Recovery of a piecewise constant lower coefficient of an elliptic equation

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Аннотация. We propose a new algorithm for the recovery of a piecewise constant lower coefficient of an elliptic problem. The inverse problem is reduced to a shape reconstruction problem. The proposed algorithm is based on the minimization of a cost functional where a control function is the right-hand side of an auxiliary elliptic equation for a level set representation of unknown shape. Numerical implementation is based on the finite element method and the open-source computing platform FEniCS and dolfin-adjoint. The performance of the algorithm is demonstrated on computationally simulated data.

Keywords: coefficient inverse problem, elliptic equation, finite element method

1. Introduction

Inverse problems of identifying unknown coefficients and the right-hand side of elliptic equations has been studied extensively [1, 2, 3, 4, 5]. Such problems arise when determining the internal structure of objects using measurements made on their boundaries. Coefficient inverse problems are often not well-posed and special algorithms are needed to numerically solve such problems. Also, the stability and uniqueness of their solution are crucial for development of stable numerical algorithms. The main results on these fundamental questions for multi-dimensional elliptic equations can be found in the review paper [6].

In this work, we focus on the recovery of a lower coefficient of two-dimensional elliptic equation. The uniqueness for this problem has been considered in many works [7, 8, 9, 10]. In addition, we assume that the desired coefficient is piecewise-constant and has only one of two possible values, which are known a priori. Therefore, our inverse problem is reduced to a shape reconstruction problem [11].

The general approach of solution of coefficient inverse problems involves minimization of an objective functional with or without some regularization term using iterative iterative algorithms [12, 13]. In [10] the lower coefficient of elliptic equations is recovered by minimizing a discrepancy functional using gradient methods. The identification of lower coefficient in elliptic equation as a shape reconstruction problem was solved using Newton and Landweber algorithms [14]. In recent years, the level set methods were extensively employed to solve shape recovery problems [15, 16]. In [17] the level set approach is applied to identify a piecewise constant conductivity coefficient in electric impedance tomography [17].

In this work, a new algorithm for identification of piecewise-constant lower coefficient of elliptic equation is proposed. The main idea is similar to the level set method and consists in solving
an auxiliary elliptic equation to find a representation of unknown shape. The algorithm is based on minimization of a cost functional with a control function taken as the right-hand side of the auxiliary equation. The paper is organised as follows. In the next section, we introduce the inverse problem. Details of reconstruction algorithm and finite element approximation of equations are discussed in section 3. The numerical verification of the proposed algorithm is presented in section 4.

2. Problem statement
Let $\Omega \subset \mathbb{R}^2$ be the bounded domain with sufficiently smooth boundary $\partial \Omega$, and $x = (x_1, x_2) \in \Omega$. We consider the following boundary value problem for second order elliptic equation

$$-\Delta u + c(x)u = 0, \quad x \in \Omega,$$

$$u(x) = \phi(x), \quad x \in \partial \Omega. \quad (2)$$

The lower coefficient $c(x)$ is defined as

$$c(x) = c_0 + c_1 \chi_D(x), \quad (3)$$

where $c_0, c_1 \geq 0$ are given constants and $\chi_D(x)$ is the characteristic function of the subdomain $D \subset \Omega$:

$$\chi_D(x) = \begin{cases} 1, & x \in D, \\ 0, & x \in \Omega \setminus D. \end{cases} \quad (4)$$

In this case, the inverse problem of the recovery of the coefficient $c(x)$ reduces to the reconstruction of the shape of the subdomain $D$ using only single measurement of the Dirichlet boundary values $\phi(x)$ and the additional Neumann boundary values

$$\frac{\partial u}{\partial n} = g(x), \quad x \in \partial \Omega, \quad (5)$$

where $n$ denotes the unit outer normal to $\partial \Omega$. Inverse problems under such conditions were investigated, for example, in [13].

