CATALYSIS OF DYNAMICAL SYMMETRY BREAKING
BY A MAGNETIC FIELD

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ABSTRACT. A constant magnetic field in 3+1 and 2+1 dimensions is a strong
catalyst of dynamical chiral symmetry breaking, leading to the generation of
a fermion dynamical mass even at the weakest attractive interaction between
fermions. The essence of this effect is the dimensional reduction $D \to D - 2$ in
the dynamics of fermion pairing in a magnetic field. The effect is illustrated in
the Nambu-Jona-Lasinio model and QED. Possible applications of this effect and
its extension to inhomogeneous field configurations are discussed.

1. INTRODUCTION

At present there are only a few firmly established non-perturbative phenomena
in 2+1 and, especially, 3+1 dimensional field theories. In this talk, I will describe
one more such phenomenon: dynamical chiral symmetry breaking by a magnetic
field. The talk is based on a series of the recent papers with V. Gusynin and
I. Shovkovy [1-5].

The problem of fermions in a constant magnetic field had been considered by
Schwinger long ago [6]. In that classical work, while the interaction with the
external magnetic field was considered in all orders in the coupling constant,
quantum dynamics was treated perturbatively. There is no spontaneous chiral
symmetry breaking in this approximation. In the papers [1-5], we reconsidered
this problem, treating quantum dynamics non-perturbatively. It was shown that
in 3+1 and 2+1 dimensions, a constant magnetic field is a strong catalyst of

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dynamical chiral symmetry breaking, leading to generating a fermion mass even at the weakest attractive interaction between fermions. I stress that this effect is universal, i.e. model independent.

The essence of this effect is the dimensional reduction $D \to D-2$ in the dynamics of fermion pairing in a magnetic field: while at $D = 2 + 1$ the reduction is $2 + 1 \to 0 + 1$, at $D = 3 + 1$ it is $3 + 1 \to 1 + 1$. The physical reason of this reduction is the fact that the motion of charged particles is restricted in directions perpendicular to the magnetic field.

The emphasis in this talk will be on 3+1 dimensional field theories. However, it will be instructive to compare the dynamics in 2+1 and 3+1 dimensions.

As concrete models for the quantum dynamics, we consider the Nambu-Jona-Lasinio (NJL) model and QED. We will show that the dynamics of the lowest Landau level (LLL) plays the crucial role in catalyzing spontaneous chiral symmetry breaking. Actually, we will see that the LLL plays here the role similar to that of the Fermi surface in the BCS theory of superconductivity [7].

As was shown in Refs. [4, 5], the dimensional reduction $D \to D-2$ is reflected in the structure of the equation describing the Nambu-Goldstone (NG) modes in a magnetic field. In Euclidean space, for weakly interacting fermions, it has the form of a two-dimensional (one-dimensional) Schrödinger equation at $D = 3 + 1$ ($D = 2 + 1$):

$$(-\Delta + m_{\text{dyn}}^2 + V(r))\Psi(r) = 0 \quad (1)$$

Here $\Psi(r)$ is expressed through the Bethe-Salpeter (BS) function of NG bosons,

$$\Delta = \frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_4^2}$$

(the magnetic field is in the $+x_3$ direction, $x_4 = it$) for $D = 3 + 1$, and $\Delta = \frac{\partial^2}{\partial x_3^2}$, $x_3 = it$, for $D = 2 + 1$. The attractive potential $V(r)$ is model dependent. In the NJL model (both at $D = 2 + 1$ and $D = 3 + 1$), $V(r)$ is a $\delta$-like potential. In (3+1)-dimensional ladder QED, the potential $V(r)$ is

$$V(r) = \frac{\alpha}{\pi \ell^2} \exp\left(\frac{r^2}{2\ell^2}\right) Ei\left(-\frac{r^2}{2\ell^2}\right), \quad r^2 = x_3^2 + x_4^2, \quad (2)$$

where $Ei(x) = -\int_{-\infty}^{\infty} dt \exp(-t)/t$ is the integral exponential function [8], $\alpha = \frac{e^2}{4\pi}$ is the renormalized coupling constant and $\ell \equiv |eB|^{-1/2}$ is the magnetic length. It is important that, as we shall show below, an infrared dynamics is responsible for spontaneous chiral symmetry breaking in QED in a magnetic field. Therefore, because the QED coupling is weak in the infrared region, the treatment of the non-perturbative dynamics can be reliable in this problem.
Since \(-m_{\text{dyn}}^2\) plays the role of energy \(E\) in this equation and \(V(r)\) is an attractive potential, the problem is reduced to finding the spectrum of bound states (with \(E = -m_{\text{dyn}}^2 < 0\)) of the Schrödinger equation with such a potential. More precisely, since only the largest possible value of \(m_{\text{dyn}}^2\) defines the stable vacuum [9], we need to find the lowest eigenvalue of \(E\). For this purpose, we can use results proved in the literature for the one-dimensional (\(d = 1\)) and two-dimensional (\(d = 2\)) Schrödinger equation [10]. These results ensure that there is at least one bound state for an attractive potential for \(d = 1\) and \(d = 2\). The energy of the lowest level \(E\) has the form:

