Compactifying the state space for alternative theories of gravity

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Abstract
In this paper, we address important issues surrounding the choice of variables when performing a dynamical systems analysis of alternative theories of gravity. We discuss the advantages and disadvantages of compactifying the state space and illustrate this using two examples. We first show how to define a compact state space for the class of LRS Bianchi type I models in \(R^n\)-gravity and compare to a non-compact expansion-normalized approach. In the second example we consider the flat Friedmann matter subspace of the previous example, and compare the compact analysis to studies where non-compact non-expansion-normalized variables were used. In both examples, we comment on the existence of bouncing or recollapsing orbits as well as the existence of static models.

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1. Introduction
Over the past decade a number of models have been proposed to account for the shortcomings of the standard model of cosmology based on general relativity (GR). In most of these models, some modification is made to GR to explain recent observations such as the cosmic acceleration. These modifications include the adding of extra dimensions like in the brane world models [1], the adding of a minimally or non-minimally coupled scalar field [2], or modifications of the underlying field equations by either adding higher order corrections to the curvature [3] or changing the equation of state [4]. In general, these modified theories of gravity have more complicated effective evolution equations and it can be more difficult to find exact analytical solutions.

The implementation of the theory of dynamical systems (see, for example, [5–7]) has proven to be useful to gain a qualitative understanding of a given class of cosmological models.
This dynamical systems approach does not require knowledge of any exact solutions. However, the equilibrium points of the dynamical system correspond to the interesting cosmological solutions. This approach helps in identifying exact solutions with special symmetries, which is particularly useful when studying complicated field equations.

In recent times, this approach has been used to investigate alternative theories of gravity such as Brans–Dicke theory [8–12], scalar–tensor theories [13–22] and higher order gravity [23–27, 31–35]. It has also proven useful in theories with nonlinear equations of state [36, 37] and brane-world models [38–43].

In this paper, we consider the various frameworks in which the dynamical systems theory can be applied to cosmology. In section 2, we discuss the characteristics of non-compact and compact state spaces in general. We point out the advantages of compactifying the state space, emphasizing the aspect of static and bounce type solutions. In section 3, we proceed to give the specific example of LRS Bianchi I models in $R^n$-gravity and compare the results of [24, 25], where compact and non-compact expansion-normalized variables were used, respectively. In section 4, we consider the flat Friedmann models in $R^n$-gravity and compare the compact formalism of [25] to that of Clifton et al [27], where non-compact non-expansion-normalized variables were used. We summarize our results in section 5, pointing out the discrepancies between the three approaches [24, 25, 27].

2. Choice of the state space

In order to perform a dynamical systems analysis on homogeneous cosmologies, one has to construct variables corresponding to the kinematic quantities as well as a time variable that together define an autonomous system of first-order differential equations. The choice of variables depends on several physical considerations: firstly, one would like to study the cosmological behaviour close to the initial singularity and the late time behaviour of the model. Secondly, we want to study the effect of matter, shear and other physical influences on the cosmological dynamics. Finally, we would like to constrain the system by making use of observations such as the cosmic microwave background.

The so-called Hubble- or expansion-normalized variables together with a Hubble-normalized time variable [44] have been used successfully to study important issues such as the isotropization of cosmological models [45]. The state space defined by the Hubble-normalized variables is compact for simple classes of ever expanding models such as the open and flat FLRW models and the spatially homogeneous Bianchi type I models in GR. In these cases, the dynamical systems variables are bounded even close to the cosmological singularity [7]. This is due to the fact that these simple classes of cosmological models do not allow for bouncing, recollapsing or static models, since there are no contributions to the Friedmann equation that would allow for the Hubble-parameter to vanish.

As soon as there are additional degrees of freedom allowing $\Theta$ to pass through zero (e.g. the simple addition of positive spatial curvature), the state space obtained from expansion-normalized variables becomes non-compact. Note that even the time variable becomes ill-defined in this case and needs to be used carefully (see below). If the expansion-normalized variables are unbounded, one has to perform an additional analysis to study the equilibrium points at infinity. This can be done using the well-known Poincaré projection [46, 47], where the points at infinity are projected onto a unit sphere. These projected equilibrium points can then be analysed in the standard way, i.e. by considering small perturbations around the points. However, it may still be difficult to determine the stability of the equilibrium points at infinity.

Alternatively, one may break up the state space into compact subsectors, where the dynamical systems and time variables are normalized differently in each sector (see e.g.
The full state space is then obtained by pasting the compact subsectors together. We will discuss these two methods in the following subsections, highlighting the advantages and disadvantages in this context.

2.1. Non-compact state spaces and the Poincaré projection

In cosmology it is not always straightforward to construct variables defining a compact dynamical system associated with the class of cosmological models of interest. This is especially true if one considers more complicated theories such as modified theories of gravity. In many of the analyses of these types of theories, the dynamical systems variables are not expansion-normalized and define a non-compact state space [8–12, 19, 20–22, 27]. These analyses make use of a conformal time, which places restrictions on the ranges of physical quantities, such as the energy density, Ricci scalar or scalar field (see section 4). The behaviour of the system at infinity can then be studied using a Poincaré projection. In this framework, the equilibrium points at infinity represent the cosmological singularities such as initial singularities or other singularities where the scale factor, scalar field or other variables of the system tend to zero. Alternatively, the original physical variables may be used [28–30], but an asymptotic analysis is still required to study initial singularities. Despite the non-compactness of the state space constructed in these two approaches, one may in principle study bouncing or recollapsing behaviour as well as static solutions, since one does not normalize with $\Theta$.

