Invariant Properties of the Ansatz of the Hirota Method for Quasilinear Parabolic equations

K.A. Volosov
Moscow Institute of Electronics and Mathematics
Contsam@dol.ru

November 5, 2018

Abstract

We propose a new method based on the invariant properties of the ansatz of the Hirota method which have been discovered recently. This method allows one to construct new solutions for a certain class the dissipative equations classified by degrees of homogeneity. This algorithm is similar to the method of “dressing” the solutions of integrable equations. A class of new solutions is constructed. It is proved that all known exact solutions of the FitzHygh–Nagumo–Semenov equation can be expressed in terms of solutions of the linear parabolic equation. This method is compared with the Miura transforms in the theory of Kortweg de Vries equations. This method allows on to create a package by using the methods of computer algebra.

1

We consider the quasilinear parabolic equation

\[
(1 + b_1 u + b_2 u^2)u_t - (h_1 + h_2 u + h_3 u^2)u_x - g_0(u_x)^2
- (k_0 + k_1 u)u_{xx} + \sum_{i=1}^{4} \varphi_i u^i = 0. \tag{1.1}
\]

where \(b_j(x,t), h_j(x,t), k_{j-1}(x,t), j = 1, 2, \varphi_i(x,t), i = 1, \ldots, 4, g_0(x,t)\) are smooth functions or constants.

It is assumed that the coefficients of \(u_t\) and \(u_{xx}\) do not vanish in the range of the function \(u\).

We seek a solution in the form of the fraction \(\mathbb{F}\) (R. Hirota):

\[
u(x,t) = \frac{G(x,t)}{F(x,t)}. \tag{1.2}
\]

After multiplication by \(F(x,t)^4\), for the functions \(G(x,t)\) and \(F(x,t)\) we receive a homogeneous equation of the fourth order of homogeneity with respect to all functions.
A special case of Eq. (1.1) has the form
\[(1 + b_1u)u_t - (h_1 + h_2u)u_x - k_0u_{xx} + \sum_{i=1}^{3} \varphi_i u^i = 0, \tag{1.3}\]

which, after the substitution of (1.2) and multiplication by \(F(x, t)^3\) becomes an equation of the third order of homogeneity with respect to the functions \(G(x, t)\) and \(F(x, t)\).

Thus this equation is of the third order of homogeneity, and the the components \(u^i\) or \((u_x)^2\) complicate the equation so that it is an equation of the fourth order of homogeneity.

Special cases of this equation are widely known (see the extensive bibliography in [1, 2, 3, 12, 13]).

If \(h_1 = 0, \varphi_i = 0, i = 1, 3\), then this is the B"urgers equation in the wave theory [1].

If \(h_1 = 0, h_2 = 0, \varphi_1 = -1, \varphi_2 = -\varphi_1, \varphi_3 = 0\), then this is the Fisher–Kolmogorov–Petrovskii–Piskynov (FKPP) equation (see the bibliography in [2, 3, 13]);

If \(h_1 = 0, h_2 = 0, \varphi_1 = -1, \varphi_2 = 0, \varphi_3 = -\varphi_1\), then this is the FitzHygh–Nagumo–Semenov (FhNS) equation in the wave theory (see bibliography in [1, 2, 3], p. 12), and the Allen–Cahn equation (which is close to the FhNS equation) in the theory of phase transitions [10] and in ecology [14, 15]. Equations in which the coefficients of the derivative \(u_t\) depend on the function \(u\) appear in models of the theory of epidemic distribution [20]. Equations in which the coefficients of the derivative \(u_t\) depending on \(u_x\) arise, for example, in models of processes of ocean water freshening [21]. The initial reasons on a theme of the given operation can be detected in operation [22].

The method proposed for constructing solutions of Eq. (1.1) ideologically goes back to the Hirota method. An essential distinguishing feature of the present paper is that a new invariance property of solutions of equations from a given class is discovered (cf. [16, 17]). The equations obtained from (1.1) after the substitution of (1.2) can be arranged in the order of homogeneity as embedded equations.

From now on, we use the following notation for the derivatives:
\[\partial_\nu(\cdot) = (\cdot)_\nu, \quad \nu = x, t\] \tag{1.4}

We choose the solution (1.2) of Eq. (1.3) in the form
\[G(x, t) = \exp(H(x, t))B(x, t) + Q(x, t), \tag{1.5}\]
\[F(x, t) = \left(1 + A(x, t)\exp(H(x, t))\right). \tag{1.6}\]

All the other representations of the solution obtained by the Hirota method can be reduced to this form. There is only one possibility to extend this notation, namely, to add sums of series in powers of the exponentials \(\exp(iH(x, t))\), \(i = 2, \ldots\). Let us substitute (1.3) first into (1.2) and then into (1.3) and equate the coefficients of equal powers of \(\exp(H(x, t))\) (these powers are 0, 1, 2, 3) with zero.

We denote the obtained system of equations by (eq0), (eq1), (eq2), (eq3).

Equation (eq0) is the same original equation (1.3) for the function \(Q(x, t)\). Hence \(Q(x, t)\) is a solution of this equation.

Let us consider the equation obtained at a power of \(\exp(3H(x, t))\), namely, Eq. (eq3):
\[B\left(B^2\varphi_3 - 2k_0(A_x)^2 + B(-b_1A_t + H_2A_x)\right)
+ A\left(B^2(\varphi_1 + \varphi_3) + 2k_0A_xB_x + B(-A_t + B_1B_t + h_1A_x - h_2B_x + k_0A_{xx})\right)
+ A^2(B\varphi_1 + B_t - h_1B_x - K_0B_{xx}) = 0. \tag{1.7}\]
The special case considered in [3], p. 51, Section 24, and in [2], p. 190, is the case of $B = (\pm)A$, where the equation has the form
\[-A^2B + B^3 - 2ABA_t - 2B A^2_x + AB A_{xx} + A^2 B_t + 2AA_xB_x - A^2B_{xx} = 0. \quad (1.8)\]
and is an identity.

First, we present a lemma in which we prove that, in the conventional ansatz form of a solution, there is considerable arbitrariness related to the “extra” function $A(x, t)$, which must be excluded. Namely, this function must be added to the function $H(x, t)$ according to the formula
\[H(x, t)_{\text{new}} = H(x, t)_{\text{old}} + \exp(\ln(A)).\]
Then the same notation $H(x, t)$ is preserved for the function $H(x, t)$.

