Global Persistence in Directed Percolation

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Abstract

We consider a directed percolation process at its critical point. The probability that the deviation of the global order parameter with respect to its average has not changed its sign between 0 and $t$ decays with $t$ as a power law. In space dimensions $d \geq 4$ the global persistence exponent $\theta_p$ that characterizes this decay is $\theta_p = 2$ while for $d < 4$ its value is increased to first order in $\varepsilon = 4 - d$. Combining a method developed by Majumdar and Sire $^2$ with renormalization group techniques we compute the correction to $\theta_p$ to first order in $\varepsilon$. The global persistence exponent is found to be a new and independent exponent. We finally compare our results with existing simulations.

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1 Motivations

1.1 Directed Percolation

At the initial time $A$ particles are placed randomly with density $\rho_0$ on the sites of a $d$-dimensional hypercubic lattice. They perform independent simple random walks with a diffusion constant $\lambda$. Multiple occupancy is allowed. The $A$ particles undergo three reaction processes: coagulation upon encounter at a rate $k$, branching at a rate $k'$, spontaneous death at a rate $\gamma$,

$$
A + A \xrightarrow{k} A \\
A \xrightarrow{k'} A + A \\
A \xrightarrow{\gamma} \emptyset
$$

As the branching rate $k'$ is decreased below a threshold value $k'_c$ (equal to $\gamma$ in mean-field), the steady state of this system exhibits a continuous transition from a state in which a finite positive density of $A$’s survive indefinitely to an absorbing $A$-free state. The order parameter of the transition is $\rho_A$, the average of the local density of $A$’s.

We have used here the language of the Schlögl reaction-diffusion process to describe directed percolation as in [2]. Various alternative formulations exist ([3, 4]). Furthermore the scope of directed percolation reaches far beyond chemical kinetics, as an overwhelmingly large class of nonequilibrium systems possessing a phase transition in their steady state fall in the same universality class (cellular automata, surface growth, reaction-diffusion processes). This makes of the process Eq. (1.1) a paradigm for non-equilibrium systems with a transition in their steady state. Our knowledge of the behavior of the system in the steady state and during the relaxation stages rests on numerical simulations (in low space dimensions, $d = 1, 2$) and on analytical techniques (short time series expansions in $d = 1, 2$, renormalization group in $d = 4 - \varepsilon$). The critical regime is characterized by a set of three independent exponents: the dynamical exponent $z$, the anomalous dimension of the order parameter $\eta$ and the correlation length exponent $\nu$. Scaling laws for $\rho_A$ can be extracted from special cases of

$$
\rho_A(t, k' - k'_c, \rho_0) = b^{-\frac{4d}{d+z}} F(b^{1/\nu}|k' - k'_c|, b^{-\zeta} t, b^{\frac{d}{d+z}} \rho_0)
$$

which holds in the limit $b \to \infty$ with the arguments of $F$ fixed. Similar scaling relations exist for correlation functions.

1.2 Global Persistence

In this article we want to focus on a property that cannot be deduced from the knowledge of the scaling properties of correlation functions alone. We first define the deviation of the global time-dependent order parameter with
respect to its average:

\[ \Psi(t) \equiv \lim_{L \to \infty} L^{-d/2} \sum_{x \in L^d} \left[ n_A(x, t) - \langle n_A(x, t) \rangle \right] \quad (1.3) \]

In Eq. (1.3) we denote by \( n_A(x, t) \) the number of A particles at site \( x \) at time \( t \), in a particular realization of the reaction-diffusion process. The brackets \( \langle \ldots \rangle \) denote an average with respect to the set of microscopic realizations consistent with the initial conditions, the rules Eq. (1.1) and the diffusion.

We define the global persistence probability as the probability that \( \Psi \) remain of constant sign between 0 and \( t \). Similar quantities have been considered (see [1, 5]) in critical dynamics of magnetic systems, \( \Psi \) simply being the total magnetization. There it was shown that, following a quench from a high temperature disordered state to the critical point, the global persistence probability decays with time as a power law characterized by a universal exponent \( \theta_p \). In critical dynamics the persistence probability is a quantity that appears naturally in the description of the system while it relaxes to its equilibrium state. Our motivation for the present work lies in the lack of both qualitative and analytical picture of the onset of long range correlations in non-equilibrium systems relaxing to their steady state. We believe that the knowledge of the global persistence probability will shape our picture of the way the system organizes at criticality.

