Symplectically invariant soliton equations from non-stretching geometric curve flows

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Abstract

Bi-Hamiltonian hierarchies of symplectically invariant soliton equations are derived from geometric non-stretching flows of curves in the Riemannian symmetric spaces $Sp(n+1)/Sp(1) \times Sp(n)$ and $SU(2n)/Sp(n)$. The derivation uses Hasimoto variables defined by a moving parallel frame along the curves. As main results, two new multi-component versions of the sine–Gordon equation and the modified Korteweg–de Vries (mKdV) equation exhibiting $Sp(2n+1)/Sp(1) \times Sp(n-1)$ invariance are obtained along with their bi-Hamiltonian integrability structure consisting of a hierarchy of symmetries and conservation laws generated by a hereditary recursion operator. The corresponding geometric curve flows in both $Sp(n+1)/Sp(1) \times Sp(n)$ and $SU(2n)/Sp(n)$ are shown to be described by a non-stretching wave map and a mKdV analogue of a non-stretching Schrödinger map.

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1. Introduction

Both the modified Korteweg–de Vries (mKdV) equation and the sine–Gordon (SG) equation are well known to have a geometric origin given by certain flows of the curvature invariant of arclength-parameterized curves in the two-dimensional geometries $\mathbb{R}^2$ and $S^2$ [1–4]. Similarly, the nonlinear Schrödinger (NLS) equation has long been known to arise from a certain flow of $U(1)$-covariants of arclength-parameterized curves in the three-dimensional geometries $\mathbb{R}^3$ and $SO(3)$, where the covariants are related to the standard curvature and torsion invariants of the curve by the famous Hasimoto transformation [5–7]. In all of these flows, the equation of motion of the curve has the geometrical properties that it preserves the arclength locally at each point on the curve (i.e. the motion is non-stretching) and that it is invariant under the action of the isometry group of the underlying Riemannian geometry. Additionally, the differential
invariant in the two-dimensional case and the differential covariants in the three-dimensional case have a direct geometrical meaning as the components of the Cartan connection in a parallel frame [8] along the curve.

A broad generalization of such results has been obtained in recent work [9] using a moving parallel frame formulation for non-stretching curve flows in Riemannian symmetric spaces $M = G/H$. These spaces describe curved generalizations of Euclidean geometries in which the Euclidean isometry group is replaced by a simple Lie group $G$ and the Euclidean frame rotation gauge group is replaced by an involutive compact Lie subgroup $H$ in $G$. In this geometric setting, the Cartan connection components in a suitably defined parallel frame along an arclength-parameterized curve represent differential covariants of the curve, which are related to standard differential invariants by a generalized Hasimoto transformation. For curves undergoing certain non-stretching geometric flows, these covariants satisfy multi-component SG and mKdV equations whose integrability structure as given by a pair of compatible Hamiltonian operators is encoded directly in the Cartan structure equations of the parallel frame. In cases where $M$ additionally has a Hermitian structure or a Lie group structure, the Hamiltonian operators also give rise to integrable multi-component NLS equations [10]. Moreover, all of these integrable multi-component equations, along with their bi-Hamiltonian structure, possess an explicit group invariance which arises from the action of the equivalence group of the parallel frame. This main result provides a geometric derivation of many known group-invariant versions of multi-component soliton equations as well as the possibility of deriving new versions that exhibit other invariance groups.

For example, there are exactly two different Riemannian symmetric spaces with the structure $G/SO(n)$, as given by $G = SO(n + 1)$ and $G = SU(n)$ (see, e.g. [11]). For curves in each of these two spaces, the components of the Cartan connection in a parallel frame yield $n - 1$ covariants that satisfy vector mKdV equations and vector SG equations with an $SO(n - 1)$ invariance group when the curve undergoes certain non-stretching geometric flows [12] (see also [13]). This derivation geometrically accounts for the two different rotationally invariant vector versions of the mKdV and SG equations obtained from symmetry-integrability classifications [14, 15].

In this paper, we geometrically derive symplectically invariant multi-component soliton equations from non-stretching curve flows in the Riemannian symmetric spaces $Sp(1 + 1)/Sp(1) \times Sp(n)$ and $SU(2n)/Sp(n)$. These two geometries happen to share the same symplectic equivalence group $Sp(1) \times Sp(n - 1)$ for parallel framings of arclength-parameterized curves. One main motivation for our work is the absence, to date, of any symmetry-integrability classifications for multi-component versions of mKdV or SG equations with symplectic invariance. Other work [16] with a similar motivation to ours has recently found multi-component symplectically invariant mKdV and SG equations of derivative type by considering certain algebraic reductions of integrable matrix systems. These derivative-type soliton equations have a different form of nonlinearity (exhibiting, in particular, a different scaling symmetry) than the multi-component soliton equations obtained from our results.

For the geometry $SU(2n)/Sp(n)$, we obtain new symplectically invariant mKdV and SG equations for a vector pair, together with their symplectically invariant bi-Hamiltonian integrability structure. For the geometry $Sp(1 + 1)/Sp(1) \times Sp(n)$, we find symplectically invariant mKdV and SG equations for a scalar pair coupled to a vector pair, which represent the component form of new quaternionic soliton equations with a quaternion bi-Hamiltonian integrability structure derived in recent work [17] (see also [18]) on non-stretching curve flows in the quaternionic projective space $\mathbb{H}^{2n} \simeq U(n + 1, \mathbb{H})/U(1, \mathbb{H}) \times U(n, \mathbb{H}) \simeq Sp(1 + 1)/Sp(1) \times Sp(n)$ (where $\mathbb{H}$ denotes Hamilton’s quaternions). The symplectic
invariance group of these new bi-Hamiltonian soliton equations arising from both geometries $SU(2n)/Sp(n)$ and $Sp(n + 1)/Sp(1) \times Sp(n)$ is given by $Sp(1) \times Sp(n - 1)$.

There are several important ways in which our results go beyond the previous literature on integrable systems connected with symmetric spaces and Lie algebras.

In [19, 20], multi-component NLS and mKdV equations are written down using a Lax pair construction based on the Lie algebra structure of Hermitian symmetric spaces. This construction does not apply to the non-Hermitian symmetric spaces $SU(2n)/Sp(n)$ and $Sp(n + 1)/Sp(1) \times Sp(n)$ considered in our work or the Riemannian symmetric spaces $SO(n+1)/SO(n)$ and $SU(n)/SO(n)$ in earlier work [13, 12]. Thus the two different rotationally invariant vector mKdV equations obtained in [12] as well as the two different mKdV equations with symplectic invariance obtained in this paper fall outside the multi-component mKdV equations constructed in [20].

Multi-component SG equations with rotational invariance and unitary invariance are derived in [21–24] using algebraic methods applied to symmetric spaces $SO(n+1)/SO(n)$, $SU(n)/SO(n)$, $SU(n+1)/U(n)$, $Sp(n)/U(n)$ with either a rotation gauge group $SO(n)$ or a unitary gauge group $U(n)$. None of these works include the symmetric spaces $SU(2n)/Sp(n)$ and $Sp(n + 1)/Sp(1) \times Sp(n)$ having symplectic gauge groups. Hence the two symplectically invariant SG equations obtained by us are different than the SG equations with rotational invariance and unitary invariance found in previous work.

Likewise, the moving frame method in [9] which we apply in this paper differs from other geometric approaches in the literature [25–32] on curve flows. The approach in [25–28] has invariance and unitary invariance found in previous work. In [29] there are several important ways in which our results go beyond the previous literature on integrable systems connected with symmetric spaces and Lie algebras.

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bi-Hamiltonian operators which are subsequently used to construct the new multi-component $Sp(1) \times Sp(n-1)$-invariant mKdV equations and SG equations obtained in sections 4.2–4.3 and 5.2–5.3 for each geometry. The corresponding geometric curve flows are worked out in sections 4.4 and 5.4 and shown to be a non-stretching wave map equation and a mKdV analogue of a non-stretching Schrödinger map equation. We conclude with some remarks in section 6.

2. Parallel moving frames and non-stretching curve flows

For a Riemannian symmetric space $M = G/H$, defined by a simple Lie group $G$ and an involutive compact Lie subgroup $H$ in $G$, any linear frame on $M$ provides a soldering identification between the tangent space $T_x M$ at points $x$ and the vector space $m = g/\mathfrak{h}$. Relative to the Cartan–Killing form and Lie bracket on $g$, there is a decomposition $g = \mathfrak{h} \oplus m$ given by a direct sum of orthogonal vector spaces

$$\langle h, m \rangle = 0$$

with the Lie bracket relations

$$[h, h] \subset \mathfrak{h}, \quad [h, m] \subset m, \quad [m, m] \subset \mathfrak{h}. \quad (2.1)$$

Geometrically, the Lie subalgebra $\mathfrak{h}$ is identified with the generators of isometries that leave fixed the origin $o$ in $M$, while the vector space $m$ is identified with the generators of isometries that carry the origin $o$ to any point $x \neq o$ in $M$. These isometries represent the action of the group $G$ on the space $M$, whereby the subgroup $H$ acts as the gauge group of the frame bundle of $M$.

The Riemannian structure of the space $M = G/H$ is naturally described [33] in terms of an $m$-valued linear coframe $e$ and an $\mathfrak{h}$-valued linear connection $\omega$ whose torsion and curvature

$$\mathcal{T} := de + [\omega, e], \quad \mathcal{R} := d\omega + \frac{1}{2} [\omega, \omega] \quad (2.3)$$

are 2-forms with respective values in $m$ and $\mathfrak{h}$, given by the following Cartan structure equations:

$$\mathcal{T} = 0, \quad \mathcal{R} = -\frac{1}{2} [e, e]. \quad (2.4)$$

Here $[\cdot, \cdot]$ denotes the Lie bracket on $g$ composed with the wedge product on $T^*_x M$. This structure together with the (negative-definite) Cartan–Killing form determines a Riemannian metric and Riemannian connection (i.e. covariant derivative) on the space $M$ as follows: for all $X, Y$ in $T^*_x M$,

$$g(X, Y) := -(e_X, e_Y), \quad e \nabla_X Y := d_X e_Y + [\omega_X, e_Y], \quad (2.5)$$

where the coframe provides a soldering identification between the tangent space $T_x M$ and the vector space $m = g/\mathfrak{h}$ as given by $e \nabla_X := e_X, e \nabla_Y := e_Y \in m$. The connection is metric compatible, $\nabla g = 0$, and torsion-free, $\nabla T = 0$, while its curvature is covariantly constant, $\nabla \mathcal{R} = 0$, as given by

$$e \nabla R(X, Y) Z = [\mathcal{R}] (X \wedge Y), e_{\varepsilon Z} = -[[e_X, e_Y], e_Z], \quad e \nabla T(X, Y) = \mathcal{T} (X \wedge Y) = 0, \quad (2.6)$$

where $T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$ is the torsion tensor and $R(X, Y) := [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$ is the curvature tensor. Note the linear coframe and linear connection have gauge freedom given by the following transformations:

$$e \longrightarrow \text{Ad}(h^{-1}) e, \quad \omega \longrightarrow \text{Ad}(h^{-1}) \omega + h^{-1} dh \quad (2.7)$$

for an arbitrary function $h : M \rightarrow H \subset G$. These gauge transformations comprise a local ($x$-dependent) representation of the linear transformation group $H^* = \text{Ad}(H)$ which defines
the gauge group [34] of the frame bundle of $M$. Both the metric tensor $g$ and curvature tensor $R$ on $M$ are gauge invariant.

Let $\gamma(x)$ be any smooth curve in $M$. A frame consists of a set of orthonormal vectors that span the tangent space $T_xM$ at each point $x$ on the curve $\gamma$. The Frenet equations of a frame yield a connection matrix consisting of the set of frame components of the covariant $x$-derivative of each frame vector along the curve [35]. A coframe consists of a set of orthonormal covectors that are dual to the frame vectors relative to the Riemannian metric $g$. Such a framing for $\gamma(x)$ is determined by the Lie-algebra components of $e$ and $\omega|_{\gamma_t}$ when an orthonormal basis is introduced for $m$ and $h$ with respect to the Cartan–Killing form, where the Frenet equations are defined by the frame components of the transport equation

$$\nabla_x e = -\text{ad}(\omega|_{\gamma_t})e$$  \hspace{1cm} (2.8)

along the curve. In particular, if $\{m_l\}_{l=1,\ldots,\dim(m)}$ is any fixed orthonormal basis for $m$, then a frame at each point $x$ along the curve is given by the set of vectors $X_l := -(e^*, m_l)$, $l = 1, \ldots, \dim(m)$. Here, $e^*$ is an $m$-valued linear frame defined as the dual to the linear coframe $e$ by the condition that $-(e^*, e) = \text{id}$ is the identity map on each tangent space $T_xM$ (cf [9, 33]).

Now consider any smooth flow $\gamma(t, x)$ of a curve in $M$. We write $X = \gamma_t$ for the tangent vector and $Y = \gamma_t$ for the evolution vector at each point $x$ along the curve. Note the flow is non-stretching provided that it preserves the $G$-invariant arclength $d\gamma = |\gamma_t|dx$, or equivalently $\nabla_t |\gamma_t| = 0$, in which case we have

$$\gamma(t, x) = |\gamma_t|^2 = 1$$  \hspace{1cm} (2.9)

without loss of generality. For flows that are transverse to the curve (such that $X$ and $Y$ are linearly independent), $\gamma(t, x)$ will describe a smooth two-dimensional surface in $M$. The pullback of the torsion and curvature equations (2.4) to this surface yields

$$D_t e_t - D_x e_t + [\omega_t, e_t] = 0,$$  \hspace{1cm} (2.10)

$$D_t \omega_t - D_x \omega_t + [\omega_t, \omega_t] = -[e_t, e_t],$$  \hspace{1cm} (2.11)

with

$$e_t := e|\gamma_t = e|\gamma_t, \quad e_t := e|\gamma_t = e|\gamma_t,$$  \hspace{1cm} (2.12)

$$\omega_t := \omega|\gamma_t = \omega|\gamma_t, \quad \omega_t := \omega|\gamma_t = \omega|\gamma_t,$$  \hspace{1cm} (2.13)

where $D_t$ and $D_x$ denote derivative operators with respect to $x$ and $t$, respectively. Remarkably, for any non-stretching curve flow, these structure equations (2.10)–(2.13) encode an explicit pair of bi-Hamiltonian operators once a specific choice of frame along $\gamma(t, x)$ is made.

We utilize a natural choice of a moving frame defined by the following two properties which are a direct algebraic generalization of a parallel moving frame in Euclidean geometry [9]:

1. $e_t$ is a constant unit-norm element lying in a Cartan subspace $a \subseteq m$ that is contained in the centralizer subspace $m_0$ of $e_t$, i.e. $D_t e_t = D_x e_t = 0$, $(e_t, e_t) = 1$, and $\text{ad}(m_0)e_t = 0$, where $m_0 \oplus m_0 = m$ and $(m_0, m_0) = 0$.

2. $\omega_t$ lies in the perp space $h_\perp$ of the Lie subalgebra $h_0 \subset h$ of the linear isotropy group $H^* \subset H^* = \text{Ad}(H)$ that preserves $e_t$, i.e. $\text{Ad}(h_0)e_t = 0$ and $\langle \omega_t, h_0 \rangle = 0$, where $h_0 \oplus h_\perp = h$ and $(h_0, h_\perp) = 0$.

Cartan subspaces of $m$ are defined as a maximal Abelian subspace $a \subseteq m$ having the property that it is the centralizer of its elements, $a = m \cap e(a)$. It is well known (see, e.g. [11]) that any two Cartan subspaces are isomorphic to one another under some linear transformation.
in $\text{Ad}(H)$ and that the action of the linear transformation group $\text{Ad}(H)$ on any Cartan subspace $a$ generates $m$. The dimension of $a$ as a vector space is equal to the rank of $m$.

A moving frame satisfying properties (1) and (2) is called $H$-parallel and its existence can be established by constructing a suitable gauge transformation (2.7) on an arbitrary frame at each point $x$ along the curve [9]. Specifically, given any $m$-valued linear coframe $\tilde{e}$ and $h$-valued linear connection matrix $\tilde{\omega}$ along $\gamma$, we can first find a gauge transformation such that $h^{-1}\tilde{e}, h = e_\gamma$ is a constant element in any Cartan subspace $a \subset m$, as a consequence of the fact $m = \text{Ad}(H)a$. The norm of $e_\gamma$ will satisfy $-\langle e_\gamma, e_\gamma \rangle = g(\gamma_t, \gamma_t) = 1$ because we have chosen an arclength parameterization (2.9) of the curve. We can then find a gauge transformation belonging to the subgroup $H^*_x$ preserving $e_\gamma$, so that $h^{-1}D_xh + h^{-1}\tilde{\omega}_xh = \omega_x$ where $h(x) \in H^*_x$ is given by solving the linear matrix ODE $D_xh + \tilde{\omega}^\parallel h = 0$ in terms of the decomposition of $\tilde{\omega}_x = \tilde{\omega}^\parallel + \tilde{\omega}^\perp$ relative to $e_\gamma$. Note the solution will depend on an arbitrary initial condition $h(x_0) \in H^*_x$, specified at some point $x = x_0$ along the curve, which represents a rigid gauge freedom (i.e. the equivalence group) in the construction of the $H$-parallel moving frame.

Underpinning this construction are the Lie bracket relations on $m$, $m_\perp$, $h_\parallel$, $h_\perp$ coming from the structure of $g$ as a symmetric Lie algebra (2.2). These relations consist of

$$[m, m] \subseteq h, \quad [m, h] \subseteq m, \quad [h, h] \subseteq h,$$

$$[h, m_\perp] \subseteq m_\perp, \quad [h, h_\perp] \subseteq h_\perp,$$

$$[m, m_\perp] \subseteq h_\perp, \quad [m, h_\perp] \subseteq m_\perp,$$

while the remaining Lie brackets

$$[m_\perp, m_\perp], \quad [h_\perp, h_\perp], \quad [m_\perp, h_\perp]$$

obey the general relations (2.2).

**Theorem 2.1.** For $e_\gamma \in a \subset m$, let $e_\gamma = h_{\parallel} + h_{\perp} \in m \oplus m_\perp$, $\omega_\gamma = \omega^\parallel + \omega^\perp \in h_\parallel \oplus h_\perp$, and $u = \omega = h_\perp \in h_\perp$. Also let $h^\perp = \text{ad}(e_\gamma)h_{\perp} \in h_{\perp}$. Then the Cartan structure equations (2.10) and (2.11) for any $H$-parallel linear coframe $e$ and linear connection $\omega$ pulled back to the two-dimensional surface $\gamma(t, x)$ in $M = G/H$ yield the flow equation [9]

$$u_\gamma = h_\parallel \omega^\perp + h_\perp = J(h^\perp),$$

where

$$h_\parallel = K|_{h_{\parallel}}, \quad J = -\text{ad}(e_\gamma)^{-1}K|_{m} \text{ad}(e_\gamma)^{-1}$$

are a bi-Hamiltonian pair of operators that act on $h_\perp$-valued functions and are invariant under $H^*_x$, as defined in terms of the linear operator

$$K := D_x + [u, \cdot]_{\perp} - [u, D_x^{-1}[u, \cdot]]_1.$$

In particular, every linear combination of $h_\parallel$ and $J^{-1}$ is a Hamiltonian operator with respect to $u$.

We emphasize that the formulation in theorem 2.1 applies to all non-stretching curve flows $\gamma(t, x)$ in $M = G/H$, with the flow being determined by specifying $h^\perp$, or equivalently $h_{\perp} = \text{ad}(e_\gamma)^{-1}h^\perp$, freely as a function of $t$ at each point $x$ along the curve. In particular, every flow equation (2.18) determines a corresponding curve flow $\gamma(t, x)$ through the geometrical relation

$$Y = -\langle e_\gamma, h_{\perp} + h_\parallel \rangle = \langle e_\gamma, \gamma(h^\perp) \rangle$$

in terms of the operator

$$\gamma := D_x^{-1}[u, \text{ad}(e_\gamma)^{-1}]_1 - \text{ad}(e_\gamma)^{-1},$$

$$= [D_x^{-1}[u, \text{ad}(e_\gamma)^{-1}]_1 - \text{ad}(e_\gamma)^{-1}],$$

$$= [D_x^{-1}[u, \text{ad}(e_\gamma)^{-1}]_1 - \text{ad}(e_\gamma)^{-1}]^{-1},$$

$$= [D_x^{-1}[u, \text{ad}(e_\gamma)^{-1}]_1 - \text{ad}(e_\gamma)^{-1}]^{-1}.$$
where $e^*$ is the linear frame dual to the linear coframe $e$ along $\gamma$, with $e_\gamma = e^*|X$. In this correspondence (2.21), $e_\gamma$ is preserved under the action of the equivalence group $H^*_\gamma$ while up to equivalence, both $e^*$ and $e$ are determined by $\omega_\gamma$ from the transport equation (2.8) along $\gamma$. The resulting equation of motion $\gamma_l = Y_l = (e^*, Y_l(h^*))$ will be $G$-invariant if and only if $h^*$ is an $H^*_\gamma$-equivariant function of $x$, $u$- and $x$-derivatives of $u$. In addition, the corresponding flow on $u(t, x)$ will have a Hamiltonian structure if and only if $\sigma^* = J^*(h^*)$ is the variational derivative of some $H^*_\gamma$-invariant Hamiltonian function of $x$, $u$ and $x$-derivatives of $u$. The following general results are established in [9].

