Viscosity of two-dimensional electrons

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The hydrodynamic regime of electron transport has been recently realized in conductors with ultra-low densities of defects. Although relaxation processes in two-dimensional (2D) fluids have been studied in many theoretical works, the viscosity of the realistic Fermi gas of 2D electrons having the quadratic energy spectrum and interacting by Coulomb’s law has not been reliably determined either in theory or in experiment up to now. Here we construct a theory of viscosity and thermal conductivity in such system. We compare the calculated viscosity of the 2D electron Fermi gas and the previously known viscosity of a 2D Fermi liquid with available experimental data extracted from the hydrodynamic negative magnetooresistance of the best-quality GaAs quantum wells. Based on this comparison, we argue that measurements of the temperature dependence of the viscosity allow to trace the transition between an electron Fermi liquid and a Fermi gas.

1. Introduction. At low density of defects, conduction electrons in solids can form a viscous fluid, which transfers heat and charge along the sample. Such hydrodynamic regime of transport was recently discovered in novel ultra-pure conductors: graphene [1–6], monovalent layered metal PdCoO$_2$ [7], bulk Weyl metal WP$_2$ [8], and high-mobility GaAs quantum wells [9–20].

In all dissipative hydrodynamic phenomena, the key physical process, controlling the magnitude of viscosity, is the relaxation of the shear stress of the electron fluid. For Fermi systems in low-defect materials, this process is mainly due to inter-particle collisions being controlled by the numbers of filled and empty quantum particle states. Therefore for the shear stress relaxation time $\tau_{ee,2}$ of 2D electrons it is reasonable to expect that $h/\tau_{ee,2} \sim T^2/\varepsilon_F$, as for the scattering times in 3D Fermi systems [21, 22] (here $T$ is temperature and $\varepsilon_F \gg T$ is the Fermi energy). However, the actual relaxation times of 2D fermions can deviate significantly from this estimate due to the kinematic constrictions in 2D electron collisions [23–27].

Depending on the strength of the interparticle interaction, an electron Fermi system in the hydrodynamic regime should be treated as a viscous gas or a viscous liquid. The possibility of propagation of the transverse sound waves can be considered as a characteristic property that distinguishes these cases. In a Fermi liquid, such modes appear at sufficiently large Landau interaction parameters [22]. For an electron Fermi liquid, this corresponds to relatively low electron densities. The transverse sound waves in 2D electron and electron-hole liquids were studied in Refs. [20, 28, 31]. For a 2D electron fluid in a magnetic field such waves can manifest itself by the viscoelastic resonance at a double cyclotron frequency [20, 31, 32], that was apparently observed in the record-quality GaAs quantum wells [13, 14, 20].

The viscosity effect in a strongly non-ideal 2D Fermi liquid was theoretically studied in Ref. [24]. It was shown that the main contribution to the collisions of the pairs of quasi-particles with the total momentum being close to zero (the “head-on collisions”). The probability of such collisions in a Fermi liquid is strongly affected by the Cooperon pairing (that is responsible for superconductivity in the case of the attractive interparticle interaction). The resulting stress relaxation rate takes the form:

$$\frac{1}{\tau_{ee,2}} \propto \frac{T^2}{\ln^2(\varepsilon_F/T)}. \tag{1}$$

Relaxation of perturbations of various types in a 2D Fermi gas was theoretically examined in Refs. [23–27]. It was shown that the head-on collisions also play the main role for relaxation of the shear stress. However, the oversimplifications in the inter-particle scattering probability were made in Refs. [23–27] not allowing a proper evaluation of the relaxation rates for the Fermi gas of 2D electrons interacting by the screened Coulomb potential.

Viscosity of 2D carriers in graphene was theoretically studied in Ref. [33]. The hydrodynamic equations were derived from the current-current response functions. In particular, it was obtained that the viscous stress relaxation rate in the limit of low temperatures is proportional to the squared temperature. The thermal conductivity of 2D electrons with Dirac and quadratic spectrums was were theoretically examined in Ref. [34]. It was shown that the heat flow relaxation time at lowest temperatures is close to the quantum lifetime of 2D electron states.

In this work we study viscosity and thermal conductivity of a Fermi gas of 2D electrons with a quadratic spectrum. We show that, in contrast to 3D Fermi systems [35, 36], the energy part of the electron distribution function describing a viscous flow has a simple structure corresponding to the relaxation time approximation for the inter-electron collision integral. Based on this result, we calculate the shear stress relaxation time $\tau_{ee,2}$, determining the viscosity:

$$\frac{\hbar}{\tau_{ee,2}} = \frac{T^2}{\varepsilon_F} r_s^2 \Lambda, \quad \Lambda = \frac{8\pi}{3} \ln \left( \frac{1}{\zeta + r_s} \right), \tag{2}$$

where $\zeta = T/\varepsilon_F$ is the dimensionless temperature, $r_s = 1/(\sqrt{\pi n a_B})$ is the interparticle interaction parameter, $n$ is the 2D electron density, $a_B$ is the Bohr radius, $\Lambda$ is

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the Coulomb logarithm, which reflects the small angle character of the most of electron-electron collisions [21].

We also estimated the time of relaxation of the heat flow, determining the thermal conductivity. Such time $\tau_{ee,h}$ is much shorter than $\tau_{ee,2}$ and is close at very low temperatures, $\zeta \ll r_s$, to the quantum lifetime $\tau_{ee,q}$, in accordance with the results of Ref. [33], while at moderately low temperatures, $r_s \ll \zeta \ll 1$, the time $\tau_{ee,h}$ turns out to be much smaller than $\tau_{ee,q}$.

We carry out an analysis of temperature dependencies of the 2D electron viscosity, which we extract from the experimental data on the giant negative magnetoresistance of the 2D electron viscosity, which we extract from the experimental data on the giant negative magnetoresistance in high-mobility GaAs quantum wells [11,12,13]. Such magnetoresistance, as it was demonstrated in Ref. [19], is proportional to the viscosity of 2D electrons in quantum wells. We show that the experimental temperature dependencies of the shear stress relaxation time $\tau_2$ are well described by laws [11 or 12], depending on the strength of the interparticle interaction determined by the electron density $n$. We conclude that studies of temperature dependencies of the viscosity allows to distinguish between the electron hydrodynamics in a Fermi gas and in a Fermi liquid [see Fig. 1(a,b)].

2. Collision integral. We study scattering of two 2D electrons with initial momenta $p_1$, $p_2$ and final momenta $p_3$, $p_4$, which are characterized by the angles $\alpha$, $\varphi$, $\theta$, and $\psi$ as shown in Fig. 2(a). The momentum and the energy conservation laws for scattering particles have the usual forms: $p_1 + p_2 = p_3 + p_4$ and $\epsilon_1 + \epsilon_2 = \epsilon_3 + \epsilon_4$. The energy spectrum of 2D electrons is supposed to be quadratic: $\epsilon_i = mv_i^2/2$, and $p_i = mv_i$, $i = 1, 2, 3, 4$.

We consider small perturbations of an electron Fermi gas, which are described by the distribution function: $f_p = f_F(\varepsilon) + \delta f_p$, where $f_F$ is the Fermi function and $\delta f_p = -f_F(\varepsilon)(\Psi, \varepsilon, \alpha)$. Relaxation of perturbations $\delta f_p$ is described by the linearized interparticle collision integral [21,22,26,32,36]

\[
\text{St}[\Psi](\epsilon_1, \alpha) = \frac{m^2}{(2\pi)^2} \int d\epsilon_2 d\epsilon_3 f_F(\epsilon_2) \times \left[1 - f_F(\varepsilon_3)[1 - f_F(\varepsilon_4)] \right] d\varphi d\theta W \left[ \Psi(\epsilon_1, \alpha) \right. \\
+ \Psi(\epsilon_2, \alpha + \varphi - \Psi(\epsilon_3, \alpha + \theta - \Psi(\epsilon_4, \alpha + \psi)) \right],
\]

where $\epsilon_4 = \epsilon_1 + \epsilon_2 - \epsilon_3$, the scattering angle $\psi = \psi(\varphi, \theta, \varepsilon)$ is calculated from the momentum conservation law (for exact form of $\psi$ see [37]), $\varepsilon = (\epsilon_1, \epsilon_2, \epsilon_3)$, and $W = W(\varphi, \theta; \varepsilon)$ is the scattering probability:

\[
W = \frac{2\pi}{\hbar} M^2 \delta \left[ \epsilon_1 + \epsilon_2 - \epsilon_3 - \frac{(p_1 + p_2 - p_3)^2}{2m} \right].
\]

The delta-function in this formula, taking into account the conservation of energy, leads to the equation for the other scattering angle $\theta = \theta(\varphi, \varepsilon)$ (see [37]). The squared matrix element $M^2 = M^2(\varphi, \theta; \varepsilon)$ in Eq. (4) for spin-unpolarized electrons at sufficiently large scattering angles, $\theta \gg \zeta$, $\psi \gg \zeta$, has the form [26]:

\[
M^2 = 2 \left[ V(q)^2 + V(w)^2 - V(q)V(w) \right].
\]

Here the processes of the forward, $1 \to 2 \to 3, 4$, and the exchange, $1 \to 2 \to 4, 3$, scattering with the transmitted momenta $q = p_3 - p_1$ and $w = p_3 - p_2$ are taken into account; and $V(q)$ is the static screened 2D Coulomb potential in the Fourier representation within the random phase approximation (RPA) [37]:

\[
V(q) = 2\pi e^2 \left[ q/\hbar + 2/a_B \right]^{-1}.
\]

Further consideration is based on the smallness of the two parameters: the dimensionless temperature, $\zeta \ll 1$, and the interaction parameter, $r_s \ll 1$. The magnitude of $\zeta$ is always very small in GaAs quantum wells at typical experiment conditions; $\zeta \sim 10^{-2}$ at $T = 1 \text{ K}$ and $n = 10^{11} \text{ cm}^{-2}$. In realistic structures the interaction parameter $r_s$ is smaller than unity only at rather high electron densities: in GaAs quantum wells with $n = 10^{11} - 10^{12} \text{ cm}^{-2}$ the magnitude of $r_s$ varies in the interval $1.7 - 0.55$. In this work we use the random phase approximation, being valid at $r_s \ll 1$ and any ratio $\zeta/r_s$. However, when discussing the case of GaAs quantum wells, we will keep in mind that $r_s$ is not too small and the case $\zeta/r_s \ll 1$ is typically realized.

The delta-function in Eq. (4) can be written as the sum of the expressions, corresponding to the two solutions $\theta_{\pm} = \theta_{\pm}(\varphi, \varepsilon)$ of the energy conservation equation:

\[
\sum_{\pm} \frac{\delta(\theta - \theta_{\pm})}{2\sqrt{\varepsilon_3} \sin(\theta_{\pm}) + \sqrt{\varepsilon_2} \sin(\theta_{\pm} - \varphi)}.
\]

Analysis [37] demonstrates that, after integration in the collision integral (3) by the angle $\theta$, expressions (7) have...
the singularities at \( \varphi \to 0 \) and \( \varphi \to \pi \), where the values of the \( \pm \) terms in Eq. (4) are much larger than at \( \varphi \sim 1 \).

The singularity in Eq. (7) at \( \varphi \to 0 \) refers to the collisions of electrons with almost collinear momenta \( p_1 \approx -p_2 \).

The formulas for the actual scattering angles \( \theta_{\pm} = \theta_{\pm}(\varphi, \varepsilon) \) and \( \psi_{\pm} = \psi_{\pm}(\varphi, \varepsilon, \bar{\varepsilon}) \) leads to the estimate: \( \zeta \lesssim |\theta_{\pm}|, |\psi_{\pm}| \ll 1 \) at the small angles of incident electron, \( \zeta \lesssim |\varphi| \ll 1 \) (see Fig. 2(b) and [37]). So the contribution from the \( \varphi \to 0 \) singularity is negligible compared to the almost-collinear small-angle scattering. In this case one should take into account the dependence of \( M^2 \) on the particle energies \( \bar{\varepsilon} \) which is neglected in Eq. [37].

The singularity at \( \varphi \to \pi \) corresponds to scattering of electrons with almost opposite directed momenta, \( p_1 \approx -p_2 \), generally speaking, on arbitrary angles, \( 0 < |\theta_{\pm}| < \pi \). Such “head-on” collisions are most important for relaxation of the distribution function describing a viscous flow [25, 26]. Due to the form of matrix element [5], the most significant are again the collisions in which one of the scattering angles is relatively small: \( \zeta \ll |\theta_{\pm}| \ll 1 \) or \( \zeta \ll |\psi_{\pm}| \ll 1 \) (see Fig. 2(b) and [37]). For such \( \theta_{\pm} \) and \( \psi_{\pm} \), one can use the static matrix element [5] with the omitted dependence on the energies [37].

3. Relaxation of shear stress and heat flow. The effect of shear viscosity consists in arising of a shear stress in a gas or in a liquid due to the presence of an inhomogeneity of the hydrodynamic velocity \( \mathbf{V}(r) \). This effect is described by the distribution function \( \Psi = p \cdot \mathbf{V} + \Psi_s \), where \( p \cdot \mathbf{V} \) is the locally equilibrium part corresponding to the mean fluid velocity \( \mathbf{V} = \mathbf{V}(r) \), while \( \Psi_s \) is the nonequilibrium part being proportional to the second angular harmonic by \( p \) and leading to a nonzero shear stress tensor \( \sigma_{ik} = \int \psi_{ik}(\varepsilon) \Psi_s(\varepsilon, \alpha) \rho p_p d^2p/(2\pi^2\hbar^2) \). For an incompressible flow, where \( \text{div} \mathbf{V} = 0 \), the kinetic equation \( \mathbf{v} \cdot \partial \mathbf{v}/\partial \mathbf{r} = \text{St}[\mathbf{f}_p] \) leads to the following form of \( \Psi_s \):

\[
\Psi_s(\varepsilon, \alpha) = \frac{mv_p^2}{4} \left( \frac{\partial V_i}{\partial x_k} + \frac{\partial V_k}{\partial x_i} \right) F_{ik},
\]

where \( F_{xx,yy} = \pm F(\varepsilon) \cos(2\alpha) \), \( F_{xy} = F(\varepsilon) \sin(2\alpha) \). The operator \( \text{St} \) is diagonal by angular harmonics, for example, \( \text{St}[F_{xx}](\varepsilon_1, \alpha) \propto \cos(2\alpha) \). Thus the kinetic equation for the function \( F(\varepsilon) \) takes the form:

\[
\text{St}[F(\varepsilon) \cos(2\phi)]|_{\alpha = 0} = -f_p'(\varepsilon_1). \tag{9}
\]

The viscosity coefficient \( \eta \) is defined as the coefficient of proportionality between the stress tensor \( \sigma_{ik} \) and the derivatives of \( \mathbf{V}(r) \): \( \sigma_{ik} = \eta (\partial V_i/\partial x_k + \partial V_k/\partial x_i), \eta = mnv_p^2\tau_{ee,2}/4 \), where \( \tau_{ee,2} \) is the shear stress relaxation time. From this formula for \( \eta \), Eq. (8), and the above formula for \( \sigma_{ik} \) we obtain: \( \tau_{ee,2} = \int d\varepsilon f_p(\varepsilon) F(\varepsilon) \). In the presence of a magnetic field, the term \( \omega_c \partial \mathbf{f}_p/\partial \alpha \) arises in the kinetic equation and the time \( \tau_{ee,2} \) also determines the magnetic field dependence of viscosity [37].

We have shown in [37] that, due to properties of scattering of 2D electrons interacting by potential [9], the collision integrals St consists of the two parts, the first of which acts only on the energy variable of \( \Psi(\varepsilon, \alpha) \), while the second one, being much smaller, acts on the angular as well as energy variable of \( \Psi(\varepsilon, \alpha) \). Such structure of St leads to the fact that the function \( F(\varepsilon) \) obtained from the solution of Eq. (9) turns out to be mainly proportional to a constant. Therefore we obtain the formula:

\[
\frac{1}{\tau_{ee,2}} = -\int d\varepsilon \{ \text{St}[\cos(2\phi)] \} |_{\alpha = 0}, \tag{10}
\]

corresponding to applicability of the relaxation time approximation for the problem of the viscous transport.

This formula together with Eqs. (3)–(7) yields the result [2] [37]. The main contribution to the rate \( 1/\tau_{ee,2} \) arises from integration of the expression in Eq. (3) by the angle \( \varphi \) over the interval \( \zeta \leq \pi - |\varphi| \leq \zeta \ll \tau_{ee,2} \) in the case of low temperatures, \( \zeta \ll \tau_{ee,2} \), or over the interval \( \zeta \leq |\varphi| \leq 1 \) at high temperatures, \( \tau_{ee,2} \ll \tau_{ee,2} \) [see Fig. 2(b)].

In [37] we also present our results for the relaxation times \( \tau_{ee,m} \) of the distribution functions proportional to the higher angular harmonics, \( \Psi_m(\varepsilon, \phi) = \cos(m\phi), m \geq 3 \), and discuss the relation between our results for the 2D electron Fermi gas and the results of Refs. 24–27 for a general 2D Fermi gas. In particular, for the relaxation rates of the odd harmonics \( \Psi_m \) we obtained

\[
\frac{\hbar}{\tau_{ee,m}} \sim \frac{T^2}{\varepsilon_F} \zeta^2 \ln(\tau_{ee,m}/\zeta^2) \tag{11}
\]

at \( \zeta \ll \tau_{ee,2} \) and \( m \ll 1/\tau_{ee,2} \). This rate differs from the one obtained in Ref. [26] by the logarithm \( \ln(\tau_{ee,m}/\zeta) \) due to taking into account the dependence of the screened Coulomb matrix element \( M^2 [4] \) on the angles and the energies.

Another effect in 2D electron fluids being controlled by the inter-particle scattering is the heat transport. The
heat flow is described by the term
\[ \Psi_h(\alpha, \varepsilon) = \mathbf{v} \cdot \nabla T \ G(\varepsilon) \] (12)
in the distribution function, where \( G(\varepsilon) \) is odd by \( \varepsilon - \mu \).
Such \( \Psi_h \) describes different directions of the velocities of the “cold” \( \varepsilon < \mu \) and the “hot” \( \varepsilon > \mu \) electrons. The solution of the kinetic equation for \( \Psi_h \) allows to find the thermal conductivity \( \kappa \) that relates the heat flux and the temperature gradient by the Fourier law: \( \mathbf{q} = -\kappa \nabla T \).