It is often useful to define expansion-normalized variables together with a dimensionless, expansion-normalized time variable in order to decouple the expansion rate from the remaining propagation equations. This approach only yields a well-defined time variable if we only study ever-expanding or ever-collapsing models; a sign change in the expansion rate would make this time variable non-monotonic. For the simple class of FLRW models in GR for example, there are no bouncing or recollapsing models, and the expanding or collapsing models can be studied separately in a well-defined compact framework. In a more general scenario however, static and recollapsing or bouncing solutions may occur, and one would have to introduce a modified normalization in order to define a state space that includes these singularities.

In some cases it may be useful to employ expansion-normalized variables, but it may not be feasible to compactify the state space. This is the case when, for example, only studying ever-expanding cosmological models. Non-compact expansion-normalized variables have been used successfully to study aspects of isotropization in higher order gravity models [24, 31]. As pointed out above, the non-compact expansion-normalized state space can only contain expanding (or, by time reversal, collapsing) solutions by construction. In particular, one cannot easily study bounce behaviour in this setup, since the expanding and collapsing subspaces would have to be pasted together at infinity, which is non-trivial. Furthermore, the time variable is ill-defined in this limit and needs careful treatment.

2.2. Compact state spaces

As mentioned above, expansion-normalized variables define a compact state space for certain simple classes of cosmologies such as the class of flat Friedmann models in GR [44]. When, for example, additionally allowing for positive spatial curvature however, this behaviour breaks down even in GR. Formally, we have a negative contribution to the Friedmann equation, allowing all the other variables to become unbounded. Physically, the reason for the non-compactness of the state space is that the positive spatial curvature allows for static and bouncing solutions which have a vanishing expansion rate at least at some point in time.
At this point in time, the simple Hubble-normalization is ill-defined, causing the expansion-normalized variables, as well as the expansion-normalized time, to diverge.

In [48] a simple formalism has been established to compactify the state space: if any negative contribution to the Friedmann equation is absorbed into the normalization, one can define compact expansion-normalized variables. If there are any quantities that may be positive or negative, one has to study each option in a separate sector of the state space and obtain the full state space by matching the various sectors along their common boundaries. In particular, this choice of normalization ensures that the accordingly normalized time variable is well defined and monotonic, and the state space obtained in this way may include static, bouncing and recollapsing models. This approach has been successfully adapted to compactify the state space corresponding to more complicated classes of cosmologies (see, for example, [38, 39, 42, 43]).

3. Example 1: LRS Bianchi I cosmologies in $R^n$-gravity

In this section, we outline how the method discussed in [25] is used to construct a compact expansion-normalized state space for the simple class of LRS Bianchi I cosmologies. We then compare the results obtained in this framework to the results obtained using the non-compact expansion-normalized setup of [24]. We will express the equilibrium points and coordinates of the compact analysis [25] with a tilde to distinguish them from the corresponding points in the non-compact analysis [24]. We end this section with a discussion of bouncing and recollapsing models based on the compact framework.

The Bianchi I models in $R^n$-gravity are fully characterized by the Raychaudhuri equation, the Gauss–Codazzi equation and the Friedmann constraint:

\begin{equation}
\frac{1}{3}\dot{\Theta} + \frac{1}{2n} \frac{\dot{R}}{R} - \frac{1}{2} R - (n - 1) \frac{\dot{R}}{R} \Theta + \frac{\mu}{n R^{n-1}} = 0,
\end{equation}

\begin{equation}
\dot{\sigma} + \left( \Theta + (n - 1) \frac{\dot{R}}{R} \right) \sigma = 0,
\end{equation}

\begin{equation}
\Theta^2 = 3\sigma^2 - 3(n - 1) \frac{\dot{R}}{R} \Theta + \frac{3(n - 1)}{2n} R + \frac{3\mu}{n R^{n-1}}.
\end{equation}

Assuming that the standard matter behaves like a perfect fluid with barotropic index $w$, the energy conservation equation is as usual given by

\begin{equation}
\dot{\mu} + (1 + w) \mu \Theta = 0.
\end{equation}

3.1. Construction of the compact state space

The LRS Bianchi I state space is compactified as discussed in detail in [25]: we introduce the dynamical variables

\begin{equation}
\begin{aligned}
\Sigma &= \sqrt{3\sigma} \frac{D}{D},
\xi &= \frac{3R\Theta}{RD^2}(1 - n),
\ddot{y} &= \frac{3R}{2nD^2}(n - 1),
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
\zeta &= \frac{3\mu}{n R^{n-1} D^2},
\hat{Q} &= \frac{\Theta}{D},
\end{aligned}
\end{equation}

and a dimensionless time variable $\tau$ defined by

\begin{equation}
\frac{d}{d\tau} \equiv \frac{1}{D} \frac{d}{dt}.
\end{equation}

4
Figure 1. Schematic construction of the compact state space of the vacuum LRS Bianchi I models. The sectors have been labelled (numbers in square boxes) according to [25]. Note that the state space is symmetric around $\tilde{\Sigma} = 0$, so that this figure can represent $\tilde{\Sigma} \geq 0$ or $\tilde{\Sigma} \leq 0$.

