A NOTE ON TAME/COMPATIBLE ALMOST COMPLEX STRUCTURES ON FOUR-DIMENSIONAL LIE ALGEBRAS

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Abstract. Four-dimensional, oriented Lie algebras $\mathfrak{g}$ which satisfy the tame-compatible question of Donaldson for all almost complex structures $J$ on $\mathfrak{g}$ are completely described. As a consequence, examples are given of (non-unimodular) four-dimensional Lie algebras with almost complex structures which are tamed but not compatible with symplectic forms.

1. Introduction

Among other interesting problems on compact 4-manifolds raised in [4], Donaldson asked the following:

Question 1.1. If $J$ is an almost complex structure tamed by a symplectic form, is $J$ also compatible with a symplectic form?

Recall that an almost complex structure $J$ is said to be tamed by a symplectic form $\omega$ (and such an $\omega$ is called $J$-tamed), if $\omega$ is positive on $J$-planes, i.e.
\[ \omega(u, Ju) > 0, \quad \text{for all vectors } u \neq 0. \]

An almost complex structure $J$ is said to be compatible with a symplectic form $\omega$ (and such an $\omega$ is called compatible with $J$, or $J$-compatible), if $\omega$ is $J$-tamed and $J$-invariant, i.e.
\[ \omega(u, Ju) > 0 \quad \text{and} \quad \omega(Jv, Jw) = \omega(v, w), \quad \text{for all vectors } u \neq 0, v, w. \]

Question [2.1] is still open for compact 4-manifolds, although important progress has been made by Taubes [25] who answered the question affirmatively for generic almost complex structures on 4-manifolds with $b^+ = 1$. There are other significant positive partial results, e.g. see [18], [19], [11], as well as results on the symplectic Calabi-Yau problem [29] [27] [28], also proposed by Donaldson in [4] and known to imply an affirmative answer to Question [1.1] for compact 4-manifolds with $b^+ = 1$.

It is also worth noting that Donaldson’s question is true locally for all almost complex 4-manifolds, but this is no longer the case in higher dimensions, as for certain $J$’s the structure of their Nijenhuis tensors becomes a local obstruction to the existence of compatible symplectic forms (see e.g. [20], [26], [15]). There are no such obstructions for integrable almost complex structures and an important version of Donaldson question ([18], p.678, and [24], Question 1.7) is whether it holds for compact complex manifolds of arbitrary dimensions. This is known to be true for compact complex surfaces [18], for higher dimensions there are known only some partial results, e.g. [8], [9], [7], [10].

Date: March 16, 2015. This note is part of an undergraduate research project the first author conducted under the direction of the second author.
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Question \[1\] has an obvious Lie algebra version, which has been already considered. Indeed, on a Lie algebra \( g \) an almost complex structure is an endomorphism \( J : g \to g \) with \( J^2 = -1 \), and we talk about symplectic (or closed, or exact) forms on \( g \) with respect to the Chevalley-Eilenberg differential \( d \) induced by the Lie bracket. Let us denote by \( Z^2 \) the space of closed 2-forms and by \( B_2 \) the space of boundary 2-vectors on \( g \). The space \( B_2 \) can be defined as the annihilator of \( Z^2 \) with respect to the natural pairing between forms and vectors, that is \( u \in B_2 \) if and only if \( \alpha(u) = 0 \), for all \( \alpha \in Z^2 \). For convenience, we also introduce the following definition. We say that an oriented Lie algebra \( g \) satisfies the tame-compatible property if the answer to Question \[1\] is affirmative for all almost complex structures \( J \) on \( g \) inducing the given orientation.

For the 4-dimensional Lie algebra version of Question \[1\] one main result of Li and Tomassini in \[17\] is the following:

**Theorem:** \((17), \text{Theorem 0.2}\) On a four-dimensional Lie algebra \( g \), if the space of boundary 2-vectors \( B_2 \) is isotropic with respect to the wedge product, that is, if \( u \wedge u = 0 \), for all \( u \in B_2 \), then \( g \) satisfies the tame-compatible property.

As pointed out in \[17\], any four-dimensional unimodular Lie algebra satisfies the condition of the theorem, thus, satisfies the tame-compatible property. A consequence is that Question \[1\] has an affirmative answer for any left-invariant almost complex structure on a compact quotient of a 4-dimensional Lie group by a discrete subgroup (Theorem 4.3, \[17\]). It is well known that if a Lie group admits a co-compact discrete subgroup then its Lie algebra must be unimodular \[21\]. Note also that the assumption in the above result is independent of the choice of orientation on \( g \), hence the conclusion is valid for both orientations. Although it covers the important unimodular case, the above result of Li and Tomassini gives only a sufficient condition for a 4-dimensional Lie algebra to satisfy the tame-compatible property. Our first observation was that the proof of Li-Tomassini will go through under a slightly weaker condition which takes into account orientation. Then we showed that our weaker condition is also necessary for the tame-compatible property.

**Theorem 1.2.** Let \( g \) be an oriented symplectic four-dimensional Lie algebra with a volume form \( \mu \). Then \( g \) satisfies the tame-compatible property if and only if the space of boundary 2-vectors \( B_2 \) is negative semi-definite with respect to the bilinear form defined by the wedge product and the volume form, that is, if and only if \( \mu(u \wedge u) \leq 0 \), for all \( u \in B_2 \).