The heat flow relaxation time is determined via the heat flow function \( \Psi_{h_1} = \cos(\alpha)(\varepsilon - \mu) / T \) corresponding to a nonzero heat flow \( \mathbf{q} \). The heat flow relaxation rate is estimated via \( \Psi_{h_1} \) as [37]:

\[ \frac{1}{\tau_{ee,h}} \sim -\int d\varepsilon \{\Psi_{h_1}(1)\} \bigr|_{\alpha=0} \ . \] (13)

Calculation [37] based on Eqs. (3)-(7) and (13) yields:

\[ \frac{\hbar}{\tau_{ee,h}} \sim T^2 \frac{1}{\varepsilon_F} \left\{ \ln \left( \frac{r_s}{\zeta} \right), \quad \zeta \ll r_s \right\} \left\{ \ln \left( \frac{r_s^2}{\zeta^2} \right), \quad \zeta \gg r_s \right\} \ . \] (14)

In the case of low temperatures, \( \zeta \ll r_s \), such rate originates from the head-on collisions with the angles \( \zeta / r_s \ll 1 \) [see Fig. 2(b)] and coincides by the order of magnitude with the quantum width of electron levels due to the electro-electron scattering, \( \hbar / \tau_{ee,q} \) [37]. In the case \( \zeta \gg r_s \), the rate \( 1/\tau_{ee,h} \) comes from intermediate angles \( |\varphi| \sim 1 \) and is much smaller than the levelwidth, which is estimates as \( (T^2 / \varepsilon_F) \ln(\zeta / r_s) r_s^2 / \zeta^2 \).

4. Comparison with experiment. When the hydrodynamic regime of transport in a 2D electron fluid is realized, the dependence of resistance on magnetic field \( g(B) \) arises due to the dependence of the diagonal viscosity coefficient \( \eta_{xx} \) of the fluid on magnetic field [19, 37]:

\[ g(B) - g(\infty) \propto \eta_{xx} \beta \frac{\partial \ln(\eta_{xx})}{\partial B} = \frac{\eta}{1 + (2\omega_c \tau_2)^2} \ . \] (15)

The width of this Lorentzian curve is controlled by the total shear stress relaxation time \( \tau_2 \).

We compare result (2) for the interparticle scattering time \( \tau_{ee,2} \) with available experimental data [9-15] on magnetotransport in high-mobility GaAs quantum wells. In real samples, the rate \( 1/\tau_2 \) contains the two contributions [19]:

\[ \frac{1}{\tau_2(T)} = \frac{1}{\tau_{imp,2}} + \frac{1}{\tau_{ee,2}(T)} \ . \] (16)

The first term is the “residual” relaxation rate of the shear stress at \( T \to 0 \) due to scattering of electrons on disorder, while the second one, being the inter-electron relaxation rate, determines the temperature dependence of \( 1/\tau_2 \). The possibility of the hydrodynamic regime in real samples, apparently, is associated with the substantial relaxation of the odd and the even harmonics of \( \Psi \) in electron scattering on disorder with the rates \( 1/\tau_{imp,m} \), being much greater than \( 1/\tau_{ee,m} \) for odd \( m \) and comparable with \( 1/\tau_{ee,m} \) for even \( m \) (see discussion in [37]).

We fitted the magnetoresistance measured in Refs. [9, 15] by the Lorentzian profile [15]. The experimental dependencies \( 1/\tau_{ee,2} \) were extracted from the analysis of the temperature dependencies of the widths of the magnetoresistance curves \( g(B) \) [37]. In Fig. 3 we present the results for the two samples with the maximal and minimal electron densities: \( n = 1.6 \cdot 10^{11} \text{ cm}^{-2} \) for one of the samples studied in Ref. [9] (sample A) and \( n = 9.1 \cdot 10^{11} \text{ cm}^{-2} \) for the other one. The parameters \( \tau_{ee,2} \) are equal to 1.38 and 0.58, respectively.

In Fig. 3 we plotted the experimental data for samples A and B together with the theoretical curves given by Eq. (16) with \( 1/\tau_{ee,2} = A_{imp} T^2 / \ln(\varepsilon_F / T^2) \) corresponding to a strongly non-ideal Fermi liquid [Eq. (1)] and with \( 1/\tau_{ee,2} = A_{gas} T^2 \) corresponding to an almost ideal Fermi gas at \( \zeta \ll r_s \) [Eq. (2)]. The values \( \tau_{imp,2} \), \( A_{gas} \), \( A_{liq} \) were the fitting parameters.

We see from Fig. 3 that the dependence \( 1/\tau_{ee,2} \) much better describes the experimental data than \( 1/\tau_{ee,2} \) for the high-density sample B, for which the interaction parameter \( r_s \) is relatively small, while for the low-density sample A, for which \( r_s \) is relatively large, the experimental points much better correspond to the curve \( 1/\tau_{ee,2} \) than to \( 1/\tau_{ee,2} \).

We also compare the amplitude \( A_{gas} \) extracted in this fitting procedure for sample B with our result (2). The fitting yields \( C = 4.0 \) for the numeric factor \( C \) defined as \( C = \hbar \varepsilon_F A_{gas} / [\tau_2 r_s^2 \ln(1/r_s)] \) [see Eq. (2)]. Although \( r_s \) for this sample is only about one half, \( r_s = 0.58 \), such value of \( C \) coincides in the order of magnitude with the result of our calculation, \( C = 8\pi / 3 = 8.4 \), corresponding to equation (2).
In the limit of lowest temperatures and high magnetic field the momentum relaxation time on bulk disorder, $\tau_{imp,1}$, can be estimated from the resistance $\rho(\infty)$ [19]. Although it is usually presumed that $\tau_{imp,1}$ and $\tau_{imp,2}$ are of the same order of magnitude, the simple fitting procedure performed in [19] (and in [19] by the method of [19]) led to a surprisingly large difference between these two times in high-mobility GaAs quantum wells, $\tau_{imp,1}$ turns out to be longer than $\tau_{imp,2}$ in 10-100 times. We believe that the resolution of this issue will provide a very important information about the nature of the 2D electron system in the high-mobility structures.

5. Conclusion and acknowledgements. We have evaluated the shear stress relaxation time and the heat flow relaxation time, those determine the viscosity and the thermal conductivity of a 2D electron Fermi gas. The first time has been compared with its value for 2D electron fluids in the best-quality GaAs quantum wells. We have succeeded in identifying of the temperature dependencies (1) and (2) in the experimental data that distinguishes between a Fermi gas and a Fermi liquid.

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Supplemental material

In this Supplemental material, first, we present the formulas for the scattering angles and the momentum transferred in the inter-particle collisions of 2D electrons in a Fermi gas; the kernel of the electron-electron interaction with taking into account the energy transferred in collisions; the asymptotic expressions for the inter-particle collision integral. Second, we develop a method of solution of the kinetic equation for weakly interacting 2D electrons and perform the calculations of the kinetic coefficients and the relaxation times of different types of perturbations. Third, we discuss the difficulties in analysis of the experiments on hydrodynamic magnetotransport within the developed theory.

1. EXPRESSIONS FOR THE CHARACTERISTICS OF INTER-PARTICLE COLLISIONS

1.1. The scattering angles ψ and θ

Below we present the formulas for the relations between the scattering angles ϕ, θ, and ψ at different initial and final electron energies ε₁, ε₂ and ε₃. Herewith we imply that all the angles are defined in the diapason (−π, π).

The energy dependence of the kernel of the collision integral (M3) is mainly determined by the Fermi function

\[ f_F(x_1) f_F(x_2) [1 - f_F(x_3)][1 - f_F(x_4)] = \frac{1}{2^4 \cosh(x_1/2) \cosh(x_2/2) \cosh(x_3/2) \cosh(x_4/2)} \]

where we introduce the dimensionless electron energies counted from the chemical potential \( \mu \): \n
\[ x_i = \frac{\varepsilon - \mu}{T}, \quad i = 1, 2, 3, 4. \]

Let us remind that \( \mu \) differs from the Fermi energy \( \varepsilon_F \) on the value of the second order by temperature, \( \sim (T/\mu)^2 \).

Above and below references (M1), (M2), (M3), and so on denote the formulas in the main text.

Due to factor (1) the initial and final electron energies \( \varepsilon_1, \varepsilon_2 \) and \( \varepsilon_3, \varepsilon_4 = \varepsilon_1 + \varepsilon_2 - \varepsilon_3 \) are close to the chemical potential \( \mu = \varepsilon_F \): \n
\[ \frac{\varepsilon_i - \mu}{\varepsilon_F} \ll \zeta, \quad \zeta = \frac{T}{\varepsilon_F}. \]

These inequalities in the dimensionless energies \( x_i \) takes the form:

\[ x_i \ll 1. \]

The momentum conservation law \( p_1 + p_2 = p_3 + p_4 \) projected on the \( y' \) axis leads to the following equation for the scattering angle \( \psi = \psi(\varphi, \theta, \varepsilon) \) [see Fig. 2(a) in the main text]:

\[ \sqrt{\varepsilon_1 + \varepsilon_2 - \varepsilon_3} \sin \psi = \sqrt{\varepsilon_2} \sin \varphi - \sqrt{\varepsilon_3} \sin \theta. \]

In the arguments of the function \( \psi(\varphi, \theta, \varepsilon) \) we introduced the shortened designation \( \varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \). The solution of Eq. (5) \( \psi = \psi(\varphi, \theta, \varepsilon) \) has the form \( \psi = \psi_0 \), where

\[ \psi_0(\varphi, \theta, \varepsilon) = \arcsin \left( \frac{\sqrt{\varepsilon_2} \sin \varphi - \sqrt{\varepsilon_3} \sin \theta}{\sqrt{\varepsilon_1 + \varepsilon_2 - \varepsilon_3}} \right), \]

\[ \psi = \pi - \psi_0, \text{ or } \psi = -\pi - \psi_0, \]

depending on the positions of the angles \( \varphi \) and \( \theta \) on the unit circle. One should choose one of these three expressions for \( \psi(\varphi, \theta, \varepsilon) \) that allow to satisfy the \( x' \) component of the momentum conservation law at given \( \varphi \) and \( \theta \).

The energy conservation law \( \varepsilon_1 + \varepsilon_2 = \varepsilon_3 + \varepsilon_4 \) in the variables \( \varphi, \theta, \) and \( \varepsilon \) takes the form:

\[ \varepsilon_3 + \sqrt{\varepsilon_1 \varepsilon_2} \cos \varphi = \sqrt{\varepsilon_1 + \varepsilon_2} \cos \theta + \sqrt{\varepsilon_3^2} \cos(\varphi - \theta). \]

It provides the relation, \( \theta = \theta_\pm(\varphi, \varepsilon) \), between the scattering angles \( \varphi \) and \( \theta \) at given energies \( \varepsilon \). This solution of Eq. (7) can be presented as:

\[ \theta_\pm(\varphi, \varepsilon) = \arctan \left( \frac{\sqrt{\varepsilon_2} \sin \varphi}{\sqrt{\varepsilon_1 + \varepsilon_2} \cos \varphi} \right) \pm \arccos \left( \frac{\sqrt{\varepsilon_3^2} + \sqrt{\varepsilon_1 \varepsilon_2} \cos \varphi}{\sqrt{\varepsilon_1 + \varepsilon_2} \cos \varphi} \right) + N_\pm, \]

where the functions \( N_\pm = N_\pm(\varphi, \varepsilon) \) takes one of the values \( 0, \pi, \) or \( -\pi \) for which the angles \( \theta_\pm(\varphi, \varepsilon) \) and the corresponding other scattering angle \( \psi(\varphi, \theta_\pm(\varphi, \varepsilon), \varepsilon) \) satisfy the momentum conservation law projected on the \( x' \) axis.

By substituting Eqs. (8) into equation (6), we obtain the angle \( \psi = \psi_\pm(\varphi, \varepsilon) \) as a function of only the angle \( \varphi \) and the energies \( \varepsilon \):

\[ \psi_\pm(\varphi, \varepsilon) = \psi[\varphi, \theta_\pm(\varphi, \varepsilon), \varepsilon]. \]

Provided Eq. (8) is fulfilled, functions (8) in the interval of the intermediate angles \( |\varphi| \sim 1 \) (more precisely, \( |\varphi| \gg 1, |\pi - \varphi| \gg \zeta \) have the asymptotes:

\[ \theta_+ = \varphi + \Delta \theta_+, \quad \theta_- = \Delta \theta_- \]

where \( \Delta \theta_\pm = \Delta \theta_\pm(\varphi, \varepsilon) \sim \zeta \). From Eqs. (10) and (8) we see that in the same interval of \( \varphi \) the similar asymptotes for \( \psi_\pm(\varphi, \varepsilon) \) take place:

\[ \psi_+ = \Delta \psi_+, \quad \psi_- = \varphi + \Delta \psi_- \]
where $\Delta \psi_{\pm} = \Delta \psi_{\pm}(\varphi, \bar{\varepsilon}) \sim \zeta$. So at the intermediate angles $\varphi, |\varphi| \sim 1$, the scattering angles $\theta$ and $\psi$ are close to 0 and $\varphi$, or visa versa.

In other words, the collision processes $(1, 2) \rightarrow (3, 4)$ at $|\varphi| \sim 1$ are the small-angle scattering,

\[(0, \varphi) \rightarrow \left( \theta = \Delta \theta_-, \psi = \varphi + \Delta \psi_- \right), \quad (12)\]

and the exchange of the two scattering electrons accompanied by a small-angle scattering,

\[(0, \varphi) \rightarrow \left( \theta = \varphi + \Delta \theta_+, \psi = \Delta \psi_+ \right). \quad (13)\]

The functions $\theta_{\pm}(\varphi; \bar{\varepsilon})$ given by Eqs. (5) and (10) are drawn in Fig. 2(b) in the main text for the simplest case when $\varepsilon_1 > \varepsilon_2, \varepsilon_3 > \varepsilon_4$, and $\Delta < 0$ Here and below we introduce the value:

$$\Delta = \Delta(\bar{\varepsilon}) = \frac{\varepsilon_3 \varepsilon_4 - \varepsilon_1 \varepsilon_2}{T^2} = x_3 x_4 - x_1 x_2. \quad (14)$$

It is noteworthy that in this case $N_\pm \equiv 0$ in Eq. (5) and the parameter $\Delta$ determines the gaps between the curves $\theta_{\pm}(\varphi, \bar{\varepsilon})$ and the vertical lines $\varphi = 0, \varphi = \pi$ [see Fig. 2(b) in the main text]. For other relations between $\varepsilon_1, \varepsilon_2$, and $\varepsilon_3$, the functions $\theta_{\pm}(\varphi, \bar{\varepsilon})$ are defined on other intervals of angles $\varphi$, including the points $\varphi = 0$ and $\varphi = \pi$ and exhibit other possible types of connectivity near the points $(0, 0), (\pi, 0)$, and $(\pi, \pi)$ on the $(\varphi, \theta)$-plane.

In the limit $\varphi \rightarrow 0$, functions (5) and (8) determining the scattering angles $\psi = \psi(\varphi, \theta, \bar{x})$ and $\theta = \theta_{\pm}(\varphi, \bar{x})$ take the forms:

$$\psi(\varphi, \theta, \bar{x}) = \varphi - \theta \quad (15)$$

and

$$\theta_{\pm}(\varphi, \bar{x}) = \frac{\varphi \pm \sqrt{\varphi^2 + \zeta^2 \Delta}}{2}. \quad (16)$$

From Eqs. (15) and (10) for the functions $\psi_{\pm}(\varphi, \bar{x})$ [9] we obtain:

$$\psi_{\pm}(\varphi, \bar{x}) = \theta_{\mp}(\varphi, \bar{x}). \quad (17)$$

We see that at $|\varphi| \ll 1$ the scattering angles $\theta$ and $\psi$ are also small: $|\theta|, |\psi| \ll 1$.

The asymptotic expressions (15)-(17) are valid until $|\varphi| \ll \sqrt{\zeta}$.

From Eqs. (6) and (8) we also derive the asymptotic formulas for the scattering angles $\psi_{\pm} = \psi_{\pm}(\varphi, \bar{x})$ and $\theta_{\pm} = \theta_{\pm}(\varphi, \bar{x})$ determining the collision integral in the limit $|\varphi| \rightarrow \pi$ in the main order by $\zeta \ll 1$. For the interval of positive $\varphi$ the formulas take the form:

$$\psi(\varphi, \theta, \bar{x}) = \pi + \theta_{\pm} \text{ or } \psi(\varphi, \theta, \bar{x}) = -\pi + \theta, \quad (18)$$

corresponding to negative and positive values of $\theta = \theta_{\pm}(\varphi, \bar{x})$, respectively [see Fig. 2(a) in the main text], and

$$\theta_{\pm}(\varphi, \bar{x}) = \arctan \left( \frac{\bar{s}}{a} \right) \pm \arccos \left( \frac{b}{\sqrt{a^2 + s^2}} \right) + N_{\pm}. \quad (19)$$

Here we introduced the notations: $s = (\pi - \varphi)/\zeta$, $a = (x_1 - x_2)/2$ and $b = (x_3 - x_4)/2$.

Formulas (18) and (19) are valid up to the angles $\varphi \sim 1$ (that is, at $s \lesssim 1/\zeta$).

1.2. The transferred momenta $q$ and $w$

The absolute value $q = q(\theta, \bar{x})$ of the transferred momentum $q = p_3 - p_1$ is expressed via the particle energies $\varepsilon_1, \varepsilon_3$ and the scattering angle $\theta$ as:

$$\frac{q}{\sqrt{2m}} = \sqrt{\varepsilon_1 + \varepsilon_3 - 2\sqrt{\varepsilon_1 \varepsilon_3} \cos \theta}. \quad (20)$$

Analogously, for the absolute value $w = w(\varphi, \theta, \bar{x})$ of the indirect transferred momenta, $w = p_4 - p_2$, entering the exchange terms in Eq. (M5), we have:

$$\frac{w}{\sqrt{2m}} = \sqrt{\varepsilon_2 + \varepsilon_3 - 2\sqrt{\varepsilon_2 \varepsilon_3} \cos(\varphi - \theta)}. \quad (21)$$

In order to derive the asymptotic expressions for the collision integral near the angles $\varphi = 0$ and $\varphi = \pi$, one needs to evaluate the values $q$ and $w$ in the zero, the first, and the second orders by the small parameter $\zeta$ in the limits $\varphi \rightarrow 0$ and $\varphi \rightarrow \pi$.