The remainder of the article is divided as follows. We first recall in Sec. 2 the well known correspondence between directed percolation and field theory. Following Majumdar and Sire [1], it is possible to obtain the global persistence probability from a careful analysis of the autocorrelation of the global order parameter. This analysis is performed in great detail in Secs. 3 and 4. In Sec. 5 we turn to the explicit calculation of the persistence exponent. In our conclusion we compare our results with existing simulations.

2 Field theoretic formulation

There are several ways of mapping directed percolation onto a field theory ([2, 3]). The resulting field theory involves a field \( \psi \) whose average is the local density of A individuals, and a conjugate field \( \bar{\psi} \); dropping terms irrelevant in the vicinity of the upper critical dimension \( d_c = 4 \) the corresponding action reads

\[ S[\psi, \bar{\psi}] = \int d^d x \ dt \left[ \bar{\psi} (\partial_t + \lambda (\sigma - \Delta)) \psi + \frac{\lambda g}{2} \psi \bar{\psi} (\psi - \bar{\psi}) - \rho_0 \delta(t) \bar{\psi} \right] \quad (2.1) \]

The parameter \( g \) can be expressed in terms of the original reaction rates \( k, k', \gamma \) and the mass in the propagator is \( \lambda \sigma = \gamma - k' \). The action Eq. (2.1) is the starting point of the subsequent analysis. Renormalization group techniques allow us to focus on scaling laws close to or at criticality, during the
relaxation process or in the steady state. From here on, as we shall eventually focus on phenomena taking place at criticality, we set $\sigma = 0$. We now summarize a few well-known results on the renormalization of the action Eq. (2.1) that can be found e.g. in [3].

One first defines renormalized parameters and fields as follows

$$
\psi = \sqrt{Z} \psi_R, \quad \bar{\psi} = \sqrt{Z} \bar{\psi}_R, \quad \lambda = Z^{-1} Z_\lambda \lambda_R, \quad \frac{g^2}{(8\pi)^{d/2}} = Z^{-2} Z^{-1} Z_u u \mu^\varepsilon \quad (2.2)
$$

where $\mu$ is a momentum scale. From the one loop expression of the two and three-point vertex functions one deduces the values of the $Z$-factors using dimensional regularization and the minimal subtraction scheme. They read

$$
Z = 1 + \frac{u}{\varepsilon}, \quad Z_\lambda = 1 + \frac{u}{2\varepsilon}, \quad Z_u = 1 + \frac{8u}{\varepsilon} \quad (2.3)
$$

The $\beta$-function has the one-loop expression

$$
\beta_u \equiv \mu \frac{du}{d\mu} = u(-\varepsilon + 2\gamma_\lambda + \gamma - \gamma_u) = u(-\varepsilon + 6u) \quad (2.4)
$$

where we have introduced the Wilson functions $\gamma_i \equiv \mu \frac{d ln Z_i}{d\mu}$, $i = \emptyset, \lambda, u$.

The $\beta$-function has a stable nontrivial fixed point $u^* = \frac{\varepsilon}{6} + O(\varepsilon^2)$. Critical exponents are then obtained from linear combinations of the $\gamma_i(u^*)$, e.g. $z = 2 - \gamma_\lambda + \gamma^*$ and $\eta = \gamma^*$.

We find it convenient to shift $\psi$ by its mean-field expression

$$
\psi_{mf}(t) = \frac{\rho_0}{1 + \frac{\lambda g}{2} \rho_0 t} \quad (2.5)
$$

Therefore the action expressed in terms of the fields $\phi \equiv \psi - \psi_{mf}$ and $\bar{\phi} \equiv \bar{\psi}$ reads

$$
S[\phi, \bar{\phi}] = \int \left[ \bar{\phi}(\partial_t + \lambda(\frac{g \rho_0}{1 + \frac{\lambda g}{2} \rho_0 t} - \Delta))\phi - \frac{\lambda g \rho_0}{2(1 + \frac{\lambda g}{2} \rho_0 t)} \bar{\phi}^2 + \frac{\lambda g}{2} \bar{\phi}\phi(\phi - \bar{\phi}) \right] \quad (2.6)
$$