\[
E(\lambda) = -m_{\text{dyn}}^2(\lambda) = -|eB|f(\lambda),
\]

where \(\lambda\) is a coupling constant (\(\lambda = \lambda = G\) in the NJL model and \(\lambda = \alpha\) in QED). While for \(d = 1\), \(f(\lambda)\) is an analytic function of \(\lambda\) at \(\lambda = 0\), for \(d = 2\), it is non-analytic at \(\lambda = 0\). Actually we found that, as \(G \to 0\),

\[
m_{\text{dyn}}^2 = \frac{N_c^2 G^2 |eB|}{4\pi^2},
\]

where \(N_c\) is the number of fermion colors, in (2+1)-dimensional NJL model [1,2], and

\[
m_{\text{dyn}}^2 = \frac{|eB|}{\pi} \exp\left(-\frac{4\pi^2 (1 - g)}{|eB| N_c G}\right),
\]

where \(g \equiv N_c G A^2 / (4\pi^2)\), in (3+1)-dimensional NJL model [3,5]. In (3+1)-dimensional ladder QED, \(m_{\text{dyn}}\) is [4,5]

\[
m_{\text{dyn}} = C \sqrt{|eB|} \exp\left[-\frac{\pi}{2} \left(\frac{\pi \sqrt{\alpha}}{2\alpha}\right)^{1/2}\right],
\]

where the constant \(C\) is of order one and \(\alpha\) is the renormalized coupling constant relating to the scale \(\mu = m_{\text{dyn}}\). The important point is that this expression for \(m_{\text{dyn}}\) is gauge invariant.

As we will discuss in Sec.6, there may exist interesting applications of this effect: in planar condensed matter systems, in cosmology, in the interpretation of the heavy-ion scattering experiments, and for understanding of the structure of the QCD vacuum. We will also discuss an extension of these results to inhomogeneous field configurations.

### 2. FERMIONS IN A CONSTANT MAGNETIC FIELD

In this section we will discuss the problem of relativistic fermions in a magnetic field in 3+1 dimensions and compare it with the same problem in 2+1 dimensions.
We will show that the roots of the fact that a magnetic field is a strong catalyst of chiral symmetry breaking are actually in this dynamics.

The Lagrangian density in the problem of a relativistic fermion in a constant magnetic field $B$ takes the form

$$\mathcal{L} = \frac{1}{2} [\bar{\psi} (i\gamma^\mu D_\mu - m) \psi] , \quad \mu = 0, 1, 2, 3,$$

where the covariant derivative is

$$D_\mu = \partial_\mu - ieA_\mu^{\text{ext}} .$$

We will use the symmetric gauge:

$$A_\mu^{\text{ext}} = -\frac{1}{2} \delta_\mu^1 B x_2 + \frac{1}{2} \delta_\mu^2 B x_1$$

(9)

The magnetic field is in the $+x_3$ direction.

The energy spectrum of fermions is [11]:

$$E_n(k_3) = \pm \sqrt{m^2 + 2|eB| n + k_3^2} , \quad n = 0, 1, 2, \ldots$$

(10)

(the Landau levels). Each Landau level is degenerate: at each value of the momentum $k_3$, the number of states is

$$dN_0 = S_{12} L_3 \frac{|eB|}{2\pi} \frac{dk_3}{2\pi}$$

at $n=0$, and

$$dN_n = S_{12} L_3 \frac{|eB|}{\pi} \frac{dk_3}{2\pi}$$

at $n \geq 1$ (where $L_3$ is the size in the $x_3$-direction and $S_{12}$ is the square in the $x_1x_2$-plane). In the symmetric gauge (9), the degeneracy is connected with the angular momentum $J_{12}$ in the $x_1x_2$-plane.