**Lemma 1.1** Suppose that the functions $B(x, t)$, $A(x, t)$ in (1.5) belong to $C^2[R \otimes R_+]$ and satisfy Eq. (eq3), and the function $B(x, t)$ has the form
\[B(x, t) = M(x, t)A(x, t). \quad (1.9)\]
Then the function $M(x, t)$ satisfies Eq. (1.3)
\[
\left(1 + b_1(x, t)M\right)M_t - \left(h_1(x, t) + h_2(x, t)M\right)M_x - k_0(x, t)M_{xx} + \sum_{i=1}^{3} \varphi_i(x, t)M^i = 0 \quad (1.10)
\]
for all functions $A(x, t)$.

**Proof.** The proof of the lemma is directly obtained by the substitution of (1.9) into (1.7). Equation (1.10), which follows from Eq. (1.7), coincides with (1.3). Hence the function $M(x, t)$ is a solution of this equation. Below $M(x, t)$ stands for some known exact solution of Eq. (1.3). \(\Box\)

It follows from Lemma 1.1 that the number of arbitrary functions in the ansatz can be decreased. Namely, we write $A = \exp(\ln(A))$ and add this summand to the function $H(x, t)$. In the present paper, solutions of the form (1.2) are called solutions of the first, second, etc. mode with respect to their complexity (the number of terms containing a power of $\exp(iH)$ as a factor).

**Definition 1.1** The set $\Omega$ of solutions of Eq. (1.3) of the form
\[u(x, t) = \frac{\exp(H(x, t))M(x, t) + Q(x, t)}{1 + \exp(H(x, t))}, \quad (1.11)\]
where $M(x, t)$, $Q(x, t)$ are some solutions of Eqs. (1.3) and $H(x, t)$ is an arbitrary function, will be called SOLUTIONS OF THE FIRST MODE; solutions of the form
\[U(x, t) = \frac{\exp(H(x, t))M(x, t) + \exp(2H(x, t))R(x, t) + Q(x, t)}{1 + Z_1(x, t)\exp(H(x, t)) + \exp(2H(x, t))Z_2(x, t)}, \quad (1.12)\]
will be called SOLUTIONS OF THE SECOND MODE; etc.

We have the following theorem.
\textbf{Theorem 1.1} Suppose that all coefficients in Eq. (1.3) are constants. The function $M$ is a solution of the equation

$$(1 + b_1 M) M_t - (h_1 + h_2 M) M_x - k_0 M_{xx} + \sum_{i=1}^{3} \varphi_i M^i = 0,$$ \hfill (1.13)

and $Q$ is a solution of the equation

$$(1 + b_1 Q) Q_t - (h_1 + h_2 Q) Q_x - k_0 Q_{xx} + \sum_{i=1}^{3} \varphi_i Q^i = 0,$$ \hfill (1.14)

In addition, they are related by the compatibility relations

$$\left( (M - Q) b_1 k_0 H_{xx}(x,t) + H_x(x,t) \left( M(b_1 h_1 - h_2) + Q(-b_1 h_1 + h_2) + 2b_1 k_0 (M_x - Q_x) \right) \right)$$

$$+ (2 + M b_1 + Q b_1) k_0 H_x(x,t)^2 + (M - Q) \left( M^2 b_1 \varphi_3 - Q^2 b_1 \varphi_3 + M(-\varphi_3 + b_1(\varphi_1 + \varphi_3)) - Q(-\varphi_3 + b_1(\varphi_1 + \varphi_3)) \right)$$

$$+ b_1(b_1(M_t - Q_t) + h_2(-M_x + Q_x))) = 0,$$ \hfill (1.15)

and, together with the function $H(x,t)$ satisfy the equation

$$-(M - Q) b_1 H_t(x,t) + H_x(x,t)(M h_2 - Q h_2 - 2k_0 H_x(x,t)) + (M - Q)^2 \varphi_3 = 0.$$ \hfill (1.16)

Then the function $u(x,t)$ is a solution of Eq. (1.3).

(The arguments of the functions are omitted here for brevity.)

\textbf{Remark 1.1} The equation (1.13) is the Riccaty equation for the functions $H^{(1,0)}(x,t)$, and equation (1.14) is the Hamilton–Jacobi equation for the function $H(x,t)$.

If $b_1 = 0$, then the statement of Theorem 1.1 becomes simpler and the function $H(x,t)$ can explicitly be written in terms of the functions $M(x,t)$ and $Q(x,t)$.

\textbf{Theorem 1.2} Let $b_1 = 0$. Suppose that all coefficients in Eq. (1.3) are constants. The function $M$ is a solution of the equation

$$M_t - (h_1 + h_2 M) M_x - k_0 M_{xx} + \sum_{i=1}^{3} \varphi_i M^i = 0,$$ \hfill (1.17)

and $Q$ is a solution of the equation

$$Q_t - (h_1 + h_2 Q) Q_x - k_0 Q_{xx} + \sum_{i=1}^{3} \varphi_i Q^i = 0.$$ \hfill (1.18)

In addition, they are related by the compatibility relations

$$\left( \int (M_t - Q_t) \, dx \right) 4(h_2 - m) - 2(M^2 - Q^2)(h_2^2 + 12k_0 \varphi_3 - h_2 m)$$

$$- 4(M - Q)(h_1 h_2 + 4k_0 \varphi_2 - h_1 m) + 16k_0 C'_0(t)$$

$$+ 4k_0(h_2 + 3m)(M_x - Q_x) = 0,$$ \hfill (1.19)
where \( m = ((\pm) \sqrt{h_2^2 + 8k_0\varphi_3}) \), and the function \( H(x, t) \) has the form

\[
H(x, t) = C_0(t) - \int \frac{(M(x, t) - Q(x, t))(-h_2 \pm \sqrt{h_2^2 + 8k_0\varphi_3})}{4k_0} \, dx.
\] (1.20)

Then the function \( u(x, t) \) is a solution of Eq. (1.3).

Proof. We give a brief proof of this theorem. We substitute (1.11) into (1.3), reduce to the common denominator, and consider the numerator of the obtained large equation.

