ON RELATIVE RATIONAL CHAIN CONNECTEDNESS OF
THREEFOLDS WITH ANTI-BIG CANONICAL DIVISORS IN
POSITIVE CHARACTERISTICS

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Abstract. In this paper we prove two results about the rational chain connectedness
for klt threefolds with anti-big canonical divisors in the relative setting.

1. INTRODUCTION

It is widely recognized that the geometry of a higher-dimensional variety is closely
related to the geometry of rational curves on it. A classical result in the early 90s by
Campana ([Cam92]) and Kollár-Miyaoka-Mori ([KMM92]) says that smooth Fano vari-
eties are rationally connected in characteristic zero and are rationally chain connected
in positive characteristics. This was later generalized by Zhang ([Zha06]) and Hacon-
McKernan ([HM07]) in characteristic zero. More recently using the minimal model pro-
gram by Hacon-Xu ([HX15]) and Birkar ([Bir13]), Gongyo-Li-Patakfalvi-Schwede-Tanaka-
Zong ([GLP+15]) proved that projective globally $F$-regular threefolds in characteristic
$\geq 11$ are rationally chain connected and this was later generalized to log Fano type three-
fold by Gongyo-Nakamura-Tanaka ([GNT15]).

The main result in [HM07] is as follows.

Theorem 1.1. [HM07, Theorem 1.2] Let $(X, \Delta)$ be a log pair, and let $f : X \to S$ be a
proper morphism such that $-K_X$ is relatively big and $-(K_X + \Delta)$ is relatively semiample.
Let $g : Y \to X$ be any birational morphism. Then the connected components of every
fiber of $f \circ g$ are rationally chain connected modulo the inverse image of the locus of log
canonical singularities of $(X, \Delta)$.

In this paper we prove a theorem similar to Theorem 1.1 for morphisms from a klt
threefold to a variety of dimension $\geq 1$. More precisely, we have

Theorem A (Theorem 3.1). Let $X$ be a normal $\mathbb{Q}$-factorial threefold over an algebraically
closed field $k$ of characteristic $\geq 7$ and $(X,D)$ a klt pair. Let $f : X \to Z$ be a proper
morphism such that $f_*\mathcal{O}_X = \mathcal{O}_Z$, $\dim(Z) = 1$ or $2$, $Z$ is klt, $-K_X$ is relatively big,
$-(K_X + D)$ is relatively semi-ample and $(X_z, D_z)$ is klt for general $z \in Z$. Let $g : Y \to X$ be any birational morphism. Then the connected components of every fiber of $f \circ g$ are rationally chain connected.

Motivated by Theorem A, we construct a global version of rational chain connectedness for threefolds.

**Theorem B** (Theorem 5.1). Let $X$ be a projective threefold over an algebraically closed field $k$ of characteristic $p > 0$, $f : X \to Y$ a projective surjective morphism from $X$ to a projective variety $Y$ such that $f_*\mathcal{O}_X = \mathcal{O}_Y$. Let $D$ be an effective $\mathbb{Q}$-divisor, and $X_\eta$ the geometric generic fiber of $f$. Assume that the following conditions hold.

1. $(X, D)$ is klt, $-K_X$ is big and $f$-ample, $K_X + D \sim_\mathbb{Q} 0$ and the general fibers of $f$ are smooth.
2. $p > \frac{2}{\delta}$, where $\delta$ is the minimum non-zero coefficient of $D$.
3. $D = E + f^*L$ where $E$ is an effective $\mathbb{Q}$-Cartier divisor such that $p \nmid \text{ind}(E)$, $(X_\eta, E|_{X_\eta})$ is globally $F$-split, and $L$ is a big $\mathbb{Q}$-divisor on $Y$.
4. $\dim(Y) = 1$ or $2$.

Then $X$ is rationally chain connected.

Here $\text{ind}(E)$ means the Cartier index of $E$.

The main ingredients of the proofs of Theorem A and Theorem B are the minimal model program constructed in [HX15], [Bir13] and [GLP+15]; some facts, especially Theorem 2.1, in [GLP+15]; some positivity results by Patakfalvi ([Pat14]) and Ejiri ([Eji15]); a canonical bundle formula constructed in Section 4 in the spirit of the paper [PS09] by Prokhorov and Shokurov. Note that the condition (3) in Theorem B is used in order to apply the result [Eji15, Theorem 1.1] of Ejiri to deduce that $-K_Y$ is big, and to apply Theorem 4.3 when $\dim Y = 2$. This creates enough rational curves on $Y$. Note that by [Eji15, Example 3.4], $(X_\eta, E|_{X_\eta})$ being globally $F$-split is equivalent to $S^0(X_\eta, E|_{X_\eta}, \mathcal{O}_{X_\eta}) = H^0(X_\eta, \mathcal{O}_{X_\eta})$.

Note that although the proof is independent, Theorem A can be implied by Theorem 4.1 of the paper [GNT15], which was put on arXiv before this paper. The proof of [GNT15, Theorem 4.1] relies on the minimal model program in dimension 3 in positive characteristic, which is only established in characteristic $\geq 7$ so far. On the other hand, Theorem B covers some cases in characteristic $< 7$. In particular it does not rely on the minimal model program and is not implied by [GNT15, Theorem 4.1].

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2. Preliminaries

We work over an algebraically closed field $k$ of characteristic $p > 0$.