As already mentioned above, for the proof of one direction, we could have slightly refined the arguments of Li-Tomassini (with small adjustments, a version of the Theorem 2.5 \[17\] still holds). However, partly to make our note self-contained and partly to present a slightly different proof, in section 3 we prefer to cast the 4-dimensional tame-compatible problem in an abstract linear algebra setting. We prove two linear algebra results (Propositions 3.2, 3.3) which might have some independent interest. Theorem 1.2 follows directly from Proposition 3.2, as shown in section 4.

Using the classification of four-dimensional symplectic Lie algebras obtained by Ovando \[22\], and her notations, there are two examples (or, rather, one and one-half!) for which \( B_2 \) is not negative semi-definite.
Corollary 1.3. On the Lie algebra \( \mathfrak{r}_2 \mathfrak{r}_2 \) endowed with either orientation, or on the Lie algebra \( \mathfrak{d}_{4,2} \) endowed with the non-complex orientation there exist almost complex structures which are tamed by symplectic forms but which are not compatible with any symplectic forms. These are the only 4-dimensional Lie algebras carrying such almost complex structures exist.

In Section 4 we give the bracket descriptions of the two Lie algebras mentioned above. Although Corollary 1.3 follows directly from Theorem 1.2, we provide in each case explicit examples of almost complex structures which are tamed but not compatible. Let us just mention here that \( \mathfrak{d}_{4,2} \) is the Lie algebra underlying the unique proper 4-dimensional example of 3-symmetric space discovered by Kowalski [14]. With one orientation this Lie algebra admits a complex (in fact, Kähler) structure, with the other orientation it does not admit complex structures. This is the orientation which we call “non-complex”. Note also that \( \mathfrak{d}_{4,2} \) does admit symplectic structures for both orientations.

To end the introduction, let us note that it was well known that the Lie algebra version of Question 1.1 can have negative answer for almost complex structures on Lie algebras of dimension 6 or higher, even in the nilpotent case. The first such examples are due to Migliorini and Tomassini [20] (see also [26], [2], [1]). It would be interesting to know if Theorem 1.2 extends in any way for Lie algebras of dimensions higher than 4. Certainly, on an abelian Lie algebra of arbitrary even-dimension all almost complex structures are tamed and compatible with symplectic forms. As a direct consequence of Lemma 3.2 in [8], we observe that this property is specific only to abelian Lie algebras (see Proposition 4.1). We leave open the question of the existence of non-abelian Lie algebras of dimension greater or equal to 6 which satisfy the tame-compatible property (for all almost complex structures) and an eventual classification of such examples.

Acknowledgments: The second author is grateful to Tian-Jun Li for encouragement to write this note and for useful comments on earlier versions. He also thanks Anna Fino for helpful observations.

2. Notations and Preliminaries

Given a Lie algebra \( \mathfrak{g} \) of dimension 4, we denote by \( \Lambda^k(\mathfrak{g}) \) and \( \Lambda^k(\mathfrak{g}^*) \), respectively, the spaces of (real) \( k \)-vectors and \( k \)-forms on \( \mathfrak{g} \). The Lie bracket induces the Chevalley-Eilenberg differential \( d \) on the spaces of forms on \( \mathfrak{g} \). On \( \Lambda^1(\mathfrak{g}^*) \), \( d \) is defined by

\[
d\alpha(u,v) = -\alpha([u,v]), \quad \alpha \in \Lambda^1(\mathfrak{g}^*), \quad u,v \in \mathfrak{g},
\]

and then is extended to \( d : \Lambda^k(\mathfrak{g}^*) \to \Lambda^{k+1}(\mathfrak{g}^*) \) by the Leibniz rule. Using the non-degenerate pairings

\[
\Psi : \Lambda^k(\mathfrak{g}) \times \Lambda^k(\mathfrak{g}^*) \to \mathbb{R}, \quad \Psi(u,\alpha) = \alpha(u),
\]

one defines the differential \( d \) for \( k \)-vectors as well, \( d : \Lambda^k(\mathfrak{g}) \to \Lambda^{k-1}(\mathfrak{g}) \). If \( u \) is a \( k \)-vector, \( du \) is defined (uniquely, since \( \Psi \) is non-degenerate) by

\[
\Psi(du,\alpha) := (-1)^{k-1}\Psi(u,\alpha), \quad \forall \alpha \in \Lambda^{k-1}(\mathfrak{g}^*).
\]

Thus one obtains the dual Chevalley-Eilenberg complexes \( (\Lambda^*(\mathfrak{g}^*), d) \), \( (\Lambda^*(\mathfrak{g}), d) \), yielding the real cohomology, respectively homology of the Lie algebra. In this note
we will only need the spaces of closed 2-forms, and boundary 2-vectors
\[ \mathcal{Z}^2 = \{ \alpha \in \Lambda^2(g^*) \mid d\alpha = 0 \}, \quad B_2 = \{ u \in \Lambda^2(g) \mid \exists w \in \Lambda^3(g), \ u = dw \}. \]
Equivalently, \( B_2 \) can be seen as the annihilator of \( \mathcal{Z}^2 \) with respect to the pairing \( \Psi \) above,
\[ B_2 = \{ u \in \Lambda^2(g) \mid \Psi(u, \alpha) = 0, \forall \alpha \in \mathcal{Z}^2 \}. \]