**Theorem 2.2.** Composition of the operators $H$ and $J$ yields a recursion operator $R = HJ$ that produces a hierarchy of $H^*_\gamma$-invariant flows (2.18) on $u$ given in terms of

$$h_{\gamma0}^\perp = R^l(u_\gamma), \quad l = 0, 1, 2, \ldots$$

Each flow in this hierarchy inherits a bi-Hamiltonian structure given by

$$h_{\gamma l}^\perp = H(\sigma_{\gamma l}^\perp) = J^{-1}(\omega_{\gamma l+1}^\perp), \quad \omega_{\gamma l}^\perp = \delta H^{(l)}/\delta u = R^l(u), \quad l = 0, 1, 2, \ldots$$

in terms of the $H^*_\gamma$-invariant Hamiltonians

$$H^{(l)} = -\frac{1}{1 + 2l}\langle e_x, h_{\gamma l}^\perp \rangle, \quad l = 0, 1, 2, \ldots,$$

where $R^* = J^*H$ is the adjoint of $R$. Moreover, the kernel of the recursion operator $R$ yields a further $H^*_\gamma$-invariant flow (2.18) on $u$ in terms of $h_{\gamma -1}^\perp$ defined by

$$J(h_{\gamma -1}^\perp) = 0.$$

This flow has a Hamiltonian structure given by

$$h_{\gamma -1}^\perp = H(\sigma_{\gamma -1}^\perp), \quad \sigma_{\gamma -1}^\perp = \delta H^{(-1)}/\delta u$$

with

$$H^{(-1)} = \langle e_x, h_{\gamma -1}^\perp \rangle.$$

The bi-Hamiltonian flows (2.23) and (2.26) have a geometrical formulation through the correspondence (2.21).

**Theorem 2.3.** The hierarchy of bi-Hamiltonian flows (2.23) corresponds to a hierarchy of non-stretching geometric curve flows in $M = G/H$ given by equations of motion

$$\gamma_l = Y_l(\gamma, \nabla_1\gamma, \ldots, \nabla^n \gamma), \quad |\gamma_l| = 1, \quad l = 0, 1, 2, \ldots$$

where $Y_l = (e^*, Y_l(h^*))$. The additional Hamiltonian flow (2.26) corresponds to the non-stretching geometric curve flow

$$\nabla_1 \gamma_l = \nabla_1 Y_{(-1)} = 0, \quad |\gamma_l| = 1,$$

with $Y_{(-1)} = (e^*, Y(h_{\gamma -1}^\perp))$. Each equation of motion ($l = -1, 0, 1, 2, \ldots$) is invariant with respect to the isometry group $G$ of $M$ and preserves the $G$-invariant arclength $x$ of the curve $\gamma(t, x)$.

3. Algebraic preliminaries

Recall, the complex symplectic group $Sp(n, \mathbb{C})$ is the group of matrices $g$ in $GL(2n, \mathbb{C})$ that leaves invariant the exterior form $z_1 \wedge z_{n+1} + \cdots + z_n \wedge z_{2n}$ in terms of coordinates $(z_1, \ldots, z_{2n}) \in \mathbb{C}^{2n}$, i.e.

$$g^*Jg = J,$$

(3.1)
where
\[ J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \tag{3.2} \]
with \( I_n \) denoting the identity matrix in \( GL(n, \mathbb{C}) \). Also recall, the complex unitary group \( U(2n) \) is the group of matrices \( g \) in \( GL(2n, \mathbb{C}) \) that leaves invariant the Hermitian form \( z_1 \bar{z}_1 + \cdots + z_{2n} \bar{z}_{2n} \), i.e.
\[ g^* \bar{g} = I_{2n}. \tag{3.3} \]

The compact symplectic group is defined by \( Sp(n) = Sp(n, \mathbb{C}) \cap U(2n) \).

For later convenience, we let \( s(n, \mathbb{C}) \) denote the vector space of symmetric matrices \( g \) in \( gl((n, \mathbb{C})) \), i.e. \( g^t = g \).

A general reference for the following material is [11, 9].

3.1. The special unitary Lie algebra \( su(2n) \)

The special unitary Lie algebra \( su(2n) \) is defined by the matrices \( g \) in \( gl(2n, \mathbb{C}) \) that are skew-Hermitian and trace-free, i.e. \( g^t = -\bar{g}, \) \( \text{tr}(g) = 0 \). There is an involutive automorphism of \( gl(2n, \mathbb{C}) \) given by
\[ \sigma(g) = JgJ^{-1} \tag{3.4} \]

preserving \( su(2n) \subset gl(2n, \mathbb{C}) \). The matrices \( h \) in \( gl(2n, \mathbb{C}) \) that are skew-Hermitian, trace-free and invariant under \( \sigma \), i.e. \( h^t = -\bar{h}, \) \( \text{tr}(h) = 0 \), \( \sigma(h) = h \), span the compact symplectic Lie algebra \( sp(n) \). This leads to the orthogonal decomposition of \( g = su(2n) \) as a symmetric Lie algebra given by the eigenspaces of \( \sigma \),
\[ h := sp(n) \subset g, \quad \sigma(h) = h \tag{3.5} \]

and
\[ m := su(2n)/sp(n) \subset g, \quad \sigma(m) = -m. \tag{3.6} \]

Lemma 3.1.

1. The matrix representations of the vector space \( m = su(2n)/sp(n) \) and the Lie subalgebra \( h = sp(n) \) in \( gl(2n, \mathbb{C}) \) are respectively given by
\[ (A, B) := \left( \begin{array}{cc} A & B \\ \bar{B} & -\bar{A} \end{array} \right) \in m, \quad B^t = -B, \quad A^t = -\bar{A}, \quad \text{tr}(A) = 0, \tag{3.7} \]
\[ (C, D) := \left( \begin{array}{cc} C & D \\ -\bar{D} & \bar{C} \end{array} \right) \in h, \quad C^t = -\bar{C}, \quad D^t = D. \tag{3.8} \]

where \( A, B, C, D \in gl(n, \mathbb{C}) \). The Lie bracket relations (2.2) have the matrix representation
\[
(A_1, B_1), (A_2, B_2) = ([A_1, A_2] + B_1 \bar{B}_2 - B_2 \bar{B}_1, A_1B_2 + B_2 \bar{A}_1 - B_1 \bar{A}_2 - A_2B_1) \in h. \tag{3.9a}
\]
\[
(A_1, B_1), (C_1, D_1) = ([A_1, C_1] - B_1 \bar{D}_1 - D_1 \bar{B}_1, A_1D_1 + D_1 \bar{A}_1 + B_1 \bar{C}_1 - C_1B_1) \in m. \tag{3.9b}
\]
\[
(C_1, D_1), (C_2, D_2) = ([C_1, C_2] - D_1 \bar{D}_2 + D_2 \bar{D}_1, C_1D_2 - D_2 \bar{C}_1 + D_1 \bar{C}_2 - C_2D_1) \in h. \tag{3.9c}
\]
(2) The restriction of the Cartan–Killing form on \( g = \mathfrak{su}(2n) \) to \( m = \mathfrak{su}(2n)/\mathfrak{sp}(n) \) yields a negative-definite inner product
\[
\langle (A_1, B_1), (A_2, B_2) \rangle = 4n(2\text{tr}(A_1A_2) + \text{tr}(B_1B_2 + B_1B_2)).
\] (3.10)

(3) The (real) dimension of \( m = \mathfrak{su}(2n)/\mathfrak{sp}(n) \) is \((n - 1)(2n + 1)\) and its rank is \( n - 1 \).

The subspace \( a \subset m \) spanned by the \( n - 1 \) matrices
\[
\left( \begin{array}{cc} E_k & 0 \\ 0 & E_{k-1} \end{array} \right), \quad E_k = \text{diag}(0, \ldots, 0, i, -i, 0, \ldots, 0), \quad k = 1, \ldots n - 1
\] (3.11)
is a Cartan subspace. A special choice of an element of \( a \) is given by
\[
e := \left( \begin{array}{cc} E & 0 \\ 0 & -E \end{array} \right) \in m = \mathfrak{su}(2n)/\mathfrak{sp}(n), \quad E = \frac{1}{\sqrt{\chi}} \text{diag}((n - 1)i, -i, \ldots, -i) = -E,
\] (3.12)
which has the distinguishing property that the centralizer subspace \( c(e) \) of \( e \) in \( g = \mathfrak{su}(2n) \) is of maximal dimension. The corresponding linear operator \( \text{ad}(e) \) induces a direct sum decomposition of the vector spaces \( m = \mathfrak{su}(2n)/\mathfrak{sp}(n) \) and \( \mathfrak{h} = \mathfrak{sp}(n) \) into centralizer spaces \( m_i \) and \( h_i \) and their orthogonal complements (perp spaces) \( m_i^\perp \) and \( h_i^\perp \) with respect to the Cartan–Killing form. Through the Lie bracket relation (2.16), this operator \( \text{ad}(e) \) maps \( h_i^\perp \) into \( m_i \), and vice versa, whence \( \text{ad}(e)^2 \) is well defined as a linear mapping of each subspace \( h_i^\perp \) and \( m_i \) into itself. The eigenvalues of this linear map can be normalized relative to the Cartan–Killing form by choosing the factor \( \chi \) so that \( e \) has unit norm,
\[
-1 = \langle e, e \rangle = 8n \text{tr}(E^2) = -8(n - 1)n^2/\chi,
\] (3.13)
which determines
\[
\chi = 8(n - 1)n^2.
\] (3.14)

**Lemma 3.2.**

(1) The matrix representations of \( m_i \) and \( m_i^\perp \) in \( m = \mathfrak{su}(2n)/\mathfrak{sp}(n) \) are given by
\[
(A_i, B_i) := \left( \begin{array}{cc} A_i & B_i \\ \overline{B_i} & -\overline{A_i} \end{array} \right) \in m_i, \quad (a_i, b_i) := \left( \begin{array}{cc} A_i & B_i \\ \overline{B_i} & -\overline{A_i} \end{array} \right) \in m_i^\perp.
\] (3.15)
in which
\[
A_i = \begin{pmatrix} -\text{tr}A_i & 0 \\ 0 & A_i \end{pmatrix}, \quad B_i = \begin{pmatrix} 0 & 0 \\ 0 & B_i \end{pmatrix}, \quad A_i^\perp = -\overline{A_i}, \quad B_i^\perp = -B_i,
\]
\[
a_i = \begin{pmatrix} 0 & a_i \\ -\overline{a_i} & 0 \end{pmatrix}, \quad b_i = \begin{pmatrix} 0 & b_i^\perp \\ -b_i^\perp & 0 \end{pmatrix},
\]
where \( A_i \in \mathfrak{u}(n-1) \), \( B_i \in \mathfrak{so}(n-1, \mathbb{C}) \), \( a_i, b_i \in \mathbb{C}^{n-1} \).

(2) The matrix representations of \( h_i \) and \( h_i^\perp \) in \( \mathfrak{h} = \mathfrak{sp}(n) \) are given by
\[
(c_i, d_i) := \left( \begin{array}{cc} C_i & D_i \\ -\overline{D_i} & \overline{C_i} \end{array} \right) \in h_i, \quad (c_i, d_i) := \left( \begin{array}{cc} C_i & D_i \\ -\overline{D_i} & \overline{C_i} \end{array} \right) \in h_i^\perp.
\] (3.16)
in which
\[
C_i = \begin{pmatrix} c_i & 0 \\ 0 & C_i \end{pmatrix}, \quad D_i = \begin{pmatrix} d_i & 0 \\ 0 & D_i \end{pmatrix}, \quad C_i^\perp = -\overline{C_i}, \quad D_i^\perp = -D_i,
\]
\[
C_i = \begin{pmatrix} 0 & c_i \\ -\overline{c_i} & 0 \end{pmatrix}, \quad D_i = \begin{pmatrix} 0 & d_i \\ -\overline{d_i} & 0 \end{pmatrix},
\]
where \( C_i \in \mathfrak{u}(n-1) \), \( D_i \in \mathfrak{so}(n-1, \mathbb{C}) \), \( c_i, d_i \in \mathbb{C}^{n-1} \), \( c_i, d_i \in \mathbb{C} \).
(3) \( \dim m_\parallel = (n - 1)(2n - 3) \), \( \dim m_\perp = \dim h_\perp = 4(n - 1) \), \( \dim h_\parallel = (n - 1)(2n - 1) + 3 \).

(4) The linear operator \( \text{ad}(e) \) acts on \( m_\parallel \) and \( h_\perp \) by

\[
\text{ad}(e)(a_\parallel, b_\parallel) = \frac{1}{\sqrt{\rho}}(ia_\parallel, ib_\parallel) \in h_\perp, \quad \text{ad}(e)(c_\perp, d_\perp) = \frac{1}{\sqrt{\rho}}(ic_\perp, id_\perp) \in m_\parallel
\]

where \( \rho = \chi/n^2 = 8(n - 1) \).

To write out the explicit Lie bracket relations on \( m = m_\parallel \oplus m_\perp \) and \( h = h_\parallel \oplus h_\perp \), we introduce the following inner products and outer products. For \( x, y \in \mathbb{C}^{n-1} \), let

\[
P(x, y) := x\overline{y} - y\overline{x} = 12 \text{Im}(x, y) \in i\mathbb{R},
\]

\[
Q(x, y) := x\overline{y} + y\overline{x} = 2 \text{Re}(x, y) \in \mathbb{R},
\]

\[
S(x, y) := xy^\dagger + yx^\dagger = (x, \overline{y}) + (\overline{y}, x) \in \mathbb{C},
\]

where

\[
\langle x, y \rangle = x\overline{y} = \frac{1}{2}Q(x, y) + \frac{1}{2}P(x, y)
\]

is the Hermitian inner product, and where

\[
\langle x, y \rangle = xy^\dagger = yx^\dagger = \frac{1}{2}S(x, y)
\]

is the standard Euclidean inner product. Also let

\[
P(x, y) = \overline{x}\overline{y} - \overline{y}\overline{x} \in u(n - 1),
\]

\[
Q(x, y) = x\overline{y} + \overline{y}\overline{x} \in so(n - 1, \mathbb{C}),
\]

\[
S(x, y) = x\overline{y} + \overline{y}\overline{x} \in so(n - 1, \mathbb{C}).
\]

The inner products (3.18)–(3.20) have the following symmetry properties:

\[
P(\mathbf{y}, \mathbf{x}) = -P(\mathbf{x}, \mathbf{y}), \quad Q(\mathbf{y}, \mathbf{x}) = Q(\mathbf{x}, \mathbf{y}), \quad S(\mathbf{y}, \mathbf{x}) = S(\mathbf{x}, \mathbf{y}),
\]

while the outer products (3.23)–(3.25) obey the following transpose, symmetry and trace properties

\[
P(\mathbf{y}, \mathbf{x})^\dagger = -\overline{P(\mathbf{x}, \mathbf{y})}, \quad P(\mathbf{y}, \mathbf{x}) = -P(\mathbf{x}, \mathbf{y}),
\]

\[
Q(\mathbf{y}, \mathbf{x})^\dagger = -\overline{Q(\mathbf{x}, \mathbf{y})}, \quad Q(\mathbf{y}, \mathbf{x}) = -\overline{Q(\mathbf{x}, \mathbf{y})},
\]

\[
S(\mathbf{y}, \mathbf{x})^\dagger = S(\mathbf{x}, \mathbf{y}), \quad S(\mathbf{y}, \mathbf{x}) = S(\mathbf{x}, \mathbf{y}),
\]

\[
tr(P(\mathbf{y}, \mathbf{x})) = P(\mathbf{y}, \mathbf{x}), \quad tr(Q(\mathbf{y}, \mathbf{x})) = 0, \quad tr(S(\mathbf{y}, \mathbf{x})) = 2\langle \mathbf{x}, \mathbf{y} \rangle.
\]

Proposition 3.3.

(1) The Lie brackets (2.14)–(2.16) are given by

\[
[(A_\parallel, B_\parallel)] = (0, 0), \quad ([A_\parallel, A_\parallel] - B_\parallel \overline{B}_\parallel + B_\parallel \overline{B}_\parallel),
\]

\[
A_\parallel B_\parallel + B_\parallel \overline{A}_\parallel = B_\parallel \overline{A}_\parallel = A_\parallel B_\parallel \in h_\parallel,
\]

\[
[(c_\parallel, d_\parallel)] = \{[C_\parallel, A_\parallel] + D_\parallel \overline{B}_\parallel + B_\parallel \overline{D}_\parallel, [C_\parallel B_\parallel - B_\parallel \overline{C}_\parallel - D_\parallel \overline{B}_\parallel - A_\parallel \overline{D}_\parallel] \subseteq m_\parallel.
\]

(2) \(|(c_\parallel, d_\parallel)| \subseteq (c_\parallel, d_\parallel), (c_\parallel, d_\parallel) \}

\[
= \{D_\parallel \overline{D}_\parallel - D_\parallel \overline{D}_\parallel, C_\parallel D_\parallel - D_\parallel \overline{C}_\parallel + D_\parallel \overline{D}_\parallel - C_\parallel \overline{D}_\parallel \}
\]

\[
\subseteq h_\parallel.
\]

(3.28c)
\[(c_1, d_1), (C_1, D_1), (a_2, b_2)] = (-a_2C_1 + c_1a_2 + b_2D_1 + d_1B_2, \]
\[-a_2D_1 - d_1\bar{C}_2 - b_2C_1 + c_1b_2) \in m_\perp, \quad (3.29a)\]
\[(c_1, d_1), (C_1, D_1), (c_2, d_2)] = (c_1c_2 - c_2C_1 - d_1\bar{C}_2 + d_2C_1, \]
\[c_1d_2 - d_2C_1 + d_1c_2 - c_2D_1) \in b_\perp, \quad (3.29b)\]
\[(A_1, B_1), (a_2, b_2)] = (-\langle A_1\rangle a_2 - a_2A_1 - b_2B_1, \]
\[\langle A_1\rangle b_2 + b_2A_1 - a_2B_1) \in b_\perp, \quad (3.30a)\]
\[(A_1, B_1), (c_2, d_2)] = (-\langle A_1\rangle c_2 - c_2A_1 - d_2B_1, \]
\[-\langle A_1\rangle d_2 + d_2A_1 - c_2B_1) \in m_\perp. \quad (3.30b)\]

(2) The remaining Lie brackets (2.17) are given by

\[\langle a_1, b_1, (a_2, b_2) = ((P(a_2, a_1) - P(b_2, b_2), -S(a_1, b_2) + S(b_1, a_2)), \]
\[P(a_1, b_1) + P(b_1, b_2, b_2), -S(b_2, a_1) + S(b_1, a_2)) \in b_\perp, \quad (3.31a)\]
\[\langle c_1, d_1, (c_2, d_2) = (P(c_2, c_1) + P(d_2, d_1), S(c_1, c_2) - S(d_1, c_2)), \]
\[P(c_1, c_2) + P(d_1, d_2) - S(d_2, c_1) + S(c_1, c_2)) \in b_\perp, \quad (3.31b)\]
\[\langle a_1, b_1, (c_2, d_2) = \langle P(c_2, c_1) + P(d_2, b_1), \]
\[-Q(d_2, c_1) - Q(b_1, c_2) \rangle) \in m_\perp. \quad (3.31c)\]

(3) The Cartan–Killing form on \(m_\perp\) is given by

\[
\langle a_1, b_1, (a_2, b_2) = -8n(Q(a_1, a_2) + Q(b_1, b_2)). \quad (3.32)
\]

The adjoint action of the Lie subalgebra \(h_\perp \subset h = sp(n)\) on \(\mathfrak{g} = su(n)\) generates the linear transformation group \(H^+_n \subset H^+ = \text{Ad}(H)\) that preserves the element 1 in the Cartan subspace \(a \subset m = su(n)/sp(n)\). This group \(H^+_n\) can be identified with the adjoint action of a symplectic group \(Sp(1) \times Sp(n-1) \subset Sp(n)\) whose matrix representation is given by

\[
\begin{pmatrix}
C & D \\
-D & C
\end{pmatrix} \in Sp(1) \times Sp(n-1) \simeq H^+_n, \quad C = \begin{pmatrix} c & 0 \\
0 & C \end{pmatrix}, \quad D = \begin{pmatrix} d & 0 \\
0 & D \end{pmatrix}, \quad (3.33)
\]

where

\[CC + DDB = I_{n-1}, \quad CDB = 0, \quad (3.34)\]
\[c + d = 1. \quad (3.35)\]