The normalization $D$ will be chosen such that it is strictly positive at all times. As in GR, we have to explicitly exclude the static flat isotropic vacuum cosmologies [48]. We define eight different sectors according to the possible signs of $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$. The first sector is characterized by $\tilde{x}$, $\tilde{y}$, $\tilde{z} \geq 0$. In this sector, we can simply choose $D = |\tilde{\Theta}| = \epsilon/\tilde{\Theta}$, where $\epsilon = \pm 1$ is defined to be the sign function of $\tilde{\Theta}$: $\epsilon = |\tilde{\Theta}|/\tilde{\Theta}$. The Friedmann equation becomes

$$1 = \tilde{\Sigma}^2 + \tilde{x} + \tilde{y} + \tilde{z}. \quad (7)$$

By construction, all the contributions to the right-hand side of (7) are positive; hence the variables $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$ have to take values in the interval $[0, 1]$, while $\tilde{\Sigma}$ must lie in $[-1, 1]$. Note that $\tilde{Q} = \epsilon = \pm 1$ is not a dynamical variable in this sector only, where we have excluded $\tilde{\Theta} = 0$ as motivated above. This means that we have to create two copies of this sector, one corresponding to the expanding models ($\tilde{Q} = \epsilon = 1$) and the other corresponding to the collapsing models ($\tilde{Q} = \epsilon = -1$). These two sectors are disconnected (see figure 1).

In the other sectors, we absorb any negative contributions to the Friedmann equation into the normalization. For example, if $\tilde{x}$, $\tilde{y} > 0$ and $\tilde{z} < 0$ (as in sector 4 of [25]), we define $D = \sqrt{\tilde{\Theta}^2 - \frac{\mu}{\tilde{\Theta}^2}}$. Note that we may now include the static and bouncing or recollapsing solutions with $\tilde{\Theta} = 0$ as long as there is matter present ($\mu \neq 0$). Again we can express

$3$ Note that the sign of these quantities is independent of the exact choice of the normalization, since $D$ is real and enters quadratically.
the Friedmann equation in terms of the normalized variables (5) and observe that \( \tilde{z} \) does not explicitly appear, but all the other contributions enter with a positive sign. This means that \( \tilde{x}, \tilde{y} \) and \( \tilde{\Sigma}^2 \) are positive and must take values in \([0, 1]\). One can easily show that \( \tilde{z} \) is bounded by the interval \([-1, 0]\) in this sector, and \( \tilde{Q} \) lies in \([-1, 1]\). The other sectors are constructed by analogy (see [25] for details).

Note that in all the sectors other than the first one, \( \tilde{Q} \) is a dynamical variable (taking values in \([-1, 1]\)) with the sign of \( \tilde{Q} \) corresponding to the sign of the Hubble factor. This means that in these sectors, we naturally include both expanding and collapsing models and do not have to artificially create two copies of the sectors. Furthermore, we point out that in all the sectors other than sectors 1 and 2, we can principally include static solutions. The exclusion of static or bouncing/recollapsing models in sector 1 has been explained above. Sector 2 is similar to sector 1 in the limit \( \tilde{\Theta}_1 = 0 \) because of the special way the variable \( x \) is defined: in this case the normalisation vanishes, and we therefore have to exclude this case.

The full state space is obtained by matching the various sectors along their common boundaries defined by \( \tilde{x}, \tilde{y}, \tilde{z} = 0 \). For simplicity, we will first address the vacuum subspace \((\tilde{z} = 0)\). This space consists of four two-dimensional compact sectors corresponding to the sign of the variables \( \tilde{x} \) and \( \tilde{y} \). As discussed above, we have to create the two copies of the first sector (labelled \( 1^+ \) and \( 1^- \)) corresponding to the disconnected expanding and collapsing parts, respectively. The full state space is then composed of five different pieces as depicted schematically in figure 1. Strictly speaking, we have to exclude the points with \( \tilde{Q} = 0 \) and \( \tilde{x}, \tilde{y} \neq 0 \), since \( \tilde{Q} = 0 \) implies \( \tilde{\Theta} = 0 \) which in turn implies \( \tilde{x} = 0 \) unless \( R = 0 \). This is indicated with a dotted line in figures 1–4, showing that the \( \tilde{Q} = 0 \) plane may only be crossed at the points with \( \tilde{x} = 0 \) or \( \tilde{y} = 0 \). We will label these points \( \tilde{M} \) and \( \tilde{N} \) respectively.

The points with \( \tilde{y} = 0 \) have to be treated with caution: these points necessarily have vanishing Ricci scalar \( R \) and the corresponding cosmological solutions can only be discussed in the limit \( R \to 0 \). This issue is addressed in detail in [25], where it was found that there only exist solutions corresponding to these points for very special values of \( n \). The same issue applies to point \( \tilde{N} \), which is a degenerate point as discussed in section 4 below.

The state space corresponding to the matter case is three-dimensional and consists of eight separate sectors. It is straightforward to construct by analogy with the vacuum case, but harder to present in a graphic visualization because of the higher dimensionality of the state space. We therefore omit a graphic representation of the matter state space.

