I. INTRODUCTION

Several model independent sum rules were derived for the forward light-by-light scattering, and were exactly verified at leading order in scalar and spinor QED [1,3]. Such sum rules are valid for the case when at least one photon is real and the other is spacelike or below the first particle production threshold, i.e. for photon virtualities \( q_1^2 \leq s_0, q_2^2 = 0 \), with \( s_0 \) the particle production threshold. Three of these sum rules have the form of a superconvergence relation, for which an integral over an experimentally measurable quantity yields zero [3]. One of these is a helicity sum rule of the form:

\[
0 = \int_{s_0}^{\infty} \frac{ds}{s} \left( \sigma_2 - \sigma_0 \right) q_2^2 = 0, \tag{1}
\]

where \( \sigma_0 \) (\( \sigma_2 \)) are the total helicity cross sections for the \( \gamma^* \gamma \rightarrow X \) processes for total helicity 0 (2) respectively, where \( X \) denotes the sum over all allowed final states. Such light-by-light sum rules have been applied within different field theories both in perturbative and non-perturbative settings [2,4]. Furthermore, their application to the \( \gamma^* \gamma \) production of light-quark mesons has been discussed in Refs. [3] [4], and the application to \( \gamma \gamma \) production of charmonium states in Ref. [6]. For the pseudo-scalar, scalar, axial-vector, and tensor mesons, where \( \gamma^* \gamma \rightarrow X \) data are available, these sum rules were shown to be verified within the 10% - 30% experimental accuracies [5].

In the present work, we will investigate the extension of such sum rules, when one of the virtual photons is replaced by a vector quarkonium state. For the conventional heavy quark \( QQ \) bound states, radiative transitions have been measured quite extensively in the past decades by collaborations at the charm and B-factories, CLEO@CESR, BaBar@PEP-II, Belle@KEKB, BESIII@BEPCII, and in the near future by Belle-II.

The study of light-by-light sum rules in the heavy quarkonium sector may also be worthwhile in light of the plethora of new states, so-called \( XYZ \) quarkonium states, which have been observed in recent years above open heavy flavor thresholds at all of these facilities, see e.g. Refs. [7–9] for some recent reviews and references therein. Such sum rule relations have the potential to reveal how much of the radiative decay strength from or into vector quarkonium states results from possible exotic mesons. An example is the \( \chi_{c}(3872) \) state with \( J^{PC} = 1^{++} \), which sits right at the \( D \bar{D}^* \) threshold, for which the radiative transitions \( \gamma J/\psi \) and \( \gamma \psi' \) have been proposed as a diagnostic tool into the nature of this state [10], shedding light on its hybrid charmonium-molecular nature. Rare decays of \( X(3872) \) will be an important part of the PANDA [11] scientific program where such studies are feasible even at the start of data taking. Also at BESIII, the first radiative transition between two exotic mesons has been observed in the process \( \gamma(4260) \rightarrow \gamma X(3872) \) [12], and detailed studies of radiative transitions can be expected from Belle-II in the near future [13].

The outline of this work is as follows. In Section II we introduce the helicity sum rule which we will study in this work for radiative transitions between quarkonium states, of which one has \( J^{PC} = 1^{-+} \) quantum numbers. In Section III we describe the potential model adopted from Refs. [14] [15] to reproduce heavy quarkonium wave functions. In Section IV we review the formalism to evaluate the leading radiative transitions between quarkonium states with defined total helicity, and make a comparison between available experimental values and theoretical results in the literature. In Section V we make use of experimental information on the radiative transitions \( \Upsilon(mS) \rightarrow \gamma \chi_{bJ}(nP) \) for \( m > n \), as well as theoretical estimates for \( \chi_{bJ}(nP) \rightarrow \gamma \Upsilon(mS) \) for \( n \geq m \), and evaluate the derived helicity sum rule. We provide a quantitative discussion for the \( \Upsilon(1S), \Upsilon(2S), \) and \( \Upsilon(3S) \) states. Finally, a summary and outlook is given in Section VI.
FIG. 1: LbL forward scattering involving one virtual (V) and one on-shell (γ) photon. We associate V with a vector quarkonium state. As a result of the optical theorem (dashed cut), intermediate quarkonium states X, with \( J^{PC} = 0^{++}, 1^{++}, 2^{++}, \ldots \) contribute to the forward scattering.

II. SUM RULE FOR QUARKONIA RADIATIVE TRANSITIONS

One can generalize the light-by-light sum rule given in Eq. (1) to the case when one photon is replaced by a vector quarkonium state. For the states below \( BB \) threshold (\( M_V^2 < s_0 = 4m_B^2 \)) a similar relation to Eq. (1) is valid

\[
\text{SR} \equiv \int_0^\infty \frac{ds}{(s-m_V^2)^2} (\text{Im} M_{++} - \text{Im} M_{++})_{q^2=0} = 4\pi^2 \frac{\alpha em}{m_V^2} a_v^2, \quad (2)
\]

where the sum extends over both bound states and open flavor states, and where the anomalous magnetic moment \( a_v \) accounts for the non-point like structure of the vector quarkonia. For bottomonia the anomalous magnetic moment is a small quantity and has been estimated as \( 10 \):

\[
a_v = \frac{2\alpha_s(m_b)}{3\pi}. \quad (3)
\]

Using such value, the sum rule of Eq. (2) should yield almost zero for bottomonia, i.e. \( \text{SR} \sim 8 \text{ mb} \).

