Two Poisson structures invariant with respect to discrete transformation in the case of arbitrary semi-simple algebras

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Abstract
Two Poisson structures invariant with respect to discrete transformation of Maximal root presented in explicit form for arbitrary semi simple algebra. Thus problem of construction of multi-component hierarchies of integrable systems is solved.

1 Introduction
In the present paper we would like to systematize results obtained in the previous papers of the author [1],[2] - [7] concerning the discrete transformations, hierarchies of systems of equations and its multi-soliton solutions of $n$ - wave problem. Now we would like to forget about the origin of the problem and consider some general property of semi-simple algebra. The problem may be formulated as the following. With each semi-simple algebra it is possible to connect two Poisson structures invariant with respect to some discrete transformation of the given form. Of course this observation was done only after consideration of the examples of the semi simple algebras of the low ranks. And we advise to the reader at first read one of the mentioned above papers in which calculations may be done by fourth action of arithmetic plus differentiation.

2 Participants of the game and its rules

2.1 Notations
$G$ - arbitrary semi-simple algebra. $R$ - the space of its positive and negative roots. The dimension of $R$ equal $2n = N - r$, $N$ - dimension of the algebra, $r$ its rank. $X_R$ are generators with commutation relations $[X_R, X_{R'}] = D_{R,R'}^{R+R'} X_{R+R'}$, where $D_{R,R'}^{R+R'}$ structure constants (some times we omitted sign $X_R \rightarrow R$). $f_R$
- the system of functions depending on one parameter $x$ - space coordinate 
(\text{and possible some other parameters}). Some times we use notation $f_{\pm R} = f_{\pm R}^R$.

Elements of Cartan sub algebra will be denoted by little latine letters $c = \sum c_i h_i, [c, X_R] = c_R X_R$. Generators of all roots of the algebra are normalized on unity ($X_R, X_{R'} = \delta_{R, -R'}$).

### 2.2 The grading of the maximal root

In what follows we use the grading of the maximal root [8]. This means that all generators of the algebra may be distributed on the subspaces with $\pm 2, \pm 1$ and $0$ grading indexes. The grading index is defined by the proper value of the Cartan generator of the maximal root $H_M = [X_M^+, X_M^-], [H_M, X_M^\pm] = \pm 2X_M^\pm$.

Thus arbitrary element of semi simple algebra may be represented as

$$f = f^{(+2)} + f^{(+1)} + f^0 + f^{(-1)} + f^{(-2)}, \quad [H_M, f^{(m)}] = mf^{(m)}$$

The subspaces $f^{(\pm 2)} = f^{(\pm)}_M X_M^{\pm}$ are one dimensional. The subspaces with $\pm 1$ graded indexes $f^{(\pm 1)} = f^{(\pm)}_+ + f^{(\pm)}_-$ may be decoupled in such way that $[f^{(\pm)}_+, f^{(\pm)}_-] = c X_M^{\pm}$ and all generators in subspaces with the same additional index \pm are mutually commutative. Elements of one graded subspaces are represented in a form $f^{(\pm)} = \sum f_{R^{(\pm)}(S^{(-1)})} X_{R^{(S^{(-1)})}}$. In what follows zero graded subspace is limited by additional condition $[f^0, X_M^{\pm}] = 0$. The zero graded subspace consists from the different elements $[X_{R^{(S^{(-1)})}}, X_{S^{(-1)}}]$ not coincide with the elements of Cartan sub algebra. The following Furie decomposition take place

$$\sum (A^{(+1)} X_{R^{-1}(S^{(-1)})}) (B^{(-1)} X_{R^{(+1)}(S^{(-1)})}) = (A^{(+1)} B^{(-1)})$$

and some modified formula with respect to summation on elements of zero order subspace

$$\sum (A^0 X_{r_0}) (B^0 X_{r_0}) = (A^0 (B^0 - \sum h_{\beta} k^{-1}_{\beta, \gamma} (h_{\gamma} B^0))) =
(A^0 B^0) - \sum (h_{\beta} B^0) k^{-1}_{\beta, \gamma} (h_{\gamma} A^0)$$

The last modification is connected with the fact that Cartan elements of the simple roots of the algebra may be among $A^0, B^0$ and they must be excluded from the result of summation on zero order subspace.

### 2.3 Discrete transformation, Frechet derivative and Poisson structure

We repeat corresponding text from [9]. The discrete invertible substitution (mapping) defined as

$$\tilde{u} = \phi (u, u', ..., u^r) \equiv \phi (u) \quad (1)$$
$u$ is $s$ dimensional vector function; $u^r$ its derivatives of corresponding order with respect to "space" coordinates.

The property of invertibility means that (1) can be resolved and "old" function $u$ may expressed in terms of new one $\tilde{u}$ and its derivatives.

Frechet derivative $\phi'(u)$ of (1) is $s \times s$ matrix operator defined as

$$\phi'(u) = \phi_u + \phi_u' D + \phi_u'' D^2 + ...$$  \hspace{1cm} (2)

where $D^m$ is operator of $m$-times differentiation with respect to space coordinates.

Let us consider equation

$$F_n(\phi(u)) = \phi'(u) F_n(u)$$ \hspace{1cm} (3)

where $F_n(u)$ is $s$-component unknown vector function, each component of which depend on $u$ and its derivatives not more than $n$ order. If equation (3) possesses at least one not trivial solution ($F_{trivial} \equiv u'$) then substitution (1) is called integrable one.

With each discrete substitution it is possible try to connect Poisson structure defined as anti symmetric matrix valued operator $J^T(u) = -J(u)$ and its invariance with the respect to discrete transformation (1) is fixed by equation it satisfied

$$\phi'(u) J(u)(\phi'(u))^T = J(\phi(u)), \quad \phi'(u) H(u)(\phi'(u))^{-1} = H(\phi(u))$$ \hspace{1cm} (4)

where $(\phi'(u))^T = \phi_u^T - D\phi_u'^T + D^2(\phi_u'')^T + ...$ and $J(u), H(u)$ are unknown $s \times s$ matrix operators, the matrix elements of which are polynomial of some finite order with respect to operator of differentiation (of its positive and negative degrees). First equation in (4) is condition of invariance of Poisson structure with respect to discrete transformation, the second one is equation for raising operator. Two different Poisson structures lead obviously to $H(u) = J(u)_1 J^{-1}(u)_2$.

3 The statement of the problem

In [1] it was shown that equations of n-th waves are invariant with respect some number of discrete transformations (coincide with $r$ - the rank of corresponding semi-simple algebra). Namely with such kind of discrete transformation we will have deal in what follows below. It will be shown that there exist two different Poisson structures (which can be used for construction of the hierarchies of integrable systems of equations). But the fact of existence of such Poisson structures is the inner property of semi-simple algebra and the written ad hoc discrete mapping of the definite form. Author don’t know and have no guess how these two objects may be connected in the frame work of group representation theory.
3.1 Discrete transformation of the maximal root

The discrete transformation of the maximal root \([8],[1]\) is the mapping realized on the space of \(f_R\) functions depending on \(r\) arbitrary parameters \([c, X_R]\) = \(cR_X\) and one space coordinate \(x\) \((\frac{\partial}{\partial x} = f')\)

\[
\begin{align*}
\frac{T_{M}^{-}}{f_{M}^{+}} & = \frac{1}{f_{M}^{-}}, & \frac{T_{M}^{-}}{f_{M}^{+}} & = \frac{[X_{M}^{+}, f^{(-1)}]}{f_{M}^{-}} = \alpha^{(+1)} \frac{1}{f_{M}^{-}}, & \frac{T_{M}^{-}}{f_{0}^{-}} & = (f^{0} + \frac{1}{2f_{M}^{-}}[\alpha^{(+1)} f^{(-1)}])h,
\end{align*}
\]

where \(\ldots)h\) means factor on Cartan sub-algebra. In other words

\[
\begin{align*}
([\alpha^{(+1)} f^{(-1)}])h & = \alpha^{(+1)} f^{(-1)} - \sum h_{\beta} h_{\beta,\gamma}(\alpha^{(+1)} f^{(-1)})
\end{align*}
\]

where \(h_{\beta}\) Cartan element of simple root of the algebra. Below we use the notation \(\theta \equiv ([X_{M}^{+}, f^{(-1)}][c, f^{(-1)}])\)

\[
\begin{align*}
\frac{T_{M}^{-}}{c, f^{(-1)}} & = (f^{(-1)})' + \frac{(f_{M}^{+})' - \frac{4\theta}{c_{M}f_{M}^{+}}[c, f^{(-1)}] + \frac{1}{6f_{M}^{-}}[c[\alpha^{(+1)} f^{(-1)} f^{(-1)}]] - [f^{(-1)}[c, f^{0}]] + f_{M}^{-}[[c, f^{(+1)}]X_{M}^{+}] - c_{M}f_{M}^{+}2f_{M}^{+} + \frac{(f_{M}^{+})'' - \frac{4\theta}{c_{M}f_{M}^{+}}[c_{M}, f^{(-1)}]' - \frac{(f_{M}^{+})' - \frac{4\theta}{c_{M}f_{M}^{+}}[c, f^{(-1)}] - \frac{1}{2}[\alpha^{(+1)} f^{(-1)}][c, f^{0}])}{2f_{M}^{-}}
\end{align*}
\]

We would like to show that transformation (5) is canonical one. That means that there exist some functional \(G\) (generating function) depending on the pairs of transformed and initial variables \(T_{M}^{-}, f_{R}, f_{R}^{+}\) in which encoded all information about (5). We will use the following identification for general impulses \(P^{M} = c_{M}f_{M}^{+}, P^{0} = [c, f^{0}]\) and coordinates \(X_{M} = f_{M}, X_{1} = f^{(+1)}, X_{0} = f^{0}\) where \(f_{R}^{0}\) means part of \(f^{0}\) connected with positive and negative spaces of zero order subspace. We choose the following dependence of mentioned above functional \(G = G(X_{M,1}, X_{M,1}; X_{0}, P_{0}^{0})\) then in the case of canonical transformation

\[
\begin{align*}
P_{M,1}^{1} = \frac{\delta G}{\delta X_{M,1}}, & \quad \frac{T_{M}^{-}}{P_{M,1}^{1}} = \frac{\delta G}{\delta X_{M,1}}, & \quad P^{0} = -\frac{\delta G}{\delta X_{0}}, & \quad \frac{T_{M}^{-}}{X_{0}} = -\frac{\delta F}{\delta X_{0}}
\end{align*}
\]

where functional derivatives coincide with the Frechet one (2). In the case of canonical transformation of the classical dynamics (and thermodynamic) functional \(G\) depend only on canonical coordinates and generalized impulses but not on its derivatives. It is not surprising that discrete transformation is connected
with generalized canonical ones. This fact was observed some times ago and proof of this fact with respect to self-dual system (and many other integrable systems) reader can find in [10].