We point out that unlike in the vacuum case, where \( \tilde{\Theta} = 0 \) was only allowed at the single points \( \tilde{M} \) and \( \tilde{N} \), the matter case allows for one additional degree of freedom. In this case static or bouncing/recollapsing models must pass through the one-dimensional lines extending \( \tilde{M} \) and \( \tilde{N} \) along the \( z \)-direction, and can therefore occur for a wider range of variables. This is, of course, due to the fact that the curvature term coupled to the matter contribution can counterbalance the other terms in the Friedmann equation.

3.2. Comparison of equilibrium points

We first look at the vacuum equilibrium points found in [24] and in the LRS Bianchi I state subspace of [25]. Since the former paper only considered expanding models, we restrict ourselves to the expanding subset of the compact LRS Bianchi I state subspace of [25] in this comparison.

In the non-compact analysis [24], one Friedmann-like equilibrium point \( A \), a line of equilibrium points \( L_1 \) corresponding to Bianchi I models, and four asymptotic equilibrium points \( A_\infty, B_\infty, C_\infty \) and \( D_\infty \) were found. The coordinates of \( A \) diverge as \( n \to 1/2 \). This means that for this bifurcation value the point moves to infinity, where it merges with the
finite isotropic points were found: the vacuum point. Equilibrium points in [24] correspond to the similarly labelled ones in [25] for all values of $B$ that the asymptotic points as done in [25], it becomes clear that $\tilde{A}$ half of the full state space was studied. When including the collapsing part of the state space $C$ in [25], labelled $\tilde{A}$ asymptotic equilibrium point $A$. Note that the two lines are in fact the same but for different signs of the variable $\tilde{\lambda}$. This means that for $n = 1/2$, asymptotic equilibrium points occur at all angles. This bifurcation was not considered in [24].

In the compact analysis [25], one Friedmann-like equilibrium point $\bar{A}$ and two Bianchi I lines of equilibrium points $\bar{L}_1$ and $\bar{L}_2$ were found in the flat vacuum subspace explored here. Note that the two lines are in fact the same but for different signs of the variable $\tilde{\lambda}$ (see below).

Table 1 summarizes the equilibrium points from the compact analysis [25] and the corresponding counterparts in the non-compact analysis [24]. We can see that the finite equilibrium points in [24] correspond to the similarly labelled ones in [25] for all values of $n$, even for the bifurcation values of $n$ for which the finite points in [24] move to infinity. We note that the asymptotic points $B_\infty$ and $C_\infty$ only have analogues in [25] for the bifurcation value $n = 1/2$. The line $L_1$ in [24] corresponds to $\tilde{L}_1$ in [25] for $\Sigma_s \in [0, 1]$ and to $\tilde{L}_2$ for $\Sigma_s > 1$, where $\Sigma_s$ parametrizes the line $L_1$. $A_\infty$ corresponds to the single static ($\tilde{\lambda}_* = 0$) point on $\tilde{L}_2$ in [25], labelled $\tilde{N}$ in this section.

The equilibrium point $D_\infty$ in [24] corresponds to the point $\tilde{M}$ in the compact analysis as shown in figures 2 and 3. Note that $\tilde{M}$ is not an equilibrium point in [25]; it only appears to be an equilibrium point in the non-compact analysis because in this case only the expanding half of the full state space was studied. When including the collapsing part of the state space as done in [25], it becomes clear that $\tilde{M}$ merely denotes the point at which orbits may cross between the expanding and contracting parts of the state space.

We now consider the matter equilibrium points. In the compact analysis [24] three finite isotropic points were found: the vacuum point $A$ and two non-vacuum points $B$ and $C$. Furthermore, the vacuum Bianchi I line of points $L_1$ was recovered. There were five asymptotic equilibrium points $A_\infty$, $B_\infty$, $C_\infty$, $D_\infty$ and $E_\infty$, and a line of equilibrium points denoted as $L_\infty$. The coordinates of $A$ diverge when $n \to 1/2$; in this case $A$ merges with $D_\infty$ when $n \to 1/2^-$ and with $E_\infty$ when $n \to 1/2^+$. Similarly, the coordinates of equilibrium point $C$ approach infinity when $n \to 0$: $C$ merges with $A_\infty$ when $n \to 0^-$ and with $B_\infty$ when $n \to 0^+$. Point $B$ on the other hand remains a finite equilibrium point for all values of $n$. As pointed out in the vacuum case, $C_\infty$ is the ‘endpoint’ of $L_1$ at infinity with $\Sigma_s \to \infty$. As in the vacuum case, there is ring of asymptotic fixed points in the $z = 0$ plane at the bifurcation value $n = 1/2$. Furthermore, there is a ring of equilibrium points in the $\Sigma = 0$ plane at the bifurcation value $n = 0$. These bifurcations have not been noted in [24].
In the compact analysis [25], the three isotropic points $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ and the two vacuum Bianchi I lines of equilibrium points $\tilde{L}_1$ and $\tilde{L}_2$ were found. As in the vacuum case, we can see the correspondence between the equilibrium points in [25] and those in [24], where the finite equilibrium points may move to infinity for certain values of $n$. We have summarized these results in table 2.