In particular, the subgroup \(Sp(n-1) \subset H^+_n\) acts on \(m_\perp\) by right multiplication,

\[
\text{Ad}(C, D) (a_1, b_1) = (a_1C + b_1D, -a_1D + b_1C) \in m_\perp, \quad (3.36)
\]

\[
\text{where} \quad (C, D) \in Sp(n-1) \text{ is defined to be the matrix } (3.33) \text{ with } c = 1 \text{ and } d = 0, \text{ and the subgroup } Sp(1) \subset H^+_n \text{ acts similarly as}
\]
\[
\text{Ad}(c, d) (a_1, b_1) = (ca_1 + d\bar{b}_1, cb_1 - d\bar{a}_1) \in m_\perp, \quad (3.37)
\]

\[
\text{where} \quad (c, d) \in Sp(1) \text{ is defined to be the matrix } (3.33) \text{ with } C = I_{n-1} \text{ and } D = 0. \text{ Composition of these subgroups } (3.36) \text{ and } (3.37) \text{ yields the group } H^+_n = \text{Ad}(Sp(1) \times Sp(n-1)) \subset \text{Ad}(Sp(n)).
\]
The symplectic Lie algebra span the compact symplectic Lie algebra $\sigma$. There is an involutive automorphism of $J. \text{Phys. A: Math. Theor.}$  

**Proposition 3.4.** The vector space $m_{\perp} \simeq C^{n-1} \oplus C^{n-1}$ is an irreducible representation of the group $H_{\perp}$ on which the linear map $\text{ad}(\epsilon)^{2}$ is a multiple of the identity 

$$\text{ad}(\epsilon)^{2}(a_{\perp}, b_{\perp}) = -\frac{1}{\rho}(a_{\perp}, b_{\perp}), \quad (3.38)$$

where 

$$\rho = 8(n-1). \quad (3.39)$$

3.2. The vector space $\mathfrak{sp}(n+1)/\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$

The symplectic Lie algebra $\mathfrak{sp}(n+1)$ consists of all matrices $g$ in $\mathfrak{gl}(2(n+1), \mathbb{C})$ satisfying 

$$gJ + Jg^{t} = 0, \quad g^{t} = -\overline{g}, \quad J = \begin{pmatrix} 0 & In_{n+1} \\ -In_{n+1} & 0 \end{pmatrix}. \quad (3.40)$$

There is an involutive automorphism of $\mathfrak{gl}(2(n+1), \mathbb{C})$ given by 

$$\sigma(g) = SgS, \quad S = \begin{pmatrix} I_{n,1} & 0 \\ 0 & I_{n,1} \end{pmatrix}, \quad I_{n,1} = \begin{pmatrix} 1 & 0 \\ 0 & -In \end{pmatrix} \quad (3.41)$$

preserving $\mathfrak{sp}(n+1) \subset \mathfrak{gl}(2(n+1), \mathbb{C})$. The matrices in $\mathfrak{sp}(n+1)$ that are invariant under $\sigma$ span the compact symplectic Lie algebra $\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$. This leads to the orthogonal decomposition of $\mathfrak{g} = \mathfrak{sp}(n+1)$ as a symmetric Lie algebra given by the eigenspaces of $\sigma$, 

$$\mathfrak{h} := \mathfrak{sp}(1) \oplus \mathfrak{sp}(n) \subset \mathfrak{g}, \quad \sigma(\mathfrak{h}) = \mathfrak{h} \quad (3.42)$$

and 

$$\mathfrak{m} := \mathfrak{sp}(n+1)/\mathfrak{sp}(1) \oplus \mathfrak{sp}(n) \subset \mathfrak{g}, \quad \sigma(\mathfrak{m}) = -\mathfrak{m}. \quad (3.43)$$

**Lemma 3.5.**

1. The matrix representation of the Lie algebra $\mathfrak{g} = \mathfrak{sp}(n+1)$ is given by 

$$\begin{pmatrix} A & B \\ -B & \overline{A} \end{pmatrix} \in \mathfrak{sp}(n+1), \quad A^{t} = -\overline{A}, \quad B^{t} = B \quad (3.44)$$

where $A, B \in \mathfrak{gl}(n+1, \mathbb{C})$. The matrix representations of the vector space $m = \mathfrak{sp}(n+1)/\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$ and the Lie subalgebra $\mathfrak{h} = \mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$ in $\mathfrak{gl}(2(n+1), \mathbb{C})$ are respectively given by 

$$\begin{pmatrix} A & B \\ -B & \overline{A} \end{pmatrix} \in \mathfrak{m}, \quad A^{t} = -\overline{A}, \quad B^{t} = B \quad (3.45)$$

$$\begin{pmatrix} C & D \\ -D & \overline{C} \end{pmatrix} \in \mathfrak{h}, \quad C^{t} = -\overline{C}, \quad D^{t} = D \quad (3.46)$$

in which 

$$A = \begin{pmatrix} 0 & a \\ -\overline{a} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & b \\ -\overline{b} & 0 \end{pmatrix} \quad (3.47)$$

$$C = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix} \quad (3.48)$$

where $a, b \in \mathbb{C}^{n}, c \in i\mathbb{R}, d \in \mathbb{C}, C \in \mathfrak{u}(n, \mathbb{C}), D \in \mathfrak{a}(n, \mathbb{C})$. 


(2) The Lie bracket relations (2.2) have the matrix representation

\[
\begin{pmatrix}
(a_1, b_1), (a_2, b_2) = ((a_2 a'_1 - a_1 a'_2 + b_2 b'_1 - b_1 b'_2, a_1 b'_2 + b_2 a'_1 - b_1 a'_2 - a_1 b'_1), \\
(a_1 a'_1 - a'_1 a_2 + b_1 b'_1 - b'_1 b_2, -b'_1 a_1 - a'_1 b_1) \in h,
\end{pmatrix}
\]

\[(3.49a)\]

\[
\begin{pmatrix}
(a_1, b_1), ((c_1, d_1), (C_1, D_1)) = (a_1 C_1 - c_1 a_1 - b_1 D_1 + d_1 b_1, \\
a_1 D_1 - d_1 a_1 + b_1 C_1 - c_1 b_1) \in m,
\end{pmatrix}
\]

\[(3.49b)\]

\[
\begin{pmatrix}
((c_1, d_1), (C_1, D_1)), ((c_2, d_2), (C_2, D_2)) = ((-d_1 C_2 - d_2 C_1, c_1 D_2 - d_2 C_1 + d_1 C_2 - c_2 D_1, \\
C_1 C_2 - C_2 C_1 - D_1 D_2 + D_2 D_1, C_1 D_2 - D_2 C_1 + D_1 C_2 - C_2 D_1) \in h.
\end{pmatrix}
\]

\[(3.49c)\]

(3) The restriction of the Cartan–Killing form on \(\mathfrak{g} = \mathfrak{sp}(n+1)\) to \(\mathfrak{m} = \mathfrak{sp}(n+1)/\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)\) yields a negative-definite inner product

\[
((a_1, b_1), (a_2, b_2)) = -4(n+2)(a_1 a'_2 + a_2 a'_1 + b_1 b'_2 + b_2 b'_1).
\]

\[(3.50)\]

(4) The (real) dimension of \(\mathfrak{m} = \mathfrak{sp}(n+1)/\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)\) is \(4n\) and its rank is 1.

The one-dimensional subspace \(\mathfrak{a} \subset \mathfrak{m} = \mathfrak{sp}(n+1)/\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)\) spanned by the matrix

\[
(e_1, 0) := \begin{pmatrix} E_1 & 0 \\ 0 & E_1 \end{pmatrix} \in \mathfrak{m}, \quad E_1 = \begin{pmatrix} 0 & e_1 \\ -e_1 & 0 \end{pmatrix}, \quad e_1 = (1, 0, \ldots, 0)_{n-1}
\]

is a Cartan subspace. The element

\[
e := \frac{1}{\sqrt{\chi}} (e_1, 0) \in \mathfrak{a}
\]

in this subspace has unit norm, where

\[-1 = \langle e, e \rangle = -8(n+2)/\chi
\]

determines

\[
\chi = 8(n+2).
\]

The corresponding linear operator \(\text{ad}(e)\) induces a direct sum decomposition of the vector spaces \(\mathfrak{m} = \mathfrak{sp}(n+1)/\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)\) and \(\mathfrak{h} = \mathfrak{sp}(n)\) into centralizer spaces \(\mathfrak{m}_\parallel\) and \(\mathfrak{h}_\parallel\) and their orthogonal complements (perp spaces) \(\mathfrak{m}_\perp\) and \(\mathfrak{h}_\perp\) with respect to the Cartan–Killing form. From the Lie bracket relation (2.16), \(\mathfrak{h}_\parallel\) is mapped into \(\mathfrak{m}_\perp\), and vice versa, under \(\text{ad}(e)\). Hence, \(\text{ad}(e)^2\) defines a linear mapping of each subspace \(\mathfrak{h}_\perp\) and \(\mathfrak{m}_\perp\) into itself.

**Lemma 3.6.**

(1) The matrix representations of \(\mathfrak{m}_\parallel\) and \(\mathfrak{m}_\perp\) in \(\mathfrak{m} = \mathfrak{sp}(n+1)/\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)\) are given by

\[
(a_1) := \begin{pmatrix} A_\parallel & B_\parallel \\ -B_\parallel & A_\parallel \end{pmatrix} \in \mathfrak{m}_\parallel, \quad ((a_\perp, b_\perp), (a_\perp, b_\perp)) := \begin{pmatrix} A_\perp & B_\perp \\ -B_\perp & A_\perp \end{pmatrix} \in \mathfrak{m}_\perp
\]

in which

\[
A_\parallel = \begin{pmatrix} 0 & a_\parallel \\ -a_\parallel & 0 \end{pmatrix}, \quad B_\parallel = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad a_\parallel \in \mathbb{R},
\]

\[
A_\perp = \begin{pmatrix} 0 & a_\perp \\ a_\perp & 0 \end{pmatrix}, \quad B_\perp = \begin{pmatrix} 0 & b_\perp \\ b_\perp & 0 \end{pmatrix}, \quad a_\perp \in \mathbb{R}, b_\perp \in \mathbb{C},
\]

where \(a_\parallel \in \mathbb{R}, a_\perp \in i\mathbb{R}, b_\perp \in \mathbb{C}, a_\perp, b_\perp \in \mathbb{C}^{n-1}\).
(2) The matrix representations of $h_\|$ and $h_\perp$ in $h = \mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$ are given by

\[
(c_\|, d_\|, \mathbf{C}_\|, \mathbf{D}_\|) := \begin{pmatrix} C_\| & D_\| \\ -\bar{D}_\| & \bar{C}_\| \end{pmatrix} \in h_\|
\]

\[
(c_\perp, d_\perp, \mathbf{C}_\perp, \mathbf{D}_\perp) := \begin{pmatrix} C_\perp & D_\perp \\ -\bar{D}_\perp & \bar{C}_\perp \end{pmatrix} \in h_\perp,
\]

in which

\[
C_\| = \begin{pmatrix} c_\| & 0 & 0 \\ 0 & c_\| & 0 \\ 0 & 0 & C_\| \end{pmatrix}, \quad D_\| = \begin{pmatrix} d_\| & 0 & 0 \\ 0 & d_\| & 0 \\ 0 & 0 & D_\| \end{pmatrix}, \quad C_\perp = -\bar{C}_\|, \quad D_\perp = D_\|
\]

\[
C_\perp = \begin{pmatrix} c_\perp & 0 & 0 \\ 0 & -c_\perp & 0 \\ 0 & 0 & C_\perp \end{pmatrix}, \quad D_\perp = \begin{pmatrix} d_\perp & 0 & 0 \\ 0 & -d_\perp & 0 \\ 0 & 0 & D_\perp \end{pmatrix}
\]

where $C_\| \in \mathfrak{u}(n-1)$, $D_\| \in \mathfrak{s}(n-1, \mathbb{C})$, $c_\|, d_\|, C_\| \in \mathbb{C}^{n-1}$, $c_\perp \in i\mathbb{R}$, $d_\perp \in \mathbb{C}$.

(3) $\dim m_\| = 1$, $\dim m_\perp = \dim h_\perp = 2n + 1$, $\dim h_\| = 2(n - 1)^2 + n + 2$.

(4) The linear operator $\text{ad}(e)$ acts on $m_\perp$ and $h_\|$ by

\[
\text{ad}(e)((a_\perp, b_\perp), (a_\perp, b_\perp)) = \frac{1}{\sqrt{\chi}}((2a_\perp, 2b_\perp), (-a_\perp, -b_\perp)) \in h_\perp,
\]

\[
\text{ad}(e)((c_\perp, d_\perp), (c_\perp, d_\perp)) = \frac{1}{\sqrt{\chi}}((-2c_\perp, -2d_\perp), (c_\perp, d_\perp)) \in m_\perp.
\]

We use the inner products (3.18)–(3.20) and outer products (3.23)–(3.25) to write out the explicit Lie bracket relations on $m = m_\| \oplus m_\perp$ and $h = h_\| \oplus h_\perp$.

Proposition 3.7.

(1) The Lie brackets (2.14)–(2.16) are given by

\[
[[a_1], [a_1]] = 0 \in h_\|
\]

\[
[[c_1, d_1], (C_1, D_1)), (a_2)] = 0 \in m_1
\]

\[
[[c_1, d_1], (C_1, D_1)), ((c_2, d_2), (C_2, D_2))] = ((d_2\bar{C}_1 - d_1\bar{D}_2, c_1d_2 + d_2c_1 - d_1c_2d_1),
\]

\[
((C_1, C_2) + D_1\bar{D}_2 - D_1\bar{D}_2, C_1\bar{D}_2 - D_2\bar{C}_2 - C_2\bar{D}_1)) \in h_1,
\]

\[
[[c_1, d_1], (C_1, D_1)), ((a_2, b_2), (a_2, b_2)) = ((-d_2\bar{b}_1 + b_2\bar{a}_1, c_1b_2 + b_2c_1 - d_1a_2 - a_2d_1),
\]

\[
(c_1a_2 - a_2c_1, -d_1\bar{b}_2 + b_2\bar{D}_1, c_1b_2 - b_2\bar{C}_1, d_1\bar{a}_2 - a_2D_1)) \in m_1
\]

\[
[[c_1, d_1], (C_1, D_1)), (c_2, d_2), (c_2, d_2)) = ((-d_1\bar{c}_2 + d_2\bar{C}_1, c_1d_2 + d_2c_1 - d_1c_2d_1),
\]

\[
(c_1c_2 - c_2c_1, -d_1\bar{d}_2 + d_2\bar{D}_1, c_1d_2 - d_2\bar{C}_1 + d_1\bar{b}_2 - c_2D_1)) \in h_1
\]
[\([a_{11}]), (a_{21}, b_{11}), (a_{21}, b_{21})]\]
\[= (2a_{11}a_{21}, 2a_{11}b_{11}, -a_{11}a_{21}, -a_{11}b_{21}) \in h_{\perp}, \quad (3.60a)\]

\[[a_{11}), (c_{21}, d_{11}), (c_{21}, d_{21})]\]
\[= (−2a_{11}c_{21}, -2a_{11}d_{11}, (a_{11}c_{21}, a_{11}d_{21})) \in m_{\perp}. \quad (3.60b)\]

(2) The remaining Lie brackets (2.17) are given by
\[[\([b_{11}), (a_{11}, b_{11}), (a_{21}, b_{21})]\]_{\mathcal{h}_{\perp}}
\[= ((−\frac{1}{2}P(a_{11}, a_{21}) − \frac{1}{2}P(b_{11}, b_{21}) − b_{11}b_{21} + b_{21}b_{11},
2a_{11}b_{21} − 2a_{21}b_{11} + \frac{1}{2}S(a_{11}, b_{21}) − \frac{1}{2}S(b_{11}, a_{21})),
(−P(a_{11}, a_{21}) + P(b_{11}, b_{21}) \frac{1}{2}, −S(b_{11}, a_{21}) + S(b_{11}, a_{21})) \in \mathcal{h}_{\perp}. \quad (3.61a)\]

\[[\([c_{11}), (c_{11}, d_{11}), (c_{21}, d_{21})]\]_{\mathcal{h}_{\perp}}
\[= ((−d_{11}d_{21} + d_{21}d_{11} − \frac{1}{2}P(c_{11}, c_{21}) − \frac{1}{2}P(d_{11}, d_{21}),
2c_{11}d_{21} − 2d_{11}c_{21} + \frac{1}{2}S(c_{11}, d_{21}) − \frac{1}{2}S(d_{11}, c_{21})),
(−P(c_{11}, c_{21}) + P(d_{11}, d_{21}) \frac{1}{2}, −S(d_{11}, c_{21}) + S(d_{11}, c_{21})) \in \mathcal{h}_{\perp}. \quad (3.61b)\]

\[[\([c_{11}), (c_{11}, d_{11}), (c_{21}), (c_{21}, d_{21})]\]_{\mathcal{h}_{\perp}}
\[= ((\frac{1}{2}P(c_{11}, c_{21}) + \frac{1}{2}P(d_{11}, d_{21}), −\frac{1}{2}S(c_{11}, d_{21}) + \frac{1}{2}S(d_{11}, c_{21})),
(−c_{11}c_{21} + c_{21}c_{11} + d_{11}d_{21} − d_{21}d_{11}, −c_{11}d_{21} + d_{21}c_{21} − c_{11}c_{21} + c_{21}d_{11})) \in \mathcal{h}_{\perp}. \quad (3.62a)\]

\[[\([a_{11}), (a_{11}, b_{11}), (a_{21}, b_{21}), (a_{21}, d_{21})]\]_{\mathcal{m}_{\perp}}
\[= (b_{11}d_{21} + d_{21}b_{11} − \frac{1}{2}Q(b_{11}, d_{21}) − 2a_{11}c_{21} − \frac{1}{2}Q(a_{11}, c_{21})) \in \mathcal{m}_{\perp}. \quad (3.63a)\]

\[[\([a_{11}), (a_{11}, b_{11}), (c_{21}, d_{21}), (c_{21}, d_{21})]\]_{\mathcal{m}_{\perp}}
\[= ((−\frac{1}{2}P(b_{11}, d_{21}) − \frac{1}{2}P(a_{11}, c_{21}), \frac{1}{2}S(a_{11}, d_{21}) − \frac{1}{2}S(b_{11}, c_{21})),
(a_{11}d_{21} − c_{21}a_{11} − b_{11}d_{21} + b_{21}d_{11}, a_{11}d_{21} − d_{21}a_{11} + b_{11}c_{21} − c_{11}b_{21})) \in \mathcal{m}_{\perp}. \quad (3.63b)\]

(3) The Cartan–Killing form on \([m_{\perp} is given by
\[[\([a_{11}), (a_{11}, b_{11}), (a_{21}), (a_{21}, b_{21})]\]
\[= −4(\pi + 2)(Q(a_{11}, a_{21}) + Q(a_{11}, a_{21}) + Q(b_{11}, b_{21}) + Q(b_{11}, b_{21})). \quad (3.64)\]

\[S C Anco and E Asadi\]
where generates the linear transformation group $H^*_n \subset H^* = \text{Ad}(H)$ that preserves the element $e$ in the Cartan subspace $a \subset m = \text{sp}(n+1)/\text{sp}(1) \oplus \text{sp}(n)$. This group $H^*_n$ can be identified with the adjoint action of a symplectic group $Sp(1) \times Sp(n-1) \subset Sp(1) \times Sp(n)$ whose matrix representation is given by

$$
\begin{pmatrix}
C & D \\
-D & C
\end{pmatrix} \in Sp(1) \times Sp(n-1) \simeq H^*_n, \quad C = \begin{pmatrix}
c & 0 & 0 \\
0 & c & 0 \\
0 & 0 & C
\end{pmatrix}, \quad D = \begin{pmatrix}
d & 0 & 0 \\
0 & d & 0 \\
0 & 0 & D
\end{pmatrix}, \quad (3.65)
$$

where

$$
C'C + D'D = I_{n-1}, \quad C'D - D'C = 0, \quad (3.66)
$$

$$
c\bar{c} + d\bar{d} = 1. \quad (3.67)
$$

In particular, the subgroup $Sp(n-1) \subset H^*_1$ acts on $m_\perp$ by right multiplication,

$$
\text{Ad}(C, D)((a_\perp, b_\perp), (a_\perp, b_\perp)) = ((a_\perp, b_\perp), (a_\perp\bar{C} - b_\perp\bar{D}, a_\perp\bar{D} + b_\perp\bar{C})) \in m_\perp, \quad (3.68)
$$

where $(C, D) \in Sp(n-1)$ is defined to be the matrix (3.65) with $c = 1$ and $d = 0$, while the subgroup $Sp(1) \subset H^*_1$ has a non-standard action on $m_\perp$ given by

$$
\text{Ad}(c, d)((a_\perp, b_\perp), (a_\perp, b_\perp)) = ((ca_\perp\bar{c} - d\bar{b}_\perp e - cb_\perp d + da_\perp d, \\
ca_\perp\bar{d} - d\bar{b}_\perp e + cb_\perp d - da_\perp d, \\
(ca_\perp\bar{D} + cb_\perp d + da_\perp d)) \in m_\perp,
$$

where $(c, d) \in Sp(1)$ is defined to be the matrix (3.65) with $C = I_{n-1}$ and $D = 0$. Composition of these subgroups (3.68) and (3.69) yields the group $H^*_1 = \text{Ad}(Sp(1) \times Sp(n-1)) \subset \text{Ad}(Sp(1) \times Sp(n))$.

