Exploring Topological Phase Transition via Quantum Walk in Coherent State Space

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Topological phase transition plays a significant role in modern condensed matter physics, since it is beyond Landau symmetry-breaking theory. Quantum walks have been demonstrated to be a powerful method in exploring topological quantum matter. Here, we investigate a special quantum walk in a line in terms of coherent state representation. By detecting average number of photons of the system, quantum phase transition can be observed, and the properties of coherent space are utilized. In addition, we propose an experimental protocol in a circuit quantum electrodynamics architecture, where a superconducting qubit is a coin while the cavity mode is used for quantum walk.

Topological order, as a fundamental phenomenon in condensed matter physics, can be divided into two categories. One is the intrinsic topological order which comes from long-range quantum entanglement and has anyonic excitations [1–3], such as fractional quantum Hall effect [4, 5] and toric code model [6]. The other is symmetry protected topological (SPT) order which is induced by system symmetry [7], for instance, time-reversal-protected topological insulators [8–11] and p-wave superconductors [11–13]. Nontrivial topological phase has two unique features, topological invariants of bulk states and gapless edge states. Besides natural systems, some artificial quantum systems can realize topological phase, for example, optical lattice [14–20]. Actually they are convenient to study such intriguing topological features of matter, since many of the parameters are adjustable.

Quantum walks, extended from classical random walks, are designed as the protocol which describes the dynamics of single spinful particle on a lattice [21–23]. Recently, Kitagawa \textit{et al.} [24] found that quantum walk is a powerful platform to explore the topological phases, especially SPT order. Subsequently, a lot of experiments have observed such topological features, including edge-bulk correspondence, topological invariants and topological phase transitions [25–29]. In addition, the non-Hermitian quantum walks are also proposed and realized experimentally [27, 30], which can be used to study topological phase. There are many methods or platforms to implement quantum walks, include optical system [25–28], trapped ions [31–33] and circuit quantum electrodynamics (c-QED) [29, 34, 35].

In this Letter, we propose coherent-state-space (CSS) quantum walks, of which walker evolves in coherent-state space, rather than normal real space. In Refs. [29, 32, 34, 35], CSS quantum walks have been investigated, but in regarding the coherent space as lattice rather than taking advantage of the inherent source of coherent state itself. Since arbitrary two coherent states are not orthogonal to each other, it in general cannot be used directly. Nevertheless, we will point out later that, on the one hand, such non-orthogonality can be cancelled by some simple operations, on the other hand, the non-orthogonality itself is a useful source to show topological phase transition, such that it can characterize nonanalytical points (or critical points). Furthermore, we do not need a lot of observables to characterize different phases, and instead, the average number of photons is enough, which can be measured easily in experiments. Then we design an experiment in superconducting quantum circuit to realize CSS quantum walks [36, 37], where the superconducting qubit couples to a microcavity with large detuning.

We start with split-step quantum walks (SSQW), which alternates two coin tosses between two spin-dependent translations. The unitary operator of each step

![FIG. 1. (Color online) The phase diagram of the Hamiltonian. Here, parameters in (1) are as, $\theta_1, \theta_2 \in [0, \pi]$. The critical points satisfy $| \frac{\tan \theta_1}{\tan \theta_2} | = 1$. When $| \frac{\tan \theta_1}{\tan \theta_2} | > 1$, the winding number is 2, i.e., topological nontrivial. When $| \frac{\tan \theta_1}{\tan \theta_2} | < 1$, it is topological trivial phase.](image-url)

$\theta_1 \quad 2 \quad 0 \quad 2 \quad \pi$
can be written as $\hat{U}_W(\theta_1, \theta_2) = \hat{T}_1 \hat{R}_x(2\theta_2) \hat{\bar{T}}_1 \hat{R}_x(2\theta_1),$ where $\hat{T}_1 = \sum_x (|x+1\rangle \otimes |\uparrow\rangle + |x-1\rangle \otimes |\downarrow\rangle) / \sqrt{2}$ is spin rotation. Using Fourier transformation $|k\rangle = 1 / \sqrt{2\pi} \sum_x e^{-ikx} |x\rangle$, the spin-dependent operator can be diagonalized as $\hat{T}_1 = e^{ik\sigma_z} \otimes |k\rangle \langle k|$, and since $\hat{U}_W(\theta_1, \theta_2) = e^{-i\epsilon_{\alpha} \sigma_z} \otimes |k\rangle \langle k|$, we can obtain the effective Hamiltonian [24]

$$H_{\text{eff}}(\theta_1, \theta_2) = \sum_k \epsilon_{\theta_1, \theta_2}(k) n_{\theta_1, \theta_2}(k) \cdot \sigma \otimes |k\rangle \langle k|.$$  