Suppose from now on that the four-dimensional Lie algebra \( g \) is oriented by a fixed volume form \( \mu \in \Lambda^4(g^*) \). An important feature of dimension 4 is that the wedge product and the fixed volume form induce non-degenerate inner products of signature \((3,3)\) on the 6-dimensional spaces of 2-vectors or 2-forms:

1. \( \Phi_\mu : \Lambda^2(g) \times \Lambda^2(g) \to \mathbb{R}, \quad \Phi_\mu(u, v) = \mu(u \wedge v), \ \forall \ u, v \in \Lambda^2(g). \)

2. \( \Phi^\mu : \Lambda^2(g^*) \times \Lambda^2(g^*) \to \mathbb{R}, \quad \alpha \wedge \beta = \Phi^\mu(\alpha, \beta)\mu, \ \forall \ \alpha, \beta \in \Lambda^2(g^*). \)

It can be easily checked that these are dual inner products, that is the Riesz maps \( \Phi_\mu \) and \( \Phi^\mu \) are dual to each other and the fixed volume form induces inner products of wedge product and the fixed volume form induce non-degenerate inner products of signature \((3,3)\) on the 6-dimensional spaces of 2-vectors or 2-forms:

- \( \Phi_\mu : \Lambda^2(g) \times \Lambda^2(g) \to \mathbb{R}, \quad \Phi_\mu(u, v) = \mu(u \wedge v), \ \forall \ u, v \in \Lambda^2(g). \)
- \( \Phi^\mu : \Lambda^2(g^*) \times \Lambda^2(g^*) \to \mathbb{R}, \quad \alpha \wedge \beta = \Phi^\mu(\alpha, \beta)\mu, \ \forall \ \alpha, \beta \in \Lambda^2(g^*). \)

Let now \( J \) be an almost complex structure on the Lie algebra \( g \), that is, an endomorphism \( J : g \to g \) with \( J^2 = -1 \). The induced action of \( J \) on the bundle of 2-forms, \( \alpha(\cdot, \cdot) \to \alpha(J\cdot, J\cdot) \), is an involution. This induces the decomposition of \( \Lambda^2(g^*) \) into \( J \)-invariant and \( J \)-anti-invariant forms, respectively, the \( \pm 1 \)-eigenspaces of the above action
\[ \Lambda^2(g^*) = \Lambda^+_J(g^*) \oplus \Lambda^-_J(g^*). \]

This decomposition is orthogonal with respect to the bilinear form \( \Phi^\mu \). The particularity of dimension 4 is that in this case \( \Lambda^-_J(g^*) \) is a two-dimensional plane positive-definite with respect to \( \Phi^\mu \). Certainly, \( \Lambda^-_J(g^*) \) with the restriction of \( \Phi^\mu \) becomes a Minkowski vector space of signature \((1,3)\) (one “+”, three “−”). Similarly, for 2-vectors, there is the decomposition
\[ \Lambda^2(g) = \Lambda^+_J(g) \oplus \Lambda^-_J(g), \]
and with the natural identifications through the Riesz maps above, we have
\[ (\Lambda^-_J(g^*))^\perp = \Lambda^+_J(g), \quad (\Lambda^+_J(g^*))^\perp = \Lambda^-_J(g). \]

As observed by Donaldson in the introduction of [4], in dimension four various geometric structures can be characterized in terms of subspace of the space of 2-forms (or 2-vectors) and their behavior with respect with the above bilinear forms. The following proposition gathers, in the Lie algebra context, some of the observations made in the introduction of [4].

**Proposition 2.1.** Let \( g \) be a four-dimensional Lie algebra, oriented by a fixed volume form \( \mu \in \Lambda^4(g^*). \)

- A Riemannian metric on \( g \) is given (up to rescaling) by a 3-dimensional subspace \( \Lambda^+ \subset \Lambda^2(g^*) \) on which \( \Phi_\mu \) is positive definite.
- A (positively oriented) symplectic form is defined by an element \( \omega \in \Lambda^2(g^*), \) with \( d\omega = 0 \) and \( \Phi_\mu(\omega, \omega) > 0. \)
- The map \( J \to \Lambda^-_J \) is a two-to-one correspondence between (positively oriented) almost complex structures \( J \) on \( g \) and 2-dimensional planes in \( \Lambda^2(g^*), \) positive definite with respect to \( \Phi^\mu. \)
Definition 3.1. (i) We say that $H$ is a tame by a symplectic form $\omega$ if $\Lambda^+_1 + \mathbb{R}\omega$ generates a 3-dimensional positive definite subspace of $\Lambda^2(g^*)$ on which $\Phi_\mu$ is positive definite.

(ii) An almost complex structure $J$ is compatible with the symplectic form $\omega$ if $J$ is tamed by $\omega$ and $\omega$ is orthogonal to $\Lambda^-_1$ with respect to $\Phi_\mu$.

