Killing Horizons: Negative Temperatures and Entropy Super-Additivity

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Abstract

Many discussions in the literature of spacetimes with more than one Killing horizon note that some horizons have positive and some have negative surface gravities, but assign to all a positive temperature. However, the first law of thermodynamics then takes a non-standard form. We show that if one regards the Christodoulou and Ruffini formula for the total energy or enthalpy as defining the Gibbs surface, then the rules of Gibbsian thermodynamics imply that negative temperatures arise inevitably on inner horizons, as does the conventional form of the first law. We provide many new examples of this phenomenon, including black holes in STU supergravity. We also give a discussion of left and right temperatures and entropies, and show that both the left and right temperatures are non-negative. The left-hand sector contributes exactly half the total energy of the system, and the right-hand sector contributes the other half. Both the sectors satisfy conventional first laws and Smarr formulae. For spacetimes with a positive cosmological constant, the cosmological horizon is naturally assigned a negative Gibbsian temperature. We also explore entropy-product formulae and a novel entropy-inversion formula, and we use them to test whether the entropy is a super-additive function of the extensive variables. We find that super-additivity is typically satisfied, but we find a counterexample for dyonic Kaluza-Klein black holes.
Contents

1 Introduction 3

2 The Gibbs Surface and Thermodynamic Geometry 8
   2.1 The Gibbs surface 8
   2.2 Thermodynamic metrics 11

3 Asymptotically Flat Black Holes 14
   3.1 The Gibbs surface for Reissner-Nordström 14
   3.2 The Gibbs surface for Kerr 18
   3.3 Kerr-Newman black holes 20
   3.4 STU black holes 21
   3.5 Five-dimensional STU supergravity 32
   3.6 Einstein-Maxwell-Dilaton black holes 35
   3.7 Two-field dilatonic black holes 37

4 Entropy Product and Inversion Laws 40

5 Asymptotically AdS and dS Black Holes 42
   5.1 Kottler 42
   5.2 Reissner-Nordström-de Sitter 44
   5.3 Kerr-Newman-de Sitter black holes 46
   5.4 Pairwise-equal charge anti-de Sitter black hole 46
   5.5 Wu black hole 47

6 Entropy and Super-Additivity 47
   6.1 STU black holes with pairwise-equal charges 49
   6.2 STU black holes with three equal non-zero charges 51
   6.3 Dyonic Reissner-Nordström 52
   6.4 A counterexample: The dyonic Kaluza-Klein black hole 53

7 Conclusions and Future Prospects 56

A Carter-Penrose Diagram for Two Horizons 59

B STU Supergravity 61
1 Introduction

Since the early days of black hole thermodynamics there have been suggestions that the thermodynamic of the inner, Cauchy, horizons of charged and or rotating black holes should be taken more seriously than it has been [1–10]. With the development of String Theory approaches these suggestions have become more insistent [11–17]. This interest increased considerably with the observation that the product of the areas and hence entropies of the inner and outer horizon takes in many examples a universal form which should be quantised at the quantum level [18–22]. Some of these papers, and others, for example [22–27], encountered the same feature first noticed in [1]: the fact that with a conventional first law of thermodynamics the temperature of the inner horizon would be negative. The authors of [22] chose to resolve this issue by defining the temperature of the inner horizon to be the absolute value of the “thermodynamic” temperature, and proposing an appropriately-modified first law on the inner horizon to compensate for this. In this paper we shall explore the consequences of adhering to the standard first law of thermodynamics for inner horizons, with the inevitable result that the temperature will be negative there.

In the derivation of the first law of black hole dynamics one finds, integrating in the region between the inner and outer horizons, that

\[ 0 = \frac{\kappa_+}{8\pi} dA_+ - \frac{\kappa_-}{8\pi} dA_- + \cdots, \]

where \( \kappa_\pm \) are the surface gravities and \( A_\pm \) the areas of the outer and inner horizons respectively. (The contributions from the angular momentum and charge(s) are represented by the ellipses in this equation.) If, as turns out to be the case in the examples we consider, the signs of \( dA_+ \) and \( dA_- \) are opposite for a given change in the black-hole parameters, then the signs of the surface gravities at \( \kappa_+ \) and \( \kappa_- \) must be opposite too. The surface gravity is defined by evaluating

\[ \ell^\mu \nabla_\mu \ell^\nu = \kappa \ell^\nu \]

on the horizon, where \( \ell^\mu \) is the future-directed null generator of the horizon, which coincides with a Killing vector \( K^\mu \) on the horizon. One then finds that whilst \( \kappa \) is positive on the outer horizon, it is negative on the inner horizon. Hawking showed that for an isolated

\[ h(r) = \frac{(r - r_+)(r - r_-)}{r^2}, \]

as in the Reissner-Nordström metric, then \( \kappa_+ = (r_+ - r_-)/(2r_+^2) > 0 \), while \( \kappa_- = -(r_+ - r_-)/(2r_-^2) < 0 \). In general, of course, the slope of \( h(r) \) must always have opposite signs at two adjacent zeros, and thus the surface gravities must have opposite signs.
event horizon in an asymptotically flat spacetime (for which in fact $\kappa$ is positive), the temperature is $\kappa/(2\pi)$. We shall discuss the extension of Hawking’s calculation to the case of inner horizons in the concluding section of this paper. In what follows, however, we shall frequently make reference to the formula

$$T = \frac{\kappa}{2\pi},$$

with the understanding that $T$ may not be a temperature measured by a physical thermometer, but rather, as we shall explain shortly, a “Gibbsian” temperature.

The occurrence of a negative $\kappa$ on an inner horizon is somewhat obscured in many discussions in the literature by the fact that the surface gravity is commonly calculated by evaluating

$$\kappa^2 = -g^{\mu\nu}(\partial_\mu K^2)(\partial_\nu K^2)$$

in the limit on the horizon. This formula is derivable from \(1.2\), but the information about the sign of $\kappa$ is lost, and commonly the positive root is assumed when calculating $\kappa$ from \(1.4\). A guaranteed correct procedure is to use the formula \(1.2\), working in a coordinate system that covers the horizon region.

Another situation where one encounters two horizons is when a positive cosmological constant $\Lambda$ is involved and one has both a black hole event horizon and a cosmological event horizon bounding a static or stationary region [28]. A number of recent studies have pointed out that the surface gravities of the black hole horizon $\kappa_B$ and the cosmological event horizon $\kappa_C$ again have opposite signs [29–31]. Most have followed the procedure adopted in [28] and taken the physical temperature to be $|\kappa|/(2\pi)$ (for example, see [32]). A similar situation arises in the case of the C-metrics, which contain both a black-hole horizon and an acceleration horizon. Their surface gravities are of opposite signs.

In order to assess the status of these suggestions, in this paper we shall re-examine the foundations of classical black hole thermodynamics from the viewpoint of the approach to classical thermodynamics advocated by Gibbs [33]. The central idea of this approach is that the physical properties of a substance are encoded into the shape of its Gibbs surface, i.e. the surface given by regarding the height of the surface as given by the total energy, regarded as a function of the remaining extensive variables. From this point of view, the temperature is given by the slope of the curve of energy versus entropy. To this end, we shall need explicit Christodoulou-Ruffini formulae, and a major goal of this paper is to obtain these for a variety of black hole solutions. As we shall see, it is a common feature of these examples that the “Gibbsian temperature” thus defined, while positive for black hole
event horizons, is negative for inner horizons (i.e. Cauchy horizons) and for cosmological horizons. We shall return to a discussion of the physical consequences for spacetimes with two horizons in the conclusions.

Let us recall that the formalism of thermodynamics, applied to classical black holes, began with two independent discoveries:

- Christodoulou’s concept of reversible and irreversible transformations such that the energy $E$ of a rotating black hole of angular momentum $J$ and momentum $P$ may be expressed as

$$M^2 = M_{\text{irr}}^2 + P^2 + \frac{J^2}{M_{\text{irr}}^2},$$

(1.5)

where the irreducible mass $M_{\text{irr}}$ is non-decreasing [34].

- Hawking’s Theorem [35, 36] that the area $A$ of the event horizon is non-decreasing.

In fact

$$A = 16\pi M_{\text{irr}}^2,$$

(1.6)

and for charged rotating Kerr-Newman black holes and dropping the momentum contribution and setting $J = |J|$, one has [37] the Christodoulou-Ruffini formula:

$$M^2 = \left(M_{\text{irr}} + \frac{Q^2}{4M_{\text{irr}}^2}\right)^2 + \frac{J^2}{M_{\text{irr}}^2}.$$  

(1.7)

The obvious analogy of some multiple of the area of the horizon with entropy became even more striking with the discovery by Smarr [38] of an analogue of the Gibbs-Duhem relation for homogeneous substances. For Kerr-Newman black holes, this reads

$$M = \frac{1}{4\pi} \kappa A + 2\Omega J + \Phi Q,$$

(1.8)

where $\kappa$ is the surface gravity, $\Omega$ the angular velocity and $\Phi$ the electrostatic potential of the event horizon. The analogy became almost complete with the formulation of three laws of black hole mechanics, including the first law

$$dM = \frac{1}{8\pi} \kappa dA + \Omega dJ + \Phi dQ,$$

(1.9)

by Bardeen, Carter and Hawking [39]. Note that the Smarr relation (1.8) follows from the first law (1.9) by differentiating the weighted homogeneity relation

$$M(\lambda^2 A, \lambda^2 J, \lambda Q) = \lambda M$$

(1.10)

with respect to $\lambda$ and then setting $\lambda = 1$. 

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The existence of a “first law” is not in itself surprising, nor does it, in itself, imply any thermodynamic consequences. Whenever one has a problem involving varying a function subject to some constraints, and considering the value of the function at critical points, one has a formula analogous to \( \text{(1.9)} \). In the case of black hole solutions of the Einstein equations, they are known to satisfy a variational principle in which the mass is extremised keeping the horizon area, angular momenta and charges fixed (see, for example, \([40–42]\)). Similar formulae arise in the theory of rotating stars (see, for example, \([43]\)). The study of these variations is sometimes referred to as comparative statics.

For homogeneous substances with pressure \( P \), volume \( V \) and internal energy \( U \), it is well known that the Gibbs-Duhem relation is equivalent to the statement that the Gibbs free energy, or thermodynamic potential,

\[
G = U - TS + PV, \tag{1.11}
\]

vanishes identically. For black holes the Smarr relation \( \text{(1.8)} \) implies that

\[
G = TS + \Omega J. \tag{1.12}
\]

Classically, a number of arguments led to the conclusion that the laws of black hole mechanics were just analogous to the laws of thermodynamics. One argument was that as perfect absorbers, classical black holes should have vanishing temperature and hence the entropy should be infinite (cf. \([44, 45]\)). Another was based on dimensional reasoning. In units where Boltzmann’s constant is taken to be unity, entropy is dimensionless, but in classical general relativity it is not obvious how to achieve this without introducing a unit of length. The obvious guess for entropy would be some multiple of the area \( A \), but why not some monotonically increasing function of the area? Despite these doubts it was conjectured by Bekenstein \([45]\) that when quantum mechanics is taken into account some multiple of \( \frac{A}{l^p} \) should correspond to the physical entropy of a black hole. This conjecture was subsequently confirmed at the semi-classical level by by Hawking \([46, 47]\), using quantum field theory in a curved background. Given this, one recognises the Christodoulou-Ruffini formula \( \text{(1.7)} \) in the form

\[
M = M(S, J, Q) \tag{1.13}
\]

as an explicit expression for the analogue of the Gibbs surface \( U = U(S, V) \) for a homogeneous substance.

To summarise, the purpose of the present paper is to re-examine these issues systematically, based on Gibbs’s geometric viewpoint of the mathematical formalism of thermodynamics \([33]\). This starts with a choice of pairs of extensive and intensive variables and
an expression for some sort of “energy,” which is regarded as a function of the extensive variables. For the black holes in asymptotically flat spacetimes that we shall consider, the energy is taken to be the ADM mass $M$, and the extensive variables $S^\mu$ are usually taken to be $S^\mu = (S, J, Q_i, P^i) = (S, s)$, where $S = \frac{1}{4} A$ and $A$ is the area of the event horizon, $J$ is the total angular momentum and $Q_i$ and $P^i$ are $2N$ electric and magnetic charges. Thus we have

$$M = M(S, J, Q_i, P^i) = M(S^\mu) .$$

The intensive variables are taken to be $T_\mu = \frac{\partial M}{\partial S^\mu} = (T, \Omega, \Phi^i, \Psi_i) = (T, t)$ where $T$ is the temperature, $\Omega$ is the angular velocity of the horizon, and $\Phi^i$ and $\Psi_i$ are the electrostatic and magnetostatic potentials.

The organisation of this paper is as follows. In section 2, we review the theory of Gibbs surfaces, and the various thermodynamic metrics with which they may be equipped. Section 3 forms the core of the paper. In it, we give many new results for the thermodynamics of a wide range of asymptotically-flat black holes. We begin in subsections 3.1, 3.2 and 3.3 by reviewing how the well-known Reissner-Nordström, Kerr and Kerr-Newman black holes fit into the Gibbsian framework. Subsection 3.4 then provide a extensive discussion of the thermodynamics of families of black holes in four-dimensional STU supergravity. In particular, we give a systematic discussion of the notion of the decomposition of the system into left-handed and right-handed sectors, and their associated thermodynamics. Subsection 3.5 has analogous results for five-dimensional STU supergravity black holes. Subsections 3.6 and 3.7 give similar results for the general family of four-dimensional Einstein-Maxwell-Dilaton (EMD) black holes, and a two-field generalisation. Included in the discussion of these two-field EMD black holes, we exhibit a new area-product formula.

A rather general feature of many asymptotically flat black holes with two horizons is that the product of the areas of the two horizons is independent of the mass, and given in terms of conserved charges and angular momenta, which may plausibly be quantised at the quantum level. In section 4, we use this area-product formula to exhibit an intriguing $S \to 1/S$ inversion symmetry of the Christodoulou-Ruffini formulae for such black holes. This symmetry of the Gibbs surface interchanges the positive and negative temperature branches.

In section 5 we extend our discussion to black holes that are asymptotically AdS, or black holes with positive cosmological constant. In the AdS case the situation for inner and outer horizons is broadly similar to that for the asymptotically flat case. For positive

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2We shall not consider scalar charges and moduli in this paper.
cosmological constant, the black hole event horizon continues to have positive Gibbsian
temperature, but that of the cosmological horizon is negative.

In section 6, we revisit an old observation, that the entropy of the Kerr-Newman solution
is a super-additive function of the extensive variables, and we its relation to Hawking’s area
theorem for black-hole mergers. We find that super-additivity holds also for a wide variety
of the asymptotically-flat examples that we considered in section 3. However, we find that
Kaluza-Klein dyonic black holes provide a counterexample, and we speculate on the reason
for this.

The paper ends with conclusions and future prospects in section 7.

2 The Gibbs Surface and Thermodynamic Geometry

2.1 The Gibbs surface

In this section we shall briefly summarise those aspects of the Gibbs surface which are
relevant for the latter part of the paper. If we think of \((S^\mu, M)\) as coordinates in \(\mathbb{R}^{3+2N}\)
then \((1.14)\) defines a hypersurface \(G \subset \mathbb{R}^{3+2N}\) whose co-normal is \((T_\mu, -1)\). Since in our
case \(M\) is a unique function of the extensive variables, the intensive variables are unique
functions of the extensive variables: \(T_\mu = T_\mu(S^\nu)\). The converse need not be true. If the
function \(M(S^\mu)\) were convex, then for fixed co-normal \((T_\mu, -1)\) the plane

\[
T_\mu S^\mu - M = 0
\]

would touch the surface at a unique point \((S^\mu, M)\). For a smooth Gibbs surface \(G\), convexity
requires that the Hessian

\[
g_W^{\mu\nu} = \frac{\partial^2 M}{\partial S^\mu \partial S^\nu} \tag{2.2}
\]

be positive definite and one may then define a positive definite metric

\[
ds^2 = g_W^{\mu\nu} dS^\mu dS^\nu, \tag{2.3}
\]

called the Weinhold metric [49]. Because one of the components of the Weinhold metric
\((2.2)\) is related to the heat capacity\(^3\) at constant \(J\) and \(Q^i\) and \(P^i\), namely

\[
g_W^{SS} = TC_s^{-1} = \left. \frac{\partial T}{\partial S} \right|_s, \tag{2.4}
\]

\(^3We use the term “heat capacity” rather than “specific heat” because the latter is defined to be per unit
mass.
and neutral black holes or black holes with small charges or angular momentum have negative heat capacities, the Gibbs surface $G$ is typically not convex and the Weinhold metric for black holes is typically Lorentzian [50].

If one defines a totally symmetric co-covariant tensor of rank three by

$$C_{\lambda\mu\nu} = \frac{\partial^3 M}{\partial S^\lambda \partial S^\mu \partial S^\nu}, \quad (2.5)$$

the Riemann and Ricci tensors and the Ricci scalar of the Weinhold metric are given by

$$R^{\alpha\beta\mu\nu} = -\frac{1}{4} \left[ C^{\alpha\mu\lambda} C^{\lambda\nu\beta} - C^{\alpha\nu\lambda} C^{\lambda\mu\beta} \right],$$

$$R_{\beta\nu} = -\frac{1}{4} \left[ C^{\alpha\beta\alpha\lambda} C^{\lambda\nu} - C^{\alpha\beta\lambda\nu} C^{\alpha\lambda\nu} \right],$$

$$R = -\frac{1}{4} \left[ C^{\alpha\alpha\lambda\nu} C^{\lambda\nu} - C^{\alpha\lambda\nu\lambda} C^{\alpha\nu\lambda} \right], \quad (2.6)$$

all indices being raised with $g_{\mu\nu}^W$, the inverse of $g_{\mu\nu}^W$. Divergences in $R$ are sometimes held to be a diagnostic for phase transitions.

The geometry of the Gibbs surface is essentially the geometry behind the first law of thermodynamics. As we remarked previously, this fits into a pattern that is more general than just the theory of black holes, and arises whenever one is considering a variational problem with constraints. Since this is not as widely known as it deserves to be, we shall pause to describe the general situation, and then we shall restrict attention to its application to black hole theory. Consider a real-valued function $f(x)$ on some space $X$ with coordinates $x$, subject to the $n$ constraints that certain functions $C^a(x) = c^a$, $1 \leq a \leq n$, where the $c^a$ are constants. Adopting the method of Lagrange multipliers, we require that

$$\delta f - \lambda_a \delta C^a = 0, \quad (2.7)$$

for all variations in $X$. Suppose the solutions of these equations lie in an $n$-dimensional sub-manifold $S$ of $X$, parameterised by the values of the constraints, $c^a$. One may restrict the variations in (2.7) to directions within the solution space $S$, in which case we obtain the formula

$$\delta f(c) = \lambda_a \delta c^a. \quad (2.8)$$

Geometrically, we can think of this situation as follows. We construct a $(2n+1)$-dimensional space with coordinates $(f, \lambda_a, c^a)$. Since, locally at least, $f$ and $\lambda_a$ may be thought of as functions of the $c^a$, we obtain an $n$-dimensional surface in this space. From (2.8), it follows that the Lagrange multipliers $\lambda_a$ are determined by the tangent planes to the surface. From now on, we shall restrict attention to the thermodynamic case, with $f$ being the total energy, or mass, $M$, and the $c^a$ being $(S, J, Q_i, P_i)$. 9
The Gibbs surface $G$ can be lifted to give a $(3+2N)$-dimensional submanifold $\mathcal{L}A : T_\mu = T_\mu(S^\nu)$ of the thermodynamic phase space, i.e. $\mathcal{L}A \subset \mathbb{R}^{6+2N}$ with coordinates $(T_\mu, S^\nu)$, equipped with the symplectic form
\[ \omega = dT_\mu \wedge dS^\mu. \] (2.9)
Since, when pulled back to $\mathcal{L}A$ we have $T_\mu dS^\mu = dM(S^\mu)$, the pull-back of $\omega$ to $\mathcal{L}A$ vanishes,
\[ \omega \big|_{\mathcal{L}A} = 0. \] (2.10)
In other words, $\mathcal{L}A$ is a Lagrangian submanifold of $\mathbb{R}^{6+3N}$.

One may go a step further and lift $\mathcal{L}A$ to $\mathbb{R}^{7+2N}$ with coordinates $(P_\mu, S^\nu, M)$, equipped with the contact form
\[ \eta = T_\mu dS^\mu - dM, \] (2.11)
as a Legendre submanifold $\mathcal{LE}$, i.e. one for which
\[ \eta \big|_{\mathcal{LE}} = 0. \] (2.12)
In most of the cases we shall be considering, for dimensional reasons $M(S, J, Q_i, P^i)$ satisfies the weighted homogeneity relation
\[ M(\lambda^2 A, \lambda^2 J, \lambda Q^i, \lambda P^i) = \lambda M. \] (2.13)
Differentiating with respect to $\lambda$ and then setting $\lambda = 1$ yields the Smarr relation
\[ M = 2TS + 2\Omega J + \Phi^i Q_i + \Psi^i P^i. \] (2.14)
The Gibbs function, or thermodynamic potential, $G$, is the total Legendre transform of the mass with respect to the extensive variables. It satisfies
\[ dG = -S^\mu dT_\mu, \] (2.15)
where
\[ G(T_\mu) = M - T_\mu S^\mu = M - TS - \Omega J - \Phi^i Q_i - \Psi^i P^i = TS + \Omega J. \] (2.16)
Note that $G$ is not necessarily a single-valued function of the intensive variables $T_\mu$, unless the Gibbs surface $G$ is convex. The Hessian of the Gibbs function with respect to the intensive variables is related to the Weinhold metric by the easily-verified formula
\[ \frac{\partial^2 G}{\partial T_\mu \partial T_\nu} \frac{\partial^2 M}{\partial S^\nu \partial S^\lambda} = -\delta^\mu_\lambda. \] (2.17)
It provides a metric on the space of intensive variables.

It is important to realise that from the point of view of the symplectic and contact structures described above, the coordinates \((S^\mu, T^\mu, M)\) have a privileged status and it makes little physical sense to consider arbitrary coordinate transformations even if they preserve the symplectic or contact structures. Only a limited number of Legendre transformations are of physical relevance. It is physically meaningful to consider positive linear combinations of the vectors \(S^\mu\), i.e. sending \(S^\mu \rightarrow D^{\mu\nu} S^\nu\), where \(D^{\mu\nu}\) is a constant diagonal matrix, and also to reverse the sign of any but the first component (i.e. the first diagonal component of \(D^{\mu\nu}\), associated with scaling the entropy itself, should be positive). In other words, physical states are future directed with respect to the first component.