In the first limit, $|\varphi| \ll 1$, for the functions $q = q(\theta, \bar{x})$ and $w = w(\varphi, \theta, \bar{x})$ we get from Eqs. (6), (8), (20), (21):

$$q = p_F \sqrt{\theta^2 + \zeta^2 (x_1 - x_3)^2 / 4}, \quad (22)$$

$$w = p_F \sqrt{\psi^2 + \zeta^2 (x_2 - x_3)^2 / 4}, \quad (23)$$

where $p_F = \sqrt{2m \varepsilon_F}$; $\psi = \psi(\varphi, \theta, \bar{x})$ is given by Eq. (6); and $\bar{x} = (x_1, x_2, x_3)$ are the dimensionless electron energies (2).

It is seen from Eqs. (6)-(10), (22) that at the actual scattering angles $\theta = \theta_{\pm}$ and $\psi = \psi_{\pm}$ we have the estimate

$$q, w \sim p_F \zeta \quad (23)$$

in the diapason of very small $\varphi$, $|\varphi| \sim \zeta$, and the estimates

$$q \sim p_F |\varphi|, \quad w \sim p_F |\varphi| \quad (24)$$

or, visa versa,

$$q \sim p_F |\varphi|, \quad w \sim p_F \zeta \quad (25)$$
in the diapason of moderately small $\varphi$, $\zeta \ll |\varphi| \ll 1$.

In the limit, corresponding to the head-on collisions, $\pi - |\varphi| \ll 1$, the functions $q = q(\theta, \vec{x})$ and $w = w(\varphi, \theta, \vec{x})$ from (20) and (21) take the forms:

$$q = p_F \sqrt{2(1 - \cos \theta) + \zeta^2(x_1 - x_3)^2/4},$$

$$w = p_F \sqrt{2(1 + \cos \theta) + \zeta^2(x_2 - x_3)^2/4}. \tag{26}$$

Depending on the sign $\pm$ in Eqs. (8), (10) and the value of $\varphi$, the first terms under the square roots in Eq. (20), $1 \mp \cos \theta$, at actual scattering angles $\theta = \theta_{\pm}(\varphi, \xi)$ and given values $\xi$ can be much greater or comparable with the second terms, $\zeta^2(x_{1,2} - x_3)^2/4$.

### 2. THE KERNEL OF THE INTER-PARTICLE INTERACTION

#### 2.1. The kernel taking into account the transferred energies

The use of the static screened potential $V(q)$ (M6) implies the neglect of the dependence of the kernel $V(q, \epsilon)$ of the inter-particle interaction on the energy transferred in a collision,

$$\epsilon = \epsilon_3 - \epsilon_1. \tag{27}$$

The approach that provides a general description of the inter-particle scattering in a 2D electron Fermi gas is the random phase approximation (RPA). In particular, this approach allows to account for the energy transfer in the scattering probability $W$ by the formula analogous to Eq. (M4). The main element in calculation of $V(q, \epsilon)$ within RPA is the polarization operator $\Pi(q, \epsilon)$ of a 2D Fermi gas. Accounting of $\Pi(q, \epsilon)$ leads to the change of the static screened potential $V(q)$ (M6) in the squared scattering matrix element $M^2$ on the kernel $V(q, \epsilon)$ [1, 2]:

$$V(q, \epsilon) = \frac{\pi e^2 a_B}{1 + \frac{a_B q}{2\hbar} - i \frac{\epsilon}{\sqrt{v_F^2 q^2 - \epsilon^2}}}. \tag{28}$$

Here and below references [1], [2], [3], and so on denote the items in the reference list at the end of this Supplemental material.

At the absence of spin polarization of 2D electrons, the squared matrix element $M^2 = M^2(\varphi, \theta, \xi)$ corresponding to the kernel (28) takes the form [1, 3]:

$$M^2 = 2 \left\{ |V(q, \epsilon)|^2 + |V(w, \epsilon')|^2 - 2 \text{Re} \left[ V(q, \epsilon) V(w, \epsilon')^* \right] \right\}, \tag{29}$$

where $\epsilon' = \epsilon_3 - \epsilon_2$ is the indirect transferred energy. In Eq. (29) the first term is the direct diagonal term, while the second and the third ones are the exchange diagonal and the exchange interference terms. The scattering probability $W$ is again expressed via $M^2$ by the Golden rule-like formula (M4).

Below in this section we will analyze at which scattering angles it is possible and in which it is impossible to neglect the dependence of the matrix element $M^2$ on the transferred energies $\epsilon$ and $\epsilon'$. We present this consideration in details only for the first direct term $2|V(q, \epsilon)|^2$ in $M^2$ (20). A similar consideration for the second and the third terms in $M^2$, containing the kernel $V(w, \epsilon')$, is performed just by the change of the values corresponding to the “3” and “4” electrons.

It follows from Eqs. (20) and (M3) that the magnitudes of $q$ and $\epsilon$ independently vary in the diapasons from 0 to $\approx 2p_F$ and from 0 to $\sim T$, respectively. For the small and the large scattering angles:

$$|\theta| \sim \zeta, \quad \zeta \ll |\theta| \ll 1, \quad |\theta| \sim 1, \tag{30}$$

we have the estimates:

$$q \sim p_F \zeta, \quad q \sim p_F \theta, \quad q \sim p_F, \tag{31}$$

respectively [see Fig. 2(a) in the main text]. Thus the absolute value of the term $-i\epsilon/\sqrt{v_F^2 q^2 - \epsilon^2}$ in the denominator of Eq. (28) is estimated as:

$$\frac{\epsilon}{\sqrt{v_F^2 q^2 - \epsilon^2}} \sim 1, \quad \frac{\zeta}{|\theta|}, \quad \zeta \tag{32}$$

in the diapasons of $\theta \gg \zeta$. Therefore for the not too small angles $\theta$, $|\theta| \gg \zeta$, term (32) can be neglected in the denominator of Eq. (28) compared with unity, while at the small angles, $|\theta| \ll \zeta$, it is about unity and cannot be neglected. [Moreover, at $|\varphi| \sim \sqrt{\zeta}$ we obtain from Eq. (10) that $\theta - \varphi \ll \zeta$ therefore term (32) turns out to be even much greater than unity].

When all the energies $\epsilon_{1,2,3}$ are close to the chemical potential, $|\epsilon_{1,2,3} - \mu| \lesssim T$, it follows from Eqs. (10), (8), and (11) that at the small and the intermediate angles $\varphi$,

$$\pi - |\varphi| \sim 1 \tag{33}$$

(in particular, $|\varphi| \sim \zeta$), the scattering angles $\theta = \theta_{\pm}(\varphi, \xi)$ and $\psi = \psi_{\pm}(\varphi, \xi)$ satisfy the conditions

$$|\theta| \lesssim \zeta, \quad |\psi - \varphi| \lesssim \zeta \tag{34}$$

[the “horizontal” part of the red curve in Fig. 2(b) in the main text] or the conditions

$$|\theta - \varphi| \lesssim \zeta, \quad |\psi| \lesssim \zeta \tag{35}$$

[the “diagonal” part of the blue curve in Fig. 2(b) in the main text]. For the “head-on” collisions, when the angle of the incident electron $\varphi$ is close to $\pm \pi$,

$$\pi - |\varphi| \ll 1, \tag{36}$$
[the “vertical” parts of the blue and the red curves in Fig. 2(b) in the main text], both the scattering angles \( \theta = \theta_\pm(\varphi, \xi) \) and \( \psi = \psi_\pm(\varphi, \xi) \) become large as compared with the small parameter \( \zeta \):

\[
|\theta|, |\psi| \gg \zeta. \tag{37}
\]

So equations (34), (35), and (37) evidence the following. First, the head-on collisions \( |\pi - |\varphi| \ll 1 \) correspond to scattering on the large angles \( \theta \) and \( \psi \) [Eq. (37)]. In view of Eq. (32), for the scattering probability on them one can use the static potential \( V(q) \) (M6). Second, the collisions with the incident electrons having a small or an intermediate angle \( \varphi \) [\( |\pi - |\varphi| \sim 1 \)] are small-angle: \( |\theta| \lesssim \zeta \) or \( |\psi| \lesssim \zeta \). Thus, owing to Eq. (32), it is necessary to use the exact inter-particle interaction kernel \( V(q, \varepsilon) \) (28) in order to properly account them.

2.2. The factor \( S \) and importance of accounting for the transferred energy at different \( \varphi \)

In this subsection, we estimate the expression \( S = S(\varphi, \xi) \) in the last square brackets in the collision integral (M3):

\[
S = \Psi(\varepsilon_1, \alpha) + \Psi(\varepsilon_2, \alpha + \varphi) - \Psi(\varepsilon_3, \alpha + \theta) - \Psi(\varepsilon_4, \alpha + \psi), \tag{38}
\]

where \( \varepsilon_4 = \varepsilon_1 + \varepsilon_2 - \varepsilon_3, \theta = \theta_\pm(\varphi, \xi), \) and \( \psi = \psi_\pm(\varphi, \xi). \) For this purpose, it is convenient to put \( \alpha = 0 \) and to consider that the function \( \Psi(\varepsilon, \phi) \) is even by \( \phi \) and is normalized such as \( \Psi(\varepsilon, \phi) \sim 1 \) at general \( \phi \sim 1 \) and \( x = (\varepsilon - \mu) / T \sim \zeta. \) Thus \( S \) becomes the function of the variables \( \varphi \) and \( \xi, S = S(\varphi, \xi) \), and should be compared with unity.

Owing to Eqs. (10) and (34)-(35), we can decompose such \( S(\varphi, \xi) = S(\varphi, \bar{x}) \) at each small or medium \( \varphi, |\pi - |\varphi| \sim 1 \), in the Taylor series by the small values \( \theta \) and \( (\psi - \varphi) \) for the “\(-\)”-solutions or by the small values \( \psi \) and \( (\theta - \varphi) \) for the “\(+\)”-solutions. It follows from Eqs. (10) and (3) that the angles \( \theta_\pm(\varphi, \bar{x}) = 0 \) and \( \psi_\pm(\varphi, \bar{x}) = 0 \) are equal to 0 and \( \varphi \) or visa versa. This means that we decompose \( S \) with respect to the deviations of the angle functions \( \theta_\pm \) and \( \psi_\pm \) from their value at zero energy variable \( \bar{x} = 0 \). In Fig. 2(b) in the main text the points \([\varphi, \theta_\pm(\varphi, 0)]\) on the \((\varphi, \theta)\)-plane are drawn by the brown thin lines; while the functions \( \theta_\pm(\varphi, \bar{x}) \) at some nonzero \( \bar{x} \) are drawn by thick blue and red curves.

We express the value \( S \) in the form being convenient for the Taylor decomposition by the small deviation of the scattering angles \( \theta \) and \( \psi \) from their values at \( \varepsilon_1 = \varepsilon \varphi \). For ease of the sign “\(-\)” in the solutions \( \theta = \theta_\pm \) and \( \psi = \psi_\pm \) we write:

\[
S_- = \Psi(x_1, 0) + \Psi(x_2, \varphi) - \Psi(x_3, 0) - \Psi(x, \varphi) + [\Psi(x_3, 0) - \Psi(x_3, \theta)] + \tag{39}
\]

\[
+ [\Psi(x_4, \varphi) - \Psi(x_4, \psi)].
\]

Such form allows to make the decomposition of \( S \) by \( \theta \) and \( (\psi - \varphi) \). In the case of the sign “\(+\)” in the solutions \( \theta = \theta_\pm \) and \( \psi = \psi_\pm \) it is convenient to write:

\[
S_+ = \Psi(x_1, 0) + \Psi(x_2, \varphi) - \Psi(x_3, \varphi) - \Psi(x_4, 0) + [\Psi(x_3, \varphi) - \Psi(x_3, \theta)] + \tag{40}
\]

\[
+ [\Psi(x_4, 0) - \Psi(x_4, \psi)],
\]

instead of Eq. (39) in order to make the Taylor decomposition by the small values \( \theta \) and \( \psi \).

If \( \Psi(\varepsilon, \alpha) \) substantially depends on the angle variable \( \alpha \) and weakly depends on \( \varepsilon \) [\( \Psi(\varepsilon, \alpha) \approx \Psi(\alpha) \)], the terms in the first lines of Eqs. (39) and (40) are close to zero and the decomposition of the other terms in \( S \) in the Taylor series leads to the following estimate. At small and intermediate \( \varphi, |\pi - |\varphi| \sim 1 \) [where the scattering angles \( \theta \) and \( \psi \) lie in the diapasons (34) and (35)], \( S \) is a linear combination of \( x_i \) in the first order by \( \zeta \) and is a quadratic function of \( x_i \) in the second order by \( \zeta \):

\[
S(\varphi, \bar{x}) \approx \zeta A_1(\varphi, \bar{x}) + \zeta^2 A_2(\varphi, \bar{x}), \tag{41}
\]

where

\[
A_1(\varphi, \bar{x}) = \sum_{i=1}^{3} a_i \bar{x}_i, \quad A_2(\varphi, \bar{x}) = \sum_{i,j=1}^{3} a_{ij} x_i x_j, \tag{42}
\]

\[
a_i = a_i(\varphi) \sim 1, \quad a_{ij} = a_{ij}(\varphi) \sim 1.
\]

For the head-on collisions corresponding to \( \varphi \) near \( \pi \) [Eq. (30)], the scattering angles are large [Eq. (37)]. Provided \( \Psi(0) \) is not close to \( -\Psi(\pi) \), the combination \( S \) cannot be decomposed in the Taylor series by \( \theta, (\theta - \varphi), \psi, \) and \( (\psi - \varphi) \) and is generally estimated as:

\[
S(\varphi, \bar{x}) \sim 1. \tag{43}
\]

Visa versa, if \( \Psi(0) = -\Psi(\pi) \), we can make the decomposition of \( S \) by \( \varphi - \pi \) and therefore the inequality

\[
S(\varphi, \bar{x}) \ll 1 \tag{44}
\]

is valid instead of Eq. (43).

From comparison of the magnitudes of \( S \) in Eqs. (41) and (43) we conclude that, for the functions \( \Psi(\varepsilon, \alpha) = \cos(m\alpha) \) with even nonzero \( m \), the contributions to the collision integral (M3) from the head-on collisions \( |\pi - |\varphi| \ll 1 \) can dominate on the contribution from the small and the intermediate angles \( \varphi \) \( |\pi - |\varphi| \sim 1 \), provided
the sufficiently large magnitude of the other factors in the operator St (M3) in the region \( \pi - |\varphi| \ll 1 \). Such other factors are the delta-function factors (M7) and the matrix element \( M^2 \). Factors (M7) diverge as \( 1/|\varphi| \) at \( \varphi \to 0 \) and as \( 1/(\pi - \varphi) \) at \( \varphi \to \pi \), while matrix element (M5) [and (29)], generally speaking, decreases at \( |\varphi| \to \pi \).

Therefore, the question of the importance of the head-on collisions at \( \varphi \to \pi \) for various \( \Psi \) is non-trivial. If they actually dominate, one should use the static potential \( V(q) \) (M6) in the corresponding main contribution in St, according to the obtained in Section 2.1 relations between \( V(q) \) (M6) and the exact kernel \( V(q, \varepsilon) \) (28).

Visa versa, from Eqs. (41) and (44) it is seen that that for distribution functions \( \Psi(\varepsilon, \alpha) = \cos(m\alpha) \) with odd \( m \) [leading to \( \Psi(0) = -\Psi(\pi) \)] the factor \( S(\varphi, \varepsilon) \) acts in some other way at determining the range of the important angles \( \varphi \). For such \( \Psi \) not only the head-on collisions, but, generally, the collisions with the angles \( |\varphi| \ll 1 \) and \( |\varphi| \sim 1 \) can be also important. In accordance with the consideration of the last subsection, in this case, one needs to use the exact kernel \( V(q, \varepsilon) \) (28). If the function \( \Psi(\varepsilon, \alpha) \) substantially depends on the energy variable, for example:

\[
\Psi_m^F(\varepsilon, \alpha) = F(\varepsilon) \cos(m\alpha),
\]

the factor \( S \) (39) at intermediate angles \( \varphi, |\varphi| \sim 1 \) as well as for the head-on collisions, \( \pi - |\varphi| \ll 1 \), is related mainly with the terms in the first line in Eq. (M3) and is estimated as unity [similarly to Eq. (43)]. At small angles, \( |\varphi| \ll 1 \), for such \( \Psi = \Psi_m^F \) we have:

\[
S(\varphi, \varepsilon) \sim F(x_1) + F(x_2) - F(x_3) - F(x_4),
\]

which is also a value of the order of unity except the cases \( F(x) = 1 \) and \( F(x) = x \). We conclude that the main part of \( S(\varphi, \varepsilon) \) is estimated as unity at any \( \varphi \) and \( F \) except \( \varphi \to 0 \) and those \( F \) corresponding to perturbations of the particle density and energy.

The terms in Eqs. (39), (40) in the second and the third lines, being much smaller than the terms in the first line, are in the case of the energy-dependent distribution function \( \Psi_m^F \) (15) determine the dependence of the factor \( S \) and the collision operator \( St[\Psi_m^F] \) on the angular harmonic number \( m \), that is the angular-dependent part of \( St[\Psi_m^F] \). This fact is important for the analysis of the general properties of collision operator (see the next Sections 3 and 4).

3. CONTRIBUTIONS TO THE COLLISION INTEGRAL FROM DIFFERENT DIAPASONS OF THE ANGLES \( \varphi \)

3.1. General structure of the collision integral

The regions \( 0 \leq \varphi \leq \pi \) and \( -\pi \leq \varphi \leq 0 \) provide identical contributions to the collision integral \( St \), given by Eq. (M3). Thus in the operator \( St \) one can calculate the integral by the variable \( \varphi \) over only one of these two intervals, for example, over \( 0 \leq \varphi \leq \pi \), and to take into account the contribution from the second interval, \( -\pi \leq \varphi \leq 0 \), just by multiplication of the first contribution on the factor two.