We equate with zero the coefficient of the exponential raised to the zero power, i.e., the coefficient of \( \exp(0) \). The equation thus obtained is exactly Eq. (1.3) for the function \( Q(x, t) \). Hence the function \( Q(x, t) \) is a solution of this equation.

We equate with zero the coefficient of the exponential \( \exp(3H(x, t)) \). The equation thus obtained will be called Eq. (eq3). Again it is exactly Eq. (1.3). Hence the function \( M(x, t) \) is a solution of this equation.

We equate with zero the coefficient of the exponential \( \exp(H(x, t)) \). The equation thus obtained will be called Eq. (eq1).

We express \( H_{xx} \) from Eq. (eq1) and substitute it into Eq. (eq2). Then we obtain the Hamilton–Jacobi equation (1.16). For \( b_1 = 0 \) this equation in Theorem 1.2 naturally determines the function \( H(x, t) \) (1.20). If we express the derivative \( H_t \) from Eq. (1.16) and substitute it into Eq. (eq1), then we obtain the compatibility condition (1.15) or (1.19), respectively. If we substitute the derivative \( H_t \) into Eq. (eq2), then we also obtain the compatibility condition (i.e., the two equations coincide). In conclusion we note that there exist two branches in the relations given in the theorem with different signs. The proof of the theorem is complete. \( \Box \)

In fact, the theorems proved above describe the procedure of constructing new solutions of the equations (an analog of the superposition of solutions), which is specified by the properties of Eq. (1.3) and the ansatz.

Let us consider two smooth solutions of Eq. (1.3): \( M(x, t) \) and \( Q(x, t) \). The following function is assign to these solutions:

\[
P(Q, M) = \frac{Q(x, t) + M(x, t) \exp(H(x, t))}{1 + \exp(H(x, t))},
\] (1.21)

where \( H(x, t) \) is determined by (1.16) or by formula (1.20) if \( b_1 = 0 \).

The following theorem states that the solutions are invariant.

**Theorem 1.3** Suppose that two pairs of functions

\[
(M_0(x, t), Q_0(x, t)) \quad \text{and} \quad (M_1(x, t), Q_1(x, t))
\]

are solutions of the equations

\[
(1 + b_1 M_j)M_{jt} - (h_1 + h_2 M_j)M_jx - k_0 M_{jxx} + \sum_{i=1}^{3} \varphi_i M_j^i = 0,
\] (1.22)
where \( j = 0, 1 \), and that they determine the solution by formula \((1.11)\). Suppose that the assumptions of Theorems 1.1 and 1.2 hold for each pair. Then the two functions

\[
Q(x, t) = \frac{\exp(H_1(x, t))M_1(x, t) + Q_1(x, t)}{1 + \exp(H_1(x, t))},
M(x, t) = \frac{\exp(H_0(x, t))M_0(x, t) + Q_0(x, t)}{1 + \exp(H_0(x, t))}
\]

are solutions of the equations

\[
(1 + b_1 M)M_t - (h_1 + h_2 M)M_x - k_0 M_{xx} + \sum_{i=1}^{3} \varphi_i M^i = 0,
\]

\[(1.25)\]

\[
(1 + b_1 Q)Q_t - (h_1 + h_2 Q)Q_x - k_0 Q_{xx} + \sum_{i=1}^{3} \varphi_i Q^i = 0,
\]

\[(1.26)\]

and the function

\[
u(x, t) = \frac{\exp(H(x, t))M(x, t) + Q(x, t)}{1 + \exp(H(x, t))}\]

is a solution of Eq. \((1.3)\).

**Proof.** Each of the functions \( M_j, Q_j, j = 0, 1, \) satisfies Eq. \((1.3)\) written for this function. Pariwise, they satisfy the corresponding compatibility condition \((1.15)\) or \((1.19)\). The theorem can be proved by a direct substitution. \(\Box\)

**Remark 1.2** After calculating the derivative with respect to \( x \), it is possible to consider the compatibility condition \((1.15)\) as a modified Bürgers equation with a potential.

**Corollary 1.1** The class of solutions \((1.27)\) is closed with respect to the procedure described in Theorem 1.3. Thus \((1.27)\) defines a surjective mapping:

\[
Vol : \Omega \mapsto \Omega.
\]

\[(1.28)\]

Similar results take place for Eq. \((1.1)\). We perform the change of the variables \( \varphi_1 = a_2 a_1, \varphi_2 = -(a_1 + a_2 + a_1 a_2), \varphi_3 = 1 + a_1 + a_2, \varphi_4 = -1 \), which is equivalent to a source-sink function written so that the four roots are explicitly distinguished: \( u(a_1 - u)(a_2 - u)(1 - u) \).

We have a lemma similar to Lemma 1.1 and the following theorem.

**Theorem 1.4** Suppose that the solution \((1.1)\) has the form \((1.11)\) and the relations hold in one of the following cases:

1) \( Q(x, t) \) or \( M(x, t) ) \equiv const = -k_0/k_1, (k_0 + k_1)(k_0 + a_1 k_1)(k_0 + 2a_1 k_1) = 0; \)
2) \( b_1 = k_1/k_0, b_2 = 0; \)
3) \( b_1 = 0, b_2 = 0, k_1 = 0. \)

Then the functions \( M \) and \( Q \) are also solutions of an equation coinciding with Eq. \((1.1)\). Moreover, these functions are related by the compatibility relations (the Riccati equation for \( H_x(x, t) \)). In addition, the Hamilton–Jacobi equation must be satisfied.