**Proposition 3.8.** The vector space $m_\perp \simeq i\mathbb{R} \oplus \mathbb{C} \oplus \mathbb{C}^{n-1} \oplus \mathbb{C}^{n-1}$ is a reducible representation of the group $H^*_1$ such that the linear map $\text{ad}(e)^2$ is given by

$$
\text{ad}(e)^2((a_\perp, b_\perp), (a_\perp, b_\perp)) = \frac{1}{\chi} \left( (-4a_\perp, -4b_\perp), (-a_\perp, -b_\perp) \right). \quad (3.69)
$$

The irreducible subspaces in this representation consist of $((a_\perp, b_\perp), (0, 0)) \simeq i\mathbb{R} \oplus \mathbb{C}$ and $((0, 0), (a_\perp, b_\perp)) \simeq \mathbb{C}^{n-1} \oplus \mathbb{C}^{n-1}$ on which $\text{ad}(e)^2$ is a multiple of the identity with respective eigenvalues $-4/\chi$ and $-1/\chi$.

**4. Bi-Hamiltonian soliton equations in SU(2n)/Sp(n)**

Let $\gamma(t, x)$ be any non-stretching curve flow in $M = SU(2n)/Sp(n)$. Employing the notation and preliminaries in sections 2 and 3.1, we introduce an $Sp(n)$-parallel framing along $\gamma$ as expressed in terms of the variables

$$
e_\gamma = \frac{1}{\sqrt{\chi}} (-i\eta_{n-1}, 0) \in u(n-1) \oplus so(n-1, \mathbb{C}) \simeq m_\parallel, \quad \chi = 8(n-1)n^2, \quad (4.1)
$$

$$
\omega_\gamma = (u_1, u_2) \in \mathbb{C}^{n-1} \oplus \mathbb{C}^{n-1} \simeq h_\perp, \quad (4.2)
$$

and

$$
h_\parallel = (H_{11}, H_{22}) \in u(n-1) \oplus so(n-1, \mathbb{C}) \simeq m_\parallel, \quad (4.3)
$$

$$
h_\perp = (h_{11}, h_{22}) \in \mathbb{C}^{n-1} \oplus \mathbb{C}^{n-1} \simeq m_\perp, \quad (4.4)
$$

$$
\omega_\gamma = ((w_{11}, w_{22}), (W_{11}, W_{22})) \in sp(1) \oplus sp(n-1) \simeq h_\parallel. \quad (4.5)
$$
\[ \omega^\perp = (w^{1\perp}, w^{2\perp}) \in \mathbb{C}^{n-1} \oplus \mathbb{C}^{n-1} \simeq \mathfrak{h}_\perp, \]

(4.6)

using the matrix identifications (3.15)–(3.16), where \( w^{1\perp} \in i\mathbb{R} \) is an imaginary (complex) scalar variable, \( w^{2\perp} \in \mathbb{C} \) is a complex scalar variable, \( u_1, u_2, w^{1\perp}, w^{2\perp}, h_{1\perp}, h_{2\perp} \in \mathbb{C}^{n-1} \) are complex vector variables, \( W^{1\perp}, H_{1\perp} \in \mathfrak{u}(n-1) \) are anti-Hermitian matrix variables, \( W^{2\perp} \in \mathfrak{s}(n-1, \mathbb{C}) \) is a complex symmetric matrix variable and \( H_{2\perp} \in \mathfrak{so}(n-1, \mathbb{C}) \) is a complex anti-symmetric matrix variable. For later use, through property (3.17) we also introduce the variable

\[ h^\perp = (h^{1\perp}, h^{2\perp}) = \text{ad}(e_\gamma) h_\perp = \frac{1}{\sqrt{\rho}} (i h^{1\perp}, i h^{2\perp}) \in \mathbb{C}^{n-1} \oplus \mathbb{C}^{n-1} \simeq \mathfrak{h}_\perp, \quad \rho = 8(n-1), \]

(4.7)

where \( h^{1\perp}, h^{2\perp} \in \mathbb{C}^{n-1} \) are complex vector variables.

Up to the rigid (\( x \)-independent) action of the equivalence group \( H^*_r = \text{Ad}(Sp(1) \times Sp(n-1)) \subset \text{Ad}(Sp(n)) \), an \( Sp(n) \)-parallel linear coframe \( e \) along \( \gamma \) is then determined by the variables (4.1) and (4.2) via the transport equation

\[ \nabla e = -\text{ad}(\omega_\gamma) e \]

(4.8)

together with the soldering relation

\[ e_\parallel | \gamma_\parallel = e_\parallel. \]

(4.9)

The resulting coframe \( e \) defines an isomorphism between \( T_\gamma \mathcal{M} \) and \( m \simeq u(n-1) \oplus \mathfrak{so}(n-1, \mathbb{C}) \oplus \mathbb{C}^{n-1} \oplus \mathbb{C}^{n-1} \), which yields a correspondence between the set of frames for \( T_\gamma \mathcal{M} \) and the set of basis vectors for \( m \), as follows. Let \( e_\parallel \) and \( e_\perp \) be the respective projections of \( e \) into \( m_\parallel \) and \( m_\perp \) given in terms of the matrix identifications (3.15)–(3.16) by

\[ e_\parallel = (A_\parallel (\cdot), B_\parallel (\cdot)) \]

(4.10)

\[ e_\perp = (a_\perp (\cdot), b_\perp (\cdot)), \]

(4.11)

where \( A_\parallel (\cdot) \) and \( B_\parallel (\cdot) \) are linear maps from \( T_\gamma \mathcal{M} \) into \( u(n-1) \) and \( \mathfrak{so}(n-1, \mathbb{C}) \) respectively, and where both \( a_\perp (\cdot) \) and \( b_\perp (\cdot) \) are linear maps from \( T_\gamma \mathcal{M} \) into \( \mathbb{C}^{n-1} \). Let \( (T_\gamma \mathcal{M})_\parallel \) and \( (T_\gamma \mathcal{M})_\perp \) be the orthogonal subspaces of \( T_\gamma \mathcal{M} \) respectively defined by the kernels of \( e_\parallel \) and \( e_\perp \), so thus

\[ e_\parallel | (T_\gamma \mathcal{M})_\perp = e_\perp | (T_\gamma \mathcal{M})_\parallel = 0 \]

and hence

\[ e_\parallel | (T_\gamma \mathcal{M})_\parallel = e_\parallel | T_\gamma \mathcal{M} = m_\parallel \simeq u(n-1) \oplus \mathfrak{so}(n-1, \mathbb{C}), \]

(4.12)

\[ e_\perp | (T_\gamma \mathcal{M})_\perp = e_\perp | T_\gamma \mathcal{M} = m_\perp \simeq \mathbb{C}^{n-1} \oplus \mathbb{C}^{n-1}. \]

(4.13)

Note that, in this notation,

\[ e_\parallel | | \gamma_\parallel = e_\parallel, \quad e_\perp | | \gamma_\parallel = 0, \]

(4.14)

\[ e_\parallel | | \gamma_\parallel = h_\parallel, \quad e_\perp | | \gamma_\parallel = h_\perp. \]

(4.15)

Now if \( \{ M^{(i)}_{\parallel j} \}, \ i = 1, \ldots, (n-1)^2, \) is a matrix basis for \( u(n-1) \) viewed as a real vector space, and \( \{ M^{(i)}_{\perp j} \}, \ j = 1, \ldots, (n-1)(n-2), \) is a matrix basis for \( \mathfrak{so}(n-1, \mathbb{C}) \) viewed as a real vector space, then \( e_\parallel \) determines a corresponding basis \( \{ X^{(i)}_{\parallel j} \}, \ i = 1, \ldots, (n-1)^2 \) and \( j = 1, \ldots, (n-1)(n-2), \) for the vector space \( (T_\gamma \mathcal{M})_\parallel \) given by

\[ (A_\parallel(X^{(i)}_{\parallel j}), B_\parallel(X^{(i)}_{\parallel j})) = (M^{(i)}_{\parallel j}; 0), \quad (A_\parallel(X^{(i)}_{\perp j}), B_\parallel(X^{(i)}_{\perp j})) = (0, M^{(i)}_{\perp j}). \]
Similarly, if \( (m_1^{(k)})_1, k = 1, \ldots, 2(n - 1), \) is a basis for \( \mathbb{C}^{n-1} \) viewed as a real vector space, then \( e_\perp \) determines a corresponding basis \( \{X_{1\perp}^{(k)}, X_{2\perp}^{(k)}\}, k, k' = 1, \ldots, 2(n - 1), \) for the vector space \( (T_\rho M)_\perp \) given by
\[
(a_\perp(X_{1\perp}^{(k)}), b_\perp(X_{2\perp}^{(k)})) = (m_{1\perp}^{(k)}, 0), \quad (a_\perp(X_{2\perp}^{(k)}), b_\perp(X_{2\perp}^{(k)})) = (0, m_{2\perp}^{(k)}).
\]
In addition, if each basis \( \{M_{1\rho}^{(i)}, M_{2\rho}^{(j)}\}, \{m_{1\rho}^{(i)}, m_{2\rho}^{(j)}\} \) is normalized such that
\[
(M_{1\rho}^{(i)}, M_{2\rho}^{(j)}) = -\delta_{ij}, \quad (M_{1\rho}^{(j)}M_{2\rho}^{(j)}) = -\delta_{jj}, \quad (m_{1\rho}^{(i)}, m_{2\rho}^{(j)}) = -\delta_{ij},
\]
then the basis for \( T_\rho M = (T_\rho M)_1 \oplus (T_\rho M)_\perp \) has the corresponding normalization
\[
g(X_{1\rho}^{(i)}, X_{1\rho}^{(j)}) = \delta_{ij}, \quad g(X_{2\rho}^{(i)}, X_{2\rho}^{(j)}) = \delta_{jj}, \quad g(X_{1\rho}^{(i)}, X_{2\rho}^{(j)}) = g(X_{2\rho}^{(i)}, X_{1\rho}^{(j)}) = \delta_{ij}.
\]
Consequently, the resulting orthonormal frame
\[
\{X_{1\rho}^{(i)}, X_{2\rho}^{(i)}, X_{1\perp}^{(k)}, X_{2\perp}^{(k)}\}
\]
(4.16)
can be shown to satisfy the Frenet equations
\[
\begin{align*}
\nabla_i X_{1\rho}^{(i)} &= \sum_k U_{i,j}^{(i,k)} X_{1\rho}^{(k)} + \sum_k U_{i,j}^{(i,k)} X_{2\rho}^{(k)}, \\
\nabla_i X_{2\rho}^{(i)} &= \sum_k U_{i,j}^{(j,k)} X_{1\rho}^{(k)} + \sum_k U_{i,j}^{(j,k)} X_{2\rho}^{(k)}, \\
\nabla_i X_{1\rho}^{(k)} &= -\sum_i U_{i,j}^{(i,k)} X_{1\rho}^{(j)} - \sum_j U_{i,j}^{(i,k)} X_{1\rho}^{(j)}, \\
\nabla_i X_{2\rho}^{(k)} &= -\sum_i U_{i,j}^{(i,k)} X_{2\rho}^{(j)} - \sum_j U_{i,j}^{(i,k)} X_{2\rho}^{(j)}
\end{align*}
\]
(4.17)
obtained from the transport equation (4.8) combined with the Lie brackets (3.30b) and (3.31c), where
\[
\begin{align*}
U_{i,j}^{(i,k)} &= \{\{M_{1\rho}^{(i)}, 0\}, (u_1, u_2)\}, \quad \{m_{1\rho}^{(k)}, 0\} = 8nQ((trM_{1\rho}^{(i)})u_1 + u_1 M_{1\rho}^{(i)}, m_{2\rho}^{(k)}) \\
U_{i,j}^{(j,k)} &= \{\{M_{2\rho}^{(j)}, 0\}, (u_1, u_2)\}, \quad \{m_{2\rho}^{(k)}, 0\} = 8nQ((trM_{2\rho}^{(j)})u_1 - u_1 M_{2\rho}^{(j)}, m_{1\rho}^{(k)}) \\
U_{i,j}^{(j,k)} &= \{\{0, M_{2\rho}^{(j)}\}, (u_1, u_2)\}, \quad \{m_{1\rho}^{(k)}, 0\} = 8nQ(u_2 M_{2\rho}^{(j)}, m_{1\rho}^{(k)}) \\
U_{i,j}^{(i,k)} &= \{\{0, M_{1\rho}^{(i)}\}, (u_1, u_2)\}, \quad \{m_{2\rho}^{(k)}, 0\} = 8nQ(u_1 M_{1\rho}^{(i)}, m_{2\rho}^{(k)})
\end{align*}
\]
(4.18)
denote the Cartan matrix components of the underlying \( Sp(n) \)-parallel linear connection (4.2) projected into the tangent space of the curve.

The geometrical meaning of this linear connection is seen through looking at the frame components of the principal normal vector
\[
N := \nabla_i X = (e^*, \text{ad}(e^*)\omega_1)
\]
given by
\[
e^* N = -\text{ad}(e_\perp)\omega_\perp = -\frac{1}{\sqrt{p}} (i u_1, i u_2) \in \mathbb{C}^{n-1} \oplus \mathbb{C}^{n-1} \simeq m_\perp, \quad \rho = 8(n - 1)
\]
(4.20)
again using relation (3.17). These components \( (i u_1, i u_2) \) are invariantly defined by the curve \( \gamma \) up to the rigid \( (x\)-independent) action of the equivalence group \( H_\gamma = \text{Ad}(Sp(1) \times Sp(n-1)) \subset \text{Ad}(Sp(n)) \) that preserves the framing at each point \( x \). Hence, in geometrical terms, the complex vector pair \( (i u_1, i u_2) \) describes a covariant of the curve \( \gamma \) relative to the group \( H_\gamma \). Moreover, \( x \)-derivatives of the pair \( (i u_1, i u_2) \) describe differential covariants of \( \gamma \) relative to \( H_\gamma \), which arise geometrically from the frame components of \( x \)-derivatives of the principal normal vector \( N \). We thus note that the geometric invariants of \( \gamma \) as defined by Riemannian inner products
of the tangent vector $X = x\gamma$, and its derivatives $N = \nabla_4 \gamma$, $\nabla_4 N = \nabla_{\gamma^2} \gamma$, etc along the curve $\gamma$ can be expressed as scalars formed from Cartan–Killing inner products of the covariant $(i\mathbf{u}_1, i\mathbf{u}_2)$ and differential covariants $(i\mathbf{u}_{1x}, i\mathbf{u}_{2x})$, etc; for example

$$g(N, N) = -g(X, \nabla^2 \gamma X) = \frac{2n}{n - 1}((|u_1|^2 + |u_2|^2)$$

yields the square of the classical curvature invariant of the curve $\gamma$. In particular, the set of invariants given by $\{g(X, \nabla^2 \gamma X), I = 1, \ldots, 2n^2 - n - 2(= \dim \mathfrak{m} - 1)\}$, generates the components of the connection matrix of a classical Frenet frame [35] determined by $\gamma$.

4.1. Hamiltonian operators and flows

The Cartan structure equations (2.10) and (2.11) for the $Sp(n)$-parallel framing of $\gamma$ expressed in terms of the variables (4.1)–(4.6) are respectively given by

$$-\frac{1}{\sqrt{\rho}} \nabla^{w} 1 \parallel = D_{\mathbf{u}} \mathbf{h}_{1 \parallel} + (\text{tr} \mathbf{H}_{1 \parallel}) \mathbf{u}_{1} + \mathbf{u}_{1} \mathbf{H}_{1 \parallel} + \mathbf{u}_{1} \mathbf{H}_{1 \parallel},$$

$$-\frac{1}{\sqrt{\rho}} \nabla^{w} 2 \parallel = D_{\mathbf{u}} \mathbf{h}_{2 \parallel} + (\text{tr} \mathbf{H}_{1 \parallel}) \mathbf{u}_{2} - \mathbf{u}_{2} \mathbf{H}_{1 \parallel} + \mathbf{u}_{1} \mathbf{H}_{1 \parallel},$$

$$D_{\gamma} \mathbf{H}_{1 \parallel} = \mathcal{P}(\mathbf{u}_{1}, \mathbf{h}_{1 \parallel}) - \mathcal{P}(\mathbf{u}_{2}, \mathbf{h}_{2 \parallel}),$$

$$D_{\gamma} \mathbf{H}_{2 \parallel} = \mathcal{Q}(\mathbf{u}_{2}, \mathbf{h}_{1 \parallel}) + \mathcal{Q}(\mathbf{u}_{1}, \mathbf{h}_{2 \parallel}),$$

and

$$\mathbf{u}_{1 \parallel} = D_{\gamma} \nabla^{w} 1 \parallel - w^{1 \parallel} \mathbf{u}_{1} + w^{2 \parallel} \mathbf{u}_{2} + \mathbf{u}_{1} \mathbf{W}^{1 \parallel} - \mathbf{u}_{2} \mathbf{W}^{2 \parallel} + \frac{i}{\sqrt{\rho}} \mathbf{h}_{1 \parallel},$$

$$\mathbf{u}_{2 \parallel} = D_{\gamma} \nabla^{w} 2 \parallel - w^{1 \parallel} \mathbf{u}_{2} - w^{2 \parallel} \mathbf{u}_{1} + \mathbf{u}_{2} \mathbf{W}^{1 \parallel} + \mathbf{u}_{1} \mathbf{W}^{2 \parallel} + \frac{i}{\sqrt{\rho}} \mathbf{h}_{2 \parallel},$$

$$D_{\gamma} \nabla^{w} 1 \parallel = \mathcal{P}(\mathbf{u}_{1}, \mathbf{W}^{1 \parallel}) + \mathcal{P}(\mathbf{u}_{2}, \mathbf{W}^{2 \parallel}),$$

$$D_{\gamma} \nabla^{w} 2 \parallel = \mathcal{S}(\mathbf{u}_{1}, \mathbf{W}^{1 \parallel}) + \mathcal{S}(\mathbf{u}_{2}, \mathbf{W}^{2 \parallel}),$$

$$D_{\gamma} \mathbf{W}^{1 \parallel} = \mathcal{P}(\mathbf{u}_{1}, \mathbf{W}^{1 \parallel}) - \mathcal{P}(\mathbf{u}_{2}, \mathbf{W}^{2 \parallel}),$$

$$D_{\gamma} \mathbf{W}^{2 \parallel} = \mathcal{S}(\mathbf{u}_{1}, \mathbf{W}^{1 \parallel}) - \mathcal{S}(\mathbf{u}_{2}, \mathbf{W}^{2 \parallel}).$$

As stated by theorem 2.1, these equations (4.21)–(4.24) directly encode a pair of compatible Hamiltonian operators. To display the operators explicitly, we first define the following operator notations in terms of the inner products (3.18)–(3.20) and outer products (3.23)–(3.25). For $x \in \mathbb{C}$, $y \in \mathbb{C}^{n-1}$, $X \in \mathfrak{gl}(n - 1, \mathbb{C})$, let

$$P_{x} y := P(x, y) \in i\mathbb{R},$$

$$Q_{x} y := Q(x, y) \in \mathbb{R},$$

$$S_{x} y := S(x, y) \in \mathbb{C},$$

$$P_{x} y := P(x, y) \in \mathfrak{u}(n - 1),$$

$$Q_{x} y := Q(x, y) \in \mathfrak{so}(n - 1, \mathbb{C}),$$

$$S_{x} y := S(x, y) \in \mathfrak{s}(n - 1, \mathbb{C}),$$

and

$$R_{x} y := x y \in \mathbb{C}^{n-1},$$

$$L_{x} y := yX \in \mathbb{C}^{n-1},$$

(4.27)

(4.28)
$\mathcal{C}X := \bar{X} \in \mathbb{C}^{n-1}$. 