Here, $\epsilon_{\theta_1, \theta_2}(k)$ characterizes the band structure while $n_{\theta_1, \theta_2}(k)$ corresponds to the eigenstates of single particle in terms of lattice momentum $k$. We can find that when $|\tan \theta_1 / \tan \theta_2| = 1$, the gap closes, which indicates topological phase transition. Furthermore, $H_{\text{eff}}(\theta_1, \theta_2)$ has time reversal, particle-hole and chiral symmetries, satisfying $T'^2 = \mathcal{P}^2 = 1$ and $\mathcal{P} = e^{-iA} \sigma_z/2$, where $A = (0, \cos \theta_1, -\sin \theta_1)$. Therefore, it belongs to class BDI, which means that its topological invariant is winding number defined as [7]

$$\gamma = \frac{1}{2\pi} \int_{-\pi}^{\pi} dk \left( n_x \times \frac{\partial n}{\partial k} \right) \cdot A. \quad (2)$$

When $\gamma > 0$, it is topological nontrivial, and the phase diagram is shown in Fig.1. Without loss of generality, in the following discussion, unless otherwise specified, $\theta_1$ is fixed to $\pi/4$. The critical points are at $\theta_2 = \pi/4$ and $3\pi/4$.

To describe phase transition, we need the first and second moments. According to Refs. [26, 28], the $j$-th moments are defined as

$$M_j(N) = \sum_x x^j P_N(x), \quad (3)$$

where $x$ represents the walker position while $P_N(x)$ is the relevant probability distribution after $N$ steps. Firstly, let us calculate $M_2$. Considering $N \to \infty$, we have

$$M_2(N)/N^2 = L(\theta_1, \theta_2) + O(1/N^2), \quad (4)$$

and

$$L(\theta_1, \theta_2) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} [V_\theta(\theta_1, \theta_2)]^2. \quad (5)$$

Here, $V_{\theta_1, \theta_2}(k) = d\epsilon_{\theta_1, \theta_2}(k) / dk$, is the associated group velocity. According to the residue theorem, we know $L(\theta_1, \theta_2)$ is nonanalytical at critical points, since gapless points are the poles of $V_{\theta_1, \theta_2}(k)$. So $M_2$ can characterize the phase transition just like the “order parameter”. Fig. 2(a) shows the relation between $M_2$ and $\theta_2$ when $N$ choose different values.

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**FIG. 2.** (Color online) The first and second moments for real-space quantum walks. The results for walk step $N = 10, 20, 50$ are shown respectively. (a) The initial state is $|0\rangle \otimes |\uparrow\rangle$. In topological nontrivial phase, $M_2$ keeps constant when varying $\theta_2$. With the increase of walk steps, the nonanalyticity at critical point is more and more distinct. (b) The initial state is $|0\rangle \otimes ((|\sqrt{2} + 1\rangle \otimes |\uparrow\rangle) + |\downarrow\rangle)/2\sqrt{\sqrt{2} + 4}$, of which the coin state is the eigenstate of $\mathcal{P}$ and the eigenvalue is 1. The equilibrium position tends to $\gamma/2 = 1$ in topological topological nontrivial phase, while in topological trivial phase, it tends to 0.

As for the first moment $M_1$, similar as in [28], we have,

$$M_1(N) = \langle \Gamma_{\downarrow} \rangle \psi_0 [L(t) + S(t)] - \langle \Gamma_{\uparrow} \rangle \psi_0 S_r(t). \quad (6)$$

Here, $\langle \psi_0 \rangle$ is the initial state and $a = \langle \sigma \rangle \psi_0$, and

$$S_{\uparrow}(t) = -\frac{1}{2} \int_{-\pi}^{\pi} \frac{dk}{2\pi} \cos(2N\epsilon) (n_x \times \frac{\partial n}{\partial k}) \cdot A, \quad (7)$$

where $\gamma$ is the winding number. If the initial spin polarization direction is orthogonal to $A$, i.e., $\langle \Gamma_{\uparrow} \rangle \psi_0 = 0$, then

$$M_1(N) = \pm \frac{\gamma}{2} \int_{-\pi}^{\pi} \frac{dk}{2\pi} \cos(2N\epsilon) (n_x \times \frac{\partial n}{\partial k}) \cdot A. \quad (8)$$

So when $N \to \infty$, the second term of Eq.(8) quickly converges to zero. It can thus show obvious topological phase transition, and see Fig. 2(b).