In the literature on thermodynamic metrics, much discussion has focused on whether or not the Ricci scalar is a good indicator of phase transitions. Because, as explained above, general coordinate transformations do not have physical significance, it is not obvious that one should be concerned with a scalar such as the Ricci scalar. In fact, what is more relevant is the behaviour of the Hessian, i.e. the thermodynamic metric. If this is not invertible then a divergence of the Ricci scalar will occur, but the value of the Ricci scalar itself does not appear in general to have any physical significance.

### 2.2 Thermodynamic metrics

It has been traditional in the literature to focus on the Ruppeiner and Weinhold metrics, and this is especially convenient if one has available an explicit Christodoulou-Ruffini formula. However, as in standard text books on thermodynamics, it is frequently convenient to introduce a variety of other thermodynamic potentials related by Legendre transformations, depending upon what quantities are being held fixed. In the context of black hole thermodynamics this corresponds to what boundary conditions are being considered. The consequent uniqueness or “No Hair” properties will depend in general on precisely what is to be held fixed. This lack of uniqueness is what is often referred to as a “phase transition,” but as in standard thermodynamics it is important to specify the physical conditions under which the phase transition takes place.

From the point of view of the Gibbs surface \(\mathcal{G}\), geometrically this should really be thought of as an \(n\)-dimensional Legendrian sub-manifold of the \((2n + 1)\)-dimensional Legendre manifold whose coordinates consist of the the total energy and the the \(n\) pairs of intensive and extensive variables. Given a choice of \(n\) coordinates chosen from these \(2n\) variables, one may locally describe the surface in terms of the associated thermodynamic
potential, and from that compute the associated Hessian metric. But \textit{globally}, it is not in general true that the Gibbs surface equipped with the choice of Hessian is a single-valued non-singular graph over the \(n\)-plane spanned by the chosen set of \(n\) coordinates. It should also be remembered that although the Hessian metrics may be thought of as the pull-back to \(G\) of a flat metric on the \(2n\)-dimensional flat hyperplane spanned by the choice \(n\) pairs of intensive and extensive variables, the signature of that flat metric depends upon that choice.

Here we review some key results on the general classes of thermodynamic metrics that were presented in [51]. Consider first the energy \(M = M(S^\mu)\), which obeys the first law

\[
dM = T_\mu dS^\mu = T dS + \Omega dJ + \Phi^i dQ_i + \cdots. \tag{2.18}
\]

One can define from this the metric

\[
ds^2(M) = dT_\mu \otimes_s dS^\mu, \tag{2.19}
\]

where \(T_\mu\) are viewed as functions of the \(S^\mu\) variables, with

\[
T_\mu = \frac{\partial M}{\partial S^\mu}; \tag{2.20}
\]

and \(\otimes_s\) denotes the symmetrised tensor product. In the usual parlance of general relativity we may simply write (2.19) as

\[
ds^2(M) = dT_\mu dS^\mu. \tag{2.21}
\]

In view of (2.20) we have

\[
ds^2(M) = \frac{\partial^2 M}{\partial S^\mu \partial S^\nu} dS^\mu dS^\nu, \tag{2.22}
\]

which is nothing but the Weinhold metric.

One can obtain a set of conformally-related metrics by dividing (2.18) by any one of the intensive variables \(T_\mu\) for \(\mu = \bar{\mu}\) where \(\bar{\mu}\) denotes the associated specific index value of the chosen intensive variable, and then constructing the thermodynamic metric \(ds^2(S^{\bar{\mu}})\) for the conjugate extensive variable by using the same procedure as before [51]. Thus, for example, if we choose \(\bar{\mu} = 0\), so that \(T\) is the chosen intensive variable and \(S\) its conjugate, then we rewrite (2.18) as

\[
dS = \frac{dM}{T} - \frac{1}{T} T_\alpha dS^\alpha, \tag{2.23}
\]

where we have split the \(\mu\) index as \(\mu = (0, a)\), and then write the associated thermodynamic
The second line was obtained by using (2.18), and the third line follows from (2.22). Thus
\[ ds^2(S) = \frac{1}{T^2} dT dM + \frac{1}{T^2} dT a dS^a - \frac{1}{T} dT a dS^a, \]
\[ = \frac{1}{T} (dT dS + dT a dS^a), \]
\[ = -\frac{1}{T} ds^2(M). \] (2.24)

Further thermodynamic metrics that are not merely conformally related to the Weinhold metric can be obtained by making Legendre transformations to different energy functions before implementing the above procedure \[51\]. For example, if one make the Legendre transform to the free energy \( F = M - TS \), for which one has the first law

\[ dF = -S dT + T a dS^a, \] (2.26)

then the associated thermodynamic metric will be

\[ ds^2(F) = -dT dS + dT a dS^a, \] (2.27)

where \( S \) and \( T a \), which are now the intensive variables, are viewed as functions of \( T \) and \( S^a \). The metric components in \( ds^2(F) \) are therefore given by the Hessian of \( F \). As observed in \[51\], the metric \( ds^2(F) \) has the property that, unlike the Weinhold or Ruppeiner metrics, its curvature is singular on the so-called Davies curve where the heat capacity diverges.

Clearly, by making different Legendre transformations, one can construct many different thermodynamic metrics, which take the form

\[ ds^2 = \sum_{\mu \geq 0} \eta_\mu dT_\mu dS^\mu, \] (2.28)

where each \( \eta_\mu \) can independently be either +1 or −1. The overall sign is of no particular importance, and so metrics related by making a complete Legendre transformation of all
the intensive/extensive pairs in a given energy definition really yields an equivalent metric. For example, the Gibbs energy \( G = M - T \mu S \) gives the metric

\[
ds^2(G) = -dT \mu dS \mu,
\]

which is just the negative of the Weinhold metric \( ds^2(M) \) in (2.22).

One further observation that was emphasised in [51] is that one is not, of course, obliged when writing a thermodynamic metric to use the associated extensive variables as the coordinates. It is sometimes the case, as we shall see in later examples, that although one can calculate the thermodynamic variables in terms of the metric parameters, one cannot explicitly invert these relations. In such cases, one can always choose to use the metric parameters as the coordinates when writing the thermodynamic metrics. Geometric invariants such as the Ricci scalar of the thermodynamic metric will be the same whether written using the thermodynamic variables or the metric parameters, since one is just making a general coordinate transformation. Thus even in cases where the relations between the thermodynamic variables and metric parameters are too complicated to allow one to find an explicit Christodoulou-Ruffini formula to define the Gibbs surface, one can still study the geometrical properties of the various thermodynamic metrics.

### 3 Asymptotically Flat Black Holes

In this section, we shall illustrate the issues raised in the previous section by listing the cases of asymptotically-flat black holes for which we have explicit formulae. Whilst the formulae for the Kerr-Newman family of black holes are well known, we first review these in some detail in preparation for our discussion of much less well known black holes, such as those that occur in supergravity or Kaluza-Klein theories.

#### 3.1 The Gibbs surface for Reissner-Nordström

The Gibbs surface \( \mathcal{G} \) for the Reissner-Nordström solution is given by the Christodoulou-Ruffini formula

\[
M = \sqrt{\frac{S}{4\pi}} + \frac{Q^2}{4} \sqrt{\frac{4\pi}{S}} = M_{\text{irr}} + \frac{Q^2}{4M_{\text{irr}}},
\]

where \( M_{\text{irr}} = \sqrt{\frac{S}{4\pi}} \). It is convenient to envisage \((M,Q,S)\) as a right-handed Cartesian coordinate system with \( M > 0 \) taken vertically and \(-\infty < Q < \infty \) and \( S > 0 \) spanning a horizontal half-plane. In \((M,Q,S)\) coordinates the surface is part of the quadratic cone

\[
M^2 = \left( \frac{S}{2\pi} - M \right)^2 + Q^2.
\]
We have
\[ M \geq |Q| , \]
(3.3)
with \( M > |Q| \) being sub-extremal black holes. Rewriting \( 3.2 \) as
\[ S^2 - 2 \pi (2 M^2 - Q^2) S + \pi^2 Q^4 = 0 , \]
(3.4)
the two solutions for \( S \) at fixed \( M \) and \( Q \) are given by
\[ \frac{S}{\pi} = 2 M^2 - Q^2 \pm 2 M \sqrt{M^2 - Q^2} , \]
(3.5)
with these corresponding to the entropies (i.e. one quarter the area) of the outer \( (S_+) \) and inner \( (S_-) \) horizons respectively. It is straightforward to see that the temperature \( T = \partial M / \partial S \) is positive on the outer horizon and negative on the inner horizon.

Equality, \( M = |Q| \), corresponds to extreme black holes. They lie on the space curve \( \gamma_{\text{extreme}} \) given by the intersection of the two surfaces
\[ M = |Q| , \quad M = \sqrt{\frac{S}{\pi}} . \]
(3.6)
The first is a plane orthogonal to the \( Q \) plane, and the second a parabolic cylinder with generators parallel to the \( Q \) axis. The projection of \( \gamma_{\text{extreme}} \) onto the \( Q - S \) plane is given by the parabola
\[ S = \pi Q^2 . \]
(3.7)
Roughly speaking, the Gibbs surface \( G \) is folded over the space curve \( \gamma_{\text{extreme}} \). Now the Weinhold metric, or equivalently the Hessian of \( M(S,Q) \), is given by
\[ ds^2_W = \sqrt{\frac{4 \pi}{S}} \left\{ \frac{1}{2} dQ^2 - \frac{Q}{2S} dQ dS + \frac{1}{16 S^2} (3Q^2 - \frac{S}{\pi}) dS^2 \right\} , \]
(3.8)
Note that \( \frac{\partial^2 M}{\partial S^2} \) changes sign, passing through zero, along the space curve \( \gamma_{\text{Davies}} \), given by
\[ S = 3 \pi Q^2 = \frac{9}{4} M^2 . \]
(3.9)
Since the heat capacity at constant charge, \( C_Q \), is given by
\[ C_Q = T \left( \frac{\partial^2 M}{\partial S^2} \right)^{-1} , \]
(3.10)
it also changes sign across the curve \( \gamma_{\text{Davies}} \), on which it diverges \[ 54 \]. This is often taken as a sign of a phase transition. In support of this interpretation, it has been shown \[ 55 \] that the single negative mode of the Lichnerowicz operator passes through zero and becomes positive as \( Q \) is increased across \( \gamma_{\text{Davies}} \).
The curve $\gamma_{\text{Davies}}$ is an example of what, in the literature on phase transitions, is often referred to as a spinodal curve, and is usually defined in terms of the vanishing of a diagonal element of the Hessian of the Gibbs function. In the present case, the Gibbs function is

$$G = M - TS - \Phi Q = \frac{(1 - \Phi^2)^2}{16\pi T}, \quad \text{(3.11)}$$

and the Hessian is given by

$$\begin{bmatrix}
\frac{\partial^2 G}{\partial T^2} & \frac{\partial^2 G}{\partial T \partial \Phi} \\
\frac{\partial^2 G}{\partial \Phi \partial T} & \frac{\partial^2 G}{\partial \Phi^2}
\end{bmatrix} = \begin{bmatrix}
\frac{(1 - \Phi^2)^2}{8\pi T^3} & \frac{(1 - \Phi^2) \Phi}{4\pi T^2} \\
\frac{(1 - \Phi^2) \Phi}{4\pi T^2} & \frac{(1 - 3\Phi^2)}{4\pi T}
\end{bmatrix}. \quad \text{(3.12)}$$

The spinodal curve is thus given by $\Phi^2 = \pm \frac{1}{3}$, which, in terms of $S$ and $Q$, coincides with (3.9).

The Weinhold metric may be written as

$$ds^2_W = \sqrt{\frac{4\pi}{S}} \left\{ \frac{1}{2} (dQ - \frac{Q}{2S} dS)^2 - \frac{1}{16S^2} \left( \frac{S}{\pi} - Q^2 \right) dS^2 \right\}, \quad \text{(3.13)}$$

and hence the Gibbs surface for sub-extremal black holes has a Hessian, or equivalently a Weinhold metric, that is non-singular but Lorentzian. Moreover the Gibbs surface for non-extreme black holes is non-convex. Expressed in terms of $S$ and the electrostatic potential

$$\Phi = \Phi(S,Q) = Q \sqrt{\frac{\pi}{S}}, \quad \text{(3.14)}$$

the Weinhold metric becomes

$$ds^2_W = \frac{1}{8\sqrt{\pi} S^{3/2}} \left[ - (1 - \Phi^2) dS^2 + 8S^2 d\Phi^2 \right]. \quad \text{(3.15)}$$

Note that the metric is non-singular when either $\Phi^2 < 1$, corresponding to the outer horizon, or $\Phi^2 > 1$, corresponding to the inner horizon. It changes signature from $(- +)$ to $(++)$ as $\Phi$ goes from $\Phi^2 < 1$ to $\Phi^2 > 1$. The heat capacity passes through infinity at $\Phi^2 = \frac{1}{3}$.

Expressed in terms of $\Phi$ and $S$, the temperature is given by $T = (1 - \Phi^2)/(4\sqrt{\pi S})$, and so the Ruppeiner metric is given by

$$ds^2_R = -\frac{1}{T} ds^2_W = -\frac{dS^2}{2S} + 4S \frac{d\Phi^2}{1 - \Phi^2} = -d\tau^2 + \tau^2 d\sigma^2_+, \quad \text{(3.16)}$$

where we have defined, for the outer horizon,

$$S = \frac{1}{2} \tau^2, \quad \Phi = \sin \frac{\sigma_+}{\sqrt{2}}. \quad \text{(3.17)}$$
The metric in the second line of (3.16) is the Milne metric on a wedge of Minkowski spacetime inside the light cone. This is made apparent by introducing new coordinates according to

\[ t = \tau \cosh \sigma_+, \quad x = \tau \sinh \sigma_+, \]  

(3.18)
in terms of which the Ruppeiner metric becomes

\[ ds^2_R = -dt^2 + dx^2, \quad S = S_+ = \frac{1}{2}(t^2 - x^2). \]  

(3.19)

Since the range of \( \sigma_+ \) is \( -\frac{\pi}{\sqrt{2}} \leq \sigma_+ \leq \frac{\pi}{\sqrt{2}} \), the extremal solutions lie on the timelike geodesics \( t = \pm \arctanh \frac{\pi}{\sqrt{2}} \). The heat capacity changes sign at \( \Phi^2 = \frac{1}{3} \).

If \( \Phi^2 > 1 \), corresponding to the inner horizon, then, if \( Q > 0 \), substituting

\[ \Phi = \cosh \frac{\sigma_-}{\sqrt{2}} \]  

(3.20)

(or \( \Phi = -\cosh(\sigma_-/\sqrt{2}) \) if \( Q < 0 \)) into (3.16) gives

\[ ds^2 = -(dr^2 + r^2 d\sigma_-^2). \]  

(3.21)

The metric in brackets is the flat metric on Euclidean space in polar coordinates, except that the range of the coordinate \( \sigma_- \) is \( 0 \leq \sigma_- \leq \infty \), so we have an infinitely branched covering of the Euclidean plane, with the branch point at the origin. The Weinhold metric is itself positive definite. Thus the Gibbs surface is convex and the entropy surface is concave for the inner horizon.

The flatness of the Ruppeiner metric for Reissner-Nordström has given rise to much comment, because singularities of the Ruppeiner metric are expected to reveal the occurrence of phase transitions. However, the geometrical significance of the change in sign of the heat capacity is that for fixed charge \( Q \), there is a maximum temperature. In fact

\[ T = T(S,Q) = \frac{1}{2S} \sqrt{\frac{S}{4\pi}} - \frac{Q^2}{8S} \sqrt{\frac{4\pi}{S}}, \]  

(3.22)

so for given \( |Q| \) and positive \( T \) less than \( \frac{\sqrt{2}}{8\pi|Q|} \), there are two positive values of \( S \) and hence two non-extreme black holes. By contrast, since the electrostatic potential \( \Phi \) satisfies (3.14), there is a unique positive value of \( S \) and hence a unique black hole for given \( Q \) and \( \Phi^2 < 1 \).

Every two-dimensional metric is conformally flat. Therefore it is not surprising that both the Weinhold and Ruppeiner metrics for Reissner-Nordström are conformally flat. It is, however, nontrivial that the Ruppeiner metric is flat. It has recently been pointed out [56] that one can also consider the Hessian of the charge \( Q \), considered as a function of the mass and entropy, as a metric \( ds^2_Q \). In fact \( ds^2_Q = -\Phi^{-1} ds^2_W \), as in (2.25). Geometrically, there
is no reason to give a preference to any of the metrics $ds^2_W$, $ds^2_R$, or $ds^2_Q$. Since $T$ and $\Phi$ are both non-singular on the curve along which the heat capacity diverges, none of the three metrics is capable of detecting the associated “phase transition.”

As was shown in [51], and we reviewed in section 2.2, the thermodynamic metric (2.27) constructed from the free energy $F = M - TS$ does exhibit a singularity on the Davies curve where the heat capacity diverges. For the Reissner-Nordström metric (2.27) is the restriction of $ds^2(F) = -dTdS + d\Phi dQ$ to the Gibbs surface, and hence we find

$$ds^2(F) = \sqrt{\frac{\pi}{S}} dQ^2 + \frac{1}{8\sqrt{\pi S^{3/2}}} (S - 3\pi Q^2) dS^2.$$  \hspace{1cm} (3.23)

A straightforward calculation shows that its Ricci scalar is given by

$$R_F = \frac{4\sqrt{\pi S^{3/2}}}{(S - 3\pi Q^2)^2},$$  \hspace{1cm} (3.24)

which does indeed diverge on the Davies curve $S = 3\pi Q^2$.

3.2 The Gibbs surface for Kerr

This is qualitatively very similar to the Reissner-Nordström case. To begin with, we shall summarise, in our notation, some results first presented by Curir [1]. One has

$$M^2 = \frac{S}{4\pi} + \frac{\pi J^2}{S},$$  \hspace{1cm} (3.25)

and $M(S, J)$ at fixed $J$ has a minimum value when

$$S = 2\pi |J|, \quad M = \sqrt{|J|}.$$  \hspace{1cm} (3.26)

This is the extreme case and, as before, the inner horizon has a negative temperature, a point made first by Curir [1]. Explicitly one has

$$T = \frac{1}{8\pi M} \left(1 - \frac{4\pi^2 J^2}{S^2}\right).$$  \hspace{1cm} (3.27)

For any given values of $J$ and of $M > 0$, there are two positive solutions, $S_+$ and $S_-$, of (3.25), where $S_+ \geq 2\pi |J|$ corresponds to one quarter of the area of the outer horizon of a sub-extremal black hole and $S_- \leq 2\pi |J|$ corresponds to one quarter of the area of the inner horizon of a sub-extremal black hole. From (3.25), they obey the entropy product formula

$$S_- S_+ = 4\pi^2 J^2.$$  \hspace{1cm} (3.28)

By (3.27), the outer horizon has a positive temperature, which we label $T_+$, and the inner horizon has a negative temperature, which we label $T_-$. One has [1]

$$T_\pm = \frac{S_\pm - S_\mp}{8\pi M S_\pm}, \quad \Omega_\pm = \frac{\pi J}{M S_\pm},$$  \hspace{1cm} (3.29)
where $\Omega_\pm = (\partial M / \partial J)_{S_\pm}$. Note that it follows from the first equation in (3.29) that

$$T_+ S_+ + T_- S_- = 0.$$  \hfill (3.30)

Note also that $M$ and $J$, which are conserved quantities defined in terms of integrals at infinity, are universal and do not carry $\pm$ labels.

In terms of $S_+$ and $S_-$, one has, from (3.25),

$$M^2 = \frac{S_+}{4\pi} + \frac{S_-}{4\pi}.$$  \hfill (3.31)

Therefore

$$M = \frac{\sqrt{S_+ + S_-}}{\sqrt{4\pi}} \leq \sqrt{\frac{S_+}{4\pi}} + \sqrt{\frac{S_-}{4\pi}}.$$  \hfill (3.32)

If one varies $M$, one has

$$dM = T_\pm dS_\pm + \Omega_\pm dJ.$$  \hfill (3.33)

There is also a modified Smarr formula

$$M = T_+ S_+ + T_- S_- + \Omega_+ J + \Omega_- J = (\Omega_+ + \Omega_-) J,$$  \hfill (3.34)

where the second equality follows from (3.30). This way of writing the first law of thermodynamics was employed in [57] for deriving a simple formula for holographic complexity. These results were interpreted in [1] as indicating that the total energy of a rotating black hole may be regarded as receiving contributions from two thermodynamic systems; one associated with the outer horizon and the other with the inner horizon. The negative temperature was interpreted in terms of Ramsey’s account of the thermodynamics of isolated spin systems [58].

Okamoto and Kaburaki [10] introduced the dimensionless parameter $h = \frac{a}{M + \sqrt{M^2 - a^2}}$ in their discussion of the energetics of Kerr black holes and noticed that it satisfies the quadratic equation

$$h^2 - \frac{2hM^2}{|J|} + 1 = 0.$$  \hfill (3.35)

It was initially assumed that only the solution of (3.35) satisfying $0 \leq h \leq 1$ has physical significance. However Abramowicz [59] drew their attention to [1] and they realised that the other root of (3.35), which satisfies $1 \leq h \leq \infty$ and is given by $h = \frac{a}{M - \sqrt{M^2 - a^2}}$, is associated with the inner horizon [10]. Expressing the thermodynamic variables in terms of $h$ they established (3.30) if $T_-$ is taken to be negative, and they also obtained the formula

$$\frac{\Omega_+}{T_+} + \frac{\Omega_-}{T_-} = 0.$$  \hfill (3.36)
3.3 Kerr-Newman black holes

Kerr-Newman black holes may have both electric and magnetic charges. By electric-magnetic duality invariance one may set the magnetic charge $P$ to zero. To restore electric-magnetic duality invariance it suffices to replace $Q^2$ by $Q^2 + P^2$ in all formulae thus producing a manifestly $O(2)$ invariant Gibbs surface.