2.1. Preliminaries on rational connected varieties and the minimal model program.

**Definition 2.1.** For a variety $X$ and a $\mathbb{Q}$-Weil divisor on $X$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $f : Y \to X$ be a log resolution of $(X, \Delta)$ and we write

$$K_Y = f^*(K_X + \Delta) + \sum a_i E_i$$

where $E_i$ is a prime divisor. We say that $(X, \Delta)$ is

- sub Kawamata log terminal (sub-klt for short) if $a_i > -1$ for any $i$.
- Kawamata log terminal (klt for short) if $a_i > -1$ for any $i$ and $\Delta \geq 0$.
- log canonical if $a_i \geq -1$ for any $i$ and $\Delta \geq 0$.

**Definition 2.2.** [Kol96, IV.3.2] Suppose that $X$ is a variety over $k$.

1. We say that $X$ is rationally chain connected (RCC) if there is a family of proper and connected algebraic curves $g : U \to Y$ whose geometric fibers have only rational components and there is a cycle morphism $u : U \to X$ such that $u(2) : \overline{U \times_k U} \to X \times_k X$ is dominant.
2. We say that $X$ is rationally connected (RC) if (1) holds and moreover the geometric fibers of $g$ in (1) are irreducible.

**Proposition 2.3.** Let $X$ be a klt $\mathbb{Q}$-factorial threefold over an algebraically closed field $k$ and $\text{char}(k) \geq 7$. Let $g : W \to X$ be a log resolution and assume that $K_W + E = g^*K_X + B$, where $E$ and $B$ are exceptional divisors and the coefficients in $E$ are all 1. Then relative minimal model for $(W, E)$ over $X$ exists. Denote this process by

$$W = W_0 \to W_1 \to \cdots \to W_N = W'.$$

Then we actually have $W' = X$. Moreover if we have a morphism $h : X \to Y$ such that every fiber of $h$ is RCC, then every fiber of $h \circ g$ is RCC.

**Proof.** The existence of this minimal model program is by [GLP+15, Theorem 3.2]. So we have a morphism $g' : W' \to X$ and we want to show that $g'$ is the identity. Denote the strict transform of $E$ by $E'$, then $K_{W'} + E' = g'^*K_X + B'$ for some exceptional $\mathbb{Q}$-divisor $B'$. By construction of the minimal model program we know that $g'^*K_X + B'$ is nef over $X$ which means that $B'$ is $g'$-nef and since $X$ is klt the support of $B'$ is the whole exceptional locus of $g'$. So we can get that $B' = 0$ by negativity lemma, and since $X$ is $\mathbb{Q}$-factorial we will get $W' = X$.

The proof of the last statement follows the proof of [GLP+15, Proposition 3.6]. Without loss of generality we can do a base change and assume that the base field $k$ is uncountable.
Define $F$ in the following way: if $f_i$ is a divisorial contraction, then let $E_0 = E$, $E_{i+1} = f_{i,*}E_i$ and $F$ an arbitrary component of $E_i$; if $f_i$ is a flip and $C$ is any flipping curve then let $F$ be a component of $E_i$ that contains $C$. Let $K_F + \Delta_F := \left( K_{W_i} + E_i - \frac{1}{n}(E_i - F) \right) |_{F}$ where $n \gg 0$. By assumption $K_{W_i} + E_i - \frac{1}{n}(E_i - F)$ is plt, then by adjunction $K_F + \Delta_F$ is klt, hence by [Tan14, Theorem 14.4] $F$ is $\mathbb{Q}$-factorial. We also know that $-(K_{W_i} + E_i)$ is $f_i$-ample by assumption, then $-(K_F + \Delta_F)$ is ample. Moreover by [Pro01, Corollary 2.2.8] the coefficients of $\Delta_F$ are in the standard set $\{1 - \frac{1}{n} | n \in \mathbb{N}\}$. Let $\tilde{F}$ be the normalization of $F$. Then by [HX15, Theorem 3.1] we know that $(\tilde{F}, \Delta_{\tilde{F}})$ is strongly $F$-regular and by [HX15, Theorem 4.1] $F$ is a normal surface.

Next we consider three cases.

Case 1: If $f_i$ is a divisorial contraction and the exceptional divisor is contracted to a point, then since $-(K_F + \Delta_F)$ is ample, by [Kaw94, Lemma 2.2] $F$ is a rational surface, in particular it is rationally connected.

Case 2: If $f_i$ is a divisorial contraction and the exceptional divisor is contracted to a curve, then let $p : F \to B$ be the Stein factorization of $f_i|_{F}$. By assumption $-(K_F + \Delta_F)$ is $f_i$-ample, so it is $p$-ample. Then for a general fiber $D$ of $p$ we have

$$(K_F + D) \cdot D = (K_F + \Delta_F + D - \Delta_F) \cdot D = (K_F + \Delta_F) \cdot D - \Delta_F \cdot D < 0.$$  

Here $D$ is reduced and irreducible by [Bäd01, Theorem 7.1]. Hence by [Tan14, Theorem 5.3] $D \cong \mathbb{P}^1$. Therefore every component of every fiber of $f_i$ is a rational curve.

Case 3: If $f_i$ is a flip, then let $C$ be an arbitrary flipping curve. By assumption we have $(K_F + \Delta_F) \cdot C < 0$, $C^2 < 0$ and $0 \leq \operatorname{coeff}_{C} \Delta F < 1$, so $(K_F + C) \cdot C < 0$. Again by [Tan14, Theorem 5.3] $C \cong \mathbb{P}^1$.

