($I = 0$). These coefficients are computed in [11] and, according to our convention, we have

$$Y(m_1, m_2, m_3) h_T^G(m_i) = \frac{\pi}{8 \sin m_4 \sin m_5 \sin m_6} \frac{1}{\sqrt{D(m_i)}}$$

(83)

with

$$D(m_i) =$$

$$\begin{vmatrix}
1 & \cos m_1 & \cos m_2 & \cos m_3 \\
\cos m_1 & 1 & \cos m_6 & \cos m_5 \\
\cos m_2 & \cos m_6 & 1 & \cos m_4 \\
\cos m_3 & \cos m_5 & \cos m_4 & 1
\end{vmatrix}$$

(84)

$D(m)$ is a Graam determinant. In the (physically interesting) case where all the masses $m_i$ are equal to the same value $m$, the Graam determinant simplifies and reads $D(m) = (1 - \cos m)^3 (3 \cos m + 1)^{-1/2}$.

In the non-gravitational case, the evaluation of the same graph is given in terms of the volume $V(m_1, \ldots, m_6)$ of a tetrahedron, whose length is fixed to the values $m_i$, by the formula

$$Y(m_1, m_2, m_3) h_T^0(m_i) = \frac{\pi}{48 m_1 m_2 m_3} \frac{1}{V(m_i)}.$$

(85)

When all the masses are fixed to the value $m$, the tetrahedron is regular and its volume is simply given by $V(m) = \sqrt{2} m^3/12$. Using formulae (83) and (85), one obtains immediately that

$$h_T^G(m, \ldots, m) \equiv h_T^G(m) = \frac{1}{G^3} \left( 1 + \frac{13}{16} m^2 G^2 + O(G^3) \right) h_T^0(m).$$

(86)
Then, quantum gravity corrections to the vertex one-loop corrections are immediately obtained from the previous expression (beware with the overall factor $G^{-6}$).

### 3.2.3. Amplitudes and invariance under braidings

To precise what we mean by invariance of the amplitudes under braidings, we start by interpreting the amplitudes computed previously as $S$ matrix elements involving in and out states we have described in section 2.2.

Let us focus on the case of the three-point function. One can naturally interpret the three-point functions as the deformed Fourier transform of a certain $S$-matrix element:

$$\langle \phi(\vec{x}_1) \phi(\vec{x}_2) \phi(\vec{x}_3) \rangle = \int \prod_{k=1}^3 d\mu(\lambda_k) e^{it^r(\lambda_k h(m) \lambda_k^{-1})} \delta \left( \prod_{k=1}^3 \lambda_k h(m) \lambda_k^{-1} \right).$$  \hspace{1cm} (87)

The $S$-matrix element $\langle \lambda_2 \otimes \lambda_3; \ell_2 \otimes \ell_3 | \lambda_1 \otimes \ell_1 \rangle$ gives the amplitude between the in-state $|\lambda_1 \otimes \ell_1 \rangle$ and the out-state $|\lambda_2 \otimes \lambda_3; \ell_2 \otimes \ell_3 \rangle$ when we have taken into account the gravitational interaction and the self-interaction. We have implicitly assumed that the observer has zero mass ($m_0 = 0$); the field has a mass $m$.

The $S$-matrix element is, up to a factor $A_G(m)$, a simple intertwining coefficient and therefore the three-point function reads

$$\langle \phi(\vec{x}_1) \phi(\vec{x}_2) \phi(\vec{x}_3) \rangle = A_G(m) \int \prod_{k=1}^3 d\mu(\lambda_k) e^{it^r(\lambda_k h(m) \lambda_k^{-1})} \delta \left( \prod_{k=1}^3 \lambda_k h(m) \lambda_k^{-1} \right).$$  \hspace{1cm} (88)

We have shown in section 2.2 that the action of the braiding on a given particles-spin-network state reduces to an action of the $R$-matrix on the state and, as simple intertwiners are ‘invariant’ under braiding, we have

$$\langle \phi(\vec{x}_1) \tau(\phi(\vec{x}_2) \phi(\vec{x}_3)) \rangle = \langle \phi(\vec{x}_1) \phi(\vec{x}_3) \phi(\vec{x}_2) \rangle.$$  \hspace{1cm} (89)

Therefore, the result of the braiding on the three-points function trivially reduces to the permutation of the arguments $\vec{x}_2$, $\vec{x}_3$; the amplitude $A_G(m)$ is unchanged. This property is illustrated in figure 4.