The mass of the Kerr-Newman black hole is given by

$$M = \left[ \frac{\pi}{4S} \left( \frac{S}{\pi} + Q^2 \right)^2 + \frac{\pi J^2}{S} \right]^{\frac{1}{2}},$$

(3.37)

and therefore it satisfies

$$M \geq \sqrt{\sqrt{J^2 + \frac{Q^4}{4}} + \frac{Q^2}{2}},$$

(3.38)

acquiring its least value on the surface $\gamma_{\text{extreme}}$ in the three dimensional space of extensive variables given by

$$S = \pi \sqrt{4J^2 + Q^4},$$

(3.39)

on which the temperature

$$T = \left( \frac{\partial M}{\partial S} \right)_{J,Q} = \frac{1}{8\pi M} \left[ 1 - \frac{\pi^2}{S^2} (4J^2 + Q^4) \right]$$

(3.40)

vanishes. If $J = 0$, then (3.38) is the usual Bogomolnyi bound \[60\]. One also has

$$\Omega = \frac{\pi J}{MS}, \quad \Phi = \frac{\pi Q^2}{2MS} \left( Q^2 + \frac{S}{\pi} \right).$$

(3.41)

The explicit formulae (3.37), (3.40) and (3.41) allow a lift of the Gibbs surface $\mathcal{G}$ to a Lagrangian submanifold $\mathcal{L}$ in $\mathbb{R}^6$ and a Legendrian submanifold in $\mathbb{R}^7$. The entropy product law becomes

$$S_+ S_- = \pi^2 (4J^2 + Q^4),$$

(3.42)

where the $-$ refers to the inner and $+$ to outer horizon.

The temperatures and angular velocities of the two horizons are given by

$$T_\pm = \frac{S_\pm - S_\mp}{8\pi MS_\pm}, \quad \Omega_\pm = \frac{\pi J}{MS_\pm},$$

(3.43)

and one has

$$S_+ T_+ + S_- T_- = 0.$$  

(3.44)

There is a conventional first law for both horizons:

$$dM = T_\pm dS_\pm + \Omega_\pm dJ + \Phi_\pm dQ,$$

(3.45)

and a modified Smarr formula

$$M = T_+ S_+ + T_- S_- + \Omega_+ J + \Omega_- J + \frac{1}{2} \Phi_+ Q + \frac{1}{2} \Phi_- Q = \left( \Omega_+ + \Omega_- \right) J + \frac{1}{2} \left( \Phi_+ + \Phi_- \right) Q.$$  

(3.46)
### 3.4 STU black holes

Four-dimensional black holes in string theory or M-theory can be described as solutions of $\mathcal{N} = 8$ supergravity. The most general black holes are supported by just four of the 28 gauge fields, in the Cartan subalgebra of $SO(8)$. The black holes can therefore be described just within the $\mathcal{N} = 2$ STU supergravity theory, which is a consistent truncation of the $\mathcal{N} = 8$ theory whose bosonic sector comprises the metric, the four gauge fields, and six scalar fields. Black holes of the STU model are parameterised by mass $M$, angular momentum $J$ and four electric $Q_i$ ($i = 1, 2, 3, 4$) and four magnetic charges $P_i$ ($i = 1, 2, 3, 4$). The most general black hole solution was obtained by Chow and Compère [61] by solution generating techniques.

We shall follow the usual conventions for STU supergravity, in which the normalisation of the gauge fields $F^{(i)}$ is such that if the scalar fields are turned off, the Lagrangian will take the form $\mathcal{L} = \sqrt{-g} \left[ R - \frac{1}{4} \sum_i (F^{(i)})^2 + \cdots \right]$ (see appendix B for a presentation of the bosonic sector of the STU supergravity Lagrangian). This contrasts with the conventional normalisation $\mathcal{L} = \sqrt{-g} (R - F^2)$, in Gaussian units, which we use when describing the pure Einstein-Maxwell theory. Since this means that the charge normalisation conventions will be different in the two cases, we shall briefly summarise our definitions here. If we consider the Lagrangian

$$\mathcal{L} = \sqrt{-g} (R - \gamma F^2), \quad (3.47)$$

one can derive by considering variations of the associated Hamiltonian that black holes will obey the first law

$$dM = \frac{\kappa}{8\pi} dA + \Phi dQ + \Omega dJ, \quad (3.48)$$

where $\kappa$ is the surface gravity, $\Phi$ is the potential difference between the horizon and infinity (with the potential being equal to $\xi^\mu A_\mu$, where $\xi^\mu$ is the future-directed Killing vector that is null on the horizon and is normalised such that $\xi^\mu \xi_\mu \rightarrow -1$ at infinity). The electric charge $Q$ is given by

$$Q = \frac{\gamma}{4\pi} \int * F, \quad (3.49)$$

Thus in Einstein-Maxwell theory, with $\mathcal{L} = \sqrt{-g} (R - F^2)$, we shall have

$$Q = \frac{1}{4\pi} \int * F, \quad (3.50)$$
while in STU supergravity we shall have (neglecting the scalar fields for simplicity) 

$$Q_i = \frac{1}{16\pi} \int *F^{(i)}.$$  \hspace{1cm} (3.51) 

The black hole solutions have two horizons, with the the product of the horizon entropies quantised: 

$$S_+ S_- = 4\pi^2 |J^2 + \Delta|,$$  \hspace{1cm} (3.52) 

where $\Delta$ is the Cayley hyperdeterminant $\Delta(Q_i, P^i)$: 

$$\Delta = 16 \left[ 4 \left( Q_1 Q_2 Q_3 Q_4 + P^1 P^2 P^3 P^4 \right) + 2 \sum_{i<j} Q_i Q_j P^i P^j - \sum_i (Q_i)^2 (P^i)^2 \right].$$  \hspace{1cm} (3.53) 

Note that eqn (3.52) has previously appeared in the literature without the absolute value symbol (for example, in [61]). We have written (3.52) with an absolute value sign since $\Delta$, and hence $\Delta + J^2$, can be negative; for example for a static Kaluza-Klein dyonic black hole. (In [61] it was proposed that $S_-$ is negative when $\Delta + J^2 < 0$, but this would contradict the fact that, for example, the area of the inner horizon of the static Kaluza-Klein dyonic black hole is positive.)

It should be noted that if $J$ vanishes and $\Delta = 0$, then $S_-$ will vanish also. In this case there is no non-singular inner horizon.

The entropy formulae (3.52) can be cast in the form 

$$S_+ = S_L + S_R, \quad S_- = |S_L - S_R|,$$  \hspace{1cm} (3.54) 

with 

$$S_L = 2\pi \sqrt{F + \Delta}, \quad S_R = 2\pi \sqrt{F - J^2},$$  \hspace{1cm} (3.55) 

where $F$ is another complicated expression that is a function of $M$, $Q_i$ and $P^i$ only [61]. Note that it follows from (3.51) that $S_+ \geq S_-$. Unlike [61], we have put an absolute value sign around $(S_L - S_R)$ in the expression for $S_-$, since, for the reasons discussed above, there can be circumstances where $S_L < S_R$, but $S_-$ should be non-negative. Note that $F + \Delta$ is always non-negative, and $F - J^2$ is non-negative provided that the black hole is 

\footnote{In general, including the scalar fields, and writing the Lagrangian as a 4-form, we shall have $\mathcal{L} = R + \frac{1}{2} R_{ij}(\Phi) * F^{(i)} \wedge F^{(j)} - \frac{1}{2} N_{ij}(\Phi) F^{(i)} \wedge F^{(j)} + \cdots$, where $F^{(i)} = dA^{(i)}$. The electric charges can be written as 

$$Q_i = -\frac{1}{16\pi} \int \frac{\delta \mathcal{L}}{\delta F^{(i)}}.$$  \hspace{1cm} (3.56) 

(Here the variational derivative is defined by $\delta X = (\delta X/\delta F) \wedge \delta F$. For example if $X = u * F \wedge F + v F \wedge F$ then $\delta X/\delta F = 2u * F + 2v F$.) The magnetic charges are given by $P^i = \frac{1}{16\pi} \int F^{(i)}$.}
not over-rotating \[61\]. The quantities \(S_L\) and \(S_R\) are both non-negative. In the extremal limit \(F - J^2 = 0\), one gets the extremal value for the entropy \(S_+ = S_- = 2\pi \sqrt{\Delta}\). This was seen for the BPS solutions \((F = 0\) and \(J^2 = 0)\) in \[13\].

Note from \((3.55)\) that while the right-moving entropy \(S_R\) is a function of all the extensive variables \((M, Q_i, P^i, J)\), the left-moving entropy \(S_L\) is a function of \((M, Q_i, P^i)\) but not \(J\) \[61\]. This was noted previously in the special case of the four-charge black holes characterised by \((M, Q_i, J)\) in \[11, 62\]. The expressions \((3.55)\) may in principle be inverted to give two different Christodoulou-Ruffini formulae:

\[
M = M(S_L, Q_i, P^i), \quad \text{and} \quad M = M(S_R, Q_i, P^i, J).
\]

The structure \((3.55)\) ensures that the two entropies \(S_+\) and \(S_-\) are solutions of the quadratic equation

\[
S^2 - S \Sigma + 4\pi^2 |J^2 + \Delta| = 0,
\]

where \(\Sigma = S_L + S_R + |S_L - S_R|\), and we employed \((3.54), (3.55)\) and \((3.52)\). Note that \(\Sigma = 2S_L\) if \(S_L > S_R\), which corresponds to \(J^2 + \Delta > 0\), whilst \(\Sigma = 2S_R\) if \(S_L < S_R\), corresponding to \(J^2 + \Delta < 0\). From \((3.57)\) we can deduce

\[
\frac{\partial M}{\partial S} \frac{1}{\Sigma} \bigg|_{(Q_i, P^i, J)} = \left[1 - \frac{4\pi^2 |J^2 + \Delta|}{S^2} \right] = \frac{1}{S} \left[S - \frac{S_+ S_-}{S} \right].
\]

Since \(S_+ \geq S_-\), the final expression in \((3.58)\) is non-negative for \(S = S_+\), and non-positive for \(S = S_-\). Since \(\frac{\partial M}{\partial S} \bigg|_{(Q_i, P^i, J)} = T\), and since \(\frac{\partial \Sigma}{\partial M} \bigg|_{(Q_i, P^i, J)}\) is independent of whether one takes \(S = S_+\) or \(S = S_-\), it then follows that

\[
S_+ T_+ + S_- T_- = 0.
\]

In particular, this implies that \(T_+\) and \(T_-\) must have opposite signs.

As well as considering the left-moving and right-moving entropies \(S_L\) and \(S_R\), one can also introduce left-moving and right-moving temperatures \(T_L\) and \(T_R\), defined by \[15\]

\[
\frac{1}{T_L} = \frac{1}{T_+} + \frac{1}{T_-}, \quad \frac{1}{T_R} = \frac{1}{T_+} - \frac{1}{T_-}.
\]

These definitions are motivated by the fact that when one calculates scattering amplitudes for test fields propagating in the black-hole background, one finds that they factorise into the product of thermal Boltzmann factors for the temperatures \(T_L\) and \(T_R\) respectively \[15\].

Using \((3.59)\), together with the expressiona for \(S_+\) and \(S_-\) in terms of \(S_L\) and \(S_R\) in \((3.54)\), it follows from \((3.60)\) that

\[
S_L \geq S_R : \quad \frac{S_L}{T_L} = \frac{S_R}{T_R}, \quad \frac{S_R}{T_L} = \frac{S_L}{T_R},
\]

\[
S_L \leq S_R : \quad \frac{S_L}{T_L} = \frac{S_R}{T_R}, \quad \frac{S_R}{T_L} = \frac{S_L}{T_R}.
\]

23
for the two cases that we described previously. From its definition, $T_R$ is obviously non-negative since $T_+ \geq 0$ and $T_- \leq 0$. It is then evident from (3.61) that $T_L$ is non-negative also, since we already know that $S_L$ and $S_R$ are non-negative.

We can also derive, from

$$\Omega = \left. \frac{\partial M}{\partial J} \right|_{(Q_i, P^i, S)} = \left. \frac{\partial M}{\partial S} \frac{\partial S}{\partial J} \right|_{(Q_i, P^i, S)},$$

and using either (3.54) or else simply writing $S_+$ and $S_-$ in terms of $S_L$ and $|J^2 + \Delta|$ by using (3.52), that in the two cases $S_L \geq S_R$ and $S_L \leq S_R$ we have

$$S_L \geq S_R : \quad \Omega_+ S_+ = \Omega_- S_-, \quad \frac{\Omega_+}{T_+} = -\frac{\Omega_-}{T_-};$$

$$S_L \leq S_R : \quad \Omega_+ S_+ = -\Omega_- S_-, \quad \frac{\Omega_+}{T_+} = \frac{\Omega_-}{T_-}.$$  \hspace{1cm} (3.63)

Note that when $S_L < S_R$, i.e. when $J^2 + \Delta < 0$, the angular velocities of the inner and outer horizons are opposite. Note also that the two cases in (3.63) can be expressed in the single universal formula

$$(S_L + S_R) \Omega_+ = (S_L - S_R) \Omega_-.$$ \hspace{1cm} (3.64)

### 3.4.1 Thermodynamics of the left-moving and right-moving sectors

The introduction of the left and right temperatures and entropies suggested the possibility of viewing the black hole as being composed of excitations in left-moving and right-moving sectors in a string or D-brane description, associated with degrees of freedom of a weakly coupled two-dimensional conformal quantum field theory. The total entropy $S_+$ of the outer horizon is viewed as the sum of the entropies $S_L$ and $S_R$ of the left-moving and right-moving sectors. It is then natural to expect that there should exist thermodynamic descriptions for these sectors, with first laws of the form

$$dE_L = T_L dS_L + \Omega_L dJ + \Phi^i_L dQ_i + \Psi_{L,i} dP^i;$$

$$dE_R = T_R dS_R + \Omega_R dJ + \Phi^i_R dQ_i + \Psi_{R,i} dP^i.$$ \hspace{1cm} (3.65)

For now, we shall focus for simplicity on the regime where $S_L \geq S_R$, i.e. $(J^2 + \Delta) \geq 0$.

Let us first consider processes where $dJ = 0$ and $dQ_i = dP^i = 0$. From the definitions of $T_L$, $T_R$, $S_L$ and $S_R$ given in (3.51) and (3.60), it straightforward to see from the first laws

$$dM = T_\pm dS_\pm + \Phi^i_\pm dQ_i + \Psi_{\pm, i} dP^i + \Omega_\pm dJ$$ \hspace{1cm} (3.66)

5The analysis the thermodynamics of asymptotically-flat black holes in terms of left-moving and right-moving degrees of freedom was first addressed in [15] for general STU black holes in five dimensions, and briefly in [16] for four charge STU black holes.
on the outer and inner horizons that we must have

\[ E_L = E_R = \frac{1}{2} M. \]  

(3.67)

In other words, the left-moving and right-moving sectors contribute equally to the mass of the black hole. (This was observed in the case of Kerr-Newman black holes in \[23, 24\].) Dividing the first laws (3.66) by \( T_\pm \) respectively and then taking the plus and minus combinations, one finds that these match with (3.65) provided that we define the left-moving and right-moving quantities as

\[
\begin{align*}
\Phi^i_L &= T_L \left( \frac{\Phi^i_+}{2T_+} + \frac{\Phi^i_-}{2T_-} \right), \quad \Psi_{L,i} = T_L \left( \frac{\Psi_{+,i}}{2T_+} + \frac{\Psi_{-,i}}{2T_-} \right), \quad \Omega_L = T_L \left( \frac{\Omega_+}{2T_+} + \frac{\Omega_-}{2T_-} \right), \\
\Phi^i_R &= T_R \left( \frac{\Phi^i_+}{2T_+} - \frac{\Phi^i_-}{2T_-} \right), \quad \Psi_{R,i} = T_R \left( \frac{\Psi_{+,i}}{2T_+} - \frac{\Psi_{-,i}}{2T_-} \right), \quad \Omega_R = T_R \left( \frac{\Omega_+}{2T_+} - \frac{\Omega_-}{2T_-} \right),
\end{align*}
\]

(3.68)

and so we have the first laws

\[
\begin{align*}
\frac{1}{2} dM &= T_L dS_L + \Omega_L dJ + \Phi^i_L dQ_i + \Psi_{L,i} dP^i, \\
\frac{1}{2} dM &= T_R dS_R + \Omega_R dJ + \Phi^i_R dQ_i + \Psi_{R,i} dP^i
\end{align*}
\]

(3.69)

for the left-moving and right-moving sectors.

In a similar fashion, we can then see that the Smarr relations

\[
M = 2T_\pm S_\pm + 2\Omega_\pm J + \Phi^i_\pm Q_i + \Psi_{\pm,i} P^i
\]

(3.70)

on the outer and inner horizons imply the Smarr relations

\[
\begin{align*}
\frac{1}{2} M &= 2T_L S_L + 2\Omega_L J + \Phi^i_L Q_i + \Psi_{L,i} P^i, \\
\frac{1}{2} M &= 2T_R S_R + 2\Omega_R J + \Phi^i_R Q_i + \Psi_{R,i} P^i
\end{align*}
\]

(3.71)

for the left-moving and right-moving sectors.

It should be noted that, from (3.63) and (3.68), the left-moving angular velocity is in fact zero:

\[
\Omega_L = 0, \quad \Omega_R = \frac{T_R}{T_+} \Omega_+.
\]

(3.72)

If we now turn to the regime where \( S_L < S_R \), we find that the roles of \( S_L \) and \( S_R \) are exchanged in both the first laws and the Smarr relations for the left-moving and right-
moving sectors, so that we have

\[
S_L < S_R : \quad \frac{1}{2} dM = T_L dS_R + \Omega_L dJ + \Phi^i_L dQ_i + \Psi_{L,i} dP^i,
\]
\[
\frac{1}{2} dM = T_R dS_L + \Omega_R dJ + \Phi^i_R dQ_i + \Psi_{R,i} dP^i,
\]
\[
\frac{1}{2} M = 2T_L S_R + 2\Omega_L J + \Phi^i_L Q_i + \Psi_{L,i} P^i,
\]
\[
\frac{1}{2} M = 2T_R S_L + 2\Omega_R J + \Phi^i_R Q_i + \Psi_{R,i} P^i.
\]

Furthermore, it follows from (3.63) and (3.68) that it is now \(\Omega_R\), rather than \(\Omega_L\), that vanishes. One possible way to make the formulae more uniform for the \(S_L < S_R\) regime would be exchange the L and R labels in the definitions of all the intensive thermodynamic variables, \(T\), \(\Phi^i\), \(\Psi_i\), \(\Omega\), when \(S_L < S_R\). This would have the merit that, with the relabelling, the left-moving angular velocity would vanish in all cases, while still retaining the property that \(S_L\) is independent of \(J\) in all cases. The left-moving and right-moving first laws and Smarr relations would then take the same forms as in (3.69) and (3.71) for both \(S_L \geq S_R\) and \(S_L < S_R\), in terms of the relabelled variables.

### 3.4.2 Four-charge STU black holes

The prospects for obtaining an explicit Christodoulou-Ruffini formulae for the general 8-charge black hole solutions are not good. The main problem is the \(F\)-invariant that appears in the expressions for \(S_L\) and \(S_R\) in eqn (3.55), whose evaluation in terms of physical charges and mass appears to be quite intractable [63]. In order to obtain more explicit, concrete expressions, we shall now focus on the specialisation to black-hole solutions carrying just four electric charges, which were found in [11].

These black holes are parameterised in terms of the non-extremality parameter \(m \geq 0\) (Kerr mass parameter), the “bare” angular momentum \(a\) (Kerr rotation parameter) and four boost parameters \(\delta_i \geq 0\) \((i = 1, 2, 3, 4)\) [11] (see also [64] for compact expressions for the metric and the other fields). In terms of these, the physical mass, charges and angular momentum are given by

\[
M = \frac{m}{4} \sum_i \cosh 2\delta_i,
\]
\[
Q_i = \frac{1}{4} m \sinh 2\delta_i,
\]
\[
J = ma(\Pi_c - \Pi_s).
\]

The black hole entropies, associated with the inner and the outer horizon, are given by
\[ S_{\pm} \equiv \frac{A_{\pm}}{4} = 2\pi m \left[ m(\Pi_c + \Pi_s) \pm (\Pi_c - \Pi_s)\sqrt{m^2 - a^2} \right] \quad (3.76) \]
\[ = 2\pi \left[ m^2(\Pi_c + \Pi_s) \pm \sqrt{m^4(\Pi_c - \Pi_s)^2 - J^2} \right]. \quad (3.77) \]

The temperatures \( T_{\pm} \), related to surface gravities \( \kappa_{\pm} \) by \( T_{\pm} = \frac{\kappa_{\pm}}{2\pi} \), and angular velocities \( \Omega_{\pm} \), which are associated with the inner and outer horizons respectively, are given by \[16\]:
\[ \frac{1}{T_{\pm}} = \frac{2\pi}{\kappa_{\pm}} = \frac{4\pi m}{\sqrt{m^2 - a^2}} \left[ \pm m(\Pi_c + \Pi_s) + (\Pi_c - \Pi_s)\sqrt{m^2 - a^2} \right], \quad (3.78) \]
\[ \Omega_{\pm} = \pm \frac{2\pi a T_{\pm}}{\sqrt{m^2 - a^2}}, \quad (3.79) \]
where
\[ \Pi_c = \prod_i \cosh \delta_i, \quad \Pi_s = \prod_i \sinh \delta_i. \quad (3.80) \]

Note that \( T_- \) is negative.\(^6\) From the above expressions one also finds
\[ S_{\pm} = \pm \frac{\sqrt{m^2 - a^2}}{2T_{\pm}}, \quad (3.81) \]

It can easily be verified that the entropies \( S_{\pm} \), temperatures \( T_{\pm} \) and angular velocities \( \Omega_{\pm} \) satisfy equation \((3.59)\) and the \( S_L \geq S_R \) equations in \((3.63)\).