One can distinguish the contributions to the collision integral from the collinear collisions (corresponding to \( |\varphi| \ll 1 \) and, at the same time, \( |\theta|, |\psi| \ll 1 \)); from head-on collisions (for which \( |\varphi| \approx \pi \)); and from the collisions with electrons having the intermediate angles (\( |\varphi| \sim 1 \)):

\[
St = St_0 + St_\pi + St_1.
\]

As in was discussed above in Section 2.2, the estimates of the factor \( S \) (38) demonstrates that depending on the type of the function \( \Psi_m^F \) (15), some of the terms, \( St_0 \), \( St_\pi \), or \( St_1 \), in Eq. (47) can be more or less substantial.

Further, it is seen from the general expressions (M3) for the collision integral that the operator \( St \) is diagonal in angular harmonics:

\[
St[\Psi_m^F(\varepsilon, \varphi)](\varepsilon_1, \alpha) = \cos(m\alpha) St(m)[F(\varepsilon)](\varepsilon_1),
\]

and, correspondingly:

\[
St_\xi[\Psi_m^F(\varepsilon, \varphi)](\varepsilon_1, \alpha) = \cos(m\alpha) St_\xi(m)[F(\varepsilon)](\varepsilon_1),
\]

where \( \xi = 0, \pi, 1 \). One can show from Eqs. (39) and (40) [see details below in this section] that for distribution functions of the general form, \( \Psi_m^F \) (15), it is possible to distinguish in each of the terms \( St_\xi \) the angular-independent part and the angular-dependent parts:

\[
St_\xi(m)[F] = St_\xi^m[F] + St_\xi^{dep,(m)}[F],
\]

Here it is implied that the angular-dependent part vanishes at zero harmonics:

\[
St_\xi^{dep,(m=0)} = 0,
\]

so that the collision integral on the functions being independent on \( \alpha \) is given by the the angular-independent part:

\[
St_\xi[c_0(\varphi)F(\varepsilon)] \equiv St_\xi^m[F(\varepsilon)],
\]

where \( c_0(\varphi) = 1 \).

We will show in next subsections that for energy-dependent functions (15) with \( F(x) \neq \text{const} \), some of the angle-independent contributions \( St_\xi^m \) are much greater than all the angular-dependent parts \( St_\xi^{dep} \). At the same time, for the functions \( \Psi_m^F \) (15) with \( F(x) = \text{const} \) and \( m \neq 0 \), the operators \( St_\xi^m \) become exact zero [see Eq. (M3)] and only the contributions \( St_\xi^{dep,(m)} \) remain.
3.2. The collision integral at $\varphi \to 0$

According to Eq. (M7), integration by the scattering angle $\theta$ of the expressions in the collision integral (M3) containing the energy delta-function yields:

$$\int_0^\pi d\theta \delta\left[ \varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \frac{(p_1 + p_2 - p_3)^2}{2m} \right] \times H(\varphi, \theta; \vec{x}) = \sum_{\pm} \frac{H(\varphi, \theta_\pm(\varphi); \vec{x})}{2 \varepsilon_F \sqrt{\varphi^2 + \zeta^2 \Delta}},$$

where $H(\varphi, \theta, \vec{x})$ is any regular function of the variables $\varphi, \theta$, and $\vec{x}$.

Substitution of Eqs. (15)-(17) into the exact matrix element (29), taking into account the energy dependence of the squared matrix element $\tilde{M}^2(\varphi, \vec{x})$ at the angles $\varphi, \theta$ such that $H \sim 1$, and the relation between $\varphi, \theta$, one needs to use the general expressions (M3), (52), and (54).

The analysis shows that the collinear collisions can provide the substantial contribution $S_{\text{coll}}$ in the angular-independent $S_{\text{in}}$ part of the operator $S_t$ (M3). It is written as:

$$S_{\text{coll}}(\varphi | x_1, \alpha) = \frac{T^2}{\varepsilon_F} \frac{1}{8\pi \hbar T} \int dx_2 dx_3 \frac{d\varphi}{f_1 f_2 f_3 f_4} \times$$

$$\int_{\varphi_{\text{min}}}^{\varphi_{\text{max}}} \frac{d\varphi}{\sqrt{\varphi^2 + \zeta^2 \Delta}} \sum_{\pm} \tilde{M}^2_{\pm}(\varphi(x_1, \alpha) + \varphi(x_2, \alpha))$$

$$- \left[ \Psi(x_3, \alpha) - \Psi(x_4, \alpha) \right],$$

where $f_i = 2 \cosh(x_i/2), \ i = 1, 2, 3, 4$; and, as usual, $x_4 = x_1 + x_2 - x_3$. Due to the singular character of the integrand in Eq. (59), such $S_{\text{coll}}$ must be independent on the upper limit $\varphi_{\text{max}}$ in the main order by the small parameters $\zeta \ll 1$ and $r_s \ll 1$.

According to the definition of the angular-independent part (50), in formula (59) integration over angular variable $\varphi$ does not affect the unknown distribution function $\Psi$. Therefore this integration can be performed explicitly.

At moderately low temperatures, $\zeta \ll r_s \ll \sqrt{\zeta}$, for the squared matrix element at the angles $\varphi_{\text{min}} < \varphi < r_s$ we have from (54):

$$\tilde{M}^2_{\pm} = 2 \left[ \frac{1}{|\chi_q|^2} \frac{1}{|\chi_w|^2} - \text{Re} \left( \frac{1}{\lambda_q \lambda_w} \right) \right].$$

At $\varphi \sim \zeta$ this expression as a function of $\varphi$ substantially depends on the energy variables $\vec{x}$, herewith we have:

$$\tilde{M}^2_{\pm} \sim 1.$$  

At the angles $\zeta \ll r_s$ one of the terms $2/|\chi_q|^2$ or $2/|\chi_w|^2$ (depending on the sign $\pm$) dominates in Eq. (60) and the squared matrix element becomes equal to $2$ in the main order by $r_s$ and $\zeta$:

$$\tilde{M}^2_{\pm} = 2, \quad \tilde{M}^2_{\mp} = 2.$$  

At the larger angles, $r_s \ll \varphi < \sqrt{\zeta}$ the matrix element $\tilde{M}^2_{\pm}$ is determined by general formula (54). A calculation shows that it decreases as $\sim 1/\varphi^2$:

$$\tilde{M}^2_{\pm} \sim r_s^2/\varphi^2.$$  

Using these estimates of $\tilde{M}^2_{\pm}$, one can calculate the integral by $d\varphi$ in the operator $S_{\text{in}}$:

$$I_0(\vec{x}) = \int_{\varphi_{\text{min}}}^{\varphi_{\text{max}}} d\varphi \tilde{M}^2_{\pm}(\varphi(x_1, \alpha) + \varphi(x_2, \alpha)) \approx \int_{\varphi_{\text{min}}}^{\varphi_{\text{max}}} d\varphi \tilde{M}^2_{\pm}(\varphi(x_3, \alpha) + \varphi(x_4, \alpha)).$$

Putting $\varphi_{\text{min}} = \zeta$ and $\varphi_{\text{max}} = r_s$, in view of the fast decrease of $\tilde{M}^2_{\pm}(\varphi)$ at $\varphi \gg r_s$ [Eq. (63)] and and keeping in mind the estimate $\Delta \sim 1$ at $x_1 \sim 1$, we have:

$$I_0 \approx 2 \int_{\zeta}^{r_s} d\varphi = 2 \ln(r_s/\zeta).$$
In the limit of very low temperatures, $\sqrt{\zeta} \ll r_s$, result (62) for the matrix element $M^2_\pm$ remains valid in the whole diapason, $0 < \varphi \ll \sqrt{\zeta}$, for which equations (15)-(17) and (54)-(58) have been derived. At larger angles, $\varphi \gtrsim \sqrt{\zeta}$, the matrix element is calculated by the general formulas (6), (8), (20), (21), and (29). We performed such calculation and obtained equation (61) for $M^2_\pm$ at $\sqrt{\zeta} \ll \varphi \ll r_s$ and equation (63) at $\varphi \gtrsim r_s$. This estimates are similar to the ones for the matrix element $M^2_\mp$ in the case $\zeta \ll r_s \ll \sqrt{\zeta}$ up to the change of Eq. (62) on Eq. (61), therefore we again arrive to result (65) for the equations (15)-(17) and (54)-(58) for the weak interaction).

In the case not too small temperatures (the moderately weak interaction), $\zeta^{3/2} \ll r_s \ll \zeta$, equation (64) leads to the following estimates of the matrix element:

$$M^2_\pm \sim r_s^2 / \zeta^2.$$ \hspace{1cm} (66)

at the angles $\zeta \lesssim \varphi \ll \zeta^2 / r_s$ and

$$M^2_\pm \sim \zeta^2 / \varphi^2.$$ \hspace{1cm} (67)

at the angles $\zeta^2 / r_s \lesssim \varphi \ll \sqrt{\zeta}$. In this case, we introduced the lower bound on the value $r_s$, $r_s \gg \zeta^{3/2}$, in order to fulfill the inequality $\zeta^2 / r_s \ll \sqrt{\zeta}$ allowing to use equations (15)-(17) and (54)-(58) for $M^2_\pm$.

Formulas (66) and (67) lead to the estimate $\varphi_{\text{max}} \sim \zeta^2 / r_s$ and therefore yield the following result for the integral $I_0$:

$$I_0 \sim \frac{r_s^2}{\zeta^2} \int_0^{\zeta} \frac{d\varphi}{\varphi} \approx \frac{r_s^2}{\zeta^2} \ln(\zeta / r_s).$$ \hspace{1cm} (68)

Integration by one of the energy variables $x_2$ and $x_3$ in Eq. (69) leads to the final form of the angular-independent contribution to $\text{St}$ from the collinear scattering:

$$\text{St}^{in}_0[\Psi](x_1, \alpha) = -\frac{T^2}{\varepsilon_F} \frac{1}{4\pi^2 T} \int dx' I_0 K_0(x_1, x') \times \left[ \Psi(x_1, \alpha) + \Psi(-x', \alpha) - 2 \Psi(x', \alpha) \right],$$ \hspace{1cm} (69)

where $I_0$ is given by Eqs. (65) or (68), the kernel $K_0(x_1, x')$ is an exponentially decreasing function at $|x_1|, |x'| \gtrsim 1$:

$$K_0(x_1, x') = \frac{1}{4 \cosh(x_1/2) \cosh(x'/2)} \times \frac{(x_1 - x')/2}{\sinh[(x_1 - x')/2]}.$$ \hspace{1cm} (70)

According to the definition (41), (52) of the angular independent part, we have for functions $\Psi_m$ (43) with $F(x) \neq \text{const}$

$$\text{St}^{in}_0[\Psi_m](x_1, \alpha) = \cos(ma) \text{St}^{in}_0[F(x)](x_1, 0).$$ \hspace{1cm} (71)

For the functions $\Psi_m(x, \alpha) = \Psi(\alpha)$, being independent on the energy variable $x$, operator $\text{St}^{in}_0$ is zero [see Eqs. (59) and (69)].

3.3. The collision integral at $\varphi \to \pi$

Integration of the expression in the collision integral $S_{\pi}$ with the energy delta-function by the angle $\varphi$ takes the form:

$$\int_0^\pi d\varphi \delta \left[ \varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \frac{(p_1 + p_2 - p_3)^2}{2m} \right] \times L(\varphi, \theta, \vec{x}) = \frac{L[\pi - \zeta s, \theta_+ (s, \vec{x}), \vec{x}]}{2 \varepsilon_F \zeta \sqrt{s^2 + \Delta}}.$$ \hspace{1cm} (72)

Here $L(\varphi, \theta, \vec{x})$ is any regular function of the variables $\varphi$, $\theta$, and $\vec{x}$. The value $\Delta = \Delta(\vec{x})$ is expressed via the variables $a$ and $b$: $\Delta = a^2 - b^2$.

As we discussed in Section 2, for the contribution $S_{\pi}$ to the collision integral $S_{\text{St}}$ from the angles $\pi - |\varphi| < 1$ the dependence of scattering probability on the transferred energy $\varepsilon$ is not substantial and one can use the matrix element $M^2$ (M5) with the static inter-particle potential $V(q)$ (M6).

The value $\cos \theta$, entering Eqs. (20) for $q$ and $w$ in the matrix element $M^2$, at the actual scattering angles $\theta = \theta_\pm$ and $s \sim 1$ is estimated as unity. In the diapason $1 \ll s \ll 1 / \zeta$ it takes the form:

$$\cos[\theta\pm(s, \vec{x})] \approx \mp \left[ 1 - \frac{(a \pm b)^2}{2 s^2} \right].$$ \hspace{1cm} (73)

Thus for the the values $[1 \pm \cos(\theta_\pm)]$, occurring in expressions (20) for $q$ and $w$, we obtain the relations:

$$\zeta^2 \ll 1 + \cos(\theta_+) \ll 1,$$ \hspace{1cm} (74)

$$\zeta^2 \ll 1 - \cos(\theta_-) \ll 1,$$ \hspace{1cm} (74)

$$1 - \cos(\theta_+) \approx 1 + \cos(\theta_-) \approx 2$$

at $1 \ll s \ll 1 / \zeta$. Therefore the term $\zeta^2(x_1 - x_3)^2 / 4$ in Eqs. (20) can be omitted, and the momenta $q$ and $w$ are expressed as: $q = (2\hbar / a_B) q_\pm$ and $w = (2\hbar / a_B) w_\pm$, where

$$q_\pm(s, \vec{x}) = \frac{\sqrt{1 - \cos(\theta_+)} \pm \sqrt{2}}{r_s} \approx \frac{\sqrt{2}}{r_s} \frac{|a - b|}{\sqrt{2 r_s s}},$$ \hspace{1cm} (75)

$$w_\pm(s, \vec{x}) = \frac{\sqrt{1 + \cos(\theta_+)} \pm \sqrt{2}}{\sqrt{2 r_s s}} \approx \frac{|a + b|}{r_s},$$ \hspace{1cm} (75)

Let us remind here that $r_s = \sqrt{2} \hbar / (p_F a_B) \ll 1$.

From Eqs. (75) we get the following inequalities between the factors in the direct and the exchange contri-
\[ \frac{1}{1 + q_+} \leq \frac{1}{1 + q_-}, \quad \frac{1}{1 + w_-} \leq \frac{1}{1 + w_+}, \]
\[ \frac{1}{(1 + q_-)(1 + w_-)} \leq \frac{1}{(1 + q_-)^2}, \quad \frac{1}{(1 + q_+)(1 + w_+)} \leq \frac{1}{(1 + w_+)^2}. \]

Based on inequalities (76), we obtain for the main terms in the dimensionless squared matrix element \( \tilde{M}^2(\varphi, \bar{x}) = M^2[\varphi, \theta(\varphi, \bar{x}), \bar{x}]/(\pi e a_B)^2 \):
\[ \tilde{M}^2(s, \bar{x}) = \frac{2}{(1 + q_-)^2} = \frac{2}{\left(1 + \frac{|a - b|}{\sqrt{2} r_s s}\right)^2}, \]
\[ \tilde{M}^2(s; \bar{x}) = \frac{2}{(1 + w_+)^2} = \frac{2}{\left(1 + \frac{|a + b|}{\sqrt{2} r_s s}\right)^2}. \]

The analysis shows that the collisions with the electrons having the angles \( \pi - \varphi \ll 1 \) can dominate both the angular-dependent and angular-independent parts of \( \text{St} \). For the sum \( \text{St}_\pi = \text{St}^{in}_\pi + \text{St}^{dep}_\pi \) of such the two contributions, \( \text{St}^{in}_\pi \) and \( \text{St}^{dep}_\pi \), in the main order by \( \alpha \) we obtained:
\[ \text{St}_\pi[\Psi](x_1, \alpha) = -\frac{T^2}{\varepsilon_F} \frac{1}{8\pi h T} \int \int dx_2 dx_3 \int_{s_{\text{min}}}^{s_{\text{max}}} ds \times \sum_{\pm} \tilde{M}^2(s; \bar{x}) \delta \left( \Psi(x_1, \alpha) + \Psi(x_2, \alpha + \pi) - \Psi\left(3 x_3, \alpha + \theta_+(s)\right) - \Psi\left(4 x_4, \alpha + \pi + \theta_-(s)\right) \right), \]
\[ \times \left\{ \left[ F(x_1) + F(x_2) - 2F(x_3) \right] + 2F(x_4) \left[ 1 - \cos(m\theta_-) \right] \right\}. \]

It follows from these functions that at such \( s \) the head-on collisions are simultaneously small-angle: \( |\theta_-| \ll 1 \) or \( |\psi_+| = |\pi - \theta_+| \ll 1 \) [see also Fig. 2(b) in the main text].

The change of variables \( x_3 \leftrightarrow x_4 \) in the terms “+” in Eq. (78) leads to the following simplification of \( \text{St}_\pi \) on the functions \( \Psi^F_m = F(x) \cos(m\alpha) \) (73) with even \( m \):
\[ \text{St}_\pi[\Psi^F_m](x_1, \alpha) = -\cos(m\alpha) \frac{T^2}{\varepsilon_F} \frac{1}{8\pi h T} \times \int \frac{dx_2 dx_3}{f_{1/1/2} f_{3/4}} \int_{s_{\text{min}}}^{s_{\text{max}}} ds \left\{ \left[ F(x_1) + F(x_2) - 2F(x_3) \right] + 2F(x_4) \left[ 1 - \cos(m\theta_-) \right] \right\}. \]

Here, in particular, we changed the interval of integration from \( s_{\text{min}} < s < 1/\zeta \) on \( 1 < s < 1/\zeta \), that is valid in the main order by \( \alpha \) due to the presence of the singular factor \( 1/\sqrt{s^2 + \Delta} \approx 1/s \) in the source expression (78).

We see that the operator \( \text{St}_\pi[\Psi^F_m] \) is indeed divided on the two parts, \( \text{St}_\pi = \text{St}^{in}_\pi + \text{St}^{dep}_\pi \), the first of which, \( \text{St}^{in}_\pi \), is the integral operator only by the energy variable \( x \), while the second one, \( \text{St}^{dep}_\pi \), acts on both the angular and the energy variables \( s \) and \( x \) and is equal to zero at \( m = 0 \), according to definition (84), (85).