We present below explicit expression for density of functional $G$. Reader can check by using only one operation of differentiation that (6) and (5) are identical

$$G = c_M \left( \frac{f_M'}{f_M} \right) - \frac{1}{f_M} \left( \alpha^{(1)}[c, f^{(-1)}] \right) - \frac{1}{2c_M f_M} \left( \frac{f_M'}{f_M} \right)^2 - \frac{1}{c_M f_M} \left( \alpha^{(1)}[c, f^{(-1)}] \right) -$$

As reader can see from (7) generating function in the case under consideration depend on the derivatives of the first order of $f$ functions and second order on $f$ one. It lead to some modification in usual theory of canonical transformation which will be noticed below.

### 4 Conservation of the Poisson brackets

The main property of canonical transformation consists in conservation of Poisson brackets between dynamical variables involved. This fact proofs automatically in the usual theory (without derivatives of dynamical variables). We will try to repeat corresponding calculation and show that this property with respect to discrete transformation of the previous section is conserved.

For this goal let us consider Jacobi matrix

$$J = D\left( \frac{T_M \to T_M \to T_M \to T_M}{P \to X_0; P_0 \to P_0 \to X} \right) = \frac{D\left( T_M \to T_M \to T_M \to T_M \to T_M \to X \right)}{D(p, x_0; p_0, x)} = \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$$

In (8) in $P$ we unite generalized impulses $P=P_M, P^1$ and the same do with corresponding coordinates $X=X_M, X^1$. We can not do the same with coordinates and impulses with zero indexes by the same reason why unite canonical transformation in usual theory my be written not in all pairs of canonical variables. For further transformation of (8) we unite "4" independent variables of
$G$ functional into two pairs $\bar{y} = (T_{\mathcal{M}} \rightarrow P_0 , T_{\mathcal{M}} \rightarrow X_0 )$, $y = (x, x_0)$. In this notations (6) can be rewritten as follows

\[
\begin{pmatrix} T_{\mathcal{M}} \rightarrow P \\ T_{\mathcal{M}} \rightarrow X_0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} G_y \equiv \sigma_2 G_{\bar{y}}, \quad \begin{pmatrix} p \\ p_0 \end{pmatrix} = -G_g
\]

After differentiation of the last equality with respect to pairs of arguments $p = (p, p_0), y = (x, x_0)$ and taking into account independent arguments of generating function $G$ we obtain

\[
\bar{y}_p = -(G_{y,\bar{y}})^{-1}, \quad \bar{y}_x = -(G_{y,\bar{y}})^{-1}G_{y,y},
\]

\[
\begin{pmatrix} T_{\mathcal{M}} \rightarrow P \\ T_{\mathcal{M}} \rightarrow X_0 \end{pmatrix}_p = -\sigma_2 G_{\bar{y},y}(G_{y,\bar{y}})^{-1},
\]

\[
\begin{pmatrix} T_{\mathcal{M}} \rightarrow P \\ T_{\mathcal{M}} \rightarrow X_0 \end{pmatrix}_y = -\sigma_2 G_{\bar{y},y}\sigma_1 + \sigma_2 G_{\bar{y},\bar{y}}(G_{y,\bar{y}})^{-1}G_{y,y}
\]

\[
(\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}).
\]

In the general case, when generating functional has arbitrary dependence on derivatives of $p, x$ arguments the last expression may be had no mean, but in the case under consideration generating functional is linear in $\bar{y}$ arguments. In this case matrix $G_{y,\bar{y}}$ depend only on generalized coordinates $X$ and impulses $P$ but not from their derivatives and all operations done above are absolutely correct. After this manipulations expression for Jacobi matrix valued operator may rewritten in two dimensional form ( we take into account condition of linearity of $G, G_{\bar{y},\bar{y}} = 0$

\[
J = \begin{pmatrix} 0 & \sigma_3 G_{\bar{y},y}\sigma_1 \\ -(G_{y,\bar{y}})^{-1} & -(G_{y,\bar{y}})^{-1}G_{y,y}\sigma_1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \delta(2 \to 3)
\]

Now by direct calculations let us check that $J \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} J^T = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}$.

Indeed

\[
\sigma_1 \delta(2 \to 3) \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} \delta(2 \to 3)^T \sigma_1^T = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}
\]

and

\[
J \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} J^T = \begin{pmatrix} 0 & \sigma_3 G_{\bar{y},y} \\ -(G_{y,\bar{y}})^{-1} & -(G_{y,\bar{y}})^{-1}G_{y,y} \end{pmatrix} \begin{pmatrix} G_{y,\bar{y}}\sigma_3 & G_{\bar{y},y}(G_{y,\bar{y}})^{-1} \\ 0 & -(G_{y,\bar{y}})^{-1} \end{pmatrix} = \delta(2 \to 3)
\]

6
\[
\begin{pmatrix}
0 & -\sigma_3 & \sigma_3 \\
-\sigma_3 & G_{y,y}^{-1} & (G_{y,y}^{-1} - G_{\bar{y},\bar{y}}^{-1})
\end{pmatrix}
\]

In the case when \( G \) don't depend on space derivatives the term in the right
down corner of the last matrix obviously equal to zero. In the case of functional
(7) the checking of this fact reader find in Appendix.

In the previous papers it was used the order of functions \( f_{M}^{+}, f_{M}^{(-1)}, f_{M}^{0}, f_{M}' \).
To come to such order from considered above it is necessary to do some number
of point like tangent transformation: change by the places \( X_0 \rightarrow P_0 \) and do
mirror reflection in \( X_0, X \) terms. After such transformations the zero degree
Poisson structure looks as
\[
\begin{pmatrix}
0 & \sigma \\
-\sigma & 0
\end{pmatrix}
\]

where \( \sigma \) is matrix \((1 + N^1 + N^0 \times 1 + N^1 + N^0)\) with different from zero unity
on its main anti diagonal.

The proof of conservation of the Poisson brackets may be obtained more
directly from the fact of existence of Lagrangian function [14] (and action for
n-wave system)
\[
L(f) = \text{Sp}(\frac{1}{2}([d, f_t] + [c, f_x]) + \frac{1}{3}(f[[[d, f][c, f]]]), S = \int dt dx L(t, x)
\]

change on the derivatives under transformation of maximal root above.

4.1 Frechet derivative

The Frechet derivative of (5) in connection of its definition (2) may be presented
in block form as \(5 \times 5\) matrix valued operator
\[
\begin{pmatrix}
0 & 0 & 0 & \phi'_{R^{-1}, S^{-1}} & \phi'_{M, -M} \\
0 & 0 & 0 & \phi'_{R^{-1}, -R} & \phi'_{R^{-1}, -M} \\
0 & \phi'_{R^{-1}, S^{-1}} & \phi'_{R^{-1}, -R} & \phi'_{R^{-1}, -S} & \phi'_{R^{-1}, -M} \\
\phi'_{M, -R} & \phi'_{M, -R} & \phi'_{M, -R} & \phi'_{M, -S} & \phi'_{M, -M}
\end{pmatrix}
\]

where \( I \) is the unity matrix with the dimension of zero grading subspace: the
notation \( M_R \) means that from the algebra valued function \( M \) necessary extract
coefficient under corresponding root \( R \) of the algebra belonging to grading spaces
with indexes \( \pm 2, \pm 1 \) and 0.

\[
\phi'_{M, -M} = -\frac{1}{(f'_M)^2}, \quad \phi'_{R^{-1}, S^{-1}} = \frac{[X^+_M, X^{-1}_{S^{-1}}]X^{-1}_{R^{-1}}}{f'_M},
\]

\[
\phi'_{R^{-1}, -M} = \frac{[X^+_M, f^{-1}_{-R}X^{-1}_{R^{-1}}]}{(f'_M)^2}, \quad \phi'_{R^{-1}, -S} = I,
\]

\[
\phi'_{R^{-1}, -M} = \frac{[[X^+_M, f^{-1}_{-S}]X^{-1}_{R^{-1}}]}{(f'_M)^2}
\]