Now if we let the walker walks in coherent space rather than real space, that is, CSS quantum walk, then what will happen? We replace real-space state $|x\rangle$ with coherent state $|x\alpha\rangle = e^{-|x\alpha|^2/2} \sum_n x^n \alpha^n / \sqrt{n!} |n\rangle$. In this case, the spin-dependent-translation operator, $\hat{T}_1(\alpha)$, satisfies $\hat{T}_1(\alpha) |x\alpha, \uparrow\rangle = (|x + 1\rangle \alpha, \uparrow\rangle)$. Thus, the final state has the form of $|\psi_f\rangle = \sum_{x,\sigma} A_{x,\sigma} |x\alpha, \sigma\rangle$. Where $\sigma$ is the spin index. So the expected photon number of the final state, $N_w$, can be obtained as

$$N_w = \langle \hat{a}^\dagger \hat{a} \rangle = \sum_{x,\sigma} \sum_{x,\sigma} \langle A_{x,\sigma} \rangle |x\alpha|^2$$

$$+ \sum_{x,\sigma} x(x+1) A_{x,\sigma}^* A_{x+1,\sigma} |x\alpha|^2 e^{-\alpha^2/2} + c.c$$

$$+ \sum_{x,\sigma} x(x+2) A_{x,\sigma}^* A_{x+2,\sigma} |x\alpha|^2 e^{-2\alpha^2} + c.c + \cdots, \quad (9)$$

where $c.c$ is complex conjugation.
First of all, we consider the case that $|\alpha|$ is large enough, that is, only the first term contributes in Eq. (9). In such case, $N_w = |\alpha|^2 M_2$. According to above discussion, we know that $M_2$ is nonanalytical at critical points, so $N_w$ can indeed characterize topological phase transition. In addition, if there are two CSS quantum walks, of which initial states are $|m\alpha\rangle \otimes |s\rangle$ and $|-m\alpha\rangle \otimes |s\rangle$, respectively, then $N_{w1} = \sum_x |\alpha|^2 (x+m)^2P_x$ while $N_{w2} = \sum_x |\alpha|^2 (x-m)^2P_x$, thus, $\Delta N_w = N_{w2} - N_{w1} = 4m|\alpha|^2 M_1$. When the polarization direction of $|s\rangle$ is orthogonal to $\Lambda$, $\Delta N_w$ is also able to characterize topological phase transition as mentioned in Eq. (8).

Nevertheless, if $|\alpha|$ is very large, then the system optical field will be so strong that may lead some problems in experiments, for instance, some nonlinear effects which we do not expect. Hence, we should consider the case that $|\alpha|$ is not very large. Now, for Eq. (9), the second and third terms (or higher terms) cannot be neglected, and we denote the $j+1$-th term as $T_j$. Consider another state in terms of real space $|\psi'\rangle = \sum_{x,\sigma} A_{x,\sigma}|x\rangle$, which can be regarded as the final state of real-space quantum walk by just replace coherent-state basis $|\alpha\rangle$ with real-space-state basis $|x\rangle$ for $|\psi\rangle$,

$$T_j = |\alpha|^2 e^{-j\alpha^2/2}\langle\psi'\rangle e^{-j\alpha^2/2} \left\{ T_1(j) + T_n(-j) \right\} |\psi'\rangle.$$  

(10)

Here, $\hat{T}(j)$ is translation operator and has the form of $e^{-ik\sigma}$. Following Eq. (10) to calculate $N_w$, we have

$$N_w = |\alpha|^2 M_2 - 2|\alpha|^2 e^{-\alpha^2/2} \left\{ (\hat{U}_W^N \hat{\partial}_x^2 \cos k\hat{U}_W^N) \psi_0 \right\}$$

$$-2|\alpha|^2 e^{-\alpha^2} \left\{ (\hat{U}_W^N \hat{\partial}_x^2 \cos 2k\hat{U}_W^N) \psi_0 \right\} + \cdots.$$  

(11)