(Here we do not write these equations, since they are very cumbersome.) The proof is similar to the proofs of the preceding theorems.
Theorem 2.1 Suppose that \( b_1(x, t) = 0 \), the function \( M(x, t) \) is a solution of the equation
\[
M_t - \left( h_1(x, t) + h_2(x, t)M \right) M_x - k_0(x, t)M_{xx} + \sum_{i=1}^{3} \varphi_i(x, t)M^i = 0,
\]
the function \( Q(x, t) \) satisfies the modified equation
\[
Q_t - \left( h_{01}(x, t) + h_{02}(x, t)Q \right) Q_x - k_{00}(x, t)Q_{xx} + \sum_{i=4}^{6} \varphi_i(x, t)Q^{i-3} = 0,
\]
and the coefficients in the equations satisfy the relations
\[
Q\left( \varphi_1(x, t) - \varphi_4(x, t) + Q(\varphi_2(x, t) - \varphi_5(x, t)) + Q^2(\varphi_3(x, t) - \varphi_6(x, t)) \right)
+ \left( h_{01}(x, t) - h_1(x, t) + (h_{02}(x, t) - h_2(x, t))Q \right) Q_x
-(k_0Q_{xx} - k_{00}Q_{xx}) = 0,
\]
and the functions \( M, Q \) and the coefficients satisfy the second compatibility equation
\[
\left( \int \partial_t \left( \frac{(-M + Q)(h_2 \pm m)}{k_0} \right) dx \right) 4k_0(-M + Q)m
+ 2(M - Q)(\pm h_3(M^2 - Q^2) + 2h_1(M - Q)((\pm)m^2 + h_2m)
+ h_2^2 \left( (M - Q)(M + Q)m + (\pm)2Mk_{0x} + (\pm)2Qk_{0x} + (\pm)6k_0(M_x - Q_x) \right)
+ 2h_2 \left( (\pm)4k_0\varphi_3(M^2 - Q^2) + (\pm)k_0h_2x(M - Q) - m(k_{0x}(M - Q) + k_0(M_x - Q_x) \right)
+ 2k_0 \left( m(2(M - Q)(2\varphi_2 + 3\varphi_3(M + Q)) - 4C'_0(t) + h_{2x}(M - Q)
- (\pm)4\varphi_3k_{0x}(M - Q) + (\pm)24\varphi_3k_0(M_x - Q_x) + (\pm)4\varphi_3^2k_0(M - Q) \right) \right) = 0,
\]
where \( m = \sqrt{(h_3^2 + 8k_0\varphi_3)} \) and the function \( H(x, t) \) has the form
\[
H(x, t) = C_0(t) - \int \left( M(x, t) - Q(x, t) \right) \left( -h_2(x, t) \pm \sqrt{h_2(x, t)^2 + 8k_0(x, t)\varphi_3(x, t)} \right) \frac{d}{dx}.
\]
Then the function \( u(x, t) \) \([1.11]\) is a solution of Eq. \([1.3]\).

Proof. The proof is similar to that of Theorem 1.2. Note that, first, in this case the set of solutions to the system of equations for the functions \( M, Q \) and the coefficients of the equation described in the theorem is not empty;

second, it is possible to construct solutions with variable roots of the source-sink function.
We substitute (1.11) into (1.3), reduce to the common denominator, and consider the numerator of the obtained equation. By equating with zero the coefficient of the exponential raised to the zero power, we obtain Eq. (eq0), which is the compatibility condition for coefficients (2.22). From this condition we express some coefficient and exclude this coefficient from the other equations. The equation obtained by equating the coefficient of the exponential \(\exp(H(x, t))\) with zero is called (eq1). The equation obtained by equating the coefficient of the exponential \(\exp(2H(x, t))\) with zero is called (eq2). Since the function \(Q\) satisfies Eq. (2.30), we exclude the derivative \(Q_t(x, t)\) from Eqs. (eq1) and (eq2). We can express \(H_{xx}(x, t)\) from Eq. (eq1) and substitute it into Eq. (eq2). Then we obtain the Hamilton–Jacobi equation, which naturally determines the function \(H(x, t)\) (2.24). The proof of the theorem is complete.

In the construction of the solution of the second mode, the transformation relates the solution of the original equation and the solutions of two modified equations, and so on.

The FitzHygh–Nagumo–Semenov equation

**EXAMPLE 1.** We apply our method to the FitzHygh–Nagumo–Semenov (FhNS) equation known in wave theory:

\[
 u_t - \varphi_3 u_{xx}/(2a^2) - \varphi_3 u + \varphi_3 u^3. \tag{2.34}
\]

where \(k_0 = \varphi_3/(2a^2)\) and \(\varphi_1 = \varphi_3\) are constants. We show that this method leads to new results and provides a new understanding of the results (see [3], p. 64, and [9]).

First we show the action of the semigroup related to the transformation (1.11) and explain how the translation constants pass into new solutions, by choosing the form of the functions (of its solutions, [1], p. 17, [3], p. 51) satisfying the assumptions of Theorem 1.2. Namely, we consider the functions \(u(x, t) = M(x, t)\) or \(u(x, t) = Q(x, t)\):

\[
 M(x, t) = -(1 + \exp(m_0 + ax + bt))^{-1}, \tag{2.35}
\]

\[
 Q(x, t) = (-1 + \exp(2ax + q_0))/(1 + \exp(2ax + q_0)), \quad b = -3\varphi_3/2.
\]

We calculate the function

\[
 H(x, t) = \ln \frac{1 + \exp(m_0 + ax - 3t\varphi_3/2)}{1 + \exp(q_0 + 2ax)}, \tag{2.36}
\]

and obtain the solution

\[
 u_1 = \frac{1 - \exp(-q_0 - 2ax + \ln(2))}{1 + \exp(-q_0 - 2ax + \ln(2)) + \exp(m_0 - q_0 - ax - 3t\varphi_3/2)}. \tag{2.37}
\]

Thus the original transition constants \(q_0, m_0\) pass into a new solution and a new constant \(\ln(2)\) appears.

Note that this is a well-known solution that describes the wave interaction ([2], p. 190, [3], p. 51, and [7]). Below we prove that this solution can be expressed via the solution of the linear parabolic equation.