We have thus gotten rid of the initial term localized at $t = 0$. We will use the following notation: $G^{(n,m)}$ denotes the $(n + m)$-point correlation function involving $n$ fields $\phi$'s and $m$ fields $\bar{\phi}$'s, as defined in Eq. (4.4), and $W^{(n,m)}$ denotes its connected counterpart. The basic ingredients for a perturbative expansion are the free propagator $G$ and the free correlator $C$, defined by the zero-loop expression of $G^{(1,1)}$ and $G^{(2,0)}$, respectively. We shall need the large time behavior of $G$ and $C$:

$$
G(k; t', t) = \Theta(t' - t) \left( \frac{t}{t'} \right)^2 \exp[-\lambda(k^2 + \sigma)(t' - t)] \quad (2.7)
$$
to determine the renormalized autocorrelation function

calculation of these diagrams combined with Eqs. (2.2) and (2.3) allows one by a one loop connected Feynman diagram shown in Fig. 1. The explicit

0
find
Using again the shorthand notations
coupling constant
g

determine
C
which merely is the autocorrelation function of the field ψ. In order to
determine C(t, t') we carry out a perturbation expansion in powers of the
coupling constant g. The first term of this expansion is of course C(k =
0; t, t'). The first non trivial corrections come in six pieces, each depicted by a one loop connected Feynman diagram shown in Fig. 1. The explicit calculation of these diagrams combined with Eqs. (2.2) and (2.3) allows one to determine the renormalized autocorrelation function

C_R(t, t') = Z^{-1}C(t, t')

(3.2)

Using again the shorthand notations t_< = min{t, t'}, t_> = max{t, t'}, we find

C_R(t, t') = \frac{1}{2}(\lambda t_< \mu^2) \hat{\Phi}\left(\frac{t_<}{t_>}\right) \left[\frac{1}{6} A \left[1 - \frac{5}{6} F\left(\frac{t_>}{t_<}\right) + O(\varepsilon^2)\right]\right]

(3.3)

which holds for t_<, t_> large with t_> / t_< finite. In Eq. (3.3) the amplitude A reads

A = 1 + \frac{\varepsilon}{6} \left(\frac{1457}{600} + \frac{\pi^2}{20} - \frac{96}{25} \ln 2\right) + O(\varepsilon^2)

(3.4)

and the function F has the expression

F(x) = -\frac{19}{30x} + \frac{11}{6} + \frac{\pi^2}{10} - \frac{96}{25} \ln 2 + \frac{12}{25} x - \frac{7}{50} x^2 - \frac{77}{50} x^3 + 

+ \ln(1 - x^{-1}) \left[\frac{101}{25} - 5x + x^2 + x^3 - \frac{26}{25} x^4 - \frac{1}{25} x^5 + \frac{1}{25x}\right]

+ \ln(1 + x^{-1}) \left[\frac{23}{50} + x + x^2 + x^3 + \frac{23}{50} x^4 - \frac{1}{25} x^5 - \frac{1}{25x}\right]

+ 2 \ln^2 x + 3 \ln x \ln(1 - x^{-1})

+ 4 \text{Li}_2(1 - x) + \text{Li}_2(1 - x^{-1}) - \frac{3}{20} (x^4 - 1) \text{Li}_2(x^{-2}) - \frac{3}{5} \text{Li}_2(x^{-1})

(3.5)
where \( \text{Li}_2(x) = -\int_0^x \frac{dt}{t} \ln(1-t)/t \) is the dilogarithm function. The limiting behavior of \( F \) is found to be

\[
F(\infty) = \frac{8329}{1200} + \frac{2}{5} \pi^2 + \frac{96}{25} \ln 2 = 0.3313...
\]

\( (3.6) \)

We have listed in the appendix the individual contributions to \( C(t, t') \) arising from the corresponding Feynman diagrams.

### 4 Short time expansion

The result of the previous section Eq. (3.3) for \( C_R(t', t) \) holds for all times \( t \) and \( t' \), with \( t/t' \) finite, but the limit \( t \ll t' \) is singular. In this section we show that for \( t \ll t' \) the autocorrelation function \( C_R(t', t) \) displays power law behaviour with respect to both time arguments and determine the corresponding exponents. In this limit the random variable \( \Psi(t) \) becomes a Markovian process for which the persistence exponent may be expressed in terms of well-known critical exponents. In the case of the Ising model the Markovian approximation for \( \Theta_p \) is already close to the values obtained by simulations [7, 5].

