A General Theory of Phase-Space Quasiprobability Distributions

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We present a general theory of quasiprobability distributions on phase spaces of quantum systems whose dynamical symmetry groups are (finite-dimensional) Lie groups. The family of distributions on a phase space is postulated to satisfy the Stratonovich-Weyl correspondence with a generalized traciality condition. The corresponding family of the Stratonovich-Weyl kernels is constructed explicitly. In the presented theory we use the concept of the generalized coherent states, that brings physical insight into the mathematical formalism.

Since the introduction of the Wigner function in 1932 \cite{1}, it has found numerous physical applications. Perhaps the most important is the phase-space formulation of quantum mechanics, that has its origins in the early work of Moyal \cite{2}. In this formulation, a function on the phase space is associated with an operator on the Hilbert space, opening the way to formally representing quantum mechanics as a statistical theory on classical phase space. Various aspects of the phase-space formalism for basic quantum systems have been developed by a number of authors (e.g., Refs. \cite{3–11}). More extensive lists of the literature on the subject can be found in review papers \cite{12–14}.

Besides the Wigner function \(W\), other phase-space functions have been considered in the literature. In particular, the Husimi \(Q\) function and the Glauber-Sudarshan \(P\) function have found extensive applications in quantum optics. Cahill and Glauber \cite{4} have shown that there exists a whole family of phase-space functions parametrized by a number \(s\); the values \(+1\), \(0\), and \(-1\) of \(s\) correspond to the \(Q\), \(W\), and \(P\) functions, respectively. These phase-space functions are known as quasiprobability distributions (QPDs), as they play in quantum mechanics a role similar to that of genuine probability distributions in classical statistical mechanics.

The phase-space formalism has been applied successfully to the description of a spinless quantum particle and a mode of the quantized radiation field (modelled by a quantum harmonic oscillator). The corresponding phase space is \(\mathbb{R}^2\) (or, equivalently, the complex plane). A generalization of this description to a set of \(N\) independent particles or harmonic oscillators in a \(p\)-dimensional world is straightforward \cite{18}. A more complicated problem is the phase-space description of spin. A number of authors have used different approaches to the construction of the Wigner function for spin \cite{19,20,21,22}. The explicit expressions for the \(Q\), \(W\), and \(P\) functions for arbitrary spin were first obtained by Agarwal \cite{7}, who used the spin coherent-state representation \cite{21,22} and the Fano multipole operators \cite{23}. Várilly and Gracia-Bondía \cite{10} have shown that the spin coherent-state approach is equivalent to a general mathematical formalism based on the Stratonovich-Weyl (SW) correspondence \cite{8} and on the concept of twisted product \cite{9}.

In the present Letter we develop a general theory of QPDs on phase spaces of quantum systems whose dynamical symmetry groups are (finite-dimensional) Lie groups. This theory can be viewed as a generalization of the Cahill-Glauber QPDs (related to the Heisenberg-Weyl group) to arbitrary Lie groups. We make clear that the structure of the family of the phase-space QPDs for a Lie group is determined by the group covariance and the traciality conditions.

Let \(G\) be a Lie group (connected and simply connected, with finite dimension \(n\)), that is the dynamical symmetry group of a given quantum system. Let \(T\) be a unitary irreducible representation of \(G\) acting on the Hilbert space \(\mathcal{H}\). By choosing a fixed normalized reference state \(|\psi_0\rangle\in\mathcal{H}\), one can define the system of coherent states \(|\psi_g\rangle\):

\[
|\psi_g\rangle = T(g)|\psi_0\rangle, \quad g \in G.
\]

(1)

The isotropy subgroup \(H\subset G\) consists of all the group elements \(h\) that leave the reference state invariant up to a phase factor,

\[
T(h)|\psi_0\rangle = e^{i\phi(h)}|\psi_0\rangle, \quad |e^{i\phi(h)}| = 1, \quad h \in H.
\]

(2)

For every element \(g\in G\), there is a unique decomposition of \(g\) into a product of two group elements, one in \(H\) and the other in the coset space \(X=G/H\),

\[
g = \Omega h, \quad g \in G, \quad h \in H, \quad \Omega \in X.
\]

(3)

It is clear that group elements \(g\) and \(g'\) with different \(h\) and \(h'\) but with the same \(\Omega\) produce coherent states which differ only by a phase factor: \(|\psi_g\rangle = e^{i\phi} |\psi_{g'}\rangle\), where \(\phi = \phi(h) - \phi(h')\). Therefore a coherent state \(|\Omega\rangle = |\psi_{\Omega}\rangle\) is determined by a point \(\Omega = \Omega(g)\) in the coset space \(G/H\). As a direct consequence of Schur’s lemma, one obtains the identity resolution in terms of the coherent states:

\[
\int_X d\mu(\Omega)\langle\Omega|\Omega\rangle = I,
\]

(4)

where \(d\mu(\Omega)\) is the invariant integration measure on \(X = G/H\), the integration is over the whole manifold \(X\), and \(I\) is the identity operator on \(\mathcal{H}\).