We denote a fiber of $h$ over $y \in Y$ by $F_{X,y}$. There is a morphism from $W_i$ to $Y$ for every $i$, and we denote the fiber of this morphism over $y$ as $F_{W_i,y}$. Then there is a rational map $F_{W_i,y} \dashrightarrow F_{W_{i+1},y}$. From the above Case 1-3 we see that compared to $F_{W_i,y}$, there are only rational curves or a rational surface generated in $F_{W_{i+1},y}$. So the RCC-ness of $F_{W_{i+1},y}$ implies the RCC-ness of $F_{W_i,y}$. By assumption $F_{X,y}$ is RCC, so $F_{W,y}$ is RCC.

**Proposition 2.4.** Let $X$ be a klt $\mathbb{Q}$-factorial threefold over an algebraically closed field $k$ and char($k$) $\geq 7$. Let $f : X \to Y$ be a morphism from $X$ to a normal surface $Y$. Suppose we run a $K_X$-minimal model program and it terminates at $g : X' \to Y$. If every fiber of $g$ is RCC then every fiber of $f$ is RCC.

**Proof.** This can be easily deduced from Proposition 2.3 by taking a common resolution of $X$ and $X'$. The proof of [GLP⁺15, Proposition 3.6] works as well.

2.2. Preliminaries on $F$-singularities. In this article, for a proper variety $X$, a $\mathbb{Q}$-divisor $\Delta$ and the line bundle $M$ we will use the concepts of strongly $F$-regular, the non $F$-pure ideal $\sigma(X, \Delta)$ and $S^0(X, \sigma(X, \Delta) \otimes M)$. The definitions of these can be found
in many papers related to $F$-singularities (e.g. [HX15]). For a pair $(X, \Delta)$ where $\Delta$ is a $\mathbb{Q}$-Cartier divisor we also follow the definition of globally $F$-split in [Eji15].

**Lemma 2.5.** Let $X$ be a surface, $D$ an effective $\mathbb{Q}$-divisor on $X$, $f : X \to C$ a morphism from $X$ to a smooth curve $C$, and $(X_c, D_c)$ is a strongly $F$-regular pair for general $c \in C$. Assume that $-K_X$ is big, $K_X + D \sim_{\mathbb{Q}} 0$, then $C \cong \mathbb{P}^1$.

**Proof.** By Kodaira’s Lemma we can write $D \sim_{\mathbb{Q}} \epsilon f^*H + E$ where $H$ is an ample $\mathbb{Q}$-divisor on $C$, $0 < \epsilon \in \mathbb{Q}$, $E$ is an effective $\mathbb{Q}$-divisor on $X$ and $(X_c, E_c)$ is also strongly $F$-regular for general $c \in C$ (since $X_c$ is a curve). Suppose that $C$ is not isomorphic to $\mathbb{P}^1$. We know that $K_{X/C} + E \sim_{\mathbb{Q}} f^*(-K_C - \epsilon H)$ is $f$-nef and $K_{X_c} + E_c$ is semi-ample for general $c \in C$, so by [Pat14, Theorem 3.16], $K_{X/C} + E = K_X - f^*K_C + E$ is nef. Since we have assumed that $g(C) > 0$ we have that $K_X + E$ is nef. However this is impossible since $K_X + E \sim_{\mathbb{Q}} -\epsilon f^*H$ where $H$ is ample and $\epsilon > 0$. \qed

2.3. **Weak positivity.** Let $Y$ be a non-singular projective variety, $\mathcal{F}$ a torsion-free coherent sheaf on $Y$. We take $i : \hat{Y} \to Y$ to be the biggest open subvariety such that $\mathcal{F}|_{\hat{Y}}$ is locally free. Let $\hat{S}^k(\mathcal{F}) := i_*S^k(i^*\mathcal{F})$.

**Definition 2.6.** [Vie83, Definition 1.2] We call $\mathcal{F}$ weakly positive, if there is an open subset $U \subseteq Y$ such that for each ample line bundle $\mathcal{H}$ on $Y$ and every positive number $\alpha$ there exists some positive number $\beta$ such that $\hat{S}^{\alpha \cdot \beta}(\mathcal{F}) \otimes \mathcal{H}^\beta$ is generated by global sections over $U$.

**Lemma 2.7.** Weakly positive line bundles are nef.

**Proof.** This easily follows from Definition 2.6. \qed

3. Relative rational chain connectedness

In this section we prove the following

**Theorem 3.1.** Let $X$ be a normal $\mathbb{Q}$-factorial threefold over an algebraically closed field $k$ of characteristic $\geq 7$ and $(X, D)$ a klt pair. Let $f : X \to Z$ be a proper morphism such that $f_*\mathcal{O}_X = \mathcal{O}_Z$, $\dim(Z) = 1$ or 2, $Z$ is klt, $-K_X$ is relatively big, $-(K_X + D)$ is relatively semi-ample and $(X_z, D_z)$ is klt for general $z \in Z$. Let $g : Y \to X$ be any birational morphism. Then the connected components of every fiber of $f \circ g$ are rationally chain connected.

**Proof.** First we observe that $(X_z, D_z)$ being klt implies that $X_z$ is normal (in particular reduced) and irreducible.

Next we prove that if every fiber of $f$ is RCC, then every fiber of $f \circ g$ is RCC. We take a log resolution of $Y$ and denote it by $p : Y' \to Y$ and let $q = g \circ p$. If we have $K_{Y'} = q^*K_X + \tilde{B}$ then $K_{Y'} - \tilde{B} = q^*K_X$ and the coefficients of $-\tilde{B}$ are $< 1$. Then we can
add another effective divisor to make all the coefficients 1, and we denote this divisor by \( \tilde{E} \). Now we run a relative \((K_{Y'} + \tilde{E})\)-MMP of \( Y' \) over \( X \). By Proposition 2.3 we see that if every fiber of \( f \) is RCC then every fiber of \( f \circ g \circ p \) is RCC, hence every fiber of \( f \circ g \) is RCC.