An appropriate method to study the correlation function for \( t \ll t' \) is the short time expansion (STE) of the field \( \psi(r, t) \) in terms of operators located at the ‘time surface’ \( t = 0 \). Since the Gaussian propagator and correlator are of the order \( t^2 \) for \( t \to 0 \) we expect that the leading term in the STE is the second time derivative \( \ddot{\psi} \) of \( \psi \), i.e.

\[
\psi(r, t) - \langle \psi(r, t) \rangle = c(t) \ddot{\psi}(r, 0) + \ldots
\]  

(4.1)

(For \( t = 0 \) the second time derivative of the response field \( \ddot{\psi} \) is equivalent to \( \ddot{\psi} \).) The function \( c(t) \) is a power of \( t \) which can be obtained from the difference of the scaling dimensions of \( \psi \) and \( \ddot{\psi} \). Naively, \( c(t) \sim t^2 \).

To compute the scaling dimension of \( \ddot{\psi} \) in an \( \varepsilon \)-expansion one could determine the additional renormalization that is necessary to render correlation functions with \( \ddot{\psi} \) insertions finite. Fortunately, it is possible to express this dimension to every order in \( \varepsilon \) in terms of other critical exponents. For the initial density \( \rho_0 = \infty \) there is a similarity between directed percolation and the semi-infinite Ising model at the normal transition (i.e., for infinite surface magnetization). In the latter case the short distance expansion of the order parameter field near the surface is governed by the stress tensor [8, 9]. Due to the translational invariance of the bulk Hamiltonian the stress tensor requires no renormalization. Here we look for an initial field which remains unrenormalized as a consequence of the translational invariance (with respect to time) of the stationary state.

Our argument applies to any dynamic field theory defined by a dynamic functional of the form

\[
S[\psi, \ddot{\psi}] = \int_0^\infty dt \int d^d r \left( \ddot{\psi} \partial_t \psi - T[\psi, \ddot{\psi}] \right)
\]  

(4.2)
We assume that $\psi$ satisfies the sharp initial condition $\psi(r,0) = \rho_0$. For directed percolation we have

\[ T[\psi, \bar{\psi}] = -\lambda \bar{\psi}(\sigma - \Delta)\psi - \frac{\lambda g}{2} \psi \bar{\psi}(\psi - \bar{\psi}) \]  

(4.3)

Correlation functions may be written in the form

\[ G^{(n,m)}(\{r, t\}) = \int \mathcal{D}[\psi, \bar{\psi}] \prod_{i=1}^{m} \bar{\psi}(\bar{r}_i, \bar{t}_i) \prod_{j=1}^{n} \psi(r_j, t_j) \exp(-S[\psi, \bar{\psi}]) \]  

(4.4)

where the functional integral runs over all histories \{\psi, \bar{\psi}\} which satisfy the initial condition.

We now introduce a new time variable $t \to t' = t + a(t)$ (with $a(t) > -1$ to maintain the time order) and the transformed fields $\bar{\psi}'$ and $\psi'$ with $\bar{\psi}'(t) = \bar{\psi}(t')$ and $\psi'(t) = \psi(t')$. At lowest order in $a(t)$ the dynamic functional becomes

\[ S[\psi, \bar{\psi}] = S[\psi', \bar{\psi}'] - \int dt \int d^d r \, \dot{a}(t) T[\psi', \bar{\psi}'] \]  

(4.5)

where $a(0) = 0$ has been assumed.

Performing the time shift in the correlation function $G^{(n,m)}$ and comparing the terms of first order in $a(t)$ on both sides of Eq. (4.4) one finds

\[ \left( \sum_{i=1}^{m} a(\bar{t}_i) \frac{\partial}{\partial \bar{t}_i} + \sum_{j=1}^{n} a(t_j) \frac{\partial}{\partial t_j} \right) G^{(n,m)}(\{r, t\}) \]  

\[ = \langle \bar{\psi}(\bar{r}_1, \bar{t}_1) \ldots \psi(r_n, t_n) \int dt \int d^d r \dot{a}(t) T[\psi, \bar{\psi}] \rangle \]  

(4.6)