The integration by \( d\varphi \) in the both parts can be explicitly calculated. The integral by \( s \) in the first angular-independent part of Eq. (81), \( \text{St}^{in}_\pi \), is
\[ I_{\pi}(\bar{x}) = \frac{1}{\zeta} \int_{s_{\text{min}}}^{s_{\text{max}}} ds \frac{s^2}{s} \left( r_s s + |a - b|/\sqrt{2}\right)^2, \]

At \( \zeta \ll r_s \) it turns out into:
\[ I_{\pi} \approx \int_{s_{\text{min}}}^{s_{\text{max}}} ds = \ln(r_s/\zeta) \cdot \]

The integral by \( s \) in the second angular-dependent part of Eq. (81), \( \text{St}^{dep}_\pi \), takes the form:
\[ I_s(\bar{x}) = \frac{1}{\zeta} \int_{s_{\text{min}}}^{s_{\text{max}}} ds \frac{s^2}{s} \left( r_s s + |a - b|/\sqrt{2}\right)^2, \]

In all the three cases \( \zeta \ll r_s, \zeta \gg r_s, \) and \( \zeta \sim r_s \) the result of its calculation is:
\[ I_s = 2r_s^2 \ln \left( \frac{1}{\zeta + r_s} \right). \]

It is seen from Eq. (84) that at \( \zeta \ll r_s \) the main contribution to \( I_s \) comes from the angles \( \zeta \ll \pi - \varphi \ll \zeta/r_s \), while at \( \zeta \gg r_s \) all the head-on collisions, \( \zeta \ll \pi - \varphi \ll 1 \), determine the magnitude of \( I_s \).
After the substitution of using Eqs. (79), (81), and (84) into Eq. (80) and the integration over one of the variables $x_2$ and $x_3$, the operator $S_{\pi}$ is written as:

$$S_{\pi}[\Psi_m^F](x_1, \alpha) = -\cos(m\alpha)\frac{T^2}{\varepsilon_F} \frac{1}{2\pi \hbar T}$$

$$\times \int dx' K_0(x_1, x') \left\{ I_{\pi} [F(x_1) + F(-x')] - 2F(x') \right\}.$$  \hspace{1cm} (85)

where the kernel $K_0(x_1, x')$ is the same as for the contribution $S_0$ (69).

It is important that the angular-dependent part $S_{\pi}^{dep}$ of the obtained operator $S_{\pi}$ (85), being proportional to $I_s \ll 1$, is much smaller than its angular-independent $S_{\pi}^{in}$, containing the factor $I_s \gg 1$.

At the very weak interparticle interaction, corresponding to moderately low temperatures, $r_s \ll \zeta$, an analysis of $S_{\pi}$ (78) applied to the functions $\Psi_m^F$ again allows to divide in the angle-independent and the angle-dependent parts. The second one again originates from the head-on collisions corresponding to $1 \ll s \ll 1/\zeta$ and is expressed by the last term in the curly brackets in Eq. (85) with the same $I_s$ (64). However, the angle-independent part of $S_{\pi}$ now mainly comes not from the diapason $\pi - \varphi \ll 1$, but from the intermediate angles, $\varphi \sim 1$ [see Eq. (81)].

That is, the asymptotic form of the angular-independent terms in $S_{\pi}$ in the limit $\varphi \to \pi$ becomes unacceptable for exact calculations. However, Eq. (85) now provides the estimation for the angular-independent part $S_{\pi}^{in}$ of the operator $S_{\pi}$ from the angles $\varphi \sim 1$. So the integral factor $I_{\pi}$ (81), becoming in fact the factor $I_1$ in the contribution $S_{\pi}^{in}$, is evaluated as:

$$I_{\pi} \sim \frac{r_s^2}{\zeta^2}.$$ \hspace{1cm} (86)

This value just corresponds to the estimate $M_{\pi}^2 \sim \frac{r_s^2}{\zeta^2}$ of the matrix element, being valid at the minimal scattering angles $\theta \sim \zeta$ and $\varphi \sim 1$, corresponding to $\varphi \sim 1$ [see Eq. (41)].

We see that the contribution $S_{\pi}^{in}$ with such $I_{\pi} \sim I_1$ (83) is much smaller as compared with the angular-independent part $S_0^{in}$ (69) from the collinear collisions, $\varphi \ll 1$.

From Eq. (85) we see the two properties of the operator $S_{\pi}$ acting on the functions $\Psi_m^F$ (15) with even $m$. First, as the angular-independent part of $S_{\pi}$ is much larger than the angular-dependent one, for the functions $\Psi_m^F = F(x) \cos(m\varphi)$ with $F(x) \neq \text{const}$ the equality takes place:

$$S_{\pi}[\Psi_m^F](x_1, \alpha) \approx \cos(m\alpha) S_{\pi}[F(x)](x_1, 0).$$ \hspace{1cm} (87)

Second, the first term in the curly brackets in $S_{\pi}$ (85), being the angular-independent part $S_{\pi}^{in}$, turns into zero for the function $F(x)$ equal to unity, $F(x) \equiv \text{const}$.

### 3.4. The collision integral at $\varphi \sim 1$ and relative roles of the contributions $S_0, S_{\pi}$, and $S_1$

To estimate the contribution $S_{\pi}$ to the collision integral $S_{\pi}$ from the intermediate angles, $\varphi \sim 1$, we note the following.

Due to the asymptotic behavior of the scattering angles at $\varphi \sim 1$ [Eq. (10)], the exact matrix element (29) in this diapason of $\varphi$ is estimated as:

$$M_{\pi}^2 \sim 1$$ \hspace{1cm} (88)

at low temperatures, $\zeta \ll r_s$, and as

$$M_{\pi}^2 \sim \frac{r_s^2}{\zeta^2}$$ \hspace{1cm} (89)

in the opposite case $\zeta \gg r_s$. This is just the typical value of $M_{\pi}^2$ or the “typically-small values” of the scattering angle $\theta$: $\theta \sim \zeta$ in the diapason $\varphi \sim 1$ [see Fig. 2(b) in the main text].

The factor $K_{\pi}$ from the energy delta-function is simply estimated as

$$\int d\theta \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4(\varphi, \theta, \zeta)) \sim \frac{1}{\varepsilon_F}$$ \hspace{1cm} (90)

compare with Eqs. (65) and (72)]. As it was discussed in Section 2, the factor $S_{\pi}$ (85) for functions that substantially depend on energy is of the order of unity [see Eq. (49)], while for functions $\Psi_m^F = \cos(m\varphi)$ it is a value of the order of $\zeta^2$ in the first non-vanishing order after integration over the energy variables $x_i$ [see Eq. (41) and (42)].

In view of these estimates and as the length of this interval is approximately equal to $\pi$, the contribution $S_{\pi}$ to the collision operator $S_{\pi}$ for $\Psi_m^F = F(x) \cos(m\varphi)$ with $F(x) \neq \text{const}$ is estimated as:

$$S_{\pi}[\Psi_m^F] \approx S_{\pi}^{in}[\Psi_m^F] \approx \frac{T^2}{\varepsilon_F} \frac{1}{2\pi \hbar T} \left\{ \begin{array}{ll}
\frac{1}{r_s^2} & \zeta \ll r_s \\
\frac{\zeta^2}{r_s^2} & \zeta \gg r_s
\end{array} \right.$$ \hspace{1cm} (91)

at $\zeta \ll r_s$ and $\zeta \gg r_s$. For the pure angular harmonics $\Psi_m^F = \cos(m\varphi)$ as well as for the angular-dependent part $S_{\pi}^{dep} \equiv S_{\pi}^{in}$ of the operator $S_{\pi}$ on the general functions $\Psi_m^F = F(x) \cos(m\varphi)$ [here we imply that $F(x) \sim 1$] we have:

$$S_{\pi}[\Psi_m^F] \approx S_{\pi}^{dep} \equiv S_{\pi}^{in} \approx \frac{T^2}{\varepsilon_F} \frac{1}{2\pi \hbar T} \left\{ \begin{array}{ll}
\frac{1}{r_s^2} & \zeta \ll r_s \\
\frac{\zeta^2}{r_s^2} & \zeta \gg r_s
\end{array} \right.$$ \hspace{1cm} (92)

An analysis based on equations (91), (92) and the results of Sections 3.2, 3.3 shows that, at $\zeta \ll r_s$, for the functions $\Psi = \Psi_m^F$ (15) with $F(x) \neq \text{const}$ the collinear and the head-on collisions provide the two main similar contributions $S_0^{in}$ and $S_{\pi}^{in}$ to the angular-independent part of $S_{\pi}$ [the terms given by Eqs. (69), (65) and (85),
transport regimes, described by Eq. (94) \[4\].

In the case \( r_s \ll \zeta \), the collinear scattering make the dominant contribution to the angular-independent part of \( \text{St}[\Psi] \) for the distribution functions \( \Psi = \Psi^F_m \) with \( F(x) \neq \text{const} \) and \( \text{St}^0 \) as given by Eqs. (93), (95), while the angular-dependent part of \( \text{St}[\Psi^F_m] \) for even \( m \) and any \( F(x) \) is again determined mainly by the head-on collisions \( \text{St}^\text{dep} \) described by equations \[55\], \[51\], despite the fact that their contribution to the angular-independent part of \( \text{St} \) becomes insignificant \( \text{compare Eqs. (93) and (94)} \).

The relative magnitudes of the angular-dependent contributions to \( \text{St} \) for the functions \( \Psi = \Psi^F_m \) being proportional to odd harmonics \( m \) will be studied in details further, in Section 5.3 and 5.4. Here we announce that at low temperatures, \( \zeta \ll r_s \), the angular-dependent part of \( \text{St}[\Psi^F_m] \) also originate from the head-on collisions, corresponding to \( \varphi \) in the vicinity of the angle \( \varphi = \pi \).

4. THE KINETIC EQUATION FOR 2D ELECTRON FERMI GAS

4.1. Transport regimes and their description by the kinetic equation

For the problems of weakly nonequilibrium transport in a 2D electron gas, the electron distribution function has the form:

\[ f_P = f_F(\varepsilon) + \delta f_P, \tag{93} \]

where the perturbed part is written as \( \delta f_P = -f_F'(\varepsilon)\Psi(\varepsilon, \alpha) \). The kinetic equation in the presence of an external electric field and a magnetic field perpendicular to the electron layer is:

\[ -f_F'(\varepsilon) \left[ \frac{\partial \Psi}{\partial t} + \mathbf{v} \cdot \nabla \Psi + \varepsilon c_0 \nabla \Psi + \frac{\partial \Psi}{\partial \alpha} - \varepsilon \mathbf{E}(r,t) \cdot \mathbf{v} \right] = \text{St}[\Psi], \tag{94} \]

where \( \omega_c \) is the cyclotron frequency and \( \mathbf{E}(r,t) \) is the electric field consisting of the external field \( \mathbf{E}_{\text{ext}}(t) \) and the internal field \( \mathbf{E}_{\text{int}}(r,t) \) due to violation of the local charge neutrality in the electron gas. The electric field term \( -e\mathbf{E}_{\text{int}} \cdot \mathbf{v} \) in Eq. (94) can be combined with the gradient term \( \nabla \Psi/\partial \alpha \) by the change \( \Psi(\varepsilon, \varphi, r, t) \rightarrow \Psi(\varepsilon, \alpha, r, t) + \nabla \Phi_{\text{el}}(r,t) \), in which the zero harmonic by the angle \( \alpha \) of the generalized distribution function \( \Psi \) takes into account the electric potential \( \Phi_{\text{el}} \) \( (\mathbf{E}_{\text{int}} = -\nabla \Phi_{\text{el}}). \)

One can distinguish the two types of transport regimes, described by Eq. (94) \[4\].

First, when the cyclotron frequency \( \omega_c \) and the frequency \( \omega \) of variation of \( \Psi(r,t) \) are slow as compared with the character value of the operator \( \text{St} \) as well as the character wavevector \( k \) of the function \( \Psi(r,t) \) is small compared with \( \text{St}/v_F \), the true hydrodynamic regime is realized. The function \( \Psi \) is divided on the two parts:

\[ \Psi = \Psi_0 + \Psi_1, \tag{95} \]

where \( \Psi_0 \) is the locally equilibrium distribution, being the correction to the Fermi function due to slowly varying parameters \( \mu, T \), or the hydrodynamic velocity \( \mathbf{v} \), while \( \Psi_1 \ll \Psi_0 \) is the nonequilibrium part, whose form depends on the considered transport problem. This part \( \Psi_1 \) describes dissipation and, for the given problem, is proportional to the fixed angular harmonic by \( \alpha \) and has the dependence on \( \varepsilon \) of a fixed type. The kinetic equation (94) leads to the two macroscopic equations. The first one is the Fourier law-like equation connecting the gradients of \( \mu, T \), or \( \mathbf{v} \), defining the function \( \Psi_0 \), with the flow values related to the nonequilibrium part \( \Psi_1 \) (the heat flow \( \mathbf{q} \), the shear stress tensor \( \sigma_{\alpha k} \), and so on) via the kinetic coefficients. The second one is the transport equation, governing the slow evolution of \( \mu, T \), or \( \mathbf{v} \) and containing the flow values \( \mathbf{q}, \sigma_{\alpha k} \).

The macroscopic equation, connecting the gradient of equilibrium values with the inequilibrium ones, in the case of low frequencies \( \omega, \omega_c \ll \text{St} \), is derived from Eq. (94) taking the form:

\[ -f_F'(\varepsilon) \mathbf{v}_P \cdot \nabla \Psi = \text{St}[\Psi]. \tag{96} \]

Here in the left part we should take into account only the locally equilibrium part \( \Psi_0 \gg \Psi_1 \), while the right part contains only the nonequilibrium part \( \text{St}[\Psi_1] \), as the collision integral conserve any locally equilibrium distribution:

\[ \text{St}[\Psi_0] = 0. \tag{97} \]

So we obtain an inhomogeneous integral equation for \( \Psi_1 \), that should be solved to find the kinetic coefficient. Adding of the time-dependent and the magnetic field terms \( \partial \Psi/\partial t \) and \( \omega_c \partial \Psi/\partial \alpha \) in Eq. (96) leads to the time dispersion and the magnetic field dependencies of kinetic coefficients, which both are relatively weak at \( \omega, \omega_c \ll \text{St} \).

For the first time, the problems of hydrodynamic-like transport regimes in 2D electron systems was considered within Eq. (96) in Refs. [4].

One of such regimes is the low-frequency viscous transport in which an inhomogeneous flow \( \mathbf{V}(r,t) \) leads to diffusion of momentum. The kinetic coefficient in this problem is the viscosity \( \eta \) defined as the coefficient in the linear relation between the shear stress tensor:

\[ \sigma_{\alpha k}(r,t) = \int \frac{d^3p}{2\pi^2} f_F'(\varepsilon) \Psi(\varepsilon, \alpha) p_i v_{ik}, \tag{98} \]
and the gradients of the velocity, ∂V/∂x_k:

\[ \sigma_{ik} = \eta \left( \frac{\partial V_i}{\partial x_k} + \frac{\partial V_k}{\partial x_i} \right). \]  

Equations (98) and (99) are written for the case of an incompressible flow for which div\\(V) = 0. The perturbed part \(\Psi\) of the distribution function has the form:

\[ \Psi(\varepsilon, \alpha) = V \cdot p + \frac{mv_F^2}{4} \left( \frac{\partial V_i}{\partial x_k} + \frac{\partial V_k}{\partial x_i} \right) F_{ik}(x, \alpha), \]  

where, as usual, \(x = (\varepsilon - \mu)/T\), the first term \(\Psi_0 = V \cdot p) describes the locally equilibrium flow, and the functions \(F_{ik}(x, \alpha)\), determining the nonequilibrium part \(\Psi_1\), are proportional to the second angular harmonics by the velocity angle \(\alpha\):

\[ F_{xx}(x, \alpha) = F(x) \cos(2\alpha), \]
\[ F_{yy}(x, \alpha) = -F(x) \cos(2\alpha), \]
\[ F_{xy}(x, \alpha) = F(x) \sin(2\alpha). \]  

The function \(F(x)\) should be found from the inhomogeneous integral equation (M9):

\[ \text{St}[F(x) \cos(2\phi)](\varepsilon_1, \alpha) = -f_{F'}(\varepsilon_1) \cos(2\alpha). \]

As the right part of this equation and the kernel of the operator \(\text{St}\) are even relative to the changes \((\varepsilon_1 - \mu) \rightarrow -(\varepsilon_1 - \mu)\) and \(x_1, x' \rightarrow -x_1, -x'\) [see Eqs. (63) and (55)], the resulting function \(F(x)\) will be also even relative to \(x \rightarrow -x\). Below we will solve it and calculate from the obtained \(F(x)\) the shear stress relaxation time \(\tau_{ee,2}\) which is determined via the viscosity as

\[ \eta = m n v_F^2 \tau_{ee,2}/4. \]

From Eqs. (98)-(100) we have for this time parameter:

\[ \tau_{ee,2} = \int dx f_{F'}(x) F(x). \]

In the low-frequency heat transport, an inhomogeneous temperature \(T(r, t) = T + \delta T(r, t)\) induces a transfer of heat due to diffusion of hot and cool particles in opposite directions. The thermal conductivity coefficient \(\kappa\) is defined as the coefficient proportionality in the Fourier law,

\[ q = -\kappa \nabla \delta T, \]

connecting the gradient \(\nabla \delta T\) and the heat flow

\[ q(r, t) = \int \frac{d^2p}{2\pi^2 p_F^2} [-f_{F'}(\varepsilon)] \Psi(\varepsilon, \alpha)(\varepsilon - \mu) v. \]

The heat flow relaxation time \(\tau_{ee,h}\) can be defined as the coefficient in the formula

\[ \kappa = T n v_F^2 \tau_{ee,h}. \]  

The distribution function for this problem has the form:

\[ \Psi(\varepsilon, \alpha) = \delta T \frac{\varepsilon - \mu}{T} + v_F \frac{\partial \delta T}{\partial x_i} G_i(x, \alpha), \]

where the first term \(\Psi_0 = \delta T (\varepsilon - \mu)/T\) is the locally equilibrium part and the inequilibrium part is determined by \(G_x(x, \alpha) = G(x) \cos \alpha\) and \(G_y(x, \alpha) = G(x) \sin \alpha\). For such function \(\Psi\), equation (99) becomes the inhomogeneous integral equation for \(G(x)\):

\[ \text{St}[G(x) \cos \phi](\varepsilon_1, \alpha) = -f_{F'}(\varepsilon_1) x_1 \cos \alpha, \]

where \(x_1 = (\varepsilon_1 - \mu)/T.\) After obtaining \(G(x)\) from it, we can calculate the time \(\tau_{ee,h}\) by the formula:

\[ \tau_{ee,h} = \int dx f_{F'}(x) x G(x). \]

Second, if the character wavevector \(k\) of the function \(\Psi(r, t)\) is small compared with the character value of \(St/v_F\) or the values \(\omega/v_F, \omega_c/v_F\), but the frequency \(\omega, \omega_c\) are not small [\(\omega, \omega_c \sim St \gg St\)], the transport regime similar to the hydrodynamic one can be realized. In it, the distribution function \(\Psi\) is not divided on the equilibrium and nonequilibrium parts. Now its form is controlled by the inequalities \(v_F k \ll \omega, \omega_c, \) or \(St\), dictating the relations between the magnitudes of the angular harmonics of \(\Psi\). However, can be still valid the decomposition \(\Psi = \Psi_0 + \Psi_1,\) similar to Eq. (15), in which the first term \(\Psi_0\) describes the kinematic (not equilibrium) motion of the electron gas in fields and is responsible the values of particle flows, while \(\Psi_1\) describes dissipative processes (2). The macroscopic kinetic relations, following from Eq. (56), between the flows proportional to \(\Psi_1\) and the gradients of the values related to \(\Psi_0\) are again valid, but become nonlocal in time.