For the third point, the correspondence is two-to-one, as $J$ and $-J$ induce the same plane of anti-invariant forms. Choosing a positive definite 2-plane $H \subset (\Lambda^2(g^*), \Phi_\mu)$, one can also determine the sign of the almost complex structure by additionally choosing one component of null cone in the Minkowski space $H^\perp$ (this amounts to choosing the simple positive $J$-vectors between $v \wedge Ju$ vs. $v \wedge (-J)u$). Certainly, the distinction $J$ vs. $-J$ is irrelevant with respect to the tame and compatible properties, as $J$ being tamed (compatible) is equivalent to $-J$ being tamed (compatible).

Because of the above proposition, it is natural to consider the extension of the Lie algebra 4-dimensional tame/compatible problem to an abstract linear algebra setting, as we do in the next section.

3. The Linear Algebra Extension

Let $(V, \langle \cdot, \cdot \rangle)$ be a pseudo-Euclidean real vector space of signature $(k, l)$, with $k \geq 2$, $l \geq 2$. In other words, $\langle \cdot, \cdot \rangle$ is a real, symmetric, non-degenerate bilinear form on $V$, which, when diagonalized by Sylvester's theorem, yields a diagonal form $(+1, \ldots, +1, -1, \ldots, -1)$ with $k$ plus ones and $l$ minus ones.

For any subspace $L \subseteq V$, denote by $q^+(L), q^-(L), q^0(L)$ the Sylvester's numbers of $(L, \langle \cdot, \cdot \rangle|_{L \times L})$, i.e. respectively the number of plus ones, the number of minus ones and the number of zeros occurring in a diagonalization of $\langle \cdot, \cdot \rangle|_{L \times L}$. We'll also denote by $\dim(L)$ the dimension of the subspace $L$, and we denote by $L^\perp$ the orthogonal subspace of $L$. For brevity, we'll call a subspace $L$ be a positive $r$-plane, if $L$ is $r$-dimensional and positive definite with respect to the inner product.

Consider now a subspace $Z \subseteq V$ with $q^+(Z) \geq 1$ and consider also a positive $(k - 1)$-plane $H \subseteq V$. In other words, $\dim(H) = q^+(H) = k - 1$, where, by assumption, $k = q^+(V)$. We introduce the following definitions:

Definition 3.1. (i) We say that $H$ is $Z$-extendable if there exists $z \in Z$ so that $H + \mathbb{R}z$ is a positive $k$-dimensional plane in $V$.

(ii) We say that $H$ is $Z$-orthogonally-extendable if there exists $z \in Z \cap H^\perp$ so that $H + \mathbb{R}z$ is a positive $k$-dimensional plane.

The goal of this section is to prove the following two results:

Proposition 3.2. Suppose $(V, \langle \cdot, \cdot \rangle)$ is a pseudo-Euclidean real vector space of signature $(k, l)$, with $k \geq 2$, $l \geq 2$, and suppose $Z \subseteq V$ is a subspace with $q^+(Z) \geq 1$. The following statements are equivalent:

(i) $q^+(Z^\perp) = 0$;

(ii) for any $H$ positive $(k - 1)$-plane, if $H$ is $Z$-extendable, then $H$ is also $Z$-orthogonally-extendable.

Proposition 3.3. Suppose $(V, \langle \cdot, \cdot \rangle)$ is a pseudo-Euclidean real vector space of signature $(k, l)$, with $k \geq 2$, $l \geq 2$, and suppose $Z \subseteq V$ is a subspace of $V$. The following statements are equivalent:
(i) \( q^+(Z) = q^+(V) = k; \)
(ii) any positive \((k-1)\)-plane \(H\) is \(Z\)-extendable;
(iii) any positive \((k-1)\)-plane \(H\) is \(Z\)-orthogonally-extendable.

3.1. A lemma for a Minkowski vector space. Let \((W, \langle \cdot, \cdot \rangle)\) be a Minkowski vector space with dimension at least 3, i.e. \(\langle \cdot, \cdot \rangle\) is a real, symmetric, non-degenerate inner product of signature \((1, l)\) with \(l \geq 2\). The convention we adopt for a Minkowski vector space is that when diagonalized the inner product has one plus one and the rest are minus ones.

Let \(C(W) = \{w \in W \mid \langle w, w \rangle \geq 0\}\) be the set of causal vectors in \(W\), i.e. time-like vectors \((w, so that \(\langle w, w \rangle > 0\)) or null vectors \((w, so that \(\langle w, w \rangle = 0\)). Note that \(C(W)\) is a closed convex cone. Without the origin, \(C(W) \setminus \{0\}\) has two connected components, which we denote \(C^+\) and \(C^-\). One notes immediately that \(C^- = -C^+\). Of course, which of the two components we denote \(C^+\) and which we denote \(C^-\) is just a convention. To give some motivation of this notation, note that if we fix an orthogonal basis for \(W\), \(\{e_0, e_1, ..., e_l\}\), with \(\langle e_0, e_0 \rangle = 1, \langle e_j, e_j \rangle = -1, for j = 1, ..., l\) and \(\langle e_a, e_b \rangle = 0\) for \(a \neq b \in \{0, 1, ..., l\}\), then we can choose \(C^+(W)\), by convention, to be the set of causal vectors \(w \in C(W)\) with \(\langle w, e_0 \rangle > 0\). Certainly the choice depends on the basis. If we replace \(e_0\) by \(-e_0\) in the fixed basis, we’ll obviously pick the other component as \(C^+\).