Unitarity allows us to relate the imaginary part of the \( \gamma V \to \gamma V \) helicity amplitude \( M_{\lambda_1 \lambda'_1 \lambda_3 \lambda_3} \) to the \( X \to \gamma V \) (for \( m_X > m_V \)) or \( V \to \gamma X \) (for \( m_V > m_X \)) transition amplitudes \( \mathcal{M}_{\lambda_X \lambda_3 \lambda_3} \)

\[
\text{Im } M_{\lambda_1 \lambda'_1 \lambda_3 \lambda_3} = \frac{1}{2} \int d\Gamma_X (2\pi)^4 \delta^4(q_1 + q_2 - p_X) (4)
\]

where \( \lambda = \lambda_3 - \lambda_3 \) denotes the helicity of the quarkonium state \( X \), with \( \lambda_3 \) (\( \lambda_3 \)) being the helicity of the vector quarkonium state \( V \) (photon). In the narrow resonance approximation, Eq. (4) can be written as

\[
\text{Im } M_{++,-} = \pi \delta (s-m_X^2) \left| \mathcal{M}_{2,0+1} \right|^2,
\]

\[
\text{Im } M_{++,+} = \pi \delta (s-m_X^2) \left| \mathcal{M}_{0,0+1} \right|^2, \quad (5)
\]

which allows us to rewrite the sum rule Eq. (2) in terms of the helicity dependent radiative widths \( \Gamma_{\Lambda=0,2} \) for either the \( X \to \gamma V \) or \( V \to \gamma X \) transitions. For the \( X \to \gamma V \) transitions, the helicity radiative widths are given by:

\[
\Gamma_0(X) = \frac{1}{4\pi} \frac{k}{m_X^2} \frac{1}{2J+1} \left| \mathcal{M}_{0,0+1} \right|^2,
\]

\[
\Gamma_2(X) = \frac{1}{4\pi} \frac{k}{m_X^2} \frac{1}{2J+1} \left| \mathcal{M}_{2,0+1} \right|^2,
\]

\[
\Gamma^{EM}(X) = \sum_\Lambda \Gamma_\Lambda(X), \quad (6)
\]

where the photon energy is given by \( k = (m_X^2 - m_V^2)/(2m_X) \), and where \( \Gamma^{EM}(X) \) is the unpolarized radiative width of the corresponding transition. The corresponding expressions for the helicity radiative widths for the \( V \to \gamma X \) transitions are obtained by analogous expressions as Eq. (6) with the replacement \( X \leftrightarrow V \).

Substituting Eqs. (5) and (6) into the sum rule Eq. (2) yields the master formula which we will use in this work

\[
\sum_X \frac{8\pi^2}{X} \left( \frac{2J+1}{m_V^2-m_X^2} \right)^2 (r_2(X) - r_0(X)) \Gamma^{EM}(X)
\]

\[
+ \sum_X \frac{8\pi^2}{X} \left( \frac{2J+1}{m_X^2-m_V^2} \right)^2 (r_2(X) - r_0(X)) \Gamma^{EM}(X)
\]

\[
\simeq 0. \quad (7)
\]

Furthermore in Eq. (7), we introduced the helicity ratios

\[
r_\Lambda(X) = \frac{\Gamma_\Lambda(X)}{\Gamma^{EM}(X)}, \quad (8)
\]

which, as will be shown below, are universal constants in case of E1 transition, and depend only on total angular momentum of the state \( X \).

III. POTENTIAL MODEL

Since the relativistic effects in bottomonia are expected to be small, the spectrum and the wave functions can be calculated with the help of the Schrödinger equation and a conventional heavy quarkonium potential,

\[
H_0 \psi(\vec{r}) = \left[ \frac{\vec{p}^2}{m_b} + V(r) \right] \psi(\vec{r}) = E \psi(\vec{r}),
\]

\[
V(r) = V_0(r) + V_{SD}(r),
\]

\[
V_{SD}(r) = V_{SS}(r) + V_{LS}(r) + V_T(r), \quad (9)
\]

where the Cornell potential \( V_0 \) is the sum of one-gluon exchange (\( V_V \)), and linear scalar confining (\( V_S \)) parts [17]:

\[
V_0(r) = V_V(r) + V_S(r) = -\frac{4\alpha_s}{3} \frac{1}{r} + br, \quad (10)
\]

while \( V_{SD} \) is a spin-dependent part which splits into spin-spin \( V_{SS} \), spin-orbit \( V_{SL} \), and tensor potential \( V_T \), re-
respectively. Up to an order $1/m_b^2$ they are given by

$$V_{SS}(r) = \frac{32\pi \alpha_S}{9m_b^2} \delta(r) \hat{s}_1 \cdot \hat{s}_2,$$

$$V_{LS}(r) = \frac{1}{2m_b^2r} \left( 3 \frac{dV_V}{dr} - \frac{dV_S}{dr} \right) \vec{L} \cdot \vec{S},$$

$$V_T(r) = \frac{1}{12m_b^2} \left( \frac{1 dV_V}{r} - \frac{d^2V_V}{dr^2} \right) H_T,$$

$$H_T = \frac{6}{r} \left( \vec{S} \cdot \vec{r} \right)^2 - 2 \vec{S}^2,$$  \hspace{1cm} (11)

where $L$ is the relative orbital momentum operator and $S = \vec{s}_1 + \vec{s}_2$ is the total spin operator of the quark anti-quark system. Typically the spin-dependent terms are treated using the leading-order perturbation theory. However, in the present work we follow Deng et al. \cite{13} and account for them non-perturbatively. In order to do that, several modifications are needed.