There exists another version of the functions for which the assumptions of Theorem 1.2 are satisfied:

\[
 M(x, t) = (1 + \exp(m_1 - ax + bt))^{-1}, \tag{2.38}
\]

\[
 Q(x, t) = (1 - \exp(q_1 - 2ax))/(1 + \exp(q_1 - 2ax)), \quad b = -3\varphi_3/2.
\]
Then the function $H(x, t)$ has the form

$$H(x, t) = \ln \frac{1 + \exp(m_1 - ax - 3t\varphi_3/2)}{1 + \exp(q_1 - 2ax)},$$  \hspace{2cm} (2.39)

and we obtain the solution

$$u_2 = \frac{1 - \exp(-q_1 + 2ax + \ln(2))}{1 + \exp(-q_1 + 2ax + \ln(2)) + \exp(m_1 - q_1 + ax - 3t\varphi_3/2)}. \hspace{2cm} (2.40)$$

For the new functions $Q(x, t) = u_1$, $M(x, t) = u_2$, we take the solutions of Eq. (2.34):

$$H(x, t) = \ln \left( \frac{2\exp(-q_1) + \exp(-2ax) + \exp(m_1 - q_1 - ax - 3t\varphi_3/2)}{1 + 2\exp(q_1 - 2ax)} \right.\left. + \exp(m_0 - q_0 - ax - 3t\varphi_3/2) \right),$$

$$u_3 = \left( -1 + \exp \left( q_0 - q_1 + 2ax + \ln \frac{2 + \exp(q_1)}{2 + \exp(q_0)} \right) \right) \times \left( 1 + \exp \left( q_0 - q_1 + 2ax + \ln \frac{2 + \exp(q_1)}{2 + \exp(q_0)} \right) \right) \times \exp \left( -q_1 + ax + \ln \frac{\exp(m_1 + q_0) + \exp(m_0 + q_1)}{2 + \exp(q_0)} - 3t\varphi_3/2 \right)^{-1}. \hspace{2cm} (2.41)$$

By analyzing this formula, it is possible to understand the structure of the translation constant. Indeed, in the variables $\tau_1 = -ax + 3t\varphi_3/2$, $\tau_2 = ax + 3t\varphi_3/2$, the solution (2.41) has the form

$$u_3(\tau_1, \tau_2) = \frac{-c_1 \exp(\tau_1) + c_2 \exp(\tau_2)}{1 + c_1 \exp(\tau_1) + c_2 \exp(\tau_2)},$$

$$c_1 = \frac{\exp(q_1)(2 + \exp(q_0))}{\exp(q_0 + m_1) + \exp(q_1 + m_0)}, \hspace{2cm} (2.42)$$

$$c_2 = \frac{\exp(q_0)(2 + \exp(q_1))}{\exp(q_0 + m_1) + \exp(q_1 + m_0)}.$$

**Lemma 2.1** The set $u(x, t)$, $M(x, t)$, $Q(x, t)$ of solutions of the FhNS equation (2.34) of the form (2.38), (2.42) forms a semigroup with operation $\odot$ (1.11) and the following properties:

(a) commutativity, $M(x, t) \odot Q(x, t) = Q(x, t) \odot M(x, t)$;
(b) associativity, $u(x, t) \odot (M(x, t) \odot Q(x, t)) = (u(x, t) \odot Q(x, t)) \odot M(x, t)$;
(c) any element is the unity for itself, $M(x, t) \odot M(x, t) = M(x, t)$.

**Proof.** Let us consider two solutions with different sets of constants $C_i$, $V_i$, $i = 1, 2$

$$Q(x, t) = \frac{-\exp(\tau_1 + C_1) + \exp(\tau_2 + C_2)}{1 + \exp(\tau_1 + C_1) + \exp(\tau_2 + C_2)}, \hspace{2cm} \tau_i = \tau_i[x, t],$$

$$M(x, t) = \frac{-\exp(\tau_1 + V_1) + \exp(\tau_2 + V_2)}{1 + \exp(\tau_1 + V_1) + \exp(\tau_2 + V_2)}, \hspace{2cm} \tau_i = \tau_i(x, t). \hspace{2cm} (2.43)$$
By using Theorem 1.2, we find the solution

\[ u(x, t) = M(x, t) \circ Q(x, t) \]
\[ = \frac{-\exp(\tau_1)(\exp(C_1) + \exp(V_1)) + \exp(\tau_2)(\exp(C_2) + \exp(V_2))}{1 + \exp(\tau_1)(\exp(C_1) + \exp(V_1)) + \exp(\tau_2)(\exp(C_2) + \exp(V_2))}, \]
\[ \tau_i = \tau_i[x, t], \] (2.44)

which implies the properties given in Lemma 2.1.

Note that the action of the semigroup is a nonlinear “time-translation” of the solution.

As was noted in [2], p. 192, [3], p. 64, [9], and [12], it is known that Eq. (2.34) has solutions with singularities (monsters, contrast structures) whose role cannot be explained.

We have the following theorem.

**Theorem 2.2** Suppose that the function \( Q(x, t) \) is a solution of the equation

\[ Q_t - k_0 Q_{xx} - \varphi_1 Q + \varphi_3 Q^3 = 0, \] (2.45)

the function \( U(x, t) \) is a solution of the linear parabolic equation

\[ U_t + \varphi_3 a_1/a_2 U_x - \varphi_3/a_2 U_{xx} = 0, \] (2.46)

and the compatibility condition

\[ Q_x + Q(a_1 - (\pm) aQ - U_x/U) = 0 \] (2.47)

is satisfied. Then the function \( u(x, t) \) is a solution of Eq. (2.34) and has the form

\[ u = \frac{Q(x, t)U(x, t) \exp(t\varphi_3(1 + a_1^2/(2a^2)))}{Q(x, t) \exp(a_1 x) - U(x, t) \exp(t\varphi_3(1 + a_1^2/(2a^2)))}. \] (2.48)

**Proof.** To prove this theorem, we substitute the ansatz of the solution of the third mode

\[ u(x, t) = \frac{\exp(H(x, t))M(x, t) + \exp(2H(x, t))R(x, t) - Q(x, t)}{1 + Z(x, t) \exp(H(x, t)) + \exp(2H(x, t))Z_1(x, t) + \exp(3H(x, t))R(x, t)}, \] (2.49)

into Eq. (2.34), collect similar terms with equal powers of the exponentials, equate them with zero, and obtain a system of nine equations. As previously, from these equations we successively exclude the second derivatives of all functions contained there. Since the ansatz is invariant, there is a remarkable fact common for the algorithm of constructing the solution. Namely, just as in the theorems presented above, the equation becomes factored at some step. This equation contains various hints, which allow us to correct the ansatz (2.50) and to pass to the next iteration.

Equation (eq8) has the form

\[ 2a^2\varphi_3 + 2a^2H_t + \varphi_3 H_x^2 - \varphi_3 H_{xx} = 0. \] (2.50)

Its solution has the the form

\[ H(x, t) = -\ln(U(x, t)) + a_1 x - (2a^2 + a_1^2)t\varphi_3/(2a^2), \] (2.51)
and thus implies (2.47).