As the fermion mass $m$ goes to zero, there is no energy gap between the vacuum and the lowest Landau level (LLL) with $n = 0$. The density of the number of states of fermions on the energy surface with $E_0 = 0$ is

$$\nu_0 = V^{-1} \frac{dN_0}{dE_0} \bigg|_{E_0=0} = S_{12}^{-1} L_3^{-1} \frac{dN_0}{dE_0} \bigg|_{E_0=0} = \frac{|eB|}{4\pi^2} ,$$

(11)

where $E_0 = |k_3|$ and $dN_0 = V \frac{|eB|}{2\pi} \frac{dk_3}{2\pi}$ (here $V = S_{12} L_3$ is the volume of the system). We will see that the dynamics of the LLL plays the crucial role in
catalyzing spontaneous chiral symmetry breaking. In particular, the density \( \nu_0 \) plays the same role here as the density of states on the Fermi surface \( \nu_F \) in the theory of superconductivity [7].

The important point is that the dynamics of the LLL is essentially (1+1)-dimensional. In order to see this, let us consider the fermion propagator in a magnetic field. It was calculated by Schwinger [6] and has the following form in the gauge (9):

\[
S(x, y) = \exp \left[ \frac{ie}{2} (x - y) \mu A^\text{ext}_\mu (x + y) \right] \tilde{S}(x - y) ,
\]

where the Fourier transform of \( \tilde{S} \) is

\[
\tilde{S}(k) = \int_0^\infty ds \exp \left[ is(k_0^2 - k_3^2 - k_\perp^2 \tan(eBs) - m^2) \right] \cdot [(k^0 \gamma^0 - k^3 \gamma^3 + m)(1 + \gamma^1 \gamma^2 \tan(eBs)) - k_\perp \gamma_\perp (1 + \tan^2(eBs))] 
\]

(13)

(here \( k_\perp = (k_1, k_2) \), \( \gamma_\perp = (\gamma_1, \gamma_2) \)). Then by using the identity \( i \tan(x) = 1 - 2 \exp(-2ix)/(1 + \exp(-2ix)) \) and the relation [8]

\[
(1 - z)^{-(\alpha+1)} \exp \left( \frac{xz}{z-1} \right) = \sum_{n=0}^\infty L^\alpha_n(x) z^n ,
\]

(14)

where \( L^\alpha_n(x) \) are the generalized Laguerre polynomials, the propagator \( \tilde{S}(k) \) can be decomposed over the Landau poles [12]:

\[
\tilde{S}(k) = i \exp \left( \frac{-k_\perp^2}{|eB|} \right) \sum_{n=0}^\infty (-1)^n \frac{D_n(eB, k)}{k_0^2 - k_3^2 - m^2 - 2|eB|n} 
\]

(15)

with

\[
D_n(eB, k) = (k^0 \gamma^0 - k^3 \gamma^3 + m) \left[ (1 - i\gamma^1 \gamma^2 \text{sign}(eB)) L_n \left( 2 \frac{k_\perp^2}{|eB|} \right) \right] \\
-(1 + i\gamma^1 \gamma^2 \text{sign}(eB)) L_{n-1} \left( 2 \frac{k_\perp^2}{|eB|} \right) + 4(k^1 \gamma^1 + k^2 \gamma^2) L_{n-1}^1 \left( 2 \frac{k_\perp^2}{|eB|} \right) ,
\]

(16)

where \( L_n \equiv L^0_n \) and \( L^\alpha_{-1} = 0 \) by definition. The LLL pole is

\[
\tilde{S}^{(0)}(k) = i \exp \left( \frac{-k_\perp^2}{|eB|} \right) \frac{k^0 \gamma^0 - k^3 \gamma^3 + m}{k_0^2 - k_3^2 - m^2} (1 - i\gamma^1 \gamma^2 \text{sign}(eB)) .
\]

(17)

This equation clearly demonstrates the (1+1)-dimensional character of the LLL dynamics in the infrared region, with \( k_\perp^2 \ll |eB| \). Since at \( m^2, k_0^2, k_3^2, k_\perp^2 \ll |eB| \) the LLL pole dominates in the fermion propagator, one concludes that the
dimensional reduction $3 + 1 \to 1 + 1$ takes place for the infrared dynamics in a strong (with $|eB| \gg m^2$) magnetic field. It is clear that such a dimensional reduction reflects the fact that the motion of charged particles is restricted in directions perpendicular to the magnetic field.