Here, the unitary operator $\hat{U}_W$ and initial state $\psi_0$ both correspond to real-space quantum walk. For convenience, let $\mathcal{I}(m) \equiv \langle \hat{U}_W^N \hat{\partial}_x^2 \cos (mk) \hat{\hat{U}}_W^N \rangle \otimes N^2$, and obviously $\mathcal{I}(0) = M_2/N^2$. For $N \to \infty$, $\mathcal{I}(m = \text{odd}) = 0$ while $\mathcal{I}(m = \text{even})$ are nonanalytical at critical points. So $\mathcal{I}(m)$ are the function of $\theta_x$ and the curves show in Fig. 3(a), when $m = 2, 4, 6$, respectively. When $|\alpha|$ is not so small, i.e., $\mathcal{I}(2)$ has no contribution yet, $N_w$ is still proportional to $M_2$. However, if $|\alpha|$ is so small, $N_w$ will deviate from $M_2$, see Fig. 3(b).

If we want to reduce further the photon number, we need to deal with the second and fourth terms. Now, we add a spin rotation operator around $z$-axis after each $T_1(\alpha)$, so that new unitary operator of one walk step is $\hat{U}_W(1, \theta_z) = R_z(2\phi)T_1(\alpha)R_z(2\phi)T_2(\alpha)T_1(\alpha)R_z(2\phi)$. Thus, it can accumulate different phases for different spin directions during each of the walk, that is, the final state of the system is $|\psi'\rangle = \sum_{x,\sigma} e^{i\phi\sigma} A_{x,\sigma}|x\alpha, \sigma\rangle$. In such case, $N_w$

![FIG. 3. (Color online) (a) The function between $\theta_2$ and $\mathcal{I}(m = \text{even})$ while $m = 2, 4, 6$ respectively. There are distinct pinacles at critical points, which are nonanalytical and represent phase transition. (b) The average number of photons with walk steps $N = 20$ and $\alpha = 1.5$, 1.0, 0.8 respectively. When $\alpha = 1.5$, $\mathcal{I}(2)$ has not contributed, so $N_w(\alpha = 1.5)$ is proportional to $M_2$. When $\alpha = 1.0$, 0.8, $N_{w,s}$ deviate from $M_2$, since the higher terms are not able to be neglected, however, at critical points, $N_{w,s}$ are also nonanalytical. (c)-(e) The average number of photons, when adding $R_z(2\phi)$, we can separate the different terms which contribute $N_{w,s}$. Here, $\tilde{N}_{w,s}, \Delta N_{w,s}$ and $\delta N_{w,s}$ are displayed with $\alpha = 0.4$ and walk steps $N = 10, 20$ and 30, respectively. For (c)-(d), the initial state is $|0\rangle \otimes |\uparrow\rangle$, while for (e), the initial states are $|2\rangle \otimes |s\rangle$ and $|-2\rangle \otimes |s\rangle$ where $|s\rangle = (|\sqrt{2} + 1\rangle |\uparrow\rangle + |\downarrow\rangle) / \sqrt{2\sqrt{2} + 4}$.

by calculating the average number of photons of these two walks, we have $N_w(\alpha = \pi/8) = |\alpha|^2 M_2 - \sqrt{2}|N|^2 e^{-\alpha^2/2} \mathcal{I}(2)$ and $N_{w,s} = 3\pi/8 = |\alpha|^2 M_2 + \sqrt{2}|N|^2 e^{-\alpha^2/2} \mathcal{I}(2)$. Define the summation and difference of two $N_{w,s}$ as

$$\tilde{N}_{w,s} = N_w(\alpha = \pi/8) + N_{w,s} = 2|\alpha|^2 M_2$$

$$\Delta N_{w,s} = N_{w,s} - N_w(\alpha = \pi/8) = -2\sqrt{2}|\alpha|^2 e^{-\alpha^2/2} \mathcal{I}(2).$$  

(13)

(14)
is coupling strength, $\hat{\text{erator}} \text{ of photons.}$

The solution to the cavity satisfy dispersive coupling, of which free evo-

sion $[29, 34, 35], we consider the superconducting qubit and spin-dependent translation $\hat{\text{walker.}}$