In that sense, we claim that $n$-point functions are invariant under braidings.
4. Discussion and generalization

This paper proposes a model for a three-dimensional Euclidean self-gravitating non-causal quantum field theory. The basic idea of our construction is to quantize first gravitational degrees of freedom (using LQG techniques) before quantizing matter field degrees of freedom: $n$-particle states in a quantum background are defined as particle-spin-network states and form a physical Hilbert space; the quantum self-gravitating field is described as usual in QFT as an operator acting on the self-gravitating Fock space, i.e. the infinite tower of $n$-particles physical Hilbert spaces. We focus only on the case of massive spinless particles. The resulting theory is a quantum field theory whose symmetry group is no longer the isometry group of the flat Euclidean space $\mathbb{E}^3$ but the quantum group $DSU(2)$ which can be viewed as a deformation of the classical group $ISU(2)$. In the Lagrangian point of view, it is clear that the theory is in fact a group field theory that can be easily written as a non-commutative quantum field theory (as it was first proposed by [6]). The non-commutativity is a consequence of quantum gravity effects. Then, we generalize the model by introducing a self-interacting potential, whose coupling constant is $\lambda$, that makes it more interesting.

The nice feature of our model is that we can explicitly compute any terms of the series in the coupling $\lambda$ defining $n$-point functions and then we can evaluate quantum gravity corrections. We have illustrated this property in the computation of lowest order terms of the propagator (two-point function) and the vertex (three-point functions) of the self-gravitating self-interacting quantum field theory. Lowest order (in the Newton constant $G$) quantum gravity corrections are explicitly computed.

Nevertheless, the model is not physical for it describes a three-dimensional Euclidean QFT. How to generalize to four-dimensional case is a completely open question even if LQG techniques work very well (at least at the kinematical level) even in four dimensions. How to construct a similar model with a Lorentzian signature of spacetime is a more suitable question. Making the theory Lorentzian is just a matter of technicality for one has to replace the quantum double $DSU(2)$ by its Lorentzian counterpart. It would be very nice to study and develop such a model for different reasons: recovering a good notion of causality, introducing a time dimension and therefore defining real dynamics for the quantum field. In the Euclidean model, there are indeed no dynamics and there is no canonical way to recover certain dynamics. In fact, our paper presents a canonical construction of a group field theory that reproduces non-causal spin-foam amplitudes of 3D gravity coupled to matter field. In that sense, one can say that our model is a topological theory and does not behave as usual causal quantum field theory. In particular, the propagator is the gravitational analogue of the Hadamard propagator of the scalar field and not the usual Feynman propagator. A natural way to recover the gravitational analogue of the Feynman propagator is to add by hand a kinetic term to the action as was done in [6] or to impose a causality relation at the level of the spin-foam model as was done in [12]. The problem is that the resulting theory is apparently non-unitary. This might come from the fact that there is no canonical way to define a causality relation for Riemannian quantum field theory. The behaviour of the Lorentzian model seems to be much nicer. We hope to present the Lorentzian model in great details in a future article.

Even in the Euclidean model, many points remain to be understood and deserve to be studied. First, we could generalize the construction to the case where the space admits a non-trivial topology: in that case, the model would admit new types of degrees of freedom and one could compute topology changing amplitudes in the presence of a matter field. Then, we could consider spinning particles and could try to describe the self-gravitating quantum field theory. Therefore, we could in principle adapt our construction to describe a self-gravitating theory for fermions, gauge vectors and coupling theory involving different types of particles. We are
currently working in that direction. Finally, we could also construct the model in the presence of a cosmological constant: in that case, the quantum group structure would be changed into a more interesting quantum group like $U_q(su(2))$ where the quantum deformation $q$ is related to the value of the cosmological constant. We are currently working in that direction.

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Et surtout . . . ma petite fille Inès.

Appendix A. Spin and statistics for 3D Euclidean QFT

We review generalities concerning the relation between spin and statistics in quantum field theory in the first section of this appendix. The second section illustrates the no spin-statistics relation in the special case of three-dimensional Euclidean quantum field theory.

A.1. Generalities

The classical configuration space $M_n$ of a system of $n$ indistinguishable particles moving on a (connected and path connected) manifold locally $\mathbb{R}^d$ is given by

$$M_n = (\mathbb{R}^{dn} - D)/S_n,$$  \hspace{1cm} (A.1)

where $S_n$ is the permutation group of $n$ elements and the ‘diagonal’ $D$ is the set of singular configurations where at least two particles coincide.

(i) Case $d = 1$: $M_n$ is clearly multiply connected (if $n > 1$ of course) and each of its connected component is topologically equivalent to the sphere $S^{dn}$.

(ii) Case $d = 2$: $M_n$ is connected but not simply connected. Its fundamental group $\pi_1(M_n)$ is non-trivial: in fact $\pi_1(M_n) = B_n(\mathbb{R}^2)$, the braid group in dimension 2. $B_n(\mathbb{R}^2)$ is generated by $n-1$ elements $\sigma_1, \ldots, \sigma_n$ which satisfy the following well-known algebraic relations:

$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{if} \quad |i - j| \neq 1, \quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}. \hspace{1cm} (A.2)$$

(iii) Case $d > 2$: $M_n$ is connected; its fundamental group is the permutation group $S_n$. Note that $S_n$ is generated by the $n - 1$ elements $\sigma_i$ of $B_n(\mathbb{R}^2)$ satisfying the same relations as the previous ones plus the condition $\sigma^2_1 = 1$.