The entropies and the inverses of the surface gravities, associated with the outer and inner horizons, have a suggestive form in terms of the left-moving and right-moving entropy and inverse temperature excitations of a weakly coupled 2-dimensional conformal field theory (2D CFT), given in \[16\]:
\[ S_L = \frac{1}{2}(S_+ + S_-) = 2\pi m^2(\Pi_c + \Pi_s), \quad (3.82) \]
\[ S_R = \frac{1}{2}(S_+ - S_-) = 2\pi m \sqrt{m^2 - a^2}(\Pi_c - \Pi_s), \quad (3.83) \]
\[ \frac{1}{T_L} = \frac{1}{T_+} + \frac{1}{T_-} = 8\pi m(\Pi_c - \Pi_s), \quad (3.84) \]
\[ \frac{1}{T_R} = \frac{1}{T_+} - \frac{1}{T_-} = \frac{8\pi m^2}{\sqrt{m^2 - a^2}}(\Pi_c + \Pi_s). \quad (3.84) \]

Note that these solutions with four electric charges have \( \Delta \geq 0 \), as can be seen from \((3.53)\), and so they have \( S_L \geq S_R \), as is evident from \((3.82)\). In this suggestive form the central charges \( C_{L,R} \) of the left-moving and right-moving sector of the 2D CFT, related to \( S_{L,R} \)

\(^6\)Note that in \[16\] the value of \( T_- \) was taken to be positive, and equal to the absolute value of the \( T_- \) given in \((3.79)\).
and \( T_{L,R} \) via Cardy relation \( S_L = \frac{\pi^2}{3} C_L T_L \) and \( S_R = \frac{\pi^2}{3} C_R T_R \), respectively, turn out to be the same and equal to:

\[
C_L = \frac{3 S_L}{\pi^2 T_L} = \frac{3}{48m^3(\Pi_c^2 - \Pi_s^2)} = \frac{3 S_R}{\pi^2 T_R} = C_R. \tag{3.85}
\]

Again the product of outer and inner horizon entropies is quantized in terms of \( J \) and \( Q_i \) (\( i = 1, 2, 3, 4 \)) only [18]:

\[
S_+ S_- = S_L^2 - S_R^2 = 4\pi^2 \left( J^2 + 64 \prod_{i} Q_i \right), \tag{3.86}
\]

which agrees with the result for Kerr-Newman black hole after equating \( Q_1 = Q_2 = Q_3 = Q_4 = \frac{1}{4} Q \):

\[
S_+ S_- = 4\pi^2 \left( J^2 + \frac{1}{4} Q^4 \right). \tag{3.87}
\]

The main challenge here is to obtain the formulae \( M = M(S, J, Q_i) \) and \( S = S(M, J, Q_i) \). As an initial step, we observe the solutions for \( S_{\pm} \), due to relation (3.82), satisfy a quadratic equation:

\[
S^2 - 2 S_L S + 4\pi^2 \left( J^2 + 64 \prod_{i=1}^4 Q_i \right) = 0, \tag{3.88}
\]

where \( S_L \), defined in (3.82), depends on \( M \) and \( Q_i \) (\( i = 1, 2, 3, 4 \)) only. Furthermore as \( S_L \geq S_R, \ S_L \geq S_- \geq 0 \), where the extremal value \( S_+ = S_- \) is achieved for \( S_R = 0 \). The extremal the case either corresponds to the BPS solution \( \delta_i \rightarrow \infty, m \sim a \rightarrow 0 \) and \( Q_i = \frac{m}{2} \exp(2\delta_i) \) - finite, or to the extremal rotating solution with \( m = a \).

Eq. (3.88) (which is a special case of (3.57) implies again that \( T_+ \) and \( T_- \) have opposite signature. By having an explicit expression for \( S_L \) we can actually obtain an explicit expression for the temperatures. Namely, we can express \( S_L \) in terms of \( m \) and \( Q_i \), by employing:

\[
4m^2(\Pi_c \pm \Pi_s) = \left( \prod_{i=1}^4 \sqrt{m^2 + 16Q_i^2} + m \pm \prod_{i=1}^4 \sqrt{m^2 + 16Q_i^2 - m} \right), \tag{3.89}
\]

and

\[
M = \frac{1}{4} \sum_{i=1}^4 \sqrt{m^2 + 16Q_i^2}. \tag{3.90}
\]

From (3.88) we obtain:

\[
\frac{\partial S_L}{\partial S} \bigg|_{Q_i} = \frac{1}{2} \left[ 1 - \frac{4\pi^2 \left( J^2 + 64 \prod_{i=1}^4 Q_i \right)}{S^2} \right], \tag{3.91}
\]

Furthermore, employing (3.89) and (3.90) we obtain:

\[
\frac{\partial S_L}{\partial S} \bigg|_{Q_i} = \frac{\partial S_L}{\partial m} \frac{\partial m}{\partial M} \frac{\partial M}{\partial S} = 4\pi m(\Pi_c - \Pi_s) \frac{\partial M}{\partial S}, \tag{3.92}
\]

28
which leads to the explicit expression for the temperature:

$$T = \frac{\partial M}{\partial S} = \frac{1}{8\pi m(\Pi_c - \Pi_s)} \left[ 1 - \frac{4\pi^2(J^2 + 64 \prod_{i=1}^{4} Q_i)}{S^2} \right],$$

(3.93)

and angular velocity:

$$\Omega = \frac{\partial M}{\partial J} = \frac{1}{m(\Pi_c - \Pi_s)} \frac{\pi J}{S} = \frac{a\pi}{S}.$$  

(3.94)

These expressions are in agreement (3.59) and (3.63), and explicitly determine $T_+ > 0$, $T_- < 0$ and $\Omega_\pm$, in agreement with direct calculations at the horizons (3.79).

The technical difficulty in obtaining an explicit Christodoulou-Ruffini mass expression is due to the fact that an explicit expression for $S_L$ in terms of $M$ and $Q_i$ is cumbersome, in general. However, we succeeded in the following special cases.

### 3.4.3 Pairwise-equal charges

The four-charge black-hole solutions simplify considerably in the special case of pair-wise equal charges (see, for example, [64]) $Q_1 = Q_3$ and $Q_2 = Q_4$ where (3.38) can be solved explicitly for $M$:

$$M^2 = \frac{\pi}{4S} \left[ \left( \frac{S}{\pi} + 16Q_1^2 \right) \left( \frac{S}{\pi} + 16Q_2^2 \right) + 4J^2 \right].$$

(3.95)

Furthermore (3.95) and (3.86) implies:

$$M^2 = \frac{S_+}{4\pi} + \frac{S_-}{4\pi} + 4Q_1^2 + 4Q_2^2.$$  

(3.96)

For $Q_2 = 0$ the result reduces to the example of rotating dilatonic black hole with the dilaton coupling $a = 1$. The result reduces to the Kerr-Newman (or Reissner-Nordström) black hole expression when $Q_1 = Q_2 = \frac{1}{4}Q$.

It becomes straightforward that the differentiation of (3.95) with respect to $S_\pm$ (with $J$ and $Q_{1,2}$ fixed), produces the expected expressions for $T_\pm$, including the sign.

### 3.4.4 Three equal non-zero charges

It turns out that for the example of three equal non-zero charges, i.e. $Q_1 = Q_2 = Q_3 = q$ and $Q_4 = 0$, which corresponds to the rotating dilatonic black hole with the dilatonic coupling $a = \frac{1}{\sqrt{3}}$, one can again obtain an explicit expression for the the Christodoulou-Ruffini mass:

$$M^2 = \frac{16q^2 + \sqrt{64q^4 + \left( \frac{S_+}{\pi} + \frac{4\pi}{S_-}J^2 \right)^2}}{32q^2 + 4\sqrt{64q^4 + \left( \frac{S_+}{\pi} + \frac{4\pi}{S_-}J^2 \right)^2}}.$$  

(3.97)

Note, however, that when the black hole is rotating, an axion in the STU supergravity is also turned on when $Q_1$ and/or $Q_2$ is non-zero (except in the case $Q_1 = Q_2$).
(As in the pairwise-equal charge case above, here too an axion is also turned on if the black hole is rotating.)

### 3.4.5 One non-zero charge

We also note in the case of only one non-zero charge (say, \( Q_1 = q = \frac{1}{4} m \sinh 2\delta \)), which corresponds to the rotating dilatonic black hole with the dilaton coupling \( a = \sqrt{3} \), the Christodoulou-Ruffini mass can be expressed in the following form:

\[
M^2 = \frac{S_L}{8\pi} \left( 3 \cosh \delta + \frac{1}{\cosh \delta} + y \right),
\]

where \( y = \frac{32\pi r^2}{S_L} \), \( S_L = \frac{1}{2} \left( S_\pm + \frac{4\pi^2 J^2}{S_\pm} \right) \), and \( \cosh \delta \) is a solution of the cubic equation \( \cosh^3 \delta - \cosh \delta - y = 0 \):

\[
\cosh \delta = A + \frac{1}{3A}, \quad A = \frac{y}{2} + \sqrt{\frac{y^2}{4} - \frac{1}{27}}.
\]

### 3.4.6 Dyonic Kaluza-Klein black hole

In all the explicit STU supergravity black holes we have discussed so far, each of the four field strengths carries a charge of a single complexion (which could be pure electric or pure magnetic). The most general possibility is where each field strength carries independent electric and magnetic charges, as described in the general 8-charge case that was constructed by Chow and Comp`ere. Although explicit, these general solutions are rather unwieldy. Here, we discuss a much simpler case, which is still rather non-trivial, and that goes beyond what we have explicitly presented so far. We consider the case where just one of the four field strengths is non-vanishing, but it carries independent electric and magnetic charges. For simplicity we shall restrict attention to the case of static black holes. The Lagrangian (in the normalisation we are using for the STU supergravities) is given by\(^8\):

\[
\mathcal{L}_4 = \sqrt{-g} \left[ R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{4} e^{-\sqrt{3} \phi} F^2 \right],
\]

---

\(^8\)This Lagrangian can also be obtained by means of a circle reduction of five-dimensional pure Einstein gravity. For this reason, the black hole solutions are sometimes referred to as Kaluza-Klein dyons.
and a convenient way [65] to present the static dyonic black hole solutions is

\[ ds^2 = -(H_1 H_2)^{-\frac{1}{2}} f dt^2 + (H_1 H_2)^{\frac{1}{2}} \left( f^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right)\right) , \]

\[ \phi = \frac{\sqrt{3}}{2} \log \frac{H_2}{H_1} , \quad f = 1 - \frac{2\mu}{r} , \]

\[ A = \sqrt{2} \left[ \frac{(1 - \beta_1 f)}{\sqrt{\beta_1 \gamma_2 H_1}} dt + \frac{2\mu \sqrt{\beta_2 \gamma_1}}{\gamma_2} \cos \theta d\varphi \right] , \]

\[ H_1 = \gamma_1^{-1} (1 - 2\beta_1 f + \beta_1 \beta_2 f^2) , \quad H_2 = \gamma_2^{-1} (1 - 2\beta_2 f + \beta_1 \beta_2 f^2) , \]

\[ \gamma_1 = 1 - 2\beta_1 + \beta_1 \beta_2 , \quad \gamma_2 = 1 - 2\beta_2 + \beta_1 \beta_2 , \quad (3.101) \]

where \( m, \beta_1 \) and \( \beta_2 \) are constants that parameterise the physical mass \( M \), electric charge \( Q \) and magnetic charge \( P \), with

\[ M = \frac{(1 - \beta_1)(1 - \beta_2)(1 - \beta_1 \beta_2) \mu}{\gamma_1 \gamma_2} , \]

\[ Q = \frac{\sqrt{\beta_1 \gamma_2} \mu}{\sqrt{2 \gamma_1}} , \quad P = \frac{\sqrt{\beta_2 \gamma_1} \mu}{\sqrt{2 \gamma_2}} . \quad (3.102) \]

A necessary condition for regularity of the black hole is \( 0 \leq \beta_i \leq 1 \). The entropy of the outer horizon, located at \( r = 2\mu \), is given by

\[ S_+ = \frac{4\pi \mu^2}{\sqrt{\gamma_1 \gamma_2}} , \quad (3.103) \]

whilst the entropy of the inner horizon, located at \( r = 0 \), is given by

\[ S_- = \frac{4\pi \beta_1 \beta_2 \mu^2}{\sqrt{\gamma_1 \gamma_2}} . \quad (3.104) \]

The product of the entropies on the outer and inner horizons is given by

\[ S_+ S_- = 64\pi^2 P^2 Q^2 . \quad (3.105) \]

Note that \( S_- \) vanishes if \( Q \) or \( P \) vanishes. Note also that the dyonic black hole is an example where the invariant \( \Delta \), defined in (3.53), is negative. Of course since the solutions we are considering here are static, \((J^2 + \Delta)\) is negative too, and so we are in the regime where \( S_L < S_R \) for these black holes, and in fact we have

\[ S_L = \frac{2\pi \mu^2 (1 - \beta_1 \beta_2)}{\sqrt{\gamma_1 \gamma_2}} , \quad S_R = \frac{2\pi \mu^2 (1 + \beta_1 \beta_2)}{\sqrt{\gamma_1 \gamma_2}} . \quad (3.106) \]

One can straightforwardly calculate the temperatures on the outer and inner horizons, finding as usual that the temperature \( T_+ \) is positive and \( T_- \) is negative. The left-moving and right-moving temperatures, defined by (3.60), then turn out to be

\[ T_L = \frac{\sqrt{\gamma_1 \gamma_2}}{8\pi \mu (1 - \beta_1 \beta_2)} , \quad T_R = \frac{\sqrt{\gamma_1 \gamma_2}}{8\pi \mu (1 + \beta_1 \beta_2)} . \quad (3.107) \]
These are both non-negative.

A special case is when the black hole is extremal, which is achieved in this parameterisation by taking a limit in which $m$ goes to zero and the $\beta_i$ go to 1. The result is that in the extremal case

$$M_{\text{ext}} = \left( Q_3^2 + P_3^2 \right)^{\frac{3}{2}}, \quad S_{\text{ext}} = 8\pi Q P. \quad (3.108)$$

By a straightforward, although somewhat intricate, procedure, one can eliminate the metric parameters $m$, $\beta_1$ and $\beta_2$ from the four equations (3.102) and (3.103) that define the physical mass, charges and entropy, thereby arriving at a Christodoulou-Ruffini type formula relating these quantities. If we first define

$$\tilde{S} = \frac{S}{\pi}, \quad (3.109)$$

we find that they and $M$ obey the relation $W(\tilde{S}, M, Q, P) = 0$ where

$$W(\tilde{S}, M, Q, P) =$$

$$\frac{16M^6 \left( P^2 + Q^2 \right) \left( P^2 Q^2 - 8PQ \tilde{S} + 4\tilde{S}^2 \right) \left( P^2 Q^2 + 8PQ \tilde{S} + 4\tilde{S}^2 \right)}{P^2 Q^2 \tilde{S}^2} +$$

$$\frac{M^4}{16P^2 Q^2 S^4}\left( P^8 Q^8 - 48P^8 Q^4 \tilde{S}^2 - 400P^6 Q^6 \tilde{S}^2 + 1152P^6 Q^2 \tilde{S}^4 - 48P^4 Q^8 \tilde{S}^2 - 2208P^4 Q^4 \tilde{S}^4 - 768P^4 Q^6 \tilde{S}^2 + 1152P^2 Q^2 \tilde{S}^4 - 6400P^2 Q^2 \tilde{S}^6 - 768Q^4 \tilde{S}^6 + 256\tilde{S}^8 \right) -$$

$$\frac{M^2 (P^2 + Q^2)}{64P^2 Q^2 \tilde{S}^4}\left( 5P^8 Q^8 - 12P^8 Q^4 \tilde{S}^2 + 40P^6 Q^6 \tilde{S}^2 + 160P^6 Q^2 \tilde{S}^4 - 12P^4 Q^8 \tilde{S}^2 - 352P^4 Q^4 \tilde{S}^4 - 192P^4 Q^2 \tilde{S}^6 + 160P^2 Q^2 \tilde{S}^4 + 640P^2 Q^2 \tilde{S}^6 - 192Q^4 \tilde{S}^6 + 1280\tilde{S}^8 \right) -$$

$$\left( P^2 + 4\tilde{S}^2 \right)^2 \left( Q^4 + 4\tilde{S}^2 \right)^2 \left( P^2 Q^2 - 4\tilde{S}^4 \right)^2 \right). \quad (3.110)$$

This defines a multinomial of 12th order in $\tilde{S}$, and $W$ is invariant under the inversion transformation $\tilde{S} \to Q^2 P^2 / (4\tilde{S})$. Note that because $M$ is invariant under the inversion, the coefficients of each separate power of $M$ in (3.110) are invariant under the inversion.

### 3.5 Five-dimensional STU supergravity

Here, we consider black hole solutions in five-dimensional STU supergravity. General solutions with mass $M$, two angular momenta $J_\phi$ and $J_\psi$, and three charges $Q_i$ were constructed in [12] by employing solution generating techniques. We use principally the conventions of [15], except that we shall use the labels $\uparrow$ and $\downarrow$ to denote the sum and difference combinations of the angular momenta and angular velocities associated with the $\phi$ and $\psi$.
azimuthal coordinates, reserving \( L \) and \( R \) to denote the combinations of inner and outer horizon quantities, analogous to the definitions used previously for the four-dimensional STU black holes. The physical mass, charges and angular momenta are given by

\[
M = m \sum_{i=1}^{i=3} \cosh 2\delta_i, \quad Q_i = m \sinh 2\delta_i, \quad J_\downarrow = m(l_1 - l_2)(\Pi_c + \Pi_s), \quad J_\uparrow = m(l_1 + l_2)(\Pi_c - \Pi_s),
\]

(3.111)

where \( \Pi_c = \prod_{i=1}^{i=3} \cosh \delta_i, \quad \Pi_s = \prod_{i=1}^{i=3} \sinh \delta_i, \quad \text{and} \quad J_\downarrow = \frac{1}{2}(J_\phi - J_\psi), \quad J_\uparrow = \frac{1}{2}(J_\phi + J_\psi). \) Here the five-dimensional Newton constant is taken to be \( G_5 = \frac{\pi}{4}. \) We shall, without loss of generality, take the rotation parameters \( l_1 \) and \( l_2 \) and the charge boost parameters \( \delta_i \) to be non-negative in what follows.

These black holes have many analogous properties to those of the four-dimensional STU black holes, except, of course, that they can carry only electric charges but not magnetic. In particular, they have two horizons, with the inner and outer horizon entropies expressed as

\[
S_+ = S_L + S_R, \quad S_- = S_L - S_R,
\]

(3.112)

where

\[
S_L = 2\pi \sqrt{2m^3(\Pi_c + \Pi_s)^2 - J_\downarrow^2}, \quad S_R = 2\pi \sqrt{2m^3(\Pi_c - \Pi_s)^2 - J_\uparrow^2}.
\]

(3.113)

(3.114)

The product of the inner and outer horizon entropies is again quantised as:

\[
S_+S_- = 4\pi^2 \left( J_\phi J_\psi + \prod_{i=1}^{i=3} Q_i \right) = 4\pi^2 \left( J_\uparrow^2 - J_\downarrow^2 + \prod_{i=1}^{i=3} Q_i \right).
\]

(3.115)

Note that as in the case of the four-dimensional STU black holes, here it would in general be necessary to use an absolute value in the expression for \( S_- \) in (3.112), and on the right-hand side of (3.115), since \( S_- \) must be non-negative while \( S_L \) and \( S_R \), which are both non-negative, could obey either \( S_L > S_R \) or \( S_L < S_R \) depending on the relative values of the charge and angular momentum parameters. However, our non-negativity assumptions stated above for the charge and rotation parameters imply that in fact \( S_L \geq S_R \) in this case, and so we can omit the absolute value in the expression for \( S_- \), as we have done in (3.112), and in (3.115).

From the above expressions it follows that \( S \) (either \( S_+ \) or \( S_- \)) again obeys a quadratic equation,

\[
S^2 - 2SS_L + 4\pi^2 \left( J_\uparrow^2 - J_\downarrow^2 + \prod_{i=1}^{i=3} Q_i \right) = 0.
\]

(3.116)
Furthermore one can analogously derive the general result that $T_+ \text{ and } T_- \text{ have opposite signs, with:}$

$$S_+ T_+ + S_- T_- = 0, \quad (3.117)$$

and similarly

$$\frac{\Omega^\uparrow_+}{T_+} + \frac{\Omega^\uparrow_-}{T_-} = 0, \quad \frac{\Omega^\downarrow_+}{T_+} - \frac{\Omega^\downarrow_-}{T_-} = 0, \quad (3.118)$$

where $\Omega^\uparrow_\pm = \frac{1}{2}(\Omega^\phi_\pm + \Omega^\psi_\pm)$ and $\Omega^\downarrow_\pm = \frac{1}{2}(\Omega^\phi_\pm - \Omega^\psi_\pm)$. (The relative signs between the terms in these two equations in (3.118) are the opposite of those given in [15], because in that paper $\kappa_-$ was taken to be positive.)

The black holes obey the usual first laws on the outer and inner horizons:

$$dM = T_\pm dS_\pm + \Omega^\uparrow_\pm dJ^\uparrow_\pm + \Omega^\downarrow_\pm dJ^\downarrow_\pm + \Phi^\pm_i dQ_i, \quad (3.119)$$

As in the four-dimensional case, the calculation of scattering amplitudes in the black-hole background shows that they factorise into left and right sectors with Boltzman factors corresponding to temperatures $T_L$ and $T_R$ given by (3.60) [15]. Together with the normalisation of $S_L$ and $S_R$, such that $S_+ = S_L + S_R$ in accordance with the interpretation of the entropy as the sum of left-moving and right-moving contributions, one can then establish by rewriting the first laws $dM = T_\pm dS_\pm + \cdots$ in terms of left and right-moving quantities that $\frac{1}{2}dM = T_L dS_L + \cdots$ and $\frac{1}{2}dM = T_R dS_R + \cdots$, and so each of the sectors contributes one half the total mass of the black hole. Matching the first laws for arbitrary variations of the parameters then allows one to read off the appropriate definitions of the left-moving and right-moving angular momenta and electric potentials. Thus one finds the first laws

$$\frac{1}{2}dM = T_L dS_L + \Omega^\uparrow_L dJ^\uparrow_L + \Omega^\downarrow_L dJ^\downarrow_L + \Phi^L_i dQ_i, \quad (3.120)$$

$$\frac{1}{2}dM = T_R dS_R + \Omega^\uparrow_R dJ^\uparrow_R + \Omega^\downarrow_R dJ^\downarrow_R + \Phi^R_i dQ_i,$$

where

$$\Phi^L_i = T_L \left( \frac{\Phi^i_+}{2T_+} + \frac{\Phi^i_-}{2T_-} \right), \quad \Omega^\uparrow_L = T_L \left( \frac{\Omega^\uparrow_+}{2T_+} + \frac{\Omega^\uparrow_-}{2T_-} \right), \quad \Omega^\downarrow_L = T_L \left( \frac{\Omega^\downarrow_+}{2T_+} + \frac{\Omega^\downarrow_-}{2T_-} \right),$$

$$\Phi^R_i = T_R \left( \frac{\Phi^i_+}{2T_+} - \frac{\Phi^i_-}{2T_-} \right), \quad \Omega^\uparrow_R = T_R \left( \frac{\Omega^\uparrow_+}{2T_+} - \frac{\Omega^\uparrow_-}{2T_-} \right), \quad \Omega^\downarrow_R = T_R \left( \frac{\Omega^\downarrow_+}{2T_+} - \frac{\Omega^\downarrow_-}{2T_-} \right). \quad (3.121)$$

In view of the relations (3.118), one finds

$$\Omega^\uparrow_L = 0, \quad \Omega^\uparrow_R = \frac{T_L}{T_+} \Omega^\uparrow_+, \quad \Omega^\downarrow_R = \frac{T_R}{T_+} \Omega^\downarrow_+, \quad \Omega^\downarrow_R = 0. \quad (3.122)$$

Thus we see that the angular momentum $J^\uparrow_+$ and the associated angular velocity $\Omega^\uparrow_+$ enters only in the right-moving first law and in $S_R$, while the angular momentum $J^\downarrow_+$ and associated
angular velocity $\Omega^i$ enters only in the left-moving first law and in $S_L$. Note that as in four dimensions, $T_L$ and $T_R$ are both non-negative.