The high-frequency viscous transport in a magnetic field is an example of such hydrodynamical-like regime. The relations between the shear stress tensor \(\sigma_{ik}(r, t)\) and the gradients of velocity \(V(r, t)\) are given in Refs. (4, 9–11). In those works, a simplified consideration neglecting the energy dependence of \(\Psi_1\) was performed. The resulting viscosity coefficients becomes strongly dependent on the frequencies \(\omega, \omega_c,\) \(\omega, \omega_c\).

Based on the exact form (55) of the part \(St_\pi\) of the collision operator \(St\), we will show in the next subsection that the inverse operator \(St^{-1}\) for the even functions \(F(x)\), \(F(x) = F(-x)\), turns out to be close to the projector on the function \(\Phi_0^2(x) = 1\). Because of this, all the terms in the kinetic equation (24) for the problems of the viscous transport have, roughly speaking, approximately the same energy dependence. This can be presented by the schematic equality (“the relaxation time approximation”):

\[ St^{-1} \rightarrow -\tau_{ee,2}. \]
Thus the simplified consideration of the viscous transport in Refs. [9-11] neglecting the energy dependence of $\Psi_1$ is, in fact, asymptotically exact.

The problem of the heat transport (in zero as well as in nonzero magnetic field) seems to cannot be simplified in a similar way. Its rigorous treatment requires the solution of the kinetic equation (94), which becomes the integral equation for $G_\epsilon(x)$ and $G_p(x)$, with the inhomogeneous term proportional to $x_1 f_p'(\epsilon_1)$, the homogeneous terms with the factors $\omega_x, \omega_z$, and the integral term $St[\Psi]$. Such inhomogeneous term is odd relative to the change $x_1 \rightarrow -x_1$. We will demonstrate in the next subsection that for the kinetic equation with the inhomogeneous term the simplification of St by the change:

$$\text{St}^{-1} \rightarrow -\tau_{ee,h},$$

is not possible. Therefore the functions $G_\epsilon(x)$, apparently, are not universal for different relations between $\omega, \omega_c, \text{and St}$, as it takes place for the high-frequency viscous transport in magnetic field (see the above paragraph). Correspondingly, the thermal conductivity cannot be expressed in a closed form as a function of $\omega$ and $\omega_c$ as it was done for the viscosity tensor $\hat{\eta}(\omega, \omega_c)$ in Refs. [9,10].

One can be interested in the problem of relaxation of different types of spatially homogenous perturbations. Such process should be studied on the base of Eq. (94) with only the first and the last terms remained:

$$-f_p'(\epsilon) \frac{\partial \Psi}{\partial t} = \text{St}[\Psi].$$

(113)

This problem can be important, for example, for determination of the limits of applicability of the hydrodynamic description with respect to flow geometry and for numerical studies of the high-frequency heat transport, when simplification (112) of St is impossible. In addition, a nontrivial transport regime related to different relaxation rates of the odd and even harmonics, the superdiffusion of electrons by the velocity angle, was considered for 2D electrons in recent publications (see, for example, Ref. [12]).

To solve this problem, one wonders the (weighted) eigenfunctions $\Psi_i$ and the eigenvalues $\lambda_i$ of St (for details of the eigenvalue problem see the next section):

$$\lambda_i \Psi_i[-f_p'(\epsilon)] = \text{St}[\Psi_i].$$

(114)

Then the general solution of Eq. (113) is written as:

$$\Psi(t) = \sum_i c_i \Psi_i e^{\lambda_i t}.$$ 

(115)

Applicability of the “relaxation time approximation” (111) for kinetic equation (113) with the distribution functions proportional to each harmonic $m$ means that St has the eigenfunction:

$$\Psi_m(x, \alpha) = \cos(m \alpha),$$

(116)

[and also $\sin(m \alpha)$] with no dependence on the energy variable $x$: $F_m(x) \equiv 1$. The corresponding eigenvalues $\lambda^{(m)}, \lambda^{(m)} \in \{\lambda_i\}$, are the rates of relaxation of the “pure angular” perturbations.

We see from Eq. (101) that such function (110) for $m = 2$ has the form of the nonequilibrium part of the distribution function $\Psi$ (104) for the viscous transport, if the rule (111) is applicable. Therefore, the problem of calculating the viscosity $\eta$ and calculation of the relaxation rate of the functions (116) at $m = 2$ are directly related. Namely, the $m = 2$ eigenvalue is $\lambda^{(2)} = -1/\tau_{ee,2}$, where $\tau_{ee,2}$ is the shear stress relaxation time, defined above as the time parameter in the viscosity coefficient $\eta = m n v_2^2 \tau_{ee,2}/4$.

4.2. The mathematical structure and the method of solution of the kinetic equation

In Section 3, we examined the contributions to the operator St from different types of collisions. In this section, we will use the obtained properties of these contributions to consistently construct the solution of the kinetic equation for the problems of the viscous transport, of the heat transport, and of relaxation of higher angular harmonics.

For the viscous transport, the distribution function $\Psi_1$ is proportional to the second angular harmonics $m = 2$: $F_{xx}(x, \alpha) = F(x) \cos(2\alpha)$ [see Eqs. (100) and (101)]. We must consider that $F(x)$ is even by $x$ due to the even inhomogeneous term in Eq. (94), but, generally speaking, $F(x)$ could substantially depend on $x$, as it takes place in the 3D case [13,14].

The main contribution to the angular-dependent part of St at low temperatures, $\zeta \ll r_s$ for such $\Psi_1$ originates from the head-on collisions [the term in Eq. (85) proportional to $I_s$]. Herewith the collinear, $|\varphi| \ll 1$, and the head-on collisions, $\pi - |\varphi| \ll 1$, give the same contributions to the angle-independent part of St [Eqs. (69) and (85) proportional to $I_0$ and $I_2$]. In the case of the weak interparticle interaction, when $\zeta \gg r_s$, the angular-dependent part of St again comes predominantly from the head-on collisions, but the angle-independent part of St originates now originates only from the collinear scattering.

The kinetic equation Eq. (96) with the collision operator $\text{St} = \text{St}_0 + \text{St}_t$ for the odd functions $F(x)$, $F(-x) = F(x)$, can be presented as: $Qf = c$, or, in the exact form:

$$\int Q(x, x') f(x') \ dx' = c(x)$$

(117)

where the function $f(x)$ is proportional to the unknown function $F(x)$:

$$f(x) = \frac{F(x)}{2 \cosh(x/2)};$$

(118)
the function \( c(x) \) is proportional to the inhomogeneous term in Eq. (119):

\[
c(x) = \frac{1}{2 \cosh(x/2)} ;
\]

the kernel of the operator \( Q \) in the case of low temperature, \( \zeta \ll r_s \), is given by the formulas:

\[
Q(x, x') = 2Q_0(x, x') + Q_1(x, x') ,
\]

where the first term is the angular-independent part:

\[
Q_0(x, x') = K(x - x') - \frac{\delta(x - x')}{c(x')} \int K(x' - x'') c(x'') dx'', \tag{121}
\]

\[
K(x) = \frac{T^2}{\varepsilon_F} \frac{I_s}{4\pi\hbar \sinh(x/2)} ,
\]

while the second term in Eq. (120) is the angular-dependent part: \( Q_1(x, x') = -\xi K(x - x') \), where

\[
\xi = \xi(m) = m^n I_s/I_\pi \ll 1 \tag{123}
\]

\([m = 2 for the viscous transport, but the further consideration is valid for any even \( m \ll I_\pi/I_s \)].\) These formulas imply that the kernel of the collision operator \( St_x \) is related to the kernel of the operator \( Q = 2Q_0 + Q_1 \) as:

\[
St_x(x, x') = [Q_0(x, x') + Q_1(x, x')] c(x) c(x') / T . \tag{124}
\]

Doubling of the operator \( Q_0 \) in \( Q \) [120] occurs due to the equal contributions from the collinear and the head-on collisions to the angle-independent part of the collision operator \( St = St_0 + St_x \) [see Eqs. (193) and (195)].

For the case of the higher temperatures, \( \zeta \gg r_s \), we should make only one change, \( 2Q_0 \rightarrow Q_0 \), in the above formula (120) for \( Q \). All other formulas (117)-(124) remain valid.

Note that the derivative of the Fermi function is connected with \( c(x) \) as:

\[
-f'_F(x) = c(x)^2 .
\]

From Eq. (124) it is seen that

\[
\int Q_0(x, x') c(x') dx' = 0 . \tag{125}
\]

This means that \( c(x) \) is the eigenfunction \( \Phi_0^0(x) \) of the operator \( Q_0 \) corresponding to the zero eigenvalue \( \lambda_0^0 = 0 \). The function \( \Phi_0^0(x) = c(x) \) is normalized on unity as:

\[
\int c(x)^2 dx = 1 . \tag{126}
\]

Owing to inequality \( \xi \ll 1 \), the operator \( Q_1 \), acting on the angular variable of \( \Psi \), is a small perturbation in the operator \( Q \) relative to the angular-independent part \( 2Q_0 \). As it seen from Section 3, this fact, being crucial for our consideration, originates from the singular character of electron-electron collisions in the 2D case reflected by the Fermi function factor [1], the energy delta-function factor (M7), and the sharp dependence of the interaction potential (M6) on the scattering angles.

So the eigenvalues and the eigenfunctions of \( Q \) can be found by the perturbation theory by \( Q_1 \). It can be proved by the analysis of the operator \( Q_0 \) in the Fourier transformed form that the eigenvalues \( \lambda_i^0 \) and the eigenfunctions \( \Phi_i^0(x) \) of the operator \( Q_0 \) are discrete and not degenerated (see analogous considerations for the similar kernels in 2D and 3D cases in Refs. [14] and [13]). For the eigenvalues \( \lambda_i \) and the eigenfunctions \( \Phi_i(x) \) of the operator \( Q \) we have in this case:

\[
\lambda_i = \lambda_i^0 + \Delta \lambda_i , \quad \Delta \lambda_i = \langle \Phi_0^i , Q_i \Phi_0^i \rangle , \tag{127}
\]

\[
\Phi_i(x) \approx \Phi_0^i(x) ,
\]

where \( \langle \Phi_1, \Phi_2 \rangle = \int \Phi_1(x) \Phi_2(x) dx \) is the scalar product in the Hilbert space \( \{ \Phi(x) \} \) of real functions \( \Phi(x) \).

In particular, for the minimal eigenvalue \( \lambda_0 \) of the operator \( Q \), corresponding to zero eigenvalue \( \lambda_0^0 = 0 \) of the operator \( Q_0 \) we have:

\[
\lambda_0 = \langle c, Q_1 c \rangle = \int dx dx' c(x) Q_1(x, x') c(x') = \int d\varepsilon_1 St_\pi [ F(x) = 1 ] |\varepsilon_1| . \tag{128}
\]

Solution of the kinetic equation (117) can be now written using the basis \( \Phi_i^0 \) of the eigenfunctions of \( Q \). The kernel \( Q_{inv}(x, x') \) of the inverse operator \( Q^{-1} \) is given by the usual formula:

\[
Q_{inv}(x, x') = \sum_i \frac{\Phi_i^0(x) \Phi_i(x')}{\lambda_i^0 + \Delta \lambda_i} . \tag{129}
\]

Thus the solution of Eq. (117) is written as:

\[
f(x) = \int \frac{\Phi_i^0(x)}{\lambda_i^0 + \Delta \lambda_i} \int \Phi_i(x') c(x') dx' . \tag{130}
\]

Due to the discussed above properties of \( \lambda_i^0 \), in Eq. (130) the term with \( i = 0 \) dominates. Thus the function \( f(x) \) is approximately expressed via the minimal eigenvalue \( \lambda_0 = \Delta \lambda_0 \) and its eigenfunction \( \Phi_0^0 \approx \Phi_0 = c \).

In this way, we have from Eqs. (126) and (130):

\[
f(x) \approx \frac{c(x)}{\Delta \lambda_0} . \tag{131}
\]

In terms of the function \( F(x) \) result (131) and takes the form:

\[
F(x) = \frac{1}{\Delta \lambda_0} . \tag{132}
\]
For the relaxation time $\tau_{ee,2}$ in the viscosity coefficient, according to Eq. (124), we obtain:

$$\tau_{ee,2} = -\frac{1}{\Delta \lambda_0} = -\left\{ \int d\xi \St_\xi [\cos(2\alpha)](\xi_1,0) \right\}^{-1} \quad \text{(133)}$$

By this way, the special character of the kernel $Q$ [the existence of the eigenfunction $\Phi_0^\alpha(x) = c(x)$ corresponding to the zero eigenvalue $\lambda_0^0 = 0$ for the dominating part $Q_0 \gg Q_1$] greatly simplifies solution the kinetic equation. Its proper solution, being asymptotically exact by the parameter $\xi \ll 1$, is a constant [see Eq. (132)].

Now we briefly discuss the ways of solution of the kinetic equations (111) and (126) for the other problems of the hydrodynamic transport: the viscous transport at nonzero frequencies $\omega$ and $\omega_c$, the heat transport, and the auxiliary problem of relaxation of angular harmonics $\Psi_m$ (116).

For the problem of high-frequency viscous transport in magnetic field, the part $\Psi_1(x,\alpha)$ of the disturbed distribution function $\Psi(x,\alpha)$ is now proportional to $\cos(2\alpha)$ and $\sin(2\alpha)$. The operator $Q$ remains the same for each term $F(x)\cos(2\alpha)$ and $F(x)\sin(2\alpha)$ as in the discussed above case when $\omega,\omega_c = 0$. The energy dependence of the inhomogeneous term in equations (124), is even, as in the case $\omega,\omega_c = 0$:

$$c(x) = c(-x) \quad \text{(134)}$$

One can again apply the operator $Q^{-1}$, expressed via formula (129), to all terms in the kinetic equation (111). As the result, all the terms of the transformed kinetic equation, $Q^{-1}c, \omega Q^{-1}f, \omega Q^{-1}\partial f/\partial \alpha$, and thus, $Q^{-1}Qf = f$ will be proportional to the eigenfunction of $\Phi_0^\alpha(x) = c(x) = 1/[2 \cosh(x/2)]$, corresponding to the zero eigenvalue $\lambda_0^0 = 0$ of the operator $Q_0$. So the energy dependence of the solution $f(x)$ will be the same, $c(x)$, as for the discussed above problem of the slow hydrodynamic at $\omega,\omega_c \to 0$.

This consideration is the proof that the operator $St$ in Eq. (94) just should be replaced by its minimal eigenvalue $\lambda_0 = -1/\tau_{ee,2}$, according to the “relaxation time approximation rule” (111). So we justify the approach of construction of the high-frequency hydrodynamic equations in magnetic field, developed in Refs. (3-11).

For the problems of low and high-frequency thermal transport in zero and nonzero magnetic field, the part $\Psi_1(x,\alpha)$ of the disturbed function $\Psi(x,\alpha)$ is proportional to the first angular harmonics $\cos \alpha$, $\sin \alpha$. The kernel $Q$ is again consists of the two parts: the main part $Q_0$, being the same as for the viscous transport, and the angular-dependent correction $Q_1^h$:

$$Q = Q_1^h + Q_0 \quad , \quad Q_1^h \ll Q_0 \quad \text{(135)}$$

being different from the one for the viscous transport. The estimate (135), as the analogous estimate $\xi(m) \ll 1$ [see Eq. (123)] for relaxation of the even harmonics, is due to special character of kinematics of 2D electron collisions. The energy factor $d(x) = x(x)$ in the inhomogeneous term in Eqs. (111) and (126) is now an odd function relative the change $x \to -x$ [see Eq. (109)]:

$$d(x) = -d(-x) \quad \text{(136)}$$

Such $d(x)$ turns out to be orthogonal to the eigenfunction $\Phi_0^\alpha(x) = c(x)$ of $Q_0$:

$$\int dx \Phi_0^\alpha(x) d(x) = 0 \quad \text{(137)}$$

Therefore, although the main term of the kernel $Q_{inv}(x,x')$ of the the inverse operator $Q^{-1}$ is proportional to $\Phi_0^\alpha(x)\Phi_0^\alpha(x')/\Delta \lambda_0$, where $\Delta \lambda_0 \sim Q_1 \sim \xi$, the solution of the kinetic equation

$$Qg = d \quad \text{(138)}$$

is not expressed the simple form similar to Eq. (130). In this case, one needs to perform solution of the kinetic equation (138) analogous to Eq. (117), but containing $d(x)$ instead of $c(x)$, by the method of the decomposition of $g(x)$ is series by some basis, for example, by the eigenfunctions $\Phi_0^\alpha(x)$ of $Q_0$. Herewith, one should find, first of all, the distribution function $g(x)$ with the neglect of the small angular-dependent term $Q_1^h \ll Q_0$ in the operator $Q$.

A similar integral equation for the thermal transport in the 3D Fermi systems was solved in Ref. [4].