Here the angular brackets indicate the average with respect to the weight $\exp(-S[\psi, \bar{\psi}])$. We may choose

\[ a(t) = a_0 \left( 1 - e^{-vt} \right) \]  

(4.7)

to obtain in the limit $v \to \infty$

\[ \left( \sum_{i=1}^{m} \frac{\partial}{\partial \bar{t}_i} + \sum_{j=1}^{n} \frac{\partial}{\partial t_j} \right) G^{(n,m)}(\{r, t\}) = \langle \bar{\psi}(\bar{r}_1, \bar{t}_1) \ldots \psi(r_n, t_n) \int d^d r T_+ \rangle \]  

(4.8)

where $T_+$ denotes the operator $T[\psi, \bar{\psi}]$ in the limit $t \to 0^+$. (We have assumed that all time arguments of the correlation function are nonzero.)

This result shows that $T_+$ remains unrenormalized to every order of the perturbation theory. Therefore its scaling dimension is given by $d(T_+) = d + z$. At the upper critical dimension $d_c = 4$ we find $d(T_+) = d(\bar{\psi}) = 6$. In fact, one can show that $T_+$ and $\bar{\psi}(0)$ differ for $\rho_0^{-1} = 0$ only by a constant prefactor. To see this we express $T[\psi, \bar{\psi}]$ in terms of the shifted field $\phi = \psi - \psi_{mf}$. Since $\phi(t), \bar{\psi}(t) \sim t^2$ for $t \to 0$ while $\psi_{mf}(t) \sim t^{-1}$ only
the term \(-\frac{\lambda g}{2}\psi_{mf}^2\bar{\psi}\) contributes. Thus \(T_+ \sim \ddot{\psi}(0) \sim \dot{\psi}(0)\), and the STE in Eq. (4.1) becomes

\[
\psi(r,t) - \langle \psi(r,t) \rangle = c(t)T_+(r) + \ldots \tag{4.9}
\]

with

\[
c(t) \sim t^{-(d(\psi)-d(T_+))/z} = t^{-(d+\eta)/2-(d+z)/z} \tag{4.10}
\]

Combining the STE with the general scaling form of the autocorrelation function one obtains

\[
C_R(t',t) \sim t'^{-\eta/z}\left(\frac{t}{t'}\right)^{1+(d-\eta)/(2z)} \tag{4.11}
\]

which holds for \(t/t' \rightarrow 0\).

5 Global persistence

5.1 A detour via the Ornstein-Uhlenbeck process

Let \(X(\tau)\) be a Gaussian process with the following autocorrelation function

\[
\langle X(\tau)X(\tau') \rangle = e^{-\theta_p^{(0)}|\tau-\tau'|} \tag{5.1}
\]

for \(\tau, \tau'\) large (but arbitrary \(\tau - \tau'\)). The random variable \(X\) is thus a Gaussian stationary Markov process (of unit variance). It satisfies a Langevin equation

\[
\frac{dX}{d\tau} = -\theta_p^{(0)}X(\tau) + \zeta(\tau), \quad \theta_p^{(0)} > 0 \tag{5.2}
\]

where \(\zeta\) is Gaussian white noise:

\[
\langle \zeta(\tau)\zeta(\tau') \rangle = 2\theta_p^{(0)}\delta(\tau - \tau') \tag{5.3}
\]

Therefore \(X\) is an Ornstein-Uhlenbeck process. For such a process the probability that \(X\) be positive between 0 and \(\tau\) decays exponentially as

\[
\text{Prob}\{\forall \tau' \in [0, \tau], X(\tau') > 0\} \propto e^{-\theta_p^{(0)}\tau} \tag{5.4}
\]

These are standard results.

5.2 Expansion around an Ornstein-Uhlenbeck process

We now consider a Gaussian stationary process \(X(\tau)\) which has the autocorrelation function

\[
\langle X(\tau)X(\tau') \rangle = e^{-\theta_p^{(0)}|\tau'-\tau|} + \epsilon f(\tau'-\tau) \tag{5.5}
\]
with \( f(0) = 0 \) and \( \epsilon \ll 1 \). Then \( X \) is not a Markovian process. Majumdar and Sire \[1\] have shown how to evaluate the probability that \( X \) be positive between 0 and \( \tau \) to first order in \( \epsilon \). They find

\[
\text{Prob}\{\forall \tau' \in [0, \tau], \ X(\tau') > 0\} \propto e^{-\theta_p \tau} \tag{5.6}
\]

where

\[
\theta_p = \theta_p^{(0)} \left[ 1 - \frac{2\theta_p^{(0)}}{\pi} \int_0^{\infty} \frac{f(\tau)}{1 - \exp(-2\theta_p^{(0)} \tau))^{3/2}} \, d\tau + O(\epsilon^2) \right] \tag{5.7}
\]

Hakim \[10\] has extended this result to \( O(\epsilon^2) \).