Equation (eq0) is exactly Eq. (2.34) for the function Q(x, t). Hence the function Q(x, t) is a solution of this equation.

Equation (eq1) at the second iteration step implies the compatibility conditions.

The second iteration of Eq. (eq1) implies the compatibility condition in the form of the Riccaty equation, in which we study only the situation corresponding to the sign “plus”.

Further, at each successive iteration, we have some versions due to the fact that there situations not considered earlier. We choose

\[
Z(x, t) = -\left(M(x, t) + Q(x, t)^2\right)/Q(x, t),
\]
\[
M(x, t) = -Q(x, t)^2/2,
\]
\[
R(x, t) = -(2Q(x, t)Z_1(x, t) + Q(x, t)^3)/2. \tag{2.52}
\]

By substituting all these expressions into the ansatz (2.50), we obtain the desired solution. The theorem is proved.

There is the following analogy with the theory of Korteweg-de Vries equations. In this context, relations (2.48) can be considered as an analog of Miura transformation.

Let us prove that all known solutions of the FhNS equation can be calculated in terms of the solutions of the linear parabolic equation. We find the solution of the Riccaty equation (2.48) and, choosing the upper sign, obtain

\[
Q(x, t) = \frac{U(x, t)\exp(-a_1x)}{C_0 - \int (aU(x, t)\exp(-a_1x)\,dx).} \tag{2.53}
\]

The compatibility conditions impose strong restrictions. Hence, for example, we have the pair of functions

\[
Q(x, t) = \frac{1 + C_1\exp(2ax)}{1 - C_1\exp(2ax)},
\]
\[
U(x, t) = C_1 + \exp(-2ax), \quad a_1 = -a. \tag{2.54}
\]

the substitution of which into (2.49) implies the following solution of Eq. (2.34):

\[
u(x, t) = \frac{1 + C_1\exp(2ax)}{-1 + C_1\exp(2ax) + \exp(ax - 3t\varphi_3/2)}. \tag{2.55}
\]

And the solution of (2.47)

\[
U(x, t) = \left(1 + 2aC_2\exp(-2ax)\right)\exp\left((a + a1)x + (a - a1)(a + a1)t\varphi_3/(2a^2)\right),
\]
\[
a_1 = Ia\sqrt{2}. \tag{2.56}
\]

gives solution of the equation (2.34)

\[
u(x, t) = \frac{2aC_2 + \exp(2ax)}{2aC_2 - \exp(2ax) + \exp(ax - 3t\varphi_3/2)}. \tag{2.57}
\]

These solutions have a specific characteristic feature, which can either appear or disappear. Such solutions were considered in [3], p. 64, [9], [12]. In what follows, we consider Example 3 constructed by this method. In this example, two singularities simultaneously appear in the modified FhNS equation.
There is the following statement: to the WELL-KNOWN SOLUTION of Eq. (2.34)
\[ u = \frac{1 - \exp(2ax)}{1 + \exp(2ax) + \exp(ax - 3t\varphi_3/2)}, \] (2.58)
there corresponds the solution of the LINEAR PARABOLIC EQUATION (2.40)
\[ U(x, t) = (-1 + \exp(2ax)) \exp((-a + a1)x + (a - a1)(a + a1)t\varphi_3/(2a^2)), \]
\[ a_1 = Ia\sqrt{2}. \] (2.59)

It turns out that the complex solutions of the linear parabolic equation can be recalculated into the solutions of the FhNS equation and vice versa. Therefore, the complex solutions of these equations must be studied.

Let us consider the FitzHygh–Naguma–Semenov (FhNS) equation (2.34) with variable coefficients.

By Theorem 2.1, for the first mode and separately for the third mode, there exist the following solutions. We have the following theorem.

**Theorem 2.3** Let \( k_0 = k_0(x, t), \varphi_1 = \varphi_1(x, t), \varphi_3 = \varphi_3(x, t) \) be smooth functions. Suppose that the function \( M(x, t) \) is a solution of the equation
\[ M_t - k_0M_{xx} - \varphi_1M + \varphi_3M^3 = 0, \] (2.60)
and the following compatibility condition imposed on its coefficients is satisfied:
\[-\sqrt{2}(\int \frac{-M\varphi_3k_0t + k_0(2\varphi_3M_t + \varphi_3M)}{k_0\sqrt{m_2}} dx)\sqrt{m_2} + 6M^2\varphi_3\sqrt{m_2} - 4C_0(t)\sqrt{m_2} + 6\sqrt{2}M_k\varphi_3 - \sqrt{2}M_k\varphi_3 + \sqrt{2}M_k\varphi_3_x = 0, \] (2.61)
where \( m_2 = k_0(x, t)\varphi_3(x, t) \) Then the function
\[ u(x, t) = \frac{\exp(f)M(x, t)}{1 - \exp(f)}, \quad f = C_0(t) + \int \frac{M(x, t)\sqrt{\varphi_3(x, t)}}{\sqrt{2k_0(x, t)}} dx. \] (2.62)
is a solution of the FhNS equation (2.34).

**Proof.** The proof follows from Theorems 1.2 and 2.1 and can be carried out by a direct substitution of the ansatz of the first mode, taking into account the fact that the coefficients in the equation are variable and \( M = 0 \). One compatibility condition is identically satisfied. One compatibility condition remains. Hence the set of solutions is not empty. The following solution illustrates this theorem.

For example, we set \( k_0 = 1, \varphi_1 = \varphi_3 \). We also assume, that we want to dress the solution \( Q = 1 \). The compatibility condition (2.62) turns into the equation for the function \( \varphi_1 \):
\[ \varphi_{1t} - 3\sqrt{2}\varphi_1\varphi_{1x} + (\varphi_1)^2/\varphi_1 + \varphi_{1xx} = 0. \] (2.63)

We choose one of its solutions, for example, \( \varphi_1 = C_2 \tanh^2(x/2\sqrt{3C_2\sqrt{2}})/3\sqrt{2} \).
The solution of the FhNS equation

\[ u_t - u_{xx} - \varphi_1(u - u^3) = 0, \]  

has the form

\[ u(x, t) = \frac{f_1}{1 - f_1}, \quad f_1 = \exp \left( \frac{C_2 t}{2\sqrt{2}} \right) \cosh^{1/3} \left( \frac{\sqrt{3C_2} x}{2^{3/4}} \right). \]  

This solution is real. If the constant is \( C_2 > 0 \), then this solution describes wave interaction. At the initial time moment, the solution has a singularity. Then the solution becomes smooth. For negative values of this constant \( C_2 < 0 \), this solution describes a periodic linear structure.