The line $\mathcal{L}_\infty$ is the higher dimensional matter analogues of point $\mathcal{D}_\infty$ from the vacuum analysis: the counterpart of $\mathcal{L}_\infty$ in [25] is not a line of equilibrium points. $\mathcal{L}_\infty$ only appears as a line of equilibrium points in [24] because the collapsing part of the state space is not included (see the above).

3.3. Solutions and stability

In both [24, 25], the exact solutions to the field equations (1)–(3) corresponding to each equilibrium point were derived.

The solutions to the finite points in [24] are the same as those obtained in [25] for their counterparts in the compact analysis except for the points with $y = 0$, which are very
special, since they necessarily have vanishing Ricci scalar $R$ and their solutions can only be obtained in a careful limiting procedure. This is discussed in great detail in [25], where
it was found that these points only have corresponding solutions for very special values of $n$. In the LRS Bianchi I state space discussed here, the point $\tilde{B}$ and the lines $\tilde{L}_{1,2}$ have vanishing Ricci scalar. As discussed in [25], $\tilde{B}$ only has a solution for the bifurcation value $n = 5/4$ and $w = 2/3$, and $\tilde{L}_{1,2}$ only admits a solution for $n \in (1, N_{\pm})$, where we abbreviate $N_{\pm} = \frac{1}{16\sqrt{w}} (13 + 9w \pm \sqrt{9w^2 + 66w + 73})$. Only two points on $\tilde{L}_{1,2}$ have corresponding cosmological solutions. These points are marked with solid circles in figures 2 and 3. This issue was not addressed in [24], where the authors did not solve the full set of field equations to obtain the exact solutions. However, this is not a problem caused by the use of non-compact variables; the results of [25] can be recovered using the setup of [24] if one carefully solves for all cosmological variables (including $R$).

The solutions for the asymptotic equilibrium points in [24] differ from those obtained in [25] for the corresponding equilibrium points. In [24], the solutions corresponding to the asymptotic vacuum points $B_{\infty}, C_{\infty}, D_{\infty}$ and the asymptotic matter points $A_{\infty}, B_{\infty}, D_{\infty}, E_{\infty}$ were all de Sitter like. In [25] on the other hand, it was shown that there are no solutions to the corresponding equilibrium points.

A careful analysis shows that the stationary solutions in [24] are not valid, since they cannot simultaneously satisfy the evolution equations and the definitions of the dimensionless expansion normalized variables in this limit. Note that these solutions can satisfy the coordinates of the asymptotic equilibrium points in the special static case. However, the static models do not satisfy all the original field equations for this class of models and therefore are not solutions as shown in [25]. This was not investigated in [24].

The solutions for the vacuum point $A_{\infty}$ and the matter point $C_{\infty}$ given in [24] were static in the appropriate limit $\Sigma_* \to \infty$. In [25] however, it is shown that the static models do not satisfy all the evolution equations and therefore do not present cosmological solutions. In this sense, we call all these equilibrium points ‘unphysical’.

We conclude that while [24, 25] found the same solutions for the finite equilibrium points with $y \neq 0$ in [24], there was disagreement with the solutions corresponding to the asymptotic points in [24] and the points with $y = 0$. This discrepancy arises from the fact that the non-compact framework is much more complicated, so that it was not noted that the given solutions indeed do not simultaneously satisfy the original equations and the coordinates of the (asymptotic) equilibrium points.

The nature of the equilibrium points remains unchanged in both formalisms, even though the time variable in [24] is strictly speaking not well defined at infinity. The reason for this agreement is that we study perturbations away from the equilibrium points, i.e. strictly speaking we never reach infinity when studying the eigenvalues. As long as the given point is actually an equilibrium point, the results from the expanding sector alone reflect the dynamical nature of the point in the entire state space correctly. However, we emphasize again that $D_{\infty}$ and $L_{\infty}$ in [24] are in fact not equilibrium points in [25]. As explained in the previous subsection, they only appear to be fixed points in the non-compact dynamical system. The stability of $D_{\infty}$ and $L_{\infty}$ only indicates the direction of the orbits in the expanding part of the compact state space in [25].

3.4. Bounce behaviours

As motivated above, the non-compact expansion normalized variables are not suitable to study trajectories that correspond to recollapsing or bouncing cosmologies. We therefore only discuss this issue in the context of [25].

Static models may be studied if one carefully takes into consideration that they have to be analysed separately in the two copies corresponding to expanding and collapsing models.
We start with the vacuum case: from the Raychaudhuri and Friedmann equations, one can see that there can be bouncing solutions for \( n \in (1/2, 1) \) and recollapsing solutions for \( n \in (0, 1/2) \) or \( n > 1 \). This is reflected in figures 2 and 3: we can see that there are trajectories corresponding to bouncing solutions for \( n \in (1/2, 1) \) and to recollapsing solutions for \( n \in (1, 5/4) \). In both cases, the bouncing or recollapsing trajectories have to go through the point ˜\( M \), which is characterized by ˜\( x = ˜Q = 0 \). The existence of these bouncing or recollapsing solutions has been confirmed by a numerical analysis. For \( n > 1 \), the recollapsing models have a negative Ricci tensor \( R \), while the bouncing or recollapsing models for \( n < 1 \) have a positive value of \( R \).