**Lemma 3.4.** (i) For any non-zero causal vectors \(u, v\) in the same connected component (e.g. \(u, v \in C^+(W)\)), we have \(\langle u, v \rangle \geq 0\) with equality if and only if \(u\) and \(v\) are proportional null vectors. If \(u, v\) are non-zero causal vectors in different components, then \(\langle u, v \rangle \leq 0\) with equality if and only if \(u\) and \(v\) are proportional null vectors.

(ii) If \(u \in W\), \(u \neq 0\), satisfies \(\langle u, v \rangle \geq 0\) for any \(v \in C^+(W)\), then \(u \in C^+(W)\).

(iii) If \(L \subseteq W\) is a subspace which contains no non-zero causal vector, then \(L^\perp\) must contain a time-like vector.

**Proof.** Part (i) follows from Cauchy-Schwarz inequality and is a standard fact for Minkowski vector spaces (e.g. see [23], p.3-4). Part (ii) can be also easily checked and we leave it to the reader. It might be also well known in the literature. For part (iii), note that if \(L\) does not contain any non-zero causal vector, it means that \(L\) is negatively defined with respect to the inner product. In particular, \(L\) is non-degenerate so \(L \oplus L^\perp = W\). Thus, \(L^\perp\) must contain a time-like vector. \(\square\)

3.2. Equivalent characterizations of the Z-extendable notions. As in the start of this section, let \((V, \langle \cdot, \cdot \rangle)\) be a pseudo-Euclidean real vector space of signature \((k, l)\), with \(k \geq 2, l \geq 2\). Let \(Z\) be a subspace of \(V\) with \(q^+(Z) \geq 1\) and let \(H\) be a positive \((k-1)\)-dimensional plane in \(V\). In this subsection we’ll prove some equivalent characterizations of the \(Z\)-extendable and \(Z\)-orthogonally extendable notions.

Note first that since \(H\) is a positive \((k-1)\)-dimensional plane, \(H\) is in particular non-degenerate, so

\[ V = H \oplus H^\perp, \]

where \(H^\perp\) with the induced inner product is a Minkowski space of signature \((1, l)\). We will apply Lemma 1 to \(H^\perp\), so \(C(H^\perp)\) will denote the set of causal vectors in \(H^\perp\) and \(C^+(H^\perp)\) will be one connected component of \(C(H^\perp) \setminus \{0\}\). Denote also

\[ \pi^H : V \to H, \quad \pi^{H^\perp} : V \to H^\perp, \]
the corresponding projections.

**Lemma 3.5.** With the notations above, the following are equivalent:

(i) $H$ is $Z$-extendable;
(ii) $q^+(H + Z) = k$;
(iii) $q^+((H + Z) \cap H^+) \geq 1$;
(iv) There exists $z \in Z$ such that $\langle z, u \rangle > 0$, for all $u \in C^+(H^+)$;
(v) $Z^+ \cap C^+(H^+) = \emptyset$.

**Proof.** The equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) are clear. We show next that (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (iii). For the first implication, assuming (iii), there exist $h \in H$, $z \in Z$, such that $h + z$ is a time-like vector in $H^+$, that is $h + z \in H^+$ and $\langle h + z, h + z \rangle > 0$. Without loss of generality, we can assume that $h + z$ is in the “positive” component $C^+(H^+)$. By Lemma 1 (i), we have

$$\langle h + z, u \rangle > 0, \quad \forall u \in C^+(H^+), \quad \text{hence} \quad \langle z, u \rangle > 0, \quad \forall u \in C^+(H^+),$$

as $h \in H$.

The implication (iv) $\Rightarrow$ (v) is immediate, since $\langle z, Z^+ \rangle = 0$.

For the last implication (v) $\Rightarrow$ (iii), note the equivalences

$$Z^+ \cap C^+(H^+) = \emptyset \Leftrightarrow (Z^+ \cap H^+) \cap C^+(H^+) = \emptyset \Leftrightarrow ((Z + H)^+ \cap H^+) \cap C^+(H^+) = \emptyset,$$

where for the last equivalence, we used that $Z^+ \cap H^+ = (Z + H)^+$. By Lemma 3.3 (iii), if $((Z + H)^+ \cap H^+) \cap C^+(H^+) = \emptyset$, then $((Z + H) \cap H^+) \cap C^+(H^+) \neq \emptyset$, so $q^+((H + Z) \cap H^+) \geq 1$.

\[\square\]

**Lemma 3.6.** With the notations above, the following are equivalent:

(i) $H$ is $Z$-orthogonally extendable;
(ii) $q^+(Z \cap H^+) \geq 1$;
(iii) $\pi^{H^+}(Z^+) \cap C^+(H^+) = \emptyset$.