The LLL dominance can, in particular, be seen in the calculation of the chiral condensate:

$$\langle 0 | \bar{\psi} \psi | 0 \rangle = - \lim_{x \to y} \frac{4m}{(2\pi)^4} \int d^4k \tilde{S}_E(k)$$

$$= - \frac{4m}{(2\pi)^4} \int d^4k \int_{1/\Lambda^2}^{\infty} ds \exp \left[ -s \left( m^2 + k_1^2 + k_3^2 + k_1^2 \frac{\tanh(eBs)}{eBs} \right) \right]$$

$$= - |eB| \frac{m}{4\pi^2} \int_{1/\Lambda^2}^{\infty} ds \frac{e^{-sm^2}}{s} \text{coth}(|eBs|) \xrightarrow{m \to 0} - |eB| \frac{m}{4\pi^2} \left( \ln \frac{\Lambda^2}{m^2} + O(m^0) \right),$$

where $\tilde{S}_E(k)$ is the image of $\tilde{S}(k)$ in Euclidean space and $\Lambda$ is an ultraviolet cutoff. As it is clear from Eqs. (15) and (17), the logarithmic singularity in the condensate appears due to the LLL dynamics.

The above consideration suggests that there is a universal mechanism for enhancing the generation of fermion masses by a strong magnetic field in $3+1$ dimensions: the fermion pairing takes place essentially for fermions at the LLL and this pairing dynamics is $(1+1)$-dimensional (and therefore strong) in the infrared region. This in turn suggests that in a magnetic field, spontaneous chiral symmetry breaking takes place even at the weakest attractive interaction between fermions in $3+1$ dimensions. This effect was indeed established in the NJL model and QED [3-5].

In conclusion, let us compare the dynamics in a magnetic field in $3+1$ dimensions with that in $2+1$ dimensions [1,2]. In $2+1$ dimensions, the LLL pole for the propagator of the four-component fermions [13] is [2,5]:

$$\tilde{S}^{(0)}(k) = i \exp \left( - \frac{k^2}{|eB|} \right) \frac{k^0 \gamma_0 + m}{k_0^2 - m^2} (1 - i \gamma^1 \gamma^2 \text{sign}(eB)).$$

Then, as $m \to 0$, the condensate $\langle 0 | \bar{\psi} \psi | 0 \rangle$ remains non-zero due to the LLL:

$$\langle 0 | \bar{\psi} \psi | 0 \rangle = - \lim_{m \to 0} \frac{m}{(2\pi)^3} \int d^3k \exp \left( - \frac{k^2}{|eB|} \right) \frac{k^2}{k_3^2 + m^2} = - \frac{|eB|}{2\pi}$$

(for concreteness, we consider $m \geq 0$). The appearance of the condensate in the flavor (chiral) limit, $m \to 0$, signals the spontaneous breakdown of the flavor (chiral) symmetry even for free fermions in a magnetic field at $D = 2 + 1$ [1,2]. As we will discuss in Section 4, this in turn provides the analyticity of the dynamical mass $m^\text{dyn}$ as a function of the coupling constant $G$ at $G = 0$ in the $(2+1)$-dimensional NJL model.
3. IS THE DIMENSIONAL REDUCTION $3+1 \rightarrow 1+1$ (2+1 → 0+1) CONSISTENT WITH SPONTANEOUS CHIRAL SYMMETRY BREAKING?

In this section we consider the question whether the dimensional reduction $3+1 \rightarrow 1+1$ (2+1 → 0+1) in the dynamics of the fermion pairing in a magnetic field is consistent with spontaneous chiral symmetry breaking. This question occurs naturally since, due to the Mermin-Wagner-Coleman (MWC) theorem [14], there cannot be spontaneous breakdown of continuous symmetries at $D = 1 + 1$ and $D = 0 + 1$. The MWC theorem is based on the fact that gapless Nambu-Goldstone (NG) bosons cannot exist in dimensions less than 2+1. This is in particular reflected in that the (1+1)-dimensional propagator of would be NG bosons would lead to infrared divergences in perturbation theory (as indeed happens in the $1/N_c$ expansion in the (1+1)-dimensional Gross-Neveu model with a continuous symmetry [15]).

However, the MWC theorem is not applicable to the present problem. The central point is that the condensate $\langle 0|\bar{\psi}\psi|0 \rangle$ and the NG modes are neutral in this problem and the dimensional reduction in a magnetic field does not affect the dynamics of the center of mass of neutral excitations. Indeed, the dimensional reduction $D \rightarrow D - 2$ in the fermion propagator, in the infrared region, reflects the fact that the motion of charged particles is restricted in the directions perpendicular to the magnetic field. Since there is no such restriction for the motion of the center of mass of neutral excitations, their propagators have $D$-dimensional form in the infrared region (since the structure of excitations is irrelevant at long distances, this is correct both for elementary and composite neutral excitations[2]). This fact was shown for neutral bound states in the NJL model in a magnetic field, in the $1/N_c$ expansion, in Refs. [2,5]. Since, besides that, the propagator of massive fermions is, though $(D-2)$-dimensional, nonsingular at small momenta, the infrared dynamics is soft in a magnetic field, and spontaneous chiral symmetry breaking is not washed out by the interactions, as happens, for example, in the (1+1)-dimensional Gross-Neveu model.