There is a consistent quantization associated with each unitary irreducible representations (irreps) of the fundamental group $\pi_1(M_n)$. Here, we are only interested with one-dimensional irreps even if higher dimensional irreps are mathematically conceivable. The statistics associated with these one-dimensional irreps are the following (we will not consider the one-dimensional case):

(i) Case $d > 2$: there exist two one-dimensional representations $\chi_{\pm}$, the trivial one $\chi_+ = 1$ and the ‘inverse’ one $\chi_- = -1$. The remarkable spin-statistics theorem claims that (upon some hypothesis we do not want to precise in details here) fields which are symmetric under permutations (i.e. permutations are represented by $\chi_+$) are integer spin-fields (bosons) and fields which are anti-symmetric under permutations (i.e. permutations are represented...
by $\chi_-$) are half-integer spin-fields (fermions). Therefore, there is a strong relation between representations of the Poincaré group and one-dimensional representations of $\pi_1(M_4)$.

(ii) Case $d = 2$: one-dimensional irreps $\chi_\theta$ are labelled by a real number $\theta \in [0, 2\pi]$ such that $\chi_\theta(\sigma_i) = e^{i\theta}$ for any $i$. These statistics contains of course the bosonic ($\theta = 0$) and the fermionic ($\theta = \pi$) ones; the other statistics ($\theta \neq 0, \pi$) are known as anyonic statistics.

There is generically no spin-statistics theorem in that case.

A.2. 3D Euclidean quantum field theory

This section aims at illustrating the no spin-statistic [14] theorem in a simple example of 3D Euclidean quantum field theory.

We start by considering massive spinless particles of mass $m$. One-particle states on a three-dimensional Euclidean space are represented by elements $\phi \in L^2(S^2, d\mu)$. For what follows, it is convenient to include in the space of states distributional states and then we introduce the bra-ket notation $|\lambda\rangle$ to label a state of pure momentum ($\lambda \in S^2 \subset SU(2)$). Such a state is defined by $\langle \phi(\lambda) | = \phi(\lambda)$. The no-particle state is denoted by $|0\rangle$ and creation and annihilation operators $a^\dagger(\lambda)$ and $a(\lambda)$ act as usual on one and no-particle states. In particular, we have $a^\dagger(\lambda)|0\rangle = |\lambda\rangle$.

Let us assume that these operators satisfy an anyonic statistic, i.e.,

$$[a(\lambda_1); a^\dagger(\lambda_2)]_q = a(\lambda_1)a^\dagger(\lambda_2) - qa^\dagger(\lambda_2)a(\lambda_1) = \delta(\lambda_1^{-1}\lambda_2) \quad (A.3)$$

$$[a(\lambda_1); a(\lambda_2)]_q = 0 = [a^\dagger(\lambda_1); a^\dagger(\lambda_2)]_q. \quad (A.4)$$

One can directly generalize these commutation relations for any $\lambda \in SU(2)$. The statistic parameter $q$ is a priori any complex number. We want to see the compatibility between this type of statistics and the value of the spin of the quantum field. To do so, we first need to define a spinning quantum field.

One can make use of spinless particles creation/annihilation operators to construct spinning particles creation/annihilation operators as follows:

$$a^\dagger(\lambda, s) \equiv \int d\mu(\theta) e^{-i\theta s} a^\dagger(\lambda h(\theta)), \quad a(\lambda, s) \equiv \int d\mu(\theta) e^{i\theta s} a(\lambda h(\theta)). \quad (A.5)$$

We have introduced the following notations: $s \in \frac{1}{2}\mathbb{Z}$ is the spin of the particle; the group element $h(\theta) = \text{diag}(e^{i\theta}, e^{-i\theta})$ in the $SU(2)$ spinorial representation and the measure is

$$d\mu(\theta) \equiv \lim_{n \to \infty} \int_{-\pi}^{n\pi} d\theta. \quad (A.6)$$

It is easy to see that spinning particles creation/annihilation operators satisfy the following commutation relations:

$$[a(\lambda_1, s_1); a^\dagger(\lambda_2, s_2)]_q = \delta(s_1 - s_2) \int d\mu(\theta) e^{-i\theta s}\delta(\lambda_1^{-1}\lambda_2 h(\theta)). \quad (A.7)$$

A straightforward calculation shows that the state $|\lambda, s\rangle = a^\dagger(\lambda, s)|0\rangle$ is a (distributional) vector of a spinning massive representation of $ISU(2)$ and therefore represents a pure momentum spinning massive particle state. To be more concrete, we have

$$|\lambda h(a), s\rangle = e^{is\theta}|\lambda, s\rangle. \quad (A.8)$$

Then, a spinning quantum field operator $\phi_\theta(x)$ is defined as a linear combination of creation and annihilation operators; and the requirement of locality and covariance implies that it takes the following form:

$$\phi_\theta(x) = \int d\mu(\lambda)(A(s) e^{i\lambda \hat{\mathbf{\lambda}} a^\dagger(\lambda, s) + B(s) e^{i\lambda \hat{\mathbf{\lambda}}} a(\lambda, s)). \quad (A.9)$$
A(s) and B(s) are complex coefficients that depend \textit{a priori} of the spin s. As our theory is Euclidean, the causality requirement is replaced by the following conditions:

\[ \Delta_0(\vec{x} - \vec{y}) = 0 \]