The Smarr formulae for the left-moving and right-moving sectors agree with the ones derived in [15]:

$$\frac{1}{2}M = \frac{3}{2}T_L S_L + \frac{3}{2}\Omega^i_L J^i_L + \Phi^i_L Q, \quad \frac{1}{2}M = \frac{3}{2}T_R S_R + \frac{3}{2}\Omega^i_R J^i_R + \Phi^i_R Q. \quad (3.123)$$

The expression for the Christodoulou-Ruffini formula in terms solely of the conserved charges, angular momenta, mass and entropy are too cumbersome to present explicitly. Even in the case of three equal charges, the mass is determined by a cubic equation.

### 3.6 Einstein-Maxwell-Dilaton black holes

There exists a more general class of black holes in the theory of Einstein-Maxwell gravity with an additional dilatonic scalar field, which is coupled to the Maxwell field with a dimensionless coupling constant $a$, with the Lagrangian

$$\mathcal{L} = \sqrt{-g} \left( R - 2(\partial \phi)^2 - e^{-2a\phi} F^2 \right), \quad (3.124)$$

The electrically-charged black-hole solution can be written as [66–68]

$$ds^2 = -\left(1 - \frac{r_+}{r}\right)^b \left(1 - \frac{r_-}{r}\right)^{b} dt^2 + \left(1 - \frac{r_+}{r}\right)^{-b} \left(1 - \frac{r_-}{r}\right)^{-b} dr^2$$

$$+ r^2 \left(1 - \frac{r_-}{r}\right)^{1-b} d\Omega^2, \quad e^{2a\phi} = \left(1 - \frac{r_-}{r}\right)^{1-b}, \quad A = \frac{Q}{r} dt, \quad (3.125)$$

where

$$b = \frac{1 - a^2}{1 + a^2}. \quad (3.126)$$

The relevant thermodynamic quantities for these black holes in this theory are given by

$$S = \pi r_+^2 \left(1 - \frac{r_-}{r_+}\right)^{1-b}, \quad T = \frac{1}{4\pi r_+} \left(1 - \frac{r_-}{r_+}\right)^b,$$

$$Q = \sqrt{\frac{r_+ + r_-}{1 + a^2}}, \quad M = \frac{1}{2}(r_+ + b r_-), \quad \Phi = \frac{1}{\sqrt{1 + a^2}} \sqrt{\frac{r_-}{r_+}}, \quad (3.127)$$

where $r_+$ is the radius of the outer horizon, and $r_-$ is a singular surface unless $a = 0$. Since by assumption $r_+ \geq r_-$, it follows that

$$M > \frac{|Q|}{\sqrt{1 + a^2}}. \quad (3.128)$$

This is consistent with the BPS bound derived in [69] using “fake supersymmetry.”
The Smarr relations continue to hold and the Gibbs free energy is again given by

\[ G = TS = \frac{1}{4}(r_+ - r_-). \] (3.129)

The coordinates \( \{r_+, r_-\} \) are now related to the coordinates \( \{T, \Phi\} \) by

\[ r_+ = \frac{1}{4\pi T} \left( 1 - (1 + a^2)\Phi^2 \right)^b \] (3.130)

and

\[ r_- = \frac{(1 + a^2)\Phi^2}{4\pi T} \left( 1 - (1 + a^2)\Phi^2 \right)^b. \] (3.131)

Thus the Gibbs energy as a function of \( \{T, \Phi\} \) is given by

\[ G = \frac{1}{16\pi T} \left( 1 - (1 + a^2)\Phi^2 \right)^{1+b}. \] (3.132)

As discussed in section 2.2, the Ricci scalar of the Helmholtz free energy metric \( ds^2(F) = -dSdT + d\Phi dQ \) will be singular on the Davies curve where the heat capacity at constant charge changes sign. It is easiest to use \( r_+ \) and \( r_- \) as the coordinate variables in this calculation, which gives

\[ R = \frac{4(1 + a^2)^2 r_+}{[(1 + a^2)r_+ - (3 - a^2)r_-]^2}. \] (3.133)

Thus the Davies curve is given by

\[ \frac{r_-}{r_+} = \frac{1 + a^2}{3 - a^2}, \] (3.134)

which implies

\[ \frac{Q^2}{M^2} = \frac{3 - a^2}{(2 - a^2)^2}. \] (3.135)

Since we must have \( r_- < r_+ \), a solution for (3.134) exists only for \( a^2 < 1 \). The spinodal curve thus projects down to the parabola in the \( S - Q \) plane given by

\[ S = (3 - a^2) \frac{1-a^2}{1+a^2} 2^{\frac{2a^2}{1+a^2}} \left( 1 - a^2 \right)^\frac{2a^2}{1+a^2} \pi Q^2. \] (3.136)

From (3.127), one can in general solve for \( r_+ \) and \( r_- \) in terms of \( M \) and \( Q \), obtaining

\[ r_+ = M + \sqrt{M^2 - (1-a^2)Q^2}, \quad r_- = \frac{1}{b} \left( M - \sqrt{M^2 - (1-a^2)Q^2} \right), \] (3.137)

and hence express \( S \) in terms of \( M \) and \( Q \) [70]:

\[ \frac{S}{\pi} = \left( M + \sqrt{M^2 - (1-a^2)Q^2} \right)^2 \left( 1 - \frac{(1+a^2)Q^2}{(M + \sqrt{M^2 - (1-a^2)Q^2})^{\frac{2a^2}{1+a^2}}} \right). \] (3.138)
If $a^2 > 0$ the entropy vanishes at extremality, namely $r_+ = r_-$ and hence $|Q| = \sqrt{1 + a^2} M$. Then $r = r_+ = r_-$ is a point-like singularity and there is no inner horizon. One can also, in general, express the entropy in terms of $r_+$ and $Q$, using

$$\left( \frac{S}{\pi r_+^2} \right)^{1 + a^2} = 1 - \frac{(1 + a^2)Q^2}{r_+^2}. \quad (3.139)$$

Particular cases include the following, which also arise as special cases of STU Black holes:

- $a = 0$ is the Reissner-Nordström case.
- $a^2 = \frac{1}{3}$ is a reduction of Einstein-Maxwell in 5 dimensions.
- $a^2 = 1$ is the so-called string case. We have

$$\frac{S}{\pi} = 4M^2 - 2Q^2, \quad M = \frac{1}{2} \sqrt{\frac{S}{\pi} + 2Q^2}. \quad (3.140)$$

The spinodal curve coincides with the $Q$-axis and the Gibbs surface is nowhere convex. It is a hyperbolic paraboloid for which the Ruppeiner metric is flat [70]. The temperature is given by

$$T = \frac{1}{4\pi \sqrt{\frac{S}{\pi} + 2Q^2}} = \frac{1}{8\pi M}, \quad (3.141)$$

and is always positive. It goes to a non-vanishing value at extremality. The heat capacity at constant charge is given by

$$C_Q = -\frac{1}{8\pi^2 (\frac{S}{\pi} + 2Q^2)^2} = -\frac{1}{64\pi^2 M^3}, \quad (3.142)$$

and is always negative, and is also non-vanishing at extremality [67].

- $a^2 = 3$ is the Kaluza-Klein black hole.

### 3.7 Two-field dilatonic black holes

Here we review a class of theories [71] which are similar to the Einstein-Maxwell-Dilaton (EMD) theory of the previous subsection, but with two field strengths rather than just one. The Lagrangian, in an arbitrary dimension $D$, is given by

$$\mathcal{L}_D = \sqrt{-g} \left( R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{4} e^{a_1 \phi} F_1^2 - \frac{1}{4} e^{a_2 \phi} F_2^2 \right). \quad (3.143)$$

The advantage of considering this extension of EMD theory is that by choosing the coupling constants $a_1$ and $a_2$ appropriately, we can find general classes of static black hole solutions...
with two horizons, and one can study the thermodynamic properties at both the outer and inner horizon.

If we turn on both the gauge fields $A_i$ independently, the theory for general $(a_1, a_2)$ does not admit explicit black hole solutions. We shall determine the condition on $(a_1, a_2)$ so that the system will give such explicit solutions. It is advantageous for later purpose that we reparameterize these dilaton coupling constants as

$$a_1^2 = \frac{4}{N_1} - \frac{2(D-3)}{D-2}, \quad a_2^2 = \frac{4}{N_2} - \frac{2(D-3)}{D-2}.$$  \quad \text{(3.144)}

(Note that $N_1$ and $N_2$ are not necessarily integers.) For the $a_i$ to be real, we must have

$$0 < N_i \leq \frac{2(D-2)}{D-3}.$$ \quad \text{(3.145)}

(If both $N_i$ are outside the range, the Lagrangian could still be made real by sending $\phi \to i\phi$, corresponding to having a ghost-like dilaton. We shall not consider this possibility here.)

Here we shall consider the case where $a_1$ and $a_2$ obey the constraint

$$a_1 a_2 = -\frac{2(D-3)}{D-2},$$ \quad \text{(3.146)}

which implies the identities

$$N_1 a_1 + N_2 a_2 = 0, \quad N_1 + N_2 = \frac{2(D-2)}{D-3}.$$ \quad \text{(3.147)}

It follows from the second identity in (3.147) that both $N_i$ can take integer values only in four and five dimensions, with $N_1 + N_2 = 3$ and $4$ respectively. The solutions with positive integers for $N_i$ are known black holes in relevant supergravities.

With $a_1$ and $a_2$ obeying (3.146), one can find black hole solutions, given by

$$ds^2 = -\left(H_1^{N_1} H_2^{N_2}\right)^{\frac{(D-3)}{D-2}} f dt^2 + \left(H_1^{N_1} H_2^{N_2}\right)^{-\frac{1}{2}} \left(f^{-1} dr^2 + r^2 d\Omega_{D-2}^2\right)$$

$$A_1 = \frac{\sqrt{N_1 c_1}}{s_1} H_1^{-1} dt, \quad A_2 = \frac{\sqrt{N_2 c_2}}{s_2} H_2^{-1} dt,$$

$$\phi = \frac{1}{2} N_1 a_1 \log H_1 + \frac{1}{2} N_2 a_2 \log H_2, \quad f = 1 - \frac{\mu}{r^{D-3}},$$

$$H_1 = 1 + \frac{\mu s_1^2}{r^{D-3}}, \quad H_2 = 1 + \frac{\mu s_2^2}{r^{D-3}},$$ \quad \text{(3.148)}

where we are using the standard notation where $s_i = \sinh \delta_i$ and $c_i = \cosh \delta_i$. The mass and charges are given by

$$M = \frac{(D-2) \mu \omega_{D-2}}{16\pi} \left(1 + \frac{D-3}{D-2} \left(N_1 s_1^2 + N_2 s_2^2\right)\right),$$

$$Q_i = \frac{(D-3) \mu \omega_{D-2}}{16\pi} \sqrt{N_i c_i s_i},$$ \quad \text{(3.149)}

38
where \( \omega_{D-2} \) is the volume of the unit \((D - 2)\)-sphere. The outer horizon is located at \( r_0 = \mu^{1/(D-3)} \), and the entropy is given by

\[
S = S_+ = \frac{1}{4} \omega_{D-2} \mu^{D-3} c_1^{N_1} c_2^{N_2}.
\] (3.150)

The inner horizon is located at \( r = 0 \), and we have

\[
S_- = \frac{1}{4} \omega_{D-2} \mu^{D-3} s_1^{N_1} s_2^{N_2}.
\] (3.151)

Multiplying the two entropies gives the product formula

\[
S_+ S_- = S_{\text{ext}}^2,
\] (3.152)

where

\[
S_{\text{ext}} = 4^{D-1}\left(\frac{\pi}{D-3}\right)^{\frac{D-2}{D-3}} \omega_{D-2} \frac{1}{\sqrt{N_1}} \left(\frac{Q_1}{\sqrt{N_1}}\right)^{\frac{1}{2}N_1} \left(\frac{Q_2}{\sqrt{N_2}}\right)^{\frac{1}{2}N_2}.
\] (3.153)

Thus the entropy product is independent of the mass.

There exists an extremal limit in which we send \( \mu \to 0 \) while keeping the charges \( Q_i \) non-vanishing. In this limit, the inner and outer horizons coalesce and the near-horizon geometry becomes AdS\(_{D-2} \times S^2\). The mass now depend only on the charges, and is given by

\[
M_{\text{ext}} = \sqrt{N_1} Q_1 + \sqrt{N_2} Q_2.
\] (3.154)

It is useful to define

\[
\tilde{M} = \frac{16\pi}{(D-2)\omega_{D-2}} M, \quad \tilde{Q}_i = \frac{8\pi}{(D-3)\omega_{D-2} \sqrt{N_i}} Q_i, \quad \tilde{S} = \frac{1}{\omega_{D-2}} S,
\] (3.155)

and then we have

\[
s_i^2 = \sqrt{\tilde{Q}_i^2 + 16\mu^2} - \frac{1}{2\mu}.
\] (3.156)

Some specific examples are as follows:

**Case 1:** \( D = 4, N_1 = N_2 = 2 \):

\[
\tilde{M}^2 - \frac{4\left(\tilde{Q}_1^2 + \tilde{S}\right)(\tilde{Q}_2^2 + \tilde{S})}{\tilde{S}} = 0.
\] (3.157)

We can define

\[
\hat{S} = \frac{\tilde{S}}{Q_1 Q_2},
\] (3.158)

and then

\[
\tilde{M}^2 - 4(Q_1^2 + \tilde{Q}_2^2) - 4\tilde{Q}_1 \tilde{Q}_2 \left(\hat{S} + \frac{1}{\hat{S}}\right) = 0.
\] (3.159)
Case 2: $D = 4$, $N_1 = 1, N_2 = 3$:

$$
\tilde{M}^6 + \frac{\tilde{M}^4 \left( \tilde{S}^4 - 3\tilde{S}^2\tilde{Q}_1^2 - 15\tilde{S}^2\tilde{Q}_1^2\tilde{Q}_2^2 + \tilde{Q}_1^2\tilde{Q}_2^6 \right)}{\tilde{S}^2\tilde{Q}_1^2} \\
- \frac{\left( 4\tilde{S}^4 + \tilde{S}^2\tilde{Q}_1^2 - 6\tilde{S}^2\tilde{Q}_1^2\tilde{Q}_2^2 - 3\tilde{S}^2\tilde{Q}_2^4 + 4\tilde{Q}_1^2\tilde{Q}_2^6 \right)^2}{\tilde{S}^4\tilde{Q}_1^2} \\
- \frac{\tilde{M}^2}{\tilde{S}^2\tilde{Q}_1^2} \left( 20\tilde{S}^4\tilde{Q}_1^2 + 12\tilde{S}^4\tilde{Q}_2^4 - 3\tilde{S}^2\tilde{Q}_1^2 - 3\tilde{S}^2\tilde{Q}_1^2\tilde{Q}_2^4 - 5\tilde{S}^2\tilde{Q}_1^2\tilde{Q}_2^4 - \tilde{S}^2\tilde{Q}_2^6 \\
+ 20\tilde{Q}_1^2\tilde{Q}_2^4 + 12\tilde{Q}_1^2\tilde{Q}_2^4 \right) = 0. 
$$

(3.160)

Case 3: $D = 5$, $N_1 = 1, N_2 = 2$:

$$
0 = \tilde{M}^4 + \frac{\tilde{M}^3 \left( 4\tilde{S}^4 + \tilde{Q}_1^2\tilde{Q}_2^4 \right)}{3\tilde{S}^2\tilde{Q}_1^2} - \frac{4\tilde{M}^2 \left( 8\tilde{Q}_1^2 + 20\tilde{Q}_1^2\tilde{Q}_2^4 - \tilde{Q}_2^4 \right)}{9\tilde{Q}_1^2} \\
- \frac{8\tilde{M} \left( 2\tilde{Q}_1^2 + \tilde{Q}_2^4 \right) \left( 4\tilde{S}^4 + \tilde{Q}_1^2\tilde{Q}_2^4 \right)}{3\tilde{S}^2\tilde{Q}_1^2} \\
- \frac{4 \left( 432\tilde{S}^8 - 64\tilde{S}^4\tilde{Q}_1^2 + 192\tilde{S}^4\tilde{Q}_1^2\tilde{Q}_2^4 + 24\tilde{S}^4\tilde{Q}_1^2\tilde{Q}_2^4 + 64\tilde{S}^4\tilde{Q}_2^8 + 27\tilde{Q}_1^2\tilde{Q}_2^8 \right)}{81\tilde{S}^4\tilde{Q}_1^2}. 
$$

(3.161)

Case 4: General $D$, but with $N_1 = N_2 = (D - 2)/(D - 3)$

These cases lie, in general, outside the realm of supergravity theories. We have

$$
\tilde{M}^2 - 4(\tilde{Q}_1^2 + \tilde{Q}_2^4) - \left( 16\tilde{S}^4\tilde{Q}_1^2\tilde{Q}_2^2\tilde{S}^2\tilde{Q}_1^2\tilde{Q}_2^2 - 16\tilde{S}^4\tilde{Q}_1^2\tilde{Q}_2^4 \right) = 0. 
$$

(3.162)

Entropy super-additivity is difficult to prove in general, but we can at least look at the case of extremal black holes, for which

$$
S_{\text{ext}} \sim \sqrt{Q_1^{N_1}Q_2^{N_2}}. 
$$

(3.163)

It seems that super-additivity will be satisfied if $N_1 + N_2 \geq 2$, and in fact, from (3.147), we have $N_1 + N_2 > 0$ in all dimensions.

4 Entropy Product and Inversion Laws

It is well known from many examples that if a black hole has two horizons then the product of the areas, or equivalently entropies, of these horizons is equal to an expression written
purely in terms of the conserved charges and angular momenta [18,20]. Thus we may write

\[ S_+ S_- = K(Q, J), \]  

(4.1)

where \( Q \) represents the complete set of charges carried by the black hole, and \( J \) represents the set of angular momenta. (Generalisations arise also if there are more than two horizons or “pseudo-horizons” (see, for example, [18]).) We also saw various examples in the previous section where there is a Christodoulou-Ruffini formula relating the entropy to the mass, charges and angular momenta, of the form

\[ W(S, M, Q, J) = 0, \]  

(4.2)

for which there was a symmetry under a certain inversion of the entropy, \( S \to S' \sim 1/S \).

Here, we make some observations about the relation between these properties of the black hole entropy. First, we note that when one derives a Christodoulou-Ruffini formula of the form (4.2), one uses properties of the metric functions that determine the horizon radius in terms of the metric parameters, and hence implicitly they determine the horizon radius in terms of \( M, Q \) and \( J \). This means that when one arrives at the Christodoulou-Ruffini relation (4.2), the expression will necessarily be valid not only when \( S = S_+ \), but also when instead \( S = S_- \). Since \( S_+ \) and \( S_- \) are related by the product formula (4.1), this means that if \( S \), the entropy of the outer horizon, obeys (4.2) then we will also have

\[ W\left(\frac{K(Q, J)}{S}, Q, J\right) = 0. \]  

(4.3)

In other words, the Christodoulou-Ruffini formula will be invariant under the inversion symmetry

\[ S \to \frac{K(Q, J)}{S}, \]  

(4.4)

where \( K(Q, J) \) is the right-hand side of the entropy-product formula (4.1).

In some cases, for example in the case of STU black holes where \( J = 0 \) and insufficiently many charges are turned on, there is only one horizon and so there is no entropy-product formula. In such cases the argument above demonstrating the existence of an inversion symmetry of the Christodoulou-Ruffini relation breaks down. Indeed, in section 3.6 we saw examples where, for this reason, the Christodoulou-Ruffini relation had no inversion symmetry.

\[ \text{Or conformally invariant, depending on how one chooses the overall multiplicative factor when defining } W(S, M, Q, J). \]
One important consequence of the inversion symmetry of the Christodoulou-Ruffini relation \( M = M(S, Q, J) \) is that the relation \( S_+ T_+ + S_- T_- = 0 \), seen, for example, for the STU black holes in (3.59), is true quite generally. Since the temperature is given by \( \partial M / \partial S \) at fixed \( Q \) and \( J \) we have

\[
T = \frac{\partial M(S, Q, J)}{\partial S} = \frac{\partial}{\partial S} M\left(\frac{K}{S}, Q, J\right)
\]

\[
= -\frac{K}{S^2} \frac{\partial M(S', Q, J)}{\partial S'} \Bigg|_{S' = K/S},
\]

where \( K = K(Q, J) \) is the numerator in the inversion formula (4.4). Taking \( S = S_+ \) we therefore have \( S' = S_- \), and so we find from (4.5) that

\[
T_+ S_+ + T_- S_- = 0
\]

whenever there is an entropy-product rule of the form (4.1) and the related inversion symmetry under (4.4).

5 Asymptotically AdS and dS Black Holes

In this section we shall extend the previous discussion to the case of a non-vanishing cosmological constant. If the cosmological constant is negative, the situation is similar to the case when it vanishes. However, if the cosmological constant is positive a new feature arises, namely the occurrence of an additional “cosmological” horizon outside the black hole event horizon. Typically the surface gravity at the cosmological horizon is negative.