In the matter case, there is one more degree of freedom. Any bouncing or recollapsing solution must now pass through the 1-dimensional extension of point ˜\( M \) in the \( z \)-direction. This means it is easier to achieve bouncing or recollapsing behaviour in the matter case. In particular, there can be a bounce or recollapse even if ˜\( y > 0 \) (if ˜\( z < ˜y \)). Note that even though at first sight we also expect bouncing behaviour through the one-dimensional extension of ˜\( N \), this line in fact corresponds to degenerate cosmological models, and orbits approaching the line can never reach or cross it as explained in detail in the following example.

4. Example 2: flat Friedmann cosmologies in \( R^n \)-gravity

In this section we consider the flat FLRW models with matter. We will compare the results of [25] with the results of Clifton et al [27], where non-compact non-expansion-normalized variables were used. We briefly summarize the approach used by [27] (which follows [8, 12]): a conformal time coordinate
\[
d\tau = \sqrt{\frac{8\pi \mu}{3R^{n-1}}} dt,
\]
and the dynamical variables
\[
X = \frac{R'}{R} \quad \text{and} \quad Y = \frac{a'}{a},
\]
are introduced, where the primes denote differentiation with respect to \( \tau \). Note that \( \tau \) is only valid when \( R^{n-1} \) is a positive real root of \( R \), which restricts the ranges of \( R \) and \( n \).

Using the evolution equations 1 and 3 (with \( \sigma = 0 \)), an autonomous set of first-order differential equations for the variables \( X \) and \( Y \) is derived. This system is non-compact, and is then analysed using standard dynamical system methods together with the Poincaré projection. Note that this approach does not exclude models with \( \theta = 0 \), which allowed the authors of [27] to study static and bouncing or recollapsing models.

The class of flat FLRW cosmologies is the isotropic subspace of the class of LRS Bianchi I models studied in the previous section. We can therefore simply take over the framework from [25] as outlined in section 3. The equilibrium points for the FLRW state space are simply the isotropic equilibrium points from the previous example. Note that, unlike in the previous example, we now include both the expanding and collapsing sectors in order to compare to [27].

4.1. Comparison of equilibrium points

The analysis in [27] yielded the two pairs of finite equilibrium points 1, 2 and 3, 4. Furthermore, three pairs of equilibrium points at infinity were found: a pair of static points 5, 6 and the two pairs 7, 8 and 9, 10 with power-law solutions. The odd and even numbers in each pair correspond to expanding and collapsing models depending on the value of \( n \). Note that points
1, 2 only have real coordinates for $n > 0$ and $w < 2/3$, while 3, 4 only have real coordinates for $n \in (N_-, N_+)$, where $N_{\pm} = \frac{1}{4(w\mp 3)} (13 + 9w \pm \sqrt{9w^2 + 66w + 73})$. The pair 1, 2 merges with 7, 8 for $n = 0$ or $w = \frac{2}{3}$. Pair 3, 4 merges with 5, 6 for $n = 0$, and 9, 10 merges with 5, 6 for $n = \frac{1}{2}$.

The compact analysis [25] yields three flat Friedmann points (see section 3): $\tilde{A}_{\pm}$, $\tilde{B}_{\pm}$ and $\tilde{C}_{\pm}$, where the expanding solutions are indicated by a plus and collapsing ones by a minus subscript. As noted in the previous section, $\tilde{B}_{\pm}$ only admits a solution at the bifurcation $n = 5/4$ and $w = 2/3$, while $\tilde{C}_{\pm}$ only has a cosmological solution for $n \in (1, N_+)$ and $w > -1$.

In table 3, we summarize the equilibrium points from [25] and the corresponding counterparts in the analysis of [27]. We note that the two matter solutions $\tilde{B}_{\pm}$ and $\tilde{C}_{\pm}$ in [25] correspond to the finite equilibrium points 1, 2 and 3, 4 in [27], while the vacuum equilibrium point $\tilde{A}_{\pm}$ in [25] corresponds to the equilibrium points at infinity in [27]. This is due to the choice of coordinates 9, which diverge for $\mu \to 0$.

We now discuss in detail for which parameter values the correspondence between the equilibrium points occurs. We find that the expanding (collapsing) point $\tilde{A}_+$ ($\tilde{A}_-$) corresponds to 9 (10) for $n \in (1/2, 1)$ or $n > 2$, and to point 10 (9) for $n \in (0, 1/2)$ or $n \in (1, 2)$. There is no dependence on the equation of state parameter $w$ in this case, since the point $\tilde{A}_{\pm}$ corresponds to a vacuum solution. The matter point $\tilde{B}_+$ ($\tilde{B}_-$) corresponds to point 1 (2) for all $n > 0$ provided $w < \frac{2}{3}$, while for $w = 2/3$ point $\tilde{B}_+$ ($\tilde{B}_-$) corresponds to point 7 (8) when $n > 1$ and to point 8 (7) when $n \in (0, 1)$. Note that the matter point $\tilde{C}_+$ ($\tilde{C}_-$) corresponds to 3 (4) over the entire allowed range of $n$.

As in the previous example, an equilibrium point at infinity in the non-compact analysis (here 5, 6 or 7, 8) only has analogues in the compact framework for the specific bifurcation values (of $n$ and in this case $w$) for which a finite equilibrium point moves to infinity and merges with the respective asymptotic point.