An important class of coherent-state systems corresponds to the coset spaces \(X = G/H\) which are homogeneous Kählerian manifolds. Then \(X\) can be considered as
the phase space of a classical dynamical system, and the mapping \( \Omega \to |\Omega\rangle \langle \Omega | \) is the quantization for this system \( \mathbb{H} \). The standard (or maximal-symmetry) systems of the coherent states correspond to the cases when an extreme state of the representation Hilbert space (e.g., the vacuum state of an oscillator or the lowest spin state) is chosen as the reference state. In general, this choice of the reference state leads to systems consisting of states with properties “closest to those of classical states” \( [23] \).

In what follows we will consider the coherent states of maximal symmetry and assume that the phase space of the quantum system is a homogeneous \( \mathbb{C} \) \( \times \) \( \mathbb{H} \) manifold \( X = G/H \), each point of which corresponds to a coherent state \( |\Omega\rangle \). In particular, the Glauber coherent states of the Heisenberg-Weyl group \( H_3 \) are defined on the complex plane \( \mathbb{C}^2 = H_3/U(1) \), and the spin coherent states are defined on the unit sphere \( S^2 = SU(2)/U(1) \).

The idea of the phase-space formalism is as follows. Let \( A \) be an operator on the Hilbert space \( \mathcal{H} \) and \( \Omega \) be the reference state. In general, this choice of the reference state leads to systems consisting of states with properties “closest to those of classical states” \( [23] \).

The linearity is taken into account, if we implement the map \( A \to F_A^{(s)}(\Omega) \) by the generalized Weyl rule

\[
F_A^{(s)}(\Omega) = \text{Tr} [A \Delta^{(s)}(\Omega)],
\]

where \{\( \Delta^{(s)}(\Omega) \)\} is a family (labelled by \( s \)) of operator-valued functions on the phase space \( X \). These operators are referred to as the SW kernels. The generalized traciality condition \( [23] \) is taken into account, if we define the inverse of the generalized Weyl rule \( \Omega \) as

\[
A = \int_X d\mu(\Omega) F_A^{(s)}(\Omega) \Delta^{(-s)}(\Omega).
\]

Now, the conditions \( [23], [24] \) of the SW correspondence for \( F_A^{(s)}(\Omega) \) can be translated into the following conditions on the SW kernel \( \Delta^{(s)}(\Omega) \):

\[
\begin{align*}
(\text{i}) & \quad \Delta^{(s)}(\Omega) = |\Delta^{(s)}(\Omega)|^2 \quad \forall \Omega \in X. \\
(\text{ii}) & \quad \int_X d\mu(\Omega) \Delta^{(s)}(\Omega) = I. \\
(\text{iii}) & \quad \Delta^{(s)}(g\Omega) = T(g) \Delta^{(s)}(\Omega) T(g)^{-1}.
\end{align*}
\]

Substituting the inverted maps \( \Omega \) for \( A \) and \( B \) into the generalized traciality condition \( [24] \), we obtain the relation between the QPDs with different values of the index \( s \):

\[
F_A^{(s)}(\Omega) = \int_X d\mu(\Omega') K_{s,s'}(\Omega,\Omega') F_A^{(s')}(\Omega'),
\]

where

\[
K_{s,s'}(\Omega,\Omega') = \text{Tr} [\Delta^{(s)}(\Omega) \Delta^{(-s')}(\Omega')].
\]

If we take in equation \( [24] \) \( s = s' \) and take into account the arbitrariness of \( A \), we obtain the following relation

\[
\Delta^{(s)}(\Omega) = \int_X d\mu(\Omega') K(\Omega,\Omega') \Delta^{(s)}(\Omega'),
\]

where the function

\[
K(\Omega,\Omega') = \text{Tr} [\Delta^{(s)}(\Omega) \Delta^{(-s)}(\Omega')]
\]

behaves like the delta function on the manifold \( X \).