First of all, one needs to take the matrix elements over operators in the $[L, S, J, j]$ basis (where $J$ is the spin projection of $J$ on a fixed axis)

$$\langle \hat{s}_1 \cdot \hat{s}_2 \rangle = \frac{1}{2} S(S+1) - \frac{3}{4},$$

$$\langle \vec{L} \cdot \vec{S} \rangle = \frac{1}{2} (J(J+1) - L(L+1) - S(S+1)), $$

$$\langle H_T \rangle = \frac{4 (S^2 L^2 - 3 \vec{L} \cdot \vec{S} - 3 (\vec{L} \cdot \vec{S})^2)}{(2L+1)(2L-1)} ,$$  \hspace{1cm} (12)

and regularize a $1/c^3$ behaviour in the terms $V_{LS}$ and $V_T$ when $r \rightarrow 0$. The most obvious way to do that is to saturate these potentials at some low-distance scale $r_C$, i.e. set $V_T(r) = V_T(r_c)$ and $V_LC(r) = V_L(r_c)$ when $r < r_C$. \hspace{1cm} (13)

Secondly, the physical hyperfine interaction corresponds to smeared $\delta$-function \cite{18}

$$\delta(r) \rightarrow \left( \frac{\sigma}{\sqrt{\pi}} \right)^3 e^{-\sigma^2 r^2},$$

where $1/\sigma$ is a radius of order $\alpha^2$. \hspace{1cm} (14)

Finally, in order to effectively account for the creation of virtual light $q\bar{q}$ pairs in the Wilson loop, one considers a screening of the confining potential at large distances $r \gg 1/\mu$:

$$b r \rightarrow \frac{b}{\mu} (1 - e^{-\mu r}) .$$  \hspace{1cm} (13)

The unknown parameters were determined in \cite{15} by fitting the spectrum. A fairly good description of the energy levels was achieved with the following choice of the parameters:

$$\alpha_S = 0.368,$$

$$m_b = 4.757 \text{ GeV},$$

$$b = 0.206 \text{ GeV}^2,$$

$$\sigma = 3.10 \text{ GeV},$$

$$\mu = 0.056 \text{ GeV},$$

$$r_C = 0.060 \text{ fm}.$$  \hspace{1cm} (14)

In our work we were able to reproduce the results of \cite{14} to an accuracy of less than 1 MeV on the energy levels, and use these results for the masses of the yet unmeasured $\chi_{00}(3P)$ and $\chi_{02}(3P)$ states, as well as to evaluate the radiative transition matrix elements.

### IV. E1 RADIATIVE TRANSITIONS

The helicity amplitudes entering Eq. (6) can be expressed, choosing the Coulomb gauge, as:

$$\mathcal{M}_{\lambda, \lambda', \lambda V} = \sqrt{2m \pi} \sqrt{2E_f} \mathcal{M}_{fi},$$  \hspace{1cm} (15)

$$\mathcal{M}_{fi} = \int d^3x \ e^{-i \hat{x} \cdot \vec{r}} \langle \psi_f | \ e^{\lambda x} \cdot \vec{J}(\vec{x}) | \psi_i \rangle,$$  \hspace{1cm} (16)

with $\vec{J}(\vec{x})$ the electromagnetic current operator, and where we introduced the labels $i$ (initial) and $f$ (final) instead of $X$ and $V$ to keep further calculations independent of the direction of the transition. Here the initial and final internal states are labeled by $|\psi_i \rangle = |n_iL_iS_iJ_i\rangle$, and $|\psi_f \rangle = |n_fL_fS_fJ_f\rangle$, where $j_i, j_f$ are the spin projections on a fixed axis. By going to the rest frame of the decaying state, we can orient the quantization axis along the photon momentum direction $\vec{k}$ and identify the spin projections in terms of helicities, e.g. for the $X \rightarrow \gamma V$ transitions, as: $j_i = \lambda = \lambda_i - \lambda_V, j_f = -\lambda_V$. Note that in the matrix element of Eq. (16) we use the covariant normalization for the electromagnetic field, but initial and final quarkonium states are normalized non-relativistically.

A multipole expansion of the electromagnetic field allows to express the matrix element of Eq. (16) as:

$$\mathcal{M}_{fi} = -\sqrt{2} \pi \sum_{j=1}^{\infty} (-i)^j \int d^3x \times \langle \psi_f | \ [\sqrt{\hat{J} + 1} j_{j-1}(k|\vec{x}|) \hat{Y}^{-\lambda_j}_{j-1}(\hat{x}) - \sqrt{\hat{J} j_{j+1}(k|\vec{x}|) \hat{Y}^{-\lambda_j}_{j+1}(\hat{x})} - \sqrt{2} \hat{J} + 1 \lambda \hat{J} j_{j}(k|\vec{x}|) \hat{Y}^{-\lambda_j}_{j}(\hat{x}) \cdot \vec{J}(\hat{x}) | \psi_i \rangle,$$  \hspace{1cm} (17)

where $j_{jL}(k|\vec{x}|)$ denote the spherical Bessel functions, and $\hat{Y}^{\lambda_j}_{j}(\hat{x})$ the vector spherical harmonic function. Furthermore, in Eq. (17) the first term, proportional to the squared bracket, corresponds with the electric multipole transitions, whereas the last term corresponds with the magnetic multipole transitions.