### 5.3 Application to the global order parameter

At any fixed time \( t \) there exists a dynamical correlation length \( \xi \sim t^{1/z} \) such that the system may be considered as a collection of effectively independent blocks of linear size \( \xi \). Hence \( \Psi \) is the sum of \((L/\xi)^d\) independent degrees of freedom, which is a Gaussian variable in the limit \( L \to \infty \). We are now in a position to apply the result Eq. (5.7) to the random variable

\[
X(\tau) \equiv \Psi(e^{\tau})/\sqrt{\text{var}\Psi(e^{\tau})} \tag{5.8}
\]

which has the autocorrelation function

\[
\langle X(\tau)X(\tau') \rangle = e^{-\theta_p^{(0)}|\tau'-\tau|} - \frac{\epsilon}{6} e^{-\theta_p^{(0)}|\tau'-\tau|} F(e^{|\tau'-\tau|}) \tag{5.9}
\]

with, after Eqs. (4.11) and (5.1)

\[
\theta_p^{(0)} = 1 + \frac{d}{2z} \tag{5.10}
\]

Substitution into Eq. (5.7) yields

\[
\theta_p = \theta_p^{(0)} \left[ 1 + \frac{2\epsilon}{3\pi} I + O(\epsilon^2) \right] \tag{5.11}
\]

where the integral

\[
I \equiv \int_1^{\infty} \frac{dx}{x^3 F(x)} \tag{5.12}
\]
has the analytic expression

\[
I = \frac{13}{200} - \frac{9}{200} C - \frac{91 \pi}{200} + \frac{3 \pi^2}{16} - \frac{41}{80} \ln 2
+ \frac{\Gamma\left(\frac{1}{4}\right)^2}{\sqrt{2 \pi}} \left(\frac{\pi}{4} - \frac{1}{8} \ln 2 + \frac{41}{80}\right) + \frac{\sqrt{2 \pi^3}}{12 \Gamma\left(\frac{1}{4}\right)^2} \left[-\frac{77 \pi}{200} + \frac{23}{12} - \frac{3}{4} \ln 2\right]
+ \frac{1}{50} _3F_2\left(\frac{1}{4}, 1, 1; \frac{3}{4}, \frac{3}{2}; 1\right) - \frac{6 \sqrt{2 \pi^3}}{5 \Gamma\left(\frac{1}{4}\right)^2} _3F_2\left(\frac{1}{4}, \frac{3}{4}, 1; \frac{5}{4}, \frac{5}{4}; 1\right)
- \frac{1}{30} \Gamma\left(\frac{1}{4}\right)^2 _3F_2\left(\frac{3}{4}, 1, 1; \frac{7}{4}, \frac{3}{2}; 1\right) + \frac{\sqrt{2 \pi^3}}{150 \Gamma\left(\frac{1}{4}\right)^2} _3F_2\left(\frac{1}{4}, \frac{3}{4}, 1; \frac{9}{4}, \frac{3}{2}; 1\right)
- \frac{1}{240} \Gamma\left(\frac{1}{4}\right)^2 _3F_2\left(\frac{3}{4}, 1, 1; \frac{3}{2}, \frac{3}{2}; 1\right) - \frac{3 \Gamma\left(\frac{1}{4}\right)^2}{8 \sqrt{2 \pi}} _3F_2\left(\frac{1}{4}, \frac{1}{2}, 1; \frac{3}{4}, \frac{3}{2}; 1\right)
+ \frac{1}{2} _3F_2\left(\frac{3}{4}, 1, 1; \frac{7}{4}, \frac{3}{2}; 1\right) - \frac{3 \Gamma\left(\frac{1}{4}\right)^2}{8 \sqrt{2 \pi}} _3F_2\left(\frac{1}{4}, \frac{1}{2}, 1; \frac{3}{4}, \frac{3}{2}; 1\right)
= 0.630237...
\]



where \( C \) denotes Catalan’s constant and \(_3F_2\) the hypergeometric function of order \((3, 2)\). The final result reads

\[
\theta_p = \theta_p^{(0)}(1 + 0.134 \varepsilon + O(\varepsilon^2)) = 2 + 0.059 \varepsilon + O(\varepsilon^2)
\]