Thus one can construct a solution of the equation with variable roots of the source-sink function. For example, if \( \varphi_1 \) is given to be a periodic function, then \( \varphi_3 \) is constructed from solutions of Mathieu and Hill type equations.

**Theorem 2.4** Let \( k_0 = k_0(x, t) \), \( \varphi_1 = \varphi_1(x, t) \), \( \varphi_3 = \varphi_3(x, t) \) be smooth functions. Suppose that one of the compatibility conditions on the coefficients is satisfied, namely, the first compatibility equation

\[ Q_x(x, t) = -2\sqrt{(2\varphi_3 k_0)(U Q)^2 - 4k_0 U Q (\varphi_1 U - U_x)} \]  

or the second compatibility condition

\[ Q_x(x, t) = -2\sqrt{k_0 U Q - 2U Q^2 \varphi_3 + 2\sqrt{k_0} U Q_x} \]  

Suppose that the function \( Q(x, t) \) is a solution of the FhNS equation

\[ Q_1 - k_0 Q_{xx} - \varphi_1 Q + \varphi_3 Q^3 = 0, \]  

and the function \( U(x, t) \) is a solution of the linear parabolic equation

\[ U_t + 2\varphi_1 k_0 U_x - k_0 U_{xx} = U \left( \int \varphi_1 \, dx + \varphi_1 + k_0 \varphi_1^2 - k_0 \varphi_1 x \right) = 0. \]  

Then the function

\[ u(x, t) = \frac{Q(x, t) U(x, t)}{Q(x, t) \exp(\int \varphi_1 \, dx) - U(x, t)}, \]  

is a solution of the FhNS equation (2.34).

**Proof.** This statement can be proved by the substitution of the ansatz of the solution of the third mode (2.49), just as in the proof of Theorem 2.2. This theorem is a partial case in the analysis of the one of the two branches, which are determined by the choice of the compatibility condition.

The method has the following advantages:

a) the algorithm is iterative;
b) at each iteration step we solve one ordinary differential equation of the first order or one algebraic equation (cf. [17], where one has to solve a system of ordinary differential equations);

c) the method allows one to obtain new solutions; it is necessary only to know which mode has nontrivial solutions; to this end, there are some considerations the discussion of which is beyond the framework of this paper.

Note that formulas (2.41), (2.42) can be also obtained in a different way, while formulas (2.49), (2.63), (2.72) are constructed by this method.

**EXAMPLE 2.** Let us consider a modified FitzHygh–Nagumo–Semenov equation, which is a special case of the equation (1.1). In this case there also exist a lot of solutions, but we write only one solution corresponding to the second mode, since it differs from all other solutions. The change (1.2) reduces our equation to an equation the fourth order of homogeneity.

Suppose that Eq. (1.1) has the form

\[(a_1 - h_1 u)u_t - (a_1 - h_1 u)u_{xx} - h_1 (a_1 + h_1 u)u_x - 2h_1 (u_x)^2 - a_1 (a_1^2 + h_1^2)u(1 - u^2)/2 = 0,\] (2.71)

The solution has the form

\[u(x, t) = \frac{1 - \exp(a_1 x)}{1 + \exp(a_1 x) + \exp((-h_1^2 - 3a_1^2) t + 2(h_1 + a_1)x)/4},\] (2.72)

and describes the development of the second wave from small perturbations.

This solution can be used for modeling phenomena related to phase transitions [10]. One of the elementary equations for which this effect is preserved for \(a_1 = 0\) has the form

\[u_t - u_{xx} + h_1 u_x + 2(u_x)^2/u = 0,\] (2.73)

**EXAMPLE 3.** We consider the modified FhNS equation (1.3):

\[u_t + 2(h_1)^2u_{xx}/\varphi_3 - H_1(1 + 4u)u_x - \varphi_3 u(1 - u^2) = 0,\] (2.74)

The solution has the form

\[u(x, t) = \frac{1 - \exp((h_1 t + x)\varphi_3/h_1)}{1 - \exp(x\varphi_3/(2h_1)) + \exp((h_1 t + x)\varphi_3/h_1)},\] (2.75)

This exact solution shows how a solution of the problem with a smooth initial condition of double singularity is developed during a finite time interval. A discontinuity arises at \(t = 1.4\). This discontinuity is of a special structure such that the left-hand discontinuity does not move, while the right-hand discontinuity moves to the right.

Note that if the sign of the parameter \(\varphi_3\) is changed (i.e., we have \(\varphi_3 = 1\)), then Eq. (1.1) becomes an inverse parabolic equation. However, the solution (1.1) exists and is bounded. Small perturbations of the initial condition relax in time.

**EXAMPLE 4.** We consider the solution corresponding to the second mode for the equation the fourth order of homogeneity. Suppose that the modified FhNS equation (1.1) has the form

\[(1 - u)u_t - (1 - u)u_{xx} + (-1/a_1 + u/a_1 - 2a_1 u\varphi_3)u_x - 2(u_x)^2 - \varphi_3 u(1 - u^2) = 0.\] (2.76)
This equation has a solution of the form

$$u[x, t] = \frac{1 - \exp(x/a)}{1 + \exp(x/a)} + \exp(x/a - t\varphi_3).$$

(2.77)

The solution shows how the structure develops from an initial state whose range is $[-0.5, 1]$ to a traveling wave $u \in [0, 1]$. Note that it is possible to construct many solutions of such type, i.e., solutions that describe how a certain wave evolves from some initial state.

**EXAMPLE 5.** We consider the modified Fisher–Kolmogorov–Petrovskii–Peskunov equation

$$u_t - k_0u_{xx} - (2k_0\varphi_2/h_2 + h_2u)u_{xx} - \varphi_1 u + \varphi_2u^2 = 0. \tag{2.78}$$

We consider an important example of solutions to the Fisher–Kolmogorov–Petrovskii–Piskunov modified equation and show that its solutions can be expressed in terms of higher transcendental functions such as Bessel functions or hypergeometric functions. We also show how they are related to the Bäcklund transformation for the Bürgers equation.