7
\[\phi'_{\rho_0,-M} = -\frac{1}{2} \left(\left[\left[X^+_{M}, f^{(-1)}\right]f^{(-1)}\left[X^+_{M}\right]\right]\right)_{\rho_0}, \quad \phi'_{\rho_{S(-1),R}^{+1}} = f_M^{c_{S(-1)}}\left([[c, X^{R^{+1}}_M]X^{-}_{M}]X^{-}_{S(-1)}, X\right)\]
\[\phi'_{\rho_{S(-1),r_0}} = -\frac{1}{c_{S(-1)}}\left([[f^{(-1)}[c, X^{\rho_0}]]X^{-}_{S(-1)}\right), \quad \phi'_{\rho_{-M},M} = \left(f_M^{c_{S(-1)}}\right)^2, \quad \phi'_{\rho_{-M},R^{+1}} = -\frac{f_M^{c_{S(-1)}}}{c_{S(-1)}}\left(f^{(-1)}[c, X^{R^{+1}}_M]\right)\]
\[\phi'_{\rho_{-M},r_0} = \frac{1}{2\rho_{M}}\left([[\left[X^+_{M}, f^{(-1)}\right]f^{(-1)}[c, X^{\rho_0}]\right], \quad \phi'_{\rho_{-M},S(-1)} = \frac{c_{S(-1)}}{c_{M}^2}\left([[\left[X^+_{M}, f^{(-1)}\right]X^{-}_{S(-1)}\right]D+\]
\[\frac{1}{6c_{S(-1)}f_{M}}\left([[\left[X^+_{M}, f^{(-1)}\right]X^{-}_{S(-1)}\right][[\left[X^+_{M}, f^{(-1)}\right]f^{(-1)}]\right]+\frac{f_{M}^{c_{S(-1)}}}{c_{M}}\left([[c, f^{(1)}_M]X^{S(-1)}\right)-\frac{1}{c_{S(-1)}}\left([[\left[X^+_{M}, f^{(-1)}\right][c, f^{(1)}]\right])\left([[\left[X^+_{M}, f^{(-1)}\right]X^{-}_{S(-1)}\right]ight)+\]
\[\frac{1}{2c_{M}(f_M^{c_{S(-1)}})^2c_M}\left([[\left[X^+_{M}, f^{(-1)}\right][c, f^{(1)}]\right]+\frac{c_{S(-1)}}{c_{M}^2}\left([[\left[X^+_{M}, f^{(-1)}\right][c, f^{(1)}]\right])\left([[\left[X^+_{M}, f^{(-1)}\right]X^{-}_{S(-1)}\right]ight)-\]
\[\frac{1}{c_{S(-1)}}\left([[\left[X^+_{S(-1)}, f^{(-1)}\right][c, f^{(1)}]\right]+\frac{1}{2c_{M}}\left([[\left[X^+_{S(-1)}, f^{(-1)}\right][c, f^{(1)}]\right])\left([[\left[X^+_{S(-1)}, f^{(-1)}\right]X^{-}_{S(-1)}\right]ight)\]
\[\phi'_{\rho_{-M},-M} = \frac{1}{(c_{M})^2D^2} - 2\left((\frac{f_{M}^{c_{S(-1)}}}{c_{M}^2f_{M}})^2\right) - 2\left(\frac{f_{M}^{c_{S(-1)}}}{c_{M}}\right)^2D + \frac{(f_M^{c_{S(-1)}})^2}{(c_{M})^2f_{M}^2}\left([[\left[\left[X^+_{S(-1)}, f^{(-1)}\right][\left[X^+_{S(-1)}, f^{(-1)}\right]\right]]\left([[\left[X^+_{S(-1)}, f^{(-1)}\right][c, f^{(1)}]\right])\right)\]

5 The main assertion

There are two Poisson structures \(\rho_{0}\) and \(\rho_{1}\), which are invariant with respect to discrete transformation of the maximal root of the previous subsection (equations (4) are satisfied) with the explicit form of its matrix elements

\[<R|\rho_{0}|R'> = \delta_{R,-R'}\frac{1}{c_{R}}, c_{R} = -c_{R} \quad (10)\]

The fact of invariance of \(\rho_{0}\) Poisson structure with respect to discrete transformation was proved in the previous section.

\[<R|\rho_{1}|R'> = -f_{R}(h_{R}, h_{R})D^{-1}f_{R'} + \frac{1}{c_{R}}[(X_{R'}, X_{R'})[c, f]]\frac{1}{c_{R'}} + \frac{1}{c_{R}}\delta_{R,-R'}D \quad (11)\]

where \(h_{R} = \sum n_{R}^{R}h_{\alpha}, n_{0}^{R} \) multiplicity simple root \(\pm \alpha\) in the root \(R\) and \(h_{\alpha}\) Cartan element of this simple root. The proof of the last (not so trivial) proposition will be presented below.
5.1 Simplest nontrivial solution of (3)

The column function \( F_R = g_R f_R \) is the solution of the equation (3). We have

\[
(\phi'(f) F)|R = \sum < R|\phi'(f)|R' > F_{R'} = \sum < R|\phi'(f)|R' > g_{R'} f_{R'} = \\
\sum \frac{\partial T_m}{\partial f_{R'}} [g, f_{R'}] = [g, T_m f_R] = g_R T_m f_{R'}
\]

Indeed \( g_R f_R \) may be considered as result of differentiation \([g, f_{R'}] \) and the discrete transformation (5) conserve the grading structure with respect to simple roots of the algebra.

5.2 Equation which it is necessary to check

Let us rewrite the equation for Poisson structure \( J(4) \) taking into account the fact of existence of \( J_0(10) \). We have

\[
\phi'(f) J = T_m\longrightarrow J (\phi'(f)^{-1} T_m\longrightarrow)^T = -(J_0 \phi'(f)^{-1} J_0^{-1} T_m\longrightarrow J)^T
\]

Let us present first term in (11) in equivalent form

\[
f_R(h_R, h_{R'}) D^{-1} f_{R'} = \sum_{i=1}^{r} F_i D^{-1} F_i^T
\]

where \( F_i = g_R^i f_R \) are \( r \) independent nontrivial solutions of the first subsection. Then the form above is equivalent to (11) after identification \((g_R, g_{R'}) = \sum_i g_R^i g_{R'}^i = (h_R, h_{R'}) \). And this fact it will necessary to prove. Now let us consider result of multiplication Frechet derivative on (column) nontrivial solution \( F_i \). Using the rule of multiplication quadratical in derivat ives operator on scalar function \( F \)

\[
(A + BD + CD^2) = AF + BF' + CF'' + (BF + 2CF')D + CFD^2
\]

and explicit form of Frechet derivative we conclude that first \( 1 + n^1 + n^0 \) lines of it does not contain operator of differentiation \( D \), next \( n^1 \) ones are linear in \( D \) and the last one is quadratical in it (we remind that 1-dimension of \( \pm 2 \) graded subspaces, \( n^1, n^0 \) dimensions of \( \pm 1,0 \) ones). We present below \( n^1 + 1 \times n^1 + 1 \) matrix in the left dawn corner of Frechet derivative which contain operator of differentiation in explicit form

\[
\begin{pmatrix}
\frac{1}{c_{s-1}} \delta_{(s-1),s-1} D \\
\frac{1}{c_M} [c, X_{(s-1)}^+] [X_{M'}^+, f^{(-1)}] D \\
\frac{1}{c_M} D^2 - 2 \frac{f_{s-1}}{c_M f_M} D \\
\frac{1}{c_M} [c, X_{(s-1)}^+] [X_{M'}^+, f^{(-1)}] D
\end{pmatrix}
\]
In connection with (3) \( T_M \rightarrow \phi(f)F_i \), (in the last formulae all operators of differentiation acts directly on \( F_i \)). And thus we have

\[
\phi(f)F_i = \frac{T_M \rightarrow}{F_i} + \left( \begin{array}{c}
0_{1+n^1+n^0} \\
-(\frac{g'_0}{c_{M}} - \frac{g'_{-1}}{c_{-1}})f_{S^{-1}}D \\
-\frac{g'_M f_{M} D^2}{c_M} + \frac{1}{c_M}([c, f^{(-1)}][X^+_M, [g^i, f^{(-1)}]])D \\
\end{array} \right) = (15)
\]

The assumed form of second Poisson structure contain terms with negative, zero and positive degrees of \( D \) (\( J^{-1}_1, J^0, J^1 \)). The (15) together with (12) and comments after leads to conclusion

\[
\phi'(f)J_i^{-1} = -\sum_{i=1}^{r} \left( \frac{T_M \rightarrow}{F_i} D^{-1}F_i^T + \Gamma_iD^{-1}F_i^T \right)
\]

But \( \Gamma_i \) is proportional to \( D \) and thus last term in the sum above contain only 0, 1 degrees of \( D \). Thus explicit expression for \(-\sum_{i=1}^{r} (\Gamma_iD^{-1}F_i^T)\) looks as

\[
\left( \begin{array}{c}
\left( \frac{(M,P)}{c_M} \right)^{0_{1+n^1+n^0}} \\
\left( \frac{(M,P)}{c_M} \right)^{-(\frac{g'_0}{c_{M}} - \frac{g'_{-1}}{c_{-1}})f_{S^{-1}}}D \\
\left( \frac{(M,P)}{c_M} \right)^{-\frac{g'_M f_{M} D^2}{c_M} + \frac{1}{c_M}([c, f^{(-1)}][X^+_M, [g^i, f^{(-1)}]])} \\
\end{array} \right) (F_P)^T
\]

In the last column different from zero only components of its \( n_1 + 1 \) last rows. In what follows we use notation \( \Theta = ([c, f^{(-1)}][X^+_M, [g^i, f^{(-1)}]]) \), \( \Theta_P = ([c, f^{(-1)}][X^+_M, [g^i, f^{(-1)}]]) \). Now let us inverse (15) presenting it in a form

\[
F_i^T = \left( \frac{T_M \rightarrow}{F_i} \right)^T (\phi^{-1}(f))^T + \Gamma_i^T(\phi^{-1}(f))^T
\]

For calculation \( (\phi'(f)^{-1})^T = J_0^{-1}\phi'(f)J_0 \) it is suitable to present \( J_0 \) as product of two matrices one diagonal one with matrix elements \( < R|d|R' > = \delta_{R,R'} \frac{1}{\gamma_R} \) and the second with different from zero unites on its main anti diagonal \( < R|ad|R' > = \delta_{R,-R'} \). Thus we obtain

\[
\phi'(f)^{-1} = J_0^{-1} \phi'(f)J_0 =
\]

\[
d^{-1} \begin{pmatrix}
\phi'_{-M,-M} & \phi'_{-M,-(R+1)'y} & \phi'_{-M,-r_0} & \phi'_{-M,-(S-1)'y} & \phi'_{-M,M} \\
\phi'_{-R+1,-M} & \phi'_{-R+1,-(R+1)'y} & \phi'_{-R+1,-r_0} & \phi'_{-R+1,-(S-1)'y} & 0 \\
\phi'_{-R_0,-M} & \phi'_{-R_0,(-R+1)'y} & I & 0 & 0 \\
\phi'_{-S,-M} & \phi'_{-S,(-R+1)'y} & 0 & 0 & 0 \\
\phi'_{M,-M} & \phi'_{M,-(R+1)'y} & 0 & 0 & 0 \\
\end{pmatrix} d
\]

\( (16) \)
Now calculation of $\sum T_{M \rightarrow F} \cdot D^{-1} \tilde{\Gamma}_i^T = \sum T_{M \rightarrow F} \cdot D^{-1} \Gamma_i^T (\phi^{-1}(f))^T$ became obvious with the result

$$
\sum T_{M \rightarrow F} \cdot D^{-1} \tilde{\Gamma}_i^T = \sum T_{M \rightarrow F} \cdot D^{-1} \Gamma_i^T (\phi^{-1}(f))^T \quad (\phi^{-1}(f))^T
$$

In the last line only first $(1 + n_1)$ terms are different from zero.