Next we eliminate $H_1, H_2$ through the torsion equation (4.22), and also eliminate $w^{1\perp}, w^{2\perp}, W_1, W_2$ through the curvature equation (4.24). We also replace $h_{1\perp}, h_{2\perp}$ respectively in terms of $h^{1\perp}, h^{2\perp}$ from equation (4.7), which leads to the following main result.

**Theorem 4.1.** The flow equations given by (4.21)–(4.24) for the pair of complex vector variables $u_1(t, x), u_2(t, x) \in \mathbb{C}^{n-1}$ have the operator form

$$
\begin{pmatrix}
  u_1 \\
  u_2
\end{pmatrix}_t = \mathcal{H} \begin{pmatrix}
  w^{1\perp} \\
  h^{1\perp} \\
  w^{2\perp} \\
  h^{2\perp}
\end{pmatrix}, \quad \begin{pmatrix}
  w^{1\perp} \\
  h^{1\perp}
\end{pmatrix} = 8(n-1)\mathcal{J} \begin{pmatrix}
  h^{1\perp} \\
  h^{2\perp}
\end{pmatrix},
$$

where

$$\mathcal{H} = \begin{pmatrix}
  D_1 - R_{u_1}D_x^{-1}P_{u_1} + R_{P_{u_1}}D_x^{-1}S_{u_1} + L_{u_1}D_x^{-1}S_{u_1} & -R_{u_1}D_x^{-1}P_{u_2} + R_{P_{u_1}}D_x^{-1}S_{u_1} \\
  -R_{u_1}D_x^{-1}P_{u_2} + R_{P_{u_1}}D_x^{-1}S_{u_1} + L_{u_1}D_x^{-1}S_{u_1} & D_1 - R_{u_1}D_x^{-1}P_{u_2} + R_{P_{u_1}}D_x^{-1}S_{u_1}
\end{pmatrix}$$

and

$$\mathcal{J} = \begin{pmatrix}
  D_x + R_{u_1}D_x^{-1}Q_{u_1} + L_{Q_{u_1}}D_x^{-1}C_{Q_{u_1}} & R_{u_1}D_x^{-1}Q_{u_2} + L_{Q_{u_1}}D_x^{-1}C_{Q_{u_2}} \\
  -R_{u_1}D_x^{-1}P_{u_1} + L_{P_{u_1}}D_x^{-1}C_{P_{u_1}} & D_x + R_{u_1}D_x^{-1}Q_{u_2} + L_{Q_{u_1}}D_x^{-1}C_{Q_{u_2}}
\end{pmatrix}$$

are compatible Hamiltonian cosymplectic and symplectic operators on the $x$-jet space of $(u_1, u_2)$.

We now explain some details about this Hamiltonian structure. Let $J^\infty$ denote the $x$-jet space of the variables $(u_1, u_2)$, and let subscripts $l, l' = 1, 2$ denote the $2 \times 2$ components of $\mathcal{H}$ and $\mathcal{J}$.

Associated with the operator $\mathcal{H}$ is the Poisson bracket

$$\{\delta_1, \delta_2\}_\mathcal{H} := \int \sum_{l=1,2} \sum_{l'=1,2} Q(\delta_1 u_l / \partial u_l, \mathcal{H}_{ll'}(\delta_2 u_{l'}) / \partial u_{l'}) \, dx,$$

where $\delta_1, \delta_2$ are real-valued functionals on $J^\infty$. The cosymplectic property of $\mathcal{H}$ means that this bracket is skew-symmetric

$$\{\delta_1, \delta_2\}_\mathcal{H} + \{\delta_2, \delta_1\}_\mathcal{H} = 0$$

and obeys the Jacobi identity

$$\{\delta_1, \{\delta_2, \delta_3\}_\mathcal{H} + \{\delta_3, \delta_2\}_\mathcal{H} + \{\delta_2, \delta_3\}_\mathcal{H} + \text{cyclic} = 0.$$ 

A dual of the Poisson bracket is the symplectic 2-form associated with the operator $\mathcal{J}$,

$$\omega(X_1, X_2)_\mathcal{J} := \int \sum_{l=1,2} \sum_{l'=1,2} Q(X_1 u_l, \mathcal{J}_{ll'}(X_2 u_{l'})) \, dx,$$

where $X_1$ and $X_2$ are vector fields $X = h^{1\perp} \cdot \partial / \partial u_1 + h^{2\perp} \cdot \partial / \partial u_2$ defined in terms of vector function pairs $(\mathcal{h}^{1\perp}, \mathcal{h}^{2\perp}) \in \mathbb{C}^{n-1} \oplus \mathbb{C}^{n-1}$ on $J^\infty$ (with $\cdot \cdot$ standing for summation with respect to vector components). The symplectic property of $\mathcal{J}$ corresponds to $\omega$ being skew-symmetric

$$\omega(X_1, X_2) + \omega(X_2, X_1) = 0$$

(4.37)
and closed

\[ \text{pr}(X_1)\omega(X_2, X_3) + \text{cyclic} = \int \sum_{l=1,2} Q \left( h^{(l)}_x, \text{pr} \left( \sum_{l=1,2} h^{(l)}_t \cdot \partial / \partial u^r \right) J_{\text{pr}}(h^{(l)}_t) \right) \text{d}x \text{+ cyclic} = 0. \]  

(4.38)

Compatibility of the operators \( \mathcal{H} \) and \( \mathcal{J} \) is the statement that every linear combination \( c_1 \mathcal{H} + c_2 \mathcal{J}^{-1} \) is a cosymplectic Hamiltonian operator, or equivalently that \( c_1 \mathcal{H}^{-1} + c_2 \mathcal{J} \) is a symplectic operator, where \( \mathcal{H}^{-1} \) and \( \mathcal{J}^{-1} \) denote formal inverse operators defined on \( \mathcal{J}^\infty \).

The following result is a consequence of theorem 2.2.

**Corollary 4.2.** The operator \( \mathcal{R} = \mathcal{H} \mathcal{J} \) generates a hierarchy of bi-Hamiltonian flows (4.30) on \( (u_1(t, x), u_2(t, x)) \), given by

\[ \left( h^{(l)}_{1x}, h^{(l)}_{2x} \right) = \mathcal{R}^k \left( u^{(l)}_{1x}, u^{(l)}_{2x} \right), \quad k = 0, 1, 2, \ldots \]  

(4.39)

and

\[ \left( w^{(l)}_{1x}, w^{(l)}_{2x} \right) = \left( \delta H^{(l)} / \delta u_1, \delta H^{(l)} / \delta u_2 \right) = \mathcal{R}^k \left( u^{(l)}_1, u^{(l)}_2 \right), \quad k = 0, 1, 2, \ldots \]  

(4.40)

in terms of the Hamiltonians

\[ H^{(k)} = \frac{1}{1 + 2k} \text{tr}(iH^{(k)}_{11}), \quad k = 0, 1, 2, \ldots \]  

(4.41)

with

\[ \text{tr}(iH^{(k)}_{11}) = D^{-1}_x \left( A(u_1, h^{(k)}_{11}) + A(u_2, h^{(k)}_{21}) \right), \]  

(4.42)

where the operator \( \mathcal{R}^* = \mathcal{J} \mathcal{H} \) is the adjoint of \( \mathcal{R} \).

The \( +k \) flow in this hierarchy (4.39) is scaling invariant under \( (u_1, u_2) \rightarrow \lambda^{-1}(u_1, u_2), \) \( x \rightarrow \lambda x, t \rightarrow \lambda^{1+2k}t. \)

### 4.2. mKdV flow

After a scaling of \( t \rightarrow t / \rho, \) where \( \rho = 8(n - 1), \) the \( +1 \) flow in the hierarchy (4.39) yields an integrable system of coupled vector mKdV equations

\[
\begin{align*}
&u_{1t} - \rho^{-1} u_{1x} = u_{1xxx} + 3(u_1 \cdot \bar{u}_1 + u_2 \cdot \bar{u}_2)u_{1x} + 3(u_2 \cdot u_{1x} - u_{2x} \cdot u_1)\bar{u}_2 \\
&\quad \quad + 3(u_1 \cdot u_1 + u_2 \cdot u_2)u_1 \\
&u_{2t} - \rho^{-1} u_{2x} = u_{2xxx} + 3(u_1 \cdot \bar{u}_1 + u_2 \cdot \bar{u}_2)u_{2x} + 3(u_2 \cdot u_{1x} - u_{2x} \cdot u_1)\bar{u}_1 \\
&\quad \quad + 3(u_1 \cdot \bar{u}_1 + u_2 \cdot \bar{u}_2)u_1,
\end{align*}
\]

(4.43)

where a dot denotes the standard Euclidean inner product (cf (3.21)–(3.22)). This system is invariant under the symplectic group \( \text{Sp}(1) \times \text{Sp}(n - 1), \) defined by the transformations (3.36)–(3.37) on the vector pair \( (u_1, u_2), \) and has the following bi-Hamiltonian structure

\[
\begin{pmatrix}
    u_1 \\
    u_2
\end{pmatrix}_t - \rho^{-1} \begin{pmatrix}
    u_1 \\
    u_2
\end{pmatrix}_x = \mathcal{H} \begin{pmatrix}
    \delta H^{(1)} / \delta u_1 \\
    \delta H^{(1)} / \delta u_2
\end{pmatrix} = \mathcal{E} \begin{pmatrix}
    \delta H^{(0)} / \delta u_1 \\
    \delta H^{(0)} / \delta u_2
\end{pmatrix}
\]

(4.44)

in terms of the Hamiltonians

\[ H^{(0)} = u_1 \cdot \bar{u}_1 + u_2 \cdot \bar{u}_2, \]  

(4.45)

\[ H^{(1)} = -u_{1x} \cdot \bar{u}_1 - u_{2x} \cdot \bar{u}_2 + (u_1 \cdot \bar{u}_1 + u_2 \cdot \bar{u}_2)^2, \]  

(4.46)

where \( \mathcal{E} = \mathcal{H} \mathcal{J} \mathcal{H} \) is a Hamiltonian cosymplectic operator compatible with \( \mathcal{H}. \)

We remark that the convective terms \( u_{1x}, u_{2x} \) on the left-hand side in the system (4.43) and (4.44) can be removed by the Galilean transformation \( t \rightarrow t, x \rightarrow x + \rho^{-1}t. \)
4.3. SG flow

The $-1$ flow connected with the hierarchy (4.39) is defined by

$$0 = \left( \begin{array}{c} w_1^\perp \\ w_2^\perp \end{array} \right) = \mathcal{J} \left( \begin{array}{c} h_1^\perp \\ h_2^\perp \end{array} \right),$$

yielding the flow equation

$$\left( \begin{array}{c} u_1 \\ u_2 \end{array} \right) = \mathcal{J} \left( \begin{array}{c} h_1^\perp \\ h_2^\perp \end{array} \right),$$

with

$$i\sqrt{\rho} D_\alpha h_1^\perp = (\text{tr} H_{1||}) u_1 + u_1 H_{1||} + u_2 H_{2||},$$

$$i\sqrt{\rho} D_\alpha h_2^\perp = (\text{tr} H_{1||}) u_2 - u_2 H_{1||} + u_1 H_{2||},$$

and

$$D_\alpha H_{1||} = \sqrt{\rho} (P(\alpha h_1^\perp, u_1) + P(\beta h_2^\perp, u_2)),$$

$$D_\alpha H_{2||} = \sqrt{\rho} (-Q(u_2, \alpha h_1^\perp) + Q(\beta h_2^\perp, u_1)).$$

Note that equations (4.49) and (4.50) will determine the variables $h_1^\perp$, $h_2^\perp$, $H_{1||}$, $H_{2||}$ as nonlocal functions of $u_1$ and $u_2$. Similar to the method used to derive the SG flow in the case $SU(n)/SO(n)$ [12], we will seek inverse local expressions for $u_1$ and $u_2$ arising from an algebraic reduction of the form

$$H_{1||} = \frac{1}{2} \alpha (P(h_1^\perp, h_1^\perp) + P(h_2^\perp, h_2^\perp)) + \beta u_{n-1}$$

and

$$H_{2||} = \gamma Q(h_2^\perp, h_1^\perp)$$

for some expressions $\alpha(h_1)$, $\beta(h_2)$, $\gamma(h_1), \gamma(h_2) \in \mathbb{R}$, where it is convenient to introduce the variable

$$h_{||} := -\text{itr} H_{||}$$

satisfying

$$D_\alpha h_{||} = -\sqrt{\rho} (Q(u_1, h_1^\perp) + Q(u_2, h_2^\perp)).$$

To proceed, we substitute expressions (4.51) and (4.52) into equation (4.50) and use equations (4.49) and (4.54) to eliminate $x$ derivatives. This yields

$$\gamma - \alpha = 0$$

and

$$D_\alpha \beta = 0.$$

By applying $D_\alpha$ to equation (4.58) and using equation (4.56) together with equation (4.49), we obtain equation (4.57). Therefore, we can just algebraically solve equation (4.58) to obtain

$$\alpha = \gamma = \frac{-n\beta \pm \sqrt{n^2\beta^2 - 4\rho(|h_1^\perp|^2 + |h_2^\perp|^2)}}{2(|h_1^\perp|^2 + |h_2^\perp|^2)},$$

where

$$|h_1^\perp|^2 := \frac{1}{2} Q(h_1^\perp, h_1^\perp), \quad |h_2^\perp|^2 := \frac{1}{2} Q(h_2^\perp, h_2^\perp).$$

To determine $\beta$, we use the conservation law

$$0 = D_\alpha \left( |h_1^\perp|^2 + |h_2^\perp|^2 + \frac{1}{\beta} (h_1^\perp + |H_{1||}|^2 + |H_{2||}|^2) \right).$$
admitted by the system of equations (4.49), (4.50), (4.54), where
\[ h_i = \alpha(|h^{1^\perp}|^2 + |h^{2^\perp}|^2) + (n - 1)\beta \] (4.62)
and
\[ |H_{1^\perp}|^2 = -\text{tr}(H_{1^\perp}^2) = \alpha^2(|h^{1^\perp}|^4 + |h^{2^\perp}|^4) + \frac{1}{2}\gamma(h^{1^\perp}, h^{2^\perp}S(h^{1^\perp}, h^{2^\perp})) \]
\[ + 2\alpha\beta(|h^{1^\perp}|^2 + |h^{2^\perp}|^2) + \beta^2(n - 1) \]
\[ |H_{2^\perp}|^2 = -\text{tr}(H_{2^\perp}^2 + H_{3^\perp}^2) = \alpha^2(|h^{1^\perp}|^2 |h^{2^\perp}|^2) - \frac{1}{2}\gamma(h^{1^\perp}, h^{2^\perp})S(h^{1^\perp}, h^{2^\perp})) \] (4.63)
are obtained from equations (4.51)–(4.62). Substitution of expressions (4.62) and (4.63) into the conservation law (4.61), followed by the use of the algebraic equation (4.58), gives
\[ |h^{1^\perp}|^2 + |h^{2^\perp}|^2 + \frac{1}{\rho} (|h_i|^2 + |H_{1^\perp}|^2 + |H_{2^\perp}|^2) \]
\[ = 1 \]
through equations (4.64) and (4.61), we see that a conformal scaling of \( t \) can be used to make \( \beta \) equal to a constant. We will put
\[ \beta = -2\sqrt{\rho/n} \] (4.65)
which simplifies expression (4.59) for \( \alpha \) and \( \gamma \),
\[ \alpha = \gamma = \sqrt{\rho} \frac{1}{2} \left(1 \pm \sqrt{1 - |h^{1^\perp}|^2 - |h^{2^\perp}|^2} \right) \] (4.66)

Local expressions for \( u_1 \) and \( u_2 \) now arise directly from substitution of expressions (4.51), (4.52), and (4.62) into equation (4.49) to obtain
\[ h_i^{1^\perp} = -\frac{\sqrt{\rho}}{\alpha} u_1 + \frac{\alpha}{\sqrt{\rho}} (S(u_1, h^{1^\perp}) + S(u_2, h^{2^\perp}))h_i^{1^\perp} + \frac{\alpha}{2\sqrt{\rho}} (S(u_1, h^{2^\perp}) - S(u_2, h^{1^\perp}))h_i^{2^\perp} \]
\[ h_i^{2^\perp} = -\frac{\sqrt{\rho}}{\alpha} u_2 + \frac{\alpha}{\sqrt{\rho}} (S(u_1, h^{1^\perp}) + S(u_2, h^{2^\perp}))h_i^{2^\perp} - \frac{\alpha}{2\sqrt{\rho}} (S(u_1, h^{2^\perp}) - S(u_2, h^{1^\perp}))h_i^{1^\perp} \] (4.67)
(4.68)

Algebraically combining equations (4.67) and (4.68), we obtain
\[ u_1 = -\frac{\sqrt{\rho}}{\alpha} h_i^{1^\perp} + \frac{\alpha^2}{\rho} (ah^{1^\perp} + b h^{2^\perp}) \]
\[ u_2 = -\frac{\sqrt{\rho}}{\alpha} h_i^{2^\perp} + \frac{\alpha^2}{\rho} (ah^{2^\perp} - b h^{1^\perp}) \] (4.69)
where, after using expression (4.66), we have
\[ a = \frac{h_i^{1^\perp} - h_i^{1^\perp} + h_i^{2^\perp} \cdot h_i^{1^\perp}}{\pm 2\sqrt{1 - |h^{1^\perp}|^2 - |h^{2^\perp}|^2}}, \quad b = \frac{h_i^{2^\perp} - h_i^{1^\perp} \cdot h_i^{2^\perp}}{\pm 2\sqrt{1 - |h^{1^\perp}|^2 - |h^{2^\perp}|^2}} \] (4.70)
with a dot denoting the standard Euclidean inner product (cf (3.21)–(3.22)).

Finally, we express the flow equation (4.48) entirely in terms of \( u_1, u_2 \) and their \( t \) derivatives. Substitution of \( h^{1^\perp} = u_1 \) and \( h^{2^\perp} = u_2 \) into equations (4.66)–(4.68) directly yields the nonlinear evolution equation
\[ \left(\frac{u_1}{u_2}\right)_t = D^n_i \left( -\frac{\sqrt{\rho}}{\alpha} u_1 + \frac{\alpha}{\sqrt{\rho}} ((u_1 \cdot \overline{u}_1) + (u_2 \cdot \overline{u}_2))u_1 + (u_1 \cdot u_2 - u_2 \cdot u_1)\overline{u}_2 \right) \]
\[ + \frac{\alpha}{\sqrt{\rho}} ((u_1 \cdot \overline{u}_1) + (u_2 \cdot \overline{u}_2))u_2 - (u_1 \cdot u_2 - u_2 \cdot u_1)\overline{u}_1 \) (4.71)
with
\[
\frac{\alpha}{\sqrt{\rho}} = \frac{1 \pm \sqrt{1 - |u_{1x}|^2 - |u_{2x}|^2}}{|u_{1x}|^2 + |u_{2x}|^2}
\] (4.72)

and
\[
\frac{\sqrt{\rho}}{\alpha} = 1 \mp \sqrt{1 - |u_{1x}|^2 - |u_{2x}|^2}.
\] (4.73)

This −1 flow equation (4.71) is equivalent to a hyperbolic system of coupled vector SG equations
\[
u_{1x} = \frac{1 \pm \sqrt{1 - |u_{1x}|^2 - |u_{2x}|^2}}{|u_{1x}|^2 + |u_{2x}|^2} (u_{1x} \cdot \bar{u}_{1x} + u_{2x} \cdot \bar{u}_{2x}) u_{1x} + (u_{1} \cdot u_{2x} - u_{2} \cdot u_{1x}) \bar{u}_{1x}
\]
\[
-\left(1 \mp \sqrt{1 - |u_{1x}|^2 - |u_{2x}|^2}\right) u_{1}
\]
\[
u_{2x} = \frac{1 \pm \sqrt{1 - |u_{1x}|^2 - |u_{2x}|^2}}{|u_{1x}|^2 + |u_{2x}|^2} (u_{1x} \cdot \bar{u}_{1x} + u_{2x} \cdot \bar{u}_{2x}) u_{2x} - (u_{1} \cdot u_{2x} - u_{2} \cdot u_{1x}) \bar{u}_{1x}
\]
\[
-\left(1 \mp \sqrt{1 - |u_{1x}|^2 - |u_{2x}|^2}\right) u_{2}
\] (4.74)

which is invariant under the symplectic group \(Sp(1) \times Sp(n-1)\), defined by the transformations (3.36)–(3.37) on the vector pair \((u_{1}, u_{2})\).