3. Reconstruction algorithm
First, we introduce an auxiliary function $q(x)$, which describes $D$ as follows

$$\begin{cases} q(x) \geq 0, \quad x \in D, \\ q(x) < 0, \quad x \in \Omega \setminus D. \end{cases} \quad (6)$$

The function $q(x)$ is similar to a level set function in the well-known level set methods [16]. To find the function $q(x)$ we solve the following elliptic equation

$$-\gamma \Delta q + q = f(x), \quad x \in \Omega,$$

$$q = 0, \quad x \in \partial \Omega, \quad (7)$$

where $\gamma = \text{const} > 0$ is the parameter. When we have the distribution of the function $q(x)$ we can define the coefficient $c(x)$

$$c(x) = c_0 + c_1 H(q) \quad (8)$$

with the Heaviside function $H(q)$ defined as

$$H(q) = \begin{cases} 1, & q(x) \geq 0, \\ 0, & q(x) < 0. \end{cases} \quad (9)$$
Our reconstruction algorithm is based on the classical approach with the minimization of the cost functional $J$ defined as

$$J(f) = \int_{\partial \Omega} \left( \frac{\partial u}{\partial n}(x, f) - g(x) \right)^2 \, ds.$$  \hfill (10)

Note that the value of functional $J$ is controlled by the right-hand side $f(x)$ of equation (7) rather than the desired coefficient $c(x)$.

Before delving into steps of reconstruction algorithm, we discuss some details of the numerical implementation. For discretization of our equations the finite element method is used. First, we define the following functional spaces:

$$V = \{ v \in H^1(\Omega) : v(x) = 0, \, x \in \partial \Omega \}, \quad Q = \{ v \in H^1(\Omega) : v(x) = \phi(x), \, x \in \partial \Omega \}$$

where $H^1(\Omega)$ is Sobolev space. We obtain a variational problem for (1), (2) by multiplying equation (1) by a test function $v \in V$ and integrating by parts: find $u \in Q$ such that

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\Omega} c u v \, dx = 0, \quad \forall v \in V. \hfill (11)$$

Similarly, we have a variational problem for the auxiliary equation (7): find $q \in V$ such that

$$\int_{\Omega} \gamma \nabla q \cdot \nabla v \, dx + \int_{\Omega} q v \, dx = \int_{\Omega} f v \, dx, \quad \forall v \in V. \hfill (12)$$

Next, we construct a computational mesh $\Omega_h = \{ \omega_1, \omega_2, \ldots, \omega_N \}$ of domain $\Omega$. Here, $N$ is the number of triangular cells, $h = \max_{\omega \in \Omega_h} h_\omega$, where $h_\omega$ is the diameter of circle inscribed in a cell $\omega$. Then, in this mesh we define discrete spaces $V_h \subset V$, $Q_h \subset Q$ and restrict the variational problems (11), (12) to these spaces: find $u_h \in Q_h$ and find $q_h \in V_h$ such that

$$\int_{\Omega_h} \nabla u_h \cdot \nabla v_h \, dx + \int_{\Omega_h} c_h u_h v_h \, dx = 0, \quad \forall v_h \in V_h. \hfill (13)$$

$$\int_{\Omega_h} \gamma \nabla q_h \cdot \nabla v_h \, dx + \int_{\Omega_h} q_h v_h \, dx = \int_{\Omega} f h v_h \, dx, \quad \forall v_h \in V_h. \hfill (14)$$

In addition, the Heaviside function $H(q)$ in (9) is numerically approximated by the following function $H_\alpha(q)$

$$H_\alpha(q) = \begin{cases} 
0, & q < 0, \\
0.5 - 0.5 \cos \left( \frac{\pi q}{\alpha} \right), & 0 \leq q < \alpha \\
1, & q \geq \alpha,
\end{cases} \hfill (15)$$

where $\alpha > 0$ is smoothing parameter.