Therefore it suffices to show that every fiber of \( f \) is RCC. We consider the cases of \( \dim(Z) = 2 \) and \( \dim(Z) = 1 \) respectively.

**Case 1: \( \dim(Z) = 2 \).**

If \( \dim(Z) = 2 \) then a general fiber of \( f \) being normal and \(-K_X\) being relatively big implies that a general fiber of \( f \) is a smooth rational curve. Next we run a relative minimal model program over \( Z \) and denote this process as:

\[
X = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{N-1}} X_n = X'.
\]

Since \(-K_X\) is relatively big we end up with a Mori fiber space \( X' \xrightarrow{h} Z' \xrightarrow{p} Z \) where \( Z' \) is also a surface. Then the general fibers of \( h \) are rational curves. Moreover since \( p_*O_{Z'} = O_Z \) we know that \( p \) is birational.

Now we prove that \( h \) is equidimensional. Suppose that this is not the case, then there is a fiber \( \tilde{F} \) of \( h \) over a point \( \tilde{z} \in Z' \) which contains a 2-dimensional irreducible component. If \( \tilde{F} \) is reducible then let \( \tilde{F}_1 \subseteq \tilde{F}_2 \) such that \( \tilde{F}_1 \cdot \tilde{C}_2 > 0 \). On the other hand if we take a general point \( z' \in Z' \) then \( h^{-1}(z') \) is an irreducible curve and \( h^{-1}(z') \cdot \tilde{F}_2 = 0 \). This is a contradiction to the fact that \( \rho(X'/Z') = 1 \). If \( \tilde{F} \) is irreducible, by Bertini’s Theorem we have a very ample divisor \( H \subseteq X' \) such that \( H \cap \tilde{F} \) is an irreducible curve which we denote by \( \tilde{C} \). We do the Stein factorization of \( h|_H \) and denote the process as:

\[
H \xrightarrow{h_1} Z'' \xrightarrow{h_2} Z',
\]

then \( h_1 \) is birational and \( \tilde{C} \) is an exceptional curve of \( h_1 \). After possibly replacing \( Z'' \) by its normalization we can assume that \( Z'' \) is normal. Now \( \tilde{F} \cdot \tilde{C} \) is equal to \( \tilde{C}^2 \) viewed as the self-intersection of \( \tilde{C} \) in \( H \), so by the Negativity Lemma it is \( < 0 \). On the other hand we can still take a general point \( z' \in Z' \) as above such that \( h^{-1}(z') \cdot \tilde{F} = 0 \). This is also a contradiction to the fact that \( \rho(X'/Z') = 1 \).

Since \( h \) is equidimensional, by [Deb01, Lemma 3.7] the components of every fiber of \( h \) are rational curves. Then by Proposition 2.4 every fiber of \( f \) is RCC.

**Case 2: \( \dim(Z) = 1 \).**

Without loss of generality we can do a base change and assume that the base field \( k \) is uncountable. By passing to the normalization of \( Z \) we can assume that \( Z \) is smooth. Then since every closed point of \( Z \) is a Cartier divisor, every fiber of \( f \) is also Cartier, hence \( f \) is equidimensional.
We first show that the general fibers of $f$ are rationally chain connected. Let $F$ be a general fiber of $f$. Since we assume that $(F, D|_F)$ is klt, by adjunction we know that

$$K_X|_F \equiv_{	ext{num}} (K_X + F)|_F = K_F + \text{Diff}_F(0),$$

where $\text{Diff}_F(0) \geq 0$ (cf. [Kol92, Proposition-Definition 16.5]). So $-(K_F + \text{Diff}_F(0))$, hence $-K_F$, is big. Therefore $\kappa(F) = -\infty$ and $F$ is birationally ruled by classification of surfaces. To prove that the general fibers of $f$ are RCC it suffices to prove that $F$ is rational. By assumption $-(K_F + D|_F) = -(K_X + D)|_F$ is semiample, so there exists an effective $\mathbb{Q}$-divisor $H$ such that $H \sim_{\mathbb{Q}} -(K_F + D|_F)$ and $(F, D|_F + H)$ is klt. We define $\Delta := D|_F + H$. Let $\pi : F' \to F$ be a minimal resolution of $(F, \text{Diff}_F(0))$, then $F'$ maps to a ruled surface $F''$ over a smooth curve $B$ via a sequence of blow-downs and we denote the morphism by $\psi$. The situation is as follows.

$$\pi \quad \psi$$

Since we have that $(F, \Delta)$ is klt, by [KM98, Thmorem 4.7] $\pi$ and $\psi$ only contract $\mathbb{P}^1$s. So $F$ is RCC if and only if $F''$ is RCC. We define $\Delta''$ on $F''$ via the following

$$K_{F''} + \Delta'' = \psi_*\pi^*(K_F + \Delta).$$

Then $(F, \Delta)$ being klt implies that $(F'', \Delta'')$ is klt.

We denote a general fiber of $q$ by $R$. By construction $R \cong \mathbb{P}^1$, so we know that $(R, \Delta''|_R)$ is klt and hence strongly $F$-regular. Then by applying Lemma 2.5 on $F''$ we know that $B = \mathbb{P}^1$. So $F$ is rational. Therefore we have proven that the general fibers of $f$ are RCC.

Since we have assumed that the base field $k$ is uncountable, by [Kol96, Ch. IV Corollary 3.5.2] we know that every fiber of $f$ is RCC.