Substituting all results obtained above into main equation (12) we come to equality which have to be checked

$$
- \sum \tilde{\Gamma}_i^T \left( \frac{(PM)}{c_M} D \frac{1}{(f_M L^2)} \Theta P \quad (\frac{c_M}{(P R^+)} \frac{(PR^+ L)}{c_{R+1} (f_{R+1} 1 + n_1 + n_0)}) \right)
$$

(17)

We pay attention the reader that in (18) the terms containing $D^{-1}$ mutually cancel and all terms in equality above contain only positive degree of operator $D$. The equation (18) is equation on matrix elements of $(2 + 2N^1 + N^0) \times (2 + 2N^1 + N^0)$ matrix. We present this matrix as a sum of 4 matrix quadratic $(1 + N_1 + N_0) \times (1 + N_1 + N_0)$ one in its upper right corner, two rectangular $(1 + N_1 \times 1 + N_1 + N_0)$ and $(1 + N_1 + N_0 \times 1 + N_1)$ in its upper left and right dawn corners correspondingly and quadratical $(1 + N_1 \times 1 + N_1)$ in its left dawn corner. Below we prove that all matrix elements of left and right sides of (18) are equal to each other. Of course this calculations are not simple one and author absolutely sure that they are not necessary but at the present moment have no idea, on what it is possible to change them.

6 Proofs of validity of the main equation (18)

Some general comments. In all calculations we will have deal with matrix elements of product of two matrices. The sum on mediate states always has a form $\sum_{P} c_{P}(K, X-P)(L, X_P), \nu = -1, 0$. When $\nu = 0$ the summation may be performed in explicit with help the comments in the second section. In the case $\nu = -1$ it is necessary find the same terms in both side of equality to be checked. The strategy of calculations below will be the same. At first we consider coefficient before positive degree of $D$. In nonlocal sums we find terms of the same structure in both sides. All non locality cancels. And summation it is possible perform in explicit form. In some cases under consideration below one of the factors in the expression under summation fixed uniquely value $\nu_P$ and thus it is possible to take this factor out of the sign of the sum with further summation in explicit form.
6.1 Checking of the elements of \((1 + N^1 + N^0)\times (1 + N^1 + N^0)\) quadratical matrix of right upper corner of (18)

For these elements \(\Gamma_i = \Gamma_j = 0\) and the main equation (18) looks as (we introduce additional notation \(J_1^1 \equiv J_1^0 + J_1^1\))

\[
\phi'(f) J_1^1 = T_{M^{-}} T_{M^{-}} ((\phi'(f)^{-1})^T
\]

or in more detail form

\[
\sum_p \frac{\phi R^P X_P X_{R'}}{c_R} \frac{1}{c_{R'}} = \frac{1}{c_R} \sum_p X_P ((\phi')^{-1})_{R,P}^T \frac{1}{c_{R'}}
\]

where \(R = M, R^{(+1)}, r_0, R' = -M, S^{-1}, r_0\).

Below we present calculations for more complicate case considering the last equality between the states \(r_0, r_0'\).

\[
(J_1^1)_{r_0, r_0'} + \sum_{S^{-1}} \phi'_{r_0, S^{-1}} (J_1^0)_{S^{-1}, r_0'} = T_{M^{-}} T_{M^{-}} (J_1^1)_{r_0, r_0'} + \sum_{R^{(+1)}} T_{M^{-}} (J_1^1)_{r_0, R^{(+1)}} ((\phi'(f)^{-1})^T)_{R^{(+1)}, r_0'}
\]

In writing the last equality we conserve all not vanishing terms in connection with definition of Frechet derivative, assumed form of \(J_1^P\) and the fact that \(\phi'(f)_{r_0, r_0'}\) is unity matrix. Now terms with derivatives in \((J_1^1)_{r_0, r_0'} = \delta_{r_0, r_0'} D\) does not depend from shifts and thus cancels in both sides of equality. Further

\[
\frac{T_{M^{-}}}{2 f_M c_r c_{r_0'}} ([X_{-r_0}, X_{-r_0'}] [c, [X_M^+, f(-1)] f(-1)]) =
\]

Now we consider sum in left hand side. Commutator \([X_{-S^{-1}}, X_{-r_0'}]\) belongs to +1 graded subspace and thus \((X_{-S^{-1}}, X_{-r_0'}) [c, f] = (c_{S^{-1}} + c_{r_0'}) ([X_{-S^{-1}}, X_{-r_0'}] f(-1))\). Using this fact and substituting explicit expression for elements of Frechet derivative we rewrite this sum in a form

\[
\frac{1}{f_M} \sum_{S^{-1}} (\frac{1}{c_{S^{-1}}} + \frac{1}{c_{r_0'}}) ([X_{-S^{-1}}, X_{-r_0'}] f(-1)) ([X_M^+, f(-1)] X_{S^{-1}} X_{-r_0})
\]

In corresponding calculations in right side it is necessary to take into account that commutator \([X_{-r_0}, X_{-R^{(+1)}}]\) belongs to -1 graded subspace and thus

\[
([X_{-r_0}, X_{-R^{(+1)}}] [c, f]) = (-c_{-R^{(+1)}} + c_{r_0}) ([X_{-r_0}, X_{R^{(+1)}}] f^{(+1)})
\]
Using this fact, the value for $f_{\Gamma_i}^{T_M \rightarrow f} = \frac{1}{f_{M}} [X_{M}^{+}, f^{-1}]$ from (5) and explicit expression for matrix element of Frechet derivative we present the sum under consideration in a form (in all places we exchange $-R^+ \rightarrow S^{-1}$ to have the same notations as in the sum in the left side calculation above)

$$\frac{1}{f_{M}} \sum_{S^{-1}} \left( \frac{1}{c_{S^{-1}}} - \frac{1}{c_{r_0}} \right) \left( [X_{-S^{-1}}, X_{-r_0}] f^{-1} \right) \left( [[X_{M}^{+}, f^{-1}]_{X_{S^{-1}}} X_{r_0}] \right)$$

The terms containing $c_{S^{-1}}$ in denominator are the same in both sides of equality and mutually cancels. In remaining terms it is possible perform summation in explicit form

$$-\frac{1}{f_{M}} \left( \frac{1}{c_{r_0}} + \frac{1}{c_{r_0}} \right) \left( [[X_{M}^{+}, f^{-1}]_{X_{r_0}}, f^{-1}]]_{X_{r_0}} \right)$$

The last term (after simple manipulations) cancels with the same obtained on the first step of computation above. Absolutely in the same manner it is possible to convince that equation (18) is satisfied for all elements of the matrix of the left upper corner.

### 6.2 Checking of the elements of $(1 + N^1 \times 1 + N^1 + N^0)$ rectangular matrix of left upper corner of (18)

For this elements $\Gamma_i = 0$, but terms with $\Gamma_{M}, \Gamma_{R^{-1}}$ will give input into left hand side of (18).

Below we present result of computation the matrix elements of two second terms of left-hand side of (18) and three terms of right hand side of this equation.

#### 6.2.1 Matrix element between states $< M |, | M >$

$$\sum_{P} \phi'_{M,P} J_{P,M} = \phi'_{M,-M} J_{-M,M} = -\frac{1}{c_{M}^{2}(f_{M})^{2}} D$$

$$\frac{T_{M}^{- \rightarrow f_{M}}}{f_{M}} \sum_{M} (g_{M}^{i} \Gamma_{M}) =$$

$$-\frac{(M,M)}{c_{M}^{2}} D \frac{1}{f_{M}} + \frac{1}{c_{M}^{2}(f_{M})^{3}} \left( [c, f^{-1}] [X_{M}^{-1}] \sum_{l} (g_{M}^{i} g_{i}^{-1}) f^{-1} \right)$$

and at last

$$\sum_{P} (T_{M}^{- \rightarrow f_{P}} J_{M,P} ((\phi')^{-1})_{P,M} = (T_{M}^{- \rightarrow f_{M}})_{M,-M} ((\phi')^{-1})_{-M,M} +$$

$$\left( T_{M}^{- \rightarrow f_{S^{-1}}} J_{M,S^{-1}} ((\phi')^{-1})_{S^{-1},M} = D \frac{1}{c_{M}^{2}(f_{M})^{2}} + \frac{1}{c_{M}^{2}(f_{M})^{3}} \left( [c, f^{-1}] [X_{M}^{+}, f^{-1}] \right) \right)$$
And at last

\[ \langle R \rangle \]

propagation of the main assertion.