Alternatively, from the flow equation (4.48) combined with relations (4.69) and (4.70), the variables \(h^{1\perp}\) and \(h^{2\perp}\) are found to obey coupled vector SG equations
\[
\left\langle \begin{array}{l}
\left( -\frac{\alpha}{\sqrt{\rho}} h^{1\perp} + \frac{\alpha^2}{\rho} (ah^{1\perp} + b\bar{h}^{1\perp}) \right) \\
\left( -\frac{\alpha}{\sqrt{\rho}} h^{2\perp} + \frac{\alpha^2}{\rho} (ah^{2\perp} - b\bar{h}^{2\perp}) \right)
\end{array} \right\rangle = \mathbf{h}^{1\perp},
\] (4.75)

with
\[
\frac{\alpha}{\sqrt{\rho}} = \frac{1 \pm \sqrt{1 - |h^{1\perp}|^2 - |h^{2\perp}|^2}}{|h^{1\perp}|^2 + |h^{2\perp}|^2}.
\] (4.76)

A Hamiltonian structure for the system (4.74) is given by
\[
\left\langle \begin{array}{l}
\v_1 \\
\v_2
\end{array} \right\rangle = \mathcal{H} \left( \frac{\delta H^{(-1)}}{\delta \v_1} \right) \frac{\delta H^{(-1)}}{\delta \v_2}
\] (4.77)
in terms of
\[
H^{(-1)} = \pm 8 \sqrt{1 - |h^{1\perp}|^2 - |h^{2\perp}|^2},
\] (4.78)

where \(h^{1\perp}\) and \(h^{2\perp}\) are implicitly determined as nonlocal functions of the variables \((\v_1, \v_2)\) (and their \(x\)-derivatives) through expressions (4.69), (4.70), (4.76).

4.4. Geometric curve flows

From theorem 2.3, the flows in the hierarchy (4.39) and (4.47) for \((u_{1}(f, x), u_{2}(f, x)) \in \mathbb{C}^{n-1} \oplus \mathbb{C}^{n-1}\) correspond to \(SU(2n)\)-invariant non-stretching geometric curve flows for \(\gamma(t, x) \in M = SU(2n)/Sp(n)\). The resulting equations of motion can be expressed covariantly in terms of \(X = \gamma_{c}, N = \nabla_{c} \gamma_{c}\) and \(\nabla_{c}\)-derivatives of \(N\), in addition to the Riemannian metric and curvature tensors on \(M\).

The SG flow (4.74) is given by \(w^{1\perp} = w^{2\perp} = 0\), which implies \(w^{1\parallel} = w^{2\parallel} = 0\) and \(W^{1\perp} = W^{2\perp} = 0\) as a consequence of the structure equation (4.24). This determines
\[
\alpha^{1\perp} = \alpha^{1\parallel} = 0.
\] (4.79)
Hence the corresponding flow vector $\gamma_t = Y_{(-1)}$ satisfies
\[
ed_i \nabla_x \gamma_t = D_x e_i + [\omega_x, e_i] = [\omega_x, e_i] = -\text{ad}(e_i)\sigma^\perp = 0
\] (4.80)
yielding the $SU(2n)$-invariant curve flow equation
\[
0 = \nabla_t \gamma_t, \quad |\gamma_t| = 1,
\] (4.81)
which is called the \textit{non-stretching wave map} on $M = SU(2n)/Sp(n)$. In addition to satisfying the non-stretching property $\nabla_x |\gamma_t| = 0$, this equation (4.81) possesses the conservation law $\nabla_x |\gamma_t| = 0$, corresponding to equation (4.61). Thus, up to a conformal scaling of $t$, the wave map equation describes a flow with unit speed $|\gamma_t| = 1$.

The mKdV flow (4.43), after $t$ has been rescaled, is given by $e_1 = -i\sqrt{\rho}u_1$, and $e_2 = -i\sqrt{\rho}u_2$, from which $H_{e_1} = -i\sqrt{\rho}(\bar{u}_1 u_1 + \bar{u}_2 u_2)$ and $H_{e_2} = -i\sqrt{\rho}(\bar{u}_1 u_2 - \bar{u}_2 u_1)$ are obtained by the structure equation (4.22). This determines
\[
(e_1)_\perp = -\sqrt{\rho}(iu_1, iu_2) \in m_\perp,
\] (4.82)
\[
(e_2)_\perp = \sqrt{\rho}\left(\frac{1}{2}P(iu_1, u_1) - \frac{1}{2}P(iu_2, u_2), Q(iu_2, u_1)\right) \in m_\perp.
\] (4.83)
Then $e_t = e_j \gamma_t$ can be expressed as follows in terms of
\[
e_j N = [\omega_x, e_i] = -\frac{1}{\sqrt{\rho}}(iu_1, iu_2) \in m_\perp,
\] (4.84)
\[
ev_\gamma N = (e_j)\gamma_t = -\frac{1}{\sqrt{\rho}}(iu_1x_i, iu_2x_i) \in m_\perp,
\] (4.85)
\[
e_j \nabla_x N = (e_j)\nabla_x N = -\frac{1}{\sqrt{\rho}}(P(iu_1, u_1) - \bar{P}(iu_2, u_2), 2Q(iu_2, u_1)) \in m_\perp.
\] (4.86)
Consider
\[
\text{ad}(e_j N)e_i = -\frac{1}{\rho}(u_1, u_2) \in h_\perp
\] (4.87)
which leads to
\[
\text{ad}(e_j N)^2 e_i = -\frac{1}{\sqrt{\rho}}(P(iu_1, u_1) - \bar{P}(iu_2, u_2), 2Q(iu_2, u_1)) \in m_\parallel
\] (4.88)
by means of the Lie brackets (3.30a) and (3.31c). Comparing equations (4.82) and (4.83) with equations (4.85), (4.86) and (4.88), we see that
\[
(e_i)_\perp = 2(e_1)\gamma_t = \rho e_j \nabla_x N,
\] (4.89)
\[
2(e_1)_\perp = -\rho \text{ad}(e_j N)^2 e_i.
\] (4.90)
This yields
\[
ev_\gamma = (e_i)_\perp + (e_2)_\perp = \rho e_j \nabla_x N - \frac{3}{2}\rho^2 e_j \text{ad}(N)^2 X,
\] (4.91)
where
\[
\text{ad}(N)^2 = -R(\cdot, N)N
\] (4.92)
is a linear map on $T_c M$. Hence the flow vector $\gamma_t = Y_{(1)}$ satisfies
\[
\gamma_t = \nabla_t^2 \gamma_t - \rho^2 \text{ad}(\nabla_x \gamma_t)^2 \gamma_t, \quad |\gamma_t| = 1,
\] (4.93)
which is an $SU(2n)$-invariant curve flow equation called the \textit{non-stretching mKdV map} on $M = SU(2n)/Sp(n)$. The simple form of the nonlinearities in this equation is due to the algebraic property that $\text{ad}(e_i)^2$ is a multiple of the identity on the vector spaces $m_\perp \cong h_\perp$, as explained by the general results in [9].
5. Bi-Hamiltonian soliton equations in \( Sp(n + 1)/Sp(1) \times Sp(n) \)

Let \( \gamma (t, x) \) be any non-stretching curve flow in \( M = Sp(n + 1)/Sp(1) \times Sp(n) \). Employing the notation and preliminaries in sections 2 and 3.2, we introduce an \( Sp(n) \times Sp(1) \)-parallel framing along \( \gamma \) as expressed in terms of the variables

\[
e_t = \frac{1}{\sqrt{\chi}}(1) \in \mathbb{R} \simeq m_{\parallel}, \quad \chi = 8(n + 2) \tag{5.1}
\]

\[
\omega_t = ((u_1, u_2), (u_1, u_2)) \in i\mathbb{R} \oplus \mathbb{C} \oplus \mathbb{C}^{n-1} \oplus \mathbb{C}^{n-1} \simeq h_{\perp}, \tag{5.2}
\]

and

\[
h_{\parallel} = (h_{\parallel}) \in \mathbb{R} \simeq m_{\parallel}, \tag{5.3}
\]

\[
h_{\perp} = ((h_{\perp, 1}, h_{\perp, 2}), (h_{\perp, 1}, h_{\perp, 2})) \in i\mathbb{R} \oplus \mathbb{C} \oplus \mathbb{C}^{n-1} \oplus \mathbb{C}^{n-1} \simeq m_{\perp}, \tag{5.4}
\]

\[
\sigma_1 = ((w^{11}, w^{21}), (W^{11}, W^{21})) \in \mathfrak{sp}(1) \oplus \mathfrak{sp}(n - 1) \simeq h_{\parallel}, \tag{5.5}
\]

\[
\sigma_1 = ((w^{11}, w^{21}), (w^{11}, w^{21})) \in \mathbb{R} \oplus \mathbb{C} \oplus \mathbb{C}^{n-1} \oplus \mathbb{C}^{n-1} \simeq h_{\perp}, \tag{5.6}
\]

using the matrix identifications (3.55) and (3.56), where \( h_{\parallel} \in \mathbb{R} \) is a real variable, \( w^{11}, w^{21}, h_{\perp, 1} \in i\mathbb{R} \) are imaginary (complex) scalar variables, \( w^{22}, w^{33}, h_{\perp, 2} \in \mathbb{C} \) are complex scalar variables, \( u_1, u_2, w^{11}, w^{22}, h_{\perp, 1}, h_{\perp, 2} \in \mathbb{C} \) are complex vector variables, \( W^{11} \in \mathfrak{u}(n - 1) \) is an anti-Hermitian matrix variable and \( W^{22} \in \mathfrak{sp}(n - 1, \mathbb{C}) \) is a complex symmetric matrix variable. For later use, through properties (3.57a) and (3.57b) we also introduce the variable

\[
h_{\perp} = ((h^{11}, h^{21}), (h^{11}, h^{21})) = \text{ad}(e_t)h_{\perp} = \frac{1}{\sqrt{\chi}}((2h_{\perp, 1}, 2h_{\perp, 2}), (-h_{\perp, 1}, -h_{\perp, 2})) \in i\mathbb{R} \oplus \mathbb{C} \oplus \mathbb{C}^{n-1} \oplus \mathbb{C}^{n-1} \simeq h_{\perp}, \quad \chi = 8(n + 2), \tag{5.7}
\]

where \( h^{11} \in i\mathbb{R} \) is an imaginary (complex) scalar variable, \( h^{22} \in \mathbb{C} \) is a complex scalar variable and \( h^{11}, h^{22} \in \mathbb{C}^{n-1} \) are complex vector variables.

Up to the rigid (\( t \)-independent) action of the equivalence group \( H^1_\gamma = \text{Ad}(Sp(1) \times Sp(n - 1)) \subset \text{Ad}(Sp(n) \times Sp(1)) \), an \( Sp(n) \times Sp(1) \)-parallel linear coframe \( e \) along \( \gamma \) is then determined by the variables (5.1) and (5.2) via the transport equation

\[
\nabla_{\gamma_0} e = -\text{ad}(\omega_t)e \tag{5.8}
\]

together with the soldering relation

\[
e_1|_{\gamma_0} = e_\perp. \tag{5.9}
\]

The resulting coframe \( e \) defines an isomorphism between \( T_\gamma M \) and \( m \simeq \mathbb{R} \oplus i\mathbb{R} \oplus \mathbb{C} \oplus \mathbb{C}^{n-1} \oplus \mathbb{C}^{n-1} \), which yields the following correspondence between the set of frames for \( T_\gamma M \) and the set of bases for \( m \). Let \( e_1 \) and \( e_{\perp} \) be the respective projections of \( e \) into \( m_{\parallel} \) and \( m_{\perp} \) given in terms of the matrix identifications (3.55) and (3.56) by

\[
e_1 = (a_{\parallel}(-)), \tag{5.10}
\]

\[
e_{\perp} = ((a_{\perp}(-), b_{\perp}(-)), (a_{\perp}(-), b_{\perp}(-))). \tag{5.11}
\]

where \( a_{\parallel}(-) \) and \( b_{\perp}(-) \) are linear maps from \( T_\gamma M \) into \( \mathbb{R}, i\mathbb{R} \) and \( \mathbb{C} \), respectively, and where both \( a_{\perp}(-) \) and \( b_{\perp}(-) \) are linear maps from \( T_\gamma M \) into \( \mathbb{C}^{n-1} \). Let \( (T_\gamma M)_1 \) and \( (T_\gamma M)_{\perp} \) be the orthogonal subspaces of \( T_\gamma M \) respectively defined by the kernels of \( e_1 \) and \( e_{\perp} \), so

\[
e_1|_{(T_\gamma M)_1} = e_{\perp}|_{(T_\gamma M)_{\perp}} = 0 \tag{5.12}
\]

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and hence
\[ e \parallel (T_p M) \parallel = e \parallel T_p M = m \parallel \simeq \mathbb{R}, \]  
\[ e \perp (T_p M) \perp = e \perp T_p M = m \perp \simeq i\mathbb{R} \oplus \mathbb{C} \oplus \mathbb{C}^{n-1} \oplus \mathbb{C}^{n-1}. \]  
(5.13)  
(5.14)

Note that, in this notation,
\[ e \parallel | y_\parallel = e_\parallel, \quad e \parallel | y_\perp = 0, \]  
(5.15)  
\[ e \perp | y_\parallel = h_\parallel, \quad e \perp | y_\perp = h_\perp. \]  
(5.16)

Now if \([m_{\parallel \mathbb{R}}]\) is a basis for \(\mathbb{R}\), then \(e_\parallel\) determines a corresponding basis \([X_{\parallel \mathbb{R}}]\) for \((T_p M) \parallel\) given by
\[ a_\parallel (X_{\parallel \mathbb{R}}) = m_{\parallel \mathbb{R}}. \]

Similarly, if \([m_{\parallel \mathbb{C}}, m'_{\parallel \mathbb{C}}]\) and \([m_{\perp \mathbb{C}}, m'_{\perp \mathbb{C}}]\) are respectively a basis for \(i\mathbb{R}\) and \(\mathbb{C}\) viewed as real vector spaces, and if \([m^{(k)}_{\perp \mathbb{C}}], k = 1, \ldots, 2(n-1),\) is a basis for \(\mathbb{C}^{n-1}\) viewed as a real vector space, then \(e_\perp\) determines a corresponding basis \([X_{\perp \mathbb{C}}, X'_{\perp \mathbb{C}}, X^{(k)}_{\perp \mathbb{C}}, X'^{(k)}_{\perp \mathbb{C}}]\), \(k', k = 1, \ldots, 2(n-1),\) for the vector space \((T_p M) \perp\) given by
\[ a_\perp (X_{\perp \mathbb{C}}, X'_{\perp \mathbb{C}}, X^{(k)}_{\perp \mathbb{C}}, X'^{(k)}_{\perp \mathbb{C}}) = m_{\perp \mathbb{C}}, m'_{\perp \mathbb{C}}, m^{(k)}_{\perp \mathbb{C}}, m'^{(k)}_{\perp \mathbb{C}}. \]

In addition, if each basis \([m_{\parallel \mathbb{R}}, m_{\parallel \mathbb{C}}, m'_{\parallel \mathbb{C}}, m^{(k)}_{\parallel \mathbb{C}}]\) is normalized such that
\[ \langle m_{\parallel \mathbb{R}}, m_{\parallel \mathbb{R}} \rangle = \langle m_{\parallel \mathbb{C}}, m_{\parallel \mathbb{C}} \rangle = -1, \quad \langle m_{\perp \mathbb{C}}, m'_{\perp \mathbb{C}} \rangle = -1, \quad \langle m_{\perp \mathbb{C}}, m'_{\perp \mathbb{C}} \rangle = 0, \]
\[ \langle m^{(k)}_{\perp \mathbb{C}}, m^{(k)}_{\perp \mathbb{C}} \rangle = -\delta_{kk}, \]

then the basis for \(T_p M = (T_p M) \parallel \oplus (T_p M) \perp\) has the corresponding normalization
\[ g(X_{\parallel \mathbb{R}}, X_{\parallel \mathbb{R}}) = g(X_{\parallel \mathbb{C}}, X_{\parallel \mathbb{C}}) = 1, \]
\[ g(X_{\perp \mathbb{C}}, X'_{\perp \mathbb{C}}) = g(X^{(k)}_{\perp \mathbb{C}}, X'^{(k)}_{\perp \mathbb{C}}) = \delta_{kk}. \]

Consequently, from the transport equation (5.8) together with the Lie brackets (3.60b), (3.63a) and (3.63b), the resulting orthonormal frame
\[ [X_{\parallel \mathbb{R}}, X_{\parallel \mathbb{C}}, X'_{\perp \mathbb{C}}, X^{(k)}_{\perp \mathbb{C}}, X'^{(k)}_{\perp \mathbb{C}}], \]  
(5.17)

can be shown to satisfy the Frenet equations
\[ \nabla \times X_{\parallel \mathbb{R}} = U_{R,\mathbb{R}}X_{\parallel \mathbb{R}} + U_{R,\mathbb{C}}X_{\parallel \mathbb{C}} + U'_{R,\mathbb{C}}X'_{\perp \mathbb{C}} + \sum_k U^{(k)}_{R,\mathbb{C}}X^{(k)}_{\perp \mathbb{C}} + \sum_k U'^{(k)}_{R,\mathbb{C}}X'^{(k)}_{\perp \mathbb{C}} \]
\[ \nabla \times X_{\parallel \mathbb{C}} = -U_{R,\mathbb{R}}X_{\parallel \mathbb{R}} + \sum_k U^{(k)}_{R,\mathbb{C}}X^{(k)}_{\perp \mathbb{C}} + \sum_k U^{(k)}_{R,\mathbb{C}}X'^{(k)}_{\perp \mathbb{C}} \]
\[ \nabla \times X'_{\perp \mathbb{C}} = -U_{R,\mathbb{C}}X_{\parallel \mathbb{R}} + \sum_k U^{(k)}_{R,\mathbb{C}}X^{(k)}_{\perp \mathbb{C}} + \sum_k U^{(k)}_{R,\mathbb{C}}X'^{(k)}_{\perp \mathbb{C}} \]
\[ \nabla \times X^{(k)}_{\perp \mathbb{C}} = -U'_{R,\mathbb{C}}X_{\parallel \mathbb{R}} + \sum_k U^{(k)}_{R,\mathbb{C}}X^{(k)}_{\perp \mathbb{C}} + \sum_k U^{(k)}_{R,\mathbb{C}}X'^{(k)}_{\perp \mathbb{C}} \]
\[ (5.18a) \]
\[ \nabla_{\ell} X_{\ell C}^{(k)} = -U_{\ell IR}^{(k)} X_{\ell IR}^{(k)} - U_{\ell IR}^{(k)} X_{\ell IR}^{(k)} - U_{\ell C}^{(k)} X_{\ell C}^{(k)} - U_{\ell C}^{(k)} X_{\ell C}^{(k)} + \sum_{\ell} U_{\ell C}^{(k)} X_{\ell C}^{(k)} + \sum_{\ell} U_{\ell C}^{(k)} X_{\ell C}^{(k)} \]
\[ \nabla_{\ell} X_{\ell C}^{(k')} = -U_{\ell IR}^{(k')} X_{\ell IR}^{(k')} - U_{\ell IR}^{(k')} X_{\ell IR}^{(k')} - U_{\ell C}^{(k')} X_{\ell C}^{(k')} - U_{\ell C}^{(k')} X_{\ell C}^{(k')} - \sum_{\ell} U_{\ell C}^{(k)} X_{\ell C}^{(k)} + \sum_{\ell} U_{\ell C}^{(k)} X_{\ell C}^{(k)}, \quad (5.18b) \]