4. A Canonical Bundle Formula for Threefolds in Positive Characteristics

In this section following the idea of the proof of [PS09] we construct a canonical bundle formula in characteristic $p$ for a morphism from a threefold to a surface, whose general fibers are $\mathbb{P}^1$. There are similar constructions in [CTX13, 6.7] and [DH15, Theorem 4.8].

Let $\overline{M}_{0,n}$ be the moduli space of $n$-pointed stable curves of genus 0, $f_{0,n} : \mathcal{U}_{0,n} \to \overline{M}_{0,n}$ the universal family, and $\mathcal{P}_1, \mathcal{P}_2, ..., \mathcal{P}_n$ the sections of $f_{0,n}$ which correspond to the marked points. Let $d_j (j = 1, 2, ..., n)$ be the rational numbers such that $0 < d_j \leq 1$ for all $j$, $\sum_j d_j = 2$ and $D = \sum_j d_j \mathcal{P}_j$. 
Lemma 4.1. [DH15, Lemma 4.6][Kaw97, Theorem 2]

1. There exists a smooth projective variety $\mathcal{U}_{0,n}^*$, a $\mathbb{P}^1$-bundle $g_{0,n} : \mathcal{U}_{0,n}^* \to \overline{\mathcal{M}}_{0,n}$, and a sequence of blowups with smooth centers

\[ \overline{\mathcal{U}}_{0,n} = \mathcal{U}^{(1)} \xrightarrow{\sigma_2} \mathcal{U}^{(2)} \xrightarrow{\sigma_3} \ldots \xrightarrow{\sigma_{n-2}} \mathcal{U}^{(n-2)} = \mathcal{U}_{0,n}^* \]

2. Let $\sigma : \overline{\mathcal{U}}_{0,n} \to \mathcal{U}_{0,n}^*$ be the induced morphism, and $\mathcal{D}^* = \sigma_* \mathcal{D}$. Then $K_{\overline{\mathcal{U}}_{0,n}} + \mathcal{D} - \sigma^*(K_{\mathcal{U}_{0,n}^*} + \mathcal{D}^*)$ is effective.

3. There exists a semi-ample $\mathbb{Q}$-divisor $\mathcal{L}$ on $\overline{\mathcal{M}}_{0,n}$ such that

\[ K_{\mathcal{U}_{0,n}^*} + \mathcal{D}^* \sim_{\mathbb{Q}} g_{0,n}^*(K_{\overline{\mathcal{M}}_{0,n}} + \mathcal{L}). \]

Definition 4.2. Let $f : X \to Y$ be a surjective proper morphism between two normal varieties and $K_X + D \sim_{\mathbb{Q}} f^* L$, where $D$ is a boundary divisor on $X$ and $L$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $Y$. Let $(X, D)$ be log canonical near the generic fiber of $f$, i.e., $(f^{-1}U, D|_{f^{-1}U})$ is log canonical for some Zariski dense open subset $U \subseteq Y$. We define

\[ D_{\text{div}} := \sum (1 - c_Q)Q, \]

where $Q \subset Z$ are prime Weil divisors on $Z$ and

\[ c_Q = \sup \{ c \in \mathbb{R} : (X, D + cf^* Q) \text{ is log canonical over the generic point } \eta_Q \text{ of } Q \}. \]

Next we define

\[ D_{\text{mod}} := L - K_Y - D_{\text{div}}, \]

so $K_X + D = f^*(K_Y + D_{\text{div}} + D_{\text{mod}})$.

Theorem 4.3. Let $f : X \to Y$ be a proper surjective morphism, where $X$ is a normal threefold and $Y$ is a normal surface over an algebraically closed field $k$ of characteristic $p > 0$. Assume that $Q = \sum_i Q_i$ is a divisor on $Y$ such that $f$ is smooth over $(Y - \text{Supp}(Q))$ with fibers isomorphic to $\mathbb{P}^1$. Let $D = \sum_i d_i D_i$ be a $\mathbb{Q}$-divisor on $X$ where $d_i = 0$ is allowed, which satisfies the following conditions:

1. $(X, D \geq 0)$ is klt on a general fiber of $f$.
2. Suppose $D = D^h + D^v$ where $D^h$ is the horizontal part and $D^v$ is the vertical part of $D$. Then $p = \text{char}(k) > \frac{2}{5}$, where $\delta$ is the minimum non-zero coefficient of $D^h$.
3. $K_X + D \sim_{\mathbb{Q}} f^*(K_Y + M)$ for some $\mathbb{Q}$-Cartier divisor $M$ on $Y$.

Then we have that $D_{\text{mod}}$ is $\mathbb{Q}$-linearly equivalent to an effective $\mathbb{Q}$-divisor. Here $D_{\text{mod}}$ is defined as in Definition 4.2. Moreover if $(X, D)$ is klt then there exists an effective $\mathbb{Q}$-divisor $\overline{\mathcal{D}}_{\text{mod}}$ on $Y$ such that $\overline{\mathcal{D}}_{\text{mod}} \sim_{\mathbb{Q}} D_{\text{mod}}$ and $(Y, D_{\text{div}} + \overline{\mathcal{D}}_{\text{mod}})$ is klt.