We now give special consideration to the points 5, 6. We observe that the two points 5 and 6 have the single analogue $\tilde{N}$ in the compact analysis, which is not an equilibrium point in [25]. The reason for this discrepancy is the following: points 5, 6 correspond to the limit $R \to 0$ (see equation (14) in [27]). As pointed out in [23], the plane $R = 0$ is invariant, so that orbits approaching this plane must turn around. Assuming $R$ starts out positive and approaches zero, it is clear that the limit from the left corresponds to $R' < 0$ and $X \to -\infty$, while the limit from the right corresponds to $R' > 0$ and $X \to \infty$. Thus 5, 6 are not equilibrium points.
in the compact analysis: while they appear as sink and source respectively in [27], they merge into the single transitory point \( \tilde{N} \) in [25], similar to the case of \( D_{\infty} \) above.

However, this point \( \tilde{N} \) represents a singular state: here \( R = \Theta = \tilde{\Theta} = 0 \) that means the field equations break down and can only be studied in a careful limiting procedure (see [25]). In particular, orbits approaching \( \tilde{N} \) asymptotically slow down and never reach or pass through the point. In this sense we recover the results of [27]: even though the two disconnected points 5, 6 have merged into the single point \( \tilde{N} \), no orbits can pass through \( \tilde{N} \) and therefore the qualitative result from [27] is maintained.

### 4.2. Solutions and stability

The solutions given in [27] have corresponding solutions in [25] but only for specific values of the parameters (see section 3.3). The solutions for points 7, 8 are the same as those found for \( B_{\pm} \) when \( n = 5/4 \) and \( w = 2/3 \). In [27], points 1, 2 have the same solutions as 7, 8 but they have no corresponding solutions in [25]. Points 3, 4 have the same solution as \( \tilde{C}_{\pm} \), and points 9, 10 have the same solutions as \( \tilde{A}_{\pm} \). The static solutions for points 5, 6 given in [27] do not satisfy all the evolution equations and therefore are strictly speaking no exact solutions. This result was also found in [25] for the points \( \tilde{A}_{\pm} \) for \( n \to 1/2 \) and \( \tilde{C}_{\pm} \) for \( n \to 0 \), which again reflects the correspondence between the points in the respective limits.

The nature of the equilibrium points in [25] agrees with the stability properties of the corresponding points in [27]. We note that the equilibrium points at infinity, 5, 6 and 7, 8, only have corresponding points in [25] for specific values of \( n \) and \( w \) respectively. These parameter values correspond to the bifurcations where the stability of the equilibrium points changes and were not analysed in detail in [25]. We therefore do not compare the two formalisms in this case.

### 4.3. Bounce behaviours

Unlike in the expansion-normalized non-compact analysis [24] studied in the previous example, bouncing or recollapsing solutions can be investigated the non-compact formalism of [27], where specific examples were given: it was shown that for matter with \( w = 0 \) recollapsing solutions occur for \( n = 1.1 \) and bouncing solutions occur for \( n = 0.9 \).⁵

We confirm this in our compact analysis, whereas in the LRS Bianchi I case, there are bouncing orbits through point \( M \) for \( n \in (1/2, 1) \) and recollapsing orbits through \( M \) for \( n > 1 \). For \( n \in (1/2, 1) \), only point \( \tilde{A}_{\pm} \) is physical, and so the physically relevant behaviour is restricted to sectors 3 and 5 (since \( y = 0 \) is invariant). The dynamics are the same as illustrated in figure 2 for sectors 3 and 5, except that there is no line of equilibrium points. In the case of \( n \in (1, N_e) \) and \( w > -1 \), point \( \tilde{C}_{\pm} \) is also physical so that we have both matter and vacuum solutions in the state space. This is the most interesting case and we will therefore concentrate the discussion below on this range of \( n \) for dust and radiation.

In figure 4 we consider \( w = 0 \) and \( n \in (1, N_e) \), and it can be seen that there are trajectories between the isotropic vacuum points \( \tilde{A}_{\pm} \), corresponding to recollapsing solutions. At a first glance there also appear to be bouncing solutions through point \( \tilde{N} \) in sectors 2 and 5. In sector 2 for example, orbits seem to move from the collapsing matter point \( \tilde{C}_{\pm} \) to its expanding counterpart \( \tilde{C}_{\pm} \). However, since \( \tilde{N} \) is a degenerate point (see section 4.2) where \( R = \Theta = \tilde{\Theta} = 0 \), orbits in the collapsing sector \( 2_{-} \) approach \( \tilde{N} \) asymptotically in the future while the orbits in the expanding sector \( 2_{+} \) approach \( \tilde{N} \) asymptotically in the past. These orbits cannot move through point \( \tilde{N} \) and therefore do not represent bounce solutions.

⁵ Exact solutions corresponding to these bounces are given in [49].
We note that in sector 5 bounce cosmologies exist in which we first have expansion towards $\mathcal{M}$, and then asymptotic collapse towards $\mathcal{N}$. As noted above, these orbits cannot cross at $\mathcal{N}$; otherwise re-expansion to $\dot{\mathcal{M}}$ with a final recollapse towards $\dot{\mathcal{A}}_-$ could have been possible. Thus cyclic universes are not possible in this scenario, since it would require passing through a degenerate state represented by the point $\bar{\mathcal{N}}$.