6 Discussion

6.1 Comparison to existing simulations

Recently Hinrichsen and Koduvely \[11\] have performed a numerical study of one-dimensional directed percolation in order to determine the asymptotic behavior of the global persistence probability. In terms of the variable \( \Psi \) defined in Eq. (1.3), they find the following results. For the probability that \( \Psi \) remain negative between 0 and \( t \) they indeed find a power law decay characterized by a universal exponent \( \theta_p \) that has the numerical value \( \theta_p = 1.50(2) \). However they find an exponential decay for the probability that \( \Psi \) remain positive between 0 and \( t \). The latter assertion is in contradiction with our finding that the global persistence probability decays algebraically irrespective of the sign of \( \Psi \). A plausible interpretation for such an asymmetry could be the following. On the one hand the global persistence exponent is well-defined in the regime in which the system has lost the memory of the initial condition. This regime takes place for times \( t \) such that \( t^{d-\eta} / \rho_0^0 \gg 1 \). On the other hand, \( \Psi \) is well defined in the limit of infinitely large systems, and then it is the sum of a large number of effectively independent contributions, which, on a lattice of size \( L \), forces \( L \gg t^{1/z} \). Hence for numerical simulations to yield acceptable results, care must be taken that the double limit \( L^2 \gg t \gg \rho_0^{-2z/(d-\eta)} \) is satisfied. Whether the simulations in \[11\] fulfill these bounds is questionable. Finally, in mapping the random process
Ψ(t) to X(τ) we have assumed that the time interval under consideration contains only times large compared with ρ₀⁻¹ so that the regime in which X is stationary be reached. Strictly speaking, we should have defined the persistence probability over a time interval [t₀, t], with t ≫ t₀ ≫ ρ₀⁻¹. In a simulation the choice t₀ = 0 leads to a persistence probability that enters the asymptotic regime after times t very large with respect to ρ₀⁻¹. This may be another problem with [11].

6.2 Some speculations

It is interesting to use the ε-expansion to speculate on the numerical value of θₚ in low space dimension. We define the improved value of θₚ, which we denote by θₚ^spec, by the product of the actual value of θₚ^(0) deduced from numerical simulations, and the O(ε) correction given by Eq. (5.7). We use recent simulations results of one and two-dimensional directed percolation carried out by Lauritsen et al. [12].

| d  | z   | ν   | η   | θₚ^(0) | θₚ^spec | θₚ |
|----|-----|-----|-----|--------|---------|-----|
| 1  | 1.581 | 1.097 | -0.496 | 1.316 | 1.8 | 1.50(2) |
| 2  | 1.764 | 0.734 | -0.409 | 1.567 | 2.0 | - |
| 4 - ε | 2 - \frac{d}{2} \left(1 + \frac{d}{z} \right) - \frac{d}{2} | 2 - \frac{d}{2} | - | 2 + 0.059ε |
| ≥ 4 | 2  | \frac{d}{2} | 0  | 2  | 2  | 2 |

(6.1)

In the first two lines the exponents z, ν and η are taken from [12]; the value of θₚ^(0) was obtained using the hyperscaling relation θₚ^(0) = 1 + d/(2z), and that of θₚ in d = 1 is taken from [11]. Of course the column θₚ^spec gives but a qualitative estimate of the true θₚ that is supposedly closer to it than that obtained by the crude ε-expansion. These predictions certainly have to be tested against numerical simulations.