The function \( \Delta_0 \) is the Hadamard propagator as defined in (12). We see immediately that there is no obstruction to define a quantum spinning field whatever the statistics is: this is the illustration of the no spin-statistics theorem in three-dimensional space. There is only one subtlety concerning spinless particle: in that case, only the bosonic statistics \( (q = -1) \) is compatible with the requirements of locality, covariance and causality.

**Appendix B. DSU(2) as a gravitational deformation of ISU(2)**

The quantum double DSU(2) has already been defined many times in the literature (see [4] for the finite group case). This section aims at presenting DSU(2) in such a way that it appears clearly as a quantum deformation of the Euclidean group ISU(2). For clarity reasons, we will not enter into mathematical details. The reader interested in such details is invited to go to [13].

**B.1. The Euclidean group ISU(2)**

For that purpose, we start by recalling that ISU(2) is the semi-product of SU(2) by \( \mathbb{R}^3 \): its elements are usually denoted \((\vec{a}, u) \in \mathbb{R}^3 \times SU(2)\) and they satisfy the following algebra and (group-like) co-algebra relations:

\[
\begin{aligned}
(\vec{a}_1, u_1) \cdot (\vec{a}_2, u_2) &= (u_1 \vec{a}_2 + \vec{a}_1, u_1 u_2), \\
\Delta(\vec{a}, u) &= (\vec{a}, u) \otimes (\vec{a}, u).
\end{aligned}
\]

The notation \( u \vec{a} \) holds for the action of the vectorial representation of \( u \) on the vector \( \vec{a} \). For latter convenience, it is useful to give the following equivalent description where the elements

\[
\begin{aligned}
(\vec{k}, u) &= \frac{1}{(2\pi)^3} \int d^3\vec{a} \ e^{i\vec{a} \cdot \vec{k}} (\vec{a}, u)
\end{aligned}
\]

are ‘Fourier’ transforms of the previous ones:

\[
\begin{aligned}
(\vec{k}_1, u_1) \cdot (\vec{k}_2, u_2) &= \delta^3(\vec{k}_1 - u_1 \vec{k}_2)(\vec{k}_1, u_1 u_2), \\
\Delta(\vec{k}, u) &= \int d^3\vec{p} (\vec{p}, u) \otimes (\vec{k} - \vec{p}, u).
\end{aligned}
\]

**B.2. The Drinfeld double DSU(2)**

The quantum double DSU(2) is defined as a coalgebra as the tensor product of SU(2) with Fun(SU(2)) where Fun(SU(2)) is a suitable set of functions on SU(2). Its elements are usually denoted \((f, u) \in \text{Fun}(SU(2)) \times SU(2)\) in terms of which the algebra and co-algebra structures are given by

\[
\begin{aligned}
(f_1, u_1) \cdot (f_2, u_2) &= (f_1 f_2 \circ ad_{u_1}, u_1 u_2), \\
\Delta(f, u) &= \sum_{(f)} (f_{(1)}, u) \otimes (f_{(2)}, u).
\end{aligned}
\]

We have \((f \circ ad_{u})(a) = f(u^{-1}au)\) and \((f_{(1)} \otimes f_{(2)})(a, b) = f(ab)\) for any \(a, b \in SU(2)\). As for the ISU(2) case, it will be convenient in what follows to deal with dual elements \((g, u) \in SU(2) \otimes SU(2)\) related to the previous one by

\[
(f, u) = \int dg f(g)(g, u).
\]
In terms of these elements, the Hopf algebra structure reads
\[
(g_1, u_1) \cdot (g_2, u_2) = \delta(g_1^{-1} u_1 g_2 u_2^{-1}) (g_1, u_1 u_2), \quad \Delta(g, u) = \int dh (gh, u) \otimes (h^{-1}, u).
\] (B.6)

**B.3. The relation between \( ISU(2) \) and \( DSU(2) \)**

Using the notations introduced previously in this appendix, it is immediate to see that there exists a one parameter family \( \phi_G \) of algebra morphisms between \( ISU(2) \) and \( DSU(2) \) defined as follows:
\[
\phi_G : ISU(2) \rightarrow DSU(2) \quad (\tilde{k}, u) \mapsto (e^{iG\tilde{k}} \tilde{\sigma}, u),
\] (B.7)
where \( \tilde{\sigma} = (\sigma_1, \sigma_2, \sigma_3) \) are the Pauli matrices. In that sense, \( DSU(2) \) is viewed as a gravitational deformation of \( ISU(2) \) and \( G \) is the Newton constant. Obviously, \( \psi_G \) is not a co-morphism; otherwise \( ISU(2) \) and \( DSU(2) \) would have been equivalent as Hopf algebra.