**Theorem 2.5** Suppose $b_1 = 0$, $\varphi_3 = 0$,

$$h_1 = -2k_0\varphi_2/h_2 \tag{2.79}$$

in equation (1.3). The function $U(x, t)$ is a solution of the linear parabolic equation

$$U_t = r(x, t)U + k_0U_{xx}, \tag{2.80}$$

with the potential

$$r(x, t) = \exp(2x/h_2 + C_1(t)). \tag{2.81}$$

The function $U(x, t)$ has the form

$$U(x, t) = v(z(x, t)), \quad Z(x, t) = \exp(\frac{x}{(2h_2) + C_1(t)/2}) \tag{2.82}$$

and the function $v(z)$ is a solution of the equation

$$2h_2^2z\varphi_2v(z) + h_2^2\varphi_1v'(z) + 2k_0\varphi_2v'(z) + 2k_0z\varphi_2v''(z). \tag{2.83}$$

Then the function $u(x, t)$ is a solution of Eq. (2.78) and has the form

$$u(x, t) = (2k_0/h_2)\partial_x(\ln(U(x, t))). \tag{2.84}$$

**Proof.** We substitute (2.84) into the original equation (2.78) and obtain a nonlinear homogeneous equation from which we exclude the derivatives $U_t$, $U_{xt}$ with the help of Eq. (2.80). Then, for the parameters specified in the assumptions of the theorem by the change (2.83), we obtain ordinary differential equations for higher transcendental functions. The proof of Theorem 2.5 is complete. □

It should be noted that the change (2.84) coincides in form with the Cole–Hopf change of variables for the Bürgers equation.

The author is grateful to V. G. Danilov and S. Yu. Dobrohotov for constant attention to his work and useful discussions and to V. P. Maslov, and A. D. Polynin for constructive advice.
References

[1] R. K. Bullough, R. J. Caudry, eds., *Solitons*, Springer-Verlag, Berlin, Heidelberg, New York, 1980; S. P. Novikov, ed., Moscow, Mir, 1983. (in Russian).

[2] V. P. Maslov, V. G. Danilov, K. A. Volosov, *Mathematical Modeling of Process of Heat-Mass Transfer* (evolution of dissipative structures), With addition by N. A. Kolobov, Moscow, Nauka, 1987 (in Russian).

[3] V. G. Danilov, V. P. Maslov, K. A. Volosov, *Mathematical Modelling of Heat and Mass Transfer Processes*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1995.

[4] A. Scott, *Active and Nonlinear Wave Propagation in Electronics*, Wiley-Interscience, 1970; S. P. Novikov, ed., Moscow, Mir, 1977 (in Russian).

[5] J. Goldstone, R. Jasciw, Phys. Rev. D, 11 (1975), 1486.

[6] R. F. Dashen, B. Hasslancher, A. Neveu, Phys. Rev. D, 10 (1974).

[7] M. J. Ablowitz, Zeppeteller Bul. Math. Biol., 41 (1979), 835–840.

[8] R. Hirota, J. Phys. Soc. Jpn., 33 (1972), 1459.

[9] V. G. Danilov, P. Yu. Subochev, *Kink interaction in the KPP-Fisher equation*, Mat. Zametki, 50 (1991), No. 3, 152–154; English transl. in Math. Notes.

[10] V. G. Danilov, G. A. Omel’yanov, E. V. Radkevich, *A justification of asymptotic solutions and a simulation of the Stefan problem*, Mat. Sbornik, 186 (1995), No. 12, 64–80; English transl. in Russian Acad. Sci. Sb. Math.

[11] M. J. Ablowitz, H. Segur, *Solitons and the Inverse Scattering Theory*, SIAM, Philadelphia, 1981.

[12] K. A. Volosov, V. G. Danilov, A. M. Loginov, *Exact self-similar two-phase solutions for systems of semilinear parabolic equations*, Teoret. Mat. Fiz., 101 (1994), No. 2, 189–199; English transl. in Theoret. and Math. Phys.

[13] R. A. Fisher, *The wave of advance of an advantageous gene*, Am Euden, 7 (1936), 355–369.

[14] N. B. Belotelov, A. I. Lobanov, *Population models of nonlinear diffusion*, Mat. Model., 9 (1997), No. 12, 43–56 (in Russian).

[15] A. I. Lobanov, T. K. Starozhilova, *Qualitative research of the initial stage of formation of nonequilibrium structures in “reaction–diffusion” type models*, Mat. Model., 9 (1997), No. 12, 23–26 (in Russian).

[16] V. V. Pukhnachev, *Equivalence transforms and hidden symmetry of evolutionary equations*, Dokl. Akad. Nauk SSSR, 294 (1987), No. 3, 535–538; English transl. in Soviet. Math. Dokl.
[17] V. A. Galaktionov, S. A. Posashkov, *Exact solutions and invariant spaces for nonlinear equations of gradient diffusion*, Zh. Vychisl. Mat. i Mat. Fiz., 34 (1994), 374–383; Engl. transl. in U.S.S.R. Comput. Math. and Math. Phys.

[18] K. A. Volosov, V. G. Danilov, N. A. Kolobov, V. P. Maslov, *Localized solitary waves*, Dokl. Akad. Nauk SSSR, 287 (1986), No. 6, 535–538; English transl. in Soviet. Math. Dokl.

[19] D. Henry, *Geometric theory of semilinear parabolic equations*. Springer-Verlag, Berlin, 1981; Moscow, Mir, 1985 (in Russian).

[20] E. M. Melnikova, *Nonlinear dynamics of epidemics distribution*, Izv. Vyssh. Uchebn. Zaved. Appl. Nonlin. Dynamics, 6 (1998), No. 2, 110–116 (in Russian).

[21] A. I. Kozhonov, *A boundary-value problem for a class of parabolic equations originating in the description of the water freshening process*, Institute of Hydrodynamics, Vol. 36, 38–46, 1978 (in Russian).

[22] K.A. Volosov, contsam@dol.ru Invariant properties ansatz of a method R. Hirota. New information process engineerings. Materials The fourth seminar. www/miem.edu.ru./rio/seminar4/ Moscow Inst.of Electron.and Math..2001 .(Russian)