5.1 Kottler

Either we regard \( \Lambda \) as a fixed constant or as an intensive variable which may be varied, in which case we obtain an analogy with a gas with positive pressure

\[
P = -\frac{\Lambda}{8\pi}.
\]

In the first case we should think of the Abbott-Deser mass \( M \) as the total energy. In the second case, we should instead think of it as the total enthalpy [72, 73]. In both cases we have

\[
2M = \left(\frac{S}{\pi}\right)^{\frac{3}{2}} - \frac{\Lambda}{3} \left(\frac{S}{\pi}\right)^{\frac{3}{2}},
\]

and in both cases

\[
T = \frac{\partial M}{\partial S} \bigg|_{\Lambda} = \frac{1}{4\pi} \left[ \sqrt{\pi} - \Lambda \sqrt{\pi} \sqrt{\frac{S}{\pi}} \right]
\]
and the heat capacity at constant pressure is given by
\[ C_\Lambda = T \left( \frac{\partial T}{\partial S} \right|_\Lambda^{-1} = \frac{2S(\Lambda S - \pi)}{\Lambda S + \pi}. \] (5.4)

We now consider the two cases where \( \Lambda < 0 \) and \( \Lambda > 0 \).

### 5.1.1 \( \Lambda < 0 \)

The temperature \( T \) is a positive, monotonic-increasing function of entropy \( S \) at fixed pressure \( P \). The isobaric curve in the \( S - M \) plane has a point of inflection at which the heat capacity changes sign when
\[ \frac{S}{\pi} = -\frac{1}{\Lambda}, \quad M = \frac{2}{3\sqrt{-\Lambda}}, \] (5.5)
where the slope, and hence the temperature, has a minimum value;
\[ T = T_{\text{min}} = \frac{1}{2\pi}\sqrt{-\Lambda}. \] (5.6)

It follows that for fixed negative \( \Lambda \) there are no black holes with temperatures less than \( T_{\text{min}} \). For temperatures above \( T_{\text{min}} \) there are two black holes, one with a mass smaller than \( \frac{2}{3\sqrt{-\Lambda}} \) and the other with a mass greater than \( \frac{2}{3\sqrt{-\Lambda}} \).

The radius \( r_H \) of the critical black hole, where the two branches coalesce, is given by
\[ r_H = \frac{3}{2}M. \] (5.7)

This is the location where the heat capacity diverges. It is connected with the Hawking-Page phase transition \[74,75\]. There is actually a region of masses \( M_{\text{HP}} > M > M_{\text{cr}} \) where the \( AdS_4 \) space is entropically favoured; however the black hole still has a positive heat capacity. As with the Reissner-Nordström black hole, it has been shown that the sign of the lowest eigenvalue of the Lichnerowicz operator changes sign as the heat capacity changes sign \[76\].

### 5.1.2 \( \Lambda > 0 \)

We have a negative pressure, \( P < 0 \). If \( M \) is assumed positive we have two horizons, a black hole horizon with
\[ 0 < S \leq \frac{\pi}{\Lambda}, \] (5.8)
and positive temperature \( T = \partial M/\partial S \), and a cosmological horizon with
\[ \frac{\pi}{\Lambda} \leq S \leq \frac{3\pi}{\Lambda}, \] (5.9)
for which $T = \partial M/\partial S < 0$, and hence the temperature is negative. The heat capacity is therefore always negative. The temperature vanishes when the two horizons coincide, that is if
\[
\frac{S}{\pi} = \Lambda, \tag{5.10}
\]
at which the mass has a maximum of
\[
M = \frac{1}{3\sqrt{\Lambda}}. \tag{5.11}
\]

In summary, we have two horizons; a black hole horizon and a cosmological horizon. The entropy of the former is smaller then or equal to the entropy of the latter. It seems most appropriate to regard $M$ as the enthalpy. In this case the black hole horizon has positive temperature and the cosmological horizon has negative temperature. This differs from the usual interpretation in which both temperatures are taken to be positive. In effect one takes $T_C = \frac{|\kappa_C|}{2\pi}$ where $\kappa_C$, where $\kappa_C$ is the surface gravity of the event horizon [28–31]. However, even if one follows the conventional interpretation it should be borne in mind that it is not an equilibrium system and there is no period in imaginary time which would produce an everywhere non-singular gravitational instanton, except when the black hole is absent as in [28, 77].

5.2 Reissner-Nordström-de Sitter

5.2.1 $\Lambda < 0$

If $r = \sqrt{\frac{S}{\pi}}$ is the radius in the area coordinate, we have
\[
2M = r + \frac{Q^2}{r} + g^2 r^3. \tag{5.12}
\]
where $\frac{4}{3} = -g^2$. Using the fact that
\[
\frac{\partial}{\partial S} = \frac{1}{2\pi r} \frac{\partial}{\partial r}, \tag{5.13}
\]
one finds that
\[
T = \frac{\partial M}{\partial S} = \frac{1}{4\pi r} \left(1 - \frac{Q^2}{r^2} + 3g^2 r^2\right) \tag{5.14}
\]
and thus $T$ vanishes at $r = r_{\text{extreme}}$ where
\[
\frac{r_{\text{extreme}}^2}{g^2} = \frac{1}{6g^2} \left(\sqrt{1 + 12Q^2 g^2} - 1\right). \tag{5.15}
\]

One has
\[
\frac{\partial^2 M}{\partial S^2} = \frac{1}{4\pi^2} \left(-\frac{1}{r^3} + \frac{Q^2}{r^5} + 3g^2 \frac{r}{r}\right) \tag{5.16}
\]
If $6|gQ| < 1$ there are two inflection points at which the heat capacity changes sign at $r = r_{\text{inflection}}$ where

$$r_{\text{inflection}}^2 = \frac{1}{6g^2}(1 + \pm \sqrt{1 - 36Q^2g^2}) \quad (5.17)$$

If we take the limit that $Q^2 \to 0$ we obtain the spinodal curve of the the Hawking-Page phase transition [74] and if we take the limit $g^2 \to 0$ we obtain the spinodal curve of the Davies phase transition [54]. The two curves meet at the critical point $6|gQ| = 1$.

5.2.2 $\Lambda > 0$

This case admits new qualitatively different phenomena since both a black hole and a cosmological horizon are present. This was extensively investigated in 1989 [78–83]. In all these references the absolute value of the surface gravity was taken and the and so the temperature of both horizons was take to be positive. For the choice $M = |Q|$ the temperatures of the black hole and cosmological horizon were observed to be equal. This allowed the construction of a gravitational instanton. To ensure that the electromagnetic field is real on the Euclidean section it is most convenient to assume that the electromagnetic field is purely magnetic which can be arranged by a duality rotation. In order to avoid confusion with pressure in what follows we replace $Q$ by $Z$ and take $Z$ to be real and positive. We have

$$-r^2g_{tt} = (r - M)^2 + Z^2 - M^2 - \frac{r^4}{l^2}, \quad (5.18)$$

and

$$2M = r + \frac{Z^2}{r} - \frac{r^3}{l^2}, \quad (5.19)$$

with $l^2 = \frac{3}{\Lambda}$.

If $M^2 = Z^2$ there are three positive values of $r$ for which $g_{tt} = 0$:

$$r_1 = \frac{l}{2}(1 + \sqrt{1 - 4\frac{M}{l}}), \quad (5.20)$$

$$r_2 = \frac{l}{2}(1 - \sqrt{1 - 4\frac{M}{l}}), \quad (5.21)$$

$$r_3 = \frac{l}{2} \left( \sqrt{1 + 4\frac{M}{l}} - 1 \right). \quad (5.22)$$

which correspond to the cosmological event horizon, the black hole horizon and its inner
horizon respectively. From the Gibbsian point of view one has \( T = \frac{\kappa}{2\pi} \) and therefore

\[
\begin{align*}
T_1 &= -\frac{1}{2\pi l}\sqrt{1-4Ml}, \\
T_2 &= \frac{1}{2\pi l}\sqrt{1-4Ml}, \\
T_3 &= -\frac{1}{2\pi l}\sqrt{1+4Ml}.
\end{align*}
\]

(5.23) \quad (5.24) \quad (5.25)

Because \(|T_1|=T_2\) we obtain a gravitational instanton by setting \( t = i\tau \) and identifying \( \tau \) modulo \( \beta = \frac{1}{T_2} \). The sign used for the period appears to have no geometrical significance and proceeding in the standard way one may argue that the two horizons are in equilibrium with respect to the exchange of thermal Hawking quanta.

It was also argued that if \(|\kappa_3| \geq |\kappa_1|\), then the Cauchy horizon should be stable.

### 5.3 Kerr-Newman-de Sitter black holes

From [84] we take the formula

\[
M = \frac{1}{2}\sqrt{\tilde{S}}\sqrt{(1 - \frac{\Lambda \tilde{S}}{3} + \frac{Q^2}{S})^2 + \frac{4J^2}{S^2}(1 - \frac{\Lambda \tilde{S}}{3})}
\]

(5.26)

where \( \tilde{S} = \frac{S}{\pi} \). Writing \( \Lambda = -3g^2 \), the formula takes the form

\[
M^2 = \frac{\pi}{4S}\left\{ \left[ \frac{\tilde{S}}{\pi}(1 + g^2 \frac{S}{\pi}) + Q^2 \right]^2 + 4J^2 \left(1 + g^2 \frac{S}{\pi}\right) \right\}.
\]

(5.27)

For \( \Lambda = 0 \) the result reduces to that of the Kerr-Newman black hole.

### 5.4 Pairwise-equal charge anti-de Sitter black hole

These solutions were obtained in [64], and they are special cases of solutions in the gauged STU supergravity model. (Those are also solutions of maximally supersymmetric four-dimensional theory, which is a consistent truncation of a Kaluza-Klein compactified eleven-dimensional supergravity on \( S^7 \).) The theory is specified by mass \( M \), angular momentum \( J \), two charges, i.e., equating \( Q_1 = Q_3 \) and \( Q_2 = Q_4 \), and cosmological constant \( \Lambda = -3g^2 \).

In [64] the solution was parameterised by the non-extremality parameter \( m \), rotational parameter \( a \), two boost parameters \( \delta_{1,2} \) and \( g^2 \). The thermodynamic quantities are of the following form:

\[
\begin{align*}
M &= \frac{m(1 + s_1^2 + s_2^2)}{\Xi^2}, \quad (5.28) \\
J &= \frac{am(1 + s_1^2 + s_2^2)}{\Xi^2}, \quad (5.29) \\
Q_i &= \frac{ms_ic_i}{2\Xi}, \quad i = 1, 2, \quad (5.30)
\end{align*}
\]
where $s_i = \sinh \delta_i$, $c_i = \cosh \delta_i$ ($i = 1, 2$). The entropy is of the form:

$$S = \frac{\pi}{\Xi} (r_1 r_2 + a^2),$$

(5.31)

where $r_i = r_+ + m s_i^2$ ($i = 1, 2$) and $r_+$ is a location of a horizon, which is a solution of the equation:

$$r^2 - 2mr + a^2 + g^2 r_1 r_2 (r_1 r_2 + a^2) = 0.$$  

(5.32)

Manipulation of the horizon equation, along with the expressions for the $M$, $J$, $Q_i$ and $S$, allows one to derive the following explicit Christodoulou-Ruffini mass:

$$M^2 = \frac{\pi}{4S} \left\{ \left[ \frac{S}{\pi} \left(1 + g^2 \frac{S}{\pi} \right) + 16Q_1^2 \right] \left[ \frac{S}{\pi} \left(1 + g^2 \frac{S}{\pi} \right) + 16Q_2^2 \right] + 4J^2 \left(1 + g^2 \frac{S}{\pi} \right) \right\}. 

(5.33)

5.5 Wu black hole

The Wu black hole [85] is 5D, three charge rotating solution with negative cosmological constant ($\propto g^2 a^2$). Employing expressions from [86] for a product of the entropy and temperature of this black hole, associated with all three horizons we obtain the following interesting expression:

$$n_1 + n_2 + n_3 + \frac{1}{2} \left( \frac{n_1 n_2}{n_3} + \frac{n_1 n_3}{n_2} + \frac{n_2 n_3}{n_1} \right) = 0,$$

(5.34)

where

$$n_1 = \frac{4 \xi_a \xi_b}{g^2 a^2} T_1 S_1 = (u_1 - u_2)(u_2 - u_3) \quad \text{&cyclic permutations}. 

(5.35)

Here $\xi_a = 1 - g^2 a^2$, $\xi_b = 1 - g^2 b^2$ and $u_i$ is the root of the horizon equation $X = g^2 (u - u_1)(u - u_2)(u - u_3)$. Note that as $g^2 \to 0$, $u_3 \to -1/g^2 \to -\infty$, and in this case the above equation reduces to the standard equation $T_1 S_1 + T_2 S_2 = 0$.

6 Entropy and Super-Additivity

The thermodynamics of equilibrium systems with a substantial contribution to the total energy from their gravitational self energy differs significantly from that of ordinary substances encountered in the laboratory. This is because of the long range nature of the Newtonian gravitational force, which cannot be screened. As a consequence the total entropy $S$ of a gravitating system need not be proportional to the total energy $M$. A consequence of this is that negative heat capacities are possible, and indeed these have long been encountered in the theory of stellar structure [87].

In the case of black holes, the long range nature of gravitational interaction expresses itself in the fact that while the individual extensive variables may be added, they do not
necessarily scale. Even if they do, they do not scale with the same power as the total energy $M$. In the case of ungauged supergravity black holes, the scaling behaviour is guaranteed, but the fact that the scaling behaviour is not homogeneous, that is, not the same for all extensive variables, leads to a modification of the standard form of the Gibbs-Duhem relation for ordinary homogeneous substances

$$G = M - TS - PV = 0,$$  \hspace{1cm} (6.1)

where $G$ is the Gibbs free energy, $V$ the volume and $P$ the pressure. By contrast, for black holes the Smarr relation (2.14) gives rise to the Gibbs function (2.16).

The requirement of homogeneous scaling plays such an important role in the thermodynamics of ordinary substances that it has been suggested that it be called the **Fourth Law of Thermodynamics** \[88, 89\]. It certainly fails for systems with significant self-gravitation and, *a fortiori*, for black holes. In fact if the matter sector is sufficiently non-linear such as in Einstein’s theory coupled to non-linear electrodynamics, even the property of weighted homogeneity ceases to hold.\footnote{A function $f(x_1, x_2, \ldots, x_n)$ of $n$ variables is said to be weighted homogeneous of weights $w_1, w_2, \ldots, w_n$ if $f(\lambda^{w_1} x_1, \lambda^{w_2} x_2, \ldots, \lambda^{w_n} x_n) = \lambda f(x_1, x_2, \ldots, x_n)$. If $w_i = 1$ for all $i$, the function is said to be homogeneous of weight one. The Fourth Law is the statement that all extensive variables have weight one and thus all intensive variables have weight zero.}

As a consequence, while the first law of black hole thermodynamics holds there is no analogue of a Smarr formula \[90\].

In the thermodynamics of ordinary substances it is usually assumed that the total energy $M$ is a convex function\footnote{A function $f(x)$ is said to be convex if $f(\lambda x_1 + (1 - \lambda) x_2) \leq \lambda f(x_1) + (1 - \lambda) f(x_2) \forall 1 \leq \lambda \leq 1$ and concave if $\leq$ is changed to $\geq$. Subject to suitable differentiability this is equivalent to negative (positive) definiteness of the Hessian $\frac{\partial^2 f}{\partial x_i \partial x_j}$. In other words, if $M$ is the total energy then the graph of the Gibbs surface along a straight line joining two equilibrium states $x_1$ and $x_2$ never lies above the straight line joining these points on the Gibbs surface.} of the extensive variables or that the $S$ is a concave function of the other extensive variables. This guarantees that the heat capacity and other susceptibilities are positive, and that the Hessians have the correct signs to render the Weinhold and Ruppeiner metrics positive definite.

Now if the extensive quantities scale in a uniform fashion, the property of concavity is equivalent to that of super-additivity\footnote{A function $f(x)$ is super-additive if $f(x_1 + x_2) \geq f(x_1) + f(x_2)$ and sub-additive if we replace $\geq$ by $\leq$.} but not necessarily if uniform scaling ceases to hold \[91, 94\]. Remarkably, it was shown long ago in a little noticed paper by Tranah and Landsberg \[93\] that while concavity fails for the entropy of Kerr-Newman black holes,
super-additivity remains true. In other words

\[ S(M_1 + M_2, J_1 + J_2, Q_1 + Q_2) \geq S(M_1, J_1, Q_1) + S(M_2, J_2, Q_3). \] (6.2)

The super-additivity inequality (6.2) is related to Hawking’s area theorem \[35, 36\]. If two black holes of areas \(A_1\) and \(A_2\) can merge to form a single black hole of area \(A_3\), then, subject to the assumption of cosmic censorship,

\[ A_3 \geq A_1 + A_2. \] (6.3)

If the angular momentum and charge of the final black hole are equal to the sums of the angular momenta and charges of the initial black holes, one has in addition

\[ S(M_3, J_1 + J_2, Q_1 + Q_2) \geq S(M_1, J_1, Q_1) + S(M_2, J_2, Q_3), \] (6.4)

where \(M_3\), the mass of the black hole final state after the merger, obeys

\[ M_3 < M_1 + M_2, \] (6.5)

since energy will be lost by gravitational radiation. It follows from the first law that at fixed charge and angular momentum, \(dM = TdS\) and so provided that the temperature is positive,

\[ S(M_1 + M_2, J_1 + J_2, Q_1 + Q_2) > S(M_3, J_1 + J_2, Q_1 + Q_2). \] (6.6)

The assumption that \(Q_3 = Q_1 + Q_2\) is reasonable for theories like Einstein-Maxwell or ungauged supergravity, where there are no particles that carry charge. The assumption that \(J_3 = J_1 + J_2\), however, is less reasonable, because both electromagnetic and gravitational waves can carry angular momentum.

In the following subsections we shall obtain generalisations of the Kerr-Newman super-additivity result of Tranah and Landsberg for various more complicated black hole solutions. We also obtain a counter-example in the case of dyonic Kaluza-Klein black holes.

### 6.1 STU black holes with pairwise-equal charges

From the formula expressing \(M\) in terms of \(S, Q_1, Q_2\) and \(J\) for pairwise-equal charged STU black holes, we have

\[ \frac{1}{\pi} S(M, Q_1, Q_2, J) = Y + \sqrt{X}, \quad Y = 2M^2 - \frac{1}{2}(Q_1^2 + Q_2^2), \quad X = Y^2 - Q_1^2 Q_2^2 - 4J^2. \] (6.7)

For regular black holes we must have \(X \geq 0\) and hence \(Y \geq \sqrt{4Q_1^2 Q_2^2 + 16J^2}\), thus implying

\[ 4M^2 \geq Q_1^2 + Q_2^2 + 4J^2. \] (6.8)
Without loss of generality, we shall assume $Q_1$, $Q_2$ and $J$ are all non-negative. Note that we also have the weaker inequality

$$M \geq \frac{1}{2}(Q_1 + Q_2),$$

(6.9)

which we shall use frequently in the following.

We wish to check whether the entropy of these pairwise-equal charged black holes obey the super-additivity inequality

$$S_{\text{tot}} \geq S + S',$$

(6.10)

where

$$S_{\text{tot}} \equiv S(M + M', Q_1 + Q_1', Q_2 + Q_2', J + J'), \quad S \equiv S(M, Q_1, Q_2, J), \quad S' \equiv S(M', Q_1', Q_2', J').$$

(6.11)

With analogous definitions for the quantities $X$ and $Y$, proving super-additivity requires proving that

$$Y_{\text{tot}} - Y - Y' + \sqrt{X_{\text{tot}}} - \sqrt{X} - \sqrt{X'} \geq 0.\quad (6.12)$$

We first note that the $Y$ functions are non-negative, and that they obey

$$Y_{\text{tot}} - Y - Y' = 4M M' - Q_1 Q_1' - Q_2 Q_2'$$

$$\geq (Q_1 + Q_2)(Q_1' + Q_2') - Q_1 Q_1' - Q_2 Q_2'$$

$$= Q_1 Q_2' + Q_2 Q_1'$$

$$\geq 0.\quad (6.13)$$

Thus, if we can show that

$$\sqrt{X_{\text{tot}}} - \sqrt{X} - \sqrt{X'} \geq 0\quad (6.14)$$

then the super-additivity inequality (6.10) will be established. To prove this, we first note that is can be re-expressed as

$$X_{\text{tot}} - (\sqrt{X} + \sqrt{X'})^2 \geq 0.\quad (6.15)$$

We now observe that the following identity holds:

$$P \ := \ \left( c \sqrt{X} - \frac{1}{c} \sqrt{X'} \right)^2 + 4 \left(c J - \frac{1}{c} J' \right)^2$$

$$= -2 \sqrt{X} \sqrt{X'} - 8J J' + 8M^2 M'^2 - 2M^2 (Q_1'^2 + Q_2'^2) - 2M'^2 (Q_1^2 + Q_2^2)$$

$$- 2Q_1 Q_2 Q_1' Q_2' + \frac{1}{2}(Q_1^2 + Q_2^2)(Q_1'^2 + Q_2'^2),$$

(6.16)
where we have defined
\[
e^2 = \frac{4M^2 - (Q_1' - Q_2')^2}{4M^2 - (Q_1 - Q_2)^2}.
\] (6.17)

We can use (6.16) to substitute for \(\sqrt{X} \sqrt{X'}\) in (6.15), thus yielding
\[
X_{\text{tot}} - X - X' - 2\sqrt{X} \sqrt{X'} =
\]
\[
P + 8(M M' - Q_2 Q_2')(M^2 + M'^2 - Q_2^2 - Q_2'^2)
\]
\[
+ 8[(M + M')^2 - (Q_2 + Q_2')^2](M M' - Q_2 Q_2'),
\]
(6.18)

where we have defined
\[
Q_\pm = \frac{1}{2}(Q_1 \pm Q_2), \quad Q_\pm' = \frac{1}{2}(Q_1' \pm Q_2').
\] (6.19)

The inequality (6.9) implies \(M \geq Q_+\) and \(M' \geq Q_+'\), and \(a \text{ fortiori} \ M \geq |Q_-|\) and \(M' \geq |Q_-'|\) (recall that we are taking all charges to be non-negative). Since \(P\), defined in (6.16), is manifestly non-negative it follows from (6.18) that the left-hand side must be non-negative, and hence the required inequality (6.14) is satisfied. Thus we have proven that the super-additivity property (6.10) is indeed obeyed by the entropy of the pairwise-equal charged black holes of STU supergravity.