### 6.2.2 Matrix element between states \( \langle M, | R^{(+)} > \)

\[
\phi_{M,-M}^{(+)} J_{M,R^{(+)}} = - \frac{1}{c_M c_{R^{(+)}} (f_M^+)^2} ([X^+_{M}, X^{-}_{R^{(+)}}]|c, f^{-1}) = \\
\frac{(c_M - c_{R^{(+)}})}{c_M c_{R^{(+)}} (f_M^+)^2} ([X^+_{M}, f^{-1}] X_{R^{(+)}}) \\
- \sum (g_M^R \Gamma_{R^{(+)}}^i) = \\
\frac{T_{M} \rightarrow f_M^+ (1 - \frac{(MM)}{c_M} + \frac{(MR^{(+)})}{c_{R^{(+)}}}) T_{M} \rightarrow f_{R^{(+)}} = (- \frac{2}{c_{R^{(+)}}} + \frac{1}{c_{R^{(+)}}} ([X^+_{M}, f^{-1}] X_{R^{(+)}})}{(f_M^+)^2} \\
\sum \sum_{P}^T_{M,| M, P((\phi')^{-1})^T_{P,R^{(+)}} = (T_{M} \rightarrow f_{R^{(+)}} = ([X^+_{M}, f^{-1}] X_{R^{(+)}})}{c_M (f_M^+)^2} \]}

And at last

\[
\sum \sum_{S^{(-1)}}^T \phi_{R^{(1)},S^{(-1)}} J_{S^{(-1)},M} + \phi_{R^{(1)},-M} J_{M,M} = \sum \sum_{P}^T_{M,| M, P((\phi')^{-1})^T_{P,R^{(+)}} = (T_{M} \rightarrow f_{R^{(+)}} = ([X^+_{M}, f^{-1}] X_{R^{(+)}})}{c_M (f_M^+)^2} \]

The sum of two last rows exactly equal to the first one and thus the main equality (18) for matrix element under consideration is satisfied.

### 6.2.3 Matrix element between states \( \langle R^{(+)} | M > \)

\[
\sum \sum_{S^{(-1)}}^T \phi_{R^{(1)},S^{(-1)}} J_{S^{(-1)},M} + \phi_{R^{(1)},-M} J_{M,M} = \sum \sum_{P}^T_{M,| M, P((\phi')^{-1})^T_{P,R^{(+)}} = (T_{M} \rightarrow f_{R^{(+)}} = ([X^+_{M}, f^{-1}] X_{R^{(+)}})}{c_M (f_M^+)^2} \]

In writing of the last equality we exchange matrix elements of \((\phi' (f)^{-1})^T\) on matrix elements of \(\phi' (f)\) in connection with (16).

Now we substitute explicit expressions for transformed values (16) and Frechet derivatives and after summation rewrite the last expression in an equivalent form

\[
- \frac{T_{M} \rightarrow f_{R^{(+)}}}{f_M^+ c_{R^{(+)}}} D - \frac{c_{R^{(+)}}}{c_M - c_{R^{(+)}} c_M} f_{R^{(+)}} = \frac{T_{M} \rightarrow f_{R^{(+)}}}{f_M^+ c_{R^{(+)}}} (D - \frac{1}{c_M f_M^+} (f_M^+)^2) + \\
\]
\[
\frac{1}{(f_M)^2} = \frac{1}{c_M} \frac{1}{f_M^2} = \frac{1}{c_M} \frac{1}{f_M^2} [X_{-R+1}, X_M^+] [(f^{-1})'] + \\
\frac{1}{c_M} \frac{1}{f_M^2} [c, f^{-1}] + \frac{1}{6 f_M} [c, [[\alpha^{(1)}, f^{-1}] f^{-1}]] - \frac{1}{f_M} [f^{-1} [c, f^{-1}]] + f_M ([c, f^{(1)}] X_M^-) - \\
\frac{1}{2 c_{R+1} c_M} \frac{1}{f_M^2} (X_{-R+1} [\alpha^{(1)}]_h [c, f^{-1}]) - \\
\frac{1}{c_{R+1} f_M} ([X_{-R+1}, X_{-M+R^{(1)}]} X_M^+] (\phi'(f))_{-M+S^{(-1)}, -M}
\]

The both sides of the last expression contain terms first and zero degree in \( D \).

The terms linear in derivative are the following one ((\( R^{(1)} M \)) = 1)

\[
\frac{T_{M \rightarrow}}{f_{R+1}^2} D = \frac{T_{M \rightarrow}}{f_{R+1}^2} D + \frac{1}{c_{R+1} c_M} \frac{1}{f_M^2} \frac{T_{M \rightarrow}}{f_{R+1}^2} D - \frac{1}{c_{R+1} c_M} \frac{1}{f_M^2} \frac{T_{M \rightarrow}}{f_{R+1}^2} D
\]

It is easy to see that coefficient before terms linear in \( D \) equal to zero.

By the same way it is possible to check that the last expression is equality.

We present below only calculations the most complicate terms of the third degree in \( f^{-1} \) functions to obtain the value \( (R^{(1)} S^{(-1)}) \) unknown up to now. 

These terms are the following ones (we conserve in mind common factor \( \frac{1}{c_{R+1} c_M} f_M^{-1} \), which will be taken into account in the end of computation)

\[
\frac{1}{2 c_M} ([X_{-R+1}, X_M^+] [c, f^{-1}]) \theta - \frac{1}{6} ([X_{-R+1}, X_M^+] [c, [\alpha^{(1)}, f^{-1}] f^{-1}])
\]

These terms arise after separation the third degree terms from \([c, f^{-1}] \) (see section Frechet derivative).

\[
- \frac{1}{2} (X_{R'} [\alpha^{(1)}] c^{[\alpha^{(1)}, f^{-1}]]}] - \frac{1}{2} (X_{-R^{'(1)}} [(\alpha^{(1)}, f^{-1})_h [c, \alpha^{(1)}])
\]

Two last terms arise from \([c, f^0 \) and \((X_{-R^{'(1)}} [(\alpha^{(1)}, f^{-1})_h [c, f^{(1)}]) \) correspondingly (about the symbol \( A_h \) see section discrete transformation, below this is very important). And at last from \((\phi'(f))_{-M} S^{(-1)}, -M \) we have in addition

\[
\frac{1}{2} ([X_{-R+1}, X_M^+] f^{-1}) \theta - \frac{c_M}{6} ([\alpha^{(1)}, f^{-1}] f^{-1}]
\]

Further transformations are connected with the single relation

\[
[c [\alpha^{(1)}, f^{-1}] f^{-1}] - c_M[[\alpha^{(1)}, f^{-1}] f^{-1}] = 3 [[\alpha^{(1)}, f^{-1}], [c, f^{-1}]] + 3 \theta f^{-1}
\]
which easy follows from the rules of commutation, definition of $\alpha^{(+1)} = [X_M^+, f^{(-1)}]$ and commutation relation $[f^{(-1)} c, f^{(-1)}] = \theta X_M^+ \equiv (\alpha^{(+1)} c, f^{(-1)}) X_M^-$. After some not complicate calculation all terms above lead to finally result

$$\frac{c_{R^{(+1)}}}{2c_M} \theta ([X_{-R^{(+1)}}^+, X_M^+] f^{(-1)})$$

or after taking into account factors omitted above we come to the finally expression for the sum of all terms presented above

$$\frac{1}{2(c_M)^2(f_M)^2} \theta X_M^+ \sum \frac{1}{T_{R^{(+1)}}}$$

Up to now we have not took into account linear in Cartan elements terms in $X_{-R^{(+1)}}, ([\alpha^{(+1)}, f^{(-1)}] h, T_{R^{(+1)}})$. Account of this term lead to result (Cartan matrix in our consideration is symmetrical)

$$\sum [(h_\beta, X_{-R^{(+1)}}) c, f^{(+1)}]) k_{\beta, \gamma}^{-1} (h_\gamma, [\alpha^{(+1)}, f^{(-1)}]) = c_{R^{(+1)}} \frac{1}{f_{R^{(+1)}}} \sum n_{\gamma}^{R^{(+1)}} (\alpha^{(+1)}, [h_\gamma, f^{(-1)}]) =$$

$$c_{R^{(+1)}} \frac{1}{f_{R^{(+1)}}} \sum (R^{(+1)} S^{(-1)}) f_{S^{(-1)}} f_{S^{(-1)}} (X_{(S^{(-1)})}, [X_M^+, X_{S^{(-1)}}])$$

where $(R^{(+1)} S^{(-1)}) = \sum n_{\gamma}^{R^{(+1)}} k_{\gamma, \beta} n_{\beta}^{S^{(-1)}}$

Now let us produce some transformation of equivalence with scalar product $\theta_P = ([c, f^{(-1)}][X_M^+, \sum (g_\beta^2 g^\beta), f^{(-1)})])$

$$\theta_P = \sum_{S^{(-1)}} c_{-M-S^{(-1)}} (PS^{(-1)}) f_{-M-S^{(-1)}} f_{S^{(-1)}} (X_{-M-S^{(-1)}} [X_M^+, X_{S^{(-1)}}]) =$$

$$-c_M \sum_{S^{(-1)}} (PS^{(-1)}) f_{-M-S^{(-1)}} f_{S^{(-1)}} (X_{-M-S^{(-1)}} [X_M^+, X_{S^{(-1)}}]) +$$

$$\sum_{S^{(-1)}} c_{-M-S^{(-1)}} (P - M - S^{(-1)}) f_{-M-S^{(-1)}} f_{S^{(-1)}} (X_{-M-S^{(-1)}} [X_M^+, X_{S^{(-1)}}]) =$$

$$-c_M \sum_{S^{(-1)}} (PS^{(-1)}) f_{-M-S^{(-1)}} f_{S^{(-1)}} (X_{-M-S^{(-1)}} [X_M^+, X_{S^{(-1)}}]) - (PM) \theta - \theta_P.$$ 