**Proof.** The equivalence (i) $\Leftrightarrow$ (ii) is obvious. We prove (ii) $\Rightarrow$ (iii). Assume there is $z \in Z \cap H^+$, with $\langle z, z \rangle > 0$. Then $\exists z \in C^+(H^+)$ and without loss of generality, assume that $z \in C^+(H^+)$. Since, moreover $z$ is not a null-vector, by Lemma 3.4 it follows that $\langle z, u \rangle > 0$, for all $u \in C^+(H^+)$. Let $w \in Z^+$ and let $w = w^H + w^{H^+}$, where $w^H = \pi^H(w)$ and $w^{H^+} = \pi^{H^+}(w)$. Since $z \in Z \cap H^+$, $\langle z, w^H \rangle = \langle z, w \rangle = 0$. Thus relation (iii) must hold.

The implication (iii) $\Rightarrow$ (ii) follows directly from Lemma 3.4 (iii), simply noting that the orthogonal complement of $\pi^{H^+}(Z^+)$ in $H^+$ is just $Z \cap H^+$.

\[\square\]

We are now ready to prove Proposition 3.2.

**Proof of Proposition 3.2.** (i) $\Rightarrow$ (ii) Let $H$ be a $(k - 1)$-dimensional positive plane. Assume that $H$ is $Z$-extendable. By Lemma 3.5 this is equivalent with $Z^+ \cap C^+(H^+) = \emptyset$. Assume also that $H$ is not $Z$-orthogonally extendable. By Lemma 3.6 this means that there exists $u \in Z^+$, so that $\pi^{H^+}(u) \in C^+(H^+)$. For simplicity, denote $u^{H^+} = \pi^{H^+}(u)$ and $u^H = \pi^H(u)$. Obviously,

$$\langle u, u \rangle = \langle u^H, u^H \rangle + \langle u^{H^+}, u^{H^+} \rangle.$$
Since $u \in Z^+$, the assumption $q^+(Z^+) = 0$ implies that $\langle u, u \rangle \leq 0$. On the other hand, $\langle u^H, u^H \rangle \geq 0$ since $H$ is positive definite and $\langle u^{H^+}, u^{H^+} \rangle \geq 0$ since $u^{H^+} \in C^+(H^+)$. Thus, we must have

$$0 = \langle u, u \rangle = \langle u^H, u^H \rangle = \langle u^{H^+}, u^{H^+} \rangle.$$ 

Since $H$ is positive definite, it follows that $u^H = 0$. Thus $u = u^{H^+} \in Z^+ \cap C^+(H^+)$.

This contradicts $Z^+ \cap C^+(H^+) = \emptyset$. Hence $H$ must be $Z$-orthogonally extendable and the first implication is proved.

\( (ii) \Rightarrow (i) \) We’ll prove the counter-positive – assume that $q^+(Z^+) \geq 1$ and construct a $(k - 1)$-positive plane which is $Z$-extendable, but not $Z$-orthogonally-extendable. Denote by $r = q^+(Z) \geq 1$. Because of the assumption $q^+(Z^+) \geq 1$, note that $r \leq k - 1$. Consider now an $r$-positive plane in $Z$ and let $\{\omega_1, \ldots, \omega_r\}$ be an orthonormal basis for this plane. Next, pick $u \in Z^+$ with $\langle u, u \rangle = 1$ (such $u$ exists by the assumption $q^+(Z^+) \geq 1$). Extend $\{\omega_1, \ldots, \omega_r, u\}$ to an orthonormal basis of a positive $k$-plane in $V$, $\{\omega_1, \ldots, \omega_r, u, \eta_{r+2}, \ldots, \eta_k\}$. This is possible since $L = \text{Span}\{\omega_1, \ldots, \omega_r, u\}$ is a positive $(r+1)$-plane in $V$, hence non-degenerate. One can then pick an orthogonal basis for $L^\perp$ and extract $\{\eta_{r+2}, \ldots, \eta_k\}$ the positive vectors in this basis. Define the positive $(k - 1)$-plane

$$H = \text{Span}\{u + \omega_1, \omega_2, \ldots, \omega_r, \eta_{r+2}, \ldots, \eta_k\}.$$ 

It is then evident that $H$ is $Z$-extendable because

$$H + \Re \omega_1 = \text{Span}\{\omega_1, \ldots, \omega_r, u, \eta_{r+2}, \ldots, \eta_k\}.$$ 

We claim that $H$ is not $Z$-orthogonally-extendable. Indeed, if $H$ were $Z$-orthogonally-extendable, then there is a vector $\omega \in Z \cap H^+$ such that $H + \Re \omega$ is a positive $k$-dimensional. By the definition of $H$ and since $\omega \in Z$, $u \in Z^+$, we have

$$0 = \langle \omega, \omega_1 \rangle \text{ for all } i \geq 2,$$

$$0 = \langle \omega, u + \omega_1 \rangle = \langle \omega, u \rangle + \langle \omega, \omega_1 \rangle = \langle \omega, \omega_1 \rangle.$$ 

Therefore, the span of $\{\omega, \omega_1, \ldots, \omega_r\}$ is an $(r + 1)$-dimensional positive plane contained in $Z$, contradicting the fact that $q^+(Z) = r$. This completes the proof of the second implication and, thus, of the theorem. \( \square \)

Next we prove Proposition 3.3.