If \( ISU(2) \) is the isometry group of the Euclidean space \( \mathbb{E}^3 \), \( DSU(2) \) can be viewed as an isometry quantum group of a deformed Euclidean space \( \mathbb{E}^3_G = DSU(2)/SU(2) \). How to recover a good notion of position in \( \mathbb{E}^3_G \)? A natural idea is to generalize the classical Fourier transform introduced in (B.2) to the deformed case in such a way that the following diagram is commutative:
\[
(k, u) \in ISU(2) \xrightarrow{FT_k} \tilde{\phi}_G = \int \mathbb{R}^3 e^{i\tilde{k}\cdot\tilde{a}} (k, u) \quad \downarrow \quad \phi_G \\
\downarrow \phi_G \\
(g, u) \in DSU(2) \xrightarrow{FT_g} \tilde{\phi}_G = \int \mathbb{R}^3 K(a, g)(g, u).
\] (B.8)
To make the diagram commutative, one can choose the kernel \( K(a, g) = e^{i\Delta(a)(g)} \) where we make the identification \( a = \tilde{a} \cdot \tilde{\sigma} \) and \( \Delta(a) \) is the trace in the fundamental \( SU(2) \) (equivalently \( su(2) \)) representation. We recover naturally the same notion of position as in [6]: \( \tilde{a} \) (in the last line of the diagram) can be viewed as position coordinates in \( \mathbb{E}^3_G \).

To precise ‘how much’ the space \( \mathbb{E}^3_G \) is non-commutative, it is much more convenient to deal with the dual space Fun\( (\mathbb{E}^3_G) \) where Fun means a suitable set of functions. The product \( \star \) between two such functions \( f_1 \) and \( f_2 \) is defined by Hopf duality as follows:
\[
(f_1 \star f_2)(\tilde{a}) = \sum_{\tilde{a}(i)} f_1(\tilde{a}(i)) f_2(\tilde{a}(j)) \quad \text{with}
\]
\[
\Delta(a) = \sum_{\tilde{a}(1)} \tilde{a}(1) \otimes \tilde{a}(2) = \int \mathbb{R}^3 dh e^{i\Delta(a)(gh)} g \otimes h.
\] (B.9)
As a consequence, the product between two non-commutative plane waves of the type \( w_g(\tilde{a}) \equiv e^{i\Delta(a)(g)} \) is the same as the one found in [6]:
\[
w_g \star w_h = w_{gh}.
\] (B.10)
Therefore, the non-commutativity is controlled by the ‘classical’ non-commutativity of the group \( SU(2) \) itself. In fact, this star product is nothing but the convolution product on the group \( SU(2) \). To see this point, we work with Fourier transformed functions defined by
\[
\hat{f} : SU(2) \rightarrow \mathbb{C} \quad \hat{f}(g) = \left( \frac{1}{2\pi} \right)^3 \int d^3\tilde{a} \hat{f}(\tilde{a}) e^{i\Delta(a)(g)}
\] (B.11)
and we see immediately that
\[
(\hat{f}_1 \star \hat{f}_2)(\tilde{a}) = \left( \frac{1}{2\pi} \right)^3 \int d^3\tilde{a} (f_1 \star f_2)(\tilde{a}) e^{i\Delta(a)(\tilde{a})} = (\hat{f}_1 \circ \hat{f}_2)(\tilde{a}),
\] (B.12)
where \( \circ \) is the convolution product.
Appendix C. The three-point function

This section is devoted to give the technical details of the computation of the classical and gravitational three-point function.

C.1. Classical case

A direct application of Feynmann rules leads to the following form of the free classical three-point function:

\[ H_{\gamma V}^0(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \int \prod_{\ell=1}^3 d^3 p_{\ell} \frac{\delta(p_{\ell} - m_{\ell})}{P(m_{\ell})} e^{i\vec{p}_\ell \cdot \vec{x}_\ell} \delta^{(3)}(\vec{p}_1 + \vec{p}_2 + \vec{p}_3). \]  

(C.1)

Replacing the function \( P(m) \) by its expression and decomposing the delta function \( \delta^{(3)} \) in plane waves, one immediately obtains that

\[ H_{\gamma V}^0(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \frac{2}{(4\pi)^{s_1 m_2 m_3}} \int d^3 \vec{x} \prod_{\ell=1}^3 \Pi_\ell(\vec{x} - \vec{x}_\ell) \]  

(C.2)

with

\[ \Pi_\ell(\vec{x}) = \int d^3 \vec{p} \delta(\|\vec{p}\| - m_{\ell}) e^{i\vec{p} \cdot \vec{x}} = 4\pi m_{\ell} \frac{\sin(m_{\ell} \|\vec{x}\|)}{\|\vec{x}\|}. \]  