Comparing to [27], we observe that the orbits connecting points 4 and 6 in [27] correspond to the orbits between points $\bar{\mathcal{C}}_-$ and $\bar{\mathcal{N}}$ in the compact analysis for the case $n \in (1, N_+)$ considered here, while the orbits connecting points 3 and 5 correspond to the orbits between points $\bar{\mathcal{C}}_+$ and $\bar{\mathcal{N}}$. We observe again that the bounce and recollapse behaviours found in [27] are recovered in this compact analysis.

The qualitative results remain unchanged when considering radiation-dominated regimes with $w = 1/3$, the only difference being that point $\bar{\mathcal{B}}_\pm$ moves closer towards the intersection of $\bar{y} = 0$ and $\bar{x} = 0$.

5. Remarks and conclusions

In this work, we compared the use of compact and non-compact variables for a dynamical systems analysis of alternative theories of gravity. We first considered state spaces where
expansion-normalized variables were used. These expansion-normalized variables were first introduced in the context of flat FLRW models in GR [44], where they define a compact state space. This compactness is desirable for determining the global behaviour of cosmological models and was one of the original reasons for introducing expansion-normalized variables in the dynamical systems theory applied to cosmology. In [48], a method to compactify the state space of more general cosmologies was introduced, which was successfully applied to a modified theory of gravity in [25].

We showed here that when non-compact expansion-normalized variables are used, we are restricted to expanding or contracting cosmological models only. Static solutions and bouncing or recollapsing type solutions lie at or approach infinity in this framework. This was illustrated in section 3, where we compared the expansion-normalized compact [25] and non-compact [24] state spaces of LRS Bianchi I models in $R^n$-gravity. While the works agree for the finite points in the non-compact analysis with $y \neq 0$, discrepancies were found for the points with $y = 0$ and the points at infinity. For $y = 0$, which corresponds to the limit of vanishing Ricci curvature, differences with respect to the existence of solutions at the given points were observed. At infinity in the non-compact analysis [24], both the occurrence of equilibrium points and the exact solutions at the equilibrium points differ in places from the results obtained in [25]. For example, we found that the asymptotic points in [24] only have analogues in the compact analysis for specific values of the parameter $n$. The asymptotic equilibrium point $D_{\infty}$ does not have a counterpart in [25] at all—it only appears to be an equilibrium point in [24] because the collapsing part of the state space is not included in this case.

We resolved these problems and found the approach used in the compact analysis [25] more straightforward to analyse: since there are no infinities in this framework, the opportunity to miss relevant information is reduced.

In section 4, we extended our comparison of formalisms by considering the non-compact non-expansion-normalized variables used in [27]. We compared the results of [27] with the compact analysis of [25] for the flat Friedmann models. As in the first example, the points at infinity only have corresponding equilibrium points in the compact analysis for specific values of $n$ and/or $w$. Unlike in [24], bounce and recollapse behaviours could be investigated in the framework of [27], and we recover these results in the compact formalism.

We note that in both non-compact formalisms [24, 27], the equilibrium points at infinity are associated with a divergence in the respective dynamical systems time variable $\tau$. When expansion-normalized variables are used, $\tau$ diverges when $\Theta \to 0$, while in framework of [27] $\tau$ diverges when the matter density becomes negligible. This divergence however does not seem to affect the stability of the equilibrium points at infinity, since the same results were found in the compact analysis.

Finally, we observe that in both non-compact analyses considered here, there are more equilibrium points found than in the corresponding compact analysis. In the non-compact analysis of [24] for example, five individual equilibrium points and a line of points were found for the expanding LRS Bianchi I vacuum models, while in the corresponding compact analysis [25] only one equilibrium point and a line of equilibrium points were found in the expanding subset. Similarly, for the flat Friedmann model, five pairs of (expanding and collapsing) equilibrium points were found in [27], while only three pairs of points were found in the compact analysis [25]. The detailed comparison in sections 3 and 4 of this paper shows that there are two main reasons for the additional equilibrium points in the non-compact analyses [24, 27]. The first is the duplication of equilibrium points at infinity: we find that in the non-compact analysis [27] two copies of the same point in the finite analysis (here points 5, 6) are created, and while static points exist only for special parameter values in [25], they have
analouges at infinity in [24] for all values of $n$ (see tables 1, 2 and 3). Secondly, some points which are classified as equilibrium points in the non-compact analysis are not equilibrium points in the compact analysis. In the examples studied here, this applies to points $D_{\infty}$ in [24] and 5, 6 in [27], which correspond to $\mathcal{M}$ and $\tilde{\mathcal{N}}$ respectively.

It is worth noting that the analysis considered here is also applicable to more general theories of gravity such as $f(R) = R + \alpha R^n$. In principle, compact variables can be constructed for such theories, but at the expense of a large number of sectors to be analysed (as can be seen from equation (18) in [31]).

In conclusion, we have shown that it is advantageous to compactify the state space whenever possible. The use of appropriately constructed compact variables allows for a clear and complete analysis including static, bouncing and recollapsing solutions and avoids the complications caused by equilibrium points at infinity.

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