The last equality allow present $Q$ in a form

$$\theta_P = -\frac{1}{2}c_M \sum_{S^{(-1)}} (PS^{(-1)}) f_{-M-S^{(-1)}} f_{S^{(-1)}} (X_{-M-S^{(-1)}} [X_M^+, X_{S^{(-1)}}]) - \frac{1}{2}(PM) \theta$$

In the case under consideration $P = R^{(+1)}, (R^{(+1)} M) = 1$ and two terms above exactly opposite the sign with the same calculated before cancels with them and thus main equation is satisfied.
6.2.4 Matrix element between states $< R^{(+1)}, |Q^{(+1)}>$

$$\sum_{S(-1)} \phi_{R^{(+1)},S(-1)}^* J_{S(-1),Q^{(+1)}} + \phi_{R^{(+1)},-M}^* J_{-M,Q^{(+1)} =}$$

$$\sum_{S(-1)} T_{R^{(+1)}} \left[ \frac{- (MR^{(+1)})}{c_M} + \frac{(Q^{(+1)}R^{(+1)})}{c_{Q^{(+1)}}} \right] =$$

$$\sum_{S(-1)} T_{R^{(+1)}} J_{1}^{R^{(+1)},S(-1)} \frac{c_{Q^{(-1)}}}{c_{Q^{(+1)}}} \phi'(f)_{-S(-1),-Q^{(+1)}} + \sum_{P^{(+1)}} T_{P^{(+1)}} J_{1}^{P^{(+1)},P^{(+1)}} \frac{c_{P^{(-1)}}}{c_{P^{(+1)}}} \phi'(f)_{-P^{(+1)},-Q^{(+1)}}$$

Terms quadratical in $f^{(-1)}$ arises in left side only from the second term in a form

$$\left( -\frac{1}{c_M} + \frac{1}{c_{Q^{(+1)}}} \right) \sum_{S(-1)} T_{R^{(+1)}} J_{1}^{R^{(+1)},F^{(+1)}}$$

The same structure have two first terms in right side. Summation on $S(-1)$ in the first sum lead to

$$\frac{c_{P^{(+1)}} + c_{Q^{(+1)}} - c_{M}}{2c_{Q^{(+1)}} c_{P^{(+1)}} (f_M)^2} ([X_{-R^{(+1)}}[[X^+_M, X_{-Q^{(+1)}}]] \alpha^{(+1), f^{(-1)}]}$$

Summation on $r_0$ in the second sum lead to (we take not into account linear in Cartan elements terms, it will be done separately later)

$$\frac{1}{2c_{Q^{(+1)}} c_{P^{(+1)}} (f_M)^2} ([X_{-R^{(+1)}}[[X^+_M, X_{-Q^{(+1)}}]] \alpha^{(+1), f^{(-1)}]}$$

Summation on $P^{(+1)}$ in the last sum lead to three terms which are containing among quadratical terms in $\phi'(f)_{-P^{(+1)},-Q^{(+1)}}$ (see section Frechet derivative). One term is coming with expression with $\delta$ after summation

$$\frac{1}{2c_{Q^{(+1)}} c_{P^{(+1)}} (f_M)^2} ([X_{-R^{(+1)}}[[X^+_M, X_{-Q^{(+1)}}]] \alpha^{(+1), f^{(-1)}]}$$

which is exactly opposite by the sign of the last term in line above. Second equal to the next term in this expression and after summation on $P^{(+1)}$ takes the form

$$-\frac{1}{c_{P^{(+1)}}} \sum_{S(-1)} T_{R^{(+1)}} J_{1}^{R^{(+1)},Q^{(+1)}}$$
and the last more complicate term after summation having the form
\[
\frac{c_M}{2c_P + c_Q} \left( [X_{-R}^+][X_M^+, X_{-Q}^+]][\alpha^{(+1), f^{(-1)}]} \right)
\]
which cancels last term of the first sum. Now by simple computation we come to equality
\[
([X_{-R}^+][X_M^+, X_{-Q}^+]][\alpha^{(+1), f^{(-1)}]} = 2(f_M^-)^2 \sum_{T_M^+ - T_M^-} T_{R^+}^+ T_{Q^+}^+ - ([\alpha^{(+1), X_{-R}^+}][\alpha^{(+1), X_{-Q}^+}])
\]
All calculated above terms are mutually cancels and now it is necessary to calculate omitted above terms linear in Cartan elements under summation on \( r_0 \). We have
\[
\frac{1}{c_P + c_Q} \sum (h_\beta, X_{-Q}^+) \left( \sum_{T_M^+ - T_M^-} \sum_{(\alpha^{(+1), f^{(-1)}})} k_\beta \gamma h_\gamma \alpha^{(+1), X_{-Q}^+} \right) = \frac{1}{c_Q} \sum_{T_M^+ - T_M^-} T_{R^+}^+ T_{Q^+}^+ \sum n_{R^+}^{(1)} k_\gamma \theta_\beta n_{\beta}^{(1)}
\]
and finally we come to equalities
\[
(M R^{(+1)}) = 1, (Q^{(+1) R^{(+1)}}) = - \sum n_{\gamma}^{(1)} k_{\gamma} \theta_\beta n_{\beta}^{(1)}
\]
which coincide with the proposition of the main assertion.

### 6.2.5 Matrix element between states \(< r_0, |M >\)

In this case the main equation looks as
\[
\frac{\phi_{r_0, S^(-1), M}^{'(1)}}{f_M} = \frac{1}{f_M c_M^2} \sum \left( \frac{1}{(f_M^3 c_M^2\theta_{r_0})} \sum T_{R^+}^+ T_{Q^+}^+ \sum n_{R^+}^{(1)} k_\gamma \theta_\beta n_{\beta}^{(1)} \right)
\]
We rewrite equation above in more detail form, fulfilling where possible summation
\[
\sum (\frac{1}{c_S(-1) c_M}) \left( [\alpha^{(+1), X_{S^(-1)}, X_{-r_0}}] \left( [X_{S^(-1)}, X_M^+] \left( c, f^{(+1)} \right) \right) - \frac{[\alpha^{(+1), f^{(-1)}, X_{-r_0}}]}{2(f_M^3 c_M^2)} \right)
\]
\[
\frac{1}{c_M c_0} \sum \left( [X_{-r_0}, X_{-R}^+][c, f^{(+1)}] \phi'(f)_{-R^{(+1)}, -M} \right)
\]
\[
= \frac{1}{2 c_M c_0} \left( [X_{-r_0}, \alpha^{(+1)}_0, f^{(-1)}][c, f^0] \right) + \frac{1}{2 c_M c_0} D \left( [\alpha^{(+1)}_0, f^{(-1)}] X_{-r_0} \right)
\]
\[
- \frac{1}{(f_M^2)^{c_M c_0}} \left( [X_{-r_0}, \alpha^{(+1)}][c, f^{-1}] \right)
\]

The terms linear in \( D \) except of written explicit above are presented also in \( \phi'(f)_{-R^{(+1)}, -M} \) in a form \( \frac{f_{R^{(+1)}}}{f_M^2 c_M} D \). Substituting this expression in corresponding place we rewrite it (after not cumbersome manipulations) in the form

\[
\frac{1}{(f_M^2)^{c_M c_0}} \left( [X_{-r_0}, f^{(-1)}][c, X_M^+, f^{(-1)}] \right) =
\]
\[
- \frac{c_M + c_0}{2 (f_M^2)^{c_M c_0}} \left( [X_M^+, f^{(-1)}] f^{(-1)} X_{-r_0} \right)
\]

Thus linear in \( D \) terms are the following ones

\[
- \frac{1}{2} \left( \frac{[X_M^+, f^{(-1)}] f^{(-1)} X_{-r_0}}{(f_M^2)^{c_M c_0}} \right) \cdots = \frac{1}{c_{-r_0} c_M f_M} \left( [X_{-r_0}, f^{(-1)}][c, f^{(+1)}] \right)
\]
\[
+ \frac{1}{2 c_{-r_0} c_M} D \left( [X_M^+, f^{(-1)}] f^{(-1)} X_{-r_0} \right) \frac{f_M}{(f_M^2)^2} \cdots
\]

having as its result \((r_0, M) = 0\).

Now we would like to compare remaining terms zero degree in \( D \) in both sides and find value for \((r_0, S^{-1})\).

First of all necessary compare all nonlocal terms in both sides. Such nonlocal terms arises as result of summation on \( S^{-1} \) in first line and summation on \( R^{(+1)} \) in the second line. It is necessary take in mind that only one term in \( \phi'(f)_{-R^{(+1)}, -M} \) is not local namely \( \frac{1}{c_{-r_0}} ([X_M^+, X_{R^{(+1)}}][c, f^{(+1)})]. \) Changing in summation \( -R^{(+1)} \rightarrow S^{-1} \) we obtain

\[
\frac{1}{c_M} \sum \left( \frac{1}{c_{S^{-1}}} + \frac{1}{c_{r_0}} \right) \left( [\alpha^{(+1)}_{S^{-1}}, X_{S^{-1}}] X_{-r_0} \right) \frac{f_M}{[c, f^{(-1)}]}
\]

Nonlocal terms in this sum and in the sum of left side are cancels and result of summation exactly opposite by the sign with the same term arising from \( [c, f^{(-1)}] \).