Proof. First we reduce the problem to the case where all components of $D^h$ are sections. Let $D_{i_0}$ be a horizontal component of $D$ and $D_{i_0} = D_{i_0}^h \to Y$ be the Stein factorization of $f|_{D_{i_0}}$. Let $Y' \to D_{i_0}^h$ be the normalization of $D_{i_0}^h$, then $Y' \to Y$ is a finite surjective
morphism of normal surfaces. Let $X'$ be the normalization of the component of $X \times_Y Y'$ dominating $Y$.

\[
\begin{array}{ccc}
X & \xrightarrow{\nu'} & X' \\
f & & f' \\
Y & \xleftarrow{\nu} & Y'
\end{array}
\]

Let $m = \deg(\mu : Y' \to Y)$ and $l$ be a general fiber of $f$. Then

\begin{equation}
(4.1) \quad m = D_i \cdot l \leq \frac{1}{d_i}(D \cdot l) = \frac{1}{d_i}(-K_X \cdot l) = \frac{2}{d_i} \leq \frac{2}{\delta} < \text{char}(k).
\end{equation}

Therefore $\nu$ is a separable and tamely ramified morphism.

Let $D'$ be the log pullback of $D$ under $\nu'$, i.e.

\[K_{X'} + D' = \nu'^*(K_X + D).\]

More precisely by [Kol92, 20.2] we have

\[D' = \sum_{i,j} d'_{ij}D'_{ij}, \quad \nu'(D'_{ij}) = D_i, \quad d'_{ij} = 1 - (1 - d_i)e_{ij},\]

where $e_{ij}$ is the ramification indices along $D'_{ij}$.

By construction $X$ dominates $Y$. Also, since $\nu$ is étale over a dense open subset of $Y$, say $\nu^{-1}U \to U$, and étale morphisms are stable under base change, $(f' \circ \nu)^{-1}U \to f^{-1}U$ is étale. Thus the ramification locus $\Lambda$ of $\nu'$ does not contain any horizontal divisor $f'$, i.e. $f'(\Lambda) \neq Y'$. Therefore $D'$ is a boundary near the generic fiber of $f'$, i.e. $D'^h$ is effective. We observe that the coefficients of $D'^h$ can be computed by intersecting with a general fiber of $f' : X' \to Y'$, hence they are equal to the coefficient of $D^h \subseteq X$. Thus the condition $p > \frac{2}{\delta}$ remains true for $D'$ on $X'$.

After finitely many such base changes we get a family $f'' : X'' \to Y''$, such that all of the horizontal components of $D''$ are rational sections of $f''$. Here $D''$ is the log pullback of $D$ via the induced finite morphism $\alpha : X'' \to X$, i.e. $K_{X''} + D'' = \alpha^*(K_X + D)$.

By construction of $\overline{\mathcal{M}}_{0,n}$ there is a generically finite rational map $Y'' \dashrightarrow \overline{\mathcal{M}}_{0,n}$. Let $\beta_0 : \tilde{Y} \to Y''$ be a morphism that resolves the indeterminacies of $Y'' \to \overline{\mathcal{M}}_{0,n}$ and $\tilde{X}$ the normalization of $X'' \times_{Y''} \tilde{Y}$. We have a morphism $\tilde{Y} \to \overline{\mathcal{M}}_{0,n}$ and let $\tilde{X} = \tilde{Y} \times_{\overline{\mathcal{M}}_{0,n}} \overline{U}_{0,n}$. Let $X^\sharp$ be a common resolution of $\tilde{X}$ and $\hat{X}$. We have the following diagram:
Let $D^\sharp$ and $\hat{D}$ be $\mathbb{Q}$-divisors on $X^\sharp$ and $\hat{X}$ respectively, defined by

$$K_{X^\sharp} + D^\sharp = \pi^*(K_X + D)$$

and

$$K_{\hat{X}} + \hat{D} = \mu_* (K_{X^\sharp} + D^\sharp).$$

We also define $D''_{\text{mod}}$ and $D''_{\text{div}}$ on $Y''$ for $(X'', D'')$ as in Definition 4.2, such that

$$K_{X''} + D'' = f''_* (K_{Y''} + D''_{\text{mod}} + D''_{\text{div}}),$$

and we define $\hat{D}_{\text{mod}}$ and $\hat{D}_{\text{div}}$ on $\hat{Y}$ in a similar way. Since $K_{X^\sharp} + D^\sharp$ is the pullback of some $\mathbb{Q}$-divisor from the base $\hat{Y}$ we get

$$K_{X^\sharp} + D^\sharp = \mu^*(K_{\hat{X}} + \hat{D}).$$

Since $D_{\text{div}}$ does not depend on the birational modification of the family (see [PS09, Remark 7.3]), we will define it with respect to $\hat{f} : \hat{X} \to \hat{Y}$.

Since $\hat{\phi}$ is generically finite and $D^*$ is horizontal it follows that $\hat{\phi}^* D^*$ is horizontal too. Since $\hat{D}^h$ is also horizontal one sees that

$$(4.2) \quad \hat{D}^h = \hat{\phi}^* D^*.$$  

From the construction of $\sigma : \Omega_{0,n} \to U_{0,n}^*$ we see that $(F, D^*|_F)$ is log canonical for any fiber $F$ of $g_{0,n} : U_{0,n}^* \to \Omega_{0,n}$. Since the fibers of $\hat{f} : \hat{X} \to \hat{Y}$ are isomorphic to the fiber of $g_{0,n}$, $(\hat{F}, \hat{D}^h|_{\hat{F}})$ is also log canonical, where $\hat{F}$ is any fiber of $\hat{f}$. Let $\hat{D}^u$ be a component of $\hat{D}^v$ and $\eta$ the generic point of $\hat{f}(\hat{D}^u)$. Then by inversion of adjunction we know that $(\hat{X}_\eta, (\hat{D}^v + \hat{D}^h)|_{\hat{X}_\eta})$ is log canonical. Since the fibers of $\hat{f}$ are reduced, the log canonical threshold of $(\hat{X}, \hat{D}; \hat{D}^u)$ over the generic point of $\hat{D}^u$ is $(1 - \text{coeff} \hat{D}^u)$. Hence we get

$$\hat{D}^v = \hat{f}^* \hat{D}_{\text{div}}. \quad \text{Note that the coefficients of } \hat{D}^v \text{ can be } > 1.$$