(C.3)

Therefore, we obtain the first expression for the classical three-point function:

\[ H_{\gamma V}^0(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \frac{1}{(2\pi)^3} \frac{1}{m_1 m_2 m_3} \int d^3 \vec{x} \prod_{\ell=1}^3 \frac{\sin(m_{\ell} \|\vec{x} - \vec{x}_\ell\|)}{\|\vec{x} - \vec{x}_\ell\|}. \]  

(C.4)

This expression is symmetric in the variables \( \vec{x}_\ell \) but not very convenient. There exist other expressions we want to give here. Let us consider (C.1) and let us integrate over the variable \( \vec{p}_3 \). We write the remaining momenta in spherical coordinates, i.e. \( \vec{p}_1 = m_1 \vec{n}_1 \) and \( \vec{p}_2 = m_2 \vec{n}_2 \) (after integrating over the delta functions involving the norms of the momenta), the expression (C.1) reduces to

\[ H_{\gamma V}^0(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \frac{1}{4\pi m_3} \int d^2 \vec{n}_1 \, d^2 \vec{n}_2 \delta(p_{12} - m_3) \, e^{i\vec{A}_\|\vec{n}_1\|}, \]  

(C.5)

where \( \vec{A}_\ell = m_{\ell}(\vec{n}_\ell - \vec{x}_\ell) \) and \( p_{12} = \sqrt{\|m_1 \vec{n}_1 - m_2 \vec{n}_2\|} \). Using the fact that

\[ \delta(p_{12} - m_3) = 2m_3 \delta(p_{12}^2 - m_3^2) = \frac{m_3}{m_1 m_2} \delta(\vec{n}_1 \cdot \vec{n}_2 - M) \]  

with \( M = \frac{m_1^2 + m_2^2 - m_3^2}{2m_1 m_2} \)  

(C.6)

we obtain the following form:

\[ H_{\gamma V}^0(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \mathcal{N}_0 \int d^2 \vec{n}_1 \, d^2 \vec{n}_2 \delta(\vec{n}_1 \cdot \vec{n}_2 - M) \, e^{i\vec{A}_\|\vec{n}_1\|}. \]  

(C.7)

\( \mathcal{N}_0^{-1} = 4\pi m_1 m_2 m_3 \). The integral vanishes if \( |M| > 1 \); in the non-vanishing case, we note \( M = \cos S_0 \) with \( S_0 \) being a positive angle. One can proceed as follows to integrate the last delta function in that expression: we decompose the vector \( \vec{n}_2 \) in the orthonormal basis \( (\vec{n}_1, \vec{b}, \vec{t}) \) such that

\[ \vec{n}_2 = \cos \theta \vec{n}_1 + \sin \theta (\cos \phi \vec{b} + \sin \phi \vec{t}) \quad \text{and} \quad d^2 \vec{n}_2 = \frac{1}{4\pi} \sin \theta \, d\theta \, d\phi; \]  

(C.8)
and we obtain that
\[
H^0_{(V)}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = \frac{N_0}{4\pi} \int d^2\tilde{n}_1 \int_0^\pi d\theta \sin \theta \delta(\cos \theta - \cos S_0) \\
\times \int_0^{2\pi} d\phi \ e^{i(\tilde{A}_1 \cos \theta \tilde{A}_2)} \tilde{n}_1 \ e^{i \sin \theta_{\tilde{A}_1 \tilde{A}_2} \tilde{A}_2} \ e^{i \sin \theta_{\tilde{A}_1 \tilde{A}_2} \tilde{A}_2}.
\] (C.9)

As a consequence, one can integrate over the angular variable $\theta$, we introduce the Bessel function of the first kind $J_0(\cdot) = (2\pi)^{-1} \int_0^{2\pi} d\theta' \ e^{i \theta' \cos \theta}$ and the expression reduces to
\[
H^0_{(V)}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = \frac{N_0}{2} \int d^2\tilde{n} \ e^{i(\tilde{A}_1 \cos S_0 \tilde{A}_2)} \tilde{n}_1 \left( \sin S_0 \sqrt{||\tilde{A}_2||^2 - (\tilde{A}_2 \cdot \tilde{n}_2)^2} \right).
\] (C.10)

We choose a system of coordinates where $\tilde{A}_2 = A_2(1, 0, 0)$, $\tilde{A}_1 = A_1(\cos U, \sin U, 0)$ and $\tilde{n} = (\cos \theta, \sin \theta \cos \phi, \sin \theta \sin \phi)$ and we obtain the final expression:
\[
H^0_{(V)}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = \frac{N_0}{4} \int_0^\pi d\theta \sin \theta \ e^{iA_1 \cos U \cos \theta} J_0(A_1 \sin U \sin \theta) \ e^{iA_2 \cos \theta} J_0(A_2 \sin S \sin \theta).
\] (C.11)