In this work we will only need the expressions for the electric dipole (E1) radiative transitions, i.e. for $J = 1$. Furthermore for the bottomonium system, we use the non-relativistic electromagnetic current operator $\vec{J}(\vec{x})$ as:

$$\vec{J}(\vec{x}) = e_Q \delta^3 \left( \vec{x} - \vec{x}_Q \right) \frac{\hat{p}_Q}{m_Q} + e_Q \delta^3 \left( \vec{x} + \vec{x}_Q \right) \frac{\hat{p}_Q}{m_Q},$$  \hspace{1cm} (18)
with \(m_Q = m_{Q_b} = m_b\), \(e_Q = -e_{Q_b} = -1/3\) the bottom quark charge, \(\vec{r}\) the relative vector between quark and anti-quark positions, and \(\vec{p}_{Q_b} (\vec{p}_{Q})\) the momentum operators for quark (anti-quark) respectively. This yields the \(E1\) matrix element from Eq. (17):

\[
\mathcal{M}^{E1}_{fi} = -\sqrt{2\pi} e_Q \int d^3 r \psi_i^*(\vec{r}) \left[ \sqrt{2} j_0 \left( \frac{kr}{2} \right) \tilde{Y}_{01}^\lambda (\vec{r}) - j_2 \left( \frac{kr}{2} \right) \tilde{Y}_{21}^\lambda (\vec{r}) \right] \cdot \frac{\vec{p}}{m_Q/2} \psi_i(\vec{r}),
\]

(19)

with \(\vec{p} = (\vec{p}_{Q_b} - \vec{p_Q})/2\) the relative momentum operator in the quarkonium system.

Furthermore, to describe the \(E1\) radiative transitions for the bottomonia states with \(n_i = n_f\), the long-wavelength limit \((kr \ll 1,\) with \(r\) the size of the bottomonium system\) provides a rather good approximation. In this limit the matrix element of Eq. (19) is given by:

\[
\mathcal{M}^{E1}_{fi} \approx -e_Q \langle \psi_f | \frac{\vec{p}}{m_Q/2} | \psi_i \rangle \cdot \vec{r} \cdot \vec{e}_{-\lambda_i},
\]

\[
= -ie_Q \langle \psi_f | [H_0, \vec{r}] | \psi_i \rangle \cdot \vec{r} \cdot \vec{e}_{-\lambda_i},
\]

\[
= ie_Q (m_i - m_f) \langle \psi_f | \vec{r} \cdot \vec{e}_{-\lambda_i} | \psi_i \rangle,
\]

(20)

which yields the non-relativistic (NRel) expression for the \(E1\) radiative transition width:

\[
\Gamma^{E1}_{\text{NRel}}(i \rightarrow j) = \frac{e_Q^2 k^3}{4\pi} \left( \frac{2E_f}{m_i} \right) \sum_{J_i} \sum_{J_f} \sum_{\lambda_i, \lambda_f} \left| \langle \psi_f | \vec{r} \cdot \vec{e}_{-\lambda_i} | \psi_i \rangle \right|^2,
\]

(21)

where the mass difference \(m_i - m_f\) was approximated by the photon energy \(k\).

As we will only consider transitions from \(L_i = 1 \rightarrow L_f = 0\) or \(L_i = 0 \rightarrow L_f = 1\) spin triplet states \((S_i = S_f = 1)\) in the following, Eq. (21) reduces to:

\[
\Gamma^{E1}_{\text{NRel}}(\chi_{bJ}(n_iP) \rightarrow \gamma Y(n_fS)) = \frac{e_Q^2 k^3}{4\pi} \left( \frac{E_f}{m_i} \right) \left| \int_0^\infty dr \ r^3 R_f(r) R_i(r) \right|^2,
\]

(22)

where \(R_i\) and \(R_f\) denote the radial wave functions of initial and final states.

One can account for finite-size corrections to the non-relativistic \(E1\) result by the replacement \([19]\):

\[
\langle \psi_f | r | \psi_i \rangle \rightarrow \frac{6}{k} \langle \psi_f | \hat{1} \left( \frac{kr}{2} \right) | \psi_i \rangle.
\]

(23)

A more systematic inclusion of relativistic effects in calculating the \(E1\) decay widths of heavy quarkonia requires to estimate besides the recoil and finite-size effects also the relativistic corrections to the wave functions. Such early estimates of the relativistic corrections to the heavy quarkonia \(E1\) decay rates in an expansion up to order \(v^2/c^2\) were performed using different potential models: for a Richardson type of potential \([20]\), for a Coulomb-plus-linear Cornell type potential for \(r \gtrsim 0.1\) fm, modified to saturate for \(r \lesssim 0.1\) fm \([21]\), and for a Buchmüller-Tye potential with scalar confining part \([22]\). To compare the first-order relativistic corrections between different approaches, we express the relativistic (Rel) calculations of the \(E1\) decay widths of bottomonia as:

\[
\Gamma^{E1}_{\text{Rel}}(i \rightarrow j) = \Gamma^{E1}_{\text{NRel}}(i \rightarrow j) (1 + \delta).
\]

(24)

In Table I we compare the relativistic correction factor \(\delta\) for the \(\chi_{bJ}(1P) \rightarrow \gamma Y(1S)\) and \(\chi_{bJ}(2P) \rightarrow \gamma Y(2S)\) decays in the three above mentioned calculations. For the decays shown in Table I, the bulk of the relativistic corrections comes from the relativistic modifications to the wave functions, and obviously depends on the choice of the potential. For example for the \(\chi_{bJ}(1P) \rightarrow \gamma Y(1S)\) decay rates, Refs. \([20, 21]\) find corrections of order \(\delta \sim +10\%\), whereas Ref. \([22]\) reported corrections of order \(\delta \sim -20\%\) to \(-15\%\) for the same transitions.