Proof of Proposition 3.3: \( (i) \Rightarrow (ii) \) Follows directly from Lemma 3.5 as for any $H$ we have $q^+(H + Z) = q^+(Z) = k$.

\( (ii) \Rightarrow (i) \) The argument is very similar to the corresponding implication in Proposition 3.2. Suppose $q^+(Z) = r \leq k - 1$. Consider a positive $(k - 1)$-plane $H$ that contains a positive $r$-dimensional subspace of $Z$. Then $H$ cannot be $Z$-extendable, as this would imply that $Z$ contains a positive $(r + 1)$-dimensional plane.

\( (i) \Rightarrow (iii) \) Observe that $q^+(Z) = k$ immediately implies $q^+(Z^+) = 0$ and now use $(i) \Rightarrow (ii)$ and Proposition 3.2.

As implication $(iii) \Rightarrow (ii)$ is obvious, the proof is complete. \( \square \)
4. Proofs of the main results

Proof of Theorem 1.2: Theorem 1.2 follows directly from Proposition 3.2. Just take \((V, \langle \cdot, \cdot \rangle)\) be \((\Lambda^2(g^*), \Phi^\mu)\) and take the subspace \(Z\) be the space of closed 2-forms \(Z^2\). Modulo sign (which is not important), an almost complex structure \(J\) is identified with its positive 2-plane of \(J\)-anti-invariant forms \(H = \Lambda_j^\perp\), and is clear from Proposition 2.1 that the \(Z\)-extendable condition in this context is equivalent to the tame condition for \(J\), while the \(Z\)-orthogonally-extendable condition is just the compatibility of \(J\) with a symplectic form. Note also that \((Z^2)^\perp\) is isomorphic with the space of boundary vectors \(B_2\), via the Riesz map induced by \(\Phi^\mu\). \(\square\)

Proof of Corollary 1.3: Using the classification of symplectic 4-dimensional Lie algebras of Ovando [22], one checks that the only cases when the condition of Theorem 1.2 is not satisfied are the Lie algebra \(\mathfrak{r}_2\mathfrak{r}_2\) endowed with either orientation and the Lie algebra \(\mathfrak{d}_{4,2}\) endowed with the non-complex orientation.

The first case, \(\mathfrak{g} = \mathfrak{r}_2\mathfrak{r}_2\) is characterized by a basis \(\{e_1, e_2, e_3, e_4\}\) such that \([e_1, e_2] = e_2, [e_3, e_4] = e_4\). (It is the Lie algebra of the group \(\text{Aff} \mathbb{R} \times \text{Aff} \mathbb{R}\), where \(\text{Aff} \mathbb{R}\) is the Lie group of affine transformations of the Euclidean line and \(\mathfrak{r}_2 = \text{aff} \mathbb{R}\) is the unique non-abelian 2-dimensional Lie algebra.) Alternatively, if \(\{e^1, e^2, e^3, e^4\}\) is the dual basis of \(\mathfrak{g}^* = \Lambda^1(\mathfrak{g}^*)\),

\[
de^1 = 0 \quad de^2 = -e^{12} \quad de^3 = 0 \quad de^4 = -e^{34}.
\]

It is easily checked that the spaces of closed 2-forms and boundary 2-vectors are given by

\[
Z^2 = \text{Span}(e^{12}, e^{34}) \quad B_2 = \text{Span}(e_{14}, e_{23}, e_{24}),
\]

Note that if the orientation on \(\mathfrak{g}\) is given by \(\mu = e^{1234}\), then \(\Phi_\mu(e_{14} - e_{23}, e_{14} - e_{23}) = -2 < 0\), and if \(\mu = -e^{1234}\), then \(\Phi_\mu(e_{14} + e_{23}, e_{14} + e_{23}) = -2 < 0\).

Here is a concrete example of an almost complex structure on \(\mathfrak{r}_2\mathfrak{r}_2\) (with orientation \(\mu = e^{1234}\)), which is tamed but not compatible. Let \(J\) be given by

\[
\Lambda_j^\perp = \text{Span} \left((e^{12} + e^{34}) + (e^{14} + e^{23}), (e^{13} - e^{24})\right).
\]

On vectors, \(J\) is explicitly given by

\[
J e_1 = \frac{1}{\sqrt{2}}(e_2 - e_4), \quad J e_2 = \frac{1}{\sqrt{2}}(-e_1 - e_3), \quad J e_3 = \frac{1}{\sqrt{2}}(e_4 + e_2), \quad J e_4 = \frac{1}{\sqrt{2}}(-e_3 + e_1).
\]

Then \(J\) is tamed by the symplectic form \(\omega_0 = e^{12} + e^{34}\). On the other hand, \(J\) is not compatible with any symplectic form. Indeed, a positively oriented symplectic form on \(\mathfrak{r}_2\mathfrak{r}_2\) with orientation \(\mu = e^{1234}\) is of the form

\[
\omega = ae^1 \wedge e^2 + be^1 \wedge e^3 + ce^3 \wedge e^4, \quad \text{with } a, b, c \in \mathbb{R}, ac > 0.
\]

But the condition that \(\omega\) be orthogonal to \(\Lambda_j^\perp\) is \(a + c = 0, b = 0\), thus, \(ac > 0\) cannot be satisfied.