Now, our problem is to find the explicit form of the SW kernel \( \Delta^{(s)}(\Omega) \) that satisfies the conditions \( [23], [24] \) and \( [24] \). We start by considering the Hilbert space \( L^2(X,\mu) \) of square-integrable functions \( u(\Omega) \) on \( X \) with the invariant measure \( d\mu \). The representation \( T \) of the Lie group \( G \) on \( L^2(X,\mu) \) is defined as
The eigenfunctions $Y_\nu(\Omega)$ of the Laplace-Beltrami operator \[26\] form a complete orthonormal basis in $L^2(X, \mu)$:

$$\sum_\nu Y_\nu^*(\Omega)Y_\nu(\Omega') = \delta(\Omega - \Omega'), \quad (14a)$$

$$\int_X d\mu(\Omega)Y_\nu^*(\Omega)Y_\nu(\Omega) = \delta_{\nu\nu'} . \quad (14b)$$

The functions $Y_\nu(\Omega)$ are called the harmonic functions [27], and $\delta(\Omega - \Omega')$ is the delta function in $X$ with respect to the measure $d\mu$. The eigenfunctions $Y_\nu(\Omega)$ are linear combinations of matrix elements $T_{\nu'\nu}(g)$. Therefore, the transformation rule for the harmonic functions is [26]

$$T(g)Y_\nu(\Omega) = Y_\nu(g^{-1}\Omega) = \sum_{\nu'} T_{\nu'\nu}(g)Y_{\nu'}(\Omega). \quad (15)$$

The function $|\langle \Omega | \Omega' \rangle|^2$ is symmetric in $\Omega$ and $\Omega'$. Therefore, its expansion in the orthonormal basis must be of the form

$$|\langle \Omega | \Omega' \rangle|^2 = \sum_\nu \tau_\nu Y_\nu^*(\Omega)Y_\nu(\Omega') = \sum_\nu \tau_\nu Y_\nu^*(\Omega')Y_\nu(\Omega), \quad (16)$$

where $\tau_\nu$ are real positive coefficients. Since $|\langle \Omega | \Omega' \rangle|^2$ is real and $Y_\nu^*(\Omega) = e^{i\omega(\nu)}Y_\nu(\Omega)$, the coefficients $\tau_\nu$ must be invariant under this index transformation: $\tau_\nu = \tau_\nu'$. Since $\langle \Omega | \Omega' \rangle = \langle g|g\Omega' \rangle$, the coefficients $\tau_\nu$ must be invariant under the index transformation of Eq. \[15\]: $\tau_\nu = \tau_\nu'$.

Let us now define the set of operators \{\$D_\nu\$\} on $\mathcal{H}$:

$$D_\nu \equiv \omega_\nu \int_X d\mu(\Omega)Y_\nu(\Omega)|\Omega\rangle\langle \Omega| , \quad (17)$$

where $\omega_\nu$ are real coefficients to be determined from the normalization condition. Using the expression \[14\], we obtain the orthogonality condition

$$\text{Tr}(D_\nu D_{\nu'}^\dagger) = \langle \tau_\nu \omega_\nu^2 \rangle \delta_{\nu\nu'} . \quad (18)$$

The proper normalization is then obtained by taking

$$\omega_\nu^2 = 1/\tau_\nu . \quad (19)$$

Note that $\omega_\nu$ is defined only up to a sign, $\omega_\nu = \pm \tau_\nu^{-1/2}$. Using \[14\], we also obtain the relation

$$\omega_\nu \langle \Omega | D_\nu | \Omega \rangle = Y_\nu(\Omega). \quad (20)$$

The coefficients $\omega_\nu$ satisfy the same invariance conditions as $\tau_\nu$ (up to a choice of the sign). Therefore, $D_\nu$ are the tensor operators whose transformation rule is the same as for the harmonic functions $Y_\nu(\Omega)$:

$$T(g)D_\nu T(g)^{-1} = \sum_{\nu'} T_{\nu'\nu}(g)D_{\nu'} . \quad (21)$$

Now we are able to find the SW kernel $\Delta^{(s)}(\Omega)$ with all the desired properties. Specifically, let us define \[28\]

$$\Delta^{(s)}(\Omega) \equiv \sum_\nu f(s; \tau_\nu)Y_\nu^*(\Omega)D_{\nu}. \quad (22)$$

Here $f(s; \tau_\nu)$ is a function of $\tau_\nu$ and of the index $s$. We assume that $f$ possesses the invariance properties of $\tau_\nu$. The reality condition \[29\] is then satisfied if $f(s; \tau_\nu)$ is a real-valued function. Therefore, we can consider only real values of the index $s$. Then it is sufficient to use the convention in which $s \in [-1, 1]$. Next we consider the standardization condition \[31\]. It can be verified that there exists some $\nu_0$ such that \[29\]

$$\int_X d\mu(\Omega)Y_{\nu_0}(\Omega) \propto \delta_{\nu_\nu_0} . \quad (23)$$

We also are able to show that $\tau_{\nu_0} = 1$. Then the standardization condition \[31\] is satisfied if

$$f(s; 1) = \omega_{\nu_0} = \pm 1, \quad \forall s. \quad (24)$$

The covariance condition \[32\] is guaranteed by virtue of the transformation rules \[13\] and \[21\] and by the invariance of $\tau_\nu$ under these index transformations.