| \(\delta\) | \([20]\) | \([21]\) | \([22]\) |
|---|---|---|---|
| \(\chi_{b0}(1P) \rightarrow \gamma Y(1S)\) | +0.09 | +0.11 | -0.14 |
| \(\chi_{b1}(1P) \rightarrow \gamma Y(1S)\) | +0.06 | +0.11 | -0.17 |
| \(\chi_{b2}(1P) \rightarrow \gamma Y(1S)\) | +0.05 | +0.11 | -0.20 |
| \(\chi_{b0}(2P) \rightarrow \gamma Y(2S)\) | +0.04 | +0.32 | -0.12 |
| \(\chi_{b1}(2P) \rightarrow \gamma Y(2S)\) | -0.06 | +0.06 | -0.24 |
| \(\chi_{b2}(2P) \rightarrow \gamma Y(2S)\) | -0.10 | -0.11 | -0.35 |

TABLE I: First-order relativistic correction \(\delta\) to the dominant \(\chi_{bJ}(nP) \rightarrow \gamma Y(nS)\) (for \(n = 1, 2\)) \(E1\) radiative widths, according to Eq. (24), in different approaches.

In view of the expected corrections in the \(10 - 20\%\) range for the lower bottomonia states, our strategy in minimizing the model uncertainties in the sum rule estimates is to use the experimental values of the \(E1\) decay widths wherever possible. The latter are available for the \(Y(2S) \rightarrow \gamma \chi_{bJ}(1P), Y(3S) \rightarrow \gamma \chi_{bJ}(2P),\) and \(Y(3S) \rightarrow \gamma \chi_{bJ}(1P)\) transitions. For the \(\chi_{bJ}(nP) \rightarrow \gamma Y(nS)\) transitions for \(n = 1, 2, 3\), for which the absolute \(E1\) decay widths are not known empirically at present, we will compare their calculated values between five different realistic models which are fit to the spectrum. The spread in the \(\Gamma^{E1}\) model predictions will be taken as an estimate of the error on the \(E1\) decay width.

To evaluate the sum rule Eq. (7), besides the unpolarized radiative width, we also need the helicity ratios \(r_{\lambda, X, 0}=0.2(X)\). We will work in the \(E1\) approximation, i.e. neglect the M2 transition for \(\chi_{b1}(nP) \rightarrow Y(nfS),\) and the M2 and E3 transitions for \(\chi_{b2}(nP) \rightarrow Y(nfS)\). In the \(E1\) approximation, the coefficients \(r_{\lambda, X}\) can be expressed as ratios of Clebsch-Gordan coefficients and do not depend on the internal structure of the mesons. Their values are shown in Table II.

Note that in the extreme non-relativistic limit where the fine-structure is neglected, i.e. when the three
| $J_X$ | $r_0(X)$ | $r_2(X)$ |
|------|------|------|
| 0    | 1    | 0    |
| 1    | $1/2$ | 0    |
| 2    | $1/10$ | $3/5$ |

TABLE II: Helicity ratios $r_{ij}(X)$ for $\Lambda = 0, 2$, and for $J_X = 0, 1, 2$.

$\chi_b(nP)$ states (for $J = 0, 1, 2$) are degenerate, the corresponding helicity radiative widths become all proportional to the same $E_1$ squared matrix element. As a consequence, the helicity-0 and helicity-2 sum rule contributions of the three $\chi_b(nP)$ states are given by:

$$
\sigma_0 \sim \left\{ r_0(\chi_{b0}) + 3r_0(\chi_{b1}) + 5r_0(\chi_{b2}) \right\} \Gamma_{E1}^{\text{NRel}},
$$

$$
= \left\{ 1 + \frac{3}{2} \right\} \Gamma_{E1}^{\text{NRel}} = 3 \Gamma_{E1}^{\text{NRel}},
$$

$$
\sigma_2 \sim 5r_2(\chi_{b2}) \Gamma_{E1}^{\text{NRel}} = 3 \Gamma_{E1}^{\text{NRel}},
$$

(25)

which shows that in this extreme limit, the sum rule holds exactly for the radiative transitions originating from each shell separately. When calculating with realistic potentials below, where the degeneracy between the three $\chi_b(nP)$ states is lifted, we will nevertheless observe an approximate cancellation between these states. However, the sum rule of Eq. (7) in general only holds when summing over all radiative transitions to or from a given $\Upsilon(nS)$ state.

V. RESULTS AND DISCUSSION

The central objects in the evaluation of the sum rule of Eq. (7) are the radiative transitions of either $X \to V\gamma$ or $V \to \gamma X$. Several of these transitions have been studied by the Crystal Ball, ARGUS, CLEO, BaBar, and Belle Collaborations, as summarized by the PDG [23]. For the transitions of $\Upsilon(mS) \to \gamma \chi_{bJ}(nP)$ states when $m > n$ absolute radiative widths are known, and given in Tables III and IV. For the opposite transitions of $\chi_{bJ}(nP) \to \gamma \Upsilon(mS)$ when $n \geq m$ only branching fractions $B(\chi_{bJ}(nP) \to \gamma \Upsilon(mS))$ have been measured. In the absence of the total widths these results cannot be converted into partial widths. For these transitions we will use the results of the potential model, based on Ref. [15], and outlined in Sections III and IV. To account for the theory uncertainty in the $E_1$ radiative transitions, we include the spread between different theory predictions as our error estimate on this quantity. For the nonrelativistic models, we include the predictions from [18, 24] and [25], which mainly differ in the form of the potential. The result of Ref. [24] corresponds with the first-order relativistically corrected wave function with a screened potential model. As for the relativized quark model we refer to [18, 29], where the spinless Salpeter equation was solved and to [27], which relies on the relativized quasipotential approach.