The second case, the Lie algebra \(\mathfrak{d}_{4,2}\) is given by the non-zero Lie brackets:

\[
\mathfrak{d}_{4,2} : \ [e_1, e_2] = e_3, \ [e_4, e_3] = e_3, \ [e_4, e_1] = 2e_1, \ [e_4, e_2] = -e_2.
\]

The spaces of closed two-forms \(Z^2\) and boundary two-vectors \(B_2\) are given respectively by

\[
Z^2 = \text{Span}(e^{12} - e^{34}, e^{14}, e^{23}), \quad B_2 = \text{Span}(e_{12} + e_{34}, e_{13}).
\]
Note that if the orientation is given by $\mu = -e^{1234}$ then $\Phi_\mu$ is non-positive definite on $B_2$, hence with this orientation $\mathcal{O}_{4,2}$ satisfies the tame-compatible property. However, if the orientation is given by $\mu = e^{1234}$, then $\Phi_\mu(e_{12} + e_{34}, e_{12} + e_{34}) > 0$, so according to Theorem 1.2 there are almost complex structures inducing this orientation which are tamed but not compatible with symplectic forms. A concrete example of such $J$ can be again given by

$$\Lambda_J = \text{Span} \left( (e^{12} + e^{34}) + (e^{14} + e^{23}), (e^{13} - e^{24}) \right).$$

As in the case above, the reader can check directly that $J$ is tamed by, but not compatible with symplectic forms on $\mathcal{O}_{4,2}$, with the orientation given by $\mu = e^{1234}$.

Remark: Using the classification of Ovando [22], in [6] it was shown that Question 1.1 has an affirmative answer for 4-dimensional Lie algebras endowed with an integrable almost complex structure $(\mathfrak{g}, J)$. Certainly Theorem 1.2 offers an alternative way of eliminating most cases, but note that $\mathfrak{t}_2\mathfrak{t}_2$ does admit complex structures. One checks separately that these complex structures are compatible with symplectic forms, so $\mathfrak{t}_2\mathfrak{t}_2$ carries Kähler structures, but also carries almost complex structures which are tamed and non-compatible.

As we mentioned in the introduction, a natural further question is if Theorem 1.2 and Corollary 1.3 have any kind of extension in higher dimensions. We do not know the answer and leave this for further work. The linear algebra setup of section 3 applies only to the 4-dimensional tame-compatible problem, as we use the description of almost complex structures via positive planes of 2-forms, which is particular to dimension 4.

We end with the observation that abelian Lie algebras are the only oriented even-dimensional Lie algebras with the property that all almost complex structures are tamed by (and compatible with) symplectic forms.

**Proposition 4.1.** If $\mathfrak{g}$ is a $2n$-dimensional ($2n \geq 4$), non-abelian, oriented Lie algebra, then there exists an almost complex structure $J$ on $\mathfrak{g}$ which is not tamed by a symplectic form.

**Proof.** If the Lie algebra $\mathfrak{g}$ does not carry symplectic forms then the statement is obvious. If the Lie algebra $\mathfrak{g}$ carries symplectic forms then $\mathfrak{g}$ must be solvable [16], hence, the center $\xi$ is nontrivial. We then use Lemma 3.2 of [8], which we restate here for the convenience of the reader. Note that in the original statement in [8] $J$ is assumed complex, but integrability is not used anywhere in the proof:

**Lemma:** (Lemma 3.2 in [8]) Let $\mathfrak{g}$ be a real Lie algebra endowed with an almost complex structure $J$ such that $J\xi \cap [\mathfrak{g}, \mathfrak{g}] \neq \{0\}$, where $\xi$ denotes the center of $\mathfrak{g}$. Then $(\mathfrak{g}, J)$ cannot admit a tamed symplectic structure.

Now Proposition 4.1 is immediate. Pick non-zero vectors $u \in \xi$, $v \in [\mathfrak{g}, \mathfrak{g}]$, define $Ju = v$, $Jv = -u$ and then extend $J$ to an almost complex structure on $\mathfrak{g}$. By the Lemma, $J$ cannot be tamed. □

Remark: In dimension 4, Proposition 4.1 can be also seen as a consequence of Proposition 3.3, with the additional observation that a 4-dimensional Lie algebra $\mathfrak{g}$ with $q^+(\mathbb{Z}^2) = 3$ must be abelian. The observation is true because the condition $q^+(\mathbb{Z}^2) = 3$ implies the existence of a triple of symplectic forms $\omega_i$ orthogonal with
respect to $\Phi^\mu$ and with $\omega^2_i = \mu$. Then Hitchin’s lemma [12] implies that $\mathfrak{g}$ must carry a hyperKähler structure. But in dimension four, the only such Lie algebra is the abelian one, as it follows from the general description of hyperKähler Lie algebras given by [3].

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