This point is intimately connected with the status of the space-translation symmetry in a constant magnetic field. In the symmetric gauge (9), the translation symmetry along the $x_1$ and $x_2$ directions is broken. However, for neutral states, all the components of the momentum of their center of mass are conserved quantum numbers (this property is gauge invariant). In order to show this in the symmetric gauge, let us introduce the following operators (generators of the group of

\[ \mathcal{D}(P) \sim (P_0^2 - C_1 P_1^2 - C_3 P_3^2)^{-1} \]

with $C_1, C_3 \neq 0$.
magnetic translations) describing the translations in first quantized theory:

\[
\hat{P}_{x_1} = \frac{1}{i} \frac{\partial}{\partial x_1} - \frac{\hat{Q}}{2} B x_2, \quad \hat{P}_{x_2} = \frac{1}{i} \frac{\partial}{\partial x_2} + \frac{\hat{Q}}{2} B x_1, \quad \hat{P}_{x_3} = \frac{1}{i} \frac{\partial}{\partial x_3}
\] (21)

(\hat{Q} is the charge operator). One can easily check that these operators commute with the Hamiltonian of the Dirac equation in a constant magnetic field. Also, we get:

\[
[\hat{P}_{x_1}, \hat{P}_{x_2}] = \frac{1}{i} \hat{Q} B, \quad [\hat{P}_{x_1}, \hat{P}_{x_3}] = [\hat{P}_{x_2}, \hat{P}_{x_3}] = 0. \quad (22)
\]

Therefore all the commutators equal zero for neutral states, and the momentum \( P = (P_1, P_2, P_3) \) can be used to describe the dynamics of the center of mass of neutral states.

4. THE NAMBU-JONA-LASINIO MODEL IN A MAGNETIC FIELD

Let us consider the (3+1)-dimensional NJL model with the \( U_L(1) \times U_R(1) \) chiral symmetry:

\[
\mathcal{L} = \frac{1}{2}[\bar{\psi}, (i \gamma^\mu D_\mu) \psi] + \frac{G}{2} [((\bar{\psi} \psi)^2 + (\bar{\psi} i \gamma^5 \psi)^2], \quad (23)
\]

where \( D_\mu \) is the covariant derivative (8) and fermion fields carry an additional “color” index \( \alpha = 1, 2, \ldots, N_c \). The theory is equivalent to the theory with the Lagrangian density

\[
\mathcal{L} = \frac{1}{2}[\bar{\psi}, (i \gamma^\mu D_\mu) \psi] - \bar{\psi}(\sigma + i \gamma^5 \pi) \psi - \frac{1}{2G}(\sigma^2 + \pi^2). \quad (24)
\]

The Euler-Lagrange equations for the auxiliary fields \( \sigma \) and \( \pi \) take the form of constraints:

\[
\sigma = -G(\bar{\psi} \psi), \quad \pi = -G(\bar{\psi} i \gamma^5 \psi), \quad (25)
\]

and the Lagrangian density (24) reproduces Eq. (23) upon application of the constraints (25).

The effective action for the composite fields is expressed through the path integral over fermions:

\[
\Gamma(\sigma, \pi) = \tilde{\Gamma}(\sigma, \pi) - \frac{1}{2G} \int d^4x (\sigma^2 + \pi^2), \quad (26)
\]

\[
\exp(i\tilde{\Gamma}) = \int [d\psi][d\bar{\psi}] \exp \left\{ \frac{i}{2} \int d^4x [\bar{\psi}, \{i \gamma^\mu D_\mu - (\sigma + i \gamma^5 \pi) \} \psi] \right\}
\]

\[
= \exp \left\{ \text{Tr ln} \left[ i \gamma^\mu D_\mu - (\sigma + i \gamma^5 \pi) \right] \right\}, \quad (27)
\]
i.e.
\[ \Gamma(\sigma, \pi) = -i \text{Tr} \ln[i\gamma^\mu D_\mu - (\sigma + i\gamma^5 \pi)] . \]  

As \( N_c \to \infty \), the path integral over the composite fields \( \sigma \) and \( \pi \) is dominated by the stationary points of the action: \( \delta \Gamma / \delta \sigma = \delta \Gamma / \delta \pi = 0 \). The dynamics in this limit was analysed [3,5] by using the expansion of the action \( \Gamma \) in powers of derivatives of the composite fields.