6.3 Final comments

We would like to add some comments on table (6.1). We find that θₚ > 2 in d < 4 which says that both the average and the variance of the time during which Ψ is of constant sign are finite. The average time depends on a parameter which has the dimension of time, but there is no time scale left in our treatment of the persistence probability. Therefore the average time must depend on a microscopic scale or on ρ₀⁻¹, which we have treated as a microscopic scale. This may also explain why the average time is infinite for critical dynamics: For zero initial magnetization a time scale such as ρ₀⁻¹ does not exist. In critical dynamics one could also define a persistence exponent for the critical relaxation from an initial state with nonzero local magnetization. In that case the persistence exponent would read in the Markovian approximation θₚ^(0) = 1 + d/(2z) > 1 (as for the problem we have treated in this article).
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7 Appendix

We introduce the notations $t_\leq \equiv \min\{t, t'\}$ and $t_\geq \equiv \max\{t, t'\}$ and the ratio $r \equiv t_\geq / t_\leq$. The one loop contributions to the function $C(t, t')$ are the following.

\begin{align*}
\text{Fig. 1(a)} &= \int d\tau d\tau' [C(0; \tau, t) \frac{d^d q}{(2\pi)^d} G(q; \tau', \tau)^2 G(0; t', \tau') ] \\
&\quad + C(0; \tau, t') \frac{d^d q}{(2\pi)^d} G(q; \tau', \tau)^2 G(0; t, \tau')] \\
&= \frac{g^2 \mu^{-\varepsilon}}{(8\pi)^d/2} r^{-2} (\mu^2 \lambda t_\leq)^{\varepsilon/2} \left( \frac{31}{2} \varepsilon + \frac{1}{\varepsilon} \ln r - \frac{11}{5} \ln \frac{1}{r} - \frac{21}{80} - \frac{1}{8} \frac{r^2}{16} - \frac{3}{8} \frac{r^3}{8} \\
&\quad - \frac{3}{8} (r^4 - 1) \ln(1 - r^{-1}) + \frac{1}{8} \ln r \ln(1 - r^{-1}) + \frac{1}{4} \ln^2 r \\
&\quad - \frac{1}{2} \text{Li}_2(1 - r^{-1}) \right) \\
&= \frac{g^2 \mu^{-\varepsilon}}{4(8\pi)^d/2} r^{-2} (\mu^2 \lambda t_\leq)^{\varepsilon/2} \left[ \frac{551}{150} - \frac{\pi^2}{10} - \frac{11}{3} \ln r - \frac{169}{150} r + \frac{3}{100} r^2 + \frac{129}{50} r^3 \\
&\quad + \ln(1 - r^{-1}) \left( \frac{21}{50} + 2r - 2r^2 - 2r^3 \\
&\quad + \frac{79}{50} r^4 + \frac{2}{25} r^5 - \frac{2}{25} r^7 \right) + \ln(1 + r^{-1}) \left( \frac{23}{25} - 2r - 2r^2 - 2r^3 \right) \\
&\quad - \frac{23}{25} r^4 + \frac{2}{25} r^5 + \frac{2}{25} r^7 \right) \\
&\quad + \frac{3}{10} (r^4 - 1) \text{Li}_2(r^{-2}) + \frac{6}{5} \text{Li}_2(r^{-1}) \right] 
\end{align*}
Fig. 1(c)+(d) = \frac{g^2 \mu^{-\varepsilon}}{(8\pi)^{d/2}} r^{-2} (\mu^2 \lambda t_{\varepsilon})^{\varepsilon/2} \left[ -\frac{1}{\varepsilon} + \frac{2}{\varepsilon} \ln r \right.

- \frac{43}{120} + \frac{3}{5} \ln r + \frac{1}{6} r + \frac{1}{4} r^2 + \frac{1}{2} r^3

- \ln(1 - r^{-1}) \left( \frac{5}{2} - 2r - \frac{1}{2} r^{-4} \right)

- \frac{1}{2} \ln^2 r - \frac{3}{2} \ln r - 2 \ln r \ln(1 - r^{-1})

- 2 \text{Li}_2(1 - r) \left. \right]\right] (7.4)

Fig. 1(e)+(f) = \frac{g^2 \mu^{-\varepsilon}}{(8\pi)^{d/2}} r^{-2} (\mu^2 \lambda t_{\varepsilon})^{\varepsilon/2} \left[ -\frac{3}{\varepsilon} \ln r + \frac{9}{4} \ln r - \frac{3}{4} \ln^2 r \right] (7.5)
Figure 1: One-loop diagrams involved in the expression of $C(t, t')$. Note that diagrams (a,d,f) are the only one loop contributions to $\langle \psi(q = 0, 0) \psi(q = 0, t) \rangle$. 
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