The superconducting qubit represents the internal spin of the $\text{walker,}$

in coherent space of cavity mode in a line while the su-

and realized experimentally. Here, we let walker walks

CSS quantum walk and detect topological phase tran-

sition of the operation sequence: $\hat{U}(t^*) \to \hat{D}(i\alpha) \to \hat{U}(3t^*)$, that is, $\hat{T}_{11}(\alpha) = \hat{U}(3t^*)\hat{D}(i\alpha)\hat{U}(t^*)$. Here, since the time of $t^*$ is too long, we can use two $\pi$ pulses (X gate) and $\hat{U}(t^*)$ to replace $\hat{U}(3t^*)$, that is, $\hat{U}(3t^*) = \hat{\sigma}_x \cdot \hat{U}(t^*) \cdot \hat{\sigma}_x$. Furthermore, this sequence has another advantage that it can realize dynamical-decoupling ef-

fect (or spin echo) which can prolong coherent time of qubit$[38]$.

As for measurement, we just need to detect the final average number of photons in cavity after several walk steps. Here, state tomography (or Q tomography) is not necessary, therefor, the complexity of experiment can be reduced greatly. In fact, we even do not need to detect the absolute number of photons, while the relative optical in-

ternacy is enough instead. Accordingly, all of the realiza-

tions corresponding to the operation sequence of walker

and the measurement are presented, which can indeed be implemented in terms of existing experimental methods.

The explicit quantum circuit is presented in Fig. 4.

In summary, we have demonstrated that CSS quantum walks can detect the topological phase transition by measuring the expected photons number of walkers. The key advantage of CSS quantum walks is that operation and measurement are both very convenient. The challenge is how to control the number of photons when walk step is too large, since it is proportional to the square of walk steps. Of course, in this Letter, only the case of closed system is discussed, and we look forward to extending this idea to dissipated systems or non-Hermitian systems. Furthermore, high dimensional CSS quantum walks are also worth exploring.

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FIG. 4. (Color online) The diagram of quantum circuit corresponding to $N$-step CSS quantum walk in c-QED. Here, the superconducting qubit is a coin while the cavity is a walker. The top one is the total circuit, including initialization, quantum walk and measurement. The bottom one is the diagram of the CSS quantum walk and measurement. The top one is the total circuit, including initialization, quantum walk and measurement. The bottom one is the detail of the quantum circuit corresponding to the operation sequence of walker and measurement. The bottom one is the detail of the quantum circuit corresponding to the operation sequence of walker and measurement.