Simplifications occur in some particular cases:

(i) $S_0 = 0$: after some simple calculations, we see that
\[
H^0_{(V)}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = \frac{N_0}{2} \int_0^1 dx \cos (A_1 \cos U + A_2 x) \ J_0(A_1 \sin U \sqrt{1 - x^2}).
\] (C.12)

Using a formula involving Bessel functions of the first kind, we simplify this expression as follows:
\[
H^0_{(V)}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = \frac{N_0 \sin(||\tilde{A}_1 + \tilde{A}_2||)}{2 \ ||\tilde{A}_1 + \tilde{A}_2||}.
\] (C.13)

Furthermore, $S_0 = 0 \iff m_2^2 = (m_1 + m_2)^2$ and then we obtain expression (67) given in the core of the paper.

(ii) $U = 0$: the same type of calculations leads in that case to the following expression:
\[
H^0_{(V)}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = \frac{N_0 \sqrt{(A_1 + A_2 \cos U)^2 + A_2^2 \sin^2 U}}{2 \ \sqrt{(A_1 + A_2 \cos U)^2 + A_2^2 \sin^2 U}}.
\] (C.14)

Using $\cos U = (m_1^2 - m_2^2 - m_3^2)/(2m_1 m_2)$ we end up with expression (68).

Before going to the gravitational case, we present other equivalent formulations of the function $H^0_{(V)}$ that could be useful. In fact, there is an identity involving Bessel functions and Gegenbauer polynomials $C_k^\alpha(t)$ defined for instance in [1]:
\[
e^{iA \cos \alpha \cos \beta} J_0(A \sin \alpha \sin \beta) = \sqrt{2\pi} \sum_{k=0}^{\infty} i^{-k} \left( \frac{1}{2} + k \right) J_{\frac{1}{2} + k}(A) C_k^{1/2}(\cos \alpha) C_k^{1/2}(\cos \beta).
\] (C.15)

$\Gamma$ is the well-known Euler function and $J_n$ are Bessel functions. This identity allows us to write $H^0_{(V)}$ as the following indefinite series:
\[
H^0_{(V)}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = \frac{\pi}{2\sqrt{A_1 A_2}} \sum_{k=0}^{\infty} \left( \frac{1}{2} + k \right) J_{\frac{1}{2} + k}(A_1) J_{\frac{1}{2} + k}(A_2) \frac{C_k^{1/2}(\cos S_0)}{C_k^{1/2}(1)} \frac{C_k^{1/2}(\cos U)}{C_k^{1/2}(1)}.
\] (C.16)

This series is convergent.
C.2. Gravitational case

In the gravitational case, the three-point function is defined by

\[ H^G_{(V)}(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \frac{1}{2\pi^2} \int \prod_{\ell=1}^{3} dg_\ell \delta_m(g_\ell) e^{i\text{tr}(x_\ell)} \delta(g_1 g_2 g_3). \]  

(C.17)

One can integrate the variable \( g_3 \) for instance. We parametrize the two remaining variables as usual by \( g_2 = \cos m\vec{n}_2 \cdot \vec{\sigma} \), and we have

\[ H^G_{(V)}(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \frac{1}{2\pi \sin m_3} \int d^2\vec{n}_1 d^2\vec{n}_2 \delta(\text{tr}(g_1 g_2) - 2 \cos m_3) e^{i\text{tr}(x_1 \vec{n}_1 + x_2 \vec{n}_2 - x_3 \vec{n}_3)}. \]  

(C.18)

Furthermore,

\[ \text{tr}(g_1 x_1) = \sin m_3 \vec{\sigma} \cdot \vec{x}_1 \]  

(C.19)

\[ \text{tr}(g_1 g_2 x_3) = (\cos m_3 \sin m_1 \vec{n}_1 + \cos m_3 \sin m_3 \vec{n}_2 + \sin m_1 \sin m_2 \vec{n}_1 \cdot \vec{n}_2) \cdot \vec{x}_3 \]  

(C.20)

\[ \delta(\text{tr}(g_1 g_2) - 2 \cos m_3) = \frac{1}{2 \sin m_1 \sin m_2} \delta \left( \vec{n}_1 \cdot \vec{n}_2 - \frac{\cos m_3 - \cos m_1 \cos m_2}{\sin m_1 \sin m_2} \right) \]  

(C.21)

and the previous expression becomes

\[ H^G_{(V)}(\vec{x}_1, \vec{x}_2, \vec{x}_3) = N_G \int d^2\vec{n}_1 d^2\vec{n}_2 \delta \left( \vec{n}_1 \cdot \vec{n}_2 - \frac{\cos m_3 - \cos m_1 \cos m_2}{\sin m_1 \sin m_2} \right) \]

\[- \times \exp \left[ i (\sin m_1 \vec{n}_1 \cdot \vec{x}_1 + \sin m_2 \vec{n}_2 \cdot \vec{x}_2) \right] \]

\[- \times \exp \left[ -i (\cos m_1 \sin m_1 \vec{n}_1 + \cos m_1 \sin m_2 \vec{n}_2 + \sin m_1 \sin m_2 \vec{n}_1 \cdot \vec{n}_2) \cdot \vec{x}_3 \right], \]  