By the same technique it is possible to convince that all terms in equation under consideration are mutually cancel. We demonstrate these calculations on two most complicate examples.
At first we calculate terms containing \( f^0 \). Matrix coefficient before before \([c, f^0] \) is the following one ( up to common numerical factors)

\[
\frac{1}{2} X_{r_0}[\alpha^{(+1)}, f^{(-1)}] + [[X_{r_0}, \alpha^{(+1)}] f^{(-1)}] = \\
\frac{1}{2} ([[X_{r_0}, \alpha^{(+1)}] f^{(-1)}] + [[X_{r_0}, f^{(-1)}] \alpha^{(+1)}]) - \\
\frac{1}{2} \sum [X_{r_0} h_\beta k_{\beta, \gamma}^*(h_\gamma[\alpha^{(+1)}, f^{(-1)}])]
\]

Expression in the second line equal 0. Indeed

\[
[[X_{r_0}, \alpha^{(+1)}] f^{(-1)}]|c, f^0] = [[X_M^+, [X_{r_0}, f^{(-1)}] f^{(-1)}]|c, f^0] = [\alpha^{(+1)} [X_{r_0}, f^{(-1)}]|c, f^0]
\]

After scalar multiplication on \([c, f^{(-1)}] \) it is necessary calculate two terms

\[
[X_{r_0} h_\beta]|c, f^0] = c_{r_0} \sum n_\gamma^r k_{\gamma, \beta} \frac{T_M \to f_0}{f_0} \\
(h_\gamma[\alpha^{(+1)}, f^{(-1)}]) = \sum_{S^{-1}, \beta} n_\beta^{S^{-1}} k_{\beta, \gamma} f_{s^{-1}} f_{M-s^{-1}} ([X_M^+, X_{M-s^{-1}}] X_{s^{-1}})
\]

Substituting these expressions we come to finally result

\[
\frac{T_M \to f_0}{f_0} \frac{1}{2c_M(f_M)^3} \sum_{S^{-1}} (S^{-1} r_0) f_{s^{-1}} f_{M-s^{-1}} ([X_M^+, X_{M-s^{-1}}] X_{s^{-1}}) = - \\
\frac{1}{(c_M)^2(f_M)^3} \frac{T_M \to f_0}{f_0} \frac{1}{\theta_r}
\]

with \((S^{-1} r_0) = - \sum n_\beta^{S^{-1}} k_{\beta, \gamma} n_\gamma^r \). The last equality written in connection with general formulae in the end of the section (6.2.4) in which \((M r_0) = 0 \).

Now we calculate terms of fourth degree in \( f^{(-1)} \) functions. They all arises as scalar product with generator \( X_{r_0} \). Up to common factor \( \frac{1}{c_{r_0} c_M(f_M)^3} \) they are the following ones

\[
- \frac{1}{6} [[[\alpha^{(+1)}, f^{(-1)}] f^{(-1)}]|c, \alpha^{(+1)}] - \frac{1}{6} [\alpha^{(+1)} [c[\alpha^{(+1)}, f^{(-1)}] f^{(-1)}]] - \\
\frac{1}{2} \alpha^{(+1)} [f^{(-1)} [c[\alpha^{(+1)}, f^{(-1)}]]]
\]

The first two terms arises after summation in corresponding terms of \( \phi^r(f)_{-R^{(+1)}, -M} \) and \([c, f^{(-1)}] \). The last one after representation \( f^0 = f^0 - \frac{1}{2} \alpha^{(+1)}, f^{(-1)} \) in
Except of these 3 terms fourth degree $f^{(-1)}$ have two terms containing $	heta$ function. We present 3 terms above to the same form after which cancels all terms of fourth order became obvious. For this goal let us consider obvious equality

$[[[\alpha^{(+1)} f^{(-1)}] f^{(-1)}] f^{(-1)}] = -[[[\alpha^{(+1)} f^{(-1)}] [\alpha^{(+1)} f^{(-1)}]] X_{M}^{-}$

and perform commutation both sides with $X_{M}^{+}$. We obtain

$[[[\alpha^{(+1)} f^{(-1)}] \alpha^{(+1)}] f^{(-1)}] + [[[\alpha^{(+1)} f^{(-1)}] \alpha^{(+1)}] f^{(-1)}] = 2[[[\alpha^{(+1)} f^{(-1)}] \alpha^{(+1)}] f^{(-1)}]$

$$= -([\alpha^{(+1)} f^{(-1)}] [\alpha^{(+1)} f^{(-1)}]) H$$

From the last result it follows that the both terms above with coefficient $\frac{1}{2}$ are equal to each other ($(X_{-r_{0}} H) = 0$) and after application to this term equality for $[c][\alpha^{(+1)} f^{(-1)}] f^{(-1)}]]$ (see middle of the page 15) we rewrite this term in a form

$$-[[\alpha^{(+1)} [\alpha^{(+1)} f^{(-1)}] c, f^{(-1)}]) - [\alpha^{(+1)} f^{(-1)}] \theta - \frac{1}{2} [\alpha^{(+1)} [f^{(-1)} c] [\alpha^{(+1)} f^{(-1)}]]] =$$

$$\frac{1}{2} [\alpha^{(+1)} f^{(-1)}] \theta$$

The same structure have two terms arises after summation in corresponding terms of $\phi'_{-R^{(1)}-M}$ and in $[c, f^{(-1)}]$. Sum of these terms (with the same common factor as above) are the following

$$\frac{1}{2c_{M}} (\alpha^{(+1)} [c, f^{(-1)}]) + [f^{(-1)} c, \alpha^{(+1)}]) \theta = \frac{1}{2} ([\alpha^{(+1)}, f^{(-1)}] \theta$$

exactly opposite by the sign with the result obtained few lines above.

This result prove validity matrix element under consideration of the main equation.

### 6.3 Checking of the elements of $(1 + N^{1} + N^{0} \times 1 + N^{1})$

rectangular matrix of right dawn corner of (18)

These calculations are absolutely of the same kind as performed in the previous section. We would like to demonstrate them on detail calculations on example of matrix element $<-M|, |S^{(-1)}>$. 

$$\phi'_{-M,M} J_{M,S^{(-1)}} \phi'_{-M,R^{(1)},-M} J_{R^{(1)},+1,S^{(-1)}} + \sum_{R^{(1)}} \phi'_{-M,R^{(1)},-M} J_{R^{(1)},+1,S^{(-1)}} + \sum_{R^{(1)}} \phi'_{-M,R^{(1)},+1,S^{(-1)}} + \sum_{Q^{(-1)}} \phi'_{-M,Q^{(-1)},S^{(-1)}}$$

$$\frac{c_{M}}{c_{S^{(-1)}}} T_{M} \to J_{1} - M,M \phi'_{-M,-S^{(-1)}} \phi'_{-R^{(1)},-S^{(-1)}} - \frac{1}{c_{S^{(-1)}c_{M}}} \sum_{R^{(1)}} T_{M} \to J_{1} - M,R^{(1)},-S^{(-1)} \phi'_{-R^{(1)},-S^{(-1)}}$$

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6.3.1 Matrix element between states $<-M|J|M>$

In this case both additional terms arising from $J^{-1}_1$ (see subsection 5.2) must be taken into account. Equation to be checked is as follows

$$\sum_{S(-1)} \phi'_{-M,S(-1)} J_{S(-1),M} + \phi'_{-M,-M} J_{-M,M} + \frac{(M,M) f_M^T F_M}{c_M^2} D f_M^T - \frac{f_M^T}{c_M^2} \Theta_M =$$

$$\sum_{R^{(1)}} \phi'_{-M,R^{(1)},-M} J_{-M,M} \phi'_{(f)_{-M,-M}} + \frac{(M,M) f_M^T D 1}{f_M^T} + \frac{f_M^T}{c_M^2} \Theta_{-M}$$

This expression contain terms of the zero, first and third degree of $D$. We will present below only checking terms of the positive degree. $J_{-M,M} = J_{M,-M} = \frac{1}{c_M^2} D$ two terms in both sides of the last expression containing this operation leads to (all terms linear in $D$ we present in the left side of equation) $-[D, \phi'_{-M,-M}]$ and terms in $\phi'_{-M,-M}$ with operation of $D$ are the following ones (see section 4.1) $\frac{1}{c_M^2} D^2 - 2 \frac{f_M^T \Theta f_M^T}{f_M^T c_M^2} D$ and thus commutator with $D$ leads to linear term $-[D, \phi'_{-M,-M}] = 2 \frac{f_M^T \Theta f_M^T}{f_M^T c_M^2} D$. Linear in $D$ terms in $\phi'_{-M,S(-1)}$ after summation lead to $\frac{(f^{-1}_{(c,f^{(1)})})}{c_M^2}$, the same with respect to term $\phi'_{-R^{(1)},-M}$ looks as $-\frac{\frac{T_{M,-M}(X_{c,f}^{-1})}{f_M^T c_M^2}}{f_M^T c_M^2}$ and thus coefficient before $D$ looks as

$$2 \left( \frac{f_M^T \Theta f_M^T}{f_M^T c_M^2} \right) + \frac{2}{c_M^2} \left( f_M^T c_M^2 \right) - \frac{(f^{-1}_{(c,f^{(1)})})}{c_M^2} - \frac{(f^{-1}_{(c,f^{(1)})})}{c_M^2} - \frac{T_{M,-M}(X_{c,f}^{-1})}{f_M^T c_M^2}$$

Using equations of discrete transformation and explicit expressions for Frechet derivatives it is convic that the last expression equal to zero.