6.2 STU black holes with three equal non-zero charges

One can also show analytically that the super-additivity property of the entropy is true for the case of STU black holes with three equal non-zero charges, say, \(Q_1 = Q_2 = Q_3 = q\), with \(Q_4 = 0\). In this case \(S = \pi(Y + \sqrt{X})\) with:
\[
Y^2 = \frac{1}{64}(3z - 2M)(z + 2M)^3,
\] (6.20)

where
\[
z = \sqrt{4M^2 - 2q^2},
\] (6.21)

and
\[
X = Y^2 - J^2.
\] (6.22)

It is straightforward to show that
\[
z_{\text{tot}}^2 - (z + z')^2 = 8M M' \left(1 - ww' - \sqrt{1 - w^2} \sqrt{1 - w'^2}\right) \geq 0,
\] (6.23)

where \(w = \frac{q}{\sqrt{2M}}\) and \(w' = \frac{q'}{\sqrt{2M'}}\). The second inequality in (6.23) is true for any value of \(\{w, w'\} \leq 1\). This result implies
\[
Y_{\text{tot}} - Y - Y' \geq 0.
\] (6.24)
It is now straightforward to show that

$$\sqrt{X_{tot}} - \sqrt{X} - \sqrt{X'} \geq 0,$$

(6.25)

thus proving the super-additivity of the entropy in this case as well.

An analytic proof of the super-additivity of the entropy for the case of one non-zero charge follows analogous steps.

While a numerical analysis indicates that the super-additivity is true for the STU black holes with four arbitrary electric charges, it would be interesting to prove this result analytically.

### 6.3 Dyonic Reissner-Nordström

In the explicit examples we have studied so far, the black hole is supported by one or more field strengths that each carry a single complexion of field (pure electric charge, or instead and equivalently, one could consider pure magnetic charge). The details of the entropy super-additivity inequality are different if we consider a case where one or more field strengths carries both electric and magnetic charge. In this subsection, we shall study the dyonic Reissner-Nordström black hole, and show that in this case too the super-additivity property is satisfied. This case, where the Lagrangian is just that of the pure Einstein-Maxwell system, can be view as STU black holes where all four field strengths are equal. By contrast, in the next subsection we shall see that in the case of STU black holes where only a single field strength is non-zero, the dyonic black holes have an entropy that violates the super-additivity property.

The Einstein-Maxwell Lagrangian $L = \sqrt{-g}(R - F^2)$ admits static dyonic black hole solutions given by

$$ds^2 = -h dt^2 + \frac{dr^2}{h} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

$$A = \frac{Q}{r} dt + P \sin \theta d\phi,$$

$$h = 1 - \frac{2M}{r} + \frac{Q^2 + P^2}{r^2},$$

(6.26)

with mass $M$, electric charge $Q$ and magnetic charge $P$. To have a black hole, these quantities must obey the inequality

$$M \geq \sqrt{Q^2 + P^2},$$

(6.27)

with extremality being attained when the inequality is saturated. The entropy is given by

$$S(M,Q,P) = \pi \left[2M^2 - Q^2 - P^2 + 2M \sqrt{M^2 - Q^2 - P^2}\right].$$

(6.28)
For super-additivity, one must have

\[ S(M + M', Q + Q', P + P') - S(M, Q, P) - S(M', Q', P') \geq 0, \tag{6.29} \]

where, as usual, we assume, without loss of generality, that the charges are all non-negative. Substituting (6.28) into this, we see that super-additivity is satisfied if

\[ 4MM' - 2QQ' - 2PP' + (M + M')\sqrt{(M + M')^2 - (Q + Q')^2 - (P + P')^2} \]
\[ -M\sqrt{M^2 - Q^2 - P^2} - M'\sqrt{M'^2 - Q'^2 - P'^2} \geq 0. \tag{6.30} \]

First, we note that the argument of the first square root is non-negative, since, after using (6.27) for the unprimed and primed quantities we have

\[ (M + M')^2 - (Q + Q')^2 - (P + P')^2 \geq 2(MM' - QQ' - PP'), \tag{6.31} \]

and since

\[ (MM')^2 - (QQ' + PP')^2 \geq (Q^2 + P^2)(Q'^2 + P'^2) - (QQ' + PP')^2 = (QP' - PQ')^2 \geq 0, \tag{6.32} \]

the non-negativity is proven.

Returning to the inequality (6.30) that we wish to establish, we see that the terms

\[ 4MM' - 2QQ' - 2PP' \]

are themselves certainly non-negative, since \(2MM' - 2QQ' - 2PP' \geq 0\) as we just demonstrated. The inequality is therefore established if we can show that

\[ M(\sqrt{(M + M')^2 - (Q + Q')^2 - (P + P')^2} - \sqrt{M^2 - Q^2 - P^2}) \geq 0, \tag{6.33} \]

together with the analogous expression with the primes and unprimed variables exchanged. The expression in parentheses is non-negative if

\[ (M + M')^2 - (Q + Q')^2 - (P + P')^2 - (M^2 - Q^2 - P^2) \geq 0 \]

is non-negative. After using (6.27) again we see that (6.34) is greater than or equal to

\[ 2(MM' - QQ' - PP'), \]

and we have already shown that this is non-negative. Thus the super-additivity property (6.29) is established for the dyonic Reissner-Nordström black holes.

### 6.4 A counterexample: The dyonic Kaluza-Klein black hole

Here, we demonstrate that dyonic Kaluza-Klein black holes that we discussed in section 6.4.6 provide counterexamples where entropy super-additivity breaks down. The phase
space for checking entropy super-additivity for these dyonic black holes is rather large, so we shall just focus on a restricted subspace within which we are able to exhibit violations. Specifically, we shall consider two black holes with the following \((M, Q, P)\) values:

\[
(P, 0, P) \quad \text{and} \quad (M', Q', 0),
\]

so the unprimed case is an extremal black hole with purely magnetic charge\(^\text{14}\) and the primed case is a (sub-extremal) black hole with purely electric charge. The masses and charges will be chosen so that the black hole with the summed mass and charges will be an extremal dyonic black hole, for which \(M_{\text{tot}} = (Q_{\text{tot}}^{2/3} + P_{\text{tot}}^{2/3})^{3/2}\). Thus

\[
M_{\text{tot}} = M + M' = P + M', \quad Q_{\text{tot}} = Q', \quad P_{\text{tot}} = P,
\]

with

\[
P + M' = \left( Q_{\text{tot}}^{2/3} + P_{\text{tot}}^{2/3} \right)^{3/2}.
\]

We shall characterise the ratio \(P/Q'\) by means of a constant \(x\), such that

\[
P = x^{3/2} Q'.
\]

We therefore have

\[
S = 0, \quad S' = \frac{\pi m^2}{\sqrt{1 - 2\beta_1}}, \quad S_{\text{tot}} = 8\pi x^{3/2} Q'^2,
\]

where the primed black hole defined above has metric parameters \(m\) and \(\beta_1\), with \(\beta_2 = 0\). This means that

\[
M' = \frac{(1 - \beta_1) m}{2(1 - 2\beta_1)}, \quad Q' = \frac{\sqrt{2\beta_1} m}{4(1 - 2\beta_1)},
\]

the entropy is given by

\[
S' = 8\pi \frac{(1 - 2\beta_1)^{3/2}}{\beta_1} Q'^2,
\]

and from (6.37) \(\beta_1\) is given in terms of \(x\) by

\[
\frac{2(1 - \beta_1)}{\sqrt{2\beta_1}} = (1 + x)^{3/2} - x^{3/2}.
\]

Let us first consider the case where \(x\) is very small, \(x = \epsilon^{3/2}\). From (6.42) we find at leading order \(\beta_1 = \frac{1}{2}(1 - \epsilon^{2/3})\), and so \(S' = 16\pi\epsilon Q'^2\). Thus we have

\[
S_{\text{tot}} - S - S' = 8\pi\epsilon Q'^2 - 0 - 16\pi\epsilon Q'^2 = -8\pi\epsilon Q'^2,
\]

\(^{14}\)Strictly speaking, the extremal configuration \((P, 0, P)\) is not a black hole, but rather a naked singularity. However, one can make an infinitesimal deformation away from extremality, to a configuration with parameters \((P + \delta, 0, P)\), and this will describe a genuine black hole. The results that we shall derive here, including the bound (6.35) on \(P\) versus \(Q'\) for obtaining violations of entropy super-additivity, are thus valid.
and so super-additivity does not hold in this region of the parameter space.

When \( x \) becomes larger, we find from numerical analysis that the ratio \( S_{\text{tot}}/(S+S') \),
which equals 2 in the limit as \( x \) goes to zero, falls monotonically. The ratio reaches unity
when \( S' = S_{\text{tot}} \), which implies
\[
x = (1 - 2\beta_1) \beta_1^2.
\]
(6.44)
Substituting into (6.42), we find that this occurs when \( \beta_1 = y^3 \) and \( y \) is the single real root
of the 9th-order polynomial
\[
17y^9 - 12y^8 + 42y^7 - 80y^6 + 39y^5 - 48y^4 - 12y^3 + 9y - 8 = 0.
\]
(6.45)
This root is given approximately by \( y = 0.698234 \), implying \( \beta_1 = 0.340411 \), and hence
\( x = 0.654681 \). Thus the parameter range where we find a violation of entropy super-
additivity is when
\[
0 < P < 0.529718 Q'.
\]
(6.46)
In other words, we have found super-additivity violation when we add an extremal purely
magnetic black hole and a non-extremal purely electric black hole, with parameters arranged
such that the “total” dyonic black hole is extremal, provided that the magnetic charge of
the original extremal black hole is sufficiently small in comparison to the electric charge of
the original non-extremal black hole.

We can give a more complete treatment by choosing two black holes with parameters
\((M, Q, P)\) of the form \((M, 0, P)\) and \((M', Q', 0)\), subject to the assumption that the total
black hole \((M_{\text{tot}}, Q_{\text{tot}}, P_{\text{tot}})\) is again extremal, obeying
\[
M_{\text{tot}} = [Q_{\text{tot}}^{2/3} + P_{\text{tot}}^{2/3}]^{3/2}.
\]
(6.47)
Thus
\[
M_{\text{tot}} = M + M', \quad Q_{\text{tot}} = Q', \quad P_{\text{tot}} = P.
\]
(6.48)
It is straightforward to show from the formulae in section 3.4.5 that for the individual black
holes that carry purely electric or purely magnetic charge, one has
\[
S = \sqrt{8 \pi \sqrt{M^4 - 20M^2 P^2 - 8P^4 + M(M^2 + 8P^2)^{3/2}}},
\]
\[
S' = \sqrt{8 \pi \sqrt{M'^4 - 20M'^2 Q'^2 - 8Q'^4 + M'(M'^2 + 8Q'^2)^{3/2}}}.
\]
(6.49)
One can then use (6.47), together with (6.48), to solve for \( M' \), and hence one can express
\( Y = S_{\text{tot}} - S - S' \), where \( S_{\text{tot}} = 8\pi P_{\text{tot}} Q_{\text{tot}} \), as a function of \( M, P \) and \( Q' \). One can then
explore the regions in the space of these parameters for which \( \mathcal{Y} \) is negative, signifying a violation of entropy super-additivity.

Of course, by continuity we expect that super-additivity violations will occur at least in some neighbourhood of the region found above when all the masses and charges are allowed to be adjusted. In other words, there will also be super-additivity violations if we consider cases where all three black holes are non-extremal, for appropriate ranges of the various masses and charges.

In our earlier remarks relating super-additivity to the Hawking area theorem, we assumed not only cosmic censorship but also that the coalescence of the two black holes was allowed physically. In the case of dyons, it should be recalled that they carry angular momentum, and moreover it is not localised within the event horizon. This, as suggested in [95], may lead to restrictions on what coalescences are allowed, and thus the non-super additivity of the entropy in this counter-example need not imply any conflict with Hawking’s area theorem. This is an interesting problem worthy of further study.

7 Conclusions and Future Prospects

We shall turn in this section to a consideration of the significance of negative surface gravities, and negative Gibbsian temperatures. We shall begin by recalling the most physically convincing argument that Schwarzschild black holes have a temperature, and hence entropy. This was given by Hawking [46,47], who coupled a collapsing black-hole metric in an asymptotically-flat spacetime to a quantum field, and showed that if the quantum field was initially in its vacuum state, then at late times it would emit particles with a thermal spectrum and temperature given by (1.3). The term “vacuum state” implied that it contained no particles having positive frequency with respect to the standard retarded time coordinate on past null infinity. This required his considering the behaviour of the quantum field as it passed through the time-dependent spacetime generated by the collapse. However, one may dispense with that region, and work with the exact vacuum Schwarzschild solution, obtaining the same result, by choosing an appropriate boundary condition for the quantum field on the past horizon. The appropriate boundary condition, which reproduces Hawking’s result, in the exterior region of the Schwarzschild solution, corresponds to requiring that the state contains no particles defined as having positive frequency with respect to a Kruskal null coordinate on the past horizon. This state is now referred to as the Unruh vacuum state. This is obviously not a state in thermal equilibrium. A different state, introduced by
Hartle and Hawking, is defined on the past horizon in the same way, but at past null infinity
the definition of positive frequency is such that it describes an ingoing flux of particles at
the Hawking temperature. Thus the Hartle-Hawking state should be regarded as a state in
thermal equilibrium.

The situation with two event horizons is more complicated. In order to discuss quantum
fields between the horizons, one needs to specify a notion of positive frequency on each past
horizon. If the region is static, and one interprets positive frequency as being with respect
to a local Kruskal coordinate on the horizons, the resulting quantum state will describe
thermal radiation entering the static region at temperatures given by $\frac{1}{2\pi} |\kappa_\pm|$. This is not
a state in thermal equilibrium. If the region between the two horizons is not static, as for
example in the Reissner-Nordström solution, one may define a similar state which would also
not be in thermal equilibrium. If, on the other hand, one considers the static region behind
the inner horizon in the Reissner-Nordström, one needs to specify boundary conditions on
the singularity at $r = 0$. If one chose the notation of positive frequency on the past inner
horizon, then whatever boundary conditions were chosen on the singularity, the quantum
state would contain radiation coming from the inner horizon with a temperature $\frac{1}{2\pi} |\kappa_\pm|$. Thus if we adopt this procedure, we see in all cases that the temperature we associate with
particles coming from the horizons is given by the absolute value of the surface gravity,
divided by $2\pi$.

An alternative way of establishing the temperature and entropy of an asymptotically-
flat black hole is to follow the procedure of [77, 96], in which one analytically continues the
metric to imaginary time, and discovers that the metric is periodic in imaginary time with
a period given by $2\pi/|\kappa|$, which is what one expects for a state in thermal equilibrium at
temperature $\frac{1}{2\pi} |\kappa_\pm|$. Of course, the period itself can have either sign, but the quantum
state would not necessarily exist if one chose a negative sign for the temperature. This
procedure will work when one has a single horizon, including an asymptotically anti-de
Sitter spacetime [74, 75]. However, this procedure will not work for a spacetime with two
horizons having differing values of $|\kappa|$. The conclusion seems to be that classically, the
sign of the temperature can only be determined by appealing to the first law, and this
provides us with a Gibbsian temperature. Quantum mechanically, which seems to be the
only physically reliable argument provided one is prepared to contemplate non-equilibrium
situations, the temperature should be taken to be positive. In other words, the temperature
is not uniquely defined by the metric, a conclusion also reached in [25].

The original suggestion that inner horizons should be assigned a negative temperature [1]
was based not quantum field theoretic considerations, but rather on a consideration of quantum mechanical systems, such as spin systems, exhibiting population inversion \[58\]. Thus one might regard the total energy of a black hole as receiving contributions both from the outer and inner horizons. The inner system would then be thought of as the analogue of a spin system. This viewpoint was supported by the existence for the Kerr-Newman black hole of the modified Smarr formula (3.34), and its variation, which may be written as

\[
dM = \frac{1}{2} (T_+ dS_+ + \Omega_+ dJ + \Phi_+ dQ) + \frac{1}{2} (T_- dS_- + \Omega_+ dJ + \Phi_+ dQ) . \tag{7.1}
\]

As we saw, these formulae generalise to the case of STU black holes with four electric charges. The addition of electric charges, which were not included in the discussion in \([1]\), suggest that the posited spin system inverted population should be supplemented by the inclusion of charged states.

In the case of four-dimensional STU black holes, the generalisation of equation (7.1) may be rewritten in terms of the left-moving and right-moving sectors (see (3.69)) as

\[
dM = (T_L dS_L + \Omega_L dJ + \Phi_L dQ^i + \Psi_{L,i} dP^i) + (T_R dS_R + \Omega_R dJ + \Phi_R dQ^i + \Psi_{R,i} dP^i) , \tag{7.2}
\]

with each sector contributing equally to \(dM\). In contrast to the proposal in \([1]\), which attempted to give a microscopic interpretation to the negative temperature on the inner horizon, here the left-moving and right-moving sectors both have positive temperatures, consistent with the proposed microscopic interpretation in terms of D-brane states \([11, 62]\). An analogous interpretation for five-dimensional STU black holes has also been given \([16]\).

This paper has been concerned exclusively with time-independent solutions; we have not discussed what happens to inner horizons when perturbations are considered. There is a widespread belief that in classical general relativity, generic perturbations will render Cauchy horizons, of the sort one finds inside black holes, singular. This is referred to as the Cosmic Censorship Hypothesis. There are various forms of this hypothesis, and the literature is at present rather inconclusive. A recent discussion can be found in \([97]\). Our motivation is largely quantum mechanical, and the relevance of these classical results to a full quantum gravitational treatment is unclear.

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A Carter-Penrose Diagram for Two Horizons

In this appendix, we summarise some facts about the Carter-Penrose diagram of asymptotically-flat spherically symmetric spacetimes with an inner and outer horizon. Consider a suitable metric of the form
\[ ds^2 = -A(r)dt^2 + \frac{dr^2}{f^2(r)A(r)} + R(r)^2d\Omega^2. \]  
(A.1)

Introducing an advanced time coordinate \( v \) by defining
\[ dv = dt + \frac{dr}{fA}, \] 
(A.2)

the metric takes the Eddington-Finkelstein form
\[ ds^2 = -Adv^2 + 2f^{-1}drdv + R^2d\Omega^2. \]  
(A.3)

The metric will be regular as long as \( A, f \) and \( R^2 \) are real, bounded, and twice differentiable, and in addition \( f \) and \( R \) are non-zero. We may take \( f \), without loss of generality, to be positive. In particular, the metric is well-behaved regardless of whether \( A \) is positive, zero or negative. Asymptotic flatness requires that \( A \) and \( f \) tend to 1 as \( R^2 \) tends to infinity. In the cases we shall consider, \( R \) tends to \( r \) at infinity. We shall assume that \( A \) is positive in the interval \( r_+ < r \leq \infty \), and negative in the the interval \( r_- < r < r_+ \), and that it vanishes on the outer horizon \( r = r_+ \) and the inner horizon \( r = r_- \). We shall also assume that \( A \) has a smooth positive extension for values of \( r < r_- \). The Killing vector \( K = \partial/\partial v \) is thus timelike for \( r_+ < r < \infty \), lightlike at \( r = r_+ \), spacelike for \( r_- < r < r_+ \), lightlike at \( r = r_- \) and timelike for \( r < r_- \). It becomes lightlike as \( v \) tends to \( \pm \infty \), and also as \( r \) tends to infinity.

If \( r_+ < r < \infty \), then as \( v \) tends to \( +\infty \) we obtain future null infinity, \( I^+ \). For \( v \) instead tending to \( -\infty \), we obtain past null infinity \( I^- \). As \( v \) tends to \( -\infty \) and \( r \) tends to \( r_+ \) we obtain the past null horizon. The Killing vector \( K \) is future-directed inside and on the boundary of this region. The inner region is bounded by a past Cauchy horizon at \( v = -\infty \) and \( r = r_+ \), and a future Cauchy horizon at \( v = +\infty \) and \( r = r_- \). It has a further boundary on the inner horizon at \( r = r_- \), with \( -\infty < v < +\infty \). Thus the Killing vector \( K \) is future directed both on this inner horizon and on the outer horizon.
If one looks at radial geodesics in this spacetime, there are two conserved quantities \( p_v \) and \( k \), where

\[
\dot{p}_v = A \ddot{v} - f^{-1} \dot{r}, \quad -A \dot{v}^2 + 2 f^{-1} \dot{r} \dot{v} = -k, \tag{A.4}
\]

and a dot denotes a derivative with respect to an affine parameter \( \lambda \). Thus radially-infalling geodesics obey

\[
\dot{r} = -f \sqrt{p_v^2 - kA}, \tag{A.5}
\]

with \( k > 0 \) and \( p_v^2 > k \) for timelike geodesics that originate at large \( r \). The constant \( p_v \) is positive. The infalling particle passes through the outer and the inner horizons before reaching a turning point at a radius \( \bar{r} < r_- \) at which \( p_v^2 = kA(\bar{r}) \).

Solving for \( \dot{v} \) one finds

\[
\dot{v} = \frac{p_v - \sqrt{p_v^2 - kA}}{A}, \tag{A.6}
\]

and so

\[
\frac{dv}{dr} = \frac{1}{f A} \left[ 1 - \frac{p_v}{\sqrt{p_v^2 - kA}} \right]. \tag{A.7}
\]

Thus one finds that \( \dot{v}, dv/dr \) and \( v \) all remain finite as the particle falls in from infinity to \( \bar{r} \). Note that \( \dot{v} \) is always positive.

In conclusion, we note that the Killing vector \( K = \partial/\partial v \) is future directed and lightlike on both the future event horizon of the exterior region, \( r = r_+ \) with \(-\infty < v < +\infty\), and on the inner horizon, \( r = r_- \) with \(-\infty < v < +\infty\).