(C.22)

with \( N_G^{-1} = 4\pi \sin m_1 \sin m_2 \sin m_3 \). After some trivial algebra, we recover expression (73) given in the core of the paper. To recover the first given expression (72) of \( H^G_{(V)} \), we just remark after some calculations that

\[ \int d^3 p_3 \delta(\|\vec{p}_3\| - \sin m_3) \delta^{(3)}(\vec{p}_3 + \cos m_2 \vec{p}_1 + \cos m_1 \vec{p}_2 + \vec{p}_1 \wedge \vec{p}_2) \]

\[ = \delta(\| \cos m_2 \vec{p}_1 + \cos m_1 \vec{p}_2 + \vec{p}_1 \wedge \vec{p}_2 \| - \sin m_3) \]

\[ = 2 \sin m_3 \delta(\| \cos m_2 \vec{p}_1 + \cos m_1 \vec{p}_2 + \vec{p}_1 \wedge \vec{p}_2 \|^2 - \sin^2 m_3) \]

\[ = 2 \sin m_3 \left[ \delta(\vec{p}_1 \cdot \vec{p}_2 - \cos m_1 \cos m_2 + \cos m_3) \right. \]

\[ + \delta(\vec{p}_1 \cdot \vec{p}_2 - \cos m_1 \cos m_2 + \cos m_3) \]

\[ \left. + \delta(\vec{p}_1 \cdot \vec{p}_2 + \cos m_1 \cos m_2 + \cos m_3) \right]. \]  

(C.23)

Therefore, expressions (72) and (73) are not strictly speaking equivalent. In the particular case where we identify \( m_3 \) and \( (\pi - m_3) \), then the two delta functions in the last line of the previous calculations are equivalent and the equality between the two expressions is true. We assume that for simplicity we do this identification. Therefore, expression (72) is proven. To recover, expression (74) of \( H^G_{(V)} \), we proceed exactly in the same way as in the classical case. The simplifications that occur in the particular cases \( S_G = 0 \) or \( U = 0 \) are easy to see and left to the reader.
Three-dimensional loop quantum gravity: towards a self-gravitating quantum field theory

C.3. First-order quantum gravity corrections

To compute first-order quantum gravity corrections to the three-point function, we start with the following expression:

\[ H^{(0)}_{(V)}(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \frac{N_G}{2} \int d^2\vec{n} \, e^{i(\vec{B}_1 + \cos S_0 \vec{B}_2) \cdot \vec{n}} \, J_0(\sin S_0 R(\vec{n})), \tag{C.24} \]

where \( \vec{B}_1, \vec{B}_2 \) and \( R(\vec{n}) \) have been introduced in the core of the paper. To compute the classical limit, we have to scale the length variables and the mass variables respectively by the Planck length and the Planck mass:

\[ \vec{x}_\ell \mapsto \vec{x}_\ell \ell_p, \quad m_\ell \mapsto m_\ell m_p \]

with \( \ell_p = \frac{G}{c} \) and \( m_p = \frac{1}{G} \). \( \tag{C.25} \)

Then, we develop the integrand of expression (C.24) at the first order and the only contribution to first order is the following (all the others create contributions at least at the second order):

\[ R(\vec{n}) \simeq \sqrt{\| \vec{A}_2 \|^2 - (\vec{A}_2 \cdot \vec{n})^2} - Gm_1m_2^2 \vec{x}_2 \wedge \vec{x}_3 \cdot \vec{n} \sqrt{\| \vec{A}_2 \|^2 - (\vec{A}_2 \cdot \vec{n})^2}. \tag{C.26} \]

Furthermore, \( N_G = G^{-3}N_0 \); as a consequence, we have

\[ H^{(0)}_{(V)}(\vec{x}_1, \vec{x}_2, \vec{x}_3) \simeq \frac{H^{(0)}_{(V)}(\vec{A}_1, \vec{A}_2, \vec{x}_3)}{G^3} + \frac{N_G m_1m_2^2}{G^2} \int d^2\vec{n} \, e^{i(\vec{A}_1 + \cos S_0 \vec{A}_2) \cdot \vec{n}} \, J_1 \left( \sin S_0 \sqrt{\| \vec{A}_2 \|^2 - (\vec{A}_2 \cdot \vec{n})^2} \right) \]

\[ \times \vec{x}_2 \wedge \vec{x}_3 \cdot \vec{n} \sqrt{\| \vec{A}_2 \|^2 - (\vec{A}_2 \cdot \vec{n})^2}. \tag{C.27} \]

We used \( J_0'(z) = -J_1(z) \). Then, the expression given in the core of the paper for first-order quantum gravity corrections (78) follows immediately.

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