6.3.2 Matrix element between states $<S^{(1)},|R^{(1)}>\$

This matrix element is the most difficult for calculations. We will present general formulae below and give some advice in what order this not trivial calculations must be performed. Terms in basic equation (18) (in both sides) are the following ones

$$\sum_{Q^{(1)}} \phi'_{S^{(1)},Q^{(1)},R^{(1)}} J_{Q^{(1)},R^{(1)}} + \sum_{r_0} \phi'_{S^{(1)},r_0} J_{r_0,R^{(1)}} + \sum_{Q^{(-1)}} \phi'_{S^{(-1)},Q^{(-1)},JQ^{(-1),R^{(1)}}} + \sum_{Q^{(-1)},r_0} \phi'_{S^{(-1)},Q^{(-1)},r_0} J_{r_0,R^{(1)}} + \sum_{r_0} \phi'_{S^{(-1)},r_0} J_{r_0,R^{(1)}}$$

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\[ \phi'_{S^{-1},-M} J_{-M,R+1} + f_{R+1} \frac{(R^{+1},M)}{c_M} - \frac{(R^{+1},S^{-1})}{c_{S^{-1}}} f_{S^{-1}} = \]
\[ T_{M \rightarrow} f_{S^{-1}} \frac{(R^{+1},M)}{c_M} - \frac{(R^{+1},S^{-1})}{c_{R+1}} f_{R+1} + \sum_{Q' \rightarrow} T_{J \rightarrow} S^{-1} Q'^{+1} \frac{c_{Q'}^{+1}}{c_{R+1}} \phi'_{Q'^{+1},-R'^{+1}} + \sum_{Q'^{+1}} T_{J \rightarrow} S^{-1} Q'^{+1} \frac{c_{Q'}^{+1}}{c_{R+1}} \phi'_{Q'^{-1},-R'^{+1}} + \]
\[ \sum_{Q'^{-1}} T_{J \rightarrow} S^{-1} Q'^{-1} \frac{c_{Q'}^{+1}}{c_{R+1}} \phi'_{Q'^{-1},-R'^{+1}} \]

Now we present the same formulae after summation in which it is only necessary substitute explicit expressions for Frechet derivatives and transformed by discrete transformation values \( f \rightarrow \) (all terms are written in the same order as above)

\[ \frac{f_{M} f_{M}^{+}}{c_{S^{-1}} c_{R+1}} \delta_{S^{-1},-R'^{+1}} - \frac{1}{c_{S^{-1}} c_{R+1}} ([X_{-S^{-1}}, f^{-1}] h X_{-R'^{+1}} [c, f^{+1}]) + \]
\[ \frac{1}{c_{R'^{+1}} c_{M}} \phi'_{S^{-1},-R'^{+1}} \delta_{S^{-1},-R'^{+1}} + \frac{1}{c_{R'^{+1}} c_{M}} \sum_{Q'^{-1}} \phi'_{S^{-1},Q'^{-1}} \frac{1}{c_{Q'^{-1}}} ([X_{-Q'^{-1}}, X_{-R'^{+1}}] [c, f^{+1}]) + \]
\[ \frac{1}{c_{R'^{+1}} c_{M}} \phi'_{S^{-1},-M} ([X_{S^{-1}}, X_{-R'^{+1}}] [c, f^{+1}]) + f_{R'^{+1}} \frac{(R^{+1} M)}{c_M} \frac{(R^{+1} S^{-1})}{c_{S^{-1}}} f_{S^{-1}} = \]
\[ T_{M \rightarrow} f_{S^{-1}} \frac{(R^{+1} M)}{c_M} - \frac{(R^{+1} S^{-1})}{c_{R+1}} f_{R+1} + \frac{1}{c_{S^{-1}} c_{R+1}} ([X_{-S^{-1}}, X_{-R'^{+1}}] [c, f^{+1}]) \phi'_{S^{-1},-R'^{+1}} - \]
\[ \frac{1}{c_{R'^{+1}} c_{M}} \phi'_{S^{-1},-R'^{+1}} \delta_{S^{-1},-R'^{+1}} + \frac{1}{c_{S^{-1}} c_{R'^{+1}}} \sum_{Q'^{-1}} ([X_{-S^{-1}}, X_{-Q'^{-1}}] [c, f^{+1}]) \phi'_{Q'^{-1},-R'^{+1}} + \]
\[ \frac{1}{c_{S^{-1}} c_{R'^{+1}}} ([X_{-S^{-1}} [\alpha^{+1}, X_{-R'^{+1}}] h] [c, f^{+1}]) + \frac{T_{M \rightarrow} f_{M}}{c_{S^{-1}} c_{R+1} f_{M}} \delta_{S^{-1},-R'^{+1}} \]

Now we would like to show that all non locality in terms above are mutually cancel. Let us rewrite two last terms of \( \phi'_{S^{-1},Q'^{-1}} \) in equivalent form

\[-\frac{1}{c_{S^{-1}}} (X_{-S^{-1}} [X_{Q'^{-1}} [c, f^{+1}]] + \frac{1}{2 f_{M}} (X_{-S^{-1}} [[\alpha^{+1}, f^{+1}] X_{Q'^{-1}}]) = \]
\[-\frac{1}{c_{S^{-1}}} (X_{-S^{-1}} [X_{Q'^{-1}} [c, f^{+1}]] + \frac{T_{M \rightarrow} f_{M}}{c_{S^{-1}} c_{R+1} f_{M}} (X_{S^{-1}} [\alpha^{+1}, f^{+1}] X_{Q'^{-1}})) \]

Now we rewrite two last terms of \( \phi'_{S^{-1},Q'^{-1}} \) in equivalent form

\[-\frac{1}{c_{S^{-1}}} (X_{-S^{-1}} [X_{Q'^{-1}} [c, f^{+1}]] + \frac{T_{M \rightarrow} f_{M}}{2 f_{M}} (X_{S^{-1}} [\alpha^{+1}, f^{+1}] X_{Q'^{-1}})) \]
From which follows only one non local term in the sum in the left side has the form

\[-\frac{1}{c_{S(-1)}c_{R(1)}} \sum_{Q(-1)} \frac{1}{c_{Q(-1)}} ([X_{-Q(-1)}, X_{-R(1)}][c, f^0])(X_{-S(-1)}[X_{Q(-1)}[c, T_{M(-1)}]][f^0]])\]

Exact the same is non local term in the right hand side of sum. All other terms allow summation and after substitution. After in principle non cumbersome calculations – it is necessary compare coefficients before terms of the same structure – we convince that expression written above is equality.

We will not present checking all other matrix elements because the tremendous number of cancelation under calculations above can not be occasional.

7 Outlook

The main positive result of the present paper - two Poisson structures in explicit form invariant with respect to discrete transformation of the maximal root of arbitrary semi simple algebra. The knowledge of these structures allow to construct equations of hierarchies of n-waves systems in the case of arbitrary semi simple algebra. And thus in this problem this paper put the finally point.

But by the author opinion the most interesting is the standing by this paper the following problem What is group representation background of the discrete transformation of maximal root? All calculations above were done using only main properties of the structure theory of semi simple algebra (properties of its root space and commutation relation in it). Discrete substitution is some new element of this construction possible taken ad hoc from the head. But it properties can be proofed only on the base of group representation theory and thus it must have some group theoretical origin. What? This is unsolved in the present paper very interesting problem.

Keeping in mind that discrete transformation of maximal root is consequence of Yang-Mills equations describing interaction of elementary particles after finding the answer on the stated above problem it will be possible to understand why equations of Y-M are excluded from other ones and why Great God have used them in the process of construction of our Universe.

Some comments about perspectives of possible further investigation. It is absolutely clear that it is possible to resolve symmetry equation in the case of four dimensional self dual system and by this way construct all integrable systems in the sense of Richard Ward. The same is true for all super symmetrical generalization of Y-M self dual equations.

Author hope return to solution of this problem in the nearest time.
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9 Appendix

The terms with derivatives in (7) are the following ones
\[
\frac{1}{2c_M} \left( \frac{X'_M}{X_M} \right)^2 + \frac{1}{c_M f_M} ([X_M^+, f^{(-1)}]([c, (f^{(-1)})']))
\]
and thus it is necessary to check connection of following mixed derivatives
\[
G_{T,M,-}^T - G_{M,-}^T, \quad G_{M,f^{(-1)}-}^T - G_{f^{(-1)},M}^T, \quad G_{f^{(-1)},f^{(-1)}-}^T - G_{f^{(-1)},f^{(-1)}-}^T.
\]
Consider the first difference under arbitrary dependence of generating functional on \((x, x')\) variables.
\[
\frac{\delta G}{\delta x} = -G'_x + G_x = -x''G'_{x',x'} - x'G'_{x',x} + G_x,
\]
\[
\frac{\delta^2 G}{\delta x, \delta x} = -G''_{x',x'} D^2 + (-x''G'_{x',x'} - x'G'_{x',x} + G_x) x' D + (-x''G'_{x',x'} - x'G'_{x',x} + G_x) x
\]
Thus
\[
G_{T,x,x}^T - G_{x,x} = -D^2 G_{x',x'} + G_{x',x'} D^2 - 2D(-x''G_{x',x'} - x'G_{x',x} + G_x) x' + (x''G_{x',x'} - x'G_{x',x} + x'G_{x',x'}) x' = -2G'_{x',x'} D - (G_{x',x'})'' - 2(-x''G_{x',x'} - x'G_{x',x} + x'G_{x',x'}) D - (-x''G_{x',x'} - x'G_{x',x} + x'G_{x',x'}) x' = 0
\]
Calculations and comparing derivatives of the second order connected with last term in (19)
\[
G_{f^{(-1)},-}^T = -(\frac{1}{f_M} ([X_M^+, f^{(-1)}][c, X_i^-])') + \frac{1}{f_M} ([X_M^+, X_i^-][c, (f^{(-1)})']) =
\]
\[
-(\frac{1}{f_M} )'([X_M^+, f^{(-1)}][c, X_i^-] - \frac{c_M}{f_M} ([X_M^+, X_i^-])(f^{(-1)}'))
\]
\[
G_{f^{(-1)}, f_j^{(-1)}}^T = -(\frac{1}{f_M} )'([X_M^+, X_j^-][c, X_i^-] - \frac{c_M}{f_M} ([X_M^+, X_i^-] [X_j^-] )D)
\]
From which equality \(G_{f^{(-1)}, f_j^{(-1)}}^T = G_{T,f_j^{(-1)}, f_i^{(-1)}}^T \) \(((AD)^T = -AD)\) follows immediately. The same calculations lead to equality \(G_{T,f^{(-1)},M}^T = G_{f^{(-1)},M}^T\).
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