$$
V = \Upsilon(2S) \quad m_{\chi_{bJ}} \quad \Gamma_{E1}^{\text{th}} \quad \Gamma_{\exp}
$$

| $V$ | $m_{\chi_{bJ}}$ | $\Gamma_{E1}^{\text{th}}$ | $\Gamma_{\exp}$ |
|-----|--------------|----------------|---------------|
| $\Upsilon(2S)$ | 9912 | 2.58 [0.04] [0.03] 2.29 ± 0.30 |
| $\Upsilon(1S)$ | 9893 | 2.28 [0.17] [0.06] 2.21 ± 0.31 |
| $\Upsilon(1S)$ | 9859 | 1.19 [0.44] [0.28] 1.22 ± 0.23 |

TABLE III: Results for bottomonium radiative transitions $\Upsilon(2S) \to \gamma \chi_{bJ}(1P)$ in quark potential model outlined in Sections 3 and 4, compared with experimental values [23]. For $\Gamma_{E1}^{\text{th}}$ we include the spread between different predictions [18, 24, 27] as our error estimate.

$$
V = \Upsilon(3S) \quad m_{\chi_{bJ}} \quad \Gamma_{E1}^{\text{th}} \quad \Gamma_{\exp}
$$

| $V$ | $m_{\chi_{bJ}}$ | $\Gamma_{E1}^{\text{th}}$ | $\Gamma_{\exp}$ |
|-----|--------------|----------------|---------------|
| $\Upsilon(3S)$ | 10269 | 3.18 [0.04] [0.03] 2.66 ± 0.57 |
| $\Upsilon(2P)$ | 10255 | 2.66 [0.04] [0.03] 2.56 ± 0.48 |
| $\Upsilon(2P)$ | 10233 | 1.31 [0.18] [0.28] 1.20 ± 0.23 |
| $\Upsilon(1P)$ | 9912 | 0.20 [0.10] [0.01] 0.20 ± 0.04 |
| $\Upsilon(1P)$ | 9893 | 0.00 [0.16] [0.00] 0.02 ± 0.01 |
| $\Upsilon(1P)$ | 9859 | 0.12 [0.03] [0.11] 0.06 ± 0.01 |

TABLE IV: Same as in Table III for the radiative transitions $\Upsilon(3S) \to \gamma \chi_{bJ}(2P, 1P)$.

We start by comparing the theoretical $E1$ radiative widths for the $\Upsilon(2S) \to \gamma \chi_{bJ}(1P)$ and $\Upsilon(3S) \to \gamma \chi_{bJ}(2P)$ transitions with their experimental results in Tables III and IV respectively. We see that for nearly all of these transitions the central values, calculated as explained in Sections III and IV, agree with experiment to within 15%. The spread between the different theoretical values is also in this range, as is illustrated in Fig. 2. We therefore feel confident that we can estimate the unknown radiative widths $\chi_{bJ}(nP) \to \gamma \Upsilon(mS)$, for $m = 1, 2, 3$, with an accuracy at the 20% level or better.

Having compared the theoretical results for the radiative widths of the bottom states with available data, we are now in the position to verify the sum rule Eq. (7) quantitatively. In Table V we show the sum rule evaluation for the $\Upsilon(1S)$ state. As for the $\chi_{bJ}(nP) \to \gamma \Upsilon(1S)$ transitions the absolute radiative widths are not known, we are using our model estimates to evaluate the sum rule. As theoretical uncertainty in the sum rule evaluation we take the spread among the different models discussed above, and tested on the known radiative widths in Fig. 2. From Table V we observe a hierarchy of the $\chi_{bJ}(nP) \to \gamma \Upsilon(1S)$ contributions for different shells $n$. As different bottomonia contribute to Eq. (7) with a weighting proportional to the inverse of the third
power of the mass difference between the participating bottomonium states, one sees that the individual sum rule contributions from the 1P states are more than an order of magnitude larger than those from the 2P states, which are about a factor 4 more important than the 3P state contributions. This observed hierarchy also shows that states with \( n \geq 4 \) are expected to contribute in the few percent range at most. One also notices that for each shell \( (n = 1, 2, 3) \) separately the sum rule is satisfied well within the theoretical error. For the dominant \( n = 1 \) transitions, the sum rule result is around 5% of the dominant helicity-2 contribution from the \( \chi_{b2}(1P) \) state. The cancellation between helicity-0 and helicity-2 contributions for each shell, was already discussed following Eq. (25), being exact in the extreme non-relativistic limit when the fine-structure is neglected. We now see that when using realistic potentials, for which the degeneracy is lifted within each shell, the cancellation is still quite accurate numerically. For the total sum rule, we have an agreement at the 10% level of the dominant helicity-2 contribution.

In Table VI, we show the corresponding results for the \( \Upsilon(2S) \) and \( \Upsilon(3S) \) states respectively. For those states, we have partial experimental information available on the radiative transitions. We are thus able to also test the sum rule in a more model independent way, when using only experimental data. Besides, for the transitions that are not known experimentally, we are using the theory estimates with their model error range as discussed above.