where

\[ U_{\ell IR}^{(k)} = \{(m, u), ((u_1, u_2), (u_1, u_2))\}, ((0, 0), 0) \}
\[ = 8(n + 2)Q(m, u_1, u_2) \]
\[ U_{\ell C}^{(k)} = \{(m, u), ((u_1, u_2), (u_1, u_2))\}, ((0, 0), 0) \}
\[ = 8(n + 2)Q(m, u_1, u_2) \]
\[ U_{\ell C}^{(k')} = \{(m, u), ((u_1, u_2), (u_1, u_2))\}, ((0, 0), 0) \}
\[ = 8(n + 2)Q(m, u_1, u_2) \]
\[ U_{\ell C}^{(k)} = \{(m, u), ((u_1, u_2), (u_1, u_2))\}, ((0, 0), 0) \}
\[ = 8(n + 2)Q(m, u_1, u_2) \]
\[ U_{\ell C}^{(k')} = \{(m, u), ((u_1, u_2), (u_1, u_2))\}, ((0, 0), 0) \}
\[ = 8(n + 2)Q(m, u_1, u_2) \]
\[ U_{\ell C}^{(k)} = \{(m, u), ((u_1, u_2), (u_1, u_2))\}, ((0, 0), 0) \}
\[ = 8(n + 2)Q(m, u_1, u_2) \]
\[ U_{\ell C}^{(k')} = \{(m, u), ((u_1, u_2), (u_1, u_2))\}, ((0, 0), 0) \}
\[ = 8(n + 2)Q(m, u_1, u_2) \]
\[ U_{\ell C}^{(k)} = \{(m, u), ((u_1, u_2), (u_1, u_2))\}, ((0, 0), 0) \}
\[ = 8(n + 2)Q(m, u_1, u_2) \]
\[ U_{\ell C}^{(k')} = \{(m, u), ((u_1, u_2), (u_1, u_2))\}, ((0, 0), 0) \}
\[ = 8(n + 2)Q(m, u_1, u_2) \]
\[ U_{\ell C}^{(k)} = \{(m, u), ((u_1, u_2), (u_1, u_2))\}, ((0, 0), 0) \}
\[ = 8(n + 2)Q(m, u_1, u_2) \]
\[ U_{\ell C}^{(k')} = \{(m, u), ((u_1, u_2), (u_1, u_2))\}, ((0, 0), 0) \}
\[ = 8(n + 2)Q(m, u_1, u_2) \]
\[ U_{\ell C}^{(k)} = \{(m, u), ((u_1, u_2), (u_1, u_2))\}, ((0, 0), 0) \}
\[ = 8(n + 2)Q(m, u_1, u_2) \]
\[ U_{\ell C}^{(k')} = \{(m, u), ((u_1, u_2), (u_1, u_2))\}, ((0, 0), 0) \}
\[ = 8(n + 2)Q(m, u_1, u_2) \]
\[ U_{\ell C}^{(k)} = \{(m, u), ((u_1, u_2), (u_1, u_2))\}, ((0, 0), 0) \]
\[ = 8(n + 2)Q(m, u_1, u_2) \]
\[ U_{\ell C}^{(k')} = \{(m, u), ((u_1, u_2), (u_1, u_2))\}, ((0, 0), 0) \]
\[ = 8(n + 2)Q(m, u_1, u_2) \]
\[ U_{\ell C}^{(k)} = \{(m, u), ((u_1, u_2), (u_1, u_2))\}, ((0, 0), 0) \]
\[ = 8(n + 2)Q(m, u_1, u_2) \]
\[ U_{\ell C}^{(k')} = \{(m, u), ((u_1, u_2), (u_1, u_2))\}, ((0, 0), 0) \]
\[ = 8(n + 2)Q(m, u_1, u_2) \]
\[ U_{\ell C}^{(k)} = \{(m, u), ((u_1, u_2), (u_1, u_2))\}, ((0, 0), 0) \]
\[ = 8(n + 2)Q(m, u_1, u_2) \]
\[ U_{\ell C}^{(k')} = \{(m, u), ((u_1, u_2), (u_1, u_2))\}, ((0, 0), 0) \]
\[ = 8(n + 2)Q(m, u_1, u_2) \]

denote the Cartan matrix components of the underlying \( Sp(n) \times Sp(1) \)-parallel linear connection (5.2) projected into the tangent space of the curve.
In this frame, the components of the principal normal vector
\[ N := \nabla_s X = \langle e^s, \text{ad}(e_s)\omega_s \rangle \] (5.20)
are given by
\[ e^s N = -\text{ad}(e_s)\omega_s = \frac{1}{\sqrt{\gamma}}((-2u_1, -2u_2), (u_1, u_2)) \in i\mathbb{R} \oplus \mathbb{C} \oplus \mathbb{C}^{n-1} \oplus \mathbb{C}^{n-1} \simeq m_\perp \] (5.21)
through relation (3.57b). These components \((-2u_1, -2u_2), (u_1, u_2)\) are invariantly defined by the curve \(\gamma\) up to the rigid (\(x\)-independent) action of the equivalence group \(H^*_\parallel = \text{Ad}(Sp(1) \times Sp(n-1)) \subset \text{Ad}(Sp(n))\) that preserves the framing at each point \(x\). Moreover, the pair of scalars \((u_1, u_2)\) and the pair of vectors \((u_1, u_2)\) belong to separate irreducible representations of this group. Hence, in geometrical terms, the complex scalar–vector pair \((-2u_1, -2u_2), (u_1, u_2)\) describes covariants of the curve \(\gamma\) relative to the group \(H^*_\parallel\), while \(x\)-derivatives of this pair describe differential covariants which arise geometrically from the frame components of \(x\)-derivatives of the principal normal vector \(N\). We thus note that the geometric invariants of \(\gamma\) as defined by Riemannian inner products of the tangent vector \(X = \gamma_t\) and its derivatives \(N = \nabla_v^t\gamma_t, \nabla_s^t\gamma_t = \nabla_v^t\gamma_t, \) etc along the curve \(\gamma\) can be expressed as scalars formed from Cartan–Killing inner products of the covariants \((-2u_1, -2u_2), (u_1, u_2)\) and the differential covariants \((-2u_1, -2u_2), (u_1, u_2)\), etc; for example
\[ g(N, N) = -g(X, \nabla^2_s X) = \frac{1}{\gamma} (4(|u_1|^2 + |u_2|^2) + |u_1|^2 + |u_2|^2) \]
yields the square of the classical curvature invariant of the curve \(\gamma\). In particular, the set of invariants given by \(g(X, \nabla^2_s X), l = 1, \ldots, 4n - 1 (= \dim m - 1)\) generates the components of the connection matrix of a classical Frenet frame [35] determined by \(\gamma_t\).

5.1. Hamiltonian operators and flows

The Cartan structure equations (2.10) and (2.11) for the \(Sp(n) \times Sp(1)\)-parallel framing of \(\gamma\) expressed in terms of the variables (5.1)–(5.6) are respectively given by
\[
\begin{align*}
 w^{1\perp} & = \frac{\sqrt{\gamma}}{2} \left( D_h h_{1\perp} - \frac{1}{2} P(u_2, h_{2\perp}) - \frac{1}{2} P(u_1, h_{1\perp}) + 2h_1 u_1 \right), \\
 w^{2\perp} & = \frac{\sqrt{\gamma}}{2} \left( D_h h_{2\perp} - \frac{1}{2} S(u_2, h_{1\perp}) + \frac{1}{2} S(u_1, h_{2\perp}) + 2h_2 u_2 \right), \\
 w^{1\perp} & = -\sqrt{\gamma} \left( D_h h_{1\perp} - h_{1\perp} u_1 + u_1 h_{1\perp} + h_{2\perp} u_2 - u_2 h_{2\perp} - h_1 u_1 \right), \\
 w^{2\perp} & = -\sqrt{\gamma} \left( D_h h_{2\perp} - h_{1\perp} u_2 + u_2 h_{1\perp} - h_{2\perp} u_1 + u_1 h_{2\perp} - h_2 u_2 \right), \\
 D_h h_{1\perp} & = Q(u_1, h_{1\perp}) - \frac{1}{2} Q(u_1, h_{1\perp}) + Q(u_2, h_{2\perp}) - \frac{1}{2} Q(u_2, h_{2\perp}),
\end{align*}
\] (5.22)
and
\[
\begin{align*}
 u_{1\perp} & = D_h u_{1\perp} + \frac{1}{2} P(u_1, w^{1\perp}) + \frac{1}{2} P(u_2, w^{2\perp}) + \overline{u}_2 w^{2\perp} - u_2 \overline{w}^{2\perp} + h^{1\perp}, \\
 u_{2\perp} & = D_h u_{2\perp} - \frac{1}{2} S(u_1, w^{2\perp}) + \frac{1}{2} S(u_2, w^{1\perp}) - 2u_2 w^{1\perp} + 2u_1 \overline{w}^{1\perp} + h^{2\perp}, \\
 u_{1\perp} & = D_h u_{1\perp} - u_1 w^{1\perp} + u_2 \overline{w}^{2\perp} - u_2 \overline{w}^{2\perp} + w^{1\perp} u_1 + w^{2\perp} u_2 + u_1 W^{1\perp} - \overline{u}_2 \overline{W}^{2\perp} + h^{1\perp}, \\
 u_{2\perp} & = D_h u_{2\perp} - u_1 w^{2\perp} + u_2 \overline{w}^{1\perp} - u_2 \overline{w}^{1\perp} + w^{1\perp} u_2 - w^{2\perp} u_1 + \overline{u}_1 W^{1\perp} + u_2 \overline{W}^{2\perp} + h^{2\perp}.
\end{align*}
\] (5.24)
\[
D_x w^{11} = -u_2 w^{2 \perp} + u_2 w^{2 \perp} + \frac{1}{2} P(u_1, w^{1 \perp}) + \frac{1}{2} P(u_2, w^{2 \perp}),
\]
\[
D_x w^{21} = 2 u_2 w^{1 \perp} - 2 u_1 w^{1 \perp} + \frac{1}{2} S(u_2, w^{1 \perp}) - \frac{1}{2} S(u_1, w^{2 \perp}),
\]
\[
D_x w^{11} = P(u_1, w^{1 \perp}) + P(u_2, w^{2 \perp}),
\]
\[
D_x w^{21} = S(u_1, w^{2 \perp}) - S(u_2, w^{1 \perp}).
\]

These equations (5.22)–(5.25) directly encode a pair of compatible Hamiltonian operators as stated by theorem 2.1. Using the operator notation (4.25)–(4.29), and eliminating \( h_2 \) through the torsion equation (5.23) and \( w^{11}, w^{21}, W^{11}, W^{21} \) through the curvature equation (5.25), as well as replacing \( h_{1 \perp}, h_{2 \perp}, h_{1 \perp}, h_{2 \perp} \) respectively in terms of \( h^{1 \perp}, h^{2 \perp}, h^{1 \perp}, h^{2 \perp} \), we obtain the following main result.

**Theorem 5.1.** For the imaginary scalar variable \( u_1 \in i \mathbb{R} \), the complex scalar variable \( u_2 \in \mathbb{C} \), and the pair of complex vector variables \( u_1(t, x), u_2(t, x) \in \mathbb{C}^{n-1} \), the flow equations given by (5.22)–(5.24) have the operator form

\[
\left( \begin{array}{c}
    u_1 \\
    u_2 \\
    u_1 \\
    u_2
\end{array} \right) = \mathcal{H} \left( \begin{array}{c}
    w^{1 \perp} \\
    w^{2 \perp} \\
    w^{1 \perp} \\
    w^{2 \perp}
\end{array} \right) + \frac{\chi}{4} \mathcal{J} \left( \begin{array}{c}
    h^{1 \perp} \\
    h^{2 \perp} \\
    h^{1 \perp} \\
    h^{2 \perp}
\end{array} \right),
\]

where \( \mathcal{H} \) and \( \mathcal{J} \) are compatible Hamiltonian cosymplectic and symplectic operators on the \( x \)-jet space of \((u_1, u_2, u_1, u_2)\). The \( 4 \times 4 \) components of these operators \( \mathcal{H} = (\mathcal{H}_{ij}) \) and \( \mathcal{J} = (\mathcal{J}_{ij}) \), with \( i, j = 1, 2, 3, 4 \), are given by

\[
\mathcal{H}_{11} = D_x - P_{u_2} D_x^{-1} S_{u_2}
\]
\[
\mathcal{H}_{12} = P_{u_1} D_x^{-1} S_{u_1}
\]
\[
\mathcal{H}_{13} = \frac{1}{4} P_{u_1} - \frac{1}{2} P_{u_2} D_x^{-1} S_{u_2}
\]
\[
\mathcal{H}_{14} = \frac{1}{4} P_{u_2} + \frac{1}{2} P_{u_2} D_x^{-1} S_{u_1}
\]
\[
\mathcal{H}_{21} = S_{u_1} D_x^{-1} S_{u_2}
\]
\[
\mathcal{H}_{22} = D_x - S_{u_2} D_x^{-1} P_{u_2} - S_{u_1} D_x^{-1} S_{u_1}
\]
\[
\mathcal{H}_{23} = \frac{1}{2} S_{u_2} - \frac{1}{2} S_{u_1} D_x^{-1} P_{u_1} + \frac{1}{2} S_{u_1} D_x^{-1} S_{u_1}
\]
\[
\mathcal{H}_{24} = -\frac{1}{2} S_{u_2} - \frac{1}{2} S_{u_1} D_x^{-1} P_{u_1} - \frac{1}{2} S_{u_1} D_x^{-1} S_{u_1}
\]
\[
\mathcal{H}_{31} = R_{u_1} + R_{u_1} D_x^{-1} S_{u_2}
\]
\[
\mathcal{H}_{32} = -R_{u_1} - R_{u_1} D_x^{-1} P_{u_2} - R_{u_1} D_x^{-1} S_{u_1}
\]
\[
\mathcal{H}_{33} = D_x - L_{u_1} - \frac{1}{2} R_{u_1} D_x^{-1} P_{u_1} + L_{u_1} D_x^{-1} P_{u_1} + \frac{1}{2} R_{u_1} D_x^{-1} S_{u_1} + L_{u_1} D_x^{-1} C S_{u_1}
\]
\[
\mathcal{H}_{34} = L_{u_1} - C - \frac{1}{2} R_{u_1} D_x^{-1} P_{u_1} + L_{u_1} D_x^{-1} C P_{u_1} - R_{u_1} D_x^{-1} S_{u_1} - L_{u_1} D_x^{-1} S_{u_1}
\]
\[
\mathcal{H}_{41} = R_{u_2} - R_{u_1} D_x^{-1} S_{u_2}
\]
\[
\mathcal{H}_{42} = R_{u_2} - R_{u_1} D_x^{-1} P_{u_2} + R_{u_1} D_x^{-1} S_{u_1}
\]
\[
\mathcal{H}_{43} = -L_{u_1} C - \frac{1}{2} R_{u_1} D_x^{-1} P_{u_1} + L_{u_1} D_x^{-1} C P_{u_1} - \frac{1}{2} R_{u_1} D_x^{-1} S_{u_1} - L_{u_1} D_x^{-1} S_{u_1}
\]
\[
\mathcal{H}_{44} = D_x - L_{u_1} - \frac{1}{2} R_{u_1} D_x^{-1} P_{u_1} + L_{u_1} D_x^{-1} P_{u_1} + \frac{1}{2} R_{u_1} D_x^{-1} S_{u_1} + L_{u_1} D_x^{-1} C S_{u_1}
\]

and

\[
\mathcal{J}_{11} = D_x + S_{u_1} D_x^{-1} Q_{u_1}
\]
\[
\mathcal{J}_{12} = S_{u_1} D_x^{-1} Q_{u_1}
\]
\[
\mathcal{J}_{13} = P_{u_1} + S_{u_1} D_x^{-1} Q_{u_1}
\]
\[
\mathcal{J}_{14} = P_{u_1} + S_{u_1} D_x^{-1} Q_{u_1}
\]
\[ J_{21} = S_{u_2}D_x^{-1}Q_{u_1} \]
\[ J_{22} = D_t + S_{u_2}D_x^{-1}Q_{u_2} \]
\[ J_{23} = S_{u_2} + S_{u_2}D_x^{-1}Q_{u_1} \]
\[ J_{24} = -S_{u_2} + S_{u_2}D_x^{-1}Q_{u_2} \]  \hspace{1cm} (5.32)
\[ J_{31} = 2R_{u_1} + 2R_{u_2}D_x^{-1}Q_{u_1} \]
\[ J_{32} = -2R_{u_2} + 2R_{u_2}D_x^{-1}Q_{u_2} \]
\[ J_{33} = 4D_t + 4L_{u_1} + 2R_{u_2}D_x^{-1}Q_{u_1} \]
\[ J_{34} = -4L_{u_2}C + 2R_{u_2}D_x^{-1}Q_{u_2} \]  \hspace{1cm} (5.33)
\[ J_{41} = 2R_{u_1} + 2R_{u_2}D_x^{-1}Q_{u_1} \]
\[ J_{42} = 2R_{u_1} + 2R_{u_2}D_x^{-1}Q_{u_2} \]
\[ J_{43} = 4L_{u_1}C + 2R_{u_2}D_x^{-1}Q_{u_1} \]
\[ J_{44} = 4D_t + 4L_{u_2} + 2R_{u_2}D_x^{-1}Q_{u_2}. \]  \hspace{1cm} (5.34)

The properties of these operators are similar to the Hamiltonian structure (4.33)–(4.38). Let \( J^\infty \) denote the \( x \)-jet space of the variables \((u_1, u_2, u_1, u_2)\). The symplectic property of \( \mathcal{H} \) means that it defines an associated Poisson bracket
\[
[\delta_1, \delta_2]_{\mathcal{H}} := \int \sum_{l=1,2} Q(\delta \delta_{1l}/\delta u_l, \mathcal{H}^{l'}(\delta \delta_{2l}/\delta u_{l'}))
+ \sum_{l=3,4} Q(\delta \delta_{1l}/\delta u_{l-2}, \mathcal{H}^{l'}(\delta \delta_{2l}/\delta u_{l'-2})) \, dx
\]  \hspace{1cm} (5.35)
which is skew-symmetric and obeys the Jacobi identity for all real-valued functionals \( \delta \) on \( J^\infty \). The symplectic property of \( J \) means that it defines an associated symplectic 2-form
\[
\omega(X_1, X_2)_{\mathcal{J}} := \int \sum_{l=1,2} Q(X_1 u_l, J^{l'}(X_2 u_{l'})) + \sum_{l=3,4} Q(X_1 u_{l-2}, J^{l'}(X_2 u_{l'-2})) \, dx
\]  \hspace{1cm} (5.36)
which is skew-symmetric and closed for all vector fields \( X = h^{1\perp} \partial/\partial u_1 + h^{2\perp} \partial/\partial u_2 + h^{1\perp} \partial/\partial u_1 + h^{2\perp} \partial/\partial u_2 \) defined in terms of scalar–vector function pairs \((h^{1\perp}, h^{2\perp}) \in \mathbb{R} \oplus \mathbb{C}, (h^{1\perp}, h^{2\perp}) \in \mathbb{C}^{n-1} \oplus \mathbb{C}^{n-1}\) on \( J^\infty \). Compatibility of the operators \( \mathcal{H} \) and \( J \) means that every linear combination \( c_1 \mathcal{H} + c_2 J^{-1} \) is a symplectic Hamiltonian operator, or equivalently that \( c_1 \mathcal{H}^{-1} + c_2 J \) is a symplectic operator, where \( \mathcal{H}^{-1} \) and \( J^{-1} \) denote formal inverse operators defined on \( J^\infty \).

As a consequence of theorem 2.2, we have the following result.

**Corollary 5.2.** The operator \( \mathcal{R} = \mathcal{H} J \) generates a hierarchy of bi-Hamiltonian flows (5.26) on \((u_1(t, x), u_2(t, x), u_1(t, x), u_2(t, x))\) given by

\[
\begin{pmatrix}
  h_{1\perp}^{(k)} \\
  h_{2\perp}^{(k)} \\
  h_{1\perp}^{(k)} \\
  h_{2\perp}^{(k)}
\end{pmatrix} = \mathcal{R}^k
\begin{pmatrix}
  u_{1,t} \\
  u_{2,t} \\
  u_{1,x} \\
  u_{2,x}
\end{pmatrix}, \quad k = 0, 1, 2, \ldots
\]  \hspace{1cm} (5.37)
\[ \begin{pmatrix} w^{(1)}_{1(k)} \\ w^{(2)}_{1(k)} \\ w^{(1)}_{2(k)} \\ w^{(2)}_{2(k)} \end{pmatrix} = \begin{pmatrix} \delta H^{(k)}/\delta u_1 \\ \delta H^{(k)}/\delta u_2 \\ \delta H^{(k)}/\delta u_1 \\ \delta H^{(k)}/\delta u_2 \end{pmatrix} = R^k \begin{pmatrix} u_1 \\ u_2 \\ u_1 \\ u_2 \end{pmatrix}, \quad k = 0, 1, 2, \ldots \quad (5.38) \]

in terms of the Hamiltonians
\[ H^{(k)} = \frac{1}{1 + 2k} h^{(k)}, \quad k = 0, 1, 2, \ldots \quad (5.39) \]

with
\[ h^{(k)} = D^{-1}_x (A(u_1, h^{(k)}_{1(k)}) + A(u_2, h^{(k)}_{2(k)}) + A(u_1, h^{(k)}_{1(k)}) + A(u_2, h^{(k)}_{2(k)})), \quad (5.40) \]

where the operator \( R^* = \mathcal{J} \mathcal{H} \) is the adjoint of \( R \).

The +k flow in this hierarchy (5.37) is scaling invariant under \((u_1, u_2, u_1, u_2) \rightarrow \lambda^{-1} (u_1, u_2, u_1, u_2), x \rightarrow \lambda x, t \rightarrow \lambda^{1+2k} t \).