Is the \( 1/N_c \) expansion reliable in this problem? The answer to this question is “yes”. It is connected with the fact, already discussed in the previous section, that the dimensional reduction \( 3+1 \to 1+1 \) by a magnetic field does not affect the dynamics of the center of mass of the NG bosons. If the reduction affected it, the \( 1/N_c \) perturbative expansion would be unreliable. In particular, the contribution of the NG modes in the gap equation, in next-to-leading order in \( 1/N_c \), would lead to infrared divergences (as happens in the \( 1+1 \) dimensional Gross-Neveu model with a continuous chiral symmetry [15]). This is not the case here. Actually, as was shown in Ref.[5], the next-to-leading order in \( 1/N_c \) yields small corrections to the whole dynamics at sufficiently large values of \( N_c \). The same also valid in the \( 2+1 \) dimensional NJL model [2].

The effective actions in the \( 2+1 \) dimensional and \( 3+1 \) dimensional NJL models in a magnetic field were derived in Refs.[1,2] and Refs.[3,5], respectively. Here we will discuss the physics underlying the expressions (4) and (5) for the dynamical mass, for weakly interacting fermions, in these models.

It is instructive to compare the relation (5) with the relations for the dynamical mass (energy gap) in the \( (1+1) \)-dimensional Gross-Neveu model and in the BCS theory of superconductivity [7].

The relation for \( m_{\text{dyn}}^2 \) in the Gross-Neveu model is
\[ m_{\text{dyn}}^2 = \Lambda^2 \exp\left(-\frac{2\pi}{N_c G^{(0)}}\right) \]  

where \( G^{(0)} \) is the bare coupling, which is dimensionless at \( D = 1 + 1 \). The similarity between relations (5) and (29) is evident: \( |eB| \) and \( |eB|G \) in Eq. (5) play the role of an ultraviolet cutoff and the dimensionless coupling constant in Eq. (29), respectively. This of course reflects the point that the dynamics of the fermion pairing in the \( (3+1) \)-dimensional NJL model in a magnetic field is essentially \( (1+1) \)-dimensional.

We recall that, because of the Fermi surface, the dynamics of the electron in superconductivity is also \( (1+1) \)-dimensional. This analogy is rather deep. In particular, the expression (5) for \( m_{\text{dyn}} \) can be rewritten in a form similar to that
for the energy gap $\Delta$ in the BCS theory: while $\Delta \sim \omega_D \exp(-\text{const.}/\nu_F G_S)$, where $\omega_D$ is the Debye frequency, $G_S$ is a coupling constant and $\nu_F$ is the density of states on the Fermi surface, the mass $m_{\text{dyn}}$ is $m_{\text{dyn}} \sim \sqrt{|eB|} \exp(-1/2G\nu_0)$, where the density of states $\nu_0$ on the energy surface $E = 0$ of the LLL is now given by expression (11) multiplied by the factor $N_c$. Thus the energy surface $E = 0$ plays here the role of the Fermi surface.

Let us now compare the relation (5) with the relation (4) in the (2+1)-dimensional NJL model in a magnetic field, in a weak coupling regime. While the expression (5) for $m_{\text{dyn}}^2$ has an essential singularity at $G = 0$, $m_{\text{dyn}}^2$ in the (2+1)-dimensional NJL model is analytic at $G = 0$. The latter is connected with the fact that in 2+1 dimensions the condensate $\langle 0|\bar{\psi}\psi|0 \rangle$ is non-zero even for free fermions in a magnetic field (see Eq. (20)). Indeed, Eq. (25) implies that $m_{\text{dyn}} = \langle 0|\sigma|0 \rangle = -G\langle 0|\bar{\psi}\psi|0 \rangle$. From this fact, and Eq. (20), we get the relation (4), to leading order in $G$. Therefore the dynamical mass $m_{\text{dyn}}$ is essentially perturbative in $G$ in this case.

This is in turn connected with the fact that, for $D = 2+1$, the dynamics of fermion pairing in a magnetic field is (0+1)-dimensional. Indeed, as was already pointed out in Introduction, the dynamics of the NG modes for $D = 2+1$ is described by the one-dimensional ($d = 1$) Schrödinger equation (1) with an attractive ($\delta$-like) potential, where $-m_{\text{dyn}}^2$ plays the role of the energy $E$. The general theorem [10] ensures that, at $d = 1$, the energy of the lowest level $E(G)$ is an analytic function of the coupling constant $G$ around $G = 0$.