By definition of $\hat{D}_{\text{mod}}$ we have

$$(4.3) \quad K_{X} + \hat{D}^h \sim_{\mathbb{Q}} \hat{f}^*(K_{\hat{Y}} + \hat{D}_{\text{mod}}).$$

Then we have

$$(4.4) \quad K_{X} + \hat{D}^h - \hat{f}^*(K_{\hat{Y}} + \phi_0^* \mathcal{L}) = K_{X/\hat{Y}} + \hat{D}^h - \hat{\phi}^* K_{U_{0,n}^*/\Omega_{0,n}} - \hat{\phi}^* D^* \sim_{\mathbb{Q}} 0,$$
where the first equality follows from (4.3) and Lemma 4.1 (3), and the second relation from (4.2) and [Liu02, Chapter 6 Theorem 4.9 (b) and Example 3.18].

Since \( \hat{f} \) has connected fibers, by (4.3) and (4.4) and projection formula we get
\[
\hat{D}_{\text{mod}} \sim_{\mathbb{Q}} \phi_0^* \mathcal{L},
\]
i.e. \( \hat{D}_{\text{mod}} \) is semi-ample.

Now since \( \alpha_0 : Y'' \to Y \) is a composition of finite morphisms of degree strictly less than \( \text{char}(k) \) and \( \beta_0 \) is a birational morphism, by [Amb99, Theorem 3.2 and Example 3.1] we get
\[
K_{Y''} + D_{\text{div}}'' \sim_{\mathbb{Q}} \alpha_0^*(K_Y + D_{\text{div}})
\]
and
\[
K_Y + \hat{D}_{\text{div}} \sim_{\mathbb{Q}} \beta_0^*(K_{Y''} + D_{\text{div}}'').
\]
So \( \alpha_0^* D_{\text{mod}} \sim_{\mathbb{Q}} D_{\text{mod}}'' \), and \( \beta_0^* D_{\text{mod}}'' \sim_{\mathbb{Q}} \hat{D}_{\text{mod}} \). By the projection formula we have
\[
D_{\text{mod}}'' \sim_{\mathbb{Q}} \beta_0^* \hat{D}_{\text{mod}}.
\]

Then since \( \alpha_0 \) is finite,
\[
\psi_{0, *} \hat{D}_{\text{mod}} \sim_{\mathbb{Q}} \alpha_0^* \beta_0^* \hat{D}_{\text{mod}} \sim_{\mathbb{Q}} \alpha_0^* \alpha_0^* \hat{D}_{\text{mod}} \sim_{\mathbb{Q}} \alpha_0^* \alpha_0^* \alpha_0^* D_{\text{mod}} \sim_{\mathbb{Q}} D_{\text{mod}}.
\]

Here we view the push-forward through \( \alpha_0 \) as push-forward of cycles. Therefore \( D_{\text{mod}} \) is \( \mathbb{Q} \)-linearly equivalent to an effective divisor.

Next we prove the second statement. Since \( \alpha \) is finite, by [Kol13, Corollary 2.42] we know that \((X'', D'')\) is klt, and as \( \beta, \lambda \) and \( \mu \) are birational we know that \((X, \hat{D})\) is sub-klt, in particular \( \hat{D}'' \) has coefficients \( < 1 \). Since \( \hat{f} \) is a \( \mathbb{P}1 \) fibration and \((\hat{Y}, \hat{D}_{\text{div}})\) is log smooth we have that \((\hat{Y}, \hat{D}_{\text{div}})\) is sub-klt. By construction \( \hat{D}_{\text{mod}} \) is semi-ample, so by [Tan15, Theorem 1] we know that \((\hat{Y}, \hat{D}_{\text{div}} + \hat{D}_{\text{mod}})\) is sub-klt up to \( \mathbb{Q} \)-linear equivalence. Then \( K_{Y''} + D_{\text{mod}}'' + D_{\text{div}}'' \sim_{\mathbb{Q}} \beta_0^*(K_{\hat{Y}} + \hat{D}_{\text{div}} + \hat{D}_{\text{mod}}) \) is also sub-klt. Finally using [Kol13, Corollary 2.42] again and the fact that \( D_{\text{mod}} + D_{\text{div}} \geq 0 \) we get that \((Y, D_{\text{mod}} + D_{\text{div}})\) is klt.

5. Global rational chain connectedness

In this section we prove the following theorem.

**Theorem 5.1.** Let \( X \) be a projective threefold over an algebraically closed field \( k \) of characteristic \( p > 0 \), \( f : X \to Y \) a projective surjective morphism from \( X \) to a projective variety \( Y \) such that \( f_* \mathcal{O}_X = \mathcal{O}_Y \). Let \( D \) be an effective \( \mathbb{Q} \)-divisor, and \( X_\eta \) the geometric generic fiber of \( f \). Assume that the following conditions hold.