5.2. mKdV flow

After a scaling of \( t \rightarrow 4t/\chi \), where \( \chi = 8(n + 2) \), the +1 flow in the hierarchy (5.37) yields an integrable system of coupled scalar–vector mKdV equations
\[ \begin{align*}
\frac{1}{\chi} u_1 &= u_{1,x} = u_{1,xx} - 3(u_{1,xx} \cdot u_1 - u_1 \cdot u_{1,xx} + u_{2,xx} \cdot u_2 - u_2 \cdot u_{2,xx}) + 6u_1(|u_1|^2 + |u_2|^2) \\
&\quad + 3u_2(u_2 \cdot u_1 - u_1 \cdot u_2) - 3u_2(u_2 \cdot u_1 - u_1 \cdot u_2)
\end{align*} \quad (5.41) \]

\[ \begin{align*}
\frac{1}{\chi} u_2 &= u_{2,x} = u_{2,xx} + 3(u_{1,xx} \cdot u_2 - u_2 \cdot u_{1,xx}) + 6u_2(|u_2|^2 + |u_1|^2) \\
&\quad + 6u_1(u_1 \cdot u_2 - u_2 \cdot u_1) + 6u_2(u_1 \cdot u_1 - u_1 \cdot u_1 + u_{2,xx} \cdot u_2 - u_2 \cdot u_{2,xx})
\end{align*} \quad (5.42) \]

where a dot denotes the standard Euclidean inner product (cf (3.21) and (3.22)).

This system is invariant under the symplectic group \( Sp(1) \times Sp(n - 1) \), defined by the transformations (3.68) and (3.69) on the scalar–vector pair \((u_1, u_2, (u_1, u_2))\), and has the following bi-Hamiltonian structure:
\[ \begin{pmatrix} u_1 \\ u_2 \\ u_1 \\ u_2 \end{pmatrix} = \chi^{-1} \begin{pmatrix} u_1 \\ u_2 \\ u_1 \\ u_2 \end{pmatrix} = \mathcal{H} \begin{pmatrix} \delta H^{(1)}/\delta u_1 \\ \delta H^{(1)}/\delta u_2 \\ \delta H^{(0)}/\delta u_1 \\ \delta H^{(0)}/\delta u_2 \end{pmatrix} = \mathcal{E} \begin{pmatrix} \delta H^{(1)}/\delta u_1 \\ \delta H^{(1)}/\delta u_2 \\ \delta H^{(0)}/\delta u_1 \\ \delta H^{(0)}/\delta u_2 \end{pmatrix} \quad (5.43) \]
in terms of the Hamiltonians
\[ H^{(0)} = 4(|u_1|^2 + |u_2|^2) + |u_1|^2 + |u_2|^2, \]  
(5.44)

\[ H^{(1)} = -(|u_1|^2 + |u_2|^2 + |u_{1x}|^2 + |u_{2x}|^2) + 2u_1(u_{1x} \cdot \overline{u}_1 - u_1 \cdot \overline{u}_{1x}) + u_2(\overline{u}_{1x} \cdot \overline{u}_2 - \overline{u}_{2x} \cdot \overline{u}_1) + \overline{u}_2(u_{1x} \cdot u_2 - u_{2x} \cdot u_1) + (|u_1|^2 + |u_2|^2)^2 + 2(|u_1|^2 + |u_2|^2)(|u_1|^2 + |u_2|^2), \]  
(5.45)

where \( \mathcal{E} = \mathcal{H}^* \mathcal{J} \mathcal{H} \) is a Hamiltonian cosymplectic operator compatible with \( \mathcal{H} \).

We remark that the convective terms \( u_{1x}, u_{2x}, u_{1x}, u_{2x} \) on the left-hand side in the system (5.41)–(5.42) and (5.43) can be removed by the Galilean transformation \( t \rightarrow t, x \rightarrow x + \chi^{-1}t \).

### 5.3. SG flow

The recursion operator \( \mathcal{R} = \mathcal{H}^* \mathcal{J} \) yields a \(-1\) flow defined by
\[ 0 = \left( \begin{array}{c}
\dot{w}_{1x} \\
\dot{w}_{2x} \\
\dot{w}_{1w} \\
\dot{w}_{2w}
\end{array} \right) = \frac{\mathcal{J}}{4} \left( \begin{array}{c}
h_{1x} \\
 h_{2x} \\
 h_{1w} \\
 h_{2w}
\end{array} \right). \]  
(5.46)

The resulting flow equations (5.26) have the form
\[ \begin{pmatrix}
u_1 \\
u_2 \\
u_1 \\
u_2
\end{pmatrix} = \begin{pmatrix} h_{1x} \\
 h_{2x} \\
 h_{1w} \\
 h_{2w}
\end{pmatrix}, \]  
(5.47)

with
\[ D_x h_{1x} = -P(u_1, h_{1x}) - P(u_2, h_{2x}) - \frac{4}{\sqrt{\chi}} h_{1x} u_1, \]  
\[ D_x h_{2x} = -S(u_2, h_{1x}) + S(u_1, h_{2x}) - \frac{4}{\sqrt{\chi}} h_{1x} u_2, \]  
\[ \begin{align*}
-D_x h_{1w} &= \frac{1}{2} h_{1x} u_1 + u_1 h_{1x} - \frac{1}{2} h_{2x} \overline{u}_2 - u_2 h_{2x} + \frac{1}{\sqrt{\chi}} h_{1w} u_1, \\
-D_x h_{2w} &= \frac{1}{2} h_{1x} u_2 + u_2 h_{1x} + \frac{1}{2} h_{2x} \overline{u}_1 + u_1 h_{2x} + \frac{1}{\sqrt{\chi}} h_{1w} u_2, \end{align*} \]  
(5.48)

and
\[ \frac{1}{\sqrt{\chi}} D_x h_{1x} = \frac{1}{2} Q(h_{1x}, u_1) + \frac{1}{2} Q(h_{2x}, u_1) + \frac{1}{2} Q(h_{2x}, u_2) + \frac{1}{2} Q(h_{2x}, u_2). \]  
(5.50)

This system of equations (5.48)–(5.50) for the variables \( h_{1x}, h_{2x}, h_{1w}, h_{2w}, h_1 \) possesses the conservation law
\[ D_x \left( \frac{1}{\chi} h_1^2 + \frac{1}{4} |h_{1x}|^2 + |h_{2x}|^2 + \frac{1}{4} |h_{1w}|^2 + |h_{1w}|^2 \right) = 0. \]  
(5.51)

Hence, after a conformal scaling of \( t \), we can put
\[ \frac{1}{\chi} h_1^2 + \frac{1}{4} |h_{1x}|^2 + |h_{2x}|^2 + \frac{1}{4} |h_{1w}|^2 + |h_{1w}|^2 = 1 \]  
(5.52)

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which yields the relation
\[
\frac{1}{\sqrt{\chi}} h_1 = \pm \sqrt{1 - \frac{1}{4}|h^{+1}|^2 - \frac{1}{4}|h^{-1}|^2 - |h^{\perp}|^2}. \tag{5.53}
\]
Substitution of \( h^{+1} = u_1 \), \( h^{-1} = u_2 \), \( h^{1\perp} = u_1 \), \( h^{2\perp} = u_2 \), into equations (5.53) and (5.48) and (5.49) produces a hyperbolic system of coupled scalar–vector SG equations
\[
u_{1tx} = u_{1t} \cdot \mathbf{u}_1 - u_1 \cdot \mathbf{u}_{1t} + u_{2t} \cdot \mathbf{u}_2 - u_2 \cdot \mathbf{u}_{2t} \mp 4u_1 \sqrt{1 - \frac{1}{4}|u_1|^2 + |u_2|^2 - |u_{1t}|^2 - |u_{2t}|^2}
\]
\[
u_{2tx} = u_{2t} \cdot \mathbf{u}_1 - u_1 \cdot \mathbf{u}_{2t} + u_{2t} \cdot \mathbf{u}_2 - u_2 \cdot \mathbf{u}_{1t} \mp \sqrt{1 - \frac{1}{4}|u_1|^2 + |u_2|^2 - |u_{1t}|^2 - |u_{2t}|^2}
\]
\[
u_{1tx} = \frac{1}{2}(u_2 \mathbf{u}_1 - u_1 \mathbf{u}_2) - u_1 \mathbf{u}_1 - u_1 \mathbf{u}_{1t} \mp \sqrt{1 - \frac{1}{4}|u_1|^2 + |u_2|^2 - |u_{1t}|^2 - |u_{2t}|^2}
\]
\[
u_{2tx} = -\frac{1}{2}(u_1 \mathbf{u}_1 + u_2 \mathbf{u}_2) - u_1 \mathbf{u}_1 - u_2 \mathbf{u}_{1t} \mp \sqrt{1 - \frac{1}{4}|u_1|^2 + |u_2|^2 - |u_{1t}|^2 - |u_{2t}|^2}
\]

with a dot denoting the standard Euclidean inner product (cf (3.21)–(3.22)). This system is invariant under the symplectic group \( SP(1) \times SP(n-1) \), defined by the transformations (3.68) and (3.69) on the scalar–vector pair \((u_1, u_2), (u_1, u_2)\).

Alternatively, we can algebraically combine equations (5.48) and (5.49) to express \( u_1, u_2, u_1, u_2 \) entirely in terms of \( h^{+1}, h^{-1}, h^{1\perp}, h^{2\perp} \), and their \( x \) derivatives. Substitution of the resulting expressions into the flow equation (5.47) yields coupled scalar–vector SG equations for the variables \( h^{+1}, h^{-1}, h^{1\perp}, h^{2\perp} \).

### 5.4. Geometric curve flows

From theorem 2.3, the flows in the hierarchy (5.37) and (5.46) for \((u_1(t,x), u_2(t,x), u_1(t,x), u_2(t,x)) \in \mathbb{I} \mathbb{R} \oplus \mathbb{C} \oplus \mathbb{C}^{n-1} \oplus \mathbb{C}^{n-1} \) correspond to \( SP(n+1) \)-invariant non-stretching geometric curve flows for \( \gamma(t,x) \in M = SP(n+1)/SP(1) \times SP(n) \). The resulting equations of motion can be expressed covariantly in terms of \( X = \gamma_s, N = \nabla \gamma_s \) and \( \nabla_x \)-derivatives of \( N \), in addition to the Riemannian metric and curvature tensors on \( M \).

The SG flow (5.54)–(5.55) is given by \( w^{+1} = w^{-1} = 0 \) and \( w^{1\perp} = w^{2\perp} = 0 \), which implies \( w^{||} = w^2 = 0 \) and \( W^{+1} = W^{2\perp} = 0 \) from the structure equation (5.25). This determines
\[
\sigma^{+1} = \sigma^{-1} = 0 \tag{5.56}
\]
and consequently the corresponding flow vector \( \gamma_s = (\gamma_{s-1}) \) satisfies
\[
\alpha \nabla_s \gamma_s = D_s e_1 + [\alpha_1, e_1] = 0, \quad \alpha = -\text{ad}(e_1) \sigma^{+1} = 0. \tag{5.57}
\]
Hence, we obtain the \( SP(n+1) \)-invariant curve flow equation
\[
0 = \nabla_s Y_s, \quad |Y_s| = 1, \tag{5.58}
\]
which is called the non-stretching wave map on \( M = SP(n+1)/SP(1) \times SP(n) \). This equation (5.58) satisfies the non-stretching property \( \nabla_s |Y_s| = 0 \) and possesses the conservation law \( \nabla_s |Y_s| = 0 \), corresponding to equation (5.51). Thus, up to a conformal scaling of \( t \), the wave map equation describes a flow with unit speed, \( |Y_s| = 1 \).

The mKdV flow (5.41)–(5.42), after \( t \) has been rescaled, is given by \( h_{1\perp} = \frac{1}{2} \sqrt{\chi} u_{1\perp}, h_{2\perp} = \frac{1}{2} \sqrt{\chi} u_{2\perp}, h_{1\perp} = -\sqrt{\chi} u_{1\perp}, h_{2\perp} = -\sqrt{\chi} u_{2\perp} \), which gives \( h_0 = \frac{1}{2} \sqrt{\chi} ([u_1]^2 + [u_2]^2 - |u_1|^2 - |u_2|^2) \) from the structure equation (5.23). This determines
\[
(e_t)_{\perp} = \sqrt{\chi} ((\frac{1}{2} u_1 + \frac{1}{2} u_2), (-u_1, -u_2)) \in m_{\perp}, \tag{5.59}
\]
Then $e_i = e_j \gamma_i$ can be expressed as follows in terms of

$$e_j^N = [\omega_x, e_j] = \frac{1}{\sqrt{X}}((-2u_1, -2u_2), (u_1, u_2)) \in m_{⊥},$$

and

$$\begin{align*}
(e_j)_| &= \frac{\sqrt{X}}{4} (Q(u_1, u_1) + Q(u_2, u_2) + Q(u_1, u_1) + Q(u_2, u_2)) \in m_{∥} \\
&= \frac{X}{4} (Q(u_1, u_1) + Q(u_2, u_2) + Q(u_1, u_1) + Q(u_2, u_2)) e_i.
\end{align*}$$

(5.60)

To proceed, it is useful to introduce the linear map defined for all $X \in T_pM$ by

$$\text{ad}(Z)^2 = -R(\cdot, Z)Z.$$ (5.64)

Now consider

$$e_1 \text{ad}(X)^{-2}N = \text{ad}(e_1)^{-2}e_1^N = -\text{ad}(e_1)^{-1} \omega_x = \sqrt{X}((-\frac{1}{2} u_1, \frac{1}{2} u_2, (-u_1, -u_2)) \in m_{⊥}.$$ (5.65)

Hence we have

$$e_1 \text{ad}(X)^{-2}N X = (\text{ad}(e_1)^{-1} \omega_x)^2 e_i \perp = [\omega_x, \text{ad}(e_1)^{-1} \omega_x] \perp$$

$$= -\frac{3\sqrt{X}}{2} ((0, 0), (u_1 u_1 - u_2 u_2, u_1 u_2 + u_2 u_1)) \in m_{⊥}$$ (5.66)

and

$$e_1 \text{ad}(e_1)^{-1} \omega_x^2 e_i \parallel = (\text{ad}(e_1) \text{ad}(X)^{-2}N)^2 e_i \parallel = [\omega_x, \text{ad}(e_1)^{-1} \omega_x] \parallel$$

$$= -\frac{\sqrt{X}}{2} (Q(u_1, u_1) + Q(u_2, u_2) + Q(u_1, u_1) + Q(u_2, u_2)) \in m_{∥}$$ (5.67)

by means of the Lie brackets (3.63a) and (3.63b). In addition, we have

$$g(N, \text{ad}(X)^{-2}N) = g(X, \text{ad}(X)^{-2}N X) = -e_1 \cdot [\omega_x, \text{ad}(e_1)^{-1} \omega_x]$$

$$= -\frac{X}{2} (Q(u_1, u_1) + Q(u_2, u_2) + Q(u_1, u_1) + Q(u_2, u_2))$$ (5.68)

since the inner product is ad-invariant.

Comparing equations (5.59)–(5.60) with equations (5.62)–(5.63) and (5.66)–(5.68), we see that

$$\begin{align*}
(e_i)_\perp + 4(e_i) &= e_1 \text{ad}(X)^{-2}\nabla_0 N - 2e_1 \text{ad}(X)^{-2}N X, \\
2(e_i) &= -g(N, \text{ad}(X)^{-2}N e_1 X).
\end{align*}$$

(5.69)

(5.70)

This yields

$$e_j \gamma_i = (e_i)_\perp + (e_i) = e_1 \text{ad}(X)^{-2}\nabla_0 N - 2e_1 \text{ad}(X)^{-2}N X + \frac{1}{2}g(N, \text{ad}(X)^{-2}N e_1 X).$$

(5.71)
Hence, the flow vector $\gamma_i = Y_{(1)}$ satisfies
\begin{equation}
\gamma_i = \text{ad}(\gamma_i)^{-1} \nabla^2 \gamma_i - 2 \text{ad}(\gamma_i)^{-1} \nabla_i \gamma_i \gamma_i + \frac{1}{2} g(\nabla_i \gamma_i, \text{ad}(\gamma_i)^{-1} \nabla_i \gamma_i) \gamma_i, \quad |\gamma_i| = 1,
\end{equation}
(5.72)

which is an $Sp(n+1)$-invariant curve flow equation called the non-stretching mKdV map on $M = Sp(n+1)/Sp(1) \times Sp(n)$. The nonlinearity in this equation are more complicated than in the mKdV map (4.93) on $M = SU(2n)/Sp(n)$ because of the algebraic property that here the vector spaces $m_{\perp} \simeq h_{\perp}$ each split into two orthogonal eigenspaces under the linear map $\text{ad}(e_i)^2$.

6. Concluding remarks

The Riemannian symmetric spaces $M = SU(2n)/Sp(n)$, $Sp(n+1)/Sp(1) \times Sp(n)$ describe curved generalizations of Euclidean geometries in which the Euclidean isometry group is replaced by the Lie group $G = SU(2n)$, $Sp(n+1)$ and the Euclidean frame rotation gauge group is replaced by a symplectic subgroup $H = Sp(n) \times Sp(n)$ in $G$. For arclength-parameterized curves in these geometries, the components of the Cartan connection in a suitably defined parallel frame [9] along the curve represent covariant differential invariants of the curve, which can be related to standard differential invariants by a generalized Hasimoto transformation. In both geometries these covariants are determined uniquely from the curve up to the action of a rigid equivalence group $H_1 = Sp(1) \times Sp(n-1)$ in $H$. In particular, when $H = Sp(n)$, the covariants transform as a pair of complex vectors having a total of $4n - 4$ real components, whereas when $H = Sp(1) \times Sp(n)$, the covariants transform as an imaginary scalar and a complex scalar, plus a pair of complex vectors, comprising $4n - 1$ real components in total.

For curves undergoing geometric flows described by the non-stretching mKdV map equation [9] and the non-stretching wave map equation, the covariants of the curve respectively satisfy bi-Hamiltonian mKdV and SG equations that exhibit invariance under the symplectic group $Sp(1) \times Sp(n-1)$. The simplest cases of these equations occur for the low-dimensional Riemannian symmetric spaces $M = SU(4)/Sp(2)$, $Sp(2)/Sp(1) \times Sp(1)$.

In the case of $M = SU(4)/Sp(2)$, the covariants reduce to a pair of complex scalars $u_1, u_2 \in \mathbb{C}$ that transform as a representation of the symplectic group $Sp(1) \times Sp(1)$. The resulting multi-component $Sp(1) \times Sp(1)$-invariant mKdV and SG equations (cf (4.43) and (4.74)) for this pair of variables are equivalent to well-known $SO(4)$-invariant equations $u_1 = u_{xx} + |u|^2 u_1$ and $u_2 = \pm \sqrt{1 - |u|^2} u$ for the four-component vector variable $u = (Re u_1, Im u_1, Re u_2, Im u_2)$. This equivalence is a consequence of the local isomorphisms $SU(4) \simeq SO(6)$ and $Sp(2) \simeq SO(5)$ which imply that $M = SU(4)/Sp(2)$ is locally isometric to $SO(6)/SO(5)$.

In the case of $M = Sp(2)/Sp(1) \times Sp(1)$, the covariants reduce to an imaginary scalar $u_1 \in \mathbb{R}$ plus a complex scalar $u_2 \in \mathbb{C}$, transforming as a representation of the symplectic group $Sp(1)$. As a consequence of the local isomorphisms $Sp(2) \simeq SO(5)$ and $Sp(1) \times Sp(1) \simeq SO(4)$, which imply $M = Sp(2)/Sp(1) \times Sp(1) \simeq SO(5)/SO(4)$, the resulting multi-component $Sp(1)$-invariant mKdV and SG equations (cf (5.41)–(5.42) and (5.54)–(5.55)) for these variables are equivalent to $SO(3)$-invariant equations $u_1 = u_{xxx} + |u|^2 u_1$, and $u_2 = \pm \sqrt{1 - |u|^2} u$ for the three-component vector variable $u = (1u_{t1}, Re u_2, Im u_2)$. This is a reduction of the four-component vector equations in the previous case.

For all other cases, the multi-component $Sp(1) \times Sp(n-1)$-invariant mKdV and SG equations that arise from the geometries $M = SU(2n)/Sp(n)$ when $n > 2$ and $M = Sp(n+1)/Sp(1) \times Sp(n)$ when $n > 1$ are new and different from each other.
As will be explained by general results presented elsewhere, no NLS equations arise from these geometries $M = SU(2n)/Sp(n)$ and $M = Sp(n+1)/Sp(1) \times Sp(n)$ since neither of them has a Hermitian structure. The same statement applies to the geometries $M = SO(n+1)/SO(n)$ and $M = SU(n)/SO(n)$ considered in earlier work [12, 36]. Nevertheless, symplectically invariant NLS equations can be derived from the corresponding Lie groups $G = SU(2n)$ and $G = Sp(n+1)$, in analogy with the derivation of unitarily invariant NLS equations from $G = SO(n+1)$ and $G = SU(n)$ carried out in [10] by means of a suitable parallel frame formulation for non-stretching geometric curve flows in these Lie groups.

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