Finally, we present the exact formulas for the problem of relaxation of space-homogeneous functions proportional to the higher angular harmonics (116).

As we noted in previous subsection, applicability of Eq. (111) to the homogeneous kinetic equation (113) means that it has the solutions $\Psi_m e^{i\chi(m)x}$, where $\Psi_m(x,\alpha) = F_m(x)\cos(m\alpha)$ with $F_m(x) \approx 1$ [Eq. (116)] and $\chi (m)$ is the eigenvalue of the problem:

$$\chi (m)f_m(x) = \int Q^{(m)}(x,x')f_m(x')dx' \quad \text{(139)}$$

Here $f_m(x) = F_m(x)/[2 \cosh(x/2)]$ and $Q^{(m)}(x,x')$ is the kernel (120) for even $m$ and is the kernel of general collision integral $St$ (M3) for odd $m$. The equality $F_m(x) \approx 1$ is the consequence of equality (111), so they both should be proved on base of properties of $Q^{(m)}(x,x')$ in the eigenproblem (139) formulated for arbitrary $F_m(x)$.

Our analysis shows that for odd $m$ the operator $Q^{(m)}$ also has the decomposition

$$Q^{(m)} = Q_0 + Q_1^{(m)} \quad , \quad Q_1^{(m)} \ll Q_0 \quad \text{(140)}$$

on the angular-independent term $Q_0$, identical with Eq. (121) for the discussed above case of even $m$, and
the smaller angular-dependent term $Q_1^{(m)}$. Therefore inversion of $Q^{(m)}$ in Eq. (139) by using Eq. (129) leads to the eigenfunction

$$f_m(x) \approx c(x) = 1/[2 \cosh(x/2)] \quad (141)$$

of the operator $Q^{(m)}$, corresponding to the minimal eigenvalue $\lambda^{(m)}$ of $Q^{(m)}$,

$$\lambda^{(m)} \sim Q_1^{(m)} \quad (142)$$

[analogous to $\Delta \lambda_0$ in the above consideration for even $m$]. Such eigenvalue provides the relaxation time $\tau_{ee,m} \equiv -1/\lambda^{(m)}$ of pure angular harmonic perturbations $\Psi_m(x,\alpha) = \cos(m\alpha)$ of odd $m$.

In this way, we conclude that the time of relaxation of even and odd harmonics is given by formula:

$$\tau_{ee,m} = -\left\{ \int d\varepsilon \, S_t[\Psi_m(\varepsilon,0)] \right\}^{-1}. \quad (143)$$

Here $S_t$ denotes the contribution to $S_t$, being most important for the given $\Psi_m = \cos(m\varphi)$. For even $m$ $S_t = S_t$, as we have demonstrated in Section 3.3. For odd $m$, the contribution $S_t[\Psi_m]$ from the angles $\varphi \approx 1$ dominates in the value of $S_t[\Psi_m]$ (see the proof of this point below in Section 5.3).

The other eigenvalues $\lambda^{(m)}_i$ of $Q^{(m)}$ for odd as well as even $m$ have significantly larger magnitudes, proportional to the value of the operator $Q_0$. They are responsible for relaxation of the nonequilibrium distribution both by angle and by energy, having the form $\cos(m\alpha)\Phi_1(x)$, where $\Phi_1(x)$ substantially depends on $x$. Such eigenfunctions describe the effect of thermal conductivity (see the previous subsection) and a thermalization of equilibrium excitations of a general type.

5. CALCULATION OF THE RELAXATION TIMES

5.1. Electron departure time (quantum lifetime)

First of all, we note that the distribution functions with $\Psi_i = \sqrt{2m\varepsilon / \cos \alpha}$, $\Psi_n = 1$, and $\Psi_m = \varepsilon - \mu$, describing a homogeneous particle flow, a perturbation of the particle density and a perturbation of the energy density, do not decay due to inter-particle collisions. Indeed, for such functions the expression $S(t)$, entering the exact collision integral $S_t$ (M3), is zero. This fact reflects the conservation of momentum, number of particles, and of energy in inter-particle collisions.

In this subsection we calculate the reciprocal departure time due to interparticle scattering, averaged by energy and determining the width of the electron energy levels in a Fermi gas:

$$\frac{1}{\tau_{ee,q}} = \int \frac{d\varepsilon}{T} \, f_F(\varepsilon_1) \sum_{2,3} W_{12,3} 1.34 f_F(\varepsilon_2) \times$$

$$\times \left[1 - f_F(\varepsilon_3)\right]\left[1 - f_F(\varepsilon_4)\right]. \quad (144)$$

Here the scattering probability $W_{12,3} \equiv W(\varphi, \theta, \varepsilon)$ is the same as in Eq. (M4) the main text.

For the first time, the value $1/\tau_{ee,q}$ for 2D electrons was apparently introduced and calculated in Ref. [10].

For details of calculation of $1/\tau_{ee,q}$ within the method of the density-density response function see Ref. [17].

From comparison of Eq. (144) with the departure terms in the collision integrals $S_{i\pi}$ and $S_{\pi}$ [93], one can see that, at low temperatures, $\varsigma \ll r_s$, the collinear and the head-on collisions provides the similar contributions to the rate $1/\tau_{ee,q}$. $1/\tau_{ee,q} = 1/\tau_{\pi,q}$ [namely, this is related to the equality $I_0 = 2I_\pi$. Due to the presence of the singular factors

$$\frac{1}{\sqrt{\varphi^2 + \varsigma^2 \Delta}} \quad \text{and} \quad \frac{1}{\sqrt{s^2 + \Delta}} \quad (145)$$

in $S_{i\pi}$ and $S_{\pi}$, the contribution from the incident electrons with the angles $|\varphi| \sim 1$ is negligibly small. Thus the departure rate $1/\tau_{ee,q}$ is calculated via the kernel $K(x) [120]$, describing the departure terms in both $S_{i\pi}$ and $S_{\pi}$:

$$\frac{1}{\tau_{ee,q}^{\pi}} = \int dx_1 \, dx' \, c(x_1) \, K(x_1 - x') \, c(x'). \quad (146)$$

For calculation of the integral over the energy variables $x_1$ and $x'$ in Eq. (146), it is convenient to express this integral via the four primary variables $x_i$, $i = 1, 2, 3, 4$, expanding the value $K(x_1 - x')$ backward as the integral of $c(x_i)$ [see, for instance, Eq. (69) adding and the energy delta function $\delta(x_1 + x_2 - x_3 - x_4)$. The departure rate (146) takes the form:

$$\frac{\hbar}{\tau_{ee,q}} = \frac{T^2}{2 \frac{\varepsilon F}{I_E}} \frac{I_E}{2\pi} \quad (147)$$

where

$$I_E = \int \delta(x_1 + x_2 - x_3 - x_4) \prod_{i=1}^4 \frac{dx_i}{2 \cosh(x_i/2)}. \quad (148)$$

Using the methods of the theory of functions of a complex variable, we obtain for this integral:

$$I_E = 2\pi^2 / 3. \quad (149)$$

Finally, from Eqs. (147) and (149) we obtain the result:

$$\frac{\hbar}{\tau_{ee,q}} = \frac{2\pi T^2}{3 \frac{\varepsilon F}{I_E}} \ln(r_s/\varsigma). \quad (150)$$
At the weak interparticle interaction (and not very low temperatures), \( \zeta^{3/2} \ll r_s \ll \zeta \), the factors \( I_0 \) and \( I_\varepsilon \) in the angular-independent parts of \( \text{St}_0 \) and \( \text{St}_\pi \) becomes strongly dependent on the interaction parameter \( r_s \) as \( \sim r_s^2 \), and thus \( 1/\tau_{ee,q} \) decrease. From equations (15) and (16) it is seen that the main contribution to \( 1/\tau_{ee,q} \) now originates from the almost collinear collisions described by \( \text{St}_0 \). Thus we obtain:

\[
\frac{\hbar}{\tau_{ee,q}} = \frac{T^2 I_0 I_E}{\varepsilon F 4\pi},
\]

(151)

that leads to

\[
\frac{\hbar}{\tau_{ee,q}} = \frac{\pi}{3} \frac{T^2 r_s^2}{\varepsilon F} \ln(\zeta/r_s).
\]

(152)

It is noteworthy that the obtained value depends on temperature only via the logarithm.

5.2. Relaxation of the shear stress

Now we complete the solution of the problem of relaxation of the shear stress. We calculate the time \( \tau_{ee,2} \) entering in the viscosity \( \eta = \frac{\hbar}{\tau_{ee,2}}/\varepsilon_F \) and being equal to the characteristic time of relaxation of the function \( \Psi_2 \equiv \Psi_{s0} \) (as in was proved in Section 4.2):

\[
\Psi_{s0}(\varepsilon, \varphi) = \cos(2\varphi).
\]

(153)

We have demonstrated in Sections 3 and 4, that the main contribution to the relaxation rate \( 1/\tau_{ee,2} \) comes from head-on collisions with electrons whose angles lie in the interval \( \zeta \ll \pi - |\varphi| \ll 1 \). The rate \( 1/\tau_{ee,2} \) is calculated just by integration of the value \( \text{St}(\Psi_{s0}(\varepsilon, \alpha)) \) over the variable \( \varepsilon_1 \) [Eq. (133)].

For a more complete understanding, here we refine the estimates of the quantity \( S \) in the whole interval of the angle \( \varphi \in (0, \pi) \).

It can be seen from Eqs. (15) and (16) that the expression \( S \) for the distribution function \( \Psi_{s0} \) is a function of the order of \( \zeta^2 \) at \( |\varphi| \ll 1 \). At the intermediate angles, \( |\varphi| \sim 1 \), we have derived the estimate \( (11) \) saying that the factor \( S \) consists of the contribution of the order \( \sim \zeta \) which linearly depends on \( \alpha \), and the contribution of the higher powers by \( \zeta \) and \( \alpha \), starting from \( \sim \zeta^2 \alpha^2 \). So the nonvanishing terms in \( S \) are estimated as \( \sim \zeta^2 \). At \( \pi - |\varphi| \ll 1 \) the value \( S \) takes the form:

\[
S_{\pm} = 4 \left( \frac{s b \pm \alpha \sqrt{s^2 + \Delta}}{s^2 + \alpha^2} \right)^2.
\]

(154)

Such factors are about unity at \( s \sim 1 \) and go to zero as \( 4(b \pm a)^2/s^2 \) at \( s \gg 1 \). The last values \( 4(b \pm a)^2/s^2 \) are much greater at \( 1 \ll s \ll 1/\zeta \) [corresponding to \( \pi - |\varphi| \ll 1 \)] than the magnitude of \( S \), \( S \sim \zeta^2 \), at the small and the intermediate angles, \( \pi - |\varphi| \sim 1 \). Note that the factors in the operators \( \text{St}_0 \) and \( \text{St}_\pi \), other than \( S \), have the similar order of magnitude as functions of \( |\varphi| \) and \( (\pi - |\varphi|) \) at \( |\varphi| \leq r_s \) for \( \text{St}_0 \) and \( (\pi - |\varphi|) \leq r_s \) for \( \text{St}_\pi \).

These estimates quantify the relative importance of head-on collisions for relaxation of the shear stress [153].

In this way, according to Eqs. (15), (133), and (148), we obtain:

\[
\frac{\hbar}{\tau_{ee,2}} = \frac{T^2 I_s^2 I_E}{\varepsilon F 2\pi},
\]

(155)

Substitution of the angular integral \( I_s \) and the energy integral \( I_E \) yields the final result for the shear stress relaxation time:

\[
\frac{\hbar}{\tau_{ee,2}} = \frac{8\pi}{3} \frac{T^2 r_s^2}{\varepsilon F} \ln \left( \frac{1}{\zeta + r_s} \right).
\]

(156)

At low temperatures and moderately weak inter-particle interaction, when \( \zeta \ll r_s \), the logarithm in Eq. (156) takes the form \( \ln(1/r_s) \) leading to the quadratic temperature dependence of the relaxation rate: \( 1/\tau_{ee,2}(T) \sim T^2 \).

At the very weak interaction, corresponding to high 2D electron densities, when \( r_s \ll \zeta \), the temperature dependence also contains the logarithmic factor: \( 1/\tau_{ee,2}(T) \sim T^2 \ln(\varepsilon_F/T) \). We remind the logarithm in Eq. (156) arises from integration of the function \( 1/(\pi - \varphi) \) over the diapason

\[
\zeta \ll \pi - \varphi \ll \zeta/r_s
\]

(157)

in the case \( r_s \gg \zeta \) and the diapason

\[
\zeta \ll \pi - \varphi \ll 1
\]

(158)

in the case \( r_s \ll \zeta \) [see Fig. 2(b) in the main text].

The rate \( 1/\tau_{ee,2} \) \( (156) \) in the case of low temperatures, \( \zeta \ll r_s \), can be obtained up to a numerical coefficient, from the results of Ref. [18], where the relaxation of distribution functions proportional to the even angular harmonics were studied for a general Fermi gas. For this purpose, we substitute the matrix element (M5) with the screened Coulomb potential (M6) into equation (11) from Ref. [18], which provides the relaxation rate \( 1/\tau_{ee,2} \) with neglect of the change of the particle energy in the momentum conservation law. However, such derivation of \( 1/\tau_{ee,2} \) is incomplete and inconclusive.

First in Ref. [18] (and in Ref. [9]) the properties of the operator \( \text{St} \) were not fully studied, in particular, it was not proved that the energy factor \( F(\varepsilon) \) in the eigenfunction \( \Psi_{s0} \) of \( \text{St} \) is asymptotically equal to a constant. The solution of the integral equation (102) is reduced to calculation of the averaged collision operator \( \int d\varepsilon \text{St}[\cos(2\varphi)] \) exactly due to the singular character of the eigenproblem for the operator \( \text{St} \), namely, due to existing of the eigenvalue

\[
\lambda^{(2)} = -1/\tau_{ee,2}.
\]

(159)
corresponding to \( F(x) \cos(2\alpha) \) with \( F(x) \approx 1 \) and being much smaller by the absolute value than the other eigenvalues \( \lambda_i^{(2)} \) corresponding to the functions \( F_i(x) \cos(2\alpha) \) with \( F_i(x) \neq 1 \). It is the facts those allow us to use the “relaxation time approximation” \( \tau_{ee, q} \) and the corresponding formula \( \tau_{\text{ee}, q} \).

Second, the neglect of the transfer of the particle energy in the momentum conservation law at calculation of the relaxation rates of the even harmonics in Refs. \( \cite{3, 18} \) does not allow to use the regularizations of the operators \( S_0 \) and \( S_{\pi} \) by the factors \( \lambda_i \), which is crucial for obtaining \( 1/\tau_{ee, 2} \) \( \cite{15} \) at medium and moderately high temperatures, \( r_s \sim \zeta \) and \( r_s \ll \zeta \).

In Ref. \( \cite{15} \) the problem of relaxation of the shear stress was solved for the case of a strongly non-ideal 2D Fermi liquid and formula \( \tau_{ee, q} \) for the shear stress relaxation time was derived. In this system the interaction parameter \( r_s \) is large:

\[
\tau_{ee, q} \sim 1 \quad \text{or even} \quad r_s \gg 1 , \quad (160)
\]

up to the point when the Wigner crystallization of 2D electrons occurs.

In Fermi liquid the kernel of the integral equation for the energy part of the distribution function \( F(x) \) in our notation \( \) is rather similar to the kernel \( Q = 2Q_0 + Q_1 \) of Eq. \( \\cite{111} \). We traced the differences between Eq. \( \\cite{111} \) and the analogous equation in Ref. \( \cite{15} \) for the Fermi liquid. Based on this analysis, we demonstrated that our result for \( \tau_{ee, 2} \) \( \cite{15} \) can be also derived from comparison of the formulas of our work and of Ref. \( \cite{15} \). This method of obtaining \( \tau_{ee, 2} \) \( \cite{15} \) is roundabout and much more complicated than our method, but is important for additional checking of Eq. \( \\cite{15} \).

The shear stress relaxation time in Ref. \( \cite{15} \) is obtained in a form of series [Eq. \( (23) \) in that work]. The parameter \( \alpha \) in the integral equation solved in Ref. \( \cite{15} \) for our case of a Fermi gas corresponds to the parameter:

\[
\alpha_N = 2 \left( 1 - \frac{8 r_s^2 \ln(1/r_s)}{\ln(1/\zeta)} \right) \quad (161)
\]

(compare Eq. \( \\cite{85} \) with Eq. \( (22) \) in Ref. \( \cite{15} \)). For such \( \alpha_N \approx 2 \), equation \( (23) \) from Ref. \( \cite{15} \) together with Eq. \( \\cite{15} \) for the scattering departure time \( \tau_{ee, q} \) in our work lead to the shear stress relaxation time \( \tau_{ee, 2} \) \( \cite{15} \).

At large values of \( r_s \), Fermi-liquid effects become important and the random phase approximation, in particular, Eq. \( \\cite{M5} \) and \( \\cite{M6} \) for the scattering probability, turns out inapplicable. Such scattering probability \( W(\varphi, \theta, \bar{\varepsilon}) \) should be calculated via the Fermi-liquid vertex function \( \Gamma(p_1, p_3, p_1 + p_2) \) taking into account the Cooperon contribution \( \\cite{15} \). The upper limit on the parameter \( r_s \) when the consideration of the current work remains valid can be obtained from comparison of the decomposition of the Fermi-liquid vertex function over the harmonics by the scattering angle \( \theta \) \{Eq. \( (15) \) in Ref. \( \cite{15} \) and the analogous decomposition of the RPA screened potential \( \\cite{M6} \).

We obtain from such procedure, using equations from Ref. \( \cite{15} \), that until \( r_s \ll r_{s, 0} \),

\[
\tau_{s, 0} \sim \frac{1}{\ln(1/\zeta)} \ln[\ln(1/\zeta)] , \quad (162)
\]

the Cooperon renormalization of the scattering probability is negligible and we can use potential \( \\cite{M6} \).