1. \((X, D)\) is klt, \(-K_X\) is big and \( f \)-ample, \( K_X + D \sim_{\mathbb{Q}} 0 \) and the general fibers of \( f \) are smooth.
2. \( p > \frac{2}{\delta} \), where \( \delta \) is the minimum non-zero coefficient of \( D \).
(3) \(D = E + f^*L\) where \(E\) is an effective \(\mathbb{Q}\)-Cartier divisor such that \(p \nmid \text{ind}(E)\), 
\((X_\eta,E|_{X_\eta})\) is globally \(F\)-split, and \(L\) is a big \(\mathbb{Q}\)-divisor on \(Y\).

(4) \(\dim(Y) = 1\) or \(2\).

Then \(X\) is rationally chain connected.

Remark 5.2. The smoothness of the general fibers of \(f\) holds in characteristic \(p \geq 11\) when \(\dim Y = 1\) by \([\text{Hir04, Theorem 5.1 (2)}]\), and in characteristic \(p \geq 5\) when \(\dim Y = 2\) by adjunction and a theorem of Tate (cf. \([\text{Lie13, Theorem 5.1}]\)).

Proposition 5.3. Let \(f : X \to Y\) be a projective surjective morphism between normal varieties with \(f_*\mathcal{O}_X = \mathcal{O}_Y\). Assume that the following conditions hold.

1. The general fibers of \(f\) are isomorphic to \(\mathbb{P}^1\).
2. \(Y\) is rationally chain connected.

Then \(X\) is rationally chain connected.

Proof. The proof is essentially the same as \([\text{GLP}^+15, \text{Lemma 3.12 and Proposition 3.13}]\). We take two general points \(x_1, x_2 \in X\) and let \(y_1 = f(x_1), y_2 = f(x_2)\), by construction \(f^{-1}(y_1) \cong f^{-1}(y_2) \cong \mathbb{P}^1\). By assumption \(y_1\) and \(y_2\) can be connected by a chain of rational curves, say \(C_1, C_2, ..., C_n\). Let \(\overline{C_i} \to C_i\) be the normalization for each \(C_i\), \(S_i := f^{-1}(C_i)\), \(\overline{S_i} := S_i \times_{\overline{C_i}} C_i\) and \(g_i : \overline{S_i} \to S_i\) the induced morphisms. Now the morphism \(\overline{S_i} \to \overline{C_i}\) is a flat projective morphism whose general fibers are \(\mathbb{P}^1\), by \([\text{dJS03, Theorem}]\) it has a section which we denote by \(\tilde{C_i}\). Then \(x_1\) and \(x_2\) is connected by \(f^{-1}(y_1), f^{-1}(y_2), g_i(\tilde{C_i})\) and the fibers of \(f\) over the intersection points of \(\{C_i\}\), which is a union of rational curves by \([\text{Deb01, Lemma 3.7}]\). \(\square\)

Proof of Theorem 5.1. We first prove the following lemma.

Lemma 5.4. Under the condition of Theorem 5.1, \(-K_Y\) is big.

Proof. By assumption \(m(K_{X_\eta} + E|_{X_\eta}) \sim_{\mathbb{Q}} 0\) for sufficiently large and divisible \(m\), in particular the \(k(\eta)\)-algebra \(\bigoplus_{m \geq 0} H^0(am(K_{X_\eta} + E|_{X_\eta}))\) is finitely generated. On the other hand since \((X_\eta,E|_{X_\eta})\) is globally \(F\)-split we have that

\[S^0(X_\eta,\sigma(X_\eta,E|_{X_\eta}) \otimes \mathcal{O}_{X_\eta}(m(K_{X_\eta} + E|_{X_\eta}))) = H^0(X_\eta,\mathcal{O}_{X_\eta}(m(K_{X_\eta} + E|_{X_\eta}))).\]

Here we would like to mention that for a line bundle \(M\) and a \(\mathbb{Q}\)-Cartier divisor \(\Delta\), the notation \(S^0(X,\Delta, M)\) is the same as the standard notation \(S^0(X,\sigma(X,\Delta) \otimes M)\) (cf. \([\text{HX15, between Lemma 2.2 and Proposition 2.3}]\)). Therefore by \([\text{Eji15, Theorem 1.1}]\) we know that

\[f_*\mathcal{O}_X(am(K_{X/Y} + E)) \cong f_*\mathcal{O}_X(f^*(-am(K_Y + L))) = \mathcal{O}_Y(-am(K_Y - L))\]

is weakly positive for \(m \gg 0\). By Lemma 2.7, \(-K_Y - L\) is nef, so \(-K_Y\) is big. \(\square\)
Next we consider the following two cases.

**Case 1: Y is 1-dimensional.**

After possibly taking the normalization of $Y$ we can assume that $Y$ is smooth. Then Lemma 5.4 implies that $g(Y) = 0$, i.e. $Y \cong \mathbb{P}^1$. Let $F$ be a general fiber of $f$. By assumption $F$ is smooth and $K_F$ is anti-ample, hence $F$ is separably rationally connected. By [dJS03, Theorem] we know that $f$ has a section which we denote by $s$. Then $s(Y)$ is a rational curve in $X$ which dominates $Y$. Therefore we get that $X$ is rationally chain connected.

**Case 2: Y is 2-dimensional.**

By assumption, a general fiber of $f$ is isomorphic to $\mathbb{P}^1$. Now by Lemma 5.4 we know that $-K_Y$ is big. On the other hand since $(X, D)$ is klt, by Theorem 4.3 there is a nonzero effective $\mathbb{Q}$-Cartier divisor $M$ on $Y$ such that $K_Y + M \sim_{\mathbb{Q}} 0$ and $(Y, M)$ is klt. Then by the proof of Case 2 of Theorem 3.1 we know that $Y$ is rational. Finally by Proposition 5.3 we get that $X$ is rationally chain connected. 

□

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