On the other hand, the same theorem ensures that the energy $E(G)$ is non-analytic at $G = 0$ at $d = 2$. Moreover, at $d = 2$ for short-range potentials, the energy $E(G) = -m_{\text{dyn}}^2$ takes the form $E(G) \sim -\exp[1/(aG)]$ (with $a$ being positive constant) as $G \to 0$ [10].

Thus the results obtained in the NJL model agree with this general theorem.

5. SPONTANEOUS CHIRAL SYMMETRY BREAKING BY A MAGNETIC FIELD IN QED

As we indicated in Introduction, in 3+1 dimensional QED, in ladder approximation, the dynamics of the NG modes is described by the two-dimensional Schrödinger equation (1) with the potential (2). The essential difference of this potential with respect to the $\delta$-like potential in the NJL model is that it is long range. Indeed, using the asymptotic relations for $Ei(x)$ [8], we get:

$$V(r) \simeq -\frac{2\alpha}{\pi} \frac{1}{r^2}, \quad r \to \infty,$$

$$V(r) \simeq -\frac{\alpha}{\pi\ell^2} \left(\gamma + \ln \frac{2\ell^2}{r^2}\right), \quad r \to 0,$$

where $\gamma$ is the Euler-Mascheroni constant.

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where $\gamma \approx 0.577$ is the Euler constant. Therefore, the theorem of Ref. [10] (asserting that, for short-range potentials, $E(\alpha) \equiv m_{\text{dyn}}^2(\alpha) \sim \exp(-1/a\alpha)$, with $a > 0$, as $\alpha \to 0$) cannot be applied to this case. As was shown in Refs. [4,5], the result in this case is:

$$m_{\text{dyn}} = C \sqrt{|eB|} \exp \left[ -\frac{\pi}{2} \left( \frac{\pi}{2a\alpha} \right)^{1/2} \right], \quad (31)$$

where the constant $C = O(1)$. Note that this expression is gauge invariant.

Since

$$\lim_{\alpha \to 0} \exp \left[ -\frac{1}{a\sqrt{\alpha}} \right] \exp \left[ -\frac{1}{a'\alpha} \right] = \infty, \quad (32)$$

at $a, a' > 0$, we see that the long-range character of the potential leads to the essential enhancement of the dynamical mass.

The present effect is based on the dynamics of the LLL, i.e., on the infrared, weakly coupling, dynamics in QED. This seems suggest that the ladder approximation is reliable for this problem. However, because of the (1+1)-dimensional form of the fermion propagator in the infrared region, there may be also relevant higher order contributions [5].

For example, let us consider the photon propagator in a strong ($|eB| >> m_{\text{dyn}}^2, |k^2||, |k^2\perp|$) magnetic field, with the polarization operator calculated in one-loop approximation [16]. One can rewrite it in the following form:

$$D_{\mu\nu} = -i \left( \frac{1}{k^2} g_{\mu\nu}^{\perp} + \frac{k_{\mu}^{\perp} k_{\nu}^{\parallel}}{k^2 k^2} + \frac{1}{k^2 + k_{\parallel}^2 \Pi(k^2_{\parallel})} \left( g_{\mu\nu}^{\parallel} - \frac{k_{\mu}^{\parallel} k_{\nu}^{\parallel}}{k^2_{\parallel}} - \frac{\lambda}{k^2_{\parallel}} \frac{k_{\mu}^{\parallel} k_{\nu}^{\parallel}}{k^2_{\parallel}} \right) \right), \quad (33)$$

$$\Pi(k^2_{\parallel}) = -\frac{\alpha |eB|}{2\pi m_{\text{dyn}}^2} \left[ \frac{4m_{\text{dyn}}^2}{k^2_{\parallel}} - \frac{8m_{\text{dyn}}^4}{k^2_{\parallel} \sqrt{(k^2_{\parallel})^2 - 4m_{\text{dyn}}^2 k^2_{\perp}}} \right] \times \ln \left[ \frac{(k^2_{\parallel})^2 - 4m_{\text{dyn}}^2 k^2_{\perp} - k^2_{\parallel}}{(k^2_{\parallel})^2 - 4m_{\text{dyn}}^2 k^2_{\perp} + k^2_{\parallel}} \right], \quad (34)$$

where the symbols $\perp$ and $\parallel$ are related to the (1, 2) and (0, 3) components, respectively, and $\lambda$ is a gauge parameter. The asymptotic behavior of $\Pi(k^2_{\parallel})$ is:

$$\Pi(k^2_{\parallel}) \to \frac{\alpha |eB|}{3\pi m_{\text{dyn}}^2} \quad \text{at} \quad |k^2_{\parallel}| << m_{\text{dyn}}^2, \quad (35)$$