At \( r_s \gg 1 \) the Cooperon renormalization of many angular harmonics of the scattering probability \( W \) may be important. Equations \( \\cite{M3} \), \( \\cite{M8} \), \( \\cite{M9} \) and the formulas for \( \theta_\pm(\varphi, \bar{\varepsilon}), \psi_\pm(\varphi, \bar{\varepsilon}) \) at \( \varphi \to \pi \) (from Section 1.1) lead to result the the Fermi liquid result \( \\\cite{15} \), if we use the probability with the Cooperon renormalization, \( W \sim 1/\ln^2(1/|\varphi - \pi|) \) \( \\cite{15} \), instead of the RPA probability given by Eqs. \( \\cite{M4} \)-\( \\cite{M6} \).

We conclude that result \( \\cite{15} \) is valid in the interval of sufficiently low \( r_s, r_s \ll r_{s, 0} \), where the marginal value \( r_{s, 0} \ll 1 \) is given by Eq. \( \\cite{162} \). Apparently, relaxation processes due to the inter-quasiparticle scattering in the intermediate interval of \( r_s \),

\[
r_{s, 0} \ll r_s \ll 1 , \quad (163)
\]

has not been studied yet. In this diapason of \( r_s \), the Cooperon renormalization of the scattering probability \( W \) seems to be also substantial, but, apparently, the consideration is much more complex than ones of the current work and of Ref. \( \cite{15} \).

5.3. Relaxation of the higher angular harmonics

As it was discussed in Section 4.1, it is instructive to study the relaxation of the distribution functions proportional to the higher harmonics by the velocity angle, \( \Psi_m(\varepsilon, \alpha) = \cos(m\alpha) \) \( \\cite{116} \) for even as well as for odd \( m \gg 2 \).

We present the results on this problem only for the case of low temperatures, \( \zeta \ll r_s \).

In Section 4.2 we demonstrated that relaxation of the even harmonics \( \Psi_m \) with \( m \geq 4 \) is mainly related to the head-on collisions and their relaxation rates \( 1/\tau_{ee,m} \) are calculated by Eq. \( \\cite{163} \) with \( St_\zeta = St_\pi \). From this formula and formula \( \\cite{85} \) for \( St_\pi \), we obtain for the rates \( 1/\tau_{ee,m} \):

\[
\frac{\hbar}{\tau_{ee,m}} \sim \frac{T^2}{\varepsilon_F} r_s^2 \ln \left( \frac{1}{r_s} \right) m^2 \quad (164)
\]

at not too large \( m, m \ll 1/r_s \), and

\[
\frac{\hbar}{\tau_{ee,m}} \sim \frac{T^2}{\varepsilon_F} r_s^2 \ln(m) \quad (165)
\]

at rather large \( m, 1/r_s \ll m \ll 1/\zeta \). Equation \( \\cite{165} \) coincides with the result obtained in Refs. \( \cite{3, 18} \) for
\(1/\tau_{ee,m}\) at \(m \ll 1/\zeta\). Result \(163\) in the interval \(m \ll 1/r_x\) was not obtained in those works because of the neglect of the angle and the energy dependencies of the scattering matrix element \(M^2_{+}(\varphi, \bar{x})\).

A more sophisticated problem is to evaluate the relaxation rates of the distribution functions proportional to the third and higher odd harmonics. In Refs. \[2\]--\[8\] it was shown that the relaxation of such perturbations is a slow process as compared with the relaxation of the even harmonics. In Ref. \[3\] it was derived that for odd \(m\), \(m \geq 3\) the collision integral \(S\) (M3) has eigenfunctions:

\[
\Psi^X_m(\varepsilon, \varphi) = X_m(\varepsilon) \cos(m\varphi),
\]

with a nontrivial dependencies on energy, given by the factor \(X_m(\varepsilon)\). The relaxation rates \(1/\tau_{ee,m}\) of the functions \(\Psi^X_m\) turn out to be much lower, in the factor \(\zeta^2\), than the relaxation rates of the even harmonics \[3\]. This result is due to the peculiarities of the 2D electron collisions related to the Fermi factors \[1\]--\[8\].

In order to estimate the rates \(1/\tau_{ee,m}\) for odd \(m\) within our approach, we below summarize the obtained above properties of \(S\) and study the behavior of \(S\) for odd \(m\) (for even \(m\) it was described in details above in Section 5.2).

First, the consideration in Section 4 demonstrates that the energy dependence of the eigenfunction of \(S\) proportional to an odd harmonic and corresponding to the minimal eigenvalue \(\lambda^{(m)}\) is close to \(\Psi^F_m(x, \alpha) = F(x) \cos(m\alpha)\), where \(F(x) \approx \text{const}\), in the limit of smallness of the angular part of the corresponding operator \(Q^{(m)}\) [Eq. \(143\)]. So the following question remains open: to what extent the taking into account of the energy dependence of the function \(X_m(\varepsilon)\) \(166\) in Ref. \[3\] corresponds or contradict to the fact that the eigenfunction \(\Psi^F_m\) calculated in limit \(140\) has the energy factor \(F(x)\) being close to constant.

Second, an analysis shows that the factor \(S\) \(38\), \(41\) for the functions \(\Psi^F_m\) \(140\) with odd \(m\) and \(F(x) \approx \text{const}\) should be estimated as \(\zeta^2 a_{ij}\) in the integrals by \(x_i\) in the interval \(\zeta \ll |\varphi| \lesssim 1\), where \(a_{ij} \sim 1\) as the term of the order of \(\zeta\) in Eq. \(111\) is asymmetric relative to the change \(x_j \leftrightarrow -x_j\). So the contribution \(S_{\text{dep}}[\Psi^F_m]\) to the collision operator from the angles \(\varphi \rightarrow \pi\) is proportional to the integral of the product of the part \(|S|_2 \sim \zeta^2 a_{ij}\) of the factor \(S\), the singular divergent factor \(1/\sqrt{s^2 + \Delta}\), and the matrix element \(M^2_{+}(\varphi, \bar{x})\) with the static screened Coulomb potential \(M(6)\). This leads to the logarithmic divergence by the variable \(\varphi\), being similar to one in Eq. \(51\) up to the factor \(\zeta^2\):

\[
I_{\text{odd}, \pi}(\bar{x}) = \frac{1/\zeta}{s} \int_{1}^{s} \frac{|S(\varphi, \bar{x})|_2}{1 + |a - b|/(\sqrt{2}r_s)} \, ds.
\]

In view of the relation \(|S|_2 \sim \zeta^2\), this integral at typical \(x_i \sim 1\) is estimated as:

\[
I_{\text{odd}, \pi} \sim \zeta^2 \ln(r_s/\zeta).
\]

We remind that in this subsection we consider only the case \(\zeta \ll r_s\).

Third, an analysis based on Eqs. \(160\) and \(177\) yields that in the vicinity of the angle \(\varphi = 0, |\varphi| \ll 1\), the factor \(S\) for the functions \(\Psi^m = \cos(m\varphi)\) becomes independent on \(\varphi\) in the main order by \(|\varphi|\) and takes the form:

\[
S(\varphi, \bar{x}) = m^2 \frac{\zeta^2 \Delta(\bar{x})}{4}.
\]

In follows from (M3) that the resulting angle-dependent contribution \(S_{\text{dep}}\) to the collision integral from the vicinity of \(\varphi = 0\) becomes proportional to the expression:

\[
m^2 \zeta^2 \int \int \int \int \frac{d\varphi}{\sqrt{\varphi^2 + \zeta^2 \Delta(\bar{x})}} = \int \frac{d\varphi}{\sqrt{\varphi^2 + \zeta^2 \Delta(\bar{x})}}.
\]

In view of result \(143\), the relaxation rate of the odd harmonics, \(1/\tau_{ee,m}\), is expressed via the integral \(\int dx_1 S_{\text{dep}}[\Psi^1_m]\), where \(S_{\text{dep}}\) contains the sum of the contributions \(S_{\text{dep}}^1\) and \(S_{\text{dep}}^2\) being proportional to factors \(167\) and \(170\). Owing to the antisymmetry of the value \(\Delta(\bar{x}) = x_3 x_4 - x_1 x_2\) relative to the change \(x_3, x_4 \leftrightarrow x_1, x_2\), the integral of expression \(170\) by \(x_1\) turns out to zero. Thus the main part of the integral \(\int dx_1 S_{\text{dep}}[\Psi^1_m]\) originates from \(S_{\text{dep}}^2\) and is proportional to the factor \(168\).

We conclude that the relaxation of the odd harmonics is mainly due to the head-on collision, corresponding to \(\zeta \ll |\varphi| \ll 1\) and being described by the operator \(S_{\text{dep}}\). The relaxation rate \(1/\tau_{ee,m}\) is calculated by equations \(75\), \(143\), and \(167\). The result of such calculation for \(\zeta \ll r_s\) and the odd harmonic number \(m\) in the diapason \(3 \leq m \ll 1/r_x\) takes the form:

\[
\frac{h}{\tau_{ee,m}} \sim \frac{T^2}{\varepsilon_F} \zeta^2 \ln(r_s/\zeta) m^2.
\]

This result differs from the one obtained in Ref. \[3\] by another way on the logarithmic factor \(\ln(r_s/\zeta)\). We think that this difference is due to taking into account of the exact dependence of the scattering matrix element \(M^2_{\pm}\) on the variables \(\varphi\) and \(\bar{x}\).

5.4. Relaxation of the heat flow

Using the results of Section 4, one can study the processes in a 2D electron gas that involves the relaxation of the heat flow: the thermal conductivity and the thermoelectric effects an zero and nonzero frequencies \(\omega\) and \(\omega_c\). Here we consider only the stationary problem of the thermal conductivity at zero magnetic field.
In section 4.1 it was shown that, in the problem of heat transport, the kinetic equation for the function $\Psi(x)$ with the nonequilibrium term $\Psi = \nu F dT/dx \cos \alpha G(x)$ is transformed into the closed form [109]. The heat flow relaxation time and the thermal conductivity coefficient, should be calculated via $\Psi_h$ by Eqs. [110] and [107].

The energy factor $G(x)$ in the nonequilibrium term $\Psi_h$ in the distribution function $\Psi(x)$ [108] can be presented in the form of series:

$$\Psi_h(x, \alpha) = G(x) \cos \alpha, \quad G(x) = \sum_{l=1}^{\infty} a_l x^l. \quad (172)$$

The first term in this formula, being proportional to $\Psi_{h1}(x, \alpha) = x \cos \alpha, \quad (173)$ and other terms with odd numbers $l$ provide the contributions to the value of the heat flow [108]. The solution of the integral equation [109] implies finding all the coefficients $a_l$ in the decomposition of $G(x)$ [172]. Unlike the case of relaxation of the “pure angular harmonics” (Sections 5.2 and 5.3), in the problem of heat transport all the terms in Eq. (172) are substantial for constructing the proper solution $G(x)$.

The main role in the collision operator $S(t)$ in the equation for $G(x)$ is played by the angular-independent part $S_{0n}^\infty$ of $S(t)$, which contains the equal contributions $S_{0n}^\infty$ and $S_{1n}^\infty$ at $\zeta \ll r_s$ and only the contribution $S_{0n}^\infty$ at $\zeta \gg r_s$ [see Eq. (135) and Section 3]. The angular-dependent part of the collision integral for $\Psi_h$ and $\Psi_{h1}$ is much smaller and is not substantial for their relaxation.

The problem of heat transport in a 3D Fermi liquid was solved in Refs. [13, 14]. The kernels of the main integral equations for $G(x)$ in the 2D and 3D cases differ only by the density of states. According to Ref. [14], the solution of Eq. [109] for $G(x)$ should be performed by using the Fourier transform of Eq. [109] by the variables $x, x_1$ and solving the resulting differential equation for the transformed function

$$G(\xi) = \frac{1}{\sqrt{2\pi}} \int dx e^{-i\xi x} G(x). \quad (174)$$

In this work, we only estimate the time $\tau_{ee,h}$ by an order of magnitude. For this purpose, we truncate the distribution function $\Psi$ [172] down to the first term, being proportional to the lowest term $\Psi_{h1}$ [173]:

$$\Psi(x, \alpha) \to a_1 \Psi_{h1}(x, \alpha). \quad (175)$$

Such function $a_1 \Psi_{h1}$ is the simplest distribution function carrying a heat flow. From Eqs. [109] and [110] we can deduce the estimates for $a_1 \Psi_{h1}$ and the corresponding relaxation time $\tau_{ee,h}$.

The resulting expression for $\tau_{ee,h}$ turns out to be similar the expression for the angular harmonics relaxation rates [133] and [143]:

$$\tau_{ee,h} \sim -\left\{ \int d\varepsilon x_1 \text{St}[\Psi_{h1}](\varepsilon_1, 0) \right\}^{-1}, \quad (176)$$

where the odd factor $x_1 = (\varepsilon_1 - \mu)/T \sim 1$ is important as the function $\Psi_{h1}(x, \alpha)$ [173] is odd by $x \to -x$ and therefore the value $\text{St}[\Psi_{h1}](\varepsilon_1, \alpha)$ is also odd relative the change $(\varepsilon_1 - \mu) \to - (\varepsilon_1 - \mu)$.

As it was shown in Section 2.2, the factor $S(\varphi, \varepsilon)$ [38] for $\Psi_{h1}$ being substantially dependent on $x$ is estimated as unity at the intermediate angles and for the head-on collisions, corresponding to $1 \leq |\varphi| < \pi$. For the collinear collisions, hen $|\varphi| \ll 1$, it follows from Eqs. (173) and (10) that:

$$S(\varphi, \varepsilon) \sim \varphi^2 \ll 1. \quad (177)$$

Based on these estimates of $S$ and Eqs. [69], [78], and [91] for the contributions $\text{St}$ we obtain that at $\zeta \ll r_s$ the main effect on the rate $1/\tau_{ee,h}$ [176] originates from the angles:

$$\zeta/r_s \ll \pi - |\varphi| \ll 1. \quad (178)$$

In this interval, the inequalities $|\theta_\pm(\varphi)| \gg \zeta$ and $|\psi_\pm(\varphi)| \gg \zeta$ are fulfilled, thus we can use the static inter-particle potential (M6) and all the formulas from Section 3.3 for the operator $S_{tr}$. As the result, the value $1/\tau_{ee,h}$ is calculated by angular-independent part $S_{0n}^\infty$ of $S_{tr} $ [85] with the factor $I_x$ [82]. Estimating the energy variables $x_1$ and $x'$ in Eq. (85) as unity, we obtain from Eq. (179):

$$\frac{h_0}{\tau_{ee,h}} \sim \frac{T^2}{\varepsilon_F} \ln(r_s/\zeta) \quad (179)$$

for the case current $\zeta \ll r_s$.

At the weak interparticle interaction, $r_s \ll \zeta$, according to the analysis in the Section 3.3, the head-on collisions, corresponding to $\pi - |\varphi| \ll 1$, do not provide the most significant contribution $S^\infty_{0n}$ in the angular-independent part of $S_{tr}$ as compared with the scattering with electrons with the angles $|\varphi| \sim 1$. In view of Eq. (177), the factor $S$ [85] for $\Psi_{h1}$ is small for the collinear collisions, when $\varphi \ll 1$, therefore they also does not provide the substantial contribution to $S_{tr}$. Thus the main contribution to $1/\tau_{ee,h}$ comes from the scattering by the incident electrons with the angles $|\varphi| \sim 1$, that is described by the contribution $S^\infty_{1n}$, estimated by Eq. (91). In view of Eqs. (91) and (176), in the case $\zeta \gg r_s$ we have the estimate:

$$\frac{h_0}{\tau_{ee,h}} \sim \frac{T^2}{\varepsilon_F} \frac{\varepsilon^2}{\zeta^2}. \quad (180)$$

Note that such rate is independent on temperature.

The derived heat flow relaxation rate $1/\tau_{ee,h}$ [179] has the magnitude of one half of the departure rate $1/\tau_{ee,q}$. 
in which electrons scatter on phonons. Obtained for the first time in Ref. [20] for bulk 3D metals, in magnetic field, similar to one in formula (M15), was phonon interaction. The viscosity of the electron fluid in pure bulk 3D metals with the strong electron-phonon interaction from the collinear collisions, $|φ| ≪ 1$, while their role in the relaxation of the heat flow function $Ψ_{h1}$ is weakened due to the particular form of the factor $S$ for $Ψ = Ψ_{h1}$ [see Eq. (177)].

6. DIFFICULTIES IN COMPARISON OF EXPERIMENT WITH THEORY: POSSIBLE TRANSITIONAL REGIMES BETWEEN HYDRODYNAMICS AND BALLISTICS

For the first time, the hydrodynamic mechanism for negative magnetoresistance, similar to the one used in the current paper to explain the experiments, was proposed in Ref. [19] in a Poiseuille flow of electron-phonon fluid in pure bulk 3D metals with the strong electron-phonon interaction. The viscosity of the electron fluid in magnetic field, similar to one in formula (M15), was obtained for the first time in Ref. [20] for bulk 3D metals, in which electrons scatter on phonons.

In Fig. 3 in the main text the experimental data on $1/τ_{ee,2}$ are presented for the samples for which the Lorentzian profile (M15) of the hydrodynamic magnetoresistance is observed very well in a wide range of magnetic fields, including the interval

$$ω_c ≲ 1/τ_2.$$  \hfill (181)

This situation corresponds, at least, to the inequality $l_2 < W$ for the width $W$ of the conducting channel [here $l_2 = v_{ee}τ_2$ is the shear stress relaxation length and $1/τ_2 = 1/τ_{imp,2} + 1/τ_{ee,2}$ is the total shear stress relaxation rate (M16)]. Magnetoresistance (M15) implies that the hydrodynamic regime of the electron flow is formed in the whole range of magnetic field.

However, if the mean free path $l_2$ is greater than the width of the conducting channel $W$, the ballistic regime is realized in small magnetic fields $B < B_0$, corresponding to

$$2R_c > W,$$  \hfill (182)

while the hydrodynamic one becomes possible only at the higher fields $B > B_0$ corresponding to

$$2R_c < W,$$  \hfill (183)

when a substantial part of electrons in the central part of the sample does not scatter on the channel edges.

In this case, only the segment of the curve $q_{xx}(B) \propto 1/B^2$ (M15) at large magnetic fields, $|B| > B_0$, can be observed, and therefore it becomes impossible to extract relaxation time $τ_2$ from the widths of magnetoresistance curve. At the fields, $|B| < B_0$, the magnetoresistance is determined by the ballistics effects controlled by the relations between the magnitudes of the scattering length, the sample width $W$ and lengths $L$, and the cyclotron radius $R_c [21, 22].$

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