For the \( \Upsilon(2S) \) state, shown in Table VI, the transitions to the \( \chi_{bJ}(1P) \) states are all known experimentally to around 15% precision. The sum rule is seen to hold experimentally for this shell at the 10% level of the dominant helicity-2 contribution (0.7 \( \mu b \) vs 9.8 \( \mu b \)). The same quality of agreement is also found for the second dominant shell in this case, \( n = 2 \), based on the theoretical estimates for the \( \chi_{bJ}(2P) \rightarrow \gamma \Upsilon(2S) \) transitions. When evaluating the sum rule for the first three shells, one finds an agreement better than 5% of the total helicity-2 contribution, concluding again that this sum rule is well satisfied within the theoretical and experimental error estimates.

Similar conclusions can also be reached for the \( \Upsilon(3S) \) state, shown in Table VII. The two dominant shells contributing in this case are \( n = 2 \) and \( n = 3 \), with the \( \Upsilon(3S) \rightarrow \gamma \chi_{bJ}(2P) \) decays widths being known experimentally, and the \( \chi_{bJ}(3P) \rightarrow \gamma \Upsilon(3S) \) being estimated theoretically. One sees for the transitions to the \( n = 2 \) shell that the sum rule is seen to hold experimentally to around 5% of the dominant helicity-2 contribution (1.3 \( \mu b \) vs 24.0 \( \mu b \)). For the \( n = 3 \) shell contribution, the theoretical estimate shows this to hold at the 20% level. When again evaluating the sum rule for the first three shells, one finds an agreement better than 10% of the total helicity-2 contribution, concluding again that this sum rule is well satisfied within the theoretical and ex-

### Table V: Bottomonium sum rule of Eq. (7) for the radiative transitions involving the \( \Upsilon(1S) \) state. Here \( m_X \) is the mass of the state \( X \), \( \Gamma \) the corresponding radiative decay width to the \( \Upsilon(1S) \) state, and \( SR \) the contribution of the corresponding transition to the sum rule. The subscript "th" indicates that the theoretical estimate is used.

| Process | \( m_X \) [MeV] | \( \Gamma \) [KeV] | \( SR \) [\( \mu b \)] |
|---------|---------------|-----------------|-----------------|
| \( \chi_{b0}(1P) \rightarrow \gamma \Upsilon(1S) \) | 9859 | 24.2^{+1.7}_{-0.4} | \( -1.6^{+0.0}_{-0.4} \) |
| \( \chi_{b1}(1P) \rightarrow \gamma \Upsilon(1S) \) | 9893 | 30.2^{+6.4}_{-2.3} | \( -2.3^{+0.1}_{-0.5} \) |
| \( \chi_{b2}(1P) \rightarrow \gamma \Upsilon(1S) \) | 9912 | 36.1^{+4.1}_{-3.5} | \( +4.0^{+0.5}_{-0.4} \) |
| Total | | | \( 0.2^{+0.5}_{-0.2} \) |
| \( \chi_{b0}(2P) \rightarrow \gamma \Upsilon(1S) \) | 10255 | 10.7^{+4.2}_{-0.3} | \( -0.14^{+0.07}_{-0.06} \) |
| \( \chi_{b1}(2P) \rightarrow \gamma \Upsilon(1S) \) | 10269 | 16.9^{+0.9}_{-0.8} | \( +0.35^{+0.00}_{-0.18} \) |
| Subtotal | | | \( 0.17^{+0.09}_{-0.21} \) |
| \( \chi_{b0}(3P) \rightarrow \gamma \Upsilon(1S) \) | 10528 | 10.7^{+0.9}_{-0.7} | \( +0.10^{+0.00}_{-0.08} \) |
| Subtotal | | | \( 0.06^{+0.03}_{-0.08} \) |
| Total | | | \( 0.4^{+0.7}_{-0.5} \) |
For the latter, we show the spread in the theoretical evaluation. We tested this sum rule on bottomonia vector quarkonium states with defined total helicity, of which one has the subscript “exp”. For the transitions where the absolute radiative transitions from PDG [23] where available, indicated by subscript “th”. For the transitions involving the Υ(2S) → γχbJ(1P) transitions of Υ(2S) → γχbJ(2P) transitions and found that the theoretical estimates agree with experiment to within 15%. We then tested the helicity sum rule for the Υ(1S), Υ(2S), and Υ(3S) states. For all three cases, we observed that, due to a cancellation between transitions involving χb0, χb1, and χb2 states, the sum rule is satisfied within experimental and theoretical error estimates. For the total sum rule, a cancellation at the 5% - 10% level of the dominant helicity-2 contribution was observed. Furthermore, we also observed that for each shell (n = 1, 2, 3) separately, the sum rule is satisfied well within the theoretical error. Having tested this sum rule for the low-lying bottomonia states, it may next be applied to charmonia, where one expects relativistic corrections to potential model results to be more important. Furthermore, as a next step such sum rule may be used as a tool to investigate the nature of exotic states in the charmonium and bottomonium spectrum, once the corresponding radiative transitions involving a vector quarkonium state, are measured.