Now, we summarize our algorithm for reconstruction of the coefficient $c(x)$. The minimization is performed by a gradient based iterative procedure:

1. Define the initial guess for the control function $f(x)$;
2. Obtain the function $q(x)$ by solving (14);
3. Modify the coefficient $c(x)$ according to (8);
4. Solve the forward problem (13);
5. Update the control function $f(x)$ by a gradient method;
6. Repeat steps 2 – 5 until convergence.

Numerical implementation of the reconstruction is performed in the Python. For solving partial differential equations (13), (14) we use an open-source computing platform FEniCS [18]. The differentiation of the functional $J$ and minimization are implemented using an open source library dolfin-adjoint based on FEniCS [19, 20].
4. Numerical experiments

In this section, to verify the proposed algorithm we will reconstruct the coefficient \( c(x) \) from computationally simulated data obtained by solving the forward problem (1), (2). We consider a unit square domain \( \Omega \). For simplicity, the constants \( c_0 \) and \( c_1 \) are taken to be equal to 0 and 1, respectively. Also, we use the boundary function \( \phi(x) = 1 \). To test stability of the algorithm we use the function \( g(x) \) with different levels of noise: 0% noise and 5% noise.

The initial guess for the control function is chosen as follows

\[
 f_0(x) = \begin{cases} 
 1.0, & x \in D_0, \\
 0.0, & x \in \Omega \setminus D_0,
\end{cases}
\]

where \( D_0 \) is the initial guess for the subdomain \( D \). It is taken as a circle \( D_0 = B_r(x_0) \) with the radius \( r = 0.2 \) and the center \( x_0 = (0.5, 0.5) \). This initial control function \( f_0(x) \) is shown in Figure 1a. Then, the initial value of the function \( q_0(x) \) obtained by solving (14) with \( \gamma = 0.005 \) is depicted in Figure 1b. The corresponding coefficients \( c_0(x) \) calculated using (8) and the Heaviside functions (15) with \( \alpha = 0.1 \) and \( \alpha = 0.01 \) are displayed in Figures 1c and 1d, respectively.

Two shapes of the subdomain \( D \) are reconstructed: a single rectangle and two circles with different sizes. The rectangle is \( 0.2 \times 0.4 \) and is located in the center of domain \( \Omega \). The radii of circles are equal to 0.1 and 0.15. The center of the smaller circle is located at \((0.3, 0.3)\) and the center of the other one is \((0.7, 0.7)\).

First, we present reconstruction from noiseless data. The reconstructed coefficient \( c(x) \) for the rectangle after 10, 15, 20, and 34 iterations of minimization process are depicted in Figure 2. The true shape is shown in black line. These images are obtained using \( \gamma = 0.005 \) and \( \alpha = 0.01 \). The minimization of the functional \( J \) is performed using the limited-memory Broyden-Fletcher-Goldfarb-Shanno (L-BFGS) algorithm implemented in SciPy [21, 22]. This algorithm showed the fastest convergence rate compared to other techniques. Note that Figure 2d is the final image after the minimization algorithm converged in 34 iterations.

The reconstruction of the two circles after 15, 30, 60, and 121 iterations are presented in Figure 3. The results are obtained using the same coefficient \( \gamma = 0.005 \). However, we used the larger smoothing parameter \( \alpha = 0.1 \). We have observed that for more complicated shapes of subdomain \( D \) with several objects the larger parameter \( \alpha \) leads to much better reconstruction results. We note that the influence of coefficient \( \gamma \) on the performance of our algorithm is subject to more thorough investigation.

Figure 4 shows the reconstruction results for both rectangle and circles in the case of boundary data \( g(x) \) with 5% noise. We can see that the true shapes are recovered reasonably well. In the
first case, L-BFGS algorithm converged in 55 iterations, for the case of circles it converged in 183 iterations.

Рис. 2: Reconstruction results for rectangle after: a) 10, b) 15, c) 20 and d) 34 iterations

Рис. 3: Reconstruction results for circles after: a) 15, b) 30, c) 60 and d) 121 iterations

Рис. 4: Reconstruction results for noisy data: a) rectangle, b) circles

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