$$\Pi(k^2_{\parallel}) \to -\frac{2\alpha |eB|}{\pi k^2_{\parallel}} \quad \text{at} \quad |k^2_{\parallel}| \gg m_{\text{dyn}}^2. \quad (36)$$
There is a strong screening effect in the \((g_{\mu\nu} - k_{\mu}k_{\nu}/k_{\parallel}^2)\)-component of the photon propagator. In particular there is a pole \(-\frac{2\alpha |eB|}{\pi k_{\parallel}^2}\) in \(\Pi(k_{\parallel}^2)\) as \(m_{\text{dyn}}^2 \to 0\): this is of course a reminiscence of the Higgs effect in the \((1 + 1)\)-dimensional massless QED (Schwinger model).

Following Refs. [4,5], one can show that, with the photon propagator (33), the expression for \(m_{\text{dyn}}\) has the form (6) with \(\alpha \to \alpha/2\).

It is a challenge to define the class of all those diagrams in QED which give a relevant contribution in this problem. Since the QED coupling constant is weak in the infrared region, this seems not to be a hopeless problem.

6. CONCLUSION

In 3+1 and 2+1 dimensions, a constant magnetic field is a strong catalyst of spontaneous chiral symmetry breaking, leading to the generation of a fermion dynamical mass even at the weakest attractive interaction between fermions. The essence of this effect is the dimensional reduction \(D \to D - 2\) in the dynamics of fermion pairing in a magnetic field.

So far we considered the dynamics in the presence of a constant magnetic field only. It would be interesting to extend this analysis to the case of inhomogeneous electromagnetic fields. In connection with this, note that in 2+1 dimensions, the present effect is intimately connected with the fact that the massless Dirac equation in a constant magnetic field admits an infinite number of normalized solutions with \(E = 0\) (zero modes) [1,2]. More precisely, the density of the zero modes

\[
\tilde{\nu}_0 = \lim_{S \to \infty} S^{-1} N(E) \bigg|_{E=0}
\]

(where \(S\) is a two-dimensional volume of the system) is finite. As has been already pointed out [2,17], spontaneous flavor (chiral) symmetry breaking in 2+1 dimensions should be catalysed by all stationary (i.e. independent of time) field configurations with \(\tilde{\nu}_0\) being finite. On the other hand, as we saw in Sec.4, the density

\[
\nu_0 = \lim_{V \to \infty} V^{-1} \frac{dN(E)}{dE} \bigg|_{E=0}
\]

of the states with \(E = 0\) (from a continuous spectrum) plays the crucial role in the catalysis of chiral symmetry breaking in 3+1 dimensions. One may expect that the density \(\nu_0\) should play an important role also in the case of (stationary) inhomogeneous configurations in 3+1 dimensions.

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As a first step in studying this problem, it would be important to extend the Schwinger results [6] to inhomogeneous field configurations. Interesting results in this direction have been recently obtained in Ref. [18].

In conclusion, let us discuss possible applications of this effect.

Since (2+1)-dimensional relativistic field theories may serve as effective theories for the description of long wavelength excitations in planar condensed matter systems [19], this effect may be relevant for such systems. It would be also interesting to take into account this effect in studying the possibility of the generation of a magnetic field in the vacuum, i.e. spontaneous breakdown of the Lorentz symmetry, in (2+1)-dimensional QED [20].

In 3+1 dimensions, one potential application of the effect can be connected with the possibility of the existence of very strong magnetic fields \( B \sim 10^{24} G \) during the electroweak phase transition in the early universe [21]. As the results obtained in this paper suggest, such fields might essentially change the character of the electroweak phase transition.

Another application of this effect can be connected with the role of chromomagnetic backgrounds as models for the QCD vacuum (the Copenhagen vacuum [22]).

Yet another potentially interesting application is the interpretation of the results of the GSI heavy-ion scattering experiments in which narrow peaks are seen in the energy spectra of emitted \( e^+e^- \) pairs [23]. One proposed explanation [24] is that a strong electromagnetic field, created by the heavy ions, induces a phase transition in QED to a phase with spontaneous chiral symmetry breaking and the observed peaks are due to the decay of positronium-like states in the phase. The catalysis of chiral symmetry breaking by a magnetic field in QED, studied in this paper, can serve as a toy example of such a phenomenon. In order to get a more realistic model, it would be interesting to extend this analysis to non-constant background fields [25].

We believe that the effect of the dimensional reduction by external field configurations may be quite general and relevant for different non-perturbative phenomena. It deserves further study.

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