In order to satisfy the relation \[11\], the function $K(\Omega, \Omega')$ of Eq. \[12\] must be the delta function in $X$ with respect to the measure $d\mu$,

$$K(\Omega, \Omega') = \sum_\nu Y_\nu^*(\Omega)Y_\nu(\Omega') = \delta(\Omega - \Omega'). \quad (25)$$

This result is valid if

$$f(s; \tau_\nu)f(-s; \tau_\nu) = 1. \quad (26)$$

This property is satisfied only by the exponential function, i.e.,

$$f(s; \tau_\nu) = \pm [f(\tau_\nu)]^s. \quad (27)$$

Note that the standardization condition \[24\] then reads $f(1) = 1$. The double-valuedness of type \[27\] was pointed out by Várilly and Gracia-Bondía [10] who considered the Wigner function for spin. The exact form of the function $f(\tau_\nu)$ can be determined if we define for $s = 1$

$$\Delta^{(1)}(\Omega) \equiv |\Omega\rangle\langle \Omega|. \quad (28)$$

Then we obtain $\pm f(\tau_\nu) = 1/\omega_\nu = \pm \tau_\nu^{1/2}$, i.e.,

$$f(\tau_\nu) = \sqrt{\tau_\nu}. \quad (29)$$

Obviously, the standardization condition $f(1) = 1$ is satisfied. This result concludes the construction of the generalized SW kernel. It is evident that the properties of the kernels are completely determined by the harmonic
functions on the corresponding manifold and by the coherent states that form this manifold. We also note that
the function $K_{s,s'}(\Omega, \Omega')$ of Eq. (10) is given by

$$K_{s,s'}(\Omega, \Omega') = \sum_\nu \tau_{\nu}^{(s-s')/2} Y_{\nu}^s(\Omega) Y_{\nu}^{s'}(\Omega'),$$  \hspace{1cm} (30)

and it clearly satisfies the condition (11).

In order to avoid the double-valuedness of type (27), we adopt the convention with sign $^+\nu$, i.e., $\omega_\nu = +\tau_{\nu}^{1/2}$. Then we can write the generalized QPDs on the phase space as

$$F_{A}^{(s)}(\Omega) = \sum_\nu \tau_{\nu}^{s/2} A_\nu Y_{\nu}(\Omega),$$  \hspace{1cm} (31)

$$A_\nu \equiv \text{Tr} (A D_{\nu}^s) = \tau_{\nu}^{-1/2} \int_X d\mu(\Omega) Y_{\nu}^s(\Omega) A(\Omega).$$  \hspace{1cm} (32)

In particular, for $s = 1$, we obtain the $Q$ function (Berezin’s covariant symbol [24]):

$$Q_A(\Omega) = F_{A}^{(1)}(\Omega) = \langle \Omega | A | \Omega \rangle.$$  \hspace{1cm} (33)

For $s = -1$, we obtain the $P$ function (Berezin’s contravariant symbol [24]):

$$P_A(\Omega) = F_{A}^{(-1)}(\Omega) = \sum_\nu \omega_\nu A_\nu Y_{\nu}(\Omega),$$  \hspace{1cm} (34)

$$A = \int_X d\mu(\Omega) P_A(\Omega) |\Omega\rangle \langle \Omega|.$$  \hspace{1cm} (35)

The functions $P$ and $Q$ are counterparts in the traciality condition (24). Perhaps the most important QPD corresponds to $s = 0$, because this function is “self-conjugate” in the sense that it is the counterpart of itself in the traciality condition (24). It is natural to call the QPD with $s = 0$ the generalized Wigner function:

$$W_A(\Omega) = F_{A}^{(0)}(\Omega) = \sum_\nu A_\nu Y_{\nu}(\Omega).$$  \hspace{1cm} (36)

In conclusion, we have developed the general group-theoretical formalism of the phase-space QPDs. More details and examples of the QPDs on phase spaces of physical systems will be presented elsewhere.

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