For the four-charge STU black holes considered in this paper, the situation when they are non-rotating is qualitatively similar to that for the Reissner-Nordström solution. The metric takes the form

\[
ds^2 = -(H_1 H_2 H_3 H_4)^{-1/2} W \, dt^2 + (H_1 H_2 H_3 H_4)^{1/2} \left( W^{-1} \, dr^2 + r^2 d\Omega^2 \right), \tag{A.8}
\]

where

\[
H_i = 1 + \frac{\mu \sinh^2 \delta_i}{r}, \quad W = 1 - \frac{\mu}{r}. \tag{A.9}
\]

The outer horizon is located at \( r_+ = \mu \), and the inner horizon at \( r_- = 0 \). There are curvature singularities at the four locations \( r = -\mu \sinh^2 \delta_i \), and the Carter-Penrose diagram will be similar to that for Reissner-Nordström, with the curvature singularity in the diagram occurring at the least negative of the four locations.
B STU Supergravity

The Lagrangian of the bosonic sector of four-dimensional ungauged STU supergravity can be written in the relatively simple form

\[
    \mathcal{L}_4 = R \ast 1 - \frac{1}{2} e\phi \ast d\phi_i \wedge d\phi_i - \frac{1}{2} e^{2\phi} \ast d\chi_i \wedge d\chi_i - \frac{1}{2} e^{-\phi} \left( e^{\phi_2 - \phi_3} \ast F_{(2)}^1 \wedge F_{(2)}^1 + e^{\phi_2 + \phi_3} \ast F_{(2)}^2 \wedge F_{(2)}^2 \right) \\
    - \chi_1 \left( F_{(2)}^1 \wedge F_{(2)}^1 + F_{(2)}^2 \wedge F_{(2)}^2 \right),
\]

(B.1)

where the index \( i \) labelling the dilatons \( \phi_i \) and axions \( \chi_i \) ranges over \( 1 \leq i \leq 3 \). The four field strengths can be written in terms of potentials as

\[
    F_{(2)}^1 = dA_{(1)1} - \chi_2 dA_{(1)}^2, \\
    F_{(2)}^2 = dA_{(1)2} + \chi_2 dA_{(1)}^1 - \chi_3 dA_{(1)1} + \chi_2 \chi_3 dA_{(1)}, \\
    F_{(2)}^1 = dA_{(1)}^1 + \chi_3 dA_{(1)}^2, \\
    F_{(2)}^2 = dA_{(1)}^2.
\]

(B.2)

The field strengths here are not in the same duality frame as the one we have assumed in our discussions in this paper however. To convert from (B.1) and (B.2) to the frame we are using, one would need to dualise the field strengths \( F_{(2)}^1 \) and \( F_{(2)}^2 \), and if then written explicitly, the resulting Lagrangian would be rather cumbersome. Alternatively, one could simply exchange the roles of the electric and magnetic charges for the field strengths \( F_{(2)}^1 \) and \( F_{(2)}^2 \), and work with (B.1) without performing any dualisations. For example, the 4-charge black hole solutions that we refer to in this paper as having four electric charges would, as solutions in terms of the fields in (B.1), instead comprise two electric and two magnetic charges. (As for example, in the presentation of these solution in [64].)

References

[1] A. Curir, Spin entropy of a rotating black hole, Il Nuovo Cimento B 52 (1979) 262-266.

[2] A. Curir and M. Francaviglia, Spin thermodynamics of a Kerr black hole, Il Nuovo Cimento B 52 (1979) 165-176.

[3] A. Curir and M. Francaviglia, On certain transformations for black-holes energetics, Atti Acad Naz. Lincei Ser VIII 61 (1976) 448.
[4] M. Calvani and M. Francaviglia, *Irreducible mass, unincreasable angular momentum and isoareal transformations for black hole physics*, Acta Phys Polonica B 9 (1978) 11-14.

[5] A. Curir and M. Francaviglia, *Isoareal transformations of the Kerr-Newman black holes*, Acta Phys Polonica B 9 (1978) 3-10.

[6] A. Curir, *On the energy emission by a Kerr black hole in the superradiant range*, Phys. Lett. 161B (1985) 310. doi:10.1016/0370-2693(85)90768-3

[7] A. Curir, *On the generalized second law for rotating black holes*, Phys. Lett. B 176 (1986) 26. doi:10.1016/0370-2693(86)90918-4

[8] A. Curir, *Convexity of thermodynamic functions and thermodynamics of rotating black holes*, Europhys. Lett. 9 (1989) 609-612.

[9] A. Curir, *Entropic forces in a Kerr geometry: a link with rotational properties*, Commun. Theor. Phys. 55 (2011) 594. doi:10.1088/0253-6102/55/4/12

[10] I. Okamoto and O. Kaburaki, *The inner-horizon thermodynamics of Kerr black holes*, Mon. Not R. Ast. Soc. 255 (1992), 539 (1992)

[11] M. Cvetiˇ c and D. Youm, *Entropy of non-extreme charged rotating black holes in string theory*, Phys. Rev. D 54, 2612 (1996) doi:10.1103/PhysRevD.54.2612 [hep-th/9603147].

[12] M. Cvetiˇ c and D. Youm, *General rotating five dimensional black holes of toroidally compactified heterotic string*, Nucl. Phys. B476, 118 (1996) doi:10.1016/0550-3213(96)00355-0 [hep-th/9603100].

[13] M. Cvetiˇ c and A.A. Tseytlin, *Solitonic strings and BPS saturated dyonic black holes*, Phys. Rev. D 53, 5619 (1996) Erratum: [Phys. Rev. D 55, 3907 (1997)] doi:10.1103/PhysRevD.53.5619, 10.1103/PhysRevD.55.3907 [hep-th/9512031].

[14] F. Larsen, *A string model of black hole microstates*, Phys. Rev. D56 (1997) 1005 doi:10.1103/PhysRevD.56.1005 [hep-th/9702153].

[15] M. Cvetiˇ c and F. Larsen, *General rotating black holes in string theory: Grey body factors and event horizons*, Phys. Rev. D 56, 4994 (1997) doi:10.1103/PhysRevD.56.4994 [hep-th/9705192].
[16] M. Cvetić and F. Larsen, *Grey body factors for rotating black holes in four-dimensions*, Nucl. Phys. B 506, 107 (1997) doi:10.1016/S0550-3213(97)00541-5 [hep-th/9706071].

[17] A. Castro, A. Maloney and A. Strominger, *Hidden conformal symmetry of the Kerr black hole*, Phys. Rev. D 82 (2010) 024008 doi:10.1103/PhysRevD.82.024008 [arXiv:1004.0996 [hep-th]].

[18] M. Cvetič, G.W. Gibbons and C.N. Pope, *Universal area product formulae for rotating and charged black holes in four and higher dimensions*, Phys. Rev. Lett. 106 (2011) 121301 doi:10.1103/PhysRevLett.106.121301 [arXiv:1011.0008 [hep-th]].

[19] M. Visser, *Area products for stationary black hole horizons*, Phys. Rev. D 88 (2013) no.4, 044014 doi:10.1103/PhysRevD.88.044014 [arXiv:1205.6814 [hep-th]].

[20] M. Cvetič, H. Lü and C.N. Pope, *Entropy-product rules for charged rotating black holes*, Phys. Rev. D 88 (2013) 044046 doi:10.1103/PhysRevD.88.044046 [arXiv:1306.4522 [hep-th]].

[21] D.N. Page and A.A. Shoom, *Universal area product for black holes: A heuristic argument*, Phys. Rev. D 92 (2015) no.4, 044039 doi:10.1103/PhysRevD.92.044039 [arXiv:1504.05581 [hep-th]].

[22] A. Castro and M.J. Rodriguez, *Universal properties and the first law of black hole inner mechanics*, Phys. Rev. D 86 (2012) 024008 doi:10.1103/PhysRevD.86.024008 [arXiv:1204.1284 [hep-th]].

[23] S.Q. Wu, *New formulations of first law of black hole thermodynamics: A “Stringy” analogy*, Phys. Lett. B 608, 251 (2005) doi:10.1016/j.physletb.2005.01.018 [gr-qc/0405029].

[24] Y.H. Wei, *Effective first law of thermodynamics of black holes with two horizons*, Chin. Phys. B 18, 821 (2009). doi:10.1088/1674-1056/18/2/068

[25] M.I. Park, *Can Hawking temperatures be negative?*, Phys. Lett. B 663, 259 (2008) doi:10.1016/j.physletb.2008.04.009 [hep-th/0610140].

[26] M.I. Park, *Thermodynamics of exotic black holes, negative temperature, and Bekenstein-Hawking entropy*, Phys. Lett. B 647, 472 (2007) doi:10.1016/j.physletb.2007.02.036 [hep-th/0602114].
M.I. Park, *Note on the area theorem*, Int. J. Mod. Phys. A **24S1**, 3111 (2009). doi:10.1142/S0217751X09044267

G.W. Gibbons and S.W. Hawking, *Cosmological wvnt horizons, thermodynamics, and particle creation*, Phys. Rev. D **15** (1977) 2738. doi:10.1103/PhysRevD.15.2738

A.V. Frolov and L. Kofman, *Inflation and de Sitter thermodynamics*, JCAP **0305** (2003) 009 doi:10.1088/1475-7516/2003/05/009 [hep-th/0212327].

B.P. Dolan, D. Kastor, D. Kubiznak, R.B. Mann and J. Traschen, *Thermodynamic volumes and isoperimetric inequalities for de Sitter black holes*, Phys. Rev. D **87** (2013) no.10, 104017 doi:10.1103/PhysRevD.87.104017

R. Gregory, D. Kastor and J. Traschen, *black hole thermodynamics with dynamical lambda*, arXiv:1707.06586 [hep-th].

R. Bousso and S.W. Hawking, *Pair creation of black holes during inflation*, Phys. Rev. D **54**, 6312 (1996), doi:10.1103/PhysRevD.54.6312, gr-qc/9606052.

J.W. Gibbs, *On the equilibrium of heterogeneous substances*, Transactions of the Connecticut Academy of Arts and Sciences, **3** (1874-1878), pages 104-248 and 343-524.

D. Christodoulou, *Reversible and irreversible transformations in black hole physics*, Phys. Rev. Lett. **25** (1970) 1596-1597, [http://dx.doi.org/10.1103/PhysRevLett.25.1596](http://dx.doi.org/10.1103/PhysRevLett.25.1596)

S.W. Hawking, *Gravitational radiation from colliding black holes*, Phys. Rev. Lett. **26** (1971) 1344. doi:10.1103/PhysRevLett.26.1344

S.W. Hawking, *Black holes in general relativity*, Commun. Math. Phys. **25** (1972) 152. doi:10.1007/BF01877517

D. Christodoulou and R. Ruffini, *Reversible transformations of a charged black hole*, Phys. Rev. D **4** (1971) 3552. doi:10.1103/PhysRevD.4.3552

L. Smarr, *Mass formula for Kerr black holes*, Phys. Rev. Lett. **30** (1973) 71 Erratum: [Phys. Rev. Lett. **30** (1973) 521]. doi:10.1103/PhysRevLett.30.71

J.M. Bardeen, B. Carter and S.W. Hawking, *The four laws of black hole mechanics*, Commun. Math. Phys. **31**, 161 (1973). doi:10.1007/BF01645742

S.W. Hawking, *A variational principle for black holes*, Commun. Math. Phys. **33**, 323 (1973). doi:10.1007/BF01646744
[41] S. Dain, *A variational principle for stationary, axisymmetric solutions of Einstein’s equations*, Class. Quant. Grav. 23, 6857 (2006), doi:10.1088/0264-9381/23/23/016, gr-qc/0508061.

[42] O.S. An, M. Cvetič and I. Papadimitriou, *Black hole thermodynamics from a variational principle: Asymptotically conical backgrounds*, JHEP 1603, 086 (2016), doi:10.1007/JHEP03(2016)086, arXiv:1602.01508 [hep-th].

[43] J.M. Bardeen, *A variational principle for rotating stars in general relativity*, Astrophys. J. 162, 71 (1970). doi:10.1086/150635

[44] Colloquium at Princeton University (Dec. 1971).

[45] J.D. Bekenstein, *Black holes and the second law*, Lett. Nuovo Cim. 4, 737 (1972). doi:10.1007/BF02757029

[46] S.W. Hawking, *Black hole explosions*, Nature 248, 30 (1974). doi:10.1038/248030a0

[47] S.W. Hawking, *Particle creation by black holes*, Commun. Math. Phys. 43, 199 (1975), Erratum: [Commun. Math. Phys. 46, 206 (1976)]. doi:10.1007/BF02345020, 10.1007/BF01608497

[48] G.W. Gibbons, R. Kallosh and B. Kol, *Moduli, scalar charges, and the first law of black hole thermodynamics*, Phys. Rev. Lett. 77 (1996) 4992 doi:10.1103/PhysRevLett.77.4992 [hep-th/9607108].

[49] F. Weinhold, *Metric geometry of equilibrium thermodynamics*, J. Chem. Phys. 63 (1975) 2479, doi:10.1063/1.431689.

[50] D.N. Page, *Thermodynamic paradoxes*, Physics Today 30, 1, 11 (1977), doi:10.1063/1.3037360.

[51] H. Liu, H. Lü, M. Luo and K.N. Shao, *Thermodynamical metrics and black hole phase transitions*, JHEP 1012, 054 (2010), doi:10.1007/JHEP12(2010)054 [arXiv:1008.4482 [hep-th]].

[52] S. Ferrara, G.W. Gibbons and R. Kallosh, *Black holes and critical points in moduli space*, Nucl. Phys. B 500 (1997) 75 doi:10.1016/S0550-3213(97)00324-6 hep-th/9702103.
[53] J. Aman, I. Bengtsson and N. Pidokrajt, *Thermodynamic metrics and black hole physics*, Entropy 17 (2015) 6503 doi:10.3390/e17096503 [arXiv:1507.06097 [gr-qc]].

[54] P.C.W. Davies, *Thermodynamics of black holes*, Proc. Roy. Soc. Lond. A 353 (1977) 499. doi:10.1098/rspa.1977.0047

[55] R. Monteiro and J.E. Santos, *Negative modes and the thermodynamics of Reissner-Nordström black holes*, Phys. Rev. D 79 (2009) 064006 doi:10.1103/PhysRevD.79.064006 [arXiv:0812.1767 [gr-qc]].

[56] S.A.H. Mansoori, B. Mirza and M. Fazel, *Hessian matrix, specific heats, Nambu brackets, and thermodynamic geometry*, JHEP 1504, 115 (2015) doi:10.1007/JHEP04(2015)115 [arXiv:1411.2582 [gr-qc]].

[57] H. Huang, X.H. Feng and H. Lü, “Holographic complexity and two identities of action growth,” Phys. Lett. B 769, 357 (2017) doi:10.1016/j.physletb.2017.04.011 [arXiv:1611.02321 [hep-th]].

[58] N.F. Ramsey, *Thermodynamics & statistical mechanics at negative absolute temperatures*, Phys. Rev. 103 (1956) 20-28.

[59] M.A. Abramowicz, private communication.

[60] G.W. Gibbons and C.M. Hull, *A Bogomolnyi bound for general relativity and solitons in N = 2 supergravity*, Phys. Lett. 109B (1982) 190. doi:10.1016/0370-2693(82)90751-1

[61] D.D.K. Chow and G. Compère, *Seed for general rotating non-extremal black holes of \( \mathcal{N} = 8 \) supergravity*, Class. Quant. Grav. 31 (2014) 022001 doi:10.1088/0264-9381/31/2/022001 [arXiv:1310.1925 [hep-th]].

[62] G.T. Horowitz, D.A. Lowe and J.M. Maldacena, *Statistical entropy of nonextremal four-dimensional black holes and U duality*, Phys. Rev. Lett. 77, 430 (1996) doi:10.1103/PhysRevLett.77.430 [hep-th/9603195].

[63] G. Sárosi, *Entropy of nonextremal STU black holes: The F-invariant unveiled*, Phys. Rev. D 93 (2016) no.2, 024036 doi:10.1103/PhysRevD.93.024036 [arXiv:1508.00667 [hep-th]].

[64] Z.-W. Chong, M. Cvetič, H. Lü and C.N. Pope, *Charged rotating black holes in four-dimensional gauged and ungauged supergravities*, Nucl. Phys. B 717, 246 (2005) doi:10.1016/j.nuclphysb.2005.03.034 [hep-th/0411045].
[65] H. Lü, Y. Pang and C.N. Pope, *AdS dyonic black hole and its thermodynamics*, JHEP **1311**, 033 (2013) doi:10.1007/JHEP11(2013)033 [arXiv:1307.6243 [hep-th]].

[66] G. W. Gibbons, *Antigravitating black hole solitons with scalar hair in N = 4 supergravity*, Nucl. Phys. B **207**, 337 (1982). doi:10.1016/0550-3213(82)90170-5

[67] G.W. Gibbons and K.i. Maeda, *Black holes and membranes in higher dimensional theories with dilaton fields*, Nucl. Phys. B **298** (1988) 741. doi:10.1016/0550-3213(88)90006-5

[68] D. Garfinkle, G. T. Horowitz and A. Strominger, *Charged black holes in string theory*, Phys. Rev. D **43**, 3140 (1991), Erratum: [Phys. Rev. D 45, 3888 (1992)]. doi:10.1103/PhysRevD.43.3140, 10.1103/PhysRevD.45.3888

[69] G. W. Gibbons, D. Kastor, L. A. J. London, P. K. Townsend and J. H. Traschen, *Supersymmetric selfgravitating solitons*, Nucl. Phys. B **416**, 850 (1994), doi:10.1016/0550-3213(94)90558-4, [hep-th/9310118].

[70] J. E. Aman, N. Pidokrajt and J. Ward, *On geometro-thermodynamics of dilaton black holes*, EAS Publ. Ser. **30**, 279 (2008), doi:10.1051/eas:0830044. [arXiv:0711.2201 [hep-th]].

[71] H. Lü, *Charged dilatonic ads black holes and magnetic AdS$_{D-2}$ × $R^2$ vacua*, JHEP **1309**, 112 (2013) doi:10.1007/JHEP09(2013)112 [arXiv:1306.2386 [hep-th]].

[72] D. Kastor, S. Ray and J. Traschen, *Enthalpy and the mechanics of AdS black holes*, Class. Quant. Grav. **26** (2009) 195011 doi:10.1088/0264-9381/26/19/195011 [arXiv:0904.2765 [hep-th]].

[73] M. Cvetić, G.W. Gibbons, D. Kubiznak and C.N. Pope, *Black Hole Enthalpy and an Entropy Inequality for the Thermodynamic Volume*, Phys. Rev. D **84** (2011) 024037 doi:10.1103/PhysRevD.84.024037 [arXiv:1012.2888 [hep-th]].

[74] S.W. Hawking and D.N. Page, *Thermodynamics of black holes in anti-de sitter space*, Commun. Math. Phys. **87** (1983) 577. doi:10.1007/BF01208266

[75] E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. **2** (1998) 253 [hep-th/9802150].
[76] T. Prestidge, *Dynamic and thermodynamic stability and negative modes in Schwarzschild-anti-de Sitter*, Phys. Rev. D **61** (2000) 084002 doi:10.1103/PhysRevD.61.084002 [hep-th/9907163].

[77] G.W. Gibbons and S.W. Hawking, *Action integrals and partition functions in quantum gravity*, Phys. Rev. D **15** (1977) 2752. doi:10.1103/PhysRevD.15.2752

[78] F. Mellor and I. Moss, *Stability of black holes in de Sitter space*, Phys. Rev. D **41** (1990) 403. doi:10.1103/PhysRevD.41.403

[79] F. Mellor and I. Moss, *Black holes and gravitational instantons*, Class. Quant. Grav. **6** (1989) 1379. doi:10.1088/0264-9381/6/10/008

[80] F. Mellor and I. Moss, *Black holes and quantum wormholes* Phys. Lett. B **222** (1989) 361. doi:10.1016/0370-2693(89)90324-9

[81] I. Moss, *Journeys beyond the Cauchy horizon*, NCL-89-TP6.

[82] P.C.W. Davies and I.G. Moss, *Journey through a black hole*, Class. Quant. Grav. **6** (1989) L173. doi:10.1088/0264-9381/6/9/004

[83] P.C.W. Davies, *Thermodynamic phase transitions of Kerr-Newman black holes in de Sitter space*, Class. Quant. Grav. **6** (1989) 1909. doi:10.1088/0264-9381/6/12/018

[84] M.M. Caldarelli, G. Cognola and D. Klemm, *Thermodynamics of Kerr-Newman-AdS black holes and conformal field theories*, Class. Quant. Grav. **17** (2000) 399 doi:10.1088/0264-9381/17/2/310

[85] S.Q. Wu, *General nonextremal rotating charged AdS black holes in five-dimensional U(1)3 gauged supergravity: A simple construction method*, Phys. Lett. B **707**, 286 (2012) doi:10.1016/j.physletb.2011.12.031 [arXiv:1108.4159 [hep-th]].

[86] T. Birkandan and M. Cvetič, *Wave equation for the Wu black hole*, JHEP **1409**, 121 (2014) doi:10.1007/JHEP09(2014)121

[87] D. Lynden-Bell, *Negative specific heat in astronomy, physics and chemistry*, Physica A **263** (1999) 293 doi:10.1016/S0378-4371(98)00518-4 [cond-mat/9812172].

[88] P.T. Landsberg, *The fourth law of thermodynamics*, Nature **238** (1972) 229.
[89] P.T. Landsberg, *Thermodynamics and black holes*, in Black Hole Physics, edited by Venzo de Sabbata and Zhenjiu Zhang. Dordrecht, Netherlands, Kluwer Academic, (1992) 99-146 (NATO Advanced Study Institute (ASI), Series C: Mathematics and Physical Sciences, v. 364)

[90] D.A. Rasheed, *Nonlinear electrodynamics: Zeroth and first laws of black hole mechanics*, [hep-th/9702087](https://arxiv.org/abs/hep-th/9702087).

[91] P.T. Landsberg and D. Tranah, *Entropies need not be convex*, Phys. Lett. A **78**(1980) 219-220.

[92] P.T. Landsberg and D. Tranah, *Thermodynamics of non-extensive Entropies I*, Collective Phenomena **3** (1980) 73-80.

[93] D. Tranah and P.T. Landsberg, *Thermodynamics of non-extensive entropies II*, Collective Phenomena **3** (1980) 81-88.

[94] P.T. Landsberg, *Is Equilibrium Always an Entropy Maximum*, J. Statistical Physics **35** (1984) 159-169.

[95] F. Larsen, *Rotating Kaluza-Klein black holes*, Nucl. Phys. B **575**, 211 (2000) doi:10.1016/S0550-3213(00)00064-X [hep-th/9909102](https://arxiv.org/abs/hep-th/9909102).

[96] J.B. Hartle and S.W. Hawking, *Path integral derivation of black hole radiance*, Phys. Rev. D **13**, 2188 (1976). doi:10.1103/PhysRevD.13.2188.

[97] M. Dafermos and J. Luk, *The interior of dynamical vacuum black holes I: The $C^0$-stability of the Kerr Cauchy horizon*, [arXiv:1710.01722](https://arxiv.org/abs/1710.01722) [gr-qc].