$\text{Qubit (Coin)}$ $\text{Cavity (Walker)}$

$\text{Qubit (Coin)}$ $\text{Cavity (Walker)}$

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particles, Il Nuovo Cimento B (1971-1996) 37, 1 (1977).
[4] D. C. Tsui, H. L. Stormer and A. C. Gossard, Two-dimensional magnetotransport in the extreme quantum limit, Phys. Rev. Lett. 48, 1559 (1982).
[5] R. B. Laughlin, Anomalous quantum Hall effect: an incompressible quantum fluid with fractionally charged excitations, Phys. Rev. Lett. 50, 1359 (1983).
[6] A. Y. Kitaev, Fault-tolerant quantum computation by anyons, Ann. Phys. (N.Y.), 303 2 (2003).
[7] C. K Chiu, J. C.Y. Teo, A. P. Schnyder and S. Ryu, Classification of topological quantum matter with symmetries, Rev. Mod. Phys. 88, 035005 (2005)
[8] C. L. Kane and E. J. Mele, Quantum spin Hall effect in graphene, Phys. Rev. Lett. 95, 226801 (2005).
[9] C. L. Kane and E. J. Mele, Z_2 Topological Order and the Quantum Spin Hall Effect, Phys. Rev. Lett. 95, 146802 (2005).
[10] M. Z. Hasan and C. L. Kane, Colloquium: Topological insulators, Rev. Mod. Phys. 82, 3045 (2010)
[11] X. L. Qi and S. C. Zhang, Topological insulators and superconductors, Rev. Mod. Phys. 83, 1057 (2011)
[12] N. Read and D. Green, Paired states of fermions in two dimensions with breaking of parity and time-reversal symmetries and the fractional quantum Hall effect, Phys. Rev. B. 61, 10267 (2000).
[13] A. Y. Kitaev, Unpaired Majorana fermions in quantum wires, Phys. Usp. 44, 131 (2001).
[14] M. Greiner, O. Mandel, Theodor W. Hnsch, and I. Bloch, Collapse and revival of the matter wave field of a Bose-Einstein condensate, Nature (London) 419, 51 (2002).
[15] I. Bloch, J. Dalibard, and S. Nascimbne, Quantum simulations with ultracold quantum gases, Nat. Phys. 8, 267 (2012).
[16] N. Flaschner, D. Vogel, M. Tarnowski, B. S. Rem, D.-S. Lee, M. Heyl, J. C. Budich, L. Mathey, K. Sengstock, and C. Weitenberg, Observation of dynamical vortices after quenches in a system with topology, Nat. Phys. 16, 265 (2012).
[17] M. Atala, M. Aidelsburger, J. T. Barreiro, D. Abanin, T. Kitagawa, E. Demler, and I. Bloch, Direct measurement of the Zak phase in topological Bloch bands, Nat. Phys. 9, 795 (2013).
[18] Z. Wu, L. Zhang, W. Sun, X.T. Xu, B.Z. Wang, S.C. Ji, Y. Deng, S. Chen, X.J. Liu, and J.W. Pan, Realization of two-dimensional spin-orbit coupling for Bose-Einstein condensates, Science 354, 83 (2016).
[19] M. Lohse, C. Schweizer, O. Zilberberg, M. Aidelsburger, and I. Bloch, A Thouless quantum pump with ultracold bosonic atoms in an optical superlattice, Nat. Phys. 12, 350 (2016).
[20] S. Nakajima, T. Tomita, S. Taie, T. Ichinose, H. Ozawa, L. Wang, M. Troyer, and Y. Takahashi, Topological Thouless pumping of ultracold fermions, Nat. Phys. 12, 296 (2016).
[21] Y. Aharonov, L. Davidovich, and N. Zagury, Quantum random walks, Phys. Rev. A. 48, 1687 (1993).
[22] J. Kempe, Quantum random walks: an introductory overview, Contemp. Phys. 44, 307 (2003).
[23] S. E. Venegas-Andraca, Quantum walks: a comprehensive review, Quantum Inf. Process. 11, 1015 (2012).
[24] T. Kitagawa, M. S. Rudner, E. Berg, and E. Demler, Exploring topological phases with quantum walks, Phys. Rev. A. 82, 033429 (2010).
[25] T. Kitagawa, M. A. Broome, A. Fedrizzi et al., Observation of topologically protected bound states in photonic quantum walks, Nat. Commun. 3, 882 (2012).
[26] F. Cardano, M. Maffei and F. Massa et al., Detecting topological invariants in nonunitary discrete-time quantum walks, Phys. Rev. Lett. 119, 130501 (2017).
[27] E. Flurin, V. V. Ramasesh and S. Hacohen-Gourgy et al., Observing Topological Invariants Using Quantum Walks in Superconducting Circuits, Phys. Rev. X. 7, 031023 (2017).
[28] M. S. Rudner, L. S. Levitov, Topological transition in a non-Hermitian quantum walk, Phys. Rev. Lett. 102, 065703 (2009).
[29] B. C. Travaglione, G. J. Milburn, Implementing the quantum random walk, Phys. Rev. A. 65, 032310 (2008).
[30] H. Schmitz, R. Matjeschk, Ch. Schneider et al., Quantum walk of a trapped ion in phase space, Phys. Rev. Lett. 103, 090504 (2009).
[31] F. Zahringer, G. Kirchmair, R. Gerritsma et al., Realization of a quantum walk with one and two trapped ions, Phys. Rev. Lett. 104, 100503 (2010).
[32] P. Xue, B. C. Sanders, A. Blais et al., Quantum walks on circles in phase space via superconducting circuit quantum electrodynamics, Phys. Rev. A. 78, 042334 (2008).
[33] V.V. Ramasesh, E. Flurin, M. Rudner et al., Direct probe of topological invariants using Bloch oscillating quantum walks, Phys. Rev. Lett. 118, 130501 (2017).
[34] B. C. Sanders, S. D. Bartlett, B. Tregenna et al., Quantum quincunx in cavity quantum electrodynamics, Phys. Rev. A. 67, 042305 (2003).
[35] J. Koch, T. M. Yu, J. Gambetta et al., Charge-insensitive qubit design derived from the Cooper pair box, Phys. Rev. A. 67, 042319 (2007).
[36] L. Viola, E. Knill, and S. Lloyd, Dynamical Decoupling of Open Quantum Systems, Phys. Rev. Lett. 82, 2417 (2018).