TABLE VI: Bottomonium sumrule of Eq. (7) for the radiative transitions involving the Υ(2S) state. We took data for the radiative transitions from PDG [23] where available, indicated by the subscript “exp”. For the transitions where the absolute radiative widths are not known, we use the predictions based on the model described in this work, used by subscript “th”. For the latter, we show the spread in the theoretical calculations as an estimate of the theoretical model error.

| Process               | mX    | Γ      | SR               |
|-----------------------|-------|--------|------------------|
| Υ(2S) → γχb0(1P)      | 9859  | 1.22 ± 0.23 | -3.3 ± 0.6 exp   |
| Υ(2S) → γχb1(1P)      | 9903  | 2.21 ± 0.31 | -5.8 ± 0.8 exp   |
| Υ(2S) → γχb2(1P)      | 9912  | 2.29 ± 0.30 | +9.8 ± 1.3 exp   |
| Subtotal              |       | 0.7 ± 1.0 | 0.6 exp          |
| χb0(2P) → γΥ(2S)      | 10233 | 13.2 ± 1.1 | -5.7 ± 0.6 exp   |
| χb1(2P) → γΥ(2S)      | 10255 | 15.3 ± 1.2 | -7.3 ± 0.6 exp   |
| χb2(2P) → γΥ(2S)      | 10269 | 16.7 ± 1.2 | +11.2 ± 0.9 exp  |
| Subtotal              |       | -1.7 ± 2.5 | -0.3 exp         |
| χb0(3P) → γΥ(2S)      | 10491 | 2.2 ± 1.1 | -0.9 ± 2.0 exp   |
| χb1(3P) → γΥ(2S)      | 10512 | 5.0 ± 1.3 | -0.2 ± 2.0 exp   |
| χb2(3P) → γΥ(2S)      | 10528 | 7.5 ± 1.3 | +0.6 ± 2.0 exp   |
| Subtotal              |       | 0.25 ± 0.12| -0.32 exp        |

Total -0.8 ± 2.4 | -3.4 exp ± 1.6 exp

VI. CONCLUSION

In this work, we generalized a forward light-by-light scattering sum rule to the case of radiative transitions between quarkonium states with defined total helicity, of which one has JPC = 1−− quantum numbers. The sum rule requires data on radiative transitions in its evaluation. We tested this sum rule on bottomonia vector states. For the transitions of Υ(mS) → γχbJ(nP) states with m > n, for which absolute radiative widths are known, we used those data in the sum rule evaluation. For the transitions of χbJ(nP) → γΥ(mS) when n ≥ m, for which only branching fractions have been measured, we provided theoretical estimates within a potential model. We considered the spread between similar approaches in the literature as an estimate for the model error. We checked the potential model on the known Υ(2S) → γχbJ(1P) and Υ(3S) → γχbJ(2P) transitions and found that the theoretical estimates agree with experiment to within 15%. We then tested the helicity sum rule for the Υ(1S), Υ(2S), and Υ(3S) states. For all three cases, we observed that, due to a cancellation between transitions involving χb0, χb1, and χb2 states, the sum rule is satisfied within experimental and theoretical error estimates. For the total sum rule, a cancellation at the 5% - 10% level of the dominant helicity-2 contribution was observed. Furthermore, we also observed that for each shell (n = 1, 2, 3) separately, the sum rule is satisfied well within the theoretical error. Having tested this sum rule for the low-lying bottomonia states, it may next be applied to charmonia, where one expects relativistic corrections to potential model results to be more important. Furthermore, as a next step such sum rule may be used as a tool to investigate the nature of exotic states in the charmonium and bottomonium spectrum, once the corresponding radiative transitions involving a vector quarkonium state, are measured.

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\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
Process & $m_X$ & $\Gamma$ & $SR$ \\
& [MeV] & [KeV] & [$\mu$b] \\
\hline
$\Upsilon(3S) \rightarrow \gamma \chi_{b0}(1P)$ & 9859.44 & 0.055 & $0.0056 \pm 0.0013_{\text{exp}}$ \\
$\Upsilon(3S) \rightarrow \gamma \chi_{b1}(1P)$ & 9892.78 & 0.018 & $0.0011 \pm 0.0007_{\text{exp}}$ \\
$\Upsilon(3S) \rightarrow \gamma \chi_{b2}(1P)$ & 9912.21 & 0.201 & $0.0142 \pm 0.0030_{\text{exp}}$ \\
Subtotal & & & $0.0075 \pm 0.0034_{\text{exp}}$ \\
$\Upsilon(3S) \rightarrow \gamma \chi_{b0}(2P)$ & 10232.50 & 1.20 & $-7.6 \pm 1.5_{\text{exp}}$ \\
$\Upsilon(3S) \rightarrow \gamma \chi_{b1}(2P)$ & 10255.46 & 2.56 & $-15.1 \pm 2.8_{\text{exp}}$ \\
$\Upsilon(3S) \rightarrow \gamma \chi_{b2}(2P)$ & 10268.65 & 2.66 & $+24.0 \pm 5.1_{\text{exp}}$ \\
Subtotal & & & $1.3 \pm 6.0_{\text{exp}}$ \\
$\chi_{b0}(3P) \rightarrow \gamma \Upsilon(3S)$ & 10491.40 & 7.6$^{+0.9}_{-0.7_{\text{th}}}$ & $-11.8^{+1.5}_{-1.4_{\text{th}}}$ \\
$\chi_{b1}(3P) \rightarrow \gamma \Upsilon(3S)$ & 10512.00 & 9.4$^{+0.6}_{-1.0_{\text{th}}}$ & $-14.3^{+1.5}_{-0.9_{\text{th}}}$ \\
$\chi_{b2}(3P) \rightarrow \gamma \Upsilon(3S)$ & 10528.24 & 11.2$^{+0.0}_{-1.9_{\text{th}}}$ & $+21.3^{+0.0}_{-3.6_{\text{th}}}$ \\
Subtotal & & & $-4.8^{+2.6}_{-5.9_{\text{th}}}$ \\
Total & & & $-3.6^{+2.6}_{-5.9_{\text{th}}} \pm 6.0_{\text{exp}}$ \\
\hline
\end{tabular}
\caption{Same as Table VII for the bottomonium sum rule of Eq. (7) for the radiative transitions involving the $\Upsilon(3S)$ state.}
\end{table}

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